A numerical study of the alpha model for two-dimensional magnetohydrodynamic turbulent flows

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We explore some consequences of the “alpha model,” also called the “Lagrangian-averaged” model, for two-dimensional incompressible magnetohydrodynamic (MHD) turbulence. This model is an extension of the smoothing procedure in fluid dynamics which filters velocity fields locally while leaving their associated vorticities unsmoothed, and has proved useful for high Reynolds number turbulence computations. We consider several known effects (selective decay, dynamic alignment, inverse cascades, and the probability distribution functions of fluctuating turbulent quantities) in magnetofluid turbulence and compare the results of numerical solutions of the primitive MHD equations with their alpha-model counterparts’ performance for the same flows, in regimes where available resolution is adequate to explore both. The hope is to justify the use of the alpha model in regimes that lie outside currently available resolution, as will be the case in particular in three-dimensional geometry or for magnetic Prandtl numbers differing significantly from unity. We focus our investigation, using direct numerical simulations with a standard and fully parallelized pseudo-spectral method and periodic boundary conditions in two space dimensions, on the role that such a modeling of the small scales using the Lagrangian-averaged framework plays in the large-scale dynamics of MHD turbulence. Several flows are examined, and for all of them one can conclude that the statistical properties of the large-scale spectra are recovered, whereas small-scale detailed phase information (such as e.g. the location of structures) is lost.

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\section{I. INTRODUCTION}

One of the most persistent difficulties in the computation of the turbulent behavior of fluids and magnetofluids is the wide range of dynamically interacting length and time scales that have to be evaluated. At large Reynolds number, many orders of magnitude in length scales are implied, for example, in the dynamical behavior of the atmosphere, the oceans, or the solar wind, to take some familiar situations. For many purposes, it might be adequate to compute only the long-wavelength components of the spectra of the fields involved if some more economical representation or model of the small scale behavior could be given which would not do violence to the accuracy with which the large scales are computed. Such topics as “large eddy simulation” and “eddy viscosity,” designed to cope with this difficulty, have generated a vast literature, one which we make no attempt to survey here (see e.g. Refs. \cite{8}).

A novel offering along these lines which has appeared in recent years is the so-called “alpha model” of Holm, Foias, Margolin, Marsden, Olson, Ratti, Titi, Wynne and especially Chen, whose comparisons with turbulent channel and pipe flow called the most attention to the alpha model’s possibilities (e.g. Refs. \cite{5}-\cite{12}; many other references could also be cited). The model is also variously called the “Lagrangian averaged model” or, in some of the earliest papers, the “Camassa-Holm” equations. This alpha model is subject to a variety of derivations, interpretations and connections, ranging from the mathematically sophisticated \cite{5} \cite{10} \cite{11} to the intuitive and simple \cite{13}. It is of interest to subject its predictions to tests against both experimental data (see Ref. \cite{8}) and numerical solutions of the relevant continuum equations to which alpha modeling has not yet been applied (see e.g. Ref. \cite{9} for the Navier-Stokes equations in three dimensions). Its extension to the case of coupling to a magnetic field in the magnetohydrodynamic (MHD) limit and in the non-dissipative case can be found in \cite{16} \cite{14}. In that context, the main purpose of this article is to carry out some of the numerical tests that have not previously been done for the case of MHD.

It is to be emphasized that there is no derivation of the alpha model that is completely systematic and deductive. Every presentation of it has involved steps that call for justification by their consequences, and that is the spirit in which we are proceeding here. In Ref. \cite{13}, which seems the most economical derivation possible, the point of view is taken that we smooth the fields (e.g. the velocity field $v$ and, in MHD, the magnetic field $B$) but not their “sources” (e.g., the vorticity field ! and, in MHD, the electric current density $j$). By “sources,” we mean here the curls of $v$ and $B$ which, given a set of boundary conditions, determine them through the Biot-Savart law or solutions to Poisson’s equation. The assumption is that the large-scale dynamics are relatively insensitive to small changes in the positions of the sources, but are sensitive to the strengths and approximate positions of them. Said another way, the large-scale fields alone are assumed to be responsible for those motions of their sources which significantly affect those large-scale fields. This assumption is by no means self-evident, but does have the advantage of reducing the derivation to a single algebraic step, in contrast to some more involved derivations which have been given, and which seem logically no more compelling. Our focus here is on neither presenting unarguable alpha-model derivations or comparing the possible variants of it, but rather on exploring the consequences of the primitive version of it given here [Eqs. \cite{11} and \cite{12} in what follows].

Modeling MHD flows with the Lagrangian-averaged methodology has been barely explored with no emphasis on the turbulence regime. This article thus focuses on several such predictions, numerically obtained, for the case of two-
dimensional magnetohydrodynamics, or hereafter 2D MHD (see e.g. for a brief review, Ref. [15]). Several effects have been studied phenomenologically, theoretically and numerically in the past, and may be identified in the literature by the names selective decay, dynamic alignment, direct and inverse cascades, and the characterization of probability distribution functions (pdfs) for the fluctuating field variables. There are several Reynolds-like numbers which can be attributed to MHD turbulent flows, since there are two possible velocities which may appear in the numerators (the flow speed and the Alfvén speed) and two diffusivities that may appear in the denominators (kinematic viscosity and magnetic diffusivity). Some length scale characteristic of the initial fields is usually present in the numerators. All these Reynolds-like numbers can be made to appear in the places of reciprocals of the transport coefficients in front of the dissipative terms in various dimensionless representations of the MHD equations. In general, the larger the values of these Reynolds-like numbers (or equivalently, the smaller the transport coefficients), the greater the required numerical resolution to follow their solutions. Values of a Reynolds number like $10^4$ usually strain available computer resources even in two space dimensions (2D), and while the attainable total number of degrees of freedom with computers has been steadily increasing over several decades, there are situations in which one might be curious about results in cases of far higher values of direct interest for geophysical flows and yet not attainable in the foreseeable future. The alpha model, if it can be verified to give correct predictions in the range of accurate, un-modelized solutions, will acquire a certain credibility in providing the behavior (at least of the long-wavelength Fourier components) in situations with Reynolds-like numbers so high as to put them presently far out of reach of direct numerical solutions (DNS), particularly so in three space dimensions (3D). Another set of regimes where modeling is needed is when widely disparate time and length scales occur, such as for either a small or large magnetic Prandtl number; this is the case for the former in liquid metals as encountered in laboratory dynamo experiments and in breeder reactors, in the core of the earth and planets, and in the convective zones of the sun and stars, or for the latter in the interstellar medium.

In Section II, we write down the alpha-model equations that we assume for incompressible, one-fluid MHD with a minimum of theoretical justification, following Ref. [13]; they include the effects of true viscous and dissipation. We provide expressions for ideal invariants that are conserved by the alpha model when the viscous and Ohmic dissipation coefficients are dropped, and decay laws for them when the dissipation coefficients are present and finite. Much of the approach to and vocabulary of the way turbulence problems have historically been formalized for MHD (a subject where computational data vastly exceeds experimental or observational data) can by now be taken for granted. In Section III, we describe results for the problem of selective decay in 2D. In Section IV, we turn our attention to that of dynamic alignment of the velocity and magnetic fields in turbulent decays. In Section V, we focus on inverse cascade computations. In all these cases, there are comparisons to be made between the alpha-modeled results and direct solutions of the primitive MHD equations. In Section VI, we address ourselves to the problem of quantitatively assigning errors to the alpha model, as compared with well-resolved full MHD as well as with unresolved MHD, of resolution comparable to that used for the alpha model. In Section VII, we sample a few effects at Reynolds-like numbers that are too high for any immediately foreseeable DNS code to approach. Finally, in Section VIII, we briefly summarize the results and suggest future problems in which the alpha model may have some utility.

