Generalized hypergeometric $G$-functions
take linear independent values

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Abstract

In this article, we show a new general linear independence criterion related to values of $G$-functions, including the linear independence of values at algebraic points of contiguous hypergeometric functions, which is not known before. Let $K$ be any algebraic number field and $v$ be a place of $K$. Let $r \in \mathbb{Z}$ with $r \geq 2$. Consider $a_1, \ldots, a_r, b_1, \ldots, b_{r-1} \in \mathbb{Q} \setminus \{0\}$ not being negative integers. Assume neither $a_k$ nor $a_k + 1 - b_j$ be strictly positive integers $(1 \leq k \leq r, 1 \leq j \leq r - 1)$. Let $\alpha_1, \ldots, \alpha_m \in K \setminus \{0\}$ with $\alpha_1, \ldots, \alpha_m$ pairwise distinct. By choosing sufficiently large $\beta \in \mathbb{Z}$ depending on $K$ and $v$ such that the points $\alpha_1/\beta, \ldots, \alpha_m/\beta$ are closed enough to the origin, we prove that the $rm + 1$ numbers :

$$
\begin{align*}
&\binom{a_1, \ldots, a_r}{b_1, \ldots, b_{r-1}}(\alpha_i/\beta), \quad \binom{a_1 + 1, \ldots, a_r + 1}{b_1 + 1, \ldots, b_{r-s} + 1, b_{r-s+1}, \ldots, b_{r-1}}(\alpha_i/\beta) \\
&(1 \leq i \leq m, 1 \leq s \leq r - 1)
\end{align*}
$$

and 1 are linearly independent over $K$. The essential ingredient is our term-wise formal construction of type II of Padé approximants together with new non-vanishing argument for the generalized Wronskian.

Key words: Generalized hypergeometric function, $G$-function, linear independence, the irrationality, Padé approximation.

1 Introduction

The generalized hypergeometric $G$-function, in the sense of C. L. Siegel, is one of central objects from analytic point of view as well as number theoretical interest. In the article, we study arithmetic properties of values of the generalized hypergeometric functions, relying on Padé approximations of type II. We provide a new general linear independence criterion for the values of the functions at several distinct points, over a given algebraic number field of any finite degree. Our statement extends previous ones due to D. V. Chudnovsky or D. V. Chudnovsky-G. V. Chudnovsky in [9, Theorem 3.1] [13 Theorem I], [14 Theorem 0.3] [15 Theorem I] and Yu. Nesterenko [34 Theorem 1] [35 Theorem 1], which all dealt with values at one point and over the rational number field. We proceed constructions of Padé approximants by our formal method, generalizing that used in [18] [19] [20]. We are inspired, together with those quoted above, by works due to A. I. Galochkin in [22] [23], V. N. Sorokin in [45], K. Väänänen in [47] and W. Zudilin in [49], which gave several linear independence criteria, either over the field of rational numbers or quadratic imaginary fields, of values those concerns polylogarithmic function or hypergeometric $G$-function. However, these previous results were either for values at only one point, or in the case where the ground field was limited. As related works, we refer to the algebraic independence announced in...
Theorem 3.4] of the two special values of Gauss’ hypergeometric functions \( _2F_1 \left( \frac{1}{2}, \frac{1}{2} \mid \alpha \right) \) and \( _2F_1 \left( -\frac{1}{2}, \frac{1}{2} \mid \alpha \right) \) when \( \alpha \) is a non-zero algebraic number supposed to be of small module, that later proved by Y. André in \([2]\) with the \( p \)-adic analogue. We also mention that the work by F. Beukers involves several algebraicity of values of the function \([6, 7]\). A historical survey for further reference is given in \([18, 19]\), with comparison which concerns earlier works.

This criterion indeed shows the linear independence of values of generalized hypergeometric functions including the contiguous ones, whose functional linear independence has been discussed in \([34, 35]\). Our contribution in the proof, if any, is an uncharted non-vanishing property for the generalized Wronskian of Hermite type, corresponding to the case of generalized hypergeometric \( G \)-function.

2 Notations and main result

We collect some notations which we use throughout the article. Let \( \mathbb{Q} \) be the rational number field and \( K \) be an algebraic number field of arbitrary degree \(|K : \mathbb{Q}| < \infty \). Let us denote by \( \mathbb{N} \) the set of strictly positive integers. We denote the set of places of \( K \) by \( \mathcal{M}_K \) (by \( \mathcal{M}_K^\infty \) for infinite places, by \( \mathcal{M}_K^f \) for finite places, respectively). For \( v \in \mathcal{M}_K \), we denote the completion of \( K \) with respect to \( v \) by \( \mathbb{K}_v \), and the completion of an algebraic closure of \( K_v \) by \( \mathbb{C}_v \) (resp. for \( v \in \mathcal{M}_K^\infty \), for \( v \in \mathcal{M}_K^f \)).

Let \( p, q \in \mathbb{N} \). Let \( a_1, \ldots, a_p, b_1, \ldots, b_q \in \mathbb{Q} \setminus \{0\} \) be non-negative integer. We define the generalized hypergeometric function by

\[
\begin{align*}
_pF_q \left( \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \mid z \right) &= \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!},
\end{align*}
\]

where \((a)_k\) is the Pochhammer symbol: \((a)_0 = 1\), \((a)_k = a(a+1) \cdots (a+k-1)\).

For a rational number \( x \), let us define

\[
\mu(x) = \prod_{q \text{ prime}} q^{\mu(x)/q-1}.
\]

Let us denote the normalized absolute value \(|\cdot|_v\) for \( v \in \mathcal{M}_K^f\):

\[
|p|_v = p^{-\left[\frac{K_v: \mathbb{Q}_p}{K: \mathbb{Q}}\right]} \quad \text{if } v \in \mathcal{M}_K^f \text{ and } v \mid p,
\]

\[
|x|_v = |\iota_v x|^{\left[\frac{K_v: \mathbb{R}}{K: \mathbb{Q}}\right]} \quad \text{if } v \in \mathcal{M}_K^\infty,
\]

where \( p \) is a prime number and \( \iota_v \) the embedding \( K \hookrightarrow \mathbb{C} \) corresponding to \( v \). On \( K_v^p \), the norm \( \|\cdot\|_v \) denotes the norm of the supremum. Then we have the product formula

\[
\prod_{v \in \mathcal{M}_K^f} |\xi|_v = 1 \quad \text{for } \xi \in K \setminus \{0\}.
\]

Let \( m \) be a positive integer and \( \beta := (\beta_0, \ldots, \beta_m) \in K^{m+1} \setminus \{0\} \). Define the absolute height of \( \beta \) by

\[
H(\beta) = \prod_{v \in \mathcal{M}_K} \max\{1, |\beta_0|_v, \ldots, |\beta_m|_v\}.
\]
and logarithmic absolute height by \( h(\beta) = \log H(\beta) \). Let \( v \in \mathfrak{M}_K \). We denote \( \log \max(1, |\beta|_v) \) by \( h_v(\beta) \). Then we have \( h(\beta) = \sum_{v \in \mathfrak{M}_K} h_v(\beta) \). For a finite set \( S \subset \overline{\mathbb{Q}} \), we define the denominator of \( S \) by

\[
\text{den}(S) = \min\{1 \leq n \in \mathbb{Z} \mid n\alpha \text{ are algebraic integer for all } \alpha \in S\}.
\]

Let \( m, r \) be strictly positive integers with \( r \geq 2 \) and \( \alpha := (\alpha_1, \ldots, \alpha_m) \in (K \setminus \{0\})^m \) whose coordinates are pairwise distinct. For \( \beta \in K \setminus \{0\} \), define a real number

\[
V_v(\alpha, \beta) = \log |\beta|_{v_0} - rmh(\alpha, \beta) - (rm + 1) \log \|\alpha\|_{v_0} + rm \log \|(\alpha, \beta)\|_{v_0} - \left( rm \log(2) + r \left( \log(rm + 1) + rm \log \left(\frac{rm + 1}{rm}\right) \right) \right) - \sum_{j=1}^{r} \left( \log \mu(a_j) + 2 \log \mu(b_j) + \frac{\text{den}(a_j)}{\varphi(\text{den}(a_j))} \frac{\text{den}(b_j)}{\varphi(\text{den}(b_j))} \right),
\]

where \( \varphi \) is the Euler’s totient function.

Now we are ready to state our main theorem.

**Theorem 2.1.** Let \( v_0 \) be a place of \( K \). Let \( a_1, \ldots, a_r, b_1, \ldots, b_{r-1} \in \mathbb{Q} \setminus \{0\} \) be non-negative integer. Assume neither \( a_k \) nor \( a_k + 1 - b_j \) be strictly positive integers \((1 \leq k \leq r, 1 \leq j \leq r - 1) \). Suppose \( V_{v_0}(\alpha, \beta) > 0 \). Then the \( rm + 1 \) numbers:

\[
_{rF_{r-1}} \left( \begin{array}{c} a_1, \ldots, a_r \\ b_1, \ldots, b_{r-1} \end{array} \right) \bigg| \frac{\alpha}{\beta} \bigg), \quad _{rF_{r-1}} \left( \begin{array}{c} a_1 + 1, \ldots, a_r + 1 \\ b_1 + 1, \ldots, b_{r-s+1}, \ldots, b_{r-1} \end{array} \right) \bigg| \frac{\alpha}{\beta}
\]

\((1 \leq i \leq m, 1 \leq s \leq r - 1) \) and 1 are linearly independent over \( K \).

We mention that a linear independence criterion for values of generalized hypergeometric \( G \)-functions with cyclic coefficients also follows from Theorem 2.1 based on the same argument in [20]. We will join it in the context of our future paper to avoid heavy calculations in the current article.

This article is organized as follows. In Section 3.1 we describe our setup for generalized hypergeometric \( G \)-functions. In Section 3.2 we proceed our construction of Padé approximants, generalizing the method used in [18, 19, 20]. Section 3 is devoted to show the non-vanishing property of the crucial determinant. In Section 6 we give the proof of Theorem 2.1. A more general statement, together with totally effective linear independence measures, is also given in this section by Theorem 6.1.

### 3 Padé approximation of generalized hypergeometric functions

Throughout this section, denote by \( K \) a field of characteristic 0. For a variable \( z \), we denote \( z \frac{d}{dz} \) by \( \theta_z \).

#### 3.1 Preliminaries

In this subsection, we introduce the generalized hypergeometric function. First let us introduce polynomials \( A(X), B(X) \in K[X] \) satisfying \( \max(\deg A, \deg B) > 0 \). Assume

\[
(1) \quad A(k)B(k) \neq 0 \quad (k \geq 0).
\]
Notice that this assumption yields $A(\theta_t + k), B(\theta_t + k) \in \text{Aut}_K(K[t])$ for any non-negative integer $k$. Next, consider a sequence $c := (c_k)_{k \geq 0}$ satisfying $c_k \in K \setminus \{0\}$ and

\[(2)\quad c_{k+1} = c_k \cdot \frac{A(k)}{B(k+1)} \quad (k \geq 0) .\]

We introduce the formal power series

$$F(z) := \sum_{k=0}^{\infty} c_k z^{k+1} ,$$

sometime also called generalized hypergeometric function.

By the recurrence relation (2), the series $F(1/z)$ is a solution of the differential equation:

$$\left( B(-\theta_z) z - A(-\theta_z) \right) f(z) = B(0) .$$

In order to construct Padé approximants of the function $F(z)$, we introduce a power series, say, contiguous to $F(z)$.

Put $r = \max(\deg A, \deg B)$ and take $\gamma_1, \ldots, \gamma_{r-1} \in K$. Let $s$ be an integer with $0 \leq s \leq r - 1$. We define the power series $F_s(z)$ by

\[(3)\quad F_0(z) = F(z), \quad F_s(z) = \sum_{k=0}^{\infty} (k + \gamma_1) \cdots (k + \gamma_s) c_k z^{k+1} \quad (1 \leq s \leq r - 1) .\]

Notice that $F_s(1/z)$ satisfies

$$F_s(1/z) = (-\theta_z + \gamma_1 - 1) \cdots (-\theta_z + \gamma_s - 1)(F_0(1/z)) .$$

**Remark 3.1.** Let $p, q \in \mathbb{N}, a_1, \ldots, a_p, b_1, \ldots, b_q \in K \setminus \{0\}$ be non-negative integer. Put $A(X) = (X + a_1 + 1) \cdots (X + a_p + 1)$, $B(X) = (X + b_1) \cdots (X + b_q)(X + 1)$ and define

$$c_k = \frac{(a_1)_{k+1} \cdots (a_p)_{k+1}}{(b_1)_{k+1} \cdots (b_q)_{k+1}(k+1)!} \quad (k \geq 0) .$$

Then $(c_k)_{k \geq 0}$ satisfies

$$c_{k+1} = c_k \cdot \frac{A(k)}{B(k+1)} .$$

For this sequence, we have

$$F(1/z) = p F_q \begin{pmatrix} a_1, \ldots, a_p \mid z \end{pmatrix} - 1 .$$

We assume $r := p - 1 = q$. Put $\gamma_1 = 1, \gamma_2 = b_{r-1}, \ldots, \gamma_{r-1} = b_2$. Then the series $F_s(1/z)$ has the expression:

\[(4)\quad F_0(1/z) = r F_{r-1} \begin{pmatrix} a_1, \ldots, a_r \mid z \end{pmatrix} - 1 ,\]

\[(5)\quad F_s(1/z) = \frac{a_1 \cdots a_r}{b_1 \cdots b_{r-s}} \cdot \frac{1}{z} r F_{r-1} \begin{pmatrix} a_1 + 1, \ldots, a_r + 1 \mid z \end{pmatrix} ,\]

for $1 \leq s \leq r - 1$. 
3.2 Construction of Padé approximants

Let $K$ be a field of characteristic 0. We define the order function $\text{ord}_\infty$ at "$z = \infty$" by

$$\text{ord}_\infty : K((1/z)) \to \mathbb{Z} \cup \{\infty\}; \quad \sum_k \frac{c_k}{z^k} \mapsto \min\{k \in \mathbb{Z} \mid c_k \neq 0\}.$$ 

We first recall the following fact (see [21]):

**Lemma 3.2.** Let $r$ be a positive integer, $f_1(z), \ldots, f_r(z) \in (1/z) \cdot K[[1/z]]$ and $n := (n_1, \ldots, n_r) \in \mathbb{N}^r$. Put $N := \sum_{i=1}^r n_i$. Let $M$ be a positive integer with $M \geq N$. Then there exists a family of polynomials $(P_0(z), P_1(z), \ldots, P_r(z)) \in K[z]^{r+1} \setminus \{0\}$ satisfying the following conditions:

(i) $\deg P_0(z) \leq M$,

(ii) $\text{ord}_\infty(P_0(z)f_j(z) - P_j(z)) \geq n_j + 1$ for $1 \leq j \leq r$.

