Synthesis of Near-Field Arrays Based on Electromagnetic Inner Products

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Abstract—Near-field (NF) antennas have been successfully adopted in several wireless applications. To exploit the high reconfigurability of array antennas, multiple synthesis techniques for arrays operating in the NF region have been proposed. Building upon previous works on eigenmode expansions of the radiated fields, two synthesis methods for the excitations of NF arrays based on the definition of an inner product on the electromagnetic fields are investigated: the “maximum norm” and “minimum error field norm” methods. The “maximum norm” method computes the array excitations that maximize either the active power flow through a target surface or the electric/magnetic energy stored in an assigned volume, depending on the adopted inner product. The performance of the maximum active power flow method is compared with one of the simpler conjugate phase methods. Furthermore, the limit solution achieved when the target surface reaches the far-field region is compared against the “maximum beam collection efficiency” method. The “minimum error norm field” method allows to synthesize a given target field. As an example, the latter method is used to find the optimal excitation of a plane wave generator with a spherical quiet zone. The effectiveness and performance of the discussed synthesis methods are validated through numerical simulations.

Index Terms—Electromagnetic (EM) inner product, Fresnel region, maximum electric energy, maximum power flow, mutual coupling, near-field (NF) array, NF array excitation synthesis, plane wave generator (PWG).

I. INTRODUCTION

NEAR-FIELD (NF) arrays have gained increasing popularity in the last decade due to their adoption in several relevant applications. These include microwave hyperthermia [1], noncontact microwave sensing [2], [3], radio frequency identification (RFID) [4], [5], [6], [7], plane wave generators (PWGs) [8], [9], [10], and wireless power transfer (WPT) systems [11], [12], [13], [14], [15], [16].

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The most simple focusing technique for NF arrays consists in the conjugate phase (CP) method [17], [18], which is based on the geometrical optics (GO) approximation. Array elements are excited with equal amplitude and a phase shift that compensates for the propagation phase delay between the element and the focal point. Although the method is suited for a single focal point, a suboptimal solution for multiple foci can be obtained by computing the phase coefficients of the excitations for each focal point separately and then adding the computed excitations with properly selected amplitude weights [19]. In [20] a Dolph–Chebyshev amplitude taper is combined with the CP method to control the sidelobe level (SLL) of the electric field in the focal plane. Furthermore, by simulating a subarray of smaller dimensions, the method can consider the mutual coupling between adjacent elements even in an electrically large structure. Bellizzi et al. [21] proposed a method to compute the excitations that give the maximum uniform amplitude of a scalar field in a fixed grid of points located in the target focal region while keeping the SLL below a predefined threshold in a different fixed grid of points. The problem is recast into a convex optimization problem, but an additional set of phase terms must be used. Thus, for a large number of points, the method becomes computationally expensive. The method is extended to vector fields in [22]. Several other constrained optimization techniques have been proposed to select the excitation of the NF array in [23].

A radically different approach to compute the excitations of NF arrays is proposed in [16]. Once the geometry of the array has been defined, one can apply the method of maximum power transmission efficiency (PTE) with a fictitious receiving antenna placed at the focal point. This technique is based on the computation of the excitation vector that maximizes the PTE between the transmitting array and the receiving antenna and requires complete knowledge of the S-parameter matrix of the system. By using multiple fictitious receiving antennas, the method can be extended to obtain multiple focal points. Furthermore, since the method is based on the S-parameter matrix, the mutual coupling between the transmitting elements is considered. In the same paper, the author describes an alternative method, called extended method for maximum PTE (EMMPTE), which does not require any fictitious antenna. Two different formulations of the EMMPTE are provided based on the electric energy stored in a set of volumes or the active power flow through a set of assigned surfaces. The advantage of these techniques is that the optimal solution...
can be computed in a closed form by solving a generalized eigenvalue problem.

Two interesting design methods for NF arrays have been proposed in [24]. Both methods are based on the expansion of the electromagnetic (EM) field in a set of modes, which correspond to a different combination of the excitations. The “Poynting-based” method guarantees the maximization of the active power flowing through a given surface, while the “field-based” one minimizes the active power flow through a surface of the difference between the excited field and the target field. In [25], the latter method is used to synthesize both Airy and Bessel beams, providing an alternative to classical Bessel beam launchers [26], which have been successfully used in WPT systems [27]. Both methods have a closed-form solution but require the computation of several surface integrals.

Building upon the work in [16] and [24], here, we extend the two methods proposed by the former to any EM inner product. The two methods, named “maximum norm” and “minimum error field norm” methods, are derived by only exploiting the three fundamental properties of an inner product. A different choice of the involved EM inner product corresponds to a different set of excitations and optimization objectives. In particular, in Section III, the “Poynting-based” method in [24] is derived as a special case of the “maximum norm” method. The solution provided by the latter method is compared with the CP method. The two solutions tend to each other when the target region is electrically small. When the target region is in the far-field (FF) region of the array, an approximate formulation of the method can be derived, which generalizes the method proposed in [28], which maximizes the beam collection efficiency (BCE), and we denote as “maximum BCE” method. In Section IV, we exploit the constrained “minimum error field norm” method to find the excitations for a PWG. More specifically, the method provides the solution that minimizes the electric energy stored in the quiet zone (QZ) by the error field, i.e., the difference between the synthesized field and the target field while keeping the ratio between the power flowing through the target region and the incident power above a given threshold. Furthermore, since the radiated EM field corresponds to the superposition of a set of modes computed in the presence of all the elements of the array, both methods are valid regardless of the mutual coupling level. The proposed methods are validated by simulation: the canonical basis of $\mathbb{R}^N$, i.e., $[\hat{e}_n]_m = \delta_{nm}$. The operators $(\cdot)^T$, $(\cdot)^H$, and $(\cdot)^*$ denote the transpose, conjugate transpose (Hermitian), and conjugate operator, respectively. $I_N$ denotes the $N \times N$ identity matrix, while $0_{N \times M}$ denotes an all zeros $N \times M$ matrix. The absolute value of a scalar is represented by $|\cdot|$, while the Euclidean norm of a vector by $\|\cdot\|$. The real and imaginary parts of a complex number are denoted by $\Re\{\cdot\}$ and $\Im\{\cdot\}$, respectively. In the following we are considering time-harmonic fields, expressed as $F(r)$, where the associated real field can be computed as $f(r, t) = \Re\{F(r)e^{j\omega t}\}$. Given $a, b \in \mathbb{C}^3$, $a \cdot b$ corresponds to $\sum_{n=1}^{3} a_n b_n$.

II. MATHEMATICAL FORMULATION OF THE PROBLEM

In this work, we consider an array with $N$ elements. Furthermore, we suppose that each element is excited by a single mode port, and thus, the system can be represented as an $N$-port network, as shown in Fig. 1. If we define with $a \in \mathbb{C}^N$ and $b \in \mathbb{C}^N$ the input and output power waves, respectively, then [31]

$$b = Sa \tag{1}$$

where $S \in \mathbb{C}^{N \times N}$ is the $S$-parameter matrix. If the system is linear, we can express the EM field generated by the excitation vector $a$ as

$$E(r) = \sum_{n=1}^{N} (\hat{e}_n^H a) E_n(r)$$

$$H(r) = \sum_{n=1}^{N} (\hat{e}_n^H a) H_n(r) \tag{2}$$

where $\{\hat{e}_n\}_{n=1}^{N}$ is an orthonormal basis for the excitation space and $(E_n, H_n)$ is the EM field generated when $a = \hat{e}_n$ (see Appendix A).

