ASYMPTOTIC AND QUENCHING BEHAVIORS OF SEMILINEAR PARABOLIC SYSTEMS WITH SINGULAR NONLINEARITIES

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Abstract. In this paper, we consider a family of parabolic systems with singular nonlinearities. We study the classification of global existence and quenching of solutions according to parameters and initial data. Furthermore, the rate of the convergence of the global solutions to the minimal steady state is given. Due to the lack of variational characterization of the first eigenvalue to the linearized elliptic problem associated with our parabolic system, some new ideas and techniques are introduced.

1. Introduction. We consider the following coupled singular parabolic system of the form

\[
\begin{aligned}
&u_t - \Delta u = \lambda \alpha(x)f(v), \quad &\text{in} \quad \Omega \times (0, T),
\vspace{1mm}
&v_t - \Delta v = \mu \beta(x)g(u), \quad &\text{in} \quad \Omega \times (0, T),
\vspace{1mm}
&u = v = 0, \quad &\text{on} \quad \partial\Omega \times (0, T),
\vspace{1mm}
&u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad &\text{in} \quad \Omega,
\end{aligned}
\]

where \( \Omega \subset \mathbb{R}^N \) is a smooth bounded domain, \( \lambda \) and \( \mu \) are positive parameters, \( \alpha(x) \) and \( \beta(x) \) are nonnegative nontrivial Hölder continuous functions in \( \bar{\Omega} \), the initial data satisfy

\[
u_0(x), v_0(x) \in C^2(\Omega) \cap C^1(\bar{\Omega}), 0 \leq u_0, v_0 < 1, \ u_0 = v_0 = 0 \text{ on } \partial\Omega,
\]

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and \( f, g \) satisfy
\[
\lim_{v \to 1^-} f(v) = \lim_{u \to 1^-} g(u) = +\infty.
\] (H2)

A typical example satisfying (H2) is \( f(v) = (1 - v)^{-p}, g(u) = (1 - u)^{-q}, p, q > 0 \), and more examples can be given as \( 1 - \ln(1 - \cdot), e^{\frac{1}{1 - \cdot}}, \) etc.

**Remark 1.1.** In (H2), we fix the blow-up level at \( u = 1, v = 1 \) for simplicity. It is easy to see that by scaling, our approach works for \( f, g \) blowing up at any positive values \( a \) and \( b \), respectively.

Problem (P) is a semilinear reaction diffusion system exhibiting a coupling on the unknowns \( u(x, t) \) and \( v(x, t) \). These can be thought of as the temperatures of two substances which constitute a combustible mixture, where heat release is described by the right-hand side of (P). The solution \( (u, v) \) of (P) is called quenching at time \( t = T < +\infty \) if
\[
\limsup_{t \to T^-} \sup_{\Omega} \{\max_{\Omega} u(\cdot, t), \max_{\Omega} v(\cdot, t)\} = 1. \tag{1.1}
\]

Since the study of quenching phenomena was begun in 1975 by Kawarada [7], a lot of works have been contributed to this subject. In these previous works, for system (P), some sufficient conditions for global existence, quenching and quenching time estimates of solutions, as well as the non-simultaneous quenching criteria are studied in [1, 10, 14, 20] and the references cited therein. However, as far as we know, the classification of global existence and quenching of solutions was not fully described yet, and the rate of the convergence of the global solutions to a steady state is not considered before.

Our work is motivated firstly by [19], where they considered the scalar equation
\[
\begin{dcases}
 u_t - \Delta u = \lambda \alpha(x) f(u), & \text{in } \Omega \times (0, T), \\
 u = 0, & \text{on } \partial \Omega \times (0, T), \\
 u(x, 0) = u_0(x) \in [0, 1) & \text{in } \Omega,
\end{dcases} \tag{1.2}
\]
as well as the associated stationary equation
\[- \Delta u = \lambda \alpha(x) f(u) \text{ in } \Omega, \ u = 0 \text{ on } \partial \Omega \tag{1.3}\]
with \( f \) satisfying (H2). More precisely, it is shown in [19] that for any given \( \alpha \geq 0 \) and \( f \) satisfying (H2), there exists a critical value \( \lambda^* > 0 \) such that if \( \lambda > \lambda^* \), then no solution of (1.3) exists, and the solution to (1.2) will reach the value 1 in finite time, i.e., the so called quenching or touchdown phenomenon occurs; while for \( \lambda \in (0, \lambda^*) \), problem (1.3) is solvable and the solution of (1.2) is global and convergent to the minimal solution of (1.3) for some \( u_0 \). Moreover, the convergence rate in some situation is obtained in [11], by using the variational characterization of the first eigenvalue of the linearized problem associated with (1.3).

In fact, besides [11, 19], for the particular case \( f(u) = (1 - u)^{-p}, p > 0 \), especially for \( p = 2 \), as the mathematical model of micro-electromechanical systems (MEMS), (1.2) has been extensively studied by many authors in recent years (cf. [3, 4, 6, 8, 9, 16–18] and references therein). MEMS device consists of an elastic membrane suspended over a rigid ground plate. For MEMS, \( u \) denotes the normalized distance between the membrane and the ground plate, \( \alpha(x) \) represents the permittivity profile. When a voltage \( \lambda \) is applied, the membrane deflects toward the ground plate and a snap-through may occur when it exceeds a certain critical value.
\( \lambda^* \) (pull-in voltage). This creates a so-called “pull-in instability”, which greatly affects the design of many devices (cf. [3] for more details).

Coming back to system \((P)\), before describing precisely the outcome of our paper, we state that similar to the considerations in single MEMS equation (cf. [3, 19]), there exists a close relationship between system \((P)\) and the associated stationary problem

\[
\begin{aligned}
-\Delta w &= \lambda \alpha(x)f(z), \quad \text{in } \Omega, \\
-\Delta z &= \mu \beta(x)g(w), \quad \text{in } \Omega, \\
w &= z &= 0, \quad \text{on } \partial \Omega.
\end{aligned}
\]

In fact, our work is also motivated by [2], where they proved that for \((E)\) with \(f(\cdot) = g(\cdot) = (1 - \cdot)^{-2}\), there exists a critical curve \(\Gamma\) splitting the positive quadrant of the \((\lambda, \mu)\)-plane into two disjoint sets \(\mathcal{O}_1\) and \(\mathcal{O}_2\) such that the elliptic problem has a smooth minimal stable solution \((w_{\lambda, \mu}, z_{\lambda, \mu})\) for \((\lambda, \mu) \in \mathcal{O}_1\), while for \((\lambda, \mu) \in \mathcal{O}_2\) there is no solution of any kind. In our paper, these results are extended to general elliptic problem \((E)\) in the following Proposition A, which can be illustrated by Figure 1.

**Proposition A.** There exist \(0 < \lambda^*, \mu^* < +\infty\), and a non-increasing continuous curve \(\mu = \Gamma(\lambda)\) connecting \((0, \mu^*)\) and \((\lambda^*, 0)\) such that the positive quadrant \(\mathbb{R}^+ \times \mathbb{R}^+\) of the \((\lambda, \mu)\)-plane is separated into two connected components \(\mathcal{O}_1\) and \(\mathcal{O}_2\). For \((\lambda, \mu) \in \mathcal{O}_1\), problem \((E)\) has a positive classical minimal solution \((w_{\lambda, \mu}, z_{\lambda, \mu})\). Otherwise, for \((\lambda, \mu) \in \mathcal{O}_2\), \((E)\) admits no weak solution.

![Figure 1. The critical curve \(\Gamma\) in \((\lambda, \mu)\)-plane](image)

Since Proposition A can be established similarly to [2], we sketch the proof in Appendix A.

We are now in a position to state our main results. Note that the local (in time) existence and uniqueness of nonnegative classical solutions of \((P)\) under conditions \((H_1)\) and \((H_2)\) are rather standard. We will concentrate here on the global existence and quenching of such solutions. More precisely, in Theorem 1.1 we give the classification of global existence and quenching of solutions according to parameters and initial data. Then the global solution’s asymptotic behavior, especially the convergence rate are further estimated in Theorem 1.2. Throughout our paper, we use \(\|\cdot\|_p\) to denote the standard norm of \(L^p(\Omega)\), and \((u, v) \leq (w, z)\) to denote \(u \leq w\) and \(v \leq z\).
Theorem 1.1 (Global existence vs quenching). Suppose (H1) and (H2) hold. Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be the connected components defined in Proposition A, and $(w_{\lambda, \mu}, z_{\lambda, \mu})$ be the minimal solution of (E).

(a) If $(\lambda, \mu) \in \mathcal{O}_1$, then there hold:

(a1) For $(u_0, v_0) \leq (w_{\lambda, \mu}, z_{\lambda, \mu})$, the unique solution $(u(x, t), v(x, t))$ to (P) exists globally and converges uniformly to $(w_{\lambda, \mu}, z_{\lambda, \mu})$ as $t \to +\infty$. If $(u_0(x), v_0(x))$ is further a subsolution of (E), then $(u(x, t), v(x, t))$ is increasing with respect to $t$.

(a2) Suppose that (E) has solutions more than one. Let $(w_1, z_1)$ be any solution different from $(w_{\lambda, \mu}, z_{\lambda, \mu})$. Then

(a21) For $(u_0, v_0) \leq (w_1, z_1)$, the unique solution $(u(x, t), v(x, t))$ to (P) exists globally and converges uniformly to $(w_{\lambda, \mu}, z_{\lambda, \mu})$ as $t \to +\infty$. If there exists $s \in (0, 1)$ such that $(u_0, v_0) = s(w_{\lambda, \mu}, z_{\lambda, \mu}) + (1-s)(w_1, z_1)$, then $(u(x, t), v(x, t))$ is decreasing with respect to $t$.

(b) If $(\lambda, \mu) \in \mathcal{O}_2$, then for any $(u_0, v_0)$ satisfying (H1), the solution $(u, v)$ to (P) will quench in finite or infinite time.

