Abstract. In this note we show that every discrete polymatroid is $M$-shellable. This gives, in a partial case, a positive answer to a conjecture of Chari and improves a recent result of Schweig where he proved that the $h$-vector of a lattice path matroid satisfies a conjecture of Stanley.

1. Introduction and Preliminaries

A matroid $M$ is a pair $(E(M), B(M))$ consisting of a finite set $E(M)$ and a collection $B(M)$ of subsets of $E(M)$, called bases of $M$, that satisfy the following two conditions:

(B1) $B(M) \neq \emptyset$, and

(B2) for each pair of distinct sets $B, B'$ in $B(M)$ and for each element $x \in B \setminus B'$, there is an element $y \in B' \setminus B$ such that $(B - x) \cup y$ is in $B(M)$.

Subsets of bases are called independent sets. The collection of independent sets of a matroid form an abstract simplicial complex, called matroid complex.

For a $(d-1)$-dimensional simplicial complex $\Delta$, let $f_i$ be the number of $(i-1)$-dimensional faces of $\Delta$ (i.e. the faces of cardinal $i$), and $\Delta = (f_0, f_1, \ldots, f_d)$ its $f$-vector. The $h$-vector $h(\Delta) = (h_0, h_1, \ldots)$ is defined by $H(y) = F(y - 1)$, where $H(y) = \sum h_i y^{d-i}$ and $F(y) = \sum f_i y^{d-i}$.

A monomial order ideal $\Gamma$ on a set $V = \{x_1, \ldots, x_n\}$ of variables is a set of monomials $x_1^{a_1} \ldots x_n^{a_n}$ such that $u \in \Gamma$ and $v | u$ imply that $v \in \Gamma$. The degree sequence of $\Gamma$ is $h(\Gamma) = (h_0, h_1, \ldots)$, where $h_i = \# \{u \in \Gamma | deg u = i\}$. We will not distinguish between a monomial order ideal and its poset (ordered by divisibility).

A pure $M$-vector is the degree sequence of an order ideal of monomials, whose maximal elements have the same degree.

The following conjecture of Stanley [5] is one of the most important conjectures on $h$-vector of matroid complexes.

Conjecture 1.1. (Stanley) The $h$-vector of a matroid complex is a pure $M$-vector.

A poset $Q$ is an $M$-poset if there exists a monomial $M$ on a finite set $E$ of indeterminates (variables) such that $Q$ is isomorphic to the poset (ordered by divisibility) on the set of monomials on $E$ that divide $M$. Equivalently, an $M$-poset is a direct product of chains.
Given two elements \( x \leq y \) of a poset \( P \), the interval \([x, y]\) is called an \( M \)-interval if it is an \( M \)-poset. A pure poset \( P \) is called \( M \)-partitionable if \( P \) can be partitioned into \( M \)-intervals \([x_1, y_1], \ldots, [x_n, y_n]\) such that for each \( 1 \leq i \leq n \), \( y_i \) is a maximal element of the poset \( P \). Such a partition is called an \( M \)-partition of the poset \( P \).

**Definition 1.2.** An \( M \)-shelling of a poset \( P \) is an \( M \)-partition of \( P \) along with an ordering of the \( M \)-intervals such that the union of the elements in any initial subsequence of \( M \)-intervals is an order ideal of \( P \). A poset \( P \) is \( M \)-shellable, if it admits an \( M \)-shelling.

Chari [2] proposed a stronger version of Stanley’s conjecture for \( h \)-vectors of matroid complexes based on the concept of \( M \)-shellability:

**Conjecture 1.3.** (Chari) The \( h \)-vector of a matroid complex is a shellable \( M \)-vector.

Recall that a pure \( M \)-vector is called shellable if it is the degree sequence of an \( M \)-shellable order ideal of monomials.

Herzog and Hibi [3] introduced discrete polymatroid, which it is a generalization of matroids. Let \( \Gamma \) be a pure monomial order ideal on the variables \( \{x_1, \ldots, x_r\} \) and for any \( m \in \Gamma \), the degree of \( x_i \) in \( m \) is denoted by \( m_i \). We say \( \Gamma \) is a discrete polymatroid if, for any two maximal monomials \( m, m' \in \Gamma \) and index \( i \) with \( m_i > m'_i \), there exists an index \( j \) such that \( m_j < m'_j \) and \( \frac{m_j}{x_j} \in \Gamma \), cf. [4, Definition 4.1.].

The aim of this paper is to show that every discrete polymatroid is \( M \)-shellable (Theorem 2.1). We apply this result to show that the \( h \)-vector of a lattice path matroid (see Section 2 for definition) satisfies Conjecture 1.3.

2. Main Theorem

**Theorem 2.1.** Every discrete polymatroid is \( M \)-shellable.

**Proof.** Let \( \Gamma \) be a discrete polymatroid on the set \( \{x_1, \ldots, x_r\} \) of variables and let \( p \) be the number of maximal elements of \( \Gamma \). The proof is by induction on \( p \). If \( p = 1 \), the basic case, then \( \Gamma \) is an \( M \)-poset and the assertion is obvious. So assume that \( p > 1 \). Then there exist an index \( j \) and two maximal elements \( m \) and \( m' \) in \( \Gamma \) with \( m_j \neq m'_j \). With no lose of generality, we assume that \( j = r \). Now, put

- \( k = \max\{m_r \mid m \in \Gamma\} \);
- \( \Gamma_1 = \{m \in \Gamma \mid x_r^k \nmid m\} \);
- \( \Gamma_2 = \Gamma - \Gamma_1 \); and
- \( \Gamma' = \{\frac{m}{x_r^k} \mid m \in \Gamma_2\} \).

