Generalized recurrence relations for two-loop propagator integrals with arbitrary masses

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Abstract

An algorithm for calculating two-loop propagator type Feynman diagrams with arbitrary masses and external momentum is proposed. Recurrence relations allowing to express any scalar integral in terms of basic integrals are given. A minimal set consisting of 15 essentially two-loop and 15 products of one-loop basic integrals is found. Tensor integrals and integrals with irreducible numerators are represented as a combination of scalar ones with a higher space-time dimension which are reduced to the basic set by using the generalized recurrence relations proposed in [1].
1 Introduction

Mass effects play an important role in confronting experimental data obtained at the high-energy colliders, like LEP and SLC, with theoretical predictions. Precise determinations of many physical parameters in the Standard Model (SM) require the evaluation of the mass dependent radiative corrections.

At the one-loop level the problem of calculating Feynman integrals with several masses was in principle solved in [2]. The history of two-loop calculations is rather long. First results were reported more than forty years ago [3], [4]. During the last years a number of approaches for the evaluation of two-loop diagrams with massive particles were proposed [5] – [12]. Integals with specific topologies or mass combinations were considered in [13] – [20]. Most results for scalar master integrals were obtained using dispersion relations. The first generalization of the method of integration by parts [27] to massive integrals was done for the on-shell diagrams with one massive particle [28]. The evaluation of the two-loop correction for the photon propagator with the aid of recurrence relations was performed in [17]. Up to now all attempts to extend the standard method of integration by parts [27] to two-loop diagrams with arbitrary external momentum and all masses different were unsuccessful. For a short review of some techniques for calculating two-loop diagrams, see for instance Ref. [29].

Despite these efforts even for propagator type two-loop diagrams no satisfactory solution similar to the one-loop case is available so far. In the SM, due to the complicated structure of integrals and the large number of diagrams (typically thousands), no complete calculation of a two-loop self-energy has been carried out. Many different species of particles with different masses have to be taken into account and this makes the evaluation of two-loop integrals a rather difficult problem. Existing numerical methods for evaluating two-loop diagrams cannot guarantee required accuracy for the sum of thousands of diagrams. More promising may be the semi-analytic method proposed in Ref. [8]. It is based on momentum expansion, conformal mapping and the use of Padé approximants.

In calculating Feynman diagrams mainly three difficulties arise: tensor decomposition of integrals, reduction of scalar integrals to several basic integrals and the evaluation of scalar basic integrals.

The first two problems were addressed in Refs. [9], [10] where an algorithm for the analytic calculation of two-loop diagrams with several masses was proposed. This algorithm, in principle, allows one to express any diagram in terms of several scalar integrals but in our opinion the problem was not solved completely. In calculating the tensor integrals, the authors of Ref. [9] have to introduce in addition to the original five propagators with different masses a fictitious massless propagator. As a consequence, in the final result there are not only integrals with the original masses but in general also integrals having an extra massless propagator. These integrals may correspond to diagrams having a topology which is not inherent to the original diagram under consideration. The minimal set of independent basic integrals was not found in [9]. In fact, the algorithm of Ref. [9] allows only partial reduction and reveals even in the simplest case with one massive particle an overcomplicated result. For example, the two-loop correction to the photon propagator in [9] is given in terms of 4 two-loop scalar integrals, though there must be only two [17]. The position of the nearest threshold
singularity of the two additional integrals is not characteristic to the two-loop photon propagator. Without further simplification of the result additional integrals may cause potentially large numerical cancellations.

Recently in Ref. [30] a framework for treating all two-loop diagrams which occur in a renormalizable theory was developed. To reduce tensor integrals to scalar ones the authors introduced as a basis ten scalar integrals. For nonrenormalizable theories they predict the impossibility to isolate a finite number of functions from which all other can be obtained by differentiation. As will be seen from our paper there is no difference between two-loop propagator integrals occurring in renormalizable and non-renormalizable theories. Any two-loop, so-called "London transport" (or "sunset") type diagram with arbitrary masses considered in [30] can be expressed as a combination of four two-loop integrals and products of one-loop tadpole integrals.

In the present paper we propose a method which completely solves for propagator diagrams the first of the two aforementioned problems. We formulate an algorithm for transforming tensor integrals into a combination of scalar ones. In fact, this is a specification to the two-loop case of the general algorithm given in Ref. [1]. We show how to reduce all scalar integrals occurring in the computation of an arbitrary two-loop diagram to a sum over a minimal set of basic integrals with rational coefficients depending on masses, momenta and the dimensionality of space-time $d$. All recurrence relations needed for the reduction of a variety of Feynman integrals to basic integrals are presented. The minimal set of basic integrals is given.

The paper is organized as follows. In Sec.2 we describe an optimal procedure for the simplification of the integrand of a diagram. In Sec. 3 a method for the representation of tensor integrals and integrals with irreducible numerators in terms of scalar ones with shifted $d$ is presented. In Sec. 4 generalized recurrence relations which are necessary for the reduction of two-loop integrals to basic ones are given. In Appendix a complete set of basic integrals is given.

2 General notation and integrand simplification

The subject of our consideration will be two-loop two-point dimensionally regulated diagrams with arbitrary masses. In principle, with the method presented in this article one can treat the diagram in tensor form. In Sec. 3 an algorithm for the tensor decomposition of individual integrals will be given. Technically the evaluation of diagrams in tensor form is more involved than the evaluation of scalar diagrams. In practice one need to know the coefficient of the particular tensor structure of the diagram. After contracting a diagram with an appropriate projection operator, performing traces we commonly encounter integrals of the form:

$$I^{(d)}(q^2) = \frac{1}{\pi^d} \int \int \frac{d^dk_1 d^dk_2}{c_1^{x_1} c_2^{x_2} c_3^{x_3} c_4^{x_4} c_5^{x_5}} N(k_1^2, k_2^2, k_1 q, k_2 q, k_1 k_2),$$

where $N(k_1^2, k_2^2, k_1 q, k_2 q, k_1 k_2)$ is a polynomial and
\[
c_1 = k_1^2 - m_1^2 + i\epsilon, \quad c_3 = (k_1 - q)^2 - m_3^2 + i\epsilon, \quad c_5 = (k_1 - k_2)^2 - m_5^2 + i\epsilon,
\]
\[
c_2 = k_2^2 - m_2^2 + i\epsilon, \quad c_4 = (k_2 - q)^2 - m_4^2 + i\epsilon.
\]

For brevity, we shall omit the “causal” \(i\epsilon\)'s below. Individual integrals can be specified by the dimensionality of the space-time, \(d\) and by the powers of denominators \(\nu_i\), called indices of the lines. Figure 1 shows a generic topology of the two-loop two-point diagram with lines labeled as indicated. We assume that to the \(i\)th line here, corresponds a factor \(1/c_i^{\nu_i}\). In the case of integer \(\nu\)'s any two-loop diagram with different topology can also be reduced to integrals of the form (1), if we use the partial fraction decomposition for denominators having the same momenta but different masses.

![Figure 1. The master two-loop two-point diagram](image)

The evaluation of the integrals (1) will be performed in several steps. First, we simplify the integrand as much as possible, by bringing it to a standard form suitable for further calculations. Second, one has to get rid off the numerator \(N(k_1^2, k_2^2, k_1q, k_2q, k_1k_2)\) by introducing scalar integrals with a higher space-time dimension whenever needed. Third, by turns, two-loop integrals with five, four and three lines having different powers of propagators will be reduced to a set of basic integrals. Fourth, integrals in higher dimensions must be reduced to a basic set of integrals in the generic dimension \(d\). Finally, one-loop propagator and tadpole integrals should be evaluated.

Putting the integrand into a useful form is just a matter of tedious algebra. We start the simplification with the repeated use of the substitutions:

\[
\begin{align*}
\frac{(k_1k_2)^\alpha}{c_5^{\nu_5}} &= \frac{(k_1k_2)^\alpha}{c_5^{\nu_5}} \left( \frac{k_1^2 - c_5 + k_2^2 - m_5^2}{2 k_1k_2} \right)^{<\alpha,\nu_5>}, \\
\frac{(k_2q)^\alpha}{c_4^{\nu_4}} &= \frac{(k_2q)^\alpha}{c_4^{\nu_4}} \left( \frac{k_2^2 - c_4 + q^2 - m_4^2}{2 k_2q} \right)^{<\alpha,\nu_4>}, \\
\frac{(k_1q)^\alpha}{c_3^{\nu_3}} &= \frac{(k_1q)^\alpha}{c_3^{\nu_3}} \left( \frac{k_1^2 - c_3 + q^2 - m_3^2}{2 k_1q} \right)^{<\alpha,\nu_3>}, \\
\frac{(k_2)^\alpha}{c_2^{\nu_2}} &= \frac{(k_2)^\alpha}{c_2^{\nu_2}} \left( \frac{c_2 + m_2^2}{k_2^2} \right)^{<\alpha,\nu_2>}, \\
\frac{(k_1)^\alpha}{c_1^{\nu_1}} &= \frac{(k_1)^\alpha}{c_1^{\nu_1}} \left( \frac{c_1 + m_1^2}{k_1^2} \right)^{<\alpha,\nu_1>},
\end{align*}
\]

where we used the notation:

\(<\alpha, \nu_i> = \min(\alpha, \nu_i)\).
After performing these substitutions scalar products in the numerator remain only in the case when at least one line is eliminated. For integrals of this kind the simplification of the integrand proceeds as follows.

The scalar product \((k_1 k_2)\) remains in the numerator only if invariant \(c_5\) is cancelled, i.e. the integral is in fact the product of one-loop tensor integrals. In this case the substitution:

\[ k_1 k_2 = A(k_1, k_2) + \frac{(k_1q)(k_2q)}{q^2}, \tag{3} \]

where

\[ A(k_1, k_2) = k_{1\mu} \left( g_{\mu\nu} - \frac{q_{\mu}q_{\nu}}{q^2} \right) k_{2\nu}, \tag{4} \]

allows one to transform the integral into a sum of products of one-loop scalar integrals. Integrals with \(A_l(k_1, k_2)\), where \(l\) is odd (i.e. with odd powers of \(k_1\) or \(k_2\) in the numerator) are zero since the transverse tensor standing in braces of Eq.(4) will be always multiplied by an external momentum \(q\). Integrals with even powers of \(A(k_1, k_2)\) are:

\[
\begin{align*}
&\int \int d^d k_1 d^d k_2 f_1(k_1, q, m_i) f_2(k_2, q, m_i) A^{2n}(k_1, k_2) = \\
&\left. \frac{\Gamma(n + \frac{d}{2}) \Gamma \left( \frac{d-1}{2} \right)}{\Gamma \left( \frac{d}{2} \right) \Gamma \left( n + \frac{d-1}{2} \right)} \right) \times \\
&\int \int d^d k_1 d^d k_2 f_1(k_1, q, m_i) f_2(k_2, q, m_i) A^n(k_1, k_1) A^n(k_2, k_2). \tag{5}
\end{align*}
\]

Once the factor \(A(k_1, k_2)\) is removed the integrations with respect to \(k_1\) and \(k_2\) are completely decoupled.

