Existence and uniqueness of Rayleigh waves in isotropic elastic Cosserat materials and algorithmic aspects

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Abstract

We discuss the propagation of surface waves in an isotropic half space modelled with the linear Cosserat theory of isotropic elastic materials. To this aim we use a method based on the algebraic analysis of the surface impedance matrix and on the algebraic Riccati equation, and which is independent of the common Stroh formalism. Due to this method, a new algorithm which determines the amplitudes and the wave speed in the theory of isotropic elastic Cosserat materials is described. Moreover, the method allows us to prove the existence and uniqueness of a subsonic solution of the secular equation, a problem which remains unsolved in almost all generalised linear theories of elastic materials. Since the results are suitable to be used for numerical implementations, we propose two numerical algorithms which are viable for any elastic material. Explicit numerical calculations are made for aluminium-epoxy in the context of the Cosserat model. Since the novel form of the secular equation for isotropic elastic material has not been explicitly derived elsewhere, we establish it in this paper, too.

Keywords: Cosserat elastic materials, Matrix analysis method, Riccati equation, Rayleigh waves, Stroh formalism, secular equation, existence and uniqueness.

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1 Introduction

Traveling waves can exist within a small depth from a free surface of an elastic continuum, while the bulk of the continuum remains almost at rest. Such waves are called Rayleigh waves named after the English physicist Lord Rayleigh, who carried out pioneering work in studies of wave propagations in isotropic elastic media [60]. Rayleigh waves are important for modelling seismic waves that are created by earthquakes on the surface of the earth. These waves are a combination of longitudinal (horizontal) and transverse (vertical) motions. The horizontal components of the displacement are parallel to the direction of propagation, whereas vertical components are directed into the half-space. Rayleigh waves move elliptically, they take place counter-clockwise near the surface and clockwise deep down. The amplitude of the waves decays exponentially with depth beneath the surface. The studies of Rayleigh waves captivated the attention of many scientists owing to its industrial application such as material characterization, nondestructive evaluation and acoustic microscopy. These waves are also employed to detect cracks and other defects in the material. In addition, many applications have been found in seismology and near surface geophysical exploration.

Rayleigh waves are particular inhomogeneous plane waves. Inhomogeneous plane waves, also known as evanescent waves, represent those waves for which the planes of constant phase are not the same as the planes of constant amplitude. Usually, these types of waves are described in terms of the slowness bivector (a complex vector) and the amplitude bivector. Hayes [29, 8] has developed the directional-ellipse method for a systematic study of all inhomogeneous plane waves that may propagate in classical linear elasticity. In classical elasticity of isotropic materials, the solution of the corresponding equation for Rayleigh surface waves speed (the secular equation) has been studied numerically, see e.g. [60, 30]. For the statement of the problem one may consult the book [1] and the works [55, 56, 59, 46, 41, 16, 69, 70, 71, 72, 73].

An important non-trivial task in classical linear elasticity is the extension of the results concerning the Rayleigh waves to anisotropic materials. Using a formalism (Stroh formalism) constructed in [63], Stroh [64] was able to avoid the complex secular equation derived by Synge [65] and he has given a real expression of it, see also Currie [14]. Another issue concerning the surface wave in an anisotropic elastic half-space is the derivation of the explicit secular equations such that a solution (analytical, if possible) can be easily found (i.e. at least with the help of a clear numerical strategy). In this respect, we mention [66, 50, 15, 16, 18, 68, 68]. For isotropic as well as for anisotropic materials, the mathematical analysis of this equation has, perhaps, at least the same importance as the derivation of the secular equations, since it is not obvious if there exists an admissible solution of the secular equation and if this solution is unique. By an admissible solution we mean a solution which takes into account all the restrictions which were imposed in the derivation of the secular equation. In many approaches this essential aspect is neglected or only conjectured. For linear isotropic materials this is explained in the book by Achenbach [1]. While strong ellipticity (Legendre–Hadamard ellipticity) guarantees the existence of plane waves, it is not obvious if seismic waves exist since three inhomogeneous body wave solutions are needed when the traction-free boundary condition specific to seismic waves is solved. For seismic waves propagating in anisotropic elastic materials, the first uniqueness result is due to Barnett et al. [6], see also [9] for more details, while for the proof of the existence of the solution of the secular equation, we mention the works by Barnett and Lothe [4], Lothe and Barnett [42] and Ingebrigtsen and Tonning [31].

Motivated by previous work of Mielke and Sprenger [48] which is more related to control theory [33] than to surface wave propagation, Fu and Mielke [24] and Mielke and Fu [47] have devised a new method for anisotropic elastic materials which is not based on the Stroh formalism, it is conceptually different from the other methods and it is mathematically well explained. They have shown that the impedance matrix [31] defining the secular equation is the solution of an algebraic Riccati equation. Using the properties of this equation, in [47, 24] it is then proven that the secular equation does not admit spurious roots. It can be said that the works by Mielke and Fu give an elegant final answer to the general case of anisotropic elastic materials, in the framework of classical linear elasticity. Since in [47, 24] the secular equation is not written explicitly, in the current paper we
have computed it for linear isotropic classical elasticity, too, and we compare it with the classical one and that resulting by using the Eringen formalism.

However, the classical theory of elasticity does not explain certain discrepancies that occur in propagation of waves at high frequency and short wavelength. It is now a common place that the response of the material to external stimuli depends heavily on the motions of its inner structure and if the ratio of the characteristic length associated with the external stimuli and the internal characteristic length is near to 1, then the response of constituent subcontinua becomes important, see [20]. Classical linear elasticity ignores these effects.

One of the first generalization of classical linear elasticity is represented by the Cosserat (micropolar) continuum theory in which the rotational degrees of freedom play a central role [22, 49]. For a quick comparison, we mention that denoting the macroscopic displacement by \(u: \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3\) and the microrotation vector field by \(\vartheta: \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3\), the elastic energy density of the isotropic linear Cosserat model read

\[
W(Du, \vartheta, D\vartheta) = \mu_e \|\text{dev}_3 \text{sym} \, Du\|^2 + \mu_e \|\text{skew} \,(Du - \text{Anti} \, \vartheta)\|^2 + \frac{2\mu_e + 3\lambda_e}{6} [\text{tr} \,(Du)]^2
\]

\[
+ \frac{\mu_e}{2} \left[ \alpha_1 \|\text{dev}_3 \text{sym} \, D\vartheta\|^2 + \alpha_2 \|\text{skew} \, D\vartheta\|^2 + \frac{2\alpha_1 + 3\alpha_3}{6} [\text{tr} \,(D\vartheta)]^2 \right], \tag{1.1}
\]

where \((\lambda_e, \mu_e), \mu_c, \lambda_c, \alpha_1, \alpha_2 \text{ and } \alpha_3\) are six isotropic elastic moduli representing the parameters related to the meso-scale, the Cosserat couple modulus, the characteristic length, and the three general isotropic curvature parameters, respectively. \((\text{Anti}(\vartheta))_{ij} = -\epsilon_{ijk} \vartheta_k\), with \(\epsilon_{ijk}\) the totally antisymmetric third order permutation tensor, see Section 2.1, while the elastic energy density of classical linear elasticity is

\[
W(Du) = \mu_e \|\text{dev}_3 \text{sym} \, Du\|^2 + \frac{2\mu_e + 3\lambda_e}{6} [\text{tr} \,(Du)]^2. \tag{1.2}
\]

In the framework of the Cosserat theory, the propagation of seismic waves in an isotropic Cosserat elastic half space was studied in [11] using the Stroh formalism [16, 64], see also [40]. The strong point of the approach in [11] is that explicit expressions of involved eigenvalue problems and explicit conditions upon the wave speed were found, as well as the exact expressions of three linear independent amplitude vectors. Then, a simple form of the secular equation is obtained, which, by comparison with other generalized forms of the secular equation for Cosserat materials [20, 23, 27, 34, 35, 36, 37, 38] does not involve the complex form of the attenuating coefficients. For a specified class of materials, i.e., for which the constitutive coefficients\(^1\) satisfy

\[
\mu_e - \mu_c + \lambda_e > 0, \quad \mu_e + \mu_c > 0, \quad \alpha_1 + \alpha_2 > 0, \tag{1.3}
\]

Chirita and Ghiba [11, Eq. (4.14)] have proved the existence of the solution of their secular equation. Using some illustrative graphics they conjectured that the solution should be also unique for this subclass of materials, but there does not exist a proof of the uniqueness in the Cosserat theory or an existence and uniqueness proof for a larger class of isotropic Cosserat materials.

It is also important to notice that in the Cosserat theory, too, it is not obvious how to avoid the spurious roots of the secular equation, as long as the Stroh formalism is used. Even if the relation between Mielke and Fu’s method [47, 24] and the Cosserat model is not evident, in our paper we show that it may be adapted to the study of propagation of seismic waves in Cosserat materials. So, our derivation will involve a matrix algebraic Riccati equation which will provide a formula for the desired solution. We will show that the new form of the secular equations, written in terms of the impedance matrix, does not admit any spurious root, i.e., there exists only one subsonic surface wave.

In the current paper we prove both the existence and the uniqueness of the wave speed of the Rayleigh wave in Cosserat materials, for the first time, and this under weaker conditions\(^2\) on the constitutive coefficients, i.e., we require only

\[
2\mu_e + \lambda_e > 0, \quad \mu_e > 0, \quad \mu_c > 0, \quad \alpha_1 + \alpha_2 > 0. \tag{1.4}
\]

For a given direction \(\xi \in \mathbb{R}^3, \xi \neq 0\) of the form \(\xi = (\xi_1, \xi_2, 0)^T\), the constitutive requirements (1.4) are equivalent to the existence of only real waves propagating in the plane normal to the direction of propagation and parallel

\(^1\)We use different notations in comparison to the Eringen notation in [11], i.e. \(\mu_c = \mu_e + \lambda_e\), \(\mu_c = \mu_e + \lambda_e\), see also [28]. In the Eringen notations these are \(\lambda_{\text{Eringen}} + \mu_{\text{Eringen}} > 0\), \(\kappa_{\text{Eringen}} > 0\), \(\alpha_1 + \alpha_2 > 0\).

\(^2\)In the sense that the conditions \(\mu_e - \mu_c + \lambda_e > 0\), \(\mu_e + \mu_c > 0\) imply \(2\mu_e + \lambda_e > 0\) but not vice versa, even when the extra conditions \(\mu_c > 0\) and \(\mu_e > 0\) are also assumed.
to the direction of propagation. These conditions do not involve the constitutive parameter \( \alpha_3 \) as it would be the case for the requirement that only real waves may propagate (in arbitrary planes) in any directions and for arbitrary wave numbers. Indeed, the propagation of real plane waves is equivalent to the constitutive inequalities

\[
2 \mu_e + \lambda_e > 0, \quad \mu_e > 0, \quad \mu_c > 0, \quad \alpha_1 + \alpha_2 > 0, \quad 2 \alpha_1 + \alpha_3 > 0.
\] (1.5)

Let us notice that while in classical linear elasticity the existence of seismic waves is guaranteed for strongly elliptic materials, i.e.,

\[
2 \mu_e + \lambda_e > 0, \quad \mu_e > 0,
\] (1.6)
in the Cosserat model the strong ellipticity conditions (Legendre–Hadamard ellipticity) [32, 61, 2] are equivalent to

\[
2 \mu_e + \lambda_e > 0, \quad \mu_e + \mu_c > 0, \quad \alpha_1 + \alpha_2 > 0, \quad 2 \alpha_1 + \alpha_3 > 0.
\] (1.7)

and the latter conditions are not sufficient to impose the existence of seismic waves. The existence of seismic waves is rather related to the propagation of real plane waves than to strong ellipticity, while the strong ellipticity conditions are useful in the study of accelerated waves [2]. We mention that the existence for only real plane waves implies the strong ellipticity conditions but not vice versa. For the Cosserat theory, strong ellipticity is related only to the propagation of acceleration waves [19, 2].

| Name | Expression | Dispersive waves/Non-dispersive waves |
|------|------------|--------------------------------------|
| the velocity of the acoustic branch of translational compression (longitudinal) plane wave | \( c_p = \sqrt{\frac{\lambda_e + 2\mu_e}{\rho}} \) | non-dispersive |
| the limit of the group/phase velocity of the acoustic branch of the shear–rotational wave at \( \omega \to 0 \) \( (k \to 0) \) | \( c_t = \sqrt{\frac{\mu_e}{\rho}} \) | dispersive |
| the limit of the group/phase velocity of the acoustic branch of the shear–rotational wave at \( \omega \to \infty \) \( (k \to \infty) \) | \( c_s = \sqrt{\frac{\mu_e + \mu_c}{\rho}} \) | dispersive |
| the group/phase velocity for the compressional rotational wave in the limit \( \omega \to \infty \) \( (k \to \infty) \) | \( c_{m,p} = \sqrt{\frac{L_2^2(2\alpha_1 + \alpha_3)}{\rho j \tau_2^2}} \) | dispersive |
| the limit of the group/phase velocity of the acoustic branch of the shear–rotational wave at \( \omega \to \infty \) \( (k \to \infty) \) | \( c_{m,s} = \sqrt{\frac{L_2^2(\alpha_1 + \alpha_2)}{\rho j \tau_2^2}} = \sqrt{\frac{L_2^2 \gamma}{\rho j \tau_2^2}} \) | dispersive |
| the limit of the optical branch (compressional-rotational and shear-rotational) at the cut-off frequency \( \omega = 2\sqrt{\frac{\mu_e}{\rho j \mu_c \tau_2^2}}, \ k = 0 \) | \( 0 \) (group velocity) /\( \infty \) (phase velocity) | dispersive |

Table 1: Some group velocities \( c = \frac{d\omega}{dk} \) or/and phase velocities \( \frac{d\omega}{dk} \) and the cut-off frequency in linear Cosserat elasticity.

Since different authors use different notation for elastic constants, we will also interpret almost all the conditions upon the constitutive parameters with the help of some relations between the velocities of compression/transversal acoustic/optical plane waves and with the help of some cut-off frequency of the optical
The velocity of the acoustic branch of translational compression (longitudinal) plane wave

\[ c_p = \sqrt{\frac{\lambda + 2\mu}{\rho}} \]

Non-dispersive waves

the limit of the group/phase velocity of the acoustic branch of the shear–rotational wave at \( \omega \to 0 \) (\( k \to 0 \))

\[ c_t = \sqrt{\frac{\mu}{\rho}} \]

Non-dispersive waves

the limit of the group/phase velocity of the acoustic branch of the shear–rotational wave at \( \omega \to \infty \) (\( k \to \infty \))

\[ c_t = \sqrt{\frac{\mu}{\rho}} \]

Non-dispersive waves

the group/phase velocity for the compressional rotational wave in the limit \( \omega \to \infty \) (\( k \to \infty \))

\( \times \)

not present

the limit of the group/phase velocity of the acoustic branch of the shear–rotational wave at \( \omega \to \infty \) (\( k \to \infty \))

\( \times \)

not present

cut-off frequency

\( \times \)

not present

| Name | Expression | Dispersive waves/Non-dispersive waves |
|------|------------|---------------------------------------|
| the velocity of the acoustic branch of translational compression (longitudinal) plane wave | \( c_p = \sqrt{\frac{\lambda + 2\mu}{\rho}} \) | non-dispersive |
| the limit of the group/phase velocity of the acoustic branch of the shear–rotational wave at \( \omega \to 0 \) (\( k \to 0 \)) | \( c_t = \sqrt{\frac{\mu}{\rho}} \) | non-dispersive |
| the limit of the group/phase velocity of the acoustic branch of the shear–rotational wave at \( \omega \to \infty \) (\( k \to \infty \)) | \( c_t = \sqrt{\frac{\mu}{\rho}} \) | non-dispersive |
| the group/phase velocity for the compressional rotational wave in the limit \( \omega \to \infty \) (\( k \to \infty \)) | \( \times \) | not present |
| the limit of the group/phase velocity of the acoustic branch of the shear–rotational wave at \( \omega \to \infty \) (\( k \to \infty \)) | \( \times \) | not present |

Table 2: The group velocities \( c = \frac{\omega}{k} \) or/and phase velocities \( \frac{d\omega}{dk} \) in classical linear elasticity.

With the help of the quantities presented in Table 1, the first two inequalities of the set of conditions (1.3) considered by Chirita and Ghiba [11] imply that the translational compressional wave is real and that the shear-rotational wave (optical branch) is real at high frequencies, the first inequality also implies that at the limit of high frequencies the translational compressional wave is faster than the shear–rotational wave (if they both exist), while the third one means that the shear–rotational wave (acoustic branch) is real at high frequencies. The inequalities (1.3) do not imply that the shear-rotational wave (optical branch) is real at low frequencies, i.e., \( \omega \to 0 \). In fact, the inequalities (1.3) do not imply that the plane waves are real, i.e., that the propagation plane waves are defined only by real frequencies.

To the contrary, the first implication of the set of conditions (1.5) means that all these waves (compressional/shear-rotational waves, acoustic/optical branch) are real. Then, we can treat and interpret further the propagation of plane waves. It is clear that under conditions (1.5) all branches of waves are real for the entire range \([0, \infty)\) of the frequency. However, we may also see directly from the first condition (1.5) that the translational compressional wave is real, the second condition (1.5) means that the acoustic branch of the shear–rotational wave is real at low frequencies and together with the third condition (1.5) means that the acoustic branch of the shear–rotational wave is real at high frequencies, the fourth inequality (1.5) implies that the optical branch of the shear–rotational wave is real at high frequencies. In addition, the third inequality (1.5) means that the optical branch of the shear–rotational wave at high frequencies has a larger velocity than the acoustic branch of the same wave at low frequencies (if they both exist, which is the case due to other conditions). We have just given the interpretation of the first four inequalities, but the fifth one expresses directly that the compressional rotational wave at high frequencies is real.

Considering the quantities from Tables 1 and 2, we may infer from (1.6) that in classical elasticity the strong ellipticity conditions correspond to the existence of real compressional and shear waves, while in the Cosserat case the corresponding conditions (1.7) (strong ellipticity conditions, Legendre-Hadamard ellipticity,
the positive definiteness of the acoustic tensor) implies the existence of the real translational compressional wave on the entire range of real frequencies, of the real shear–rotational waves (both branches) at high frequencies, and of real rotational compressional waves at high frequencies, but at lower frequencies the latter waves may not be real since (1.7) does not guarantee that $\frac{c_1}{c_2}$ is real. To the contrary, the conditions (1.5) imply that all these branches and types of plane wave are real, i.e., the group/phase velocities are real on the entire range of possible frequencies.

The structure of the present paper is now the following. In Section 2, after a short introduction of our notation, we present the linear Cosserat model for isotropic elastic materials as a special case of the relaxed micromorphic model. This comparison establishes the relations between these models and it is also useful for further studies, where the results obtained in the linear relaxed micromorphic model will be compared with those established in the Cosserat theory of linear elastic materials. Then, since a self contained study of the propagation of real waves in isotropic Cosserat elasticity is still missing, as well as the explicit form of the conditions on the constitutive coefficient which imply it, we dedicate Subsection 2.3 to it. In Subsection 2.4 we provide the explicit form of the secular equations for isotropic linear elastic materials, since its explicit form allows to give the main result of the paper, i.e., the first proof in the literature of the existence and uniqueness of the subsonic speed of the secular equation, in the framework of isotropic linear Cosserat elastic models. In Section 6 we provide two numerical algorithms which can be implemented for any material once the constitutive coefficients are known. We present effective numerical results for an aluminium-epoxy composite. In Section 7 we allow to give the main result of the paper, i.e., the first proof in the literature of the existence and uniqueness of possible frequencies.

2 Statement of the problem

2.1 Notation

We consider that the mechanical behaviour of a body occupying the unbounded regular region of three dimensional Euclidean space is modelled with the help of the Cosserat theory of linear isotropic elastic materials. We denote by $n$ the outward unit normal on $\partial \Omega$. The body is referred to a fixed system of rectangular Cartesian axes $Ox_i (i = 1, 2, 3)$; $\{e_1, e_2, e_3\}$ being the unit vectors of these axes.

In the following, we recall some useful notations for the present work. For $a, b \in \mathbb{R}^{3 \times 3}$ we let $\langle a, b \rangle_{\mathbb{R}^3}$ denote the scalar product on $\mathbb{R}^3$ with associated vector norm $\|a\|^2 = \langle a, a \rangle$. We denote by $\mathbb{R}^{3 \times 3}$ the set of real $3 \times 3$ second order tensors, written with capital letters. Matrices will be denoted by bold symbols, e.g., $X \in \mathbb{R}^{3 \times 3}$, while $X_{ij}$ will denote its component. The standard Euclidean product on $\mathbb{R}^{3 \times 3}$ is given by $\langle X, Y \rangle_{\mathbb{R}^{3 \times 3}} = \text{tr}(XY^T)$, and thus, the Frobenius tensor norm is $\|X\|^2 = \langle X, X \rangle_{\mathbb{R}^{3 \times 3}}$. In the following we omit the index $\mathbb{R}^3, \mathbb{R}^{3 \times 3}$. The identity tensor on $\mathbb{R}^{3 \times 3}$ will be denoted by $\mathbb{I}$, so that $\text{tr}(X) = \langle X, \mathbb{I} \rangle$. We let Sym denote the set of symmetric tensors. We adopt the usual abbreviations of Lie-algebra theory, i.e., $\mathfrak{so}(3) := \{ A \in \mathbb{R}^{3 \times 3} | A^T = -A \}$ is the Lie-algebra of skew-symmetric tensors and $\mathfrak{sl}(3) := \{ X \in \mathbb{R}^{3 \times 3} | \text{tr}(X) = 0 \}$ is the Lie-algebra of traceless tensors. For all $X \in \mathbb{R}^{3 \times 3}$ we set $\text{sym}X = \frac{1}{2}(X^T + X) \in \text{Sym}$, $\text{skew}X = \frac{1}{2}(X - X^T) \in \mathfrak{so}(3)$ and the deviatoric (trace-free) part $\text{dev}X = X - \frac{1}{3}\text{tr}(X)\mathbb{I} \in \mathfrak{sl}(3)$ and we have the orthogonal Cartan-decomposition of the Lie-algebra $\mathfrak{gl}(3) = \{ \mathfrak{sl}(3) \cap \text{Sym}(3) \} \oplus \mathfrak{so}(3) \oplus \mathbb{R} \cdot \mathbb{I}$, $X = \text{dev}X + \text{skew}X + \frac{1}{3}\text{tr}(X)\mathbb{I}$. We use the canonical identification of $\mathbb{R}^3$ with $\mathfrak{so}(3)$, and, for

$$A = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \in \mathfrak{so}(3)$$ (2.1)
we consider the operators $axl : so(3) \to \mathbb{R}^3$ and $\text{Anti} : \mathbb{R}^3 \to so(3)$ through
\[ (axl \mathbf{A})_k = -\frac{1}{2} \epsilon_{ijk} \mathbf{A}_{ij} = \frac{1}{2} \epsilon_{kij} A_{ij}, \quad (\text{Anti}(v))_{ij} = -\epsilon_{ijk} v_k, \quad \forall v \in \mathbb{R}^3, \]
where $\epsilon_{ijk}$ is the totally antisymmetric third order permutation tensor.

