Introduction

In this paper, we solve the problem of overcoming some limitations of atomic functions which found wide applications in solving the boundary value problems of electromagnetics and image processing and possess important properties of locality and representation of algebraic polynomials but does not possess sufficient flexibility to allow to take into account nonuniform and inhomogeneous character of the data of the objects of research – complex geometry and variable coefficients. The properties and parameters of the systems which are the objects of analysis and processing in electromagnetics, distant sensing, processing of multidimensional signals often undergo rapid changes and are described by differential equations with variable coefficients in the domains with complex geometry.

The aim of this paper is to introduce some generalization of atomic functions (AF) – so called atomic functions (TF) which while retaining most important advantages of AF take into account this inhomogeneity, variability of the behaviour of the solutions of the problems encountered in applications.

1. Formulation of the problem: Birkhoff or lacunary interpolation

The main task of this paper is starting from ideas and machinery of atomic functions offer the solution of the lacunary interpolation problem by construction the new class of function – atomic functions.

As one example we consider a solution with the help of this new apparatus of the constructive theory of functions of the problems of Birkhoff or lacunary interpolation. In the well-known Newton or Lagrange interpolation in order to reconstruct an unknown function we use its known values in some points. In Hermite interpolation in addition to the values of the function in some points the derivatives to some order in the same points are used. The Birkhoff (or lacunary) interpolation uses the known values of the derivatives of some order in points where the values of the function are unknown.

Let us here make a remark that the opinion that it is more easy to determine the values of a variable than to find the values of the derivatives of this variable is not always true. It is in fact very often absolutely wrong.

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Let us give some examples. Consider such frequently met functions as \( \ln x \), \( \arctg x \). Their derivatives are correspondingly

\[
\frac{1}{x}, \quad \frac{1}{1+x^2}.
\]

It is obvious that in this case the derivatives are computed much easier. One more frequently used in probability theory and its applications function is

\[
F(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \exp(-t^2/2)dt.
\]

The tabulated computed values of this function are contained in practically every probability and statistics textbook. Its derivative is

\[
F'(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)
\]

– this is an elementary function. Let us recollect that the computation of integrals is often much harder task than computation of the derivatives. The derivative of an elementary function is always an elementary function while antiderivative (primitive) of an elementary function is not as a rule. The problems of solving the differential equations are problems of finding the function given some information about its derivatives and are called the integration of the differential equation. Now let us Consider such mechanical quantities as displacement (position), velocity, acceleration, jerk, snap etc. If displacement as function of time is denoted by \( d(t) \), velocity as \( v(t) \), acceleration \( a(t) \), jerk \( j(t) \), snap \( sp(t) \), then by definition we have

\[
\frac{d(v(t))}{dt} = a(t); \quad \frac{d(a(t))}{dt} = j(t); \quad \frac{d(j(t))}{dt} = sp(t).
\]

Determining of the displacement as function of time at given moment of time is considerably more difficult than determining of the velocity – first derivative of the displacement because for determining the velocity we need only local frame of reference while to determine displacement relative to some distant initial point of space we need global frame of reference, determining of the velocity is more difficult than determining of the acceleration in view of the Newton law \( F = ma \) and we can determine the acceleration even without local reference frame.

