THE LARGE AMPLITUDE SOLUTION OF THE BOLTZMANN EQUATION WITH SOFT POTENTIAL

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Abstract. In this paper, we deal with the (angular cut-off) Boltzmann equation with soft potential ($-3 < \gamma < 0$). In particular, we construct a unique global solution in $L^\infty_{t,x,v}$ which converges to global equilibrium asymptotically provided that initial data has a large amplitude but with sufficiently small relative entropy. Because frequency multiplier is not uniformly positive anymore, unlike hard potential case, time-involved velocity weight will be used to derive sub-exponential decay of the solution. Motivated by recent development of $L^2$-$L^\infty$ approach also, we introduce some modified estimates of quadratic nonlinear terms. Linearized collision kernel will be treated in a subtle manner to control singularity of soft potential kernel.

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1. Introduction

The Boltzmann equation is one of the fundamental mathematical models for collisional rarefied gas theory. The equation describes behavior of the system via distribution function $F(t,x,v)$ where $(t,x,v) \in [0,\infty) \times \Omega \times \mathbb{R}^3$ where $\Omega$ is spatial domain and $\mathbb{R}^3$ is a domain for velocity variable. In this paper we choose periodic domain $\Omega = \mathbb{T}^3$. The Boltzmann equation takes the following form:

\[
\partial_t F + v \cdot \nabla_x F = Q(F,F),
\]

where the bilinear Boltzmann collision operator acting only on velocity variables $v$ is given by

\[
Q(G,F)(v) = \int_{\mathbb{R}^3} \int_{S^2} B(v-u,\omega)G(u')F(v') \, d\omega du - \int_{\mathbb{R}^3} \int_{S^2} B(v-u,\omega)G(u)F(v) \, d\omega du \tag{1.2}
\]

Here the post-collisional velocity $(u',v')$ and the pre-collisional velocity $(u,v)$ satisfy the relation

\[ u' = u + [(v-u) \cdot \omega] \omega, \quad v' = v - [(v-u) \cdot \omega] \omega, \]

with $\omega \in S^2$, according to conservation laws of momentum and energy of two particles for an elastic collision

\[ u + v = u' + v', \quad |u|^2 + |v|^2 = |u'|^2 + |v'|^2. \]
For the collision kernel $B(v-u, \omega)$, it depends only on the relative velocity $|v-u|$ and $\cos \theta := (v-u) \cdot \omega / |v-u|$. Throughout the paper, we assume Grad’s angular cut-off and soft potential:

$$B(v-u, \omega) = |v-u|^\gamma q_0(\theta), \quad -3 < \gamma < 0, \quad 0 \leq q_0(\theta) \leq C|\cos \theta|. \quad (1.3)$$

Since we have removed angular singularity (cutoff assumption), it is convenient to use $Q_\pm(G, F)$ to denote the gain and loss terms in (1.2), respectively. We also consider the Boltzmann equation (1.1) with the following initial data

$$F(0, x, v) = F_0(x, v), \quad (x, v) \in \mathbb{T}^3 \times \mathbb{R}^3. \quad (1.4)$$

Our aim is to construct the unique global solution $F(t, x, v) \geq 0$ of (1.1) and (1.4), converging in large time to the global Maxwellian

$$\mu(v) = \frac{1}{(2\pi)^{\frac{3}{2}}} e^{-\frac{|v|^2}{2}} \quad (1.5)$$

imposing initial data $F_0$ which satisfies mass and energy conservation (initial compatibility condition)

$$\int_{\mathbb{T}^3 \times \mathbb{R}^3} (F - \mu) dx dv = 0, \quad \int_{\mathbb{T}^3 \times \mathbb{R}^3} (F - \mu)|v|^2 dx dv = 0. \quad (1.6)$$

Meanwhile, one of the most important quantity in the Boltzmann theory is entropy

$$H(F)(t) := \int_{\mathbb{T}^3 \times \mathbb{R}^3} F \log F dx dv$$

which has non-increasing in time property by celebrated H-theorem. Since we expect the global solution to converges to global Maxwellian (1.5), $\mu$ is the state with minimal entropy. To measure difference of entropy between $F$ and $\mu$, we define relative entropy

$$\mathcal{E}(F) := \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \left( \frac{F}{\mu} \ln \frac{F}{\mu} - \frac{F}{\mu} + 1 \right) \mu dv dx, \quad (1.7)$$

which is identical to $\mathcal{E}(F) = \int \int F \ln \frac{F}{\mu} dv dx$ under mass conservation and to $H(F) - H(\mu)$ under both mass and energy conservation (1.6). Note that the integrand in (1.7) is nonnegative since it holds that $a \ln a - a + 1 \geq 0$ for any $a > 0$.

Because of its importance, there have been lots of mathematical studies for the Boltzmann equation. In celebrated work \cite{5}, the authors proved global in time existence of renormalized solution for very general large data $F_0 > 0$ without any smallness assumptions. However, in the aspect of statistical physics, one of the most important question for the Boltzmann equation is its asymptotic behavior, e.g., convergence to Maxwellian (equilibrium state). For this issue, \cite{4} proved convergence to equilibrium of the Boltzmann equation using the results about Cercignani’s conjecture \cite{29} under the global in time high order regularity assumption.

Meanwhile, well-posedness results, including uniqueness, have been widely developed in near perturbation framework. In \cite{19, 18}, Guo developed high order regularity framework with small data to give well-posedness of asymptotic behavior of the equation. We also refer to \cite{28, 6}. Despite its practical importance, however, there were few results about well-posedness of boundary problems of the Boltzmann equation such as specular reflection, diffuse reflection, etc. (See \cite{27}). To the best of author’s knowledge, high order regularity solution of the Boltzmann equation with many physical boundary conditions are not known yet. To overcome this difficulty, Guo \cite{20} suggested low regularity $L^\infty$ mild solution of the boundary problem. In particular, he developed $L^2-L^\infty$ bootstrap argument to derive $L^\infty$ decay from $L^2$ decay. In fact, low regularity approach looks quite crucial, considering \cite{23} which imply limitedness of high order regularity solution.

The low regularity approach has been widely used after \cite{20}. In \cite{25}, with his collaborator, the second author removed strong analyticity assumption of \cite{20} for specular reflection boundary condition case and extended the result to general $C^3$ convex domain. In the specular reflection boundary condition, convexity condition is fairly important to handle linear trajectory. In the following study, convexity condition was partially removed in \cite{24} and Vlasov-Poisson-Boltzmann model is also studied in \cite{6}. All these works deal with hard potential cases. For soft potential case (1.3), we refer \cite{26}, where the author treated both diffuse and specular boundary conditions. We also refer to other important works: \cite{4} for polynomial tail (weight).
and [22] for Maxwell boundary condition.

The forementioned works with low regularity approach treated small data problem, i.e., initial data $F_0(x, v)$ and $\mu$ are sufficiently close in $L^\infty$ sense with some velocity weight. Global well-posedness and convergence to equilibrium for large data looks extremely hard and is far beyond our knowledge. To extend small data results to broader class of initial data, [11] suggested large amplitude solution of the Boltzmann equation. In their work, initial data $F_0(x, v)$ can be artificially far from $\mu$ in $L^\infty$ sense but still very close in $L^1_t L^\infty_x$ sense. This large amplitude result was adopted to diffuse boundary condition problem in [13] with small $L^2_x L^\infty_t$ data. And recently, combining with idea of [25], the large amplitude problem with specular reflection in $C^3$ convex domain with small relative entropy [17] has been resolved in [14].

In this paper, we solve the large amplitude solution of the Boltzmann equation with soft potential in $\mathbb{T}^3$ domain with small relative entropy solution. We note that [11] already treated $\mathbb{T}^3$ soft potential case, but we impose relative entropy condition instead of strong $L^1_t L^\infty_x$ condition. Moreover, the result of [11] gives only decay of $f = \frac{F - \mu}{\sqrt{\mu}}$, instead of weighted $w(v)f = w(v)\frac{F - \mu}{\sqrt{\mu}}$, even though we should impose initial condition for weighted one.

Lastly, we briefly mention about loss of velocity weight in the soft potential problem in torus $\mathbb{T}^3$. Unlike to hard potential problem, it seems natural to lose some velocity weight as we see in [2] and [28]. For example, in [28], one loses whole exponential weight to obtain sub-exponential decay. Otherwise, one lose only some polynomial degree of velocity weight to get polynomial decay to Maxwellian. Using time-involved velocity weight in this paper, we lose some velocity weight, but we do not lose whole exponential weight while keeping sub-exponential decay in the case of large amplitude problem. In this sense, it is very interesting to note the result of [12] where they did not lose any weight using advantage of diffuse boundary condition! It looks that the diffuse boundary condition yields even stronger decaying property than torus. It is not clear for us, however, if the specular boundary condition also preserve whole velocity weight. Of course, we strongly expect that the time-involved weight can give large amplitude solution for the specular boundary condition, in the case of analytic boundary case, at least.

1.1. Perturbation framework. We rewrite the Boltzmann equation (1.1) in standard perturbation with Gaussian tail, $F(t, x, v) = \mu + \sqrt{\mu} f(t, x, v)$. Then, the Boltzmann equation (1.1) can be expressed as

$$
\partial_t f + v \cdot \nabla_x f + Lf = \Gamma(f, f),
$$

(1.8)

with the usual notations on the linearized operator

$$
Lf := -\frac{1}{\sqrt{\mu}} \{Q(\mu, \sqrt{\mu} f) + Q(\sqrt{\mu} f, \mu)\} = \nu(v) f - Kf,
$$

(1.9)

$$
Kf = -\frac{1}{\sqrt{\mu}} \{Q_+(\mu, \sqrt{\mu} f) + Q_+(\sqrt{\mu} f, \mu) + Q_-(\sqrt{\mu} f, \mu)\} := \int_{\mathbb{R}^3}^{\gamma} k(v, u) f(u) du,
$$

(1.10)

(we abbreviated time and space variable of $f$ for simplicity) and the quadratic nonlinear term

$$
\Gamma(g, f) := \frac{1}{\sqrt{\mu}} Q(\sqrt{\mu} g, \sqrt{\mu} f).
$$

Here, we have used $\nu(v)$ to denote the collision frequency

$$
\nu(v) := \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |v - u|^{\gamma} q_0(\theta) \mu(u) d\omega du, \quad -3 < \gamma < 0.
$$

(1.11)

It is well-known that there exist generic positive constants $C_{1, \gamma}, C_{2, \gamma} > 0$ such that

$$
C_{1, \gamma}(1 + |v|^2)^{\gamma/2} \leq \nu(v) \leq C_{2, \gamma}(1 + |v|^2)^{\gamma/2}, \quad -3 < \gamma < 0.
$$

(1.12)

The integral operator $Kf$ in (1.9) can be further split into $Kf = K_2f - K_1f$ with

$$
(K_1f)(v) = \int_{\mathbb{R}^3 \times \mathbb{S}^2} |v - u|^{\gamma} q_0(\theta) \sqrt{\mu(u)} \sqrt{\mu(v)} f(u) d\omega du := \int_{\mathbb{R}^3} k_1(v, u) f(u) du,
$$

$$
(K_2f)(v) = \int_{\mathbb{R}^3 \times \mathbb{S}^2} |v - u|^{\gamma} q_0(\theta) \sqrt{\mu(u)} \left[ \sqrt{\mu(v')} f(u') + \sqrt{\mu(u')} f(v') \right] d\omega du := \int_{\mathbb{R}^3} k_2(v, u) f(u) du,
$$

(1.13)
where \( k(v, u) := k_2(v, u) - k_1(v, u) \) by (1.10).

We also note that (1.11) satisfy mass and energy conservations (1.6), and in our perturbation framework, those are rewritten by

\[
\int_{T^3} \int_{\mathbb{R}^3} \sqrt{\mu} f(t, x, v) \, dv \, dx = 0, \quad \int_{T^3} \int_{\mathbb{R}^3} |v|^2 \sqrt{\mu} f(t, x, v) \, dv \, dx = 0, \quad t \geq 0. \tag{1.13}
\]

1.2. Main result and scheme of the proof. We solve (1.8) with (1.3) in torus domain \( T^3 \) for given initial data \( f_0 \). The following theorem is the main result of this paper.

**Theorem 1.1.** Let \((t, x, v) \in \mathbb{R}_+ \times T^3 \times \mathbb{R}^3 \) and assume initial data \( f_0 \) satisfies (1.12). Define a weight function

\[
w_{q, \vartheta, \beta}(v, t) = (1 + |v|^2)^2 \exp \left\{ \frac{q}{8} \left( 1 + \frac{1}{(1 + t)^\vartheta} \right) |v|^2 \right\}, \tag{1.14}
\]

where \( \beta \geq 7/2, \, 0 \leq \vartheta < \frac{3}{\gamma} \) and \( q \) satisfies (2.11). Then for any \( M_0 > 0 \), there exists \( \epsilon_0 > 0 \) such that if \( f_0 \) satisfies

\[
\| w_{q, \vartheta, \beta} f_0 \|_{L^\infty} \leq M_0, \quad \mathcal{E}(F_0) \leq \epsilon_0, \tag{1.15}
\]

there exists a unique global solution \( F(t, x, v) = \mu(v) + \sqrt{\mu(v)} f(t, x, v) \geq 0 \) to Boltzmann equation (1.1) with soft potential (1.3) on \( T^3 \times \mathbb{R}^3 \). Moreover, \( f \) satisfies

\[
\| w_{q, \vartheta, \beta} f(t) \|_{L^\infty} \leq \left( 4CM_0 \exp \left\{ \frac{4CM_0}{\lambda_0 \rho} \right\} + 4M_0 \right) e^{-\lambda_0 t^\vartheta},
\]

for some \( \lambda_0 > 0 \), where \( \rho - 1 = \frac{(1 + \vartheta)\gamma}{2 - \gamma} > 0 \).

One of the main difference to hard potential is behavior of collision frequency. Let us recall uniform estimate of collision frequency

\[
\nu(v) := \int_{\mathbb{T}^3 \times \mathbb{R}^3} B(v-u, \omega) \mu(u) d\omega du \approx (v)^\gamma := (1 + |v|^2)^\frac{\gamma}{2}, \quad v \in \mathbb{R}^3. \tag{1.16}
\]

If we read (1.8) with (1.9) along linear trajectory

\[
X(s; t, x, v) = x - v(t - s), \quad V(s; t, x, v) = v,
\]

(Here, \( X(s; t, x, v) \) and \( V(s; t, x, v) \) mean position and velocity of a particle at time \( s \) which was at \( t, x, v) \)), we have linearized exponential decay \( e^{-\nu(v)|v|^2} \) along the trajectory. In the case of hard potential \( 0 \leq \gamma \leq 1 \), (1.16) has a uniformly positive lower bound and hence we get uniform exponential decay factor. However, with soft potential \( -3 < \gamma < 0 \), (1.16) is not uniformly positive anymore since \( |v| \rightarrow \infty \). To resolve this problem without abandoning weight, we adopt time-involved velocity weight

\[
w_{q, \vartheta, \beta}(v) = (1 + |v|^2)^2 \exp \left\{ \frac{q}{8} \left( 1 + (1 + t)^{-\vartheta} \right) |v|^2 \right\}, \quad \beta \geq 7/2, \quad 0 < q < 1, \quad 0 \leq \vartheta < \frac{3}{\gamma}, \tag{1.17}
\]

which was introduced in [7, 8, 9, 10]. Note that exponent factor \( \frac{q}{8} (1 + (1 + t)^{-\vartheta}) \) is uniform for all \( 0 \leq t < \infty \). Putting time dependence we get extra coercive factor in the equation of \( h := w_{q, \vartheta, \beta} f \) :

\[
\partial_t h + v \cdot \nabla h + \left( \nu(v) + \frac{\partial q |v|^2}{8(1 + t)^{\vartheta + 1}} \right) h = w_{q, \vartheta, \beta} K f + w_{q, \vartheta, \beta} \Gamma(f, f).
\]

Applying Young’s inequality, we are able to deduce a lower bound depending on time for a new collision frequency \( \tilde{\nu}(t, v) \)

\[
\tilde{\nu}(t, v) := \nu(v) + \frac{\partial q |v|^2}{8(1 + t)^{\vartheta + 1}} \geq C_{q, \vartheta} (1 + t)^{(1 + \vartheta)\gamma/2 - \gamma}, \quad (1.18)
\]

by (3.5) which implies the subexponential time decay property. Specifically, from \( -3 < \gamma < 0 \) and \( 0 \leq \vartheta < \frac{3}{\gamma} \),

\[
0 > \frac{(1 + \vartheta)\gamma}{2 - \gamma} > \frac{\gamma - 2}{2 - \gamma} = -1.
\]
In large amplitude problem, we cannot bound nonlinear quadratic $\Gamma(f, f)$ as $\|wf\|_{L^\infty}$ since $L^\infty$ could be artificially large in general. Instead, we perform $L^\infty \times L^p$ type estimate

$$|w\Gamma_\pm(f, f)| \lesssim \langle v \rangle^\gamma \|wf\|_\infty \left( \int |\xi(u)| |wf(u)|^p du \right)^{1/p}, \tag{1.19}$$

for nonlinear $\Gamma(f, f)$ where $\xi(u)$ has proper decay. (Lemma 2.4) Unlike to hard potential case ([13], [14]), however, $\Gamma_-$ can be treated in another way. Nonlinear $\Gamma_-$ contains local term $f(v)$, so we see factor $\langle v \rangle^\gamma$. In the case of hard potential this gives growth in $|v|$ so $\Gamma_-$ was combined with collision frequency and treated separately. In soft potential case, $\langle v \rangle^\gamma$ gives polynomial decay in $|v|$ and there is no big problem in dealing with $\Gamma_-$.

