Foundations of generalized Prabhakar-Hilfer fractional calculus with applications

GEORGE A. ANASTASSIOU

1 Department of Mathematical Sciences
University of Memphis, Memphis,
TN 38152, U.S.A.
ganastss@memphis.edu

ABSTRACT

Here we introduce the generalized Prabhakar fractional calculus and we also combine it with the generalized Hilfer calculus. We prove that the generalized left and right side Prabhakar fractional integrals preserve continuity and we find tight upper bounds for them. We present several left and right side generalized Prabhakar fractional inequalities of Hardy, Opial and Hilbert-Pachpatte types. We apply these in the setting of generalized Hilfer calculus.

RESUMEN

Introducimos el cálculo fraccionario generalizado de Prabhakar y también lo combinamos con el cálculo generalizado de Hilfer. Demostramos que las integrales fraccionarias generalizadas de Prabhakar izquierda y derecha preservan la continuidad y encontramos cotas superiores ajustadas para ellas. Presentamos diversas desigualdades fraccionarias generalizadas de Prabhakar izquierda y derecha de tipos Hardy, Opial y Hilbert-Pachpatte. Aplicamos estos resultados en el contexto del cálculo generalizado de Hilfer.

Keywords and Phrases: Prabhakar fractional calculus, Hilfer fractional calculus, fractional Hardy, Opial and Hilbert-Pachpatte inequalities.

2020 AMS Mathematics Subject Classification: 26A33, 26D10, 26D15.
1 Background

During the last 50 years fractional calculus due to its wide applications to many applied sciences has become a main trend in mathematics. Its predominant kinds are the old Riemann-Liouville fractional calculus and the newer one of Caputo type. Around these two versions have been built a plethora of other variants and all of these involve singular kernels. More recently researchers presented also new fractional calculi of non-singular kernels.

The recent Hilfer fractional calculus unifies the Riemann-Liouville and Caputo fractional calculi and the Prabhakar fractional calculus unifies both singular and non-singular kernel fractional calculi.

Finally the newer Hilfer-Prabhakar fractional calculus is the most general one unifying all trends and for different values of its parameters we get the particular fractional calculi. In this article we present and employ unifying advanced and generalized versions of Prabhakar and Hilfer-Prabhakar fractional calculi and we establish related unifying fractional integral inequalities of the following types: Hardy, Opial and Hilbert-Pachpatte. The advantage of this unification is the uniform action taken in describing the various natural phenomena.

We are inspired by [7], [6] and [1]. We start by introducing our own generalized $\psi$-Prabhakar type of fractional calculus, then mixing it with the $\psi$-Hilfer fractional calculus. Then, we prove a variety of generalized Hardy, Opial and Hilbert-Pachpatte type left and right fractional integral inequalities related to $\psi$-Hilfer ([8]) and $\psi$-Prabhakar fractional calculi. We involve several functions.

We consider the Prabhakar function (also known as the three parameter Mittag-Leffler function), (see [4, p. 97]; [3])

$$E_{\alpha,\beta}^\gamma (z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{k!} \Gamma (\alpha k + \beta) z^k, \quad (1.1)$$

where $\Gamma$ is the gamma function; $\alpha, \beta, \gamma \in \mathbb{R} : \alpha, \beta > 0$, $z \in \mathbb{R}$, and $(\gamma)_k = \gamma (\gamma + 1) \cdots (\gamma + k - 1)$.

It is $E_{\alpha,\beta}^0 (z) = \frac{1}{\Gamma (\beta)}$.

Let $a, b \in \mathbb{R}$, $a < b$ and $x \in [a, b]$; $f \in C ([a, b])$. Let also $\psi \in C^1 ([a, b])$ which is increasing. The left and right Prabhakar fractional integrals with respect to $\psi$ are defined as follows:

$$\left( e_{\mu,\omega,a}^{\rho,\psi} \right) (x) = \int_a^x \psi' (t) \left( \psi (x) - \psi (t) \right)^{\mu-1} E_{\rho,\mu}^\gamma [\omega (\psi (x) - \psi (t))] f (t) \, dt, \quad (1.2)$$

and

$$\left( e_{\rho,\omega,b}^{\mu,\psi} \right) (x) = \int_x^b \psi' (t) \left( \psi (t) - \psi (x) \right)^{\mu-1} E_{\rho,\mu}^\gamma [\omega (\psi (t) - \psi (x))] f (t) \, dt, \quad (1.3)$$

where $\rho, \mu > 0$; $\gamma, \omega \in \mathbb{R}$.

Functions (1.2) and (1.3) are continuous, see Theorem 3.1.

Next, additionally, assume that $\psi' (x) \neq 0$ over $[a, b]$.

Let $\psi, f \in C^N ([a, b])$, where $N = \lceil \mu \rceil$, ($\lceil \cdot \rceil$ is the ceiling of the number), $0 < \mu \in \mathbb{N}$. We define the
ψ-Prabhakar-Caputo left and right fractional derivatives of order \( \mu \) as follows (\( x \in [a, b] \)):

\[
\left( C D_{\rho,\mu,\omega,a+}^{\gamma,\psi} \right) (x) = \int_a^x \psi' (t) (\psi (x) - \psi (t))^{N-\mu-1} dt,
\]

\[
E_{\rho,N-\mu}^{-\gamma} \left[ \psi (x) - \psi (t) \right]^\rho \left( \frac{1}{\psi' (t)} \right)^N f (t) dt,
\]

and

\[
\left( C D_{\rho,\mu,\omega,b-}^{\gamma,\psi} \right) (x) = \int_x^b \psi (t) (\psi (x) - \psi (t))^{N-\mu-1} dt,
\]

\[
E_{\rho,N-\mu}^{-\gamma} \left[ \psi (x) - \psi (t) \right]^\rho \left( \frac{1}{\psi' (t)} \right)^N f (t) dt.
\]

One can write (see (1.4), (1.5))

\[
\left( C D_{\rho,\mu,\omega,a+}^{\gamma,\psi} \right) (x) = \left( e_{\rho,N-\mu-1,\omega,a+}^{\gamma,\psi} f \right) (x),
\]

and

\[
\left( C D_{\rho,\mu,\omega,b-}^{\gamma,\psi} \right) (x) = \left( e_{\rho,N-\mu-1,\omega,b-}^{\gamma,\psi} f \right) (x),
\]

where

\[
f_{\psi}^{(N)} (x) = f_{\psi}^{(N)} := \left( \frac{1}{\psi' (x)} \right)^N f (x),
\]

\( \forall x \in [a, b] \).

Functions (1.6) and (1.7) are continuous on [a, b].

