Asymptotic behaviour of a differential operator
with a finite number of transmission conditions

Erdoğan Şen*, Oktay Mukhtarov#

*Department of Mathematics, Faculty of Science and Letters, Namık Kemal University, 59030
Tekirdağ, Turkey.
*Department of Mathematics Engineering, Istanbul Technical University, Maslak, 34469 İstanbul,
Turkey
#Gaziosmanpaşa Üniversitesi, Fen Fakültesi, Matematik Bölümü, 60250, Tokat

e-mails: erdogan.math@gmail.com, omukhtarov@yahoo.com

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ABSTRACT
In this paper following the same methods in [M. Kadakal, O. Sh. Mukhtarov, Sturm-Liouville problems with discontinuities at two points, Comput. Math. Appl., 54 (2007) 1367-1379] we investigate discontinuous two-point boundary value problems with eigenparameter in the boundary conditions and with transmission conditions at the finitely many points of discontinuity. A self-adjoint linear operator $A$ is defined in a suitable Hilbert space $H$ such that the eigenvalues of such a problem coincide with those of $A$. We obtain asymptotic formulas for the eigenvalues and eigenfunctions. Also we show that the eigenfunctions of $A$ are complete in $H$.

1 Introduction

The theory of discontinuous Sturm-Liouville type problems mainly has been developed by Mukhtarov and his students (see [1-11]). It is well-known that many topics in mathematical physics require the investigation of eigenvalues and eigenfunctions of Sturm-Liouville type boundary value problems. In recent years, more and more researchers are interested in the discontinuous Sturm-Liouville problems and its applications in physics (see [1-28]).

Discontinuous Sturm-Liouville problems with supplementary transmission conditions at the point(s) of discontinuity have been investigated in [6-9, 23-27].

In this study, we examine eigenvalues and eigenfunctions of the differential
equation
\[ \tau u := -u''(x) + q(x)u(x) = \lambda u(x) \quad (1) \]
on \([-1, h_1) \cup (h_1, h_2) \cup ... \cup (h_m, 1] \), with boundary conditions
\begin{align*}
\tau_1 u &:= \alpha_1 u(-1) + \alpha_2 u'(-1) = 0, \\
\tau_2 u &:= (\beta_1^2 \lambda + \beta_2) u(1) + (\beta_2^2 \lambda + \beta_2) u'(1) = 0
\end{align*}
(2) (3)
and transmission conditions at the points of discontinuity \( x = h_i \ (i = 1, m) \),
\begin{align*}
\tau_{2i+1} u &:= u(h_i - 0) - \delta_i u(h_i + 0) = 0, \\
\tau_{2i+2} u &:= u'(h_i - 0) - \delta_i u'(h_i + 0) = 0
\end{align*}
(4) (5)
where \(-1 < h_1 < h_2 < ... < h_m < 1\), \(q(x)\) is a given real-valued function continuous in \([-1, h_1), (h_1, h_2), ..., (h_m, 1]\) and has finite limits \(q(h_i \pm 0) = \lim_{x \to h_i \pm 0} q(x)\) \((i = 1, m): \lambda\) is a complex eigenvalue parameter; \(\delta_i \ (i = 1, m)\), \(\alpha_j, \beta_j, \beta_j'\) \((j = 1, 2)\) are real numbers; \(|\alpha_1| + |\alpha_2| \neq 0\) and \(\delta_i \neq 0 \ (i = 1, m)\). As following [6] every where we assume that \(\rho = \beta_1^2 \beta_2 - \beta_1^2 \beta_2' > 0\).

2 Operator Formulation

In the Hilbert space \(H = L_2(-1, 1) \oplus \mathbb{C}\) we define an inner product by
\[
\langle F, G \rangle := \int_{-1}^{h_1} f(x) g(x) dx + \delta_1^2 \int_{h_1}^{h_2} f(x) g(x) dx + \delta_1^2 \delta_2^2 \int_{h_2}^{h_3} f(x) g(x) dx + ... + \prod_{i=1}^{m} \delta_i^2 \int_{h_m}^{1} f(x) g(x) dx + \frac{\prod_{i=1}^{m} \delta_i^2}{p} f_1 g_1
\]
for
\[
F := \left( \begin{array}{c} f(x) \\ f_1 \end{array} \right), \quad G := \left( \begin{array}{c} g(x) \\ g_1 \end{array} \right) \in H
\]
following (7) for convenience we put
\[
R_1(u) := \beta_1 u(1) - \beta_2 u'(1), \\
R'_1(u) := \beta_1' u(1) - \beta_2' u'(1).
\]
For functionals \(f(x)\), which defined on \([-1, h_1) \cup (h_1, h_2) \cup ... \cup (h_m, 1]\) and has finite limits \(f(h_i \pm 0) := \lim_{x \to h_i \pm 0} f(x) \ (i = 1, m)\). By \(f_i(x) \ (i = 1, m + 1)\)
we denote the functions

\[ f_1 (x) := \begin{cases} f(x), & x \in [-1, h_1), \\ \lim_{x \to h_1^-} f(x), & x = h_1, \end{cases} \quad f_2 (x) := \begin{cases} \lim_{x \to h_1^+} f(x), & x = h_1, \\ f(x), & x \in (h_1, h_2), \end{cases} \]

\[ f_m (x) := \begin{cases} \lim_{x \to h_{m-1}^+} f(x), & x = h_{m-1}, \\ f(x), & x \in (h_{m-1}, h_m), \\ \lim_{x \to h_m^+} f(x), & x = h_m, \end{cases} \quad f_{m+1} (x) := \begin{cases} \lim_{x \to h_m^+} f(x), & x = h_m, \\ f(x), & x \in (h_m, 1] \end{cases} \]

which are defined on \( \Omega_1 := [-1, h_1], \Omega_2 := [h_1, h_2], \ldots, \Omega_m := [h_{m-1}, h_m], \Omega_{m+1} := [h_m, 1] \) respectively. We can rewrite the considered problem (1) − (5) in the operator formulation as

\[ AF = \lambda F \]

where

\[ F := \begin{pmatrix} f(x) \\ -R_1 (f) \end{pmatrix} \in D (A) \]

and with

\[ D (A) := \left\{ F \in H \mid f_i (x), f'_i (x) \text{ are absolutely continuous in } \Omega_i \ (i = 1, m+1), \tau f \in L^2 [-1, 1], \tau_{2i+1} u := u (h_i - 0) - \delta_i u (h_i + 0) = 0, \tau_{2i+2} u := u' (h_i - 0) - \delta_i u' (h_i + 0) = 0 \right\} \]

and

\[ AF = \begin{pmatrix} \tau f \\ -R_1 (f) \end{pmatrix}. \]

Consequently, the problem (1) − (5) can be considered as the eigenvalue problem for the operator \( A \). Obviously, we have

Lemma 2.1. The eigenvalues of the boundary value problem (1)-(5) coincide with those of \( A \), and its eigenfunctions are the first components of the corresponding eigenfunctions of \( A \).

Lemma 2.2. The domain \( D (A) \) is dense in \( H \).

Proof. Let \( F = \begin{pmatrix} f(x) \\ f_1 \end{pmatrix} \in H, F \perp D (A) \) and \( \widetilde{C}_0^\infty \) be a functional set such that

\[ \phi (x) = \begin{cases} \phi_1 (x), x \in [-1, h_1), \\ \phi_2 (x), x \in (h_1, h_2), \\ \vdots \\ \phi_m (x), x \in (h_{m-1}, h_m), \\ \phi_{m+1} (x), x \in (h_m, 1] \end{cases} \]

for \( \phi_1 (x) \in \widetilde{C}_0^\infty [-1, h_1), \phi_2 (x) \in \widetilde{C}_0^\infty (h_1, h_2), \ldots, \phi_{m+1} (x) \in \widetilde{C}_0^\infty (h_m, 1] \). Since \( \widetilde{C}_0^\infty \oplus 0 \subset D (A) \ (0 \in \mathbb{C}), \) any \( U = \begin{pmatrix} u(x) \\ 0 \end{pmatrix} \in \widetilde{C}_0^\infty \oplus 0 \) is orthogonal to \( F \),
\[ \langle F, U \rangle = \int_{-1}^{h_1} f(x) u(x) dx + \delta_1^2 \int_{h_1}^{h_2} f(x) u(x) dx + \delta_2^3 \int_{h_2}^{h_3} f(x) u(x) dx + \ldots + \prod_{i=1}^{m} \delta_i^2 \int f(x) u(x) dx = \langle f, u \rangle_{1}. \]

We can learn that \( f(x) \) is orthogonal to \( \tilde{C}_0^{\infty} \) in \( L^2[-1, 1] \), this implies \( f(x) = 0 \).

So for all \( G = \left( \begin{array}{c} g_1 \\ g_1 \end{array} \right) \in D(A), \) \( \langle F, G \rangle = \prod_{i=1}^{m} \delta_i^2 \left( R'_{g_1}(g) - R_{g_1}(g) \right) = 0. \) Thus \( f_1 = 0 \) since \( g_1 = R'_2(g) \) can be chosen arbitrarily. So \( F = \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \), which proves the assertion.

**Theorem 2.1.** The linear operator \( A \) is symmetric in \( H \).

**Proof.** Let \( F, G \in D(A) \). By two partial integrations, we get

\[
\langle AF, G \rangle = \langle F, AG \rangle + \delta_1^2 (W(f, \overline{g}; h_2 - h_1)) + \delta_1^2 (W(f, \overline{g}; h_2 - h_1)) + \ldots + \prod_{i=1}^{m} \delta_i^2 (W(f, \overline{g}; h_m - h_1))
\]

\[
+ \delta_1^2 \left( R'_1(f) R_{1}(g) - R'_1(f) R_{1}(g) \right) \tag{6}
\]

where

\[
W(f, \overline{g}, x) = f(x) \overline{g}(x) - f'(x) \overline{g}(x)
\]

denotes the Wronskians of the functions \( f \) and \( g \). Since \( f \) and \( \overline{g} \) satisfy the boundary condition (2), it follows that

\[
W(f, \overline{g}, -1) = 0. \tag{7}
\]

From the transmission conditions (4)-(5) we get

\[
\begin{align*}
W(f, \overline{g}; h_1 - 0) &= \delta_1^2 W(f, \overline{g}; h_1 + 0), \\
W(f, \overline{g}; h_2 - 0) &= \delta_2^3 W(f, \overline{g}; h_2 + 0), \\
&\vdots \\
W(f, \overline{g}; h_m - 0) &= \delta_m^2 W(f, \overline{g}; h_m + 0). 
\end{align*} \tag{8}
\]
Furthermore,

\[ R'_1 (f) R_1 (g) - R_1 (f) R'_1 (g) \]
\[ = \left( \beta'_1 f (1) - \beta'_2 f' (1) \right) \left( \beta_1 g (1) - \beta_2 g' (1) \right) - \left( \beta_1 f (1) - \beta_2 f' (1) \right) \left( \beta'_1 g (1) - \beta'_2 g' (1) \right) \]
\[ = \beta_2 \beta'_1 f (1) - \beta_2 \beta'_1 f' (1) \]
\[ = \rho \left( f' (1) g (1) - f (1) g' (1) \right) = -\rho W (f, g; 1). \]

Finally, substituting (7)-(9) in (6) then we get
\[ \langle AF, G \rangle = \langle F, AG \rangle \quad (F, G \in D (A)) \quad (10) \]

Now we can write the following theorem with the help of Theorem 2.1, Naimark’s Patching Lemma \[15\] and using the similar way in \[6\]

**Theorem 2.2.** The linear operator \( A \) is self-adjoint in \( H \).

**Corollary 2.1.** All eigenvalues of the problem (1)-(5) are real.

We can now assume that all eigenfunctions are real-valued.

**Corollary 2.2.** If \( \lambda_1 \) and \( \lambda_2 \) are two different eigenvalues of the problem (1)-(5), then the corresponding eigenfunctions \( u_1 \) and \( u_2 \) of this problem satisfy the following equality:

\[ \int_{-1}^{h_1} u_1 (x) u_2 (x) \, dx + \delta^2 \int_{h_1}^{h_2} u_1 (x) u_2 (x) \, dx + \delta^2 \int_{h_2}^{h_3} u_1 (x) u_2 (x) \, dx + \cdots + \prod_{i=1}^{m} \int_{h_i}^{h_{i+1}} u_1 (x) u_2 (x) \, dx = \frac{\prod \delta^2_i}{\rho} R'_1 (u_1) R'_1 (u_2). \]

In fact this formula means the orthogonality of eigenfunctions \( u_1 \) and \( u_2 \) in the Hilbert space \( H \).

We need the following lemma, which can be proved by the same technique as in \[4\].

**Lemma 2.3.** Let the real-valued function \( q (x) \) be continuous in \([-1, 1]\) and \( f (\lambda), g (\lambda) \) are given entire functions. Then for any \( \lambda \in \mathbb{C} \) the equation

\[ -u'' + q (x) u = \lambda u, \quad x \in [-1, 1] \]

has a unique solution \( u = u (x, \lambda) \) satisfies the initial conditions

\[ u (-1) = f (\lambda), \quad u' (-1) = g (\lambda) \quad \text{(or} \quad u (1) = f (\lambda), \quad u' (1) = g (\lambda) \text{)}. \]

For each fixed \( x \in [-1, 1], \ u (x, \lambda) \) is an entire function of \( \lambda \).
We shall define two solutions

\[
\phi_{\lambda} (x) = \begin{cases} 
\phi_{1 \lambda} (x), & x \in [-1, h_1), \\
\phi_{2 \lambda} (x), & x \in (h_1, h_2), \\
\vdots \\
\phi_{(m+1) \lambda} (x), & x \in (h_m, 1], 
\end{cases}
\]

and \( \chi_{\lambda} (x) = \begin{cases} 
\chi_{1 \lambda} (x), & x \in [-1, h_1), \\
\chi_{2 \lambda} (x), & x \in (h_1, h_2), \\
\vdots \\
\chi_{(m+1) \lambda} (x), & x \in (h_m, 1], 
\end{cases} \)

of the Eq. (1) as follows: Let \( \phi_{1 \lambda} (x) := \phi_1 (x, \lambda) \) be the solution of Eq. (1) on \([-1, h_1] \), which satisfies the initial conditions

\[
u (-1) = \alpha_2, \quad u' (-1) = -\alpha_1. \tag{11}\]

By virtue of Lemma 2.1, after defining this solution, we may define the solution \( \phi_2 (x, \lambda) := \phi_{2 \lambda} (x) \) of Eq. (1) on \([h_1, h_2] \) by means of the solution \( \phi_1 (x, \lambda) \) by the initial conditions

\[
u (h_1) = \delta_{1}^{-1} \phi_1 (h_1, \lambda), \quad u' (h_1) = \delta_{1}^{-1} \phi_1' (h_1, \lambda). \tag{12}\]

After defining this solution, we may define the solution \( \phi_3 (x, \lambda) := \phi_{3 \lambda} (x) \) of Eq. (1) on \([h_2, h_3] \) by means of the solution \( \phi_2 (x, \lambda) \) by the initial conditions

\[
u (h_2) = \delta_{2}^{-1} \phi_2 (h_2, \lambda), \quad u' (h_2) = \delta_{2}^{-1} \phi_2' (h_2, \lambda). \tag{13}\]

Continuing in this manner, we may define the solution \( \phi_{(m+1)} (x, \lambda) := \phi_{(m+1) \lambda} (x) \) of Eq. (1) on \([h_m, 1] \) by means of the solution \( \phi_m (x, \lambda) \) by the initial conditions

\[
u (h_m) = \delta_{m}^{-1} \phi_m (h_m, \lambda), \quad u' (h_m) = \delta_{m}^{-1} \phi_m' (h_m, \lambda). \tag{14}\]

Therefore, \( \phi (x, \lambda) \) satisfies the Eq. (1) on \([-1, h_1) \cup (h_1, h_2) \cup \ldots \cup (h_m, 1], \) the boundary condition (3), and the transmission conditions (4)-(5).

