Generalized Tonnetz and discrete Abel-Jacobi map

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May 1, 2020

Abstract
Motivated by classical Euler’s Tonnetz, we introduce and study the combinatorics and topology of more general simplicial complexes $Tonn^{n,k}(L)$ of Tonnetz type. Our main result is that for a sufficiently generic choice of parameters the generalized tonnetz $Tonn^{n,k}(L)$ is a triangulation of a $(k-1)$-dimensional torus $T^{k-1}$. In the proof we construct and use the properties of a discrete Abel-Jacobi map, which takes values in the torus $T^{k-1} ≃ \mathbb{R}^{k-1}/\Lambda$ where $\Lambda ≃ A_{k-1}^*$ is the permutohedral lattice.

Keywords: generalized Tonnetz, discrete Abel-Jacobi map, permutohedral lattice, simplicial complexes, polyhedral combinatorics, triangulated manifolds.
MSC2010: 14H40, 52B05, 52B20, 52B70, 52C07, 57Q15

1 Introduction

In his seminal work on music theory “Tentamen novae theoriae musicae ex certissimis harmoniae principiis dilucide expositae” (1739), Leonhard Euler introduced a lattice diagram – Tonnetz – representing the classical tonal space. In more recent interpretations this diagram is identified as a triangulation of a torus with 24 triangles representing all the major and minor chords. If the equal tempered scale is identified with $\mathbb{Z}_{12}$, the Tonnetz can be described as

$$Tonnetz = \left\{ \{x, x+3, x+7\} \mid x \in \mathbb{Z}_{12} \right\} \cup \left\{ \{x, x+4, x+7\} \mid x \in \mathbb{Z}_{12} \right\}.
$$

Notice that if $\{x, y, z\} \in Tonnetz$ then $\{x - y, y - z, z - x\} = \pm \{3, 4, 5\}$. This serves as an inspiration to introduce and study more general complexes of “Tonnetz type.”

This work was supported by the Serbian Ministry of Education, Science and Technological Development through Mathematical Institute of the Serbian Academy of Sciences and Arts.
1.1 Generalized Tonnetz

Suppose that \( L = \{l_i\}_{i=1}^k \) is a collection of \( k \) positive integers which add up to \( n \)

\[
l_1 + l_2 + \cdots + l_k = n.
\]

We say that a collection \( L \) is generic if for each pair \( I, J \in 2^{[n]} \) of subsets of \([n]\)

\[
\sum_{i \in I} l_i = \sum_{j \in J} l_j \Rightarrow I = J.
\] (1.1)

A collection \( L \) is reduced if the largest common divisor of all \( l_i \) is 1,

\[
\langle l_1, l_2, \ldots, l_k \rangle = 1.
\] (1.2)

Caveat: From here on we identify elements of \([n]\) with the corresponding elements (congruence classes) in the additive group \( \mathbb{Z}_n = \{0, 1, \ldots, n-1\} \) of integers, modulo \( n \). A standard geometric model for this set is \( \mathbb{V}_n = \{\varepsilon \mid \varepsilon^n = 1\} \), the set of vertices of a regular \( n \)-gon. For this reason we in principle assume a counterclockwise orientation on the unit circle \( S^1 \) where the \( n \)-gon is inscribed. However for some readers it may be more natural to use (occasionally) more traditional presentation of the (classical) Tonnetz, with clockwise orientation and \( n = 12 \) occupying its “usual” place.

Each ordered pair \((x, y)\) of elements in \( \mathbb{Z}_n \) (respectively \([n]\) or \( \mathbb{V}_n \)) defines an interval

\[
I_{x,y} = \{x, x+1, \ldots, y\} \subset \mathbb{Z}_n.
\]

The length of this interval is \( \mathcal{L}(I_{x,y}) = |I_{x,y}| - 1 = y - x \in \mathbb{Z}_n \).

If a set \( \tau = \{v_0, v_1, \ldots, v_t\} \) is a subset of \( \mathbb{Z}_n \) we always assume the cyclic order \( v_0 < v_1 < \cdots < v_t \) of its elements, i.e. \( v_j - v_i \in \{1, 2, \ldots, n-1\} \) for each pair \( i < j \) of indices.

Definition 1.3. A generalized tonnetz \( \text{Ton}^{n,k}(L) \subseteq 2^{[n]} \) is a \((k-1)\)-dimensional simplicial complex whose maximal simplices are

\[
\Delta(x; \sigma) = \{x, x + l_\sigma(1), x + l_\sigma(1) + l_\sigma(2), \ldots, x + l_\sigma(1) + \cdots + l_\sigma(k-1)\}
\] (1.4)

where \( x \in \mathbb{Z}_n \) and \( \sigma \in \Sigma_k \) is a permutation.

Our main result is the following theorem which claims that a (sufficiently generic) generalized Tonnetz is also a triangulation of a torus, as its classical counterpart.

Theorem 1.5. Suppose that \( L = \{l_i\}_{i=1}^k \) is generic and reduced in the sense of (1.1) and (1.2). Then the generalized tonnetz \( \text{Ton}^{n,k}(L) \) is a triangulation of a \((k-1)\)-dimensional torus \( T^{k-1} := (S^1)^{k-1} \).

The central idea in the proof of Theorem 1.5 is to identify the (triangulated) torus \( T^{k-1} \) as a combinatorial Jacobian, i.e. as the target of a (discrete) Abel-Jacobi map \( J : \text{Ton}^{n,k}(L) \rightarrow \mathbb{R}^{k-1}/\Lambda \) where \( \Lambda \cong A_{k-1}^* \) is a permutohedral lattice.
1.2 Discrete Abel-Jacobi map

The classical Abel-Jacobi map is a map from an algebraic curve $S$ (Riemann surface of genus $g$) into the torus $\mathbb{C}^g/\Lambda$ where $\Lambda \subset \mathbb{C}^g$ is the lattice of periods. More explicitly there exist $g$ linearly independent holomorphic differentials $\omega_1, \ldots, \omega_g$ on $S$ and if $\{c_j\}_{j=1}^{2g} \subset H_1(S; \mathbb{Z})$ is a collection of basic cycles then the vectors $v_j = \langle c_j, \omega \rangle = (\int_{c_j} \omega_1, \ldots, \int_{c_j} \omega_g)$ form a basis of a lattice $\Lambda$. Then the Abel-Jacobi map $J : S \to \mathbb{C}^g/\Lambda$ is defined by

$$J(p) = (\int_{p_0}^p \omega_1, \ldots, \int_{p_0}^p \omega_g) \mod \Lambda. \quad (1.6)$$

In the special case when the curve $S$ is elliptic the Jacobi map is an isomorphism.

In analogy with this construction, we describe explicit simplicial cocycles $\omega_i (1 \leq i \leq k)$ on $\text{Tonn}^{n,k}(L)$, which play the role of holomorphic differentials and allow us to construct the corresponding “discrete Abel-Jacobi map.” This can be compared to the use of discrete Abel-Jacobi maps in the construction of standard realizations of maximal abelian covers of graphs in topological crystallography, see [8, 9, 10] and [2].

2 $\text{Tonn}^{n,k}(L)$ is a manifold

We begin our analysis of complexes of Tonnetz type by showing that the irreducibility condition (1.2) can always be assumed, without an essential loss of generality.

**Proposition 2.1.** The (geometric realization of the) complex $\text{Tonn}^{n,m,k}(pL) \subseteq 2^{[pm]}$, where $pL := \{pl_1, pl_2, \ldots, pl_k\}$, is homeomorphic to the disjoint union of $p$ copies of $\text{Tonn}^{n,k}(L)$.