To anticipate the conclusions of the paper, we note some features of the DNS solutions that the alpha model apparently finds out of reach. For instance, the location, in the plane, of specific features of evolving turbulent fields such as contour plots of vorticity or vector potential are virtually never accu-
rately reproduced after short times (contours of constant magnetic vector potential in 2D are magnetic field lines). Likewise, the spectral details at the small scales are not accurate and are not expected to be, since it is modifications of the dynamics at small scales that make the alpha model possible in the first place. But as far as the long wavelength component behavior for the turbulent kinetic and magnetic spectra is concerned, the alpha model seems to recover the main features of MHD turbulent flows in two space dimensions.

One technical feature of the computations peculiar to two-dimensional (2D) MHD should be commented upon. Because of the inherent tendency of 2D magnetic excitations to migrate to longer wavelengths, treatment of initial value problems requires beginning with excitations located in intermediate length scales, rather than at the longest wavelengths. In wavenumber space, this means that any filtering that is done must be done above wavenumbers corresponding to shorter wavelengths than those in the initial conditions. For this reason, very low resolution alpha-model calculations are not possible, unless the initial conditions themselves are to be left outside the basic box in Fourier space. This contrasts with the situation in three-dimensional hydrodynamics, where the emphasis is typically on cascades to shorter length scales, and where for the above reasons, it is possible to attempt large eddy simulations (LES) with very low maximum wavenumbers, and the initial excitations may all reside at the largest scales. While the ratio of our maximum retained wavenumber to the wavenumber where the filtering begins is about 8, it should be noted that larger ratios are feasible for three-dimensional Navier-Stokes LES and alpha model computations.

II. THE ALPHA MODEL FOR MHD

A. The equations

We write the three dimensional version of the equations first for the primitive incompressible MHD equations, and then for the alpha model. We will then specialize them to two dimensions for the purposes of this paper. The basic variables are the velocity field \( \mathbf{v} \) and the magnetic field \( \mathbf{B} \), functions of space and time coordinates \((x; t)\). In dimensionless (Alfvénic) units, the equations are:

\[
\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \mathbf{j} \times \mathbf{B} + \frac{1}{\Re} \nabla \times (\nabla \times \mathbf{v}) \quad (1)
\]

\[
\frac{\partial \mathbf{B}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{B} = \mathbf{B} \times \mathbf{v} + \frac{\mathbf{j} \times \mathbf{B}}{\mu} \quad (2)
\]

with \( \nabla \cdot \mathbf{v} = 0 \) and \( \nabla \cdot \mathbf{B} = 0 \).

The velocity field may be considered to be expressed in units of an r.m.s. value of the initial fluctuating velocity field, which we typically take to be 1. The magnetic field is made dimensionless by solving for the magnetic field value that would lead to an Alfven speed equal to the r.m.s. velocity field and dividing the magnetic field in laboratory units by that. The mechanical pressure is \( p \), which has first been divided by the mass density and then expressed in the units of the dimensionless velocity. The mass density is assumed to be constant and uniform. The viscosity and the magnetic diffusivity can be considered to be reciprocals of the mechanical and magnetic Reynolds numbers, respectively, in these units. Anticipating that the computations will be carried out inside a periodic box of edge \( 2L \), the unit of length will in general be taken to be equal to unity, or about 1–6 of a box dimension.

In the dimensionless units, the curl of the velocity is \( \mathbf{j} \), the vorticity field, and the curl of the magnetic field is \( \mathbf{j} \times \mathbf{B} \), the electric current density. The magnetic field \( \mathbf{B} \) can be written as the curl of a vector potential \( \mathbf{A} \), which, removing a curl from Eq. (2), obeys

\[
\frac{\partial \mathbf{A}}{\partial t} = \nabla \times \mathbf{B} = \mathbf{j} \times \mathbf{B} \quad (3)
\]

where the scalar potential is \( \phi \), can be determined by taking the divergence of Eq. (3), imposing the Coulomb gauge on \( \mathbf{A} \) (i.e. writing \( \nabla \cdot \mathbf{A} = 0 \)), and solving the resulting Poisson equation for \( \phi \), involving \( \mathbf{v} \) and \( \mathbf{B} \) in the source term. (In a similar way, the pressure \( p \) can be found by taking the divergence of Eq. (1), using the vanishing of the divergence of the time derivative of the velocity field \( \nabla \times \mathbf{v} \), and solving the resulting Poisson equation for the pressure. These Poisson solutions are easy to solve in Fourier space.)

To obtain the alpha-model version of Eqs. (1–4), we may divide \( \mathbf{v} \) and \( \mathbf{B} \) into smoothed values plus fluctuations about those values. Thus

\[
\mathbf{v} = \mathbf{v}_s + \mathbf{v} \quad (4)
\]

and

\[
\mathbf{B} = \mathbf{B}_s + \mathbf{B} \quad (5)
\]

Here, the smoothed values of the fields, \( \mathbf{v}_s \) and \( \mathbf{B}_s \), are defined by

\[
\mathbf{v}_s = c^3 \mathbf{x} \cdot \frac{\exp \left[ -k \mathbf{x} \cdot \mathbf{v}_s \right]}{4 \mathbf{k} \cdot \mathbf{x}^2} \mathbf{v}(\mathbf{x}_s; t) \quad (6)
\]

\[
\mathbf{B}_s = c^4 \mathbf{x} \cdot \frac{\exp \left[ -k \mathbf{x} \cdot \mathbf{B}_s \right]}{4 \mathbf{k} \cdot \mathbf{x}^2} \mathbf{B}(\mathbf{x}_s; t) \quad (7)
\]

with \( c \) at this point an arbitrary length; \( c^3 \) is typically to be chosen as larger than the wavenumbers whose behavior it is desired to reproduce accurately.

In general, we use the same values of \( c \) for smoothing \( \mathbf{v} \) and \( \mathbf{B} \), though we note the possibility of assigning \textit{a priori} one value for \( \mathbf{v} \) and a different value for \( \mathbf{B} \) (respectively, \( \kappa \) and \( \eta \)). Another possibility is to choose \( \eta = 0 \) for \( \mathbf{B} \), which in that case will leave us with an unsmoothed magnetic field. We will show such an example later.

We now take the curl of Eq. (1) to obtain the equation of motion in the vorticity representation,

\[
\frac{\partial \mathbf{A}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{A} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \frac{1}{\Re} \nabla \times (\nabla \times \mathbf{v}) \quad (8)
\]
FIG. 2: (a) Magnetic energy and (b) kinetic energy spectra, for selective decay runs 1-4 (see Table I), at $t = 5$; and (c) magnetic energy and (d) kinetic energy spectra at $t = 100$. The vertical line gives $k = 1$. The crosses (+) indicate the place on the spectrum where an under-resolved DNS run ($256^2$ grid points) departs significantly from the resolved computed DNS spectra. This convention will also be used in subsequent figures: crosses always indicate part of the under-resolved DNS spectrum for the same initial conditions and times.

