Restricted extension of sparse partial edge colorings of hypercubes

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Abstract

We consider the following type of question: Given a partial proper $d$-edge coloring of the $d$-dimensional hypercube $Q_d$, and lists of allowed colors for the non-colored edges of $Q_d$, can we extend the partial coloring to a proper $d$-edge coloring using only colors from the lists? We prove that this question has a positive answer in the case when both the partial coloring and the color lists satisfy certain sparsity conditions.

1 Introduction

The chromatic index $\chi'(G)$ of a (simple) graph $G$ is far simpler in terms of its possible values than the chromatic number; Vizing’s theorem [Viz64] tells us that in order to properly color the edges of $G$ we need either $\Delta(G)$ or $\Delta(G) + 1$ colors, where $\Delta(G)$ denotes the maximum degree of $G$, and by König’s edge coloring theorem, $\chi'(G) = \Delta(G)$ if $G$ is bipartite [Kon16].

This simplicity quickly disappears in many of the natural variations on the basic edge coloring problem, e.g. the precoloring extension problem, where some of the edges of a graph have been (properly) colored and we want to determine if this partial coloring can be extended to a proper edge coloring of the full graph using no extra colors; indeed this problem is NP-complete already for 3-regular bipartite graphs [Fia03].

One of the earlier references explicitly discussing the problem of extending a partial edge coloring is [MS90]; there a simple necessary condition for the existence of an extension is given and the authors find a class of graphs where this condition is also sufficient. More recently the question of extending a precoloring where the precolored edges form a matching has gathered interest; in [EGv+14] a number of positive results and conjectures are given. In particular, it is conjectured that for every graph $G$, if $\varphi$ is an edge precoloring of a matching $M$ in $G$ using $\Delta(G) + 1$ colors, and any two edges in $M$ are at distance at least 2 from each other, then $\varphi$ can be extended to a proper $(\Delta(G) + 1)$-edge coloring of $G$; this was first conjectured in [AM01], but then with distance 3 instead. By the distance between two edges $e$ and $e'$ here we mean the number of edges in a shortest path between an endpoint of $e$ and an endpoint of $e'$; a distance-$t$ matching is a matching where any two edges are at distance at least $t$ from each other. The $t$-neighborhood of an edge $e$ is the graph induced by all edges of distance at most $t$ from $e$.
Note that the conjecture in [EGv+14] on distance-2 matchings is sharp both with respect to the distance between precolored edges, and in the sense that $\Delta(G) + 1$ can in general not be replaced by $\Delta(G)$, even if any two precolored edges are at arbitrarily large distance from each other [EGv+14]. In [EGv+14], it is proved that this conjecture hold for e.g. bipartite multigraphs and subcubic multigraphs, and in [GK16] it is proved that a version of the conjecture with the distance increased to 9 holds for general graphs.

However, for one specific family of graphs, the balanced complete bipartite graphs $K_{n,n}$, the edge precoloring extension problem was studied far earlier than in the above-mentioned references. Here the extension problem corresponds to asking whether a partial latin square can be completed to a latin square. In this form the problem appeared already in 1960, when Evans [Eva60] stated his now classic conjecture that for every positive integer $n$, if $n - 1$ edges in $K_{n,n}$ have been (properly) colored, then this partial coloring can be extended to a proper $n$-edge-coloring of $K_{n,n}$. This conjecture was solved for large $n$ by Häggkvist [Hág78] and later for all $n$ by Smetaniuk [Sme81], and independently by Andersen and Hilton [AH83].

Generalizing this problem, Daykin and Häggkvist [DHS84] proved several results on extending partial edge colorings of $K_{n,n}$, and they also conjectured that much denser partial colorings can be extended, as long as the colored edges are spread out in a specific sense: a partial $n$-edge coloring of $K_{n,n}$ is $\epsilon$-dense if there are at most $\epsilon n$ colored edges from $\{1, \ldots, n\}$ at any vertex and each color in $\{1, \ldots, n\}$ is used at most $\epsilon n$ times in the partial coloring. Daykin and Häggkvist [DHS84] conjectured that for every positive integer $n$, every $\frac{1}{4}$-dense partial proper $n$-edge coloring can be extended to a proper $n$-edge coloring of $K_{n,n}$, and proved a version of the conjecture for $\epsilon = o(1)$ (as $n \to \infty$) and $n$ divisible by 16. Bartlett [Bar13] proved that this conjecture holds for a fixed positive $\epsilon$, and recently a different proof which improves the value of $\epsilon$ was given in [BKL+16].

For general edge colorings of balanced complete bipartite graphs, Dinitz conjectured, and Galvin proved [Gal95], that if each edge of $K_{n,n}$ is given a list of $n$ colors, then there is a proper edge coloring of $K_{n,n}$ with support in the lists. Indeed, Galvin’s result was a complete solution of the well-known List Coloring Conjecture for the case of bipartite multigraphs (see e.g. [HC92] for more background on this conjecture and its relation to the Dinitz’ conjecture).

Motivated by the Dinitz’ problem, Häggkvist [Hág89] introduced the notion of $\beta n$-arrays, which correspond to list assignments $L$ of forbidden colors for $E(K_{n,n})$, such that each edge $e$ of $K_{n,n}$ is assigned a list $L(e)$ of at most $\beta n$ forbidden colors from $\{1, \ldots, n\}$, and at every vertex $v$ each color is forbidden on at most $\beta n$ edges adjacent to $v$; we call such a list assignment for $K_{n,n}$ $\beta$-sparse. If $L$ is a list assignment for $E(K_{n,n})$, then a proper $n$-edge coloring $\varphi$ of $K_{n,n}$ avoids the list assignment $L$ if $\varphi(e) \notin L(e)$ for every edge $e$ of $K_{n,n}$; if such a coloring exists, then $L$ is avoidable. Häggkvist conjectured that there exists a fixed $\beta > 0$, in fact also that $\beta = \frac{1}{3}$, such that for every positive integer $n$, every $\beta$-sparse list assignment for $K_{n,n}$ is avoidable. That such a $\beta > 0$ exists was proved for even $n$ by Andrén in her PhD thesis [And10], and later for all $n$ in [ACO13].

Combining the notions of extending a sparse precoloring and avoiding a sparse list assignment, Andrén et al. [ACM16] proved that there are constants $\alpha > 0$ and $\beta > 0$, such that for every positive integer $n$, every $\alpha$-dense partial edge coloring of $K_{n,n}$ can be extended to a proper $n$-edge-coloring avoiding any given $\beta$-sparse list assignment $L$, provided that no edge $e$ is precolored by a color that appears in $L(e)$. In contrast to this, it was proved in [EGv+14] that there are bipartite graphs $G$ with a precolored matching of size 2, which is not
extendable to a proper $\Delta(G)$-edge coloring. These examples have edge densities converging to some constant $0 < c \leq \frac{1}{2}$, and many of the proof methods used in the papers mentioned above rely on the high edge density of the complete bipartite graph. It is thus natural to ask if the good behaviour seen for $K_{n,n}$ will hold for well-structured graphs of lower densities.

The aim of this paper is to show that some generalizations of this type are possible. We will demonstrate that results similar to those from [ACM16] hold for the family of $d$-dimensional hypercubes $Q_d$; these graphs have a degree which is logarithmic in the number of vertices, rather than linear, so we are now looking at a family of graphs with vanishing asymptotic density. Instead of bounding the global density of precolored edges, as in the case of complete bipartite graphs, for hypercubes we shall bound the number of precolored edges appearing in neighborhoods of given size. Our results are stated in terms of 27-neighborhoods; this size of neighborhoods is solely due to proof technical reasons.

To state our main theorem, we need some terminology: a dimensional matching $M$ of $Q_d$ is a perfect matching of $Q_d$ such that $Q_d - M$ is isomorphic to two copies of $Q_{d-1}$; evidently there are precisely $d$ dimensional matchings in $Q_d$. An edge precoloring of $Q_d$ with colors $1, \ldots, d$ is called $\alpha$-dense if

(i) there are at most $\alpha d$ precolored edges at each vertex;

(ii) for every 27-neighborhood $W$ of an edge $e$ of $Q_d$, there are at most $\alpha d$ precolored edges with color $i$ in $W$, $i = 1, \ldots, d$;

(iii) for every 27-neighborhood $W$, and every dimensional matching $M$, at most $\alpha d$ edges of $M$ are precolored in $W$.

Here, and in the following, all $t$-neighborhoods are taken with respect to edges. A list assignment $L$ for $E(Q_d)$ is $\beta$-sparse if the list of each edge is a (possibly empty) subset of $\{1, \ldots, d\}$, and

(i) $|L(e)| \leq \beta d$ for each edge $e \in E(Q_d)$;

(ii) for every vertex $v \in V(Q_d)$, each color in $\{1, \ldots, d\}$ occurs in at most $\beta d$ lists of edges incident to $v$;

(iii) for every 27-neighborhood $W$, and every dimensional matching $M$, any color appears at most $\beta d$ times in lists of edges of $M$ contained in $W$.