(c) For any given $\lambda > 0$ and $\mu > 0$, once $\int_{\Omega} u_0 \phi dx > (\lambda_1 \int_{\Omega} \phi/\alpha dx)/(\lambda f(0))$ or $\int_{\Omega} v_0 \phi dx > (\lambda_1 \int_{\Omega} \phi/\beta dx)/(\mu g(0))$, the solution $(u, v)$ of (P) will quench in finite time. Here $\lambda_1 > 0$ is the first eigenvalue of $-\Delta$ in $H^1_0(\Omega)$ and $\phi$ is the corresponding eigenfunction satisfying $\int_{\Omega} \phi dx = 1$.

Remark 1.2. If $1 \leq n \leq 3$, then all the convergences in Theorem 1.1 further hold in $C^1$ norm (see Proposition 4.3).

Remark 1.3. As far as we know, the multiplicity of solution to (E) is still an open problem.

Remark 1.4. When the minimal solution $(w_{\lambda, \mu}, z_{\lambda, \mu})$ is the only solution to (E), except for the case in Theorem 1.1 (c), the behavior of solution $(u(x, t), v(x, t))$ to (P) with initial data above $(w_{\lambda, \mu}, z_{\lambda, \mu})$ is unknown.

Corollary 1.1. Under the conditions of Theorem 1.1, we have

$$\int_{\Omega} w_{\lambda, \mu} \phi dx \leq \left( \lambda_1 \int_{\Omega} \phi/\alpha dx \right)/(\lambda f(0))$$

and

$$\int_{\Omega} z_{\lambda, \mu} \phi dx \leq \left( \lambda_1 \int_{\Omega} \phi/\beta dx \right)/(\mu g(0));$$

if (E) has another solution $(w_1, z_1)$, we further have

$$\int_{\Omega} w_1 \phi dx \leq \left( \lambda_1 \int_{\Omega} \phi/\alpha dx \right)/(\lambda f(0))$$

and

$$\int_{\Omega} z_1 \phi dx \leq \left( \lambda_1 \int_{\Omega} \phi/\beta dx \right)/(\mu g(0)).$$

Next, before giving the rate of the convergence of global solutions to the minimal solution $(w_{\lambda, \mu}, z_{\lambda, \mu})$ of (E) as time goes to infinity, we stress a related result about the linear stability of $(w_{\lambda, \mu}, z_{\lambda, \mu})$. In fact, for the linearized elliptic problem around
\( (w_{\lambda,\mu}, z_{\lambda,\mu}) \)

\[
\begin{aligned}
-\Delta \varphi - \lambda \alpha(x)f'(z_{\lambda,\mu})\psi &= \nu \varphi, & \text{in } \Omega, \\
-\Delta \psi - \mu \beta(x)g'(w_{\lambda,\mu})\varphi &= \nu \psi, & \text{in } \Omega, \\
\varphi = \psi = 0, & \text{on } \partial \Omega,
\end{aligned}
\tag{1.4}
\]

it has been proved in [12, Theorem 1.5] and [2, p10] that

\( (1.4) \) has a first eigenvalue \( \nu_1 > 0 \) with strictly positive smooth eigenfunction

\[
(\varphi_1, \psi_1).
\tag{1.5}
\]

Here \( \nu_1 > 0 \) means that the minimal solution \( (w_{\lambda,\mu}, z_{\lambda,\mu}) \) is stable. The result (1.5) is useful to obtain the convergence rate in the following Theorem 1.2.

**Theorem 1.2** (Convergence rate). Suppose (H1) and (H2) hold. For cases (a1) and (a21) in Theorem 1.1, if \( 1 \leq n \leq 3 \), the rate of the convergence of the global solutions to the minimal steady state can be estimated by

\[
\| u(x, t) - w_{\lambda,\mu}(x) \|_2 + \| v(x, t) - z_{\lambda,\mu}(x) \|_2 \\
\leq C_0 \exp \left( -\min \left\{ \lambda_1, \frac{\nu_1}{4} \right\} t \right), \quad t > T_0
\tag{1.6}
\]

for some \( T_0 > 0 \), where \( C_0 \) is a constant depending at most on \( u_0, v_0, w_{\lambda,\mu}, z_{\lambda,\mu}, w_1, z_1, \varphi_1, \psi_1 \). Here \( \lambda_1 > 0 \) is the first eigenvalue of \( -\Delta \) on \( H_0^1(\Omega) \), \( \nu_1 > 0 \) is the first eigenvalue of linearized elliptic system (1.4), and \( (\varphi_1, \psi_1) \) is the corresponding strictly positive eigenfunction defined in (1.5).

**Remark 1.5.** Assume further that \( (u_0, v_0) \leq (w_{\lambda,\mu}, z_{\lambda,\mu}) \) and \( (u_0, v_0) \) is a subsolution of (E), or that \( (w_{\lambda,\mu}, z_{\lambda,\mu}) \leq (u_0, v_0) \neq (w_1, z_1) \) (if \( (w_1, z_1) \) exists) and \( (u_0, v_0) \) is a supersolution of (E). Then the exponential convergence further holds in \( H^1 \) norm. More precisely, there holds

\[
\| u(x, t) - w_{\lambda,\mu}(x) \|_{H^1} + \| v(x, t) - z_{\lambda,\mu}(x) \|_{H^1} \\
\leq C \exp \left( -\min \left\{ \frac{\lambda_1}{4}, \frac{\nu_1}{4}, \frac{1}{2} \right\} t \right), \quad t > T_0
\tag{1.7}
\]

for some \( T_0 > 0 \). Here, \( C \) is a constant depending at most on \( u_0, v_0, w_{\lambda,\mu}, z_{\lambda,\mu}, w_1, z_1, \varphi_1, \psi_1 \); \( \lambda_1 \) and \( \nu_1 \) are defined as in Theorem 1.2.

We remark that as one of the main contributions of our paper, obtaining the convergence rate (1.6) needs new ideas and techniques. In fact, for single parabolic MEMS equation (1.2), in the process of obtaining the convergence rate of global solution (see [11]), it plays an important role that the first eigenvalue of the associated linearized elliptic equation having a variational characterization. However, as pointed out in [2], no such analogous formulation is available for our system (E). Hence, we have to find a new way to overcome this difficulty. In our approach, in addition to taking advantage of the linear stability of the minimal steady state, another key idea relies on the \( C^1 \)-convergence of the global solutions to the minimal steady state. Note that the \( C^1 \)-convergence (see Proposition 4.3) can not be obtained directly by the same way as obtaining the uniform convergence in Theorem 1.1. In our paper, we get the \( C^1 \)-convergence by higher-order estimates (see Proposition 4.2) for global solutions of system (P).

This paper is organized as follows. In Section 2, we provide some preliminary results. In Section 3, we prove Theorem 1.1. In Section 4, we prove Remark 1.2, Theorem 1.2 and Remark 1.5. At last, we prove Proposition A in Appendix A.
2. Preliminary results. In this section, we provide some useful results. In Proposition 2.1, we study the structure of the stationary solution set. In Proposition 2.2, we establish the monotonicity respect to \( t \) of solutions to (P).

Proposition 2.1. There cannot be a triple \((w_i, z_i)(i = 1, 2, 3)\) of solutions to \((E)\) with \((w_1, z_1) \ll (w_2, z_2) \ll (w_3, z_3)\). Here \( w_1 \ll w_2 \) means that \( \gamma \rho \leq w_2 - w_1 (x \in \Omega) \) for some positive number \( \gamma, \rho(x) \) being the distance from \( x \) to \( \partial \Omega \).

Proof. Suppose by contradiction that there exists a triple \((w_i, z_i)(i = 1, 2, 3)\) of solutions to \((E)\), i.e.,

\[
\begin{align*}
&-\Delta w_i = \lambda \alpha(x)f(z_i), \quad \text{in } \Omega, \\
&-\Delta z_i = \mu \beta(x)g(w_i), \quad \text{in } \Omega, \\
&w_i = z_i = 0, \quad \text{on } \partial \Omega.
\end{align*}
\]

(2.1)

It then follows that

\[
\begin{align*}
&-\Delta(w_3 - w_2) = \lambda \alpha(x)(f(z_3) - f(z_2)), \quad \text{in } \Omega, \\
&-\Delta(z_3 - z_2) = \mu \beta(x)(g(w_3) - g(w_2)), \quad \text{in } \Omega, \\
&-\Delta(w_2 - w_1) = \lambda \alpha(x)(f(z_2) - f(z_1)), \quad \text{in } \Omega, \\
&-\Delta(z_2 - z_1) = \mu \beta(x)(g(w_2) - g(w_1)), \quad \text{in } \Omega.
\end{align*}
\]

(2.2)

If we multiply the first equation in (2.2) by \( z_2 - z_1 \), the second equation in (2.2) by \( w_2 - w_1 \) and integrating over \( \Omega \), then by (2.2) there exist \( \xi_w \in (w_2, w_3), \xi_z \in (z_2, z_3), \eta_w \in (w_1, w_2), \eta_z \in (z_1, z_2) \) such that

\[
\int_{\Omega} \mu \beta(x)(w_3 - w_2)(w_2 - w_1)(g'(\eta_w) - g'(\xi_w))dx \\
+ \int_{\Omega} \lambda \alpha(x)(z_3 - z_2)(z_2 - z_1)(f'(\eta_z) - f'(\xi_z))dx = 0.
\]

(2.3)

This is impossible, since by the strict convexity of \( f, g \) there hold \( g'(\eta_w) - g'(\xi_w) < 0 \) and \( f'(\eta_z) - f'(\xi_z) < 0 \) by noting that \( \eta_w < \xi_w, \eta_z < \xi_z \) in \( \Omega \). The proof is completed.

\[\square\]

Remark 2.1. Suppose that \((w_i, z_i)(i = 1, 2)\) are solutions to \((E)\). If \((w_1, z_1) \neq (w_2, z_2)\), then we have \((w_1, z_1) \ll (w_2, z_2)\) by Hopf Lemma.

