CONICAL SQUARE FUNCTIONS ASSOCIATED WITH BESSEL, LAGUERRE AND SCHRÖDINGER OPERATORS IN UMD BANACH SPACES

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ABSTRACT. In this paper we consider conical square functions in the Bessel, Laguerre and Schrödinger settings where the functions take values in UMD Banach spaces. Following a recent paper of Hytönen, van Neerven and Portal [31], in order to define our conical square functions, we use γ-radonifying operators. We obtain new equivalent norms in the Lebesgue-Bochner spaces $L^p((0,\infty),\mathbb{B})$ and $L^p(\mathbb{R}^n,\mathbb{B})$, $1 < p < \infty$, in terms of our square functions, provided that $\mathbb{B}$ is a UMD Banach space. Our results can be seen as Banach valued versions of known scalar results for square functions.

1. INTRODUCTION

In this paper we obtain equivalent norms in the Lebesgue-Bochner space $L^p(\mathbb{R}^n,\mathbb{B})$, $1 < p < \infty$, where $\mathbb{B}$ is a UMD Banach space. In order to do this we consider conical square functions defined via fractional derivatives of Poisson semigroups associated with Bessel, Laguerre and Schrödinger operators. According to the ideas developed by Hytönen, van Neerven and Portal [31] we use appropriate tent spaces using γ-radonifying operators (or, in other words, methods of stochastic analysis in a Banach valued setting).

We denote by $P_t(z)$, the classical Poisson kernel in $\mathbb{R}^n$, that is,

$$P_t(z) = c_n \frac{t}{(|z|^2 + t^2)^{(n+1)/2}}, \quad t > 0 \text{ and } z \in \mathbb{R}^n,$$

where $c_n = \Gamma((n+1)/2)/\pi^{(n+1)/2}$.

Segovia and Wheeden [43] introduced fractional derivatives as follows. Suppose that $\beta > 0$ and $m \in \mathbb{N}$ is such that $m - 1 \leq \beta < m$. If $F : \Omega \times (0,\infty) \rightarrow \mathbb{C}$ is a reasonable nice function, where $\Omega \subset \mathbb{R}^n$, the $\beta$-th derivative with respect to $t$ of $F$ is defined by

$$\partial_t^\beta F(x,t) = \frac{e^{-i\pi(m-\beta)}}{\Gamma(m-\beta)} \int_0^\infty \partial_t^m F(x,t+s)s^{m-\beta-1}ds \quad t > 0 \text{ and } x \in \Omega.$$

In [43] this fractional derivative was used to get characterizations of classical Sobolev spaces.

As in [50] we define the $\beta$-conical square function $S_\beta$ by

$$S_\beta(f)(x) = \left( \int_{\Gamma(x)} |\partial_t^\beta P_t(f)(y)|^2 \frac{dydt}{t^{n+1}} \right)^{1/2}, \quad x \in \mathbb{R}^n,$$

where $P_t(f)$ denotes the Poisson integral of $f$, that is,

$$P_t(f)(x) = \int_{\mathbb{R}^n} P_t(x-y)f(y)dy, \quad x \in \mathbb{R}^n, t > 0,$$

and, for every $x \in \mathbb{R}^n$, $\Gamma(x) = \{(y,t) \in \mathbb{R}^n \times (0,\infty) : |x-y| < t\}$. According to [50] Theorems 5.3 and 5.4 the square function $S_\beta$ defines an equivalent norm in $L^p(\mathbb{R}^n)$, $1 < p < \infty$.

Theorem A. Let $1 < p < \infty$ and $\beta > 0$. Then, there exists $C > 0$ such that

$$1 \leq \|f\|_{L^p(\mathbb{R}^n)} \leq \|S_\beta(f)\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}, \quad f \in L^p(\mathbb{R}^n).$$

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The equivalence in Theorem A for \( \beta \in \mathbb{N} \) can also be encountered in [31, 36] and [45].

Coifman, Meyer and Stein [14] introduced a family of spaces called tent spaces. These tent spaces are well adapted to certain questions related to harmonic analysis. Suppose that \( 1 \leq p, q < \infty \).

The tent space \( T^q_p(\mathbb{R}^n) \) consists of all those measurable functions \( g \) on \( \mathbb{R}^n \) \( \times (0, \infty) \) such that \( A_q(g) \in L^p(\mathbb{R}^n) \), where

\[
A_q(g)(x) = \left( \int_{\Gamma(x)} |g(y,t)|^q \frac{dydt}{t^{n+1}} \right)^{1/q}, \quad x \in \mathbb{R}^n.
\]

The norm \( \| \cdot \|_{T^q_p(\mathbb{R}^n)} \) in \( T^q_p(\mathbb{R}^n) \) is defined by \( \| g \|_{T^q_p(\mathbb{R}^n)} = \| A_q(g) \|_{L^p(\mathbb{R}^n)} \), \( g \in T^q_p(\mathbb{R}^n) \).

More recently, Harboure, Torrea and Viviani [29] have simplified some proofs of properties in [17] by using vector valued harmonic analysis techniques. Note that the result in Theorem A can be rewritten in terms of tent spaces as follows. If \( 1 < p < \infty \) and \( \beta > 0 \), then, for every \( f \in L^p(\mathbb{R}^n) \), \( t^\beta \partial_t^\beta P_t(f) \in T^q_p(\mathbb{R}^n) \) and

\[
\frac{1}{C} \| f \|_{L^p(\mathbb{R}^n)} \leq \| t^\beta \partial_t^\beta P_t(f) \|_{T^q_p(\mathbb{R}^n)} \leq C \| f \|_{L^p(\mathbb{R}^n)},
\]

where \( C > 0 \) does not depend on \( f \).

Assume that \( \mathbb{B} \) is a Banach space. In order to show a version of Theorem A for the Lebesgue-Bochner space \( L^p(\mathbb{R}^n, \mathbb{B}) \), the most natural definition of the \( \beta \)-conical square function \( S_{\beta,\mathbb{B}} \) is the following

\[
S_{\beta,\mathbb{B}}(f)(x) = \left( \int_{\Gamma(x)} \| t^\beta \partial_t^\beta P_t(f)(y) \|^q \frac{dydt}{t^{n+1}} \right)^{1/2}, \quad f \in L^p(\mathbb{R}^n, \mathbb{B}), \quad 1 < p < \infty.
\]

This type of Banach valued conical function has been considered in [36, 50] and [56]. Since a Banach space \( \mathbb{B} \) has Lusin type 2 and Lusin cotype 2 if, and only if, \( \mathbb{B} \) is isomorphic to a Hilbert space, from [50] Theorems 5.3 and 5.4 we can deduce the following result.

**Theorem B.** Assume that \( \mathbb{B} \) is a Banach space, \( 1 < p < \infty \) and \( \beta > 0 \). The following assertions are equivalent:

\( (i) \) \( \mathbb{B} \) is isomorphic to a Hilbert space.

\( (ii) \) There exists \( C > 0 \) such that

\[
\frac{1}{C} \| f \|_{L^p(\mathbb{R}^n, \mathbb{B})} \leq \| S_{\beta,\mathbb{B}}(f) \|_{L^p(\mathbb{R}^n)} \leq C \| f \|_{L^p(\mathbb{R}^n, \mathbb{B})}, \quad f \in L^p(\mathbb{R}^n, \mathbb{B}).
\]

In order to extend the equivalence (2) to \( L^p(\mathbb{R}^n, \mathbb{B}) \), \( 1 < p < \infty \), when \( \mathbb{B} \) is a Banach space which is not isomorphic to a Hilbert space, Hytönen, van Neerven and Portal [31] introduced new Banach valued tent function spaces. They considered UMD Banach spaces and \( \gamma \)-radonifying operators.

As it is well-known, the Hilbert transform \( \mathcal{H} \) defined by

\[
\mathcal{H}(f)(x) = \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_{|x-y|< \varepsilon} \frac{f(y)}{x-y} \, dy, \quad a.e. \ x \in \mathbb{R},
\]

is a bounded operator from \( L^p(\mathbb{R}) \) into itself, \( 1 < p < \infty \), and from \( L^1(\mathbb{R}) \) into \( L^{1,\infty}(\mathbb{R}) \).

Suppose that \( \mathcal{B} \) is a Banach space. The Hilbert transform can be defined in \( L^p(\mathbb{R}) \otimes \mathcal{B} \), \( 1 < p < \infty \), in the obvious way. It is said that \( \mathcal{B} \) is UMD, provided that

\[
\| \mathcal{H}(f) \|_{L^p(\mathbb{R}) \otimes \mathcal{B}} \leq C_p \| f \|_{L^p(\mathbb{R}) \otimes \mathcal{B}}, \quad f \in L^p(\mathbb{R}) \otimes \mathcal{B}.
\]

The main properties of UMD Banach spaces were established by Bourgain [15], Burkholder [16] and Rubio de Francia [12].

Assume that \( \{ \gamma_j \}_{j=1}^\infty \) is a sequence of independent standard normal variables defined on some probabilistic space \( (\Omega, \mathcal{F}, \mathbb{P}) \). Let \( H \) be a Hilbert space and let \( \mathcal{B} \) be a Banach space. We say that a linear operator \( T : H \to \mathcal{B} \) is \( \gamma \)-summing (shortly \( T \in \gamma^\infty(H, \mathcal{B}) \)) when

\[
\| T \|_{\gamma^\infty(H, \mathcal{B})} = \sup \left( \mathbb{E} \left\| \sum_{j=1}^k \gamma_j T(h_j) \right\|_{\mathcal{B}}^2 \right)^{1/2} < \infty,
\]

where the supremum is taken over all finite orthonormal family \( h_1, \ldots, h_k \in H \). Here, by \( \mathbb{E} \) we denote the expectation with respect to \( \mathbb{P} \). The space \( \gamma^\infty(H, \mathcal{B}) \) becomes a Banach space when it is endowed with the norm \( \| \cdot \|_{\gamma^\infty(H, \mathcal{B})} \). The space of \( \gamma \)-radonifying operators (shortly, \( \gamma(H, \mathcal{B}) \)) is the closure in \( \gamma^\infty(H, \mathcal{B}) \) of the subspace spanned by the finite rank operators from \( H \) into \( \mathcal{B} \).
According to [51] Proposition 3.15, if $H$ is separable and \{b_j\}_{j=1}^{\infty}$ is an orthonormal basis in $H$, then $T \in \gamma(H, \mathbb{B})$ if, and only if, the series $\sum_{j=1}^{\infty} \gamma_j T b_j$ converges in $L^2(\Omega, \mathbb{B})$ and, in this case,

$$\|T\|_{\gamma(H, \mathbb{B})} = \left( \mathbb{E} \left\| \sum_{j=1}^{\infty} \gamma_j T b_j \right\|_{\mathbb{B}}^2 \right)^{1/2}.$$  

We will write $\|T\|_{\gamma(H, \mathbb{B})} = \|T\|_{\gamma(H, \mathbb{B})}$, $T \in \gamma(H, \mathbb{B})$.

Hoffman-Jorgensen [30] and Kwapień [34] established that $\gamma(H, \mathbb{B}) = \gamma(H, \mathbb{B})$ provided that the Banach space $\mathbb{B}$ does not contain any closed subspace isomorphic to $c_0$. We recall that UMD Banach spaces satisfy this property.

Suppose that $(M, \mathcal{M}, \mu)$ is a measure space and $H = L^2(M, \mathcal{M}, \mu)$. A function $f : M \to \mathbb{B}$ is said to be weakly $L^2$ when, for every $L \in \mathcal{B}^*$, the function $L \circ f \in H$. Then, there exists a bounded and linear operator $T_f : H \to \mathbb{B}$ (shortly $T_f \in \mathcal{L}(H, \mathbb{B})$) such that, for every $L \in \mathcal{B}^*$,

$$(L, T_f(h))_{\mathbb{B}^*, \mathbb{B}} = \int_M \langle L, f(t) \rangle_{\mathbb{B}^*, \mathbb{B}} h(t) d\mu(t), \quad h \in H,$$

provided that $f$ is weakly $L^2$. We say that $f \in \gamma(M, \mu; \mathbb{B})$ provided that $T_f \in \gamma(H, \mathbb{B})$. If $\mathbb{B}$ does not contain any closed subspace isomorphic to $c_0$, then $\gamma(M, \mu; \mathbb{B})$ is a dense subspace of $\gamma(H, \mathbb{B})$ ([31] Remark 2.16).

Assume that $\mathbb{B}$ is a UMD Banach space and $1 < p < \infty$. Hytönen, van Neerven and Portal [31] Definition 4.1] defined the tent space $T^2_p(\mathbb{R}^n, \mathbb{B})$ as the completion of $C^\infty_c(\mathbb{R}^n \times (0, \infty)) \otimes \mathbb{B}$, where $C^\infty_c(\mathbb{R}^n \times (0, \infty))$ denotes the space of smooth functions with compact support in $\mathbb{R}^n \times (0, \infty)$, with respect to the norm

$$\|f\|_{T^2_p(\mathbb{R}^n, \mathbb{B})} = \|Jf\|_{L^p(\mathbb{R}^n, \gamma(H, \mathbb{B}))},$$

where, from now on,

$$H = L^2(\mathbb{R}^n \times (0, \infty), \frac{dydt}{t^{n+1}}),$$

the functional $J$ is defined by

$$J : f \to [x \to [(y,t) \mapsto \chi_{B(x,t)}(y)f(y,t)]$$

and $B(x,t) = \{y \in \mathbb{R}^n : |x-y| < t\}, \quad x \in \mathbb{R}^n$ and $t > 0$.

By taking into account that $\gamma(H, \mathbb{C}) \cong H$, it is clear that $T^2_p(\mathbb{R}^n, \mathbb{C}) = T^2_p(\mathbb{R}^n)$. Then, the tent space $T^2_p(\mathbb{R}^n, \mathbb{B})$ can be seen as a Banach valued extension of the classical tent space $T^2_p(\mathbb{R}^n)$. The main properties of the space $T^2_p(\mathbb{R}^n, \mathbb{B})$ were established in [31], where Banach valued tent spaces associated with certain bisectorial operators were defined. An alternative and equivalent definition for tent spaces $T^2_p(\mathbb{R}^n, \mathbb{B})$ can be encountered in [33].

In [31] Theorem 8.2 (see also [33] Example, Section 4) it was proved a vectorial extension of [2] by using the tent spaces $T^2_p(\mathbb{R}^n, \mathbb{B})$. In the following we extend in some sense (by considering any positive order of derivatives) the result in [31] Theorem 8.2.

**Theorem 1.** Let $1 < p < \infty$ and $\beta > 0$. Assume that $\mathbb{B}$ is a UMD Banach space. Then, there exists $C > 0$ such that

$$\frac{1}{C} \|f\|_{L^p(\mathbb{R}^n, \mathbb{B})} \leq \|t^\beta \partial_t^2 P_t(f)\|_{T^2_p(\mathbb{R}^n, \mathbb{B})} \leq C \|f\|_{L^p(\mathbb{R}^n, \mathbb{B})}, \quad f \in L^p(\mathbb{R}^n, \mathbb{B}).$$

Since $T^2_p(\mathbb{R}^n, \mathbb{C}) = T^2_p(\mathbb{R}^n)$ Theorem [2] is an extension of Theorem [1] to Lebesgue-Bochner spaces $L^p(\mathbb{R}^n, \mathbb{B})$, $1 < p < \infty$, provided that $\mathbb{B}$ is a UMD. Note that a UMD Banach space is not necessarily isomorphic to a Hilbert space.

We consider the Schrödinger operator $L_V$ in $\mathbb{R}^n$ defined by

$$L_V = -\Delta + V,$$

where $\Delta$ represents the usual Laplacian operator, that is, $\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$. We assume that the potential $V \not\equiv 0$ is a nonnegative measurable function for which there exist $s > n/2$ and $C > 0$ such that, for every ball $B$ in $\mathbb{R}^n$,

$$\left( \int_B V(x)^s dx \right)^{1/s} \leq C \int_B V(x) dx.$$

When $V$ satisfies [3] we say that $V$ verifies the $s$-reverse Hölder inequality and we write $V \in RH_s(\mathbb{R}^n)$. 


In a precise way our Schrödinger operator $\mathbf{L}_V$ is defined as follows. We consider the sesquilinear form $Q_V$ given by

$$ Q_V[f,g] = \int_{\mathbb{R}^n} \nabla f(x) \overline{\nabla g(x)} dx + \int_{\mathbb{R}^n} V(x) f(x) g(x) dx, \quad (f,g) \in D(Q_V), $$

where $\nabla$ denotes the usual gradient. The domain $D(Q_V)$ of $Q$ is the product $\mathcal{D}_V \times \mathcal{D}_V$, where

$$ \mathcal{D}_V = \{ f \in L^2(\mathbb{R}^n) : |\nabla f| \in L^2(\mathbb{R}^n) \text{ and } V^{1/2} f \in L^2(\mathbb{R}^n) \}. $$

The Schrödinger operator $\mathbf{L}_V$ is the unique selfadjoint operator such that its domain is $\mathcal{D}_V$ and

$$ \langle \mathbf{L}_V f, g \rangle = Q_V[f,g], \quad f, g \in \mathcal{D}_V. $$

It is clear that $C^\infty_c(\mathbb{R}^n)$, the space of smooth functions with compact support in $\mathbb{R}^n$, is contained in $\mathcal{D}_V$ and $\mathbf{L}_V = L_V$ on $C^\infty_c(\mathbb{R}^n)$. $\mathbf{L}_V$ is a positive operator.

The semigroup of operators $\{W_t^{\mathbf{L}_V}\}_{t > 0}$ generated by $-\mathbf{L}_V$ in $L^2(\mathbb{R}^n)$ can be written as

$$ W_t^{\mathbf{L}_V}(f) = \int_0^\infty e^{-t\lambda} \mathbf{L}_V (d\lambda)f, \quad f \in L^2(\mathbb{R}^n) \text{ and } t > 0, $$

where $\mathcal{E}_{\mathbf{L}_V}$ denotes the spectral measure for $\mathbf{L}_V$.

For every $t > 0$ there exists a measurable function $W_t^{E_{\mathbf{L}_V}}(x,y), x, y \in \mathbb{R}^n$, such that for each $f \in L^2(\mathbb{R}^n)$,

$$ W_t^{E_{\mathbf{L}_V}}(x,y) = \int_{\mathbb{R}^n} W_t^{\mathbf{L}_V}(x,y) f(y) dy. $$

Moreover, according to the Feynman-Kac formula ([22, p. 280]), we have that

$$ \left| W_t^{E_{\mathbf{L}_V}}(x,y) \right| \leq C e^{-|x-y|^2/4t}, \quad x, y \in \mathbb{R}^n \text{ and } t > 0. $$

Then, the integral in (4) is absolutely convergent for every $f \in L^p(\mathbb{R}^n), 1 \leq p \leq \infty$. The family $\{W_t^{E_{\mathbf{L}_V}}\}_{t > 0}$, where $W_t^{E_{\mathbf{L}_V}}, t > 0$, is defined by (4), is a positive bounded semigroup in $L^p(\mathbb{R}^n), 1 \leq p < \infty$, generated by $-\mathbf{L}_V$. $\{W_t^{E_{\mathbf{L}_V}}\}_{t > 0}$ is not Markovian because $V \neq 0$.

The semigroup of operators $\{P_t^{\mathbf{L}_V}\}_{t > 0}$ subordinated to $\{W_t^{E_{\mathbf{L}_V}}\}_{t > 0}$, also called Poisson semigroup associated to $\mathbf{L}_V$, is defined by

$$ P_t^{\mathbf{L}_V}(f)(x) = \frac{t}{2\sqrt{\pi}} \int_0^\infty e^{-t^2/4u} W_u^{E_{\mathbf{L}_V}}(f)(x) du, \quad f \in L^p(\mathbb{R}^n), 1 \leq p < \infty \text{ and } t > 0. $$

An important special case of Schrödinger operator is the Hermite operator $\mathcal{H}$ (also called harmonic oscillator) that appears when $V(x) = |x|^2, x \in \mathbb{R}^n$.

Harmonic analysis associated with Schrödinger and Hermite operators has been developed in the last years by several authors ([2, 7, 13, 14, 20, 21, 22, 23, 24, 44, 45] and others).

Our second result establishes the equivalence in Theorem 2 when the classical Poisson semigroup is replaced by the Poisson semigroups $\{P_t^{E_{\mathbf{L}_V}}\}_{t > 0}$ or $\{P_t^{E_{\mathbf{H}_V}}\}_{t > 0}$.

