Quasistatic problems for piecewise-continuously growing solids with integral force conditions on surfaces expanding due to additional material influx

D A Parshin\textsuperscript{1,2} and A V Manzhirov\textsuperscript{1,2,3,4}

\textsuperscript{1} Laboratory of Modelling in Mechanics of Solids, Institute for Problems in Mechanics of the Russian Academy of Sciences, Prospekt Vernadskogo 101-1, Moscow 119526, Russia
\textsuperscript{2} Department of Applied Mathematics, Bauman Moscow State Technical University, 2-ya Baumanskaya ulitza 5, Moscow 105005, Russia
\textsuperscript{3} Department of Higher Mathematics, National Research Nuclear University MEPhI, Kashirskoye Shosse 31, Moscow 115409, Russia
\textsuperscript{4} Department of Higher Mathematics, Moscow Technological University (MIREA), Prospekt Vernadskogo 78, Moscow 119454, Russia

E-mail: parshin@ipmnet.ru, manzh@inbox.ru

Abstract. The piecewise continuous processes of additive forming of solids are studied. The being formed solids exhibit properties of deformation heredity and aging. The approaches of linear mechanics of growing solids in the framework of the theory of viscoelasticity of the homogeneously aging isotropic media are applied. Nonclassical boundary-value problems for describing the mentioned processes with the integral satisfaction of force conditions on some expanding due to the influx of additional material parts of the formed solid surface are investigated. A proposition about the commutativity of the time-derived integral operator of viscoelasticity with a limit depending on the solid point with the integration over an arbitrary, expanding due to the growth, surface inside or on the boundary of the growing solid is given. This proposition provides a way to construct the solution of the corresponding growing solids mechanics problem on the basis of Saint-Venant principle. The solution will retrace the evolution of the stress-strain state of the solid under consideration during and after the process of its additive formation. An example of applying the announced technic to modelling the processes of additive forming solids of conical shape under simultaneous action of end loads that are statically equivalent to an axial time-varying force is demonstrated.

1. Introduction

The additive formation of solids is realized in a wide variety of natural and technological processes. Many of these processes can be considered as continuous growing processes, such that during the formation of a solid an infinitely thin layer of additional material joins to its surface each infinitely small period of time. In the course of additive processes different factors influence on solids being formed and cause their deformation. The development of stress-strain state of such solids is impossible to describe within the framework of classical concepts of continuum mechanics in principle. This is due to the lack of any configuration of the continuously growing solid which could be associated with introduction of the strain measures. An adequate description of mechanical behavior of solids deforming in processes of their continuous growing...
can be given on the basis of approaches and methods of mechanics of growing solids being actively developed nowadays [1–11].

In the proposed study we consider the situation when the solid being additively formed exhibits the properties of deformation heredity (viscoelasticity) and aging (weakening the deformation properties over time regardless stresses existing in the solid), and therefore, during pauses in the growing process as well as after the final cessation of growth the solid continues to change its stress-strain state. This situation is quite difficult to simulate as rheological manifestations in the deformation response of the material continuously interact with mechanical reactions of the solid on the developing in time process of adding new material elements to it.

2. The material properties and its description
We consider the homogeneously aging isotropic linearly viscoelastic material described by the equation of state [12]

\[ \mathbf{T}(\mathbf{r}, t) = \mathcal{H}^{-1}_{\tau_0(r)} \left[ 2\mathbf{E}(\mathbf{r}, t) + \chi \mathbf{I} \mathbf{E}(\mathbf{r}, t) \right] \]

(1)

were \( \tau_0(\mathbf{r}) \) is the time of occurrence of stresses at the point of the solid with the radius-vector \( \mathbf{r} \); \( \mathbf{T} \) and \( \mathbf{E} \) are the stress and small strain tensors at the point \( \mathbf{r} \) at the time instant \( t \), \( \mathbf{I} \) is the unit tensor of the second rank; \( \chi = 2\nu/(1-2\nu) \), and \( \nu \) = const is the Poisson’s ratio. The linear time-operator \( \mathcal{H}^{-1}_{\tau} \) = \( G(t)(\mathbf{I} + \mathcal{N}) \) is inverse to the linear operator \( \mathcal{H} = (\mathbf{I} - \mathcal{L})G(t)^{-1} \) with the real parameter \( s \geq 0 \). Here \( G(t) \) is the elastic shear modulus, \( \mathbf{I} \) is the identity operator, \( \mathcal{L} \) is the operator of the first kind of the integral of stresses over the added volume \( \mathbf{V} \) at the time instant \( t \), \( \mathcal{N} \) is the operator of the first kind of time-derivative of stresses at the point \( \mathbf{r} \) at the time \( \tau \), \( f(\mathbf{r}, t) \) is the function of time and space.

