COMPACTNESS RESULTS FOR STATIC AND DYNAMIC CHIRAL SKYRMIONS NEAR THE CONFORMAL LIMIT

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Abstract. We examine lower order perturbations of the harmonic map problem from $\mathbb{R}^2$ to $S^2$ including chiral interaction in form of a helicity term that prefers modulation, and a potential term that enables decay to a uniform background state. Energy functionals of this type arise in the context of magnetic systems without inversion symmetry. In the almost conformal regime, where these perturbations are weighted with a small parameter, we examine the existence of relative minimizers in a non-trivial homotopy class, so-called chiral skyrmions, strong compactness of almost minimizers, and their asymptotic limit. Finally we examine dynamic stability and compactness of almost minimizers in the context of the Landau-Lifshitz-Gilbert equation including spin-transfer torques arising from the interaction with an external current.

1. Introduction and main results

Isolated chiral skyrmions are homotopically nontrivial field configurations $m: \mathbb{R}^2 \to S^2$ occurring as relative energy minimizers in magnetic systems without inversion symmetry. In such systems the leading-order interaction is Heisenberg exchange in terms of the Dirichlet energy

$$D(m) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla m|^2 \, dx.$$ 

Chiral interactions, in magnetism known as antisymmetric exchange or Dzyaloshinskii-Moriya interactions, are introduced in terms of Lifshitz invariants, the components of the tensor $\nabla m \times m$. A prototypical form is obtained by taking the trace, which yields the helicity functional

$$H(m) = \int_{\mathbb{R}^2} m \cdot (\nabla \times m) \, dx,$$

well-defined for moderately smooth $m$ that decay appropriately to a uniform background state. Extensions to the canonical energy space will be discussed later.

Chiral interactions are sensitive to independent rotations and reflections in the domain $\mathbb{R}^2$ and the target $S^2$, and therefore select specific field orientations. The helicity prefers curling configurations. The uniform background state $m(x) \to \hat{e}_3$ as $|x| \to \infty$ is fixed by a potential energy $V(m) = V_p(m)$ depending on a power $2 \leq p \leq 4$ with

$$V_p(m) = \frac{1}{2p} \int_{\mathbb{R}^2} |m - \hat{e}_3|^p \, dx.$$ 

The borderline case $p = 2$ corresponds to the classical Zeeman interaction with an external magnetic field. The case $p = 4$ turns out to play a particular mathematical role in connection with helicity. From the point of view of physics, since $\frac{1}{4} |m - \hat{e}_3|^4 = |m - \hat{e}_3|^2 + (m \cdot \hat{e}_3)^2 - 1$, the case $p = 4$ features a specific combination of...
Zeeman and in-plane anisotropy interaction. Upon scaling, the governing energy functional

\[ E_\varepsilon(m) = D(m) + \varepsilon(H(m) + V(m)) \]

only depends on one coupling constant \( \varepsilon > 0 \). For \( p = 2 \) variants of this functional have been examined in physics literature, see e.g. \([3, 4, 11]\), predicting the occurrence of specific topological defects, so-called chiral skyrmions, arranged in a regular lattice or as isolated topological soliton. In our scaling, tailored towards an asymptotic analysis, the parameter \( \varepsilon \) corresponds to the inverse of the renormalized strength of the applied field. The almost conformal regime \( 0 < \varepsilon \ll 1 \) features the ferromagnetic phase of positive energies, where \( H \) is dominated by \( D \) and \( V \), i.e.

\[ E_\varepsilon(m) \gtrsim D(m) + \varepsilon V(m). \]

In this case the configuration space

\[ \mathcal{M} = \{ m : \mathbb{R}^2 \to S^2 : D(m) + V(m) < \infty \}, \]

admits the structure of a complete metric space (see below). In the ferromagnetic regime, \( m \equiv \hat{e}_3 \) is the unique global energy minimizer, while chiral skyrmions are expected to occur as relative energy minimizers in a nontrivial homotopy class. In the case \( p = 2 \) and for \( 0 < \varepsilon \ll 1 \) this has been proven in \([21]\).

Homotopy classes are characterized by the topological charge (Brouwer degree)

\[ Q(m) = \frac{1}{4\pi} \int_{\mathbb{R}^2} m \cdot (\partial_1 m \times \partial_2 m) \, dx \in \mathbb{Z}, \]

which decomposes the configuration space into its path-connected components, the topological sectors. In view of the background state \( \hat{e}_3 \), the specific topological charge \( Q(m) = -1 \) is energetically selected by the presence of a chiral interaction. In fact, for all \( 2 \leq p \leq 4 \) we have

\[ \inf \{ E_\varepsilon(m) : m \in \mathcal{M} \text{ with } Q(m) = -1 \} < 4\pi \quad \text{for } \varepsilon > 0, \]

less than the classical topological lower bound for the Dirichlet energy, while

\[ \inf \{ E_\varepsilon(m) : m \in \mathcal{M} \text{ with } Q(m) \notin \{0, -1\} \} > 4\pi \quad \text{for } \varepsilon \ll 1, \]

a consequence of the energy bounds provided in Section 2.

These properties are in contrast to two-dimensional versions of the classical Skyrme functional (see e.g. \([21, 2]\)) featuring full rotation and reflection symmetry. Here, the helicity term is replaced by the the Skyrme term

\[ S(u) = \frac{1}{4} \int_{\mathbb{R}^2} |\partial_1 u \times \partial_2 u|^2 \, dx, \]

a higher order perturbation of \( D(u) \), which prevents a finite energy collapse of the topological charge due to concentration effects. In particular, the energy functional

\[ D(u) + \lambda S(u) + \mu V(u), \]

for positive coupling constants \( \lambda, \mu \), has an energy range above \( 4\pi \) in every non-trivial homotopy class. In the case \( p = 4 \), the attainment of least energies for unit charge configurations and topologically non-trivial configurations has been examined in \([17, 18, 16]\) and \([18]\), respectively. Explicit minimizers arise for \( p = 8 \), see \([24]\). We shall recover this situation in the chiral case for \( p = 4 \).

Our first result confirms existence of (global) minimizers of \( E_\varepsilon \) in \( \mathcal{M} \), subject to the constraint \( Q = -1 \), extending the result in \([21]\) for \( p = 2 \) to the whole range \( 2 \leq p \leq 4 \) of exponents:
Theorem 1 (Existence of minimizers). Suppose $2 \leq p \leq 4$ and $0 < \varepsilon \ll 1$. Then the infimum of $E_{\varepsilon}$ in $\mathcal{M}$ subject to the constraint $Q = -1$ is attained by a continuous map $m_{\varepsilon}$ in this homotopy class such that

$$4\pi(1 - 4\varepsilon) \leq E_{\varepsilon}(m_{\varepsilon}) \leq 4\pi(1 - 2(p - 2)\varepsilon).$$

For $p = 2$ and $0 < \varepsilon \ll 1$, we have, more precisely,

$$E_{\varepsilon}(m_{\varepsilon}) \leq 4\pi \left(1 - (4 + o(1))\frac{\varepsilon}{|m_{\varepsilon}|}\right).$$

If $p = 4$, minimizers are characterized by the equation

$$D_1 m + m \times D_2 m = 0 \quad \text{where} \quad D_j m = \partial_j m - \frac{1}{2} \hat{e}_i \times m.$$

For $2 \leq p < 4$, Theorem 1 is obtained by a concentration-compactness argument similar to [21, 17]. Provided “vanishing” holds, we prove that the helicity functional becomes negligible, so that the energy of a minimizing sequence approaches $4\pi$, which contradicts the upper bound coming from Lemma 3 below. If “dichotomy” holds, the cut-off result Lemma 8 (see Appendix) yields a comparison function with an energy well below the global minimum in its homotopy class. Hence, neither vanishing nor dichotomy appear.

The case $p = 4$ is special in the sense that vanishing can no longer be ruled out within our approach. However, upper and lower energy bounds match, so that an explicit energy-minimizer in form of a specifically adapted stereographic map $m_0$ is available. It follows that $m_0$ belongs to the class

$$\mathcal{C} := \{m : \mathbb{R}^2 \to S^2 : D(m) = 4\pi, Q(m) = -1, m(\infty) = \hat{e}_3\}$$

consisting of anti-conformal (harmonic) maps of minimal energy. Recall that harmonic maps on $\mathbb{R}^2$ with finite energy extend to harmonic maps on $S^2$ (cf. [25]) with a well-defined limit as $x \to \infty$.

Anti-conformal maps are characterized by the equation $\partial_1 m - m \times \partial_2 m = 0$, a geometric version of the Cauchy-Riemann equation. Hence, identifying $\mathbb{R}^2 \simeq \mathbb{C}$, the moduli space of $\mathcal{C}$ is $\mathbb{C} \setminus \{0\} \times \mathbb{C}$. More precisely, $\mathcal{C}$ agrees with the two-parameter family of maps $m_0(z) = \Phi(az + b)$ for $z \in \mathbb{C}$, where $(a, b) \in \mathbb{C} \setminus \{0\} \times \mathbb{C}$ and $\Phi : \mathbb{R}^2 \simeq \mathbb{C} \to S^2$ is a stereographic map of negative degree with $\Phi(\infty) = \hat{e}_3$, cf. [5, Lemma A.1]. Note that $\mathcal{C} \cap \mathcal{M}$ is empty in the limit case $p = 2$.

In the context of the energies $E_{\varepsilon}$, the degeneracy of a map $m_0 \in \mathcal{C}$ with respect to the complex scaling parameter $a$ is lifted if

i) it satisfies the Bogolomov type equation (2), i.e. is also an energy minimizer subject to $Q = -1$ for $p = 4$ and $\varepsilon > 0$ arbitrary, or

ii) it is obtained from a family of chiral skyrmions $\{m_{\varepsilon}\}_{\varepsilon \ll 1}$, which we prove for $2 < p < 4$ and conjecture in the limit cases $p \in \{2, 4\}$:

Theorem 2 (Compactness of almost minimizers). Suppose $2 < p < 4$ and $\{m_{\varepsilon}\}_{\varepsilon \ll 1} \subset \mathcal{M}$ is a family such that

$$Q(m_{\varepsilon}) = -1 \quad \text{and} \quad E_{\varepsilon}(m_{\varepsilon}) \leq 4\pi - C_0 \varepsilon$$

for some constant $C_0 > 0$. Then, we have:

i) There exists $m_0 \in \mathcal{C}$ so that for $\varepsilon \to 0$, up to translations and a subsequence,

$$\nabla m_{\varepsilon} \to \nabla m_0 \quad \text{strongly in} \ L^2(\mathbb{R}^2)$$

and

$$\hat{e}_3 \cdot (m_{\varepsilon} - m_0) \to 0 \quad \text{weakly in} \ L^2(\mathbb{R}^2).$$
ii) If \(\{m_\varepsilon\}_{\varepsilon \leq 1}\) satisfies the more restrictive upper bound
\[
E_\varepsilon(m_\varepsilon) \leq 4\pi + \varepsilon \min_{m \in \mathcal{C}} (H(m) + V(m)) + o(\varepsilon) \quad \text{for } \varepsilon \to 0,
\]
then, modulo translations, the whole family converges to a unique limit \(m_0 \in \mathcal{C}\), which is determined by
\[
H(m_0) + V(m_0) = \min_{m \in \mathcal{C}} (H(m) + V(m)) = -8\pi(p-2),
\]
such that \(\hat{\epsilon}_3 \cdot (m_\varepsilon - m_0) \to 0\) strongly in \(L^2(\mathbb{R}^2)\). Moreover,
\[
\lim_{\varepsilon \to 0} \varepsilon^{-1}(E_\varepsilon(m_\varepsilon) - 4\pi) = \min_{m \in \mathcal{C}} (H(m) + V(m)).
\]

In particular, Theorem 2 applies to the family \(\{m_\varepsilon\}_{\varepsilon \geq 0}\) of minimizers that has been constructed in Theorem 1. Fixing the adapted stereographic map
\[
\Phi: \mathbb{R}^2 \to \mathbb{S}^2, \quad \Phi(x) = \left(\frac{2x^1}{1+|x|^2}, -\frac{1-|x|^2}{1+|x|^2}\right),
\]
so that \(Q(\Phi) = -1\) and \(\Phi(\infty) = \hat{\epsilon}_3\), we have
\[
m_0(x) = \Phi\left(\frac{x}{2(p-2)}\right) \quad \text{for } x \in \mathbb{R}^2. \tag{3}
\]

It remains an open question whether for positive \(\varepsilon\) the minimizers \(m_\varepsilon\) of \(E_\varepsilon\) in the homotopy class \(\{Q = -1\}\) are actually unique (up to translations) and axially symmetric. As a first step and for \(2 < p < 4\), Theorem 2 implies that \(m_\varepsilon\) is at least close in \(H^1\) and \(L^p\) to the unique, axially symmetric vector field \(m_0\) given above.