Determining of the acceleration is more complex than determining of the jerk – the derivative of acceleration. What is measured by the sensors of plants and animals (including human beings) -temperature or its change, loudness of sound or its change, brightness of light or its change? Turning to computer science – what is more memory efficient for storage of the information on a function – 1) to store the values directly or 2) to store only some values with large steps and differences or both differences and second differences and third differences with diminishing small steps? The books containing very precise tables of logarithms and trigonometric functions with 10 digit accuracy of the pre-computer era used the second approach – values with large steps on the left side of a page, to the right to the column of values the column of the first differences with smaller step, and next to the right the column of second differences with the smallest step. But the differences by the Lagrange theorem are derivatives multiplied by the powers of the steps. And when the steps are small and constant the powers of steps are even more small, so in order to provide the needed accuracy of the values of the function, we need less digits to store the values of derivatives than to store the values of the function itself. Those are arguments in favour of lacunary or Birkhoff interpolation and one observes a lot of papers on lacunary spline interpolation which were published recently. Lacunary interpolation was invented by George Birkhoff (the same Birkhoff who proved the ergodic hypothesis and created the dynamic system theory) at the beginning of the twentieth century but was not very successful because mathematical tools he used were algebraic polynomials and an algebraic polynomial \( P_n(x) \) of degree \( n \) has not more than \( n \) real roots, its derivative – not more than \( n - 1 \) roots, its second derivative not more than \( n - 2 \) roots and so on, and the derivative of the order \( n \) is constant not equal to zero, so has no roots at all. As well- known saying goes “new wine needs new skin bags”. To develop lacunary interpolation new tools were needed – the splines [12, 16-18]. Numerous examples of applications of polynomial splines to lacunary interpolation are in [25-36]. But splines of degree \( n \) are functions of finite smoothness – only first \( n - 1 \) derivatives are continuous. To construct the lacunary interpolation series we need the atomic functions. The generalized atomic Taylor expansion proposed in 1991 by V. A. Rvachev (V. O. Rvachov in Ukrainian) is an example of such lacunary interpolation of infinite order. Atomic and tomic functions are the tools for such kind of interpolation. Examples in medicine, geophysics, image processing, remote sensing [19-21, 23-25]. Now we will state reasons in favor of lacunary (Birkhoff) interpolation. There are many applications where robot motion with abrupt changes of jerk is not wanted, such as in transportation of people and goods where dropouts and breakages may easily occur. Limiting jerk in robot trajectories also contributes to
extended life of robot joints and thus to more precise trajectory tracking. A technique for time-jerk optimal planning of robot trajectories. The trajectory planning problem is a fundamental one in Robotics. It may be formulated thus: define a temporal motion law along a given geometric path, such as certain requirements set on the trajectory properties are fulfilled. Hence, the aim of trajectory planning is to generate the reference inputs for the control system of the manipulator, in order to be able to execute the motion. The inputs of any trajectory planning algorithm are: the geometric path, the kinematic and dynamic constraints; and the output is the trajectory of the joints (or of the end effector), expressed as a time sequence of position, velocity and acceleration values. Usually, the geometric path is specified in the operating space, i.e. with reference to the end effector of the robot. Standard generalized Taylor expansions on the basis of up(x) function which were introduced by V. A. Rvachev in 1981 use values of the function and its first derivative in integer points, the values of derivative of the order 2 in half integer points. The values of the derivatives of the order in points of form $k2^{-n+1}$. In places where the unknown functions are varying rapidly the discretization step should be made smaller and alternately in regions where those functions are slow varying the step could be made larger. So, generalised atomic Taylor expansion (GATE) expansions should be made with different steps in different regions. But then the problem of smooth transition from one region to the neighbour must be solved. To solve this problem we have to introduce new smooth compactly supported functions similar to the atomic functions but which have zeroes of the derivatives placed non-uniformly or in other words the widths of the intervals between two neighbouring zeroes of the derivatives (parts of the derivative on these intervals which could be named hills and holes) should be different in different regions. Atomic function which satisfy functional differential equation (FDE) of pantograph type with some fixed compression coefficient? The hills and holes of the derivative of a given order have equal widths. Let us remind here that atomic functions by definition are compactly supported solutions of the equations of the form

$$L y(x) = \sum_{k=1}^{n} c_k y(ax - b_k),$$

where $L$ is linier differential operator of order $m$ with constant coefficients. Their Fourier transforms are of the form

$$F_{AF}(t) = \prod_{k=0}^{\infty} P_n(e^{it/a^k}),$$

where $P_n(t), Q_m(t)$ are algebraic polynomials. If we denote

$$S_{nc}(x) = \frac{\sin x}{x},$$

then Fourier transform of the up(x) function will be

$$F_{up}(t) = \prod_{k=0}^{\infty} S_{nc}(t/2^k),$$

$$Ly(x) = \sum_{k=1}^{m} c_k y(ax + b_k).$$

Convolution of up(x) with itself is needed for application in variational and projection methods. To compute the convolution $u_{pp} = up(x) * up(x)$, we expand it into Fourier series on interval $[-2, 2]$. Coefficients of this series are values of the square of the Fourier transform of the function $u_{pp}(x)$ in points $k\pi/2$, where

