Erlang circular model motivated by inverse stereographic projection

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Abstract. The Erlang distribution is a special case of the Gamma distribution with the shape parameter is an integer. This paper proposed a new circular model used inverse stereographic projection. The inverse stereographic projection which is a mapping that projects a random variable from a real line onto a circle can be used in circular statistics to construct a distribution on the circle from real domain. From the circular model, then can be derived the characteristics of the Erlang circular model such as the mean resultant length, mean direction, circular variance and trigonometric moments of the distribution.

Key words: Inverse stereographic projection, Erlang, Circular

1. Introduction
Erlang which is a member of Exponential family is a special distribution of Gamma distribution. Erlang distribution is a Gamma distribution with parameters which are positive integers. Erlang distribution is used nowadays in the field of stochastic processes and biomathematics. In addition, the implementation of the Erlang distribution can be used in the field of insurance. For example, if the amount of insurance claim $X$ have distribution of Erlang. In this illustration, Erlang works on real domain. Directional statistics is sub statistics field where measurement in direction ways. This paper will be proposed a new model circular distribution using inverse stereographic projection in circular measurement. As we know, at least there are four methods of obtaining circular distribution [7]. Using stereographic projection, we will apply Erlang distribution with real line domain then identify such real random variable on the circumference of the circle. There are many paper talk about circular stereographic distribution. Phani, Girija and Dattatreya [10] proposed circular model induced by inverse stereographic projection on extreme value distribution. Abe, Shimizu and Pewsey [1] also introduced a symmetric unimodal models such as t distribution motivated by inverse stereographic projection. Getut [4] also derived exponential circular distribution, Chi Square circular and Chi Square semi circular [5] and the latest paper is Beta circular [6].

2. Materials and Methods

2.1 Circular and Semicircular Data
Mostly, measurement in any field is regarded as real number. Actually, in many diverse field, any observation can be measured as a direction. For example, circular data. Wind direction or direction of migrating birds as a circular data can be measured the compass and the clock [8]. Other example, such as
component lifetime, orientations of certain organism, direction of a pollutant also can be regarded as directional observation. A set of such observation on direction is referred to as directional data [5]. Direction can be represented as points on the circumference of a unit circle centre or a unit vectors connecting the origin to these points [6]. Thus, observation on two dimensional directions are called as circular data and in three dimensions are called as spherical data. Circular data which can be represented by an angle \( \theta \in [-\pi, \pi] \) or \( \theta \in [0, 2\pi) \). The angle is periodic, i.e. \( \theta = \theta + 2\pi m, m \in \mathbb{Z} \). Beside circular data, a random variable which having values on a semicircle and have the domain in \( \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \) or \( [0, \pi) \) can be called as axial or semi-circular data [2].

2.2 Circular Models
Circular models is the model of circular random variable. The circular models can be modelled by its distribution function. Mardia and Jupp [8] describe some families of circular models such as Poisson, Uniform, Von Mises, Cardioid, Wrapped Normal, Wrapped Poisson and Wrapped Cauchy. The distribution function of circular is defined as the function on the whole real line and the distribution can be regarded as that of a random angle \( \theta \). As other word we can say that a circular distribution can be represent as a probability distribution. The distribution of this circular variable also can be described by its characteristic function which is defined only at integer values, the next term it’s called as \( p \)-th trigonometric moment of \( \theta \) and can be write as (1) [7].

\[
\varphi_p(p) = \int_0^{2\pi} e^{ip\theta} dF(\theta), \quad p = 0, \pm 1, \pm 2, \ldots
\]  

Circular models can be generated from any known probability distribution, [7] describe a few methods such as: a) By wrapping a linear distribution around the unit circle. Rao, Girija and Devaraaj also derived the characteristics of wrapped gamma [3]. Shogo Kato and M.C Jones make an extended family of circular distribution related to wrapped Cauchy distributions via Brownian motion [12], b) By characterizing properties, c) offset distribution, and d) stereographic projection

