ON THE ESSENTIAL DIMENSION OF COHERENT SHEAVES

INDRANIL BISWAS, AJNEET DHILLON, AND NORBERT HOFFMANN

Abstract. We characterize all fields of definition for a given coherent sheaf over a projective scheme in terms of projective modules over a finite-dimensional endomorphism algebra. This yields general results on the essential dimension of such sheaves. Applying them to vector bundles over a smooth projective curve, we obtain an upper bound for the essential dimension of their moduli stack. The upper bound is sharp if the conjecture of Colliot-Thélène, Karpenko and Merkurjev holds. Consequently, this Artin stack also has the genericity property studied by Brosnan, Reichstein and Vistoli.

1. Introduction

The essential dimension of an algebraic object was introduced in [6]. Roughly speaking, it is the number of algebraically independent parameters needed to define the object; the precise definition is recalled below. This notion has been studied intensively, leading to many interesting connections with several areas of algebra and algebraic geometry, as the recent surveys [23] and [22] show.

The essential dimension of a moduli stack is the supremum of the essential dimensions of the objects it parameterizes. For smooth Deligne-Mumford stacks, it suffices to consider generic objects, according to the genericity theorem of Brosnan, Reichstein and Vistoli [7]. They use it to determine the essential dimension of the moduli stack of curves. In an appendix to [7], Fakhruddin does likewise for the moduli stack of abelian varieties. The genericity theorem is generalized to smooth Artin stacks with reductive automorphism groups in [24].

The subject of this article is the essential dimension of coherent sheaves over a projective scheme. We relate it to the essential dimension of projective modules over a finite-dimensional algebra, and study the latter systematically. The essential dimension also involves the number of moduli. In order to count moduli of coherent sheaves, we express those with nilpotent endomorphisms as iterated extensions.

We then apply our general results to the special case of vector bundles over a smooth projective curve. Theorem 7.3 gives the essential dimension of the moduli stack in this case, modulo the now famous conjecture of Colliot-Thélène, Karpenko and Merkurjev [8]. Our result improves the upper bounds on this essential dimension given in [9] and in [4]. Although the moduli stack is not Deligne-Mumford, and its automorphism groups are in general not reductive, we find that it still has the genericity property mentioned above. Our methods specifically address non-reductive automorphism groups, by focussing on nilpotent endomorphisms.

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Let $X \hookrightarrow \mathbb{P}_k^N$ be a projective scheme over a base field $k$. Let $K$ be a field containing $k$, and let $E$ be a coherent sheaf over the base change $X_K := X \otimes_k K$. One says that $E$ is \emph{defined over a field $K'$} with $k \subseteq K' \subseteq K$ if there is a coherent sheaf $E'$ over $X_{K'}$ with $E' \otimes_{K'} K \cong E$. The \emph{essential dimension} of $E$ is

$$\text{ed}_k(E) := \min_{K'} \text{trdeg}_k K'$$

where the minimum is taken over all fields $K'$ with $k \subseteq K' \subseteq K$ such that $E$ is defined over $K'$.

Let $k(E) \subseteq K$ denote the \emph{field of moduli} for the coherent sheaf $E$ over $X_K$; cf. Remark 5.1. Since $k(E) \subseteq K'$ for every field of definition $K'$ for $E$, we have

$$\text{ed}_k(E) = \text{trdeg}_k k(E) + \text{ed}_{k(E)}(E).$$

The essential dimension of $E$ over $k(E)$ measures how far $E$ is from being defined over $k(E)$. This defect is caused by $\text{Aut}(E)$, since an object without automorphisms is usually defined over its field of moduli. We make use of the fact that $\text{Aut}(E)$ is the group of units of the finite-dimensional $K$-algebra $\text{End}(E)$. Theorem 5.3 describes the obstruction against defining $E$ over $k(E)$ in terms of modules over such algebras. This is the basis of our results on the essential dimension of $E$ over $k(E)$. We also deduce that every vector bundle over an elliptic curve is defined over its field of moduli.

We then have to estimate the transcendence degree of $k(E)$. This is more subtle if $E$ has nilpotent endomorphism. Our estimates are based on Theorem 6.1, which describes sheaves with nilpotent endomorphisms as iterated extensions.

These two theorems are our main technical tools. We formulate them for coherent sheaves over projective schemes, but the method generalizes to other kinds of objects as long as they have finite-dimensional endomorphism algebras.

The structure of this paper is as follows. Section 2 deals with projective modules over right-artinian rings, in particular over finite-dimensional algebras. Section 3 studies the essential dimension of such modules, and reduces this question to the case of central simple algebras, which is well-studied.

Section 4 deals with endomorphism algebras of coherent sheaves. Section 5 relates the fields of definition for $E$ to those for some module over an endomorphism algebra, and deduces information on the essential dimension of $E$ over $k(E)$.

Section 6 contains the moduli count, in particular for sheaves with nilpotent endomorphisms. Section 7 puts all this together in the case of vector bundles over a curve, and contains our results on their essential dimension.

\section{Projective Modules over Right-Artinian Rings}

Let $R$ be a ring. Our rings are always associative, and they always have a unit, but they are not necessarily commutative. By an $R$-module, we mean a right $R$-module, unless stated otherwise. Let $n \subseteq R$ be a nilpotent two-sided ideal.

\begin{lemma}
Every element $q \in R/n$ with $q^2 = q$ admits a lift $p \in R$ with $p^2 = p$.
\end{lemma}

\begin{proof}
By assumption, there is an integer $n \geq 1$ such that $n^2 = 0$. Using induction over $n$, we may assume $n^2 = 0$ without loss of generality.

Let $p \in R$ be any lift of $q$. Then $p^2 \equiv p$ modulo $n$, and hence $(p^2 - p)^2 = 0$. Therefore, $p' := -2p^3 + 3p^2 \in R$ is another lift of $q$, and

$$(p')^2 = 4p^6 - 12p^5 + 9p^4 = (p^2 - p)^2(4p^2 - 4p - 3) - 2p^3 + 3p^2 = p'. \quad \square$$
Corollary 2.2. Let $N$ be a finitely generated projective $(R/n)$-module. Then there is a finitely generated projective $R$-module $M$ such that $M/Mn \cong N$. The finitely generated projective $R$-module $M$ is unique up to isomorphisms.

Proof. By assumption, $N$ is isomorphic to a direct summand of a free module $(R/n)^r$ for some $r \in \mathbb{N}$. Therefore, $N$ is isomorphic to the image of a matrix

$$q \in \text{Mat}_{r \times r}(R/n)$$

with $q^2 = q$. Using Lemma 2.1, we can lift $q$ to a matrix

$$p \in \text{Mat}_{r \times r}(R)$$

with $p^2 = p$. The image of $p$ is a finitely generated projective $R$-module $M$ with $M/Mn \cong N$. For the uniqueness, suppose that $M'$ is another finitely generated projective $R$-module with $M'/M'n \cong N$. Then there are $(R/n)$-linear maps

$$g_1 : M/Mn \rightarrow M'/M'n \quad \text{and} \quad g_2 : M'/M'n \rightarrow M/Mn$$

with $g_1 \circ g_2 = \text{id}$ and $g_2 \circ g_1 = \text{id}$. Since $M$ and $M'$ are direct summands of free modules, we can lift $g_1$ and $g_2$ to $R$-linear maps

$$f_1 : M \rightarrow M' \quad \text{and} \quad f_2 : M' \rightarrow M.$$

They satisfy $f_1 \circ f_2 \equiv \text{id}$ and $f_2 \circ f_1 \equiv \text{id}$ modulo $n$. Therefore, $f_1 \circ f_2$ and $f_2 \circ f_1$ are automorphisms. This shows that $M'$ is isomorphic to $M$. $\square$

We will only need rings that are finite-dimensional algebras over a field. These satisfy the descending chain condition for right ideals, so they are right-artinian.