Our main result is the following.

**Theorem 1.1.** There are constants $\alpha > 0$ and $\beta > 0$ such that for every positive integer $d$, if $\varphi$ is an $\alpha$-dense $d$-edge precoloring of $Q_d$, $L$ a $\beta$-sparse list assignment for $Q_d$, and $\varphi(e) \notin L(e)$ for every edge $e \in E(Q_d)$, then there is a proper $d$-edge coloring of $Q_d$ which agrees with $\varphi$ on any precolored edge and which avoids $L$.

As a corollary of our main theorem we note that a version of the conjecture on precolored distance-2 matchings from [EGv+13], with $\Delta(G)$ in place of $\Delta(G) + 1$, but with a weaker distance requirement, holds for the family of hypercubes.
Corollary 1.2. There is a constant $\beta > 0$ such that if $L$ is a $\beta$-sparse list assignment $L$ for $Q_d$ and $\varphi$ is a $d$-edge precoloring of a distance-$t$ matching in $Q_d$, where $t > 55$, then $\varphi$ can be extended to a proper $d$-edge-coloring of $Q_d$ which avoids $L$.

This follows from the fact that a precolored distance-$t$ matching is an $\alpha$-dense precoloring if $t > 55$. A precolored matching has a much more restricted structure than a general $\alpha$-dense $d$-edge precoloring, so this corollary is most likely far from optimal in terms of the lower bound on $t$; it would be interesting to see how far this can be improved.

If we place both the precolored edges and those with a list of forbidden colors on them on a matching, then our proof method in fact trivially yields the following.

Theorem 1.3. Let $\varphi$ be a $d$-edge precoloring and $L$ a list assignment for the edges of $Q_d$. If every edge $e$ which is either precolored or satisfies $L(e) \neq \emptyset$ belongs to a distance-3 matching $M$ in $Q_d$, then there is a $d$-edge coloring which agrees with $\varphi$ on any precolored edge, and which avoids $L$.

This proves that a slightly stronger version, with $d$ rather than $d + 1$ colors, of the earlier mentioned conjecture from [AM01] holds for the family of hypercubes.

The rest of the paper is organized as follows. In Section 2 we introduce some terminology and notation and also outline the proof of Theorem 1.1. Section 3 contains the proof of a slightly reformulated version of Theorem 1.1, we also indicate how Theorem 1.3 can be deduced from the proof of Theorem 1.1. In Section 4 we give some concluding remarks; in particular, we give an example indicating what numerical values of $\alpha$ and $\beta$ in Theorem 1.1 might be best possible. At the beginning of Section 3 we shall present numerical values of $\alpha$ and $\beta$ for which our main theorem holds, provided that $n$ is large enough. Finally, throughout the paper, the base of the natural logarithm is denoted by $e$.

## 2 Terminology, notation and proof outline

Given an edge precoloring $\varphi$ (or just precoloring, or partial edge coloring) of a graph $G$ with $\Delta(G)$ colors, an extension of $\varphi$ is a proper $\Delta(G)$-edge coloring of $G$ which agrees with $\varphi$ on every precolored edge; if such a coloring of $G$ exists, then $\varphi$ is extendable.

For a vertex $u \in Q_d$, we denote by $E_u$ the set of edges with one endpoint being $u$, and for a (partial) edge coloring $f$ of $Q_d$, let $f(u)$ denote the set of colors on edges in $E_u$ under $f$. If two edges $xy$ and $zt$ of $Q_d$ are in a cycle of length 4 in $Q_d$, and all the vertices $x, y, z, t$ are distinct, then the edges $xy$ and $zt$ are parallel.

As noted above, $Q_d$ decomposes into precisely $d$ dimensional matchings. Note that a dimensional matching in $Q_d$ contains precisely $2d^{-1}$ edges. For a $d$-edge coloring $f$ of $Q_d$, a dimensional matching $M$ is called a dimensional matching of color $c$ (under $f$) if there are more edges of $M$ colored $c$ than by any other color.

For the proof of Theorem 1.1, we shall use the standard $d$-edge coloring $h$ of $Q_d$ where all edges of the $i$th dimensional matching in $Q_d$ is colored $i$, $i = 1, \ldots, d$. A cycle is 2-colored if its edges are colored by two distinct colors from $\{1, \ldots, d\}$. The following property is crucial for our proof of Theorem 1.1.

Property 2.1. In the standard $d$-edge coloring $h$, every edge of $Q_d$ is in exactly $d-1$ 2-colored 4-cycles.
Let $\varphi$ be an $\alpha$-dense precoloring of $Q_d$. Edges of $Q_d$ which are colored under $\varphi$, are called prescribed (with respect to $\varphi$). For the edge coloring $h$ (or an edge coloring obtained from $h$), an edge $e$ of $Q_d$ is called requested (under $h$ with respect to $\varphi$) if $h(e) = c$ and $e$ is adjacent to an edge $e'$ such that $\varphi(e') = c$.

Consider a $\beta$-sparse list assignment $L$ for $Q_d$. For the edge coloring $h$ (or an edge coloring obtained from $h$), an edge $e$ of $Q_d$ is called a conflict edge (of $h$ with respect to $L$) if $h(e) \in L(e)$. An allowed cycle (under $h$ with respect to $L$) of $Q_d$ is a 4-cycle $C = uvztu$ in $Q_d$ that is 2-colored under $h$, and such that interchanging colors on $C$ yields a proper $d$-edge coloring $h_1$ of $Q_d$ where none of $uv, vz, zt, tu$ is a conflict edge. We call such an interchange a swap in $h$.

Instead of proving Theorem 1.1 we shall in fact prove the following theorem, which is easily seen so imply Theorem 1.1

**Theorem 2.2.** There are constants $\alpha > 0$, $\beta > 0$ and $d_0$, such that for every positive integer $d \geq d_0$, if $\varphi$ is an $\alpha$-dense precoloring of $Q_d$ and $L$ a $\beta$-sparse list assignment for $Q_d$, and $\varphi(e) \notin L(e)$ for every edge $e \in E(Q_d)$, then there is an extension of $\varphi$ which avoids $L$.

Below we outline the proof of Theorem 2.2. Let $h$ be the standard proper $d$-edge coloring of $Q_d$ defined above, $\varphi$ an $\alpha$-dense precoloring of $Q_d$, and $L$ a $\beta$-sparse list assignment for $E(Q_d)$.

**Step I.** Given the standard $d$-edge coloring $h$ of $Q_d$, find a permutation $\rho$ of the elements of the set $\{1, \ldots, d\}$ such that in the proper $d$-edge-coloring $h'$ obtained by applying $\rho$ to the colors used in $h$, locally, each dimensional matching in $Q_d$ contains “sufficiently few” conflict edges with $L$, as well as “sufficiently few” requested edges with respect to $\varphi$. Moreover, we require that each vertex $u$ of $Q_d$ satisfies that $E_u$ contains “sufficiently few” conflict and requested edges, and that each edge of $Q_d$ belongs to “many” allowed cycles under $h'$. These conditions shall be more precisely articulated below.

**Step II.** From the precoloring $\varphi$ of $Q_d$, define a new edge precoloring $\varphi'$ such that an edge $e$ of $Q_d$ is colored under $\varphi'$ if and only if $e$ is colored under $\varphi$ or $e$ is a conflict edge of $h'$ with respect to $L$. We shall also require that locally, each of the colors in $\{1, \ldots, d\}$ is used a bounded number of times under $\varphi'$.

**Step III.** From $h'$, construct a proper $d$-edge coloring $h''$ of $Q_d$ such that under $h''$, no edge in $Q_d$ is both requested and prescribed (with respect to $\varphi'$); this is done by swapping on a set of disjoint allowed 4-cycles. We also require that each requested edge $e$ of $h''$ is adjacent to at most one edge $e'$ such that $h''(e) = \varphi'(e')$.

**Step IV.** For each edge $e$ of $Q_d$ that is prescribed with respect to $\varphi'$, construct a subset $T_e \subseteq E(Q_d)$, such that performing a series of swaps in $h''$ on allowed cycles, all edges of which are in $T_e$, yields a coloring $h'_1$ where $h''_1(e) = \varphi'(e)$. Moreover, if $e$ and $e'$ are prescribed edges of $Q_d$, then the sets $T_{e'}$ and $T_e$ will be disjoint. Thus, performing the series of swaps on all the sets $T_e$ associated with prescribed edges $e$ yields a proper $d$-edge coloring $\hat{h}$ which is an extension of $\varphi'$ (and thus $\varphi$), and which avoids $L$. 