Similar to the existence of minimizers of \(E_\varepsilon\), Theorem 2 is proven by means of P. L. Lions’ concentration-compactness principle. However, since the minimal energy tends to \(4\pi\) as \(\varepsilon \to 0\), the argument of Theorem 1 needs to be modified in a suitable way. In fact, in order to rule out “dichotomy”, we will use the boundedness of the lower-order correction \(H + V\) to the Dirichlet energy \(D\), which comes from the matching upper and lower \(a\)-priori bounds to the minimal energy and is preserved by the cut-off result Lemma 8. As a consequence, we obtain a comparison vector-field of non-zero degree with Dirichlet energy strictly below \(4\pi\), contradicting the classical topological lower bound \(D(m) \geq 4\pi|Q(m)|\). “Vanishing”, on the other hand, would imply that the helicity functional becomes negligible along a sequence of (almost-)minimizers, which is again ruled out by the \(a\)-priori bounds.

The second part of this paper addresses the dynamic stability of spin-current driven chiral skyrmions in the almost conformal regime \(\varepsilon \ll 1\). This is ultimately a question of regularity for the Landau-Lifshitz-Gilbert equation, for which finite time blow-up, typically accompanied by topological changes, has to be expected if energy accumulates to the critical threshold of \(4\pi\). In the presence of an in-plane spin-velocity \(v \in \mathbb{R}^2\) the Landau-Lifshitz-Gilbert equation is given by
\[
\partial_t m + (v \cdot \nabla)m = m \times \left[\alpha \partial_t m + \beta(v \cdot \nabla)m - h_\varepsilon(m)\right] \tag{4}
\]
where \(\alpha\) and \(\beta\) are positive constants and
\[
h_\varepsilon(m) = -\text{grad}E_\varepsilon(m)
\]
is the effective field, see \([26, 28, 14]\) and \([8, 15, 22]\) for a mathematical account. In the Galilean invariant case \(\alpha = \beta\) traveling wave solutions are obtained by transporting equilibria \(m \times h_{\text{eff}} = 0\) along \(c = v\). In the conformal case \(\varepsilon = 0\), as observed in
[14], traveling wave solutions are obtained for arbitrary \( \alpha \) and \( \beta \) by transporting conformal or anti-conformal equilibria of unit degree along \( c \in \mathbb{R}^2 \) determined by the free Thiele equation

\[
(c - v)^\perp = \alpha c - \beta v.
\]

We are interested in the regime \( 0 < \varepsilon \ll 1 \) for that case \( p = 4 \). Taking into account the asymptotic behavior of almost minimizers, it is natural to pass to the moving frame

\[
m(x, t) \rightarrow m(x + ct, t) \quad \text{where} \quad (c - v)^\perp = \alpha c - \beta v.
\]

After a rigid rotation in space (see Appendix C), this yields the pulled back equation

\[
(\partial_t - \nu \partial_z)m = m \times \left[ \alpha(\partial_t - \nu \partial_z)m - h_z(m) \right]
\]

with effective coupling parameter

\[
\nu = \frac{2(\alpha - \beta)v}{1 + \alpha^2},
\]

where \( v > 0 \) is now the intensity of the spin current, and with the Cauchy-Riemann operator

\[
\partial_z m = \frac{1}{2} (\partial_1 m - m \times \partial_2 m).
\]

revealing the conformal character of (4).

Observe that any \( m \in \mathcal{C} \), which is also an equilibrium for the energy, is a static solution for the pulled back dynamic equation, i.e. a traveling wave profile for (4). For \( \varepsilon = 0 \), the pure Heisenberg model, every \( m \in \mathcal{C} \) is a minimizer, hence an equilibrium, recovering the observation from [14]. For \( p = 4 \) and \( \varepsilon > 0 \) the matching upper energy bound characterizes \( m(x) = \Phi(x/4) \) with \( \Phi \) given by (3) not only as explicit energy minimizer within the class \( \{Q = -1\} \) but also as an explicit static solution of (6), i.e. an explicit traveling wave profile of (4).

**Theorem 3** (Existence, stability, compactness). Suppose \( p = 4 \) and \( 0 < \varepsilon \ll 1 \).

i) There exists \( m \in \mathcal{C} \) independent of \( \varepsilon \), which minimizes the energy in its homotopy class and is a static solution of (6) and therefore a traveling wave profile for (4).

ii) Suppose \( \{m_\varepsilon\}_{\varepsilon \ll 1} \subset \mathcal{M} \) is a family of initial data with \( \nabla m_\varepsilon \in H^2(\mathbb{R}^2) \) and such that for a constant \( c > 0 \) independent of \( \varepsilon \)

\[
Q(m_\varepsilon) = -1 \quad \text{and} \quad E_\varepsilon(m_\varepsilon) \leq 4\pi - \varepsilon c.
\]

Then there exists a unique family \( \{m_\varepsilon\}_{\varepsilon \ll 1} \subset C^0([0; T]; \mathcal{M}) \) of local smooth solutions of (6) with initial data \( m_\varepsilon(t = 0) = m_\varepsilon \) for every \( t \in [0, T] \).

iii) If \( \nabla m_\varepsilon \rightarrow \nabla m_0 \) strongly in \( L^2(\mathbb{R}^2) \) for some \( m_0 \in \mathcal{M} \) as \( \varepsilon \rightarrow 0 \), then \( \nabla m_\varepsilon(t) \rightarrow \nabla m_0 \) in \( L^2(\mathbb{R}^2) \) for every \( t \in [0, T] \).
Outline of the paper. The remainder of the paper is structured as follows: First, in Section 2 we prove the upper and lower bounds (1) to the minimal energy $E_\varepsilon$ in the homotopy class $\{Q = -1\}$, i.e. Lemmas 2 and 3. In particular, we obtain the equation (2) characterizing minimizers in the case $p = 4$.

In Section 3, we exploit the energy bounds and derive the first two main results, i.e. Theorems 1 and 2. In fact, both will be rather straightforward corollaries of a separate concentration-compactness result in the spirit of [21], i.e. Proposition 1.

Section 4 contains the proof of Theorem 3. The main point are regularity arguments in the spirit of [29], which exploit the energy bounds to rule out blow-up on a uniform time interval.

Finally, in the Appendix, we provide a few supplementary, technical results: A cut-off lemma similar to the ones used for example in [21, 17], which enters the proof of Proposition 1; the explicit construction of a “stream function” that is needed in the upper-bound construction in Lemma 3 for $p = 2$; and the derivation of (6).

Notation and preliminaries. Throughout the paper, we shall use the convention

$$\nabla \times m = \begin{pmatrix} \nabla \times m_1 \\ \nabla \times m_2 \\ \nabla \times m_3 \end{pmatrix}$$

for $m = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}$.

where

$$\nabla \times m = \partial_1 m_2 - \partial_2 m_1 \quad \text{and} \quad \nabla \times m_3 = -\nabla \hat{e}_3 = \begin{pmatrix} \partial_2 m_3 \\ -\partial_1 m_3 \end{pmatrix}.$$
parts formula above holds true. Accordingly the energy $E_\varepsilon(m) = D(m) + \varepsilon(H(m) + V(m))$, initially defined on $\mathcal{M}_0$, extends to a continuous integral functional on $\mathcal{M}$

$$E_\varepsilon(m) = \int_{\mathbb{R}^2} e_\varepsilon(m) \, dx$$

with integrable density

$$e_\varepsilon(m) = \frac{1}{2} |\nabla m|^2 + \varepsilon \left( (m - \hat{e}_3) \cdot \nabla \times m + \frac{1}{\varepsilon^2} |m - \hat{e}_3|^p \right).$$

For later purpose it will be convenient to introduce the topological charge density

$$\omega(m) = m \cdot (\partial_1 m \times \partial_2 m)$$

entering the definition of topological charge

$$Q(m) = \frac{1}{4\pi} \int_{\mathbb{R}^2} \omega(m) \, dx \in \mathbb{Z}$$

for $m \in \mathcal{M}_0$, which uniquely extends to $\mathcal{M}$ by virtue of Wente’s inequality \[32, 13\], and satisfies the classical topological lower bound $D(m) \geq 4\pi |Q(m)|$ for all $m \in \mathcal{M}$.

2. ENERGY BOUNDS

Both the treatments of the static and dynamic problem rely on good upper and lower bounds to the energy $E_\varepsilon$ in terms of $0 < \varepsilon \ll 1$. In fact, a major problem in extending our analysis to the physically relevant case $p = 2$ consists in the lack of a lower bound that matches the logarithmic upper bound in Theorem [1]. Due to the quadratic decay of the stereographic map $\Phi$ for $|x| \gg 1$, which leads to a logarithmically growing potential energy $V$ if $p = 2$, we conjecture the logarithmic upper bound to be optimal in terms of scaling.

From the above representations of $H$ and $V$ it follows

$$\left( H(m) \right)^2 \leq 32D(m) V(m) \quad \forall m \in \mathcal{M}.$$ 

By Young’s inequality we immediately infer the following lower energy bound:

**Lemma 1** (Boundedness in $\mathcal{M}$). Suppose $2 \leq p \leq 4$ and $\varepsilon > 0$. Then,

$$E_\varepsilon(m) \geq (1 - 16\varepsilon) D(m) + \frac{\varepsilon}{2} V(m) \quad \text{for any } m \in \mathcal{M}.$$ 

Using the helical derivatives \[9\], we can further improve the lower bound:

**Lemma 2** (Lower bound). Suppose $2 \leq p \leq 4$, $\varepsilon > 0$ and $m \in \mathcal{M} \setminus \{m \equiv \hat{e}_3\}$. Then

$$E_\varepsilon(m) \geq 4\pi Q(m) + \varepsilon \left( 1 - 2\varepsilon \frac{V_4(m)}{V_p(m)} \right) V_p(m)$$

and

$$E_\varepsilon(m) \geq \left( 1 - 2\varepsilon \frac{V_4(m)}{V_p(m)} \right) D(m) + 8\pi \varepsilon \frac{V_4(m)}{V_p(m)} Q(m).$$

The second lower bound is attained if and only if

$$\partial^*_i m + m \times \partial^*_2 m = 0,$$

where $\partial^*_i m = \partial_i m - \kappa \hat{e}_i \times m$,

holds for $\kappa = V_4(m) / 2V_p(m)$. In particular, for $Q(m) = -1$

$$E_\varepsilon(m) \geq D(m) \left( 1 - 4\varepsilon \frac{V_4(m)}{V_p(m)} \right) \geq 4\pi (1 - 4\varepsilon).$$
A corresponding upper bound in the homotopy class $Q(m) = -1$ is obtained by rescaling the stereographic map $\Phi$ appropriately. For $p = 2$, an additional cut-off procedure is needed.

Lemma 3 (Upper bound). Suppose $2 \leq p \leq 4$ and $\varepsilon > 0$. Then, there exists a smooth representative $\tilde{m} \in M$ in the homotopy class $Q = -1$ such that

$$\inf \{ E_\varepsilon(m) : m \in M, Q(m) = -1 \} \leq E_\varepsilon(\tilde{m})$$

$$= \begin{cases} 4\pi(1 - 2(p - 2)\varepsilon), & \text{if } 2 < p \leq 4, \\ 4\pi \left(1 - (4 + o(1))\frac{\varepsilon}{\|m\|^2}\right), & \text{if } p = 2 \text{ and } 0 < \varepsilon \ll 1. \end{cases}$$

For $p = 4$, upper and lower bounds match, so that the vector field $\tilde{m}$ actually is a minimizer of $E_\varepsilon$ in the homotopy class $Q = -1$.

Proof of Lemma 3. As in [21] we will employ the helical derivatives $D_1^\varepsilon$ as given in (9) and appeal to the following relation from [21, Proof of Lemma 3.2]:

Step 1: For any $m \in M$, we have

$$\frac{1}{2}\|\nabla m\|^2 - \omega(m) + \kappa((m - \tilde{e}_3) \cdot \nabla m + 2\kappa\frac{1}{4}(1 - m_3)^2)$$

$$= |D_1^\varepsilon m + m \times D_2^\varepsilon m|^2 \geq 0.$$

Indeed, using $|D_1^\varepsilon m + m \times D_2^\varepsilon m|^2 = |D_1^\varepsilon m|^2 + |D_2^\varepsilon m|^2 + 2D_1^\varepsilon m \cdot (m \times D_2^\varepsilon m)$, the claim immediately follows from

$$|D_1^\varepsilon m|^2 + |m \times D_2^\varepsilon m|^2 = \|\nabla m\|^2 + \kappa^2(1 + m_3^2) + 2\kappa m \cdot \nabla \times m$$

and

$$D_1^\varepsilon m \cdot (m \times D_2^\varepsilon m) = -\omega(m) - \kappa^2 m_3 - \kappa \tilde{e}_3 \cdot \nabla \times m.$$

Step 2: Conclusion. Recall that for $2 \leq p \leq 4$

$$V(m) = V_p(m) = \int_{\mathbb{R}^2} \left(\frac{1}{2}(1 - m_3)\right) \Phi \, dx.$$

Choosing $\kappa = \varepsilon$ in Step 1 and integrating over $\mathbb{R}^2$, the first claim follows as in [21].

