REPRESENTING A POINT AND THE DIAGONAL AS ZERO LOCI IN
FLAG MANIFOLDS

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Abstract. The zero locus of a generic section of a vector bundle over a manifold defines a
submanifold. A classical problem in geometry asks to realise a specified submanifold in this
way. We study two cases; a point in a generalised flag manifold and the diagonal in the direct
product of two copies of a generalised flag manifold. These cases are particularly interesting
since they are related to ordinary and equivariant Schubert polynomials respectively.

1. Introduction

Let $N$ be a manifold of dimension $2n$. Consider a smooth function $f : N \to \mathbb{C}^m$ having
$0 \in \mathbb{C}^m$ as a regular value. Then, $M = f^{-1}(0) \subset N$ is a submanifold of codimension $2m$.
Conversely, we can ask if a submanifold $M \subset N$ of codimension $2m$ can be realised in this way,
or more generally, as the zero locus of a generic section of a rank $m$ complex vector bundle
$\xi \to N$. Here by a generic section, we mean it is transversal to the zero section. We say
$M$ is represented by $\xi$ if such a bundle $\xi$ exists.

The following example tells that even for the simplest case the question is not as trivial as it
may appear to be.

Example 1.1. Consider the representability of a point in $S^2$. Identify $S^2 = \mathbb{C}P^1$ and let
$\gamma^* \to \mathbb{C}P^1$ be the dual of the tautological bundle
$$
\gamma = \{ ([x, y], (cx, cy)) \mid x, y, c \in \mathbb{C} \}.
$$
We can define its generic section by the projection $(cx, cy) \mapsto cx$ whose zero locus is exactly
the south-pole $[0, y] \in \mathbb{C}P^1$. Since $S^2$ is homogeneous with a transitive $SO(3)$ action, for any
pair of points $x, y \in S^2$, there is an element $g \in SO(3)$ such that $gx = y$. By choosing $g$
appropriately, we can represent any point in $S^2$ by $g^*(\gamma^*)$.

On the other hand, consider the representability of a point in $S^{2n}$ for $n > 2$. Bott’s integrality
theorem tells that the top Chern class $c_n$ of any rank $n$ complex vector bundle on $S^{2n}$ is divisible
by $(n - 1)!$ (see, for example, [8, Proposition 6.1]). However, if there is a rank $n$ bundle with
a generic section whose zero locus is a point, the top Chen class of the bundle has to be the
generator of $H^{2n}(S^{2n})$. Hence, there is no such bundle.

From now on, all spaces are assumed to be based, and the base points are denoted by $pt$.
The following two submanifolds are particularly interesting (see [12] and references therein):

1. the base point $\{pt\} \subset X$
2. the diagonal $\Delta(X) \subset X \times X$.

Note that the choice of the base point does not make any difference since for any pair of points
$x, y \in X$ there exists a diffeomorphism $f : X \to X$ satisfying $f(y) = x$ so that the bundle and
the section for the representability of the point $x$ are pulled back to represent the point $y$. Note
also that when $\Delta(X) \subset X \times X$ is represented by $\xi$, then $\{pt\} \subset X$ is represented by $\iota^*(\xi)$,
where $\iota$ is the inclusion $N \hookrightarrow N \times N$ defined by $\iota(x) = (x, pt)$.
In [12] analogous problems in different settings are considered; in an algebraic setting and in topological settings with complex bundles, real bundles, and real oriented bundles. In this note, we focus on the following topological variant:

**Problem 1.2.** Let \( X \) be a (generalised) flag manifold \( G/P \), where \( G \) is a simple complex Lie group and \( P \) is its parabolic subgroup. Find a rank \( \dim \mathbb{C}(X) \) complex bundle \( \xi \to X \) (resp. \( \xi \to X \times X \)) with a smooth generic section which vanishes exactly at the base point (resp. along \( \Delta(X) \)).

In the language of [12], we study the properties \((D_c)\) and \((P_c)\) for flag manifolds. The problem is related to Schubert calculus. The Poincaré dual to the fundamental class of the base point defines a cohomology class, which corresponds to the top Schubert class. Similarly, the class of the diagonal can be thought of as a certain restriction of the torus equivariant top Schubert class (see §3).