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3 Proofs

In this section we prove Theorem 2.2. In the proof we shall verify that it is possible to perform Steps I-IV described above to obtain a proper $d$-edge-coloring of $Q_d$ that is an extension of $\varphi$ and which avoids $L$. This is done by proving a lemma in each step.

We will not specify the value of $d_0$ in the proof but rather assume that $d$ is large enough whenever necessary. Since the proof will contain a finite number of inequalities that are valid if $d$ is large enough, this suffices for proving Theorem 2.2.

The proof of Theorem 2.2 involves a number of functions and parameters: $\alpha, \beta, \gamma, \kappa, \epsilon, \epsilon_0, \tau$, and a number of inequalities that they must satisfy. For the reader's convenience, explicit choices for which the proof holds are presented here:

$$\alpha = 10^{-622}, \beta = 2 \cdot 10^{-622}, \gamma = 2^{-11}, \kappa = 9/2^{11}$$

$$\epsilon = 2^{-3}, \epsilon_0 = 2^{-8}, \tau = 2^{-7}.$$

We remark that since the numerical values of $\alpha$ and $\beta$ are not anywhere near what we expect to be optimal, we have not put an effort into choosing optimal values for these parameters. Finally, for simplicity of notation, we shall omit floor and ceiling signs whenever these are not crucial.

Proof of Theorem 2.2. Let $\varphi$ be an $\alpha$-dense precoloring of $Q_d$, and let $L$ be a $\beta$-sparse list assignment for $Q_d$. Moreover, let $h$ be the standard $d$-edge coloring defined above.

Step I: We use the following lemma for constructing a required $d$-edge-coloring $h'$ from $h$.

Lemma 3.1. Let $\gamma, \tau < 1$ be constants such that $0 < \alpha, \beta \leq \gamma, 2^{1/2+1}e\alpha < \gamma, 2^7e\beta < \gamma/3$ and $2^{\gamma/2-2}\epsilon_0 < e_{\beta < 2\gamma - 2\beta}$. There is a permutation $\rho$ of $\{1, \ldots, d\}$, such that applying $\rho$ to the set of colors $\{1, \ldots, d\}$ used in $h$, we obtain a $d$-edge coloring $h'$ of $Q_d$ satisfying the following:

(a) For every 26-neighborhood $W$, and every dimensional matching $M$, at most $\gamma d$ edges of $M \cap E(W)$ are requested.

(b) For every 27-neighborhood $W$, and every dimensional matching $M$, at most $\gamma d$ edges of $M \cap E(W)$ are conflict.

(c) No vertex $u$ in $Q_d$ satisfies that $E_u$ contains more than $\gamma d$ requested edges.

(d) No vertex $u$ in $Q_d$ satisfies that $E_u$ contains more than $\gamma d$ conflict edges.

(e) Each edge in $Q_d$ belongs to at least $(1 - \tau)d$ allowed cycles.

Proof. Let $A, B, C, D$ and $E$ be the number of permutations which do not fulfill the conditions (a), (b), (c), (d) and (e), respectively. Let $X$ be the number of permutations satisfying the five conditions (a), (b), (c), (d) and (e). There are $d!$ ways to permute the colors, so we have

$$X \geq d! - A - B - C - D - E$$

We will now prove that $X$ is greater than 0.
• Recall that an edge $e$ is requested if $e$ is adjacent to an edge $e'$ such that $h(e) = \varphi(e')$. Let $M'$ be a dimensional matching, and consider a subset $M \subseteq M'$ of all edges in $M'$ that are contained in a given 26-neighborhood $W_1$. Then every edge of $M$ and every edge adjacent to an edge of $M$ is contained in a 27-neighborhood $W_2$ containing $W_1$. Since all edges in $M$ have the same color in any edge coloring obtained from $h$ by permuting colors, and there are at most $\alpha d$ precolored edges with color $i$ in $W_2$, $i = 1, \ldots, d$, the maximum number of requested edges in $M$ is $\alpha d$. In other words, no subset of a dimensional matching contained in a 26-neighborhood contains more than $\alpha d$ requested edges. Since $\gamma \geq \alpha$, this means that all permutations satisfy condition (a) or $A = 0$.

• Since all edges that are in the same dimensional matching have the same color under $h$ and for every 27-neighborhood $W$, and every dimensional matching $M$, any color appears at most $\beta d$ times in lists of edges of $M$ contained in $W$, we have that the maximum number of conflict edges in a subset of a given dimensional matching contained in a 27-neighborhood is $\beta d$. Since $\gamma \geq \beta$, this means that all permutations satisfy condition (b) or $B = 0$.

• To estimate $C$, let $u$ be a fixed vertex of $Q_d$, and let $S$ be a set of size $\gamma d$ of edges of $E_u$. There are $\binom{d}{\gamma d}$ ways to choose $S$. For a vertex $v$ adjacent to $u$, if $uv$ is a requested edge, then the colors used in $h$ should be permuted in such a way that in the resulting coloring $h'$, $uv$ is colored by some color in the set $\{(\varphi(u) \cup \varphi(v)) \setminus \varphi(uv)\}$. Since $|\varphi(u)| \leq \alpha d$ and $|\varphi(v)| \leq \alpha d$, there are at most $(2\alpha d)^\gamma$ ways to choose which colors from $1, 2, \ldots, d$ to assign to the edges in $S$ so that all edges in $S$ are requested. The rest of the colors can be arranged in any of the $(d - \gamma d)!$ possible ways. In total this gives at most

$$\binom{d}{\gamma d} (2\alpha d)^\gamma (d - \gamma d)! = \frac{d!(2\alpha d)^\gamma}{(\gamma d)!}$$

permutations that do not satisfy condition (c) on vertex $u$.

There are $2^d$ vertices in $Q_d$, so we have

$$C \leq 2^d \frac{d!(2\alpha d)^\gamma}{(\gamma d)!}$$

• To estimate $D$, let $u$ be a fixed vertex of $Q_d$, and let $S$ be a set of size $\gamma d$ ($|S| = \gamma d$) of edges from $E_u$. For a vertex $v$ adjacent to $u$, if $uv$ is a conflict edge, then the colors used in $h$ should be permuted in such a way that in the resulting coloring $h'$, the color of $uv$ is in $L(uv)$. Since $|L(uv)| \leq \beta d$, there are at most $(\beta d)^\gamma$ ways to choose which colors from $\{1, 2, \ldots, d\}$ to assign to the edges in $S$ so that all edges in $S$ are conflict. The rest of the colors can be arranged in any of the $(d - \gamma d)!$ possible ways. In total this gives at most

$$\binom{d}{\gamma d} (\beta d)^\gamma (d - \gamma d)! = \frac{d!(\beta d)^\gamma}{(\gamma d)!}$$

permutations that do not satisfy condition (d) on vertex $u$. There are $2^d$ vertices in $Q_d$, so we have

$$D \leq 2^d \frac{d!(\beta d)^\gamma}{(\gamma d)!}$$
To estimate $E$, let $uv$ be a fixed edge of $Q_d$. Each cycle $C = uvztu$ containing $uv$ is uniquely defined by an edge $zt$ which is parallel with $uv$. Moreover, a permutation $\zeta$ is in $E$ if and only if there are more than $\tau d$ choices for $zt$ so that $C$ is not allowed. We shall count the number of ways $\zeta$ could be constructed for this to happen. First, note that for each choice of color $c_1$ from $\{1, \ldots, d\}$, for the dimensional matching which contains $uv$, there are up to $2\beta d$ cycles that are not allowed because of this choice. This follows from the fact that there are at most $\beta d$ choices for $t$ (or $z$) such that $L(ut)$ (or $L(vz)$) contains $c_1$. So for a permutation $\zeta$ to belong to $E$, $\zeta$ must satisfy that at least $(\tau - 2\beta)d$ cycles containing $uv$ are forbidden because of the color assigned to the dimensional matching containing $ut$ and $vz$.

Let $S$ be a set of edges, $|S| = (\tau - 2\beta)d$, such that for every edge $zt \in S$, the cycle $C = uvztu$ is not allowed because of colors assigned to $ut$ and $vz$. There are $\binom{d-1}{(\tau-2\beta)d}$ ways to choose $S$. Furthermore, $L(uv)$ and $L(vz)$ contain at most $\beta d$ colors each, so there are at most $2\beta d$ choices for a color for the dimensional matching containing $ut$ and $vz$ that would make $C$ disallowed because of the color assigned to this dimensional matching. The remaining colors can be permuted in $(d - 1 - (\tau - 2\beta)d)! = ((1 - \tau + 2\beta)d - 1)!$ ways.