FIG. 3: Square vector potential energy spectrum shown at $t = 300$, for the same selective decay runs as in Fig. 2. By this late time, essentially all of $A$ is concentrated in $k = 1$.

and then substitute into Eqs. 1 and 8 the fields expressed in Eqs. 4 and 5. Note that we do not smooth the vorticity $\omega$, which can be regarded as the source, in a Poisson or Biot-Savart sense, of $v$. Nor do we smooth $j$ which bears the same mathematical relation to $B$ as $\omega$ does to $v$. The result is:
\[
\frac{\partial \mathbf{v}}{\partial t} + \mathbf{u}_s \cdot \nabla \mathbf{v} - \mathbf{v} \cdot \nabla \mathbf{u}_s = \mathbf{f}
\]

and

\[
\frac{\partial \mathbf{B}_s}{\partial t} + (\mathbf{u}_s + \mathbf{v}) \cdot \nabla \mathbf{B}_s - \mathbf{B}_s \cdot \nabla \mathbf{u}_s - \mathbf{B}_s \cdot \nabla \mathbf{v} = \mu_0 \mathbf{J}_s + \mathbf{r}_{\mathbf{B}}
\]

upon which no approximations have as yet been made. That is, they are equivalent to Eqs. (11), (12).

Taking a modeling or heuristic point of view, the essence of the alpha model is to neglect the fluctuations \(\mathbf{v}\) and \(\mathbf{B}\) in relation to the smoothed fields \(\mathbf{u}_s\) and \(\mathbf{B}_s\) in Eqs. (9) and (10), while leaving the source terms of \(\mathbf{v}\) and \(\mathbf{B}\) alone. This is one way of looking at the alpha approximation. Its relation to other, more complicated derivations will not be discussed here, since our intent is to test the alpha model rather than to justify it from anything like first principles. Further discussion of the above approximation, which is the only one in our formulation, can be expected elsewhere.

The alpha model equations are then (removing a curl from Eq. (9))

\[
\frac{\partial \mathbf{v}}{\partial t} + \mathbf{u}_s \cdot \nabla \mathbf{v} = \mathbf{f}
\]

and

\[
\frac{\partial \mathbf{B}_s}{\partial t} + (\mathbf{u}_s + \mathbf{v}) \cdot \nabla \mathbf{B}_s = \mathbf{r}_{\mathbf{B}}
\]

Note that the smoothed quantities bear the subscript letter \(s\), and the unsmoothed ones do not. We shall follow this convention throughout. Eq. (12) could be viewed alternatively as a hyper-resistivity approximation on \(\mathbf{B}_s\). The connection between the smoothed and unsmoothed fields may be stated in differential form as

\[
\mathbf{v} = (1 - \mathbf{r}^2) \mathbf{u}_s
\]

and

\[
\mathbf{B} = (1 - \mathbf{r}^2) \mathbf{B}_s
\]

We may associate smoothed values of \(\mathbf{v}\) and \(\mathbf{B}\) with the unsmoothed ones according to the same recipe; even though they do not enter directly into the dynamical equations, they are at some points convenient to think and talk about. Thus \(\mathbf{r}^2 \mathbf{u}_s\) !, similarly \(\mathbf{r}^2 \mathbf{B}_s\) \(j\), and \(\mathbf{r}^2 \mathbf{A}_s\) \(B\). A smoothed vector potential \(\mathbf{A}_s\) may be regarded as having a curl \(\mathbf{B}_s\), while obeying a Poisson relation to \(\mathbf{j}\), namely \(\nabla^2 \mathbf{A}_s = \mathbf{j}\). We stress that \(\mathbf{j}\) and \(\mathbf{A}_s\) do not enter the alpha model equations we use.

Specialization to two dimensions is achieved by taking the curl of Eq. (13) or specializing Eq. (13) and Eq. (5) to the two-dimensional geometry in which there are only two \((x; y)\) non-zero components of \(\mathbf{v}\) and \(\mathbf{B}\), and only one component \((z)\) of \(\mathbf{r}\) or \(\mathbf{j}\) and carrying out the smoothing approximations so described. All fields are independent of the \(z\) coordinate.

Noting that only one component of \(\mathbf{A}\), the \(z\)-component, is relevant to two dimensions, the result is:

\[
\frac{\partial \mathbf{v}}{\partial t} + \mathbf{u}_s \cdot \nabla \mathbf{v} = \mathbf{f}_s \quad \text{Eq. (15)}
\]

and

\[
\frac{\partial \mathbf{B}_s}{\partial t} + (\mathbf{u}_s + \mathbf{v}) \cdot \nabla \mathbf{B}_s = \mathbf{j}_s \quad \text{Eq. (16)}
\]

where there are stream functions \(\mathbf{A}_z\) and vector potentials \(\mathbf{A}_z\) that bear Poisson relations to their sources, both for the smoothed and unsmoothed versions:

\[
r^2 \mathbf{A}_z = \mathbf{j} \quad \text{Eq. (17)}
\]

To re-iterate, the principal intent of this paper is to compare typical solutions of Eqs. (15) and (16) with solutions, for the same initial and boundary conditions, of the well-known 2D MHD equations, unsmoothed.

### B. The invariants

There are three ideal invariants for both sets of equations: the total energy \(E\), the total cross helicity \(H_C\), and the total mean-square vector potential \(\mathbf{A}\). The alpha model expressions for these are, respectively,

\[
E = \frac{1}{2} \mathbf{d}^2 \times (\mathbf{u}_s \mathbf{v} + \mathbf{B}_s \mathbf{B})
\]

and

\[
H_C = \frac{1}{2} \mathbf{d}^2 \times \mathbf{v} \mathbf{B}_s
\]

Note that the energy invariant \(E\) involves both the smoothed and unsmoothed velocity and magnetic field, whereas only the
| Run | $a^1$ | $k^1$ | N  | R    | Figs. |
|-----|-------|-------|-----|------|-------|
| 1   | 1     | 1     | 1024 | 215  | 15    |
| 2   | 40    | 40    | 1024 | 235  |       |
| 3   | 40    | 40    | 512  | 240  |       |
| 4   | 40    | 40    | 256  | 240  |       |
| 5   | 1     | 1     | 512  | 280  | 39    |
| 6   | 20    | 20    | 512  | 305  |       |
| 7   | 20    | 20    | 128  | 300  |       |
| 8   | 1     | 1     | 256  | 28   | 10-15 |
| 9   | 20    | 20    | 256  | 30   |       |
| 10  | 30    | 30    | 256  | 29   |       |
| 11  | 1     | 20    | 256  | 28   |       |
| 12  | 30    | 30    | 128  | 30   |       |
| 13  | 1     | 1     | 1024 | 1,150| 15,18 |
| 14  | 50    | 50    | 256  | 1,170|       |
| 15  | 1     | 50    | 256  | 940  |       |
| 16  | 1     | 1     | 256  | 980  |       |
| 17  | 300   | 300   | 2048 | 5,200|       |

TABLE I: Main characteristics of the runs. $a^1$ and $k^1$ are the reciprocal of the alpha lengths for the velocity and the magnetic field; $N$ is the grid resolution before dealiasing, $R$ is the Taylor Reynolds number (see Eq. 25) at peak dissipation, and the last column gives the figures relating to the different runs, namely: runs 1–4 for selective decay, runs 5–7 for dynamic alignment, runs 8–12 for the inverse cascade of magnetic potential and runs 13–17 for large-scale turbulence. In the figures, solid lines are for fully resolved DNS (runs 1, 5, 8 & 13), dashed lines for runs 2, 6, 9 & 14, dashed-triple dots for runs 3, 11 & 15, dotted lines for runs 4, 7, 12 & 16, and a dash-dot line for run 10.

smoothed magnetic variables appear in the expressions of $H_C$ and $A_z$, due to the linearity of the induction equation, once the velocity field is given; however, the decay rates of $H_C$ and $A_z$ involve both the smoothed and unsmoothed magnetic variables whereas the decay rate of energy only involves the unsmoothed current density (see below).

