Motivic characteristic classes for singular schemes and a singular Gauss-Bonnet formula

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Abstract

We construct a class in motivic Borel-Moore homology, extending the Euler class of the tangent bundle from smooth to singular schemes, and refining Aluffi’s approach for pro-CSM class on Chow groups. We deduce a singular extension to the motivic Gauss-Bonnet formula.

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1 Introduction

One possible starting point for defining characteristic classes on singular schemes is Macpherson’s work [Mac], in which a natural transformation is defined between the functor of constructible functions and the singular homology functor on a compact complex algebraic variety. For $X$ a compact complex algebraic variety we have a natural transformation between constructible functions and singular homology

$$Con(X) \to H_*(X).$$

Taking the image of the constant constructible function $1_X$ we obtain a class $c_{SM}(X) \in H_*(X)$, which coincides with the total Chern class in the case when $X$ is smooth. This last condition characterises the natural transformation uniquely, making the classes $c_{SM}(X)$ a natural way to extend Chern classes for singular varieties. The class $c_{SM}$ constructed by MacPherson coincides with a definition by Schwartz, and is called the Chern-Schwartz-MacPherson class of $X$. This can be defined in an algebraic setting to have values in the Chow ring $CH_*(X)$ instead of singular homology.
Motivic homotopy theory gives us a natural way to define and work with a wide array of Borel-Moore homology theories on schemes, other than Chow groups. Each motivic ring spectrum $E \in \text{SH}(k)$, defines a Borel-Moore homology, with the choice $E = H\mathbb{Z}$ defining the Chow group homology theory. It is possible to assign to each smooth scheme $X$ the Euler class of the tangent bundle of $X$, which has its own meaning in every such theory $E$. In the case of Chow groups this is the top Chern class of $X$. One can then seek to extend the Euler class of the tangent bundle to a class for singular varieties, even though they do not have a tangent bundle, and get what may be called ‘Euler-Schwartz-MacPherson classes’ defined in a general motivic theory $E$. Such classes satisfy a cut-and-paste relation with respect to subvarieties. After presenting the preliminaries and properties of the motivic Euler class of the tangent bundle in chapter 2 and 3, we discuss in chapter 4 how to define such classes for a general motivic Borel-Moore homology theory. In chapter 5 we extend Aluffi’s approach for constructing pro-CSM classes in pro-Chow group setting, to Borel Moore homology theories. Our main construction is a class

$$e\{X\} \in \hat{E}_0(X).$$

This class coincides with the Euler class of the tangent bundle for a smooth and proper $X$. In the case of Chow groups this is the top degree of the CSM-class defined by Aluffi. We follow the same approach as Aluffi, the main difference is that where Aluffi uses the Chern class of the sheaf of differential forms with poles along a divisor, we use an alternating sum of Euler classes of strata of an normal crossing divisor. On each good closure $\bar{U}/U$ (see Definition 5.1), with $U$ smooth, the class is defined by

$$e_{\bar{U}} = \sum_{I \subset P(\{1, \ldots, r\})} (-1)^{|I|} i_{D_I} e(D_I)$$

where $D = \bar{U} \setminus U = \bigcup_{i=1}^n D_i$ is a simple normal crossings divisor, $D_I = \bigcap_{i \in I} D_i$, $i_{D_I} : D_I \to \bar{U}$ are the inclusions, and $e(D_I)$ is the Euler class of the tangent bundle of $D_I/k$. Our main theorem shows that this gives a well defined class.

**Theorem 1.1** (Definition 5.16). Let $k$ be a field of characteristic 0 and $E \in \text{SH}(k)$ a motivic ring spectrum. Then we have uniquely defined classes $e\{X\} \in \hat{E}_0(X)$ for each variety $X/k$. This definition extends the Euler class of the tangent bundle in $E_0(X) = \hat{E}_0(X)$ for smooth and proper $X/k$ and satisfies the cut and paste property.

The key property of the Euler class, allowing this definition, is additivity along blow-up squares, see Axiom 4.3. In chapter 6 we deduce an extension of the motivic Gauss-Bonnet formula of [DJK], for singular schemes, using the pro classes constructed.

**Theorem 1.2** (Singular motivic Gauss-Bonnet formula, Theorem 6.4). Let $\pi : X \to \text{Spec} \ k$ be a proper variety over a field of characteristic 0, and $E$ a motivic ring spectrum. Let $u : \mathbb{1}_k \to E$ be the unit map in SH$(k)$. Then

$$u_* \chi(E)(X/k) = \pi_* e^E\{X\}$$

Note that in a recent work [JSY], Jin, Sun and Yang give independently a different construction of a pro-class in motivic Borel-Moore homology which agrees with Aluffi’s construction in pro-Chow groups.

**Notation** Let $k$ be a perfect field, let $\text{Sch}_k$ denote the category of separated finite type $k$-schemes, let $\text{Sch}_k^{\text{red}}$ denote the full subcategory of reduced separated finite type $k$-schemes, we call objects in this category varieties. Let $\text{Sch}_k^{\text{sp}}$ be the subcategory of $\text{Sch}_k^{\text{red}}$ with the same objects, where only proper morphisms are considered. Let $\text{Sm}_k$ denote the full subcategory of $\text{Sch}_k$ of separated finite type smooth $k$-schemes. Let $\text{Sm}_k^{\text{sp}}$ denote the full subcategory of $\text{Sm}_k$ spanned by schemes which are also proper over $k$. By a morphism of schemes we always mean a separated morphism of finite type. For a quasi-compact quasi-separated scheme $S$, we denote by $\text{SH}(S)$ the stable $\infty$-category of motivic $S$-spectra. Throughout the paper we assume our field $k$ to admit resolution of singularities and weak factorisation, see section 2.2 for details.

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2 Preliminaries

2.1 $\mathbb{A}^1$-bivariant homology

We use the formalism and notation of [DJK]. Let $S$ be a quasi compact quasi separated scheme and consider the category $\text{SH}(S)$. For $X \in \text{Sm}_S$ with a vector bundle $V \to X$, and zero section $s : X \to V$, we denote by $Th_X(V)$ the space $\Sigma_\infty^\otimes V/(V \setminus s(X))$ in $\text{SH}(X)$. This definition can be extended to any virtual vector bundle on $X$, that is an element $v$ in the K-theory space $K(X)$, giving a $\otimes$-invertible object $Th_X(v) \in \text{SH}(X)$, see [DJK, 2.1.5].

**Definition 2.1.** Let $E \in \text{SH}(S)$ be a motivic ring spectrum, let $p : X \to S$ be an $S$-scheme, with a virtual vector bundle $v \in K(X)$. Define the bivariant theory (also called Borel-Moore homology) spectrum with coefficients in $E$, on the scheme $X$ with twist $v$ to be the mapping spectrum

$$E(X/S, v) = Maps_{\text{SH}(X)}(Th_X(v), p^! E);$$

and the $n$-th homology group by

$$E_n(X/S, v) = \pi_nE(X/S, v) = [Th_X(v)[n], p^! E]_{\text{SH}(X)} \simeq [pTh_X(v)[n], E]_{\text{SH}(S)}.$$

We also set $E_n(X/S) = E_n(X/S, 0)$, and $H_n(X/S, v) = 1_n(X/S, v)$.

For a proper morphism $f : X \to Y$, $X, Y \in \text{Sch}_k$ there is a direct image morphism ([DJK, 2.2.7]),

$$f_* : E_n(X/k, f^*(V)) \to E_n(Y/k, V). \quad (2.1)$$

For a quasi-projective lci morphism $f : Y \to X$ we have the Gysin map [DJK, 4.3.4]

$$f^! : E_n(X/S, e) \to E_n(Y/S, f^*(e) + \langle L_f \rangle) \quad (2.2)$$

defined by a fundamental class $\eta_f \in E_n(Y/X, \langle L_f \rangle)$. Here $L_f$ is the virtual vector bundle arising from the relative cotangent complex for $f$.

For a motivic ring spectrum $E$, the unit map $u : 1_\mathcal{S} \to E$, defines the $\mathbb{A}^1$-regulator natural transformation,

$$u_* : H_n(X/S, e) \to E_n(X/S, e).$$

2.2 Resolution of singularities and weak factorization

Let $X$ be in $\text{Sm}_k$ and let $D$ be a reduced effective divisor on $X$ with irreducible components $D_1, \ldots, D_r$. For $I \subset \{1, \ldots, r\}$ we have the corresponding stratum of $D$, $D_I := \cap_{i \in I} D_i$; let $|I|$ denote the cardinality of $I$. Recall that $D$ is a simple normal crossing divisor on $X$ if each $D_I$ is a smooth closed subscheme of $X$ of codimension $|I|$, in other words, the components $\{D_i \mid i \in I\}$ intersect properly and transversely on $X$. Let $W$ be an irreducible smooth closed subscheme of $X$, and let $I \subset \{1, \ldots, r\}$ be the index of maximal cardinality such that $W \subset D_I$. We say that $W$ intersects $D$ with normal crossing if for each $J \subset \{1, \ldots, r\} \setminus I$, $D_J$ intersects $W$ properly and transversely. Equivalently, letting $D_W$ be the divisor $\Sigma_{J \in \{1, \ldots, r\} \setminus I} D_J \cap W$, $D_W$ is a simple normal crossing divisor on $W$. For $W$ an arbitrary smooth closed subscheme of $X$, we say $W$ intersects $D$ with normal crossing if each irreducible component of $W$ does so.