A remarkable result for the problem is obtained by Fulton, which works in an algebraic setting:

**Theorem 1.3** ([4, Proposition 7.5]). For a type \( A \) flag manifold \( X = \text{SL}(k + 1)/P \) with any parabolic subgroup \( P \), there is a rank \( \dim \mathbb{C}(X) \) holomorphic bundle \( \xi \to X \times X \) with a holomorphic section \( s \) whose zero locus is \( \Delta(X) \).

For full flag manifolds (when \( P = B \) is the Borel subgroup) of other Lie types, P. Pragacz and the author showed:

**Theorem 1.4** ([6, Theorem 17]). The base point in \( G/B \) (and hence, the diagonal in \( G/B \times G/B \)) is not representable unless \( G \) is of type \( A \) or \( C \).

Indeed, the base point in \( G/B \) is representable if and only if \( G \) is of type \( A \) or \( C \) (see Proposition 3.2). Moreover, we see in Theorem 3.3 that the base point in \( G/P \) is not representable for any proper parabolic subgroup \( P \) when \( G \) is of exceptional type. Thus, the remaining cases are those of flag manifolds of type \( B, C, \) and \( D \). In [12, Theorem 12], non representability of the diagonal is shown for the odd complex quadrics, which are partial flag manifolds of type \( B \). Naturally, we may ask if there is any flag manifold where the base point is representable but the diagonal is not. The main result of this note is to give such an example. Namely, we show in Theorem 1.2 that the complex Lagrangian Grassmannian \( \text{Lag}_\omega(\mathbb{C}^{2k}) \) of maximal isotropic subspaces in the complex symplectic vector space \( \mathbb{C}^{2k} \) for \( k \equiv 2 \mod 4 \) provides such an example. We also see the base point is not representable for many type \( B \) and \( D \) partial flag manifolds (Proposition 4.1 and Remark 4.6).

Throughout this note, \( H^*(X) \) stands for the singular cohomology \( X \) with integer coefficients. Denote by \( M \subset N \) a closed oriented submanifold \( M \) of codimension \( 2m \) embedded in a closed oriented \( 2n \)-manifold \( N \). The cohomology class which is Poincaré dual to the fundamental class of \( M \) is denoted by \([M] \in H^{2m}(N)\).

### 2. Criteria for Representability

We begin with trivial but useful criteria for the representability of submanifolds in general.

**Proposition 2.1.** Let \( N \) be a closed oriented manifold of dimension \( 2n \). Assume that \( \xi \to N \) represents a submanifold \( M \subset N \) of codimension \( 2m \).

1. The top Chern class \( c_m(\xi) \in H^{2m}(N) \) is equal to the class \([M] \).
2. The restriction \( \xi|_M \) is isomorphic to the normal bundle \( \nu(M) \) of \( M \subset N \).

Note that the converse to (1) does not hold; the equality of classes \([c_m(\xi)] = [M] \) does not necessarily mean that we can find a generic section whose zero locus is exactly \( M \). However, when \( M \) is a point, we can pair zeros with opposite orientations of any generic section to cancel out. This means:
**Lemma 2.2.** The base point in a homogeneous $2n$-manifold $N$ is representable if and only if there is a rank $n$ complex bundle whose top Chern class is the generator of $H^{2n}(N)$.

The complex $K$-theory $K^0(N)$ can be identified with the set of stable equivalence classes of vector bundles over $N$, where

$$\xi_1 \sim \xi_2 \iff \xi_1 \oplus \mathbb{C}^l \simeq \xi_2 \oplus \mathbb{C}^l$$

for some $l \geq 0$.

The Chern class $c_m(\xi) \in H^{2m}(N)$ of a bundle $\xi \to N$ depends only on the stable equivalence class of $\xi$ in $K^0(N)$. Therefore, if for a complete set of representatives of $K^0(N)$ there exists no bundle whose $m$-th Chern class is equal to the class $[M] \in H^{2m}(N)$, we can conclude that $M$ is not representable.