Once we have $L^\infty \times L^p$ type estimate (1.19), we can treat $\Gamma_\pm$ terms similar as linear $Kf$ except for extra $\|wf\|_\infty$. Motivated by Duhamel iteration and $L^2$-$L^\infty$ scheme, we will derive $\|f\|_{L^2_{t,v}}$ in double Duhamel iteration. Since relative entropy control $L^2$ and $L^1$ of $f$ partially, (see Lemma 2.6) we can close proper apriori estimate and bootstrap argument allows us to extend the time interval until $\|wf\|_\infty$ becomes sufficiently small so that small data theory becomes valid.

Lastly, we briefly comment about some delicate issues of treating kernel $k(v, u)$ of linear operator $K$ in (1.2). Because of singular behavior of kernel near $v \approx u$ ($|v - u|^{-\gamma}$ is not square integrable locally), we adopt cutoff kernel [24, 28] to treat singular region separately,

$$k(v, u) = k^\chi(v, u) + k^{1-\chi}(v, u)$$

where $k^\chi(v, u)$ has support away from singularity and $k^{1-\chi}(v, u)$ corresponds to very small region near singularity. When we perform Duhamel iteration twice, the most delicate issue comes when we have interaction between $k^\chi(v, u)$ in the first iteration and $k^{1-\chi}(u, u')$ in the second iteration. (We have similar issue when we play with $\Gamma$, but by structure (1.19), it will be resolved by similar manner.) In fact we have to control

$$\int_0^t e^{-\int_0^t \hat{\nu}(v, \tau) d\tau} \int_{u'} k^\chi_{\hat{w}}(v, u) \int_0^\infty e^{-\int_0^\tau \hat{\nu}(u, \tau') d\tau'} \int_{u''} k^{1-\chi}_{\hat{w}}(u, u') h(u') d\tau' \tag{1.20}$$

and, unlike hard potential case, time integration does not help us, because of polynomial growth $(-3 < \gamma < 0)$

$$\int_0^t e^{-\int_0^t \hat{\nu}(v, \tau) d\tau} \sim \nu^{-1}(v) \quad \text{and} \quad \int_0^\infty e^{-\int_0^\tau \hat{\nu}(u, \tau') d\tau'} \sim \nu^{-1}(u).$$

Fortunately, from (2.8), we obtain Gaussian decaying factor and we can obtain small control

$$\int_0^t e^{-\int_0^t \hat{\nu}(v, \tau) d\tau} \sim \nu^{-1}(v) \int_{u'} k^\chi_{\hat{w}}(v, u) \nu^{-1}(u) \int_{u''} k^{1-\chi}_{\hat{w}}(u, u') h(u') d\tau' \int v^{-1}(v) \int u^{-1}(u) \mu(u)$$

$$\sim \nu^{-3} \|h\|_{L^\infty} \nu^{-1}(v) \int k(v, u) \frac{(1 + |v|^2)^{2\beta}}{(1 + |u|^2)^{2\beta}} \frac{(1 + |v|^2)^{2\beta}}{(1 + |u'|^2)^{2\beta}} \nu^{-1}(u) \mu(u)$$

$$\sim \nu^{-3} \|h\|_{L^\infty} \nu^{-1}(v) \int u^{-1}(u) \mu(u)$$

$$\sim \nu^{-3} \|h\|_{L^\infty}.$$
Lemma 2.2. 

where $\delta$ is a generic constant and $h$ is a symmetric integral kernel of $K$ for $i = 1, 2$.

Lemma 2.1. The integral kernel $k_1$ and $k_2$ of (2.1) satisfy

$$0 \leq k_1(v, u) \leq C|v - u|e^{-\frac{|u|^2}{8}}e^{-\frac{|u|^2}{32}},$$

and

$$0 \leq k_2(v, u) \leq \frac{C_1}{|v - u|^2}e^{-\frac{|v - u|^2}{8}}e^{-\frac{|v - u|^2}{32}},$$

where $C > 0$ is a generic constant and $C_1$ is a constant depending on $\gamma$.

To treating the singularity of $K$, we introduce modified kernel with smooth cutoff:

$$\chi(|v - u|) = \begin{cases} 1, & \text{if } |v - u| \geq 2\epsilon, \\ 0, & \text{if } |v - u| \leq \epsilon. \end{cases}$$

We split the operator $K$ using cutoff function $\chi$:

$$Kf = K_1^x f + K_1^{1-x} f, \quad K_2 f = K_2^x f + K_2^{1-x} f, \quad \text{and} \quad K_1 f = K_1^x f + K_1^{1-x} f.$$}

Specifically, for $i = 1, 2$, we define

$$K_1^x f = \int_{\mathbb{R}^3} \chi(|v - u|)k_i(v, u)f(u)du = \int_{\mathbb{R}^3} k_i^x(v, u)f(u)du,$$

$$K_1^{1-x} f = \int_{\mathbb{R}^3} (1 - \chi(|v - u|))k_i(v, u)f(u)du = \int_{\mathbb{R}^3} k_i^{1-x}(v, u)f(u)du,$$

$$K_2^x f = \int_{\mathbb{R}^3} \chi(|v - u|)k_i(v, u)f(u)du = \int_{\mathbb{R}^3} k_i^x(v, u)f(u)du,$$

$$K_2^{1-x} f = \int_{\mathbb{R}^3} (1 - \chi(|v - u|))k_i(v, u)f(u)du = \int_{\mathbb{R}^3} k_i^{1-x}(v, u)f(u)du.$$

Lemma 2.2. There are constants $C > 0$ and $C_1 > 0$ depending on $\epsilon$ such that

$$|k_1^x(v, u)| \leq C\epsilon^{-1}\exp\left(-\frac{1}{8}|v - u|^2 - \frac{1}{8}(\frac{|u|^2 - |v-u|^2)^2}{|v-u|^2}\right),$$

or

$$|k_2^x(v, u)| \leq C\epsilon^{-1}\exp\left(-\frac{3}{8}|v - u|^2 - \frac{3}{8}(\frac{|u|^2 - |v-u|^2)^2}{|v-u|^2}\right),$$

for any $0 < s_1 < s_2 < 1$.

With a weight introduced in (1.14), we have the following linearized operator estimates.
Lemma 2.3.20 There exists a constant \( C > 0 \) such that
\[
 w_{q,\vartheta,\beta}(v)K^{1-x}f \leq C\mu(v)^{\frac{1}{1-x}}\epsilon^{\gamma+3}\|w_{q,\vartheta,\beta}f\|_{L^\infty},
\]
and
\[
 w_{q,\vartheta,\beta}(v)\int_{\mathbb{R}^3}K(v,u)e^{\epsilon|v-u|^2}|f(u)|\,du \leq C_q\epsilon(v)^{\gamma-2}\|w_{q,\vartheta,\beta}f\|_{L^\infty}.
\]

Proof. Note that
\[
 w_{q,\vartheta,\beta}K^{1-x}f = w_{q,\vartheta,\beta}K_1^{1-x}f + w_{q,\vartheta,\beta}K_2^{1-x}f.
\]
We first consider \( K_1^{1-x} \) in (2.10).
\[
 w_{q,\vartheta,\beta}(v)K_1^{1-x}f \leq Cw_{q,\vartheta,\beta}(v)\sqrt{\mu(v)}\int_{|v-u| \leq 2\epsilon} |v-u|^{\gamma}\sqrt{\mu(u)}\frac{|w_{q,\vartheta,\beta}f(u)|}{w_{q,\vartheta,\beta}(u)}\,du
\]
\[
 \leq C(1+|v|^2)^{\beta}\exp\left(\frac{9}{8}(1+(1+t)^{-\theta})|v|^2\right)\sqrt{\mu(v)}\epsilon^{\gamma+3}\|w_{q,\vartheta,\beta}f(t)\|_{L^\infty}
\]
\[
 \leq C\mu(v)^{\frac{1}{1-x}}\epsilon^{\gamma+3}\|w_{q,\vartheta,\beta}f\|_{L^\infty}.
\]
Next, we will deal with the remaining part \( K_2^{1-x} \) in (2.10).
\[
 w_{q,\vartheta,\beta}(v)K_2^{1-x}f \leq Cw_{q,\vartheta,\beta}(v)\int_{|v-u| \leq 2\epsilon} |v-u|^{\gamma}\sqrt{\mu(u)}\left[\sqrt{\mu(u')}\frac{|w_{q,\vartheta,\beta}f(u')|}{w_{q,\vartheta,\beta}(u')} + \sqrt{\mu(u')}\frac{|w_{q,\vartheta,\beta}f(u')|}{w_{q,\vartheta,\beta}(u')}\right]\,du.
\]
Under \(|v-u| \leq 2\epsilon, \)
\[
|v'| = |v| + [(u-v) \cdot \omega] |v| > |v| - |v-u| \geq |v| - 2\epsilon,
|v'| = |v| + u - v - [(u-v) \cdot \omega] |v| > |v| - 2|v-u| \geq |v| - 4\epsilon,
|u| = |v| + u - v \geq |v| - |v-u| \geq |v| - 2\epsilon,
\]
which implies
\[
\begin{align*}
\frac{\sqrt{\mu(u)}\sqrt{\mu(u')}}{\sqrt{\mu(u')}\sqrt{\mu(u')}} & \leq e^{-\frac{(u-u')^2}{4}}e^{-\frac{(u-u')^2}{4}} \leq e^{-\frac{|v|^2}{2}}e^{3\epsilon|v|} \leq C\sqrt{\mu(v)}, \\
\frac{\sqrt{\mu(u)}\sqrt{\mu(u')}}{\sqrt{\mu(u')}\sqrt{\mu(u')}} & \leq e^{-\frac{(u-u')^2}{4}}e^{-\frac{(u-u')^2}{4}} \leq e^{-\frac{|v|^2}{2}}e^{2\epsilon|v|} \leq C\sqrt{\mu(v)},
\end{align*}
\]
where we used \( \epsilon < 1 \) and the following fact
\[
C\epsilon|v| = \left(\frac{|v|}{2}\right)^2(2C\epsilon) \leq \frac{|v|^2}{4} + C\epsilon.
\]
From (2.13), it holds that
\[
 w_{q,\vartheta,\beta}(v)K_2^{1-x}\left(\frac{|w_{q,\vartheta,\beta}f|}{w_{q,\vartheta,\beta}}\right)(t,x,v) \leq C_q\mu(v)^{\frac{1}{1-x}}\epsilon^{\gamma+3}\|w_{q,\vartheta,\beta}f(t)\|_{L^\infty}.
\]
Summing (2.10), (2.11), and (2.13) yields
\[
 w_{q,\vartheta,\beta}(v)K^{1-x}f \leq C_q\mu(v)^{\frac{1}{1-x}}\epsilon^{\gamma+3}\|w_{q,\vartheta,\beta}f(t)\|_{L^\infty}.
\]
For (2.10), we first consider the part \( K_1^n \):
\[
 w_{q,\vartheta,\beta}(v)\int_{\mathbb{R}^3}K_1^n(v,u)e^{\epsilon|v-u|^2}|f(u)|\,du
\]
\[
 \leq Cw_{q,\vartheta,\beta}(v)\sqrt{\mu(v)}\int_{\mathbb{R}^3}\chi(|v-u||v-u)^{\gamma}\sqrt{\mu(u)}e^{\epsilon|v-u|^2}|w_{q,\vartheta,\beta}f(u)|\,du
\]
\[
 \leq C_q\int_{\mathbb{R}^3}|v-u|^{\gamma}\mu(v)^{\frac{1}{1-x}}\sqrt{\mu(u)}e^{\epsilon|v-u|^2}|w_{q,\vartheta,\beta}f(t)|\,du
\]
\[
 \leq C_q\epsilon(v)^{\gamma}\mu(v)^{\frac{1}{1-x}}\|w_{q,\vartheta,\beta}f(t)\|_{L^\infty},
\]
where the last inequality comes from $\int_{\mathbb{R}^3} |v - u|^\gamma \mu(u)^{1/4} \, du \leq C(v)^\gamma$.

It remains to check the part $K_2^v$ for (2.9). Recall (2.7) and take $s_0 = \min\{s_1, s_2\}$. Then,

$$
w_{q, \vartheta, \beta}(v) \int_{\mathbb{R}^3} k_2^v(v, u) \left( \frac{e^{x|v-u|^2} w_{q, \vartheta, \beta}(u)}{w_{q, \vartheta, \beta}(u)} \right) \, du
$$

$$
\leq C_v \|w_{q, \vartheta, \beta} f(t)\|_{L^\infty} \langle v \rangle^{\gamma - 1} w_{q, \vartheta, \beta}(v) \int_{\mathbb{R}^3} \exp \left( -\frac{s_0}{8} |v - u|^2 - \frac{s_0}{8} \frac{(v^2 - |u|^2)^2}{|v - u|^4} \right) e^{x|v-u|^2} \, du. \tag{2.16}
$$

From definition of the weight function (1.17), we can deduce that

$$
\frac{w_{q, \vartheta, \beta}(v)}{w_{q, \vartheta, \beta}(u)} \leq C_\beta (1 + |v - u|^2)^{\beta} e^{-\frac{q}{4}|w^2 - |v|^2|},
$$

where $\tilde{q} = \frac{q}{2} (1 + (1 + t) - \theta)$. Notice that $\frac{q}{2} < \tilde{q} \leq q$. Let $v - u = \eta$ and $u = v - \eta$ in the integral of (2.16). We now compute the total exponent in $k_2^v(v, u) w_{q, \vartheta, \beta}(v)$.

$$
- \frac{s_0}{8} |\eta|^2 - \frac{s_0}{8} \frac{|\eta|^2 - 2 v \cdot \eta|^2}{|\eta|^2} - \frac{\tilde{q}}{4} (|v - \eta|^2 - |v|^2)
$$

$$
= - \frac{s_0}{4} |\eta|^2 + \frac{s_0}{2} v \cdot \eta - \frac{s_0}{2} \frac{|v \cdot \eta|^2}{|\eta|^2} - \frac{\tilde{q}}{4} (|\eta|^2 - 2 v \cdot \eta)
$$

$$
= \frac{1}{4} (\tilde{q} + s_0) |\eta|^2 + \frac{1}{2} (s_0 + \tilde{q}) v \cdot \eta - \frac{s_0}{2} \frac{(v \cdot \eta)^2}{|\eta|^2}.
$$

Setting

$$
0 < \tilde{q} \leq q < s_0 < 1,
$$

where $\tilde{q} = \frac{q}{2} (1 + (1 + t) - \theta)$ and $s_0 = \min\{s_1, s_2\}$, the discriminant of the above quadratic form of $|\eta|$ and $\frac{v \cdot \eta}{|\eta|}$ is

$$
\Delta = \frac{1}{4} (s_0 + \tilde{q})^2 - (\tilde{q} + s_0) \frac{s_0}{2} = \frac{1}{4} (\tilde{q}^2 - s_0^2) < 0.
$$

Thus, we have, for $\epsilon > 0$ sufficiently small and $q < s_0$, that there is $C_q > 0$ such that

$$
- \frac{s_0 - 8 \epsilon}{8} |\eta|^2 - \frac{s_0}{8} \frac{|\eta|^2 - 2 v \cdot \eta|^2}{|\eta|^2} - \frac{\tilde{q}}{4} (|\eta|^2 - 2 v \cdot \eta)
$$

$$
\leq - C_q \left\{ |\eta|^2 + \frac{|v \cdot \eta|^2}{|\eta|^2} \right\}
$$

$$
= - C_q \left\{ \frac{|\eta|^2}{2} + \left( \frac{|\eta|^2}{2} + \frac{|v \cdot \eta|^2}{|\eta|^2} \right) \right\}
$$

$$
\leq - C_q \left\{ \frac{|\eta|^2}{2} + |v \cdot \eta| \right\}. \tag{2.18}
$$

Substituting (2.17) into (2.19), one then has

$$
\|w_{q, \vartheta, \beta} f(t)\|_{L^\infty} \langle v \rangle^{\gamma - 1} w_{q, \vartheta, \beta}(v) \int_{\mathbb{R}^3} \exp \left( -\frac{s_0}{8} |v - u|^2 - \frac{s_0}{8} \frac{(v^2 - |u|^2)^2}{|v - u|^4} \right) e^{x|v-u|^2} \, du
$$

$$
\leq C_{q, \epsilon} \langle v \rangle^{\gamma - 1} \|w_{q, \vartheta, \beta} f(t)\|_{L^\infty} \int_{\mathbb{R}^3} \frac{(1 + |\eta|^2) \beta}{|\eta|} \exp \left\{ - C_q \left\{ \frac{|\eta|^2}{4} + |v \cdot \eta| \right\} \right\} \, d\eta \tag{2.19}
$$

$$
\leq C_{q, \epsilon} \langle v \rangle^{\gamma - 1} \|w_{q, \vartheta, \beta} f(t)\|_{L^\infty} \int_{\mathbb{R}^3} \frac{1}{|\eta|} \exp \left\{ - C_q \left\{ \frac{|\eta|^2}{4} + |v \cdot \eta| \right\} \right\} \, d\eta.
$$
For $|v| \geq 1$, we change the variables $\eta_\parallel = \left\{ \eta \cdot \frac{v}{|v|} \right\}$, and $\eta_\perp = \eta - \eta_\parallel$ so that $|v \cdot \eta| = |v| \times |\eta_\parallel|$, which implies that

\[
\int_{\mathbb{R}^3} \frac{1}{|\eta|} \exp \left\{ -C_\gamma \left( \frac{|\eta|^2}{4} + |v \cdot \eta| \right) \right\} \, d\eta \\
\leq C \int_{\mathbb{R}^2} \frac{1}{|\eta_\perp|} e^{-C_\gamma \frac{|\eta_\perp|^2}{4}} \left\{ \int_{-\infty}^{\infty} e^{-C_\gamma |v| \times |\eta_\parallel|} \, d|\eta_\parallel| \right\} \, d\eta_\perp \\
\leq C \int_{|v|} \frac{1}{|\eta_\perp|} e^{-C_\gamma |\eta_\perp|^2} \left\{ \int_{-\infty}^{\infty} e^{-C_\gamma |v| \times |\eta_\parallel|} \, d|\eta_\parallel| \right\} \, d\eta_\perp \\
\leq \frac{C}{1 + |v|}.
\]  

(2.20)

On the other hand, for $|v| \leq 1$,

\[
\int_{\mathbb{R}^3} \frac{1}{|\eta|} \exp \left\{ -C_\gamma \left( \frac{|\eta|^2}{4} + |v \cdot \eta| \right) \right\} \, d\eta \\
\leq C \int_{\mathbb{R}^2} \frac{1}{|\eta_\perp|} e^{-C_\gamma \frac{|\eta_\perp|^2}{4}} \left\{ \int_{-\infty}^{\infty} e^{-C_\gamma |v| \times |\eta_\parallel|} \, d|\eta_\parallel| \right\} \, d\eta_\perp \\
\leq C \int_{|v|} \frac{1}{|\eta_\perp|} e^{-C_\gamma |\eta_\perp|^2} \left\{ \int_{-\infty}^{\infty} e^{-C_\gamma |v| \times |\eta_\parallel|} \, d|\eta_\parallel| \right\} \, d\eta_\perp \\
\leq \frac{C}{1 + |v|}.
\]  

Summing (2.16), (2.19) and (2.20), we obtain

\[
w_{q,\alpha,\beta}(v) \int_{\mathbb{R}^3} k_y^2(v,u) \left( e^{|v-u|^2} \frac{|w_{q,\alpha,\beta} f(u)|}{w_{q,\alpha,\beta}(u)} \right) \, du \leq C_{q,\gamma}(v) \gamma^{-2} \|w_{q,\alpha,\beta} f(t)\|_{L^\infty}.
\]  

(2.21)

Thus, we complete the proof of Lemma 2.3.

