Non-abelian convexity by symplectic cuts

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Abstract
In this paper we extend the results of Kirwan et alii on convexity properties of the moment map for Hamiltonian group actions, and on the connectedness of the fibers of the moment map, to the case of non-compact orbifolds.

Our motivation is twofold. First, the category of orbifolds is important in symplectic geometry because, generically, the symplectic quotient of a symplectic manifold is an orbifold. Second, our proof is conceptually very simple since it reduces the non-abelian case to the abelian case.

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In this paper we prove the following theorem, which extends a result of Kirwan (Ki2, Ki1) to the case of orbifolds which need not be compact. Recall that a subset $\Delta$ of a vector space $V$ is polyhedral if it is the intersection of finitely many closed half-spaces. Recall also that $\Delta \subset V$ is locally polyhedral if for any point $x \in \Delta$ there is a neighborhood $U$ of $x$ in $V$ and a polyhedral set $P$ in $V$ such that $U \cap \Delta = U \cap P$.

**Theorem 1.1** Let $(M, \omega)$ be a connected symplectic orbifold with a Hamiltonian action of a compact Lie group $G$ and a proper moment map $\Phi : M \to g^*$. 

1. Let $t_+^*$ be a closed Weyl chamber for the Lie group $G$ considered as a subset of $g^*$. The moment set $\Phi(M) \cap t_+^*$ is a convex locally polyhedral set. In particular, if $M$ is compact then the moment set is a convex polytope.

2. Each fiber of the moment map $\Phi$ is connected.

Kirwan’s theorem has numerous applications in symplectic geometry: For example, it is used in the classification of Hamiltonian $G$-orbifolds, in geometric quantization, and in the study of the existence of invariant Kähler structures.

Convexity theorems in symplectic geometry have a long history. For the action of a maximal torus on a coadjoint orbit Theorem 1.1 was proved by Kostant [K], extending previous results of Schur and Horn. This was generalized to the actions of subgroups by Heckman [H]. For Hamiltonian torus actions on manifolds the theorem was proved independently by Atiyah [A] and by Guillemin and Sternberg [GS1]. In the non-abelian case Guillemin and Sternberg proved that the moment set is a union of convex polytopes and that it is a single polytope for a Kähler manifold. The convexity in the Kähler case was independently proved by Mumford [NM]. The first complete proof for Hamiltonian actions of non-abelian groups on manifolds was given by Kirwan [Ki2], using Morse theory of the Yang-Mills functional and results of Guillemin and Sternberg. Several alternative proofs of Kirwan’s result have appeared: CDM, HNP (based on CDM) and S (based on B).

Our motivation for offering this proof is twofold. First, Theorem 1.1 generalizes the result from manifolds to orbifolds; this category is important in symplectic geometry because, generically, the symplectic quotient of a symplectic manifold is an orbifold. Secondly, our proof is conceptually very simple. The first step is to generalize the abelian version of the theorem to certain non-compact orbifolds. The key idea is to apply the technique of symplectic cutting [L] to reduce to
the compact case. In the second step we reduce the non-abelian case to the abelian case by means of a symplectic cross-section: We show that there is a unique open wall $\sigma$ of the Weyl chamber $t_+^*$ such that $\Phi(M) \cap \sigma$ is dense in the moment set $\Delta = \Phi(M) \cap t_+^*$, and that the preimage $\Phi^{-1}(\sigma)$ is a $T$-invariant symplectic suborbifold.

Under assumptions similar to those of the main theorem, the result holds for Hamiltonian actions of loop groups provided the preimage of an alcove under the moment map is finite dimensional. This will be discussed in a forthcoming paper.

In the last section, using similar methods we extend to symplectic orbifolds a result of Sjamaar [Sj] which says that the moment set near a point $x$ can be read off from local data near an arbitrary point in the fiber $\Phi^{-1}(x)$:

**Theorem 1.2** Let $(M, \omega)$ be a connected symplectic orbifold with a Hamiltonian action of a compact Lie group $G$ and a proper moment map $\Phi : M \to g^*$. For every $m \in \Phi^{-1}(t_+^*)$, and every $G$-invariant neighborhood $U$ of $m$ in $M$, the image $\Phi(U) \cap t_+^*$ is a relatively open neighborhood of $\Phi(m)$ in $\Phi(M) \cap t_+^*$.

2 Background

In this section, we extend some well-known results in symplectic geometry from manifolds to orbifolds. These extensions are straightforward. Readers who are familiar with the standard versions of these results and not too skeptical may wish to skip this section.

2.1 Hamiltonian actions on orbifolds

An orbifold $M$ is a topological space $|M|$, together with an atlas of uniformizing charts $(\tilde{U}, \Gamma, \varphi)$, where $\tilde{U}$ is open subset of $\mathbb{R}^n$, $\varphi(\tilde{U})$ is an open subset of $|M|$, $\Gamma$ is a finite group which acts linearly on $\tilde{U}$ and fixes a set of codimension at least two, and $\varphi : \tilde{U} \to |M|$ induces a homeomorphism from $\tilde{U}/\Gamma$ to $\varphi(\tilde{U}) \subset |M|$. Just as for manifolds, these charts must cover $|M|$; they are subject to certain compatibility conditions; and there is a notion of when two atlases of charts are equivalent. For more details, see Satake [Sa].

A smooth function on $M$ is a collection of smooth invariant functions on each uniformizing chart $(\tilde{U}, \Gamma, \varphi)$ which agree on overlaps of the images $\varphi(\tilde{U})$. Differential forms, vectors fields, and other objects can be similarly defined. There is also a notion of morphisms (maps) of orbifolds.

Let $x$ be a point in an orbifold $M$, and let $(\tilde{U}, \Gamma, \varphi)$ be a uniformizing chart with $x \in \tilde{U}/\Gamma$. The (orbifold) structure group of $x$ is the isotropy group $\Gamma_x$ of $\tilde{x} \in \tilde{U}$, where $\varphi(\tilde{x}) = x$. The group $\Gamma_x$ is well defined as an abstract group. The tangent space to $\tilde{x}$ in $\tilde{U}$, considered as a representation of $\Gamma_x$ is called the uniformized tangent space at $x$, and denoted by $\tilde{T}_xM$. The quotient $\tilde{T}_xM/\Gamma_x$ is $T_xM$, the fiber of the tangent bundle of $M$ at $x$.

Let $G$ be a compact connected Lie group with Lie algebra $g$. A smooth action $a$ of $G$ on an orbifold $M$ is a smooth orbifold map $a : G \times M \to M$ satisfying the usual laws for an action. Given
an action of $G$ on $M$, every vector $\xi \in \mathfrak{g}$ induces a vector field $\xi_M$ on $M$.

A symplectic orbifold is an orbifold $M$ with a closed non-degenerate 2-form $\omega$. A group $G$ acts symplectically on $(M, \omega)$ if the action preserves $\omega$. A moment map for a symplectic action of a group $G$ is an equivariant map $\Phi : M \to \mathfrak{g}^*$ such that

$$i_{\xi_M} \omega = -\langle d\Phi, \xi \rangle$$

for all $\xi \in \mathfrak{g}$.

If a moment map exists, we say that the action of $G$ on $(M, \omega)$ is Hamiltonian and refer to $(M, \omega)$ as a Hamiltonian $G$-space. If $b \in \mathfrak{g}^*$ is a regular value of the moment map $\Phi$, then the orbifold version of the Marsden-Weinstein-Meyer theorem says that the quotient $M_b := \Phi^{-1}(b)/G$ is a symplectic orbifold, called the symplectic reduction of $M$ at $b$. For a proof, see [LT].