Definition 2.3. A projective module $M$ over a right-artinian ring $R$ has rank $r \in \mathbb{Q} > 0$ if the direct sum $M \oplus n$ is free of rank $nr$ for some $n \in \mathbb{N}$ with $nr \in \mathbb{Z}$.

Example 2.4. Let $R$ be a simple right-artinian ring. Wedderburn’s theorem states

$$R \cong \text{Mat}_{n \times n}(D)$$

for some division ring $D$ and some integer $n \geq 1$. There is a projective $R$-module $M$ of rank $r \in \mathbb{Q} > 0$ if and only if $nr \in \mathbb{Z}$, and any such module $M$ satisfies

$$M \cong \text{Mat}_{n \times nr}(D).$$

Proposition 2.5. Let $R$ be a right-artinian ring. Then all projective $R$-modules $M$ of the same rank $r \in \mathbb{Q} > 0$ are isomorphic.

Proof. Let $j$ denote the Jacobson radical of $R$; this is by definition the smallest two-sided ideal such that $R/j$ is semisimple. The ideal $j \subseteq R$ is known to be nilpotent (see for example Theorem 14.2 in [10]). Therefore, Corollary 2.2 allows us to replace $R$ by the semisimple right-artinian ring $R/j$ without loss of generality.

We may thus assume $R \cong \prod_i R_i$ for some simple right-artinian rings $R_i$. Then $M$ is isomorphic to a product of projective $R_i$-modules $M_i$ of rank $r$. Each $M_i$ is unique up to isomorphisms according to Example 2.4. $\square$

Corollary 2.6. If $R$ is right-artinian, and $M$ is a projective $R$-module of rank $r \in \mathbb{Q} > 0$, then the direct sum $M \oplus n$ is free of rank $nr$ for every $n \in \mathbb{N}$ with $nr \in \mathbb{Z}$.
3. Essential Dimension and Finite-Dimensional Algebras

Let $k$ be a field. Let $\text{Fields}/k$ denote the category of fields $K \supseteq k$. Let a functor $F : \text{Fields}/k \to \text{Sets}$ be given. If an element $a \in F(K)$ is the image of an element $a' \in F(K')$ for some intermediate field $k \subseteq K' \subseteq K$, then $a$ is said to be defined over $K'$.

**Definition 3.1** (Merkurjev).

i) The essential dimension of an element $a \in F(K)$ is

$$\text{ed}_k(a) := \inf_{K'} \text{trdeg}_k K'$$

where the infimum is over all fields $k \subseteq K' \subseteq K$ over which $a$ is defined.

ii) The essential dimension of the functor $F$ is

$$\text{ed}_k(F) := \sup_a \text{ed}_k(a)$$

where the supremum is over all fields $K \supseteq k$ and all elements $a \in F(K)$. We put $\text{ed}_k(F) = -\infty$ if $F(K) = \emptyset$ for all $K$.

iii) The essential dimension of a stack $\mathcal{M}$ over $k$ is the essential dimension of the functor $\text{Fields}/k \to \text{Sets}$ that sends each field $K \supseteq k$ to the set of isomorphism classes in the groupoid $\mathcal{M}(K)$.

Given a finite-dimensional $k$-algebra $A$ and a number $r \in \mathbb{Q}_{>0}$, we denote by

$$\text{Mod}_{A,r} = \text{Mod}_{k,A,r} : \text{Fields}_k \to \text{Sets}$$

the functor that sends each field $K \supseteq k$ to the set $\text{Mod}_{A,r}(K)$ of isomorphism classes of projective $(A \otimes_k K)$-modules of rank $r$. Each of these sets $\text{Mod}_{A,r}(K)$ has at most one element by Proposition 2.5.

This section deals with $\text{ed}_k(\text{Mod}_{A,r})$.

The following four propositions will allow us to assume that $A$ is semisimple, simple, central simple, and a central division algebra, respectively.

**Proposition 3.2.** If $n \subseteq A$ is a nilpotent two-sided ideal, then

$$\text{ed}_k(\text{Mod}_{A,r}) = \text{ed}_k(\text{Mod}_{A/n,r}).$$

**Proof.** Corollary 2.2 states that the canonical map

$$\text{Mod}_{A,r}(K) \to \text{Mod}_{A/n,r}(K)$$

is bijective for every field $K \supseteq k$. \hfill \Box

**Proposition 3.3.** If $A$ is isomorphic to a product of $k$-algebras $A_i$, then

$$\text{ed}_k(\text{Mod}_{A,r}) \leq \sum_i \text{ed}_k(\text{Mod}_{A_i,r}).$$

**Proof.** For each field $K \supseteq k$, we have a canonical bijection

$$\prod_i \text{Mod}_{A_i,r}(K) \to \text{Mod}_{A,r}(K)$$

which sends each sequence of projective $(A_i \otimes_k K)$-modules $M_i$ to their product $M$. If each $M_i$ is defined over some intermediate field $k \subseteq K_i' \subseteq K$, then $M$ is defined over the compositum $K' \subseteq K$ of all $K_i'$. This shows $\text{ed}_k(M) \leq \sum_i \text{ed}_k(M_i).$ \hfill \Box
Proposition 3.4. If the center of $A$ contains a field $l \supseteq k$, then
\[ \text{ed}_k(\text{Mod}_{A,r}) \leq [l : k] \cdot \text{ed}_l(\text{Mod}_{A,r}). \]

Proof. Let $K \supseteq k$ be a field with $\text{Mod}_{A,r}(K) \neq \emptyset$. Let $k'$ be the algebraic closure of $k$ in $K$. If $r$ denotes the radical of the commutative $k$-algebra $l \otimes_k k'$, then

\[ (l \otimes_k k')/r \cong \prod l'_i \]

for some fields $l'_i \supseteq k'$. We will construct the following diagrams of fields:

\[ l \rightarrow l'_i \rightarrow L'_i \rightarrow L''_i \rightarrow L_i \]

\[ k \rightarrow k' \rightarrow K' \rightarrow K \]

Let us start with $L_i := l'_i \otimes_{k'} K$. This is a field, because $l'_i$ is finite over $k'$, and $k'$ is algebraically closed in $K$. We have $\text{Mod}_{A,r}(L_i) \neq \emptyset$, since $A \otimes_k L_i$ is a quotient of $A \otimes_k K$. Consequently, there is an intermediate field $l \subseteq L'_i \subseteq L_i$ with

\[ \text{Mod}_{A,r}(L'_i) \neq \emptyset \]

such that $\text{trdeg}_k(L_i) \leq \text{ed}_l(\text{Mod}_{A,r})$. We assume $L'_i \subseteq L_i$ without loss of generality.