2.2. Nonlinear term $\Gamma(f,f)$. We control nonlinear $\Gamma_\pm$ by a product form of $L^\infty$ and $L^p$.

**Lemma 2.4.** [10] Let $p$ be a positive number satisfying

\[
p > 1, \quad p\gamma > -3.
\]  

(2.22)

There are constants $C_\gamma$ depending only on $\gamma$ such that

\[
|w_{q,\alpha,\beta}(v) \Gamma_-(f,f)(v)| \leq C_\gamma(v) \|w_{q,\alpha,\beta} f(t)\|_{L^\infty} \left( \int_{\mathbb{R}^3} |f(u)|^{p'} \, du \right)^{1/p'},
\]  

(2.23)

and

\[
|w_{q,\alpha,\beta}(v) \Gamma_+(f,f)(v)| \leq C_\gamma(v) \|w_{q,\alpha,\beta} f(t)\|_{L^\infty} \left( \int_{\mathbb{R}^3} (1 + |u|)^{-2\beta p' + 16} |w_{q,\alpha,\beta} f(u)|^{p'} \, du \right)^{1/p'}
\]  

where $p' = \frac{5p}{p+1}$.

**Proof.** By the definition of $\Gamma_-(f,f)$, it holds that

\[
|w_{q,\alpha,\beta}(v) \Gamma_-(f,f)(v)| \leq C \|w_{q,\alpha,\beta} f\|_{L^\infty} \int_{\mathbb{R}^3} |v - u| \gamma \sqrt{\mu(u)} |f(u)| \, du.
\]  

(2.25)
From (2.29) and Hölder’s inequality, it holds that
\[ |w_{q, \vartheta, \beta}(v)\Gamma_-(f, f)(v)| \]
\[ \leq C\|w_{q, \vartheta, \beta}f\|_{L^\infty} \left( \int_{\mathbb{R}^3} |v-u|^{p\gamma} \sqrt{\mu(u)} \, du \right)^{1/p} \left( \int_{\mathbb{R}^3} \sqrt{\mu(u)} |f(u)|^{p'/p} \, du \right)^{1-\frac{1}{p}} \]
\[ \leq C\nu(v)\|w_{q, \vartheta, \beta}f\|_{L^\infty} \left( \int_{\mathbb{R}^3} (\mu(u))^{5/8} \, du \right)^{\frac{1}{p} - \frac{1}{p'}} \left( \int_{\mathbb{R}^3} |f(u)|^{p/p'} \, du \right)^{\frac{2p}{p'}} \]
\[ \leq C\nu(v)\|w_{q, \vartheta, \beta}f\|_{L^\infty} \left( \int_{\mathbb{R}^3} |f(u)|^{p'} \, du \right)^{1/p'}, \]
where \( p' = \frac{5p}{p-1} \). Thus, we obtain (2.30). Next, we consider the gamma term \( \Gamma_+(f, f) \). We notice
\[ \frac{1}{2} |v|^2 \leq |v'|^2 \quad \text{or} \quad \frac{1}{2} |v|^2 \leq |u'|^2, \]
which comes from \( |v|^2 \leq |v'|^2 + |u'|^2 \). Hence, one gets that
\[ |w_{q, \vartheta, \beta}(v)\Gamma_+(f, f)(v)| \]
\[ \leq w_{q, \vartheta, \beta}(v) \int_{\mathbb{R}^3 \times S^2} B(v-u, \omega) \sqrt{\mu(u)} |f(u')f(v')| \left( \frac{1}{2} |v| \leq |v'| \leq |u'| \right) \, d\omega du \]
\[ + w_{q, \vartheta, \beta}(v) \int_{\mathbb{R}^3 \times S^2} B(v-u, \omega) \sqrt{\mu(u)} |f(u')f(v')| \left( \frac{1}{2} |v'| \leq |v| \leq |u'| \right) \, d\omega du \]
\[ \leq C\beta \int_{\mathbb{R}^3 \times S^2} B(v-u, \omega) \sqrt{\mu(u)} \left| w_{q, \vartheta, \beta}(u')f(u') \right| \exp \left( \frac{q}{8} (1 + (1 + t)^{\vartheta}) |v'|^2 \right) f(v') \, d\omega du \]
\[ + C\beta \int_{\mathbb{R}^3 \times S^2} B(v-u, \omega) \sqrt{\mu(u)} \left| w_{q, \vartheta, \beta}(v')f(v') \right| \exp \left( \frac{q}{8} (1 + (1 + t)^{\vartheta}) |u'|^2 \right) f(u') \, d\omega du \]
\[ := J_1 + J_2. \]
For \( J_1 \), as in (17), we rewrite the variables as
\[ V = u - v, \quad V_{\parallel} = (V \cdot \omega) \omega, \quad V_{\perp} = V - V_{\parallel}, \quad \eta = v + V_{\parallel}, \]
which gives
\[ v' = v + V_{\parallel}, \quad \eta = \nu + V_{\parallel}, \]
Hence, it holds that
\[ J_1 \leq C\beta \|w_{q, \vartheta, \beta}f\|_{L^\infty} \int_{\mathbb{R}^3 \times S^2} B(v-u, \omega) \sqrt{\mu(u)} \exp \left( \frac{q}{8} (1 + (1 + t)^{\vartheta}) |v'|^2 \right) f(v') \, d\omega du \]
\[ \leq C\beta \|w_{q, \vartheta, \beta}f\|_{L^\infty} \left( \int_{\mathbb{R}^3} |v-u|^{p\gamma} \sqrt{\mu(u)} \, du \right)^{1/p} \left( \int_{\mathbb{R}^3 \times S^2} \sqrt{\mu(u)} \exp \left( \frac{q}{8} (1 + (1 + t)^{\vartheta}) |v'|^2 \right) f(v') \right)^{rac{p}{p'}} \, d\omega du \]
\[ \leq C\beta \nu(v)\|w_{q, \vartheta, \beta}f\|_{L^\infty} \left( \int_{\mathbb{R}^3 \times S^2} e^{-\frac{1}{2} |v+V_{\parallel}|^2} \exp \left( \frac{q}{8} (1 + (1 + t)^{\vartheta}) |v+V_{\parallel}|^2 \right) f(v+V_{\parallel}) \right)^{rac{p}{p'}} \, d\omega dV \]
\[ \leq C\beta \nu(v)\|w_{q, \vartheta, \beta}f\|_{L^\infty} \left( \int_{\mathbb{R}^3 \times S^2} \frac{1}{|V_{\parallel}|^2} e^{-\frac{1}{2} |v+V_{\parallel}|^2} \exp \left( \frac{q}{8} (1 + (1 + t)^{\vartheta}) |v+V_{\parallel}|^2 \right) f(v+V_{\parallel}) \right)^{rac{p}{p'}} \, dV_{\perp} dV_{\parallel} \]
\[ = C\beta \nu(v)\|w_{q, \vartheta, \beta}f\|_{L^\infty} \left( \int_{\mathbb{R}^3 \times S^2} \frac{1}{|\eta-v|^2} e^{-\frac{1}{2} |\eta|^2} \exp \left( \frac{q}{8} (1 + (1 + t)^{\vartheta}) |\eta|^2 \right) f(\eta) \right)^{rac{p}{p'}} \, d\nu d\eta \]
\[ \leq C\beta \nu(v)\|w_{q, \vartheta, \beta}f\|_{L^\infty} \left( \int_{\mathbb{R}^3} (1 + |\eta|)^{-4} e^{-\frac{1}{2} |\eta|^2} \exp \left( \frac{q}{8} (1 + (1 + t)^{\vartheta}) |\eta|^2 \right) f(\eta) \right)^{rac{p}{p'}} \, d\eta \]
\[ \leq C\beta \nu(v)\|w_{q, \vartheta, \beta}f\|_{L^\infty} \left( \int_{\mathbb{R}^3} (1 + |\eta|)^{-2p+16} |w_{q, \vartheta, \beta}f(\eta)|^{p'} \, d\eta \right)^{1/p'}, \]
where \( p' = \frac{5p}{p-1} \). Notice that \( J_2 \) has a similar form to \( J_1 \) if the velocities \( u' \) and \( v' \) are interchanged. Hence, similar to the way of treating \( J_1 \), we get the same results in \( J_2 \) as well. Thus, the proof of Lemma 2.4 is complete.

2.3. Relative entropy. Recall the definition of the relative entropy (1.7)

\[
\mathcal{E}(F) = \int_{T^3 \times \mathbb{R}^3} \left( \frac{F}{\mu} \ln \frac{F}{\mu} - \frac{F}{\mu} + 1 \right) \mu \, dv \, dx.
\]

The below lemma is the global in time a priori estimate related with the relative entropy.

**Lemma 2.5.** Assume that \( F \) satisfies the Boltzmann equation (1.1).

\[
\mathcal{E}(F) \leq \mathcal{E}(F_0),
\]

for any \( t \geq 0 \).

**Proof.** Define a function

\[
h(s) = s \ln s - s + 1,
\]

for \( s > 0 \). Then, \( h \) is a nonnegative and convex function on \((0, \infty)\) with

\[
h'(s) = \ln s.
\]

From (1.1), one can deduce that

\[
\partial_t \left[ \mu h \left( \frac{F}{\mu} \right) \right] + \nabla_x \cdot [v \mu h \left( \frac{F}{\mu} \right)] = Q(F, F) \ln \frac{F}{\mu}.
\]

Taking integration for \( v \in \mathbb{R}^3 \) and then for \( x \) in \( T^3 \), we get

\[
\frac{d}{dt} \int_{T^3} \int_{\mathbb{R}^3} h \left( \frac{F}{\mu} \right) \mu \, dv \, dx = \int_{T^3} \int_{\mathbb{R}^3} Q(F, F) \ln F \, dv \, dx.
\]

From

\[
\int_{T^3} \int_{\mathbb{R}^3} Q(F, F) \ln F \, dv \, dx \leq 0,
\]

we can deduce that

\[
\frac{d}{dt} \int_{T^3} \int_{\mathbb{R}^3} h \left( \frac{F}{\mu} \right) \mu \, dv \, dx \leq 0.
\]

Furthermore, we have

\[
\int_{T^3} \int_{\mathbb{R}^3} h \left( \frac{F}{\mu} \right) \mu \, dv \, dx \leq \int_{T^3} \int_{\mathbb{R}^3} h \left( \frac{F_0}{\mu} \right) \mu \, dv \, dx.
\]

**Lemma 2.6.** Assume that \( F \) satisfies the Boltzmann equation (1.1). Then, it holds that

\[
\int_{T^3 \times \mathbb{R}^3} \frac{1}{4\mu} |F - \mu|^2 1_{|F - \mu| \leq \mu} \, dv \, dx + \int_{T^3 \times \mathbb{R}^3} \frac{1}{4} |F - \mu| 1_{|F - \mu| > \mu} \, dv \, dx \leq \mathcal{E}(F_0),
\]

for any \( t \geq 0 \). Furthermore, if we write \( F \) in terms of the standard perturbation \( f \), then

\[
\int_{T^3 \times \mathbb{R}^3} \frac{1}{4} f^2 1_{|f| \leq \sqrt{\pi}} \, dv \, dx + \int_{T^3 \times \mathbb{R}^3} \frac{\sqrt{\pi}}{4} f 1_{|f| > \sqrt{\pi}} \, dv \, dx \leq \mathcal{E}(F_0).
\]

**Proof.** It is noticed that

\[
F \ln F - \mu \ln \mu = (1 + \ln \mu)(F - \mu) + \frac{1}{2F} |F - \mu|^2,
\]

where \( \tilde{F} \) is between \( F \) and \( \mu \) form Taylor expansion. Then, we compute

\[
\frac{1}{2F} |F - \mu|^2 = F \ln F - \mu \ln \mu - (1 + \ln \mu)(F - \mu)
\]

\[
= h \left( \frac{F}{\mu} \right) \mu.
\]
Thus,
\[
\int_{T^3 \times \mathbb{R}^3} \frac{1}{2F} |F - \mu|^2 \, d\nu dx = \int_{T^3 \times \mathbb{R}^3} h \left( \frac{F}{\mu} \right) \mu \, d\nu dx,
\]
which is uniformly in time bounded in terms of Lemma 3.1. For the left-hand side, we write
\[
1 = 1_{|F - \mu| \leq \mu} + 1_{|F - \mu| > \mu}.
\]
Over \{\|F - \mu\| > \mu\}, we have \(F > 2\mu\) and hence
\[
\frac{|F - \mu|}{F} = \frac{F - \mu}{F} = \frac{F - \frac{1}{2}F}{F} = \frac{1}{2}.
\]
Over \{\|F - \mu\| \leq \mu\}, we have \(0 \leq F \leq 2\mu\) and hence
\[
\frac{1}{F} \geq \frac{1}{2\mu}.
\]
Therefore, we obtain from Lemma 2.5
\[
\int_{T^3 \times \mathbb{R}^3} \frac{1}{4\mu} |F - \mu|^2 1_{|F - \mu| \leq \mu} \, d\nu dx + \int_{T^3 \times \mathbb{R}^3} \frac{1}{4} |F - \mu| 1_{|F - \mu| > \mu} \, d\nu dx
\]
\[
\leq \int_{T^3 \times \mathbb{R}^3} h \left( \frac{F}{\mu} \right) \mu \, d\nu dx \leq \int_{T^3 \times \mathbb{R}^3} h \left( \frac{F_0}{\mu} \right) \mu \, d\nu dx = \mathcal{E}(F_0),
\]
for any \(t \geq 0\). □

3. A PRIORI ESTIMATE

Let \(F\) satisfy (1.1), and denote that \(F(t, x, v) = \mu(v) + \sqrt{\mu(v)} f(t, x, v)\). Then we can rewrite the Boltzmann equation (1.1) for \(h = w_{q, \vartheta, \beta} f\) as
\[
\partial_t h(t, x, v) + v \cdot \nabla_x h(t, x, v) + \tilde{\nu}(v, t) h(t, x, v) = K w h(t) + w \Gamma(f, f)(t),
\]
where
\[
\tilde{\nu}(v, t) := \nu(v) + \frac{\partial q |v|^2}{8(1 + t)^{\alpha + 1}}.
\]
Let us fix \(T > 0\) and assume a priori bound
\[
\sup_{0 \leq t \leq T} \|h(t)\|_{L^\infty} \leq \bar{M}.
\]
To get the sub-exponential time-decay property of the linearized solution operator \(G_v(t, s)\), we consider the linearized equation as follow:
\[
\partial_t h + v \cdot \nabla_x h + \tilde{\nu} h = 0,
\]
where \(\tilde{\nu}\) is defined in (1.1S). Then the solution of (3.3) can be written by
\[
h(t, x, v) = e^{-\int_0^t \tilde{\nu}(v, \tau) d\tau} h_0(x - tv, v).
\]
We can define the solution operator \(G_v(t, s)\) as follow:
\[
G_v(t, s) := e^{-\int_s^t \tilde{\nu}(v, \tau) d\tau}.
\]

