Reduced Density Matrix of Identical Particles from Three Aspects: the First Quantization, Exterior Products, and GNS representation

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We probe the theoretical connection among three different approaches to analyze the entanglement of identical particles, i.e., the first quantization language (1QL), exterior and elementary-symmetric products (which has the mathematical equivalence to no-labeling approaches), and the algebraic approach based on the GNS construction. Among several methods to quantify entanglement of identical particles, we focus on the computation of reduced density matrix, which can be achieved by the concept of symmetrized partial trace defined in 1QL. We show that the symmetrized partial trace corresponds to the interior product between exterior (or elementary symmetric) state vectors, which also corresponds to the subalgebra restriction in the algebraic approach based on GNS representation. By expressing the state of identical particles as exterior and elementary symmetric state vectors, we can also prove that GHJW theorem holds for identical particles.

Introduction.— Quantum entanglement is one of the crucial quantum concepts which reveals the essential feature of quantum physics. It implies the possibility of composite systems that cannot be described as a simple collection of individual subsystems, even when the subsystems locate far from each other.\textsuperscript{1,}\textsuperscript{2} It is also exploited as a crucial resource to enable several tasks with quantum speedup.\textsuperscript{3} One of the insightful approaches to analyze the entanglement of a given quantum system is to use the partial trace technic. Intuitively, if a multipartite quantum system is entangled, a subsystem of the quantum system has some nonlocal (measurement-dependent) correlation with the “outer world” in the total system. The information on the correlation is encoded in the reduced density matrix, a quantum state acquired by partial tracing the outer part of the total state.

For the case of non-identical particles, in which each particle resides in a distinguished Hilbert space, the concept of the partial trace is well-defined. On the other hand, in the case of identical particles (both bosons and fermions), individual particles do not reside in independent Hilbert spaces, by which the concept of partial trace seems inappropriate to obtain reduced density matrices suitable for analyzing the entanglement of identical particles. Several alternatives have been suggested to overcome this problem.

The algebraic approach (AA)\textsuperscript{4,}\textsuperscript{5} is based on Gel’fand-Naimark-Segal construction, in which state vectors and Hilbert spaces are emergent concepts. They demonstrated that, instead of partial trace, a restriction to a chosen subalgebra of observables provides a sound tool to compute the entanglement entropy (the importance of subalgebras for interpreting the entanglement of identical particles in the second quantization language (2QL) is also pointed out in Ref.\textsuperscript{4,5}). On the other hand, a non-standard approach (NSA)\textsuperscript{6,}\textsuperscript{7} describes the identical particles without introducing particle pseudo-labels. By extracting the transition relations of wave functions from the symmetric properties of identical particles, partial trace can be defined in the no-labeling formalism (the same process was reproduced in the second quantization language in Ref.\textsuperscript{12}). Also, the concept of symmetrized partial trace for identical particles in the first quantization language (1QL) was introduced in Ref.\textsuperscript{13}. By applying the symmetrization principle for identical particles not only to wavefunctions but also to detectors, the reduced density matrix that preserves the particle label symmetry can be derived (for more discussions on the entanglement of identical particles from various viewpoints, see Refs.\textsuperscript{14,29}).

Hence, one can state that there exists a seemingly incompatible standpoint on the validity of the partial trace in the problem of entanglement of identical particles. However, considering that all the above theories deal with the same physical systems, one can surmise that this seeming inconsistency must be reconciled by examining the resultant quantities or the formal structures. In this work, we show how the aforementioned different approaches can translate and provide fundamentally equivalent results to each other.

In the course of delving into the issue, we utilize the exterior product (for fermions) and elementary symmetric product (for bosons) to demonstrate the indistinguishability and projection rules for identical particles. It will be shown that the symmetrized partial trace of identical particles is equivalent to the interior products between exterior (for fermions) and elementary symmetric (for bosons) state vectors, which is equivalent to the restriction to a subalgebra on GNS representation. Our discussion in the main text will focus on the fermionic case that corresponds to the exterior product. The definition of elementary symmetric products and its application to the bosonic entanglement case will be discussed in Supplemental Materials.
Symmetrized partial trace of fermions in 1QL.— In 1QL, the wavefunction of a particle with a pseudo-lable A (which is unphysical, i.e., unable to be detected in principle) with a physical state Ψ (which has the information on the location and internal state of the particle) is described by |Ψ⟩A. Then, since identical fermions are anti-symmetric under particle exchanges, an N-fermion total pure state |Ψ⟩f is expressed in 1QL with

