Forced waves of parabolic-elliptic Keller-Segel models in shifting environments

Wenxian Shen* and Shuwen Xue
Department of Mathematics and Statistics
Auburn University, AL 36849, USA

Abstract
The current paper is concerned with the forced waves of Keller-Segel chemotaxis systems in shifting environments of the form,

\[
\begin{aligned}
&\frac{du}{dt} = u_{xx} - \chi u v_x x + u(r(x - ct) - bu), \quad x \in \mathbb{R} \\
&0 = v_{xx} - \nu v + \mu u, \quad x \in \mathbb{R},
\end{aligned}
\]

where \(\chi, b, \nu, \mu\) are positive constants, \(c \in \mathbb{R}\), the resource function \(r(x)\) is globally Hölder continuous, bounded, \(r^* = \sup_{x \in \mathbb{R}} r(x) > 0\), \(r(\pm \infty) := \lim_{x \to \pm \infty} r(x)\) exist, and either \(r(-\infty) < 0 < r(\infty)\), or \(r(\pm \infty) < 0\). Assume that \(b > 2\chi \mu\). In the case that \(r(-\infty) < 0 < r(\infty)\), it is shown that (1.1) has a forced wave solution connecting \((r^*, 0, 0)\) and \((0, 0)\) with speed \(c\) provided that \(c > \frac{\sqrt{\frac{\nu}{2\chi}} + b - 2\chi \mu}{2\chi}\). In the case that \(r(\pm \infty) < 0\), it is shown that (1.1) has a forced wave solution connecting \((0, 0)\) and \((0, 0)\) with speed \(c\) provided that \(\chi\) is sufficiently small and \(\lambda_\infty > 0\), where \(\lambda_\infty\) is the generalized principal eigenvalue of the operator \(u(\cdot) \mapsto u_{xx}(\cdot) + cu_x(\cdot) + r(\cdot)u(\cdot)\) on \(\mathbb{R}\) in certain sense. Some numerical simulations are also carried out. The simulations indicate the existence of forced wave solutions in some parameter regions which are not covered in the theoretical results, induce several problems to be further studied, and also provide some illustration of the theoretical results.

Key words. Parabolic-elliptic chemotaxis system, spreading speeds, persistence, forced wave, shifting environment.

1 Introduction
This work is concerned with the forced wave solutions of the attraction Keller-Segel chemotaxis models in shifting environments of the form

\[
\begin{aligned}
&\frac{du}{dt} = u_{xx} - (\chi uv_x) x + u(r(x - ct) - bu), \quad x \in \mathbb{R} \\
&0 = v_{xx} - \nu v + \mu u, \quad x \in \mathbb{R},
\end{aligned}
\]

where \(b, \nu, \mu\) and \(\chi\) are positive constants, \(c \in \mathbb{R}\) and \(u(t, x)\) and \(v(t, x)\) represent the densities of a mobile species and a chemical substance, respectively. Biologically, the positive constant \(\chi\)

*Partially supported by the NSF grant DMS–1645673
measures the effect on the mobile species by the chemical substance which is produced overtime by the mobile species; the reaction term \( u(r(x - ct) - bu) \) in the first equation of (1.1) describes the local dynamics of the mobile species which depends on the density \( u \) and on the shifting habitat with a fixed speed \( c \); \( \nu \) represents the degradation rate of the chemical substance; and \( \mu \) is the rate at which the mobile species produces the chemical substance.

System (1.1) with \( r(\cdot) \) being a constant function is a simplified version of the chemotaxis system proposed by Keller and Segel in their works [19, 20]. Chemotaxis describes the oriented movements of biological cells and organisms in response to chemical gradient which they may produce themselves over time and is crucial for many phenomena such as the location of food sources, avoidance of predators and attracting mates, slime mold aggregation, tumor angiogenesis, and primitive streak formation. Chemotaxis is also crucial in macroscopic process such as population dynamics and gravitational collapse. The reader is referred to [15, 16] for some detailed introduction into the mathematics of Keller-Segel chemotaxis models.

In this paper, we consider (1.1) with \( c \neq 0 \) and \( r(\cdot) \) being a sign changing function. In particular, we will consider the following two cases:

**Case 1.** Favorable and unfavorable habitats are separated in the sense that \( r(x) \) is globally Hölder continuous, bounded, \( r(\pm \infty) := \lim_{x \to \pm \infty} r(x) \) exist and are finite, and \( r(-\infty) < 0 < r(\infty) \), \( r(-\infty) \leq r(x) \leq r(\infty) \), \( \forall x \in \mathbb{R} \).

**Case 2.** Favorable habitat is surrounded by unfavorable habitat in the sense that \( r(x) \) is globally Hölder continuous, bounded, \( \sup_{x \in \mathbb{R}} r(x) > 0 \), the limits \( r(\pm \infty) := \lim_{x \to \pm \infty} r(x) \) exist and are finite, and \( r(\pm \infty) < 0 \), \( \min\{r(\infty), r(-\infty)\} \leq r(x) \), \( \forall x \in \mathbb{R} \).

As described in [42], in **Case 1**, \( r(x - ct) \) divides the spatial domain into two regions: the region with good-quality habitat suitable for growth \( \{x \in \mathbb{R}: r(x - ct) > 0\} \) and the region with poor-quality habitat unsuitable for growth \( \{x \in \mathbb{R}: r(x - ct) < 0\} \). The edge of the habitat suitable for species growth is shifting at a speed \( c \). In **Case 2**, \( r(x - ct) \) still divides the spatial domain into two regions: one favorable for growth \( \{x \in \mathbb{R}: r(x - ct) > 0\} \) and one unfavorable for growth \( \{x \in \mathbb{R}: r(x - ct) < 0\} \). The favorable habitat is bounded and surrounded by the unfavorable habitat. The favorable habitat is shifting at a speed \( c \).

Consider (1.1). It is important to know whether the species will become extinct or persist; whether it spreads into larger and larger regions, and if so, how fast it spreads; whether the system has so called forced wave solutions. A positive solution \( (u(t, x), v(t, x)) \) of (1.1) is called a forced wave solution if it is defined for all \( t \in \mathbb{R}, x \in \mathbb{R} \), and \( (u(t, x), v(t, x)) = (\phi(x - ct), \psi(x - ct)) \) for some one variable functions \( \phi(\cdot) \) and \( \psi(\cdot) \).

There are many studies on these problems for (1.1) in the absence of the chemotaxis (i.e. \( \chi = 0 \)). Note that in the absence of the chemotaxis \( (\chi = 0) \), the first equation in (1.1) is decoupled from the second equation, and the dynamics of (1.1) is determined by the first equation of (1.1), that is,

\[
\begin{align*}
    u_t &= u_{xx} + u(r(x - ct) - bu), & x \in \mathbb{R}.
\end{align*}
\]

(1.2)

For example, in **Case 1**, Li et al. [26] studied the spatial dynamics of system (1.2) for the case \( b = 1 \) and they showed that the persistence and spreading dynamics of (1.2) depend on the speed of the shifting habitat edge \( c \) and a number \( c^* \), where \( c^* = 2\sqrt{r(\infty)} \) for (1.2). More precisely, they proved that if \( c > c^* \), then the species will become extinct in the habitat, and if \( 0 < c < c^* \), the species will persist and spread along the shifting habitat at the asymptotic spreading speed.
Recently, Hu and Zou [18] demonstrated that in the case \( b = 1 \), for any given speed \( c > 0 \) of the shifting habitat edge, (1.2) admits a nondecreasing traveling wave solution \( u(t, x) = \phi(x - ct) \) connecting 0 and \( r(\infty) \) (i.e. \( \phi(-\infty) = 0 \) and \( \phi(\infty) = r(\infty) \)) with the speed \( c \) agreeing to the habitat shifting speed, which accounts for an extinction wave. Very recently, Wang and Zhao [45] obtained the uniqueness of the forced wave of (1.2) by using the sliding technique and established the global exponential stability of the forced wave via the monotone semiflows approach combined with the method of super- and subsolutions (see [45, Theorem 2.3]).

In Case 2, Berestycki et al. [2] proposed to use the following reaction-diffusion equation with a forced speed \( c > 0 \) to study the influence of climate change on the population dynamics of biological species:

\[
 u_t = u_{xx} + f(x - ct, u), \quad x \in \mathbb{R}.
\]  

A typical \( f \) considered in [2] is

\[
 f(x, u) = \begin{cases} 
 au(1 - \frac{u}{K}), & 0 \leq x \leq L, \\
 -ru, & x < 0 \text{ and } x > L 
\end{cases}
\]

for some positives constants \( a, r, K, L \). They first considered this special case and derived an explicit condition for the persistence of species by gluing phase portraits. Then they established a strict qualitative dichotomy for a large class of models by the rigorous PDE methods. More precisely, they showed that if \( \lambda_\infty \), defined to be the generalized principal eigenvalue of the operator \( u \rightarrow u_{xx} + cu_x + f_u(x, 0)u \) on \( \mathbb{R} \), is less than or equal to zero, then (1.3) has no forced wave solution and every positive solution of (1.3) converges to zero as \( t \to \infty \), uniformly in \( x \). If \( \lambda_\infty > 0 \), (1.3) has a unique forced wave solution and every nontrivial positive solution of (1.3) converges to this unique forced wave solution as \( t \to \infty \), uniformly in \( x \).

It should be pointed out that the paper [35] addresses the same question as in [2], but focuses on the effect of a moving climate on the outcome of competitive interactions between two species. There are also many studies on forced wave solution of (1.3) with different shifting habitats. For example, Fang, Lou and Wu [10] established the existence and nonexistence of forced waves and pulse waves of (1.3) with \( f \) of the form \( f(x - ct, u) = u(r(x - ct) - u) \) where \( r(-\infty) > 0 > r(\infty) \). Berestycki and Fang [3] obtained the complete existence and multiplicity of forced waves as well as their attractivity except for some critical cases under the condition \( f : (s, u) \in \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R} \) is asymptotically of KPP type. For other related works on climate change problem with different shifting habitats for certain reaction-diffusion equations, nonlocal dispersal equations, lattice differential equations, as well as integro-difference equations, we refer the readers to [9, 17, 23, 24, 27, 28, 41, 45, 49] and the references therein.

Observe that, when \( r(x) \equiv r \), (1.2) becomes

\[
 u_t = u_{xx} + u(r - bu), \quad x \in \mathbb{R}.
\]  

Due to the pioneering works of Fisher [11] and Kolmogorov, Petrovsky, Piskunov [21] on traveling wave solutions and take-over properties of (1.4), (1.4) is also referred to as the Fisher-KPP equation. The following results are well known about the traveling wave solutions and spreading speeds of (1.4). Equation (1.4) has traveling wave solutions \( u(t, x) = \phi(x - ct) \) connecting \( \frac{x}{b} \) and 0 (i.e. \( \phi(-\infty) = \frac{x}{b} \) and \( \phi(\infty) = 0 \)) of all speeds \( c \geq 2\sqrt{r} \) and has no such traveling wave solutions of slower speed. Such a traveling wave solution is unique up to a translation. For any nonnegative
solution \( u(t, x) \) of (1.4), if at time \( t = 0 \), \( u(0, x) = u_0(x) \) is \( \frac{r}{b} \) for \( x \) near \(-\infty\) and 0 for \( x \) near \( \infty \), then

\[
\limsup_{x \geq ct, t \to \infty} u(t, x) = 0 \quad \forall c > 2\sqrt{r}
\]

and

\[
\limsup_{x \leq ct, t \to \infty} |u(t, x) - \frac{r}{b}| = 0 \quad \forall c < 2\sqrt{r}.
\]

In literature, \( c^*_0 = 2\sqrt{r} \) is called the spreading speed for (1.4). Since the pioneering works by Fisher [11] and Kolmogorov, Petrovsky, Piscunov [21], a huge amount research has been carried out toward the front propagation dynamics of reaction diffusion equations of the form,

\[
u_t = \Delta u + uf(t, x, u), \quad x \in \mathbb{R}^N,
\]

where \( f(t, x, u) < 0 \) for \( u > 1 \), \( \partial_u f(t, x, u) < 0 \) for \( u \geq 0 \) (see [1, 4, 5, 6, 7, 12, 13, 29, 30, 31, 32, 33, 40, 41, 46, 47, 50], etc.).

Comparing with the Fisher-KPP equation, traveling wave solutions and spreading speeds of chemotaxis models have only been studied recently. For example, Salako and Shen studied in [36, 37, 38] the spatial spreading dynamics of (1.1) with constant growth rate \( r > 0 \) and obtained several fundamental results. Some lower and upper bounds for the propagation speeds of solutions with compactly supported initial functions were derived, and some lower bound for the speeds of traveling wave solutions was also derived. It is proved that all these bounds converge to the spreading speed \( c^*_0 = 2\sqrt{r} \) of (1.4) as \( \chi \to 0 \). Assume that \( 2\chi \mu < b \). It’s also proved that there is a positive constant \( c^*(\chi, r, b, \nu, \mu) > c^*_0 = 2\sqrt{r} \) such that for any \( c > c^*(\chi, a, b, \lambda, \mu) \), (1.1) with constant growth rate \( r > 0 \) has a traveling wave solution \((u(t, x), v(t, x)) = (U(x - ct), V(x - ct))\) with speed \( c \) connecting the constant solutions \((\frac{r}{b}, \frac{r}{b})\) and \((0, 0)\) (i.e. \( U(-\infty) = \frac{r}{b} \) and \( U(\infty) = 0 \)).

The reader is also referred to [14] for the lower and upper bounds of propagation speeds of (1.1) with constant growth rate.

Very recently, the authors of the paper [39] improved the results in [37]. It is proved that in the case of constant growth rate \( r > 0 \), if \( b > \chi \mu \) and \((1 + \frac{1}{2} \frac{(\sqrt{r^*} - \sqrt{\nu^*})}{(\sqrt{r^*} + \sqrt{\nu^*})}) \chi \mu \leq b \) hold, then \( 2\sqrt{r} \) is the spreading speed of the solutions \((u(t, x; u_0), v(t, x; u_0))\) of (1.1) with nonnegative compactly supported continuous initial functions \( u_0 \), that is,

\[
\limsup_{t \to \infty} u(t, x; u_0) = 0 \quad \forall c > 2\sqrt{r}
\]

and

\[
\liminf_{t \to \infty} \inf_{|x| \leq ct} u(t, x; u_0) > 0 \quad \forall 0 < c < 2\sqrt{r}.
\]

It is also proved that, if \( b > 2\chi \mu \) and \( \nu \geq r \) hold, then \( c^*_0 = 2\sqrt{r} \) is the minimal speed of the traveling wave solutions of (1.1) with constant growth rate \( r > 0 \) connecting \((\frac{r}{b}, \frac{r}{b})\) and \((0, 0)\), that is, for any \( c \geq c^*_0 \), (1.1) with constant growth rate \( r > 0 \) has a traveling wave solution connecting \((\frac{r}{b}, \frac{r}{b})\) and \((0, 0)\) with speed \( c \), and (1.1) with constant growth rate \( r > 0 \) has no such traveling wave solutions with speed less than \( c^*_0 \), where \((u(t, x; u_0), v(t, x; u_0))\) is the unique global classical solution of (1.1) with \( u(0, x; u_0) = u_0(x) \).