Proposition 3.1. Let \(h(t, x, v)\) satisfy the equation (3.1) and \(\rho - 1 = \frac{(1 + \alpha)N}{2 - 7}\). Assume that (1.17) holds. Let \(0 < t \leq T < \infty\). Then it holds that
\[
\|h(t)\|_{L^\infty} \leq C e^{-\frac{1}{2}(1 + t)^{\rho}} \|h_0\|_{L^\infty} \left( \int_0^t \|h(s)\|_{L^\infty} \, ds + 1 \right) + D,
\]
where \(0 < \delta \ll 1, 0 < \epsilon \ll 1,\) and \(N \gg 1\) can be chosen arbitrary small and large and,
\[
D := C_M \epsilon^{\gamma + 3} + C_M \left( \frac{\gamma}{N^\gamma} + \frac{1}{N^\gamma + 1} \right) + C_M \epsilon^\delta + C_{M, N, \epsilon, \delta} \left( \mathcal{E}(F_0)^{\frac{5}{7}} + \mathcal{E}(F_0)^{\frac{5}{7}} \right).
\]
\textbf{Proof.} Take \((t, x, v) \in (0, T] \times \mathbb{T}^3 \times \mathbb{R}^3\). By Duhamel’s principle, it holds that
\[ |h(t, x, v)| \leq |G_v(t, 0)h_0(x - tv, v)| + \int_0^t |G_v(t, s)||K_w h(s, x - (t - s)v, v)| + |w \Gamma_+(f, f)(s, x - (t - s)v, v)| + |w \Gamma_-(f, f)(s, x - (t - s)v, v)|ds = I_1 + I_2 + I_3 + I_4. \]

First of all, note that \(G_v(t, s) \leq e^{-\nu(t-s)}\) and we can get
\[ \int_0^t G_v(t, s)\nu(v)ds \leq \int_0^t e^{-\nu(t-s)}\nu(v)ds = 1 - e^{-\nu(t)} \leq 1. \]

By extracting the collision frequency \(\nu\) from the other terms, we will deal with the time terms \(G_v(t, s)\) and \(G_u(s, s')\) as above except for the terms including \(h_0\). For (1.18), notice that \((1 + |v|)1_{|v| \leq 1} \leq 2\) and \(1_{|v| \geq 1} \leq |v|^2\). Then it follows that
\[ 1 + |v|^2 + \nu(v) = 1_{|v| \leq 1} + 1_{|v| \geq 1} + |v|^2 + \nu(v) \leq 2^{-\gamma}(1 + |v|) + |v|^2 + |v|^2 + \nu(v) \leq C(|v|^2 + \nu(v)). \]

Denote \(a = \frac{\gamma - 2}{\gamma}\) and \(b = \frac{2 - \gamma}{2}\). Then \(1/a + 1/b = 1\) and \(a > 1, b > 1\). Using the above inequality and Young’s inequality, we gain
\[ \tilde{\nu}(v, t) = \frac{\partial q|v|^2}{8(1 + t)^{\gamma+1}} + \nu(v) \geq C \left\{ (1 + t)^{-\frac{\gamma}{\gamma+1}}(1 + t)^{-\frac{\gamma}{\gamma+1}} + (1 - (1 + t)^{-\frac{\gamma}{\gamma+1}})\nu(v) \right\} \geq C \left\{ (1 + t)^{-\frac{\gamma}{\gamma+1}}(1 + |v|^2 + \nu(v)) + (1 - (1 + t)^{-\frac{\gamma}{\gamma+1}})\nu(v) \right\} \geq C \left\{ (1 + t)^{-\frac{\gamma}{\gamma+1}}(1 + |v|^2 + \nu(v)) + (1 - (1 + t)^{-\frac{\gamma}{\gamma+1}})\nu(v) \right\} \geq C \left\{ (1 + t)^{-\frac{\gamma}{\gamma+1}}(1 + |v|^2 + \nu(v)) + (1 - (1 + t)^{-\frac{\gamma}{\gamma+1}})\nu(v) \right\} = C(1 + t)^{-\frac{\gamma}{\gamma+1}}. \]

For \(I_1\), by the \(\tilde{\nu}\) estimate (3.5), we obtain
\[ I_1 \leq e^{-\int_0^t C(1+t)^{\gamma-1}dr}\|h_0\|_{L^\infty} \leq Ce^{-\lambda(1+t)^\gamma}\|h_0\|_{L^\infty}, \]

where \(\lambda = \frac{C}{\rho} > 0\). For \(I_2\), we divide \(I_2\) into four cases where i) \(|v| \geq N\), ii) \(|v| \leq N\) and \(|u| \geq 2N\), iii) \(|v| \leq N\), \(|u| \leq 2N\), and \(|u^*| \geq 3N\), and iv) \(|v| \leq N\), \(|u| \leq 2N\), and \(|u^*| \leq 3N\).

\textbf{(Case 1)} \(|v| \geq N\)

See that \(\nu(v) \sim (1 + v)^\gamma\) and \(\nu(v)^{-1}(1 + v)^{-\frac{\gamma\gamma}{\gamma+1}}\) is bounded for \(|v| \geq N\). By Lemma 2.3 \(I_2 1_{|v| \geq N}\) is bounded by
\[ I_2 1_{|v| \geq N} \leq \int_0^t e^{-\nu(t-s)}1_{|v| \geq N} \left[ |K_w h(s)| + |K_w h(s)| \right]ds \leq \int_0^t e^{-\nu(t-s)}\nu(v)1_{|v| \geq N} \left[ C\nu^{-1}(v)\mu^\frac{1}{1-N^2}(v)\nu^{\gamma+3} + C\nu^{-1}(v)\nu^{-2} \right]\|h(s)\|_{L^\infty}ds \leq \int_0^t e^{-\nu(t-s)}\nu(v) \left[ C\mu^\frac{1}{1-N^2}(v)\nu^{\gamma+3} + \frac{C}{1+N^2} \right]\|h(s)\|_{L^\infty}ds \leq \left( C\nu^{\gamma+3} + \frac{C}{N^2} \right) \sup_{0 \leq s \leq T} \|h(s)\|_{L^\infty}. \]

\textbf{\textit{Case 2}} \(|v| \leq N\) and \(|u| \geq 2N\)
Observe that \(|u - v| \geq N\) and (2.20) holds although there is no \(\frac{1}{|\eta|}\) in (2.20). It follows from Lemma 2.1, (2.18), (2.19), and (2.20) that

\[
\int_{|u| \geq 2N} k_{u,2}(v, u) \mathbf{1}_{\{|v| \leq N\}} du \leq \int_{|u| \geq 2N} \frac{C_\gamma}{\sqrt{\pi}} \mathbf{1}_{\{|v| \leq N\}} e^{-\frac{|u-v|^2}{s(u-v)^2}} \frac{w_{q,\theta,\beta}(v)}{w_{q,\theta,\beta}(u)} du \\
\leq \frac{C_\gamma}{N^{\frac{3}{2}}} \int_{|u| \geq 2N} e^{-\frac{|u-v|^2}{s(u-v)^2}} \frac{w_{q,\theta,\beta}(v)}{w_{q,\theta,\beta}(u)} du \\
\leq \frac{C_\gamma}{N^{\frac{3}{2}}}(1 + |v|).
\]

Note that \(\nu(v)^{-1}(1 + |v|^2)^{\beta} e^{-\frac{|v|^2}{s(u-v)^2}}\) is bounded for \(|v| \leq N\). Then we can gain by Lemma 2.1 and (3.8)

\[
I_2^{1}_{\{|v| \leq N, |u| \geq 2N\}} \leq \int_0^t e^{-\nu(v)(t-s)} \mathbf{1}_{\{|v| \leq N\}} \int_{|u| \geq 2N} (|k_{u,1}(v, u)h(s)| + |k_{u,2}(v, u)h(s)|) dus \\
\leq \int_0^t e^{-\nu(v)(t-s)} \|h(s)\|_{L^\infty} \mathbf{1}_{\{|v| \leq N\}} \\
\times \int_{|u| \geq 2N} C(1 + |v|^2)^{\beta} |v - u|^2 e^{-\frac{|v|^2}{s(u-v)^2}} e^{-\frac{|u|^2}{s(u-v)^2}} + \frac{C_\gamma}{|v-u|^2} e^{-\frac{|u-v|^2}{s(u-v)^2}} \frac{w_{q,\theta,\beta}(v)}{w_{q,\theta,\beta}(u)} dus \\
\leq \int_0^t e^{-\nu(v)(t-s)} \nu(v) \|h(s)\|_{L^\infty} \mathbf{1}_{\{|v| \leq N\}} \\
\times \left( \int_{|u| \geq 2N} CN^{\gamma} \nu(v)^{-1}(1 + |v|^2)^{\beta} e^{-\frac{|v|^2}{s(u-v)^2}} e^{-\frac{|u|^2}{s(u-v)^2}} du + \frac{C_\gamma \nu(v)^{-1}}{N^{\frac{3}{2}}(1 + |v|)} \right) ds \\
\leq \int_0^t e^{-\nu(v)(t-s)} \nu(v) \|h(s)\|_{L^\infty} \left( CN^{\gamma} \int_{|u| \geq 2N} e^{-\frac{|u|^2}{s(u-v)^2}} du + \frac{C_\gamma}{(1 + |v|)N^{\frac{3}{2}}(1 + |v|)} \right) ds \\
\leq \left( CN^{\gamma} + \frac{C_\gamma}{N^{\frac{3}{2}}(1 + |v|)} \right) \sup_{0 \leq t \leq T} \|h(s)\|_{L^\infty},
\]

where \(\gamma < 0\) and \(\frac{\gamma + 3}{2} > 0\).

**Case 3** \(|v| \leq N, |u| \leq 2N, \text{ and } |u'| \geq 3N\),

Denote \(h(s') = h(s', x - (t-s)v - (s-s')u, u')\). We can split \(I_2^{1}_{\{|v| \leq N, |u| \leq 2N, |u'| \geq 3N\}}\) by Lemma 2.1 into

\[
I_2^{1}_{\{|v| \leq N, |u| \leq 2N, |u'| \geq 3N\}} = \int_0^t \int_{|u| \leq 2N} \int_0^s G_v(t, s)G_u(s, u')k_{u,2}(v, u) \mathbf{1}_{\{|v| \leq N\}} \\
\times \|K_v h(s')\| + |w_{\Omega_+}(f, f)(s')| + |w_{\Omega_-}(f, f)(s')| ds' dus \\
\leq J_1 + J_2 + J_3,
\]
where
\[
J_1 := \int_0^t \int_{|u| \leq 2N} \int_{|u^*| \geq 3N} e^{-\nu(v)(t-s)} e^{-\nu(u)(s-s')} 1_{\{|v| \leq N\}} k_w(v, u) k_w(u, u^*) |h(s')| du^* ds' dsuds,
\]
\[
J_2 := C \int_0^t \int_{|u| \leq 2N} \int_{|u^*| \geq 3N} e^{-\nu(v)(t-s)} e^{-\nu(u)(s-s')} \nu(u) k_w(v, u) 1_{\{|v| \leq N\}} \|h(s')\|_{L^\infty} \times \left( \int_{|u^*| \geq 3N} (1 + |u^*|)^{-2\beta'} + 16 \|h(u^*)\|^p \|du^*\| \right)^{1/p} \ ds' duds,
\]
\[
J_3 := C \int_0^t \int_{|u| \leq 2N} \int_{|u^*| \geq 3N} e^{-\nu(v)(t-s)} e^{-\nu(u)(s-s')} \nu(u) k_w(v, u) 1_{\{|v| \leq N\}} \|h(s')\|_{L^\infty} \times \left( \int_{|u^*| \geq 3N} \frac{1}{w_{\eta, \alpha, \beta}(u^*)} \|h(u^*)\|^p \|du^*\| \right)^{1/p} \ ds' duds.
\]
For \(J_1\), notice that \(|u^* - u| \geq N, |u - v|^{-\gamma} \leq (1 + |u - v|)^{-\gamma}\) and \((2.19)\) holds although \((1 + |\eta|)^{\beta'}\) is replaced to \((1 + |\eta|)^{\beta'} - \gamma\). Then we can get by Lemma 2.1.2 (2.18), (2.19), and (2.20)
\[
\int_{|u^*| \geq 3N} k_w.2(u, u^*) 1_{\{|u| \leq 2N\}} du^* \leq \int_{|u^*| \geq 3N} 1_{\{|u| \leq 2N\}} C_\gamma |u^* - u|^{-\gamma} e^{-\frac{|u^* - u|}{s} - \frac{||u^2 - u|^2| s}{8 |u^* - u|^2}} \frac{w_{\eta, \alpha, \beta}(u)}{w_{\eta, \alpha, \beta}(u^*)} du^*
\]
\[
\leq \frac{C_\gamma}{N^{\frac{1}{2}}} \int_{|u^*| \geq 3N} |u^* - u|^{-\gamma} e^{-\frac{|u^* - u|}{s} - \frac{||u^2 - u|^2| s}{8 |u^* - u|^2}} \frac{w_{\eta, \alpha, \beta}(u)}{w_{\eta, \alpha, \beta}(u^*)} du^*
\]
\[
\leq \frac{C_\gamma}{N^{\frac{1}{2}}} (1 + |u|).
\]
Recall that \(k_w(v, u)\) is integrable on \(|u| \leq 2N\). From (3.8) and Lemma 2.1 \(J_1\) is bounded similarly to the estimate of \(I_2\) for \(|u| \leq N\) and \(|u| \geq 2N\) by

\[
J_1 = \int_0^t \int_{|u| \leq 2N} \int_{|u^*| \geq 3N} e^{-\nu(v)(t-s)} e^{-\nu(u)(s-s')} 1_{\{|v| \leq N\}} k_w(v, u) k_w(u, u^*) |h(s')| du^* ds' dsuds
\]
\[
\leq \int_0^t \int_{|u| \leq 2N} \int_{|u^*| \geq 3N} e^{-\nu(v)(t-s)} e^{-\nu(u)(s-s')} 1_{\{|v| \leq N\}} k_w(v, u) \int_{|u^*| \geq 3N} (k_{w,1}(u, u^*) + k_{w,2}(u, u^*)) |h(s')| du^* ds' dsuds
\]
\[
\leq \int_0^t \int_{|u| \leq 2N} \int_{|u^*| \geq 3N} e^{-\nu(v)(t-s)} e^{-\nu(u)(s-s')} 1_{\{|v| \leq N\}} k_w(v, u)
\]
\[
\times \left( \int_{|u^*| \geq 3N} C(1 + |u|^2) |u^* - u|^{-\gamma} e^{-\frac{|u^* - u|}{s}} |h(s')| du^* + \frac{C_\gamma}{N^{\frac{1}{2}}} (1 + |u|) \sup_{0 \leq s' \leq T} \|h(s')\|_{L^\infty} \right) ds' dsuds
\]
\[
\leq C \int_0^t \int_{|u| \leq 2N} \int_{|u^*| \geq 3N} \nu(v) e^{-\nu(v)(t-s)} \nu(u) e^{-\nu(u)(s-s')} 1_{\{|v| \leq N\}} k_w(v, u)
\]
\[
\times \nu(u)^{-1} \nu(v)^{-1} \left( \frac{N^\gamma \nu(v) e^{-\frac{\gamma N^2}{8}}}{N^{\frac{2}{N^2}}} \int_{|u^*| \geq 3N} e^{-\frac{|u^*|^2}{8}} |h(s')| du^* + \frac{C_\gamma}{N^{\frac{1}{2}}} \sup_{0 \leq s' \leq T} \|h(s')\|_{L^\infty} \right) ds' dsuds
\]
\[
\leq C \int_0^t \int_{|u| \leq 2N} \int_{|u^*| \geq 3N} \nu(v) e^{-\nu(v)(t-s)} \nu(u) e^{-\nu(u)(s-s')} k_w(v, u)
\]
\[
\times \left\{ \left( 1 + \frac{1}{N} \right)^\gamma e^{-\frac{\gamma N^2}{8}} + \frac{\left( 1 + N \right) \left( 1 + 2N \right)}{N^{\frac{2}{N^2}} \gamma} \right\} \sup_{0 \leq s' \leq T} \|h(s')\|_{L^\infty} ds' dsuds
\]
\[
\leq C \int_0^t \int_{|u| \leq 2N} \nu(v) e^{-\nu(v)(t-s)} k_w(v, u) \left( e^{-\frac{\gamma N^2}{8}} + \frac{1}{N^{\frac{1}{2}}} \right) \sup_{0 \leq s' \leq T} \|h(s')\|_{L^\infty} dsuds
\]
\[
\leq C \left( e^{-\frac{\gamma N^2}{8}} + \frac{1}{N^{\frac{1}{2}}} \right) \sup_{0 \leq s' \leq T} \|h(s')\|_{L^\infty},
\]
(3.11)
because \((1 + |u|^2)^{\frac{3}{2}} e^{-\frac{|u|^2}{2}} \leq C \nu(u)\) for \(|u| \leq 2N\). For \(J_2\), denote that \(h(u^*) = h(s', x - (t-s)v - (s-s')u, u^*)\) and we use Lemma \(2.4\) to get

\[
J_2 = C \int_0^t \int_{|v| \leq 2N} \int_0^s \nu(v) e^{-\nu(v)(t-s)} e^{-\nu(u)(s-s')} \nu(u) k_w(v, u) 1_{\{|v| \leq N\}} \|h(s')\|_{L^\infty} \frac{du^*}{2} \] 
\[
\times \left( \int_{|u^*| \geq 3N} (1 + |u^*|)^{-2\beta' + 16} |h(u^*)|^{p'} du^* \right)^{1/p'} ds' duds
\]
\[
\leq C \int_0^t \int_{|v| \leq 2N} \int_0^s \nu(v) e^{-\nu(v)(t-s)} e^{-\nu(u)(s-s')} \nu(u) k_w(v, u) 1_{\{|v| \leq N\}} \|h(s')\|_{L^\infty} \]
\[
\times \left( \int_{|u^*| \geq 3N} (1 + |u^*|)^{-2\beta' + 16} |h(u^*)|^{p'} du^* \right)^{1/p'} ds' duds
\]
\[
\leq C \left(1 + 3N\right)^3 \sup_{0 \leq s' \leq T} \|h(s')\|^2_{L^\infty},
\]

since \(p' \geq 5\) and \(\beta \geq 7/2\) yield \(-(2\beta - 3)p' + 16 < -3\) and \(k_w(u, v)\) is integrable on \(|v| \leq 2N\). For \(J_3\), denote that \(h(u^*) = h(s', x - (t-s)v - (s-s')u, u^*)\) and we can get

\[
J_3 = C \int_0^t \int_{|v| \leq 2N} \int_0^s \nu(v) e^{-\nu(v)(t-s)} e^{-\nu(u)(s-s')} \nu(u) k_w(v, u) 1_{\{|v| \leq N\}} \|h(s')\|_{L^\infty} \]
\[
\times \left( \int_{|u^*| \geq 3N} \frac{1}{w_q, q, \beta(u^*)} |h(u^*)|^{p'} du^* \right)^{1/p'} ds' duds
\]
\[
\leq C \int_0^t \int_{|v| \leq 2N} \int_0^s \nu(v) e^{-\nu(v)(t-s)} e^{-\nu(u)(s-s')} \nu(u) k_w(v, u) 1_{\{|v| \leq N\}} \|h(s')\|_{L^\infty} \]
\[
\times \left( \int_{|u^*| \geq 3N} (1 + |u^*|)^{-2\beta' + 16} |h(u^*)|^{p'} du^* \right)^{1/p'} ds' duds.
\]