Analogically, first we define the solution \( \chi_{(m+1) \lambda} (x) := \chi_{(m+1)} (x, \lambda) \) on \([h_m, 1] \) by the initial conditions

\[
u (1) = \beta_2 \lambda + \beta_2, \quad u' (1) = \beta_1 \lambda + \beta_1. \tag{15}\]

Again, after defining this solution, we may define the solution \( \chi_{m \lambda} (x) := \chi_m (x, \lambda) \) of the Eq. (1) on \([h_{m-1}, h_m] \) by the initial conditions

\[
u (h_m) = \delta_{m} \chi_{m+1} (h_m, \lambda), \quad u' (h_m) = \delta_{m} \chi_{m+1}' (h_m, \lambda). \tag{16}\]

Continuing in this manner, we may define the solution \( \chi_{1 \lambda} (x) := \chi_1 (x, \lambda) \) of the Eq. (1) on \([-1, h_1] \) by the initial conditions

\[
u (h_1) = \delta_1 \chi_2 (h_1, \lambda), \quad u' (h_1) = \delta_1 \chi_2' (h_1, \lambda). \tag{17}\]

Therefore, \( \chi (x, \lambda) \) satisfies the Eq. (1) on \([-1, h_1) \cup (h_1, h_2) \cup \ldots \cup (h_m, 1], \) the boundary condition (3), and the transmission conditions (4)-(5).

It is obvious that the Wronsians

\[
\omega_i (\lambda) := W_\lambda (\phi_i, \chi_i; x) := \phi_i (x, \lambda) \chi_i' (x, \lambda) - \phi_i' (x, \lambda) \chi_i (x, \lambda), \quad x \in \Omega_i \quad (i = 1, m + 1)
\]
are independent of \(x \in \Omega_i\) and entire functions.

**Lemma 2.4.** For each \(\lambda \in \mathbb{C}\), \(\omega_1 (\lambda) = \delta_1^2 \omega_2 (\lambda) = \delta_1^2 \delta_2^2 \omega_3 (\lambda) = \ldots = \left( \prod_{i=1}^{m} \delta_i^2 \right) \omega_{m+1} (\lambda)\).

**Proof.** By the means of (12), (13), (14), (16) and (17), the short calculation gives

\[
W_\lambda (\phi_1, \chi_1; h_1) = \delta_1^2 W_\lambda (\phi_2, \chi_2; h_1) = \delta_1^2 \delta_2^2 W_\lambda (\phi_3, \chi_3; h_2) = \delta_1^2 \delta_2^2 \delta_3^2 W_\lambda (\phi_4, \chi_4; h_3) = \ldots = \left( \prod_{i=1}^{m} \delta_i^2 \right) W_\lambda (\phi_{m+1}, \chi_{m+1}; h_m)
\]

so \(\omega_1 (\lambda) = \delta_1^2 \omega_2 (\lambda) = \delta_1^2 \delta_2^2 \omega_3 (\lambda) = \ldots = \left( \prod_{i=1}^{m} \delta_i^2 \right) \omega_{m+1} (\lambda)\).

Now we may introduce the characteristic function

\[
\omega (\lambda) := \omega_1 (\lambda) = \delta_1^2 \omega_2 (\lambda) = \delta_1^2 \delta_2^2 \omega_3 (\lambda) = \ldots = \left( \prod_{i=1}^{m} \delta_i^2 \right) \omega_{m+1} (\lambda).
\]

**Theorem 2.3.** The eigenvalues of the problem (1)-(5) are the zeros of the function \(\omega (\lambda)\).

**Proof.** Let \(\omega (\lambda_0) = 0\). Then \(W_\lambda (\phi_1, \chi_1; x) = 0\) and therefore the functions \(\phi_1 \lambda_0 (x)\) and \(\chi_1 \lambda_0 (x)\) are linearly dependent, i.e.

\[
\chi_1 \lambda_0 (x) = k_1 \phi_1 \lambda_0 (x), \quad x \in [-1, h_1]
\]

for some \(k_1 \neq 0\). From this, it follows that \(\chi (x, \lambda_0)\) satisfies also the first boundary condition (2), so \(\chi (x, \lambda_0)\) is an eigenfunction of the problem (1)–(5) corresponding to this eigenvalue \(\lambda_0\).

Now we let \(u_0 (x)\) be any eigenfunction corresponding to eigenvalue \(\lambda_0\), but \(\omega (\lambda_0) \neq 0\). Then the functions \(\phi_1, \chi_1, \phi_2, \chi_2, \ldots, \phi_{m+1}, \chi_{m+1}\) would be linearly independent on \([-1, h_1], [h_1, h_2]\) and \([h_m, 1]\) respectively. Therefore \(u_0 (x)\) may be represented in the following form

\[
u_0 (x) = \begin{cases} 
  c_1 \phi_1 (x, \lambda_0) + c_2 \chi_1 (x, \lambda_0), & x \in [-1, h_1), \\
  c_3 \phi_2 (x, \lambda_0) + c_4 \chi_2 (x, \lambda_0), & x \in (h_1, h_2), \\
  \vdots \\
  c_{2m+1} \phi_{m+1} (x, \lambda_0) + c_{2m+2} \chi_{m+1} (x, \lambda_0), & x \in (h_m, 1].
\end{cases}
\]

where at least one of the constants \(c_1, c_2, \ldots, c_{2m+2}\) is not zero. Considering the equations

\[
\tau_v (u_0 (x)) = 0, \quad v = 1, 2m + 2
\]

as the homogenous system of linear equations of the variables \(c_1, c_2, c_{2n+2}\) and taking (12), (13), (14), (16) and (17) into account, it follows that the
Thus we get a contradiction, which completes the proof.

Proof. Suppose, if possible, that $k_1 = k_2 = \ldots = k_{m+1}$. Then by virtue of Theorem 2.2, we can derive that $\phi(x, \lambda_0)$ and $\chi(x, \lambda_0)$ are linearly dependent.

**Lemma 2.5.** If $\lambda = \lambda_0$ is an eigenvalue, then $\phi(x, \lambda_0)$ and $\chi(x, \lambda_0)$ are linearly dependent.

**Proof.** Let $\lambda = \lambda_0$ be an eigenvalue. Then by virtue of Theorem 2.2

$$W(\phi_{i\lambda_0}, \chi_{i\lambda_0}; x) = \omega_i(\lambda_0) = 0$$

and hence

$$\chi_{i\lambda_0}(x) = k_i \phi_{i\lambda_0}(x) \quad (i = 1, m + 1) \quad (19)$$

for some $k_1 \neq 0, k_2 \neq 0, \ldots, k_{m+1} \neq 0$. We must show that $k_1 = k_2 = \ldots = k_{m+1}$. Suppose, if possible, that $k_m \neq k_{m+1}$.

Taking into account the definitions of the solutions $\phi_i(x, \lambda)$ and $\chi_i(x, \lambda)$ from the equalities (19), we have

$$\tau_{2m+1}(\chi_{\lambda_0}) = \chi_{\lambda_0}(h_m - 0) - \delta_m \chi_{\lambda_0}(h_m + 0) = \chi_{m\lambda_0}(h_m) - \delta_m \chi_{m+1\lambda_0}(h_m) = k_m \phi_m(h_m) - \delta_m \phi_{m+1}(h_m) = k_m \delta_m \phi_{m+1}(h_m) - \delta_m k_{m+1} \phi_{m+1}(h_m) = 0.$$

Since $\tau_{2m+1}(\chi_{\lambda_0}) = 0$ and $\delta_m (k_m - k_{m+1}) \neq 0$, it follows that

$$\phi_{(m+1)\lambda_0}(h_m) = 0. \quad (20)$$

By the same procedure from $\tau_{2m+2}(\chi_{\lambda_0}) = 0$ we can derive that

$$\phi'_{(m+1)\lambda_0}(h_m) = 0. \quad (21)$$

From the fact that $\phi_{(m+1)\lambda_0}(x)$ is a solution of the differential equation (1) on $[h_m, 1]$ and satisfies the initial conditions (20) and (21), it follows that $\phi_{(m+1)\lambda_0}(x) = 0$ identically on $[h_m, 1]$ because of the well-known existence
and uniqueness theorem for the initial value problems of the ordinary linear differential equations. Making use of (14), (19) and (20), we may also derive that
\[ \phi_{m, k_0} (h_m) = \phi'_{m, k_0} (h_m) = 0 \]  
(22)
Continuing in this matter, we may also find that
\[ \begin{aligned}
\phi_{(m-1), k_0} (h_{m-1}) &= \phi'_{(m-1), k_0} (h_{m-1}) = 0. \\
& \vdots \\
\phi_{1, k_0} (h_1) &= \phi'_{1, k_0} (h_1) = 0.
\end{aligned} \]  
(23)
identically on \([h_{m-1}, h_m], \ldots, [-1, h_1]\) respectively. Hence \(\phi(x, \lambda_0) = 0\) identically on \([-1, h_1] \cup (h_1, h_2) \cup \ldots \cup (h_m, 1]\). But this contradicts with (11). Hence \(k_m = k_{m+1}\). Analogically we can prove that \(k_{m-1} = k_m, \ldots, k_2 = k_3\) and \(k_1 = k_2\).

\textbf{Corollary 2.3.} If \(\lambda = \lambda_0\) is an eigenvalue, then both \(\phi(x, \lambda_0)\) and \(\chi(x, \lambda_0)\) are eigenfunctions corresponding to this eigenvalue.

\textbf{Lemma 2.6.} All eigenvalues \(\lambda_n\) are simple zeros of \(\omega(\lambda)\).

\textbf{Proof.} Using the Lagrange’s formula (cf. [25], p. 6-7), it can be shown that
\[ (\lambda - \lambda_n) \left( \int_{h_1}^{h_m} \phi_\lambda (x) \phi_{\lambda_n} (x) \, dx + \delta_1^2 \int_{h_1}^{h_2} \phi_\lambda (x) \phi_{\lambda_n} (x) \, dx + \delta_1^2 \delta_2^2 \int_{h_1}^{h_3} \phi_\lambda (x) \phi_{\lambda_n} (x) \, dx + \ldots + \prod_{i=1}^{m} \delta_i^2 \right) W (\phi_\lambda, \phi_{\lambda_n}; 1) \]  
(24)
for any \(\lambda\). Recall that
\[ \chi_{\lambda_n} (x) = k_n \phi_{\lambda_n} (x), \quad x \in [-1, h_1] \cup (h_1, h_2) \cup \ldots \cup (h_m, 1] \]
for some \(k_n \neq 0, n = 1, 2, \ldots\). Using this equality for the right side of (24), we have
\[ W (\phi_\lambda, \phi_{\lambda_n}; 1) = \frac{1}{k_n} W (\phi_\lambda, \chi_{\lambda_n}; 1) = \frac{1}{k_n} \left( \lambda_n R'_1 (\phi_\lambda) + R_1 (\phi_\lambda) \right) \]
\[ = \frac{1}{k_n} \left[ \omega (\lambda) - (\lambda - \lambda_n) R'_1 (\phi_\lambda) \right] \]
\[ = (\lambda - \lambda_n) \frac{1}{k_n} \left[ \frac{\omega (\lambda)}{\lambda - \lambda_n} - R'_1 (\phi_\lambda) \right]. \]

Substituting this formula in (24) and letting \(\lambda \to \lambda_n\), we get
\[ \int_{-1}^{h_1} (\phi_{\lambda_n} (x))^2 \, dx + \delta_1^2 \int_{h_1}^{h_2} (\phi_{\lambda_n} (x))^2 \, dx + \delta_1^2 \delta_2^2 \int_{h_1}^{h_3} (\phi_{\lambda_n} (x))^2 \, dx + \ldots + \prod_{i=1}^{m} \delta_i^2 \int_{h_m} (\phi_{\lambda_n} (x))^2 \, dx \]
\[ = \frac{1}{k_n} \left( \omega (\lambda_n) - R'_1 (\phi_{\lambda_n}) \right). \]  
(25)
Now putting
\[ R'_1 (\phi_{\lambda_n}) = \frac{1}{k_n} R'_1 (\chi_{\lambda_n}) = \frac{\rho}{k_n} \]
in (25) we get \( \omega' (\lambda_n) \neq 0. \]

**Definition 2.1.** The geometric multiplicity of an eigenvalue \( \lambda \) of the problem (1)-(5) is the dimension of its eigenspace, i.e., the number of its linearly independent eigenfunctions.

**Theorem 2.4.** All eigenvalues of the problem (1)-(5) are geometrically simple.

**Proof.** If \( f \) and \( q \) are two eigenfunctions for an eigenvalue \( \lambda_0 \) of (1)-(5) then (2) implies that \( f(-1) = cg(-1) \) and \( f'(-1) = cg'(-1) \) for some constant \( c \in \mathbb{C} \). By the uniqueness theorem for solutions of ordinary differential equation and the transmission conditions (4)-(5), we have that \( f = cg \) on \([-1, h_1] \) and \([h_m, 1]\). Thus the geometric multiplicity of \( \lambda_0 \) is one. \( \blacksquare \)

### 3 Asymptotic approximate formulas of \( \omega (\lambda) \) for four distinct cases

We start by proving some lemmas.

**Lemma 3.1.** Let \( \phi (x, \lambda) \) be the solutions of Eq. (1) defined in Section 2, and let \( \lambda = s^2 \). Then the following integral equations hold for \( k = 0, 1 \):

\[
\begin{cases}
\phi^{(k)}_{1\lambda} (x) = \alpha_2 (\cos s (x + 1))^{(k)} - \alpha_1 \frac{1}{s} (\sin s (x + 1))^{(k)} \\
+ \frac{1}{s^2} \int_{-1}^{-1} (\sin s (x - y))^{(k)} q (y) \phi_{1\lambda} (y) \ dy, \\
\phi^{(k)}_{2\lambda} (x) = \frac{1}{s} \phi_{1\lambda} (h_1) (\cos s (x - h_1))^{(k)} + \frac{1}{s^2} \phi'_{1\lambda} (h_1) (\sin s (x - h_1))^{(k)} \\
+ \frac{1}{s^2} \int_{h_1}^{h_1} (\sin s (x - y))^{(k)} q (y) \phi_{2\lambda} (y) \ dy, \\
\vdots \\
\phi^{(k)}_{(m+1)\lambda} (x) = \frac{1}{s^m} \phi_{m\lambda} (h_m) (\cos s (x - h_m))^{(k)} + \frac{1}{s^m} \phi'_{m\lambda} (h_m) (\sin s (x - h_m))^{(k)} \\
+ \frac{1}{s^m} \int_{h_m}^{h_m} (\sin s (x - y))^{(k)} q (y) \phi_{(m+1)\lambda} (y) \ dy,
\end{cases}
\]  
\[ (26) \]

where \( (\cdot)^{(k)} = \frac{d^k}{dx^k} (\cdot) \).

**Proof.** It is enough to substitute \( s^2 \phi_{1\lambda} (y) + \phi_{1\lambda}'' (y), s^2 \phi_{2\lambda} (y) + \phi_{2\lambda}'' (y), \ldots, s^2 \phi_{(m+1)\lambda} (y) + \phi_{(m+1)\lambda}'' (y) \) instead of \( q (y) \phi_{1\lambda} (y), q (y) \phi_{2\lambda} (y), q (y) \phi_{(m+1)\lambda} (y) \) in the integral terms of the (26), respectively, and integrate by parts twice. \( \blacksquare \)

**Lemma 3.2.** Let \( \lambda = s^2 \), \( \text{Im} s = t \). Then the functions \( \phi_{k\lambda} (x) \) have the following asymptotic formulas for \( |\lambda| \to \infty \), which hold uniformly for \( x \in \Omega_i \),
(for $i = 1, m + 1$ and $k = 0, 1.$):

$$
\phi_{1\lambda}^{(k)} (x) = \alpha_2 \cos s (x + 1))^{(k)} + O \left( |s|^{-k-1} e^{t|x+1|} \right),
$$

$$
\phi_{2\lambda}^{(k)} (x) = \frac{\alpha_2}{\delta_1} \cos s (x + 1))^{(k)} + O \left( |s|^{-k-1} e^{t|x+1|} \right),
$$

$$
\vdots
$$

$$
\phi_{(m+1)\lambda}^{(k)} (x) = \frac{\alpha_2}{m} \prod_{i=1}^{m} \cos s (x + 1))^{(k)} + O \left( |s|^{-k-1} e^{t|x+1|} \right) \quad (27)
$$

if $\alpha_2 \neq 0$,

$$
\phi_{1\lambda}^{(k)} (x) = -\frac{\alpha_1}{s} \sin s (x + 1))^{(k)} + O \left( |s|^{-k-2} e^{t|x+1|} \right),
$$

$$
\phi_{2\lambda}^{(k)} (x) = -\frac{\alpha_1}{\delta_1} \sin s (x + 1))^{(k)} + O \left( |s|^{-k-2} e^{t|x+1|} \right),
$$

$$
\vdots
$$

$$
\phi_{(m+1)\lambda}^{(k)} (x) = -\frac{\alpha_1}{s \prod_{i=1}^{m} \delta_i} \sin s (x + 1))^{(k)} + O \left( |s|^{-k-2} e^{t|x+1|} \right) \quad (28)
$$

if $\alpha_2 = 0$.