### 3. Flag manifolds

From now on, we focus on flag manifolds $G/P$ with the base point taken to be the identity coset $eP$. We assume $G$ is a simple complex Lie group with a fixed Borel subgroup $B$ containing a maximal compact torus $T \subset B$, and its Weyl group is denoted by $W(G)$. A parabolic subgroup $P$ is a subgroup of $G$ containing $B$. Parabolic subgroups are in one-to-one correspondence with the subgraphs of the Dynkin diagram of $G$. Denote by $K$ the maximal compact subgroup of $G$ containing $T$ and by $H$ its subgroup $P \cap K$. We have a diffeomorphism $K/H \simeq G/P$ by the Iwasawa decomposition, and in particular, $K/T \simeq G/B$. We use notations $G/P$ and $K/H$ interchangeably. We also have a diffeomorphism $K/H \simeq \tilde{K}/p^{-1}(H)$, where $p : \tilde{K} \to K$ is the universal covering. So we can assume $K$ is simply-connected if necessary.

The universal flag bundle is denoted by $K/T \xrightarrow{c} BT \to BK$, where $BK$ is the classifying space of $K$. More generally, we have the universal partial flag bundle $K/H \to BH \to BK$. We say a bundle $K/H \to E \to X$ is a flag bundle if it is a pullback of the universal (partial) flag bundle via a map $X \to BK$. The Atiyah-Hirzebruch homomorphism $H^*(BT) \to K^0(K/T)$ is defined by assigning to a character $\lambda \in \text{hom}(T, \mathbb{C}^* \simeq H^2(BT))$ the line bundle $L_\lambda := K \times_T \mathbb{C}_\lambda$ over $K/T$ and extending multiplicatively. Here denoted by $\mathbb{C}_\lambda$ is the space $\mathbb{C}$ acted by $T$ via $\lambda$. This map is known to be surjective when $K$ is simply-connected (cf. [9]).

We first note how the representability of the base point and the diagonal is related to Schubert polynomials. One way to look at Schubert polynomials [1] is that they are elements in $H^*(BT)$ which pullback via $c : K/T \to BT$ to the classes of Schubert varieties in $K/T$. In other words, they are polynomials in the first Chern classes of line bundles on $K/T$ representing the Schubert classes. The top Schubert polynomial represents the class of the base point and it is known by [1] that it “produces” all the other Schubert polynomials when applied the divided difference operators. So in a sense, the top Schubert polynomial carries the information of the whole $H^*(K/T)$. This is why we are interested in representing the base point. A similar story goes for the (Borel) $T$-equivariant cohomology $H^*_T(K/T)$, the top double Schubert polynomial, and the diagonal, as is explained below.

Let $EK$ be the universal $K$-space; that is, $EK$ is contractible on which $K$ acts freely. Consider the following commutative diagram:

$$
\begin{array}{ccc}
K/T \times K/T & \xrightarrow{i_1} & K/T \\
\downarrow & & \downarrow c \\
K/T & \xrightarrow{i_2} & EK \times_T K/T \\
\downarrow p_1 & & \downarrow p_2 \\
K/T & \xrightarrow{c} & BT \\
\downarrow & & \downarrow BK,
\end{array}
$$

where the outer and the lower-right squares are pullbacks, $p_1([x, gT]) = [x]$, and $p_2([x, gT]) = [xg^{-1}]$. Here, $[x, gT] = [xt, tgT] \in EK \times_T K/T$ for $t \in T$ and $BT = EK/T$. We have the sequence of maps $K/T \xrightarrow{i} EK \xrightarrow{\cdot g_1} K/T \xrightarrow{i} BT \times BT$, where $i(g_1 T, g_2 T) = [pt \cdot g_1, g_1 g_2^{-1} T]$.
and $p = (p_1, p_2)$. The class of the equivariant point $EK \times_T eT/T$ pulls back via $i$ to the class of the diagonal in $K/T \times K/T$. The class of $EK \times_T eT/T$ corresponds to the class of the top Schubert variety in the equivariant cohomology, which in turn corresponds to the top double Schubert polynomial.

