Worldline fuzziness and spacetime noncommutativity

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Abstract. We here offer a perspective on results obtained in collaboration with Amelino-Camelia and Rosati, which were reported in Refs.[6, 7, 8]. We advocate a formalization of spacetime noncommutativity within the manifestly covariant formulation of quantum mechanics. Using as illustrative example the case of the much-studied $\kappa$-Minkowski noncommutative spacetime, we show how such a setup may lead to a crisp characterization of relative locality and of worldline fuzziness in a quantum spacetime.

1. Spacetime noncommutativity and covariant formulation of quantum mechanics
One of the proposal to implement quantum effects in the description of spacetime is that of non-commutative spacetimes, in which coordinates of classical relativistic mechanics are promoted to operators that, in addition, do not commute between them.

One of the most studied such a proposal is the $\kappa$-Minkowski noncommutative spacetime[1, 2, 3, 4, 5],
\begin{equation}
[x_j, x_0] = i\ell x_j , \quad [x_j, x_k] = 0 ,
\end{equation}
with the only non trivial commutator between the time coordinate and the spatial ones.

Once given such a commutation rule one would like to give also some more “physically characterizing” effects of such spacetimes, but to implement the non trivial commutation rules in the known formalism of classical or even quantum mechanics is quite difficult.

Evidently classical mechanics cannot be our choice, since its formalization provides no room for noncommutativity of coordinates. And even giving a formulation of $\kappa$-Minkowski spacetime noncommutativity in a quantum-mechanics setup is not straightforward. This is due to the fact that in $\kappa$-Minkowski the time coordinate should be an operator that does not commute with the spatial-coordinate operators, but in the standard setup of quantum mechanics we are not in the situation of time being described by an operator that commutes with the spatial-coordinate operators: in the standard setup of quantum mechanics time is not an observable at all, it just plays the role of evolution parameter.

We believe that it was indeed because of this mismatch between the nature of time in quantum mechanics and the properties of the $\kappa$-Minkowski time coordinate that progress did not materialize for formulating some of the observable spacetime consequences of $\kappa$-Minkowski that we are here interested in. We here summarize and offer an overall perspective on our results reported in Refs. [6, 7, 8] whose main starting point is recent progress on a manifestly-covariant formulation of quantum mechanics, which matured significantly over the last decade [9, 10, 11, 12].
This formulation gives us the perfect environment to implement our non trivial commutators, because space and time coordinates are treated on the same grounds, are both non trivial operators on an hilbert space, so we can seek for a representation of our non-commutative spacetime as a deformation of a standard covariant formulation of quantum mechanics.

In our analysis we will restrict ourselves to a $1+1$-dimensional spacetime, for ease of explanation. In this case the "kinematical hilbert space" [11] of our theory is simply the space of square-integrable functions of two variables. In the standard, commutative case spatial and time coordinates $q_0, q_1$ are well-defined multiplicative operators on this hilbert space, and one also has standard momenta $\pi_0, \pi_1$ conjugate to the spacetime coordinates:

$$[[\pi_0, q_0] = i, \quad [\pi_1, q_1] = -i, \quad [\pi_1, q_0] = [\pi_0, q_1] = 0 \quad (2)$$

We want to represent the $\kappa$-Minkowski defining commutators,

$$[x_1, x_0] = i\ell x_1 \quad (3)$$
on the above defined hilbert space.

We can achieve this goal by posing a relationship between $\kappa$-Minkowski coordinates and the phase-space observables of the undeformed, standard covariant formulation of quantum mechanics (the ones of Eq. (2), here viewed merely as formal auxiliary operators [6]):

$$x_1 = e^{\ell \pi_0} q_1, \quad x_0 = q_0. \quad (4)$$

Besides a meaningful description of the noncommutativity of the $\kappa$-Minkowski coordinates, this setup leads straightforwardly [6, 8] also to a description of translation transformations in $\kappa$-Minkowski, and to the deformation of the whole Poincaré group that suits well as a symmetry group for our non-commutative spacetime - the so-called $\kappa$-Poincaré (that we do not show here, [6]):

$$T_\alpha \ni f(x_0, x_1) \leftrightarrow f(q_0, q_1 e^{\ell \pi_0} - i\alpha \left[ \pi_\mu, f(q_0, q_1 e^{\ell \pi_0}) \right], \quad (5)$$