**Theorem 2.** Let $1 < p < \infty$ and $\beta > 0$. Assume that $\mathbb{B}$ is a UMD Banach space.

(i) If $V \in RH_s(\mathbb{R}^n)$ for some $s > n/2$, and $n \geq 3$, then there exists $C > 0$ such that

$$ \frac{1}{C} \|f\|_{L^p(\mathbb{R}^n, \mathbb{B})} \leq \|t^\beta \partial_t^\beta P_t^{E_{\mathbf{L}_V}}(f)\|_{T_{p}^2(\mathbb{R}^n, \mathbb{B})} \leq C \|f\|_{L^p(\mathbb{R}^n, \mathbb{B})}, \quad f \in L^p(\mathbb{R}^n, \mathbb{B}). $$

(ii) For every $n \in \mathbb{N}$, there exists $C > 0$ such that

$$ \frac{1}{C} \|f\|_{L^p(\mathbb{R}^n, \mathbb{B})} \leq \|t^\beta \partial_t^\beta P_t^{E_{\mathbf{L}_V}}(f)\|_{T_{p}^2(\mathbb{R}^n, \mathbb{B})} \leq C \|f\|_{L^p(\mathbb{R}^n, \mathbb{B})}, \quad f \in L^p(\mathbb{R}^n, \mathbb{B}). $$

Since $T_{p}^2(\mathbb{R}^n, \mathbb{C}) = T_{p}^2(\mathbb{R}^n), 1 < p < \infty$, the scalar results can be deduced as special cases of Theorem 2.

We now define the space $T_{p}^2((0,\infty), \mathbb{B}), 1 < p < \infty$, where $\mathbb{B}$ is again a UMD Banach space. Let $1 < p < \infty$. The tent space $T_{p}^2((0,\infty), \mathbb{B})$ is the completion of $C^\infty_c((0,\infty)) \otimes \mathbb{B}$ with respect to the norm

$$ \|f\|_{T_{p}^2((0,\infty), \mathbb{B})} = \|j_x f\|_{L^p((0,\infty), \gamma(H_x, \mathbb{B}))}, $$

where, from now on,

$$ H_x = L^2((0,\infty)^2, \frac{dydt}{t^2}), $$

and
the functional $J_+$ is defined by

$$J_+: f \mapsto [x \mapsto [(y, t) \mapsto \chi(x, y, t) f(y, t)]]$$

and $B_+(x, t) = \{ y \in (0, \infty) : |x - y| < t \}, x, t \in (0, \infty)$. Here, by $C_c^\infty(0, \infty)$ we denote the space of smooth functions with compact support on $(0, \infty)$.

We will use the tent space $T^p((0, \infty), \mathbb{B})$ to get equivalent norms in the Lebesgue-Bochner space $L^p((0, \infty), \mathbb{B})$, for every $1 < p < \infty$ and every UMD Banach space $\mathbb{B}$. Our new norms (see Theorems 3 and 4 below) involve Poisson semigroups associated with Bessel and Laguerre operators.

Harmonic analysis in the Bessel setting began with the deep paper of Muckenhoupt and Stein [39]. Recently, operators related to the harmonic analysis (Riesz transform, Littlewood-Paley functions, maximal operators, multipliers, ...) in the Bessel context have been investigated (see, for instance, [3], [9], [10], [12] and [54]). We consider the Bessel operator on $(0, \infty)$,

$$B_\alpha = -x^\alpha D_x 2x^\alpha x^{-\lambda} = -\frac{d^2}{dx^2} + \frac{\lambda(\lambda - 1)}{x^2},$$

where $\lambda > 0$. The Hankel transform $h_\lambda$ is defined by

$$h_\lambda(f)(x) = \int_0^\infty \sqrt{xy} J_{\lambda-1/2}(xy) f(y) dy, \quad f \in L^1(0, \infty),$$

where $J_\alpha$ denotes the Bessel function of the first kind and order $\alpha$. $h_\lambda$ plays with respect to the Bessel operator the same role as the Fourier transform with respect to the classical Laplacian operator. $h_\lambda$ can be extended from $L^1(0, \infty) \cap L^2(0, \infty)$ to $L^2(0, \infty)$ as an isometry in $L^2(0, \infty)$ ([39] Ch. VIII]). By using well-known properties of the Bessel function $J_\alpha$ we can deduce that, for every $f \in C_c^\infty(0, \infty)$,

$$h_\lambda(B_\lambda f)(x) = x^2 h_\lambda(f)(x), \quad x \in (0, \infty),$$

([57] Lemma 5.4-1(5)]. We define the operator $\mathcal{B}_\lambda$ by

$$\mathcal{B}_\lambda(f) = h_\lambda(x^2 h_\lambda(f)), \quad f \in D(\mathcal{B}_\lambda),$$

where the domain $D(\mathcal{B}_\lambda)$ of $\mathcal{B}_\lambda$ is given by

$$D(\mathcal{B}_\lambda) = \{ f \in L^2(0, \infty) : y^2 h_\lambda(f) \in L^2(0, \infty) \}.$$

Since $h_\lambda^{-1} = h_\lambda$ in $L^2(0, \infty)$, $C_c^\infty(0, \infty) \subset D(\mathcal{B}_\lambda)$ and $\mathcal{B}_\lambda f = B_\lambda f, f \in C_c^\infty(0, \infty)$. The operator $-\mathcal{B}_\lambda$ generates a positive and bounded semigroup of operators $\{W_t^{\mathcal{B}_\lambda}\}_{t \geq 0}$ in $L^p(0, \infty)$, for every $1 \leq p < \infty$. Moreover, the Poisson semigroup $\{P_t^{\mathcal{B}_\lambda}\}_{t \geq 0}$ associated with the Bessel operator $\mathcal{B}_\lambda$ can be written as

$$P_t^{\mathcal{B}_\lambda}(f)(x) = \int_0^\infty P_t^{\mathcal{B}_\lambda}(x, y) f(y) dy, \quad t \in (0, \infty),$$

for every $f \in L^p(0, \infty), 1 \leq p < \infty$. Here the Poisson kernel $P_t^{\mathcal{B}_\lambda}(x, y), t, x, y \in (0, \infty)$, is given by ([39] (16.4), [55])

$$P_t^{\mathcal{B}_\lambda}(x, y) = \frac{2 \lambda(x+y)^\lambda}{\pi} \int_0^\pi \frac{(\text{sin} \theta)^{2\lambda-1}}{(t^2 + (x-y)^2 + 2xy(1 - \text{cos} \theta))^{(\lambda+1)/2}} d\theta, \quad t, x, y \in (0, \infty).$$

**Theorem 3.** Let $1 < p < \infty$ and $\beta, \lambda > 0$. Assume that $\mathbb{B}$ is a UMD Banach space. Then, there exists $C > 0$ such that

$$\frac{1}{C} \| f \|_{L^p((0, \infty), \mathbb{B})} \leq \| t^{\beta} \partial_t^{\beta} P_t^{\mathcal{B}_\lambda}(f) \|_{T^p((0, \infty), \mathbb{B})} \leq C \| f \|_{L^p((0, \infty), \mathbb{B})}, \quad f \in L^p((0, \infty), \mathbb{B}).$$

Muckenhoupt ([37] and [38]) began the study of harmonic analysis associated to Laguerre operators. Later, Dinger [13] established $L^p$-boundedness properties for the maximal operator defined by the $n$-dimensional heat semigroup in the Laguerre context. In the last years a lot of authors have investigated harmonic analysis operators related to Laguerre operators (see, for instance, [11], [19], [25], [28], [29], [41], [51] and [53]).

We consider the Laguerre operator on $(0, \infty)$

$$L_\alpha = -\frac{d^2}{dx^2} + \frac{\alpha^2 - 1/4}{x^2} + x^2,$$

where $\alpha > -1/2$. For every $k \in \mathbb{N}$, we have that

$$L_\alpha \varphi_k^\alpha = 2(2k + \alpha + 1) \varphi_k^\alpha.$$
where
\[ \varphi_k^\alpha(x) = \left(\frac{2\Gamma(k+1)}{\Gamma(k+\alpha+1)}\right)^{1/2} e^{-x^2/2} x^{\alpha+1/2} \ell_k^\alpha(x^2), \quad x \in (0, \infty), \]
and \( \ell_k^\alpha \) represents the \( k \)-th Laguerre polynomial of order \( \alpha \) ([17] p. 100–102). The sequence \( \{\varphi_k^\alpha\}_{k \in \mathbb{N}} \) is an orthonormal basis in \( L^2(0, \infty) \).

We define the operator \( \mathcal{L}_\alpha \) as follows
\[ \mathcal{L}_\alpha f = 2 \sum_{k=0}^{\infty} (2k + \alpha + 1) c_k^\alpha(f) \varphi_k^\alpha, \quad f \in D(\mathcal{L}_\alpha), \]
where, for every \( k \in \mathbb{N} \),
\[ c_k^\alpha(f) = \int_0^\infty \varphi_k^\alpha(x) f(x) dx, \quad f \in L^2(0, \infty), \]
and the domain \( D(\mathcal{L}_\alpha) \) of \( \mathcal{L}_\alpha \) is given by
\[ D(\mathcal{L}_\alpha) = \{ f \in L^2(0, \infty) : \sum_{k=0}^{\infty} (2k + \alpha + 1)^2 |c_k^\alpha(f)|^2 < \infty \}. \]
The operator \( -\mathcal{L}_\alpha \) generates a positive and bounded semigroup \( \{W_t^\mathcal{L}_\alpha\}_{t > 0} \) in \( L^2(0, \infty) \), given by
\[ W_t^\mathcal{L}_\alpha(f) = \sum_{k=0}^{\infty} e^{-2(2k+\alpha+1)t} c_k^\alpha(f) \varphi_k^\alpha, \quad f \in L^2(0, \infty) \quad \text{and} \quad t > 0. \]

Mehler’s formula for Laguerre polynomials ([48] p. 8]) allows us to write, for each \( f \in L^2(0, \infty) \),
\[ W_t^\mathcal{L}_\alpha(f)(x) = \int_0^{\infty} W_t^\mathcal{L}_\alpha(x, y) f(y) dy, \quad t, x \in (0, \infty), \]
where
\[ W_t^\mathcal{L}_\alpha(x, y) = \left(\frac{2e^{-2t}}{1 - e^{-4t}}\right)^{1/2} \left(\frac{2xye^{-2t}}{1 - e^{-4t}}\right)^{1/2} I_\alpha \left(\frac{2xye^{-2t}}{1 - e^{-4t}}\right) \exp\left(-\frac{1}{2}x^2 + \frac{y^2}{1 + e^{-4t}}\right), \]
and \( I_\alpha \) represents the modified Bessel function of the first kind and order \( \alpha \).

Moreover, if \( W_t^\mathcal{L}_\alpha \) is defined by (5), for every \( t > 0 \), then \( \{W_t^\mathcal{L}_\alpha\}_{t > 0} \) is a positive and bounded semigroup of operators in \( L^p(0, \infty) \), \( 1 \leq p < \infty \).

As usual the Poisson semigroup \( \{P_t^\mathcal{L}_\alpha\}_{t > 0} \) associated with \( \mathcal{L}_\alpha \) is defined as the one subordinated to \( \{W_t^\mathcal{L}_\alpha\}_{t > 0} \), that is, for every \( t > 0 \),
\[ P_t^\mathcal{L}_\alpha(f)(x) = \frac{t}{2\sqrt{\pi}} \int_0^{\infty} e^{-x^2/4u} W_u^\mathcal{L}_\alpha(f)(x) du, \quad f \in L^p(0, \infty) \quad \text{and} \quad 1 \leq p < \infty. \]

**Theorem 4.** Let \( 1 < p < \infty \) and \( \alpha, \beta > 0 \). Assume that \( \mathcal{B} \) is a UMD Banach space. Then, there exists \( C > 0 \) such that
\[ \frac{1}{C} \|f\|_{L^p((0, \infty), \mathcal{B})} \leq \|t^\beta \partial^\beta P_t^\mathcal{L}_\alpha(f)\|_{T^p_2((0, \infty), \mathcal{B})} \leq C \|f\|_{L^p((0, \infty), \mathcal{B})}, \quad f \in L^p((0, \infty), \mathcal{B}). \]

In the following sections we prove Theorems [1], [2], [3] and [4].

Throughout this paper by \( C \) and \( c \) we always denote positive constants that can change in each occurrence.

### 2. Proof of Theorem [1]

We split the proof of Theorem 1 in the Lemmas [2] and [3] below. We are going to use the arguments developed in [31] Theorem 8.2.

**Lemma 2.1.** Let \( \mathcal{B} \) be a UMD Banach space, \( 1 < p < \infty \) and \( \beta > 0 \). Then, there exists \( C > 0 \) such that
\[ \|t^\beta \partial^\beta P_t^\mathcal{L}_\alpha(f)\|_{T^p_2(\mathbb{R}^n, \mathcal{B})} \leq C \|f\|_{L^p(\mathbb{R}^n, \mathcal{B})}, \quad f \in L^p(\mathbb{R}^n, \mathcal{B}). \]

**Proof.** Let \( f \in L^p(\mathbb{R}^n, \mathcal{B}) \). It is not hard to see that, for every \( k \in \mathbb{N} \),
\[ \partial^k_t P_t(f)(x) = \int_{\mathbb{R}^n} \partial^k_t P_t(x - y) f(y) dy, \quad x \in \mathbb{R}^n \quad \text{and} \quad t > 0. \]
Assume that $m \in \mathbb{N}$ is such that $m - 1 \leq \beta < m$. We have that
\[
\partial_t^\beta P_t(f)(x) = \frac{e^{-i\pi(m-\beta)}}{\Gamma(m-\beta)} \int_0^\infty s^{m-\beta-1} \int_{\mathbb{R}^n} \partial_t^m P_{t+s}(x-y)f(y)dyds \\
= \frac{e^{-i\pi(m-\beta)}}{\Gamma(m-\beta)} \int_{\mathbb{R}^n} f(y) \int_0^\infty s^{m-\beta-1}\partial_t^m P_{t+s}(x-y)dsdy \\
= \int_{\mathbb{R}^n} \partial_t^\beta P_t(x-y)f(y)dy, \quad x \in \mathbb{R}^n \text{ and } t > 0.
\]
The interchange in the order of integration is justified because
\[
\int_0^\infty \int_{\mathbb{R}^n} \|f(y)\|s^{m-\beta-1} |\partial_t^m P_{t+s}(x-y)|dyds < \infty, \quad x \in \mathbb{R}^n \text{ and } t > 0.
\]
Indeed, according to Fa di Bruno’s formula (26 (4.6)) we can write, for each $n \geq 2$, $t > 0$ and $z \in \mathbb{R}^n$,
\[
\partial_t^m \left[ \frac{t}{(t^2 + |z|^2)^{(n+1)/2}} \right] = \frac{1}{1-n} \partial_t^{m+1} \left[ \frac{1}{(t^2 + |z|^2)^{(n-1)/2}} \right] \\
= \frac{1}{1-n} \sum_{\ell=0}^{\lfloor (m+1)/2 \rfloor} \left( -\frac{1}{2} \right)^{m+1-\ell} (n-1)(n+1) \ldots (n-1+2(m-\ell)) E_{m,\ell} \frac{t^{m+1-2\ell}}{(t^2 + |z|^2)^{(n+2(m-\ell))/2}}.
\]
and
\[
\partial_t^m \left[ \frac{t}{(t^2 + |z|^2)^{(n+1)/2}} \right] = \frac{1}{2} \partial_t^{m+1} \ln(t^2 + |z|^2) = \frac{1}{2} \sum_{\ell=0}^{\lfloor (m+1)/2 \rfloor} (-1)^{m+1-\ell}(m+1)!E_{m,\ell} \frac{t^{m+1-2\ell}}{(t^2 + |z|^2)^{(n+2(m-\ell))/2}},
\]
where $E_{m,\ell} = 2^{m+1-2\ell}(m+1)!/(\ell!(m+1-2\ell)!)$.

From (10) and (11) we deduce that
\[
\left| \partial_t^m \left[ \frac{t}{(t^2 + |z|^2)^{(n+1)/2}} \right] \right| \leq C \frac{1}{(t + |z|)^{m+n}}, \quad z \in \mathbb{R}^n \text{ and } t > 0.
\]
Then,
\[
\int_0^\infty s^{m-\beta-1} \left| \partial_t^m \left[ \frac{t + s}{(t^2 + |z-y|^2)^{(n+1)/2}} \right] \right| ds \leq C \int_0^\infty \frac{s^{m-\beta-1}}{(t + s + |z-y|)^{m+n}} ds \\
\leq C \frac{1}{(t + |z-y|)^{n+\beta}}, \quad x, y \in \mathbb{R}^n, \quad t > 0,
\]
and Hölder inequality allows us to obtain (9).

Also, we have that
\[
|t^\beta \partial_t^\beta P_t(x-y)| \leq C \frac{t^\beta}{(t + |x-y|)^{n+\beta}}, \quad x, y \in \mathbb{R}^n, \quad t > 0.
\]

To simplify we write
\[
(Sf)(t, y) = \int_{\mathbb{R}^n} k(y, z, t)f(z)dz, \quad f \in L^p(\mathbb{R}^n, \mathcal{B}),
\]
where $k(y, z, t) = t^\beta \partial_t^\beta P_t(y-z), \quad y, z \in \mathbb{R}^n \text{ and } t > 0$. Our objective is to see that $S$ is a bounded operator from $L^p(\mathbb{R}^n, \mathcal{B})$ into $T^r_p(\mathbb{R}^n, \mathcal{B})$. In order to do this we use [31, Theorem 4.8].

We consider the operator
\[
(\mathcal{S}g)(t, y) = \int_{\mathbb{R}^n} k(y, z, t)g(z)dz, \quad g \in L^2(\mathbb{R}^n).
\]

As usual, for every $g \in L^2(\mathbb{R}^n)$, we denote by $\widehat{g}$ the Fourier transform of $g$ and by $\widetilde{g}$ the inverse Fourier transform of $g$. Let $g \in L^2(\mathbb{R}^n)$. It is well-known that
\[
\partial_t^k P_t(g)(y) = \frac{(-1)^k |z|^k e^{-t|z|} \widehat{g}}{2\pi_i^{k+n/2}} \int_{\mathbb{R}^n} e^{iyz} |z|^k e^{-t|z|} \widetilde{g}(z)dz.
\]
where

\begin{align}
\partial_t^\beta P_t(g)(y) &= \frac{(-1)^m e^{-i\pi (m-\beta)}}{(2\pi)^{n/2} \Gamma(m-\beta)} \int_0^\infty s^{m-\beta-1} \int_{\mathbb{R}^n} e^{iyz} |s|^{m-\beta} e^{-(t+s)|z|^2} \tilde{g}(z) dz ds \\
&= \frac{e^{i\pi \beta}}{(2\pi)^{n/2} \Gamma(m-\beta)} \int_{\mathbb{R}^n} e^{iyz} |z|^{m-\beta} \tilde{g}(z) \int_0^\infty e^{-s|z|^2} s^{m-\beta-1} ds dz \\
&= \frac{e^{i\pi \beta}}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{iyz} |z|^{m-\beta} \tilde{g}(z) dz = e^{i\pi \beta}(|z|^{m-\beta} \tilde{g}(z))^\beta.
\end{align}

(13)

The interchange of the order of integration is justified by the absolute convergence of the integral. Thus, Plancherel equality leads to

\[
\|g\|_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \int_{\Gamma(x)} |(\mathcal{S}g)(y,t)|^2 \frac{dydt}{t^{n+1}} dx = \int_{\mathbb{R}^n} \int_{\Gamma(x)} |t^\beta \partial_t^\beta P_t(g)(y)|^2 \frac{dydt}{t^{n+1}} dx \\
= \int_{\mathbb{R}^n} \int_{\Gamma(x)} \left|\int (e^{-t|z|} (t|z|)^\beta \tilde{g}(z))(y) dy\right|^2 \frac{dx}{t^{n+1}} dx \\
= v_n \int_{\mathbb{R}^n} \int_{\Gamma(x)} \left|\int (e^{-t|z|} (t|z|)^\beta \tilde{g}(z))(y) dy\right|^2 \frac{dt}{t} = v_n \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |e^{-t|z|} (t|z|)^\beta \tilde{g}(z)|^2 \frac{dz dt}{t} \\
= v_n \frac{\Gamma(2\beta)}{2\beta} \int_{\mathbb{R}^n} \|	ilde{g}\|^2 dz = v_n \frac{\Gamma(2\beta)}{2\beta} \|g\|^2_{L^2(\mathbb{R}^n)},
\]

where \(v_n\) is the volume of the unit ball in \(\mathbb{R}^n\). Hence, \(\mathcal{S}\) is a bounded operator from \(L^2(\mathbb{R}^n)\) into \(T^2_{\beta}(\mathbb{R}^n)\).