The integrand can be further simplified by applying the substitutions:

\[
\begin{align*}
&\frac{(k^2_1)^\alpha}{c_3^\alpha} = \frac{(k^2_2)^\alpha}{c_3^\alpha} = \frac{(c_3 + 2k_1q - q^2 + m_3^2)}{k_1^2} <^{\alpha,\nu_1}> , \tag{6} \\
&\frac{(k_2^2)^\alpha}{c_4^\alpha} = \frac{(k^2_2)^\alpha}{c_4^\alpha} = \frac{(c_4 + 2k_2q - q^2 + m_4^2)}{k_2^2} <^{\alpha,\nu_4}> . \tag{7}
\end{align*}
\]

After these substitutions have been made, \(k_1^2, k_2^2\) will remain in the numerator only in bubble-like integrals. Integrals with irreducible numerators reveal themselves as integrals having scalar products \((k_1q)^\alpha\) and/or \((k_2q)^\beta\) in the numerator. In principle it is possible to eliminate \((k_1q)\), \((k_2q)\) at the expense of \(k_1^2, k_2^2\). As we shall see later, integrals with \((k_1q)\), \((k_2q)\) will produce scalar integrals with a smaller shift of \(d\) (and therefore simpler expressions for further computation) than integrals with \(k_1^2, k_2^2\).

If \(c_3\) and \(c_4\) are absent in the integrand then the scalar products \((k_1q)\), \((k_2q)\) can be eliminated. In this case, for an arbitrary scalar function \(f(k_1, k_2)\),

\[
\int \int d^d k_1 d^d k_2 f(k_1, k_2)(k_1q)^{\nu_1} (k_2q)^{\nu_2} = v(\nu_1, \nu_2), \tag{8}
\]

is zero if \(\nu_1 + \nu_2\) is odd. For \(\nu_1 + \nu_2\) even, the scalar products \((k_1q)\), \((k_2q)\) can be eliminated from the integrand applying the recurrence relations derived in [31]:

\[
(d + \nu_1 + \nu_2 - 2)v(\nu_1, \nu_2) = q^2 \{(\nu_1 - 1)k_1^2 1^- + \nu_2(k_1k_2)2^- \} 1^- \circ v(\nu_1, \nu_2), \quad (\nu_1 > 1) \tag{9}
\]
(d + \nu_1 + \nu_2 - 2)v(\nu_1, \nu_2) = q^2\{(\nu_2 - 1)k_1^2 2^{-} + \nu_1 (k_1 k_2) 1^{-}\} 2^{-} \circ v(\nu_1, \nu_2), \quad (\nu_2 > 1), \quad (10)

where $1^\pm v(\nu_1, \nu_2) \equiv v(\nu_1 \pm 1, \nu_2)$ etc., and the $\circ$ sign means that factors $k_1^2, k_2^2, k_1 k_2$ in braces must be considered under the integral sign in $v$. To simplify one-loop tadpole-like integrals, rather frequently, application of the simple formula

$$\int \frac{d^d k_1}{(k_1^2 - m_1^2)^{\nu_1}} (2k_1 q)^{2\nu_1} = \frac{(2\nu_1)!}{(d/2)^{\nu_1} (\nu_1)!} (q^2)^{\nu_1} \int \frac{d^d k_1}{(k_1^2 - m_1^2)^{\nu_2}} (k_1^2)^{\nu_1}, \quad (11)$$

turns out to be useful.

Using the formulas presented in this section, a scalar contribution (11) will be transformed into a sum of integrals with irreducible numerators $(k_1 q)^a (k_2 q)^b$ having at maximum four lines and a variety of scalar integrals with different indices. Integrals with irreducible numerators $(k_1 q), (k_2 q)$ will be considered in the next section. Notice that, the integrals (8) can also be regarded as integrals with irreducible numerators and they can be treated according to the formalism discussed in the next section.

### 3 Tensor integrals and irreducible numerators

Tensor integrals can be written as a combination of scalar integrals with shifted space-time dimension multiplied by tensor structures made from external momenta and the metric tensor. This was shown in Ref. [32] for the one-loop case and in Ref. [1] for an arbitrary case. In the approach of Ref. [1], the reduction of tensor integrals to such a representation is performed by applying a tensor operator to a scalar integral. For the two-loop tensor integrals of interest the following relation holds:

$$\int \int \frac{d^d k_1}{c_1 \nu_1 c_2 \nu_2 c_3 \nu_3 c_4 \nu_4 c_5 \nu_5} \frac{d^d k_2}{c_1 \nu_1 c_2 \nu_2 c_3 \nu_3 c_4 \nu_4 c_5 \nu_5} k_{1\mu_1} \ldots k_{1\mu_r} k_{2\lambda_1} \ldots k_{2\lambda_s} = \mathcal{T}_{\mu_1 \ldots \lambda_s}(q, \{\partial_j\}, d^+) \int \int \frac{d^d k_1}{c_1 \nu_1 c_2 \nu_2 c_3 \nu_3 c_4 \nu_4 c_5 \nu_5}, \quad (12)$$

where

$$\partial_j = \frac{\partial}{\partial m_j^2},$$

and $d^+$ is the operator shifting the value of the space-time dimension of the integral by two-units: $d^+ I^{(d)} = I^{(d+2)}$. On the right-hand side of Eq. (12) it is assumed that, at the beginning, invariants $c_i$ have different nonzero masses and after differentiation with respect to $m_i^2$ they must be set to their original values.

To illustrate the method of Ref. [1] we will derive here an explicit expression for the tensor operator $\mathcal{T}_{\mu_1 \ldots \lambda_s}(q, \{\partial_j\}, d^+)$. The main ingredients of the derivation are independent auxiliary vectors $a_1, a_2$ and the use of the $\alpha$-parametric representation [33]. The tensor structure of the integrand on the left-hand side of (12) can be written as

$$k_{1\mu_1} \ldots k_{1\mu_r} k_{2\lambda_1} \ldots k_{2\lambda_s} = \frac{1}{i^{r+s}} \frac{\partial}{\partial a_{1\mu_1}} \ldots \frac{\partial}{\partial a_{1\mu_r}} \frac{\partial}{\partial a_{2\lambda_1}} \ldots \frac{\partial}{\partial a_{2\lambda_s}} \exp [i(a_1 k_1 + a_2 k_2)] \bigg|_{a_i=0} \quad (13)$$

To convert the integral

$$G^{(d)}(q^2) = \int \int \frac{d^d k_1 d^d k_2}{c_1 c_2 c_3 c_4 c_5} \exp [i(a_1 k_1 + a_2 k_2)]. \quad (14)$$
into the $\alpha$-parametric representation we apply standard methods (see for example, [33]). Transforming all propagators into a parametric form

$$\frac{1}{(k^2 - m^2 + i\epsilon)^\nu} = \frac{i^{\nu - 1}}{\Gamma(\nu)} \int_0^\infty d\alpha \, \alpha^{-\nu - 1} \exp \left[ i\alpha(k^2 - m^2 + i\epsilon) \right],$$

(15)

and using the $d$-dimensional Gaussian integration formula

$$\int d^d k \exp \left[ i(Ak^2 + 2(pk)) \right] = i \left( \frac{\pi}{iA} \right)^{d/2} \exp \left[ -i\frac{p^2}{A} \right],$$

(16)

we can easily evaluate the integrals over loop momenta. The final result is:

$$G^{(d)}(q^2) = i^2 \left( \frac{\pi}{7} \right)^5 \prod_{j=1}^5 \frac{i^{-\nu_j}}{\Gamma(\nu_j)} \int_0^\infty \cdots \int_0^\infty \prod_{l=1}^5 \frac{d\alpha_l \alpha_l^{-1}}{[D(\alpha)]^{5/4}} \exp \left[ i \left( \frac{Q(\alpha, a_1, a_2)}{D(\alpha)} - \sum_{l=1}^5 \alpha_l (m_l^2 - i\epsilon) \right) \right],$$

(17)

where

$$D(\alpha) = \alpha_5(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) + (\alpha_1 + \alpha_3)(\alpha_2 + \alpha_4),$$

(18)

$$Q(\alpha, a_1, a_2) = (\alpha_1 + \alpha_2)(\alpha_3 + \alpha_4) + \alpha_1 \alpha_2(\alpha_3 + \alpha_4) + \alpha_3 \alpha_4(\alpha_1 + \alpha_2) + q^2 + (qa_1)Q_1 + (qa_2)Q_2 + a_1^2Q_{11} + a_2^2Q_{22} + (a_1a_2)Q_{12},$$

(19)

and

$$Q_1 = \alpha_3 \alpha_5 + \alpha_2 \alpha_4 + \alpha_2 \alpha_3 + \alpha_3 \alpha_4,$$

$$Q_2 = \alpha_3 \alpha_5 + \alpha_3 \alpha_5 + \alpha_1 \alpha_4 + \alpha_3 \alpha_4,$$

$$-4Q_{11} = \alpha_2 + \alpha_4 + \alpha_5,$$

$$-4Q_{22} = \alpha_1 + \alpha_3 + \alpha_5,$$

$$-2Q_{12} = \alpha_5.$$ 

(20)

It is straightforward to work out from (12), (13) and the explicit expression (17) that

$$T_{\mu_1 \ldots \mu_n}(q, \{\partial\}, d^+) = \frac{1}{i^{\nu + s} \prod_{j=1}^r \frac{\partial}{\partial a_{1\mu_j}} \cdots \prod_{n=1}^s \frac{\partial}{\partial a_{2\lambda_n}}}$$

$$\times \exp \left[ i \left( (qa_1)Q_1 + (qa_2)Q_2 + a_1^2Q_{11} + a_2^2Q_{22} + (a_1a_2)Q_{12} \right) \rho \right] \bigg|_{\rho = a_{j=0} \frac{1}{\pi^2} d^+}.$$ 

(21)

This operator is a particular case of the more general one derived in Ref. [1]. Formula (12) can be used for the direct evaluation of two-loop tensor integrals. Notice that in order to obtain the tensor decomposition of the integral no contractions with external momenta and the metric tensor and no solution of a linear system of equations are needed.

Integrals with irreducible numerators

$$I_{rs}^{(d)}(q, k_1, k_2) = \int d^dk_1 d^dk_2 \frac{1}{c_1^{\nu_1} c_2^{\nu_2} c_3^{\nu_3} c_4^{\nu_4} c_5^{\nu_5}} (k_1q)^r (k_2q)^s,$$

(22)
can be regarded as a contraction of the tensor integral (12) with $q_{\mu_1} \ldots q_{\nu_s}$. For the scalar integrals (22) a somewhat simpler formula, analogous to (12), can be derived by introducing auxiliary scalar parameters $\beta_j$. Similar to (13) we write

$$\left( k_1 q \right)^s (k_2 q)^s = \frac{\partial^r}{(i\partial\beta_1)^r (i\partial\beta_2)^s} \exp \{ i [\beta_1 (k_1 q) + \beta_2 (k_2 q)] \} \bigg|_{\beta_i=0}.$$  

Carrying through the steps which were leading us to (21), we find the relation:

$$I_{rs}^{(d)} = T_{rs}(q, \{ \partial_j \}, d^+) I_{00}^{(d)},$$  

where

$$T_{rs}(q, \{ \partial_j \}, d^+) = \frac{1}{i^{r+s}} \frac{\partial^r}{\partial \beta_1^r} \frac{\partial^s}{\partial \beta_2^s} \times \exp \left\{ i q^2 [Q_1 \beta_1 + Q_2 \beta_2 + Q_{11} \beta_1^2 + Q_{22} \beta_2^2 + Q_{12} \beta_1 \beta_2] \rho \right\} \bigg|_{\beta_i=0, \alpha_j=0, \rho=-\frac{1}{\pi^2} d^+}.$$  

with $Q_i, Q_{ij}$ given in (20).

