A Lattice Fermion Doublet With A Generalization Of The Ginsparg-Wilson Relation

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Abstract

We present a new staggered discretization of the Dirac operator. In comparison with standard staggered fermions, real and imaginary parts are located in different nodes. Doubling gives only a doublet of Dirac fermions which we propose to interpret as a physical (lepton or quark) doublet. Contrary to usual staggered fermions, we have no exact chiral symmetry but obtain a generalization of the Ginsparg-Wilson relation.

1 Introduction

In this paper we present a new discretization of the Dirac equation. In comparison with staggered fermions it creates not four but only two flavours of Dirac fermions. This has been reached by placing not only different spin components, but also their real and imaginary parts into different nodes. These sixteen steps of freedom (two fermions) can be understood, in some sense, as the result of “doubling” of a real scalar step of freedom $\varphi(n)$ on the lattice.

Moreover, these two fermions live on different sub-lattices ($\psi_o$ on “odd” nodes, $\psi_e$ on “even” nodes), thus, we obtain also a single fermion (eight steps of freedom) on the lattice by omitting half of the lattice nodes. But, if the neutrino is a Dirac particle, all fermions appear in doublets of Dirac particles. In this context, the appearance of a fermion doublet in this discretization may

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be not a bug but a feature which allows to explain why Dirac particles appear in such doublets.

In our approach the complex structure is an operator among others. Moreover, there is no natural complex structure, but, instead, a quaternionic structure. The nature of this structure is a topic of future research.

Once a complex structure is defined, we have to define the operator $\gamma^5$ on the grid. Now, there is no exact chiral symmetry. Instead, we obtain formulas for some operators $\gamma^5$ on the grid which define a generalization of the Ginsparg-Wilson relation.

## 2 A Real Representation Of The Dirac Algebra

We forget – for some time – about the complex structure. Instead of the usual representations with four complex fields, we use an eight-dimensional real representation of the operators $\gamma^\mu$ defined here by their linear combination with $\partial_\mu$: $\gamma^0 \partial_0 - \gamma^i \partial_i = \text{def} \begin{pmatrix} \partial_0 & \partial_1 & \partial_2 & \partial_3 \\ -\partial_1 & -\partial_0 & \partial_2 & \partial_3 \\ -\partial_2 & -\partial_0 & -\partial_1 & \partial_3 \\ -\partial_3 & -\partial_2 & -\partial_0 & -\partial_1 \end{pmatrix}$ (1)

In the context of this representation, it seems also natural to define (by their linear combination with scalar parameters $m_i$) the following operators $\beta^i$:

$m_i \beta^i = \text{def} \begin{pmatrix} m_1 & m_2 & m_3 \\ m_1 & m_2 & m_3 \\ m_2 & -m_1 & m_3 \\ m_2 & -m_1 & m_3 \\ m_3 & -m_1 & -m_2 \\ m_3 & -m_1 & -m_2 \\ m_3 & -m_2 & m_1 \\ m_3 & -m_2 & m_1 \end{pmatrix}$ (2)
The following operator equation holds:

\[(\gamma^0 \partial_0 - \gamma^i \partial_i + m_i \beta^i)^2 = -\Box + \delta^{ij} m_i m_j \] (3)

This can be easily seen – this operator iterates three times, in each coordinate direction, the same trick:

\[
\begin{pmatrix}
A & (m_i + \partial_i)I \\
(m_i - \partial_i)I & -A
\end{pmatrix}^2 = (A^2 + (m_i + \partial_i)(m_i - \partial_i)I) \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \] (4)

As follows immediately, the \(\gamma^\mu\) define a representation of the Dirac matrices, and the matrices \(\beta^i\) fulfil the following anticommutation relations:

\[\beta^i \beta^j + \beta^j \beta^i = \delta^{ij}\] (5)

and anticommute with all \(\gamma^\mu\):

\[\beta^i \gamma^\mu + \gamma^\mu \beta^i = 0\] (6)

It is also easy to see (and to generalize to arbitrary dimension) that

\[\gamma^0 (\gamma^1 \beta^1)(\gamma^2 \beta^2)(\gamma^3 \beta^3) = 1.\] (7)

### 3 Complex Structures

In this representation we have not yet defined a complex structure. There are several things in the standard approach to Dirac fermions which depend on them.