**Proof.** Since the proof of the formulae for $\phi_{2\lambda} (x)$ is identical to Titchmarsh’s proof to similar results for $\phi_\lambda (x)$ (see [26], Lemma 1.7 p. 9-10), we may formulate them without proving them here.

Since the proof of the formulae for $\phi_{2\lambda} (x)$ and $\phi_{3\lambda} (x)$ are identical to Kadakal’s and Mukhtarov’s proof to similar results for $\phi_\lambda (x)$ (see [11], Lemma 3.2 p. 1373-1375), we may formulate them without proving them here. But the similar formulae for $\phi_{4\lambda} (x), ..., \phi_{(m+1)\lambda} (x)$ need individual consideration, since the last solutions are defined by the initial conditions of these special nonstandart forms. We shall only prove the formula (27) for $k = 0$ and $m = 3$.

Let $\alpha_2 \neq 0$. Then according to (27) for $m = 2$

$$
\phi_{3\lambda} (h_3) = \alpha_2 \left\{ \frac{1}{\delta_2} \cos s (h_3 - h_2) \left[ \frac{1}{\delta_1} \cos s (h_2 - h_1) \cos s (h_1 + 1) \right. 
\right.
$$

$$
\left. - \frac{1}{\delta_1} \sin s (h_2 - h_1) \sin s (h_1 + 1) \right. 
$$

$$
\left. - \frac{1}{\delta_2} \sin s (h_3 - h_2) \left[ \frac{1}{\delta_1} \sin s (h_2 - h_1) \cos s (h_1 + 1) 
\right. 
\right.
$$

$$
\left. + \frac{1}{\delta_1} \cos s (h_2 - h_1) \sin s (h_1 + 1) \right) \right\} + O \left( |s|^{-1} e^{t|[(h_3-h_2)+(h_2-h_1)+(h_1+1)]} \right)$$

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we have

\[ \phi_3' (h_3) = \alpha_2 \left\{ -\frac{s}{\delta_2} \sin s (h_3 - h_2) \left[ \frac{1}{\delta_1} \cos s (h_2 - h_1) \cos s (h_1 + 1) \right. \right. \]
\[ \left. \left. - \frac{1}{\delta_1} \sin s (h_2 - h_1) \sin s (h_1 + 1) \right] - \frac{s}{\delta_2} \cos s (h_3 - h_2) \left[ \frac{1}{\delta_1} \sin s (h_2 - h_1) \cos s (h_1 + 1) \right. \right. \]
\[ \left. \left. \left. + \frac{1}{\delta_1} \cos s (h_2 - h_1) \sin s (h_1 + 1) \right] + O \left( e^{\mid t \mid (h_3 - h_2) + (h_2 - h_1) + (h_1 + 1)} \right) \right\} \]

Substituting these asymptotic expressions into (26), we get

\[ \phi_4 (x) = \alpha_2 \left\{ \frac{1}{\delta_3} \cos s (x - h_3) \right\} \left\{ \frac{1}{\delta_1} \cos s (h_3 - h_2) \right\} \left\{ \frac{1}{\delta_1} \cos s (h_2 - h_1) \cos s (h_1 + 1) \right\} \]
\[ - \frac{1}{\delta_1} \sin s (h_2 - h_1) \sin s (h_1 + 1) \right\} - \frac{1}{\delta_1} \sin s (h_2 - h_1) \cos s (h_1 + 1) \]
\[ \times \left[ \frac{1}{\delta_1} \cos s (h_2 - h_1) \cos s (h_1 + 1) \right. \left. - \frac{1}{\delta_1} \sin s (h_2 - h_1) \sin s (h_1 + 1) \right] + \frac{1}{\delta_1} \]
\[ \times \cos s (h_3 - h_2) \left[ \frac{1}{\delta_1} \sin s (h_3 - h_1) \cos s (h_1 + 1) \right. \left. + \frac{1}{\delta_1} \cos s (h_2 - h_1) \sin s (h_1 + 1) \right] \}
\[ + \frac{1}{s} \int_{h_3}^{x} \sin s (x - y) q (y) \phi_4 (y) dy + O \left( \left\lvert s \right\rvert^{-1} e^{\mid t \mid (x - h_3) + (h_3 - h_2) + (h_2 - h_1) + (h_1 + 1)} \right) \right) \]

Multiplying through by \( e^{-\mid t \mid (x - h_3) + (h_3 - h_2) + (h_2 - h_1) + (h_1 + 1)} \), and denoting

\[ F_4 (x) := e^{-\mid t \mid (x - h_3) + (h_3 - h_2) + (h_2 - h_1) + (h_1 + 1)} \phi_4 (x) \]

we have

\[ \left\lvert F_4 (x) - e^{-\mid t \mid (x - h_3) + (h_3 - h_2) + (h_2 - h_1) + (h_1 + 1)} \phi_4 (x) \left\rvert \right. \]
\[ \left. \right\rvert \leq \left\lvert \left\{ \frac{1}{\delta_3} \cos s (x - h_3) \right\} \left\{ \frac{1}{\delta_1} \cos s (h_3 - h_2) \right\} \left\{ \frac{1}{\delta_1} \cos s (h_2 - h_1) \cos s (h_1 + 1) \right\} \right. \]
\[ \left. \left. - \frac{1}{\delta_1} \sin s (h_2 - h_1) \sin s (h_1 + 1) \right] - \frac{1}{\delta_1} \sin s (h_3 - h_2) \left[ \frac{1}{\delta_1} \sin s (h_2 - h_1) \cos s (h_1 + 1) \right. \right. \]
\[ \left. \left. \left. \left. \left. \left. + \frac{1}{\delta_1} \cos s (h_2 - h_1) \sin s (h_1 + 1) \right] + \frac{1}{\delta_1} \right. \right. \]
\[ \times \left[ \frac{1}{\delta_1} \cos s (h_2 - h_1) \cos s (h_1 + 1) \right. \left. - \frac{1}{\delta_1} \sin s (h_2 - h_1) \sin s (h_1 + 1) \right] + \frac{1}{\delta_1} \]
\[ \times \cos s (h_3 - h_2) \left[ \frac{1}{\delta_1} \sin s (h_3 - h_1) \cos s (h_1 + 1) \right. \left. + \frac{1}{\delta_1} \cos s (h_2 - h_1) \sin s (h_1 + 1) \right] \]
\[ + \frac{1}{s} \int_{h_3}^{x} \sin s (x - y) q (y) e^{-\mid t \mid (x - h_3) + (h_3 - h_2) + (h_2 - h_1) + (h_1 + 1)} F_4 (y) dy + O \left( \left\lvert s \right\rvert^{-1} \right) \].
Denoting \( M := \max_{x \in [h_3, 1]} |F_{4\lambda} (x)| \) from the last formula, it follows that

\[
M (\lambda) \leq |\alpha_2| \frac{2}{|\beta_1|} \frac{2}{|\beta_2|} \frac{2}{|\beta_3|} + \frac{M (\lambda)}{|s|} \int_{h_3}^{1} q (y) \, dy + \frac{M_0}{|s|}
\]

for some \( M_0 > 0 \). From this, it follows that \( M (\lambda) = O (1) \) as \( \lambda \to \infty \), so

\[
\phi_{4\lambda} (x) = O \left( e^{\left| t \right| (x-h_3)+(h_3-h_2)+(h_2-h_1)+(h_1+1)} \right).
\]

Substituting this back into the integral on the right side of (29) yields (27) for \( k = 0 \) and \( m = 3 \). The other cases may be considered analogically.

**Theorem 3.1.** Let \( \lambda = s^2 \), \( t = \text{Im} \, s \). Then the characteristic function \( \omega (\lambda) \) has the following asymptotic formulas:

**Case 1:** If \( \beta_2' \neq 0 \), \( \alpha_2 \neq 0 \), then

\[
\omega (\lambda) = \beta_2' \alpha_2 s^3 \left( \prod_{i=1}^{m} \delta_i^2 \right) \sin 2s + O \left( |s|^{2} e^{2|t|} \right). \tag{30}
\]

**Case 2:** If \( \beta_2' \neq 0 \), \( \alpha_2 = 0 \), then

\[
\omega (\lambda) = -\beta_2' \alpha_1 s^2 \left( \prod_{i=1}^{m} \delta_i^2 \right) \cos 2s + O \left( |s|^{2} e^{2|t|} \right). \tag{31}
\]

**Case 3:** If \( \beta_2' = 0 \), \( \alpha_2 \neq 0 \), then

\[
\omega (\lambda) = \beta_1' \alpha_2 s^2 \left( \prod_{i=1}^{m} \delta_i^2 \right) \cos 2s + O \left( |s|^{2} e^{2|t|} \right). \tag{32}
\]

**Case 4:** If \( \beta_2' = 0 \), \( \alpha_2 = 0 \), then

\[
\omega (\lambda) = -\beta_1' \alpha_1 s \left( \prod_{i=1}^{m} \delta_i^2 \right) \sin 2s + O \left( |s|^{2} e^{2|t|} \right). \tag{33}
\]

**Proof.** The proof is completed by substituting (27) and (28) into the represen-
tation
\[ \omega (\lambda) = \left( \prod_{i=1}^{m} \delta_{i} \right) \omega_{m+1} (\lambda) = \left( \prod_{i=1}^{m} \delta_{i} \right) \left[ \phi_{(m+1)\lambda} (1) \chi_{(m+1)\lambda} (1) - \phi'_{(m+1)\lambda} (1) \chi'_{(m+1)\lambda} (1) \right] = \]
\[ = \left( \prod_{i=1}^{m} \delta_{i} \right) \left[ (\lambda \beta'_{1} + \beta_{1}) \phi_{(m+1)\lambda} (1) - (\lambda \beta'_{2} + \beta_{2}) \phi'_{(m+1)\lambda} (1) \right] \]
\[ = \left( \prod_{i=1}^{m} \delta_{i} \right) \left[ \lambda \left( \beta'_{1} \phi_{(m+1)\lambda} (1) - \beta'_{2} \phi'_{(m+1)\lambda} (1) \right) + \left( \beta_{1} \phi_{(m+1)\lambda} (1) - \beta_{2} \phi'_{(m+1)\lambda} (1) \right) \right] \]
\[ = -\lambda \left( \prod_{i=1}^{m} \delta_{i} \right) \beta_{2} \phi_{(m+1)\lambda} (1) + \lambda \left( \prod_{i=1}^{m} \delta_{i} \right) \beta'_{1} \phi_{(m+1)\lambda} (1) + \left( \prod_{i=1}^{m} \delta_{i} \right) - \beta_{2} \phi'_{(m+1)\lambda} (1) \]
\[ + \left( \prod_{i=1}^{m} \delta_{i} \right) \beta_{1} \phi_{(m+1)\lambda} (1). \]  

\[ \text{(34)} \]

**Corollary 3.1.** The eigenvalues of the problem (1)-(5) are bounded below.

**Proof.** Putting \( s = it \) \((t > 0)\) in the above formulas, it follows that \( \omega (-t^{2}) \to \infty \) as \( t \to \infty \). Therefore, \( \omega (\lambda) \neq 0 \) for \( \lambda \) negative and sufficiently large. \( \blacksquare \)

4 Asymptotic formulas for eigenvalues and eigenfunctions

Now we can obtain the asymptotic approximation formulae for the eigenvalues of the considered problem (1)-(5).

Since the eigenvalues coincide with the zeros of the entire function \( \omega_{m+1} (\lambda) \), it follows that they have no finite limit. Moreover, we know from Corollaries 2.1 and 3.1 that all eigenvalues are real and bounded below. Hence, we may relabel them as \( \lambda_{0} \leq \lambda_{1} \leq \lambda_{2} \leq ... \), listed according to their multiplicity.

**Theorem 4.1.** The eigenvalues \( \lambda_{n} = s_{n}^{2} \), \( n = 0, 1, 2, ... \) of the problem (1)-(5) have the following asymptotic formulae for \( n \to \infty \):

**Case 1:** If \( \beta'_{2} \neq 0, \alpha_{2} \neq 0 \), then
\[ s_{n} = \frac{\pi (n - 1)}{2} + O \left( \frac{1}{n} \right). \]  

**Case 2:** If \( \beta'_{2} \neq 0, \alpha_{2} = 0 \), then
\[ s_{n} = \frac{\pi (n - \frac{1}{2})}{2} + O \left( \frac{1}{n} \right). \]  

**Case 3:** If \( \beta'_{2} = 0, \alpha_{2} \neq 0 \), then
\[ s_{n} = \frac{\pi (n - \frac{1}{2})}{2} + O \left( \frac{1}{n} \right). \]  

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Case 4: If \( \beta_2' = 0, \alpha_2 = 0 \), then

\[
s_n = \frac{\pi n}{2} + O\left(\frac{1}{n}\right).
\] (38)

**Proof.** We shall only consider the first case. The other cases may be considered similarly. Denoting \( \omega_1(s) \) and \( \omega_2(s) \) the first and \( O \)-term of the right of (42) respectively, we shall apply the well-known Rouché's theorem, which asserts that if \( f(s) \) and \( g(s) \) are analytic inside and on a closed contour \( C \), and \( |g(s)| < |f(s)| \) on \( C \), then \( f(s) \) and \( f(s) + g(s) \) have the same number zeros inside \( C \), provided that each zero is counted according to their multiplicity. It is readily shown that \( |\omega_1(s)| > |\omega_2(s)| \) on the contours

\[
C_n := \left\{ s \in \mathbb{C} \mid ||s|| = \frac{(n + \frac{1}{2}) \pi}{2} \right\}
\]

for sufficiently large \( n \).

Let \( \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq ... \) be zeros of \( \omega(\lambda) \) and \( \lambda_n = s_n^2 \). Since inside the contour \( C_n \), \( \omega_1(s) \) has zeros at points \( s = 0 \) and \( s = \frac{k\pi}{4}, \ k = \pm 1, \pm 2, ..., \pm n \).

\[
s_n = \frac{(n - 1) \pi}{2} + \delta_n \] (39)

where \( \delta_n = O(1) \) for sufficiently large \( n \). By substituting this in (30), we derive that \( \delta_n = O\left(\frac{1}{n}\right) \), which completes the proof. 

The next approximation for the eigenvalues may be obtained by the following procedure. For this, we shall suppose that \( q(y) \) is of bounded variation in \([-1, 1]\).