In the following, we present two methods to select the excitation vector $a$ according to two different optimization criteria. The first method maximizes the norm of the EM field, whereas the second method minimizes the norm of the error field. Both methods are a generalization of the ones proposed in [24] to any EM inner product.

A. Maximum Norm Method

Once an inner product has been selected, we can derive the excitation vector $a \in \mathbb{C}^N$ that maximizes the induced norm for a given total incident power $a^H Ba$, with $B = I_N/2$. 

Fig. 1. Schematic of an $N$ port array.
Otherwise, one can maintain the total input power fixed as in [16], with the matrix \( B \) corresponding to \((I_N - S^H S)/2\). If the array elements are well matched to the reference impedance and mutual coupling can be neglected, then the total input and incident power can be assumed to be equal. By using the expressions in (2) for the electric and magnetic field, the norm squared can be expressed as

\[
\|(E, H)\|^2 = (\langle E, H \rangle, \langle E, H \rangle)
\]

\[
= \sum_{m=1}^{N} \sum_{n=1}^{N} \langle \hat{x}_{m}^H a \rangle \langle \hat{x}_{n} a \rangle (\langle E_{m}, H_{m} \rangle, (E_{n}, H_{n})),
\]

\[
= \sum_{m=1}^{N} \sum_{n=1}^{N} \langle a^H \hat{x}_{n} \hat{x}_{m}^H \rangle \langle \hat{x}_{n} a \rangle (\langle E_{m}, H_{m} \rangle, (E_{n}, H_{n}))
\]

\[
= a^H \left( \sum_{m=1}^{N} \sum_{n=1}^{N} \hat{x}_{m} \hat{x}_{m}^H \right) a
\]

\[
= a^H A a
\]

(3)

where we have defined matrix \( A \) as

\[
A \triangleq \sum_{m=1}^{N} \sum_{n=1}^{N} \hat{x}_{m} \hat{x}_{m}^H (\langle E_{m}, H_{m} \rangle, (E_{n}, H_{n})) = X M X^H
\]

(4)

where \( X \triangleq [\hat{x}_1, \ldots, \hat{x}_N] \in \mathbb{C}^{N \times N} \) is a unitary matrix and \( M \in \mathbb{C}^{N \times N} \) is a Hermitian matrix with \([M]_{nm} \triangleq \langle \langle E_{n}, H_{m} \rangle, (E_{n}, H_{n}) \rangle \). The matrix \( M \) depends on the set \([\langle E_{n}, H_{m} \rangle]_{n=1}^N \) that in turn depends on the choice of the orthonormal basis \([\hat{x}_n]_{n=1}^N \); thus, \( X \) and \( M \) are set once an orthonormal basis has been chosen. By the definition of norm, (3) must be positive for any \( a \in \mathbb{C}^N \), so matrix \( A \) must be positive definite. Since \( X \) is a unitary matrix, it follows that \( M \) is positive definite too.

Observation: If \([\hat{\phi}_n, \lambda_n]_{n=1}^N \) is the set of orthonormal eigenvectors and eigenvalues of \( A \), then its eigenvalue decomposition corresponds to

\[
A = \Phi A \Phi^H.
\]

(5)

If we choose \( X = \Phi \) and compare (4) with (5), it follows that \( M = \Lambda \). If \((\tilde{E}_{n}, \tilde{H}_{n})\) is the field generated when \( a = \hat{\phi}_n \), then we can derive the following orthogonality relationship:

\[
\lambda_n \delta_{nm} = [A]_{nm} = [M]_{nm} = \langle \tilde{E}_{n}, \tilde{H}_{m} \rangle, \langle \tilde{E}_{n}, \tilde{H}_{n} \rangle).
\]

(6)

Now, we can solve the following constrained optimization problem:

\[
a_{\text{opt}} = \arg \max_{a \in \mathbb{C}^N} a^H A a
\]

s.t.

\[
a^H B a = \tilde{p}
\]

(7a)

(7b)

whose solution is

\[
a_{\text{opt}} = \frac{\tilde{p}}{\tilde{p}_{\text{max}}} \cdot \hat{\theta}_{\text{max}}
\]

(8)

where \( \hat{\theta}_{\text{max}} \) corresponds to the unitary eigenvector associated with the maximum eigenvalue (\( \mu_{\text{max}} \)) of the following generalized eigenvalue problem:

\[(A - \mu B) a = 0.
\]

(9)

A detailed proof can be found in Appendix B.

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### Algorithm 1 Algorithm to Solve Optimization Problem (12)

1. \( a_U \leftarrow A^{-1} X v \)
2. if \( a_U^H C a_U - h \leq 0 \) then
3. \( a_C \leftarrow a_U \)
4. else
5. Solve (52) for \( \xi \)
6. NormSquared \( \leftarrow +\infty \)
7. for \( \xi \in \Psi_{\xi}^+ \) do
8. \( a \leftarrow (A + \xi C)^{-1} X v \)
9. if \( \| (\Delta E, \Delta H) \|^2 < \text{NormSquared} \) then
10. \( a_C \leftarrow a \)
11. \( \text{NormSquared} \leftarrow \| (\Delta E, \Delta H) \|^2 \)
12. end if
13. end for
14. end if

---

### B. Minimum Error Field Norm Method

In this section, we are interested in deriving the excitation vector \( a \) that radiates the closest EM field to a target field \((\tilde{E}(r), \tilde{H}(r))\) in a specific region. Formally, we want to find the \( a \) vector that minimizes the norm of the error field, which is defined as

\[
\Delta E(r) \triangleq \tilde{E}(r) - \sum_{n=1}^{N} \langle \hat{x}_{n} a \rangle E_{n}(r)
\]

(10)

\[
\Delta H(r) \triangleq \tilde{H}(r) - \sum_{n=1}^{N} \langle \hat{x}_{n} a \rangle H_{n}(r).
\]

The solution to the optimization problem corresponds to

\[
a_U = \arg \min_{a \in \mathbb{C}^N} \| (\Delta E, \Delta H) \|^2 = A^{-1} X v = X M^{-1} v
\]

(11)

where we have defined \([v]_n \triangleq \langle \tilde{E}, \tilde{H}, (E_{n}, H_{n}) \rangle, n = 1, \ldots, N\).

The main drawback of the proposed solution consists in considering the target region only, without considering the behavior of the fields in the surroundings. To overcome this issue, the following constrained optimization problem can be considered:

\[
a_C = \arg \min_{a \in \mathbb{C}^N} \| (\Delta E, \Delta H) \|^2 \quad \text{s.t.} \quad a^H C a - h \leq 0
\]

(12a)

(12b)

where \( h \in \mathbb{R} \) and \( C \in \mathbb{C}^{N \times N} \) is a Hermitian matrix. The family of constraints expressed as (12b) includes several physically meaningful ones. One specific example will be provided in Section IV where we compute the optimal excitation array for a PWG. Algorithm 1 describes the steps to find the solution to the constrained optimization problem (12). In words, after computing the optimal solution of the unconstrained optimization problem \( a_U \), the algorithm checks whether it satisfies the constraint (12b); if it is satisfied, then the optimal solution corresponds to \( a_U \); otherwise, the algorithm performs the following steps. After numerically solving (52), the algorithm performs a for loop over the subset of positive real solutions \( \Psi_{\xi}^+ \). For each value of \( \xi \), the associated excitation vector \( a \) is computed as \( (A + \xi C)^{-1} X v \), and the associated
In the neighborhood of containing all the EM fields that satisfy the Maxwell equations in Fig. 2. In this case, the set the goodness of the solution. However, the “minimum error field method” guarantees that the solution is the one minimizing the norm of the error field. In Section IV-A, we introduce a metric to evaluate the goodness of the solution.