The evaluation of the scalar integrals in the form (24) is more efficient than the use of its tensor analog (12). Notice that, as we mentioned in the previous section, irreducible numerators can appear only if at least one of lines 1 to 4 is contracted. In this case the operator $T_{rs}(q, \{ \partial_j \}, d^+)$ becomes simpler because one can set equal to zero those $\alpha$ parameters which correspond to contracted lines. One can easily build up transformation operators $T_{\mu_1 \ldots \lambda_4}(q, \{ \partial_j \})$ or $T_{rs}(q, \{ \partial_j \}, d^+)$ by using any computer algebra system. Running FORM [34] on a PC Pentium 90 it takes usually several seconds (or minutes in complicated cases) to construct these operators.

As an application of the formalism presented in this section, let us consider a typical integral with an irreducible numerator:

$$I_{11} = \int \int \frac{d^d k_1 d^d k_2}{c_2 c_3 c_5} (k_1 q)(k_2 q).$$  

For this particular case, dropping irrelevant terms with $\beta^2_1, \beta^2_2$ in (23), the transformation operator reads:

$$T_{11} = -\frac{\partial}{\partial \beta_1} \frac{\partial}{\partial \beta_2} \exp \left\{ i q^2 [\beta_1 Q_1 + \beta_2 Q_2 + \beta_1 \beta_2 Q_{12}] \rho \right\} \bigg|_{\beta_i=0, \alpha_j=0, \rho=-\frac{1}{\pi^2} d^+}.$$  

Since $c_1$ and $c_4$ are absent in the integrand (i.e. corresponding lines are contracted), we have to set $\alpha_1 = \alpha_4 = 0$ and thus obtain from (20):

$$Q_1 = \alpha_3 (\alpha_2 + \alpha_5), \quad Q_2 = \alpha_3 \alpha_5, \quad Q_{12} = -\frac{1}{2} \alpha_5.$$  

Substituting these expressions into (27) we get

$$T_{11} = \frac{q^2}{2 \pi^2} d^+ \partial_5 + \frac{q^4}{\pi^4} (d^+) \partial_5^2 \partial_5 (\partial_2 + \partial_5).$$  

(29)
With this operator (24) leads to the desired relation:
\[
\int d^d k_1 d^d k_2 \frac{(k_1 q) (k_2 q)}{c_2 c_3 c_5} = \frac{q^2}{2\pi^2} \int \frac{d^{d+2} k_1 d^{d+2} k_2}{c_2 c_3 c_5^2} + \frac{q^4}{\pi^4} \int d^{d+4} k_1 d^{d+4} k_2 \left[ \frac{2}{c_2 c_3 c_5^2} + \frac{4}{c_2 c_3 c_5^3} \right].
\] (30)
Integrals on the right-hand side of (30) can be reduced to basic ones in the generic dimension \(d\) by using recurrence relations given in the next section. Notice that for these integrals at arbitrary \(d\) an analytic expression in terms of Lauricella functions is available [19] and therefore formula (23) shows that the “sunset” type diagrams with any number of scalar products in the numerator are always expressible in terms of restricted number of the Lauricella functions.

4 Recurrence relations

In the present section we will be concerned with generalized recurrence relations. Applying methods presented in previous sections we have reduced the problem of calculating two-loop diagrams to that of evaluating scalar integrals without numerator and having different shifts of \(d\). The reduction of scalar integrals with different powers of propagators (or different indices) to a sum over a minimal set of basic integrals in an arbitrary dimension will be done by means of generalized recurrence relations. The method of their derivation was described in Ref. [1]. For the two-loop general mass case the derivation of these relations is rather tedious and for brevity of the presentation will be omitted in the present paper. We will give only the final formulae and describe their application. To avoid overcomplication by irrelevant indices it is convenient to introduce a separate notation for two-loop integrals with five, four and three propagators:

\[
F^{(d)}_{\nu_1 \nu_2 \nu_3 \nu_4 \nu_5} = \frac{1}{\pi^d} \int \frac{d^d k_1 d^d k_2}{[k_1^2 - m_1^2]^{\nu_1} [k_2^2 - m_2^2]^{\nu_2}} \times \frac{1}{[(k_1 - q)^2 - m_3^2]^{\nu_3} [(k_2 - q)^2 - m_4^2]^{\nu_4} [(k_1 - k_2)^2 - m_5^2]^{\nu_5}},
\]

\[
V^{(d)}_{\nu_1 \nu_2 \nu_3 \nu_4} = \frac{1}{\pi^d} \int \frac{d^d k_1 d^d k_2}{[k_1 - k_2]^2 - m_1^2} \times \frac{1}{[k_2^2 - m_2^2]^{\nu_2} [(k_1 - q)^2 - m_3^2]^{\nu_3} [(k_2 - q)^2 - m_4^2]^{\nu_4}},
\]

\[
J^{(d)}_{\nu_1 \nu_2 \nu_3} (q^2) = \frac{1}{\pi^d} \int \frac{d^d k_1 d^d k_2}{[k_1 - k_2]^2 - m_1^2} \times \frac{1}{[k_1^2 - m_1^2]^{\nu_1} [(k_1 - k_2)^2 - m_2^2]^{\nu_2} [(k_1 - q)^2 - m_3^2]^{\nu_3}}.
\] (31)

One-loop integrals will be denoted as

\[
G^{(d)}_{\nu_1 \nu_2} = \frac{1}{\pi^d} \int \frac{d^d k_1}{[k_1^2 - m_2^2]^{\nu_1} [(k_1 - q)^2 - m_2^2]^{\nu_2}},
\] (32)

\[
T^{(d)}_{\nu_1} = \frac{1}{\pi^d} \int \frac{d^d k_1}{[k_1^2 - m_1^2]^{\nu_1}}.
\] (33)
Where no confusion can arise, we simply refer to the above functions as $F^{(d)}$, $V^{(d)}$, ... etc.

The application of recurrence relations to $F^{(d)}$ will produce $V^{(d)}$, $J^{(d)}$ and more simple one-loop integrals. In turn integrals $V^{(d)}$ will produce $J^{(d)}$ plus one-loop integrals. Thus one should first apply the recurrence relations to $F^{(d)}$, then to $V^{(d)}$ and after that to $J^{(d)}$.

### 4.1 Integrals with five propagators $F^{(d)}$

The integrals $F^{(d)}$ originate from diagrams with the topology given in Fig. 1. The first step will be the reduction of indices $\nu_1, \ldots, \nu_4$ by using the recurrence relations which derive from the generalization of the method of integration by parts [27] to massive integrals. It is particularly convenient to represent the recurrence relations in terms of shift operators decreasing or increasing indices of the integral by one unit [27]:

$$1^\pm F^{(d)}_{\nu_1\nu_2\nu_3\nu_4\nu_5} = F^{(d)}_{\nu_1\pm1\nu_2\nu_3\nu_4\nu_5}, \quad \text{etc.} \quad (34)$$

with a similar convention for $V^{(d)}$, $J^{(d)}$ and other integrals. Notice that an analogous notation we already employed in Section 2.

The indices $\nu_1, \ldots, \nu_4$ can be reduced to 0 or 1 by iteratively applying the relation:

$$2\nu_1\Delta 1^+ F^{(d)}_{\nu_1\nu_2\nu_3\nu_4\nu_5} = \left\{ (d - 2\nu_1 - \nu_3 - \nu_5)\Delta_1 + \Delta_{345}[\nu_5 5^+ (2^- - 1^-) - \nu_3 3^+ 1^-] \right. \right.$$

$$+ \Delta_2 [\nu_1 1^+ (5^- - 2^-) + \nu_3 3^+ (5^- - 4^-) + \nu_5 - \nu_1]$$

$$\left. + \Delta_6 [\nu_1 1^+ 3^- + \nu_5 5^+ (3^- - 4^-) + \nu_3 - \nu_1] \right\} F^{(d)}_{\nu_1\nu_2\nu_3\nu_4\nu_5}, \quad (35)$$

along with relations for $2^\pm F^{(d)}$, $3^\pm F^{(d)}$, $4^\pm F^{(d)}$ following from (35) by replacements:

$$2^+ F : (\nu_1, \nu_3, m_1, m_3, 1^\pm, 3^\pm) \leftrightarrow (\nu_2, \nu_4, m_2, m_4, 2^\pm, 4^\pm),$$

$$3^+ F : (\nu_1, \nu_2, m_1, m_2, 1^\pm, 2^\pm) \leftrightarrow (\nu_3, \nu_4, m_3, m_4, 3^\pm, 4^\pm),$$

$$4^+ F : (\nu_1, \nu_2, m_1, m_2, 1^\pm, 2^\pm) \leftrightarrow (\nu_3, \nu_4, m_3, m_4, 4^\pm, 3^\pm), \quad (36)$$

which are due to the symmetry of the integral. In the above formula we used the following abbreviations:

$$\Delta = q^4 m_3^2 \min\{m_1^2, m_2^2, m_3^2\} m_5^2 \min\{(m_1^2 - m_3^2)(m_2^2 - m_4^2), (m_3^2 - m_4^2)(m_2^2 - m_1^2)\} q^2$$

$$+ (m_2^2 - m_4^2)(m_1^2 - m_3^2) m_5^2 + (m_1^2 - m_2^2 - m_3^2 + m_4^2)(m_1^2 m_4^2 - m_2^2 m_3^2), \quad (37)$$

$$\Delta = \frac{\partial \Delta}{\partial m_3^2},$$

$$\Delta_6 = \frac{\partial \Delta}{\partial q^2} = 2q^2 m_3^2 - m_3^2 (m_1^2 + m_2^2 + m_3^2 + m_4^2 - m_5^2) + (m_4^2 - m_3^2)(m_2^2 - m_1^2),$$
\[ \Delta_{ijk} = m_i^4 + m_j^4 + m_k^4 - 2(m_i^2 m_j^2 + m_i^2 m_k^2 + m_j^2 m_k^2) \\
= (m_i + m_j + m_k)(m_i + m_j - m_k)(m_i - m_j + m_k)(m_i - m_j - m_k) \\
= -u_{ijk}(u_{ijk} + u_{kij}) - u_{ijk}u_{kij}, \]
(38)

\[ u_{ijk} = \frac{1}{2} \partial_i \Delta_{ijk} = m_i^2 - m_j^2 - m_k^2, \]

\[ \Delta_{ij6} = m_i^4 + m_j^4 + q^4 - 2(m_i^2 m_j^2 + m_i^2 q^2 + m_j^2 q^2). \]
(39)