**Proof:** As a consequence of (1.4) if $\sigma = \{v_1, v_2, \ldots, v_k\} \in \text{Tonn}^{n,m,k}(pL)$ then $v_i \equiv v_j \mod p$ for each $i, j \in [k]$. It follows that $\text{Tonn}^{n,m,k}(pL)$ is a disjoint union of its $p$ subcomplexes $T_j \cong \text{Tonn}^{n,k}(L)$ where $T_j$ is spanned by vertices in the same $\mathbb{Z}_n$-coset of the group $\mathbb{Z}_{pn}$. \hfill \square

Unlike the irreducibility condition (1.2), the genericity condition (1.1) is essential for the proof of the following proposition.

**Proposition 2.2.** $\text{Tonn}^{n,k}(L)$ is a connected, combinatorial manifold if the “length vector” $L = (l_1, \ldots, l_k)$ is both generic and reduced, in the sense of (1.1) and (1.2). Moreover, the links of vertices are isomorphic to boundaries of simplicial polytopes dual to $(k-1)$-dimensional permutohedra.

**Proof:** By definition $\tau = \{v_0, v_1, \ldots, v_t\} \in \text{Tonn}^{n,k}(L)$ if and only there is a partition $[k] = I_0 \sqcup I_1 \sqcup \cdots \sqcup I_t$ such that for each $i$ the length of the interval $I_{v_i,v_{i+1}} (v_{t+1} := v_0)$ is

$$\mathcal{L}(I_{v_i,v_{i+1}}) = \sum_{j \in I_i} l_j.$$
For a chosen vertex $v = v_0$ the face poset of the star $Star(v_0) \subseteq \text{Tonn}^{n,k}(L)$ is isomorphic to the poset of all ordered partitions of $[k]$ (here we use the genericity of $L$), where the top dimensional simplices in $Star(v_0)$ correspond to the finest ordered partitions of $[k]$. Note that the finest ordered partitions of $[k]$ are in 1-1 correspondence with permutations of $[k]$.

Recall [13, Example 0.10] that the face poset of a $(k - 1)$-dimensional permutahedron $\text{Perm}_{k-1}$ is also the poset of all ordered partitions of $[k]$, but with the reversed ordering. (The finest partitions/permutations correspond to the vertices of the permutahedron.) It immediately follows that $\text{Link}(v_0) \cong \partial Q_{k-1}$ where $Q_{k-1} = \text{Perm}^\circ_{k-1}$ is the (simplicial) polytope polar to the permutahedron.

More generally, the link $\text{Link}(\tau)$ of $\tau = \{v_0, v_1, \ldots, v_t\}$ is isomorphic to the join

$$\text{Link}(\tau) = \partial Q_{s_0-1} * \cdots * \partial Q_{s_t-1}$$

where $s_j = |I_j|$ is the cardinality of the set $I_j$. Consequently, $\text{Tonn}^{n,k}(L)$ is indeed a manifold. To show that it is connected, it is sufficient to show that consecutive vertices $x$ and $x + 1$ are connected. Indeed, since $L$ is reduced we obtain the relation

$$a_1l_1 + \ldots + a_kl_k = 1$$

for some $a_1, \ldots, a_k \in \mathbb{Z}$, which describes a sequence of edges connecting $x$ and $x + 1$. □

**Remark 2.3.** The following geometric model for the complex $Star(v_0)$ can be used for an alternative proof of Proposition 2.2. Let $c_i \in \mathbb{R}^{k-1}$ ($i = 1, \ldots, k$) be a spanning set of vectors such that $c_1 + c_2 + \cdots + c_k = 0$. Let $Z = [0, c_1] + \cdots + [0, c_k] \subset \mathbb{R}^{k-1}$ be the Minkowski sum of line segments $I_j = [0, c_j]$. Then the zonotope $Z$ admits a triangulation where the maximal simplices $\Sigma_\pi$ ($\pi \in S_k$), indexed by permutations, are the following

$$\Sigma_\pi = \text{Conv}\{c_\pi(1), c_\pi(2), \ldots, c_\pi(1) + \cdots + c_\pi(k)\}.$$  \hfill (2.4)

This triangulation of $Z$ is isomorphic to $Star(v_0)$ which can be proved by comparing (1.4) and (2.4).

A very special case of Theorem 1.5 can be established by an elementary, direct argument.

**Proposition 2.5.** Let $L = (l_1, l_2, l_3)$ be a reduced, generic length vector. Then the associated, 2-dimensional generalized Tonnetz $\text{Tonn}^{n,3}(L)$ is a triangulation of the 2-dimensional torus $T^2 = (S^1)^2$.

**Proof:** In light of Propositions 2.1 and 2.2 $\text{Tonn}^{n,3}(L)$ is a connected 2-manifold. Assume $l_1 < l_2 < l_3$. It is not difficult to see that the $f$-vector of $T = \text{Tonn}^{n,3}(L)$ is $f(T) = (n, 3n, 2n)$, hence $\chi(T) = 0$.

The complex $\text{Tonn}^{n,3}(L)$ is orientable. Indeed, all triangles in $\text{Tonn}^{n,3}(L)$ fall into two classes. Generalized “major triads” are the triangles $\tau = \{v_0, v_1, v_2\}$ where $v_1 - v_0 = l_1, v_2 - v_1 = l_2$ and $v_0 - v_2 = l_3$ (for some circular order of vertices of $\tau$). These simplices are

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positively oriented. Negatively oriented are generalized “minor triads”, i.e. the triangles $\tau = \{v_0, v_1, v_2\}$ where $v_1 - v_0 = l_1, v_2 - v_1 = l_2$ and $v_0 - v_2 = l_2$.

Summarizing, $Ton_{n,3}(L)$ is a connected, orientable 2-manifold with vanishing Euler characteristic, hence it must be the torus $T^2$. □

2.1 Euler characteristic and $f$-vector of $Ton_{n,k}(L)$

Proposition 2.6. Suppose that $L$ is generic and reduced and let $f(T) = (f_0, f_1, \ldots, f_{k-1})$ be the $f$-vector of the generalized Tonnetz $T = Ton_{n,k}(L)$. Then

$$f_{m-1} = n P(k, m) = n \frac{m!}{m} S(k, m)$$

(2.7)

where $P(k, m)$ is number of ordered partitions partition $I_1 \sqcup \ldots \sqcup I_m = [k]$ and $S(k, m)$ is a Stirling number of the second kind.

Proof: If $S$ is a $(m - 1)$-dimensional face of $Ton_{n,k}(L)$ then there exists $x \in \mathbb{Z}_n$ and a partition $[n] = I_1 \sqcup \ldots \sqcup I_m$ such that

$$S = \{x, x + \mu_L(I_1), x + \mu_L(I_1 \sqcup I_2), \ldots, x + \mu_L(I_1 \sqcup \ldots \sqcup I_{m-1})\},$$

where $\mu_L(I) = \sum_{i \in I} l_i$. The first equality in the formula (2.7) is an immediate consequence. (Since $L$ is generic if $\mu_L(I) = \mu_L(J)$ then $I = J$.) The second follows from the equality $P(k, m) = m! S(k, m)$, where $S(k, m)$ is a Stirling number of the second kind. □

Proposition 2.8. If the vector $L$ is generic and reduced then the Euler characteristic of $Ton_{n,k}(L)$ is 0.