Next we define the ψ-Prabhakar-Riemann-Liouville left and right fractional derivatives of order \( \mu \) as follows (\( x \in [a, b] \)):

\[
\left( RL D_{\rho,\mu,\omega,a+}^{\gamma,\psi} \right) (x) = \left( \frac{1}{\psi' (x)} \right)^N \int_a^x \psi (t) (\psi (x) - \psi (t))^{N-\mu-1} \]

\[
E_{\rho,N-\mu}^{-\gamma} \left[ \psi (x) - \psi (t) \right]^\rho f (t) dt,
\]

and

\[
\left( RL D_{\rho,\mu,\omega,b-}^{\gamma,\psi} \right) (x) = \left( \frac{1}{\psi' (x)} \right)^N \int_x^b \psi (t) (\psi (x) - \psi (t))^{N-\mu-1} \]

\[
E_{\rho,N-\mu}^{-\gamma} \left[ \psi (x) - \psi (t) \right]^\rho f (t) dt.
\]

That is we have

\[
\left( RL D_{\rho,\mu,\omega,a+}^{\gamma,\psi} \right) (x) = \left( e_{\rho,N-\mu-1,\omega,a+}^{\gamma,\psi} f \right) (x),
\]

and

\[
\left( RL D_{\rho,\mu,\omega,b-}^{\gamma,\psi} \right) (x) = \left( e_{\rho,N-\mu-1,\omega,b-}^{\gamma,\psi} f \right) (x),
\]

\( \forall x \in [a, b] \).
We define also the $\psi$-Hilfer-Prabhakar left and right fractional derivatives of order $\mu$ and type $0 \leq \beta \leq 1$, as follows

$$
\left( H^\gamma,\beta;\psi \right)_{\rho,\mu,\omega,a+f} (x) = e^{-\gamma;\beta;\psi}_{\rho,\mu,(N-\mu),\omega,a} \left[ \frac{d}{dx} \right]^N e^{-\gamma;1-\beta;\psi}_{\rho,(1-\beta)(N-\mu),\omega,a+f} (x),
$$

(1.13)

and

$$
\left( H^\gamma,\beta;\psi \right)_{\rho,\mu,\omega,b-f} (x) = e^{-\gamma;\beta;\psi}_{\rho,\beta(N-\mu),\omega,b} \left[ \frac{d}{dx} \right]^N e^{-\gamma;1-\beta;\psi}_{\rho,(1-\beta)(N-\mu),\omega,b-f} (x),
$$

(1.14)

$\forall x \in [a, b]$.

When $\beta = 0$, we get the Riemann-Liouville version, and when $\beta = 1$, we get the Caputo version.

We call $\xi = \mu + \beta (N - \mu)$, we have that $N - 1 < \mu \leq \mu + \beta (N - \mu) \leq \mu + N - \mu = N$, hence $[\xi] = N$.

We can easily write that

$$
\left( H^\gamma,\beta;\psi \right)_{\rho,\mu,\omega,a+f} (x) = e^{-\gamma;\beta;\psi}_{\rho,\xi-\mu,\omega,a+}^{RLD} e^{-\gamma;1-\beta;\psi}_{\rho,\xi,\omega,a+} f (x),
$$

(1.15)

and

$$
\left( H^\gamma,\beta;\psi \right)_{\rho,\mu,\omega,b-f} (x) = e^{-\gamma;\beta;\psi}_{\rho,\xi-\mu,\omega,b-}^{RLD} e^{-\gamma;1-\beta;\psi}_{\rho,\xi,\omega,b-} f (x),
$$

(1.16)

$\forall x \in [a, b]$.

## 2 Main results

We start with a left $\psi$-Prabhakar fractional Hardy type integral inequality involving several functions.

**Theorem 2.1.** Here $i = 1, \ldots, m$; $f_i \in C ([a, b])$, $\psi \in C^1 ([a, b])$ and $\psi$ is increasing. Let $\rho_i, \mu_i > 0$, $\gamma_i, \omega_i \in \mathbb{R}$. Also let $r_1, r_2, r_3 > 1 : \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = 1$, and assume that $\mu_i > \frac{1}{r_2} + \frac{1}{r_3}$, for all $i = 1, \ldots, m$.

Then

$$
\left| \prod_{i=1}^{m} e_{\rho_i,\mu_i,\omega_i,a+f_i}^{\gamma_i;\psi} \right|_{L^{r_1} ([a,b],\psi)} \leq
$$

$$
\frac{(\psi (b) - \psi (a)) \left[ \sum_{i=1}^{m} \mu_i - m + \frac{1}{r_2} + \frac{1}{r_3} \right]}{r_1 r_3 \left( \sum_{i=1}^{m} \mu_i - m \right) + m r_3 + 1} \frac{\prod_{i=1}^{m} (r_1 (\mu_i - 1) + 1)}{r_1^{r_1}}
$$

$$
\left\{ \int_{a}^{b} \left[ \prod_{i=1}^{m} \left( \int_{a}^{x} \left[ E_{\rho_i,\mu_i}^{\mu_i} \left( \omega_i (\psi (x) - \psi (t))^{\rho_i} \right) \right]^{r_2} d\psi (t) \right)^{r_1} d\psi (x) \right] \right\}^{\frac{1}{r_1 r_2}}
$$

$$
\left( \prod_{i=1}^{m} \left\| f_i \right\|_{L^{r_1} ([a,b],\psi)} \right).
$$

(2.1)
Proof. By (1.2) we have
\[
\left( e_{\beta_i,j_i}^\gamma \right)_i (x) = \int_a^x \psi' (t) (\psi (x) - \psi (t))^{\mu_i - 1} E_{\rho_i,j_i}^{\gamma_i} \left[ \omega_i (\psi (x) - \psi (t))^{\rho_i} \right] f_i (t) \, dt, \tag{2.2}
\]
i = 1, \ldots, m; \forall x \in [a, b].

By Hölder’s inequality and (2.2) we obtain
\[
\left| \left( e_{\beta_i,j_i}^\gamma \right)_i (x) \right| \leq \left( \int_a^x \psi' (t) (\psi (x) - \psi (t))^{\mu_i - 1} \left| E_{\rho_i,j_i}^{\gamma_i} \left[ \omega_i (\psi (x) - \psi (t))^{\rho_i} \right] \right| f_i (t) \right) dt \leq \left( \int_a^x (\psi (x) - \psi (t))^{\rho_i} d\psi (t) \right)^{\frac{1}{\rho_i} - \frac{1}{\rho_i + 1}} \left( \int_a^x f_i (t)^{\rho_i} d\psi (t) \right)^{\frac{1}{\rho_i + 1}} \| f_i \|_{L_{\rho_i} ([a, b], \psi)}, \tag{2.3}
\]
for any \( \rho_i > \frac{1}{\rho_i + 1} + \frac{1}{\rho_i}, \) for any \( i = 1, \ldots, m. \)

Hence it holds
\[
\left( \prod_{i=1}^m \left( e_{\beta_i,j_i}^\gamma \right)_i (x) \right)^{\frac{r_1}{\rho_i}} \leq \frac{\left( \psi (x) - \psi (a) \right)^{\rho_i} \sum_{i=1}^m \mu_i - m r_1 + m}{\left( \prod_{i=1}^m (r_1 (\mu_i - 1) + 1) \right)^{\frac{r_1}{\rho_i}}}, \tag{2.4}
\]
\( \forall x \in [a, b]. \)

Therefore we obtain
\[
\int_a^b \left( \prod_{i=1}^m \left( e_{\beta_i,j_i}^\gamma \right)_i (x) \right)^{\frac{r_1}{\rho_i}} d\psi (x) \leq \frac{\left( \prod_{i=1}^m \| f_i \|_{L_{\rho_i} ([a, b], \psi)} \right)^{\frac{r_1}{\rho_i}}}{\left( \prod_{i=1}^m (r_1 (\mu_i - 1) + 1) \right)^{\frac{r_1}{\rho_i}}}, \tag{2.5}
\]
for any \( \rho_i > \frac{1}{\rho_i + 1} + \frac{1}{\rho_i}, \) for any \( i = 1, \ldots, m. \)
We continue with a right \( \psi \)-Prabhakar fractional Hardy type integral inequality involving several functions.