**Definition 3.3.** For the family of polynomials $(P_0(z), P_1(z), \ldots, P_r(z)) \in K[z]^{r+1}$ satisfying the properties (i) and (ii) of Lemma 3.2, let us call it, weight $n$ and degree $M$ Padé type approximants of $(f_1, \ldots, f_r)$. For such $(P_0(z), P_1(z), \ldots, P_r(z))$, of $(f_1, \ldots, f_r)$, consider the family of formal Laurent series $(P_0(z)f_j(z) - P_j(z))_{1 \leq j \leq r}$. We call it weight $n$ degree $M$ Padé type approximations of $(f_1, \ldots, f_r)$.

**Notation 3.4.**

(i) For $\alpha \in K$, denote by $\text{Eval}_\alpha$ the linear evaluation map $K[t] \to K, P \mapsto P(\alpha)$.

Whenever there is an ambiguity in a setting of variables, we will denote the map by $\text{Eval}_{t \mapsto \alpha}$.

(ii) For $P \in K[t]$, we denote by $[P]$ the multiplication by $P$ (the map $Q \mapsto PQ$).

(iii) For a $K$-automorphism $\varphi$ of a $K$-module $M$ and an integer $k$, put

$$\varphi^k = \begin{cases} 
\varphi \circ \cdots \circ \varphi & \text{if } k > 0 \\
\text{id}_M & \text{if } k = 0 \\
\varphi^{-1} \circ \cdots \circ \varphi^{-1} & \text{if } k < 0 
\end{cases}.$$ 

Now we explicitly construct Padé approximants of generalized hypergeometric functions at distinct points. The following lemma is a key ingredient.

**Lemma 3.5.** Let $k$ be a non-negative integer.

(i) Let $H(X) \in K[X]$. We have $[t^k] \circ H(\theta_t) = H(\theta_t - k) \circ [t^k]$.

(ii) Let $A, B \in K[X]$ be polynomials with (1). Let $c := (c_k)_{k \geq 0}$ be a sequence satisfying $c_k \in K \setminus \{0\}$ together with (2) for $A, B$. Define $\mathcal{T}_c \in \text{Aut}_K(K[t])$ by

$$\mathcal{T}_c : K[t] \to K[t]; \quad t^k \mapsto \frac{t^k}{c_k}.$$ 

Then we have the relation, in the ring $\text{End}_K(K[t])$,

$$[t^k] \circ \mathcal{T}_c = \mathcal{T}_c \circ A(\theta_t - 1) \circ \cdots \circ A(\theta_t - k) \circ B(\theta_t)^{-1} \circ \cdots \circ B(\theta_t - k + 1)^{-1} \circ [t^k].$$

**Proof.** (i) Let $n$ be a non-negative integer. We may assume $H(X) = X^n$. For any non-negative integer $m$, we have

$$[t^k] \circ \theta_t^n(t^m) = m^n t^{m+k}.$$
On the other hand, we have
\[(\theta_t - k)^n \circ [t^k](t^m) = (k + m - k)^n t^{m+k} = m^n t^{m+k} .\]

By (i) and the above identity, we obtain the assertion.

(ii) Let \( m \) be a non-negative integer. The recurrence relation (2) yields
\[ \frac{1}{c_{m+k}} = \frac{B(m+k) \cdots B(m+1)}{A(m+k-1) \cdots A(m)} \cdot \frac{1}{c_m} , \]
hence we obtain
\[ [t^k] \circ T_c(t^m) = \frac{t^{k+m}}{c_m} = \frac{1}{c_{m+k}} \frac{A(m+k-1) \cdots A(m)}{B(m+k) \cdots B(m+1)} t^{k+m} \]
\[ = T_c \circ A(\theta_t - 1) \circ \cdots \circ A(\theta_t - k) \circ B(\theta_t)^{-1} \circ \cdots \circ B(\theta_t - k + 1)^{-1} \circ [t^k](t^m) . \]
which achieves the proof of (ii).

We are now ready for our construction of Padé approximants, of the hypergeometric functions at distinct points. Let \( e := (c_k)_{k \geq 0} \) be a sequence satisfying \( c_k \in K \setminus \{0\} \) together with (2) for polynomials \( A, B \in K[X] \). Put \( r = \max(\deg A, \deg B) \). Let us fix \( \gamma_1, \ldots, \gamma_{r-1} \in K \). We denote by \( F_s(z) \) the power series defined in (3) for \( \gamma_1, \ldots, \gamma_{r-1} \in K \). Let \( m \) be a strictly positive integer and \( \alpha_1, \ldots, \alpha_m \in K \setminus \{0\} \) which are pairwise distinct. For \( 0 \leq s \leq r-1 \), we shall introduce a \( K \)-homomorphism \( \psi_{i,s} \in \text{Hom}_K(K[t], K) \) by
\[ \psi_{i,s} : K[t] \longrightarrow K; t^k \mapsto (k + \gamma_1) \cdots (k + \gamma_s) c_k \alpha_i^{k+1} , \]
where \( (k + \gamma_1) \cdots (k + \gamma_s) = 1 \) for \( s = 0 \) and \( k \geq 0 \).

**Proposition 3.6.** (confer (14) Theorem 5.5) We use the notations as above. For a non-negative integer \( \ell \), we define polynomials:
\[ P_{t}(z) = \left[ \frac{1}{(n-1)!^n} \right] \circ \text{Eval}_z \circ T_c \circ \text{Eval}_t + 1 \ B(\theta_t + j) \left( t^m \prod_{i=1}^{m}(t - \alpha_i)^{r_1} \right) , \]
\[ P_{t,i,s}(z) = \psi_{i,s} \left( \frac{P_{t}(z) - P_{t}(t)}{z-t} \right) \text{ for } 1 \leq i \leq m, 0 \leq s \leq r-1 , \]
where \( T_c \in \text{Aut}_K(K[t]) \) defined in (4). Then \( (P_{t}(z), P_{t,i,s}(z))_{1 \leq i \leq m, 0 \leq s \leq r-1} \) forms a weight \( (n, \ldots, n) \in \mathbb{N}^m \) and degree \( rmn + \ell \) Padé type approximants of \( (F_s(\alpha_i/z))_{1 \leq i \leq m, 0 \leq s \leq r-1} \).

**Proof.** By the definition of \( P_{t}(z) \), we have
\[ \deg P_{t}(z) = rmn + \ell . \]

Hence the required condition on the degree is verified. By the definition of \( T_c \) and \( \psi_{i,s} \), we have
\[ \psi_{i,s} = \psi_{i,0} \circ (\theta_t + \gamma_1) \circ \cdots \circ (\theta_t + \gamma_s) , \]
\[ \psi_{i,0} \circ T_c = [\alpha_i] \circ \text{Eval}_{\alpha_i} \text{ for } 1 \leq i \leq m . \]
Put \( R_{t,i,s}(z) = P_{t}(z)F_s(\alpha_i/z) - P_{t,i,s}(z) \). Then, by the definition of \( R_{t,i,s}(z) \), we obtain
\[ R_{t,i,s}(z) = P_{t}(z)\psi_{i,s} \left( \frac{1}{z-t} \right) - P_{t,i,s}(z) = \psi_{i,s} \left( \frac{P_{t}(t)}{z-t} \right) = \sum_{k=0}^{\infty} \psi_{i,s}(t^k P_{t}(t)) \frac{z^k}{z^{k+1}} . \]
Let $k$ be an integer with $0 \leq k \leq n - 1$. By Lemma 3.8 (ii), we have

$$(n - 1)^r t^k P_t(t) = [t^k] \cdot T_e \bigcirc_{j=1}^{n-1} B(\theta + j) \left( t^f \prod_{i=1}^{m} (t - \alpha_i)^{r_n} \right)$$

$$= T_e \bigcirc_{j=1}^{k} A(\theta - j') \bigcirc_{j'=0}^{k-1} B(\theta - j' - 1) \bigcirc_{j=1}^{n-1} B(\theta + j - k) \left( t^{f+k} \prod_{i=1}^{m} (t - \alpha_i)^{r_n} \right)$$

$$= T_e \bigcirc_{j=1}^{k} A(\theta - j') \bigcirc_{j'=0}^{k-1} B(\theta + j) \left( t^{f+k} \prod_{i=1}^{m} (t - \alpha_i)^{r_n} \right),$$

where $\bigcirc_{j=1}^{k} A(\theta - j') = \text{id}_K[t]$ if $k = 0$. Therefore we have

$$\psi_{i,s}((n-1)^r t^k P_t(t)) = \psi_{i,s} \cdot T_e \bigcirc_{j'=1}^{s-1} A(\theta + j) \bigcirc_{j=0}^{n-1-k} B(\theta + j) \left( t^{f+k} \prod_{i=1}^{m} (t - \alpha_i)^{r_n} \right)$$

$$= \psi_{i,0} \cdot T_e \bigcirc_{j'=1}^{s-1} A(\theta + j) \bigcirc_{j=0}^{n-1-k} B(\theta + j) \left( t^{f+k} \prod_{i=1}^{m} (t - \alpha_i)^{r_n} \right)$$

$$= [\alpha_i] \cdot \text{Eval}_{\alpha_i} \bigcirc_{j'=1}^{s-1} A(\theta + j) \bigcirc_{j=0}^{n-1-k} B(\theta + j) \left( t^{f+k} \prod_{i=1}^{m} (t - \alpha_i)^{r_n} \right).$$

Note that, in (12), (13), we use (10) and (11) respectively. Since we have

$$\deg \left( \prod_{j'=1}^{s} (X + \gamma_{j''}) \right) \bigcirc_{j'=1}^{k} A(\theta - j') \bigcirc_{j=1}^{n-1-k} B(\theta + j) \leq s + rk + r(n - 1 - k) \leq rn - 1$$

thanks to the Leibniz rule, the polynomial $\bigcirc_{j'=1}^{s-1} A(\theta + j) \bigcirc_{j=0}^{n-1-k} B(\theta + j) \left( t^{f+k} \prod_{i=1}^{m} (t - \alpha_i)^{r_n} \right)$ is contained in the ideal $(t - \alpha_i) = \ker \text{Eval}_{\alpha_i}$. Consequently we have

$$\psi_{i,s}(t^k P_t(t)) = 0 \text{ for } 0 \leq k \leq n - 1, 1 \leq i \leq m, 0 \leq s \leq r - 1.$$ 

By the above expansion of $R_{\ell,i,s}(z)$, we obtain

$$\text{ord}_{\infty} \cdot R_{\ell,i,s}(z) \geq n + 1 \text{ for } 1 \leq i \leq m, 0 \leq s \leq r - 1,$$

hence Proposition 3.8 follows.

We should mention that this construction was also considered by D. V. Chudnovsky and G. V. Chudnovsky in [17, Theorem 5.5], but without arithmetic application. See also a related work in [31].

**Remark 3.7.** The polynomial $P_t(z)$ does not depend on the choice of $\gamma_1, \ldots, \gamma_{r-1} \in K$. By contrast, the polynomials $P_{\ell,i,s}(z)$ depend on these choice.

**Remark 3.8.** Let $r, m$ be strictly positive integers. Let $x \in K$, supposed to be non-negative integer and $\alpha_1, \ldots, \alpha_m \in K \setminus \{0\}$ be pairwise distinct. Put $A(X) = B(X) = (X + x + 1)^r$ and $c_k = 1/(k + x + 1)^r$. Then we have

$$c_{k+1} = c_k \cdot \frac{A(k)}{B(k+1)}.$$

Put $\gamma_1 = \cdots = \gamma_{r-1} = x + 1$. This gives us

$$F_s(\alpha_i/z) = \sum_{k=0}^{\infty} \frac{1}{(k + x + 1)^r - s} \cdot \frac{\alpha_i^{k+1}}{z^{k+1}} = \Phi_{r-s}(x, \alpha_i/z) \quad (1 \leq i \leq m, 0 \leq s \leq r - 1),$$

hence Proposition 3.8 follows. \qed
where $\Phi_s(x, 1/z)$ is the $s$-th Lerch function (generalized polylogarithmic function; confer [19]). In this case, we have $T_s = \frac{(\theta_i + x + 1)^r}{(x + 1)^r}$ and

$$P_t(z) = \left[ \frac{1}{(x + 1)^r \cdot (n - 1)!} \right] \circ \text{Eval}_z \circ \prod_{j=1}^{n} (\theta_i + x + j)^r \left( t^\ell \prod_{i=1}^{m} (t - \alpha_i)^{rn} \right).$$

The polynomial $\frac{(x+1)^r}{n!} P_t(z)$ gives Padé type approximants of this Lerch functions in [13, Theorem 3.8].