The following theorem is due in the manifold case to Atiyah [A] and to Guillemin and Sternberg [GS1]. The proofs in the orbifold case are in [LT]. Let $T$ be a torus, $t$ its Lie algebra, $t^*$ the dual of $t$ and $\ell = \ker\{\exp : t \to T\}$ the integral lattice. A polytope in $t^*$ is rational if the faces of the polytope are cut out by hyperplanes whose normal vectors are in the lattice $\ell$.

**Theorem 2.1** Let $(M, \omega)$ be a compact, connected symplectic orbifold, and let $\Phi : M \to t^*$ be a moment map for a Hamiltonian torus action on $M$.

1. The image $\Phi(M)$ is a rational convex polytope, and
2. each fiber of $\Phi$ is connected.

**2.2 Symplectic Cuts**

Symplectic cutting is a technique which allows one to naturally construct a symplectic structure on a subquotient of a symplectic orbifold. It was introduced in [L]. We will use symplectic cuts to produce compact orbifolds out of non-compact ones.

Let $(M, \omega)$ be a symplectic orbifold with a Hamiltonian circle action and a moment map $\mu : M \to \mathbb{R}$. Suppose that $\epsilon$ is a regular value of the moment map. Consider the disjoint union

$$M_{[\epsilon, \infty)} := \mu^{-1}((\epsilon, \infty)) \cup M_{\epsilon},$$

obtained from the orbifold with boundary $\mu^{-1}([\epsilon, \infty))$ by collapsing the boundary under the $S^1$-action. We claim that $M_{[\epsilon, \infty)}$ admits a natural structure of a symplectic orbifold, in such a way that the embeddings of $\mu^{-1}((\epsilon, \infty))$ and $M_{\epsilon}$ are symplectic. Moreover, the induced circle action on $M_{[\epsilon, \infty)}$ is Hamiltonian, with a moment map coming from the restriction of the original moment map $\mu$ to $\{m \in M : \mu(m) \geq \epsilon\}$.

**Definition 2.2** We call the symplectic orbifold $M_{[\epsilon, \infty)}$ the symplectic cut of $M$ with respect to the ray $[\epsilon, \infty)$ (with symplectic form and moment map understood).
To see that the claim holds consider the product $M \times \mathbb{C}$ of the orbifold with a complex plane. It has a natural (product) symplectic structure: $\omega + (-i)dz \wedge d\bar{z}$. The function $\nu : M \times \mathbb{C} \to \mathbb{R}$ given by $\nu(m, z) = \mu(m) - |z|^2$ is a moment map for the diagonal action of the circle (it commutes with the original action of the circle on $M$). The point $\epsilon$ is a regular value of $\mu$ if and only if it is a regular value of $\nu$. The map 

$$\{m \in M : \mu \geq \epsilon\} \to \nu^{-1}(\epsilon), \quad m \mapsto (m, \sqrt{\mu(m) - \epsilon})$$

descends to a homeomorphism from the cut space $M_{[\epsilon, \infty)}$ to the reduced space $\nu^{-1}(\epsilon)/S^1$. Moreover the homeomorphism is $S^1$-equivariant and it is a symplectomorphism on the set $\{\mu > \epsilon\}$.

**Example 2.3** Consider the complex plane $\mathbb{C}$ with its standard symplectic form $(-i)dz \wedge d\bar{z}$, and let a circle $U(1) = \{z \in \mathbb{C} : |z| = 1\}$ act by multiplication. The moment map is $\mu(z) = -|z|^2$. The symplectic cut of $\mathbb{C}$ at $\epsilon < 0$ is a two-sphere.

This construction generalizes to torus actions as follows.\footnote{See [W], [M] for the generalization to non-abelian actions.} Let $\mu : M \to t^*$ be a moment map for an effective action of a torus $T$ on a symplectic orbifold $(M, \omega)$ and let $\ell \subset t$ denote the integral lattice. Choose $N$ vectors $v_j \in \ell$, $1 \leq j \leq N$. The form $\omega - i \sum_{j=1}^{N} dz_j \wedge d\bar{z}_j$ is a symplectic form on the orbifold $M \times \mathbb{C}^N$. The map $\nu : M \times \mathbb{C}^N \to \mathbb{R}^N$ with $j$th component 

$$\nu_j(m, z) = \langle \mu(m), v_j \rangle - |z_j|^2$$

is a moment map for a $(S^1)^N$ action on $M \times \mathbb{C}^N$. For any $b \in \mathbb{R}^N$, define a convex rational polyhedral set 

$$P = \{x \in t^* \mid \langle x, v_i \rangle \geq b_i \text{ for all } 1 \leq i \leq N\}.$$ 

The **symplectic cut of $M$ with respect to a rational polyhedral set $P$** is the reduction of $M \times \mathbb{C}^N$ at $b$. We denote it by $M_P$.

**Remark 2.4** If $b$ is a regular value of $\nu$, then $M_P$ is a symplectic orbifold. Note that regular values are generic. Thus, for an intersection $P$ of finitely many rational half-spaces of $t^*$ in general position the cut space $M_P$ is a symplectic orbifold. Note that if $P$ is a compact polytope, then the fact that $P$ is generic implies that $P$ is **simple**: the number of codimension one faces meeting at a given vertex is the same as the dimension of $P$.

There is a natural decomposition of $M_P$ into a union of symplectic suborbifolds, indexed by the open faces $F$ of $P$: 

$$M_P = \bigcup_F \mu^{-1}(F)/T_F,$$

where $T_F \subset T$ is the torus **perpendicular** to $F$. That is, the Lie algebra of $T_F$ is the annihilator of the linear space defined by the face $F$. Thus, one can think of $M_P$ as $\mu^{-1}(P)$, with its boundary
collapsed by means of the $T_F$-actions. Under this identification, the $T$-action on $M$ descends to a $T$-action on $M_P$. One can show that this action is smooth and Hamiltonian, with moment map $\mu_P$ induced from the restriction $\mu|\mu^{-1}(P)$. In particular,

$$\mu_P(M_P) = \mu(M) \cap P.$$ 

It follows immediately that the cut space $M_P$ is compact exactly if $\mu^{-1}(P)$ is. Similarly $M_P$ is connected if and only if $\mu^{-1}(P)$ is connected. To summarize, we have the following result.

**Proposition 2.5** Let $\mu : M \to \mathfrak{t}^*$ be a moment map for an effective action of a torus $T$ on a symplectic orbifold $(M, \omega)$. Let $P \subset \mathfrak{t}^*$ be a generic rational polyhedral set. Let $F$ be the set of all open faces of $P$. Then the topological space $M_P$ defined by

$$M_P = \bigcup_{F \in F} \mu^{-1}(F)/T_F,$$

where $T_F \subset T$ is the subtorus of $T$ perpendicular to $F$, is a symplectic orbifold with a natural Hamiltonian action of the torus $T$. Moreover, the map $\mu_P : M_P \to \mathfrak{t}^*$, induced by the restriction $\mu|\mu^{-1}(P)$, is a moment map for this action. Consequently,

1. the cut space $M_P$ is connected if and only if $\mu^{-1}(P)$ is connected;
2. the fibers of $\mu_P$ are connected if and only if the fibers of $\mu|\mu^{-1}(P)$ are connected;
3. $M_P$ is compact if and only if $\mu^{-1}(P)$ is compact.