First, in the standard approach a Hermitean scalar product \(\langle ., . \rangle\) is widely used. We have only a standard Euclidean scalar product \((., .)\) yet. Now, for a complex structure \(i\) these notions are closely related in a simple way:

The Hermitean scalar product defines an Euclidean scalar product by

\[(\psi, \phi) = \frac{1}{2}(\langle \psi, \phi \rangle + \langle \phi, \psi \rangle)\] (8)

\footnote{This observation also suggests how to iterate this construction to arbitrary dimension.}
so that \((i\psi, i\phi) = (\psi, \phi)\). For a complex structure \(i\), \(i^2 = -1\), with this property this Hermitean scalar product is defined by the Euclidean scalar product by defines the Hermitean scalar product

\[
\langle \psi, \phi \rangle = (\psi, \phi) - i(\psi, i\phi).
\]  

(9)

Thus, we should not care about the Hermitean scalar product, the Euclidean scalar product is all we need. Note also that for a complex linear operator \(A\), which is a real linear operator with \([A, i] = 0\), the Hermitean adjoint operator \(A^+\) and the Euclidean adjoint operator \(A^*\) coincide: \(\langle A^*\psi, \phi \rangle = \langle \psi, A\phi \rangle\). As a consequence, the classical properties of the \(\gamma\)-matrices

\[
(\gamma^\mu)^+ = \gamma^0\gamma^\mu\gamma^0
\]  

(10)

are equivalent to

\[
(\gamma^\mu)^* = \gamma^0\gamma^\mu\gamma^0.
\]  

(11)

These properties are fulfilled in our representation for the standard Euclidean scalar product \(\langle ., . \rangle\) in \(\mathbb{R}^8\).

Next, the “classical” operator \(\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3\) also depends on the complex structure. A natural replacement which does not depend on it – the expression \(\gamma^0\gamma^1\gamma^2\gamma^3\) – we denote with \(\iota\):

\[
\iota = \text{def} \gamma^0\gamma^1\gamma^2\gamma^3 = \beta^1\beta^2\beta^3 \quad \iota\gamma^\mu + \gamma^\mu\iota = 0 \quad (\iota)^2 = -1
\]  

(12)

But, of course, we have to introduce a complex structure if we want to connect the Dirac fermion in the usual way with gauge fields. The properties we need for a complex structure \(i\) are \(i^{-1} = i^* = -i\) and \([\gamma^\mu, i] = 0\). Now, an interesting point is that there are several candidates for such a structure:

\[
i = \beta^1\beta^2 = i\beta^3
\]  

(13)

\[
j = \beta^2\beta^3 = i\beta^1
\]  

(14)

\[
k = \beta^3\beta^1 = i\beta^2
\]  

(15)

which together define a quaternionic structure:

\footnote{The classical representation \(ij = k\) can be obtained using reverse signs for \(i, j, k\), but we prefer this sign convention because it gives \(\gamma^5 = \beta^3\).}
\[ ij = -ji = -k; \quad jk = -kj = -i; \quad ki = -ik = -j; \quad i^2 = j^2 = k^2 = -1 \quad (16) \]

For each candidate \( i \) for a complex structure, we obtain an own operator \( \gamma^5 = -i \). Especially for \( i = \iota \beta^3 \) we obtain \( \gamma^5 = -\iota \beta^3 t = \beta^3 \). Thus, it seems that to fix the complex structure we somehow have to break spatial symmetry, prefer one spatial direction.

4 Discretization Of The Dirac Equation

Our representation is appropriate for a discretization of the Dirac equation on a regular hyper-cubic lattice. It can be obtained in a quite simple way: We start with a naive central difference approximation

\[ \partial_t \psi(n) \rightarrow \frac{1}{2a_t} (\psi(n + a_t) - \psi(n - a_t)). \quad (17) \]

This naive discretization leads to the problem of “fermion doubling”. The continuous limit of this set of discrete equations gives not only the original Dirac equation, but also additional, highly oscillating components, the “doublers”. In classical computations such doublers may be often ignored, but in quantum computations, where the number of steps of freedom is important (Pauli principle) this is no longer possible. We obtain in each direction a factor to, thus, \( 2^4 = 16 \) doublers. Fortunately, eight pairs of doublers decouple in a really simple way: It is sufficient to hold only one of the eight real components \( \psi^a \) per node. On the three-dimensional reference cube \( (\varepsilon_1, \varepsilon_2, \varepsilon_3) \), \( \varepsilon_i \in \{0, 1\} \) we obtain the following locations for the eight components:

\[
\begin{align*}
\psi^0 & \text{ located at } (0, 0, 0), & \psi^4 & \text{ located at } (0, 0, 1); \\
\psi^1 & \text{ located at } (1, 0, 0), & \psi^5 & \text{ located at } (1, 0, 1); \\
\psi^2 & \text{ located at } (0, 1, 0), & \psi^6 & \text{ located at } (0, 1, 1); \\
\psi^3 & \text{ located at } (1, 1, 0), & \psi^7 & \text{ located at } (1, 1, 1); 
\end{align*}
\]