4 Non-vanishing of the generalized Wronskian of Hermite type

Let $K$ be a field of characteristic $0$ and $A(X), B(X) \in K[X]$ satisfying (11). From this section to last, we assume $\deg A = \deg B > 0$ and put $\deg A = r$. We shall choose a sequence $c := (c_k)_{k \geq 0}$ satisfying $c_k \in K \setminus \{0\}$ and (2) for the given polynomials $A(X), B(X)$. Let $\alpha := (\alpha_1, \ldots, \alpha_m) \in (K \setminus \{0\})^m$ whose coordinates are pairwise distinct and $\gamma_1, \ldots, \gamma_r - 1 \in K$. Let us fix a positive integer $n$. For a non-negative integer $\ell$ with $0 \leq \ell \leq rm$, recall the polynomials $P_\ell(z), P_{\ell,i,s}(z)$ defined in (8) and (9). We define column vectors $\bar{p}_\ell(z) \in K[z]^{rm + 1}$ by

$$\bar{p}_\ell(z) = \ell \left( P_\ell(z), P_{\ell,1,r-1}(z), \ldots, P_{\ell,1,0}(z), \ldots, P_{\ell,m,r-1}(z), \ldots, P_{\ell,m,0}(z) \right),$$

and put

$$\Delta_n(z) = \Delta(z) = \det \left( \bar{p}_0(z) \cdots \bar{p}_{rm}(z) \right).$$

The aim of this section is to prove the following proposition.

**Proposition 4.1.** The determinant $\Delta(z)$ satisfies $\Delta(z) \in K \setminus \{0\}$.

4.1 First Step

**Lemma 4.2.** We have $\Delta(z) \in K$.

**Proof.** We denote the remainder function $R_{\ell,i,s}(z) := P_\ell(z)F_s(\alpha_i/z) - P_{\ell,i,s}(z)$ $(0 \leq \ell \leq rm, 1 \leq i \leq m, 0 \leq s \leq r - 1)$. For the matrix in $\Delta(z)$, multiplying the first row by the $F_s(\alpha_i/z)$ and adding it to the $(i - 1)r + s + 1$-th row $(1 \leq i \leq m, 0 \leq s \leq r - 1)$, we obtain

$$\Delta(z) = (-1)^{rm} \det \begin{pmatrix}
P_0(z) & \cdots & P_{rm}(z) \\
R_{0,1,r-1}(z) & \cdots & R_{rm,1,r-1}(z) \\
\vdots & \ddots & \vdots \\
R_{0,1,0}(z) & \cdots & R_{rm,1,0}(z) \\
\vdots & \ddots & \vdots \\
R_{0,m,r-1}(z) & \cdots & R_{rm,m,r-1}(z) \\
\vdots & \ddots & \vdots \\
R_{0,m,0}(z) & \cdots & R_{rm,m,0}(z)
\end{pmatrix}. $$

We denote the $(s,t)$-th cofactor by $\Delta_{s,t}(z)$ of the matrix in the right hand side above. Then we have, by developing along the first row:

$$\Delta(z) = (-1)^{rm} \sum_{\ell=0}^{rm} P_\ell(z) \Delta_{1,\ell+1}(z).$$

(15)
Since we have
\[
\text{ord}_\infty R_{\ell,i,s}(z) \geq n + 1 \text{ for } 0 \leq \ell \leq rm, \ 1 \leq i \leq m \text{ and } 0 \leq s \leq r - 1,
\]
we obtain
\[
\text{ord}_\infty \Delta_{1,\ell+1}(z) \geq (n + 1)rm.
\]
The fact \( \deg P_\ell(z) = rmn + \ell \) with the lower bound above yields
\[
P_\ell(z)\Delta_{1,\ell+1}(z) \in (1/z) \cdot K[[1/z]] \text{ for } 0 \leq \ell \leq rm - 1,
\]
and
\[
\text{(16)} \quad P_{rm}(z)\Delta_{1,rm+1}(z) \in K[[1/z]].
\]
In the relation above, the constant term of \( P_{rm}(z)\Delta_{1,rm+1}(z) \) equals to:
\[
\text{Coefficient of } z^{rm(n+1)} \text{ of } P_{rm}(z) \times \text{Coefficient of } 1/z^{rm(n+1)} \text{ of } \Delta_{1,rm+1}(z).
\]
thanks to the fact that \( \Delta(z) \) in (15) is a polynomial of non-positive valuation in \( z \) with respect to \( \text{ord}_\infty \), it is necessarily to be a constant. Moreover, the terms of strictly negative valuation in \( z \), they have to cancel out, hence
\[
\text{(17)} \quad \Delta(z) = (-1)^{rm} \cdot \left( \sum_{\ell=0}^{rm} P_\ell(z)\Delta_{1,\ell+1}(z) \right) = (-1)^{rm} \times \text{Constant term of } P_{rm}(z)\Delta_{1,rm+1}(z) \in K.
\]
This completes the proof of Lemma 4.2.

4.2 Second step

We now start the second procedure, by factoring \( \Delta \) as an element of \( K(\alpha_1, \ldots, \alpha_m) \). We use the same notations as in the proof of Lemma 4.2. By the equalities (16) and (17), we have
\[
\text{(18)} \quad \Delta(z) = (-1)^{rm} \times \text{Coefficient of } z^{rm(n+1)} \text{ of } P_{rm}(z) \times \text{Coefficient of } 1/z^{rm(n+1)} \text{ of } \Delta_{1,rm+1}(z).
\]
Define a column vector \( \vec{q}_0 \in K^{rm} \) by
\[
\vec{q}_\ell = \left( \psi_{1,r-1}(t^n P_\ell(t)), \ldots, \psi_{1,0}(t^n P_\ell(t)), \ldots, \psi_{m,r-1}(t^n P_\ell(t)), \ldots, \psi_{m,0}(t^n P_\ell(t)) \right).
\]
By the definition of \( \Delta_{n,1,rm+1}(z) \) with the identities
\[
R_{\ell,i,s}(z) = \sum_{k=n}^{\infty} \frac{\psi_{i,s}(t^k P_\ell(t))}{z^{k+1}} \quad \text{for } 0 \leq \ell \leq rm, \ 1 \leq i \leq m \text{ and } 0 \leq s \leq r - 1,
\]
we have
\[
\text{Coefficient of } 1/z^{rm(n+1)} \text{ of } \Delta_{n,1,rm+1}(z) = \det \left( \vec{q}_0 \cdots \vec{q}_{rm-1} \right).
\]
By (18) with the above identity, we have
\[
\text{(19)} \quad \Delta(z) = (-1)^{rm} \left( \frac{1}{(rmn + rm)!} \left( \frac{d}{dz} \right)^{rmn+rm} P_{m,rm}(z) \right) \cdot \det \left( \vec{q}_0 \cdots \vec{q}_{rm-1} \right).
\]
Note that, by the definition of \( P_{rm}(z) \), we have \( \deg P_{rm} = (n + 1)rm \) and thus
\[
\frac{1}{(rmn + rm)!} \left( \frac{d}{dz} \right)^{rmn+rm} P_{m,rm}(z) \neq 0.
\]
4.3 Third step

Relying on (19), we study here the values

\[ \Theta = \det\left( \hat{q}_0 \cdots \hat{q}_{rn-1} \right). \]

From this subsection, we specify the choice of \( \gamma_1, \ldots, \gamma_{r-1} \in K \) as follows. Replacing \( K \) by an appropriate finite extension, we may assume \( A(X), B(X) \) be decomposable in \( K \). Put

\[ A(X) = (X + \eta_1) \cdots (X + \eta_r), \quad B(X) = (X + \zeta_1) \cdots (X + \zeta_r), \]

where \( \eta_1, \ldots, \eta_r, \zeta_1, \ldots, \zeta_r \in K \setminus \{0\} \), being non-negative integer. Take a sequence \( \gamma_i \) of \( K \) with \( \gamma_1 = \zeta_r, \ldots, \gamma_r = \zeta_1 \). For each \( 0 \leq s \leq r - 1 \), there exists a sequence \( (a_{k,s})_{0 \leq k \leq rn} \in K^{rn+1} \)

\[ \prod_{j=1}^{n} A(X - j) = \sum_{k=0}^{rn} a_{k,s} \prod_{u=1}^{k} (X + \gamma_{r-s+1+u}), \]

where it read \( \prod_{u=1}^{k}(X + \gamma_{r-s+1+u}) = 1 \) if \( k = 0 \). We now simplify the determinant \( \Theta \) using the quantities \( a_{0,s} \) to prove the non-vanishing property of \( \Theta \).

**Lemma 4.3.** Put \( H_{\ell}(t) = t^{\ell} \prod_{i=1}^{m} (t - \alpha_i)^{rn} \) for \( 0 \leq \ell \leq rm - 1 \). Then we have

\[ \Theta = \frac{\prod_{i=1}^{m} \alpha_i^{r} \prod_{j=0}^{rn} \theta_j^{rn}}{(n-1)!^{rn}}, \quad \det \{ \text{Eval}_{\alpha_i} \circ \bigcirc_{w=0}^{s} (\theta_t + \gamma_{r-s+w})^{-1}(t^{a} H_{\ell}(t)) \}_{0 \leq \ell \leq rm-1} \}_{1 \leq s \leq rn, 0 \leq s \leq r-1}. \]

**Proof.** Using (11) and (21), we have

\[ \psi_{i,r-s} \circ T_c \circ _{j=1}^{n} A(\theta_t - j) \circ B(\theta_t)^{-1} = \sum_{k=0}^{s} a_{k,s} \psi_{i,0} \circ T_c \circ _{w=0}^{s} \theta_t + \gamma_{w} \circ B(\theta_t)^{-1} \]

\[ + \sum_{k=s+1}^{rn} a_{k,s} \psi_{i,0} \circ T_c \circ B(\theta_t) \circ _{w=r+1}^{s} (\theta_t + \gamma_{w}) \circ B(\theta_t)^{-1}. \]

Since

\[ \deg \prod_{w=r+1}^{r-s+1+k} (X + \gamma_{w}) = k - s - 1 \leq rn - 1 \]

\( (s + 1 \leq k \leq rn) \), by the Leibniz rule, the polynomial \( \Theta_{w=r+1}^{s} (\theta_t + \gamma_{w})(t^{a} H_{\ell}(t)) \) belongs to the ideal \( (t - \alpha_i) = \ker \text{Eval}_{\alpha_i} \). Therefore, using (11), we obtain

\[ \sum_{k=s+1}^{rn} a_{k,s} \alpha_i \circ \text{Eval}_{\alpha_i} \circ _{w=r+1}^{s} (\theta_t + \gamma_{w})(t^{a} H_{\ell}(t)) = 0. \]

By the above equality with (22), we have

\[ \psi_{i,r-1-s} \circ T_c \circ _{j=1}^{n} A(\theta_t - j) \circ B(\theta_t)^{-1}(t^{a} H_{\ell}(t)) = \sum_{k=0}^{s} a_{k,s} \psi_{i,0} \circ T_c \circ _{w=0}^{s} \theta_t + \gamma_{w} \circ B(\theta_t)^{-1}(t^{a} H_{\ell}(t)) \]

\[ = \sum_{k=0}^{s} a_{k,s} \alpha_i \circ \text{Eval}_{\alpha_i} \circ _{w=k}^{s} (\theta_t + \gamma_{r-s+w})(t^{a} H_{\ell}(t)) \].

Interpreting the relations above as linear manipulations of lines, the columns let the determinant unchanged. This completes the proof of Lemma 4.3.
We now study when the quantity $\prod_{s=0}^{r-1} a_{0,s}^n$ does not vanish. The following lemma will be used to calculate each $a_{0,s}$.

**Lemma 4.4.** Let $u$ be a strictly positive integer and $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_u, \tilde{\eta}_1, \ldots, \tilde{\eta}_u \in K$. Denote

$$(X + \tilde{\eta}_1) \cdots (X + \tilde{\eta}_u) = b_{u,0} + \sum_{k=1}^{u} b_{u,k} (X + \tilde{\gamma}_1) \cdots (X + \tilde{\gamma}_k),$$

with $b_{u,0}, b_{u,1}, \ldots, b_{u,u} \in K$. Then we have $b_{u,0} = (\tilde{\eta}_1 - \tilde{\gamma}_1) \cdots (\tilde{\eta}_u - \tilde{\gamma}_1)$.

**Proof.** We prove the lemma by induction on $u$. In the case of $u = 1$, we have

$$X + \tilde{\eta}_1 = \tilde{\eta}_1 - \tilde{\gamma}_1 + (X + \tilde{\gamma}_1).$$

This shows $b_{1,0} = \tilde{\eta}_1 - \tilde{\gamma}_1$ then yields the assertion. Suppose that the current lemma be true for $u \geq 1$. We show its validity for $u + 1$. In this case we get

$$(X + \tilde{\eta}_1) \cdots (X + \tilde{\eta}_u)(X + \tilde{\eta}_{u+1}) = \left[ b_{u,0} + \sum_{k=1}^{u} b_{u,k} (X + \tilde{\gamma}_1) \cdots (X + \tilde{\gamma}_k) \right] (X + \tilde{\eta}_{u+1})$$

$$= b_{u,0} (X + \tilde{\gamma}_1 + \tilde{\eta}_{u+1} - \tilde{\gamma}_1) + \sum_{k=1}^{u} b_{u,k} (X + \tilde{\gamma}_1) \cdots (X + \tilde{\gamma}_k) (X + \tilde{\gamma}_{k+1} + \tilde{\eta}_{u+1} - \tilde{\gamma}_{k+1}).$$

The above identity yields $b_{u+1,0} = b_{u,0}(\tilde{\eta}_{u+1} - \tilde{\gamma}_1)$. By induction hypothesis for $b_{u,0}$, we conclude

$$b_{u+1,0} = (\tilde{\eta}_1 - \tilde{\gamma}_1) \cdots (\tilde{\eta}_u - \tilde{\gamma}_1)(\tilde{\eta}_{u+1} - \tilde{\gamma}_1).$$

This completes the proof of Lemma 4.4.