\[ K(t, \tau) = G(\tau) \frac{\partial \Delta(t, \tau)}{\partial \tau} \]

\[ \Delta(t, \tau) = \frac{1}{G(\tau)} + \omega(t, \tau) \]

\( K(t, \tau) \) and \( R(t, \tau) \) are the kernels of creep and relaxation, \( \Delta(t, \tau) \) and \( \omega(t, \tau) \) are the specific strain function and the creep measure for pure shear \( (t \geq \tau \geq 0) \). It is accepted by definition that \( \omega(\tau, \tau) \equiv 0 \).

We denote \( g^0(\mathbf{r}, t) = \mathcal{H}_{\tau_0(\mathbf{r})} g(\mathbf{r}, t) \) for arbitrary function \( g(\mathbf{r}, t) \) of solid point \( \mathbf{r} \) and time \( t \), and we denote \( f^0(t) = \mathcal{H}_{\tau_0} f(t) \) for arbitrary function of time \( f(t) \) which is not associated with specific points of considered solid.

Note that the defining relations written out above were developed especially for the description of processes of concrete deformation. However, they are also well suited to describe the mechanical behaviour of some rocks, as well as polymers, soils, ice.

3. Commutativity of the time-derived viscoelasticity operator with the integration over an expanding surface
The state equation (1) is used in the present work to describe the mechanical behaviour of a solid which is built up with additional material being loaded simultaneously with its attaching to the solid. Obviously, in this case, the function \( \tau_0(\mathbf{r}) \) in (1) will be determined in the following way. In the originally existing (before accreting) part of the solid it will be identically equal to the time moment \( t_0 \) of loading of this part. In the additional part of the solid, formed during accreting, it will coincide with the distribution \( \tau_s(\mathbf{r}) \) of moments of attaching particles \( \mathbf{r} \) of additional material to the solid.

We investigate mechanical problems for growing solids in quasistatic statement and in the approximation of small strains and displacements. The latter let us consider the time-variable space domain occupied with the growing solid to be known at any time instant and prescribed by a specific simulated growing process. So the part of the growing solid surface to which the additional material inflows during this process moves in the space in a known manner. We will name this part of the solid surface the growth surface.
We suppose also that the problem in question contains integral force conditions on some expanding due to the influx of additional material parts of the growing solid surface. We can prove the following proposition:

Let \( \Omega_0 \) and \( \Omega_A \) be two arbitrary limited surfaces inside or on the boundary of a solid, subordinated to the state equation (1) and formed in a process of piecewise-continuous accretion in \( N \) stages \( t \in [t_{2k-1}, t_{2k}) (k = 1, \ldots, N) \) of continuous growth with arbitrary long pauses between them. The surface \( \Omega_0 \) lies entirely within the original (existing before accreting) part of the solid considered. The surface \( \Omega_A \) lies entirely in the additional part of the solid and is obtained by motion in space of a curve \( \Gamma(t) \), \( t \in [t_1, +\infty) \), which belongs to the current growth surface of the solid at every moment \( t \) of its continuous accreting and is fixed in the pauses between the stages of continuous accreting, i.e. outside the time intervals \( [t_{2k-1}, t_{2k}) \). Let \( g(\mathbf{r}, t) \) be an arbitrary function defined in the points \( \mathbf{r} \) of surfaces \( \Omega_0 \) and \( \Omega_A \) for \( t \geq \tau_0(\mathbf{r}) \). Assume that \( g(\mathbf{r}, \tau_0(\mathbf{r})) = 0 \). Then, when \( t > t_1 \) the formula

\[
\frac{d}{dt} \left( \int_{\Omega(t)} g(\mathbf{r}, t) dS \right)^\circ = \int_{\Omega(t)} \frac{\partial g^\circ(\mathbf{r}, t)}{\partial t} dS
\]

will be fair, where the expanding in time (disconnected in general) surface \( \Omega(t) \) combines the surface \( \Omega_0 \) and that part of the surface \( \Omega_A \), which has already been formed by the time \( t \geq t_0 \): \( \Omega(t) = \Omega_0 \) if \( t \in [t_0, t_1] \), and \( \Omega(t) = \Omega_0 \cup \{ \Gamma(\tau) \mid t_1 \leq \tau \leq t \} \) if \( t \in (t_1, +\infty) \).