What remains are sixteen steps of freedom (eight steps of freedom on two time steps which we need because of our use of central differences in time) which corresponds to a doublet of Dirac fermions\(^3\). Note that our discretization may be interpreted as a way to discretize the d’Alembert equation \(^3\). This is a variant of a well-known approach to solve the “fermion doubling” problem – so-called staggered fermions. The standard staggered grid approach reduces the doublers only by a factor four. Because we ignore the complex structure of the standard representation, we are free to place “real” and “imaginary” part of the complex fields into different nodes. This gives the additional reduction by factor two.
for a single scalar step of freedom $\varphi(n)$ with central differences, which gives $2^4 = 16$ doublers.

Now, the last doublet decouples too, but in a slightly less trivial way: We can distinguish “even” and “odd” nodes on the full space-time lattice. The central difference equations on even (odd) nodes connects only values on odd (even) nodes. Thus, we obtain two fermions $\psi_e$ and $\psi_o$ on even resp. odd nodes. On the four-dimensional reference cube $(\varepsilon_0, \varepsilon_1, \varepsilon_2 \varepsilon_3)$, $\varepsilon_i \in \{0, 1\}$ we have

$$
\begin{align*}
\psi_e^0 & \text{ located at } (0, 0, 0, 0), & \psi_e^4 & \text{ located at } (1, 0, 0, 1); \\
\psi_e^1 & \text{ located at } (1, 1, 0, 0), & \psi_e^5 & \text{ located at } (0, 1, 0, 1); \\
\psi_e^2 & \text{ located at } (1, 0, 1, 0), & \psi_e^6 & \text{ located at } (0, 0, 1, 1); \\
\psi_e^3 & \text{ located at } (0, 1, 1, 0), & \psi_e^7 & \text{ located at } (1, 1, 1, 1);
\end{align*}
$$

$$
\begin{align*}
\psi_o^0 & \text{ located at } (1, 0, 0, 0), & \psi_o^4 & \text{ located at } (0, 0, 0, 1); \\
\psi_o^1 & \text{ located at } (0, 1, 0, 0), & \psi_o^5 & \text{ located at } (1, 1, 0, 1); \\
\psi_o^2 & \text{ located at } (0, 0, 1, 0), & \psi_o^6 & \text{ located at } (1, 0, 1, 1); \\
\psi_o^3 & \text{ located at } (1, 1, 1, 0), & \psi_o^7 & \text{ located at } (0, 1, 1, 1);
\end{align*}
$$

But, instead of removing one sub-mesh to describe a single Dirac fermion, we propose to accept above doublers as a way to describe a physically meaningful flavour doublet. Remarkably, if the neutrino is a standard Dirac particle, then all fermions of the standard model appear in doublets. The appearance of a fermion doublet in our approach may be, therefore, not a bug but a feature which allows to explain the existence of these doublets.

### 4.1 Lagrange Formalism

To define the Lagrange formalism on the lattice note that the equations for the $\psi_o$ are located at the nodes of $\psi_e$ and reverse. Therefore, we can use the Lagrangian

$$
L = \sum \psi_e(D\psi_o) = - \sum \psi_o(D\psi_e) \tag{21}
$$

and obtain the equation for $D\psi_{o/e} = 0$ as the Euler-Lagrange equation for $\psi_{e/o}$. We can also rewrite this Lagrangian as

$$
L = \frac{1}{2} \sum \varphi(-1)^{\epsilon_0} \gamma^0(D\varphi) \tag{22}
$$
4.2 Fermion Families and Lattice Distortions

We have observed that to fix the complex structure we somehow have to break spatial symmetry. But there is a simple way out of this. Instead of one scalar step of freedom \( \varphi(n) \) on the lattice, we can consider a vector field – thus, three components \( \varphi^i(n) \). Now, each component has a natural “preferred direction” and, therefore, a natural complex structure.

Moreover, a vector on a lattice is a quite natural step of freedom. It is, for example, the natural way to describe lattice distortions with a shift vector field \( u^i(n) \).

On the other hand, in the standard model we have three fermion families – three copies of each fermion. This suggests to explain on the kinematic level the three fermion families using the hypothesis that the fundamental steps of freedom of the “theory of everything” are three-dimensional vector fields \( u^i(n) \) of spatial distortions.

5 Chiral Symmetry On The Lattice

One problem with standard staggered fermions \([7]\) is that they have the wrong number of doublers (four) to allow a natural physical interpretation in the standard model. In our approach, we have only a pair of Dirac fermions, and pairs of fermions appear in the standard model as quark pairs as well as lepton pairs (if the neutrino is a Dirac particle).