With the choice of $\kappa = \frac{V_4(m)}{2V_4(m)}$, it follows that

$$D(m) - 4\pi Q(m) + \frac{V_4(m)}{2V_4(m)} (H(m) + V_p(m)) \geq 0,$$

i.e.

$$H(m) + V(m) \geq -2\frac{V_4(m)}{V_p(m)} (D(m) - 4\pi Q(m)).$$

Hence, we obtain the second lower bound:

$$E_\varepsilon(m) = D(m) + \varepsilon (H(m) + V(m)) \geq D(m) - 2\varepsilon \frac{V_4(m)}{V_p(m)} (D(m) - 4\pi Q(m)).$$

In particular, Step 1 implies that the inequality is sharp if and only if [4] holds for $\kappa = \frac{V_4(m)}{2V_4(m)}$. 
If $Q(m) = -1$, we can use the classical topological lower bound $D(m) \geq 4\pi |Q(m)| = 4\pi$ to conclude

$$E_\varepsilon(m) \geq D(m) - 2\varepsilon \frac{V_4(m)}{V_p(m)} (D(m) - 4\pi Q(m)) \geq \left(1 - 4\varepsilon \frac{V_4(m)}{V_p(m)} \right) D(m) \geq 4\pi(1 - 4\varepsilon).$$

□

Proof of Lemma 3. If $2 < p \leq 4$, we may just define

$$\tilde{m}: \mathbb{R}^2 \rightarrow S^2, \quad \tilde{m}(x) := \Phi(\lambda x),$$

for $\lambda > 0$ yet to be determined. Since $D(\Phi) = 4\pi$, $H(\Phi) = -8\pi$, $V(\Phi) = 2\pi/(p - 2)$ and

$$E_\varepsilon(\Phi_{\lambda^*}) = \min_{\lambda > 0} E_\varepsilon(\Phi_{\lambda}) = D(\Phi) - \frac{\varepsilon H(\Phi)^2}{4V(\Phi)}, \quad \lambda^* = -\frac{2V(\Phi)}{H(\Phi)},$$

by a simple scaling argument, we obtain the claim with $\lambda = \lambda^* = (2(p - 2))^{-1}$.

For $p = 2$, however, $\Phi \notin M$, since the potential energy $V(\Phi)$ diverges logarithmically. Thus, $\Phi$ needs to be cut off in a suitable way. To this end, for $R \gg 1$ to be chosen later, we fix a smooth function $f_R: [0, \infty) \rightarrow \mathbb{R}$ (see Figure 1 and the Appendix for an explicit construction) so that

$$0 \leq f_R'(r) \leq \frac{2}{1 + r^2}, \quad 0 \leq -f_R''(r) \leq \frac{C}{1 + r^2},$$

for all $r \geq R$.

Then, we define a smooth vector field $\Phi_R: \mathbb{R}^2 \rightarrow S^2$ via

$$\Phi_R(x) := \left( f_R'(|x|) \frac{x}{|x|}, \text{sgn}(|x| - 1) \sqrt{1 - (f_R'(|x|))^2} \right)^T, \quad x \in \mathbb{R}^2.$$

Note that $\Phi_R = \Phi$ on $B_R$ and $\Phi_R = \hat{e}_3$ on $\mathbb{R}^2 \setminus B_{2R}$. On $A_R := B_{2R} \setminus B_R$, we have

$$|\nabla \Phi_R(x)|^2 \leq \frac{C}{|x|^4}, \quad |\Phi_R(x) - \hat{e}_3|^2 \leq \frac{C}{|x|^2}, \quad x \in A_R.$$
Hence, we compute in polar coordinates

\[
\int_{B_R} \frac{1}{4} |\nabla \Phi_R|^2 dx = 4\pi \int_0^R \frac{2r}{(1+r^2)^2} dr \leq 4\pi,
\]

\[
\int_{B_R} \frac{1}{4} |\Phi_R - \varepsilon_3|^2 dx = \pi \int_0^R \frac{2r}{1+r^2} dr = \pi \ln(1+R^2),
\]

\[
\int_{A_R} \frac{1}{4} |\nabla \Phi_R|^2 dx \leq C \int_0^{2R} \frac{1}{r} dr = \frac{C}{r},
\]

\[
\int_{A_R} \frac{1}{4} |\Phi_R - \varepsilon_3|^2 dx \leq C \int_0^{2R} \frac{1}{r} dr = C.
\]

The region $\mathbb{R}^2 \setminus B_{2R}$ does not contribute to the energy. In particular, we have

\[
|Q(\Phi) - Q(\Phi_R)| \leq C \int_{B_R} |\nabla \Phi|^2 + |\nabla \Phi_R|^2 dx \ll 1 \quad \text{provided } R \gg 1,
\]

so that $Q(\Phi_R) = Q(\Phi_R) = -1$.

In order to estimate the contribution from the helicity, we exploit that

\[
\left( \Phi_{1,R} + \Phi_{2,R} \right)(x) \cdot \nabla \times \Phi_{3,R}(x) = \text{sgn}(|x| - 1) \frac{\left(f_R'(|x|)\right)^2 f_R''(|x|)}{\sqrt{1 - \left(f_R'(|x|)\right)^2}}
\]

\[
\begin{cases}
= -8\frac{|x|^2}{(1+|x|^2)^2}, & \text{for } 0 \leq |x| \leq R, \\
\leq 0, & \text{for } |x| \geq R.
\end{cases}
\]

Hence, using $\frac{d}{dr} \frac{r^4}{(1+r^2)^2} = 4\frac{r^3}{(1+r^2)^2}$, we find

\[
H(\Phi_R) = 2 \int_{\mathbb{R}^2} \left( \Phi_{1,R} + \Phi_{2,R} \right) \cdot \nabla \times \Phi_{3,R} dx \leq -32\pi \int_0^R \left( \frac{r}{1+r^2} \right)^3 dr = -8\pi \frac{R^4}{(1+R^2)^2}.
\]

Summarizing, for sufficiently large $R \gg 1$, we have obtained

\[
D(\Phi_R) \leq 4\pi + \frac{C}{r},
\]

\[
H(\Phi_R) \leq -8\pi \left( \frac{R^2}{R^2+1} \right)^2 \leq -8\pi + \frac{C}{r},
\]

\[
V(\Phi_R) \leq \pi \ln(1+R^2) + C.
\]

Defining

\[
\hat{m} : \mathbb{R}^2 \to \mathbb{S}^2, \quad \hat{m}(x) = \Phi_R(\lambda x),
\]

where $\lambda > 0$ will be chosen below, and rescaling, we arrive at

\[
E_\varepsilon(\hat{m}) = D(\hat{m}) + \varepsilon \lambda^{-1} \left( H(\hat{m}) + \lambda^{-2} V(\hat{m}) \right)
\]

\[
\leq 4\pi + \frac{C}{r} + \varepsilon \lambda^{-1} (-8\pi + \lambda^{-1} \pi \ln(1+R^2) + C(R^{-2} + \lambda^{-1})).
\]

Now, choose $R = \varepsilon^{-\frac{1}{2}} |\ln \varepsilon|$ and let $\lambda = L |\ln \varepsilon|$ for $L > 0$ fixed and $0 < \varepsilon \ll 1$. Then,

\[
E_\varepsilon(\hat{m}) \leq 4\pi + \frac{C}{|\ln \varepsilon|} \left( -\frac{8\pi}{L^2} + \frac{C}{L^2} + o(1) \right) \quad \text{for } 0 < \varepsilon \ll 1,
\]

which turns into the claim for $L = \frac{1}{4}$.
3. Compactness and proofs of Theorems 1 and 2

In this section, we prove existence of minimizers $m_\varepsilon$ of $E_\varepsilon$ under the constraint $Q = -1$, and their strong convergence to a unique harmonic map $m_0 \in C$ as $\varepsilon \to 0$. In fact, both results rely on P. L. Lions’ concentration-compactness principle. We state the common part as a separate compactness result – Proposition 1 – from which Theorems 1 and 2 can be deduced easily:

**Proposition 1.** Suppose $2 \leq p < 4$ and consider positive numbers $\{\varepsilon_k\}_{k \in N} \subset \mathbb{R}$ so that $\varepsilon_\infty := \lim_{k \to \infty} \varepsilon_k$ exists and satisfies $0 \leq \varepsilon_\infty \ll 1$. Define

$$I := \inf_{m \in M} E_{\varepsilon_\infty}(m) \begin{cases} 4\pi, & \text{if } \varepsilon_\infty = 0, \\ < 4\pi, & \text{if } \varepsilon_\infty > 0. \end{cases}$$

Moreover, let $\{m_k\}_k \subset M$ be asymptotically minimizing in the homotopy class $Q = -1$; that is, suppose that

$$Q(m_k) = -1 \quad \text{and} \quad \lim_{k \to \infty} E_{\varepsilon_k}(m_k) = I.$$  

Finally, assume

$$\liminf_{k \to \infty} (-H(m_k)) > 0 \quad \text{as well as} \quad \limsup_{k \to \infty} (V(m_k) - H(m_k)) < \infty.$$  

Then, up to translations and a subsequence, there exists $m_\infty \in M$ with $Q(m_\infty) = -1$ so that

$$\nabla m_k \rightharpoonup \nabla m_\infty \quad \text{weakly in } L^2(\mathbb{R}^2),$$  

$$m_k \rightharpoonup m_\infty \quad \text{weakly in } L^q(\mathbb{R}^2) \quad \text{for all } p \leq q < \infty,$$

$$1 - m_{3,k} \rightharpoonup 1 - m_{3,\infty} \quad \text{weakly in } L^q(\mathbb{R}^2) \quad \text{for all } \frac{q}{2} \leq q < \infty,$$

$$m_k \to m_\infty \quad \text{strongly in } L^q_{\text{loc}}(\mathbb{R}^2) \quad \text{for all } 1 \leq q < \infty,$$

and

$$\liminf_{k \to \infty} E_{\varepsilon_k}(m_k) \geq E_{\varepsilon_\infty}(m_\infty).$$

In particular, the infimum $I$ is attained by $m_\infty \in M$.

In the case $p = 2$ with $\varepsilon_\infty = 0$, the above result does not apply to families of minimizers $\{m_\varepsilon\}_\varepsilon$ of $E_\varepsilon$, since we are unable to verify the bounds on $-H(m_\varepsilon)$ and $V(m_\varepsilon)$ as $\varepsilon \to 0$ (in fact, in the given scaling, we expect $H(m_\varepsilon) \to 0$ as $\varepsilon \to 0$). For $p = 4$, on the other hand, the proof fails, since we cannot exclude “vanishing” in the concentration-compactness alternative – in the derivation of Theorem 1 we will instead exploit the matching upper and lower bounds to $E_\varepsilon$.

Before turning to the proof of Proposition 1 however, we will deduce both Theorem 1 and Theorem 2

**Proof of Theorem 1.**

**Step 1 (The case $p = 4$):** For $p = 4$, we may appeal to the matching upper and lower bounds Lemma 2 and 3. That is,

$$m_\varepsilon : \mathbb{R}^2 \to S^2, \quad m_\varepsilon(x) := \Phi\left(\frac{x}{\varepsilon^{\frac{p}{2(p-2)}}}\right),$$

is a minimizer of $E_\varepsilon$ in the homotopy class $Q = -1$. Moreover, by Lemma 2 any minimizer $\tilde{m} \in M$ of $E_\varepsilon$ must satisfy (2) for $\kappa = \frac{V_4(m)}{V_4(\tilde{m})} = \frac{1}{2}$.

**Step 2 (The case $2 \leq p < 4$):** When $V = V_p$ represents the classical Zeeman interaction, that is for $p = 2$, the existence of a minimizer $m_\varepsilon$ of $E_\varepsilon$ in the homotopy
class \( Q = -1 \) has been shown in \cite{21}. However, the same approach can be used for the whole range \( 2 \leq p < 4 \): Consider a minimizing sequence \( \{ m_k \}_{k \in \mathbb{N}} \subset \mathcal{M} \) for \( E_\varepsilon \) with \( Q(m_k) = -1 \), and let \( 0 < \varepsilon_k \equiv \varepsilon \ll 1 \). Lemma \ref{lem:bound} yields for \( 2 < p < 4 \)
\[
\lim_{k \to \infty} E_\varepsilon(m_k) = \inf \{ E_\varepsilon(m) : m \in \mathcal{M}, \; Q(m) = -1 \} \leq 4\pi(1 - 2(p - 2)\varepsilon).
\]
Hence, using that \( p < C \) the constant 0 lower bound yield \( \pi \) class \( H \) harmonic map with an \( L^2 \) perturbation as considered in \cite{22} (see also \cite{13}). Hence, \( Q \) constructed before is a local minimizer of \( E \) so that \( \lim_{k \to \infty} \frac{\varepsilon_k H(m_k)}{4} = 8\pi(1 - 2(p - 2)\varepsilon) = 8\pi(p - 2)\varepsilon > 0 \).

If \( p = 2 \), we can use the upper bound \( 4\pi(1 - (4 + o(1))^{1/2}) < 4\pi \) to arrive at the same conclusion \( \lim_{k \to \infty} (-H(m_k)) > 0 \).