A continuous random variable, \( X \) on the real line is said to have the Erlang distribution [11] with scale parameter \( \lambda > 0 \) and shape parameter \( r \), with notation \( X \sim \text{ERL}(\alpha, r) \) and the probability density function of random variable, \( f(x) \) defined as (2)

\[
f(x, \lambda, r) = \frac{1}{\lambda^r (r-1)!} x^{r-1} e^{-x/\lambda}, \quad x > 0
\]  

The random variable \( X \) that equals the interval length until \( r \) counts occur in a Poisson has an Erlang random variable for \( r = 1, 2, \ldots \)

2.3 Measure of Location and Dispersion
The distribution of the circular data can be modelled as the probability density function (pdf) and the cumulative density function (cdf). If \( f(\theta) \) denote the pdf of the random variable \( \theta \) and \( F(\theta) \) denote the cdf of the random variable \( \theta \) then for finding measure of location and dispersion of the distribution, we can work with the characteristic function [2]. As on the real line, directional data also have any model that be described via it’s distribution. We can call it then as a circular distribution. Thus, circular distribution is a probability distribution whose total probability is concentrated on the circumference of a unit circle [7]. The trigonometric moment write by (1) which can be wrote as (3).

\[
\varphi_p = \alpha_p + i\beta_p
\]  

...
Where \( \alpha_p = E[\cos p\theta] \), \(|\alpha_p| \leq 1\)
And \( \beta_p = E[\sin p\theta] \), \(|\beta_p| \leq 1\)

The characteristic function of a random angle \( \theta \) is the doubly infinite sequence of complex numbers [7].

For \( p = 0 \), (3) can be write as (4)

\[
\varphi_p = \rho_p + e^{i\mu_p}
\]

(4)

For the value of \( p = 1 \). The \( \rho_1 = \rho \) is called the mean resultant length and the \( \mu_1 = \mu \) is called the mean direction [8]. Mean resultant length and the mean direction can be obtained from (5).

\[
\rho_p = \sqrt{\alpha_p^2 + \beta_p^2}
\]

(5)

And

\[
\mu_p = \arctan \left( \frac{\beta_p}{\alpha_p} \right)
\]

(6)

The dispersion measure of circular variable random can be called as circular variance, \( \nu \). The circular variance of a random angle \( \theta \) is defined as [7]:

\[
\nu = 1 - \rho
\]

(7)

2.4 Stereographic Projection

Stereographic projection or Mobius (bilinear) transformation was proposed by Minh and Farnum [9]. Stereographic projection is defined as (8).

\[
\theta = 2\tan^{-1} \left( \frac{x-u}{v} \right)
\]

(8)

Stereographic projection is used to induce distribution on the circle. if \( x \) is random variable chosen on real line and \( x \in (-\infty, \infty) \) then the inverse stereographic projection defined by a one to one mapping given by

\[
x = u + v \tan \left( \frac{\theta}{2} \right)
\]

(9)

Abe et al [1] defined \( \theta \) as random variable angle on the circle of radius \( v \), \( u \in \mathbb{Z} \) and \( v > 0 \), \( v = \sqrt{m} \), \( m = 2n + 1 \), \( n = 0,1,2,.. \)

Radhika et al [11] proposed a model for periodic data whose density is given by (10).

\[
g(\theta) = vl \left[ 1 + \tan^2 \left( \frac{\theta}{2} \right) \right] f \left( u + v \tan \left( \frac{\theta}{2} \right) \right)
\]

(10)

For \( \nu > 0, l \in \mathbb{N} \)

3. Results

3.1 The Stereographic Erlang Circular Distribution

If \( g(\theta) \) denote the probability density function (pdf) of the random angle \( \theta \) on the unit circle. Let \( F(x), f(x) \) denote the cumulative density function and (pdf) of the random variable \( x \) on the real line.

