Two torsion, the endomorphism field, and the finitude of endomorphism algebras for a class of abelian varieties

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Abstract

A conjecture attributed to Robert Coleman states that there should only be finitely many isomorphism classes for endomorphism algebras of abelian varieties of fixed dimension over a fixed number field.

In this paper, we establish new cases of this conjecture, given conditions on the 2-torsion field. In fact, for the abelian varieties considered, we provide an explicit list of possible endomorphism algebras. This list is then refined by studying the endomorphism field. In particular, we find conditions which guarantee it is contained in the 2-torsion field.

1 Introduction

A conjecture attributed to Robert Coleman states that for any positive integer $g \geq 1$ there should only be finitely many isomorphism classes for $\text{End}^0(A)$ among all abelian varieties $A/\mathbb{Q}$ of dimension $g$. Here $\text{End}^0(A) = \text{End}(A) \otimes \mathbb{Q}$ and $\text{End}(A)$ denotes the ring of all endomorphisms of $A$ defined over a fixed algebraic closure of the ground field.

This conjecture is currently only known to hold in full generality for $g = 1$. For $g = 2$, recent work of Fité and Guitart [FG20] shows the conjecture holds when $A$ is geometrically split. However, the case of geometrically simple abelian surfaces over $\mathbb{Q}$ remains completely open.

The theorem below proves a particular case of this conjecture.

Theorem 1.1 (Corollary 2.6). Let $A/\mathbb{Q}$ be an abelian variety of dimension $g \geq 1$ with $p = 2g + 1$ prime. Suppose $\text{Gal}(\mathbb{Q}(A[2])/\mathbb{Q}) \cong C_p$. Then either

- $\text{End}^0(A)$ is a proper subfield of $\mathbb{Q}(\zeta_p)$; or
- $p \in \{7, 11, 19, 43, 67, 163\}$ and $A$ is isogenous over $\bar{\mathbb{Q}}$ to the power of an elliptic curve with complex multiplication by $\mathbb{Q}(\sqrt{-p})$.

In particular there are only finitely many possibilities for $\text{End}^0(A)$.

\footnote{Here we have presented a weaker version than usually stated. For the full-strength version see, for example, [BFGKR06] pg 384.}
Let $K$ be a number field. In the proof of the above theorem we study the minimal extension $L/K$ over which all endomorphisms of an abelian variety $A/K$ are defined. Following Guralnick and Kedlaya [GK17], we call $L$ the endomorphism field.

By exploiting the connection of $L/K$ to Sato-Tate groups, strong restrictions on the degree $[L:K]$ (for $g$ fixed, $A$ and $K$ varying) have been found [FKRS12, GK17, FKS21, top of page 3]. See also [Goo21, Theorem 2.5] for a partial converse to the main theorem of [GK17].

The results we present on $L/K$ here are in the same vein as those of Silverberg, who showed $L$ is contained in $K(A[2])$ for $n \geq 3$ [Sil92, Theorem 2.4]. The elliptic curve $y^2 = x^3 - x$, which has CM by $\mathbb{Z}[i]$, shows this result is not true in general for $n = 2$. However, we have the following:

**Theorem 1.2** (Theorem 2.1). Suppose $E = \text{End}^0(A)$ is a (finite) Galois extension of $\mathbb{Q}$ and $L \not\subseteq K(A[2])$. The following hold:

- $\text{Gal}(E/\mathbb{Q})$ has a non-trivial normal elementary abelian 2-subgroup;
- if $\text{End}(A)$ is 2-maximal in $E$, then 2 is wildly ramified in $E/\mathbb{Q}$.

In particular, if $E/\mathbb{Q}$ is Galois, $\text{End}(A)$ is a 2-maximal order and 2 is not wildly ramified, then $L \subseteq K(A[2])$.

An order $O \subseteq \text{End}^0(A)$ is said to be 2-maximal if for any other order $O' \subseteq \text{End}^0(A)$ satisfying $O \otimes \mathbb{Z}_2 \subseteq O' \otimes \mathbb{Z}_2$ we have $O \otimes \mathbb{Z}_2 = O' \otimes \mathbb{Z}_2$.

**Example 1.3.** The condition that $\text{End}(A)$ is 2-maximal cannot be removed. Indeed, the elliptic curve $y^2 = (x + 2)(x^2 - 2x - 11)$ has CM by $\mathbb{Z}[\sqrt{-3}]$ and its 2-torsion field is $\mathbb{Q}(\sqrt{3})$, see [Sil94, Appendix A §3].

**Remark 1.4.** By Proposition 2.2 of [Goo21] $\text{End}(A)$ is a 2-maximal order and 2 is totally inert in $E$ when the absolute Galois group $G_L$ acts irreducibly on $A[2]$. For a more explicit criterion, see [Goo21, Theorem 2.9].

Theorem 1.2 along with Remark 1.4 facilitates finding the endomorphism field in explicit examples. However, our main motivation comes from questions of the form “How does $\text{Gal}(K(A[2])/K)$ influence $\text{End}(A) = \text{End}_L(A)$?” We note this question may be made explicit for hyperelliptic jacobians. Indeed, if $J/K$ is the jacobian of a hyperelliptic curve determined by $y^2 = f(x)$, $f \in K[x]$, then $\text{Gal}(f) := \text{Gal}(K(f)/K) \cong \text{Gal}(K(J[2])/K)$, see for example [Zar01, Theorem 4.1]. Let us list a few further corollaries of Theorems 1.1 and 1.2 in this direction.

