On a General Sextic Equation Solved by the Rogers-Ramanujan Continued fraction

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Abstract

In this article we solve a general class of sextic equations. The solution follows if we consider the \( j \)-invariant and relate it with the polynomial equation’s coefficients. The form of the solution is a relation of Rogers-Ramanujan continued fraction. The inverse technique can also be used for the evaluation of the Rogers-Ramanujan continued fraction, in which the equation is not now the depressed equation but another quite more simplified equation.

1 Introductory Definitions

We will solve the following equation
\[
\frac{b^2}{20a} + bX^3 + aX^6 = C_1X^5 : (eq)
\]
or equivalent
\[
\frac{b^2}{20a} + bX + aX^2 = C_1X^{5/3} \tag{1}
\]
using the \( j \)-invariant and the Rogers-Ramanujan continued fraction.

For \(|q| < 1\), the Rogers Ramanujan continued fraction (RRCF) (see [2],[3],[4]) is defined as
\[
R(q) := \frac{q^{1/5}}{1+ \frac{q^1}{1+ \frac{q^{2}}{1+ \frac{q^{3}}{\ddots}}}} \tag{2}
\]
From the Theory of Elliptic functions the \( j \)-invariant (see [5],[8]) is

\[
j_r := \left[ \left( \frac{\eta(\frac{1}{2} \sqrt{-r})}{\eta(\sqrt{-r})} \right)^{16} + 16 \left( \frac{\eta(\sqrt{-r})}{\eta(\frac{1}{2} \sqrt{-r})} \right) \right]^{3},
\]

where

\[
\eta(\tau) := e^{\pi i \tau / 12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau})
\]

is the Dedekind’s eta function and

\[
\tau = \frac{1 + \sqrt{-r}}{2}, \quad \tau = \sqrt{-r}, \quad r \text{ positive real.}
\]

We have also in the \( q \)-notation

\[
f(-q) := \prod_{n=1}^{\infty} (1 - q^n).
\]

In what follows we use the following known result (see Wolfram pages for ‘Rogers-Ramanujan Continued Fraction’ and [17]):

If

\[
R = R(e^{-2\pi \sqrt{r}}),
\]

then:

\[
j_r = -\frac{(R^{20} - 228R^{15} + 494R^{10} + 228R^5 + 1)^3}{R^5 (R^{10} + 11R^5 - 1)^5}
\]

From ([3],[4]) we have

\[
\frac{1}{R^5(q)} - 11 - R^5(q) = \frac{f^5(-q)}{qf^6(-q^5)}
\]

The general hypergeometric function is defined as

\[
_pF_q \left[ \{ a_1, a_2, \ldots, a_p \}, \{ b_1, b_2, \ldots, b_q \} , x \right] = \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n, \ldots, (a_p)_n x^n}{(b_1)_n(b_2)_n, \ldots, (b_q)_n n!}
\]

where \((c)_n = c(c + 1) \ldots (c + n - 1)\), hence \((1)_n = n!\).

The standard definition of the elliptic integral of the first kind (see [7],[8],[15]) is:

\[
K(x) = \int_{0}^{\pi/2} \frac{dt}{\sqrt{1 - x^2 \sin^2(t)}}
\]

\[
K(x) = \frac{\pi}{2} _2F_1 \left( \{ 1/2, 1/2 \}; \{ 1 \}; x^2 \right) = \frac{\pi}{2} _2F_1 \left( 1/2, 1/2; 1; x^2 \right)
\]

In the notation of Mathematica we have

\[
K(x) = \text{EllipticK}[x^2]
\]
The elliptic singular modulus $k = k_r$ is defined to be the solution of the equation:

$$\frac{K(\sqrt{1-k^2})}{K(k)} = \sqrt{r}. \quad (11)$$

In Mathematica’s notation

$$k = k_r = k[r] = \text{InverseEllipticNomeQ}[e^{-\pi \sqrt{r}}]^{1/2}. \quad (12)$$

The complementary modulus is given by $k'_r = \sqrt{1-k_r^2}$. (For evaluations of $k_r$ see [5],[15],[16]).

Also we call $w_r := \sqrt{k_r k_{25r}}$ noting that if one knows $w = w_r$ then (see [2]), knows $k_r$ and $k_{25r}$.

2 Theorems

**Proposition 1.** (see [2])

If $q = e^{-\pi \sqrt{r}}$ and $r$ real positive then we define

$$A = A_r := \frac{f^6(-q^2)}{q^2 f^6(-q^{10})} = R(q^2)^{-5} - 11 - R(q^2)^5 \quad (13)$$

then

$$A_r = a_{4r} = \frac{(k_r k'_r)^2}{(w_r w'_r)^2} \left( \frac{w_r + w'_r}{k_r} - \frac{w_r w'_r}{k_r k'_r} \right)^3. \quad (14)$$

**Theorem 1.**

Let $a$, $b$, $C_1$ be constants. One can solve the equation

$$\frac{b^2}{20a} + bX + aX^2 = C_1 X^{5/3}, \quad (15)$$

finding $r > 0$ such that

$$j_r = 250C_1^3a^{-2}b^{-1}. \quad (16)$$

Then (15) have solution

$$X = \frac{b}{250a}A_r = \frac{b}{250a} \frac{f(-e^{-2\pi \sqrt{r}})^6}{e^{-2\pi \sqrt{r}} f(-e^{-10\pi \sqrt{r}})^6}. \quad (17)$$

**Proof.**

For to solve the equation (15) find $r$ such that

$$j_r^{1/3} = 5 \cdot 21^{1/3}C_1 \frac{a^{-2/3}b^{1/3}}{a^{2/3}b^{1/3}} \quad (18)$$

Consider also the transformation of the constants

$$3125m = \frac{b^2}{20a}, \quad 250ml^{-1} = \frac{b^2}{250a} \left( b + \frac{b^2}{20a} \right)^{-1}$$
and

\[ ml^{-2} = \frac{b^2}{62500} \left( b + \frac{b^2}{20a} \right)^{-2}, \]

with inverse

\[ l = \frac{b(20a + b)}{20a}, \quad m = \frac{b^2}{62500a}. \]

Then

\[ X = \frac{250m}{l(l - 3125m)} x_1 = \frac{250m}{l - 3125m} x = \frac{b}{250a} x, \]

where \( x_1 \) satisfies

\[ 3125m + 250x_1ml^{-1} + x_1^2 ml^{-2} = ml^{-5/3}j^{1/3}x_1^{5/3} \]

If we set \( x_1 = lx_1 \), then it is

\[ 3125 + 250x_1 + x_1^2 = j^{1/3}x_1^{5/3} \]

or equivalently

\[ 3125 + 250A_r + A_r^2 = j^{1/3}A_r^{5/3} \]

Relation (19) is equivalent to equation (6), in view of (7). Hence from Proposition 1

\[ X = A_r = a_4 = \frac{b}{250a} \frac{f(-e^{-2\pi\sqrt{r}})^6}{f(-e^{-10\pi\sqrt{r}})^6} = \]

\[ = \frac{b}{250a} \left( \frac{k_r k_r'}{w_r} \right)^2 \left( \frac{w_r}{k_r} + \frac{w_r'}{k_r'} - \frac{w_r w_r'}{k_r k_r'} \right)^3 = \frac{b}{250a} \left( R^{-5}(q^2) - 11 - R^5(q^2) \right) \]

and the proof is complete.