$$u_{pp}(x) = \int_{-\infty}^{\infty} u_{pp}(x-t)u_{pp}(t)dt,$$

$$u_{pp}(x) = \int_{-\infty}^{\infty} e^{it} u_{pp}^2(t) dt,$$

where

$$F_{up}(t) = \prod_{k=1}^{\infty} \sin t2^{-k}.$$
2. Pantograph-type equations

Atomic functions are solutions with a compact support of linear functional differential equations with a linearly transformed argument.

Equations of this type are often called pantograph-type equations.

The generalized pantograph equation is

\[ y'(t) = a(t)y(t) + b(t)y(\alpha t) , \]

where \( \alpha > 1 \), has numerous applications. Such equations describe the absorption of light by an interstellar medium; they are found in the theory of electrical materials, mathematical cell biology uses similar equations to describe the number of cells in the process of division. Similar equations in physics are called Fokker-Planck equations. Fokker-Planck equation is

\[ y'(x) + by'(x) + py(x) + qy(\alpha x) = 0. \]

Here, the coefficient shows how many parts the cell divides during division (mitosis), i.e. almost always equal to 2. But in nuclear physics, the coefficient can be greater than 2, since the nucleus can be divided into a larger number of particles - neutrons and protons.

3. Linear spaces generated by shifts of function \( up(x) \)

In approximation theory and computational mathematics, spaces of linear combinations of function \( up(x) \) shifts are used [1-3, 7, 8]

\[ UP_n = \{ \sum_{k=1}^{m} c_k up(x - k2^{-n}) \}. \]

It makes sense to consider a wider set of linear spaces

\[ UP_n (m) = \{ \sum_{k=1}^{m} c_k up(2^m x - k2^{m-n}) \}, \quad n, m \in \mathbb{Z} \]

since the operations of differentiation and integration of functions from spaces transform them into functions from. Note that here the limits in the sums can also be taken as infinite, since for each only a finite number of terms are different from 0 due to the compactness of the support of the function. Important elements of space are functions \( fup_n (x) \). These functions among the elements of \( UP_n \) have minimal support (the support of a function is the set − the closure of the set of points where the function is not equal to 0).
Namely, the length of the function $\text{fup}_n(x)$ support is equal to $(n + 2)2^{-n}$. Function $\text{fup}_n(x)$ is $\text{up}(x)$ this. The shifts of these functions form a basis in space $\text{UP}_n$, consisting of functions with the smallest support. It is obvious that

$$\lim_{n \to \infty} (n + 2)2^{-n} = 0.$$ 

For comparison the length of the support of Shoenberg basic spline $B_n(x)$ степені $n$ with step $h = 2^{-n}$ is $(n + 1)2^{-n}$, that is somewhat less, but the form of it is different for different $n$ and it is of finite smoothness.

The letter $f$ in notation of this function is taken from the first letter of the word fundamental (basic). There are two variants $\text{FuP}_n(x)$ and $\text{fup}_n(x)$. The initial variant had capital $F$ and means the function normed by condition

$$\int F_{\text{up}}(x) dx = 1.$$ 

The variant with $f$ small means the function the maximum value of which is 1. This is convenient in applications of atomic functions in collocation methods (E. A. Fedotova, Gotovac (Chroatia (Split))). It is easily seen that

$$\text{UP}_n \subset \text{UP}_{n+1}.$$ 

Denote the orthogonal complement to $\text{UP}_n$ in $\text{UP}_{n+1}$ by $\text{OUP}_n$. In the linear space $\text{UP}_n$ there is a base of shifts of the function $\text{fup}_n(x)$ of the form $\text{fup}_n(x - k2^{-n})$, in the linear space $\text{UP}_{n+1}$ there is a base of shifts $\text{fup}_{n+1}(x - k2^{-n-1})$. The supports of these functions are of lengths $(n + 1)2^{-n}$ and $(n + 2)2^{-n-1}$ respectively. In the space $\text{OUP}_n$ a base of shifts of a function $\text{ofup}_n(x)$ the length of the support of which is $(2n + 1)2^{-n}$. As the space $\text{UP}_n$ contains subspace of polynomials of degree $n$ the function $\text{ofup}_n(x)$ has zero moments of degree less or equal to $n$. This is an important fact for applications in border element methods (Boundary integral equations) because the solutions of Laplace equation and similar equations in half plane or circle which have zero moments of degree less or equal to given integer are small far from the boundary and the coefficients of the linear algebraic systems are accordingly small.