For a regular enough function $f(t, x_1, x_2, x_3)$, $f_t$ denotes the derivative with respect to the time $t$, while $\frac{\partial f}{\partial x_i}$ denotes the $i$-component of the gradient $\nabla f$. For vector fields $u = (u_1, u_2, u_3)^T$ with $u_i \in H^1(\Omega) = \{u_i \in L^2(\Omega) | \nabla u_i \in L^2(\Omega)\}$, $i = 1, 2, 3$, we define $Du := (\nabla u_1 | \nabla u_2 | \nabla u_3)^T$. The corresponding Sobolev-space will be also denoted by $H^1(\Omega)$. In addition, for a tensor field $\mathbf{P}$ with rows in $H(\text{div}; \Omega)$, i.e., $\mathbf{P} = (\mathbf{P}^{\mathbf{T}, e_1} | \mathbf{P}^{\mathbf{T}, e_2} | \mathbf{P}^{\mathbf{T}, e_3})^T$ with $(\mathbf{P}^{\mathbf{T}, e_1})^T \in H(\text{div}; \Omega) := \{v \in L^2(\Omega) | \text{div} v \in L^2(\Omega)\}$, $i = 1, 2, 3$, we define $\text{Div} \mathbf{P} := (\text{div} (\mathbf{P}^{\mathbf{T}, e_1})^T | \text{div} (\mathbf{P}^{\mathbf{T}, e_2})^T | \text{div} (\mathbf{P}^{\mathbf{T}, e_3})^T)^T$ while for tensor fields $\mathbf{P}$ with rows in $H(\text{curl}; \Omega)$, i.e., $\mathbf{P} = (\mathbf{P}^{\mathbf{T}, e_1} | \mathbf{P}^{\mathbf{T}, e_2} | \mathbf{P}^{\mathbf{T}, e_3})^T$ with $(\mathbf{P}^{\mathbf{T}, e_1})^T \in H(\text{curl}; \Omega) := \{v \in L^2(\Omega) | \text{curl} v \in L^2(\Omega)\}$, $i = 1, 2, 3$, we define $\text{Curl} \mathbf{P} := (\text{curl} (\mathbf{P}^{\mathbf{T}, e_1})^T | \text{curl} (\mathbf{P}^{\mathbf{T}, e_2})^T | \text{curl} (\mathbf{P}^{\mathbf{T}, e_3})^T)^T$.

### 2.2 Cosserat theory of isotropic elastic solids as particular case of the relaxed micromorphic model

In this subsection we show that the dynamic Cosserat model for isotropic materials [20, 49] is not only a special case of the most general micromorphic model, but also a special case of the relaxed micromorphic model [52, 44, 43, 45, 51]. In the micromorphic theory, the micro-distortion tensor $\mathbf{P} = (\mathbf{P}_{ij}) : \Omega \times [0, T] \to \mathbb{R}^{3 \times 3}$ describes the substructure of the material which can rotate, stretch, shear and shrink, while $u = (u_i) : \Omega \times [0, T] \to \mathbb{R}^3$ is the displacement of the macroscopic material points.

In the relaxed micromorphic model, in which the Cosserat modulus $\mu_c > 0$ is related to the isotropic Eringen-Claus model for dislocation dynamics [12, 21, 13], the free energy is given by
\[
W_{\text{relax}} = \mu_c \|\text{sym}(Dw - \mathbf{P})\|^2 + \mu_c \|\text{skew}(Dw - \mathbf{P})\|^2 + \frac{\lambda_c}{2} |\text{tr}(Dw - \mathbf{P})|^2 + \mu_{\text{micro}} \|\text{sym} \mathbf{P}\|^2 + \frac{\lambda_{\text{micro}}}{2} |\text{tr} \mathbf{P}|^2
+ \frac{\mu_c}{2} \left[ a_1 \|\text{dev} \text{Curl} \mathbf{P}\|^2 + a_2 \|\text{skew} \text{Curl} \mathbf{P}\|^2 + a_3 \frac{3}{3} |\text{tr}(\text{Curl} \mathbf{P})|^2 \right],
\]
where $(\mu_c, \lambda_c), (\mu_{\text{micro}}, \lambda_{\text{micro}}), \mu_c, L_c$ and $(a_1, a_2, a_3)$ are the elastic moduli representing the parameters related to the meso-scale, the parameters related to the micro-scale the Cosserat couple modulus, the characteristic length, and the three general isotropic curvature parameters (weights), respectively. Formally, letting $L_c \to \infty$ means a “zoom” into the micro-structure while $L_c \to 0$ means considering arbitrary large bodies while retaining the size of the unit-cell or keeping the dimensions of the body fixed while reducing the dimensions of the unit cell to zero, or, in other words, “no special effects of the microstructure taking into account” (classical elasticity).

In the internal energy is positive definite in terms of the independent constitutive variables $Dw - \mathbf{P}$, $\text{sym} \mathbf{P}$, $\text{Curl} \mathbf{P}$ if and only if
\[
\mu_c > 0, \quad \kappa_c := \frac{2 \mu_c + 3 \lambda_c}{3} > 0, \quad \mu_c > 0, \quad \mu_{\text{micro}} > 0, \quad \kappa_{\text{micro}} := \frac{2 \mu_{\text{micro}} + 3 \lambda_{\text{micro}}}{3} > 0, \quad \mu_c > 0, \quad (a_1, a_2, a_3) > 0.
\]

The complete system of linear partial differential equations in terms of the kinematical unknowns $u$ and $P$ is given by
\[
\rho u_{tt} = \text{Div} \left[ 2 \mu_c \text{sym}(Dw - \mathbf{P}) + 2 \mu_c \text{skew}(Dw - \mathbf{P}) + \lambda_c \text{tr}(Dw - \mathbf{P}) \cdot \mathbf{I} \right] + f,
\]
the non-symmetric force-stress tensor
\[
\rho \eta \tau^2 P_{tt} = -\mu_c L_c^2 \text{Curl} \left[ a_1 \text{dev} \text{Curl} \mathbf{P} + a_2 \text{skew} \text{Curl} \mathbf{P} + a_3 \frac{3}{3} \text{tr}(\text{Curl} \mathbf{P}) \cdot \mathbf{I} \right] + \text{the second-order moment stress tensor}
+ 2 \mu_c \text{sym}(Dw - \mathbf{P}) + \lambda_c \text{tr}(Dw - \mathbf{P}) \cdot \mathbf{I} - 2 \mu_{\text{micro}} \text{sym} \mathbf{P} - \lambda_{\text{micro}} \text{tr} \mathbf{P} \cdot \mathbf{I} + \mathbf{M} \quad \text{in} \ \Omega \times [0, T],
\]
where \( f : \Omega \times [0, T] \to \mathbb{R}^3 \) describes the external body force, \( M : \Omega \times [0, T] \to \mathbb{R}^{3 \times 3} \) describes the external body moment, \( \rho \) is the mass density and \( \eta \tau_c^2 \) is the inertia coefficient, with \( \eta > 0 \) a weight parameter and \( \tau_c \) the internal characteristic time [20, page 163].

In the Cosserat theory we assume that the micro-distortion tensor is skew-symmetric, i.e. \( P = A \in \mathfrak{so}(3) \). Using the Curl-Daxl identities, (see [54], Nye’s formula [58])

\[
-\text{Curl} \ A = (D \text{axl} \ A)^T - \text{tr}((D \text{axl} \ A)^T) \cdot 1, \quad D \text{axl} \ A = -((Curl \ A)^T + \frac{1}{2} \text{tr}((Curl \ A)^T) \cdot 1, \quad (2.6)
\]

for all matrix fields \( A \in \mathfrak{so}(3) \), it is easy to obtain that the total energies admits the form (identifying \( \vartheta = \text{axl} \ A \))

\[
\mathcal{L}(u, \vartheta, Du - A, \text{axl} \ A) = \int_{\Omega} \left( \frac{1}{2} \rho \|u_t\|^2 + \rho \eta \tau_c^2 \|\text{axl} \ A\|_t^2 + \mu_c \|\text{sym} Du\|^2 + \mu_c \|\text{skew}(Du - A)\|^2 + \frac{\lambda_c}{2} [\text{tr}(Du)]^2 \\
+ \frac{\mu_c L_c^2}{2} \left[ a_1 \|\text{dev sym}(D \text{axl} \ A)\|^2 + a_2 \|\text{skew}(D \text{axl} \ A)\|^2 + \frac{4a_3}{3} [\text{tr}(D \text{axl} \ A)]^2 \right] \right) dv
\]

and lead to Euler-Lagrange equations which are equivalent to those derived from the Curl-formulation.

The power functional is given by

\[
\Pi(t) = \int_{\Omega} (\langle f, u_t \rangle + (M, A_t)) dv = \int_{\Omega} ((f, u_t) + 2\langle \text{axl} \text{skew} M, \text{axl} \vartheta \rangle) dv . \quad (2.8)
\]

We introduce the action functional of the considered system to be defined as

\[
\mathcal{A} = \int_0^T \int_{\Omega} \left( \frac{1}{2} \rho \|u_t\|^2 + \rho \eta \tau_c^2 \|\text{axl} \ A\|_t^2 - \left[ \mu_c \|\text{sym} Du\|^2 + \mu_c \|\text{skew}(Du - A)\|^2 + \frac{\lambda_c}{2} [\text{tr}(Du)]^2 \right] \\
- \frac{\mu_c L_c^2}{2} \left[ a_1 \|\text{dev sym}(D \text{axl} \ A)\|^2 + a_2 \|\text{skew}(D \text{axl} \ A)\|^2 + \frac{4a_3}{3} [\text{tr}(D \text{axl} \ A)]^2 \right] \right) dv dt \\
+ \int_0^T \int_{\Omega} (\langle f, u \rangle + 2\langle \text{axl} \text{skew} M, \text{axl} \vartheta \rangle) dv dt . \quad (2.9)
\]

The condition of vanishing first variation of the action functional can thus be written as

\[
- \int_0^T \int_{\Omega} (\langle f, \delta u \rangle + 2\langle \text{axl} \text{skew} M, \text{axl} (\delta \vartheta) \rangle) dv dt \\
= \int_0^T \int_{\Omega} \left( \rho \langle u_{tt}, (\delta u)_t \rangle + 2 \rho \eta \tau_c^2 \langle (\text{axl} \ A)_t, (\text{axl} (\delta \vartheta))_t \rangle \\
- 2 \mu_c \langle \text{sym} Du, \text{sym} \delta u \rangle - \mu_c \langle \text{skew}(Du - A), \text{skew}(D \delta u - \delta \vartheta) \rangle - \lambda_c \langle \text{tr}(Du), \text{tr}(D \delta u) \rangle \\
- \frac{\mu_c L_c^2}{2} a_1 \langle \text{dev sym}(D \text{axl} \ A), \text{dev sym}(D \text{axl} (\delta \vartheta)) \rangle \\
- \frac{\mu_c L_c^2}{2} \frac{4a_3}{3} [\text{tr}(D \text{axl} \ A)] [\text{tr}(D \text{axl} (\delta \vartheta))] \right) dv dt \\
= \int_0^T \int_{\Omega} \left( \rho \langle u_{tt}, (\delta u)_t \rangle + 2 \rho \eta \tau_c^2 \langle (\text{axl} \ A)_t, (\text{axl} (\delta \vartheta))_t \rangle \\
- \langle 2 \mu_c \text{sym} Du + \mu_c \text{skew}(Du - A) + \lambda_c [\text{tr}(Du)] \cdot 1, D \delta u \rangle - 2 \mu_c \langle \text{skew}(Du - A), \delta \vartheta \rangle \\
- \frac{\mu_c L_c^2}{2} (a_1 \text{dev sym}(D \text{axl} \ A) + a_2 \text{skew}(D \text{axl} \ A) + \frac{4a_3}{3} [\text{tr}(D \text{axl} \ A)] \cdot 1, D \text{axl} (\delta \vartheta)) \right) dv dt \\
= \int_0^T \int_{\Omega} \left( - \rho \langle u_{tt}, \delta u \rangle - 2 \rho \eta \tau_c^2 \langle (\text{axl} \ A)_t, (\text{axl} (\delta \vartheta)) \rangle \\
+ \langle \text{Div}[2 \mu_c \text{sym} Du + \mu_c \text{skew}(Du - A) + \lambda_c [\text{tr}(Du)] \cdot 1], \delta u \rangle - 4 \mu_c \langle \text{axl} \text{skew}(Du - A), \text{axl} (\delta \vartheta) \rangle \\
+ \langle \text{Div}[2 \mu_c \text{sym} Du + \mu_c \text{skew}(Du - A) + \lambda_c [\text{tr}(Du)] \cdot 1], \delta u \rangle - 4 \mu_c \langle \text{axl} \text{skew}(Du - A), \text{axl} (\delta \vartheta) \rangle \right) dv dt .
\]
In the absence of external body forces and of external body moment, the PDE-system of the model is

\[ \text{Cosserat theory, and within the framework of the linear isotropic hyperelastic theory, the elastic energy density of the} \]

\[ \text{definitions for the independent constitutive variables} \]

\[ \text{where we have used the notations for the weight parameters} \]

\[ \text{for all virtual displacements} \]

\[ \text{results.} \]

\[ \text{However, our entire subsequent analysis will be made under weaker conditions on the constitutive parameters.} \]

\[ \text{Therefore, in the Cosserat theory of linear elastic materials, two vector fields are used to describe the macro-} \]

\[ \text{due to the orthogonal Cartan-decomposition of the Lie-algebra} \]

\[ \text{Summarizing, in view of} \]

\[ \text{Due to the orthogonal Cartan-decomposition of the Lie-algebra} \]

\[ \text{Therefore, in the Cosserat theory of linear elastic materials, two vector fields are used to describe the macro-} \]

\[ \text{where} \]

\[ \text{where} \]

\[ \text{where} \]

\[ \text{Hence, the stress-strain relations for the homogeneous isotropic Cosserat elastic solid are} \]

\[ \text{where} \]

\[ \text{with} \]

\[ \text{Due to the orthogonal Cartan-decomposition of the Lie-algebra} \]

\[ \text{However, our entire subsequent analysis will be made under weaker conditions on the constitutive parameters.} \]

\[ \text{In the following we assume} \]

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2.3 Real plane waves in isotropic Cosserat elastic solids

We say that there exists real plane waves in the direction $\xi = (\xi_1, \xi_2, \xi_3)$, $\|\xi\|^2 = 1$, if for every wave number $k > 0$ the system of partial differential equations (2.15) admits a solution in the form:

$$u(x_1, x_2, x_3, t) = \begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \end{pmatrix} e^{i(k \xi \cdot x_3 - \omega t)},$$  \hspace{1cm} (2.17)

$$\vartheta(x_1, x_2, x_3, t) = \begin{pmatrix} \hat{\vartheta}_1 \\ \hat{\vartheta}_2 \\ \hat{\vartheta}_3 \end{pmatrix} e^{i(k \xi \cdot x_3 - \omega t)},$$

$$\hat{u}, \hat{\vartheta} \in \mathbb{C}^3, \quad (\hat{u}, \hat{\vartheta})^T \neq 0,$$

only for real frequencies $\omega \in \mathbb{R}$, where $i = \sqrt{-1}$ is the complex unit. The plane wave is called "real" since it is defined by real values of $\omega$. Note that we take $i \vartheta$ since this choice will lead us in the end only to real valued matrices. Otherwise, we would have to deal with complex valued matrices in the linear Cosserat theory.

The functions (2.17) are a solution of (2.15) if and only if the following system is satisfied:

$$-\omega^2 \rho \hat{u}_i = -k^2 (\mu_c + \mu_e) \hat{u}_i - k^2 (\mu_c - \mu_e + \lambda_e) \sum_{l=1}^{3} \hat{u}_l \xi_i \xi_l - 2 k \mu_c \sum_{l,s=1}^{3} \varepsilon_{i ls} \hat{\vartheta}_s \xi_l,$$

$$-i \omega^2 \rho j \mu_c \tau_c^2 \hat{\vartheta}_i = \mu_c L_c^2 \xi_i \hat{\vartheta}_i - i k^2 (\alpha_1 + \alpha_2) \hat{\vartheta}_i - i k^2 (\alpha_1 - \alpha_2 + \alpha_3) \sum_{l=1}^{3} \hat{\vartheta}_l \xi_i \xi_l$$  \hspace{1cm} (2.18)

$$- 2 i k \mu_c \sum_{l,s=1}^{3} \varepsilon_{i ls} \hat{\vartheta}_s \xi_l - 4 i \mu_c \hat{\vartheta}_i, \quad i = 1, 2, 3.$$

However, since our formulation is isotropic, by demanding real plane waves in any direction $\xi = (\xi_1, \xi_2, \xi_3)$, $\|\xi\| = 1$, it is equivalent to demand real plane waves in the direction $e_1 = (1, 0, 0)$ which means that for all $k > 0$ the system

$$u(x_1, x_2, t) = \begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \end{pmatrix} e^{i(k x_1 - \omega t)}, \quad \vartheta(x_1, x_2, t) = \begin{pmatrix} \hat{\vartheta}_1 \\ \hat{\vartheta}_2 \\ \hat{\vartheta}_3 \end{pmatrix} e^{i(k x_1 - \omega t)},$$

$$\hat{u}, \hat{\vartheta} \in \mathbb{C}^3, \quad (\hat{u}, \hat{\vartheta})^T \neq 0$$  \hspace{1cm} (2.19)

admits non-trivial solutions only for real positive values $\omega^2$.

Inserting (2.19) into (2.15) we see that $\hat{u}_1, \hat{u}_2$ and $\hat{\vartheta}_3$ have to satisfy the following linear algebraic equations

$$-\omega^2 \rho \hat{u}_1 = -k^2 (\mu_c + \mu_e) \hat{u}_1 - k^2 (\mu_c - \mu_e + \lambda_e) \hat{u}_1,$$

$$-\omega^2 \rho \hat{u}_2 = -k^2 (\mu_c + \mu_e) \hat{u}_2 + 2 k \mu_c \hat{\vartheta}_3,$$

$$-i \omega^2 \rho j \mu_c \tau_c^2 \hat{\vartheta}_3 = -i k^2 \mu_c L_c^2 (\alpha_1 + \alpha_2) \hat{\vartheta}_3 - 2 i k \mu_c \hat{\vartheta}_2 - 4 i \mu_c \hat{\vartheta}_3,$$  \hspace{1cm} (2.20)

while $\hat{u}_3, \hat{\vartheta}_1$ and $\hat{\vartheta}_2$ have to satisfy the system of linear equations

$$-\omega^2 \rho \hat{u}_3 = -k^2 (\mu_c + \mu_e) \hat{u}_3 - 2 k \mu_c \hat{\vartheta}_2,$$

$$-i \omega^2 \rho j \mu_c \tau_c^2 \hat{\vartheta}_1 = -i k^2 \mu_c L_c^2 (\alpha_1 + \alpha_2) \hat{\vartheta}_1 - i k^2 \mu_c L_c^2 (\alpha_1 - \alpha_2 + \alpha_3) \hat{\vartheta}_1 - 4 i \mu_c \hat{\vartheta}_1,$$

$$-i \omega^2 \rho j \mu_c \tau_c^2 \hat{\vartheta}_2 = -i k^2 \mu_c L_c^2 (\alpha_1 + \alpha_2) \hat{\vartheta}_2 + 2 i k \mu_c \hat{\vartheta}_3 - 4 i \mu_c \hat{\vartheta}_2.$$  \hspace{1cm} (2.21)

Hence, there exist real plane wave if for every wave number $k > 0$ the following systems of equations (2.15) admit non-trivial solutions:

$$[Q_1(e_1, k) - \omega^2 I] w = 0 \quad w = (\hat{u}_1, \hat{u}_2, \hat{u}_3)^T,$$  \hspace{1cm} (2.22)
It is now easy to remark that the positive definiteness of $\mathbf{Q}_2(e_1, k) - \omega^2 \mathbf{I}$ implies that all the diagonal elements are positive, i.e., we also have that

$$\omega^2 \mathbf{I} - \mathbf{Q}_2(e_1, k) \mathbf{w} = 0, \quad \mathbf{w} = (\hat{u}_3, \hat{v}_1, \hat{v}_2)^T$$

only for real frequencies $\omega \in \mathbb{R}$, where

$$\mathbf{Q}_1(e_1, k) = \begin{pmatrix} k^2(2 \mu_e + \lambda_e) & 0 & 0 \\ 0 & k^2(\mu_e + \mu_c) & -2k \mu_c \\ 0 & -2k \mu_c & k^2 \mu_c L^2_c (\alpha_1 + \alpha_2) + 4 \mu_c \end{pmatrix}, \quad (2.23)$$

$$\mathbf{Q}_2(e_1, k) = \begin{pmatrix} k^2(\mu_e + \mu_c) & 0 & 2k \mu_c \\ 0 & k^2 \mu_e L^2_c (2 \alpha_1 + \alpha_3) + 4 \mu_c & 0 \\ 2k \mu_c & 0 & k^2 \mu_e L^2_c (\alpha_1 + \alpha_2) + 4 \mu_c \end{pmatrix}, \quad (2.24)$$

$$\hat{\mathbf{I}} = \begin{pmatrix} \rho & 0 & 0 \\ 0 & \rho & 0 \\ 0 & \rho j \mu_e \tau_c^2 \end{pmatrix}, \quad \tilde{\mathbf{I}} = \begin{pmatrix} \rho & 0 & 0 \\ 0 & \rho j \mu_e \tau_c^2 & 0 \\ 0 & 0 & \rho j \mu_e \tau_c^2 \end{pmatrix}. \quad (2.25)$$

In this form, since $\hat{\mathbf{I}} \neq \mathbf{I}$ and $\tilde{\mathbf{I}} \neq \mathbf{I}$, these are not eigenvalue problems. However, the system (2.22) is equivalent to

$$\left[ \hat{\mathbf{I}}^{-1/2} \mathbf{Q}_1(e_1, k) \hat{\mathbf{I}}^{-1/2} - \omega^2 \mathbf{I} \right] \mathbf{d} = 0, \quad d = \hat{\mathbf{I}}^{1/2} \begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \end{pmatrix}. \quad (2.26)$$

Hence, the system (2.22) is equivalent to the eigenvalue problem

$$\begin{pmatrix} k^2 \rho \mu_e + \lambda_e & 0 & 0 \\ 0 & k^2 \mu_e + \mu_c & -2k \frac{\mu_c}{\rho \sqrt{\mu_e \tau_c^2}} \\ 0 & -2k \frac{\mu_c}{\rho \sqrt{\mu_e \tau_c^2}} & k^2 \mu_e L^2_c (\alpha_1 + \alpha_2) + 4 \frac{\mu_c}{\rho j \mu_e \tau_c^2} \end{pmatrix} - \omega^2 \mathbf{I} \mathbf{0} = 0. \quad (2.27)$$

Thus, asking that for all $k > 0$ the system (2.22) admits non trivial solutions $\mathbf{w} = (\hat{u}_1, \hat{u}_2, \hat{u}_3)^T \neq 0$ only for real positive values $\omega^2$ is equivalent to the positive definiteness of $\mathbf{Q}_1(e_1, k)$ for all $k > 0$. Using the Sylvester criterion this means the following set of conditions

$$k^2 (2 \mu_e + \lambda_e) > 0, \quad k^2 (\mu_e + \mu_c) > 0, \quad (2.28)$$

$$\left( \det \mathbf{Q}_1(e_1, k) > 0 \quad \iff \quad \mu_e L^2_c (\alpha_1 + \alpha_2) k^4 (\mu_e + \mu_c) + k^2 4 \mu_c \mu_e > 0 \right) \quad \forall \ k > 0,$$

but it also implies that all the diagonal elements are positive, i.e., we also have that

$$k^2 \mu_e L^2_c (\alpha_1 + \alpha_2) + 4 \mu_c > 0 \quad \forall \ k > 0. \quad (2.29)$$

