Prime Clocks

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Abstract

Physical implementations of digital computers began in the latter half of the 1930’s and were first constructed from various forms of logic gates. Based on the prime numbers, we introduce prime clocks and prime clock sums, where the clocks utilize time and act as computational primitives instead of gates. The prime clocks generate an infinite abelian group, where for each \( n \), there is a finite subgroup \( S \) such that for each Boolean function \( f : \{0,1\}^n \rightarrow \{0,1\} \), there exists a finite prime clock sum in \( S \) that can represent and compute \( f \). A parallelizable algorithm, implemented with a finite prime clock sum, is provided that computes \( f \). In contrast, the negation \( \neg \), conjunction \( \land \), and disjunction \( \lor \) operations generate a Boolean algebra. In terms of computation, Boolean circuits computed with logic gates NOT, AND, OR have a depth. This means that a completely parallel computation of Boolean functions is not possible with these gates. Overall, some new connections between number theory, Boolean functions and computation are established.

1 Introduction

1.1 Notation and Preliminaries

Symbol \( \mathbb{Z} \) denotes the integers and \( \mathbb{N} \) the non-negative integers. For any \( n \in \mathbb{N} \) such that \( n \geq 2 \) and \( a \in \mathbb{N} \) such that \( 0 \leq a \leq n-1 \), consider the equivalence class \( [a] = \{a + kn : k \in \mathbb{Z}\} \) that is a subset of \( \mathbb{Z} \). Let \( \mathbb{Z}_n = \{[0],[1],\ldots,[n-1]\} \). \( a \mod n \) is the remainder when \( a \) is divided by \( n \). In the standard manner, \((\mathbb{Z}_n, +_n)\) is an abelian group, where binary operator \(+_n\) is defined as \([a] +_n [b] = [(a+b) \mod n]\). The brackets are sometimes omitted and \([a] \in \mathbb{Z}_n\) is represented with the integer \( a \), satisfying \( 0 \leq a \leq n-1 \). The set of all functions \( f : \mathbb{N} \rightarrow \mathbb{Z}_n \) is denoted as \( \mathbb{Z}_n^\mathbb{N} \). Symbol \( c \) is the constant function \( f : \mathbb{N} \rightarrow \mathbb{N} \) where \( f(m) = c \) for all \( m \in \mathbb{N} \). The set of all \( n \)-bit strings is \( \{0,1\}^n \).

It is convenient to identify the 2 bits in \( \{0,1\} \) with the elements \([0] \) and \([1]\) in \( \mathbb{Z}_2 \). The least common multiple of positive integers \( a \) and \( b \) is \( \text{lcm}(a,b) \). Let \( p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7, \ldots \) where the \( n \)th prime number is \( p_n \). Let \( p \) be an odd prime. \( p \) is called a 3 mod 4 prime if \( \frac{p-1}{2} \) is odd. \( p \) is called a 1 mod 4 prime if \( \frac{p-1}{2} \) is even.

1.2 Intuition and Motivation for Prime Clocks

Physical implementations of digital computers began in the latter half of the 1930’s and early designs were based on various implementations of logic gates [1, 4, 5, 17, 18, 19] (e.g., mechanical switches, electro-mechanical devices, vacuum tubes). The transistor was conceptually invented [9, 10] in the late 1920’s, but the first working prototype [2, 13] was not demonstrated until 1947. Transistors act
as building blocks for logic gates when they operate above threshold [11]. The transistor enabled the invention of the integrated circuit [8, 12], which is the physical basis for modern digital computers.

As an alternative to gates, prime clocks are based on the prime numbers and the notion of a common clock. Consider the prime number 2 and the clock [2, 0]. The 2 means that the clock has two states \(\{0, 1\}\) and the 0 means that the clock starts ticking from state 0 at time 0. Shown in column 2 of table 1, the clock [2, 0] ticks 0, 1, 0, 1, and so on. In column 3 of table 1, the clock [3, 1] has 3 states \(\{0, 1, 2\}\) and ticks 1, 2, 0, 1, 2, 0 and so on.

| Time | [2, 0] | [3, 1] | [2, 0] ⊕ [3, 1] | [7, 3] | [13, 6] | [7, 3] ⊕ [13, 6] |
|------|-------|-------|----------------|-------|--------|-----------------|
| 0    | 0     | 1     | 1              | 3     | 6      | 1               |
| 1    | 1     | 2     | 1              | 4     | 7      | 1               |
| 2    | 0     | 0     | 0              | 5     | 8      | 1               |
| 3    | 1     | 1     | 0              | 6     | 9      | 1               |
| 4    | 0     | 2     | 0              | 0     | 10     | 0               |
| 5    | 1     | 0     | 1              | 1     | 11     | 0               |

Expressed as \(\oplus\) in table 1, two or more prime clocks can be added and their sum can be projected into \(\mathbb{Z}_2^N\). The fourth column of table 1 shows the sum of clocks [2, 0] and [3, 1], projected into \(\mathbb{Z}_2^N\). This paper primarily focuses on prime clock sums, projected into \(\mathbb{Z}_2^N\), since they can compute Boolean functions. These sums have a mathematical property that has a practical application. This property is formally stated in theorem 3.6: for every natural number \(n\), every Boolean function \(f : \{0, 1\}^n \rightarrow \{0, 1\}\) can be computed with a finite prime clock sum that lies inside the infinite abelian group \((\mathbb{Z}_2^N, \oplus)\). This means prime clocks can act as computational primitives instead of gates [14, 15]. A computer can be built from physical devices that implement prime clock sums.

Prime clock addition \(\oplus\) is associative and commutative. These two group properties enable prime clocks to compute in parallel, while gates do not have this favorable property. For example, \(\neg(x \land y) \neq (\neg x) \land y\) because \(\neg(0 \land 0) = 1\) while \((\neg 0) \land 0 = 0\). The unary operation \(\neg\), conjunction operation \(\land\), and disjunction operation \(\lor\) form a Boolean algebra [7], so circuits built from the NOT, AND, and OR gates must have a depth.

![Gate-based circuit](image)

Figure 1: A gate-based circuit that computes \([7, 3] \oplus [13, 6]\) on \(\{0, 1\}^4\).