4. Achievements of atomic function theory and applications

Atomic functions were successfully applied to various problems of mathematical analysis including approximation theory, for solving the problem of representation of infinitely differentiable function by Taylor-Birkhoff expansions, in numerical methods for ordinary differential equation (ODE), FDE, partial differential (PDE), for signal and image processing [1-11]. Sometimes someone asks – why atomic functions are needed? Are not already used function classes sufficient for all purposes? The answer is following. The analytic functions are not sufficient and spline functions (which are piecewise analytic) are now widely applied. Splines have the advantage of being local – we can change a spline on small interval not changing elsewhere. But splines are only piecewise smooth and at the joining points of the pieces are differentiable only several times so that approximation rate is not very high. The approximation by atomic functions which are infinitely differentiable but nonanalytic allows to change the approximating aggregate on small intervals nevertheless providing high rate of approximation if the function that is approximated is very smooth. But approximation spaces generated by atomic functions have some limitations due to insufficient flexibility and refinement. As noted above the main task of this paper is starting from ideas and machinery of atomic functions offer the solution of the lacunary interpolation problem by construction the new class of function – atomic functions.

5. Necessity and indispensability of atomic function - motivations and definitions

Shortcoming of AF Disadvantage of using the atomic functions consists in the fact that that the zeroes of derivatives of an atomic function are spaced uniformly, equidistantly and hills and holes of the derivatives have equal widths. If the behaviour of the analyzed object – be it solution of a boundary value problem for a partial derivatives equation or the signal or image to be processed in different locations differs considerably – varies slow or fast – to take into account these variations the size of hills and holes of the derivatives of the function which describes this object ought to be accordingly variable. We certainly can try to divide the domain under exploration into more or less homogeneous parts and choose for each part the atomic functions of suitable size of the support and then sew together the different atomic expansions but we will need nevertheless functions with compact support and heterogeneous location of zeroes of the derivatives (the size of hills and holes of the derivatives) for area of transition. Introduc-
tion of tomic functions targets exactly this problem –
the construction of expansions of the solutions of the
problem which possess different behaviour in different
parts of the domain of definition. The word atom means
indivisible and the term tomic function means divisible
function but to avoid mixing with other uses of the
words such as division, divisor in different branches of
mathematics we introduce the term tomic for our pur-
pose.

6. Definition and construction

Tomic function of order 1 is a function \( f(x) \)
equal 0 outside the interval \( [a,b] \), one time continu-
ously differentiable, positive inside \( [a,b] \) and which has
unique point of strict maximum \( c, a < c < b \) (where the
derivative is zero \( f'(c) = 0 \)). If \( f(x) \) is twice continu-
ously differentiable, i.e. \( f(x) \in C^2[\mathbb{R}] \) and let

\[
1(x) = \begin{cases} 
  f'(x), & x \in [a,c]; \\
  0, & x \not\in [a,c]; 
\end{cases}
\]

and

\[
r(x) = \begin{cases} 
  f'(x), & x \in [c,b]; \\
  0, & x \not\in [c,b]. 
\end{cases}
\]