Choose a transcendence basis $(t_{ij})$ of $L'_i$ over $l$. Let the polynomial

\[ x^{d_{ij}} + \sum_{m=0}^{d_{ij}-1} a_{ijm} x^m \in K[x] \]

be the minimal polynomial of $t_{ij} \in L_i$ over $K$. Let $K' \subseteq K$ be the algebraic closure of the subfield $k'(a_{ijm}) \subseteq K$ generated by $k'$ and all the coefficients $a_{ijm}$. Then $L''_i := l'_i \otimes_{k'} K'$ is algebraically closed in $L_i$. This implies that $L'_i \subseteq L''_i$, and hence

\[ \text{Mod}_{A,r}(L''_i) \neq \emptyset. \]

Proposition 3.5. If $A \cong \text{Mat}_{n \times n}(B)$ for a central simple $k$-algebra $B$, then

\[ \text{ed}_k(\text{Mod}_{A,r}) = \text{ed}_k(\text{Mod}_{B,1/d}), \]

where the integer $d \geq 1$ is the denominator of the rational number $nr > 0$.

Proof. Let a field $K \supseteq k$ be given. We have

\[ B \otimes_k K \cong \text{Mat}_{m \times m}(D) \quad \text{and} \quad A \otimes_k K \cong \text{Mat}_{mn \times mn}(D) \]

for some central division algebra $D$ over $K$ and some integer $m \geq 1$. Therefore

\[ \text{Mod}_{B,1/d}(K) \neq \emptyset \iff m/d \in \mathbb{Z} \iff mnr \in \mathbb{Z} \iff \text{Mod}_{A,r}(K) \neq \emptyset \]

according to Example 2.4 and the choice of $d$. \qed
Let $A$ be a central simple $k$-algebra. Recall that the degree of $A$ is

$$\deg A := \sqrt{\dim_k A} \in \mathbb{N}.$$ 

If $A \cong \text{Mat}_{n \times n}(D)$ for a central division algebra $D$ over $k$, then the index of $A$ is

$$\text{ind } A := \deg D = \deg A / n.$$ 

Example 2.4 shows $\text{ed}_k(\text{Mod}_{A,r}) = -\infty$ whenever $r \deg A \not\in \mathbb{Z}$.

An upper bound for $\text{ed}_k(\text{Mod}_{A,1/\dim_k A})$ is proved in [8]. The arguments readily generalize to $\text{Mod}_{A,r}$. For the convenience of the reader, we give the details here.

**Proposition 3.6.** If $A$ is a central simple $k$-algebra, and $0 < r < 1$, then

$$\text{ed}_k(\text{Mod}_{A,r}) \leq r(1 - r) \dim_k A.$$ 

**Proof.** We assume that $r \deg A \in \mathbb{Z}$, since there is nothing to prove otherwise. Let $\text{SB}(r, A)$ denote the generalized Severi-Brauer variety over $k$ that parameterizes right ideals $a \subset A$ which are projective of rank $r$ over $A$. Then

$$\dim_k \text{SB}(r, A) = r \deg A(\deg A - r \deg A),$$

because $\text{SB}(r, A)$ is a form of the Grassmannian that parameterizes linear subspaces of dimension $r \deg A$ in a vector space of dimension $\deg A$. Suppose

$$\text{Mod}_{A,r}(K) \neq \emptyset$$

for some field $K \supseteq k$. According to Example 2.4 this implies that

$$A \otimes_k K \cong \text{Mat}_{n \times n}(D)$$

for some integer $n \geq 1$ with $nr \in \mathbb{Z}$. Then there is a right ideal $a \subset A \otimes_k K$ which is projective of rank $r$ over $A$. The ideal $a$ corresponds to a $K$-valued point in $\text{SB}(r, A)$. Let $K' \subseteq K$ be the residue field of that point in $\text{SB}(r, A)$. Then

$$\text{Mod}_{A,r}(K') \neq \emptyset$$

because the ideal $a$ is already defined over $K'$, and

$$\text{trdeg}_k(K') \leq \dim_k \text{SB}(r, A) = r(1 - r) \dim_k A.$$ 

**Corollary 3.7.** If $A \cong \text{Mat}_{n \times n}(B)$ for a simple $k$-algebra $B$, then

$$\text{ed}_k(\text{Mod}_{A,r}) < nr \dim_k B.$$ 

**Proof.** Let $l \supseteq k$ be the center of $B$. Proposition 3.5 and Proposition 3.6 yield

$$\text{ed}_l(\text{Mod}_{A,r}) = \text{ed}_l(\text{Mod}_{B,1/d}) \leq \frac{1}{d}(1 - \frac{1}{d}) \dim_l B,$$

where $d \in \mathbb{N}$ is the denominator of $nr \in \mathbb{Q}$. Since $1/d \leq nr$ and $1 - 1/d < 1$, we get

$$\text{ed}_l(\text{Mod}_{A,r}) < nr \dim_l B.$$ 

Now apply Proposition 3.3. 

Given a prime number $p$ and an integer $n \geq 1$, we denote by $v_p(n)$ the $p$-adic valuation of $n$. Therefore, $p^{v_p(n)}$ is the largest power of $p$ that divides $n$. 

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Corollary 3.8. If $D$ is a central division algebra over $k$, and $d$ divides $\deg D$, then
\[
ed_k(\text{Mod}_{D,1/d}) \leq \sum_{p | \deg D} p^{2v_p(\deg D/d)}(p^{v_p(d)} - 1).
\]

Proof. There are central division algebras $D_p$ over $k$ such that $D \cong \bigotimes_{p | \deg D} D_p$ and $\deg D_p = p^{v_p(\deg D)}$. We put $d_p := p^{v_p(d)}$. Example 2.4 shows that $\text{Mod}_{D,1/d}(K) = \prod_{p | \deg D} \text{Mod}_{D_p,1/d_p}(K)$ for every field $K \supseteq k$. Hence we conclude that
\[
ed_k(\text{Mod}_{D,1/d}) \leq \sum_{p | \deg D} \ned_k(\text{Mod}_{D_p,1/d_p}).
\]

Using Proposition 3.6 to bound each summand from above, the result follows. □

Karpenko [17, 18] has proved that this bound is sharp if $\deg D$ is a prime power:

Theorem 3.9 (Karpenko).
If $D$ is a central division algebra over $k$ with $\deg D = p^n$, and $1 \leq m \leq n$, then
\[
ed_k(\text{Mod}_{D,1/p^m}) = p^{2(n-m)}(p^m - 1).
\]

Colliot-Thélène, Karpenko and Merkurjev [8] have conjectured that the above bound is sharp in the case $d = \deg D$:

Conjecture 3.10 (Colliot-Thélène, Karpenko, Merkurjev).
If $D$ is a central division algebra over $k$, then
\[
ed_k(\text{Mod}_{D,1/\deg D}) = \sum_{p | \deg D} (p^{v_p(\deg D)} - 1).
\]

4. Endomorphism Algebras of Coherent Sheaves

Let $X \hookrightarrow \mathbb{P}^N_k$ be a projective scheme over the base field $k$. We put $X_S := X \times_k S$ for each $k$-scheme $S$, and $X_K := X \otimes_k K$ for each field $K \supseteq k$. Let $E$ be a coherent sheaf over $X_K$. Then $E$ is a quotient of a vector bundle $V$ over $X_K$, so
\[
\dim_K \text{End}(E) \leq \dim_K \text{Hom}(V, E) = \dim_K H^0(X, V^* \otimes E) < \infty.
\]
Therefore, the theory of finite-dimensional algebras applies to $\text{End}(E)$. Let $j(E)$ denote the Jacobson radical of $\text{End}(E)$. Wedderburn’s Theorem states
\[
\text{End}(E)/j(E) \cong \prod_i \text{Mat}_{n_i \times n_i}(D_i)
\]
for some finite-dimensional division algebras $D_i$ over $K$ and some integers $n_i \geq 1$.