Studying the whole symmetry algebra of $\kappa$-Minkowski, one can obtain two other strongly-characterizing results. The first result [6, 8] concerns the deformation of the mass casimir, that in the covariant formalism is the tool we use to enforce dynamics on our kinematical space. The corresponding operator (Hamiltonian constraint [6, 8, 11]) for the $\kappa$-Poincaré group is:

$$H_\ell = \left( \frac{2}{\ell} \right)^2 \sinh^2 \left( \frac{\ell \pi_0}{2} \right) - e^{-\ell \pi_0} \pi_1^2. \quad (6)$$

The second result [6, 8] concerns the measure for integration over momenta, needed for evaluating scalar products in the “momentum representation” of our hilbert space; it turns out that the right, invariant measure of integration in the non-commutative case is:

$$d\pi_0 d\pi_1 \rightarrow d\pi_0 d\pi_1 e^{-\ell \pi_0} \quad (7)$$

2. Fuzzy points, translation transformations and relative locality
States in the kinematical hilbert space are distribution over spacetime, that can be interpreted as the probability amplitude to find a particle in a given point of spacetime. In the standard commutative case we can have - at the kinematical level - states that are arbitrarily close to points: a dirac delta-like distribution in the commutative hilbert space have a sharply definite value for both the time and the spatial coordinates.

This is no more the case in the deformed, noncommutative case: spatial and time coordinates no longer commute, so we cannot expect them to be diagonalizable at the same time.
For the first time in our setting we can give in addition a precise, quantitative statement of what the consequence of this non-commutativity are. A class of states which is well suited for exploring the properties of $\kappa$-Minkowski fuzziness is the one of gaussian states.

We denote them by $\Psi_{\bar{q}_0,\bar{q}_1}(\pi_\mu,\bar{\pi}_\mu,\sigma_\mu)$ and they are specified by functions of the variables $\pi_\mu$ parametrized by $\bar{\pi}_\mu$, $\sigma_\mu$, and $\bar{q}_\mu$:

$$
\Psi_{\bar{q}_0,\bar{q}_1}(\pi_\mu,\bar{\pi}_\mu,\sigma_\mu) = Ne^{-\frac{(\pi_0-\bar{\pi}_0)^2}{4\sigma_0^2}} e^{i\pi_0\bar{q}_0-\pi_1\bar{q}_1},
$$

where $N$ is a normalization constant. Of course, our main focus of attention will be on establishing how the $\kappa$-Minkowski scale $\ell$ affects the results, since this is going to be our indicator of the difference between classical Minkowski spacetime and $\kappa$-Minkowski.

Following the analysis we reported in Ref. [6] one finds that the $\kappa$-Minkowski scale $\ell$ turns out to play a particularly significant role in the properties of the coordinate $x_1$ for which we find

$$
\langle x_1 \rangle = \langle q_1 \rangle \left( e^{i\pi_0} \right) = q_1 e^{i\ell_0} e^{-\frac{\ell_0^2}{2\sigma_0^2}}, \quad \delta x_1 = e^{i\pi_0} \left[ \frac{1}{4\sigma_0^2} + q_1^2 \left( 1 - e^{-\ell_0^2} \right) \right]^{1/2},
$$

whereas for the $\kappa$-Minkowski time coordinate $x_0$ one has

$$
\langle x_0 \rangle = \bar{q}_0 - i\frac{\ell}{2}, \quad \delta x_0 = \frac{1}{2\sigma_0},
$$

(comments on the imaginary contribution $-i\ell/2$ to $(x_0)$ are given in Ref. [6]).

We argue that in our non-commutative spacetime the notion of point itself is changed: before imposing any dynamic, already at the kinematical level, there are no more sharply definite spacetime points, but they are replaced by non-local distributions on spacetime, as it is intuitively suggested by a spacetime foam picture. In the following, when we talk of points, is to these distributions that we are referring, being them the best replace available to classical points.

Our next task is to act with a translation transformation on our gaussian state and see what changes as a result. For the expectation values here of interest this is equivalently done by acting with the translation transformation on the operators $x_1$ and $x_0$. Following again our work in Ref. [6] one finds that

$$
\langle T_\alpha^\mu \triangleright x_0 \rangle = \bar{q}_0 - a_0 - i\frac{\ell}{2}, \quad \delta \left( T_\alpha^\mu \triangleright x_0 \right) = \frac{1}{2\sigma_0},
$$

and

$$
\langle T_\alpha^\mu \triangleright x_1 \rangle = (\bar{q}_1 - a_1) e^{i\pi_0} e^{-\frac{\ell_0^2}{2\sigma_0^2}}, \quad \delta \left( T_\alpha^\mu \triangleright x_1 \right) = e^{i\pi_0} \left[ \frac{1}{4\sigma_0^2} + (\bar{q}_1 - a_1)^2 \left( 1 - e^{-\ell_0^2} \right) \right]^{1/2}.
$$