Proof: Let $T = Ton_{n,k}(L)$. Its Euler characteristic is

$$\chi(T) = n \sum_{m=1}^k (-1)^{m+1} \frac{m!}{m} S(k, m).$$

Using the well-known recurrence for Stirling numbers

$$S(k, m) = m S(k-1, m) + S(k-1, m-1)$$

we obtain

$$\chi(T) = n \left( \sum_{m=1}^k (-1)^{m+1} \frac{m!}{m} S(k-1, m) + \sum_{m=1}^k (-1)^{m+1} \frac{m!}{m} S(k-1, m-1) \right)$$

$$= n \left( \sum_{m=1}^k (-1)^{m+1} m! S(k-1, m) + \sum_{q=0}^{k-1} (-1)^q q! S(k-1, q) \right)$$

$$= n \left( (-1)^{k+1} k! S(k-1, k) + S(k-1, 0) \right)$$

$$= 0$$

□
2.2 Fundamental group of $Ton^{n,k}(L)$

A consequence of Theorem 1.5 is that the fundamental group of a generalized Tonnetz $Ton^{n,k}(L)$ is free abelian of rank $k - 1$, provided the vector $L$ is generic and reduced. Proposition 2.12 is a key step in the direction of this result. Before we commence the proof, let us make some general observation about the edge-path groupoid of the Tonnetz $Ton^{n,k}(L)$.

Each edge-path connecting vertices $a = v_0$ and $b = v_m$ is of the form $\alpha = X_1 X_2 \cdots X_m$ where $X_i = \overrightarrow{v_{i-1}v_i}$ is an oriented edge (1-simplex) in $Ton^{n,k}(L)$.

Recall (Section 1.1) that $I_X = I_{u,v} \subset \mathbb{Z}_n$ is the (oriented) interval, corresponding to $X = \overrightarrow{uv}$. (With a slight abuse of language we use the same notation for the corresponding arc in $S^1$.)

Let the $L$-type $X^L$ of $X$ be defined as the unique non-empty subset $I \subset [k]$ such that $\mathcal{L}(I_{u,v}) = \sum_{j \in I} l_j$.

We say that $X = \overrightarrow{uv}$ is atomic if either $v = u + l_i$ or $u = v + l_i$ for some $i \in [k]$. If $X$ is positively oriented, i.e. if $v = u + l_i$, then we call it $\oplus$-atomic (similarly $\ominus$-atomic if $u = v + l_i$).

Note that the $L$-type of an $\oplus$-atomic oriented 1-simplex $X = \overrightarrow{uv}$ is a singleton $X^L = \{i\}$ (we say that $X$ is of type $i$), while the $L$-type of the associated $\ominus$-atomic 1-simplex $X^{-1} = \overrightarrow{vu}$ is $[k] \setminus \{i\}$.

The following lemma is an immediate consequence of Definition 1.3.

**Lemma 2.9.** Each oriented 1-simplex $X = \overrightarrow{uv}$ is homotopic $X \cong Y_1 Y_2 \cdots Y_t$ (relative to the end-points $u$ and $v$) to a product of $\oplus$-atomic 1-simplices $Y_j$. Moreover, one can read off the $L$-type of $X$ from this representation as,

$$X^L = \{Y_1^L, Y_2^L, \ldots, Y_t^L\} .$$

(2.10)

The following lemma (see Figure 2 for a visual proof) shows that we can rearrange and group $\oplus$-atomic 1-simplices according to their type.

**Lemma 2.11.** An edge-path which is a product $XY$ of two $\oplus$-atomic 1-simplices $X$ and $Y$, respectively of type $i$ and $j$ (where $i \neq j$) is homotopic (rel the end points) to a product $Y'X'$ of two $\oplus$-atomic 1-simplices, where type($Y'$) = $j$ and type($X'$) = $i$.

**Proposition 2.12.** If the length vector $L$ is generic and reduced then the fundamental group $\pi_1(Ton^{n,k}(L))$ of the generalized Tonnetz is abelian.

**Proof:** Suppose that $v_0$ is the chosen base point and assume that $\alpha$ and $\beta$ are two edge-paths (loops) based at $v_0$. We are supposed to show that the edge paths $\alpha \beta$ and $\beta \alpha$ are homotopic (rel $v_0$).

By Lemma 2.9 we are allowed to assume that both $\alpha = Y_1 Y_2 \cdots Y_t$ and $\beta = Z_1 Z_2 \cdots Z_t$ are products of $\oplus$-atomic 1-simplices.
Use Lemma 2.11 to rearrange atoms in the product $\alpha\beta$ (similarly $\beta\alpha$) and write it as a product $\alpha\beta = A_1A_2\ldots A_k$, where $A_i$ is the product of $\oplus$-atoms of type $i$. (Some $A_i$ may be empty words.)

Observe that (as a consequence of Lemma 2.11) the length of the word $A_i$ is equal to the number of type $i$ $\oplus$-simplices in the product $\alpha\beta$.

If $\beta\alpha = A'_1A'_2\ldots A'_k$ is the corresponding regrouped presentation of $\beta\alpha$ we observe that $A_j = A'_j$ for each $j$. This completes the proof of the proposition. \hfill \Box

3 Canonical cycles and cocycles in $Tonnn^{n,k}(L)$

We already know (Proposition 2.2) that a generalized Tonnetz $T = Tonnn^{n,k}(L)$ is a connected complex. Since, according to Proposition 2.12, the fundamental group $\pi_1(T)$ is
abelian, it is isomorphic to the first homology group $H_1(T; \mathbb{Z}) := \mathbb{Z}/B_1$, where $Z_1$ and $B_1$ are the corresponding groups of cycles and boundaries.

When working with the homology group it is more customary to use additive notation. For example the $\oplus$-atom decomposition $X \cong Y_1Y_2 \cdots Y_t$ from Lemma 2.11 can be rewritten as the following equality (in homology), $X = \sum_{i=1}^t Y_i$.

In this section the emphasis is on (co)homology so here we follow the additive notation.

**Definition 3.1.** The cochains $\theta_{i,j} \in C^1 = \text{Hom}(C_1; \mathbb{Z})$, where $1 \leq i \neq j \leq k$, are defined on $\oplus$-atomic 1-simplices as follows:

$$\theta_{i,j}(Y) = \begin{cases} +1 & \text{if } Y \text{ is of } L\text{-type } i \\ -1 & \text{if } Y \text{ is of } L\text{-type } j \\ 0 & \text{if the } L\text{-type of } Y \text{ is neither } i \text{ nor } j. \end{cases}$$

If $X = \overrightarrow{uv}$ is an oriented 1-simplex and $X \cong Y_1Y_2 \cdots Y_t$ its $\oplus$-atom decomposition from Lemma 2.9 then by definition

$$\theta_{i,j}(X) = \sum_{m=1}^t \theta_{i,j}(Y_m). \quad (3.2)$$

On other oriented 1-chains they are extended by linearity.

**Proposition 3.3.** The cochain $\theta_{i,j}$ is well defined. Moreover, it is a cocycle which defines an element of $H^1(T; \mathbb{Z})$. These classes (cocycles) are referred to as “elementary classes” defined on $T = \text{Tonn}^{n,k}(L)$.

**Proof:** We check that $\theta_{i,j}$ is well defined by showing that possibly different ways to extend $\theta_{i,j}$ lead to the same result. Essentially the only case when this happens is when we evaluate $\theta_{i,j}(-X) = \theta_{i,j}(X^{-1})$, where $X = \overrightarrow{uv}$ and $X^{-1} = \overrightarrow{vu}$ (by formula (3.2)) expecting to obtain the result $-\theta_{i,j}(X)$.

