Asynchronous Decentralized Optimization in Directed Networks

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Abstract

A popular asynchronous protocol for decentralized optimization is randomized gossip where a pair of neighbors concurrently update via pairwise averaging. In practice, this creates deadlocks and is vulnerable to information delays. It can also be problematic if a node is unable to respond or has only access to its private-preserved local dataset. To address these issues simultaneously, this paper proposes an asynchronous decentralized algorithm, i.e. APPG, with directed communication where each node updates asynchronously and independently of any other node. If local functions are strongly-convex with Lipschitz-continuous gradients, each node of APPG converges to the same optimal solution at a rate of $O(\lambda^k)$, where $\lambda \in (0, 1)$ and the virtual counter $k$ increases by 1 no matter on which node updates. The superior performance of APPG is validated on a logistic regression problem against state-of-the-art methods in terms of linear speedup and system implementations.

I. INTRODUCTION

As data gets larger and more distributed, decentralized optimization over a network of computing nodes (aka workers) has found numerous applications in machine learning as it provides ample opportunities to improve the speed and accuracy of optimization \[\text{[1]}-\text{[3]}.\] To achieve the best possible convergence rate, it is crucial to design a decentralized algorithm that is robust to slow nodes. Recently, asynchronous decentralized algorithms have emerged mostly for the undirected peer-to-peer network, see Fig. [1](a) in [2], and converge considerably faster than their synchronous counterparts in heterogenous environment.

Among them, a popular asynchronous protocol is randomized gossip where at each iterate a pair of neighbors exchange values and set their new values to the pairwise average of their previous values \[\text{[2]}\]. Although this protocol is very simple and relatively easy to theoretically evaluate convergence performance, it suffers from 1) deadlocks in practice \[\text{[4]}\], especially in large-scale decentralized optimization problems where the number of data transmissions is non-trivial, and 2) failure to exactly train models in machine learning over distributed dataset where each node is associated with a private-preserved local dataset (see Fig. [1], and 3) vulnerability to information delays.

Moreover, there are many applications that communication between nodes is only directed, e.g., a low-level node can only pull data from an upper-level node but is not allowed to push data back to the same node or a direction of the communication link between neighbors is broken. In the directed case, one may argue to use the gossip
protocol in directed networks, see Fig. 1(b), where only the receiver of a node computes its update per iterate. This works only if 1) the so-called gossip matrix is doubly stochastic, which further requires global coordination among nodes and is a complicated design problem at the scale of hundreds of nodes [5], or 2) the frequency of computing updates in each node is of the same, which is impossible for any asynchronous system.

To solve these issues simultaneously, this work proposes an asynchronous push-pull gradient algorithm (APPG) for decentralized optimization, that is easy to implement in directed peer-to-peer networks with distributed dataset and robust to bounded transmission delays, while maintaining a linear convergence rate if local functions are strongly-convex with Lipschitz-continuous gradients. In particular, we adopt the general asynchronous setting [6] where nodes in APPG are free to transmit values of their iterates, either via gossip or broadcast with directed communication, and compute local updates using delayed or stale information, without expecting a response from any neighbor. Note that in decentralized optimization problems, the amount of communication is usually non-trivial which inevitably introduces transmission delays. The APPG is robust to any bounded delays with a convergence rate of $O(\lambda^k)$ where $\lambda \in (0, 1)$ depends on the network structure and the virtual counter $k$ increases by 1 no matter on which node updates, and supports uneven update frequencies among nodes.

We implement APPG in MPI against the state-of-the-art algorithms on a multi-core server to solve a multi-class logistic regression problem over the real-world Covertype dataset. The result shows its empirical convergence rate in running time is faster than the existing algorithms, and achieves a linear speedup with respect to (w.r.t.) the number of cores. We also show its strong robustness to the scenario where nodes have different update frequencies, which clearly is very common in the large-scale network.

Besides advantages of APPG in practical implementation and fast convergence, we develop an augmented system approach to explicitly evaluate the performance of asynchronous decentralized algorithms, which is substantially different from that of the gossip based protocol [2]. In the later, they usually assume that the gossip process is independently and identically distributed, which cannot hold in real implementation. From this point of view, their theoretical results are unable to exactly reflect the practical performance of algorithms. There is no such an issue via our novel approach.

II. RELATED WORK

Centralized parallel optimization using a master-slave architecture has been widely adopted to train models [7]. In this architecture, each slave pulls the shared parameters from the master, computes its own gradient or stochastic...
gradient, and then pushes the gradient back to the master, where gradients from all slaves are aggregated to update parameters. This process can be either synchronous [8] or asynchronous [9]. [10]. However, there are two main drawbacks: 1) the bandwidth bottleneck limits its scalability to large-scale networks, and 2) the system stops working if the master breaks down.

To overcome these issues, decentralized parallel optimization over a peer-to-peer network is an attractive alternative and allows each node to talk with only a subset of nodes. Under a connected graph, each node maintains a local copy of the training model and updates it by using its own gradient or stochastic gradient and the information received from its neighbors, after which the updated model is sent to neighbors. In general, each node only talks with a few number of neighbors even in a large-scale network, which makes it very scalable and robust. It has also been widely studied in the control community, see e.g. [11] for a comprehensive review, and novel algorithms such as DGD [12], DDA [13] and EXTRA [14] have been developed. Recently, it has been demonstrated faster convergence in decentralized training of machine learning models with stochastic gradients, such as D-PSGD [1], MSDA [15], MSPD [3], COLA [16] and D² [17], which are proposed only for undirected networks.

There are some synchronous algorithms for directed networks [18], [19] where fast nodes can only start to compute updates after waiting for slow nodes, which results in much idle time and thus makes it less efficient in large networks. The AllReduce based decentralized algorithms adopt a ring graph instead of a central node to aggregate gradients from n nodes [20], [21]. At each iterate, a node receives information from its predecessor, and sends updated information to its successor. All nodes shall collect some global information after n − 1 iterates. However, each iterate of AllReduce must be synchronized and thus also suffers a relatively poor scalability.

Asynchronous decentralized parallel optimization solves the problem by breaking the synchronization in each iterate. [22] proposed an asynchronous algorithm for undirected graphs. The seminal work [23] and recent work [24] focus on asynchronous coordinate descent algorithms. Recently, [2] proposed an algorithm called AD-PSGD using stochastic gradients, which is an asynchronous implementation of D-PSGD [1]. However, it assumes all workers have access to the whole dataset or the global dataset can be split according to update frequencies of nodes, which is restrictive in practice. Moreover, it also needs an undirected network.

To our best knowledge, there are few works on asynchronous decentralized algorithms over directed networks, except for the two recent works. However, the algorithm in [25] cannot ensure exact convergence to an optimal solution if nodes have different update frequencies. [26] addresses this issue by designing a novel adaptive learning rate, but the algorithm only has a sublinear convergence rate. In contrast, APPG converges linearly.

We finally briefly review some synchronous decentralized algorithms over directed networks. The algorithm in [18] is based on the push-sum methods, but only achieves sub-linear convergence rate. [27] improves it to the linear convergence rate with a gradient tracking technique, but the nonlinear operator in its update rule degrades the practical performance. [19] and [28] modify it with a linear updating rule, which is key to our work.

**Notation:** Throughout this paper, we use the following notation and definitions:

- \( a, \mathbf{a}, A, \) and \( \mathcal{A} \) are used to denote a scalar, column vector, matrix, and set, respectively.
- \( \mathbf{a}^\top \) and \( A^\top \) are transposes of \( \mathbf{a} \) and \( A \), respectively.
- \( \mathbb{R}^n \) and \( \mathbb{N} \) denote the set of \( n \)-dimensional real numbers and natural numbers, respectively.
• \| \cdot \|_2 denotes the \( l_2 \)-norm of a vector or matrix. \( \| \cdot \|_F \) denotes the matrix Frobenius norm.
• \( \cdot \) denotes the vector with all 1. Its dimension depends on the context.
• \( \nabla f(x) \) denotes the gradient of a function \( f \) at \( x \).
• \( A \) is called a row-stochastic matrix if each element of \( A \) is nonnegative and \( A1 = 1 \). \( A \) is column-stochastic if \( A^T \) is row-stochastic. \( A \) is doubly-stochastic if \( A \) is both row- and column-stochastic.

### III. Problem and the APPG Algorithm

In this section, we formulate the decentralized optimization problem and then propose the APPG algorithm in directed networks.

#### A. The decentralized optimization problem

We aim to solve the following problem in a directed network,

\[
\min_{x \in \mathbb{R}^m} f(x) := \sum_{i=1}^{n} f_i(x) \tag{1}
\]

where \( n \) is the number of nodes. The local objective function \( f_i(x) \) is only known by node \( i \). In machine learning, it often takes the form

\[
f_i(x) := \sum_{\xi \in D_i} F_i(x; \xi)
\]

where \( D_i \) is a private-preserved local dataset in node \( i \) and \( F_i(x; \xi) \) is the cost of a single sample \( \xi \).

A directed network (digraph) is denoted by \( G = (V, E) \), where \( V = \{1, 2, \cdots, n\} \) is the set of nodes and \( E \in V \times V \) is the set of edges. The directed edge \((i, j) \in E\) if node \( i \) can directly send information to node \( j \). Let \( \mathcal{N}_i^\text{in} = \{ j | (j, i) \in E \} \cup \{ i \} \) denote the set of in-neighbors of node \( i \) and \( \mathcal{N}_i^\text{out} = \{ j | (i, j) \in E \} \cup \{ i \} \) denote the set of out-neighbors of \( i \). A path from node \( i \) to node \( j \) is a sequence of consecutively directed edges from node \( i \) to node \( j \). Then, \( G \) is \textit{strongly connected} if there exists a directed path between any pair of nodes in the digraph.

Each node \( i \) has a local state \( x_i \), and the objective is to ensure all local states \( x_i, i \in V \) converge to an optimal solution of (1) via directed interactions with neighbors. We make the following assumptions.

**Assumption 1.**

1) The digraph \( G \) is strongly connected.

2) The local function \( f_i(x) \) is \( \alpha \)-strongly convex and \( \beta \)-strongly smooth for all \( i \in V \), i.e., there exist some positive \( \alpha > 0, \beta > 0 \) such that for all \( x, y \in \mathbb{R}^m \),

\[
f_i(y) \geq f_i(x) + \nabla f_i(x)^T (y - x) + \frac{\alpha}{2} \| y - x \|_2^2,
\]

\[
\beta \| x - y \|_2 \geq \| \nabla f_i(x) - \nabla f_i(y) \|_2.
\]

Assumption [1] is standard and necessary to design a linearly convergent algorithm [14], [19], [27]. The strongly convex property can often be satisfied in machine learning by adding a regularization term. Under Assumption [1], \( f(x) \) has a unique minimizer, which is denoted by \( x^* \), i.e.

\[
f^* := f(x^*) = \min f(x).
\]
B. The APPG algorithm

The APPG of this paper is given in Algorithm 1 where we do not introduce any global index to emphasize the fact of asynchronous implementation.