The interpretation here of course is such that the $x_0, x_1$ are operators characterizing the distance of a given (fuzzy) point from the frame origin of some observer Alice, and then $T_\alpha^\mu \triangleright x_0, T_\alpha^\mu \triangleright x_1$ are the operators that characterize the distance of that point from the frame origin of an observer Bob, purely translated with respect to Alice. Accordingly one can deduce the relation between the mean values and uncertainties in positions among two distant observers in relative rest by comparing (10) to (11) and comparing (9) to (12). This we did in detail in Ref. [6]. We here summarize the main message of that analysis, focusing on the case of two fuzzy points in $\kappa$-Minkowski as described by two distant observers: one of the points is near observer Alice, while the other one is near observer Bob, purely translated with respect to Alice.
(10) to (11) and comparing (9) to (12) one then finds two main features [6]:

(i) the same point appears to be more fuzzy to a distant observer than to a nearby observer,
(ii) the point at Alice is not described as being at Alice in the coordinatization of spacetime of observer Bob, and vice versa the point at Bob is not described as being at Bob in the coordinatization of spacetime of observer Alice.

The second feature, (ii), is essentially already known from previous studies of relative locality in the classical limit [14, 15]: one can have consistently relativistic theories where pairs of points found to be coincident by a nearby observer (or, as in the case here considered, a point found to coincide with the origin of that observer) are instead described as noncoincident if one uses the inferences about those points by a distant observer. Feature (i) was established for the first time in our Ref. [6], and is a feature of relative locality for the fuzziness of points in a quantum spacetime. The emerging picture is fully relativistic (though in the sense of deformed relativistic symmetries [16, 17, 18, 19]): all observers in $\kappa$-Minkowski perceive their origin as the point of lowest fuzziness and attribute to distant points fuzziness proportional to the distance.

![Figure 1.](image)

Figure 1. We illustrate the features of relative locality we uncovered for the $\kappa$-Minkowski quantum spacetime by considering the case of two distant observers, Alice and Bob, in relative rest (with synchronized clocks). In figure we have only two points in $\kappa$-Minkowski, each described by a gaussian state in our Hilbert space. One of the points is at Alice (centered in the spacetime origin of Alice’s coordinatization) while the other point is at Bob. The left panel reflects Alice’s description of the two points, which in particular attributes to the distant point at Bob larger fuzziness than Bob observes (right panel). And in Alice’s coordinatization the distant point is not exactly at Bob. Bob’s description (right panel) of the two points is specular, in the appropriately relativistic fashion, to the one of Alice. The magnitude of effects shown would require the distance $L$ to be much bigger than drawable. And for definiteness in figure we assumed $\pi_0 \simeq 2\sigma_0$ and $\sigma_1 \simeq \sigma_0$.

3. Fuzziness of $\kappa$-Minkowski worldlines
In the previous section we worked on the kinematical Hilbert space. For our next (and here last) task we must progress to the level of the physical Hilbert space [6, 8, 11]. We want now to use $\kappa$-Minkowski noncommutativity as a way to model the influence of quantum-gravity effects on the fuzziness of worldlines. This should be viewed in the context of attempts to describe the dynamics of matter particles as effectively occurring in an “environment” of short-distance quantum-gravitational degrees of freedom: for propagating particles with wavelength much larger than the Planck length, when it may be appropriate to integrate out these quantum-
gravitational degrees of freedom, the main residual effect of short-distance gravity could indeed be an additional contribution to the fuzziness of worldlines, which one could model with spacetime noncommutativity.

For working on the physical Hilbert space we follow (as we did in Ref. [8]) the prescription adopted in Ref. [11]: we impose the restriction for our states to belong to the kernel of the Hamiltonian constraint $\mathcal{H}_\ell$, Eq.(6), by introducing a new scalar product that projects all the orbit of the gauge transformation generated by $\mathcal{H}_\ell$ on the same state. For massless particles, on which we here focus, this amounts to inserting the Hamiltonian constraint as follows

$$\langle \psi|\phi \rangle_{\mathcal{H}_\ell} = \langle \psi|\delta (\mathcal{H}_\ell) \Theta(\pi_0)|\phi \rangle$$

where $\langle \rangle$ is the scalar product on the kinematical Hilbert space and $\Theta(\pi_0)$ specifies a restriction [11] to positive-energy solutions of the constraint.