The formulated proposition gives a way to solve the corresponding problem for the growing solid on the basis of Saint-Venant principle as we show below.

We omit the proof of the proposition due to the limited volume of this paper. Note that the surfaces \( \Omega_0 \) and \( \Omega_A \) considered in the proposition may have arbitrary curvature. Meanwhile their boundaries may not have common points. In the special case it is possible that \( \Omega_0 = \emptyset \). Forming a surface \( \Omega_A \) curves \( \Gamma(t) \) can be both closed and unclosed. In particular, the surface \( \Omega_A \) may “circle” original part of the solid or form a “tube” enveloping only the material of the additional part of the having been formed solid.

4. An example of solving a problem with integral force conditions on the expanding surface of a viscoelastic growing solid

Applying the above announced proposition to solving the growing solids mechanics problems with integral force conditions is demonstrated in the present work on the example of modelling additive processes of formation of the relatively long in the axial direction conical solids. It is assumed that in the process of formation of the solid its end surfaces are acted with loading statically equivalent to the axial tension–compression force which can change over time. Forming the solid under consideration is carried out by means of its thickening in the radial direction due to the influx of additional material to the conic side surface. This process is piecewise continuous, i.e. consists of arbitrary number of stages of continuous accreting alternating with arbitrary long pauses during which the influx of the material does not take place. With the help of the given proposition the solution of the boundary-value problem for the considered mechanical model is constructed in the presented work in a closed analytical form.

4.1. Description of the problem

Let there be a conical solid of rotation which length \( l \) significantly exceeds its transverse dimensions. It is made from isotropic homogeneous aging linearly viscoelastic material subordinated to the constitutive equation (1). Take the moment of this material nucleation be the start of timing \( t \).
At the moment \( t = t_0 \) a load is applied to the ends of the existing solid. We believe that at every moment of time \( t \geq t_0 \) it is statically equivalent to axial forces acting in the central points of the ends and varying with time following the law \( P(t) \). We will consider positive the magnitude of tensile end force.

Some time after the application loading at the time \( t = t_1 \) we start the process of gradual axisymmetric thickening of the considered conical solid by adding the additional material to its lateral initially free from stresses surface. Thickening occurs in such a way that in each time moment the accreted body maintains the shape of a right circular truncated cone of length \( l \). This process is piecewise continuous in time, i.e. it consists of \( N \) consecutive phases of continuous accreting \( t \in [t_{2k-1}, t_{2k}] \) \((k = 1, \ldots, N)\), separated by pauses of arbitrary duration. At the stages of continuous accreting an infinitely thin layer of material attaches to the solid each infinitely small period of time. The added material is supposed identical to the original one. In pauses the influx of additional material to the solid does not take place and its lateral surface is free from stresses. In the process of piecewise continuous accreting and after its completion time-varying central axial forces \( P(t) \) continue to act to the end surfaces of the cone.

We will investigate the evolution of stress-strain state of the considered conical solid under specified conditions of loading before the start, during and upon the completion of the described process of accreting.

Changing the geometry of the considered conical solid due to its piecewise-continuous accreting is completely defined obviously by defining laws of increasing the radii of its ends in time. Denote them by \( a(t) \) and \( b(t) \), \( t \geq t_0 \). These functions are continuous, non-decreasing and constant outside intervals \([t_{2k-1}, t_{2k})\).