**Definition 2.2.** We say that $k$ admits resolution of singularities if

1. For each $X$ in $\text{Sch}_k^{\text{red}}$, there is a proper birational morphism $p : X' \to X$ with $X'$ smooth over $k$, with $p$ a sequence of blow-ups with smooth centers lying over the singular locus $X_{\text{sing}}$ of $X$. 


2. Let \( X \) be in \( \text{Sch}_k^{\text{red}} \) and let \( U \subset X \) be a dense open subscheme. Suppose that \( U \) is smooth over \( k \). Then there exists a proper birational morphism \( p : X' \to X \) with \( X' \) smooth over \( k \) such that \( p \) a sequence of blow-ups with smooth center lying over \( X \setminus U \) and such that the complement \( X' \setminus p^{-1}(U) \) is a simple normal crossing divisor on \( X' \).

3. Let \( f : Y \to X \) be a rational map with \( X, Y \in \text{Sch}_k^{\text{red}} \), and suppose that \( f \) is a morphism on a smooth dense open subscheme \( U \) of \( Y \). Then there is proper birational morphism \( p : Y' \to Y \) such that \( p \) a sequence of blow-ups with smooth centers lying over \( Y \setminus U \) and such that \( f \circ p \) is a morphism.

4. Let \( X \) be in \( \text{Sm}_k \) and let \( D \) be a simple normal crossing divisor on \( X \). Let \( W \subset X \) be a closed subscheme, none of whose components is contained in \( D \), such that \( W \setminus D \) is smooth. Then there exists a proper birational morphism \( p : X' \to X \), with \( X' \) smooth over \( k \), such that \( p \) a sequence of blow-ups with smooth center lying over \( D \cap W \) and such that the strict transform \( W' := p^{-1}[W] = p^{-1}(W \setminus D) \) is smooth over \( k \). In addition, letting \( D' = p^{-1}(D)_{\text{red}} \), \( D' \) is a simple normal crossing divisor on \( X' \) and \( W' \) intersects \( D' \) with normal crossing.

**Definition 2.3.** We say that \( k \) admits weak factorization if given \( X, Y \in \text{Sm}_k \), both proper over \( k \), \( U \subset X \) and \( V \subset Y \) dense open subsets, and a rational map \( f : X \to Y \) which restricts to an isomorphism \( f_{|U} : U \to V \), there is a sequence of rational maps \( p_i : X_i \to X_{i+1}, \, i = 0, \ldots, N - 1 \), such that

a. \( X_0 = X, \, X_N = Y \) and for each \( i \), either \( p_i \) or \( p_i^{-1} \) is a morphism. If \( p_i \) is a morphism, then \( p_i \) the morphism given by the blow up of smooth center \( F_{i+1} \subset X_{i+1} \); if \( p_i^{-1} \) a morphism, then \( p_i^{-1} \) is the morphism given by the blow up of smooth center \( F_i \subset X_i \).

b. There is an index \( r \) such that for each \( i \leq r \), the induced rational map \( g_i : X_i \to X \) is a morphism, and for each \( i > r \), the induced rational map \( f_i : X_i \to Y \) is a morphism. Moreover, for \( i \leq r \), \( g_i \) is an isomorphism over \( U \) and for \( i > r \), \( f_i \) is an isomorphism over \( V \).

c. Suppose that \( X \setminus U \) and \( Y \setminus V \) are simple normal crossing divisors. Then for \( i \leq r \), \( D_i := X_i \setminus g_i^{-1}(U) \) is a simple normal crossing divisor on \( X_i \) and for \( i > r \), \( D_i := X_i \setminus f_i^{-1}(V) \) is a simple normal crossing divisor on \( X_i \). Moreover, each smooth center \( F_j \subset X_j \) is contained in \( D_j \) and intersects \( D_j \) with normal crossing.

**Remark 2.4.** Every field of characteristic 0 admits resolution of singularities ([add citation]) and weak factorization [AKMW, Theorem 0.0.1].

### 3 The motivic Euler class of the tangent bundle

**Definition 3.1.** Let \( E \in \text{SH}(S) \) be a motivic ring spectrum. Let \( X \in \text{Sch}_S \) and \( V \) a vector bundle on \( X \). Denote by \( e^E(V) \) the Euler class of \( V \), defined as the composite

\[
\mathbb{I}_X \simeq \sum_i \mathbb{I}_V \to \sum_i \mathbb{I}_{V/(V \setminus 0)} = \text{Th}_X(V) \to \text{Th}_X(V) \otimes p^* E
\]

In the case where \( p : X \to S \) is smooth, \( p^* \simeq \text{Th}_X(-S_p) \circ p^* \), and therefore the Euler class defines a map \( \mathbb{I}_X \to \text{Th}_X(V - T_p) \otimes p^* E \), hence a class in \( E_0(X/S, T_p - V) \). If also \( V = T_p \), the tangent bundle of \( p : X \to S \) then we get a class \( e^E(T_p) \) in \( E_0(X/S) \) which also denote by \( e^E(X) \).

**Remark 3.2.** In the case \( E = H\mathbb{Z} \) the class we get is the Euler class of the tangent bundle in Chow groups, and that is the top Chern class of the scheme. The Chern-Schwarz-MacPherson class is an extension of the Chern class to a class defined for singular schemes. For other motivic ring spectra \( E \) we don’t have a notion of Chern class but we can still consider the Euler class of the tangent bundle \( e^E(X) \) in the theory \( E \). Our purpose is to extend this class to a class for singular schemes in \( E \)-Borel-Moore homology.

**Remark 3.3.** We mention below the fact that for an \( SL \)-oriented homology theory we may also view the Euler class as a restriction of the Thom class to the scheme \( X \).

**Definition 3.4.** Let \( E \in \text{SH}(S) \) be a motivic ring spectrum. We denote by \( (V, \rho) \) pairs consisting of a vector bundle \( p : V \to X \) and an isomorphism \( \rho : \text{detV} \overset{\sim}{\to} \mathcal{O}_X \).

An \( SL \)-orientation on \( E \) is an assignment of an element \( \text{th}(V, \rho) \in E_0(V, \text{det}^{-1}(p^*V^*)) \) for each such pair \( (V, \rho) \), satisfying the axioms of [LR, Definition 3.4]. An \( SL \)-oriented ring spectrum \( E \) is a motivic ring spectrum \( E \in \text{SH}(S) \) together with a fixed \( SL \)-orientation \( \text{th}(-, -) \).
Let $E \in \text{SH}(S)$ be an $SL$-oriented motivic ring spectrum, and $X \in \text{Sm}_S$ a proper scheme over $k$. Let $T_{X/S} \to X$ be the tangent bundle of $X$ over $k$. Let $s : X \to T_{X/S}$ be the zero section. Then

$$e^E(X) = s^*s_*(1_X) = s^*\text{th}(T_{X/S}) \in E_0(X/S).$$

### 3.1 Additivity of Euler class in blow-up squares

We follow the description in [JY]. By $\text{SH}_c(S)$ we denote the full subcategory of $\text{SH}(S)$ spanned by constructible objects [add definition]. Given a scheme $f : Y \to \text{Spec}(k)$, an object $M \in \text{SH}_c(X)$ and an endomorphism $u : M \to M$, the characteristic class $C_Y(M, u) \in H^0(X/k, 0)$ is defined as the composition

$$1_X \xrightarrow{u} \text{Hom}(M, M) \to \mathbb{D}(M) \otimes M \cong M \otimes \mathbb{D}(M) \to K_Y$$

where $K_Y := f^!1_k$, $\mathbb{D}(M) := \text{Hom}(M, K_Y)$ [add details later].

We describe those maps explicitly in the case $M = p^!1_X$, where $p : X \to S$ is a proper morphism, following [JY, Proposition 5.1.15]: It is the composite

$$1_S \to p_*1_X \xrightarrow{u} p_*1_X \xrightarrow{p_*e(T_{X/k})} p_*K_X \to K_S$$

(3.1)

the last map is the counit map of the adjunction $p_*p^! \cong p^!p_* \to id$ composed with $K_S = f^!1_k$. We denote $C_X(M) := C_X(M, id_M)$.

We are going to use the following result for the next theorem.