From a single node’s point of view, the implementation of APPG is easy. A node just keeps receiving messages from its in-neighbors and storing them to the corresponding local buffers, until it is activated to start to compute a new update, which is completed by using (2). Then, the node only needs to broadcast the updated state to its out-neighbors. A node stops updating if a local stopping criteria, i.e., \(\|\mathbf{y}_i\|_2 < \epsilon\) is satisfied, and the whole algorithm terminates after all nodes stop. The stopping criteria in Algorithm 1 can be locally executed and shall be explained in Section III-C. It is worthy mentioning that how to stop a decentralized algorithm _locally_ is also an important problem and so far has not been well addressed.

In machine learning, the full gradient of local objective function \(\nabla f_i(x_i)\) is usually replaced by stochastic gradient \(\nabla F_i(x_i, \xi)\), where \(\xi\) is sampled from the local private-preserved dataset \(D_i\), see Appendix B for an example.

Different from synchronous algorithms, APPG does not require any global clock or synchronization among nodes, and nodes do not wait for others but independently start new updates completely using the local information available. For example, a node can simply start to compute a new update once it completes the current one. Obviously, there is no deadlock problem. Moreover, each node is allowed (and is likely) to use the delayed or stale information from neighbors for computing update, which is unavoidable when nodes are connected via high-latency networks in practice. Finally, we note that only the learning rate \(\rho\) needs to be tuned in APPG, and the local buffers are bounded, both of which guarantee the ease of implementation in practice.

Although APPG applies to any strongly connected network, it is essential to design a suitable network topology to fully exploit computational resources. However, this problem is very involved and should be designed in view of the specific application. For a low-latency network, a dense network is preferred, which leads to fast convergence both theoretically and empirically. For a high-latency network, a sparse network can reduce the communication overhead and the information staleness, and thus may result in a better performance.

Remark 1. Compared with AD-PSGD [2]. APPG has the following advantages in implementation

- AD-PSGD essentially adopts the random gossip protocol, which involves a pair of neighbors to concurrently compute updates. This is difficult to implement without a shared memory and may lead to deadlocks. In APPG, each active node updates its own state without coordination with any other node.
- AD-PSGD only works on undirected graphs, which is a special case of directed graphs of this work.
- AD-PSGD assumes either nodes have the same update frequency or the dataset in each node is manually divided according to the update frequencies of nodes. Firstly, the update frequency of a node is not a prior information, and is usually unavailable to the designer. Secondly, distributed datasets in some applications are privately preserved for each node and will not be shared among nodes. In APPG, we do not need any information on the update frequencies of nodes and also apply to distributed datasets.
Algorithm 1 The asynchronous push-pull gradient algorithm (APPG) — from the view of node $i$

- **Initialization:** Each node $i$ initializes $x_i$ as an arbitrary real vector in $\mathbb{R}^m$, computes $y_i = \nabla f_i(x_i)$, and creates local buffers $X^{\text{rec}}_i$ and $Y^{\text{rec}}_i$. Then it broadcasts $\tilde{x}_i := x_i$ and $\tilde{y}_i := y_i/|N_{\text{out}}^i|$ to its out-neighbors.

- **Repeat**
  1: Keep receiving $\tilde{x}_j$ and $\tilde{y}_j$ from in-neighbors of node $i$ and copy to $X^{\text{rec}}_i$ and $Y^{\text{rec}}_i$ respectively, until node $i$ is activated to update.
  2: Update $x_i$ and $y_i$ by
    \[
    x_i \leftarrow \text{avg}(X^{\text{rec}}_i), \\
    g_i^- \leftarrow g_i, \quad g_i \leftarrow \nabla f_i(x_i) \\
    y_i \leftarrow \text{sum}(Y^{\text{rec}}_i) + g_i - g_i^- \\
    \tilde{x}_i \leftarrow x_i - \rho y_i
    \]  
    \[(2)\]
    where $\text{avg}(X^{\text{rec}}_i)$ returns the average of vectors in $X^{\text{rec}}_i$, $\text{sum}(Y^{\text{rec}}_i)$ takes the sum of vectors in $Y^{\text{rec}}_i$, $\tilde{x}_i$, $g_i^-$ and $g_i$ are three auxiliary vectors.
  3: Broadcast $\tilde{x}_i$ and $\tilde{y}_i := y_i/|N_{\text{out}}^i|$ to all out-neighbors of $i$, after which empty both $X^{\text{rec}}_i$ and $Y^{\text{rec}}_i$.

- **Until** a stopping criteria is satisfied. e.g., node $i$ stops if $\|y_i\|_2 < \epsilon$ for some predefined $\epsilon > 0$.
- **Return** $x_i$.

C. Correctness of the APPG

We briefly discuss how the *synchronous* version of APPG works by adding a synchronization barrier after step 3 in Algorithm 1 and has also been studied in [19], [28].

Let $X(k), Y(k)$ and $\nabla f(X(k))$ be the stacked local states and gradients at the $k$-th iterate, i.e.,
\[
X(k) = [x_1(k), x_2(k), \ldots, x_n(k)]^T \in \mathbb{R}^{n \times m} \\
Y(k) = [y_1(k), y_2(k), \ldots, y_n(k)]^T \in \mathbb{R}^{n \times m} \\
\nabla f(X(k)) = [\nabla f_1(x_1(k)), \ldots, \nabla f_n(x_n(k))]^T \in \mathbb{R}^{n \times m}
\]

Then, it follows that
\[
X(k + 1) = A(X(k) - \rho Y(k)) \tag{3a} \\
Y(k + 1) = BY(k) + \nabla f(X(k + 1)) - \nabla f(X(k)) \tag{3b}
\]

where the row-stochastic matrix $A$ and column-stochastic matrix $B$ are from the $\text{avg}(\cdot)$ operator and the $\text{sum}(\cdot)$ operator in [2], respectively. $\rho$ is the learning rate.

For the celebrated DGD [12] and its stochastic gradient version, e.g. D-PSGD [1], we have
\[
X(k + 1) = WX(k) - \rho \nabla f(X(k)) \tag{4}
\]

1The name ‘PPG’ comes from [19].
where $W$ must be a doubly-stochastic matrix. In comparison, (4) only converges at a rate of $O(1/k)$ (or $O(1/\sqrt{k})$ in the stochastic gradient case), while (3) converges linearly. The main reason is that we further introduce $y_i$ to track the global gradient of $f(x)$. To illustrate it, we left multiply (3b) with $1^T$, use the column-stochasticity of $B$ and notice $Y(0) = \nabla f(X(0))$. Then,

$$1^T Y(k) = 1^T \nabla f(X(k)).$$

(5)

Now suppose $X(k)$ and $Y(k)$ have already converged to $X^\infty$ and $Y^\infty$, respectively. Then (3b) becomes $Y^\infty = BY^\infty$, which combined with (5) implies

$$y^\infty_i = \pi^B_i (1^T \nabla f(X^\infty))^T$$

(6)

where $\pi^B = [\pi^B_1, \ldots, \pi^B_n]^T$ is the Perron vector of $B$, i.e. $B\pi^B = \pi^B$. On the other hand, the row-stochasticity of $A$ ensures that $X^\infty$ has the form $1(x^\infty)^T$, i.e., all $x_i$ finally converge to $x^\infty$, and then

$$(1^T \nabla f(X^\infty))^T = (1^T \nabla f(1(x^\infty)^T))^T = \nabla f(x^\infty)$$

(7)

Therefore, (6) and (7) show that $y_i$ converges to some scaled global gradient, i.e., $\pi^B_i \nabla f(x^\infty)$.

Substituting $X^\infty$ and $Y^\infty$ into (3a) implies

$$X(k + 1) = A(X^\infty - \rho \pi^B \nabla f(x^\infty)^T)$$

Let $\pi^A$ be Perron vector of $A$, i.e., $A^T \pi^A = \pi^A$ and $1^T \pi^A = 1$. We left multiply (3a) with $(\pi^A)^T$ and notice that $X^\infty = 1(x^\infty)^T$. Then,

$$x(k + 1) = x^\infty - \rho (\pi^A)^T \pi^B \nabla f(x^\infty)$$

$$= x^\infty - \gamma \nabla f(x^\infty)$$

(8)

where $\gamma = \rho (\pi^A)^T \pi^B$. Clearly, (8) is a gradient descent update, which converges linearly under Assumption 1.

Eq. (8) shows that the limit point $x^\infty$ must be an optimal point $x^*$. Thus, $y_i$ converges to $y_i^\infty = \pi^B_i \nabla f(x^\infty) = \pi^B_i \nabla f(x^*) = 0$. Moreover, the smaller the $y_i$, the closer $x_i$ to an optimal solution. Therefore, $y_i$ can serve as a stopping criteria for Algorithm 1.

To the contrary, suppose that $X(k)$ in (4) has achieved the optimum $X^* := 1(x^*)^T$. Then, the next update would be

$$X(k + 1) = X^* - \rho \nabla f(1(x^*)^T).$$

(9)

Note that $\nabla f_i(x^*)$ is not 0 in general, so does $\nabla f(1(x^*)^T)$. Hence the update (9) will drive $x_i(k + 1)$ away from $x^*$. In fact, (4) converges only if $\rho$ tends to zero, which clearly results in a slow convergence rate. In contrast, (3) converges with a constant learning rate.

A formal proof of (3) is provided in [19], [28]. As a non-trivial asynchronous extension of (3), APPG is much more difficult as it involves a complicated multi-timescale problem. In Appendix C, we develop a novel augmented graph approach to prove it.
IV. Theoretical result

Two conditions on the communication delays and update frequencies of nodes are needed.

**Assumption 2** (Bounded activation time interval). Let $t_i$ and $t_i^+$ be any two consecutive activation time of node $i$, then there exist $\underline{\tau}, \bar{\tau}$ such that $0 < \underline{\tau} \leq |t_i^+ - t_i| \leq \bar{\tau} < \infty$ for all $i \in V$.

Assumption 2 is easily satisfied and desirable in practice. In fact, both the lower and upper bound exist naturally since computing update consumes time and should be finished in finite time. If violated, e.g., some node is broken, then the information from this node’s dataset can no longer be accessed. Then, it is impossible to find an optimal solution for any algorithm.

**Assumption 3** (Bounded transmission delays). For any $(i,j) \in E$, the transmission delay from node $i$ to node $j$ is bounded by a constant $\tau > 0$.

Transmission delays can be time-varying, and boundedness is also reasonable in practice. Note that all the above parameters $\tau, \bar{\tau}, \underline{\tau}$ are not needed for implementing APPG.

Let $T = \{t(k)\}_{k \geq 1}$ be an increasing sequence of updating time of all nodes, i.e. $t \in T$ if some node starts to update at time $t$. Denote the most recent state of node $i$ just before $t(k)$ by $x_i(k)$ and $y_i(k)$. The following theorem is the main theoretical contribution of this paper.