Let the reference plane of a cylindrical polar coordinate system coincides with that end of the cone which radius was denoted by \( a(t) \). Place the beginning of coordinates \( O \) in the center of this end and extend coordinate axis \( Oz \) perpendicular to it inside the cone. Denote the polar radius and the angle as \( \rho \) and \( \varphi \). If \( \{e_\rho, e_\varphi, k\} \) is normalized local basis of the introduced cylindrical coordinate system \( (\rho, \varphi, z) \), then the radius-vector of an arbitrary point of the solid can be represented in the form \( r = e_\rho(\varphi) \rho + k z \).

Moving due to the influx of additional material (accreting) the lateral surface of the cone under consideration is described by the equation \( \rho = \Lambda(z, t) \), where \( \Lambda(z, t) = a(t) \cdot (1 - z/l) + b(t) \cdot z/l \). The trace of its passing in the space forms an additional part of the considered solid. At time moments \( t \in [t_{2k-1}, t_{2k}] \) \((k = 1, \ldots, N)\) the lateral surface represents the actual growing surface of the cone, i.e. it is the level surface \( t \) of the function \( \tau_s(r) \). Unit vectors of the external (directed from the axis of the cone) normal line to this surface form a vector field \( n(r) = e_\rho(\varphi) \cos \alpha(\tau_s(r)) - k \sin \alpha(\tau_s(r)) \), in the additional part of the solid, where \( \alpha(t) = \arctan(b(t) - a(t))/l \) is the current polarstar angle of the growing cone.

4.2. Boundary problem on the stage before accreting

Before the start of accretion the stress-strain state of the considered conical solid can be determined on the basis of the theory of viscoelasticity of homogeneously aging isotropic solids [13] and the principle of Saint-Venant from the solution of the following classical mechanical boundary value problem with integral force condition on its end surface, \( t_0 \leq t \leq t_1 \):

\[
\nabla \cdot T = 0, \quad 0 \leq \rho < \Lambda(z, t_0); \quad H(t_0) T = 2E + \chi 1 \text{tr} E, \quad E = (\nabla u^T + \nabla u)/2; \\
\n\int_{\{z=t\}} k \cdot T \left\| e_\rho \rho \times (k \cdot T) \right\| \, dS = \left\| k \frac{P(t)}{0} \right\|.
\]

(2)

Here \( u(r, t) \) is the vector field of displacements. To exclude displacement components not causing deformation of the solid we are to impose the conditions of fixing the neighborhood of the center point of one of its end surfaces: \( u = 0 \) and \( \nabla \times u = 0 \) for \( \rho = 0, z = 0 \). We require these conditions to be satisfied after the start of the process of the considered solid accretion as well.
The boundary value problem (2) can be reformulated for values $u, E, T^\circ, t_0 \leq t \leq t_1$:

$$\nabla \cdot T^\circ = 0, \quad 0 \leq \rho < \Lambda(z, t_0); \quad T^\circ = 2E + \chi 1 \text{tr } E, \quad E = (\nabla u^T + \nabla u)/2;$$

$$n \cdot T^\circ = 0, \quad \rho = \Lambda(z, t_0); \quad \int_{\{z = l\}} \left\| \frac{k \cdot T^\circ}{e_{p, \rho} \times (k \cdot T^\circ)} \right\| dS = \left\| \frac{k P^\circ(t)}{0} \right\|. \quad (3)$$

In the boundary value problem (3) time $t$ is not a significant variable but acts only as a parameter. We can see that the problem (3) turned out to be mathematically equivalent to the similar classical mechanical problem of the equilibrium of a linearly elastic truncated circular cone of permanent composition with free lateral surface $\rho = \Lambda(z, t_0), z \in [0, l]$, being under the action of axial forces centrally applied to its ends. The radii of the ends of the cone and the value of forces acting on it depend on a real parameter $t$. This formal coincidence is gained by substituting in the problem (3) the values $P^\circ$ to the value of tensile force related to the shear modulus, and the tensor $T^\circ$ to the stress tensor related to the shear modulus. We can construct the closed analytical solution of the described classical problem if we combine known specific solutions of the theory of elasticity [14]. We will not trace out the constructed analytical solution in this succinct paper.