**Proposition 3.5** (additivity of characteristic classes). [JY, 5.1.5]

Given a distinguished triangle in $\text{SH}_c(X)$

$$L \to M \to N \to$$

The following formula holds

$$C_X(M) = C_X(L) + C_X(N).$$

**Theorem 3.6** (additivity of Euler class on blow-ups). The Euler class $e(X)$ for $X$ smooth and proper satisfies Axiom 4.3. That is, let

$$E \xrightarrow{i_E} \tilde{X} \xrightarrow{q} \tilde{U}$$

be a blow-up square with $X$ and $C$ smooth. Then

$$e(X) - i_{C*}e(C) = q_*\left(e(\tilde{X}) - i_{E*}e(E)\right).$$

**Proof.** Consider the following diagram composed of pullback squares:

$$E \xrightarrow{i} \tilde{X} \xrightarrow{q} \tilde{U}$$

$$C \xrightarrow{i} X \xrightarrow{j} U$$

Using the gluing property for motivic spectra for the bottom line of the diagram, i.e. the triangle

$$j\#j^* \to id \to i_*i^* \to$$

and applying it to $1_X \in \text{SH}(X)$ we get:

$$j\#j^*1_X \to 1_X \to i_*i^*1_X$$

so

$$j\#1_U \to 1_X \to i_*1_C$$

(3.2)
Example 4.2. Let functor $f$ for additive $A$ Borel-Moore homology functor over square $c$ the following condition which we refer to as Axiom 4.3. We assume we are given a class $4.2$ Classes in Borel-Moore Homology functor $a$ Borel-Moore homology functor $[DJK]$. We call such homology functor a motivic Borel Moore homology. The map $j$ such that filtered projective limits exist in $A$ Let $A$ Let $A_{Borel-Moore}$ smooth and proper variety and would like to extend it to a well defined class on all varieties. This is based on Bittner’s theorem [Bitt] and is noted by Brasselet, Schurmann and Yukora in [BSY, Corollary 0.1]. We repeat this here, in order to apply it for a general $E$-Borel-Moore homology theory.

4 Extending characteristic classes to singular schemes by cut and paste relations

The structure of Grothendieck’s group of varieties is such that a class defined on smooth varieties with certain compatibilities can be extended to a definition of class over any variety. This is based on Bittner’s theorem [Bitt] and is noted by Brasselet, Schurmann and Yukora in [BSY, Corollary 0.1]. We repeat this here, in order to apply it for a general $E$-Borel-Moore homology theory.

4.1 Borel-Moore Homology

**Definition 4.1.** Let $\text{Sch}_k$ be the subcategory of $\text{Sch}_k^{red}$ with morphisms the proper maps. Let $\mathcal{A}$ be an additive category such that filtered projective limits exist in $\mathcal{A}$. Let $E : \text{Sch}_k \to \mathcal{A}$ be a functor. For a morphism $f : Y \to X$ in $\text{Sch}_k$, denote by $f_* : E(Y) \to E(X)$ the morphism $E(f)$. We say that $E$ is an additive functor if for $X \in \text{Sch}_k^{red}$ a disjoint union, $X = U \sqcup V$, with inclusions $j_U : U \to X$, $j_V : V \to X$, the map $j_{U*} + j_{V*} : E(U) \oplus E(V) \to E(X)$ is an isomorphism.

A Borel-Moore homology functor over $k$ with values in $\mathcal{A}$ is an additive functor $E : \text{Sch}_k \to \mathcal{A}$.

**Example 4.2.** Let $E \in \text{SH}(k)$ be a motivic ring spectrum and let $n \in \mathbb{Z}$, then $X \mapsto E_n(X/k)$ defines a Borel-Moore homology functor $[DJK]$. We call such homology functor a motivic Borel Moore homology functor.

4.2 Classes in Borel-Moore Homology

Our main method for producing characteristic classes for varieties from classes defined for schemes will be the following condition which we refer to as Axiom 4.3. We assume we are given a class $c(X)$ for every smooth and proper variety and would like to extend it to a well defined class on all varieties.

**Axiom 4.3.** Let $c(X) \in E(X)$ be a family of classes for each proper $X \in \text{Sm}_k$. Then for each blow-up square

\[
\begin{array}{ccc}
E & \xrightarrow{j} & \tilde{X} \\
\downarrow & & \downarrow q \\
C & \xrightarrow{i} & X
\end{array}
\]
with $X$ and $C$ smooth, $X$ proper over $k$, $\bar{X} = Bl_C(X)$, $E = q^{-1}(X)$,
\[c(X) - i_\ast c(C) = q_\ast (c(\bar{X}) - j_\ast c(E)).\]

Note that by Theorem 3.6, this axiom is satisfied for the Euler class.

**Proposition 4.4.** [BSY, Corollary 0.1] Let $E : \text{Sch}_k^* \to A$ be a Borel-Moore homology functor as in 4.1. Assume we have for each smooth $X \in \text{Sch}_{k}^{\text{red}}$ a class $c(X) \in E(X)$, such that for each isomorphism $h : X \to X'$, $h_\ast (c(X)) = c(X')$, and axiom 4.3 is satisfied. Then $c$ can be extended to a system of compatible group homomorphisms $c\{-\} : K_0(\text{Var}/X) \to E(X)$ that commute with proper pushforward and satisfy:
\[c\{f : Y \to X\} = f_\ast c(Y) \text{ for } Y \text{ smooth over } S, \text{ and each } f : Y \to X.\]

**Proof.** This is a consequence of Bittner’s theorem [Bitt, Theorem 5.1], characterising the group $K_0(\text{Var}/S)$ as the group generated by smooth and proper $S$-schemes, with relations just according to axiom 4.3. □

**Example 4.5.** Let $E_0(-/k)$ be the Borel-Moore homology functor defined by a motivic ring spectrum $E \in \text{SH}(k)$. We have well defined classes $e(X) \in E_0(X/k)$ for each $X \in \text{Sm}_k$. By Theorem 3.6 they satisfy axiom 4.3. Therefore they extend to a class $e\{X\} \in E_0(X/k)$ for all $X \in \text{Var}_k$.

**Example 4.6.** Let $E_0(-/k)$ be the Borel-Moore homology functor defined by a motivic ring spectrum $E \in \text{SH}(k)$ as in the previous example. In Section 5 we construct a Borel-Moore homology functor in pro-groups $E_0(-/k)$, with Euler classes $\hat{e}(X) \in E_0(X/k)$ for each $X \in \text{Sm}_k^*$. By Theorem 5.15 they satisfy axiom 4.3. Therefore they extend to a class $\hat{e}\{X\} \in E_0(X/k)$ for each $X$.

**Remark 4.7** (A comparison between the two above constructions). The pro-Borel Moore homology theory in the second example refines the one in the first example, as we have compatible natural homomorphisms $E_0(X/k) \to E_0(X/k)$ (ref). Under these natural homomorphisms the classes $\hat{e}\{X\}$ constructed in the second example map to the classes $e\{X\}$ of the first example. Therefore we can think of the second example of Euler classes as more refined objects than in the first example. However for proper $X \in \text{Sm}_k^*$, as $\hat{E}_0(X/k) \simeq E_0(X/k)$, we get that both classes agree. The motivic Gauss Bonnet formula we prove in section 6 is valid for both classes, with the class in the second example giving the stronger result.

5 The pro Euler class

5.1 Motivic pro-homology

In this section we define pro-homology groups for a bivariant theory, and describe how to get well-defined classes in them. The discussion here is identical to [Alu06, sections 2.3], where here we use a general Borel-Moore homology theory $E_n(-)$ instead of the Chow groups; the existence of a good theory of proper pushforward for $E$-Borel-Moore homology allows us to use Aluffi’s arguments in this more general setting.

5.1.1 Closures and completions

We repeat here definitions by Aluffi [Alu06, Section 2]

**Definition 5.1.** Let $X$ be in $\text{Sch}_k^{\text{red}}$.

1. A morphism $i : X \to Z^i$ in $\text{Sch}_k$ with $Z^i$ proper over $k$ and $i$ an open immersion is called a completion of $X$; if $i(X)$ is dense in $Z^i$, we call $i$ a closure of $X$. If $i : X \to Z^i$ is a closure of $X$ such that $Z^i$ is smooth over $k$ and the complement $Z^i \setminus i(X)$ is a simple normal crossing divisor, we call $i$ a good closure of $X$.

2. Let $C^s(X)$ be the category with objects the completions $i : X \to Z^i$ and where a morphism $\pi : i \to j$ is a morphism $\pi : Z^i \to Z^j$ such that the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{i} & Z^i \\
\downarrow{j} & & \downarrow{\pi} \\
& & Z^j
\end{array}
\]
commutes. Let $C(X)$ be the full subcategory of $C^*(X)$ with objects the closures of $X$ and let $C^0(X)$ be the full subcategory of $C(X)$ with objects the good closures of $X$.