III. MAXIMUM ACTIVE POWER FLOW THROUGH A SURFACE

In this section, we apply the “maximum norm” method described in Section II-A with the inner product defined in Section III-A, whose induced norm corresponds to the active power flow through a surface. As a consequence, the “maximum norm” method provides the excitation array that maximizes the power flow through a given surface and corresponds to the “Poynting-based” method in [24]. Under the assumption that the surface belongs to the FF region of the array, in Section III-C, we derive an explicit formulation of matrix \( A \) as a function of the active pattern of each element. Furthermore, by considering ideal isotropic radiators, the above method converges to the “maximum BCE” method in [28]. In Sections III-B and III-D, the results are validated via numerical simulation.

A. Maximum Active Power Flow: General Case

First, we start by defining the following set.

**Definition 1 (O Set):** Given a surface \( S \) and its unit normal vector \( \hat{n} \), we define the set

\[
O(S) \triangleq \left\{ (E, H) \neq (0, 0) : \frac{1}{2} \int_S \Re\{E \times H^*\} \cdot \hat{n} d\Sigma > 0 \right\}
\]

containing all the EM fields that satisfy the Maxwell equations in the neighborhood of \( S \), which have a positive active power flow through \( S \) in the direction given by \( \hat{n} \).

As an example, we consider a volume \( V \) and its boundary \( \partial V \) with normal unit vector in the outward direction, as shown in Fig. 2. In this case, the set \( O(\partial V) \) contains all the EM fields that are generated by sources inside the volume \( V \). Now, we can define the following inner product.

**Definition 2 (EM Inner Product):** Given two sets of solutions of Maxwell equations \( (E_1, H_1), (E_2, H_2) \in \mathcal{O}(S) \), the following operator:

\[
\langle (E_1, H_1), (E_2, H_2) \rangle \triangleq \frac{1}{4} \int_S (E_1 \times H_2^* + E_2^* \times H_1) \cdot \hat{n} d\Sigma
\]

satisfies the properties of an inner product.

**Proof:** see Appendix D.

This is one possible choice of an inner product on EM fields. The norm induced from this inner product corresponds to

\[
\| (E, H) \|^2 = \langle (E, H), (E, H) \rangle = \frac{1}{4} \int_S (E \times H^* + E^* \times H) \cdot \hat{n} d\Sigma
\]

and

\[
[M]_{mm} = \frac{1}{4} \int_S (E_m \times \mathcal{H}_n^* + E_n^* \times \mathcal{H}_m) \cdot \hat{n} d\Sigma
\]

where \( A = X M X^H \). Thus, the optimal excitation vector in (8) corresponds to the one that maximizes the radiated power through the surface \( S \) when the total incident power is \( \bar{P} \).

This is exactly the same result as the one derived in [24], where the authors refer to it as the “Poynting-based” technique. Furthermore, the orthogonality relationship in (6) becomes

\[
\frac{1}{4} \int_S (\tilde{E}_m \times \tilde{H}_n^* + \tilde{E}_n^* \times \tilde{H}_m) \cdot \hat{n} d\Sigma = \lambda_n \delta_{mn}
\]

that is the same as in [24, eq. (18)], apart from a factor 2 that is accounted by \( \lambda_n \) in this article.

If, instead, we consider the following inner product:

\[
\langle (E_1, H_1), (E_2, H_2) \rangle \triangleq \frac{1}{4} \int_{\bigcup_{p=1}^P S_p} W(r) [E_1(r) \times H_2^*(r) + E_2^*(r) \times H_1(r)] \cdot \hat{n} d\Sigma
\]

\[
= \frac{1}{4} \sum_{p=1}^P \int_{S_p} W(r) [E_1(r) \times H_2^*(r) + E_2^*(r) \times H_1(r)] \cdot \hat{n} d\Sigma
\]

where \( \{S_p\}_{p=1}^P \) is a set of separate target surfaces, we obtain the EMMMPTE proposed in [16, Sec. III-B]. \( W(r) \in \mathbb{R} \) is an optional weighting function and must be chosen in such a way that the inner product in (18) is positive-definite.

B. Maximum Active Power Flow: General Case Simulation Results

In this section, we present the results obtained by simulating the configuration shown in Fig. 3. The array consists of \( N_x \times N_y \) half-wave dipoles placed \( \lambda_0/4 \) above an infinite ground plane resonating at 1 GHz. The dipoles are 142.1 mm long with radius equal to 30 \( \mu \)m and excitation gap 0.3 mm. \( d_x \) and \( d_y \) represent the interelement spacing along the \( x \)- and
The method for two different array configurations. Table I lists power. The "maximum norm" method is compared to the CP method, The surface \( S \) corresponds to the region where the radiated power must be maximized.

![Fig. 3. y-oriented half-wave dipole array placed above an infinite ground plane. The square surface \( S \) corresponds to the region where the radiated power must be maximized.](image)

![Fig. 4. Plot of the efficiency \( \eta = P_S / P_{\text{inc}} \) as a function of the square surface \( S \) side length \( L \). The solid curves correspond to Configuration 1 (negligible coupling effects), and the dashed curves correspond to Configuration 2 (stronger coupling effects).](image)

...y-directions, respectively. The surface \( S \) is a square of length \( L \) in the plane \( z = z_0 = 2\lambda_0 \). Fig. 4 shows the efficiency \( \eta = P_S / P_{\text{inc}} \), where \( P_S \) is the active power flow through the surface \( S \) in the \( z \)-direction and \( P_{\text{inc}} \) is the total incident power. The "maximum norm" method is compared to the CP method [18] for two different array configurations. Table I lists the parameters of the two configurations: both arrays occupy the same area, but in Configuration 2, the coupling between each element and the adjacent ones in the \( x \)-direction is above \(-10 \) dB, while in Configuration 1, the highest coupling experienced by an element corresponds to \(-15 \) dB. When the sides of the surface \( S \) are small compared to a wavelength, the performance of the CP method approaches the one of the "maximum norm" method. When \( L = 0.2\lambda_0 \), the difference between the efficiency of the two methods is 0.22 and 0.65 dB for Configurations 1 and 2, respectively. The larger difference for Configuration 2 is due to the stronger effects of coupling, which is neglected by the CP method. The "maximum norm" method outperforms the CP method in both configurations. Since the CP method focuses the field at \((0, 0, z_0)\), while the "maximum norm" method maximizes the power flow through \( S \), the difference in performance between the two synthesis techniques increases for larger arrays. Due to reflection losses induced by a stronger EM coupling, Configuration 2 loses around 4 dB compared to Configuration 1.

