GENERIC FRIEDMANN DOMAINS AND THE
QUESTION OF SYNCHRONIZATION

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August 2022

Abstract

We study the evolution of Friedmann domains in generic inhomogeneous spacetime
placing special emphasis on their synchronization. We show that a single causally
(inter-) connected region will bifurcate to future states of exchanged stability, and
subsequently desynchronize during its evolution. We then show that two causally
disjoint isotropic domains generically desynchronize unless they are tuned initially
to identical states. These results imply that synchronization of evolving spatial
regions may require mechanisms other than causality-based homogenization, or an
adaptation of special initial conditions.

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1 Introduction

It is a mainstream assumption of theoretical cosmology that the universe’s evolving history is made up of a number of different epochs such as the vacuum, inflationary, radiation, matter, and other periods, coherently joined to produce the standard cosmological scenario from some early phase to the future. This scenario usually assumes a cosmic fluid with an equation of state given by $p = (\gamma - 1)\rho$, relating the fluid density $\rho$ to its pressure through the fluid parameter $\gamma$, specific values of which correspond to special epochs of cosmological evolution, cf. e.g., [1], chap. 4, [2].

It is also well-known that in studies of cosmological models that admit no isometries, a general approach is usually adopted in which each point is evolving as a separate universe having some given symmetry. For instance, in studies of the Belinski-Khalatnikov-Lifshitz (‘BKL’) conjecture the universe proceeds in the direction of the initial singularity in such a way that each spatial point evolves as a separate Mixmaster universe [3, 4]. If we denote by $X(t, x^i)$ the normalized vector field describing a generic inhomogeneous state, $t, x^i$ being the time and spatial coordinates, then in the standard $(3 + 1)$-approach we end up with evolution equations and constraints for $X$, and by fixing $x^i = \text{const.}$ we may describe individual orbits in a finite-dimensional subset of the full state space [4].

In Refs. [5], [6], an application of this approach to the question of chaotic synchronization of different Mixmaster universes was first considered, and we direct the reader to these references for more details, general approaches and results, as well as the motivation of the introduction of synchronization (‘sync’ hereafter) in cosmology. Below we study the simpler problem where each spatial point evolves as a separate Friedmann universe. We compare ‘adjacent’ Friedmann universes evolving along their $x^i = \text{const.}$ orbits in generic inhomogeneous spacetime parametrized by their fluid parameters and satisfying the usual Friedmann equations in different variables. This leads to an alternative way of studying Friedmann cosmologies, in particular, how their global properties depend on parameters and especially their possible tendency for synchronization.

The structure of this paper is as follows. In the next Section, we examine whether or
not a single parametric domain consisting of causally connected subregions may evolve in
time to an observable synchronized universe as the one we observe today. In Section 3, we
consider two causally disjoint domains and ask whether sync is possible between them
during their future time evolution. We further discuss repercussions of these results
in the last Section. In an Appendix, we present a short discussion of the method of
studying separate evolving domains in an unspecified background and how to test them
for synchronization.

2 Causal domains

We consider a single Friedmannian domain and assume that all points (or subregions)
inside it are initially synced with each other, possibly due to a previous inflationary
era and associated causal exchanges of signals that made them causally connected or
‘homogenized’.

Since these subregions are initially homogeneous, their evolution can be described by
the same variables and equations, which are taken to be the usual ones, cf. [1], chap. 4: if
we use a dimensionless time variable $\tau$ defined by $dt/d\tau = 1/H$, with $t$ being the proper
time, and $H$ is the Hubble parameter defined using the scale factor $a$, $H = \dot{a}/a$, and
also the density parameter $\Omega = \rho/3H^2$, and the fluid density $\rho$ related to the pressure
by $p = (\gamma - 1)\rho$, the defining equations are given by,

$$\frac{d\Omega}{d\tau} = -\mu\Omega + \mu\Omega^2$$

(2.1)

$$\frac{dH}{d\tau} = -H - \frac{1}{2}\mu H\Omega.$$ (2.2)

Here $q = \mu\Omega/2$ denotes the deceleration parameter, and we have set $\mu = 3\gamma - 2$. Our
overall approach to this problem is to search for possible bifurcation properties of the
dynamical solutions $(\Omega, H; \mu)$ with respect to the parameter $\mu$, cf. [7], chaps. 18.2, 19.2.