### 2.3 Principal orbit type

Consider a connected orbifold $M$, together with an action of a compact connected Lie group $G$. For each $m \in M$, let $\mathfrak{g}_m \subset \mathfrak{g}$ be the corresponding isotropy Lie algebra. Clearly, $\mathfrak{g}_{g \cdot m} = \text{Ad}(g)(\mathfrak{g}_m)$ for all $g \in G$. The set of subalgebras

$$(\mathfrak{g}_m) = \{ \text{Ad}(g)(\mathfrak{g}_m) \mid g \in G \}$$

is called the (infinitesimal) orbit type of $m$. As in the case of manifolds, each point $m \in M$ has a neighborhood $U$ such that the number of orbit types $\mathfrak{g}_{m'}$, for $m' \in U$ is finite and each $\mathfrak{g}_{m'}$ is subconjugate to $\mathfrak{g}_m$.

**Proposition 2.6** There exists a unique orbit type $(\mathfrak{h})$ (called the principal orbit type) such that the set

$$M_{(\mathfrak{h})} = \{ m \in M \mid (\mathfrak{g}_m) = (\mathfrak{h}) \}$$

of points of orbit type $(\mathfrak{h})$ is open, dense and connected.
Proof. The proof is an adaptation of the proof for manifold to orbifolds. The key point is that slices (Definition 3.1) exist for actions of compact groups on orbifolds [LT]. □

**Definition 2.7** For an action of a compact connected Lie group $G$ on a connected orbifold $M$, we define the principal stratum $M_{\text{prin}}$ to be the intersection of the set $M_{(h)}$ of points of principal orbit type with the set $M_{\text{smooth}}$ of smooth points of $M$.

**Remark 2.8** By definition, the set $M_{\text{sing}}$ of singular points of an orbifold $M$ is a union of submanifolds of codimension at least 2. Therefore $M_{\text{smooth}}$ is open, dense and connected, and so is $M_{\text{prin}}$.

**Remark 2.9** Let $M$ be a connected Hamiltonian $G$-orbifold. As in the case of manifolds, the definition of the moment map, equation (1), implies that the image of the differential of the moment map at a point $m$ is the annihilator of the corresponding isotropy Lie algebra $g_m$. Consequently the restriction of the moment map to $M_{\text{prin}}$ has constant rank.

3 The principal cross-section

In this section, we define cross-sections and show that they have the properties needed to reduce the non-abelian case to the abelian case.

**3.1 The cross-section theorem**

Theorem 3.3 is a version of the cross section theorem of Guillemin and Sternberg [GS2, Theorem 26.7] adapted to orbifolds. This version of the theorem and its relationship to fibrations of coadjoint orbits and representation theory are discussed in [GLS]. Since [GLS] is not yet published, we thought that it would be useful to present the proof.

**Definition 3.1** Suppose that a group $G$ acts on an orbifold $M$. Given a point $m$ in $M$ with isotropy group $G_m$, a suborbifold $U \subset M$ containing $m$ is a slice at $m$ if $U$ is $G_m$-invariant, $G \cdot U$ is a neighborhood of $m$, and the map

$$G \times_{G_m} U \to G \cdot U, \quad [a,u] \mapsto a \cdot u$$

is an isomorphism. In other words, $G \cdot y \cap U = G_m \cdot y$ and $G_y \subset G_m$ for all $y \in U$.

**Remark 3.2** Consider the coadjoint action of a compact connected Lie group $G$ on $g^*$. For all $x \in g^*$, there is a unique largest open subset $U_x \subset g^*_x \subset g^*$ which is a slice at $x$. We refer to $U_x$ as the natural slice at $x$ for the coadjoint action. In order to describe the natural slice, we may
assume without loss of generality that \( x \in t_\tau^* \). Let \( \tau \subset t_\tau^* \) be the open wall of \( t_\tau^* \) containing \( x \) and let \( G_\tau \) denote the isotropy Lie group of \( x \) (the group is the same for all points of \( \tau \)). Then

\[
U_\tau = G_\tau \cdot \{ y \in t_\tau^* \mid G_y \subset G_\tau \} = G_\tau \cdot \bigcup_{\tau' \subset \tau} \tau'
\]

is an open subset of \( g_\tau^* \), and is equal to the natural slice \( U_x \).

**Theorem 3.3 (Cross-section)** Let \( (M, \omega) \) be a symplectic orbifold with a moment map \( \Phi : M \to g^* \) arising from an action of a compact Lie group \( G \). Let \( x \) be a point in \( g^* \) and let \( U \) be the natural slice at \( x \) (see above). Then the cross-section \( R := \Phi^{-1}(U) \) is a \( G_x \)-invariant symplectic suborbifold of \( M \), where \( G_x \) is the isotropy group of \( x \). Furthermore the restriction \( \Phi | R \) is a moment map for the action of \( G_x \) on \( R \).

**Proof.** By definition of the slice, coadjoint orbits intersect \( U \) transversally. Since the moment map is equivariant, it is transversal to \( U \) as well. Hence the cross-section \( R = \Phi^{-1}(U) \) is a suborbifold. Since the slice \( U \) is preserved by the action of \( G_x \) and the moment map is equivariant, the cross-section is preserved by \( G_x \). It remains to show that for all \( r \in R \), the uniformized tangent space \( \tilde{T}_r R \) is a symplectic subspace of \( \tilde{T}_r M \). Let \( y = \Phi(r) \), and let \( m \) be the \( G_x \)-invariant complement of \( g_x \) in \( g \). Then \( T_y U \) is the annihilator of \( m \) for any \( y \in U \). We claim that

(a) the tangent space \( \tilde{T}_r R \) is symplectically perpendicular to \( m_M(r) = \{ \xi_M(r) : \xi \in m \} \), the subspace of the tangent space to the orbit through \( r \) spanned by \( m \);

(b) \( m_M(r) \) is a symplectic subspace of \( \tilde{T}_r M \).

Together the two claims establish the theorem because \( \tilde{T}_r M = \tilde{T}_r R \oplus m_M(r) \). To see that (a) holds observe that for \( v \in \tilde{T}_r R \) and \( \xi \in m \),

\[
\omega_r(\xi_M(r), v) = \langle \xi, d\Phi_r(v) \rangle = 0
\]

since \( d\Phi_r(v) \in T_y U = m^0 \) and \( \xi \in m \).

To see that (b) is true, observe that for \( \xi, \eta \in g \),

\[
\omega_r(\xi_M(r), \eta_M(r)) = \langle \xi, d\Phi_r(\eta_M(r)) \rangle = \langle \xi, ad^\dagger(\eta) \cdot \Phi(r) \rangle = \langle [\xi, \eta], y \rangle.
\]

Thus \( m_M(r) \) is symplectic if and only if \( ad^\dagger(m)y \) is a symplectic subspace of the tangent space \( T_y(G \cdot y) \). Since \( G_x \cdot y \subset U \) and since \( m = (T_y U)^0 \), for any \( \xi \in m \) and any \( \zeta \in g_x \) we have

\[
\langle [\xi, \zeta], y \rangle = \langle \xi, ad^\dagger(\zeta)y \rangle = 0,
\]

i.e., \( T_y(G_x \cdot y) \) and \( ad^\dagger(m)y \) are symplectically perpendicular in \( T_y(G \cdot y) \). Since \( T_y(G \cdot y) = T_y(G_x \cdot y) \oplus ad^\dagger(m)y \), it remains to show that the orbit \( G_x \cdot y \) is a symplectic submanifold of the coadjoint orbit \( G \cdot y \). Since the natural projection \( \pi : g^* \to g_x^* \) is \( G_x \)-equivariant, \( \pi(G_x \cdot y) = G_x \cdot \pi(y) \).
By the definition of the symplectic forms on a coadjoint orbit the restriction of the symplectic form on \( G \cdot y \) to \( G_x \cdot y \) is the pull-back by \( \pi \) of the symplectic form on the \( G_x \) coadjoint orbit \( G_x \cdot \pi(y) \).

\[ \square \]

**Remark 3.4** For the theorem to hold for a non-compact Lie group \( G \) one has to assume that a slice \( U \) exists at \( x \) and that the differential of the restriction to \( U \) of the projection \( g^* \to g_x^* \) is surjective.