**C. Maximum Active Power Flow: FF Case**

In this section, we suppose that the surface \( S \) is located in the FF region of the array. Furthermore, we assume that the mutual coupling between the array elements can be neglected. Finally, we consider the canonical basis as the orthonormal basis, so \( X = I_N \). In the following, we use the symbol \((\hat{\cdot})\) to denote a quantity evaluated using the canonical basis \( a = \hat{e}_n \).

By referring to Fig. 5, \( J_n (a, r_n) \) and \( M_n (a, r_n) \) denote the electric and magnetic current densities within volume \( V_n' \) at the \( n \)-th array element in its local coordinate system when the excitation vector is equal to \( a \), respectively. Thus, \( J_n (r_n) \equiv J_n (\hat{e}_n, r_n) \) and \( M_n (r_n) \equiv M_n (\hat{e}_n, r_n) \). In the general case, when the EM coupling effects cannot be neglected, the electric and magnetic currents of element \( n \) depend on the whole excitation vector \( a \). The electric field radiated by the \( n \)-th array element at \( r \), which lies in the FF region of the whole array, in the global coordinate system reference can be expressed as [32]

\[
\tilde{\mathbf{E}}_n (r) = j \omega [ -\hat{r} \times \mathbf{\hat{A}}_n (r) \times \mathbf{\hat{r}} + \zeta \mathbf{\hat{r}} \times \mathbf{\hat{F}}_n (r) ]
\]

\[
= j \omega \frac{k}{4\pi r} \int_{V_n'} \int_{V_n'} \left[ -\hat{r} \times \mathbf{\hat{J}}_n (r_n') \times \mathbf{\hat{r}} + \frac{1}{\zeta} \hat{r} \times \mathbf{\hat{M}}_n (r_n') \right] e^{jk \mathbf{\hat{r}} \cdot \mathbf{r}_n'n'} d \tau' + \frac{1}{\zeta} \hat{r} \times \mathbf{\hat{M}}_n (r_n') \right] e^{jk \mathbf{\hat{r}} \cdot \mathbf{r}_n'n'} d \tau'
\]

where we have defined

\[
\tilde{\mathbf{g}}_n (\mathbf{\hat{r}}) \triangleq \int_{V_n'} \int_{V_n'} \left[ -\hat{r} \times \mathbf{\hat{J}}_n (r_n') \times \mathbf{\hat{r}} + \frac{1}{\zeta} \hat{r} \times \mathbf{\hat{M}}_n (r_n') \right] e^{jk \mathbf{\hat{r}}' \cdot \mathbf{r}_n'n'} d \tau'
\]

and \( \mathbf{\hat{H}}_n (r) = |\mathbf{\hat{r}} \times \mathbf{\hat{E}}_n (r)| / \zeta \). \( \mathbf{\hat{A}}_n (r) \) and \( \mathbf{\hat{F}}_n (r) \) are the electric and magnetic vector potentials, respectively, and \( k \),

---

**Table I**

| Configuration | \( N_x \) | \( d_x \) | \( N_y \) | \( d_y \) |
|---------------|----------|----------|----------|----------|
| 1             | 4        | 0.6\( \lambda_0 \) | 4        | 0.6\( \lambda_0 \) |
| 2             | 19       | 0.1\( \lambda_0 \) | 4        | 0.6\( \lambda_0 \) |

Fig. 5. Relative coordinate system centered at the position of the \( n \)-th array element \( \rho_n \). \( V' \) is a volume that bounds all the sources of the array in the global coordinate system. \( V_n' \) is a volume that bounds the sources of the \( n \)-th array element in its relative coordinate system.

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**Footnotes:**

[18] \[ \text{Ref. 18} \]

[32] \[ \text{Ref. 32} \]
λ, and ζ are the wavenumber, wavelength, and characteristic impedance of free space, respectively. \( r \triangleq \| r \| \) and \( \hat{r} \triangleq r / r \).

By substituting the expressions of \( \mathbf{E}_n \) and \( \mathbf{H}_n \) into (16) and (4), we obtain

\[
[A]_{nm} = [M]_{nm} = \frac{1}{4} \int_S \left( \mathbf{E}_m \times \mathbf{H}_n^* + \mathbf{E}_m^* \times \mathbf{H}_n \right) \cdot \hat{n} d\Sigma
= \frac{1}{2\zeta} \int_S \left( \mathbf{E}_m \cdot \hat{r} \right) \left( \hat{r} \cdot \hat{n} \right) d\Sigma
= \frac{\zeta}{8\lambda^2} \int_S \mathbf{g}_m(\hat{r}) \cdot \mathbf{g}_n^*(\hat{r}) e^{jk(\rho_s - \rho_n)} \cdot \hat{n} d\Sigma. \tag{21}
\]

By considering a planar array in the \( xy \) plane, (21) can be expressed in spherical coordinates as

\[
[A]_{nm} = \frac{\zeta}{8\lambda^2} \int_{\Omega} e^{jk\sin(\theta)} \left[ (\xi_n - x_n \cos(\phi) + (\eta_n - y_n) \sin(\phi)) \right] 
\mathbf{g}_m(\theta, \phi) \cdot \mathbf{g}_n^*(\theta, \phi) \sin(\theta) d\theta d\phi \tag{22}
\]

where \( \rho_n = x_n \hat{x} + y_n \hat{y} \), \( n = 1, \ldots, N \), and \( \Omega \) is the subtended solid angle.

Finally, to verify that the FF approximation of the “maximum norm” method coincides with the “maximum BCE” method in [28], we assume that all the elements are ideal isotropic radiators and there are no losses. Under the above assumptions, the same results as in [28] are obtained

\[
[A]_{nm} = \frac{\zeta}{8\lambda^2} \left\| \mathbf{g}_n \right\|^2 \int_{\Omega} e^{jk\sin(\theta)} \left[ \Delta x_{nm} \cos(\phi) + \Delta y_{nm} \sin(\phi) \right] 
\sin(\theta) d\theta d\phi
\]

\[
[B]_{nm} = \frac{\zeta}{8\lambda^2} \left\| \mathbf{g}_n \right\|^2 \int_0^{2\pi} \int_0^\pi e^{jk\sin(\theta)} \left[ \Delta x_{nm} \cos(\phi) + \Delta y_{nm} \sin(\phi) \right] 
\sin(\theta) d\theta d\phi
\]

\[
= \frac{\zeta}{8\lambda^2} \left\| \mathbf{g}_n \right\|^2 2\pi \int_0^\pi \sin \left( k \sqrt{\Delta x_{nm}^2 + \Delta y_{nm}^2} \right) 
\frac{\sin(k \sqrt{\Delta x_{nm}^2 + \Delta y_{nm}^2})}{k \sqrt{\Delta x_{nm}^2 + \Delta y_{nm}^2}} \tag{23}
\]

where we have defined \( \Delta x_{nm} \triangleq x_m - x_n \) and \( \Delta y_{nm} \triangleq y_m - y_n \). By computing the solution in (8) with the expressions in (23) for \( A \) and \( B \), this corresponds to the “maximum BCE” method. Thus, the “maximum BCE” method is a special case of the “maximum norm” method. Since the multiplicative term before the integrand in (8) is the same for \( A \) and \( B \), the optimal solution does not depend on its value. The “maximum BCE” method does not require any information about the radiating properties of the elements of the array and depends solely on the selected surface \( S \).