On the other hand, we may use Lemma \ref{lem:bound} to obtain
\[
\sqrt{\varepsilon} \limsup_{k \to \infty} |H(m_k)| \lesssim \limsup_{k \to \infty} (D(m_k) + \varepsilon V(m_k)) \leq \limsup_{k \to \infty} E_\varepsilon(m_k) \leq 4\pi,
\]
i.e.
\[
\limsup_{k \to \infty} (V(m_k) - H(m_k)) < \infty.
\]
Hence, we may apply Proposition \ref{prop:prop} to obtain convergence (up to a subsequence and translations) of \( \{ m_k \}_{k \in \mathbb{N}} \) to a limit \( m_\infty \in \mathcal{M} \) with \( Q(m_\infty) = -1 \) and
\[
I = \lim_{k \to \infty} E_\varepsilon(m_k) = E_\varepsilon(m_\infty) \geq I.
\]
Thus, \( m_\infty \) minimizes \( E_\varepsilon \) in the class \( \mathcal{M} \), subject to the constraint \( Q = -1 \). By the \( H^1 \) continuity of the topological charge \( Q(m) \), the constrained minimizer \( m_\infty \in \mathcal{M} \) constructed before is a local minimizer of \( E_\varepsilon(m) \) in \( \mathcal{M} \) and as such an almost harmonic map with an \( L^2 \) perturbation as considered in \cite{22} (see also \cite{13}). Hence, \( m_\infty \) is Hölder continuous. \hfill \( \square \)

Proof of Theorem \ref{thm:main}. By the lower bound Lemma \ref{lem:bound} we may assume w.l.o.g. that the constant \( 0 < C_0 < \infty \) satisfies
\[
(10) \quad 4\pi - C_0^{-1}\varepsilon \leq E_\varepsilon(m_\varepsilon) \leq 4\pi - C_0\varepsilon.
\]

Step 1 (Verification of the assumptions of Proposition \ref{prop:prop}): We prove
\[
\lim_{\varepsilon \to 0} D(m_\varepsilon) = 4\pi, \quad \liminf_{\varepsilon \to 0} (-H(m_\varepsilon)) > 0 \; \text{and} \; \limsup_{\varepsilon \to 0} (V(m_\varepsilon) - H(m_\varepsilon)) < \infty.
\]

Indeed, we have
\[
-H(m_\varepsilon) = \frac{1}{\varepsilon} \left( D(m_\varepsilon) + \varepsilon V(m_\varepsilon) - E_\varepsilon(m_\varepsilon) \right) \geq \frac{1}{\varepsilon} \left( 4\pi - (4\pi - C_0\varepsilon) \right) = C_0,
\]
so that \( \liminf_{\varepsilon \to 0} (-H(m_\varepsilon)) > 0 \). On the other hand, Lemma \ref{lem:bound} and the topological lower bound yield
\[
4\pi \leq D(m_\varepsilon) \leq \frac{1}{\varepsilon} E_\varepsilon(m_\varepsilon) \leq \frac{4\pi - C_0\varepsilon}{1 - 4\varepsilon} \to 4\pi \; \text{as} \; \varepsilon \to 0.
\]
Hence, \( D(m_\varepsilon) \to 4\pi \) for \( \varepsilon \to 0 \).
Due to (8), it remains to prove that \( V(m) \) is bounded uniformly in \( 0 < \varepsilon \ll 1 \).
Indeed, from Lemma 1 we obtain

\[
\frac{\pi}{2} V(m) \leq \frac{E(m)}{2} - 4\pi (1 - 16\varepsilon) \leq 64\pi \varepsilon \quad \forall \ 0 < \varepsilon \ll 1.
\]

Thus, \( \limsup_{k \to \infty} \left( V(m_k) - H(m_k) \right) < \infty \).

**Step 2** (Proof of part i)): By Step 1, we may apply Proposition 1. Hence, there exists \( m_0 \in M \) with \( Q(m_0) = -1 \) so that in the limit \( \varepsilon \to 0 \), along a subsequence and up to translations (not relabeled):

\[
\begin{align*}
\nabla m_\varepsilon \to & \quad \nabla m_0 \quad \text{weakly in } L^2(\mathbb{R}^2), \\
m_\varepsilon \to & \quad m_0 \quad \text{weakly in } L^q(\mathbb{R}^2) \text{ for all } p \leq q < \infty, \\
1 - m_{3,\varepsilon} \to 1 - m_{3,0} \quad \text{weakly in } L^q(\mathbb{R}^2) \text{ for all } \frac{2}{q} \leq q < \infty, \\
m_\varepsilon \to & \quad m_0 \quad \text{strongly in } L^q_{\text{loc}}(\mathbb{R}^2) \text{ for all } 1 \leq q < \infty,
\end{align*}
\]

Since, by Step 1 and \( Q(m_0) = -1 \), we have \( 4\pi = \liminf_{\varepsilon \to 0} D(m_\varepsilon) \geq D(m_0) \geq 4\pi \), weak convergence \( \nabla m_\varepsilon \to \nabla m_0 \) upgrades to strong convergence in \( L^2(\mathbb{R}^2) \). In particular, \( m_0 \in C \), which proves the first part of the claim.

**Step 3** (Proof of part ii)): Assume that

\[
E_\varepsilon(m) \leq 4\pi + \varepsilon \min_{m \in C} (H(m) + V(m)) + o(\varepsilon)
\]

holds as \( \varepsilon \to 0 \), i.e.

\[
\limsup_{\varepsilon \to 0} \varepsilon^{-1}(E_\varepsilon(m) - D(m_0)) \leq \min_{m \in C} (H(m) + V(m)).
\]

By Step 2, we have \( \nabla m_\varepsilon \to \nabla m_0 \) strongly in \( L^2(\mathbb{R}^2) \) and \( 1 - m_{3,\varepsilon} \to 1 - m_{3,0} \) weakly in \( L^q(\mathbb{R}^2) \) and \( L^q(\mathbb{R}^2) \) along a suitable subsequence as \( \varepsilon \to 0 \). Thus, we obtain

\[
\begin{align*}
\lim_{\varepsilon \to 0} H(m_\varepsilon) &= H(m_0), \\
\liminf_{\varepsilon \to 0} V(m_\varepsilon) &\geq V(m_0),
\end{align*}
\]

and, using that \( D(m_\varepsilon) \geq 4\pi = D(m_0) \),

\[
\liminf_{\varepsilon \to 0} \varepsilon^{-1}(E_\varepsilon(m) - D(m_0)) \geq H(m_0) + V(m_0) \geq \min_{m \in C} (H(m) + V(m)).
\]

Therefore,

\[
\varepsilon^{-1}(E_\varepsilon(m) - D(m_0)) \to \min_{m \in C} (H(m) + V(m)) = H(m_0) + V(m_0) \quad \text{as } \varepsilon \to 0.
\]

In particular, we obtain

\[
\lim_{\varepsilon \to 0} \varepsilon^{-1}(D(m_\varepsilon) - D(m_0)) = 0 \quad \text{and} \quad \lim_{\varepsilon \to 0} V(m_\varepsilon) = V(m_0).
\]

Hence, \( m_\varepsilon \to m_0 \) strongly in \( L^p(\mathbb{R}^2) \), i.e. \( d(m_\varepsilon, m_0) \to 0 \) as \( \varepsilon \to 0 \), up to translations and a suitable subsequence.

Recall that (with the identification \( \mathbb{R}^2 \simeq C \)) \( m \in C \) may be represented as

\[
m(x) = m^{(\rho, \varphi)}(x) = \Phi(ax + b) = e^{i\varphi} \Phi(\rho x + \tilde{b})
\]

for two complex numbers \( a = \rho e^{i\varphi} \neq 0 \) and \( b \), with \( \tilde{b} = a^{-1}b \). Thus, dropping \( b \) due to the translation invariance of the problem, minimization is a finite dimensional problem; in fact, we have

\[
H(m^{(\rho, \varphi)}) + V(m^{(\rho, \varphi)}) = \frac{\cos \varphi}{\rho} H(\Phi) + \frac{V(\Phi)}{\rho^2} = -\frac{8\pi \cos \varphi}{\rho} + \frac{2\pi}{\rho^2 (\rho - 2)}.
\]
which obviously is minimized by \( \varphi \in 2\pi\mathbb{Z} \) and \( \rho = \frac{1}{2(p-2)} \). Hence, up to translation, the unique minimizer of \( H + V \) in \( C \) is given by

\[
m_0(x) = \Phi(\rho x) \quad \text{with} \quad \rho = \frac{1}{2(p-2)} = -2 \frac{V(\Phi)}{\mu(\Phi)}.
\]

In particular, the whole sequence \( \{m_\varepsilon\}_{\varepsilon > 0} \) converges with respect to \( d \), up to translations, to the unique limit \( m_0 \). \( \square \)

It remains to prove Proposition \( \dagger \)

**Proof of Proposition \( \dagger \)** We first remark that in view of \( \natural \) and Lemma \( \natural \), the assumptions also imply

\[
\limsup_{k \to \infty} D(m_k) < \infty \quad \text{and} \quad \liminf_{k \to \infty} V(m_k) > 0.
\]

Moreover, we will use the symbol \( \lesssim \) to indicate that an inequality holds up to a universal, multiplicative constant that may change from line to line.

**Step 1:** We prove:

\[
|H(m_k)| \lesssim \left( \sup_{y \in \mathbb{R}^2} \left( \int_{B_1(y)} |\nabla m_k|^2 \, dx \right)^{\frac{1}{2}} + \sup_{y \in \mathbb{R}^2} \left( \int_{B_1(y)} |\nabla m_k|^2 \, dx \right)^{\frac{3}{2} - \frac{1}{2}} \right) \\
\times \left( D(m_k) + V(m_k) \right) \quad \forall k \in \mathbb{N}.
\]

Indeed, choose \( \delta > 0 \) so that \( \bigcup_{y \in \mathbb{R}^2} B_1(y) = \mathbb{R}^2 \). Then, we have

\[
\left| \int_{\mathbb{R}^2} (1 - m_{3,k})(\nabla \times m_k) \, dx \right| \lesssim \sum_{y \in \mathbb{Z}^2} \left( \int_{B_1(y)} (m_{3,k} - 1)^2 \, dx \right)^{\frac{1}{2}} \left( \int_{B_1(y)} |\nabla m_k|^2 \, dx \right)^{\frac{1}{2}}.
\]

Moreover, the Sobolev embedding theorem and Jensen’s inequality yield

\[
\left( \int_{B_1(y)} |m_k - \hat{e}_3|^4 \, dx \right)^{\frac{1}{4}} \lesssim \int_{B_1(y)} |\nabla m_k|^2 + |m_k - \hat{e}_3|^2 \, dx
\]

\[
\lesssim \int_{B_1(y)} |\nabla m_k|^2 \, dx + \left( \int_{B_1(y)} \frac{1}{\varepsilon^p} |m_k - \hat{e}_3|^p \, dx \right)^{\frac{2}{p}},
\]

so that, using Young’s inequality in the last step,

\[
|H(m_k)| \lesssim \sum_{y \in \mathbb{Z}^2} \left( \int_{B_1(y)} |\nabla m_k|^2 \, dx \right)^{\frac{1}{2}}
\]

\[
+ \sum_{y \in \mathbb{Z}^2} \left( \int_{B_1(y)} |\nabla m_k|^2 \, dx \right)^{\frac{3}{2}} \left( \int_{B_1(y)} \frac{1}{\varepsilon^p} |m_k - \hat{e}_3|^p \, dx \right)^{\frac{2}{p}}
\]

\[
\leq \sup_{y \in \mathbb{R}^2} \left( \int_{B_1(y)} |\nabla m_k|^2 \, dx \right)^{\frac{1}{2}} D(m_k)
\]

\[
+ \sup_{y \in \mathbb{R}^2} \left( \int_{B_1(y)} |\nabla m_k|^2 \, dx \right)^{\frac{3}{2} - \frac{1}{2}} \left( D(m_k) + V(m_k) \right),
\]

which is the claim.

**Step 2** (Concentration-compactness): We consider the full energy density \( \natural \) to define \( \rho_k := \varepsilon_\natural(m_k) \geq 0 \). Note that we have

\[
\rho_k \gtrsim |\nabla m_k|^2 + \varepsilon_k \frac{1}{\varepsilon^p} |m_k - \hat{e}_3|^p \quad \forall k \in \mathbb{N}
\]
COMPACTNESS RESULTS OF CHIRAL SKYRMIONS

\[ \lim_{k \to \infty} \int_{\mathbb{R}^2} \rho_k \, dx = I > 0. \]

Hence, we may apply the concentration-compactness lemma (see, e.g., [19]) to the sequence \( \{\rho_k\}_{k \in \mathbb{N}} \) of non-negative densities and obtain that, up to a subsequence, one of the following holds:

- **Compactness**: There exists a sequence \( \{y_k\}_{k \in \mathbb{N}} \subset \mathbb{R}^2 \) so that
  \[ \forall \delta > 0: \exists \mathcal{R} < \infty: \int_{\mathbb{R}^2 \setminus B_{\mathcal{R}}(y_k)} \rho_k \, dx \leq \delta. \]

- **Vanishing**: We have
  \[ \lim_{k \to \infty} \sup_{y \in \mathbb{R}^2} \int_{B_{\mathcal{R}}(y)} \rho_k \, dx = 0 \quad \forall \mathcal{R} < \infty. \]

- **Dichotomy**: There exist \( a^{(1)}, a^{(2)} > 0 \) so that \( a^{(1)} + a^{(2)} = I \) and for all \( \delta > 0 \), there exist \( k_0 \in \mathbb{N}, \{y_k\}_{k \in \mathbb{N}} \subset \mathbb{R}^2, R < \infty \), and a sequence \( R_k \to \infty \), so that for \( k \geq k_0 \):
  \[ |a^{(1)} - \int_{B_{\mathcal{R}}(y_k)} \rho_k \, dx| + |a^{(2)} - \int_{\mathbb{R}^2 \setminus B_{R_k}(y_k)} \rho_k \, dx| + \left| \int_{B_{R_k}(y_k) \setminus B_{\mathcal{R}}(y_k)} \rho_k \, dx \right| \leq \delta. \]

In order to conclude, we need to rule out vanishing and dichotomy.

