MIRROR SYMMETRY AND SELF-DUAL MANIFOLDS

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Abstract. We introduce self-dual manifolds and show that they can be used to encode mirror symmetry for affine-Kähler manifolds and for elliptic curves. Their geometric properties, especially the link with special lagrangian fibrations and the existence of a transformation similar to the Fourier-Mukai functor, suggest that this approach may be able to explain mirror symmetry also in other situations.

1. Introduction

Starting with the paper [SYZ] by Strominger, Yau and Zaslow, the search for a geometric counterpart to mirror symmetry has been directed mainly at special lagrangian fibrations of Calabi-Yau manifolds. In recent years however this approach has come under some criticism, because it appears unlikely that such fibrations will exist in general, at least of the well behaved kind required for mirror symmetry. Much of the research has therefore usually assumed the existence of such well behaved fibrations (see for example [Gross]), and has studied the behaviour "in the large complex structure limit" ([GW], [KS]). In this paper we propose that the "right" object to associate to a mirror pair of Calabi-Yau manifolds should not be a pair of lagrangian fibrations, but a self-dual manifold. As will become clear later, the special lagrangian fibrations then come back into the picture as a "Gromov-Hausdorff limits" of foliations on the self-dual manifold, and as such may be very singular and badly behaved in general. Instead, the self-dual manifold is expected to be smooth, and to contain in its structure the tools to explain mirror symmetry. We prove that this picture is correct in the case for complex dimension $n = 1$ (elliptic curves), and for affine-Kähler manifolds of any dimension. Although we did not attempt to include them here for reasons of space, we believe that the cases of abelian varieties and of $K3$’s should be within the reach of the techniques that we introduce.

The basic tool the following geometric data: a smooth Riemannian manifold $(X, g)$ of (real) dimension $3n$, together with two smooth differential forms of degree 2 on $X$, $\omega_1$ and $\omega_2$, which are compatible with the metric, in the sense that at all points $p \in X$ there is an orthonormal coframe

$$dx_1, \ldots, dx_n, dy_1^1, \ldots, dy_n^1, dy_1^2, \ldots, dy_n^2$$

such that $\omega_j = \sum_i dx_i \wedge dy_i^j$. Self-dual manifolds are objects of the above kind, with two more conditions on the data. For the precise definition, see the next section.

Of course, the final goal of explaining mirror symmetry in terms of self-dual manifolds, even if achievable, will require a lot of effort. In the present paper we provide some clues as to why we think our approach should work.

A self-dual manifold of real dimension $3n$ should be a way to interpolate between two mirror dual manifolds of complex dimension $n$, which can be recovered as natural Gromov-Hausdorff limits. Again, we prove this statement only for affine-Kähler manifolds and for elliptic curves. In general, this interpolation property is

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expected to happen at limit points in the moduli spaces of the mirror pair. However, in the case of elliptic curves and of affine-Kähler manifolds we show that this holds at all points.

One of the advantages of self-dual manifolds over the traditional approach via special lagrangian fibrations is that while in general the fibrations are expected to exist only in the limit, you just expect a self-dual manifold also at finite points; the original Calabi-Yaus are then quotients (with respect to foliations), which near the boundary of the moduli spaces became Gromov-Hausdorff limits.

To build the smooth self-dual manifold associated to a mirror pair we start from the fibre product of dual special lagrangian fibrations, which in our case don’t have singular fibres. It should be possible to use this method also when the fibrations have isolated singularities. We did not elaborate on this in the present paper.

Another advantage of self-dual manifolds is that their structure can be significantly weakened (to a polysymplectic structure) or strengthened (to a 2-Kähler structure). While polysymplectic manifolds share with symplectic ones the absence of local moduli, 2-Kähler manifolds are in a sense similar to hyperkähler ones (although they have dimension 3n). The rich algebraic structure of the cohomology of 2-Kähler manifolds is what brought us to their study in the first place, although we were not very successful in constructing smooth compact ones (except in the homogeneous cases). This might be just a temporary limitation, or might be due to some actual obstructions. In any case, we expect that the \(\mathfrak{sl}(4,\mathbb{R})\) representation which exists on the cohomology of compact 2-Kähler manifolds will be useful for the study of the cohomology of self-dual manifolds near boundary points of their moduli spaces.

An aspect which we did not develop in the present paper is the connection of self-dual manifolds with other constructions unrelated to mirror symmetry. For example, contact structures and Seifert fibrations come into play when studying self-dual manifolds of dimension 3. In a future paper we plan to investigate the relationship of 3n dimensional self-dual manifolds with \(c = 3n\) (super) conformal field theory and the geometry of PDE’s with target an \(n\)-dimensional manifold. On this last subject some material can already be found in the first part of [G].

Let us now come to a brief description of the contents of the various sections:

In section 2 we introduce our main object of study, self-dual manifolds. To do that, we choose to introduce first the weaker notions of polysymplectic manifold and of almost 2-Kähler manifold, because they will play a role later. Almost s-Kähler manifolds are polysymplectic manifolds together with a compatible metric. We show that in the case \(s = 2\) you have a natural dualizing form. When this form is closed, and the leaves of a certain foliation have all Riemannian volume one, you have self-dual manifolds. We show some natural ways of deforming almost 2-Kähler (and self-dual) manifolds, and finally we introduce a transformation which is similar in nature to the Fourier-Mukai transform, and will play a role if one will want to use self-dual manifolds to prove mirror symmetry.

In section 3 we show that fibre products of Riemannian lagrangian fibrations of almost Kähler manifolds over the same base give rise to almost 2-Kähler manifolds. We show that this applies to the significant case of special lagrangian tori fibrations of Calabi-Yau manifolds. Of course, you do not expect to obtain self-duality unless you start from a mirror pair.

In section 4 we apply the notions developed in the previous sections to show that self-dual manifolds can indeed be used to characterize mirror symmetry for affine-Kähler manifolds.

In section 5 we do the same that we did in the previous section, this time for elliptic curves. We also formulate a conjecture which deals with what to expect in the case of \(K3\) surfaces.
In section 6 we show that polysymplectic manifolds have no local moduli, and we prove that the space of metrics compatible with a given polysymplectic structure is non-empty and contractible. We then introduce 2-Kähler manifolds, and prove a characterization of them which generalizes a classical one for Kähler ones. We prove that a 2-Kähler manifold automatically has one of the two properties of self-dual manifolds (namely the dualizing form is closed), and that the natural deformations introduced in section 2 carry over to the 2-Kähler case.

In section 7 we show that there is a natural action of the Lie algebra \( \mathfrak{sl}(4, \mathbb{R}) \) on the forms of an almost 2-Kähler manifold, which in the 2-Kähler case carries over to an action on the harmonic forms.

Notation

For a form \( \alpha \), we indicate with \( \alpha^\perp \) the space of vectors which contract to zero with it. We indicate with \( T^*M \) the cotangent bundle to the manifold \( M \) and its total space. If \( \Gamma \) is a lattice inside an euclidean space, its dual lattice (with respect to the metric) is indicated with \( \Gamma^\vee \).

2. Self-dual manifolds

While the crucial notion is that of self-dual manifold (Definition 2.6), we first introduce polysymplectic and almost \( s \)-Kähler manifolds, which generalize symplectic and almost Kähler manifolds respectively.

Definition 2.1 (\[G\], Definition 2.1 page 12).

1) A polysymplectic structure on a vector space \( V \) of dimension \( n(s + 1) \) is given by \( s \) elements \( \omega_1, ..., \omega_s \) of \( \Lambda^2 V^* \) which, in some basis \( v_1, ..., v_n, w_1^*, ..., w^*_n \) for \( V \), have the polysymplectic normal form

\[
\omega_j = \sum_{i=1}^{n} v_i^* \wedge (w_i^j)^*
\]

Any such basis is called standard polysymplectic.

2) A polysymplectic manifold is given by a smooth manifold \( X \) of dimension \( n(s + 1) \), together with \( s \) smooth differential forms \( \omega_1, ..., \omega_s \) of rank 2 such that:

a) The forms \( \omega_j \) are closed,

b) At all points \( p \in X \) the forms \( (\omega_1)_p, ..., (\omega_s)_p \) determine a polysymplectic structure on \( T_pX \),

c) The distribution \( \sum_j \omega_j^\perp \) is integrable.

The notion of polysymplectic manifold reduces to that of symplectic manifold for \( s = 1 \), and in that case condition 2c) is automatically true. The case relevant for mirror symmetry is \( s = 2 \), and in this case condition 2c) does not follow from the other ones.

Example 2.2. Let \( M \) be a smooth manifold, and let \( T^*M \) indicate the cotangent bundle of \( M \). If

\[
X = T^*M \times_M \cdots \times_M T^*M \text{ (s times)},
\]

\( \pi_i : X \to T^*M \) is the projection on the \( i \)th factor, and \( \omega \) the canonical symplectic form on \( T^*M \), let \( \omega_i := \pi_i^*\omega \). We have then that \( (X, \omega_1, ..., \omega_s) \) is polysymplectic.

Definition 2.3 (\[G\], Definition 6.1 page 30). Let \( (X, \omega_1, ..., \omega_s) \) be a polysymplectic manifold, and let \( g \) be a Riemannian metric on \( X \). We say that \( g \) is compatible with the polysymplectic structure if for every \( p \in X \) there exists a standard polysymplectic basis of \( T_pX \) which is also orthonormal with respect to \( g \). In that case, we say that \( (X, \omega_1, ..., \omega_s, g) \) is an almost \( s \)-Kähler manifold.

Again, for \( s = 1 \) the previous notion reduces to the classical one of almost Kähler manifold.
Definition 2.4. Let $X = (X, \omega_1, \omega_2, g)$ be an almost 2-Kähler manifold. The dualizing form $\omega_D$ is the differential form of degree 2 defined at the point $p \in X$ as

$$\sum_{i=1}^{n} (w_i^1)^* \wedge (w_i^2)^*$$

for any orthonormal standard polysymplectic basis $v_1, \ldots, v_n, w_1, \ldots, w_n$ on $T_pX$.

Remark 2.5. The form $\omega_D$ is well defined, as if $\tilde{v}_1, \ldots, \tilde{v}_n, \tilde{w}_1, \ldots, \tilde{w}_n$ is another orthonormal standard polysymplectic basis, it is easy to see that we must have

$$(\tilde{w}_i^1)^* = \sum_j a_{ij} (w_j^1)^*, \quad (\tilde{w}_i^2)^* = \sum_j a_{ij} (w_j^2)^*$$

for some orthogonal matrix $(a_{ij})$, and therefore

$$\sum_i (\tilde{w}_i^1)^* \wedge \tilde{w}_i^2 = \sum_{i,j,k} a_{ij} a_{ik} (w_i^1)^* \wedge (w_k^2)^* = \sum_i (w_i^1)^* \wedge (w_i^2)^*$$

Definition 2.6. Let $X = (X, \omega_1, \omega_2, g)$ be an almost 2-Kähler manifold. We say that $X$ is a self-dual almost 2-Kähler manifold (or more briefly a self-dual manifold) if:

1) The differential form $\omega_D$ is closed.
2) The leaves of the foliation $\omega_1^1 + \omega_2^1$ have all (Riemannian) volume equal to one.