Shown in the last column of table 1 the clock sum \([7, 3] \oplus [13, 6]\), helps illustrate the disparity between the parallelization of prime clock sums versus the circuit depth of gates. Figure 1 shows
a gate-based circuit with depth 5 that computes \([7, 3] \oplus [13, 6]\) on \([0, 1]^4\). This circuit computes Boolean function \(h : \{0, 1\}^4 \rightarrow \{0, 1\}\), where \(h(x_0, x_1, x_2, x_3) = \left((\neg x_0 \land x_1) \land (\neg x_2) \land x_3\right) \lor \left[x_0 \land x_1 \land x_2 \land (\neg x_3)\right] \lor \left((\neg x_2) \land (\neg x_3)\right)\). Note \(([7, 3] \oplus [13, 6])(m) = h(x_0, x_1, x_2, x_3)\), whenever \(m = x_0 + 2x_1 + 4x_2 + 8x_3\).

This disparity enlarges for Boolean functions \(f : \{0, 1\}^n \rightarrow \{0, 1\}\) as \(n\) increases. Informally, Shannon’s theorem [14] implies that most functions \(f : \{0, 1\}^n \rightarrow \{0, 1\}\) require on the order of \(2^n\) gates. More precisely, let \(\beta(\epsilon, n)\) be the number of distinct functions \(f : \{0, 1\}^n \rightarrow \{0, 1\}\) that can be computed by circuits with at most \((1 - \epsilon)2^n\) gates built from the \textsc{not}, \textsc{and}, and \textsc{or} gates. Shannon’s theorem states for any \(\epsilon > 0\)

\[
\lim_{n \to \infty} \frac{\beta(\epsilon, n)}{2^n} = 0.
\]

Let the gates of a circuit be labeled as \(\{g_1, g_2, \ldots, g_m\}\) where \(m\) is about \(\frac{2^n}{n}\). The graph connectivity of the circuit specifies that the output of gate \(g_1\) connects to the input of gate \(g_{k_1}\), and so on. Shannon’s theorem implies that for most of these Boolean functions the graph connectivity requires an exponential (in \(n\)) amount of information. This is readily apparent after comparing the number of symbols used in \([7, 3] \oplus [13, 6]\) versus the symbolic expression \([(\neg x_0 \land x_1) \land (\neg x_2) \land x_3] \lor [x_0 \land x_1 \land x_2 \land (\neg x_3)] \lor [(\neg x_2) \land (\neg x_3)]\).

Consider a cryptographic application that uses a function \(h : \{0, 1\}^{20} \rightarrow \{0, 1\}^{20}\), where \(h = (h_0, \ldots, h_{19})\) and each \(h_i : \{0, 1\}^{20} \rightarrow \{0, 1\}\) is highly nonlinear [6]. Then over 1 million gates can be required to compute \(h\), since \(\frac{2^{20}}{20} = 52428\) and there are 20 distinct \(h_i\) functions. Using the first 559 prime numbers (i.e., all primes \(\leq 4051\)), finite prime clock sums can compute any function \(f_{20} : \{0, 1\}^{20} \rightarrow \{0, 1\}\) even though there are \(2^{20} = 2^{1048576}\) distinct \(f_{20}\) functions. This means a physical realization\(^1\) with prime clocks may use the first 599 prime numbers to implement an arbitrary \(h : \{0, 1\}^{20} \rightarrow \{0, 1\}^{20}\).

Lastly, the structure of our paper is summarized. Section 2 provides formal definitions of a prime clock, prime clock sums, and some results about the periodicity of finite prime clock sums. Section 3 covers prime clock sums projected into \(\mathbb{Z}_2^n\), where the main theorem is that any Boolean function \(f : \{0, 1\}^n \rightarrow \{0, 1\}\) can be computed with a finite prime clock sum. Section 4 provides a parallelizable algorithm for computing a Boolean function with prime clock sums.

2 Prime Clocks

Definition 2.1. Prime Clocks

Let \(p\) be a prime number. Let \(t \in \mathbb{N}\) such that \(0 \leq t \leq p - 1\). Define \([p, t] : \mathbb{N} \rightarrow \mathbb{N}\) as \([p, t](m) = (m + t) \mod p\). Function \([p, t]\) is called a \(p\)-clock that starts ticking with its hand pointing to \(t\).

Herein the expression \(\text{prime clock} [p, t]\) always assumes that \(0 \leq t \leq p - 1\). Thus, if \(p \neq q\) or \(s \neq t\), then prime clock \([p, s]\) is not equal to \([q, t]\); equivalently, if \(p = q\) and \(s = t\), then \([p, s] = [q, t]\). For the \(n\)th prime \(p_n\), let \(\mathcal{P}_n = \{[p_n, 0], [p_n, 1], \ldots, [p_n, p_n - 1]\}\) be the distinct \(p_n\)-clocks. The set of all prime clocks is defined as

\[
\mathcal{P} = \bigcup_{n=1}^{\infty} \mathcal{P}_n
\]

\(^1\)Physical realizations of prime clocks are beyond the scope of this paper.
For \( n \geq 2 \), let \( \Omega_n = \mathbb{Z}_n^N \). Define \( \pi_n : \mathcal{P} \rightarrow \Omega_n \) as the projection of each \( p \)-clock into \( \Omega_n \) where 
\[ \pi_n([p,t](m)) = ([p,t](m)) \mod n. \]

**Definition 2.2.**

Let \( n \in \mathbb{N} \) such that \( n \geq 2 \). On the set \( \mathcal{P} \) of all prime clocks, define the binary operator \( \oplus_n \) as 
\[ ([p,s] \oplus_n [q,t])(m) = ([p,s](m) + [q,t](m)) \mod n, \]
where + is computed in \( \mathbb{Z} \). Observe that 
\[ [p,s] \oplus_n [q,t] \in \Omega_n. \]

**Definition 2.3.** *Finite Prime Clock Sum*

Similarly, with prime clocks \([q_1,t_1], [q_2,t_2] \ldots \) and \([q_l,t_l]\), the function \([q_1,t_1] \oplus_n [q_2,t_2] \oplus_n \cdots \oplus_n [q_l,t_l] : \mathbb{N} \rightarrow \mathbb{Z}_n \) can be constructed. For each \( m \in \mathbb{N} \), define 
\[ ([q_1,t_1] \oplus_n [q_2,t_2] \oplus_n \cdots \oplus_n [q_l,t_l])(m) = ([q_1,t_1](m) + [q_2,t_2](m) + \cdots + [q_l,t_l](m)) \mod n, \]
where + is computed in \( \mathbb{Z} \). \([q_1,t_1] \oplus_n [q_2,t_2] \oplus_n \cdots \oplus_n [q_l,t_l]\) is called a *finite prime clock sum* in \( \Omega_n \).