Hence, the total number of permutations $\sigma$ with not enough allowed cycles for a given edge is bounded from above by

$$d \left(\frac{d-1}{(\tau-2\beta)d}\right) (2\beta d)^{(\tau-2\beta)d}((1 - \tau + 2\beta)d - 1)! = \frac{d! (2\beta d)^{(\tau-2\beta)d}}{((\tau-2\beta)d)!}$$

and the total number of permutation $\sigma$ that have too few allowed cycles for at least one edge is bounded from above by

$$2^{d-1} d \frac{d! (2\beta d)^{(\tau-2\beta)d}}{((\tau-2\beta)d)!}$$

Hence,

$$X \geq d! - 2^d \frac{d! (2\alpha d)^{\gamma d}}{(\gamma d)!} - 2^d \frac{d! (\beta d)^{\gamma d}}{(\gamma d)!} - 2^{d-1} d \frac{d! (2\beta d)^{(\tau-2\beta)d}}{((\tau-2\beta)d)!}$$

Using Stirling’s approximation, $n! \geq n^n e^{-n}$ and $2^{d-1} d < \frac{2^d}{3}$, we have

$$X \geq d! \left(1 - 2^d e^{\gamma d} (2\alpha d)^{\gamma d} / (\gamma d)^{\gamma d} - 2^d e^{\gamma d} (\beta d)^{\gamma d} / (\gamma d)^{\gamma d} - 2^{2d} e^{(\tau-2\beta)d} (2\beta d)^{(\tau-2\beta)d} / 3((\tau-2\beta)d)^{(\tau-2\beta)d}\right)$$

$$X \geq d! \left(1 - \left(\frac{2^{\frac{1}{2}+1} e\alpha}{\gamma}\right)^{\gamma d} - \left(\frac{2^{\frac{1}{2}} e\beta}{\gamma}\right)^{\gamma d} - \frac{1}{3} \left(\frac{2^{\frac{1}{2}+1} e\beta}{\tau-2\beta}\right)^{(\tau-2\beta)d}\right)$$

Using the conditions $\frac{2^{\frac{1}{2}+1} e\alpha}{\gamma} < \frac{1}{3}$, $\frac{2^{\frac{1}{2}} e\beta}{\gamma} < \frac{1}{3}$, and $\frac{2^{\frac{1}{2}+1} e\beta}{\tau-2\beta} < 1$, we have

$$\left(\frac{2^{\frac{1}{2}+1} e\alpha}{\gamma}\right)^{\gamma d} < \frac{1}{3} \quad \text{and} \quad \left(\frac{2^{\frac{1}{2}} e\beta}{\gamma}\right)^{\gamma d} < \frac{1}{3} \quad \text{and} \quad \frac{1}{3} \left(\frac{2^{\frac{1}{2}+1} e\beta}{\tau-2\beta}\right)^{(\tau-2\beta)d} < \frac{1}{3}$$

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This implies $X > 0$. 

**Step II:** Let $h'$ be the proper $d$-edge coloring satisfying conditions (a)-(e) of Lemma 3.1 obtained in the previous step.

We use the following lemma for extending $\varphi$ to a proper $d$-edge precoloring $\varphi'$ of $Q_d$, such that an edge $e$ of $Q_d$ is colored under $\varphi'$ if and only if $e$ is precolored under $\varphi$ or $e$ is a conflict edge of $h'$ with $L$.

**Lemma 3.2.** Let $\alpha', \epsilon_0, \gamma, \kappa$ be constants such that $\alpha' = \max(\alpha + \gamma, \alpha + \epsilon_0)$, $\kappa = \max(\alpha + \gamma, \alpha + \epsilon_0, \gamma + \epsilon_0)$ and

$$d - \beta d - 2\alpha d - 2\gamma d - \frac{4\gamma}{\epsilon_0} d - \frac{\alpha}{\epsilon_0} d \geq 1.$$ 

There is a proper $d$-edge precoloring $\varphi'$ of $Q_d$ satisfying the following:

(a) $\varphi'(uv) = \varphi(uv)$ for any edge $uv$ of $Q_d$ that is precolored under $\varphi$.

(b) For every conflict edge $uv$ of $h'$, $uv$ is colored under $\varphi'$ and $\varphi'(uv) \notin L(uv)$.

(c) There are at most $\alpha'd$ precolored edges at each vertex of $Q_d$ under $\varphi'$.

(d) For every 12-neighborhood $W$ in $Q_d$, there are at most $\alpha'd$ precolored edges with color $i$ in $W$, $i = 1, \ldots, d$, under $\varphi'$.

(e) For every 12-neighborhood $W$ in $Q_d$, and every dimensional matching $M$, at most $\alpha'd$ edges of $M$ are precolored under $\varphi'$ in $W$.

Furthermore, the edge coloring $h'$ of $Q_d$ and the precoloring $\varphi'$ of $Q_d$ satisfy that

(f) For every 11-neighborhood $W$ in $Q_d$, and every dimensional matching $M$, at most $\kappa d$ edges of $M \cap E(W)$ are requested.

(g) No vertex $x$ in $Q_d$ satisfies that $E_x$ contains more than $\kappa d$ requested edges.

**Proof.** Consider the edge coloring $h'$ and the precoloring $\varphi$; for each 26-neighborhood $W$, no dimensional matching in $Q_d$ contains more than $\gamma d^2$ requested edges that are in $W$, so the total number of requested edges in $W$ is not greater than $\gamma d^2$. Similarly, the total number of conflict edges in each 27-neighborhood $W$ is not greater than $\gamma d^2$.

We shall construct the coloring $\varphi'$ by assigning a color to every conflict edge; this is done by iteratively constructing a $d$-edge precoloring $\phi$ of the conflict edges of $Q_d$; in each step we color a hitherto uncolored conflict edge, thereby transforming a conflict edge to a prescribed edge. At each step of transforming a conflict edge $uv$ into prescribed edge, the number of requested edges will increase by 2. Hence, after constructing the proper $d$-edge precoloring $\varphi'$, the total number of requested edges of each 26-neighborhood is at most $\gamma d^2 + 2\gamma d^2 = 3\gamma d^2$.

Suppose now that we have constructed the precoloring $\varphi'$. A vertex $u$ in $Q_d$ is $\varphi'$-overloaded if $E_u$ contains at least $\epsilon_0 d$ requested edges; note that no more than $\frac{3\gamma d^2}{\epsilon_0} = \frac{3\gamma d}{\epsilon_0}$ vertices of each 25-neighborhood are $\varphi'$-overloaded.
A color $c$ is $\varphi'$-overloaded in a $t$-neighborhood $W$ if $c$ appears on at least $\epsilon_0d$ edges in $W$ under $\varphi'$; note that at most
\[
\frac{\gamma d^2}{\epsilon_0d} + \frac{\alpha d^2}{\epsilon_0d} = \frac{\gamma + \alpha}{\epsilon_0}d
\]
colors are $\varphi'$-overloaded in each 25-neighborhood $W$. These upper bounds hold for any choice of the precoloring $\varphi'$ obtained from $\varphi$ by coloring the conflict edges of $Q_d$.

Let $G$ be the subgraph of the hypercube $Q_d$ induced by all conflict edges of $Q_d$. Let us now construct the $d$-edge coloring $\phi$ of $G$. We color the edges of $G$ by steps, and in each step we define a list $L(e)$ of allowed colors for a hitherto uncolored edge $e = uv$ of $G$ by for every color $c \in \{1, \ldots, d\}$ including $c$ in $L(e)$ if

- $c \notin L(uv)$,
- $c$ does not appear in $\varphi(u)$ or $\varphi(v)$, or on any previously colored edge of $G$ that is adjacent to $e$.
- $c$ is distinct from the color of the edge $uu'$ (or $vv'$) under $h'$ if $u'$ (or $v'$) is $\varphi'$-overloaded.
- $c$ is not $\varphi'$-overloaded in the 25-neighborhood of $e$.

Our goal is then to pick a color $\phi(e)$ from $L(e)$ for $e$. Given that this is possible for each edge of $G$, this procedure clearly produces a $d$-edge-coloring $\phi$ of $G$, so that $\phi$ and $\varphi$ taken together form a proper $d$-edge precoloring of $Q_d$.

Using the estimates above and the facts that $G$ has maximum degree $\gamma d$, and $|\varphi(v)| \leq \alpha d$ for any vertex $v$ of $Q_d$, we have
\[
L(e) \geq d - \beta d - 2\alpha d - 2\gamma d - \frac{3\gamma}{\epsilon_0}d - \frac{\gamma + \alpha}{\epsilon_0} d,
\]
for every edge $e$ of $G$ in the process of constructing $\phi$, and by assumption $L(e) \geq 1$. Thus, we conclude that we can choose an allowed color for each conflict edge so that the coloring $\phi$ satisfies the above conditions. This implies that taking $\phi$ and $\varphi$ together we obtain a proper $d$-precoloring $\varphi'$ of the edges of $Q_d$. Let us now prove that the precoloring $\varphi'$ satisfy the conditions in the lemma.