**Theorem 2.2.** All as in Theorem 2.1. It holds

\[
\left[ \prod_{i=1}^{m} \left( \int_{a}^{b} |E_{\rho_i, \mu_i}^{\gamma_i} \left[ \omega_i \left( \psi (x) - \psi (t) \right) \right]^{\nu_i} \, d\psi (t) \right)^{\frac{r_1}{r_2}} \, d\psi (x) \right]^{\frac{1}{r_2}}
\]

(again by Hölder’s inequality)

\[
\leq \left( \prod_{i=1}^{m} \left\| f_i \right\|_{L_{r_2}([a, b], \omega)} \right)^{\frac{r_1}{r_2}} \left( \int_{a}^{b} (\psi (x) - \psi (a))^{r_1 r_3 \sum_{i=1}^{m} \mu_i - m r_1 r_3 + m r_3} \, d\psi (x) \right)^{\frac{1}{r_2}}
\]

\[
\left\{ \int_{a}^{b} \left[ \prod_{i=1}^{m} \left( \int_{a}^{x} \left| E_{\rho_i, \mu_i}^{\gamma_i} \left[ \omega_i \left( \psi (x) - \psi (t) \right) \right]^{\nu_i} \right|^{\nu_i} \, d\psi (x) \right)^{\frac{r_1}{r_2}} \, d\psi (x) \right\}^{\frac{1}{r_2}}
\]

(2.7)

where \( \mu_i > \frac{1}{r_2} + \frac{1}{r_3}, \ i = 1, \ldots, m. \)

The claim is proved. \( \square \)

Next we apply Theorems 2.1 and omitted.

We give the related Hardy type inequalities:
Theorem 2.3. Here $i = 1, \ldots, m$; $f_i \in C^{N_i}(a, b)$, where $N_i = \lceil \mu_i \rceil$, $0 < \mu_i \notin \mathbb{N}$; $\theta := \max\{N_1, \ldots, N_m\}$, $\psi \in C^0([a, b])$ with $\psi' \neq 0$ and increasing. Let $\rho_i > 0$, $\gamma_i, \omega_i \in \mathbb{R}$. Also let $r_1, r_2, r_3 > 1 : \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = 1$, and assume that $N_i - \mu_i > \frac{1}{r_2} + \frac{1}{r_3}$, for all $i = 1, \ldots, m$.

Then

\[ i) \quad \left\| \prod_{i=1}^{m} C^{D_{\rho_i,\mu_i,\omega_i,a}f_i}_{L_{r_1}([a,b],\psi)} \right\|_{L_{r_3}([a,b],\psi)} \leq \frac{(\psi(b) - \psi(a))^{\left(\sum_{i=1}^{m} (N_i - \mu_i) - m + \frac{m}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \right)}{r_1 r_3 \left( \prod_{i=1}^{m} (r_1 (N_i - \mu_i - 1) + 1) \right)^{\frac{1}{r_1}}}
\]

\[
\left\{ \int_{a}^{b} \left[ \int_{a}^{x} \left| E_{\rho_i,\mu_i,\omega_i,b}^{-\gamma_i} [\omega_i (\psi(x) - \psi(t))^{\rho_i}] \right|^{r_2} d\psi(t) \right]^{r_1} d\psi(x) \right\} \left( \prod_{i=1}^{m} \left\| f_i \right\|_{L_{r_3}([a,b],\psi)} \right),
\]

(2.9)

\[ \text{and} \]

\[ ii) \quad \left\| \prod_{i=1}^{m} C^{D_{\rho_i,\mu_i,\omega_i,b}f_i}_{L_{r_1}([a,b],\psi)} \right\|_{L_{r_3}([a,b],\psi)} \leq \frac{(\psi(b) - \psi(a))^{\left(\sum_{i=1}^{m} (N_i - \mu_i) - m + \frac{m}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \right)}{r_1 r_3 \left( \prod_{i=1}^{m} (r_1 (N_i - \mu_i - 1) + 1) \right)^{\frac{1}{r_1}}}
\]

\[
\left\{ \int_{a}^{b} \left[ \int_{a}^{x} \left| E_{\rho_i,\mu_i,\omega_i,b}^{-\gamma_i} [\omega_i (\psi(t) - \psi(x))^{\rho_i}] \right|^{r_2} d\psi(t) \right]^{r_1} d\psi(x) \right\} \left( \prod_{i=1}^{m} \left\| f_i \right\|_{L_{r_3}([a,b],\psi)} \right),
\]

(2.10)

Proof. By (1.6), (1.7) and Theorems 2.1, 2.2.

We also present other Hardy type related inequalities:

Theorem 2.4. Here $i = 1, \ldots, m$; $f_i \in C^{N_i}(a, b)$, where $N_i = \lceil \mu_i \rceil$, $0 < \mu_i \notin \mathbb{N}$; $\theta := \max\{N_1, \ldots, N_m\}$, $\psi \in C^0([a, b])$, $\psi' \neq 0$, and $\psi$ is increasing. Let $\rho_i > 0$, $\gamma_i, \omega_i \in \mathbb{R}$, $0 \leq \beta_i \leq 1$, $\xi_i = \mu_i + \beta_i (N_i - \mu_i)$. Also let $r_1, r_2, r_3 > 1 : \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = 1$, and assume that $\xi_i - \mu_i > \frac{1}{r_2} + \frac{1}{r_3}$, for all $i = 1, \ldots, m$.

Also assume that $RL^{\gamma_i(1-\beta_i)}_{\mu_i,\omega_i,a}f_i$, $RL^{\gamma_i(1-\beta_i)}_{\mu_i,\omega_i,b}f_i \in C([a, b])$, $i = 1, \ldots, m$.

Then
\[ \left| \prod_{i=1}^{m} H^{\gamma_i, \beta_i, \psi}_{p_i, \mu_i, \omega_i, a+1} f_i \right|_{L_{r_1}([a, b], \psi)} \leq \frac{(\psi(b) - \psi(a)) \left( \sum_{i=1}^{m} (\xi_i - \mu_i) - m + m + \frac{1}{r_1} - \frac{1}{r_1/2} \right)}{(r_1 r_3 \left( \sum_{i=1}^{m} (\xi_i - \mu_i) - m + m r_3 + 1 \right))^{\frac{1}{r_1}}} \left( \prod_{i=1}^{m} (r_1 (\xi_i - \mu_i - 1) + 1) \right)^{\frac{1}{r_1}} \]  