**Corollary 1.5** (Corollary 2.7). Let $C: y^2 = f(x)$ be an elliptic curve defined over a number field with a real embedding. If $\text{Gal}(f) \cong C_3$, then $C$ does not have complex multiplication.
We let \( \text{End}_F(A) \) denote the endomorphisms of \( A/K \) defined over a field \( F/K \) and write \( \text{End}_F^0(A) = \text{End}_F(A) \otimes \mathbb{Q} \).

**Corollary 1.6** (= Corollary 2.8). Let \( A/\mathbb{Q} \) be an abelian surface. Suppose \( \text{Gal}(\mathbb{Q}(A[2])/\mathbb{Q}) \cong C_5 \). Then either \( \text{End}(A) = \mathbb{Z} \) or \( \text{End}_Q^0(A) = \mathbb{Q}(\sqrt{5}) \).

**Example 1.7.** It is well known the first case occurs and examples are easily found. The Jacobian \( J \) of the hyperelliptic curve \( y^2 = x^{(x^5 - 4x^4 + 2x^3 + 5x^2 - 2x - 1)} \) has \( \text{End}_Q(J) = \text{End}(J) \cong \mathbb{Z} \left[ \frac{1+\sqrt{5}}{2} \right] \) by [Wil00, Proposition 1] and \( x^5 - 4x^4 + 2x^3 + 5x^2 - 2x - 1 \) has Galois group isomorphic to \( C_5 \). This shows the second case may also occur.

Combining Theorems 1.2, 2.12 with Theorems 1.5 and 2.9 of [Goo21] along with the main theorem of [Zar00] (for the final statement). We obtain the following:

**Theorem 1.8.** Let \( f(x) \in K[x] \) be a polynomial of degree 5.

1. Suppose \( \text{Gal}(f) \cong F_5 \). The following hold:
   
   (i) If \( \text{End}_Q^0(J) \) is a (real) quadratic field, then \( L = K(\sqrt{\Delta_f}) \), the unique degree 2 extension of \( K \) contained in \( K(f) \).
   
   (ii) If \( \text{End}_Q^0(J) \cong E \) is a degree 4 CM field, then it is cyclic and \( L \) is the unique degree 4 extension of \( K \) contained in \( K(f) \). Moreover, \( L = EK \).

2. Suppose \( \text{Gal}(f) \cong D_5 \). If \( \text{End}_Q^0(J) \) is a degree 4 CM field, then \( L \) is the unique degree 2 extension of \( K \) contained in \( K(f) \).

3. Suppose \( \text{Gal}(f) \cong C_5 \). Then \( L = K \). Furthermore, if \( \text{End}_Q^0(J) \) is a degree 4 CM field, then \( K \) contains a real quadratic field with discriminant congruent to 5 modulo 8.

In particular, if \( K = \mathbb{Q} \), \( J \) has CM and \( \text{Gal}(f) \) contains an element of order 5, then \( \text{Gal}(f) \cong F_5 \).

Motivated by a question of Ciaran Schembri and John Voight, linked to their work on an analogue of Mazur’s Theorem for abelian surfaces defined over \( \mathbb{Q} \) with QM (see §3 for a definition), we prove analogues of Theorem 1.2 for certain quaternion algebras, see Theorems 3.3 and 3.5

**Notation.** We denote by \( C_n, D_n, F_n \) the cyclic group of order \( n \), the dihedral group of order \( 2n \), and the Frobenius group isomorphic to \( F_n \times F_n \) (for \( n \) a prime power) respectively.

A primitive \( n \)-th root of unity is denoted by \( \zeta_n \). For a polynomial \( f \in K[x] \), we denote its splitting field by \( K(f) \), and write \( \text{Gal}(f) \) for \( \text{Gal}(K(f)/K) \).
Throughout $A$ will denote an abelian variety of dimension $g$ defined over a number field $K$ (often $\mathbb{Q}$). Moreover, we denote by $L$ the smallest extension of $K$ over which all the endomorphisms of $A$ are defined.

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## 2 Number fields

Silverberg has shown the endomorphism field $L/K$ of an abelian variety $A/K$ is contained in the $n$-torsion field for every $n \geq 3$ [Sil92, Theorem 2.4]. The case $n = 4$ implies $\operatorname{Gal}(L/L \cap K(A[2]))$ is an elementary abelian 2-group, since $\operatorname{Gal}(K(A[4])/K(A[2]))$ is too, for details see [Goo21, Proposition 3.9].

Below we provide conditions on $\operatorname{End}(A)$ which guarantee $L \subseteq K(A[2])$.

**Theorem 2.1.** Suppose $E = \operatorname{End}^0(A)$ is a (finite) Galois extension of $\mathbb{Q}$ and $L \nsubseteq K(A[2])$. The following hold:

- $\operatorname{Gal}(E/\mathbb{Q})$ has a non-trivial normal elementary abelian 2-subgroup;
- if $\operatorname{End}(A)$ is 2-maximal in $E$, then 2 is wildly ramified in $E/\mathbb{Q}$.

In particular, if $E/\mathbb{Q}$ is Galois, $\operatorname{End}(A)$ is a 2-maximal order and 2 is not wildly ramified, then $L \subseteq K(A[2])$.