If only condition one holds, $X = (X, \omega_1, \omega_2, g)$ is a weakly self-dual manifold

Example 2.7. Let $l \in \mathbb{R}^+$, and let

$$X = \mathbb{R}/l\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \times \mathbb{R}/m\mathbb{Z}$$

Call $y^1$ (resp. $x$, $y^2$) the coordinate induced by $\mathbb{R}$ on the first (resp. second, third) factor. With this choice, and with

$$g = (dx)^2 + (dy^1)^2 + (dy^2)^2, \quad \omega_1 = dx \wedge dy^1, \quad \omega_2 = dx \wedge dy^2$$

we have that $(X, \omega_1, \omega_2, g)$ is weakly self-dual. If $lm = 1$ it is also self-dual.

Weakly self-dual manifolds have a very rich structure, and as we will see in the following, are rather easy to construct if you do not insist on them being compact. For now, let us just point out a feature which may look like a hyperkähler property (although, as the above example shows, there are compact self-dual manifolds of dimension 3)

Remark 2.8. Let $X = (X, \omega_1, \omega_2, g)$ be a weakly self-dual manifold. Then, if $\omega_D$ is its dualizing form and the distribution $\omega_D^1 + \omega_2^1$ is integrable, also $(X, \omega_D, \omega_1, g)$ is a weakly self-dual manifold. If the distribution $\omega_D^1 + \omega_2^2$ is integrable, also $(X, \omega_2, \omega_D, g)$ is a weakly self-dual manifold.

In all the examples that we will build in this paper, both the integrability conditions of the previous remark hold. In those cases, there are three different structures of weakly self-dual manifold on the same underlying Riemannian manifold.

Proposition 2.9. Let $\nabla$ be the Levi-Civita connection associated to the metric. If $\nabla \omega_1 = \nabla \omega_2 = 0$, then $\nabla \omega_D = 0$ and hence the manifold is weakly self-dual.

Proof If the two forms $\omega_1, \omega_2$ are covariant constant, then parallel transport, which is also orthogonal, will send any orthonormal standard polysymplectic basis to an orthonormal standard polysymplectic basis. From this and the same reasoning of Remark 2.8, we conclude that $\omega_D$ is sent to itself. □
There are various ways of deforming an almost 2-Kähler manifold. In the following definition we single out three of the most relevant ones for questions concerning Mirror Symmetry.

Definition 2.10. Let $X = (X, \omega_1, \omega_2, g)$ be an almost 2-Kähler manifold, and let $t \in \mathbb{R}^+$. We then define:

1) $\alpha_t(X) = (X, t\omega_1, \omega_2, \alpha_t(g))$

where $\alpha_t(g)$ is such that given any orthonormal standard polysymplectic basis $v_1, \ldots, v_n, w_1, \ldots, w_n$ of $T_pX$ (with respect to the given almost 2-Kähler structure on $X$), it assigns length squared $t$ to all the $v_i, w_i$ and length squared $t^{-1}$ to the $w_i^2$, for $i = 1, \ldots, n$.

2) $\beta_t(X) = (X, \omega_1, t\omega_2, \beta_t(g))$

is defined in the same way as $\alpha_t$, only with the indices 1 and 2 in the definition of the metric exchanged.

3) $\lambda_t(X) = (X, t\omega_1, t\omega_2, f(g))$

We omit the easy proof of the following proposition.

Proposition 2.11. Let $X$ be an almost 2-Kähler manifold. Then

1) $\alpha_t(X), \beta_t(X), \lambda_t(X)$ are almost 2-Kähler manifolds.
2) The deformations $\alpha_t$ and $\beta_t$ leave the dualizing form $\omega_D$ unchanged. In particular, if $X$ is (weakly) self-dual, then also $\alpha_t(X), \beta_t(X)$ are (weakly) self-dual.
3) If $\omega_1$ is (Levi-Civita) covariant constant with respect to $g$, then it is so also with respect to $\alpha_t(g), \beta_t(g)$.

The following maps are one of the ingredients of the Mirror correspondence, and are similar in nature and in behaviour to the Fourier-Mukai functor of [Muk].

Definition 2.12. Let $(X, \omega_1, \omega_2, g)$ be a self-dual manifold, and assume that there are surjections $\pi_1 : X \to X_1, \pi_2 : X \to X_2$ and $\pi_B : X \to B$ with compact fibres equal to leaves of the foliations $\omega_1^\perp, \omega_2^\perp$ and $\omega_1^\perp + \omega_2^\perp$ respectively. Let $\alpha \in \Omega^1(X_1)$. Define $S^{1 \to 2,j}(\alpha) \in \Omega^{1+2j-n}(X_2)$ as

$$S^{1 \to 2,j}(\alpha)_p(v_1, \ldots, v_{i+2j-n}) = \int_{\pi_2^{-1}(p)} \tilde{v}_1 \cdots \tilde{v}_{i+2j-n} \omega_D^{j} \wedge \pi_1^\perp \alpha$$

for $p \in X_2$, $v_k \in T_pX_2$ and $\tilde{v}_k$ the lifting of $v_k$ to a vector field along $\pi_2^{-1}(p)$ which projects to $v_k$ and is orthogonal to the fibre. Define also $S^{1 \to 2} = \sum_j S^{1 \to 2,j}$. Define $S^{2 \to 1,j}$ and $S^{2 \to 1}$ similarly, but with the indices 1 and 2 interchanged.

Although the previous definition has many similarities with that of the Fourier-Mukai functor, note that we are not assuming that the fibres of $X \to B$ are tori, or flat with the induced metric. We are not using an almost complex structure on the fibres, and the natural one induced by $\omega_D$ and by the metric is ”wrong” (in the case of elliptic fibrations of K3’s one would say that it is ”rotated” with respect to the one in which the Poincaré bundle is defined). Also note that we did not attempt to find the right sign in the definition of $S^{1 \to 2}$. 
3. Special lagrangian fibrations

We now build examples of almost 2-Kähler manifolds starting from Riemannian lagrangian fibrations of almost Kähler manifolds. When one starts from mirror dual semi-flat special lagrangian tori fibrations of Calabi-Yau manifolds, the almost 2-Kähler manifolds thus obtained (actually a small deformation of them) are conjectured to be self-dual. We will prove that this is the case at least in some situations in the next two sections.

The following conditions on a submersion have been already considered in the literature:

**Definition 3.1.** Let \((X, g)\) be a Riemannian manifold, let \(B\) be a smooth manifold, and let \(f : X \to B\) be a smooth submersion.

1) We say that \(f\) is Riemannian if there exists a (necessarily unique) Riemannian metric on \(B\) such that \(df\) is an isometry from \(\text{Ker}(df_p)\) to \(T_f(p)B\) for all \(p \in X\).
2) We say that \(f\) is covariant constant if it is Riemannian, and \(df\) commutes with parallel transport, i.e. if \(\gamma(t), t \in [0, 1]\) is a path in \(X\), \(G_X : T_{\gamma(0)}X \to T_{\gamma(1)}X\) is parallel transport in \(X\) along \(\gamma\), and \(G_B : T_{\gamma(0)}B \to T_{\gamma(1)}B\) is parallel transport in \(B\) along \(f(\gamma)\), then \(G_B(df_{\gamma(0)})(v)) = df_{\gamma(1)}(G_X(v))\) for all \(v \in T_pX\).

Recall that an almost Kähler manifold is a symplectic manifold together with a compatible Riemannian metric. We are now ready to state the main result of this section:

**Theorem 3.2.** Let \((X_i, \omega_{X_i}, g_{X_i})\) be almost Kähler manifolds of (real) dimension \(2n\), for \(i = 1, \ldots, s\). Let \(B\) be a smooth manifold of dimension \(n\), and let \(f_i : X_i \to B\) be surjections which are also lagrangian fibrations (with respect to the Kähler forms). Consider \(X = X_1 \times_B \cdots \times_B X_s\), with the metric \(g\) induced from \(X_1 \times \cdots \times X_s\) and with the \(2\)-forms \((\omega_1, \ldots, \omega_s)\), where \(\omega_i\) is \(\sqrt{s}\) times the pull-back of the Kähler form of \(X_i\), under the natural projection \(\pi_i : M \to X_i\). We then have that:

1) \((X, \omega_1, \ldots, \omega_s)\) is a polysymplectic manifold.
2) If all the \(f_i\) are Riemannian with respect to the same metric on \(B\), then \(g\) is compatible with the polysymplectic structure \(\omega_1, \ldots, \omega_s\). In other words, \((M, \omega_1, \ldots, \omega_s, g)\) is an almost \(s\)-Kähler manifold.
3) If moreover the \(\omega_{X_i}\) are covariant constant on the respective \(X_i\) and all the \(f_i\) are covariant constant, then all the \(\omega_j\) are covariant constant with respect to the metric \(g\) on \(M\).

**Proof**

1) The proof amounts to proving that the forms induce a polysymplectic structure pointwise, and that the distribution \(\sum_j \omega_j^\perp\) is integrable. The first fact is an easy linear algebra observation. For the second one, let \(F : X_1 \times_B \cdots \times_B X_s \to B\) be the induced map, which is a fibration. Then the required integrability follows from the fact that

\[\omega_1^\perp + \omega_2^\perp = \text{Ker}(dF)\]

2) Given \(p \in X\), we will show that there is an orthonormal polysymplectic basis of \(T_pM\). Pick an orthonormal basis \(v_1, \ldots, v_n\) of \(T_{f(p)}B\), and let \(z_1^1, \ldots, z_n^s\) be a set of vectors in \(\text{Ker}(df_{j(p)})\) (were \(p = (p_1, \ldots, p_s) \in X \subset X_1 \times \cdots \times X_s\)), such that \(df_j(z_i^j) = v_i\) for all \(i, j\). Because the \(f_j\) are Riemannian, it follows that the \(z_i^j\) are orthonormal (for fixed \(j\)). Define

\[w_i = \frac{1}{\sqrt{s}}(z_1^i, \ldots, z_s^i) \in T_p(X_1 \times \cdots \times X_s)\]
From their definition, it follows that the $w_i$ lie actually in $T_p M$. Moreover, $(w_1, w_m) = \delta_{lm}$. Define also

$$w_i^j = (0, ..., Jz_i^j, 0, 0) \ (j^{th} \ place)$$

in $T_p M$. We are indicating with $J$ the almost complex structure on the various $X_i$ (or the induced one on $X_1 \times \cdots \times X_s$, which is the same). The fact that the $w_i^j \in T_p M$ follows from the fact that $Jz_i^j \in \ker(d(f_j)_p)$, which is a consequence of the Lagrangian condition. It is now very easy to verify that $w_1, ..., w_n, w_1^j, ..., w_n^j$ is an orthonormal polysymplectic basis at $p$ with respect to the polysymplectic structure $\omega_1, ..., \omega_s$.

3) The forms $\omega_j$ are covariant constant on $X_1 \times \cdots \times X_s$ (because they are by hypothesis covariant constant on their respective $X_j$'s). If all the $f_j$ are covariant constant, then parallel transport on $X$ is then the restriction of parallel transport on $X_1 \times \cdots \times X_s$, and hence the $\omega_j$ are constant also on $X$.