This is indeed the case since $\sum_{m=1}^k \theta_{i,j}(Y_m) = 0$, where $Y_m$ is a $\oplus$-atomic 1-simplex of type $m$, for each $m \in [k]$. Similarly we obtain that the coboundary

$$\delta \theta_{i,j}(\tau) = \theta_{i,j}(\overrightarrow{u_0u_1}) + \theta_{i,j}(\overrightarrow{u_1u_2}) + \theta_{i,j}(\overrightarrow{u_2u_0}) = 0$$

is zero for each (oriented) 2-simplex $\tau = \{u_0, u_1, u_2\}$. \qed

**Definition 3.4.** For each $i \in [n]$ let $\omega_i$ be the cocycle defined by $\omega_i = \sum_{j \neq i} \theta_{i,j}$. More explicitly, $\omega_i$ is the unique 1-cocycle defined on $\text{Tonn}^{n,k}(L)$ such that for each $\oplus$-atomic 1-simplex $Y$

$$\omega_i(Y) = \begin{cases} k - 1 & \text{if } Y \text{ is of type } i \\ -1 & \text{if } Y \text{ is of type } j \neq i. \end{cases}$$

These cocycles are referred to as the “canonical” cocycles defined on $\text{Tonn}^{n,k}(L)$.  

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As an immediate consequence of the definition we obtain the following relation

$$\omega_1 + \omega_2 + \cdots + \omega_k = 0.$$  \hfill (3.5)

The complex $Ton(n,k)(L)$ also has naturally defined 1-cycles.

**Definition 3.6.** For $i \in [k]$ let $c_i$ be the 1-cycle defined by $c_i := \sum_{x \in [n]} E_x^i$ where $E_x^i = \overrightarrow{xy}$ (for $x \in [n]$ and $i \in [k]$) is the $\oplus$-atomic 1-simplex of type $i$ with end-points $x$ and $y = x + l_i$.

**Proposition 3.7.** If $[c_i] \in H_1(Ton(n,k)(L); \mathbb{Z})$ is the homology class of the cycle $c_i$ then

$$[c_1] + [c_2] + \cdots + [c_k] = 0.$$ \hfill (3.8)

**Proof:** Informally, the cycle $c_i$ is the sum of all $\oplus$-atomic 1-simplices of type $i$. They can be concatenated to form $d$ irreducible cycles of length $q$, where $n = qd$ and $d = (n, l_i)$. For each $x \in \mathbb{Z}_n$ the cycle $E_x = E_x^1 + E_x^2 + l_1 + C = E_x^k + \cdots + l_{k-1}$ is trivial by Definition 1.3. Since $\sum_{i=1}^k c_k = \sum_{x \in \mathbb{Z}_n} E_x$ the equality (3.8) is an immediate consequence.

The following proposition implies that aside from (3.5) and (3.8) there are essentially no other relations among $\{\omega_i\}_{i \in [k]}$ and $\{[c_i]\}_{i \in [k]}$.

**Proposition 3.9.** Let $\langle \cdot, \cdot \rangle$ be the pairing between the cohomology and homology classes and let $M = [m_{i,j}]_{i,j=1}^{k-1}$ be a $(k-1) \times (k-1)$-matrix where $m_{i,j} = \langle \omega_i, c_j \rangle$. Then $det(M) = n(nk)^{k-2}$.

**Proof:** By direct calculation we have

$$\langle \omega_i, c_j \rangle = \sum_{\nu \neq i} \langle \theta_{i,\nu}, c_j \rangle = \left\{ \begin{array}{cl} n(k-1) & \text{if } i = j \\ -n & \text{if } i \neq j \end{array} \right.$$ \hfill (3.10)

It follows that $M$ is a circulant matrix with the associated polynomial equal to $f(x) = n(k-1) - n(x + x^2 + \cdots + x^{k-2})$. Recall that the determinant of the circulant matrix with the associated polynomial $f(x)$ is equal to $\prod_{j=1}^{k-1} f(\epsilon_j)$, where $\epsilon_j$ are solutions of the equation $x^{k-1} - 1 = 0$. From here it immediately follows that $det(M) = n(nk)^{k-2}$.

4 Homology $H_1(Tonn^{n,k}(L))$

In this section we complete the analysis and summarize our knowledge about the homology group $H_1(T; \mathbb{Z})$ of the generalized Tonnetz $T = Ton^{n,k}(L)$.

We already know (Section 2) that each homological 1-cycle has a (multiplicative) representation $X = Y_1 Y_2 \cdots Y_t$ where $Y_i$ are $\oplus$-atomic 1-simplices. We also write $X = v_0 Y_1 Y_2 \cdots Y_t$ when we want to emphasize that the initial vertex of $Y_1$ (playing the role of the base point of the loop $X$) is $v_0$. 

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If $Y_j$ is of $L$-type $i_j$ then we can also (symbolically) record this information as the word
\[ X = E_{i_1}E_{i_2}\ldots E_{i_l} = v_0E_{i_1}E_{i_2}\ldots E_{i_l}, \] (4.1)
where $E_j$ denotes a step of length $l_j$ in the positive direction. Most of the time we can safely remove the base point $v_0$ from the notation. For example the representation $X = E_2E_1E_1E_3E_3 = E_2E_1^3E_3^2$ describes an edge-path which begins at $v_0$, makes one step of type 2, then three steps of type 1 and finally two steps of type 3.

Lemma 2.11 can be interpreted as a symbolic (edge-path) relation $E_iE_j = E_jE_i$ which says that one can interchange two consecutive steps (as in Fig. 2) without changing the homotopy type of the edge-path (rel end-points). From here we easily deduce the following proposition.

**Proposition 4.2.** Each homological cycle $X$ has a representation $X = E_1^{p_1}E_2^{p_2}\ldots E_k^{p_k}$ where $p_1,\ldots,p_k$ are non-negative integers such that $p_1l_1 + p_2l_2 + \cdots + p_kl_k = p_0n$ for some $p_0 \geq 0$. Moreover, if $Y = E_1^{p'_1}E_2^{p'_2}\ldots E_k^{p'_k}$ has a similar representation, where $p'_1l_1 + p'_2l_2 + \cdots + p'_kl_k = p'_0n$, then the cycles $X$ and $Y$ are homologous if and only if
\[ (p_1,\ldots,p_k) - (p'_1,\ldots,p'_k) \in \mathbb{Z} \mathbb{1}. \] (4.3)
where $\mathbb{1} = (1,\ldots,1) \in \mathbb{Z}^k$ and $\mathbb{Z} \mathbb{1} = \{m\mathbb{1} | m \in \mathbb{Z}\}$.

**Proof:** If the relation (4.3) is satisfied then $X$ and $Y$ are clearly homologous since $E = E_1E_2\ldots E_k$ is a trivial cycle.

Conversely, suppose that $X$ and $Y$ are homologous. Then,
\[ p_i - p_j = \theta_{i,j}(X) = \theta_{i,j}(Y) = p'_i - p'_j \]
for each $i < j$ and the relation (4.3) follows. \hfill $\square$

As an immediate consequence we obtain the following representation of the first homology group of the generalized Tonnetz $Tonn^{n,k}(L)$ as a lattice of rank $(k-1)$ in a hyperplane $H_0^L \subset \mathbb{R}^k$.

**Theorem 4.4.** Let $H_0^L = \{x \in \mathbb{R}^k \mid \langle x, L \rangle = x_1l_1 + \cdots + x_kl_k = 0\}$ be the central hyperplane in $\mathbb{R}^k$, orthogonal to $L$. Then there is an isomorphism
\[ H_1(Tonn^{n,k}(L);\mathbb{Z}) \rightarrow H_0^L \cap \mathbb{Z}^k \] (4.5)
where $H_0^L \cap \mathbb{Z}^k$ is a free abelian group (lattice) of rank $(k-1)$.