To verify Proposition 2.2 below, we need to borrow a comparison principle for the parabolic system as follows, which can be derived from [15, Theorem 13, Chapter 3].

Lemma 2.1 (Comparison Principle). Suppose that \( u = (u_1, u_2, \ldots, u_k) \) satisfies the following uniformly parabolic system of inequalities

\[
\begin{align*}
&\frac{\partial u_1}{\partial t} - \Delta u_1 - \sum_{i=1}^{k} h_{1i} u_i \leq 0, \\
&\frac{\partial u_2}{\partial t} - \Delta u_2 - \sum_{i=1}^{k} h_{2i} u_i \leq 0, \\
&\quad \quad \vdots \\
&\frac{\partial u_k}{\partial t} - \Delta u_k - \sum_{i=1}^{k} h_{ki} u_i \leq 0.
\end{align*}
\]

(2.4)

in \( \Omega \times (0, T) \). If \( u \leq 0 \) at \( t = 0 \) and on \( \partial \Omega \times (0, T) \), \( h_{ij} \) is bounded and satisfies

\[
h_{ji} \geq 0 \text{ for } i \neq j, i, j = 1, 2, \ldots, k,
\]

(2.5)
then \( u \leq 0 \) in \( \Omega \times (0, T) \). Moreover, if there exists \( i_0 \) such that \( u_{i_0} = 0 \) at an interior point \((x_0, t_0)\), then \( u_t = 0 \) for \( t > t_0 \). Here, we use the notation \( u \leq 0 \) to mean that every component \( u_i, i = 1, 2, \ldots, k \) is nonpositive.

**Proposition 2.2.** Let \( (u, v) \) satisfy

\[
\begin{align*}
    u_t - \Delta u &= f(x, v) > 0, & \text{in} \; \Omega \times (0, T), \\
    v_t - \Delta v &= g(x, u) > 0, & \text{in} \; \Omega \times (0, T), \\
    u &= v = 0, & \text{on} \; \partial \Omega \times (0, T), \\
    u(x, 0) &= u_0, & v(x, 0) = v_0, \quad \text{for} \; x \in \Omega.
\end{align*}
\]

(2.6)

Suppose \( \partial f / \partial v \) and \( \partial g / \partial u \) are positive and locally bounded. Then if \( (u_0(x), v_0(x)) \) is a strictly subsolution (supersolution) of the corresponding stationary system to (2.6), there hold \( u_t > 0, v_t > 0 \) \((u_t < 0, v_t < 0)\) in \( \Omega \).

**Proof.** We omit the proof here, since it can be easily deduced by Lemma 2.1. \( \square \)

3. **Global existence vs quenching.** This section is devoted to proving Theorem 1.1. Some ideas are borrowed from [5].

3.1. **Proof of Theorem 1.1 (a).**

3.1.1. **Proof of Theorem 1.1 (a).** For the special case \((u_0, v_0) = (0, 0)\), the global existence can be deduced directly by sub-super solution method in [13, Chapter 8]. Obviously, the unique global solution is bounded by the unique minimal solution \((w_{\lambda, \mu}, z_{\lambda, \mu})\) of (E). Note that \((0, 0)\) is a subsolution to (P). By Proposition 2.2 and assumption (H2), we conclude that \( u_t > 0, \; v_t > 0 \). These imply that \( u(x, t) \) and \( v(x, t) \) converge pointwise as \( t \to +\infty \) to some functions \( \bar{u}(x) \) and \( \bar{v}(x) \), respectively, which satisfy \( \bar{u} \leq w_{\lambda, \mu} < 1, \; \bar{v} \leq z_{\lambda, \mu} < 1 \) in \( \Omega \).

Let \( \varphi(x) \in C^2(\Omega) \) and \( \varphi|_{\partial \Omega} = 0 \). Multiplying the equations in (P) by \( \varphi \) and integrating over \( \Omega \), we arrive at

\[
\begin{align*}
    \frac{d}{dt} \int_{\Omega} u \varphi dx - \int_{\Omega} u \Delta \varphi dx &= \int_{\Omega} \lambda \alpha(x) \varphi f(v) dx, \\
    \frac{d}{dt} \int_{\Omega} v \varphi dx - \int_{\Omega} v \Delta \varphi dx &= \int_{\Omega} \mu \beta(x) \varphi g(u) dx.
\end{align*}
\]

(3.1)

Operating on both sides with \( \frac{1}{T} \int_{0}^{T} \), it follows that

\[
\begin{align*}
    \int_{\Omega} u(x, T) - u_{0}(x) \varphi dx + \int_{\Omega} (\Delta \varphi - \varphi) \frac{1}{T} \int_{0}^{T} u(x, t) dt dx &= \int_{\Omega} \lambda \alpha(x) \varphi \frac{1}{T} \int_{0}^{T} f(v) dt dx, \\
    \int_{\Omega} v(x, T) - v_{0}(x) \varphi dx + \int_{\Omega} (\Delta \varphi - \varphi) \frac{1}{T} \int_{0}^{T} v(x, t) dt dx &= \int_{\Omega} \mu \beta(x) \varphi \frac{1}{T} \int_{0}^{T} g(u) dt dx.
\end{align*}
\]

(3.2)

Note that for every \( x \in \Omega \), there hold

\[
\begin{align*}
    \lim_{T \to +\infty} \frac{u(x, T) - u_0(x)}{T} &= 0, & \lim_{T \to +\infty} \frac{v(x, T) - v_0(x)}{T} &= 0, \\
    \lim_{T \to +\infty} \frac{1}{T} \int_{0}^{T} u(x, t) dt = \bar{u}(x), & \lim_{T \to +\infty} \frac{1}{T} \int_{0}^{T} v(x, t) dt = \bar{v}(x), \\
    \lim_{T \to +\infty} \frac{1}{T} \int_{0}^{T} g(u) dt = g(\bar{u}), & \lim_{T \to +\infty} \frac{1}{T} \int_{0}^{T} f(v) dt = f(\bar{v}).
\end{align*}
\]

(3.3)
Therefore, by the Lebesgue dominated convergence theorem we get that as $T \to +\infty$

\begin{align}
\int_{\Omega} \bar{u}(-\Delta \varphi) dx &= \int_{\Omega} \lambda \alpha(x) \varphi f(\bar{v}) dx, \\
\int_{\Omega} \bar{v}(-\Delta \varphi) dx &= \int_{\Omega} \mu \beta(x) \varphi g(\bar{u}) dx,
\end{align}

(3.4)

which implies $(\bar{u}, \bar{v})$ is a weak solution of (E). By the $L^p$ estimates, the Sobolev embedding theorem, and the classical Schauder estimates, we obtain that $(\bar{u}, \bar{v})$ is a classical solution of (E), and hence $(\bar{u}, \bar{v}) = (w_{\lambda,\mu}, z_{\lambda,\mu})$. Since $u_t > 0$, $v_t > 0$ and $u, v, w_{\lambda,\mu}, z_{\lambda,\mu}$ are continuous in $x$, by Dini Theorem the convergence of the unique global solution $(u(x,t), v(x,t))$ to $(w_{\lambda,\mu}(x), z_{\lambda,\mu}(x))$ is further uniform in $x$, i.e.,

$$
\lim_{t \to \infty} (\|u(x,t) - w_{\lambda,\mu}(x)\|_\infty + \|v(x,t) - z_{\lambda,\mu}(x)\|_\infty) = 0.
$$

(3.5)

By the comparison principle in Lemma 2.1, the convergence (3.5) also holds for general case $(u_0, v_0) \leq (w_{\lambda,\mu}, z_{\lambda,\mu})$. Therefore, we complete the proof of Theorem 1.1 (a1). Furthermore, if $(u_0(x), v_0(x))$ is further a subsolution of (E), by Proposition 2.2 $(u(x,t), v(x,t))$ is increasing with respect to $t$.

3.1.2. Proof of Theorem 1.1 (a2). We first prove Theorem 1.1 (a21). Without loss of generality, we assume that there exists $s \in (0,1)$ such that

$$
(u_0, v_0) \leq (sw_1, s_1 + (1-s)z_{\lambda,\mu}).
$$

(3.6)

Otherwise by the Hopf Lemma there exists $t_0 > 0$ such that $(u(x,t_0), v(x,t_0))$ satisfies the assumption.

For case $(u_0, v_0) = (sw_1, s_1 + (1-s)z_{\lambda,\mu})$, by direct calculations, we can get that $(sw_1 + (1-s)w_{\lambda,\mu}, s_1 + (1-s)z_{\lambda,\mu})$ is a supersolution of (E). By Proposition 2.2 we conclude that $u_t < 0$, $v_t < 0$ in $\Omega$. Similar to the proof in Theorem 1.1 (a1), this implies that $(u, v)$ converges uniformly to $(w_{\lambda,\mu}, z_{\lambda,\mu})$ as $t \to +\infty$ and the convergence is monotone decreasing. At last, the convergence holding for general initial data satisfying $(0,0) \leq (u_0, v_0) \leq (w_1, z_1)$ follows directly by the the comparison principle in Lemma 2.1. The proof of Theorem 1.1 (a21) is completed.

