LARGE-SCALE REGULARITY IN STOCHASTIC HOMOGENIZATION WITH DIVERGENCE-FREE DRIFT

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We provide a proof of stochastic homogenization for random environments with a mean zero, divergence-free drift. We prove that the environment homogenizes weakly in $H^1$ if the drift admits a stationary $L^2$-integrable stream matrix, and we prove that the two-scale expansion converges strongly in $H^1$ if the drift admits a stationary $L^{d/(2+\delta)}$-integrable stream matrix. Additionally, under this stronger integrability assumption, we show that the environment almost surely satisfies a large-scale Hölder regularity estimate and first-order Liouville principle.

1. Introduction. In this paper, we prove the quenched homogenization of the equation

$$-\nabla \cdot a(x/\varepsilon, \omega) \nabla u^\varepsilon + \varepsilon^{-1} b(x/\varepsilon, \omega) \cdot \nabla u^\varepsilon = f \text{ in } U \text{ with } u^\varepsilon = g \text{ on } \partial U,$$

for a uniformly elliptic matrix $a$ and a mean zero, divergence-free drift $b$. The coefficients are jointly-measurable, stationary, and ergodic random variables defined on some probability space $(\Omega, \mathcal{F}, P)$. Stationarity asserts that the random environment is statistically homogenous: there exists a measure preserving transformation group $\{\tau_x : \Omega \rightarrow \Omega\}_{x \in \mathbb{R}^d}$ such that

$$a(x, \omega), b(x, \omega) = (A(\tau_x \omega), B(\tau_x \omega))$$

for random variables $A : \Omega \rightarrow \mathbb{R}^{d \times d}, B : \Omega \rightarrow \mathbb{R}^d$.

The ergodicity is a qualitative form of mixing: for $g \in L^\infty(\Omega),

$$g(\omega) = g(\tau_x \omega) \text{ almost surely for every } x \in \mathbb{R}^d \text{ if and only if } g \text{ is almost surely constant.}$$

In terms of the coefficients, we will assume that the matrix $A$ is bounded and uniformly elliptic: there exist $\lambda, \Lambda \in (0, \infty)$ such that, almost surely for every $\xi \in \mathbb{R}^d$,

$$|A\xi| \leq \Lambda |\xi| \text{ and } A\xi \cdot \xi \geq \lambda |\xi|^2.$$

We will assume that, for some $p \in [2, \infty)$ specified in the statements below, the random drift $B \in L^p(\Omega; \mathbb{R}^d)$ admits a stationary $L^p$-integrable stream matrix: there exists a skew-symmetric $S \in L^p(\Omega; \mathbb{R}^{d \times d})$ which satisfies

$$\nabla \cdot S = B \text{ for } (\nabla \cdot S)_i = \partial_k S_{ik},$$

fixed by the choice of gauge

$$-\Delta S_{ij} = \partial_i B_j - \partial_j B_i.$$

For an $L^p$-integrable stream matrix to exist in $d \geq 3$, for some $p \in [2, \infty)$, it is sufficient that $B$ is $L^p$-integrable and satisfies a finite range of dependence (see Kozlov [41], Kozma and Tóth [42, Proposition 4, Proposition 5], and Proposition 2.7 below). A stationary stream matrix does not exist in general if $d = 2$, and for this reason the homogenization of (1) in $d = 2$ remains largely an open problem.

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In the symmetric case, for sufficiently regular coefficients and a sufficiently regular domain, solutions of (1) are related by the Feynman-Kac formula to a rescaling of the stochastic differential equation
\[ dX_t = \sigma(X_t, \omega) \, dB_t + (\nabla \cdot a^t(X_t, \omega) - b(X_t, \omega)) \, dt, \]
for \( \sigma = 2a \). Precisely, for the exit time \( \tau^\varepsilon \) from the dilated domain \( U/\varepsilon \),
\[ u^\varepsilon(x) = E_x^{\varepsilon, \omega} \left[ g(\varepsilon X_{\tau^\varepsilon}) + \varepsilon^2 \int_0^{\tau^\varepsilon} f(\varepsilon X_s) \, ds \right] \]
for every \( x \in U \).

The homogenization of (1) is therefore equivalent to characterizing the asymptotic behavior in law of the exit distributions and exit times of (7) from large domains. Furthermore, in the case \( A = I \), equation (7) is the passive tracer model
\[ dX_t = \sqrt{2} \, dB_t + b(X_t, \omega) \, dt, \]
which is a simple approximation for the transport of a passive tracer particle in a turbulent, incompressible flow. This model has applications to hydrology, meteorological sciences, and oceanography, and we point the reader, for instance, to Csanady [18], Frish [31], and Monin and Yaglom [45, 46] for more details.

The stream matrix allows equation (1) to be rewritten in the form
\[ -\nabla \cdot (a^\varepsilon + s^\varepsilon) \cdot \nabla u^\varepsilon = f \quad \text{in } U \quad \text{with } u^\varepsilon = g \quad \text{on } \partial U, \]
for \( s^\varepsilon(x, \omega) = S(\tau^\varepsilon, \omega) \). The transformation (8) formally justifies the two-scale expansion familiar from the homogenization of divergence form equations without drift. That is, for the standard orthonormal basis \( \{e_i\}_{i \in \{1, \ldots, d\}} \) of \( \mathbb{R}^d \), we expect the corrector \( \phi_i \) to be defined by a stationary gradient \( \nabla \phi_i \) that solves
\[ -\nabla \cdot (A + S)(\nabla \phi_i + e_i) = 0, \]
and we expect the homogenized coefficient \( \overline{a} \in \mathbb{R}^{d\times d} \) to be defined by
\[ \overline{a} e_i = E[(A + S)(\nabla \phi_i + e_i)]. \]
This is the case if \( S \) is bounded, for which the methods of Papanicolaou and Varadhan [50] and Osada [49] yield readily that, for the solution
\[ -\nabla \cdot \overline{a} \nabla v = f \quad \text{in } U \quad \text{with } v = g \quad \text{on } \partial U, \]
the solutions \( u^\varepsilon \) almost surely converge to \( v \) weakly in \( H^1 \), and the two-scale expansion almost surely satisfies, for the rescaled correctors \( \phi_i^\varepsilon = \phi_i(\varepsilon^{-1} \cdot, \omega) \),
\[ \lim_{\varepsilon \to 0} \| u^\varepsilon - (v + \varepsilon \phi_i^\varepsilon \partial_i v) \|_{H^1(U)} = 0, \]
where here and throughout the paper we use Einstein’s summation convention over repeated indices.

The case of an unbounded stream matrix \( S \) is fundamentally different. Proving the existence of a solution to (9) is essentially straightforward, arguing by approximation and the skew-symmetry of \( S \). Uniqueness is however not clear and was posed as an open problem in Avellaneda and Majda [10]. The reason for this is that, while the equation defines \( S \nabla \phi_i \) as an element of the dual for any solution \( \nabla \phi_i \), it is not clear that this rule extends to a skew-symmetric operator on the solution space. Issues related to this fact explain the strong regularity assumptions used in Oelschläger [47], which imply almost surely that \( s \) is locally bounded on \( \mathbb{R}^d \), and form the technical core of the more recent works Kozma and Tóth [42] and Tóth [57]. In [42], assuming the existence of a square integrable stream matrix, the
authors prove that the analogous discrete random walk in random environment satisfies a central limit theorem in probability with respect to the environment. In [57], using an adaptation of Nash’s moment bound, the author proves a quenched invariance principle for such environments under the assumption of an $L^{2+\delta}$-integrable stream matrix. There is no essential difficulty in extending these techniques to the continuous setting provided the coefficients are regular enough to define the underlying diffusion process. In this way the PDE-based techniques of this paper provide a new approach to the discrete results of [42, 57] and extend the results of [47] to a regime for which the underlying diffusion cannot be defined. These techniques also provide an answer to the question originally posed by [10] under their higher $L^{d\nu(2+\delta)}$-integrability assumption, and establish a large-scale regularity estimate and Liouville theorem.

In our first result, assuming the existence of a square integrable stream matrix, we prove that (1) almost surely homogenizes weakly in $H^1$. Together with the a priori estimate of Proposition 3.2 below, this result implies homogenization strongly in $L^p$ for all $p \in [1, \infty)$, and while quenched is similar to proving homogenization with respect to the probability of the environment [42]. However, we do not obtain homogenization in $L^\infty$, which is most analogous to the results of [57] that were obtained under the stronger $L^{2+\delta}$-integrability assumption on the stream matrix. A second minor point is that the results of this paper apply without change to the elliptic and parabolic problems, and so also characterize formally the asymptotic behavior of exit distributions and exit times from arbitrary domains.

Our methods are based on an adaptation of the perturbed test function method. We prove the existence of homogenization correctors in Proposition 2.4 below. The essential point of the argument is to prove that the correctors satisfy the energy estimate
\[
\mathbb{E} [A \nabla \phi_i \cdot \nabla \phi_i] = -\mathbb{E} [(A + S)e_i \cdot \nabla \phi_i],
\]
which is shown in Proposition 2.9 to imply the skew-symmetry of the operator $S$ on the solution space. The modification of the perturbed test function method relies on the a priori estimates of Proposition 3.2 below, a stationary approximation of the homogenization correctors, and the regularity of the stream matrix. To simplify the statements, we introduce a steady assumption.

Assume (2), (3), (4), (5), and (6).

THEOREM (cf. Theorem 3.3 below). Assume (12) for $S \in L^2(\Omega; \mathbb{R}^{d \times d})$, let $U \subseteq \mathbb{R}^d$ be a bounded $C^{2,\alpha}$-domain for some $\alpha \in (0, 1)$, let $f \in L^q(U)$ for some $q \in (2 \vee \frac{d}{2}, \infty)$, and let $g \in W^{1,\infty}(\partial U)$. For every $\varepsilon \in (0, 1)$ let $u^\varepsilon \in H^1(U)$ be the unique solution of (1) and let $v \in H^1(U)$ be the unique solution of (11). Then, almost surely as $\varepsilon \to 0$,
\[
u^\varepsilon \to w \text{ weakly in } H^1(U).
\]

The second purpose of this paper is to explain the higher $L^{d\nu(2+\delta)}$-integrability assumption on the stream matrix introduced in [10]. Under this higher integrability assumption, our second result proves the strong convergence of the two-scale expansion in $H^1$. This is done by introducing the homogenization flux correctors $\sigma_i$ satisfying
\[
\nabla \cdot \sigma_i = (a + s)(\nabla \phi_i + e_i) - \bar{a}e_i.
\]
The flux correction was used originally in the context of stochastic homogenization by Gloria, Neukamm, and Otto [34], and allows the residuum of the two-scale expansion to be written in the form
\[
- \nabla \cdot (a^\varepsilon + s^\varepsilon) \nabla (u^\varepsilon - v - \varepsilon \phi_i \partial_i v) = \nabla \cdot [(\varepsilon \phi_i^\varepsilon (a^\varepsilon + s^\varepsilon) - \varepsilon \sigma_i^\varepsilon) \nabla (\partial_i v)].
\]
The $L^{d\nu(2+\delta)}$-integrability of $S$ is exactly the threshold which guarantees that the righthand side of (14) vanishes strongly in $L^2$ as $\varepsilon \to 0$, using the sublinearity of $\phi_i$ and $\sigma_i$ proven in Proposition 2.1 below.
THEOREM (cf. Theorem 4.1 below). Assume (12) for $S \in L^d(\Omega; \mathbb{R}^{d \times d})$ if $d \geq 3$ and for $S \in L^{2+\delta}(\Omega; \mathbb{R}^{d \times d})$ for some $\delta \in (0, 1)$ if $d = 2$, let $\alpha \in (0, 1)$, let $U \subseteq \mathbb{R}^d$ be a bounded $C^{2, \alpha}$-domain, let $f \in C^\alpha(U)$, and let $g \in C^{2, \alpha}(\partial U)$. For every $\varepsilon \in (0, 1)$ let $u^\varepsilon \in H^1(U)$ be the unique solution of (1) and let $v \in H^1(U)$ be the unique solution of (11). Then, almost surely as $\varepsilon \to 0$,

$$
\lim_{\varepsilon \to 0} \|u^\varepsilon - v - \varepsilon \phi_\varepsilon \partial_i v\|_{H^1(U)} = 0.
$$

The third main result of this work is an almost sure large-scale $\alpha$-Hölder regularity estimate for whole space solutions $u \in H^1_{\text{loc}}(\mathbb{R}^d)$ of the equation

$$
- \nabla \cdot (a + s)\nabla u = 0 \quad \text{in} \quad \mathbb{R}^d.
$$

Following [34], based on the equivalence of Morrey, Campanato, and Hölder spaces (see, for example, Giaquinta and Martinazzi [32]), we introduce a version of the large-scale $\alpha$-Hölder semi-norm defined with respect to the intrinsic $(a + s)$-harmonic coordinates $(x_i + \phi_i)$: the excess $\text{Exc}(u; R)$ is defined by

$$
\text{Exc}(u; R) = \inf_{\xi \in \mathbb{R}^d} \left( R^{-2\alpha} \int_{B_R} |\nabla u - \xi - \nabla \phi_\xi|^2 \right),
$$

for $\phi_\xi = \xi_i \phi_i$. The following theorem proves that there exists an almost surely finite radius $R_0 \in (0, \infty)$ after which point the solutions of (15) enter the regime of $\alpha$-Hölder regularity. The radius $R_0$ is quantified precisely by the sublinearity of the correctors in Proposition 5.5 below.

THEOREM (cf. Proposition 5.5, Theorem 5.6 below). Assume (12) for $S \in L^d(\Omega; \mathbb{R}^{d \times d})$ if $d \geq 3$ and for $S \in L^{2+\delta}(\Omega; \mathbb{R}^{d \times d})$ for some $\delta \in (0, 1)$ if $d = 2$. On a subset of full probability, for every $\alpha \in (0, 1)$ there exists a random radius $R_0 \in (0, \infty)$ and a deterministic $c \in (0, \infty)$ depending on $\alpha$ such that, for every weak solution $u \in H^1_{\text{loc}}(\mathbb{R}^d)$ of (15), for every $R_1 < R_2 \in (R_0, \infty)$,

$$
R_1^{-2\alpha} \text{Exc}(u; R_1) \leq c R_2^{2\alpha} \text{Exc}(u; R_2),
$$

for the excess $\text{Exc}(u; R)$ defined in (16).

As a consequence of the large-scale regularity estimate, we establish the following first-order Liouville theorem: the $(a + s)$-harmonic coordinates $(x_i + \phi_i)$ are the linear functions in the random geometry of the space, and every subquadratic $(a + s)$-harmonic function is a corrector. The sublinearity is quantified with respect to the $L^2$-norm, for $2s > 2$ defined below, as opposed to the $L^2$-norm used in [34]. This stronger condition is necessary for our arguments due to the unboundedness of the stream matrix. In combination, the Liouville theorem and Proposition 2.1 below prove the quenched uniqueness of the homogenization correctors and thereby provide a strong answer to the original question of [10] on the physical space.

THEOREM (cf. Theorem 6.2 below). Assume (12) for $S \in L^d(\Omega; \mathbb{R}^{d \times d})$ if $d \geq 3$ and for $S \in L^{2+\delta}(\Omega; \mathbb{R}^{d \times d})$ for some $\delta \in (0, 1)$ if $d = 2$, let $g_d = d \vee (2 + \delta)$, and let $1/2_s = 1/2 - 1/g_d$. Then, on a subset of full probability, every weak solution $u \in H^1_{\text{loc}}(\mathbb{R}^d)$ of (15) that is strictly subquadratic in the sense that, for some $\alpha \in (0, 1)$,

$$
\lim_{R \to \infty} \frac{1}{R^{1+\alpha}} \left( \int_{B_R} |u|^2 \right)^{\frac{1}{2s}} = 0,
$$

satisfies $u = c + \xi \cdot x + \phi_\xi$ in $H^1_{\text{loc}}(\mathbb{R}^d)$ for some $c \in \mathbb{R}$ and $\xi \in \mathbb{R}^d$. 

1.1. The organization of the paper. In Section 2, we describe how equations (5), (6), and (9) are lifted to the probability space: in Section 2.1 we construct the homogenization correctors, in Section 2.2 we construct the homogenization flux correctors, in Section 2.3 we prove the existence of a stationary stream matrix, and in Section 2.4 we characterize the homogenized coefficient. In Section 3 we prove homogenization weakly in $H^1$ under the assumption of an $L^2$-integrable stream matrix. In Section 4, we prove the convergence of the two-scale expansion strongly in $H^1$ assuming the existence of an $L^{d/2-(2+\delta)}$-stream matrix. In Section 5, we first obtain an energy estimate for the homogenization error in Proposition 5.4 and then prove the large-scale regularity estimate. We prove the Liouville theorem in Section 6, which is a consequence of the large-scale regularity estimate and a version of the Caccioppoli inequality adapted to the divergence-free setting.

1.2. Overview of the literature. The foundational theory of homogenization for elliptic and parabolic equations with periodic coefficients can be found in the references Bensoussan, Lions, and Papanicolaou [15] and Jikov, Kozlov, and Oleinik [36]. The stochastic homogenization of divergence form equations, and non-divergence form equations without drift, was initiated by Papanicolaou and Varadhan [50, 51], Osada [49], and Kozlov [41]. In the absence of additional assumptions on the drift, the general question of stochastic homogenization for diffusion equations of the type

$$- \nabla \cdot a(x/\varepsilon, \omega) \nabla u^\varepsilon + \varepsilon^{-1} b(x/\varepsilon, \omega) \cdot \nabla u^\varepsilon = f \text{ in } U \text{ with } u^\varepsilon = g \text{ on } \partial U$$

remains open. The difficulty lies in constructing the invariant measure for the process from the point of view of the particle. Thus far, the construction of this measure has required additional assumptions on the drift, such as the case when $b = \nabla U$ is the gradient of a stationary field, which has been treated, for instance, by Olla in [48], and the case when $b$ is divergence-free, which will be discussed in detail below. The only other known results apply to a perturbative, strongly mixing, and isotropic regime in $d \geq 3$, which have been obtained in the discrete case by Bricmont and Kupiainen [17], Bolthausen and Zeitouni [16], Baur and Bolthausen [12], and Baur [11] and in the continuous case by Sznitman and Zeitouni [56] and the author [26, 27, 28] where [27] constructs the invariant measure. A general overview can be found in the reference [48] and the book Komorowski, Landim, and Olla [38, Chapter 9].

The homogenization of (17) with divergence-free drift was initiated by [49], who considered the case of a bounded stream matrix, and Oelschlager [47], who proved an invariance principle and the homogenization of equations like (1) in probability on the whole space assuming the existence of an $L^2$-integrable, $C^2$-smooth stream matrix. More recently, in the discrete case, Toth and Kozma [42] have proven an invariance principle in probability with respect to the environment for the analogous discrete random walk under the so-called $H_{-1}$-condition, which is equivalent to the existence of a stationary, $L^2$-integrable stream matrix. Toth [57] subsequently proved a quenched central limit theorem in this setting assuming the existence of an $L^{2+\delta}$-integrable stream matrix, using an adaptation of Nash’s moment bound. The higher $L^{d/2-(2+\delta)}$-integrability assumption was introduced in Avellaneda and Majda [10] to prove the quenched homogenization of the parabolic version of (1) on the whole space. In [10] correctors are constructed by approximation, and therefore lack an intrinsic characterization. Related problems under more restrictive integrability assumptions have been considered by Fannjiang and Komorowski [23], and time-dependent problems have been considered by Landim, Olla, and Yau [43], Fannjiang and Komorowski [24, 25], and Komorowski and Olla [39]. Komorowski and Olla [40] have provided a counterexample to the annealed homogenization of equations like (1) on the whole space for drifts that do not admit a square-integrable stream matrix.

The results and methods of this paper are closely related to those used in the study of invariance principles for random walks in random environments. The random walk $X_n$ is
often decomposed as \( X_n = M_n + \phi(X_n) \) for a martingale \( M_n \) and the vector-valued corrector \( \phi = (\phi_1, \ldots, \phi_d) \) defined analogously to (9). The diffusive limit can then be characterized by proving an a priori tail estimates for the process, establishing the almost sure sublinearity of the corrector, and applying the martingale central limit theorem. For a complete discussion of these techniques and for a detailed account of the substantial literature on the subject we point the reader to [38, Chapter 3] and Zeitouni [59].

We mention as well here, in addition to [42, 57], the work of Kipnis and Varadhan [37], Sidoravicius and Sznitman [52] who establish a quenched invariance principle for random walks on supercritical percolation clusters in \( d \geq 4 \) and a quenched invariance principle for the random conductance model with uniformly elliptic, i.i.d. conductances in an arbitrary dimension, Andres, Barlow, Deuschel, and Hambly [1] who establish a general invariance principle for the random conductance model assuming that the conductances almost surely percolate, and Deuschel and Kösters [19] who establish an invariance principle for random walks satisfying the bounded cycle condition. We also mention that [40] and Tóth and Válko [58] have shown that a random walk failing to satisfy the \( H_{-1} \)-condition can exhibit superdiffusive behavior.

The relationship between Schauder estimates and Liouville theorems for constant coefficient elliptic equations without drift was shown by Simon [53]. In the context of the periodic homogenization of divergence form elliptic equations without drift Avellaneda and Lin [9] obtained a full hierarchy of Liouville theorems based on the large-scale regularity theory in Hölder- and \( L^p \)-spaces of the same authors Avellaneda and Lin [7, 8]. Armstrong and Smart [6] first adapted the approach of [8] to the stochastic case and obtained a large-scale regularity theory for environments satisfying a finite range of dependence. Their methods are based on the variational characterization of solutions and quantify the convergence of certain sub- and super-additive energies. Armstrong and Mourrat [5] extended the results of [6] to more general mixing conditions and these works have given rise to a significant literature on the subject. A complete account of these developments can be found in the monograph Armstrong, Kuusi and Mourrat [4], which includes applications to percolation clusters Armstrong and Dario [3] and time-dependent environments Armstrong, Bordas, and Mourrat [2].