It is worthwhile to note that the repeatedly appearing expression \( (37) \) is the well known Cayley kinematical determinant:

\[ \Delta = -\frac{1}{2} \begin{vmatrix} 
0 & 1 & 1 & 1 \\
1 & 0 & q^2 & m_4^2 \\
1 & q^2 & 0 & m_2^2 \\
1 & m_2^2 & m_3^2 & 0 \\
1 & m_3^2 & m_4^2 & 0 
\end{vmatrix}. \]
(40)

To reduce the index \( \nu_5 \) another recurrence relation is needed. It has a form similar to \( (35) \):

\[ 2\nu_5 \Delta \Sigma^+ F^{(d)}_{\nu_1 \nu_2 \nu_3 \nu_4 \nu_5} = \left\{ \Delta_{136}[\nu_3 3^+(4^- - 5^-) + \nu_1 1^+(2^- - 5^-)] + (d - \nu_1 - \nu_3 - 2\nu_5) \Delta_5 + \Delta_2[\nu_5 5^+(1^- - 2^-) + \nu_3 3^+ 1^- + \nu_1 - \nu_5] + \Delta_4[\nu_1 1^+ 3^- + \nu_5 5^+(3^- - 4^-) + \nu_3 - \nu_5] \right\} F^{(d)}_{\nu_1 \nu_2 \nu_3 \nu_4 \nu_5}. \]
(41)

If for some mass values \( \Delta = 0 \), then the formulas \( (33), (41) \) are of no use. This case needs a separate treatment. For arbitrary \( q^2 \), the relation \( \Delta = 0 \) can be fulfilled only at

\[ m_5^2 = 0, \quad m_4^2 = m_2^2, \quad m_2^2 = m_1^2. \]
(42)

If this condition is satisfied, substituting \( (42) \) into \( (35) \) (or \( (41) \)) yields the simpler relations:

\[ (d - \nu_1 - \nu_3 - 2\nu_5) F^{(d)}_{\nu_1 \nu_2 \nu_3 \nu_4 \nu_5} = [\nu_1 1^+(5^- - 2^-) + \nu_3 3^+(5^- - 4^-)] F^{(d)}_{\nu_1 \nu_2 \nu_3 \nu_4 \nu_5}, \]
(43)

\[ (d - \nu_2 - \nu_4 - 2\nu_5) F^{(d)}_{\nu_1 \nu_2 \nu_3 \nu_4 \nu_5} = [\nu_2 2^+(5^- - 1^-) + \nu_4 4^+(5^- - 3^-)] F^{(d)}_{\nu_1 \nu_2 \nu_3 \nu_4 \nu_5}. \]
(44)

By repeated use of these relations, the integrals \( F^{(d)} \) can be reduced to a combination of integrals \( J^{(d)} \) and one-loop ones. Prior to applying \( (13), (44) \), in some cases, it may be more efficient to first use the following relation which reduces the sum of indices of the integral

\[ \Delta_{136} 1^+ F^{(d)}_{\nu_1 \nu_2 \nu_3 \nu_4 \nu_5} = \left\{ u_{613} [\nu_1 1^+ 3^- + \nu_3 5^+(3^- - 4^-)] \\
- 2u_{136} [\nu_2 2^+(5^- - 1^-) + \nu_4 4^+(5^- - 3^-) - \Sigma_5 - \nu_1 + 3\nu_3] \\
+ 2m_3^2 [\nu_5 5^+(1^- - 2^-) + \nu_3 3^+ 1^- + \nu_1 - \nu_3] \right\} F^{(d)}_{\nu_1 \nu_2 \nu_3 \nu_4 \nu_5}. \]
(45)
Analogous recurrence relations for $2^+ F^{(d)}$, $3^+ F^{(d)}$, $4^+ F^{(d)}$ follow from (13) and substitutions (30). Here, and in the sequel, we use the notation

$$\Sigma_i = 3d - 2 \sum_{j=0}^{i} \nu_j - 2. \quad (46)$$

Also for particular values of the external momentum squared $F^{(d)}$ can be reduced to simpler integrals. Again this happens if the determinant (31) vanishes. If $m_5 \neq 0$, then $\Delta = 0$ is a quadratic equation for $q^2$ with solutions

$$q_{\pm}^2 = m_i^2 + m_3^2 - \frac{1}{2m_5^2} [u_{215}u_{435} \pm \sqrt{\Delta_{215}\Delta_{435}}]. \quad (47)$$

Furthermore, if $m_5^2 = 0, m_1 \neq m_2, m_3 \neq m_4$ then we have $\Delta = 0$ at

$$q^2 = \frac{(m_1^2 - m_2^2 - m_3^2 + m_4^2)(m_1^2 m_4^2 - m_2^2 m_3^2)}{(m_2^2 - m_4^2)(m_3^2 - m_4^2)}. \quad (48)$$

At $\Delta = 0$ the left-hand sides of Eqs. (33), (41) are zero and we are left with recurrence relations of a form similar to (13), (14). By repeated application of this kind of relations $F^{(d)}$ again is reducible to $J^{(d)}$ and one-loop integrals. For example, the special point $m_1 = m_4 = 0, m_2 = m_3 = m_5 = m$ is the on-shell value of the momentum

$$q^2 = m^2.$$ 

This kind of integrals are encountered in calculation of the on-shell diagrams for the fermion propagator of QED [28]. Simplification of integrals at some specific values of $q^2$ can be useful, for example, in establishing gauge invariance of physical amplitudes. Considering special cases we have been tacitly assuming that the dimensional regularization does not break down at the considered specific values of $q^2$.

Now we proceed with a few remarks concerning a reduction of the space-time dimension $d$. If we simplify the numerator of a scalar integral according to the rules described in Sec. 2 then the integrals $F^{(d)}$ will occur in the final result without change of dimension. Decomposing tensor integrals we will encounter in general $F^{(d)}$’s with shifted $d$. The connection between $F^{(d)}_1$ and $F^{(d-2)}_1$ which suffices to reduce these integrals to generic dimension can be worked out from the relation derived in Ref. [11]:

$$F^{(d-2)}_{\nu_1 \nu_2 \nu_3 \nu_4 \nu_5} = D(\partial)F^{(d)}_{\nu_1 \nu_2 \nu_3 \nu_4 \nu_5}. \quad (49)$$

Here $D(\partial)$ is a differential operator obtained from (18) by converting $\alpha_j$ into $\partial_j$. By applying the recurrence relations already presented in this section Eq. (49) can be written in the following form:

$$\Delta F^{(d-2)}_{11111} = -q^2(d - 3)(d - 4)F^{(d)}_{11111} + (P_1 1^- + P_2 2^- + P_3 3^- + P_4 4^- + P_5 5^-) F^{(d)}_{11111}, \quad (50)$$

where $P_i$ are polynomials in masses, $q^2$, $d$ and operators $1^+, \ldots, 5^+$. The polynomial $P_i$ does not depend on the operator $i^+$. The final formula connecting $F^{(d)}_{11111}$ and $F^{(d-2)}$ with all masses arbitrary is rather long and for this reason it will be not given here. The reader can easily obtain it by using the recurrence relations given in the present
article. Notice that at $\Delta = 0$ formula (50) gives us a relation for $F^{(d)}_{11111}$ in terms of simpler integrals.

To conclude this subsection, we note that in the case of arbitrary masses and external momenta by applying (35), (41), integrals $F^{(d)}_{\nu_1\nu_2\nu_3\nu_4\nu_5}$ with integer $\nu_j > 0$ will be expressed in terms of $F^{(d)}_{11111}$, $V^{(d)}_{\nu_1\nu_2\nu_3\nu_4\nu_5}$, $J^{(d)}_{\nu_1\nu_2\nu_3}$ and more simple ones. Notice that only one integral with five propagators, namely $F^{(d)}_{11111}$ is irreducible with respect to more simple ones. It will be the first representative of our minimal basis of two-loop integrals. As mentioned previously in the special case of (42) all integrals $F^{(d)}$ and $V^{(d)}$ are reducible to $J^{(d)}$ and one-loop ones by means of (43).

4.2 Integrals with four propagators $V^{(d)}$

Now we shall consider the integrals $V^{(d)}$ which at first sight look simpler than the integrals $F^{(d)}$, but in fact their reduction causes more problems. The reason is twofold. First, the $V^{(d)}$’s are less symmetrical than the $F^{(d)}$’s. Therefore the number of different relations needed is larger. Second, there are more cases to be distinguished because different mass combinations lead to different recurrence relations. The topology of the diagram corresponding to the integrals $V^{(d)}$ and the labeling of lines are shown in Fig. 2.

![Two-loop diagram with four propagators](image)

Fig. 2. Two – loop diagram with four propagators

In many cases the most efficient way to reduce integrals is to first apply the recurrence relations which reduce the sum of indices of the integral. Relations of this kind, which reduce $\nu_{1,2}$ to 0 or 1, are

$$\Delta_{134}\nu_1^{1+}V^{(d)}_{\nu_1\nu_2\nu_3\nu_4} = \left\{2m_3^2\nu_3^3(1^- - 4^-) + u_{413}\nu_1^{1+}(3^- - 4^-) + u_{134}(d - \nu_1 - 2\nu_3) + 2m_3^2(\nu_1 - \nu_3)\right\}V^{(d)}_{\nu_1\nu_2\nu_3\nu_4},$$

(51)

$$\Delta_{246}\nu_2^{2+}V^{(d)}_{\nu_1\nu_2\nu_3\nu_4} = \left\{2m_1^2\nu_1^{1+}(4^- - 2^-) + 2m_3^2\nu_3^{3+}(4^- - 2^-) - 2m_2^2\nu_2^{2+}4^- + (2m_2^2 - u_{426})\nu_2^{2+}4^- - (\Sigma_4 + 2)(4^- - 2^-) + u_{246}(d - 3\nu_2) + 2m_4^2(\nu_4 - \nu_2)\right\}V^{(d)}_{\nu_1\nu_2\nu_3\nu_4},$$

(52)

The index $\nu_3$ is lowered to zero or one by using the relation which follows from (51) by interchanging

$$\nu_1 \leftrightarrow \nu_3, \quad m_1 \leftrightarrow m_3, \quad 1^\pm \leftrightarrow 3^\pm.$$

(53)
An alternative way of reducing $\nu_3$ is to use first the relation

$$2m_3^2\nu_3^3 V_{\nu_2\nu_3\nu_4}^{(d)} = [u_{413}\nu_1^1 + \nu_1^1 (4^- - 3^-) + d - \nu_1 - 2\nu_3] V_{\nu_1\nu_2\nu_3\nu_4}^{(d)}, \quad (54)$$

and then to apply (51).