Firstly we consider the case \( \beta_2' \neq 0 \) and \( \alpha_2 \neq 0 \). Putting \( x = h_1, x = h_2, ..., x = h_m \) in (26) and then substituting in the expression of \( \phi_{(m+1)\lambda}' \), we get that

\[
\phi_{(m+1)\lambda}'(1) = \delta \frac{\alpha_2}{\prod_{i=1}^{m} \delta_i} \sin 2s - \frac{\alpha_1}{\prod_{i=1}^{m} \delta_i} \cos 2s + \left(\prod_{i=1}^{m} \delta_i\right) \int_{h_1}^{h_2} \cos (s (1 - y)) q(y) \phi_{1\lambda}(y) dy + \left(\prod_{i=1}^{m} \delta_i\right) \int_{h_1}^{h_2} \cos (s (1 - y)) q(y) \phi_{2\lambda}(y) dy + \left(\prod_{i=1}^{m} \delta_i\right) \int_{h_1}^{h_2} \cos (s (1 - y)) q(y) \phi_{3\lambda}(y) dy + ... + \left(\prod_{i=1}^{m} \delta_i\right) \int_{h_1}^{h_2} \cos (s (1 - y)) q(y) \phi_{m\lambda}(y) dy.
\]

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Substituting (27) into the right side of the last integral equality the n gives
\[ \phi'_{(m+1)\lambda}(1) = -\frac{s\alpha_2}{m} \sin 2s - \frac{\alpha_1}{m} \cos 2s + \frac{\alpha_2}{m} \int_{h_1}^{h_2} \cos (s (1 - y)) \cos (s (1 + y)) q(y) \, dy \]
\[ + \frac{\alpha_2}{m} \int_{h_1}^{h_2} \cos (s (1 - y)) \cos (s (1 + y)) q(y) \, dy + \]
\[ ... + \frac{\alpha_2}{m} \int_{h_m}^{h_2} \cos (s (1 - y)) \cos (s (1 + y)) q(y) \, dy + O\left(|s|^{-1} e^{2|t|}\right). \]

On the other hand, from (27), it follows that
\[ \phi_{(m+1)\lambda}(1) = \frac{\alpha_2}{m} \cos 2s + O\left(|s|^{-1} e^{2|t|}\right). \]

Putting these formulas into (34), we have
\[ \omega(\lambda) = \frac{s^3 \beta_1^2 \alpha_2}{m} \sin 2s + s^2 \left[ \left( \frac{\beta_1^2 \alpha_2 + \beta_2^2 \alpha_1}{m} \prod_{i=1}^{\delta_i} \right) \cos 2s \right. \]
\[ - \frac{\beta_2^2}{m} \int_{h_1}^{h_2} \cos (s (1 - y)) q(y) \phi_{1\lambda}(y) \, dy - \frac{\beta_2^2}{m} \int_{h_1}^{h_2} \cos (s (1 - y)) q(y) \phi_{2\lambda}(y) \, dy - \]
\[ ... - \frac{\beta_2^2}{\delta_m} \int_{h_m}^{h_2} \cos (s (1 - y)) q(y) \phi_{m\lambda}(y) \, dy - \beta_2^2 \int_{h_m}^{h_2} \cos (s (1 - y)) q(y) \phi_{(m+1)\lambda}(y) \, dy \]
\[ + O\left(|s|^{-1} e^{2|t|}\right). \]

Putting (39) in the last equality we find that
\[ \sin (2\delta_n) = -\frac{\cos (2\delta_n)}{s_n} \left[ \frac{\beta_1^2}{\beta_2^2} + \frac{\alpha_1}{\alpha_2} - \frac{1}{\prod_{i=1}^{\delta_i}} \int q(y) \, dy - \frac{1}{\prod_{i=1}^{\delta_i}} \int \cos (2s_n y) q(y) \, dy \right] \]
\[ + O\left(|s_n|^{-2}\right) \] (40)
Recalling that $q(y)$ is of bounded variation in $[-1, 1]$, and applying the well-known Riemann-Lebesgue Lemma (see [27], p. 48, Theorem 4.12) to the second integral on the right in (40), this term is $O\left(\frac{1}{n}\right)$. As a result, from (40) it follows that

$$
\delta_n = -\frac{1}{\pi (n-1)} \left[ \frac{\beta_1'}{\beta_2'} + \frac{\alpha_1}{\alpha_2} - \frac{1}{2 \prod_{i=1}^{m} \delta_i} \int_{-1}^{1} q(y) \, dy \right] + O\left(\frac{1}{n^2}\right).
$$

Substituting in (40), we have

$$
s_n = \frac{\pi (n-1)}{2} - \frac{1}{\pi (n-1)} \left[ \frac{\beta_1'}{\beta_2'} + \frac{\alpha_1}{\alpha_2} - \frac{1}{2 \prod_{i=1}^{m} \delta_i} \int_{-1}^{1} q(y) \, dy \right] + O\left(\frac{1}{n^2}\right).
$$

Similar formulas in the other cases are as follows:

In case 2:

$$
s_n = \frac{\pi (n - \frac{1}{2})}{2} + \frac{1}{\pi (n - \frac{1}{2})} \left[ \frac{\beta_2}{\beta_1'} - \frac{\alpha_1}{\alpha_2} + \frac{1}{2 \prod_{i=1}^{m} \delta_i} \int_{-1}^{1} q(y) \, dy \right] + O\left(\frac{1}{n^2}\right).
$$

In case 3:

$$
s_n = \frac{\pi (n - \frac{1}{2})}{2} + \frac{1}{\pi (n - \frac{1}{2})} \left[ \frac{\beta_2}{\beta_1'} - \frac{\alpha_1}{\alpha_2} + \frac{1}{2 \prod_{i=1}^{m} \delta_i} \int_{-1}^{1} q(y) \, dy \right] + O\left(\frac{1}{n^2}\right).
$$

In case 4:

$$
s_n = \frac{\pi n}{2} + \frac{1}{\pi n} \left[ \frac{\beta_2}{\beta_1'} + \frac{1}{2 \prod_{i=1}^{m} \delta_i} \int_{-1}^{1} q(y) \, dy \right] + O\left(\frac{1}{n^2}\right).
$$

Recalling that $\phi(x, \lambda_n)$ is an eigenfunction according to the eigenvalue $\lambda_n$ and by putting (35) into the (27) we obtain that

$$
\phi_{1\lambda_n}(x) = \alpha_2 \cos \left(\frac{\pi (n-1)(x + 1)}{2}\right) + O\left(\frac{1}{n}\right),
$$

17
\[
\phi_{2\lambda_n}(x) = \frac{\alpha_2}{\delta_1} \cos \left( \frac{\pi (n-1)(x+1)}{2} \right) + O \left( \frac{1}{n} \right),
\]

\[
\vdots
\]

\[
\phi_{(m+1)\lambda_n}(x) = \frac{\alpha_2}{\prod_{i=1}^{m} \delta_i} \cos \left( \frac{\pi (n-1)(x+1)}{2} \right) + O \left( \frac{1}{n} \right)
\]
in the first case. Consequently, if \( \beta_2 \neq 0 \) and \( \alpha_2 \neq 0 \), then the eigenfunction \( \phi (x, \lambda_n) \) has the following asymptotic formulae:

\[\phi (x, \lambda_n) = \left\{
\begin{array}{l}
\alpha_2 \cos \left( \frac{\pi (n-1)(x+1)}{2} \right) + O \left( \frac{1}{n} \right), \quad x \in [-1, h_1) \\
-\frac{2\alpha_1}{\pi (n-\frac{1}{2})} \sin \left( \frac{\pi (n-\frac{1}{2})(x+1)}{2} \right) + O \left( \frac{1}{n^2} \right), \quad x \in (h_1, h_2) \\
\vdots \\
-\frac{2\alpha_1}{\prod_{i=1}^{m} \delta_i} \sin \left( \frac{\pi (n-\frac{1}{2})(x+1)}{2} \right) + O \left( \frac{1}{n^m} \right), \quad x \in (h_m, 1]. 
\end{array}\right.\]

which holds uniformly for \( x \in [-1, h_1) \cup (h_1, h_2) \cup \ldots \cup (h_m, 1] \).

Similar formulas in the other cases are as follows:

In case 2

\[\phi (x, \lambda_n) = \left\{
\begin{array}{l}
\alpha_2 \cos \left( \frac{\pi (n-\frac{1}{2})(x+1)}{2} \right) + O \left( \frac{1}{n} \right), \quad x \in [-1, h_1) \\
-\frac{2\alpha_1}{\pi (n-\frac{1}{2})} \sin \left( \frac{\pi (n-\frac{1}{2})(x+1)}{2} \right) + O \left( \frac{1}{n^2} \right), \quad x \in (h_1, h_2) \\
\vdots \\
-\frac{2\alpha_1}{\prod_{i=1}^{m} \delta_i} \sin \left( \frac{\pi (n-\frac{1}{2})(x+1)}{2} \right) + O \left( \frac{1}{n^m} \right), \quad x \in (h_m, 1]. 
\end{array}\right.\]

In case 3

\[\phi (x, \lambda_n) = \left\{
\begin{array}{l}
\alpha_2 \cos \left( \frac{\pi (n-\frac{1}{2})(x+1)}{2} \right) + O \left( \frac{1}{n} \right), \quad x \in [-1, h_1) \\
-\frac{2\alpha_1}{\pi (n-\frac{1}{2})} \sin \left( \frac{\pi (n-\frac{1}{2})(x+1)}{2} \right) + O \left( \frac{1}{n^2} \right), \quad x \in (h_1, h_2) \\
\vdots \\
-\frac{2\alpha_1}{\prod_{i=1}^{m} \delta_i} \sin \left( \frac{\pi (n-\frac{1}{2})(x+1)}{2} \right) + O \left( \frac{1}{n^m} \right), \quad x \in (h_m, 1]. 
\end{array}\right.\]
Theorem 5.1

It suffices to prove that if $x$ is not an eigenvalue of $A$, then $\eta \in \sigma (A)$.

Since $A$ is self-adjoint, we only consider a real $\eta$. We investigate the equation

$$(A - \eta) Y = F \in H,$$

where $F = \left( \begin{array}{c} f \\ f_1 \end{array} \right)$.

Let us consider the initial-value problem

$$
\begin{cases}
\tau y - \eta y = f, \quad x \in [-1, h_1) \cup (h_1, h_2) \cup ... \cup (h_m, 1), \\
\alpha_1 y (-1) + \alpha_2 y' (-1) = 0, \\
y (h_i - 0) - \delta_i y (h_i + 0) = 0, \\
y' (h_i - 0) - \delta_i y' (h_i + 0) = 0.
\end{cases}
$$

(41)

Let $u(x)$ be the solution of the equation $\tau u - \eta u = 0$ satisfying

$$
\begin{cases}
u (-1) = \alpha_2, \quad u' (-1) = -\alpha_1, \\
u (h_i - 0) - \delta_i u (h_i + 0) = 0, \\
u' (h_i - 0) - \delta_i u' (h_i + 0) = 0.
\end{cases}
$$

In fact,

$$
\begin{array}{c}
u (x) = \\
u_1 (x), \quad x \in [-1, h_1), \\
u_2 (x), \quad x \in (h_1, h_2), \\
\vdots \\
u_{m+1} (x), \quad x \in (h_m, 1] \end{array}
$$

where $u_1 (x)$ is the unique solution of the initial-value problem

$$
\begin{cases}
-u'' + q(x) u = \eta u, \quad x \in [-1, h_1), \\
u (-1) = \alpha_2, \quad u' (-1) = -\alpha_1;
\end{cases}
$$

$u_2 (x)$ is the unique solution of the problem

$$
\begin{cases}
-u'' + q(x) u = \eta u, \quad x \in (h_1, h_2), \\
u (h_1 - 0) = \delta_1 u (h_1 + 0), \\
u' (h_1 - 0) = \delta_1 u' (h_1 + 0); \end{cases}
$$
Let

\[ u_m (x) \text{ is the unique solution of the problem} \]

\[
\begin{align*}
-u'' + q(x)u &= \eta u, \quad x \in (h_2, h_3), \\
u (h_2 - 0) &= \delta_2 u (h_2 + 0), \\
u' (h_2 - 0) &= \delta_2 u' (h_2 + 0), \\
\end{align*}
\]

and \( u_{m+1} (x) \) is the unique solution of the problem

\[
\begin{align*}
-u'' + q(x)u &= \eta u, \quad x \in (h_m, 1), \\
u (h_m - 0) &= \delta_m u (h_m + 0), \\
u' (h_m - 0) &= \delta_m u' (h_m + 0). \\
\end{align*}
\]

Let

\[
w (x) = \begin{cases} \\
    w_1 (x), & x \in [-1, h_1), \\
    w_2 (x), & x \in (h_1, h_2), \\
    \vdots \\
    w_{m+1} (x), & x \in (h_m, 1], \\
\end{cases}
\]

be a solution of \( \tau w - \eta w = f \) satisfying

\[
\begin{align*}
\alpha_1 w (-1) + \alpha_2 w' (-1) &= 0, \\
\beta_1 (h_1 - 0) &= \delta_1 w (h_1 + 0), \\
\beta_1 (h_1 - 0) &= \delta_1 w' (h_1 + 0). \\
\end{align*}
\]

Then, (41) has the general solution

\[
y (x) = \begin{cases} \\
    du_1 + w_1, & x \in [-1, h_1), \\
    du_2 + w_2, & x \in (h_1, h_2), \\
    \vdots \\
    du_{m+1} + w_{m+1}, & x \in (h_m, 1], \\
\end{cases}
\]

where \( d \in \mathbb{C} \).

Since \( \eta \) is not an eigenvalue of the problem (1) – (5), we have

\[ \eta \left[ \beta' u_{m+1} (1) + \beta' u_{m+1} (1) \right] + \left[ \beta_1 u_{m+1} (1) + \beta_2 u_{m+1} (1) \right] \neq 0. \]  (43)

The second component of \( (A - \eta) Y = F \) involves the equation

\[ -R_1 (y) - \eta R_1' (y) = f_1, \]

namely,

\[ [-\beta_1 y (1) - \beta_2 y' (1)] - \eta [\beta_1 y (1) + \beta_2 y' (1)] = f_1. \]  (44)

Substituting (42) into (44), we get

\[
\begin{align*}
(-\beta_2 u_{m+1} (1) - \beta_1 u_{m+1} (1) - \eta \beta_2 u_{m+1} (1) - \eta \beta_1 u_{m+1} (1)) d \\
= f_1 + \beta_1 u_{m+1} (1) + \beta_2 u_{m+1} (1) + \eta \beta_1 u_{m+1} (1) + \eta \beta_2 u_{m+1} (1)
\end{align*}
\]
In view of (43), we show that \( d \) is uniquely solvable. Therefore, \( y \) is uniquely determined.

The above arguments show that \((A - \eta I)^{-1}\) is defined on all of \( H \), where \( I \) is identity matrix. We obtain that \((A - \eta I)^{-1}\) is bounded by Theorem 2.2 and the Closed Graph Theorem. Thus, \( \eta \in \sigma(A) \). Therefore, \( \sigma(A) = \sigma_p(A) \). ■

The following lemma may be easily proved.

**Lemma 5.1** The eigenvalues of the boundary value problem (1) – (5) are bounded below, and they are countably infinite and can cluster only at \( \infty \).

For every \( \delta \in \mathbb{R} \setminus \sigma_p(A) \), we have the following immediate conclusion.

**Lemma 5.2** Let \( \lambda \) be an eigenvalue of \( A - \delta I \), and \( V \) a corresponding eigenfunction. Then, \( \lambda^{-1} \) is an eigenvalue of \((A - \delta I)^{-1}\), and \( V \) is a corresponding eigenfunction. The converse is also true.

On the other hand, if \( \mu \) is an eigenvalue of \( A \) and \( U \) is a corresponding eigenfunction, then \( \mu - \delta \) is an eigenvalue of \( A - \delta I \), and \( U \) is a corresponding eigenfunction. The converse is also true. Accordingly, the discussion about the completeness of the eigenfunctions of \( A \) is equivalent to considering the corresponding property of \((A - \delta I)^{-1}\).

By Lemma 1.1, Lemma 3.1 and Corollary 1.1, we suppose that \( \{\lambda_n; n \in \mathbb{N}\} \) is the real sequence of eigenvalues of \( A \), then \( \{\lambda_n - \delta; n \in \mathbb{N}\} \) is the sequence of eigenvalues of \( A - \delta I \). We may assume that

\[
|\lambda_1 - \delta| \leq |\lambda_2 - \delta| \leq ... \leq |\lambda_n - \delta| \leq ... \to \infty.
\]

Let \( \{\mu_n; n \in \mathbb{N}\} \) be the sequence of eigenvalues of \((A - \delta I)^{-1}\). Then \( \mu_n = (\lambda_n - \delta)^{-1} \) and

\[
|\mu_1| \geq |\mu_2| \geq ... \geq |\mu_n| \geq ... \to 0.
\]

Note that 0 is not an eigenvalue of \((A - \delta I)^{-1}\).

**Theorem 5.2** The operator \( A \) has compact resolvents, i.e., for each \( \delta \in \mathbb{R} \setminus \sigma_p(A) \), \((A - \delta I)^{-1}\) is compact on \( H \).

**Proof.** Let \( \{\mu_1, \mu_2, \ldots\} \) be the eigenvalues of \((A - \delta I)^{-1}\), and \( \{P_1, P_2, \ldots\} \) the orthogonal projections of finite rank onto the corresponding eigenspaces. Since \( \{\mu_1, \mu_2, \ldots\} \) is a bounded sequence and all \( P_n \)'s are mutually orthogonal, we have \( \sum_{n=1}^{\infty} \mu_n P_n \) is strongly convergent to the bounded operator \((A - \delta I)^{-1}\), i.e., \((A - \delta I)^{-1} = \sum_{n=1}^{\infty} \mu_n P_n \). Because for every \( \alpha > 0 \), the number of \( \mu_n \)'s satisfying \(|\mu_n| > \alpha \) is finite, and all \( P_n \)'s are of finite rank, we obtain that \((A - \delta I)^{-1}\) is compact. ■

In terms of the above statements and the spectral theorem for compact operators, we obtain the following theorem.