We now prove that

\[
|\nabla z k(y, z, t)| \leq C \frac{t^\beta}{(t + |y - z|)^{n+\beta+1}}, \quad y, z \in \mathbb{R}^n \text{ and } t > 0.
\]

Let \(j = 1, \ldots, n\). According to (10) and (11) we obtain

\[
|\frac{t}{(t^2 + |z|^2)^{(n+1)/2}}| = \sum_{\ell=0}^{\lfloor(m+1)/2\rfloor} a_\ell \frac{t^{m+1-2\ell} z_j}{(t^2 + |z|^2)^{(m+2\ell-m-1)/2}}, \quad z \in \mathbb{R}^n \text{ and } t > 0,
\]

for certain \(a_\ell \in \mathbb{R}\). Then,

\[
|t^{\beta} \partial_t^m \left[ \frac{t}{(t^2 + |z|^2)^{(n+1)/2}} \right]| \leq C \frac{t^\beta}{(t + |z|)^{n+m+1}}, \quad z \in \mathbb{R}^n \text{ and } t > 0.
\]

By proceeding as in the proof of (12) we get

\[
|\partial_j k(y, z, t)| \leq C \frac{t^\beta}{(t + |y - z|)^{n+\beta+1}}, \quad y, z \in \mathbb{R}^n \text{ and } t > 0.
\]

Thus (14) is established.

From (14) and by using the mean value theorem we can deduce that

\[
|k(y, z, t) - k(y, z', t)| \leq C \frac{t^\beta |z - z'|}{(t + |y - z|)^{n+\beta+1}},
\]

provided that \(t > 0, y, z, z' \in \mathbb{R}^n\) and \(|y - z| + t > 2|z - z'|\).

Moreover, we can write

\[
\int_{\mathbb{R}^n} k(y, z, t) dz = t^\beta \partial_t^\beta \int_{\mathbb{R}^n} P_t(y - z) dz = 0, \quad t > 0 \text{ and } y \in \mathbb{R}^n.
\]

According to [31] Theorem 4.8], the operator \(S_B = \mathcal{S} \otimes I_B\) can be extended from \(C_c^\infty(\mathbb{R}^n) \otimes \mathcal{B}\) to \(L^p(\mathbb{R}^n, \mathcal{B})\) as a bounded operator \(\widetilde{S_B}\) from \(L^p(\mathbb{R}^n, \mathcal{B})\) into \(T^2_{\beta}(\mathbb{R}^n, \mathcal{B})\). Also, as a special case, the operator \(S\) can be extended from \(C_c^\infty(\mathbb{R}^n)\) to \(L^p(\mathbb{R}^n)\) as a bounded operator \(\widetilde{S}\) from \(L^p(\mathbb{R}^n)\) into \(T^2_{\beta}(\mathbb{R}^n)\). We are going to show that \(S = \widetilde{S_B}\) on \(L^p(\mathbb{R}^n, \mathcal{B})\).

In order to get a better understanding of the proof we see firstly that \(\widetilde{S}\) is bounded on \(L^p(\mathbb{R}^n)\), where

\[
(\mathcal{S}g)(t, y) = \int_{\mathbb{R}^n} k(y, z, t)g(z) dz, \quad g \in L^p(\mathbb{R}^n).
\]
Let \( f \in L^p(\mathbb{R}^n) \). We choose a sequence \((f_k)_{k \in \mathbb{N}} \) in \( C_c^\infty(\mathbb{R}^n) \) such that
\[
f_k \to f, \quad \text{as } k \to \infty, \text{ in } L^p(\mathbb{R}^n).
\]
Then,
\[
Sf_k \to Sf, \quad \text{as } k \to \infty, \text{ in } T^2_p(\mathbb{R}^n).
\]
Hence, \((J(Sf_k))_{k \in \mathbb{N}} \) converges to a function \( g \in L^p(\mathbb{R}^n, H) \). There exists an increasing sequence \((k_t)_{t \in \mathbb{N}} \subset \mathbb{N} \) and a subset \( \Omega \subset \mathbb{R}^n \) such that \(|\mathbb{R}^n \setminus \Omega| = 0 \) and, for every \( x \in \Omega \),
\[
(15) \quad [J(Sf_{k_t})](x) \to g(x), \text{ as } \ell \to \infty, \text{ in } H.
\]
On the other hand, according to \([12]\) we have that, for every \( \ell \in \mathbb{N} \),
\[
\left| S(f - f_{k_t})(t, x) \right| \leq \int_{\mathbb{R}^n} |f(y) - f_{k_t}(y)||t^\beta \partial_1^\beta P_1(y - x)|dy \leq C \int_{\mathbb{R}^n} |f(y) - f_{k_t}(y)| \frac{\ell^\beta}{(t + |x - y|)^{n+\beta}}dy \leq C\|f - f_{k_t}\|_{L^p(\mathbb{R}^n)}t^\beta \left( \int_{\mathbb{R}^n} \frac{1}{(t + |z|)^{(n+\beta)p'}}dz \right)^{1/p'} \leq Ct^{-n/p}\|f - f_{k_t}\|_{L^p(\mathbb{R}^n)}, \quad x \in \mathbb{R}^n \text{ and } t > 0.
\]
Here \( p' = p/(p - 1) \). From \([16]\) we deduce that, for each \( \ell \in \mathbb{N} \) and \( \varepsilon > 0 \),
\[
\left\| [Jf - Sf_{k_t}](x) \right\|^2_{L^2(\mathbb{R}^n \times (\varepsilon, \infty), \frac{dydt}{t^{n+1}})} \leq C\|f - f_{k_t}\|^2_{L^p(\mathbb{R}^n)} \int_{\varepsilon}^\infty \int_{B(x,t)} t^{-2n/p}dydt \leq C\|f - f_{k_t}\|^2_{L^p(\mathbb{R}^n)}e^{-2n/p}, \quad x \in \mathbb{R}^n.
\]
Thus, for every \( \varepsilon > 0 \) and \( x \in \mathbb{R}^n \),
\[
(17) \quad J(Sf_{k_t})(x) \to J(Sf)(x), \quad \text{as } \ell \to \infty, \text{ in } L^2 \left( \mathbb{R}^n \times (\varepsilon, \infty), \frac{dydt}{t^{n+1}} \right).
\]
From \((15)\) and \((17)\) it follows that, for every \( x \in \Omega \), \( J(\mathcal{S}f)(x) = g(x) \) as elements of \( H \). We conclude that
\[
J(Sf_k) \to J(\mathcal{S}f), \quad \text{as } k \to \infty, \text{ in } L^p(\mathbb{R}^n, H).
\]
Thus, we prove that
\[
J(Sf_k) \to J(\mathcal{S}f), \quad \text{as } k \to \infty, \text{ in } L^p(\mathbb{R}^n, \gamma(H, \mathbb{B})).
\]
Hence, \( \mathcal{S}f = \widetilde{S}f \).

Almost the same proof works for every \( f \in L^p(\mathbb{R}^n, \mathbb{B}) \). Let \( f \in L^p(\mathbb{R}^n, \mathbb{B}) \). We choose a sequence \((f_k)_{k \in \mathbb{N}} \subset C_c^\infty(\mathbb{R}^n) \otimes \mathbb{B} \) such that
\[
f_k \to f, \quad \text{as } k \to \infty, \text{ in } L^p(\mathbb{R}^n, \mathbb{B}).
\]
Then,
\[
Sf_k \to Sf, \quad \text{as } k \to \infty, \text{ in } T^2_p(\mathbb{R}^n, \mathbb{B}).
\]
Hence, there exists \( G \in L^p(\mathbb{R}^n, \gamma(H, \mathbb{B})) \) such that
\[
J(Sf_k) \to G, \quad \text{as } k \to \infty, \text{ in } L^p(\mathbb{R}^n, \gamma(H, \mathbb{B})).
\]
There exists an increasing sequence \((k_t)_{t \in \mathbb{N}} \subset \mathbb{N} \) and a set \( \Omega \subset \mathbb{R}^n \) such that \(|\mathbb{R}^n \setminus \Omega| = 0 \) and
\[
J(Sf_{k_t})(x) \to G(x), \quad \text{as } \ell \to \infty, \text{ in } \gamma(H, \mathbb{B}),
\]
for every \( x \in \Omega \).

By proceeding as above we can see that, for every \( \varepsilon > 0 \) and \( x \in \mathbb{R}^n \),
\[
J(Sf_{k_t})(x) \to J(Sf)(x), \quad \text{as } \ell \to \infty, \text{ in } L^2 \left( \mathbb{R}^n \times (\varepsilon, \infty), \frac{dydt}{t^{n+1}}, \mathbb{B} \right).
\]
Since \( \gamma(H, \mathbb{B}) \) is continuously contained in the space \( \mathcal{L}(H, \mathbb{B}) \) of bounded linear operators from \( H \) into \( \mathbb{B} \), we have that, for every \( x \in \Omega \),
\[
J(Sf_{k_t})(x) \to G(x), \quad \text{as } \ell \to \infty, \text{ in } \mathcal{L}(H, \mathbb{B}).
\]
Assume that \( h \in C_c^\infty(\mathbb{R}^n \times (0, \infty)) \) and \( L \in \mathbb{B}^* \). Then, for every \( x \in \Omega \),
\[
\langle L, [J(Sf_{k_t})(x)](h) \rangle_{\mathbb{B}^*, \mathbb{B}} \to \langle L, [G(x)](h) \rangle_{\mathbb{B}^*, \mathbb{B}}, \quad \text{as } \ell \to \infty,
\]
being
\[ [J(S_h f_k)](x)](h) = \int_{\mathbb{R}^n \times (0, \infty)} [J(S_h f_k)](x)(y, t) h(y, t) \frac{dydt}{t^{n+1}}, \quad \ell \in \mathbb{N}. \]

Suppose that \( g = \sum_{m=1}^{s} b_m g_m \), where \( s \in \mathbb{N} \), \( b_m \in \mathbb{B} \) and \( g_m \in C_c^\infty(\mathbb{R}^n) \), \( m = 1, \ldots, s \). There exists a set \( W \subset \mathbb{R}^n \) such that \( |\mathbb{R}^n \setminus W| = 0 \) and, for every \( x \in W \),
\[ [J(S_h g)](x)(y, t) = \chi_{B(x, t)}(y, t) (S_h g_m)(y, t) = \sum_{m=1}^{s} b_m \chi_{B(x, t)}(y, t) (S g_m)(y, t), \quad (y, t) \in \mathbb{R}^n \times (0, \infty), \]
is in \( L^2\left(\mathbb{R}^n \times (0, \infty), \frac{dydt}{t^{n+1}}; \mathbb{B}\right) \) and
\[ \langle L, [J(S_h g)](x) \rangle_{\mathbb{B}', \mathbb{B}} = \sum_{m=1}^{s} \langle L, b_m \rangle_{\mathbb{B}', \mathbb{B}} \int_{\Gamma(x)} (S g_m)(y, t) h(y, t) \frac{dydt}{t^{n+1}} \]
\[ = \int_{\mathbb{R}^n \times (0, \infty)} \langle L, [J(S_h g)](x)(y, t) \rangle_{\mathbb{B}', \mathbb{B}} h(y, t) \frac{dydt}{t^{n+1}}, \quad x \in W. \]

For every \( \ell \in \mathbb{N} \), we denote by \( W_\ell \) the set in \( \mathbb{R}^n \) associated with \( f_k \) as above. We define \( \Omega_\ell = \Omega \cap (\cap_{\ell \in \mathbb{N}} W_\ell) \). We have that \( |\mathbb{R}^n \setminus \Omega_\ell| = 0 \).

Let \( x \in \Omega_\ell \). We can write
\[ (L, [G(x)](h))_{\mathbb{B}', \mathbb{B}} = \lim_{\ell \rightarrow \infty} \langle L, [J(S_h f_k)](x) \rangle_{\mathbb{B}', \mathbb{B}} = \lim_{\ell \rightarrow \infty} \int_{\mathbb{R}^n \times (0, \infty)} \langle L, [J(S_h f_k)](x)(y, t) \rangle_{\mathbb{B}', \mathbb{B}} h(y, t) \frac{dydt}{t^{n+1}} \]
\[ = \int_{\mathbb{R}^n \times (0, \infty)} \langle L, [J(S_f)](x)(y, t) \rangle_{\mathbb{B}', \mathbb{B}} h(y, t) \frac{dydt}{t^{n+1}}, \quad x \in \Omega_\ell. \]

We conclude that \( (L, [J(S_f)](x)(\cdot, \cdot))_{\mathbb{B}', \mathbb{B}} \in H, \quad x \in \Omega_\ell \). Then, for every \( x \in \Omega_\ell \),
\[ G(x) = [J(S_f)](x)(\cdot, \cdot), \]
as elements of \( \gamma(H, \mathbb{B}) \).

Hence, \( Sf = \tilde{S}_f \). Thus, \( \langle 8 \rangle \) is proved. \( \square \)

Before establishing the first inequality in Theorem 1 we need the following polarization formula.

**Lemma 2.2.** Let \( f \in C_c^\infty(\mathbb{R}^n) \odot \mathbb{B} \) and \( g \in C_c^\infty(\mathbb{R}^n) \odot \mathbb{B}^* \). Then,
\[ \int_{\mathbb{R}^n} \int_{\Gamma(x)} (t^2 \partial^2_t P_t(y)) \cdot t^2 \partial^2_t P_t(y) \frac{dydt}{t^{n+1}} dx = v_n \Gamma(2\beta) \frac{2^{2\beta}}{2^{2\beta} - 1} \int_{\mathbb{R}^n} (g(y), f(y))_{\mathbb{B}', \mathbb{B}} dy, \]
where \( v_n \) denotes the volume of the unit ball in \( \mathbb{R}^n \).

**Proof.** Suppose firstly that \( f, g \in C_c^\infty(\mathbb{R}^n) \). According to \( \langle 13 \rangle \) and by using Plancherel equality we get
\begin{align*}
\int_{\mathbb{R}^n} \int_{\Gamma(x)} t^2 \partial^2_t P_t(y) \cdot t^2 \partial^2_t P_t(y) \frac{dydt}{t^{n+1}} dx & = \int_{\mathbb{R}^n} \int_{\Gamma(x)} t^2 \partial^2_t P_t(f(y)) \cdot t^2 \partial^2_t P_t(f(y)) \frac{dydt}{t^{n+1}} dx \\
& = \int_{\mathbb{R}^n} \int_{\Gamma(x)} (||t^2 \partial^2_t P_t(f(y))||^2_{\mathbb{B}'}) \frac{dydt}{t^{n+1}} dx \\
& = v_n \int_0^\infty \int_{\mathbb{R}^n} (||t^2 \partial^2_t P_t(f(y))||^2_{\mathbb{B}'}) \frac{dydt}{t^{n+1}} dx \\
& = v_n \Gamma(2\beta) \frac{2^{2\beta}}{2^{2\beta} - 1} \int_{\mathbb{R}^n} f(y) g(y) dy.
\end{align*}

The interchanges in the order of integration are justified because the integrals are absolutely convergent. In order to see this it is sufficient recall that the Littlewood-Paley-Stein function defined by
\[ g_s(f)(x) = \left( \int_0^\infty |t^2 \partial^2_t P_t(f(x))|^2 \frac{dt}{t} \right)^{1/2} \]
is bounded from \( L^2(\mathbb{R}^n) \) into itself.

From \( \langle 13 \rangle \) we easily conclude the proof of this lemma. \( \square \)

**Lemma 2.3.** Let \( \mathbb{B} \) be a UMD Banach space, \( 1 < p < \infty \) and \( \beta > 0 \). Then, there exists \( C > 0 \) such that
\[ \|f\|_{L^p(\mathbb{R}^n, \mathbb{B})} \leq C \|t \partial^2_t P_t(f)\|_{L^2(\mathbb{R}^n, \mathbb{B})}, \quad f \in L^p(\mathbb{R}^n, \mathbb{B}). \]
Proof. Let \( f \in C_c^\infty(\mathbb{R}^n) \otimes \mathbb{B} \) and \( g \in C_c^\infty(\mathbb{R}^n) \otimes \mathbb{B}^* \). Since \( \mathbb{B} \) is a UMD Banach space, \( \mathbb{B}^* \) has also the UMD property. According to Lemma 2.1, Lemma 2.2 and Proposition 2.4 we can write

\[
\left| \int_{\mathbb{R}^n} \langle g(y), f(y) \rangle_{\mathbb{B}^*, \mathbb{B}} dy \right| \\
\leq C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n \times (0, \infty)} \left| \chi_{B(x,t)}(y,t) t^\beta \partial_t^\beta P_t(f)(y) \right|_{\mathbb{B}^*, \mathbb{B}} dy dt dx \\
\leq C \int_{\mathbb{R}^n} \| \chi_{B(x,t)}(y,t) t^\beta \partial_t^\beta P_t(f)(y) \|_{L^1(\mathbb{R}^n, \mathbb{B})} dy \int_{\mathbb{R}^n} \| \chi_{B(x,t)}(y,t) t^\beta \partial_t^\beta P_t(g)(y) \|_{\mathbb{B}} dx \\
\leq C \int_{\mathbb{R}^n} \| \chi_{B(x,t)}(y,t) t^\beta \partial_t^\beta P_t(f)(y) \|_{L^2(\mathbb{R}^n, \mathbb{B})} \| \chi_{B(x,t)}(y,t) t^\beta \partial_t^\beta P_t(g)(y) \|_{L^2(\mathbb{R}^n, \mathbb{B})} dx \\
\leq C \int_{\mathbb{R}^n} \| \chi_{B(x,t)}(y,t) t^\beta \partial_t^\beta P_t(f)(y) \|_{L^2(\mathbb{R}^n, \mathbb{B})} \| \chi_{B(x,t)}(y,t) t^\beta \partial_t^\beta P_t(g)(y) \|_{L^2(\mathbb{R}^n, \mathbb{B})} dx.
\]

Then, since \( C_c^\infty(\mathbb{R}^n) \otimes \mathbb{B} \) is a dense subspace of \( L^p(\mathbb{R}^n, \mathbb{B}) \), by taking into account Lemma 2.1 we conclude the proof of this lemma.

3. Proof of Theorem 2

3.1. Suppose that \( V \in RH_s(\mathbb{R}^n) \), where \( s > n/2 \) and \( n \geq 3 \). We are going to show that, for every \( f \in L^p(\mathbb{R}^n, \mathbb{B}) \),

\[
\| t^\beta \partial_t^\beta P_t^{L^p}(f) \|_{L^2(\mathbb{R}^n, \mathbb{B})} \leq C \| f \|_{L^p(\mathbb{R}^n, \mathbb{B})}.
\]

In order to see this we cannot proceed as in the proof of Lemma 2.1. In this Schrödinger case Theorem 4.8 does not apply because \( \partial_t P_t^{L^p}(1) \neq 0 \). Note that, since \( \lim_{t \to 0^+} P_t^{L^p}(1)(x) = 1, x \in \mathbb{R}^n \), if \( \partial_t P_t^{L^p}(1)(x) = 0, t > 0 \) and \( x \in \mathbb{R}^n \), then \( P_t^{L^p}(1)(x) = 1, t > 0 \) and \( x \in \mathbb{R}^n \), and we could infer that

\[
0 = (\partial_t - \mathcal{L}_V) P_t^{L^p}(1)(x) = -V(x) P_t^{L^p}(1)(x) = -V(x).
\]

Hence, since \( V \neq 0 \), we have that \( \partial_t P_t^{L^p}(1)(x) \neq 0 \).

We define, for every \( x \in \mathbb{R}^n \),

\[
\rho(x) = \sup \left\{ r > 0 : \frac{1}{r^{n-2}} \int_{B(x,r)} V(y) dy \leq 1 \right\}.
\]

The function \( \rho \) (usually called critical radius) plays an important role in the development of the harmonic analysis associated with \( \mathcal{L}_V \) (see, for instance, [20], [22], [24] and [44]). The main properties of \( \rho \) can be encountered in [44] Section 1).