**Remark 3.5** The cross-section \( R \) need not be a slice for an action of \( G \) on \( M \) since the group \( G_x \) need not appear as an isotropy group of any point in the cross-section. However, the set \( G \cdot R \) of orbits through the cross-section is an open subset of the manifold \( M \) and it is equivariantly diffeomorphic to the associated bundle \( G \times_{G_x} R \) over the coadjoint orbit \( G \cdot x \). This is because \( U \) is a slice at \( x \in g^* \), \( R = \Phi^{-1}(U) \) and \( \Phi \) is equivariant.

**Remark 3.6** Let \( G \) be a compact connected group, and \( M \) a Hamiltonian \( G \)-orbifold, with moment map \( \Phi : M \to g^* \). In various applications in this paper, we will use symplectic cross-sections to reduce statements about \( G \)-orbits in \( M \) to the case that the orbit is contained in the zero level set \( \Phi^{-1}(0) \). The general argument is as follows. Let \( m \in \Phi^{-1}(t^*_+ \Phi) \), and let \( \tau \subset t^*_+ \) be the open wall containing \( x = \Phi(m) \). Let \( U_\tau \subset g^* \) be the natural slice and \( R_\tau \) the corresponding natural cross-section, which is a Hamiltonian \( G_\tau \)-space. Since \( G_\tau \) contains the maximal torus, there is a unique \( G_\tau \)-invariant decomposition

\[ g = z(g_\tau) \oplus [g_\tau, g_\tau] \oplus m_\tau, \]

where \( z(g_\tau) \) is the center of \( g_\tau \), \([g_\tau, g_\tau]\) its semi-simple part and \( m_\tau \) a complement in \( g \). Notice that \( z(g_\tau) \) can be characterized as the fixed point set of the \( G_\tau \)-action on \( g \). It follows that the linear span of \( \tau \) is equal to \( z(g_\tau)^* \). Since \( x = (\Phi|R_\tau)(m) \in \tau \subset z(g_\tau)^* \), one can shift the moment map \( \Phi|R_\tau \) by \( x \) to obtain a new moment map \( \Phi' \) for the \( G_\tau \)-action on \( R_\tau \) for which \( m \in (\Phi')^{-1}(0) \).

### 3.2 The principal wall and the corresponding cross-section

The main result of this subsection is:

**Theorem 3.7** Let \( G \) be a compact connected Lie group, and \( M \) a connected Hamiltonian \( G \)-orbifold, with moment map \( \Phi : M \to g^* \).

1. There exists a unique open wall \( \sigma \) of the Weyl chamber \( t^*_+ \) with the property that \( \Phi(M) \cap \sigma \) is dense in \( \Phi(M) \cap t^*_+ \).

2. The preimage \( Y = \Phi^{-1}(\sigma) \) is a connected symplectic \( T \)-invariant suborbifold of \( M \), and the restriction \( \Phi_Y \) of \( \Phi \) to \( Y \) is a moment map for action of the maximal torus \( T \).
3. The set $G \cdot Y = \{ g \cdot m \mid g \in G, \ m \in Y \}$ is dense in $M$.

We refer to $\sigma$ as the principal wall and to $Y = \Phi^{-1}(\sigma)$ as the principal cross-section. Lemma 3.8 below is used to show the existence of the principal wall.

**Lemma 3.8** Let $G$ be a compact, connected Lie group, and $M$ a connected Hamiltonian $G$-orbifold, with moment map $\Phi : M \rightarrow g^*$.

1. For all $m \in M_{prin}$, the isotropy Lie algebra $g_m$ is an ideal in $g_{\Phi(m)}$, i.e., $[g_m, g_{\Phi(m)}] \subset g_m$.

2. All points in the intersection $\Phi^{-1}(t^*_+) \cap M_{prin}$ have the same isotropy Lie algebra $\mathfrak{h} \subset g$.

3. Given $\alpha \in \Phi(M_{prin}) \cap t^*_+$ let $S$ be the affine subspace $(\alpha + \mathfrak{h}^0) \cap t^*$, where $\mathfrak{h}^0$ is the annihilator of $\mathfrak{h}$ in $g^*$. The intersection $\Phi(M_{prin}) \cap t^*_+$ is a connected, relatively open subset of $S \cap t^*_+$.

**Proof.**

A. Consider first the case when $\Phi(M_{prin})$ intersects $\mathcal{Z}(g)^*$, the fixed point set for the coadjoint action of $G$. Let $m$ be a point in $\Phi^{-1}(\mathcal{Z}(g)^*)$ and let $\mathfrak{h}$ be its isotropy Lie algebra. Since the restriction $\Phi|_{M_{prin}}$ has constant rank by Remark 2.9, there exists a $G$-invariant open neighborhood $U \subset M_{prin}$ of $m$, such that $\Phi(U) \subset g^*$ is a submanifold, and $T_{\Phi(m)}\Phi(U) = d\Phi_m(T_m M) = \mathfrak{h}^0$. Since $\Phi(m)$ is fixed by the coadjoint action, the tangent space $T_{\Phi(m)}\Phi(U)$ is $G$-invariant, which proves that $\mathfrak{h}$ is $G$-invariant. Therefore, the isotropy Lie algebras of all points in the principal stratum $M_{prin}$ are the same. This proves part 2 and 3 of the lemma in the special case, and also part 1 for the case $\Phi(m) = 0$.

B. We now reduce the general case to part A, using symplectic cross-sections. Given $m \in \Phi^{-1}(t^*_+) \cap M_{prin}$, let $\tau$ be the open wall containing $\Phi(m)$, and $R_\tau$ be the corresponding natural cross-section. Denote by $N$ the connected component of $R_\tau$ containing $m$. Then $N$ is a Hamiltonian $G_\tau$-space, with moment map the restriction $\Phi|_N$, and $m \in N_{prin} = N \cap M_{prin}$. Since $\Phi(m) \in \tau \subset \mathcal{Z}(g^*_\tau)^*$, part A shows that $g_m$ is an ideal in $g_{\Phi(m)}$, and that every point in $N_{prin}$ has the same isotropy algebra. In particular, the isotropy algebra of points in $\Phi^{-1}(t^*_+) \cap M_{prin}$ is locally constant. We claim that $\Phi^{-1}(t^*_+) \cap M_{prin}$ is connected. Indeed, consider the natural surjective map

$$\pi : \Phi^{-1}(t^*_+) \cap M_{prin} \rightarrow M_{prin}/G, \ m \mapsto G \cdot m.$$ 

The fiber of $\pi$ over every point $G \cdot m \in M_{prin}/G$ is equal to the intersection $G \cdot m \cap \Phi^{-1}(t^*_+) \cong G_{\Phi(m)} \cdot m$, which is connected since $G_{\Phi(m)}$ is connected. Since $M_{prin}$ is connected, it follows that $M_{prin}/G$ is connected. Thus $\pi$ has connected target space and connected fibers, and it follows that $\Phi^{-1}(t^*_+) \cap M_{prin}$ is connected. This rest of the Lemma is an easy consequence of the fact that the image of $d\Phi_m$ is the annihilator of $g_m$. \hfill $\Box$