The \( j \)-invariant is connected with the singular modulus from the equation

\[ j_r = \frac{256(k_r^2 + k_r'^2)\left( k_r k_r' \right)^4}{(k_r k_r')^4}. \]

We can solve (21) and express \( k_r \) in radicals to an algebraic function of \( j_r \).

The 5th degree modular equation which connects \( k_{25r} \) and \( k_r \) is (see [3]):

\[ k_r k_{25r} + k_r' k_{25r}' + 2 \cdot 4^{1/3}(k_r k_{25r} k_r' k_{25r}')^{1/3} = 1 \]

We will evaluate the root of (1) first with parametrization and second with Rogers-Ramanujan continued fraction and the Elliptic-\( K \) function.

For this it have been showed (see [19]) that if

\[ k_{25r} k_r = w_r^2 = w^2, \]

setting the following parametrization of \( w \):

\[ w = \sqrt{\frac{L(18 + L)}{6(64 + 3L)}}. \]
we get

\[
\frac{(k_{25r})^{1/2}}{w^{1/2}} = \frac{w^{1/2}}{(k_r)^{1/2}} = \frac{1}{2} \sqrt{4 + \frac{2}{3} \left( \frac{L^{1/6}}{M^{1/6}} - \frac{4M^{1/6}}{L^{1/6}} \right)^2 + \frac{1}{2} \sqrt{\frac{2}{3} \left( \frac{L^{1/6}}{M^{1/6}} - \frac{4M^{1/6}}{L^{1/6}} \right)^3}}
\]

where

\[
M = \frac{18 + L}{64 + 3L}
\]

From the above relations we get also

\[
-\frac{k_r - w}{\sqrt{k_r w}} = \frac{k_{25r} - w}{\sqrt{k_{25r} w}} = \sqrt{\frac{2}{3} \left( \frac{L^{1/6}}{M^{1/6}} - \frac{4M^{1/6}}{L^{1/6}} \right)^3}
\]

Hence we can consider the above equations as follows: Taking an arbitrary number \(L\) we construct an \(w\). Now for this \(w\) we evaluate the two numbers \(k_{25r}\) and \(k_r\). Thus when we know the \(w\), the \(k_r\) and \(k_{25r}\) are given from (24),(25),(26).

The result is: We can set a number \(L\) and from this calculate the two inverse elliptic nome's. But we don’t know the \(r\). One can see (from the definition of \(k_r\)) that the \(r\) can evaluated from equation

\[
r = \frac{K^2(\sqrt{1-k_r^2})}{K^2(k_r)}
\]

Hence we define

\[
r = k^{(-1)}(x) := \frac{K^2(\sqrt{1-x^2})}{K^2(x)}
\]

However is very difficult to evaluate the \(r\) in a closed form, such as roots of polynomials or else when a number \(x\) is given. Some numerical evaluations indicate us that even if \(x\) are algebraic numbers, (not trivial as with \(k^{(-1)}(2^{-1/2}) = 1\) or the cases \(x = k_r, r = 1, 2, 3, \ldots\)) the \(r\) are not rational and may even not algebraics.

The algebraic representation of \(X\)

We know that (see [2]):

\[
X = X(L) = \frac{b}{250a} \frac{x_L^2(1-x_L^2)}{(w_L w'_L)^2} \left( \frac{w_L}{x_L} + \frac{w'_L}{\sqrt{1-x_L^2}} - \frac{w_L w'_L}{x_L \sqrt{1-x_L^2}} \right)^3
\]

where \(x_L = k_r\) is the singular modulus which corresponds to some \(L\).

\[
C_1 = \frac{a^{2/3}b^{1/3}}{5 \cdot 2^{1/3} \cdot 3^{1/3}}.
\]
The procedure is to select a number \( L \) and from (24),(25) evaluate \( w_L, x_L \) and

\[
\sqrt{\frac{1 - \frac{w_L^4}{x_L^2}}{1 - x_L^2}}. \quad (31)
\]

The solution \( X = X(L) \) of (1) is (29) and for this \( L \) holds

\[
r_L = k^{(-1)}(x_L) \quad (32)
\]

\[
X = \frac{b}{250a} \frac{f \left( -e^{-2\pi \sqrt{k^{(-1)}(x_L)}} \right)^6}{e^{-2\pi \sqrt{k^{(-1)}(x_L)}} f \left( -e^{-10\pi \sqrt{k^{(-1)}(x_L)}} \right)^6} \quad (33)
\]

\[
j_{rL} = 250 C_1^3 a^{-2} b^{-1} \quad (34)
\]

\[
j_{rL} = \frac{256(x_L^2 + (1-x_L^2)^2)^3}{x_L^4 (1-x_L^2)^2} = 250 C_1^3 a^2 b \quad (35)
\]

Hence we get the next:

**Theorem 2.**

One can find parametric solutions of (1) if for a given \( L \) construct the \( w_L, x_L \) and the complementary \( w_L' \) (these values are given from (23),(24),(25),(31)). Also \( x_L' = \sqrt{1 - x_L^2} \). The \( C_1 \) must be

\[
C_1 = 3 \sqrt{a^2 b j_{rL}} \quad (36)
\]

The solution \( X = X_L \) is given from

\[
X = X(L) = \frac{b}{250a} \frac{x_L^2 (1-x_L^2)}{(w_L w_L')^2} \left( \frac{w_L}{x_L} + \frac{w_L'}{\sqrt{1-x_L^2}} - \frac{w_L w_L'}{x_L \sqrt{1-x_L^2}} \right)^3 \quad (37)
\]

**Note.**

i) The above solution (37) works for parametric solutions (setting a \( L \), as also for solutions which we know \( r, k_r \) and \( k_{25r} \). (For a related method on solving the quintic see Wolfram pages 'Quintic Equation')

ii) In [16] it have been shown that when one knows for some \( r_0 \) the \( k_{r_0} \) and \( k_{25r_0} \) then can evaluate any \( k_{25^nr_0} \) in radicals closed form for all \( n \) positive integers. But in general the values \( k_r \) and \( k_{25r} \) can given from tables or with a simple PC (see [4],[5],[13],[15],[17]).