In the recent work [42], the authors of the current paper investigated the persistence and spreading speeds of (1.1) when \( r(x) \) is as in Case 1 or Case 2. Assume \( b > \chi \mu \) and \( b \geq (1 + \frac{1}{2} \frac{(\sqrt{r^*} - \sqrt{\nu^*})}{(\sqrt{r^*} + \sqrt{\nu^*})}) \chi \mu \), where \( r^* = \sup_{x \in \mathbb{R}} r(x) \). In the Case 1, it is shown that if the moving speed
for every $x \geq 0$, we denote by $(\lambda, \nu)$,\footnote{Note that, in Case 2, it is shown that if $|c| > c^*$, then the species will become extinct in the habitat. If $\lambda_{\infty} > 0$, then the species will persist surrounding the good habitat.} whenever (1.1) both theoretically and numerically.

The objective of the current paper is to investigate the existence of forced wave solutions of (1.1) with given initial functions. Let $u \in C(\mathbb{R})$ be the principal eigenvalue of the operator $u \rightarrow u_{xx} + cu_x + r(x)u$ on $\mathbb{R}$, is negative and the degradation rate $\nu$ of the chemical substance is greater than or equal to the number $\nu^* := (\sqrt{4r^* + c^* + |c|})^2$, then the species will also become extinct in the habitat. If $\lambda_{\infty} > 0$, then the species will persist surrounding the good habitat.

The following proposition states the existence and uniqueness of solutions of (1.1) with given initial functions.

**Proposition 1.1.** Suppose that $r(x)$ is globally Hölder continuous and bounded. For every nonnegative initial function $u_0 \in C^b(\mathbb{R})$, there is a unique maximal time $T_{\max} > 0$, such that $(u(t, x; u_0), v(t, x; u_0))$ is defined for every $x \in \mathbb{R}$ and $0 \leq t < T_{\max}$. Moreover if $\chi \mu < b$ then $T_{\max} = \infty$ and the solution is globally bounded.

The above proposition can be proved by similar arguments as those in ([38, Theorem 1.1 and Theorem 1.5]).

Throughout this paper, we assume that $r(x)$ is as in Case 1 or Case 2. We put

$$r_* = \inf_{x \in \mathbb{R}} r(x), \quad r^* = \sup_{x \in \mathbb{R}} r(x), \quad c^* = 2\sqrt{r^*}. \quad (1.6)$$

Note that, in Case 1, $r_* = r(-\infty)$ and $r^* = r(+\infty)$, and in Case 2, $r_* = \min\{r(-\infty), r(+\infty)\}$ and $r^* = \max_{x \in \mathbb{R}} r(x)$.

Let $\lambda_L(r(\cdot))$ be the principal eigenvalue of

$$\begin{cases}
\phi_{xx} + c\phi_x + r(x)\phi = \lambda\phi, & -L < x < L \\
\phi(-L) = \phi(L) = 0.
\end{cases} \quad (1.7)$$

Note that $\lambda_L(r(\cdot))$ is increasing as $L$ increases. Let $\lambda_{\infty}(r(\cdot)) = \lim_{L \to \infty} \lambda_L(r(\cdot))$.  

\begin{align*}
c > c^* := 2\sqrt{r^*}, & \text{ then the species becomes extinct in the habitat. If the moving speed } -c^* \leq c < c^*, \\
& \text{ then the species will persist and spread along the shifting habitat at the asymptotic spreading speed } c^*.
\end{align*}
When and \( \phi \) Theorem 1.2. Suppose that \( b > 2\chi\mu \) and \( c > \frac{\chi \mu r}{\sqrt{r(b-2\chi\mu)}} - 2\sqrt{r(b-2\chi\mu)} \).

(H1) \( b > 2\chi\mu \) and \( c > -2\sqrt{r} \).

(H2) \( b \geq \frac{3}{2}\chi\mu \) and \( \lambda_\infty(r(\cdot)) > 0 \).

Note that \( \lambda_\infty(r(\cdot)) > 0 \) implies that \(-2\sqrt{r} < c < 2\sqrt{r}\).

A positive solution \((u(t,x),v(t,x))\) of (1.1) is called a forced wave solution if it is defined for all \( t \in \mathbb{R}, x \in \mathbb{R} \) and \((u(t,x),v(t,x)) = (\phi(x-ct), \psi(x-ct))\) for some one variable functions \( \phi(\cdot) \) and \( \psi(\cdot) \). It is clear that \((u,v) = (\phi(x), \psi(x))\) is a stationary solution of

\[
\begin{align*}
\frac{u_t}{u} = u_{xx} + cu_x - (\chi uv_x)_x + u(r(x) - bu), & \quad x \in \mathbb{R} \\
0 = v_{xx} - \nu v + \mu u, & \quad x \in \mathbb{R}.
\end{align*}
\]

We say that a positive forced wave solution \((u(t,x),v(t,x)) = (\phi(x-ct), \psi(x-ct))\) of (1.1) connects \((u^*, v^*) \) and \((u^+, v^+)*\) if \((\phi(\pm \infty), \psi(\pm \infty)) = (u^*_{\pm}, v^*_{\pm})\).

Our main theoretical results are stated in the following two theorems.

**Theorem 1.1.** (1) Suppose that \( r(x) \) is as in Case 1, and (H1) holds. Then there is a forced wave solution \((u(t,x),v(t,x)) = (\phi(x-ct), \psi(x-ct))\) connecting \((u^*, v^*)\) and \((0,0)\).

(2) Suppose that \( r(x) \) is as in Case 1, and (H1)' holds. Then there exists a number \( \eta > 0 \) such that for any \( 0 < \chi < \min\left\{ \frac{2\chi \mu r}{\sqrt{r(b-2\chi\mu)}}, \frac{b\eta}{\mu r^2+\mu \eta} \right\} \), there is a forced wave solution \((u(t,x),v(t,x)) = (\phi(x-ct), \psi(x-ct))\) connecting \((u^*, v^*)\) and \((0,0)\).

**Theorem 1.2.** Suppose that \( r(x) \) is as in Case 2, and (H2) holds. Then there exists a number \( \eta > 0 \) such that for any \( 0 < \chi < \min\left\{ \frac{2\chi \mu r}{\sqrt{r(b-2\chi\mu)}}, \frac{b\eta}{\mu r^2+\mu \eta} \right\} \), there is a forced wave solution \((u(t,x),v(t,x)) = (\phi(x-ct), \psi(x-ct))\) connecting \((0,0)\) and \((0,0)\), that is, \( \phi(x) > 0 \) for all \( x \in \mathbb{R} \) and \( \phi(\pm \infty) = 0 \).

**Remark 1.1.** (1) When \( \chi = 0, (H1) \) becomes \( c > -2\sqrt{r} \). Hence Theorem 1.1 recovers [18, Theorem 1.1].

(2) When \( \chi = 0, \) Theorem 1.2 recovers [2, Theorem 4.8].

(3) Thanks to the presence of chemotaxis, the comparison principle for parabolic equations cannot be applied directly to (1.1) or (1.2), and the techniques used in the study of (1.1) and (1.3) are difficult to be applied to the study of (1.1). We used Schauder’s fixed point theorem together with sub- and super-solutions in the study of the existence of forced wave solutions of (1.1). We point out that the construction of some appropriate sub-solutions is highly nontrivial in both cases.

(4) The condition \( b > 2\chi\mu \) in (H1) and (H1)' or \( b \geq \frac{3}{2}\chi\mu \) in (H2) indicates that the logistic damping is large relative to the chemotaxis sensitivity. The reader is referred to [18] for the pioneering study on the global existence of classical solutions of the chemotaxis models on bounded domains with relatively large logistic damping.
Observe that the conditions in Theorem 1.1 (resp. in Theorem 1.2) are sufficient conditions for the existence of forced wave solutions. To see whether (1.1) still has forced wave solutions when these sufficient conditions are not satisfied, some numerical simulations are carried out and are described in the following (see section 5 for detail).

In the case that \( r(\cdot) \) is as in Case 1, consider
\[
\begin{align*}
  u_t &= u_{xx} + cu_x - (\chi uv_x)_x + u(r(x) - bu), \quad -L < x < L \\
  0 &= v_{xx} - \nu v + \mu u, \quad -L < x < L \\
  u(t, -L) &= v(t, -L) = 0 \\
  \frac{\partial u}{\partial x}(t, L) &= \frac{\partial v}{\partial x}(t, L) = 0.
\end{align*}
\]

Note that, if (1.9) has a positive stationary solution \((u_L(x), v_L(x))\) for all \( L \gg 1 \), then there is \( L_k \to \infty \) such that \( \lim_{k \to \infty} (u_{L_k}(x), v_{L_k}(x)) \) exists locally uniformly in \( x \in \mathbb{R} \) and \((u^*(x), v^*(x)) = \lim_{k \to \infty} (u_{L_k}(x), v_{L_k}(x))\) is a nonnegative stationary solution of (1.8). If \((u^*(x), v^*(x))\) is positive, we then have a forced wave solution of (1.1). We will numerically analyze the existence of positive stationary solutions of (1.8) by looking at the long time behavior of numerical solutions of (1.9) with a given nonnegative initial function for sufficiently large \( L \).

We observe from the numerical experiments 1-4 in section 5.1 that the assumptions in Theorem 1.1 can be weakened. We conjecture that if \( b > \chi \mu \) and \( c > -2\sqrt{r^*} \), there is a forced wave solution \((u(t, x), v(t, x)) = (\phi(x - ct), \psi(x - ct))\) of (1.1) connecting \((\frac{r^*}{b}, \frac{\mu r^*}{b})\) and \((0, 0)\). If \( b > \chi \mu \) and \( c < -2\sqrt{r^*} \), there is no forced wave solution \((u(t, x), v(t, x)) = (\phi(x - ct), \psi(x - ct))\) connecting \((\frac{r^*}{b}, \frac{\mu r^*}{b})\) and \((0, 0)\).

In the case that \( r(\cdot) \) is as in Case 2, consider the following cut-off problem of (1.8),
\[
\begin{align*}
  u_t &= u_{xx} + cu_x - (\chi uv_x)_x + u(r(x) - bu), \quad -L < x < L \\
  0 &= v_{xx} - \nu v + \mu u, \quad -L < x < L \\
  u(t, -L) &= v(t, -L) = 0 \\
  u(t, L) &= v(t, L) = 0.
\end{align*}
\]

We will numerically analyze the existence of positive stationary solutions of (1.8) by looking at the long time behavior of numerical solutions of (1.10) with a given nonnegative initial function for sufficiently large \( L \). We also observe from the numerical experiments 1-3 in section 5.2 that \( \chi \) is not necessarily small and the assumption \( b \geq \frac{3}{2} \chi \mu \) can also be weakened for the existence of forced waves. We conjecture that if \( b > \chi \mu \) and \( \lambda_\infty(r(\cdot)) > 0 \), there is a forced wave solution \((u(t, x), v(t, x)) = (\phi(x - ct), \psi(x - ct))\) of (1.1) connecting \((0, 0)\) and \((0, 0)\). If \( b > \chi \mu \) and \( |c| > c^* \), there is no forced wave solution \((u(t, x), v(t, x)) = (\phi(x - ct), \psi(x - ct))\) connecting \((0, 0)\) and \((0, 0)\).

We plan to provide some further study on the scenarios observed numerically somewhere else.

The rest of the paper is organized as follows. In section 2, we present some preliminary lemmas to be used in the proofs of the main results. In section 3, we study the existence of forced wave solutions of (1.1) with \( r(x) \) being as in Case 1 and prove Theorem 1.1. In section 4, we study the existence of forced wave solutions of (1.1) with \( r(x) \) being as in Case 2 and prove Theorem 1.2. In section 5, we present some numerical simulations.
2 Preliminary lemmas

In this section, we present some preliminary lemmas to be used in the proofs of the main theorems in later sections.

Note that, by the second equation in (1.1), \( v_{xx} = \nu v - \mu u \). Hence the first equation in (1.1) can be written as

\[
 u_t = u_{xx} - \chi v_x u_x + u(r(x - ct) - \chi \nu v - (b - \chi \mu) u), \quad x \in \mathbb{R}.
\]

By the comparison principle for parabolic equations, if \( b > \chi \mu \), then for any \( u_0 \in C_{\text{unif}}^b(\mathbb{R}) \) with \( u_0 \geq 0 \),

\[
 0 \leq u(t,x;u_0) \leq \max \{ \| u_0 \|_{\infty}, \frac{r^*}{b - \chi \mu} \} \quad \forall \ t \geq 0, \ x \in \mathbb{R}.
\]

Consider

\[
 \begin{cases}
  u_t = u_{xx} - \chi (uv_x)_x + u(r^* - bu), & x \in \mathbb{R} \\
  0 = v_{xx} - \nu v + \mu u, & x \in \mathbb{R}.
\end{cases}
\] (2.1)

Lemma 2.1. Assume that \( b > 2 \chi \mu \). Then for any \( u_0 \in C_{\text{unif}}^b(\mathbb{R}) \) with \( \inf_{x \in \mathbb{R}} u_0(x) > 0 \),

\[
 \lim_{t \to \infty} \| u(t, \cdot; u_0) - \frac{r^*}{b} \|_{\infty} = 0,
\]

where \( (u(t,x;u_0), v(t,x;u_0)) \) is the solution of (2.1) with \( u(0,x;u_0) = u_0(x) \).

Proof. It follows from [38, Theorem 1.8]. \( \square \)

For every \( u \in C_{\text{unif}}^b(\mathbb{R}) \), let

\[
 \Psi(x;u) = \mu \int_0^\infty \int_{\mathbb{R}} e^{-\nu s} e^{-\frac{|y-x|^2}{4s}} u(y) dy ds.
\] (2.2)

It is well known that \( \Psi(x;u) \in C_{\text{unif}}^2(\mathbb{R}) \) and solves the elliptic equation

\[
 \frac{d^2}{dx^2} \Psi(x;u) - \nu \Psi(x;u) + \mu u = 0.
\]

Lemma 2.2.

\[
 \Psi(x;u) = \frac{\mu}{2\sqrt{\nu}} \int_\mathbb{R} e^{-\sqrt{\nu} |x-y|} u(y) dy
\] (2.3)

and

\[
 \frac{d}{dx} \Psi(x;u) = -\frac{\mu}{2} e^{-\sqrt{\nu} x} \int_\mathbb{R} e^{\sqrt{\nu} y} u(y) dy + \frac{\mu}{2} e^{\sqrt{\nu} x} \int_\mathbb{R} e^{-\sqrt{\nu} y} u(y) dy.
\] (2.4)

Proof. The lemma is proved in [39, Lemma 2.1]. \( \square \)

Lemma 2.3. Suppose that \( b > \chi \mu \). For every \( u \in C_{\text{unif}}^b(\mathbb{R}) \), \( 0 \leq u(x) \leq \frac{r^*}{b - \chi \mu} \), it holds that

\[
 \Psi(x;u) \leq \frac{\mu r^*}{\nu (b - \chi \mu)} \quad \text{and} \quad \frac{d}{dx} \Psi(x;u) \leq \frac{\mu r^*}{2\sqrt{\nu (b - \chi \mu)}}.
\]

Proof. It follows from a direct calculation. \( \square \)
3 Forced wave solutions in Case 1

In this section, we study the existence of forced wave solutions of (1.1) with \( r(x) \) being as in Case 1 and prove Theorem 1.1.3. We first present a lemma. Throughout this section, we assume that \( r(x) \) is as in Case 1.