4.3. **Boundary problem on the stage of piecewise-continuous accretion**

Due to the objective lack of natural (unstressed) configuration in the growing solid the kinematic description of the process of its deformation that is traditional in the mechanics of deformable solids is not suitable for this solid. However, it is clear that the particles of the new material after the attaching to the surface of growth continue to move as a part of continuous, even though growing, solid. This means that in the region of space occupied by the whole growing solid at this time, the enough smooth velocity field $v(r, t)$ of the motion of its particles is uniquely determined. Therefore, the problem of such a body deformation can be put in terms of velocity. In this case in the formulation of the defining relations of the material a tensor of velocities of deformation

$$D(r, t) = (\nabla v^T + \nabla v)/2 \quad (4)$$

may play a part of the deforming process characteristics. The adopted equation of state (1) can be rewritten by using this tensor in the form [15]:

$$S = 2D + \chi 1 \text{tr } D, \quad (5)$$

where we have introduced the so-called tensor of velocities of operator stresses $S(r, t) = \partial T^\circ/\partial t$.

The approach requires knowing the whole story of changing the state of additional material elements up to their inclusion in the composition of the solid considered. In the studied in the present work process of accreting the additional material is supposed to be initially free of stresses (see Subsection 4.1). In other words, we believe that the additional material begins to deform directly in the time of its attaching to the formed body, and the attaching layers of additional material to the surface of the body does not cause the appearance nonzero stresses in the formed solid near the surface of its growth:

$$T = 0, \quad \rho = \Lambda(z, t), \quad t \in [t_{2k-1}, t_{2k}) \quad (k = 1, \ldots, N). \quad (6)$$

Note that condition (6) provides the equality to zero of the stress vector $n \cdot T$ at the current growth surface, i.e. unload of this surface.

The condition of instantaneous local equilibrium in the growing body has obviously the same form as in the classical solid of permanent composition. In the considered case of mass forces absence this condition is expressed by the standard equation $\nabla \cdot T = 0$. It is possible
to show [15] that for the simulated growth process (in the absence of load on the future and the actual surface of solid growth during the whole process of its deformation) this equation generates similar differential equation for the tensor $\mathbf{S}$:

$$\nabla \cdot \mathbf{S} = 0.$$  \hfill (7)

Equation (7) is fair at every moment of time $t > t_1$ in the region of space occupied by the whole growing body at this moment. It should be emphasized that this equation is not a trivial consequence of the standard equilibrium equation, as in the case of growing the body the integral operator $H_{\tau_0(r)}$ and the operator of divergence $(\nabla \cdot )$ do not commute in general because of the principal dependence of time $\tau_0$ of the occurrence of stresses in the growing solid from the point of this solid $\mathbf{r}$.

One can also show, following [15] that from the specific boundary condition (6) on the moving surface of growth $\rho = \Lambda(z,t)$ the condition on the components of the tensor $\mathbf{S}$ implies for every $k$th step of continuous accreting which is similar in appearance to the standard boundary condition for the stresses:

$$\mathbf{n} \cdot \mathbf{S} = 0, \quad \rho = \Lambda(z,t), \quad t \in [t_{2k-1}, t_{2k}).$$  \hfill (8)

In the pauses between stages of continuous growth and after the completion of growing the non-traditional condition (6) on the lateral surface of the cone should be replaced by the classical condition of equality to zero of the stress vector on this surface: $\mathbf{n} \cdot \mathbf{T} = 0$. Acting on this condition with the operator $H_{\tau_0(r)}$ and differentiating the result by time $t$, we see that the boundary condition (8) saves force even out of time intervals $[t_{2k-1}, t_{2k})$. However, it has a completely different mechanical nature in this case.

The above-formulated relations (7), (5), (4), (8) for the quantities $\mathbf{v}$, $\mathbf{D}$, $\mathbf{S}$ are to be complemented with the same integral force conditions as the boundary problem (2) contains — but here for the cone end surface $\{z = l\}$ expanding over time $t$ due to the additional material influx. After that we get the boundary problem that describes the process of deforming the considered conical solid on all the temporary beam $t > t_1$ after the beginning of its accreting. The main difficulty by solving this problem is that the integral conditions are formulated for the stress tensor $\mathbf{T}$, not for the tensor of velocities of operator stresses $\mathbf{S}$.