The reduction of $\nu_4$ is more problematic. It can be done at the expense of increasing $\nu_1$ and $\nu_2$ by applying the relation

$$2m_4^2\nu_4^4 V_{\nu_1\nu_2\nu_3\nu_4}^{(d)} = \left\{ u_{314}\nu_1^1 + u_{624}\nu_2^2 + \nu_1^1 (3^- - 4^-) - \nu_2^2 4^- + d - \nu_1 - \nu_2 - 2\nu_4 \right\} V_{\nu_1\nu_2\nu_3\nu_4}^{(d)}, \quad (55)$$

In turn, $\nu_{1,2}$ are to be lowered by means of (51), (52). As an alternative we may use the more symmetrical relation:

$$2m_4^2\nu_4^4 V_{\nu_1\nu_2\nu_3\nu_4}^{(d)} = \left\{ -2m_1^2\nu_1^1 + u_{624}\nu_2^2 - 2\nu_3^2 3^- - \nu_2^2 4^- + 2d - 2\nu_1 - \nu_2 - 2\nu_3 - 2\nu_4 \right\} V_{\nu_1\nu_2\nu_3\nu_4}^{(d)}, \quad (56)$$

which increases not only $\nu_{1,2}$ but also $\nu_3$.

For arbitrary $q^2$ there are two special cases for which some of the above relations do not work. Relation (51) which reduces $\nu_1$ and the analogous one reducing $\nu_3$ are nonapplicable if the masses $m_1, m_3, m_4$ are subject to the condition

$$\Delta_{134} = (m_1 + m_3 + m_4)(m_1 + m_3 - m_4)(m_1 - m_3 + m_4)(m_1 - m_3 - m_4) = 0. \quad (57)$$

Similarly the relation reducing the index $\nu_4$ cannot be used if

$$m_4 = 0. \quad (58)$$

These cases require additional investigations.

When all masses are different from zero but satisfy $\Delta_{134} = 0$ we found several recurrence relations which allow us to reduce all $V^{(d)}$-integrals to simpler ones with at least one line contracted. For example, one possible relation of this kind

$$[2m_1^2u_{134}(d - \nu_1 - 2\nu_3) + (\nu_3 - 1) u_{413}^2 + 4m_1^2m_3^2(\nu_1 - \nu_3)] V_{\nu_1\nu_2\nu_3\nu_4}^{(d)} =$$

$$[u_{413}\nu_3^3 (4^- - 1^-) - 4m_1^2m_3^2\nu_3^3 1^- + (4m_1^2m_3^2 + u_{413}^2)\nu_3^3 4^-$$

$$+ u_{413}(\nu_3 - 1) 1^- - (d - 2\nu_1 - \nu_3 + 1) u_{413} 3^-$$

$$+ (d - 2\nu_1 - \nu_3 + 1) u_{413} 4^-] V_{\nu_1\nu_2\nu_3\nu_4}^{(d)}. \quad (59)$$

follows from Eq. (51) with $\Delta_{134} = 0$ and using relation

$$2m_1^2\nu_1^1 V_{\nu_1\nu_2\nu_3\nu_4}^{(d)} = [u_{413}\nu_3^3 + \nu_3^3 (4^- - 1^-) + d - \nu_3 - 2\nu_1] V_{\nu_1\nu_2\nu_3\nu_4}^{(d)}, \quad (60)$$
which is obtained from (54) performing substitutions (53).

Before applying relation (59) the indices \( \nu_{1,2,4} \) should be lowered to 1 with the help of the relations (52), (55) and (60). Then applying the relation (59) as often as needed, one line in \( V^{(d)} \) will be eliminated. Notice that, the term \( 3^+ 4^- 4^- \) in (59) produces factors in the numerator which are supposed to be treated by the method proposed in Section 3.

If one of \( m_1^2, m_3^2, m_4^2 \) is zero then \( \Delta_{134} = 0 \) requires two other masses to be equal. There are two distinct cases:

Case 1: at \( m_1^2 = 0 \) (\( m_3^2 = 0 \)) and \( m_3^2 = m_4^2 \) (\( m_1^2 = m_4^2 \)) any integral with four lines can be reduced to a combination of simpler integrals with three lines by using the relation:

\[
(d - 2\nu_1 - \nu_3)V_{\nu_1\nu_2\nu_3\nu_4}^{(d)} = \nu_3 3^+ (1^- - 4^-) V_{\nu_1\nu_2\nu_3\nu_4}^{(d)}.
\]

(61)

Case 2: if \( m_3^2 = 0 \) and \( m_1^2 = m_3^2 \) then \( \Delta_{134} = 0 \) and the index \( \nu_2 \) should be lowered to one by means of (52). Then the relation

\[
(q^2 - \nu_1)^2 (d - 2\nu_2 + 2\nu_3 + 2\nu_4) 4^+ V_{\nu_1\nu_2\nu_3\nu_4}^{(d)} = \{ \nu_1 (m_1^2 - u_{612}) 1^+ + 4m_2^2 \nu_2 2^+ \\
+ 2m_1^2 \nu_3 3^+ + [\Sigma_4 - 2m_1^2 \nu_1 1^+ - 2m_1^2 \nu_3 3^+] 2^+ 4^+ \\
+ \nu_1 (q^2 - m_2^2) 1^+ 4^+ 3^- - \Sigma_4 \} V_{\nu_1\nu_2\nu_3\nu_4}^{(d)},
\]

is applied until one of the indices \( \nu_{2,3,4} \) will be zero. After each iteration it is reasonable to reduce \( \nu_2 \) again, in order to avoid a zero on the left-hand side of (52).

Let us consider now the case \( m_1^2 = 0 \) and \( \Delta_{134} \neq 0 \). Since \( \Delta_{134} \neq 0 \) the relation (51) and its analog for \( 3^+ V^{(d)} \) may be used to reduce \( \nu_{1,3} \) to 1. Then, instead of (52), the relation

\[
(q^2 - \nu_1)^2 [(m_3^2 - m_1^2) (d - 2\nu_2 - 2\nu_3 + 2\nu_4) + 2m_3^2 (\nu_3 - \nu_1)] V_{\nu_1\nu_2\nu_3\nu_4}^{(d)} = \\
\{ (m_1^2 - m_3^2) [(2m_1^2 \nu_1 1^+ + 2m_3^2 \nu_3 3^+ - \Sigma_4 - 2)(2^- - 4^-) - 4m_2^2 \nu_2 2^+ 4^-] \\
+ 2(q^2 - m_2^2) [m_3^2 \nu_3 3^+ (1^- - 4^-) + m_2^2 \nu_1 1^+ (4^- - 3^-)] \} V_{\nu_1\nu_2\nu_3\nu_4}^{(d)}
\]

(63)

must be applied until one of the \( \nu_i \) will become zero. After each application of (53), it is reasonable to reduce \( \nu_{1,3} \) to 1, if they have been increased after the use of (52).

Again the integrals \( V^{(d)} \) at some special values of \( q^2 \) admit a reduction to simpler integrals. These special values are determined by the equation

\[
\Delta_{246} = 0,
\]

(64)

which has the simple solution

\[
q_+^2 = (m_2 \pm m_4)^2.
\]

(65)

At \( \Delta_{246} = 0 \) the left-hand side of (52) is zero. With the help of equation (55) we exclude the term \( 2^+ 4^- \) from the right-hand side of (52) and obtain a relation which can be used to eliminate one line in \( V^{(d)} \). After each application of this relation one should reduce indices \( \nu_{1,3,4} \) to 1 if they have been increased by the previous iteration.
So far, we have considered only relations connecting integrals with the same space-time dimension. The whole concept of the method presented in this article would not work if we would not able to express integrals with shifted dimension in terms of those in a generic dimension. In fact there are several possibilities to reduce the dimensionality of an integral. As we shall see, one possibility always exists and it amounts to first express the integrals $V^{(d+2k)}_{\nu_1\nu_2\nu_3\nu_4}$ in terms of $V^{(d+2k)}_{1111}$ and simpler ones and then to express all $V^{(d+2k)}_{1111}$ in terms of $V^{(d)}_{1111}$ plus simpler ones.

The other possibility is to reduce simultaneously the indices and the space-time dimension. We found several recurrence relations of this kind:

\begin{equation}
4q^2\nu_1\nu_2\nu_31^+2^+3^+4^+V^{(d+2)}_{\nu_1\nu_2\nu_3\nu_4} = \\
\bigg\{6d - 6\nu_1 - 5\nu_2 - 6\nu_3 - 4\nu_4 - 6m_1^2\nu_11^+ + (u_{624} - 4m_2^2)\nu_22^+ \\
- 6m_3\nu_33^+ + 2(u_{624} - m_2^2)\nu_44^+ - \nu_22^+4^- - 2\nu_44^+2^-\bigg\} V^{(d)}_{\nu_1\nu_2\nu_3\nu_4}.
\end{equation}

\begin{equation}
4q^2\nu_1(\nu_1 + 1)\nu_21^+1^+2^+V^{(d+2)}_{\nu_1\nu_2\nu_3\nu_4} = \\
\bigg\{6d - 6\nu_1 - 5\nu_2 - 6\nu_3 - 4\nu_4 - 2(u_{246} + 3m_1^2)\nu_11^+ \\
+ (u_{624} - 4m_1^2)\nu_22^+ - 6m_3\nu_33^+ + 2(u_{624} - m_2^2)\nu_44^+ \\
+ 2\nu_11^+(4^- - 2^-) - \nu_22^+4^- - 2\nu_44^+2^-\bigg\} V^{(d)}_{\nu_1\nu_2\nu_3\nu_4}.
\end{equation}

\begin{equation}
4q^2m_2^2\nu_1\nu_2(\nu_2 + 1)1^+2^+2^+V^{(d+2)}_{\nu_1\nu_2\nu_3\nu_4} = \\
\bigg\{[(m_2^2 - m_3^2)(2d - 2\nu_1 - \nu_2 - 2\nu_3 - 4\nu_4) + m_4^2(2\nu_1 + \nu_2 - 2\nu_3)] \\
- u_{134}[2m_1^2\nu_11^+ + u_{246}\nu_22^+] + u_{314}[2m_3^2\nu_33^+ + \nu_22^+4^-] \\
- 2m_2^2(\nu_22^+ + \nu_44^+)(3^- - 1^-)\bigg\} V^{(d)}_{\nu_1\nu_2\nu_3\nu_4}.
\end{equation}

Exploiting the symmetry of $V^{(d)}$, two additional relations follow from (37), (38) and the interchangepments (33). Whenever possible, the above mixed recurrence relations should be used first. Their application is more efficient than the step by step reduction of all $V^{(d)}$s to master integrals in different dimensions and then the reduction of master integrals to the generic dimension.