It is now easy to remark that the positive definiteness of $\mathbf{Q}_1(e_1, k)$ for all $k > 0$ is equivalent to the non-redundant set of inequalities

$$2 \mu_e + \lambda_e > 0, \quad \mu_e > 0, \quad \mu_c > 0, \quad \alpha_1 + \alpha_2 > 0. \quad (2.30)$$

In a similar way, we find that for all $k > 0$ the system (2.22) admits non trivial solutions $\mathbf{w} = (\hat{u}_3, \hat{v}_1, \hat{v}_2)^T \neq 0$ only for real positive values $\omega^2$ if and only if the following eigenvalue problem admits only real solutions

$$\begin{pmatrix} k^2 \frac{\mu_e + \mu_c}{\rho} & 0 & -2k \frac{\mu_c}{\rho \sqrt{\mu_e \tau_c^2}} \\ 0 & k^2 \mu_e L^2_c \frac{2 \alpha_1 + \alpha_2}{\rho j \mu_e \tau_c^2} + 4 \frac{\mu_c}{\rho j \mu_e \tau_c^2} & 0 \\ -2k \frac{\mu_c}{\rho \sqrt{\mu_e \tau_c^2}} & 0 & k^2 \mu_e L^2_c \frac{\alpha_1 + \alpha_2}{\rho j \mu_e \tau_c^2} + 4 \frac{\mu_c}{\rho j \mu_e \tau_c^2} \end{pmatrix} - \omega^2 \mathbf{I} \mathbf{0} = 0. \quad (2.31)$$
with \( f = \bar{1}^{1/2} \begin{pmatrix} \bar{u}_1 \\ \bar{u}_2 \end{pmatrix} \), i.e., if and only if for all \( k > 0 \)

\[
\begin{align*}
& k^2 (\mu_e + \mu_c) > 0, \\
& k^2 \left( \det \tilde{Q}_2(e_1, k) > 0 \right) \iff k^6 \left( \mu_e + \mu_c \right) \\
& \left. \left( (\alpha_1 + \alpha_2) \left( 2 \alpha_1 + \alpha_3 \right) \right) \right) L^4_e \mu_e \left( \mu_e + \mu_c \right) \\
& + 4 k^4 L^2_e \mu_e \left[ \left( \alpha_1 + \alpha_2 \right) \mu_e \left( \mu_e + \mu_c \right) + \left( 2 \alpha_1 + \alpha_3 \right) \mu_c \right] + 16 k^2 \mu_e^2 > 0 \right) ,
\end{align*}
\]

(2.32)

but it also implies that all the diagonal elements are positive, i.e., we also have that

\[
k^2 \mu_e L^2_e (\alpha_1 + \alpha_2) + 4 \mu_c > 0.
\]

(2.33)

All together this shows that the positive definiteness of \( \tilde{Q}_2(e_1, k) \) is equivalent to

\[
\mu_e > 0, \quad \mu_c > 0, \quad 2 \alpha_1 + \alpha_3 > 0, \quad \mu_e (\alpha_1 + \alpha_2) > 0, \quad \mu_c > 0.
\]

(2.34)

Therefore, the non-redundant inequalities from (2.34) are

\[
\mu_e > 0, \quad \mu_c > 0, \quad 2 \alpha_1 + \alpha_3 > 0, \quad (\alpha_1 + \alpha_2) > 0.
\]

(2.35)

The analysis presented in this subsection is similar to that used in [53]. Then, for \( k > 0 \) and due to the isotropy, extrapolating to all directions of propagation, we have

**Proposition 2.1.** The necessary and sufficient conditions for existence of real planar waves in any direction \( \xi \in \mathbb{R}^3 \), \( \xi \neq 0 \), in the framework of the linear isotropic elastic Cosserat theory are

\[
2 \mu_e + \lambda_e > 0, \quad \mu_e > 0, \quad \mu_c > 0, \quad \alpha_1 + \alpha_2 > 0, \quad 2 \alpha_1 + \alpha_3 > 0.
\]

(2.36)

**Proof.** The proof is given by the previous calculations. ■

**Remark 2.2.** The conditions (2.36) from Proposition 2.1 have also some direct interpretations, i.e.,

i) the first implication of the set of conditions (2.36) means that all these waves (compressional/shear-rotational waves, acoustic/optical branch) are real; Once this aspect is clarified, we can treat and interpret further the propagation of plane waves;

ii) under conditions (2.36) all branches of waves are real for the entire range \([0, \infty)\) of the frequency;

iii) we can also see directly from the first condition that the translational compressional wave is real;

iv) the second means that the acoustic branch of shear–rotational wave is real at low frequencies and together with the third means that the acoustic branch of shear–rotational wave is real at high frequencies;

v) the fourth implies that the optical branch of the shear–rotational wave is real at high frequencies;

vi) the third one means that the optical branch of the shear–rotational wave at high frequencies has a larger velocity than the acoustic branch of the same wave at low frequencies (if they both exist, which is the case due to other conditions);

vii) the fifth one expresses directly that the compressional rotational wave at high frequencies is real.

**Remark 2.3.**

i) In linear isotropic classical elasticity, the necessary and sufficient conditions for existence of real planar waves in any direction \( \xi \in \mathbb{R}^3 \), \( \xi \neq 0 \) are

\[
2 \mu_e + \lambda_e > 0, \quad \mu_e > 0,
\]

(2.37)

and they are equivalent to the strong ellipticity conditions (Legendre-Hadamard ellipticity).
ii) The necessary and sufficient conditions for existence of a real planar wave are slightly different compared to the strong ellipticity conditions (Legendre–Hadamard ellipticity) (2.36) for the Cosserat (micropolar) model investigated in [2, 19, 61] and which are connected to acceleration waves. In our notation, the strong ellipticity condition for Cosserat media is represented by the inequality [2, 19, 61]

$$\frac{d}{d\tau}W(e + \tau \xi \otimes \eta, \mathbf{R} + \tau \zeta \otimes \eta)\bigg|_{\tau=0} > 0 \quad \forall \eta, \xi, \zeta \in \mathbb{R}^3, \quad ||\eta|| = ||\xi|| = ||\zeta|| = 1. \quad (2.38)$$

and it is satisfied if and only if

$$2 \mu_c + \lambda_c > 0, \quad \mu_c + \mu_c > 0, \quad \alpha_1 + \alpha_2 > 0, \quad 2 \alpha_1 + \alpha_3 > 0. \quad (2.39)$$

The explicit calculations in our notations are made in [61]. The absence of a coupling between $e$ and $\mathbf{R}$ in the strain energy leads to a simplification of the calculations.

iii) The conditions (2.39) (strong ellipticity conditions, Legendre-Hadamard ellipticity, the positive definiteness of the acoustic tensor) imply the existence of the real translational compressional waves in the entire range of real frequencies, of the real shear rotational waves (both branches) at high frequencies, and of real rotational compressional wave at high frequencies, but at lower frequencies the latter waves may not be real since (2.39) does not guaranty that $c_t = \frac{\omega}{\omega}$ is real. To the contrary, the conditions (2.36) imply that all these branches and types of plane wave are real, i.e., the group/phase velocities are real on the entire range of possible frequencies.

The strong ellipticity conditions (2.39) are weaker than the conditions (2.36) in the sense that they are implied by the necessary and sufficient conditions for existence of a real planar wave (i.e., they imply the strong ellipticity and, therefore, the considered PDEs system is not unstable) but not vice versa. However, the strong ellipticity conditions (2.39) are not sufficient for the application of our solution method, and we believe that they are also not suitable for any approach regarding the propagation of Rayleigh waves in Cosserat solids.

In the end of this section we mention that Eringen [20, pages 149-151] affirmed that one has to impose (in addition) that

$$\frac{L_c^2}{\tau_c^2} (\alpha_1 + \alpha_2) > \frac{\mu_c + \mu_c}{\rho j}, \quad (2.40)$$

in order that there exist four real $\omega$ which lead to plane waves solutions (real plane waves). This seems to be only a consequence of the representation formula and the method he used to construct the plane wave solution and seems to be not a necessary condition for real plane waves propagation. Condition (2.40) required by Eringen means that the shear rotational wave at high frequencies is faster for the acoustic branch than for the optical branch. Eringen [20, pages 151] also claimed that this condition is in accordance to the lattice dynamical calculations but no further explanations are given.

Nevertheless, this Eringen-type condition (2.40) seems to have sense when we are going back to the classical linear elasticity model by considering $\mu_c \to 0$ and large values of $\frac{L_c}{\tau_c}$ ($L_c \to \infty$ or $\tau_c \to 0$), since for $\xi = e_1$ the eigenvalue problems characterising the possible real values of $\omega$ become

$$\begin{bmatrix} k^2 \frac{\mu_c + \lambda_c}{\rho j} & 0 & 0 \\ 0 & k^2 \frac{\mu_c}{\rho j} & 0 \\ 0 & 0 & k^2 L_c^2 \frac{\alpha_1 + \alpha_2}{\rho j} \end{bmatrix} - \omega^2 \mathbb{1} \end{bmatrix} d = 0, \quad d = \vec{u}^{1/2} \begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \frac{\partial \hat{u}}{\partial j} \end{pmatrix} \quad (2.41)$$

and

$$\begin{bmatrix} k^2 \frac{\mu_c}{\rho j} & 0 & 0 \\ 0 & k^2 \frac{\mu_c}{\rho j} \frac{2 \alpha_1 + \alpha_2}{\rho j} & 0 \\ 0 & 0 & k^2 L_c^2 \frac{\alpha_1 + \alpha_2}{\rho j} \end{bmatrix} - \omega^2 \mathbb{1} \end{bmatrix} f = 0, \quad f = \vec{v}^{-1/2} \begin{pmatrix} \hat{v}_3 \\ \frac{\partial \hat{v}}{\partial j} \end{pmatrix}. \quad (2.42)$$
In classical linear elasticity, it is natural to impose that plane waves propagate only with speeds \( \sqrt{\frac{2\mu_e + \lambda_e}{\rho}} \) and \( \sqrt{\frac{\mu_e}{\rho}} \) and no other “exotic” plane wave arises. Indeed, for \( \mu_e \to 0 \) and large values of \( \frac{L^2}{c_e^2} \), by imposing

\[
\frac{L^2}{c_e^2} \min \left\{ \frac{2\alpha_1 + \alpha_3}{\rho j}, \frac{\alpha_1 + \alpha_2}{\rho j} \right\} > \max \left\{ \frac{2\mu_e + \lambda_e}{\rho}, \frac{\mu_e}{\rho} \right\},
\]

the unique propagating speeds which lead to non-vanishing displacements for plane waves are only those from classical elasticity. Condition (2.43) means that both shear–rotational acoustic branch and rotational compressional wave at high frequencies are faster than the shear–rotational acoustic branch at low frequencies and the translational compressional wave. More precisely, under this additional restriction upon the curvature coefficients, in the limit case \( \mu_e \to 0 \), we determine the amplitudes

\[
\begin{align*}
\text{for } \omega^2 &= k^2 \frac{2\mu_e + \lambda_e}{\rho} : & \hat{u} \times e_1 &= 0, & \hat{\theta} &= 0, \\
\text{for } \omega^2 &= k^2 \frac{\mu_e}{\rho} : & \langle \hat{u}, e_1 \rangle &= 0, & \hat{\theta} &= 0, \\
\text{for } \omega^2 &= k^2 \frac{L^2}{c_e^2} \frac{2\alpha_1 + \alpha_3}{\rho j} : & \hat{u} &= 0, & \hat{\theta} \times e_1 &= 0, \\
\text{for } \omega^2 &= k^2 \frac{L^2}{c_e^2} \frac{\alpha_1 + \alpha_2}{\rho j} : & \hat{u} &= 0, & \langle \hat{\theta}, e_1 \rangle &= 0.
\end{align*}
\]

This means that real plane waves for which \( \hat{u} \neq 0 \) (i.e., only what real plane waves in classical elasticity means) are possible only for \( \omega^2 = k^2 \sqrt{\frac{2\mu_e + \lambda_e}{\rho}} \) and \( \omega^2 = k^2 \sqrt{\frac{\mu_e}{\rho}} \), situation in which the microrotation vector \( \hat{\theta} \) vanishes. Moreover, the extremely high values of the frequency which lead to non-vanishing \( \hat{\theta} \) are beyond the framework of classical elasticity and belong rather to quantum mechanics or relativistic mechanics.

It is important to note that when studying plane waves, we have to limit the plausible domains of the frequency (i.e., the speed is less than the speed of light) in the framework of classical mechanics, for seismic waves this is already done once the ansatz is chosen, since only subsonic speeds are admissible. We will explain this aspect in more details in the following sections.

### 2.4 The setup for the propagation of Rayleigh waves

In the framework of the Rayleigh wave, we consider the region \( \Omega \) to be the half space

\[
\Sigma := \{(x_1, x_2, x_3) \mid x_1, x_3 \in \mathbb{R}, x_2 \geq 0\}.
\]

The boundary of the homogeneous and isotropic half-space is free of surface traction, i.e.,

\[
\sigma \cdot n = 0, \quad m \cdot n = 0 \quad \text{for} \quad x_2 = 0.
\]

In addition, the solution has to satisfy the following decay condition

\[
\lim_{x_2 \to \infty} \{u_1, u_2, \theta_3, \sigma_{12}, \sigma_{21}, \sigma_{22}, m_{23}\}(x_1, x_2, t) = 0 \quad \forall x_1 \in \mathbb{R}, \quad \forall t \in [0, \infty).
\]

In isotropic solids and in the context of Rayleigh wave propagation, the surface particles move in the planes normal to the surface \( x_2 = 0 \) and parallel to the direction of propagation \( e_1 = (1, 0, 0)^T \), see Figure 1. Accordingly to these characteristics of the seismic waves, we consider the following plain strain ansatz as a first step in our process of construction of the solution

\[
\begin{align*}
\mathbf{u}(x_1, x_2, t) &= \begin{pmatrix} u_1(x_1, x_2, t) \\ u_2(x_1, x_2, t) \\ 0 \end{pmatrix}, & \mathbf{D} \mathbf{u}(x_1, x_2, t) &= \begin{pmatrix} \frac{\partial u_1}{\partial x_1}(x_1, x_2, t) & \frac{\partial u_1}{\partial x_2}(x_1, x_2, t) \\ \frac{\partial u_2}{\partial x_1}(x_1, x_2, t) & \frac{\partial u_2}{\partial x_2}(x_1, x_2, t) \\ 0 & 0 \end{pmatrix}, \\
\mathbf{A}(x_1, x_2, t) &= \begin{pmatrix} 0 & -\vartheta_3(x_1, x_2, t) \\ \vartheta_3(x_1, x_2, t) & 0 \\ 0 & 0 \end{pmatrix}, & \mathbf{D} \vartheta(x_1, x_2, x_3, t) &= \mathbf{ax} \mathbf{l} \mathbf{A} = \begin{pmatrix} 0 \\ 0 \\ \vartheta_3(x_1, x_2, t) \end{pmatrix}.
\end{align*}
\]
Besides the Rayleigh waves, in the full isotropic Cosserat medium there is also another (transversal, “Love-like”) surface wave described by \((u_1, u_2, \theta_3)^T = 0\) and \((u_3, \theta_1, \theta_2)^T \neq 0\), see [39], while in the reduced Cosserat medium \((L_c \to 0)\) there exist non-propagating transversal oscillations of the same kind, see [35]. However, this is not the purpose of the present work and the completed proof of the existence of these “Love-like” surface waves will be considered in the future.

Corresponding to our ansatz, we deduce the following form of the stress tensor

\[
\sigma = \begin{pmatrix}
(2\mu_e + \lambda_e) \frac{\partial u_1}{\partial x_1} + \lambda_e \frac{\partial u_2}{\partial x_2} & (\mu_e + \mu_c) \frac{\partial u_1}{\partial x_2} + 2\mu_c \theta_3 + (\mu_e - \mu_c) \frac{\partial u_2}{\partial x_1} & 0 \\
(\mu_e + \mu_c) \frac{\partial u_2}{\partial x_1} - 2\mu_c \theta_3 + (\mu_e - \mu_c) \frac{\partial u_3}{\partial x_2} & (2\mu_e + \lambda_e) u_{2,2} + \lambda_e \frac{\partial u_1}{\partial x_1} & 0 \\
0 & 0 & \lambda_e \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right)
\end{pmatrix},
\]

and of the couple stress tensor

\[
m = \begin{pmatrix}
0 & 0 & \mu_e L_c^2 (\alpha_1 - \alpha_2) \frac{\partial \theta_3}{\partial x_1} \\
0 & 0 & \mu_e L_c^2 (\alpha_1 - \alpha_2) \frac{\partial \theta_3}{\partial x_2} \\
\mu_e L_c^2 (\alpha_1 + \alpha_2) \frac{\partial \theta_3}{\partial x_1} & \mu_e L_c^2 (\alpha_1 + \alpha_2) \frac{\partial \theta_3}{\partial x_2} & 0
\end{pmatrix},
\]

while the equation of motion are reduced to

\[
\rho \frac{\partial^2 u_1}{\partial t^2} = \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2},
\rho \frac{\partial^2 u_2}{\partial t^2} = \frac{\partial \sigma_{22}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2},
\rho j \mu_c \frac{\partial^2 \theta_3}{\partial t^2} = \frac{\partial m_{31}}{\partial x_1} + \frac{\partial m_{32}}{\partial x_2} + 2\mu_c \frac{\partial u_2}{\partial x_1} - 2\mu_c \frac{\partial u_1}{\partial x_2} - 4\mu_c \theta_3,
\]

subjected to the aforementioned boundary conditions (2.45), which turn out to be

\[
\sigma_{12} = 0, \quad \sigma_{22} = 0, \quad m_{32} = 0 \quad \text{at} \quad x_2 = 0.
\]

Therefore, the aim of this paper is to give an explicit solution \((u, \theta)\) of the following system

\[
\rho \frac{\partial^2 u_1}{\partial t^2} = (2\mu_e + \lambda_e) \frac{\partial^2 u_1}{\partial x_1^2} + \lambda_e \frac{\partial^2 u_2}{\partial x_2 \partial x_1} + (\mu_e + \mu_c) \frac{\partial^2 u_1}{\partial x_2^2} + 2\mu_c \frac{\partial \theta_3}{\partial x_1} + (\mu_e - \mu_c) \frac{\partial^2 u_2}{\partial x_1 \partial x_2},
\rho \frac{\partial^2 u_2}{\partial t^2} = (\mu_e + \mu_c) \frac{\partial^2 u_2}{\partial x_1^2} - 2\mu_c \frac{\partial \theta_3}{\partial x_1} + (\mu_e - \mu_c) \frac{\partial^2 u_1}{\partial x_2 \partial x_1} + (2\mu_e + \lambda_e) \frac{\partial^2 u_2}{\partial x_2^2} + \lambda_e \frac{\partial^2 u_1}{\partial x_1 \partial x_2},
\]

subjected to the boundary conditions (2.45), which turn out to be

\[
\sigma_{12} = 0, \quad \sigma_{22} = 0, \quad m_{32} = 0 \quad \text{at} \quad x_2 = 0.
\]
\[ \rho j \mu_e \varepsilon_e \frac{\partial^2 \varrho}{\partial t^2} = \mu_e L_c^2 \gamma \frac{\partial^2 \varrho}{\partial x_1^2} + \mu_e L_c^2 \gamma \frac{\partial^2 \varrho}{\partial x_2^2} + 2 \mu_e \frac{\partial u_2}{\partial x_1} - 2 \mu_e \frac{\partial u_1}{\partial x_2} - 4 \mu_e \varrho, \]

which satisfies the boundary conditions at \( x_2 = 0 \)

\[
(\mu_e + \mu_c) \frac{\partial u_1}{\partial x_2} + 2 \mu_c \varrho_3 = 0, \\
(2 \mu_e + \lambda_e) \frac{\partial u_2}{\partial x_2} + \lambda_e \frac{\partial u_1}{\partial x_1} = 0, \\
\mu_e L_c^2 \gamma \frac{\partial \varrho_3}{\partial x_2} = 0,
\]

(2.53)

where \( \gamma = \alpha_1 + \alpha_2 \), and which has the asymptotic behaviour (2.46).

Even if until now we have considered the propagation of surface waves with the direction \( e_1 = (1, 0, 0)^T \), i.e. some horizontal direction, \( x_2 \) being the vertical direction orthogonal to the surface along which the wave decays, in one point of our method (see Proposition 3.3 and its implications in Subsection 5) we need to consider a general direction of wave propagation \( \xi = (\xi_1, \xi_2, 0)^T \) and to characterize where the wave is a “real” bulk wave. Therefore, it is useful to know for which conditions on the constitutive parameters (for every wave number \( k > 0 \) and in the direction \( \xi = (\xi_1, \xi_2, 0)^T \) with \( ||\xi||^2 = 1 \)) the system of partial differential equations (2.15) admits a non trivial solution in the form

\[
\begin{align*}
\hat{u}_1(t, \xi_1) & = e^{i(k(\xi_1)\xi_1^2 - \omega t)}, \\
\hat{u}_2(t, \xi_1) & = \frac{1}{\mu_e} \hat{u}_1(t, \xi_1), \\
\hat{\varrho}_3(t, \xi_1) & = 1 - \frac{\mu_e}{\mu_e + \lambda_e} \hat{u}_1(t, \xi_1),
\end{align*}
\]

(2.54)

only for real positive values \( \omega^2 \). According to the results given in Subsection 2.3, see (2.20), (2.27) and (2.30), we have the following result

**Proposition 2.4.** Let \( \xi \in \mathbb{R}^3, \xi \neq 0 \) be any direction of the form \( \xi = (\xi_1, \xi_2, 0)^T \). The necessary and sufficient conditions for existence of a non trivial solution of the system of partial differential equations (2.15) of the form given by (2.54) are

\[
2 \mu_e + \lambda_e > 0, \quad \mu_e > 0, \quad \mu_c > 0, \quad \alpha_1 + \alpha_2 > 0.
\]

(2.55)

These restriction (2.55) are the restrictions on the constitutive parameters that we will impose for the rest of the paper. For an interpretation of the conditions (2.55), see Remark 2.2. These conditions do not involve the constitutive parameter \( \alpha_3 \) and do not imply the existence of real waves in any directions. However, for a given direction \( \xi \in \mathbb{R}^3, \xi \neq 0 \) of the form \( \xi = (\xi_1, \xi_2, 0)^T \), the restrictions (2.55) imply the existence of only real waves defined by expressions of the form (2.54).