The construction of the above theorem is natural enough to deserve a name. We will actually normalize the metric along the ”horizontal” directions for a reason which will be made clear by the remark following the definition.

**Definition 3.3.** Let

$$X_1 = (X_1, \omega_{X_1}, g_{X_1}), \ X_2 = (X_2, \omega_{X_2}, g_{X_2})$$

be Kähler manifolds, and let $f_i : (X_i, g_{X_i}) \rightarrow (B, g_B)$ be smooth Riemannian surjections which are also lagrangian fibrations, with $\dim(B) = n$. We then define the almost 2-Kähler manifold $X_1 \times_B X_2$ as

$$X_1 \times_B X_2 = \left( X_1 \times_B X_2, (f_1^* \omega_{X_1}, f_2^* \omega_{X_2}), g_{X_1 \times_B X_2} \right)$$

The metric $g_{X_1 \times_B X_2}$ is $\alpha \frac{1}{\sqrt{2}} \left( \beta i^* \left( i^*(g_{X_1} \times g_{X_2}) \right) \right)$ where $i^*$ is the pull-back along the inclusion, $g_{X_1} \times g_{X_2}$ is the product metric on $X_1 \times X_2$ and $\alpha, \beta$ are with respect to the almost 2-Kähler structure $(\sqrt{2} f_1^* \omega_{X_1}, \sqrt{2} f_2^* \omega_{X_2}, i^*(g_{X_1} \times g_{X_2}))$ given by the previous theorem.

The reason we adopted the definition above is that then we have the following.

**Remark 3.4.** Assume that $X_i, f_i : (X_i, g_{X_i}) \rightarrow (B, g_B)$ are as in the previous definition. Then the naturally induced maps

$$X_1 \times X_2 \rightarrow X_1, \ X_1 \times_B X_2 \rightarrow X_2, \ X_1 \times_B X_2 \rightarrow B$$

are smooth Riemannian surjections.

One could also use the previous remark (plus the fact that the forms $\omega_i$ are pull-backs of the Kähler forms on the $X_i$) to characterize $X_1 \times_B X_2$.

To put the condition of being Riemannian into perspective, and to make contact with Mirror Symmetry, we relate it with the semi-flatness condition of [SYZ], or rather with one of its consequences. We start by recalling the following standard definition:

**Definition 3.5.** Let $(X, \omega, g, \Omega)$ be a Calabi-Yau manifold of complex dimension $n$ (where $\omega$ is the Kähler form, $g$ the Kähler metric and $\Omega$ the globally defined nondegenerate holomorphic $n$-form).

1) We say that a submanifold $L \subset X$ is Special Lagrangian if it is Lagrangian (of maximal dimension) with respect to $\omega$, and there exists a complex number of the form $e^{i\theta}$ such that $\operatorname{Im}(e^{i\theta} \Omega)|_L = 0$. Such a $\theta$ is called the phase of the special lagrangian submanifold.

2) We say that a smooth map $f : X \rightarrow B$ to a smooth manifold $B$ of (real) dimension $n$ is a Special Lagrangian Fibration if $f$ is a submersion and for all
$q \in B$ the submanifold $L_q = f^{-1}(q) \subset X$ is a special lagrangian submanifold of $(X, \omega, g, \Omega)$. We require also that the phase of the fibres is constant.

3) A special lagrangian fibration is said semi-flat if the induced metrics on the fibres are flat.

**Proposition 3.6.** Let $(X, \omega_x, g_x, \Omega)_X$ be a Calabi-Yau manifold of complex dimension $n$. Let $f : X \to B$ be a special lagrangian fibration with compact connected fibres, such that the metric of $X$ restricted to any fibre is flat. Then $f$ is Riemannian.

**Proof.**

In view of the description of deformations of special lagrangian manifolds of [ML], it is enough to observe that harmonic forms on a flat manifold are covariant constant, and also their dual vector fields are covariant constant. As parallel transport is an isometry on any Riemannian manifold, and the complex endomorphism is also an isometry, this implies that on each fibre you have an orthonormal frame of vector fields, whose transformations under the complex involution give a complete set of first order normal deformations of the fibre itself. This clearly implies that $f$ is Riemannian.

**Remark 3.7.** In the situation of the above proposition, [ML] proves also that the map $f$ is a surjection to a smooth manifold of dimension $n$.

**Corollary 3.8.** Let $f_1 : X_1 \to B$ and $f_2 : X_2 \to B$ be semi-flat special lagrangian fibrations of Calabi-Yau manifolds. There is then a natural structure of almost 2-Kähler manifold on $X_1 \times_B X_2$. If the fibrations are covariant constant, then the forms $\omega_1, \omega_2$ and $\omega_D$ associated to this almost 2-Kähler structure are (Levi-Civita) covariant constant (and therefore, in particular, $(X_1 \times_B X_2, \omega_1, \omega_2, g)$ is weakly self-dual).

**Remark 3.9.** One does not expect actual special lagrangian fibrations to be covariant constant, except in flat cases (arising, from example, from special lagrangian fibrations of abelian varieties).

**Conjecture 3.1.** If we start from a mirror pair $X,Y$ of Calabi-Yau manifolds, then for each point near the large complex structure limit of $X$ there is a point near the large Kähler structure of $Y$ for which there are lagrangian tori fibrations of (dense open subsets of) $X$ and $Y$ over the same basis $B$ such that there is a ("small") deformation $h$ of the metric $g$ on $X \times_B Y$ for which $(X \times_B Y, \omega_1, \omega_2, h)$ is self-dual, where $\omega_1$ and $\omega_2$ are $\sqrt{2}$ times the pull-backs of the Kähler forms of $X$ and $Y$ respectively.

**Remark 3.10.** In general, we cannot expect to have fibrations of all of $X$ and $Y$ over the same $B$. We do expect however that $(X \times_B Y, \omega_1, \omega_2, h)$ admits a natural compactification (as a self-dual manifold).

Although the conjecture is not established in general, in the next two sections we will prove it in the "limit" situation of affine-Kähler manifolds, studied for example in the papers [KS] and [GW], and then for elliptic curves (in a refined form). In these two situations the behaviour is actually simpler than the general expected one, because we have dual special lagrangian fibrations over all of $X$ and $Y$ at all mirror pairs of points from their respective moduli spaces.

**Remark 3.11.** We do not expect the induced metric on $X \times_B Y$ to be self-dual (of even almost 2-Kähler) at finite points in the moduli spaces. The corrections
necessary in order to make it self-dual could be thought of as "quantum corrections" although, be warned, the deformed metric does not necessarily induce one either on $X$ or on $Y$.

The (conjectural) recipe to find the self-dual manifolds associated to mirror pairs is then the following: find "mirror dual" lagrangian fibrations on big enough open subsets $X$ (resp. $Y$) of the manifolds over the same basis $B$ by going near large complex (resp. Kähler) structure points of the moduli spaces. Put a self-dual metric on $X \times_B Y$, and then compactify what you obtained in the category of self-dual manifolds.

4. AFFINE-KÄHLER MANIFOLDS

In this section we will show how one can build self-dual manifolds starting from affine-Kähler manifolds. We first need to recall a definition and two lemmas from \cite{KS}:

**Definition 4.1** (\cite{KS}, Definition 2, page 17). An affine-Kähler manifold is a triple $(Y, g, \nabla)$ where $(Y, g)$ is a smooth Riemannian manifold with the metric $g$, and $\nabla$ is a flat connection on $TY$ such that:

a) $\nabla$ defines an affine structure on $Y$

b) Locally in affine coordinates $(x_1, \ldots, x_n)$ the matrix $(g_{ij})$ of $g$ is given by $g_{ij} = \partial^2 K/\partial x_i \partial x_j$ for some smooth real-valued function $K$ on $Y$.

If moreover one has that

c) The Monge-Ampère equation $\det(g_{ij}) = \text{const}$ is satisfied then $(Y, g, \nabla)$ is called an Monge-Ampère manifold.

**Lemma 4.2** (\cite{KS}, Proposition 2, section 3.2). For a given affine-Kähler manifold $(Y, g_Y, \nabla_Y)$ there is a canonically defined dual affine-Kähler manifold $(Y^\vee, g_{Y^\vee}, \nabla_{Y^\vee})$ such that $(Y, g_Y)$ is identified with $(Y^\vee, g_{Y^\vee})$ as Riemannian manifolds, and the local system $(T_Y^\vee, \nabla_{Y^\vee})$ is naturally isomorphic to the local system dual to $(TY, \nabla_Y)$.

**Lemma 4.3** (\cite{KS}, Corollary 1, section 3.2). If $\nabla_Y$ defines an integral affine structure on $Y$, then $\nabla_{Y^\vee}$ defines an integral affine structure on $Y^\vee$. As the dual covariant lattice one takes the lattice $(TY^{\ast})^\vee$ dual to $TY^\ast$ with respect to the metric $g_Y$.

**Theorem 4.4.** Let $(Y, g_Y, \nabla_Y)$ be a affine-Kähler manifold, such that $\nabla_Y$ defines an integral affine structure. Let $(Y^\vee, g_Y^\vee, \nabla_{Y^\vee}) = (Y, g_Y, \nabla_Y)$ be its dual affine-Kähler manifold. We then have that there is a canonically induced almost 2-Kähler structure $(\omega_1, \omega_2, h)$ on

$$X_Y = (TY/TY^\ast) \times_Y (TY/(TY^{\ast})^\vee)$$

and with this structure $X_Y$ is self-dual.

**Proof** To build the almost 2Kähler structure we first put a Riemannian metric on $TY$, using the flat connections given by the affine structure to select the orthogonal complements to the fibres of the projection to $Y$. The metric $g = g_Y$ on $Y$ then induces via the projection the metric on these horizontal distributions, and by translation that on the fibre directions. By construction we get that the fibration $TY \to Y$ is Riemannian. The Kähler form is the pull-back of the standard symplectic form of $T_Y^\ast$ to $TY$ via the map induced by the metric on $Y$. If we choose coordinates $x_1, \ldots, x_n$ on (an open set inside) $Y$, we have induced coordinates $(x_1, \ldots, x_n, y_1 = dx_1, \ldots, y_n = dx_n)$ on $TY$, and in these coordinates the symplectic form is

$$\omega_{TY} = \sum_{i,k} g_{ik}(x_1, \ldots, x_n) dx_i \wedge dy_k$$
It is immediate to verify that this form is compatible with the metric, and defines an almost complex structure which is integrable (this Kähler structure can be identified with that of $\mathbb{K}^S$). paragraph 5.2, once we identify $TY$ with $T_Y$ via the metric $g$.

The projection to $Y$ is Riemannian and lagrangian. This is all that is needed in Theorem 3.2 and Definition 3.3, and we get therefore an almost 2-Kähler structure also on the quotient

$$X_Y = TY \times_Y TY/\left(\pi_1^{-1}(TY^Z) \times \pi_2^{-1}(TY^{Z'})\right)$$

We continue to call $\pi_Y$ the projection from $X_Y$ to $Y$, which is Riemannian also with respect to $h$, and induces on $Y$ the metric $g$.