**Proof.** If $\operatorname{Gal}(L/K)$ acts faithfully on $\operatorname{End}(A) \otimes \mathbb{Z}/2\mathbb{Z}$, then we may recuperate Lemma 2.1 in [Sil92] with $N = 2$, $\Lambda = \operatorname{End}(A)$ and $G$ the image of $\operatorname{Gal}(L/K)$ in $\operatorname{Aut}(A)$. Following through the proofs of Propositions 2.2, 2.3 and Theorem 2.4 in [Sil92] shows $L \subseteq K(A[2])$. Hence we may suppose the action of $\operatorname{Gal}(L/K)$ on $\operatorname{End}(A) \otimes \mathbb{Z}/2\mathbb{Z}$ has a non-trivial element $\gamma$ in its kernel.

Since $\operatorname{Gal}(L/K)$ acts on $\operatorname{End}(A)$ by $\mathbb{Z}$-linear automorphisms of $\operatorname{End}(A)$, we may view its image as a subgroup of $\operatorname{Gal}(E/\mathbb{Q})$. Identify $\gamma$ with its image in $\operatorname{Gal}(E/\mathbb{Q})$. Let $d$ be the rank of $\operatorname{End}(A)$ as a free $\mathbb{Z}$-module and view $\operatorname{Gal}(E/\mathbb{Q})$ as a (finite) subgroup of $\operatorname{GL}_d(\mathbb{Z})$ via its action on $\operatorname{End}(A)$. The reduction map $\operatorname{GL}_d(\mathbb{Z}) \to \operatorname{GL}_d(\mathbb{Z}/4\mathbb{Z})$ is injective on finite groups and the kernel of $\operatorname{GL}_d(\mathbb{Z}/4\mathbb{Z}) \to \operatorname{GL}_d(\mathbb{Z}/2\mathbb{Z})$ is an elementary abelian 2-group (for more details see [Goo21, Prop. 3.9]). Thus the kernel of the map...
Gal(E/Q) → Aut(End(A) ⊗ ℤ/2ℤ) ≅ GL_d(ℤ/2ℤ) is an elementary abelian 2-group which is non-trivial since it contains γ.

Assume now End(A) is a 2-maximal order in E. As γ acts trivially on the order End(A) modulo 2, it also acts trivially on End(A) modulo every prime above 2 in E. Since End(A) is 2-maximal, we find γ belongs to the inertia group I_p for every prime p above 2 in E. By the above γ has order two, and hence 2 is wildly ramified in E/Q.

Remark 2.2. The condition that End(A) is 2-maximal cannot be removed. Indeed, the elliptic curve y^2 = x^3 − 15x − 22 has CM by ℤ[√−3] and its 2-torsion field is Q(√3), see [Si84, Appendix A §3].

We now prove finitude results for endomorphism algebras of certain abelian varieties and apply the above theorem to restrict the list of possible endomorphism algebras.

Proposition 2.3. Let A/K be an abelian variety of dimension g ≥ 1 with p = 2g + 1 prime. Suppose q is a prime of bad reduction for A and there is an element of order p in the image of the inertia group I_q on A[ℓ] for some q | ℓ. Then either

- p does not divide [L : K] and End^0(A) is a subfield of Q(ζ_p); or
- p divides [L : K], g ≥ 3 and A is isogenous over ℚ to the power of an absolutely simple abelian variety with complex multiplication by a proper subfield of Q(ζ_p).

In particular there are only finitely many possibilities for End^0(A).

Proof. Suppose p divides [L : Q]. Then by Theorem 2.5 of [Go21], g ≥ 3 and A is isogenous over Q to the power of an absolutely simple abelian variety with complex multiplication by a proper subfield F of Q(ζ_p). Hence we may suppose p does not divide [L : K].

The criterion of Néron-Ogg-Shafarevich implies that the image of the inertia group I_q on the Tate module T_ℓ(A) for any prime ℓ, not divisible by q, contains an element τ of order p. Furthermore, as the trace of τ is an integer [St68, Thm. 2] its eigenvalues are the primitive p-th roots of unity.

By Dirichlet’s theorem on arithmetic progressions, we can find a prime ℓ, not divisible by q, which is a primitive root modulo p. The reduction of τ modulo ℓ lands in Gal(K(A[ℓ])/K) and has order p. This allows us to apply [Go21, Theorem 2.9] and deduce E := End^0(A) is a field.

As p does not divide [L : K], the element τ lies in the image of G_L. The decomposition E ⊗ Q_ℓ = ∏_λ E_λ induces a decomposition T_ℓ(A) = ∏_λ T_λ(A) giving representations G_L → GL_n(E_λ) where n = [2g/(deg)] see §2 of [Rib76] for further details. Let τ_λ ∈ GL_n(E_λ) be the projection of τ onto T_λ(A). By the above, the eigenvalues of τ_λ are distinct primitive p-th roots.
of unity. Taking the trace of $\tau_\lambda$ we deduce $E$ contains a subfield of $\mathbb{Q}(\zeta_p)$ of degree $[E : \mathbb{Q}]$. In other words, $E \subseteq \mathbb{Q}(\zeta_p)$.

By finding a condition on the ray class groups of primes above 2 in $K$, we deduce an explicit version of the above:

**Theorem 2.4.** Let $A/K$ be an abelian variety of dimension $g \geq 1$ with $p = 2g + 1$ prime. Suppose $\text{Gal}(K(A[2])/K) \cong C_p$ and $p$ divides neither the class number of $K$, nor the multiplicative order of the residue field of any prime above 2. Then either

- $p$ does not divide $[L : K]$ and $\text{End}^0(A)$ is a subfield of $\mathbb{Q}(\zeta_p)$; or
- $p$ divides $[L : K]$, $g \geq 3$ and $A$ is isogenous over $\mathbb{Q}$ to the power of an absolutely simple abelian variety with complex multiplication by a proper subfield of $\mathbb{Q}(\zeta_p)$. 