### 3 The ansatz for the solution and the limiting speed

We look for a solution of (2.52) and (2.53) having the form\(^3\)

\[
\mathcal{U}(x_1, x_2, t) = \begin{pmatrix} u_1(x_1, x_2, t) \\ u_2(x_1, x_2, t) \\ \varrho_3(x_1, x_2, t) \end{pmatrix} = \text{Re} \left( \begin{pmatrix} z_1(x_2) \\ z_2(x_2) \\ z_3(x_2) \end{pmatrix} e^{i(k(x_1 - vt))} \right),
\]

(3.1)

where \( v \) is the propagation speed (the phase velocity). If \( z_i, i = 1, 2, 3 \), are solutions of the following systems

\[
\begin{pmatrix} \mu_e + \mu_c \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} z'_1(x_2) \\ z'_2(x_2) \\ z'_3(x_2) \end{pmatrix} + \begin{pmatrix} k^2 (2 \mu_e + \lambda_e) - \rho k^2 v^2 \\ 0 \\ -2 \mu_e k \end{pmatrix} \begin{pmatrix} z_1(x_2) \\ z_2(x_2) \\ z_3(x_2) \end{pmatrix} = 0,
\]

(3.2)

\(^3\)We take \( z_3 \) since this choice leads us, in the end, only to real matrices.
and (from the boundary conditions)
\[
\begin{pmatrix}
\mu_e + \mu_c & 0 & 0 \\
0 & 2\mu_e + \lambda_e & 0 \\
0 & 0 & \mu_e L_e^2 \gamma
\end{pmatrix}
\begin{pmatrix}
z_1'(0) \\
z_2'(0) \\
z_3'(0)
\end{pmatrix}
+ \frac{1}{k} \begin{pmatrix}
0 & k(\mu_e - \mu_c) & 2\mu_c \\
k \lambda_e & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
z_1(0) \\
z_2(0) \\
z_3(0)
\end{pmatrix} = 0,
\]

where \(z'\) denotes the derivative with respect to \(x_2\), then \(U\) given by the ansatz (3.1) satisfies (2.52) and (2.53).

In a more compact notation, the above equations admit the following equivalent form
\[
\frac{1}{k^2} T z''(x_2) + \frac{1}{k} (R + R^T) z'(x_2) - Q z(x_2) + k^2 v^2 \mathbb{I} z(x_2) = 0,
\]

where
\[
\begin{align*}
T &= k^2 \begin{pmatrix}
\mu_e + \mu_c & 0 & 0 \\
0 & 2\mu_e + \lambda_e & 0 \\
0 & 0 & \mu_e L_e^2 \gamma
\end{pmatrix}, \\
R &= k \begin{pmatrix}
0 & k(\mu_e - \mu_c) & 0 \\
0 & 0 & 0 \\
k \lambda_e & 0 & 0
\end{pmatrix}, \\
Q &= \begin{pmatrix}
\mu_e + \mu_c & 0 & 0 \\
0 & 2\mu_e + \lambda_e & 0 \\
0 & 0 & \mu_e L_e^2 \gamma
\end{pmatrix}.
\end{align*}
\]

The system (3.3) has a similar structure to that from classical linear elasticity [24] but we still have to rewrite it in order to make it manageable for our analysis. In this respect, the solution \(z\) of (3.3) is equivalent to find a solution \(y\) of
\[
\frac{1}{k^2} \tilde{T} \tilde{y}''(x_2) + \frac{1}{k} \tilde{R} \tilde{y}'(x_2) - \tilde{Q} \tilde{y}(x_2) + k^2 v^2 \mathbb{I} \tilde{y}(x_2) = 0,
\]

where \(y(x_2) := \tilde{T}^{-1/2} z(x_2)\). Therefore we use the modified matrices
\[
\begin{align*}
\mathcal{T} &:= k^2 \begin{pmatrix}
\mu_e + \mu_c & 0 & 0 \\
0 & 2\mu_e + \lambda_e & 0 \\
0 & 0 & \mu_e L_e^2 \gamma
\end{pmatrix}, \\
\mathcal{R} &:= k \begin{pmatrix}
0 & k(\mu_e - \mu_c) & 0 \\
0 & 0 & 0 \\
k \lambda_e & 0 & 0
\end{pmatrix}, \\
\mathcal{Q} &:= \begin{pmatrix}
\mu_e + \mu_c & 0 & 0 \\
0 & 2\mu_e + \lambda_e & 0 \\
0 & 0 & \mu_e L_e^2 \gamma
\end{pmatrix},
\end{align*}
\]

and the following equivalent form of the system (3.3)
\[
\frac{1}{k^2} \mathcal{T} y''(x_2) + \frac{1}{k} (\mathcal{R} + \mathcal{R}^T) y'(x_2) - \mathcal{Q} y(x_2) + k^2 v^2 \mathbb{I} y(x_2) = 0,
\]

Lemma 3.1. If the constitutive coefficients satisfy the conditions (2.55), then the matrices \(\mathcal{Q}\) and \(\mathcal{T}\) are symmetric and positive definite.

Proof. Symmetry is clear. It is easy to see that
\[
\begin{align*}
T_{11} &= k^2(\mu_e + \mu_c), \\
T_{12} - T_{21} &= k^4(2\mu_e + \lambda_e)(\mu_e + \mu_c), \\
det T &= k^6(2\mu_e + \lambda_e)(\mu_e + \mu_c)\mu_e L_e^2 \gamma.
\end{align*}
\]

Therefore, our constitutive hypothesis imply that \(T\) is positive-definite. In addition, \(\mathcal{Q}\) is positive-definite if and only if the principal minors are positive, namely
\[
\begin{align*}
Q_{11} &= k^2(2\mu_e + \lambda_e), \\
Q_{12} - Q_{21} &= k^4(2\mu_e + \lambda_e)(\mu_e + \mu_c),
\end{align*}
\]

(3.10)
$$\det(Q) = k^4(2\mu_c + \lambda_c)[(\mu_c + \mu_c)k^2\mu_c L_c^2\gamma + 4\mu_c\mu_c],$$

i.e. under the hypothesis of the lemma. Since $T$ and $Q$ are positive definite, so there are $T$ and $Q$ defined by (3.7), and the proof is complete.

We now seek a solution $y$ of the differential system (3.8) in the form

$$y(x_2) = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} e^{i r k x_2}, \quad \text{Im} r > 0,$$

(3.11)

where $r \in \mathbb{C}$ is a complex parameter, $d = (d_1, d_2, d_3)^T \in \mathbb{C}^3$, $a \neq 0$ is the amplitude and Im $r$ is the coefficient of the imaginary part of $r$. The condition Im $r > 0$ ensures the asymptotic decay condition (2.46). Inserting (3.11) in (3.3) we obtain the systems of algebraic equations

$$[r^2 T + r(\mathbf{R} + \mathbf{R}^T) + Q - k^2 v^2 \mathbb{1}] d = 0, \quad [r^2 T + \mathbf{R}^T] d = 0. \quad (3.12)$$

The characteristic equation corresponding to the eigenvalue problem (3.12), i.e., the condition to have a non-trivial solution of $d = (d_1, d_2, d_3)^T$, is

$$\det[r^2 T + r(\mathbf{R} + \mathbf{R}^T) + Q - k^2 v^2 \mathbb{1}] = 0,$$

(3.13)

which gives six roots of the eigenvalue $r$. The associated eigenvectors $a$ can be determined for the corresponding eigenvalues.

**Definition 3.2.** By the limiting speed we understand a speed $\hat{v} > 0$, such that for all wave speeds satisfying $0 \leq v < \hat{v}$ (subsonic speeds) the roots of the characteristic equation (3.13) are not real and vice versa, i.e., if the roots of the characteristic equation (3.13) are not real then they correspond to a wave speeds $v$ satisfying $0 \leq v < \hat{v}$.

**Proposition 3.3.** If the constitutive coefficients satisfy the conditions (2.55), then there exists a limiting speed $\hat{v} > 0$. Furthermore, if one root $r_v$ of the characteristic equation (3.13) is real then it corresponds to a speed $v > \hat{v}$ (non-admissible).

**Proof.** Assume that there exists a real $r_v$ as solution of the characteristic equation (3.13), then $\exists \theta \in (-\pi/2, \pi/2)$ such that $r_v = \tan \theta$. Therefore, corresponding to (3.11), $U$ given by (3.1) and defined by $v_r$ turns into

$$U(x_1, x_2, t) = \begin{pmatrix} u_1(x_1, x_2, t) \\ u_2(x_1, x_2, t) \\ \theta_3(x_1, x_2, t) \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ i d_3 \end{pmatrix} e^{i k(x_1 + \tan \theta x_2 - \tau t)} = \begin{pmatrix} d_1 \\ d_2 \\ i d_3 \end{pmatrix} e^{\frac{i k}{\cos \theta} (\cos \theta x_1 + \sin \theta x_2 - \cos \theta \tau t)}, \quad (3.14)$$

which means that $U(x_1, x_2, t)$ is a non-trivial plane wave solution with wave number $\frac{k}{\cos \theta}$, the speed $\bar{v}_0 = v \cos \theta$ and propagation in the direction $n_\theta$ where $n_\theta = (\cos \theta, \sin \theta, 0)$. A direct substitution of (3.14) into (2.18) implies the existence of a non-trivial solution $(d_1, d_2, d_3) \neq 0$ of the algebraic system

$$k^2 \sin^2 \theta (\mu_c + \mu_c) d_1 + k \sin \theta \cos \theta [k \lambda_c + k (\mu_c - \mu_c)] d_2 + 2 \mu_c \sin \theta \cos \theta d_3 + \cos^2 \theta [k^2(2\mu_c + \lambda_c) - k^2 \rho v^2] d_1 = 0,$$

$$k^2 \sin^2 \theta (2\mu_c + \lambda_c) d_2 + k \sin \theta \cos \theta [k \lambda_c + k (\mu_c - \mu_c)] d_1 - 2 \mu_c k \cos^2 \theta d_3 + \cos^2 \theta [k^2(\mu_c + \mu_c) - k^2 \rho v^2] d_2 = 0,$$

$$k^2 \sin^2 \theta \mu_c L_c^2 \gamma d_2 + 2 \mu_c k \sin \theta \cos \theta d_1 - 2 \mu_c k \cos^2 \theta d_2 + \cos^2 \theta [k^2 \mu_c L_c^2 \gamma + 4 \mu_c - \rho j \mu_c \tau_c^2 k^2 v^2] d_3 = 0,$$

(3.15)

which in matrix form gives

$$\begin{pmatrix} \sin^2 \theta \begin{pmatrix} \mu_c + \mu_c & 0 & 0 \\ 0 & 2 \mu_c + \lambda_c & 0 \\ 0 & 0 & \mu_c L_c^2 \gamma \end{pmatrix} & \theta_1 \\ \theta_2 & \sin \theta \cos \theta \begin{pmatrix} 0 & k (\mu_c - \mu_c + \lambda_c) & 2 \mu_c \\ 2 \mu_c & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \theta_3 \\ \theta_4 & \cos^2 \theta \begin{pmatrix} k^2 (2 \mu_c + \lambda_c) - \rho k^2 v^2 & 0 & 0 \\ 0 & k^2 (\mu_c + \mu_c) - \rho k^2 v^2 & -2 \mu_c k \\ -2 \mu_c k & k^2 \mu_c L_c^2 \gamma + 4 \mu_c - \rho j \mu_c \tau_c^2 k^2 v^2 & 0 \end{pmatrix} & \theta_5 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = 0,$$

(3.16)

\[\text{Since the quadratic equation does not have real solutions, there is a complex solution } r \text{ for which } \text{Im } r > 0, \text{ since the complex solutions are pair-conjugated. Therefore, the existence of such a solution } r, \text{ i.e., } \text{Im } r > 0, \text{ implies the existence of a wave propagating in the direction } x_1 \text{ with the phase velocity } v \text{ and decaying exponentially in the direction } x_2.\]
and this is equivalent to
\[
\left[ k^2 \sin^2 \theta \hat{1}^{-1/2} \left( \begin{array}{cccc}
\mu_0 + \mu_c & 0 & 0 & 0 \\
0 & 2 \mu_c + \lambda_c & 0 & 0 \\
0 & 0 & 2 \mu_c + \lambda_c & 0 \\
\mu_c L^2 & 0 & 0 & 2 \mu_c
\end{array} \right) \hat{1}^{-1/2} + k \sin \theta \cos \theta \hat{1}^{-1/2} \left( \begin{array}{cccc}
k(\mu_0 - \mu_c + \lambda_c) & 0 & 0 & 0 \\
0 & 2 \mu_c & 0 & 0 \\
0 & 0 & 2 \mu_c & 0 \\
0 & 0 & 0 & 2 \mu_c
\end{array} \right) \hat{1}^{-1/2} \\
+ \cos^2 \theta \hat{1}^{-1/2} \left( \begin{array}{cccc}
k^2 (2 \mu_c + \lambda_c) & 0 & 0 & 0 \\
0 & - 2 \mu_c & 0 & 0 \\
0 & 0 & - 2 \mu_c & 0 \\
0 & 0 & 0 & - 2 \mu_c
\end{array} \right) \hat{1}^{-1/2} \right] \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = 0,
\]
where \( \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} \neq 0. \)

Therefore, the assumption from the beginning of the proof implies that there exists a solution \((f_1, f_2, f_3) \neq 0\) of the algebraic system written in matrix format
\[
\left[ \sin^2 \theta T + \sin \theta \cos \theta (\mathcal{R} + \mathcal{R}^T) + \cos^2 \theta Q - k^2 v_0^2 \cos^2 \theta \mathbb{I} \right] \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = 0.
\]

Let us observe that equation (3.16) is actually the propagation condition for plane waves in isotropic Cosserat materials, in the fixed direction \(n_\theta = (\cos \theta, \sin \theta, 0)\). Since the constitutive coefficients satisfy the conditions (2.55), according to Proposition 2.4, for the direction \(n_\theta = (\cos \theta, \sin \theta, 0)\) in particular, the system of partial differential equations (2.15) admits a non trivial solution in the form
\[
\begin{pmatrix} u_1 \\ u_2 \\ \theta \\ 0 \\ \vartheta \\ \lambda \\ \tau \end{pmatrix} = 0,
\]
only for real positive values \(\omega\). To each \(\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})\) we associate these real frequencies \(\omega_\theta\) satisfying
\[
\det \{ \sin^2 \theta T + \sin \theta \cos \theta (\mathcal{R} + \mathcal{R}^T) + \cos^2 \theta Q - k^2 v_0^2 \cos^2 \theta \mathbb{I} \} = 0.
\]
Then, each \(\omega_\theta\) defines a \(v_\theta\) such that \(\omega_\theta = v_\theta \cos \theta\). We define \(\tilde{v}\) as the minimum of the values of \(v_\theta \cos \theta\) for all \(\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})\). Hence, this means that we find all solutions \(v_\theta\) of
\[
\det \{ \sin^2 \theta T + \sin \theta \cos \theta (\mathcal{R} + \mathcal{R}^T) + \cos^2 \theta Q - k^2 v_\theta^2 \cos^2 \theta \mathbb{I} \} = 0,
\]
and we define \(^5\)
\[
\tilde{v} = \inf_{\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})} v_\theta.
\]

In conclusion, if there exists a value of \(v\) such that the equation (3.13) admits a real solution \(v\), then \(v\) must satisfy \(v \geq \tilde{v}\). Thus, if \(v\) is such that \(0 \leq v < \tilde{v}\), then \(r\) can not be real and if \(r\) is real, then \(v \geq \tilde{v}\).

**Proposition 3.4.** If the constitutive coefficients satisfy the conditions (2.55), then for all \(\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})\) and \(k > 0\), the tensor \(Q_\theta := \sin^2 \theta T + \sin \theta \cos \theta (\mathcal{R} + \mathcal{R}^T) + \cos^2 \theta Q\) is positive definite.

**Proof.** Since the constitutive coefficients satisfy the conditions (2.55), according to Proposition 2.4, it follows that there exists only real numbers \(\omega\) such that the following system \(^6\) admits non trivial solutions
\[
-\omega^2 \rho \tilde{u}_i = -k^2 (\mu_0 + \mu_c) \tilde{u}_i + k^2 (\mu_0 - \mu_c + \lambda_c) \sum_{i=1}^2 \tilde{u}_i \tilde{\xi}_l \tilde{\xi}_l - 2 k \mu_0 \sum_{i=1}^2 \epsilon_{i3a} \tilde{\vartheta}_3 \xi_l, \quad i = 1, 2,
\]
\[
-\omega^2 \rho j \mu \tau^2 \tilde{\vartheta}_3 = -i k^2 \mu_0 L^2 (\alpha_1 + \alpha_2) \sum_{i=1}^2 \tilde{\vartheta}_3 \xi_l^2 - 2 i k \mu_0 \sum_{i=1}^2 \epsilon_{i3a} \tilde{\vartheta}_3 \xi_l - 4 i \mu_0 \tilde{\vartheta}_3 \sum_{i=1}^2 \xi_l^2.
\]

\(^5\)In others words, we can take the solution \(v_\theta(\theta)\) of all possible plane body waves propagating in the direction \(\hat{\theta}\) and we take
\[
\tilde{v} = \inf_{\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})} v_\theta(\theta).
\]

\(^6\)This system follows directly from (2.18). Inserting (2.17) into (2.15) we see that if \(\tilde{u}_1, \tilde{u}_2 \) and \(\tilde{\vartheta}_3\) have to satisfy the following partial differential equations
\[
-\omega^2 \rho \tilde{u}_i = -k^2 (\mu_0 + \mu_c) \tilde{u}_i + k^2 (\mu_0 - \mu_c + \lambda_c) \sum_{i=1}^2 \tilde{u}_i \tilde{\xi}_l \tilde{\xi}_l - 2 k \mu_0 \sum_{i=1}^2 \epsilon_{i3a} \tilde{\vartheta}_3 \xi_l, \quad i = 1, 2,
\]
\[
-\omega^2 \rho j \mu \tau^2 \tilde{\vartheta}_3 = -i k^2 \mu_0 L^2 (\alpha_1 + \alpha_2) \sum_{i=1}^2 \tilde{\vartheta}_3 \xi_l^2 - 2 i k \mu_0 \sum_{i=1}^2 \epsilon_{i3a} \tilde{\vartheta}_3 \xi_l - 4 i \mu_0 \tilde{\vartheta}_3 \sum_{i=1}^2 \xi_l^2.
\]
\[ w = (\hat{u}_1, \hat{u}_2, \hat{u}_3)^T \neq 0 \] for real positive numbers \( \omega^2 \), i.e.,

\[ [\mathbf{Q}_1(\xi, k) - \omega^2 \mathbb{I}] w = 0, \quad (3.23) \]

where

\[
\mathbf{Q}_1(\xi, k) = \begin{pmatrix}
  k^2(2\mu_c + \lambda_c)\xi_1^2 + k^2(\mu_c + \mu_c)\xi_2^2 & k^2(\mu_c - \mu_c + \lambda_c)\xi_1\xi_2 & 2k\mu_c\xi_1 \\
  k^2(\mu_c - \mu_c + \lambda_c)\xi_1\xi_2 & k^2(\mu_c + \mu_c)\xi_1^2 + k^2(2\mu_c + \lambda_c)\xi_2^2 & -2k\mu_c\xi_1 \\
  2k\mu_c\xi_2 & -2k\mu_c\xi_1 & k^2\mu_cL_2^2(\alpha_1 + \alpha_2)(\xi_1^2 + \xi_2^2) + 4\mu_c(\xi_1^2 + \xi_2^2)
\end{pmatrix}.
\]

But this is equivalent to the positive definiteness of the matrix

\[
\mathbf{\bar{Q}}_1(\xi, k) = \mathbb{I}^{-1/2} \begin{pmatrix}
  k^2(2\mu_c + \lambda_c)\xi_1^2 + (\mu_c + \mu_c)\xi_2^2 & k^2(\mu_c - \mu_c + \lambda_c)\xi_1\xi_2 & 2k\mu_c\xi_1 \\
  k^2(\mu_c - \mu_c + \lambda_c)\xi_1\xi_2 & (\mu_c + \mu_c)\xi_1^2 + k^2(2\mu_c + \lambda_c)\xi_2^2 & -2k\mu_c\xi_1 \\
  2k\mu_c\xi_2 & -2k\mu_c\xi_1 & k^2(\alpha_1 + \alpha_2 + 4\mu_c)
\end{pmatrix} \mathbb{I}^{-1/2}. \quad (3.25)
\]

Since the conditions (2.55) imply the positive definiteness of the matrix \( \mathbf{\bar{Q}}_1(\xi, k) \) for all \( \xi = (\xi_1, \xi_2, 0) \in \mathbb{R}^3 \), \( ||\xi|| = 1 \), taking \( \xi = (\cos \theta, \sin \theta, 0) \) we deduce the desired result.

**Remark 3.5.** The Legendre–Hadamard ellipticity condition is not sufficient for the positive definiteness of \( \mathbf{Q}_\theta \), since they are equivalent to (1.7) which does not imply (2.36). Instead, we will see that the Legendre–Hadamard ellipticity condition leads to the acoustic tensor and its positive definiteness. Note that there does not exist a direction \( \xi \in \mathbb{R}^3 \) such that \( \mathbf{Q}_\theta \) equals the acoustic tensor. Moreover, even if in classical linear elasticity this is well known, in the Cosserat theory it seems to be impossible (or at least it is not obvious to us) to arrive from the Legendre–Hadamard ellipticity condition (the positive definiteness of the acoustic tensor) to the structure of the tensor \( \mathbf{Q}_\theta \) or to its positive definiteness.