**Proof:** Let $P \subseteq \mathbb{R}^k$ be the closed, convex cone
\[ P = \{(x_1,\ldots,x_k) \in \mathbb{R}^k \mid (\forall i) x_i \geq 0 \text{ and } (\exists x_0 \geq 0) x_0n = x_1l_1 + \cdots + x_kl_k\}. \] (4.6)
Let $W = P \cap \mathbb{Z}^k$ be the abelian semigroup of all lattice points in $P$. Obviously $\mathbb{1} = (1,\ldots,1)$ is in $W$. Let $\mathbb{Z}_{\geq 0} \mathbb{1} = \{m\mathbb{1} | m \in \mathbb{Z}_{\geq 0}\} \subset W$ be the subsemigroup of $W$ generated by the

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vector $1$. Then, as a consequence of Proposition 4.2, there is an isomorphism of abelian groups
\[ H_1(\text{Ton}^{n,k}(L); \mathbb{Z}) \cong W/(\mathbb{Z}_{\geq 0} \cdot 1). \]

It is not difficult to see that the map $p : P \to H_0^L$, which sends $x \in P$ to $p(x) = x - x_0 \cdot 1$ (see (4.6)), induces an isomorphism $W/(\mathbb{Z}_{\geq 0} \cdot 1) \to H_0^L \cap \mathbb{Z}^k$. For example it induces an epimorphism since for each lattice point $y \in H_0^L \cap \mathbb{Z}^k$ the vector $y + m \cdot 1$ is in $W$ for a sufficiently large positive integer $m$.

Let $\lambda := l_1 l_2 \cdots l_k$ and $\lambda_i := \lambda/l_i$. The vectors $z^{(j)} = (z_1^{(j)}, \ldots, z_k^{(j)})$, where $z_j^{(j)} = n\lambda_j$ and $z_i^{(j)} = 0$ for $i \neq j$, clearly belong to $W$. The corresponding vectors $\{z^{(j)} := z^{(j)} - \lambda \cdot 1\} \in H_0^L \cap \mathbb{Z}^k$ span the hyperplane $H_0^L$. From here we deduce that $\text{rank}(H_0^L \cap \mathbb{Z}^k) = k - 1$. □

We conclude this section by some observations about the cycles $c_i$, introduced in Definition 3.6.

Suppose that $l_i$ is not relatively prime to $n$, say $n = qd$ and $l_i = pd$, where $d \geq 2$ and $p$ and $q$ are relatively prime. In this case the cycle $c_i$ can be decomposed as a sum of $d$ (irreducible) cycles, each of length $q$. The following proposition claims that all these cycles determine the same homology class.

**Proposition 4.7.** All cycles of the form $v E_i^q$ are homologous.

**Proof:** Let $u$ be another base point such that $v = u + l_j$. Then the cycle $u E_j^q E_j^{-1}$ is clearly homologous to the cycle $v E_i^q$. On the other hand
\[ u E_j^q E_j^{-1} = u E_i^q E_j E_j^{-1} = u E_i^q. \]
By iterating this argument we see that \( x \in q_i \) and \( x + z \in q_i \) are homologous for any integer \( z \) which can be written in the form \( z = p_0 n + p_1 l_1 + \cdots + p_k l_k \). Since the vector \( L \) is reduced we see that \( z = 1 \) for some choice of parameters \( p_0, p_1, \ldots, p_k \), which completes the proof of the proposition. □

5 Proof of Theorem 1.5

We already know that the fundamental group of the generalized tonnetz \( T = \text{Ton}^{n,k}(L) \) is free abelian of rank \( k - 1 \) (Theorem 4.4). For the continuation of the proof of Theorem 1.5 a natural step would be to show that \( \text{Ton}^{n,k}(L) \) is an aspherical manifold, in the sense that its all higher homotopy groups are trivial. Note however that asphericity alone is not sufficient to guarantee that such a manifold is covered by an euclidean space, see [6] for examples.

We offer a direct proof that the universal covering \( \tilde{T} \) of a generalized Tonnetz \( T = \text{Ton}^{n,k}(L) \) is homeomorphic to \( \mathbb{R}^{k-1} \) (with a lattice \( \Lambda_L \subset \mathbb{R}^{k-1} \) of rank \( (k - 1) \) as a group of deck transformations) which implies that \( T \) is homeomorphic to a \( (k - 1) \)-dimensional torus. To this end we construct a discrete Abel-Jacobi map

\[
\Omega : \tilde{T} \longrightarrow D_\Lambda
\]

where \( H_0 = \{ x \in \mathbb{R}^k \mid x_1 + \cdots + x_k = 0 \} \), \( \Lambda := H_0 \cap \mathbb{Z}^k \) is a lattice isomorphic to the permutohedral lattice \( A^{*}_{k-1} \) and \( D_\Lambda \) is the associated Delone triangulation.

5.1 Simplicial universal covering

It is well known that each finite simplicial complex \( K \) admits an universal covering \( \tilde{K} \to K \) in the simplicial category.

By a classical construction, see Seifert-Threlfall [7], the vertices of \( \tilde{K} \) are (combinatorial) homotopy classes of (simplicial) edge-paths \( \alpha = \alpha_x = v_0 \alpha_x \) in \( \tilde{K} \), connecting the base-point \( v_0 \) with a (variable) vertex \( x \in K \).

By definition a simplex in \( \tilde{K} \) is a collection of edge-paths \( \{ \alpha_{i \cdot} \}_{i=0}^d \) (or rather their homotopy classes) such that the end-points form a simplex \( \tau = \{ x_0, \ldots, x_d \} \) in \( K \) and for each \( i \neq j \) the edge-paths \( \alpha_{i \cdot} \) and \( \alpha_{j \cdot} \) are neighbours in the sense of the following definition.

**Definition 5.1.** Two edge paths \( \alpha_x \) and \( \beta_y \) are neighbors if \( \{ x, y \} \) is an edge \( e \in K \) and the edge-path \( \alpha_x e \beta_y^{-1} \) is a homotopically trivial loop based at \( v_0 \).

We emphasize that the homotopy always refers to combinatorial homotopy. In particular two edge-paths \( a \alpha_b \) and \( a \beta_b \) are homotopic means that one can be obtained from the other by a sequence of elementary modifications (moves) which replace one side of a triangle by the remaining two sides (or vice versa).
## 5.2 Discrete Abel-Jacobi map

Following the notation from Section 4, each edge-path \( \alpha = v_0 \alpha_x \), which emanates from the base point \( v_0 \) and ends at a vertex \( x \), is homotopic (rel the end-points) to an edge-path of the form

\[
\alpha = v_0 E_1^{p_1} E_2^{p_2} \cdots E_k^{p_k} \quad (5.2)
\]

where \( p_i \geq 0 \) for each \( i \in [k] \).

The canonical cocycles \( \omega_i \), introduced in Section 3, together define a vector valued 1-cocycle \( \omega = (\omega_1, \omega_2, \ldots, \omega_k) \) on the generalized tonnetz \( \text{Tonn}^{n,k}(L) \) which in light of (3.5) takes values in the subspace \( H_0 = \{ y \in \mathbb{R}^k \mid y_1 + \cdots + y_k = 0 \} \subset \mathbb{R}^k \). More precisely, the cocycle \( \omega \) takes values in the lattice

\[
\Lambda = \{ x \in \mathbb{Z}^k \mid \sum_{i=1}^k x_i = 0 \text{ and all } x_i \text{ are in the same (mod } k \text{) congruence class} \}.
\]

This is one of incarnations of the lattice of type \( A_{k-1}^* \) (the dual of the root lattice \( A_{k-1} \)), which can be also described as the projection on \( H_0 \) of the \( k \)-fold dilatation \( k \mathbb{Z}^k \) of the cubical lattice \( \mathbb{Z}^k \), along the main diagonal \( D = \{(t, t, \ldots, t) \}_{t \in \mathbb{R}} \subset \mathbb{R}^k \).

