Every planar graph without $i$-cycles adjacent simultaneously to $j$-cycles and $k$-cycles is DP-4-colorable when $\{i, j, k\} = \{3, 4, 5\}$.

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Abstract

DP-coloring is a generalization of a list coloring in simple graphs. Many results in list coloring can be generalized in those of DP-coloring. Kim and Ozeki showed that planar graphs without $k$-cycles where $k = 3, 4, 5$, or 6 are DP-4-colorable. Recently, Kim and Yu extended the result on 3- and 4-cycles by showing that planar graphs without triangles adjacent to 4-cycles are DP-4-colorable. Xu and Wu showed that planar graphs without 5-cycles adjacent simultaneously to 3-cycles and 4-cycles are 4-choosable. In this paper, we extend the result on 5-cycles and triangles adjacent to 4-cycles by showing that planar graphs without $i$-cycles adjacent simultaneously to $j$-cycles and $k$-cycles are DP-4-colorable when $\{i, j, k\} = \{3, 4, 5\}$. This also generalizes the result of Xu and Wu.

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1 Introduction

Every graph in this paper is finite, simple, and undirected. Embedding a graph $G$ in the plane, we let $V(G)$, $E(G)$, and $F(G)$ denote the vertex set, edge set, and face set of $G$. For $U \subseteq V(G)$, we let $G[U]$ denote the subgraph of $G$ induced by $U$. For $X, Y \subseteq V(G)$ where $X$ and $Y$ are disjoint, we let $E_G(X, Y)$ be the set of all edges in $G$ with one endpoint in $X$ and the other in $Y$.

The concept of choosability was introduced by Vizing in 1976 [10] and by Erdős, Rubin, and Taylor in 1979 [5], independently. A $k$-list assignment $L$ of a graph $G$ assigns a list $L(v)$ (a set of colors) with $|L(v)| = k$ to each vertex $v$. A graph $G$ is $L$-colorable if there is a proper coloring $f$ where $f(v) \in L(v)$. If $G$ is $L$-colorable for every $k$-assignment $L$, then we say $G$ is $k$-choosable.

Dvořák and Postle [4] introduced a generalization of list coloring in which they called a correspondence coloring. But following Bernshteyn, Kostochka, and Pron [3], we call it a DP-coloring.

Definition 1. Let $L$ be an assignment of a graph $G$. We call $H$ a cover of $G$ if it satisfies all the followings:

(i) The vertex set of $H$ is $\bigcup_{u \in V(G)} (\{u\} \times L(u)) = \{(u, c) : u \in V(G), c \in L(u)\}$;

(ii) $H[u \times L(u)]$ is a complete graph for every $u \in V(G)$;

(iii) For each $uv \in E(G)$, the set $E_H(\{u\} \times L(u), \{v\} \times L(v))$ is a matching (maybe empty).

(iv) If $uv \notin E(G)$, then no edges of $H$ connect $\{u\} \times L(u)$ and $\{v\} \times L(v)$.

Definition 2. An $(H, L)$-coloring of $G$ is an independent set in a cover $H$ of $G$ with size $|V(G)|$. We say that a graph is DP-$k$-colorable if $G$ has an $(H, L)$-coloring for every $k$-assignment $L$ and every cover $H$ of $G$. The DP-chromatic number of $G$, denoted by $\chi_{DP}(G)$, is the minimum number $k$ such that $G$ is DP-$k$-colorable.

If we define edges on $H$ to match exactly the same colors in $L(u)$ and $L(v)$ for each $uv \in E(H)$, then $G$ has an $(H, L)$-coloring if and only if $G$ is $L$-colorable. Thus DP-coloring is a generalization of list coloring. This also implies that $\chi_{DP}(G) \geq \chi_l(G)$. In fact, the difference of these two chromatic numbers can be arbitrarily large. For graphs with average degree $d$, Bernshteyn [2] showed that $\chi_{DP}(G) = \Omega(d/\log d)$, while Alon [1] showed that $\chi_l(G) = \Omega(\log d)$.