The probability density function (pdf) of the random variable Erlang circular can be induced by an inverse stereographic projection follows with Radhika et al [11] methodology. Later we can called the model as stereographic Erlang circular.

Let the pdf of Erlang random variable \( X \) on real line, \( f(x) \). Then applying inverse stereographic projection defined one to one mapping (8), (9) and (10) we can find the pdf of stereographic Erlang circular random variable, \( g(\theta) \).
First, we looking for the value of \( g(\theta) \), i.e.

\[
g(\theta) = v \left[ 1 + \tan^2 \left( \frac{\theta}{2} \right) \right] f \left( u + v \tan \left( \frac{\theta}{2} \right) \right)\]
\[
g(\theta) = \frac{v}{2} \sec^2 \left( \frac{\theta}{2} \right) f \left( u + v \tan \left( \frac{\theta}{2} \right) \right)
\]

Let \( u = 0 \) we can find,

\[
f \left( v \tan \left( \frac{\theta}{2} \right) \right) = \frac{1}{\lambda(r-1)!} \left( v \tan \left( \frac{\theta}{2} \right) \right)^{r-1} e^{-\frac{v \tan \left( \frac{\theta}{2} \right)}{\lambda}}, \lambda > 0, r = 1,2, ...
\]

So,

\[
g(\theta) = \frac{v}{2} \sec^2 \left( \frac{\theta}{2} \right) \frac{1}{\lambda(r-1)!} \left( v \tan \left( \frac{\theta}{2} \right) \right)^{r-1} e^{-\frac{v \tan \left( \frac{\theta}{2} \right)}{\lambda}}, \lambda > 0, r = 1,2, ...
\]

Let \( k = \frac{1}{r} \)

Because of \( v > 0 \) and \( \lambda > 0 \) then \( k > 0 \) and the pdf of random variable as follow (11)

\[
g(\theta) = \frac{k^r \sec^2 \left( \frac{\theta}{2} \right) \tan^{r-1} \left( \frac{\theta}{2} \right) e^{-k \tan \left( \frac{\theta}{2} \right)}}{2
\]

\[
(11)
\]

Where

\( k > 0, r = 1,2, ..., \theta \in [-\pi, \pi) \) or \( \theta \in [0,2\pi) \)

If \( r = 1, k = 1/4, 1/2 \) and \( 3/4 \) then the probability density function from stereographic Erlang circular can be graphed as figure 1.

**Figure 1.** \( r = 1, k = 1/4, 1/2, 3/4 \) for \( \theta \in \left[ -\frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \pi \right) \)

Figure 1 is pdf of Erlang circular with different scale parameters, \( k \). From figure 1, we can that the probability density function of Erlang circular will be going up as well as it’s scale.

If \( k = 1/2, r = 1,2,3 \) then the probability density function can be graphed as figure 2.
Figure 2. k=1/2, r =1, 2 and 3

Figure 2 is the pdf of Erlang circular with different shape parameters, r. Opposite with the figure 1, the probability density function will be down as the shape goes up. From figure 1 and figure 2, the shape of probability density function of stereographic Erlang circular looks opposite with the Exponential circular stereographic [4].

The descriptive measure of any random variable can be measured from the center and dispersion of measurements. And in circular random variable, both measurement can showed by each mean resultant length, mean direction and measure of dispersion. For $k=0.1, 0.3, 0.5, 0.75, 1$ and $r=1$, we can find the mean resultant length $\rho$, mean direction, $\mu$ and measure of dispersion $\upsilon$ as Table 1.

| $k$  | $\alpha_1$  | $\beta_1$  | $\rho$     | $\mu$     | $\upsilon$ |
|------|--------------|-------------|------------|-----------|------------|
| 0.1  | -580.442     | -22.8789    | 580.8927   | 0.039396  | -579.893   |
| 0.3  | -1.97E-08    | -6.5415E-06 | 6.54E-06   | 1.567792  | 0.999993   |
| 0.5  | -6.6475E-13  | -2.1593E-12 | 2.26E-12   | 1.272149  | 1          |
| 0.75 | -5.4213E-20  | -1.741E-19  | 1.82E-19   | 1.268923  | 1          |
| 1    | -4.4213E-27  | -1.412E-26  | 1.48E-26   | 1.267344  | 1          |

From the Table 1 above, we can see that the mean resultant length of the circular random variable will be going small whenever $k$ is large. Thus, the mean direction looks fluctuate when the measure of dispersion is large as the value of $k$.