Table 2 shows a finite prime clock sum in \( \Omega_5 \).

**Table 2:** Some Prime Clocks and their Sum in \( \Omega_5 \)

| Time | [5, 3] | [7, 6] | [11, 3] | [13, 0] | [5, 3] \( \oplus_5 \) [7, 6] | [5, 3] \( \oplus_5 \) [11, 3] | [13, 0] |
|------|--------|--------|--------|--------|----------------|----------------|--------|
|      | 0      | 3      | 6      | 3      | 0              | 2              | 4      |
| 1    | 4      | 0      | 4      | 1      | 4              |                |        |
| 2    | 0      | 1      | 5      | 2      | 3              |                |        |
| 3    | 1      | 2      | 6      | 3      | 2              |                |        |
| 4    | 2      | 3      | 7      | 4      | 1              |                |        |
| 5    | 3      | 4      | 8      | 5      | 0              |                |        |
| 6    | 4      | 5      | 9      | 6      | 4              |                |        |
| 7    | 0      | 6      | 10     | 7      | 3              |                |        |
| 8    | 1      | 0      | 0      | 8      | 4              |                |        |
|      |        |        |        |        | \ldots         |                |        |

**Definition 2.4.** Let \( r_1, \ldots r_k \) be \( k \) prime numbers and \( q_1, \ldots q_r \) be \( r \) prime numbers. Let \( f = [r_1, s_1] \oplus_n [r_2, s_2] \cdots \oplus_n [r_k, s_k] \). Let \( g = [q_1, t_1] \oplus_n [q_2, t_2] \cdots \oplus_n [q_l, t_l] \). Define \( f \oplus_n g \) in \( \Omega_n \) as \( (f \oplus_n g)(m) = f(m) +_n g(m) \), where \( +_n \) is the binary operator in the group \( (\mathbb{Z}_n, +_n) \).

Definition 2.4 is well-defined with respect to definition 2.3 (i.e., \( f \oplus_n g = [r_1, s_1] \oplus_n [r_2, s_2] \cdots \oplus_n [r_k, s_k] \oplus_n [q_1, t_1] \oplus_n [q_2, t_2] \cdots \oplus_n [q_l, t_l] \)) because \( (m_1 + m_2) \mod n = ((m_1 \mod n) + (m_2 \mod n)) \mod n \) for any \( m_1, m_2 \in \mathbb{N} \).

**Remark 2.1.** \( (m_1 + m_2) \mod n = ((m_1 \mod n) + (m_2 \mod n)) \mod n \) for any \( m_1, m_2 \in \mathbb{N} \).

**Proof.** Euclid’s division algorithm implies \( m_1 = k_1 n + r_1 \) and \( m_2 = k_2 n + r_2 \), where \( 0 \leq r_1, r_2 < n \). Thus, \( (m_1 + m_2) \mod n = ((k_1 + k_2)n + r_1 + r_2) \mod n = (r_1 + r_2) \mod n = ((m_1 \mod n) + (m_2 \mod n)) \mod n \)

The binary operator \( \oplus_n \) can be extended to all of \( \Omega_n \). For any \( f, g \in \Omega_n \), define \( (f \oplus_n g)(m) = f(m) +_n g(m) \). The associative property \( (f \oplus_n g) \oplus_n h = f \oplus_n (g \oplus_n h) \) follows immediately from the fact that \( +_n \) is associative. The zero function \( \mathbf{0} \), where \( \mathbf{0}(m) = 0 \) in \( \mathbb{Z}_n \), is the identity in \( \Omega_n \). For any \( f \) in \( \Omega_n \), its unique inverse \( f^{-1} \) is defined as \( f^{-1}(m) = -f(m) \), where \( -f(m) \) is the
Definition 2.5. Periodic Functions

We focus on subgroups of $\Omega_2$. Namely, $\{f \in \Omega_n : f \circ f = f\}$ for all $x \in \Omega_n$. Let $\Omega_n$ be a subset of the prime clocks $\mathcal{P}$. Using the projection $\pi_n$ of $\Omega_n$, define $S_Q = \{H : H \supseteq \pi_n(Q) \text{ and } H \text{ is a subgroup of } \Omega_n\}$. The subset $Q$ generates a subgroup of $(\Omega_n, \oplus_n)$. Namely,

$$\bigcap_{H \in S_Q} H$$

We focus on subgroups of $\Omega_2$, generated by a finite number of prime clocks; consequently, the more natural symbol $\oplus$ is used instead of $\oplus_2$.

Definition 2.5. Periodic Functions

$f \in \Omega_n$ is a periodic function if there exists a positive integer $b$ such that for every $m \in \mathbb{N}$, then $f(m) = f(m + b)$. Furthermore, if $a$ is the smallest positive integer such that $f(m) = f(m + a)$ for all $m \in \mathbb{N}$, then $a$ is called the period of $f$. After $k$ substitutions of $m + a$ for $m$, this implies for any $m \in \mathbb{N}$ that $f(m) = f(m + ka)$ for all positive integers $k$.

Table 3: Some 2-Clocks, 3-Clocks and Sums in $\Omega_2$

| Time | [2,0] | [2,1] | [3,0] | [3,1] | [2,0] $\oplus$ [3,0] | [2,1] $\oplus$ [3,0] | [2,0] $\oplus$ [3,1] |
|------|-------|-------|-------|-------|-----------------|-----------------|-----------------|
| 0    | 0     | 0     | 0     | 0     | 0               | 0               | 0               |
| 1    | 1     | 0     | 1     | 2     | 0               | 1               | 1               |
| 2    | 0     | 0     | 2     | 0     | 0               | 1               | 0               |
| 3    | 1     | 0     | 0     | 1     | 1               | 0               | 0               |
| 4    | 0     | 1     | 1     | 2     | 1               | 0               | 0               |
| 5    | 1     | 0     | 2     | 0     | 1               | 0               | 1               |
| 6    | 0     | 1     | 0     | 1     | 0               | 1               | 1               |
| 7    | 1     | 0     | 1     | 2     | 0               | 1               | 1               |

...
Remark 2.2. If $a$ is the period of $f$ and $b$ is a positive integer such that $f(m) = f(m + b)$ for all $m \in \mathbb{N}$, then $a$ divides $b$.