The equilibria are at $(0, 0)$ and $(1, 0)$, but since we are interested in synchronization
we need to consider the behaviour of the solutions near $(0, 0)$. The Jacobian of the
system (2.1)-(2.2) at \((\Omega, H; \mu) = (0, 0; 0)\) is,

\[
J_{(0,0)} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix},
\]

(2.3)

with eigenvalues a 0, −1. The sets \(\Omega = 0, H = 0\) are invariant. It follows from general theorems that a center manifold exists for all values of \(\mu\) sufficiently close to \(\mu = 0\), and all bifurcating solutions of the full system (2.1)-(2.2) will be on that lower-dimensional center manifold given by,

\[
W_{c \text{ loc}} = \{ (\Omega, \mu, H) \in \mathbb{R}^3 \mid H = h(\Omega, \mu), |\Omega| < \delta_1, |\mu| < \delta_2, h(0, 0) = 0, Dh(0, 0) = 0 \},
\]

(2.4)

and \(\delta_1, \delta_2\) sufficiently small. This represents a graph over the \(\Omega, \mu\) variables. The vector field on \(W_{c \text{ loc}}\) is given by,

\[
\frac{d\Omega}{d\tau} = -\mu \Omega + \mu \Omega^2
\]

(2.5)

\[
\dot{\mu} = 0
\]

(2.6)

and we can find \(h(\Omega, \mu)\) by direct calculation. Taking the time derivative of the center manifold \(H = h(\Omega, \mu)\), leads to \(\dot{H} = D_{\Omega}h \dot{\Omega}\). Further, using (2.1)-(2.2) and substituting,

\[
h(\Omega, \mu) = a\Omega^2 + b\mu \Omega + c\mu^2,
\]

(2.7)

the tangency condition reads,

\[
D_{\Omega}h (-\mu \Omega + \mu \Omega^2) + h + \frac{1}{2} \mu \Omega h = 0,
\]

(2.8)

and so equating terms of like powers, and neglecting terms of \(O(\mu^2), O(\Omega^2 \mu), O(\mu^3), \ldots\), we find that \(a = b = 0\) This means that the center manifold is the \(H = 0\) axis, that is the \(\Omega\)-line. This was to be expected because the \(\Omega\) equation (2.5) decouples.

Since Eq. (2.5) holds true on the center manifold and the other eigenvalue is negative, it follows that the unstable manifold \(W_{u \text{ loc}}\) is empty and all solutions of the full system are stably attracted by (or rapidly decay to) the center manifold. The global evolution of
the system is thus characterized by the different signs of the $\mu$-parameter: when $\mu < 0$, the equilibrium $\left(0, 0\right)$ is unstable and $\left(1, 0\right)$ is stable, when $\mu > 0$, $\left(0, 0\right)$ is stable and $\left(1, 0\right)$ is unstable, while at $\mu = 0$ we have that $\Omega = 0$ and the two equilibria coalesce at $\Omega = 0$. Hence, there is an exchange of stability at $\mu = 0$, which leads us to suspect the existence of a transcritical bifurcation. This is indeed true as we now show.