Let us look at the concrete example of $K = U(k)$. Note that $U(k)/T \simeq SU(k)/T'$ with $T' = T \cap SU(k)$. Although $U(k)$ is not simple, we consider $U(k)$ for convenience. Lascoux and Schützenberger’s top double Schubert polynomial $[10]$

\[ \mathcal{S}_{\omega_0}(x, y) = \prod_{1 \leq i < j \leq k} (x_i - y_j) \]

can be considered as an element in $H^*(BT \times BT) \simeq H^*(BT) \otimes H^*(BT)$ which pulls back via $p$ to the class of the equivariant point $EK \times_T eT/T$ in the equivariant cohomology $H^*(EK \times_T K/T) \simeq H^*_T(K/T)$. This class further pulls back via $i$ to the class in $H^*(K/T \times K/T)$. For a character $\lambda \in H^2(BT)$, let $\hat{L}_\lambda$ be the line bundle $BT \times_T C^\lambda \rightarrow BT$. As $\mathcal{S}_{\omega_0}(x, y)$ is a product of linear terms, we can define the rank $\dim C(K/T) = k(k - 1)/2$ bundle

\[ \xi = \bigoplus_{1 \leq i < j \leq k} \hat{L}_{x_i} \otimes \hat{L}_{y_j} \rightarrow BT \times BT \]

such that its top Chern class is equal to $\mathcal{S}_{\omega_0}(x, y)$. We have $c_k(i^*p^*(\xi)) = [\Delta(K/T)] \in H^{k(k-1)}(K/T \times K/T)$.

Similarly for $K = Sp(k)$, consider the rank $\dim C(K/T) = k^2$ bundle

\[ \xi = \bigoplus_{1 \leq i < j < k} \hat{L}_{x_i} \otimes \hat{L}_{y_j} \bigoplus_{1 \leq i < j < k} \hat{L}_{x_i} \otimes \hat{L}_{y_j} \rightarrow BT \times BT. \]

Then, the top Chern class of $p^*(\xi)$ is the equivariant top Schubert class $[3] [8]$. We have $c_{k^2}(i^*p^*(\xi)) = [\Delta(K/T)] \in H^{2k^2}(K/T \times K/T)$. This means that there is a generic section $s$ of $\xi$ such that $[Z(s)] = [\Delta(K/T)]$ but the equation holds only in the cohomology. We will show that the diagonal of $Sp(k)/T$ is actually representable.

For this, we recall the following slight generalisation of Fulton’s theorem (Theorem 1.3) in the current smooth setting.

**Proposition 3.1** ([6] Theorem 14)). If a point in (resp. the diagonal of) $X$ is representable, then so is any point in (resp. the diagonal of) the total space $E$ of any flag bundle of type $A$:

\[ U(k + 1)/H \hookrightarrow E \rightarrow X, \]

where $H = U(k + 1) \cap P$ for any parabolic $P$ that contains a maximal torus of $U(k + 1)$.

**Proposition 3.2.** The base point in $G/B$ is representable if and only if $G$ is of type $A$ or $C$. The same is true for the diagonal in $G/B \times G/B$.

**Proof.** By Theorems 1.4 and 1.3, we have only to show that the diagonal of the type $C_k$ full flag manifold $F_{1,2,\ldots,k}^k$ is representable. We will actually show that the diagonal of every partial flag manifold $F_{1,2,\ldots,k}^{k'}$ of type $C_k$ corresponding to the parabolic subgroup of type $C_{k'}$ for $k' < k$ is representable.