We now choose integral affine coordinates $x_1, ..., x_n$ on (an open set inside) $Y$, and indicate with $y^1_1, ..., y^1_n$ (resp. $y^2_1, ..., y^2_n$) the induced coordinates on the first copy (resp. the second copy) of $TY$. This determines coordinates

$$x_1, ..., x_n, y^1_1, ..., y^1_n, y^2_1, ..., y^2_n$$

on $TY \times_Y TY$, which can also be used locally on $X_Y$. With these coordinates,

$$\omega_j = \sum_{ik} g_{ik}(x_1, ..., x_n) dx_i \wedge dy^j_k$$

The affine coordinates $z_1, ..., z_n$ on $Y$ dual to $x_1, ..., x_n$ satisfy (by definition)

$$\frac{\partial z_i(x_1, ..., x_n)}{\partial x_k} = g_{ik}(x_1, ..., x_n)$$

and therefore the coordinates $w_1, ..., w_n$ in the fibre directions associated to $z_1, ..., z_n$ satisfy the relation $w_k = \sum_l g_{lk} y_l$. It follows that the leaves of the horizontal distribution associated to the connection dual to $\nabla_Y$ are described (locally) by $\{d(x_1, ..., x_n, \sum_l g^{ll} w_l, ..., \sum_l g^{nn} w_n)\}$, for numbers $w_1, ..., w_n$. The dual horizontal distribution at the point $(x_1, ..., y_n)$ of $TY$ is therefore generated by the vectors

$$\frac{\partial}{\partial x_i} + \sum_{lkm} y_m g_{mk} \frac{\partial g^{lk}}{\partial x_i} \frac{\partial}{\partial y_l}$$

The metric $h$ makes the vectors $v_l = (\frac{\partial}{\partial y^l}, \sum_{lkm} y_m g_{mk} \frac{\partial g^{lk}}{\partial y^l})$ orthogonal to the fibre directions of both the projections $\pi_1$ and $\pi_2$. Moreover,

$$\omega_1(\frac{\partial}{\partial y^1}, v_k) = -g_{ik} = \omega_2(\frac{\partial}{\partial y^2}, v_k)$$

and therefore the map from $\omega_2^1$ to $\omega_1^1$ induced by the almost 2-Kähler structure is simply induced by the correspondence $\frac{\partial}{\partial y^1} \to \frac{\partial}{\partial y^2}$. The form $\omega_D$ is simply the differential form associated to the composition of this map with the natural map in the dual of $\omega_1^0$ built with the metric, and therefore it will be of the form $\sum_i d\omega_l^i \wedge \alpha_i$, where $\alpha_i$ is the 1-form annihilating $\omega_2^1 + V$ and dual to $\frac{\partial}{\partial y^1}$ inside $\omega_1^0$, where $V$ is the orthogonal complement to the subspace $\omega_1^1 + \omega_2^1$. For the choice

$$\alpha_i = \sum_j g_{ij} dy^j - \sum_{lmkj} y_m g_{mk} \frac{\partial g^{lk}}{\partial x_j} dx_j$$
\[ \alpha_i(v_h) = \sum_{ilm} g_{ij} g_{mk} \frac{\partial g_{lk}}{\partial x_h} - \sum_{lmk} g_{jm} g_{mk} \frac{\partial g_{lk}}{\partial y_j} = 0, \quad \alpha_i\left(\frac{\partial}{\partial y_j}\right) = g_{ij} \]

and therefore

\[ \omega_D = \sum_i d y_i^1 \wedge \alpha_i = \sum_{ij} g_{ij} d y_i^1 \wedge d y_j^2 - \sum_{ijlmk} g_{jm} g_{mk} \frac{\partial g_{lk}}{\partial x_j} d y_i^1 \wedge d x_j \]

To show that \( X_Y \) is weakly self-dual, we must prove that \( d \omega_D = 0 \).

\[ d \omega_D = \sum_{ijkm} g_{ij} d y_i^1 \wedge d y_j^2 \wedge d x_k - \sum_{ijkmn} g_{jm} g_{mk} \frac{\partial g_{lk}}{\partial x_j} d y_i^1 \wedge d y_j^1 \wedge d x_j - \sum_{ijkmn} g_{jm} g_{mk} \frac{\partial g_{lk}}{\partial x_j} d y_i^1 \wedge d x_j \wedge d x_n \]

By changing name to the summation indices the vanishing of the first line is equivalent to that of

\[ \sum_{ij} \left( \frac{\partial g_{ij}}{\partial x_k} + \sum_{lm} g_{jm} g_{il} \frac{\partial g_{lm}}{\partial x_k} \right) d y_i^1 \wedge d y_j^2 \]

Multiplying with \( g^{ir} \) and summing over \( j \) we get that the vanishing of the first line is equivalent to the vanishing of \( \sum_j g^{ir} \frac{\partial g_{ij}}{\partial x_k} + \sum_i g_{il} \frac{\partial g^{ir}}{\partial x_k} = \frac{\partial}{\partial x_k} \left( \sum_j g^{ir} g_{ij} \right) = \frac{\partial}{\partial x_k} \delta_{ir} \)

and this is clearly zero.

Coming to the second line, it is clear that we need the vanishing of

\[ \sum_{jmn} \sum_{kl} \left( \frac{\partial g_{km}}{\partial x_n} g_{il} \frac{\partial g_{lk}}{\partial x_j} + g_{mk} \frac{\partial g_{il}}{\partial x_n} \frac{\partial g_{lk}}{\partial x_j} + g_{mk} \frac{\partial g_{il}}{\partial x_n} \frac{\partial g_{lk}}{\partial x_j} \right) d x_j \wedge d x_n \]

We are therefore reduced to proving the symmetry in \( j, n \) of

\[ \sum_{kl} \left( \frac{\partial g_{km}}{\partial x_n} g_{il} \frac{\partial g_{lk}}{\partial x_j} + g_{mk} \frac{\partial g_{il}}{\partial x_n} \frac{\partial g_{lk}}{\partial x_j} \right) \]

By multiplying by \( g^{ir} g^{ms} \) and summing over \( i \) and \( m \) we reduce this to

\[ \sum_{likm} \left( g^{ms} g^{ir} \frac{\partial g_{km}}{\partial x_n} g_{il} \frac{\partial g_{lk}}{\partial x_j} + g^{ms} g^{ir} g_{mk} \frac{\partial g_{il}}{\partial x_n} \frac{\partial g_{lk}}{\partial x_j} \right) = \sum_{km} g^{ms} \frac{\partial g_{mk}}{\partial x_n} \frac{\partial g^{rk}}{\partial x_j} + \]

\[ \sum_{li} g^{ir} \frac{\partial g_{il}}{\partial x_n} \frac{\partial g^{ls}}{\partial x_j} = - \sum_{km} g^{ms} \frac{\partial g^{rk}}{\partial x_j} - \sum_{li} g^{il} \frac{\partial g^{ls}}{\partial x_j} \]

and the last expression is clearly symmetric in \( j, n \).

We have therefore that \((X_Y, \omega_1, \omega_2, h)\) is weakly self-dual. To prove that it is self-dual it remains to be shown that the fibres of the projection to \( Y \) have all Riemannian volume one. This however is clear, as they are all of the form \( T \times T^\vee \), where \( T \) is a torus and \( T^\vee \) is its dual with respect to the metric, and it is a general fact that in this case \( \text{vol}(T \times T^\vee) = \text{vol}(T) \cdot \text{vol}(T^\vee) = 1 \).

5. Elliptic curves

Before going into the characterization of mirror symmetry for elliptic curves in terms of self-dual manifolds, we need to define it. In this case there are no ambiguities, and all is clear and settled by now. First we recall some terminology from [D1] (or equivalently from [PZ] or [D3]).

**Definition 5.1** (See for example [D1], pages 152-153). Let \((\tau, t) \in \mathbb{H} \times \mathbb{H} \), where \( \mathbb{H} \) is the upper half plane inside \( \mathbb{C} \). We associate to \((\tau, t)\) the complex manifold \( E_{\tau} = \mathbb{C}/\mathbb{Z} \tau \oplus \mathbb{Z} \) with the complexified Kähler form

\[ \omega^{\tau, t} = -\frac{t}{2 \text{Im}(\tau)} dz \wedge d\bar{z} \]
Such a pair \((E_\tau, \omega^{\tau,t})\) is also indicated with \(E_{\tau,t}\).

Notice that our complexified Kähler class must be multiplied by \(2\pi\) to recover that of \([D1]\). The imaginary part of \(\omega^{\tau,t}\) (multiplied by \(2\pi\)) is what is usually called the \(B\)-field, while the real part is a Kähler form on \(E_\tau\).

**Definition 5.2.** The elliptic curve with a complexified Kähler class \(E_{\tau,t}\) is mirror dual to the elliptic curve with a complexified Kähler class \(E_{t,\tau}\).

For a justification of the above definition, see for example \([PZ]\) or \([D1]\) and the references therein. We will not get into this justification here.

**Remark 5.3.** The natural projection map \(E_\tau \to B = S^1\) induced by the projection \(\mathbb{C} \to i\mathbb{R}\) is (special) lagrangian and Riemannian (with respect to the flat metric \(\frac{1}{\tau_2}(dx_1 + dy_1)\) on \(\mathbb{C}\)). With the induced metric the basis has length \(\sqrt{\tau_2}\).

**Definition 5.4.** Let \((\tau,t) = (\tau_1 + it_1, \tau_2 + it_2) \in \mathbb{H} \times \mathbb{H}\), where \(\mathbb{H}\) is the upper half plane inside \(\mathbb{C}\). We associate to \((\tau,t)\) the almost 2-Kähler manifold

\[
X_{\tau,t} = (X_{\tau,t}, \omega_1^{\tau,t}, \omega_2^{\tau,t}, g^{\tau,t})
\]

defined as \(X_{\tau,t} = E_\tau \times_B E_t\), with the almost 2-Kähler structure induced as in Definition 5.3 by the Kähler structures \(-i\frac{\tau_1}{\tau_2} dz \wedge d\bar{z}\) and \(-i\frac{\tau_2}{\tau_2} dz \wedge d\bar{z}\) on \(E_\tau\) and \(E_t\) respectively.

**Lemma 5.5.** For any choice of \((\tau,t) \in \mathbb{H} \times \mathbb{H}\), the almost 2-Kähler manifold \(X_{\tau,t}\) is self-dual.

**Proof** The forms \(\omega_1\) and \(\omega_2\) are covariant constant, therefore the manifold is automatically weakly self-dual. To check that the leaves of \(\omega_1^\perp + \omega_2^\perp\) have Riemannian volume one, observe that the leaves of \(E_\tau \to B\) are of the form \(\mathbb{R}/\mathbb{Z}\) with metric \(\frac{1}{\tau_2} dx\), while those of \(E_t \to B\) are of the form \(\mathbb{R}/\mathbb{Z}\) with metric \(\frac{1}{\tau_2} dx\). This proves that the volume of the product is one, as desired.