Both \( l(x) \) and \( -r(x) \) are tomic functions of order 1,
then the function \( f(x) \) is called tomic function of or-
der 2. If functions \( l(x),-r(x) \) are tomic functions of
order 2, then \( f(x) \) is called tomic function of order 3.
Similarly tomic function of any positive integer order is
defined. Tomic function which for every positive inte-
ger \( k \) is tomic function of order \( k \) is called tomic func-
tion of infinite order. The point \( c \) is called the middle
point of order 1. Middle points \( c_1,c_2 \) of functions
\( l(x),-r(x) \) are called middle points of order 2 and so
on. The number of middle points of the order \( n \) of a
tomic function of order \( n \) will be \( 2^{n-1} \). Naturally two
problems concerning tomic functions of infinite order -
the problem of existence and the problem of uniqueness
arise. It is obvious that function \( u(x) \) is tomic function of
infinite order. So tomic functions of infinite order exist.
From the theorem of uniqueness of restoration of function
\( u(x) \) by zeroes of its derivatives which was
proved in paper by V. A. Rvachev (V. O. Rvachov) in
the paper [2] (Compactly supported solutions of func-
tional-differential equations and their applications) fol-
lows that when middle points are in the middle. That is
when ratio of lengths of supports of functions
\( l(x),r(x) \) at each step and in each part is 1 the existence
and uniqueness of the tomic function takes place. One
can prove the existence and uniqueness of tomic func-
tion of infinite order if the ratio of lengths of supports of
functions \( l(x),r(x) \) at each step and at each part takes
place only starting with some order of derivatives. We
will consider only such functions now because they are
sufficient to take care of heterogeneity for most
problems. There are such two new unproven hypothe-
ses. Hypothesis A. The existence and uniqueness of
tomic function with given zeroes of derivatives and be-
ing normalized by some condition, such as its maximal
value equals 1 holds if the ratio of the lengths of sup-
ports \( l(x)/r(x) \to 1 \) at each step and in each part. Hy-
pothesis B. The existence and uniqueness of tomic func-
tion with given zeroes of derivatives and being normal-
ized by some condition, such as its maximal value
equals 1 holds if the ratio of the lengths of supports
\( l(x)/r(x) \) at each step and in each part is bounded
\( 0 < a < l(x)/r(x) < b < +\infty \) where \( a,b \) do not depend on
the order of a derivative and number of a part on which
the support of the function is partitioned by the zeroes
of the derivative.

7. Coordinated (coherent) systems
of tomic functions

Spaces of atomic functions or Atomic spaces are
by definition linear spaces of linear combinations of
shifts (translates) of the function \( u(x) \) of the form
\( u(x - k 2^{-n}) \) (in other words generated by the
shifts of function \( u(x) \) with constant step \( 2^{-n} \)). The
contain algebraic polynomials of degree \( n \),These spac-
es can be considered to be spaces of smoothed poly-
nomial splines. In order to build the spaces of linear com-
binations of tomic functions (which will be called the
tomic spaces), the zeroes of derivatives of which are not
uniformly spaced we cannot use shifts (or translates) of
a single tomic function. Here some construction of co-
herent systems of tomic functions is needed to satisfy
the condition that the linear combinations of the func-
tions belonging to such system must contain all algebra-
ic polynomials of degrees not greater than some \( n \). This
condition provides for good approximation properties of
tomic spaces. This can be done in the following way:
subdivide the interval \([a,b]\), on which we want to build
a coherent set of tomic functions with given steps be-
tween the nodes at which the senior derivatives of re-
quired order \( r \) are prescribed, by the required quantity
of nodes from \( x_1,\ldots,x_M \). Then we add at the left and
at the right sides additional \( 2^r \) nodes so that total num-

ber of nodes will be \( 2^M + 2^{r+1} \) rename all the nodes from left to right as

\[ z_1 < z_2 < \ldots < z_N, \]

where \( N = 2^M + 2^{r+1} \). Now we start construction of the sought coherent system of tomic functions \( \text{tofu}_j(x) \) in the following way: on each interval \([z_k, z_{k+1}]\) we place the function of the form \( \text{up}(a_k x - b_k) \) in such a way that its support coincided with the interval and denote it \( g_k(x) \). Simple calculations give

\[ a_k = \frac{2}{z_{k+1} - z_k}, \quad b_k = \frac{z_{k+1} + z_k}{z_{k+1} - z_k}. \]

For each \( j \) consider the function \( \varphi_j(x) \) defined on the interval \([z_j, z_{j+2^r}]\)

\[ \varphi_j(x) = \sum_{k=j}^{j+2^r} c_k g_k(x). \]