However we can overcome the mentioned difficulty by using the proposition given in Section 3. As a surface $\Omega(t)$ in the being solved problem of accreting a conical solid it is necessary to consider the flat surface constituting one end side of the growing cone $z = l$ for $t > t_0$. The surface $\Omega_A$ in this case is annular, and its forming curves $\Gamma(t)$ are concentric circles $\rho = b(t)$. The surface $\Omega_0$ is a circle $0 \leq \rho \leq b_0$. Then according to the proposition we have

$$\frac{\partial}{\partial t} \left[ \int_{\{z=l\}} \left\| \mathbf{k} \cdot \mathbf{T} \right\| e_\rho \times (\mathbf{k} \cdot \mathbf{T}) dS \right] = \int_{\{z=l\}} \left\| \mathbf{k} \cdot \mathbf{S} \right\| e_\rho \times (\mathbf{k} \cdot \mathbf{S}) dS, \quad t > t_1.$$  \hfill (9)

As a result we can supply the following value problem on the stage of piecewise-continuous accretion of the considered conical solid, $t > t_1$:

$$\nabla \cdot \mathbf{S} = 0, \quad 0 \leq \rho < \Lambda(z,t); \quad \mathbf{S} = 2\mathbf{D} + \chi \mathbf{1 tr D}, \quad \mathbf{D} = (\nabla \mathbf{v}^T + \nabla \mathbf{v})/2;$$

$$\mathbf{n} \cdot \mathbf{S} = 0, \quad \rho = \Lambda(z,t); \quad \int_{\{z=l\}} \left\| \mathbf{k} \cdot \mathbf{S} \right\| e_\rho \times (\mathbf{k} \cdot \mathbf{S}) dS = \left\| k \frac{\partial P^o(t)}{\partial t} \right\| \mathbf{0}.$$  \hfill (9)

When solving the problem (9) we are to provide rigid fixing the neighbourhood of the coordinates origin $O$ throughout the whole process of deformation of considered growing solid, i.e. to meet
the following conditions for the vector field of velocities \( \mathbf{v}(\mathbf{r}, t) \) in this neighbourhood: \( \mathbf{v} = \mathbf{0} \) and \( \nabla \times \mathbf{v} = \mathbf{0} \) for \( \rho = 0, z = 0. \)

We can see that the boundary problem (9) contains the time variable \( t \) only as a real parameter and also is mathematically identical to the problem (3): the analogues of the desired quantities \( \mathbf{S}, \mathbf{D}, \mathbf{v}, \mathbf{E}, \mathbf{u}, \) and of the known function \( \Lambda(z, t) \) and \( \partial P^o(t)/\partial t \) in the problem (9) are the desired quantities \( \mathbf{T}^o, \mathbf{E}, \mathbf{u}, \) and the known function \( \Lambda(z, t_0) \) and \( P^o(t) \) in the problem (3). Therefore, we can obtain the closed-form analytical solution of the problem (9) (see Subsection 4.2).

4.4. Determination of the stress evolution in the accreted solid
After we have solved the boundary problem (9) we know the evolution of the velocity vector \( \mathbf{v} \) and the tensor \( \mathbf{S} \) of velocities of operator stresses at each point \( \mathbf{r} \) of the considered piecewise continuously accreted aging viscoelastic conical solid on the time beam

\[
t > \tau_1(\mathbf{r}) = \begin{cases} t_1, & 0 \leq \rho < \Lambda(z, t_0), \\ \tau_*(\mathbf{r}), & \Lambda(z, t_0) \leq \rho < \Lambda(z, t_{2N}). \end{cases}
\]

This beam covers the entire history of deformation of the neighborhood of a given point in the composition of the formed solid after the beginning of the process of the solid accretion.

After solution of the problem (3) we know the evolution of the displacement vector \( \mathbf{u} \) and the operator stress tensor \( \mathbf{T}^o \) at the points of the original part of the considered solid on the time segment \( t \in [t_0, t_1] \), i.e. before the beginning of the accretion process.

So the evolution of the operator stress tensor \( \mathbf{T}^o \) at any point \( \mathbf{r} \) of the solid for all \( t \geq \tau_1(\mathbf{r}) \) can be recovered by using the integration procedure:

\[
\mathbf{T}^o(\mathbf{r}, t) = \mathbf{T}^o(\mathbf{r}, \tau_1(\mathbf{r})) + \int_{\tau_1(\mathbf{r})}^{t} \mathbf{S}(\mathbf{r}, \tau) d\tau.
\]

We have \( \mathbf{T}^o(\mathbf{r}, \tau_1(\mathbf{r})) = \mathbf{0} \) in the additional part of the solid in consequence of the condition (6).