Next, we verify Theorem 1.1 (a22). Suppose by contradiction that for some initial value $(u_0, v_0) \geq (w_1, z_1)$, the solution $(u, v)$ to (P) is bounded above by $1 - \delta$ for all $t \geq 0$ and some $0 < \delta < 1$. Similar to the proof of Theorem 1.1 (a21), without loss of generality, we assume that there exists $\epsilon > 0$ such that

$$
(u_0, v_0) := ((1 + \epsilon)w_1 - \epsilon w_{\lambda,\mu}, (1 + \epsilon)z_1 - \epsilon z_{\lambda,\mu}) \leq (u_0, v_0).
$$

Secondly, by the convexity of $f$ and $g$ we have

$$
\Delta u_0 + \lambda \alpha(x) f(u_0) \geq 0, \quad \Delta v_0 + \mu \beta(x) g(v_0) \geq 0,
$$

(3.7)

which imply by Proposition 2.2 that $u_0 > 0$, $v_0 > 0$ in $\Omega$. Here $(u, v)$ is the solution to (P) with $(u(x,0), v(x,0)) = (u_0, v_0)$. Note that $(u, v)$ is also bounded by $1 - \delta$ for all $t > 0$ by the comparison principle in Lemma 2.1. We have that $(u, v)$ converges increasingly to some $(\bar{w}, \bar{z})$. Note that it can be checked similarly to Theorem 1.1 (a1) that $(\bar{w}, \bar{z})$ is a classical solution to (E). We have $(w_{\lambda,\mu}, z_{\lambda,\mu}) \leq (w_1, z_1) \leq (\bar{w}, \bar{z})$, which contradicts Proposition 2.1. The proof of Theorem 1.1 (a22) is completed.
3.2. Proof of Theorem 1.1 (b). We will only prove the case \((u_0,v_0) = (0,0)\), then the holding for general nonnegative initial data follows directly by the comparison principle in Lemma 2.1.

Let \((\lambda,\mu) \in \mathcal{O}_2\). Suppose on the contrary that the local solution \((u,v)\) exists globally, i.e. \(0 \leq u < 1, 0 \leq v < 1\) for all \(t \geq 0\). Take \(\delta > 1, a = \lambda/\delta, b = \mu/\delta\). Since \(U = u/\delta < u, V = v/\delta < v\), it then indicates that \(U \leq 1/\delta < 1, V \leq 1/\delta < 1\), and by the monotone increasing of \(f, g\) there holds

\[
\begin{aligned}
U_t - \Delta U &= \lambda \alpha(x)f(v)/\delta \geq \alpha(x)f(V), & \text{in } \Omega \times (0,T), \\
V_t - \Delta V &= \mu \beta(x)g(u)/\delta \geq b \beta(x)g(U), & \text{in } \Omega \times (0,T), \\
U &= V = 0, & \text{on } \partial\Omega \times (0,T), \\
U(x,0) = 0, & \text{for } x \in \Omega.
\end{aligned}
\] (3.8)

Hence \((U,V)\) is a supersolution of

\[
\begin{aligned}
U_t - \Delta U &= a \alpha(x)f(V), & \text{in } \Omega \times (0,T), \\
V_t - \Delta V &= b \beta(x)g(U), & \text{in } \Omega \times (0,T), \\
U &= V = 0, & \text{on } \partial\Omega \times (0,T), \\
U(x,0) = 0, & \text{for } x \in \Omega.
\end{aligned}
\] (3.9)

Therefore (3.9) has a global classical solution \((U(x,t),V(x,t))\), since \(0 \leq U \leq 1/\delta < 1, 0 \leq V \leq 1/\delta < 1\). Note that there further hold

\[
\lim_{t \to +\infty} (\|U\|_\Omega^2 + \|V\|_\Omega^2) = 0 \quad \text{and} \quad \sup_{t>1} (\|U\|_{H^2(\Omega)} + \|V\|_{H^2(\Omega)}) < +\infty,
\] (3.10)

which can be proved similarly to Proposition 4.2. By Sobolev embedding theorem, one can have that there exists a subsequence \(\{t_j\}_{j=1}^\infty\), \(t_j \to +\infty\) such that \((U(\cdot,t_j),V(\cdot,t_j))\) converges strongly in \(H^1_0(\Omega)\) to \((U_\infty,V_\infty)\). Now take \(\phi \in H^1_0(\Omega)\).

Multiplying (3.9) by \(\phi\) and integrating by parts with respect to \(x\) yield,

\[
\begin{aligned}
\int_\Omega \phi U_t(\cdot,t_j) dx + \int_\Omega \nabla U(\cdot,t_j) \nabla \phi dx &= \int_\Omega a \alpha(x) \phi f(V(\cdot,t_j)) dx, \\
\int_\Omega \phi V_t(\cdot,t_j) dx + \int_\Omega \nabla V(\cdot,t_j) \nabla \phi dx &= \int_\Omega b \beta(x) \phi g(U(\cdot,t_j)) dx.
\end{aligned}
\] (3.11)

Let \(t_j \to +\infty\) and we will get that \((U_\infty,V_\infty)\) is a weak solution of

\[
\begin{aligned}
-\Delta w &= a \alpha(x)f(z), & \text{in } \Omega, \\
-\Delta z &= b \beta(x)g(w), & \text{in } \Omega, \\
u &= v = 0, & \text{on } \partial\Omega.
\end{aligned}
\] (3.12)

Chose \(\delta\) close to 1 such that \((a,b) \in \mathcal{O}_2\). Then we get a contradiction with Proposition A. The proof of this proposition is therefore completed.

3.3. Proof of Theorem 1.1 (c). Suppose that \((u,v)\) is the solution of (P) for \(0 \leq t < T\). Then there holds \(0 \leq u, v < 1\) for \(0 \leq t < T\). Define \(F(t) = \int_\Omega u \phi dx\). Multiplying the first equation in (P) by \(\phi\) and integrating over \(\Omega\) yield

\[
F'(t) = \int_\Omega u_t \phi dx = \int_\Omega (\Delta u + \lambda \alpha(x)f(v)) \phi dx
= - \lambda_1 \int_\Omega u \phi dx + \lambda \int_\Omega \alpha(x) \phi f(v) dx \text{ for } t \in [0,T).
\] (3.13)
Note that there holds
\[ F(t) = \int_{\Omega} u \phi dx \leq \int_{\Omega} \frac{\sqrt{f(v)\sqrt{\phi}}}{\sqrt{f(0)}} \phi dx \]
\[ \leq \left( \int_{\Omega} \frac{\alpha f(v)\phi}{f(0)} dx \right)^{\frac{1}{2}} \left( \int_{\Omega} \phi/\phi dx \right)^{\frac{1}{2}} \text{ for } t \in [0, T) \] (3.14)
by (H2) and \( u < 1 \). Consequently, \( F^2(t) \leq (\int_{\Omega} \frac{\alpha f(v)\phi}{f(0)} dx)(\int_{\Omega} \phi/\phi dx) \). Combining this inequality with (3.13) yields
\[ F'(t) + \lambda_1 F(t) \geq (\lambda f(0) F^2(t)) / \left( \int_{\Omega} \phi/\phi dx \right). \] (3.15)
Thus \( \frac{d}{dt}(-\frac{1}{e^{\lambda t} F(t)}) \geq (\lambda f(0) e^{-\lambda t}) / (\int_{\Omega} \phi/\phi dx) \). By integrating on both sides there holds
\[ e^{-\lambda_1 t} \left( 1 - \frac{1}{F(t)} \right) + \frac{1}{\lambda_1} \int_{\Omega} \phi/\phi dx \geq \frac{\lambda f(0)}{\lambda_1} \int_{\Omega} \phi/\phi dx (1 - e^{-\lambda_1 t}), \] (3.16)
and then
\[ e^{-\lambda_1 t} \left( \frac{\lambda f(0)}{\lambda_1} \int_{\Omega} \phi/\phi dx - \frac{1}{F(t)} \right) \geq \frac{\lambda f(0)}{\lambda_1} \int_{\Omega} \phi/\phi dx - \frac{1}{F(0)}. \] (3.17)
Note that \( F(0) = \int_{\Omega} u_0 \phi dx > (\lambda_1 \int_{\Omega} \phi/\phi dx) / (\lambda f(0)) \). We have
\[ \frac{\lambda f(0)}{\lambda_1} \int_{\Omega} \phi/\phi dx - \frac{1}{F(t)} \geq e^{\lambda_1 t} \left( \frac{\lambda f(0)}{\lambda_1} \int_{\Omega} \phi/\phi dx - \frac{1}{F(0)} \right) \geq 0. \] (3.18)
Combining (3.17) and (3.18) yields
\[ e^{-\lambda_1 t} \geq \frac{\lambda f(0) - \left( \lambda_1 \int_{\Omega} \phi/\phi dx \right) / F(0)}{\lambda f(0) - \left( \lambda_1 \int_{\Omega} \phi/\phi dx \right) / F(t)}, \] (3.19)
which implies by \( F(t) \leq \|\phi\|_1 = 1 \) that
\[ t \leq \frac{1}{\lambda_1} \ln \frac{\lambda f(0) - \left( \lambda_1 \int_{\Omega} \phi/\phi dx \right) / F(t)}{\lambda f(0) - \left( \lambda_1 \int_{\Omega} \phi/\phi dx \right) / F(0)} \]
\[ \leq \frac{1}{\lambda_1} \ln \frac{\lambda f(0) - \lambda_1 \int_{\Omega} \phi/\phi dx}{\lambda f(0) - \lambda_1 \int_{\Omega} \phi/\phi dx} \text{ for any } t \in [0, T). \] (3.20)
Hence, \( u \) must quench at finite time \( T_u^* \) and \( T_u^* \leq \frac{1}{\lambda_1} \ln \frac{\lambda f(0) - \lambda_1 \int_{\Omega} \phi/\phi dx}{\lambda f(0) - \lambda_1 \int_{\Omega} \phi/\phi dx} / F(0) \).

Similarly, for any given \( \mu > 0 \), once \( \int_{\Omega} v_0 \phi dx \geq \lambda_1 \int_{\Omega} \phi/\beta dx/\mu g(0) \), \( v(x, t) \) must quench at a finite time \( T_v^* \) and \( T_v^* \leq \frac{1}{\lambda_1} \ln \frac{\mu g(0) - \lambda_1 \int_{\Omega} \phi/\beta dx}{\mu g(0) - \lambda_1 \int_{\Omega} \phi/\beta dx}/G(0) \). The proof is therefore completed.
4. Convergence rate. This section is devoted to proving Theorem 1.2. Recall that for single parabolic MEMS equation (1.2), in the process of obtaining the convergence rate of global solution (see [11]), it plays an important role that the first eigenvalue of the associated linearized elliptic equation has a variational characterization. However, as pointed out in [2], no such analogous formulation is available for our system (E). To overcome this difficulty, we have to apply some new ideas and techniques. In fact, in our approach, in addition to taking advantage of the linear stability of the minimal steady state, another key idea relies on the \(C^1\)-convergence (see Proposition 4.3) of the global solutions to the minimal steady state. Note that the \(C^1\)-convergence can not be obtained directly by the same way as obtaining the uniform convergence in Theorem 1.1. We get the \(C^1\)-convergence by higher-order estimates (see Proposition 4.2) for global solutions of system (P).