Proof. First, verify that $a \sim b$. By the definition of period, $a \leq b$ and for all $m \in \mathbb{N}$, then $f(m + b - a) = f(m + a + b - a) = f(m + b) = f(m)$. From the prior observation, $a$ lies in $[0]$ and $b$ also lies in $[0]$. Thus, $b = ma$ for some positive integer $m$.

Lemma 2.1. If $f, g \in \Omega_n$ are periodic, then $f \oplus_n g$ is periodic. Further, if the period of $f$ is a and the period of $g$ is $b$, then $f \oplus_n g$ has a period that divides $\text{lcm}(a, b)$.

Proof. Let $a$ be the period of $f$ and $b$ the period of $g$. Let $l_{a,b} = \text{lcm}(a, b)$. $l_{a,b} = ia$ and $l_{a,b} = jb$ for positive integers $i, j$. For any $m \in \mathbb{N}$, $(f \oplus_n g)(m) = f(m) +_n g(m) = f(m + ia) +_n g(m + jb) = f(m + l_{a,b}) +_n g(m + l_{a,b}) = (f \oplus_n g)(m + l_{a,b})$. Thus, $f \oplus_n g$ is periodic and remark 2.2 implies its period divides $l_{a,b}$.

In regard to lemma 2.1 if $g = -f$, then the period of $f \oplus_n g$ is 1.

Remark 2.3. There are $n^n$ distinct periodic functions $f \in \Omega_n$ whose period divides $a$.

Proof. Since $f$ is periodic and its period divides $a$, the values of $f(0)$, $f(1)$, ..., $f(a - 1)$ uniquely determine $f$. There are $n$ choices for $f(0)$. There are $n$ choices for $f(1)$, and so on.

Remark 2.4. Let $p$ be prime. There are $n^p - n$ distinct periodic functions $f \in \Omega_n$ with period $p$.

Proof. Consider a finite sequence $c_0$, $c_1$, ..., $c_{p-1}$ of length $p$ where each $c_i \in \mathbb{Z}_n$. This sequence uniquely determines a periodic $f$ such that $f(m + p) = f(m)$ for all $m \in \mathbb{N}$. In particular, $f(0) = c_0$, $f(1) = c_1$, ..., $f(p - 1) = c_{p-1}$. There are $n^p$ periodic functions with a period that divides $p$. If the period of $f$ is less than $p$, then remark 2.2 implies $f$ has period 1 since $p$ is prime. There are $n$ distinct, constant (period 1) functions in $\Omega_n$. Thus, the remaining $n^p - n$ periodic functions have period $p$.

Remark 2.5. The prime clock $[p, t]$, projected into $\Omega_n$, has period $p$.

Proof. Since $p$ is prime, this follows immediately from remark 2.2.

Theorem 2.2. Finite Prime Clock Sums are Periodic

Any finite sum of prime clocks $[q_1, t_1] \oplus_n [q_2, t_2] \oplus_n \cdots \oplus_n [q_l, t_l]$ is periodic.

Proof. Use induction and apply remark 2.5 and lemma 2.1.

3 Prime Clock Sums in $\Omega_2$

Remark 3.1. $[p, t] \oplus [p, t] = \overline{0}$ for any prime clock $[p, t]$.

Per definition 2.2, $(\langle p, k \rangle \oplus \langle p, k \rangle)(m) = \langle (p, k)(m) + (p, k)(m) \rangle \mod 2 = 0$ in $\mathbb{Z}_2$.

Remark 3.2. Let $p$ be an odd prime. If $p$ is a 3 mod 4 prime, then $[p, 0] \oplus [p, 1] \oplus \cdots \oplus [p, p - 1] = 1$. If $p$ is a 1 mod 4 prime, then $[p, 0] \oplus [p, 1] \oplus \cdots \oplus [p, p - 1] = \overline{0}$.

Proof. $([p, 0] \oplus [p, 1] \oplus \cdots \oplus [p, p - 1])(0) = (0 + 1 + \cdots + p - 1) \mod 2 = \frac{1}{2}(p - 1)p \mod 2$. For each $m > 0$, $([p, 0] \oplus [p, 1] \oplus \cdots \oplus [p, p - 1])(m)$ is a permutation of the sum inside $(0 + 1 + \cdots + p - 1) \mod 2$. □
For the special case \( p = 2 \), observe that \([2, 0] \oplus [2, 1] = \mathbb{T} \).

**Definition 3.1.** A finite sum \([q_1, t_1] \oplus [q_2, t_2] \oplus \cdots \oplus [q_l, t_l]\) of prime clocks is **non-repeating** if \(i \neq j\) implies \([q_i, t_i]\) is not equal to \([q_j, t_j]\).

**Remark 3.3.** Any finite sum \([q_1, t_1] \oplus [q_2, t_2] \oplus \cdots \oplus [q_l, t_l]\) of prime clocks in \(\Omega_2\) can be reduced to a non-repeating finite sum \([q_1, t_1] \oplus [q_2, t_2] \oplus \cdots \oplus [q_r, t_r]\), where \(r \leq l\) such that for any \(m \in \mathbb{N}\), 

\[
[q_1, t_1] \oplus [q_2, t_2] \oplus \cdots \oplus [q_r, t_r](m) = ([q_1, t_1] \oplus [q_2, t_2] \oplus \cdots \oplus [q_r, t_r])(m).
\]

**Proof.** Since \((\Omega_2, +_2)\) is abelian, if necessary, rearrange the order of \([q_1, t_1] \oplus [q_2, t_2] \oplus \cdots \oplus [q_l, t_l]\), so that the prime clocks are ordered using the dictionary order. If two or more adjacent prime clocks are equal, then the associative property and remark 3.1 enables the cancellation of even numbers of equal prime clocks. This reduction can be performed a finite number of times so that the resulting sum is non-repeating. \(\square\)

**Definition 3.2.** Let \( p \) be a prime. A finite sum of prime clocks \([p, t_1] \oplus [p, t_2] \oplus \cdots \oplus [p, t_l]\) is called a **\( p \)-clock sum** of length \( l \) if for each \( 1 \leq i \leq l \), the clock \([p, t_i]\) is a \( p \)-clock and the sum is non-repeating. The non-repeating condition implies \( l \leq p \).