Notice that unknown coefficients in fact depend on \( j \) \( c_k = c_k(j) \), i.e. for different basic function \( \text{tofu}_j(x) \) the coefficients \( c_k = c_k(j) \) are different unlike the case of generalized Taylor series on the basis of the function \( \text{up}(x) \). We assume that the function \( \varphi_j(x) \) is to be the derivative of the order \( r \) of the function \( \text{tofu}_j(x) \), that is \( \text{tofu}_j^{(r)}(x) = \varphi_j(x) \). To find the unknown coefficients \( c_k = c_k(j) \) we have some algorithmic way. How do equations of this system for determining a basic function \( \text{tofu}_j(x) \) look like? The sought function \( \text{tofu}_j(x) \) the zeroes of derivatives must be in prescribed points. For the derivative of the order \( r \) zeroes are all points \( z_k \). As we assumed that the function \( \varphi_j(x) \) is the derivative of order \( r \) of the function \( \text{tofu}_j(x) \), so the conditions on the zeroes of the order \( r \) are satisfied by its definition. Unknown coefficients \( c_k \) we find from the conditions of vanishing of the derivatives of orders from 1 to \( r-1 \) in prescribed points and the condition that the function itself either equals to 1 in the middle points of the range or the condition that its integral equals 1 (2 different normalizations which are convenient in different applications). For the derivative of the order \( r-1 \) the zeroes are to be in points \( z_k \) with \( k = j + 2s \). For the derive of the order \( r-2 \) the zeroes are to be in points \( z_k \) with \( k = j + 4s \) and so on. The derivatives of the order less than \( r \) are found by successive integration (finding the primitive) of the derivative of the order \( r \) which is \( \varphi_j(x) = \sum_{k=j}^{j+2^r} c_k g_k(x) \) in limits from \( z_j \) to \( x \). Therefore

\[ (\text{tofu}_j(x))^{(r-1)} = \int_{z_j}^{x} \sum_{k=j}^{j+2^r} c_k g_k(t)dt = \sum_{k=j}^{j+2^r} c_k \int_{z_j}^{x} g_k(t)dt. \]

Integrals contained in these equations are successive primitives of compressed and shifted function \( \text{up}(x) \) and as a consequence of the functional differential equation for this function satisfies, also are functions of the form \( \text{up}(\beta x + \gamma) \) on left side of the support and further to the right an algebraic polynomial of the degree 1 for derivative \( r-1 \). Here it is convenient to make use the Cauchy formula

\[ f^{(-n)}(x) = \frac{1}{(n-1)!} \int_{a}^{x} (t-x)^{n-1} f(t)dt. \]

Derivative of a negative order (antiderivative) is just what we need here - the operation of successive multiple integration to obtaining the primitives. With the help of this formula the equations for finding the coefficients \( c_k \) obtain a simple look. Now we see that expanding expressions \( (t-x)^{n-1} \) in this integral it remains to compute the integrals of the form \( \int_{z_j}^{z_{j+1}} t^r g_j(t)dt \). And as \( g_j(x) = \text{up}(a_j x - b_j) \), such integrals are expressed via the known moments of the function \( \text{up}(x) \). Moments of the \( \text{up}(x) \) functions are computed by the recursive formula

\[ a_n = \int_{-1}^{1} x^n \text{up}(x)dx, \quad a_{2n+1} = 0, \quad a_0 = 1, \]

\[ a_{2n} = \frac{(2n)!}{2^{2n} \cdot n!} \sum_{k=0}^{n} \frac{a_{2n-2k} \cdot (2n-2k)!}{(2k+1)!}. \]

Integrals of the form (half-moments)

\[ b_{2n+1} = \int_{0}^{1} x^{2n+1} \text{up}(x)dx \]

are computed by the recursive formula

\[ b_{2n+1} = \frac{1}{(n+1)2^{2n+3}} \sum_{k=0}^{n+1} a_{2n+2-2k} C_{2n+2}^{2k}. \]
In particular, $a_2 = \frac{1}{9}$, $a_4 = \frac{19}{3^2}$, $a_6 = \frac{583}{3^55^2}$, $b_1 = \frac{5}{36}$, $b_3 = \frac{143}{8 \cdot 27 \cdot 25}$, $b_5 = \frac{1153}{64 \cdot 3^6 \cdot 49}$.

