A Comment on Fisher Information and Quantum Algorithms

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Abstract

We show that Grover’s algorithm defines a geodesic in quantum Hilbert space with the Fubini-Study metric. From statistical point of view Grover’s algorithm is characterized by constant Fisher’s function. Quantum algorithms changing complexity class as Shor’s factorization does not preserve constant Fisher’s information. An adiabatic quantum factorization algorithm in non polynomial time is presented to exemplify the result.

1.- Recently a lot of attention has been paid to the problem of defining quantum algorithms [1] [2] [3]. Generically a quantum algorithm defines a discrete path in a quantum Hilbert space with the end point of the path corresponding to a quantum state that, after an appropriated measurement, will eventually provide, with high probability, the answer to a given problem. In this note we will work out some geometrical aspects of quantum algorithms. We will consider first the example of Grover’s algorithm [4]. In this case it can be shown that the path - in the quantum Hilbert space- associated with the algorithm, is a geodesic in Fubini-Study metric [4]. Geodesics in quantum Hilbert space are intimately connected with Fisher information function [5]. Using Fisher’s function we will define a formal Lagrangian on probability space such that their trajectories coincide with the quantum algorithm path. Next we will work out Shor’s factorization algorithm. We will show that, in this case as in any other involving change in complexity class (from NP to P problem), the Fisher’s function does not remain constant. However it is possible to design a factorization algorithm using Grover’s scheme. In this factorization algorithm Fisher’s function remains constant but “computing time” is non poly-
mial (the process is adiabatic with respect to Fisher’s “entropy”). This strongly indicates that changes in complexity class require no conservation of Fisher’s function - i.e. non unitarity quantum state projection.

2.- Grover’s algorithm provides a way to find, by means of a quantum computer [1], one particular item - in a set of N items randomly ordered - after approximately \( \pi \sqrt{\frac{N}{4}} \) iterations. This algorithm is known to be optimal [6]. In order to define the algorithm let us introduce the quantum state:

\[
|\psi\rangle = \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} |i\rangle
\]

The algorithm is determined by a set of states \( |\psi_j\rangle \):

\[
|\psi_j\rangle = k_j |0\rangle + \sum_{i=1}^{N-1} l_j |i\rangle
\]

with:

\[
k_{j+1} = \frac{N-2}{N} k_j + 2 \frac{N-1}{N} l_j
\]

\[
l_{j+1} = -\frac{2}{N} k_j + \frac{N-2}{N} l_j
\]

where the state \(|0\rangle\) represents the item we are looking for [7]. We will think of \( |\psi\rangle \) as the discrete path defining Grover’s algorithm. Let us approximate the discrete path \( |\psi\rangle \) by a path:

\[
|\psi\rangle(\phi) = \sum_{j=0}^{N-1} c_j(\phi) |j\rangle
\]

These probabilities define a path on ”probability space”. Transitions from \(|\psi\rangle(\phi)\) to \(|\psi\rangle(\phi + \delta\phi)\) are associated with the quantum computer operations performed by means of quantum gates. If these transformations are unitary we get:

\[
\langle \psi | \dot{\psi} \rangle = \frac{1}{4} \sum_{j=1}^{N} \frac{\dot{p}_j^2}{p_j} = 1
\]

provided we normalize the state \(|\psi\rangle\), and where \( \dot{p}_j = \frac{dp_j}{d\phi} \). Equation (3) is our first contact with Fisher’s information function. In fact defining [3]:

\[
\mathcal{F}(\phi) = \sum_{i=1}^{N} \frac{\dot{p}_i^2}{p_i}
\]

we notice that a path of states generated by unitary transformations are associated with a one parameter family of probability distributions of a constant Fisher function of value equal to four.

3.- Introducing quantum phases by \( c_j = \sqrt{p_j} e^{i\phi} \), the Fubini-Study metric on Hilbert space is given by:

\[
ds_{FS}^2 = \frac{1}{4} \sum_{j=1}^{N} \frac{dp_j^2}{p_j} + \left[ \sum_{j=1}^{N} p_j d\varphi_j^2 - \left( \sum_{j=1}^{N} p_j d\varphi_j \right)^2 \right].
\]

The induced metric on a path \( (p_j(\phi), \varphi_j(\phi)) \) is given by:

\[
ds_{ind.}^2 = \frac{1}{4} \left( \mathcal{F}(\phi) + 4 \sigma^2 \right) d\phi^2
\]

with \( \sigma = \frac{d\varphi}{d\phi} \), and \( \mathcal{F}(\phi) \) the Fisher function defined in (7). For a path with
\[ \phi = 0^3, \] as the one defined by Grover’s algorithm, the geodesic is given by minimizing

\[ S = \frac{1}{2} \int_{A}^{B} (F(\phi))^{1/2} \ d\phi \] (10)

with the constraint:

\[ \sum_{i=1}^{N} p_i = 1. \] (11)

Defining new variables \( p_i = x_i^2 \) the equations of motion for the "Lagrangian"

\[ \frac{1}{2} (F(\phi))^{1/2} \] are:

\[ \ddot{x}_i - \left( \frac{\dot{F}(\phi)}{F(\phi)} \right) \dot{x}_i + \frac{F(\phi)}{4} x_i = 0. \] (12)

For any quantum algorithm performed by successive unitary transformations, we know \( F(\phi) = cte. \) reducing (12) to the harmonic oscillator equation:

\[ \ddot{x}_i + \frac{F(\phi)}{4} x_i = 0 \] (13)

where the natural frequency is given by \( \omega^2 = \frac{F(\phi)}{4} = 1. \) It is now easy to check that Grover’s path (3) is in fact solution to (13). Thus, we conclude that Grover’s algorithm defines a geodesic path in quantum Hilbert space.