Suppose that \( f = \sum_{j=1}^m b_j f_j \), where \( b_j \in \mathbb{B} \) and \( f_j \in C_c^\infty(\mathbb{R}^n) \), \( j = 1, \ldots, m \). It is clear that

\[
t^\beta \partial_t^\beta P_t^{L^p}(f)(y) = \sum_{j=1}^m b_j t^\beta \partial_t^\beta P_t^{L^p}(f_j)(y), \quad t > 0 \quad \text{and} \quad y \in \mathbb{R}^n.
\]

Moreover, for almost every \( x \in \mathbb{R}^n \), we have that \( \chi_{\Gamma(x)}(y,t) t^\beta \partial_t^\beta P_t^{L^p}(f)(y) \in H \otimes \mathbb{B} \). Indeed, let \( g \in L^2(\mathbb{R}^n) \), we can write

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left| t^\beta \partial_t^\beta P_t^{L^p}(g)(y) \right|^2 \frac{dy dt}{t^{n+1}} = v_n \int_{\mathbb{R}^n} \int_0^\infty \left| t^\beta \partial_t^\beta P_t^{L^p}(g)(y) \right|^2 \frac{dy}{t^{n+1}} dt,
\]

where once again \( v_n \) denotes the volume of the unit ball in \( \mathbb{R}^n \). Then, according to [44] Theorem A, it follows that

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left| t^\beta \partial_t^\beta P_t^{L^p}(g)(y) \right|^2 \frac{dy dt}{t^{n+1}} dx \leq C \| g \|_{L^2(\mathbb{R}^n)}^2.
\]

Hence, for almost \( x \in \mathbb{R}^n \),

\[
t^\beta \partial_t^\beta P_t^{L^p}(g)(y) \chi_{\Gamma(x)}(t,y) \in H \otimes \mathbb{B}.
\]

To simplify, we write

\[
K^{L^p}(f)(x,y,t) = t^\beta \partial_t^\beta P_t^{L^p}(f)(y) \chi_{\Gamma(x)}(t,y), \quad x, y \in \mathbb{R}^n \quad \text{and} \quad t > 0.
\]

We decompose the operator \( K^{L^p} \) as follows:

\[
K^{L^p}(f)(x,y,t) = K_{loc}^{L^p}(f)(x,y,t) + K_{glob}^{L^p}(f)(x,y,t), \quad x, y \in \mathbb{R}^n \quad \text{and} \quad t > 0,
\]

where

\[
K_{loc}^{L^p}(f)(x,y,t) = K^{L^p}(f \chi_{B(x,\rho(x))})(x,y,t), \quad x, y \in \mathbb{R}^n \quad \text{and} \quad t > 0.
\]
The operators $K_{loc}^{2V}$ and $K_{glob}^{2V}$ are usually called local and global part of $K^{2V}$, respectively. We also consider the operators

$$K(f)(x; y,t) = t^\beta \partial_\Gamma^\beta P_t(f)(y,t)\chi_{\Gamma}(x), \quad x, y \in \mathbb{R}^n \text{ and } t > 0,$$

and

$$K_{loc}(f)(x;y,t) = \mathbb{K}(f\chi_{B(x,\rho(x))})(x;y,t), \quad x, y \in \mathbb{R}^n \text{ and } t > 0.$$ 

We can write

$$K^{2V}(f)(x;y,t) = D^{2V}(f)(x;y,t) + K_{loc}^{2V}(f)(x;y,t) + K_{glob}^{2V}(f)(x;y,t), \quad x, y \in \mathbb{R}^n \text{ and } t > 0,$$

where

$$D^{2V}(f)(x;y,t) = K_{loc}^{2V}(f)(x;y,t) - K_{loc}(f)(x;y,t), \quad x, y \in \mathbb{R}^n \text{ and } t > 0.$$ 

The three terms in the right hand side of (19) are studied separately in the following lemmas.

**Lemma 3.1.** Let $B$ be a Banach space and $1 < p < \infty$. Then,

$$\|K_{glob}^{2V}(f)\|_{L^p(x^n,\gamma(H;B))} \leq C\|f\|_{L^p(x^n,\mathbb{R})}, \quad f \in C^\infty_{loc}(x^n) \cap B.$$  

**Proof.** Let $f = \sum_{j=1}^m b_j f_j$, where $b_j \in B$ and $f_j \in C^\infty_{loc}(x^n)$, $j = 1, \ldots, m$. Firstly, we show that for every $x \in \mathbb{R}^n$, $K_{glob}^{2V}(f)(x;\cdot,\cdot) \in \gamma(H,B)$. Fix $x \in \mathbb{R}^n$. According to the subordination formula, we have

$$P_t^{2V}(y,z) = \frac{t}{\sqrt{4\pi}} \int_0^\infty e^{-t^2/4u} W_u^{2V}(y,z) du, \quad y, z \in \mathbb{R}^n \text{ and } t > 0.$$ 

By [3, Lemma 4]

$$|\partial_\Gamma^\beta \partial_t^{\beta} P_t^{2V}(y,z)| \leq C e^{-t^2/8u} u^{(1-\beta)/2}, \quad t, u \in (0, \infty).$$

Also, by tacking into account the Feynman-Kac formula (20, 2.2)) and (20) we can interchange the order of integration and differentiate under the integral sign to get

$$\partial_\Gamma^\beta P_t^{2V}(y,z) = \frac{1}{\sqrt{4\pi}} \int_0^\infty \partial_\Gamma^\beta |\partial_t^{\beta} P_t^{2V}(y,z)| du, \quad x, y \in \mathbb{R}^n \text{ and } t > 0.$$ 

Then, by using (20, 2.3)) and (20) we obtain

$$|t^\beta \partial_\Gamma^\beta P_t^{2V}(y,z)| \leq Ct\beta \rho(y) \int_0^\infty e^{-t^2/8u} u^{-(\beta+n)/2} e^{-c(y-z)^2/u} du \leq C \frac{t^\beta \rho(y)}{(t + |y-z|)^{n+\beta+1}}, \quad x, y \in \mathbb{R}^n \text{ and } t > 0.$$ 

Estimation (21) justifies the differentiation under the integral sign, and we get

$$\|K_{glob}^{2V}(f)(x;y,t)\|_B \leq C \chi_{\Gamma}(x)(y,t) \int_{x^n \setminus B(x,\rho(x))} \frac{t^\beta \rho(y)}{(t + |y-z|)^{n+\beta+1}} \|f(z)\|_B dz$$

$$\leq Ct\beta \rho(y) \chi_{\Gamma}(x)(y,t) \|f\|_{L^p(x^n,\mathbb{R})} \left(\int_{x^n \setminus B(x,\rho(x))} \frac{dz}{(t + 2|x-y| + |y-z|)^{(n+\beta+1)p'}}\right)^{1/p'}$$

$$\leq Ct\beta \rho(y) \chi_{\Gamma}(x)(y,t) \|f\|_{L^p(x^n,\mathbb{R})} \left(\int_{x^n \setminus B(x,\rho(x))} \frac{dz}{(t + |x-y| + |z-x|)^{(n+\beta+1)p'}}\right)^{1/p'}$$

$$\leq Ct\beta \rho(y) \chi_{\Gamma}(x)(y,t) \|f\|_{L^p(x^n,\mathbb{R})} \left(\int_{\rho(x)}^{\infty} \frac{r^{n-1} dr}{(t + |x-y| + \rho(x))^{(n+\beta+1)p'}}\right)^{1/p'}$$

$$\leq C \frac{t^\beta \rho(y) \chi_{\Gamma}(x)(y,t)}{(t + |x-y| + \rho(x))^{(n+\beta+1+1-n)/p'}} \|f\|_{L^p(x^n,\mathbb{R})}, \quad x, y \in \mathbb{R}^n \text{ and } t > 0.$$ 

According to [20, Proposition 1] we deduce that

$$\rho(y) \leq C \rho(x) \left(1 + \frac{t}{\rho(x)}\right)^\gamma, \quad |x-y| < t, \quad x, y \in \mathbb{R}^n \text{ and } t > 0,$$

(22)
Then, for every
\[ \| K_{g\text{glob}}^U(f)(x; y, t) \|_{L^p(\mathbb{R}^n, \mathcal{B})} \leq \| f \|_{L^p(\mathbb{R}^n, \mathcal{B})} \]
\[ \sum_{j=1}^{\infty} \gamma_j \int_{\mathbb{R}^n \setminus B(x, \rho(x))} \int_{\mathbb{R}^n \times (0, \infty)} f(z) \chi_{\Gamma(x)}(y, t) t^{\beta \Delta} \partial_t^{\Delta} T_t^{\Delta}(y, z) h_j(y, t) \frac{dydt}{t^{n+1}} d\Gamma(z) \]
\[ = \left( \mathbb{E} \left[ \sum_{j=1}^{\infty} \gamma_j \int_{\mathbb{R}^n \setminus B(x, \rho(x))} \int_{\mathbb{R}^n \times (0, \infty)} f(z) \chi_{\Gamma(x)}(y, t) t^{\beta \Delta} \partial_t^{\Delta} T_t^{\Delta}(y, z) h_j(y, t) \frac{dydt}{t^{n+1}} d\Gamma(z) \right] \right)^{1/2}, \quad x \in \mathbb{R}^n.

The interchange of the order of integration is justified because by proceeding as above we can show that the integrals are norm convergent.

Then, from (21) and (22) it follows that
\[ \| G(x) \|_{\gamma(\mathbb{H}, \mathcal{B})} \leq C \int_{\mathbb{R}^n \setminus B(x, \rho(x))} \| f(z) \|_{\mathcal{B}} \left( \mathbb{E} \left[ \sum_{j=1}^{\infty} \gamma_j \int_{\Gamma(x)} t^{\beta \Delta} \partial_t^{\Delta} T_t^{\Delta}(y, z) h_j(y, t) \frac{dydt}{t^{n+1}} \right] \right)^{1/2} d\Gamma(z) \]
\[ \leq C \int_{\mathbb{R}^n \setminus B(x, \rho(x))} \| f(z) \|_{\mathcal{B}} \rho(x)^2 \left( 1 + \frac{t}{\rho(x)} \right)^{2} \left( 1 + \frac{t}{\rho(x)} \right)^{2(n+\beta+1)} d\Gamma(z) \]
\[ \leq C \int_{\mathbb{R}^n \setminus B(x, \rho(x))} \| f(z) \|_{\mathcal{B}} \left( \int_{\Gamma(x)} \rho(x)^2 \frac{dydt}{(t + |y - z|)^{2(n+\beta+1)}} \right)^{1/2} d\Gamma(z) \]
Lemma 3.2. Let

\[ M \]

where

\[ \beta \]

Proof. According to Kato-Trotter's formula \[23, (2.10)\] we have that

\[ \int \left( 1 + t/\rho(x) \right)^{2\gamma} \frac{dt}{t^{2\beta-1}} \]
\[ \left\| \frac{\partial^\beta [e^{-t^2/4u}]}{u^{3/2}} \right\| \leq \frac{1}{(t+|y-z|^2)^{n+\beta/2-n+1}} \int_0^{\rho(z)^2} \frac{e^{-c(t^2+|y-z|^2)/u}}{u^{1/2}} \, du, \]

for \( y, z \in \mathbb{R}^n \) such that \( |y-z| \leq 2\rho(x) \) and \( |x-z| < \rho(x) \). Estimates (20) and (23) lead to

\[ \left\| \frac{\partial^\beta [e^{-t^2/4u}]}{u^{3/2}} \right\| \leq \frac{C}{\rho(x)^{2\beta}} \int_0^{\rho(z)^2} \frac{e^{-c(t^2+|y-z|^2)/u}}{u^{1/2}} \, du, \]

for \( y, z \in \mathbb{R}^n \) such that \( |y-z| \leq 2\rho(x) \) and \( |x-z| < \rho(x) \). On the other hand, (20) and (23) imply that

\[ \left\| \frac{\partial^\beta [e^{-t^2/4u}]}{u^{3/2}} \right\| \leq \frac{1}{(t+|y-z|^2)^{n+\beta/2-n+1}} \int_0^{\rho(z)^2} \frac{e^{-c(t^2+|y-z|^2)/u}}{u^{1/2}} \, du, \]

for \( y, z \in \mathbb{R}^n \) such that \( |y-z| \leq 2\rho(x) \) and \( |x-z| < \rho(x) \).

We are going to see that \( D^2x (f)(x; \cdot, \cdot) \in H \), for every \( x \in \mathbb{R}^n \). Fix \( x \in \mathbb{R}^n \). We can write

\[ D^2x (f)(x; y, t) = \chi_{B(x, r)}(y, t) \int_{B(x, r)} f(z) \frac{t^\beta}{\sqrt{4\pi}} \int_0^{\rho(z)^2} \frac{\partial^\beta [e^{-t^2/4u}]}{u^{3/2}} (W^x_u (y, z) - W_u (y - z)) \, du \, dz \]

\[ + \chi_{B(x, r)}(y, t) \int_{B(x, r)} f(z) \frac{t^\beta}{\sqrt{4\pi}} \int_0^{\rho(z)^2} \frac{\partial^\beta [e^{-t^2/4u}]}{u^{3/2}} (W^x_u (y, z) - W_u (y - z)) \, du \, dz \]

\[ = D^2x (f)(x; y, t) + D^2x (f)(x; y, t), \quad y \in \mathbb{R}^n \] and \( t > 0 \).

Minkowski’s inequality and (23) leads to

\[ \| D^2x (f)(x; \cdot, \cdot) \|_{L^2(\mathbb{R}^n \times (0, \infty))} \]

\[ \leq C \int_{B(x, r)} \| f(z) \|_2 \left( \int_0^{\rho(z)^2} \frac{\partial^\beta [e^{-t^2/4u}]}{u^{3/2}} \, du \right) \| W^x_u (y, z) - W_u (y - z) \|_{H} \, dz \]

\[ \leq C \int_{B(x, r)} \| f(z) \|_2 \left( \int_0^{\rho(z)^2} \frac{1}{u^{1/2}} \left( \int_0^\infty e^{-c(t^2+|y-z|^2)/u} \, dt \right)^{1/2} \, du \right) \]

\[ \leq C \int_{B(x, r)} \| f(z) \|_2 \left( \int_0^{\rho(z)^2} \frac{1}{u^{1/2}} \left( \int_0^\infty e^{-c|y-z|^2/2\beta} \, dt \right)^{1/2} \, du \right) \]

\[ \leq C \int_{B(x, r)} \| f(z) \|_2 \left( \int_0^{\rho(z)^2} \frac{1}{u^{1/2}} \, du \right) \leq C \frac{1}{\rho(x)^{n/2}} \int_{B(x, r)} \| f(z) \|_2 \, dz \leq C. \]

We have taken into account that \( \rho(z) \sim \rho(x) \) because \( |x-z| < \rho(x) \).

We now decompose \( D^2x (f) \) as follows

\[ D^2x (f) = D^2x (f) + D^2x (f), \]
where
\[ D_{1,1}^{\beta}(f)(x; y, t) = D_1^{\beta}(f \chi_{B(y, 2\rho(x))})(x; y, t), \quad y \in \mathbb{R}^n, \; t > 0. \]

By using again the Minkowski’s inequality and (26) we get
\[
\|D_{1,1}^{\beta}(f)(x; \cdot, \cdot)\|_{L^2(\mathbb{R}^n \times (0, \infty), \frac{dx dy dt}{t^{n+1}})} \leq C \int_{B(x, \rho(x))} \|f(z)\|_{\mathbb{B}} \left( \int_0^{\infty} \int_{|x-y| < t} \frac{e^{-|z|^2/a}}{u^{1-\beta/2}} du \right)^{1/2} dz
\]
\[
\leq C \int_{B(x, \rho(x))} \|f(z)\|_{\mathbb{B}} \left( \int_0^{\infty} \int_{|x-y| < t} \frac{e^{-|z|^2/a}}{u^{1-\beta/2}} \frac{\|f(z)\|_{\mathbb{B}}}{\rho(x)^{2\beta}} du \right)^{1/2} dz \leq C \mathcal{M}(\|f\|_{\mathbb{B}})(x).
\]

Now, estimations (20) and (24) imply that
\[
\|D_{1,1}^{\beta}(f)(x; \cdot, \cdot)\|_{L^2(\mathbb{R}^n \times (0, \infty), \frac{dx dy dt}{t^{n+1}})} \leq C \sup_{u > 0} \int_{\mathbb{R}^n} \frac{e^{-|z|^2/a}}{u^{n/2}} \|f(z)\|_{\mathbb{B}} dz \int_0^{\infty} \int_{|x-y| < t} \frac{\|f(z)\|_{\mathbb{B}}}{\rho(x)^{2\beta}} du \frac{\|f(z)\|_{\mathbb{B}}}{\rho(x)^{2\beta}} \left( \int_0^{\infty} \int_{|x-y| < t} \frac{e^{-|z|^2/a}}{u^{1-\beta/2}} \frac{\|f(z)\|_{\mathbb{B}}}{\rho(x)^{2\beta}} du \right)^{1/2} dz
\]
\[
\leq C \sup_{u > 0} \int_{\mathbb{R}^n} \frac{e^{-|z|^2/a}}{u^{n/2}} \|f(z)\|_{\mathbb{B}} dz \int_0^{\infty} \int_{|x-y| < t} \frac{\|f(z)\|_{\mathbb{B}}}{\rho(x)^{2\beta}} du \frac{\|f(z)\|_{\mathbb{B}}}{\rho(x)^{2\beta}} \left( \int_0^{\infty} \int_{|x-y| < t} \frac{e^{-|z|^2/a}}{u^{1-\beta/2}} \frac{\|f(z)\|_{\mathbb{B}}}{\rho(x)^{2\beta}} du \right)^{1/2} dz \leq C W_*(\|f\|_{\mathbb{B}})(x),
\]

where \( W_*(g) = \sup_{t > 0} |W_t(g)|, \quad g \in L^q(\mathbb{R}^n), \quad 1 \leq q \leq \infty. \)

We conclude that
\[
\|D_{1,1}^{\beta}(f)(x; \cdot, \cdot)\|_{L^2(\mathbb{R}^n \times (0, \infty), \frac{dx dy dt}{t^{n+1}})} \leq C \left( W_*(\|f\|_{\mathbb{B}})(x) + \mathcal{M}(\|f\|_{\mathbb{B}})(x) \right).
\]

Also from the above estimations we get
\[
\|D_{1,1}^{\beta}(f)(x; \cdot, \cdot)\|_{L^2(\mathbb{R}^n \times (0, \infty), \frac{dx dy dt}{t^{n+1}})} \leq C \|f\|_{L^\infty(\mathbb{R}^n, \mathbb{B})}, \quad x \in \mathbb{R}^n.
\]

We can now proceed as in the study of \( K_{\text{glob}}^{\beta} \) to obtain that
\[
\|D_{1,1}^{\beta}(f)\|_{L^p(\mathbb{R}^n, \gamma(H, B))} \leq C \|f\|_{L^p(\mathbb{R}^n, \mathbb{B})},
\]
where \( C \) does not depend on \( f \), because \( \mathcal{M} \) and \( W_* \) are bounded operators from \( L^p(\mathbb{R}^n) \) into itself. \( \square \)
Lemma 3.3. Let $\mathcal{B}$ be a UMD Banach space and $1 < p < \infty$. Then,

$$\|\mathcal{K}_{\textrm{loc}}(f)\|_{L^p(\mathbb{R}^n, \gamma(\mathcal{H};\mathcal{B}))} \leq C \|f\|_{L^p(\mathbb{R}^n, \mathcal{B})}, \quad f \in C^\infty_c(\mathbb{R}^n) \otimes \mathcal{B}. $$

Proof. Let $f = \sum_{j=1}^m b_j f_j$, where $b_j \in \mathcal{B}$ and $f_j \in C^\infty_c(\mathbb{R}^n)$, $j = 1, \ldots, m$. We are going to use the ideas developed in the proof of [1, Theorem 3.7].

Our first goal is to see that $\mathcal{K}_{\textrm{loc}}(f)(x; \cdot, \cdot) \in L^2 \left( \mathbb{R}^n \times (0, \infty), \frac{dy dt}{t^n}; \mathcal{B} \right)$, for every $x \in \mathbb{R}^n$. According to [22] Lemma 2.3 we consider a sequence $(x_k)_{k=1}^\infty$ in $\mathbb{R}^n$ such that if $Q_k = B(x_k, \rho(x_k))$, $k \in \mathbb{N}$, we have that

(i) $\bigcup_{k=1}^\infty Q_k = \mathbb{R}^n$,

(ii) There exists $N = N(\rho)$ such that, for every $k \in \mathbb{N}$, card$\{j \in \mathbb{N} : Q_j^{**} \cap Q_k^{**} \neq \emptyset\} \leq N$, where $Q_j^{**} = B(x_j, 4\rho(x_j))$, $j \in \mathbb{N}$. Fix $x \in \mathbb{R}^n$. We define the operators

$$S(f)(x; y, t) = \sum_{k=1}^\infty \chi_{Q_k}(x) \mathcal{K}(f \chi_{Q_k^*})(x; y, t), \quad y \in \mathbb{R}^n \text{ and } t > 0, $$

where $Q_k^* = B(x_k, 2\rho(x_k))$, $k \in \mathbb{N}$, and

$$S(f)(x; y, t) = \sum_{k=1}^\infty \chi_{Q_k}(x) \mathcal{K}_{\textrm{loc}}(f)(x; y, t) - S(f)(x; y, t), \quad y \in \mathbb{R}^n \text{ and } t > 0. $$

We can write for every $y \in \mathbb{R}^n$ and $t > 0$,

$$S(f)(x; y, t) = \chi_{1}(x)(y, t) \int_{\mathbb{R}^n} \sum_{k=1}^\infty \chi_{Q_k}(x) t^\beta \partial^n_P(y - z) \left( \chi_{B(x, \rho(z))}(z) - \chi_{Q_k^*}(z) \right) f(z) dz. $$

According to [13] Lemma 1.4, (a) we deduce that, if $\chi_{Q_k}(x) \left( \chi_{B(x, \rho(z))}(z) - \chi_{Q_k^*}(z) \right) \neq 0$, for some $k \in \mathbb{N}$; then $\frac{1}{\rho(x)} |x - z| \leq C_1 \rho(x)$, for a certain $C_1 > 0$. By [22] it follows that

$$\left\| t^\beta \partial^n_P(y - z) \chi_{1}(x)(y, t) \right\|_{L^p(\mathbb{R}^n \times (0, \infty), \frac{dy dt}{t^n}; \mathcal{B})} \leq C \left( \int_0^\infty \int_{|x-y| < t} \frac{t^{2\beta - n-1}}{(t + |y-z|)^{2(n+\beta)}} dy dt \right)^{1/2} \leq \frac{1}{\rho(x)^n} \int_{|x-z| \leq C_1 \rho(x)} \|f(z)\|_p dz \leq C \|f\|_{L^p(\mathbb{R}^n, \mathcal{B})}. $$

Note that by virtue of (ii), $C$ does not depend on $x \in \mathbb{R}^n$. By proceeding as in the study of $K_{\gamma(\mathcal{H};\mathcal{B})}$ we conclude that

$$\|S(f)\|_{L^p(\mathbb{R}^n, \gamma(\mathcal{H};\mathcal{B}))} \leq C \|f\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}.$$
Also, we have that
\[
\|K_{\text{loc}}(f)\|_{L^p(\mathbb{R}^n, \gamma(\mathbb{H}, \mathbb{B}))} \sim \left\| \sum_{k=1}^{\infty} \chi_{Q_k} K_{\text{loc}}(f) \right\|_{L^p(\mathbb{R}^n, \gamma(\mathbb{H}, \mathbb{B}))}.
\]
Then, we conclude that
\[
\|K_{\text{loc}}(f)\|_{L^p(\mathbb{R}^n, \gamma(\mathbb{H}, \mathbb{B}))} \leq C\|f\|_{L^p(\mathbb{R}^n, \mathbb{B})},
\]
where \(C\) does not depend on \(f\).