**Theorem 1.** Suppose that Assumptions 1-3 hold. If the learning rate $\rho$ satisfies

$$0 < \rho < \frac{\alpha \theta}{4b(n + \sqrt{n})^2(2\sqrt{nb\beta} + 1)\omega},$$

there exists a positive $\lambda$ such that

$$\|x_i(k) - x^*\|_2 \leq c\lambda^k, \|y_i(k)\|_2 \leq c\lambda^k, \forall i \in V, k \in \mathbb{N}$$

(10)

where

$$\max \left\{ \sqrt{\frac{1}{2} + \theta^{-1} \mu n}, \sqrt{1 - \frac{\rho \alpha \theta n}{2}} \right\} < \lambda < 1,$$

and $\alpha, \beta$ are defined in Assumption 1, $b = n(\bar{\tau} + \tau)/\underline{\tau}, \theta, \bar{\gamma}$ are positive constants in Lemmas 2 and 3 in Appendix C, $\omega$ and $c$ are given in (24) and (25) in Appendix C respectively.

Thus, if the learning rate $\rho > 0$ is sufficiently small, then $x_i$ in APPG converges to an optimum $x^*$ at a linear rate $O(\lambda^k)$ and $y_i$ converges to 0 at the same rate. Note that the virtual counter $k$ in (10) increases by 1 no matter on which node updates. Therefore, the more the nodes, the faster $k$ increases. To some extent, this suggests that the convergence rate would be proportional to the number of cores, which is also pointed out in [1], [2]. In fact, we indeed observe that APPG achieves a linear speedup in experiments.

The convergence rate in Theorem 1 is obtained in the worst case, where the difference of update rates of nodes and the communication delays are based on the upper bound in Assumptions 2 and 3. The performance is often
much better in practice. For example, when using homogenous GPUs connected via fast Ethernet to train a model, their update frequencies will be very close, and the communication delay is often low.

To prove Theorem 1, we develop an augmented system approach to address asynchrony and delays, which enables us to transform APPG to the form of a synchronous algorithm on the augmented digraph. From this point of view, it also forms the technical contribution of this work. The complete proof is provided in Appendix C.

V. Experiments

We use APPG to train a multi-class logistic regression classifier on the Covertype dataset from the UCI Machine Learning Repository [29], where the objective function takes the following form

\[
f(X) = -\sum_{i=1}^{n_s} \sum_{j=1}^{n_c} l_i^j \log \left( \frac{\exp(x_j^T s_i)}{\sum_{j'=1}^{n_c} \exp(x_j'^T s_i)} \right) + \frac{\gamma}{2} \|X\|_F^2.
\]

Here \( n_s = 581012 \) is the number of training instances, \( n_c = 7 \) is the number of classes, \( n_f = 55 \) is the number of features, \( s_i \in \mathbb{R}^{55} \) is the feature vector of the \( i \)-th instance, \( l^i = [l^i_1, ..., l^i_7]^T \) is the label vector of the \( i \)-th instance using the one-hot encoding, \( X = [x_1, ..., x_7] \in \mathbb{R}^{n_f \times n_c} \) is the weighting matrix to be optimized, \( \gamma = 20 \) is a regularization factor.

**Environment:** APPG is implemented in Python 3.6 with OpenMPI 1.10 on Ubuntu 14.04. The hardware is a server with 28 Xeon E5-2660 cores. Each core serves as a computing node.

**Distributed Data:** We firstly normalize non-categorical features by subtracting the mean and dividing by the standard deviation in the whole dataset. Then, we randomly divide the dataset to \( n \) parts. Each node (core) only has access to one part. That is, we are dealing with distributed dataset.

**Gradient:** At each iterate, a node computes a full gradient using all samples in its local dataset. APPG with stochastic gradient is also implemented in Appendix B.

**Topology:** The default network of nodes is similar to [2] except that we use directed edges. Each node \( i \) sends messages to node \( \text{mod}(2^j + i, n) \), where \( j \in \mathbb{N} \cap [0, \log_2(n)) \) and \( \text{mod}(a, b) \) returns the remainder after division of \( a \) by \( b \). Thus, each node has \( O(\log(n)) \) out-neighbors, which results in a relatively sparse directed networks. We also implement APPG over other networks in Section V-B.
Learning rate: The default learning rate is set to be $\rho = 0.5/n_s$. We also test other learning rates in Section V-B.

Local Termination: Node $i$ stops locally if the value of $y_i$ in last $n$ consecutive iterates are less than $300/n_s$.

A. Convergence performance and linear speedup

We implement APPG over $n = 1, 6, 12, 18, 24$ nodes. The training loss w.r.t. running time is plotted in Fig. 2(a) which shows that the training time is significantly reduced with more cores.

Fig. 2(b) depicts the training loss w.r.t. the number of iterates (epochs). We find that the number of iterates required to achieve a certain accuracy is close to each other for different number of cores, which is consistent with [2]. Since the time to compute an update in a node is proportional to the size of its local dataset, which is roughly inversely proportional to the number of cores, this explains why linear speedup is possible from this point of view.

Fig. 2(c) plots the training loss of a synchronous version of APPG, which is done by adding a barrier after each update. That is, nodes start to update simultaneously (see also Section III-C). The result shows its convergence rate is slower than APPG.

Fig. 3 roughly shows that the APPG achieves a linear speedup in convergence rate w.r.t. the number of nodes, where $T_n$ is the running time of the APPG with $n$ core(s) when the training loss decays to 0.005. One can also find
that the synchronous version of APPG has an approximately linear speedup when the number of cores is small, but it decreases fast when the number of cores is relatively large. This behavior has also been observed in [2], [26] and illustrates the advantage of asynchronous algorithms.

Ideally, the speedup would be $n$ when using $n$ cores. However, the communication among nodes introduces delays and staleness to the algorithm, which degrades the convergence rate. In practice, a higher speedup than Fig. 3 can be achieved by using a network with larger bandwidth.

B. Effect of network topology and learning rate

Selections of communication topology and the learning rate are important to implement the APPG. We show their effect to the convergence performance in this subsection.

We test the APPG under the following directed graphs.

1) log topology (default): Node $i$ sends messages to node mod($2^j + i, n$), where $j \in \mathbb{N} \cap [0, \log_2(n))$.
2) sqrt topology: Node $i$ sends messages to node mod($j^2 + i + 1, n$), where $j \in \mathbb{N} \cap [0, \sqrt{n})$.
3) linear topology: Node $i$ sends messages to node mod($5j + i + 1, n$), where $j \in \mathbb{N} \cap [0, n/5)$.
4) fully topology: Fully connected graph, a node sends information to all the rest nodes.

The log topology has the sparsest edges while the fully topology is the densest one.

Fig. 4(a) shows the convergence rate in running time of 24 cores over these topologies, and Fig. 4(b) depicts the speedup. For log, sqrt or linear topologies, the convergence rate is slightly faster if the graph is denser, which is because a denser graph accelerates the information mixing speed. However, there is a sharp reduction in convergence rate when the graph is too dense as in the fully topology. The reason is such a dense graph results in large amount of communication data at each iteration, which heavily increases the communication overhead and the staleness in gradient computation. In practice, an appropriate topology should be designed according to the network bandwidth.

We then test the APPG with different learning rates. The result is plotted in Fig. 4(a) which shows that the algorithm converges faster by using a larger learning rate in the feasible range.

C. Comparison with the state of the art

We compare the APPG with AllReduce decentralized gradient descent algorithm (AllReduce-DGD), asynchronous centralized parallel gradient algorithm (A-PGD) and AD-PSGD [2]. The AD-PSGD is originally designed to use stochastic gradients, and we modify it here to use the full local gradients with a constant learning rate. The implementation details of these algorithms can be found in Appendix A.

Fig. 5 shows the convergence performance of these algorithms with different number of cores. We have the following observations:

1) APPG converges faster than A-PGD in all cases, which agrees with the result in [1], [2] that decentralized algorithms can outperform centralized algorithms.
Fig. 5. Convergence performance of the APPG, AllReduce-DGD, A-PGD, AD-PSGD with different number of cores. We modify the AD-PSGD to use the full local gradient instead of stochastic gradient at each iteration to make the comparison fair.

Fig. 6. Speedup of APPG and the synchronous implementation of APPG when one core slows down.

2) APPG outperforms the AD-PSGD with full gradients. This is consistent with Section III-C which also explains why AD-PSGD has a larger oscillation around the optimum. In fact, it can only converge to a neighborhood of the optimum with a constant learning rate.

3) APPG is slightly slower than AllReduce-DGD when the number of cores is small, but is faster when the number of cores is large. This is because AllReduce method collects information more efficiently in small-scale networks, but its synchrony property slows it down in relatively large-scale networks.

D. Robustness of APPG to slow cores

We evaluate the robustness of APPG by making one core in the network slow down. This is achieved by adding an artificial time delay (20ms, a normal iteration takes about 15ms with 24 cores) after each local iteration of a core, which simulates either the slow computation or slow communication.

Fig. 6 shows the speedup of running time of the APPG and the synchronous implementation of APPG in this scenario. It indicates that the synchronous counterpart of APPG has a sharp reduction in convergence rate even when only 1 core slows down. In contrast, APPG can still keep an almost linear speedup. This result is also consistent with that in [2], [26].

Introducing the slowing core also brings an easily overlooked problem of asynchronous algorithms, that is, the cores have uneven update rates. To show its effect to the performance, we compare the proposed algorithm to
AD-PSGD with full local gradients. Fig. 7 shows the result over 12 cores and 24 cores, where the AD-PSGD fails to converge to the accurate optimum. In contrast, APPG converges to the exact optimum in the environment of uneven update rates despite the convergence rate is reduced a bit.

VI. CONCLUSION

This paper proposes an asynchronous decentralized parallel optimization algorithm called APPG. It can work over a directed communication topology of workers, and allows workers to have local datasets sampled from different distributions. Linear theoretic convergence rate of APPG is guaranteed under some mild assumptions. The practical performance of APPG is also demonstrated via a logistic regression problem. Future work may focus on the extension of APPG to stochastic gradients case.

APPENDIX A

IMPLEMENTATION DETAILS OF ALGORITHMS IN SECTION V-C

For AllReduce-DGD, at each iteration, a node first computes the local gradient using the local dataset. Then, it performs a sum reduction using the MPI allreduce function, and receives the sum of all nodes’ local gradients. Finally, it updates its local state using the received information. The process repeats until the norm of received gradient is smaller than a threshold.

For A-PGD, we let the first core be the master node and other cores are slave nodes. A slave node computes its local gradient and then sends it to the master node. The master node updates the parameter once it receives a local gradient from some slave node, and then sends the updated parameter back to the node. The process repeats until be terminated by running time.

For AD-PSGD, the implementation is based on the Appendix A of [2], except that
1) Each worker uses a full local gradient instead of a stochastic gradient at each update. The full local gradient is computed using all samples in the local dataset.
2) We do not use GPUs, and thus the communications and computations have to run in serial instead of in parallel.
3) Compared with the APPG, we make the directed network undirected by adding an inverse communication link to each link in the graph. That is, if \((i, j) \in \mathcal{E}\), then we add \((j, i)\) to \(\mathcal{E}\).

The AD-PSGD is also terminated by running time.