In particular there are only finitely many possibilities for $\text{End}^0(A)$.

**Proof.** Let $F/K$ be an odd degree extension unramified outside of 2. Then by class field theory, any prime dividing $[F : K]$ divides either the class number of $K$, or the multiplicative order of the residue field of a prime above 2 in $K$.

It follows that $K(A[2])/K$ is ramified at some prime $q \nmid 2$. This allows us to apply the above proposition and conclude.

The following lemma is well-known.

**Lemma 2.5.** Let $A/K$ be an absolutely simple abelian variety with CM by a Galois extension $E/\mathbb{Q}$. Then $L = E^*K$ and $E \supseteq E^*$, where $E^*$ is the reflex field of $E$. Furthermore if $E/\mathbb{Q}$ is abelian, then $E = E^*$.

**Proof.** As $E/\mathbb{Q}$ is Galois, the reflex field $E^*$ is a subfield of $E$ [Shi98, Prop. 28, pg 62]. Moreover, since $A$ is absolutely simple its CM type is primitive. Proposition 30 on page 65 of [Shi98] applies to tell us the endomorphism field $L$ equals $E^*K$. Finally, if $E/\mathbb{Q}$ is an abelian extension, then $E = E^*$ by Example (1) on page 63 of [Shi98].

**Corollary 2.6.** Let $A/\mathbb{Q}$ be an abelian variety of dimension $g \geq 1$ with $p = 2g + 1$ prime. Suppose $\text{Gal}(\mathbb{Q}(A[2])/\mathbb{Q}) \cong C_p$. Then either

- $\text{End}^0(A)$ is a proper subfield of $\mathbb{Q}(\zeta_p)$; or
- $p \in \{7, 11, 19, 43, 67, 163\}$ and $A$ is isogenous over $\mathbb{Q}$ to the power of an elliptic curve with complex multiplication by $\mathbb{Q}(\sqrt{-p})$.

In particular there are only finitely many possibilities for $\text{End}^0(A)$. 

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Proof. Clearly, we may apply Theorem 2.4. Suppose first $p$ divides $[L : \mathbb{Q}]$. Then $g \geq 3$ and $A$ is isogenous over $\overline{\mathbb{Q}}$ to the power of an absolutely simple abelian variety $B$ with complex multiplication by a proper subfield $F$ of $\mathbb{Q}(\zeta_p)$. The above lemma applies to $B$ and shows $L$, the endomorphism field of $A$, contains $F$.

Proposition 3.9 of [Goo21] (recalled at the beginning of this section) shows $\text{Gal}(L/\mathbb{Q})$ has order $2^m p$ for some $m$. Moreover, as the quotient $\text{Gal}(\mathbb{Q}(A[2])/\mathbb{Q}) \cong \text{Gal}(L/\mathbb{Q})/C_2^m$ has odd order $p$, any element of even order in $\text{Gal}(L/\mathbb{Q})$ has either order two or $2p$.

Since $p$ does not divide $[F : \mathbb{Q}]$, the cyclic quotient $\text{Gal}(F/\mathbb{Q})$ of $\text{Gal}(L/\mathbb{Q})$ has order two (it is not trivial since $F$ is a CM field). Using $\text{Gal}(F/\mathbb{Q})$ only ramified at $p$, we find $F = \mathbb{Q}(\sqrt{-p})$ where $p \equiv 3 \mod 4$. In particular $g$ is odd.

We show $F$ has class number one, which by the Baker–Heegner–Stark Theorem will conclude the proof in this case. As $F/\mathbb{Q}$ is a quadratic extension, $B$ is an elliptic curve. Thus by CM theory the compositium of $F$ and the field of definition of $B$ contains $H$ the Hilbert class field of $F$. In particular, $L$ contains $H$, so $[H : F]$ divides $2^m p$ for some $m$. On the other hand, [FGIS] Corollary 2.16 tells us every element of $\text{Gal}(H/F)$ has order dividing $g$. As $g$ is odd and less than $p$, we find $\text{Gal}(H/F) = 1$ as claimed.

We now suppose $p$ does not divide $[L : \mathbb{Q}]$ and show $E \neq \mathbb{Q}(\zeta_p)$. Suppose we had equality, then applying Lemma 2.5 we find $L = \mathbb{Q}(\zeta_p)$. If $p = 3$, then we apply Theorem 2.9 of [Goo21] and Theorem 2.1 to obtain a contradiction. Else, $p \geq 5$ and Proposition 3.9 of [Goo21] provides us with a contradiction, since $\text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ is not an elementary abelian 2-group.

Let us record some easy corollaries of Theorem 2.9 of [Goo21], Theorems 2.1 2.6 and Lemma 2.5.

**Corollary 2.7.** Let $A : y^2 = f(x)$ be an elliptic curve defined over a number field with a real embedding. If $\text{Gal}(f) \cong C_3$, then $A$ does not have complex multiplication.

**Corollary 2.8.** Let $A/\mathbb{Q}$ be an abelian surface. Suppose $\text{Gal}(\mathbb{Q}(A[2])/\mathbb{Q}) \cong C_5$. Then either $\text{End}(A) = \mathbb{Z}$ or $\text{End}^0(\mathbb{Q})(A) = \text{End}^0(A) = \mathbb{Q}(\sqrt{5})$.