Suppose that \( b > \chi\mu \). Fix \( r_1 \) with \( r(-\infty) < r_1 < 0 \). Let \( x_1 \) be given satisfying that \( r(x) \leq r_1 \) for any \( x \leq x_1 \). Let \( \theta_1 \) be the positive root of the equation \( \theta^2 + c\theta + r_1 = 0 \). Define

\[
U_1^+(x) = \min\left\{ \frac{r^*}{b - \chi\mu}, \frac{r^*}{b - \chi\mu} e^{\theta_1(x-x_1)} \right\},
\]
and consider the set

\[
\mathcal{E}_1^+ = \{ u \in C^{b}_{\text{unif}}(\mathbb{R}) : 0 \leq u(x) \leq U_1^+(x), \forall x \in \mathbb{R} \}.
\]

For every \( u \in \mathcal{E}_1^+ \), consider the operator

\[
A_u(U)(x) = U_{xx}(x) + (c - \chi \Psi_x(x; u)) U_x(x) + (r(x) - \chi \nu \Psi(x; u) - (b - \chi \mu) U(x)) U(x),
\]

where \( \Psi(x; u) \) is given by (2.2).

**Lemma 3.1.** Suppose that \( b \geq \frac{3}{2} \chi\mu \). For every \( u \in \mathcal{E}_1^+ \), it holds that \( A_u(\frac{r^*}{b - \chi\mu})(x) \leq 0 \) for \( x \in \mathbb{R} \) and \( A_u(\frac{r^*}{b - \chi\mu} e^{\theta_1(x-x_1)})(x) \leq 0 \) for \( x \in (-\infty, x_1) \).

**Proof.** Let \( u \in \mathcal{E}_1^+ \) be given. First, we have

\[
A_u(\frac{r^*}{b - \chi\mu})(x) = \frac{r^*}{b - \chi\mu} (r(x) - \chi \nu \Psi(x; u) - (b - \chi \mu) \frac{r^*}{b - \chi\mu})
= \frac{r^*}{b - \chi\mu} (r(x) - \chi \nu \Psi(x; u) - r^*)
\leq 0, \forall x \in \mathbb{R}.
\]

Next, for \( x \in (-\infty, x_1) \), we have \( r(x) \leq r_1 \), and hence

\[
A_u(\frac{r^*}{b - \chi\mu} e^{\theta_1(x-x_1)})(x)
= \frac{r^*}{b - \chi\mu} e^{\theta_1(x-x_1)} + (c - \chi \Psi_x(x; u)) \theta_1 \frac{r^*}{b - \chi\mu} e^{\theta_1(x-x_1)}
+ \frac{r^*}{b - \chi\mu} e^{\theta_1(x-x_1)} (r(x) - \chi \nu \Psi(x; u) - (b - \chi \mu) \frac{r^*}{b - \chi\mu} e^{\theta_1(x-x_1)})
= \frac{r^*}{b - \chi\mu} e^{\theta_1(x-x_1)} \left( \theta_1^2 + c \theta_1 + r(x) - \chi \theta_1 \Psi_x(x; u) - \chi \nu \Psi(x; u) - r^* e^{\theta_1(x-x_1)} \right)
\leq \frac{r^*}{b - \chi\mu} e^{\theta_1(x-x_1)} \left( \theta_1^2 + c \theta_1 + r_1 - \chi \theta_1 \Psi_x(x; u) - \chi \nu \Psi(x; u) - r^* e^{\theta_1(x-x_1)} \right)
= \frac{r^*}{b - \chi\mu} e^{\theta_1(x-x_1)} \left( -\chi \theta_1 \Psi_x(x; u) - \chi \nu \Psi(x; u) - r^* e^{\theta_1(x-x_1)} \right).
\]

It then follows from Lemma 2.2 and (3.5) that

\[
A_u(\frac{r^*}{b - \chi\mu} e^{\theta_1(x-x_1)})(x) \leq \frac{r^*}{b - \chi\mu} e^{\theta_1(x-x_1)} \left( \frac{\chi \mu}{2} (\theta_1 - \sqrt{c}) e^{-\sqrt{c} x} \int_{-\infty}^{x} e^{\sqrt{c} y} u(y) dy - r^* e^{\theta_1(x-x_1)} \right).
\]
If $\theta_1 \leq \sqrt{\nu}$, we then have
\[
A_u\left(\frac{r^*}{b-\chi \mu} e^{\theta_1(-x_1)}\right)(x) \leq 0 \quad \forall x \in (-\infty, x_1).
\]

If $\theta_1 > \sqrt{\nu}$, we then have
\[
A_u\left(\frac{r^*}{b-\chi \mu} e^{\theta_1(-x_1)}\right)(x)
\leq \frac{r^*}{b-\chi \mu} e^{\theta_1(x-x_1)} \left(\frac{\chi \mu r^*}{2(b-\chi \mu)} (\theta_1 - \sqrt{\nu}) e^{-\sqrt{\nu}x} \int_{-\infty}^{x} e^{\sqrt{\nu}y} e^{\theta_1(y-x_1)} dy - r^* e^{\theta_1(x-x_1)}\right)
\leq 0 \quad \forall x < x_1.
\]

The lemma thus follows.

3.1 Proof of Theorem 1.1(1)

In this subsection, we prove Theorem 1.1(1). First, we prove some lemmas.

Suppose $b > 2\chi \mu$. For any $0 < \varepsilon \ll 1$, define an ignition nonlinearity by
\[
f_\varepsilon(u) = \begin{cases} u \left( r^* - \varepsilon - \frac{\chi \mu r^*}{b-\chi \mu} - (b-\chi \mu)u \right), & \text{if } u \geq 0, \\
0, & \text{if } -\varepsilon \leq u < 0. \end{cases} \tag{3.6}
\]

Consider the equation
\[
u u_t = u_{xx} + f_\varepsilon(u), \quad x \in \mathbb{R}. \tag{3.7}
\]

Equation (3.7) has a decreasing traveling wave solution $\phi_\varepsilon(x - \tilde{c}_\varepsilon t)$ connecting $(r^*-\varepsilon)(b-\chi \mu) - \chi \mu r^*/(b-\chi \mu)^2$ and $-\varepsilon$ with speed $0 < \tilde{c}_\varepsilon < 2\sqrt{\frac{r^*(b-2\chi \mu)}{b-\chi \mu}}$ and $\lim_{\varepsilon \to 0^+} \tilde{c}_\varepsilon = 2\sqrt{\frac{r^*(b-2\chi \mu)}{b-\chi \mu}}$ (see [8]), that is, $(\phi_\varepsilon, \tilde{c}_\varepsilon)$ satisfies
\[
\begin{align*}
-\tilde{c}_\varepsilon \phi_\varepsilon' &= \phi_\varepsilon'' + f_\varepsilon(\phi_\varepsilon), \\
\phi_\varepsilon(-\infty) &= \frac{(r^*-\varepsilon)(b-\chi \mu) - \chi \mu r^*/(b-\chi \mu)^2}{(b-\chi \mu)^2}, \quad \phi_\varepsilon(\infty) = -\varepsilon, \quad \phi_\varepsilon' < 0.
\end{align*} \tag{3.8}
\]

Let $\psi_\varepsilon(x) = \phi_\varepsilon(-x)$ for any $x \in \mathbb{R}$. It then follows from (3.8) that
\[
\begin{align*}
\tilde{c}_\varepsilon \psi_\varepsilon' &= \psi_\varepsilon'' + f_\varepsilon(\psi_\varepsilon), \\
\psi_\varepsilon(\infty) &= \frac{(r^*-\varepsilon)(b-\chi \mu) - \chi \mu r^*/(b-\chi \mu)^2}{(b-\chi \mu)^2}, \quad \psi_\varepsilon(-\infty) = -\varepsilon, \quad \psi_\varepsilon' > 0.
\end{align*} \tag{3.9}
\]

Without loss of generality, we can assume that $\psi_\varepsilon(x_0) = 0$, $r(x) \geq r^* - \varepsilon$ if $x > x_0$, and $x_0 > x_1$. This can be realized by some appropriate translation of $\psi_\varepsilon(x)$ if necessary.

**Lemma 3.2.** Suppose that (H1) holds. For every $u \in E_1^+$ and $0 < \varepsilon \ll 1$, $U_1^-(x) = \max\{\psi_\varepsilon(x), 0\}$ satisfies that $A_u(U_1^-\cdot)(x) \geq 0$ for any $x \neq x_0$. Moreover, $U_1^-(x) < U_1^+(x)$ for all $x \in \mathbb{R}$.

**Proof.** Let $\delta = c - \frac{\chi \mu r^*}{2\sqrt{\nu(b-\chi \mu)}} + 2\sqrt{\frac{r^*(b-2\chi \mu)}{b-\chi \mu}}$. By (H1), $\delta > 0$. Since $\lim_{x \to 0^+} \tilde{c}_\varepsilon = 2\sqrt{\frac{r^*(b-2\chi \mu)}{b-\chi \mu}}$, it then follows that for $0 < \varepsilon \ll 1$, we have $\tilde{c}_\varepsilon > 2\sqrt{\frac{r^*(b-2\chi \mu)}{b-\chi \mu}} - \frac{\delta}{2}$.
Fix such $\varepsilon$. For every $u \in \mathcal{E}_1^+$. If $x > x_0$, $U_1^-(x) = \psi_\varepsilon(x) > 0$ and $U_{1x}^-(x) > 0$. By Lemma 2.3 and (3.9), we have

$$
\mathcal{A}_u(\psi_\varepsilon(\cdot))(x) = \psi_\varepsilon'' + (c - \chi \Psi_x(x; u))\psi_\varepsilon' + \psi_\varepsilon(r(x) - \chi \nu \Psi(x; u) - (b - \chi \mu)\psi_\varepsilon)
\geq \psi_\varepsilon'' - \tilde{c}_\varepsilon\psi_\varepsilon' + (c - \chi \Psi_x(x; u) + \tilde{c}_\varepsilon)\psi_\varepsilon' + \psi_\varepsilon(r^* - \varepsilon - \chi \nu \Psi(x; u) - (b - \chi \mu)\psi_\varepsilon)
\geq \psi_\varepsilon'' - \tilde{c}_\varepsilon\psi_\varepsilon' + (c - \chi \frac{\mu r^*}{2\sqrt{\nu(b - \chi \mu)}} + 2\sqrt{\frac{r^*(b - 2\chi \mu)}{b - \chi \mu} - \frac{\delta}{2}}\psi_\varepsilon'
+ \psi_\varepsilon(r^* - \varepsilon - \chi \mu \frac{r^*}{b - \chi \mu} - (b - \chi \mu)\psi_\varepsilon)
= (c - \chi \frac{\mu r^*}{2\sqrt{\nu(b - \chi \mu)}} + 2\sqrt{\frac{r^*(b - 2\chi \mu)}{b - \chi \mu} - \frac{\delta}{2}}\psi_\varepsilon'
\geq 0.
$$

(3.10)

If $x < x_0$, $U_1^-(x) = 0$. Then $\mathcal{A}_u(U_1^-)(x) = 0$.

Since $x_1 < x_0$, it is clear that $U_1^-(x) < U_1^+(x)$ for all $x \in \mathbb{R}$. The lemma is thus proved. $\square$

Let

$$\mathcal{E}_1 = \{u \in C^0_{\text{unif}}(\mathbb{R}) : U_1^-(x) \leq u(x) \leq U_1^+(x), \forall x \in \mathbb{R}\}.$$

For any $u \in \mathcal{E}_1$, let $U(t, x; u)$ be the solution of the following parabolic equation

$$
\begin{cases}
U_t = \mathcal{A}_u(U), & t > 0, \ x \in \mathbb{R} \\
U(0, x; u) = U_1^+(x).
\end{cases}
$$

(3.11)

**Lemma 3.3.** Suppose that (H1) holds. For any $u \in \mathcal{E}_1$, $U_1^+(x; u) = \lim_{t \to \infty} U(t, x; u)$ exists and satisfies the elliptic equation

$$
0 = U_{xx} + (c - \chi \Psi_x(x; u))U_x + (r(x) - \chi \nu \Psi(x; u) - (b - \chi \mu)U) \forall x \in \mathbb{R}.
$$

(3.12)

Moreover, $U_1^+(\cdot; u) \in \mathcal{E}_1$.

**Proof.** First, thanks to Lemma 3.1, it follows from the comparison principle for parabolic equations that

$$U(t_2, x; u) \leq U(t_1, x; u) \leq U_1^+(x), \ \forall x \in \mathbb{R}, \ 0 < t_1 < t_2, \ u \in \mathcal{E}_1.$$

Thus the function

$$U_1^+(x; u) = \lim_{t \to \infty} U(t, x; u), \ \forall u \in \mathcal{E}_1
$$

(3.13)

is well defined. Moreover, by a priori estimates for parabolic equations, it is not difficult to see that $U_1^+(\cdot; u) \in \mathcal{E}_1^+$ and $U_1^+(x; u)$ satisfies (3.12).

Next, it follows from Lemma 3.2 and the comparison principle for parabolic equations that

$$U_1^-(x) \leq U(t, x; u), \ \forall x \in \mathbb{R}, \ t > 0, \ u \in \mathcal{E}_1.
$$

(3.14)

Hence,

$$U_1^-(x) \leq U_1^+(x; u), \ \forall x \in \mathbb{R}, \ \forall u \in \mathcal{E}_1.
$$

(3.15)

Therefore, $U_1^+(\cdot; u) \in \mathcal{E}_1$. The lemma is thus proved. $\square$
Lemma 3.4. Suppose that (H1) holds. For any \( u \in \mathcal{E}_1 \), suppose that \( U_{1*}(x;u) \) is also a solution of (3.12) in \( \mathcal{E}_1 \). Then

\[
\lim_{x \to \infty} \frac{U_{1*}(x;u)}{U_{1*}^1(x;u)} = 1. 
\] (3.16)

Proof. First of all, by Lemma 2.3 \( \sup_{x \in \mathbb{R}} |\Psi(x;u)| < \infty \) and \( \sup_{x \in \mathbb{R}} |\frac{\partial}{\partial x} \Psi(x;u)| < \infty \). By \( \Psi_{xx}(x;u) = \nu \Psi(x;u) - \mu u \), we have \( \sup_{x \in \mathbb{R}} |\Psi_{xx}(x;u)| < \infty \). This implies that for any \( \{x_n\}_{n=1}^\infty \subset \mathbb{R} \), there is \( \{x_{nk}\}_{k=1}^\infty \subset \{x_n\}_{n=1}^\infty \) such that \( \lim_{k \to \infty} \Psi(x + x_{nk};u) \) and \( \lim_{k \to \infty} \Psi_x(x + x_{nk};u) \) exist locally uniformly on \( \mathbb{R} \).

Next, note that \( \frac{U_{1*}(x;u)}{U_{1*}^1(x;u)} \leq 1 \) for all \( x \in \mathbb{R} \). It then suffices to prove that \( \liminf_{x \to \infty} \frac{U_{1*}(x;u)}{U_{1*}^1(x;u)} \geq 1 \).