The results of this work are most closely related to those of Gloria, Neukamm and Otto [34], who established a large-scale regularity theory and first-order Liouville principle for divergence form equations without drift under the qualitative assumption of ergodicity. In particular, the homogenization flux-correction introduced in [34] is used essentially in the proof of several results of this work, and their introduction of an intrinsic excess decay is used to obtain the large-scale regularity estimate and Liouville theorem. Marahrens and Otto [44] had previously obtained a Liouville theorem assuming a quantified form of ergodicity. Fischer and Otto [29, 30] extended the results of [34] to obtain a full hierarchy of Liouville theorems under a mild quantification of ergodicity. Degenerate environments were considered by Bella, the author, and Otto [14] and time-dependent environments by Bella, Chiarini, and the author [13]. The work [34] has similarly given rise to a substantial literature on the subject including, for instance, Gloria and Otto [35] and Duerinckx, Gloria, and Otto [20].

2. The extended homogenization corrector. In this section, we will describe how the equations (5), (6), and (9) are lifted to the probability space. Following [50], the transformation group \( \{\tau_x\}_{x \in \mathbb{R}^d} \) is used to define so-called horizontal derivatives \( \{D_i\}_{i \in \{1, \ldots, d\}} \): for each \( i \in \{1, \ldots, d\} \),

\[
D(D_i) = \{ f \in L^2(\Omega) : \lim_{h \to 0} f(\tau_{hei}^\omega) - f(\omega)/h \text{ exists strongly in } L^2(\Omega) \},
\]

and \( D_i : D(D_i) \to L^2(\Omega) \) is defined by \( D_i f = \lim_{h \to 0} f(\tau_{hei}^\omega) - f(\omega)/h \). The \( D_i \) are closed, densely defined operators on \( L^2(\Omega) \). We define \( \mathcal{H}^1(\Omega) = \cap_{i=1}^d D(D_i) \) and we will write
particular, to handle the less standard case $q$ matrix. The proof of sublinearity is essentially well-known, but we include details here, in Lemma 3.27 and extends [47, Lemma 3.27] to the case of a general stationarity gradient satisfying (18). The proof of Proposition 2.4 is strongly motivated by [47, Proposition 2.2] below. Finally, in Proposition 2.4 below, we prove that there exists a unique stability for the homogenization correctors in Proposition 2.1 below. We analyze (19) in Proposition 2.5 below. We will now use this framework to lift equations like (5), (6), and (9) to the probability space.

The subsection is organized as follows. We will first present a general proof of sublinearity for the homogenization corrector. We will construct the homogenization corrector as a stationary gradient $\Phi_i$ in $L^2_{\text{pot}}(\Omega)$ satisfying

$$-D \cdot (A + S)(\Phi_i + e_i) = 0 \text{ in } D'(\Omega).$$

The solution is identified by approximation. We first prove that for every $\alpha \in (0, 1)$ there exists a unique $\Phi_{i,\alpha} \in H^1(\Omega)$ satisfying the equation

$$\alpha \Phi_{i,\alpha} - D \cdot (A + S)(D\Phi_{i,\alpha} + e_i) = 0,$$

where here, in comparison to (18), the proof of uniqueness is simpler and relies crucially on the stationarity of $\Phi_{i,\alpha}$ itself. We will then show that the $D\Phi_{i,\alpha}$ converge along the full sequence $\alpha \to 0$ in $L^2_{\text{pot}}(\Omega)$ to the unique solution of (18).

The subsection is organized as follows. We will first present a general proof of sublinearity for the homogenization correctors in Proposition 2.1 below. We analyze (19) in Proposition 2.2 below. Finally, in Proposition 2.4 below, we prove that there exists a unique stationary gradient satisfying (18). The proof of Proposition 2.4 is strongly motivated by [47, Lemma 3.27] and extends [47, Lemma 3.27] to the case of a general $L^2$-integrable stream matrix. The proof of sublinearity is essentially well-known, but we include details here, in particular, to handle the less standard case $q = p_*$.

**Proposition 2.1.** Assume (12), let $p \in (1, \infty)$, let $F \in L^p(\Omega; \mathbb{R}^d)$ satisfy

$$D_i F_j = D_j F_i \text{ for every } i, j \in \{1, \ldots, d\} \text{ and } \mathbb{E}[F] = 0,$$

and we will write $D(\Omega)$ for the space of all such functions. The space $D(\Omega)$ is dense in $L^p(\Omega)$ for every $p \in [1, \infty)$. We will write $D'(\Omega)$ for the dual of $D(\Omega)$, and we will understand distributional inequalities in $D'(\Omega)$ in the sense that, for $f \in L^1(\Omega)$,

$$D_i f = 0 \text{ if and only if } \mathbb{E}[f D_i \psi] = 0 \text{ for every } \psi \in D(\Omega).$$

For a vector field $V = (V_i)_{i \in \{1, \ldots, d\}} \in L^2(\Omega; \mathbb{R}^d)$ we define the distributional divergence $D \cdot V = D_j V_j$. The space of vector fields $L^2(\Omega; \mathbb{R}^d)$ then admits the following Helmholtz decomposition. The space of potential or curl-free fields on $\Omega$ is defined by

$$L^2_{\text{pot}}(\Omega) = \{D\psi \in L^2(\Omega; \mathbb{R}^d); \psi \in H^1(\Omega)\}.$$

This is the $L^2(\Omega; \mathbb{R}^d)$-closure of the space of $H^1$-gradients. The space of solenoidal or divergence-free fields is defined by

$$L^2_{\text{sol}}(\Omega) = \{V \in L^2(\Omega; \mathbb{R}^d); D \cdot V = 0\}.$$

The space $L^2(\Omega; \mathbb{R}^d)$ then admits the orthogonal decomposition

$$L^2(\Omega; \mathbb{R}^d) = L^2_{\text{pot}}(\Omega) \oplus L^2_{\text{sol}}(\Omega),$$

which can be deduced from Proposition 2.5 below. We will now use this framework to lift equations like (5), (6), and (9) to the probability space.
and let $\phi : \mathbb{R}^d \times \Omega \to \mathbb{R}$ almost surely satisfy $\phi \in W^{1,p}_\text{loc}(\mathbb{R}^d)$ with $\nabla \phi(x,\omega) = F(\tau_x \omega)$. If $p < d$ and $1/p = 1/p - 1/d$ we have almost surely that, for every $q \in [1,p_*]$,

$$\lim_{R \to \infty} \frac{1}{R} \left( \int_{B_R} |\phi|^q \right)^{\frac{1}{q}} = 0. \tag{20}$$

If $p \geq d$ then (20) holds for every $q \in [1,\infty)$. If $p > d$ then $\lim_{R \to \infty} \|\phi\|_{L^\infty(B_R)/R} = 0$.

**PROOF.** We will first consider the case $p < d$ and $q = p_*$. After rescaling we observe that

$$\limsup_{R \to \infty} \frac{1}{R} \left( \int_{B_R} |\phi|^{p_*} \right)^{\frac{1}{p_*}} = \limsup_{\varepsilon \to 0} \left( \int_{B_1} \left| \phi^{\varepsilon} - \int_{B_1} \phi^{\varepsilon} \right|^{p_*} \right)^{\frac{1}{p_*}}, \tag{21}$$

for $\phi^{\varepsilon}(x) = \varepsilon \phi(\varepsilon^2 x)$. We will first prove that

$$\limsup_{\varepsilon \to 0} \left( \int_{B_1} \left| \phi^{\varepsilon} - \int_{B_1} \phi^{\varepsilon} \right|^{p_*} \right)^{\frac{1}{p_*}} = 0, \tag{22}$$

and then show that (22) implies (20). For every $\delta \in (0,1)$ let $\rho^{\delta}$ be a standard convolution kernel of scale $\delta$, and for every $\varepsilon, \delta \in (0,1)$ let $\phi^{\varepsilon,\delta} = \phi^{\varepsilon} * \rho^{\delta}$. The triangle inequality proves that, for every $\varepsilon, \delta \in (0,1)$,

$$\left( \int_{B_1} \left| \phi^{\varepsilon} - \int_{B_1} \phi^{\varepsilon} \right|^{p_*} \right)^{\frac{1}{p_*}} \leq \left( \int_{B_1} \left| \phi^{\varepsilon,\delta} - \int_{B_1} \phi^{\varepsilon,\delta} \right|^{p_*} \right)^{\frac{1}{p_*}} + \left( \int_{B_1} \left| \phi^{\varepsilon} - \phi^{\varepsilon,\delta} \right|^{p_*} \right)^{\frac{1}{p_*}} + \left( \int_{B_1} \left| \phi^{\varepsilon,\delta} - \int_{B_1} \phi^{\varepsilon,\delta} \right|^{p_*} \right)^{\frac{1}{p_*}}.$$

The Sobolev inequality proves that there exists $c \in (0,\infty)$ such that, for each $\varepsilon, \delta \in (0,1)$,

$$\int_{B_{1}} \left| \phi^{\varepsilon} - \int_{B_{1}} \phi^{\varepsilon} \right|^{p_*} \leq c \left( \int_{B_{1}} \left| \nabla \phi^{\varepsilon,\delta} \right|^{p_*} \right)^{\frac{1}{p_*}} + \int_{B_{1}} \left| \nabla \left( \phi^{\varepsilon} - \phi^{\varepsilon,\delta} \right) \right|^{p_*} \right)^{\frac{1}{p_*}}. \tag{23}$$

For the first term on the right hand side of (23), since Jensen’s inequality proves that, for every $x \in B_1$ and $\varepsilon, \delta \in (0,1)$,

$$\left| \nabla \phi^{\varepsilon,\delta}(x,\omega) \right|^p = \left| \int_{\mathbb{R}^d} \nabla \phi^{\varepsilon}(y) \rho^{\delta}(y - x) \, dy \right|^p \leq \left\| \rho^{\delta} \right\|_{L^\infty(\mathbb{R}^d)} \int_{B_2} |\nabla \phi^{\varepsilon}|^p,$$

the ergodic theorem and $F \in L^p(\Omega;\mathbb{R}^d)$ prove that, almost surely for each $\delta \in (0,1)$,

$$\sup_{\varepsilon \in (0,1)} \left( \sup_{x \in B_1} \left| \nabla \phi^{\varepsilon,\delta}(x,\omega) \right| \right) < \infty. \tag{24}$$

Since the ergodic theorem and $\mathbb{E}[F] = 0$ prove almost surely that, as $\varepsilon \to 0$,

$$\nabla \phi^{\varepsilon}(x,\omega) \to 0 \text{ weakly in } L^p_{\text{loc}}(\mathbb{R}^d;\mathbb{R}^d),$$

we have, almost surely for every $\delta \in (0,1)$ and $x \in B_1$,

$$\lim_{\varepsilon \to 0} \left| \nabla \phi^{\varepsilon,\delta}(x) \right|^p = \lim_{\varepsilon \to 0} \left| \int_{\mathbb{R}^d} \nabla \phi^{\varepsilon}(y) \rho^{\delta}(y - x) \, dy \right|^p = 0. \tag{25}$$

The dominated convergence theorem, (24), and (25) then prove that, almost surely for every $\delta \in (0,1)$,

$$\limsup_{\varepsilon \to 0} \left( \int_{B_1} \left| \nabla \phi^{\varepsilon,\delta} \right|^{p_*} \right)^{\frac{1}{p_*}} = 0. \tag{26}$$
For the second term on the righthand side of (23), the ergodic theorem proves almost surely for every $\delta \in (0, 1)$ that
\[
\lim_{\varepsilon \to 0} \left( \int_{B_{1}} |\nabla (\phi^{\varepsilon} - \phi^{\varepsilon, \delta})|^{p} \right)^{\frac{1}{p}} = \mathbb{E} \left[ |F - F^{\delta}|^{\frac{1}{p}} \right],
\]
for $F^{\delta}(\omega) = \int_{\mathbb{R}^{d}} F(\tau_{y}\omega) \rho^{\delta}(y) \, dy$. Returning to (23), it follows from (26) and (27) that, for every $\delta \in (0, 1)$,
\[
\limsup_{\varepsilon \to 0} \left( \int_{B_{1}} |\phi^{\varepsilon} - \int_{B_{1}} \phi^{\varepsilon} |^{p_{*}} \right)^{\frac{1}{p_{*}}} \leq \mathbb{E} \left[ |F - F^{\delta}|^{\frac{1}{p}} \right].
\]
It follows from $F \in L^{p}(\Omega; \mathbb{R}^{d})$ that $\lim_{\delta \to 0} \mathbb{E} \left[ |F - F^{\delta}|^{p} \right] = 0$, which complete the proof of (22).

It remains to prove that (22) implies (20). The following argument appears in [14, Lemma 2]. Due to the equivalence (21), almost surely for every $\delta \in (0, 1)$ there exists $R_{0} \in (0, \infty)$ such that, for every $R \geq R_{0}$,
\[
\left( \int_{B_{R}} |\phi - \int_{B_{R}} \phi |^{p_{*}} \right)^{\frac{1}{p_{*}}} \leq R\delta.
\]
By the triangle inequality, for every $R \in [R_{0}, 2R_{0}]$, for $c \in (0, \infty)$ independent of $R$,
\[
\left| \int_{B_{R}} \phi - \int_{B_{R_{0}}} \phi \right| \leq \left( \int_{B_{R_{0}}} |\phi - \int_{B_{R_{0}}} \phi |^{p_{*}} \right)^{\frac{1}{p_{*}}} + \left( \int_{B_{R_{0}}} |\phi - \int_{B_{R_{0}}} \phi |^{p_{*}} \right)^{\frac{1}{p_{*}}} \leq \left( \frac{R}{R_{0}} \right)^{\frac{1}{p_{*}}} R\delta + R_{0}\delta \leq \left( 2^{(\frac{1}{p_{*}} + 1)} + 1 \right) R_{0}\delta = cR_{0}\delta.
\]
Therefore, for every $R \in [R_{0}, 2R_{0}]$,
\[
\left| \frac{1}{R} \int_{B_{R}} \phi \right| \leq \left( \frac{R_{0}}{R} \right) \left| \frac{1}{R_{0}} \int_{B_{R_{0}}} \phi \right| + c \left( \frac{R_{0}}{R} \right) \delta.
\]
It then follows inductively that, for every $R \in [2^{k-1}R_{0}, 2^{k}R_{0}]$,
\[
\left| \frac{1}{R} \int_{B_{R}} \phi \right| \leq \left( \frac{2^{k-1}R_{0}}{R} \right) \left| \frac{1}{2^{k-1}R_{0}} \int_{B_{2^{k-1}R_{0}}} \phi \right| + c \left( \frac{2^{k-1}R_{0}}{R} \right) \delta \leq \left| \frac{1}{R} \int_{B_{R_{0}}} \phi \right| + c \sum_{j=0}^{\infty} 2^{-j} \delta = \left| \frac{1}{R} \int_{B_{R_{0}}} \phi \right| + 2c\delta.
\]
Since $\delta \in (0, 1)$ was arbitrary, we have almost surely that
\[
\limsup_{R \to \infty} \left| \frac{1}{R} \int_{B_{R}} \phi \right| = 0.
\]
The triangle inequality, (22), and (28) prove almost surely that
\[
\limsup_{R \to \infty} \left( \frac{1}{R} \int_{B_{R}} |\phi|^{p_{*}} \right)^{\frac{1}{p_{*}}} \leq \limsup_{R \to \infty} \left( \frac{1}{R} \int_{B_{R}} |\phi - \int_{B_{R}} \phi |^{p_{*}} \right)^{\frac{1}{p_{*}}} + \limsup_{R \to \infty} \left| \frac{1}{R} \int_{B_{R}} \phi \right| = 0,
\]
which completes the proof for the case $p < d$ and $q = p_{*}$. The fact that (20) holds for the cases $p < d$ and $q \in [1, p_{*})$ and $p \geq d$ and $q \in [1, \infty)$ is then a consequence of Hölder’s inequality. If $p > d$, returning to (22), the Sobolev embedding theorem implies that the sequence
{φ^ε - \int_{B_1} φ^ε} \in C^α(B_1) for α = 1 - d/p. The Arzelà-Ascoli theorem, (25), and (28) then prove almost surely that

$$\lim_{\varepsilon \to 0} \|φ^ε\|_{L^\infty(B_1)} = \lim_{R \to \infty} \frac{1}{R} \|φ\|_{L^\infty(\Omega)} = 0.$$

**PROPOSITION 2.2.** Assume (12) for some \( S \in L^2(\Omega; \mathbb{R}^{d \times d}) \), let \( F \in L^2(\Omega; \mathbb{R}^d) \), and let \( α \in (0, 1) \). Then there exists a unique \( \Phi \in \mathcal{H}^1(\Omega) \) satisfying the equation

$$α \Phi - D \cdot (A + S)D\Phi = -D \cdot F \text{ in } \mathcal{D}'(\Omega).$$

Furthermore, \( \Phi \) satisfies the energy identity

$$\mathbb{E} [α\Phi^2 + AD\Phi \cdot D\Phi] = \mathbb{E} [F \cdot D\Phi].$$

**PROOF.** We will write \( S = (S_{i j})_{i,j \in \{1,...,d\}} \in L^2(\Omega; \mathbb{R}^{d \times d}) \) and for every \( n \in \mathbb{N} \) we define

$$S_n = ((S_{i j} \wedge n) \vee (-n))_{i,j \in \{1,...,d\}}.$$

The Lax-Milgram theorem proves that there exists a unique \( \Phi_n \in \mathcal{H}^1(\Omega) \) which satisfies

$$α\Phi_n - D \cdot (A + S_n)D\Phi_n = -D \cdot F \text{ in } \mathcal{H}^{-1}(\Omega).$$

The boundedness and anti-symmetry of \( S_n \), the uniform ellipticity of \( A \), Hölder’s inequality, and Young’s inequality prove that there exists \( c \in (0, \infty) \) such that, for each \( n \in \mathbb{N} \),

$$\mathbb{E} [α\Phi_n^2 + |D\Phi_n|^2] \leq c\mathbb{E} [|F|^2].$$

It follows from (31) that there exists \( \Phi \in \mathcal{H}^1(\Omega) \) such that, after passing to a subsequence, as \( n \to \infty \),

$$\Phi_n \rightharpoonup \Phi \text{ weakly in } \mathcal{H}^1(\Omega).$$

Since \( S_n \to S \) strongly in \( L^2(\Omega; \mathbb{R}^{d \times d}) \), it follows from (32) and \( \mathcal{D}(\Omega) \subseteq \mathcal{H}^1(\Omega) \) that \( \Phi \) solves

$$α\Phi - D \cdot (A + S)D\Phi = -D \cdot F \text{ in } \mathcal{D}'(\Omega).$$

Uniqueness is an immediate consequence of the linearity, the uniform ellipticity of \( A \), and the energy estimate. Therefore, it remains only to prove the energy estimate (30). The skew-symmetry of \( S \) and \( D \cdot (D \cdot S) = 0 \) prove that, for every \( \psi \in \mathcal{D}(\Omega) \),

$$\mathbb{E} [SD\Phi \cdot D\psi] = -\mathbb{E} [(D \cdot S) \cdot D\psi] = \mathbb{E} [(D \cdot S) \cdot D\Phi] \psi.$$

For each \( n \in \mathbb{N} \) let \( \Phi_n = (\Phi \wedge n) \vee (-n) \) and for every \( \varepsilon \in (0, 1) \) let \( \rho^\varepsilon \) denote a standard convolution kernel of scale \( \varepsilon \in (0, 1) \). For every \( n \in \mathbb{N} \) and \( \varepsilon \in (0, 1) \) let

$$\Phi_{n, \varepsilon}(\omega) = \int_{\mathbb{R}^d} \Phi_n(\tau_\varepsilon \omega) \rho^\varepsilon(x) \, dx.$$

It follows by definition that the \( \Phi_{n, \varepsilon} \) are admissible test functions for (33) and, after using the boundedness of \( \Phi_n \) to pass to the limit \( \varepsilon \to 0 \), it follows from (33) and (34) that

$$\mathbb{E} [α\Phi_{n, \varepsilon} + AD\Phi \cdot D\Phi_{n, \varepsilon}] = -\mathbb{E} [(D \cdot S)(D\Phi)\Phi_n] + \mathbb{E} [F \cdot D\Phi_n].$$

Since the distributional equality \( D\Phi_n = D\Phi 1_{\{\Phi \leq n\}} \) proves the distributional equality \( (D\Phi)\Phi_n = D(\Phi_{n, \varepsilon} - 1/2\rho^\varepsilon_n) \), the boundedness of \( \Phi_n \) and \( D \cdot (D \cdot S) = 0 \) prove that

$$\mathbb{E} [(D \cdot S)(D\Phi)\Phi_n] = \mathbb{E} [(D \cdot S) \cdot D(\Phi_{n, \varepsilon} - 1/2\rho^\varepsilon_n)] = 0.$$
It then follows from (35), (36), and \( D\Phi_n = D\Phi 1_{\{|\Phi| \leq n\}} \) that
\[
E \left[ \alpha \Phi_n + AD \Phi \cdot D\Phi 1_{\{|\Phi| \leq n\}} \right] = E \left[ F \cdot D\Phi 1_{\{|\Phi| \leq n\}} \right].
\]
The energy estimate (30) then follows by the dominated convergence theorem, after passing to the limit \( n \to \infty \) in (37). This completes the proof.

**Remark 2.3.** We emphasize that the approximate correctors constructed in Proposition 2.2 are themselves stationary functions in \( H^1(\Omega) \) that satisfy equation (29). However, this is not the case for the exact correctors. The exact correctors constructed in Proposition 2.4 are identified by a stationary potential field \( \Phi \in L^2_{\text{pot}}(\Omega) \), which does not imply that there exists some \( \psi \in H^1(\Omega) \) satisfying \( D\psi = \Phi \). In general, the correctors themselves will not exist as stationary functions, despite the fact that their gradients are always stationary.