Dvořák and Postle [4] showed that $\chi_{DP}(G) \leq 5$ for every planar graph $G$. This extends a seminal result by Thomassen [8] on list colorings. On the other hand, Voigt [11] gave an
example of a planar graph which is not 4-choosable (thus not DP-4-colorable). It is of interest to obtain sufficient conditions for planar graphs to be DP-4-colorable. Kim and Ozeki \cite{6} showed that planar graphs without $k$-cycles are DP-4-colorable for each $k = 3, 4, 5, 6$. Kim and Yu \cite{7} extended the result on 3- and 4-cycles by showing that planar graphs without triangles adjacent to 4-cycles are DP-4-colorable. In this paper, we extend the result on 5-cycles and triangles adjacent to 4-cycles by showing that planar graphs without $i$-cycles adjacent simultaneously to $j$-cycles and $k$-cycles are DP-4-colorable when $\{i, j, k\} = \{3, 4, 5\}$.

Theorem 1. Every planar graph without $i$-cycles adjacent simultaneously to $j$-cycles and $k$-cycles is DP-4-colorable when $\{i, j, k\} = \{3, 4, 5\}$.

Theorem \ref{thm:1} generalizes the result of Xu and Wu \cite{9} as follows.

Corollary 3. Every planar graphs without 5-cycles adjacent simultaneously to 3-cycles and 4-cycles is 4-choosable.

2 Structure Obtained from Condition on Cycles

First, we introduce some notations and definitions. A $k$-vertex ($k^+$-vertex, $k^-$-vertex, respectively) is a vertex of degree $k$ (at least $k$, at most $k$, respectively. The same notations are applied to faces. A $(d_1, d_2, \ldots, d_k)$-face $f$ is a face of degree $k$ where all vertices on $f$ have degree $d_1, d_2, \ldots, d_k$ in any arbitrary order. A $(d_1, d_2, \ldots, d_k)$-vertex $v$ is a vertex of degree $k$ where all faces incident to $v$ have degree $d_1, d_2, \ldots, d_k$ in any arbitrary order. A graph $C(m, n)$ is a plane graph obtained from an $(m+n-2)$-cycle with one chord such that internal faces have length $m$ and $n$. A graph $C(l, m, n)$ is a plane graph obtained from an $(l+m+n-4)$-cycle with two chords such that internal faces have length $l, m,$ and $n$ where a middle face has length $m$. Note that each $C(l, m, n)$ is not necessary unique. For example, there are two non isomorphic graphs that are $C(3, 4, 3)$.

Let $G$ be a graph without $i$-cycles adjacent simultaneously to $j$-cycles and $k$-cycles where $\{i, j, k\} = \{3, 4, 5\}$ The following property is straightforward.

Proposition 4. $G$ does not contain $C(3, 4), C(3, 3, 3), \text{ and } C(3, 3, 5)$.

Proposition \ref{prop:4} yields the following immediate consequence.

Proposition 5. If $v$ be an $n$-vertex in $G$, then $v$ in $G$ is incident to at most $n - 2$ 3-faces.
3 Structure of Minimal Non DP-4-colorable Graphs

Definition 6. Let $H$ be a cover of $G$ with list assignment $L$. Let $G' = G - F$ where $F$ is an induced subgraph of $G$. A list assignment $L'$ is a restriction of $L$ on $G'$ if $L'(u) = L(u)$ for each vertex in $G'$. A graph $H'$ is a restriction of $H$ on $G'$ if $H' = H[\{v \times L(v) : v \in V(G')\}]$. Assume $G'$ has an $(H', L')$-coloring with an independent set $I'$ in $H'$ such that $|I'| = |V(G)| - |V(F)|$.

A residual list assignment $L^*$ of $F$ is defined by

$$L^*(x) = L(x) - \bigcup_{ux \in E(G)} \{c' \in L(x) : (u, c)(x, c') \in M_{L,ux} \text{ and } (u, c) \in I'\}$$

for each $x \in V(F)$.

A residual cover $H^*$ is defined by $H^* = H[\{x \times L^*(x) : x \in V(F)\}]$.

From above definitions, we have the following fact.

Lemma 7. A residual cover $H^*$ is a cover of $F$ with an assignment $L^*$. Furthermore, if $F$ is $(H^*, L^*)$-colorable, then $G$ is $(H, L)$-colorable.