By combining Lemmas 3.1, 3.2, and 3.3 we obtain
\[
\|K^{\mathbb{H}}_t(\mathbb{f})\|_{L^p(\mathbb{R}^n, \gamma(\mathbb{H}, \mathbb{B}))} \leq C\|f\|_{L^p(\mathbb{R}^n, \mathbb{B})}, \quad f \in C_c^\infty(\mathbb{R}^n) \otimes \mathbb{B},
\]
or, in other words,
\[
\left\|t^\beta \partial_t^\beta P^{\mathbb{H}}_t(f)\right\|_{T^2_2(\mathbb{R}^n, \mathbb{B})} \leq C\|f\|_{L^p(\mathbb{R}^n, \mathbb{B})}, \quad f \in C_c^\infty(\mathbb{R}^n) \otimes \mathbb{B},
\]
provided that \(\mathbb{B}\) is a UMD Banach space. Here \(C > 0\) does not depend on \(f\).

We define the operator
\[
T(f) = t^\beta \partial_t^\beta P^{\mathbb{H}}_t(f), \quad f \in C_c^\infty(\mathbb{R}^n) \otimes \mathbb{B}.
\]
Since \(C_c^\infty(\mathbb{R}^n) \otimes \mathbb{B}\) is a dense subspace of \(L^p(\mathbb{R}^n; \mathbb{B})\), \(T\) can be extended from \(C_c^\infty(\mathbb{R}^n) \otimes \mathbb{B}\) to \(L^p(\mathbb{R}^n, \mathbb{B})\) as a bounded operator \(\tilde{T}\) from \(L^p(\mathbb{R}^n, \mathbb{B})\) into \(T^2_2(\mathbb{R}^n, \mathbb{B})\). The same argument developed in Lemma 2.1 allows us to obtain that
\[
\tilde{T}f = t^\beta \partial_t^\beta P^{\mathbb{H}}_t(f), \quad f \in L^p(\mathbb{R}^n, \mathbb{B}),
\]
and then,
\[
\left\|t^\beta \partial_t^\beta P^{\mathbb{H}}_t(f)\right\|_{T^2_2(\mathbb{R}^n, \mathbb{B})} \leq C\|f\|_{L^p(\mathbb{R}^n, \mathbb{B})}, \quad f \in L^p(\mathbb{R}^n, \mathbb{B}).
\]

3.2. We now prove that, there exists \(C > 0\) for which
\[
\|f\|_{L^p(\mathbb{R}^n, \mathbb{B})} \leq C\left\|t^\beta \partial_t^\beta P^{\mathbb{H}}_t(f)\right\|_{T^2_2(\mathbb{R}^n, \mathbb{B})}, \quad f \in L^p(\mathbb{R}^n, \mathbb{B}).
\]

According to [3] Proposition 2.1, (ii) we have that, for every \(f, g \in L^2(\mathbb{R}^n)\),
\[
\int_{\mathbb{R}^n} \int_{0}^{\infty} t^\beta \partial_t^\beta P^{\mathbb{H}}_t(f)(x)t^\beta \partial_t^\beta P^{\mathbb{H}}_t(g)(x)^2 \frac{dt}{t} \frac{dx}{t} = \frac{\Gamma(2\beta)}{2^{2\beta}} \int_{\mathbb{R}^n} f(x)g(x) \, dx.
\]
Then, for every \(f, g \in L^2(\mathbb{R}^n)\),
\[
\int_{\mathbb{R}^n} \int_{\Gamma(x)} t^\beta \partial_t^\beta P^{\mathbb{H}}_t(f)(y)t^\beta \partial_t^\beta P^{\mathbb{H}}_t(g)(y) \frac{dy}{t} \frac{dt}{t} = \int_{\mathbb{R}^n} \int_{0}^{\infty} t^\beta \partial_t^\beta P^{\mathbb{H}}_t(f)(y)t^\beta \partial_t^\beta P^{\mathbb{H}}_t(g)(y) \frac{dy}{t} \frac{dt}{t}
\]
\[
= \int_{\mathbb{R}^n} f(y)g(y) \, dy.
\]
Hence, for every \(f \in L^2(\mathbb{R}^n) \otimes \mathbb{B}\) and \(g \in L^2(\mathbb{R}^n) \otimes \mathbb{B}^*\), we get
\[
\left(\int_{\mathbb{R}^n} \int_{\Gamma(x)} t^\beta \partial_t^\beta P^{\mathbb{H}}_t(f)(y)t^\beta \partial_t^\beta P^{\mathbb{H}}_t(g)(y) \frac{dy}{t} \frac{dt}{t}\right)_{\mathbb{B} \otimes \mathbb{B}^*} = \int_{\mathbb{R}^n} f(y)g(y) \, dy.
\]
Now (3.2) can be established by proceeding as in the proof of Lemma 2.3

4. Proof of Theorem 2.1 (ii)

For every \(k \in \mathbb{N}\) we consider the \(k\)-th Hermite function defined by
\[
h_k(x) = (\sqrt{\pi} 2^k k!)^{-1/2} e^{-x^2/2} H_k(x), \quad x \in \mathbb{R},
\]
where \(H_k\) represents the \(k\)-th Hermite polynomial ([17, pp. 105–106]). If \(k = (k_1, \ldots, k_n) \in \mathbb{N}^n\), the \(k\)-th Hermite function \(h_k\) in \(\mathbb{R}^n\) is defined by
\[
h_k(x) = \prod_{j=1}^{n} h_{k_j}(x_j), \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n.
\]
For every \(k \in \mathbb{N}^n\), \(h_k\) is an eigenfunction of the Hermite operator \(\mathcal{H}\) satisfying
\[
\mathcal{H} h_k = (2|k| + n) h_k,
\]
where \(|k| = k_1 + \ldots + k_n\). The system \(\{h_k\}_{k \in \mathbb{N}^n}\) is an orthonormal basis in \(L^2(\mathbb{R}^n)\).
The heat semigroup associated with \( \{h_k\}_{k \in \mathbb{N}^n} \) is defined by
\[
W_t^H(f)(x) = \sum_{k \in \mathbb{N}^n} e^{-t(2k_1 + n)} c_k(f)h_k(x), \quad f \in L^2(\mathbb{R}^n),
\]
where
\[
c_k(f) = \int_{\mathbb{R}^n} h_k(y)f(y)dy, \quad k \in \mathbb{N}^n.
\]
According to the Mehler’s formula \([18, (1.1.36)]\) we can write, for every \( t > 0 \),
\[
W_t^H(f)(x) = \int_{\mathbb{R}^n} W_t^H(x,y)f(y)dy, \quad f \in L^2(\mathbb{R}^n),
\]
where, for every \( x, y \in \mathbb{R}^n \) and \( t > 0 \),
\[
W_t^H(x,y) = \frac{1}{n^{n/2}} \left( \frac{e^{-2t}}{1 - e^{-4t}} \right)^{n/2} \exp \left[ -\frac{1}{4} \left( |x-y|^2 \frac{1 + e^{-2t}}{1 - e^{-2t}} + |x+y|^2 \frac{1 - e^{-2t}}{1 + e^{-2t}} \right) \right].
\]

We define, for each \( \pi \),
\[
\frac{1}{1 + |x|}, \quad |x| \geq 1
\]
\[
\frac{1}{2}, \quad |x| < 1.
\]

We are going to see that in this context we can establish properties that allow us to prove Theorem \([2] \) (ii), by proceeding as in the proof of Theorem \([2] \) (i). Note that now \( n \) can be any nonnegative integer.

Firstly, according to Feynman-Kac formula we have that
\[
|W_t^H(x,y)| \leq Ce^{-|x-y|^2/4t} \frac{1}{n^{n/2}}, \quad x, y \in \mathbb{R}^n \quad \text{and} \quad t > 0.
\]

On the other hand, we have that
\begin{itemize}
  \item If \( x, y \in \mathbb{R}^n \), \( x \cdot y > 0 \), then \( |x+y| \geq |y| \) and
  \[
  |W_t^H(x,y)| \leq Ce^{-\frac{2t}{1-e^{-4t}}} \frac{n/2}{\exp \left[ -\frac{1}{4} \left( |x-y|^2 \frac{1 + e^{-2t}}{1 - e^{-2t}} + |y|^2 \frac{1 - e^{-2t}}{1 + e^{-2t}} \right) \right]} \leq Ce^{-c|x-y|^2/t} \frac{1 + e^{-2t}}{1 - e^{-2t}} \frac{1}{|y|} \leq Ce^{-c|x-y|^2/t} \frac{\rho(y)}{\sqrt{|y|}}, \quad |y| > 1 \quad \text{and} \quad t > 0.
  \]
  \item If \( x, y \in \mathbb{R}^n \), \( x \cdot y < 0 \), then \( |x-y| \geq |y| \) and
  \[
  |W_t^H(x,y)| \leq Ce^{-\frac{ct}{1-e^{-4t}}} \frac{n/2}{\exp \left[ -\frac{1}{8} \left( |x-y|^2 \frac{1 + e^{-2t}}{1 - e^{-2t}} + |y|^2 \frac{1 - e^{-2t}}{1 + e^{-2t}} \right) \right]} \leq Ce^{-ct} \frac{1}{|y|} \frac{1 + e^{-2t}}{1 - e^{-2t}} \frac{1}{|y|} \leq Ce^{-c|x-y|^2/t} \frac{\rho(y)}{\sqrt{|y|}}, \quad |y| > 1 \quad \text{and} \quad t > 0.
  \]
  \item If \( x, y \in \mathbb{R}^n \) and \( |y| < 1 \), then
  \[
  |W_t^H(x,y)| \leq Ce^{-c|x-y|^2/t} \frac{1}{\sqrt{|y|}} \leq Ce^{-c|x-y|^2/t} \frac{\rho(y)}{\sqrt{|y|}}, \quad |y| \leq 1 \quad \text{and} \quad t > 0.
  \]
\end{itemize}

We conclude that
\[
|W_t^H(x,y)| \leq Ce^{-c|x-y|^2/t} \frac{\rho(y)}{\sqrt{|y|}}, \quad x, y \in \mathbb{R}^n \quad \text{and} \quad t > 0.
\]
The Poisson semigroup \( \{P_t^\Phi\}_{t>0} \) associated with the Hermite operator is defined, as usual, by subordination

\[
P_t^\Phi(f)(x) = \frac{t}{\sqrt{4\pi}} \int_0^\infty e^{-t^2/4u} W_u^\Phi(f)(x)du, \quad f \in L^p(\mathbb{R}^n) \text{ and } t > 0.
\]

The Poisson kernel \( P_t^\Phi(x,y) \) can be written as

\[
P_t^\Phi(x,y) = \frac{t}{\sqrt{4\pi}} \int_0^\infty e^{-t^2/4u} W_u^\Phi(x,y)du, \quad x,y \in \mathbb{R}^n \text{ and } t > 0.
\]

By using (20) and (29) we obtain

\[
|t^\beta \partial_t^\beta P_t^\Phi(z,y)| \leq C \frac{\rho(y)|t^\beta|}{(t + |y - z|)^{3+\nu}}, \quad z,y \in \mathbb{R}^n \text{ and } t > 0.
\]

We also have that

\[
\int_{\mathbb{R}^n} |z|^2 e^{-\frac{|w|}{s^a/2}} \frac{e^{-|w|^2/2}}{s} dw = \int_{\mathbb{R}^n} (|y|^2 + |w|^2) e^{-|w|^2/2} dw \leq C \frac{s}{\rho(y)^2}, \quad 0 < s < \rho(y)^2.
\]

Note that \( \rho(y) \leq 1, y \in \mathbb{R}^n \).

By proceeding as in the proof of [44, Lemma 1.4] we can see that there exists \( 1/2 \leq \gamma < 1 \) such that

\[
\rho(y) \leq C \rho(x) \left( 1 + \frac{|x - y|}{\rho(x)} \right) ^\gamma, \quad x,y \in \mathbb{R}^n.
\]

Also, we can find a sequence \( \{x_k\}_{k \in \mathbb{N}} \) in \( \mathbb{R}^d \) such that if \( Q_k = B(x_k, \rho(x_k)) \) and \( Q_k^{**} = B(x_k, 4\rho(x_k)) \), \( k \in \mathbb{N} \), the following properties hold ([22, Lemma 2.3])

1. \( \cup_{k \in \mathbb{N}} Q_k = \mathbb{R}^n \),
2. There exists \( N \in \mathbb{N} \) such that, for every \( k \in \mathbb{N} \), \( \text{card}\{j \in \mathbb{N} : Q_j^{**} \cap Q_k^{**} \neq \emptyset\} \leq N \).

These properties of the sequence of balls \( \{Q_k\}_{k \in \mathbb{N}} \) and the estimates (28)–(32) allow us to show Theorem [2, (ii)], by proceeding as in the proof of Theorem [2, (i)].

5. Proof of Theorem [3]

5.1. Our first objective is to show that

\[
||t^\beta \partial_t^\beta P_t^{\Phi,\lambda}(f)||_{T^\beta_p((0,\infty),B)} \leq C ||f||_{L^p((0,\infty),B)}, \quad f \in L^p((0,\infty),B).
\]

We can write

\[
B_\lambda = -\frac{d^2}{dx^2} + V(x), \quad x \in (0,\infty),
\]

where \( V(x) = \lambda(\lambda - 1)/x^2 \). Then, \( \partial_t P_t^{\Phi,\lambda}(1) \neq 0 \), and in order to prove (33) we cannot use [31, Theorem 4.8]. We are going to proceed as in Section [3] by comparing the operator \( t^\beta \partial_t^\beta P_t^{\Phi,\lambda}(f) \) with the one related to the one-dimensional classical Poisson semigroup given by (1). In the following lemmas we collect some estimates that will be very helpful for our purposes.

**Lemma 5.1.** Let \( \beta > 0 \). Then,

\[
||t^\beta \partial_t^\beta P_t(y \pm z)\chi_{\Gamma_+}(x)(y,t)||_{H^\epsilon} \leq C \frac{c_k}{|x \pm z|}, \quad x,z \in (0,\infty),
\]

where \( \Gamma_+ = \{(y,t) \in (0,\infty)^2 : |x - y| < t\} \).

**Proof.** In [3, Lemma 2] it was established that

\[
t^\beta \partial_t^\beta P_t(z) = \sum_{k=0}^{(m+1)/2} \frac{c_k}{t} \varphi^k(\frac{z}{t}), \quad z \in \mathbb{R} \text{ and } t > 0,
\]

where \( m \in \mathbb{N} \) is such that \( m - 1 \leq \beta < m \), and, for every \( k \in \mathbb{N}, 0 \leq k \leq (m+1)/2 \), \( c_k \in \mathbb{C} \) and

\[
\varphi^k(z) = \int_0^\infty (1 + v)^{m+1-2k} v^{m-\beta-1} \frac{dv}{((1+v)^2 + z^2)^{m-k+1}}, \quad z \in \mathbb{R}.
\]
Let $k \in \mathbb{N}$, $0 \leq k \leq (m+1)/2$. We can write
\[
\frac{1}{t} \varphi^k \left( \frac{y + z}{t} \right) = t^{2(m-k)+1} \int_0^\infty \frac{(1 + v)^m + 2k v^m - 1}{((1 + v)^2 + (y + z)^2)^{m-k+1}} \, dv, \quad t, y, z \in (0, \infty).
\]
By using Minkowski’s inequality we obtain
\[
\left( \int_{\Gamma_+(x)} \left( \frac{1}{t} \varphi^k \left( \frac{y + z}{t} \right) \right)^2 \, dy dt \right)^{1/2} \leq C \int_0^\infty (1 + v)^{m+1 - 2k} v^{m-1} \left( \int_0^\infty \frac{t^{4(m-k)}}{(t + y + z + tv)^{4(m-k+1)}} \, dt \right)^{1/2} \, dv \leq C \int_0^\infty (1 + v)^{m+1 - 2k} v^{m-1} \left( \int_0^\infty \frac{t^{4(m-k)}}{(t + y + z + tv)^{4(m-k+1)}} \, dt \right)^{1/2} \, dv \leq \frac{C}{x + z} \int_0^\infty \frac{v^{m-1}}{(1 + v)^m} \, dv, \quad x, z \in (0, \infty).
\]
Hence,
\[
\left\| t^\beta \partial_t^\beta P_t(y + z) \chi_{\Gamma_+(x)}(y, t) \right\|_{H_+} \leq \frac{C}{x + z}, \quad x, z \in (0, \infty).
\]
By taking into account that $|x - y| + |y - z| \geq |x - z|$, $x, y, z \in (0, \infty)$, the above arguments allow us to obtain that
\[
\left\| t^\beta \partial_t^\beta P_t(y - z) \chi_{\Gamma_+(x)}(y, t) \right\|_{H_+} \leq \frac{C}{|x - z|}.
\]
\[
\square
\]
It is common to decompose the Bessel-Poisson kernel as follows
\[
P_t^{2\lambda}(y, z) = \frac{2\lambda(yz)\lambda t}{\pi} \left( \int_0^{\pi/2} + \int_{\pi/2}^{\pi} \right) \frac{(sin \theta)^{2\lambda-1}}{(t^2 + (y + z)^2 + 2yz(1 - cos \theta))^{\lambda+1}} \, d\theta
\approx P_t^{2\lambda,1}(y, z) + P_t^{2\lambda,2}(y, z), \quad t, y, z \in (0, \infty).
\]
**Lemma 5.2.** Let $\beta, \lambda > 0$. Then,
\[
\left\| t^\beta \partial_t^\beta P_t^{2\lambda,1}(y, z) \chi_{\Gamma_+(x)}(y, t) \right\|_{H_+} \leq \frac{C}{x + z}, \quad x, z \in (0, \infty),
\]
and
\[
\left\| t^\beta \partial_t^\beta P_t^{2\lambda,2}(y, z) \chi_{\Gamma_+(x)}(y, t) \right\|_{H_+} \leq \frac{C}{|x - z|}, \quad x, z \in (0, \infty).
\]
**Proof.** By \([27]\) we have that, for each $t, x, y \in (0, \infty),$
\[
(35) \quad t^\beta \partial_t^\beta P_t^{2\lambda}(x, y) = \sum_{k=0}^{(m+1)/2} \frac{b_k^{\lambda}}{t^{2\lambda+1}} (xy)^\lambda \int_0^\pi (sin \theta)^{2\lambda-1} \varphi^{\lambda,k} \left( \frac{\sqrt{(y - z)^2 + 2xy(1 - cos \theta)}}{t} \right) \, d\theta,
\]
where $m \in \mathbb{N}$ is such that $m - 1 \leq \beta < m$ and, for every $k \in \mathbb{N}$, $0 \leq k \leq (m+1)/2$,
\[
\varphi^{\lambda,k}(z) = \int_0^\infty \frac{(1 + v)^{m+1 - 2k} v^m - 1}{((1 + v)^2 + z^2)^{\lambda+m-k+1}} \, dv, \quad z \in (0, \infty),
\]
and
\[
b_k^{\lambda} = \frac{2\lambda(\lambda + 1) \cdots (\lambda + m - k)}{(m - k)!} c_k.
\]
Here $c_k$ is as in \([34]\). By \((35)\) we get
\[
t^\beta \partial_t^\beta P_t^{2\lambda,2}(y, z) = \sum_{k=0}^{(m+1)/2} \frac{b_k^{\lambda}}{t^{2\lambda+1}} (yz)^\lambda \int_0^\pi (sin \theta)^{2\lambda-1} \varphi^{\lambda,k} \left( \frac{\sqrt{(y - z)^2 + 2yz(1 - cos \theta)}}{t} \right) \, d\theta,
\]
for each \( t, y, z \in (0, \infty) \). Let \( k \in \mathbb{N} \), \( 0 \leq k \leq (m + 1)/2 \). We have that, for every \( t, y, z \in (0, \infty) \),

\[
\frac{(yz)^\lambda}{t^{2\lambda+1}} \int_{\pi/2}^{\pi} (\sin \theta)^{2\lambda-1} \left| t^{\lambda+m-2k} (1 + v)^{m+1-2k} v^{m-\beta-1} \right| dv d\theta \\
\leq C(z)^\lambda y^{2m-2k+1} \int_{\pi/2}^{\pi} (\sin \theta)^{2\lambda-1} \left| (1 + v)^{m+1-2k} v^{m-\beta-1} \right| dv d\theta.
\]