**Proposition 2.4.** Assume (12) for some \( S \in L^2(\Omega; \mathbb{R}^{d \times d}) \) and let \( F \in L^2(\Omega; \mathbb{R}^d) \). Then there exists a unique \( \Phi \in L^2_{\text{pot}}(\Omega) \) which satisfies the equation
\[
-D \cdot (A + S) \Phi = -D \cdot F \quad \text{in} \quad D'(\Omega).
\]
Furthermore, \( \Phi \) satisfies the energy identity
\[
E [A\Phi \cdot \Phi] = E [F \cdot \Phi].
\]

**Proof.** For every \( \alpha \in (0, 1) \) let \( \Phi_\alpha \in H^1(\Omega) \) be the unique solution of
\[
\alpha \Phi_\alpha - D \cdot (A + S) D\Phi_\alpha = -D \cdot F \quad \text{in} \quad D'(\Omega).
\]
It follows from (30), the uniform ellipticity, Hölder’s inequality, and Young’s inequality that, for some \( c \in (0, \infty) \) independent of \( \alpha \in (0, 1) \),
\[
E \left[ \alpha (\Phi_\alpha)^2 + |D\Phi_\alpha|^2 \right] \leq c E \left[ |F|^2 \right].
\]
Therefore, after passing to a subsequence \( \alpha \to 0 \), there exists \( \Phi \in L^2_{\text{pot}}(\Omega) \) such that
\[
\alpha \Phi_\alpha \to 0 \quad \text{strongly in} \quad L^2(\Omega) \quad \text{and} \quad D\Phi_\alpha \rightharpoonup \Phi \quad \text{weakly in} \quad L^2_{\text{pot}}(\Omega).
\]
It follows from (40) and (41) that
\[
-D \cdot (A + S) \Phi = -D \cdot F \quad \text{in} \quad D'(\Omega).
\]
This completes the proof of existence.

We will now prove the energy identity (39). Since \( \Phi \in L^2_{\text{pot}}(\Omega) \) is curl-free and satisfies (42), by integration let \( \phi : \mathbb{R}^d \times \Omega \to \mathbb{R} \) be the unique function almost surely satisfying that \( \int_{B_r} \phi = 0 \), that \( \phi \in H^1_{\text{loc}}(\mathbb{R}^d) \) with \( \nabla \phi(x,o) = \Phi(\tau_o \omega) \), and that, for every \( \psi \in C^\infty_c(\mathbb{R}^d) \),
\[
\int_{\mathbb{R}^d} (a + s) \nabla \phi \cdot \nabla \psi = \int_{\mathbb{R}^d} f \cdot \nabla \psi,
\]
for \( a(x,\omega) = A(\tau_o \omega), s(x,\omega) = S(\tau_o \omega) \), and \( f(x,\omega) = F(\tau_o \omega) \). Since \( \nabla \cdot (\nabla \cdot s) = 0 \) almost surely on \( \mathbb{R}^d \), a repetition of the argument from Proposition 2.2 proves that, for every \( \psi \in C^\infty_c(\mathbb{R}^d) \),
\[
\int_{\mathbb{R}^d} s \nabla \phi \cdot \nabla \psi = \int_{\mathbb{R}^d} (\nabla \cdot s) \cdot \nabla \phi \psi.
\]
Therefore, almost surely for every \( \psi \in C^\infty_c(\mathbb{R}^d) \),
\[
\int_{\mathbb{R}^d} a \nabla \phi \cdot \nabla \psi + (\nabla \cdot s) \cdot \nabla \phi \psi = \int_{\mathbb{R}^d} f \cdot \nabla \psi.
\]
Let \( \eta : \mathbb{R}^d \to [0, 1] \) be a smooth function satisfying \( \eta = 1 \) on \( \overline{B}_1 \) and \( \eta = 0 \) on \( \mathbb{R}^d \setminus B_2 \), and for every \( R \in (0, \infty) \) let \( \eta_R(x) = \eta(x/R) \). For every \( \varepsilon \in (0, 1) \) let \( \rho^\varepsilon \in C_0^\infty(\mathbb{R}^d) \) be a standard convolution kernel of scale \( \varepsilon \in (0, 1) \). For every \( n \in \mathbb{N} \) let \( \phi_n = (\phi \land n) \lor (-n) \). Then for every \( R \in (0, \infty), \varepsilon \in (0, 1), \) and \( n \in \mathbb{N} \), the function \( (\phi_n \ast \rho^\varepsilon) \eta_R \) is an admissible test function for (43). Using the boundedness of \( \phi_n \) to pass to the limit \( \varepsilon \to 0 \), we have almost surely for every \( n \in \mathbb{N} \) that
\[
\int_{\mathbb{R}^d} a \nabla \phi \cdot \nabla \phi_{Rn} \eta_R + a \nabla \phi \cdot \nabla \eta_R \phi_n + ((\nabla \cdot s) \cdot \nabla \phi) \phi_n \eta_R = \int_{\mathbb{R}^d} f \cdot \nabla \phi_{Rn} \eta_R + f \cdot \nabla \eta_R \phi_n.
\]
The distributional equality \((\nabla \phi) \phi_n = \nabla (\phi \phi_n - 1/2 \phi_n^2)\), the fact that \((\nabla \cdot s)\) is divergence-free, and the skew-symmetry of \( s \) prove that
\[
\int_{\mathbb{R}^d} ((\nabla \cdot s) \cdot \nabla \phi) \phi_n \eta_R = -\int_{\mathbb{R}^d} ((\nabla \cdot s) \cdot \nabla \eta_R) (\phi \phi_n - 1/2 \phi_n^2) = \int_{\mathbb{R}^d} (s \nabla \phi \cdot \nabla \eta_R) \phi_n.
\]
It then follows from (44), (45), and the distributional equality \( \nabla \phi_n = \nabla \phi 1_{\{|\phi| \leq n\}} \) that
\[
\int_{\mathbb{R}^d} a \nabla \phi \cdot \nabla \phi 1_{\{|\phi| \leq n\}} \eta_R - \int_{\mathbb{R}^d} f \cdot \nabla \phi 1_{\{|\phi| \leq n\}} \eta_R
\]
\[
= \int_{\mathbb{R}^d} f \cdot \nabla \eta_R \phi_n - a \nabla \phi \cdot \nabla \eta_R \phi_n - (s \nabla \phi \cdot \nabla \eta_R) \phi_n.
\]
For each \( R \in (0, \infty) \) let \( c_R = \int_{\mathbb{R}^d} \eta_R \). It follows from the definition of \( \eta_R \) that \(|B_R| \leq c_R \leq \|B_{2R}\|\). We now make the choice \( n = R \). It then follows from the definition of \( \eta_R \), the definition of \( c_R \), the definition of \( \phi_n \), the uniform ellipticity, (46), and Hölder’s inequality that, for some \( c \in (0, \infty) \) independent of \( R \),
\[
\left| c_R^{-1} \int_{\mathbb{R}^d} a \nabla \phi \cdot \nabla \phi 1_{\{|\phi| \leq R\}} \eta_R - c_R^{-1} \int_{\mathbb{R}^d} f \cdot \nabla \phi 1_{\{|\phi| \leq R\}} \eta_R \right|
\]
\[
\leq c \left( \frac{1}{R} \left( \int_{B_{2R}} |\phi|^2 \right)^{\frac{1}{2}} \left( \int_{B_{2R}} |f|^2 + |\nabla \phi|^2 \right)^{\frac{1}{2}} + \frac{1}{R} \left( \int_{B_{2R}} |s|^2 \right)^{\frac{1}{2}} \left( \int_{B_{2R}} |\nabla \phi|^2 + 2 \phi_R \right)^{\frac{1}{2}} \right).
\]
The difficulty in the proof is that, since \( \phi \) is not itself stationary, it is not obvious for instance that
\[
\lim_{R \to \infty} c_R^{-1} \int_{\mathbb{R}^d} a \nabla \phi \cdot \nabla \phi 1_{\{|\phi| \leq R\}} \eta_R = \mathbb{E} [a \Phi \cdot \Phi],
\]
as is formally suggested by the ergodic theorem. We will prove (48) using the sublinearity of \( \phi \). For each \( R \in (0, \infty) \) we write
\[
c_R^{-1} \int_{\mathbb{R}^d} a \nabla \phi \cdot \nabla \phi 1_{\{|\phi| \leq R\}} \eta_R = c_R^{-1} \int_{\mathbb{R}^d} a \nabla \phi \cdot \nabla \phi \eta_R - c_R^{-1} \int_{\mathbb{R}^d} a \nabla \phi \cdot \nabla \phi 1_{\{|\phi| > R\}} \eta_R.
\]
Since the ergodic theorem proves almost surely that
\[
\mathbb{E} [A \Phi \cdot \Phi] = \lim_{R \to \infty} c_R^{-1} \int_{\mathbb{R}^d} a \nabla \phi \cdot \nabla \phi R,
\]
it remains only to prove almost surely that
\[
\lim_{R \to \infty} c_R^{-1} \int_{\mathbb{R}^d} a \nabla \phi \cdot \nabla \phi 1_{\{|\phi| > R\}} \eta_R = 0.
\]
Chebyshev’s inequality and the definition of \( \eta_R \) prove that, for \( c \in (0, \infty) \) independent of \( R \in (0, \infty) \),

\[
c_R^{-1} \left\| \{ \left| \phi \right| > R \} \cap \text{Supp}(\eta_R) \right\| \leq c_R^{-1} \left\| \{ \left| \phi \right| > R \} \cap B_{2R} \right\| \leq \frac{c}{R^2} \int_{B_{2R}} |\phi|^2,
\]

from which we almost surely conclude using Proposition 2.1 that

\[
\limsup_{R \to \infty} c_R^{-1} \left\| \{ \left| \phi \right| > R \} \cap \text{Supp}(\eta_R) \right\| \leq \limsup_{R \to \infty} \frac{c}{R^2} \int_{B_{2R}} |\phi|^2 = 0.
\]

We now exploit the stationarity of \( \nabla \phi \). For each \( R \in (0, \infty) \) and \( K \in \mathbb{N} \), the uniform ellipticity and the definitions of \( \eta_R \) and \( c_R \) prove that, for some \( c \in (0, \infty) \) independent of \( R \) and \( K \),

\[
\left\| c_R^{-1} \int_{\mathbb{R}^d} a \nabla \phi \cdot \nabla \phi 1_{\{|\phi| > R\}} \eta_R \right\| \leq \left\| c_R^{-1} \int_{\mathbb{R}^d} a \nabla \phi \cdot \nabla \phi 1_{\{|\phi| > R, |\nabla \phi| > K\}} \eta_R \right\| + c \left\| \int_{B_{2R}} |\nabla \phi|^2 1_{\{|\nabla \phi| > K\}} \right\|.
\]

After applying the dominated convergence theorem, the ergodic theorem, the stationarity of \( \nabla \phi \), and (50) to (51), we have almost surely for every \( K \in \mathbb{N} \) that

\[
\limsup_{R \to \infty} c_R^{-1} \left\| \int_{\mathbb{R}^d} a \nabla \phi \cdot \nabla \phi 1_{\{|\phi| > R\}} \eta_R \right\| \leq c \mathbb{E} \left( |\Phi|^2 1_{\{|\phi| > K\}} \right).
\]

Therefore, since \( \Phi \in L^2_{\text{pot}}(\Omega) \), after passing to the limit \( K \to \infty \) we conclude the proof of (49) and therefore the proof of (48). The identical proof shows almost surely that

\[
\lim_{R \to \infty} c_R^{-1} \int_{\mathbb{R}^d} f \cdot \nabla \phi 1_{\{|\phi| \leq R\}} = \mathbb{E} [F \cdot \Phi].
\]

It remains to treat the two terms on the righthand side of (47). Proposition 2.1, \( F, \Phi \in L^2(\Omega; \mathbb{R}^d) \), and the ergodic theorem prove almost surely that

\[
\lim_{R \to \infty} \left[ \frac{1}{R} \left( \int_{B_{2R}} |\phi|^2 \right)^{\frac{1}{2}} \left( \int_{B_{2R}} |\nabla \phi|^2 \phi_R^2 \right)^{\frac{1}{2}} \right] = 0.
\]

For the final term on the righthand side of (47), it follows from the definition of \( \phi_R \) and the triangle inequality that, for each \( K \in \mathbb{N} \) and \( R \in (0, \infty) \),

\[
\frac{1}{R} \left( \int_{B_{2R}} |s|^2 \right)^{\frac{1}{2}} \left( \int_{B_{2R}} |\nabla \phi|^2 \phi_R^2 \right)^{\frac{1}{2}} \leq \frac{1}{R} \left( \int_{B_{2R}} |s|^2 \right)^{\frac{1}{2}} \left( \int_{B_{2R} \cap \{|\nabla \phi| > K\}} |\nabla \phi|^2 \phi_R^2 \right)^{\frac{1}{2}} + \frac{1}{R} \left( \int_{B_{2R}} |s|^2 \right)^{\frac{1}{2}} \left( \int_{B_{2R} \cap \{|\nabla \phi| > K\}} |\nabla \phi|^2 \phi_R^2 \right)^{\frac{1}{2}} \leq \frac{K}{R} \left( \int_{B_{2R}} |s|^2 \right)^{\frac{1}{2}} \left( \int_{B_{2R}} \phi^2 \right)^{\frac{1}{2}} + \left( \int_{B_{2R} \cap \{|\nabla \phi| > K\}} |\nabla \phi|^2 \right)^{\frac{1}{2}} \frac{1}{R} \left( \int_{B_{2R} \cap \{|\nabla \phi| > K\}} |\nabla \phi|^2 \right)^{\frac{1}{2}}.
\]
The ergodic theorem, the stationarity of $\nabla \phi$, $S \in L^2(\Omega; \mathbb{R}^{d \times d})$, and Proposition 2.1 then prove almost surely that, for some $c \in (0, \infty)$ independent of $K \in \mathbb{N}$,

$$
\limsup_{R \to \infty} \frac{1}{R} \left( \frac{1}{|B_{2R}|} \int_{B_{2R}|} |s|^2 \right)^{\frac{1}{2}} \left( \frac{1}{|B_{2R}|} \int_{B_{2R}|} |\nabla \phi|^2 \phi_R^2 \right)^{\frac{1}{2}} \leq c \mathbb{E} \left[ |\Phi|^2 1_{\{|\Phi| > K\}} \right].
$$

Since $\Phi \in L^2_{\text{pot}}(\Omega)$, after passing to the limit $K \to \infty$ we conclude almost surely that

$$
\limsup_{R \to \infty} \frac{1}{R} \left( \frac{1}{|B_{2R}|} \int_{B_{2R}|} |s|^2 \right)^{\frac{1}{2}} \left( \frac{1}{|B_{2R}|} \int_{B_{2R}|} |\nabla \phi|^2 \phi_R^2 \right)^{\frac{1}{2}} = 0.
$$

In combination (47), (48), (52), (53), and (54) prove that

$$
\mathbb{E} [A \Phi \cdot \Phi] = \mathbb{E} [F \cdot \Phi],
$$

which complete the proof of the energy identity.

It remains only to prove uniqueness. Suppose that $\Phi_1, \Phi_2 \in L^2_{\text{pot}}(\Omega)$ satisfy (38) and (39). Then by linearity the difference $\Phi_1 - \Phi_2$ satisfies both (38) and (39) with $F = 0$. The uniform ellipticity and the energy identity (55) prove that

$$
\lambda \mathbb{E} \left[ |\Phi_1 - \Phi_2|^2 \right] \leq \mathbb{E} [A \cdot (\Phi_1 - \Phi_2) \cdot (\Phi_1 - \Phi_2)] = 0,
$$

which proves that $\Phi_1 = \Phi_2$ in $L^2_{\text{pot}}(\Omega)$ and completes the proof. \qed

2.2. The homogenization flux corrector. In this section, we will assume that the stream matrix $S \in L^{d/(2+\delta)}(\Omega; \mathbb{R}^{d \times d})$, for some $\delta \in (0, 1)$, and construct the skew-symmetric flux correctors $\sigma_i$ satisfying (13). Let $p_d \in (1, 2)$ denote the integrability exponent

$$
p_d = \frac{2d}{d+2} \quad \text{if} \quad d \geq 3 \quad \text{and} \quad p_d = \frac{4 + 2\delta}{4 + \delta} \quad \text{if} \quad d = 2,
$$

and for each $i \in \{1, \ldots, d\}$, using Hölder’s inequality, let $Q_i \in L^{p_d}(\Omega; \mathbb{R}^d)$ be defined by

$$
Q_i = (A + S)(\Phi_i + e_i),
$$

for the corrector fields $\Phi_i \in L^2_{\text{pot}}(\Omega)$ satisfying (18) constructed in Proposition 2.4. We will identify the flux correctors $\sigma_i = (\sigma_{ijk})$ by their stationary gradients $\Sigma_{ijk}$ satisfying the equation

$$
-D \cdot \Sigma_{ijk} = D_j Q_{ik} - D_k Q_{ij} \quad \text{in} \quad \mathcal{D}'(\Omega).
$$

We construct the $\Sigma_{ijk}$ in Proposition 2.5 below. In Proposition 2.6 below, we prove that the resulting skew-symmetric matrices $\sigma_i$ defined on $\mathbb{R}^d$ by integration almost surely satisfy $\nabla \cdot \sigma_i = q_i$ for $q_i(x, \omega) = Q_i(\tau_{x,\omega}) - \mathbb{E}[Q_i]$. The proof of existence and uniqueness for the flux correctors is an extension of [14, Lemma 1].

**Proposition 2.5.** Assume (12), let $p \in (1, \infty)$, and let $F \in L^p(\Omega; \mathbb{R}^d)$. Then there exists a unique weak solution $\Phi \in L^p(\Omega; \mathbb{R}^d)$ of the equation

$$
-D \cdot \Phi = -D \cdot F \quad \text{in} \quad \mathcal{D}'(\Omega),
$$

with $\mathbb{E}[\Phi] = 0$ and such that, for every $i, j \in \{1, 2, \ldots, d\}$,

$$
D_i \Phi_j = D_j \Phi_i \quad \text{in} \quad \mathcal{D}'(\Omega).
$$
PROOF. Let $p \in (1, \infty)$. We will first consider a smooth righthand side $F = (F_1, \ldots, F_d) \in D(\Omega)^d$. The Lax-Milgram theorem proves that, for every $\alpha \in (0, 1)$, there exists a unique solution $\Phi_\alpha \in L^2_{\text{pot}}(\Omega)$ of the equation
\[
\alpha \Phi_\alpha - D \cdot D \Phi_\alpha = -D \cdot F \text{ in } D'(\Omega).
\]
Since $\Phi_\alpha \in L^2_{\text{pot}}(\Omega)$ is mean zero and curl-free, define by integration $\phi_\alpha : \mathbb{R}^d \times \Omega \to \mathbb{R}$ which almost surely satisfies $\phi_\alpha \in H^1_{\text{loc}}(\mathbb{R}^d)$, $\nabla \phi_\alpha(x, \omega) = \Phi_\alpha(\tau_{x, \omega})$, and
\[
a \phi_\alpha - \Delta \phi_\alpha = -\nabla \cdot f \text{ in } \mathbb{R}^d,
\]
for $f(x, \omega) = F(\tau_{x, \omega})$. Due to the $C^1$-boundedness of $f$, it follows from the Feynman-Kac formula that
\[
\phi_\alpha(x, \omega) = \int_0^\infty \int_{\mathbb{R}^d} e^{-\alpha s} (4\pi s)^{-d/2} \exp(-|x-y|^2/4s) (-\nabla_y \cdot f(y, \omega)) \, dy \, ds,
\]
from which a direct computation proves almost surely that, for $c \in (0, \infty)$ independent of $\alpha \in (0, 1)$,
\[
\|\nabla \phi_\alpha\|_{L^\infty(\mathbb{R}^d, \mathbb{R}^d)} \leq \frac{c}{\alpha}.
\]
Therefore, almost surely,
\[
\nabla \phi_\alpha(x, \omega) = \int_0^\infty \int_{\mathbb{R}^d} e^{-\alpha s} (4\pi s)^{-d/2} \nabla_x \nabla_y (\exp(-|x-y|^2/4s)) f(y, \omega) \, dy \, ds
\]
\[= \int_{\mathbb{R}^d} \nabla_x \nabla_y K_\alpha(x, y) f(y, \omega) \, dy,
\]
for $K_\alpha(x, y) = \int_0^\infty e^{-\alpha s} (4\pi s)^{-d/2} \exp(-|x-y|^2/4s) \, ds$. A direct computation proves that, for $c \in (0, \infty)$ independent of $\alpha \in (0, 1)$,
\[
|\nabla_x K_\alpha(x, y)| + |\nabla_y K_\alpha(x, y)| \leq c |x-y|^{1-d} e^{-\sqrt{\alpha}|x-y|},
\]
and, for $c \in (0, \infty)$ independent of $\alpha \in (0, 1)$,
\[
|\nabla_x \nabla_y K_\alpha(x, y)| \leq c |x-y|^{-d} e^{-\sqrt{\alpha}|x-y|}.
\]
Therefore, $\nabla_x \nabla_y K_\alpha(x, y)$ defines a Calderon-Zygmund kernel (see, for example, Stein [54]). Let $\eta : \mathbb{R}^d \to [0, 1]$ be a smooth function satisfying $\eta = 1$ on $B_1$ and $\eta = 0$ on $\mathbb{R}^d \setminus B_2$, and for every $R \in (0, \infty)$ let $\eta_R(x) = \eta(x/R)$. For each $R \in (0, \infty)$ let
\[
\nabla \phi_{\alpha,R}(x, \omega) = \int_{\mathbb{R}^d} \nabla_x \nabla_y K_\alpha(x, y) \eta_R(y) f(y, \omega) \, dy.
\]
It follows almost surely from (60) that, for constants $c_1, c_2 \in (0, \infty)$ independent of $\alpha \in (0, 1)$,
\[
\sup_{x \in B_{R/2}} |\nabla \phi_{\alpha}(x, \omega) - \nabla \phi_{\alpha,R}(x, \omega)| \leq c_1 \|F\|_{L^\infty(\Omega)} (\sqrt{\alpha}R)^{-1} \exp(-c_2 \sqrt{\alpha}R).
\]
It follows from (60) and the Calderon-Zygmund estimate (see, for example, [54]) that there exists $c \in (0, \infty)$ depending on $p$ and $d$ such that
\[
\int_{\mathbb{R}^d} |\nabla \phi_{\alpha,R}|^p \leq c \int_{\mathbb{R}^d} |\eta_R f|^p.
\]
In combination (61), (62), and the definition of \( \eta_R \) prove almost surely that, for every \( R \in (0, \infty) \), for \( c \in (0, \infty) \) depending on \( p \) and \( d \) but independent of \( R \in (0, \infty) \),

\[
\int_{B_R} |\nabla \phi_\alpha|^p \leq c \int_{B_R} |f|^p + c_1 \|F\|_{L^\infty(\Omega)} (\sqrt{\alpha R})^{-1} \exp(-c_2 \sqrt{\alpha R}).
\]

Therefore, after passing to the limit \( R \to \infty \), the ergodic theorem and (59) prove that, for \( c \in (0, \infty) \) depending on \( p \) and \( d \),

\[
\mathbb{E} [\|D \Phi_\alpha\|^p] \leq c \mathbb{E} [\|F\|^p].
\]

Then, after passing to a subsequence \( \Phi \to 0 \), the weak lower- semicontinuity of the Sobolev norm proves that there exists \( \hat{\Phi} \in L^p_{\text{pot}}(\Omega) \) satisfying, for \( c \in (0, \infty) \) depending on \( p \) and \( d \),

\[-D \cdot \Phi = -D \cdot F \text{ in } D'(\Omega) \text{ with } \mathbb{E} [\|\Phi\|^p] \leq c \mathbb{E} [\|F\|^p].\]

The proof of existence for \( F \in L^p(\Omega; \mathbb{R}^d) \) then follows from the density of \( D(\Omega) \) in \( L^p(\Omega) \), the definition of \( L^p_{\text{pot}}(\Omega) \), and the weak lower- semicontinuity of the Sobolev norm. It remains to prove uniqueness.