In this section, since only the case \((\lambda, \mu) \in \mathcal{O}_1\) is considered, problem (E) always has a minimal solution \((w_{\lambda, \mu}, z_{\lambda, \mu})\) by Proposition A. Before verifying Theorem 1.2, we introduce a lemma and three propositions as follows.

**Lemma 4.1.** Given a smooth bounded domain \(\Omega\) in \(\mathbb{R}^N\). Suppose \(a(x, t) \in C([0, +\infty), C^1(\Omega))\), \(a \geq 0\) in \(\Omega \times [0, +\infty)\), \(b(x) \in C^1(\Omega)\), \(b > 0\) in \(\Omega\), \(\alpha = b = 0\) on \(\partial \Omega\), \(\frac{\partial b}{\partial \nu} < 0\) on \(\partial \Omega\), \(\lim_{t \to +\infty} \|a(\cdot, t)\|_{C^1} = 0\). Then there exists \(T_0 > 0\) such that \(a(x, t) \leq b(x)\) in \(\Omega\) for all \(t > T_0\). Here \(\nu\) denotes the outward unit normal vector on \(\partial \Omega\).

**Proof.** Since \(\frac{\partial b}{\partial \nu} < 0\) and \(b(x) \in C^1(\Omega)\), there exists a constant \(\varepsilon > 0\) such that for all \(x \in \Omega_{\varepsilon} := \{x \in \Omega; \text{dist}(x, \partial \Omega) \leq \varepsilon\}\), there holds \(b(x) = b(x) - b(x_0) \geq C_0|x-x_0|\), where \(x_0 \in \partial \Omega\) satisfying \(x \in \Omega\) \(\|\nu\) and \(C_0 > 0\) is a constant independent of \(x\). On the other hand, for all \(x \in \Omega_{\varepsilon}\), there also holds \(a(x, t) - a(x_0, t) \leq \|a(\cdot, t)\|_{C^1}|x-x_0|\). Note that \(\lim_{t \to +\infty} \|a(\cdot, t)\|_{C^1} = 0\). Therefore, there holds \(\|a(\cdot, t)\|_{C^1} \leq C_0\) for \(t\) large enough and it follows that \(a(x, t) \leq b(x)\) on \(\Omega\) for \(t\) large enough. At last, it is obviously that for any given subset \(\Omega \subset \Omega\), \(a(x, t) \leq b(x)\) on \(\Omega\) for \(t\) large enough. Hence, we conclude this lemma.

**Proposition 4.1.** Assume that \((u_0, v_0) \leq (w_{\lambda, \mu}, z_{\lambda, \mu})\) and \((u_0, v_0)\) is a subsolution of (E) (or \((w_{\lambda, \mu}, z_{\lambda, \mu}) \leq (u_0, v_0)\) if \((w_{1, z_1})\) exists and \((u_0, v_0)\) is a supersolution of (E)). Let \((u, v)\) be the unique global solution of (P), then there exist \(c_1, c_2, c_3, c_4 \in \mathbb{R}^+\), such that

\[
u_1 \geq c_1 \nu_t > 0 \quad \text{and} \quad \nu_t \geq c_2 \nu_x > 0 \quad \text{(or} \quad u_t \leq c_3 v_t < 0 \quad \text{and} \quad v_t \leq c_4 u_t < 0. \quad (4.1)
\]

**Proof.** We only give the proof for \((u_0, v_0) \leq (w_{\lambda, \mu}, z_{\lambda, \mu})\) and \((u_0, v_0)\) being a subsolution of (E), since the proof for the other case is totally similar. For any given small number \(h > 0\), introduce

\[
\hat{u}(x, t) = u(x, t + h) - u(x, t), \quad \hat{v}(x, t) = v(x, t + h) - v(x, t),
\]

\[
\hat{f} = (f(v(x, t + h)) - f(v(x, t)))/\hat{v}, \quad \hat{g} = (g(u(x, t + h)) - g(u(x, t)))/\hat{u}.
\]

Let

\[
R(x, t) = \hat{u} - c_1 \hat{v} = (u - c_1 v)(x, t + h) - (u - c_1 v)(x, t).
\]

It can be deduced that

\[
\left\{ \begin{array}{l}
R_t - \Delta R + \alpha \hat{f} \hat{g} R = (\lambda \alpha \hat{f} - c_4^2 \beta \hat{g}) \hat{v}, \\
R|_{\partial \Omega} = 0, \\
R(x, 0) = (u(x, h) - u(x, 0)) - c_1 (v(x, h) - v(x, 0)).
\end{array} \right. \tag{4.2}
\]
Applying the comparison principle in Lemma 2.1, we get $R \geq 0$ provided that
\[ c_1 \leq \min \left\{ \frac{\lambda}{\mu} \inf_{\Omega} \alpha(x) \frac{f'(0)}{\beta(x) g'(\|w_{\lambda,\mu}\|_{\infty})}, \inf_{\Omega} \frac{\Delta u_0 + \lambda \alpha(x) f(v_0)}{\Delta v_0 + \mu \beta(x) g(u_0)} \right\}, \] (4.3)
by noting that
\[ \lim_{h \to 0} u(x, h) - u(x, 0) = u_t(x, 0) = \frac{\Delta u_0 + \lambda \alpha f(v_0)}{\Delta v_0 + \mu \beta g(u_0)} \]
and
\[ \lim_{h \to 0} \frac{\hat{f}'(v(x, t))}{g'(u(x, t))} = \frac{\hat{f}'(0)}{g'(\|w_{\lambda,\mu}\|_{\infty})}. \]
Here $0 \leq u, v < 1$, the nonnegativity of $\lambda, \mu, \alpha, \beta, u_t, v_t, f'(s), g'(s)$ for $0 \leq s < 1$ and the monotonicity of $f'(s), g'(s)$ are used. Note that $R \geq 0$ implies $u_t \geq v_t$. Similarly, it can be proved that $v_t - c_2 u_t \geq 0$ provided
\[ c_2 \leq \min \left\{ \frac{\mu}{\lambda} \inf_{\Omega} \beta(x) \frac{g'(0)}{\alpha(x) f'(\|w_{\lambda,\mu}\|_{\infty})}, \inf_{\Omega} \frac{\Delta v_0 + \mu \beta(x) g(u_0)}{\Delta u_0 + \lambda \alpha(x) f(v_0)} \right\}. \] (4.4)
Together with Proposition 2.2, we complete the proof of (4.1).

Without causing confusion, for convenience we use $(w, z)$ instead of $(w_{\lambda,\mu}, z_{\lambda,\mu})$ to denote the minimal solution of problem (E) in the rest part of this section.

**Proposition 4.2.** Assume that $(u_0, v_0) \leq (w, z)$ and $(u_0, v_0)$ is a subsolution of (E), or $(w, z) \leq (u_0, v_0) \leq (w_1, z_1)$ (if $(w_1, z_1)$ exists) and is a supersolution of (E). Let $(u, v)$ be the unique global solution of (P), then we have
\[ \lim_{t \to +\infty} \|u_t\|_2 = \lim_{t \to +\infty} \|v_t\|_2 = 0 \] (4.5)
and
\[ \|u\|_{H^1} + \|v\|_{H^1} \leq C(\delta), \text{ for all } t \geq \delta > 0. \] (4.6)

**Proof.** First we claim that
\[ \|\nabla u\|_2 + \|\nabla v\|_2 \leq C(u_0, v_0, w, z), \] (4.7)
where $C$ is a constant independent of time $t$. To prove this claim, we denote $\xi = u - w, \eta = v - z$. Then it follows from systems (P) and (E) that
\[
\begin{cases}
\xi_t - \Delta \xi = \lambda \alpha(x) (f(v) - f(z)), & \text{in } \Omega \times (0, T), \\
\eta_t - \Delta \eta = \mu \beta(x) (g(u) - g(w)), & \text{in } \Omega \times (0, T), \\
\xi = \eta = 0, & \text{on } \partial \Omega \times (0, T), \\
\xi(x, 0) = u_0(x) - w(x), & \eta(x, 0) = v_0(x) - z(x), \text{ for } x \in \Omega.
\end{cases}
\] (4.8)

Multiplying the first equation in (4.8) by $\xi_t$ yields
\[
\frac{1}{2} \frac{d}{dt} \|\nabla \xi\|_2^2 + \|\xi_t\|_2^2 = \lambda \int_{\Omega} \alpha(x) |f(v) - f(z)| \xi_t dx \leq 0,
\] (4.9)
where assumption (H2) and Proposition 2.2 are used. The above inequality then implies that
\[
\frac{d}{dt} \|\nabla \xi\|_2^2 \leq 0
\] (4.10)
and hence
\[
\|\nabla \xi\|_2^2 \leq \|\nabla \xi(x, 0)\|_2 \leq C(u_0, v_0, w, z).
\] (4.11)
Note that \( \|\nabla \eta\|_2 \leq C(u_0, v_0, w, z) \) can be obtained similarly. Then (4.7) follows by
\[
\|\nabla u\|_2 + \|\nabla v\|_2 \leq \|\nabla \xi\|_2 + \|\nabla w\|_2 + \|\nabla \eta\|_2 + \|\nabla z\|_2 \leq C(u_0, v_0, w, z). \tag{4.12}
\]
Next, we show that
\[
\int_0^{+\infty} (\|u_t\|^2 + \|v_t\|^2)dt \leq C. \tag{4.13}
\]
After multiplying the equations in (P) by \( u_t \) and \( v_t \), respectively, adding them up and integrating over \( \Omega \), we can see that problem (P) admits a Lyapunov function
\[
E(u, v) = \int_{\Omega} (\nabla u \nabla v - F(x, v) - G(x, u))dx, \tag{4.14}
\]
where \( F(x, v) = \lambda \alpha(x) \int_0^v f(s)ds, \ G(x, u) = \mu \beta(x) \int_0^u g(s)ds \), and there holds
\[
\frac{d}{dt}E(u, v) + 2 \int_{\Omega} F(x, C_1)dx + \int_{\Omega} G(x, C_2)dx \leq C, \tag{4.16}
\]
where \( C_1 = \max_{\Omega} z \) and \( C_2 = \max_{\Omega} w \) for \( (u_0, v_0) \leq (w, z) \), \( C_1 = \max_{\Omega} z \) and \( C_2 = \max_{\Omega} w \) for \( (w, z) \leq (u_0, v_0) \). Then (4.13) can be concluded by (4.16), Proposition 2.2 and Proposition 4.1.