Proof. One can check from the definition of a cover and residual cover that $H^*$ is a cover of $F$ with an assignment $L^*$.

Suppose that $F$ is $H^*(L^*)$-colorable. Then $H^*$ has an independent set $I^*$ with $|I^*| = |F|$. It follows from Definition 6 that no edges connect $H^*$ and $I'$. Additionally, $I'$ and $I^*$ are disjoint. Altogether, we have that $I = I' \cup I^*$ is an independent set in $H$ with $|I| = (|V(G)| - |V(F)|) + |V(F)| = |V(G)|$. Thus $G$ is $(H, L)$-colorable.

From now on, let $G$ be a minimal non DP-4-colorable graph.

Lemma 8. Every vertex in $G$ has degree at least 4.

Proof. Suppose to the contrary that $G$ has a vertex $x$ degree at most 3. Let $L$ be a 4-assignment and let $H$ be a cover of $G$ such that $G$ has no $(H, L)$-coloring. By the minimality of $G$, the subgraph $G' = G - x$ admits where $L'$ ($H'$) is a restriction of $L$ ($H$) in $G'$. Thus there is an independent set $I'$ with $|I'| = |G'|$ in $H'$. Consider a residual list assignment $L^*$ on $x$. Since $|L(x)| = 4$ and $d(x) \leq 3$, we obtain $|L^*(x)| \geq 1$. Clearly, $\{(x, c)\}$ is an independent set in $G[x]$ where $c \in L^*(x)$. Thus $G[x]$ is $(H^*, L^*)$-colorable. It follows from Lemma 7 that the graph $G$ is $(H, L)$-colorable, a contradiction.

Lemma 9. If $F$ is an induced subgraph of $G$ obtained from a cycle $x_1x_2 \ldots x_m x_1$ and $k$ chords $x_1x_i, x_1x_i, \ldots, x_1x_k$, then $d(x_1) \geq k + 4$ or $d(x_i) \geq 5$ for some $i \in \{2, 3, \ldots, m\}$
Proof. Suppose to the contrary that \( d(x_i) \leq k + 3 \) and \( d(x_i) \leq 4 \) for \( i = 2, 3, \ldots, m \).
Let \( L \) be a 4-assignment and let \( H \) be a cover of \( G \) such that \( G \) has no \((H, L)\)-coloring.
By the minimality of \( G \), the subgraph \( G' = G - F \) admits an \((H', L')\)-coloring where \( L' \) \((H')\) is a restriction of \( L \) \((H)\) in \( G' \). Thus there is an independent set \( I' \) with \(|I'| = |G'|\) in \( H' \). Consider a residual list assignment \( L^* \) on \( x \). Since \(|L(v)| = 4\) for every \( v \in V(G) \), we have \(|L^*(x_1)|, |L^*(x_i)|, |L^*(x_k)|, \ldots, |L^*(x_k)| \geq 3\) and \(|L^*(x_i)| \geq 2\) for remaining vertices in \( H \). Let \( H^* \) be an residual cover of \( F \). We can choose a color \( c \) from \( L^*(x_1) \) such that \(|L^*(x_m) - \{\ f' : (x_1, c)(x_m, c') \in H^* \}| \geq 2\). By choosing colors of \( x_2, x_3, \ldots, x_m \) in this order, we obtain an independent set \( I^* \) with \(|I^*| = m = |F|\). Thus \( F \) is \((H^*, L^*)\)-colorable. It follows from Lemma 7 that the graph \( G \) is \((H, L)\)-colorable, a contradiction. \( \square \)

By Lemma 9 we obtain the lower bound of the number of incident faces of a 6-vertex that are incident to at least two \( 5^+ \)-vertices.

Corollary 10. Every 6-vertex \( v \) in \( G \) has at least two incident faces of \( v \) that are incident to at least two \( 5^+ \)-vertices.

4 Main Result

Theorem 2. Every planar graph without \( i \)-cycles is adjacent simultaneously to \( j \)-cycles and \( k \)-cycles is DP-4-colorable when \( \{i, j, k\} = \{3, 4, 5\} \).