**Remark 2.9.** The above result is the best possible. Indeed, it is well known the first case occurs and examples are easily found. The jacobian $J$ of the hyperelliptic curve $y^2 = x(x^5 - 4x^4 + 2x^3 + 5x^2 - 2x - 1)$ has $\text{End}(J) = \text{End}(J) \cong \mathbb{Z} \left[\frac{1 + \sqrt{5}}{2}\right]$ by [W300] Proposition 1 and $x^5 - 4x^4 + 2x^3 + 5x^2 - 2x - 1$ has Galois group isomorphic to $C_5$. This provides an example for the second case.

We present a variant of Theorem 2.4 for abelian varieties over imaginary quadratic fields.
Theorem 2.10. Let $g \geq 2$ be an integer and suppose $p = 2g + 1$ is prime. Let $K$ be an imaginary quadratic field of class number coprime to $p$. Let $A/K$ be an abelian variety of dimension $g$. Suppose $\text{Gal}(K(A[2])/K) \cong C_p$. Then either
\begin{itemize}
  \item $\text{End}^0(A)$ is a proper subfield of $\mathbb{Q}(\zeta_p)$; or
  \item $A$ is isogenous over $\mathbb{Q}$ to the power of an elliptic curve with complex multiplication by $\mathbb{Q}(\sqrt{-p})$ where $p \equiv 3 \pmod{4}$ is such that the Hilbert class field of $\mathbb{Q}(\sqrt{-p})$ is contained in $K(\sqrt{-p})$.
\end{itemize}

In particular there are only finitely many possibilities for $\text{End}^0(A)$.

Proof. Clearly, we may apply Theorem 2.4. Suppose first $p$ divides $[L : K]$. Then $A$ is isogenous over $K$ to the power of an absolutely simple abelian variety $B$ with complex multiplication by a proper subfield $F$ of $\mathbb{Q}(\zeta_p)$. The above lemma applies to $B$ and shows $L$, the endomorphism field of $A$, contains $F$.

Proposition 3.9 of [Goo21] (recalled at the beginning of this section) shows $\text{Gal}(L/K)$ has order $2^mp$ for some $m$. Moreover, as the quotient $\text{Gal}(K(A[2])/K) \cong \text{Gal}(L/K)/C_m^p$ has odd order $p$, any element of even order in $\text{Gal}(L/K)$ has either order two or $2p$. Since $p$ does not divide $[F : \mathbb{Q}]$, the cyclic quotient $\text{Gal}(FK/K)$ of $\text{Gal}(L/K)$ has order dividing two.

In turn we deduce $[F : \mathbb{Q}]$ divides 4 and is equal to 4 only if $F \supseteq K$. Being a CM field, $[F : \mathbb{Q}] = 4$ would imply $F$ also contains a real quadratic field, whence $\text{Gal}(F/\mathbb{Q}) \cong C_2 \times C_2$ giving a contradiction. Thus $[F : \mathbb{Q}] = 2$.

Using again that $F$ is a CM field only ramified at $p$, we deduce $F = \mathbb{Q}(\sqrt{-p})$ where $p \equiv 3 \pmod{4}$.

Let $H$ be the Hilbert class field of $F$. By [FG18, Theorem 2.14] the group $\text{Gal}(HK/FK)$ has order dividing $g$, thus arguing as in the proof of Theorem 2.6, we deduce $[HK : FK] = 1$. In other words, $H \subseteq FK = K(\sqrt{-p})$.

We now suppose $p$ does not divide $[L : K]$ and show $E \neq \mathbb{Q}(\zeta_p)$. Suppose we had equality, then by Lemma 2.5 $L = K(\zeta_p)$. Proposition 3.9 of [Goo21] then implies $\text{Gal}(L/K) = \text{Gal}(K(\zeta_p)/K) \cong \text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}(\zeta_p) \cap K)$ has order at most 2. Hence $[\mathbb{Q}(\zeta_p) : \mathbb{Q}]$ divides 4, being equal to 4 only if $K$ is contained in $\mathbb{Q}(\zeta_p)$. But $[\mathbb{Q}(\zeta_p) : \mathbb{Q}] = 4$ implies $p = 5$ and $\mathbb{Q}(\zeta_5)$ does not have an imaginary subfield. Whence $[\mathbb{Q}(\zeta_5) : \mathbb{Q}] = 2$ and $p = 3$. This in turn implies $g = 1$ which we have ruled out by assumption.

\begin{remark}
The condition on the class number of $K$ cannot be removed. Indeed, the polynomial $f(x) = x^5 - 19x^4 + 107x^3 + 95x^2 + 88x - 16$ has Galois group $D_5$ and its splitting field is the Hilbert class field of $K := \mathbb{Q}(\sqrt{-131})$. The jacobian $J_f$ of the hyperelliptic curve defined by $y^2 = f(x)$, has endomorphism algebra isomorphic to $\mathbb{Q}(\sqrt{13})$ and the group $\text{Gal}(K(J_f[2])/K)$ is isomorphic to $C_5$.
\end{remark}
However, notice $13 \equiv 5 \mod 8$ as predicted by [Goo21, Theorem 2.10]. Also note $J_f$ could not have had CM by combining Theorem 2.1 and Theorem 2.10 of [Goo21].

Before proving Theorem 1.8 we give the following extension of [Goo21, Theorem 3.5].

**Theorem 2.12.** Let $f \in K[x]$ be a polynomial of odd degree $n$ with Galois group isomorphic to a Frobenius group $G$ of order $n(n-1)$. Let $J_f$ be the jacobian associated to the hyperelliptic curve defined by $y^2 = f(x)$.