**Claim:** \( \Gamma_1 \) and \( \Gamma' \) are discrete polymatroids.

**Proof of Claim:** We only show that \( \Gamma_1 \) is discrete polynomial. A similar argument works for \( \Gamma' \). First note that \( \Gamma_1 \) is a monomial order ideal. Since, for \( m \in \Gamma_1 \) and \( u \mid m \) we get that \( x_r^k \nmid u \) which implies that \( u \in \Gamma_1 \). To prove the purity of \( \Gamma_1 \), we assume that this is not the case and get a contradiction. By assumption, there exist a maximal element \( m \in \Gamma_1 \) and an element \( m' \in \Gamma_2 \) such that \( m \mid m' \). So \( m' = x_r^t m \), for some \( t > 0 \). Let \( m'' \) be a maximal elements in \( \Gamma \) with \( m''_r < k \). Then there exists an
index $j$ such that $\frac{x_i}{x_j}m' = x_jx_i^{-1}m \in \Gamma$ which is contradict $m$ is a maximal element of $\Gamma$. Thus $\Gamma_1$ is pure. To complete the proof we assume that $m$ and $m'$ be two monomials in $\Gamma_1$ with $m_i > m'_i$, for some $i$. Then there exists an index $j$ such that $m_j < m'_j$ and $\frac{x_i}{x_j}m \in \Gamma$, since $\Gamma$ is a discrete polymatroid. If $j \neq r$, then $x_k^s \notin \frac{x_i}{x_j}m$, since $x_r^k \notin m$. For $j = r$ we have $m_r < m'_r < k$ and then $(\frac{x_i}{x_j}m)_r < k$. Therefore $\Gamma_1$ is a discrete polymatroid. This complete the proof of the claim.

By induction hypothesis, there exist the following $M$-shelling orders for $\Gamma_1$ and $\Gamma'$:

$$\Gamma_1 = [a_1, b_1] \cup \cdots \cup [a_n, b_n] \quad \text{and} \quad \Gamma' = [c_1, d_1] \cup \cdots \cup [c_l, d_l].$$

We claim that the following order

$$\Gamma = [a_1, b_1] \cup \cdots \cup [a_n, b_n] \cup [x^k_c, x^{k_d}] \cup \cdots \cup [x^k_c, x^{k_d}]$$

is an $M$-shelling for $\Gamma$. It suffices to show that every initial subsequence $A = \Gamma_1 \cup [x^k_c, x^{k_d}] \cup \cdots \cup [x^k_c, x^{k_d}]$ ($s < l$) is an order ideal. Assume the contrary. Then there exist $m \in A - \Gamma_1$ and $u \in \Gamma - A$ with $u \mid m$. Therefore, $\frac{m}{x^k} \in [c_1, d_1] \cup \cdots \cup [c_l, d_l]$. Since $\frac{m}{x^k} \mid \frac{m}{x^k}$, and $\Gamma'$ is $M$-shellable. It contradicts $u \in \Gamma - A$. Now the proof is complete. 

Note that the converse of Theorem 2.1 does not hold. As a counterexample, one can consider the monomial order ideal $\Sigma$ with maximal elements $xy$ and $z^2$. It is easy to see that $\Sigma$ is $M$-shellable but it is not a discrete polymatroid.

A sequence $(h_0, h_1, \ldots, h_r)$ is called a $PM$-vector if it is the degree sequence of some discrete polymatroid. Clearly, every $PM$-vector is a pure $M$-vector. But Theorem 2.1 gives the following generalization of this fact.

**Corollary 2.2.** Every $PM$-vector is a shellable $M$-vector.

The $h$-vector of $\Sigma$ in the example before Corollary 2.2 is $(1, 3, 2)$. It shows that $(1, 3, 2)$ is a shellable $M$-vector, but it is indeed a $PM$-vector (take the discrete polymatroid with maximal elements $xy$ and $yz$). However we guess these two classes of vectors are very closed.

We end the paper by a result on lattice path matroids. Fix two lattice paths $P = p_1p_2 \ldots p_{m+r}$ and $Q = q_1q_2 \ldots q_{m+r}$ from $(0, 0)$ to $(m, r)$ with $P$ never going above $Q$. For every lattice path $R$ between $P$ and $Q$, let $\mathcal{N}(R)$ be the set of $R$’s north steps.

In [4], the authors showed that $M[P, Q] = \{ \mathcal{N}(R) : R \text{ is a path between } Q \text{ and } P \}$ is a matroid. $M[P, Q]$ is called a lattice path matroid.

Schweig [4, Theorem 3.6.] showed that lattice path matroids satisfy Conjecture [1.1] Even more, he proved that the $h$-vector of a lattice path matroid is a $PM$-vector. [4, Corollary 4.5]. This result of Schweig and Corollary 2.2 together imply the following result, which says that lattice path matroids satisfy Conjecture [1.3]

**Corollary 2.3.** The $h$-vector of a lattice path matroid is a shellable $M$-vector.
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