**Theorem 5.3** The eigenfunctions of the problem (1) – (5), augmented to become eigenfunctions of \( A \), are complete in \( H \), i.e., if we let

\[
\left\{\Phi_n = \left( \frac{\phi_n(x)}{R_1(\phi_n)} \right); \ n \in \mathbb{N}\right\}
\]

be a maximum set of orthonormal eigenfunctions of \( A \), where \( \{\phi_n(x); n \in \mathbb{N}\} \) are eigenfunctions of (1) – (5), then for all \( F \in H \),

\[
F = \sum_{n=1}^{\infty} (F, \Phi_n) \Phi_n.
\]
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Asymptotic behaviour of a differential operator with a finite number of transmission conditions

Erdoğan Şen and O. Sh. Mukhtarov

Department of Mathematics Engineering, Istanbul Technical University, Maslak, 34469 Istanbul, Turkey
Department of Mathematics, Istanbul Technical University, Maslak, 34469 Istanbul, Turkey
*Department of Mathematics, Faculty of Arts and Science, Gazişmanpaşa University, 60250 Tokat, Turkey.
e-mail: erdogan.math@gmail.com, *omukhtarov@yahoo.com

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ABSTRACT
In this paper following the same methods in [M. Kadakal, O. Sh. Mukhtarov, Sturm-Liouville problems with discontinuities at two points, Comput. Math. Appl., 54 (2007) 1367-1379] we investigate discontinuous two-point boundary value problems with eigenparameter in the boundary conditions and with transmission conditions at the finitely many points of discontinuity. A self-adjoint linear operator \( A \) is defined in a suitable Hilbert space \( H \) such that the eigenvalues of such a problem coincide with those of \( A \). We obtain asymptotic formulas for the eigenvalues and eigenfunctions. Also we show that the eigenfunctions of \( A \) are complete in \( H \).

1 Introduction

The theory of discontinuous Sturm-Liouville type problems mainly has been developed by Mukhtarov and his students (see [1-11]). It is well-known that many topics in mathematical physics require the investigation of eigenvalues and eigenfunctions of Sturm-Liouville type boundary value problems. In recent years, more and more researchers are interested in the discontinuous Sturm-Liouville problems and its applications in physics (see [1 – 28]).

Discontinuous Sturm-Liouville problems with supplementary transmission conditions at the point(s) of discontinuity have been investigated in [6-9, 23-27].

In this study, we examine eigenvalues and eigenfunctions of the differential
equation

\[ \tau u := -u''(x) + q(x)u(x) = \lambda u(x) \]  \hspace{1cm} (1)

on \([-1, h_1) \cup (h_1, h_2) \cup \ldots \cup (h_m, 1]\), with boundary conditions

\[ \tau_1 u := \alpha_1 u(-1) + \alpha_2 u'(1) = 0, \] \hspace{1cm} (2)

\[ \tau_2 u := (\beta_1' \lambda + \beta_1) u(1) + (\beta_2' \lambda + \beta_2) u'(1) = 0 \] \hspace{1cm} (3)

and transmission conditions at the points of discontinuity \( x = h_i \) (\( i = 1, m \))

\[ \tau_{2i+1} u := u(h_i - 0) - \delta_i u(h_i + 0) = 0, \] \hspace{1cm} (4)

\[ \tau_{2i+2} u := u'(h_i - 0) - \delta_i u'(h_i + 0) = 0, \] \hspace{1cm} (5)

where \(-1 < h_1 < h_2 < \ldots < h_m < 1\), \( q(x) \) is a given real-valued function continuous in \([-1, h_1), (h_1, h_2), \ldots, (h_m, 1]\) and has finite limits \( q(h_i \pm 0) = \lim_{x \to h_i \pm 0} q(x) \) \( (i = 1, m) \); \( \lambda \) is a complex eigenvalue parameter; \( \delta_i \) \( (i = 1, m) \), \( \alpha_j, \beta_j, \beta_j' \) \( (j = 1, 2) \) are real numbers; \( |\alpha_1| + |\alpha_2| \neq 0 \) and \( \delta_i 
eq 0 \) \( (i = 1, m) \). As following [6] every where we assume that \( \rho = \beta_1' \beta_2 - \beta_1 \beta_2' > 0 \).

\section{Operator Formulation}

By using the method, introduced in [6] we shall define direct sum of Hilbert spaces but with the usual inner product replaced by appropriate multiples as follows. In the Hilbert space \( H = L_2(-1, 1) \oplus \mathbb{C} \) we define an inner product by

\[ \langle F, G \rangle := \int_{-1}^{h_1} f(x) \overline{g(x)} \, dx + \delta_1^2 \int_{h_1}^{h_2} f(x) \overline{g(x)} \, dx + \delta_2^2 \int_{h_2}^{h_3} f(x) \overline{g(x)} \, dx + \ldots + \prod_{i=1}^{m} \delta_i^2 \int_{h_m}^{1} f(x) \overline{g(x)} \, dx + \sum_{i=1}^{m} \int_{h_i}^{h_{i+1}} \frac{1}{\rho} f(x) \overline{g(x)} \, dx \]

for

\[ F := \left( \begin{array}{c} f(x) \\ f_1 \end{array} \right) \quad G := \left( \begin{array}{c} g(x) \\ g_1 \end{array} \right) \in H \]

following (7) for convenience we put

\[ R_1(u) := \beta_1 u(1) - \beta_2 u'(1), \]

\[ R_1'(u) := \beta_1' u(1) - \beta_2' u'(1). \]

For functionals \( f(x) \), which defined on \([-1, h_1) \cup (h_1, h_2) \cup \ldots \cup (h_m, 1]\) and has finite limits \( f(h_i \pm 0) = \lim_{x \to h_i \pm 0} f(x) \) \( (i = 1, m) \). By \( f_i(x) \) \( (i = 1, m + 1) \)
we denote the functions

\[
\begin{align*}
1 \mapsto f_1(x) := \begin{cases} f(x), & x \in [-1, h_1), \\ \lim_{x \to h_1^-} f(x), & x = h_1, \end{cases} & \quad f_2(x) := \begin{cases} \lim_{x \to h_2^+} f(x), & x = h_1, \\ f(x), & x \in (h_1, h_2), \end{cases} \\
, \mapsto f_m(x) := \begin{cases} \lim_{x \to h_{m-1}^-} f(x), & x = h_{m-1}, \\ f(x), & x \in (h_{m-1}, h_m), \\ \lim_{x \to h_m^+} f(x), & x = h_m, \end{cases} & \quad f_{m+1}(x) := \begin{cases} \lim_{x \to h_m^+} f(x), & x = h_m, \\ f(x), & x \in (h_m, 1] \end{cases}
\end{align*}
\]

which are defined on \( \Omega_1 := [-1, h_1], \Omega_2 := [h_1, h_2], ..., \Omega_m := [h_{m-1}, h_m], \Omega_{m+1} := [h_m, 1] \) respectively. We can rewrite the considered problem (1) – (5) in the operator formulation as

\[ AF = \lambda F \]

where

\[ F := \begin{pmatrix} f(x) \\ -R_1(f) \end{pmatrix} \in D(A) \]

and with

\[ D(A) := \left\{ F \in H \mid f_i(x), f_i'(x) \text{ are absolutely continuous in } \Omega_i \ (i = 1, m + 1), \tau f \in L^2[-1, 1], \tau_{2i} u := u(h_i - 0) - \delta u(h_i + 0) = 0, \tau_{2i+1} u := u'(h_i - 0) - \delta u'(h_i + 0) = 0 \right\} \]

and

\[ AF = \begin{pmatrix} \tau f \\ -R_1(f) \end{pmatrix}. \]

Consequently, the problem (1) – (5) can be considered as the eigenvalue problem for the operator A. Obviously, we have

**Lemma 2.1.** The eigenvalues of the boundary value problem (1)-(5) coincide with those of A, and its eigenfunctions are the first components of the corresponding eigenfunctions of A.

**Lemma 2.2.** The domain \( D(A) \) is dense in \( H \).

**Proof.** Let \( F = \begin{pmatrix} f(x) \\ f_1 \end{pmatrix} \in H, F \perp D(A) \) and \( C_0^\infty \) be a functional set such that

\[ \phi(x) = \begin{cases} \phi_1(x), x \in [-1, h_1), \\ \phi_2(x), x \in (h_1, h_2), \\ \vdots \\ \phi_{m+1}(x), x \in (h_m, 1] \end{cases} \]

for \( \phi_1(x) \in \overline{C_0^\infty} [-1, h_1], \phi_2(x) \in \overline{C_0^\infty} (h_1, h_2), ..., \phi_{m+1}(x) \in \overline{C_0^\infty} (h_m, 1] \). Since \( \overline{C_0^\infty} \oplus 0 \subset D(A) \ (0 \in \mathbb{C}), \) any \( U = \begin{pmatrix} u(x) \\ 0 \end{pmatrix} \in \overline{C_0^\infty} \oplus 0 \) is orthogonal to \( F \),
namely
\[
\langle F, U \rangle = \int_{-1}^{h_1} f(x) u(x) dx + \delta_1^2 \int_{h_1}^{h_2} f(x) u(x) dx + \delta_1^2 \delta_2^2 \int_{h_2}^{h_3} f(x) u(x) dx + \ldots + \prod_{i=1}^{m} \delta_i^2 \int_{h_m}^{1} f(x) u(x) dx = \langle f, u \rangle_1.
\]

We can learn that \( f(x) \) is orthogonal to \( \tilde{C}_0^\infty \) in \( L^2 [-1, 1] \), this implies \( f(x) = 0 \).

So for all \( G = \left( g(x) \right) \in D(A), \langle F, G \rangle = \prod_{i=1}^{m} \delta_i^2 f_1 g_1 = 0 \). Thus \( f_1 = 0 \) since \( g_1 = R'_1 (g) \) can be chosen arbitrarily. So \( F = \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \), which proves the assertion. ■

**Theorem 2.1.** The linear operator \( A \) is symmetric in \( H \).

**Proof.** Let \( F, G \in D(A) \). By two partial integrations, we get
\[
\langle AF, G \rangle = \langle F, AG \rangle + \left( W (f, \overline{g}; h_1 - 0) - W (f, \overline{g}; -1) \right) + \delta_1^2 (W (f, \overline{g}; h_2 - 0) - W (f, \overline{g}; h_1 + 0))
\]
\[
\quad + \delta_1^2 \delta_2^2 (W (f, \overline{g}; h_3 - 0) - W (f, \overline{g}; h_2 + 0)) + \ldots + \prod_{i=1}^{m} \delta_i^2 (W (f, \overline{g}; 1) - W (f, \overline{g}; h_m + 0))
\]
\[
\quad + \prod_{i=1}^{m} \delta_i^2 \left( R'_1 (f) R_1 (\overline{g}) - R_1 (f) R'_1 (\overline{g}) \right)
\]
where
\[
W (f, \overline{g}, x) = f (x) \overline{g}' (x) - f' (x) \overline{g} (x)
\]
denotes the Wronskians of the functions \( f \) and \( \overline{g} \). Since \( f \) and \( \overline{g} \) satisfy the boundary condition (2), it follows that
\[
W (f, \overline{g}, -1) = 0.
\]

From the transmission conditions (4)-(5) we get
\[
\begin{align*}
W (f, \overline{g}; h_1 - 0) &= \delta_1^2 W (f, \overline{g}; h_1 + 0), \\
W (f, \overline{g}; h_2 - 0) &= \delta_2^2 W (f, \overline{g}; h_2 + 0), \\
&\vdots \\
W (f, \overline{g}; h_m - 0) &= \delta_m^2 W (f, \overline{g}; h_m + 0).
\end{align*}
\]
Furthermore,
\[ R_1 (f) R_1 (g) - R_1 (f) R_1 (g) = (\beta_1 f (1) - \beta_2 f' (1)) (\beta_1 g (1) - \beta_2 g' (1)) - (\beta_1 f (1) - \beta_2 f' (1)) (\beta_1 g (1) - \beta_2 g' (1)) \]
\[ = (\beta_2 \beta_1 - \beta_2 \beta_1) f' (1) g (1) + (\beta_2 \beta_1 - \beta_2 \beta_1) f (1) g' (1) \]
\[ = \rho \left( f' (1) g (1) - f (1) g' (1) \right) = -\rho W (f, g) . \]

Finally, substituting (7)-(9) in (6) then we get
\[ \langle AF, G \rangle = \langle F, AG \rangle \quad (F, G \in D (A)) \quad (10) \]

Now we can write the following theorem with the help of Theorem 2.1, Naimark’s Patching Lemma [15] and using the similar way in [6]

**Theorem 2.2.** The linear operator \( A \) is self-adjoint in \( H \).

**Corollary 2.1.** All eigenvalues of the problem (1)-(5) are real.

We can now assume that all eigenfunctions are real-valued.

**Corollary 2.2.** If \( \lambda_1 \) and \( \lambda_2 \) are two different eigenvalues of the problem (1)-(5), then the corresponding eigenfunctions \( u_1 \) and \( u_2 \) of this problem satisfy the following equality:

\[ \int_{-1}^{1} u_1 (x) u_2 (x) dx + \int_{-1}^{1} u_1 (x) u_2 (x) dx + \int_{-1}^{1} \cdots + \int_{-1}^{1} u_1 (x) u_2 (x) dx = -\rho \prod_{i=1}^{m} R_1 (u_1) R_1 (u_2) . \]

In fact this formula means the orthogonality of eigenfunctions \( u_1 \) and \( u_2 \) in the Hilbert space \( H \).

We need the following lemma, which can be proved by the same technique as in [4].

**Lemma 2.3.** Let the real-valued function \( q (x) \) be continuous in \([-1, 1]\) and \( f (\lambda), g (\lambda) \) are given entire functions. Then for any \( \lambda \in \mathbb{C} \) the equation

\[ -u'' + q (x) u = \lambda u, \quad x \in [-1, 1] \]

has a unique solution \( u = u (x, \lambda) \) satisfies the initial conditions

\[ u (-1) = f (\lambda), u' (-1) = g (\lambda) \quad \text{or} \quad u (1) = f (\lambda), u' (1) = g (\lambda) . \]

For each fixed \( x \in [-1, 1] \), \( u (x, \lambda) \) is an entire function of \( \lambda \).
We shall define two solutions

\[
\phi_{\lambda}(x) = \begin{cases} 
\phi_{1\lambda}(x), & x \in [-1, h_1), \\
\phi_{2\lambda}(x), & x \in (h_1, h_2), \\
\vdots \\
\phi_{(m+1)\lambda}(x), & x \in (h_m, 1], 
\end{cases} \quad \text{and} \quad \chi_{\lambda}(x) = \begin{cases} 
\chi_{1\lambda}(x), & x \in [-1, h_1), \\
\chi_{2\lambda}(x), & x \in (h_1, h_2), \\
\vdots \\
\chi_{(m+1)\lambda}(x), & x \in (h_m, 1], 
\end{cases}
\]

of the Eq. (1) as follows: Let \( \phi_{1\lambda}(x) := \phi_1(x, \lambda) \) be the solution of Eq. (1) on \([-1, h_1]\), which satisfies the initial conditions

\[
u(-1) = \alpha_2, \quad \nu'(-1) = -\alpha_1.
\]

By virtue of Lemma 2.1, after defining this solution, we may define the solution \( \phi_2(x, \lambda) := \phi_{2\lambda}(x) \) of Eq. (1) on \([h_1, h_2]\) by means of the solution \( \phi_1(x, \lambda) \) by the initial conditions

\[
u(h_1) = \delta_1^{-1} \phi_1(h_1, \lambda), \quad \nu'(h_1) = \delta_1^{-1} \phi'_1(h_1, \lambda).
\]

After defining this solution, we may define the solution \( \phi_3(x, \lambda) := \phi_{3\lambda}(x) \) of Eq. (1) on \([h_2, h_3] \) by means of the solution \( \phi_2(x, \lambda) \) by the initial conditions

\[
u(h_2) = \delta_2^{-1} \phi_2(h_2, \lambda), \quad \nu'(h_2) = \delta_2^{-1} \phi'_2(h_2, \lambda).
\]

Continuing in this manner, we may define the solution \( \phi_{(m+1)}(x, \lambda) := \phi_{(m+1)\lambda}(x) \) of Eq. (1) on \([h_m, 1]\) by means of the solution \( \phi_m(x, \lambda) \) by the initial conditions

\[
u(h_m) = \delta_m^{-1} \phi_m(h_m, \lambda), \quad \nu'(h_m) = \delta_m^{-1} \phi'_m(h_m, \lambda).
\]

Therefore, \( \phi(x, \lambda) \) satisfies the Eq. (1) on \([-1, h_1) \cup (h_1, h_2) \cup \ldots \cup (h_m, 1]\), the boundary condition (3), and the transmission conditions (4)-(5).