By linearity, it suffices to prove that the only \( \Phi \in L^p(\Omega; \mathbb{R}^d) \) satisfying (57) and (58) with \( F = 0 \) is \( \Phi = 0 \). Since \( \Phi \) is mean zero and curl-free let \( \phi: \mathbb{R}^d \times \Omega \to \mathbb{R} \) be the unique function that almost surely satisfies \( \int_{\Omega} \phi = 0 \), that \( \phi \in W^{1,p}(\mathbb{R}^d) \) with \( \nabla \phi(x, \omega) = \Phi(x, \omega) \), and that \( \phi \) is a weak solution of \( -\Delta \phi = 0 \) on \( \mathbb{R}^d \).

For every \( \epsilon \in (0, 1) \) let \( \rho^\epsilon \in C_0^\infty(\mathbb{R}^d) \) be a standard convolution kernel of scale \( \epsilon \) and let \( \phi^\epsilon = u \ast \rho^\epsilon \). Then \( \phi^\epsilon \) is almost surely harmonic on \( \mathbb{R}^d \) and the Feynman-Kac formula proves that there exists \( c \in (0, \infty) \) such that, for every \( \epsilon \in (0, 1) \) and \( t \in (0, \infty) \),

\[
|\nabla \phi^\epsilon(0)| = \left| \int_{\mathbb{R}^d} \phi^\epsilon(y) \phi^\epsilon(y)(4\pi t)^{-d/2} y/2 \exp\left(-|y|^2/4t\right) dy \right| \leq c \left( \int_{\mathbb{R}^d} \frac{\phi^\epsilon(\sqrt{t}y)}{\sqrt{t}} |y| \exp\left(-|y|^2/4t\right) dy \right).
\]

For each \( R \in (0, \infty) \), there exists \( c \in (0, \infty) \) independent of \( R \) such that

\[
|\nabla \phi^\epsilon(0)| \leq c \left( \int_{\mathbb{R}^d} \frac{\phi^\epsilon(\sqrt{t}y)}{\sqrt{t}} dy + \int_{R} \left( \frac{\phi^\epsilon(0)}{\sqrt{t}} + \int_{B_R} \left| \nabla \phi^\epsilon(\sqrt{t}y) \right| dy \right) r^{2d} e^{-\frac{r^2}{4t}} dr \right).
\]

Proposition 2.1, the ergodic theorem, and \( \Phi \in L^p(\Omega; \mathbb{R}^d) \) prove almost surely for some \( c \in (0, \infty) \) that, after passing to the limit \( t \to \infty \),

\[
|\nabla \phi^\epsilon(0)| \leq c \mathbb{E} [\|\Phi\|^p] \int_{R} \int_{R} r^{2d} e^{-\frac{r^2}{4t}} dr,
\]

for \( \Phi^\epsilon(\omega) = \int_{\mathbb{R}^d} \Phi(x, \omega) \rho^\epsilon(x) dx \). After passing to the limit \( R \to \infty \), we conclude almost surely that \( |\nabla \phi^\epsilon(0)| = 0 \) and therefore by stationarity that \( \Phi^\epsilon = 0 \). After passing to the limit \( \epsilon \to 0 \), we conclude that \( \Phi = 0 \). This completes the proof. \( \square \)

**Proposition 2.6.** Assume (12) for \( S \in L^d \) if \( d \geq 3 \) and for \( S \in L^{2+\delta} \) for some \( \delta \in (0, 1) \) if \( d = 2 \) and let \( p_d \in (1, \infty) \) be defined in (56). For every \( i, j, k \in \{1, \ldots, d\} \) let \( \Sigma_{ijk} \in L^{p_d}(\Omega; \mathbb{R}^d) \) be the unique solution of

\[-D \cdot \Sigma_{ijk} = D_j Q_{ik} - D_k Q_{ij} \text{ in } D'(\Omega),\]
defined in Proposition 2.5 and let \( \sigma_{ijk} : \mathbb{R}^d \times \Omega \to \mathbb{R} \) be the unique function that almost surely satisfies \( \int_{B_1} \sigma_{ijk} = 0, \sigma_{ijk} \in W^{1,p_d}(\mathbb{R}^d) \) with \( \nabla \sigma(x, \omega) = \Sigma_{ijk}(\tau_x \omega) \), and, for every \( \psi \in C^\infty_c(\mathbb{R}^d), \)

\[
\int_{\mathbb{R}^d} \nabla \sigma_{ijk} \cdot \nabla \psi = \int_{\mathbb{R}^d} \partial_k \psi q_{ij} - \partial_j \psi q_{ik},
\]

for \( q_i(x, \omega) = Q_i(\tau_x \omega) - \mathbb{E}[Q_i] \). Then for every \( i \in \{1, \ldots, d\} \) the matrix \( \sigma_i = (\sigma_{ij}) \) is skew-symmetric and almost surely satisfies

\[
(66) \quad \nabla \cdot \sigma_i = q_i \quad \text{in} \quad \mathbb{R}^d \quad \text{for} \quad (\nabla \cdot \sigma_i)_j = \partial_k \sigma_{ijk}.
\]

**Proof.** Let \( i, j, k \in \{1, \ldots, d\} \). It follows from the uniqueness of Proposition 2.5 that \( \Sigma_{ijk} = -\Sigma_{ikj} \) and therefore it follows from the definition of \( \sigma_{ijk} \) that \( \sigma_{ijk} = -\sigma_{ikj} \). This proves that \( \sigma_i \) is skew-symmetric. It remains only to prove the equality (66). This will follow from the distributional equality, for every \( j \in \{1, \ldots, d\} \),

\[
\Delta ( (\nabla \cdot \sigma_i)_j - q_{ij} ) = 0.
\]

Indeed, using the equation satisfied by the \( \sigma_{ijk} \) and the fact that \( q_i \) is divergence-free, for each \( j \in \{1, \ldots, d\} \) we have as distributions that

\[
\Delta ( (\nabla \cdot \sigma_i)_j - q_{ij} ) = \partial_s \partial_s (\partial_k \sigma_{sj} \sigma_{kj} - q_{ij}) = \Delta q_{ij} - \partial_k \partial_j q_{ik} - \Delta q_{ij} = -\partial_j (\nabla \cdot q_{ik}) = 0.
\]

Equation (67) proves that, for standard convolution kernels \( \rho^\varepsilon \in C^\infty_c(\mathbb{R}^d) \) of scale \( \varepsilon \in (0, 1) \), for every \( j \in \{1, \ldots, d\} \) and \( \varepsilon \in (0, 1) \),

\[
\Delta [( (\nabla \cdot \sigma_i)_j - q_{ij}) * \rho^\varepsilon] = 0 \quad \text{in} \quad \mathbb{R}^d.
\]

A repetition of the arguments leading to (64) and (65) in the proof of Proposition 2.5 proves that, for each \( j \in \{1, \ldots, d\} \) there exists \( c^j_\varepsilon \in L^\infty(\Omega) \) such that almost surely

\[
[( (\nabla \cdot \sigma_i)_j - q_{ij}) * \rho^\varepsilon](x, \omega) = c^j_\varepsilon(\omega) \quad \text{for every} \quad x \in \mathbb{R}^d.
\]

Since the gradient fields \( \Sigma_{ijk} \) are mean zero, the stationarity of the gradient, the stationarity of the flux, and the definition of the \( q_i \) prove almost surely with the ergodic theorem that, for every \( j \in \{1, \ldots, d\} \),

\[
0 = \lim_{R \to \infty} \int_{B_R} ( (\nabla \cdot \sigma_i)_j - q_{ij}) * \rho^\varepsilon = c^j_\varepsilon(\omega).
\]

After passing to the limit \( \varepsilon \to 0 \), we have almost surely that

\[
\nabla \cdot \sigma_i = q_i \quad \text{in} \quad \mathbb{R}^d.
\]

2.3. The stream matrix. In Proposition 2.7 below, we will prove that every mean zero, divergence-free, \( L^p \)-integrable vector field \( B \) satisfying a finite-range of dependence admits an \( L^p \)-integrable stream matrix provided \( p \in [2, \infty) \) and the dimension \( d \geq 3 \). We assume a finite range of dependence for simplicity: that is, there exists \( R \in (0, \infty) \) such that for subsets \( A_1, A_2 \subseteq \mathbb{R}^d \) the sigma algebras

\[
\sigma(B(\tau_x \omega) : x \in A_1) \quad \text{and} \quad \sigma(B(\tau_x \omega) : x \in A_2)
\]

are independent whenever \( d(A_1, A_2) \geq R \).

In the case \( p = 2 \), for instance, the same proof yields the existence of a stationary stream matrix provided the spatial correlations of \( B \) decay faster than a square. This result is a small modification of the analogous results in [41] and [42, Proposition 4, Proposition 5], and extends these results to the case \( p \in (2, \infty) \).
PROPOSITION 2.7. Assume (12), let $d \in [3, 4, \ldots)$, let $p \in [2, \infty)$, and let $B \in L^p(\Omega; \mathbb{R}^d)$ satisfy $\mathbb{E}[B] = 0$, $D \cdot B = 0$, and, for some $R \in (0, \infty)$, for every $A_1, A_2 \subseteq \mathbb{R}^d$,

$$\sigma(B(\tau_x\omega): x \in A_1) \text{ and } \sigma(B(\tau_x\omega): x \in A_2)$$

are independent whenever $d(A_1, A_2) \geq R$. Then there exists skew-symmetric matrix $S = (S_{jk})_{j,k \in \{1, \ldots, d\}} \in L^p(\Omega; \mathbb{R}^{d \times d})$ that satisfies

$$D \cdot S = B \text{ in } L^p(\Omega; \mathbb{R}^d).$$

PROOF. Let $\mathcal{F}_B$ denote the sigma algebra generated by $B$. It follows from (68) that every $\mathcal{F}_B$-measurable random variable satisfies a finite range of dependence. Let $X = (X_i)_{i \in \{1, \ldots, d\}} \in L^\infty(\Omega; \mathbb{R}^d)$ be $\mathcal{F}_B$-measurable and for every $\alpha \in (0, 1)$ let $S_{\alpha} \in \mathcal{H}^1(\Omega)$ denote the unique Lax-Milgram solution of the equation

$$\alpha S_{\alpha} - D \cdot D S_{\alpha} = D \cdot X.$$

Due to the boundedness of $X$, we have the representation

$$S_{\alpha}(\omega) = \int_0^\infty \int_{\mathbb{R}^d} (4\pi s)^{-\frac{d}{2}} e^{-\alpha s - \frac{|x|^2}{2s}} \cdot X(\tau_x\omega) \, dx \, ds$$

$$= (4\pi)^{-\frac{d}{2}} \int_0^\infty \int_{\mathbb{R}^d} |x|^{1-d} s^{-(\frac{d}{2}+1)} e^{-\alpha s|x|^2 - \frac{1}{2} \frac{x}{s} \cdot X(\tau_x\omega)} \, dx \, ds.$$ 

Let $q \in \{2, 4, 6, \ldots\}$ be a nonzero even integer and let $\mathcal{I}_q$ denote the collection of partitions of $\{1, 2, \ldots, q\}$ of the form

$$\mathcal{I}_q = \{\beta = (\beta_1, \ldots, \beta_{N(\beta)}): \beta_j \in \{2, 3, \ldots\} \forall j \in \{1, \ldots, N(\beta)\}, \text{ and } \sum_{j=1}^{N(\beta)} \beta_j = q\},$$

which are exactly the partitions of $\{1, 2, \ldots, q\}$ that contain no singletons. We define for every $\beta \in \mathcal{I}_q$ the integral

$$I_{\beta} = \prod_{j=1}^{N(\beta)} \int_{\mathbb{R}^d} \int_{B_{\nu}(x_{j,k})} \beta_j \cdot |x_k|^{1-d} \, dx_{\beta_j} \, dx_{\beta_{j-1}} \ldots \, dx_{\beta_1}$$

and observe from the assumption $d \geq 3$ and the fact that $\beta_j \geq 2$ for every $j \in \{1, \ldots, N(\beta)\}$ that, for some $c \in (0, \infty)$ independent of $\alpha \in (0, 1)$, for every $\beta \in \mathcal{I}_q$,

$$I_{\beta} \leq c \prod_{j=1}^{N(\beta)} \int_{\mathbb{R}^d} \left(1 \wedge |x_{\beta_j}|^{1-d}\right)^{\beta_j} \, dx_{\beta_j} \leq c \left(\int_0^\infty (1 \wedge r^{(1-d)}) \, dr\right)^{N(\beta)} < \infty.$$ 

It then follows from the $\mathcal{F}_B$-measurability of $X$, (68), the fact that transformation group preserves the measure, Hölder’s inequality, and an explicit calculation based on (69) that, for some $c \in (0, \infty)$ independent of $\alpha \in (0, 1)$,

$$\mathbb{E}[S_{\alpha}^q] \leq c \mathbb{E}[|X|^q] \sum_{\beta \in \mathcal{I}_q} I_{\beta} \leq c \mathbb{E}[|X|^q].$$ 

Since it follows from (63) that, for some $c \in (0, \infty)$ independent of $\alpha \in (0, 1)$,

$$\mathbb{E}[|D S_{\alpha}|^q] \leq c \mathbb{E}[|X|^q],$$

it follows after passing to a subsequence $\alpha \to 0$ that there exists $S \in L^q(\Omega) \cap \mathcal{H}^1(\Omega)$ with $S_{\alpha} \rightharpoonup S$ weakly in $L^q(\Omega)$ and $DS_{\alpha} \rightharpoonup DS$ weakly in $L^q(\Omega; \mathbb{R}^d)$.

$$S_{\alpha} \rightharpoonup S \text{ weakly in } L^q(\Omega) \text{ and } D S_{\alpha} \rightharpoonup D S \text{ weakly in } L^q(\Omega; \mathbb{R}^d).$$
It follows from Proposition 2.5, (70), (71), and the weak lower-semicontinuity of the Sobolev norm that \( DS \in L^q(\Omega; \mathbb{R}^d) \) is the unique curl-free, mean zero solution of
\[
- D \cdot DS = D \cdot X \quad \text{in} \quad D'(\Omega),
\]
and that, for some \( c \in (0, \infty) \) depending on \( q \in \{2, 4, 6, \ldots\} \) but independent of \( X \),
\[
\mathbb{E} [ |S|^q + |DS|^q] \leq c \mathbb{E} [ |X|^q].
\]

The density of bounded functions in \( L^q(\Omega) \) for every \( q \in \{2, 4, 6, \ldots\} \) proves that, for every \( F_B \)-measurable \( X \in L^q(\Omega; \mathbb{R}^d) \) there exists a unique \( S \in L^q(\Omega) \cap \mathcal{H}^1(\Omega) \) with \( DS \in L^q(\Omega; \mathbb{R}^d) \) that satisfies (72) and (73). Finally, since \( q \in \{2, 4, 6, \ldots\} \) was arbitrary, it follows from the Riesz-Thorin interpolation theorem applied to the spaces \( L^p(\Omega, \mathcal{F}_B) \) for \( p \in [2, \infty) \) that for every \( F_B \)-measurable \( X \in L^p(\Omega; \mathbb{R}^d) \) there exists a unique \( S \in L^p(\Omega) \cap \mathcal{H}^1(\Omega) \) with \( DS \in L^p(\Omega; \mathbb{R}^d) \) satisfying (72) and (73).

Now let \( B = (B_i)_{i \in \{1, \ldots, d\}} \in L^p(\Omega; \mathbb{R}^d) \) for some \( p \in [2, \infty) \) be mean zero and divergence-free in the sense that \( \mathbb{E} [B] = 0 \) and \( D \cdot B = 0 \), and let \( B \) satisfy a finite range of dependence. For every \( j, k \in \{1, \ldots, d\} \) let \( S_{jk} \in L^2(\Omega) \cap \mathcal{H}^1(\Omega) \) be the unique solution of
\[
- D \cdot DS_{jk} = D_j B_k - D_k B_j.
\]
The uniqueness proves that \( S_{jk} = -S_{kj} \) for every \( j, k \in \{1, \ldots, d\} \) and it follows from Proposition 2.6 and \( \mathbb{E} [B] = 0 \) that for \( S = (S_{jk})_{j,k \in \{1, \ldots, d\}} \) we have \( D \cdot S = B \) in \( L^p(\Omega; \mathbb{R}^d) \). This completes the proof.

2.4. The homogenized coefficient field. In the final subsection of this section, we will summarize the essential properties of the homogenized coefficient \( \bar{\pi} \in \mathbb{R}^{d \times d} \) defined in (10). Precisely, for every \( i \in \{1, \ldots, d\} \), let \( \Phi_i \in L^2_{\text{pot}}(\Omega) \) be the unique solution of
\[
- D \cdot (A + S)(\Phi_i + e_i) = 0 \quad \text{in} \quad L^2_{\text{pot}}(\Omega).
\]
The homogenized coefficient is the unique element \( \bar{\pi} \in \mathbb{R}^{d \times d} \) that satisfies, every \( i \in \{1, \ldots, d\} \),
\[
\bar{\pi} e_i = \mathbb{E} [(A + S)(\Phi_i + e_i)].
\]

We prove in Proposition 2.8 that \( \bar{\pi} \) is uniformly elliptic, and in Proposition 2.9 we will show that the transpose \( \bar{\pi}' \) is characterized by the formula, for every \( i \in \{1, \ldots, d\} \),
\[
\bar{\pi}' e_i = \mathbb{E} [(A' - S)(\Phi_i' + e_i)],
\]
for \( \Phi_i' \) the unique solution of the equation
\[
- D \cdot (A' - S)(\Phi_i' + e_i) = 0 \quad \text{in} \quad L^2_{\text{pot}}(\Omega).
\]

Observe, in particular, the important role of the energy equality (39) in proving both of these statements. The energy inequality is precisely what guarantees that the operator \( S \) is skew-symmetric on the solution space.

**Proposition 2.8.** Assume (12) for some \( S \in L^2(\Omega; \mathbb{R}^{d \times d}) \). Let \( \bar{\pi} \in \mathbb{R}^{d \times d} \) be defined by (74). Then, for every \( \xi \in \mathbb{R}^d \),
\[
|\bar{\pi}\xi| \leq 2 \left( A + \mathbb{E} \left[ |S|^2 \right] \right)^{\frac{1}{2}} \left( \sum_{i=1}^d \mathbb{E} [ |\Phi_i + e_i|^2] \right)^{\frac{1}{2}} |\xi| \quad \text{and} \quad \bar{\pi} \xi : \xi \geq \lambda |\xi|^2.
\]
PROOF. The uniform ellipticity, Hölder’s inequality, the linearity, and the definition of $\overline{a}$ prove that, for every $\xi \in \mathbb{R}^d$,

$$|\overline{a}\xi| = |\xi|\mathbb{E}[(A + S)(\Phi_i + e_i)] \leq 2 \left(A + \mathbb{E}[|S|^2]\right)^{\frac{1}{2}} |\xi| \mathbb{E} \left[|\Phi_i + e_i|^2\right]^{\frac{1}{2}} \leq 2 \left(A + \mathbb{E}[|S|^2]\right)^{\frac{1}{2}} \left(\sum_{i=1}^d \mathbb{E}[|\Phi_i + e_i|^2]\right)^{\frac{1}{2}} |\xi|.$$ 

Similarly it follows by definition that

$$\overline{a}\xi \cdot \xi = \xi^2 \mathbb{E}[(A + S)(\Phi_i + e_i) \cdot e_i],$$

and it follow from the skew-symmetry of $S$ and the corrector equation that

$$\overline{a}\xi \cdot \xi = \xi^2 \mathbb{E}[(A + S)(\Phi_i + e_i) \cdot (\Phi_i + e_i)] \geq \xi^2 \mathbb{E}[A(\Phi_i + e_i) \cdot (\Phi_i + e_i)] + \xi^2 \mathbb{E}[S(\Phi_i + e_i) \cdot (\Phi_i + e_i)] = \xi^2 \mathbb{E}[A(\Phi_i + e_i) \cdot (\Phi_i + e_i)] + \xi^2 \mathbb{E}[S\Phi_i \cdot \Phi_i],$$

where the energy identity (39) proves that $\mathbb{E}[S\Phi_i \cdot \Phi_i] = 0$. The uniform ellipticity, Jensen’s inequality, and $\mathbb{E}[\Phi_i] = 0$ then prove that

$$\overline{a}\xi \cdot \xi \geq \lambda \xi^2 \mathbb{E}[|\Phi_i + e_i|^2] \geq \lambda \xi^2 \mathbb{E}[|\Phi_i + e_i|^2] = \lambda |\xi|^2,$$

which completes the proof. \hfill \Box

**Proposition 2.9.** Assume (12) for some $S \in L^2(\Omega; \mathbb{R}^{d \times d})$ and for every $i \in \{1, \ldots, d\}$ let $\Phi_i \in L^2_{\text{reg}}(\Omega)$ be the unique solution of the equation

$$-D \cdot (A^t - S)(\Phi_i^t + e_i) = 0 \text{ in } L^2_{\text{reg}}(\Omega).$$

Then $\overline{a} \in \mathbb{R}^{d \times d}$ defined in (74) satisfies, for every $i \in \{1, \ldots, d\}$,

$$\overline{a}^t e_i = \mathbb{E}[(A^t - S)(\Phi_i^t + e_i)],$$

for $\overline{a}^t$ the transpose of $\overline{a}$.