**Proof.** The Legendre–Hadamard ellipticity condition \( [2, 19, 61] \) demands the acoustic tensor

\[
\mathbf{\bar{Q}} = \begin{pmatrix} \mathbf{Q}_1 & 0 \\ 0 & \mathbf{Q}_2 \end{pmatrix}
\]

defined through

\[
D^2W(\mathbf{e}_1, \mathbb{R}).((\xi \otimes \eta, \zeta \otimes \eta), (\xi \otimes \eta, \zeta \otimes \eta)) = D^2W_1(\mathbf{e}_1)(\xi \otimes \eta, \xi \otimes \eta) + D^2W_2(\mathbb{R})(\xi \otimes \eta, \zeta \otimes \eta)
\]

\[
= \langle \eta, \mathbf{\bar{Q}}_1(\xi) \eta \rangle + \langle \eta, \mathbf{\bar{Q}}_2(\zeta) \eta \rangle
\]

is strictly positive definite for any nonzero wave directions \( \xi \in \mathbb{R}^3 \) and \( \zeta \in \mathbb{R}^3 \). Hence, let us identify the matrix \( \mathbf{\bar{Q}}_1 \) and \( \mathbf{\bar{Q}}_2 \). First identify \( \mathbf{\bar{Q}}_1 \). Since

\[
D^2W_1(\mathbf{e}_1)(\xi \otimes \eta, \xi \otimes \eta) = \mu_c ||\text{sym} \xi \otimes \eta||^2 + \mu_c ||\text{skew} \xi \otimes \eta||^2 + \frac{\lambda_c}{2} (\text{tr} (\xi \otimes \eta))^2
\]

is given by

\[
\mathbf{\bar{Q}}_1(\xi) = \frac{1}{2} \begin{pmatrix}
  (2\mu_c + \lambda_c)\xi_1^2 + (\mu_c + \mu_c)(\xi_2^2 + \xi_3^2) & (\mu_c - \mu_c + \lambda_c)\xi_1\xi_2 & (\mu_c - \mu_c + \lambda_c)\xi_1\xi_3 \\
  (\mu_c - \mu_c + \lambda_c)\xi_1\xi_2 & (2\mu_c + \lambda_c)\xi_2^2 + (\mu_c + \mu_c)(\xi_1^2 + \xi_3^2) & (\mu_c - \mu_c + \lambda_c)\xi_2\xi_3 \\
  (\mu_c - \mu_c + \lambda_c)\xi_1\xi_3 & (\mu_c - \mu_c + \lambda_c)\xi_2\xi_3 & (2\mu_c + \lambda_c)\xi_3^2 + (\mu_c + \mu_c)(\xi_1^2 + \xi_2^2)
\end{pmatrix}.
\]

(3.29)
Next, we identify \( \tilde{Q}_2 \). Since

\[
D^2W_2(\mathcal{R}).(\zeta \otimes \eta, \zeta \otimes \eta) = \alpha_1 \|\text{sym} \zeta \otimes \eta\|^2 + \alpha_2 \|\text{skew} \zeta \otimes \eta\|^2 + \frac{\alpha_3}{2} [\text{tr} (\zeta \otimes \eta)]^2
\]

\[
= \frac{\alpha_1 + \alpha_2}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} \zeta_i^2 \eta_j^2 + \frac{\alpha_1 - \alpha_2 + \alpha_3}{2} \left( \sum_{i=1}^{3} \zeta_i^2 \eta_i^2 + 2 \sum_{i,k=1}^{3} \zeta_i \zeta_k \eta_i \eta_k \right),
\]

the matrix \( \tilde{Q}_2 \) which satisfies

\[
D^2W_2(\mathcal{R}).(\zeta \otimes \eta, \zeta \otimes \eta) = \langle \eta, \tilde{Q}_2(\zeta) \eta \rangle
\]

is given by

\[
\tilde{Q}_2(\zeta) = \frac{1}{2} \begin{pmatrix}
(2 \alpha_1 + \alpha_3) \zeta_1^2 + (\alpha_1 + \alpha_2) (\zeta_1^2 + \zeta_2^2)
(\alpha_1 - \alpha_2 + \alpha_3) \zeta_1 \zeta_2
(\alpha_1 - \alpha_2 + \alpha_3) \zeta_1 \zeta_3
(\alpha_1 - \alpha_2 + \alpha_3) \zeta_2 \zeta_3
(2 \alpha_1 + \alpha_3) \zeta_2^2 + (\alpha_1 + \alpha_2) (\zeta_2^2 + \zeta_3^2)
(\alpha_1 - \alpha_2 + \alpha_3) \zeta_2 \zeta_3
(\alpha_1 - \alpha_2 + \alpha_3) \zeta_3^2 + (\alpha_1 + \alpha_2) (\zeta_1^2 + \zeta_3^2)
\end{pmatrix}.
\]

Actually, our entire approach is based on the assumption that the matrix \( \tilde{Q}_1(\xi,k) \) defined by (3.24) is positive definite. So, if the Legendre-Hadamard condition implies this fact, or it is equivalent with, then the Legendre-Hadamard condition would be also suitable for our approach. However, it seems that this is not possible because we did not find appropriate values for \( \xi \) and \( \zeta \) such that, for some particular values of them the positive definiteness of \( \tilde{Q}_1(\xi,k) \) \( \tilde{Q}_2(\xi) \) would imply the positive definiteness of the matrix \( Q_1(\xi,k) \), as we need in our approach.

Even when we consider the Legendre-Hadamard ellipticity condition on the set of admissible solutions of the form

\[
u(x_1, x_2, t) = \begin{pmatrix}
u_1(x_1, x_2, t)
\nu_2(x_1, x_2, t)
0
\end{pmatrix}, \quad \vartheta(x_1, x_2, x_3, t) = \begin{pmatrix}0
0
\vartheta_3(x_1, x_2, t)
\end{pmatrix},
\]

it is not clear how the positive definiteness of the matrix \( \tilde{Q}_1(\xi,k) \) can possibly be a consequence of the Legendre-Hadamard ellipticity condition.

**Proposition 3.6.** If the constitutive coefficients satisfy the conditions (2.55), then for all \( \theta \in (-\frac{\pi}{4}, \frac{\pi}{4}) \), \( k > 0 \) and \( 0 \leq v < \hat{v} \), the tensor \( Q_\theta := \sin^2 \theta \mathcal{T} + \sin \theta \cos \theta \mathcal{R} + \mathcal{R}^T + \cos^2 \theta Q - k^2 v^2 \cos^2 \theta \mathbf{1} \) is positive definite.

**Proof.** Since \( Q_\theta := \sin^2 \theta \mathcal{T} + \sin \theta \cos \theta \mathcal{R} + \mathcal{R}^T + \cos^2 \theta Q \) is positive definite, it admits only positive eigenvalues. We need to prove that for all \( \theta \in (-\frac{\pi}{4}, \frac{\pi}{4}) \), \( k > 0 \) and for all \( 0 \leq v < \hat{v} \) the matrix \( Q_\theta := Q_\theta - k^2 v^2 \cos^2 \theta \mathbf{1} \) is positive definite, which is equivalent to the property that all eigenvalues \( Q_\theta \) are larger than those of the matrix \( k^2 v^2 \cos^2 \theta \mathbf{1} \), i.e., than \( k^2 v^2 \cos^2 \theta \). Assuming that there exist \( \theta_0 \in (-\frac{\pi}{4}, \frac{\pi}{4}) \) and \( v_0 \in [0, \hat{v}) \) for which there is an eigenvalue \( \lambda_{\theta_0} \) of \( Q_\theta \) such that \( \lambda_{\theta_0} < k^2 v_0^2 \cos^2 \theta_0 \), then

\[
v_{\theta_0} := \sqrt{\frac{\lambda_{\theta_0}}{k^2 \cos^2 \theta_0}} < v_0 < \hat{v}
\]

is solution of (3.21), i.e., for fixed \( \theta_0 \) we have that \( v_{\theta_0} < \hat{v} \) verifies

\[
\det \{\sin^2 \theta_0 \mathcal{T} + \sin \theta_0 \cos \theta_0 \mathcal{R} + \mathcal{R}^T + \cos^2 \theta_0 Q - k^2 v_{\theta_0}^2 \cos^2 \theta_0 \mathbf{1}\} = 0.
\]

This is in contradiction to the definition of the limiting speed and Proposition 3.22, since \( \hat{v} \) is the smallest speed having this property. Therefore, it remains that for all \( \theta \in (-\frac{\pi}{4}, \frac{\pi}{4}) \), \( k > 0 \) and for all \( 0 \leq v < \hat{v} \), all the eigenvalues of \( Q_\theta \) are larger than \( k^2 v^2 \cos^2 \theta \) and the proof is complete.

## 4 The common method to construct the solution using the Stroh formalism

In this subsection we present the main steps of the common method using the Stroh formalism, since in this point of our approach it is still possible to switch to the Stroh formalism and vice versa, without using conceptually different methods.
On one hand, using the ansatz \( z(x_2) = \hat{1}^{1/2} \left( \begin{array}{c} d_1 \\ d_2 \\ d_3 \end{array} \right) e^{r k x_2} \) in (3.3), we are led to define a vector \( b \neq 0 \in \mathbb{C}^3 \) as

\[
b = [r T + R T] d.
\]

Since \( T \) is symmetric and positive definite there exists \( T^{-1} \). Hence from (4.1) we get

\[
r d = -T^{-1} R T d + T^{-1} b
\]

and

\[
r b = [r^2 T + r R T] d.
\]

On the other hand, from (3.11) and considering the same ansatz for \( z(x_2) \) as before, we deduce

\[
(r^2 T + r R T) d + (r R + \tilde{Q}) d = 0 \quad \Leftrightarrow \quad r b = -[r R + \tilde{Q}] d,
\]

where \( \tilde{Q} = Q - k^2 v^2 \hat{1} \). Making use of (4.2) in (4.4) we obtain

\[
r b = -\tilde{Q} d - R \{ -T^{-1} R d + T^{-1} b \} = [-\tilde{Q} + R T^{-1} R T] d - R T^{-1} b.
\]

From (4.2) and (4.5) it is possible to indicate the relation of our analysis with the study given in [11], i.e., the scalar variable \( r \) appearing in our ansatz is a solution of the following eigenvalue problem

\[
\left( \begin{array}{cc}
-T^{-1} R T & T^{-1} R T \\
R T^{-1} R T - \tilde{Q} & -R T^{-1}
\end{array} \right) \left( \begin{array}{c}
d \\
\tilde{b}
\end{array} \right) = r \left( \begin{array}{c}
d \\
\tilde{b}
\end{array} \right) \quad \Leftrightarrow \quad N \tilde{V} = r \tilde{V},
\]

where \( N \in \mathbb{R}^{6 \times 6} \) is called the Stroh matrix [67, 11].

For Cosserat elastic materials [11] it is possible to find a suitable structure of the characteristic equation corresponding to the above eigenvalue problem in the form\(^7\)

\[
r^6 + P_1 r^4 + P_2 r^2 + P_3 = 0,
\]

where

\[
P_1 = C_1 + C_2 + C_3,
\]

\[
P_2 = C_1 C_2 + C_2 C_3 + C_3 C_1 - \frac{4 \rho \mu_c^2 v^2}{\mu e L_c^2 \gamma (\mu e + \mu c)^2},
\]

\[
P_3 = C_1 C_2 C_3 - \frac{4 \rho \mu_c^2 v^2}{\mu e L_c^2 \gamma (\mu e + \mu c)^2} C_1,
\]

\[
c_t = 1 - \frac{\rho v^2}{2 \mu e + \lambda e}, \quad c_l = 1 - \frac{\rho v^2}{\mu e + \mu c}, \quad c_m = 1 + \frac{4 \mu e \mu_c}{\mu e L_c^2 \gamma (\mu e + \mu c)} - \frac{\rho j \mu_c \tau_c^2 v^2}{\mu e L_c^2 \gamma}.
\]

Then, due to this structure, it is possible to factorise it as

\[
(r^2 + c_t)(r^2 + c_l)(r^2 + c_m) - \frac{4 \rho \mu_c^2 v^2}{\mu e L_c^2 \gamma (\mu e + \mu c)^2} = 0,
\]

and to find the analytical form of its solutions

\[
r_1^2 = -c_t,
\]

\[
r_2^2 = \frac{1}{2} \left[ -(c_t + c_m) + \sqrt{(c_t + c_m)^2 + \frac{8 \rho \mu_c^2 v^2}{\mu e L_c^2 \gamma (\mu e + \mu c)^2}} \right],
\]

\[
r_3^2 = \frac{1}{2} \left[ -(c_t + c_m) - \sqrt{(c_t + c_m)^2 + \frac{8 \rho \mu_c^2 v^2}{\mu e L_c^2 \gamma (\mu e + \mu c)^2}} \right].
\]

Since the explicit analytical form of the roots \( r_v \) of the characteristic equation (4.7) as function of \( v \) are known, in [11] it is shown that

\(^7\)We have rewritten everything in the notation of the present paper.
Proposition 4.1. If the constitutive coefficients satisfy the conditions
\[ 2\mu_e + \lambda_e > 0, \quad \mu_e > 0, \quad \alpha_1 + \alpha_2 > 0, \quad (4.11) \]
then the roots \( r_e \) of the characteristic equation (4.7) are not real if and only if
\[ 0 \leq v < \min \{ c_p, c_s, c_{m1}, c_{m2} \}, \quad (4.12) \]
where
\[ c_p = \sqrt{\frac{2\mu_e + \lambda_e}{\rho}}, \quad c_{m1} = \sqrt{\frac{\mu_e L_e^2 \gamma}{\rho j \mu_e \tau_e^2} \left[ 1 + \frac{4\mu_e \mu_e}{\mu_e L_e^2 \gamma k^2 (\mu_e + \mu_e)} \right]}, \]
\[ c_s = \sqrt{\frac{\mu_e + \mu_e}{\rho}}, \quad c_{m2} = \sqrt{\frac{1}{2} \left( c_s^2 + c_{m1}^2 + \frac{4\mu_e^2}{k^2 \rho j \mu_e \tau_e^2 (\mu_e + \mu_e)} - \sqrt{\Delta} \right) + \left( \frac{4\mu_e^2}{k^2 \rho j \mu_e \tau_e^2 (\mu_e + \mu_e)} \right)^2 > 0. \quad (4.13) \]

Remark 4.2. The inequalities (4.11) from Proposition 4.1 have the following interpretations:

i) the first two inequalities of the set of conditions (1.3) considered by Chirita and Ghiba [11] imply that the translational compressional wave is real and that the shear-rotational wave (optical branch) is real at high frequencies;

ii) the first inequality also implies that at the limit of high frequencies the translational compressional wave is faster than the shear-rotational wave (if they both exist);

iii) the third one means that the shear-rotational wave (acoustic branch) is real at high frequencies.

In addition,

iv) the inequalities (1.3) do not imply that the shear-rotational wave (optical branch) is real at low frequencies, i.e., \( \omega \to 0 \);

v) the inequalities (1.3) do not imply that the plane waves are real, i.e., that the propagating plane waves are defined only by real frequencies.

Notice that we have slightly changed the results obtained in [11], and we do not assume that \( c_p \geq c_s \) because all the calculations from [11] are indeed valid without this additional assumption. Actually, the inequality \( c_p \geq c_s \) holds true once the internal energy is assumed to be positive definite, i.e., when \( 2\mu_e + 3\lambda_e > 0, \mu_e > 0 \). However, as we show in the present paper, the existence of the seismic waves is true for weaker conditions on the constitutive parameters.

We also mention that under the hypothesis of the above proposition, the following inequalities are valid
\[ c_{m2} < c_{m1}, \quad \text{and} \quad c_{m2} < c_s, \quad (4.14) \]
since
\[ -\sqrt{\Delta} \leq - \left( c_s^2 - c_{m1}^2 \right) + \left( \frac{4\mu_e^2}{k^2 \rho j \mu_e \tau_e^2 (\mu_e + \mu_e)} \right), \]
\[ -\sqrt{\Delta} \leq - \left( c_{m1}^2 - c_{m2}^2 \right) + \left( \frac{4\mu_e^2}{k^2 \rho j \mu_e \tau_e^2 (\mu_e + \mu_e)} \right), \]
\[ c_{m2}^2 - c_{m1}^2 = \frac{1}{2} \left( c_s^2 - c_{m1}^2 + \frac{4\mu_e^2}{k^2 \rho j \mu_e \tau_e^2 (\mu_e + \mu_e)} - \sqrt{\Delta} \right) < 0, \quad (4.15) \]
\[ c_{m2}^2 - c_s^2 = \frac{1}{2} \left( c_{m1}^2 - c_s^2 + \frac{4\mu_e^2}{k^2 \rho j \mu_e \tau_e^2 (\mu_e + \mu_e)} - \sqrt{\Delta} \right) < 0. \]
Therefore, if the constitutive parameters satisfy the conditions 4.11, then the speeds \( c_i, i = 1, 2, 3, 4 \) defined in (4.13), are ordered as follows
\[
\epsilon_{m_2} < \min\{\epsilon_{m_1}, \epsilon_s\}.  \tag{4.16}
\]
This inequality does not mean that the conditions 4.11 impose (or is in contradiction with) a sort of a priori order of Eringen-type (2.40) between the speed of microscopic real waves and the speed of macroscopic real waves, since in the definition of \( \epsilon_{m_2} \) both the microscopic and macroscopic constitutive parameters are involved.

Using this last remark and combining Proposition 4.1 and Proposition 3.3, we conclude

**Proposition 4.3.** For \( 2\mu_e + \lambda_e > 0, \mu_e > 0, \mu_e > 0, \mu_e L_e^2 \gamma > 0 \), the limiting speed in Cosserat elastic materials is given by

\[
\hat{v} := \inf_{\theta \in (-\pi, \pi]} v_\theta \equiv \min \{\epsilon_p, \epsilon_{m_2}\}.  \tag{4.17}
\]

The usual Stroh algorithm to construct the solution of the seismic wave propagation problem is based on the possibility to explicitly know the analytical form of the solution \( r_v \) of equation (4.7) as function of \( v \). Then, the next step in the common methods is to let \( r_k, k = 1, 2, 3 \) be the eigenvalues that satisfy (4.7) and the associated eigenvector are given by \( \lambda^{(k)} = \left( d^{(k)} \right) \) with

\[
\begin{align*}
\lambda^{(1)} &= \left( \frac{1}{k} \sqrt{2r_1} \right), &\lambda^{(2)} &= \left( \frac{2\mu_e}{(\mu_e + \mu_c)k^2} r_2 \right), &\lambda^{(3)} &= \left( \frac{2\mu_e}{(\mu_e + \mu_c)k^2} r_3 \right), \\
\lambda^{(2)} &= \left( 2\mu_e + \lambda_e \right) \left( r_2^2 + c_1 \right), &\lambda^{(3)} &= \left( \frac{2\mu_e}{(\mu_e + \mu_c)k^2} r_3 \right).
\end{align*}  \tag{4.18}
\]

Without loss of generality, we may assume that the \( r_k \) are distinct and, in consequence, \( \lambda^{(k)} \) are linearly independent\(^8\) [11]. Then a general solution of (2.46) having a proper decay is taken in the form

\[
z = \sum_{k=1}^{3} q_k d^{(k)} e^{i r_k x},  \tag{4.19}
\]

where \( q_k \) are constants to be determined by the boundary condition (3.3)\_2, i.e., we will have the system

\[
\sum_{k=1}^{3} q_k b^{(k)} = B q = 0,  \tag{4.20}
\]

where

\[
B = (b^{(1)} \mid b^{(2)} \mid b^{(3)}), \quad b^{(1)} = p_k \tau^{(k)} + R^{(k)} d^{(k)}, \quad q = (q_1, q_2, q_3)^T.
\]

We have nonzero solutions \( q \) if and only if

\[
det B = 0.  \tag{4.21}
\]

We remark that the expression of \( B \) contains \( v \), so the condition \( det B = 0 \) is in fact a condition to determine \( v \) and it is called the secular equation. Its explicit form \( \forall v \in [0, \hat{v}] \) was determined in [11] in the following form

\[
s(v) = \sqrt{P_3(v) \left( C_2(v) + C_3(v) + 2\sqrt{P(v)} \right) 4\mu_e^2 - \left( C_3(v) + \sqrt{P(v)} \right) \left[ \lambda_e - (2\mu_e + \lambda_e) C_1(v) \right]^2} = 0,  \tag{4.22}
\]

where

\[
P(v) = P_2(v) - C_1(v) C_2(v) - C_1(v) C_3(v)  \tag{4.23}
\]

and the notations (4.8) are used. In [11, Eq. (4.14)] the following result is established

\(^8\)If \( r_k \) are not distinct, then we would find \( d^{(k)} \) eigenvectors associated to \( r_k \) which are independent.
Theorem 4.4. If the constitutive coefficients satisfy
\[ \mu_e - \mu_c + \lambda_e > 0, \quad \mu_c + \mu_e > 0, \quad \alpha_1 + \alpha_2 > 0 \] (these conditions imply \( 2 \mu_e + \lambda_e > 0 \)) \hspace{1cm} (4.24)
then, the secular equation (4.22) has an admissible solution, i.e., a solution \( v \) such that \( 0 \leq v < \hat{v} \).

Remark 4.5.

- The very important aspect in analysing the secular equation is to prove that it has an admissible solution, i.e., there exists at least one solution \( v \) such that \( 0 \leq v < \hat{v} \), since otherwise the constructed wave ansatz does not satisfy the asymptotic decay condition (2.46). This is the essential condition in modelling seismic waves and it will validate or not the entire approach used for the construction of the solution.

- As always in the modelling process, the uniqueness of the desired solution is also a very important aspect, because if the uniqueness is not clearly stated then the question of which solution has to be effectively chosen arises.

However, in almost all studies concerning the propagation of seismic waves in generalized theories of solid mechanics, the complete study of the existence and uniqueness problem of an admissible solution of the corresponding secular equation is often left unsolved. This is also true for the case of the linear Cosserat theory. This question is completely settled by the present paper.

5 The new secular equation. Existence and uniqueness of Rayleigh waves

5.1 Derivation and matrix analysis of the algebraic Riccati equation

The main ingredient of the method used by Fu and Mielke [24] is to look at (3.8) as an initial value problem and to search for a solution in the form
\[ y(x_2) = e^{-k x_2} \mathcal{E} y(0), \] \hspace{1cm} (5.1)
where \( \mathcal{E} \in \mathbb{C}^{3 \times 3} \) is to be determined. On substituting (5.1) into (3.12), we get
\[ [\mathcal{T} \mathcal{E}^2 - i (\mathcal{R} + \mathcal{R}^T) \mathcal{E} - \mathcal{Q} + k^2 v^2 \mathbb{1}] y(x_2) = 0, \quad (-\mathcal{T} \mathcal{E} + i \mathcal{R}^T) y(0) = 0. \] \hspace{1cm} (5.2)
It is clear that in order to have a proper decay the eigenvalues of \( \mathcal{E} \) have to be such that their real part is positive. We anticipate and we mention that this will be the case if \( 0 \leq v < \hat{v} \), as we will see in Theorem 5.2.

For the linear Cosserat model, we introduce the so called surface impedance matrix
\[ \mathcal{M} = -(-\mathcal{T} \mathcal{E} + i \mathcal{R}^T) \iff \mathcal{E} = \mathcal{T}^{-1}(\mathcal{M} + i \mathcal{R}^T). \] \hspace{1cm} (5.3)
It seems that this matrix was first introduced by Ingebrigsten and Tonning [31] for the classical elastic model, by Mielke and Sprenger [48] on a topic indirectly connected to the surface wave problem, and then by Fu and Mielke [47, 24] in order to prove the existence and uniqueness of the surface-wave speed for linear anisotropic elastic materials. We may argue the utility of this replacement of \( \mathcal{E} \) to \( \mathcal{M} \) in order to convert the equation (5.2) into an equation for a Hermitian matrix \( \mathcal{M} \). On substituting (5.3) into (5.2), we obtain
\[ \{\mathcal{T} \mathcal{E} - i (\mathcal{R} + \mathcal{R}^T)\}\mathcal{E} - \mathcal{Q} + k^2 v^2 \mathbb{1} = 0, \]
\[ (\mathcal{M} + i \mathcal{R}^T) \mathcal{T}^{-1}(\mathcal{M} + i \mathcal{R}^T) - i \mathcal{R}^T \mathcal{T}^{-1}(\mathcal{M} + i \mathcal{R}^T) - \mathcal{Q} + k^2 v^2 \mathbb{1} = 0, \] \hspace{1cm} (5.4)
Hence, in terms of the surface impedance matrix, equations (5.2) become
\[ (\mathcal{M} - i \mathcal{R}) \mathcal{T}^{-1}(\mathcal{M} + i \mathcal{R}^T) - \mathcal{Q} + k^2 v^2 \mathbb{1} = 0, \quad \mathcal{M} y(0) = 0. \] \hspace{1cm} (5.5)
We call equation (5.5) the algebraic Riccati equation for the linear Cosserat model. For the classical anisotropic model, the same form of this equation is present in the paper by Mielke and Sprenger [48] for \( v = 0 \).
and in the papers by Fu and Mielke [47, 24] in the case of general $v$, see also the earlier work by Biryukov [7]. For this reason, our expectation is that the entire approach presented by Fu and Mielke [47, 24] should be suitable in the Cosserat theory, too.

Since we are interested in a nontrivial solution $y$, we impose $y(0) \neq 0$, so that the matrix $\mathbf{M}$ has to satisfy

$$
\det \mathbf{M} = 0.
$$

The equation (5.6) is called the **secular equation** for the linear Cosserat model in terms of the **impedance matrix** $\mathbf{M}$.