The number of conditions (the numbers of equations) is equal to the number of unknowns and the structure of the matrix of this linear algebraic system is block diagonal. If the order of system we denote by $N = 2^n + 2^{r+1}$, then we have $N/2$ equations with two different unknowns in each, $N/4$ equations with 4 different unknowns in each, $N/8$ equations with 8 different unknowns in each and so on,... ending with 2 equations with half different unknowns in each corresponding to the 2 zeroes of first derivative and 1 equation for one zero at the right end for all unknowns and 1 equation with all unknowns to satisfy normalization condition. It is obvious that the matrix of the system is invertible. The solution exists and is unique. From the construction of this coherent system of tomic follows that the linear combinations of the elements of it contain algebraic polynomials of the order $r-1$, but the proof of it being straightforward is rather lengthy and is omitted here. It can be calculated that due to the special block-diagonal structure of the matrix of the linear algebraic system for finding the $2^r$ coefficients $c_k$ we need only $Cr^2$ arithmetic operations (ao) (per one function). For $r=5$ to find 33 $c_k$ coefficients for one function $\psi_j(x)$ takes 160 ao and for coherent system of 1000 functions – 200000 ao. To store 33 thousand of coefficient – adequate amount of memory. For $r=10$ to compute 1025 coefficients per one tomic function we need 10 thousand ao and for computation of such coefficients for the coherent system of 10 thousand functions -100 million ao and storage of 10 million coefficients – the adequate amount of memory. If the norm $C^f$ is insufficient we can break each interval $[z_k, z_{k+1}]$ by half and add to already built coherent system of tomic functions additional tomic functions. The tomic functions are analogues of shifts (translations) of the function $\up(x)$ – the functions $\up(x-k2^{-n})$. But in applications to the methods of solution of the boundary value problems for PDE of the finite element type or boundary element type we use not the shifts of the function $\up(x)$, but shifts of the functions $\text{fup}_n(x)$) because the supports of shifts of $\up(x)$ are too wide and supports of $\text{fup}_n(x)$ are minimal possible. For GATE – Birkhoff interpolation series, where we do not integrate but collocate the shifts of $\up(x)$ are optimal, but for methods where we integrate we need to build analogues of shifts of $\text{fup}_n(x)$ in the space of tomic functions. They $\text{fup}_{r,j}(x)$ are constructed in the form of the antiderivative of the order $r$ of the sums of the form $\psi_j(x) = \sum_{k=j}^{j+r+2} c_k g_k(x)$. With conditions only at the right end $z_{j+r+3}$ antiderivatives of $\psi_j(x)$ of the order from 1 to $r$ being 0, and possess minimal possible supports. They are analogues of the function $\text{fup}_n(x)$ for tomic expansions. Glueing (matching) two homogeneous atomic expansions with different steps depends on the kind of the problem we solve. In case of Birkhoff interpolation we need to construct tomic functions for interval of $2^r + r$ steps to the left and to the right of the transition point because we need wide tomic function which possess zeroes of derivatives in needed places. But in variational problems where we use orthogonal systems and in collocation we use the function $\text{fup}_{r,j}(x)$ - analogues of B-splines and functions $\text{fup}_n(x)$ the transition interval (where we need tomic functions) is only $2r + 3$ steps wide.

Conclusions

In this paper we introduce the tomic functions in order to transfer the application of atomic functions to solving problems with sharp geometric inhomogeneities and rapidly variable medium properties of the objects under study. The tomic functions are constructed on the basis of atomic function $\up(x)$ by iterative procedure and are of two kinds, namely, designed for lacunary interpolation – analogues of the basic functions of atomic generalized Taylor series $\text{bfu}_{n,k}(x)$ and designed for variational and collocation methods in boundary value problems for partial differential equations – analogues of atomic functions $\text{fup}_n(x)$.

References (GOST 7.1:2006)

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У цій статті розглядаємо і вирішуємо завдання побудови так званих томік функцій і лакунарні інтерполаційні ряди у крайових задачах для рівнянь з частинними похідними та обробці зображень.