**APPENDIX B**

**APPG WITH STOCHASTIC GRADIENTS**

We test the APPG with stochastic gradient by replacing the local gradient \(\nabla f_i(x_i)\) in Algorithm 1 with a stochastic gradient \(\sum_{j=1}^{p} F_i(x_i; \xi^j_i)\), where \(p = 128\) is the batch size and \(\{\xi^1_i, \cdots , \xi^p_i\}\) is a batch randomly sampled from \(\mathcal{D}_i\). The learning rate is set to be a constant \(\rho = 0.01/n_s\). We compare the APPG to AD-PSGD with the same batch size and learning rate. No momentum is used in both algorithms.

Fig. 8 depicts the result, which shows the convergence rate of APPG is slightly faster than that of AD-PSGD. Moreover, APPG achieves higher accuracy than AD-PSGD as illustrated in Fig. 8(c). This is because AD-PSGD requires nodes to have the same update frequencies, which is difficult to satisfy in practice.

![Fig. 8. Convergence performance of the APPG and AD-PSGD with stochastic gradients.](image)

To further illustrate the effect of update rates, we implement the two algorithms when one core slows down as in Section V-D. This causes nodes to have different update frequencies. The result is plotted in Fig. 9, which shows that APPG is almost not affected by the uneven update rates, while the accuracy of AD-PSGD has a degradation.

![Fig. 9. Convergence performance of the APPG and AD-PSGD with stochastic gradients and 1 slow core.](image)

**APPENDIX C**

**PROOF OF THEOREM 1**

We add the following notations and definitions for the proof.
The following statements hold.

Lemma 1. The Augmented System

The convergence of the transformed synchronous algorithm in Section C-B.

Proof. (a) Suppose that node \( i \) finishes an update at time \( t(p), p \in \mathbb{N} \) but is activated at \( t(q) + 1 \). It follows from Assumption 2 that \( t(q) - t(p) \leq \bar{\tau} \). Moreover, any other node can be activated at most \( |(t(q) - t(p))/\tau| \leq |\bar{\tau}/\tau| \) times during the time interval \( (t(p), t(q)) \), which implies \( q - p \leq (n - 1)|\bar{\tau}/\tau| \). Hence the first part of the result follows.

(b) Suppose that node \( i \) sends information at time \( t(q), q \in \mathbb{N} \) and node \( j \) receives it in the time interval \( (t(q), t(q) + 1), q \in \mathbb{N} \). It follows from Assumption 3 that \( t(q) - t(p) \leq \tau \). Moreover, Assumption 2 implies that any node can be activated at most \( |\tau/\tau| \) times during the time interval \( [t(p), t(q)] \), i.e., \( q - p + 1 \leq n|\tau/\tau| \), and hence \( q + 1 \leq p + n|\tau/\tau| \). The result follows by letting \( p = k \). Jointly with Lemma 1(a), the rest of the results follow immediately.

1) Construction of the Augmented Digraph: Let \( T_i \subseteq T \) be the sequence of activation time of node \( i \), i.e., \( t \in T_i \) if node \( i \) finishes an update at time \( t \). Then, it is clear that

\[
[x_i(k+1), y_i(k+1), g_i(k+1), g_i^{-1}(k+1)] = [x_i(k), y_i(k), g_i(k), g_i^{-1}(k)], \quad \forall t(k) \notin T_i.
\]
To handle bounded time-varying transmission delays and asynchrony, we design an augmented system. Firstly, we associate each node $i$ with two types of virtual nodes, and each type has $b$ virtual nodes, where $b$ is defined in Lemma 1(b). We denote the two types of virtual nodes by $\{v^{(1)}_{x,i}, \ldots, v^{(b)}_{x,i}\}$ and $\{v^{(1)}_{y,i}, \ldots, v^{(b)}_{y,i}\}$, respectively. We call the first type virtual nodes with subscript $x$ $x$-type nodes, which is to deal with the staleness of the state $x_i$, $i \in \mathcal{V}$. The second type with subscript $y$ is called $y$-type nodes, which is to handle the staleness of $y_i$, $i \in \mathcal{V}$.

Then, we use an augmented digraph $\tilde{G}(k) = (\tilde{V}, \tilde{E}(k))$ to represent the communication topology of all these nodes at time $t(k)$, where $\tilde{V}$ contains $n(2b + 1)$ nodes, including the original $n$ nodes and $2nb$ virtual nodes. The edge set $\tilde{E}(k)$ in the augmented graph is described as follows.

We first note that there is no edge between any $x$-type node and any $y$-type node since they are introduced for different functions. For the $x$-type virtual nodes, the edges $(i, v^{(1)}_{x,i}), (v^{(2)}_{x,i}, v^{(b)}_{x,i}), \ldots, (v^{(b-2)}_{x,i}, v^{(b-1)}_{x,i})$ and $(v^{(b-1)}_{x,i}, v^{(b)}_{x,i})$ always exist for all $k \in \mathbb{N}$ and $i \in \mathcal{V}$ (c.f. Fig. 10). If $(i, j) \in \mathcal{E}$ in $\mathcal{G}$ and node $j$ receives $x_i$ at time $t(k)$, then some of edges $(v^{(1)}_{x,i}, j), (v^{(2)}_{x,i}, j), \ldots, (v^{(b)}_{x,i}, j)$ and $(i, j)$ shall be included in $\mathcal{E}(k)$ (c.f. Fig. 11(a)), which depends on the transmission delay of the received message. For example, if node $j$ received the delayed information $x_i(t - u)$ and $x_i(t - v)$ at time $t(k)$ for some $u, v > 1$, then $(v^{(u-1)}_{x,i}, j), (v^{(v-1)}_{x,i}, j) \in \tilde{E}(k)$. If $u = 1$, which means that there is no communication delay, then $(i, j) \in \tilde{E}(k)$. Fig. 11(a) illustrates such an augmented graph.

The topology of the $y$-type virtual nodes is similarly developed, but most edges’ directions are reversed (c.f. Fig. 12(a)), which is the reason why we must use two types of virtual nodes. Firstly, edges $(v^{(1)}_{y,j}, i), (v^{(2)}_{y,j}, i), \ldots, (v^{(b-1)}_{y,j}, i)$ and $(v^{(b)}_{y,j}, i)$ are always included in $\mathcal{E}(k)$. Note that the directions are reversed compared with the $x$-type nodes, see Fig. 10. Secondly, if $(i, j) \in \mathcal{E}$ in $\mathcal{G}$ and $k \in \mathcal{T}$, then one and only one of edges $(i, v^{(1)}_{y,j}), (i, v^{(2)}_{y,j}), \ldots, (i, v^{(b)}_{y,j})$ and $(i, j)$ shall be included in $\mathcal{E}(k)$, which depends on the transmission delay of $\tilde{y}_i$ sent from node $i$ to node $j$.

Specifically, at time $t(k)$, suppose that node $i$ sends $y_i(k)$ to node $j$, which is received at $t(k + u)(u > 1)$, then $(i, v^{(u-1)}_{y,j}) \in \tilde{E}(k)$ and the delay is $u$. Similarly, if there is no communication delay, i.e. $u = 1$, then $(i, j) \in \tilde{E}(k)$. Fig. 12(a) illustrates such an augmented graph.

A simple example is given to further illustrate the augmented graph approach. Consider that node $i$ sends $x_i(k)$ and $\tilde{y}_i(k) = y_i(k)/|\mathcal{N}_i| + 1$ to node $j$ at time $t(k)$, and node $j$ receives it at time $t(k + 2)$. Hence, the delay is 1. In the augmented graph, this can be viewed as node $i$ directly sends $x_i(k)$ to the virtual node $v^{(1)}_{x,i}$ and sends $\tilde{y}_i(k)$ to the virtual node $v^{(1)}_{y,j}$ at time $t(k)$ (note the different subscripts). $v^{(1)}_{x,i}$ and $v^{(1)}_{y,j}$ respectively receive $x_i(k)$ and $\tilde{y}_i(k)$ at time $t(k + 1)$, and immediately send them to node $j$ at time $t(k + 1)$. Finally, node $j$ receives $x_i(k)$ and $\tilde{y}_i(k)$ at time $t(k + 2)$. Clearly, all non-virtual nodes in $\tilde{G}$ receive the same information as that in $\mathcal{G}$ and hence their updates appear to be synchronous and without delays.

With the help of the augmented graph, we shall transform the APPG to a synchronous algorithm in the next subsection.

2) **Compact Form of APPG with the Augmented Digraph:** Let $\mathbf{x}_i^{(u)}(k)$ and $\mathbf{y}_i^{(u)}(k)$ respectively denote the states of virtual node $v^{(u)}_{x,i}$ and $v^{(u)}_{y,i}$ just after time $t(k)$, and let $\tilde{n} = n(2b + 1)$. Then, the APPG can be expressed as a

---

2The idea of adding virtual nodes to address staleness or delays is firstly adopted in [30] to study consensus problems in multi-agent system. Moreover, [30] only uses one type virtual nodes, which is not sufficient here.
The original graph

Virtual nodes

Fig. 10. An augmented graph with virtual nodes to address delays of the original graph.

(a) The topology of the $x$-type virtual nodes in the augmented graph at some time $t(k)$. At this time, node 1 uses the 2-steps delayed information $x_3(k-3)$ and the latest information $x_1(k-1)$ to compute $x_1(k)$. Node 2 uses $x_2(k-1)$ and the 1-step delayed information $x_1(k-2)$ and $x_3(k-2)$ to compute $x_2(k)$. Node 3 use the latest information $x_3(k-1)$ and $x_3(k-1)$ to compute $x_3(k)$. (b) The corresponding row-stochastic matrix $\tilde{A}(k)$ in \ref{eq:row-stochastic-x}.

(b) The corresponding column-stochastic matrix $\tilde{B}(k)$ in \ref{eq:column-stochastic-y}.