**Proposition 4.5.** The following two properties are equivalent.

(i) The value $\prod_{s=0}^{r-1} a_{0,s}^n$ is non-zero.

(ii) For $1 \leq i, j \leq r$ and $1 \leq k \leq n$, we have $\eta_i - k - \zeta_j \neq 0$.

**Proof.** Let $s$ be an integer with $0 \leq s \leq r - 1$. Applying Lemma 4.4 with $u = rn$ and

$$(\tilde{\gamma}_1, \ldots, \tilde{\gamma}_{rn}) = (\gamma_{r-s}, \ldots, \gamma_{(n+1)\gamma_{r-s}-1}), \quad (\tilde{\eta}_1, \ldots, \tilde{\eta}_{rn}) = (\eta_i - k)_{1 \leq i \leq r, 1 \leq k \leq n},$$

we get

$$a_{0,s} = \prod_{i=1}^{r} \prod_{k=1}^{\eta_i - k - \gamma_{r-s}}, \quad (0 \leq s \leq r - 1).$$

Since $\gamma_{r-s} = \zeta_{s+1}$ for $0 \leq s \leq r - 1$, the proposition follows.

In the following, we assume

$$\eta_i - \zeta_j \text{ not be strictly positive integers for } 1 \leq i, j \leq r.$$  

4.4 **Fourth step**

Now, we take the ring $K[t_{i,s}]_{1 \leq i \leq m, 0 \leq s \leq r-1}$, the ring of polynomials in $rn$ variables over $K$. Recall the polynomial $B(X)$ is decomposed as $B(X) = (X + \zeta_1) \cdots (X + \zeta_r)$ with $\zeta_i \in K$ which are not negative integer. We choose $(\gamma_i)_{1 \leq i \leq r} \in K^r$ by $\gamma_i = \zeta_r, \ldots, \gamma_r = \zeta_1$. For each variable $t_{i,s}$, one has a well defined map for $\alpha \in K$:

$$\tilde{\psi}_{\alpha, i, s} = \text{Eval}_{t_{i,s} \rightarrow \alpha} \cap_{w = 0}^{\gamma_{r-s}+w} (\theta_{i,s} + \gamma_{r-s+w})^{-1} : K[t_{i,s}]_{0 \leq s \leq r-1} \rightarrow K[t_{i',s'}]_{0 \leq s' \leq r-1} ; \quad t_{i,s}^{k} \rightarrow \alpha^k \prod_{w=0}^{(\gamma_{r-s+w})}.$$
Using the definition above where \( K[t_{i,s}]_{1 \leq i \leq m, 0 \leq s \leq r-1} \) is seen as the one variable polynomial ring \( K'[t_{i,s}] \) over \( K' = K[t_{i',s'}]_{(i',s') \neq (i,s)} \).

We now define for non-negative integers \( n, u \)

\[
\hat{P}_{n,u}(t_{i,s}) = \prod_{i=1}^{m} \prod_{s=0}^{r-1} t_{i,s}^{u} \prod_{j=1}^{m} (t_{i,s} - \alpha_j)^{rn} \prod_{(i_1,s_1) < (i_2,s_2)} (t_{i_2,s_2} - t_{i_1,s_1}) ,
\]

where the order \((i_1, s_1) < (i_2, s_2)\) means lexicographical order. In the following of this section, the index \( n \) will be conveniently omitted, to be easier to read.

Also set (when no confusion is deemed to occur, we omit the subcripts \( \alpha = (\alpha_1, \ldots, \alpha_m) \)):

\[
\Psi = \Psi_{\alpha} := \bigcirc_{i=1}^{m} \bigcirc_{s=0}^{r-1} \hat{\psi}_{\alpha_i, i, s} .
\]

Note that, by the definition of \( \Theta \), we have

\[
\Theta = \prod_{i=1}^{m} \alpha_i \prod_{s=0}^{r-1} a_{0,s} \psi(\hat{P}_{n,n}) .
\]

Let \( u \) be a non-negative integer, we study the value

\[
C_{n,u,m} = C_{u,m} := \Psi(\hat{P}_u) .
\]

The following of subsection, we occupy the proof of the following property of \( C_{u,m} \).

**Proposition 4.6.** There exists a constant \( c_{u,m} \in K \) with

\[
C_{u,m} = c_{u,m} \prod_{i=1}^{m} \alpha_i^{ru+r^2n+\binom{r}{2}} \prod_{1 \leq i_1 < i_2 \leq m} (\alpha_{i_2} - \alpha_{i_1})^{(2n+1)r^2} .
\]

It is also easy to see that since all the variables \( t_{i,s} \) have been specialized, \( C_{u,m} \in K \) is a polynomial in the \( \alpha_i \). The statement is then about a factorization of this polynomial. To prove of Proposition 4.6 we are going to perform the following steps:

(a) Show that \( C_{u,m} \) is homogeneous of degree \( m[ru+ru^2n+\binom{r}{2}] + \binom{m}{2}(2n+1)r^2 \).

(b) Show that \( \prod_{i=1}^{m} \alpha_i^{ru+ru^2n+\binom{r}{2}} \) divides \( C_{u,m} \).

(c) Show that \( \prod_{1 \leq i_1 < i_2 \leq m} (\alpha_{i_2} - \alpha_{i_1})^{(2n+1)r^2} \) divides \( C_{u,m} \).

We first prove (a) and (b).

**Lemma 4.7.** \( C_{u,m} \) is homogeneous of degree \( m[ru+ru^2n+\binom{r}{2}] + \binom{m}{2}(2n+1)r^2 \) and is divisible by \( \prod_{i=1}^{m} \alpha_i^{ru+ru^2n+\binom{r}{2}} \).

**Proof.** First the polynomial \( \hat{P}_u(t) \) is a homogeneous polynomial with respect to the variables \( \alpha_i, t_{i,s} \) of degree \( m[ru+ru^2n+\binom{r}{2}] + \binom{m}{2}(2n+1)r^2 \). By the definition of \( \Psi \), it is easy to see that \( C_{u,m} = \Psi(\hat{P}_u(t)) \) is a homogeneous polynomial with respect to the variables \( \alpha_i \) of degree \( m[ru+ru^2n+\binom{r}{2}] + \binom{m}{2}(2n+1)r^2 \). Second we show the later assertion. By linear algebra \( \hat{\psi}_{\alpha_i, i, s}(B(t_{i,s})) = \alpha_i \hat{\psi}_{\alpha_i, i, s}(B(t_{i,s})) \) (i.e. the variable \( t \) specializes in 1, confer Lemma 4.5 (ii) below) for any integer \( \ell \) and any polynomial \( B(t_{i,s}) \in K[t_{i,s}] \). So, by composition, the same holds for \( \Psi \), and, putting \( 1 = (1, \ldots, 1) \), one gets

\[
C_{u,m} = \prod_{i=1}^{m} \alpha_i^{ru+ru^2n+\binom{r}{2}} \left( \Psi_1(\hat{P}_u(\alpha_i t_{i,s})) \right) .
\]
We now compute
\[ \hat{P}_u(\alpha_i t_{i,s}) = \prod_{i=0}^{m} \alpha_i^{n+ru+\binom{r}{2}} \cdot Q_u(t), \]
where
\[ Q_u(t) = Q_n, u, m(t) = \left( \prod_{i=1}^{m} \prod_{s=0}^{r-1} \left[ t_{i,s}^u \prod_{j \neq i} (\alpha_j t_{i,s} - \alpha_j)^{rn} (t_{i,s} - 1)^{rn} \right] \right) \cdot \prod_{1 \leq i_1 < i_2 < m} \prod_{0 \leq s_1, s_2 \leq r-1} \prod_{i=1}^{m} (t_{i,s_2} - t_{i,s_1}), \]
by linearity, we obtain
\[ C_{u, m} = \prod_{i=1}^{m} \alpha_i^{r(u+1) + ru + \binom{r}{2}} \left( \Psi_1(Q_u) \right). \]
This concludes the proof of the lemma. \(\square\)

Now we consider (c). Since the statement is trivial for \( m = 1 \), we can assume \( m \geq 2 \). We need to show that \( (\alpha_j - \alpha_i)^{(2n+1)} r^2 \) divides \( C_{u, m} \). Without loss of generality, after renumbering, we can assume that \( j = 2, i = 1 \). To ease notations, we are going to take advantage of the fact that \( m \geq 2 \), and set \( X_s = t_{1,s}, Y_s = t_{2,s}, \alpha_1 = \alpha, \alpha_2 = \beta \) and \( \tilde{\psi}_{\alpha, 1, s} = \tilde{\psi}_{\alpha, s}, \tilde{\psi}_{\alpha, 2, s} = \tilde{\psi}_{\beta, s} \). So our polynomial \( \hat{P}_u \) rewrites as
\[ \hat{P}_u(X_s) = \prod_{s=0}^{r-1} [(X_s Y_s)^u [(X_s - \alpha)(X_s - \beta)(Y_s - \alpha)(Y_s - \beta)]^{rn}] \cdot \prod_{0 \leq i < j < r-1} (X_j - X_i) \prod_{0 \leq i < j < r-1} (Y_j - Y_i) \prod_{0 \leq i,j \leq r-1} (t_{i,s} - X_i)(t_{k,s} - Y_j) , \]
where \( c(t_{i,s}) := \prod_{i=0}^{m} \prod_{s=0}^{r-1} \left[ \alpha_i t_{i,s}^{rn} \right] \prod_{(i_1, s_1) \neq (i_2, s_2), i_1, i_2 \geq 3} (t_{i_2, s_2} - t_{i_1, s_1}) \) (the precise value of \( c \) does not actually matter as it is treated as a scalar by the operators \( \tilde{\psi}_{\alpha, s}, \tilde{\psi}_{\beta, s} \)).

We set \( \Psi_x = \bigcirc_{s=0}^{r-1} \tilde{\psi}_{\alpha, s} \) and \( \Psi_{\beta} = \bigcirc_{s=0}^{r-1} \tilde{\psi}_{\beta, s} \) respectively. One has \( \Psi = \Psi_\alpha \circ \Psi_\beta \circ \tilde{\psi} \) where \( \tilde{\psi} = \bigcirc_{i \geq 3} \bigcirc_{s=0}^{r-1} \tilde{\psi}_{\alpha, i, s}. \)

**Lemma 4.8.** (i) The morphisms \( \tilde{\psi}_{\alpha, s}, \tilde{\psi}_{\beta, s} \) pairwise commute for \( 0 \leq s_1, s_2 \leq r - 1 \).

(ii) The operator \( \frac{\partial}{\partial \alpha} \) commutes with any \( \tilde{\psi}_{\beta, s} \) and hence with \( \Psi_{\beta} \) and with \( \tilde{\psi} \).

(iii) We have
\[ \frac{\partial}{\partial \alpha} \Psi_\alpha = \Psi_\alpha \frac{\partial}{\partial \alpha} + \frac{1}{\alpha} \sum_{s=0}^{r-1} \Psi_\alpha \circ \theta_{X_s} . \]

**Proof.** The assertion (i) follows from the definition since both multiplication by a scalar, specialization of one variable or integration with respect to a given variable all pairwise commute, (ii) follows from commutation of integrals with respect to a parameter with differentiation with respect to that parameter.

Finally, we prove (iii). If \( h(X_0, \ldots, X_{r-1}) \in K[\alpha, 1/\alpha][X_0, \ldots, X_{r-1}] \), we write \( h(X_0, \ldots, X_{r-1}) = \sum_{\xi} a_\xi(\alpha) X_\xi^{r-1} \). By definition, we have
\[ \Psi_\alpha(h) = \sum_{\xi} a_\xi(\alpha) \frac{\alpha^{\xi}}{\prod_{s=0}^{r-1} (i_s + \gamma_{r-s}) \cdots (i_s + \gamma_r)} , \]
where \(|\bar{u}| = \sum_{s=0}^{r-1} i_s\). Then
\[
\begin{align*}
\frac{\partial}{\partial \alpha}(\Psi_\alpha(h)) = \sum_{i} \frac{\partial}{\partial \alpha}(a_i(\alpha)) \frac{\alpha \bar{u}}{\prod_{s=0}^{r-1} (i_s + \gamma_{r-s}) \cdots (i_s + \gamma_s)} + \sum_{i} \frac{a_i(\alpha) |\bar{u}|}{\prod_{s=0}^{r-1} (i_s + \gamma_{r-s}) \cdots (i_s + \gamma_s)} .
\end{align*}
\]

First term in the sum is easily seen to be equal to \(\Psi_\alpha \circ \frac{\partial}{\partial \alpha}(h)\). So the claim (iii) reduces to the statement
\[
\sum_{s=0}^{r-1} \Psi_\alpha \circ \theta_{X_s}(X^0_s \cdots X^{r-1}_r) = |\bar{u}| \frac{\alpha \bar{u}}{\prod_{s=0}^{r-1} (i_s + \gamma_{r-s}) \cdots (i_s + \gamma_s)} .
\]

But left hand side is
\[
\sum_{s=0}^{r-1} \Psi_\alpha \circ \theta_{X_s}(X^0_s \cdots X^{r-1}_r) = \sum_{s=0}^{r-1} \frac{i_s \alpha \bar{u}}{\prod_{s'=0}^{r-1} (i_{s'} + \gamma_{r-s'}) \cdots (i_{s'} + \gamma_r)} = |\bar{u}| \frac{\alpha \bar{u}}{\prod_{s'=0}^{r-1} (i_{s'} + \gamma_{r-s'}) \cdots (i_{s'} + \gamma_r)} .
\]
This completes the proof of this lemma. \(\square\)

We introduce a specialization morphism for the variable \(\alpha\). Set
\[
\Delta = \Delta_\alpha : \mathbb{Q}[\alpha, 1/\alpha, \beta, \alpha_2, \ldots, \alpha_m] \longrightarrow \mathbb{Q}(\alpha); \quad \Delta(P(\alpha, \beta, \alpha_2, \ldots, \alpha_m)) = P(\alpha, \alpha, \alpha_2, \ldots, \alpha_m) .
\]
Note that \(\tilde{\psi}\) and \(\Psi_\alpha\) commute so, it is enough to prove that
\[
\frac{\partial^\ell}{\partial \alpha^\ell} \Delta_\alpha \left(\Psi_\alpha \circ \Psi_\beta(\tilde{P}_u)\right) = 0 \quad \text{for } 0 \leq \ell \leq (2n+1)r^2 - 1 .
\]
We postpone the end of the proof of (c) and start with a few preliminaries. We set
\[
\begin{align*}
f_{n,u}(\alpha, \beta, X, Y) = f(\alpha, \beta, X, Y) = c(t_{i,s}) \prod_{k \geq 3} \prod_{s=0}^{r-1} \prod_{1 \leq i, j \leq r} (t_{k,s} - X_i)(t_{k,s} - Y_j) \\
\quad \quad \cdot \prod_{s=0}^{r-1} (X_s Y_s)^n [(X_s - \alpha)(X_s - \beta)(Y_s - \alpha)(Y_s - \beta)]^n ,
\end{align*}
\]
and
\[
\begin{align*}
g(X, Y) = g(\alpha, \beta, X, Y) = \prod_{0 \leq i, j \leq r-1} (X_i - Y_j) \prod_{0 \leq i < j \leq r-1} [(X_j - X_i)(Y_j - Y_i)] .
\end{align*}
\]
So that \(\tilde{P}_u = \tilde{P} = fg\) (for the rest of the proof, the index \(u\) will not play any role and may be conveniently left off to ease reading).