D. Maximum Active Power Flow: FF Case Simulation Results

In this section, we validate the results found in Section III-C by simulating a square array of \( 5 \times 5 \) half-wave dipoles as the ones in Section III-B. The interelement spacing is equal to 0.6\( \lambda_{0} \) in both directions, and the coupling between the elements is below \(-15 \) dB. We consider the efficiency of various methods when the surface \( S \) moves along the \( z \)-direction while keeping its subtended solid angle \( \Omega \) constant, as shown in Fig. 7. The length of the side of the square surface \( S \) at \( z \) can be computed as \( L = 2z\tan(\theta_0) \). The \( \theta_0 \) value has been chosen as the first null angle of the array factor of a uniform planar array (UPA), i.e., \( \theta_0 = \arcsin(\lambda_0/(N_4 d_4)) \). The Fraunhofer distance is approximately \( D_F = 2((N_4 d_4)^2 + (N_4 d_4)^2)/\lambda_0 = 36\lambda_0/ \). Fig. 7 shows the performance in terms of efficiency of the following four algorithms: the general “maximum norm,” the “maximum BCE,” the UPA, and the CP method with an assigned focal point at the center of the surface. When the surface \( S \) is closer to the array plane, the general “maximum norm” method and the CP method have better performance compared to the “maximum PTE” and the UPA since the latter methods are based on the FF approximation of the fields. However, when the distance grows, the “maximum BCE” method performance surpasses the CP one. Since the “maximum BCE” method does not consider the element factor, as described in Section III-C, there is still a gap between its performance and the one of the “maximum norm” method when the surface \( S \) is in the FF region. The performance of the UPA tends to the one of the CP method in the FF, as expected. As a final remark, the performance gap between the “maximum norm” (“maximum BCE”) and the CP (UPA) method in the NF (FF) increases when considering a wider solid angle \( \Omega \). This is confirmed by the results in Section III-B for the NF region.

In addition, we solved the “Maximum norm” optimization problem in Section II-A numerically with a genetic algorithm (GA) to validate the analytical closed-form solution. Fig. 8 shows the efficiency \( \eta \) of the best individual of each generation of a single run of the GA for three different values of \( z \), i.e., the distance between the surface \( S \) and the infinite ground plane. All the curves show a similar behavior: the efficiency...
η associated with the best individual of each generation of the GA is a monotonically nondecreasing function, which converges to the value of the analytical closed-form solution. While the solid and dotted curves have almost converged at the third generation, the dashed curve needs one more. Since we are considering a single run of the GA, this behavior depends solely on the set of parameters of the GA, and by rerunning the simulation, the number of generations before convergence is reached may change.

IV. MINIMUM ERROR FIELD NORM METHOD: PWG EXCITATIONS SYNTHESIS

In this section, we exploit the "minimum error field norm" method to select the optimal excitations for the synthesis of a PWG. This consists in an array that generates a planar wavefront in the QZ, which is a region in the NF of the array. By placing the antenna under test (AUT) in the QZ, it is possible to perform FF measurements in the NF region of the PWG. This property allows to drastically reduce the anechoic chamber size and, consequently, its cost. This is particularly interesting for the measurement of 5G base station antennas, which are large compared to the wavelength and require tens of meters to directly measure their radiation characteristics. The PWG must be large enough to generate a QZ, whose size is larger than the maximum size of the AUT, since the latter must completely lie inside the former. Some design criteria can be found in [10]. In this work, we assume that the geometrical properties of the PWG have been determined, and the objective is to find the optimal excitations. Since obtaining a linearly polarized plane wave in a spherical region of radius R and centered at (0, 0, z₀) is our objective, we exploit the following volumetric inner product:

\[
\langle (E_1, H_1), (E_2, H_2) \rangle \triangleq \iiint_V \frac{1}{4} E_1(r) \cdot E_2^*(r) d\tau
\]

(24)

where V is the target spherical region or QZ. The target field corresponds to a linearly polarized plane wave traveling in the direction normal to the planar array, which we assume to be the z-direction, i.e., \( \mathbf{E} = E_0 e^{-jk(z-z_0)} \mathbf{z} \) and \( \mathbf{H} = -E_0 e^{-jk(z-z_0)} \mathbf{z} / \eta \). As a consequence, the elements of \( M \) and \( v \), as defined in Sections II-A and II-B respectively, correspond to

\[
[M]_{lm} = \langle (E_m, \mathcal{H}_m), (E_n, \mathcal{H}_n) \rangle = \iiint_V \frac{E_m(r) \cdot E_n^*(r)}{4} d\tau
\]

(25)

and \( A = XMX^H \). These are all the elements needed to compute the excitation vector with the "minimum error field norm" method without constraints, as described in Section II-B. The main drawback of this technique is that the optimization problem depends only on the properties of the solution in the QZ, without considering the fields generated in the surroundings. To overcome this problem, we consider the solution that satisfies the following constraint: the active power flow through the circular surface S, which is obtained by cutting the spherical QZ with a plane parallel to the array plane passing through the QZ center, must be greater than the incident power divided by a factor \( \alpha > 1 \). As in Section III-A, the power flowing through the surface \( S \) can be expressed as

\[
P_S = a^H C_1 a \text{ with } C_1 = XM_1 X^H
\]

and

\[
[M_1]_{lm} = \frac{1}{4} \iiint_S (E_m \times \mathcal{H}_m^* + \mathcal{H}_n \times \mathcal{H}_n^*) \cdot \mathbf{n} d\Sigma
\]

(26)

where \( C_1 \) and \( M_1 \) correspond to \( A \) and \( M \) in Section III-A. The incident power corresponds to \( P_{inc} = a^H Ba \) with \( B = I_N/2 \), and thus, we can express the constraint as \( P_S \geq P_{inc}/\alpha \) or equivalently

\[
P_{inc} - \alpha P_S = a^H (B - \alpha C_1) a = a^H Ca \leq 0
\]

(27)

where we have defined the matrix \( C \triangleq B - \alpha C_1 \). As a consequence, the constraint belongs to the family defined by \( (12b) \) with \( h = 0 \). Since quantities related to power and energy can be expressed as bilinear forms of \( a \) in a linear system, it is now clear why the considered family of constraints contains several physically relevant ones. It is worth mentioning that, given the array geometry, the \( \alpha \) value cannot be chosen arbitrarily. A lower bound on \( \alpha \) can be found from the following inequality chain:

\[
\alpha_{min} \leq \frac{a^H Ba}{a^H C_1 a} \leq \alpha
\]

(28)

where the last inequality follows from (27) and the first one follows from the properties of Rayleigh quotients. Since the expression in the middle corresponds to a generalized Rayleigh quotient, then \( \alpha_{min} \) is the smallest eigenvalue of the generalized eigenvalue problem \( Ba = \mu C_1 a \). The physical meaning of the lower bound on \( \alpha \) can be explained as follows: once the geometry of the array and the surface \( S \) are assigned, there is a maximum amount of active power that can flow through the surface for a given input power, as discussed in Section III. Likewise, the upper bound on \( \alpha \) corresponds to \( \alpha_{max} = (a^H_{U} Ba_{U}) / (a^H_{U} C_{1} a_{U}) \). In fact, if we select \( \alpha > \alpha_{max} \), then

\[
a^H_{U} Ca_{U} = a^H_{U} (B - \alpha C_1) a_{U} < a^H_{U} (B - \alpha_{max} C_1) a_{U} = 0.
\]