Hence, Minkowski’s inequality leads to

\[
\left( \int_{\Gamma_+(x)} \left| \frac{(yz)^\lambda}{t^{2\lambda+1}} \int_{\pi/2}^{\pi} (\sin \theta)^{2\lambda-1} \phi(t, y, z) \right|^2 \frac{dydt}{t^2} \right)^{1/2} \leq C \left( \int_{\Gamma_+(x)} \left| \frac{(yz)^\lambda}{t^{2\lambda+1}} \int_{\pi/2}^{\pi} (\sin \theta)^{2\lambda-1} \phi(t, y, z) \right|^2 \frac{dydt}{t^2} \right)^{1/2}\]

\[
\leq C \left( \int_{\Gamma_+(x)} \left| \frac{(yz)^\lambda}{t^{2\lambda+1}} \int_{\pi/2}^{\pi} (\sin \theta)^{2\lambda-1} \phi(t, y, z) \right|^2 \frac{dydt}{t^2} \right)^{1/2} \leq C \frac{1}{x + z}, \quad x, z \in (0, \infty).
\]

Then,

\[
\left( \int_{\Gamma_+(x)} \left| t^\beta \partial_t^\beta P_t^\lambda (y, z) \right|^2 \frac{dydt}{t^2} \right)^{1/2} \leq \frac{C}{x + z}, \quad x, z \in (0, \infty).
\]

In a similar way we can see that

\[
\left( \int_{\Gamma_+(x)} \left| t^\beta \partial_t^\beta P_t^\lambda (y, z) \right|^2 \frac{dydt}{t^2} \right)^{1/2} \leq \frac{C}{|x - z|}, \quad x, z \in (0, \infty).
\]

\[\square\]

**Lemma 5.3.** Let \( \beta, \lambda > 0 \). Then,

\[
\left\| t^\beta \partial_t^\beta [P_t^\lambda(y, z) - T_t(y, z)] \mathcal{L}_{\Gamma_+(x)}(y, t) \right\|_{H^+} \leq \frac{C}{z} \left( 1 + \log_+ \frac{z}{|x-z|} \right), \quad 0 < \frac{x}{2} < z < 2x.
\]

**Proof.** We use the following decomposition

\[
P_t^\lambda(y, z) = \frac{2\lambda(yz)^\lambda}{\pi} \int_0^{\pi/2} (\sin \theta)^{2\lambda-1} - \theta^{2\lambda-1} \frac{d\theta}{t^2 + (y - z)^2 + 2yz(1 - \cos \theta)}
\]

\[
+ \frac{2\lambda(yz)^\lambda}{\pi} \int_0^{\pi/2} \theta^{2\lambda-1} \frac{d\theta}{(t^2 + (y - z)^2 + 2yz(1 - \cos \theta))^{\lambda+1}} - \frac{1}{(t^2 + (y - z)^2 + yz\theta^2)^{\lambda+1}}
\]

\[
+ \frac{2\lambda(yz)^\lambda}{\pi} \int_0^{\pi/2} \frac{d\theta}{(t^2 + (y - z)^2 + yz\theta^2)^{\lambda+1}}, \quad t, y, z \in (0, \infty).
\]

On the other hand, we get (see [8], p. 485) for every \( t, y, z \in (0, \infty) \),

\[
\int_0^{\pi/2} \frac{\theta^{2\lambda-1}}{(t^2 + (y - z)^2 + yz\theta^2)^{\lambda+1}} d\theta = \left( \int_0^{\infty} - \int_{\pi/2}^{\pi} \right) \frac{\theta^{2\lambda-1}}{(t^2 + (y - z)^2 + yz\theta^2)^{\lambda+1}} d\theta
\]

\[
= \frac{\pi}{2\lambda(yz)^\lambda} t P_t(y, z) - \int_{\pi/2}^{\infty} \frac{\theta^{2\lambda-1}}{(t^2 + (y - z)^2 + yz\theta^2)^{\lambda+1}} d\theta.
\]

Hence,

\[
t^\beta \partial_t^\beta [P_t^\lambda(y, z) - T_t(y, z)] = \sum_{j=1}^3 S_j(y, z, t), \quad t, y, z \in (0, \infty),
\]

where

\[
S_1(y, z, t) = t^\beta \partial_t^\beta \left( \frac{2\lambda(yz)^\lambda}{\pi} \int_0^{\pi/2} \frac{(\sin \theta)^{2\lambda-1} - \theta^{2\lambda-1}}{(t^2 + (y - z)^2 + 2yz(1 - \cos \theta))^{\lambda+1}} d\theta \right),
\]

\[
S_2(y, z, t) = \frac{2\lambda(yz)^\lambda}{\pi} \int_0^{\pi/2} \frac{d\theta}{(t^2 + (y - z)^2 + 2yz(1 - \cos \theta))^{\lambda+1}}
\]

\[
- \frac{2\lambda(yz)^\lambda}{\pi} \int_0^{\pi/2} \frac{d\theta}{(t^2 + (y - z)^2 + yz\theta^2)^{\lambda+1}}.
\]
where we obtain

\[ S_2(y, z, t) = t^3 \partial_t^3 \left( \frac{2\lambda(yz)^\lambda t}{\pi} \int_0^{\pi/2} \frac{\theta^{2\lambda-1}}{(t^2 + (y - z)^2 + 2yz(1 - \cos \theta))^{\lambda+1}} \right. \]

and

\[ S_3(y, z, t) = -t^3 \partial_t^3 \left( \frac{2\lambda(yz)^\lambda t}{\pi} \int_{\pi/2}^{\infty} \frac{\theta^{2\lambda-1}}{(t^2 + (y - z)^2 + yz\theta^2)^{\lambda+1}} d\theta \right). \]

Assume that 0 < x/2 < z < 2x. We are going to analyze \( S_1, S_2 \) and \( S_3 \) separately.

From [35] we deduce that, for every \( t \in (0, \infty) \),

\[ S_1(y, z, t) = \sum_{k=0}^{(m+1)/2} b_k^{yz} (yz)^{\lambda} \int_0^{\pi/2} \frac{\theta^{2\lambda-1}}{(t^2 + (y - z)^2 + 2yz(1 - \cos \theta))^{\lambda+m-k+1}} d\theta, \]

where \( m \in \mathbb{N} \) is such that \( m - 1 \leq \beta < m \). Let \( k \in \mathbb{N}, 0 \leq k \leq (m+1)/2 \). Since

\[ |(\sin \theta)^{2\lambda-1} - \theta^{2\lambda-1}| \leq C \theta^{2\lambda+1} \quad \text{and} \quad |1 - \cos \theta| \geq C \theta^2, \quad \theta \in [0, \pi/2], \]

from [39] p. 60–61 we obtain

\[ \int_0^{\pi/2} \frac{(\sin \theta)^{2\lambda-1} - \theta^{2\lambda-1}}{(t^2 + (y - z)^2 + 2yz(1 - \cos \theta))^{\lambda+m-k+1}} d\theta \leq C \int_0^{\pi/2} \frac{\theta^{2\lambda+1}}{(t^2 + (y - z)^2 + 2yz(1 - \cos \theta))^{\lambda+m-k+1}} d\theta \]

\[ \leq C \frac{(yz)^{-\lambda-1}}{(t^2 + (y - z)^2 + 2yz(1 - \cos \theta))^{\lambda+k}} \left\{ \frac{1}{(t^2 + (y - z)^2 + 2yz(1 - \cos \theta))^{\lambda+k}} \right\} \]

\[ \leq C(yz)^{-\lambda-1} \left\{ \frac{1}{(t^2 + (y - z)^2 + 2yz(1 - \cos \theta))^{\lambda+k}} \right\} \]

\[ \leq C(yz)^{-\lambda-1} \left\{ \frac{1}{(t^2 + (y - z)^2 + 2yz(1 - \cos \theta))^{\lambda+k}} \right\} \]

\[ \leq C \frac{(yz)^{-\lambda-1}}{(t^2 + (y - z)^2 + 2yz(1 - \cos \theta))^{\lambda+k}} \]

\[ \leq C \frac{(yz)^{-\lambda-1}}{(t^2 + (y - z)^2 + 2yz(1 - \cos \theta))^{\lambda+k}} \]

\[ \leq C \frac{(yz)^{-\lambda-1}}{(t^2 + (y - z)^2 + 2yz(1 - \cos \theta))^{\lambda+k}} \]

for every \( t, z, y \in (0, \infty) \). Notice that, since \( 0 \leq k \leq (m+1)/2; k = m \) if, and only if, \( k = m = 1 \).

Suppose that \( m > k \). We have that

\[ \left( \int_{\Gamma_+(z)} (yz)^{\lambda} \int_0^{\pi/2} \frac{(\sin \theta)^{2\lambda-1} - \theta^{2\lambda-1}}{2\lambda+1} d\theta \right)^{1/2} \]

\[ \leq C \int_0^{\infty} (1 + v)^{m+1-2k} u^{m-\beta-1} \]

\[ \times \left( \int_{\Gamma_+(z)} \frac{(yz)^{2\lambda+1}}{(t^2 + (y - z)^2 + 2yz(1 - \cos \theta))^{\lambda+1}} \right)^{1/2} \]

\[ \leq C \int_0^{\infty} (1 + v)^{m+1-2k} u^{m-\beta-1} \left[ \left( \int_0^{\pi/2} \int_{\frac{\pi}{2}}^{\infty} \right) \right. \]

\[ \left. \times \int_{|x|}^{\infty} \frac{(yz)^{2\lambda+4(m-k)}}{(t^2 + (y - z)^2 + 2yz(1 - \cos \theta))^{2(m-k)}} \right]^{1/2} \]

\[ = \sum_{j=1}^{3} I_j(x, z). \]

We can write

\[ I_1(x, z) + I_3(x, z) \]
Hence,

\[
\int_{\Gamma_+} \left( \frac{(yz)^{2\lambda/3}}{t^2 + 1} \right) dt dt \leq C \frac{1}{t^2}
\]

Assume now \( k = m = 1 \). Then, \( 0 < \beta < 1 \). We have that

\[
\left( \int_{\Gamma_+} \left( \frac{(yz)^{\lambda}}{t^{2\lambda+1}} \right)^{1/2} \right) dt dt \leq C \frac{1}{t^2}
\]

We can write

\[
J_4(x, z) + J_3(x, z) \leq C \int_0^\infty v^{-\beta} \left[ \int_0^{z/2} + \int_{z/2}^{2z} \right] \int_{|y-z|+v(1+v)^2 + (y-z)^2 + (y+z)^2}^{2\lambda+2} dt dt \leq C \frac{1}{t^2}
\]
and
\[ J_2(x, z) \leq C \int_0^\infty v^{-\beta} \left[ \int_{z/2}^{2z} \frac{(yz)^{2\lambda}}{(t^2(1+v)^2 + (y-z)^2 + yz)^{2\lambda+2}} \right] \frac{1}{t^2} \left[ \int_{z/2}^{2z} \frac{(yz)^{2\lambda}}{(t^2(1+v)^2 + (y-z)^2 + yz)^{2\lambda+2}} \right] dt dy d\theta \]

Hence
\[ \left( \int_{\Gamma_+(x)} \left( \frac{(yz)^{\lambda}}{t^{2\lambda+1}} \int_0^{\pi/2} \left[ (\sin \theta)^{2\lambda-1} - \theta^{2\lambda-1} \right] \varphi \lambda \left( \frac{\sqrt{(y-z)^2 + 2yz(1-\cos \theta)}}{t} \right)^{\frac{2}{t^2}} \right)^{1/2} \right) \leq C \left( 1 + \log + \frac{z}{|x-z|} \right). \]

We conclude that
\[ \left( \int_{\Gamma_+(x)} \left| S_1(y, z, t) \right|^2 \frac{dy dt}{t^2} \right)^{1/2} \leq C \left( 1 + \log + \frac{z}{|x-z|} \right). \]

Next we treat \( S_2 \). Let \( k \in \mathbb{N}, 0 \leq k \leq (m+1)/2 \). By using the mean value theorem we obtain
\[ \left| \frac{1}{((1+v)^2 + (y-z)^2 + yz(1-\cos \theta)))^{\lambda+m-k+1} - \frac{1}{((1+v)^2 + (y-z)^2 + yz(1-\cos \theta))^{\lambda+m-k+1}} \right| \leq C t^{2\lambda+m-k+1} \]
\[ \left| \frac{1}{(t^2(1+v)^2 + (y-z)^2 + yz(1-\cos \theta))^{\lambda+m-k+1}} \right| \leq C t^{2\lambda+m-k+1} \]
\[ \left| \frac{yz}{(t^2(1+v)^2 + (y-z)^2 + yz(1-\cos \theta))^{\lambda+m-k+1}} \right| \leq C t^{2\lambda+m-k+1} \]
\[ \theta^{2\lambda+1} \]

because \( |1 - \cos \theta - \theta^2/2| \leq C \theta^4, \theta \in (0, \pi/2), t, y \in (0, \infty) \),

By proceeding as in the previous case we get
\[ \left( \int_{\Gamma_+(x)} \left| S_2(y, z, t) \right|^2 \frac{dy dt}{t^2} \right)^{1/2} \leq C \left( 1 + \log + \frac{z}{|x-z|} \right). \]

Finally we consider \( S_3 \). From (35) it follows that
\[ S_3(y, z, t) = - \sum_{k=0}^{\frac{m+1}{2}} \frac{b_k^k}{t^{2\lambda+1}} \varphi \lambda \int_{\pi/2}^{\infty} \theta^{2\lambda-1} \varphi \lambda \left( \frac{\sqrt{(y-z)^2 + yz^2}}{t} \right) d\theta, \quad y, t \in (0, \infty), \]
where $m \in \mathbb{N}$ is such that $m - 1 \leq \beta < m$. Let $k \in \mathbb{N}$, $0 \leq k \leq (m + 1)/2$. We can write

$$
\int_{\pi/2}^{\infty} \theta^{2\lambda - 1} \varphi_{\lambda,k} \left( \frac{\sqrt{(y-z)^2 + yz\theta^2}}{t} \right) d\theta
\leq \int_{0}^{\infty} \theta^{2\lambda - 1} \varphi_{\lambda,k} \left( \frac{\sqrt{(y-z)^2 + yz\theta^2}}{t} \right) d\theta
\leq \int_{0}^{\infty} \int_{\pi/2}^{\infty} \frac{\theta^{2\lambda - 1} \varphi_{\lambda,k} \left( \frac{\sqrt{(y-z)^2 + yz\theta^2}}{t} \right) d\theta}{(t^2(1+v)^2 + (y-z)^2 + yz\theta^2)^{\lambda - m + k + 1}} d\theta dv, \quad t, y \in (0, \infty).
$$

Then,

$$
\int_{\Gamma_+} \left( \frac{(yz)^\lambda}{t^{2\lambda + 1}} \int_{\pi/2}^{\infty} \theta^{2\lambda - 1} \varphi_{\lambda,k} \left( \frac{\sqrt{(y-z)^2 + yz\theta^2}}{t} \right) d\theta \right)^2 dy dt \leq \int_{0}^{\infty} \int_{\pi/2}^{\infty} \frac{\theta^{2\lambda - 1} \varphi_{\lambda,k} \left( \frac{\sqrt{(y-z)^2 + yz\theta^2}}{t} \right) d\theta}{(t^2(1+v)^2 + (y-z)^2 + yz\theta^2)^{\lambda - m + k + 1}} d\theta dv
\leq \int_{0}^{\infty} \int_{\pi/2}^{\infty} \frac{\theta^{2\lambda - 1} \varphi_{\lambda,k} \left( \frac{\sqrt{(y-z)^2 + yz\theta^2}}{t} \right) d\theta}{(t^2(1+v)^2 + (y-z)^2 + yz\theta^2)^{\lambda - m + k + 1}} d\theta dv
\leq \int_{0}^{\infty} \frac{\varphi_{\lambda,k} \left( \frac{\sqrt{(y-z)^2 + yz\theta^2}}{t} \right) d\theta}{(t^2(1+v)^2 + (y-z)^2 + yz\theta^2)^{\lambda - m + k + 1}} d\theta dv.
$$

Since $x/2 < z < 2x$, if $y < z$, then $y < x/2$ and we have that

$$
\int_{0}^{\infty} \frac{\varphi_{\lambda,k} \left( \frac{\sqrt{(y-z)^2 + yz\theta^2}}{t} \right) d\theta}{(t^2(1+v)^2 + (y-z)^2 + yz\theta^2)^{\lambda - m + k + 1}} d\theta dv
\leq \int_{0}^{\infty} \frac{\varphi_{\lambda,k} \left( \frac{\sqrt{(y-z)^2 + yz\theta^2}}{t} \right) d\theta}{(t^2(1+v)^2 + (y-z)^2 + yz\theta^2)^{\lambda - m + k + 1}} d\theta dv
\leq \int_{0}^{\infty} \frac{\varphi_{\lambda,k} \left( \frac{\sqrt{(y-z)^2 + yz\theta^2}}{t} \right) d\theta}{(t^2(1+v)^2 + (y-z)^2 + yz\theta^2)^{\lambda - m + k + 1}} d\theta dv.
$$

Also we get

$$
\int_{\infty}^{\infty} \frac{\varphi_{\lambda,k} \left( \frac{\sqrt{(y-z)^2 + yz\theta^2}}{t} \right) d\theta}{(t^2(1+v)^2 + (y-z)^2 + yz\theta^2)^{\lambda - m + k + 1}} d\theta dv
\leq \int_{0}^{\infty} \frac{\varphi_{\lambda,k} \left( \frac{\sqrt{(y-z)^2 + yz\theta^2}}{t} \right) d\theta}{(t^2(1+v)^2 + (y-z)^2 + yz\theta^2)^{\lambda - m + k + 1}} d\theta dv.
$$

Hence,

$$
\left( \int_{\Gamma_+} \left( \frac{(yz)^\lambda}{t^{2\lambda + 1}} \int_{\pi/2}^{\infty} \theta^{2\lambda - 1} \varphi_{\lambda,k} \left( \frac{\sqrt{(y-z)^2 + yz\theta^2}}{t} \right) d\theta \right)^2 dy dt \right)^{1/2}
\leq C \int_{0}^{\infty} \frac{\varphi_{\lambda,k} \left( \frac{\sqrt{(y-z)^2 + yz\theta^2}}{t} \right) d\theta}{(t^2(1+v)^2 + (y-z)^2 + yz\theta^2)^{\lambda - m + k + 1}} d\theta dv
\leq C \left( \int_{0}^{\infty} \frac{\varphi_{\lambda,k} \left( \frac{\sqrt{(y-z)^2 + yz\theta^2}}{t} \right) d\theta}{(t^2(1+v)^2 + (y-z)^2 + yz\theta^2)^{\lambda - m + k + 1}} d\theta dv \right)^{1/2}
\leq C \int_{0}^{\infty} \frac{\varphi_{\lambda,k} \left( \frac{\sqrt{(y-z)^2 + yz\theta^2}}{t} \right) d\theta}{(t^2(1+v)^2 + (y-z)^2 + yz\theta^2)^{\lambda - m + k + 1}} d\theta dv.
$$

We have obtained that

$$
(38) \quad \left( \int_{\Gamma_+} |S \delta \psi |^2 dy dt \right)^{1/2} \leq C \int_{\Gamma_+} \left( \frac{\varphi_{\lambda,k} \left( \frac{\sqrt{(y-z)^2 + yz\theta^2}}{t} \right) d\theta}{(t^2(1+v)^2 + (y-z)^2 + yz\theta^2)^{\lambda - m + k + 1}} d\theta dv \right)^{1/2}
\leq C \frac{z}{z}.
$$

By combining (36), (37) and (38) we conclude the proof of this lemma.