Fig. 11. (a) The topology of the $y$-type virtual nodes in the augmented graph at some time $t(k)$, which represents that node 1 sends $\tilde{y}_1(k)$ to node 2 and node 2 shall use it at $t(k+3)$ to compute $y_2(k+3)$. Similarly for other edges. (b) The corresponding column-stochastic matrix $\tilde{B}(k)$ in \ref{eq:column-stochastic-y}.
synchronous algorithm over \( \tilde{G}(k) \),

\[
\begin{align*}
\tilde{X}(k+1) &= \tilde{A}(k)\tilde{X}(k) - \rho I_k^a \tilde{Y}(k), \\
\tilde{Y}(k+1) &= \tilde{B}(k)\tilde{Y}(k) + I_k^a (\nabla(k+1) - \nabla(k)) \\
&= \tilde{B}(k)\tilde{Y}(k) + \nabla(k+1) - \nabla(k)
\end{align*}
\] (11)

where

\[
\tilde{X}(k) = [X(k); X^{(1)}(k); \cdots; X^{(b)}(k)] \in \mathbb{R}^{\tilde{n} \times m}
\]

\[
X^{(u)}(k) = [x_1^{(u)}(k), \cdots, x_{n_i}^{(u)}(k)]^T
\]

\[
\tilde{Y}(k) = [Y(k); Y^{(1)}(k); \cdots; Y^{(b)}(k)] \in \mathbb{R}^{\tilde{n} \times m}
\]

\[
Y^{(u)}(k) = [y_1^{(u)}(k), \cdots, y_{n_i}^{(u)}(k)]^T
\]

\[
\nabla(k) = [\nabla f(X(k)); \mathbf{0}_{(\tilde{n} - n) \times m}]
\]

with initial condition

\[
\tilde{X}(0) = [X(0); \mathbf{0}_{(\tilde{n} - n) \times m}]
\]

\[
\tilde{Y}(0) = [\nabla(0); \mathbf{0}_{(\tilde{n} - n) \times m}]
\]

and the matrices \( \tilde{A}(k), \tilde{B}(k), I_k^a \in \mathbb{R}^{\tilde{n} \times \tilde{n}} \) are defined by

\[
[\tilde{A}(k)]_{ij} = \begin{cases} 
\frac{1}{|\mathcal{X}_{iu}^{(i)}(k)|}, & \text{if } i, v \in \mathcal{V}, j = nu + v, t(k+1) \in \mathcal{T}_i, \text{ and node } i \text{ receives } x_v(k - u) \text{ at } t(k+1) \\
1, & \text{if } i \in \mathcal{V}, t(k+1) \notin \mathcal{T}_i \text{ and } j = i \\
1, & \text{if } i \notin \mathcal{V} \text{ and } j = i - n \\
0, & \text{otherwise}
\end{cases}
\]

\[
[\tilde{B}(k)]_{ji} = \begin{cases} 
\frac{1}{|\mathcal{X}_{iu}^{(i)}|}, & \text{if } i, v \in \mathcal{V}, j = nu + v, t(k+1) \in \mathcal{T}_i, \text{ and node } v \text{ shall receive } x_i(k) \text{ at } t(k+u) \\
1, & \text{if } i \in \mathcal{V}, t(k+1) \notin \mathcal{T}_i \text{ and } j = i \\
1, & \text{if } i \notin \mathcal{V} \text{ and } j = i - n \\
0, & \text{otherwise}
\end{cases}
\]

and

\[
[I_k^a]_{ij} = \begin{cases} 
1, & \text{if } i = j, i \in \mathcal{V}, \text{ and } t(k+1) \in \mathcal{T}_i \\
0, & \text{otherwise}
\end{cases}
\]

where \( |\mathcal{X}_{iu}^{(i)}(k)| \) is the number of elements in the buffer \( \mathcal{X}_{iu}^{(i)} \) at time \( t(k+1) \).

An example of \( \tilde{A}(k) \) and \( \tilde{B}(k) \) is illustrated in Fig. 11(b) and Fig. 12(b) respectively. \( I_k^a \) is a diagonal matrix with its \( i \)-th diagonal element be 1 if node \( i \) is activated at time \( t(k+1) \). The third equality in (11) holds because by definition \( \nabla(k+1) = \nabla(k) \) for any \( i \in \{i|I_k^a]_{ii} = 0 \} \). An important fact is that \( \tilde{A}(k) \) is a row-stochastic matrix and \( \tilde{B}(k) \) is a column-stochastic matrix, which results from the usage of two types virtual nodes. Moreover,

\[
1_{\tilde{n}}^T \tilde{Y}(k) = 1_{\tilde{n}}^T \nabla(k) = 1_{\tilde{n}}^T f(X(k))
\] (13)

which is obtained by left multiplying the second equation of (11) with \( 1_{\tilde{n}}^T \).
Note again that (11) generates the same sequences of the states $x_i$ and $y_i$ as that in APPG. Hence, it is sufficient to study the convergence of $\tilde{X}(k)$ and $\tilde{Y}(k)$ in (11). To this end, we define

$$\Phi^A(k) = \tilde{A}(k + t - 1)\tilde{A}(k + t - 2) \cdots \tilde{A}(k + 1)\tilde{A}(k)$$

$$\Phi^B(k) = \tilde{B}(k + t - 1)\tilde{B}(k + t - 2) \cdots \tilde{B}(k + 1)\tilde{B}(k).$$

where $k, t \in \mathbb{N}$, and we adopt the convention that $\Phi^A_0(k) = \Phi^B_0(k) = I$ and $\Phi^A_t(k) = \Phi^B_t(k) = 0$ for any $k \in \mathbb{N}$ and $t < 0$.

The following lemma states that $\Phi^A_t(k)$ and $\Phi^B_t(k)$ linearly converge to rank-one matrices.

**Lemma 2.** Under Assumptions 1, 2 and 3, the following statements are in force.

(a) There exist stochastic vectors $\phi^A_t(k)$ and $\phi^B_t(k)$ such that

$$\|\Phi^A_t(k) - 1\phi^A_t(k)^T\|_F \leq 2\gamma^t$$

$$\|\Phi^B_t(k) - \phi^B_t(k)1^T\|_F \leq 2\gamma^t$$

for all $k, t \in \mathbb{N}$, where

$$\gamma = (1 - \theta)^{\frac{1}{\tilde{n}n}} < 1, \quad \theta = \left(\frac{1}{\tilde{n}}\right)^{d_g} \in (0, 1),$$

$b$ is defined in Lemma 2(b), $d_g$ is the diameter of $G$ and $\tilde{n} = n(b + 1)$.

(b) $\sum_{j=1}^n[\Phi^B_t(0)]_{ij} \geq n\theta, \forall i \in \mathcal{V}, j \in \tilde{\mathcal{V}}, t \in \mathbb{N}$.

Proof. In view of Lemma 1 both $\Phi^A_0(k)$ and $\Phi^B_0(k)$ are primitive for all $k$. With this result, the proof of the first part can be found in the Lemma 5 of [30] by the definition of $d_g$ and the fact that each nonzero element of $\tilde{A}(k)$ and $\tilde{B}(k)$ is larger than $1/\tilde{n}$.

To prove (b), two cases are separately studied. If $t < d_gb$, then $[\Phi^B_t(0)]_{ii} \geq [\tilde{B}(t-1)]_{ii} [\tilde{B}(t-2)]_{ii} \cdots [\tilde{B}(0)]_{ii} \geq (1/\tilde{n})^{d_gb-1} = \tilde{n}\theta \geq n\theta$, and hence the result is obtained.

If $t \geq d_gb$, it follows from a similar argument with the Lemma 2(b) in [30] that $[\Phi_{d_gb}(k)]_{ij} \geq \theta$ for all $i \in \mathcal{V}$ and $j \in \tilde{\mathcal{V}}$. Then,

$$[\Phi_{d_gb+1}(k-1)]_{ij} = \sum_{u=1}^{\tilde{n}}[\Phi_{d_gb}(k)]_{iu} [\tilde{B}(k)]_{uj} \geq \theta \sum_{u=1}^{\tilde{n}}[\tilde{B}(k)]_{uj} \geq \theta. \forall i \in \mathcal{V}, j \in \tilde{\mathcal{V}}.$$

where the last inequality follows from the column-stochasticity of $\tilde{B}(k)$. The desired result is obtained by induction.

The following lemma is a direct result of Lemma 2 which introduces variables $\mu$ and $\tilde{\mu}$ used in Theorem 1 and the next section.

**Lemma 3.** Under Assumptions of Lemma 2 and given $\mu \in (0, \theta/(2n))$, let $\tilde{\mu} \in \mathbb{N}$ be a number such that $2\gamma^\tilde{\mu} \leq \mu$, then for all $k \in \mathbb{N}$,

$$\|\Phi^A_t(k) - 1\phi^A_t(k)^T\|_F \leq \mu < \theta/(2n) < 1$$

$$\|\Phi^B_t(k) - \phi^B_t(k)1^T\|_F \leq \mu < \theta/(2n) < 1$$
Finally, we introduce an important concept called absolute probability sequence from [31], which is also used in [32] to study synchronous PPG over time-varying graphs.

**Lemma 4.** For any sequence of row-stochastic matrices \( \{A(k)\} \), there exists a sequence of stochastic vectors \( \{\pi(k)\} \) satisfying

\[
\pi(k + 1)^T A(k) = \pi(k)^T, \quad \forall k \in \mathbb{N}.
\]

The sequence \( \{\pi(k)\} \) is called an absolute probability sequence of \( \{A(k)\} \).

The proof of Lemma 4 can be found in the Theorem 4.2 of [31]. In the sequel, we shall use \( \pi(k) \in \mathbb{R}^{\tilde{n}} \) to denote a absolute probability sequence of \( \tilde{A}(k) \), i.e.,

\[
\pi(k + 1)^T \tilde{A}(k) = \pi(k)^T, \quad \forall k \in \mathbb{N}.
\]

Clearly, we have \( \pi(k + t)^T \Phi(\tilde{A})^k = \pi(k)^T, \forall k, t \in \mathbb{N} \).

**B. Proof of the Main Result**

1) **Outline of the Proof:** We first introduce a definition from [27] to study the convergence rate of a sequence. Let \( \{p(k)\} \) be a nonnegative sequence and \( \lambda \in (0, 1) \), we define the \( \lambda \)-sequence of \( p(k) \) be

\[
p^{\lambda,k} = \sup_{t \in \mathbb{N}, t \leq k} \frac{p(t)}{\lambda^t}.
\]

Clearly, if \( p^{\lambda,k} \) is bounded above by some constant \( c \) for all \( k \), then \( \{p(k)\} \) converges to 0 at a linear rate, i.e., \( p(k) \leq c\lambda^k \) for all \( k \). Our method to prove Theorem 1 is to show the boundedness of \( p^{\lambda,k}, k \in \mathbb{N} \) in (16) for some carefully designed sequences \( p(k) \).

Under Assumptions [1][2][3] and the recursion (11), the proof of Theorem 1 relies on the following four key claims:

**Claim 1:** Let \( \pi(k) \) be defined in (15),

\[
Q(k) = I_{\tilde{n}} - \tilde{\pi}(k - 1)^T, \quad \|Q(k)\|_Q = \|Q(k)\tilde{X}(k)\|_F, \quad \|X\|_Q^\lambda \quad \text{be the } \lambda \text{-sequence of } \|X(k)\|_Q \quad (\text{c.f. (16))}. \text{ If } \lambda > (1 + 2\mu)/2, \text{ where } \tilde{\pi} \text{ and } \mu \text{ are defined in Lemma [3]} \text{ then}
\]

\[
\|X\|_Q^\lambda \leq \rho \omega_1 \|Y\|_F^\lambda + c_1, \quad \omega_1 = \frac{\sqrt{2\tilde{n}t}}{1 - 2\mu},
\]

where \( \|Y\|_F^\lambda \) is the \( \lambda \)-sequence of \( \|Y(k)\|_F \) and \( c_1 \) given by (32) is not related to \( k \).

In fact, \( \|X(k)\|_Q \) represents a weighted disagreement among nodes’ states \( x_i(k) \), and (17) shows that it can be bounded by the gradient estimate \( \|Y(k)\|_F \).