Let $\alpha' = \max(\alpha + \gamma, \alpha + \epsilon_0)$. Then $\varphi'$ satisfies the following:

- If $uv$ is precolored under $\varphi$, then $\varphi'(uv) = \varphi(uv)$. For every conflict edge $uv$, there is a precolor $\varphi'(uv)$ such that $\varphi'(uv) \notin L(uv)$.
- There are at most $\alpha'd$ precolored edges at each vertex.

Let us next prove that the precoloring $\varphi'$ satisfies conditions (d) and (e) of the lemma. Suppose that some 12-neighborhood $W$ in $Q_d$ contains more than $\alpha'd$ precolored edges with color $i$, for some $i \in \{1, \ldots, d\}$. Consider an edge $e$ in $W$ with $\phi(e) = i$. By the construction of $\phi$, in the 25-neighborhood $W'$ of $e$ no color is $\varphi'$-overloaded. Note further that every 12-neighborhood in $Q_d$ that $e$ lies in is contained in $W'$; thus $W$ is contained in $W'$, so the color $i$ is $\varphi'$-overloaded in $W'$, a contradiction. We conclude that condition (d) holds. A similar argument shows that condition (e) holds as well.
Let us now turn to conditions (f) and (g). There are at most $\alpha'd$ precolored edges with color $i$, $i = 1, \ldots, d$, in every 12-neighborhood in $Q_d$, and all edges that are in the same dimensional matching have the same color under $h'$. This implies that for each 11-neighborhood $W$ and every dimensional matching $M$, the maximum number of requested edges in $M$ that are in $W$ is $\alpha'd$. Since

$$\kappa \geq \max\{\alpha + \gamma, \alpha + \epsilon_0, \gamma + \epsilon_0\},$$

condition (f) holds. Similarly, at each step of transforming a conflict edge into a prescribed edge under $\phi$, we create 2 new requested edges, 1 at each vertex which is incident with the conflict edge. Since the maximum degree in $G$ is $\gamma d$, and no vertex is $\varphi'$-overloaded, no vertex $x$ in $Q_d$ satisfies that $E_x$ contains more than $\epsilon_0 d + \gamma d$ requested edges. Thus every vertex $x$ in $Q_d$ satisfies that $E_x$ contains at most $\kappa d$ requested edges.

**Step III:** Let $\varphi'$ be the proper $d$-precoloring of $Q_d$ obtained in the previous step and $h'$ the $d$-edge coloring of $Q_d$ obtained in Step I. By a *clash edge* (of $h'$) in $Q_d$ we mean an edge which is both prescribed and requested (under $\varphi'$). We use the following lemma for constructing, from $h'$, a proper $d$-edge coloring $h''$ of $Q_d$ with no clash edge. The coloring $h''$ will also have the property that every requested edge $e$ of $h''$ is adjacent to at most one prescribed edge $e'$ such that $h''(e) = \varphi'(e')$.

**Lemma 3.3.** Let $\kappa, \epsilon, \mu, \tau, \alpha' = \max(\alpha + \gamma, \alpha + \epsilon_0)$ be constants such that $\mu = 3\kappa + \epsilon + 1$ and

$$d - \tau d - 9\kappa d - 3\alpha' d - 3\epsilon d - \frac{12\kappa}{\epsilon} d - 3 > 0.$$  

By performing a sequence of swaps on disjoint allowed 2-colored 4-cycles in $h'$, we obtain a proper $d$-edge coloring $h''$ of $Q_d$ satisfying the following:

(a) There is no clash edge in $h''$.

(b) For each requested edge $e$ of $h''$, $e$ is adjacent to at most one edge $e'$ satisfying that $h''(e) = \varphi'(e')$.

(c) For each vertex $u \in V(Q_d)$, at most $2\kappa d + \epsilon d + 1$ edges incident with $u$ appears in swaps for constructing $h''$ from $h'$.

(d) For every 3-neighborhood $W$ of $Q_d$, and every dimensional matching $M$, at most $2\kappa d + \epsilon d + 1$ edges of $E(W) \cap M$ appears in swaps for constructing $h''$ from $h'$.

(e) For every 3-neighborhood $W$ in $Q_d$, and every dimensional matching $M$, there are at most $\mu d$ requested edges in $M \cap E(W)$.

(f) No vertex in $Q_d$ is incident with more than $\mu d$ requested edges.
Proof. An unexpected edge of \( h' \) is a clash edge or a requested edge \( e \) of \( h' \) that is adjacent to more than one edge \( e' \) satisfying that \( h'(e) = \varphi'(e') \). For constructing \( h'' \) from \( h' \), we will perform a number of swaps on 2-colored 4-cycles, and we shall refer to this procedure as \( S\)-swap. In more detail, we are going to construct a set \( S \) of disjoint allowed 4-cycles, each such cycle containing exactly one unexpected edge in \( h' \). An edge that belongs to a cycle in \( S \) is called used in \( S\)-swap.

Let us first deduce some properties that our set \( S \), which is yet to be constructed, will satisfy.

By Lemma 3.2 for every 11-neighborhood \( W \) in \( Q_d \), and every dimensional matching \( M \), the number of unexpected edges in \( E(W) \cap M \) is not greater than \( \kappa d \). Suppose we have included a 4-cycle \( C \) in \( S \). Every edge in \( C \) is at distance at most 1 from the unexpected edge contained in \( C \); this implies that for every 10-neighborhood \( W \) of \( Q_d \), the total number of edges in \( W \) that are used in \( S\)-swap is at most \( 4 \kappa d^2 \).

A vertex \( u \) in \( Q_d \) is \( S\)-overloaded if \( E_u \) contains at least \( \epsilon d \) edges that are used in \( S\)-swap; note that no more than \( 4 \kappa d^2 \) vertices of each 9-neighborhood are \( S\)-overloaded. A dimensional matching \( M \) in \( Q_d \) is \( S\)-overloaded in a \( t \)-neighborhood \( W \) if \( M \cap E(W) \) contains at least \( \epsilon d \) edges that are used in \( S\)-swap; note that for each 10-neighborhood \( W \), no more than \( 4 \kappa \epsilon d \) dimensional matchings of \( Q_d \) are \( S\)-overloaded in \( W \).

Using these facts, let us now construct our set \( S \) by steps; at each step we consider an unexpected edge \( e \) and include an allowed 2-colored 4-cycle containing \( e \) in \( S \). Initially, the set \( S \) is empty. Next, for each unexpected edge \( e = uv \) in \( Q_d \), there are at least \( d - \tau d \) allowed cycles containing \( e \). We choose an allowed cycle \( uvztu \) which contains \( e \) and satisfies the following:

1. \( z \) and \( t \) and the dimensional matching that contains \( vz \) and \( ut \) are not \( S\)-overloaded in the 9-neighborhood \( W_e \) of \( e \); this eliminates at most \( \frac{12 \kappa}{\epsilon} d \) choices.

   Note that with this strategy for including 4-cycles in \( S \), after completing the construction of \( S \), every vertex is incident with at most \( 2 \kappa d + \epsilon d + 1 \) edges that are used in \( S\)-swap; that is, condition (c) holds.

   Furthermore, after we have constructed the set \( S \), no dimensional matching is \( S\)-overloaded in a 3-neighborhood of \( Q_d \); this follows from the fact that every 3-neighborhood \( W' \) in \( Q_d \) that \( ut, vz \) or \( zt \) belongs to is contained in \( W_e \). Moreover, this implies that condition (d) holds.

2. None of the edges \( vz, zt, ut \) are prescribed, or requested, or used before in \( S\)-swap.

   All possible choices for these edges are in the 3-neighborhood \( W_e \) of \( e \) in \( Q_d \). By Lemma 3.2, no vertex in \( W_e \), or subset of a dimensional matching that is in \( W_e \), contains more than \( \kappa d \) requested edges or \( \alpha'd \) prescribed edges. Moreover, \( S\)-swap uses at most \( 2 \kappa d + \epsilon d + 1 \) edges at each vertex and in each subset of a dimensional matching contained in \( W_e \). Hence, these restrictions eliminate at most \( 3(\kappa d + \alpha'd) + 3(2 \kappa d + \epsilon d + 1) \) or \( 9 \kappa d + 3 \alpha'd + 3 \epsilon d + 3 \) choices.
It follows that we have at least
\[ d - \tau d - 9kd - 3\alpha' d - 3\epsilon d - \frac{12\kappa}{\epsilon}d - 3 \]
choices for an allowed cycle \(uvztu\) which contains \(uv\). By assumption, this expression is greater than zero, so we conclude that there is a cycle satisfying these conditions, and thus we may construct the set \(S\) by iteratively adding disjoint allowed 2-colored 4-cycles such that each cycle contains an unexpected edge.