**Remark 5.6.** Let \(X = (X, \omega_1, \omega_2, g) \cong X_{\tau,t}\) as almost 2-Kähler manifold. We then have that the (a priori non-commutative) quotients

\[
E_1 = X/\omega_1, \quad E_2 = X/\omega_2, \quad B = X/\omega_1^\perp + \omega_2^\perp
\]

are smooth manifolds, and the natural projection maps \(\pi_i : X \to E_i\) are smooth Riemannian. Moreover, \((E_1, \pi_1, \omega_1, \pi_1, g)\) and \((E_2, \pi_2, \omega_1, \pi_2, g)\) are elliptic curves both fibred (with lagrangian Riemannian maps) onto \(B\). \(X\) can be recovered as \(E_1 \times_B E_2\) (with the induced almost 2-Kähler structure).

**Lemma 5.7.** The fibration \(X_{\tau,t}/\omega_1^\perp \to X_{\tau,t}/\omega_1^\perp + \omega_2^\perp\) is a principal fibration with group \(S^1\) and monodromy \(t_2 \in S^1\) around the generator of \(\pi_1\) of the basis. Similarly, the fibration \(X_{\tau,t}/\omega_2^\perp \to X_{\tau,t}/\omega_1^\perp + \omega_2^\perp\) is a principal fibration with group \(S^1\) and monodromy \(t_1 \in S^1\) around the generator of \(\pi_1\) of the basis.

**Proof** We are simply considering the fibration \(\mathbb{C}/\mathbb{Z} \to \mathbb{R}/it_2\mathbb{Z}\) induced by the fibration \(\mathbb{C} \to i\mathbb{R}\), with a multiple of the flat metric. The statement is then clear. The second statement is proved in the same way.

**Lemma 5.8.** Let \(X = X_{\tau,t}\) be the self-dual manifold associated to the pair \((\tau,t)\). We then have that:

1) The length of the manifold \(X/\omega_1^\perp + \omega_2^\perp\) (with the induced metric) is \(\sqrt{t_2\tau_2}\).
2) The length of the fibre of the fibration \(X \to X/\omega_1^\perp\) is \(\sqrt{\tau_1}\).
2) The length of the fibre of the fibration \(X \to X/\omega_2^\perp\) is \(\sqrt{\tau_2}\).
Proof All the statements are easy calculations. We omit the details.

The lemmas above imply the following

Theorem 5.9. The self-dual manifold $X_{\tau,t}$ determines (by a well defined procedure) the elliptic curves with a complexified Kähler class $E_{\tau,t}$ and $E_{t,\tau}$

The result above should not come as a surprise, as the self-dual manifold determines the elliptic curves together with a metric and a special lagrangian fibration.

Definition 5.10. We will indicate with

\[ X_{\tau,t} \to_1 E_{\tau,t}, \quad X_{\tau,t} \to_2 E_{t,\tau} \]

the content of the previous theorem

This definition allows us to state more precisely what has been proved:

Theorem 5.11. Let $E_1, E_2$ be elliptic curves together with complexified Kähler classes. Then the following are equivalent:
1) $E_1$ and $E_2$ form a mirror pair
2) There is a self-dual manifold of the form $X_{\tau,t}$ such that

\[ X \to_1 E_1, \quad X \to_2 E_2 \]

From our point of view, the situation of elliptic curves (and very likely of abelian varieties in general) is a degenerate one, in which the description in terms of self-dual manifolds and that in terms of $B$-fields are equivalent. In general, we expect that the description in terms of $B$-fields and special lagrangian fibrations holds only “in the limit”, while self-dual manifolds exist also at finite points, and converge to the limit situation near the boundary of the moduli space.

The following remark might be useful to recover another part of the classical terminology.

Remark 5.12. \[
(X_{\tau,t}, g^{\tau,t}) \to (E_{\tau}, g_{E_{\tau,t}})
\]

in the sense of Gromov-Hausdorff, for \( \frac{\text{Im}(\tau)}{\text{Im}(t)} \to +\infty \).

For reasons of space (and time) we do not analyze in detail what happens for elliptic fibrations of $K3$ surfaces, which (together with abelian varieties) would be the next natural step to take. However, to give the reader something to think about, we formulate a simple conjecture which relates the above constructions to those of [GW].

Let $X \to B$ and $\hat{X} \to B$ be mirror dual semi-flat special lagrangian fibrations of a general mirror pair of $K3$ surfaces over the same basis $B$ (take your favourite definition for what that is). This fibration will have for a general $K3$ exactly 24 singular fibres. Call $B^0$ the complement of the singular set inside $B$. As we have proven in Theorem 3.2, we then have an almost 2-Kähler structure on $X \times_{B^0} \hat{X}$.

Conjecture 5.1. In the situation described above, and with the induced almost 2-Kähler structure, $X \times_{B^0} \hat{X}$ is self-dual. Moreover, for all $b \in B$ the class of the form $\omega_D$ restricted to the fibre over $b$ is a constant (independent of $b$) multiple of the first Chern class of the Poincaré bundle of that fibre (once you rotate the complex structure of $X$ and $\hat{X}$ to make the fibrations to $B$ holomorphic). The self-dual manifold $X \times_{B^0} \hat{X}$ admits a natural compactification to a smooth compact self-dual manifold.

This conjecture should shed some light on the nature of the form $\omega_D$, and on why we expect it to be closed in situations arising from mirror pairs.
6. Polysymplectic and 2-Kähler manifolds

In this section we describe some of what happens if we weaken (in the polysymplectic case) or strengthen (in the 2-Kähler case) the condition of self-duality. These two notions were introduced in [3], where the reader can go to find a more detailed study. As the following will be general considerations, we will not need to stick to the case \( s = 2 \). Remember however that we defined self-dual manifolds only for \( s = 2 \) (although it would be easy to generalize the definition to general \( s > 1 \)).

**Theorem 6.1** (Polysymplectic normal form). Let \((X, \omega_1, \ldots, \omega_s)\) be a smooth polysymplectic manifold and \( p \in X \). Assume given elements \( \phi_1, \ldots, \phi_n, \psi^1, \ldots, \psi_s^p \) of \( T^*_p X \) such that for all \( j = 1, \ldots, s \) one has \( (\omega_j)_p = \sum_i \phi_i \wedge \psi^i_j \). Then there are a neighborhood \( U \subset X \) of \( p \in X \), a neighborhood \( V \subset \mathbb{R}^{\dim(X)} \) of \( 0 \in \mathbb{R}^{\dim(X)} \) and an isomorphism of polysymplectic manifolds

\[
\phi : (U, \omega_1, \ldots, \omega_s) \to \left( V, \sum_i dx_i \wedge dy^i_1, \ldots, \sum_i dx_i \wedge dy^i_s \right)
\]

where we indicated the coordinates on \( \mathbb{R}^{\dim(X)} \) with \( x_1, \ldots, x_n, y^1_1, \ldots, y^n_s \). With this notation, one can also assume \((dx_i)_p = \phi, (dy^j_p)_p = \psi^j_i \).

**Proof**

If \( V \) is a vector space, given an element of \( \alpha \in \mathcal{A}^1(V) \) we indicate with \( C(\alpha) \) the smallest subspace \( W \subset V \) such that \( \alpha \in \mathcal{A}^1 D \). A priori, the \( C(\omega_j) \) are only "generalized Pfaffian systems", as defined for example in [1, Page 382]. From Darboux’s Reduction Theorem, in the form stated for example in [FUB, Bryant, Page 103], we see that \( C(\omega_j) \) is a vector bundle (of rank \( 2n \)) for any \( j = 1, \ldots, s \), with local coframes given by closed 1-forms.

Define \( C^X = \bigcap_j C(\omega_j) \). Then \( C^X = \left( \sum_j \omega^j_1 \right)^\perp \) is a constant rank distribution of subspaces of \( T^* X \), and by the definition of a polysymplectic structure and Frobenius it is locally generated by closed 1-forms. From the constant rank property, we may assume that there are (locally) \( n \) functions \( x_1, \ldots, x_n \) such that \( dx_1, \ldots, dx_n \) are independent, and for all \( q \) in the open set considered

\[
\langle (dx_1)_p, \ldots, (dx_1)_p \rangle = C^X
\]

By acting if necessary with a constant transformation matrix we can assume that \( \forall i \ (dx_i)_p = \phi_i \).

Fix now an index \( j \in \{1, \ldots, s\} \). From Darboux’s reduction theorem, we can find coordinates \( z_1, \ldots, z_d \) such that \( \omega_j \) is expressed only in terms of \( z_{d-2n+1}, \ldots, z_d \), and such that \( \frac{\partial}{\partial z_k} \) is in \( C(\omega_j)^\perp \) for \( k = 1, \ldots, d-2n \) (and therefore one has also \( < dz_{d-2n+1}, \ldots, dz_d > = C(\omega_j) \)). From their definition, it follows that \( \frac{\partial}{\partial z_k} = 0 \) for all \( i \), and for \( k = 1, \ldots, d-2n \). Therefore, we can apply the theorem of Carathéodory-Jacobi-Lie (see [LM, Page 136]) to conclude that there are functions \( y^i_j \) (depending only on the \( z_{d-2n+1}, \ldots, z_d \)) such that \( dy^i_j \in C(\omega_j) \) and \( \omega_j = \sum_i dx_i \wedge dy^i_j \). Because \( < dx_1, \ldots, dx_n, dy^1_j, \ldots, dy^n_j > = C(\omega_j) \), by an invertible linear transformation inside \( C(\omega_j) \) (with constant coefficients) leaving all the \( dx_i \) fixed we can also assume that \( dy^i_j = \psi^j_i \).

After repeating the procedure for all \( j \), we end up with functions \( x_1, \ldots, x_n, y^1_1, \ldots, y^n_s \) near \( p \in X \). The \( x_1, \ldots, x_n, y^1_1, \ldots, y^n_s \) form a system of coordinates because the \( dx_1, \ldots, dx_n, dy^1_1, \ldots, dy^n_s \) are independent forms.

One could try to give a more conceptual proof, similar to Moser’s proof of the theorem of Darboux for symplectic manifolds, using the techniques of [3]. This however would have taken us too far away from the theme of the present work.
Corollary 6.2. Let \( M \) be a smooth manifold, and \( \omega_1, \ldots, \omega_2 \) be smooth 2-forms on it. The following are then equivalent:
1) \((M, \omega_1, \ldots, \omega_s)\) is a polysymplectic manifold.
2) For all \( p \in M \) there are coordinates \( x_1, \ldots, x_n, y_1^1, \ldots, y_n^s \) near \( p \) such that
   \[
   \forall j \omega_j = \sum_i dx_i \wedge dy_i^j
   \]

This characterization of polysymplectic manifolds makes clear why we consider them a natural generalization of symplectic ones.

Theorem 6.3. Let \((M, \omega_1, \ldots, \omega_s)\) be a polysymplectic manifold. The space of Riemannian metrics on \( M \) compatible with the polysymplectic structure is non-empty and contractible.

For the purposes of this proof, we give the following definition.