Suppose $E = \text{End}^0(J_f)$ is a number field of dimension $s$ over $\mathbb{Q}$. Then $E/\mathbb{Q}$ is Galois with $\text{Gal}(E/\mathbb{Q})$ isomorphic to a quotient of $H$ the Frobenius complement of $G$.

Furthermore, $L/K$ is an extension of degree $s$ contained in $K(f)$, and as abstract groups, $\text{Gal}(L/K) \cong \text{Gal}(E/\mathbb{Q})$. Moreover, if $s = n-1$, then $L = EK$.

Finally, if $\text{End}(J_f)$ is $2$-maximal, then $E$ is unramified at $2$.

**Proof.** This result is [Goo21, Theorem 3.5] with the extra assertion that if $s = n-1$, then $L = EK$. If $s = n-1$, then $J_f$ is an absolutely simple abelian variety with CM by $E$. We may therefore apply Lemma 2.5 to find $L = E^*K$ and $E^* \subseteq E$. Hence $[E^* : \mathbb{Q}] \geq [L : K] = s = [E : \mathbb{Q}]$. It follows we have equality $E^* = E$. 

**Example 2.13.** For $f(x) = x^5 - 2$, it is well known the endomorphism algebra of $J_f$ is $\mathbb{Q}(\zeta_5)$ and $\mathbb{Q}(\zeta_5)$ is the unique degree 4 extension contained in the splitting field $\mathbb{Q}(f)$.

A more interesting example is given by the genus 2 curve with LMFDB label 28561.a.371293.1 first found in [BSS+16] (though we use an odd degree model computed using Magma [BCP97]). Here $f(x) = 52x^5 + 104x^4 + 104x^3 + 52x^2 + 12x + 1$, $\text{Gal}(f) \cong F_5$ and $J_f$ has CM by the number field $K$ defined by $x^4 - x^3 + 2x^2 + 4x + 3$. The unique degree 4 extension contained in $\mathbb{Q}(f)$ is given by $K$.

This field is totally inert at 2 (as predicted by [Goo21, Theorem 2.9]) and unramified outside 13. We note its class number is one, but in line with the theorems presented above, $\mathbb{Q}(f)/K$ is a degree 5 extension ramified only at 2.

Theorem 1.8 is a direct consequence of Theorems 2.1, 2.12 and Theorems 1.5, 2.9 of [Goo21] with the final statement then following from the main theorem of [Zar00].

### 3 Quaternion algebras

Let $B = \mathbb{Q} + i\mathbb{Q} + j\mathbb{Q} + k\mathbb{Q}$ be an indefinite quaternion algebra generated by $i, j$ satisfying $i^2 = D/m, j^2 = m$ and $k = ij$, where $D$ is a positive squarefree
integer and \(m|D\). We note \(0 > -m = n(j)\), the (reduced) norm of \(j\).

An order of \(B\) is said to be hereditary if it has squarefree (reduced) discriminant (for alternate, equivalent, definitions see [DR04, Voi21]). We shall take \(O\) to be a (hereditary) order in \(B\) of discriminant \(D\).

An abelian surface \(A/K\) is said to have quaternion multiplication by \(O\) if there is an isomorphism \(\iota : O \cong \text{End}(A)\). Any such abelian surface admits a principal polarisation over \(\bar{K}\) [Voi21, 43.6.6, pg 818]. Equivalently, there is an element \(\mu \in O\) satisfying \(\mu^2 + D = 0\). Fix such an \(A, \mu\) and \(\iota\).

Following [DR04, Definition 3.3], we say \(\chi \in B\) is a twist of \((O, \mu)\) if it lies in both \(O\) and the normaliser \(N_{B^\ast}(O)\), has trace zero, \(n(\chi)\) divides \(D\), and \(\mu \chi = -\chi \mu\). The existence of such an element can be verified by a finite computation, we refer the reader to the remark at the bottom of page 9 of [DR04] for further details.

Owing in part to the fact \((O, \mu)\) has been fixed, there are only two possible values \(\mu \chi\) can take and the product of these values is equal to \(D\).

We call \((O, \mu, \chi)\) a twisted principally polarised order, and say \((O, \mu, \chi)\) is of discriminant \(D\) and norm \(-n(\chi)\).

As before, we let \(L/K\) denote the endomorphism field of \(A\). Dieulefait and Rotger [DR04, Theorem 1.3] showed \(\text{Gal}(L/K)\) is isomorphic to one of the trivial group, \(C_2\), or \(C_2 \times C_2\). Moreover, they proved in each case \(\text{End}_{K}^0(A)\) is respectively isomorphic to \(B\); one of \(\mathbb{Q}(\mu), \mathbb{Q}(\chi), \mathbb{Q}(\mu \chi)\); or \(\mathbb{Q}\).

This determines (and heavily restricts) the possible images of

\[
\text{Gal}(L/K) \to \text{Aut}(\text{End}(A)) \cong N_{B^\ast}(O)/\mathbb{Q}^\ast
\]

arising from the natural action of \(\text{Gal}(L/K)\) on \(\text{End}(A)\).

To gain information on the intersection \(L \cap K(A[2])\), we will use a description of orders of discriminant \(D\) in \(B\).