A nonzero coherent sheaf $E$ over $X_K$ is called indecomposable if $E \cong E' \oplus E''$ implies that either $E' = 0$ or $E'' = 0$. Then $\text{End}(E)/j(E)$ is a division ring $D$, according to Lemma 6 in [1]. We will use the following slightly more general fact.
Lemma 4.1. In the notation of [2], the coherent sheaf $E$ admits a decomposition

$$E \cong \bigoplus_i E_i \oplus n_i$$

into indecomposable coherent sheaves $E_i$ with $\text{End}(E_i)/\text{j}(E_i) \cong D_i$.

Proof. Suppose that there is an isomorphism of $K$-algebras

$$\text{End}(E)/\text{j}(E) \cong A' \times A''.$$  

Then $(1, 0) \in A' \times A''$ corresponds to an element $q \in \text{End}(E)/\text{j}(E)$ with $q^2 = q$.

Lemma 2.1 allows us to lift $q$ to an element $p \in \text{End}(E)$ with $p^2 = p$. Therefore,

$$E = E' \oplus E''$$

with $E' := \text{im} p \subseteq E$ and $E'' := \text{im}(1 - p) \subseteq E$. We have

$$\text{End}(E')/\text{j}(E') \cong A' \quad \text{and} \quad \text{End}(E'')/\text{j}(E'') \cong A'',$$

since $\text{End}(E') = p \text{End}(E)p$ and $\text{End}(E'') = (1 - p) \text{End}(E)(1 - p)$.

The above argument allows us to assume that $\text{End}(E)/\text{j}(E)$ is simple, say

$$\text{End}(E)/\text{j}(E) \cong \text{Mat}_{n \times n}(D).$$

Corollary 2.2 allows us to lift the projective module $\text{Mat}_{n \times 1}(D)$ over this algebra to a projective module $M$ of rank $1/n$ over $\text{End}(E)$. The coherent sheaf

$$E_1 := M \otimes_{\text{End}(E)} E$$

over $X_K$ satisfies $E_1^{\oplus n} \cong E$, and therefore $\text{End}(E_1)/\text{j}(E_1) \cong D$. The latter implies that $E_1$ is indecomposable. $\square$

Lemma 4.2. Suppose that the scheme $X$ is connected and has a rational point $P \in X(k)$. Let $E$ be an indecomposable vector bundle over $X_K$. Then we have

$$\dim_K \text{End}(E)/\text{j}(E) \leq \text{rank}(E).$$

Proof. Note that $X_K$ is still connected, because each connected component of it contains the point $P$. Therefore, the rank of $E$ is constant over $X_K$.

Since $E$ is indecomposable, $\text{End}(E)/\text{j}(E)$ is a division algebra $D$ by Lemma 6 in [11]. The fiber $E_P$ of $E$ at $P$ is a nonzero left module over $\text{End}(E)$, and hence

$$\dim_K(D) \leq \dim_K(E_P) = \text{rank}(E).$$  $\square$

Recall that the projective $k$-scheme $X$ is an elliptic curve if $X$ is a connected smooth curve of genus one with a rational point $P \in X(k)$.

Proposition 4.3. Suppose that $X$ is an elliptic curve. Let $E$ be an indecomposable vector bundle over $X_K$. Then $\text{End}(E)/\text{j}(E)$ is a commutative field.

Proof. Since $E$ is indecomposable, Lemma 6 in [11] implies that the $K$-algebra

$$\text{End}(E)/\text{j}(E) =: D$$

is a division algebra. Let $L \supseteq K$ be the center of $D$. Then $D$ is a simple quotient of $D \otimes_K L$. Hence $D$ is also a simple quotient of $\text{End}(E) \otimes_K L$. Using Lemma 1.1 we thus obtain a direct summand $F$ of the vector bundle $E \otimes_K L$ over $X_L$ with

$$\text{End}(F)/\text{j}(F) \cong D.$$  

Let $L^{\text{alg}}$ denote the algebraic closure of $L$. The nilpotent two-sided ideal

$$\text{j}(F) \otimes_L L^{\text{alg}} \subseteq \text{End}(F) \otimes_L L^{\text{alg}}$$

has the simple quotient $D \otimes_L L^{\text{alg}} \cong \text{Mat}_{n \times n}(L^{\text{alg}})$ with $n := \deg D$. Therefore, this two-sided ideal coincides with the Jacobson radical

$$j(F \otimes_L L^{\text{alg}}) \subseteq \text{End}(F \otimes_L L^{\text{alg}})$$

for the vector bundle $F \otimes_L L^{\text{alg}}$ over $X_{L^{\text{alg}}}$. Using Lemma 4.1, we thus obtain

$$F \otimes_L L^{\text{alg}} \cong V \otimes^n$$

for some indecomposable vector bundle $V$ over $X_{L^{\text{alg}}}$.

Atiyah’s classification [2] of such vector bundles $V$ shows that $V/j(V) \cdot V$ is a line bundle over $X_{L^{\text{alg}}}$. Therefore, $F/j(F) \cdot F$ is a vector bundle of rank $n$ over $X_L$. But its fiber at the point $P$ is also a left module over the division algebra $D$ of dimension $n^2$ over $L$. This shows $n = 1$. Hence $D$ is commutative. □

5. Fields of Definition for Coherent Sheaves

As before, $X$ is a projective scheme over a field $k$. Let $\mathcal{Coh}_X$ denote the moduli stack of coherent sheaves over $X$ (cf. [21] and [15] for moduli spaces of sheaves). The stack $\mathcal{Coh}_X$ is given by the following groupoid $\mathcal{Coh}_X(S)$ for each $k$-scheme $S$:

- An object in $\mathcal{Coh}_X(S)$ is a coherent sheaf $E$ over $X_S$ which is flat over $S$. 
- A morphism in $\mathcal{Coh}_X(S)$ is an isomorphism of coherent sheaves.

Théorème 4.6.2.1 in [20] states that $\mathcal{Coh}_X$ is an Artin stack, and that it is locally of finite type over $k$ (cf. also [11] or [13] for the curve case).

We consider a point of $\mathcal{Coh}_X$, in the sense of [20, Section 5]. Let $\mathcal{G}$ be the residue gerbe of this point, and let $k(\mathcal{G})$ denote its residue field. Théorème 11.3 in [20] states that $\mathcal{G}$ is an Artin stack of finite type over the field $k(\mathcal{G})$.

**Remark 5.1.** Any coherent sheaf $E$ over $X_K$ for a field $K \supseteq k$ defines a point of $\mathcal{Coh}_X$. The residue gerbe $\mathcal{G}$ of this point parameterizes forms of $E$. The residue field $k(\mathcal{G}) \subseteq K$ is known as the field of moduli for $E$. It is also denoted by $k(E)$.