3. Let $f : Y \to X$ be a morphism in $\text{Sch}_k$, let $i : X \to Z^i$, $j : Y \to W^j$ be closures of $X$ and $Y$, respectively. A morphism $\pi : W^j \to Z^i$ is a morphism of closures over $f$ if the diagram

$$
\begin{array}{ccc}
  Y & \xrightarrow{j} & W^j \\
  f \downarrow & & \downarrow \pi \\
  X & \xrightarrow{i} & Z^i
\end{array}
$$

commutes.

**Lemma 5.2.** 1. $C^*(X)$ is left-filtering and $C(X) \subset C^*(X)$ is a left cofinal subcategory.

2. For each pair of closures $i : X \to Z^i$, $j : X \to Z^j$ there is at most one morphism $\pi : i \to j$ in $C(X)$.

3. Let $f : Y \to X$ be a morphism in $\text{Sch}_k$ and let $i : X \to Z^i$ be a closure of $X$. For $j : Y \to W^j$ a closure of $Y$, there is at most one morphism of closures $\pi : W^j \to Z^i$ over $f$. Moreover, the subcategory $C(Y)/(i,f)$ of $C(Y)$ with objects the closures $j : Y \to W^j$ such that there exists a morphism of closures $j \to i$ over $f$ is left cofinal in $C(Y)$.

4. Suppose that $\text{Sch}_k^{\text{red}}$ admits resolution of singularities and $X$ is smooth over $k$. Then (1), (2) and (3) hold with $C(X)$ replaced by $C^0(X)$.

**Proof.** The compactification theorem of Nagata (see e.g. [Stacks Project Theorem 38.33.8]) implies there exists a closure of $X$ so $C(X)$ and $C^*(X)$ are non-empty. Given completions $i : X \to Z^i$, $j : X \to Z^j$ and two morphisms $f_1, f_2 : i \to j$, let $Z^{h} \subset Z^i$ be the closure of $i(X)$, with $h : X \to Z^h$, $g : Z^h \to Z^i$ the induced morphisms. Then $h$ is a closure of $X$ and $g$ defines a morphism of completions $g : h \to i$. Since $X$ is dense in $Z^h$ and $Z^i$ is separated over $k$, we have $f_1 \circ g = f_2 \circ g$. This proves (1). A similar argument proves (2) and gives the uniqueness assertion in (3). Given $f : Y \to X$, $i : X \to Z^i$ and $j : Y \to W^j$ as in (3), let $W^j$ be the closure of $(j \circ f)(Y)$ in $W^j \times_k Z^i$, then the induced map $h : Y \to W^h$ is a closure of $Y$ with morphism $p_2 : W^h \to Z^i$ of closures over $f$ and morphism $p_1 : W^h \to W^j$ of closures of $Y$. This proves the second part of (3).

For (4), resolution of singularities says that for each closure $i : X \to Z^i$ of $X$ there is a good closure $j : X \to Z^j$ and a morphism of closures $\pi : i \to j$. Then assertion (4) follows from this together with (1)-(3). 

**5.1.2 Borel Moore pro motivic homology**

Let $E \in \text{SH}(k)$ be a motivic ring spectrum. For each $X \in \text{Sch}_k$, $E$ defines Borel Moore homology groups $E_*(X/k)$. We define now pro-homology groups as the limit of system of Borel Moore groups of the completion, similar to the way Aluffi defines pro-Chow groups.

**Definition 5.3.** Let $E \in \text{SH}(k)$ be a motivic ring spectrum and let $X \in \text{Sch}_k$ be reduced. Sending a completion $i : X \to Z^i$ to $E_n(Z^i/k)$ and a morphism $\pi : (j : X \to Z^j) \to (i : X \to Z^i)$ to the map $\pi_* : E_n(Z^j/k) \to E_n(Z^i/k)$ defines a functor

$$E_n(-/k) : C^*(X) \to \text{Ab}.$$  

Define the pro-$E$ Borel-Moore homology group of $X$ by

$$E_n(X) = \lim_{\longleftarrow i \in C(X)} E_n(Z^i/k).$$

In the next lemma we explain that it is enough to consider only closures instead of all completions, and that we have proper pushforward maps on pro homology groups.

**Lemma 5.4.** 1. The canonical map

$$\lim_{\longleftarrow i \in C(X)} E_n(Z^i/k) \to \hat{E}_n(X)$$

is an isomorphism. If $X$ is smooth over $k$, and $\text{Sch}_k^{\text{red}}$ admits resolution of singularities, the canonical map

$$\lim_{\longleftarrow i \in C^0(X)} E_n(Z^i/k) \to \lim_{\longleftarrow i \in C(X)} E_n(Z^i/k)$$
is an isomorphism as well.

2. Given a morphism \( f : Y \to X \) in \( \text{Sch}_k \), the morphisms \( \pi_* : E_n(W^j/k) \to E_n(Z^i/k) \) for each morphism \( \pi \) of \( j : Y \to W^j \) to \( i : X \to Z^i \) give rise to a well-defined morphism

\[ f_* := \hat{E}_n(f) : \hat{E}_n(Y) \to \hat{E}_n(X). \]

Moreover, we have \((fg)_* = f_*g_*\) for composable morphism \( g : W \to Y, f : Y \to X\).

3. If \( X \) is proper over \( k \), there is a canonical isomorphism \( \hat{E}_n(X) \cong E_n(X) \).

4. If \( f : Y \to X \) is a proper map of proper schemes over \( k \), the induced homomorphism

\[ f_* : E_n(Y) \cong \hat{E}_n(Y) \to \hat{E}_n(X) \cong E_n(X) \]

is the usual pushforward in motivic Borel Moore homology.

5. For \( E = HZ \), \( E_n(X) = A_n(X) \) give the proChow groups defined by Aluffi [Alu06, Def. 2.2].

Proof. For 1. and 2. use Lemma 5.2. For 3. and 4. note that when \( X \) is smooth and proper over \( k \), \( id : X \to X \) is an isomorphism in the category \( \mathcal{C}(X) \). 5. is exactly Aluffi’s definition, using the fact that \( HZ_n(X/k) = A_n(X/k) \), and the pushforward for proper maps in \( E_n \) agrees with proper pushforward for Chow groups.

This defines the functor \( \hat{E}_n(-/k) : \text{Sch}_k^{ed} \to \text{Ab} \).

5.2 Defining pro-classes

Lemma 5.5. Suppose that \( k \) admits weak factorization, and let \( U \in \text{Sm}_k \). In order to define an element \( c \in \hat{E}_n(U) \), it suffices to assign a family of elements \( c^i \in E_n(Z^i) \) for each good closure \( U \to Z^i \), such that for each good closure of good closures \( \pi : i \to j \), with \( \pi : Z^j \to Z^i \) being a blow-up along a smooth centre, intersecting \( Z_j \setminus j(U) \) with normal crossing, we have \( \pi_*(c^i) = c^j \).

Proof. By definition of \( \hat{E}_n(U) \), an element \( c \in \hat{E}_n(X) \) is defined by a class \( c^i \in E_n(Z^i) \) for each good closure \( X \to Z^i \), satisfying for each morphism of that \( f_*(c^i) = c^j \). It remains to see that we can check this compatibility condition only on morphisms \( \pi \) which are blow-ups as in the lemma.

Consider the following situation: \( U \to Z^i, i = 1, 2, 3 \) are good closures, with proper morphisms \( \alpha : (U, Z_2) \to (U, Z_1), \beta : (U, Z_2) \to (U, Z_3) \) and \( \gamma : (U, Z_1) \to (U, Z_3) \), satisfying \( \beta = \gamma \circ \alpha \). Assume we have \( c^i \in E_n(Z^i) \) for each \( i \). Then if \( \alpha_*c^2 = c^1 \) and \( \beta_*c^2 = c^3 \), we evidently have \( \gamma_*c^1 = \gamma_*\alpha_*c^2 = \beta_*c^2 = c^3 \).

Now let \( f \) be a morphism of good closures \( f : (i : U \to Z) \to (j : U \to W) \). Then \( f : Z \to W \) is a proper birational map and \( f \circ i = j \). By weak factorization, retaining notation from 2.3, we can decompose \( f \) as a composition of birational maps \( p_{N-1} \circ \ldots \circ p_0 \circ p_0 \) such that either \( p_i : X_i \to X_{i+1} \) is the blow-up of \( X_{i+1} \) along a smooth center \( F_{i+1} \subset X_{i+1} \) or \( p_{i-1} : X_{i-1} \to X_i \) is the blow-up of \( X_i \) along a smooth center \( F_i \subset X_i \); here \( Z = X_0 \) and \( W = X_N \). Moreover, the induced rational map \( U \dashrightarrow X_i \) is a morphism, defining a good closure \( \alpha_i : U \to X_i \); let \( D_i \) be the simple normal crossing divisor \( X_i \setminus \alpha_i(U) \). Finally, if \( p_i \) is a morphism, then \( F_{i+1} \subset D_{i+1} \) and intersects \( D_{i+1} \) with normal crossing, and if \( p_i^{-1} \) is a morphism, then \( F_i \subset D_i \) and intersects \( D_i \) with normal crossing.