Recall that $F_{1,2,\ldots,k'}^{k}$ is identified with the set of isotropic flags with respect to a symplectic form in $\mathbb{C}^{2k}$:

\[ F_{1,2,\ldots,k'}^{k} = \{ 0 \subset U_1 \subset U_2 \subset \cdots \subset U_{k'} \subset U_{k'}^1 \subset \cdots \subset U_1^1 \subset \mathbb{C}^{2k} \mid \dim C(U_i) = i \}. \]

Denote the tautological bundle on $F_{1,2,\ldots,k'}^{k}$ corresponding to $U_i$ by $\mathcal{U}_i$. By dropping $U_{k'}$, we obtain a projection $p : F_{1,2,\ldots,k'}^{k} \rightarrow F_{1,2,\ldots,k'-1}^{k}$, which makes $F_{1,2,\ldots,k'}^{k}$ the projectivisation of $\mathcal{U}_{k'-1}/\mathcal{U}_{k'-1}$ over $F_{1,2,\ldots,k'-1}^{k}$. By Proposition 3.2, if the diagonal of $F_{1,2,\ldots,k'-1}^{k}$ is representable, so is that of $F_{1,2,\ldots,k'}^{k}$. This procedure can be iterated to $F_1^{k} = \mathbb{C}P^{2k-1}$, of which the diagonal is representable by Theorem 1.3.

For exceptional Lie groups, the arguments in [6] 56 extend to show:
Theorem 3.3. When $G$ is of exceptional type, the base point in $G/P$ is not representable for any (proper) parabolic subgroup $P$.

Proof. By taking the universal covering, we can assume $K$ is simply-connected. Let $H = P \cap K$. We shall see that there is no bundle $\xi$ with $c_n(\xi) = u_{2n} \in H^{2n}(K/H)$, where $u_{2n}$ is the generator of the top degree cohomology. The flag bundle $H/T \hookrightarrow K/T \to K/H$ induces isomorphisms

$$H^*(K/H) \simeq H^*(K/T)^{W(H)}, \quad K^0(K/H) \simeq K^0(K/T)^{W(H)},$$

where $W(H)$ is the Weyl group of $H$. The universal flag bundle $K/T \hookrightarrow BT \to BK$ induces a map $c^*: H^*(BT) \to H^*(K/T)$, which is compatible with the action of $W(K)$. The Atiyah-Hirzebruch homomorphism $H^*(BT) \to K^0(K/T)$ is also compatible with the action of $W(K)$ and it restricts to a surjection $H^*(BT)^{W(H)} \to K^0(K/T)^{W(H)} \simeq K^0(K/H)$. This asserts that any bundle over $K/H$ stably splits into line bundles when pulled back via $K/T \to K/H$, and hence, its Chern classes are polynomials in the elements of $H^2(K/T) \simeq c^*(H^2(BT))$. Let $\tau_{K/H}$ be the least positive integer such that $\tau_{K/H} \cdot u_{2n}$ is in the image of $c^*: H^*(BT)^{W(H)} \to H^*(K/T)^{W(H)} \simeq H^*(K/H)$.

induced by $c^*: H^*(BT) \to H^*(K/T)$. Consider the flag bundle

$$H/T \hookrightarrow K/T \to K/H.$$

There is a class $v \in H^*(K/T)$ which restricts to the class of the base point in $H^*(H/T)$. Since the class of the base point in $H^{2n}(K/T)$ is the product of the pullback of $u_{2n}$ with $v$, we have $\tau_{K/T} \leq \tau_{H/T} \cdot \tau_{K/H}$.

On the other hand, it is known (see [13])

$$\tau_{G_2/T} = 2, \tau_{E_6/T} = 6, \tau_{E_7/T} = 12, \tau_{E_8/T} = 2880.$$

Parabolic subgroups are in one-to-one correspondence with subgraphs of the Dynkin diagram. So for any (proper) parabolic subgroup of an exceptional Lie group, we have $\tau_{K/T} > \tau_{H/T}$. Therefore, $t_{K/H} > 1$ and $u_{2n}$ cannot be the Chern class of a bundle. \hfill \Box

4. Grassmannian manifolds

An argument similar to the one in the previous section also works for some $G/P$ with $G$ of classical types.

Proposition 4.1. The base point in the complex Lagrangian Grassmannian $\text{Lag}(\mathbb{C}^{2k})$ of maximal isotropic subspaces in the complex quadratic vector space $\mathbb{C}^{2k}$ is not representable when $k > 3$.