**Lemma 3.1.** Suppose \(2|D\) and \(m \equiv 3 \mod 4\). Then

\[O = \mathbb{Z} + \frac{1}{2}(1 + j + k)\mathbb{Z} + \frac{1}{2}(1 + j - k)\mathbb{Z} + \frac{1}{2}(i + k)\mathbb{Z}\]

is an order of discriminant \(D\) in \(B\). Moreover, any order \(O'\) of discriminant \(D\) which contains \(\mathbb{Z}[1, mi, j, k]\), satisfies \(O' \otimes \mathbb{Z}_2 = O \otimes \mathbb{Z}_2\).

**Proof.** It is a routine calculation to show \(O\) is an order of discriminant \(D\) (for an example, see [Vig80, pg. 85 - 86]).

This follows from [Rot94, Lemmas 3.5 and 3.7]. Indeed, in the notation of the paper, we have \(F = \mathbb{Q}\) and \(D \neq 3\), as \(D\) is divisible by an even number of primes. This forces \(\omega_{\text{odd}} = 1\). The cited lemmas in turn show \(C_2 \cong U_0 \leq V_0 \cong C_2 \times C_2\). The non-trivial element of \(U_0\) can be represented by \(\mu = \sqrt{-D}\). This allows us to write \(V_0 = [\mu, |\chi|]\) where the representative \(\chi\) may be taken to have reduced norm \(m|D\).

Moreover, any representative of \([\chi]\) has reduced norm \(m\) up to a rational square. Likewise, representatives of the class \([\mu \chi]\) have reduced norm \(D/m\) up to rational squares.  

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We now show the final statement. As $m$ is odd we have $\mathbb{Z}[1, mi, j, k] \otimes \mathbb{Z}_2 = \mathbb{Z}[1, i, j, k] \otimes \mathbb{Z}_2$. Thus we may assume $O'$ contains $\mathbb{Z}[1, i, j, k]$. The discriminant of $\mathbb{Z}[1, i, j, k]$ equals $4D$ and thus any element contained in $O'$ but not in $\mathbb{Z}[1, i, j, k]$ is of the form $\frac{1}{4}a$ with $a \in \mathbb{Z}[1, i, j, k]$.

By considering the (monic) minimal polynomial of $\frac{1}{4}a$, we find $a \in 2\mathbb{Z}[1, i, j, k]$. Hence it suffices to check elements of the form $\frac{1}{2}(a+bi+cj+dk)$ with $a, b, c, d \in \{0, 1\}$. One finds $\frac{1}{2}(1+j+k)$, $\frac{1}{2}(1+i+j)$, $\frac{1}{2}(i+k)$ are the only integral such that they must all belong to $O'$.

Lemma 3.2. Suppose $m \equiv 1 \mod 4$. Then

$$O = \mathbb{Z} + \frac{1}{2}(1+j)\mathbb{Z} + k\mathbb{Z} + \frac{1}{2}(i+k)\mathbb{Z}$$

is an order of discriminant $D$ in $B$. Furthermore, if $D \equiv 1 \mod 4$ then

$$O_1 = \mathbb{Z} + \frac{1}{2}(1+i)\mathbb{Z} + j\mathbb{Z} + \frac{1}{2}(j+k)\mathbb{Z}$$

is an order of discriminant $D$ in $B$. Likewise, if $D \equiv 3 \mod 4$ then

$$O_3 = \mathbb{Z} + \frac{1}{2}(1+k)\mathbb{Z} + j\mathbb{Z} + \frac{1}{2}(i+j)\mathbb{Z}$$

is an order of discriminant $D$ in $B$.

Moreover, any order $O'$ of even discriminant $D$ which contains $\mathbb{Z}[1, mi, j, k]$ satisfies $O' \otimes \mathbb{Z}_2 = O \otimes \mathbb{Z}_2$. Any order $O'$ of discriminant $D \equiv t \mod 4$ with $t \in \{1, 3\}$, which contains $\mathbb{Z}[1, mi, j, k]$, satisfies either $O' \otimes \mathbb{Z}_2 = O \otimes \mathbb{Z}_2$ or $O' \otimes \mathbb{Z}_2 = O_t \otimes \mathbb{Z}_2$.

Proof. The proof follows the same strategy as for Lemma 3.1.

Theorem 3.3. Let $A/K$ be an abelian surface with QM by a twisted principally polarised order $(O, \mu, \chi)$ of discriminant $D$ and norm $m$, where $O \subseteq B$.

If $2|D$ and $m \equiv 3 \mod 4$, then $L \subseteq K(A[2])$.

Proof. As $\text{End}(A)$ is a hereditary order, the results of [DR04, Theorem 1.3] apply to show a non-trivial element of $\text{Gal}(L/K)$ acts on $\text{End}(A)$ (possibly after scaling) by conjugation as one of $\mu, \chi$ or $\mu\chi$.

We look to determine the action of these elements on $\text{End}(A) \otimes \mathbb{Z}/2\mathbb{Z}$. To do so we reduce a $\mathbb{Z}$-basis of $\text{End}(A)$ modulo 2, thus we may work with $\text{End}(A) \otimes \mathbb{Z}_2$ in place of $\text{End}(A)$. By considering the algebraic relations they satisfy, we may assume $\mu = k$, $\chi = j$ and $\mu\chi = mi$. Hence Lemma 3.1 allows us to take $\text{End}(A) = \mathbb{Z} + \frac{1}{2}(1+j+k)\mathbb{Z} + \frac{1}{2}(1+j-k)\mathbb{Z} + \frac{1}{2}(i+k)\mathbb{Z}$.