\[ |Ψ⟩f = \sum_{a} \left| Ψ_{1}^{a}, \ldots, Ψ_{N}^{a} \right⟩^{f} \]  

where

\[ \left| Ψ_{1}^{a}, \ldots, Ψ_{N}^{a} \right⟩^{f} = \frac{1}{\sqrt{N!}} \sum_{σ \in S_{N}} sgn(σ) \left| Ψ_{1}^{σ(1)} \right⟩ \left| Ψ_{2}^{σ(2)} \right⟩ \cdots \left| Ψ_{N}^{σ(N)} \right⟩ \]  

(1)

The summation is over all the possible permutations σ (∈ SN), and sgn(σ) denotes the sign of each permutation σ. The pure state of a subsystem that contain n (≤ N) identical particles with physical states (Ψ1, ..., Ψn) is then written as

\[ |Ψ_{1}, \ldots, Ψ_{n}⟩^{f} = \frac{1}{\sqrt{n!}} \sum_{a_{1}, \ldots, a_{n} = 1}^{N} e^{iθ_{a_{1} \cdots a_{n}}} \] 

× \left( \sum_{σ \in S_{n}} sgn(σ) \left| Ψ_{1}^{σ(1)} \right⟩ \cdots \left| Ψ_{n}^{σ(σ)} \right⟩ \right) \]  

(2)

Note that θa1...an for each {a1, ..., an} does not affect the anti-symmetric property of the subsystem state, hence can be chosen arbitrarily.

The symmetrized partial trace of a given N-fermion state |Ψ⟩f over a subsystem with n-fermions can be defined with Eq. (3). If a subsystem S contains n particles, the identity matrix for the subsytem can be written as

\[ I_{S} = \sum_{q} \left| Ψ_{1}^{q}, \ldots, Ψ_{n}^{q} \right⟩ \left( \left| Ψ_{1}^{q}, \ldots, Ψ_{n}^{q} \right⟩ \right) ^{q} \]  

is the complete basis set for the subsystem). Then the reduced density matrix ρS for the complementary subsystem S is obtained by the symmetrized partial trace as

\[ ρ_{S} = TR_{S} |Ψ⟩⟨Ψ|^{f} = \sum_{q} \left( Ψ_{1}^{q}, \ldots, Ψ_{n}^{q} \right⟩ \left| Ψ_{1}^{q}, \ldots, Ψ_{n}^{q} \right⟩^{f} \]  

(3)

\[ \left( TR_{S} \text{denotes the symmetrized partial trace over S}. \right) \]  

An (N, n) = (3, 1) example is given in Supplemental Materials.

Symmetrized partial trace as the interior products of exterior state vectors.— From the anti-symmetric property of fermions, one can express fermionic states in the language of exterior products. Let \{e1, ..., eN\} be a basis set of a N-dimensional vector space V and \{e1, ..., eN\} its dual, so that the inner product among them is given by \langle e_{a}, e_{b} \rangle = δ_{a}^{b}. Defining |Ψ⟩₁ = \sum_{n=1}^{N} |Ψ⟩_{A_{n}} e_{n}, which is a vector in V that is invariant under the exchange of pseudolables, one can see that a N-particle state |Ψ₁, ..., Ψₙ⟩ (see Eq. (2)) can be rewritten as

\[ |Ψ₁, ..., Ψₙ⟩^{f} = N_{N} \sum_{k_{1} < \cdots < k_{N}} e^{-iθ_{k_{1} \cdots k_{N}}} |Ψ₁⟩ \wedge \cdots \wedge |Ψₙ⟩, \]  

(4)

where ∧ denotes the exterior product and (v, w) is the inner product between the given k-vectors v and w (see, e.g., Ref. [23]), by which |Ψ₁⟩ ∧ \cdots ∧ |Ψₙ⟩ is projected as an anti-symmetric N-vector. Nₚ is the normalization factor. Likewise, |Ψ₁, ..., Ψₙ⟩ is rewritten as

\[ |Ψ₁, ..., Ψₙ⟩^{f} = N_{(N,n)} \sum_{k_{1} < \cdots < k_{N}} e^{-iθ_{k_{1} \cdots k_{N}}} |Ψ₁⟩ \wedge \cdots \wedge |Ψₙ⟩, \]  