The inverse functions method
From the analysis in [2], the solution $X$ of (1) can reduced with inverse functions as follows: Consider the function

$$U(x) = \frac{256(x^2 + (1 - x^2)^2)^3}{x^4(1 - x^2)^2}, \quad (38)$$

The equation $U(x) = t$ have known solution with respect to $x$, which we will call $x = U^{(-1)}(t)$. Hence

$$\frac{256(k_r^2 + (1 - k_r^2)^2)^3}{k_r^4(1 - k_r^2)^2} = 250 \frac{C_1^3}{a^2b}, \quad (39)$$

or

$$U(k_r) = 250 \frac{C_1^3}{a^2b},$$

$$k_r = U^{(-1)}\left(250 \frac{C_1^3}{a^2b}\right)$$

or

$$r = k^{(-1)} \left(U^{(-1)}\left(250 \frac{C_1^3}{a^2b}\right)\right)$$

The function $k^{(-1)}(x)$ is that of (28).

**Theorem 3.**

The equation (1) have solution

$$X = \frac{b}{250a} \left[R\left(e^{-2\pi\sqrt{k^{(-1)}(\alpha)}}\right)^{-5} - 11 - R\left(e^{-2\pi\sqrt{k^{(-1)}(\alpha)}}\right)^5\right] \quad (40)$$

where

$$\alpha = U^{(-1)}\left(250 \frac{C_1^3}{a^2b}\right) \quad (41)$$

**Notes.**

1) Observe here that we don’t need the value of $w$ and the class invariant $j$.
2) From [10] we have

$$R(e^{-x}) = e^{-x/5} \frac{\vartheta_4(3ix/4, e^{-5x/2})}{\vartheta_4(ix/4, e^{-5x/2})}, \forall x > 0$$

Hence the solution can expressed also in theta functions. That is if $\alpha = k_r$, $r = 1, 2, 3, \ldots$ then the solution of (1) reduced to that of evaluation of Rogers-Ramanujan continued fraction $R(q)$ with $q = e^{-\pi \sqrt{r}}$. In view of [10] we have

$$X = \frac{b}{250a} \left[e^{2\pi\sqrt{r}} \left(\frac{\vartheta_4\left(3i\pi \sqrt{r}/2, e^{-5\pi \sqrt{r}}\right)}{\vartheta_4\left(i\pi \sqrt{r}/2, e^{-5\pi \sqrt{r}}\right)}\right)^{-5} - 11 - e^{-2\pi\sqrt{r}} \left(\frac{\vartheta_4\left(3i\pi \sqrt{r}/2, e^{-5\pi \sqrt{r}}\right)}{\vartheta_4\left(i\pi \sqrt{r}/2, e^{-5\pi \sqrt{r}}\right)}\right)^5\right]$$
Example 1.
The equation
\[ X^2 + 3X + \frac{9}{20} = \frac{26}{5\sqrt{3}}X^{5/3} \]
have
\[ \alpha = U^{(-1)} \left( \frac{35152}{9} \right) = \frac{\sqrt{3}}{2} \]
hence a solution is
\[ X = \frac{3}{250} \left( R \left( e^{-2\pi\sqrt{r}} \right)^{-5} - 11 - R \left( e^{-2\pi\sqrt{r}} \right)^5 \right) \]
where
\[ r = \frac{K \left( \frac{1}{3} \right)^2}{K \left( \frac{\sqrt{3}}{2} \right)^2} \]
For this \( r \) the \( X \) is a solution.

Continuing one can set to
\[ X^5 = \frac{b^2}{20aC_1} + \frac{b}{C_1}X^3 + \frac{a}{C_1}X^6 \] (42)
any value \( X = X_0 \) and \( C_1 = 1 \) then evaluate
\[ a = \frac{-5bX_0^3 + 5X_0^5 + \sqrt{5}\sqrt{4b^2X_0^6 - 10bX_0^8 + 5X_0^{10}}}{10X_0^6} \] (43)
equation (42) holds always and we get that
\[ R \left( e^{-2\pi\sqrt{r}} \right)^{-5} - 11 - R \left( e^{-2\pi\sqrt{r}} \right)^5 = \]
\[ = 25 \left( -5b + 5X_0^2 + \sqrt{5}\sqrt{4b^2 - 10bX_0^2 + 5X_0^4} \right) \frac{bX_0^2}{b \left( -5b + 5X_0^2 + \sqrt{5}\sqrt{4b^2 - 10bX_0^2 + 5X_0^4} \right)^2} \]
where \( j_r = 250a^{-2}b^{-1} \).
\[ j_r = \frac{25000X_0^6}{b \left( -5b + 5X_0^2 + \sqrt{5}\sqrt{4b^2 - 10bX_0^2 + 5X_0^4} \right)^2} \] (44)
The result is the following parametrized evaluation of the Rogers-Ramanujan continued fraction

Theorem 4.
\[ A_r = R \left( e^{-2\pi\sqrt{r}} \right)^{-5} - 11 - R \left( e^{-2\pi\sqrt{r}} \right)^5 = \]
\[ \frac{25 \left( -5b + 5t^2 + \sqrt{5\sqrt{4b^2 - 10bt^2 + 5t^4}} \right)}{b} \]  

(45)

and

\[ j_r = \frac{25000t^6}{b \left( -5b + 5t^2 + \sqrt{5\sqrt{4b^2 - 10bt^2 + 5t^4}} \right)^2} \]  

(46)

Corollary.

If \( \sqrt[3]{A^2 j_r} = \text{rational} \)

then \( A_r \) is of the form

\[ A_r = \frac{A + B\sqrt{D}}{C} \]

where \( A, B, C, D \) rationals

**Theorem 5.**

If for a certain \( r > 0 \) we know the value of \( R(e^{-\pi \sqrt{r}}) \) in radicals, then we can evaluate both \( k_r \) and \( k_{25r} \) and the opposite.