The decay laws for these quantities are, in periodic boundary conditions,

$$\frac{dE}{dt} = Z \sum \frac{\partial^2}{\partial x_j^2} A_{s_z} \tag{22}$$

$$\frac{dH_C}{dt} = Z \sum \frac{\partial^2}{\partial x_j^2} A_{s_z} \tag{23}$$

and

$$\frac{dA_z}{dt} = \frac{Z}{2} \frac{\partial^2}{\partial x_j^2} A_{s_z} \frac{\partial^2}{\partial x_j^2} \frac{\partial^2}{\partial x_k^2} \tag{24}$$

These three invariants will be at the core of the numerical tests to be reported in the next three Sections. Finally, it should be noted that Eqs. (19)-(24) differ from their full MHD equivalents in detail, but approach them as $\alpha \to 0$.

For convenience and later reference, the main characteristics of all the runs described in this paper are given in Table I together with the number of the figures related to the different category of runs. $N$ is the grid resolution before dealiasing, using the standard $2^3$ rule in all the runs described in this paper (hence, for a run on a grid of $N$ points, the maximum wavenumber attainable is equal to $N=3$). Finally, $R$ is the Taylor Reynolds number defined as

$$R = \frac{V_{rms}}{\nu}; \tag{25}$$

it is based on the $r.m.s.$ velocity and on the Taylor scale

$$\frac{\nu}{\nu_x} = \frac{2}{\nu} = \frac{< |\nu|^2 >}{< |\nu_x|^2 >}; \tag{26}$$

computed at the peak of the dissipation. The viscosity is equal to $10^{-4}$ for the selective decay runs 1–4, with initial conditions with non-vanishing Fourier coefficients (see Eq. (27)) for wavenumbers between $k_1 = 10$ and $k_2 = 30$. Runs 5–7 are for dynamic alignment, with $\alpha = 10^{-3}$ and $k_1 = 5$; $k_2 = 10$. Runs 8–12 are for the inverse cascade of magnetic potential, with $\alpha = 10^{-3}$, the forcing occurring in the interval $k_1 = 18$; $k_2 = 22$. Runs 13–16 deal with large-scale turbulence with $\alpha = 5 \times 10^{-4}$ and the initial conditions confined between $k_1 = 1$ and $k_2 = 3$, whereas for run 17, $\alpha = 2 \times 10^{-3}$. In the figures, solid lines are for fully resolved DNS (runs 1, 5, 8 & 13), dashed lines for runs 2, 6, 9 & 14, dashed-triple dots for runs with $k \in (0; \infty)$; for runs 3, 11 & 15); finally, dotted lines are for runs 4, 7, 12 & 16, and a dash-dot line for run 10. Note that all runs have unit magnetic Prandtl number; the initial conditions are such that kinetic and magnetic energies are equal and of order unity with random phases. All runs are decaying (i.e. no forcing), except for the inverse cascade runs, which have zero initial conditions and forcing in the induction equation only.

III. SELECTIVE DECAY

By “selective decay,” we mean turbulent processes (see e.g. [16]-[18]) in which one or more ideal invariants are dissipated rapidly relative to another, due to the transfer of the dissipated quantities to short wavelengths where the dissipation coefficients become effective. In 2D MHD, with negligible cross-helicity, the selectively dissipated quantity is energy, while the nearly-conserved quantity is mean square vector potential. The limit defines a variational problem which seeks the state in which the dissipated quantities are as close to zero as they can be for the surviving value of the nearly-conserved quantity. There are no constraints on the cascade of kinetic energy.
to short wavelengths, so the asymptotic state is expected to be one that is largely magnetic and has the surviving magnetic excitations peaked at the longest wavelengths. In particular, the vector potential spectrum should have a sharp maximum at the lowest wavenumber of the computation, here \( k_{\text{min}} = \frac{1}{\Delta x} \). This effect has been demonstrated repeatedly in the past ([16]-[18]).

In this Section, we compare the full MHD (0) results for a selective decay run of a familiar type with the consequences of the alpha model for the same initial conditions but finite. We specify the initial conditions in Fourier space, with \( \nu \) and \( B \) represented as the Fourier series,

\[
\nu(x; t) = \sum_k \phi_k(t) e^{ikx}, \quad B(x; t) = \sum_k \hat{B}_k(t) e^{ikx},
\]

with similar decompositions for the other vector fields. The non-vanishing initial Fourier coefficients are confined to a ring in \( k \)-space between \( k_1 = 10 \) and \( k_2 = 30 \). The amplitudes in

FIG. 4: Contour plots of vector potential (left) and current density (right) at \( t = 400 \) for the DNS run (top) and alpha run 3 (middle) and 4 (bottom), with positive and negative values respectively in solid and dashed lines. Large scales are almost in the form of bars parallel to the axes.
FIG. 5: Pdfs at $t = 15$ of (a) the current density, and (b) $\ln\left(\nu^2 + j^2\right)$, for all selective decay runs (see Table I). Note the larger values of current density for the alpha runs compared to the DNS (solid line).

FIG. 6: (a) Temporal evolution of magnetic energy (top curves) and kinetic energy (bottom curves), and (b) of normalized cross helicity, for all dynamic alignment runs (see Table I). Note that only the first five units of time for the run are shown in (a); the solid line is for the DNS run.

Reynolds numbers, kinetic and magnetic, are then formally equal respectively to 2000, based on a unit length scale in the basic box of edge 2, whereas the Taylor Reynolds number at peak of dissipation is equal to 215 for the DNS run, and slightly larger for the alpha runs (see Table I). The time will be measured in units that are defined by the ratio of unit length to the initial r.m.s. velocity; based on the energy containing scale, one time unit can be several initial eddy turnover times, a number which may increase or decrease as the kinetic energy is dissipated.

Figure II displays the computed magnetic energy (upper curves) and kinetic energy (lower curves) versus time, showing the ultimate dominance of the magnetic energy over the kinetic, by an order of magnitude at this time.

The solid lines are the results of the full MHD computa-
FIG. 7: (a) Magnetic and (b) kinetic energy spectra, for the three dynamic alignment runs (see Table I), at \( t = 2 \); (c) Magnetic and (d) kinetic energy spectra at \( t = 30 \). As elsewhere, crosses (+) indicate points on the computed under-resolved DNS spectrum (128\(^2\) grid points) for the same initial conditions, and indicate the values of \( k \) where the first significant departures occur from the well-resolved DNS.

tion (\textit{i.e.}, \( \rho = 0 \)), with 1024\(^2\) grid points. The dashed lines are the results of the alpha model with 1024\(^2\) grid points and with \( \rho = 1=40 \). The dotted line (barely distinguishable from the dashed one) shows the results for an alpha-model run with 256\(^2\) grid points and the same value of \( \rho \). Note that a full MHD run with the lower number of grid points, \textit{i.e.} an under-resolved computation of MHD turbulence, would display disagreement with the other three runs. We shall discuss this question in Section VI.