(5)

where N_{(N,n)} is the normalization factor. Note that the phase ambiguity of Eq. (5) can be restricted to the complex coefficients of the n-dimensional basis vectors e_{A_{k_{1}} \cdots A_{k_{n}}} in Eq. (4). Considering the phase ambiguity is a purely mathematical feature that has no physical implication, one can state that the exterior state vectors |Ψ₁⟩ ∧ \cdots ∧ |Ψₙ⟩ (1 ≤ n ≤ N) have the complete physical information on the total and sub-states for a set of identical fermions. Hence, any subset of a given N identical fermions can be exactly represented as an exterior product of single particle vectors |Ψ₁⟩ that is invariant under pseudolable exchanges. From now on, we call such exterior vectors “exterior state vectors.”

The connection of fermionic states to the exterior state vectors provides intriguing insight on our physical system. Since an exterior state vector |Ψ₁⟩ ∧ \cdots ∧ |Ψₙ⟩ (1 ≤ iₐ ≤ N, ∀ a = 1, ..., n) is in the nth exterior power \bigwedge^{n}(V) (1 ≤ n ≤ N), all the possible subsets of identical fermions construct the complete set of the graded structure, i.e., \{ |Ψ₁⟩ ∧ \cdots ∧ |Ψₙ⟩ \}_{n=1}^{N} \cong \bigoplus_{n=1}^{N} \bigwedge^{n}(V). Moreover, considering the bra states (Ψ₁, ..., Ψₙ) \equiv (Ψ₁, ..., Ψₙ)\dagger are expressed likewise with the exterior products of |Ψ⟩ = \sum_{k} |Ψ⟩^{A_{k}} e_{A_{k}} one can see that the projection of |Ψ₁, ..., Ψₙ⟩ onto |Ψ₁, ..., Ψₙ⟩ (1 ≤ n ≤ m ≤ N) is replaced with the interior products of \langle Ψ₁, ..., Ψₙ⟩ \rangle \equiv (Ψ₁, ..., Ψₙ)\dagger are given by

\[ \langle Ψ₁, ..., Ψₙ⟩ \rangle \equiv \sum_{j} (-1)^{j-1} \langle Ψ₁, ..., Ψₙ⟩ \rangle |Ψ_j⟩ \wedge \cdots \wedge (|Ψ_j⟩ \rangle) \wedge |Ψ_m⟩ \]  

(7)
(here (|Ψ⟩⟩) means that Ψj is absent in the exterior state vector). For n = 2, we have

\[
\langle \hat{\Phi}_1 | \wedge \langle \hat{\Phi}_2 | \cdot |\Psi_1\rangle \wedge \cdots \wedge |\Psi_m\rangle \\
= \langle \hat{\Phi}_1 | \cdot (\langle \hat{\Phi}_2 | \cdot |\Psi_1\rangle \wedge \cdots \wedge |\Psi_m\rangle ) \\
= \sum_{j} (-1)^{j-1} \langle \hat{\Phi}_2 | \cdot |\Psi_1\rangle \langle \hat{\Phi}_1 | \cdot (|\Psi_j\rangle \wedge \cdots \wedge (|\Psi_j\rangle) \wedge |\Psi_m\rangle ) \\
= \sum_{j,k} (-1)^{j-k} \langle \hat{\Phi}_1 | \Psi_j \rangle \langle \hat{\Phi}_2 | \Psi_k \rangle \\
\times \left[ |\Psi_1\rangle \wedge \cdots \wedge (|\Psi_j\rangle) \wedge \cdots \wedge (|\Psi_j\rangle) \wedge \cdots \wedge |\Psi_m\rangle \right].
\]

(8)

The same operation can be applied to an arbitrary n. Then from the projected vectors such as Eqs. (7) and (5) we obtain the reduced density matrix in a given subsystem (an (N, n) = (3, 1) example is given in Supplemental Materials). The algebraic relation of states defined in NSA [11] is equal to the interior product Eq. (7), which means that the physical states in NSA are operationally equivalent to the exterior symmetric vectors in \( \Lambda^N(V) \).