The key relationship between $V^{(d)}_{1111}$ and $V^{(d-2)}_{1111}$, which is necessary for the $d-$ recurrences, follows from the equation (see Ref. [1]):

\begin{equation}
V^{(d-2)}_{1111} = (\partial_1\partial_2 + \partial_1\partial_3 + \partial_1\partial_4 + \partial_2\partial_3 + \partial_3\partial_4) V^{(d)}_{1111}.
\end{equation}

To simplify the right-hand side of Eq. (39) we use the recurrence relations (21), (22), (54), (55). After some additional algebra a relatively cumbersome formula is obtained:

\begin{equation}
(3d - 10)(d - 3)^2m_1^2V^{(d)}_{1111} = \frac{1}{4q^2} \left\{(3d - 10)\Delta_{134}[\Delta_{246} + u_{246}(2^- - 4^-)] \\
- 8(d - 4)^2q^2m_3^2m_4^24^-\right\} V^{(d-2)}_{1111}
\end{equation}
\[ + \{(d - 3)m_4^2[4m_1^2(d - 4)1^+1^+ - 8(d - 3)m_4^22^+2^+ + (3d - 10)(d - 3)2^+] \]
\[ + u_{6244}(3d - 10)u_{134}(1^+ - 3^+) + (d - 2)(d - 3)m_4^21^+|2^+ \]
\[ - m_4^2((d - 4)^2u_{134}2^+ + (3d - 10)u_{413}3^+]1^+ \} 4^- V_{1111}^{(d)} \]
\[ - (d - 3)m_4^2 \{4(d - 4)m_1^21^+ + (d - 2)[u_{6244}2^+ - 2m_4^24^+] \} 1^+3^- V_{1111}^{(d)} \]
\[ + \frac{1}{2}(3d - 10)u_{134}[u_{6244}2^+ - 2m_4^24^+] (1^-3^+ - 1^+3^-) V_{1111}^{(d)}. \]  

Notice that all terms with \( V_{1111}^{(d)} \) on the right-hand side of (70) are accompanied by shift operators contracting one of the lines i.e. producing simpler integrals. We kept \( V_{1111}^{(d)} \) for compactness of the formula, instead of specifying particular integrals obtained from it by contracting lines. It is worthwhile to note that in the relation (70) when \( \Delta_{134} = 0 \) or \( \Delta_{246} = 0 \), the term \( V_{1111}^{(d-2)} \) with uncontracted lines drops out and we obtain expression for \( V_{1111}^{(d)} \) in terms of simpler integrals.

Recall that if \( m_4^2 = 0 \), all \( V^{(d)} \) for arbitrary \( d \) are reducible to integrals \( J^{(d)} \) plus simpler ones and therefore in this case the problem of reducing \( d \) is transferred to simpler integrals.

Concluding this subsection it is worthwhile to mention that in the general mass case, after applying appropriate recurrence relations, the only \( V^{(d)} \) integral which will remain is \( V_{1111}^{(d)} \). This will be the second integral in our minimal basis of integrals. In fact in calculating diagrams with the topology given in Fig.1. four integrals \( V_{1111}^{(d)} \) corresponding to different mass distributions will occur (see Appendix ). All these integrals are to be included in the basis.

### 4.3 Integrals with three propagators \( J^{(d)} \)

The remaining nontrivial integrals \( J^{(d)} \) (see Fig.3), which are of interest in this subsection, are rather symmetrical though the corresponding recurrence relations are more cumbersome than in the previous cases.

![Fig. 3. Diagrams with three lines](image)

To reduce integrals with three lines to a basic set of integrals two kind of recurrence relations are needed. The first one reduces the sum of indices of the integral if at least
two lines have indices exceeding 1. The relation applicable when \( \nu_1 > 1 \) and \( \nu_2 > 1 \) has the form:

\[
2\nu_1\nu_2 D_{123} 1^2 + 2^+ J_{\nu_1\nu_2\nu_3}^{(d)}(q^2) = [2\nu_1 h_{123} 1^+ + 2\nu_2 h_{213} 2^+ + 4\nu_3 m_3^2 \sigma_{123} 3^+ \\
+ \nu_2 m_3^2 \phi_{213} 1^- 2^+ 3^+ + \nu_1 \nu_3 m_3^2 \phi_{123} 1^- 2^+ 3^- - 2\nu_1 \nu_2 \rho_{123} 1^+ 2^+ 3^- \\
+ \frac{1}{2} \Sigma_3 (d - \nu_1 - \nu_2 - \nu_3) \phi_{321} J_{\nu_1\nu_2\nu_3}^{(d)}(q^2),]
\]  

(71)

where

\[
D_{ijk} = q^6 - 4q^6(m_i^2 + m_j^2 + m_k^2) + q^4[6(m_i^4 + m_j^4 + m_k^4) \\
+ 4(m_i^2 m_j^2 + m_j^2 m_k^2 + m_k^2 m_i^2)] - 4q^2[m_i^6 + m_j^6 + m_k^6 - m_i^2(m_j^4 + m_k^4) \\
- m_j^2(m_i^4 + m_k^4) - m_k^2(m_i^4 + m_j^4) + 10m_i^2 m_j^2 m_k^2] + \Delta^2_{ijk},
\]  

(72)

\[
\rho_{ijk} = -\frac{1}{4} \frac{\partial D_{ijk}}{\partial q^2} = -q^6 + 3q^4(m_i^2 + m_j^2 + m_k^2) \\
- q^2[3(m_i^4 + m_j^4 + m_k^4) + 2(m_i^2 m_j^2 + m_j^2 m_k^2 + m_k^2 m_i^2)] + m_i^6 + m_j^6 + m_k^6 \\
- m_i^2(m_j^4 + m_k^4) - m_j^2(m_i^4 + m_k^4) - m_k^2(m_i^4 + m_j^4) + 10m_i^2 m_j^2 m_k^2,
\]  

(73)

\[
\phi_{ijk} = \frac{1}{2} \frac{\partial}{\partial m_i^2} \left( \frac{\partial}{\partial m_j^2} + \frac{\partial}{\partial m_k^2} + \frac{\partial}{\partial q^2} \right) D_{ijk} = \\
4[q^4 + 2q^2 (m_i^2 - m_j^2 - m_k^2) + (m_j^2 - m_k^2)^2 + m_i^2 (2m_j^2 + 2m_k^2 - 3m_i^2)] \\
= 4[q^4 + 2q^2 u_{ijk} + u_{ijk} u_{jik} + u_{ijk} u_{kij} - u_{jik} u_{kij}],
\]  

(74)

\[
\sigma_{ijk} = \frac{1}{4} (d - \nu_i - 2\nu_j) \phi_{ijk} - \frac{1}{4} (d - 2\nu_i - \nu_j) \phi_{jik} \\
- \frac{1}{4} (2d - 2\nu_i - 2\nu_j - \nu_k - 1) \phi_{kij},
\]  

(75)

\[
h_{ijk} = \frac{1}{2} (d - 2\nu_j - \nu_k) m_k^2 \phi_{ijk} - \frac{1}{2} (2d - 2\nu_j - 2\nu_k - 1) m_i^2 \phi_{kij}
\]

\[
\frac{1}{2} (d - \nu_j - 2\nu_k) \rho_{ijk}.
\]  

(76)

Two more recurrence relations follow from (71) by the interchanges

\[
1^+ 3^+ J^{(d)} : \nu_2, m_2, 2^\pm \leftrightarrow \nu_3, m_3, 3^\pm,
\]

\[
2^+ 3^+ J^{(d)} : \nu_1, m_1, 1^\pm \leftrightarrow \nu_3, m_3, 3^\pm.
\]  

(77)

The above recurrence relations are of no use if two lines have an index equal to one. In this case the second set of recurrence relation is needed. To reduce the index of the first line the following relation applies:

\[
2m_1^2 D_{123} \nu_1 (\nu_1 + 1) 1^+ 1^+ J_{\nu_1\nu_2\nu_3}^{(d)}(q^2) = \{- \Sigma_3 (d - \nu_1 - \nu_2 - \nu_3) \rho_{123}
\]
+m_2^2m_3^2\phi_{123}\nu_2\nu_31^{-2^+} 3^+ + m_1^2m_3^2\phi_{213}\nu_1\nu_31^{-2^+} 3^+ + m_1^2m_2^2\phi_{312}\nu_1\nu_21^{-2^+} 3^-
\\
+(d-2-\nu_1)D_{123}\nu_11^+ + m_1^2S_{123}\nu_11^+ + m_2^2S_{213}\nu_22^+ + m_3^2S_{312}\nu_33^+) J^{(d)}_{\nu_1\nu_2\nu_\nu} (q^2), \quad (78)
\\
where
\\
S_{ijk} = -(d-2\nu_j-\nu_k)m_2^2\phi_{ijk} - (d-\nu_j-2\nu_k)m_3^2\phi_{kij}
+ 2(2d-\nu_i-2\nu_j-2\nu_k-1)\rho_{ijk}. \quad (79)
\\
Recurrence relations for the reduction of \nu_{2,3} can be easily deduced from (78) by symmetrical interchanges similar to (77).

We see from (78) that if two indices are equal to 1 then it is impossible to reduce the third index to 1. This means that our minimal set of integrals will include not only \(J^{(d)}_{111}\) but also \(J^{(d)}_{211}, J^{(d)}_{121}\) and \(J^{(d)}_{112}\). Notice that the last three integrals are just derivatives of \(J^{(d)}_{111}\) with respect to different masses. From Fig. 1 we see that there are two possibilities to obtain the integrals \(J^{(d)}\). One possibility corresponds to a contraction of the lines 1 and 4 and the other one to a contraction of the lines 2 and 3. Altogether, the representation of the integrals \(J^{(d)}(q^2)\) requires us to include eight integrals in our minimal basis (see Appendix).

For arbitrary \(q^2\) the second set of recurrence relations needs a modification when some masses are equal to zero. If, for instance, \(m_1^2 = 0\), then substituting this value into Eq. (78) we deduce the simpler relation:
\\
\nu_1(d-2\nu_1-2)\Delta_{236}1^+ J^{(d)}_{\nu_1\nu_2\nu_3} (q^2) = \{-4m_2^2m_3^2\nu_2\nu_31^{-2^+} 3^+
+ 2m_2^2((q^2-m_2^2)(2d-\nu_1-\nu_2-\nu_3-1) - m_3^2(2\nu_1-\nu_2-1))\nu_22^+
+ 2m_3^2((q^2-m_3^2)(2d-\nu_1-\nu_2-\nu_3-1) - m_2^2(2\nu_1-\nu_3-1))\nu_33^+
- \Sigma_3(d-\nu_1-\nu_2-\nu_3)u_{623}\} J^{(d)}_{\nu_1\nu_2\nu_3} (q^2), \quad (80)
\\
which can be used to lower the exponent of the massless propagator down to 1. Therefore, in this case, the number of basic integrals will be lower than in the general case. Each massless line reduces the number of basic \(J^{(d)}\) type integrals by 1.