**Claim 2:** Let

\[
\begin{align*}
\mathbf{v}(k + 1) &= \tilde{B}(k)\mathbf{v}(k), \quad \mathbf{v}(0) = [1_{\tilde{n}}; 0_{\tilde{n}-n}], \\
V(k) &= \text{diag}(\mathbf{v}(k)), \quad Y\mathbf{v}(k) = V(k)^\dagger Y(k)
\end{align*}
\]

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where $V(k)^\dagger$ is the pseudo inverse of $V(k)$, i.e.,

$$[V(k)^\dagger]_{ij} = \begin{cases} 1/[V(k)]_{ii}, & \text{if } i = j \text{ and } [V(k)]_{ii} > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then, let $\tilde{I}(k) = V(k)V(k)^\dagger$, $\tilde{I}(k) = \tilde{I}(k)\mathbf{1}_n$ and

$$S(k) = \tilde{I}(k) - \frac{1}{n} \tilde{I}(k)\tilde{V}(k)^T, \quad \|Y_\nu(k)\|_S = \|S(k)Y_\nu(k)\|_F$$

(18)

Define the corresponding $\lambda$-sequence $\|Y_\nu\|_{S,k}^\lambda$ (c.f. (16)). If $\lambda \tilde{t} > (1 + 2\theta^{-1}\mu n)/2$, where $\tilde{t}$ and $\mu$ are defined in Lemma 3, then

$$\|Y_\nu\|_{S,k}^\lambda \leq \rho \omega_2 \|\tilde{Y}\|_{f}^\lambda + c_2, \quad \omega_2 = \frac{2\beta \theta^{-1} \sqrt{n}(\sqrt{\nu_1} + 1)\tilde{t}}{1 - 2\theta^{-1}\mu n}.$$  

(19)

where $\beta$ is given in Assumption 1, $\theta, \tilde{t}, \mu, \bar{n}$ are defined in Lemmas 2 and 3, and $c_2$ given by (21) is not related to $k$. Similarly, $\|Y_\nu(k)\|_S$ measures the difference between the weighted gradient estimates of different nodes, which is also bounded by $\|\tilde{Y}(k)\|_F$.

Claim 3: Let $\pi(k)$ be defined in (15), and let

$$x_\pi(k) = \pi(k)^T\bar{X}(k)$$

(20)

Define $\|x_\pi - x^*\|_{f}^\lambda$ be the corresponding $\lambda$-sequence of $\|x_\pi(k) - x^*\|_F$. If $\rho \leq 1/(nb\beta)$ and $\lambda \geq 1 - \alpha\rho\theta n/2$. Then,

$$\|x_\pi - x^*\|_{f}^\lambda \leq \frac{2b}{\alpha} \left((2n + \sqrt{n})\beta \|\bar{X}\|_{Q}^\lambda + \|Y_\nu\|_{S,k}^\lambda\right) + c_3$$

(21)

where $\alpha, \beta$ are given in Assumption 1, $b$ is defined in Lemma 1, $\theta$ is defined in Lemma 2, and $c_3$ given by (48) is not related to $k$.

$x_\pi(k)$ is a weighted average of nodes’ states $x_i(k)$, and $\|x_\pi(k) - x^*\|_F$ is the distance between the weighted average and the optimal point $x^*$. Clearly, if $x_i(k)$ converges to $x^*$ for all $i$, then $\|x_\pi(k) - x^*\|_F$ will converge to 0. Eq. (20) shows that $\|x_\pi(k) - x^*\|_F$ can be bounded by $\|\bar{X}\|_{Q}^\lambda$ and $\|Y_\nu\|_{S,k}^\lambda$.

Claim 4: With the above-defined $\|\bar{X}\|_{Q}^\lambda$, $\|Y_\nu\|_{S,k}^\lambda$ and $\|x_\pi - x^*\|_{f}^\lambda$, we have

$$\|\tilde{Y}\|_{f}^\lambda \leq 2\rho \omega_2 \|\tilde{X}\|_{Q}^\lambda + n\|Y_\nu\|_{S,k}^\lambda + 2n\sqrt{n}\beta \|\bar{X}\|_{Q}^\lambda + 2\|x_\pi - x^*\|_{f}^\lambda.$$  

(22)

where $\beta$ is given in Assumption 1, $\rho$ is defined in Lemma 1, and $b$ is defined in Lemma 1.

We shall give the proof of the four claims in following subsections. Now we use them to prove Theorem 1 in this subsection.

Note that $\lambda$ defined in Theorem 1 satisfies all the conditions on $\lambda$ of the above four claims and $b$ in Theorem 1 satisfies the condition on $\rho$ of Claim 3. Thus, (17), (19), (21) and (22) hold. Substitute (22) into (17), (19) and combine (21), we obtain that for all $k \in \mathbb{N}$,

$$\begin{bmatrix} \|\tilde{X}\|_{Q}^\lambda & \|Y_\nu\|_{S,k}^\lambda & \|x_\pi - x^*\|_{f}^\lambda \end{bmatrix} \leq M \begin{bmatrix} \|\tilde{X}\|_{Q}^\lambda & \|Y_\nu\|_{S,k}^\lambda & \|x_\pi - x^*\|_{f}^\lambda \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

(23)
Lemma 5. Let \( \nabla \) be the element-wise less-than sign and \( M \) is a nonnegative matrix:

\[
M = \begin{bmatrix}
2pn(\sqrt{n} + 1)\beta\sqrt{b}\omega_1 & pn\omega_1 & 2pn\sqrt{b}\sqrt{b}\omega_1 \\
2pn(\sqrt{n} + 1)\beta\sqrt{b}\omega_2 & pn\omega_2 & 2pn\sqrt{b}\sqrt{b}\omega_2 \\
2b(2n + \sqrt{n})/\alpha\theta & 2b/\alpha\theta & 0
\end{bmatrix}
\]

It follows from (23) that if the spectral radius of \( M \) (denoted by \( \rho(M) \)) is less than 1, then \( \|X\|^\lambda_{\|Q\|}^k, \|Y\|^\lambda_{\|S\|}^k \) and \( \|X_{n}(k) - x^*\|^\lambda_{\|P\|} \) are all bounded for all \( k \in \mathbb{N} \). Note that we can choose \( \rho \) to make the first two rows of \( M \) arbitrarily close to 0, and hence \( \rho(M) \leq 1 \) for sufficiently small \( \rho \).

Now we provide a bound of \( \rho \) to make \( \rho(M) < 1 \). This can be achieved by bounding \( \|M^2\|_\infty \). With some mathematical computations, one can obtain that if \( \rho \) satisfies

\[
\rho < \frac{\alpha\theta}{2nb(2n + \sqrt{n} + 1)(\omega_1 + \omega_2)(2(2\sqrt{n} + 1)\beta\sqrt{b} + 1)}
\]

\[
< \frac{\alpha\theta}{4b(n + \sqrt{n})^2(2\sqrt{nb}\beta + 1)\omega}
\]

where

\[
\omega = \omega_1 + \omega_2 = \frac{4\sqrt{nb}((\theta - 1)\sqrt{bd} + \beta\theta - 1)}{(1 - 2\mu)(1 - 2\mu\theta - 1)},
\]

then \( \|M^2\|_\infty < 1 \).

Define

\[
c := \frac{2}{1 - c(\rho)} \max\{c_1, c_2, c_3\} < \infty
\]

where \( c(\rho) = \|M^2\|_\infty \) can be directly computed given \( \rho \), and \( c_1, c_2 \) and \( c_3 \) are given in (32), (42) and (48), respectively. We have

\[
\|x_i - x^*\|^\lambda_{\|Q\|} \leq 2\|X\|^\lambda_{\|Q\|} + \|x_n - x^*\|^\lambda_{\|P\|} \leq c, \forall i \in \mathcal{V}.
\]

where \( \|x_i - x^*\|^\lambda_{\|P\|} \) is the \( \lambda \)-sequence of \( \|x_i(k) - x^*\|_2 \). The desired result of Theorem 1 then follows by the definition of \( \lambda \)-sequence.

The following subsections are dedicated to the proofs of Claims 1-4.

2) Two Important Lemmas: In this subsection we provide two important lemmas used in the proof of the four claims. The first one shows some properties of \( \lambda \)-sequence and the second one recalls the contraction relation of gradient methods.

Lemma 5. Let \( \{p(k)\}, \{q(k)\} \) be nonnegative sequences satisfying

\[
p(t + j) \leq rp(t) + \sum_{i=0}^{j-1} q(t + i)
\]

where \( r \in [0, 1) \) is a scalar. If we choose \( \lambda \) such that \( \lambda^j \in (r, 1) \), then the \( \lambda \)-sequences \( p^\lambda_{\|k\|} \) and \( q^\lambda_{\|k\|} \) defined in (16) satisfy

\[
p^\lambda_{\|k\|} \leq \frac{j}{\lambda^j - r} q^\lambda_{\|k\|} + c_\lambda, \forall k \in \mathbb{N},
\]

where \( c_\lambda = \frac{\lambda^j}{\lambda^j - r} \sum_{t=1}^{\lambda^j} \lambda^{-t} p(t) \) is a constant not related to \( k \). In particular, if \( \lambda^j \in (\frac{1 + r}{2}, 1) \), we have

\[
p^\lambda_{\|k\|} \leq \frac{2j}{1 - r} q^\lambda_{\|k\|} + c_\lambda, \forall k \in \mathbb{N},
\]
Proof. It follows from (26) that
\[
\lambda^{-(t+j)} p(t+j) \leq \frac{r}{\lambda^j} \lambda^{-t} p(t) + \sum_{i=0}^{j-1} \frac{1}{\lambda^{j-i}} \lambda^{-(t+i)} q(t+i).
\]

This gives \(k\) inequalities for \(t = 1, \cdots, k\). On the other hand, we have
\[
\lambda^{-t} p(t) \leq \lambda^{-t} p(t)
\]
which gives another \(j\) inequalities for \(t = 1, \cdots, j\). Take the maximum on both sides of the \(k+j\) inequalities and use the definition of \(\lambda\)-sequence, we obtain
\[
p^{\lambda,k+j} \leq \frac{r}{\lambda^j} p^{\lambda,k} + \max \left\{ q^{\lambda,k} \sum_{i=0}^{j-1} \frac{1}{\lambda^{j-i}} \max_{t=1,\cdots,j} \lambda^{-t} p(t) \right\}
\]
(28)

If \(\lambda^j \in (r, 1)\), then (28) becomes
\[
p^{\lambda,k+j} \leq \frac{\lambda^j \sum_{i=0}^{j-1} \frac{1}{\lambda^{j-i}}}{\lambda^j - r} q^{\lambda,k+j} + \frac{\lambda^j}{\lambda^j - r} \sum_{t=1}^{j} \lambda^{-t} p(t)
\]
\[
= \frac{1 - \lambda^j}{(\lambda^j - r)(1 - \lambda)} q^{\lambda,k+j} + \frac{\lambda^j}{\lambda^j - r} \sum_{t=1}^{j} \lambda^{-t} p(t)
\]
\[
\leq \frac{j}{\lambda^j - r} q^{\lambda,k+j} + \frac{\lambda^j}{\lambda^j - r} \sum_{t=1}^{j} \lambda^{-t} p(t)
\]
for all \(k+j \in \mathbb{N}\), where we have used that \(1 - \lambda^j \leq j(1 - \lambda)\). The result is obtained immediately. For \(\lambda^j \in (\frac{1+r}{2}, 1)\), we have \(\lambda^j - r \geq (1 - r)/2\), and hence (27) follows.