Assume by contraction that

\[
\liminf_{x \to \infty} \frac{U_{1*}(x;u)}{U_{1*}^1(x;u)} < 1.
\]

Then there are \( 0 < \delta < 1 \) and \( x_n \to \infty \) such that

\[
\frac{U_{1*}(x;u)}{U_{1*}^1(x;u)} \leq 1 - \delta \quad \forall \ n = 1, 2, \ldots
\]

Let

\[
U_{n,1*}(x;u) = U_{1*}(x + x_n;u), \quad \Psi_n(x;u) = \Psi(x + x_n;u).
\]

Without loss of generality, we may assume that there are \( \underline{U}_*(x;u) \), \( \bar{U}_*^*(x;u) \), and \( \Psi_*(x;u) \) such that

\[
\lim_{n \to \infty} U_{n,1*}(x;u) = \underline{U}_*(x;u), \quad \lim_{n \to \infty} U_{n,1*}^*(x;u) = \bar{U}_*^*(x;u), \quad \text{and} \quad \lim_{n \to \infty} \Psi_n(x;u) = \Psi_*(x;u)
\]

locally uniformly on \( \mathbb{R} \). This implies that both \( \underline{U}_*(x;u) \) and \( \bar{U}_*^*(x;u) \) are solutions of

\[
0 = U_{xx} + (c - \chi \Psi_*(x;u))U_x + (r - \nu \Psi_*(x;u) - (b - \chi \mu)U)U.
\] (3.17)

We now claim that \( \underline{U}_*(x;u) \equiv \underline{U}_*(x;u) \). Indeed, note that

\[
0 < \inf_{x \in \mathbb{R}} \underline{U}_*(x;u) \leq \inf_{x \in \mathbb{R}} \bar{U}_*(x;u), \quad \text{and} \quad \sup_{x \in \mathbb{R}} \underline{U}_*(x;u) \leq \sup_{x \in \mathbb{R}} \bar{U}_*(x;u) < \infty.
\]

This implies that the following set is not empty,

\[
\{ \gamma \geq 1 \mid \frac{\gamma}{\gamma} \bar{U}_*(x;u) \leq \underline{U}_*(x;u) \leq \gamma \bar{U}_*(x;u) \quad \forall \ x \in \mathbb{R} \}.
\]

Hence we can define

\[
\rho(\underline{U}_*, \bar{U}_*) = \inf \{ \ln \gamma \mid \frac{\gamma}{\gamma} \bar{U}_*(x;u) \leq \underline{U}_*(x;u) \leq \gamma \bar{U}_*(x;u) \quad \forall \ x \in \mathbb{R} \}.
\]

Note that \( \rho(\underline{U}_*, \bar{U}_*) \) is the so called part metric between \( \underline{U}_* \) and \( \bar{U}_* \). Assume that \( \gamma > 1 \). Then by the arguments of [22] Proposition 3.4], there is \( \delta_0 > 0 \) such that

\[
\rho(\underline{U}_*, \bar{U}_*) \leq \rho(\underline{U}_*, \bar{U}_*) - \delta_0,
\]

which is a contradiction. Hence \( \gamma = 1 \), and then \( \underline{U}_*(x;u) = \bar{U}_*(x;u) \) for all \( x \in \mathbb{R} \). But, by the assumption,

\[
\underline{U}_*(0;u) \neq \bar{U}_*(0;u),
\]

which is a contradiction. The lemma thus follows. \( \square \)
Lemma 3.5. Suppose that (H1) holds. For any \( u \in E_1 \), \( U_1^*(x;u) \) is the unique positive solution of (3.12) in \( E_1 \).

Proof. Suppose that \( U_1^*(x;u) \) is any positive solution of (3.12) in \( E_1 \). It suffices to prove that \( U_1^*(x;u) = U_1^*(x;u) \).

For any \( \epsilon > 0 \), let

\[
K_\epsilon = \{ k \geq 1 \mid kU_1^*(x;u) \geq U_1^*(x;u) - \epsilon \quad \forall x \in \mathbb{R} \}.
\]

By Lemma 3.4 and the fact \( \lim_{x \to -\infty} U_1^*(x;u) = \lim_{x \to -\infty} U_1^*(x;u) = 0 \), \( K_\epsilon \neq \emptyset \). Let

\[
k_\epsilon = \inf K_\epsilon.
\]

Then \( k_\epsilon \geq 1 \) and

\[
k_\epsilon U_1^*(x;u) \geq U_1^*(x;u) - \epsilon \quad \forall x \in \mathbb{R}.
\]

(3.18)

For any \( 0 < \epsilon_1 < \epsilon_2 \), since

\[
k_{\epsilon_1} U_1^*(x;u) \geq U_1^*(x;u) - \epsilon_1 > U_1^*(x;u) - \epsilon_2 \quad \forall x \in \mathbb{R},
\]

it then follows that \( k_{\epsilon_1} \geq k_{\epsilon_2} \). Thus, \( k_\epsilon \) is nonincreasing in \( \epsilon > 0 \). If \( k_\epsilon = 1 \) for any \( \epsilon > 0 \), clearly, we have that \( U_1^*(x;u) \equiv U_1^*(x;u) \).

Assume that there exists \( \epsilon_0 > 0 \) such that \( k_{\epsilon_0} > 1 \). Then

\[
k_\epsilon \geq k_{\epsilon_0} > 1 \quad \text{for any } 0 < \epsilon \leq \epsilon_0.
\]

(3.19)

For any given \( 0 < \epsilon \leq \epsilon_0 \), since \( k_\epsilon > 1 \), there exists \( \delta > 0 \), such that \( k_\epsilon - \delta > 1 \). By

\[
\lim_{x \to -\infty} \frac{U_1^*(x;u)}{U_1^*(x;u)} = 1,
\]

we have that for such given \( \epsilon > 0 \),

\[
\frac{U_1^*(x;u)}{U_1^*(x;u)} \geq 1 - \frac{\epsilon}{U_1^*(x;u)} \quad x \gg 1.
\]

Hence,

\[
(k_\epsilon - \delta)U_1^*(x;u) \geq U_1^*(x;u) - \epsilon \quad x \gg 1.
\]

(3.20)

Since \( \lim_{x \to -\infty} \frac{U_1^*(x;u)-\epsilon}{U_1^*(x;u)} = -\infty \), it then clear that

\[
(k_\epsilon - \delta)U_1^*(x;u) \geq U_1^*(x;u) - \epsilon \quad x \ll -1.
\]

(3.21)

It then follows from (3.20), (3.21) and the definition of \( k_\epsilon \) that there is \( x_\epsilon \in \mathbb{R} \) such that

\[
k_\epsilon U_1^*(x_\epsilon;u) = U_1^*(x_\epsilon;u) - \epsilon.
\]

(3.22)

By (3.16), \( x_\epsilon \) is bounded above.

We claim that \( x_\epsilon \) is bounded from below. In fact, at \( x_\epsilon \), we have

\[
\partial_{xx}(k_\epsilon U_1^*(x_\epsilon;u) - U_1^*(x_\epsilon;u)) \geq 0, \quad \partial_x(k_\epsilon U_1^*(x_\epsilon;u) - U_1^*(x_\epsilon;u)) = 0,
\]

(3.10)
and hence
\[ 0 \geq k_n U_{1*}(x_n)(r(x_n) - \chi_n \Psi(x_n; u) - (b - \chi(x_n))U_{1*}(x_n)) - U_{1*}^+(x_n)(r(x_n) - \chi_n \Psi(x_n; u) - (b - \chi(x_n))U_{1*}^+(x_n)) \]
\[ \geq k_n U_{1*}(x_n)(r(x_n) - \chi_n \Psi(x_n; u) - (b - \chi(x_n))U_{1*}^+(x_n)) - U_{1*}^+(x_n)(r(x_n) - \chi_n \Psi(x_n; u) - (b - \chi(x_n))U_{1*}^+(x_n)) \]
\[ = -\epsilon(r(x_n) - \chi_n \Psi(x_n; u) - (b - \chi(x_n))U_{1*}^+(x_n)). \]

This implies that \( r(x_n) \geq 0 \) and hence \( x_n \) is bounded from below.

Therefore, \( x_n \) is bounded both from below and above. By (3.22),
\[ k_n = \frac{U_{1*}'(x_n; u) - \epsilon}{U_{1*}'(x_n; u)}. \]

Hence \( k_n \) is bounded, and there is \( \epsilon_n \to 0 \) such that \( x_{n\epsilon_n} \to x^* \) and \( k_{n\epsilon_n} \to k^*(\geq k_{\epsilon_0} > 1) \) as \( n \to \infty \). This together with (3.22) implies that \( k^* U_{1*}(x^*; u) = U_{1*}'(x^*; u) \). By (3.18),
\[ k^* U_{1*}(x; u) \geq U_{1*}'(x; u) \quad \forall x \in \mathbb{R}. \]

Since \( k^* > 1 \), by (3.16), \( k^* U_{1*}(x; u) \neq U_{1*}'(x; u) \). Then by the comparison principle for parabolic equations, we must have
\[ U_{1*}'(x; u) < k^* U_{1*}(x; u) \quad \forall x \in \mathbb{R}, \]
which is a contraction. Therefore, \( k_n = 1 \) for \( 0 < \epsilon \ll 1 \) and then \( U_{1*}(x; u) \equiv U_{1*}'(x; u) \).

We now prove Theorem 1.1(1).

**Proof of Theorem 1.1(1).** Consider the mapping \( U_{1*}'(\cdot; u) : \mathcal{E}_1 \ni u \mapsto U_{1*}'(x; u) \in \mathcal{E}_1 \) as defined by (3.13). It follows from the arguments of the proof of Theorem 3.1 and the Lemma 3.2 that this function is continuous and compact in the compact open topology. Hence it has a fixed point \( u^* \) by the Schauder’s fixed point Theorem. Taking \( v^*(x) = \Psi(x; u^*) \), we have from (3.12), that \( (u(t, x), v(t, x)) = (u^*(x - ct), v^*(x - ct)) \) is an entire solution of (1). Moreover, since \( U_{1*}'(x) \leq u^*(x) \leq U_{1*}'(x) \), it follows that \( \lim_{x \to -\infty} u^*(x) = 0. \)

In the following we show that
\[ \lim_{x \to -\infty} u^*(x) = \frac{r^*}{b}. \]
(3.23)

Suppose on the contrary that this is false. Then, there is a constant \( \delta > 0 \) and a sequence \( \{x_n\}_{n \in \mathbb{N}} \) such that \( x_n \to \infty \) and
\[ |u^*(x_n) - \frac{r^*}{b}| \geq \delta, \quad \forall n \geq 1. \]
(3.24)

Consider the sequence of functions
\[ u^n(t, x) = u(t, x + x_n) \quad \text{and} \quad v^n(t, x) = v(t, x + x_n). \]

By a priori estimate for parabolic equations, without loss of generality, we may assume that there is \( (u^{**}(t, x), v^{**}(t, x)) \in C^{1,2}(\mathbb{R} \times \mathbb{R}) \) such that \( (u^n, v^n)(t, x) \to (u^{**}(t, x), v^{**}(t, x)) \) locally uniformly in \( C^{1,2}(\mathbb{R} \times \mathbb{R}) \) as \( n \to \infty \). Furthermore, the function \( (u^{**}(t, x), v^{**}(t, x)) \) is an entire solution of the following equation
\[
\begin{align*}
\begin{cases}
\frac{\partial u}{\partial t} &= \chi(uv_x)_x + u(r^* - bu), \quad x \in \mathbb{R} \\
0 &= v_{xx} - \nu v + \mu u, \quad x \in \mathbb{R},
\end{cases}
\end{align*}
(3.25)
\]
Note that
\[ u^{**}(t, x) = \lim_{n \to \infty} u^n(t, x) \geq \lim_{n \to \infty} U_1^-(x + x_n - ct) = U_1^-(\infty) > 0, \quad \forall \ x \in \mathbb{R}, \ t \in \mathbb{R}. \]
So \( \inf_{(t, x) \in \mathbb{R} \times \mathbb{R}} u^{**}(t, x) > 0. \)

Therefore, since \( \chi \mu < \frac{b}{2} \), it follows from Lemma 2.1 that \( u^{**}(t, x) = \frac{r^2}{b} \) for every \( (t, x) \in \mathbb{R} \times \mathbb{R}. \)

In particular, \( \frac{r^2}{b} = u^{**}(0, 0) = \lim_{n \to \infty} u^n(0, 0) = \lim_{n \to \infty} u(0, x_n) = \lim_{n \to \infty} u^*(x_n) \), which contradicts to (3.24). Therefore, (3.23) must hold.

\[ \square \]

### 3.2 Proof of Theorem 1.1(2)

In this subsection, we prove Theorem 1.1(2). First, we prove some lemmas.

Throughout this subsection, we assume that \((\mathbf{H}_1)'\) holds, that is, \( b > 2\chi \mu \) and \( c > -2\sqrt{r^*} \).

For any given \( 0 < \epsilon < 2\sqrt{r^*} \), fix \( \bar{r} < r^* \) such that
\[ 4\bar{r} - \bar{c}^2 \geq \epsilon \sqrt{r^*} \quad \forall \quad -2\sqrt{r^*} + \epsilon \leq \bar{c} \leq 2\sqrt{r^*} - \epsilon. \] (3.26)

Let
\[ l = \frac{2\pi}{(\epsilon \sqrt{r^*})^2} \] (3.27) and
\[ \lambda(\bar{c}, \bar{r}) = \frac{4\bar{r} - \bar{c}^2 - \frac{\pi^2}{4}}{4}. \] (3.28)

Then \( \lambda(\bar{c}, \bar{r}) \geq \frac{3\sqrt{r^*}}{16} > 0 \) for any \( -2\sqrt{r^*} + \epsilon \leq \bar{c} \leq 2\sqrt{r^*} - \epsilon. \)

**Lemma 3.6.** For given \( 0 < \epsilon < 2\sqrt{r^*} \), let \( \bar{r} \) and \( l \) be as in (3.26) and (3.27). Then for any \( -2\sqrt{r^*} + \epsilon \leq \bar{c} \leq 2\sqrt{r^*} - \epsilon \), \( \lambda(\bar{c}, \bar{r}) \) which is defined as in (3.28) is the principal eigenvalue of

\[
\begin{align*}
\phi_{xx} + \bar{c}\phi_x + \bar{r}\phi &= \lambda \phi, & -l < x < l \\
\phi(-l) = \phi(l) &= 0,
\end{align*}
\] (3.29)

and \( \phi(x; \bar{c}, \bar{r}) = e^{\frac{\bar{r}}{2}\cos \frac{\pi}{2l} x} \) is a corresponding positive eigenfunction.

**Proof.** This is [42] Lemma 3.1. \( \square \)

Consider
\[
\begin{align*}
\begin{cases}
u_t &= u_{xx} + \bar{c}u_x + u(r^* - \xi - B(t, x) - (\bar{b} - \chi \mu)u), & -l < x < l \\
\nu(t, -l) &= \nu(t, l) = 0,
\end{cases}
\end{align*}
\] (3.30)

where \( 0 < \xi \ll 1 \) is such that \( r^* - \xi > \bar{r} \), and \( B(t, x) \) is globally Hölder continuous in \( t \in \mathbb{R} \) and \( x \in [-l, l] \) with Hölder exponent \( 0 < \alpha < 1 \) and \( \| B(\cdot, \cdot) \|_\infty < \infty \).

**Lemma 3.7.** For given \( 0 < \epsilon < 2\sqrt{r^*} \), let \( \bar{r} \) and \( l \) be as in (3.26) and (3.27). There is \( \eta_0 > 0 \) such that for any \( B(\cdot, \cdot) \) with \( \| B(\cdot, \cdot) \|_\infty < \eta_0 \), any \( \bar{c} \in [-2\sqrt{r^*} + \epsilon, 2\sqrt{r^*} - \epsilon] \), and any \( \bar{b} \in (\chi \mu, \infty) \), (3.30) has a unique positive bounded entire solution \( u(t, x; \bar{c}, \bar{b}, B) \) satisfying that
\[ \inf_{-l+\delta \leq x \leq -l-\delta, t \in \mathbb{R}, \bar{c} \in [-2\sqrt{r^*} + \epsilon, 2\sqrt{r^*} - \epsilon]} u(t, x; \bar{c}, \bar{b}, B) > 0 \quad \forall \ 0 < \delta < l. \] (3.31)
Proof. It can be proved by the similar arguments as those used in the proof of [12, Lemma 3.2].