(2.11)

\[ \begin{aligned} &\left\{ \int_a^b \left| \prod_{i=1}^{m} \left( \int_a^b E^{-\gamma_i, \beta_i}_{p_i, \xi_i - \mu_i} [\omega_i (\psi(t) - \psi(x))^p]^{1/2} \psi(t) \right) \right|^{r_1} d\psi(t) \right\}^{\frac{1}{r_1/2}} \\
&\left( \prod_{i=1}^{m} \left\| RL D^{\gamma_i (1-\beta_i)}_{p_i, \xi_i, \omega_i, a+1} f_i \right\|_{L_{r_1}([a, b], \psi)} \right), \end{aligned} \]

and

\[ \left| \prod_{i=1}^{m} H^{\gamma_i, \beta_i, \psi}_{p_i, \mu_i, \omega_i, b+1} f_i \right|_{L_{r_1}([a, b], \psi)} \leq \frac{(\psi(b) - \psi(a)) \left( \sum_{i=1}^{m} (\xi_i - \mu_i) - m + m + \frac{1}{r_1} - \frac{1}{r_1/2} \right)}{(r_1 r_3 \left( \sum_{i=1}^{m} (\xi_i - \mu_i) - m + m r_3 + 1 \right))^{\frac{1}{r_1}}} \left( \prod_{i=1}^{m} (r_1 (\xi_i - \mu_i - 1) + 1) \right)^{\frac{1}{r_1}} \]  

(2.12)

\[ \begin{aligned} &\left\{ \int_a^b \left| \prod_{i=1}^{m} \left( \int_a^b E^{-\gamma_i, \beta_i}_{p_i, \xi_i - \mu_i} [\omega_i (\psi(t) - \psi(x))^p]^{1/2} \psi(t) \right) \right|^{r_1} d\psi(t) \right\}^{\frac{1}{r_1/2}} \\
&\left( \prod_{i=1}^{m} \left\| RL D^{\gamma_i (1-\beta_i)}_{p_i, \xi_i, \omega_i, a+1} f_i \right\|_{L_{r_1}([a, b], \psi)} \right). \end{aligned} \]

Proof. By (1.15), (1.16) and Theorems 2.1, 2.2.

From now on all entities are according and respectively to Section 1. Background.

Next we give Opial type inequalities related to Prabhakar fractional calculus.

A left side one follows:

**Theorem 2.5.** Let \( p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1 \). Then

\[ \int_a^x \left| E^{\gamma, \psi}_{p, \mu, \omega, a+1} f (w) \right| \left| f (w) \right| \psi'(w) \, dw \leq 2^{-\frac{1}{q}} \]

\[ \left( \int_a^x \left( \int_a^w (\psi(t) - \psi(x))^p \right) \left| E^{\gamma, \psi}_{p, \omega} [\omega (\psi(w) - \psi(t))^p]^{1/2} \omega(t) \right|^{p-1} dt \right) d\psi(t) \right)^{\frac{1}{p}} \]

\[ \left( \int_a^x \left| f (w) \right|^p \left( \psi'(w) \right)^q \, dw \right)^{\frac{1}{q}}, \]  

(2.13)

\( \forall x \in [a, b] \).
Proof. By (1.2), using Hölder’s inequality, we have
\[
\left| \left( e^{r \gamma \psi} \right) (x) \right| \leq \int_a^x \left( \psi' (t) (\psi (x) - \psi (t))^{\mu-1} \right) |E^{\gamma}_{\rho, \mu} [\omega (\psi (x) - \psi (t))] | f (t) | dt
\]
\[
\leq \left( \int_a^x (\psi (x) - \psi (t))^{p(\mu-1)} |E^{\gamma}_{\rho, \mu} [\omega (\psi (x) - \psi (t))] |^p dt \right)^{\frac{1}{p}}
\]
\[
\left( \int_a^x (\psi' (t) |f (t)|)^q dt \right)^{\frac{1}{q}}.
\]
(2.14)

Call
\[
\phi (x) = \int_a^x (\psi' (t) |f (t)|)^q dt, \quad \phi (a) = 0.
\]
(2.15)

Thus
\[
\phi' (x) = (\psi' (x) |f (x)|)^q \geq 0,
\]
(2.16)

and
\[
(\phi' (x))^{\frac{1}{q}} = (\psi' (x) |f (x)|) \geq 0, \quad \forall x \in [a, b].
\]
(2.17)

Consequently, we get
\[
\left| \left( e^{r \gamma \psi} \right) (w) \right| \psi' (w) |f (w)| \leq
\]
\[
\left( \int_a^w (\psi (w) - \psi (t))^{p(\mu-1)} |E^{\gamma}_{\rho, \mu} [\omega (\psi (w) - \psi (t))] |^p dt \right)^{\frac{1}{p}}
\]
\[
\left( \phi (w) \phi' (w) \right)^{\frac{1}{q}}, \quad \forall w \in [a, b].
\]
(2.18)

Then, by applying again Hölder’s inequality:
\[
\int_a^x \left| \left( e^{r \gamma \psi} \right) (w) \right| |f (w)| \psi' (w) dw \leq
\]
\[
\int_a^x \left\{ \int_a^w (\psi (w) - \psi (t))^{p(\mu-1)} |E^{\gamma}_{\rho, \mu} [\omega (\psi (w) - \psi (t))] |^p dt \right\}^{\frac{1}{p}}
\]
\[
\left( \phi (w) \phi' (w) \right)^{\frac{1}{q}} dw \leq
\]
\[
\left[ \int_a^x \left\{ \int_a^w (\psi (w) - \psi (t))^{p(\mu-1)} |E^{\gamma}_{\rho, \mu} [\omega (\psi (w) - \psi (t))] |^p dt \right\} dw \right]^{\frac{1}{p}}
\]
\[
\left( \int_a^x \phi (w) d\phi (w) \right)^{\frac{1}{q}} =
\]
\[
\left[ \int_a^x \left\{ \int_a^w (\psi (w) - \psi (t))^{p(\mu-1)} |E^{\gamma}_{\rho, \mu} [\omega (\psi (w) - \psi (t))] |^p dt \right\} dw \right]^{\frac{1}{p}}
\]
\[
\left( \frac{\phi^2 (x)}{2} \right)^{\frac{1}{q}} = 2^{-\frac{1}{q}}
\]
\[
\left[ \int_a^x \left\{ \int_a^w (\psi (w) - \psi (t))^{p(\mu-1)} |E^{\gamma}_{\rho, \mu} [\omega (\psi (w) - \psi (t))] |^p dt \right\} dw \right]^{\frac{1}{p}}
\]
\[
\left( \int_a^x (\psi' (w) |f (w)|)^q dw \right)^{\frac{1}{q}}.
\]
(2.19)

The theorem is proved. \(\square\)
The right side Opial inequality follows:

**Theorem 2.6.** Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then

\[
\int_{x}^{b} \left( e^{\gamma \psi \left( \int_{x}^{w} \left( \psi (t) - \psi (w) \right) dt \right)} \right) \left| f (w) \right| \left| \psi' (w) \right| dw \leq 2^{-\frac{1}{q}}
\]

\[
\left[ \int_{x}^{b} \left( \int_{w}^{b} \left( \psi (t) - \psi (w) \right)^{p(N-\mu-1)} \left| E_{\rho, \mu, \nu}^{-\gamma} \left[ \omega \left( \psi (t) - \psi (w) \right) \right] ^{p} dt \right) \right) \right]^{\frac{1}{p}}
\]

\[
\left( \int_{x}^{b} \left| f (w) \right|^{q} \left( \psi' (w) \right)^{q} dw \right)^{\frac{2}{q}}, \tag{2.20}
\]