Proof. Suppose that \( G \) is a minimal counterexample. Then each vertex in \( G \) is a \( 4^+ \)-vertex by Lemma 8. The discharging process is as follows. Let the initial charge of a vertex \( u \) in \( G \) be \( \mu(u) = 2d(u) - 6 \) and the initial charge of a face \( f \) in \( G \) be \( \mu(f) = d(f) - 6 \). Then by Euler’s formula \(|V(G)| - |E(G)| + |F(G)| = 2\) and by the Handshaking lemma, we have

\[
\sum_{u \in V(G)} \mu(u) + \sum_{f \in F(G)} \mu(f) = -12.
\]

Now, we establish a new charge \( \mu^*(x) \) for all \( x \in V(G) \cup F(G) \) by transferring charge from one element to another and the summation of new charge \( \mu^*(x) \) remains \(-12\). If the final charge \( \mu^*(x) \geq 0 \) for all \( x \in V(G) \cup F(G) \), then we get a contradiction and the proof is completed.

Let \( w(v \to f) \) be the charge transferred from a vertex \( v \) to an incident face \( f \). A 4-vertex \( v \) is flaw if \( v \) is a \((3, 3, 5, 5^+)\)-vertex. The discharging rules are as follows.

(R1) Let \( f \) be a 3-face.
\(w(v \rightarrow f) = \begin{cases} 
0.6, & \text{if } v \text{ is flaw and } f \text{ is a } (4, 5^+, 5^+)-\text{face,} \\
0.8, & \text{if } v \text{ is flaw and } f \text{ is a } (4, 4^+)-\text{face,} \\
1, & \text{otherwise.} 
\end{cases}\)

\((R1.2)\) For a 5\(^+\)-vertex \(v\),
\[w(v \rightarrow f) = \begin{cases} 
1.4, & \text{if } f \text{ is a } (4, 4, 5^+)-\text{face with two incident flaw vertices,} \\
1.2, & \text{if } f \text{ is a } (4, 4^+, 5^+)-\text{face with exactly one flaw vertex,} \\
1, & \text{otherwise.} 
\end{cases}\]

\((R2)\) Let \(f\) be a 4-face.
For a 4\(^+\)-vertex \(v\), \(w(v \rightarrow f) = 0.5\).

\((R3)\) Let \(f\) be a 5-face.
\((R3.1)\) For a 4-vertex \(v\),
\[w(v \rightarrow f) = \begin{cases} 
0, & \text{if } v \text{ is flaw vertex with four 4-neighbors,} \\
0.1, & \text{if } v \text{ is flaw vertex with exactly one } 5^+\text{-neighbor,} \\
0.2, & \text{if } v \text{ is flaw vertex with at least two } 5^+\text{-neighbors,} \\
0.2, & \text{if } v \text{ is a } (3, 5, 5, 4)-\text{vertex,} \\
1/3, & \text{otherwise.} 
\end{cases}\]

\((R3.2)\) For a 5-vertex \(v\),
\[w(v \rightarrow f) = \begin{cases} 
0.7, & \text{if } f \text{ is a } (4, 4, 4, 4, 5)-\text{face with five adjacent } 4^-\text{-faces,} \\
0.6, & \text{if } f \text{ is a } (4, 4, 4, 4, 5)-\text{face and at least one adjacent } 5^+\text{-face,} \\
0.4, & \text{if } f \text{ is a } (4, 4, 4, 5, 5^+)-\text{face with both incident } 5^+\text{-vertices are adjacent} \\
0.3, & \text{otherwise.} 
\end{cases}\]

\((R3.3)\) For a 6\(^+\)-vertex \(v\),
\[w(v \rightarrow f) = \begin{cases} 
0.8, & \text{if } f \text{ is a } (4, 4, 4, 4, 6^+)-\text{face,} \\
0.4, & \text{if } f \text{ is incident to a } 5^+\text{-vertex other than } v. 
\end{cases}\]

It remains to show that resulting \(\mu^\ast(x) \geq 0\) for all \(x \in V(G) \cup F(G)\). Moreover, it is evident for each 6\(^+\)-face \(f\) that \(\mu^\ast(f) \geq 0\).