Note that the role of the tensor product \( \otimes \) of non-identical particles is replaced with the exterior product \( \wedge \) of identical fermions. The identical particle version of GHJW theorem [33, 34] (a pedagogic explanation on the theorem for distinguishable particles is given, e.g., in Ref. [35]) is also possible with this replacement.

**Theorem 1. (GHJW theorem for identical particles)**

Suppose two fermions locate in two subsystems L and R with internal states \( a = 0, \ldots, s - 1 \). The total state of the fermions \( |\Psi\rangle \) is a vector in \( \Lambda^{\otimes 2} \mathcal{H} \) with \( \rho_L = \mathcal{T} \mathcal{R}_L |\Psi\rangle \langle \Psi| \). For any convex summation form of \( \rho_L = \sum_a f_a |(L, a)\rangle \langle (L, a)| \) \( (f_a \geq 0, \sum_a f_a = 1) \), there exists an orthonormal set \( \{(R, a)\}\) of the subsystem R such that

\[
|\Psi\rangle = \sum_a \sqrt{f_a} |(L, a), (R, a)\rangle.
\]

(9)

The proof is in Supplemental Materials. An extension of the theorem to the arbitrary (N, n) case is also possible.

One can apply what we have discussed so far to the bosonic case by introducing the concept of elementary symmetric products. For the definition of elementary symmetric products and their relation to the bosonic wave functions, see Supplemental Materials.

**Dictionary.** — As we have discussed so far, the identical fermions (bosons) are described with the formalism of exterior products (elementary symmetric products). The relation of the physical systems of identical particles with tensor algebras is summarized in the following table.

| a set of identical particles | tensor algebra |
|-----------------------------|----------------|
| fermionic state             | exterior vector |
| (bosonic state)             | (elementary symm. vector) |
| symmetric partial trace     | interior product |
| subsystems                  | graded structure |

The tensor product \( \otimes \) for the entanglement of non-identical particles is replaced with the exterior (elementary symmetric) product \( \wedge \) for fermions (bosons). On the other hand, the entanglement of identical particles is a detector dependent quantity, which is determined by the spatial relation of particle wavefunctions to orthogonal detectors (which can be interpreted as coherence [12]). Hence, the separability of a given state is not completely determined by the mathematical structure of the wavefunction itself. More concretely, if a state of N identical particles is written as a single state vector in the detector basis, it is separable. But the inverse is not true.

**Symmetrized partial trace as the subalgebra restriction in GNS construction.** — Any state of identical particles has intrinsic correlations among single-particle spaces. In other words, for a single particle Hilbert space \( \mathcal{H}^{(1)} \), the anti-symmetrization (symmetrization) of fermionic (bosonic) wavefunctions sends the total Hilbert space \( \mathcal{H} = \mathcal{H}^{(1)} \otimes N \) to \( \mathcal{H} = \bigvee^N \mathcal{H}^{(1)} \). Since the total Hilbert space is invariant under the action of the algebra \( \mathcal{A} \) of observables, the observables also must be invariant under the symmetrizations. Therefore, the partial trace defined as the formalism of non-identical particles is no more valid. It was the motivation of Ref. [12] to introduce the symmetrized partial trace, and also the motivation of Refs. [4, 5] to suggest the concept of restrictions to subalgebras as the replacement of partial trace. Therefore, it is natural to ask about the relation between the symmetrized partial trace and subalgebra restriction. Indeed, one can show that the restriction to subalgebras is equivalent to the symmetrized partial trace for the case of identical particles.

Instead of a Hilbert space \( \mathcal{H} \) and linear operators acting on it, quantum systems can be described with an abstract algebra of physical observable, i.e., \( C^* \) algebra, in which the algebra \( \mathcal{A} \) and state \( \omega \) describe a given quantum system. By Gelfand, Naimark, and Segal (GNS) [36, 37] construction, the data \( (\mathcal{A}, \omega) \) can reconstruct the corresponding Hilbert space \( \mathcal{H}_\omega \). The relation between Hilbert space and GNS representation of quantum physics is listed in the following table.