Each \( X_i \) is a good closure of \( U \), so we have a well-defined class \( c^i \in E_n(X_i) \). Let \( r, 0 \leq r \leq N \) be as in 2.3. We have the induced morphisms \( g_i : X_i \to Z \) for \( i \leq r \) and \( f_i : X_i \to W \) for \( i \geq r \). For \( i \leq r \), \( g_i \) defines a morphism of good closures \( \alpha_i \to \alpha_0 \) and for \( i \geq r \), \( f_i \) defines a morphism of good closures \( f_i : \alpha_i \to \alpha_N \). Take \( i \leq r - 1 \) and suppose that \( p_i : X_i \to X_{i+1} \) is a morphism. Then we have the morphism of good closures \( p_i : \alpha_i \to \alpha_{i+1} \) which is the blow-up along a smooth center intersecting \( X_{i+1} \setminus \alpha_{i+1}(U) \) with normal crossing, so by assumption \( p_i*(c^i) = c^{i+1} \). Thus

\[ g_{i+1}*(c^{i+1}) = g_{i+1} \circ p_{i+1}*(c^i) = g_{i+1}*(c^i) \]

If \( i \leq r \) and \( p_i^{-1} \) is a morphism, we similarly have \( g_i*(c^i) = g_{i-1}*(c^{i-1}) \); by induction, we thus have \( g_{r}*(c^i) = c^i = c_i \).

Similarly, if \( i \geq r \), we have \( f_i*(c^i) = c^N = c_j \). Our original map \( f \) is a morphism and we have the factorization \( f \circ g_r = f_r \), thus

\[ f_*(c^i) = f_*(g_r*(c^r)) = (f \circ g_r)_*(c^r) = f_r*(c^r) = c_j. \]
Take $U \in \text{Sm}_k$. Suppose we have defined a system of classes $\{c^i \in E_n(Z^i)\}_i$ for each good closure $i : U \to Z^i$, compatible with respect to blow-ups along smooth centers as in the statement of Lemma 5.5. The family $\{c^i \in E_n(Z^i)\}_i$ thus gives us a well-defined class $c\{U\} \in \hat{E}_n(U)$. We wish to extend the definition of these classes for smooth $U$ to yield a well defined class $c\{X\}$ for each $X \in \text{Sch}_k^{red}$. Assuming that the classes $c\{U\}$ satisfy an additivity property with respect to a decomposition into an open subscheme and closed complement, this can be accomplished by taking a stratification of $X$ into smooth locally closed subschemes, $X := \amalg_j i_j : X_j \hookrightarrow X$, and defining $c\{X\} := \sum_j i_j c\{X_j\}$. The main point is that under this additivity assumption, the resulting class in $\hat{E}_n(X/k)$ is independent of the choice of smooth stratification of $X$. We proceed to fill in the details of this construction.

**Definition 5.6.** Take $X \in \text{Sch}_k^{red}$.

1. A *stratification* of $X$ is a choice of finite family of locally closed subschemes $\{U_i\}_{i \in I}$ of $X$ such that $X$ is the disjoint union of the $U_i$. A stratification is called a smooth stratification if each of the $U_i$ is smooth over $k$.

2. A *refinement* of a stratification $\{U_i\}_{i \in I}$ is a stratification $\{V_j\}_{j \in J}$ such that each $V_j$ is contained in some $U_i$.

3. Given a stratification $\{U_i\}_{i \in I}$ of $X$ and a reduced locally closed subscheme $Y$ of $X$, the *induced stratification* of $Y$ is $\{U_i \cap Y\}_{i \in I}$.

**Remark 5.7.** If $k$ is a perfect field, then each $X \in \text{Sch}_k^{red}$ admits a smooth stratification, and each pair of smooth stratifications $\{U_i\}_{i \in I}$, $\{V_j\}_{j \in J}$ admits a common smooth refinement. Indeed, the assumption that $k$ is perfect implies that the smooth locus of $X$ is a dense open subscheme, and the existence of a smooth stratification of $X$ containing $U$ as member follows by noetherian induction. Similarly, given smooth stratifications $\{U_i\}_{i \in I}$, $\{V_j\}_{j \in J}$ of $X$, these have the common refinement $\{U_i \cap V_j\}_{i \in I, j \in J}$ and then taking a smooth stratification of each $U_i \cap V_j$ yields a smooth common refinement.

If $k$ admits resolution of singularities or weak factorization, then $k$ is automatically perfect; for the remainder of the paper, we will assume that our base-field $k$ is perfect.

Suppose we are given, for each $U \in \text{Sm}_k$, an element $c\{U\} \in \hat{E}_n(U/k)$. Given an $X \in \text{Sch}_k^{red}$ and a stratification $\mathcal{U} := \{U_i\}_{i \in I}$ of $X$, we then have the element

$$c_{\mathcal{U}} \in \hat{E}_n(X)$$

defined as

$$c_{\mathcal{U}} := \sum_{i \in I} \alpha_i c\{U_i\},$$

where $\alpha_i : U_i \to X$ is the inclusion. *A priori* $c_{\mathcal{U}}$ depends on the choice of stratification $\mathcal{U}$ of $X$; here we describe a criterion that suffices to ensure that $c_{\mathcal{U}}$ is independent of the choice of stratification $\mathcal{U}$.

**Proposition 5.8.** Suppose that for each $U \in \text{Sm}_k$ we have a class $c\{U\} \in \hat{E}_n(U/k)$. Suppose in addition that for each smooth $U$ with a smooth closed subscheme $i : Z \hookrightarrow U$, and with open complement $j : U \setminus Z \hookrightarrow U$, we have

$$c\{U\} = i_* c\{Z\} + j_* c\{U \setminus Z\}. \quad (5.1)$$

Then for $X \in \text{Sch}_k^{red}$, the element $c_{\mathcal{U}}\{X\} \in \hat{E}_n(X)$ is independent of the choice of smooth stratification $\mathcal{U}$ of $X$; we denote $c_{\mathcal{U}}\{X\}$ by $c\{X\}$. Moreover, if we have an open subscheme $U \to X$ with closed complement $i : Z \to X$, then $c\{X\} = i_* c\{Z\} + j_* c\{U\}$.

**Proof.** The last assertion follows from the first by taking a smooth stratification $\mathcal{V} = \{V_j \subset Z\}$ of $Z$, a smooth stratification $\mathcal{U} := \{U_i \subset U\}$ of $U$ and using $\mathcal{V} \coprod \mathcal{U}$ as smooth stratification of $X$.

Suppose that $X$ is smooth and can be written as a disjoint union of open subsets $X = U_1 \coprod U_2$, let $\mathcal{U} = \{U_1, U_2\}$ and let $\alpha_1 : U_1 \to X$ be the inclusion. By our hypothesis (5.1), we have $c\{X\} = \alpha_{1, *}(c\{U_1\}) + \alpha_{2, *}(c\{U_2\})$. This shows that the result holds for $X$ smooth and $\mathcal{U}$ a stratification of $X$ such that each $U \in \mathcal{U}$ is a union of irreducible components of $X$. In particular, the Proposition holds for $X$ of dimension 0. We prove the result in general by induction on $\text{dim}_kX$, that is, the maximum of the dimension of the irreducible components of $X$. Also, since the result holds in the special case of a stratification of a smooth $X$ via finite unions of irreducible components, it suffices to prove the Proposition for smooth stratifications $\mathcal{U}$ of $X$ such that each $U \in \mathcal{U}$ is irreducible.
Take $X$ in $\text{Sch}_{k}^{red}$ of dimension $d > 0$ and assume the result holds for all $X'$ of dimension $< d$. $U := \{U_{i} \mid i \in I\}$ and $V := \{V_{j} \mid j \in J\}$ be two smooth stratifications of $X$. Since every two stratifications have a common smooth refinement we may assume that $V$ is a refinement of $U$, and we may assume that each $U \in U$ is irreducible.

For each $i \in I$, let $J_{i} \subset J$ be the set of $j \in J$ such that $V_{j} \subset U_{i}$. Then $V_{i} := \{V_{j} \mid j \in J_{i}\}$ is a smooth stratification of $U_{i}$; for $j \in J_{i}$, let $\alpha_{V_{j}}^{i} : V_{j} \rightarrow U_{i}$ be the inclusion. By the functoriality of pushforward, it suffices to show that

$$c\{U_{i}\} = \sum_{j \in J_{i}} \alpha_{V_{j}}^{i}(c\{V_{j}\}) \quad (5.2)$$

for each $i \in I$. By our induction hypothesis, $(5.2)$ holds if $\dim_{k}U_{i} < d$, so it suffices to prove $(5.2)$ for the $U_{i}$ of dimension $d$.