Definition 6.4. Let \((V, \omega_1, \ldots, \omega_s)\) be a vector space with a non-degenerate polysymplectic structure, \( s > 1 \). A Riemannian metric \( g \) on \( V \) is block-compatible with the polysymplectic structure if there exists a polysymplectic basis \( e_1, \ldots, e_n, f_1^1, \ldots, f_n^s \) such that for all \( i, m, j, k \) (with \( j \neq k \))
   \[
   g(e_i, f_m^j) = g(f_k^j, f_m^j) = 0
   \]

Lemma 6.5. Let \((V, \omega_1, \ldots, \omega_s)\) be a vector space with a polysymplectic structure, \( s > 1 \), and let \( g_1 \) and \( g_2 \) be two Riemannian metrics on \( V \) block-compatible with the polysymplectic structure, and such that their restrictions to the span of the spaces \( \omega_1^j \) coincide. If \( t \in [0, 1] \), then the Riemannian metric \( tg_1 + (1-t)g_2 \) is also block-compatible with the polysymplectic structure.

Proof

In view of the block-compatibility of the two metrics with the polysymplectic structure there are vectors \( d_1, \ldots, d_n, f_1, \ldots, f_n, f_1^1, \ldots, f_n^s \) such that \( e_1, \ldots, e_n, f_1, \ldots, f_n \) and \( d_1, \ldots, d_n, f_1^1, \ldots, f_n^s \) are polysymplectic bases, and for \( j \neq k \)
   \[
   g_1(e_i, f_m^j) = g_1(f_k^j, f_m^j) = 0, g_2(d_i, f_m^j) = g_2(f_k^j, f_m^j) = 0
   \]

Moreover, we can take for all \( j \) bases \( h_1^j, \ldots, h_n^j \) of the span of \( f_1^j, \ldots, f_n^j \), orthonormal with respect to \( g_1 \) (and therefore also with respect to \( g_2 \)). We do not require such bases \( h_1^j, \ldots, h_n^j \) to be part of a polysymplectic basis. Such a basis exists because of the hypothesis on the behaviour of the two metrics on the span of \( f_1^j, \ldots, f_n^j \).

Observe first that if we define the vectors
   \[
   f_i(t) = e_i + \sum_{k,m} (t-1)g_2(e_i, h_m^k)h_m^k,
   \]
then for all \( i, j, m \)
   \[
   (tg_1 + (1-t)g_2)(f_i(t), h_m^j) = 0
   \]

We now observe that there must be \( \eta_{ik}^m \) such that \( d_i = e_i + \sum_{k,m} \eta_{ik}^m h_m^k \). From the fact that \( g_2(d_i, h_m^k) = 0 \), we deduce that \( \eta_{ik}^m = -g_2(e_i, h_m^k) \). This shows that \( f_i(t) = t e_i + (1-t)d_i \) for all \( t \), or in other words \( f_i(t) = e_i + (t-1) \sum_{k,m} \eta_{ik}^m h_m^k \), from which it is easy to deduce that \( f_1(t), \ldots, f_n(t), f_1^1, \ldots, f_n^s \) is a polysymplectic basis for all \( t \). This polysymplectic basis shows that \( tg_1 + (1-t)g_2 \) is block-compatible with the polysymplectic structure.

Lemma 6.6. Let \((M, \omega_1, \ldots, \omega_s)\) be a (non-degenerate) polysymplectic manifold. There exists a Riemannian metric on \( M \) block-compatible point by point with the polysymplectic structure.
Proof
Pick a covering of $M$ by polysymplectic coordinate sets $U_\alpha$, and a partition of unity \{\alpha_\} subordinated to the covering.

Observe first that if $g_1$ and $g_2$ are two Riemannian metrics on $M$ such that for all points $p \in M$ and for any polysymplectic basis $e_1, ..., e_n, f_1, ..., f_n$ of $T_p M$, for $j \neq k$, $g_1(f_j^k, f_m^k) = 0 = g_2(f_j^k, f_m^k)$, then also $t g_1 + (1-t) g_2$ has this property. Therefore, by using the polysymplectic coordinates on the sets $U_\alpha$, and the partition of unity to sum, we can easily define a Riemannian metric $g$ on all of $M$ which has the property above at all points $p \in M$. Define now a family $g_\alpha$ of block-compatible metrics on any fixed open set $U_\alpha$, with the property that $g_\alpha$ coincides with the fixed $g$ on the span of $f_1^1, ..., f_n^1$ and for some, and therefore any, polysymplectic basis. Using the partition of unity, and the previous lemma, we see that we can sum all these metrics to provide a globally defined block-compatible Riemannian metric. \hfill $\blacksquare$

**Lemma 6.7.** Let $(M, \omega_1, ..., \omega_s)$ be a (non-degenerate) polysymplectic manifold. There is then a one to one correspondence between the following data:
1) A Riemannian metric on $M$, compatible with the polysymplectic structure.
2) A positive definite non-degenerate symmetric bilinear form $g^1$ on $\bigcap_{j \geq 1} \omega_j^\perp$, plus a constant rank distribution of subspaces $W$ of $TM$, such that at each point $p \in M$ and for some polysymplectic basis $e_1, ..., e_n, f_1, ..., f_n$ of $T_p M$, $g^1|_{T_p M}$ is supported on the span of $f_1^1, ..., f_n^1$, and $W_p = \langle e_1, ..., e_n \rangle$.

In the direction from 1) to 2) the correspondence sends a metric $g$ to the bilinear form $g^1$ and the subspace $W$ defined for any $p$ and any polysymplectic basis $e_1, ..., e_n, f_1, ..., f_n$ of $T_p M$ as $g^1|_{T_p M} = g|_{\langle f_1^1, ..., f_n^1 \rangle}$ and $W_p = \langle f_1^1, ..., f_n^1 \rangle$ respectively.

**Proof**
In the direction from 1) to 2), to check that the correspondence is well defined it is enough to observe that $W_p = \langle e_1, ..., e_n \rangle$ for any orthonormal polysymplectic basis $e_1, ..., e_n, f_1, ..., f_n$. In the direction from 2) to 1), to define $g|_{T_p M}$ choose any polysymplectic basis $e_1, ..., e_n, f_1, ..., f_n$ such that $W_p = \langle e_1, ..., e_n \rangle$, and $f_1, ..., f_n$ is $g^1|_{T_p M}$-orthonormal. Then declare any such basis to be $g$-orthonormal. To check that this definition is correct, suppose given any other polysymplectic basis with the same property. Then it is immediate to check, using the observation that if a matrix is orthogonal also the transpose of its inverse is so (and actually coincides with it), that the transition matrix from one basis to the other is orthogonal, and therefore $g$ is well defined. By construction, the metric $g$ is Riemannian, and compatible with the polysymplectic structure point by point. The verification that the metric defined varies smoothly as $p$ varies in $M$ is straightforward, and left to the reader.

Both the correspondences thus defined are one to one and onto, as they are one the inverse of the other. \hfill $\blacksquare$

**Proof of the theorem**
Pick any globally defined block-compatible Riemannian metric $g_0$ on $M$, which exists from Lemma 6.6. At any given point $p \in M$, pick any polysymplectic basis $e_1, ..., e_n, f_1, ..., f_n$, and consider the bilinear form $g^1|_{T_p M} = g|_{\langle f_1^1, ..., f_n^1 \rangle}$ and the subspace $W_p = \langle f_1^1, ..., f_n^1 \rangle$. The bilinear form $g^1$ and the distribution of subspaces $W$ thus defined determine uniquely a Riemannian metric compatible with the polysymplectic structure, in view of Lemma 6.7.

To see that the space of compatible metrics is contractible, pick any metric $g_0$ in it. Using Lemma 6.7, it is easy to see that there is a canonical way to interpolate between $g_0$ and any other metric $g$ compatible with the polysymplectic structure, and that this interpolation procedure provides a retraction of the space of compatible metrics to its point $g_0$. \hfill $\blacksquare$
Definition 6.8 ([3], Definition 7.2 page 35). A smooth manifold \( M \) of dimension \( n(s + 1) \) together with a Riemannian metric \( g \) and 2-forms \( \omega_1, \ldots, \omega_s \) is s-Kähler if the data satisfies the following property: For each point of \( M \) there exist an open neighborhood \( U \) of \( p \) and a system of coordinates \( x_i, y^i_j, i = 1, \ldots, n, j = 1, \ldots, s \) on \( U \) such that:

1) \( \forall \ j \ \omega_j = \sum_i dx_i \wedge dy^i_j \),
2) \( g_{(x,y)} = \sum_i dx_i \otimes dx_i + \sum_{i,j} dy^i_j \otimes dy^i_j + O(2) \).

Any such system of coordinates is called standard (s-Kähler).

Theorem 6.9. Let \( (X, \omega_1, \ldots, \omega_s) \) be a polysymplectic manifold, and let \( g \) be a Riemannian metric on \( X \), compatible with the polysymplectic structure. The following are then equivalent:

1) \( (X, \omega_1, \ldots, \omega_s, g) \) is an s-Kähler manifold.
2) \( \nabla \omega_j = 0 \) for all vector fields \( X \) and \( j = 1, \ldots, s \).

Proof The case \( s = 1 \) is classical, and we therefore omit the proof.

\( s \geq 2 \):

Let now \( M \) be a smooth manifold of dimension \( n(s + 1) \), with \( s > 1 \), and let \( \omega_1, \ldots, \omega_s \) and \( g \) be as defined in condition 2). Let \( p \) be a point of \( M \). Pick any standard polysymplectic coordinate system \( x_i, y^i_j, i = 1, \ldots, n, j = 1, \ldots, s \) centered at \( p \), defined on a neighborhood \( U \) of \( p \) and such that:

1) \( \forall \ j \ \omega_j = \sum_i dx_i \wedge dy^i_j \),
2) \( g_p = \sum_i dx_i dx_i + \sum_{i,j} dy^i_j dy^i_j \)

i.e. such that the induced coframe on \( T_p M \) is orthonormal. Such a coordinate system exists from the definition of almost s-Kähler manifold and from Theorem 5.3.

From the fact that \( \nabla \omega_j = 0 \) for all \( j \), we deduce that parallel transport preserves the polysymplectic structure, and therefore it must preserve also the standard subspaces associated to it, among which are the

\[
\langle \frac{\partial}{\partial y^1_i}, \ldots, \frac{\partial}{\partial y^1_n} \rangle, \ldots, \langle \frac{\partial}{\partial y^s_1}, \ldots, \frac{\partial}{\partial y^s_n} \rangle
\]

From this we deduce that for any vector field \( X \)

\[
\nabla_X \frac{\partial}{\partial y^1_i} = \sum_l \frac{\partial}{\partial x_l} \left( \nabla_X \frac{\partial}{\partial y^1_i} \right) \frac{\partial}{\partial y^1_l}, \ldots, \nabla_X \frac{\partial}{\partial y^s_i} = \sum_l \frac{\partial}{\partial y^s_l} \left( \nabla_X \frac{\partial}{\partial y^s_l} \right) \frac{\partial}{\partial y^s_l}
\]

As a consequence, \( \nabla \omega_j \wedge dx_i = -\sum_m \Gamma_{im}^l dx_m \), where \( \Gamma_{im}^l = dx_l \left( \nabla \frac{\partial}{\partial y^i_m} \right) \) are the usual Christoffel symbols. We will use the index notation \( 1, \ldots, n, (1), \ldots, (ns) \) to indicate the \( n(s + 1) \) indices for the coordinates \( x_i, y_i^j, i = 1, \ldots, n, j = 1, \ldots, s \).