The other problem is that there is exact chiral \( \gamma^5 \) symmetry on the lattice. As a consequence, the doublers appear in pairs with reverse chiral charge. This does not fit the situation in the standard model (cf. \([4]\)). Now, in our approach we do not have exact chiral \( \gamma^5 \) symmetry. Instead, we have a replacement for this symmetry. This replacement fulfills properties which define a generalization of the famous Ginsparg-Wilson (GW) relation \([3]\).

Let’s assume now that one of the complex structures, namely \( i = \imath \beta^3 \), has been chosen. To understand chiral symmetry we have to define the operator \( \gamma^5 = \beta^3 \) on the lattice. Note that it cannot be anymore a pointwise operator as for Wilson fermions and staggered fermions – it connects components which are located in different points. Now, we propose to consider the following operator as a candidate for \( \gamma^5 \) on the lattice:

\[
(\gamma^5 \phi)(n_{even}) = \phi(n_{even} - h_z), \quad (\gamma^5 \phi)(n_{odd}) = \phi(n_{odd} + h_z)
\]

(23)

It is easy to see that it approximates the continuous \( \gamma^5 \). More interesting is that some exact properties remain valid:
\[(\gamma^5)^* = \gamma^5, \quad (\gamma^5)^2 = 1\] (24)

We can also define, as an alternative, the operator \(\tilde{\gamma}^5\) by

\[(\tilde{\gamma}^5 \phi)(n_{\text{even}}) = \phi(n_{\text{even}} + h_z), \quad (\tilde{\gamma}^5 \phi)(n_{\text{odd}}) = \phi(n_{\text{odd}} - h_z)\] (25)

Similarly, we obtain

\[(\tilde{\gamma}^5)^* = \tilde{\gamma}^5, \quad (\tilde{\gamma}^5)^2 = 1\] (26)

If we define the operators \(V, O\) by \(\tilde{\gamma}^5 = \gamma^5 V = \gamma^5(1 - aO)\) we obtain the Ginsparg-Wilson (GW) relation for \(O\):

\[O\gamma^5 + \gamma^5 O = aO\gamma^5 O\] (27)

Moreover, we have also the following important commutation properties with \(D\)

\[\tilde{\gamma}^5 D + D\gamma^5 = 0, \quad \gamma^5 D + D\tilde{\gamma}^5 = 0,\] (28)

\[VD - DV = 0, \quad OD - DO = 0.\] (29)

This allows to define two sets of chiral projector operators

\[\tilde{P}_\pm = \frac{1}{2}(1 \pm \tilde{\gamma}^5), \quad P_\pm = \frac{1}{2}(1 \pm \gamma^5).\] (30)

Similar pairs of projectors play a central role in approaches to chiral gauge theory based on the GW relation (4), (9) and its generalizations (3), as domain wall fermions (10), Neuberger’s overlap operator (8), proposals by Fujikawa (2) and Chiu (1).

On the other hand, it should be noted that our approach does not fit exactly into these schemes. The operators \(V, O\) do not have the spectral properties of the similar operators considered, for example, by (4), (3). It seems that these differences may be understood as caused by different aims. The aim of the standard GW approach is to obtain a single Weyl fermion on
the lattice, without any doublers. In our approach we allow some doubling which gives a pair of Dirac fermions on the lattice, but we want nontrivial chiral symmetry to allow nontrivial chiral interactions between the doublers.

If the properties of $\gamma^5, \tilde{\gamma}^5$ we have found are sufficient to develop a consistent chiral gauge theory on this lattice remains to be shown. Note that the situation is further complicated by the non-pointwise character of the complex structure which remains to be understood.

References

[1] T.-W. Chiu, Phys.Lett. B 521, 429 (2001)
[2] K. Fujikawa, Nucl.Phys.B 589, 487 (2000)
[3] P.H.Ginsparg, K.G.Wilson, Phys.Rev.D 25, 2649 (1982)
[4] M. Golterman, Lattice chiral gauge theories, hep-lat/0011027
[5] R. Gupta, Introduction to lattice QCD, hep-lat/9807028
[6] W. Kerler, More chiral operators on the lattice, hep-lat/0204008
[7] J. Kogut, L. Susskind, Phys. Rev. D11, 395 (1975)
[8] H. Neuberger, Phys.Lett. B 417, 141, (1998)
[9] M. Lüscher, hep-th/0102028
[10] Y. Shamir, Nucl. Phys. B 406, 90 (1993)