(29)

As a consequence, the constraint would be inactive as it is already satisfied by the unconstrained excitation vector \( a_{U} \).
A. PWG: Simulation Results

In this section, we present the results obtained by simulating an array of 40 elements arranged uniformly in four concentric rings as the one described in [8] and shown in Fig. 9. Let \( n \) denote the index of each ring starting from the center. Then, the number of elements per ring is equal to \( 4n \), and the ring radius is \( R_n = nR_1 \) with \( R_1 = 80 \text{ mm}, n = 1, \ldots, 4 \). The array element is a \( y \)-oriented half-wave dipole placed \( \lambda_0/4 \) above an infinite ground plane resonating at 3.5 GHz with length 39.5 mm, radius \( 10^{-3}\lambda_0 \), and excitation gap \( 10^{-3}\lambda_0 \). The target region is a sphere of radius \( R_0 \). Once the geometry of the system has been assigned, the \( \alpha \) value belongs to the interval \([\alpha_{\text{min}}, \alpha_{\text{max}}]\). For this configuration, \( \alpha_{\text{min}} \) is equal to 3.0 and \( \alpha_{\text{max}} \) is equal to 30.8. To evaluate the performance of the PWG, we consider the following metric:

\[
\varepsilon^2(E, \bar{E}) \triangleq \frac{\| (\Delta E, \Delta H) \|^2}{\| (\bar{E}, \bar{H}) \|^2} = \frac{\iint_V \frac{1}{2} \| \Delta E \|^2 d\tau}{\iint_V \| \bar{E} \|^2 d\tau}
= \frac{\iint_V \| E - \bar{E} \|^2 d\tau}{\iint_V \| \bar{E} \|^2 d\tau} \quad (30)
\]

that we call relative error and represent the ratio between the electric field \( E \) and the target field \( \bar{E} \). This metric is significant for any target field, not only uniform ones, and considers both the phase and amplitude of the field. Two other common metrics used to evaluate the performance of a PWG are the maximum amplitude and phase deviation of the electric field. Fig. 10 shows the relative error as a function of \( \alpha \), with both quantities expressed in dB. Since \( \alpha \) represents the ratio between the incident power \( P_{\text{inc}} \) and the active power flow through the surface \( S \), the higher \( \alpha \), the lower the efficiency. It is evident that the relative error decreases monotonically when \( \alpha \) increases. To better understand this behavior, we consider Fig. 12, which shows the amplitude and phase of the \( y \) component of the electric field along the \( y \)-axis when \( x = 0 \) and \( z = z_0 = 950 \text{ mm} \) for four values of \( \alpha \). When \( \alpha \) is equal to 5.7 dB, the field is more concentrated inside the QZ, but both the amplitude and phase deviation are higher compared to the other cases. When \( \alpha \) increases, the field decreases less sharply outside the QZ, and both amplitude and phase ripples decrease. When \( \alpha \) is 13 dB, on one hand, the peak of the field is outside the QZ, meaning that most of the power flows outside the QZ; on the other hand, the ripple amplitude is the lowest. Fig. 13 shows the amplitude and phase of the \( y \) component of the electric field in the \( xy \) and \( yz \) planes cutting the QZ at its center for \( \alpha = 8.1 \text{ dB} \). The results for the \( xz \) plane are similar to the ones in the \( yz \) plane and have been omitted for brevity. Considering the three cutting planes passing through the QZ center, the maximum amplitude deviation is 0.84 dB, and the maximum phase deviation is 7.1°. These results are in good agreement with the ones presented in [9]. Although the two array geometries are similar, in [9], a wideband antenna is used as an element, while in the present work, a half-wave dipole is used.

To prove numerically the validity of the solution of the constrained optimization problem in Section II-B, we compared the analytical solution obtained with the “minimum error field norm” method against the one obtained with a GA for the first three values of \( \alpha \). Fig. 11 shows the evolution of the relative error \( \varepsilon \) of the best individual of each generation of a single run of the GA. Considering the curve corresponding to \( \alpha = 5.19 \text{ dB} \), it is worth mentioning that, although the best individuals in the generations between 3 and 23 have a lower relative error compared to the analytical solution, this is due to the fact that the GA has not converged yet, and these solutions are not satisfying the constraint, so their \( \alpha \) is higher.
Fig. 12. Plot of the amplitude (dB) and phase (°) of the y component of the PWG electric field along the y-axis (x = 0, z = 0.95 m) for multiple values of $\alpha = \frac{P_{inc}}{P_s}$. The two black vertical lines at ±R = ±0.24 m delimit the QZ. (a) Amplitude. (b) Phase.

Fig. 13. Plot of the amplitude (dB) and phase (°) of the y component of the PWG electric field in the $x\overline{y}$ and $y\overline{z}$ planes passing through the QZ center at (0, 0, 0.95 m) for $\alpha = \frac{P_{inc}}{P_s} = 8.1$ dB. The plots in the $xz$ plane are omitted since they do not significantly differ from the ones in the $yz$ plane. The black dashed circle delimits the QZ. (a) Amplitude ($xy$ plane). (b) Amplitude ($yz$ plane). (c) Phase ($xy$ plane). (d) Phase ($yz$ plane).

Finally, it is worth noting that Fig. 10 could be used for the design of the PWG excitations, as a tradeoff between the PWG performance in the assigned QZ and the percentage of power that is actually flowing through the target region.
Similar curves can be obtained by considering the maximum amplitude and phase deviation as a function of $\alpha$. Once the specifications for the field ripple in the QZ have been assigned, one can easily find the minimum $\alpha$ value that allows to meet them.

V. CONCLUSION

The concept of inner products has been applied to EM fields to synthesize the excitations of NF arrays. From the linearity of the problem, the EM field generated by the array can be expanded into a basis obtained by feeding the array with a set of orthogonal excitations. By exploiting the properties of inner products and their induced norm, two different methods to compute the excitations have been introduced and discussed. The “maximum norm” method provides the excitation vector that maximizes the induced norm for a given input power. In Section III, a proper selection of the EM inner product led to the maximum power flow method. The latter has been compared with the CP method, and the two give very close performance when the target surface is electrically small, as expected. Since the maximum power flow method can be applied to a surface located in any region of the antenna surrounding, it has been verified that it converges to the “maximum BCE” method when applied to a target surface in the FF region of the array. The “minimum error field norm” method can be applied for the synthesis of a target field. In Section IV, as an example, the constrained variant of the method has been applied to optimize the excitations of a PWG.

On the one hand, both methods provide a closed-form solution for the array excitations; on the other hand, to obtain the matrices and vectors needed to compute the solution, one has to numerically solve surface or volume integrals, depending on the chosen inner product, involving the EM fields of the basis. If the target region is in the FF of the array and the effects of mutual coupling can be neglected, then analytical expressions for the radiated fields can be used, as described in Section III-C. Otherwise, the EM field must be obtained via simulations or measurements, which may be quite time-consuming. Finally, the synthesis methods here discussed are based on global properties of the EM fields, such as the maximization of the active power flow through a surface, rather than on specific local conditions assigned at a grid of points in the antenna NF region.