In the sequel we will need a more precise description of Delone cells (simplices) of this lattice. Following [4], for \( i = 0, 1, \ldots, k \) let \( [i] = (j^i, (-i)^j) = (j, \ldots, j, -i, \ldots, -i) \in \mathbb{R}^k \), where \( i + j = k \) and in the vector \( [i] \) there are \( i \) occurrences of \( j \) (respectively \( j \) occurrences of \( -i \)). Similarly if \( \pi \in S_k \) is a permutation then \( [i] \pi = \pi([i]) \) is obtained from \([i]\) by permuting the coordinates.

**Remark 5.3.** Denote \( a_i := \omega(E_i) \). Then \( a_1 + a_2 + \cdots + a_k = 0 \) and \( \{a_i\}_{i \neq j} \) is a basis of the lattice \( \Lambda \) for each \( j \in [k] \). For \( I = \{i_1, \ldots, i_r\} \subseteq [k] \) let \( a_I := \sum_{j \in I} a_i \) and \( E_I = E_{i_1} \cdots E_{i_r} \). It is easily checked that the vector \( [i] \) (in the traditional notation [4]) is the same as the vector \( a_{[i]} = \omega(E_1 \cdots E_i) = \omega(E_{[i]}) \).

**Proposition 5.4.** ([11, Theorem 4.5], [4] Chapters 4 and 21) The Delone cells of the lattice \( \Lambda \cong A_{k-1}^* \) are \((k-1)\)-simplices, which are related via permutations of coordinates and translation to the canonical simplex whose vertices are \([0] = [k], [1], \ldots, [k-1] \).

**Remark 5.5.** In the notation of Remark 5.3 the vertices of a Delone cell are the vectors \( a_{[j]} = [j] \). Each element of the lattice \( A_{k-1}^* \) has a representation \( z = p_1 a_1 + p_2 a_2 + \cdots + p_k a_k \) where \( p_j \in \mathbb{Z} \). (This representation is unique if \( p_1 + \cdots + p_k = 0 \).) Let \( \text{Star}_{D_{\Lambda}}(z) \) be the union of all Delone cells which have \( z \) as a vertex (the star of \( z \) in the Delone complex \( D_{\Lambda} \)). Then the vertices of simplices in \( \text{Star}_{D_{\Lambda}}(z) \) are the vectors \( z + a_I \) for all subsets \( I \subseteq [k] \) and each simplex \( \sigma \in \text{Star}_{D_{\Lambda}}(z) \) is of the form

\[
\sigma = \text{Conv}\{z + a_{I_1}, z + a_{I_2}, \ldots, z + a_{I_s}\}
\]

where \( I_1 \subseteq I_2 \subseteq \cdots \subseteq I_s \).
The vector valued cocycle $\omega$ can be extended to edge-paths (5.2) by the formula
\[
\omega(\alpha) = \omega(E_1^{p_1} E_2^{p_2} \cdots E_k^{p_k}) = \sum_{i=1}^{k} p_i \omega(E_i) = \sum_{i=1}^{k} p_i a_i \in \Lambda.
\] (5.6)

Since $\omega$ is a cocycle, $\omega(\alpha a_b) = \omega(\alpha \beta_b)$ for each two homotopic edge-paths (with the same end-points). The following proposition says that $\omega$ takes different values on non-homologous cycles.

**Proposition 5.7.** The map $\omega$, described by the formula (5.6), induces a monomorphism
\[
\tilde{\omega} : H_1(\text{Ton}_n^k(L); \mathbb{Z}) \rightarrow \Lambda.
\]

**Proof:** It is sufficient to show that if $\alpha = E_1^{p_1} E_2^{p_2} \cdots E_k^{p_k}$ is a loop such that $\tilde{\omega}(\alpha) = 0$ then $\alpha$ is trivial in the homology group. However, if $\omega_i(\alpha) = 0$ for each $i \in [k]$ then
\[
k\theta_{i,j}(\alpha) = \omega_i(\alpha) - \omega_j(\alpha) = 0
\]
for each $i \neq j$. In turn $\theta_{i,j}(\alpha) = 0$ for each $i \neq j$ which implies $p_1 = p_2 = \cdots = p_k$ and, in light of Proposition 4.2, $\alpha$ is a trivial cycle. \qed

**Proposition 5.8.** Suppose that two edges paths $\alpha = a_1 \alpha_x$ and $\beta = a_1 \beta_y$, which share the same initial point $a$, satisfy the equality $\omega(\alpha) = \omega(\beta)$. Then $x = y$, i.e. they have the same end-point as well.

**Proof:** Suppose that $\alpha = a_1 E_1^{p_1} E_2^{p_2} \cdots E_k^{p_k} x$ and $\beta = a_1 E_1^{q_1} E_2^{q_2} \cdots E_k^{q_k} y$. By assumption
\[
\omega(\alpha) = \sum_{i=1}^{k} p_i a_i = \sum_{i=1}^{k} q_i a_i = \omega(\beta).
\]

It follows that $p_i - q_i$ does not depend on $i$ and the result follows. \qed

The following proposition refines Proposition 5.7. It says that a vector $b \in \text{Image}(\tilde{\omega})$ cannot be “very short” (unless it is zero).

**Proposition 5.9.** Suppose that $\xi = E_1^{p_1} E_2^{p_2} \cdots E_k^{p_k}$ is cycle representing a non-trivial homology class in $H_1(\text{Ton}_n^k(L))$, where $p_i \geq 0$ for each $i \in [k]$. Then $p_i \geq 3$ for some $i$. Moreover, its image $\omega(\xi) \in \Lambda$ in the lattice $\Lambda$ cannot be expressed as a difference $a_i - a_j$ of vectors described in Remark 5.3, where $I$ and $J$ are subsets of $[k]$.

**Proof:** Since $\xi_0 = E_1 E_2 \cdots E_k$ is a trivial cycle, by factoring out from $\xi$ the power $(\xi_0)''$, where $\nu := \min\{p_j\}_{j=1}^{k}$, we can assume that $p_j = 0$ for some $j$.

Since $\xi$ is a cycle we know that $p_1 l_1 + \cdots + p_k l_k = p_0 n$ is divisible by $n$. If $p_0 \geq 2$ then $p_i \geq 3$ for some $i \in [k]$ and we are done. Otherwise $p_0 = 1$ and $0 \leq p_i \leq 2$ for each $i$. This is not possible since the equality
\[
(p_1 l_1 + \cdots + p_k l_k) - (l_1 + \cdots + l_k) = n - n = 0
\]
would contradict the genericity of the vector \( L \). As an immediate consequence we see that the equality \( \omega(\xi) = a_I - a_J = a_I - (a_{[k]} - a_{J'}) = a_I + a_{J'} \) is not possible.

Formula (5.6) can be used for the definition of a simplicial map \( \Omega : \tilde{T} \rightarrow \mathcal{D}_\Lambda \) where \( \mathcal{D}_\Lambda \) is the Delone triangulation of the \((k-1)\)-dimensional, affine space \( H_0 \subset \mathbb{Z}^k \), associated to the lattice \( \Lambda \).

More explicitly, if \( \tilde{\tau} = \{\alpha_{i, x_i}\}_{i=0}^d \) is a simplex in \( \tilde{T} \), then \( \Omega(\tilde{\tau}) = \{\omega(\alpha_{0, x_0}), \ldots, \omega(\alpha_{d, x_d})\} \).

**Proposition 5.10.** The map \( \Omega : \tilde{T} \rightarrow \mathcal{D}_\Lambda \) is an isomorphism of simplicial complexes.

**Proof:** The map \( \Omega \) is clearly an epimorphism on vertices. Indeed, if \( z = p_1 a_1 + \cdots + p_k a_k \) is a vertex of \( \mathcal{D}_\Lambda \) then \( z = \Omega(E_1^{p_1} \cdots E_k^{p_k}) \).

