Gödel, Relativity, and Mind

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Abstract. Gödel’s acquaintance with Einstein led him to discover, by use of novel techniques, an exotic cosmological model which flouted many preconceived notions, such as the role of Mach’s principle in general relativity and the nature of time. Gödel also invoked it in speculations concerning the question of minds.

1. Gödel and Einstein

In his later years, when Gödel was at the Institute for Advanced Study at Princeton, he spent a considerable time in conversations with Einstein, as they accompanied each other on their walks to and from the Institute[*fn1. See Goldstein 2005, Yourgrau 2005.]. Though they had very different personalities, Einstein being effusive, enjoying jokes and personal relationships, whereas Gödel was reclusive, serious, suspicious of human encounters, and very much keeping to himself. Yet the two got on well together, for in addition to having a common mother tongue and fairly similar European backgrounds, they shared the deep common philosophical viewpoint that truth is something objective, in ironic contrast to a common public perception of the greatest achievements of each of these two great thinkers.

In Einstein’s case, this perception arose partly from the name “relativity” that Einstein himself attached to his revolutionary viewpoint concerning the nature of space and time. Whatever Einstein’s viewpoint might have been originally, in putting forward his formulation of special relativity, when he subsequently became convinced of the value of Minkowski’s 4-dimensional space-time viewpoint—a viewpoint which was central to his later development of general relativity—he took the view that this space-time had to have an objective reality. The things that were “relative” about relativity theory, were the individual concepts of “space” and “time” separately, these being described differently by differently moving observers, whereas the full space-time continuum was something whose structure was completely independent of the observers’ states of motion or other personal perspectives.

With regard to the common perception of Gödel’s great contributions to mathematical logic, it is frequently taken that his theorems establish that there are “unprovable” mathematical assertions, that will forever lie beyond the scope of human reason. Accordingly, the truth or falsehood of such a mathematical assertion might be a matter of personal opinion, depending upon what “formal system” is chosen as the basis of that particular person’s mathematical standpoint. On this view, Gödel’s theorems would be taken as having established the subjectivity of mathematics itself, so that a “relativistic” attitude to mathematical truth would appear to be appropriate.

Yet, this was very far from what Gödel actually established. What he really showed was fully consistent with a completely objective attitude to mathematical truth and, indeed, may be regarded as a strong argument in support of such an objective attitude. If we take advantage of Turing’s later
“computational” form of Gödel’s incompleteness results*[^ftnt. See, for example, Penrose 1994, §2.5.], we can paraphrase Gödel’s theorems in the following form: “If \( R \) is a system of computationally checkable proof procedures, where the strict application of the rules of \( R \) is trusted always to result in a true mathematical assertion, then there is a particular mathematical assertion \( G(R) \) whose truth must also be trusted, although it lies beyond the scope of \( R \) itself.” Thus, although the Gödel proposition \( G(R) \), for the system \( R \) is “unprovable” by means of the accepted proof procedures of this \( R \), it is nevertheless perceived to be actually true on the basis of the same underlying mathematical insights that tell us to trust the results that are “proved” using the procedures \( R \). Thus, if it is our belief that it is trustworthy actually to use \( R \) as a valid proof procedure, then Gödel’s theorems enable us to transcend \( R \) and accept the validity also of \( G(R) \), despite the fact that \( G(R) \) is now seen to be not provable by the means of proof that \( R \) actually provides us with.

In particular, we can see how this applies in the case of Gödel’s second incompleteness theorem. This theorem interprets an particular formulation of \( G(R) \) as asserting the consistency of the formal mathematical system from which the rules \( R \) are obtained. A trust in the actual consistency of the system is equivalent to a belief in the truth of the mathematical proposition \( G(R) \), since this is what \( G(R) \) now actually asserts. Yet, as Gödel shows, this \( G(R) \) is actually beyond the scope of \( R \). Nevertheless, our trust in the validity of the means of proof that \( R \) provides us with necessitates our trust in this actual consistency. For if the system were actually inconsistent, then it would enable us to “prove” any proposition that can be stated within the system, such as “2=3”, for example. If we are prepared to trust the use this formal system as a means of valid mathematical proof, then we must also accept the truth of this \( G(R) \).