APPENDIX A

EM FIELD OF A LINEAR SYSTEM

If we consider an N-port antenna system, we can explicitly express the dependence of the EM field on the excitation vector $\mathbf{a} \in \mathbb{C}^N$ using the notation $\mathbf{E}(\mathbf{a}, \mathbf{r})$, $\mathbf{H}(\mathbf{a}, \mathbf{r})$. If we restrict the discussion to linear systems, then we can express the electric field generated by the excitation vector $\mathbf{a} = \sum_{n=1}^{N} \zeta_n \mathbf{x}_n$ as

$$\mathbf{E}(\mathbf{a}, \mathbf{r}) = \sum_{n=1}^{N} \zeta_n \mathbf{E}(\mathbf{x}_n, \mathbf{r})$$

where $\{\zeta_n \in \mathbb{C}\}_{n=1}^{N}$ is a set of scalar complex values and $\{\mathbf{x}_n \in \mathbb{C}^N\}_{n=1}^{N}$ is a set of linearly independent vectors in $\mathbb{C}^N$. Since any set of $N$ linearly independent vectors forms a basis in $\mathbb{C}^N$, then we can express any vector in the space as a linear combination of the basis

$$\mathbf{a} = \sum_{n=1}^{N} \zeta_n \mathbf{x}_n = \mathbf{X} \zeta$$

with $\mathbf{X} \triangleq [\mathbf{x}_1, \ldots, \mathbf{x}_N] \in \mathbb{C}^{N \times N}$ and $\zeta \triangleq [\zeta_1, \ldots, \zeta_N]^T \in \mathbb{C}^N$. From the linear independence of its column vectors, it follows that $\mathbf{X}$ is invertible, so:

$$\zeta = \mathbf{X}^{-1} \mathbf{a}.$$  (33)

Substituting (33) into (31) and defining $\mathcal{E}_n(\mathbf{r}) \triangleq \mathbf{E}(\mathbf{x}_n, \mathbf{r})$ for $n = 1, \ldots, N$, we obtain

$$\mathbf{E}(\mathbf{a}, \mathbf{r}) = \sum_{n=1}^{N} ([\mathbf{X}^{-1}]_n \mathbf{a}) \mathcal{E}_n(\mathbf{r}).$$

If we consider an orthonormal basis $\{\hat{\mathbf{x}}_n\}_{n=1}^{N}$, then $\mathbf{X} = [\hat{\mathbf{x}}_1, \ldots, \hat{\mathbf{x}}_N]$ is a unitary matrix, and (34) becomes

$$\mathbf{E}(\mathbf{a}, \mathbf{r}) = \sum_{n=1}^{N} (\hat{\mathbf{x}}_n^H \mathbf{a}) \mathcal{E}_n(\mathbf{r}).$$

By following the same steps with the magnetic field, we obtain

$$\mathbf{H}(\mathbf{a}, \mathbf{r}) = \sum_{n=1}^{N} (\hat{\mathbf{x}}_n^H \mathbf{a}) \mathcal{H}_n(\mathbf{r})$$

with $\mathcal{H}_n(\mathbf{r}) \triangleq \hat{\mathbf{H}}(\hat{\mathbf{x}}_n, \mathbf{r})$, $n = 1, \ldots, N$.

APPENDIX B

SOLUTION OF THE OPTIMIZATION PROBLEM IN SECTION II-A

In this appendix, we derive the solution of the constrained optimization problem in Section II-A.

Proof: To solve the constrained optimization problem, we exploit the method of Lagrange multipliers. The Lagrangian function can be expressed as

$$\mathcal{L}(\mathbf{a}, \mathbf{\mu}) = \mathbf{a}^H \mathbf{A} \mathbf{a} - \mathbf{\mu}^T (\mathbf{a}^H \mathbf{B} \mathbf{a} - \mathbf{P})$$

$$= \mathbf{a}^H (\mathbf{A} - \mathbf{\mu B}) \mathbf{a} + \mathbf{\mu}^T \mathbf{P}$$

(37)

which is a real function of the complex variable $\mathbf{a}$ since the Lagrange multiplier $\mathbf{\mu}$ is a real value, and $\mathbf{A}$ and $\mathbf{B}$ are Hermitian matrices. If we define $\Re(\mathbf{a}) \triangleq \mathbf{a}_r$ and $\Im(\mathbf{a}) \triangleq \mathbf{a}_i$, the Lagrangian function becomes

$$\mathcal{L}(\mathbf{a}_r, \mathbf{a}_i, \mathbf{\mu}) = (\mathbf{a}_r^T - j\mathbf{a}_i^T) (\mathbf{A} - \mathbf{\mu B}) (\mathbf{a}_r + j\mathbf{a}_i) + \mathbf{\mu}^T \mathbf{P}.$$  (38)

Under the above assumptions, the partial derivative with respect to each element of $\mathbf{a}_r$ and $\mathbf{a}_i$ is

$$\frac{\partial}{\partial [\mathbf{a}_r]_n} \mathcal{L}(\mathbf{a}_r, \mathbf{a}_i, \mathbf{\mu}) = 2\Re \{\mathbf{e}_n^T (\mathbf{A} - \mathbf{\mu B}) (\mathbf{a}_r + j\mathbf{a}_i)\} = 0$$

$$\frac{\partial}{\partial [\mathbf{a}_i]_n} \mathcal{L}(\mathbf{a}_r, \mathbf{a}_i, \mathbf{\mu}) = 2\Im \{\mathbf{e}_n^T (\mathbf{A} - \mathbf{\mu B}) (\mathbf{a}_r + j\mathbf{a}_i)\} = 0$$

(39)

for $n = 1, \ldots, N$, which can be expressed in the following more compact form:

$$(\mathbf{A} - \mathbf{\mu B}) \mathbf{a} = \mathbf{0}.$$  (40)

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Equation (40) corresponds to a generalized eigenvalue problem, and \(\{(\hat{\vartheta}_n, \mu_n)\}_{n=1}^N\) are the associated eigenvectors and eigenvalues. By left-multiplying (40) by \(a^H\) and considering the constraint (7b), it follows that
\[
a^H A a = \mu (a^H B a) = \mu \hat{a}.
\]
and thus, the excitation vector that maximizes (7a) must be \(a_{\text{opt}} = c\hat{\vartheta}_{\text{max}}, c \in \mathbb{C}\), where \(\hat{\vartheta}_{\text{max}}\) is the eigenvector associated with the highest eigenvalue \(\mu_{\text{max}}\). By imposing the constraint (7b), we find that \(c\) must satisfy the following equation:
\[
|c| = \sqrt{\frac{P}{\hat{\vartheta}_{\text{max}}^H B \hat{\vartheta}_{\text{max}}}}
\]
so (8) is a suitable solution.

**APPENDIX C**

**SOLUTION OF THE OPTIMIZATION PROBLEM IN SECTION II-B**

In this appendix, we derive the solution of the optimization problem in Section II-B.