Proof. Recall from [11, §1.7] that the connected component of $\text{Lag}(\mathbb{C}^{2k})$ containing the identity is diffeomorphic to the flag manifold $SO(2k)/U(k) \simeq SO(2k-1)/U(k-1)$. If the base point in $\text{Lag}(\mathbb{C}^{2k})$ is representable, so would be in $SO(2k)/T$ by Proposition 3.1 applied to the flag bundle

$$U(k)/T \hookrightarrow SO(2k)/T \to SO(2k)/U(k).$$

However, this contradicts Theorem 3.1 when $k > 3$. Note the low rank equivalences $SO(2) \simeq S^1$, $SO(4) \simeq SU(2) \times SU(2)$, and $SO(6) \simeq SU(4)$. So for $k \leq 3$, the base point in $\text{Lag}(\mathbb{C}^{2k})$ is representable. \hfill \Box

The base point is representable in $G/P$ if the diagonal is representable in $G/P \times G/P$. The following example shows that the converse is not always true.

Theorem 4.2. Let $\text{Lag}_\omega(\mathbb{C}^{2k}) \simeq Sp(k)/U(k)$ be the complex Lagrangian Grassmannian of maximal isotropic subspaces in the complex symplectic vector space $\mathbb{C}^{2k}$ (see [11] §1.7).

1. The base point in $\text{Lag}_\omega(\mathbb{C}^{2k})$ is representable.
2. When $k \equiv 2 \mod 4$, the diagonal in $\text{Lag}_\omega(\mathbb{C}^{2k}) \times \text{Lag}_\omega(\mathbb{C}^{2k})$ is not representable.
Proof. The assertion follows from the standard isomorphism
\[ T(K/H) = \bigoplus_{\beta \in \Pi^+ \setminus \Pi^+_H} L_{\beta}, \]
where \( \Pi^+ \) (resp. \( \Pi^+_H \)) is the set of positive roots of \( K \) (resp. \( H \)). In particular for \( Sp(k)/U(k) \), \( \Pi^+ = \{ 2x_i \mid 1 \leq i \leq k \} \cup \{ x_i \pm x_j \mid 1 \leq i < j \leq k \} \) and \( \Pi^+_H = \{ x_i - x_j \mid 1 \leq i < j \leq k \} \), hence we have
\[ T(Sp(k)/U(k)) \simeq \bigoplus_{i \leq j} L_{2x_i} \bigoplus_{i<j} L_{x_i+x_j}. \]

\textbf{Lemma 4.3.} The tangent bundle of a flag manifold \( K/H \) is
\[ T(K/H) = \bigoplus_{\beta \in \Pi^+ \setminus \Pi^+_H} L_{\beta}, \]
where \( \Pi^+ \) (resp. \( \Pi^+_H \)) is the set of positive roots of \( K \) (resp. \( H \)). In particular for \( Sp(k)/U(k) \), \( \Pi^+ = \{ 2x_i \mid 1 \leq i \leq k \} \cup \{ x_i \pm x_j \mid 1 \leq i < j \leq k \} \) and \( \Pi^+_H = \{ x_i - x_j \mid 1 \leq i < j \leq k \} \), hence we have
\[ T(Sp(k)/U(k)) \simeq \bigoplus_{i \leq j} L_{2x_i} \bigoplus_{i<j} L_{x_i+x_j}. \]

\textbf{Proof.} The assertion follows from the standard isomorphism \( T(K/H) \simeq K \times_H (L(K)/L(H)) \), where \( L(K) \) and \( L(H) \) are Lie algebras of \( K \) and \( H \) respectively. \( \square \)

Let \( 2n = \dim(Sp(k)/U(k)) = k(k+1) \).

\textbf{Lemma 4.4.} Let \( c_i \) (resp. \( q_i \)) be elementary symmetric functions in \( x_j \) (resp. \( x_j^2 \)), where \( H^*(BT) = \mathbb{Z}[x_1, \ldots, x_k] \). Then,
\[ H^*(Sp(k)/U(k)) \simeq \frac{\mathbb{Z}[c_1, c_2, \ldots, c_k]}{(\mathbb{Z}[q_1, q_2, \ldots, q_k])^+}, \]
where \( (\mathbb{Z}[q_1, q_2, \ldots, q_k])^+ \) is the ideal of positive degree polynomials in \( q_j \). In particular,
\[ u_{2n} = \prod_{i} x_i \prod_{i < j} (x_i + x_j) = \prod_{i=1}^{k} c_i \]
\[ u_{2n-2} = \prod_{i=2}^{k} c_i \]
\[ u_1 = c_1 \]
are generators of \( H^{2n}(Sp(k)/U(k)), H^{2n-2}(Sp(k)/U(k)), \) and \( H^2(Sp(k)/U(k)) \) respectively.