We continue by showing that \( \Omega \) is a local isomorphism of simplicial complexes. (In particular \( \Omega \) is a simplicial map.) Recall that \([i] = \omega(E_1 E_2 \cdots E_i) = a_{[i]} \) for each \( i \) (including the case \( i = 0 \) when we have the empty word). Similarly \([i, \pi] = \omega(E_{\pi(1)} E_{\pi(2)} \cdots E_{\pi(i)}) \) for each permutation \( \pi \in S_k \).

Let \( \alpha = v_0 \alpha_x \) be an edge-path describing a vertex in \( \tilde{T} \) (connecting the base point \( v_0 \) with a vertex \( x \) in \( T \)). Then in light of Proposition 2.2 (see also Remark 2.3) the star \( \text{Star}_{\tilde{T}}(\alpha) \) of this vertex is the union of \( k! \) simplices (one for each \( \pi \in S_k \))

\[
\tilde{\tau}_\pi = \{\alpha, \alpha E_{\pi(1)}, \alpha E_{\pi(1)} E_{\pi(2)}, \ldots, \alpha E_{\pi(1)} E_{\pi(2)} E_{\pi(k-1)}\}.
\]

In light of (5.6) the \( \Omega \)-image of this simplex is

\[
\Omega(\tilde{\tau}_\pi) = \{\omega(\alpha), \omega(\alpha) + [1]_\pi, \omega(\alpha) + [2]_\pi, \ldots, \omega(\alpha) + [k-1]_\pi\}.
\]

It follows from Propositions 2.2 (Remark 2.3) and Proposition 5.4 that \( \Omega \) maps bijectively the star \( \text{Star}_{\tilde{T}}(\alpha) \) of \( \alpha \) in \( \tilde{T} \) to the star \( \text{Star}_{\mathcal{D}_\Lambda}(\omega(\alpha)) \) of \( \omega(\alpha) \) in the Delone triangulation of \( H_0 \).

The map \( \Omega \) is actually a covering projection. For this it is sufficient to show that for each \( z \in \Lambda \) the open star \( \text{OpStar}_{\mathcal{D}_\Lambda}(z) = \text{Int}(\text{Star}_{\mathcal{D}_\Lambda}(z)) \) is evenly covered by open stars in \( \tilde{T} \). More precisely we demonstrate that the inverse image

\[
\Omega^{-1}(\text{OpStar}_{\mathcal{D}_\Lambda}(z)) = \bigcup_{\omega(\alpha) = z} \text{OpStar}_{\tilde{T}}(\alpha)
\]

is a disjoint union of open stars in \( \tilde{T} \). Let \( \alpha = _\alpha \alpha_x \) and \( \beta = _\beta \beta_y \) be two edge-paths representing two vertices in \( \tilde{T} \). We want to show that if \( \omega(v_0 \alpha_x = z = \omega(v_0 \beta_y) \) and \( \text{Star}_{\tilde{T}}(\alpha) \cap \text{Star}_{\tilde{T}}(\beta) \neq \emptyset \) then \( \alpha \) and \( \beta \) represent the same vertex in \( \tilde{T} \).

Assume the opposite. As a consequence of Proposition 5.8 we know that \( x = y \), i.e. \( \alpha \) and \( \beta \) share the same end-point. It follows that \( \xi = \alpha - \beta \) is a cycle in \( T \) which defines a non-trivial homology class (otherwise \( \alpha \) and \( \beta \) would represent the same vertex in \( \tilde{T} \)).

The intersection \( K = \text{Star}_{\tilde{T}}(\alpha) \cap \text{Star}_{\tilde{T}}(\beta) \) is a subcomplex of both stars. If this intersection is non-empty then it contains a vertex \( e \) of both stars, hence \( \omega(e) = \omega(\alpha) + a_I = \omega(\beta) + a_J \) for some subsets \( I \) and \( J \) of \([k]\). This implies that

\[
\Omega(\xi) = \omega(\xi) = \omega(\alpha) - \omega(\beta) = a_I - a_J = a_I + a_{J'}.
\]
However, Proposition 5.9 says that this is not possible. In other words the cycle $\xi$ has too small image $\omega(\xi)$ for a non-trivial homology class. Hence the cycle $\alpha - \beta$ is trivial and the edge paths $\alpha$ and $\beta$ represent the same vertex in the universal cover $\tilde{T}$.

Finally, since $\tilde{T}$ is connected and $D_\Lambda$ is simply connected, we conclude that the covering map $\Omega$ must be an isomorphism of simplicial complexes. □

**Completion of the proof of Theorem 1.5:** The isomorphism $\Omega$ is clearly $\Gamma$-equivariant, where $\Gamma = H_1(Tonn^{n,k}(L);\mathbb{Z})$ acts on $D_\Lambda$ via the monomorphism $\tilde{\omega}$ from Proposition 5.7 (see also the formula (5.6)). It immediately follows that $Tonn^{n,k}(L)$ isomorphic to the simplicial complex $D_\Lambda/\Lambda_L$ where $\Lambda_L := \tilde{\omega}(\Gamma) \subset \Lambda$ is a free abelian group of rank $k - 1$. □

### 6 Examples and concluding remarks

The isomorphism $\Omega$, described in Proposition 5.10, can be used for comparison of combinatorial types of different complexes of Tonnetz type.

Note that each automorphism of the Delone simplicial complex $D_\Lambda$ induces an isometry on the ambient euclidean space $H_0 \subset \mathbb{R}^k$. If two simplicial complexes $T_1$ and $T_2$ of Tonnetz type are combinatorially isomorphic then their universal covers $\tilde{T}_1$ and $\tilde{T}_2$ are also combinatorially isomorphic.

From these two observation we conclude that each Tonnetz inherits a canonical metric from the euclidean space $H_0$ which is an invariant of its combinatorial type.

![Figure 4](image)

**Figure 4:** Combinatorially non-isomorphic complexes of Tonnetz type.

For illustration the classical Tonnetz, exhibited in Figure 4, is non-isometric (and therefore combinatorially non-isomorphic) to the “Tonnetz” shown in the same figure on the right.
This can be proved by comparing the shortest closed, non-contractible geodesics (systoles) of both complexes. For example the systole on the left has the length 3, while on the right the length is $\sqrt{7}$.

The complex $T_{12,3}(2,3,7)$ and the complex $T_{12,3}(1,2,9)$ (exhibited in Figure 5) are isometric, in particular have systoles of the same length. Moreover, they are combinatorially isomorphic. Indeed, if we cut out the parallelogram $5-3-4-6$ from Figure 5(b) and glue it on the opposite side, we obtain a fundamental domain of the “Tonnetz” $T_{12,3}(1,2,9)$ which, by an automorphism of the planar Delone complex $D_\Lambda$, can be mapped to the Figure 5(a).

Figures 4 and 5 were originally generated by lifting the triangulations from a Tonnetz $T$ to its universal cover $\tilde{T}$. Informally speaking, they are obtained by gradually unfolding the complex $T$ in the plane until the picture becomes periodic.

Results from Section 4 and 5, as summarized in the following proposition, allow us to generate these and related pictures (for an arbitrary $T_{n,k}(L)$) directly from the input length vector $L = (l_1, l_2, \ldots, l_k)$.

**Proposition 6.1.** The lattice $\Lambda_L := \mathcal{\tilde{\omega}}(\Gamma) \subset \Lambda$, which appears in the isomorphism $T_{n,k}(L) \cong D_\Lambda/\Lambda_L$, has the following explicit description

$$\Lambda_L = \{ p_1 a_1 + \cdots + p_k a_k \mid (\forall i) p_i \in \mathbb{Z} \text{ and } (\exists p_0 \in \mathbb{Z}) p_1 l_1 + \cdots + p_k l_k = p_0 n \}$$

where $a_i = \omega(E_i) \in \Lambda$ are the vectors introduced in Remark 5.3.