Fig. 1b shows the cross helicity (which remains very small, relative to the energy, because the random phases for the two fields imply negligible correlation between them) as a function of time. The alpha model computations disagree with the exact MHD run, but since all the quantities are so small, this disagreement is not deemed to be significant, but rather occurs as fluctuations around values close to zero. We should remark that in both cases, the comparison between the full MHD quantities and their alpha-model analogues has been done after a rescaling of initial data which makes the energies agree exactly at \( t = 0 \) (with the full, unsmoothed MHD values); the original MHD energy is not quite the same as the energy integrals defined in Section II involving the smoothed fields, because of the smoothing, though the difference is only a few percent.

Figures 2a,b show the omni-directional energy spectra for the magnetic and kinetic energies (as defined in Section II when \( \rho \) is non-zero), respectively, at \( t = 100 \). The same conventions adopted in Fig. 1 (\textit{i.e.}, solid lines mean full MHD, dashed lines mean alpha model with \( \rho = 1=40 \) and 1024\(^2\) grid points, and dotted lines mean runs done with the same value of \( \rho \) but with 256\(^2\) grid points) will be followed throughout. The alpha-model spectra that are plotted result from taking the spectral density of the invariants in Equations (19)-(21). The vertical line indicates the wavenumber \( k \) corresponding to the length \( \lambda \). Note that the under-resolved spectra begin to differ at \( k \approx 2 \), but the -model and well-resolved DNS agree up to \( k \approx 1 \).

Figure 3 displays a spectrum in log-lin scales, of the vector potential at very late times, when the selective decay is nearly complete and the magnetic excitations are concentrated in the longest wavelength allowed by the boundary conditions (\( k_{m \in \Omega} = 1 \)). The alpha model has reproduced this feature, with only a small disagreement in the values at \( k = 1 \) (see also Section VI for a more complete discussion of errors).

The suppression of the small scales is quite apparent for both quantities, but the large scales, like the global energies exhibited in Fig. 1a, do not appear to be significantly affected. The lower resolution alpha model run reproduces the same result. This is what can be realistically hoped for from the alpha model, although we note that the disagreement between
the true MHD run and the alpha runs starts at a scale roughly twice as large as $1$.

Figures 4 display contour plots of curves of constant vector potential $A_x$ (Fig. 4 left) and constant current density $j$ (Fig. 4 right) at time $t = 100$. The top panel is for full MHD, the middle one for $\epsilon = 1/40$ and $512^2$ grid points, and the bottom one for $\epsilon = 1/40$ and $256^2$ grid points. The flow may evolve toward a state reminiscent of those found in [19] in the case of 2D Navier-Stokes turbulence, with structures parallel to either axis. While there are marked similarities in the kinds of structures present in the DNS and in the alpha runs, there are clearly no one-to-one correspondences as to specific features, either as to location, orientation, or intensity. From these and many similar figures we have looked at, we have concluded that while the alpha model does an excellent job of reproducing long-wavelength spectra, the pointwise details of the solution are not well tracked by it, at least in the absence of constraining material boundaries.
FIG. 9: Pdfs at \( t = 50 \) of (a) the current density, and (b) \( \ln (\nu \omega^2 + \eta j^2) \), for all dynamic alignment runs (see Table II). Note again the high values of \( j \) for the alpha runs.

In Figs. 5, we display normalized probability distribution functions (pdfs) of the current density \( j \) in Fig. 5a, and in Fig. 5b of the spatial density of the dissipation rate of energy given in Eq. (22); note that the local (spatial) dissipation of kinetic energy differs in its expression, involving the symmetrized velocity gradient instead of the local squared vorticity density. The conventions with the lines are the same as those in the preceding three figures. It is apparent that the pdfs of the alpha model do a good job for the lower values of \( j \) but do not reproduce the tails accurately, in particular at lower resolution, i.e., intermittency is not fully reproduced, and is not expected to be (for a study of intermittency in the context of

FIG. 10: (a) Temporal evolution of the magnetic energy (top curves) and kinetic energy (bottom curves) until \( t = 50 \), and (b) of the squared vector potential until \( t = 400 \), for all inverse cascade runs (see Table II runs 8-12). Whereas energies are in good agreement with the DNS (shown as usual with a solid line), the growth of squared magnetic potential is slower for all alpha runs (see also Fig. 13).
LES, see e.g. Ref. [20]). The same is true of the dissipation density, although discrepancies appear smaller.

IV. DYNAMIC ALIGNMENT

A perfectly “aligned” solution to the ideal version of Eqs. (1) and (2) results whenever $v = +B$ or $v = -B$ everywhere. Previous computations [21]-[23], inspired by observations in the quiet solar wind [24], have shown that MHD turbulence in which a significant degree of initial alignment, or correlation between the $v$ and $B$ fields, exists will evolve toward a state of greater and greater alignment as time goes on. The physical origins of this process are not completely clear, except that we may note that an aligned state involves no spectral transfer to higher wave numbers where viscous and Ohmic dissipation are effective, so that those patches where alignment exists initially may have a tendency simply to outlive the more active, unaligned patches where spectral transfer makes dissipation more likely. Similar alignment, this time between velocity and vorticity, can be observed for three-dimensional Navier-Stokes flows (see e.g. Ref. [25] for an experimental study).

A useful index of the global degree of alignment of $v$ and $B$ may be taken as the “fractional alignment”: it is defined taking $2H_C$ and dividing it by the square root of $<u_s>v <B>$, where angle brackets mean spatial averages. When this ratio is unity, the fields may be regarded as perfectly aligned. In Fig. 6, we display the results of a run which starts with the Fourier amplitudes chosen to be equal in a ring with $k=10$, with unit r.m.s. values of $v$ and $B$ and with phases chosen so that the fractional alignment is initially 0.3. Finally, $t=10^{-3}$ and $=10^{-3}$; the Reynolds numbers are equal to 1000, based upon unit length, unit r.m.s. velocity at $t=0$ and the transport coefficient, and the Taylor Reynolds number at peak dissipation is equal to 280 for the DNS run, and again slightly higher for the alpha runs (see Table I).

In Fig. 6, we show that the energies, magnetic (top curves) and kinetic (lower curves), as functions of time, have comparable evolutions. The fields, however, become progressively more aligned during their decay as can be seen in Fig. 6, where the alignment index gradually increases from 0.3 to about 0.85. As before, the $=0$, or full MHD (with 512$^2$ grid points), results are exhibited as solid lines, the dashed lines are the results for $=128$ grid points, while the dotted line is for the same alpha but with only 128$^2$ grid points. The rather large discrepancies for long times for the correlation coefficient may come from the fact that it involves a normalization; the cross-helicity $H_C$ themselves do not differ significantly (not shown); it also may imply that small scales, which are modified by the alpha modeling process, play a role in the growth of large-scale correlations between the velocity and the magnetic field.

The spectra in Figs. 7 show that in the aligning situation, the alpha model continues to do a good job of reproducing the long wavelength components of the MHD spectrum. Figure 8 shows contour plots of the unsmoothed stream function (right) at $t=60$, for full MHD (top), for the alpha model with $=120$ and 512$^2$ grid points (middle), and for the alpha model with $=128$ grid points (bottom); Figs. 8 (left) show the corresponding contours for the smoothed vector potential. The pointwise alignment is more visible in the top contours (i.e. for the full MHD run) than in the finite runs.