**Lemma 3.1.** Let \( p \) be a prime. A \( p \)-clock sum with length \( l \) has period 1. A \( p \)-clock sum with length \( l \) such that \( 1 \leq l < p \) has period \( p \).

**Proof.** When \( p = 2 \), the 2-clock sum \([2, 0] \oplus [2, 1]\) has period 2 and the 2-clock sum \([2, 1] \oplus [2, 1]\) also has period 2. Recall that \([2, 0] \oplus [2, 1] = \mathbb{T}\). For the remainder of the proof, it is assumed that \( p \) is an odd prime.

Let \([p, t_1] \oplus [p, t_2] \oplus \cdots \oplus [p, t_{l-1}] \oplus [p, t_l]\) be a \( p \)-clock sum. When \( l = p \), remark 3.2 implies that \([p, t_1] \oplus [p, t_2] \oplus \cdots \oplus [p, t_{l-1}] \oplus [p, t_l]\) has period 1. Lemma 2.1 and remark 2.5 imply that \([p, t_1] \oplus [p, t_2] \oplus \cdots \oplus [p, t_{l-1}] \oplus [p, t_l]\) has period \( p \) or period 1. The rest of this proof shows that \( 1 \leq l \leq p - 1 \) implies that the \( p \)-clock sum cannot have period 1.

Thus, it suffices to show that \( 1 \leq l < p \) implies that \(([p, t_1] \oplus [p, t_2] \oplus \cdots \oplus [p, t_l]) (m) \neq ([p, t_1] \oplus [p, t_2] \oplus \cdots \oplus [p, t_l]) (m + 1)\) for some \( m \in \mathbb{N}\). If needed, the \( p \)-clock sum may be permuted so that \([p, s_1] \oplus [p, s_2] \oplus \cdots \oplus [p, s_l] = [p, t_1] \oplus [p, t_2] \oplus \cdots \oplus [p, t_l]\) and the \( s_i \) are strictly increasing. (Strictly increasing means \( 0 \leq s_1 < s_2 < \cdots < s_{l-1} < s_l \leq p - 1\)).

**Case A.** \( l \) is odd. If \( s_l < p - 1 \), then \(([p, s_1] \oplus [p, s_2] \oplus \cdots \oplus [p, s_l])(0) = \sum_{i=1}^{l} s_i \) mod 2 \( \neq \sum_{i=1}^{l} (s_i + 1) \) mod 2 = \(([p, s_1] \oplus [p, s_2] \oplus \cdots \oplus [p, s_l])(1)\) because \( l \) is odd. Otherwise, \( s_l = p - 1 \). Set \( s_0 = 0 \). (The auxiliary index \( s_0 = 0 \) handles the case \( s_{k+1} - s_k \) for all \( k \) such that \( 1 \leq k < l \)). Set \( m = \max \{ k \in \mathbb{N} : (s_{k+1} - s_k) \geq 2 \text{ and } 0 \leq k < l \} \). Since \( s_0 = 0 \) and \( 1 \leq l < p \), the pigeonhole principle implies \( m \) exists. Before the mod 2 step, the difference between \( \sum_{i=1}^{l} ((s_i + l - m + 1) \mod p) \) and \( \sum_{i=1}^{l} ((s_i + l - m) \mod p) \) equals \( l \). Hence, \(([p, s_1] \oplus [p, s_2] \oplus \cdots \oplus [p, s_l])(l - m) \neq ([p, s_1] \oplus [p, s_2] \oplus \cdots \oplus [p, s_l])(l + m)\).

**Case B.** \( l \) is even. Set \( j = (p - 1) - s_l \). Before the mod 2 step, the sum \( \sum_{i=1}^{l} ((s_i + j) \mod p) \) differs from the sum \( \sum_{i=1}^{l} ((s_i + j + 1) \mod p) \) by an odd number. Thus, \(([p, s_1] \oplus \cdots \oplus [p, s_l])(j) \neq ([p, s_1] \oplus \cdots \oplus [p, s_l])(j + 1)\).\(\square\)
Lemma 3.1 implies that

This is a contradiction, so $\{7, 0\} \oplus \{7, 2\} \oplus \{7, 3\}$ is distinct from $\{7, 1\} \oplus \{7, 2\} \oplus \{7, 3\}$. Table 4 shows that these distinct 7-clock sums are not equal.

Table 4: Two distinct 7-clock sums that are not equal in $\Omega_2$

| Time | $\{7, 0\}$ | $\{7, 1\}$ | $\{7, 2\}$ | $\{7, 3\}$ | $\{7, 0\} \oplus \{7, 2\} \oplus \{7, 3\}$ | $\{7, 1\} \oplus \{7, 2\} \oplus \{7, 3\}$ |
|------|-------------|-------------|-------------|-------------|---------------------------------------------|---------------------------------------------|
| 0    | 0           | 1           | 0           | 1           | 0                                           | 1                                           |
| 1    | 1           | 0           | 1           | 0           | 0                                           | 1                                           |
| 2    | 0           | 1           | 0           | 1           | 1                                           | 0                                           |
| 3    | 1           | 0           | 1           | 0           | 0                                           | 1                                           |
| 4    | 0           | 1           | 0           | 0           | 1                                           | 1                                           |
| 5    | 1           | 0           | 0           | 1           | 0                                           | 1                                           |
| 6    | 0           | 0           | 1           | 0           | 1                                           | 1                                           |
|      | ...         |             |             |             |                                             |                                             |

Theorem 3.2. For any 3 mod 4 prime $p$, if two p-clock sums are distinct, then they are not equal in $\Omega_2$. The theorem also holds for $p = 2$.

Proof. The special case $p = 2$ can be verified by examining the second and third columns of Table 3.