**PROOF.** Let $\tilde{a} = (\tilde{a}_{ij}) \in \mathbb{R}^{d \times d}$ be the coefficient field defined for every $i \in \{1, \ldots, d\}$ by

$$(75) \quad \tilde{a} e_i = \mathbb{E}[(A^t - S)(\Phi_i^t + e_i)].$$

It follows from the corrector equation that, for $\overline{a} = (\overline{a}_{ij})$, for every $i, j \in \{1, \ldots, d\}$,

$$\overline{a}_{ij} = \mathbb{E}[(A + S)(\Phi_j + e_j)] \cdot e_i = \mathbb{E}[(A + S)(\Phi_j + e_j) \cdot (\Phi_i^t + e_i)].$$

It then follows from the skew-symmetry of $S$ and the uniform ellipticity of $A$ that

$$(76) \quad \overline{a}_{ij} = \mathbb{E}[A^t(\Phi_i^t + e_i) \cdot (\Phi_j + e_j)] - \mathbb{E}[S e_i \cdot (\Phi_j + e_j)] - \mathbb{E}[S \Phi_j \cdot \Phi_i^t] + \mathbb{E}[S \Phi_j \cdot \Phi_i^t] + \mathbb{E}[S \Phi_j \cdot \Phi_i^t].$$

The essential point is therefore to prove that $\mathbb{E}[S \Phi_j \cdot \Phi_i^t] = -\mathbb{E}[S \Phi_j^t \cdot \Phi_i]$ which is obvious if $S$ is bounded but not otherwise. To prove this, we observe using the respective corrector equations that the difference $\Phi_j - \Phi_i^t$ satisfies the equation

$$-D \cdot (A + S)(\Phi_j - \Phi_i^t) = D \cdot (A + S)e_j + D \cdot A\Phi_i^t + D \cdot S\Phi_i^t$$

$$= D \cdot (A + S)e_i + D \cdot A\Phi_i^t + D \cdot A\Phi_i^t + D \cdot (A^t - S)e_i$$

$$= D \cdot G,$$
for $G = (A + S)e_j + (A + A^t)\Phi^t_i + (A^t - S)e_i \in L^2(\Omega; \mathbb{R}^d)$. Since we have that

$$\mathbb{E} \left[ S(\Phi_j - \Phi^t_i) \cdot (\Phi_j - \Phi^t_i) \right] = \mathbb{E}[S\Phi_j \cdot \Phi_j] + \mathbb{E} \left[ S\Phi^t_i \cdot \Phi^t_i \right] - \mathbb{E} \left[ S\Phi_j \cdot \Phi^t_i \right] - \mathbb{E} \left[ S\Phi^t_i \cdot \Phi_j \right],$$

and since it follows from the energy inequality (38) as applied in Proposition 2.8 that

$$\mathbb{E} \left[ S(\Phi_j - \Phi^t_i) \cdot (\Phi_j - \Phi^t_i) \right] = \mathbb{E}[S\Phi_j \cdot \Phi_j] = \mathbb{E} \left[ S\Phi^t_i \cdot \Phi^t_i \right] = 0,$$

we conclude that

$$\mathbb{E} \left[ S\Phi_j \cdot \Phi^t_i \right] = -\mathbb{E} \left[ S\Phi^t_i \cdot \Phi_j \right].$$

Returning to (76), it follows from (75), (77), and the respective corrector equations that

$$\overline{a}_{ij} = \mathbb{E} \left[ (A^t - S)(\Phi^t_i + e_i) \cdot (\Phi_j + e_j) \right] = \overline{a}_{ji},$$

which completes the proof. 

\[\square\]

3. Weak convergence and the perturbed test function method. In this section, we prove that the solutions of

$$-\nabla \cdot (a^\varepsilon + s^\varepsilon)\nabla u^\varepsilon = f \text{ in } U \text{ with } u^\varepsilon = g \text{ on } \partial U,$$

homogenize weakly in $H^1$ to the solution of (11). The proof is based on a variation of the perturbed test function method that relies on an a priori $L^\infty$-estimate for the solutions, a stationary approximation of the correctors, and the regularity of the stream matrix. We prove the well-posedness of (78) in Proposition 3.1, the $L^\infty$-estimate in Proposition 3.2, and the homogenization in Theorem 3.3.

**Proposition 3.1.** Let $U \subseteq \mathbb{R}^d$ be a bounded $C^{2,\alpha}$-domain for some $\alpha \in (0, 1)$, let $a \in L^\infty(U; \mathbb{R}^{d \times d})$ be uniformly elliptic, and let $s = (s_{jk}) \in H^1(U; \mathbb{R}^{d \times d})$ be skew-symmetric. Then for every $f_1 \in L^2(U)$, $f_2 \in L^2(U; \mathbb{R}^d)$, and $g \in W^{1,\infty}(\partial U)$ there exists a unique weak solution $u \in H^1(U)$ of the equation

$$-\nabla \cdot (a + s)\nabla u = f_1 + \nabla \cdot f_2 \text{ in } U \text{ with } u = g \text{ on } \partial U.$$

Furthermore, there exists $c \in (0, \infty)$ depending on $U$ such that

$$\|u\|_{H^1(U)} \leq c \left( \|f_1\|_{L^2(U)} + \|f_2\|_{L^2(U; \mathbb{R}^d)} + \|g\|_{W^{1,\infty}(U)} \right).$$

**Proof.** The regularity of the domain $U$ and the tubular neighborhood theorem prove that there exists a globally Lipschitz continuous function $\overline{g}: \mathbb{R}^d \to \mathbb{R}$ such that $\overline{g}|_{\partial U} = g$. Then by considering $\tilde{u} = u - \overline{g}$ it follows that $u \in H^1(U)$ solves (79) if and only if $\tilde{u} \in H^1_0(U)$ solves

$$-\nabla \cdot (a + s)\nabla \tilde{u} = f_1 + \nabla \cdot \tilde{f}_2 \text{ in } U \text{ with } \tilde{u} = 0 \text{ on } \partial U,$$

for $\tilde{f}_2 = f_2 + a\nabla \overline{g} + s\nabla \overline{g} \in L^2(U; \mathbb{R}^d)$. It is therefore sufficient to consider the case $g = 0$.

Let $f_1 \in L^2(U)$ and $f_2 \in L^2(U; \mathbb{R}^d)$ and for each $n \in \mathbb{N}$ let $s_n = (s_{jk})$ be defined by $s_{jk}^n = (s_{jk} \wedge n) \vee (-n)$. The Lax-Milgram theorem proves for every $n \in \mathbb{N}$ that there exists a unique solution $u_n \in H^1_0(U)$ of the equation

$$-\nabla \cdot (a + s_n)\nabla u_n = f_1 + \nabla \cdot f_2 \text{ in } U \text{ with } u_n = 0 \text{ on } \partial U,$$

which due to the skew-symmetry of $s_n$, the uniform ellipticity, and the Poincaré inequality satisfies the energy inequality, for some $c \in (0, \infty)$ independent of $n$,

$$\int_U |\nabla u_n|^2 \leq c \left( \int_U |f_1|^2 + |f_2|^2 \right).$$
Therefore, after passing to a subsequence \( n \to \infty \), it follows from (81), (82), and the strong convergence of \( s_n \) to \( s \) in \( L^2(U; \mathbb{R}^{d \times d}) \) that there exists \( u \in H^1_0(U) \) such that \( u_n \rightharpoonup u \) weakly in \( H^1_0(U) \) and such that, for every \( \psi \in C^\infty_c(U) \),

\[
\int_U (a + s) \nabla u \cdot \nabla \psi = \int_U f \psi - f_2 \cdot \nabla \psi.
\]

Estimate (80) is a consequence of (82), the weak lower semi-continuity of the Sobolev norm, and the Poincaré inequality. It remains to prove the uniqueness of \( u \). By linearity, it suffices to prove that the only \( u \in H^1_0(U) \) that solves (83) with \( f_1 = 0 \) and \( f_2 = 0 \) is \( u = 0 \). Since \( s \in H^1(U; \mathbb{R}^{d \times d}) \) is skew symmetric and since \( \nabla \cdot s \) is divergence-free, we have, for every \( \psi \in C^\infty_c(U) \),

\[
\int_U s \nabla u \nabla \psi = - \int_U (\nabla \cdot s) \cdot \nabla \psi u = \int_U (\nabla \cdot s) \cdot (\nabla u) \psi.
\]

Therefore, for every \( \psi \in C^\infty_c(U) \),

\[
(83) \quad \int_U a \nabla u \cdot \nabla \psi + \int_U (\nabla \cdot s)(\nabla u) \psi = 0.
\]

It follows as in the proof of Proposition 2.4 that for each \( n \in \mathbb{N} \) the function \( u_n = (u \wedge n) \vee (-n) \) is an admissible test function for (83). The distributional equalities \( \nabla u_n = \nabla u 1_{\{|u| \leq n\}} \) and \( (\nabla u) u_n = \nabla (u u_n - 1/2u_n^2) \) and the fact that \( \nabla \cdot s \) is divergence-free then prove, for each \( n \in \mathbb{N} \),

\[
\int_U a \nabla u \cdot \nabla u 1_{\{|u| \leq n\}} = 0.
\]

After passing to the limit \( n \to \infty \), we conclude using the uniform ellipticity and the monotone convergence theorem that \( \nabla u = 0 \) and therefore that \( u = 0 \). This completes the proof. \( \square \)

**Proposition 3.2.** Assume (12) for \( S \in L^2(\Omega; \mathbb{R}^{d \times d}) \), let \( q \in (2 \vee d/2, \infty) \), let \( f \in L^q(U) \), and let \( g \in W^{1,\infty}(\partial U) \). Then there exists \( c = c(d, U, \lambda, \Lambda, q) \in (0, \infty) \) such that the solution \( u \in H^1_0(U) \) of the equation

\[
-\nabla \cdot (a + s) \nabla u = f \quad \text{in} \quad U \quad \text{with} \quad u = g \quad \text{on} \quad \partial U,
\]

satisfies

\[
\|u\|_{L^\infty(U)} \leq \|g\|_{L^\infty(\partial U)} + c \|f\|_{L^q(U)}.
\]

**Proof.** We will write the proof for the case \( d \geq 3 \), with the case \( d = 2 \) requiring only minor modifications in the application of the Sobolev embedding theorem. By linearity and the comparison principle, it suffices to prove that there exists \( c = c(d, U, \lambda, \Lambda, q) \in (0, \infty) \) such that the solution \( u \in H^1_0(U) \) of the equation

\[
-\nabla \cdot (a + s) \nabla u = f \quad \text{in} \quad U \quad \text{with} \quad u = 0 \quad \text{on} \quad \partial U,
\]

satisfies

\[
\|u\|_{L^\infty(U)} \leq c \|f\|_{L^q(U)}.
\]

Let \( q \in (2 \vee d/2, \infty) \) and let \( \theta \in (0, 1) \) be such that

\[
\frac{d - 2}{d\theta} + \frac{1}{q} = 1.
\]
For every \( \alpha \in [1, \infty) \), testing the equation with \( u^{[\alpha]} = |u|^\alpha u \) yields the energy estimate

\[
\alpha \int_U |u|^{\alpha-1} |\nabla u|^2 = \frac{4\alpha}{(\alpha + 1)^2} \int_U |\nabla u|^{\frac{\alpha+1}{\alpha}} \leq \frac{1}{\lambda} \int_U |u|^{\alpha} |f|,
\]

which is justified analogously to (43) above. It then follows from the Sobolev embedding theorem and Hölder’s inequality that, for some \( \tau_1 \in (0, \infty) \) independent of \( \alpha \),

\[
\|u\|^{\alpha+1}_{L^{\frac{d\alpha}{d-2}}(U)} \leq \tau_1 (\alpha + 1) \|u\|^{\alpha}_{L^{\frac{d\alpha}{d-2}}(U)} \|f\|_{L^q(U)}.
\]

This estimate will form the basis for a Moser-type iteration. The base case is \( \alpha_1 = 1 \), for which we have that

\[
\|u\|_{L^{\frac{d\alpha_1}{d-2}}(U)} \leq \left( \tau_1 2 \|u\|_{L^{\frac{d\alpha_1}{d-2}}(U)} \|f\|_{L^q(U)} \right)^{\frac{1}{2}}.
\]

For \( k \in \{2, 3, \ldots\} \) let \( \alpha_k \in (1, \infty) \) be defined by \( \frac{\alpha_k d\theta}{d-2} = \frac{(1+\alpha_{k-1})d}{d-2} \) so that \( \alpha_k = \frac{(\alpha_{k-1}+1)}{\theta} \) and so that

\[
\|u\|_{L^{\frac{d\alpha_k}{d-2}}(U)} \leq \left( \tau_1 (\alpha_k + 1) \|u\|_{L^{\frac{d\alpha_{k-1}}{d-2}}(U)} \|f\|_{L^q(U)} \right)^{\frac{1}{\alpha_k+1}}.
\]

It then follows by induction that, for every \( k \in \mathbb{N} \),

\[
\|u\|_{L^{\frac{d\alpha_k}{d-2}}(U)} \leq \left( \prod_{j=1}^k \left( \tau_1 (\alpha_j + 1) \|f\|_{L^q(U)} \right)^{\left( \prod_{m=j+1}^{k} \frac{\alpha_m}{\alpha_m + 1} \right)^{\frac{1}{\alpha_j+1}}} \right) \|u\|_{L^{\frac{d\theta}{d-2}}(U)}.
\]

Since it follows by definition that \( \alpha_k \geq \theta^{-(k-1)} \) for every \( k \in \mathbb{N} \), it follows (for instance, by taking a logarithm) that there exists \( \beta \in (0, 1/2) \) such that

\[
\lim_{k \to \infty} \left( \prod_{j=1}^k \frac{\alpha_j}{1 + \alpha_j} \right) = \beta,
\]

and that

\[
\lim_{k \to \infty} \left( \sum_{j=1}^k \left( \prod_{m=j+1}^{k} \frac{\alpha_m}{\alpha_m + 1} \right) \frac{1}{\alpha_j + 1} \right) = 1 - \beta.
\]

Similarly, for every \( k \in \mathbb{N} \),

\[
\log \left( \prod_{j=1}^k (\alpha_j + 1) \left( \prod_{m=j+1}^{k} \frac{\alpha_m}{\alpha_m + 1} \right)^{\frac{1}{\alpha_j+1}} \right) \leq \sum_{j=1}^k (\alpha_j + 1)^{-1} \log(1 + \alpha_j)
\]

\[
\leq 2^{-1} \log(2) + \sum_{j=2}^k (k-1)\theta^{k-1},
\]

and therefore it follows that

\[
\lim_{k \to \infty} \left( \prod_{j=1}^k (\alpha_j + 1) \left( \prod_{m=j+1}^{k} \frac{\alpha_m}{\alpha_m + 1} \right)^{\frac{1}{\alpha_j+1}} \right) = \tau_2 \in (0, \infty).
\]
It follows from $\lim_{p \to \infty} \|u\|_{L^p(U)} = \|u\|_{L^\infty(U)}$, (85), (86), (87), and (88) that, after passing to the limit $k \to \infty$ in (85),
\[
\|u\|_{L^\infty(U)} \leq \|f\|_{L^1(U)}^{(1-\beta)} \|f\|_{L^\infty(U)}^{\beta} \|u\|_{L^{\frac{d\theta}{d\delta}}(U)}^{\beta}.
\]

Finally, the boundedness of $U$, Hölder’s inequality, Young’s inequality, and (84) prove that, for some $c \in (0, \infty)$,
\[
\|u\|_{L^{\frac{d\theta}{d\delta}}(U)} \leq c \|u\|_{L^{\frac{d\theta}{d\delta}}(U)} \leq c \left( \|u\|_{L^{\frac{d\theta}{d\delta}}(U)} \|f\|_{L^q(U)} \right)^{\frac{1}{2}} \leq \frac{1}{2} \|u\|_{L^{\frac{d\theta}{d\delta}}(U)} + c \|f\|_{L^q(U)},
\]
and therefore there exists $c \in (0, \infty)$ such that
\[
\|u\|_{L^{\frac{d\theta}{d\delta}}(U)} \leq c \|f\|_{L^q(U)}.
\]

In combination (89) and (90) complete the proof. \qed

**Theorem 3.3.** Assume (12) for $S \in L^2(\Omega; \mathbb{R}^{d \times d})$, let $U \subseteq \mathbb{R}^d$ be a bounded $C^{2,\alpha}$-domain for some $\alpha \in (0, 1)$, let $f \in L^q(U)$ for some $q \in (2 \vee \frac{d}{\alpha}, \infty)$, and let $g \in W^{1,\infty}(\partial U)$. For every $\varepsilon \in (0, 1)$ let $u^\varepsilon \in H^1(U)$ be the unique solution of (78) and let $v \in H^1(U)$ be the unique solution of (11). Then, almost surely as $\varepsilon \to 0$,
\[
u^\varepsilon \to w \text{ weakly in } H^1(U).
\]

**Proof.** We will first define the perturbed test function. For every $\delta \in (0, 1)$ and $i \in \{1, \ldots, d\}$ let $\hat{\phi}_{i}^{(\delta,t)} \in H^1(\Omega)$ be the unique Lax-Milgram solution of
\[
\delta \hat{\phi}_{i}^{(\delta,t)} - D \cdot (A^t - S)(D \hat{\phi}_{i}^{(\delta,t)} + e_i) = 0 \text{ in } H^1(\Omega),
\]
and for every $i \in \{1, \ldots, d\}$ let $\Phi_i^t \in L^2_{\text{pot}}(\Omega)$ be the unique solution of
\[
-D \cdot (A^t - S)(\Phi_i^t + e_i) = 0 \text{ in } L^2_{\text{pot}}(\Omega).
\]
It was show in Proposition 2.4 that, as $\delta \to 0$,
\[
\hat{\phi}_{i}^{(\delta,t)} \to 0 \text{ strongly in } L^2(\Omega) \text{ and } D \hat{\phi}_{i}^{(\delta,t)} \to \Phi_i^t \text{ weakly in } L^2_{\text{pot}}(\Omega).
\]
For every $\delta \in (0, 1)$ let the $\{\hat{\phi}_{i}^{(\delta,t)}\}_{i \in \{1, \ldots, d\}}$ be almost surely defined as the unique $H^1_{\text{loc}}(\mathbb{R}^d)$ functions satisfying, for every $i \in \{1, \ldots, d\}$,
\[
\nabla \hat{\phi}_{i}^{(\delta,t)}(x, \omega) = D \hat{\phi}_{i}^{(\delta,t)}(\tau_\omega) \text{ and } \int_{B_1} \hat{\phi}_{i}^{(\delta,t)}(x, \omega) \, dx = 0,
\]
from which it follows that, almost surely,
\[
\hat{\phi}_{i}^{(\delta,t)} - \nabla \cdot (\alpha^t - s)(\nabla \hat{\phi}_{i}^{(\delta,t)} + e_i) = 0 \text{ in } \mathbb{R}^d.
\]
Let $\psi \in C^\infty_c(U)$ and for every $\varepsilon, \delta \in (0, 1)$ let $\psi^{\varepsilon, \delta}$ be the perturbed test function defined by
\[
\psi^{\varepsilon, \delta} = \psi + \varepsilon \hat{\phi}_{i}^{(\delta,t)}(\varepsilon/\varepsilon) \partial_i \psi.
\]
It then follows from estimate (80) and the Sobolev embedding theorem that there almost surely exists a random subsequence still denoted $\varepsilon \to 0$ and a random $v \in H^1(U)$ such that
\[
u^\varepsilon \to v \text{ weakly in } H^1(U), \text{ strongly in } L^2(U), \text{ and almost everywhere in } U.
\]
It follows from the estimates of Proposition 3.2, \( \psi \in C^\infty(U) \), the definition of \( \psi^{\epsilon, \delta} \), and an approximation argument that, for every \( \epsilon \in (0, 1) \), for \( a^{t, \epsilon}(x) = a^t(x/\epsilon, \omega) \),

\[
(93) \int_U (a^\epsilon + s^\epsilon) \nabla u^\epsilon \cdot \nabla \psi^{\epsilon, \delta} = \int_U \nabla u^\epsilon \cdot (a^{t, \epsilon} - s^\epsilon)((e_i + \nabla \phi_i^{t, \delta}(x/\epsilon)) \partial_i \psi) + \int_U \epsilon \phi_i^{t, \delta}(x/\epsilon) a^\epsilon \nabla u^\epsilon \cdot \nabla (\partial_i \psi) + \int_U \epsilon \phi_i^{t, \delta}(x/\epsilon) s^\epsilon \nabla u^\epsilon \cdot \nabla (\partial_i \psi) = \int_U f \psi^{\epsilon, \delta}.
\]