**Proof.**

Suppose we know for a certain \( r > 0 \) the value of \( R(e^{-\pi \sqrt{r}}) \), (the correspondence between \( R(e^{-\pi \sqrt{r}}) \) and \( R(e^{-2\pi \sqrt{r}}) \) is given by (96) below or see [13]). Then from (6) we know the value of \( j_r \) and from (21) we know \( k_r \). Let also \( q = e^{-\pi \sqrt{r}}, \)

\( r > 0 \) and \( v_r = R(q) \), then it have been proved by Ramanujan that

\[ v_{r/25}^5 = v_r \frac{1 - 2v_r + 4v_r^2 - 3v_r^3 + v_r^4}{1 + 3v_r + 2v_r^2 + v_r^3 + v_r^4} \]

Hence we can get the value of \( R(e^{-\pi \sqrt{r}/5}) \). Hence again form (6) we find \( j_{r/25} \) and from (21) the value of \( k_{r/25} \). But from relation (53) below knowing \( k_r \) and \( k_{r/25} \) we can evaluate all \( k_{25n_r}, n = 1, 2, \ldots \) and consequently \( k_{25r} \) as a special case. The opposite follow from Proposition 1.

**Theorem 6.**

The solution \( U_0 \) of the equation

\[ U_0 = j_r^{1/3} \left( 125 - \sqrt{12500 + U_0} \right)^{5/3} \]  

(47)

is

\[ U_0 = U(j_r) = \sum_{n=1}^{\infty} \frac{j_r^{n/3}}{n!} \left[ \frac{d^{n-1}}{da^{n-1}} \left( 125 - \sqrt{12500 + a} \right)^{5n/3} \right]_{a=0} \]

If \( X = x_0 \) is root of \( U_0 = X^2 + 250X + 3125 \)

then

\[ 3125 + 250x_0 + x_0^2 = j_r^{1/3} (-1)^{1/3} x_0^{5/3} \]  

(47a)
and
\[ x_0 = X = X_r = A_r = R(e^{-2\pi\sqrt{r}})^{-5} - 11 - R(e^{-2\pi\sqrt{r}})^5 \quad (47b) \]

**Proof.**
Consider (1), then make the change of variable \( U_0 = X^2 + 250X + 3125 \), we arrive to (47). The Legendre inversion theorem states that the solution of \( y = af(y) \) (see [7] pg.132-133) is
\[
y = \sum_{n=1}^{\infty} \frac{a^n}{n!} \left[ \frac{d^{n-1}}{dx^{n-1}} f(x)^n \right]_{x=0}
\]
This theorem works for \( j_r \) small, for example with \( j_1 = 1728 \) it converges very slowly but for \( r \) such that \( j_r = 800 \), \((r\text{-complex})\) we get numerical evaluations and hence also theoretical.

**Theorem 7.**
If
\[
c_n := \left[ \frac{d^{n-1}}{da^{n-1}} \left( 125 - \sqrt{12500 + a} \right)^{5n/3} \right]_{a=0}
\]
then
\[
c_n = \frac{5^6}{3} (-1)^{n+1} n \cdot 10^{-5n/3} \binom{5n}{n+1} \binom{2n+3}{3} \Gamma(2+2n/3)
\]

**Proof.**
Recall a theorem of Euler (see [18] pg.306-307). If the root of
\[ aqx^p + x^q = 1 \]
is \( x \), then
\[
x^n = \frac{n}{q} \sum_{k=0}^{\infty} \frac{\Gamma((n+pk)/q)(-qa)^k}{\Gamma((n+pk)/q-k+1)!}
\]
Hence from the fact that
\[
w = \frac{250(125 - \sqrt{12500 + x})}{3125 - x}
\]
is solution of
\[ acb^{-2}w^2 + w = 1 \; , \; a = 1 \; , \; b = -250 \; , \; c = -3125 + x \]
we get
\[
(-125 + \sqrt{12500 + x})^n = \frac{n}{250^n} \sum_{k=0}^{\infty} \frac{\Gamma(n+2k)(-1)^k}{\Gamma(n+k+1)62500^k k!} (-3125 + x)^{k+n} \quad (48)
\]
Using the formula
\[
f^{(\nu)}(x_0) = \sum_{n=0}^{\infty} \frac{f^{(\nu+n)}(x_0)}{n!} (-x_0)^n
\]
The result follows.

Theorem 7 is for numerical evaluations since the hypergeometric series can more easily computed than the \((n - 1)\)th derivative of the \(5n/3\) power of \(125 - \sqrt{12500 + x}\).

**Corollary.**
For every 'suitable' value of \(x_0\) such that \(X_r = x_0\), \(X_r\) is of (47b) exists a \(r\) solution of (a) such that

\[
X_r^2 + 250X_r + 3125 = 3^{-1} \cdot 5^6 \sum_{n=1}^{\infty} (-1)^{n+1} n \frac{\Gamma(5n/3)}{\Gamma(2 + 2n/3)} F_1 \left[ \frac{5n}{6}, \frac{5n + 3}{6}; \frac{2(n + 3)}{3}; \frac{1}{5} \right] \frac{(10^{-5}j_r)^{n/3}}{n!}
\]

**Example 2.**
For \(X_r = x_0 = -12\), we have \(r = -0.186710441\ldots - i0.251574161\ldots\) and

\[
X_r^2 + 250X_r + 3125 = 269 = \sum_{n=1}^{\infty} \frac{(-j_r)^{n/3}}{n!} \left[ \frac{d^{n-1}}{dz^{n-1}} \left( 125 - \sqrt{12500 + z} \right)^{5n/3} \right]_{z=0}
\]

**Example 3.**
Consider the equation

\[
X^2 + 250X + 3125 = 2(-1)^{1/3}10^{2/3}X^{5/3}
\]

Then clearly \(j = j_r = 800\) and a solution is

\[
X = x_0 = -125 + \sqrt{12500 + \sum_{n=1}^{\infty} \frac{(2\sqrt{100})^n}{n!} \left[ \frac{d^{n-1}}{dz^{n-1}} \left( 125 - \sqrt{12500 + z} \right)^{5n/3} \right]_{z=0}}
\]

**3 Applications**

**Example 4.**
Set \(L = 1/3\) then

\[
w_N = w_1(L) = w_1 \left( \frac{1}{3} \right) = \frac{1}{3} \sqrt{\frac{11}{78}}
\]

and

\[
k_N = x_L = x_1(L) = x_1 \left( \frac{1}{3} \right) = \frac{1}{3} \sqrt{\frac{11}{78}}
\]

\[
= \left( \frac{-4(11)^{1/6} + (11)^{1/6}}{\sqrt{6}} \right)^2 + \frac{1}{2} \sqrt{4 + \frac{2}{3} \left( -4 \left( \frac{11}{17} \right)^{1/6} + \left( \frac{12}{17} \right)^{1/6} \right)^2}
\]
and

\[ k_{25N} = x_2(L) = x_2\left(\frac{1}{3}\right) = \]

\[ = \frac{1}{3} \sqrt{\frac{11}{78}} \left( -4\left(\frac{11}{13}\right)^{1/6} + \left(\frac{13}{11}\right)^{1/6} \right) \sqrt{\frac{3}{4} + 2 \left( -4 \left(\frac{11}{13}\right)^{1/6} + \left(\frac{13}{11}\right)^{1/6} \right)^2} \]

where the \( N \) is given by

\[ N = r_L = r_{1/3} = \frac{K^2 \left( \sqrt{1 - x_1 \left(\frac{1}{3}\right)^2} \right)}{K^2 \left( x_1 \left(\frac{1}{3}\right) \right)} \]

From the value of \( x_L \) we obtain \( j_r \) and hence the corresponding \( C_1 \) in radicals-closed form and hence \( X = X_L \) from (37) and (31). The numbers \( a, b \) take arbitrary values.