We now concentrate on a few elementary properties of the maps \(\tilde{\psi}\) which we regroup here and will be useful for the rest :

Now we prepare new notations. Let \(\xi_s := (\xi_{s,k})_{k \geq 1}\) for \(0 \leq s \leq r-1\) be infinite sequences of elements of \(K\). Put \(\Xi = (\xi_s)_{0 \leq s \leq r-1}\). For \(\ell := (\ell_0, \ldots, \ell_{r-1}) \in \mathbb{Z}^r\) with \(\ell_s \geq 0\), we put
\[
\begin{align*}
\tilde{\psi}_{\alpha, s, \ell_s} & \quad := \tilde{\psi}_{\alpha, s} \circ \theta_{X_s} + \xi_{s,w} \quad \text{for } 0 \leq s \leq r-1 , \\
\Psi_{\alpha, \Xi, \ell} & \quad := \circ_{s=0}^{r-1} \tilde{\psi}_{\alpha, s, \ell_s} \quad \text{for } \ell_0 \geq 0 ,
\end{align*}
\]
where \(\circ_{s=1}^{r-1} (\theta_{X_s} + \xi_{s,w})\) if \(\ell_s = 0\). We remark that, in the case of \(\ell = (0, \ldots, 0) \in \mathbb{Z}^r\), we have \(\Psi_{\alpha, \Xi, \ell} = \Psi_\alpha\) for any \(\Xi\).
Lemma 4.9. Let $\xi_s := (\xi_{s,k})_{k \geq 1}$ and $\xi'_s := (\xi'_{s,k})_{k \geq 1}$ for $0 \leq s \leq r - 1$ be infinite sequences of elements of $K$ and $\ell := (\ell_0, \ldots, \ell_{r-1}), \ell' := (\ell'_0, \ldots, \ell'_{r-1}) \in \mathbb{Z}$ with $\ell_i, \ell'_j \geq 0$. Put $\Xi := (\xi_s)s, \Xi' := (\xi'_s)s$. Assume there exist $\ell_i, \ell'_j$ with

\begin{equation}
\bigcirc_{w=1}^{\ell_i} (\theta_t + \gamma_{r-i+w})^{-1} \circ (\theta_t + \xi_{i,t_i}) = \bigcirc_{w'=0}^{\ell'_j} (\theta_t + \gamma_{r-j+w'})^{-1} \circ (\theta_t + \xi'_{j,t'_j}) ,
\end{equation}

and the polynomial $P \in K[x, y]$ is antisymmetric (any odd permutation of the variables $X_i, Y_j$ changes $P$ in its opposite). Then we have

$$\Delta \circ \Psi_\alpha, \xi \circ \Psi_\beta, \xi' (P) = 0 .$$

Similarly, if there exist $\ell_i, \ell_j$ for $0 \leq i < j \leq r - 1$ with

\begin{equation}
\bigcirc_{w=1}^{\ell_i} (\theta_t + \gamma_{r-i+w})^{-1} \circ (\theta_t + \xi_{i,t_i}) = \bigcirc_{w'=0}^{\ell_j} (\theta_t + \gamma_{r-j+w'})^{-1} \circ (\theta_t + \xi_{j,t_j}) ,
\end{equation}

we have

$$\Psi_\alpha, \xi (P) = 0 .$$

Proof. Let $0 \leq i, j \leq r - 1$. Let $\tau$ be the transposition $\tau(X_i) = Y_j, \tau(Y_j) = X_i$ leaving all the other variables invariant. Then $\tau$ acts on $K[x, y]$ by permutation of the variables. Then we have $\tau(P) = -P$ by antisymmetry. We compute

$$\Delta \circ \Psi_\alpha, \xi \circ \Psi_\beta, \xi' (P) = \Delta \bigcirc_{s=0}^{\ell_i} \tilde{\psi}_{\alpha,s, \xi_{i,t_i}} \bigcirc_{s=0}^{\ell'_j} \tilde{\psi}_{\beta,s, \xi'_{j,t'_j}} (P)$$

$$= \Delta \bigcirc_{s=0}^{\ell_i} \tilde{\psi}_{\alpha,s, \xi_{i,t_i}} \bigcirc_{s=0}^{\ell'_j} \tilde{\psi}_{\beta,s, \xi'_{j,t'_j}} (\tau P) = -\Delta \circ \Psi_\alpha, \xi \circ \Psi_\beta, \xi' (P) .$$

Note that the second equality is obtained by the assumption \ref{equation:33}. Thus we obtain the first assertion. The second statement is a variation of the same argument. \hfill \Box

Remark 4.10. Later, in Lemma \ref{lem:11} we use the first assertion of Lemma \ref{lem:9} only to the case of $\ell = (0, \ldots , 0)$. Namely, we apply Lemma \ref{lem:9} to the case of $\psi_\beta, \xi' = \Psi_\beta$.

Lemma 4.11. Let $P \in K[x, y]$ be a polynomial such that $(X_s - \alpha)^T | P$ for some $T \geq 1$ and $0 \leq \ell \leq T - 1$ an integer. Let $\xi_1, \ldots, \xi_{\ell} \in K$ (if $\ell = 0$, we mean $\{\xi_1, \ldots, \xi_{\ell}\} = \emptyset$). Then we have

$$\tilde{\psi}_{\alpha,s} \bigcirc_{w'=0}^{\ell} (\theta_{X_s} + \gamma_{r-s+w'}) \bigcirc_{w=1}^{\ell} (\theta_{X_s} + \xi_w) (P) = 0 .$$

Proof. Indeed, writing $P = (X_i - \alpha)^T Q$, with $Q \in K[x, y]$, and noting that

$$\tilde{\psi}_{\alpha,s} \bigcirc_{w'=0}^{\ell} (\theta_{X_s} + \gamma_{r-s+w'}) \bigcirc_{w=1}^{\ell} (\theta_{X_s} + \xi_w) (P) = \text{Eval}_{X_s \rightarrow \alpha} \bigcirc_{w=1}^{\ell} (\theta_{X_s} + \xi_w) (P) ,$$

By the Leibniz formula and the hypothesis $\ell \leq T - 1$, $\bigcirc_{w=1}^{\ell} (\theta_{X_s} + \xi_w) (P)$ belongs to the ideal $(X_s - \alpha)$ and so $\text{Eval}_{X_s \rightarrow \alpha} \bigcirc_{w=1}^{\ell} (\theta_{X_s} + \xi_w) (P) = 0 . \hfill \Box

Lemma 4.12. Let $P \in K[x, y]$ be a polynomial such that $((X_s - \alpha)^T (X_s - \beta)^T) | P$ for some non-negative integers $T_1, T_2$ with either $T_2$ or $T_2$ is greater than $1$ and $0 \leq \ell \leq T_1 + T_2 - 1$ an integer. Let $\xi_1, \ldots, \xi_{\ell} \in K$ (if $\ell = 0$, we mean $\{\xi_1, \ldots, \xi_{\ell}\} = \emptyset$). Then, we have

$$\Delta \circ \tilde{\psi}_{\alpha,s} \bigcirc_{w'=0}^{\ell} (\theta_{X_s} + \gamma_{r-s+w'}) \bigcirc_{w=1}^{\ell} (\theta_{X_s} + \xi_w) (P) = 0 .$$

Proof. This is a variation of the previous lemma, indeed, specialization at $\beta = \alpha$ doubles the multiplicity and commutation of specialization along $\beta$ commutes with $\tilde{\psi}_{\alpha,s} \bigcirc_{w'=0}^{\ell} (\theta_{X_s} + \gamma_{r-s+w'}) \bigcirc_{w=1}^{\ell} (\theta_{X_s} + \xi_w) (\text{variation of Lemma \ref{lem:8} (i)}). \hfill \Box
Now, let us compute what comes out by iteration of property (iii) of Lemma 4.8. We define infinite sequences of elements of $K$, $\xi_s = (\xi_{s,k})_{k \geq 1}$, with

$$\xi_{s,k} = \begin{cases} \gamma_{r-s-k} & \text{if } 1 \leq k \leq s + 1 \\ 0 & \text{if } k > s + 1 \end{cases},$$

and put $\Xi = (\xi_s)_{s=0,\ldots,r-1}$. For a non-negative integer $\ell$, there exists a sequence $(b_{s,k,\ell})_{k=0,\ldots,\ell \in K^\ell+1}$ with $X^\ell = \sum_{k=0}^\ell b_{s,k,\ell} \prod_{w=1}^k (X + \xi_{s,w})$ where $\prod_{w=1}^k (X + \xi_{s,w}) = 1$ if $k = 0$.

Let $\ell, k$ be non-negative integers with $\ell \geq k$. We define a set of differential operators

$$X_{\ell,k} = \{ V = \partial_1 \circ \cdots \circ \partial_{\ell} | \partial_i \in \{1/\alpha, \frac{\partial}{\partial w}\}, \#\{1 \leq i \leq \ell, \partial_i = 1/\alpha \} = k \}.$$

One gets that

$$\frac{\partial^\ell}{\partial \alpha^\ell} (\Psi_\alpha (\hat{P})) = \sum_{\ell=(\ell_0,\ldots,\ell_{r-1}) \in \mathbb{Z}^r} \sum_{\ell_i \geq 0, |\ell| \leq \ell} \Psi_\alpha \bigcirc_{s=0}^{r-1} \theta_{X_s} (V(\hat{P}))$$

$$= \sum_{\ell=(\ell_0,\ldots,\ell_{r-1}) \in \mathbb{Z}^r} \sum_{\ell_i \geq 0, |\ell| \leq \ell} \left( \sum_{k_0=0}^{\ell_0} \cdots \sum_{k_{r-1}=0}^{\ell_{r-1}} \prod_{s'=0}^{r-1} \theta_{X_s} (V(\hat{P})) \right)$$

$$= \sum_{\ell=(\ell_0,\ldots,\ell_{r-1}) \in \mathbb{Z}^r} \sum_{\ell_i \geq 0, |\ell| \leq \ell} \left( \sum_{k=(k_0,\ldots,k_{r-1}) \in \mathbb{Z}^r} \prod_{s'=0}^{r-1} \theta_{X_s} (V(\hat{P})) \right),$$

where $k \leq \ell$ means $k_i \leq l_i$ for each $0 \leq i \leq r - 1$.

By the Leibniz formula, for $V \in X_{\ell_i,|\ell|}$, $V(\hat{P})$ is a linear combination (over $K[1/\alpha]$) of the derivatives $\frac{\partial^j}{\partial \alpha^j} (\hat{P})$ for $0 \leq j \leq \ell - |\ell|$. Since $\hat{P} = fg$ (recall the definition of $f$ and $g$ in (27) and (28) respectively), it is a linear combination of $g \frac{\partial f}{\partial \alpha} (f)$, for $0 \leq j \leq \ell - |\ell|$.

We now perform the combinatorics argument:

**Lemma 4.13.** We use the notations as above. Let $\ell$ be a non-negative integer and $\ell = (\ell_0, \ldots, \ell_{r-1}) \in \mathbb{Z}^r$ with $\ell_i \geq 0$ such that $|\ell| \leq \ell$. Assume further either of these three to be true

(i) There exist $0 \leq i \leq r - 1$ with $\ell_i < i + 1$.

(ii) There exist $0 \leq i < j \leq r - 1$ with $\ell_i \geq i + 1$, $\ell_j \geq j + 1$ and $\ell_i - (i + 1) = \ell_j - (j + 1)$.

(iii) There exists an index $0 \leq s \leq r - 1$ such that $0 \leq \ell_s - (s + 1) < 2rn - \ell + |\ell|$.

Then, $\Delta \circ \Psi_{\alpha,\Xi_k} \circ \Psi_{\beta} (g \frac{\partial^\ell}{\partial \alpha^\ell}) = 0$ for all $0 \leq j \leq \ell - |\ell|$.