After this process terminates we have a set \(S\) of disjoint allowed cycles; we swap on all the cycles in \(S\) to obtain the coloring \(h''\). Note that for the cycle \(uvztu\) constructed above, since none of the edges \(vz, zt, ut\) are prescribed or requested, \(\{\varphi'(z) \cup \varphi'(t)\}\) does not contain the color \(h'(uv)\); so after swapping colors on the cycle \(uvztu\), none of the edges edges \(uv, vz, zt, ut\) are unexpected edges in the obtained coloring; that is, condition (a) and (b) hold.

Let us finally verify that conditions (e) and (f) hold. As noted above, for every dimensional matching \(M\) and every 3-neighborhood \(W\), \(S\)-swap uses at most \(2kd + \epsilon d + 1\) from \(E(W) \cap M\). Moreover, by Lemma 3.2, \(E(W) \cap M\) contains at most \(kd\) requested edges under \(h'\) with respect to \(\varphi'\). Thus the proper coloring \(h''\) satisfies that for every dimensional matching \(M\) and for every 3-neighborhood \(W\) in \(Q_d\), at most \(\mu d\) requested edges are contained in \(E(W) \cap M\). Similarly, no vertex \(x\) in \(Q_d\) satisfies that \(E_x\) contains more than \(\mu d\) requested edges.

\[ \Box \]

**Step IV:** Let \(h''\) be the proper \(d\)-edge coloring of \(Q_d\) obtained in the previous step and let \(\varphi'\) be the precoloring of \(Q_d\) obtained in Step II. Then \(h''\) and \(\varphi'\) satisfies (a)-(f) of Lemma 3.3, and also the following:

- each vertex of \(Q_d\) is incident with at most \(\alpha' d\) edges that are precolored under \(\varphi'\);
- for every 12-neighborhood \(W\) and every dimensional matching \(M\) in \(Q_d\), at most \(\alpha' d\) edges of \(M\) are precolored under \(\varphi'\) in \(W\);
- for every 12-neighborhood \(W\), there are at most \(\alpha' d\) edges that are precolored \(i\) under \(\varphi'\) in \(W\), \(i = 1, \ldots, d\).

As in the proof of Lemma 3.3, we say that an edge \(e\) in \(Q_d\) with \(h'(e) \neq h''(e)\) is used in \(S\)-swap. Note that since in every 12-neighborhood, the number of edges that are precolored \(i\) \((i \in \{1, \ldots, d\})\) is at most \(\alpha' d\), and since the number of precolored edges in a subset of a dimensional matching of \(Q_d\) that is contained in a 12-neighborhood is also bounded, there is a bounded number of edges colored \(i\) under \(h''\) that have been used in \(S\)-swap in each 3-neighborhood. Moreover, since \(S\)-swap uses a bounded number of edges at each vertex, and in the intersection of every dimensional matching and 3-neighborhood (by condition (c) and (d) in Lemma 3.3), most edges in \(Q_d\) are in a large number of allowed 2-colored 4-cycles under \(h''\). Those two properties are central for completing the proof of Theorem 2.2 in Step IV; this is done by proving the following lemma.

**Lemma 3.4.** Let \(\kappa, \epsilon, \tau, \mu = 3\kappa + \epsilon + 1, \alpha' = (\alpha + \gamma, \alpha + \epsilon_0)\) be constants such that
\[ d - 64\mu d - 64\alpha' d - 32\kappa d - 32\epsilon d - 10\beta d - 3\tau d - \frac{266\alpha'}{\epsilon}d - 86 > 0. \]

There is a proper \(d\)-edge coloring of \(Q_d\) that is an extension of \(\varphi'\) and which avoids \(L\).
Proof. If \( h''(e) = \varphi'(e) \) for all precolored edges \( e \) then we do nothing; \( h'' \) is the required proper edge coloring. Else, we construct a set \( T \subseteq E(Q_d) \), such that performing a sequence of swaps on allowed 2-colored 4-cycles of the subgraph of \( Q_d \) induced by \( T \), we obtain the required extension of \( \varphi' \). We refer to this construction as \( T \)-swap. For each \( \varphi' \)-precolored edge \( e \), the set \( T \) will contain a subset \( T_e \) of edges associated with \( e \); if \( e \) and \( e' \) are distinct \( \varphi' \)-precolored edges of \( Q_d \), then we will have \( T_e \cap T_{e'} = \emptyset \). An example of a subset \( T_e \) can be seen in Figure 1, where \( \mathrm{v}_2 \mathrm{v}_3 \) is a prescribed edge, and \( \mathrm{v}_1 \mathrm{v}_2 \) and \( \mathrm{v}_3 \mathrm{v}_4 \) are requested. Since distinct sets \( T_e \) and \( T_{e'} \) are disjoint, every requested edge is in at most one set \( T_e \); this property is ensured by Lemma 3.3 (b).

An edge that belongs to \( T \) is called used in \( T \)-swap. By Lemma 3.2, for every 12-neighborhood \( W \) in \( Q_d \), and every dimensional matching \( M \), at most \( \alpha' d \) edges of \( M \cap E(W) \) are precolored under \( \varphi' \). For each configuration \( T_e \) in \( T \), every edge of \( T_e \) is at distance at most 2 from the prescribed edge; this implies that for every 10-neighborhood \( W \) in \( Q_d \), the total number of edges in \( W \) that are used in \( T \)-swap is at most \( 19\alpha' d^2 \).

A vertex \( u \) in \( Q_d \) is \( T \)-overloaded if at least \( \epsilon d \) edges from \( E_u \) are used in \( T \)-swap; note that no more than \( \frac{19\alpha' d^2}{\epsilon} \) vertices of each 9-neighborhood are \( T \)-overloaded. A dimensional matching \( M \) in \( Q_d \) is \( T \)-overloaded in a \( t \)-neighborhood \( W \) if \( M \cap E(W) \) contains at least \( \epsilon d \) edges that are used in \( T \)-swap; note that for each 10-neighborhood \( W \) no more than \( \frac{19\alpha' d}{\epsilon} \) dimensional matchings are \( T \)-overloaded in \( W \).

Consider the setup in Figure 1. We now describe how to construct the set \( T_e \) for the prescribed edge \( e = \mathrm{v}_2 \mathrm{v}_3 \). Suppose that \( \varphi'(\mathrm{v}_2 \mathrm{v}_3) = c_2 \neq h''(\mathrm{v}_2 \mathrm{v}_3) = c_1 \). Since every vertex in \( Q_d \) has degree \( d \) we initially have at least \( d - 3 \) choices for a subgraph as in Figure 1.
Let $v_1v_2$ and $v_3v_4$ be the edges adjacent to $v_2v_3$ that are colored $c_2$. The set $T_{v_2v_3}$ will consist of edges incident with 14 vertices $v_1, \ldots, v_{14}$. We shall choose the vertices $v_5, \ldots, v_{14}$ such that they satisfy a number of properties:

1. $v_5, \ldots, v_{14}$ and the dimensional matchings that contain $v_1v_5, v_2v_6, v_3v_7, v_4v_8, v_5v_9, v_6v_{10}, v_6v_{11}, v_7v_{12}, v_7v_{13}, v_8v_{14}$ are not $T$-overloaded in the 9-neighborhood $W_e$ of $e$.

These edges are in at most four dimensional matchings and we select 10 new vertices, so this eliminates at most $\frac{19 \times 14 \alpha'}{\epsilon}d$ or $\frac{266 \alpha'}{\epsilon}d$ choices.

Moreover, every 2-neighborhood $W$ in $Q_d$ that one of these edges lie in is contained in $W_e$, so with this strategy and with the bounds on the number of requested and prescribed edges under $h''$, in the process of choosing the set $T$, for every dimensional matching $M$ and every 2-neighborhood $W$ in $Q_d$, $E(W) \cap M$ contains at most

$$3\mu d + 3\alpha'd + (\epsilon d - 1) + 4 = 3\mu d + 3\alpha'd + \epsilon d + 3$$

edges that are used in $T$-swap. Similarly, every vertex is incident with at most

$$3\mu d + 3\alpha'd + (\epsilon d - 1) + 5 = 3\mu d + 3\alpha'd + \epsilon d + 4$$

edges that are used in $T$-swap; these upper bounds follow from the facts that the maximum number of edges of $T_e$ incident with one vertex is 5, and the maximum number of edges from a given dimensional matching in $T_e$ is 4.