We note here that in future application of this method one must tabulate values of \( (r, w_r) \) and not \( j_r \) or \( k_r \) which follow from these of \( w_r \). This can be done in some cases using the Main Theorem in [16] and the solution (37) of Theorem 2 of the present paper.

Form [16] we have if

\[ Q(x) = \frac{\left( -1 - \alpha^y + e^{\alpha y} \right)^5 \left( e^{\alpha y} - e^{\beta y} - 2e^{\gamma y} + 3e^{\delta y} + 5e^y + 2e^{\varepsilon y} + e^{\zeta y} + e^{\eta y} \right)}{\left( e^{\alpha y} + 2e^{\beta y} + 3e^{\gamma y} + 5e^y + 3e^{\delta y} + 2e^{\varepsilon y} + e^{\zeta y} + e^{\eta y} \right)} \]  

\[ y = \text{arcsinh} \left( \frac{11 + x}{2} \right) \]

\[ Y = U_0(X) = \sqrt{-\frac{5}{3X^2} + \frac{25}{3X^2h(X)} + X^4 \frac{h(X)}{3X^2}} + \frac{h(X)}{3X^2} \]  

\[ h(x) = \left( -125 - 9x^6 + 3\sqrt{3} \sqrt{-125x^6 - 22x^{12} - x^{18}} \right)^{1/3} \]

\[ U_1(Y) = X = \sqrt{-\frac{1}{2Y^2} + \frac{Y^4}{2} + \frac{\sqrt{1 + 18Y^6} + Y^{12}}{2Y^2}} \]

and

\[ P(x) = P[x] = U_0[Q^{1/6}[U_1[x]]] \] and \( P^{(n)}(x) = (P \circ \ldots \circ P)(x) \)  

then

\[ k_{25n} = \sqrt{1/2 - 1/2 \left( 1 - 4 \left( k_{n}k_{25} \right)^2 \prod_{j=1}^{n} k_{25}^{(j)} \right) \left[ \frac{k_{n}k_{25}^{(j)}}{k_{n}k_{25}^{(j)} / k_{25}} \right]^{24}} \]
Example 5.

\[ k_{1/5} = \sqrt[5]{\frac{9 + 4\sqrt{5} + 2\sqrt{38} + 17\sqrt{5}}{18 + 8\sqrt{5}}} \]

\[ k_5 = \sqrt[5]{\frac{9 + 4\sqrt{5} - 2\sqrt{38} + 17\sqrt{5}}{18 + 8\sqrt{5}}} \]

\[ k_{125} = \sqrt[5]{\frac{1}{2} - \frac{1}{2}\sqrt{1 - (9 - 4\sqrt{5})P[1]^2}} \]  \hspace{1cm} (54)

Example 6.

It is

\[ k_1 = \frac{1}{\sqrt{2}} \]

\[ k_{25} = \frac{1}{\sqrt{2}(51841 + 23184\sqrt{5} + 12\sqrt{37325880} + 16692641\sqrt{5})} \]

Hence

\[ k_{625}k'_{625} = \frac{1}{2(161 + 72\sqrt{5})}P\left[161 - 72\sqrt{5}\right] \]

and hence

\[ k_{625} = \sqrt[5]{\frac{1}{2} - \frac{1}{2}\sqrt{1 - \left(\frac{P\left[161 - 72\sqrt{5}\right]}{161 + 72\sqrt{5}}\right)^2}} \] \hspace{1cm} (55)

By this way we can evaluate every \( k_r \) which is \( r = 4^l\,9^m\,25^n\,r_0 \) when \( k_{r_0} \) and \( k_{r_0/25} \) are known, \( l, m, n \in \mathbb{N} \).

Note. In the case that

\[ \left(\frac{L}{M}\right)^{1/6} = A = \frac{f}{g} = 3^p\frac{2p + 1}{2h + 1}, \] \hspace{1cm} (56)

where \( f, g \) positive integers, with \( f < g \) and \( p, h \neq 0 \pmod{4} \), \( (a \in \mathbb{Z} - \{-1, 0, 1\}) \),

we can find \( w \) from a given \( x = k_r \) which is of the form

\[ x = k_r = \frac{t_1\sqrt{t_2}}{(t_3 + \sqrt{t_4})^2} \]

where \( t_i, i = 1, 2, 3, 4 \) rationals.

In view of (25) and the action of the command recognize, (which is needed to put number \( x \) into his form) the output will be an octic equation with step 2 containing nested square roots:

\[ \text{<< NumberTheory\Recognize} \]
\[ \text{Solve}[\text{Reduce}[\text{N}[x,1000],16,v] == 0,v] \]

The smallest root it will be

\[ \sqrt{D} = \sqrt{4096 + 88 \left( \frac{3^a 2p + 1}{2h + 1} \right)^6 + \left( \frac{3^a 2p + 1}{2h + 1} \right)^{12}} \]

One can see that for these \( x \)'s the \( f \) and \( g \) are given from the Diofantine equation

\[ 9D = g^{12} \left( 4096 + 88 \frac{f^6}{g^6} + \frac{f^{12}}{g^{12}} \right) \]  \hspace{1cm} (57)

hence the number \( A \) will be known and

\[ w^2 = \frac{4096 - 20A^6 + A^{12} + (-64 + A^6) \sqrt{4096 + 88A^6 + A^{12}}}{108A^6} \]  \hspace{1cm} (58)

Hence we have the value of \( X \) in radicals.