As before, let $\mathcal{G}$ be a residue gerbe of $\mathcal{Coh}_X$, with residue field $k(\mathcal{G})$. Hilbert’s Nullstellensatz allows us to choose a field extension $l \supseteq k(\mathcal{G})$ with

$$d := [l : k(\mathcal{G})] < \infty$$

such that $\mathcal{G}(l) \neq \emptyset$. We choose a coherent sheaf $F$ over $X_l$ which is an object in the groupoid $\mathcal{G}(l)$. Denoting by $\pi : X_l \rightarrow X_{k(\mathcal{G})}$ the canonical projection, we put

$$A := \text{End}(\pi_*F).$$

This section will relate the residue gerbe $\mathcal{G}$ to the endomorphism algebra $A$.

**Example 5.2.** A coherent sheaf $E$ over $X_K$ for some field $K \supseteq k$ is called simple if $\text{End}(E) = K$.

Let $\mathcal{G}$ be a residue gerbe of $\mathcal{Coh}_X$ that parameterizes simple sheaves. Then $\mathcal{G}$ is a gerbe with band $\mathbb{G}_m$ over $k(\mathcal{G})$, and $A$ is a central simple algebra of degree $d$ over $k(\mathcal{G})$. Both define the same element in the Brauer group of $k(\mathcal{G})$.

**Theorem 5.3.** In the situation preceding the example, consider a field $K \supseteq k(\mathcal{G})$. Then the following two categories are equivalent:

- the category of coherent sheaves $E$ over $X_K$ which are objects in $\mathcal{G}(K)$, and
- the category of projective modules $M$ of rank $1/d$ over $A_K := A \otimes_{k(\mathcal{G})} K$. 


Proof. We will describe mutually inverse functors between these two categories.

In one direction, we send a coherent sheaf \( E \) over \( X_K \) to the \( A_K \)-module
\[
M := \text{Hom} \left( (\pi_* F) \otimes_{k(\mathcal{G})} K, E \right).
\]
Suppose that \( E \) is an object in \( \mathcal{G}(K) \). As \( \mathcal{G} \) is a gerbe over \( k(\mathcal{G}) \), there is a field extension \( L \supseteq k(\mathcal{G}) \) containing \( l \) and \( K \) such that \( E \otimes_K L \) and \( F \otimes_l L \) are isomorphic over \( X_L \); we may assume that \( [L : K] < \infty \). From this we conclude
\[
M \otimes_K L \cong \text{Hom} \left( (\pi_* F) \otimes_{k(\mathcal{G})} L, F \otimes_l L \right).
\]
Therefore the \( A_L \)-module \( (M \otimes_K L)^{\oplus d} \) is free of rank one, by the projection formula. Consequently, its underlying \( A_K \)-module \( M^{\oplus d} \cdot [L : K] \) is free of rank \([L : K]\). This shows that the \( A_K \)-module \( M \) is projective of rank \( 1/d \).

In the opposite direction, we send an \( A_K \)-module \( M \) to the quasicoherent sheaf
\[
E := M \otimes_A \pi_* F
\]
over \( X_K \). Suppose that \( M \) is projective of rank \( 1/d \). Then \( E \) is coherent. We choose a field extension \( L \supseteq k(\mathcal{G}) \) containing \( l \) and \( K \). Proposition 2.5 implies
\[
M \otimes_K L \cong \text{Hom} \left( (\pi_* F) \otimes_{k(\mathcal{G})} L, F \otimes_l L \right),
\]
since both are projective \( A_L \)-modules of rank \( 1/d \). Therefore \( E \otimes_K L \) and \( F \otimes_l L \) are isomorphic as coherent sheaves over \( X_L \). Hence \( E \) is an object in \( \mathcal{G}(K) \). \( \square \)

Corollary 5.4. Let \( \mathcal{G} \) be a residue gerbe of the moduli stack \( \text{Coh}_X \) as above.

i) Given a field \( K \supseteq k(\mathcal{G}) \), all objects in the groupoid \( \mathcal{G}(K) \) are isomorphic.

ii) For the \( k(\mathcal{G}) \)-algebra \( A \) in (4) and the integer \( d \) in (3), we have
\[
ed_k(\mathcal{G}) = ed_{k(\mathcal{G})}(\text{Mod}_{A,1/d}).
\]
Proof. Just combine Theorem 5.3 with Proposition 2.5 \( \square \)

Corollary 5.5. Suppose that \( X \) is connected and has a \( k \)-rational point. If \( E \) is a vector bundle of rank \( r \geq 1 \) over \( X_K \) for some field \( K \supseteq k \), then
\[
ed_k(E) \leq r - 1.
\]
Proof. In the above, we take for \( \mathcal{G} \) the residue gerbe of the point given by \( E \). Then the chosen coherent sheaf \( F \) over \( X_l \) is a vector bundle of rank \( r \). According to Wedderburn’s Theorem, we have
\[
\text{End}(\pi_* F)/j(\pi_* F) \cong \prod_i A_i \quad \text{with} \quad A_i \cong \text{Mat}_{n_i \times n_i}(D_i)
\]
for some division algebras \( D_i \) over \( k(E) \). Using Lemma 4.1 we obtain that
\[
\pi_* F \cong \bigoplus_i E_i^{\oplus n_i} \quad \text{with} \quad \text{End}(E_i)/j(E_i) \cong D_i
\]
for some vector bundles \( E_i \) over \( X_k(E) \). Corollary 3.7 and Lemma 4.2 imply that
\[
ed_k(E) \cdot \text{Mod}_{A,1/d} < \left\lfloor \frac{n_i}{d} \text{ dim}_{k(E)} D_i \right\rfloor \leq \left\lfloor \frac{n_i}{d} \text{ rank}(E_i) \right\rfloor.
\]
Using Proposition 3.3 and Proposition 3.2 we conclude that
\[
ed_k(E) \cdot \text{Mod}_{\text{End}(\pi_* F),1/d} < \frac{1}{d} \text{ rank}(\pi_* F) = \text{ rank}(F) = r.
\]
According to Corollary 5.3 this means \( ed_k(E) < r \), as claimed. \( \square \)
Corollary 5.6. If \( X \) is an elliptic curve over \( k \), and \( E \) is a vector bundle over \( X_K \) for some field \( K \supseteq k \), then \( E \) is defined over its field of moduli \( k(E) \subseteq K \).

**Proof.** Hilbert’s Nullstellensatz implies that some pullback of \( E \) is already defined over some extension field of finite degree over \( k(E) \). Therefore, Corollary 5.4 allows us to assume without loss of generality that \( K \) has finite degree over \( k(E) \).

Let \( d \) denote the degree of \( K \) over \( k(E) \). Let \( \pi : X_K \to X_{k(E)} \) be the canonical projection. Lemma 4.11 and Proposition 4.3 together imply that

\[
\text{End}(\pi_* E)/\text{End}(\pi_* E) \cong \prod_i \text{Mat}_{n_i}(K_i)
\]

for some (commutative!) fields \( K_i \supseteq k(E) \) and some integers \( n_i \geq 1 \).