**Step 2a (Ruling out “Vanishing”)**: Suppose vanishing holds. Since \( \rho_k \) controls \( \frac{1}{2} |\nabla m_k|^2 \), while \( V(m_k) \) is bounded by assumption, Step 1 yields \( \lim_{k \to \infty} H(m_k) = 0 \), contradicting the assumption \( \lim \inf_{k \to \infty} (-H(m_k)) > 0 \).

**Step 2b (Ruling out “Dichotomy”)**: Suppose dichotomy holds. In particular, for fixed \( 0 < \delta \ll 1 \) (to be specified later), we have

\[ \int_{B_{R_k} \setminus B_{\mathcal{R}}} |\nabla m_k|^2 + \varepsilon_k \frac{1}{p} |m_k - \hat{e}_3|^p \, dx \lesssim \int_{B_{R_k} \setminus B_{\mathcal{R}}} \rho_k \, dx \leq \delta. \]

W.l.o.g., we may assume that \( R^2 \varepsilon_k^{\frac{2}{p-2}} \geq 1 \) and \( k \gg 1 \), so that \( R_k \geq 4R \).

If \( \varepsilon_{\infty} = 0 \), we may apply Lemma 8 with \( \sigma = 0 \), otherwise with \( \sigma = 1 \), and define \( m_k^{(i)} \in M, i = 1, 2 \), so that for some constant \( C(\delta, R) \) and \( c_k \in [R, 2R] \):

\[ m_k^{(1)} = m_k \quad \text{on } B_{c_k}, \quad V(m_k^{(1)}) \lesssim C(\delta, R), \]
\[ m_k^{(2)} = m_k \quad \text{on } \mathbb{R}^2 \setminus B_{2c_k}, \quad V(m_k^{(2)}) \lesssim V(m_k) + C(\delta, R), \]

and

\[ \int_{\mathbb{R}^2 \setminus B_{2c_k}} |\nabla m_k^{(1)}|^2 + \varepsilon_k \frac{1}{p} |m_k^{(1)} - \hat{e}_3|^p \, dx \]
\[ + \int_{B_{2c_k}} |\nabla m_k^{(2)}|^2 + \varepsilon_k \frac{1}{p} |m_k^{(2)} - \hat{e}_3|^p \, dx \]
\[ \lesssim \delta + \sigma \left( \frac{\varepsilon_{\infty}}{R} \right)^{2/p} \lesssim \delta. \]
In particular, we have
\[
|Q(m_k^{(1)}) + Q(m_k^{(2)}) - Q(m_k)| \leq \left| \frac{1}{4\pi} \int_{B_{2r_k} \setminus B_{r_k}} \omega(m_k) \, dx \right|
+ \left| Q(m_k^{(1)}) - \frac{1}{4\pi} \int_{B_{r_k}} \omega(m_k) \, dx \right| + \left| Q(m_k^{(2)}) - \frac{1}{4\pi} \int_{\mathbb{R}^2 \setminus B_{2r_k}} \omega(m_k) \, dx \right| \leq \delta.
\]
Hence, since \(Q(m_k) = -1\) and \(Q(m_k^{(i)}) \in \mathbb{Z}, i = 1, 2\), we obtain
\[
Q(m_k^{(1)}) + Q(m_k^{(2)}) = Q(m_k) = -1.
\]
Moreover, using the estimate
\[
|H(m_k^{(i)})| \lesssim \left( D(m_k^{(i)}) \sqrt{|V(m_k^{(i)})|} \right)^{\frac{1}{2}},
\]
which also holds localized to \(B_{2r_k}\) and \(\mathbb{R}^2 \setminus B_{r_k}\), respectively, the “dichotomy” condition yields
\[
(11) \quad E(m_k^{(i)}) \leq a^{(i)} + C\sqrt{\delta} < I \leq 4\pi \quad \text{if} \ \delta \ll 1, \ \text{for} \ i = 1, 2.
\]
If \(|Q(m_k^{(i)})| \geq 2\) for some \(i \in \{1, 2\}\), Lemma 8 and (8), and note that
\[
\pi > E(m_k^{(i)}) \geq 3\pi |Q(m_k^{(i)})| \geq 6\pi \quad \text{if} \ 0 < \varepsilon \ll 1.
\]
Moreover, \(Q(m_k^{(i)}) = 1\) for some \(i \in \{1, 2\}\) yields \(Q(m_k^{(3-i)}) = -2\), which leads to the same contradiction as above.

Thus, we have \(Q(m_k^{(i)}) \in \{-1, 0\}\) for \(i = 1, 2\), i.e. there exists \(i_0 \in \{1, 2\}\) with \(Q(m_k^{(i_0)}) = -1\).

If \(\varepsilon = 0\), we directly obtain a contradiction, since \(m_k^{(i_0)}\) is admissible in the variational problem \(I\), hence
\[
I \leq E(m_k^{(i_0)}) \ll I. \quad \text{4}
\]
If \(\varepsilon > 0\), we use that \(H(m_k^{(i_0)}) + V(m_k^{(i_0)})\) remains bounded by construction (see Lemma 8 and 9, and note that \(R\) and hence also \(C(\delta, R)\) depend on \(\delta\), but not on \(k\)), and thus
\[
4\pi > a^{(i_0)} + C\sqrt{\delta} \geq \liminf_{k \to \infty} E_{c_k}(m_k^{(i_0)}) \geq \liminf_{k \to \infty} D(m_k^{(i_0)}) \geq 4\pi. \quad \text{4}
\]
Therefore, dichotomy cannot occur.

**Step 3 (Conclusion):** By Step 2, we may assume that compactness holds in the concentration-compactness alternative. W.l.o.g., \(y_k = 0\) for all \(k \in \mathbb{N}\). By passing to a subsequence and using Rellich’s theorem, we may assume that there exists \(m_\infty \in \dot{H}^1(\mathbb{R}^2; \mathbb{S}^2)\) such that
\[
\nabla m_k \rightharpoonup \nabla m_\infty \quad \text{weakly in} \ \mathcal{L}^2(\mathbb{R}^2),
\]
\[
m_k \rightharpoonup m_\infty \quad \text{weakly in} \ \mathcal{L}^p(\mathbb{R}^2),
\]
\[
1 - m_{3,k} \rightharpoonup 1 - m_{3,\infty} \quad \text{weakly in} \ \mathcal{L}^\frac{p}{2}(\mathbb{R}^2),
\]
\[
m_k \rightharpoonup^* m_\infty \quad \text{weak-* in} \ \mathcal{L}^\infty(\mathbb{R}^2),
\]
\[
m_k \to m_\infty \quad \text{strongly in} \ \mathcal{L}^p_{\text{loc}}(\mathbb{R}^2) \quad \text{for} \ 1 \leq p < \infty.
\]
Since compactness holds, we have (see [21, Lemma 4.1])
\[ I = \liminf_{k \to \infty} (E_{\varepsilon_k}(m_k) + 4\pi Q(m_k)) + 4\pi \geq E_{\varepsilon_{\infty}}(m_{\infty}) + 4\pi Q(m_{\infty}) + 4\pi. \]

If \( \varepsilon_{\infty} > 0 \), i.e. \( I < 4\pi \), we may immediately exclude \( Q(m_{\infty}) \geq 0 \). On the other hand, Lemma 4 in form of the inequality \( E_{\varepsilon_{\infty}}(m_{\infty}) \geq 4\pi(1 - 16\varepsilon_{\infty})|Q(m_{\infty})| \) rules out \( |Q(m_{\infty})| \geq 2 \), if \( \varepsilon_{\infty} \) is sufficiently small. Hence, we have \( Q(m_{\infty}) = -1 \).

If \( \varepsilon_{\infty} = 0 \), i.e. \( I = 4\pi \), we may argue similarly to obtain \( Q(m_{\infty}) \in \{-1, 0\} \). Moreover, if \( Q(m_{\infty}) = 0 \), we obtain \( E_{\varepsilon_{\infty}}(m_{\infty}) = D(m_{\infty}) = 0 \), i.e. \( m_{\infty} \) is const. In particular, using the “compactness” condition and the initial assumption \( \limsup_{k \to \infty} V(m_k) < \infty \) to reduce the problem to a bounded set, we obtain \( H(m_k) \to 0 \). Hence, also for \( \varepsilon_{\infty} = 0 \), we have \( Q(m_{\infty}) = -1 \). □

4. REGULARITY OF THE DYNAMIC PROBLEM AND PROOF OF THEOREM

Let us now consider the pulled back Landau-Lifshitz-Gilbert equation
\[ (\partial_t - \nu \partial_x) m = \left[ \alpha(\partial_t - \nu \partial_x) m - h_{\text{eff}}(m) \right] \]
on \( \mathbb{R}^2 \times [0, T] \) as motivated in the introduction. The effective field reads
\[ h_{\text{eff}}(m) = \Delta m - \varepsilon (2\nabla \times m + f(m)). \]

According to our choice of potential energy we have for \( p = 4 \)
\[ f(m) = \frac{1}{4} |m - \hat{e}_3|^2(m - \hat{e}_3) \]
which is smooth.

**Local well-posedness.** Starting from spatial discretization as in [30, 17] or spectral truncation as in [20, 31] one obtains for initial conditions \( m^0 \in M \) such that \( \nabla m^0 \in H^2(\mathbb{R}^2) \) a local solution \( m : \mathbb{R}^2 \times [0, T^*) \to S^2 \) for some terminal time \( T^* > 0 \), which is bounded below in terms of \( \| \nabla m^0 \|_{H^2} \), such that for all \( T < T^* \)
\[ E_{\varepsilon}(m) \in L^\infty(0, T) \quad \text{and} \quad \nabla m \in L^\infty(0, T; H^2(\mathbb{R}^2)) \cap L^2(0, T; H^3(\mathbb{R}^2)). \]

Initial data \( m^0 \) and \( \nabla m^0 \) are continuously attained in \( M \) and \( H^3(\mathbb{R}^2) \), respectively, see [31]. As \( \nabla m \in W^{1, \infty}(0, T; L^2(\mathbb{R}^2)) \), interpolation and Sobolev embedding yield uniform Hölder continuity of \( \nabla m \) in \( \mathbb{R}^2 \times [0, T] \). Uniqueness in this class can be shown by means of a Gronwall argument as in [20, 31]. Due to the slow decay of \( m - \hat{e}_3 \), the conventional \( L^2 \)-distance is replaced by a suitably weighted \( L^2 \)-distance, e.g.
\[ \| u \|_{L^2}^2 := \int_{\mathbb{R}^2} \frac{|u(x)|^2}{1 + |x|^2} \, dx \lesssim \| u \|_{L^4}^2. \]

As \( \nabla m(t) \in H^3(\mathbb{R}^2) \) for almost every \( t < T^* \), uniqueness and a bootstrap argument imply \( \nabla m \in L^\infty_{loc}(0, T^*; H^k(\mathbb{R}^2)) \) for arbitrary \( k \in \mathbb{N} \), in particular \( m \) is smooth. Now one may deduce the following Sobolev estimate (which equally holds true for approximate equations)
\[ \sup_{0 \leq t \leq T} \| \nabla m(t) \|_{H^k}^2 \lesssim \int_0^T \| \nabla m(t) \|_{H^{k+1}}^2 \, dt \leq c \left( 1 + \sup_{t \in [0, T]} \| \nabla m(t) \|_{L^\infty}^2 \right) \int_0^T \| \nabla m(t) \|_{H^k}^2 \, dt \]
for all $0 \leq k \leq 2$ and $0 < T < T^\ast$ (cf. Lemma \[\text{[4]}\] below). Hence, if $T^\ast < \infty$, then
\[ \limsup_{t \to T^\ast} \| \nabla m(t) \|_{L^\infty} = \infty. \]

**Local Sobolev estimates.** Due to lower order perturbations, \[\text{[6]}\] is translation-orthogonal but not dilation-invariant. However, with respect to transformations $\tilde{m}(x, t) = m(x_0 + \lambda x, t_0 + \lambda^2 t)$ the parameters $\epsilon$ and $\nu$ exhibit the following scaling behavior $\tilde{\epsilon} = \lambda \epsilon$ and $\tilde{\nu} = \lambda \nu$ while $\tilde{f}(m) = \lambda f(m)$. Hence, the coefficients of the lower order perturbations are uniformly bounded in the blow-up regime $\lambda \leq 1$. In this case we shall call $\tilde{m}$ a blow-up solution. We shall need a localized version of the a priori estimates from \[\text{[20]}\] that led to the existence result. Here and in the sequel let
\[ P_R = B_R \times (-R^2, 0), \]
the parabolic cylinder in space-time $\mathbb{R}^2 \times (-\infty, 0]$.