Let $p$ be a 3 mod 4 prime. Assume p-clock sum $[p, s_1] \oplus \cdots \oplus [p, s_l]$ is distinct from p-clock sum $[p, t_1] \oplus \cdots \oplus [p, t_m]$. By reductio absurdum, suppose

$$[p, s_1] \oplus \cdots \oplus [p, s_l] = [p, t_1] \oplus \cdots \oplus [p, t_m].$$  

(4)

For each $s_i \in \{t_1, \ldots, t_m\}$, the operation $\oplus [p, s_i]$ in $\Omega_2$ can be applied to both sides of equation 4. Similarly, for each $t_j \in \{s_1, \ldots, s_l\}$, the operation $\oplus [p, t_j]$ can be applied to both sides of equation 4. Since $(\Omega_2, \oplus)$ is an abelian group, equation 4 can be simplified to $[p, s_1] \oplus \cdots \oplus [p, s_l] = [p, t_1] \oplus \cdots \oplus [p, t_m]$ such that $\{s_1, \ldots, s_l\} \cap \{t_1, \ldots, t_m\} = \emptyset$ and $M + L \leq p$.

Set $f = [p, s_1] \oplus \cdots \oplus [p, s_l]$. Apply $f$ to both sides of $[p, s_1] \oplus \cdots \oplus [p, s_l] = [p, t_1] \oplus \cdots \oplus [p, t_m]$. This simplifies to $f \oplus [p, t_1] \oplus \cdots \oplus [p, t_m] = \emptyset$. Lemma 3.1 implies that $L + M = p$. Since $L + M = p$ and $\{s_1, \ldots, s_l\} \cap \{t_1, \ldots, t_m\} = \emptyset$ and $p$ is a 3 mod 4 prime, remark 3.2 implies that $f \oplus [p, t_1] \oplus \cdots \oplus [p, t_m] = \emptyset$. This is a contradiction, so $[p, s_1] \oplus \cdots \oplus [p, s_l]$ is not equal to $[p, t_1] \oplus \cdots \oplus [p, t_m]$ in $\Omega_2$. \qed

Let $S_l$ be the set of all $p$-clock sums of length $l$, where $1 \leq l \leq p$. There are $\binom{p}{l}$ distinct $p$-clock sums in each set $S_l$. Set $G_p = \bigcup_{l=1}^{p} S_l \cup \{\emptyset\}$. For any $f, g \in G_p$, remark 3.1 implies $f \oplus g^{-1}$ in $G_p$. Thus, $(G_p, \oplus)$ is an abelian subgroup of $\Omega_2$. Set $B_p = \{0, 1\}^p$. For any $a_1 \ldots a_p \in B_p$ and $b_1 \ldots b_p \in B_p$, define $a_1 \ldots a_p +_2 b_1 \ldots b_p = c_1 \ldots c_p$, where $c_i = (a_i + b_i) \mod 2$. $(B_p, +_2)$ is an abelian group with $2^p$ elements. When $p$ is a 3 mod 4 prime, define the function $\phi : G_p \to B_p$ where $\phi(\emptyset) = 0 \ldots 0 \in B_p$ and $\phi([p, t_1] \oplus [p, t_2] \oplus \cdots \oplus [p, t_l]) = c_1 \ldots c_p$ where $c_i = ([p, t_1] \oplus [p, t_2] \oplus \cdots \oplus [p, t_l])(i)$. We reach theorem 3.3 because $\phi$ is a group isomorphism.
Theorem 3.3. Let $p$ be a $3 \mod 4$ prime. The subgroup $G_p$ of $\Omega_2$, generated by the $p$-clocks $[p,0], [p,1], \ldots [p,p-1]$ has order $2^p$ and is isomorphic to $(B_p,+2)$.

Proof. Theorem 3.2 implies $\phi$ is a group isomorphism. 

Table 5: The 5-clocks projected into $\Omega_2$

| Time | 5,0 | 5,1 | 5,2 | 5,3 | 5,4 | 5,0 $\oplus$ 5,1 | 5,2 $\oplus$ 5,3 $\oplus$ 5,4 |
|------|-----|-----|-----|-----|-----|------------------|------------------|
| 0    | 0   | 1   | 0   | 1   | 0   | 5,0 $\oplus$ 5,1 | 1,0 $\oplus$ 1,1 |
| 1    | 1   | 0   | 1   | 0   | 1   | 1,0 $\oplus$ 1,1 | 1,0 $\oplus$ 1,1 |
| 2    | 0   | 1   | 0   | 0   | 1   | 5,0 $\oplus$ 5,1 | 1,0 $\oplus$ 1,1 |
| 3    | 1   | 0   | 0   | 1   | 0   | 1,0 $\oplus$ 1,1 | 1,0 $\oplus$ 1,1 |
| 4    | 0   | 0   | 1   | 0   | 1   | 5,0 $\oplus$ 5,1 | 1,0 $\oplus$ 1,1 |
| 5    | 0   | 1   | 0   | 0   | 1   | 5,0 $\oplus$ 5,1 | 1,0 $\oplus$ 1,1 |

Theorem 3.2 does not hold when $p$ is a $1 \mod 4$ prime. Table 5 shows $[5,0] \oplus [5,1]$ equals $[5,2] \oplus [5,3] \oplus [5,4]$ in $\Omega_2$.

Theorem 3.4. For any $1 \mod 4$ prime $p$, if two $p$-clock sums are distinct and their respective lengths $L$ and $M$ are both $\leq \frac{p-1}{2}$, then these two $p$-clock sums are not equal in $\Omega_2$.

Proof. The proof is almost the same as the proof in theorem 3.2. The conditions $L \leq \frac{p-1}{2}$ and $M \leq \frac{p-1}{2}$ and the reduction $[p,s_1] \oplus \cdots \oplus [p,s_L] \oplus [p,t_1] \oplus \cdots \oplus [p,t_M] = \emptyset$ leads to an immediate contradiction: $L + M \leq p - 1$ and $\{s_1, \ldots, s_L\} \cap \{t_1, \ldots, t_M\} = \emptyset$ means lemma 3.1 implies $[p,s_1] \oplus \cdots \oplus [p,s_L] \oplus [p,t_1] \oplus \cdots \oplus [p,t_M]$ has period $p$.