4. Obviously we can always transform a quantum algorithm of Grover’s type into a one parameter family of probability distributions \( p_i(\phi) \) with \( i \) running over the Hilbert space basis. What we have pointed out in this note, is that this family of probability distributions is completely determined by unitarity and the condition of minima for "Fisher’s information action" (10). Notice that in our definition of Fisher’s information function the "computing time" \( \phi \) is playing the statistical role of an statistical estimator. In particular with respect to this "computing time", \( F(\phi) \) is constant as a consequence of unitarity. Hence, in Grover’s algorithm, the "input information" at the starting point \( \phi = \phi_0 \) of the computation is given by:

\[ \sum_{i=0}^{N-1} \left( \frac{\partial p_i(\phi)}{\partial \phi} \right)^2 \] (14)

and it is this quantity the one that remains constant in the process. This is in contrast to the evolution of the standard Fisher’s information -contained in \( \{p_i(\phi)\} \)- concerning where is the item we are looking for. Obviously this second form of information increases in the process until reaching its maximum corresponding to the point where we find the desired solution. It is interesting to observe that the "input information" (14) is determined by quantum unitarity and can not be smaller or bigger. In summary, we conclude that from the information theory point of view quantum computations of Grover’s type appears as equivalent to classical statistical processes governed by minimum Fisher’s action.

5. Next let us consider Shor’s factorization algorithm [2]. As it is well known, classical algorithms for number factorization require exponential time \( \exp(c(\log N)^{1/3} (\log \log N)^{2/3}) \) where \( N \) is the integer we want to factorize an \( c \) is some constant. From complexity theory, number factorization is considered a NP-problem. Given a number \( N \) we can reduce

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3This in particular means that entanglement remains constant
4Use of Fisher’s function to define variational problems has been also considered in [8]
the problem of factorizing $N$ to find the period $r$ of the function $f(a) = y^a \mod N$ for a random number $y$ smaller than $N$ and coprime with $N$. In Shor’s quantum algorithm the period of $f(a)$ is obtained in two steps. First we define the quantum register state:

$$|\Psi\rangle = \frac{1}{\sqrt{q}} \sum_{a=0}^{q-1} |a\rangle \ | y^a \mod N\rangle$$

(15)

with $N^2 < q < 2N^2$. Then, we measure the value of $| y^a \mod N\rangle$. For each eigenvalue $l$ we get the state:

$$|\chi_l\rangle = \sqrt{\frac{r}{q}} \sum_{n=0}^{q/r-1} | l + nr\rangle.$$  

(16)

The next step is to proceed by a discrete Fourier transform to wash out the dependence on $l$. At the end of the process we get the desired period $r$ in polynomial time: $O((\log N)^2 (\log \log N)(\log \log \log N))$. In this algorithm there are series of unitary transformations we can model out in terms of standard quantum gates and a typically non unitary process consisting in the measurement projecting from the register state (15) to state (16). As it is clear from our previous discussion, Fisher information will be conserved during the unitary discrete Fourier transform but will generically change in the non unitary measurement process. This change is, as we will see in a moment, related to the change in complexity class achieved by Shor’s quantum algorithm. In order to visualize this more clearly let us design a way to find the period of $f(a)$ using Grover’s type of algorithm. We start with the quantum register state (17). Let us define the following transformation:

$$C[| a\rangle \ | f(a)\rangle] = 1 \text{ if } f(a) = f(1)$$

$$C[| a\rangle \ | f(a)\rangle] = 0 \text{ otherwise.}$$

(17)

Grover’s loop of transformations is then defined by rotating a $\pi$ angle the state $|a\rangle$ if $C[| a\rangle \ | y^a \mod N\rangle] = 1$ and doing nothing otherwise. Once we do that we do the inversion about the average as in Grover’s algorithm. At the end of $O(N)$ steps we will get:

$$|\eta\rangle = \frac{1}{\sqrt{\tau}} \sum_{j=0}^{\tau-1} | 1 + jr\rangle$$

(18)

where $\tau$ is the greatest integer less than $q/r$. So, just doing three measurement operations over the state $|\eta\rangle$ one finds [4], with high probability, the period $r$. As discussed in the first part of this note the whole Grover’s process is unitary preserving constant the Fisher information function. In terms of time it takes an exponential time of $O(2^{\log N})$. The difference between the fast projection from (15) to (16) performed in Shor’s algorithm and the adiabatic slow one using Grover’s loop defined above is that in the adiabatic one the complexity class is not changed and Fisher’s function remains constant, playing the classical role of entropy. The quantum adiabatic algorithm using Grover’s loop is certainly more efficient than the classical one and very likely more robust with respect to quantum decoherence problems than the faster Shor’s algorithm. Technologically is more feasible using for instance the recent implementation of Grover’s algorithm [9], [10].

6.- To finish we would like to suggest the following general conjecture:

Changes in complexity class should involve no conservation of Fisher’s information function and reciprocally constant Fisher’s
information will not change the complexity class.

Our exercise also shows that the typical non unitary quantum projection from (15) to (16) used by Shor’s algorithm can be, for the practical purposes of quantum computation, done using only unitary transformations. The bill you have to paid for adiabaticity is longer time and not change of complexity class.

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