\textbf{Proof.} Let \( X = Sp(k)/U(k) \). Since \( H_*(Sp(k)) \) has no torsion, by [3] we have
\[ H^*(X) \simeq \frac{H^*(BT)^W(U(k))}{(H^*(BT)^W(Sp(k)))}, \]
where \( (H^*(BT)^W(Sp(k))) \) is the ideal generated by the positive degree Weyl group invariants. Since \( W(U(k)) \subset H^*(BT) \) is permutation and \( W(Sp(k)) \subset H^*(BT) \) is signed permutation, we have
\[ H^*(X) \simeq \frac{\mathbb{Z}[c_1, c_2, \ldots, c_k]}{(\mathbb{Z}[q_1, q_2, \ldots, q_k])^+}. \]
By the degree reason, it is easy to see \( \prod_{i=1}^{k} c_i \in H^{2n}(X) \), \( \prod_{i=2}^{k} c_i \in H^{2n-2}(X) \), and \( c_1 \in H^2(X) \) are generators. The Euler characteristic \( \chi(X) \) is equal to \( \frac{[W(Sp(k))]}{[W(U(k))]} \) as the cells in the Bruhat decomposition of \( X \) are indexed by the cosets \( W(Sp(k))/W(U(k)) \). Since
\[ \prod_{i}(2x_i) \prod_{i<j} (x_i + x_j) = c_n(TX) = \chi(X)u_{2n} = \frac{[W(Sp(k))]}{[W(U(k))]}u_{2n} = 2^ku_{2n}, \]
we have \( u_{2n} = \prod_{i} x_i \prod_{i<j} (x_i + x_j) \). \( \square \)

\textbf{Lemma 4.5.} When \( k \equiv 2 \mod 4 \), any bundle \( \xi \to Sp(k)/U(k) \) representing the base point in \( Sp(k)/U(k) \) is spin.
Proof. We show \( c_n(\xi) = \pm u_{2n} \) implies \( c_1(\xi) \equiv w_2(\xi) = 0 \mod 2 \), where \( w_2(\xi) \) is the second Steifel-Whiteny class. Since \( H^*(Sp(k)/U(k)) \) has no torsion, \( H^*(Sp(k)/U(k); \mathbb{Z}/2\mathbb{Z}) \simeq H^*(Sp(k)/U(k); \mathbb{Z}) \otimes \mathbb{Z}/2\mathbb{Z} \). We use the same symbol for an integral class and its mod 2 reduction, and the equations below are meant to hold in \( H^*(Sp(k)/U(k); \mathbb{Z}/2\mathbb{Z}) \). By Wu’s formula, we have \( Sq^2(c_i) = c_1 c_i + (2i - 1)(i - 1)c_{i+1} \). Since \( c_i^2 \in \mathbb{Z}/2\mathbb{Z} \), by Lemma 4.4 we have

\[
Sq^2 u_{2n-2} = Sq^2 \left( \prod_{i=2}^k c_i \right) = (k - 1) \prod_{i=1}^k c_i = u_{2n} = u_{2n}.
\]

Set \( c_1(\xi) = au_1 \) and \( c_{n-1}(\xi) = bu_{n-1} \) for some \( a, b \in \mathbb{Z} \). Since \( k \equiv 2 \mod 4 \), we have \( n \equiv 1 \mod 2 \). Again by Wu’s formula, we have

\[
bu_n = Sq^2(c_{n-1}(\xi)) = c_1(\xi)c_{n-1}(\xi) + c_n(\xi) = (ab + 1)u_n.
\]

So \( b(a + 1) \equiv 1 \), and hence, \( a \equiv 0 \mod 2 \). \( \Box \)

Proof of Theorem 4.2. Denote \( Sp(k)/U(k) \) by \( X \).