**Proof of Theorem 3.7** Let $S \subset t^*$ be the affine subspace described in Lemma 3.8. Let $\sigma \subset t^*_+$ be the lowest dimensional wall such that $S \cap t^*_+ = S \cap \sigma$. Since the moment map is continuous, the
moment set $\Phi(M) \cap t^*_+ \subset \bar{\tau}$ is contained in the closure of $\Phi(M_{\text{prin}}) \cap t^*_+$. By Lemma 3.9, $\Phi(M_{\text{prin}}) \cap t^*_+$ is an open subset of $S \cap t^*_+$. It follows that $\Phi(M_{\text{prin}}) \cap \sigma$ is non-empty, and that the closure of $\sigma$ contains the moment set $\Phi(M) \cap t^*_+$. It is the smallest wall with this property: for any wall $\tau$ with $\Phi(M) \cap t^*_+ \subset \bar{\tau}$ we have $\sigma \subset \tau$. Consequently $\Phi(M) \cap \sigma = \Phi(M) \cap U_\sigma$ where $U_\sigma$ is the natural slice (Remark 3.2). Therefore, by the cross-section theorem,

$$Y := \Phi^{-1}(\sigma) = \Phi^{-1}(U_\sigma)$$

is a symplectic $G_\sigma$-invariant suborbifold of $M$ and $\Phi_Y := \Phi|Y : Y \to \sigma \subset U_\sigma$ is a moment map for the action of $G_\sigma$. Let $\mathfrak{h}$ be the stabilizer algebra of points in $M_{\text{prin}} \cap \Phi^{-1}(t^*_+)$. The linear space spanned by $\sigma$ can be identified with the fixed point set $3(\mathfrak{g}_\sigma)^*$ for the action of $G_\sigma$ on $\mathfrak{g}_\sigma^*$, that is, with the annihilator of the semi-simple part, $[\mathfrak{g}_\sigma, \mathfrak{g}_\sigma]^0 \cap \mathfrak{g}_\sigma^*$. By construction of $\sigma$, we have $\mathfrak{h}^0 \cap \mathfrak{g}_\sigma^* \subset [\mathfrak{g}_\sigma, \mathfrak{g}_\sigma]^0 \cap \mathfrak{g}_\sigma^*$, i.e. $[\mathfrak{g}_\sigma, \mathfrak{g}_\sigma] \subset \mathfrak{h}$. This shows that the action on $Y$ of the semi-simple part of $G_\sigma$ is trivial, and that the moment map $\Phi_Y$ is a moment map for the action of the identity component of the center of $G_\sigma$. That is, the principal cross-section $Y$ is a Hamiltonian torus orbifold.

We have proved all the assertions of the theorem, except for the fact that $G \cdot Y$ is dense in $M$ and that $Y$ is connected. The complement to $G \cdot Y \cap M_{\text{prin}} = \Phi^{-1}(G \cdot \sigma) \subset M_{\text{prin}}$ in $M_{\text{prin}}$ is equal to the union of $\Phi^{-1}(G \cdot \tau) \cap M_{\text{prin}}$ over all $\tau$ such that $\tau \subset \sigma$ and $\tau \neq \sigma$. By Lemma 3.9 below, these are all submanifolds of codimension at least three. It follows that removing these sets from $M_{\text{prin}}$ leaves it dense and connected. Hence $G \cdot Y$ is connected and dense in $M$. Since the quotient map $\mathfrak{g}^* \to t^*_+$ defined by $x \mapsto G \cdot x \cap t^*_+$ is continuous, $\Phi(Y)$ is dense in $\Phi(M) \cap t^*_+$. Since $G \cdot Y = G \times_{G_\sigma} Y$ (Remark 3.5) and since the coadjoint orbit $G/G_\sigma$ is simply connected, it follows that $Y$ is connected.

In the proof we used the following Lemma.

**Lemma 3.9** Let $G$ be a compact, connected Lie group, and $M$ a connected Hamiltonian $G$-orbifold, with moment map $\Phi : M \to \mathfrak{g}^*$. Let $\tau$ be a wall of $t^*_+$ which is different from the principal wall $\sigma$. Then the intersection $\Phi^{-1}(G \cdot \tau) \cap M_{\text{prin}}$ is either empty or is a submanifold of codimension at least 3 in $M_{\text{prin}}$.

**Proof.** Let $R_\tau = \Phi^{-1}(U_\tau)$ be the natural cross-section corresponding to $\tau$. Let $3(\mathfrak{g}_\tau)$ be the Lie algebra of the center of $G_\tau$. Let $N$ be a connected component of $R_\tau \cap M_{\text{prin}}$ such that $\Phi(N) \cap \tau \neq \emptyset$. Let $m \in N$, and $\mathfrak{h} = \mathfrak{g}_m$. By Lemma 3.9, $\Phi|N$ is a submersion onto an open subset of the affine space $\mathfrak{g}_m^* \cap (\Phi(m) + \mathfrak{h}^0)$, which by assumption is not contained in $3(\mathfrak{g}_\tau)^*$. The fixed point set for the $G_\tau$-action on $\mathfrak{g}_m^* \cap (\Phi(m) + \mathfrak{h}^0)$ is the proper affine subspace $3(\mathfrak{g}_\tau)^* \cap (\Phi(m) + \mathfrak{h}^0)$. Since $G_\tau$ is non-abelian, this subspace has codimension at least three. Notice that $3(\mathfrak{g}_\tau)^*$ can be identified with the linear space spanned by $\tau$. Since $\Phi|N$ has constant rank, it follows that $\Phi^{-1}(\tau) \cap M_{\text{prin}}$ has codimension greater or equal than 3 in $R_\tau$. By Remark 3.5 this implies that

$$\Phi^{-1}(G \cdot \tau) \cap M_{\text{prin}} = G \times_{G_\tau} (\Phi^{-1}(\tau) \cap M_{\text{prin}}) \subset G \times_{G_\tau} R_\tau$$

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has codimension at least 3 in $M_{prin}$. 

**Remark 3.10** In the proof of Theorem 3.7, we have shown that the semi-simple part $[g_\sigma, g_\sigma]$ of $g_\sigma$, where $\sigma$ is the principal wall, is contained in the principal stabilizer algebra $h$ for points in $M_{prin} \cap \Phi^{-1}(t_\ast^\vee)$. We also have $h \subset g_\sigma$, by equivariance of the moment map. Therefore, the commutators $[h, h]$ and $[g_\sigma, g_\sigma]$ are equal. It follows that the principal stabilizer algebra uniquely determines the principal wall.

### 4 Hamiltonian torus actions on non-compact orbifolds

As we have seen, the principal cross-section need not be compact even if the original manifold is compact. Thus we need to generalize Theorem 2.1 to include a class of torus actions on non-compact orbifolds with not necessarily proper moment maps. Instead, we require that the moment map $\Phi : M \to t^\ast$ is proper as a map into a convex open set $\sigma \subset t^\ast$, i.e., that $\Phi(M) \subset \sigma$ and that for every compact $K \subset \sigma$ the preimage $\Phi^{-1}(K)$ is compact. This criterion is motivated by the following fact: if a Lie group acts on a symplectic orbifold with a proper moment map, then the induced moment map on the principal cross section is proper as a map into the principal wall. We extend Theorem 2.1 to this case by using symplectic cuts to “compactify” $M$.

First we make an elementary observation.

**Lemma 4.1** Let $(M, \omega)$ be a compact, connected symplectic orbifold, and let $\Phi : M \to t^\ast$ be a moment map for a Hamiltonian torus action on $M$. Assume that $\Phi$ is proper as a map into a convex open set $\sigma \subset t^\ast$. Then for any compact set $K \subset \sigma$ there exists a generic rational polytope $P \subset \sigma$, such that $K$ is contained in the interior of $P$.