| Hilbert space | GNS |
|--------------|-----|
| observables  | \( \mathcal{O}(= \mathcal{O}^\dagger) \) | \( \alpha \in \mathcal{A} \) |
| state        | \( \rho (\text{Tr} \rho = 1 \text{ and } \rho = \rho^\dagger) \) | \( \omega (\alpha) \in \mathbb{C} \) |
| expectation value | \( \langle \mathcal{O}\rangle_\omega = \text{Tr} \rho \mathcal{O} \in \mathbb{C} \) | |

For a more thorough explanation on GNS construction, see Refs. [4, 5].
In the GNS construction, the notion of partial trace can be replaced with the restriction \( \omega_0 := \omega|_{A_0} \) of a state \( \omega \) on \( A \) to a subalgebra \( A_0 \). Suppose \( \omega \) is represented as a density matrix \( \rho_\omega \), i.e., \( \omega(\alpha) = \text{Tr}(\rho_\omega \alpha) \) (\( \alpha \in A \)). Then for a subalgebra \( A_0 \) of \( A \), we can define a restriction of \( \omega \) to \( A_0 \) as a state \( \omega|_{A_0} : A_0 \to \mathbb{C} \), i.e., from \( \alpha_0 \in A_0 \) to \( \omega|_{A_0}(\alpha_0) \) so that

\[
\omega|_{A_0}(\alpha_0) = \omega(\alpha_0) \quad (10)
\]

holds \(^4\). A simple example is a bipartite system of non-identical particles \( A \) and \( B \) in the Hilbert space \( \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \). For a given vector state \( |\Psi\rangle \in \mathcal{H} \) and the subalgebra \( A_0 = \{ \alpha_0 \in A | \alpha_0 = K_A \otimes 1_B \} \) (\( K_A \) is an observable on \( \mathcal{H}_A \)), we can see that the following equality holds:

\[
\omega|_{A_0}(\alpha_0) = \text{Tr}_{A_0}(\rho_A K_A) = \text{Tr}_A(\rho_{0_A}) = \omega(\alpha). \quad (11)
\]

where \( \rho_A \equiv \text{Tr}_{A_0}(\rho) \). For this special case, the reduced density matrix \( \rho_A \) is equivalent to the restriction of \( \rho \) to \( A_0 \).

However, when particles are identical, a single particle observable is not included in a single Hilbert space \( \mathcal{H}^{(1)} \) in the total Hilbert space \( \mathcal{H} \). Along with the permutation symmetry of identical particles, the observables also must be invariant under the permutations of single Hilbert spaces. As pointed out in Refs. \(^4\), \(^5\), this permutation symmetry of observables causes the discrepancy between the subalgebra restriction and the traditional form of partial trace that is valid for non-identical particles. Hence the authors of Refs. \(^4\), \(^5\) claimed that (instead of the partial trace) the restriction of states to subalgebras is appropriate for analyzing the entanglement of identical particles.

On the other hand, we can see that the newly defined symmetrized partial trace of identical particles plays the role of subalgebra restriction.

**Theorem 2.** The symmetrized partial trace for identical particles maps a density matrix \( \rho \) and a state \( \omega \) on the algebra \( A \) to the restrictions \( \rho_0 \) and \( \omega_0 \) on the subalgebra \( A_0 \) defined in a subsystem \( S \), i.e.,

\[
\omega|_{A_0}(\alpha_0) = \mathcal{TR}_S(\rho_0 \omega_0) = \mathcal{TR}(\rho_0 \omega). \quad (12)
\]

**Proof.** First, we will show that Eq. \((12)\) holds for the simplest case, i.e., \((N, n) = (2, 1)\). Suppose a subalgebra \( A_0 \) divides the wave functions \( \Psi_i \) into two parts, i.e., \( \Psi_a \) and \( \Psi_b \) (states with latin indices form a complete orthonormal basis set of the subsystem \( L \) and those with greek indices form a complete orthonormal basis set of the subsystem \( R \). \( \Psi_a \) are rotated to each other by operators in \( A_0 \)). Then an observable \( \alpha_0 \in A_0 \) for 2-particle states in \( A \) is written as

\[
\alpha_0 = \sum_{a, b, \mu} \alpha_{ab}(\left< \Psi_a | \left< \Psi_a \right> \right) (\left< \Psi_b | \left< \Psi_b \right> \right) \equiv \sum_{a, b} \alpha_{ab} |\Psi_b \rangle \left< \Psi_b \right| \left< \Psi_b \right| \left< \Psi_b \right). \quad (13)
\]