$\forall x \in [a, b].$

**Proof.** As it is similar to the proof of Theorem 2.5, is omitted. \qed

We continue with more interesting Opial type Prabhakar-Caputo fractional inequalities:

**Theorem 2.7.** Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then

i) \[
\int_{x}^{\bar{x}} \left| \left( C D_{\mu, \omega, \nu}^{-\gamma} \psi \left( \int_{x}^{w} \left( \psi (t) - \psi (w) \right) dt \right) \right) \left| f_{\psi}^{[N]} (w) \right| \psi' (w) \right| dw \leq 2^{-\frac{1}{q}}
\]

\[
\left[ \int_{x}^{\bar{x}} \left( \int_{x}^{w} \left( \psi (w) - \psi (t) \right)^{p(N-\mu-1)} \left| E_{\rho, \mu, \nu}^{-\gamma} \left[ \omega \left( \psi (w) - \psi (t) \right) \right] ^{p} dt \right) \right) \right]^{\frac{1}{p}}
\]

\[
\left( \int_{x}^{\bar{x}} \left| f_{\psi}^{[N]} (w) \right|^{q} \left( \psi' (w) \right)^{q} dw \right)^{\frac{2}{q}}, \tag{2.21}
\]

and

ii) \[
\int_{\bar{x}}^{b} \left| \left( C D_{\mu, \omega, \nu}^{-\gamma} \psi \left( \int_{x}^{w} \left( \psi (t) - \psi (w) \right) dt \right) \right) \left| f_{\psi}^{[N]} (w) \right| \psi' (w) \right| dw \leq 2^{-\frac{1}{q}}
\]

\[
\left[ \int_{\bar{x}}^{b} \left( \int_{w}^{b} \left( \psi (t) - \psi (w) \right)^{p(N-\mu-1)} \left| E_{\rho, \mu, \nu}^{-\gamma} \left[ \omega \left( \psi (t) - \psi (w) \right) \right] ^{p} dt \right) \right) \right]^{\frac{1}{p}}
\]

\[
\left( \int_{\bar{x}}^{b} \left| f_{\psi}^{[N]} (w) \right|^{q} \left( \psi' (w) \right)^{q} dw \right)^{\frac{2}{q}}, \tag{2.22}
\]

$\forall x \in [a, b].$

**Proof.** By Theorems 2.5, 2.6 and (1.6)-(1.8). \qed

Next come $\psi$-Hilfer-Prabhakar left and right Opial type fractional inequalities:
Theorem 2.8. Let \( p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1 \). Additionally here assume that
\[
\text{RLD}_{\rho, \xi, \omega, a}^{\gamma(1-\beta)} \psi f, \text{RLD}_{\rho, \xi, \omega, b}^{\gamma(1-\beta)} \psi f \in C([a, b]).
\]
Then

i)
\[
\int_{a}^{x} \left( H D_{\rho, \mu, \omega, a}^{\gamma\beta, \psi} (w) \right) \left( \text{RLD}_{\rho, \xi, \omega, a}^{\gamma(1-\beta)} \psi f (w) \right) \psi' (w) \, dw \leq 2^{-\frac{1}{q}}
\]
\[
\left[ \int_{a}^{x} \left( \int_{a}^{w} (\psi (w) - \psi (t))^p (\xi - 1) \left[ E_{\rho, \xi, \mu} \left[ \omega (\psi (w) - \psi (t))^q \right] \right]^{\frac{p}{q}} \, dt \right) \right]^{\frac{1}{q}}
\]
\[
\left( \int_{a}^{x} \left( \text{RLD}_{\rho, \xi, \omega, a}^{\gamma(1-\beta)} \psi (w) \right)^q (\psi' (w)) q \, dw \right)^{\frac{1}{q}},
\]  \hspace{1cm} (2.23)

and

ii)
\[
\int_{x}^{b} \left( H D_{\rho, \mu, \omega, b}^{\gamma\beta, \psi} (w) \right) \left( \text{RLD}_{\rho, \xi, \omega, b}^{\gamma(1-\beta)} \psi f (w) \right) \psi' (w) \, dw \leq 2^{-\frac{1}{q}}
\]
\[
\left[ \int_{x}^{b} \left( \int_{w}^{b} (\psi (t) - \psi (w))^p (\xi - 1) \left[ E_{\rho, \xi, \mu} \left[ \omega (\psi (t) - \psi (w))^q \right] \right]^{\frac{p}{q}} \, dt \right) \right]^{\frac{1}{q}}
\]
\[
\left( \int_{x}^{b} \left( \text{RLD}_{\rho, \xi, \omega, b}^{\gamma(1-\beta)} \psi (w) \right)^q (\psi' (w)) q \, dw \right)^{\frac{1}{q}},
\]  \hspace{1cm} (2.24)

\( \forall \, x \in [a, b] \).

Proof. By Theorems 2.5, 2.6 and (1.15), (1.16).

Next we give several Prabhakar Hilbert-Pachpatte fractional inequalities. We start with a left side one.

Theorem 2.9. Let \( p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1 \); \( i = 1, 2 \). Let \([a_i, b_i] \subset \mathbb{R}, \psi_i \in C^1 ([a_i, b_i]) \) and strictly increasing, \( f_i \in C ([a_i, b_i]); \rho_i, \mu_i > 0, \gamma_i, \omega_i \in \mathbb{R} \). Then
\[
\int_{a_1}^{b_1} \int_{a_2}^{b_2} \left( e_{\rho_1, \mu_1, \omega_1, a_1 + f_1}^{\gamma_1 \psi_1} (x_1) \right) \left( e_{\rho_2, \mu_2, \omega_2, a_2 + f_2}^{\gamma_2 \psi_2} (x_2) \right) \, dx_1 dx_2
\]
\[
\left( \frac{f_1}{f_2} \left( (\psi_1 (x_1) - \psi_1 (t_1))^{\mu_1 - 1} \left[ E_{\rho_1, \mu_1, \omega_1} \left[ \omega_1 (\psi_1 (x_1) - \psi_1 (t_1))^q \right] \right] \right)^{\frac{1}{p}} \right) \left( \frac{f_2}{f_2} \left( (\psi_2 (x_2) - \psi_2 (t_2))^{\mu_2 - 1} \left[ E_{\rho_2, \mu_2, \omega_2} \left[ \omega_2 (\psi_2 (x_2) - \psi_2 (t_2))^q \right] \right] \right)^{\frac{1}{q}} \right)
\]
\[
\leq (b_1 - a_1) (b_2 - a_2) \| \psi_1' f_1 \|_q \| \psi_2' f_2 \|_p.
\]  \hspace{1cm} (2.25)

Proof. We have that \( i = 1, 2 \)
\[
\left( e_{\rho_1, \mu_1, \omega_1, a_1 + f_1}^{\gamma_1 \psi_1} (x_1) \right) (1.2)
\]
\[
\int_{a_i}^{b_i} (\psi_i (x_i) - \psi_i (t_i))^{\mu_i - 1} E_{\rho_i, \mu_i, \omega_i} \left[ \omega_i (\psi_i (x_i) - \psi_i (t_i))^q \right] f_i (t_i) \, dt_i,
\]  \hspace{1cm} (2.26)
∀ \( x_i \in [a_i, b_i] \), where \( \rho_i, \mu_i > 0; \gamma_i, \omega_i \in \mathbb{R} \).