2. None of the edges $v_1v_5, v_2v_6, v_3v_7, v_4v_8$ are prescribed or requested or used before in $T$-swap.

Since all the possible choices for these edges are in the 2-neighborhood $W_e$ of $e$ in $Q_d$, and under $h''$ no vertex contains more than $\alpha'd$ prescribed edges and $\mu d$ requested edges, $T$-swap uses at most $3\mu d + 3\alpha'd + \epsilon d + 4$ edges at each vertex, this condition eliminates at most

$$4 \times (\mu d + \alpha'd) + 4 \times (3\mu d + 3\alpha'd + \epsilon d + 4) = 16\mu d + 16\alpha'd + 4\epsilon d + 16$$

choices.

3. $v_1v_5, v_2v_6, v_3v_7, v_4v_8$ are not used before in $S$-swap.

Note that this condition ensures that $h''(v_1v_5) = h''(v_2v_6) = h''(v_3v_7) = h''(v_4v_8) = c_3$, where $c_3$ is the most common color of the dimensional matching that contains $v_1v_5$. This eliminates at most $8\kappa d + 4\epsilon d + 4$ choices based on the conditions (c) and (d) of Lemma 3.3.

4. The three cycles $v_1v_2v_5v_1, v_2v_3v_7v_6v_2, v_3v_4v_8v_7v_3$ are allowed before $S$-swap.

Each of the edges $v_1v_2, v_2v_3, v_3v_4$ belongs to at most $\tau d$ non-allowed cycles, so this eliminates at most $3\tau d$ choices.

5. $c_2 \notin \{L(v_1v_5) \cup L(v_4v_8)\}$.

Color $c_2$ appears at most $\beta d$ times in lists of the set of edges incident to $v_1$ (or $v_4$), so this eliminates at most $2\beta d$ choices.
(6) \(c_1 \notin \{L(v_2v_6) \cup L(v_3v_7)\}\).

Similarly, this eliminates at most \(2\beta d\) choices.

(7) If \(v_1v_2\) is in a dimensional matching of color \(c_2\) of \(h''\), then we choose \(v_5, v_6\) such that \(v_5v_6\) is not used in \(S\)-swap or \(T\)-swap, and \(v_5v_6\) is not prescribed or requested. Note that the conditions imply that \(h''(v_5v_6) = c_2\); in this case we choose \(v_9, v_{10}\) arbitrarily. Based on the restriction of edges used in each dimensional matching and \(v_5v_6 \neq v_1v_2\), these restrictions eliminate at most
\[
2\kappa d + \epsilon d + 1 + (3\mu d + 3\alpha' d + \epsilon d + 4) + 1 = 2\kappa d + 3\alpha' d + 2\epsilon d + 3\mu d + 6
\]
choices.

Else, we choose \(v_5, v_6, v_9, v_{10}\) such that

(a) \(v_5v_6\) is not used in \(S\)-swap (and also \(v_5v_6 \neq v_1v_2\)); this eliminates at most \(2\kappa d + \epsilon d + 1\) choices. We can assume that \(v_5v_6\) is in a dimensional matching of color \(c_4\).

(b) \(v_5v_6\) is in an allowed cycle \(v_5v_6v_{10}v_9v_5\) with color \(c_2\) (which means \(h''(v_5v_6) = h''(v_5v_{10}) = c_2\), \(h''(v_9v_{10}) = h''(v_5v_6) = c_4\), \(c_4 \notin \{L(v_5v_9) \cup L(v_6v_10)\}\) and \(c_2 \notin \{L(v_5v_6) \cup L(v_9v_{10})\}\).

In the 3-neighborhood \(W\) of \(e\) in \(Q_d\), \(S\)-swap uses at most \(2\kappa d + \epsilon d + 1\) edges in the dimensional matching of color \(c_2\), so this eliminates at most \(4\kappa d + 2\epsilon d + 2\) choices for \(v_5v_9\) and \(v_6v_{10}\). We also require that \(v_9v_{10} \neq v_5v_6\) and that \(v_9v_{10}\) is not used in \(S\)-swap to make sure \(h''(v_9v_{10}) = h''(v_5v_6) = c_4\); this eliminates at most \(2\kappa d + \epsilon d + 2\) choices.

Since \(c_2\) occurs \(\beta d\) times in the subset of the dimensional matching of color \(c_4\) contained in the 27-neighborhood \(W\) of \(e\) in \(Q_d\), and \(c_4\) occurs \(\beta d\) times in the subset of the dimensional matching of color \(c_2\) contained in \(W\), the two conditions \(c_4 \notin \{L(v_5v_9) \cup L(v_6v_{10})\}\) and \(c_2 \notin \{L(v_5v_6) \cup L(v_9v_{10})\}\) eliminate at most \(2\beta d\) choices.

(c) \(v_5v_6, v_5v_9, v_9v_{10}, v_6v_{10}\) are not prescribed or requested or used before in \(T\)-swap.

Since all the possible choices for these edges are in the 2-neighborhood \(W\) of \(e\) in \(Q_d\), this eliminates at most \(4 \times (\mu d + \alpha' d) + 4 \times (3\mu d + 3\alpha' d + \epsilon d + 4)\) choices.

So in both cases, the choosing process eliminates at most
\[
16\mu d + 16\alpha' d + 8\kappa d + 8\epsilon d + 2\beta d + 21
\]
choices.

(8) \(v_7, v_8, v_{14}, v_{13}\) is chosen with same strategy as \(v_5, v_6, v_9, v_{10}\).

Similarly, this eliminates at most \(16\mu d + 16\alpha' d + 8\kappa d + 8\epsilon d + 2\beta d + 21\) choices.

(9) \(v_6, v_7, v_{11}, v_{12}\) is chosen with same strategy with \(v_5, v_6, v_9, v_{10}\) but the color \(c_2\) is replaced by \(c_1\).

Again, this eliminates at most \(16\mu d + 16\alpha' d + 8\kappa d + 8\epsilon d + 2\beta d + 21\) choices.
Summing up, we conclude that in total, there are at most
\[64\mu d + 64\alpha'd + 32\kappa d + 32\epsilon d + 10\beta d + 3\tau d + \frac{266\alpha'}{\epsilon}d + 83\]
forbidden choices for the configuration \(T_e\).

This implies that we have
\[d - 3 - 64\mu d - 64\alpha'd - 32\kappa d - 32\epsilon d - 10\beta d - 3\tau d - \frac{266\alpha'}{\epsilon}d - 83\]
or
\[Z = d - 64\mu d - 64\alpha'd - 32\kappa d - 32\epsilon d - 10\beta d - 3\tau d - \frac{266\alpha'}{\epsilon}d - 86\]
choices for a configuration \(T_{e'}\) in the process of constructing \(T\), whenever \(e'\) is a prescribed edge.

By assumption, \(Z > 0\), so there is a set \(T_{v_2v_3}\) that satisfies all the above conditions. We add this set to \(T\) and apply this procedure for all prescribed edges \(uv\) with \(h''(uv) \neq \varphi'(uv)\). Since the resulting subsets of \(T\) are disjoint, we can do the following transformation for each subset \(T_{v_2v_3}\) as above.

- If \(h''(v_5v_6) \neq c_2\), then interchange colors of the cycle \(v_5v_6v_{10}v_9v_5\).
- If \(h''(v_6v_7) \neq c_1\), then interchange colors of the cycle \(v_6v_7v_{12}v_{11}v_6\).
- If \(h''(v_7v_8) \neq c_2\), then interchange colors of the cycle \(v_7v_8v_{14}v_{13}v_7\).
- Next, interchange colors of the cycles \(v_1v_2v_6v_5v_1\) and \(v_3v_4v_8v_7v_3\).
- Finally, interchange colors of the cycle \(v_2v_3v_7v_6v_2\).

In the resulting edge coloring obtained from \(h''\), \(v_2v_3\) is colored \(c_2\). Moreover, it follows from conditions (3), (4), (5), (7) that we do not create any new conflict edges by performing these swaps. We thus conclude that by repeating this swapping procedure for every prescribed edge, we obtain a new proper \(d\)-edge coloring which agrees with the precoloring \(\varphi'\).

We have proved that it is possible to complete all the steps I-IV outlined in Section 2, thereby obtaining an extension of \(\varphi\) that avoids \(L\); this completes the proof of Theorem 2.2.

Let us now turn to the proof of Theorem 1.3. As we shall see, Property 2.1 of the standard edge coloring \(h\) of \(Q_d\) trivially yields the result.

Proof of Theorem 1.3 (sketch). Let \(\varphi\) be a \(d\)-edge precoloring of \(Q_d\), and \(L\) a \(\beta\)-sparse list assignment for the non-precolored edges of \(Q_d\), such that any edge \(e\) which is either precolored or satisfies \(L(e) \neq \emptyset\) belongs to a distance-3 matching \(M\) in \(Q_d\). Let \(h\) be the standard \(d\)-edge coloring of \(Q_d\) defined above.