The next problem we must face concerns the identification of an observable suitable for the characterization of the fuzziness of the worldline. The apparently obvious choices, $x_1$ and $x_0$, are not suitable for this task, since they are not self-adjoint operators on the physical Hilbert space (they do not commute with $\mathcal{H}_\ell$). We propose to remedy this by focusing on the following “intercept operator” $\mathcal{A}$:

$$\mathcal{A} = e^{\ell x_0} \left( q_1 - \mathcal{V}_q 0 - \frac{1}{2} [q_0, \mathcal{V}] \right)$$

where $\mathcal{V}$ is short-hand for $\mathcal{V} \equiv (\partial \mathcal{H}_\ell / \partial \pi^0)^{-1} \partial \mathcal{H}_\ell / \partial \pi^1$.

One may notice that $\mathcal{A}$ is describable as an $\ell$-deformed Newton-Wigner operator [20]. And it is well known that within special-relativistic quantum mechanics there is no better estimator of localization than the Newton-Wigner operator (it can only be questioned for localization comparable to the Compton wavelength of the particle [20], but this merely conceptual limit of ideal localization is not relevant for our purposes here [8]).

Let us focus, for conceptual clarity, on the analysis of the properties of $\mathcal{A}$ for the case of $\Psi_{0,0}$, i.e. for $\bar{q}_0 = 0, \bar{q}_1 = 0$. One then easily finds that

$$\langle \Psi_{0,0}|\mathcal{A}|\Psi_{0,0} \rangle_{\mathcal{H}_\ell} = 0$$

so this is a case where the particle intercepts the observer Alice in her origin.

The fuzziness of this intercept, which reflects the fuzziness of the worldline [7, 8] described by $\Psi_{0,0}$, is characterized by [8]

$$\delta \mathcal{A}^2_{[\ell]} = \left( \langle \Psi_{0,0}|\mathcal{A}^2|\Psi_{0,0} \rangle_{\mathcal{H}_\ell} \right)_{[\ell]} \approx \ell (\pi_0) \sigma^{-2} / 2$$

where for simplicity we focused on the leading $\ell$-dependent contributions (index $[\ell]$) and we assumed that $\sigma_1$ is small enough, in comparison to $\sigma_0$, $\bar{\pi}_1$, to allow a saddle point approximation in the $\pi_1$ integration; then $\sigma$ (without indices) is the effective gaussian width after the saddle point approximation in $\pi_1$: $\sigma^{-2} \equiv \sigma_1^{-2} + \langle \mathcal{V} \rangle > 2 \sigma_0^{-2}$.

In the interpretation we proposed in Ref. [8] we describe Eq. (15) as the fuzziness of the worldline “at Alice” (at the point of crossing the origin of Alice’s reference frame). It is then interesting to establish whether observers reached by the particle at cosmological distances from Alice will observe bigger fuzziness. We characterize such observers as those who are connected to Alice by a pure translation, so that for them the state of the particle is $\Psi_{a_0,a_1}$, and are, like Alice, such that $\langle A \rangle = 0$, i.e. $\langle \Psi_{a_0,a_1}|\mathcal{A}|\Psi_{a_0,a_1} \rangle_{\mathcal{H}_\ell} = 0$. Unsurprisingly this requires $a^1 = \langle \mathcal{V} \rangle a^0$. And for these observers one finds

$$\delta \mathcal{A}^2_{[\ell]} = \left( \langle \Psi_{a^0,0}|\mathcal{A}^2|\Psi_{a^0,0} \rangle_{\mathcal{H}_\ell} \right)_{[\ell]} \approx \left( \frac{\ell (\pi_0)}{2 \sigma^2} + \ell^2 a_0^2 \sigma^2 \right)$$
From this we see that the approach we developed in Refs. [6, 7, 8] leads to the first example of a quantum-spacetime picture (and of an interpretation of relativistic quantum mechanics in such a spacetime) providing the main ingredient of scenarios [21, 22, 23, 24, 25, 26] motivating the idea that quantum-gravity effects could affect worldline fuzziness in ways that inevitably lead to an increase of this fuzziness as the particle propagates. One can in fact interpret our observer Alice, the observer on the worldline for whom the fuzziness of the intercept takes the minimum value, as the observer at the source (where the particle is produced), and then the intercept of the particle worldline with the origin of the reference frames of observers distant from Alice (where the particle could be detected) has bigger uncertainty.

As we stressed in Refs. [7, 8] this formalization of gravity’s contribution to worldline fuzziness could be relevant for an ongoing phenomenological effort aimed at finding experimental evidence of this sort of spacetime-fuzziness effects [21, 22, 23, 24, 25, 26].

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