In this point of our analysis we do not have additional informations about the properties of the impedance matrix $\mathbf{M}$. Looking at the equations (5.5) and (5.6) we remark that equation (5.5)$_1$ leads to a mapping $v \mapsto \mathbf{M}_v$, where $\mathbf{M}_v$ is the solution of (5.5)$_1$ for a fixed $v$. If it is possible to construct this mapping, then equation (5.6) becomes the equation which determines the wave speed $v$. However, the construction of this mapping is possible only if we are very careful with the following aspects. The first yet unresolved problem is: does the solution of the other equation (5.6), with $\mathbf{M}_v$ expressed as function of $v$, belong (if it exists) to the domains of those $v$ for which (5.5)$_1$ admits a unique solution. The second question is: does the solution of the other equation (5.6), with $\mathbf{M}_v$ expressed as function of $v$, belong (if it exists) to the domains of those $v$ for which (5.5)$_1$ admits a solution? The last but not the least important aspect is that it is not sufficient to prove that there is a solution $\mathbf{M}_v$ of the Riccati equation (5.5), since an acceptable $\mathbf{M}_v$ has to be such that $\Re(\text{spec} \, \mathbf{E})$ is positive, where $\Re(\text{spec} \, \mathbf{E})$ means the “real part of spectra of $\mathbf{E}$”.

Similar arguments as for the classical linear anisotropic elastic materials lead us to the following

**Lemma 5.1.** If the constitutive coefficients satisfy the conditions (2.55) and $0 \leq v < \hat{v}$, then the eigenvalues of any solution of (5.5) have a non-zero real part.

**Proof.** Let $\mathbf{E}$ be a solution of (5.5), $\lambda$ be an eigenvalue and $a$ an associated eigenvector, i.e. $\mathbf{E} a = \lambda a$. Hence, if $\mathbf{E}$ is a solution of (5.5), then

$$
\{\lambda^2 \mathbf{T} - i \lambda (\mathbf{R} + \mathbf{R}^T) \mathbf{E} - \mathbf{Q} + k^2 v^2 \mathbf{I}\} d = 0,
$$

and $\lambda$ is a solution of the equation

$$
\det \{\lambda^2 \mathbf{T} - i \lambda (\mathbf{R} + \mathbf{R}^T) \mathbf{E} - \mathbf{Q} + k^2 v^2 \mathbf{I}\} = 0.
$$

Thus, $\lambda = i \lambda$ is solution of the equation

$$
\det \{-r^2 \mathbf{T} - r (\mathbf{R} + \mathbf{R}^T) \mathbf{E} - \mathbf{Q} + k^2 v^2 \mathbf{I}\} = 0 \iff \det \{r^2 \mathbf{T} + r (\mathbf{R} + \mathbf{R}^T) \mathbf{E} + \mathbf{Q} - k^2 v^2 \mathbf{I}\} = 0.
$$

Assuming that there exists an eigenvalue $\lambda$ of a solution $\mathbf{E}$ of (5.5) with a non-zero real part, it follows that the equation

$$
\det \{r^2 \mathbf{T} + r (\mathbf{R} + \mathbf{R}^T) \mathbf{E} + \mathbf{Q} - k^2 v^2 \mathbf{I}\} = 0
$$

admits a non-real solution. But we have shown in Proposition 3.3 that this is not possible if the wave speed is smaller than the limiting speed $\hat{v}$. ■

Under the assumption that $\mathbf{T}$ and $\mathbf{Q}$ are symmetric and positive definite matrices, many aspects from the above discussions are purely mathematical questions, and they are not specific to Cosserat elastic materials. Notice that we were careful to obtain a specific formulation such that many mathematical results obtained by Mielke and Fu [47, 24] can be directly applied in the Cosserat theory, too. For instance, since $\mathbf{T}$, $\mathbf{Q}$ and $\hat{\mathbf{I}}$ are symmetric real matrices, the following result established in [47, 24] remains valid in the framework of the Cosserat elastic materials

**Theorem 5.2.** If the constitutive coefficients satisfy the conditions (2.55) and $0 \leq v < \hat{v}$, the matrix problem

$$
\mathbf{T} \mathbf{E}^2 - i (\mathbf{R} + \mathbf{R}^T) \mathbf{E} - \mathbf{Q} + k^2 v^2 \mathbf{I} = 0, \quad \Re(\text{spec} \, \mathbf{E}) > 0,
$$

has a unique solution for $\mathbf{E}_v$ and the corresponding matrix $\mathbf{M}_v$ obtained from (5.3)$_2$ is Hermitian.

**Proof.** The reader may consult the paper by Fu and Mielke [24] or the Appendix, where we have rewritten the proof in our notation. ■
The new matrices $M$ has a unique subsonic solution and their proof is based on the following properties of the impedance matrix $\hat{M}$, answer, the analyses would be incomplete.

Moreover, according to the definition of the limiting speed, regarding Proposition 3.3 and Proposition 3.6, the surface impedance matrix $M_v$ from the classical linear elastic model, that hold for a subsonic solution $v$:

1. The surface impedance matrix $M_v$ is Hermitian,
2. The matrix $\frac{dM_v}{dv}$ is negative definite,
3. $\langle M_v, \mathbb{1} \rangle \geq 0$, and $\langle w, M_v w \rangle \geq 0$ for all real vectors $w$.

In the following we show that for subsonic wave speeds $v$, the impedance matrix $M_v$, solution of (5.5), satisfies the above properties 1–3 for isotropic elastic Cosserat materials, too. It is straightforward to prove the second property using the same arguments as in the proof given by Fu and Mielke [24] in the context of anisotropic classical elastic materials, since the main problem is in fact a purely mathematical question, independent of the considered theory. Indeed, the proof remains unchanged in the context of isotropic elastic Cosserat materials considered in our paper.

**Theorem 5.3.** Assume the constitutive coefficients satisfy the conditions (2.55) and $0 \leq v < \hat{v}$. Let $M_v$ and $E_v$ be the same as in the conclusion of Theorem 5.2. Then the matrix $\frac{dM_v}{dv}$ is negative definite.

**Proof.** The reader may consult the paper by Fu and Mielke [24] or the Appendix, where we have rewritten the proof in our notations. ■

In the following, we prove that the impedance matrix $M_v$ satisfies also the 3rd property. To this aim we follow again Fu and Mielke’s technique [24] and we show that this method is also applicable to the linear isotropic elastic Cosserat model. Hence, in order to establish the 3rd property, we first define matrices $\tilde{Q}_\theta, \mathcal{T}_\theta$ and $\mathcal{R}_\theta \in \mathbb{R}^{3 \times 3}$ by

$$
\begin{bmatrix}
\tilde{Q}_\theta & \mathcal{R}_\theta \\
\mathcal{R}_\theta^T & \mathcal{T}_\theta
\end{bmatrix}
= \begin{bmatrix}
\cos \theta \mathbb{1} & \sin \theta \mathbb{1} \\
-\sin \theta \mathbb{1} & \cos \theta \mathbb{1}
\end{bmatrix}
\begin{bmatrix}
\tilde{Q} & \mathcal{R} \\
\mathcal{R}^T & \mathcal{T}
\end{bmatrix}
\begin{bmatrix}
\cos \theta \mathbb{1} - \sin \theta \mathbb{1} \\
\sin \theta \mathbb{1} & \cos \theta \mathbb{1}
\end{bmatrix},
$$

(5.13)

where we write $\tilde{Q} = Q - kv^2 \mathbb{1}$, and $\theta$ is an arbitrary angle. These matrices may be seen as the counterpart of $\mathcal{T}, \tilde{Q}$ and $\mathcal{R}$ and they are obtained by rotation of the old coordinate system about $e_3$ by an angle $\theta$

$$
\mathcal{T}_\theta = \cos^2 \theta \mathcal{T} - \sin \theta \cos \theta (\mathcal{R} + \mathcal{R}^T) + \sin^2 \theta \tilde{Q},
$$
$$
\mathcal{R}_\theta = \cos^2 \theta \mathcal{R} + \sin \theta \cos \theta (\mathcal{T} - \tilde{Q}) - \sin^2 \theta \mathcal{R}^T,
$$
$$
\tilde{Q}_\theta = \cos^2 \theta \tilde{Q} + \sin \theta \cos \theta (\mathcal{R} + \mathcal{R}^T) + \sin^2 \theta \mathcal{T}.
$$

(5.14)

The new matrices $\mathcal{T}$ and $Q$ remain symmetric and $\tilde{Q}_\theta, \mathcal{T}_\theta$ and $\mathcal{R}_\theta$ are periodic in $\theta$ with periodicity $\pi$ and $\pi/2$

$$
\tilde{Q}_\theta(\theta + \frac{\pi}{2}) = \mathcal{T}_\theta, \quad \mathcal{R}_\theta(\theta + \frac{\pi}{2}) = -\mathcal{R}_\theta^T, \quad \mathcal{T}_\theta(\theta + \frac{\pi}{2}) = \tilde{Q}_\theta.
$$

(5.15)

Moreover, according to the definition of the limiting speed, regarding Proposition 3.3 and Proposition 3.6, the limiting velocity $\hat{v}$ is in fact the lowest velocity for which the matrices $Q_\theta$ and $T(\theta)$ become singular for some angle $\theta$ and $Q_\theta$ is positive definite for $0 \leq v < \hat{v}$. In view of (5.15) so is $\mathcal{T}_\theta$, too. Thus by the definition of the
limiting speed \( \hat{v} \) both \( \overline{Q}_\theta \) and \( T_\theta \) are positive definite or positive semi-definite depending on \( \theta \) (for \( v = \hat{v} \) there is at least one \( \theta \) at which \( T(\theta) \) has an eigenvalue 0, and likewise \( \overline{Q}_\theta \)).

We shall use \( E \) exclusively to denote the unique solution of (5.2), and likewise we define \( E_\theta \) to be the unique solution of the matrix problem

\[
T_\theta E_\theta^2 - (\mathcal{R}_\theta + \mathcal{R}_\theta^T) E_\theta - \overline{Q}_\theta = 0 \quad \text{Re spec } E_\theta > 0, \tag{5.16}
\]

while the matrix \( M_\theta \) has a form such that

\[
E_\theta = T_\theta^{-1} (M_\theta + i \mathcal{R}_\theta^T). \tag{5.17}
\]

In terms of the matrix \( M_\theta \) the equation (5.16) reads

\[
(M_\theta - i \mathcal{R}_\theta) T_\theta^{-1} (M_\theta + i \mathcal{R}_\theta^T) - \overline{Q}_\theta^T = 0. \tag{5.18}
\]

The following results established in [24] remain valid in our framework, too.

**Theorem 5.4.** Assume the constitutive coefficients satisfy the conditions (2.55) and \( 0 \leq v < \hat{v} \). Then,

i) The Hermitian matrix \( M_\theta \) defined above is independent of \( \theta \).

ii) Denoting by \( M \) and \( E \) the corresponding values of \( M_\theta \), and \( E_0 \) for \( \theta = 0 \), respectively, then

\[
E_\theta = (\cos \theta E + i \sin \theta) E^{-1}. \tag{5.19}
\]

iii) \( \int_0^\pi E_\theta \, d\theta = \pi I \). \( \Box \)

Proof. The reader may consult the paper by Fu and Mielke [24] or the Appendix, where we have rewritten the proof in our notations.

From a computational point of view, the decisive advantage given by the above result is that, since \( T_\theta \) and \( \mathcal{R}_\theta \) depend on the wave speed \( v \), we obtain the explicit form of the secular equation \( \det M_v = 0 \) without a priori knowing the analytical expressions (as function of the wave speed ) of the eigenvalues that satisfy (3.13) and the associated eigenvector \( d^{(k)} \), which is the main difficulty in almost all the generalised models, with exception of some models for which this task is straightforward, e.g., classical isotropic linear elasticity [1, 30], or the theory of materials with voids [57, 10, 62] after imposing restrictive conditions upon the constitutive coefficients. After the secular equation is solved, the task of finding the eigenvalues that satisfy (3.13) and the associated eigenvector \( d^{(k)} \) becomes a purely numerical task, avoiding symbolic (analytical) computations.

### 5.2 The main result: Existence and uniqueness of Rayleigh waves

Moreover, Fu and Mielke’s method has another advantage since we are able to show the existence and uniqueness of a subsonic wave speed, solution of the secular equation, which ensures in the end that there exists an acceptable \( M_v \), such that \( \text{Re spec } E \) is positive, i.e. the solution satisfies both the boundary conditions (2.45) and the decay conditions (2.46). As we will explain in the following, the matrix \( M_v \), determined by (5.19) satisfies the condition 3., i.e. \( \text{tr}(M_v) \geq 0, \) and \( \langle w, M_v w \rangle \geq 0 \) for all real vectors \( w \), and this will imply the existence of a unique subsonic solution of the secular equation.

---

9This solution exists since \( T_\theta \) and \( \overline{Q}_\theta \) are positive definite. Then, Theorem 5.2 will be used since it is valid for \( v = 0 \), too.
Theorem 5.5. [The main result of this paper] Assume the constitutive coefficients satisfy the conditions
\[ 2\mu_e + \lambda_e > 0, \quad \mu_e > 0, \quad \mu_c > 0, \quad \alpha_1 + \alpha_2 > 0, \]
then the secular equation
\[ \det \mathcal{M}_v = 0, \]
where $\mathcal{M}_v$ is given by (5.19), has a unique admissible solution $0 \leq v < \hat{v}$. In other words there exists a unique Rayleigh wave propagating in the Cosserat medium.

Proof. First, we explain why, if $\mathcal{E}_v$ solves (5.11), then the corresponding $\mathcal{M}_v$ obtained from (5.3) has the following properties

1. $\mathcal{M}_v$ is Hermitian,
2. $\frac{d\mathcal{M}_v}{dv}$ is negative definite,
3. $\text{tr}(\mathcal{M}_v) \geq 0$, and $\langle w, \mathcal{M}_v w \rangle \geq 0$ for all real vectors $w$ for all $0 \leq v \leq \hat{v}$,
4. $\mathcal{M}_v$ is positive definite for all $0 \leq v < \hat{v}$.

To this aim, we can use the arguments explained in [47, page 13]. Since due to Theorem 5.2 we know that $\mathcal{M}_v$ is Hermitian, $H_v$ and $S_v$ are both real matrices and $H_v$ is symmetric, it follows that $H_v^{-1}S_v$ is skew-symmetric. Hence, $\text{tr}(\mathcal{M}_v) = \text{tr}(H_v^{-1})$ and $\langle w, \mathcal{M}_v w \rangle = \langle H_v^{-1} w, w \rangle \forall w \in \mathbb{R}^3$, since $H_v^{-1}S_v$ is skew-symmetric. Since $H_v^{-1}$ is positive semi-definite for all $0 \leq v \leq \hat{v}$ and positive definite for all $0 \leq v < \hat{v}$, it follows that $\mathcal{M}_v$, determined by (5.19), satisfies the condition 3 and moreover $\mathcal{M}_v$ is positive definite for all $0 \leq v < \hat{v}$. Note that at $v = \hat{v}$ at least one of the eigenvalues of $H_v^{-1}$ must vanish.

Figure 2: A plot of $\det \mathcal{M}_v$ with respect to the surface waves speed $v$ for the aluminum-epoxy composite for a set of equidistant values in the interval $[0, \hat{v})$. This curve illustrates that $\det \mathcal{M}_v$ is a decreasing function of the wave speed $v$.

The rest of the proof is clearly explained in [24, page 2531] as in the following. Since $\frac{d\mathcal{M}_v}{dv}$ is negative definite, the eigenvalues of $\mathcal{M}_v$ are monotone decreasing functions of $v$ defined on $[0, \hat{v})$. Let us remark that at $v = 0$ the eigenvalues of $\mathcal{M}_0$ are positive, since $\mathcal{M}_0$ is positive definite at $v = 0$, that $\det \mathcal{M}_v = \lambda_1 \lambda_2 \lambda_3$, where $\lambda_1, \lambda_2, \lambda_3$ denote the eigenvalues of $\mathcal{M}_v$, that $\det \mathcal{M}_0 > 0$ and that the map $v \mapsto \det \mathcal{M}_v$ is monotone decreasing on $[0, \hat{v})$, too. Thus, there exists a solution of the secular equation $\det \mathcal{M}_v = 0$ only if an eigenvalue of $\mathcal{M}_v$ decreases at zero at $0 < v = v_R < \hat{v}$. Moreover, if such a $v_R$ exists it is unique, in the sense that only one eigenvalue of $\mathcal{M}_v$ may decrease to zero for a value $0 < v = v_R < \hat{v}$ of the wave speed, since if two eigenvalues would share this property, then at $v = \hat{v}$ the matrix $\mathcal{M}_v$ should have two\footnote{Note that $H_v$ is not well defined for $v = \hat{v}$ since for this value of the wave speed there exists an angle $\theta \in [0, \pi]$ such that $\mathcal{T}_v$ has a zero eigenvalue. However, $H_v^{-1}$ is well-defined for the limit $v \to \hat{v}$ and at this limit it admits a zero eigenvalue, since considering the contrary it follows that $H_v$ is well-defined for the limit $v \to \hat{v}$. This will imply that $\mathcal{T}_v^{-1}$ is defined for the limit $v \to \hat{v}$ and that $\mathcal{T}_v$ and $Q_v$ are positive definite for the limit $v \to \hat{v}$, too, a fact that is contrary to the property of the limiting speed.} eigenvalues which are

negative which will violate the positive semi-definiteness\(^{12}\) of \(H_c^{-1}\) at \(v = \hat{v}\) since at least one eigenvalue of \(H_c^{-1}\) at \(v = \hat{v}\) is zero. In the same manner we argue that at \(v = v_R\) zero there is not a repeated eigenvalue of \(\mathcal{M}_v\). We conclude the proof by pointing out that there exists a unique \(v_R\) such that an eigenvalue of \(\mathcal{M}_v\) decreases to zero at \(0 < v = v_R < \hat{v}\), and therefore that \(\det \mathcal{M}_{v_R} = 0\), while \(\det \mathcal{M}_v > 0\) for all \(0 < v < v_R\) and \(\det \mathcal{M}_v < 0\) for all \(v_R < v < \hat{v}\).

Here, we also illustrate the statements from the above paragraph numerically for the aluminum-epoxy composite considered by Gauthier [26] and Eringen [20]. Since the integral representation (5.19) is an explicit expression for the surface impedance matrix, we simply increase \(v\) in small steps from 0 to the determined \(\hat{v}\) at every step. Plotting the secular equation \(\det \mathcal{M}_v\) with respect to the wave speed \(v\) gives the curve shown in Figure 2. We observe how simple it is to find an approximation of the wave speed, once the integral representation (5.19) is given.

Comparing with the left hand side of the secular equation presented in [11, Eq. (4.7)] the left hand side of our new secular equation involve a strictly decreasing function, see Figure 2 and [11, Fig. 1]. Moreover, we have analytically proven that the left hand side defines a decreasing function, see Figure 2 and [11, Fig. 1].

In our analysis we have considered \(L_c > 0\). Thus, the results are valid for the full Cosserat medium. Part of our results and estimates may be immediately applicable to the case of the reduced Cosserat model (considered in [27, 35]) by simply taking \(L_c \to 0\). However, since we do not know explicitly how the analytical form of the solution \(v\) of the secular equation depends on \(L_c\), we are not able to obtain the form of the solution \(v\) for the reduced Cosserat model by simply letting \(L_c \to 0\) in the expression of the solution \(v\) for the full Cosserat model. But, all our calculations may be adapted to the case \(L_c = 0\).

6 Numerical implementation

In this section we consider \(k = 1\) mm\(^{-1}\) and the constitutive coefficients obtained by Gauthier [26], see also [20], for aluminum-epoxy composite. According to [20, pages 164-165], in the Eringen notations [28], for such a material we have

\[
\lambda_{\text{Eringen}} = 7.59\,\text{GPa}, \quad \mu_{\text{Eringen}} = 1.89\,\text{GPa}, \quad \kappa_{\text{Eringen}} = 0.00788\mu_{\text{Eringen}},
\]

while in our notation the same constitutive coefficients are represented by

\[
\lambda_c = 7.59\,\text{GPa}, \quad \mu_c = \frac{\kappa}{2} = 0.0074466\,\text{GPa}, \quad \mu_c = \mu_{\text{Eringen}} + \frac{\kappa}{2} = 1.89745\,\text{GPa},
\]

\[
\rho = 2.22287\,\frac{g}{\text{mm}^3}, \quad \mu_c \tau_c^2 = 0.0196\,\text{mm}^2, \quad \mu_c L_c^2 \gamma = 0.263383\,\text{GPa} \times \text{mm}.
\]

According to the results presented in the previous sections, for known numerical values of all constitutive parameters, we identify an algorithm to approximate numerically the problem of the propagation of seismic waves:

I. A first algorithm:

Step 1: Identify the limiting speed \(\hat{v}\). Since the constitutive coefficients satisfy (4.11), using Proposition 4.1 and after direct substitution in its analytical form, we find the value of the limiting speed \(\hat{v} = 0.925507\). Let us notice that it defers from that established in [11], because after reverification we have remarked that all the numerical computations in [11] are done for other values of the constitutive parameters, which do not match those proposed by Gauthier [26] for aluminum-epoxy composite. Due to a common misunderstanding of the notations, in [11] it is considered that \(\mu_c L_c^2 \gamma = 5.8546\) and \(J = \rho \mu \tau_c^2 = 0.4357\). However, the numerical calculation given in [11] are correct, but for the before mentioned values of \(\mu_c L_c^2 \gamma\) and \(J\). We have repeated the calculations in the sense of the approach given in [11] and we have found a complete agreement with the value of the limiting speed considered in the current paper, i.e., \(\hat{v} = 0.925507\).

\(^{12}\)We recall that the positive semi-definiteness of \(H_c^{-1}\) is equivalent to the positive semi-definiteness of \(\mathcal{M}_v\).
Step 2: Find the solution of the secular equation. We have applied formula (5.19) to compute $\mathcal{M}_v$ on a set of 50 values in the interval $[0, \tilde{v}]$. We have to compute numerically the needed integrals from (5.19) since we did not reach the symbolic values of them. For these values we have computed $\det\mathcal{M}_v$, too. We consider these values of $\det\mathcal{M}_v$ for a set of equidistant values in $[0, \tilde{v})$ and we use interpolation to find an approximation function $v \mapsto f(v)$ of the function $v \mapsto \det\mathcal{M}_v$, on $[0, \tilde{v})$.

After that, we find the root of $f$ on $[0, \tilde{v})$. Since this root given by mathematical software does not lead to a vanishing $\det\mathcal{M}_v$, we are looking in the neighbourhood of this root for a $v$ such that $\det\mathcal{M}_v$ is close to zero. Such a value is $v_R = 0.8730352$. This is the approximate value we have found for the wave speed, i.e. the approximate solution of our secular equation. We work with 7 decimals since the values obtained for the approximations of $\det\mathcal{M}_v$ are very sensitive to small changes of $v$.

We have considered the same coefficients in the secular equation established in [11] and we have remarked, see Figure 3, that both functions defining the secular equation in our form and the form given in [11], respectively, vanish in the same value in the interval $[0, \tilde{v})$. We have approximated also the solution of the secular equation $s(v) = 0$ from [11] and we have found the approximation of the corresponding admissible wave speed to be $0.87296$, which is not far from the approximate wave speed given by our secular equation $\det\mathcal{M}_v = 0$, i.e., $v_R = 0.8730352$. We mention we cannot obtain the precise value of $v_R$ and that a new numerical strategy or a better mathematical software could lead to a better accuracy.

Step 3: Construct the amplitudes and the solution. For $v = v_R$, we find $y(0)$ as solution of the algebraic system (5.5). Then we construct $y$ from (5.1). Finally, we construct the solution $U$ from (3.1).