**CASE 1:** A 4-vertex \(v\).
We use \((R1.1)\), \((R2)\), and \((R3.1)\) to prove this case.

**SUBCASE 1.1:** \(v\) is a flaw vertex, that is a \((3, 3, 5, 5^+)-\text{vertex.}\)
If each adjacent vertex of \(v\) is a 4-vertex, then \(\mu^\ast(v) \geq \mu(v) - 2 \times 1 = 0\). If \(v\) is
adjacent to exactly one $5^+$-vertex, then we obtain $\mu^*(v) \geq \mu(v) - 1 - 0.82 \times 0.1 = 0$. Now, assume $v$ is adjacent to at least two $5^+$-vertices. If $v$ is incident to a $(4,5^+,5^+)$-face, then $\mu^*(v) \geq \mu(v) - 1 - 0.6 - 2 \times 0.2 = 0$, otherwise, $\mu^*(v) \geq \mu(v) - (2 \times 0.8 + 2 \times 0.2) = 0$.

**SUBCASE 1.2:** $v$ is not a flaw vertex.

If $v$ is not incident to any 3-face, then $\mu^*(v) \geq \mu(v) - (4 \times 0.5 = 0$. It follows from Proposition 4 that any 4-face, then $\mu^*(v) \geq \mu(v) \geq \mu(v) - 1 - 0.6 - 2 \times 0.2 = 0$. This implies that $v$ is incident to at least two 3-faces by Corollary 10. Thus $\mu^*(v) \geq \mu(v) - 1 - 3 \times 1/3 = 0$. If $v$ is incident to two 3-faces, then the remaining two incident faces are $6^+$ by Proposition 4. Thus $\mu^*(v) \geq \mu(v) - 2 \times 1 = 0$.

**CASE 2:** Consider a 5-vertex $v$.

We use (R1.2), (R2.2), and (R3.2) to prove this case.

If $v$ is incident to some 3-face in $C(3,3)$, then two incident faces of $v$ are $6^+$-faces by Proposition 4. Thus $\mu^*(v) \geq \mu(v) - 1 - 4 \times 0.6 > 0$. Now, assume $v$ is not incident to $C(3,3)$. This implies that $v$ is adjacent to at most two 3-faces.

**SUBCASE 2.1:** $v$ is incident to at most one 3-face.

If $v$ is not incident to any 4-face, then each incident 5-face of $v$ is adjacent to some 5-face. Thus $\mu^*(v) \geq \mu(v) - 1 - 4 \times 0.6 > 0$. If $v$ is incident to at least one 4-face, then $\mu^*(v) \geq \mu(v) - 1 - 3 \times 0.7 - 0.5 = 0$.

**SUBCASE 2.2:** $v$ is incident to two 3-faces.

Let $v$ be incident to faces $f_1, f_2, \ldots, f_5$ in a cyclic order where $f_1$ and $f_3$ are 3-faces. It follows from Proposition 4 that $f_2, f_4,$ and $f_5$ are $5^+$-faces. Assume $f_1$ and $f_3$ are $(4,4,5)$-faces. It follows from Corollary 10 that $f_2$ is incident to at least two non-adjacent $5^+$-vertices. If either $f_4$ or $f_5$ is incident to one $5^+$-vertex, then the other is incident to at least one non-adjacent two $5^+$-vertices by Corollary 10. Thus $\mu^*(v) \geq \mu(v) - 2 \times 1.4 - 0.6 - 2 \times 0.3 > 0$. If both $f_4$ and $f_5$ are incident to at least two $5^+$-vertices, then $\mu^*(v) \geq \mu(v) - 2 \times 1.4 - 2 \times 0.4 - 0.3 > 0$. If $f_1$ (or $f_3$) is a $(5,5^+,5^+)$-face, then $f_2$ and $f_5$ (or $f_4$) are incident to at least two $5^+$-vertices. Thus $\mu^*(v) \geq \mu(v) - 1.4 - 1 - 0.6 - 2 \times 0.4 > 0$.