By the above estimate, we bound \(J_3\) similarly to an estimate of \(J_2\) by

\[
J_3 \leq \frac{C}{(1 + 3N)^3 + \gamma} \sup_{0 \leq s' \leq T} \|h(s')\|^2_{L^\infty}. \tag{3.13}
\]

Combining \((3.11), (3.12), \) and \((3.13)\) altogether, we obtain

\[
J_2 1_{\{|v| \leq N, |u| \leq 2N, |u^*| \geq 3N\}} \leq C_M \left( e^{-\frac{\gamma}{N^2}} + \frac{1}{N^{\gamma + \frac{\gamma}{2}}} + \frac{1}{(1 + 3N)^{3 + \gamma}} \right).
\tag{3.14}
\]

\(\textbf{(Case 4)} \ |v| \leq N, |u| \leq 2N, \) and \(|u^*| \leq 3N\)
By Duhamel’s principle and Lemma 2.4, we can separate $I_2 1_{\{|v| \leq N, |u| \leq 2N, |u^*| \leq 3N\}}$ into

$$I_2 1_{\{|v| \leq N, |u| \leq 2N, |u^*| \leq 3N\}} = \int_0^t G_v(t, s) 1_{\{|v| \leq N\}} |K_w h(s)| ds$$

$$\leq \int_0^t G_v(t, s) 1_{\{|v| \leq N\}} |K_w^{-1} \chi h(s)| + |K_w h(s)| ds$$

$$\leq I_{20} + I_{21} + I_{22} + I_{23} + I_{24} + I_{25},$$

where

$$I_{20} := \int_0^t e^{-\nu(v)(t-s)} 1_{\{|v| \leq N\}} |K_w^{-1} \chi h(s)| ds,$$

$$I_{21} := \int_0^t \int_{|u| \leq 2N} G_v(t, s) G_u(s, 0) 1_{\{|v| \leq N\}} k_w(v, u)|h_0| duds,$$

$$I_{22} := \int_0^t \int_{|u| \leq 2N} \int_{|u^*| \leq 3N} \int_0^{s-\delta} e^{-\nu(v)(t-s)} e^{-\nu(u)(s-s')} 1_{\{|v| \leq N\}} k_w(v, u) k_w(u, u^*) |h(s')| du^* ds'duds,$$

$$I_{23} := C \int_0^t \int_{|u| \leq 2N} \int_{|u^*| \leq 3N} e^{-\nu(v)(t-s)} 1_{\{|v| \leq N\}} k_w(v, u) \nu(u) e^{-\nu(u)(s-s')} |h(s')|_L^\infty$$

$$\times \left( \int_{|u^*| \leq 3N} (1 + |u^*|)^{-2\beta p' + 16} |h(u^*)|^p |du^*| \right)^{1/p'} ds'duds,$$

$$I_{24} := C \int_0^t \int_{|u| \leq 2N} \int_{|u^*| \leq 3N} e^{-\nu(v)(t-s)} e^{-\nu(u)(s-s')} 1_{\{|v| \leq N\}} k_w(v, u) \nu(u) |h(s')|_L^\infty$$

$$\times \left( \int_{|u^*| \leq 3N} w_{q, a, \beta}(u^*)^{-p'} |h(u^*)|^p |du^*| \right)^{1/p'} ds'duds,$$

$$I_{25} := \int_0^t \int_{|u| \leq 2N} \int_{|u^*| \leq 3N} e^{-\nu(v)(t-s)} e^{-\nu(u)(s-s')} 1_{\{|v| \leq N\}} k_w(v, u)$$

$$\times (|K_w h(s')| + |w\Gamma +(f, f)(s')| + |w\Gamma -(f, f)(s')|) ds'duds.$$

For $I_{20}$, it follows from Lemma 2.3 that

$$I_{20} = \int_0^t e^{-\nu(v)(t-s)} 1_{\{|v| \leq N\}} |K_w^{-1} \chi h(s)| ds$$

$$\leq C \int_0^t e^{-\nu(v)(t-s)} \mu(v) \|h(s)\|_{L^\infty} ds$$

$$\leq C \gamma^3 \sup_{0 \leq s \leq T} \|h(s)\|_{L^\infty}. \tag{3.15}$$

For $I_{21}$, recollect that $k_w(v, u) \leq k_w(v, u)$ and $k_w(v, u)$ is integrable on $\{|u| \leq 2N\}$. Then, by 3.3, we obtain

$$I_{21} = \int_0^t \int_{|u| \leq 2N} G_v(t, s) G_u(s, 0) 1_{\{|v| \leq N\}} k_w(v, u)|h_0| duds$$

$$\leq \int_0^t \int_{|u| \leq 2N} G_v(t, s) G_u(s, 0) k_w(v, u)|h_0| duds$$

$$\leq \int_0^t \int_{|u| \leq 2N} e^{-\lambda(1+t)^\nu -(1+s)^\nu} e^{-\lambda(1+s)^\nu} k_w(v, u)|h_0| duds$$

$$\leq C \int_0^t \int_{|u| \leq 2N} e^{-\lambda(1+t)^\nu} k_w(v, u)|h_0| duds$$

$$\leq C e^{-\lambda(1+t)^\nu} \|h_0\|_{L^\infty}. \tag{3.16}$$
because $te^{-\frac{A}{2}(1+t)^{\rho}}$ is bounded for $t \geq 0$. For $I_{22}$, we can divide $I_{22}$ into

$$I_{22} = \int_0^t \int_{|u| \leq 2N} \int_0^{s-\delta} \int_{|u^*| \leq 3N} e^{-\nu(v)(t-s)} e^{-\nu(u)(s-s')} 1_{\{|v| \leq N\}} k_w^\chi (v, u) k_w(u, u^*) |h(s')| du^* ds' ds' ds duds$$

$$= \int_0^t \int_{|u| \leq 2N} \int_0^{s-\delta} \int_{|u^*| \leq 3N} e^{-\nu(v)(t-s)} e^{-\nu(u)(s-s')} 1_{\{|v| \leq N\}} k_w^\chi (v, u) k_w^1 (u, u^*) |h(s')| du^* ds' ds duds$$

$$+ \int_0^t \int_{|u| \leq 2N} \int_0^{s-\delta} \int_{|u^*| \leq 3N} e^{-\nu(v)(t-s)} e^{-\nu(u)(s-s')} 1_{\{|v| \leq N\}} k_w^\chi (v, u) k_w^\chi (u, u^*) |h(s')| du^* ds' ds duds$$

$$= I_{221} + I_{222}.$$

For $I_{221}$, recall that $k_w^\chi (v, u) \leq k_w(v, u)$ and $k_w(v, u)$ is integrable on $\{|u| \leq 2N\}$. Then by Lemma $2.3$ we have

$$I_{221} = \int_0^t \int_{|u| \leq 2N} \int_0^{s-\delta} \int_{|u^*| \leq 3N} e^{-\nu(v)(t-s)} e^{-\nu(u)(s-s')} 1_{\{|v| \leq N\}} k_w^\chi (v, u) \int_{|u^*| \leq 3N} k_w^1 (u, u^*) |h(s')| du^* ds' ds duds$$

$$\leq C \int_0^t \int_{|u| \leq 2N} \int_0^{s-\delta} \int_{|u^*| \leq 3N} e^{-\nu(v)(t-s)} e^{-\nu(u)(s-s')} k_w^\chi (v, u) \mu(u) \frac{1}{\mu(u)} \|h(s')\|_L^\infty ds' ds duds$$

$$= C e^{\gamma+3} \int_0^t \int_{|u| \leq 2N} \int_0^{s-\delta} \nu(v) e^{-\nu(v)(t-s)} \nu(u) e^{-\nu(u)(s-s')} k_w^\chi (v, u) \mu(u) \frac{1}{\mu(u)} \|h(s')\|_L^\infty ds' ds duds$$

$$\leq C e^{\gamma+3} \int_0^t \int_{|u| \leq 2N} \int_0^{s-\delta} \nu(v) e^{-\nu(v)(t-s)} \nu(u) e^{-\nu(u)(s-s')} k_w^\chi (v, u) \mu(u) \frac{1}{\mu(u)} \|h(s')\|_L^\infty ds' ds duds$$

$$\leq C e^{\gamma+3} \sup_{0 \leq s' \leq T} \|h(s')\|_L^\infty,$$

(3.17)

since $(1 + |u|^2)^{\beta+3} \nu(u)^{-1} \mu(u)^{-1}$ is bounded for $|u| \leq 2N$ and $(1 + |v|^2)^{-3} \leq C \nu(v)$ for $|v| \leq N$. For $I_{222}$, we can get

$$I_{222} \leq \int_0^t \int_{|u| \leq 2N} \int_0^{s-\delta} \nu(v) e^{-\nu(v)(t-s)} \nu(u) e^{-\nu(u)(s-s')} 1_{\{|v| \leq N\}} \int_{|u^*| \leq 3N} k_w^\chi (v, u) k_w^\chi (u, u^*) |h(s')| du^* ds' ds duds$$

$$\leq (1 + 2N)^{-7} \int_0^t \int_{|u| \leq 2N} \int_0^{s-\delta} \nu(v) e^{-\nu(v)(t-s)} \nu(u) e^{-\nu(u)(s-s')} 1_{\{|v| \leq N\}} \sup_{0 \leq s' \leq s-\delta} \left( \int_{|u| \leq 2N} \int_{|u^*| \leq 3N} k_w^\chi (v, u) k_w^\chi (u, u^*) |h(s')| du^* du \right) ds$$

$$\leq (1 + 2N)^{-7} \int_0^t \int_{|u| \leq 2N} \int_0^{s-\delta} e^{-\nu(v)(t-s)} 1_{\{|v| \leq N\}} \times \sup_{0 \leq s' \leq s-\delta} \|k_w^\chi (v, u) k_w^\chi (u, u^*)\|_{L^2_{w,u^*}} \left( \int_{|u| \leq 2N} \int_{|u^*| \leq 3N} |h(s', x - (t-s)v - (s-s')u)|^2 du^* du \right)^{1/2} ds,$$

in that $k_w^\chi (v, u) k_w^\chi (u, u^*) \in L^2\{|u| \leq 2N\} \times \{|u^*| \leq 3N\}$. We change the variables as follow and denote $x' = x - (t-s)v$. Then we have

$$y = x - (t-s)v - (s-s')u = x' - (s-s')u, \quad \left| \det \frac{\partial (y, u^*)}{\partial (u, u^*)} \right| = \frac{1}{|s-s'|^3}.$$
Besides, $\|h(s')\|_{L^r_xL^\infty_y(|u^*| \leq 3N)}$ can be bounded from Lemma 2.6 by

$$
\int_{y \in \mathbb{T}^3} \int_{|u^*| \leq 3N} |h(s', y, u^*)|^2 du^* dy
\leq \int_{y \in \mathbb{T}^3} \int_{|u^*| \leq 3N} |h(s', y, u^*)|^2 1_{|f| \leq \sqrt{N}} du^* dy + \int_{y \in \mathbb{T}^3} \int_{|u^*| \leq 3N} |h(s', y, u^*)|^2 1_{|f| \geq \sqrt{N}} du^* dy
\leq \int_{y \in \mathbb{T}^3} \int_{|u^*| \leq 3N} w_{q, \theta, \beta}(u^*) |f(s', y, u^*)|^2 1_{|f| \leq \sqrt{N}} du^* dy
+ \sup_{0 \leq s' \leq T} \|h(s')\|_{L^\infty} \int_{y \in \mathbb{T}^3} \int_{|u^*| \leq 3N} w_{q, \theta, \beta}(u^*) |f(s', y, u^*)| 1_{|f| \geq \sqrt{N}} du^* dy$$

$$\leq C_N \int_{y \in \mathbb{T}^3} \int_{|u^*| \leq 3N} |f(s', y, u^*)|^2 1_{|f| \leq \sqrt{N}} du^* dy
+ C_N \sup_{0 \leq s' \leq T} \|h(s')\|_{L^\infty} \int_{y \in \mathbb{T}^3} \int_{|u^*| \leq 3N} \sqrt{\mu(u^*)} |f(s', y, u^*)| 1_{|f| \geq \sqrt{N}} du^* dy$$
$$\leq C_{N, \tilde{N}} \mathcal{E}(F_0),$$

because $w_{q, \theta, \beta}(u^*)$ and $\sqrt{\mu(u^*)}$ are bounded for $|u^*| \leq 3N$. By the change of the variables (3.18) and (3.19), it follows that

$$I_{222} \leq (1 + 2N)^{-\gamma} \int_0^t e^{-\nu(v)(t-s)} 1_{|v| \leq 2N}$$
$$\times \sup_{0 \leq s' \leq s - \delta} \|k_w^\gamma(v, u)k_{\nu, u}^\gamma(u^*, u^*)\|_{L^2_{x,u^*}} \left( \int_{|u| \leq 2N} \int_{|u^*| \leq 3N} |h(s', x' - (s - s')u, u^*)|^2 du^* du \right)^{1/2} ds$$
$$\leq C_\epsilon (1 + 2N)^{-\gamma} \nu^{-1}(v) \int_0^t \nu(v)e^{-\nu(v)(t-s)} 1_{|v| \leq 2N}$$
$$\times \sup_{0 \leq s' \leq s - \delta} (s - s')^{-\frac{5}{4}} \left( \int_{y \in \mathbb{T}^3} \int_{|u^*| \leq 3N} |h(s', y, u^*)|^2 du^* dy \right)^{1/2} ds$$
$$\leq C_{\epsilon, N, \tilde{N}} (1 + 2N)^{-\gamma}(1 + N)^{-\gamma} \delta^{-\frac{5}{4}} \mathcal{E}(F_0)^{\frac{1}{2}}.$$

For $I_{23}$, the Hölder conjugate $r$ of $p'$ satisfies

$$1 \leq r \leq \frac{5}{4}, \quad \|k_w^\gamma(v, u)\|_{L^r_{x}((|u| \leq 2N))} \leq C_\epsilon.$$

(3.21)
Then $I_{23}$ enjoys

\[
I_{23} = C \int_0^t \int_{|u| \leq 2N} \int_0^{s-\delta} e^{-\nu(v)(t-s)} 1_{\{v \leq N\}} k_\nu^X(v, u) \nu(u) e^{-\nu(u)(s-s')} \|h(s')\|_{L^\infty} \\
\times \left( \int_{|u| \leq 3N} (1 + |u^*|)^{-2\beta p' + 16} |h(u^*)|^{p'} du^* \right)^{1/p'} ds' dus \\
\leq C \int_0^t e^{-\nu(v)(t-s)} 1_{\{v \leq N\}} \\
\times \sup_{0 \leq s' \leq s-\delta} \left[ \|h(s')\|_{L^\infty} \int_{|u| \leq 2N} k_\nu^X(v, u) \left( \int_{|u| \leq 3N} (1 + |u^*|)^{-2\beta p' + 16} |h(u^*)|^{p'} du^* \right)^{1/p'} \right] ds \\
\leq C_\delta \int_0^t e^{-\nu(v)(t-s)} 1_{\{v \leq N\}} \\
\times \sup_{0 \leq s' \leq s-\delta} \left[ \|h(s')\|_{L^\infty} \left( \int_{|u| \leq 2N} \int_{|u| \leq 3N} (1 + |u^*|)^{-2\beta p' + 16} |h(u^*)|^{p'} du^* du \right)^{1/p'} \right] ds \\
\leq C_\delta (2N)^{\frac{3}{2p'}} \int_0^t e^{-\nu(v)(t-s)} 1_{\{v \leq N\}} \\
\times \sup_{0 \leq s' \leq s-\delta} \left[ \|h(s')\|_{L^\infty} \left( \int_{|u| \leq 2N} \int_{|u| \leq 3N} |h(s', x' - (s-s')u, u^*)|^2 du^* du \right)^{1/2} \right]^{1/p'} ds.
\]

By the change of the variables \eqref{2.13} and \eqref{3.15}, it follows that

\[
I_{23} \leq C_\delta (2N)^{\frac{3}{2p'}} \int_0^t e^{-\nu(v)(t-s)} 1_{\{v \leq N\}} \\
\times \sup_{0 \leq s' \leq s-\delta} \left[ \|h(s')\|_{L^\infty} \left( \int_{|u| \leq 2N} \int_{|u| \leq 3N} |h(s', x' - (s-s')u, u^*)|^2 du^* du \right)^{1/2} \right]^{1/p'} ds \\
\leq C_\delta N^{\frac{3}{2p'}} v^{-1}(v) \int_0^t \nu(v) e^{-\nu(v)(t-s)} 1_{\{v \leq N\}} \\
\times \sup_{0 \leq s' \leq s-\delta} \left[ \|h(s')\|_{L^\infty} \left( \int_{y \in T^3} \int_{|u| \leq 3N} |h(s', y, u^*)|^2 dy du \right)^{1/2} \right]^{1/p'} ds \\
\leq C_\delta N^{\frac{3}{2p'}} (1 + N)^{-\gamma} \delta^{-\frac{3}{2p'}} \sup_{0 \leq s' \leq T} \|h(s')\|_{L^\infty}^{\frac{1}{p'}} \mathcal{E}(F_0)^{\frac{1}{p'}}.
\]