Setting $f(\Omega, \mu) = \mu \Omega (\Omega - 1)$ for the right-hand side of the equation (2.5), we find,

$$f(0, 0) = 0, \quad \frac{\partial f}{\partial \mu}(0, 0) = 0, \quad \frac{\partial f}{\partial \Omega}(0, 0) = 0, \quad \frac{\partial^2 f}{\partial \Omega \partial \mu}(0, 0) = -1 \neq 0, \quad \frac{\partial^2 f}{\partial \Omega^2}(0, 0) = 0.$$  (2.9)

The first two equations mean that the equilibrium is non-hyperbolic (there is a zero eigenvalue), while the remaining equations can be used to find the slope of the curve of fixed points different from the origin where $\Omega = 0$. This is found to be,

$$\frac{d\mu}{d\Omega}(0) = \frac{\frac{\partial^2 f}{\partial \Omega \partial \mu}(0, 0)}{\frac{\partial^2 f}{\partial \Omega^2}(0, 0)} = 0,$$  (2.10)

and leads to the bifurcation diagram shown in Fig. 1 where we only consider the ‘physical region’ $\Omega \geq 0$.

This diagram provides an alternative way of describing the evolution of the system (2.5), (2.6) on the center manifold. Any point in the $(\mu - \Omega)$-plane corresponds to a subregion in the causally connected parent Friedmann domain. Let us denote the left vertical axis ($\mu < 0$) by $\Omega_{-1}$, the right vertical axis by $\Omega_{+1}$ ($\mu > 0$), and consider the evolution of a pair of points $A, B$ lying on either side of the equilibrium at $(1, 0)$ on both of these axes. We see that on the $\Omega_{-1}$ axis, the pair approaches the equilibrium at $(1, 0)$ while that on the $\Omega_{+1}$ axis recedes from it.

We now introduce the sync function $\omega_\mu = |\Omega_{\mu,A} - \Omega_{\mu,B}|$ which describes the overall difference in the density functions during the evolution of the domain pair $A, B$ for different values of $\mu$. Since $\omega_{-1} \to 0$, while $\omega_{+1}$ does not, this implies the ‘desynchronization’ of the two subregions when the pair passes from an epoch with $\mu < 0$ to one with $\mu > 0$. Same conclusions apply to any other pair of subregions lying at different places and eras on the 1-dimensional phase space.
Figure 1: Bifurcation diagram for the Friedmannian domain evolution containing causally connected subregions. The red lines show instability and the horizontal black ones signify stability. The Ω phase lines are the lines orthogonal to the μ-axis.

We can further obtain a more detailed description of this effect by drawing phase diagrams for different μ-values for the system (2.1)-(2.2). Some examples of these are shown in Fig. 2.

From these results we are led to an important conclusion: any parent domain with μ < 0 can never keep their subdomains synced with regard to a future self-copy having μ > 0. Since the bifurcation value μ = 0 will necessarily be crossed during their evolution, any two phase points in Fig. 2a as compared to their corresponding evolution in Fig. 2c will desynchronize in time as a result of stability exchange.

3 Uncorrelated domains

We now move on to the problem of describing the evolution of two totally causally disjoint Friedmannian spatial domains. Since causality restrictions are absent and no causal correlation exists between them (unlike the known situation in a general multi-fluid problem, cf. [1], pp. 60-2), it is not entirely obvious whether some kind of sync between the two could be achieved in this case. Therefore the question arises as to whether two causally disjoint domains could become synced during their evolution starting from
initial states where no synchronization between them was present. This is examined below.

We assume that the two causally disjoint spatial domains $\mathcal{B}$ and $\mathcal{E}$ evolve according to separate dynamical laws given by:

**Domain $\mathcal{B}$**:

\[
\frac{dH_1}{d\tau_1} = -(1 + q_1)H_1 \quad (3.1)
\]
\[
\frac{d\Omega_1}{d\tau_1} = -(3\gamma_1 - 2)(1 - \Omega_1)\Omega_1 \quad (3.2)
\]
\[
q_1 = \frac{1}{2}(3\gamma_1 - 2)\Omega_1, \quad (3.3)
\]