**Proof.** If the first condition is satisfied, we have

$$(\theta_{\ell} + \gamma_{\ell-i})^{-1} \circ \cdots \circ (\theta_{\ell} + \gamma_{\ell})^{-1} \circ (\theta_{\ell} + \xi_{i,1}) \circ \cdots \circ (\theta_{\ell} + \xi_{i,\ell_i}) = (\theta_{\ell} + \gamma_{\ell-i+\ell_i})^{-1} \circ \cdots \circ (\theta_{\ell} + \gamma_{\ell})^{-1}.$$

Thus, by antisymmetry of $g$, the first assertion of Lemma 4.9 ensures vanishing.

If the second conditions are satisfied, we have

$$(\theta_{\ell} + \gamma_{\ell-i})^{-1} = \bigcirc_{\ell=0}^{\ell} (\theta_{\ell} + \gamma_{\ell-i+\ell}^{-1})^{-1} \circ (\theta_{\ell} + \xi_{i,1}) \circ \cdots \circ (\theta_{\ell} + \xi_{i,\ell_i})$$

$$= \bigcirc_{\ell=0}^{\ell} (\theta_{\ell} + \gamma_{\ell-j+\ell})^{-1} \circ (\theta_{\ell} + \xi_{j,1}) \circ \cdots \circ (\theta_{\ell} + \xi_{j,\ell_j}) = \theta_{\ell}^{\ell-j}.$$
By antisymmetry of $g$, the second assertion of Lemma 4.14 ensures vanishing.

If the third condition is satisfied, Lemma 4.12 ensures that

$$\Delta \circ \tilde{\psi}_{\alpha,s} \circ \varphi_{w=1}^k (\theta X, + \xi_{s,w})(V(\hat{P})) = \Delta \circ \tilde{\psi}_{\alpha,s} \circ \varphi_{w=1}^{k+1} (\theta X, + \gamma_{r-s} \varphi_{w=1}^{-1})(V(\hat{P})) ,$$

itself vanishes for all $V \in H_1|\ell|$ since $[(X_s - \alpha)(X_s - \beta)]^\ell = f (so \frac{\partial^\ell}{\partial \alpha^\ell}(\hat{P})$ vanishes at $\alpha = \beta$ at order at least $2rn - j \geq 2rn - \ell + |\ell| > \ell_s - (s + 1)$.

**Lemma 4.14.** The smallest integer $\ell$ for which there exists $\ell = (\ell_0, \ldots, \ell_{r-1})$ with $|\ell| \leq l$ with none of the conditions of Lemma 4.13 are satisfied is $(2n + 1)^2$.

**Proof.** Assume conditions (i), (ii) and (iii) are false, then the set $\{\ell_s - s - 1\} = \{2rn - \ell + |\ell|; \ldots; 2rn - \ell + |\ell| + r - 1\}$ and $\sum_{s=0}^{r-1}(\ell_s - s - 1) \geq r(2rn - \ell + |\ell|) + r(r - 1)/2$, that is $|\ell| + r(\ell - |\ell|) \geq 2n^2 + n + r^2$. Since $\ell - |\ell| \geq 0$, the lemma follows.

**End of the proof of Proposition 4.6 (c):**

Lemma 4.13 ensures that

$$\frac{\partial^\ell}{\partial \alpha^\ell} \Delta_{\alpha}(C_{u,m}) = 0 \quad \text{for all } 0 \leq \ell \leq (2n + 1)^2 - 1.$$

This completes the proof of Proposition 4.6.

### 4.5 Last step

We shall reduce by induction the non-vanishing of $c_{u,m}$ to the non-vanishing of $c_{u,0}$ (which is obviously equal to 1). First, we prove,

**Lemma 4.15.** Set $\mathfrak{A}(t) = \prod_{s=0}^{r-1} [t_{m,s}^{n}(t_{m,s} - 1)]^{\ell_n} \cdot \prod_{1 \leq s < s', \leq r} (t_{m,s} - t_{m,s'})$ and $L_m = \prod_{1 \leq s \leq r} \tilde{\psi}_{1,m,s}$. Then,

$$c_{u,m} = (-1)^{r^2 n(m-1)} c_{u+r(m+1),m-1} \cdot L_m (\mathfrak{A}(t)) .$$

**Proof.** Set $\hat{\mathcal{L}} = \prod_{i=1}^{m+1} \tilde{\psi}_{1,m,i} \cdot L_m$ so that $\Psi_1 = \hat{\mathcal{L}} \circ L_m$ and recall that by (20),

$$D_{u,m} := \frac{c_{u,m}}{\prod_{i=1}^{m} \alpha_i^{r(u+1)+r^2n+i}} = c_{u,m} \prod_{1 \leq i < j \leq m} (\alpha - \alpha_i)^{2n+1}) \cdot \Psi_1 (Q_{u,m}) .$$

We are going to evaluate $D_{u,m}$ at $\alpha_m = 0$ and thus separate the variables in $Q_{u,m}$ first. By definition, one has

$$Q_{u,m}(t) = Q_{u,m-1}(t) \cdot \mathfrak{A}(t) \mathfrak{B}(t) ,$$

where

$$\mathfrak{B}(t) = \prod_{s=0}^{r-1} \prod_{j \neq m} (\alpha_m t_{m,s} - \alpha_j)^{\ell_n} \cdot \prod_{i=1}^{m-1} \prod_{s=0}^{r-1} (\alpha_i t_{i,s} - \alpha_m)^{\ell_n} \cdot \prod_{1 \leq i < j \leq m} (\alpha_m t_{m,s} - \alpha_i t_{i,s}) .$$

Note that $Q_{u,m-1}, \mathfrak{A}$ do not depend on $\alpha_m$, and $\Psi_1$ treats $\alpha_m$ as a scalar. Hence,

$$D_{u,m}|_{\alpha_m=0} = c_{u,m} \prod_{i=1}^{m-1} (-\alpha_i)^{2n+1}) \cdot \prod_{1 \leq i < j \leq m-1} (\alpha_j - \alpha_i)^{2n+1})^2 \cdot \Psi_1 (Q_{u,m-1}(t) \mathfrak{A}(t) \mathfrak{B}(t)|_{\alpha_m=0}) .$$

(31)
But
\[
\mathcal{B}(t)|_{\alpha_m=0} = \prod_{j=1}^{m-1} (-\alpha_j)^{r_n} \prod_{i=1}^{m-1} \prod_{s=0}^{\gamma_r-1} (\alpha_i t_i, s)^{r_n} \prod_{i=1}^{m-1} (-\alpha_i t_i, s)^{r} = (-1)^{r_2(m-1)(m+1)} \prod_{i=1}^{m-1} \alpha_i^{(2n+1)r^2} \prod_{i=1}^{m-1} t_i^{r(n+1)}.
\]

We now note that \(\theta_m\) treats the variables \(t_{i,s}, 1 \leq i \leq m - 1\) as scalars and \(\hat{\mathcal{L}}\) treats variables \(t_{m,s}\) as scalars and remark
\[
Q_{u,m-1}(t) \mathcal{B}(t)|_{\alpha_m=0} = (-1)^{r_2(m-1)(m+1)} \prod_{i=1}^{m-1} \alpha_i^{(2n+1)r^2} Q_{u+r(n+1),m-1}(t).
\]

Thus
\[
\Psi_1(Q_{u,m-1}(t) \mathcal{A}(t) \mathcal{B}(t)|_{\alpha_m=0}) = (-1)^{r_2(m-1)(m+1)} \prod_{i=1}^{m-1} \alpha_i^{(2n+1)r^2} \hat{\mathcal{L}}(Q_{u+r(n+1),m-1}(t)) \mathcal{L}_m(\mathcal{A}(t)).
\]

Using the relation (31), taking into account \(D_{u+r(n+1),m-1} = \Psi_1(Q_{u+r(n+1),m-1}(t))\) and simplifying,
\[
c_{u,m} = (-1)^{r_2(n+1)} c_{u+r(n+1),m-1} \cdot \mathcal{L}_m(\mathcal{A}(t)).
\]

This completes the proof of Lemma 4.15.

By Lemma 4.15 to prove the non-vanishing of the value \(c_{u,m}\), it is enough to show \(\mathcal{L}_m(\mathcal{A}(t)) \neq 0\). Denote the cardinality of the set \(\{\zeta_1, \ldots, \zeta_r\}\) by \(d\). If we need, by changing the order, we may assume \(\{\zeta_1, \ldots, \zeta_r\} = \{\zeta_1, \ldots, \zeta_d\}\) and
\[
(\zeta_1, \ldots, \zeta_r) = (\zeta_1, \ldots, \zeta_{r_1}, \ldots, \zeta_{r_d}, \ldots, \zeta_d),
\]
where \(r_j\) is the multiplicity of \(\zeta_j\) for \(1 \leq j \leq d\). For an integer \(s\), we define the \(K\)-homomorphism \(\varphi_{\zeta_j,s}\) by
\[
\varphi_{\zeta_j,s} : K[t] \rightarrow K; t^k \mapsto \frac{1}{(k + \zeta_j)^s}.
\]

**Lemma 4.16.** There exists \(E \in K \setminus \{0\}\) with
\[
\mathcal{L}_m(\mathcal{A}(t)) = E \cdot \det \left( \varphi_{\zeta_j,s}(t^{u+f(t-1)^{r_n}}) \right)_{0 \leq t \leq r - 1}.
\]

Especially, the value \(\mathcal{L}_m(\mathcal{A}(t))\) is not zero.

**Proof.** Define
\[
\psi_s : K[t] \rightarrow K; t^k \mapsto \frac{1}{(k + \zeta_{r-s}) \cdots (k + \zeta_r)} = \frac{1}{(k + \zeta_1) \cdots (k + \zeta_{s+1})},
\]
for \(0 \leq s \leq r - 1\). Then we have
\[
\mathcal{L}_m(\mathcal{A}(t)) = \det(\psi_s(t^{u+f(t-1)^{r_n}}))_{0 \leq t \leq r - 1}.
\]

For \(0 \leq s \leq r - 1\), there exist \(1 \leq w \leq d\) and \(1 \leq s_w \leq r_w\) with
\[
s + 1 = r_1 + \cdots + r_{w-1} + s_w.
\]
Put
\[
P_{j,k} = \begin{cases} 
\frac{1}{(r_j - k)!} d^{r_j - k} \frac{1}{\prod_{1 \leq j' \leq w-1} (X + \zeta_j')^{r_j'} (X + \zeta_w)^{s_w}} |_{X=-\zeta_j} & \text{if } 1 \leq j \leq w-1, 1 \leq k \leq r_j, \\
\frac{1}{(s_w - k)!} d^{s_w - k} \frac{1}{\prod_{j=1}^{w-1} (X + \zeta_j)^{r_j}} |_{X=-\zeta_w} & \text{if } j = w, 1 \leq k \leq s_w.
\end{cases}
\]

Then we have
\[
p_{w,s_w} = \frac{1}{\prod_{j=1}^{w-1} (\zeta_j - \zeta_w)^{r_j}} \neq 0
\]
and
\[
\psi_s = \sum_{j=1}^{w-1} \sum_{k=1}^{r_j} p_{j,k} \varphi_{\zeta_j,k} + \sum_{k=1}^{s_w} p_{w,k} \varphi_{\zeta_w,k}.
\]

Put \( E = \prod_{0 \leq s \leq r-1} p_{w,s_w} \neq 0 \) where \((w, s_w)\) is the pair of integers defined as in (35) for \( s+1 \). Then by equalities (33), (36) and the linearity of the determinant, we obtain (32). The non-vanishing of the determinant
\[
\det \left( \varphi_{\zeta_j, n_j} (t^{u+\ell}(t-1)^{r_n}) \right)_{0 \leq j \leq r-1, 1 \leq j \leq d, 1 \leq s_j \leq r_j}
\]
has been obtained in [20, Proposition 4.12]. □

5 Estimates

In this subsection, we use the following notations. Let \( K \) an algebraic number field and \( v \) be a place of \( K \).
Denote by \( K_v \) the completion of \( K \) at \( v \), \(| |_v \) the absolute value corresponding to \( v \). Let \( \eta_1, \ldots , \eta_r, \zeta_1, \ldots , \zeta_r \) be strictly positive rational numbers with \( \eta_i - \zeta_j \notin \mathbb{N} \) for \( 1 \leq i, j \leq r \). Put \( A(X) = (X + \eta_1) \cdots (X + \eta_r) \) and \( B(X) = (X + \zeta_1) \cdots (X + \zeta_r) \). We shall choose a sequence \( c := (c_k)_{k \geq 0} \) satisfying \( c_k \in K \setminus \{ 0 \} \) and (2) for the given polynomials \( A(X), B(X) \). Let \( \alpha := (\alpha_1, \ldots , \alpha_m) \in (K \setminus \{ 0 \})^m \) whose coordinates are pairwise distinct and \( n \) be a non-negative integer. We choose \( \gamma_1 = \zeta_1, \ldots , \gamma_r-1 = \zeta_2 \). For non-negative integer \( \ell \) with \( 0 \leq \ell \leq rm \), recall the polynomials \( P_\ell(z) \), \( P_{\ell,i}(z) \) defined as in (3) and (4) for the given data.

Throughout the section, the small \( o \)-symbol \( o(1) \) and \( o(n) \) refer when \( n \) tends to infinity. Put \( \varepsilon_v = 1 \) if \( v \nmid \infty \) and \( 0 \) otherwise.