We find the approximate solution of (5.5) to be

$$y(0) = \begin{pmatrix} \zeta \\ -1.66731i\zeta \\ -0.0120298i\zeta \end{pmatrix}, \quad \zeta \in \mathbb{C}. \quad (6.3)$$

Since $\mathcal{M}_{v_R}$ is approximated by

$$\mathcal{M}_{v_R} = \begin{pmatrix} 1.01413 & -0.608012i & 0.00513355i \\ 0.608012i & 0.365425 & -0.0463072 \\ -0.00513355i & -0.0463072 & 6.00576 \end{pmatrix},$$

the matrix $\mathcal{E}_{v_R} = T^{-1}(\mathcal{M}_{v_R} + i\mathcal{R})$ is numerically approximated by

$$\mathcal{E}_{v_R} = \begin{pmatrix} 1.18322 & 0.282484i & 0.00598908i \\ 0.785418i & 0.0712845 & -0.00904174 \\ 0.00706743i & -0.00766037 & 0.993447 \end{pmatrix} \quad (6.4)$$
and, using (5.1), the function \( y(x_2) \) is determined. Then, using \( y(x_2) := \begin{pmatrix} \frac{1}{\sqrt{\rho}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{\rho}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{j\mu \tau c}} \end{pmatrix}^{-1} z(x_2) \)

we find \( z(x_2) \). In the end, from (3.1) we find the solution

\[
\mathcal{U}(x_1, x_2, t) = \left( \begin{array}{c} u_1(x_1, x_2, t) \\ u_2(x_1, x_2, t) \\ \vartheta_3(x_1, x_2, t) \end{array} \right) = \text{Re} \left[ \left( \begin{array}{c} z_1(x_2) \\ z_2(x_2) \\ i z_3(x_2) \end{array} \right) e^{i k (x_1 - v t)} \right],
\]

(6.5)

represented in Figure 4.

![Figure 4](a) Plot of the \( u_1 \)-component of the displacement. (b) Plot of the \( u_2 \)-component of the displacement. (c) Plot of \( \vartheta_3 \)-component of the micro-rotation vector.

Figure 4: The plot of the solution at time \( t = 1 \) and for the choice \( \varsigma = i \).

In Figure 5 we present the dependence of the wave speed on the wave number in the framework of the linear Cosserat theory and classical linear elasticity. In contrast to classical elasticity where the wave speed does not depend on the wave number, in the linear Cosserat model there is a dispersion curve describing such a dependency. Moreover, it seems that the speed goes asymptotically to a finite value of the wave speed for large values of the wave number. The computations for linear classical elasticity as limit case of the approach which is done in this paper for linear Cosserat elasticity are presented in more details in Section 7.

![Figure 5](a) The upper bound value of the wave speed in linear Cosserat elasticity (dispersion curve) vs. linear classical elasticity (no dispersion) for a linear elastic material having the parameters (\( \lambda = 7.59 \text{ GPa}, \mu = \frac{\tau}{2} = 0.0074466 \text{ GPa}, \mu_c = \mu_{\text{Eringen}} + \frac{\tau}{2} = 1.89745 \text{ GPa}, \rho = 2.22287 \frac{\text{g}}{\text{mm}^3}, j \mu_c \tau_c^2 = 0.0196 \text{ mm}^2, \mu_c L_c^2 \gamma = 0.263383 \text{ GPa} \times \text{mm}) vs. for a linear elastic material having the parameters (\( \lambda = 7.59 \text{ GPa}, \mu = 1.89745 \text{ GPa}, \rho = 2.22287 \frac{\text{g}}{\text{mm}^3} \)) For the considered material parameters, the speed goes to 0.87327989 for large values of the wave speed.

From Figure 6 we observe that the frequency increases as function of the wave number since the wave speed increases as function of the wave number (see Figure 5) and \( \omega(k) = k \nu(k) \). Moreover, since \( 0 \leq \nu(k) < \hat{\nu} \), the
Cosserat classical elasticity
the bound defined by the limiting speed
0.2 0.4 0.6 0.8 1.0
0.2
0.4
0.6
0.8
0.5000 0.5002 0.5004 0.5006 0.5008 0.5010
0.4345 0.4350 0.4355 0.4360 0.4365 0.4370
0.9000 0.9002 0.9004 0.9006 0.9008 0.9010
0.782 0.783 0.784 0.785 0.786 0.787

Figure 6: The dependence of the wave frequency on the wave number for aluminum-epoxy composite ($\lambda_e = 7.59$ GPa, $\mu_e = \frac{\pi}{2} = 0.0074466$ GPa, $\mu_c = \kappa_2 = 0.0074466$ GPa, $\mu_e = \kappa_2 = 1.89745$ GPa, $\rho = 2.22287$ $\text{g mm}^{-3}$) vs. for a linear elastic material having the parameters ($\lambda = 7.59$ GPa, $\mu = 1.89745$ GPa, $\rho = 2.22287$ $\text{g mm}^{-3}$). From the magnified windows and since the orange curve is linear (expressing the dependency on the wave frequency as function of the wave number in classical elasticity) we observe that the wave frequency as function of the wave number is not linear and we have dispersion. However, there are no band-gap. It is clear that dispersion curve lie under (on the right side of) all the dispersion curves of the real plane waves, since the $\omega(k) = k v(k)$ with $0 \leq v(k) < \tilde{v}(k)$ and the limiting speed $\tilde{v}(k)$ represent the minimum slope of all the possible dispersion curve at $k$, for the real plane waves.

II. The second algorithm:

Step 1: Identify the limiting speed $\tilde{v}$: from (4.17) in our case, since this limiting speed was previously found by Chirită and Ghiba in [11].

Step 2: Consider the hermitian matrix $\mathcal{M}_v$ in the form

$$
\mathcal{M}_v = \begin{pmatrix}
\mathcal{M}_1 & \mathcal{M}_3 + i \mathcal{M}_4 & \mathcal{M}_5 + i \mathcal{M}_6 \\
-\mathcal{M}_3 & \mathcal{M}_2 & \mathcal{M}_7 + i \mathcal{M}_8 \\
-\mathcal{M}_5 & \mathcal{M}_6 & \mathcal{M}_7 - i \mathcal{M}_8 & \mathcal{M}_9
\end{pmatrix}
$$

and solve both the Riccati equation and the secular equation

$$
(\mathcal{M} - i \mathcal{R}) T^{-1} (\mathcal{M} + i \mathcal{R}^T) - Q + k^2 v^2 \mathbb{I} = 0,
$$

$$
det \mathcal{M}_v = 0.
$$

With the condition (6.7) this matrix has two positive eigenvalues and one is a zero eigenvalue. Writing separately the real and imaginary part of the equation yields nine quadratic equation for the $\mathcal{M}_i$ ($i = 1,2...9$). Together with the secular equation we have an algebraic system of 10 equations for the 10 unknowns $\mathcal{M}_1, \mathcal{M}_2, ..., \mathcal{M}_9, v$.

i. Solve this nonlinear system.
the group velocity: $\frac{d\omega}{dk}$

**Figure 7:** The dependence of the wave frequency on the wave number for aluminum-epoxy composite ($\lambda_0 = 7.59$ GPa, $\mu_0 = \frac{G}{2} = 0.0074466$ GPa, $\mu_0 = \mu_{\text{Eringen}} + \frac{G}{2} = 1.89745$ GPa, $\rho = 2.22287 \frac{\text{g}}{\text{mm}^3}$, $j \mu_0 \tau_c^2 = 0.0196 \text{mm}^2$, $\mu_0 L_c^2 \gamma = 0.263383$ GPa mm) vs. for a linear elastic material having the parameters ($\lambda_0 = 7.59$ GPa, $\mu = 1.89745$ GPa, $\rho = 2.22287 \frac{\text{g}}{\text{mm}^3}$). From the magnified windows and since the orange curve is linear (expressing the dependency on the wave frequency as function of the wave number in classical elasticity) we observe that the wave frequency as function of the wave number is not linear and we have dispersion. However, there are no band-gap.

**Step 3:**

1. Choose only those solutions $(v, M_v)$ for which $0 < v < \hat{v}$, $\text{tr} M_v > 0$ and $M_v$ is positive definite.

2. Take $(v_R, M_{v_R})$ as solution of the nonlinear system of equations. It has to be only one solution $(v_R, M_{v_R})$ with these properties.

3. To avoid possible numerical errors, check again that the corresponding matrix $E$ computed with (5.3) satisfies $\text{Re } \text{spec } E > 0$.

**Step 4:** Construct the solution $U$ from (3.1).

### 7 From linear Cosserat theory to classical linear elasticity: A consistency check and comparison of the results

In this section we rediscover the results from classical linear elasticity as a limit case of the results obtained in linear Cosserat theory. First, we remark that

$$\|Daxl A\|^2 = 0$$

implies that $axl A = $ constant in $\Omega$,

which corresponds to a time dependent rigid (macroscopic) movement of the entire body. In addition, under Dirichlet homogeneous boundary conditions on $axl A$ we find that $axl A$ vanishes in the entire body at any time.

In fact, by looking at the expression of the total energy, we observe that the energy due to the microrotation is

$$\rho j \mu_0 \tau_c^2 \|(axl A)_t\|^2 + \mu_c \|\text{skew}(D u - A)\|^2$$

and that the situation described by (7.1) is formally equivalent to the case

$$L_c \to \infty,$$
since the total energy has to remain finite. Moreover, assuming also that
\[
\mu_c \to 0 \quad \text{and} \quad \tau_c \to 0, \quad \text{too,}
\] (7.5)
the entire energy due to the microrotation vanishes and we are back in the framework of classical linear elasticity, i.e., the elastic energy density is
\[
W(Du) = \mu_c \|\text{dev}_3 \text{sym } Du\|^2 + \frac{2 \mu_c + 3 \lambda_c}{6} [\text{tr}(Du)]^2.
\] (7.6)

In the following we explain that all the results concerning the propagation of the seismic wave in the framework of Cosserat theory are still valid for Cosserat couple modulus \(\mu_c \to 0\), and we rediscover the results concerning the propagation of the seismic waves from classical linear elasticity without assuming further conditions on the internal length scale \(L_c\) or on the internal time scale \(\tau_c\) (as was the case for real plane waves), since the domain of the admissible speeds is already bounded and cannot reach large values defined by large values of the internal length scale \(L_c\) or small values of the internal time scale \(\tau_c\).

Letting \(\mu_c \to 0\), the matrices involved in the Riccati equation become
\[
T := k^2 \begin{pmatrix}
\frac{\mu_{ec}}{\rho} & 0 & 0 \\
0 & \frac{2 \mu_c + \lambda_c}{\rho} & 0 \\
0 & 0 & \frac{\mu_{ec} c}{\rho} \frac{\tau_c}{\mu_c}
\end{pmatrix}, \quad R := k \begin{pmatrix}
0 & \frac{k \mu_c}{\rho} & 0 \\
\frac{k \mu_c}{\rho} & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad Q := \begin{pmatrix}
k^2 \frac{2 \mu_c + \lambda_c}{\rho} - k^2 \nu^2 & 0 & 0 \\
0 & k^2 \frac{\mu_c}{\rho} - k^2 \nu^2 & 0 \\
0 & 0 & k^2 \frac{\mu_{ec} c}{\rho} \frac{\tau_c}{\mu_c} - k^2 \nu^2
\end{pmatrix}.
\] (7.7)

Thus, we have
\[
T_\theta = \begin{pmatrix}
k^2 \frac{\mu_c \cos^2(\theta) + \lambda_c + 2 \mu_c - \rho \nu^2}{\rho} & \frac{1}{2} \sin(2\theta) \left( k^2 \frac{\lambda_c}{\rho} + \frac{k^2 \mu_c}{\rho} \right) \\
\frac{1}{2} \sin(2\theta) \left( k^2 \frac{\lambda_c}{\rho} + \frac{k^2 \mu_c}{\rho} \right) & 0 \\
0 & 0
\end{pmatrix}, \quad R_\theta = \begin{pmatrix}
0 & \frac{1}{2} \sin(2\theta) \left( k^2 \frac{\lambda_c \cos^2(\theta)}{\rho} - k^2 \mu_{ec} \sin^2(\theta) \right) \\
\frac{1}{2} \sin(2\theta) \left( k^2 \frac{\lambda_c \cos^2(\theta)}{\rho} - k^2 \mu_{ec} \sin^2(\theta) \right) & 0 \\
0 & 0
\end{pmatrix}, \quad Q_\theta = \begin{pmatrix}
0 & \frac{1}{2} \sin(2\theta) \left( k^2 \frac{\lambda_c \cos^2(\theta)}{\rho} + \frac{k^2 \mu_c}{\rho} \right) \\
\frac{1}{2} \sin(2\theta) \left( k^2 \frac{\lambda_c \cos^2(\theta)}{\rho} + \frac{k^2 \mu_c}{\rho} \right) & 0 \\
0 & 0
\end{pmatrix}.
\] (7.8)

Consequently, we deduce
\[
T_\theta^{-1} = \begin{pmatrix}
\rho \left( \frac{\lambda_c}{\rho} \cos^2(\theta) + \frac{1}{2} \mu_{ec} \cos(2\theta) + \frac{\rho \nu^2}{\rho} \sin^2(\theta) \right) & k^2 \frac{\nu^2 \cos^2(\theta) - \mu_{ec \theta} \cos(2\theta) + \frac{\rho \nu^2}{\rho} \sin^2(\theta)}{\rho \sin^2(\Theta)(\lambda_c + \mu_{ec \theta})} \\
\frac{\rho \nu^2 \sin^2(\theta) - \mu_{ec \theta} \cos(2\theta) + \frac{\rho \nu^2}{\rho} \sin^2(\theta)}{\rho \sin^2(\Theta)(\lambda_c + \mu_{ec \theta})} & 0 \\
0 & \frac{\nu^2 \cos^2(\theta) - \mu_{ec \theta} \cos(2\theta) + \frac{\nu^2}{\rho} \sin^2(\theta)}{\nu^2 \cos^2(\theta) - \mu_{ec \theta} \cos(2\theta) + \frac{\nu^2}{\rho} \sin^2(\theta)}
\end{pmatrix}, \quad R_\theta^{-1} = \begin{pmatrix}
\frac{\rho \lambda_c}{\rho} \cos^2(\theta) + \frac{1}{2} \mu_{ec} \cos(2\theta) + \frac{\rho \nu^2}{\rho} \sin^2(\theta) & 0 \\
0 & \frac{\nu^2 \cos^2(\theta) - \mu_{ec \theta} \cos(2\theta) + \frac{\nu^2}{\rho} \sin^2(\theta)}{\nu^2 \cos^2(\theta) - \mu_{ec \theta} \cos(2\theta) + \frac{\nu^2}{\rho} \sin^2(\theta)}
\end{pmatrix}
\] (7.9)

\[
T_\theta^{-1} R_\theta T_\theta^{-1} = \begin{pmatrix}
\frac{\sin(2\theta)}{\rho} \left( \frac{\lambda_c}{\rho} + \frac{\rho \nu^2}{\rho} \sin^2(\theta) \right) + \frac{\rho \nu^2}{\rho} \sin^2(\theta) & 2 \mu_{ec} \cos^2(\theta) + \lambda_c \left( \mu_{ec} \cos(2\theta) + \rho \nu^2 \sin^2(\theta) \right) \\
2 \mu_{ec} \cos^2(\theta) + \lambda_c \left( \mu_{ec} \cos(2\theta) + \rho \nu^2 \sin^2(\theta) \right) & 0 \\
0 & \frac{\sin(2\theta)}{\rho} \left( \frac{\lambda_c}{\rho} + \frac{\rho \nu^2}{\rho} \sin^2(\theta) \right) + \frac{\rho \nu^2}{\rho} \sin^2(\theta)
\end{pmatrix}.
\]
Hence, the matrices $H_v$ and $S_v$ from (5.19) are

$$H_v = \begin{pmatrix}
\frac{1}{\mu} \sqrt{\frac{(\mu v - c^2)}{\mu v}} & 0 & 0 \\
0 & \frac{1}{\mu} \sqrt{\frac{(\mu v - c^2)}{\mu v}} & 0 \\
0 & 0 & \frac{2}{\mu} \sqrt{\frac{\mu v - c^2}{\mu v}}
\end{pmatrix},$$

(7.10)

$$S_v = \begin{pmatrix}
\frac{c_t}{\sqrt{c_t^2 - v^2}} & 0 & 0 \\
0 & \frac{c_t}{\sqrt{c_t^2 - v^2}} & 0 \\
0 & 0 & \frac{2 c_t \sqrt{(c_t^2 - v^2)} - c_t (2 c_t^2 - v^2)}{c_t^2 - v^2}
\end{pmatrix},$$

(7.11)

where

$$c_l = \sqrt{\frac{2 \mu c + \lambda c}{\rho}}, \quad c_t = \sqrt{\frac{\mu c}{\rho}}, \quad c_m = \sqrt{\frac{I_c^2 c}{\rho j T^2}}.$$  

Note that for $\mu c \to 0$ we have

$$c_p = c_l, \quad c_s = c_t, \quad c_{m_1} = c_m, \quad c_{m_2} = \min\{c_s, c_{m_1}\} = \min\{c_t, c_m\},$$

and that a similar proof with that of Proposition 4.3 holds true in the case $\mu c \to 0$, too.

**Proposition 7.1.** For $2 \mu c + \lambda c > 0$, $\mu c > 0$, $\gamma > 0$, the limiting speed in Cosserat elastic materials is given by

$$\hat{v} := \inf_{\theta \in (-\pi, \pi)} v_\theta \equiv \min\{v_l, c_l, c_t, c_m\}.$$  

(7.12)

By considering $\mu c \to 0$, Theorem 5.4 allows us to affirm that if the constitutive coefficients satisfy the conditions $2 \mu c + \lambda c > 0$, $\mu c > 0$, $\gamma > 0$, and $0 \leq v < \hat{v} = \inf_{\theta \in (-\pi, \pi)} v_\theta \equiv \min\{c_l, c_t, c_4\}$, then the unique solution of the algebraic Riccati equation (5.5) that satisfies $\Re\{\sigma(T^{-1}(M + iR^T))\} > 0$ given by (5.19) takes the form

$$M_v = \begin{pmatrix}
\frac{c_t k^2 v^2 \sqrt{c_t^2 - c^2}}{c_t c_t - c_t^2 - v^2 \sqrt{c_t^2 - c^2}} & \frac{i c_t k^2 (2 c_t \sqrt{(v^2 - c^2) (v^2 - c_t^2)} + c_l (v^2 - c_t^2))}{c_t c_t - c_t^2 - v^2 \sqrt{c_t^2 - c^2}} & 0 \\
i c_t k^2 (2 c_t \sqrt{(v^2 - c_t^2) (v^2 - c_t^2)} + c_l (v^2 - c_t^2)) & \frac{c_t k^2 v^2 \sqrt{c_t^2 - c^2}}{c_t c_t - c_t^2 - v^2 \sqrt{c_t^2 - c^2}} & 0 \\
0 & 0 & \frac{2 k^2 \sqrt{c_m^2 - v^2}}{c_m}
\end{pmatrix}. \quad (7.13)$$
The secular equation in the Mielke-Fu’s form is

\[
0 = \det \mathcal{M}_v = \frac{2k^2 \sqrt{c_m^2 - v^2}}{c_m} c_l c_t v^4 \sqrt{c_t^2 - v^2} \sqrt{c_t^2 - v^2} - c_t^2 \left(2c_t \sqrt{(c_t^2 - v^2)(c_t^2 - v^2)} + c_l (v^2 - 2c_t^2)\right)^2 \left(c_l c_t - \sqrt{c_t^2 - v^2} \sqrt{c_t^2 - v^2}\right)^2. \tag{7.14}
\]

An admissible wave speed has to satisfy \(0 \leq v < \hat{v} = \inf_{\theta \in (-\pi, \pi)} v_\theta \equiv \min \{c_l, c_t, c_m\}\). Therefore, the secular equation in the Mielke-Fu’s form is equivalent to

\[
0 = s_{\text{Mielke–Fu}}(v) \equiv \frac{c_l c_t v^4 \sqrt{c_t^2 - v^2} \sqrt{c_t^2 - v^2} - c_t^2 \left(2c_t \sqrt{(c_t^2 - v^2)(c_t^2 - v^2)} + c_l (v^2 - 2c_t^2)\right)^2}{\left(c_l c_t - \sqrt{c_t^2 - v^2} \sqrt{c_t^2 - v^2}\right)^2}. \tag{7.15}
\]

We mention that Mielke and Fu did not write explicitly this equation for elastic isotropic material. Since

\[
\lim_{v \to 0} \frac{c_l c_t v^4 \sqrt{c_t^2 - v^2} \sqrt{c_t^2 - v^2} - c_t^2 \left(2c_t \sqrt{(c_t^2 - v^2)(c_t^2 - v^2)} + c_l (v^2 - 2c_t^2)\right)^2}{\left(c_l c_t - \sqrt{c_t^2 - v^2} \sqrt{c_t^2 - v^2}\right)^2} = -4 c_t^4 k^4 \tag{7.16}
\]

the value \(v = 0\) is not a singular point of \(\det \mathcal{M}_v\), and the secular equation reduces to

\[
c_l c_t v^4 \sqrt{c_t^2 - v^2} \sqrt{c_t^2 - v^2} - c_t^2 \left(2c_t \sqrt{(c_t^2 - v^2)(c_t^2 - v^2)} + c_l (v^2 - 2c_t^2)\right)^2 = 0. \tag{7.17}
\]

The secular equation (7.15) and its equivalent form (7.17) do not coincide with the classical well known form of the secular equation, i.e.

\[
0 = s_{\text{classic}} \equiv \sqrt{\left(1 - \frac{v^2}{c_t^2}\right) \left(1 - \frac{v^2}{c_t^2}\right) - \left(2 - \frac{v^2}{c_t^2}\right)^2}. \tag{7.18}
\]

Indeed, even when the Stroh formalism is used [17, 16], the obtained secular equation does not coincide with the classical form, and it is given by

\[
s_{\text{Stroh}} \equiv \frac{\rho v^2 \sqrt{c_{11} - \rho v^2 \sqrt{c_{66} - \rho v^2}} - (\rho v^2 - c_{66}) (\rho v^2 - c_{66})}{c_{66}} \tag{7.19}
\]

where

\[
c_{12} = \lambda_e, \quad c_{66} = \mu_e, \quad c_{11} = \lambda_e + 2\mu_e, \quad c_0 = c_{11} - \frac{c_{12}^2}{c_{11}}. \tag{7.20}
\]

However, all these three forms of the secular equations are equivalent in the sense that they predict the same wave speed in the same domain of admissible wave speeds. We illustrate this numerically, see Figure 8, for a specific material which is related to aluminium-epoxy. We point out again that there is no dependence of the wave speed on the wave number \(k\), contrary with what happens when the Cosserat theory is considered, see Figure 5.