By \( c > -2\sqrt{r^*} \), we can choose \( 0 < \epsilon \ll \min\{1, 2\sqrt{r^*}\} \) and \( 0 < \eta \leq \eta_0 \), such that
\[
-2\sqrt{r^*} + \epsilon < c - \eta.
\]

Fix such \( \eta \). In the following, we assume that
\[
0 < \chi < \min\left\{ \frac{b\eta}{\mu r^* + \mu \eta}, \frac{2\sqrt{r^*} \eta}{\mu r^* + 2\sqrt{r^*} \eta \mu} \right\} \left( \leq \frac{b\eta_0}{\mu r^* + \mu \eta_0} \right). \tag{3.32}
\]

Let \( U_1^+ \) and \( \mathcal{E}_1^+ \) be as defined in (3.1) and (3.2), respectively. For any \( u \in \mathcal{E}_1^+ \), let \( B(t, x; u) = \chi \nu \Psi(x; u) \) where \( \Psi(x; u) \) is given by (2.2). By Lemma 2.3 and (3.32), we have
\[
\|B(\cdot, ; u)\|_\infty \leq \frac{\chi \mu r^*}{b - \chi \mu} < \eta_0.
\]

It then follows from Lemma 3.7 that for any \( u \in \mathcal{E}_1^+ \) (3.31) with \( B(t, x) = \|B(\cdot, ; u)\|_\infty \) has a unique positive bounded entire solution \( u(t, x; \bar{c}, \bar{b}, u) := u(t, x; \bar{c}, \bar{b}, \|B(\cdot, ; u)\|_\infty) \) with
\[
\inf_{t \in \mathbb{R}, t + \delta \leq t \leq t - \delta, x \in [-2\sqrt{r^*} + \epsilon, 2\sqrt{r^*} - \epsilon]} u(t, x; \bar{c}, \bar{b}, u) > 0 \quad \forall \ 0 < \delta < l. \tag{3.33}
\]

By the proof of [12, Lemma 3.2], we have that
\[
\inf_{t \in \mathbb{R}, t + \delta \leq t \leq t - \delta, x \in [-2\sqrt{r^*} + \epsilon, 2\sqrt{r^*} - \epsilon], u \in \mathcal{E}_1^+} u(t, x; \bar{c}, \bar{b}, u) > 0 \quad \forall \ 0 < \delta < l. \tag{3.34}
\]

Choose \( L \gg 1 \) such that \( r(x) > r^* - \xi \) for any \( x \geq L \). Fix such \( L \). Let \( \bar{u}_1(\cdot) \in C^0_{\text{unif}}([-L, \infty)) \setminus \{0\} \) be such that
\[
\bar{u}_1(L) = 0, \quad \bar{u}_1'(x) > 0, \quad \text{and} \quad \bar{u}_1(x) < U^+(x), \quad \forall \ x \geq L.
\]

Choose \( -2\sqrt{r^*} + \epsilon \leq \bar{c} \leq \min\{2\sqrt{r^*} - \epsilon, c - \eta\} \). For any \( u \in \mathcal{E}_1^+ \), let \( \bar{u}(t, x; u) \) be the solution of
\[
\begin{cases} 
\bar{u} = \bar{u}_{xx} + \bar{c} \bar{u}_{x} + u(r^* - \xi - \|B(\cdot, ; u)\|_\infty - (b - \chi \mu) u), & t > 0, \ x > L \\
\bar{u}(t, L; u) = 0 \\
\bar{u}(0, x; u) = \bar{u}_1(x).
\end{cases} \tag{3.35}
\]

Choose \( \bar{b} > b \) and also \( \bar{b} \gg 1 \) such that
\[
u(0, x - L - l; \bar{c}, \bar{b}, u) < \bar{u}(0, x + kl; u) \quad \forall \ L \leq x \leq L + 2l, \ k \in \mathbb{N}, \ u \in \mathcal{E}_1^+.
\]

Fix such \( \bar{b} \). It follows from the comparison principle for parabolic equations that
\[
u(t, x - L - l; \bar{c}, \bar{b}, u) < \bar{u}(t, x + kl; u) \quad \forall \ t > 0, \ L < x < L + 2l, \ k \in \mathbb{N}, \ u \in \mathcal{E}_1^+.
\]

This together with (3.34) implies that
\[
\inf_{x \geq L + \delta, t \geq 0, u \in \mathcal{E}_1^+} \bar{u}(t, x; u) > 0, \quad \forall \ \delta > 0.
\]

Define
\[
\bar{U}_1^-(x) = \begin{cases} 
\inf_{t \geq 0, u \in \mathcal{E}_1^+} \bar{u}(t, x; u) & \text{if } x > L, \\
0 & \text{if } x \leq L.
\end{cases}
\]

Then, \( \bar{U}_1^-(x) \neq 0 \) for any \( x \in \mathbb{R} \) and \( \inf_{x \geq L + \delta} \bar{U}_1^-(x) > 0 \).
Lemma 3.8. Assume that (H1)' and (3.32) hold. Then

\[ \widetilde{U}_1^-(x) < U_1^+(x) \quad \forall x \in \mathbb{R}, \]

and

\[ \widetilde{U}_1^-(x) < U(t, x; u) \quad \forall x \in \mathbb{R}, t > 0, u \in \mathcal{E}_1^+, \] (3.36)

where \( U(t, x; u) \) is the solution of (3.11).

Proof. First, note that \( U_1^+(x) = \frac{r^*}{b - \chi \mu} \) for any \( x \geq L \). By the comparison principle for parabolic equations, we have that

\[ u(t, x; u) < \frac{r^*}{b - \chi \mu} \quad \forall t > 0, x \geq L, u \in \mathcal{E}_1^+. \]

Then, we have \( \widetilde{U}_1^-(x) < U_1^+(x) \) for any \( x \in \mathbb{R} \).

Next, note that \( u_1(x) > 0 \) for any \( x \in \mathbb{R} \). Hence for any \( u \in \mathcal{E}_1^+ \), \( u_1(t, x; u) > 0 \) for any \( t > 0, x > L \). By Lemma 2.3 and Lemma 3.3, we have

\[ \| \chi \Psi(x; u) \|_\infty \leq \frac{\chi \mu r^*}{2b - \chi} < \eta. \]

Thus \( u(t, x) := u(t, x; u) \) satisfies

\[ \begin{aligned}
   u_t &= u_{xx} + c u_x + u(r^* - \xi - \|B(\cdot, \cdot; u)\|_\infty - (b - \chi \mu)u) \\
   &\leq u_{xx} + (c - \eta) u_x + u(r^* - \xi - \chi \nu \Psi(x; u) - (b - \chi \mu)u) \\
   &\leq u_{xx} + (c - \chi \nu \Psi(x; u)) u_x + u(r^* - \xi - \chi \nu \Psi(x; u) - (b - \chi \mu)u) \quad \forall t > 0, x > L. \quad (3.37)
\end{aligned} \]

This together with the comparison principle for parabolic equations implies that

\[ u(t, x; u) < U(t, x; u), \quad \forall x > L, t > 0, u \in \mathcal{E}_1^+. \quad (3.38) \]

(3.36) thus follows. \hfill \square

By Lemma 3.1 and Lemma 3.3

\[ U(t, x; u) \leq U_1^+(x), \quad \forall x \in \mathbb{R}, t > 0, u \in \mathcal{E}_1^+, \] (3.39)

and

\[ \widetilde{U}_1^*(x; u) = \lim_{t \to \infty} U(t, x; u) \]

exists and satisfies the elliptic equation (3.12). By (3.36) and (3.39),

\[ \widetilde{U}_1^- (x) \leq \widetilde{U}_1^*(x; u) \leq U_1^+(x) \quad \forall x \in \mathbb{R}, u \in \mathcal{E}_1^+. \] (3.40)

Let

\[ \mathcal{E}_1 = \{ u \in C_{\text{unif}}^b (\mathbb{R}) : \widetilde{U}_1^- (x) \leq u(x) \leq U_1^+(x), \forall x \in \mathbb{R} \}. \]

By (3.40), for any \( u \in \mathcal{E}_1 \), \( \widetilde{U}_1^*(\cdot; u) \in \mathcal{E}_1 \).

Lemma 3.9. Suppose that (H1)' holds. For any \( u \in \mathcal{E}_1 \), suppose that \( \widetilde{U}_1^*(x; u) \) is a positive solution of (3.12) in \( \mathcal{E}_1 \), then \( \lim_{x \to \infty} \frac{\widetilde{U}_1^*(x; u)}{\widetilde{U}_1^*(x; u)} = 1. \)
Proof. It can be proved by the same arguments as those used in the proof of Lemma 3.3.

Lemma 3.10. Suppose that (H1)' holds. For any $u \in \mathcal{E}_1$, $\overline{U}_1(x; u)$ is the unique positive solution of (3.12) in $\mathcal{E}_1$.

Proof. It can be proved by the same arguments as those used in the proof of Lemma 3.5.

We now prove Theorem 1.1(2).

Proof of Theorem 1.1(2). It can be proved by the same arguments as those used in the proof of Theorem 1.1(1).

4 Forced wave solutions in Case 2

In this section, we study the existence of forced wave solutions of (1.1) with $r(x)$ being as in Case 2 and prove Theorem 1.2. We first present some lemmas. Throughout this section, we assume that $r(x)$ is as in Case 2 and (H2) holds.

Fix a $\bar{r}$ with $\max\{r(-\infty), r(\infty)\} < \bar{r} < 0$. Choose $\bar{x}$ such that the inequality $r(x) < \bar{r}$ holds for all $x < \bar{x}$. Let $\bar{\theta}$ be the positive solution of $\bar{\theta}^2 + c\bar{\theta} + \bar{r} = 0$. Choose $\bar{x}$ such that the inequality $r(x) < \bar{r}$ holds for all $x > \bar{x}$. Let $\bar{\theta}$ be the positive solution of $\bar{\theta}^2 - c\bar{\theta} + \bar{r} = 0$.

Define

$$U_2^+(x) = \begin{cases} \frac{r^*}{b - \chi\mu} e^{\bar{\theta}(x - \bar{x})} & \text{if } x < \bar{x}, \\ \frac{r^*}{b - \chi\mu} e^{-\bar{\theta}(x - \bar{x})} & \text{if } x > \bar{x}, \\ \frac{r^*}{b - \chi\mu} & \text{if } \bar{x} \leq x \leq \bar{x}, \end{cases}$$

and consider the set

$$\mathcal{E}_2^+ = \{u \in C_{\text{uni}}^b(\mathbb{R}) : 0 \leq u(x) \leq U_2^+(x), \forall x \in \mathbb{R} \}.$$ 

For every $u \in \mathcal{E}_2^+$, consider the operator

$$A_u(U)(x) = U_{xx}(x) + (c - \chi\Psi(x; u))U_x(x) + (r(x) - \chi\nu\Psi(x; u) - (b - \chi\mu)U(x))U(x),$$

where $\Psi(x; u)$ is given by (2.2).

Lemma 4.1. Suppose that $b \geq \frac{3\mu}{2}$. For every $u \in \mathcal{E}_2^+$, it holds that $A_u(\frac{r^*}{b - \chi\mu}e^{\bar{\theta}(-\bar{x})})(x) \leq 0$ for $x \in (-\infty, \bar{x})$, $A_u(\frac{r^*}{b - \chi\mu})(x) \leq 0$ for $x \in \mathbb{R}$ and $A_u(\frac{r^*}{b - \chi\mu}e^{-\bar{\theta}(-\bar{x})})(x) \leq 0$ for $x \in (\bar{x}, \infty)$.

Proof. It can be proved by the similar arguments as those used in the proof of Lemma 3.1.

Consider

$$\begin{cases} u_t = u_{xx} + cu_x - A(t, x)u_x + u(r(x) - B(t, x) - (b - \chi\mu)u), \quad -L < x < L \\ u(t, -L) = u(t, L) = 0, \end{cases} \quad (4.1)$$

where $A(t, x)$ and $B(t, x)$ are both globally Hölder continuous in $t \in \mathbb{R}$ and $x \in [-L, L]$ with Hölder exponent $0 < \alpha < 1$ and $\|A(\cdot, \cdot)\|_\infty < \infty$, $\|B(\cdot, \cdot)\|_\infty < \infty$. 

18
**Lemma 4.2.** Suppose that \((H2)\) holds. Then there are \(L^* > 0\) and \(\eta > 0\) such that for any \(L \geq L^*\), any \(A(\cdot, \cdot), B(\cdot, \cdot)\) with \(\|A(\cdot, \cdot)\|_{\infty} < \eta\), \(\|B(\cdot, \cdot)\|_{\infty} < \eta\), and any \(\bar{b} > \chi \mu\), \((4.1)\) has a unique positive bounded entire solution \(u^*(t, x; \bar{b}, A(\cdot, \cdot), B(\cdot, \cdot))\) with

\[
\inf_{t \in \mathbb{R}, -L + \delta \leq x \leq L - \delta, \|A(\cdot, \cdot)\|_{\infty} < \eta, \|B(\cdot, \cdot)\|_{\infty} < \eta} u^*(t, x; \bar{b}, A(\cdot, \cdot), B(\cdot, \cdot)) > 0 \quad \forall 0 < \delta < L. \tag{4.2}
\]

**Proof.** It can be proved by the similar arguments as those used in the proof of \([42, \text{Lemma 3.2}]\). \(\square\)

Let \(L = L^*\) and \(\eta\) be fixed. For every \(u \in \mathcal{E}^+_2\), let \(A(t, x) = \chi \Psi_x(x; u), B(t, x) = \chi \nu \Psi(x; u)\). By Lemma \([2, \text{Lemma 2.3}]\) \(\Psi_x(x; u)\) and \(\Psi_{xx}(x; u)\) are both bounded for any \(u \in \mathcal{E}^+_2\), we then have \(A(t, x)\) and \(B(t, x)\) are both globally Hölder continuous in \(x \in [-L, L]\).