Thus, for $k=0.1, 0.3, 0.5$ and $1$, we can graph the doubly infinite sequence of complex numbers $\left(\alpha_p, \beta_p\right)$ as Figure 3.
Figure 3. Doubly Infinite Sequence for $k = 0.1, 0.3, 0.5, 1$

In same domain $\theta \in \left[\frac{-3\pi}{4}, \frac{-\pi}{4}\right]$, we also can graph the trigonometric moment with same value of $k = 0.1, 0.3, 0.5, 1$ as figure 4.

Figure 4. Trigonometric Moment for $k = 0.1, 0.3, 0.5, 1$

From figure 3 and 4 we can see that the trigonometric moment can graphed as doubly infinity sequence and the trigonometric moment will be large as the value of $k$.

4. Conclusion
From the discussion above, it can be concluded that using stereographic projection, we can derive a new circular model, viz stereographic Erlang circular. From the descriptive measure of the model, we can conclude that the mean resultant length will be small whenever $k$ is large. Thus, the mean direction looks fluctuate when the measure of dispersion will be large as the value of $k$. Thus, the trigonometric moments will be large as the value of the scale parameter of the stereographic Erlang circular.

5. References
[1] Abe T, Shimizu K and Pewsey A 2010 Symmetric unimodal models for directional data motivated by inverse stereographic projection *Journal of the Japan Statistical Society* **40**(1) 45–61
[2] Byoung J A and Hyoung M K A 2008 New family of semicircular models: The semicircular Laplace Distribution *Communications of the Korean Statistical Society* **15**(5) 775-81
[3] Dattatreya Rao, S V S Girija and V J Devaraaj 2013 On Characteristics of Wrapped Gamma Distribution IRACST- Engineering Science and Technology; An International Journal (ESTIJ), ISSN : 2250-3493(2) April

[4] Getut Pramesti and Yanming Jin Exponential circular distribution motivated by inverse stereographic projection 2016 International Journal of Applied Mathematics and Statistics ISSN:0973-7545 54(2) 114-22

[5] Getut Pramesti The Stereographic Semicircular Chi Square Models 2015 Far East Journal of Theoretical Statistics 51(3) 49-58

[6] Getut Pramesti The Characteristics of Stereographic Beta Circular Model 2017 Far East Journal of Probability and Statistic 53 (2), 77-88

[7] Jammalamadaka SR and SenGupta A 2001 Topics in Circular Statistics (New York: World Scientific)

[8] Mardia K V and Jupp PE 2000 Directional Statistics (England: John Wiley & Sons)

[9] Minh D L P and Farnum N R 2000 Commun. Stat.-Theory Meth. 32 1-9

[10] Phani Y, Girija S V S and Dattatreya R Circular Model Induced by Inverse Stereographic On Extreme – Value Distribution 2012 Engineering Science and Technology An International Journal 2(5) 881-88

[11] A J V Radhika, Y Phani, S V S Girija and A V Dattatreya Rao On Stereographic Semicircular Gamma 2013 Proceedings of the National Conference on Recent Trends in Mathematical Computing- NCRTMC. ISBN 978-93-82338-68-0

[12] Shogo Kato and M C Jones 2013 An Extended family of circular distributions related to wrapped Cauchy distribution via Brownian motion Bernoulli 19(1) 154-171 doi : 10.3150/11-BEJ397