Remark 3.4. Let $p$ be a $1 \mod 4$ prime. Let $f = [p,s_1] \oplus \cdots \oplus [p,s_l]$ for some $1 \leq l \leq \frac{1}{2}(p-1)$. Set $T = \{0,1,\ldots,p-1\} - \{s_1, \ldots, s_l\}$. Now $T = \{t_1, \ldots, t_m\}$, where $l+m = p$. Set $g = [p,t_1] \oplus \cdots \oplus [p,t_m]$. Then $f = g$ in $\Omega_2$.

Proof. Since $p$ is a $1 \mod 4$ prime, $(f \oplus g)(0) = \sum_{0}^{p-1} k \mod 2 = 0$ in $Z_2$. When $k > 1$, the sum of the elements of $f \oplus g$ before projecting into $\Omega_2$ is a permutation of the elements $\{0,1,\ldots,p-1\}$. Hence, for all $k > 1$, $(f \oplus g)(k) = 0$ in $Z_2$. This means $g = f^{-1}$. Lastly, $f = f^{-1}$ in $\Omega_2$, so $f = g$ in $\Omega_2$.

Let $p$ be a $1 \mod 4$ prime. Set $H_{p-1} = \frac{1}{2}(p-1) \cup \cup_{i=1}^{l} S_i \cup \{0\}$. Observe that $|H_{p-1}| = \frac{1}{2}(p-1) + 1 = 2^{p-1}$. To verify that $(H_{p-1}, \oplus)$ is a subgroup of $(\Omega_2, \oplus)$, let $f, g \in H_{p-1}$. Since $g = g^{-1}$ in $(\Omega_2, \oplus)$, it suffices to show that $f \oplus g$ lies in $H_{p-1}$. If $f$ or $g$ equals $\emptyset$, closure in $(H_{p-1}, \oplus)$ holds. Otherwise, $f = [p,s_1] \oplus \cdots \oplus [p,s_l]$ for some $1 \leq l \leq \frac{1}{2}(p-1)$ and $g = [p,t_1] \oplus \cdots \oplus [p,t_m]$ for some $1 \leq m \leq \frac{1}{2}(p-1)$. As mentioned before, the sum $f \oplus g$ may be reduced to $[p,s_1] \oplus \cdots \oplus [p,s_L] \oplus [p,t_1] \oplus \cdots \oplus [p,t_M]$, where $\{s_1, \ldots, s_L\} \cap \{t_1, \ldots, t_M\} = \emptyset$ and $L + M \leq p$. If $L + M \leq \frac{1}{2}(p-1)$, closure in $(H_{p-1}, \oplus)$ holds. Otherwise, if $L + M > \frac{1}{2}(p-1)$, Remark 3.4 implies that there is a $p$-clock sum $h = f \oplus g$, where $h$’s length is $p - (L + M)$ and $p - (L + M) \leq \frac{1}{2}(p-1)$.

Similar to the group isomorphism $\phi$, define $\psi : H_{p-1} \rightarrow B_{p-1}$ such that $\psi(\emptyset) = 0 \ldots 0 \in B_p$. For each $p$-clock sum in $S_i$, where $1 \leq l \leq \frac{1}{2}(p-1)$, define $\psi([p,t_1] \oplus [p,t_2] \oplus \ldots \oplus [p,t_l]) = c_1 \ldots c_{p-1}$ where


\[ c_i = ([p, t_1] \oplus [p, t_2] \oplus \ldots [p, t_i]) \] (i). It is straightforward to verify that \( \psi \) is a group isomorphism onto \( B_{p^{-1}} \). The group isomorphism \( \psi : H_{p^{-1}} \rightarrow B_{p^{-1}} \) leads to the following theorem.

**Theorem 3.5.** Let \( p \) be a 1 mod 4 prime. The subgroup \( H_{p^{-1}} \) of \( \Omega_2 \), generated by the p-clocks \([p, 0], [p, 1], \ldots [p, p-1]\) has order \( 2^{p-1} \) and is isomorphic to \((B_{p^{-1}}, +_2)\).

**Theorem 3.6.** For positive integer \( n \) and any function \( f : \{0, 1\}^n \rightarrow \{0, 1\} \), there exists a finite sum of prime clocks in \( \Omega_2 \) that can compute \( f \).

**Proof.** Euclid’s second theorem implies there is a prime \( p > 2^n \), where \( p \) is a 3 mod 4 or 1 mod 4 prime. Hence, theorem 3.3 or 3.5 completes the proof. \( \square \)

Furthermore, finding a finite prime clock sum that computes \( f \) is Turing computable and there are efficient Turing computable algorithms that can decide whether a natural number \( n \) is prime [3].

## 4 Prime Clock Sums Compute Boolean Functions in \( \Omega_2 \)

Let \( F_n \) denote the set of all Boolean functions in \( n \) variables. Formally, the set \( F_n = \{ f \mid f : \{0, 1\}^n \rightarrow \{0, 1\} \} \) and \( F_n \) contains \( 2^{2^n} \) distinct functions. For prime clock sums, it is convenient to think of \( f \in F_n \) as a binary string of length \( 2^n \), called the truth-table of \( f \). Table 6 shows all 16 Boolean functions in \( F_2 \), their truth tables and corresponding prime clock sums that compute each function.

| Boolean Function | Truth Table | Prime Clock Sum |
|------------------|-------------|-----------------|
| \( f_1(x, y) = 1 \) | 1111        | \([2, 0] \oplus [2, 1]\) |
| \( f_2(x, y) = 0 \) | 0000        | \([2, 0] \oplus [2, 0]\) |
| \( f_3(x, y) = x \) | 0011        | \([2, 1] \oplus [3, 1]\) |
| \( f_4(x, y) = y \) | 0101        | \([2, 0]\) |
| \( f_5(x, y) = \overline{x} \) | 1100        | \([2, 0] \oplus [3, 1]\) |
| \( f_6(x, y) = \overline{y} \) | 1010        | \([2, 1]\) |
| \( f_7(x, y) = x \land y \) | 0001        | \([2, 0] \oplus [3, 0]\) |
| \( f_8(x, y) = x \lor y \) | 0111        | \([2, 0] \oplus [3, 2]\) |
| \( f_9(x, y) = \overline{x} \land y \) | 1101        | \([3, 0] \oplus [3, 1]\) |
| \( f_{10}(x, y) = x \lor \overline{y} \) | 1011        | \([3, 1] \oplus [3, 2]\) |
| \( f_{11}(x, y) = (x \land y) \lor \overline{(x \lor y)} \) | 1001        | \([3, 1]\) |
| \( f_{12}(x, y) = (x \lor y) \land \overline{(x \land y)} \) | 0110        | \([3, 0] \oplus [3, 2]\) |
| \( f_{13}(x, y) = \overline{(x \lor y)} \) | 1000        | \([2, 1] \oplus [3, 2]\) |
| \( f_{14}(x, y) = \overline{(x \land y)} \) | 1110        | \([2, 1] \oplus [3, 0]\) |
| \( f_{15}(x, y) = \overline{x} \land y \) | 0100        | \([3, 0]\) |
| \( f_{16}(x, y) = x \land \overline{y} \) | 0010        | \([3, 2]\) |

The truth table for \( \{0, 1\}^2 \) is ordered as \( \{00, 01, 10, 11\} \).