If for example

\[ x = k_r = \frac{-37754085\sqrt{3} + 3\sqrt{476791023769787}}{77\sqrt{2}(-435 + \sqrt{224799})^2} \]

then for all \( C_1, a, b \) such that

\[ j_r = \frac{256(x^2 + (1 - x^2)^2)^3}{x^4(1 - x^2)^2} = 250 \frac{C_1^3}{a^2b} \]

with Mathematica and the package 'Recognize' we evaluate

\[ \text{Solve}[\text{Recognize}[\text{N}[x,1000],16,v] == 0,v] \]

which gives the value of \( x \) in the desired form. The solution that corresponds to \( x \) have smallest square root

\[ \sqrt{D} = \sqrt{1430373071309361} \]

the command 'Reduce' give us the \( f \) and \( g \)

\[ \text{Reduce}[9D == (4096 + 88(f/g)^6 + (f/g)^{12})g^{12}, \{f, g\}, \text{Integers}] \]

Hence we get the values \( f = 7, g = 11 \) and \( w \). The solution (29) is

\[ X = \frac{A}{35153041^3} [5579801448 - 11724990\sqrt{224799} + \]

\[ + \sqrt{6362897839 \left( 9487950991 - 20011160\sqrt{224799} \right)^3} \]

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where
\[
A = A_1 + B_1 - \frac{1}{2} \sqrt{A_2 + B_2 \sqrt{224799}}
\]

\[
A_1 = \frac{93573266991461291403517623659291588}{29148873138738228269392700625}
\]
\[
B_1 = \frac{572443990137}{2336301255738105953557025}
\]
\[
A_2 = \frac{7047382179949155201445598248892391809778079434882871423635278599557376}{8465680652825501138737121862040714266412150030875390625}
\]
\[
B_2 = \frac{125788720856386739402463890994615000261088619948544424950380544}{72342001299127714890367919848480814326642158367890625}
\]

**Theorem 8.**
When \((L/M)^{1/6}\) is rational, we can always find \(k_{25r}\) from \(kr\).

**Example 7.**
Set \(a = (kr, k'_r)^2, b = (ww')\), then equation (1) have solution

\[
X = \frac{m^3_5}{250}
\]

where \(m_5\) is the multiplier (see [3]).

Hence

\[
3125\frac{(ww')^4}{(kr, k'_r)^2} + 250(wu')^2m_5^3 + (kr, k'_r)^2m_5^6 = (kr, k'_r)^{4/3}(wu')^{2/3}j_{r^{1/3}}m_5^5
\]

**Example 8.**
We will find a solution of the equation

\[
\frac{3125}{16} + 125X + 4X^2 = 132X^{5/3}
\]

in radicals using Theorem 3.

**Solution.**
It is \(a = 4, b = 125\), and we have to solve \(j_r = 287496\) or equivalently \(r = 4\).

Hence a solution of (59) is:

\[
X = \frac{125e^{4x}}{250 \cdot 4 f(-e^{-20\pi})^6} \cdot \frac{1}{8} \left( R(e^{-4\pi})^{-5} - 11 - R(e^{-4\pi})^5 \right)
\]

The exact root in radicals can be found but is very large and complicated with our method. We give a way how one can obtain it:

It is known that

\[
R(e^{-2\pi}) = -\frac{1 + \sqrt{5}}{2} + \sqrt{\frac{5 + \sqrt{5}}{2}}
\]
But from the duplication formula (see [4],[13]):
If \( u = R(q) \) and \( \nu = R(q^2) \), then
\[
\frac{\nu - u^2}{\nu + u^2} = u\nu.
\]
(62)
Hence we find the value of \( R(e^{-4\pi}) \) in radicals and hence the solution of (59) using (60),(61),(62).
The root using the program Mathematica is
\[
X = \frac{143375}{16} + \frac{64125\sqrt{5}}{16} + \frac{1}{2} \sqrt{\frac{20553203125}{32} + \frac{9191671875\sqrt{5}}{32}}.
\]
In this case it is more convenient to use Mathematica’s command Solve. But in other cases these solutions can not found.

From the above result we have shown that
\[
\frac{1}{8} \left( R(e^{-4\pi}) - 5 - 11 - R(e^{-4\pi})^5 \right) = \frac{143375}{16} + \frac{64125\sqrt{5}}{16} + \frac{1}{2} \sqrt{\frac{20553203125}{32} + \frac{9191671875\sqrt{5}}{32}}.
\]
One can see that if we set
\[
Y_\tau := \frac{b}{250a} \left( R(e^{2\pi i \tau}) - 5 - 11 - R(e^{2\pi i \tau})^5 \right) \quad : (a)
\]
Then if
\[
\tau = \frac{1 + \sqrt{-r}}{2} \text{ or } \tau = \sqrt{-r},
\]
where \( r \) positive integer in some cases we can evaluate \( Y_\tau \) solving directly the equation (1), with parameters \( a = 4 \), \( b = 125 \) and \( C_1 \) depended on \( j_\tau \).
Some examples are
\[
\frac{1}{16} \left[ R\left(e^{i(1-\sqrt{17})}\right)^5 - 11 - R\left(e^{i(1-\sqrt{17})}\right)^5 \right] = \frac{-125}{16} \left[ 5541103 + 1343914\sqrt{17} + \sqrt{61407604829690 + 14893531819350\sqrt{17}} + 4\sqrt{\frac{1}{61407604829690 + 14893531819350\sqrt{17}}(94272348104055803848937570 + 22864402871059934148609270\sqrt{17} + 2063164169063100077\sqrt{170}(61407604829690 + 1489353181935\sqrt{17}) + 850664379203654023\sqrt{61407604829690 + 14893531819350\sqrt{17}}) \right].
\]
(63)
\[ Y \sqrt{-1/5} = \frac{5\sqrt{5}}{8}. \quad (64) \]
\[ Y \sqrt{-2/5} = \frac{5}{8} \left( 5 + 2\sqrt{5} \right). \quad (65) \]
\[ Y \sqrt{-3/5} = \frac{5}{16} \left( 25 + 11\sqrt{5} \right). \quad (66) \]
\[ Y \sqrt{-4/5} = \frac{5}{16} \left( 25 + 13\sqrt{5} + 5\sqrt{58 + 26\sqrt{5}} \right). \quad (67) \]
\[ Y \sqrt{-5/5} = \frac{125}{8} \left( 2 + \sqrt{5} \right). \quad (68) \]
\[ Y \sqrt{-6/5} = \frac{5}{8} \left( 50 + 35\sqrt{2} + 3\sqrt{5 \left( 99 + 70\sqrt{2} \right)} \right). \quad (69) \]
\[ Y \sqrt{-7/5} = \frac{5}{8} \left( 225 + 104\sqrt{5} + 10\sqrt{1047 + 468\sqrt{5}} \right). \quad (70) \]
\[ Y \sqrt{-8/5} = \frac{5}{8} \left( 1690 + 975\sqrt{3} + 29\sqrt{6755 + 3900\sqrt{3}} \right). \quad (71) \]
\[ Y \sqrt{-9/5} = \frac{5}{8} \left( 1850 + 585\sqrt{10} + 7\sqrt{5 \left( 27379 + 8658\sqrt{10} \right)} \right). \quad (72) \]
\[ Y \sqrt{-10/5} = \frac{5}{8} \left( 5360 + 585\sqrt{85} + 4\sqrt{3613670 + 391950\sqrt{85}} \right). \quad (73) \]

We describe the method bellow.