Analogically, first we define the solution \( \chi_{(m+1)\lambda}(x) := \chi_{(m+1)}(x, \lambda) \) on \([h_m, 1]\) by the initial conditions

\[
u(1) = \beta_2 \lambda + \beta_2, \quad \nu'(1) = \beta_1 \lambda + \beta_1.
\]

Again, after defining this solution, we may define the solution \( \chi_{m\lambda}(x) := \chi_m(x, \lambda) \) of the Eq. (1) on \([h_{m-1}, h_m]\) by the initial conditions

\[
u(h_m) = \delta_m \chi_{m+1}(h_m, \lambda), \quad \nu'(h_m) = \delta_m \chi'_{m+1}(h_m, \lambda).
\]

Continuing in this manner, we may define the solution \( \chi_{1\lambda}(x) := \chi_1(x, \lambda) \) of the Eq. (1) on \([-1, h_1]\) by the initial conditions

\[
u(h_1) = \delta_1 \chi_2(h_1, \lambda), \quad \nu'(h_1) = \delta_1 \chi'_2(h_1, \lambda).
\]

Therefore, \( \chi(x, \lambda) \) satisfies the Eq. (1) on \([-1, h_1) \cup (h_1, h_2) \cup \ldots \cup (h_m, 1]\), the boundary condition (3), and the transmission conditions (4)-(5).

It is obvious that the Wronskians

\[
\omega_i(\lambda) := W_\lambda(\phi_i, \chi_i; x) := \phi_i(x, \lambda) \chi'_i(x, \lambda) - \phi'_i(x, \lambda) \chi_i(x, \lambda), \quad x \in \Omega_i \quad (i = 1, m+1)
\]
are independent of $x \in \Omega_i$ and entire functions.

**Lemma 2.4.** For each $\lambda \in \mathbb{C}$, $\omega_1 (\lambda) = \delta_1^2 \omega_2 (\lambda) = \delta_1^2 \delta_2 \omega_3 (\lambda) = ... = \left( \prod_{i=1}^{m} \delta_i^2 \right) \omega_{m+1} (\lambda)$.

**Proof.** By the means of (12), (13), (14), (16) and (17), the short calculation gives

\[ W_\lambda (\phi_1, \chi_1; h_1) = \delta_1^2 W_\lambda (\phi_2, \chi_2; h_1) = \delta_1^2 W_\lambda (\phi_3, \chi_3; h_2) = \delta_1^2 \delta_2 W_\lambda (\phi_3, \chi_3; h_2) \]

\[ = \delta_1^2 \delta_2 W_\lambda (\phi_4, \chi_4; h_3) = ... = \left( \prod_{i=1}^{m} \delta_i^2 \right) W_\lambda (\phi_{m+1}, \chi_{m+1}; h_m) \]

so $\omega_1 (\lambda) = \delta_1^2 \omega_2 (\lambda) = \delta_1^2 \delta_2 \omega_3 (\lambda) = ... = \left( \prod_{i=1}^{m} \delta_i^2 \right) \omega_{m+1} (\lambda)$.

Now we may introduce the characteristic function

\[ \omega (\lambda) := \omega_1 (\lambda) = \delta_1^2 \omega_2 (\lambda) = \delta_1^2 \delta_2 \omega_3 (\lambda) = ... = \left( \prod_{i=1}^{m} \delta_i^2 \right) \omega_{m+1} (\lambda). \]

**Theorem 2.3.** The eigenvalues of the problem (1)-(5) are the zeros of the function $\omega (\lambda)$.

**Proof.** Let $\omega (\lambda_0) = 0$. Then $W_{\lambda_0} (\phi_1, \chi_1; x) = 0$ and therefore the functions $\phi_{\lambda_0} (x)$ and $\chi_{\lambda_0} (x)$ are linearly dependent, i.e.

\[ \chi_{\lambda_0} (x) = k_1 \phi_{\lambda_0} (x), \quad x \in [-1, h_1] \]

for some $k_1 \neq 0$. From this, it follows that $\chi (x, \lambda_0)$ satisfies also the first boundary condition (2), so $\chi (x, \lambda_0)$ is an eigenfunction of the problem (1) – (5) corresponding to this eigenvalue $\lambda_0$.

Now we let $u_0 (x)$ be any eigenfunction corresponding to eigenvalue $\lambda_0$, but $\omega (\lambda_0) \neq 0$. Then the functions $\phi_1, \chi_1, \phi_2, \chi_2, ..., \phi_{m+1}, \chi_{m+1}$ would be linearly independent on $[-1, h_1], [h_1, h_2]$ and $[h_m, 1]$ respectively. Therefore $u_0 (x)$ may be represented in the following form

\[ u_0 (x) = \begin{cases} 
  c_1 \phi_1 (x, \lambda_0) + c_2 \chi_1 (x, \lambda_0), & x \in [-1, h_1), \\
  c_3 \phi_2 (x, \lambda_0) + c_4 \chi_2 (x, \lambda_0), & x \in (h_1, h_2), \\
  \vdots \\
  c_{2m+1} \phi_{m+1} (x, \lambda_0) + c_{2m+2} \chi_{m+1} (x, \lambda_0), & x \in (h_m, 1].
\end{cases} \]

where at least one of the constants $c_1, c_2, ..., c_{2m+2}$ is not zero. Considering the equations

\[ \tau_v (u_0 (x)) = 0, \quad v = 1, 2m + 2 \]  

as the homogenous system of linear equations of the variables $c_1, c_2, c_{2n+2}$ and taking (12), (13), (14), (16) and (17) into account, it follows that the
Thus we get a contradiction, which completes the proof.

**Proof.**

Suppose, if possible, that \( \phi(x, \lambda_0) \) and \( \chi(x, \lambda_0) \) are linearly dependent.  

The determinant of this system is

\[
\begin{vmatrix}
0 & \omega_1(\lambda_0) & 0 & \cdots & \cdots & \cdots & 0 & 0 \\
\phi_{1 \lambda_0}(h_1) & \chi_{1 \lambda_0}(h_1) & -\delta_1 \phi_{2 \lambda_0}(h_1) & -\delta_1 \phi_{2 \lambda_0}(h_1) & \cdots & \cdots & 0 & 0 \\
\phi'_{1 \lambda_0}(h_1) & \chi'_{1 \lambda_0}(h_1) & -\delta_1 \phi'_{2 \lambda_0}(h_1) & -\delta_1 \phi'_{2 \lambda_0}(h_1) & \cdots & \cdots & \cdots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 0 & \cdots & \cdots & 0 & -\delta_m \phi_{(m+1) \lambda_0}(h_m) & -\delta_m \chi_{(m+1) \lambda_0}(h_m) & 0 \\
0 & 0 & \cdots & \cdots & 0 & -\delta_m \phi'_{(m+1) \lambda_0}(h_m) & -\delta_m \chi'_{(m+1) \lambda_0}(h_m) & 0 \\
\end{vmatrix}
\]

\[
= - \left( \prod_{i=1}^{m} \delta_i^2 \omega_i(\lambda_0) \right) \omega_{m+1}^{\lambda_0}(\lambda_0) \neq 0.
\]

Therefore, the system (18) has only the trivial solution \( c_i = 0 \) \((i = 1, 2m + 2)\). Thus we get a contradiction, which completes the proof. \(\blacksquare\)

**Lemma 2.5.** If \( \lambda = \lambda_0 \) is an eigenvalue, then \( \phi(x, \lambda_0) \) and \( \chi(x, \lambda_0) \) are linearly dependent.

**Proof.** Let \( \lambda = \lambda_0 \) be an eigenvalue. Then by virtue of Theorem 2.2

\[
W(\phi_{i \lambda_0}, \chi_{i \lambda_0}; x) = \omega_i(\lambda_0) = 0
\]

and hence

\[
\chi_{i \lambda_0}(x) = k_i \phi_{i \lambda_0}(x) \quad (i = 1, m + 1)
\]

(19)

for some \( k_1 \neq 0, k_2 \neq 0, \ldots, k_{m+1} \neq 0 \). We must show that \( k_1 = k_2 = \ldots = k_{m+1} \).

Suppose, if possible, that \( k_m \neq k_{m+1} \).

Taking into account the definitions of the solutions \( \phi_i(x, \lambda) \) and \( \chi_i(x, \lambda) \) from the equalities (19), we have

\[
\tau_{2m+1}(\chi_{\lambda_0}) = \chi_{\lambda_0}(h_m - 0) - \delta_m \chi_{\lambda_0}(h_m + 0) = \chi_{m \lambda_0}(h_m) - \delta_m \chi_{(m+1) \lambda_0}(h_m)
\]

\[
= k_m \phi_m(h_m) - \delta_m k_{m+1} \phi_{m+1}(h_m) = k_m \delta_m \phi_{m+1}(h_m) - \delta_m k_{m+1} \phi_{m+1}(h_m)
\]

\[
= \delta_m (k_m - k_{m+1}) \phi_{m+1}(h_m) = 0.
\]

since \( \tau_{2m+1}(\chi_{\lambda_0}) = 0 \) and \( \delta_m (k_m - k_{m+1}) \neq 0 \), it follows that

\[
\phi_{(m+1) \lambda_0}(h_m) = 0.
\]

By the same procedure from \( \tau_{2m+2}(\chi_{\lambda_0}) = 0 \) we can derive that

\[
\phi'_{(m+1) \lambda_0}(h_m) = 0.
\]

From the fact that \( \phi_{(m+1) \lambda_0}(x) \) is a solution of the differential equation (1) on \([h_m, 1]\) and satisfies the initial conditions (20) and (21), it follows that \( \phi_{(m+1) \lambda_0}(x) = 0 \) identically on \([h_m, 1]\) because of the well-known existence
and uniqueness theorem for the initial value problems of the ordinary linear differential equations. Making use of (14), (19) and (20), we may also derive that

\[ \phi_{m,\lambda_0}(h_m) = \phi'_{m,\lambda_0}(h_m) = 0 \]  

(22)

Continuing in this matter, we may also find that

\[
\begin{cases}
\phi_{(m-1),\lambda_0}(h_{m-1}) = \phi'_{(m-1),\lambda_0}(h_{m-1}) = 0. \\
\vdots \\
\phi_{1,\lambda_0}(h_1) = \phi'_{1,\lambda_0}(h_1) = 0.
\end{cases}
\]

(23)

identically on \([h_{m-1}, h_m], \ldots, [-1, h_1]\) respectively. Hence \(\phi(x, \lambda_0) = 0\) identically on \([-1, h_1] \cup (h_1, h_2) \cup \ldots \cup (h_m, 1]\). But this contradicts with (11). Hence \(k_m = k_{m+1}\). Analogically we can prove that \(k_{m-1} = k_m, \ldots, k_2 = k_3\) and \(k_1 = k_2\).

\[ \text{Corollary 2.3. If } \lambda = \lambda_0 \text{ is an eigenvalue, then both } \phi(x, \lambda_0) \text{ and } \chi(x, \lambda_0) \text{ are eigenfunctions corresponding to this eigenvalue.} \]

\[ \text{Lemma 2.6. All eigenvalues } \lambda_n \text{ are simple zeros of } \omega(\lambda). \]

**Proof.** Using the Lagrange’s formula (cf. [25], p. 6-7), it can be shown that

\[
(\lambda - \lambda_n) \left( \int_{-1}^{h_1} \phi_\lambda(x) \phi_{\lambda_n}(x) \, dx + \delta_1 \int_{h_1}^{h_2} \phi_\lambda(x) \phi_{\lambda_n}(x) \, dx + \delta_1^2 \delta_2 \int_{h_2}^{h_3} \phi_\lambda(x) \phi_{\lambda_n}(x) \, dx + \cdots + \left( \prod_{i=1}^{m} \delta_i^2 \right) \int_{h_{m-1}}^{h_m} \phi_\lambda(x) \phi_{\lambda_n}(x) \, dx \right) = \left( \prod_{i=1}^{m} \delta_i^2 \right) W(\phi_\lambda, \phi_{\lambda_n}; 1) \]

(24)

for any \(\lambda\). Recall that

\[
\chi_{\lambda_n}(x) = k_n \phi_{\lambda_n}(x), \quad x \in [-1, h_1] \cup (h_1, h_2) \cup \ldots \cup (h_m, 1]
\]

for some \(k_n \neq 0, n = 1, 2, \ldots\). Using this equality for the right side of (24), we have

\[
W(\phi_\lambda, \phi_{\lambda_n}; 1) = \frac{1}{k_n} W(\phi_\lambda, \chi_{\lambda_n}; 1) = \frac{1}{k_n} \left( \lambda_n R_1'(\phi_\lambda) + R_1(\phi_\lambda) \right)
\]

\[
= \frac{1}{k_n} \left[ \omega(\lambda) - (\lambda - \lambda_n) R_1'(\phi_\lambda) \right] = (\lambda - \lambda_n) \frac{1}{k_n} \left[ \omega(\lambda) - (\lambda - \lambda_n) R_1'(\phi_\lambda) \right].
\]

Substituting this formula in (24) and letting \(\lambda \to \lambda_n\), we get

\[
\int_{-1}^{h_1} (\phi_{\lambda_n}(x))^2 \, dx + \delta_1 \int_{h_1}^{h_2} (\phi_{\lambda_n}(x))^2 \, dx + \delta_1^2 \delta_2 \int_{h_2}^{h_3} (\phi_{\lambda_n}(x))^2 \, dx + \cdots + \left( \prod_{i=1}^{m} \delta_i^2 \right) \int_{h_{m-1}}^{h_m} (\phi_{\lambda_n}(x))^2 \, dx
\]

\[
= \frac{1}{k_n} \left( \omega'(\lambda_n) - R_1'(\phi_{\lambda_n}) \right).
\]

(25)
Now putting 
\[ R'_1 (\phi_{\lambda_n}) = \frac{1}{k_n} R'_1 (\chi_{\lambda_n}) = \frac{\rho}{k_n} \]
in (25) we get \( \omega' (\lambda_n) \neq 0. \)

**Definition 2.1.** The geometric multiplicity of an eigenvalue \( \lambda \) of the problem (1)-(5) is the dimension of its eigenspace, i.e. the number of its linearly independent eigenfunctions.

**Theorem 2.4.** All eigenvalues of the problem (1)-(5) are geometrically simple.

**Proof.** If \( f \) and \( g \) are two eigenfunctions for an eigenvalue \( \lambda_0 \) of (1)-(5) then (2) implies that \( f(-1) = cg(-1) \) and \( f'(-1) = cg'(-1) \) for some constant \( c \in \mathbb{C} \). By the uniqueness theorem for solutions of ordinary differential equation and the transmission conditions (4)-(5), we have that \( f = cg \) on \([-1, h_1], [h_1, h_2] \) and \([h_m, 1] \). Thus the geometric multiplicity of \( \lambda_0 \) is one. \( \blacksquare \)

## 3 Asymptotic approximate formulas of \( \omega(\lambda) \) for four distinct cases

We start by proving some lemmas.

**Lemma 3.1.** Let \( \phi(x, \lambda) \) be the solutions of Eq. (1) defined in Section 2, and let \( \lambda = \alpha \). Then the following integral equations hold for \( k = 0, 1 \):

\[
\left\{ \begin{array}{l}
\phi_{1\lambda}^{(k)} (x) = \alpha_2 \left( \cos s (x + 1) \right)^{(k)} - \frac{1}{2} \left( \sin s (x + 1) \right)^{(k)} \\
\quad \quad \quad + \frac{1}{x} \int_{-1}^{x} \left( \sin s (x - y) \right)^{(k)} q(y) \phi_{1\lambda} (y) \ dy, \\
\phi_{2\lambda}^{(k)} (x) = \frac{1}{s \alpha} \phi_{1\lambda} (h_1) \left( \cos s (x - h_1) \right)^{(k)} + \frac{1}{s} \phi_{1\lambda}' (h_1) \left( \sin s (x - h_1) \right)^{(k)} \\
\quad \quad \quad + \frac{1}{h_1} \int_{h_1}^{x} \left( \sin s (x - y) \right)^{(k)} q(y) \phi_{2\lambda} (y) \ dy, \\
\phi_{(m+1)\lambda}^{(k)} (x) = \frac{1}{s \alpha} \phi_{m\lambda} (h_m) \left( \cos s (x - h_m) \right)^{(k)} + \frac{1}{s} \phi_{m\lambda}' (h_m) \left( \sin s (x - h_m) \right)^{(k)} \\
\quad \quad \quad + \frac{1}{h_m} \int_{h_m}^{x} \left( \sin s (x - y) \right)^{(k)} q(y) \phi_{(m+1)\lambda} (y) \ dy,
\end{array} \right. 
\]  \( (26) \)

where \( (\cdot)^{(k)} = \frac{d^k}{dx^k} (\cdot) \).