Now take $i \in I$ with $\dim_{k}U_{i} = d$. Since $U_{i}$ is by assumption irreducible, there is a unique $j_{0} \in J_{i}$ such that $V_{j_{0}}$ contains the generic point of $U_{i}$; $V_{j_{0}}$ is thus an open subscheme of $U_{i}$. Let $Y = U_{i} \setminus V_{j_{0}}$. Then $Y$ has smooth stratification $V_{Y} = \{V_{j} \mid j \in J_{i} \setminus \{j_{0}\}\}$ and since $U_{i}$ is irreducible, we have $\dim Y < d$. Using our induction hypothesis again, we have

$$c_{V}(Y) = c_{V_{Y}}(Y) \quad (5.3)$$

for any smooth stratification $W$ of $Y$.

Since $k$ is perfect, $Y$ has the “terminal” smooth filtration $W$, namely, let $Y_{0} = Y$, let $W_{0}$ be the smooth locus of $Y_{0}$, let $Y_{1} = Y \setminus W_{0}$, let $W_{1}$ be the smooth locus of $Y_{1}$, and so on, until we finish with the smooth closed subset $Y_{N} = W_{N}$. The resulting smooth stratification $W := \{W_{j} \mid j \in \{0, \ldots, N\}\}$ of $Y$ has property that for each $n = 0, \ldots, N$, the subset $\bigcup_{j \geq n} W_{j}$ is the closed subset $Y_{n} \subset Y$ and $Y_{N}$ is smooth.

Let $\beta_{Y}^{j} : W_{j} \rightarrow Y$, $\beta_{j} : W_{j} \rightarrow U_{i}$, and $\iota : Y \rightarrow U_{i}$ be the inclusions; for $j \in J_{i} \setminus \{j_{0}\}$, let $\alpha_{V_{j}}^{Y} : V_{j} \rightarrow Y$ be the inclusion. By $(5.3)$, we have

$$\sum_{j \in J_{i} \setminus \{j_{0}\}} \alpha_{V_{j}}^{Y}(c\{V_{j}\}) = \sum_{j = 0}^{N} \beta_{j}^{Y}(c\{W_{j}\}) \in \hat{E}_{n}(Y/k),$$

and thus

$$\sum_{j \in J_{i}} \alpha_{V_{j}}^{i}(c\{V_{j}\}) = \alpha_{V_{j_{0}}}(c\{V_{j_{0}}\}) + \sum_{j = 0}^{N} \beta_{j}^{Y}(c\{W_{j}\}) \in \hat{E}_{n}(U_{i}/k).$$

Thus, we reduce to proving $(5.2)$ for the smooth stratification $W \cup \{V_{j_{0}}\}$ of $U_{i}$.

Changing notation, we may reduce the proof to the following case. We have a smooth $k$-scheme $U$ of dimension $d$ and a smooth stratification $V = \{V_{j} \mid j \in J := \{0, \ldots, m\}\}$ of $U$, such that for each $n \in J$, the closure $\bar{V}_{n}$ satisfies

$$\bar{V}_{n} = \Pi_{j = n}^{m} V_{j}.$$

Let $\alpha_{V_{j}}^{i} : V_{j} \rightarrow U$ be the inclusion. Then it suffices to show that

$$c\{U\} = \sum_{j \in J} \alpha_{V_{j}}^{i}(c\{V_{j}\}). \quad (5.4)$$

We prove $(5.4)$ by induction on $m$; the case $m = 0$ is trivially true.

In general, $V_{m} \subset U$ is a smooth closed subset. Let $U' = U \setminus V_{m}$, with induced smooth stratification $V' = \{V_{j} \mid j \in J' := \{0, \ldots, m - 1\}\}$ and open immersion $\beta : U' \rightarrow U$. By our hypothesis $(5.1)$, we have

$$c\{U\} = \alpha_{V_{m}}^{U}(c\{V_{m}\}) + \beta_{\iota}(c\{U'\})$$

and by our induction hypothesis, we have

$$c\{U'\} = \sum_{j = 0}^{m-1} \alpha_{V_{j}}^{U'}(c\{V_{j}\}).$$

These last two identities together with the functoriality of pushforward yield $(5.4)$. This completes the proof of the Proposition.

\qed
We now formulate a condition on a family \( \{ c_U^j \in E_n(\bar{U}/k) \}_{U \in \text{Sch}_k^{nd}} \), that will ensure we can define \( c \{ X \} \) for every \( X \in \text{Sch}_k^{nd} \).

**Definition 5.9** (good local data). Suppose we are given, for each \( U \in \text{Sm}_k \) and each good compactification \( i : U \to \bar{U} \) an element \( c_U^j \in E_n(\bar{U}/k) \); if \( U \) happens to be proper over \( k \), we write \( c(U) \) for \( c(U, id_U) \). We suppose in addition that if \( U \) is a disjoint union, \( \bar{U} = \bar{U}_1 \coprod \bar{U}_2 \), with inclusions \( j_i : \bar{U}_i \to \bar{U} \), then letting \( U_i = U \cap \bar{U}_i \), we have

\[
 c_{\bar{U}}^j = j_1*(c_{\bar{U}_1}^j) + j_2*(c_{\bar{U}_2}^j) \quad (5.5)
\]

Let \( (U, \bar{U}, W) \) be a triple with

- \( \bar{U} \in \text{Sm}_k \) a smooth variety,
- \( U \) a dense open subscheme of \( \bar{U} \),
- \( W \) a smooth closed subvariety of \( \bar{U} \),

such that \( D := \bar{U} \setminus U \) is a simple normal crossing divisor on \( \bar{U} \) and \( W \) intersects \( D \) with normal crossing.

Let \( \pi : \bar{V} = Bl_W(\bar{U}) \to \bar{U} \) be the blow-up of \( \bar{U} \) along \( W \), with exceptional divisor \( F \), and let \( w : W \to \bar{U} \) be the inclusion. Let \( \bar{D} \) be the proper transform \( \pi^{-1}[D] \). Since \( W \) intersects \( D \) with normal crossing, \( \bar{D} + F \) is a simple normal crossing divisor on \( \bar{V} \). Let \( Z := W \cap \bar{U} \), let \( V := \pi^{-1}(U) \), let \( E := \pi^{-1}(Z) = F \cap V \). Note that \( \bar{V} \) is a good compactification of \( U \), \( \bar{V} \) is a good closure of \( V \setminus E \), so \( c_{\bar{V}}^j \) and \( c_{\bar{V} \setminus E}^j \) are defined. Letting \( W_Z \subset W \) be the closure of \( Z \) in \( W \), \( W_Z \) is the union of the irreducible components of \( W \) that are not contained in \( D \), and \( Z \subset W_Z \) is a good closure of \( Z \). We write \( c_{\bar{V} \setminus E}^j \) for the image of \( c_{\bar{V} \setminus E}^j \) in \( E_n(W/k) \) via the pushforward for the inclusion \( W_Z \to W \); if \( Z = \emptyset \), we set \( c_{\bar{V} \setminus E}^j = 0 \).

We say the family \( \{ c_U^j \in E_n(\bar{U}/k) \}_{U \in \text{Sm}_k, U \in \mathcal{C}(U)} \) forms good local data if for each triple \( (U, \bar{U}, W) \) as above, we have

\[
 c_U^j = \pi_*c_{\bar{V} \setminus E}^j + w_*c_{\bar{W}}^j \in \bar{E}_n(\bar{U}). \quad (5.6)
\]

**Remark 5.10.** One could also frame Definition 5.9 with the additional requirement that \( W \) and \( U \) are irreducible. In fact, given a \( \{ c_U^j \in E_n(\bar{U}/k) \}_{U \in \text{Sm}_k, U \in \mathcal{C}(U)} \) that satisfies this restricted set of conditions, it follows from the assumed additivity of the classes \( c_U^j \) with respect to disjoint union, that this family forms good local data in the sense of the full Definition 5.9. It is however easier to apply Definition 5.9 to the problem at hand if we include the case of reducible \( U \) and \( W \) in the definition.

**Lemma 5.11.** Take \( U \in \text{Sm}_k \) and let \( i : Z \to U \) be a closed immersion with \( Z \in \text{Sm}_k \). Let \( i_U : U \to U' \) be a good closure. Then there is a triple \((U, \bar{U}, W)\) satisfying the conditions of Definition 5.9 with \( Z = W \cap U \) dense in \( W \), such that the rational map \( \bar{U} \dashrightarrow U' \) induced by the identity map on \( U \) is a morphism, and defines a map of good closures \( (U \leftarrow \bar{U}) \to U \) → \( U' \).