Let $X = \frac{1}{2}(1+j+k)$, $Y = \frac{1}{2}(1+j-k)$ and $Z = \frac{1}{2}(i+k)$. Let us examine the action of $i, j$ and $k$ on the basis of $\text{End}(A)$ given by $1, X, Y, Z$. Each of $i, j$ and $k$ fix 1. For $i$ we have $iX^{-1} = 1 - X$, $iY^{-1} = 1 - Y$ and $iZ^{-1} = Z + Y - X$. For $j$ we have $jX^{-1} = Y$, $jY^{-1} = X$, $jZ^{-1} = -Z$. Looking at the coefficients, we see the action remains faithful on $\text{End}(A) \otimes \mathbb{Z}/2\mathbb{Z}$. By proceeding as in the proof of Theorem 2.1 we are now done.
Example 3.4. The hereditary assumption is necessary. The following example shows not only $L$ need not be contained in $\mathbb{Q}(A[2])$ for a non-hereditary order, but also the result of Dieulefait and Rotger fails.

Let $J$ be the Jacobian defined by the hyperelliptic curve associated to $y^2 + y = 6x^5 + 9x^4 - x^3 - 3x^2$ with LMFDB label 20736.l.373248.1 This surface has QM by an order of (reduced) discriminant $6^2$ in $\mathbb{Q}(\sqrt{2}, \sqrt{3})$. In particular, the order is not hereditary. The endomorphism field has defining polynomial $x^8 + 4x^6 + 10x^4 + 24x^2 + 36$ and the two torsion field of $J$ is $\mathbb{Q}(\sqrt{2}, \sqrt{3})$.

This curve, made easily available on the LMFDB \cite{LMFDB}, was first found in \cite{BSS+16}. Computations linked to its endomorphism algebra were carried out in \texttt{Magma} using code from \cite{CMSV19}.

For ease of notation let $F = K(A[2])$.

Theorem 3.5. Let $A/K$ be an abelian surface with QM by a twisted principally polarised order $(\mathcal{O}, \mu, \chi)$ of discriminant $D$ and norm $m$, where $\mathcal{O} \subseteq B$.

If $m \equiv 1 \pmod{4}$, then $\text{End}^0_F(A)$ contains at least one of $\mathbb{Q}(\sqrt{m}), \mathbb{Q}(\sqrt{-D})$ and $\mathbb{Q}(\sqrt{D/m})$. Moreover, if $D$ is even, then $\text{End}^0_F(A)$ contains at least one of $\mathbb{Q}(\sqrt{-D})$ and $\mathbb{Q}(\sqrt{D/m})$.

Proof. As $\text{End}(A)$ is a hereditary order, the results of \cite{DR04} apply to show a non-trivial element of $\text{Gal}(L/K)$ acts on $\text{End}(A)$ (possibly after scaling) by conjugation as one of $\mu, \chi$ or $\mu \chi$.

We look to determine the action of these elements on $\text{End}(A) \otimes \mathbb{Z}/2\mathbb{Z}$. To do so we reduce a $\mathbb{Z}$-basis of $\text{End}(A)$ modulo 2, thus we may work with $\text{End}(A) \otimes \mathbb{Z}_2$ in place of $\text{End}(A)$. By considering the algebraic relations they satisfy, we may assume $\mu = k, \chi = j$ and $\mu \chi = m$. Lemma 3.2 allows us to take $\text{End}(A)$ equal to one of the three given orders. As these orders differ by permuting $i, j$ and $k$, we only give details for the case $\text{End}(A) = \mathbb{Z} + \frac{1}{2}(1 + j)\mathbb{Z} + k\mathbb{Z} + \frac{1}{2}(i + k)\mathbb{Z}$.

Let $X = \frac{1}{2}(1 + j), Y = k$ and $Z = \frac{1}{2}(i + k)$. Let us examine the action of $i, j$ and $k$ on the basis of $\text{End}(A)$ given by $1, X, Y, Z$. Each of $i, j$ and $k$ fix 1. For $i$ we have $iXi^{-1} = 1 - X, iYi^{-1} = -Y$ and $iZi^{-1} = Z - Y$. For $j$ we have $jXj^{-1} = X, jYj^{-1} = -Y, jZj^{-1} = -Z$. Looking at the coefficients, we see $j$ acts trivially on $\text{End}(A) \otimes \mathbb{Z}/2\mathbb{Z}$ whereas $i$ and $k$ act by the same involution. Proceeding as in the proof of Theorem 2.1 we find at least one of $\mathbb{Q}(i) \cong \mathbb{Q}(\sqrt{D/m})$ and $\mathbb{Q}(k) \cong \mathbb{Q}(\sqrt{-D})$ is contained in $\text{End}^0_F(A)$. \hfill \Box

Example 3.6. The Jacobian $J$ of the hyperelliptic curve $y^2 = -2x^6 - 12x^5 - 21x^4 - 10x^3 - 3x^2 + 6x + 1$ has QM by the maximal order of $B = \mathbb{Q}(\sqrt{3}, \sqrt{-3})$ which has discriminant 15. The endomorphism field of $J$ is $L = \mathbb{Q}(\sqrt{3}, \sqrt{-3})$, we have $L \cap \mathbb{Q}(A[2]) = \mathbb{Q}(\sqrt{-3})$ and $\text{End}^0_{\mathbb{Q}(\sqrt{-3})}(J) = \mathbb{Q}(\sqrt{-15})$. These calculations were performed in \texttt{Magma} using code from \cite{CMSV19}. The curve comes from \cite{LY20}.
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