**Proof.** It is enough to substitute \( s^2 \phi_{1\lambda} (y) + \phi_{1\lambda}' (y), s^2 \phi_{2\lambda} (y) + \phi_{2\lambda}' (y), \ldots, s^2 \phi_{(m+1)\lambda} (y) + \phi_{(m+1)\lambda}' (y) \) instead of \( q(y) \phi_{1\lambda} (y), q(y) \phi_{2\lambda} (y), \ldots, q(y) \phi_{(m+1)\lambda} (y) \) in the integral terms of the (26), respectively, and integrate by parts twice. \( \blacksquare \)

**Lemma 3.2.** Let \( \lambda = \alpha \). Im \( s = t \). Then the functions \( \phi_{1\lambda} (x) \) have the following asymptotic formulas for \( |\lambda| \to \infty \), which hold uniformly for \( x \in \Omega \).
\( (\text{for } i = 1, m + 1 \text{ and } k = 0, 1.) \):

\[
\phi_{1\lambda}^{(k)}(x) = \alpha_2 \cos s(x + 1)^{(k)} + O \left( |s|^{k-1} e^{|t|(x+1)} \right), \\
\phi_{2\lambda}^{(k)}(x) = \frac{\alpha_2}{\delta_1} \cos s(x + 1)^{(k)} + O \left( |s|^{k-1} e^{|t|(x+1)} \right), \\
\vdots \\
\phi_{(m+1)\lambda}^{(k)}(x) = \frac{\alpha_2}{m \prod_{i=1}^{\delta_i}} \cos s(x + 1)^{(k)} + O \left( |s|^{k-1} e^{|t|(x+1)} \right)
\]

(27)

\[ \text{if } \alpha_2 \neq 0, \]

\[
\phi_{1\lambda}^{(k)}(x) = -\frac{\alpha_1}{s} \sin s(x + 1)^{(k)} + O \left( |s|^{k-2} e^{|t|(x+1)} \right), \\
\phi_{2\lambda}^{(k)}(x) = -\frac{\alpha_1}{s \delta_1} \sin s(x + 1)^{(k)} + O \left( |s|^{k-2} e^{|t|(x+1)} \right), \\
\vdots \\
\phi_{(m+1)\lambda}^{(k)}(x) = -\frac{\alpha_1}{s m \prod_{i=1}^{\delta_i}} \sin s(x + 1)^{(k)} + O \left( |s|^{k-2} e^{|t|(x+1)} \right)
\]

(28)

\[ \text{if } \alpha_2 = 0. \]

**Proof.** Since the proof of the formulae for \( \phi_{1\lambda}(x) \) is identical to Titchmarsh’s proof to similar results for \( \phi_{2\lambda}(x) \) (see [26], Lemma 1.7 p. 9-10), we may formulate them without proving them here.

Since the proof of the formulae for \( \phi_{2\lambda}(x) \) and \( \phi_{3\lambda}(x) \) are identical to Kadakal’s and Mukhtarov’s proof to similar results for \( \phi_{3\lambda}(x) \) (see [11], Lemma 3.2 p. 1373-1375), we may formulate them without proving them here. But the similar formulae for \( \phi_{4\lambda}(x), \ldots, \phi_{(m+1)\lambda}(x) \) need individual consideration, since the last solutions are defined by the initial conditions of these special nonstandard forms. We shall only prove the formula (27) for \( k = 0 \) and \( m = 3 \).

Let \( \alpha_2 \neq 0 \). Then according to (27) for \( m = 2 \)

\[
\phi_{3\lambda}(h_3) = \alpha_2 \left\{ \frac{1}{\delta_2} \cos s(h_3 - h_2) \left[ \frac{1}{\delta_1} \cos s(h_2 - h_1) \cos s(h_1 + 1) \right.ight.
\]

\[
- \frac{1}{\delta_1} \sin s(h_2 - h_1) \sin s(h_1 + 1) \left[ 1 + \frac{1}{\delta_1} \sin s(h_2 - h_1) \cos s(h_1 + 1) \right] + \frac{1}{\delta_1} \cos s(h_2 - h_1) \sin s(h_1 + 1) \left. \right\} + O \left( |s|^{-1} e^{|t|[h_3-h_2]+(h_2-h_1)+(h_1+1)]} \right)
\]
and

\[ \phi_{3\lambda} (h_3) = \alpha_2 \left\{ -\frac{s}{\delta_2} \sin s (h_3 - h_2) \left[ \frac{1}{\delta_1} \cos s (h_2 - h_1) \cos s (h_1 + 1) \right] - \frac{1}{\delta_1} \sin s (h_2 - h_1) \sin s (h_1 + 1) \right\} - \frac{s}{\delta_2} \cos s (h_3 - h_2) \left[ \frac{1}{\delta_1} \sin s (h_2 - h_1) \cos s (h_1 + 1) \right] + \frac{1}{\delta_1} \cos s (h_2 - h_1) \sin s (h_1 + 1) \right\} + O \left( e^{\|t\|[h_3-h_2]+(h_2-h_1)+(h_1+1)]} \right) \]

Substituting these asymptotic expressions into (26), we get

\[ \phi_{4\lambda} (x) = \alpha_2 \left\{ \frac{1}{\delta_3} \cos s (x - h_3) \left[ \frac{1}{\delta_2} \cos s (h_3 - h_2) \right] \left[ \frac{1}{\delta_1} \cos s (h_2 - h_1) \cos s (h_1 + 1) \right] - \frac{1}{\delta_1} \sin s (h_2 - h_1) \sin s (h_1 + 1) \right\} - \frac{1}{\delta_2} \sin s (x - h_3) \left[ \frac{1}{\delta_1} \sin s (h_2 - h_1) \cos s (h_1 + 1) \right] + \frac{1}{\delta_1} \cos s (h_2 - h_1) \sin s (h_1 + 1) \right\} \]

\[ \times \cos s (h_3 - h_2) \left[ \frac{1}{\delta_1} \sin s (h_2 - h_1) \cos s (h_1 + 1) \right] + \frac{1}{\delta_1} \sin s (h_2 - h_1) \sin s (h_1 + 1) \right\} + \frac{1}{s} \int_{h_3}^{x} \sin s (x - y) q (y) \phi_{4\lambda} (y) dy + O \left( |s|^{-1} e^{\|t\|[x-h_3]+(h_3-h_2)+(h_2-h_1)+(h_1+1)]} \right) \]

(29)

Multiplying through by \( e^{-|t|[(x-h_3)+(h_3-h_2)+(h_2-h_1)+(h_1+1)]} \), and denoting

\[ F_{4\lambda} (x) := e^{-|t|[(x-h_3)+(h_3-h_2)+(h_2-h_1)+(h_1+1)]} \phi_{4\lambda} (x) \]

we have

\[ F_{4\lambda} (x) := \alpha_2 e^{-|t|[(x-h_3)+(h_3-h_2)+(h_2-h_1)+(h_1+1)]} \left\{ \frac{1}{\delta_3} \cos s (x - h_3) \left[ \frac{1}{\delta_2} \cos s (h_3 - h_2) \right] \left[ \frac{1}{\delta_1} \cos s (h_2 - h_1) \cos s (h_1 + 1) \right] - \frac{1}{\delta_1} \sin s (h_3 - h_2) \right\} - \frac{1}{\delta_2} \sin s (x - h_3) \left[ \frac{1}{\delta_1} \sin s (h_2 - h_1) \cos s (h_1 + 1) \right] + \frac{1}{\delta_1} \cos s (h_2 - h_1) \sin s (h_1 + 1) \right\} \]

\[ \times \left[ \frac{1}{\delta_1} \cos s (h_3 - h_2) \left[ \frac{1}{\delta_1} \cos s (h_2 - h_1) \cos s (h_1 + 1) \right] - \frac{1}{\delta_1} \sin s (h_2 - h_1) \sin s (h_1 + 1) \right\] + \frac{1}{\delta_1} \cos s (h_3 - h_2) \left[ \frac{1}{\delta_1} \sin s (h_2 - h_1) \cos s (h_1 + 1) \right] + \frac{1}{\delta_1} \cos s (h_2 - h_1) \sin s (h_1 + 1) \right\} + \frac{1}{s} \int_{h_3}^{x} \sin s (x - y) q (y) e^{-|t|[(x-h_3)+(h_3-h_2)+(h_2-h_1)+(h_1+1)]} F_{4\lambda} (y) dy + O \left( |s|^{-1} \right) . \]

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Denoting $M := \max_{x \in [h_3, 1]} |F_{4\lambda} (x)|$ from the last formula, it follows that

$$M (\lambda) \leq |\alpha_2| \frac{2}{|\delta_1|} \frac{2}{|\delta_2|} \frac{2}{|\delta_3|} + \frac{M (\lambda)}{|s|} \int_{h_3}^1 q (y) \, dy + \frac{M_0}{|s|}$$

for some $M_0 > 0$. From this, it follows that $M (\lambda) = O (1)$ as $\lambda \to \infty$, so

$$\phi_{4\lambda} (x) = O \left( e^{\frac{|s|}{2} (x - h_3 + (h_3 - h_2 + (h_2 - h_1) + (h_1 + 1))} \right).$$

Substituting this back into the integral on the right side of (29) yields (27) for $k = 0$ and $m = 3$. The other cases may be considered analogically. ■

**Theorem 3.1.** Let $\lambda = s^2$, $t = \text{Im} \, s$. Then the characteristic function $\omega (\lambda)$ has the following asymptotic formulas:

**Case 1:** If $\beta'_2 \neq 0$, $\alpha_2 \neq 0$, then

$$\omega (\lambda) = \beta'_2 \alpha_2 s^3 \left( \prod_{i=1}^m \delta_i^2 \right) \sin 2s + O \left( |s|^2 e^{2|t|} \right). \quad (30)$$

**Case 2:** If $\beta'_2 \neq 0$, $\alpha_2 = 0$, then

$$\omega (\lambda) = -\beta'_2 \alpha_1 s^2 \left( \prod_{i=1}^m \delta_i^2 \right) \cos 2s + O \left( |s|^2 e^{2|t|} \right). \quad (31)$$

**Case 3:** If $\beta'_2 = 0$, $\alpha_2 \neq 0$, then

$$\omega (\lambda) = \beta'_1 \alpha_2 s^2 \left( \prod_{i=1}^m \delta_i^2 \right) \cos 2s + O \left( |s|^2 e^{2|t|} \right). \quad (32)$$

**Case 4:** If $\beta'_2 = 0$, $\alpha_2 = 0$, then

$$\omega (\lambda) = -\beta'_1 \alpha_1 s \left( \prod_{i=1}^m \delta_i^2 \right) \sin 2s + O \left( |s|^2 e^{2|t|} \right). \quad (33)$$

**Proof.** The proof is completed by substituting (27) and (28) into the represen-
Corollary 3.1. The eigenvalues of the problem (1)-(5) are bounded below.

Proof. Putting $s = it$ ($t > 0$) in the above formulas, it follows that $\omega(-t^2) \to \infty$ as $t \to \infty$. Therefore, $\omega(\lambda) \neq 0$ for $\lambda$ negative and sufficiently large.

4 Asymptotic formulas for eigenvalues and eigenfunctions

Now we can obtain the asymptotic approximation formulae for the eigenvalues of the considered problem (1)-(5).

Since the eigenvalues coincide with the zeros of the entire function $\omega_{m+1}(\lambda)$, it follows that they have no finite limit. Moreover, we know from Corollaries 2.1 and 3.1 that all eigenvalues are real and bounded below. Hence, we may renumber them as $\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq ...$, listed according to their multiplicity.

Theorem 4.1. The eigenvalues $\lambda_n = s_n^2$, $n = 0, 1, 2, ...$ of the problem (1)-(5) have the following asymptotic formulae for $n \to \infty$:

Case 1: If $\beta_2' \neq 0, \alpha_2 \neq 0$, then

$$s_n = \frac{\pi (n - 1)}{2} + O \left( \frac{1}{n} \right).$$

Case 2: If $\beta_2' \neq 0, \alpha_2 = 0$, then

$$s_n = \frac{\pi (n - \frac{1}{2})}{2} + O \left( \frac{1}{n} \right).$$

Case 3: If $\beta_2' = 0, \alpha_2 \neq 0$, then

$$s_n = \frac{\pi (n - \frac{1}{2})}{2} + O \left( \frac{1}{n} \right).$$
Case 4: If $\beta_2' = 0, \alpha_2 = 0$, then

$$s_n = \frac{\pi n}{2} + O\left(\frac{1}{n}\right).$$  \hfill (38)

**Proof.** We shall only consider the first case. The other cases may be considered similarly. Denoting $\omega_1(s)$ and $\omega_2(s)$ the first and $O$-term of the right of (42) respectively, we shall apply the well-known Rouché’s theorem, which asserts that if $f(s)$ and $g(s)$ are analytic inside and on a closed contour $C$, and $|g(s)| < |f(s)|$ on $C$, then $f(s)$ and $f(s) + g(s)$ have the same number zeros inside $C$, provided that each zero is counted according to their multiplicity. It is readily shown that $|\omega_1(s)| > |\omega_2(s)|$ on the contours

$$C_n := \left\{ s \in \mathbb{C} \mid |s| = \frac{(n + \frac{1}{2}) \pi}{2} \right\}$$

for sufficiently large $n$.

Let $\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \ldots$ be zeros of $\omega(\lambda)$ and $\lambda_n = s_n^2$. Since inside the contour $C_n$, $\omega_1(s)$ has zeros at points $s = 0$ and $s = \frac{k\pi}{4}$, $k = \pm 1, \pm 2, \ldots, \pm n$.

$$s_n = \frac{(n - 1) \pi}{2} + \delta_n \hfill (39)$$

where $\delta_n = O(1)$ for sufficiently large $n$. By substituting this in (30), we derive that $\delta_n = O\left(\frac{1}{n}\right)$, which completes the proof. ■

The next approximation for the eigenvalues may be obtained by the following procedure. For this, we shall suppose that $q(y)$ is of bounded variation in $[-1, 1]$.

Firstly we consider the case $\beta_2' \neq 0$ and $\alpha_2 \neq 0$. Putting $x = h_1, x = h_2, \ldots, x = h_m$ in (26) and then substituting in the expression of $\phi_{(m+1)\lambda}$, we get that

$$\phi'_{(m+1)\lambda}(1) = \frac{\alpha_2}{\prod_{i=1}^{m} \delta_i} \sin 2s - \frac{\alpha_1}{\prod_{i=1}^{m} \delta_i} \cos 2s + \frac{1}{\prod_{i=1}^{m} \delta_i} \int_{h_1}^{h_2} \cos (s(1-y)) q(y) \phi_{1\lambda}(y) \, dy$$

$$+ \frac{1}{\prod_{i=2}^{m} \delta_i} \int_{h_1}^{h_2} \cos (s(1-y)) q(y) \phi_{2\lambda}(y) \, dy + \frac{1}{\prod_{i=3}^{m} \delta_i} \int_{h_1}^{h_2} \cos (s(1-y)) q(y) \phi_{3\lambda}(y) \, dy$$

$$+ \ldots + \frac{1}{\delta_m} \int_{h_m}^{h_m} \cos (s(1-y)) q(y) \phi_{m\lambda}(y) \, dy + \frac{1}{\prod_{i=m+1}^{m} \delta_i} \int_{h_m}^{h_m} \cos (s(1-y)) q(y) \phi_{(m+1)\lambda}(y) \, dy.$$
Substituting (27) into the right side of the last integral equality then gives

\[
\phi_{(m+1)\lambda} (1) = \frac{s\alpha_2}{m} \prod_{i=1} \delta_i \sin 2s - \frac{\alpha_1}{m} \prod_{i=1} \delta_i \cos 2s + \frac{\alpha_2}{m} \prod_{i=1} \delta_i \left( \int_{h_1} \cos (s (1 - y)) \cos (s (1 + y)) q(y) \, dy \right)
\]

\[
+ \left( \prod_{i=1} \delta_i \right) \int_{h_1} \cos (s (1 - y)) \cos (s (1 + y)) q(y) \, dy +
\]

\[
\ldots + \left( \prod_{i=1} \delta_i \right) \int_{h_m}^{1} \cos (s (1 - y)) \cos (s (1 + y)) q(y) \, dy + O \left( |s|^{-1} e^{2|t|} \right).
\]

On the other hand, from (27), it follows that

\[
\phi_{(m+1)\lambda} (1) = \frac{\alpha_2}{m} \prod_{i=1} \delta_i \cos 2s + O \left( |s|^{-1} e^{2|t|} \right).
\]