For the first term on the right-hand side of (93), it follows from the equation satisfied by \( \phi_i^{t, \delta} \), \( \psi \in C^\infty_c(U) \), and the equality \( \partial_i \psi \nabla u^\epsilon = \nabla (u^\epsilon \partial_i \psi) - u^\epsilon \nabla (\partial_i \psi \psi) \) that

\[
(94) \int_U \nabla u^\epsilon \cdot (a^{t, \epsilon} - s^\epsilon)((e_i + \nabla \phi_i^{t, \delta}(x/\epsilon)) \partial_i \psi) = -\int_U \delta \phi_i^{t, \delta}(x/\epsilon) \nabla (u^\epsilon \partial_i \psi) - \int_U u^\epsilon (a^{t, \epsilon} - s^\epsilon)(e_i + \nabla \phi_i^{t, \delta}(x/\epsilon)) \cdot \nabla (\partial_i \psi).
\]

The second term on the right-hand side of (93) is left as it is, and for the final term it follows from the skew-symmetry of \( s \) that

\[
(95) \int_U \epsilon \phi_i^{t, \delta}(x/\epsilon) s^\epsilon \nabla u^\epsilon \cdot \nabla (\partial_i \psi) = -\int_U u^\epsilon (\nabla \phi_i^{t, \delta}(x/\epsilon) s^\epsilon + \phi_i^{t, \delta}(x/\epsilon)(\nabla \cdot s^\epsilon)) \cdot (\nabla (\partial_i \psi)).
\]

For the terms in (94), it follows from the ergodic theorem (see, for example, Jikov, Kozlov, and Oleinik [36, Theorem 7.2]), the \( L^\infty \)-estimate of Proposition 3.2, Egorov’s theorem, and (92) that, almost surely along a subsequence \( \epsilon \to 0 \),

\[
(96) \lim_{\epsilon \to 0} \int_U u^\epsilon (a^{t, \epsilon} - s^\epsilon)(e_i + \nabla \phi_i^{t, \delta}(x/\epsilon)) \cdot \nabla (\partial_i \psi) = \int_U v E \left[ (A^t - S)(D\overrightarrow{\phi_i^{t, \delta}} + e_i) \right] \cdot \nabla (\partial_i \psi),
\]

and it follows from the stationarity of \( \phi_i^{t, \delta} \), \( \psi \in C^\infty_c(U) \), the ergodic theorem, the estimates of Proposition 3.1, and Hölder’s inequality that, for some \( c \in (0, \infty) \),

\[
(97) \limsup_{\epsilon \to 0} \left| \int_U \delta \phi_i^{t, \delta}(x/\epsilon) \nabla (u^\epsilon \partial_i \psi) \right| \leq c \left\| \delta \phi_i^{t, \delta} \right\|_{L^2(U)} \left( \| f \|_{L^2(U)} + \| g \|_{W^{1, \infty}(U)} \right).
\]

For the second term on the right-hand side of (93), it follows Hölder’s inequality, the uniform ellipticity, and the ergodic theorem that

\[
(98) \limsup_{\epsilon \to 0} \left| \int_U \epsilon \phi_i^{t, \delta}(x/\epsilon) a^\epsilon \nabla u^\epsilon \cdot \nabla (\partial_i \psi) \right| = 0.
\]

For (95), it follows from the ergodic theorem, the \( L^\infty \)-estimate of Proposition 3.2, Egorov’s theorem, and (92) that, almost surely along a subsequence \( \epsilon \to 0 \),

\[
(99) \lim_{\epsilon \to 0} \int_U u^\epsilon (\nabla \phi_i^{t, \delta}(x/\epsilon) s^\epsilon + \phi_i^{t, \delta}(x/\epsilon)(\nabla \cdot s^\epsilon)) \cdot \nabla (\partial_i \psi) = \int_U v E \left[ D \cdot (\phi_i^{t, \delta} S) \right] \cdot \nabla (\partial_i \psi) = 0,
\]

where the final inequality follows from the fact that the expectation of the horizontal derivative is zero. Finally, it follows from \( \psi \in C^\infty_c(U) \) and \( \overrightarrow{\phi_i} \in H^1(\Omega) \) that, almost surely as \( \epsilon \to 0 \),

\[
(100) \psi^{\epsilon, \delta} \to \psi \text{ strongly in } L^2(\Omega).
\]
Returning to (93), it follows from (96), (97), (98), (99), and (100) that there exists \( c \in (0, \infty) \) such that, almost surely for every \( \delta \in (0, 1) \),

\[
(101) \quad \int_U v \mathbb{E} \left[ (A^\epsilon - S)(D\phi_i^\epsilon + e_i) \right] \cdot \nabla (\partial_t \psi) - \int_U f \psi \leq c \left\| \delta \phi_i^\epsilon \right\|_{L_2(\Omega)} \left( \|f\|_{L^2(U)} + \|g\|_{W^{1,\infty}(U)} \right).
\]

It then follows from (91), (101), and Proposition 2.9 that, almost surely after passing to the limit \( \delta \to 0 \),

\[
\int_U v \pi^\epsilon e_i \cdot \nabla (\partial_t v) = \int_U \pi \nabla v \cdot \nabla \psi = \int_U f \psi,
\]

which uniquely characterizes the limit and completes the proof. \( \square \)

4. Strong convergence of the two-scale expansion. In this section, we will prove the strong homogenization of the equation

\[
(102) \quad - \nabla \cdot (a^\epsilon + s^\epsilon \nabla u^\epsilon) = f \text{ in } U \text{ with } u^\epsilon = g \text{ on } \partial U.
\]

The proof is based on estimating the energy of the two-scale expansion

\[
w^\epsilon = u^\epsilon - v - \epsilon \phi_i^\epsilon \partial_t v,
\]

where, for the gradient fields \( \Phi_i \in L^2_{\text{per}}(\Omega) \) constructed in Proposition 2.4, the physical correctors \( \phi_i \) are the unique functions that almost surely satisfy \( \int_{B_1} \phi_i = 0 \), \( \phi_i \in H^1_{\text{loc}}(\mathbb{R}^d) \) with \( \nabla \phi_i(x, \omega) = \Phi_i(\tau x, \omega) \), and, for every \( \psi \in C_0^\infty(\mathbb{R}^d) \),

\[
(103) \quad \int_{\mathbb{R}^d} (a + s)(\nabla \phi_i + e_i) \cdot \nabla \psi = 0,
\]

and where \( \phi_i^\epsilon(x, \omega) = \phi(x/\epsilon, \omega) \). The limit \( v \in H^1(U) \) solves the homogenized equation

\[
(104) \quad - \nabla \cdot \pi \nabla v = f \text{ in } U \text{ with } u = f \text{ on } \partial U,
\]

for the homogenized coefficient field \( \pi \in \mathbb{R}^{d\times d} \) defined (74). Motivated by the analogous computation in [34], after introducing the flux correctors \( \sigma_i \) we will prove that, up to boundary terms,

\[
- \nabla \cdot (a^\epsilon + s^\epsilon \nabla w^\epsilon) = \nabla \cdot \left[ (\epsilon \phi_i^\epsilon (a^\epsilon + s^\epsilon) - \epsilon \sigma_i^\epsilon) \nabla (\partial_t v) \right].
\]

The strong convergence of \( \nabla w^\epsilon \) to zero in the \( \epsilon \to 0 \) limit then follows formally from the \( L^{d\vee(2+\delta)} \)-integrability of the stream matrix, Proposition 2.1, and the regularity of \( \pi \)-harmonic functions. We prove the strong homogenization of (102), and the strong convergence of the two-scale expansion in Theorem 4.1 below.

**Theorem 4.1.** Assume (12) for \( S \in L^d(\Omega; \mathbb{R}^{d\times d}) \) if \( d \geq 3 \) and for \( S \in L^{2+\delta}(\Omega; \mathbb{R}^{d\times d}) \) for some \( \delta \in (0, 1) \) if \( d = 2 \), let \( \alpha \in (0, 1) \), let \( U \subseteq \mathbb{R}^d \) be a bounded \( C^{2,\alpha} \)-domain, let \( f \in C^\alpha(U) \), and let \( g \in C^{2,\alpha}(\partial U) \). For every \( \epsilon \in (0, 1) \) let \( u^\epsilon \in H^1(U) \) be the unique solution of (102) and let \( v \in H^1(U) \) be the unique solution of (104). Then, almost surely as \( \epsilon \to 0 \),

\[
\lim_{\epsilon \to 0} \|u^\epsilon - v - \epsilon \phi_i^\epsilon \partial_t v\|_{H^1(U)} = 0.
\]
PROOF. We will essentially study the equation satisfied by the homogenization error
\begin{equation}
(105) \quad w^\epsilon = u^\epsilon - v - \varepsilon \phi^\epsilon_i \partial_i v,
\end{equation}
after introducing a cutoff to ensure that \(w^\epsilon\) vanishes along the boundary. Using the fact that \(U \subseteq \mathbb{R}^d\) is a bounded \(C^{2,\alpha}\)-domain, for each \(\rho \in (0,1)\) we define \(\eta_\rho : U \to [0,1]\) to be a smooth cutoff function satisfying \(\eta_\rho(x) = 1\) if \(d(x, \partial U) \geq 2\rho\), \(\eta_\rho(x) = 0\) if \(d(x, \partial U) < \rho\), and \(|\nabla \eta_\rho(x)| \leq c/\rho\) for some \(c \in (0, \infty)\) independent of \(\rho \in (0,1)\). For each \(\varepsilon, \rho \in (0,1)\) we define
\begin{equation}
(106) \quad w^{\varepsilon,\rho} = w^\epsilon - v - \varepsilon \phi^\epsilon_i \eta_\rho \partial_i v \in H_0^1(U).
\end{equation}
It follows by definition that
\begin{equation}
\nabla w^{\varepsilon,\rho} = \nabla w^\epsilon - \nabla v - \eta_\rho \partial_i v \nabla \phi^\epsilon_i - \varepsilon \phi^\epsilon_i \nabla (\eta_\rho \partial_i v).
\end{equation}
Distributionally, using the equation satisfied by \(u^\epsilon\),
\begin{equation}
- \nabla \cdot (a^\epsilon + s^\epsilon) \nabla w^{\varepsilon,\rho} = f + \nabla \cdot (a^\epsilon + s^\epsilon) \nabla v + \nabla \cdot (a^\epsilon + s^\epsilon) (\eta_\rho \partial_i v \nabla \phi^\epsilon_i + \varepsilon \phi^\epsilon_i \nabla (\eta_\rho \partial_i v)),
\end{equation}
and, using the equation satisfied by \(v\),
\begin{equation}
(106) \quad - \nabla \cdot (a^\epsilon + s^\epsilon) \nabla w^{\varepsilon,\rho} = \nabla \cdot [(1 - \eta_\rho) ((a^\epsilon + s^\epsilon) - \overline{a}) \nabla v] + \nabla \cdot [(a^\epsilon + s^\epsilon)((\nabla \phi^\epsilon_i + e_i) - \overline{a} e_i) \eta_\rho \partial_i v] + \nabla \cdot [(a^\epsilon + s^\epsilon) \varepsilon \phi^\epsilon_i \nabla (\eta_\rho \partial_i v)].
\end{equation}
The second term on the righthand side of (106) is defined for each \(i \in \{1, \ldots, d\}\) by \(q^\varepsilon_i(x, \omega) = Q_i(\tau_{\varepsilon/\rho}, \omega) - \mathbb{E}[Q_i]\) for the flux \(Q_i\) defined by
\begin{equation}
Q_i = (A + S)(\Phi_i + e_i) \in L^{p_d}(\Omega; \mathbb{R}^d),
\end{equation}
for \(p_d\) defined in (56). The \(q^\varepsilon_i\) do not vanish in a strong sense as \(\varepsilon \to 0\), and it is for this reason that we introduce the flux correctors defined in Proposition 2.6. For each \(i \in \{1, \ldots, d\}\) let \(\sigma_i = (\sigma_{ijk}) \in W^{1,p_d}_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^{d \times d})\) be as in Proposition 2.6 and let \(\sigma^\varepsilon_i(x, \omega) = \sigma_i(\varepsilon/\rho, \omega)\). Then, for every \(\psi \in C^{\infty}_c(U)\),
\begin{align*}
\int_{\mathbb{R}^d} q^\varepsilon_i \eta_\rho \partial_i v \cdot \nabla \psi = \int_{\mathbb{R}^d} (\eta_\rho \partial_i v) q^\varepsilon_i \partial_j \psi = \int_{\mathbb{R}^d} (\eta_\rho \partial_i v) \partial_k (\varepsilon \sigma^\varepsilon_{ijk}) \partial_j \psi \\
= - \int_{\mathbb{R}^d} \varepsilon \sigma^\varepsilon_{ijk} \partial_k (\eta_\rho \partial_i v) \partial_j \psi,
\end{align*}
where the final inequality relies on the skew-symmetry. So, as distributions on \(\mathbb{R}^d\),
\begin{equation}
\nabla \cdot [q^\varepsilon_i \eta_\rho \partial_i v] = - \nabla \cdot [\varepsilon \sigma^\varepsilon_i \nabla (\eta_\rho \partial_i v)].
\end{equation}
Returning to (106), we conclude that
\begin{equation}
(107) \quad - \nabla \cdot (a^\epsilon + s^\epsilon) \nabla w^{\varepsilon,\rho}
= \nabla \cdot [(1 - \eta_\rho) ((a^\epsilon + s^\epsilon) - \overline{a}) \nabla v] + \nabla \cdot [(\varepsilon \phi^\epsilon_i (a^\epsilon + s^\epsilon) - \varepsilon \sigma^\varepsilon_i) \nabla (\eta_\rho \partial_i v)].
\end{equation}
The uniform ellipticity, Hölder’s inequality, Young’s inequality, and the definition of \(\eta_\rho\) prove that, for some \(c \in (0, \infty)\) independent of \(\varepsilon, \rho \in (0,1)\), for \(q_d = d \vee (2 + \delta) \) and \(1/2 = 1/2 - 1/q_d\),
\begin{align*}
\int_U |\nabla w^{\varepsilon,\rho}|^2 &\leq c \|\nabla v\|_{L^\infty(U; \mathbb{R}^d)}^2 \left( \int_U (1 - \eta_\rho)^2 \left( |a^\epsilon|^2 + |s^\epsilon|^2 \right) \right) \\
&+ c \|\nabla (\eta_\rho \partial_i v)\|_{L^\infty(U; \mathbb{R}^d)}^2 \left( \int_U |a^\epsilon|^{q_d} + |s^\epsilon|^{q_d} \right) \left( \int_U |\varepsilon \phi^\epsilon_i|^2 \right)^{\frac{2}{q_d}} \\
&+ c \|\nabla (\eta_\rho \partial_i v)\|_{L^\infty(U; \mathbb{R}^d)}^2 \left( \int_U |\varepsilon \sigma^\varepsilon_i|^2 \right).
\end{align*}
The regularity of the domain and Schauder estimates (see, for example, Gilbarg and Trudinger [33, Chapter 6]) prove that, for some \( c \in (0, \infty) \) depending on \( U \),
\begin{equation}
\|v\|_{C^{2,\alpha}(U)} \leq c \left( \|f\|_{C^0(U)} + \|g\|_{C^{2,\alpha}(\partial U)} \right).
\end{equation}
It follows almost surely from Proposition 2.1, (108), \( \Phi_i \in L^2_{pot}(\Omega), \Sigma_{ijk} \in L^{p_i}(\Omega; \mathbb{R}^{d \times d}), S \in L^{q_i}(\Omega; \mathbb{R}^{d \times d}) \), the uniform ellipticity, the ergodic theorem, and the definition of \( \eta_\rho \) that, for each \( \rho \in (0, 1) \), for \( c \in (0, \infty) \) depending on \( U \) but independent of \( \rho \in (0, 1) \),
\begin{equation}
\limsup_{\varepsilon \to 0} \int_U |\nabla w^{\varepsilon, \rho}|^2 \leq c\rho \left( \|\nabla v\|_{L^\infty(U; \mathbb{R}^d)}^2 \mathbb{E} \left[ |A|^2 + |S|^2 \right] \right).
\end{equation}
Then for each \( \varepsilon \in (0, 1) \) let \( w^{\varepsilon} \in H^1(U) \) be defined by (105) and for every \( \rho \in (0, 1) \) observe that
\[ \nabla w^{\varepsilon} = \nabla w^{\varepsilon, \rho} + \nabla \phi_i^\varepsilon (1 - \eta_\rho) \partial_i v + \varepsilon \phi_i^\varepsilon \nabla ((1 - \eta_\rho) \partial_i v). \]
It follows from (109), the triangle inequality, and Young’s inequality that, for \( c \in (0, \infty) \) independent of \( \varepsilon, \rho \in (0, 1) \),
\begin{equation}
\int_U |\nabla w^{\varepsilon}|^2 \leq c \left( \int_U |\nabla w^{\varepsilon, \rho}|^2 \right) + c \left( \int_U (1 - \eta_\rho)^2 |\nabla \phi_i^\varepsilon|^2 + \|\nabla ((1 - \eta_\rho) \partial_i v)\|_{L^\infty(U)}^2 \right). 
\end{equation}
Proposition 2.1, (108), the definition of \( \eta_\rho \), and the ergodic theorem therefore prove almost surely that for every \( \rho \in (0, 1) \), for \( c \in (0, \infty) \) depending on \( U \) but independent of \( \rho \),
\begin{equation}
\limsup_{\varepsilon \to 0} \int_U |\nabla w^{\varepsilon}|^2 \leq c\rho \left( \|\nabla v\|_{L^\infty(U; \mathbb{R}^d)}^2 \mathbb{E} \left[ |A|^2 + |S|^2 \right] + \|\partial_i v\|_{L^\infty(U)}^2 \mathbb{E} \left[ \Phi_i|^2 \right] \right).
\end{equation}
Passing to the limit \( \rho \to 0 \), we conclude that, almost surely as \( \varepsilon \to 0 \),
\begin{equation}
\nabla w^{\varepsilon} \to 0 \text{ strongly in } L^2(U; \mathbb{R}^d).
\end{equation}
Finally, since Proposition 2.1 and \( \Phi_i \in L^2_{pot}(\Omega) \) prove that, almost surely as \( \varepsilon \to 0 \),
\[ \varepsilon \phi_i^\varepsilon \partial_i v \to 0 \text{ strongly in } L^2(U), \]
it follows from the uniform boundedness of the \( u^\varepsilon - v \) in \( H^1_0(U) \), the Sobolev embedding theorem, and (110) that, almost surely as \( \varepsilon \to 0 \),
\begin{equation}
w^{\varepsilon} \to 0 \text{ strongly in } L^2(U).
\end{equation}
In combination (110) and (111) complete the proof.

5. The large-scale regularity estimate. In this section, motivated by the methods of [34], we will establish an almost sure intrinsic large-scale \( C^{1,\alpha} \)-regularity estimate for solutions \( u \in H^1_{loc}(\mathbb{R}^d) \) of
\begin{equation}
- \nabla \cdot (a + s)\nabla u = 0 \text{ in } \mathbb{R}^d.
\end{equation}
In analogy with the the characterization of Hölder spaces by Morrey and Campanato, for each \( \alpha \in (0, 1) \) and \( R \in (0, \infty) \) we define the excess \( \text{Exc}(u; R) \) to be the large-scale \( C^{1,\alpha} \)-Campanato semi-norm with respect to the intrinsic \( (a + s) \)-harmonic coordinates \( (x_i + \phi_i) \):
\begin{equation}
\text{Exc}(u; R) = \inf_{\xi \in \mathbb{R}^d} \frac{1}{R^{2\alpha}} \int_{B_R} \left| \nabla u - \xi - \nabla \phi \xi \right|^2.
\end{equation}
Formally the homogenization of \((112)\) in \(H^1(U)\) and the ergodic theorem imply that the excess is well-controlled for large radii \(R\) by the regularity of an \(\bar{\alpha}\)-harmonic function and the energy of the random gradient fields \(\Phi_i\). The arguments of this section make this precise. The section is organized as follows. In Propositions 5.2 and 5.3 below we recall some standard results from constant-coefficient elliptic regularity theory. We estimate the energy of the two-scale expansion in Proposition 5.4 below. We then prove the large-scale Hölder estimate and excess decay in Proposition 5.5 and Theorem 5.6 below. The proof of excess decay is most closely related to the methods of [34] in the uniformly elliptic setting, and shares aspects of the work [14] in the degenerate elliptic setting. Here, in analogy with the degenerate setting, the regularity estimate comes into effect after controlling both the sublinearity of the correctors and the large-scale averages of the unbounded state matrix. In this way Propositions 5.4 and 5.5 are wholly analytic and essentially deterministic, taking as input only this large-scale behavior. Theorem 5.6 combines these statements with the probabilistic input of Proposition 2.1 and the ergodic theorem to obtain the complete statement.

**Remark 5.1.** In this section, we will write \(a \lesssim b\) if \(a \leq cb\) for a constant \(c\) depending only on the dimension and ellipticity constants.