Let \( I \) be a non-empty finite set of indices, \( R = K[\alpha_i]_{i \in I}[z,t] \) be a polynomial ring in indeterminate \( \alpha_i, z, t \). For a non-negative integer \( n \) and \( \zeta \in \mathbb{Q} \setminus \{ 0 \} \), we define
\[
S_{n, \zeta} : K[t] \longrightarrow K[t]: t^k \mapsto \frac{(k + \zeta + 1)n}{n!} t^k.
\]

We set \( \| P \|_v = \max \{ |a|_v \} \) where \( a \) runs in the coefficients of \( P \). Thus \( R \) is endowed with a structure of normed vector space. If \( \phi \) is an endomorphism of \( R \), we denote by \( \| \phi \|_v \) the endomorphism norm defined in a standard way \( \| \phi \|_v = \inf \{ M \in \mathbb{R}, \forall x \in R, \| \phi(x) \|_v \leq M \| x \|_v \} = \sup \{ \| \phi(x) \|_v : 0 \neq x \in R \} \).
This norm is well defined provided \( \phi \) is continuous. Unfortunately, we will have to deal also with non-continuous morphisms. In such a situation, we restrict the source space to some appropriate sub-vector space \( E \) of \( R \) and talk of \( \| \phi \|_v \) with \( \phi \) seen as \( \phi : E \longrightarrow R \) on which \( \phi \) is continuous. In case of perceived ambiguity, it will be denoted by \( \| \phi \|_{E,v} \). The degree of an element of \( R \) is as usual the total degree.

For a rational number \( x \), we denote by \( \lfloor x \rfloor \) the greatest integer less than or equal to \( x \).
Lemma 5.1. (confer [28] Lemma 4.1 (ii)) Let $a, b \in \mathbb{Q}$ which are not negative integers. For a non-negative integer $n$, put
\[
D_n = \text{den}\left(\frac{(a)_0}{(b)_0}, \ldots, \frac{(a)_n}{(b)_n}\right).
\]
Then we have
\[
\limsup_{n \to \infty} \frac{1}{n} \log D_n \leq \log \mu(a) + \frac{\text{den}(b)}{\varphi(\text{den}(b))},
\]
where $\varphi$ is the Euler’s totient function.

Proof. Put
\[
D_{1,n} = \text{den}\left(\frac{(a)_0}{0!}, \ldots, \frac{(a)_n}{n!}\right), \quad D_{2,n} = \text{den}\left(\frac{0!}{(b)_0}, \ldots, \frac{n!}{(b)_n}\right).
\]
Then we have $D_n \leq D_{1,n} \cdot D_{2,n}$. The assertion is deduced from
\[
\limsup_{n \to \infty} \frac{1}{n} \log D_{1,n} \leq \log \mu(a), \quad \limsup_{n \to \infty} \frac{1}{n} \log D_{2,n} \leq \frac{\text{den}(b)}{\varphi(\text{den}(b))}. \tag{37}
\]
First inequality is proved in [7] Lemma 2.2. Second inequality is shown in [28] Lemma 4.1, however, we explain here this proof in an abbreviated form, to let our article be self-contained. This proof is originally indicated by Siegel [44, p.81]. Put $d = \text{den}(b)$, $c = d \cdot b$. Set $N_k := c(c + d) \cdots (c + (k - 1)d)$ for a non-negative integer $k$. Let $p$ be a prime number with $p | N_k$. Then the following properties hold.

(a) The integers $p, d$ are coprime and, for any integers $i, \ell$ with $\ell > 0$, there exists exactly one integer $\nu$ with $0 \leq \nu \leq p^\ell - 1$ with $p^\ell | c + (i + \nu)d$.

(b) Let $\ell$ be a strictly positive integer with $|c| + (k - 1)d < p^\ell$. Then $N_k$ is not divisible by $p^\ell$.

(c) Set $C_{p,k} := \lceil \log(|c| + (k - 1)d)/\log(p) \rceil$. Then we have
\[
\nu_p(k!) = \sum_{\ell=1}^{C_{p,k}} \left\lceil \frac{k}{p^\ell} \right\rceil \leq \nu_p(N_k) \leq \sum_{\ell=1}^{C_{p,k}} \left(1 + \left\lceil \frac{k}{p^\ell} \right\rceil \right) = \nu_p(k!) + C_{p,k},
\]
where $\nu_p$ denotes the $p$-adic valuation. These relations imply
\[
\log \left| \frac{k!}{(\beta)_k} \right|_p \leq \begin{cases} C_{p,k} \log(p) & \text{if } p \mid N_k \\ 0 & \text{otherwise}, \end{cases}
\]
and thus
\[
\log D_{2,n} = \sum_{p \text{ prime}} \max_{0 \leq \ell \leq n} \log \left| \frac{k!}{(\beta)_k} \right|_p \leq \log(|c| + (n - 1)d) \pi_{|c|,d}(|c| + (n - 1)d),
\]
where $\pi_{|c|,d}(x) := \# \{ p : \text{prime} \mid p \equiv |c| \mod d, p < x \}$ for $x > 0$. Finally, Dirichlet’s prime number theorem for arithmetic progressions conclude (37).

Lemma 5.2. We have the following norm estimates (we do hope the similarity of notations is not cause of confusion):

\[
C_{p,k} \log(p) \quad \text{if } p \mid N_k \\
0 \quad \text{otherwise,}
\]

and thus
\[
\log D_{2,n} = \sum_{p \text{ prime}} \max_{0 \leq \ell \leq n} \log \left| \frac{k!}{(\beta)_k} \right|_p \leq \log(|c| + (n - 1)d) \pi_{|c|,d}(|c| + (n - 1)d),
\]
where $\pi_{|c|,d}(x) := \# \{ p : \text{prime} \mid p \equiv |c| \mod d, p < x \}$ for $x > 0$. Finally, Dirichlet’s prime number theorem for arithmetic progressions conclude (37).
(i) Let $E_N$ be the subspace of $R$ consisting of polynomials of degree at most $N$ in $t$. Then for all $n \geq 1$ and strictly positive rational number $\zeta$, the morphism $S_{n, \zeta}$ satisfies

$$\|S_{n, \zeta}\|_{E_N, v} \leq \begin{cases} 1 & \text{if } v \nmid \infty \text{ and } |\zeta|_v \leq 1 \\ \frac{(N + \zeta + 1)_n}{n!} & \text{otherwise} \end{cases}.$$  

It acts diagonally on $R$ in the sense that each element of the canonical basis consisting of all monomials is an eigenvector for $S_{n, \zeta}$. This map conserves degrees.

(ii) Let $\zeta$ be a strictly positive rational number. Then the morphism $\theta_t + \zeta$ satisfies

$$\|\theta_t + \zeta\|_{E_N, v} \leq \begin{cases} 1 & \text{if } v \text{ is non-archimedean and } |\zeta|_v \leq 1 \\ |N + \zeta|_v & \text{otherwise} \end{cases}.$$  

It acts diagonally on $R$ in the sense that each element of the canonical basis consisting of all monomials is an eigenvector for $\theta_t + \zeta$. This map conserves degrees.

(iii) Let $c = (c_k)_{k \geq 0}$ satisfying $c_k \in \mathbb{Q} \setminus \{0\}$ and (38) for $A(X) = (X + \eta_1) \cdots (X + \eta_r), B(X) = (X + \zeta_1) \cdots (X + \zeta_r)$. For a non-negative integer $N$, we put

$$D_{c, N} = \text{den} \left( \frac{(1 + \zeta_1)_k \cdots (1 + \zeta_r)_k}{(\eta_1)_k \cdots (\eta_r)_k} \right)_{0 \leq k \leq N}.$$  

The morphism $T_c$ which is defined in (38) satisfies

$$\|T_c\|_{E_N, v} \leq \begin{cases} e^{o(N)} & \text{if } v \nmid \infty \\ |c_0^{-1} D_{c, N}|_v^{-1} & \text{otherwise} \end{cases}$$

for $N \to \infty$. It acts diagonally on $R$ in the sense that each element of the canonical basis consisting of all monomials is an eigenvector for $T_c$. This map conserves degrees.

**Proof.** (i) and (ii) follow from the very definition of $S_{n, \zeta}$ and $\theta_t + \zeta$. We proof (iii). Since we have

$$T_c(t^k) = \frac{1}{c_0} \frac{B(1) \cdots B(k)}{A(0) \cdots A(k-1)} = \frac{1}{c_0} \frac{(1 + \zeta_1)_k \cdots (1 + \zeta_r)_k}{(\eta_1)_k \cdots (\eta_r)_k},$$

we get

$$\|T_c\|_{E_N, v} \leq \max_{0 \leq k \leq N} \left( \frac{1}{c_0} \frac{(1 + \zeta_1)_k \cdots (1 + \zeta_r)_k}{(\eta_1)_k \cdots (\eta_r)_k} \right)_{v}.$$  

Let $v$ be an archimedean valuation. Since we have

$$\frac{(1 + \zeta_j)_k}{(\eta_j)_k} \leq \frac{k}{\eta_j} \left( \frac{\zeta_j}{k} \right) (k \geq 0),$$

we obtain

$$\|T_c\|_{E_N, v} \leq \left| \frac{N^r}{c_0 \cdot \eta_1 \cdots \eta_r} \frac{N + \left[ \zeta_1 \right]}{N} \cdots \frac{N + \left[ \zeta_r \right]}{N} \right|_v = e^{o(N)} (N \to \infty).$$

For a non-archimedean place $v$, by (38) and the definition of $D_{c, N}$, we get the desire estimate. \qed

From the preceding lemma, we deduce:

\[
\text{(38)}
\]
LEMMA 5.3. For a strictly positive integer $N$, we put

$$D_{c,N} = \text{den} \left( \frac{(1 + \zeta_1)k \cdots (1 + \zeta_r)k}{(\eta_1)k \cdots (\eta_r)k} \right)_{0 \leq k \leq N}, \quad D'_{c,N} = \text{den} \left( \frac{(\eta_1)k \cdots (\eta_r)k}{(1 + \zeta_1)k \cdots (1 + \zeta_r)k} \right)_{0 \leq k \leq N}.$$ 

We denote by $w$ the place of $\mathbb{Q}$ such that $v \mid w$. One has:

(i) The polynomial $P_t(z) = P_{n,t}(\alpha|z)$ satisfies

$$\|P_t(z)\|_v \leq \begin{cases} \exp \left( \frac{n[K : v | Q_w]}{[K : \mathbb{Q}]} \left[ r \log(2) + r \left( \log(rm + 1) + r \log \left( \frac{rm + 1}{rm} \right) \right) + o(1) \right] \right) & \text{if } v \mid \infty \\ e^{o(1)}|D_{c,rmn}|_{v}^{-1} \cdot \prod_{j=1}^{r} |\mu_{n-1}(\zeta_j)|_{v}^{-1} & \text{if } v \mid p \end{cases},$$

where $o(1) \longrightarrow 0$ for $n \to \infty$. Recall that $P_t(z)$ is of degree $rn$ in each variable $\alpha_i$, of degree $rmn + \ell$ in $z$ and constant in $t$.

(ii) The polynomial $P_{i,t,s}(z) = P_{i,t,s}(\alpha|z)$ satisfies

$$\|P_{i,t,s}(z)\|_v \leq \begin{cases} \exp \left( \frac{n[K : v | Q_w]}{[K : \mathbb{Q}]} \left[ r \log(2) + r \left( \log(rm + 1) + r \log \left( \frac{rm + 1}{rm} \right) \right) + o(1) \right] \right) & \text{if } v \mid \infty \\ e^{o(1)}|D_{c,rmn} \cdot D'_{c,rmn}|_{v}^{-1} \cdot \prod_{j=1}^{r} |\mu_{n-1}(\zeta_j)|_{v}^{-1} & \text{if } v \mid p \end{cases}.$$

Also, $P_{i,t,s}(z)$ is of degree $\leq rmn + \ell$ in $z$, of degree $rn$ in each of the variables $\alpha_j$ except for the index $i$ where it is of degree $rn + 1$ (recall that $\psi_{i,s}$ involves multiplication by $|\alpha_i|$).

(iii) For any integer $k \geq 0$, the polynomial $\psi_{i,s} \circ [t^{k+n}](P_t(t))$ satisfies

$$\left| \psi_{i,s} \circ [t^{k+n}](P_t(t)) \right|_v \leq \begin{cases} \exp \left( \frac{n[K : v | Q_w]}{[K : \mathbb{Q}]} \left[ r \log(2) + r \left( \log(rm + 1) + r \log \left( \frac{rm + 1}{rm} \right) \right) + o(1) \right] \right) & \text{if } v \mid \infty \\ e^{o(1)}|D_{c,rmn} \cdot D'_{c,rmn}|_{v}^{-1} \cdot \prod_{j=1}^{r} |\mu_{n-1}(\zeta_j)|_{v}^{-1} & \text{otherwise} \end{cases}.$$

By definition, it is a homogeneous polynomial in just the variables $\alpha$ of degree $\leq rmn + \ell + k + n + 1$.

**Proof.** Let $I$ be of cardinality $m$, $E_N$ be the sub-vector space of $K[y_1, \ldots, y_m, z, t]$ consisting of polynomials of degree at most $N$ in the variables $y_i$ and $\Gamma : E_N \longrightarrow R$ the morphism defined by $\Gamma(Q(y_1, \ldots, y_m, z, t)) = Q(t - \alpha_1, \ldots, t - \alpha_m, z, t)$. Set $B_{n,t}(y, t) = t^\ell \prod_{i=1}^{m} y_i^{rn}$, since $B_{n,t}$ is a monomial, its norm $\|B_{n,t}\|_v = 1$. By definition, one has

$$P_t(z) = \text{Eval}_{z \to z} \circ T_e \circ \bigcirc_{j=1}^{r} S_{n-1, \z_j} \circ \Gamma(B_{n,t}),$$

and thus, by sub-multiplicatively of the endomorphism norm,

$$\|P_t(z)\|_v \leq 2^{\epsilon_v(rmn[K:v | Q_w]/[K:Q])} \cdot \left( \prod_{j=1}^{r} |(rmn + \ell + 1 + \z_j)n-1\rangle \right)_{v}^{-1} \cdot \left( \prod_{j=1}^{r} |(rmn + \ell + 1 + \z_j)n-1\rangle \right)_{v}^{-1},$$

where $\delta_v(\z_j) = 1$ if $|\z_j|_v > 1$ and 0 otherwise (one can choose $N = rn$ while using [18, Lemma 5.2 (iv)] and $N = r(n+1)m + \ell$ for Lemma 5.2 (i) and (iii) using $\ell \leq rm$, and note that the original polynomial is a constant in $z$ so the evaluation map is an isometry).
In the ultrametric case, we have the claimed result
\[
\frac{(rmn + rm + 1 + \zeta_j)_{n-1}}{(n-1)!} \leq |\mu_{n-1}(\zeta_j)|^{-1}_v,
\]
where \(\mu_n(\zeta_j) := \prod_{q \text{ prime}} q^{n+\left\lfloor n/(q-1) \right\rfloor} \) (confer [7, Lemma 2.2]) and Lemma 5.2 (iii) with \(N = rmn + rm\).