**Proof.** Let \( W' \subset U' \) be the closure of \( i_U(Z) \) and let \( D' = U' \setminus U \). By resolution of singularities, Definition 2.2(4), there is a proper birational morphism \( \pi : \bar{U} \to U' \) that is a sequence of blowups along smooth centers lying over \( W' \cap D' \), such that \( D := \pi^{-1}(D')_{\text{red}} \) is a simple normal crossing divisor on \( \bar{U} \), the proper transform \( W := \pi^{-1}[W'] \) is smooth, and \( W \) intersects \( D \) with normal crossing. In other words, the triple \((U, \bar{U}, W)\) satisfies the conditions of Definition 5.9, and \( Z = W \cap U \) is dense in \( W \). By construction the rational map \( \bar{U} \dashrightarrow U' \) induced by the identity map on \( U \) is a morphism, and thus define maps of good compactifications \( (U \leftarrow \bar{U}) \to U \) → \( U' \).

**Proposition 5.12.** Suppose that \( k \) admits resolution of singularities and weak factorization. Suppose we are given for each \( U \in \text{Sm}_k \) and each good closure \( i : U \to \bar{U} \), an element \( c_U^j \in E_n(\bar{U}/k) \) such that the family \( \{ c_U^j \in E_n(\bar{U}/k) \}_{U \in \text{Sm}_k, U \in \mathcal{C}(U)} \) forms good local data. Then

1. For each \( U \in \text{Sm}_k \), the family of elements \( \{ c_U^j \}_{U \in \mathcal{C}(U)} \) defines an element \( c \{ U \} \in \bar{E}_n(U/k) \).
2. For \( X \) in \( \text{Sch}_k^{nd} \) choose a smooth stratification \( \mathcal{U} := \{ U_i \}_{i \in I} \) with inclusions \( \alpha_{U_i} : U_i \to X \). Then \( c_\mathcal{U}(X) := \sum_{i \in I} a_{U_i}(c(U_i)) \in \bar{E}_n(X/k) \) is independent of the choice of \( \mathcal{U} \), giving rise to the well-defined element \( c \{ X \} := c_\mathcal{U}(X) \in \bar{E}_n(X/k) \).
Proof. For (1), we show that the family $\{c_U^j\}_{j \in \mathcal{C}_c(U)}$ satisfies the hypothesis of Lemma 5.5. Given a map of good closures $f : (i : U \to Z^i) \to (j : U \to Z^j)$ such that $f$ is the blow up of $Z^i$ along a smooth centre $W$ which intersects $Z^j \setminus j(U)$ with normal crossing, we take $U = Z^j$. This gives us the triple $(U, \bar{U}, W)$ satisfying the conditions of Definition 5.9, which has $Z = W \cap U$ and with $Z$ dense in $W$, gives a cofinal subcategory of $\mathcal{C}_c(U)$. Thus, the collection of identities (5.6) for each such triple $(U, \bar{U}, W)$ suffice to show that $c(U) = i_* c(Z) + j_* c(U \setminus Z)$ in $E_n(U/k)$. By Proposition 5.8 this shows that the classes $c_U(X) \in E_n(X/k)$ for $U$ a smooth stratification of $X \in \text{Sch}_k^{ed}$ are independent of the choice of $U$, proving (2).

5.3 Motivic pro-Euler class

5.3.1 Blow-up additive classes

In this section we start with given classes $c(U)$ for each smooth and proper $U$ over $k$. We formulate a condition, axiom 4.3, which ensures that those classes extend to define a class $c(X)$ for each $X \in \text{Sch}_k^{ed}$.

Definition 5.13. Let $\text{Sm}_k^p \subset \text{Sm}_k$ be the full subcategory of smooth and proper $k$-schemes. Suppose we are given an element $c(X) \in E_n(X/k)$ for each $X \in \text{Sm}_k^p$. We suppose in addition that if $X = X_1 \amalg X_2$ is a disjoint union of open subschemes with inclusion maps $i_j : X_j \to X$, then $c(X) = i_1(c(X_1)) + i_2(c(X_2))$. We call such a family $\{c(X) \in E_n(X/k)\}_{X \in \text{Sm}_k^p}$ an additive family of characteristic classes for smooth, proper $k$-schemes.

Let $(U, \bar{U})$ a pair of $U \in \text{Sch}_k$ and a good closure $i : U \to \bar{U}$. Let $D_1, \ldots, D_r$ be the irreducible components of $D$. For $\emptyset \neq I \subset \{1, \ldots, r\}$, define $D_I = \cap_{i \in I} D_i$, with closed immersion $i_{D_I} : D_I \to \bar{U}$. Set $D_\emptyset = U$ and let $|I|$ denote the cardinality of $I$. We define

$$c_U^0 = \sum_{I \subset \{1, \ldots, r\}} (-1)^{|I|} i_{D_I}^* c(D_I) \in E_n(\bar{U}).$$

Lemma 5.14. Let $\{c(X) \in E_n(X/k)\}_{X \in \text{Sm}_k^p}$ an additive family of characteristic classes for smooth projective $k$-schemes, and form the classes $c_U^0 \in E_n(\bar{U})$ following Definition 5.13. Take $V \in \text{Sm}_k^p$ and let $V \to \bar{V}$ be a good closure. Let $D = V \setminus \bar{V}$, and let $F \subset V$ be an smooth divisor such that $F + D$ is a simple normal crossing divisor on $V$; equivalently, $F$ intersects $D$ with normal crossing. Then $E = V \cap F$. Then $V$ is a good closure of $V \setminus E$, $F$ is a good closure of $E$, and we have

$$c_{V \setminus E} = c_V - i_* c_F.$$

Proof. We have $V \setminus (V \setminus E) = F + D$, a simple normal crossing divisor. Similarly, $F \setminus E = F \cap D$, and since $F$ intersects $D$ with normal crossing, $F \cap D$ is a simple normal crossing divisor on $F$. Thus $\bar{V}$ is a good closure of $V \setminus E$ and $F$ is a good closure of $E$.

Suppose $D$ has irreducible components $D_1, \ldots, D_r$. We consider the smooth divisor $F$ as the “component” $G_0$ of the simple normal crossing divisor $G := F + D$, and separate the strata $G_I$ of $G$ into those which involve $F$ and those which do not. This gives

$$c_{V \setminus E} = \sum_{I \subset \{1, \ldots, r\}} (-1)^{|I|} i_{D_I}^* c(D_I) + \sum_{I \subset \{1, \ldots, r\}} (-1)^{|I|+1} i_{D_I \cap F}^* c(D_I \cap F) = c_V - i_* c_F.$$

The additivity of the classes $c(-)$ with respect to disjoint union justifies the first identity.

Theorem 5.15. Suppose that $k$ admits resolution of singularities and weak factorization. Let $\{c(X) \in E_n(X/k)\}_{X \in \text{Sm}_k^p}$ be an additive family of characteristic classes for smooth, proper $k$-schemes, giving by
Definition 5.13\: the family of classes \( \{ c_U^i \in E_n(\bar{U}/k) \}_{U \in \text{Sm}_k, \bar{U} \in C^q(U)} \). Suppose that the family \( \{ c(X) \in E_n(X/k) \}_{X \in \text{Sm}_k} \) satisfies Axiom 4.3. Then

1. For each \( U \in \text{Sm}_k \), there is a unique element \( c(U) \in \bar{E}_n(U/k) \) such that, for each \( \bar{U} \in C^q(U) \), the \((U, U)\)-component of \( c(U) \) is \( c_U^i \in E_n(U/k) \).
2. For each \( X \in \text{Sch}^\text{red}_k \), there is a unique element \( c(X) \in \bar{E}_n(X/k) \) such that, for each smooth stratification \( U = \{ \alpha_i : U_i \to X \mid i \in I \} \) of \( X \), we have
   \[
   c(X) = \sum_{i \in I} \alpha_i \ast (c(U)) \in \bar{E}_n(X/k).
   \]

Proof. To prove (1) and (2), it suffices by Proposition 5.12 to show that the family of classes \( \{ c_U^i \in E_n(\bar{U}/k) \}_{U \in \text{Sm}_k, \bar{U} \in C^q(U)} \) forms good local data. Retaining the notation of Definition 5.9, we need to verify the additivity property (5.5) and the identity (5.6).

The additivity property (5.5) follows immediately from the additivity of the family \( \{ c(X) \in E_n(X/k) \}_{X \in \text{Sm}_k} \). To verify (5.6), let \((U, \bar{U})\) be a triple as in Definition 5.9, let \( \pi : V \to U \) be the blowup \( Bl_W \bar{U} \) with exceptional divisor \( F \subset V \), let \( w : W \to \bar{U} \), \( f : F \to V \) be the inclusions, let \( Z = U \cap W \), \( V = \pi^{-1}(U) \), and let \( E = V \cap F \). Let \( D \) be the simple normal crossing divisor \( \bar{U} \setminus U \) on \( \bar{U} \) and let \( \bar{D} \) be the proper transform \( \pi^{-1}[D] \).