Figure 9 shows pdfs of (a) the current density $j$ and (b) $\ln(\ j^2+\ T^2)$ i.e. the spatial density of the energy dissipation rate; the units scales are lin-log; note the exponential depen-
computer.

In inverse cascades, it is the small scales which are there to inject excitations in some field or another. These terms are typically band-limited in Fourier space, so that only a narrow range of $k$-values, well above the energy-containing scales, are considered as externally excited or stirred. The excitations can be injected either into the mechanical or the magnetic part of the dynamics. In studies of the three-dimensional dynamo problem, the injection is typically into the velocity field, and involves the conversion of mechanical helicity into magnetic helicity, which is then transferred back into the long-wavelength part of the spectrum (see, e.g., Refs. [28-31]).

In two dimensions, the helicities are identically zero, and the anti-dynamo theorem prohibits the generation of persistent magnetic excitations by mechanical means, so the band-limited injections are magnetic and typically are considered to be the addition of mean square vector potential or magnetic flux, added randomly at the small scales (see, e.g., Refs. [32-37]). Here, we write a random forcing function on the right hand side of Eq. (16), which may be described as follows. We adopt a random forcing $f$ only in the induction equation for the vector potential, of the form

$$f(k; t) = \sum_{k} F(k; t) e^{ik \cdot x};$$

The sum runs from $k_1 = 18$ to $k_2 = 22$. The amplitudes of all the coefficients $F(k; t)$ in this ring are chosen equal, but the phases at each $k$ are changed randomly with a correlation time larger than the time step, but smaller than the eddy turnover time. The phases are uniformly distributed between $2 \pi$ and $2 \pi$. For the runs we will discuss in this Section, the correlation time was $\approx 2 \times 10^2$. One would expect that the relation of the forcing band of wave numbers to the reciprocal of would be a sensitive one in the outcome of an inverse vector potential cascade computation. This proves to be the case, and only in the situation where the forcing band lies at lower wave numbers than the reciprocal of are recognizable results achieved. Even there, as will be seen in what follows, the agreement is less satisfactory than it has been for the selective decay and dynamic alignment tests.

Figures 10a,b show the time histories of a magnetically forced run that started from an otherwise empty spectrum, with a time step $t = 2 \times 10^2$, and $= 10^3$; the Taylor Reynolds numbers at peak dissipation is for all runs 30. The upper curves in Fig. 10a are magnetic energies and the lower set are kinetic energies. The curves in Fig. 10b are mean square vector potentials as functions of time. The solid lines are for a full MHD run ($= 0$) with $256^2$ grid points. The dashed lines are for $= 1=20$ and $256^2$ grid points. The dashed-dotted lines are for $= 1=30$ and $256^2$ grid points. The dashed-triple-dotted lines are for the mechanical $= 1=20$, but with the alpha parameter appearing in the induction equation $\sum$ set equal to zero (i.e. the magnetic variables are unsmoothed). The dotted lines are for $128^2$ grid points and both alphas $= 1=30$. The forcing functions are identical in all cases. Both kinetic and magnetic energies are similar in amplitudes. The biggest disparity will be noted in Fig. 10b, where the growth rates of the global mean-square vector potential differ significantly, resulting in $A$ for the DNS run remaining about a factor of 2 larger than for any of the alpha approximations at the end of the runs.

Most of the discrepancy is accounted for by the values of...
FIG. 13: Contour plots of the (smoothed) vector potential at $t = 400$ for inverse cascades (top: DNS; middle and bottom: runs 11 and 10 respectively). By that time, similar large-scale structures have formed in all runs, but note that they have different locations in the DNS and alpha runs.
FIG. 14: Residual energy spectrum $E_k(k)$ normalized by the total energy $E_k(k) + E_m(k)$ at $t = 400$, with plotting conventions as in Fig. 11. The two vertical arrows correspond to the two values of alpha, one of which is in the middle of the forcing band (run 9, dashed line). Note the strong dominance of kinetic energy in the small scales for all runs except the DNS run (solid line) and for the alpha run in which no smoothing occurs for the magnetic field, \( m = 0 \) (dash-triple dotted line).

![FIG. 14](image)

TABLE II: Errors (see eq. (33)) for selective decay alpha runs 14–16, the subscripts \( m \) and \( k \) indicating respectively, the error computed on the magnetic and kinetic energy spectra.

| run  | 16_m | 14_m | 15_m | 16_k | 14_k | 15_k |
|------|------|------|------|------|------|------|
| \( E_1 \) | 0.15 | 0.08 | 0.15 | 0.17 | 0.12 | 0.15 |
| \( E_2 \) | 0.03 | 0.005 | 0.03 | 0.03 | 0.02 | 0.02 |

Note first that in all the runs, the fundamental mode is completely dominated by the magnetic energy, and in all the alpha-runs except the one without smoothing of the magnetic field \( m = 0 \), the small scales are completely dominated by the velocity; this effect is enhanced by the normalization, but little energy resides in the smallest scales, beyond alpha, so that the inverse cascade can still take place in alpha runs, albeit at a slower pace. In other words, we see that in the framework of the alpha model, this energy defect is modified from the usual MHD case, and more importantly it even changes sign for the larger values of \( \alpha \) at high wavenumber, becoming positive and thus indicative of too high a dissipation assumption of scale separation. In other words, it represents non-linear non-local interactions in Fourier space, but it does not say anything about the local interactions themselves. In Figure 15 we plot $R(k) = E_k(k) + E_m(k)$ for several runs, with as usual the solid line for the MHD run and the dash-triple-dot line for the alpha run without smoothing of the magnetic field. Note first that in all the runs, the fundamental mode is completely dominated by the magnetic energy, and in all the alpha-runs except the one without smoothing of the magnetic field \( m = 0 \), the small scales are completely dominated by the velocity; this effect is enhanced by the normalization, but little energy resides in the smallest scales, beyond alpha, so that the inverse cascade can still take place in alpha runs, albeit at a slower pace. In other words, we see that in the framework of the alpha model, this energy defect is modified from the usual MHD case, and more importantly it even changes sign for the larger values of \( \alpha \) at high wavenumber, becoming positive and thus indicative of too high a dissipation.
FIG. 16: Averaged (a) magnetic energy and (b) kinetic energy spectra for the large-scale turbulence runs 13–16 (see Table I) between $t = 3$ and $t = 7$ (first and second peak of the enstrophy). The wavenumber corresponding to $k = 1$ is indicated by the vertical line, and the under-resolved run is shown with a dotted line. At these early times, spectra are in close agreement except at small scales.

VI. QUANTITATIVE ESTIMATES OF ERROR

Two separate questions of accuracy are addressed in this Section. First, we attempt quantitative comparisons of alpha model computations run on a $256^2$ grid with the standard provided by a well-resolved MHD run at a resolution of $1024^2$ for the same initial conditions and the same Reynolds numbers. Secondly, we compare the same alpha-model runs with DNS solutions on a $256^2$ grid in which no smoothing has been applied and which are undoubtedly unresolved, in that the dissipation wavenumbers (estimated on the basis of viscous and Ohmic enstrophy dissipation) exceed the maximum values of $k$ retained in the computation; such runs correspond to under-resolved computations. This latter test is necessary if one is to argue that accuracy has been improved by the alpha model in computations of comparable resolution. Error estimates are provided first for freely-decaying turbulence at early times, near the peaks of the dissipations (runs 13–16 of Table I), and then for the situations described in Secs. IV and V, where later times (and hence greater accumulated errors) are involved.