В. О. Ричков, Т. В. Ричкова, Є. П. Томілова

У цій статті розглядаємо і вирішуємо завдання побудови так званих томік функцій – систем несхожих диференційованих функцій, які, зберігаючи багато важливих властивостей зсуїв атомарної функції ur(x) таких як локальність і зображення алгебраїчних множин і засновані на атомарних функціях, проте мають неоднорідну структуру і отже дозволяють враховувати неоднорідний і мінливий характер даних, що зустрічаються в задачах реального світу, зокрема в крайових задачах для рівнянь з частинними похідними з змінними коефіцієнтами і складною геометрією об'єктів, в яких ці крайові задачі вирішуються. Той же клас томік функцій може застосовуватися для обробки, усунення шумів і економічного зберігання сигналів та зображення за допомогою лакунарної інтерполації. Лакунарна або віртексова інтерполація функцій, в якій функція відноситься до значений похідних порядку 2 в точках, яких значення функції i її похідних порядок к-т невідомо, має велике значення в багатьох реальних задачах, таких, наприклад, як дистанційне зондування. Методи лакунарної інтерполації, що використовують томік функції, мають важливі переваги у порівнянні з широко використовуваною лакунарною спайн-інтерполацією через несхожість гладкості томік функцій. Томік функції також можуть застосовуватися для з'єднання (швидень) атомарних розкладів з різним кроком на різних інтервалах, зберігаючи гладкість і оптимальні апроксимаційні властивості. Отримані рівняння для побудови томік функції tof(x) – аналог
Томик функции и лакунарные интерполяционные ряды
в краевых задачах для уравнений с частными производными и обработке изображений

В. А. Рвачёв, Т. В. Рвачёва, Е. П. Томилова

В этой статье мы рассматриваем и решаем задачу построения так называемых томик функций – систем бесконечно дифференцируемых функций, которые, сохраняя много важных свойств сдвигов атомарной функции \( u(x) \) таких как локальность и представление алгебраических многочленов и основанные на атомарных функциях, тем не менее не имеют неоднородный характер и следовательно позволяют учитывать неоднородный и изменчивый характер данных в задачах реального мира, в частности в краевых задачах для уравнений с частными производными с переменными коэффициентами и сложной геометрией областей, в которых эти краевые задачи решаются. Тот же класс томик функций может применяться для обработки, устранения шумов и экономного хранения сигналов и изображений с помощью лакунарной интерполяции. Лакунарная или Биркгоффова интерполяция функций, в которой функция восстанавливается по значениям производных порядка \( k \) в точках, в которых значения функции и ее производных порядка \( k \) неизвестны, имеет большое значение во многих реальных задачах, таких как дистанционное зондирование.

Методы лакунарной интерполяции, использующие томик функции, обладают важными преимуществами по сравнению с широко используемой лакунарной сплайн-интерполяцией ввиду бесконечной гладкости томик функций. Томик функции также могут применяться для соединения (сплайнирование) атомарных разложений с различным шагом на разных интервалах, сохраняя гладкость и оптимальные аппроксимационные свойства. Получены уравнения для построения томик функций \( \mathcal{T}(x) \) – аналогов базисных функций обобщенных атомарных рядов Тейлора, которые нужны для лакунарной (Биркгоффовой) интерполяции. Матрицы линейных алгебраических систем для вычисления коэффициентов томик функций имеют специальную блок-диагональную структуру и легко обращаются. Для приложений в вариационных и коллокационных методах решения краевых задач для уравнений с частными производными и интегральных уравнений получены томик функции \( \mathcal{T}(x) \). Томик функции \( \mathcal{T}(x) \) является аналогами Б-сплайнов и атомарных функций \( \mathcal{T}(x) \). Используя подобные методы, можно построить томик функции, основанные на других атомарных функциях, таких как \( \mathcal{T}(x) \).

Ключевые слова: атомарные функции; томик функции; лакунарная интерполяция; Биркгоффова интерполяция; обработка и хранение изображений; вариационный метод; метод коллокации.

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