**Proposition 5.2.** Let \(\pi \in \mathbb{R}^{d \times d}\) be uniformly elliptic and let \(v \in H^1_{\text{loc}}(\mathbb{R}^d)\) be a weak solution of the equation
\[(113)\]
\[-\nabla \cdot \pi \nabla v = 0 \text{ in } B_1.\]
Then, for each \(r_1 < r_2 \in (0, 1)\) and \(c \in \mathbb{R},\)
\[
\int_{B_{r_1}} |\nabla v|^2 \lesssim \frac{1}{(r_2 - r_1)^2} \int_{B_{r_2}} (v - c)^2.
\]
And, for every \(\rho \in (0, 1),\)
\[
\sup_{B_{(1-\rho)}} \left( |\nabla^2 v|^2 + \frac{1}{\rho^2} |\nabla v|^2 \right) \lesssim \frac{1}{\rho^{2(d+1)}} \int_{B_1} |\nabla v|^2.
\]

**Proof.** Let \(r_1 < r_2 \in (0, 1)\) and let \(\eta: \mathbb{R}^d \to [0, 1]\) be a smooth cutoff function satisfying \(\eta = 1\) on \(\overline{B}_{r_1}\) and \(\eta = 0\) on \(\mathbb{R}^d \setminus B_{r_2}\) with \(|\nabla \eta| \leq 2/(r_2 - r_1).\) After testing \((113)\) with \(\eta^2(v - c),\)
\[
\int_{B_1} (\pi \nabla v \cdot \nabla \eta) \eta^2 = -2 \int_{B_1} (\pi \nabla v \cdot \nabla \eta)(v - c) \eta.
\]
The uniform ellipticity, Hölder’s inequality, and Young’s inequality prove using the definition of \(\eta\) that
\[(114)\]
\[
\int_{B_{r_1}} |\nabla v|^2 \lesssim \frac{1}{(r_2 - r_1)^2} \int_{B_{r_2}} (v - c)^2.
\]
Let \(K \in \mathbb{N}\) and \(\rho \in (0, 1).\) Since for every multi-index \(\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d\) the partial derivative \(\partial_{\alpha_1} \ldots \partial_{\alpha_d} v\) satisfies \((113),\) a repeated application of \((114)\) on the subintervals of length \(\rho/2K\) proves that, for every \(k \in \{1, \ldots, K\},\)
\[
\int_{B_{(1-\rho)}} |\nabla^{k} v|^2 \lesssim \frac{4K^2}{\rho^2} \int_{B_{(1-\rho + \frac{\rho}{2K})}} |\nabla^{k-1} v|^2 \lesssim \cdots \lesssim \frac{(4K)^{k-1}}{\rho^{2(k-1)}} \int_{B_{(1-\rho + \frac{\rho}{2K})}} |\nabla v|^2.
\]
After choosing \(K = d + 2\) the Sobolev embedding theorem (see, for example, Evans [21, Chapter 5, Theorem 6]) proves that
\[
\sup_{B_{(1-\rho)}} \left( |\nabla^2 v|^2 + \frac{2}{\rho^2} |\nabla v|^2 \right) \lesssim \frac{1}{\rho^{2(d+1)}} \int_{B_1} |\nabla v|^2.\]
PROPOSITION 5.3. Let \( \bar{\alpha} \in \mathbb{R}^{d \times d} \) be uniformly elliptic, let \( \psi \in C^\infty(\partial B_1) \), and let \( v \in H^1(B_1) \) be a weak solution of the equation

\[ -\nabla \cdot \bar{\alpha} \nabla v = 0 \text{ in } B_1 \text{ with } v = \psi \text{ on } \partial B_1. \]

Then, for every \( p \in [2, \infty) \) there exists \( c_1, c_2 \in (0, \infty) \) depending on \( p \) such that

\[ \|\nabla v\|_{L^p(B_1)} \leq c_1 \|\nabla \tan \psi\|_{L^p(\partial B_1)} \leq c_2 \|\nabla \tan \psi\|_{L^\infty(\partial B_1)}, \]

where \( \nabla \tan \) denotes the tangential derivative on \( \partial B_1 \).

PROOF. We may assume without loss of generality that \( \int_{\partial B_1} \psi = 0 \) since subtracting a constant does not change the gradient. Let \( \eta: \mathbb{R} \to [0, 1] \) be a smooth function satisfying \( \eta = 1 \) on \([3/4, \infty)\) and \( \eta = 0 \) on \((-\infty, 1/4]\). Then \( \hat{\psi}(x) = \psi(x/|x|)\eta(|x|) \) is a smooth extension of \( \psi \) into \( B_1 \) which satisfies, using Fubini’s theorem, the definition of \( \eta \), the fact that \( \hat{\psi} \) has average zero on \( \partial B_1 \), the Poincaré inequality, and \( \|\nabla (\hat{\psi}/|x|)\| = 1 \) for \( x \neq 0 \).

\begin{equation}
\|\nabla \hat{\psi}\|_{L^p(B_1)} \leq \left( \int_0^1 \left( \int_{\partial B_{r}} |\nabla \eta(r)|^p \psi(x/|x|)^p \right. \left. + \eta^p(r) |\nabla \tan \hat{\psi}(x/|x|)|^p \right) dr \right)^{\frac{1}{p}}
\end{equation}

\[ \lesssim \|\psi\|_{L^p(\partial B_1)} + \|\nabla \tan \hat{\psi}\|_{L^p(\partial B_1)} \leq \|\nabla \tan \hat{\psi}\|_{L^p(\partial B_1)} \leq \|\nabla \tan \hat{\psi}\|_{L^\infty(\partial B_1)}, \]

where the final inequality follows from Hölder’s inequality and the boundedness of \( \partial B_1 \). It then follows from [32, Theorem 7.1], (115), and \( p \geq 2 \) that, for some \( c, c_1, c_2 \in (0, \infty) \) depending on \( p \),

\[ \|\nabla v\|_{L^p(B_1)} \leq c \|\nabla \hat{\psi}\|_{L^p(B_1)} \leq c_1 \|\nabla \tan \hat{\psi}\|_{L^p(\partial B_1)} \leq c_2 \|\nabla \tan \hat{\psi}\|_{L^\infty(\partial B_1)}. \]

PROPOSITION 5.4. Assume (12) for \( S \in L^d(\Omega; \mathbb{R}^{d \times d}) \) if \( d \geq 3 \) and \( S \in L^{2+\delta}(\Omega; \mathbb{R}^{d \times d}) \) for some \( \delta \in (0, 1) \) if \( d = 2 \), let \( R \in (0, \infty) \), and let \( u \in H^1(B_R) \) be a distributional solution of

\[ -\nabla \cdot (a + s) \nabla u = 0 \text{ in } B_R. \]

Then there exists \( c \in (0, \infty) \) so that for every \( \varepsilon \in (0, 1) \) there exists an \( \bar{\alpha} \)-harmonic function \( v^\varepsilon \in H^1(B_{1/2}) \) such that, for every \( \rho \in (0, 1/4) \), for \( q_d = d \vee (2 + \delta) \) and \( 1/2 = 1/2 - 1/q_d \),

\[ \int_{B_{1/2}} |\nabla (u - v^\varepsilon - \varphi_i \partial_i v^\varepsilon)|^2 \]

\[ \leq c \left( \varepsilon + \varepsilon^{1 - \frac{d-1}{q_d}} \left( \int_{B_R} |s|^{q_d} \right)^\frac{1}{q_d} \right) \int_{B_R} |\nabla u|^2 \]

\[ + c\varepsilon^{-(d-1)} \rho^\frac{1}{q_d} \left( 1 + \left( \int_{B_R} |s|^{q_d} \right)^\frac{2}{q_d} \right) \int_{B_R} |\nabla u|^2 \]

\[ + c\rho^{-2(d+1)} R^{-2} \left[ \left( 1 + \left( \int_{B_R} |s|^{q_d} \right)^\frac{2}{q_d} \right) \left( \int_{B_R} |\varphi_i|^2 \right)^\frac{2}{q_d} + \left( \int_{B_R} |\sigma_i|^2 \right) \right] \int_{B_R} |\nabla u|^2. \]

PROOF. We will first consider the case \( R = 1 \) and obtain the general result by scaling. Let \( u \in H^1_{\text{loc}}(B_1) \) be an arbitrary distributional solution of the equation

\[ -\nabla \cdot (a + s) \cdot \nabla u = 0 \text{ in } B_1. \]
We will first prove that for every $\varepsilon \in (0, 1)$ there exists an $\overline{\alpha}$-harmonic function $u^\varepsilon \in H^1(B_{1/2})$ such that the homogenization error $u^\varepsilon = u - u^\varepsilon - \phi_i \partial_i v^\varepsilon$ satisfies, for every $\rho \in (0, 1/4)$,

\begin{align}
(116) \quad \int_{B_{1/4}} |\nabla u^\varepsilon|^2 & \lesssim (\varepsilon + \varepsilon^{-d-1} (\int_{B_1} |s|^q) \frac{1}{r^d}) \int_{B_1} |\nabla u|^2 \\
& + \varepsilon^{-(d-1)} \rho^\frac{1}{2} \left( 1 + \left( \int_{B_r} |s|^q \right)^{\frac{2}{q^d}} \right) \int_{B_1} |\nabla u|^2 \\
& + \rho^{-2(d+1)} \left[ \left( 1 + \left( \int_{B_r} |s|^q \right)^{\frac{2}{q^d}} \right) \left( \int_{B_r} |\phi_i|^2 \right)^{\frac{2}{2^*}} + \left( \int_{B_r} |\sigma_i|^2 \right) \right] \int_{B_1} |\nabla u|^2,
\end{align}

for the correctors $\phi_i$ defined in (103). Since it follows from Fubini’s theorem that

\[ \int_{B_1} |\nabla u|^2 = \int_0^1 \left( \int_{\partial B_r} |\nabla u|^2 \right) \, dr, \]

arguing by contradiction there must exist a henceforth fixed $r \in (1/2, 3/4)$ such that

\begin{align}
(117) \quad \int_{\partial B_r} |\nabla u|^2 & \leq 4 \int_{B_1} |\nabla u|^2 \quad \text{and} \quad \int_{\partial B_r} |s|^q \leq 4 \int_{B_1} |s|^q,
\end{align}

and for every $\varepsilon \in (0, 1)$ let $\nu^\varepsilon$ denote a standard convolution of scale $\varepsilon$ of $u$ on $\partial B_r$. For each $\varepsilon \in (0, 1)$ let $v^\varepsilon \in H^1(B_r)$ solve

\[ -\nabla \cdot \overline{\alpha} \nabla v^\varepsilon = 0 \quad \text{in} \quad B_r \quad \text{with} \quad v^\varepsilon = u^\varepsilon \quad \text{on} \quad \partial B_r. \]

It then follows from Dirichlet-to-Neumann estimates Fabes, Jodeit and Rivière [22, Theorem 2.4] and Stein [55, Chapter 7], (117), and the fact that the convolution preserves the $L^2$-norm that

\[ \int_{\partial B_r} |\nu \cdot \nabla v|^2 \leq \int_{\partial B_r} |\nabla v^\varepsilon|^2 \leq \int_{\partial B_r} |\nabla u|^2 \leq \int_{B_1} |\nabla u|^2, \]

for the outward unit normal $\nu$ to $\partial B_r$. Finally, for each $\rho \in (0, 1/4)$ let $\eta_\rho : \mathbb{R}^d \to [0, 1]$ be a smooth function satisfying $\eta_\rho = 1$ on $B_{1-\rho}$, satisfying $\eta_\rho = 0$ on $\mathbb{R}^d \setminus B_{1-\rho/2}$, and satisfying $|\eta_\rho(x)| \leq c/\rho$ for some $c \in (0, \infty)$ independent of $\rho \in (0, 1/4)$. The first step will be to estimate the energy of the homogenization error $w^\varepsilon,\rho \in H^1_0(B_r)$ defined by

\[ w^\varepsilon,\rho = u - u^\varepsilon - \phi_i \eta_\rho \partial_i v^\varepsilon. \]

A repetition of the derivation leading to (107) proves that

\[ -\nabla \cdot (a + s) \nabla w^\varepsilon,\rho = \nabla \cdot \left[ (1 - \eta_\rho)((a + s) - \overline{a}) \nabla v^\varepsilon \right] + \nabla \cdot \left[ \phi_i (a + s - \sigma_i) \nabla (\eta_\rho \partial_i v^\varepsilon) \right] \]

in $B_r$ with boundary condition $w^\varepsilon,\rho = u - u^\varepsilon$ on $\partial B_r$, for the flux correctors $\sigma_i$ defined in Proposition 2.6. It follows from Hölder’s inequality, Young’s inequality, the triangle inequality, the uniform ellipticity, the definition of $\eta_\rho$, and a repetition of the argument leading to (83) that

\begin{align}
(119) \quad \int_{B_r} |\nabla w^\varepsilon,\rho|^2 & \lesssim \int_{\partial B_r} (u - u^\varepsilon) \nu \cdot ((a + s) \nabla u - \overline{a} \nabla v) \\
& + \rho^{\frac{1}{2}} \left( 1 + \left( \int_{B_r} |s|^q \right)^{\frac{2}{q^d}} \right) \left( \int_{B_1} |\nabla v^\varepsilon|^2 \right)^{\frac{1}{2^*}} \\
& + \sup_{B_{(1-\rho/2)}} (|\nabla (\partial_i v^\varepsilon)|^2 + \frac{1}{\rho^2} |\partial_i v^\varepsilon|^2) \left[ (1 + \left( \int_{B_r} |s|^q \right)^{\frac{2}{q^d}}) \left( \int_{B_r} |\phi_i|^2 \right)^{\frac{2}{2^*}} + \left( \int_{B_r} |\sigma_i|^2 \right) \right].
\end{align}
Hölder’s inequality proves that, for the first term on the righthand side of (119),

\[
\left| \int_{\partial B_r} (u - u^\varepsilon) \nu \cdot ((a + s) \nabla u - \bar{a} \nabla v) \right|
\]

\[
\lesssim \left( \left( \int_{\partial B_r} |\nabla u|^2 \right)^{\frac{1}{2}} + \left( \int_{\partial B_r} |\nabla v|^2 \right)^{\frac{1}{2}} \right) \left( \int_{\partial B_r} |u - u^\varepsilon|^2 \right)^{\frac{1}{2}} + \left( \int_{\partial B_r} |s|^{q_d} \right)^{\frac{1}{q_d}} \left( \int_{\partial B_r} |\nabla u|^2 \right)^{\frac{1}{2}} \left( \int_{\partial B_r} |u - u^\varepsilon|^2 \right)^{\frac{1}{2}}.
\]

Since for each \( p \in [1, \infty) \) we have the convolution estimate

\[
\left( \int_{\partial B_r} |u - u^\varepsilon|^p \right)^{\frac{1}{p}} \lesssim \varepsilon \left( \int_{\partial B_r} |\nabla^{\text{tan}} u|^p \right)^{\frac{1}{p}},
\]

it follows from (117) that the first term on the righthand side of (120) is bounded by

\[
\left( \left( \int_{\partial B_r} |\nabla u|^2 \right)^{\frac{1}{2}} + \left( \int_{\partial B_r} |\nabla v|^2 \right)^{\frac{1}{2}} \right) \left( \int_{\partial B_r} |u - u^\varepsilon|^2 \right)^{\frac{1}{2}} \lesssim \varepsilon \int_{\partial B_r} |\nabla u|^2 \lesssim \varepsilon \int_{B_1} |\nabla u|^2.
\]

For the second term on the righthand side of (120), since the definition of the convolution proves that \( \int_{B_r} (u - u^\varepsilon) = 0 \), it follows from the Sobolev inequality, the fact that the convolution does not increase \( L^p \)-norms for \( p \in [1, \infty) \), and the triangle inequality that

\[
\left( \int_{\partial B_r} |u - u^\varepsilon|^2 \right)^{\frac{1}{2}} \lesssim \left( \int_{\partial B_r} |\nabla^{\text{tan}} (u - u^\varepsilon)|^q \right)^{\frac{1}{q}} \lesssim \varepsilon \left( \int_{\partial B_r} |\nabla u|^q \right)^{\frac{1}{q}},
\]

for \( q \in (1, 2) \) defined by \( \frac{1}{q} = \frac{1}{2} + \frac{1}{d+1} \). Interpolating between the the convolution estimate with \( p = 2_2 \) and (122) proves with (117) that

\[
\left( \int_{\partial B_r} |u - u^\varepsilon|^2 \right)^{\frac{1}{2}} \lesssim \varepsilon^{1 - \frac{d-1}{2d}} \left( \int_{\partial B_r} |\nabla u|^2 \right)^{\frac{1}{2}} \lesssim \varepsilon^{1 - \frac{d-1}{2d}} \left( \int_{B_1} |\nabla u|^2 \right)^{\frac{1}{2}}.
\]

In combination (117) and (123) prove that the second term on the righthand side of (120) satisfies

\[
\left( \int_{\partial B_r} |s|^{q_d} \right)^{\frac{1}{q_d}} \left( \int_{\partial B_r} |\nabla u|^2 \right)^{\frac{1}{2}} \left( \int_{\partial B_r} |u - u^\varepsilon|^2 \right)^{\frac{1}{2}} \lesssim \varepsilon^{1 - \frac{d-1}{2d}} \left( \int_{B_1} |s|^{q_d} \right)^{\frac{1}{q_d}} \int_{B_1} |\nabla u|^2.
\]

Returning to (120), it follows from (121) that

\[
\left| \int_{\partial B_r} (u - u^\varepsilon) \nu \cdot ((a + s) \nabla u - \bar{a} \nabla v) \right| \lesssim \left( \int_{\partial B_r} |s|^{q_d} \right)^{\frac{1}{q_d}} \left( \int_{\partial B_r} |\nabla u|^2 \right)^{\frac{1}{2}} \lesssim \varepsilon^{1 - \frac{d-1}{2d}} \left( \int_{B_1} |s|^{q_d} \right)^{\frac{1}{q_d}} \int_{B_1} |\nabla u|^2.
\]

For the second term on the righthand side of (119), since it follows from Hölder’s inequality, the definition of the convolution kernel, and (117) that

\[
\sup_{\partial B_r} |\nabla^{\text{tan}} u^\varepsilon| \lesssim \varepsilon^{-\frac{d-1}{2}} \left( \int_{\partial B_r} |\nabla^{\text{tan}} u|^2 \right)^{\frac{1}{2}} \lesssim \varepsilon^{-\frac{d-1}{2}} \left( \int_{B_1} |\nabla u|^2 \right)^{\frac{1}{2}},
\]

it follows from Proposition 5.2 and (125) that

\[
\left( \int_{B_r} |\nabla v^\varepsilon|^{2+2} \right)^{\frac{1}{2+2}} \lesssim \varepsilon^{-(d-1)} \int_{B_1} |\nabla u|^2.
\]
Therefore, using (126), the second term on the righthand side of (119) is bounded by
\begin{equation}
\rho^{\frac{1}{2d}} \left( 1 + \left( \int_{B_r} |s|^{q_d} \right)^{\frac{1}{q_d}} \right) \left( \int_{B_r} |\nabla v^\varepsilon|^2 \right)^{\frac{1}{2}} 
\end{equation}
\begin{equation}
\lesssim \varepsilon^{-(d-1)} \rho^{\frac{1}{2d}} \left( 1 + \left( \int_{B_r} |s|^{q_d} \right)^{\frac{1}{q_d}} \right) \int_{B_1} |\nabla u|^2.
\end{equation}

For the final term of (119), it follows from Proposition 5.2 that
\begin{equation}
\sup_{B_{(1-\rho^2)}} (|\nabla (\partial_i v^\varepsilon)|)^2 + \frac{1}{\sigma^2} |\partial_i v^\varepsilon|^2 \geq \left[ (1 + \left( \int_{B_r} |s|^{q_d} \right)^{\frac{1}{q_d}} \right) \int_{B_r} |\phi_i|^2 \right]^{\frac{2}{d}} + \int_{B_r} |\sigma_i|^2
\end{equation}
\begin{equation}
\lesssim \rho^{-2(d+1)} \left[ \left( 1 + \left( \int_{B_1} |s|^{q_d} \right)^{\frac{1}{q_d}} \right) \int_{B_1} |\nabla u|^2 \right].
\end{equation}

For every \( \varepsilon \in (0, 1) \) let \( w^\varepsilon = u - v^\varepsilon - \phi_i \partial_i v^\varepsilon \). It follows from \( \rho \in (0, 1/4) \) and the definition of \( \eta_p \) that \( w^\varepsilon = w^{\varepsilon-p} \) in \( B_{1/4} \), and therefore it follows from (119), (124), (127), and (128) that, for each \( \rho \in (0, 1/4) \),
\begin{equation}
\int_{B_{1/4}} |\nabla w^\varepsilon|^2 \lesssim \left( \varepsilon + \varepsilon^{1 - \frac{d-1}{q_d}} \left( \int_{B_1} |s|^{q_d} \right)^{\frac{1}{q_d}} \right) \int_{B_1} |\nabla u|^2
\end{equation}
\begin{equation}
+ \varepsilon^{-(d-1)} \rho^{\frac{1}{2d}} \left( 1 + \left( \int_{B_1} |s|^{q_d} \right)^{\frac{1}{q_d}} \right) \int_{B_1} |\nabla u|^2
\end{equation}
\begin{equation}
+ \rho^{-2(d+1)} \left[ \left( 1 + \left( \int_{B_1} |s|^{q_d} \right)^{\frac{1}{q_d}} \right) \int_{B_1} |\nabla u|^2 \right],
\end{equation}
which completes the proof of (116) with \( v^\varepsilon \in H^1(B_{1/2}) \) defined by (118). It then follows by scaling that, for each \( R \in (0, \infty) \), for any \( u \in H^1(B_{R}) \) that is a weak solution of
\[- \nabla \cdot (a + s) \cdot \nabla u = 0 \quad \text{in} \quad B_R,
\]
there exists for every \( \varepsilon \in (0, 1) \) a \( \overline{\varepsilon} \)-harmonic function \( v^\varepsilon \in H^1(B_{R/2}) \) such that the homogenization error \( w = u - v^\varepsilon - \phi_i \partial_i v^\varepsilon \) satisfies, for every \( \rho \in (0, 1/4) \), for some \( \varepsilon \in (0, \infty) \) independent of \( R, \varepsilon, \) and \( \rho,
\begin{equation}
\int_{B_{R/4}} |\nabla w^\varepsilon|^2 \lesssim \varepsilon \left( \varepsilon + \varepsilon^{1 - \frac{d-1}{q_d}} \left( \int_{B_R} |s|^{q_d} \right)^{\frac{1}{q_d}} \right) \int_{B_R} |\nabla u|^2
\end{equation}
\begin{equation}
+ \varepsilon^{-d-1} \rho^{\frac{1}{2d}} \left( 1 + \left( \int_{B_R} |s|^{q_d} \right)^{\frac{1}{q_d}} \right) \int_{B_R} |\nabla u|^2
\end{equation}
\begin{equation}
+ \rho^{-2(d+1)} R^{-2} \left[ \left( 1 + \left( \int_{B_R} |s|^{q_d} \right)^{\frac{1}{q_d}} \right) \int_{B_R} |\nabla u|^2 \right].
\end{equation}