Putting these formulas into (34), we have

\[
\omega (\lambda) = s^3 \beta' \alpha_2 \prod_{i=1} \delta_i \left( \sin 2s + \frac{1}{2} \left( \frac{\beta_1' \alpha_2 + \beta_2' \alpha_1}{\prod_{i=1} \delta_i} \right) \cos 2s \right)
\]

\[
- \frac{\beta_2'}{\prod_{i=1} \delta_i} \int_{h_1}^{1} \cos (s (1 - y)) q(y) \phi_1\lambda (y) \, dy - \frac{\beta_2'}{\prod_{i=1} \delta_i} \left( \int_{h_1}^{1} \cos (s (1 - y)) q(y) \phi_2\lambda (y) \, dy \right)
\]

\[
\ldots - \frac{\beta_2'}{\delta_{h_{m-1}}} \int_{h_m}^{1} \cos (s (1 - y)) q(y) \phi_{m\lambda} (y) \, dy - \frac{\beta_2'}{\delta_{h_{m}}} \int_{h_m}^{1} \cos (s (1 - y)) q(y) \phi_{(m+1)\lambda} (y) \, dy
\]

\[
+ O \left( |s|^{-1} e^{2|t|} \right).
\]

Putting (39) in the last equality we find that

\[
\sin (2\delta_n) = \frac{\cos (2\delta_n)}{s_n} \left[ \frac{\beta_1'}{\beta_2'} + \frac{\alpha_1}{\alpha_2} \left( \int_{h_1}^{1} q(y) \, dy \right) - \frac{1}{2 \prod_{i=1} \delta_i} \left( \int_{h_1}^{1} \cos (2s_n y) q(y) \, dy \right) \right]
\]

\[
+ O \left( (s_n)^{-2} \right) \quad (40)
\]
Recalling that $q(y)$ is of bounded variation in $[-1, 1]$, and applying the well-known Riemann-Lebesgue Lemma (see [27], p. 48, Theorem 4.12) to the second integral on the right in (40), this term is $O\left(\frac{1}{n}\right)$. As a result, from (40) it follows that

$$
\delta_n = -\frac{1}{\pi (n-1)} \left[ \frac{\beta_1'}{\beta_2'} + \frac{\alpha_1}{\alpha_2} - \frac{1}{2} \left( \prod_{i=1}^{m} \delta_i \right)^{-1} \int_{-1}^{1} q(y) \, dy \right] + O\left(\frac{1}{n^2}\right).
$$

Substituting in (40), we have

$$
s_n = \frac{\pi (n-1)}{2} - \frac{1}{\pi (n-1)} \left[ \frac{\beta_1'}{\beta_2'} + \frac{\alpha_1}{\alpha_2} - \frac{1}{2} \left( \prod_{i=1}^{m} \delta_i \right)^{-1} \int_{-1}^{1} q(y) \, dy \right] + O\left(\frac{1}{n^2}\right).
$$

Similar formulas in the other cases are as follows:

In case 2:

$$
s_n = \frac{\pi (n - \frac{1}{2})}{2} - \frac{1}{\pi (n-\frac{1}{2})} \left[ \frac{\beta_1'}{\beta_2'} + \frac{1}{2} \left( \prod_{i=1}^{m} \delta_i \right)^{-1} \int_{-1}^{1} q(y) \, dy \right] + O\left(\frac{1}{n^2}\right).
$$

In case 3:

$$
s_n = \frac{\pi (n - \frac{1}{2})}{2} + \frac{1}{\pi (n-\frac{1}{2})} \left[ \frac{\beta_2}{\beta_1'} - \frac{\alpha_1}{\alpha_2} + \frac{1}{2} \left( \prod_{i=1}^{m} \delta_i \right)^{-1} \int_{-1}^{1} q(y) \, dy \right] + O\left(\frac{1}{n^2}\right).
$$

In case 4:

$$
s_n = \frac{\pi n}{2} + \frac{1}{\pi n} \left[ \frac{\beta_2}{\beta_1'} + \frac{1}{2} \left( \prod_{i=1}^{m} \delta_i \right)^{-1} \int_{-1}^{1} q(y) \, dy \right] + O\left(\frac{1}{n^2}\right).
$$

Recalling that $\phi(x, \lambda_n)$ is an eigenfunction according to the eigenvalue $\lambda_n$ and by putting (35) into the (27) we obtain that

$$
\phi_{1\lambda_n}(x) = \alpha_2 \cos \left( \frac{\pi (n-1) (x+1)}{2} \right) + O\left(\frac{1}{n}\right),
$$

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\[ \phi_{2\lambda_n}(x) = \frac{\alpha_2}{\delta_1} \cos \left( \frac{\pi (n - 1) (x + 1)}{2} \right) + O \left( \frac{1}{n} \right), \]

\[ \vdots \]

\[ \phi_{(m+1)\lambda_n}(x) = \frac{\alpha_2}{\prod_{i=1}^m \delta_i} \cos \left( \frac{\pi (n - 1) (x + 1)}{2} \right) + O \left( \frac{1}{n} \right) \]

in the first case. Consequently, if \( \beta_2 \neq 0 \) and \( \alpha_2 \neq 0 \), then the eigenfunction \( \phi(x, \lambda_n) \) has the following asymptotic formulae

\[ \phi(x, \lambda_n) = \begin{cases} 
\alpha_2 \cos \left( \frac{\pi (n - 1) (x + 1)}{2} \right) + O \left( \frac{1}{n} \right), & x \in [-1, h_1) \\
\frac{\alpha_2}{\delta_1} \cos \left( \frac{\pi (n - 1) (x + 1)}{2} \right) + O \left( \frac{1}{n} \right), & x \in (h_1, h_2) \\
\vdots \\
\frac{\alpha_2}{\prod_{i=1}^m \delta_i} \cos \left( \frac{\pi (n - 1) (x + 1)}{2} \right) + O \left( \frac{1}{n} \right), & x \in (h_m, 1] 
\end{cases} \]

which holds uniformly for \( x \in [-1, h_1) \cup (h_1, h_2) \cup \ldots \cup (h_m, 1] \).

Similar formulas in the other cases are as follows:

In case 2

\[ \phi(x, \lambda_n) = \begin{cases} 
-\frac{2\alpha_1}{\pi (n - \frac{1}{2})} \sin \left( \frac{\pi (n - \frac{1}{2}) (x + 1)}{2} \right) + O \left( \frac{1}{n^2} \right), & x \in [-1, h_1) \\
-\frac{2\alpha_1}{\delta_1 \pi (n - \frac{1}{2})} \sin \left( \frac{\pi (n - \frac{1}{2}) (x + 1)}{2} \right) + O \left( \frac{1}{n^2} \right), & x \in (h_1, h_2) \\
\vdots \\
-\frac{2\alpha_1}{\prod_{i=1}^m \delta_i} \sin \left( \frac{\pi (n - \frac{1}{2}) (x + 1)}{2} \right) + O \left( \frac{1}{n^2} \right), & x \in (h_m, 1] 
\end{cases} \]

In case 3

\[ \phi(x, \lambda_n) = \begin{cases} 
\alpha_2 \cos \left( \frac{\pi (n - \frac{1}{2}) (x + 1)}{2} \right) + O \left( \frac{1}{n} \right), & x \in [-1, h_1) \\
\frac{\alpha_2}{\delta_1} \cos \left( \frac{\pi (n - \frac{1}{2}) (x + 1)}{2} \right) + O \left( \frac{1}{n} \right), & x \in (h_1, h_2) \\
\vdots \\
\frac{\alpha_2}{\prod_{i=1}^m \delta_i} \cos \left( \frac{\pi (n - \frac{1}{2}) (x + 1)}{2} \right) + O \left( \frac{1}{n} \right), & x \in (h_m, 1] 
\end{cases} \]
In case 4

\[ \phi(x, \lambda_n) = \begin{cases} 
- \frac{2\alpha_1}{\pi n} \sin \left( \frac{\pi n(x+1)}{2} \right) + O \left( \frac{1}{n^2} \right), & x \in [-1, h_1) \\
- \frac{2\alpha_1}{\delta_1 \pi n} \sin \left( \frac{\pi n(x+1)}{2} \right) + O \left( \frac{1}{n^2} \right), & x \in (h_1, h_2) \\
\vdots & \vdots \\
- \frac{2\alpha_1}{\prod_{m} \delta_m} \sin \left( \frac{\pi n(x+1)}{2} \right) + O \left( \frac{1}{n^2} \right), & x \in (h_m, 1]. 
\end{cases} \]

All these asymptotic formulas hold uniformly for \( x \in [-1, h_1) \cup (h_1, h_2) \cup \ldots \cup (h_m, 1]. \)

5 Completeness of eigenfunctions

**Theorem 5.1** The operator \( A \) has only point spectrum, i.e., \( \sigma(A) = \sigma_p(A) \).

**Proof.** It suffices to prove that if \( \eta \) is not an eigenvalue of \( A \), then \( \eta \in \sigma(A) \). Since \( A \) is self-adjoint, we only consider a real \( \eta \). We investigate the equation \( (A - \eta) Y = F \in H \), where \( F = \left( \frac{f}{f_1} \right) \).

Let us consider the initial-value problem

\[
\begin{cases}
\tau y - \eta y = f, & x \in [-1, h_1) \cup (h_1, h_2) \cup \ldots \cup (h_m, 1], \\
\alpha_1 y(-1) + \alpha_2 y'(-1) = 0, \\
y(h_i - 0) - \delta_i y(h_i + 0) = 0, \\
y'(h_i - 0) - \delta_i y'(h_i + 0) = 0.
\end{cases}
\]

Let \( u(x) \) be the solution of the equation \( \tau u - \eta u = 0 \) satisfying

\[
\begin{cases}
u(-1) = \alpha_2, & u'(-1) = -\alpha_1, \\
u(h_i - 0) - \delta_i u(h_i + 0) = 0, \\
u'(h_i - 0) - \delta_i u'(h_i + 0) = 0.
\end{cases}
\]

In fact,

\[ u(x) = \begin{cases} u_1(x), & x \in [-1, h_1), \\
u_2(x), & x \in (h_1, h_2), \\
\vdots \\
u_{m+1}(x), & x \in (h_m, 1], \end{cases} \]

where \( u_1(x) \) is the unique solution of the initial-value problem

\[
\begin{cases}
-u'' + q(x) u = \eta u, & x \in [-1, h_1), \\
u(-1) = \alpha_2, & u'(-1) = -\alpha_1;
\end{cases}
\]

\( u_2(x) \) is the unique solution of the problem

\[
\begin{cases}
-u'' + q(x) u = \eta u, & x \in (h_1, h_2), \\
u(h_1 - 0) = \delta_1 u(h_1 + 0), \\
u'(h_1 - 0) = \delta_1 u'(h_1 + 0); 
\end{cases}
\]
$u_3 (x)$ is the unique solution of the problem
\[
\left\{ \begin{array}{l}
-u'' + q(x)u = \eta u, \quad x \in (h_2, h_3), \\
 u (h_2 - 0) = \delta_2 u (h_2 + 0), \\
 u' (h_2 - 0) = \delta_2 u' (h_2 + 0), \\
\end{array} \right.
\]
and $u_{m+1} (x)$ is the unique solution of the problem
\[
\left\{ \begin{array}{l}
-u'' + q(x)u = \eta u, \quad x \in (h_m, 1), \\
 u (h_m - 0) = \delta_m u (h_m + 0), \\
 u' (h_m - 0) = \delta_m u' (h_m + 0). \\
\end{array} \right.
\]
Let
\[
w (x) = \left\{ \begin{array}{l}
w_1 (x), \quad x \in [-1, h_1), \\
w_2 (x), \quad x \in (h_1, h_2), \\
\vdots \\
w_{m+1} (x), \quad x \in (h_m, 1], \\
\end{array} \right.
\]
be a solution of $\tau w - \eta w = f$ satisfying
\[
\begin{align*}
\alpha_1 w (-1) + \alpha_2 w' (-1) &= 0, \\
 w (h_i) &= \beta_i w (h_i + 0), \\
w' (h_i - 0) &= \beta_i w' (h_i + 0).
\end{align*}
\]
Then, (41) has the general solution
\[
y (x) = \left\{ \begin{array}{l}
du_1 + w_1, \quad x \in [-1, h_1), \\
du_2 + w_2, \quad x \in (h_1, h_2), \\
\vdots \\
du_{m+1} + w_{m+1}, \quad x \in (h_m, 1], \\
\end{array} \right. \tag{42}
\]
where $d \in \mathbb{C}$.

Since $\eta$ is not an eigenvalue of the problem (1) – (5), we have
\[
\eta \left[ \beta_1 u_{m+1} (1) + \beta_2 u'_{m+1} (1) \right] + \left[ \beta_1 u_{m+1} (1) + \beta_2 u'_{m+1} (1) \right] \neq 0. \tag{43}
\]
The second component of $(A - \eta) Y = F$ involves the equation
\[
-R_1 (y) - \eta R_1' (y) = f_1,
\]
namely,
\[
[-\beta_1 y (1) - \beta_2 y' (1)] - \eta \beta_1 y (1) + \beta_2 y' (1)] = f_1. \tag{44}
\]
Substituting (42) into (44), we get
\[
\begin{align*}
(-\beta_2 u'_{m+1} (1) - \beta_1 u_{m+1} (1) - \eta \beta_2 u'_{m+1} (1) - \eta \beta_1 u_{m+1} (1)) d \\
&= f_1 + \beta_1 w_{m+1} (1) + \beta_2 w'_{m+1} (1) + \eta \beta_1 w_{m+1} (1) + \eta \beta_2 w'_{m+1} (1)
\end{align*}
\]
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In view of (43), we show that $d$ is uniquely solvable. Therefore, $y$ is uniquely determined.

The above arguments show that $(A - \eta I)^{-1}$ is defined on all of $H$, where $I$ is identity matrix. We obtain that $(A - \eta I)^{-1}$ is bounded by Theorem 2.2 and the Closed Graph Theorem. Thus, $\eta \in \sigma (A)$. Therefore, $\sigma (A) = \sigma_p(A)$. ■

The following lemma may be easily proved.

**Lemma 5.1** The eigenvalues of the boundary value problem (1) − (5) are bounded below, and they are countably infinite and can cluster only at $\infty$.

For every $\delta \in \mathbb{R} \setminus \sigma_p(A)$, we have the following immediate conclusion.

**Lemma 5.2** Let $\lambda$ be an eigenvalue of $A - \delta I$, and $V$ a corresponding eigenfunction. Then, $\lambda^{-1}$ is an eigenvalue of $(A - \delta I)^{-1}$, and $V$ is a corresponding eigenfunction. The converse is also true.

In view of (43), we know that $\rho_n = \sigma (A)$. Then, $\lambda = \sigma (A)$ is determined.

**Theorem 5.2** The operator $A$ has compact resolvents, i.e., for each $\delta \in \mathbb{R} \setminus \sigma_p(A)$, $(A - \delta I)^{-1}$ is compact on $H$.

**Proof.** Let $\{\mu_1, \mu_2, \ldots\}$ be the eigenvalues of $(A - \delta I)^{-1}$, and $\{P_1, P_2, \ldots\}$ the orthogonal projections of finite rank onto the corresponding eigenspaces. Since $\{\mu_1, \mu_2, \ldots\}$ is a bounded sequence and all $P_n$s are mutually orthogonal, we have $\sum_{n=1}^{\infty} \mu_n P_n$ is strongly convergent to the bounded operator $(A - \delta I)^{-1}$, i.e., $(A - \delta I)^{-1} = \sum_{n=1}^{\infty} \mu_n P_n$. Because for every $\alpha > 0$, the number of $\mu_n$s satisfying $|\mu_n| > \alpha$ is finite, and all $P_n$s are of finite rank, we obtain that $(A - \delta I)^{-1}$ is compact. ■

In terms of the above statements and the spectral theorem for compact operators, we obtain the following theorem.

**Theorem 5.3** The eigenfunctions of the problem (1) − (5), augmented to become eigenfunctions of $A$, are complete in $H$, i.e., if we let

$$\left\{ \Phi_n = \begin{pmatrix} \phi_n(x) \\ R_1(\phi_n) \end{pmatrix} ; \ n \in \mathbb{N} \right\}$$

be a maximum set of orthonormal eigenfunctions of $A$, where $\{\phi_n(x) ; n \in \mathbb{N}\}$ are eigenfunctions of (1) − (5), then for all $F \in H$, $F = \sum_{n=1}^{\infty} (F, \Phi_n) \Phi_n$.  

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