Consider \([p, s] \oplus [q, t] \) in \( \Omega_2 \). The first \( 2^n \) elements of \([p, s] \oplus [q, t] \) refer to the bit string \(([p, s] \oplus [q, t])(0), ([p, s] \oplus [q, t])(1), \ldots, ([p, s] \oplus [q, t])(2^n - 1) \) of length \( 2^n \). The first \( 2^n \) elements of
\[ p, s \oplus q, t \] represent a Boolean function \( f \in \mathbb{F}_n \). In the general case, if \( q_1, \ldots, q_L \) are primes, the first \( 2^n \) elements of \([q_1, t_1] \oplus [q_2, t_2] \oplus \cdots \oplus [q_L, t_L] \) also represent a Boolean function \( f_n \in \mathbb{F}_n \). Consider the first \( 2^n \) elements of prime clock sum \([q_1, t_1] \oplus [q_2, t_2] \oplus \cdots \oplus [q_L, t_L] \). Algorithm 1 computes the \( i \)th element of this truth table in \( \mathbb{F}_n \).

**Algorithm 1.** A Prime Clock Sum in \( \Omega_2 \) Computes a Boolean Function

**INPUT:**
- \( i \)

**OUTPUT:**
- \( y \)

**Example 1.** We demonstrate 2-bit multiplication with prime clock sums, computed with algorithm 1. In table 7, for each \( u \in \{0, 1\}^2 \) and each \( l \in \{0, 1\}^2 \), the product \( u \times l \) is shown in each row, whose 4 columns are labelled by \( \mathcal{M}_3 \), \( \mathcal{M}_2 \), \( \mathcal{M}_1 \) and \( \mathcal{M}_0 \). With input \( i \) of 4 bits (i.e., \( u \) concatenated with \( l \)), the output of the 2-bit multiplication is a 4-bit string \( \mathcal{M}_3(i) \mathcal{M}_2(i) \mathcal{M}_1(i) \mathcal{M}_0(i) \), shown in each row of table 7.

One can verify that, according to algorithm 1, prime clock sum \([2, 0] \oplus [7, 3] \oplus [7, 4] \oplus [7, 5] \oplus [11, 10] \) computes function \( \mathcal{M}_0 : \{0, 1\}^2 \times \{0, 1\}^2 \rightarrow \{0, 1\} \). Similarly, \([2, 0] \oplus [2, 1] \oplus [3, 0] \oplus [5, 2] \oplus [11, 0] \oplus [11, 1] \) computes function \( \mathcal{M}_1 \). Prime clock sum \([5, 0] \oplus [7, 0] \oplus [7, 2] \oplus [11, 4] \) computes function \( \mathcal{M}_2 \). Lastly, \([2, 1] \oplus [5, 0] \oplus [11, 1] \oplus [11, 6] \) computes function \( \mathcal{M}_3 \).

Table 7: 2-Bit Multiplication.

| \( u \) | \( l \) | \( \mathcal{M}_3 \) | \( \mathcal{M}_2 \) | \( \mathcal{M}_1 \) | \( \mathcal{M}_0 \) |
|-------|-------|-------|-------|-------|-------|
| 00    | 00    | 0     | 0     | 0     | 0     |
| 00    | 01    | 0     | 0     | 0     | 0     |
| 00    | 10    | 0     | 0     | 0     | 0     |
| 00    | 11    | 0     | 0     | 0     | 0     |
| 01    | 00    | 0     | 0     | 0     | 0     |
| 01    | 01    | 0     | 0     | 0     | 1     |
| 01    | 10    | 0     | 0     | 1     | 0     |
| 01    | 11    | 0     | 0     | 1     | 1     |
| 10    | 00    | 0     | 0     | 0     | 0     |
| 10    | 01    | 0     | 0     | 1     | 0     |
| 10    | 10    | 0     | 1     | 0     | 0     |
| 10    | 11    | 0     | 1     | 1     | 0     |
| 11    | 00    | 0     | 0     | 0     | 0     |
| 11    | 01    | 0     | 0     | 1     | 1     |
| 11    | 10    | 0     | 1     | 1     | 0     |
| 11    | 11    | 1     | 0     | 0     | 1     |
The $i$th element of \([q_1, t_1] \oplus [q_2, t_2] \oplus \cdots \oplus [q_L, t_L]\)'s truth table is stored in the variable $y$ when algorithm 1 halts. Algorithm 1 is presented in a serial form. Nevertheless, the computation of the $L$ instructions set $r_k = (t_k + i) \text{ mod } q_k$, where $1 \leq k \leq L$, can be computed in parallel when there is a separate physical device for each of these $L$ prime clocks $[q_1, t_1]$, $[q_2, t_2] \ldots [q_L, t_L]$. Subsequently, the parity of $y$ can be determined in a second computational step that executes a parallel add of $r_1 + r_2 + \cdots + r_L$, followed by setting $y$ to the least significant bit of the sum $r_1 + r_2 + \cdots + r_L$.

As an alternative implementation of algorithm 1 when there is a more suitable physical device for prime clocks, the $k$th clock can compute the $k$th bit $b_k = ((t_k + i) \text{ mod } q_k) \text{ mod } 2$ and then a parallel exclusive-or can be applied to the $L$ bits $b_1, b_2, \ldots, b_L$. In contrast, a gate-based Boolean circuit requires at least $d$ computational steps where $d$ is the depth of the circuit.

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