For a given state \( |\Psi\rangle = \sum_{c, \mu} c_{c\mu} |\Psi_c \rangle \left< \Psi_\mu \right| \left< \Psi_\mu \right| \left< \Psi_\mu \right) \), we have

\[
\mathcal{TR}_L(\rho_L \alpha_0) = \sum_{a, b, c, d} \alpha_{ab} X_{cd} |\Psi_c \rangle \left< \Psi_b \right| \left< \Psi_b \right| \left< \Psi_b \right) = \sum_{a, b} \alpha_{ab} X_{ba}. \quad (15)
\]

On the other hand,

\[
\mathcal{TR}(\alpha_0 |\Psi\rangle \left< \Psi \rangle) = \sum_{a, b, \mu, c, \nu, d, \omega} \alpha_{ab} c_{c\nu} c_{\omega d} |\Psi_c \rangle \left< \Psi_b \right| \left< \Psi_b \right| \left< \Psi_b \right) \times (\left< \Psi_\mu | \left< \Psi_b \right| \left< \Psi_b \right| \left< \Psi_b \right) \equiv \sum_{a, b} \alpha_{ab} X_{ba}. \quad (16)
\]

hence Eq. \((12)\) holds for \((N, n) = (2, 1)\).  The extension of this method to the general \((N, n)\) case is straightforward. \(\blacksquare\)

In the above proof, one can see that possible states for a given subsystem (which is determined by the implementation of detectors) determines the selection of subalgebra in GNS representation. This means that the detector dependence of entanglement is equivalent to the subalgebra dependence of entanglement mentioned in Ref. \(^3\). An \( N = 2 \) example with \( \mathcal{H}^{(1)} = \mathbb{C}^4 \) is given in Supplemental Materials.

Since the computation of symmetrized partial trace is a straightforward process, it is more convenient to obtain the same reduced density matrix for a set of identical particles using the method of symmetrized partial trace than using the subalgebra restriction technique \(^3\).

**Conclusions.**— We have discussed the concept of symmetrized partial trace in three types of formalisms, i.e., the first quantization language, exterior and elementary symmetric product (which correspond to the no-labeling approach), and the subalgebra restriction in the GNS representation. Our current work bridges several viewpoints that use different languages to understand the
quantum correlation of identical particles. We also expect that it can be applied to the entanglement properties of quantum field theory (e.g., Tomita-Takeshaki theory [24–26] and symmetric product orbifold CFT [42]).

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30. Here Ψa′ are all in the complete orthogonal basis set of the detectors in the total system. For the example of a bipartite system with detectors L and R which locate far from each other, if each particle has an internal state (pseudospin) s, Ψa ∈ {(L, s), (R, s′)}.
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Supplemental Materials

$$(N, n) = (3, 1)$$ EXAMPLE IN 1QL AND EXTERIOR VECTOR FORMALISM

Suppose

$$
(\{|\Psi_1\rangle, |\Psi_2\rangle, |\Psi_3\rangle, |\Psi_4\rangle\rangle = (|L, 0\rangle, |L, 1\rangle, |L, 2\rangle, |L, 3\rangle),

(|\Psi_5\rangle, |\Psi_6\rangle, |\Psi_7\rangle, |\Psi_8\rangle) = (|R, 0\rangle, |R, 1\rangle, |R, 2\rangle, |R, 3\rangle),

$$

(1)

where $L$ and $R$ denote the spatial location of two detectors that place far enough from each other, i.e., $\langle L|R\rangle = 0$

and $(0, 1, 2, 3)$ denote the internal states (pseudospins). Then for the subsystem of $R$, we have

$$
\mathbb{I}_R = |R, 0\rangle\langle R, 0| + |R, 1\rangle\langle R, 1| + |R, 2\rangle\langle R, 2| + |R, 3\rangle\langle R, 3|