Consider the following auxiliary problem:
\[
\begin{aligned}
P_t - \Delta P &= \lambda \alpha(x) f'(v_0)Q, & \quad \text{in } \Omega \times (0, T), \\
Q_t - \Delta Q &= \mu \beta(x) g'(u_0)P, & \quad \text{in } \Omega \times (0, T), \\
P = Q &= 0, & \quad \text{on } \partial \Omega \times (0, T), \\
P(x, 0) &= \Delta u_0 + \lambda \alpha(x) f(v_0), & \quad Q(x, 0) = \Delta v_0 + \mu \beta(x) g(u_0), & \quad \text{in } \Omega.
\end{aligned} \tag{4.17}
\]
Note that it can be proved similarly to Proposition 4.1 that
\[
0 < C_1 \leq P/Q \leq C_2 \tag{4.18}
\]
for some positive constants \( C_1 \) and \( C_2 \). Multiplying the first equation in (4.17) by \( P \) and integrating over \( \Omega \), by assumption (H2) yield
\[
\frac{1}{2} \frac{d}{dt} \|P\|^2 + \|\nabla P\|^2_2 = \int_{\Omega} \lambda \alpha f'(v) PQ dx \leq C \|P\|^2_2. \tag{4.19}
\]
Hence, by Young’s inequality we have
\[
\frac{d}{dt} \|P\|^2_2 \leq C_1 \|P\|^2_2 + C_2. \tag{4.20}
\]
Then by (4.13) and [21, Lemma 6.2.1], it follows that \( \lim_{t \to +\infty} \|P\|_2 = 0 \). Meanwhile, \( \lim_{t \to +\infty} \|Q\|_2 = 0 \) can be obtained similarly. Let
\[
\dot{u} = u_0 + \int_0^t P(\tau)d\tau, \quad \dot{v} = v_0 + \int_0^t Q(\tau)d\tau. \tag{4.21}
\]
Clearly, \( \dot{u}, \dot{v} \) are the solution to (P). By the uniqueness of solutions to (P), it holds \( (u, v) = (\dot{u}, \dot{v}) \).

In conclusion, we have
\[
(u_t, v_t) = (P, Q), \tag{4.22}
\]
Suppose that the conditions in Proposition 4.17 are satisfied. For $4.18$, we have
\[
\int_0^t \|\nabla P\|_2^2 d\tau \leq C, \tag{4.23}
\]
which implies obviously
\[
\int_0^t \|\nabla P\|_2^2 d\tau \leq C. \tag{4.24}
\]
Multiplying the first equation in (4.17) by $-\Delta P$ and integrating over $\Omega$, by (4.18) we have
\[
\frac{1}{2} \frac{d}{dt} \|\nabla P\|_2^2 + \|\Delta P\|_2^2 = \lambda \int_\Omega \alpha f'(v) Q(-\Delta P) dx \leq C \|P\|_2 \|\Delta P\|_2 \leq C \|P\|_2^2 + \frac{1}{2} \|\Delta P\|_2^2, \tag{4.25}
\]
which yields
\[
\frac{d}{dt} \|\nabla P\|_2^2 + \|\Delta P\|_2^2 \leq C \|P\|_2^2. \tag{4.26}
\]
Multiplying (4.26) by $t$, then integrating with respect to $t$ in $[0, t]$, by (4.13) and (4.24) there holds
\[
t \|\nabla P\|_2^2 + \int_0^t \|\Delta P\|_2^2 d\tau \leq t \int_0^t \|\nabla P\|_2^2 d\tau + Ct \int_0^t \|P\|_2^2 d\tau \leq C_1 + C_2 t. \tag{4.27}
\]
Thus, for $t \geq \delta > 0$, we have
\[
\|\nabla P\|_2 \leq C_1 + C_2 \leq \frac{C_1}{\delta} + C_2 \tag{4.28}
\]
and it follows by (4.22) that
\[
\|u_t\|_{H^1} = \|P\|_{H^1} \leq C(\delta) \text{ for } t \geq \delta. \tag{4.29}
\]
Now we can deduce from the equation in (P) and the regularity theory for the elliptic problem (see e.g. [21])
\[
\begin{cases}
-\Delta u = \lambda \alpha(x) f'(v) - u_t, & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega,
\end{cases} \tag{4.30}
\]
that
\[
\|u(\cdot, t)\|_{H^3} \leq C(\|f(v)\|_{H^1} + \|u_t\|_{H^1}) \leq C(\delta)(1 + \|f'(v)\|_{L^2}) \leq C(\delta)(1 + \|\nabla v\|_{L^2}) \leq C(\delta) \tag{4.31}
\]
by (4.29) and (4.7). Note that $\|v(\cdot, t)\|_{H^3}$ can be treated similarly. In conclusion, we obtain (4.6). The proof of Proposition 4.2 is completed.

**Proposition 4.3.** Suppose that the conditions in Proposition 4.2 are satisfied. For $1 \leq n \leq 3$, there holds
\[
\lim_{t \to \infty} \|u(\cdot, t) - w\|_{C^1} = \lim_{t \to \infty} \|\xi(\cdot, t)\|_{C^1} = 0,
\lim_{t \to \infty} \|v(t) - z\|_{C^1} = \lim_{t \to \infty} \|\eta(t)\|_{C^1} = 0. \tag{4.32}
\]

**Proof.** Note by (4.6) that $\xi, \eta \in H^3(\Omega)$, and $H^3(\Omega) \hookrightarrow C^1(\Omega)$ for $1 \leq n \leq 3$ by Sobolev compact embedding theorem. Thanks to (3.5), (4.32) follows by the relative compactness of $\xi(t), \eta(t)$ in $C^1$ and the uniqueness of the limits.

Now we present the proof of Theorem 1.2 as follows.
Proof of Theorem 1.2. Multiplying the equations in (4.8) by $\xi$ and $\eta$, respectively, adding them up and integrating over $\Omega$ yield
\[
\frac{d}{dt} \int_\Omega \left( \frac{1}{2} \xi^2 + \frac{1}{2} \eta^2 \right) dx + \| \nabla \xi \|^2 + \| \nabla \eta \|^2
= \int_\Omega \left( \lambda \alpha [f(z) - f(v)](-\xi) + \mu \beta [g(w) - g(u)](-\eta) \right) dx.
\] (4.33)

Rewrite equations in (4.8) as
\[
\begin{cases}
\xi_t - \Delta \xi - \lambda \alpha f'(z) \eta = \lambda \alpha (f(v) - f(z) - f'(z) \eta), & \text{in } \Omega \times (0, T), \\
\eta_t - \Delta \eta - \mu \beta g'(w) \xi = \mu \beta (g(u) - g(w) - g'(w) \xi), & \text{in } \Omega \times (0, T).
\end{cases}
\] (4.34)

By the convexity of $f$ and $g$, it is easy to deduce that $f(v) - f(z) - f'(z) \eta \geq 0$ and $g(u) - g(w) - g'(w) \xi \geq 0$. Thus it follows that
\[
\begin{cases}
\xi_t - \Delta \xi - \lambda \alpha f'(z) \eta \geq 0, & \text{in } \Omega \times (0, T), \\
\eta_t - \Delta \eta - \mu \beta g'(w) \xi \geq 0, & \text{in } \Omega \times (0, T).
\end{cases}
\] (4.35)

Step 1: Consider $(u_0, v_0) = (0, 0)$.

Let’s keep in mind that $(0, 0)$ is a subsolution to (E). Multiplying the inequalities in (4.35) by $\psi_1$ and $\varphi_1$, respectively, adding them up and integrating over $\Omega$ yield
\[
\int_\Omega (\psi_1 \xi + \varphi_1 \eta) dx + \nu_1 \int_\Omega (\psi_1 \xi + \varphi_1 \eta) dx \geq 0.
\] (4.36)

Here, $\nu_1$ is the principal eigenvalue of problem (1.4) and $(\varphi_1, \psi_1)$ is the corresponding positive eigenfunction. Multiplying (4.36) by $-1$, then adding it to (4.33) yield
\[
\begin{align*}
\frac{d}{dt} & \int_\Omega \left( \frac{1}{2} \xi^2 + \frac{1}{2} \eta^2 + [\psi(-\xi) + \varphi_1(-\eta)] \right) dx + \| \nabla \xi \|^2 \\
& \quad + \| \nabla \eta \|^2 + \nu_1 \int_\Omega \left[ \psi_1(-\xi) + \varphi_1(-\eta) \right] dx \\
& \leq \int_\Omega \left( \lambda \alpha [f(z) - f(v)](-\xi) + \mu \beta [g(w) - g(u)](-\eta) \right) dx.
\end{align*}
\] (4.37)

Now we claim that there exists $T_0 > 0$ such that for any $t > T_0$, there holds
\[
f(z) - f(v) \leq \frac{\nu_1}{2\lambda \|\alpha\|_\infty} \psi_1, \quad g(w) - g(u) \leq \frac{\nu_1}{2\mu \|\beta\|_\infty} \varphi_1.
\] (4.38)

In fact, recalling (1.4), by (1.5) we have
\[
\begin{cases}
-\Delta \varphi_1 = f'(z) \psi_1 + \nu_1 \varphi_1 \geq 0, & \text{in } \Omega, \\
-\Delta \psi_1 = g'(w) \varphi_1 + \nu_1 \psi_1 \geq 0, & \text{in } \Omega, \\
\varphi_1 = \psi_1 = 0, & \text{on } \partial \Omega.
\end{cases}
\] (4.39)

Thus, by Hopf lemma there hold $-\frac{\partial \varphi_1}{\partial n} \geq \varepsilon_0$, $-\frac{\partial \psi_1}{\partial n} \geq \varepsilon_0$ on $\partial \Omega$ for some $\varepsilon_0 > 0$. Then (4.38) follows by (1.5), Lemma 4.1 and Proposition 4.3.