Assume that both $f_1$ and $f_3$ are $(4,5,5^+)$-faces where $5^+$-vertex other than $v$ are $x$ and $y$. If $x$ and $y$ are not in $f_2$, then both $f_4$ and $f_5$ are adjacent to at least two $5^+$-vertices. Thus $\mu^*(v) \geq \mu(v) - 2 \times 1.2 - 0.7 - 2 \times 0.4 > 0$. If $x$ and $y$ are in $f_2$, then $\mu^*(v) \geq \mu(v) - 2 \times 1.2 - 2 \times 0.6 - 0.3 > 0$. If exactly one of $x$ and $y$ is in $f_2$, then $\mu^*(v) \geq \mu(v) - 2 \times 1.2 - 0.6 - 2 \times 0.4 > 0$.

Assume that exactly one of $f_1$ and $f_3$ is a $(4,5,5^+)$-face where $5^+$-vertex other than $v$ is
$w$ and $w$ is in $f_2$. It follows from Corollary 10 that at least one of $f_4$ and $f_5$ is incident to at least two $5^+$-vertices. If $f_4$ or $f_5$ is incident to at least two $5^+$-vertices and two of them are not adjacent, then $\mu^*(v) \geq \mu(v) - 1.4 - 1.2 - 0.6 - 0.4 - 0.3 > 0$, otherwise $f_4$ and $f_5$ are incident to at least two $5^+$-which implies $\mu^*(v) \geq \mu(v) - 1.4 - 1.2 - 3 \times 0.4 > 0$.

Assume that exactly one of $f_1$ and $f_5$ is a $(4, 5, 5^+)$-face where $5^+$-vertex other than $v$ is $w$ and $w$ is in $f_5$. It follows that one of $f_2$ and $f_4$ is incident to at least two $5^+$-vertices. If $f_2$ is incident at least two $5^+$-vertices, then two of them are not adjacent. Thus $\mu^*(v) \geq \mu(v) - 1.4 - 1.2 - 0.6 - 0.4 - 0.3 > 0$. If $f_5$ is incident to at least two $5^+$-vertices, then two of them are not adjacent or two of them are not not adjacent. Thus $\mu^*(v) \geq \mu(v) - 1.4 - 1.2 - 0.7 - 0.4 - 0.3 = 0$.

**CASE 3:** A 6-vertex $v$.

We use (R1.2), (R2.2), and (R3.3) to prove this case.

**SUBCASE 3.1:** $v$ is incident to $C(3, 3)$.

It follows from Proposition 1(b) that $v$ is incident to at least two $6^+$-faces. Then $\mu^*(v) \geq \mu(v) - 4 \times 1.4 > 0$.

**SUBCASE 3.2:** $v$ is not incident to $C(3, 3)$.

It follows that $v$ is incident at most three 3-faces. If $v$ is incident to at most two 3-faces, then $\mu^*(v) \geq \mu(v) - 2 \times 1.4 - 4 \times 0.8 = 0$. Assume $v$ is incident to three 3-faces. By Proposition 1(a), $v$ is not incident to any 4-face. If $v$ is incident to at least one 6$^+$-face, then $\mu^*(v) \geq \mu(v) - 3 \times 1.4 - 2 \times 0.8 > 0$. Now, it remains to consider the case that $v$ is a $(3, 3, 3, 5, 5, 5)$-vertex. By Corollary 10, two of incident faces of $v$, say $f_1$ and $f_2$ (not necessary adjacent), are incident to at least two $5^+$-vertices. If $f_1$ and $f_2$ are $5^+$-faces, then $\mu^*(v) \geq \mu(v) - 3 \times 1.4 - 0.8 - 2 \times 0.4 > 0$. If $f_1$ or $f_2$ is a 3-face, then $\mu^*(v) \geq \mu(v) - 2 \times 1.4 - 1.2 - 2 \times 0.8 - 0.4 = 0$.

**CASE 4:** A $k$-vertex $v$ with $k \geq 7$.

We use (R1.2), (R2.2), and (R3.3) to prove this case.

**SUBCASE 4.1:** $v$ is incident to $C(3, 3)$.

It follows from Proposition 1 that $v$ is incident to at least two $6^+$-faces. Thus $\mu^*(v) \geq \mu(v) - (k - 2) \times 1.4 > 0$ for $k \geq 6$.