Now we use Theorem 5.3, Example 2.4, and Corollary 2.2. They allow us to conclude that since \( E \) is defined over \( K \), each integer \( n_i \) is divisible by \( d \), and therefore \( E \) is already defined over \( k(E) \). \( \square \)

### 6. Moduli of Sheaves with Nilpotent Endomorphisms

As before, \( X \) is a projective scheme over a field \( k \). Let \( \mathcal{N}^n_X \) denote the moduli stack of coherent sheaves \( E \) over \( X \) and morphisms \( \varphi : E \to E \) with \( \varphi^n = 0 \). This stack is given by the following groupoid \( \mathcal{N}^n_X(S) \) for each \( k \)-scheme \( S \):

- An object in \( \mathcal{N}^n_X(S) \) consists of a coherent sheaf \( \mathcal{E} \) over \( X_S \) and a morphism \( \varphi : \mathcal{E} \to \mathcal{E} \) with \( \varphi^n = 0 \) such that \( \varphi \) and all \( \text{coker}(\varphi^i) \) are flat over \( S \).
- A morphism in \( \mathcal{N}^n_X(S) \) from \( (\mathcal{E}, \varphi) \) to \( (\mathcal{F}, \psi) \) is an isomorphism of coherent sheaves \( \alpha : \mathcal{E} \to \mathcal{F} \) with \( \alpha \circ \varphi = \psi \circ \alpha \).

If \( (\mathcal{E}, \varphi) \) is an object in \( \mathcal{N}^n_X(S) \), then \( \text{im}(\varphi)/\text{im}(\varphi^i) \) is flat over \( S \) for each \( i \), because \( \mathcal{E}/\text{im}(\varphi) \) and \( \mathcal{E}/\text{im}(\varphi^i) \) are so by assumption. The forgetful 1-morphism

\[
\mathcal{N}^n_X \longrightarrow \mathcal{Coh}_X, \quad (\mathcal{E}, \varphi) \longmapsto \mathcal{E}
\]

is representable and is of finite type. Therefore \( \mathcal{N}^n_X \) is an Artin stack, and it is locally of finite type over \( k \). We have \( \mathcal{N}^n_X = \text{Spec}(k) \) and \( \mathcal{N}^1_X = \mathcal{Coh}_X \). We will describe \( \mathcal{N}^n_X \) for \( n \geq 2 \) using the following moduli stacks of extensions.

Let \( \mathcal{M}^{\bullet \rightarrow \bullet}_X \) be the moduli stack of morphisms \( \varphi : E_1 \to E_2 \) of coherent sheaves over \( X \). It is given by the following groupoid \( \mathcal{M}^{\bullet \rightarrow \bullet}_X(S) \) for each \( k \)-scheme \( S \):

- An object in \( \mathcal{M}^{\bullet \rightarrow \bullet}_X(S) \) is a morphism \( \varphi : \mathcal{E}_1 \to \mathcal{E}_2 \) of coherent sheaves over \( X_S \) such that \( \text{coker}(\varphi), \text{im}(\varphi) \) and \( \ker(\varphi) \) are all flat over \( S \).
- A morphism in \( \mathcal{M}^{\bullet \rightarrow \bullet}_X(S) \) from \( (\mathcal{E}_1, \varphi) \to (\mathcal{F}_1, \psi) \) consists of two isomorphisms \( \alpha_1 : \mathcal{E}_1 \to \mathcal{F}_1 \) such that \( \alpha_2 \circ \varphi = \psi \circ \alpha_1 \).

We can also view an object \( \varphi : \mathcal{E}_1 \to \mathcal{E}_2 \) in \( \mathcal{M}^{\bullet \rightarrow \bullet}_X(S) \) as a pair of extensions

\[
0 \to \ker(\varphi) \to \mathcal{E}_1 \to \text{im}(\varphi) \to 0, \quad 0 \to \text{im}(\varphi) \to \mathcal{E}_2 \to \text{coker}(\varphi) \to 0.
\]

In particular, \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) are also flat over \( S \). The forgetful 1-morphism

\[
\mathcal{M}^{\bullet \rightarrow \bullet}_X \longrightarrow \mathcal{Coh}_X \times_k \mathcal{Coh}_X, \quad (\varphi : \mathcal{E}_1 \to \mathcal{E}_2) \longmapsto (\mathcal{E}_1, \mathcal{E}_2)
\]

is representable and is of finite type. Therefore \( \mathcal{M}^{\bullet \rightarrow \bullet}_X \) is an Artin stack, and it is locally of finite type over \( k \).

Let \( \mathcal{M}^{\bullet \rightarrow \bullet \rightarrow \bullet}_X \) be the moduli stack of pairs \( E_1 \subset E_2 \) of coherent sheaves over \( X \). We can identify it with the open substack in \( \mathcal{M}^{\bullet \rightarrow \bullet \rightarrow \bullet}_X \) where \( \varphi \) is injective.

Let \( \mathcal{M}^{\bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet}_X \) be the moduli stack of triples \( E_1 \subset E \supset E_2 \) of coherent sheaves over \( X \). It is given by the following groupoid \( \mathcal{M}^{\bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet}_X(S) \) for each \( k \)-scheme \( S \):
• An object in \( \mathcal{M}_X^{n\to m} \) is a triple \( (S, \mathcal{E}_1 \subseteq \mathcal{E} \subseteq \mathcal{E}_2) \) of coherent sheaves over \( X \) such that \( \frac{\mathcal{E}}{\mathcal{E}_1} \cong \frac{\mathcal{E}_2}{\mathcal{E}_1} \) and \( \mathcal{E}_1 \cap \mathcal{E}_2 \) are all flat over \( S \).

A morphism in \( \mathcal{M}_X^{n\to m}(S) \) from \( (E_1 \subseteq E \subseteq E_2) \) to \( (F_1 \subseteq F \subseteq F_2) \) is an isomorphism of coherent sheaves \( \alpha : \mathcal{E} \to \mathcal{F} \) with \( \alpha(E_1) = F_i \) for both \( i \).

The forgetful 1-morphism
\[
\mathcal{M}_X^{n\to m} \to \text{Coh}_X, \quad (E_1 \subseteq E \subseteq E_2) \mapsto E
\]
is representable and locally of finite type. Therefore \( \mathcal{M}_X^{n\to m} \) is an Artin stack, and it is locally of finite type over \( k \). We will use the natural 1-morphisms
\[
\text{pr}_n : \mathcal{M}_X^{n+1} \to \mathcal{M}_X^n, \quad (E, \varphi) \mapsto (\text{im } \varphi, \varphi|_{\text{im } \varphi})
\]
isomorphic to a composition of pullbacks of the natural 1-morphisms
\[
\text{pr}_{\text{sub}} : \mathcal{M}_X^{\bullet \to \bullet} \to \text{Coh}_X, \quad (E_1 \subseteq E_2) \mapsto E_1 \quad \text{and}
\text{pr}_{\text{ext}} : \mathcal{M}_X^{\bullet \to \bullet} \to \mathcal{M}_X^{\bullet \to \bullet}, \quad (E_1 \subseteq E_2 \subseteq E) \mapsto (E_2 \to E \to E/E_1).
\]