The above considerations then amount to the fact that \( \Gamma_{im}^{(ij)} = 0 \) for any index \( \alpha \), any numbers \( i, m \) in the set \( \{1, \ldots, n\} \) and any number \( j \) in the set \( \{1, \ldots, s\} \).

Consider now a coordinate change of the form

\[
\tilde{x}_i = x_i + \sum_{mp} b_{ij}^m x_m x_p, \quad \tilde{y}_i^j = y_i^j (x_1, \ldots, x_n, y_1^j, \ldots, y_s^j)
\]

where the functions \( \tilde{y}_i^j \) are determined according to the Theorem of Carathéodory-Jacobi-Lie ([LM] Theorem 13.4 Page 136), so that

\[
\omega_j = \sum_i dx_i \wedge dy^i_j, \quad \tilde{y}_i^j (0, \ldots, 0) = 0
\]

Note that it is crucial that the functions \( \tilde{x}_i \) are in involution with respect to the Poisson structures associated (in the respective \( x_1, \ldots, x_n, y_1^j, \ldots, y_s^j \) spaces) to the various symplectic forms \( \omega_1, \ldots, \omega_s \). In view of the previous considerations, we see that also in the new coordinates we have \( \nabla \omega_j \wedge \tilde{x}_i = \sum_{mp} \Gamma_{im}^{(ij)} \tilde{x}_p \), if the \( \Gamma \) are the
Christoffel symbols in the new coordinates, and moreover
\[ \nabla_{\frac{\partial}{\partial x^m}} d\tilde{x}_l = \nabla_{\frac{\partial}{\partial x^m}} d\tilde{x}_l + O(1) = \]
\[ \nabla_{\frac{\partial}{\partial x^m}} \left( d\tilde{x}_l + \sum_{ip} b^i_{lp} x_d d\tilde{x}_p \right) + O(1) = \sum_p \left( -\Gamma^l_{mp} + b^l_{mp} \right) d\tilde{x}_p + O(1). \]
As it is also the case that \( \nabla_{\frac{\partial}{\partial x^m}} = -\sum_p \tilde{\Gamma}^l_{mp} d\tilde{x}_p \), if we choose \( b^l_{mp} = \Gamma^l_{mp}(0) \) (which we can do as the connection is torsion-free), we see that the symbols \( \tilde{\Gamma}^l_{mp} \) in the new coordinate system vanish at the origin. For simplicity, we will indicate the new coordinates with \( x_i, y_i^j \), and the Christoffel symbols associated to them with \( \Gamma \), dropping the tilde everywhere. We know also that for any index \( \alpha \), and indicating with \( (\cdot)_0 \) the evaluation of a form at 0,
\[ 0 = (\nabla_\alpha \omega_j)_0 = \left( \nabla_\alpha \sum_i d\tilde{x}_i \wedge dy_i^j \right)_0 = \left( \sum_i \left( dx_i \wedge (\nabla_\alpha dy_i^j) \right)_0 \right). \]
\[ -\sum_i \left( dx_i \wedge (\sum_{mk} \Gamma^{(ij)}_{\alpha(mk)}(0) dy_k^m + \sum_m \Gamma^{(ij)}_{\alpha m}(0) dx_m) \right)_0. \]
From this we deduce that \( \Gamma^{(ij)}_{\alpha(mk)}(0) = 0 \) and \( \Gamma^{(ij)}_{\alpha m}(0) = \Gamma^{(mj)}_{\alpha i}(0) \) for all \( i, j, k, m, \alpha \).
We consider therefore the change of coordinates
\[ \tilde{y}_i^j = y_i^j + \sum_{mp} \Gamma^{(ij)}_{mp}(0) x_m x_p, \quad \tilde{x}_i = x_i \]
In the new coordinates we have
\[ \sum_i dx_i \wedge d\tilde{y}_i^j = \sum_i dx_i \wedge (dy_i^j + \sum_{mp} \Gamma^{(ij)}_{mp}(0) x_m dx_p) = \omega_j, \]
as we showed before that \( \Gamma^{(ij)}_{mp}(0) = \Gamma^{(pj)}_{mi}(0) \). All the equations for the Christoffel symbols that we have deduced so far still hold, because we did not make any assumption on the \( y_i^j \) when we obtained them, apart from the fact that we were in polysymplectic coordinates. Moreover, we have that
\[ \frac{\partial}{\partial x^i} \left( \nabla_\alpha dy_i^j + \sum_{mp} \Gamma^{(ij)}_{mp}(0) \nabla_\alpha x_m dx_p \right) = \]
\[ -\sum_m \Gamma^{(ij)}_{lm}(0) dx_m + \sum_p \Gamma^{(ij)}_{lp}(0) dx_p = 0. \]
From the previous equation, the symmetry of the Christoffel symbols coming from the fact that the connection is torsion-free, and the vanishing properties proved above, we see that all the Christoffel symbols vanish at 0.
We know from the compatibility of the polysymplectic structure with the metric that there is a linear change of coordinates which sends the given coframe at 0 to an orthonormal (but still polysymplectic) one. It follows that the same linear change, applied to the functions \( x_i, y_i^j \) will preserve the polysymplectic property, and will make the coframe at 0 orthonormal. Moreover, will not disrupt the vanishing property (at 0) of the Christoffel symbols.
On the other hand, from the vanishing at the origin of all the Christoffel symbols (and the fact that the coordinate coframe at 0 is orthonormal) it is straightforward to deduce that \( g = \sum_i dx_i \otimes dx_i + \sum_{i,j} dy_i^j \otimes dy_i^j + O(2) \).

**Proposition 6.10.** If \( X \) is 2-Kähler and \( t \in \mathbb{R}^+ \), then also \( \alpha_t(X) \), \( \beta_t(X) \) and \( \lambda_t(X) \) are 2-Kähler.

**Proof**
The statement can be proved locally, where it is clear, using any one of the characterizations of 2-Kähler manifolds.

From Remark 2.8 and Proposition 2.9 we obtain the following two remarks:
Remark 6.11. Let \( X = (X, \omega_1, \omega_2, g) \) be an almost 2-Kähler manifold with a dualizing form \( \omega_D \). If \( X \) is 2-Kähler (i.e. if \( \omega_1, \omega_2 \) are covariant constant), then \( \omega_D \) is covariant constant. In particular, \((X, \omega_1, \omega_2, g)\) is weakly self-dual.

Remark 6.12. On a 2-Kähler manifold there are three different 2-Kähler structures, which are all, from the previous remark, weakly self-dual.

We expect that 2-Kähler manifolds will show up as limits of self-dual ones, at limit point of the moduli space where there is some control on the diameter of the manifold. For this reason we expect that the representation on cohomology of 2-Kähler manifolds described in the next section should be preserved on the monodromy invariant part of the cohomology near well-behaved singularities of almost 2-Kähler manifolds.

Remark 6.13. The following question arises naturally from the above results: given a self-dual manifold \((X, \omega_1, \omega_2, g)\), are there obstructions to deforming \( g \) to a new metric \( h \) for which \((X, \omega_1, \omega_2, h)\) is 2-Kähler?

7. A REPRESENTATION OF \( \mathfrak{sl}(4, \mathbb{R}) \)

In this section we define a family of operators (together with their adjoints and associated commutators) which generalize to \( s \geq 1 \) the standard Lefschetz operator of Kähler manifolds. Throughout the first part of this section, we assume fixed an almost 2-Kähler manifold \((X, \omega_1, \omega_2, g)\).

Definition 7.1. The operators \( L_0, L_1, L_2 \) acting on \( \Lambda^s T^* X \), for an almost 2-Kähler manifold \((X, \omega_1, \omega_2, g)\) are defined as

\[
L_0(\alpha) = \omega_D \wedge \alpha, \quad L_1(\alpha) = \omega_1 \wedge \alpha, \quad L_2(\alpha) = \omega_2 \wedge \alpha
\]

In the first part of this section we will prove the following:

Theorem 7.2. The operators \( L_0, L_1, L_2, L_0^*, L_1^*, L_2^* \) generate a Lie algebra naturally isomorphic to \( \mathfrak{sl}(4, \mathbb{R}) \) acting on the bundle \( \Lambda^s T^* X \)

Remark 7.3. To define the adjoint \( L_i^* \) to \( L_i \), we simply used the pointwise definition \( \forall \alpha, \beta \in \Lambda^s T^*_p X \) \( g(L_{i\alpha}, \beta) = g(\alpha, L_i^* \beta) \).

Before going into the proof, let us remark that the same methods that we will use would show that on an almost \( s \)-Kähler manifold the similarly defined operators generate the real Lie algebra associated to \( D_{s+1} \) on the fibres (and the smooth global sections) of \( \Lambda^s T^* X \). Note also that the methods of proof of this section are similar to the ones that are used to show that on the complex cotangent bundle of a Kähler manifold you have a \( \mathfrak{sl}(2) \)-action.

Definition 7.4. 1) For \( k \in \{0, 1, 2\} \) the operators \( E^k_i, E_i^k \) on \( \Lambda^s T^*_p X \), for an orthonormal standard polysymplectic coordinate coframe \( dx_1, \ldots, dx_n, dy_1, \ldots, dy_n \) at \( p \), are defined as

\[
E^0_i(\alpha) = dx_i \wedge \alpha, \quad E^k_i(\alpha) = dy^k_i \wedge \alpha \quad \text{for} \ k \in \{1, 2\},
\]

\[
E^{\bar{0}}_i = \frac{\partial}{\partial x_i} \rightarrow \alpha, \quad E^{\bar{k}}_i(\alpha) = \frac{\partial}{\partial y^k_i} \rightarrow \alpha \quad \text{for} \ k \in \{1, 2\}
\]

2) The operators \( L_{i\alpha} \) on \( \Lambda^s T^*_p X \) with \( \alpha, \beta \in \{0, 1, 2, \bar{0}, \bar{1}, \bar{2}\} \) are defined as

\[
L_{i\alpha}(\phi) = \sum_i E^0_i E^{\bar{0}}_i(\phi) \quad \text{for} \ \alpha, \beta \in \{0, 1, 2, \bar{0}, \bar{1}, \bar{2}\}
\]
Lemma 7.5. The operators $L_{\alpha \beta}$ are well defined independently of a choice of a basis, and are therefore also well defined as operators acting on the bundle $\Lambda^* T^* X$.

Proof The almost 2-Kähler structure determines canonical isomorphisms of subspaces

$$< dx_1, ..., dx_n > \cong < dy_1, ..., dy_n > \cong < dy_1^2, ..., dy_n^2 >$$

and their duals. The operators $L_{\alpha \beta}$ can be interpreted as canonical symplectic forms on spaces of the form $V \times V'$ using these identifications, and are therefore well defined independently of a choice of an orthonormal standard polysymplectic basis.