**Domain $\mathcal{E}$**:

\[
\frac{dH_2}{d\tau_2} = -(1 + q_2)H_2 \quad (3.4)
\]
\[
\frac{d\Omega_2}{d\tau_2} = -(3\gamma_2 - 2)(1 - \Omega_2)\Omega_2 \quad (3.5)
\]
\[
q_2 = \frac{1}{2}(3\gamma_2 - 2)\Omega_2. \quad (3.6)
\]
Here the two fluid-filled domains have Hubble parameters $H_i$, density parameters $\Omega_i = \rho_i / 3H_2^2$, equations of state $p_i = w_i \rho_i$, $i = 1, 2$, $w_i = \gamma_i - 1$, and we use the dimensionless times $\tau_i$ instead of the proper time $t$, given by,

$$\frac{dt}{d\tau_i} = \frac{1}{H_i}, \quad (3.7)$$

defined through the scale factors by

$$a_i = a_i,0 e^{\gamma_i}. \quad (3.8)$$

We note that in this notation, vacuum in the form of a cosmological constant occurs at $\gamma = 0$ (i.e., $w = -1$). Below we assume that $\gamma_1 \neq \gamma_2$ for the two fluids in the domains $B, E$, because the case $\gamma_1 = \gamma_2$ implies that,

$$H_1 = H_2 \Leftrightarrow \Omega_1 = \Omega_2, \quad (3.9)$$

and we are back to the problem considered in the previous Section.

As discussed in the Appendix, to examine for sync we assume that the spatial domains $B, E$ transmit part of the information of their state to the common future point $G$. We take the transmitting part from $B$ and $E$ to $G$ to be $H_1, H_2$. We then simply set $H_2 = H_1$ in (3.4), so that we can write $\tau = \tau_1 = \tau_2$ for a common dimensionless time in (3.7), and end up with a system of two remaining equations, that is the $\Omega$-equations (3.2), (3.5). Then we say that the two domains will sync provided $\omega = \Omega_1 - \Omega_2 \to 0$.

Various phase portraits in the $(\Omega_1,\Omega_2)$ plane are shown in Fig. 3 for different choices of the $\gamma$ parameters. These describe the evolution of the two spatial, causally uncorrelated domains through the equations (3.2), (3.5). From the results of these diagrams we conclude that except for the trivial case of two identical uncorrelated domains (as in Figs. 3a, 3b, a problem considered in the previous Section), the sync function $\omega$ does not generically approach 0, and so the two uncorrelated domains $B$ and $E$ will not sync during the evolution if they were not so initially.

In particular, there is no sync when the fluid parameters $w_i = \gamma_i - 1$ belong to the sets,
Figure 3: Examples of evolution of the two causally disjoint, uncorrelated Friedmannian domains $B$, $E$ for a variety of fluids in the $(\Omega_1, \Omega_2)$ phase plane. In general, the geometry of the orbits is such that the sync function does not vanish asymptotically.

(a) $B$: vacuum - $E$: vacuum.
(b) $B$: Radiation - $E$: radiation.
(c) $B$: dust - $E$: radiation.
(d) $B$: vacuum - $E$: scalar field.
(e) $B$: Vacuum - $E$: radiation.
(f) $B$: vacuum - $E$: ($\gamma = -1$).
(g) $B$: ($\gamma = -1/2$) - $E$: ($\gamma = -1$).
(h) $B$: ($\gamma = -5/3$) - $E$: ($\gamma = -1$).
(i) $B$: ($\gamma = -2/3$) - $E$: ($\gamma = -5/3$).
Sync between the two causally uncorrelated domains appears to occur only when \( w_1, w_2 < 0 \), and in that case the two regions would sync at the common \( \Omega_i = 1 \) (flat) future state independently of the initial conditions (cf. Figs. 3f-3i). However, any change in the sign of the \( w_i \)'s (as this necessarily happens in cosmological evolution) will result in their desynchronization. (We note that the direction of the arrows in these diagrams denotes future evolution, that is as \( \tau \to +\infty \), and so all these results are reversed in the past direction.)