**Lemma 4.** Suppose $0 \leq k \leq 2$ and $m$ is a blow-up solution in a neighborhood of $P_R$ for some $R \geq 1$. Then
\[ \| \nabla m(0) \|^2_{H^k(B_{R/2})} + \int_{-R/2}^0 \| \nabla m(t) \|^2_{H^{k+1}(B_{R/2})} dt \leq c \left( 1 + \| \nabla m \|^2_{L^\infty(B_R)} \right) \int_{-R^2}^0 \| \nabla m(t) \|^2_{H^k(B_R)} dt \]
for a constant $c$ that only depends on the parameters $\alpha, \nu, \epsilon$. In particular,
\[ |\nabla m(0, 0)|^2 \leq C \int_{-R^2}^0 \| \nabla m(t) \|^2_{L^2(B_R)} dt \]
for a constant $C$ that only depends on the parameters $\alpha, \nu, \epsilon$ and $\| \nabla m \|_{L^\infty(B_R)}$.

**Sketch of proof.** The Landau-Lifshitz form of the equation reads
\[ (1 + \alpha^2) \partial_t m = \alpha (\Delta m + |\nabla m|^2 m) - \nabla \cdot (m \times \nabla m) + F(m, m), \]
for a smooth tangent field $F$ that is linear in $\nabla m$. The standard procedure uses test functions $\partial^\nu (\phi^2 \partial^\nu m)$, where $\nu$ is a multi index of length $1 \leq |\nu| \leq k + 1$, and $\phi(x, t) = \varphi(x) \eta(t)$ is an appropriate space-time cut-off function $0 \leq \phi \leq 1$ where $\varphi \in C_0^\infty(B_1)$ with $\varphi|_{B_{1/2}} = 1$ and $\eta \in C_0^\infty(\mathbb{R})$ with $\eta(t) = 0$ for $t < -1$ and $\eta(t) = 1$ for $t > -1/4$. In the case $R > 1$ one uses suitable rescalings of $\varphi$ and $\eta$.

Let us only estimate the contribution from the non-coercive term of second order $\nabla \cdot (m \times \nabla m)$:
\[ I = (\partial^\nu (m \times \nabla m), \nabla (\phi^2 \partial^\nu m)) = ((m \times \partial^\nu \nabla m + R_\nu), (\phi^2 \partial^\nu \nabla m + 2\phi \nabla \phi \partial^\nu m)), \]
which is bounded by
\[ \| \phi \partial^\nu \nabla m \|_{L^2} \left( 2\| \nabla \phi \partial^\nu m \|_{L^2} + \| \phi R_\nu \|_{L^2} \right) + 2\| \phi R_\nu \|_{L^2} \| \nabla \phi \partial^\nu m \|_{L^2}, \]
where $|R_\nu| \lesssim \sum_{|\ell| + |\mu| = |\nu| - 1} \| \nabla^\ell (\nabla m) \otimes \nabla^\mu (\nabla m) \|$. Hence for $t \in [-1, 0]$ fixed
\[ \| \phi R_\nu \|_{L^2} \leq \| \phi R_\nu \|_{L^2(B_1)} \leq c \| \nabla m \|_{L^\infty(B_1)} \| \nabla m \|_{H^k(B_1)}. \]

In fact, by Sobolev extension (preserving $L^\infty$ bounds) of $\nabla m|_{B_1}$ to a map $g \in L^\infty \cap H^k(\mathbb{R}^2; \mathbb{R}^6)$ with an equivalent $L^\infty \cap H^k$ bound, Moser’s product estimate applies. Hence for arbitrary $\delta > 0$
\[ |I| \leq \delta \| \phi \partial^\nu \nabla m \|^2_{L^2} + C(\delta) \left( 1 + \| \nabla m \|^2_{L^\infty(P_1)} \right) \| \nabla m(t) \|^2_{H^k(B_1)} \]
so that the first term can be absorbed for $\delta \lesssim \alpha$. \hfill \Box

**Energy estimates.** In proving Theorem 3 we shall argue on the level of energy. We have the following energy inequality for regular solutions $m = m_\varepsilon$ of (6) on a time interval $[0, T]$.

**Lemma 5 (Energy inequality).** There exists a universal constant $\lambda > 0$ such that for $\varepsilon \geq 0$ and $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$ smooth with compactly supported gradient

$$
\frac{\alpha}{2} \int_0^T \int_{\mathbb{R}^2} |\partial_\varepsilon m|^2 \varphi^2 dx dt + \left[ \int_{\mathbb{R}^2} e_\varepsilon(m(t)) \varphi^2 dx \right]_{t=0}^T 
\leq \frac{\lambda}{\alpha} \int_0^T \int_{\mathbb{R}^2} \left[ (1 + \alpha^2) \nu^2 |\partial_\varepsilon m(t)|^2 \varphi^2 + (|\nabla m|^2 + \varepsilon^2 |m - \varepsilon|)^2 \right] |\nabla\varphi|^2 dx dt.
$$

**Proof.** The claim follows from a standard argument based on the identity

$$
\alpha |\partial_\varepsilon m^2| - \nu (\alpha \partial_\varepsilon m + m \times \partial_\varepsilon m) \cdot \partial_\varepsilon m = h_\varepsilon(m) \cdot \partial_\varepsilon m,
$$

where the right hand side produces the time derivative of the density up to a divergence. The corresponding identity for the helicity term reads

$$
(\nabla \times m) \cdot \partial_\varepsilon m = \partial_t [(m_3 - 1) \nabla \times m] - \nabla \times [(m_3 - 1) \partial_t m].
$$

Integration by parts and Young’s inequality implies the claim. \hfill \Box

If $\varphi \equiv 1$ one can take $\lambda = \frac{1}{\alpha}$ and obtains in the case $Q(m) = -1$

$$
\frac{\alpha}{2} \int_0^T \int_{\mathbb{R}^2} |\partial_\varepsilon m|^2 dx dt + \left[ E_\varepsilon(m(t)) \right]_{t=0}^T \leq \frac{(1 + \alpha^2) \nu^2}{4\alpha} \int_0^T \left[ D(m(t)) - 4\pi \right] dt
$$

where we used that

$$
2 \int_{\mathbb{R}^2} |\partial_\varepsilon m(t)|^2 dx = D(m) - 4\pi.
$$

Lemma 2 implies for $\varepsilon \leq 1/8$ and $E_\varepsilon(m) < 4\pi$ that

$$
D(m) - 4\pi < 32\pi \varepsilon.
$$

**Proposition 2.** Suppose $0 < \varepsilon \leq 1/8$ and $E_\varepsilon(m(0)) \leq 4\pi - c\varepsilon$, then

$$
E_\varepsilon(m(T)) < 4\pi \quad \text{for all} \quad 0 < T < \frac{c\alpha}{32\pi(1 + \alpha^2)\nu^2}.
$$

Moreover as $\varepsilon \rightarrow 0$

$$
\sup_{0 \leq t \leq T} \int_{\mathbb{R}^2} |\partial_\varepsilon m(t)|^2 dx = O(\varepsilon) \quad \text{and} \quad \int_0^T \int_{\mathbb{R}^2} |\partial_\varepsilon m|^2 dx dt = O(\varepsilon).
$$

Next we show that the energy density $e_\varepsilon(m(t)) : \mathbb{R}^2 \rightarrow [0, \infty)$ remains concentrated along the flow. To this end we invoke Lemma 5 with $\varphi_R(x) = \varphi(x/R)$, where $\varphi(x) = 1$ for $|x| \geq 2$ and $\varphi(x) = 0$ for $|x| \leq 1$. By virtue of Hölder’s inequality we obtain the estimate

$$
\frac{\alpha}{2} \int_0^T \int_{\mathbb{R}^2} |\partial_\varepsilon m|^2 \varphi_R^2 dx dt + \left[ \int_{\mathbb{R}^2} e_\varepsilon(m(t)) \varphi_R^2 dx \right]_{t=0}^T
\leq c \int_0^T \nu^2 \left[ D(m) - 4\pi \right] + R^{-2} E_\varepsilon(m) dt
$$

for generic constants $c$ that only depend on $\alpha$ and $\varphi$ from which we obtain:
Lemma 6. There exists a constant $c = c(\alpha)$ such that
\[
\int_{\{ |x| > 2R \}} e^x(m(t)) \, dx \leq \int_{\{ |x| > R \}} e^x(m(0)) \, dx + c \left( 1 + \varepsilon(\nu/R)^2 \right) T/R^2
\]
for all $0 \leq t \leq T$, $R > 0$ and $\varepsilon > 0$.

Small energy regularity. The main strategy for proving regularity has been developed in the context of harmonic map heat flows and is well-established \cite{29,10,12}. The terminal time $T^*$ depends on the initial data and the parameters $\varepsilon$ and $\nu$. The only possible scenario of finite time blow-up is $|\nabla m(x_k, t_k)| \to \infty$ for some sequence $x_k \in \mathbb{R}^2$ and $t_k \not\to T^*$. We shall show that for moderately small $\varepsilon$, this scenario can be ruled out as long as $E_\varepsilon(m(t)) < 4\pi$.

Proposition 3. There exists $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$ and $E_\varepsilon(m(t)) < 4\pi$ for all $t < T^*$ and $T^* < \infty$, then
\[
\limsup_{t \to T^*} E_\varepsilon(m(t)) = 4\pi.
\]

It is customary to prove small-energy regularity using Schoen’s trick, which is well-established for harmonic maps and flows.

Lemma 7. There exists $\delta_0 > 0$ such that if $m$ is a blow-up solution in $\overline{P_2}$ with
\[
\int_{B_2(0)} |\nabla m(s)|^2 \, dy < \delta_0 \quad \text{for all} \quad s \in (-4, 0)
\]
then
\[
|\nabla m| \leq 2 \quad \text{in} \quad \overline{P_1(0)}.
\]

Proof. There exists $\rho \in [0, 2)$ such that
\[
(1 - \rho)^2 \sup_{P_\rho} |\nabla m|^2 = \max_{\sigma \in [0, 2]} (1 - \sigma)^2 \sup_{P_\sigma} |\nabla m|^2.
\]
We set $s_0 = |\nabla m(z_0)| = \sup_{P_\rho} |\nabla m|$ for some $z_0 \in \overline{P_\rho(0)}$ and claim
\[
(2 - \rho)^2 s_0^2 \leq 4.
\]
Then it follows that $\sup_{P_1} |\nabla m|^2 \leq (2 - \rho)^2 s_0^2 \leq 4$, which implies the claim.

If otherwise $(2 - \rho)^2 s_0^2 > 4$, then in particular $s_0 > \frac{2}{2 - \rho} \geq 1$. So $\lambda = 1/s_0$ is an admissible scaling parameter. For $(x, t) \in P_1$ we consider the blow-up solution
\[
\tilde{m}(x, t) = m(x_0 + s_0^{-1}x, t_0 + s_0^{-2}t),
\]
for which
\[
\sup_{P_1} |\nabla \tilde{m}|^2 \leq s_0^{-2} \sup_{P_{s_0^{-1}(x_0)}} |\nabla m|^2 \leq s_0^{-2} \sup_{P_{s_0^{-2}(x_0)}} |\nabla m|^2 \leq s_0^{-2} \frac{(2 - \rho)^2 s_0^2}{\left( \frac{1}{2} (2 - \rho) \right)^2} \leq 4.
\]
Hence it follows from Sobolev embedding $H^2(B_{1/4}) \hookrightarrow L^\infty(B_{1/4})$ and Lemma 3 applied twice to $\tilde{m}$ (being a blow-up solution) that for a generic constant $c$
\[
1 = |\nabla \tilde{m}(0, 0)|^2 \leq c \|\nabla \tilde{m}(0)\|_{H^2(B_{1/4})}^2 \leq c \int_{-1}^{0} \|\nabla \tilde{m}(t)\|_{L^2(B_1)}^2 \, dt.
\]
But then $1 \leq c \int_{-4}^{0} \|\nabla m(t)\|_{L^2(B_2)}^2 \, dt < 4c\delta_0$, impossible for appropriate $\delta_0 > 0$. □
Proof of Proposition \[\text{[Proposition]}\]: Suppose \(T^* < \infty\). It follows from Lemma \[\text{[Lemma]}\] that there exist \(R_0 > 0\) and \(\varepsilon_0 > 0\) such that
\[
\int_{|x| > 2R_0} |\nabla m(t)|^2 \, dx < \delta_0 \quad \text{for all} \quad 0 < t < T^*
\]
if \(\varepsilon < \varepsilon_0\) and \(m = m_\varepsilon\) is a solution with \(E_\varepsilon(m(t)) < 4\pi\) for all \(0 < t < T^*\). Hence for fixed \(\varepsilon < \varepsilon_0\), according to Lemma \[\text{[Lemma]}\] \(|\nabla m_\varepsilon(x, t)|\) is uniformly bounded for \(|x| > 3R_0\) and \(0 < t < T^*\). It follows that blow-up can only occur in a finite domain, and it remains to perform a bubbling analysis as in \[\text{[29]}\].