**Proof.** For any point $x \in \sigma$ there exists a polytope $P_x \subset \sigma$ with rational vertices which contains $x$ in the interior. The collection $\{\text{int}(P_x) : x \in K\}$ is a cover of $K$. Since $K$ is compact there exists a finite subcover $\text{int}(P_{x_1}), \ldots, \text{int}(P_{x_s})$. Take the convex hull $P$ of the union $P_{x_1} \cup \ldots \cup P_{x_s}$. If $P$ is generic, it is the desired polytope. If it is not, perturb it to be generic. 

**Theorem 4.2** Let $(M, \omega)$ be a connected symplectic orbifold, and let $\Phi : M \to t^\ast$ be a moment map for a Hamiltonian torus action on $M$. If $\Phi$ is proper as a map into a convex open set $\sigma \subset t^\ast$, then

1. the image $\Phi(M)$ is convex,
2. each fiber of $\Phi$ is connected, and
3. if for every compact subset $K$ of $t^\ast$, the list of isotropy algebras for the $T$-action on $\Phi^{-1}(K)$ is finite, then the image $\Phi(M)$ is the intersection of $\sigma$ with a rational locally polyhedral set.
Proof. 1. Consider $m_0, m_1 \in M$. Since $M$ is a connected orbifold, there exists a path $\gamma : [0, 1] \to M$ such that $\gamma(0) = m_0$ and $\gamma(1) = m_1$. Since $\gamma([0, 1])$ is compact, by Lemma 4.1 there exists a generic rational polytope $P \subset \sigma$ such that $\Phi(\gamma(t))$ is in the interior of $P$ for all $t \in [0, 1]$. The points $m_0$ and $m_1$ are contained in the same connected component $N$ of the cut space $M_P$. Let $\Phi_N : N \to t^*$ be the induced moment map. Since $\Phi_N(N)$ is a convex polytope by Theorem 2.1, it contains the line segment joining $\Phi(m_0)$ to $\Phi(m_1)$. Since $\Phi_N(N) \subset \Phi(M)$, this proves that $\Phi(M)$ is convex.

2. A similar argument shows that if $\Phi(m_0) = \Phi(m_1) = x$, then $m_0$ and $m_1$ are contained in the same connected component of $\Phi^{-1}(x)$, since the fibers of $\Phi_N$ are connected by Theorem 2.1 and since for points $x$ in the interior of $P$ the fibers of $\Phi$ and of $\Phi_P : M_P \to t^*$ are the same (cf. Proposition 2.3). Therefore the fibers of $\Phi$ are connected.

3. For each $x \in \Phi(M)$, let

$$C_x = \{x + t(y - x) | y \in \Phi(M) \text{ and } t \geq 0\}$$

be the cone over $\Phi(M)$ with vertex $x$. Define $A$ to be the intersection of all the cones $C_x$ for $x \in \Phi(M)$, $A := \cap_{x \in \Phi(M)} C_x$. Since $\Phi(M)$ is convex and is relatively closed in $\sigma$, Lemma 4.3 below (applied to the closure $X$ of $\Phi(M)$ and to $S = \sigma$) shows that $\Phi(M) = A \cap \sigma$.

We claim that $A$ is the desired rational locally polyhedral set. First we show that all cones $C_x$ are rational polyhedral cones. Let $x \in \Phi(M)$, and let $P \subset \sigma$ be a generic polytope containing $x$ in its interior. Since $\Phi(M)$ is convex, $C_x$ is also the cone over $\Phi_P(M_P)$ with vertex $x$. In particular, $C_x$ is a rational polyhedral cone, and the tangent space of each facet is of the form $t_i^\sigma$, where $t_i$ is an isotropy Lie algebra for the action of the torus $T$ in a neighborhood of $\Phi^{-1}(x)$.

Suppose next that $K \subset t^*$ is a compact convex subset with non-empty interior. Since the number of isotropy algebras for the $T$-action on $\Phi^{-1}(K)$ is finite, it follows that up to translation, the list of cones $C_x$ for $x \in K \cap \Phi(M)$ is finite. Moreover, if $C_x$ is a translation of $C_y$, then $C_x = C_y$ since by convexity of $\Phi(M)$ the line segment $\overline{xy}$ is contained in both $C_x$ and $C_y$. It follows that the collection of cones $C_x, x \in K \cap \Phi(M)$ itself is a finite list $C_1, \ldots, C_N$. This shows that $A$ is a rational locally polyhedral set.

Lemma 4.3 Let $V$ be a vector space, and $X, S \subset V$ convex subsets with $X$ closed. For every $x \in X$, let $C_x = \{x + t(y - x) | y \in X, t \geq 0\}$. Then

$$X \cap S = (\bigcap_{x \in X \cap S} C_x) \cap S.$$

Proof. The inclusion “$\subset$” is obvious. To prove the opposite inclusion, assume that $(X \cap S)$ and $S - (X \cap S)$ are non-empty (since otherwise there is nothing to prove). Let $y \in S - (X \cap S)$. We have to show that $y \notin \bigcap_{x \in X \cap S} C_x$. Let $r$ be a ray with vertex $y$ which intersects $X \cap S$
nontrivially. Since $X$ is closed and convex $r \cap X$ is either a closed ray or a closed line segment. Let $x$ be the point in $r \cap X$ closest to $y$. Then $y$ does not lie in $C_x$. Since $S$ is convex, $x$ is in $X \cap S$. □

In the last section, we will use the following corollary to Theorem 4.2.

**Corollary 4.4** Let $(M, \omega)$ be a connected symplectic orbifold, and let $\Phi : M \to t^*$ be the moment map for a Hamiltonian torus action on $M$. If $\Phi$ is proper as a map into a convex open set $\sigma$, then for every $\xi \in t$, every local minimum of the function $\Phi^\xi$ is a global minimum, where $\Phi^\xi(m) := (\Phi(m), \xi)$.

**Proof.** Since the moment set $\Phi(M)$ is convex, its intersection with the affine hyperplanes $\{x \in t^* | \xi(x) = a\}$ is connected for all $a \in \mathbb{R}$. Since the fibers of $\Phi$ are connected this implies that

$$(\Phi^\xi)^{-1}(a) = \Phi^{-1}(\{x \in t^* | \xi(x) = a\})$$

is connected for all $a$. The result follows. □

## 5 Proof of non-abelian convexity

By now, we have done all the necessary hard work. With a little point set topology, we bring together the results in the previous two sections to prove our main theorem.

**Theorem 1.1** Let a compact, connected Lie group $G$ act on a connected symplectic orbifold $(M, \omega)$ with a proper moment map $\Phi : M \to g^*$.

1. Let $t^*_+ \subset G$ be a closed Weyl chamber for $G$. The moment set $\Phi(M) \cap t^*_+$ is a convex rational locally polyhedral set. In particular, if $M$ is compact then the moment set is a convex rational polytope.

2. Each fiber of the moment map $\Phi$ is connected.

**Proof.** Let $\sigma$ be the principal wall. By Theorem 3.7, the principal cross section $Y := \Phi^{-1}(\sigma)$ is a connected symplectic orbifold, and the restriction of $\Phi$ to $Y$ is a moment map for the action of the maximal torus of $Y$. Since $\Phi : M \to g^*$ is proper, the restriction $\Phi|Y : Y \to t^*$ is proper as a map into the open convex set $\sigma$. Therefore, by Theorem 1.2, the image $\Phi(Y)$ is a convex set and is the intersection of $\sigma$ with a locally polyhedral set $P$, that is, $\Phi(Y) = \sigma \cap P$. By Theorem 3.7, $\Phi(M) \cap t^*_+ = \Phi(Y)$. Since the closure of a convex set is convex, the moment set $\Phi(M) \cap t^*_+$ is convex. Since the closure of $\sigma$ is a polyhedral cone and since the intersection of the interior of the locally polyhedral set $P$ with $\sigma$ is nonempty, the closure of the intersection $\sigma \cap P$ is the intersection $\sigma \cap P$. Therefore $\Phi(M) \cap t^*_+ = \sigma \cap P = \sigma \cap P$ is a locally polyhedral set. Since both $P$ and $\sigma$ are rational, the moment set is rational.