Then

\[
\left| \left( e^{\gamma_i \psi_i} \right) (x) \right| \leq \int_{a_i}^{x_i} \left( \left( \psi_i (x_i) - \psi_i (t_i) \right)^{\mu_i - 1} \left| E_{\gamma_i, \mu_i}^{\gamma_i} [\omega_i (\psi_i (x_i) - \psi_i (t_i))] \right| \right) |f_i(t_i)| \, dt_i,
\]

(2.27)
i = 1, 2, \forall \ x_i \in [a_i, b_i].

By applying Hölder’s inequality twice we get:

\[
\left| \left( e^{\gamma_1 \psi_1} \right) (x) \right| \leq \left( \int_{a_1}^{x_1} \left( \left( \psi_1 (x_1) - \psi_1 (t_1) \right)^{\mu_1 - 1} \left| E_{\gamma_1, \mu_1}^{\gamma_1} [\omega_1 (\psi_1 (x_1) - \psi_1 (t_1))] \right| \right)^p \, dt_1 \right)^{\frac{1}{p}} \left( \int_{a_1}^{x_1} \left( \left( \psi_1^\prime (t_1) \right)^q \right) \, dt_1 \right)^{\frac{1}{q}},
\]

(2.28)

∀ \( x_1 \in [a_1, b_1] \), and

\[
\left| \left( e^{\gamma_2 \psi_2} \right) (x) \right| \leq \left( \int_{a_2}^{x_2} \left( \left( \psi_2 (x_2) - \psi_2 (t_2) \right)^{\mu_2 - 1} \left| E_{\gamma_2, \mu_2}^{\gamma_2} [\omega_2 (\psi_2 (x_2) - \psi_2 (t_2))] \right| \right)^q \, dt_2 \right)^{\frac{1}{q}} \left( \int_{a_2}^{x_2} \left( \left( \psi_2^\prime (t_2) \right)^p \right) \, dt_2 \right)^{\frac{1}{p}},
\]

(2.29)

∀ \( x_2 \in [a_2, b_2] \).

Hence we have (by (2.28), (2.29))

\[
\left| \left( e^{\gamma_1 \psi_1} \right) (x) \right| \left| \left( e^{\gamma_2 \psi_2} \right) (x) \right| \leq \left( \int_{a_1}^{x_1} \left( \left( \psi_1 (x_1) - \psi_1 (t_1) \right)^{\mu_1 - 1} \left| E_{\gamma_1, \mu_1}^{\gamma_1} [\omega_1 (\psi_1 (x_1) - \psi_1 (t_1))] \right| \right)^p \, dt_1 \right)^{\frac{1}{p}} \left( \int_{a_1}^{x_1} \left( \left( \psi_1^\prime (t_1) \right)^q \right) \, dt_1 \right)^{\frac{1}{q}} \left\| \psi_1 f_1 \right\|_q \left\| \psi_2^\prime f_2 \right\|_p,
\]

(2.30)

(using Young’s inequality for \( a, b \geq 0; a^\frac{1}{p} b^\frac{1}{q} \leq \frac{a}{p} + \frac{b}{q} \))

\[
\left\{ \left( \int_{a_1}^{x_1} \left( \left( \psi_1 (x_1) - \psi_1 (t_1) \right)^{\mu_1 - 1} \left| E_{\gamma_1, \mu_1}^{\gamma_1} [\omega_1 (\psi_1 (x_1) - \psi_1 (t_1))] \right| \right)^p \, dt_1 \right) \right\}^{\frac{1}{p}} + \left\{ \left( \int_{a_2}^{x_2} \left( \left( \psi_2 (x_2) - \psi_2 (t_2) \right)^{\mu_2 - 1} \left| E_{\gamma_2, \mu_2}^{\gamma_2} [\omega_2 (\psi_2 (x_2) - \psi_2 (t_2))] \right| \right)^q \, dt_2 \right) \right\}^{\frac{1}{q}} \left\| \psi_1 f_1 \right\|_q \left\| \psi_2^\prime f_2 \right\|_p.
\]
∀ x_i ∈ [a_i, b_i], i = 1, 2.

So far we have

\[
\left| \left( \mathcal{E}_{\mu_1, \gamma_1} \psi_1 \right)(x_1) \right| \left( \mathcal{E}_{\mu_2, \gamma_2} \psi_2 \right)(x_2) \leq \left\| \psi_1' f_1 \right\|_q \left\| \psi_2' f_2 \right\|_p, \tag{2.31}
\]

∀ x_i ∈ [a_i, b_i], i = 1, 2.

The denominator in (2.31) can be zero only when x_1 = a_1 and x_2 = a_2. Therefore we obtain (2.25) by integrating (2.31) over [a_1, b_1] × [a_2, b_2].

It follows the corresponding to (2.25) right side inequality.

**Theorem 2.10.** All as in Theorem 2.9. Then

\[
\int_{a_1}^{b_1} \int_{a_2}^{b_2} \left\| \left( \mathcal{E}_{\mu_1, \gamma_1} \psi_1 \right)(x_1) \right\| \left( \mathcal{E}_{\mu_2, \gamma_2} \psi_2 \right)(x_2) \left|dx_1 dx_2\right|
\]

\[
\left[ \int_{a_1}^{b_1} \left\{ \left( \psi_1(x_1) - \psi_1(t_1) \right)^{\mu_1 - 1} \left| \mathcal{E}_{\mu_1, \gamma_1} \psi_1 \right|^p dt_1 \right\} \right]^{\frac{1}{p}} + \left[ \int_{a_2}^{b_2} \left\{ \left( \psi_2(x_2) - \psi_2(t_2) \right)^{\mu_2 - 1} \left| \mathcal{E}_{\mu_2, \gamma_2} \psi_2 \right|^p dt_2 \right\} \right]^{\frac{1}{p}}
\]

\[
\leq (b_1 - a_1) (b_2 - a_2) \left\| \psi_1' f_1 \right\|_q \left\| \psi_2' f_2 \right\|_p. \tag{2.32}
\]

**Proof.** As similar to the proof of Theorem 2.9 is omitted.

We continue with applications of Theorems 2.9, 2.10.