Remark 7.6. 1) For any differential form $\phi \in \Omega^* X$, we have

$$L_{01}(\phi) = L_1(\phi) = \omega_1 \wedge \phi, \quad L_{02}(\phi) = L_2(\phi) = \omega_2 \wedge \phi, \quad L_{12}(\phi) = L_0(\phi) = \omega_D \wedge \phi$$

2) $L_{\alpha \beta}^* = L_{\bar{\beta} \bar{\alpha}}$ for $\alpha, \beta \in \{0, 1, 2, 0, 1, 2\}$

The reasoning in the proofs that follow in this section is very similar to the one that applies to Kähler manifolds, used for example in [GH] Pages 106-114.

Lemma 7.7. The following relations hold among the operators $E_\alpha^\alpha$:

1) $E_i^\alpha E_j^\beta = -E_j^\beta E_i^\alpha$, for $(i, \alpha) \neq (j, \bar{\beta})$
2) $E_i^\alpha E_i^\alpha + E_i^\beta E_i^\beta = Id$ for all $i \in \{1, ..., n\}, \alpha \in \{0, 1, 2, 0, 1, 2\}$

Proof These identities are easily verified, using the anti-commutativity property of the wedge product.

Lemma 7.8. For $\{\alpha, \beta, \gamma\} \subset \{0, 1, 2, 0, 1, 2\}$,

1) $L_{\alpha \beta} = -L_{\beta \alpha}$ when $\alpha \neq \gamma$
2) $[L_{\alpha \beta}, L_{\bar{\gamma} \bar{\delta}}] = L_{\bar{\delta} \bar{\gamma}} - L_{\alpha \gamma}$ when $\alpha \neq \beta$
3) $[L_{\alpha \gamma}, L_{\bar{\beta} \bar{\delta}}] = L_{\alpha \beta} \bar{\gamma}$ when $\alpha \neq \gamma, \beta \neq \gamma$
4) $[L_{\alpha \beta}, L_{\gamma \delta}] = 0$ when $\{\alpha, \beta\} \cap \{\gamma, \delta\} = \emptyset$
5) $[L_{\alpha \gamma}, L_{\bar{\beta} \bar{\delta}}] = 0$ when $\alpha \neq \gamma, \beta \neq \gamma$

Proof We can restrict to $\Lambda^* T_p^* X$ for $p \in X$, and use the operators $E_i^\alpha$ with respect to some (any) orthonormal standard polysymplectic basis.

1) Immediate from Lemma 7.7.

2) $[L_{\alpha \beta}, L_{\bar{\gamma} \bar{\delta}}] = \left( \sum_i E_i^\alpha E_i^\delta \right) \left( \sum_j E_j^\beta E_j^\bar{\gamma} \right) - \left( \sum_j E_j^\beta E_j^\bar{\delta} \right) \left( \sum_i E_i^\alpha E_i^\bar{\gamma} \right) = \sum_{i \neq j} \left( E_i^\alpha E_i^\beta E_j^\bar{\gamma} - E_j^\beta E_j^\bar{\gamma} E_i^\alpha \right) + \sum_i \left( E_i^\alpha E_i^\beta E_i^\bar{\gamma} - E_j^\beta E_j^\bar{\gamma} E_i^\alpha \right) = \sum_i \left( E_i^\beta E_i^\bar{\gamma} - E_i^\beta E_i^\bar{\delta} + E_i^\beta E_i^\bar{\delta} \right) E_i^\alpha = \sum \left( E_i^\beta E_i^\bar{\gamma} - E_i^\beta E_i^\bar{\delta} \right) E_i^\alpha$

3) $[L_{\alpha \gamma}, L_{\bar{\beta} \bar{\delta}}] = \left( \sum_i E_i^\alpha E_i^\gamma \right) \left( \sum_j E_j^\beta E_j^\bar{\gamma} \right) - \left( \sum_j E_j^\beta E_j^\gamma \right) \left( \sum_i E_i^\alpha E_i^\bar{\gamma} \right) = \sum_{i \neq j} \left( E_i^\alpha E_i^\beta E_j^\gamma - E_j^\beta E_j^\gamma E_i^\alpha \right) + \sum_i \left( E_i^\alpha E_i^\beta E_i^\gamma - E_j^\beta E_j^\gamma E_i^\alpha \right) = \sum_i \left( E_i^\alpha E_i^\beta E_i^\gamma - E_j^\beta E_j^\gamma E_j^\alpha \right) = \sum_i \left( E_i^\alpha E_i^\beta E_i^\gamma - E_i^\beta E_i^\bar{\gamma} \right) = \sum_i L_i^\alpha = L_{\alpha \gamma}$

4) Immediate from Lemma 7.7.

5) $[L_{\alpha \gamma}, L_{\bar{\beta} \bar{\delta}}] = \left( \sum_i E_i^\alpha E_i^\gamma \right) \left( \sum_j E_j^\beta E_j^\bar{\gamma} \right) - \left( \sum_j E_j^\beta E_j^\gamma \right) \left( \sum_i E_i^\alpha E_i^\bar{\gamma} \right) = \sum_{i \neq j} \left( E_i^\alpha E_i^\beta E_j^\gamma - E_j^\beta E_j^\gamma E_i^\alpha \right) + \sum_i \left( E_i^\alpha E_i^\beta E_i^\gamma - E_j^\beta E_j^\gamma E_i^\alpha \right) = 0$
Proof of Theorem 7.2
To identify the Lie algebra generated by the $L_{\alpha, \beta}$ with $\mathfrak{sl}(4, \mathbb{R})$ we first write a Chevalley basis $e_0, e_1, e_2, f_0, f_1, f_2, h_0, h_1, h_2$ for $\mathfrak{sl}(4, \mathbb{R})$ satisfying

$$[e_i, f_j] = \delta_{ij}h_i, \quad [e_i, h_j] = a_{ij}e_i, \quad [f_i, h_j] = -a_{ij}f_i$$

(where $(a_{ij})$ is the Cartan matrix for $A_3$) in terms of the $L_{\alpha, \beta}$. Let

$$e_0 = L_{01}, \quad e_1 = L_{12}, \quad e_2 = L_{10},$$

$$f_0 = L_{01}, \quad f_1 = L_{12}, \quad f_2 = L_{10},$$

$$h_0 = L_{00} - L_{11}, \quad h_1 = L_{11} - L_{22}, \quad h_2 = L_{11} - L_{00}.$$  

We then have from Lemma 7.8 (2) that $[e_i, f_j] = h_i$ and from 4) $[e_i, f_j] = 0$ for $i \neq j$.

Moreover,

$$[e_0, h_0] = [L_{01}, L_{00} - L_{11}] = -[L_{10}, L_{00}] + [L_{01}, L_{11}] = -L_{10} + L_{01} = 2e_0$$

$$[e_1, h_0] = [L_{12}, L_{00} - L_{11}] = L_{21} - L_{21} = 0$$

$$[e_2, h_0] = [L_{10}, L_{00} - L_{11}] = L_{01} + L_{10} = 0$$

$$[e_0, h_1] = [L_{01}, L_{01} - L_{22}] = -L_{01} = -e_0$$

$$[e_1, h_1] = [L_{12}, L_{11} - L_{22}] = L_{21} - L_{21} = 0$$

$$[e_2, h_1] = [L_{10}, L_{11} - L_{22}] = L_{10} = 0$$

$$[e_0, h_2] = [L_{01}, L_{11} - L_{00}] = L_{01} = 0$$

$$[e_1, h_2] = [L_{12}, L_{11} - L_{00}] = L_{11} = 0$$

$$[e_2, h_2] = [L_{10}, L_{11} - L_{00}] = -L_{01} + L_{10} = 2e_2$$

We now complete the set of identities which we began to describe in Lemma 7.8. These last identities will allow us to show that we have a representation of the Lie algebra $\mathfrak{sl}(4, \mathbb{R})$ on the cohomology of an $s$-Kähler manifold, induced by the representation on the space of forms described in Theorem 7.2. This will be done showing that the Laplacian $\Delta_d$ commutes with the action of $\mathfrak{sl}(4, \mathbb{R})$.

**Theorem 7.9** (2-Kähler identities).

Let $(X, \omega_1, \omega_2, g)$ be an oriented $s$-Kähler manifold. Then we have that:

1) $$[L_{hk}, d] = 0 \forall \{h, k\} \subset \{0, 1, 2\}$$

2) If we define $d^c_{hk} := [L_{hk}, d^c]$, we have that

$$dd^c_{hk} + d^c_{hk}d = 0 \forall \{h, k\} \subset \{0, 1, 2\}$$

3) $$[L_{hk}, \Delta_d] = [L_{h\bar{k}}, \Delta_d] = 0 \forall \{h, k\} \subset \{0, 1, 2\}$$

where $\Delta_d$ is the $d$-Laplacian relative to the metric $g$ and to the orientation.

4) $$[L_{\alpha, \beta}, \Delta_d] = 0 \forall \{\alpha, \beta\} \subset \{0, 1, 2, \bar{0}, \bar{1}, \bar{2}\}$$

**Proof**

1) This equation follows immediately from the fact that $d\omega_1 = d\omega_2 = d\omega_d = 0$.

2) If we write down the expression for $d^c_{hk}$ in standard 2-Kähler coordinates centered at a point $p \in X$, we see that no derivative of the metric appears. Therefore, when we write down the expression for $dd^c_{hk} + d^c_{hk}d$, only the first derivatives of the metric are involved. We skip the details, as they are completely analogous to those of, for example, [31], Pages 111-115].

It follows, as in the classical case of Kähler manifolds, that to prove the equation it is enough to reduce to the case of a constant metric. When the metric is flat, however, the equation is easily seen to be equivalent (using 1) to $[L_{hk}, \Delta_d] = 0$, which with a flat metric follows immediately from the fact that $L_{hh}$ is constant in
flat (orthonormal) coordinates.
3) The second equation is the adjoint of the first. The first one, once written down explicitly in terms of \( d \) and \( d^* \), follows immediately from points 1) – 2).
4) This follows from the previous point, the Jacobi identity and the fact that the Lie algebra of the \( \mathbf{L}_{\alpha,\beta} \) is generated by \( \{ \mathbf{L}_{\alpha \beta} \} \cup \{ \mathbf{L}_{\alpha \bar{\beta}} \} \)

**Corollary 7.10.** Let \((X, \omega_1, \omega_2, g)\) be an oriented 2-Kähler manifold. Then there is a canonical representation of the simple Lie algebra \( \mathfrak{sl}(4, \mathbb{R}) \) on the space \( \mathcal{H}^*(X, \mathbb{R}) \) of harmonic forms on \( X \)

There is a clear similarity between the representation of \( \mathfrak{sl}(4, \mathbb{R}) \) described in this section and the representations described in [LL]. Namely, in both cases one obtains a semi-simple Lie algebra starting from an abelian set of generators, by adding their "\( \mathfrak{sl}(2) \) adjoints", which still commute among each other. And the space on which these operators act is itself a graded algebra. However, in our case it seems that the representation that you obtain is not a Jordan-Lefschetz module (see [LL] for the definition), because, even if separately the operators satisfy a form of Lefschetz duality, there does not seem to exist a unique grading associated to the dualities of all of them. In any case, it would be interesting to investigate the connections with the cited work.

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