The proof follows by considering the rescalings \( u^R(x) = R^{-1} u(Rx), \phi_i^R(x) = R^{-1} \phi_i(Rx), \) and \( \sigma_i^R(x) = R^{-1} \sigma_i(x) \) and by repeating the argument leading to (129) to obtain an \( \varepsilon \)-harmonic function \( \tilde{v}^\varepsilon \) in \( H^1(B_{1/4}) \) such that the homogenization error \( w^{R,\varepsilon}(x) = u^R - v^\varepsilon - \phi_i^R \partial_i \tilde{v}^\varepsilon \) satisfies (129) with \( \phi_i^R, \sigma_i^R, \) and \( s^R(x) = s(Rx) \). We then define \( v^\varepsilon(x) = R \tilde{v}^\varepsilon(x/R) \) and obtain (130) from (129) after rescaling. This completes the proof.
Proposition 5.5. Assume (12) for $S \in L^d(\Omega; \mathbb{R}^{d \times d})$ if $d \geq 3$ and $S \in L^{2+\delta}(\Omega; \mathbb{R}^{d \times d})$ for some $\delta \in (0, 1)$ if $d = 2$ and let $q_d = d \vee (2 + \delta)$. For every $\alpha \in (0, 1)$ there exist $c_\alpha, C_\alpha \in (0, \infty)$ such that if for any $R_1 < R_2 \in (0, \infty)$ we have, for every $R \in [R_1, R_2]$ and $i \in \{1, \ldots, d\}$,

$$\frac{1}{R} \left( \int_{B_R} |\phi_i|^2 \right)^{\frac{1}{2^*}} + \frac{1}{R} \left( \int_{B_R} |\sigma_i|^2 \right)^{\frac{1}{2}} \leq 1/C_\alpha,$$

and we have, for every $R \in [R_1, R_2]$,

$$\left( \int_{B_R} |s|^{q_d} \right)^{\frac{1}{q_d}} \leq \mathbb{E} \left[ |S|^{q_d} \right]^{\frac{1}{q_d}} + 1 \quad \text{and} \quad \sum_{i=1}^d \int_{B_R} |\Phi_i + \epsilon_i|^2 \leq \sum_{i=1}^d \left[ |\Phi_i + \epsilon_i|^2 \right] + 1,$$

then every weak solution $u \in H^1_{\text{loc}}(\mathbb{R}^d)$ of the equation

$$-\nabla \cdot (a + s) \nabla u = 0 \quad \text{in} \quad \mathbb{R}^d,$$

satisfies

$$R_1^{-2\alpha} \text{Exc}(u; R_1) \leq c_\alpha R_2^{-2\alpha} \text{Exc}(u; R_2).$$

Proof. Let $u \in H^1_{\text{loc}}(\mathbb{R}^d)$ be a distributional solution of $-\nabla \cdot (a + s) \nabla u = 0$ in $\mathbb{R}^d$, let $R \in (0, \infty)$, and for each $\epsilon \in (0, 1)$ let $w^{\epsilon}$ be defined in (130) on $B_{R/2}$. We will first show that, for each $\delta_0 \in (0, 1)$, there exists $C_1 \in (1, \infty)$ depending on $\delta_0$ such that if

$$\left( \int_{B_R} |s|^{q_d} \right)^{\frac{1}{q_d}} \leq \mathbb{E} \left[ |S|^{q_d} \right]^{\frac{1}{q_d}} + 1,$$

and such that if

$$\sum_{i=1}^d \left( \frac{1}{R^2} \left( \int_{B_R} |\phi_i|^2 \right)^{\frac{1}{2^*}} + \frac{1}{R^2} \left( \int_{B_R} |\sigma_i|^2 \right)^{\frac{1}{2}} \right) \leq \frac{1}{C_1},$$

then there exists a deterministic $\epsilon_0 \in (0, 1)$ depending on $\delta_0$ such that

$$\int_{B_{R/4}} |\nabla w^{\epsilon_0}|^2 \leq \delta_0 \int_{B_R} |\nabla u|^2,$$

for $w^{\epsilon_0}$ defined in (130). First fix $\epsilon_0 \in (0, 1)$ such that

$$\bar{c} \left( \epsilon_0 + \epsilon_0^{\frac{d-1}{q_d}} \left( \mathbb{E} \left[ |S|^{q_d} \right]^{\frac{1}{q_d}} + 1 \right) \right) < \delta_0/3.$$

Then fix $\rho_0 \in (0, 1/4)$ such that

$$\bar{c} \rho_0^{-(d-1)} \left( 2 + \mathbb{E} \left[ |S|^{q_d} \right]^{\frac{1}{q_d}} \right) < \delta_0/3.$$

Finally fix $C_1 \in (1, \infty)$ satisfying

$$\bar{c} \rho_0^{-2(d+1)} \left( 2 + \mathbb{E} \left[ |S|^{q_d} \right]^{\frac{1}{q_d}} \right) < C_1 \delta_0/3.$$

The claim (131) then follows from Proposition 5.4, (132), (133), and (134).

We will now prove that there exists $\theta_0 \in (0, 1/4)$ and $C_2 \in (1, \infty)$ such that for any $R \in (0, \infty)$ satisfying, for every $r \in [\theta_0 R, R]$,

$$\sum_{i=1}^d \frac{1}{r^d} \left( \int_{B_r} |\phi_i|^2 \right)^{\frac{1}{2^*}} + \frac{1}{r^d} \left( \int_{B_r} |\sigma_i|^2 \right)^{\frac{1}{2}} \leq 1/C_2,$$
Proposition 5.2 proves that
\[ (139) \]
for each \( \xi^\varepsilon = \nabla v^\varepsilon(0) \in \mathbb{R}^d \) and observe that
\[ u - \xi^\varepsilon - \nabla \phi \xi^\varepsilon = \nabla w^\varepsilon + (\nabla v^\varepsilon - \nabla v^\varepsilon(0)) + \nabla \phi_i (\partial_i v^\varepsilon - \partial_i v^\varepsilon(0)) + \phi_i \nabla (\partial_i v^\varepsilon) \] on \( B_{\theta R/2} \),
for \( w^\varepsilon \) defined in (130). For every \( r \in (0, R/4) \) the triangle inequality, Young’s inequality, and the mean value theorem prove that
\[
\begin{align*}
(\int_{B_r} |u - \xi^\varepsilon - \nabla \phi \xi^\varepsilon|^2) &\leq \int_{B_{\theta R/4}} |\nabla w^\varepsilon|^2 + r^2 \sup_{B_r} |\nabla \phi \xi^\varepsilon|^2 \int_{B_r} |\varepsilon_i + \nabla \phi \xi^\varepsilon|^2 + \sup_{B_r} |\nabla \partial_i v^\varepsilon|^2 \int_{B_r} |\phi_i|^2.
\end{align*}
\]
Proposition 5.2 proves that
\[
\begin{align*}
\int_{B_r} |u - \xi^\varepsilon - \nabla \phi \xi^\varepsilon|^2 &\leq \int_{B_{\theta R/4}} |\nabla w^\varepsilon|^2 + (\frac{r^2}{R^2} \int_{B_r} |\varepsilon_i + \nabla \phi \xi^\varepsilon|^2 + \sup_{B_r} |\nabla \partial_i v^\varepsilon|^2 \int_{B_r} |\phi_i|^2) \int_{B_R} |\nabla u|^2.
\end{align*}
\]
After dividing by \( r^d \), for some \( \varepsilon \in (0, \infty) \) independent of \( \varepsilon, r, \) and \( R, \)
\[
\begin{align*}
(\int_{B_r} |u - \xi^\varepsilon - \nabla \phi \xi^\varepsilon|^2 &\leq \varepsilon \left( \frac{R^d}{r^d} \int_{B_{\theta R/4}} |\nabla w^\varepsilon|^2 + \left( \frac{r^2}{R^2} \left( \int_{B_r} |\varepsilon_i + \nabla \phi \xi^\varepsilon|^2 + \sup_{B_r} |\nabla \partial_i v^\varepsilon|^2 \int_{B_r} |\phi_i|^2 \right) \int_{B_R} |\nabla u|^2 \right) \right).
\end{align*}
\]
Since \( \alpha \in (0, 1) \) fix \( \theta_0 \in (0, 1/4) \) such that
\[
(137) \quad \varepsilon \theta_0^2 \left( \sum_{i=1}^d \mathbb{E} \left| \Phi_i + e_i \right|^2 + 2 \right) \leq \theta_0^{2\alpha}/2,
\]
then fix \( \delta_0 \in (0, 1) \) such that
\[
(138) \quad \varepsilon \theta_0^{-d} \delta_0 \leq \theta_0^{2\alpha}/2,
\]
and let \( C_2 \in (1, \infty) \) and \( \varepsilon_0 \in (0, 1) \) satisfy the conclusion of (131) for this \( \delta_0 \). The definition of the excess, \( C_2 \in (1, \infty), (131), (135), \) and (136) then prove after choosing \( r = \theta_0 R \) that
\[
(139) \quad \text{Exc}(u; \theta_0 R) \leq \varepsilon \left( \delta_0 \theta_0^{-d} + \theta_0^2 \left( \sum_{i=1}^d \mathbb{E} \left| \Phi_i + e_i \right|^2 + 2 \right) \int_{B_R} |\nabla u|^2. \right.
\]
In combination (137), (138), and (139) prove that
\[
(140) \quad \text{Exc}(u; \theta_0 R) \leq \theta_0^{2\alpha} \int_{B_R} |\nabla u|^2.
\]
We now observe that for every $\xi \in \mathbb{R}^d$ the function $u - \xi \cdot x - \phi_\xi \in H^1_{\text{loc}}(\mathbb{R}^d)$ solves
\begin{equation}
- \nabla \cdot (a + s)\nabla (u - \xi \cdot x - \phi_\xi) = 0 \text{ in } \mathbb{R}^d,
\end{equation}
and by definition of the excess and linearity we have
\[ \text{Exc}(u; \theta_0 R) = \text{Exc}(u - \xi \cdot x - \phi_\xi; \theta_0 R) \text{ for every } \xi \in \mathbb{R}^d. \]
Therefore, since $u \in H^1_{\text{loc}}(\mathbb{R}^d)$ solving (141) was arbitrary, we have from (140) and the definition of the excess that
\begin{equation}
\text{Exc}(u; \theta_0 R) = \inf_{\xi \in \mathbb{R}^d} \text{Exc}(u - \xi \cdot x - \phi_\xi; \theta_0 R) 
\leq \theta_0^{2\alpha} \inf_{\xi \in \mathbb{R}^d} \left( \int_{B_R} |\nabla u - \xi - \nabla \phi_\xi|^2 \right) = \theta_0^{2\alpha} \text{Exc}(u; R).
\end{equation}
We will now use the exact inequality (142) to conclude. For $R_1 \leq R_2 \in (0, \infty)$ suppose that (135) and (136) are satisfied for every $r \in [R_1, R_2]$. We will prove that there exists $c \in (0, \infty)$ depending on $\alpha$ but independent of $R_1, R_2 \in (0, \infty)$ such that
\[ \text{Exc}(u; R_1) \leq c (R_1/R_2)^{2\alpha} \text{Exc}(u; R_2). \]
For $\theta_0 \in (0, 1/4)$ defined in (138), if $R_1/R_2 \geq \theta_0$ then by definition of the excess
\[ \text{Exc}(u; R_1) \leq (R_2/R_1)^d \text{Exc}(u; R_2) \leq \theta_0^{-d} \text{Exc}(u; R_2) \leq \theta_0^{-(d+2\alpha)} (R_1/R_2)^{2\alpha} \text{Exc}(u; R_2). \]
If $R_1/R_2 < \theta_0$ let $n \in \mathbb{N}$ be the unique positive integer satisfying $\theta_0^n \leq R_1/R_2 < \theta_0^{n-1}$ and observe by induction, the previous step, and (142) that
\[ \text{Exc}(u; R_1) \leq \theta_0^{-(d+2\alpha)} \text{Exc}(u; \theta_0^{n-1} R_2) \leq \theta_0^{-(d+2\alpha)} \theta_0^{2\alpha} \text{Exc}(u; \theta_0^{n-2} R_2) \]
\[ \leq \theta_0^{-(d+2\alpha)} (\theta_0^{n-1})^{2\alpha} \text{Exc}(u; R_2) \leq \theta_0^{-(d+4\alpha)} (R_1/R_2)^{2\alpha} \text{Exc}(u; R_2), \]
which completes the proof.

**Theorem 5.6.** Assume (12) for $S \in L^d(\Omega; \mathbb{R}^{d \times d})$ if $d \geq 3$ and for $S \in L^{2+\delta}(\Omega; \mathbb{R}^{d \times d})$ for some $\delta \in (0, 1)$ if $d = 2$. On a subset of full probability, for every $\alpha \in (0, 1)$ there exists a deterministic $c \in (0, \infty)$ and a random radius $R_0 \in (0, \infty)$ such that, for every weak solution $u \in H^1_{\text{loc}}(\mathbb{R}^d)$ of the equation
\[ -\nabla \cdot (a + s)\nabla u = 0 \text{ in } \mathbb{R}^d, \]
for every $R_1 < R_2 \in (R_0, \infty)$,
\[ R_1^{-2\alpha} \text{Exc}(u; R_1) \leq c R_2^{-2\alpha} \text{Exc}(u; R_2). \]

**Proof.** The proof is an immediate consequence of the ergodic theorem, Proposition 2.1, and Proposition 5.5.

**6. The Liouville theorem.** In this section, we will prove the first-order Liouville theorem for subquadratic solutions $u \in H^1_{\text{loc}}(\mathbb{R}^d)$ of the equation
\[ -\nabla \cdot (a + s)\nabla u = 0 \text{ in } \mathbb{R}^d. \]
That is, in analogy with the Liouville theorem for harmonic functions on Euclidean space, the space of subquadratic $(a + s)$-harmonic functions is spanned by the $(a + s)$-harmonic coordinates. The section is organized as follows. We prove in Proposition 6.1 below a version of the Caccioppoli inequality adapted to the divergence-free setting. We then prove the Liouville theorem in Theorem 6.2 below, which is a consequence of the large-scale regularity estimate of Theorem 5.6 and the Caccioppoli inequality. These methods are motivated by the analogous results in [14, 34] from the elliptic setting.
PROPOSITION 6.1. Assume (12) for $S \in L^d(\Omega; \mathbb{R}^{d \times d})$ if $d \geq 3$ and $S \in L^{2+\delta}(\Omega; \mathbb{R}^{d \times d})$ for some $\delta \in (0, 1)$ if $d = 2$, let $q_d = d \vee (2 + \delta)$, let $1/2_\ast = 1/2 - 1/q_d$, and let $u \in H^1_{\text{loc}}(\mathbb{R}^d)$ be a weak solution of
\begin{equation}
- \nabla \cdot (a + s) \nabla u = 0 \text{ in } \mathbb{R}^d. \tag{143}
\end{equation}
Then, for every $R \in (0, \infty)$, for some $c \in (0, \infty)$ independent of $R$,
\[
\int_{B_R} \left| \nabla u \right|^2 \leq \frac{c}{R^2} \left( \int_{B_{2R}} |u|^2 + \left( \int_{B_{2R}} |s|^{q_d} \right)^{\frac{2}{q_d}} \left( \int_{B_{2R}} |u|^{2q_d} \right)^{\frac{2}{q_d}} \right).
\]

PROOF. Let $\eta: \mathbb{R}^d \to \mathbb{R}$ be a smooth cutoff function satisfying $\eta = 1$ on $B_1$, satisfying $\eta = 0$ on $\mathbb{R}^d \setminus B_2$, and for each $R \in (0, \infty)$ define $\eta_R(x) = \eta(x/R)$. A repetition of the argument leading to (83) proves that, after testing (143) with $\eta_R^2 u_n$ for $u_n = (u \wedge n) \vee (-n)$ and passing to the limit $n \to \infty$,
\[
\int_{B_R} a \nabla u \cdot (\nabla \eta)^2 \eta_R^2 = -2 \int_{B_R} a \nabla u \cdot \nabla \eta_R (u \eta_R) - 2 \int_{B_R} s \nabla u \cdot \nabla \eta_R (u \eta_R).
\]
It then follows by definition of $\eta_R$, the uniform ellipticity, Hölder’s inequality, and Young’s inequality that, for some $c \in (0, \infty)$ independent of $R$,
\[
\int_{B_R} \left| \nabla u \right|^2 \leq \frac{c}{R^2} \left( \int_{B_{2R}} |u|^2 + \left( \int_{B_{2R}} |s|^{q_d} \right)^{\frac{2}{q_d}} \left( \int_{B_{2R}} |u|^{2q_d} \right)^{\frac{2}{q_d}} \right). \tag{144}
\]

THEOREM 6.2. Assume (12). Let $q_d = d \vee (2 + \delta)$ and $1/2_\ast = 1/2 - 1/q_d$. Then almost surely every weak solution $u \in H^1_{\text{loc}}(\mathbb{R}^d)$ of the equation
\begin{equation}
- \nabla \cdot (a + s) \nabla u = 0 \text{ in } \mathbb{R}^d, \tag{145}
\end{equation}
that is strictly subquadratic in the sense that, for some $\alpha \in (0, 1)$,
\[
\lim_{R \to \infty} \frac{1}{R^{1+\alpha}} \left( \int_{B_R} |u|^{2_\ast} \right)^{\frac{1}{2_\ast}} = 0,
\]
satisfies $u = c + \xi \cdot x + \phi_\xi$ on $\mathbb{R}^d$ for some $c \in (0, \infty)$ and $\xi \in \mathbb{R}^d$.

PROOF. By the ergodic theorem and $\mathbb{E}[\Phi_i] = 0$ for each $i \in \{1, \ldots, d\}$ let $\Omega_1 \subseteq \Omega$ be the subset of full probability satisfying, for every $\xi \in \mathbb{R}^d$,
\begin{equation}
\lim_{R \to \infty} \int_{B_R} |\nabla \phi_\xi + \xi|^2 = \mathbb{E} \left[ |\Phi_\xi|^2 \right] + |\xi|^2 \geq |\xi|^2 \text{ and } \lim_{R \to \infty} \int_{B_R} |s|^{q_d} = \mathbb{E} \left[ |S|^{q_d} \right]. \tag{146}
\end{equation}
Let $\Omega_2 \subseteq \Omega$ be the subset of full probability satisfying the conclusion Theorem 5.6, and let $\Omega_3 = \Omega_1 \cap \Omega_2$. For $\omega \in \Omega_3$ let $R_0 \in (0, \infty)$ be such that every weak solution $u \in H^1_{\text{loc}}(\mathbb{R}^d)$ of (144) satisfies, for every $R_1 < R_2 \in (R_0, \infty)$, for a deterministic $c \in (0, \infty)$ depending on $\alpha$,
\begin{equation}
R_1^{-2\alpha} \text{Exc}(u; R_1) \leq c R_2^{-2\alpha} \text{Exc}(u; R_2), \tag{147}
\end{equation}
and such that, for every $R \in (R_0, \infty)$ and $\xi \in \mathbb{R}^d$,
\begin{equation}
\int_{B_R} |\nabla \phi_\xi + \xi|^2 \geq |\xi|^2 / 2. \tag{148}
\end{equation}
The definition of the excess, $u \in H^1_{\text{loc}}(\mathbb{R}^d)$, and (148) prove that, for every $R \in (R_0, \infty)$,
\begin{equation}
\text{Exc}(u; R) = \inf_{\xi \in \mathbb{R}^d} \left( R^{-2\alpha} \int_{B_R} \nabla u - \xi - \nabla \phi_\xi^2 \right) = \min_{\xi \in \mathbb{R}^d} \left( R^{-2\alpha} \int_{B_R} \nabla u - \xi - \nabla \phi_\xi^2 \right). \tag{149}
\end{equation}
Fix \( R_1 \in (R_0, \infty) \). We have by definition of the excess, Proposition 6.1, and (147) that, for every \( R \in (R_1, \infty) \),

\[
R_1^{-2\alpha} \text{Exc}(u; R_1) \leq c R^{-2\alpha} \int_{B_R} |\nabla u|^2 \\
\leq c R^{-2(1+\alpha)} \left( \int_{B_{2R}} |u|^2 + \left( \int_{B_{2R}} |s|^{q_4} \right)^{\frac{2}{q_4}} \left( \int_{B_{2R}} |u|^{2^*} \right)^{\frac{2}{2^*}} \right).
\]

Hölder’s inequality, the ergodic theorem, \( 2^* \in (2, \infty) \), (145), and (146) prove almost surely that

\[
R_1^{-2\alpha} \text{Exc}(u; R_1) \\
\leq c \limsup_{R \to \infty} R^{-2(1+\alpha)} \left( \int_{B_{2R}} |u|^2 + \left( \int_{B_{2R}} |s|^{q_4} \right)^{\frac{2}{q_4}} \left( \int_{B_{2R}} |u|^{2^*} \right)^{\frac{2}{2^*}} \right) = 0.
\]

It then follows from (149) that there exists \( \xi_{R_1} \in \mathbb{R}^d \) such that

\[
\nabla u - \xi_{R_1} = 0 \quad \text{in} \quad L^2(B_{R_1}; \mathbb{R}^d),
\]

and therefore there exists \( c_{R_1} \in \mathbb{R} \) such that

\[
u = c_{R_1} + \xi_{R_1} \cdot x + \phi_{\xi_{R_1}} \quad \text{in} \quad H^1(B_{R_1}).
\]

Since the linearity and (148) prove that \( c_{R_1} = c_{R_2} \) and \( \xi_{R_1} = \xi_{R_2} \) whenever \( R_1 \leq R_2 \in (R_0, \infty) \), there exists \( \xi \in \mathbb{R}^d \) and \( c \in \mathbb{R} \) such that

\[
u = c + \xi \cdot x + \phi_{\xi} \quad \text{in} \quad H^1_{\text{loc}}(\mathbb{R}^d).
\]

\[ \Box \]

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