**Theorem 2.11.** Let p, q > 1 : \( \frac{1}{p} + \frac{1}{q} = 1; i = 1, 2 \). Let [a_i, b_i] ⊂ \( \mathbb{R} \), \( \psi_i \in C^{\max(N_1, N_2)}([a_i, b_i]), \psi_i' \neq 0 \), and strictly increasing; \( f_i \in C^{N_i}([a_i, b_i]) \), where \( N_i = [\mu_i], 0 < \mu_i \notin \mathbb{N} \). Here \( \rho_i > 0; \gamma_i, \omega_i \in \mathbb{R} \). Then

\[
\int_{a_1}^{b_1} \int_{a_2}^{b_2} \left| \left( \mathcal{D}_{\rho_1, \gamma_1} \psi_1 \right)(x_1) \right| \left( \mathcal{D}_{\rho_2, \gamma_2} \psi_2 \right)(x_2) \left|dx_1 dx_2\right|
\]

\[
\left[ \int_{a_1}^{b_1} \left\{ \left( \psi_1(x_1) - \psi_1(t_1) \right)^{\mu_1 - 1} \left| \mathcal{D}_{\rho_1, \gamma_1} \psi_1 \right|^p dt_1 \right\} \right]^{\frac{1}{p}} + \left[ \int_{a_2}^{b_2} \left\{ \left( \psi_2(x_2) - \psi_2(t_2) \right)^{\mu_2 - 1} \left| \mathcal{D}_{\rho_2, \gamma_2} \psi_2 \right|^p dt_2 \right\} \right]^{\frac{1}{p}}
\]

\[
\leq (b_1 - a_1) (b_2 - a_2) \left\| \psi_1' f_1^{[N_1]} \right\|_q \left\| \psi_2' f_2^{[N_2]} \right\|_p. \tag{2.33}
\]

**Proof.** By Theorem 2.9 and (1.2), (1.6).

We also give
Theorem 2.12. All as in Theorem 2.11. Then
\[
\int_{a_1}^{b_1} \int_{a_2}^{b_2} \left[ C D_{\rho_1,\xi_1,\omega_1,\beta_1}^{\gamma_1,\psi_1} \left( C D_{\rho_2,\xi_2,\omega_2,\beta_2}^{\gamma_2,\psi_2} f_1 \right) \right] \left[ C D_{\rho_3,\xi_3,\omega_3,\beta_3}^{\gamma_3,\psi_3} \left( C D_{\rho_4,\xi_4,\omega_4,\beta_4}^{\gamma_4,\psi_4} f_2 \right) \right] \, dx_1 dx_2 
\]
\[
\leq (b_1 - a_1) (b_2 - a_2) \left\{ \left\| \psi_1 \right\|_q \left\| \psi_2 \right\|_p \right\}.
\]
(2.34)

Proof. By Theorem 2.10 and (1.3), (1.7).

We present

Theorem 2.13. Let \( p, q \geq 1 \): \( \frac{1}{p} + \frac{1}{q} = 1 \); \( i = 1, 2 \). Let \( [a_i, b_i] \subset \mathbb{R} \), \( \psi_i \in C^{\max(N_i,N_2)}([a_i, b_i]) \), \( \psi_i' \neq 0 \), and strictly increasing; \( f_i \in C^{N_i}([a_i, b_i]) \), where \( N_i = [\mu_i] \), \( 0 < \mu_i \notin \mathbb{N} \). Here \( \rho_i > 0 \); \( \gamma_i, \omega_i \in \mathbb{R} \) and \( \xi_i = \mu_i + \beta_i (N_i - \mu_i) \), \( i = 1, 2 \), where \( 0 \leq \beta_i \leq 1 \). Then
\[
\int_{a_1}^{b_1} \int_{a_2}^{b_2} \left[ H D_{\rho_1,\xi_1,\omega_1,\beta_1}^{\gamma_1,\psi_1} \left( H D_{\rho_2,\xi_2,\omega_2,\beta_2}^{\gamma_2,\psi_2} f_1 \right) \right] \left[ H D_{\rho_3,\xi_3,\omega_3,\beta_3}^{\gamma_3,\psi_3} \left( H D_{\rho_4,\xi_4,\omega_4,\beta_4}^{\gamma_4,\psi_4} f_2 \right) \right] \, dx_1 dx_2 
\]
\[
\leq (b_1 - a_1) (b_2 - a_2) \left\{ \left\| \psi_1 \right\|_q \left\| \psi_2 \right\|_p \right\}.
\]
(2.35)

Proof. By Theorem 2.9 and (1.15).

We also give

Theorem 2.14. All as in Theorem 2.13. Then
\[
\int_{a_1}^{b_1} \int_{a_2}^{b_2} \left[ H D_{\rho_1,\xi_1,\omega_1,\beta_1}^{\gamma_1,\psi_1} \left( H D_{\rho_2,\xi_2,\omega_2,\beta_2}^{\gamma_2,\psi_2} f_1 \right) \right] \left[ H D_{\rho_3,\xi_3,\omega_3,\beta_3}^{\gamma_3,\psi_3} \left( H D_{\rho_4,\xi_4,\omega_4,\beta_4}^{\gamma_4,\psi_4} f_2 \right) \right] \, dx_1 dx_2 
\]
\[
\leq (b_1 - a_1) (b_2 - a_2) \left\{ \left\| \psi_1 \right\|_q \left\| \psi_2 \right\|_p \right\}.
\]
(2.36)

Proof. By Theorem 2.10 and (1.16).
3 Appendix

We give the following important fundamental results:

**Theorem 3.1.** Let $\rho, \mu > 0$; $\gamma, \omega \in \mathbb{R}$; and $\psi \in C^1([a, b])$ increasing, $f \in C([a, b])$. Then $(e^{\gamma t})_{\rho, \mu, \omega, a+f}, (e^{\gamma t})_{\rho, \mu, \omega, b-f} \in C([a, b]).$

**Proof.** We only prove $(e^{\gamma t})_{\rho, \mu, \omega, a+f} \in C([a, b])$. We skip the proof for the other is similar.

We consider the power series

$$E^\gamma_{\rho, \mu} (z) = \sum_{k=0}^{\infty} \frac{|(\gamma)_k|}{k! \Gamma((\rho+1)(\rho+\mu)+(\rho+\mu))} z^k, \; z \in \mathbb{R}.$$  \hspace{1cm} (3.1)

We form

$$R^{-1} := \lim_{k \to \infty} \frac{\frac{(\gamma + k)!}{(\gamma + k + 1)!}}{\frac{(\rho + \mu + k)!}{(\rho + \mu + k + 1)!}} = \lim_{k \to \infty} \frac{(\gamma + k)!}{(\gamma + k + 1)!} \frac{1}{\Gamma((\rho+1)(\rho+\mu)+(\rho+\mu))} = \left( \frac{\rho + \mu}{(\rho + \mu + \rho)} \right).$$  \hspace{1cm} (3.2)

Therefore, we have that its radius of convergence is

$$R = \lim_{k \to \infty} \frac{\frac{1}{(\gamma + k)!}}{\frac{1}{(\gamma + k + 1)!}} = \lim_{k \to \infty} \frac{1}{\Gamma((\rho+1)(\rho+\mu)+(\rho+\mu))} = \lim_{k \to \infty} \frac{(k + 1)!}{(\rho + k + 1)!} \Gamma((\rho + 1)(\rho + \mu)) = \infty,$$

because (1.1) is an entire function.

Notice that

$$\lim_{k \to \infty} \left( \frac{\rho + \mu}{(\rho + \mu + \rho)} \right) = 1.$$  \hspace{1cm} (3.4)

From (1.1) we have that its radius of convergence is

$$R = \lim_{k \to \infty} \frac{\frac{1}{(\gamma + k)!}}{\frac{1}{(\gamma + k + 1)!}} = \lim_{k \to \infty} \frac{1}{\Gamma((\rho+1)(\rho+\mu)+(\rho+\mu))} = \lim_{k \to \infty} \frac{(k + 1)!}{(\rho + k + 1)!} \Gamma((\rho + 1)(\rho + \mu)) = \infty,$$

Consequently by (3.3), (3.4), we get that $X = 0$. Thus $R^{-1} = 0$ and the radius of convergence of $E^\gamma_{\rho, \mu} (z)$, see (3.1), is $R = \infty$, hence (3.1) is convergent everywhere.