First we examine the case of freely decaying turbulence at an early time, which could be considered to be the early stages of a selective decay computation. The turbulence initially is confined to a band of wavenumbers between $k_1 = 1$ and $k_2 = 3$. In Figure 18 we show the computed mean-square vorticity (a), mean square current density (b), and mean square dissipation $<|!|^2> + <|j|^2>$ as functions of time. We show runs under all four circumstances, starting from the same initial conditions: fully-resolved $1024^2$ MHD, an alpha model run at $256^2$ with $\kappa = 1=50$ for both fields, an alpha model run with an unsmoothed magnetic field ($m = 0$) and $k = 1=50$ for the velocity field at $256^2$, and an under-resolved but unsmoothed DNS run at $256^2$; we see that time scales of growth of field gradients, and the amplitudes of such gradients are comparable but not identical.

of the magnetic field in the presence of alpha-smoothing of small scales, i.e. with $m = 0$. Even though alpha modeling is about the dynamical evolution of scales larger than alpha, the scales smaller than alpha nevertheless participate into the dynamics of the flow including at large scale and are responsible for the discrepancy in the growth rate of squared magnetic potential we observe here (see Fig 10b). Thus, such an effect presumably linked to the fact that the magnetic field is decaying with a scale dependency that is leaning more heavily on the small scales (like an effective $k^4$-type hyper-resistivity), may be at the source of the different behavior we observe between the different alpha-runs and the DNS run; if only non-local transfer (in Fourier space) is active in the inverse cascade in the case of the alpha runs, this could be at the origin of the slower growth of $A$. This point will require further investigation in the three-dimensional case, in relation with the large-scale dynamo problem, since the growth rate associated with the inverse cascade of magnetic helicity is the relative (kinetic helicity minus magnetic current helicity) in the small scales \[29\]. In the limit of very large $k$, the dynamics become trivially simple; see Ilyin and Titi \[39\].
TABLE III: Errors as defined in eq. (33) for the inverse cascade alpha runs 9–12 (see Table |); $E_m$ and $E_k$ indicate respectively the magnetic and energy spectra. Note the larger errors for run 9 corresponding to the alpha wavenumber ($k = 20$) embedded in the forcing band.

| run | $E_m$: 12 | $E_m$: 9 | $E_m$: 10 | $E_m$: 11 | $E_k$: 12 | $E_k$: 9 | $E_k$: 10 | $E_k$: 11 |
|-----|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| $E_1$ | .30       | .49       | .31       | .28       | .28       | .33       | .26       | .26       |
| $E_2$ | .13       | .41       | .14       | .11       | .09       | .13       | .08       | .09       |

Note that for the fully resolved DNS run, $k_{m,ax} = 341$ to be contrasted with the dissipation wavenumber based on the magnetic variables (computed using a Kolmogorov spectrum) of $k_{diss} = 280$ at the peak of dissipation i.e. for $t = 7$ (and $k_{diss} = 250$ when based on the velocity); this shows that the run is well resolved at all times, and indeed no “bottleneck” appears in the compensated spectra, where by bottleneck is meant an accumulation of energy in the small scales, seen as a bump in the compensated spectra for $k > k_{m,ax}$; this phenomenon has sometimes appeared in the presence of under-resolved computations, or when using hyper-viscous or hyper-resistive dissipative operators [26]. At the same time under-resolved computations, or when using hyper-viscous or hyper-resistive dissipative operators 

The same line conventions as in Fig. 16 have been adopted.

Thus a small value of $E_1$ or $E_2$ will indicate a closeness on the part of the MHD approximations (alpha-modeled or unresolved) to the full MHD DNS results.

$E_1$ and $E_2$ are exhibited as Table III for the freely decaying runs just described (and indicated by the superscript “D”). The conventions used in the Table are that the number of the run is followed by a subscript, where $m$ stands for magnetic spectra whose error is being assessed, and $k$ for kinetic ones. Time averages have been performed from $t = 3$ to $t = 7$, the vicinity of the main peaks in the dissipation rates. This is thought to be the time when the turbulence is of its most broad-band character, when the alpha model would be having its strongest impact.

The errors so defined are smaller for these early times for this freely decaying situation than they are for the situations studied in Sections III, IV, and V which examine late-time evolutions. We note that the lowest error (by a factor of 2 compared to the unresolved run) occurs for the alpha run with both alphas equal (run 14, see Table |). Note also that Fig. 17 displays the pdfs of (a) the energy error, whereas the errors in Table III are normalized by the total energy (truncated at 1= ) and hence are smaller.

In Table III we show the errors $E_1$ and $E_2$ for the inverse cascade situation of Sec. V and as indicated by the superscript I; the runs are the same as those in Sec. V, and given by their number (see Table |), with $E_m$ and $E_k$ standing as usual for magnetic and kinetic energy spectra. These errors are computed at late times, when the true solution may be expected to have drifted further from the $1024^2$ run and hence lead to larger errors than in the freely decaying runs of Table III. Moreover, as expected, when the alpha cut-off is too close to the forcing band, the errors are larger, both for the kinetic and the magnetic spectra.

Finally, in Table IV the $E_1$ and $E_2$ errors are shown for the dynamic alignment runs 6 and 7 of Sec. IV, with a superscript “DA” and a supplementary index “e” or “i”, in order to indicate early or late times in the run; specifically, early signifies that the average is taken for for $5 < t < 10$, and late for $55 < t < 60$. Again, the type of spectrum for which the error is displayed (either $E_m$ or $E_k$) is given before the run number. The low-resolution computation (run 7) does not have significantly higher errors at early times, but errors accumulate at later times, more so for the lower resolution computation (run 7). The reason for time averaging the magnitude of the errors is that, when plotted as functions of time, the error curves cross each other repeatedly. Time intervals can be found when
either one is smaller than another. Time averaging, over intervals long enough to contain many of these crossings but short compared to the duration of the runs, has seemed to provide the most objective number for addressing which error is “typically” smaller.

VII. VERY HIGH REYNOLDS NUMBERS

The eventual utility of the alpha model if it can be justified will be that it will permit explorations of Reynolds number regimes that are far above those that can be obtained from direct numerical solutions of the MHD equations. It may be noted that estimates have been given for the number of degrees of freedom of the Navier-Stokes alpha model [40]; see also [12]. Whereas most of this paper has been devoted to regimes in which direct MHD solutions can be compared to alpha model solutions, we have thought it interesting to show one alpha model computation that goes beyond what can be contemplated from unsmoothed solutions presently. We do this without any definitive claims for accuracy, but just as a suggestion of what the alpha model might provide in the way of future predictions. A detailed analysis of high Reynolds number runs at higher resolutions than what is performed here and using the alpha model will be presented elsewhere.