**SUBCASE 4.2:** $v$ is not incident to $C(3, 3)$.

It follows that $v$ is incident to at most $\frac{d(v)}{2}$ 3-faces. Thus $\mu^*(v) \geq \mu(v) - \frac{k}{2} \times 1.4 - \frac{k}{2} \times 0.8$ for $k \geq 7$.

**CASE 5:** A 3-face.

We use (R1.1) and (R1.2) to prove this case.
If each vertex of $f$ is not a flaw vertex, then $\mu^*(f) = \mu(f) + 3 \times 1 = 0$.

Now, assume that at least one incident 4-vertex of $f$ is flaw. If $f$ is a $(4,4,5^+)$-face with two incident flaw vertices, then $\mu^*(f) = \mu(f) + 1.4 + 2 \times 0.8 = 0$. If $f$ is a $(4,4,5^+)$-face with exactly one incident flaw vertex, then $\mu^*(f) = \mu(f) + 1 + 0.8 = 0$. If $f$ is a $(4,5^+,5^+)$-face, then $\mu^*(f) = \mu(f) + 2 \times 1.2 + 0.6 = 0$.

**CASE 6:** A 4-face $f$.

We obtain $\mu^*(f) \geq \mu(f) + 4 \times 0.5 = 0$ by (R2).

**CASE 7:** A 5-face $f$.

We use (R3.1), (R3.2), and (R3.3) to prove this case.

**SUBCASE 7.1:** $f$ is incident to at least three $5^+$-vertices.

It follows that each 4-vertex in $f$ is adjacent to at least one $5^+$-vertex. Then $\mu^*(f) \geq \mu(f) + 3 \times 0.3 + 2 \times 0.1 > 0$.

**SUBCASE 7.2:** $f$ is incident to two $5^+$-vertices, say $x$ and $y$.

If $x$ and $y$ is not adjacent, then each 4-vertex in $f$ is adjacent to at least one $5^+$-vertex. Additionally, one of them is adjacent to at least two $5^+$-vertices. Thus $\mu^*(f) \geq \mu(f) + 2 \times 0.3 + 0.2 + 2 \times 0.1 = 0$. If $x$ and $y$ are adjacent, then two incident 4-vertices of $f$ are adjacent to at least one $5^+$-vertex. Thus $\mu^*(f) \geq \mu(f) + 2 \times 0.4 + 2 \times 0.1 = 0$.

**SUBCASE 7.3:** $f$ is incident to one $5^+$-vertex.

If $f$ is a $(4,4,4,4,6^+)$-face, then two 4-vertices in $f$ are adjacent to at least one $6^+$-vertex. Thus $\mu^*(f) \geq \mu(f) + 0.8 + 2 \times 0.1 = 0$. Assume $f$ is a $(4,4,4,4,5)$-face with five incident vertices $x_1, x_2, \ldots, x_5$ in a cyclic order with $d(x_5) = 5$. If each adjacent face of $f$ is a 4$^-$-face and both $x_1$ and $x_4$ are flaw, then there is a a $(3,5,5^+)$-face adjacent to $f$ and incident to $x_5$ by Corollary 10. Consequently, $x_1$ or $x_4$ is adjacent to at least two $5^+$-vertices. Thus $\mu^*(f) \geq \mu(f) + 0.7 + 0.2 + 0.1 = 0$. If each adjacent face of $f$ is a 4$^-$-face but $x_1$ or $x_4$ is not flaw, then $\mu^*(f) \geq \mu(f) + 0.7 + 0.2 + 0.1 = 0$. If one adjacent face of $f$ is a 5$^+$-face, then one of 4-vertex in $f$ is not either flaw or $(3,5,5,4)$-vertex. Thus $\mu^*(f) \geq \mu(f) + 0.6 + \frac{1}{5} + 0.1 > 0$.

**SUBCASE 7.4:** $f$ is a $(4,4,4,4,4)$-face.

It follows from Lemma 9 that each adjacent 3-face of $f$ is a $(4,4,5^+)$-face. This implies that each flaw vertex in $f$ is adjacent to at least two $5^+$-vertex. Consequently, each vertex in $f$ send charge at least 0.2 to $f$. Thus $\mu^*(f) \geq \mu(f) + 5 \times 0.2 = 0$.

This completes the proof. \qed
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