(3.22)
For an estimate of $I_{24}$, we use Lemma 2.4 to get
\[
I_{24} = \frac{1}{\nu r} \int_0^t \int_{u \leq 2N} \int_0^{s-\delta} e^{-\nu(v)(t-s)} e^{-\nu(u)(s-s')} 1_{\{|v| \leq N\}} k_w^{r}(v, u) \nu(u) \|h(s')\|_{L^\infty} \times \\
\times \left( \int_{|u^*| \leq 3N} w_{q, \alpha, \beta}(u^*)^{-p'} |h(u^*)|^p' du^* \right)^{1/p'} ds'duds \\
\leq C \int_0^t \int_{u \leq 2N} \int_0^{s-\delta} e^{-\nu(v)(t-s)} \nu(u) e^{-\nu(u)(s-s')} 1_{\{|v| \leq N\}} k_w^{r}(v, u) \|h(s')\|_{L^\infty} \times \\
\times \left( \int_{|u^*| \leq 3N} (1 + |u^*|)^{-2\beta p' + 16} |h(u^*)|^p' du^* \right)^{1/p'} ds'duds.
\]

Therefore we can estimate $I_{24}$ similarly to the estimate of $I_{23}$:
\[
I_{24} \leq C_{N, M} N^2 \rho^{r}(1 + N)^{-\gamma} \delta^{-\frac{2\beta}{p'}} \sup_{0 \leq s' \leq T} \|h(s')\|_{L^\infty}^{2-\frac{1}{p'}} E(F_0)^{\frac{1}{p'}}. \tag{3.23}
\]

From Lemma 2.4, we can get
\[
|w\Gamma_{-(f, g)}(t)| \leq C_{\nu} \|w_{q, \alpha, \beta} f(t)\|_{L^\infty} \|w_{q, \alpha, \beta} g(t)\|_{L^\infty} \left( \int_{\mathbb{R}^3} \frac{1}{w_{q, \alpha, \beta}(u)^{p'}} du \right)^{1/p'} \\
\leq C_{\nu} \|w_{q, \alpha, \beta} f(t)\|_{L^\infty} \|w_{q, \alpha, \beta} g(t)\|_{L^\infty},
\]
\[
|w\Gamma_{+(f, g)}(t)| \leq C_{\nu} \|w_{q, \alpha, \beta} f(t)\|_{L^\infty} \|w_{q, \alpha, \beta} g(t)\|_{L^\infty} \left( \int_{\mathbb{R}^3} (1 + |u|)^{-2\beta p' + 16} du \right)^{1/p'}
\]

because $\frac{1}{w_{q, \alpha, \beta}(u)^{p'}}$ and $(1 + |u|)^{-2\beta p' + 16}$ are integrable on $\mathbb{R}^3$. For an estimate of $I_{25}$, note that $\int_{s-\delta}^{s} e^{-\nu(u)(s-s')} ds' \leq \delta$. By (2.4) and (3.21), we can obtain
\[
I_{25} = \int_0^t \int_{u \leq 2N} \int_0^{s-\delta} e^{-\nu(v)(t-s)} k_w^{r}(v, u) e^{-\nu(u)(s-s')} 1_{\{|v| \leq N, |u^*| \leq 3N\}} \\
\times \left[ \|K_w h(s')\|_{L^\infty} + |w\Gamma_{-(f, f)}(s')| + |w\Gamma_{-(f, f)}(s')| \right] ds'duds \\
\leq C \int_0^t \int_{u \leq 2N} \int_0^{s-\delta} e^{-\nu(v)(t-s)} k_w^{r}(v, u) e^{-\nu(u)(s-s')} 1_{\{|v| \leq N\}} \left[ \|h(s')\|_{L^\infty} + 2 \|h(s')\|_{L^\infty}^2 \right] ds'duds \\
\leq C_{\delta} \int_0^t \int_{u \leq 2N} \int_0^{s-\delta} e^{-\nu(v)(t-s)} k_w^{r}(v, u) 1_{\{|v| \leq N\}} \sup_{s-\delta \leq s' \leq s} \left[ \|h(s')\|_{L^\infty} + \|h(s')\|_{L^\infty}^2 \right] ds'duds \\
\leq C_{\delta} \nu^{-1} \int_0^t \nu(v) e^{-\nu(v)(t-s)} 1_{\{|v| \leq N\}} \sup_{s-\delta \leq s' \leq s} \left[ \|h(s')\|_{L^\infty} + \|h(s')\|_{L^\infty}^2 \right] ds \\
\leq C_{\delta} (1 + N)^{-\gamma} \sup_{0 \leq s' \leq T} \|h(s')\|_{L^\infty} + \|h(s')\|_{L^\infty}^2.
\]

Get together (3.14), (3.16), (3.17), (3.20), (3.22), (3.23), and (3.25). Then we gain
\[
I_2 1_{\{|v| \leq N, |u| \leq 2N, |u^*| \leq 3N\}} \leq C_M \epsilon^{\gamma+3} + Ce^{-\frac{(1-t)}{N} \nu 0} \|h_0\|_{L^\infty} + C_{\epsilon, N, M, \delta} (E(F_0)^{\frac{1}{2}} + E(F_0)^{\frac{1}{2p'}}) + C_{M, N, \delta}. \tag{3.26}
\]

Join (3.7), (3.9), (3.14), and (3.20) altogether. Then it follows that
\[
I_2 \leq Ce^{-\frac{(1-t)}{N} \nu 0} \|h_0\|_{L^\infty} + C_M \epsilon^{\gamma+3} + C_M \left( \frac{1}{N^{2\gamma+1}} + N^\gamma + e^{\frac{N}{2N^2}} + \frac{1}{(1+3N^3)^{3\gamma}} \right) + C_{M, \epsilon, N, M} (E(F_0)^{\frac{1}{2}} + E(F_0)^{\frac{1}{2p'}}).
\tag{3.27}
\]
For $I_3$, denote that $h(u) = h(s, x', u)$ and, by Duhamel’s principle and Lemma 2.24, we split $I_3$ into

$$I_3 = \int_0^t G_v(t, s) w \Gamma_+(f, f)(s) ds$$

$$\leq C \int_0^t G_v(t, s) \nu(v) \|h(s)\|_{L^\infty} \left( \int_{\mathbb{R}^3} (1 + |u|)^{-2\beta p' + 16} |h(u)|^p' du \right)^{1/p'} ds$$

$$\leq C \int_0^t G_v(t, s) \nu(v) \|h(s)\|_{L^\infty}$$

$$\times \left( \int_{\mathbb{R}^3} (1 + |u|)^{-2\beta p' + 16} \left| G_u(s, 0) h_0 + \int_0^s G_u(s, s') \left[ K_w h(s') + w \Gamma_+(f, f)(s') + w \Gamma_-(f, f)(s') \right] ds' \right|^p' du \right)^{1/p'} ds$$

$$\leq I_{31} + I_{32} + I_{33} + I_{34} + I_{35},$$

where

$$I_{31} := C \int_0^t G_v(t, s) \nu(v) \|h(s)\|_{L^\infty} \left( \int_{\mathbb{R}^3} (1 + |u|)^{-2\beta p' + 16} |G_u(s, 0) h_0|^p' du \right)^{1/p'} ds,$$

$$I_{32} := C \int_0^t G_v(t, s) \nu(v) \|h(s)\|_{L^\infty} \left\{ \int_{\mathbb{R}^3} (1 + |u|)^{-2\beta p' + 16} \left( \int_{s-\delta}^s G_u(s, s') |K_w h(s')| ds' \right)^{p'} du \right\}^{1/p'} ds,$$

$$I_{33} := C \int_0^t G_v(t, s) \nu(v) \|h(s)\|_{L^\infty} \left\{ \int_{\mathbb{R}^3} (1 + |u|)^{-2\beta p' + 16} \left( \int_{s-\delta}^s G_u(s, s') |w \Gamma_+(f, f)(s')| ds' \right)^{p'} du \right\}^{1/p'} ds,$$

$$I_{34} := C \int_0^t G_v(t, s) \nu(v) \|h(s)\|_{L^\infty} \left\{ \int_{\mathbb{R}^3} (1 + |u|)^{-2\beta p' + 16} \left( \int_{s-\delta}^s G_u(s, s') |w \Gamma_-(f, f)(s')| ds' \right)^{p'} du \right\}^{1/p'} ds,$$

$$I_{35} := C \int_0^t G_v(t, s) \nu(v) \|h(s)\|_{L^\infty}$$

$$\times \left( \int_{\mathbb{R}^3} (1 + |u|)^{-2\beta p' + 16} \left| G_u(s, s') \left[ K_w h(s') + w \Gamma_+(f, f)(s') + w \Gamma_-(f, f)(s') \right] ds' \right|^p' du \right)^{1/p'} ds.$$
where

\[
\begin{align*}
I_{321} & := C \int_0^t G_v(t, s) \nu(v) \|h(s)\|_{L^\infty}^p \times \left\{ \int_{|u| \geq N} (1 + |u|)^{-2\beta p' + 16} \left( \int_{|u| \geq N} G_u(s, s') \left| \int_{u^*}^{s-\delta} k_w(u, u^*) h(s') du^* \right| \right) ds' \right\}^{1/p'} ds, \\
I_{322} & := C \int_0^t G_v(t, s) \nu(v) \|h(s)\|_{L^\infty}^p \times \left\{ \int_{|u| \leq N} (1 + |u|)^{-2\beta p' + 16} \left( \int_{|u^*| \geq 2N} G_u(s, s') \left| \int_{u^*}^{s-\delta} k_w(u, u^*) h(s') du^* \right| ds' \right) \right\}^{1/p'} ds, \\
I_{323} & := C \int_0^t G_v(t, s) \nu(v) \|h(s)\|_{L^\infty}^p \times \left\{ \int_{|u| \leq N} (1 + |u|)^{-2\beta p' + 16} \left( \int_{|u^*| \leq 2N} G_u(s, s') \left| \int_{u^*}^{s-\delta} k_w(u, u^*) h(s') du^* \right| ds' \right) \right\}^{1/p'} ds, \\
I_{324} & := C \int_0^t G_v(t, s) \nu(v) \|h(s)\|_{L^\infty}^p \times \left\{ \int_{|u| \leq N} (1 + |u|)^{-2\beta p' + 16} \left( \int_{|u^*| \leq 2N} G_u(s, s') \left| \int_{u^*}^{s-\delta} k_w(u, u^*) h(s') du^* \right| ds' \right) \right\}^{1/p'} ds.
\end{align*}
\]

For \(I_{321}\), notice that \((1 + |u|)^{-2\beta p' + 16}\) is integrable on \(|u| \geq N\). By Lemma 2.3, we can get similar to the estimate of \(I_2\) for \(|v| \geq N\):

\[
\begin{align*}
I_{321} & = C \int_0^t G_v(t, s) \nu(v) \|h(s)\|_{L^\infty}^p \times \left\{ \int_{|u| \geq N} (1 + |u|)^{-2\beta p' + 16} \left( \int_{|u| \geq N} G_u(s, s') \left| \int_{u^*}^{s-\delta} (k_{w}^{1-\gamma}(u, u^*) + k_{w}^{\gamma}(u, u^*)) h(s') ds' \right| ds' \right) \right\}^{1/p'} ds, \\
& \leq C \int_0^t G_v(t, s) \nu(v) \|h(s)\|_{L^\infty}^p \times \left\{ \int_{|u| \geq N} (1 + |u|)^{-2\beta p' + 16} \left( \int_{|u| \geq N} e^{-\nu(u)(s-s')} \nu(u) \|h(s')\|_L^\infty \nu^{-1}(u) \left( \frac{1}{s} \mu(u) + C_{\gamma}(u) \gamma^{-2} \right) ds' \right) \right\}^{1/p'} ds, \\
& \leq C \int_0^t G_v(t, s) \nu(v) \|h(s)\|_{L^\infty}^p \times \left\{ \int_{|u| \geq N} (1 + |u|)^{-2\beta p' + 16} \left( e^{\nu(u)(s-s')} \nu(u) \|h(s')\|_L^\infty \sup_{0 \leq s' \leq s-\delta} \|h(s')\|_{L^\infty} ds' \right) \right\}^{1/p'} ds, \\
& \leq \left( C_{\gamma}^{\gamma + 3} + \frac{C_{\gamma}}{N} \right) \int_0^t e^{-\nu(v)(t-s')} \nu(v) \|h(s)\|_{L^\infty} \sup_{0 \leq s' \leq s-\delta} \|h(s')\|_{L^\infty} ds \\
& \leq \left( C_{\gamma}^{\gamma + 3} + \frac{C_{\gamma}}{N} \right) \sup_{0 \leq s \leq T} \|h(s)\|^2_{L^\infty}.
\end{align*}
\]

(3.29)
For $I_{322}$, from Lemma 2.1 and (3.8), we can obtain similar to the estimate of $I_2$ for $|v| \leq N$ and $|u| \geq 2N$

\[
I_{322} = C \int_0^t G_v(t, s) \nu(v) \|h(s)\|_{L^\infty} \left[ \int_{|u| \leq N} (1 + |u|)^{-2\beta p' + 16} \left( \int_0^{s-\delta} G_u(s, s') \left| \int_{|u'| \geq 2N} k_w(u, u^*) h(s') du^* \right| ds' \right) \right] \frac{1}{p'} ds \\
\leq C \int_0^t G_v(t, s) \nu(v) \|h(s)\|_{L^\infty} \left\{ \int_{|u| \leq N} (1 + |u|)^{-2\beta p' + 16} \left( \int_0^{s-\delta} e^{-\nu(u)(s-s')} \left| \int_{|u'| \geq 2N} (k_{w,1} + k_{w,2})(u^*, u) h(s') du^* \right| ds' \right) \right\} \frac{1}{p'} ds \\
\leq C \left( N^\gamma + \frac{1}{N^{\frac{1}{2p}}} \right) \sup_{0 \leq s \leq T} \|h(s)\|_{L^\infty}^2, \tag{3.30}
\]

because $-2\beta p' + 16 < -3$. For $I_{323}$, we use Lemma 2.3 to get

\[
I_{323} = C \int_0^t G_v(t, s) \nu(v) \|h(s)\|_{L^\infty} \left[ \int_{|u| \leq N} (1 + |u|)^{-2\beta p' + 16} \left( \int_0^{s-\delta} G_u(s, s') \left| \int_{|u'| \leq 2N} k_w^{1-\chi}(u, u^*) h(s') du^* \right| ds' \right) \right] \frac{1}{p'} ds \\
\leq C \int_0^t e^{-\nu(v)(s-s')} \nu(v) \|h(s)\|_{L^\infty} \left\{ \int_{|u| \leq N} (1 + |u|)^{-2\beta p' + 16} \left( \int_0^{s-\delta} e^{-\nu(u)(s-s')} \nu(u) \nu(u)^{-1} \mu \frac{1}{s^2}(u) e^{\gamma + 3} \|h(s')\|_{L^\infty} \right) \right\} \frac{1}{p'} ds \\
\leq C \int_0^t e^{-\nu(v)(s-s')} \nu(v) \|h(s)\|_{L^\infty} \left\{ \int_{|u| \leq N} (1 + |u|)^{-2\beta p' + 16} e^{p' \gamma + 3} \left( \sup_{0 \leq s' \leq s - \delta} \|h(s')\|_{L^\infty} \right) \right\} \frac{1}{p'} ds \\
\leq C e^{\gamma + 3} \sup_{0 \leq s \leq T} \|h(s)\|_{L^\infty}^2 ds \tag{3.31}
\]
For $I_{324}$, the Hölder conjugate $r$ of $p'$ satisfies $(3.21)$. Note that $\|(1 + |u|)^{-2\beta p' + 16}\|_{L^r_s} < \infty$ and $I_{324}$ can be bounded by

$$I_{324} = C \int_0^t G_v(t, s) \nu(v) \|h(s)\|_{L^\infty} \times \left\{ \int_{|u| \leq N} (1 + |u|)^{-2\beta p' + 16} \left( \int_0^s G_u(u, u') \left| k_{\gamma}^\chi(u, u^*)h(s')du^* \right| \right)^{p'} du \right\}^{1/p'} ds \leq C(1 + N)^{-\gamma} \int_0^t G_v(t, s) \nu(v) \|h(s)\|_{L^\infty} \times \sup_{0 \leq s' \leq s - \delta} \left\{ \int_{|u| \leq N} (1 + |u|)^{-2\beta p' + 16} \left( \int_{|u^*| \leq 2N} k_{\gamma}^\chi(u, u^*)|h(s')|du^* \right)^{p'} du \right\}^{1/p'} ds \leq C(1 + N)^{-\gamma} \int_0^t G_v(t, s) \nu(v) \|h(s)\|_{L^\infty} \times \sup_{0 \leq s' \leq s - \delta} \left\{ \|h(s')\|_{L^p_{\infty}}^{p'-1} \int_{|u| \leq N} \left( \int_{|u^*| \leq 2N} (1 + |u|)^{-2\beta p' + 16}|h(s')|du^* \right) \right\}^{1/2}^{1/p'} ds \leq C(1 + N)^{-\gamma} \int_0^t G_v(t, s) \nu(v) \|h(s)\|_{L^\infty} \times \sup_{0 \leq s' \leq s - \delta} \left\{ (2N)^{\frac{1}{2}} \|h(s')\|_{L^2_{u, u^*}}^{p'-1} \int_{|u| \leq N} \left( \int_{|u^*| \leq 2N} (1 + |u|)^{-2\beta p' + 16} \left| h(s', x' - (s - s')u, u^*) \right|^2du^*du \right) \right\}^{1/2}^{1/p'} ds.$