Moreover, after solving the system \(\mathcal{M}_{yR} y(0) = 0\) we obtain that the last component of \(y(0)\) vanishes, since \(\frac{2k^2 \sqrt{c_m^2 - v_R^2}}{c_m} \neq 0\) due to the fact the solution is in the admissible set, which implies that \(v_R < c_m\). This means that the micro-rotation vanishes and only macroscopic displacements govern the movement of the half-space. Therefore, from (5.1), the wave propagation solution will be

\[
y(x_2) = e^{-k x_2} E_{yR} \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} \tag{7.21}
\]

where \(E_{yR} \in \mathbb{C}^{3 \times 3}\) is given by

\[
E_{yR} = \mathcal{T}^{-1} (\mathcal{M}_{yR} + i \mathcal{R}^T), \tag{7.22}
\]

and \(y_1(0), y_2(0)\) are solutions of the algebraic system

\[
\begin{pmatrix}
c_k \frac{k^2 v^2}{c_t^2} \sqrt{c_t^2 - v^2} \\ -i c_t k^2 \left(2c_t \sqrt{(v^2 - c_t^2)(v^2 - c_t^2)} + c_l (v^2 - 2c_t^2)\right) \\ i c_t k^2 \left(2c_t \sqrt{(v^2 - c_t^2)(v^2 - c_t^2)} + c_l (v^2 - 2c_t^2)\right)
\end{pmatrix}
\begin{pmatrix}
y_1(0) \\ y_2(0) \\ y_3(0)
\end{pmatrix} = 0.
\]
Figure 8: All the alternative forms of the secular equations are equivalent. For a linear elastic material having the parameters $\lambda_e = 7.59$ GPa, $\mu_e = 1.89745$ GPa, $\rho = 2.22287 \frac{g}{mm^3}$ they give us the same value of the wave speed, i.e., $v_R = 0.868832$.

8 Final remarks

In this paper we have shown that the approach proposed by Mielke and Fu [24, 47] for the study of the propagation of seismic waves in anisotropic linear elastic materials can be used in the framework of the isotropic Cosserat elastic materials, too. One of the big advantages of this new approach, compared with the classical methods like Stroh formalism, is that it leads to the proof of the existence and uniqueness of the solution of the obtained secular equation. While the existence of solutions was proven before for several problems concerning the propagation of seismic waves in materials with microstructure [10, 11], but using restrictive conditions upon the constitutive coefficients, the uniqueness of the solution was, in the best situation, only conjectured in such generalized theories. Indeed, the problem of existence and uniqueness of the solution of the secular equation remains unsolved in almost all generalized linear theories from elasticity. This is the case since the explicit forms of the corresponding secular equations are not completely analytically written. It is not the case when Mielke and Fu’s approach is used [24, 47], the secular equation being written as $\det \mathbf{M}_v = 0$, where the hermitian matrix $\mathbf{M}_v$ has a known integral form. Moreover, due to the form of the hermitian matrix, it is possible to prove that the secular equation $\det \mathbf{M}_v = 0$ has a unique admissible solution. Therefore, with this paper we close the problem of the propagation of seismic waves in isotropic linear Cosserat elastic materials and we propose two numerically viable strategies to solve this problem for specific materials.

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A An overview of the proofs from Fu and Mielke in our notation

In this section, we will carry out a matrix algebraic analysis of the Riccati equation (5.5) of the linear Cosserat model to show that the properties listed in the previous section hold true for Cosserat isotropic materials, too. The content of this Appendix is almost entirely based on the results presented by Fu and Mielke in [25, 24] and we do not claim any merit in finding (and proving) them. The only aim of this appendix is to follow step by step the results from [25, 24] in order see that they can be applied in the framework of the linear isotropic Cosserat model, too.

Proposition A.1. (Exactly as from Fu and Mielke’s paper) The matrix problem
\[ T \mathcal{E}^2 - i (R + R^T) E - Q + k^2 v^2 I = 0, \]
where Re spec \( \mathcal{E} \) means the real part of spectra of \( \mathcal{E} \) and \( T, T, R \) are defined in (3.7), has a unique solution for \( \mathcal{E} \).

Proof. Let \( \mathcal{E} \) be the solution of (A.1). Let \( r \) be the eigenvalue of \( \mathcal{E} \) and \( a \) be the associated eigenvector (so that \( \mathcal{E} a = ra \)). It follows from (A.1) that \( r \) and \( a \) must satisfy the eigenvalue problem
\[ \{ r^2 T - i r (R + R^T) - Q + k^2 v^2 I \} a = 0, \quad \det \{ r^2 T - i r (R + R^T) - Q + k^2 v^2 I \} = 0. \]  
(A.2)

In the explicit form, the characteristic equation (A.2) takes the form
\[ r^6 + P_1 r^4 + P_2 r^2 + P_3 = 0, \]
where \( P_1, P_2, P_3, P_4 \) are given by (4.8). We observe that characteristic equation is a six degree polynomial with real coefficients. If \( r \) is the root of the characteristic equation then so is \( -r \). Thus, we conclude that the characteristic equation has three roots with positive real parts. Collecting the corresponding eigenspaces defines \( \mathcal{E} \) uniquely.

Theorem A.2. (Exactly as from Fu and Mielke’s paper) If \( \mathcal{E} \) solves (A.1), then \( \mathcal{M} \) obtained from (5.3)_2 is Hermitian.

Proof. By taking transpose and complex conjugate of (5.5)_1 we obtain
\[ (\mathcal{M}^T + i R) T^{-1} (\mathcal{M} - i R^T) - Q^T + k^2 v^2 I = 0 \]
On subtracting (A.4) from (A.4)_2, we get
\[ (\mathcal{M} - \mathcal{M}^T) E + \mathcal{E}^T (\mathcal{M} - \mathcal{M}^T) = 0, \]
which has the form of a Liapunov matrix equation
\[ XX^T E + \mathcal{E}^T X = B. \]  
(A.6)

The equation (A.6) has a unique solution \( X \). For Re spec(\( \mathcal{E} \)) > 0 it is given [3] by
\[ X = \int_0^\infty e^{-t \mathcal{E}^T} B e^{-t \mathcal{E}} dt. \]  
(A.7)

Since we have \( B = 0 \) the unique solution of (A.6) is \( X = 0 \) and hence \( \mathcal{M} = \mathcal{M}^T \).

Remark A.3. Every Hermitian matrix has real eigenvalues. The determinant of Hermitian matrix is a real number. A Hermitian matrix is said to be positive if it has positive eigenvalues.

In the proof of the following theorem, there is only a small difference compared to the proof given by Fu and Mielke [24].

Theorem A.4. (Exactly as from Fu and Mielke’s paper) Let \( \mathcal{M} \) and \( \mathcal{E} \) be the same as in Theorem A.2. Then the matrix \( \frac{d\mathcal{M}}{dv} \) is negative definite.

Proof. Differentiation (5.5)_1 w.r.t \( v \) gives
\[ \frac{d}{dv} (\mathcal{M} - i R) E + (\mathcal{M} - i R) \frac{d}{dv} E - \frac{d}{dv} Q + \frac{d}{dv} (k^2 v^2 I) = 0, \]
\[ \frac{d}{dv} \mathcal{M} E + (\mathcal{M} - i R) \frac{d}{dv} E = -2 v k^2 I, \]
\[ \frac{d}{dv} \mathcal{M} E + (\mathcal{M} - i R) \frac{d}{dv} (T^{-1} (\mathcal{M} + i R^T)) = -2 v k^2 I, \]
\[ \frac{d}{dv} \mathcal{M} E + (\mathcal{M} - i R) T^{-1} \frac{d}{dv} \mathcal{M} = -2 v k^2 I, \]
\[ \frac{d}{dv} \mathcal{M} E + (\mathcal{M} - i R) T^{-1} \frac{d}{dv} \mathcal{M} = -2 v k^2 I, \]
\[ \mathcal{M} = \mathcal{M}^T. \]
(A.8)

Finally, we obtain
\[ \frac{d}{dv} \mathcal{M} E + \mathcal{E}^T \frac{d}{dv} \mathcal{M} = -2 k^2 v I. \]  
(A.9)

Equation (A.9) has a unique solution for \( \mathcal{M} \) given by
\[ \frac{d}{dv} \mathcal{M} = \int_0^\infty e^{-t \mathcal{E}^T} (-2 \rho k^2 v^2 I) e^{-t \mathcal{E}} dt = -2 k^2 v \int_0^\infty e^{-t \mathcal{E}^T} 1 e^{-t \mathcal{E}} dt. \]  
(A.10)
Thus, for arbitrary non-zero vector $\eta \in \mathbb{C}^n$ we have
\[
\langle \eta, \frac{d}{dt} \mathbf{M} \eta \rangle = -2k^2 v \langle \eta, \int_0^\infty e^{-\tau \mathbf{E}^T} \mathbf{E} \, d\tau \eta \rangle_{\mathbb{C}^n} = -2k^2 v \int_0^\infty \langle e^{-\tau \mathbf{E}}, e^{-\tau \mathbf{E}} d\eta \rangle_{\mathbb{C}^n},
\]
\[
= -2k^2 v \int_0^\infty \langle \zeta(t), \zeta(t) \rangle_{\mathbb{C}^n} dt = -2k^2 v \int_0^\infty \| \zeta(t) \|_2^2 dt,
\]
where $\zeta(t) = e^{-t \mathbf{E}} \eta$. (A.11)

Since $\zeta(0) = \eta \neq 0$ and $\zeta(t)$ is continuous at $t = 0$ (so that $\zeta(t)$ is non-zero at least in a small but finite interval), we have $\langle \eta, \frac{d}{dt} \mathbf{M} \eta \rangle < 0$ and hence $\frac{d}{dt} \mathbf{M}$ is negative definite.

**Theorem A.5.** (Exactly as from Fu and Mielke’s paper) The Hermitian matrix $\mathbf{M}_\theta$ satisfying (5.16) is independent of $\theta$.

**Proof.** Only in this proof, we denote by $f'(\theta) = \frac{df}{d\theta}(\theta)$. On differentiating (5.14) w.r.t $\theta$ we get
\[
\mathcal{T}_\theta = 2 \sin \theta \cos(\tilde{Q} - \mathbf{T}) - \cos^2 \theta(\mathbf{R} + \mathbf{R}^T) + \sin^2 \theta(\mathbf{R} + \mathbf{R}^T).
\]
It is obvious to see
\[
\mathbf{R}^T_\theta = \cos^2 \theta \mathbf{R}^T - \sin^2 \theta \mathbf{R} + \sin \theta \cos \theta(\mathbf{T} - \tilde{Q}),
\]
and
\[
-\mathbf{R}_\theta - \mathbf{R}^T_\theta = 2 \sin \theta \cos \theta(\tilde{Q} - \mathbf{T}) - \cos^2 \theta(\mathbf{R} + \mathbf{R}^T) + \sin^2 \theta(\mathbf{R} + \mathbf{R}^T),
\]
\[
\mathcal{T}_\theta = -\mathbf{R}_\theta - \mathbf{R}^T_\theta.
\]
Similarly,
\[
\mathbf{R}_\theta = -2 \cos \theta \sin \theta(\mathbf{R} + \mathbf{R}^T) + (\cos^2 \theta - \sin^2 \theta)(\mathbf{T} - \tilde{Q}),
\]
and we see that
\[
\mathcal{T}_\theta - \mathbf{Q}_\theta = (\cos^2 \theta - \sin^2 \theta)(\mathbf{T} - \tilde{Q}) - 2 \sin \theta \cos \theta(\mathbf{R} + \mathbf{R}^T).
\]
Therefore
\[
\mathbf{R}_\theta = \mathcal{T}_\theta - \mathbf{Q}_\theta,
\]
and
\[
\mathbf{Q}_\theta = 2 \sin \theta \cos(\mathbf{T} - \tilde{Q}) + \cos^2 \theta(\mathbf{R} + \mathbf{R}^T) - \sin^2 \theta(\mathbf{R} + \mathbf{R}^T).
\]
It is also easy to see
\[
\mathbf{Q}_\theta = \mathbf{R}_\theta + \mathbf{R}^T_\theta,
\]
and after differentiating $\mathcal{T}_\theta \mathbf{T}_\theta^{-1} = 1$, we also get
\[
(\mathbf{T}_\theta^{-1})' = -\mathbf{T}_\theta^{-1} \mathbf{T}_\theta \mathbf{T}_\theta^{-1}.
\]
On differentiating (5.5) w.r.t $\theta$ we get
\[
\mathbf{M}_\theta - i \mathbf{R}_\theta)' \mathbf{E}_\theta + (\mathbf{M}_\theta - i \mathbf{R}_\theta) \mathbf{E}_\theta' - \mathbf{M}_\theta = 0,
\]
\[
(\mathbf{M}_\theta - i \mathbf{R}_\theta)' \mathbf{E}_\theta + (\mathbf{M}_\theta - i \mathbf{R}_\theta) \mathbf{E}_\theta' - \mathbf{M}_\theta = 0,
\]
\[
(\mathbf{M}_\theta - i \mathbf{R}_\theta)' \mathbf{E}_\theta + (\mathbf{M}_\theta - i \mathbf{R}_\theta) \mathbf{E}_\theta' - \mathbf{M}_\theta = 0,
\]
\[
\mathbf{M}_\theta' \mathbf{E}_\theta + \mathbf{E}_\theta' \mathbf{M}_\theta' - i \mathbf{R}_\theta' \mathbf{E}_\theta + i \mathbf{E}_\theta' \mathbf{R}_\theta' \mathbf{E}_\theta - \mathbf{Q}_\theta = 0,
\]
\[
\mathbf{M}_\theta' \mathbf{E}_\theta + \mathbf{E}_\theta' \mathbf{M}_\theta' - i (\mathbf{T}_\theta - \tilde{Q}_\theta) \mathbf{E}_\theta + i \mathbf{E}_\theta' (\mathbf{T}_\theta - \tilde{Q}_\theta) + \mathbf{E}_\theta' (\mathbf{R}_\theta + \mathbf{R}_\theta^T) \mathbf{E}_\theta - (\mathbf{R}_\theta + \mathbf{R}_\theta^T) = 0,
\]
and using (A.13),(A.14) and (A.15) in (A.17), yields
\[
\mathbf{M}_\theta' \mathbf{E}_\theta + \mathbf{E}_\theta' \mathbf{M}_\theta' - i (\mathbf{T}_\theta - \tilde{Q}_\theta) \mathbf{E}_\theta + i \mathbf{E}_\theta' (\mathbf{T}_\theta - \tilde{Q}_\theta) + \mathbf{E}_\theta' (\mathbf{R}_\theta + \mathbf{R}_\theta^T) \mathbf{E}_\theta - (\mathbf{R}_\theta + \mathbf{R}_\theta^T) = 0.
\]
On replacing first $\mathbf{Q}_\theta$ in (A.18) by $\mathbf{E}_\theta' (\mathbf{M}_\theta + i \mathbf{R}_\theta^T)$ and the second $\mathbf{Q}_\theta$ by $(\mathbf{M}_\theta - i \mathbf{R}_\theta)' \mathbf{E}_\theta$, see (5.18), we deduce
\[
\mathbf{M}_\theta' \mathbf{E}_\theta + \mathbf{E}_\theta' \mathbf{M}_\theta' - i (\mathbf{T}_\theta - \tilde{Q}_\theta) \mathbf{E}_\theta + i \mathbf{E}_\theta' (\mathbf{T}_\theta - \tilde{Q}_\theta) + \mathbf{E}_\theta' (\mathbf{R}_\theta + \mathbf{R}_\theta^T) \mathbf{E}_\theta - (\mathbf{R}_\theta + \mathbf{R}_\theta^T) = 0,
\]
\[
\mathbf{M}_\theta' \mathbf{E}_\theta + \mathbf{E}_\theta' \mathbf{M}_\theta' - i \mathbf{T}_\theta \mathbf{E}_\theta + i \mathbf{E}_\theta' \mathbf{T}_\theta - \mathbf{E}_\theta' \mathbf{R}_\theta \mathbf{E}_\theta - \mathbf{E}_\theta' \mathbf{R}_\theta^T \mathbf{E}_\theta + \mathbf{E}_\theta' (\mathbf{R}_\theta + \mathbf{R}_\theta^T) \mathbf{E}_\theta - (\mathbf{R}_\theta + \mathbf{R}_\theta^T) = 0,
\]
\[
\mathbf{M}_\theta' \mathbf{E}_\theta + \mathbf{E}_\theta' \mathbf{M}_\theta' - i \mathbf{T}_\theta \mathbf{E}_\theta + i \mathbf{E}_\theta' \mathbf{T}_\theta - \mathbf{E}_\theta' \mathbf{R}_\theta \mathbf{E}_\theta - \mathbf{E}_\theta' \mathbf{R}_\theta^T \mathbf{E}_\theta + \mathbf{E}_\theta' (\mathbf{R}_\theta + \mathbf{R}_\theta^T) \mathbf{E}_\theta - (\mathbf{R}_\theta + \mathbf{R}_\theta^T) = 0,
\]
finally we obtain,
\[
\mathbf{M}_\theta' \mathbf{E}_\theta + \mathbf{E}_\theta' \mathbf{M}_\theta' = 0
\]
which is another homogeneous Liapunov matrix equation. It then follows that $\mathbf{M}_\theta = 0$, (see (A.6)) so $\mathbf{M}_\theta$ is independent of $\theta$. ■
We have observed that $E_\theta$ reduces to $E$ defined in Theorem 2.1 when $\theta = 0$, we have $M_\theta \equiv M$, where $M$ is the corresponding $N$ given in Theorem (2.1). Thus

$$E_\theta = T_0^{-1}(M + i R_0^2).$$

(A.21)

On differentiating (A.21) with respect to $\theta$, we get

$$E_\theta' = (T_0^{-1})'(M + i R_0^2) + T_0^{-1}(M + i R_0^2)' = (T_0^{-1})'(M + i R_0^2) + T_0^{-1}(i R_0^2)',$$

(A.22)

making use of (A.16), we obtain

$$E_\theta' = -T_0^{-1}T_0^{-1}(M + i R_0^2) + i T_0^{-1}(T_0 - Q_0) = T_0^{-1}(R_\theta + R_0^2) + i (1 - T_0^{-1} Q_0).$$

(A.23)

From (A.21) it is easy to see

$$R_\theta^2 = i M - i T_0 E_\theta$$

(A.24)

which after substituting (A.24) in (A.23), leads us to

$$E_\theta' = T_0^{-1}(R_\theta + i (M - T_0 E_\theta) E_\theta + i (1 - T_0^{-1} Q_0)) = T_0^{-1}(i (M - i R_\theta) - i T_0 E_\theta) E_\theta + i (1 - T_0^{-1} Q_0)$$

(A.25)

$$= T_0^{-1}(i Q - i T_0 E_\theta) E_\theta + i (1 - T_0^{-1} Q_0) = i T_0^{-1} Q - i E_\theta^2 + 11 - i T_0^{-1} Q_0 = 11 - i E_\theta^2.$$  

After integrating this matrix differential equation subject to the condition $E_0 = E$, we obtain

**Proposition A.6.** (Exactly as from Fu and Mielke’s paper) We have

$$E_\theta = (\cos \theta + i \sin \theta E)^{-1}(\cos \theta E + i \sin \theta).$$

(A.26)

Proof. Since, $J(\theta) J^{-1}(\theta) = 1$ after differentiating, it can be observed that for any matrix $J(\theta)$, we have

$$(J^{-1}(\theta))' = -J^{-1}(\theta) J'(\theta) J^{-1}(\theta),$$

(A.27)

it then follows that

$$(J^{-1}(\theta) J'(\theta))' = -(J^{-1}(\theta) J'(\theta))^2 + J^{-1}(\theta) J''(\theta),$$

or equivalently

$$(-i J^{-1}(\theta) J'(\theta))' = -i (-i J^{-1}(\theta) J'(\theta))^2 - i J^{-1}(\theta) J''(\theta).$$

(A.28)

This suggests the transformation

$$E_\theta = -i J^{-1}(\theta) J'(\theta)$$

(A.29)

since on substituting (A.30) in (A.25), we arrive at

$$J''(\theta) + J(\theta) = 0,$$

(A.31)

the general solution of (A.31) is

$$J(\theta) = \cos \theta c_1 + i \sin \theta c_2,$$

where, $c_1$ and $c_2$ are constant matrices. From the condition $E_0 = E$ we obtain $E = -i c_1 c_2$. Thus, $J(\theta) = c_1 (\cos \theta + i \sin \theta E)$ and

$$E_\theta = -i J^{-1}(\theta) J(\theta) = (\cos \theta + i \sin \theta E)^{-1}(\cos \theta E + i \sin \theta)$$

(A.32)

and the proof is complete.

If $\lambda$ is an eigenvalue of $E$, then by (A.1) the corresponding eigenvalues of $E_\theta$ is

$$\lambda(\theta) = \Phi(\lambda c_1, \theta),$$

(A.33)

where $\Phi(\lambda, \theta) = \frac{\lambda \cos \theta + i \sin \theta}{\cos \theta + i \lambda \sin \theta}$ and $d\theta = \frac{d}{d\theta} \ln(\cos \theta + i \lambda \sin \theta)$. The real part of $\lambda(\theta)$ is indeed positive. It then follows

$$\int_0^\pi \lambda(\theta) d\theta = -i \int_0^\pi \frac{d}{d\theta} \ln(\cos \theta + i \lambda \sin \theta) d\theta = -i \ln(\cos \theta + i \lambda \sin \theta)\bigg|_0^\pi = \pi.$$

(A.34)

So this leads to

**Proposition A.7.** (Exactly as from Fu and Mielke’s paper) We have

$$\int_0^\pi E_\theta d\theta = \pi 1$$

(A.35)

Proof. We use the spectral calculus of matrices. By choosing any closed curve $\Gamma$ in the complex plane surrounding all the eigenvectors of $E$ with positive real parts and lying in the open half plane $\text{Re} \lambda > 0$. Then by Proposition 2.1 we have

$$\int_0^\pi E_\theta d\theta = \frac{1}{2\pi i} \oint_{\Gamma} \Phi(\lambda, \theta)(\lambda 1 - E)^{-1} d\lambda d\theta = \frac{1}{2\pi i} \oint_{\Gamma} \Phi(\lambda c_1, \theta) d\theta (\lambda 1 - E)^{-1} d\lambda c_1,$$

(A.36)

$$= \frac{1}{2\pi i} \oint_{\Gamma} (\lambda 1 - E)^{-1} d\lambda c_1.$$  

The conclusion of the proposition follows then since $\int_0^\pi \lambda(\theta) d\theta = \pi$. 

\[\square\]
On integrating (A.21), we get
\[
\int_0^\pi E_\theta \, d\theta = \int_0^\pi T^{-1}_\theta (\mathcal{M} + i\mathcal{R}^T_\theta) \, d\theta
\quad \Leftrightarrow \quad
\int_0^\pi T^{-1}_\theta \mathcal{M} \, d\theta = \int_0^\pi E_\theta \, d\theta - i \int_0^\pi T^{-1}_\theta \mathcal{R}^T_\theta \, d\theta,
\]
\[
\quad \Leftrightarrow \quad (\int_0^\pi T^{-1}_\theta \, d\theta) \mathcal{M} = \pi 1 - i \int_0^\pi T^{-1}_\theta \mathcal{R}^T_\theta \, d\theta.
\]
(A.36)

Finally we obtain

**Theorem A.8.** (Exactly as from Fu and Mielke’s paper) The unique solution of the algebraic Riccati equation (5.5) that satisfies
\[
\text{Re} \, \text{spec} (T^{-1}(\mathcal{M} + i\mathcal{R}^T)) > 0
\]
is given explicitly by
\[
\mathcal{M} = (\int_0^\pi T^{-1}_\theta \, d\theta)^{-1} (\pi 1 - i \int_0^\pi T^{-1}_\theta \mathcal{R}^T_\theta \, d\theta).
\]
(A.37)

**Proof.** We multiply the relation (A.36) by \( (\int_0^\pi T^{-1}_\theta \, d\theta)^{-1} \).

\[\blacksquare\]