It remains to prove that the fiber $\Phi^{-1}(x)$ is connected for all $x \in g^*$. Since the fibers of $\Phi_Y = \Phi|Y$ are connected, we know that the fibers of the restriction $\Phi|G \cdot Y$ are connected. Since
$G \cdot Y$ is dense in $M$, we would like to conclude that all fibers of $\Phi$ are connected. However, this does not immediately follow.\footnote{Consider the map $f$ from $S^2 \subset \mathbb{R}^3$ to $S^1 \subset \mathbb{C}$ given by $f(x_1, x_2, x_3) = e^{ix_3}$. The map $f$ is proper and all fibers are connected, except for the fiber over $z = -1$.}

First, observe that $\Phi^{-1}(x)$ is connected for all $x \in g^*$ if and only if $\Phi^{-1}(G \cdot x)$ is connected for all $G \cdot x \in g^*/G$: ($\Rightarrow$) is true since $\Phi^{-1}(G \cdot x) = G \cdot \Phi^{-1}(x)$ and the group $G$ is connected; ($\Leftarrow$) is true since $\Phi^{-1}(G \cdot x)/G = \Phi^{-1}(x)/G_x$ and $G_x$ is connected.

Now, we'll show that the preimage-s of orbits $\Phi^{-1}(G \cdot x)$ are connected. For a point $x \in \Phi(M) \cap t^*_+$, let $B$ be an open ball (with respect to a Weyl group invariant metric) in $t^*$ centered at $x$. Then $G \cdot (B \cap t^*_+)$ is an open set containing the orbit $G \cdot x$. It enough to show that the preimage $\Phi^{-1}(G \cdot (B \cap t^*_+))$ is connected (see Lemma\ref{lem:connected}). The intersection $\Phi^{-1}(G \cdot (B \cap t^*_+)) \cap G \cdot Y$ is dense in $\Phi^{-1}(G \cdot (B \cap t^*_+))$ and $\Phi^{-1}(G \cdot (B \cap t^*_+)) \cap G \cdot Y = G \cdot \Phi^{-1}(B \cap \sigma)$, where $\sigma$ is the principal wall.

Since $B \cap \sigma$ is connected and $\Phi^{-1}(y)$ is connected for any $y \in \sigma$, the set $G \cdot \Phi^{-1}(B \cap \sigma)$ is connected. Since $G \cdot \Phi^{-1}(B \cap \sigma)$ is dense in $\Phi^{-1}(G \cdot (B \cap t^*_+))$, the set $\Phi^{-1}(G \cdot (B \cap t^*_+))$ is connected. This proves that the fibers of the moment map are connected. \hfill \Box

In this proof, we used the following topological fact:

\begin{lemma}
Let $X$ be a metric space, $f : X \to \mathbb{R}^n$ a proper continuous map, $\{U_i\}_{i=1}^\infty$ a sequence of bounded open sets in $\mathbb{R}^n$ with $U_{i+1} \subset U_i$, $f^{-1}(U_i)$ connected and $C = \bigcap U_i$ nonempty. Then $f^{-1}(C)$ is also connected.
\end{lemma}

\begin{proof}
Suppose not. Then there are open sets $V$ and $W$ with $V \cap W = \emptyset$ and $f^{-1}(C) \subset V \cup W$. Since $f^{-1}(U_i)$ is connected there exists $x_i \in f^{-1}(U_i)$ with $x_i \notin V \cup W$. Since the $U_i$'s are bounded, the sequence $f(x_i)$ has a convergent subsequence and its limit $y$ lies in $C$. Since $f$ is proper we may assume, by passing to subsequences, that $f(x_i) \to y$ and that $x_i \to x$ for some $x \in f^{-1}(y) \subset f^{-1}(C) \subset V \cup W$. This contradicts the construction of the sequence $\{x_i\}$. \hfill \Box
\end{proof}

\begin{remark}
The properness condition on the moment map may be relaxed to require only that there is an invariant open set $V$ of $g^*$ with $\Phi(M) \subset V$, $\Phi : M \to V$ is proper and and $V \cap t^*_+$ is convex.
\end{remark}

\section{Local moment cones}

A result of Sjamaar \cite{Sjamaar} says that the moment set near a point $x \in \Delta$ can be read off from local data near an arbitrary point in the fiber $\Phi^{-1}(x)$. Later Yael Karshon provided a different proof of Sjamaar’s theorem \cite{Karshon}. In this section, we extend Sjamaar’s results to the orbifold setting, using symplectic cuts. Our proof is short and has the advantage that it works in the orbifold setting.
**Theorem 6.1** Let $\Phi : M \to \mathfrak{g}^*$ be a proper moment map for an action of a compact connected Lie group $G$ on a connected symplectic orbifold $(M, \omega)$. For every point $m \in M$ and every $G$-invariant neighborhood $U$ of $m$ in $M$, there exists a $G$-invariant neighborhood $\mathcal{V}$ of $\Phi(m)$ in $\mathfrak{g}^*$ such that

$$\mathcal{V} \cap \Phi(U) \cap t_+^* = \mathcal{V} \cap \Phi(M) \cap t_+^*.$$  

We will deduce Theorem 6.1 from the following result, which does not require properness of the moment map.

**Theorem 6.2** Let $\Phi : M \to \mathfrak{g}^*$ be a moment map for an action of a compact connected Lie group $G$ on a symplectic orbifold $(M, \omega)$. For every point $m \in \mathfrak{g}^*$, there exists a rational polyhedral cone $C_m \subset t_+^*$ with vertex at $\Phi(m)$, such that for every sufficiently small $G$-invariant neighborhood $U$ of $m$, the image $\Phi(U) \cap t_+^*$ is a neighborhood of the vertex of $C_m$. The cone $C_m$ is called the local moment cone for $m$.

We first prove Theorem 6.1, using Theorem 6.2.

**Proof.** It suffices to check for arbitrarily small $G$-invariant neighborhoods. By Theorem 6.2, there exists an open neighborhood $V$ of $\Phi(M)$ and a rational polyhedral cone $C_m \subset t_+^*$ with vertex at $\Phi(m)$, such that for every sufficiently small $G$-invariant neighborhood $U$ of $m$, the image $\Phi(U) \cap t_+^*$ is a neighborhood of the vertex of $C_m$. The cone $C_m$ is called the local moment cone for $m$.

To prove Theorem 6.2, we will use the orbifold version of the local normal form theorem due to Marle [Ma] and, independently, to Guillemin and Sternberg [GS]. The theorem asserts that an invariant neighborhood of a point $m$ in a Hamiltonian $G$-manifold $M$ is completely determined (up to equivariant symplectomorphism) by two pieces of data: (1) the value of the moment map $\Phi$ at $m$ and (2) the symplectic slice representation of the isotropy group $G_m$ of the point. Recall that the symplectic slice at a point $m$ is the largest symplectic subspace in the fiber at $m$ of the normal bundle to the orbit $G \cdot m$. The analogous result holds for Hamiltonian $G$-orbifolds (cf. [LT]) — the only difference is that the symplectic slice is no longer a vector space. Instead it is the quotient of a vector space by a linear action of a finite group.