Turning next to \(d\)-recurrence relations, we note that the reduction of \(d\) for \(J^{(d)}\) is a bit simpler than that of \(V^{(d)}\). The main reason is the existence of systematic recurrence relations which reduce simultaneously some indices and \(d\). If two indices, say \(\nu_1 > 1\) and \(\nu_2 > 1\), one can use the relation:
\\
q^2(\Sigma_3 + 2)\nu_1\nu_21^+ 2^+ J^{(d+2)}_{\nu_1\nu_2\nu_3} (q^2) = \{q^2(d-2\nu_3) + m_1^2(d-\nu_1-2\nu_2)
+ m_2^2(d-2\nu_1-\nu_2) - 2m_2^2(d-\nu_1-\nu_2-\nu_3) - 2m_3^2(q^2-m_3^2)\nu_33^+
+ m_1^2(q^2-m_1^2-3m_2^2+3m_3^2-2^- 3^-)\nu_11^+}
Relations for \( 1^+ 3^+ J^{(d+2)} \), \( 2^+ 3^+ J^{(d+2)} \) are derivable from (81) by performing appropriate interchanges (77). Relations reducing the index of one line look more complicated

\[
q^2 m_1^2 (\Sigma_3 + 2) \nu_1 (\nu_1 + 1) 1^+ J^{(d+2)}_{\nu_1 \nu_2 \nu_3} (q^2) = \left\{ \begin{array}{l} \frac{1}{2} q^2 (\Sigma_3 + 4) \nu_1 1^+ J^{(d+2)}_{\nu_1 \nu_2 \nu_3} (q^2) - \frac{1}{2} q^4 (\Sigma_3 + 2) (6 \nu_1 + 6 \nu_2 + 6 \nu_3 - 7d) \\
+ m_2^2 (7d - 2\nu_1 - 4\nu_2 - 10\nu_3) + m_3^2 (7d - 2\nu_1 - 10\nu_2 - 4\nu_3) \\
+ m_2^4 (d - \nu_1 - \nu_2) + m_3^4 (d - \nu_1 - \nu_3) + m_2^2 m_3^2 (d - \nu_1 - \nu_2 - \nu_3) \\
- m_2^2 [2q^4 - q^2 (2m_2^2 + m_3^2) + m_1^2 (m_2^2 + m_3^2) + 3(m_2^2 - m_3^2)^2] \nu_1 1^+ \\
+ m_3^2 [q^4 - (q^2 - m_3^2) (3m_1^2 - 5m_3^2 + 1^\nu 3^-) - m_3^4] \nu_2 2^+ \\
+ m_3^2 [q^4 + (q^2 - m_3^2) (3m_1^2 - 5m_2^2 + 1^\nu 2^-) - m_3^4] \nu_3 3^+ \\
- \frac{1}{2} q^2 (\Sigma_3 + 2) (1^- - 2^- - 3^-) - m_1^2 (m_2^2 - m_3^2) \nu_1 1^+ (2^- - 3^-) \right\} J^{(d)}_{\nu_1 \nu_2 \nu_3} (q^2). \tag{82}
\]

Two additional relations follow from (82) by interchanges similar to (77).

If \( m_1 = 0 \) Eq. (82) takes the simpler form

\[
q^2 (\Sigma_3 + 2) (\Sigma_3 + 4) \nu_1 1^+ J^{(d+2)}_{\nu_1 \nu_2 \nu_3} (q^2) = \left\{ \begin{array}{l} -q^4 (d + 2\nu_1 - 2\nu_2 - 2\nu_3) \\
+ q^2 [m_2^2 (7d - 2\nu_1 - 4\nu_2 - 10\nu_3) + m_3^2 (7d - 2\nu_1 - 10\nu_2 - 4\nu_3)] \\
+ 2m_2^4 (d - 2\nu_1 - \nu_2) + 2m_3^4 (d - 2\nu_1 - \nu_3) - 8m_2^2 m_3^2 (d - \nu_1 - \nu_2 - \nu_3) \\
+ 2m_2^2 [q^4 - (q^2 - m_2^2) (5m_3^2 - 1^- + 3^-) - m_3^4] \nu_2 2^+ \\
+ 2m_3^2 [q^4 - (q^2 - m_3^2) (5m_2^2 - 1^- + 2^-) - m_3^4] \nu_3 3^+ \\
- q^2 (\Sigma_3 + 2) (1^- - 2^- - 3^-) \right\} J^{(d)}_{\nu_1 \nu_2 \nu_3} (q^2). \tag{83}
\]

Again it is always possible to reduce \( J^{(d)} \) with different shifts in \( d \) to a basic sets of integrals in different dimensions and then to transform all master integrals to the
generic dimension. The relation connecting \(d-2\) and \(d\) dimensional integrals \(J_{\nu_1\nu_2\nu_3}^{(d)}(q^2)\) follow from the differential relationship given in Ref.\[1\]. Here we give it in terms of shift operators:

\[
J_{\nu_1\nu_2\nu_3}^{(d-2)}(q^2) = (\nu_1\nu_2 1 + 2^+ + \nu_1\nu_3 1^+ 3^+ + \nu_2\nu_3 2^+ 3^+) J_{\nu_1\nu_2\nu_3}^{(d)}(q^2).
\]  

(84)

The right-hand side of this equation can be simplified by using the recurrence relations given previously in this subsection. As we already mentioned \(J_{111}^{(d)}\) and \(\partial_b J_{111}^{(d)}\) are elements of our minimal set of basic integrals. For practical applications, however, it useful to use separate formulas for the reduction of \(J_{111}^{(d)}\) and \(\partial_b J_{111}^{(d)}\). Substituting \(\nu_i = 1\) into Eq.\[84\] and simplifying the right-hand side we obtain the relation:

\[
D_{123} J_{111}^{(d-2)}(q^2) = (d - 3) \left\{ (3d - 8) [3q^4 - 2q^2 t_1 - \Delta_{123}] + \tau_1 1^+ + \tau_2 2^+ + \tau_3 3^+ \right\} J_{111}^{(d)}(q^2) + \frac{1}{4} \left[ (\partial_1 D_{123}) J_{022}^{(d)}(q^2) + (\partial_2 D_{123}) J_{202}^{(d)}(q^2) + (\partial_3 D_{123}) J_{220}^{(d)}(q^2) \right],
\]  

(85)

where

\[
D_{123} = (q^2 - r_1)(q^2 - r_2)(q^2 - r_3)(q^2 - r_4) = q^8 - 4t_1q^6 + 2t_2q^4 + 4t_3q^2 + t_4,
\]

\[
\tau_i = -2(q^2 - m_i^2)[q^4 - 2q^2(t_1 - 4m_i^2) + \Delta_{123}],
\]

\[
r_1 = (m_1 + m_2 + m_3)^2, \quad r_2 = (m_1 + m_2 - m_3)^2,
\]

\[
r_3 = (m_1 - m_2 + m_3)^2, \quad r_4 = (m_1 - m_2 - m_3)^2,
\]

\[
t_1 = m_1^2 + m_2^2 + m_3^2 = -u_{213} - u_{231} + u_{312},
\]

\[
t_2 = 3(m_1^4 + m_2^4 + m_3^4) + 2(m_1^2 m_2^2 + m_1^2 m_3^2 + m_2^2 m_3^2)
\]

\[
= 3((m_1 u_{213} + u_{213} u_{312} + u_{213} u_{312}) + 2(u_{213} + u_{213} + u_{312})) + 2(u_{213} + u_{213} + u_{312}),
\]

\[
t_3 = m_1^2(m_1^4 - m_2^4 - m_3^4) + m_2^2(m_2^4 - m_1^4 - m_3^4) + m_3^2(m_3^4 - m_1^4 - m_2^4)
\]

\[
+ 10m_1^2 m_2^2 m_3^2
\]

\[
= -u_{123} u_{213} u_{312} - u_{123} u_{213} u_{312} - u_{213} u_{213} u_{312} - u_{213} u_{213} u_{312} - u_{312} u_{123} - u_{312} u_{213} u_{312},
\]

\[
t_4 = \Delta_{123}.
\]  

(86)

By differentiating \(85\) with respect to masses, and upon simplifying the right-hand side, we get a system of four equations connecting \(J_{111}^{(d-2)}, \partial_b J_{111}^{(d-2)}\) with \(J_{111}^{(d)}, \partial_b J_{111}^{(d)}\). The solution of this system is straightforward but tedious and has the form

\[
3q^2(d - 3)(d - 4)(3d - 8)(3d - 10) J_{111}^{(d)}(q^2) = \left\{ (d - 4)q^6
\right. - 2q^4 t_1(d - 4)(6d - 23) + q^2 \left[ 5t_1^2 (15d^2 - 117d + 224) - t_2 (42d^2 - 331d + 640) \right]
\]

\[
- \frac{1}{4} (d - 5) \left[ t_3 (27d - 90) - t_1 t_2 (3d - 2) - 2t_1^2 (5d - 26) \right]
\]

\[
+ f(m_1^2, m_2^2, m_3^2) 1^+ + f(m_1^2, m_2^2, m_3^2) 2^+ + f(m_1^2, m_2^2, m_3^2) 3^+ \right\} J_{111}^{(d-2)}(q^2)
\]

\[
+ g(m_1^2, m_2^2, m_3^2) J_{220}^{(d-2)}(q^2) + g(m_1^2, m_3^2, m_2^2) J_{202}^{(d-2)}(q^2)
\]

\[
+ g(m_2^2, m_3^2, m_1^2) J_{022}^{(d-2)}(q^2),
\]  

(87)
where

$$f(m_1^2, m_2^2, m_3^2) = m_1^2(q^2 - m_1^2) \left\{ -2(d - 4)q^4 + q^2 \left[ 4t_1(5d - 18) - 24m_1^2(2d - 7) \right] ight.$$  

$$- 2(4d - 13)t_1^2 + 2(9d - 31)t_2 - 24m_2^2m_3^2(4d - 13) - 24m_1^2(2d - 7) \right\},$$  

$$g(m_1^2, m_2^2, m_3^2) = \frac{m_1^2m_2^2}{(d - 4)} \left[ 4(d - 4)q^4 - 4(7d - 24)q^2(3m_3^2 - 2t_1) ight.$$  

$$- t_1^2(23d - 80) + t_2(9d - 32) - 12m_1^2(d - 4) + 12m_1^2m_2^2(7d - 24) \right] ,$$

$$3q^2(d - 3)(d - 4)(3d - 10)1^+ J_{111}^{(d)}(q^2) =$$

$$\left\{ -m_1^2[q^4(7d - 24) - 2q^2((4d - 15)t_1 - (d - 5)m_1^2) ight.$$  

$$+ \frac{3}{2}(d - 5)t_1^2 + \frac{5}{2}(d - 3)t_2 - 2(5d - 17)m_4^1 - 2(13d - 45)m_2^2m_3^2]1^+$$  

$$+ 2m_2^2(q^2 - m_3^2)((d - 3)(q^2 + m_3^2) + (7d - 25)m_1^2)2^+$$  

$$+ 2m_3^2(q^2 - m_2^2)((d - 3)(q^2 - 5m_2^2 + m_3^2) + (7d - 25)m_1^2)3^+$$  

$$- (d - 3)(d - 4)q^4 + q^2[(7d - 30)(d - 3)t_1 - (7d - 31)(3d - 10)m_1^2]$$  

$$+ \frac{1}{4}(d - 5)[(17d - 66)t_1^2 - (3d - 14)t_2 - 4(3d - 10)(m_1^4 + 5m_2^2m_3^2)] J_{111}^{(d - 2)}(q^2)$$  

$$+ \frac{2}{d - 4}[(q^2 + m_3^2)(7d - 24) + (d - 4)m_1^2 - (5d - 18)m_3^2]m_1^4m_2^2J_{220}^{(d - 2)}(q^2)$$  

$$+ \frac{2}{d - 4}[(q^2 + m_3^2)(7d - 24) + (d - 4)m_1^2 - (5d - 18)m_3^2]m_1^2m_3^2J_{202}^{(d - 2)}(q^2)$$  

$$- \frac{4}{d - 4}[(q^2 + m_3^2)(d - 3) + (d - 4)m_1^2]m_2^2m_3^2J_{222}^{(d - 2)}(q^2).$$

Relations for $2^+ J^{(d)}$ and $3^+ J^{(d)}$ in terms of $d - 2$ dimensional integrals follow from (90) by substitutions similar to (77).