1. Consider the bundle

\[
\hat{\xi} = \bigoplus_i L_{x_i} \bigoplus_{i<j} L_{x_i + x_j}
\]

over \( Sp(k)/T \). Since \( \hat{\xi} \) is invariant under the action of \( W(U(k)) \), there is a bundle \( \xi \) over \( Sp(k)/U(k) \) which pulls back to \( \hat{\xi} \) via the projection \( Sp(k)/T \to Sp(k)/U(k) \). Then, \( c_n(\xi) = \prod_i x_i \prod_{i<j} (x_i + x_j) = u_{2n} \) is a generator of the top degree cohomology by Lemma 4.4. By Lemma 2.2, the base point is represented by \( \xi \).

2. Assume that \( \xi' \to X \times X \) represents the diagonal \( \Delta(X) \). By Proposition 2.1 (2), the pullback of \( \xi' \) along \( \Delta : X \times X \to X \times X \) is isomorphic to the normal bundle \( \nu(\Delta) \), which is isomorphic to \( TX \). On the other hand, the pullback of \( \xi' \) along the inclusion to each factor \( i_1, i_2 : X \to X \times X \) represents the class of the base point, where \( i_1(x) = (x, pt) \) and \( i_2(x) = (pt, x) \). Since \( i_1^* \otimes i_2^* : H^2(X \times X) \simeq H^2(X) \otimes H^2(X) \), we see

\[
c_1(TX) = c_1(\Delta^*(\xi)) = \Delta^*(c_1(\xi)) = c_1(i_1^*(\xi)) + c_1(i_2^*(\xi)) \equiv 0 \mod 2
\]

by Lemma 4.5. However, \( c_1(TX) = (k + 1)u_1 \) by Lemma 4.3 and it contradicts that \( k \) is even.

\( \Box \)

Remark 4.6. A result of Totaro [14] shows \( t_{\text{Spin}(2k+1)/T} = t_{\text{Spin}(2k+2)/T} = 2^{u(k)} \), where \( u(k) \) is either \( k - \lfloor \log_2((k+1)/2) \rfloor \) or that expression plus one. Since any parabolic subgroup in \( \text{Spin}(2k + 1) \) (resp. \( \text{Spin}(2k + 2) \)) is of type \( B \) (resp. of type \( D \)) or of type \( A \), the base point is not representable in \( \text{Spin}(2k + 1)/P \) and \( \text{Spin}(2k + 2)/P \) for any \( P \) when \( u(k - 1) < u(k) \) by the arguments in the proofs of Theorem 3.3 and Proposition 4.1. Note that \( u(k - 1) = u(k) \) rarely occurs.

For example, let \( Q_l = \{ x \in \mathbb{C}P^{l+1} \mid x_1^2 + \cdots + x_{l+2}^2 = 0 \} \) be the complex quadric. In [12, Theorem 12], it is shown that the diagonal in \( Q_l \) is not representable for any odd \( l \). Since \( Q_l \) is isomorphic to the real oriented Grassmannian (cf. [7, p.280])

\[
\bar{G}r_2(\mathbb{R}^{l+2}) := SO(l + 2)/SO(2) \times SO(l),
\]

the base point in \( Q_l \) is not representable for most \( l \). However, there are some exceptions; e.g. the base point in \( Q_7 \) is representable by \( L_{\omega_4} \oplus L_{\omega_2} \oplus L_{\omega_3} \oplus L_{\omega_4} \), where \( \omega_i \in H^2(BT) \) are the fundamental weights. Note also the low rank equivalences \( Q_1 = \mathbb{C}P^1, Q_2 = \mathbb{C}P^1 \times \mathbb{C}P^1, Q_3 = Sp(2)/U(2) = \text{Lag}_2(\mathbb{C}^4), Q_4 = Gr_2(\mathbb{C}^4) \) and \( Q_6 = SO(8)/U(4) = \text{Lag}(\mathbb{C}^8) \).
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