By the change of the variables $(3.18)$ and $(3.19)$, $I_{324}$ satisfies

$$I_{324} \leq C(1 + N)^{-\gamma} \int_0^t G_v(t, s) \nu(v) \|h(s)\|_{L^\infty} \times \sup_{0 \leq s' \leq s - \delta} \left\{ (2N)^{\frac{1}{2}} \|h(s')\|_{L^2_{u, u^*}}^{p'-1} \int_{|u| \leq N} \left( \int_{|u^*| \leq 2N} (1 + |u|)^{-2\beta p' + 16} \left| h(s', x' - (s - s')u, u^*) \right|^2du^*du \right) \right\}^{1/2}^{1/p'} ds \leq C(1 + N)^{-\gamma} \int_0^t G_v(t, s) \nu(v) \|h(s)\|_{L^\infty} \times \sup_{0 \leq s' \leq s - \delta} \left\{ (2N)^{\frac{1}{2}} \|h(s')\|_{L^2_{u, u^*}}^{p'-1} \int_{|u| \leq N} \left( \int_{|u^*| \leq 2N} (1 + |u|)^{-2\beta p' + 16} \left| h(s', y, u^*) \right|^2du^*dy \right) \right\}^{1/2}^{1/p'} ds \leq C_{e, \eta,N}(1 + N)^{-\gamma} N^{\frac{1}{2} - \frac{1}{p'}} \sup_{0 \leq s \leq T} \|h(s)\|_{L^\infty}^{2 - \frac{1}{p'}} \mathcal{E}(F_0) \frac{1}{t'}.$$

(3.32)
Combining (3.30), (3.31), (3.32), and (3.33), we can get

\[ I_{32} \leq C_M \gamma^3 + C_M \left( N \gamma + \frac{1}{N^{3/2}} \right) + \frac{C_M}{N^2} + C_{M, \varepsilon, N, \delta} \mathcal{E}(F_0)^{\frac{1}{p'}}. \]  

(3.33)

For \( I_{33} \), it follows from Lemma 2.4 that

\[ I_{33} = C \int_0^t \int_{\mathbb{R}^3} G_v(t, s) \nu(v) \| h(s) \|_{L^\infty} \left\{ \int_{\mathbb{R}^3} (1 + |u|)^{-2\beta p' + 16} \left( \int_{s - \delta}^{s - \delta} G_u(s, s') |wGamma f(s') ds' \right)^{p'} du \right\}^{1/p'} ds \]

\[ \leq C \int_0^t \int_{\mathbb{R}^3} G_v(t, s) \nu(v) \| h(s) \|_{L^\infty} \times \left\{ \int_{\mathbb{R}^3} (1 + |u|)^{-2\beta p' + 16} \left( \int_{s - \delta}^{s - \delta} e^{-\nu(u)(s'-s')} \nu(u) \| h(s') \|_{L^\infty} \left[ \int_{\mathbb{R}^3} (1 + |u^*|)^{-2\beta p' + 16} |h(u^*)|^{p'} du^* \right]^{1/p'} du \right\}^{1/p'} ds \]

\[ \leq C \int_0^t \int_{\mathbb{R}^3} G_v(t, s) \nu(v) \| h(s) \|_{L^\infty} \times \left( \sup_{0 \leq s' \leq s - \delta} \| h(s') \|_{L^\infty}^{p'} \int_{\mathbb{R}^3} (1 + |u|)^{-2\beta p' + 16} \int_{|u^*| \geq 3N} (1 + |u^*|)^{-2\beta p' + 16} |h(u^*)|^{p'} du^* ds du \right)^{1/p'} ds 
\]

\[ \leq I_{331} + I_{332}, \]

where

\[ I_{331} := C \int_0^t \int_{\mathbb{R}^3} G_v(t, s) \nu(v) \| h(s) \|_{L^\infty} \times \left( \sup_{0 \leq s' \leq s - \delta} \| h(s') \|_{L^\infty}^{p'} \int_{\mathbb{R}^3} (1 + |u|)^{-2\beta p' + 16} \int_{|u^*| \geq 3N} (1 + |u^*|)^{-2\beta p' + 16} |h(u^*)|^{p'} du^* ds du \right)^{1/p'} ds, \]

\[ I_{332} := C \int_0^t \int_{\mathbb{R}^3} G_v(t, s) \nu(v) \| h(s) \|_{L^\infty} \times \left( \sup_{0 \leq s' \leq s - \delta} \| h(s') \|_{L^\infty}^{p'} \int_{\mathbb{R}^3} (1 + |u|)^{-2\beta p' + 16} \int_{|u^*| \leq 3N} (1 + |u^*|)^{-2\beta p' + 16} |h(u^*)|^{p'} du^* ds du \right)^{1/p'} ds. \]
For $I_{331}$, we can obtain

$$I_{331} = C \int_0^t G_v(t, s) \nu(v) \|h(s)\|_{L^\infty} \left| \int (1 + |u|)^{−2β'|+16} \int_{|u^*| \leq 3N} (1 + |u^*|)^{-2β'|+16} |h(u^*)| |u^*|^{1/p'} du^* ds \right| ds$$

$$\leq C \int_0^t G_v(t, s) \nu(v) \|h(s)\|_{L^\infty} \left( \sup_{0 \leq s' \leq s - \delta} \|h(s')\|_{L^\infty} \right) \left| \int (1 + |u|)^{−2β'|+16} \int_{|u^*| \leq 3N} (1 + |u^*|)^{-2β'|+16} |h(u^*)|^{1/p'} du^* ds \right| ds$$

$$\leq C \int_0^t G_v(t, s) \nu(v) \|h(s)\|_{L^\infty} \left( \sup_{0 \leq s' \leq s - \delta} \|h(s')\|_{L^\infty} \right) \left( \sup_{0 \leq s' \leq s - \delta} \|h(s')\|_{L^\infty} \right)$$

since $(-p')(2β - 1) + 16 < -3$. By the change of the variables (3.18) and (3.19), $I_{332}$ enjoys

$$I_{332} = C \int_0^t G_v(t, s) \nu(v) \|h(s)\|_{L^\infty} \left| \int (1 + |u|)^{−2β'|+16} \int_{|u^*| \leq 3N} (1 + |u^*|)^{-2β'|+16} |h(u^*)|^{1/p'} du^* ds \right| ds$$

$$\leq C \int_0^t G_v(t, s) \nu(v) \|h(s)\|_{L^\infty} \left( \sup_{0 \leq s' \leq s - \delta} \|h(s')\|_{L^\infty} \right) \left( \sup_{0 \leq s' \leq s - \delta} \|h(s')\|_{L^\infty} \right) \left( \sup_{0 \leq s' \leq s - \delta} \|h(s')\|_{L^\infty} \right)$$

(3.34)
Collecting (3.34) and (3.35), we can get

\[ I_{33} \leq \frac{C_M}{1 + 3N} + C_{M,\delta,N} \mathcal{E}(F_0)^{\frac{1}{p'}}. \]

(3.36)

For \( I_{34} \), it follows that

\[
I_{34} = C \int_0^t G_v(t, s)\nu(v)\|h(s)\|_{L^\infty} \left\{ \int_{\mathbb{R}^3} (1 + |u|)^{-2\beta p' + 16} \left( \int_0^{s-\delta} G_u(s', s')|w \Gamma_- (f, f)(s')| ds' \right)^{p'} du \right\}^{1/p'} ds
\]

\[
\leq C \int_0^t G_v(t, s)\nu(v)\|h(s)\|_{L^\infty} \left\{ \int_{\mathbb{R}^3} (1 + |u|)^{-2\beta p' + 16} \left( \int_0^{s-\delta} \frac{1}{\gamma} |h(u^*)|^{p'} du^* \right)^{1/p'} ds' \right\}^{1/p'} du \right\}^{1/p'} ds
\]

(3.37)

Therefore we can gain similar to the estimate of \( I_{33} \)

\[ I_{34} \leq \frac{C_M}{1 + 3N} + C_{N,M,\delta} \mathcal{E}(F_0)^{\frac{1}{p'}}. \]

(3.38)

For \( I_{35} \), notice that \( \int_{s-\delta}^s G_u(s, s')ds' \leq \int_{s-\delta}^s ds' = \delta \) and it follows from (3.24) that

\[
I_{35} = C \int_0^t G_v(t, s)\nu(v)\|h(s)\|_{L^\infty}
\]

\[
\times \left\{ \int_{s-\delta}^s G_u(s, s')K_u h(s') + w \Gamma_+(f, f)(s') + w \Gamma_-(f, f)(s') ds' \right\}^{1/p'} ds
\]

\[
\leq C \int_0^t G_v(t, s)\nu(v)\|h(s)\|_{L^\infty} \left\{ \sup_{s-\delta \leq s' \leq s} \delta \left( \|h(s')\|_{L^\infty}^{p'} + \|h(s')\|_{L^\infty}^{2p'} \right) \right\}^{1/p'} ds
\]

\[
\leq C\delta^{\frac{1}{3}} \left[ \sup_{0 \leq s \leq T} \|h(s)\|_{L^\infty}^{p'} + \sup_{0 \leq s \leq T} \|h(s)\|_{L^\infty}^{2p'} \right].
\]

(3.39)

Get together (3.28), (3.34), (3.36), (3.38), and (3.39). Then we obtain

\[
I_3 \leq C e^{-\frac{1}{2}(1+\varepsilon)|t|} \|h_0\|_{L^\infty} \int_0^t \|h(s)\|_{L^\infty} ds + C_M \varepsilon^{\gamma + 3} + C_M \delta^{\frac{1}{3}} + C_{\varepsilon,\delta,N,M} \mathcal{E}(F_0)^{\frac{1}{p'}} + C_M \left( \frac{1}{N^{\frac{1}{2}}} + N^{\gamma} + \frac{1}{1 + N} \right) + C_{\varepsilon,M} \frac{1}{N^2}.
\]

(3.40)

For \( I_4 \), we bound \( I_4 \) from Lemma [2.4] by

\[
I_4 = \int_0^t G_v(t, s)w \Gamma_-(f, f)(s) ds
\]

\[
\leq C \int_0^t G_v(t, s)\nu(v)\|h(s)\|_{L^\infty} \left( \int_{\mathbb{R}^3} \frac{1}{\gamma} |h(u)|^{p'} du \right)^{1/p'} ds
\]

\[
\leq C \int_0^t G_v(t, s)\nu(v)\|h(s)\|_{L^\infty} \left( \int_{\mathbb{R}^3} (1 + |u|)^{-2\beta p' + 16} |h(u)|^{p'} du \right)^{1/p'} ds.
\]
Therefore we can get similar to the estimate of $I_3$

$$I_3 \leq C e^{-\frac{1}{2}(1+\theta)\gamma} \|h_0\|_{L^\infty} \int_0^t \|h(s)\|_{L^\infty} ds + C_M e^{\gamma + 3} + C_M e^{(\gamma + 1)} e^{\gamma} + C_{\epsilon,\delta,N,N} F_0 \|F_0\|_{L^\infty} \tag{3.41}$$

Notice that for $0 < \delta < 1$ and sufficiently large $N > 1$,

$$\delta \leq \delta \frac{N^2}{(1 + N)^{\gamma + 1}} \leq \frac{1}{N + 1} \leq \frac{1}{N + 1}.$$

Combining (3.40), (3.43), (3.44), and (3.41), we can get

$$\|h(t)\|_{L^\infty} \leq C e^{-\frac{1}{2}(1+\theta)\gamma} \|h_0\|_{L^\infty} \left( \int_0^t \|h(s)\|_{L^\infty} ds + 1 \right) + C_M e^{\gamma + 3} + C_M \delta + C_{M,\epsilon,\delta,N} (\|F_0\|_{L^\infty} e^{\gamma} + \|F_0\|_{L^\infty} e^{\gamma})$$

$$\|h(t)\|_{L^\infty} \leq (1 + N)^{3 + \gamma} + 1 + \frac{1}{N + 1} + C_M e^{\gamma} \delta + C_M e^{\gamma} \delta + C_{M,\epsilon,\delta,N} (\|F_0\|_{L^\infty} e^{\gamma} + \|F_0\|_{L^\infty} e^{\gamma})$$

$$\leq C e^{-\frac{1}{2}(1+\theta)\gamma} \|h_0\|_{L^\infty} \left( \int_0^t \|h(s)\|_{L^\infty} ds + 1 \right) + C_M e^{\gamma + 3} + C_M e^{\gamma} \delta + C_{M,\epsilon,\delta,N} (\|F_0\|_{L^\infty} e^{\gamma} + \|F_0\|_{L^\infty} e^{\gamma})$$

$$+ C_M e^{\gamma} \delta + C_{M,\epsilon,\delta,N} (\|F_0\|_{L^\infty} e^{\gamma} + \|F_0\|_{L^\infty} e^{\gamma}).$$

\[\square\]

4. Proof of the main theorem

We need to recall the theorem in \[26\] because global existence for the solution in the large amplitude Boltzmann equation depends on the theorem in the small amplitude Boltzmann equation.

**Proposition 4.1.** \[26\] Let $0 < \theta < -\frac{2}{4}, -3 < \gamma < 0$, and $\rho - 1 = \frac{3+\gamma}{2-\gamma}$. Assume that the phase space is $\mathbb{T}_2^3 \times \mathbb{R}_3$ and $f_0$ satisfies (1.13). If $F(t, x, v) = \mu(v) + \sqrt{\nu} f(t, x, v)$ and $\|w_{\epsilon,\delta,\beta} f_0\|_{L^\infty} \leq \epsilon_1$ sufficiently small, there exists a unique solution $F(t, x, v) = \mu(v) + \sqrt{\nu} f(t, x, v) \geq 0$ to the Boltzmann equation (1.1) on $\mathbb{T}_2^3 \times \mathbb{R}_3$ and $f$ satisfies

$$\|w_{\epsilon,\delta,\beta} f(t)\|_{L^\infty} \leq C e^{-\lambda t \epsilon} \|w_{\epsilon,\delta,\beta} f_0\|,$$

for some $\lambda_1 > 0$.

**Remark 4.2.** In Proposition 4.1, unlike [26, Theorem 1.2, page 469], the domain is $\mathbb{T}^3$ and the polynomial term $(1 + |v|^2)^3$ is added to the weight function $w_{\epsilon,\delta,\beta}$. Thus, it is not exactly the same as Theorem 1.2 in [26]. Although there are some differences as above, we could get the same result. Let us introduce the process of obtaining $L^\infty$ decay property. Firstly, if we ignore the boundary effects in [26], we get $\|P f\|_{L^\infty} \leq \|(I - P) f\|_{L^\infty}$ in $\mathbb{T}_2^3$, which implies that

$$\frac{d}{dt} \|f(t)\|_{L^2_{x,v}} + \|f(t)\|_{L^2_{x,v}} \leq 0.$$

Moreover, by using the similar argument of the proof in [26, Lemma 4.3, Eq. (4.13), page 524], we obtain the following $L^2$ estimate

$$\|f(t)\|_{L^2_{x,v}} + \frac{d}{dt} \|f(t)\|_{L^2_{x,v}} \leq e^{-\lambda t} \|f(t)\|_{L^2_{x,v}} \leq e^{-\lambda t} \|w_{\epsilon,\delta,\beta} f_0\|_{L^2_{x,v}}.$$

Secondly, we consider the $L^2 - L^\infty$ bootstrap argument. Since we already developed the $K_w$ estimate containing the new weight function $w_{\epsilon,\delta,\beta}$ in Lemma 3.3, we can derive

$$\|w_{\epsilon,\delta,\beta} f(t)\|_{L^\infty_{x,v}} \leq e^{-\lambda t} \|w_{\epsilon,\delta,\beta} f_0\|_{L^\infty_{x,v}} + \int_0^t \|f(s)\|_{L^2_{x,v}} ds,$$

from similar arguments in the proof of Lemma 3.1. Here, $f$ is the solution to the linearized Boltzmann equation

$$\partial_t f + v \cdot \nabla_x f + \nu(v) f = K f \quad \text{for} \quad (t, x, v) \in \mathbb{R}_+ \times \mathbb{T}_2^3 \times \mathbb{R}^3.$$
Combining $L^2$ estimate and $L^2\text{-}L^\infty$ estimate yields
\[ \|w_{q,\varrho,\beta}f(t)\|_{L^\infty} \lesssim e^{-\lambda_t t} \|w_{q,\varrho,\beta}f_0\|_{L^\infty}. \]
For the nonlinear Boltzmann equation, we could derive from Lemma 2.4
\[ |w_{q,\varrho,\beta}(v)\Gamma(f, f)(v)| \lesssim v(t)\|w_{q,\varrho,\beta}f(t)\|_{L^\infty}^2. \]
Finally, the $t$ estimate above and the sequential argument of the proof in [26, page 537] gives the global existence and time sub-exponential decay of the nonlinear Boltzmann equation.