### 4.4 Two-loop bubble integrals

Rather frequently we will encounter two-loop bubble integrals. They can be treated as $V_{\nu_1 \nu_2 \nu_3}^{(d)}$ or as a value of $J_{\nu_1 \nu_2 \nu_3}^{(d)}(q^2)$ at $q^2 = 0$. It is more reasonable to consider these integrals separately. We found several recurrence relations most useful for practical applications:

$$(d - 2)\nu_1 1^+ J_{\nu_1 \nu_2 \nu_3}^{(d)}(0) = \left\{ -u_{123} - 1^+ + 2^+ + 3^+ \right\} J_{\nu_1 \nu_2 \nu_3}^{(d - 2)}(0),$$

$$(d - 2)\nu_2 \nu_3 2^+ J_{\nu_1 \nu_2 \nu_3}^{(d)}(0) = \left\{ -2m_1^2\nu_1 1^+ + (d - 2 - 2\nu_1) \right\} J_{\nu_1 \nu_2 \nu_3}^{(d - 2)}(0),$$
\[(d - 2)(d - \nu_1 - \nu_2 - \nu_3)J_{\nu_1\nu_2\nu_3}^{(d)}(0) = \]
\[- \left\{ \Delta_{123} + u_{123}1^- + u_{213}2^- + u_{312}3^- \right\} J_{\nu_1\nu_2\nu_3}^{(d-2)}(0), \]
\[
\Delta_{123}\nu_1 1^+ J_{\nu_1\nu_2\nu_3}^{(d)}(0) = \left\{ u_{123}(d - \nu_1 - 2\nu_2) + 2m^2_2(\nu_1 - \nu_2) + u_{312}\nu_1(2^- - 3^-) + 2m^2_2\nu_2 2^+(1^- - 3^-) \right\} J_{\nu_1\nu_2\nu_3}^{(d)}(0). \] (91)

These relations along with those obtained from \[(71)\] by making interchanges like \[(77)\]
allow one to reduce any \(J_{\nu_1\nu_2\nu_3}^{(d+2l)}(0)\) with integer \(\nu_i \geq 0\) and integer \(l\) to a combination of \(J_{\nu_1}^{(d)}(0)\) and products of different one-loop tadpole integrals. Calculating integrals \[(1)\]
we will encounter only two \(J_{\nu_1}^{(d)}(0)\) with different mass distributions (see Appendix).

In the case when \(\Delta_{123} = 0\) the recurrence relations are simpler. If all masses are different from zero, without loss of generality, we can assume that \(m^2_1 = (m_2 + m_3)^2\) and then by using the relation
\[2m_2m_3(m_2 + m_3)(d - 2\nu_1 - 2\nu_2 - 2\nu_3 - 1)1^+ J_{\nu_1\nu_2\nu_3}^{(d)}(0) = \]
\[
\left\{ m_2[(d - \nu_1 - \nu_2 - 2\nu_3 - 1)(1^+2^+ - 1) + (\nu_1 - \nu_2 + 1)1^+3^-] \right\} + \]
\[
m_3[(d - \nu_1 - 2\nu_2 - \nu_3 - 1)(1^+3^- - 1) + (\nu_1 - \nu_3 + 1)1^+2^-] \right\} J_{\nu_1\nu_2\nu_3}^{(d)}(0). \] (92)

the integral can be reduced to a combination of products of one-loop bubble integrals. If \(m_1 = 0\) and \(m_3 = m_2\) then \(J^{(d)}\) also can be reduced to a product of one-loop tadpoles by applying the relation:
\[2m_2^2(d - 2\nu_1 - 2)(d - 2\nu_1 - \nu_2 - \nu_3 - 1)(d - 2\nu_1 - \nu_2 - \nu_3)1^+ J_{\nu_1\nu_2\nu_3}^{(d)}(0) = \]
\[
(d - 2\nu_1 - 2\nu_3)(d - 2\nu_1 - 2\nu_2)(d - \nu_1 - \nu_2 - \nu_3)J^{(d)}_{\nu_1\nu_2\nu_3}(0). \] (93)

This relation is in agreement with the exact result first obtained in Ref. \[35\].

### 4.5 One-loop integrals

The reduction of one-loop integrals does not cause any serious problems. By applying the recurrence relation
\[\Delta_{126}\nu_1 1^+ G_{\nu_1\nu_2}^{(d)} = \]
\[
\left\{ u_{612}[\nu_11^+2^- - d + \nu_1 + 2\nu_2] + 2m^2_2[\nu_2 2^+1^- - d + 2\nu_1 + \nu_2] \right\} G_{\nu_1\nu_2}, \] (94)

along with the relation for \(2^+G_{\nu_1\nu_2}^{(d)}\) which follows from \[(94)\] by interchanging \(\nu_1, m_1, 1^+\) and \(\nu_2, m_2, 2^+\) any \(G_{\nu_1\nu_2}^{(d)}(q^2)\) with integer \(\nu_i > 0\) can be reduced to a combination of \(G_{11}^{(d)}(q^2)\) and tadpoles \(G_{0\nu_2}^{(d)}(q^2)\) and \(G_{\nu_1\nu_2}^{(d)}(q^2)\). Thus, one-loop integrals \(G_{\nu_1\nu_2}^{(d)}\) will require only \(G_{11}^{(d)}\) in our set of basic integrals. The reduction of scalar integrals \[(1)\] will produce
two integrals $G^{(d)}_{11}$. One will be made from the lines 1 and 3 and the other one made from the lines 2 and 4 (see Appendix). To lower the space-time dimension of $G^{(d)}_{\nu_1 \nu_2}$ two relations can be used:

$$2q^2\nu_1 1^+ G^{(d)}_{\nu_1 \nu_2} = \left\{ u_{126} + 1^- - 2^- \right\} G^{(d-2)}_{\nu_1 \nu_2}, \quad (95)$$

$$2q^2(d - \nu_1 - \nu_2 - 1)G^{(d)}_{\nu_1 \nu_2} = \left\{ \Delta_{126} + u_{126} 1^- + u_{216} 2^- \right\} G^{(d-2)}_{\nu_1 \nu_2}. \quad (96)$$

As in the case of the two-loop bubble integrals, it is convenient to consider one-loop bubble integrals separately. In principle, one can use the explicit expressions for one-loop tadpole integrals, however, it is more efficient to treat them on equal footing with the other integrals i.e. to apply recurrence relations for their evaluation. Only two recurrence relations are needed for all kinds of reductions:

$$2m^2_1 \nu_1 1^+ T^{(d)}_{\nu_1}(m^2_1) = (d - 2\nu_1)T^{(d)}_{\nu_1}(m^2_1) \quad (97)$$

$$(d - 2\nu_1)T^{(d)}_{\nu_1}(m^2_1) = -2m^2_1 T^{(d-2)}_{\nu_1}(m^2_1) \quad (98)$$

Five integrals $T^{(d)}_{\nu_1}(m^2_i), i = 1, \ldots, 5$ have to be taken as elements of our basic set of integrals. Different products of one loop integrals $G^{(d)}_{11}$ and/or $T^{(d)}_{1}$ which are to be included in the minimal set of basic integrals are given in the Appendix.

## 5 Summary and conclusions

In the present article we described a procedure for the systematic reduction of two-loop diagrams to a set of basic ones. We have shown how a scalar contribution of the form (1), obtained from any two-loop propagator diagram, can be reduced to a sum over 30 basic structures $I^{(d)}_j(q^2)$

$$I(q^2) = \sum_{j=1}^{30} R_j(q^2, \{m^2_i\}, d) I^{(d)}_j(q^2), \quad (99)$$

with $R_j(q^2, \{m^2_i\}, d)$ being ratios of polynomials in $q^2, \{m^2_i\}$ and $d$. All $I^{(d)}_j(q^2)$ are pictured in the Appendix. The choice of the elements of the basis is not unique. One can take some other 30 independent functions. We hasten to remark that in order to know the two-loop integrals of the basis one needs to deal with only three integrals $F^{(d)}_{111111}, V^{(d)}_{1111}$ and $J^{(d)}_{111}(q^2)$. The other members will be obtained from those by changing masses or by differentiating. Two-loop bubble integrals are just the value of $J^{(d)}_{111}(q^2)$ at $q^2 = 0$. The number of basic structures strongly depends on the mass values. If some masses are equal to zero or there are equal masses then the number of basic structures substantially diminishes. For example, in the case of QED the number of relevant two-loop basic integrals for the photon propagator is 2, and for the fermion propagator it is 5.

The evaluation of integrals from the basis is a separate problem. Two-loop bubble and one-loop integrals are in a sense trivial. Analytic expressions for them are widely
known (see for example, [3], [22], [36]). As for the nontrivial two-loop integrals the situation is the following. Integrals with three propagators are related to the Lauricella functions [19]. As was shown in [21] integral with four propagators $V^{(d)}_{1111}$ can be written in terms of multiple hypergeometric series. Similar series for $F^{(d)}_{11111}$ with arbitrary masses is not yet known. In the general mass case a one-fold integral representation for $F^{(d)}_{11111}$ is given in [20]. Master integrals for propagator integrals occurring in QED and QCD were investigated in [15], [17]. Some particular cases of two-loop diagrams with massive particles were considered in [23]. We can recommended the reader for the general mass case to use either the one-fold integral representation [20] or to use the method of Refs. [8], [37].

We implemented the recurrence relations presented in this article in a FORM [34] package. In a general mass case, required execution time for different diagrams varies from several seconds to several minutes. The only difficulty we encountered in calculating diagrams with all masses different was the lengthy expressions we obtained for the coefficients $R_j(q^2, \{m_i^2\}, d)$ and which are difficult to simplify by using FORM. We see two possibilities to solve this problem. The simplest solution will be to use other computer algebra system, like Maple, Reduce, etc. to simplify $R_j(q^2, \{m_i^2\}, d)$. The other solution would be to assign numerical values to the masses and then to perform calculations using FORM. To test the algorithm we have reproduced two-loop result [17] for the photon self-energy.

Several remarks concerning the possibility to extend the presented algorithm to a more complicated set of Feynman diagrams. Our preliminary investigation shows that a recurrence algorithm similar to the one described in the present article can be worked out also for the two-loop vertex (three-point) functions. Unfortunately, for the general mass case kinematical determinants which appear repeatedly in recurrence relations are much more cumbersome than for the propagator case. The implementation of these relations on the presently available computers is very problematic. For some simplified kinematical cases, when say, some momenta or masses are equal to zero, or several masses are equal, the recurrence relations are simpler and can be implemented on a computer.

For integrals with several external momenta and several masses it may be reasonable to solve the recurrence relations numerically. It was done before in [37] using multiprecision package [38] for the small momentum expansion. It was noticed that in this approach one needs huge arrays that frequently cause problems with a computer memory. However, if we do not consider small momentum expansion and work with unexpanded diagrams then encountered indices in integrals will be not so large and therefore we will not meet problems just mentioned.

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7 Appendix

In this appendix we present the complete set of integrals resulting from the application of recurrence relations for diagrams with arbitrary masses.

Fig. 1. A complete set of basic integrals

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