Note that by Lemma \[\text{[Lemma]}\] and Proposition \[\text{[Proposition]}\] the singular set must be finite. Hence after translation and dilation we may assume \(m \in C^\infty(P_2 \setminus \{(0,0)\})\) and claim that if \(\varepsilon\) is sufficiently small and \(m\) has a singularity in the origin, then
\[
\limsup_{t \to 0} E_\varepsilon(m(t); B_2(0)) \geq 4\pi.
\]
If \((0,0)\) is a singularity then by virtue of Lemma \[\text{[Lemma]}\]
\[
\int_{|x| < 0} |\nabla m(x)|^2 \, dx = \sup_{(x,t) \in B_1 \times (-1,t)} \int_{|x| < 0} |\nabla m(t)|^2 \, dy = \frac{\delta_0}{2}
\]
for suitable sequences \(x_k \to 0\), \(t_k \uparrow 0\) and \(r_k \searrow 0\). The blow-up solution
\[
m_k(x,t) = m(x + r_k x, t + r_k^2 t)
\]
defined for \(x \in \mathbb{R}^2\) and \(-1/r_k^2 \leq t \leq 0\) solves the perturbed Landau-Lifshitz-Gilbert equation
\[
\partial_t m_k = m_k \times (\alpha \partial_t m_k - \Delta m_k) + f_k
\]
for a field \(f_k \perp m_k\) with
\[
|f_k| \lesssim r_k |\nabla m_k| + r_k^2 |m_k - \hat{e}_3|^3
\]
hence \(\|f_k(t)\|_{L^2} = O(r_k)\) uniformly for all admissible \(t\). According to Lemma \[\text{[Lemma]}\] and iterations of Lemma \[\text{[Lemma]}\] \(m_k\) satisfies uniform higher order regularity bounds in \(P_{1/r_k}\). It follows from the energy inequality for \(m\) that \(\int_{-1}^{0} \int_{\mathbb{R}^2} |\partial_t m_k|^2 \, dx dt \to 0\) as \(k \to \infty\), hence \(v_k = (\partial_t m_k)(\tau_k)\) and \(w_k = f_k(\tau_k)\) converge to zero in \(L^2(\mathbb{R}^2)\) for some sequence \(\tau_k \uparrow 0\). Note that \(u_k = m_k(\tau_k)\) is an almost harmonic map in the sense that
\[
u_k \times \Delta u_k = \alpha u_k \times v_k - \hat{v}_k + \hat{w}_k
\]
and subconvergence strongly in \(H^1_{\text{loc}}(\mathbb{R}^2)\) to a harmonic map \(u\) of finite energy in \(\mathbb{R}^2\). To show that \(u\) is non-constant we invoke the local energy equality for \(m_k\)
\[
\int_{B_1} |\nabla m_k(0)|^2 \, dx - \int_{B_2} |\nabla m_k(\tau_k)|^2 \, dx \leq c \int_{\tau_k}^{0} \int_{B_2} (|\nabla m_k|^2 + |f_k|^2) \, dx dt = O(\tau_k),
\]
which implies that
\[
\int_{B_2} |\nabla u_k|^2 \, dx = \int_{B_2} |\nabla m_k(\tau_k)|^2 \, dx \geq \frac{\delta}{2} + O(\tau_k).
\]
By strong convergence \(\int_{B_2} |\nabla u|^2 \, dx > 0\), and by virtue of well-known theory about harmonic maps \(\frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 \, dx = 4\pi\). The rescaled energy densities
\[
e_{\varepsilon,k}(u) := \frac{|\nabla u|^2}{2} + \varepsilon r_k \left( (u - \hat{e}_3) \cdot (\nabla \times u) + \frac{r_k}{16} |u - \hat{e}_3|^4 \right)
\]
Lemma 8. Proposition 1: family as \( \varepsilon \) which implies strong convergence. Finally we deduce convergence of the whole sequence \( \varepsilon \) for \( k > k_0 \) depending on \( R_0 \), and \( \int_{B_{R_0}} e_{\varepsilon,k}(u_k) \, dx = \frac{1}{2} \int_{B_{R_0}} |\nabla u_k|^2 \, dx + O(r_k) \) as \( k \to \infty \) which implies the claim by strong convergence.

Proof of Theorem 3. The first claim has been discussed in the forefront of the theorem. The second follows from Proposition 2 and Proposition 3. For the third claim we deduce from Lemma 2 as in the proof of Theorem 2 that \( \limsup_{\varepsilon \to 0} V(m_{\varepsilon}) < \infty \) and \( \lim_{\varepsilon \to 0} D(m_{\varepsilon}) = 4\pi \), hence \( m_0 \in \mathcal{C} \). Moreover, it follows from Proposition 2 that for every sequence \( \varepsilon_k \downarrow 0 \) the corresponding solutions \( m_{\varepsilon_k} \) subconverge weakly to a weak solution of \( m \) of the standard Landau-Lifshitz-Gilbert equation

\[
\partial_t m = \alpha m \times \partial_t m - \nabla \cdot (m \times \nabla m) \quad \text{with} \quad m(0) = m_0.
\]

Since \( \partial_t m = 0 \) by Proposition 2 it follows that \( m \equiv m_0 \). Now for every \( t \in [0,T] \) the sequence \( \nabla m_{\varepsilon_k}(t) \) converges weakly to \( \nabla m_0 \) with \( \lim_{k \to \infty} D(m_{\varepsilon_k}(t)) = 4\pi \), which implies strong convergence. Finally we deduce convergence of the whole family as \( \varepsilon \downarrow 0 \).

Appendix A. Cut-off lemma

The following cut-off result in the spirit of [9, 17, 21] is crucial for the proof of Proposition 3.

Lemma 8. Suppose \( m : \mathbb{R}^2 \to \mathbb{S}^2 \) satisfies \( \int_{\mathbb{R}^2} |\nabla m|^2 \, dx < \infty \) and

\[
\int_{B_{4R} \setminus B_R} |\nabla m|^2 \, dx + \sigma \int_{B_{4R} \setminus B_R} \frac{1}{r^2} |m - \hat{e}_3|^p \, dx < \delta
\]

for some \( 0 < \delta \ll 1, R \geq 1, \sigma \in \{0,1\} \). Then, there exist \( m^{(1)}, m^{(2)} : \mathbb{R}^2 \to \mathbb{S}^2 \) with \( \int_{\mathbb{R}^2} |\nabla m^{(i)}|^2 \, dx < \infty \) for \( i = 1,2 \), some \( c \in [R, 2R] \) and a constant \( C = C(\delta, R) < \infty \) so that

\[
m^{(1)} = m \quad \text{on} \quad B_{c}, \quad V(m^{(1)}) \lesssim C,
\]

\[
\int_{\mathbb{R}^2 \setminus B_{c}} |\nabla m^{(1)}|^2 \, dx + \sigma \int_{\mathbb{R}^2 \setminus B_{c}} \frac{1}{r^2} |m^{(1)} - \hat{e}_3|^p \, dx \lesssim \delta + \sigma \left( \frac{\delta}{r^2} \right)^{2/p},
\]

and

\[
m^{(2)} = m \quad \text{on} \quad \mathbb{R}^2 \setminus B_{2c}, \quad V(m^{(2)}) \lesssim V(m) + C,
\]

\[
\int_{B_{2c}} |\nabla m^{(2)}|^2 \, dx + \sigma \int_{B_{2c}} \frac{1}{r^2} |m^{(2)} - \hat{e}_3|^p \, dx \lesssim \delta + \sigma \left( \frac{\delta}{r^2} \right)^{2/p}.
\]

Proof. We proceed in several steps. The symbol \( \lesssim \) will denote an inequality that holds up to a generic, universal multiplicative constant that may change from line to line.
Step 1 (Choice of radius $c$): We consider $m$ in polar coordinates and write $m(x) = m(r, \theta)$. Moreover, we define

$$g: [R, 4R] \to \mathbb{R}, \quad g(r) := \int_0^{2\pi} \left( | \partial_r m |^2 + | \frac{1}{r} \partial_\theta m |^2 + \sigma \frac{1}{r^2} | m - \hat{e}_3 |^p \right) d\theta.$$  

Poincaré's inequality yields

$$\| m(r, \cdot) - \bar{m}(r) \|_\infty^2 \lesssim \int_0^{2\pi} | \partial_\theta m(r, \theta) |^2 d\theta \quad \forall r > 0,$$

where $\bar{m}(r) := \int_0^{2\pi} m(r, \theta) d\theta$. Hence, we may choose $c \in [R, 2R]$ so that

$$\frac{\delta}{\pi} \geq \frac{1}{\pi} \int_{B_{4R} \setminus B_R} | \nabla m |^2 + \sigma \frac{1}{r^2} | m - \hat{e}_3 |^p dx = \frac{1}{\pi} \int_R^{4R} g(r) r \, dr \geq \int_R^{2R} \left( g(r) + g(2r) \right) r \, dr \geq (g(c) + g(2c)) c.$$

By definition of $g$, we obtain

$$1 - |\bar{m}(r)|^2 = \int_0^{2\pi} |m(r, \theta) - \bar{m}(r)|^2 d\theta \lesssim \| m(r, \cdot) - \bar{m}(r) \|_\infty^2 \lesssim \int_0^{2\pi} | \partial_\theta m(r, \theta) |^2 d\theta \lesssim Rg(r)r \lesssim \delta \quad \text{for} \quad r = c, 2c,$$

and

$$\sigma \left( 1 - m_3(r) \right) = \sigma \int_0^{2\pi} \left( 1 - m_3(r, \theta) \right) d\theta \lesssim \sigma \left( \int_0^{2\pi} \left( 1 - m_3(r, \theta) \right)^2 d\theta \right)^{\frac{p}{2}} \lesssim \sigma \left( g(r) \right)^{\frac{p}{2}} \lesssim \sigma \left( \frac{\delta}{\pi} \right)^{\frac{p}{2}} \quad \text{for} \quad r = c, 2c.$$

In particular, we may assume $|\bar{m}(c)| \geq \frac{\delta}{2}$.

Step 2 (Definition of $m^{(1)}$):

Let

$$e := \left\{ \begin{array}{ll} \bar{m}(c), & \sigma = 0 \\ \hat{e}_3, & \sigma = 1 \end{array} \right\} \in \mathbb{S}^2,$$

so that for $\sigma = 0$ we have

$$\| m(c) - e \|_\infty^2 \lesssim \| m(c, \cdot) - \bar{m}(c) \|_\infty^2 + \| \bar{m}(c) - e \|_\infty^2 \lesssim \delta.$$

If $\sigma = 1$, we may modify the second estimate as follows:

$$| \bar{m}(c) - \hat{e}_3 |^2 \lesssim \int_0^{2\pi} \left( m(c, \theta) - \hat{e}_3 \right)^2 d\theta \lesssim (1 - m_3(c)) \lesssim \left( \frac{\delta}{\pi} \right)^2.$$  

Hence, in either situation,

$$\| m(c, \cdot) - e \|_\infty^2 \lesssim \delta + \sigma \left( \frac{\delta}{\pi} \right)^2 \ll 1.$$  

We will define $m^{(1)}: \mathbb{R}^2 \to \mathbb{S}^2$ in two steps:
Step 2a (Definition of $m^{(1)}$ on $B_{2c}$): Let $\eta : \mathbb{R} \to [0, 1]$ be a smooth cut-off function with $\eta(s) = 1$ for $s \leq 0$ and $\eta(s) = 0$ for $s \geq 1$. We define

$$m^{(1)}(r, \theta) = \begin{cases} \eta(\frac{r}{c})m(c, \theta) + (1 - \eta(\frac{r}{c}))e, & c < r < 2c, \\ m(r, \theta), & 0 \leq r \leq c. \end{cases}$$

so that $m^{(1)}$ has a well-defined trace across $\partial B_c$. Using the inequality

$$|\partial_i(\rho m^{(1)})|^2 = \rho^2|\partial_i m^{(1)}|^2 + |\partial_i \rho|^2 \geq \frac{1}{2}|\partial_i m^{(1)}|^2, \quad i = r, \theta,$$

where

$$\rho = |\eta(\frac{r}{c})m(c, \theta) + (1 - \eta(\frac{r}{c}))e| \geq \frac{1}{2},$$

we obtain for $c \leq r \leq 2c$ that

$$|\partial, m^{(1)}(r, \theta)|^2 \lesssim \frac{1}{\rho} |\eta(\frac{r}{c})m(c, \theta) - e|^2 \lesssim \frac{1}{\rho} \|m(c, \cdot) - e\|_{\infty}^2 \lesssim \frac{\delta + (\delta R^{-2})^2}{r^2}.$$ 

and

$$\left|\frac{1}{\rho} \partial_\theta m^{(1)}(r, \theta)\right|^2 \lesssim \left|\frac{1}{\rho} \partial_\theta m(c, \theta)\eta(\frac{r}{c})\right|^2 \lesssim \frac{1}{\rho} \|\partial_\theta m(c, \theta)\|^2.$$

Hence,

$$\int_c^{2c} \int_0^{2\pi} \left(|\partial, m^{(1)}(r, \theta)|^2 + \left|\frac{1}{\rho} \partial_\theta m^{(1)}(r, \theta)\right|^2\right) d\theta r dr \lesssim \int_c^{2c} \int_0^{2\pi} \left(\delta + \left(\frac{\delta}{\rho^2}\right)^2 + \left|\partial_\theta m(c, \theta)\right|^2\right) d\theta r dr \lesssim \delta + \left(\frac{\delta}{\rho^2}\right)^2.$$ 