**Lemma 5.4.** Let $B$ be a UMD Banach space, $1 < p < \infty$ and $\beta, \lambda > 0$. Then,

$$
\left\| \partial_\beta^\lambda P_t^B \right\|_{L^2((0,\infty),B)} \leq \|f\|_{L^p((0,\infty),B)}, \quad f \in C_c^\infty(0,\infty) \odot B.
$$


Proof. Assume that \( f = \sum_{j=1}^{n} f_j b_j \), where \( f_j \in C_c^\infty(0, \infty) \) and \( b_j \in \mathcal{B}, \ j = 1, \ldots, n \). We denote by \( f_o \) the odd extension of \( f \) to \( \mathbb{R} \); that is,
\[
  f_o(x) = \begin{cases} 
    f(x), & x > 0, \\
    -f(-x), & x \leq 0.
  \end{cases}
\]

We have that
\[
P_t(f_o)(y) = \int_{-\infty}^{+\infty} P_t(y - z)f(z)dz = \int_{0}^{\infty} (P_t(y - z) - P_t(y + z))f(z)dz, \quad y \in \mathbb{R} \text{ and } t > 0.
\]
According to Lemmas 5.1, 5.2 and 5.3 it follows that
\[
  \|t^3 \partial_t^3 |P_t|_2^\mathbb{R}(y, z) - P_t(y - z)|\chi_{\Gamma^+}(y, t)\|_{H^+} \leq CK(x, z),
\]
where
\[
  K(x, z) = \begin{cases} 
    \frac{1}{x}, & 0 < z < \frac{x}{2}, \\
    \frac{1}{2} \left(1 + \log_+ \frac{z}{|x - z|}\right), & 0 < x/2 < z < 2x, \\
    \frac{1}{z}, & 0 < 2x < z.
  \end{cases}
\]

For every \( x \in (0, \infty) \),
\[
  d\mu_+(z) = \frac{1}{C_0} \left(1 + \log_+ \frac{z}{|x - z|}\right) \chi_{\{(x/2,x)\}}(z) \frac{dz}{z},
\]
is a probability measure on \((0, \infty)\), where
\[
  C_0 = \int_{1/2}^2 \left(1 + \log_+ \frac{u}{|1 - u|}\right) \frac{du}{u}.
\]

Then, by using Jensen inequality and by invoking Hardy inequalities ([58, p. 20]), we deduce that
\[
  \|t^3 \partial_t^3 |P_t|_2^\mathbb{R}(f) - P_t(f_o)|\chi_{\Gamma^+}(y, t)\|_{L^p((0, \infty), L^2((0, \infty)^2, \frac{dx dy dt}{|x|})}, \quad 1 < q < \infty.
\]
(39)

where
\[
  K(g)(x) = \int_{0}^{\infty} K(x, z)g(z)dz, \quad x \in (0, \infty),
\]
for every \( g \in L^p(0, \infty) \).

Note that, in particular, according to Lemma 2.1 \( t^3 \partial_t^3 P_t(f)\chi_{\Gamma^+}(x) \in L^q((0, \infty), H^+) \otimes \mathcal{B} \), \( 1 < q < \infty \).

It is clear that if \{\{h_j\}_{j=1}^{\infty}\} is an orthonormal system in \( H^+ \) and, for every \( j \in \mathbb{N} \), we define \( \tilde{h}_j \) by
\[
  \tilde{h}_j(y, t) = \begin{cases} 
    h_j(y, t), & y, t > 0, \\
    0, & t > 0, y \leq 0,
  \end{cases}
\]
then, \{\{\tilde{h}_j\}_{j=1}^{\infty}\} is an orthonormal system in \( L^2(\mathbb{R} \times (0, \infty), \frac{dx dy dt}{|x|}) \). Hence, since \( \mathcal{B} \) is UMD we have that
\[
  \left\|t^3 \partial_t^3 P_t(f_o)(y)\chi_{\Gamma^+}(y, t)\right\|_{H^+} \leq C\left\|\sum_{j=1}^{\infty} \gamma_j [t^3 \partial_t^3 P_t(f_o)(y)\chi_{\Gamma^+}(y, t)](h_j)\right\|_{\mathcal{B}}^{1/2}
\]
\[
  = \sup \left(\mathbb{E} \sum_{j=1}^{\ell} \gamma_j [t^3 \partial_t^3 P_t(f_o)(y)\chi_{\Gamma^+}(y, t)](h_j)\right)^{1/2}
\]
\[
  \leq \sup \left(\mathbb{E} \sum_{j=1}^{\ell} \gamma_j [t^3 \partial_t^3 P_t(f_o)(y)\chi_{\Gamma^+}(y, t)](h_j)\right)^{1/2}
\]
\[
  \leq \left\|t^3 \partial_t^3 P_t(f_o)(y)\chi_{\Gamma^+}(y, t)\right\|_{L^2(\mathbb{R} \times (0, \infty), \frac{dx dy dt}{|x|})}, \quad a.e. \ x \in (0, \infty),
\]
where the two first supremum are taken over all orthonormal systems \{\{h_j\}_{j=1}^{\ell}\} in \( H^+ \) and the last one over all orthonormal systems \{\{e_j\}_{j=1}^{\ell}\} in \( L^2(\mathbb{R} \times (0, \infty), \frac{dx dy dt}{|x|}) \).
By Lemma 2.1

\[ \|t^\beta \partial_\beta^2 P_{t}^{B\lambda}(f)\|_{L^2((0,\infty),\mathbb{B})} \leq C f \|_{L^p((0,\infty),\mathbb{B})} = C \|f\|_{L^p((0,\infty),\mathbb{B})} \]

Also, as in (39), since \( \gamma(H_+,C) = H_+ \), we get

\[ \|t^\beta \partial_\beta^2 P_{t}^{B\lambda}(f)\|_{L^2((0,\infty),\mathbb{B})} \leq C t^\lambda \|f\|_{L^p((0,\infty),\mathbb{B})} \]

\[ \leq C \| \Lambda(\{f\mu\}) \|_{L^p((0,\infty),\mathbb{B})} \]

Combining (40) and (41) we conclude the proof of this lemma.

From Lemma 5.4 and using the estimate below (Lemma 5.5), we can proceed as in the proof of Lemma 2.1 to obtain (33).

**Lemma 5.5.** Let \( \beta, \lambda > 0 \). Then,

\[ \|t^\beta \partial_\beta^2 P_{t}^{B\lambda}(x,y)\| \leq C \frac{t^\lambda}{(t + |x - y|)^{\beta + 1}}, \quad t, x, y \in (0, \infty). \]

**Proof.** Let \( m \in \mathbb{N} \) such that \( m - 1 \leq \beta < m \) and \( k \in \mathbb{N} \), verifying \( 0 \leq k \leq (m + 1)/2 \). According to the formula (35), it is enough to estimate the following expression,

\[ \int_0^\pi \left( \frac{(xy)z}{t^{2\lambda+1}} \right)^2 (\sin \theta)^{2\lambda-1} e^{\lambda, k} \left( \frac{\sqrt{(x - y)^2 + 2xy(1 - \cos \theta)}}{t} \right) d\theta \]

\[ \leq \int_0^\pi (\sin \theta)^{2\lambda-1} \int_0^{\infty} \frac{(1 + v)^{m-1-2k}v^{\lambda - 1}}{(t^2 + 2xy(1 - \cos \theta))^\lambda - m - k + 1} d\lambda d\theta \]

\[ \leq C \int_0^\pi (\sin \theta)^{2\lambda-1} \int_0^{\infty} \frac{(t^2 + 2xy(1 - \cos \theta))^\lambda - m - k + 1} d\lambda d\theta \]

\[ \leq C \int_0^\pi \frac{1}{(t^2 + 2xy(1 - \cos \theta))^\lambda - m - k + 1} d\lambda d\theta \]

\[ \geq C \frac{t^\beta}{(t + |x - y|)^{\beta + 1}}, \quad t, x, y \in (0, \infty). \]

\[ \square \]

5.2. Now we are going to show that

\[ \|f\|_{L^p((0,\infty),\mathbb{B})} \leq C \|t^\beta \partial_\beta^2 P_{t}^{B\lambda}(f)\|_{L^2((0,\infty),\mathbb{B})}, \quad f \in L^p((0,\infty),\mathbb{B}) \]

If \( f \in C^\infty_c(0, \infty) \) then, according to [5] Lemma 3.1] we have that

\[ h_\lambda(t^\beta \partial_\beta^2 P_{t}^{B\lambda}(f))(x) = e^{i\beta t} (tx)^\beta e^{-t^2} h_\lambda(f)(x), \quad t, x \in (0, \infty). \]

Suppose that \( f, g \in C^\infty_c(0, \infty) \). We denote by \( J_t(y) = \{ x \in (0, \infty) : |x - y| < t \}, t, y \in (0, \infty) \). Plancherel equality for Hankel transform leads to

\[ \int_0^\infty \int_{J_t(y)} t^\beta \partial_\beta^2 P_{t}^{B\lambda}(f)(y) \partial_\beta^2 P_{t}^{B\lambda}(g)(y) \frac{dy dt}{t |J_t(y)|} dx \]

\[ = \int_0^\infty \int_0^\infty h_\lambda e^{i\beta t} (tx)^\beta e^{-t^2} h_\lambda(f)(y) h_\lambda e^{i\beta t} (tx)^\beta e^{-t^2} h_\lambda(g)(y) \int_{J_t(y)} dx \frac{dy}{t |J_t(y)|} dt \]

\[ = \int_0^\infty \int_0^\infty e^{2i\beta t} (ty)^\beta e^{-2ty} h_\lambda(f)(y) h_\lambda(g)(y) \frac{dy dt}{t} \]

\[ = e^{2i\beta t} \int_0^\infty h_\lambda(f)(y) h_\lambda(g)(y) y^\beta \int_0^\infty e^{-2ty} t^{2\beta - 1} dt dy \]
Suppose now that \( f \in C_c^\infty(0, \infty) \otimes B \) and \( g \in C_c^\infty(0, \infty) \otimes B^* \). From [42] it follows that
\[
\int_0^\infty \langle g(x), f(x) \rangle_{B^* B} dx = \frac{e^{-2i\pi \beta}}{\Gamma(2\beta)} \int_0^\infty \int_{\Gamma_+(x)} \langle t^\beta \partial_t^\beta P_t^{2\lambda}(g)(y), t^\beta \partial_t^\beta P_t^{2\lambda}(f)(y) \rangle_{B^* B} \frac{dy}{y} dt dx.
\]
Then, since \( |J_t(y)| \geq t \), for every \( t, y \in (0, \infty) \), according to [32] Proposition 2.2 we get
\[
\left| \int_0^\infty \langle g(x), f(x) \rangle_{B^* B} dx \right| \leq C \int_0^\infty \int_{\Gamma_+(x)} \| t^\beta \partial_t^\beta P_t^{2\lambda}(g)(y) \|_{\gamma(H^+, B)} \| t^\beta \partial_t^\beta P_t^{2\lambda}(f)(y) \|_{\gamma(H^+, B^*)} \frac{dy}{y} dt dx.
\]
\[
\leq C \| t^\beta \partial_t^\beta P_t^{2\lambda}(f) \|_{T^2((0, \infty), B)} \| t^\beta \partial_t^\beta P_t^{2\lambda}(g) \|_{T^2((0, \infty), B^*)}.
\]
where \( p' = \frac{p}{p-1} \).

Since \( B^* \) is UMD, by using [33] where \( p' \) is replaced by \( p' \), we obtain
\[
\left| \int_0^\infty \langle g(x), f(x) \rangle_{B^* B} dx \right| \leq C \| t^\beta \partial_t^\beta P_t^{2\lambda}(f) \|_{T^2((0, \infty), B)} \| g \|_{L^{p'}((0, \infty), B^*)}.
\]
From [24] Lemma 2.3 and by taking into account that \( C_c^\infty(0, \infty) \otimes B \) is dense in \( L^{p'}((0, \infty), B^*) \) we get
\[
\| f \|_{L^{p'}((0, \infty), B)} \leq C \| t^\beta \partial_t^\beta P_t^{2\lambda}(f) \|_{T^2((0, \infty), B)}, \quad f \in C_c^\infty(0, \infty) \otimes B.
\]
By using again [33] and the fact that \( C_c^\infty(0, \infty) \otimes B \) is dense in \( L^p((0, \infty), B) \) we conclude that
\[
\| f \|_{L^p((0, \infty), B)} \leq C \| t^\beta \partial_t^\beta P_t^{2\lambda}(f) \|_{T^2((0, \infty), B)}, \quad f \in L^p((0, \infty), B).
\]

6. PROOF OF THEOREM 4

6.1. We are going to show that
\[
\| t^\beta \partial_t^\beta P_t^{2\lambda}(f) \|_{T^2((0, \infty), B)} \leq C \| f \|_{L^p((0, \infty), B)}, \quad f \in L^p((0, \infty), B).
\]
We have that
\[
L_\alpha = - \frac{d^2}{dx^2} + V(x), \quad x \in (0, \infty),
\]
where \( V(x) = x^2 + (\alpha^2 - 1/4) x^2 \). Then, \( \partial_t P_t^{2\alpha}(1) \neq 0 \) and [31] Theorem 4.8 can not be applied to prove [33]. The strategy will be the same as in previous sections: comparing with an operator whose boundedness property is already known. This time we are going to relate the operator \( t^\beta \partial_t^\beta P_t^{2\alpha} \) with \( t^\beta \partial_t^\beta P_t^{2\alpha} \), studied in Section 6.

Recall (7) for the definition of the Poisson Laguerre semigroup. It is given via subordination with respect to the heat Laguerre semigroup \( W_t^{2\alpha} \), in which it is involved the modified Bessel function \( I_\alpha \) (see [6]). We will use the following properties of \( I_\alpha \), that can be encountered in [35] Ch. 5],
\[
I_\alpha(z) \sim \frac{z^\alpha}{2^\alpha \Gamma(\alpha + 1)}, \quad \text{as } z \to 0^+,
\]
\[
\sqrt{2} I_\alpha(z) = \frac{e^z}{\sqrt{2\pi}} \left( 1 + O\left( \frac{1}{z} \right) \right), \quad z \in (0, \infty).
\]
From (44) and (45) we easily deduce that
\[
0 \leq W_t^{2\alpha}(x, y) \leq C e^{-(x-y)^2/2t}, \quad t, x, y \in (0, \infty).
\]
Let \( f \in C_c^\infty(0, \infty) \otimes B \). We define the measurable function \( f_0 \) by
\[
f_0(x) = \begin{cases} f(x), & x > 0 \\ 0, & x \leq 0. \end{cases}
\]
By using (29), (25), (46) and Hölder’s inequality, it is not difficult to justify that
\[
t^\beta \partial_t^\beta P_t^{2\alpha}(f)(y) = \int_0^\infty t^\beta \partial_t^\beta P_t^{2\alpha}(y, z) f(z) dz, \quad t, y \in (0, \infty),
\]
and
\[
t^\beta \partial_t^\beta P_t^{2\alpha}(f_0)(y) = \int_0^\infty t^\beta \partial_t^\beta P_t^{2\alpha}(y, z) f(z) dz, \quad t, y \in (0, \infty).
\]
Here, $P_t^\mathcal{H}(x, y)$ denotes the Poisson kernel associated with the Hermite operator on $\mathbb{R}$.
For every $x \in (0, \infty)$, we write the following decomposition

$$t^\beta \partial_t^\beta [P_t^\mathcal{H}(f)](y) = P_t^\mathcal{H}(f)(y)\chi_{\Gamma_+(x)}(y, t)$$

Let $\gamma_1$ be a Banach space and $\gamma_2$ be a Banach space and

$$\|K_{\text{glob}}^j(f)\|_{L^p((0, \infty), \gamma(H_+, \mathbb{B}))} \leq C\|f\|_{L^p((0, \infty), \mathbb{B})}, \quad f \in C_c^\infty(0, \infty) \otimes \mathbb{B}.$$ 

Proof. Let $f \in C_c^\infty(0, \infty) \otimes \mathbb{B}$. By using (20), (28) and (46) we deduce that

$$|t^\beta \partial_t^\beta [P_t^\mathcal{H}(y, z) - P_t^\mathcal{H}(y, z)]| \leq C \int_0^\infty \left| t^\beta \partial_t^\beta [u - t^{\beta/4}] \right| |W_t^\mathcal{H}(y, z) - W_t^\mathcal{H}(y, z)| du$$

Hence we obtain

$$\|t^\beta \partial_t^\beta [P_t^\mathcal{H}(y, z) - P_t^\mathcal{H}(y, z)]\chi_{\Gamma_+(x)}(y, t)\|_{H_+} \leq C \left( \int_0^\infty \int_{|y - z| < \xi} (t + |y - z|)^{\beta+2} dy dt \right)^{1/2}$$

Then, since $\gamma(H_+, \mathbb{C}) = H_+$, we get

$$\|K_{\text{glob}}^1(f)(x, y, t)\|_{L^p((0, \infty), \gamma(H_+, \mathbb{B}))} + \|K_{\text{glob}}^2(f)(x, y, t)\|_{L^p((0, \infty), \gamma(H_+, \mathbb{B}))}$$

By using Hardy inequalities (28, p. 20) we conclude that

$$\|K_{\text{glob}}^j(f)\|_{L^p((0, \infty), \gamma(H_+, \mathbb{B}))} \leq C\|f\|_{L^p((0, \infty), \mathbb{B})}, \quad j = 1, 2.$$ 

Lemma 6.2. Let $\mathbb{B}$ be a Banach space and $1 < p < \infty$. Then,

$$\|K_{\text{loc}}(f)\|_{L^p((0, \infty), \gamma(H_+, \mathbb{B}))} \leq C\|f\|_{L^p((0, \infty), \mathbb{B})}, \quad f \in C_c^\infty(0, \infty) \otimes \mathbb{B}.$$ 

Proof. Let $f \in C_c^\infty(0, \infty) \otimes \mathbb{B}$. To simplify notation, we call

$$\xi = \xi(u, y, z) = \frac{2yz e^{-2u}}{1 - e^{-4u}}, \quad u, y, z \in (0, \infty).$$

We make the following decomposition

$$t^\beta \partial_t^\beta [P_t^\mathcal{H}(y, z) - P_t^\mathcal{H}(y, z)]$$

$$= \frac{t^\beta}{2 \sqrt{\pi}} \left( \int_{\{u \in (0, \infty) : \xi \leq 1\}} + \int_{\{u \in (0, \infty) : \xi \geq 1\}} \right) \partial_t^\beta [u - t^{\beta/4}] \left[ W_t^\mathcal{H}(y, z) - W_t^\mathcal{H}(y, z) \right] du$$

$$= I_1(y, z, t) + I_2(y, z, t), \quad t, y, z \in (0, \infty).$$
According to [20] and [44] we get

\[ |I_1(y, z, t)| \leq C t^{\frac{\beta}{2}} \int_{\{u \in (0, \infty) : \xi < 1\}} \frac{e^{-t^2/8u}}{u^{(\beta+2)/2}} \left( \frac{e^{-2u}}{1 - e^{-4u}} \right)^{1/2} \exp \left( -\frac{1}{2} (y^2 + z^2) \frac{1 + e^{-4u}}{1 - e^{-4u}} \right) \left[ u^{\alpha+1/2} + e^{\xi} \right] du \]

\[ \leq C t^{\frac{\beta}{2}} \int_{0}^{\infty} \frac{e^{-t^2/8u} e^{-u e^{-c(y-z)^2/u}}}{u^{(\beta+2)/2}} \frac{u^1/2}{\xi^{1/2}} \xi \] \( t, z, y \in (0, \infty). \)

By proceeding as in [48] we obtain

\[ \|I_1(y, z, t)\chi_{\Gamma_+(x)}(y, t)\|_{H^+} \leq \frac{C}{x} \quad 0 < x/2 < z < 2x. \]

On the other hand, [20] and [45] lead to

\[ |I_2(y, z, t)| \leq C t^{\frac{\beta}{2}} \int_{\{u \in (0, \infty) : \xi < 1\}} \frac{e^{-t^2/8u} W_{\mbox{c}}(y, z)}{u^{(\beta+2)/2}} \frac{u^{1/2}}{\xi^{1/2}} \xi \]

\[ \leq C t^{\frac{\beta}{2}} \int_{\{u \in (0, \infty) : \xi < 1\}} \frac{e^{-t^2/8u} e^{-u e^{-c(y-z)^2/u}}}{u^{(\beta+2)/2}} \frac{u^1/2}{\xi^{1/2}} \xi \]

\[ \leq C t^{\frac{\beta}{2}} \int_{0}^{\infty} \frac{e^{-t^2/8u} e^{-u e^{-c(y-z)^2/u}}}{u^{(\beta+2)/2}} \frac{u^1/2}{\xi^{1/2}} \xi \] \( t, z, y \in (0, \infty). \)