**Proof:** Let us start by explicating the norm squared of the error field as follows:
\[
\| (\Delta E, \Delta H) \|^2 = \| (\hat{E}, \hat{H}) - \sum_{n=1}^N (\hat{x}^H_n a)(E_n, H_n) \|^2
\]
\[
= \| (\hat{E}, \hat{H}) \|^2 + \sum_{n=1}^N \| (\hat{x}^H_n a)(E_n, H_n) \|^2
\]
\[
-2\Re \left\{ \langle (\hat{E}, \hat{H}), \sum_{n=1}^N (\hat{x}^H_n a)(E_n, H_n) \rangle \right\}
\]
(43)
where we exploited the following equality:
\[
\| u_1 - u_2 \|^2 = \langle u_1 - u_2, u_1 - u_2 \rangle
\]
\[
= \| u_1 \|^2 + \| u_2 \|^2 - \langle u_1, u_2 \rangle - \langle u_2, u_1 \rangle
\]
\[
= \| u_1 \|^2 + \| u_2 \|^2 - 2\Re \{\langle u_1, u_2 \rangle\}
\]
(44)
that holds for any pair \((u_1, u_2)\) belonging to an inner product space. We now proceed by expressing the second and third terms in the last equality in (43) in a vector form. The second term is exactly equal to the one in (3), while the third term corresponds to
\[
\langle (\hat{E}, \hat{H}), \sum_{n=1}^N (\hat{x}^H_n a)(E_n, H_n) \rangle = \sum_{n=1}^N \langle \hat{x}^H_n a \rangle \langle (\hat{E}, \hat{H}), (E_n, H_n) \rangle
\]
\[
= a^H X v.
\]
(45)
Substituting (3) and (45) into (43) yields
\[
\| (\Delta E, \Delta H) \|^2 = \| (\hat{E}, \hat{H}) \|^2 + a^H A a - 2\Re \{a^H X v\}.
\]
(46)
The solution to the constrained optimization problem has to satisfy the following conditions (Karush–Kuhn–Tucker conditions):
1) stationarity of \(\mathcal{L}(a, \xi)\);
2) \(a^H Ca - h \leq 0\);
3) \(\xi \geq 0\);
4) \(\xi (a^H Ca - h) = 0\),
where, we have defined the following function:
\[
\mathcal{L}(a, \xi) \triangleq \| (\Delta E, \Delta H) \|^2 + \xi (a^H Ca - h)
\]
\[
= a^H (A + \xi C) a - 2\Re \{a^H X v \} - \xi h + \| (\hat{E}, \hat{H}) \|^2.
\]
(47)
In order to find the stationary points of \(\mathcal{L}(a, \xi)\), we have to compute its partial derivatives with respect to the real and imaginary parts of each element of \(a\). The procedure is almost identical to the one found in Appendix B and leads to the following vectorial equation:
\[
(A + \xi C) a = X v,
\]
so Condition (1) is equivalent to Condition (1’)
\[
a = (A + \xi C)^{-1} X v
\]
(49)
if \(A + \xi C\) is invertible. In order to satisfy Condition (4) either \(\xi = 0\) first, and the case \(\xi > 0\) later.

Case 1 \((\xi = 0)\): Since \(\xi = 0\) Condition (3) and (4) are satisfied, and Condition (1’) becomes
\[
a_U = A^{-1} X v
\]
(50)
which is exactly the solution to the unconstrained optimization problem 11. Substituting the latter expression in Condition (2) leads to the following inequality:
\[
a_U^H Ca_U - h = (A^{-1} X v)^H C (A^{-1} X v) - h \leq 0.
\]
(51)
This case corresponds to the one where the solution of the unconstrained optimization problem already satisfies the constraint.

Case 2 \((\xi > 0)\): Since \(\xi > 0\), Condition (3) is satisfied. Conditions (2) and (4) are satisfied if and only if \(a^H Ca - h = 0\). By substituting Condition (1’) in the latter equation, we obtain the following nonlinear equation for \(\xi\):
\[
0 \overset{(a)}{=} v^H X^H (A + \xi C)^{-1} C (A + \xi C)^{-1} X v - h
\]
\[
\overset{(b)}{=} v^H X^H G^{-1} (G^{-1} A G^{-1} + \xi I_N)^{-2} G^{-1} X v - h
\]
\[
\overset{(c)}{=} v^H X^H G^{-1} V (D + \xi I_N)^{-2} V^{-1} G^{-1} X v - h
\]
\[
\overset{(d)}{=} w_1^H (D + \xi I_N)^{-2} w_2 - h = \sum_{n=1}^N \left| \langle w_1_{\text{in}} | w_2_{\text{in}} \rangle \right|^2 - h
\]
(52)
where we have used the fact that \(A + \xi C\) is Hermitian in (a). Since \(C\) is Hermitian, its eigenvalue decomposition corresponds to \(V^D V^H\), where \(V\) is a unitary matrix and \(D\) is a real diagonal matrix. In (b), we have defined \(G \triangleq V^D (1/2) V^H\), so \(C = G G^*\), where \(D^{(1/2)}\) is a diagonal matrix with \(D^{(1/2)}_{1,n} = \sqrt{D_{1,n}}\). Since the square root admits two solutions, the matrix \(G\) is not uniquely defined, but either choice leads to the same solution, except for numerical errors. In (c), we have used the eigenvalue decomposition \(G^{-1} A G^{-1} = V D^H V^{-1}\). Finally, in (d), we have defined the two vectors \(w_1 \triangleq V^H (G^{-1})^H X v\) and \(w_2 \triangleq V^{-1} G^{-1} X v\).
Equation 52 can be recast into a polynomial equation of order $2N$ by multiplying both terms by $\prod_{n=1}^{N} (D_{n,n} + \xi)^2$. As a consequence, the set of solutions $\Psi^e_\xi$ contains $2N$ complex values. Let $\Psi^+ \subseteq \Psi^e_\xi$ be the subset containing the real positive solutions. Since $\xi$ must satisfy Condition (3), the optimal value corresponds to the one that minimizes $\|\Delta E, \Delta H\|^2$ among the ones in $\Psi^+$. Once the optimal $\xi$ value is found, the associated excitation vector can be computed using (49).

APPENDIX D

PROOF OF THE PROPERTIES OF THE EM INNER PRODUCT

In this appendix, we prove that the operator in Definition 2 satisfies all the properties of an inner product. Given $(E_1, H_1), (E_2, H_2) \in \mathcal{E}(S)$ and $\lambda_1, \lambda_2 \in \mathbb{C}$, the three following properties must be satisfied [33].

1) Conjugate Symmetry:
$$\langle (E_1, H_1), (E_2, H_2) \rangle = \langle (E_2, H_2), (E_1, H_1) \rangle^*.$$  

2) Linearity in the First Argument:
$$\langle \lambda_1 (E_1, H_1) + \lambda_2 (E_2, H_2), (E, H) \rangle = \lambda_1 \langle (E_1, H_1), (E, H) \rangle + \lambda_2 \langle (E_2, H_2), (E, H) \rangle.$$  

3) Positive Definiteness:
$$\| (E, H) \|^2 = \langle (E, H), (E, H) \rangle \geq 0$$  
where the equality $\| (E, H) \|^2 = 0$ holds if and only if $(E, H) = (0, 0)$, $\forall \mathbf{r} \in S$.

Proof: Properties 1 and 2 are a direct consequence of the presence of the conjugate operator in the second argument and the linearity of the integral operator, respectively, while property 3 follows from the fact that:
$$\| (E, H) \|^2 = \frac{1}{2} \iint_S |\mathbf{E} \times \mathbf{H}^*| \cdot d\mathbf{a} \Sigma \geq 0$$  
where the last inequality holds by Definition 1.

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