The following lemma illustrates the linear convergence rate of standard gradient methods, the proof of which can be found in Lemma 10 of [33].

Lemma 6. Suppose \(f(x)\) is a \(\alpha\)-strongly convex and \(\beta\)-strongly smooth function with minimizer \(x^*\), let \(\eta \in (0, \frac{2}{\beta})\), then
\[
\|x - \eta \nabla f(x) - x^*\|_2 \leq \sigma \|x - x^*\|, \quad \forall x \in \mathbb{R}^n,
\]
where \(\sigma = \max(|1 - \alpha \eta|, |1 - \beta \eta|)\).

We end this subsection with two inequalities which are commonly used in the following subsections without specific notifications. For any \(A, B \in \mathbb{R}^{n \times n}\),
\[
\|AB\|_F \leq \|A\|_2 \|B\|_F, \quad \|A\|_2 \leq \sqrt{n} \|A\|_\infty \|A\|_1.
\]
Moreover, \(\|A\|_F \leq \sqrt{n}\) and \(\|A\|_2 \leq \sqrt{2}\) for any row-stochastic matrix \(A\).
3) Proof of Claim 1: Let \( \tilde{t} \) be defined in Lemma 3. It follows from (11) and (14) that
\[
\| \tilde{X}(k + \tilde{t}) \|_Q = \| Q(k + \tilde{t})\tilde{X}(k + \tilde{t}) \|_F \\
\leq \| Q(k + \tilde{t})\Phi^A(k)\tilde{X}(k) \|_F + \rho \sum_{t=0}^{\tilde{t}-1} \| Q(k + \tilde{t})\Phi^A_{k-t}(k+t)\bar{T}_{k+t} \|_F
\]
(29)
By introducing the stochastic vector \( \phi^A(k) \) from Lemma 3, (29) implies that
\[
\| \tilde{X}(k + \tilde{t}) \|_Q \leq \| Q(k + \tilde{t})(\Phi^A(k) - 1\phi^A_{k-1}(k)^T)Q(k)\tilde{X}(k) \|_F + \rho \sum_{t=0}^{\tilde{t}-1} \| Q(k + \tilde{t})\|_2\| \Phi^A_{k-t}(k+t)\bar{T}_{k+t} \|_F \\
\leq \| Q(k + \tilde{t})\|_2\| \Phi^A(k) - 1\phi^A_{k-1}(k)^T \|_F\| Q(k)\tilde{X}(k) \|_F + 2\rho \sum_{t=0}^{\tilde{t}-1} \| \Phi^A_{k-t}(k+t) \|_2\| \bar{T}(k + t) \|_F \\
\leq 2\rho \sqrt{2n} \sum_{t=0}^{\tilde{t}-1} \| \bar{T}(k + t) \|_F
\]
(30)
where the first inequality used the fact that \( \| Q(k) \|_2 \leq 2, \forall k \), \( \Phi^A(k) \) is row-stochastic and
\[
Q(k + t)(A - 1\pi(k - 1)^T)Q(k) \\
= Q(k + t)AQ(k) - Q(k)1\pi(k - 1)^TQ(k) \\
= Q(k + t)(A - 1\tilde{\pi}(k - 1)^T) - Q(k + t)(1\tilde{\pi} - 1\tilde{\pi}(k - 1)^T) \\
= Q(k + t)A - Q(k + t)1\tilde{\pi}(k - 1)^T = Q(k + t)A
\]
for any row-stochastic matrix \( A \) and \( k \in \mathbb{N} \). The last inequality of (30) follows from Lemma 3 and that \( \Phi^A(k) \|_2 \leq \sqrt{n}/2 \).

Note that \( \mu < 1/2 \). In view of Lemma 3 and (30), we obtain that
\[
\| \tilde{X} \|_Q^{\lambda^k} \leq \frac{\rho \sqrt{2n}(1 - \lambda^{\tilde{t}})}{(\lambda^t - 2\mu)(1 - \lambda)} \| \bar{T} \|_Q^{\lambda^k} + c_1 \\
\leq \frac{\rho \sqrt{2n}t}{1 - 2\mu} \| \bar{T} \|_Q^{\lambda^k} + c_1 
\]
for any \( \lambda^{\tilde{t}} > (1 + 2\mu)/2 \), where
\[
c_1 = \frac{\rho \sqrt{2n}t}{\lambda^t - \mu} \sum_{i=1}^{\tilde{t}} \lambda^{-i} \| \tilde{X}(t) \|_Q.
\]
(32)

4) Proof of Claim 2: Let \( v(k) = [v_1(k), \cdots, v_n(k)]^T \). Note that \( v_i(k) \geq n\theta \) by Lemma 2 for all \( i \in \mathcal{V}, k \in \mathbb{N} \). We use \( \mathcal{I}_V(k) \) to denote the set of indices \( i \) such that \( v_i(k) = |V(k)|_i = 0 \). It can be shown that the \( i \)-th row of \( \bar{Y}(k) \) is \( \Omega_i^\mu \) for all \( i \in \mathcal{I}_V(k) \), and thus \( \bar{Y}(k) = V(k)YV(k) \).

Let \( R(k) = \nabla(k + 1) - \nabla(k) \). It then follows from (11) that
\[
Y_V(k + 1) = \tilde{B}_V(k)Y_V(k) + V(k + 1)^\dagger R(k)
\]
(33)
where \( \tilde{B}_V(k) = V(k + 1)^\dagger \tilde{B}(k)V(k) \) and one can prove that each row except the \( i \)-th (\( i \in \mathcal{I}_V(k) \)) row of \( \tilde{B}_V(k) \) has row sum 1 using a similar arguments as in the Lemma 4 of \([34] \) and \([27] \). Using the definition of \( \nabla(k) \) in (12) and Assumption 1, we have
\[
\| R(k) \|_F = \| \nabla(k + 1) - \nabla(k) \|_F \leq \beta \| \tilde{X}(k + 1) - \tilde{X}(k) \|_F \leq \beta \| \tilde{X}(k + 1) - \tilde{X}(k) \|_F
\]
(34)
where we have used Assumption 1.

Notice that

\[\|\tilde{X}(k+1) - \tilde{X}(k)\|_F = \|\tilde{A}(k)\tilde{X}(k) - \rho\tilde{I}^T\tilde{Y}(k) - \tilde{X}(k)\|_F\]

\[\leq \|\tilde{A}(k) - I\|_F \|\tilde{X}(k)\|_F + \rho\|\tilde{Y}(k)\|_F\]

\[\leq 2\sqrt{n}\|\tilde{X}(k)\|_F + \rho\|\tilde{Y}(k)\|_F\]

(35)

where the second inequality follows from the row-stochasticity of \(\tilde{A}(k)\), and the last inequality is from that \(\|\tilde{A}(k) - I\|_F \leq \|\tilde{A}(k)\|_2 + \|I\|_2 \leq \sqrt{\|\tilde{A}(k)\|_1} + 1 \leq \sqrt{n} + 1 \leq 2\sqrt{n} \).

By Combining (34) and (35), we obtain

\[\|R(k)\|_F \leq 2\beta\sqrt{n}\|\tilde{X}(k)\|_Q + \beta\|\tilde{Y}(k)\|_F\]

(36)

To analyze the sequence \(\{Y_t(k)\}\), we define

\[\Phi_t(k) := \tilde{B}_t(k+1)\tilde{B}_t(k+2)\cdots\tilde{B}_t(k+1)\tilde{B}_t(k)\]

\[= V(k + t)^1 \prod_{l=1}^{t} \tilde{B}(k+l+1)\tilde{I}(k+l) \tilde{B}(k)\]

(37)

where

\[[\tilde{I}(k)]_{ij} = [V(k)V(k)^1]_{ij} = \begin{cases} 1, & \text{if } i = j, i \notin \mathcal{I}_t(k), \\ 0, & \text{otherwise}. \end{cases}\]

and the last equality follows from that

\[\tilde{I}(k + 1)\tilde{B}(k)V(k) = \tilde{B}(k)V(k), \quad \forall k \in \mathbb{N}\]

where we used the fact that \([\tilde{B}(k)V(k)1_n] = v_i(k + 1) = 0\) for any \(i \in \mathcal{I}_t(k) + 1\), and thus the \(i\)-th row of \(\tilde{B}(k)V(k)\) is \(0_m^T\).

We know from Lemma 2 and Lemma 3 that \(\Phi_t^B(k)\) can be written as

\[\Phi_t^B(k) = \phi_t^B(k)1^T + \Delta\Phi_t(k)\]

(38)

where \(\|\Delta\Phi_t(k)\|_F \leq 2\gamma^1\) and \(\|\Delta\Phi_t(k)\|_F \leq \mu < 1\). Hence,

\[v(k + t) = \Phi_t^B(k)v(k) = \phi_t^B(k)1^Tv(k) + \Delta\Phi_t(k)v(k)\]

\[= \phi_t^B(k)1^Tv(0) + \Delta\Phi_t(k)v(k)\]

\[= n\phi_t^B(k) + \Delta\Phi_t(k)v(k)\]

which implies that

\[\phi_t^B(k) = \frac{1}{n}(v(k + t) - \Delta\Phi_t(k)v(k))\]

(39)

It then follows from (37), (38) and (39) that

\[\tilde{\Phi}_t(k) = \frac{1}{n}V(k + t)^1v(k + t)1^TV(k) + V(k + t)^1\Delta\Phi_t(k)(I - \frac{1}{n}v(k)1_n^TV(k)\]

\[= \frac{1}{n}\tilde{I}(k + t)v(k)^T + C_t(k)\]
where \( \bar{1}(k) = \bar{1}(k) I \), \( C_t(k) = V(k + t) \Delta \Phi_t(k)(I - \frac{1}{n} V(k)^T I V(k)) V(k) \) and

\[
\|C_t(k)\|_F \leq \|V(k + t)^T\|_2 \Delta \Phi_t(k)\|_F \|I - \frac{1}{n} V(k)^T I V(k)\|_F < \theta^{-1} \mu \cdot n = 1.
\]

where we used the fact that all entries of \( V(k)^T \) are less than \( \theta^{-1} \) by Lemma 2. Thus

\[
\left\| \Phi_t(k) - \frac{1}{n} \bar{1}(k + t) V(k)^T \right\|_F \leq \theta^{-1} \mu n < \frac{1}{2}.
\]

Now we turn to the sequence \( \{Y_\nu(k)\} \). It follows from (18) and (33) that

\[
\|Y_\nu(k + \tilde{t})\|_s = \|S(k + \tilde{t}) Y_\nu(k + \tilde{t})\|_F
\leq \|S(k + \tilde{t}) \Phi_t(k) Y_\nu(k)\|_F + \sum_{t=1}^{\tilde{t}} \|S(k + t) \Phi_{t-i}(k + t) V(k + t)^T R(k + t - 1)\|_F
\leq \|S(k + \tilde{t}) \Phi_t(k) Y_\nu(k)\|_F + \sum_{t=1}^{\tilde{t}} \|S(k + t) V(k + t)^T \Phi_{t-i}(k + t) R(k + t - 1)\|_F
\]

Similar to (31), we have

\[
S(k + \tilde{t}) \Phi_t(k) = S(k + \tilde{t}) (\Phi_t(k) - \frac{1}{n} \bar{1}(k + t) V(k)^T) S(k).
\]

Following a similar argument as in (30) and using (36), equation (40) implies that

\[
\|Y_\nu(k + \tilde{t})\|_s \leq 2\theta^{-1} \mu n \|Y_\nu(k)\|_s + 2\sqrt{n} \sum_{t=0}^{\tilde{t}-1} \|V(k + t + 2)^T R(k + t)\|_F
\leq 2\theta^{-1} \mu n \|Y_\nu(k)\|_s + 2\theta^{-1} \sqrt{n} \sum_{t=0}^{\tilde{t}-1} \left( 2\sqrt{n} \|\bar{X}(k)\|_Q + \rho \|\bar{Y}(k)\|_F \right)
\]

where we use the relation \( \|V(k)^T\|_2 \leq \theta^{-1}, \forall k \) and (36).