In the following, we assume that

\[
0 < \chi < \min\{\frac{2\sqrt{n}b}{\mu r^* + 2\sqrt{n}b}, \frac{b\eta}{\mu r^* + \mu \eta}\}. \tag{4.3}
\]

Then by Lemma \([2, \text{Lemma 2.3}]\) we have

\[
\|A(\cdot, \cdot)\|_{\infty} \leq \frac{\chi \mu r^*}{2\sqrt{n}(b - \chi \mu)} < \eta \quad \text{and} \quad \|B(\cdot, \cdot)\|_{\infty} \leq \frac{\chi \mu r^*}{b - \chi \mu} < \eta.
\]

It then follows from Lemma \([4.2, (4.1)]\) with \(A(t, x) = \chi \Psi_x(x; u)\) and \(B(t, x) = \chi \nu \Psi(x; u)\) has a unique positive bounded entire solution \(u^*(t, x; \bar{b}, u) = u^*(t, x; \bar{b}, \chi \Psi_x(x; u), \chi \nu \Psi(x; u))\) with

\[
\inf_{t \in \mathbb{R}, -L + \delta \leq x \leq L - \delta} u^*(t, x; \bar{b}, u) > 0 \quad \forall 0 < \delta < L. \tag{4.4}
\]

Fix \(\bar{b} \gg b\) such that \(u^*(t, x; \bar{b}, u) < U^+_2(x)\) for any \(-L \leq x \leq L\), any \(t \in \mathbb{R}\), any \(u \in \mathcal{E}^+_2\). By the proof of \([42, \text{Lemma 3.2}]\), we have that

\[
\inf_{t \in \mathbb{R}, -L + \delta \leq x \leq L - \delta, u \in \mathcal{E}^+_2} u^*(t, x; \bar{b}, u) > 0 \quad \forall 0 < \delta < L. \tag{4.5}
\]

Define

\[
U^-_2(x) = \begin{cases} \inf_{t \in \mathbb{R}, u \in \mathcal{E}^+_2} u^*(t, x; \bar{b}, u) & \text{if } -L < x < L, \\ 0 & \text{if } x \geq L, x \leq -L \end{cases}
\]

Then, \(U^-_2(x) \neq 0\) and \(\inf_{-L + \delta \leq x \leq L - \delta} U^-_2(x) > 0\), and \(U^-_2(x) < U^+_2(x)\) for any \(x \in \mathbb{R}\).

**Lemma 4.3.** For any \(u \in \mathcal{E}^+_2\),

\[
U^-_2(x) < U(t, x; u) \quad \forall x \in \mathbb{R}, \; t > 0,
\]

where \(U(t, x; u)\) is the solution of the following parabolic equation

\[
\begin{cases} U_t = A_u(U), & t > 0, \; x \in \mathbb{R} \\ U(0, x; u) = U^+_2(x), & x \in \mathbb{R}. \end{cases} \tag{4.6}
\]
Proof. Observe that
\[
    u^s_{xx} + (c - \chi \Psi_x(x; u))u^s_x + (r(x) - \chi \nu \Psi(x; u) - (b - \chi \mu)u^s)u^s - u^s_t = (\bar{b} - b)u^s > 0 \quad \forall - L < x < L.
\] (4.7)

Hence, by the comparison principle for parabolic equations, we get that
\[
    u^s(t, x; \bar{b}, u) < U(t, x; u), \quad \forall - L < x < L, \quad t > 0, \quad u \in \mathcal{E}_2^+.
\] (4.8)

This implies that
\[
    \inf_{t \in \mathbb{R}, u \in \mathcal{E}_2^+} u^s(t, x; \bar{b}, u) < U(t, x; u), \quad \forall - L < x < L, \quad t > 0, \quad u \in \mathcal{E}_2^+.
\]

The lemma then follows. \( \square \)

Note that, by Lemma 4.1 and the comparison principle for parabolic equations,
\[
    U(t_2, x; u) \leq U(t_1, x; u) \leq U^+_2(x), \quad \forall x \in \mathbb{R}, \quad 0 < t_1 < t_2, \quad u \in \mathcal{E}_2^+.
\]

Thus the function
\[
    U^s_2(x; u) = \lim_{t \to \infty} U(t, x; u), \quad \forall u \in \mathcal{E}_2^+\] (4.9)

is well defined, and
\[
    U^-_2(x) \leq U^s_2(x; u) \leq U^+_2(x), \quad \forall x \in \mathbb{R}, \quad u \in \mathcal{E}_2^+.
\] (4.10)

Let
\[
    \mathcal{E}_2 = \{ u \in C^b_{\text{unif}}(\mathbb{R}) : U^-_2(x) \leq u(x) \leq U^+_2(x), \quad \forall x \in \mathbb{R} \}. \]

For any \( u \in \mathcal{E}_2 \), it follows from (4.10) that \( U^s_2(\cdot; u) \in \mathcal{E}_2 \). Moreover, by a priori estimates for parabolic equation, we have that \( U^s_2(x; u) \) satisfies
\[
    0 = U_{xx} + (c - \chi \Psi_x(x; u))U_x + (r(x) - \chi \nu \Psi(x; u) - (b - \chi \mu)U) \quad \forall x \in \mathbb{R}.
\] (4.11)

Since \( U^-_2(x) \geq 0 \) for any \( x \in \mathbb{R} \) and \( U^-_2(x) \neq 0 \) for any \( x \in \mathbb{R} \), it follows from the comparison principle for parabolic equations that
\[
    U^s_2(x; u) > 0 \quad \forall x \in \mathbb{R}, \quad u \in \mathcal{E}_2^+.
\] (4.12)

**Lemma 4.4.** For any given \( u \in \mathcal{E}_2 \), (4.11) has a unique solution \( U^s_2(\cdot; u) \in \mathcal{E}_2 \).

**Proof.** Let \( U_1(x; u) \), \( U_2(x; u) \) be two solutions of (4.11) in \( \mathcal{E}_2 \). Note that \( U_i(x; u) > 0 \) for \( x \in \mathbb{R} \) and every \( i = 1, 2 \). For any \( \epsilon > 0 \), let
\[
    K_{\epsilon} = \{ k \geq 1 \mid kU_2(x; u) \geq U_1(x; u) - \epsilon \quad \forall x \in \mathbb{R} \}.
\]

\( K_{\epsilon} \) is not empty because
\[
    \lim_{x \to \pm \infty} \frac{U_1(x; u) - \epsilon}{U_2(x; u)} = -\infty.
\] (4.13)
Let \( k_\epsilon = \inf K_\epsilon \). Then \( k_\epsilon \geq 1 \) and
\[
k_\epsilon U_2(x; u) \geq U_1(x; u) - \epsilon \quad \forall x \in \mathbb{R}.
\]
Note that \( k_\epsilon \) is nonincreasing in \( \epsilon > 0 \). Following the similar arguments as those used in the proof of Lemma 3.5, we have \( k_\epsilon = 1 \) for \( 0 < \epsilon \ll 1 \). Therefore,
\[
U_2(x; u) \geq U_1(x, u) \quad \forall x \in \mathbb{R}.
\]
Similarly, we have
\[
U_1(x; u) \geq U_2(x, u) \quad \forall x \in \mathbb{R}.
\]
The lemma thus follows.

We now prove Theorem 1.2.

Proof of Theorem 1.2. Consider the mapping \( U_2^*(\cdot, \cdot) : \mathcal{E}_2 \ni u \mapsto U_2^*(x; u) \in \mathcal{E}_2 \) as defined by (4.9).

It follows from the arguments of [37, Theorem 3.1] and Lemma 4.4 that this function is continuous and compact in the compact open topology. Hence it has a fixed point \( u^* \) by the Schauder’s fixed point Theorem. Taking \( v^*(x) = \Psi(x; u^*) \), we have from (4.11), that \((u(t, x), v(t, x)) = (u^*(x - ct), v^*(x - ct))\) is an entire solution of (1.1). Moreover, by (4.12), \( u^*(x) > 0 \) for all \( x \in \mathbb{R} \). Since \( U_2^-(x) \leq u^*(x) \leq U_2^+(x) \), it follows that \( \lim_{x \to \pm \infty} u^*(x) = 0 \). The theorem is thus proved.

5 Numerical Simulations

In this section, we present some numerical investigation on the existence of forced wave solutions. All the numerical simulations were conducted using programming software MATLAB.

5.1 Numerical simulations in Case 1

In this subsection, we present some numerical simulations in Case 1 by the finite difference method. It should be pointed out that the authors in [48] provided some numerical study for the vanishing and spreading dynamics of chemotaxis systems with logistic source and a free boundary by the finite difference method.

First, we describe the numerical scheme. To numerically investigate the existence of forced wave solutions in Case 1, we use the finite difference method to compute the solution of

\[
\begin{align*}
\frac{u_t}{u_{xx}} + cu_x - (\chi u v_x)x + u(r(x) - bu), & \quad -L < x < L \\
0 = v_{xx} - \nu v + \mu u, & \quad -L < x < L \\
u(0, x) = u_0(x), & \quad -L \leq x \leq L \\
u(t, -L) = v(t, -L) = 0 \\
\frac{\partial u}{\partial x}(t, L) = \frac{\partial v}{\partial x}(t, L) = 0
\end{align*}
\]

(5.1)

for reasonable large \( L > 1 \), where \( u_0(x) \) is piece-wise linear,

\[
u_0(x) = \begin{cases} 
0 & \text{if } x \leq -1, \\
\frac{r^*}{2b}x + \frac{r^*}{2b} & \text{if } -1 < x < 1, \\
\frac{r^*}{6} & \text{if } x \geq 1.
\end{cases}
\]

Let \((u_L(t,x;u_0), v_L(t,x;u_0))\) be the solution of (5.1) with \(u_L(0,x;u_0) = u_0(x)\). Observe that if \((u_L(x), v_L(x)) := \lim_{t \to -\infty} (u_L(t,x;u_0), v_L(t,x;u_0))\) exists, then \((u_L(x), v_L(x))\) is a stationary solution of (5.1). If \((w_\infty(x), v_\infty(x)) := \lim_{x \to \infty} (u_L(x), v_L(x))\) exists, and \(u_\infty(-\infty) = 0\) and \(u_\infty(\infty) = v^*_\infty\), then \((u_\infty(x), v_\infty(x))\) is a forced wave solution of (1.1) connecting \(\left(\frac{r_2}{p}, \frac{w_2^*}{h}\right)\) and \((0,0)\). We will then compute the numerical solution of (5.1) on a reasonable large time interval \([0,T]\) for several choices of \(L\).

Note that, by the second equation in (5.1), \(v_{xx} = \nu v - \mu u\). Hence the first equation in (5.1) can be written as

\[
\frac{\partial u}{\partial t} = u_{xx} + (c - \chi v_x)u_x + u(r(x) - \chi vv - (b - \chi \mu)u), \quad -L < x < L. \tag{5.2}
\]

To find the numerical solution of (5.1) on the interval \([0,T]\), we divide the space interval \([-L,L]\) into \(M\) subintervals with equal length and divide the time interval \([0,T]\) into \(N\) subintervals with equal length. Then the space step size is \(h = \frac{2L}{M}\) and the time step size is \(\tau = \frac{T}{N}\). For simplicity, we denote the approximate value of \(u(t_j, x_i)\) by \(u(j,i)\), \(r(x_i)\) by \(r(i)\) and \(v(t_j, x_i)\) by \(v(j,i)\) with \(t_j = (j-1)\tau, 1 \leq j \leq N + 1\), and \(x_i = -L + (i-1)h, 1 \leq i \leq M + 1\).

Using the central approximation for the second spatial derivative \(v_{xx}(t_j, x_i)\),

\[
v_{xx}(t_j, x_i) \approx \frac{v(j,i-1) - 2v(j,i) + v(j,i+1)}{h^2},
\]

the second equation in (5.1) can be discretized as

\[
\frac{v(j,i-1) - 2v(j,i) + v(j,i+1)}{h^2} - \nu v(j,i) + \mu u(j,i) = 0, \quad 2 \leq i \leq M. \tag{5.3}
\]

We use the backward approximation for the spatial derivative \(\frac{\partial v}{\partial x}(t_j, x_{M+1})\),

\[
\frac{\partial v}{\partial x}(t_j, x_{M+1}) \approx \frac{v(j, M+1) - v(j, M)}{h}.
\]

By \(v(t, -L) = 0\) and \(\frac{\partial v}{\partial x}(t, L) = 0\), we set

\[
v(j,1) = 0 \quad \text{and} \quad v(j, M + 1) = v(j, M). \tag{5.4}
\]

By (5.3), for each \(j\), we have \(M-1\) equations which form a system of linear algebraic equations for \((v(j,2), \cdots, v(j,M))\). It is nonsingular and there exists a unique solution \((v(j,2), \cdots, v(j,M))\).

By the forward approximation of the time derivative

\[
\frac{\partial u}{\partial t}(t_j, x_i) \approx \frac{u(j+1,i) - u(j,i)}{\tau},
\]

and the central approximation of the spatial derivative

\[
u_{xx}(t_j, x_i) \approx \frac{u(j,i-1) - 2u(j,i) + u(j,i+1)}{h^2},
\]

\[
u_x(t_j, x_i) \approx \frac{u(j,i+1) - u(j,i-1)}{2h},
\]
equation (5.2) can be discretized as

\[
\frac{u(j + 1, i) - u(j, i)}{\tau} = u(j, i - 1) - 2u(j, i) + u(j, i + 1)
\]

\[
+ \left( c - \frac{\nu}{2h} v(j, i + 1) - v(j, i - 1) \right) \frac{h^2}{2h} u(j, i + 1) - u(j, i - 1)
\]

\[
+ u(j, i) \left( r(i) - \chi \nu v(j, i) - (b - \chi \mu) u(j, i) \right), \quad 1 \leq j \leq N, \quad 2 \leq i \leq M.
\]

Simplify and reorder the above equations, we get the following explicit formulas for \(u(j + 1, i)\) \((1 \leq j \leq N, 2 \leq i \leq M)\),

\[
u
u
\]

\[
u
\]

\[
u
\]

\[
\frac{\tau}{h^2} \left( c - \frac{\nu}{2h} v(j, i + 1) - v(j, i - 1) \right) u(j, i - 1)
\]

\[
+ \left( 1 - \frac{2\tau}{h^2} + \tau r(i) - \tau \chi \nu v(i) \right) u(j, i) - \tau (b - \chi \mu) u(j, i)^2
\]

\[
+ \left( \frac{\tau}{h^2} + \frac{\tau}{2h} \left( c - \frac{\nu}{2h} v(j, i + 1) - v(j, i - 1) \right) \right) u(j, i + 1),
\]

\[
1 \leq j \leq N, \quad 2 \leq i \leq M.
\]

Similarly, by \(u(t, -L)\) and \(\frac{\partial u}{\partial t}(t, L) = 0\), we set

\[
u
\]

\[
\frac{\partial u}{\partial t}(j, i) = 0 \quad \text{and} \quad u(j + 1, M + 1) = u(j + 1, M) \quad 1 \leq j \leq N.
\]

Thus, for each \(j\), the values \(u(j + 1, i), 1 \leq i \leq M + 1\) are obtained.

Next, we present our numerical simulations. We fix the parameter values \(\chi = 0.1, \mu = 1, \nu = 0.05\), and choose \(r(x)\) to be the piece-wise linear function

\[
r(x) = \begin{cases} 
-1 & \text{if } x \leq -8, \\
11x + 87 & \text{if } -8 < x < -7, \\
10 & \text{if } x \geq -7.
\end{cases}
\]

For this choice of \(r(x)\), \(r^* = 10\) and \(-c^* = -2\sqrt{r^*} \approx -6.325\). We will do four numerical experiments for different values of \(b\) and \(c\). In these four numerical experiments, we will use the same space step size \(h = 0.1\) and the same time step size \(\tau = 0.002\).

**Numerical Experiment 1.** Let \(b = 1\) and \(c = 1\). In this case, \(c > \frac{\chi^2 r^*}{2\sqrt{\nu(b - \chi \mu)}} - 2\sqrt{\frac{r^*(b - 2\chi \mu)}{b - \chi \mu}}\) becomes \(c > \frac{5}{9\nu 0.05} - 2\sqrt{\frac{8}{0.3}} \approx -3.478\). So for these choices of \(b\) and \(c\), the assumption (H1) holds.