4 Discussion

The results in this paper show why synchronization of two spatial Friedmann subregions cannot be solely based on causality considerations, or an examination of spatial regions only during their Friedmannian evolution. Suppose that the two subregions were causally disjoint at an early stage and made ‘homogeneous’, and proceeded as one causally connected domain possibly due to a causal mechanism of signal exchanges between them. We have shown that their states could not have been sustained in sync during their entire time development. In a sense this is because by turning causally homogeneous, the two regions are never really ‘absorbed’, but gradually desync as they continue to drift apart on their own and there can be no ‘ratcheting mechanism’ to get them closer to sync (cf. Refs. [9, 10] for a discussion of absorption in a different context).

A causality-driven homogenization of spatial domains leads to a divergent behaviour of the sync function. As it follows from the results of Section 2, a homogeneous and isotropic domain that depends on a single fluid parameter and containing a large number of causally connected subregions, desynchronizes during the course of its evolution even if it started as a completely synced, multi-domain region. In particular, the evolution of a single causal FRW domain experiences a transcritical bifurcation and proceeds
to exchanging stability between its different parts, leading to totally desynchronized late stages. Hence, our observed universe is not the result of evolution of any single Friedmannian causal domain.

For two causally disjoint, initially desynchronized, isotropic domains on the other hand, we showed in Section 3 that sync will occur only for special, or finely chosen, intervals of fluid parameter values, and will not sync during their evolution for generic choices of their fluid parameters (unless they were already synced to each other initially). The only case leading to sync is that of negative $\gamma$ parameters, but this is however unstable and their sync will only last until one of the two $\gamma$’s turns positive, in which case the two fluids will desynchronize.

A ‘absorption’ mechanism of generating dynamically synced, uncorrelated spatial domains will be discussed elsewhere \[11\].

Acknowledgments

I am grateful to John Miritzis for important comments and discussions.

A Appendix

A very brief description of dynamical synchronization in presented in this Appendix. For more details and generality, the reader may consult \[6\] and refs. therein. For the vector variable $x = (\Omega, H) \in \mathbb{R}^2$, consider the dynamical system, called the ‘transmitter’:

$$\dot{x} = f(x),$$

with the $\cdot \equiv d/d\tau$, and $f(x) = (-\mu\Omega + \mu\Omega^2, -H - \mu H\Omega/2)$. In our problem we have two transmitters, $x_i = (\Omega_i, H_i), i = 1, 2$, the domains $\mathcal{B}, \mathcal{E}$, each sending signals eventually arriving at the point $G$ in their future as introduced in the main text. A dynamical method to examine the two domains for sync is as follows:
Step 1: Split each transmitter in two subsystems $a, b$, by setting $x_i = (u_i, v_i)$, with $u_i = \Omega_i, v_i = H_i$, so that (A.1) becomes,

$$\dot{u}_i = f_1(u_i, v_i), \quad \text{subsystems-}a_B, a_E \quad (A.2)$$

$$\dot{v}_i = f_2(u_i, v_i), \quad \text{subsystems-}b_B, b_E. \quad (A.3)$$

Step 2: The receiver at $G$ receives only the ‘$v$-part’ of the signals from the transmitters, and we assume that at $G$,

$$v_1 = v_2 = v, \quad \text{(or, } H_1 = H_2 = H). \quad (A.4)$$

Step 3: The receiver then checks the difference,

$$\omega = u_1 - u_2, \quad (A.5)$$
called the sync function, using the remaining equations, the ‘$u$-part’ of the system (A.2), (A.3), namely,

$$\dot{u}_1 = f_1(u_1, v) \quad (A.6)$$
$$\dot{u}_2 = f_1(u_2, v). \quad (A.7)$$

Step 4: The two domains $B, E$ sync provided the orbits of the system (A.6), (A.7) satisfy,

$$\omega \to 0, \text{ as } \tau \to \infty. \quad (A.8)$$

In this case both domains evolve in perfect unison.

The almost magical property of systems that have the ability to sync with each other is that although the receiver has received only part of the information of the transmitter (the ‘$v$-part’), it somehow manages to reconstruct the remaining piece, so that $u_1 = u_2$.

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