More precisely, the commutative diagram of stacks
\[
\begin{array}{ccc}
\mathcal{M}_X^{n+1} & \xrightarrow{(E, \varphi) \mapsto (\text{im } \varphi, \varphi|_{\text{im } \varphi})} & \mathcal{M}_X^{n \to m} \\
\text{pr}_n & & \text{pr}_m \\
\mathcal{M}_X^n & \xrightarrow{(\mathcal{E}, \psi) \mapsto (\psi : \mathcal{F} \to \mathcal{F})} & \mathcal{M}_X^{m \to m} \\
\end{array}
\]
is cartesian, and the fibered product of stacks
\[
\begin{array}{ccc}
\mathcal{M}_X^{n\to m} \times_{\text{Coh}_X} \mathcal{M}_X^{m\to m} & \to & \mathcal{M}_X^{n\to m} \\
\downarrow & & \downarrow \text{pr}_{\text{sub}} \\
\mathcal{M}_X^{n\to m} & \to & \text{Coh}_X \\
\end{array}
\]
makes the following commutative diagram of stacks cartesian as well:
\[
\begin{array}{ccc}
\mathcal{M}_X^{n\to m} \times_{\text{Coh}_X} \mathcal{M}_X^{m\to m} & \xrightarrow{(E_1 \subseteq E \subseteq E_2) \mapsto (E_1 \subseteq E_1 + E_2 \subseteq E)} & \mathcal{M}_X^{n\to m} \\
\downarrow \text{pr}_n \times \text{pr}_m & & \downarrow \text{pr}_{\text{ext}} \\
\mathcal{M}_X^{n\to m} \times_{\text{Coh}_X} \mathcal{M}_X^{m\to m} & \xrightarrow{(\psi : \mathcal{F} \to \mathcal{F}, \text{ker } \psi \subseteq \mathcal{F}) \mapsto (\psi + 0 : F' \oplus F'' \to F)} & \mathcal{M}_X^{n\to m} \\
\end{array}
\]
Here \( F' \oplus F'' \to \text{ker } \psi \) is the pushout of \( F' \) and \( F'' \) along their common subsheaf \( \text{ker } \psi \).
Proof: We start with diagram (5). Let an object \((F, \psi)\) in \(\mathcal{N}il_X^S(S)\) be given, together with an object \(E_1 \subset E \supset E_2\) in \(M_X^{*\rightarrow*}(S)\). Let

\[
\begin{array}{ccc}
F & \xrightarrow{\psi} & F \\
\alpha \downarrow & & \downarrow \beta \\
E_2 & \longrightarrow & E_2/E_1
\end{array}
\]

be an isomorphism between the two images in \(M_X^{*\rightarrow*}(S)\). Then the composition

\[
\varphi : E \rightarrow E/E_1 \xrightarrow{\beta^{-1}} F \xrightarrow{\alpha} E_2 \hookrightarrow E
\]

has image \(E_2\) and restriction \(\varphi|_{E_2} = \alpha \circ \psi \circ \alpha^{-1}\). Since \(E/E_2\) and all \(\text{coker}(\psi^i)\) are flat over \(S\), we conclude that all \(\text{coker}(\varphi^i)\) are flat over \(S\) as well.

Therefore \((E, \varphi)\) is an object in \(\mathcal{N}il_X^{g+1}(S)\) which gives back the given objects. This construction is functorial and shows that the diagram (5) is cartesian.

It remains to consider diagram (6). Let an object \(E_1 \subset E_3 \subset E\) in \(M_X^{*\rightarrow*}\) be given, together with an object \(\psi : F' \rightarrow F\) in \(M_X^{*\rightarrow*}(S)\) and an object \(\ker \psi \subset F''\) in \(M_X^{*\rightarrow*}(S)\) that have the same image \(\ker \psi\) in \(\text{Coh}_X(S)\). Let

\[
\begin{array}{ccc}
F'' & \xrightarrow{\psi+0} & F \\
\alpha' + \alpha'' \downarrow & & \downarrow \beta \\
E_3 & \longrightarrow & E_3/E_1
\end{array}
\]

be an isomorphism between the two images in \(M_X^{*\rightarrow*}(S)\).

Comparing the cokernels and the kernels of the two horizontal maps, we see that \(E/E_3 \cong \ker \psi\) and \(E_1 = \alpha''(F'')\) in \(E_3\). We put \(E_2 := \alpha'(F')\) in \(E_3\).

As \(\alpha' + \alpha''\) is an isomorphism from the pushout of \(F'\) and \(F''\) to \(E_3\), the diagram

\[
\begin{array}{ccc}
\ker \psi & \longrightarrow & F'' \\
\downarrow & & \downarrow \alpha'' \\
F' & \xrightarrow{\alpha'} & E_3
\end{array}
\]

is cocartesian. This implies that \(E_3 = E_1 + E_2\), and \(E_1 \cap E_2 = \alpha'(\ker \psi) = \alpha''(\ker \psi)\).

Therefore \((E_1 \subset E \supset E_2)\) is an object in \(M_X^{*\rightarrow*}(S)\) which gives back the given objects. This shows that the diagram (6) is cartesian as well. \(\square\)

Now let \(C\) be a smooth projective curve of genus \(g\) over the base field \(k\). Then the stack \(\mathcal{N}il_{C,n}\) will turn out to be smooth of the expected dimension. In the very similar case of Higgs fields instead of endomorphisms, Laumon has already proved this in [19, Corollaire 2.10], using more local arguments.

Corollary 6.2. The stack \(\mathcal{N}il_{C,n}\) is smooth over \(k\). Its dimension at the \(K\)-valued point given by a coherent sheaf \(E\) over \(C_K\) and \(\varphi \in \text{End}(E)\) with \(\varphi^n = 0\) is

\[
\dim_{(E, \varphi)} \mathcal{N}il_{C,n} = (g - 1) \sum_{i=1}^{n} r_i^2,
\]

where \(r_i\) denotes the rank of \(\text{im}(\varphi^{i-1})/\text{im}(\varphi^i)\) over the generic point of \(C_K\).
vector bundles

Let $E$ be an indecomposable vector bundle over $C_K$ for an algebraically closed field $K \supseteq k$. If $r_i$ denotes the generic rank of $\text{im}(\varphi^{i-1})/\text{im}(\varphi^i)$ for a general element $\varphi$ of the Jacobson radical $j(E)$ in $\text{End}(E)$, then

$$\text{trdeg}_k k(E) \leq 1 + (g-1) \sum_i r_i^2.$$  

**Proof.** Since $E$ is indecomposable, Lemma 6 in [1] implies that $\text{End}(E)/j(E) \cong K$.

Let $C \subset \text{Coh}_X$ be the closure of the point given by $E$. It satisfies

$$\dim_k C = \text{trdeg}_k k(E) - \dim_K \text{End}(E).$$

Choose $n \in \mathbb{N}$ with $j(E)^n = 0$. Let $N \subset \text{Nil}_{X,n}$ be the closure of all points $(E, \varphi)$ with $\varphi \in j(E)$ such that each $\text{im}(\varphi^{i-1})/\text{im}(\varphi^i)$ has generic rank $r_i$. It satisfies

$$\dim_k N \leq (g-1)(r_1^2 + \cdots + r_n^2)$$

due to Corollary 6.2. The fiber of the forgetful 1-morphism $N \to C$ over the dense point $E : \text{Spec}(K) \to C$ contains a dense open subscheme of $j(E)$, so

$$\dim_k N \geq \dim_k C + \dim_K j(E) = \text{trdeg}_k k(E) - 1.$$  

**Corollary 6.4.** Let $E$ be a vector bundle of rank $r$ over $C_K$ for a field $K \supseteq k$. If $E$ is not simple, and the curve $C$ has genus $g \geq 2$, then

$$\text{trdeg}_k k(E) \leq (g-1)(r^2 - r) + 2.$$  

**Proof.** Since $k(E) = k(E \otimes_K L)$ for any field $L \supseteq K$, we may assume that $K$ is algebraically closed. We can express $E$ as a direct sum of some indecomposable vector bundles $E_j$ of rank $r_j \geq 1$ over $C_K$. Corollary 6.3 states

$$\text{trdeg}_k k(E_j) \leq 1 + (g-1) \sum_i r_{ij}^2.$$
for some integers $r_{ij} \geq 1$ with $\sum_i r_{ij} = r$. From this we conclude that

$$\text{trdeg}_k k(E) \leq \sum_j \text{trdeg}_k k(E_j) \leq \sum_{i,j} (g-1) r_{ij}.$$  