= |\Psi_5\rangle\langle \Psi_5| + |\Psi_6\rangle\langle \Psi_6| + |\Psi_7\rangle\langle \Psi_7| + |\Psi_8\rangle\langle \Psi_8|

$$

(2)

where $|\Psi_i\rangle = \frac{1}{\sqrt{2}}(|\Psi_i\rangle + e^{i\theta_i}|\Psi_i\rangle) + e^{i\eta_i}|\Psi_i\rangle$ (see Eq. (3)). If a given state is written as

$$
|\Psi\rangle = \frac{1}{\sqrt{2}}(|\Psi_1\rangle + e^{i\theta_i}|\Psi_i\rangle) + e^{i\eta_i}|\Psi_i\rangle,

$$

(3)

the reduced density matrix $\rho_L$ for the subsystem $L$ is obtained by the symmetrized partial trace as

$$
\rho_L = \frac{1}{2}(|\Psi_1\rangle\langle \Psi_1| + |\Psi_2\rangle\langle \Psi_2| + |\Psi_3\rangle\langle \Psi_3|).

(6)

The von Neumann entropy $S(|\Psi\rangle)$ is 1.

The same computation is possible in the exterior vector formalism by expressing

$$
|\Psi\rangle = \frac{1}{\sqrt{2}}(|\tilde{\Psi}_1\rangle \wedge |\tilde{\Psi}_2\rangle \wedge |\tilde{\Psi}_3\rangle + |\tilde{\Psi}_1\rangle \wedge |\tilde{\Psi}_3\rangle \wedge |\tilde{\Psi}_6\rangle).

(7)

Then using Eq. (7), the reduced density matrix is given by

$$
\rho_L = \frac{1}{2}(|\tilde{\Psi}_1\rangle \wedge |\tilde{\Psi}_2\rangle \langle \tilde{\Psi}_1| + |\tilde{\Psi}_1\rangle \wedge |\tilde{\Psi}_3\rangle \langle \tilde{\Psi}_1|).

(8)

$$

PROOF OF GHJW THEOREM

Lemma 1. $|\Psi\rangle$ and $|\Psi\rangle'$ are vectors in $\wedge^{\otimes 2} \mathcal{H}$ and locate in two orthogonal subsystems $L$ and $R$. If $\mathcal{T} |\Psi\rangle \langle \Psi| = \mathcal{T} \mathcal{R}_R |\Psi\rangle \langle \Psi|'$, then there exists an operation in the subsystem $\alpha = \mathbb{I}_L \wedge \alpha_R$ that satisfies $|\Psi\rangle = \alpha |\Psi\rangle'$.

Proof. The reduced density matrix can be written as

$$
\mathcal{T} \mathcal{R}_R |\Psi\rangle \langle \Psi| = \sum \omega_a (L, a) \langle L, a|.

$$

For any complete orthonormal basis set $\{\nu\}^{n}_n = 0$ of $R$, we have

$$
|\Psi\rangle = \sum_a \omega_a (L, a) \langle L, a|.

$$

(9)
where
\[
S_e^{(k)}(\hat{e}_{i_1} \otimes \cdots \otimes \hat{e}_{i_k}) = \frac{1}{k!} \sum_{a_1, \cdots, a_k=1}^{k} |\epsilon_{i_1 \cdots i_k}| (\hat{e}_{a_1} \otimes \cdots \otimes \hat{e}_{a_k})
\]
(10)

(|\epsilon_{i_1 \cdots i_k}| is the absolute value of the Levi-Civita symbol, which vanishes when \(i_{a_k} = i_{a_l}\) for any \(k\) and \(l\)).

Note that the elementary symmetrizing map \(S_e^{(k)}\) projects out any elements of \(x\) that satisfy \(i_a = i_b\) (\(1 \leq a, b \leq k\)). For example, when \(x = \sum_{i,j}^N x^{ij}(\hat{e}_i \otimes \hat{e}_j)\), we have
\[
S_e^{(2)}(x) = \frac{1}{2} \sum_{i,j} x^{ij} |\epsilon_{i,j}| (\hat{e}_i \otimes \hat{e}_j + \hat{e}_j \otimes \hat{e}_i)
\]
\[
= \sum_{i \neq j} \left( \frac{x^{ij} + x^{ji}}{2} \right) (\hat{e}_i \otimes \hat{e}_j + \hat{e}_j \otimes \hat{e}_i).
\]

The elementary symmetrizing map is simply denoted with the the elementary symmetric product \(\vee\) as
\[
v \vee w \equiv S_e^{(k)}(v \otimes w)
\]
(12)

where \(v \in T^d V\) and \(w \in T^{(k-1)} V\).