Consequently it holds

$$\sum_{k=0}^{\infty} \frac{|(\gamma)_k| |\omega|^k}{k! \Gamma((\rho+1)(\rho+\mu)+(\rho+\mu))} < \infty,$$

$$\forall \; x \in [a, b].$$

We notice that

$$\sum_{k=0}^{\infty} \frac{|(\gamma)_k| |\omega|^k}{k! \Gamma((\rho+1)(\rho+\mu)+(\rho+\mu))} \int_a^b \psi' (t) (\psi(x) - \psi(t))^{(\rho+\mu)-1} |f(t)| dt \leq$$
Consequently, by (3.7) we derive that

\[ \|f\|_{\infty} \sum_{k=0}^{\infty} \frac{|(\gamma)_k| \omega_k^k (\psi(x) - \psi(a))^{\rho k + \mu}}{\rho k + \mu} \leq (3.6) \]

\[ \|f\|_{\infty} (\psi(b) - \psi(a))^\mu \sum_{k=0}^{\infty} \frac{|(\gamma)_k| (\rho (\psi(x) - \psi(t))^{\rho})^k}{k! \Gamma(\rho k + \mu) (\rho k + \mu)} < \infty. \]

Consequently, by [5, p. 175], we derive

\[ \left( e^{\gamma \psi}_{\rho, \mu, \omega, a + f} \right)(x) \overset{(1.2)}{=} \int_a^x \psi'(t) (\psi(x) - \psi(t))^{\mu-1} \left( \sum_{k=0}^{\infty} \frac{(\gamma)_k}{k! \Gamma(\rho k + \mu)} (\omega (\psi(x) - \psi(t))^{\rho})^k \right) f(t) \, dt \]

\[ = \sum_{k=0}^{\infty} \frac{(\gamma)_k \omega_k^k}{k! \Gamma(\rho k + \mu)} \int_a^x \psi'(t) (\psi(x) - \psi(t))^{(\rho k + \mu)^{-1}} f(t) \, dt, \]

\[ \forall x \in [a, b]. \]

By [2, p. 98], we obtain that the function

\[ \lambda_{\rho, \mu}^{(k)}(f, x) = \int_a^x \psi'(t) (\psi(x) - \psi(t))^{(\rho k + \mu)^{-1}} f(t) \, dt, \]

\[ x \in [a, b], \] is absolutely continuous for \( \rho k + \mu \geq 1 \) and continuous for \( \rho k + \mu \in (0, 1); \) \( \psi \in C^1 ([a, b]) \) and increasing.

That is always \( \lambda_{\rho, \mu}^{(k)}(|f|, x) \in C ([a, b]), \) for all \( k = 0, 1, \ldots \)

By (3.5), one can derive that

\[ \sum_{k=0}^{\infty} \frac{|(\gamma)_k| \omega_k^k}{k! \Gamma(\rho k + \mu)} \lambda_{\rho, \mu}^{(k)}(|f|, x) \leq \|

\[ \|f\|_{\infty} (\psi(b) - \psi(a))^\mu \sum_{k=0}^{\infty} \frac{|(\gamma)_k| (\omega (\psi(b) - \psi(a))^{\rho})^k}{k! \Gamma(\rho k + \mu) (\rho k + \mu)} < \infty. \]

Notice that

\[ \left| \lambda_{\rho, \mu}^{(k)}(f, x) \right| \leq \lambda_{\rho, \mu}^{(k)}(|f|, x) = \int_a^x \psi'(t) (\psi(x) - \psi(t))^{(\rho k + \mu)^{-1}} |f(t)| \, dt \]

\[ \leq \|f\|_{\infty} \frac{(\psi(b) - \psi(a))^{(\rho k + \mu)}}{(\rho k + \mu)}, \quad k = 0, 1, \ldots \]

(3.9)

And even more we get:

\[ \frac{|(\gamma)_k| \omega_k^k}{k! \Gamma(\rho k + \mu)} \left| \lambda_{\rho, \mu}^{(k)}(f, x) \right| \leq \frac{|(\gamma)_k| \omega_k^k}{k! \Gamma(\rho k + \mu)} \lambda_{\rho, \mu}^{(k)}(|f|, x) \leq \]

\[ \frac{\|f\|_{\infty} (\psi(b) - \psi(a))^{(\rho k + \mu)}}{(\rho k + \mu)} =: M_k, \quad k = 0, 1, \ldots \]

(3.10)

and by (3.8) that \( \sum_{k=0}^{\infty} M_k < \infty, \) converges.

By Weierstrass M-test we get that \( \sum_{k=0}^{\infty} \frac{(\gamma)_k \omega_k^k}{k! \Gamma(\rho k + \mu)} \lambda_{\rho, \mu}^{(k)}(f, x) \) is uniformly and absolutely convergent for \( x \in [a, b], \)

Consequently by (3.7) we derive that \( \left( e^{\gamma \psi}_{\rho, \mu, \omega, a + f} \right) \in C ([a, b]). \) The proof is completed. \( \square \)
We finish with

**Corollary 3.2.** All as in Theorem 3.1. We have that

$$\left\| e^{\gamma;\psi;\rho,\mu,\omega,a+(b-)} f \right\|_{\infty} \leq \left( \sum_{k=0}^{\infty} \frac{|(\gamma)_{k}| \omega^{k} (\psi(b) - \psi(a))^{\rho k + \mu}}{k! \Gamma(\rho k + \mu + 1)} \right) \| f \|_{\infty} < +\infty. \quad (3.11)$$

That is $e^{\gamma;\psi;\rho,\mu,\omega,a+(b-)}$ are bounded linear operators and positive operators if $\gamma, \omega > 0$.

**Proof.** By (3.7), (3.8). \qed
References

[1] G. A. Anastassiou, *Fractional differentiation inequalities*, New York: Springer-Verlag, 2009.

[2] G. A. Anastassiou, *Intelligent Computations: abstract fractional calculus inequalities, approximations*, Cham: Springer, 2018.

[3] A. Giusti, I. Colombo, R. Garra, R. Garrappa, F. Polito, M. Popolizio and F. Mainardi, “A practical guide to Prabhakar fractional calculus”, Fract. Calc. Appl. Anal., vol. 23, no. 1, pp. 9–54, 2020.

[4] R. Gorenflo, A. Kilbas, F. Mainardi and S. Rogosin, *Mittag-Leffler functions, related topics and applications*, Heidelberg: Springer, 2014.

[5] E. Hewitt and K. Stromberg, *Real and abstract analysis. A modern treatment of the theory of functions of a real variable*, New York: Springer, 1965.

[6] F. Polito and Ž. Tomovski, “Some properties of Prabhakar-type fractional calculus operators”, Fract. Differ. Calc., vol. 6, no. 1, pp. 73–94, 2016.

[7] T. R. Prabhakar, “A singular integral equation with a generalized Mittag-Leffler function in the kernel”, Yokohama Math. J., vol. 19, pp. 7–15, 1971.

[8] J. Vanterler da C. Sousa, E. Capelas de Oliveira, “On the ψ-Hilfer fractional derivative”, Commun. Nonlinear Sci. Numer. Simul., vol. 60, pp. 72–91, 2018.