We display results for a run with \( \Reynolds = 2 \times 10^5 \), a time step of \( \Delta t = 2.5 \times 10^{-4} \), and 2048\(^2\) grid points. The value of \( \alpha \) is chosen to be 0.030, with \( \beta_{max} = 682 \) for this run. The initial (equal) kinetic and magnetic energies are loaded with random phases into the ring from \( k_1 = 1 \) to \( k_2 = 3 \) in Fourier space, and the r.m.s. values of \( v \) and \( B \) are unity. Computed at \( t = 7 \), i.e. close to the maximum of dissipation (see Fig. 19b), the Taylor Reynolds number \( \Reynolds \approx 5200 \). This is roughly comparable with what can be accomplished with a DNS on a grid of more than \( 10^8 \) points.

Fig. 19 shows the evolution of the total kinetic energy (dashed line) and total magnetic energy (solid line), referred as usual to unit volume, as functions of time. Fig. 19 shows the evolution of the mean square vorticity (dashed line) and mean square current density (solid line) as functions of time. The qualitative behavior of all quantities in Figs. 19 will be seen as not significantly different from that observed at lower resolutions, although oscillations in the kinetic and magnetic energies are persistent until \( t \approx 6 \). But the idea has been to extend the inertial range as much as possible. It will be noted that the magnetic and kinetic energies are far from equipartitioned, with magnetic energy in excess by a factor 3 at the end of the run. The peak value of mean dissipation at \( t \approx 7 \) (\( 0.074 \)) is comparable with that at the lower Reynolds number (runs 13-16), confirming previous results (see e.g. Fig. 7 in Ref. [41]) of lack of dependency of with Reynolds number, at least at a magnetic Prandtl number of unity.

Compensated energy spectra are displayed in Figs. 20. Fig. 20(a) is the kinetic energy spectrum, multiplied by \( k^{5/3} \), and averaged between \( t = 2 \) and \( t = 6 \), i.e. the times at which the energies are oscillating with little dissipation yet. Fig 20(b) shows the magnetic energy spectrum averaged over the same time interval and similarly multiplied by \( k^{5/3} \). Fig. 20(c) shows the total energy spectrum, similarly compensated and time averaged over the same time interval. In all three figures, the horizontal dotted line has zero slope, and so would coincide with a \( k^{5/3} \) Kolmogorov inertial range spectrum so compensated. In all three figures, the dashed line has slope 1=6, and so would be tangent to a \( k^{3/2} \) spectrum (as proposed by Iroshnikov and Kraichnan [42,43]) which had been multiplied by \( k^{5/3} \). It would appear from these figures that...
\( k^{5} = 3 \) would fit the computed spectra significantly better than \( k^{3/2} \) would. We do not attach any finality or conclusiveness to this observation, because intermittency is known to steepen energy spectra obtained from dimensional analysis, and it is known that high-order structure functions computed for statistically steady 2D-MHD flows display a behavior that differs from that of turbulent neutral (3D) fluids [44]. The total energy spectrum computed with the alpha model agrees with other findings (see e.g. [44, 45, 26]) at lower Taylor Reynolds numbers. In that light, we conclude that the alpha model does not alter previously known results, suggesting that intermittency is worth further investigation in the context of the alpha model, both in two and three dimensions. In the same spirit, we note the absence of any “bottle-neck” in the spectra, even at these high Reynolds numbers, in contrast to what is found in [26].

Finally, it is worth noting that, computed at the maximum of dissipation at \( t = 7 \), the Taylor wavenumber in the alpha run is \( k = 55 \), i.e. well below the alpha wavenumber of 300, whereas the dissipation wavenumber, based on the alpha-enstrophy \( < ! ! \delta > \) and the square current \( < j^2 > \), are respectively 1300 and 1500, i.e. well beyond the largest resolved wavenumber \( (k_{m, a} = 682) \) for this computation; however, even when plotting the dissipation spectrum \( k^2 [E_{k}(k) + E_{m}(k)] \), no bottle-neck is observed (not shown).

**VIII. SUMMARY AND DISCUSSION**

The intent of this article has been an empirical study of the extent to which the alpha model, or Lagrangian-averaged model, equations predict the results of computed incompressible 2D MHD behavior in which no further modeling or approximations are made. Our primary motivation has been to acquire confidence in the alpha model in hopes that it can be used for problems with such high Reynolds-like numbers that they cannot be computed in the framework of the primitive MHD equations. A useful starting point has seemed to be to work on classical problems in rectangular periodic conditions where some information about solutions has accumulated over the last thirty years. These problems include those often grouped under the terms selective decay, dynamic alignment, direct and inverse cascades, and tabulating frequency distributions or pdfs for fluctuating field quantities. We have compared the results of direct numerical solutions for these problems with the solutions of alpha-model equations for the same initial and boundary conditions, both for freely decaying and for forced turbulence. In order to do so at sufficiently high Reynolds numbers, we have kept our comparisons to the two-dimensional geometry for which reasonable resolutions can be obtained without overly taxing available computer resources.

The principal success we have to report is in the satisfactory reproduction of the long-wavelength spectral behavior for magnetic and velocity field evolution, in nearly all cases. Also, characteristic length scales of the flow, as the Taylor scale, are well reproduced by the alpha modeled simulations. The only exception is that some discrepancies remain for the longest-wavelength behavior in the case of driven cascades of mean-square magnetic vector potential at late times. The model has not proved accurate in the detailed reproduction of specific spatial features of the turbulence at earlier times. Nor does the alpha model reproduce accurately the pdfs of intermittent fluctuations of large amplitude, and likely because of the deliberate suppression of small spatial scales, is not ex-
FIG. 19: (a) Magnetic and kinetic energy, and (b) enstrophy and square current as a function of time for run 17 (see Table I) with 2048\(^2\) grid points and \(\nu^1 = 300\). Note the several peaks in the enstrophy and the mean square current density, as well as the excess in magnetic excitation, both at large scale and at small scale.

It should be remarked that there is the more difficult problem of obtaining a clear physical understanding of the nature of the alpha approximation itself. It was originally arrived at by mathematics of considerable sophistication and complexity, which left intuitive gaps in just what was being assumed. This is a very different perspective than the one used in Ref. [13] or Section II of this paper, where the recipe of smoothing the \(v\) and \(B\) fields but not their sources, and then neglecting the fluctuations in those fields about their averages was invoked. Neither prescription seems clear enough to us at a physical level to argue for it strenuously on any basis other than its satisfactory consequences. Nor is it clear why two such apparently different procedures should end at the same place, despite the happy fact that they seem to do so. Clarifying the conceptual foundations of such modeling at an intuitive physical level is a serious but stimulating challenge.

In summary, our judgment of the alpha model is that it reproduces satisfactorily the time development given by well-resolved DNS computations for spectra up to about \(k^1 = \) and does well enough at reproducing the probability distribution functions of fluctuations. What it does not do satisfactorily is to reproduce the locations, trajectories, and shapes of structures in configuration space.

We see several directions in which to extend this work. For example, there is the natural one of three-dimensional computations, still in rectangular periodic boundary conditions, where such problems as the inverse cascade of magnetic helicity (an inherent part of the large-scale dynamo problem), the spectral anisotropy induced by the presence of a dc magnetic field [46, 47], and the small magnetic Prandtl regime remain
to be investigated (see for recent studies in the latter case \textit{e.g.} [23, 24]). Finally, the questions of material boundaries with non-ideal boundary conditions and departure from rectangular to spherical or cylindrical symmetry seem necessary as well. The only work published so far involving material boundaries and the alpha model seems to be that of Chen et al. [25, 26]. It is our intent to move in these directions in the near future.

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