An \(N\)-boson total state \(|\Psi^b\rangle = |\Psi_1, \cdots, \Psi_N\rangle^b\) is expressed by the exchange symmetry as
\[
|\Psi^b\rangle = \sum_{\sigma} |\Psi_1\rangle_{A_{\sigma(1)}} |\Psi_2\rangle_{A_{\sigma(2)}} \cdots |\Psi_N\rangle_{A_{\sigma(N)}}
\]
\[
= \sum_{k_1 < \cdots < k_N=1}^{N} (\hat{e}_{k_1} \vee \cdots \vee \hat{e}_{k_N}, |\Psi_1\rangle \vee \cdots \vee |\Psi_N\rangle)
\]
(13)

(note that the equalities hold up to normalization). And

the \(n\)-boson subsystem state is written as
\[
|\Psi_{1, \cdots, n}\rangle^b
\]
\[
= \sum_{a_1 < \cdots < a_n}^{N} e^{i\theta_{a_1 \cdots a_n}} \sum_{\sigma \in S_n} |\Psi_{1}\rangle_{A_{\sigma(a_1)}} \cdots |\Psi_{n}\rangle_{A_{\sigma(a_n)}}
\]
\[
= \sum_{k_1 < \cdots < k_n=1}^{N} e^{i\theta_{k_1 \cdots k_n}} (\hat{e}_{A k_1} \vee \cdots \vee \hat{e}_{A k_n}, |\Psi_1\rangle \vee \cdots \vee |\Psi_n\rangle).
\]
(14)

The reduced density matrix for identical bosons is defined by replacing exterior products with elementary symmetric products and taking the interior products, which is equal to that defined in Ref. [13].

\section*{Two Fermions with} \(H^{(1)} = C^4\)

Suppose \(H^{(1)} = C^4\) consists of an orthonormal basis set \(\{\Psi_1, \Psi_2, \Psi_3, \Psi_4\}\). Then the two-fermion Hilbert space \(\Lambda^{\otimes 2} H^{(1)} \cong C^6\) consists of \(\{|\Psi_i\rangle \wedge |\Psi_j\rangle\}_{i,j=1}^4\). If given an exterior state vector
\[
|\Psi\rangle = \cos \theta |\hat{\Psi}_1\rangle \wedge |\hat{\Psi}_4\rangle + \sin \theta |\hat{\Psi}_2\rangle \wedge |\hat{\Psi}_3\rangle,
\]
(15)
one can take a symmetrized partial trace according to a choice of the corresponding subspaces (particle detectors) \(L\) and \(R\) \((\langle L|R \rangle = 0\)). By imposing
\[
(|\Psi_1\rangle, |\Psi_2\rangle, |\Psi_3\rangle, |\Psi_4\rangle) = (|L, \uparrow\rangle, |L, \downarrow\rangle, |R, \uparrow\rangle, |R, \downarrow\rangle),
\]
(16)an identity matrix for \(R\) is given by \(\mathbb{1}_R = |\hat{\Psi}_3\rangle \langle \hat{\Psi}_3| + |\hat{\Psi}_4\rangle \langle \hat{\Psi}_4|\). Then we have
\[
\rho_L = \mathcal{T}_R (\mathbb{1}_R |\Psi\rangle \langle \Psi|) = \cos^2 \theta |\hat{\Psi}_1\rangle \langle \hat{\Psi}_1| + \sin^2 \theta |\hat{\Psi}_2\rangle \langle \hat{\Psi}_2|\]
(17)and the von Neumann entropy is given by
\[
S(\Psi) = - \cos^2 \theta \log(\cos^2 \theta) - \sin^2 \theta \log(\sin^2 \theta)
\]
(18)(the state is entangled whenever \(\theta \neq 0\) and \(\theta \neq \pi/2\)). This is equivalent to the choice of the subalgebra with the relevant generators \(M_{ij} = |\Psi_i\rangle \langle \Psi_j|\) \((1 \leq i, j \leq 2)\) and the Hilbert space decomposition \(\mathcal{H} \cong C^2 \oplus C^2\) in Sec. IV D.1 of Ref. [3], which obtains the same reduced density matrix as Eq. (17).