Left to prove is the archimedian case, we put \(Y = \max_j \{[\zeta_j]\}.\) Then we have:
\[
\frac{(rmn + rm + 1 + \zeta_j)_{n-1}}{(n-1)!} \leq \frac{(rm + 1)n + rm + Y}{n - 1},
\]
and taking into account the standard Stirling formula, we get
\[
\left(39\right)
\]
and putting these together, one gets
\[
\frac{1}{n} \log \left(\frac{(rm + 1)n + rm + Y}{n - 1}\right) = \log(rm + 1) + rm \log \left(\frac{rm + 1}{rm}\right) + o(1),
\]
and putting these together, one gets
\[
\|P_t(z)\|_v \leq \exp \left(\frac{n[K_v : Q_w]}{[K : Q]} \left[rm \log(2) + r \left(\log(rm + 1) + rm \log \left(\frac{rm + 1}{rm}\right)\right)\right] + o(1)\right),
\]
where \(o(1) \to 0\) \((n \to +\infty)\).

Let \(E_0 = K[\alpha_i, z]\) the sub-vector space of \(R\) consisting of constants in \(t\). Define
\[
\Theta : E_0 \to R; Q \mapsto \frac{Q(\alpha_i, z) - Q(\alpha_i, t)}{z - t}.
\]
By definition, \(P_{t, i, s}(z) = \psi_{i,s} \circ \Theta(P_t(z))\) and
\[
\psi_{i,s} = [\alpha_i] \circ \text{Eval}_{t-\alpha_i} \circ T^{-1}_{c} \circ (\theta_t + \zeta_s) \circ \cdots \circ (\theta_t + \zeta_{r-s+1}).
\]
Using [18, Lemma 5.2 (i), (ii)] with \(N = rn\) and [18, Lemma 5.2 (iii)] and Lemma 5.2 (iii) for \(N = rm(n + 1)\) and since \(rn = \exp(n \cdot o(1))\), one gets (ii). Finally, we have
\[
\psi_{i,s} \circ [t^{k+n}](P_t(t)) = [\alpha_i] \circ \text{Eval}_{t-\alpha_i} \circ T^{-1}_{c} \circ (\theta_t + \zeta_s) \circ \cdots \circ (\theta_t + \zeta_{r-s+1}) \circ [t^{k+n}](P_t(t)).
\]
Again, using [18, Lemma 5.2] and Lemma 5.2 one gets (iii).

Recall that if \(P\) is a homogeneous polynomial in some variables \(y_i, i \in I\), for any point \(\alpha = (\alpha_i)_{i \in I} \in K^{\text{Card}(I)}\) where \(I\) is any finite set, and \(\| \cdot \|_v\) stands for the sup norm in \(K^{\text{Card}(I)}\), with
\[
C_v(P) = (\deg(P) + 1) \frac{[K_v : Q_w]}{[K : Q]},
\]
one has
\[
\left(39\right)
\]
So, the preceding lemma yields trivially estimates for the \(v\)-adic norm of the above given polynomials.

**Lemma 5.4.** Let \(n\) be a positive integer, \(\beta \in K\) with \(\|\alpha\|_v < |\beta|_v\). Then we have for all \(1 \leq i \leq m, 0 \leq \ell \leq rm, 0 \leq s \leq r - 1,\)
\[
|R_{\ell,i,s}(\beta)|_v \leq \frac{\|\alpha\|_v^{rn(n+1)}}{|\beta|_v^{\ell+1} \cdot \left(\frac{\|\alpha\|_v^{n+1}}{|\beta|_v} \cdot \left(\frac{\varepsilon_v |\beta|_v}{|\beta|_v - \|\alpha\|_v} + (1 - \varepsilon_v)\right) \exp \left(\frac{n [K_v : Q_w]}{[K : Q]} \left[rm \log(2) + r \left(\log(rm + 1) + rm \log \left(\frac{rm + 1}{rm}\right)\right) + o(1)\right]\right)\right)\right) \phantom{\cdots}\)
\[
\left\{\begin{align*}
&\exp \left(\frac{n [K_v : Q_w]}{[K : Q]} \left[rm \log(2) + r \left(\log(rm + 1) + rm \log \left(\frac{rm + 1}{rm}\right)\right) + o(1)\right]\right) & \text{if } v \mid \infty \\
&e^{o(1)} |D_{e,rmn} \cdot D'_{e,rmn}|_v^{-1} \cdot \prod_{j=1}^n |\mu_{n-1}(\zeta_j)|_v^{-1} & \text{otherwise}.
\end{align*}\right.
\]
Using the triangle inequality, the fact that \( \ell \leq rm \) and Lemma 5.3 (iii) and inequality 59, we have

\[
|R_{\ell,i,s}(\beta)|_v \leq \|\alpha\|_v^{rm(n+1)} \sum_{k=0}^{\infty} \left( \frac{\|\alpha\|_v}{\|\beta\|_v} \right)^{n+1+k} \exp \left( n \frac{|K_v : Q_v|}{|K : Q|} \left[ rm \log(2) + r \left( \log(rm+1) + rm \log \left( \frac{rm+1}{rm} \right) \right) + o(1) \right] \right) \text{ if } v \mid \infty
\]

\[
= \exp(\mu(\eta, \zeta, \alpha, \beta)) \left[ \prod_{j=1}^{r} \left( \frac{\log(2)}{\log(\nu_j)} + 1 \right) \right] \text{ otherwise },
\]

and the lemma follows using geometric series summation.

\[
\square
\]

6 Proof of Theorem 2.1

We use the same notations as in Section 5. To prove Theorem 2.1 we shall prove the following theorem.

**Theorem 6.1.** For \( v \in \mathfrak{M}_K \), we define the constants

\[
c(x, v) = \varepsilon_v \frac{|K_v : Q_v|}{|K : Q|} \left( rm \log(2) + r \left( \log(rm+1) + rm \log \left( \frac{rm+1}{rm} \right) \right) \right) + (1 - \varepsilon_v) \sum_{j=1}^{r} \log |\mu(\zeta_j)|_v^{-1},
\]

where \( p_v \) is the rational prime under \( v \) if \( v \) is non-archimedean. We also define

\[
\mathcal{A}_v(\eta, \zeta, \alpha, \beta) = \log |\beta|_{v_0} - (rm+1) \log \|\alpha\|_{v_0} - c(x, v_0) - (1 - \varepsilon_v) \limsup_{n \to \infty} \frac{1}{n} \log |D_{c,rmn} \cdot D'_{c,rmn}|_v^{-1},
\]

\[
U_v(\eta, \zeta, \alpha, \beta) = rmh_v(\alpha, \beta) + c(x, v) - (1 - \varepsilon_v) \limsup_{n \to \infty} \frac{1}{n} \log |D_{c,rmn}|_v,
\]

and

\[
V_v(\eta, \zeta, \alpha, \beta) = \log |\beta|_{v_0} - rmh(\alpha, \beta) - (rm+1) \log \|\alpha\|_{v_0} - c(x, v_0) - (1 - \varepsilon_v) \sum_{j=1}^{r} \log |\mu(\eta_j) + 2 \log |\mu(\zeta_j)|_v^{-1} \cdot \left( \frac{\log(2)}{\log(\nu_j)} + \frac{\text{den}(\zeta_j)}{\phi(\text{den}(\zeta_j))} \right).
\]

Let \( v_0 \) be a place in \( \mathfrak{M}_K \), either archimedean or non-archimedean, such that \( V_v(\eta, \zeta, \alpha, \beta) > 0 \). Then the functions \( F_s(\alpha_i / \beta), 0 \leq s \leq r-1 \) converge around \( \alpha_j / \beta \) in \( K_{v_0}, 1 \leq j \leq m \) and for any positive number \( \varepsilon \) with \( \varepsilon < V_{v_0}(\eta, \zeta, \alpha, \beta) \), there exists an effectively computable positive number \( H_0 \) depending on \( \varepsilon \) and the given data such that the following property holds. For any \( \lambda := (\lambda_0, \lambda_i, s)_{1 \leq i \leq m} \in K^{rm+1} \setminus \{0\} \) satisfying \( H_0 \leq H(\lambda) \), then we have

\[
\left| \lambda_0 + \sum_{i=1}^{m} \sum_{s=0}^{r-1} \lambda_i,s F_s(x, \alpha_i / \beta) \right|_{v_0, v} > C(\eta, \zeta, \alpha, \beta, \varepsilon) H_{v_0}(\lambda)^{-\mu(\eta, \zeta, \alpha, \beta, \varepsilon)},
\]

where

\[
\mu(\eta, \zeta, \alpha, \beta, \varepsilon) := \frac{A_{v_0}(\eta, \zeta, \alpha, \beta) + U_{v_0}(\eta, \zeta, \alpha, \beta)}{V_{v_0}(\eta, \zeta, \alpha, \beta) - \varepsilon},
\]

\[
C(\eta, \zeta, \alpha, \beta, \varepsilon) = \exp \left( - \left( \frac{\log(2)}{V_{v_0}(\eta, \zeta, \alpha, \beta) - \varepsilon} + 1 \right) (A_{v_0}(\eta, \zeta, \alpha, \beta) + U_{v_0}(\eta, \zeta, \alpha, \beta)) \right).
\]
PROOF. By Proposition 4.1 the matrix \(M_n = \begin{pmatrix} P_t(\beta) \\ P_{t,i,s}(\beta) \end{pmatrix}\) with entries in \(K\) is invertible. By Lemma 5.3 (i) together with inequality (39),

\[
\log \|P_t(\beta)\|_v \leq \varepsilon_v \left( n \left[ \frac{[K_v : Q_v]}{[K : Q]} \right] \left[ rm \log(2) + r \left( \log(rm + 1) + rm \log \left( \frac{rm + 1}{rm} \right) \right) + o(1) \right] \right) \\
+ (1 - \varepsilon_v) \left( \log |D_{c,rmn}|^{-1} \sum_{j=1}^r \log |\mu_n(\zeta_j)|^{-1} \right) + (rm + \ell)h_v(\alpha, \beta)
\]

\[
\leq n (rmh_v(\alpha, \beta) + c(x, v)) + o(n)
\]

= \(U_v(\eta, \zeta, \alpha, \beta)n + o(n)\).

Similarly, using this time Lemma 5.3 (ii) and inequality (39),

\[
\log \|P_{t,i,s}(\beta)\|_v \leq n (rmh_v(\alpha, \beta) + c(x, v)) + f_v(n),
\]

where

\[
f_v : \mathbb{N} \to \mathbb{R}_{\geq 0}; \quad n \mapsto rmh_v(\alpha, \beta) + (1 - \varepsilon_v) \log |D_{c,rmn} \cdot D_{c,rmn}'|^{-1}.
\]

We define

\[
F_v(\alpha, \beta) : \mathbb{N} \to \mathbb{R}_{\geq 0}; \quad n \mapsto n (rmh_v(\alpha, \beta) + c(x, v)) + f_v(n).
\]

Since on the other hand, Lemma 5.4 ensures

\[
-\log |R_{t,i,s}(\beta)|_{v_0} \leq n \log |\alpha|_{v_0} - (rm + 1)n \log \|\alpha\|_{v_0} - nc(x, v_0) + (1 - \varepsilon_v) \lim_{n \to \infty} \frac{1}{n} \log |D_{c,rmn} \cdot D_{c,rmn}'|^{-1} + o(n)
\]

\[
= \mathbb{A}_{v_0}(\eta, \zeta, \alpha, \beta)n + o(n).
\]

Using Lemma 5.1 we have

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{v \in \mathfrak{M}_K} f_v(n) = \sum_{j=1}^r \left( \log \mu(\eta_j) + \log \mu(\zeta_j) + \frac{\text{den}(\zeta_j)\text{den}(\eta_j)}{\varphi(\text{den}(\zeta_j))\varphi(\text{den}(\eta_j))} \right),
\]

\[
\sum_{v \in \mathfrak{M}_K} c(x, v) = rm \log(2) + r \left( \log(rm + 1) + rm \log \left( \frac{rm + 1}{rm} \right) \right) + \sum_{j=1}^r \log \mu(\zeta_j),
\]

we conclude

\[
\mathbb{A}_{v_0}(\eta, \zeta, \alpha, \beta) - \lim_{n \to \infty} \frac{1}{n} \sum_{v \neq v_0} F_v(\alpha, \beta)(n) = V(\eta, \zeta, \alpha, \beta).
\]

Applying \(\mathbb{E} \) Proposition 5.6 for \(\{\theta_{i,s} := F_s(\alpha_i/\beta)\}_{1 \leq i \leq m, 0 \leq s \leq r - 1}\) and the above data, we obtain the assertions of Theorem 6.1.

**Proof of Theorem 2.1** We use the same notations as in Theorem 2.1 and Theorem 6.1. Put \(\eta = (a_1 + 1, \ldots, a_r + 1), \zeta = (b_1, \ldots, b_{r-1}, 1)\). Then we have

\[
V_{v_0}(\alpha, \beta) = V_{v_0}(\eta, \zeta, \alpha, \beta).
\]

Combining with \(\mathbb{E}\) and \(\mathbb{H}\), Theorem 6.1 yields the assertion of Theorem 2.1.
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