Now, by arbitrarily picking a color from the set \(\{1, \ldots, d\} \setminus L(e')\) for each conflict edge \(e'\), we can construct a precoloring \(\varphi'\) from \(\varphi\) such that an edge \(e\) of \(Q_d\) is precolored under \(\varphi'\) if and only if \(e\) is precolored under \(\varphi\) or \(e\) is a conflict of \(h\) with \(L\). Furthermore, any 2-colored
4-cycle $C$ with colors $c_1$ and $c_2$ under $h$, and satisfying that there is an edge $e \in E(C)$ with $\varphi'(e) = c_1$ and $h(e) = c_2$ is allowed. Moreover, since edges that are precolored under $\varphi'$ are at distance at least 3 from each other, two 4-cycles containing distinct precolored edges are disjoint. Now, by Property 2.1, every edge in $Q_d$ is contained in $d - 1$ 2-colored 4-cycles under $h$; thus, we may complete the proof by simply swapping on a suitable set of disjoint 2-colored 4-cycles.

4 Upper bounds and further problems

We have proved that there are constants $\alpha$ and $\beta$ such that every $\alpha$-dense $d$-edge precoloring of $Q_d$ can be extended to a proper $d$-edge coloring avoiding any given $\beta$-sparse list assignment for $Q_d$. The values we have found for $\alpha$ and $\beta$ are quite small, to a large extent due to the calculations in Lemma 3.1.

Let us briefly compare our results obtained in this paper with corresponding results for complete bipartite graphs. Recall that a list assignment $L$ for $K_{n,n}$ is $\beta$-sparse if each edge $e$ of $K_{n,n}$ is assigned a list $L(e)$ of at most $\beta n$ forbidden colors from $\{1, \ldots, n\}$, and at every vertex $v$ each color appears in lists of at most $\beta n$ edges adjacent to $v$; similarly an $n$-edge precoloring of $K_{n,n}$ is $\alpha$-dense if every color is used at most $\alpha n$ times in the precoloring and at every vertex $v$ at most $\beta n$ edges precolored incident to $v$ are precolored. For $K_{n,n}$ Daykin and Häggkvist [DH84] conjectured that $\alpha = 1/4$ is the optimal value, and Häggkvist conjectured that $\beta = 1/3$ is optimal. The currently best value for $\beta$ is given in [ACO13] is far smaller, due to probabilistic tools. That one can simultaneously take $\alpha$ and $\beta$ to be positive was proven in [ACM16].

For the hypercube $Q_d$, the following general proposition yields an upper bound on the values of $\alpha$ and $\beta$ in Theorem 1.1.

**Proposition 4.1.** Let $G$ be a $d$-regular $d$-edge-colorable graph.

(i) If every $d$-edge precoloring of $G$, satisfying that each vertex of $G$ is incident to at most $\alpha d$ precolored edges, is extendable, then $\alpha < 1/2$.

(ii) If every list assignment $L$, such that $|L(e)| \leq \beta d$ for each edge $e \in E(G)$, and for each vertex $v$ each color appears in at most $\beta d$ lists of edges incident with $v$, is avoidable, then $\beta < 1/2$.

(iii) If every precoloring as in (i) is extendable to a coloring avoiding any list assignment as in (ii), then $\alpha + \beta < 1/2$.

**Proof.** (i) Let $u_1 u_2$ be an edge of $G$. We define an edge precoloring $\varphi$ of $G$ by coloring $[d/2]$ edges incident with $u_1$ and distinct from $u_1 u_2$ by colors $1, \ldots, [d/2]$; next, color $[d/2]$ edges incident to $u_2$ and distinct from $u_1 u_2$ by colors $[d/2] + 1, \ldots, d$. This yields an edge $d$-precoloring which is not extendable to a proper $d$-edge coloring, so necessarily $\alpha < 1/2$.

(ii) Let $u_1 u_2$ be an edge of $G$. Next, to $[d/2]$ edges incident with $u_1$, but not $u_2$, assign identical color lists containing all the colors $1, \ldots, [d/2]$. Similarly assign to $[d/2]$ edges
incident with \( u_2 \), but not \( u_1 \), identical color lists containing all the colors \([d/2]+1,\ldots,d\). Now, since apart from \( u_1 u_2 \), there are at most \([d/2] - 1\) edges incident with \( u_1 \), where colors \( 1,\ldots,[d/2] \) are not forbidden, we must have that \( u_1 u_2 \) is colored with a color from \( 1,\ldots,[d/2] \) in any proper \( d \)-edge coloring of \( G \) avoiding the list assignment; similarly by the restrictions at \( u_2 \), \( u_1 u_2 \) must be colored with a color from \([d/2] + 1,\ldots,d\) in any coloring of \( G \) avoiding the list assignments at \( u_2 \). This is clearly not possible, so the list assignment is unavoidable, and thus \( \beta < 1/2 \).

(iii) The precoloring and list-assignments defined above can be combined in the following way (we assume that \( 1/2 > \beta > \alpha \) and that \( \alpha d \) and \( \beta d \) are integers):

Let \( u_1 u_2 \) be an edge as above, let \( H_1 \) be the star induced by \( u_1 \) and its neighbors except for \( u_2 \), and \( H_2 \) the corresponding star for \( u_2 \).

We now consider the assignment where in \( H_1 \) there are \( \alpha d \) precolored edges incident with \( u_1 \) using colors \( d - \alpha d + 1,\ldots,d \); moreover, exactly \( \beta d \) edges in \( H_1 \) incident with \( u_1 \), distinct from the precolored ones, are assigned identical lists with colors \( 1,\ldots,\beta d \). Similarly, in \( H_2 \) there are \( \alpha d \) precolored edges incident with \( u_2 \) using colors \( 1,\ldots,\alpha d \); moreover, there are precisely \( \beta d \) edges in \( H_2 \) incident with \( u_2 \), distinct from the precolored edges, all of which are assigned identical color lists containing colors \( d - \beta d + 1,\ldots,d \).

Now, for any proper \( d \)-edge coloring \( f \) of \( G \) which is an extension of the precoloring and which avoids the the list assignment, the colors \( 1,\ldots,\beta d \) must appear on edges incident with \( u_1 \) which are neither precolored nor are assigned a non-empty list of forbidden colors. By a similar argument for \( H_2 \), we must have that for any coloring \( f \) which is an extension of the precoloring and also avoids the list assignment, colors \( d - \beta d + 1,\ldots,d \) must appear on edges incident with \( u_2 \) which are neither precolored nor are assigned a non-empty list of forbidden colors. Note that both \( u_1 \) and \( u_2 \) are incident with exactly \( d - \beta d - \alpha d \) edges which are neither precolored nor contain a non-empty list of forbidden colors. Thus if \( d - \alpha d - \beta d \leq \beta d \), then the edge \( u_1 u_2 \) must receive a color both from the set \( \{1,\ldots,\beta d\} \) and from the set \( \{d - \beta d + 1,\ldots,d\} \) under \( f \). Moreover, if \( \beta d \leq d - \beta d + 1 \), then these sets are disjoint, implying that there is no extension of the precoloring which avoids the given list assignment. Here, by choosing \( \beta \) close to \( 1/2 \) and \( \alpha \) small, we can make the sum \( \alpha + \beta \) arbitrarily close to \( 1/2 \).

\( \square \)

Returning to the setup of Theorem 1.1, we have attempted to find constructions which yield better upper bounds for \( \alpha \) and \( \beta \) for the hypercubes, but have not been able to do so. Moreover, the conditions (ii) and (iii) for a precoloring of \( Q_d \) to be \( \alpha \)-dense are not probably not best possible in terms of size of the neighborhoods. Those conditions are required in our proof, but might be far stronger than what is actually needed in order for a compatible edge coloring to exist. Nonetheless it would be interesting to see how far Theorem 1.1 can be improved in its current form (possibly with decreased size of the neighborhoods).

**Problem 4.2.** What are the optimal values for \( \alpha \) and \( \beta \) in Theorem 1.1?
show that in order to get results similar to those for $K_{n,n}$, and those given in this paper, one must impose some structural conditions on the considered family of graphs. Both $K_{n,n}$ and $Q_d$ are well connected bipartite graphs and it would be interesting to see how far Proposition 4.1 can be improved for this general class of graphs.

**Problem 4.3.** Given a precoloring and a list assignment as in Proposition 4.1, what are the optimal values for $\alpha$ and $\beta$ for the family of $d$-regular, $d$-edge connected, bipartite graphs?

Here the cases closest to our results are of course those where $d$ is a function of the number of vertices in the graph.

Finally, as mentioned in the introduction, our proof method easily give us Theorem 1.3 where the edges which are precolored or have non-empty lists of forbidden colors on them are forced to lie in a distance-3 matching. Here it is natural to ask if this result holds for distance-2 matchings as well.

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