Then, by choosing 0 < \( \varepsilon < \min\{2/3, 2\beta\} \) we can write

\[ \|I_2(y, z, t)\chi_{\Gamma_+(x)}(y, t)\|_{H^+} \leq C \left( \int_{\Gamma_+(x)} \frac{\xi^{2\beta-2}}{(y^2/3(t + |y-z|))^{2\beta+1/3}} dt dy \right)^{1/2} \]

\[ \leq C \left( \int_{0}^{z/4} + \int_{z/4}^{\infty} \int_{-y}^{\infty} \frac{\xi^{2\beta-2}}{(y^2/3(t + |y-z|))^{2\beta+1/3}} dt dy \right)^{1/2} \]

\[ \leq C \left( \int_{0}^{z/4} \int_{y/2}^{\infty} \frac{\xi^{2\beta-2}}{(y^2/3(2\beta+3))} + \int_{z/4}^{\infty} \int_{-y}^{\infty} \frac{\xi^{2\beta-2}}{(y^2/3(t + |y-z|))^{2\beta+1/3}} dt dy \right)^{1/2} \]

\[ \leq C \left( \int_{0}^{z/4} \frac{\xi^{2\beta-2}}{(y^2/3(2\beta+3))} + \int_{z/4}^{\infty} \frac{\xi^{2\beta-2}}{(y^2/3(t + |y-z|))^{2\beta+1/3}} dt dy \right)^{1/2} \]

\[ \leq C \left( \frac{1}{x^{\varepsilon/3/2}} + \frac{x^{\varepsilon/3/2}}{x^{\varepsilon/3/2}} \right)^{1/2} \leq C \left( 1 + \left( \frac{x}{|x-z|} \right)^{(3\varepsilon+2)/6} \right) \quad 0 < x/2 < z < 2x. \]

By combining (49) and (50) we get

\[ \|t^\beta \partial_t^\beta [P_t^{\mbox{c}}(y, z) - P_t^{\mbox{c}}(y, z)] \chi_{\Gamma_+(x)}(y, t)\|_{H^+} \leq C \left( 1 + \left( \frac{x}{|x-z|} \right)^{(3\varepsilon+2)/6} \right) \quad 0 < x/2 < z < 2x. \]

For every \( x \in (0, \infty), \)

\[ d_{\mu_0}(x) = \frac{1}{C_0} \left( 1 + \left( \frac{x}{|x-z|} \right)^{(3\varepsilon+2)/6} \right) \chi_{(z, 2x)}(z) \frac{dz}{z} \]

is a probability measure on \( (0, \infty) \) when

\[ C_0 = \int_{1/2}^{2} \left( 1 + \frac{1}{(1-u)^{(3\varepsilon+2)/6}} \right) du. \]

Then, since \( \gamma(H^+, \mathbb{C}) = H^+, \) Jensen’s inequality leads to

\[ \|K_{\mbox{loc}}(f)(x, \cdot, \cdot)\|_{H^+} \leq C \left( \int_{x/2}^{2x} \|f(z)\| \left[ \partial_t^\beta \partial_t^\beta [P_t^{\mbox{c}}(y, z) - P_t^{\mbox{c}}(y, z)] \chi_{\Gamma_+(x)}(y, t)\|_{H^+} \right] dz \right)^p \]

\[ \leq C \left( \frac{1}{x} \int_{x/2}^{2x} \|f(z)\| \left( 1 + \left( \frac{x}{|x-z|} \right)^{(3\varepsilon+2)/6} \right) dz \right)^p \]
Finally, as in the proof of Lemma 2.1, we can deduce that
\[ \text{Lemma } 6.3. \]
As in Section 5.2, by duality and density arguments, it is enough to have the following identity.

Moreover, according to Theorem 2, \( \| \partial^3 \partial^3_t P_t^{Z^\alpha} (f_0) \|_{T^2_2((0, \infty), \mathcal{B})} \) \( \leq \| f \|_{L^p((0, \infty), \mathcal{B})} \), \( f \in C_c((0, \infty)) \otimes \mathcal{B} \).

Hence,
\[ \| \partial^3 \partial^3_t P_t^{Z^\alpha} (f) \|_{T^2_2((0, \infty), \mathcal{B})} \leq \| f \|_{L^p((0, \infty), \mathcal{B})}, \quad f \in C_c((0, \infty)) \otimes \mathcal{B}. \]

By proceeding as in \( [47] \) we get
\[ |\partial^3 \partial^3_t P_t^{Z^\alpha} (y, z)| \leq C \frac{t^\beta}{(t + |y - z|)^{3\beta + 1}}, \quad t, y, z \in (0, \infty). \]

Finally, as in the proof of Lemma 2.1 we can deduce that
\[ \| t^{\beta} \partial^3 \partial^3_t P_t^{Z^\alpha} (f) \|_{T^2_2((0, \infty), \mathcal{B})} \leq \| f \|_{L^p((0, \infty), \mathcal{B})}, \quad f \in L^p((0, \infty), \mathcal{B}). \]

6.2. In this paragraph we establish that
\[ \| f \|_{L^p((0, \infty), \mathcal{B})} \leq C \| t^{\beta} \partial^3 \partial^3_t P_t^{Z^\alpha} (f) \|_{T^2_2((0, \infty), \mathcal{B})}, \quad f \in L^p((0, \infty), \mathcal{B}). \]

As in Section 5.2 by duality and density arguments, it is enough to have the following identity.

**Lemma 6.3.** Let \( \alpha, \beta > 0 \) and \( f, g \in C_c((0, \infty)) \). Then,
\[ \int_0^\infty \int_{J_t(y)} t^\beta \partial^3 \partial^3_t P_t^{Z^\alpha} (f)(y) t^\beta \partial^3 \partial^3_t P_t^{Z^\alpha} (g)(y) \frac{dy dt}{t J_t(y)} dx = \frac{e^{2 \pi i \beta \Gamma(2 \beta)}}{2^{2 \beta}} \int_0^\infty f(x) g(x) dx, \]
where \( J_t(y) = \{ x \in (0, \infty) : |x - y| < t \} \).

**Proof.** We can write
\[ P_t^{Z^\alpha} (f)(x) = \sum_{k=0}^\infty e^{-t \sqrt{2k + \alpha + 1}} c_k^\alpha(f) \varphi_k^\alpha(x), \quad t, x \in (0, \infty). \]

According to \( [40] \) p. 1124 there exists \( C > 0 \) such that \( |\varphi_k^\alpha(x)| \leq C, \ x \in (0, \infty) \). Moreover, for every \( m \in \mathbb{N} \) there exists \( C > 0 \) such that \( |c_k^\alpha(f)| \leq C(1 + k)^{-m}, \ k \in \mathbb{N} \). Hence, the series in \( [51] \) converges in \( L^2(0, \infty) \) and uniformly in \( (t, x) \in (0, \infty) \times (0, \infty) \). Also, for every \( m \in \mathbb{N} \), we get
\[ \partial^m_t P_t^{Z^\alpha} (f)(x) = \sum_{k=0}^\infty (-1)^m (2k + \alpha + 1)^{m/2} e^{-t \sqrt{2k + \alpha + 1}} c_k^\alpha(f) \varphi_k^\alpha(x), \quad x, t \in (0, \infty). \]

If \( m \in \mathbb{N} \) is such that \( m - 1 \leq \beta < m \), then
\[ \partial^m_t P_t^{Z^\alpha} (f)(x) = \frac{e^{-i \pi (m - \beta)}}{\Gamma(m - \beta)} \int_0^\infty \partial^m_t P_t^{Z^\alpha} (f)(x) s^{m - \beta - 1} ds \]
\[ = \frac{e^{-i \pi (m - \beta)}}{\Gamma(m - \beta)} \int_0^\infty s^{m - \beta - 1} \left( \sum_{k=0}^\infty (-1)^m (2k + \alpha + 1)^{m/2} e^{-t (s + \sqrt{2k + \alpha + 1})} c_k^\alpha(f) \varphi_k^\alpha(x) \right) \int_0^\infty s^{m - \beta - 1} e^{-s \sqrt{2k + \alpha + 1}} ds \]
\[ = \frac{e^{-i \pi (m - \beta)}}{\Gamma(m - \beta)} \sum_{k=0}^\infty (-1)^m (2k + \alpha + 1)^{m/2} e^{-t (s + \sqrt{2k + \alpha + 1})} c_k^\alpha(f) \varphi_k^\alpha(x) \int_0^\infty s^{m - \beta - 1} e^{-s \sqrt{2k + \alpha + 1}} ds \]
The interchange between the series and the integral is justified because \((c_k^j(f))_{k \in \mathbb{N}}\) is rapidly decreasing as \(k \to \infty\). We have that

\[
\begin{align*}
&\int_0^\infty \int_{\mathbb{R}^+}(x) t^2 \frac{\partial}{\partial t^2} P_t Z_n(f)(y) t^2 \frac{\partial}{\partial t^2} P_t Z_n(g)(y) \frac{dy dt}{\Gamma(t)} - \int_0^\infty \int_{\mathbb{R}^+}(x) t^2 \frac{\partial}{\partial t^2} P_t Z_n(f)(y) t^2 \frac{\partial}{\partial t^2} P_t Z_n(g)(y) \frac{dy dt}{\Gamma(t)} \\
&= \int_0^\infty \int_{\mathbb{R}^+}(x) t^2 \frac{\partial}{\partial t^2} P_t Z_n(f)(y) t^2 \frac{\partial}{\partial t^2} P_t Z_n(g)(y) \frac{dy dt}{\Gamma(t)} - \int_0^\infty \int_{\mathbb{R}^+}(x) t^2 \frac{\partial}{\partial t^2} P_t Z_n(f)(y) t^2 \frac{\partial}{\partial t^2} P_t Z_n(g)(y) \frac{dy dt}{\Gamma(t)} \\
&= \frac{2i\pi j}{2j} \sum_{k=0}^\infty c_k^j(f) c_k^j(g) \int_0^\infty (t\sqrt{2k + \alpha + 1})^2 e^{-t\sqrt{2k + \alpha + 1}} c_k^n(f) \varphi_k^n(y) \frac{dy dt}{\Gamma(t)} \\
&= \frac{2i\pi j}{2j} \sum_{k=0}^\infty c_k^j(f) c_k^j(g) \int_0^\infty (t\sqrt{2k + \alpha + 1})^2 e^{-t\sqrt{2k + \alpha + 1}} c_k^n(f) \varphi_k^n(y) \frac{dy dt}{\Gamma(t)} \\
&= \frac{2i\pi j}{2j} \sum_{k=0}^\infty c_k^j(f) c_k^j(g) \int_0^\infty \int_0^\infty f(x) g(x) dx.
\end{align*}
\]

\(\square\)

**References**

[1] I. Abu-Falahah, P. R. Stinga, and J. L. Torrea, Square functions associated to Schrödinger operators, Studia Math., 203 (2011), pp. 171–194.

[2] P. Auscher and B. Ben Ali, Maximal inequalities and Riesz transform estimates on \(L^p\) spaces for Schrödinger operators with nonnegative potentials, Ann. Inst. Fourier (Grenoble), 57 (2007), pp. 1975–2013.

[3] J. J. Betancor, A. J. Castro, J. Curbera, J. C. Fariña, and L. Rodríguez-Mesa, Square functions in the Hermite setting for functions with values in \(UMD\) spaces. To appear in Ann. Mat. Pura Appl. [DOI: 10.1007/s10231-013-0335-9]

[4] J. J. Betancor, A. J. Castro, and L. Rodríguez-Mesa, Characterization of uniformly convex and smooth Banach spaces by using Carlson measures in Bessel settings, J. Convex Anal., 20 (2013), pp. 763–811.

[5] J. J. Betancor, Square functions and spectral multipliers for Bessel operators in \(UMD\) spaces. Preprint 2013 [arXiv:1303.3159v1]

[6] J. J. Betancor, A. Chicco Ruiz, J. C. Fariña, and L. Rodríguez-Mesa, Odd \(BMO(\mathbb{R})\) functions and Carlson measures in the Bessel setting, Integral Equations Operator Theory, 66 (2010), pp. 463–494.

[7] J. J. Betancor, R. Crescimbeni, J. C. Fariña, P. R. Stinga, and J. L. Torrea, A \(T\) criterion for Hermite-Calderón-Zygmund operators on the \(BMO_\mathbb{R}(\mathbb{R}^n)\) space and applications, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), 12 (2013), pp. 157–187.

[8] J. J. Betancor, J. C. Fariña, L. Rodríguez-Mesa, R. Testoni, and J. L. Torrea, Fractional square functions and potential spaces, J. Math. Anal. Appl., 386 (2012), pp. 487–504.

[9] J. J. Betancor, J. C. Fariña, T. Martínez, and L. Rodríguez-Mesa, Higher order Riesz transforms associated with Bessel operators, Ark. Mat., 46 (2008), pp. 219–250.

[10] J. J. Betancor, J. C. Fariña, T. Martínez, and J. L. Torrea, Riesz transforms and \(g\)-function associated with Bessel operators and their appropriate Banach spaces, Israel J. Math., 157 (2007), pp. 259–282.

[11] J. J. Betancor, J. C. Fariña, L. Rodríguez-Mesa, A. Sanabria, and J. L. Torrea, Transference between Laguerre and Hermite settings, J. Funct. Anal., 254 (2008), pp. 826–850.

[12] J. J. Betancor, E. Harboure, A. Nowak, and B. Viviani, Mapping properties of fundamental operators in harmonic analysis related to Bessel operators, Studia Math., 197 (2010), pp. 101–140.

[13] B. Bongioanni, E. Harboure, and O. Salinas, Riesz transforms related to Schrödinger operators acting on \(BMO\) type spaces, J. Math. Anal. Appl., 357 (2009), pp. 115–131.

[14] B. Bongioanni and J. L. Torrea, Sobolev spaces associated to the harmonic oscillator, Proc. Indian Acad. Sci. Math. Sci., 116 (2006), pp. 337–360.

[15] J. Bourgain, Some remarks on Banach spaces in which martingale difference sequences are unconditional, Ark. Mat., 21 (1983), pp. 163–168.

[16] D. L. Burkholder, A geometrical characterization of Banach spaces in which martingale difference sequences are unconditional, Ann. Probab., 9 (1981), pp. 997–1011.

[17] R. Coifman, Y. Meyer, and E. M. Stein, Some new function spaces and their applications to harmonic analysis, J. Funct. Anal., 62 (1985), pp. 304–335.

[18] U. Dinger, Weak type \((1,1)\) estimates of the maximal function for the Laguerre semigroup in finite dimensions, Rev. Math. Iberoamericana, 8 (1992), pp. 93–120.
[19] J. Dziubański, *Hardy spaces for Laguerre expansions*, Constr. Approx., 27 (2008), pp. 269–287.

[20] J. Dziubański, G. Garrigós, T. Martínez, J. L. Torrea, and J. Zienkiewicz, *BMO spaces related to Schrödinger operators with potentials satisfying a reverse Hölder inequality*, Math. Z., 249 (2005), pp. 329–356.

[21] J. Dziubański and P. Głowacki, *Sobolev spaces related to Schrödinger operators with polynomial potentials*, Math. Z., 262 (2009), pp. 881–894.

[22] J. Dziubański and J. Zienkiewicz, *Hardy space $H^1$ associated to Schrödinger operator with potential satisfying reverse Hölder inequality*, Rev. Mat. Iberoamericana, 15 (1999), pp. 279–296.

[23] J. Dziubański, G. Garrigós, T. Martínez, J. L. Torrea, and J. Zienkiewicz, *Higher order Riesz transforms, fractional derivatives, and Sobolev spaces for Laguerre expansions*, J. Math. Pures Appl. (9), 84 (2005), pp. 375–405.

[24] L. Grafakos, L. Liu, and D. Yang, *Vector-valued singular integrals and maximal functions on spaces of homogeneous type*, Math. Scand., 104 (2009), pp. 296–310.

[25] C. E. Gutiérrez, A. Incognito, and J. L. Torrea, *Riesz transforms, g-functions, and multipliers for the Laguerre semi-group*, Houston J. Math., 27 (2001), pp. 579–592.

[26] E. Harboure, J. L. Torrea, and B. Viviani, *Vector-valued extensions of operators related to the Ornstein-Uhlenbeck semi-group*, J. Anal. Math., 91 (2003), pp. 1–29.

[27] J. Hoffmann-Jørgensen, *Sums of independent Banach space valued random variables*, Studia Math., 52 (1974), pp. 159–186.

[28] T. Hytönen, J. Van Neerven, and P. Portal, *Conical square function estimates in UMD Banach spaces and applications to $H^\infty$-functional calculi*, J. Funct. Anal., 256 (2009), pp. 317–351.

[29] T. Hytönen and L. Weis, *The Banach space-valued BMO, Carleson’s condition, and paraproducts*, J. Fourier Anal., 16 (2010), pp. 495–513.

[30] M. Kemppainen, *Vector-valued tent spaces and Hardy spaces associated with non-negative self-adjoint operators*. Preprint 2014 [arXiv:1402.2686v1]

[31] S. Kwapień, *Isomorphic characterizations of inner product spaces by orthogonal series with vector valued coefficients*, Studia Math., 44 (1974), pp. 583–595.

[32] N. N. Lebedev, *Special functions and their applications*, Dover Publications Inc., New York, 1972.

[33] T. Martínez, J. L. Torrea, and Q. Xu, *Vector-valued Littlewood-Paley-Stein theory for semigroups*, Adv. Math., 203 (2006), pp. 430–475.

[34] B. Muckenhoupt, *Hermite conjugate expansions*, Trans. Amer. Math. Soc., 139 (1969), pp. 243–260.

[35] B. Muckenhoupt and D. W. Webb, *Two-weight norm inequalities for the Cesàro means of Hermite expansions*, Trans. Amer. Math. Soc., 354 (2002), pp. 4525–4537.

[36] A. Nowak and K. Stempak, *Riesz transforms for multi-dimensional Laguerre function expansions*, Adv. Math., 215 (2007), pp. 642–678.

[37] J. L. Rubio de Francia, *Martingale and integral transforms of Banach space valued functions*, in Probability and Banach spaces (Zaragoza, 1985), vol. 1221 of Lecture Notes in Math., Springer, Berlin, 1986, pp. 195–222.

[38] C. Segovia and R. L. Wheeden, *On certain fractional area integrals*, J. Math. Mech., 19 (1969/1970), pp. 247–262.

[39] J. Wang, *LP estimates for Schrödinger operators with certain potentials*, Ann. Inst. Fourier (Grenoble), 45 (1995), pp. 513–546.

[40] E. M. Stein, *Topics in harmonic analysis related to the Littlewood-Paley theory*, Annals of Mathematics Studies, No. 63, Princeton University Press, Princeton, NJ, 1970.

[41] K. Stempak and J. L. Torrea, *Poisson integrals and Riesz transforms for Hermite function expansions with weights*, J. Funct. Anal., 202 (2003), pp. 443–472.

[42] G. Szegő, *Orthogonal polynomials*, American Mathematical Society, Providence, R.I., fourth ed., 1975. American Mathematical Society, Colloquium Publications, Vol. XXIII.

[43] S. Thangavelu, *Lectures on Hermite and Laguerre expansions*, vol. 42 of Mathematical Notes, Princeton University Press, Princeton, NJ, 1993.

[44] E. C. Titchmarsh, *Introduction to the theory of Fourier integrals*, Chelsea Publishing Co., New York, third ed., 1986.

[45] J. L. Torrea and C. Zhang, *Fractional vector-valued Littlewood-Paley-Stein theory for semigroups*. Preprint 2011 [arXiv:1105.6022v3]

[46] J. Van Neerven, *γ-radonifying operators—a survey*, in The AMSI-ANU Workshop on Spectral Theory and Harmonic Analysis, vol. 44 of Proc. Centre Math. Appl. Austral. Nat. Univ., Canberra, 2010, pp. 1–61.

[47] J. Van Neerven, M. C. Veraar, and L. Weis, *Stochastic integration in UMD Banach spaces*, Ann. Probab., 35 (2007), pp. 1438–1478.

[48] T. Szarek, *Littlewood-Paley-Stein type square functions based on Laguerre semigroups*, Acta Math. Hungar., 131 (2011), 59–109.

[49] M. Villani, *Riesz transforms associated to Bessel operators*, Illinois J. Math., 52 (2008), pp. 77–89.

[50] A. Weinstein, *Discontinuous integrals and generalized potential theory*, Trans. Amer. Math. Soc., 63 (1948), pp. 342–354.
[56] Q. Xu, Littlewood-Paley theory for functions with values in uniformly convex spaces, J. Reine Angew. Math., 504 (1998), pp. 195–226.
[57] A. H. Zemanian, Generalized integral transformations, Dover Publications Inc., New York, second ed., 1987.
[58] A. Zygmund, Trigonometric series. Vol. I, II, Cambridge Mathematical Library, Cambridge University Press, Cambridge, third ed., 2002.

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