Gödel interprets this as showing that our means of ascertaining mathematical truth cannot be encapsulated in any particular formal mathematical system that we trust, since this very trust enables us to transcend that particular system. Our mathematical beliefs are, therefore, not simply arbitrary things which are defined in terms of the following of some chosen set of rules \( R \). Our access to mathematical truth is not such an arbitrary matter as this, as Gödel shows. It is not “up to us” whether to accept \( G(R) \), if we have chosen to adopt \( R \). For in those circumstances, we must accept \( G(R) \) also.

Of course, Gödel’s theorems do not force us to accept the objectivity of the truth or falsehood of every mathematical assertion, since a great many such assertions will not have the form of a “\( G(R) \)”. Yet Gödel’s incompleteness theorems, rather than telling against such a “Platonic” objectivist stance, are broadly in support of it. Indeed, Gödel’s actual beliefs were strongly in favour of such Platonism. His world of mathematics—being a Platonic world—is “real” rather than merely the product of our minds, and it has a reality that is comparable with that of Einstein’s objective space-time.

2. Gödel’s cosmological models

It appears that Gödel’s many discussions with Einstein led him to become seriously involved with general relativity. Whatever it was that inspired him to venture into the area of cosmology, and to introduce into this subject a highly original collection of innovative ideas, he did this with extraordinary skill, especially remarkable for one whose previous expertise had been in the area of mathematical logic, rather than in the very different area of differential geometry and its application to the, at that time, still youthful general theory of general relativity. Presumably, it was from Einstein that he learnt the basic principles of the subject, but he developed for himself his own highly original approach to a relativistic cosmology. He employed “moving frame” techniques, apparently developed from what Cartan had used in differential geometry but these methods had not at the time been in common use in General relativity. Gödel’s techniques indeed provided the ideal tools for developing the strange ideas that was to introduce into cosmology.
What were these strange and revolutionary ideas? Basically there were two of them, not immediately obviously related to each other. One of these had to do with “Mach’s principle”, which had been one of the driving ideas that set Einstein on his course to the discovery of general relativity. According to Mach’s principle, the local dynamics—and, in particular, the local inertial frames—should be determined by the mass distribution of all the matter of the universe. Most specifically, the local non-rotating frames, as defined by the absence of centrifugal and Coriolis forces (as could be illustrated in terms of Newton’s “bucket experiment” or a Foucault pendulum) are the same as those in which the distant stars or galaxies appear not to rotate. In the cosmological model that Gödel proposed, however, this “Machian” condition is explicitly violated, and the matter substratum (as determined by the galaxies), with its local notion of a non-rotating frame, is itself everywhere rotating with respect to the global matter distribution.

More alarming is the presence, in Gödel’s first rotating cosmological model [Gödel 1949a.], of closed timelike curves. We recall that a timelike curve in space-time, in relativity theory, represents the possible history of an ordinary (classical) massive particle—or the history of a physical observer, if we are thinking in terms of macroscopic physics. Accordingly, it is normally taken, in the relativity community, that solutions of Einstein’s equations possessing closed timelike curves are unphysical, as they allow, in principle, that a space traveller can make a journey—following such a path—into his/her own past, and thereby influence that past in a way that is inconsistent with the traveller’s own former experiences. This problem is sometimes referred to, in colloquial terms, as the “grandfather paradox”, as it is fancifully imagined that such a space traveller would have the potential to murder his paternal grandfather before his own father had been conceived, thereby removing the very possibility of his own existence!

Gödel was well aware of this kind of problem, but he appears not to have regarded it as rendering his cosmological model physically unrealistic, for he actually performed a calculation estimating the amount of fuel that the undertaking of such a journey would require (noting that the closed timelike curves in this cosmology are very far from being geodesics [Gödel 1949b, footnote.], their curvatures providing a measure of the accelerations—and therefore of rocket action—that would be involved). He obtained a value for the amount of fuel required that was so absurdly enormous that he felt that he was able to reject the possibility of any space traveller ever making such a trip.