By Lemma 5 we have for any \( \lambda^i > (1 + 2\theta^{-1} \mu n)/2 \) that

\[
\|Y_\nu\|_{s, \lambda^i} \leq \frac{2\theta^{-1} n (1 - \lambda^i)}{(\lambda^i - 2\theta^{-1} \mu n) (1 - \lambda)} \|\bar{X}\|_{Q, \lambda^i} + \frac{2\rho \theta^{-1} \sqrt{n} (1 - \lambda^i)}{(\lambda^i - 2\theta^{-1} \mu n) (1 - \lambda)} \|\bar{Y}\|_{F, \lambda^i} + c_2
\]

(41)

where

\[
c_2 = \frac{2\theta^{-1} \sqrt{n} \lambda^i}{\lambda^i - 2\theta^{-1} \mu n} \sum_{t=1}^{\tilde{t}} \lambda^{-t} \left( \sqrt{n} \|\bar{X}(t)\|_Q + \rho \|\bar{Y}(t)\|_F \right)
\]

Eq. (41) together with (17) implies the desired result (19) where

\[
c_2 = c_2' + \frac{2\theta^{-1} \sqrt{n} \lambda^i}{(\lambda^i - 2\theta^{-1} \mu n) (1 - \lambda)} c_1.
\]

(42)

5) Proof of Claim 3: It follows from (11), (15) and (20) that

\[
x_\pi(k + b) = \pi(k + b)^T \bar{X}(k + b)
\]

\[
= \pi(k + b)^T \Phi_b(k) \bar{X}(k) - \rho \pi(k)^T \sum_{t=0}^{b-1} \Phi_{b-t}(k + t + 1) R_{k+t} V(k + t) Y_\nu(k + t)
\]

\[
= \pi(k)^T \bar{X}(k) - \frac{1}{n} \sum_{t=0}^{b-1} \eta_k(t) 1_n^T \nabla(k + t) - \sum_{t=0}^{b-1} r_k(t) (Y_\nu(k + t) - \frac{1}{n} \bar{1}(k + t) 1_n^T \nabla(k + t))
\]

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where $V(k)$ and $\bar{I}(k)$ are defined in Claim 2, $r_k(t) = \rho \pi(k)^T \Phi_{k-t}^A(k+t+1)I_{k+t}^oV(k+t)$ and $\eta_k(t) = r_k(t)\bar{I}(k+t)$. Note that

$$\eta_k(t) \leq \eta := \rho n, \quad \forall k, t \in \mathbb{N}$$

and the sum of any row of $\sum_{t=0}^{b-1} \Phi_{k-t}^A(k+t+1)I_{k+t}^o$ is not smaller than 1. Hence,

$$\sum_{t=0}^{b-1} \eta_k(t) \geq \rho n \pi(k)^T \sum_{t=0}^{b-1} \Phi_{k-t}^A(k+t+1)I_{k+t}^o \nu(k+t) \geq \rho \theta n,$$

$$\sum_{t=0}^{b-1} \eta_k(t) \leq b \eta \leq \rho n \beta < 1, \quad \forall k.$$  \hspace{1cm} (44)

By introducing an auxiliary term $\sum_{t=0}^{b-1} \eta_k(t)\nabla f(x_\pi(k))^T$, Eq. \textbf{(43)} becomes

$$x_\pi(k+b) = x_\pi(k) - \sum_{t=0}^{b-1} \eta_k(t)\nabla f(x_\pi(k))^T$$

$$- \sum_{t=0}^{b-1} r_k(t)(Y_\nu(k+t) - \frac{1}{n} \bar{I}(k+t)\nu(k+t))^TY_\nu(k+t))$$

$$+ \sum_{t=0}^{b-1} \eta_k(t)(\nabla f(x_\pi(k))^T - \frac{1}{n} 1_n^T \nu(k+t))$$

where we have used the relation $1_n^T \nu(k) = 1_n^T \bar{Y}(k) = \nu(k)^T Y_\nu(k)$ obtained from \textbf{(13)}.

We now bound the last term of \textbf{(45)}. Recall that $\nabla f(x) = 1_n^T \nabla f(1_nx^T)$, $\forall x \in \mathbb{R}^m$ and $1_n^T \nabla(k) = [1_n, 0]^T \nabla(k) = 1_n^T \nabla(X(k)), \forall k$. We have

$$\left\| \sum_{t=0}^{b-1} \eta_k(t)(\nabla f(x_\pi(k))^T - 1_n^T \nabla(k+t)) \right\|_F$$

$$\leq \sqrt{n} \beta \eta \sum_{t=0}^{b-1} \left\| 1_n x_\pi(k) - X(k+t) \right\|_F$$

$$\leq \sqrt{n} \beta \eta \left( \sum_{t=0}^{b-1} \left\| 1_n x_\pi(k) + 1_n \pi(k+t+1)^T \bar{X}(k+t) \right\|_F + \sum_{t=0}^{b-1} \left\| 1_n \pi(k+t+1)^T \bar{X}(k+t) - \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} \bar{X}(k+t) \right\|_F \right)$$

$$\leq \sqrt{n} \beta \eta \left( \sum_{t=0}^{b-1} \left\| 1_n \pi(k+t+1)^T \bar{X}(k+t) - \bar{X}(k+t) \right\|_F + \sum_{t=0}^{b-1} \left\| 1_n (\pi(k+t+1) - \pi(k+t+1))^T Q(k+t) \bar{X}(k+t) \right\|_F \right)$$

$$\leq (2n + \sqrt{n}) \beta \eta \sum_{t=0}^{b-1} \left\| \bar{X}(k+t) \right\|_Q$$

By combining \textbf{(45)} and \textbf{(46)}, we obtain

$$\left\| x_\pi(k+b) - x^* \right\|_F \leq \sigma \left\| x_\pi(k) - x^* \right\|_F + \sum_{t=1}^{b} \left\| r_k(t) \right\|_2 \left\| Y_\nu(k+t) \right\|_S + (2n + \sqrt{n}) \beta \eta \sum_{t=0}^{b-1} \left\| \bar{X}(k+t) \right\|_Q$$

$$\leq \sigma \left\| x_\pi(k) - x^* \right\|_F + \rho n (2n + \sqrt{n}) \beta \sum_{t=0}^{b-1} \left\| \bar{X}(k+t) \right\|_Q + \rho n \sum_{t=0}^{b-1} \left\| Y_\nu(k+t) \right\|_S$$

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where the first inequality follows from (44) and Lemma 6 with
\[
\sigma = 1 - \alpha \sum_{t=1}^{b} \eta_k(t) \leq 1 - \alpha \rho \theta n < 1.
\]
The last inequality of (47) follows from that \(\|r_k(t)\|_2 \leq \rho n, \forall k, t\).

Use the condition that \(\lambda^b > (1 + \sigma)/2\) and Lemma 5 we obtain from (47) that
\[
\|x_k - x^*\|_{\lambda F}^{\lambda_k} \leq \frac{\rho n \lambda^b}{\lambda^b - \sigma} \left( (2n + \sqrt{n}) \beta \|\bar{X}\|_Q^{\lambda_k} + \|y\|_S^{\lambda_k} \right) + c_3
\]
\[
\leq \frac{2b}{\alpha \theta} \left( (2n + \sqrt{n}) \beta \|\bar{X}\|_Q^{\lambda_k} + \|y\|_S^{\lambda_k} \right) + c_3
\]
where
\[
c_3 = \frac{\rho n \lambda^b}{\lambda^b - \sigma} \sum_{t=0}^{b-1} \lambda^{-\tau} \left( (2n + \sqrt{n}) \beta \|\bar{X}(t)\|_Q + \|y(t)\|_S \right)
\]

(48)

6) Proof of Claim 4: Since
\[
\bar{Y}(k) = V(k)Y(k) = V(k)S(k)Y(k) + \frac{1}{n} V(k)\bar{I}(k)v(k)^T Y(k)
\]
we have
\[
\|\bar{Y}(k)\|_F^r \leq \|V(k)\|_2 \left( \|Y(k)\|_S + \frac{1}{n} \|\bar{I}(k)v(k)^T Y(k)\|_F \right)
\]
\[
\leq n\|Y(k)\|_S + \|\bar{I}(k)1_n^T \nabla(k)\|_F^r.
\]

(49)

Note that
\[
\|\bar{I}(k)1_n^T \nabla(k)\|_F = \|\bar{I}(k) (1_n^T \nabla f(X(k)) - 1_n^T \nabla f(1_n(x^*)^T))\|_F
\]
\[
\leq \frac{\|\bar{I}(k)1_n^T\|_2}{n} \|\nabla f(X(k)) - \nabla f(1_n(x^*)^T)\|_F
\]
\[
\leq n\beta \sqrt{b} + 1 \|X(k) - 1_n(x^*)^T\|_F
\]
\[
= n\beta \sqrt{b} + 1 \left\| \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} \bar{X}(k) - \begin{bmatrix} 1_n \\ 0 \end{bmatrix} (x^*)^T \right\|_F
\]
\[
\leq 2n\beta \sqrt{b} \left\| \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} \bar{X}(k) - \begin{bmatrix} 1_n \\ 0 \end{bmatrix} (x^*)^T \right\|_F + 2n\beta \sqrt{b} \left\| \begin{bmatrix} 1_n \\ 0 \end{bmatrix} (x^*)^T \bar{X}(k) - \begin{bmatrix} 1_n \\ 0 \end{bmatrix} \right\|_F
\]
\[
\leq 2n\beta \sqrt{b} \left( \|\bar{X}(k)\|_Q + \sqrt{n} \|x(k) - x^*\|_F \right)
\]
\[
\leq 2n\beta \sqrt{b} \left( \|\bar{X}(k)\|_Q + \sqrt{n} \|x(k) - x^*\|_F \right)
\]

(50)

Substituting (50) into (49) yields
\[
\|\bar{Y}(k)\|_F^r \leq 2n(\sqrt{n} + 1)\beta \sqrt{b} \|\bar{X}(k)\|_Q + n\|Y(k)\|_S + 2n\sqrt{n} \beta \sqrt{b} \|x(k) - x^*\|_F
\]

The desired result is obtained by the definition of \(\|\bar{Y}\|_{\lambda,F}^{\lambda_k}\).
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