Combining (4.37), (4.38) with the Poincaré inequality $\|u\|_2 \leq \frac{1}{\lambda_1} \|\nabla u\|_2$ for any $u \in H^1_0(\Omega)$ with $\lambda_1 > 0$ being the first eigenvalue of $-\Delta$ on $H^1_0(\Omega)$, we get
\[
\begin{align*}
\frac{d}{dt} & \int_\Omega \left( \frac{1}{2} \xi^2 + \frac{1}{2} \eta^2 + [\psi_1(-\xi) + \varphi_1(-\eta)] \right) dx + \lambda_1 \|\xi\|^2 \\
& \quad + \lambda_1 \|\eta\|^2 + \nu_1 \int_\Omega \left[ \psi_1(-\xi) + \varphi_1(-\eta) \right] dx \leq 0.
\end{align*}
\] (4.40)
Let \( Y = \int_\Omega (\xi^2 + \eta^2 + 2[\psi_1(-\xi) + \varphi_1(-\eta)])dx \). Note that
\[
\int_\Omega [\psi_1(-\xi) + \varphi_1(-\eta)]dx \geq 0. \tag{4.41}
\]

By (4.40) there holds
\[
\frac{dY}{dt} + \gamma Y \leq 0, \quad \gamma = \min \left\{ 2\lambda_1, \frac{\nu_1}{2} \right\}, \tag{4.42}
\]
which yields \( Y \leq Y(0)e^{-\gamma t} \). Then by noting (4.41) again it follows that
\[
\|u(x, t) - w(x)\|^2 + \|v(x, t) - z(x)\|^2 \leq Y(t) \leq C_1 \exp \left( -\min \left\{ 2\lambda_1, \frac{\nu_1}{2} \right\} t \right), \quad \text{for } t > T_0 \tag{4.43}
\]
with
\[
C_1 = Y(0) = \|w_{\lambda, \mu}\|^2_2 + \|z_{\lambda, \mu}\|^2_2 + 2\|\psi_1w_{\lambda, \mu} + \varphi_1z_{\lambda, \mu}\|_1. \tag{4.44}
\]

**Step 2:** Consider \((u_0, v_0) = (sw_1 + (1 - s)w_{\lambda, \mu}, sz_1 + (1 - s)z_{\lambda, \mu})\) for some \(0 < s < 1\).

Let’s keep in mind that this \((u_0, v_0)\) is a supersolution to (E). Combining (4.33) with (4.34) yields
\[
\frac{d}{dt} \int_\Omega \left( \frac{1}{2} \xi^2 + \frac{1}{2} \eta^2 + [\psi_1 \xi + \varphi_1 \eta] \right)dx + \|\nabla \xi\|^2_2 + \|\nabla \eta\|^2_2 + \nu_1 \int_\Omega [\psi_1 \xi + \varphi_1 \eta]dx = \int_\Omega \lambda_1 [f(v) - f(z)]\xi + \mu_\beta [g(u) - g(w)]\eta \right)dx + \int_\Omega \lambda_1 [f(v) - f(z)] - f'(z)\eta\psi_1dx + \int_\Omega \mu_\beta (g(u) - g(w)) - f'(w)\xi \varphi_1dx. \tag{4.45}
\]

Similar to (4.38), there hold
\[
f(v) - f(z) \leq \frac{\nu_1}{4\lambda_1\|\alpha\|_\infty}\psi_1, \quad g(u) - g(w) \leq \frac{\nu_1}{4\mu_\beta\|\beta\|_\infty}\varphi_1 \tag{4.46}
\]
for \(t\) sufficiently large. Meanwhile, by Proposition 4.3 one can obtain that for \(t\) sufficiently large,
\[
\lambda_1 (f(v) - f(z))\psi_1 \leq \frac{1}{4} \nu_1 \eta_1 \varphi_1, \quad \mu_\beta (g(u) - g(w)) - g'(w)\xi \varphi_1 \leq \frac{1}{4} \nu_1 \xi \psi_1. \tag{4.47}
\]

Combining (4.45)-(4.47) with again the Poincaré inequality yields
\[
\frac{d}{dt} \int_\Omega \left( \frac{1}{2} \xi^2 + \frac{1}{2} \eta^2 + [\psi_1 \xi + \varphi_1 \eta] \right)dx + \lambda_1 \|\xi\|^2_2 + \lambda_1 \|\eta\|^2_2 + \frac{\nu_1}{2} \int_\Omega [\psi_1 \xi + \varphi_1 \eta]dx \leq 0. \tag{4.48}
\]

Let \( Y = \int_\Omega (\xi^2 + \eta^2 + 2[\psi_1 \xi + \varphi_1 \eta])dx \). Note that
\[
\int_\Omega [\psi_1 \xi + \varphi_1 \eta]dx \geq 0. \tag{4.49}
\]

By (4.48) there holds
\[
\frac{dY}{dt} + \gamma Y \leq 0, \quad \gamma = \min \left\{ 2\lambda_1, \frac{\nu_1}{2} \right\}, \tag{4.50}
\]
which again yields \( Y \leq Y(0)e^{-\gamma t} \). Then by noting (4.49) again it follows that
\[
\|u(x,t) - w(x)\|^2 + \|v(x,t) - z(x)\|^2 \leq C_2 \exp \left( -\min \left\{ 2\lambda_1, \frac{\nu_1}{2} \right\} t \right),
\]
for large \( t \) with
\[
C_2 = \|w_1 - w_{\lambda,\mu}\|^2 + \|z_1 - z_{\lambda,\mu}\|^2 + 2\|\psi_1(w_1 - w_{\lambda,\mu}) + \varphi_1(z_1 - z_{\lambda,\mu})\|_1.
\]  

**Step 3: Consider more general \((u_0, v_0)\).**

Note the comparison principle in Lemma 2.1. If \((0,0) \leq (u_0, v_0) \leq (w_{\lambda,\mu}, z_{\lambda,\mu})\), then (4.43) holds. If \((w_{\lambda,\mu}, z_{\lambda,\mu}) \leq (u_0, v_0) \leq (w_1, z_1)\), then (4.51) holds. In conclusion, for more general \((0,0) \leq (u_0, v_0) \leq (w_1, z_1)\), by applying the elementary inequality \(\sqrt{a} + \sqrt{b} \leq \sqrt{2(a^2 + b^2)}\), (1.6) holds for \(C_0 = \sqrt{2(C_1 + C_2)}\) with \(C_1\) and \(C_2\) defined in (4.44) and (4.52), respectively. The proof of Theorem 1.2 is therefore completed.

At the last of this section, we present the proof of Remark 1.5 as follows.

**Proof of Remark 1.5.** We only show the proof for case \((u_0, v_0) \leq (w, z)\) and is a subsolution of (E), since the proof for other case is similar. By Theorem 1.1 (a1), for any small enough \(\delta > 0\), there hold \(\|\eta\|_\infty < \delta\) and \(\|\xi\|_\infty < \delta\) for \(t\) large enough. Then it can be deduced that
\[
\int_\Omega \lambda_0(f(v) - f(z) - f'(z)\eta)\xi_t dx \leq C\|\eta\|_\infty \|\eta\|_2 \|\xi_t\|_2 \leq \frac{C\delta}{2} \|\eta\|_2^2 + \frac{C\delta}{2} \|\xi_t\|_2^2.
\]
and
\[
\int_\Omega \mu \beta(g(u) - g(w) - g'(w)\xi)\eta_t dx \leq C\|\xi\|_\infty \|\xi\|_2 \|\eta_t\|_2 \leq \frac{C\delta}{2} \|\xi\|_2^2 + \frac{C\delta}{2} \|\eta_t\|_2^2.
\]
Multiplying the equations in (4.34) by \(\xi_t\) and \(\eta_t\), respectively, adding them up and integrating over \(\Omega\), by (4.53) and (4.54) we have
\[
\frac{1}{2} \frac{d}{dt} \left( \|\nabla \xi\|^2 + \|\nabla \eta\|^2 \right) + \|\xi_t\|^2 + \|\eta_t\|^2 - \int_\Omega \lambda_0 f'(z)\eta\xi_t dx - \int_\Omega \mu \beta g'(w)\xi\eta_t dx \leq C\delta (\|\eta\|_2^2 + \|\xi_t\|_2^2 + \|\xi\|_2^2 + \|\eta_t\|_2^2).
\]
Note that (4.37) and (4.38) still hold now. By Poincaré inequality again we have
\[
\frac{d}{dt} \int_\Omega \left( \frac{1}{2} \xi^2 + \frac{1}{2} \eta^2 + [\psi_1(-\xi) + \varphi_1(-\eta)] \right) dx
+ \frac{1}{2} \|\nabla \xi\|_2^2 + \|\nabla \eta\|_2^2 + \frac{\lambda_1}{2} (\|\xi\|_2^2 + \|\eta\|_2^2)
+ \nu_1 \int_\Omega [\psi_1(-\xi) + \varphi_1(-\eta)] dx \leq 0.
\]
Adding (4.55) and (4.56) up, by choosing \(\delta\) small enough such that \(\frac{C\delta}{2} \leq \min\{\frac{1}{2}, \frac{\lambda_1}{2}\}\), and noting \(\eta_t < 0, \xi_t < 0\), there holds
\[
\frac{d}{dt} \int_\Omega \left( \xi^2 + \eta^2 + \|\nabla \xi\|^2 + \|\nabla \eta\|^2 + 2[\psi_1(-\xi) + \varphi_1(-\eta)] \right) dx
+ \|\nabla \xi\|_2^2 + \|\nabla \eta\|_2^2 + \frac{\lambda_1}{2} (\|\xi\|_2^2 + \|\eta\|_2^2)
+ \nu_1 \int_\Omega [\psi_1(-\xi) + \varphi_1(-\eta)] dx \leq 0.
\]
Let \( Y = \int_\Omega (\xi^2 + \eta^2 + \|\nabla \xi\|^2 + \|\nabla \eta\|^2 + 2[\psi_1(-\xi) + \varphi_1(-\eta)]) \, dx \). By (4.41) again, there holds
\[
\frac{dY}{dt} + \gamma Y \leq 0, \quad \gamma = \min \left\{ \frac{\lambda_1}{2}, \frac{\nu_1}{2}, 1 \right\},
\] (4.58)
which follows
\[
\|\xi\|_{H^1} + \|\eta\|_{H^1} \leq C \exp \left( - \min \left\{ \frac{\lambda_1}{4}, \frac{\nu_1}{4}, \frac{1}{2} \right\} t \right)
\] (4.59)
with \( C = \|w_{\lambda,\mu} - u_0\|_2^2 + \|z_{\lambda,\mu} - v_0\|_2^2 + 2\|\psi_1(w_{\lambda,\mu} - u_0) + \varphi_1(z_{\lambda,\mu} - v_0)\|_1 \). The proof of Remark 1.5 is completed. \( \square \)