When we know the complete evolution of the tensor \( \mathbf{T}^o \) at a point \( \mathbf{r} \) of the considered solid, i.e., the values of this tensor since the moment \( t = \tau_0(\mathbf{r}) \) of occurrence of stresses at a given point, we can find the complete evolution of the stresses tensor \( \mathbf{T} \) at this point by using the inverse to \( \mathcal{H}_{\tau_0(\mathbf{r})} \) transformation \( \mathcal{H}_{\tau_0(\mathbf{r})}^{-1} \):

\[
\frac{\mathbf{T}(\mathbf{r}, t)}{G(t)} = \mathbf{T}^o(\mathbf{r}, t) + \int_{\tau_0(\mathbf{r})}^{t} \mathbf{T}^o(\mathbf{r}, \tau) R(t, \tau) d\tau.
\]

If we use a particular approximation for the creep kernel \( K(t, \tau) \) the expression for the respective relaxation kernel \( R(t, \tau) \) may not be known in the closed form or be too bulky. Then the procedure of reconstructing the evolution of the tensor \( \mathbf{T} \) by numerical treatment of the Volterra equation

\[
\frac{\mathbf{T}(\mathbf{r}, t)}{G(t)} = \int_{\tau_0(\mathbf{r})}^{t} \frac{\mathbf{T}(\mathbf{r}, \tau)}{G(\tau)} K(t, \tau) d\tau = \mathbf{T}^o(\mathbf{r}, t),
\]

for example, by the method of quadratures [16], will be less expensive from a computational point of view and may be even more precise.

Acknowledgments
This work was supported by the Russian Foundation for Basic Research (under grants Nos. 18-01-00920-a, 18-01-00770-a, and 17-51-45054-IND_a), and by the Department of Energetics, Mechanical Engineering, Mechanics and Control Processes of the Russian Academy of Sciences (Program No. 12 OE).
References

[1] Manzhirov A V 2013 Mechanics of growing solids and phase transitions *Key Engineering Materials* **535–536** 89–93

[2] Manzhirov A V and Lychev S A 2014 Mathematical modeling of additive manufacturing technologies *Lecture Notes in Engineering and Computer Science: Proc. World Congr. on Engineering 2014, WCE 2014, 2–4 July 2014 London* pp 1404–1409

[3] Manzhirov A V and Parshin D A 2006 Accretion of a viscoelastic ball in a centrally symmetric force field *Mech. Solids* **41** (1) 51–64

[4] Manzhirov A V and Parshin D A 2006 Modeling the accretion of cylindrical bodies on a rotating mandrel with centrifugal forces taken into account *Mech. Solids* **41** (6) 121–134

[5] Manzhirov A V and Parshin D A 2015 Arch structure erection by an additive manufacturing technology under the action of the gravity force *Mech. Solids* **50** (5) 559–570

[6] Manzhirov A V and Parshin D A 2015 Influence of the erection regime on the stress state of a viscoelastic arched structure erected by an additive technology under the force of gravity *Mech. Solids* **50** (6) 657–675

[7] Manzhirov A V and Parshin D A 2016 Accretion of spherical viscoelastic objects under self-gravity *Lecture Notes in Engineering and Computer Science* **2224** (1) 1131–1135

[8] Parshin D A and Hakobyan V N 2017 Additive manufacturing of a cylindrical arch of viscoelastic aging material under gravity action at various modes of the process *Procedia IUTAM* **23** 66–77.

[9] Parshin D A and Hakobyan V N 2017 Application of prestressed structural elements in the erection of heavy viscoelastic arched structures with the use of an additive technology *Mech. Solids* **51** (6) 692–700

[10] Parshin D A 2017 Analytic solution of the problem of additive formation of an inhomogeneous elastic spherical body in an arbitrary nonstationary central field of forces *Mech. Solids* **52** (5) doi: 10.3103/S0025654417050089

[11] Polyanin A D and Manzhirov A V 2008 *Handbook of Integral Equations* 2nd ed (Chapman & Hall/CRC Press)