Now we can prove Theorem [1.1] applying Proposition [3.1]

Proof of main theorem. Recall that
\[ \sup_{0 \leq s \leq T} \|h(s)\|_{L^\infty} \leq M. \]
By Proposition [3.1] it holds that
\[ \|h(t)\|_{L^\infty} \leq CM_0 e^{-\frac{4}{5}(1+t)^{\rho}} \left( 1 + \int_0^t \|h(s)\|_{L^\infty} \right) + D, \]
where
\[ D := C_{M_0} \epsilon \beta + \bar{C}_M \left( N \gamma + \epsilon N + 1 \right) + \bar{C}_{M,N,\varrho} + \bar{C}_{M,N,\varrho,\beta} \left( \mathcal{E}(F_0)^{\frac{1}{2}} + \mathcal{E}(F_0)^{\frac{1}{2\varrho}} \right). \]
Define
\[ G(t) := 1 + \int_0^t \|h(s)\|_{L^\infty} ds. \]
Then we can rewrite
\[ G'(t) - CM_0 e^{-\frac{4}{5}(1+t)^{\rho}} G(t) \leq D. \] (4.1)
Note that $(1 + t)^{1-\rho} e^{-\frac{4}{5}(1+t)^{\rho}}$ is bounded and $G(t) > 0$ for $t \geq 0$. Then for all $0 < t \leq T$, we can obtain
\[ G'(t) - CM_0 e^{-\frac{4}{5}(1+t)^{\rho}} G(t) = G'(t) - CM_0 e^{-\frac{4}{5}(1+t)^{\rho}} G(t)(1 + t)^{\rho-1}(1 + t)^{1-\rho} \]
\[ \geq G'(t) - CM_0 e^{-\frac{4}{5}(1+t)^{\rho}} G(t)(1 + t)^{\rho-1} \]
\[ = \frac{d}{dt} \left( G(t) \exp \left\{ -\frac{4CM_0}{\lambda \rho} \left( 1 - e^{-\frac{4}{5}(1+t)^{\rho}} \right) \right\} \right). \] (4.2)
(4.1) and (4.2) yield
\[ \int_0^t \frac{d}{ds} \left( G(s) \exp \left\{ -\frac{4CM_0}{\lambda \rho} \left( 1 - e^{-\frac{4}{5}(1+s)^{\rho}} \right) \right\} \right) ds \leq \int_0^t D ds. \]
Then it follows that
\[ G(t) \exp \left\{ -\frac{4CM_0}{\lambda \rho} \left( 1 - e^{-\frac{4}{5}(1+t)^{\rho}} \right) \right\} \leq Dt + G(0) \exp \left\{ -\frac{4CM_0}{\lambda \rho} \left( 1 - e^{-\frac{4}{5}} \right) \right\} \leq 1 + Dt. \]
$G(t)$ is bounded for all $0 < t \leq T$ by
\[ G(t) \leq (1 + Dt) \exp \left\{ \frac{4CM_0}{\lambda \rho} \left( 1 - e^{-\frac{4}{5}(1+t)^{\rho}} \right) \right\} \leq (1 + Dt) \exp \left\{ \frac{4CM_0}{\lambda \rho} \right\}. \] (4.3)
To make $D$ sufficiently small, we can choose sufficiently small $\epsilon > 0$ depending on $M$, sufficiently large $N$ depending on $M$ and $\epsilon$, and sufficiently small $\delta > 0$ depending on $M$, $\epsilon$, and $N$. We will determine $M$ to depend only on $M_0$, $\epsilon$, $\delta$, and $N$ can be chosen depending only on $M_0$. We can take sufficiently small $\epsilon_0 \in (0, 1)$ to make
\[ D \leq \min \left\{ \frac{1}{4}M, \frac{1}{4}\epsilon_1, 1 \right\}, \]
where ǫ₁ was introduced in Proposition 4.1. Note 1 + Dt ≤ 1 + t and \((1 + t)e^{-\frac{1}{4}(1+t)^\nu} is bounded for all t ≥ 0.

Then we can get for all 0 < t ≤ T
\[
\| h(t) \|_{L^\infty} \leq CM_0 e^{-\frac{1}{4}(1+t)^\nu} (1 + Dt) \exp \left\{ \frac{4CM_0}{\lambda \rho} \right\} + D
\]
\[
\leq CM_0 e^{-\frac{1}{4}(1+t)^\nu} \exp \left\{ \frac{4CM_0}{\lambda \rho} \right\} + D
\]
\[
\leq \frac{1}{4} \tilde{M} e^{-\frac{1}{4}(1+t)^\nu} + D
\]
\[
\leq \frac{1}{4} \tilde{M} e^{-\frac{1}{4}t^\nu} + D,
\]
where \(\tilde{M}\) is defined as
\[
\tilde{M} := 4CM_0 \exp \left\{ \frac{4CM_0}{\lambda \rho} \right\} + 4M_0.
\]

Then it follows from (4.4) that for all 0 < t ≤ T,
\[
\| h(t) \|_{L^\infty} \leq \frac{1}{4} \tilde{M} + \frac{1}{4} \tilde{M} = \frac{1}{2} \tilde{M}.
\]

Therefore, we have proven that if a priori assumption holds,
\[
\sup_{0 \leq t \leq T} \| h(t) \|_{L^\infty} \leq \frac{1}{2} \tilde{M}.
\]

Next we should extend local existence of solution to global existence and check that
\[
\| h(T) \|_{L^\infty} \leq \epsilon_1.
\]

By Lemma A.1 there exists a time \(\hat{t}_0 > 0\) such that the solution \(f(t, x, v)\) of the Boltzmann equation exists for \(t \in [0, \hat{t}_0]\) and satisfies
\[
\sup_{0 \leq t \leq \hat{t}_0} \| w_{q, \vartheta, \beta} f(t) \|_{L^\infty} \leq 2 \| w_{q, \vartheta, \beta} f_0 \| \leq \frac{1}{2} \tilde{M}.
\]

Considering \(\hat{t}_0\) as the initial time, by Lemma A.1 for some \(\tilde{t} > 0\), it holds that
\[
\sup_{\hat{t}_0 \leq t \leq \hat{t}_0 + \tilde{t}} \| w_{q, \vartheta, \beta} f(t) \|_{L^\infty} \leq 2 \| w_{q, \vartheta, \beta} f(\hat{t}_0) \|_{L^\infty} \leq \tilde{M},
\]

implying, by Lemma 4.5
\[
\sup_{0 \leq t \leq \hat{t}_0 + \tilde{t}} \| w_{q, \vartheta, \beta} f(t) \|_{L^\infty} \leq \frac{1}{2} \tilde{M}.
\]

Define
\[
T := \left( \frac{1}{\lambda} \left[ \ln \tilde{M} + \ln \epsilon_1 \right] \right)^{\frac{1}{\nu}},
\]
where \(\epsilon_1\) was introduced in Proposition 4.1. We can extend the local existence of the solution to 0 ≤ t ≤ T as above and we can use (4.4) to gain
\[
\| w_{q, \vartheta, \beta} f(T) \|_{L^\infty} \leq \frac{1}{4} \tilde{M} e^{-\frac{1}{4}T^\nu} + D
\]
\[
\leq \frac{1}{4} \epsilon_1 + \frac{1}{4} \epsilon_1 < \epsilon_1.
\]

Therefore we show the global existence and uniqueness of the solution to the Boltzmann equation by Proposition 4.1. For all t ≥ T, we obtain from Proposition 4.1
\[
\| h(t) \| \leq C \| h(T) \|_{L^\infty} e^{-\lambda(t-T)} \rho \leq C \epsilon_1 e^{-\lambda t^\nu}.
\]

Taking \(\lambda_0 := \min \left\{ \frac{1}{4}, \lambda_1 \right\}\), we have that
\[
\| w_{q, \vartheta, \beta} f(t) \|_{L^\infty} \leq C \tilde{M} e^{-\lambda_0 t^\nu} \leq \left( 4CM_0 \exp \left\{ \frac{4CM_0}{\lambda_0 \rho} \right\} + 4M_0 \right) e^{-\lambda_0 t^\nu},
\]
for all \( t \geq 0 \).

\[ \square \]

### Appendix A. Local Existence and Uniqueness

**Lemma A.1.** Let \( 0 < q < 1 \) and \( 0 \leq \vartheta < -\frac{2}{7} \) be fixed in the weight function (1.17). If \( F_0(x,v) = \mu(v) + \sqrt{\mu(v)} f_0(x,v) \geq 0 \) and \( \|w_{q,\vartheta,\beta} f_0\| < \infty \), then there exists a time \( \hat{t}_0 > 0 \) such that the initial value problem (1.1) and (1.4) has a unique non-negative solution \( F(t,x,v) = \mu(v) + \sqrt{\mu(v)} f(t,x,v) \) for \( t \in [0, \hat{t}_0] \), satisfying

\[
\sup_{0 \leq t \leq \hat{t}_0} \left\| w_{q,\vartheta,\beta} f(t) \right\|_{L^\infty} \leq 2 \left\| w_{q,\vartheta,\beta} f_0 \right\|_{L^\infty}.
\]  

(A.1)

**Proof.** For the local existence of non-negative solution of the Boltzmann equation (1.1), we consider the following iteration:

\[
\begin{cases}
\partial_t F^{n+1} + v \cdot \nabla_x F^{n+1} + F^{n+1} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v-u, \omega) F^n(u) \, d\omega du = Q_+ (F^n, F^n) \\
F(0,x,v) = F_0(x,v) \geq 0, F^n(0,t,x,v) = \mu(v).
\end{cases}
\]

(A.2)

By induction on \( n \), we can prove that all \( F^n \) is non-negative for all \( n > 0 \).

Define \( F^n(t,x,v) := \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v-u, \omega) F^n(u) \, d\omega du \). For \( n = 1 \), by our assumption that \( F_0 \) is nonnegative,

\[
F^1(t,x,v) = e^{-\nu v t} F_0(x-t,v) + \int_0^t e^{-\nu v(t-s)} \nu(v) \mu(v) ds \geq 0,
\]

because \( \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(u-v, \omega) \mu(u) d\omega du = \nu(u) \). Then, we suppose that \( F^n \) is nonnegative for \( n = 1, 2, \cdots, k \).

By Duhamel’s principle, we get

\[
F^{k+1} = e^{-\int_0^t (\vartheta, v, x, \omega) ds} F_0 + \int_0^t e^{-\int_0^s (\vartheta, v, x, \omega) ds} Q_+ (F^k, F^k) ds \geq 0,
\]

since \( F_0 \geq 0 \) and \( Q_+ (F^n, F^n) \geq 0 \). Therefore \( F^n \) is nonnegative for all \( n > 0 \).

Hence we can rewrite the above iteration (A.2) for \( h = w_{q,\vartheta,\beta} f \) as follow:

\[
\begin{cases}
(\partial_t + v \cdot \nabla_x + \hat{\nu}) h^{n+1}(t) = K_w h^n(t) + w \Gamma_+ (f^n, f^n) - w \Gamma_- (f^n, f^{n+1}) \\
h^{n+1}(0,x,v) = h_0(x,v), h^0 = 0.
\end{cases}
\]

(A.3)

We will show that there exists \( \hat{t}_1 > 0 \) such that (A.3) has a solution over \([0, \hat{t}_1] \) satisfying

\[
\sup_{0 \leq t \leq \hat{t}_1} \left\| h^n(t) \right\|_{L^\infty} \leq 2 \left\| h_0 \right\|_{L^\infty},
\]  

(A.4)

for all \( n > 0 \) and \( t \in [0, \hat{t}_1] \). For \( n = 1 \), \( h^1(t) = G(t,0) h_0 \), implying that (A.4) holds for \( n = 1 \). Then we suppose that (A.4) holds for \( n = 1, 2, \cdots, k \). By Duhamel’s principle and (5.22),

\[
|h^{k+1}(t)| \leq G(t,0) \|h_0\|_{L^\infty} + \int_0^t G(t,s) \left[ |K_w h^k(s)| + |w \Gamma_+ (f^k, f^k)(s)| + |w \Gamma_- (f^k, f^{k+1})(s)| \right] ds
\]

\[
\leq \|h_0\|_{L^\infty} + C \hat{t}_1 \left\{ \sup_{0 \leq s \leq \hat{t}_1} \left\| h^k(s) \right\|_{L^\infty} + \sup_{0 \leq s \leq \hat{t}_1} \left\| h^k(s) \right\|_{L^2} + \sup_{0 \leq s \leq \hat{t}_1} \left\| h^k(s) \right\|_{L^\infty} \right\}
\]

\[
\leq \|h_0\|_{L^\infty} + C \hat{t}_1 \|h_0\|_{L^\infty} (\|h_0\|_{L^\infty} + 1) + C \hat{t}_1 \|h_0\|_{L^\infty} \sup_{0 \leq s \leq \hat{t}_1} \left\| h^{k+1}(s) \right\|_{L^\infty}.
\]

Taking \( \hat{t}_1 \leq \min \left\{ \frac{1}{4} (C (\|h_0\|_{L^\infty} + 1))^{-1}, \frac{1}{4} (C (\|h_0\|_{L^\infty})^{-1}) \right\} \), we can get

\[
\frac{2}{3} \sup_{0 \leq s \leq \hat{t}_1} \left\| h^{k+1}(s) \right\|_{L^\infty} \leq \frac{4}{3} \|h_0\|_{L^\infty}.
\]

By induction on \( n \), (A.4) holds for all \( n > 0 \) and \( t \in [0, \hat{t}_1] \). For proving the convergence of \( \{h^n\} \), we consider \((h^{n+1} - h^n)\). \((h^{n+1} - h^n)\) is the solution of the following equation:

\[
\begin{cases}
(\partial_t + v \cdot \nabla_x + \hat{\nu})(h^{n+1} - h^n) = K_w (h^n - h^{n-1}) + w \Gamma_+ (f^{n-1}, f^n) \\
- w \Gamma_+ (f^n, f^{n-1}) - w \Gamma_- (f^n, f^{n+1}) + w \Gamma_+ (f^{n-1}, f^n)
\end{cases}
\]

\((h^{n+1} - h^n)(0) = 0\).
Hence it holds that
\[ w\Gamma(f, g) - w\Gamma(h, l) = w\Gamma(f, g) - w\Gamma(h, g) + w\Gamma(h, g) - w\Gamma(h, l) = w\Gamma(f - h, g) + w\Gamma(h, g - l). \tag{A.5} \]

\[ (A.5) \] holds although \( w\Gamma \) is replaced to \( w\Gamma_- \) or \( w\Gamma_+ \). Applying Duhamel’s principle and \( (A.5) \), we have
\[
|h^{n+1}(t) - h^n(t)| \leq \int_0^t G(t, s) \left[ |K_w(h^n - h^{n-1})(s)| + |w\Gamma_+(f^n - f^{n-1}, f^n)(s)| + |w\Gamma_+(f^n - f^{n-1}, f^{n-1})(s)| + |w\Gamma_-(f^n - f^{n-1}, f^{n-1})(s)| \right] ds
\leq C \hat{t}_0 \sup_{0 \leq s \leq t} \|h^n(s) - h^{n-1}(s)\|_{L^\infty} + \sup_{0 \leq s \leq t} \|h^{n-1}(s)\|_{L^\infty} + \sup_{0 \leq s \leq t} \|h^{n+1}(s)\|_{L^\infty} \sup_{0 \leq s \leq t} \|h^n(s) - h^{n-1}(s)\|_{L^\infty}
\leq C_1 \hat{t}_0 \left(1 + \|h_0\|_{L^\infty}\right) \sup_{0 \leq s \leq t} \|h^n(s) - h^{n-1}(s)\|_{L^\infty} + \|h_0\|_{L^\infty} \sup_{0 \leq s \leq t} \|h^{n+1}(s) - h^n(s)\|_{L^\infty},
\]
where \( C_1 = 4C \). Take \( C' = \max\{C_1, C_2\} \) and \( \hat{t}_0 \leq \min\{\hat{t}_1, \frac{1}{3} \{C'(\|h_0\|_{L^\infty} + 1)\}^{-1}, \frac{1}{3} (C'\|h_0\|_{L^\infty})^{-1}\} \), where \( C_2 \) will be determined later in \( (A.7) \). Then it follows that
\[
\frac{2}{3} \sup_{0 \leq s \leq \hat{t}_0} \|h^{n+1}(s) - h^n(s)\|_{L^\infty} \leq \frac{1}{3} \sup_{0 \leq s \leq \hat{t}_0} \|h^n(s) - h^{n-1}(s)\|_{L^\infty}.
\]
Therefore \( \{h^n\} \) is a convergent sequence and we can denote \( h^n \to h \), and \( F^n \to F \) as \( n \to \infty \). Since all \( F^n \) is non-negative, \( F \) is non-negative for \( t \in [0, \hat{t}_0] \), and \( (A.4) \) implies \( (A.1) \). For the uniqueness of the local solution, suppose that there is another solution \( g \) to the Boltzmann equation with the same initial condition as \( f \) satisfying
\[
\sup_{0 \leq t \leq \hat{t}_0} \|w_{q, \vartheta, \beta}g(t)\|_{L^\infty} \leq 2\|w_{q, \vartheta, \beta}f_0\|_{L^\infty}, \tag{A.6}
\]
and set \( h_1 := w_{q, \vartheta, \beta}g \). Then by \( (A.6) \), we obtain
\[
|h_1(t) - h(t)| \leq \int_0^t G(t, s) \left[ |K_w(h_1 - h)(s)| + |w\Gamma(f - g, f)(s)| + |w\Gamma(g, f - g)(s)| \right] ds
\leq C \hat{t}_0 \sup_{0 \leq s \leq \hat{t}_0} (\|h_1(s)\|_{L^\infty} + \|h(s)\|_{L^\infty} + 1) \sup_{0 \leq s \leq \hat{t}_0} \|h_1(s) - h(s)\|_{L^\infty}
\leq C_2 \hat{t}_0 (\|h_0\|_{L^\infty} + 1) \sup_{0 \leq s \leq \hat{t}_0} \|h_1(s) - h(s)\|_{L^\infty}
\leq \frac{1}{2} \sup_{0 \leq s \leq \hat{t}_0} \|h_1(s) - h(s)\|_{L^\infty},
\]
where \( C_2 = 4C \), implying that \( h_1 = h \).

\begin{proof}

\end{proof}

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