**Example 6.2.** Let us explicitly describe the group $\Lambda_L$ for $T_{12,3}(2,3,7)$. The lattice

$$\Gamma_L = \{(x, y, z) \in \mathbb{Z}^3 \mid 2x + 3y + 7z = 0\}$$
has a parametric presentation
\[ \Gamma_L = \{(x, y, z) \in \mathbb{Z}^3 \mid (\exists r, s \in \mathbb{Z}) \mid x = -3r - 5s, y = 2r + s, z = s \}. \]

By choosing \((r, s) = (1, 0)\) and \((r, s) = (0, 1)\) we obtain that \(\{(-3, 2, 0), (-5, 1, 1)\}\) is a basis for \(\Gamma_L\). It follows that the corresponding generators of the lattice \(\Lambda_L\) are \(b_1 = -3a_1 + 2a_2\) and \(b_2 = -5a_1 + a_2 + a_3 = -6a_1\). By interpreting (in Figure 4) \(a_1\) and \(a_2\) as the vectors connecting the vertex (labeled by) 10 by the neighbouring vertices 0 and 1, we easily check that vectors \(b_1\) and \(b_2\) preserve the labeling of this lattice. They actually generate this lattice since \(\langle b_1, b_2 \rangle\) is a sublattice of \(\Lambda\) of index 12.

6.1 Irrational Tonnetz

It is natural to extend the definition of the tonnetz to the case \(n = +\infty\), interpreted as the limit case when \(k\) is fixed and \(n\) approaches infinity. Informally, vertices are points on a circle \(C\) with circumference 1 while simplices are finite subsets \(I \subset C\) which are \(L\)-admissible in the sense of the following definition.

**Definition 6.3.** Let \(C = \mathbb{R}/\mathbb{Z} = [0, 1]/\langle 0 \simeq 1 \rangle\) be a circle with induced group structure and the corresponding (circular) order. Suppose that \(L = (l_1, l_2, \ldots, l_k)\) is a collection of positive real numbers such that:

1. The numbers \(l_i\) add up to one, \(l_1 + l_2 + \cdots + l_k = 1\).
2. \(L\) is irrational in the sense that \(p_1l_1 + \cdots + p_kl_k \neq 0\) for each \(p \in \mathbb{Z}^k \setminus \{0\}\).

A subset \(I \subset C\) is \(L\)-admissible if there exists \(x \in C\) and a permutation \(\pi \in S_k\) such that

\[ \Delta(x; \sigma) = \{x, x + l_{\sigma(1)}, x + l_{\sigma(1)} + l_{\sigma(2)}, \ldots, x + l_{\sigma(1)} + \cdots + l_{\sigma(k-1)}\}. \quad (6.4) \]

Full (irrational) tonnetz \(F\)-Ton\(\infty, k\)(\(L\)) is the \((k-1)\)-dimensional simplicial complex of all \(L\)-admissible subsets of \(C\). The irrational tonnetz \(Ton\infty, k\)(\(L\)) is a connected component of the full Tonnetz \(F\)-Ton\(\infty, k\)(\(L\)).

**Remark 6.5.** A rotation of the circle \(C\) induces an automorphism of the full tonnetz \(F\)-Ton\(\infty, k\)(\(L\)). Moreover, the group \(C\) acts transitively on its connected components. It follows that all connected components of the irrational tonnetz are isomorphic.

Note that the condition (2) in Definition 6.3 guarantees that the length vector \(L\) is generic in the sense that numbers \(l_i\) satisfy the condition \((1.1)\).

**Theorem 6.6.** Irrational tonnetz \(Ton\infty, k\)(\(L\)) is isomorphic to the Delone triangulation of the vector space \(\mathbb{R}^{k-1}\) associated to the permutohedral lattice \(A^*_{k-1}\),

\[ Ton\infty, k(L) \cong D_\Lambda. \]
Proof: Many concepts introduced in Section 3, such as atomic 1-simplices $E_i$, cocycles $\theta_{i,j}$, canonical cocycles $\omega_i$ etc., preserve their meaning in the case of the infinite tonnetz $\text{Tonn}^\infty(k)(L)$. An exception are canonical cycles $c_i$ whose existence is ruled out by the condition (2) from Definition 6.3. Proposition 2.12 still holds with essentially the same proof so the fundamental group of the infinite tonnetz is always abelian. Let us show that it is actually a trivial group.

As before each 1-chain has a representation $X = E_1^{p_1} E_2^{p_2} \cdots E_k^{p_k}$. If this is a cycle (with winding number $p_0$) then $p_1 l_1 + \cdots + p_k l_k = p_0$. In light of the condition (2) (Definition 6.3) this is possible only if $p_1 = \cdots = p_k = p_0$ in which case $X$ is a boundary.

The end of the proof follows closely the idea of the proof of Proposition 5.10. The isomorphism $\Omega : \text{Tonn}^\infty(k)(L) \to \mathcal{D}_\Lambda$ is again defined with the aid of formula (5.6). □

6.2 Genericity condition (1.1)

The genericity condition (1.1) plays a central role in many arguments. For illustration a generalized tonnetz may not be a manifold without this condition, as visible from the classification of all 2-dimensional (not necessarily generic) complexes of Tonnetz type, see [3, Section 6].

The smallest examples $(k \geq 3)$ of length vectors which are generic are:

- $(1, 2, 4)$ for $(n, k) = (7, 3)$
- $(1, 2, 5)$ for $(n, k) = (8, 3)$
- $(1, 2, 6), (2, 3, 4)$ for $(n, k) = (9, 3)$
- $(1, 2, 7), (1, 3, 6)$ for $(n, k) = (10, 3)$
- $(1, 2, 8), (1, 3, 7), (1, 4, 6), (2, 3, 6), (2, 4, 5)$ for $(n, k) = (11, 3)$
- $(1, 2, 9), (1, 3, 8), (1, 4, 7), (2, 3, 7), (3, 4, 5)$ for $(n, k) = (12, 3)$, etc.

It is not difficult to construct examples of families of generic vectors as illustrated by $(1, q, q^2, \ldots, q^{k-1})$ for $q \geq 2$.

When $k$ is fixed, asymptotically (when $n \to \infty$) almost all vectors are generic. This can be deduced by observing that generic vectors are positive integer vectors in a simplex with vertices $ne_i (i = 1, \ldots, k)$ outside the union of the hyperplane arrangement $\mathcal{H}_k = \{H_{I,J}\}$ where for two disjoint, non-empty subspaces $I, J \subset [k]$

$$H_{I,J} = \{x \in \mathbb{R}^k \mid \sum_{i \in I} x_i = \sum_{j \in J} x_j\}.$$ 

6.3 Other generalizations of the Tonnetz

There are other generalizations of the classical Tonnetz, see for example [3, 5, 11] or [12]. The authors of these papers usually put more emphasis on combinatorial and geometric aspects of the musical theory and see mathematics primarily as a useful tool. These papers do not overlap with our exposition with an exception of [3] where the author introduced and studied the complexes of Tonnetz type in the case $k = 3$ (without the genericity condition (1.1)). In particular our Proposition 2.5 is included in [3, Theorem 23]. Moreover the
author provides the list of all 2-dimensional complexes which in the non-generic case can arise as Tonnetz-type complexes.

**Acknowledgements:** We would like to acknowledge valuable remarks and kind suggestions of the anonymous referee which helped us improve the presentation of results in the paper.

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