**Appendix A.** We will give the proof of Proposition A in this Appendix. First we will prove that the elliptic problem (E) has a classical solution for \( \lambda \) and \( \mu \) small enough, while (E) has no solution for \( \lambda \) or \( \mu \) large enough. More precisely, we will prove that the set
\[
\Lambda := \{ (\lambda, \mu) \in \mathbb{R}^+ \times \mathbb{R}^+ : (E) \text{ has a classical minimal solution} \}
\] (A.1)
is nonempty and bounded.

**Lemma A.1.** \( \Lambda \) is bounded, and there exist \( \lambda_0 > 0, \mu_0 > 0 \) such that \( (0, \lambda_0] \times (0, \mu_0] \subseteq \Lambda \).

**Proof.** Let \( \gamma \in H^1_0(\Omega) \) be the regular solution of \( -\Delta \gamma = 1 \) in \( \Omega \). It is then easy to verify that there exists \( \alpha \in (0, 1/\|\gamma\|_\infty) \) such that \( (\alpha \gamma, \alpha \gamma) \) is a supersolution of (E) if
\[
\lambda < \frac{1}{\sup_{x \in \Omega} \alpha(x)} \sup_{0 < s < 1/\|\gamma\|_\infty} s \int f(s\|\gamma\|_\infty) \, dx =: \lambda_0
\]
and
\[
\mu < \frac{1}{\sup_{x \in \Omega} \beta(x)} \sup_{0 < s < 1/\|\gamma\|_\infty} s \int g(s\|\gamma\|_\infty) \, dx =: \mu_0.
\]
As \( (0, 0) \) is a subsolution and \( \alpha \gamma > 0 \) in \( \Omega \), (E) admits a regular solution for \( \lambda \in (0, \lambda_0] \) and \( \mu \in (0, \mu_0] \). In fact, for these \( \lambda, \mu \), using (H1) and the monotone iteration for \( n \in \mathbb{N} \),
\[
\begin{cases}
  w_0 = z_0 = 0, \\
  -\Delta w_{n+1} = \lambda\alpha(x) f(z_n), \quad &\text{in } \Omega, \\
  -\Delta z_{n+1} = \mu \beta(x) g(w_n), \quad &\text{in } \Omega, \\
  w_{n+1} = z_{n+1} = 0, \quad &\text{on } \partial \Omega,
\end{cases}
\] (A.2)
we get the minimal solution \( (w_{\lambda,\mu}, z_{\lambda,\mu}) = \lim_{n \to +\infty} (w_n, z_n) \). Therefore, \( \Lambda \) is nonempty.

On the other hand, take a positive first eigenfunction \( \varphi \) of \( -\Delta \) in \( H^1_0(\Omega) \) with the first eigenvalue \( \lambda_1 \) such that \( \int_\Omega \varphi^2 \, dx = 1 \). By (E) and \( w < 1, z < 1 \), we arrive at
\[
\begin{cases}
  \lambda_1 \geq \lambda_1 \int_\Omega w \varphi \, dx = \int_\Omega \varphi (-\Delta w) \, dx \\
  = \lambda \int_\Omega \alpha(x)f(z) \varphi \, dx \geq \lambda \int_\Omega \alpha(x)f(0) \varphi \, dx, \\
  \lambda_1 \geq \lambda_1 \int_\Omega z \varphi \, dx = \int_\Omega \varphi (-\Delta z) \, dx \\
  = \lambda \int_\Omega \beta(x)g(w) \varphi \, dx \geq \mu \int_\Omega \beta(x)g(0) \varphi \, dx.
\end{cases}
\] (A.3)
So \( \Lambda \) is bounded and \( \Lambda \subseteq (0, \frac{\lambda_1}{\int_\Omega \alpha(x) f(0) \varphi \, dx}] \times (0, \frac{\lambda_1}{\int_\Omega \beta(x) g(0) \varphi \, dx}] \). \( \square \)
Denote \( \mu = \Gamma(\lambda) \) as the critical curve such that if \( 0 \leq \mu < \Gamma(\lambda) \), then \((\lambda, \mu) \in \Lambda\); if \( \mu > \Gamma(\lambda) \), then \((\lambda, \mu) \in (\mathbb{R}^+ \times \mathbb{R}^+) \setminus \Lambda\). By Lemma A.1, there further hold \( 0 < \mu^* := \Gamma(0) < +\infty \) and \( 0 < \lambda^* := \Gamma^{-1}(0) < +\infty \).

Next we state that the critical curve \( \mu = \Gamma(\lambda) \) is non-increasing. More precisely,

**Lemma A.2.** If \( 0 \leq \lambda' \leq \lambda, 0 \leq \mu' \leq \mu \) for some \((\lambda, \mu) \in \Lambda\), then \((\lambda', \mu') \in \Lambda\).

**Proof.** Indeed, the solution associated to \((\lambda, \mu)\) turns out to be a super-solution to (E) with \((\lambda', \mu')\).

**Proof of Proposition A.** Define \( \mathcal{O}_1 = \Lambda \setminus \Gamma \). For \((\lambda_1, \mu_1), (\lambda_2, \mu_2) \in \mathcal{O}_1\), there exist \( \theta_1, \theta_2 > 0 \) such that \( \mu_1 = \theta_1 \lambda_1 \) and \( \mu_2 = \theta_2 \lambda_2 \). Using Lemma A.2, we can define a path linking \((\lambda_1, \mu_1)\) to \((0, 0)\) and another path linking \((0, 0)\) to \((\lambda_2, \mu_2)\), which implies that \( \mathcal{O}_1 \) is connected. Now, define \( \mathcal{O}_2 = (\mathbb{R}^+ \times \mathbb{R}^+) \setminus (\Lambda \cup \Gamma) \). Let \((\lambda_1, \mu_1), (\lambda_2, \mu_2) \in \mathcal{O}_2\). Then by Lemma A.2 again that \((\lambda_{\text{max}}, \mu_{\text{max}}) \in \mathcal{O}_2\), where \( \lambda_{\text{max}} = \max\{\lambda_1, \lambda_2\} \) and \( \mu_{\text{max}} = \max\{\mu_1, \mu_2\} \). We can take a path linking \((\lambda_1, \mu_1)\) to \((\lambda_{\text{max}}, \mu_{\text{max}})\) and another path linking \((\lambda_{\text{max}}, \mu_{\text{max}})\) to \((\lambda_2, \mu_2)\), which follows that \( \mathcal{O}_2 \) is connected.

At last, it is reduced to prove that problem (E) admits no weak solution for \((\lambda, \mu) \in \mathcal{O}_2\). Suppose on the contrary that \((w, z)\) is a weak solution to (E). By the monotonicity of \( f, g \), it is easy to verify that for any \( \delta > 1 \), \((\tilde{w}, \tilde{z}) = (w/\delta, z/\delta)\) is a weak super-solution for problem

\[
\begin{aligned}
-\Delta w &= \frac{\lambda}{\delta} \alpha(x)f(z), & & \text{in } \Omega, \\
-\Delta z &= \frac{\mu}{\delta} \beta(x)g(w), & & \text{in } \Omega, \\
w &= z = 0, & & \text{on } \partial\Omega,
\end{aligned}
\]

then the monotone iteration will enable us to get a weak solution \((\tilde{w}, \tilde{z})\) of (E) satisfying \( 0 \leq \tilde{w} \leq \bar{w} \leq 1/\delta < 1\), and \( 0 \leq \tilde{z} \leq \bar{z} \leq 1/\delta < 1\). The regularity theory implies that \((\tilde{w}, \tilde{z})\) is a regular solution of (E). This means that \((\lambda/\delta, \mu/\delta) \in \mathcal{O}_1 \cup \Gamma\). Let \( \delta \) tend to 1, we get \((\lambda, \mu) \in \mathcal{O}_1 \cup \Gamma\), which contradicts with the assumption. Therefore, no weak solution exists for \((\lambda, \mu) \in \mathcal{O}_2\) and the proof of Proposition A is completed.

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