**Lemma 6.3** Let $G$ be a compact Lie group, let $(M, \omega)$ be a Hamiltonian $G$-orbifold, with moment map $\Phi$, and $m \in \Phi^{-1}(0)$ a point in the zero level set. Let $\Gamma$ be the orbifold structure group of $m$ and
$G_m \times V/\Gamma \to V/\Gamma$ be the symplectic slice representation at $m$. For every $G_m$-equivariant splitting $\mathfrak{g}^* \cong \mathfrak{g}^*_m \oplus \mathfrak{g}^*_n$, there is a $G$-invariant symplectic form on the orbifold $F = G \times_{G_m} (\mathfrak{g}^*_m \times V/\Gamma)$ such that the moment map $\Phi_F : F \to \mathfrak{g}^*$ for the $G$-action is given by

$$\Phi_F([g, \eta, v]) = g \cdot (\eta + \phi_{V/\Gamma}(v)),$$

where $\phi_{V/\Gamma} : V/\Gamma \to \mathfrak{g}^*_m$ is the moment map for the slice representation of $G_m$. Moreover, the embedding of the orbit $G \cdot m$ into $F$ as the zero section is isotropic and the symplectic slice at $[1, 0, 0]$ is $V/\Gamma$. Consequently there exists a $G$-equivariant symplectomorphism $\lambda$ from a neighborhood $U$ of $G \cdot m$ in $M$ to a neighborhood $U'$ of the zero section in $F$, and $\lambda^* \Phi_F = \Phi$ over $U$.

We are now ready to prove Theorem 6.2.

**Proof.** 1. We begin by considering the special case $\Phi(m) = 0$. Because we are considering arbitrarily small neighborhoods, by Lemma 6.3 it suffices to consider neighborhoods of the zero section of the model space $F$. Let $V/\Gamma$ be the symplectic slice at $m$. Choose a $G_m$-invariant complex structure on $V/\Gamma$ which is compatible with the symplectic form; let $\rho$ denote the norm squared of the induced metric. The model $F$ is a complex orbi-bundle over $G \times_{G_m} \mathfrak{g}^*_m$. The multiplication action of $U(1)$ on the fibers of $F \to G \times_{G_m} \mathfrak{g}^*_m$ is a Hamiltonian action on the orbifold $F$, with moment map being the function $\rho$ defined above. The action of $U(1)$ commutes with the action of $G$. Consequently the moment map $\hat{\Phi}_F : F \to \mathbb{R} \times \mathfrak{g}^*$ for the action of $U(1) \times G$ on $F$ is given by

$$\hat{\Phi}_F([g, \eta, v]) = \left(\rho([g, \eta, v]), g \cdot (\eta + \phi_{V/\Gamma}(v))\right).$$

Since $\hat{\Phi}_F$ is proper, we can apply Theorem 1.1. Therefore, $\hat{\Phi}_F(F) \cap (\mathbb{R} \times t^*_m)$ is a convex rational locally polyhedral set. In fact, since $\hat{\Phi}_F$ is homogeneous (i.e., equivariant with respect to the action of $\mathbb{R}_+$ which on $F$ is given by $t \cdot [g, \eta, v] = [g, t\eta, \sqrt{t}v]$ and on $\mathfrak{g}^*$ by multiplication) and since the number of orbit types in $F$ is finite, $\hat{\Phi}_F(F) \cap (\mathbb{R} \times t^*_m)$ is a convex rational polyhedral cone. Since $\Phi_F(F) \cap t^*_m$ is the image of $\hat{\Phi}_F(F) \cap (\mathbb{R} \times t^*_m)$ under $\pi : \mathbb{R} \times t^*_m \to t^*_m$, it is also a convex rational polyhedral cone.

2. Choose a $G$-invariant metric on $\mathfrak{g}^*$, and let $\hat{\rho} : F \to \mathbb{R}$ be defined by $\hat{\rho}([g, \eta, v]) = \rho(v) + ||\eta||$. Then $\hat{\rho}$ is homogeneous and $\hat{\rho}^{-1}(0)$ is the zero section of $F$. Choose $\epsilon > 0$ sufficiently small so that the set $\{x \in F \mid \hat{\rho}(x) < \epsilon\}$ is contained in $U$. Let $\lambda$ be the lower bound on $\hat{\rho}^{-1}(\epsilon)$ of the function

$$f : F \to \mathbb{R}, \quad f(x) = \rho(x) + ||\Phi_F(x)||$$

on $F$. Since $f \geq \hat{\rho}$, we have $\lambda \geq \epsilon$. Let $\hat{\mathcal{V}} := \{(t, \alpha) \in \mathbb{R} \times \mathfrak{g}^* \mid |t| + ||\alpha|| < \lambda\}$. Since $||\hat{\Phi}_F(x)|| \geq \lambda$ if $\hat{\rho}(x) \geq \epsilon$, one has $\hat{\Phi}_F(U) \supset \hat{\Phi}_F(F) \cap \hat{\mathcal{V}}$, which is a $G$-invariant neighborhood of $0 \in \hat{\Phi}_F(F)$. Since $\hat{\Phi}_F(F) \cap (\mathbb{R} \times t^*_m)$ is a convex rational polyhedral cone, the image of every neighborhood of the vertex of this cone under the projection $\pi : \mathbb{R} \times \mathfrak{g}^* \to \mathfrak{g}^*$ is a neighborhood of the vertex of $\Phi_F(F) \cap t^*_m = \pi(\hat{\Phi}_F(F) \cap \mathbb{R} \times t^*_m)$.

It follows that $G \cdot \pi(\hat{\mathcal{V}} \cap (\mathbb{R} \times t^*_m))$ is a $G$-invariant neighborhood of $0$ in $\mathfrak{g}^*$ with the required property.
3. Finally, we use cross sections to reduce the general case to the case that \( \Phi(m) = 0 \). Let \( \tau \) be the (open) wall containing \( \Phi(m) \). Let \( U_\tau \) be the corresponding natural slice and \( R_\tau = \Phi^{-1}(U_\tau) \) the natural cross-section. By equivariance of the moment map, we have

\[
\Phi(G \cdot R_\tau) \cap t^*_\tau = \Phi(R_\tau) \cap t^*_\tau.
\]

The Cartan subalgebras of \( G \) and \( G_\tau \) are equal, but a Weyl chamber \((t^*_\tau)_+\) of \( G_\tau \) is the union of certain Weyl chambers of \( G \). Nonetheless, \( \Phi(R_\tau) \cap (t^*_\tau)_+ = \Phi(R_\tau) \cap t^*_\tau \), which means that we can apply Remark 3.6 to reduce to the case \( \Phi(m) = 0 \).

\[\square\]

**Remark 6.4** A corollary to the above Theorems (or rather their proofs) is that if \( M \) is a Hamiltonian \( G \)-orbifold, with moment map \( \Phi \), the local moment cone \( C_m \) for a point \( m \in \Phi^{-1}(t^*_\tau) \) does not vary as \( m \) varies in a connected component of a fixed fiber of \( \Phi \). Indeed, if \( M \) is connected and \( \Phi \) is proper this follows from Theorem 6.1 and Theorem 6.2, because the local moment cone is equal to the cone with vertex at \( \Phi(m) \) over the moment set \( \Delta \). For the general case, we note that as above, we can use cross-sections to reduce to the case \( \Phi(m) = 0 \), and since the statement is local, it is enough to check it for the local normal form, \( F \). Now the moment map \( \Phi_F \) is not proper. However, we may replace \( F \) by its cut \( F_{[-\infty,\epsilon]} \) with respect to the \( S^1 \)-action generated by \( \rho \). Since the moment map for the cut space is proper, the claim follows.

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