We compute the numerical solution of (5.1) with \(L = 15, 20, 25, 30\), and \(40\) on the time interval \([0, 10]\). For all the choices of \(L\), we observe that the numerical solution of (5.1) changes very little after \(t = 3\), which indicates that the numerical solution converges to a stationary solution of (5.1) as \(t \to \infty\). We also observe that the numerical solution \(u(t, x)\) at \(t = 10\) changes very little and \(u(10, L)\) is very close to \(\frac{x}{b} = 10\) as \(L\) increases, which indicates the stationary solution of (5.1) converges to a stationary solution of (1.8) connecting \(\left( \frac{x}{b}, \frac{k}{\nu b} \right)\) and \((0, 0)\) or a forced wave solution of (1.1) connecting \(\left( \frac{x}{b}, \frac{k}{\nu b} \right)\) and \((0, 0)\) as \(L \to \infty\), whose existence is proved in Theorem 1.1. Hence the numerical results for the choices \(b = 1\) and \(c = 1\) match the theoretical results.

We demonstrate the numerical solutions of (5.1) for the cases \(L = 20\) and \(L = 40\) in Figure 1 and Figure 2, respectively. Figure 1(a) is the surface plot of the numerical solution of the system.
\(5.1\) on the interval \([-20, 20]\) as time evolves. We plot the profile of the numerical solution at time \(t = 0, 1, 2, 3, 7, 10\) in Figure 1(b). Figure 2(a) is the surface plot of the numerical solution of the system \(5.1\) on the interval \([-40, 40]\) as time evolves. We plot the profile of the numerical solution at time \(t = 0, 1, 2, 3, 7, 10\) in Figure 2(b).

Figure 1: (a) Evolution of numerical solution of \(5.1\) on the interval \([-20, 20]\) with \(b = 1\) and \(c = 1\). (b) numerical solution of \(5.1\) on the interval \([-20, 20]\) at time \(t = 0, 1, 2, 3, 7, 10\) with \(b = 1\) and \(c = 1\).

Figure 2: (a) Evolution of numerical solution of \(5.1\) on the interval \([-40, 40]\) with \(b = 1\) and \(c = 1\). (b) numerical solution of \(5.1\) on the interval \([-40, 40]\) at time \(t = 0, 1, 2, 3, 7, 10\) with \(b = 1\) and \(c = 1\).

Numerical Experiment 2. Let \(b = 1, c = -6\). For these choices of \(b\) and \(c\), \(b\) and \(c\) satisfy \(b > 2\chi\mu\) and \(c > c^*\), but the assumption \(c > \frac{\chi\mu c^*}{2\sqrt{\mu(b-\chi\mu)}} - 2\frac{\mu(b-2\chi\mu)}{b-\chi\mu}\) does not hold.

We compute the numerical solution of \(5.1\) with \(L = 15, 20, 25, 30, \) and \(40\) on the time interval \([0, 15]\). For all the choices of \(L\), we observe that the numerical solution of \(5.1\) changes very little after \(t = 7\), which indicates that the numerical solution converges to a stationary solution of \(5.1\) as \(t \to \infty\). We also observe that the numerical solution \(u(t, x)\) at \(t = 15\) changes very little and \(u(15, L)\) is very close to \(\frac{r^*}{b} \frac{\mu r^*}{b} \) as \(L\) increases, which indicates the stationary solution of \(5.1\) converges to a stationary solution of \(1.8\) connecting \((\frac{r^*}{b}, \frac{\mu r^*}{b})\) and \((0, 0)\) or a forced wave.
solution of (1.1) connecting \((r^*, b^*, \mu, \nu)\) and \((0, 0)\) as \(L \rightarrow \infty\). The numerical results indicate that when \(c > -c^*\) and \(b > 2\chi\mu\), (1.1) still has a forced wave solution. We demonstrate the numerical solutions of (5.1) for the case \(L = 20\) and \(L = 40\) in Figure 3 and Figure 4, respectively.

**Figure 3:** (a) Evolution of numerical solution of (5.1) on the interval \([-20, 20]\) with \(b = 1\) and \(c = -6\). (b) numerical solution of (5.1) on the interval \([-20, 20]\) at time \(t = 0, 1, 3, 7, 13, 15\) with \(b = 1\) and \(c = -6\).

**Figure 4:** (a) Evolution of numerical solution of (5.1) on the interval \([-40, 40]\) with \(b = 1\) and \(c = -6\). (b) numerical solution of (5.1) on the interval \([-40, 40]\) at time \(t = 0, 1, 3, 7, 13, 15\) with \(b = 1\) and \(c = -6\).

**Numerical Experiment 3.** Let \(b = 0.15\), \(c = -6\). For these choices of \(b\) and \(c\), \(b\) and \(c\) satisfy \(b > \chi\mu\) and \(c > -c^*\).

We compute the numerical solution of (5.1) with \(L = 35, 40, 45, 50, \) and \(60\) on the time interval \([0, 60]\). For all the choices of \(L\), we observe that the numerical solution of (5.1) changes very little after \(t = 50\), which indicates that the numerical solution converges to a stationary solution of (5.1) as \(t \rightarrow \infty\). We also observe that the numerical solution \(u(t, x)\) at \(t = 60\) changes very little and \(u(60, L)\) is very close to \(\frac{r^*}{b} = \frac{10}{0.15} \approx 66.67\) as \(L\) increases, which indicates the stationary solution of (5.1) converges to a stationary solution of (1.8) connecting \((\frac{r^*}{b}, \frac{\mu}{b}, \frac{\nu}{b})\) and \((0, 0)\) or a forced wave solution of (1.1) connecting \((\frac{r^*}{b}, \frac{\mu}{b}, \frac{\nu}{b})\) and \((0, 0)\) as \(L \rightarrow \infty\). The numerical results
indicate that when \( c > -c^* \) and \( b > \chi \mu \), (1.1) still has a forced wave solution.

We demonstrate the numerical solutions of (5.1) for the case \( L = 40 \) and \( L = 60 \) in Figure 5 and Figure 6, respectively.

Figure 5: (a) Evolution of numerical solution of (5.1) on the interval \([-40, 40]\) with \( b = 0.15 \) and \( c = -6 \). (b) numerical solution of (5.1) on the interval \([-40, 40]\) at time \( t = 0, 5, 10, 20, 30, 40, 50, 60 \) with \( b = 0.15 \) and \( c = -6 \).

Figure 6: (a) Evolution of numerical solution of (5.1) on the interval \([-60, 60]\) with \( b = 0.15 \) and \( c = -6 \). (b) numerical solution of (5.1) on the interval \([-60, 60]\) at time \( t = 0, 5, 10, 20, 30, 40, 50, 60 \) with \( b = 0.15 \) and \( c = -6 \).

**Numerical Experiment 4.** Let \( b = 1, c = -6.5 \). For these choices of \( b \) and \( c \), \( b \) and \( c \) satisfy \( b > 2\chi \mu \) and \( c < -c^* \).

We compute the numerical solution of (5.1) with \( L = 15 \) on the time interval \([0, 50]\), with \( L = 20 \) on the time interval \([0, 60]\), with \( L = 25 \) on the time interval \([0, 70]\), with \( L = 30 \) on the time interval \([0, 90]\), and with \( L = 40 \) on the time interval \([0, 140]\). For all the choices of \( L \), we observe that the numerical solution of (5.1) becomes very small after certain time, which indicates that the numerical solution converges to the zero solution of (5.1) as \( t \to \infty \), and also indicates that (1.8) has no positive stationary solutions or (1.1) has no forced wave solutions in the case that \( c < -c^* \).
We demonstrate the numerical solutions of (5.1) for the case $L = 20$ and $L = 40$ in Figure 7 and Figure 8, respectively.

Figure 7: (a) Evolution of numerical solution of (5.1) on the interval $[-20, 20]$ with $b = 1$ and $c = -6.5$. (b) Numerical solution of (5.1) on the interval $[-20, 20]$ at time $t = 0, 20, 30, 40, 50, 60$ with $b = 1$ and $c = -6.5$.

Figure 8: (a) Evolution of numerical solution of (5.1) on the interval $[-40, 40]$ with $b = 1$ and $c = -6.5$. (b) Numerical solution of (5.1) on the interval $[-40, 40]$ at time $t = 0, 40, 80, 100, 120, 140$ with $b = 1$ and $c = -6.5$.

Remark 5.1. (1) The numerical simulations above illustrate our Theorem 1.1 and also shows that the assumptions in Theorem 1.1 can be weakened. Based on these numerical simulations, we conjecture that if $b > \chi \mu$, and $c > -2\sqrt{r^*}$, there is a forced wave solution $(u(t, x), v(t, x)) = (\phi(x - ct), \psi(x - ct))$ connecting $(r^*, \frac{\mu r^*}{b})$ and $(0, 0)$, that is, $\phi(\infty) = \frac{r^*}{b}$ and $\phi(-\infty) = 0$. If $b > \chi \mu$ and $c < -2\sqrt{r^*}$, there is no forced wave solution $(u(t, x), v(t, x)) = (\phi(x - ct), \psi(x - ct))$ connecting $(r^*, \frac{\mu r^*}{b})$ and $(0, 0)$, that is, $\phi(\infty) = \frac{r^*}{b}$ and $\phi(-\infty) = 0$.

(2) In the above four numerical experiments, we used the same space step size $h = 0.1$ and the same time step size $\tau = 0.002$. They satisfy the numerical stable condition $\frac{h^2}{\tau} < \frac{1}{2}$. We do not give the accuracy analysis of the simulations in this paper. To see the reliability of
the numerical results, we also tried different values of \( h \) and \( \tau \) to simulate the existence of forced wave solutions. For the above four experiments, let \( h = 0.1 \) be fixed, choose \( \tau = 0.001, 0.002, 0.004 \) respectively, the graphs we got do not have a big difference. Fix \( h = 0.05 \), let \( \tau = 0.001, 0.0005, 0.00025 \) respectively, the graphs we got also do not have a big difference.

(3) We also tried to use different initial conditions to simulate the forced wave. For example, let \( u_0(x) = 0 \) for \( x < -1 \), \( u(x) = x + 1 \) for \(-1 \leq x \leq 1 \), and \( u_0(x) = 2 \) for \( x > 1 \). We see similar dynamical scenarios. We then conjecture that the forced wave solution of (1.1) is unique and stable in certain parameter region.

5.2 Numerical simulations in Case 2

In this subsection, we study the numerical simulations of the forced wave solutions in Case 2. To this end, we consider

\[
\begin{aligned}
&u_t = u_{xx} + cu_x - (\chi uv_x)_x + u(r(x) - bu), \quad -L < x < L \\
&0 = v_{xx} - \nu v + \mu u, \quad -L < x < L \\
&u(0, x) = u_0(x), \quad -L \leq x \leq L, \\
&u(t, -L) = v(t, -L) = 0 \\
&u(t, L) = v(t, L) = 0
\end{aligned}
\]

(5.5)

for reasonable large \( L > 1 \), where \( u_0(x) = \begin{cases} 0 & \text{if } |x| > 1, \\ (x + 1)(1 - x) & \text{if } -1 \leq x \leq 1. \end{cases} \)

Similarly, to compute the solution of (5.5) on a time interval \([0, T]\) numerically, we divide the space interval \([-L, L]\) into \( M \) subintervals with equal length and divide the time interval \([0, T]\) into \( N \) subintervals with equal length. The space step size is \( h = \frac{2L}{M} \) and the time step size is \( \tau = \frac{T}{N} \). For simplicity, we denote the approximate value of \( u(t_j, x_i) \) by \( u(j, i) \), \( r(x_i) \) by \( r(i) \) and \( v(t_j, x_i) \) by \( v(j, i) \) with \( t_j = (j - 1)\tau, 1 \leq j \leq N + 1 \), and \( x_i = -L + (i - 1)h, 1 \leq i \leq M + 1 \).

By the same procedure as in Case 1, the second equation in (5.5) can be discretized as

\[
\frac{v(j, i - 1) - 2v(j, i) + v(j, i + 1)}{h^2} - \nu v(j, i) + \mu u(j, i) = 0, \quad 2 \leq i \leq M. \quad (5.6)
\]

Together with the boundary conditions \( v(j, 1) = v(j, M + 1) = 0 \), we can solve for the solution \( (v(j, 1), v(j, 2), \ldots, v(j, M + 1)) \) for each \( j \). By the same numerical discretization for the first equation of (5.5) as in Case 1, we have an explicit formula for \( u(j + 1, i) \),

\[
\begin{aligned}
u(j + 1, i) = &\left(\frac{\tau}{h^2} - \frac{\tau}{2h} \left( c - \chi \frac{v(j, i + 1) - v(j, i - 1)}{2h} \right) \right) u(j, i - 1) \\
+ &\left(1 - \frac{2\tau}{h^2} + \tau r(i) - \tau \chi \nu \right) u(j, i) - \tau (b - \chi \mu) u(j, i)^2 \\
+ &\left(\frac{\tau}{h^2} + \frac{\tau}{2h} \left( c - \chi \frac{v(j, i + 1) - v(j, i - 1)}{2h} \right) \right) u(j, i + 1), \quad 1 \leq j \leq N, \quad 2 \leq i \leq M.
\end{aligned}
\]

By the boundary condition, we set

\[
u(j + 1, 1) = u(j + 1, M + 1) = 0 \quad 1 \leq j \leq N.
\]

28
Thus, the values \( u(j + 1, i), 1 \leq i \leq M + 1 \) are obtained.

We choose \( \mu = 1, \nu = 1 \) and the following growth rate function

\[
 r(x) = \begin{cases} 
 -1 & \text{if } |x| \geq 8, \\
 11x + 87 & \text{if } -8 < x < -7, \\
 10 & \text{if } -7 \leq x \leq 7, \\
 -11x + 87 & \text{if } 7 < x < 8.
\end{cases}
\]

For this choice of \( r(x) \), \( r^* = 10 \) and \( c^* := 2\sqrt{r^*} \approx 6.325 \). We will do three numerical experiments for different values of \( b \), \( c \) and \( \chi \). In these three numerical experiments, we will use the same space step size \( h = 0.1 \) and the same time step size \( \tau = 0.002 \).

**Numerical Experiment 1.** Choose \( c = 1 \), then \( \lambda_{-7}(r(\cdot)) = \frac{40 - 1 - \frac{3}{4}x^2}{4} > 0 \). Since \( \lambda_L(r(\cdot)) > \lambda_{-7}(r(\cdot)) \) for \( L > -7 \), we have \( \lambda_\infty(r(\cdot)) \geq \lambda_{-7}(r(\cdot)) > 0 \). Choose \( b = 1 \) and \( \chi = 0.6 \). Then \( b \geq \frac{3c\chi}{2} \).

We compute the numerical solution of (5.5) with \( L = 15, 20, 25, 30, \) and 40 on the time interval \([0, 10]\). In all the cases, we observe that the numerical solution changes very little after \( t = 3 \) and stays away from 0 on some fixed interval, which indicates that the numerical solution converges to a positive stationary solution of (5.5) as \( t \to \infty \). We also observe that the numerical solution \( u(t, x) \) at \( t = 10 \) changes very little as \( L \) increases, which indicates that the stationary solution of (5.5) converges as \( L \to \infty \) to a stationary solution of (1.8) or a forced wave solution of (1.1) connecting \((0, 0)\) and \((0, 0)\). We demonstrate the numerical solutions of (5.5) for the cases \( L = 20 \) and \( L = 40 \) in Figure 9 and Figure 10, respectively.