Let \( D_1, \ldots, D_r \) be the irreducible components of \( D \) and let \( \bar{D}_i \) be the proper transform \( \pi^{-1}[D_i] \). For \( I \subset \{1, \ldots, r\} \) we have the strata \( D_I := \cap_{i \in I} D_i \) and \( \bar{D}_I := \cap_{i \in I} \bar{D}_i \). Let \( w_I : W \cap D_I \to D_I \), \( f_I : F \cap \bar{D}_I \to \bar{D}_I \) be the inclusions and let \( \pi_I : \bar{D}_I \to D_I \) be the restriction of \( \pi \). We claim that
   \[
   c(D_I) - w_{I*}c(D_I \cap W) = \pi_I * (c(\bar{D}_I) - f_{I*}c(\bar{D}_I \cap F))
   \]
for each \( I \subset \{1, \ldots, r\} \). Indeed, for \( I = \emptyset \), we have the blowup square
   
   \[
   \begin{array}{ccc}
   F & \xrightarrow{j} & \bar{V} \\
   \downarrow{r} & & \downarrow{\pi} \\
   W & \xrightarrow{w} & \bar{U}
   \end{array}
   \]

and since \( D_{\emptyset} = \bar{U}, \bar{D}_{\emptyset} = \bar{V}, \pi_{\emptyset} = \pi, f_{\emptyset} = f \), the identity follows from Axiom 4.3 applied to this blowup square. For non-empty \( I \), the fact that \( W \) intersects \( D \) with normal crossing implies that
   
   \[
   \begin{array}{ccc}
   \bar{D}_I \cap F & \xrightarrow{j} & \bar{D}_I \\
   \downarrow{r} & & \downarrow{r} \\
   D_I \cap W & \xrightarrow{w} & D_I
   \end{array}
   \]

is also a blowup square. Thus Axiom 4.3 applied to this blowup square gives the identity (5.7).

Since \( \bar{V} \) is a good closure of \( V \setminus E \) with \( \bar{D} + F = \bar{V} \setminus (V \setminus E) \), we have
   
   \[
   c_{\bar{V}} = \sum_{I \subset \{1, \ldots, r\}} (-1)^{|I|} i_{\bar{D}_I *} c(\bar{D}_I),
   \]
where \( i_{\bar{D}_I} : \bar{D}_I \to \bar{V} \) is the inclusion. Similarly,
   
   \[
   c_{\bar{U}} = \sum_{I \subset \{1, \ldots, r\}} (-1)^{|I|} i_{\bar{D}_I} c(\bar{D}_I),
   \]
where \( i_{\bar{D}_I} : \bar{D}_I \to \bar{U} \) is the inclusion and
   
   \[
   c_{\bar{Z}} = \sum_{I \subset \{1, \ldots, r\}} (-1)^{|I|} i_{D_I \cap W} c(D_I \cap W),
   \]
where \( i_{D_I} : D_I \to \bar{U} \) is the inclusion and

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where \( i_{D_I \cap W} : D_I \cap W \to W \) is the inclusion.

Applying Lemma 5.14 to the blowup square for \( D_I \cap W \subset D_I \) gives the identity

\[
\pi_I \ast (e(D_I) - f_I \ast (e(D_I \cap F))) = c(D_I) - w_I \ast (e(D_I \cap W))
\]

for each \( I \subset \{1, \ldots, r\} \). Using the functoriality of pushforward, we thus have

\[
\pi \ast (c^V - f \ast (c^E)) = \sum_{I \subset \{1, \ldots, r\}} (-1)^{|I|} i_{D_I} \ast (\pi_I \ast (c(D_I) - w_I \ast (e(D_I \cap W))
\]

This verifies the identity (5.6).

\[\square\]

5.3.2 pro-Euler class

Here we show that the blow-up additivity axiom 4.3 is satisfied for the Euler class of the tangent bundle in every motivic ring spectrum, allowing to define pro-Euler class \( e \{X\} \in \tilde{E}_0(X) \) for every variety \( X \in \text{Sch}_k^{red} \), by Proposition 4.4.

**Definition 5.16.** Let \( X \in \text{Sch}_k^{red} \) and let \( E \in \text{SH}(k) \). We define the pro-Euler class of \( X, e^E \{X\} \), as the unique class in \( \tilde{E}_0(X/k) \) extending the Euler class of the tangent bundle for smooth and proper schemes. This is well defined by using theorem 5.15 and Theorem 3.6, and according to Proposition 4.4.

**Remark 5.17.** This extends the class defined by Aluffi in [Alu06, Definition 4.1, Definition 4.4]. Let \( E = HZ \), then for \( X \) smooth and proper our \( e^{HZ} \{X\} \in \tilde{E}_0(X/k) \simeq E_0(X/k) = CH_0(X/k) \) is the Euler class of the tangent bundle, which the is top Chern class, the same as Aluffi’s class. For other \( X \), the class \( c^X \in CH_0(X/k) \) for a good compactification is the same as the one defined by Aluffi by the cut-and-paste property (see [Alu06, Proposition 3.2]). Therefore the class \( e^{HZ} \{X\} \) agrees with Aluffi’s \( \{X\}_0 \) ([Alu06, Definition 4.4]) for any \( X \).

**Remark 5.18.** In the recent preprint [JSY] Jin, Sun and Yang give an alternative approach for constructing pro-CSM class. By Proposition 4.4 this class should agree with ours, hence also agree with Aluffi’s class on pro-Chow groups as they independently prove.

6 Singular Gauss-Bonnet formula

Here we define the motivic Euler characteristic with compact support, and prove how to extend the motivic

**Definition 6.1.** Let \((\mathcal{C}, \otimes, \mathbb{I}_C)\) be a symmetric monoidal category. Let \( x \) be a strongly dualisable object of \( \mathcal{C} \), with \( x^! \) its dual, and \( \delta_x : \mathbb{I}_C \to x \otimes x^! \), \( ev_x : x^! \otimes x \to \mathbb{I}_C \) the co-evaluation and evaluation maps respectively.

For \( f : x \to x \) an endomorphism, the trace of \( f \) is the element \( tr(f) \in \text{End}_C(\mathbb{I}_C) \) defined as the composition

\[
\mathbb{I}_C \overset{\delta_x}{\longrightarrow} x \otimes x^! \overset{f \otimes id}{\longrightarrow} x \otimes x^! \overset{tr}{\longrightarrow} x^! \otimes x \overset{ev_x}{\longrightarrow} \mathbb{I}_C.
\]

In particular, taking \( f = id_x \), we have the categorical Euler characteristic \( \chi_C(x) := tr_x(id_x) \).

**Definition 6.2.** Let \( k \) be a field of characteristic 0, \( q : X \to k \) a \( k \)-scheme and \( \alpha \in \text{SH}(X) \) a constructible object. Then \( \chi_C(\alpha/k) \) is defined to be the categorical Euler characteristic of \( q_! \alpha \) in \( \text{SH}(k) \):

\[
\chi_C(\alpha/k) := \chi_{\text{SH}(k)}(q_! \alpha).
\]

In particular we define

\[
\chi_C(X/k) := \chi_C(\mathbb{I}_X/k) = \chi(q_! \mathbb{I}_X) := tr(id_{q_! \mathbb{I}_X})
\]

for every \( k \)-scheme \( q : X \to \text{Spec} \ k \). We write \( \chi_C(\alpha) \) for \( \chi_C(\alpha/k) \) when the base-field \( k \) is clear from the context.
From the definitions and above discussion we deduce

**Proposition 6.3** (cut and paste).

(a) $\chi_c(X \cup Y) = \chi_c(X) + \chi_c(Y) - \chi_c(X \cap Y)$.

(b) $e\{X \cup Y\} = i_X^* e\{X\} + i_Y^* e\{Y\} - i_{XY}^* e\{X \cap Y\}$ in $E_0(X \cup Y/k)$.

**Theorem 6.4** (Singular Gauss-Bonnet). Let $\pi : X \to \text{Spec } k$ a proper algebraic variety over a field of characteristic zero. Let $E \in SH(k)$ be a motivic spectrum and $i : \mathbb{I}_k \to E$ the unit map. Then

$$i_* \chi_c(X) = \pi_* e\{X\}.$$  

**Proof.** For $X$ smooth and proper the statement is the motivic Gauss-Bonnet theorem of [DJK, Theorem 4.6.1], also [LR] for $SL$-oriented theories. For a general variety $X$ it follows from the smooth and proper case, the existence of a smooth stratification, and the cut-and-paste formulas above.

**Definition 6.5.** Let $A$ be an abelian group. Given a constructible function $f : X \to A$ write

$$f = \sum n_i 1_{Z_i}$$

with $Z_i$ locally closed subvarieties. Define

$$e\{f\} = \sum n_i e\{Z_i\}$$

and

$$\chi_c(X, f) = \sum n_i \chi_c(Z_i).$$

**Corollary 6.6.** For a constructible function $f : X \to A$

$$i_* \chi_c(X, f) = \pi_* \{f\}.$$  

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