Finally, since

$$1 = |m| = |(1 - \eta)m + \eta e + \eta(m - e)| \leq |(1 - \eta)m + \eta e| + |m - e|$$

implies

$$1 - |(1 - \eta)m + \eta e| \leq |m - e|,$$

we obtain for $\rho$ as above

$$|m^{(1)} - e| \leq |m^{(1)} - \rho m^{(1)}| + \left|\left(\eta e + (1 - \eta)\eta m(c, \theta)\right) - e\right| \lesssim |m(c, \theta) - e|.$$

Hence, in the case $\sigma = 1$

$$\int_c^{2c} \int_0^{2\pi} \frac{1}{\rho^2} |m^{(1)} - \hat{e}_3|^2 d\theta r dr \lesssim \int_c^{2c} \int_0^{2\pi} |m(c, \theta) - \hat{e}_3|^2 d\theta r dr \lesssim \int_c^{2c} \int_0^{2\pi} \left(1 - m_3(c, \theta)\right)^2 d\theta c dr \lesssim \frac{\delta}{\pi}.$$ 

Therefore, we have

$$\int_{B_{2c}} |\nabla m^{(1)}|^2 \, dx + \sigma \int_{B_{2c}} \frac{1}{\rho^2} |m^{(1)} - \hat{e}_3|^2 \, dx \lesssim \delta + \sigma \left(\frac{\delta}{\pi}\right)^2.$$
Step 2b (Definition of $m^{(1)}$ on $\mathbb{R}^2 \setminus B_{2c}$): If $\sigma = 1$, there is nothing left to be done and we may just set $m^{(1)} = \hat{e}_3$ on $\mathbb{R}^2 \setminus B_{2c}$. Otherwise, we will define $m^{(1)}$ on $(2c, 2c + L)$ for some $L \gg 2c$ (to be chosen later) by interpolating $e$ with $\hat{e}_3$. Indeed, let $\gamma : [0, 1] \to S^2$ denote a smooth curve that connects $\gamma(0) = e$ with $\gamma(1) = \hat{e}_3$. Assume w.l.o.g. that $|\frac{d}{ds}\gamma(s)| \lesssim 1$ independently of $e \in S^2$. We introduce a logarithmic cut-off function

$$ \eta_L : [2c, 2c + L] \to [0, 1], \quad \eta_L(r) := \frac{\ln(\frac{r}{L + 2c})}{\ln(\frac{2c + L}{2c})}, $$

and let

$$ m^{(1)}(r, \theta) = \begin{cases} \gamma(\eta_L(r)), & 2c \leq r \leq 2c + L \\ \hat{e}_3, & 2c + L < r. \end{cases} $$

Then, $m^{(1)}$ has a well-defined trace both across $\partial B_{2c}$ and $\partial B_{2c+L}$, and

$$ \frac{d}{dr} m^{(1)}(r) = \frac{(\frac{d}{dr}\gamma)(\eta_L(r))}{r \ln(\frac{2c + L}{2c})}. $$

Hence, $\partial_r m^{(1)} = 0$ and

$$ \int_0^2 \int_0^{2\pi} \frac{1}{\ln(1 + \frac{c}{r})} \frac{r}{\ln(\frac{2c + L}{2c})} dx \lesssim \frac{1}{\ln(\frac{2c + L}{2c})} \int_0^2 \int_0^{2\pi} dx \lesssim \delta, $$

if $L = 2c(e^\frac{1}{2} - 1)$.

Thus, we may conclude for $\sigma \in \{0, 1\}$:

$$ \int_{\mathbb{R}^2 \setminus B_{2c}} |\nabla m^{(1)}|^2 dx + \sigma \int_{\mathbb{R}^2 \setminus B_{2c}} |m^{(1)} - \hat{e}_3|^p dx \lesssim \sigma(\frac{\delta}{\sigma^2})^\frac{p}{2}, $$

and

$$ V(m^{(1)}) = \int_{B_{2c+L}} \frac{1}{\ln(1 + \frac{2c + L}{2c})} \leq 1 \int_{B_{2c+L}} |m^{(1)} - \hat{e}_3|^p dx \lesssim (2c + L)^2 =: C(\delta, R). $$

Step 3 (Definition of $m^{(2)}$): In order to define $m^{(2)}$, we proceed as in Step 2. Let

$$ e := \frac{m^{(2c)}(2c)^2}{|m^{(2c)}(2c)|} \in S^2. $$

Then

$$ \|m(2c, \cdot) - e\|^2 \lesssim \delta + \sigma(\frac{\delta}{\sigma^2})^\frac{p}{2} \ll 1, $$

and, using the same cut-off function $\eta$, we may define $m^{(2)} : \mathbb{R}^2 \to S^2$ as

$$ m^{(2)}(r, \theta) := \begin{cases} e, & r \leq c, \\ \eta(r, \theta) e + (1 - \eta(r, \theta)) m^{(2c, \theta)}, & c < r < 2c, \\ m(r, \theta), & r \geq 2c, \end{cases} $$

so that $m^{(2)}$ has a well-defined trace across $\partial B_c$ and $\partial B_{2c}$.

As before, we estimate for $c < r < 2c$

$$ |\partial_r m^{(2)}(r, \theta)|^2 \lesssim \frac{\delta + (\delta L - 2)^2}{r^2} \quad \text{and} \quad \frac{1}{r^2} |\partial_{\theta} m^{(2)}(r, \theta)|^2 \lesssim \frac{1}{r^2} |\partial_{\theta} m^{(2c, \theta)}|^2, $$

and $m^{(2)}$ satisfies

$$ \partial_r m^{(2)}(r, \theta)^2 \lesssim \frac{\delta}{r^2} \quad \text{and} \quad \frac{1}{r^2} |\partial_{\theta} m^{(2)}(r, \theta)|^2 \lesssim \frac{1}{r^2} |\partial_{\theta} m^{(2c, \theta)}|^2, $$

and

$$ |\partial_{\theta} m^{(2)}(r, \theta)|^2 \lesssim \frac{\delta + (\delta L - 2)^2}{r^2} \quad \text{and} \quad \frac{1}{r^2} |\partial_{\theta} m^{(2)}(r, \theta)|^2 \lesssim \frac{1}{r^2} |\partial_{\theta} m^{(2c, \theta)}|^2, $$

and

$$ V(m^{(2)}) = \int_{B_{2c+L}} \frac{1}{\ln(1 + \frac{2c + L}{2c})} \leq 1 \int_{B_{2c+L}} |m^{(2)} - e|^p dx \lesssim (2c + L)^2 =: C(\delta, R). $$
so that
\[ \int_c^{2c} \int_0^{2\pi} \left( |\partial_r m^{(2)}(r, \theta)|^2 + \left| \frac{1}{r} \partial_\theta m^{(2)}(r, \theta) \right|^2 \right) d\theta \, dr \lesssim \delta + \left( \frac{C}{1+r^2} \right)^2. \]
Moreover, by the same argument as in Step 2, for \( \sigma = 1 \):
\[ \int_c^{2c} \int_0^{2\pi} \left| \frac{1}{r} \partial_\theta m^{(2)} - \hat{e}_3 \right|^p d\theta \, dr \lesssim \delta. \]
Hence, we may conclude for \( \sigma \in \{0, 1\} \):
\[ \int_{B_{2c}} |\nabla m^{(2)}|^2 \, dx + \sigma \int_{B_{2c}} \left| \frac{1}{r} \partial_\theta m^{(2)} - \hat{e}_3 \right|^p \, dx \lesssim \delta + \sigma \left( \frac{C}{1+r^2} \right)^2, \]
and
\[ V(m^{(2)}) = \int_{\mathbb{R}^2 \setminus B_{2c}} \left| \frac{1}{r} m - \hat{e}_3 \right|^p \, dx + \int_{B_{2c}} \left| \frac{1}{r} m^{(2)} - \hat{e}_3 \right|^p \, dx \lesssim V(m) + \left( 2c \right)^2. \]

**APPENDIX B. CONSTRUCTION OF A STREAM FUNCTION**

**Lemma 9.** Given \( R > 1 \), there exists a smooth function \( f_R : [0, \infty) \to \mathbb{R} \) so that
\[ f_R(r) = \begin{cases} \ln(1 + r^2), & \text{for } 0 \leq r \leq R, \\ \text{const.}, & \text{for } r \geq 2R, \end{cases} \]
and
\[ 0 \leq f_R'(r) \leq \frac{2r}{1+r^2}, \quad 0 \leq -f_R''(r) \leq \frac{C}{1+r^2} \quad \text{for all } r \geq R. \]

**Proof.** Let \( h : [0, \infty) \to \mathbb{R} \) be given by (in fact, \( h \) is a regularization of the function \( y \mapsto \min(y, 0) \))
\[ h(y) = \int_0^y \eta(s) \, ds, \]
where \( \eta : \mathbb{R} \to [0, 1] \) is a smooth, non-increasing function with
\[ \eta(s) = 1 \quad \text{for } s \leq 0, \quad \eta(s) = 0 \quad \text{for } s \geq \frac{1}{2}, \quad 0 \leq -\eta'(s) \leq C \quad \forall s \in \mathbb{R}. \]
Then,
\[ f_R(r) := h \left( \ln(1 + r^2) - \ln(1 + R^2) \right) + \ln(1 + R^2), \quad r \geq 0, \]
satisfies the claim.
Indeed, we have \( h(y) = y \) for \( y \leq 0 \) and \( h(y) = \int_0^y \eta(s) \, ds \) for \( y \geq \frac{1}{2} \). Since \( \ln(1 + r^2) - \ln(1 + R^2) \leq 0 \) for \( r \leq R \), we therefore obtain \( f_R(r) = \ln(1 + r^2) \). On the other hand, \( r \geq 2R \geq 2 \) yields \( \ln(1 + r^2) - \ln(1 + R^2) \geq \ln(\frac{1+4R^2}{1+R^2}) \geq \ln(\frac{5}{2}) \geq \frac{1}{2} \), so that \( f_R(r) = \int_0^\infty \eta(s) \, ds \).
Finally, we have
\[ f_R'(r) = \eta \left( \ln(1 + r^2) - \ln(1 + R^2) \right) \frac{2r}{1+r^2} \]
and
\[ f_R''(r) = \eta' \left( \ln(1 + r^2) - \ln(1 + R^2) \right) \left( \frac{2r}{1+r^2} \right)^2 + \eta \left( \ln(1 + r^2) - \ln(1 + R^2) \right) \frac{2(1-r^2)}{(1+r^2)^2}. \]
In particular, \(0 \leq f''_H(r) \leq \frac{2\nu}{1+\nu^2}\) for \(r \geq R\) and \(0 \leq -f''_H(r) \leq \frac{C}{1+\nu^2}\). \square

**Appendix C. Pulled back Landau-Lifshitz-Gilbert equation**

We shall argue on the level of the Landau-Lifshitz form
\[
(1 + \alpha^2)\partial_t m + (1 + \alpha \beta)(v \cdot \nabla)m = -[(\alpha - \beta)m \times (v \cdot \nabla)m + m \times h_{\text{eff}} + \alpha m \times m \times h_{\text{eff}}],
\]
see e.g. [22], rather than the Gilbert form [6]. Solving Thiele’s equation we have
\[
(1 + \alpha^2)c = (1 + \alpha \beta)v - (\alpha - \beta)v^k.
\]
Now we compute
\[
(1 + \alpha^2)\frac{d}{dt} m(x + c t, t) = (1 + \alpha^2)\partial_t m + (1 + \alpha^2)(c \cdot \nabla)m
\]
\[
= (1 + \alpha^2)\partial_t m + (1 + \alpha \beta)(v \cdot \nabla)m - (\alpha - \beta)(v \times \nabla)m
\]
\[
= -[(\alpha - \beta)\Psi - m \times h_{\text{eff}} + \alpha m \times m \times h_{\text{eff}}],
\]
where with the notation \(v \times \nabla = v_1 \partial_2 - v_2 \partial_1\)
\[
\Psi = (v \times \nabla)m + m \times (v \cdot \nabla)m
\]
\[
= v_1 (\partial_2 m + m \times \partial_1 m) - v_2 (\partial_1 m - m \times \partial_2 m)
\]
\[
= 2v_1 m \times \partial_z m - 2v_2 \partial_z m.
\]
where \(\partial_z m = \frac{1}{2}(\partial_1 m - m \times \partial_2 m)\). Upon the transformation \(m(x + c t, t) \mapsto m(x, t)\) and with effective coupling parameters \(\nu_i = \frac{2(\alpha - \beta)\nu_i}{1+\alpha^2}\) this can be written as
\[
(1 + \alpha^2)\left(\partial_t m + v_1 m \times \partial_2 m - v_2 \partial_2 m\right) + m \times h_{\text{eff}} + \alpha m \times m \times h_{\text{eff}} = 0.
\]
A rigid rotation yields for \(\nu = \sqrt{\nu_1^2 + \nu_2^2}\)
\[
(1 + \alpha^2)\left(\partial_t m - \nu \partial_2 m\right) + m \times h_{\text{eff}} + \alpha m \times m \times h_{\text{eff}} = 0,
\]
which easily recasts into [6].

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