For some \( r \) positive rational we find the value of \( j_{r/5} \); this can be done with the command 'Recognize' of the program Mathematica (if \( j_{r/5} \) is root of a small degree algebraic polynomial equation). Then we find \( C_1 \) (from (16)) and for the values \( a = 4 \), \( b = 125 \) there will be

\[ Y_r = \text{root of equation (1)}. \]

In many cases of such \( r \), equation (1) can solved in radicals with Mathematica (we have not find the reason yet), but still in others not. Hence we get relations like (63)−(73).

### 4 More Theorems and Results

**Theorem 9.** (Conjecture)

For every positive real \( r \), we have

\[ Y \sqrt{-r/5} Y \sqrt{-r^{-1}/5} = \frac{125}{64}. \quad (74) \]
If $l$, $m$, $t$ and $d$ are integers and
\[ Y \sqrt{\frac{-r}{5}} = \frac{l + m\sqrt{d}}{t} \]  
then
\[ l^2 - m^2d = t^2 \frac{125}{64} \]  
(76)

In general we conjecture that

**Theorem 10.** (Conjecture)

If $r = a_1/b_1$ with $a_1, b_1 \in \mathbb{N}$ and $\text{GCD}(a_1, 5) = 1$, $\text{GCD}(b_1, 5) = 1$ then
\[ \text{deg} \left( Y \sqrt{\frac{-r}{5}} \right) = \text{deg} \left( j \sqrt{\frac{-r}{5}} \right) \]  
(77)

For example if $\text{deg} \left( Y \sqrt{\frac{-r}{5}} \right) = 4$, then
\[ Y \sqrt{\frac{-r}{5}} = A + B\sqrt{D} \]  
(78)

where $\text{deg}(A) = \text{deg}(D) = 2$ and
\[ A^2 - B^2D = \frac{125}{64}U \]  
(79)

where $\text{deg}(U) = 2$ or $U = 1$. If $U \neq 1$ then $U = l + m\sqrt{d}$ and also if $j \sqrt{\frac{-r}{5}}$ have smallest nested square root $\sqrt{d}$, then $UU^* = l^2 - m^2d = 1$. The symbol $*$ denotes the algebraic conjugate.

Hence for example if $r = 6$ then $d = 2$ and
\[ j \sqrt{\frac{-6}{5}} = 8640\left[25551735275 - 18067805280\sqrt{2} - 
-196\sqrt{10 \left(339905814008707 - 2403497060447490\sqrt{2}\right)}\right] \]

then $U = l + m\sqrt{2}$ with
\[ l^2 - 2m^2 = 1. \]

We solve the above Pell’s equation. The solution we looking for, taking the smallest to higher order solutions, for this example with $r = 6$ is $l_1 = 99$ and $m_1 = 70$. Hence $A^2 - B^2D = \frac{125}{64}(99 + 70\sqrt{2})$.

Now we assume that $A = k_1 + l_1\sqrt{d}$, again with $d = 2$ and $D = k_2 + l_2\sqrt{d}$, etc...

We proceed solving Pell’s equations.
$j \sqrt{-r/5}$ is $\sqrt{d}$ then we can evaluate the Rogers-Ramanujan continued fraction with integer parameters.

i) In the case $\deg \left(Y \sqrt{-r/5}\right) = 2$ then

$$Y \sqrt{-r/5} = \frac{l + m \sqrt{d}}{t}$$  \hspace{1cm} (80)

where

$$t^2 - m^2 d = 1 \text{ and } l, m, d \in \mathbb{N}$$  \hspace{1cm} (81)

ii) In the case $\deg \left(Y \sqrt{-r/5}\right) = 4$ we have

a) If $U \neq \frac{125}{64}$, then

$$Y \sqrt{-r/5} = \frac{5}{8} \sqrt{\left(a_0 + \sqrt{-1 + a_0^2}\right) \left(\sqrt{5 + p} - \sqrt{p}\right)}$$  \hspace{1cm} (82)

where

$$Y \sqrt{-r/5} \cdot Y^* \sqrt{-r/5} = \frac{125}{64} \left(a_0 + \sqrt{a_0^2 - 1}\right),$$  \hspace{1cm} (83)

with $a_0$ positive integer, is solution of $t^2 - m^2 d = 1$. Hence $l = a_0$ and $m = d^{-1/2} \sqrt{a_0^2 - 1}$ is positive integer. The parameter $p$ is positive rational can be found from the numerical value of $Y \sqrt{-r/5}$.

b) If $U = \frac{125}{64}$, then

$$Y \sqrt{-r/5} = A + \frac{1}{8} \sqrt{-125 + 64 A^2},$$  \hspace{1cm} (84)

where we set $A = k + l \sqrt{d}$. Then a starting point for the evaluation of the integers $k$, $l$ will be the relation

$$l^2 = \frac{(A - k)^2}{d} = \text{square of integer}$$  \hspace{1cm} (85)

iii) If $\deg \left(Y \sqrt{-r_d/4 - 15 - 1}\right) = 4$, then we can evaluate $Y \sqrt{-r_d/4 - 15 - 1}$.

It holds $\deg \left(Y \sqrt{-r_d/4 - 15 - 1}\right) = 8$, the minimal polynomial of $Y \sqrt{-r_d/4 - 15 - 1}/Y \sqrt{-r_d/4 - 15 - 1}$ is of degree 4 or 8 and symmetric. Hence it can be reduced in at most 4th degree polynomial, hence it is solvable. Thus it remains the evaluation of $Y \sqrt{-r_d/4 - 15 - 1}$, which can be done with the help of step (ii).