However, it seems to me that such considerations do not really come close to removing the fundamental difficulties that the presence of closed timelike curves can give rise to. Very serious consistency problems arise merely from the possibility of an observer sending a reliable signal into the past, rather than actually travelling into the past. For example, we can consider a different form of the grandfather paradox, which I shall refer to as the “dictator paradox”. Here we imagine that an observer has experienced, and survived, the terrible rule of a particularly vicious dictator who has caused the deaths of millions of people, whereupon this observer sends a trustworthy signal into the past, to a time before the dictator had acquired any power, to someone in a position to put an end to that potential dictator at an early stage, thereby eliminating the situation which elicited the sending of the signal in the first place! This sort of paradox is well known to writers and consumers of science fiction. In any case, quite apart from fanciful considerations of this explicit kind, there are serious considerations of consistency of the solutions of hyperbolic partial differential equations (of the kind that refer to the standard equations of physics) in circumstances where there are closed timelike curves.

Two further points of relevance should be mentioned here. The first is that the significance of Gödel’s cosmological model with closed timelike curves was not so much the presence of such curves, but the fact that these curves cannot be eliminated by taking a covering space of the model. It is very easy to produce models of space-times with closed timelike curves which are removed by passing to a covering space. The simplest example is obtained by taking the standard flat Minkowski space of special relativity, with standard Minkowski coordinates \((t,x,y,z)\), with \(t\) the time coordinate, and to
“roll it up” in the time dimension by identifying each point \((t,x,y,z)\) with the corresponding points \((t+nT,x,y,z)\), for every integer \(n\), where \(T\) is a fixed positive number. Clearly the curves of constant \(x, y, z\) are closed timelike curves, but the (universal) covering space, which removes the identifications, is simply Minkowski space, which is free of closed timelike curves. Passing to the covering space here assigns a different “reality” to each “experienced event” for an observer who follows a closed timelike curve, round and round. However, such a re-interpretation will not work for Gödel’s original cosmological model, because it is simply-connected. It thus has no non-trivial covering space, and the closed timelike curves cannot be removed (or re-interpreted) in this way. It is this which is remarkable about Gödel’s model.

The second point to mention is that Gödel’s original cosmological model is static and so cannot represent the expanding universe of actual observation. To deal with this problem, Gödel later produced an expanding rotating model. However, this later model does not actually possess any closed timelike curves, so the conundrums raised in the preceding paragraphs do not arise. But conflict with actual observation remains, as the rotation that is characteristic of all the Gödel models is not actually observed.

3. Gödel, time, and the mind

Despite what has sometimes been suggested, Gödel’s cosmological models were well appreciated by the relativity community, for example, Ozváth and Schücking 1962, Hawking and Ellis 1973, Kramer et al. 1980. In particular, Hawking explicitly cited Gödel’s example, when he addressed the possibility that the presence of closed timelike curves might supply an escape route from the singularity theorems, the conclusion being that closed timelike curves do not appear to supply such an escape route. The general view appears to be that it is not unreasonable to dismiss such closed timelike curves as unphysical, but there are some dissenting opinions. Gödel’s own view seems to have been to take these closed timelike curves very seriously. He appears to have regarded his work in this area as providing a demonstration that, according to Einstein’s general relativity, the passage of time is some kind of illusion. He also appears to have regarded this as providing some kind of link with his attitude to the relation between the mind and the physical universe.

We have seen in §1 that his incompleteness theorems appear to tell us that our access to mathematical truth cannot be described in terms of purely computational processes. At least this seems to have been Gödel’s own clear viewpoint, though he was cautious about making any definitive claim that his incompleteness theorems actually established this in a rigorous way. He apparently accepted the conventional view that the action of the physical brain must, on the other hand, be something that could in principle be described in entirely computational terms. Accordingly, he believed that the action of the mind must be something beyond the physical brain. In accordance with such a view, it would be allowable for there to be “spirits” that were beyond physical description, and therefore not subject to the normal notions of time. The presence of closed timelike curves in his original cosmological model was somehow to be tied in with the (potential?) presence of such “spirits”, but my own understanding of what Gödel might have meant by ideas of this kind is severely limited. It would therefore be inappropriate for me to attempt to venture any further opinions on this matter.

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