Because $E$ is not simple, the sum $\sum_{i,j} r_{ij} = r$ has at least two summands. Hence

$$\text{trdeg}_k k(E) \leq r + (g-1)(r^2 - 2r + 2) = (g-1)(r^2 - r) + 2 - (g-2)(r-2)$$

due to Lemma 6.5 below. Since $g \geq 2$ and $r \geq 2$, the result follows. \hfill \Box

**Lemma 6.5.** If $r_1, \ldots, r_n$, $n \geq 2$, are positive integers with $r_1 + \cdots + r_n = r$, then

$$r_1^2 + \cdots + r_n^2 \leq r^2 - 2r + 2.$$  

**Proof.** Since $r_2^2 + \cdots + r_n^2 \leq (r_2 + \cdots + r_n)^2$, it suffices to treat the case $n = 2$. In this case, the claim follows from $r_1^2 + r_2^2 = r^2 - 2r_1 r_2$ and $r_1 r_2 \geq r - 1$. \hfill \Box

7. **Essential Dimension of Vector Bundles on a Curve**

Let $C$ be a smooth projective irreducible curve of genus $g$ over the field $k$. Assume that $C$ has a $k$-rational point. We consider the open substack

$$\mathcal{B} \mathcal{W}_{C,r,d} \subseteq \mathcal{C} \mathcal{H}$$

that parameterizes vector bundles of rank $r \geq 1$ and degree $d \in \mathbb{Z}$ over $C$.

**Proposition 7.1.** If the curve $C$ has genus $g = 0$, then $\text{ed}_k(\mathcal{B} \mathcal{W}_{C,r,d}) = 0$.

**Proof.** The assumptions imply $C \cong \mathbb{P}^1$. Let $E$ be a vector bundle over $\mathbb{P}^1_K$ for some field $K \supseteq k$. Grothendieck’s splitting theorem states

$$E \cong \mathcal{O}_{\mathbb{P}^1_K}(n_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1_K}(n_r)$$

for some $n_1, \ldots, n_r \in \mathbb{Z}$. Therefore $E$ is already defined over $k$, so $\text{ed}_k(E) = 0$. \hfill \Box

**Proposition 7.2.** If the curve $C$ has genus $g = 1$, then $\text{ed}_k(\mathcal{B} \mathcal{W}_{C,r,d}) = r$.

**Proof.** Let $E$ be a vector bundle of rank $r$ over $C_k$ for some field $K \supseteq k$. We have

$$\text{ed}_k(E) = \text{trdeg}_k k(E)$$

due to Corollary 5.6. Since $k(E) = k(E \otimes_K L)$ for any field $L \supseteq K$, we may assume that $K$ is algebraically closed. We can express $E$ as a direct sum of some indecomposable vector bundles $E_j$ over $C_k$. Corollary 6.3 states

$$\text{trdeg}_k k(E_j) \leq 1$$

for all $j$. This implies that $\text{trdeg}_k k(E) \leq r$, and hence $\text{ed}_k(E) \leq r$.

Atiyah’s classification of vector bundles over an elliptic curve, [2], implies the inequality $\dim_K \text{Aut}(E) \geq r$ for every vector bundle $E$ of rank $r$ over $C_k$. If $E$ maps to the generic point of $\mathcal{B} \mathcal{W}_{C,r,d}$, then we obtain

$$\text{trdeg}_k k(E) = \dim_k(\mathcal{B} \mathcal{W}_{C,r,d}) + \dim_K \text{Aut}(E) \geq r$$

and hence $\text{ed}_k(E) \geq r$ in this case. This shows that $\text{ed}_k(\mathcal{B} \mathcal{W}_{C,r,d}) = r$. \hfill \Box

**Theorem 7.3.** If the curve $C$ has genus $g \geq 2$, then

$$\text{ed}_k(\mathcal{B} \mathcal{W}_{C,r,d}) \leq (g-1)r^2 + 1 + \sum_{p|h} (p^{v_p(h)} - 1)$$

for $h := \gcd(r,d)$. One has equality here if Conjecture 3.10 holds.
Proof. Let $E$ be a vector bundle of rank $r$ and degree $d$ over $C_K$ for some field $K \supseteq k$. If $E$ is not simple, then Corollary 6.4 and Corollary 5.5 imply
\[
ed_k(E) = \text{trdeg}_k k(E) + \ed_k(E)(E) \leq (g-1)(r^2 - r) + 2 + (r - 1) \leq (g-1)r^2 + 1.
\]
Now suppose that $E$ is simple. Then Corollary 6.4 implies that
\[
(7) \quad \text{trdeg}_k k(E) \leq (g-1)r^2 + 1.
\]
Let $G$ denote the residue gerbe of the point $E : \text{Spec}(K) \to \text{Bun}_{C,r,d}$. The residue field of this point is $k(E)$. Since $E$ is simple, Corollary 5.4 implies that $\ed_k(E)(G) = \ed_k(E)(\text{Mod}_A,1/\deg A)$ for some central simple algebra $A$ over $k(E)$. The index of $A$ divides $h = \gcd(r,d)$, because its Brauer class coincides by Example 5.2 with the Brauer class $\psi_G$ of the $\mathbb{G}_m$-gerbe $G$, and $\text{ind} \psi_G$ divides $h$ for example by Corollary 3.6 in [14]. Hence
\[
(8) \quad \ed_k(E)(G) \leq \sum_{p|h}(p^{v_p(h)} - 1)
\]
according to Proposition 3.5 and Corollary 3.8. This proves the inequality.

Suppose moreover that $E$ maps to the generic point of $\text{Bun}_{C,r,d}$. Then we have equality in (7). Assuming Conjecture 3.10, we also have equality in (5), because $\text{ind} \psi_G = h$ in this situation. For a proof of the latter, see Proposition 5.1 in [10], or Corollary 6.6 in [12], or Theorem 1.8 in [3].

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School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India

E-mail address: indranil@math.tifr.res.in

Department of Mathematics, University of Western Ontario, London, Ontario N6A 5B7, Canada

E-mail address: adhil3@uwo.ca

Department of Mathematics and Computer Studies, Mary Immaculate College, South Circular Road, Limerick, Ireland

E-mail address: norbert.hoffmann@mic.ul.ie