$$Y \sqrt{-r_d/4 - 15 - 1} = \frac{5}{8} \sqrt{a_0 + \sqrt{-1 + a_0^2} \left(\sqrt{p + 5} - \sqrt{p}\right) 2^{-1} \left(\sqrt{x + 4} - \sqrt{x}\right)}$$  \hspace{1cm} (86)

where $x = a_1 + b_1 \sqrt{d} + c \sqrt{a_2 + b_2 \sqrt{d}}$, $a_1$, $b_1$, $a_2$, $b_2$, $c$ integers and

$$Y \sqrt{-r_d/4 - 15 - 1} = \frac{5}{8} \sqrt{a_0 + \sqrt{-1 + a_0^2} \left(\sqrt{p + 5} - \sqrt{p}\right)}$$
Example 9.
For \( r = 68 = 4 \cdot 17 \) and from (73) we have \( d = 85 \)
\[
x = a_1 + b_1 \sqrt{85} + c \sqrt{a_2 + b_2 \sqrt{85}}
\]
\[
Y \sqrt{-68/5} / Y \sqrt{-17/5} = 2^{-1} (\sqrt{x} + 4 - \sqrt{x})
\]
a_1 = 2891581250, b_1 = 313636050, c = 12960
a_2 = 99557521554, b_2 = 10798529365

hence
\[
Y \sqrt{-68/5} = Y \sqrt{-17/5} 2^{-1} (\sqrt{x} + 4 - \sqrt{x}) = \\
\frac{5}{16} \left( 5360 + 585 \sqrt{5} + 4 \sqrt{3613670 + 391950 \sqrt{85}} \right) (\sqrt{x} + 4 - \sqrt{x})
\]

Theorem 12.
If \( r = a_1/b_1 \) with \( \deg(j_{r/5}) = \nu \leq 4 \), then \( \deg(A_{r/5}) = \nu \) and equation (1) (with \( a, b \) rationals) can solved in radicals.

Application.
If \( r = 3/4 \) then \( \deg(j_{3/20}) = 4 \) and \( A_{r/5} \) is solution of
\[
15625 - 2112500v + 443375v^2 - 16900v^3 + v^4 = 0
\]
hence
\[
A_{3/20} = R \left( e^{-\pi \sqrt{3/5}} \right)^5 - 11 - R \left( e^{-\pi \sqrt{3/5}} \right)^5 = \\
= \frac{5}{2} \left( 1690 - 975\sqrt{3} + 29\sqrt{6755} - 3900\sqrt{3} \right)
\]

Theorem 13.
If \( Q(x) := x^5 \) then
\[
\frac{1}{j_{3/10}^{1/3}} \left[ R \left( e^{2\pi i r} \right)^5 - 11 - R \left( e^{2\pi i r} \right)^5 \right]^{1/3} = \sqrt[3]{\frac{-125}{j_r} + \frac{12500}{j_r^2} + Q \left( \sqrt[5]{\frac{-125}{j_r} + \ldots} \right)}
\]

Proof.
Equation (1) for \( a = 1, b = 250j_r^{-1}, C_1 = 1 \) can be written in the form
\[
(X^5 - a_1)^2 - b_1 = X^5 + c_1,
\]
where \( a_1 = -125j_r^{-1}, b_1 = 12500j_r^{-2}, c_1 = 0 \)
Hence \( Y_r \) we can be expressed in nested periodical functions. This completes the proof.
Example 10.
If
\[ C_1^3 = 32 a^2 b \]
then
\[ X = \frac{b}{250a} \left( R(e^{-2\pi \sqrt{2}})^{-5} - 11 - R(e^{-2\pi \sqrt{2}})^5 \right) \]

Equation (1) and the Derivative of Rogers-Ramanujan Continued fraction

From [11] it is known that if
\[ N(q) = q^{5/6} f(-q)^{-4} \frac{R'(q)}{R(q)} \tag{89} \]
and \( N(q^2) = u(q) = u, N(q^3) = h(q) = h \) and \( N(q) = v(q) = v \), then
\[ 5u^6 - u^2v^2 - 125u^4v^4 + 5v^6 = 0 \tag{90} \]
and
\[ 125h^{12} + h^3v^3 + 1125h^9v^9 + 1953125h^{39}v^{12} - 125v^{12} = 0 \tag{91} \]
which are solvable. But from [12] we have
\[ \frac{5R'(q)}{R(q) (R(q)^{-5} - 11 - R(q)^5)^{1/6}} = f^4(-q)q^{-5/6} \tag{92} \]
or
\[ N(q) = \frac{1}{5} (R(q)^{-5} - 11 - R(q)^5)^{1/6} \tag{93} \]
Hence the solution of (1) can also given in the form
\[ X = X_r = \frac{125b}{2a} N(q^2)^6 \tag{94} \]
and from (85) we have
\[ 2^{2/3} a^{1/3} b^{1/3} (X_r X_{4r})^{1/3} + \frac{10 \cdot 21^{1/3}}{b^{1/3}} (X_r X_{4r})^{2/3} - 2a^{2/3} (X_r + X_{4r}) = 0 \tag{95} \]

Note.
One can prove relation (90) using (89),(93) and the duplication formula (see [13]):
\[ \frac{R(q^2) - R^2(q)}{R(q^2) + R^2(q)} = R(q) R^2(q^2) \tag{96} \]
The same method can work and with other higher modular equations of the derivative but the evaluations are very difficult even for a program.

Another interesting note that can simplify the problem is the singular moduli of the fifth base (see [14],[15]):

\[ u(x) = 2F_1 \left( \frac{1}{6}, \frac{5}{6}; 1; x \right). \] (97)

In this case we have

\[ j_r = \frac{432}{\beta_r(1-\beta_r)} = \frac{250C_1^3}{a^2b}, \] (98)

where \( \beta_r \) is the solution of

\[ \frac{2F_1 \left( \frac{1}{6}, \frac{5}{6}; 1 - \beta_r \right)}{2F_1 \left( \frac{1}{6}, \frac{5}{6}; 1; \beta_r \right)} = \sqrt{r} \] (99)

The moduli \( \beta_r \) can evaluated from \( k_r \) and the opposite from the relation

\[ \frac{256(k_r^2 + (1-k_r^2)^2)^3}{k_r^4(1-k_r^2)^2} = \frac{432}{\beta_r(1-\beta_r)} \] (100)

**Proposition 2.**

The equation (1) have solution

\[ X = \frac{b}{250a} \left[ R \left( e^{-2\pi \sqrt{\beta_r(1-x)}} \right)^{-5} - 11 - R \left( e^{-2\pi \sqrt{\beta_r(1-x)}} \right)^5 \right], \] (101)

where

\[ \alpha = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{216a^2b}{125C_1^3}}, \beta_r(x) = \left( \frac{u(1-x)}{u(x)} \right)^2. \]

**Corollary.**

The equation

\[ aX^2 + bX + \frac{b^2}{20a} = \frac{6a^{2/3}b^{1/3}}{5\beta_r^{1/3}(1-\beta_r)^{1/3}}X^{5/3} \]

admits solution \( X = A_r \).
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