![Figure 9](image1)

(a)

![Figure 9](image2)

(b)

Figure 9: (a) Evolution of numerical solution of (5.5) on the interval \([-20, 20]\) with \( c = 1, b = 1 \) and \( \chi = 0.6 \). (b) numerical solution of (5.5) on the interval \([-20, 20]\) at time \( t = 0, 1, 2, 3, 7, 10 \) with \( c = 1, b = 1 \) and \( \chi = 0.6 \).
Figure 10: (a) Evolution of numerical solution of (5.5) on the interval $[-40, 40]$ with $c = 1$, $b = 1$ and $\chi = 0.6$. (b) numerical solution of (5.5) on the interval $[-40, 40]$ at time $t = 0, 1, 2, 3, 7, 10$ with $c = 1$, $b = 1$ and $\chi = 0.6$.

**Numerical Experiment 2.** Choose $c = 1$ (then $\lambda_\infty(r(\cdot)) > 0$). Choose $b = 0.7$ and $\chi = 0.6$ (then $\chi \mu < b < \frac{3\chi \mu}{2}$).

We compute the numerical solution of (5.5) with $L = 15, 20, 25, 30$, and $40$ on the time interval $[0, 10]$. In all the cases, we observe that the numerical solution changes very little after $t = 3$ and stays away from 0 on some fixed interval, which indicates that the numerical solution converges to a positive stationary solution of (5.5) as $t \to \infty$. We also observe that the numerical solution $u(t, x)$ at $t = 10$ changes very little as $L$ increases, which indicates that the stationary solution of (5.5) converges as $L \to \infty$ to a stationary solution of (1.8) or a forced wave solution of (1.1) connecting $(0, 0)$ and $(0, 0)$. We demonstrate the numerical solutions of (5.5) for the cases $L = 20$ and $L = 40$ in Figure 11 and Figure 12 respectively.

Figure 11: (a) Evolution of numerical solution of (5.5) on the interval $[-20, 20]$ with $c = 1$, $b = 0.7$ and $\chi = 0.6$. (b) numerical solution of (5.5) on the interval $[-20, 20]$ at time $t = 0, 1, 2, 3, 7, 10$ with $c = 1$, $b = 0.7$ and $\chi = 0.6$. 
Numerical Experiment 3. Let $c = 6.5$ (hence $c > c^*$). Let $b = 1$ and $\chi = 0.6$ (hence $b > \frac{3\chi\mu}{2}$).

We compute the numerical solution of (5.5) with $L = 15, 20, 25, 30, 40$ on the time interval $[0, 30]$. For all the choices of $L$, we observe that the numerical solution of (5.5) becomes very small after $t = 20$, which indicates that the numerical solution converges to the zero solution of (5.5) as $t \to \infty$, and also indicates that (1.8) has no positive stationary solutions or (1.1) has no forced wave solutions in the case that $c > c^*$ and $b > \frac{3}{2}\chi\mu$ which matches the theoretical result [42, Theorem 1.3(1)]. We demonstrate the numerical solutions of (5.5) for the case $L = 20$ and $L = 40$ in Figure 13 and Figure 14, respectively.
Figure 14: (a) Evolution of numerical solution of \((5.5)\) on the interval \([-40, 40]\) with \(c = 6.5, b = 1\) and \(\chi = 0.6\). (b) numerical solution of \((5.5)\) on the interval \([-40, 40]\) at time \(t = 0, 5, 10, 15, 20, 30\) with \(c = 6.5, b = 1\) and \(\chi = 0.6\).

Similarly, if \(c = -6.5, b = 1\) and \(\chi = 0.6\), we observe that the numerical solution of \((5.5)\) becomes very small after certain time, which indicates that the numerical solution converges to the zero solution of \((5.5)\) as \(t \to \infty\), and also indicates that \((1.3)\) has no positive stationary solutions or \((1.1)\) has no forced wave solutions in the case that \(c < -c^*\) and \(b > \frac{3}{2} \chi \mu\) which matches the theoretical result [42, Theorem 1.3(1)].

**Remark 5.2.**

1. The numerical simulations above supports our Theorem 1.2 and also tells us that the assumptions in Theorem 1.2 may be weakened. Based on these numerical simulations, we conjecture that if \(b > \chi \mu\) and \(\lambda_\infty(r(\cdot)) > 0\), there is a forced wave solution \((u(t,x), v(t,x)) = (\phi(x - ct), \psi(x - ct))\) connecting \((0,0)\) and \((0,0)\), that is, \(\phi(x) > 0\) for all \(x \in \mathbb{R}\) and \(\phi(\pm\infty) = 0\). If \(b > \chi \mu\) and \(|c| > c^*\), there is no forced wave solution \((u(t,x), v(t,x)) = (\phi(x - ct), \psi(x - ct))\) connecting \((0,0)\) and \((0,0)\), that is, \(\phi(x) > 0\) for all \(x \in \mathbb{R}\) and \(\phi(\pm\infty) = 0\).

2. In these three numerical simulations, we used the same space step size \(h = 0.1\) and the same time step size \(\tau = 0.002\), which satisfy the numerical stable condition \(\frac{\tau}{h^2} < \frac{1}{2}\). Again, we do not give the accuracy analysis of the simulations in this paper. To see the reliability of the numerical results, we also tried different values of \(h\) and \(\tau\). For example, let \(h = 0.1\) be fixed, let \(\tau = 0.001, 0.002, 0.004\) respectively; let \(h = 0.2\) be fixed, let \(\tau = 0.01, 0.005, 0.0025\) respectively; let \(h = 0.05\) be fixed, let \(\tau = 0.001, 0.0005, 0.00025\) respectively. All the graphs we got do not change much.

3. We also tried to use different initial conditions to simulate the existence of forced wave solutions. For example, let

\[
u_0(x) = \begin{cases} 
0 & \text{if } x \leq -1, \\
\sin(x + 1) & \text{if } -1 < x < \pi - 1, \\
0 & \text{if } x \geq \pi - 1.
\end{cases}
\]

We see similar dynamical scenarios. We then also conjecture that the forced wave solution of \((1.1)\) is unique and stable in certain parameter region.
References

[1] D. G. Aronson and H. F. Weinberger, Multidimensional nonlinear diffusions arising in population genetics, *Adv. Math.*, 30 (1978), pp. 33-76.

[2] H. Berestycki, O. Diekmann, C. J. Nagelkerke, and P. A. Zegeling, Can a species keep pace with a shifting climate? *Bull. Math. Biol.*, 71 (2009), no. 2, 399-429.

[3] H. Berestycki, J. Fang, Forced waves of the Fisher-KPP equation in a shifting environment, *J. Differ. Equ.* 264 (2018), pp. 2157-2183.

[4] H. Berestycki, F. Hamel and G. Nadin, Asymptotic spreading in heterogeneous diffusive excitable media, *J. Funct. Anal.*, 255 (2008), 2146-2189.

[5] H. Berestycki, F. Hamel, and N. Nadirashvili, The speed of propagation for KPP type problems, I - Periodic framework, *J. Eur. Math. Soc.*, 7 (2005), 172-213.

[6] H. Berestycki, F. Hamel, and N. Nadirashvili, The speed of propagation for KPP type problems, II - General domains, *J. Amer. Math. Soc.*, 23 (2010), no. 1, 1-34.

[7] H. Berestycki and G. Nadin, Asymptotic spreading for general heterogeneous Fisher-KPP type equations, preprint, (2015).

[8] H. Berestycki, B. Nicolaenko and B. Scheurer, Traveling wave solutions to combustion models and their singular limits, *SIAM J. Math. Anal.*, 16 (1985), no. 6, 1207-1242.

[9] Y. Du, L. Wei, and L. Zhou, Spreading in a shifting environment modeled by the diffusive logistic equation with a free boundary, *J. Dynam. Differential Equations*, 30 (2018), no. 4, pp. 1389-1426.

[10] J. Fang, Y. Lou and J. Wu, Can pathogen spread keep pace with its host invasion? *SIAM J. Appl. Math.*, 76 (2016), pp. 1633-1657.

[11] R. Fisher, The wave of advance of advantageous genes, *Ann. of Eugenics*, 7 (1937), 335-369.

[12] M. Freidlin, On wave front propagation in periodic media. *In: Stochastic analysis and applications, ed. M. Pinsky, Advances in Probability and Related Topics*, 7 (1984), 147-166.

[13] M. Freidlin and J. Gärtner, On the propagation of concentration waves in periodic and random media, *Soviet Math. Dokl.*, 20 (1979), 1282-1286.

[14] F. Hamel and C. Henderson, Propagation in a Fisher-KPP equation with non-local advection, preprint, (2017), https://arxiv.org/abs/1709.00923.

[15] T. Hillen and K.J. Painter, A Users Guide to PDE Models for Chemotaxis, *J. Math. Biol.*, 58 (2009) (1), 183-217.

[16] D. Horstmann, From 1970 until present: the Keller-Segel model in chemotaxis and its consequences, *Jahresber. Dtsch. Math.-Ver.*, 105 (2003), 103-165.

[17] C. Hu and B. Li, Spatial dynamics for lattice differential equations with a shifting habitat, *J. Differential Equations*, 259 (2015), 1967-1989.

[18] H. Hu and X. Zou, Existence of an extinction wave in the Fisher equation with a shifting habitat, *Proc. Am. Math. Soc.*, 145, (2017), 4763-4771.

[19] E.F. Keller and L.A. Segel, Initiation of slime mold aggregation viewed as an instability, *J. Theoret. Biol.*, 26 (1970), 399-415.
[20] E.F. Keller and L.A. Segel, A Model for chemotaxis, *J. Theoret. Biol.*, 30 (1971), 225-234.

[21] A. Kolmogorov, I. Petrowsky, and N. Piskunov, A study of the equation of diffusion with increase in the quantity of matter, and its application to a biological problem, *Bjul. Moskovskogo Gos. Univ.*, 1 (1937), pp. 1-26.

[22] L. Kong and W. Shen, Liouville type property and spreading speeds of KPP equations in periodic media with localized spatial inhomogeneity, *J. Dynam. Differential Equations*, 26 (2014), no. 1, 181-215.

[23] P. De Leenheer, W. Shen, and A. Zhang, Persistence and extinction of nonlocal dispersal evolution equations in moving habitats, *Nonlinear Anal. Real World Appl.*, 54 (2020), 103110.

[24] C. Lei and Y. Du, Asymptotic profile of the solution to a free boundary problem arising in a shifting climate model, *Discrete Contin. Dyn. Syst. Ser. B*, 22 (2017), no. 3, 895-911.

[25] M. A. Lewis, N. G. Marculis, and Z. Shen, Integrodifference equations in the presence of climate change: persistence criterion, travelling waves and inside dynamics, *J. Math. Biol.*, 77 (2018), pp 1649-1687.

[26] B. Li, S. Bewick, J. Shang, and W. F. Fagan, Persistence and spread of a species with a shifting habitat edge, *SIAM J. Appl. Math.*, 5 (2014), 1397-1417.

[27] B. Li, S. Bewick, M. Barnard, and W. Fagan, Persistence and spreading speeds of integro-difference equations with an expanding or contracting habitat, *Bull. Math. Biol.*, 78 (2016), no. 7, 1337-1379.

[28] W. T. Li, J. B. Wang and X. Q. Zhao, Spatial dynamics of a nonlocal dispersal population model in a shifting environment, *J. Nonlinear Sci.*, 4 (2018), 1189-1219

[29] X. Liang and X.-Q. Zhao, Asymptotic speeds of spread and traveling waves for monotone semiflows with applications, *Comm. Pure Appl. Math.*, 60 (2007), no. 1, 1-40.

[30] X. Liang and X.-Q. Zhao, Spreading speeds and traveling waves for abstract monostable evolution systems, *Journal of Functional Analysis*, 259 (2010), 857-903.

[31] G. Nadin, Traveling fronts in space-time periodic media, *J. Math. Pures Anal.*, 92 (2009), 232-262.

[32] J. Nolen, M. Rudd, and J. Xin, Existence of KPP fronts in spatially-temporally periodic advection and variational principle for propagation speeds, *Dynamics of PDE*, 2 (2005), 1-24.

[33] J. Nolen and J. Xin, Existence of KPP type fronts in space-time periodic shear flows and a study of minimal speeds based on variational principle, *Discrete and Continuous Dynamical Systems*, 13 (2005), 1217-1234.

[34] K.J. Painter, Mathematical models for chemotaxis and their applications in self organisation phenomena, *Journal of Theoretical Biology*, 481 (2019), 162-182.

[35] A. B. Potapov and M. A. Lewis, Climate and competition: the effect of moving range boundaries on habitat invasibility *Bull. Math. Biol.*, 66 (5) (2004), 975-1008.

[36] R. B. Salako and Wenxian Shen, Existence of Traveling wave solution of parabolic-parabolic chemotaxis systems, *Nonlinear Analysis: Real World Applications* Volume 42, (2018), 93-119.
[37] R. B. Salako and Wenxian Shen, Spreading Speeds and Traveling waves of a parabolic-elliptic chemotaxis system with logistic source on $\mathbb{R}^N$, *Discrete and Continuous Dynamical Systems - Series A*, 37 (2017), pp. 6189-6225.

[38] R. B. Salako and Wenxian Shen, Global existence and asymptotic behavior of classical solutions to a parabolic-elliptic chemotaxis system with logistic source on $\mathbb{R}^N$, *J. Differential Equations*, 262 (2017) 5635-5690.

[39] R. B. Salako, W. Shen, and S. Xue, Can chemotaxis speed up or slow down the spatial spreading in parabolic-elliptic Keller-Segel systems with logistic source?, *J. Math. Biol.*, 79 (2019), 1455-1490.

[40] W. Shen, Variational principle for spatial spreading speeds and generalized propagating speeds in time almost and space periodic KPP models, *Trans. Amer. Math. Soc.*, 362 (2010), 5125-5168.

[41] W. Shen, Existence of generalized traveling waves in time recurrent and space periodic monostable equations, *J. Appl. Anal. Comput.*, 1 (2011), 69-93.

[42] W. Shen and S. Xue, Persistence and spreading speeds of parabolic-elliptic Keller-Segel models in shifting environments, *J. Differential Equations*, 269 (2020) 6236-6268.

[43] J. I. Tello and M. Winkler, A Chemotaxis System with Logistic Source, *Communications in Partial Differential Equations*, 32 (2007), 849-877.

[44] Hoang-Hung Vo, Persistence versus extinction under a climate change in mixed environments, *J. Differential Equations* 259 (2015), no. 10, 4947-4988.

[45] J. Wang and X. Zhao, Uniqueness and global stability of forced waves in a shifting environment, *Proc. Am. Math. Soc.*, 147, (2019), no. 4, 1467-1481.

[46] H. F. Weinberger, Long-time behavior of a class of biology models, *SIAM J. Math. Anal.*, 13 (1982), 353-396.

[47] H. F. Weinberger, On spreading speeds and traveling waves for growth and migration models in a periodic habitat, *J. Math. Biol.*, 45 (2002), 511-548.

[48] L. Yang and L. Bao, Numerical study of vanishing and spreading dynamics of chemotaxis systems with logistic source and a free boundary, *Discrete and Continuous Dynamical Systems-Series B*, doi: 10.3934/dcdsb.2020154.

[49] Y. Zhou and M. Kot, Discrete-time growth-dispersal models with shifting species ranges, *Theor. Ecol.*, 4 (2011), 13-25.

[50] A. Zlatoš, Transition fronts in inhomogeneous Fisher-KPP reaction-diffusion equations, *J. Math. Pures Appl.* (9) 98 (2012), no. 1, 89-102.