Quantum affine symmetry in vertex models

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Abstract. We study the higher spin analogs of the six vertex model on the
basis of its symmetry under the quantum affine algebra $U_q(\widehat{\mathfrak{sl}_2})$. Using the
method developed recently for the XXZ spin chain, we formulate the space of
states, transfer matrix, vacuum, creation/annihilation operators of particles,
and local operators, purely in the language of representation theory. We find
that, regardless of the level of the representation involved, the particles have
spin $1/2$, and that the $n$-particle space has an RSOS-type structure rather than
a simple tensor product of the 1-particle space. This agrees with the picture
proposed earlier by Reshetikhin.
1. Introduction
The spin-1/2 XXZ model has the Hamiltonian

\[ H = -\frac{1}{2} \sum_{l \in \mathbb{Z}} (\sigma_{l+1}^x \sigma_l^x + \sigma_{l+1}^y \sigma_l^y + \Delta \sigma_{l+1}^z \sigma_l^z) \]  

(1.1)

where \( \sigma_{l}^x, \sigma_{l}^y, \sigma_{l}^z \) are the Pauli matrices acting on the \( l \)-th component of the infinite tensor product

\[ \cdots \otimes V \otimes V \otimes V \otimes \cdots, \quad V = \mathbb{C}^2. \]  

(1.2)

Integrable generalizations are known in the representation theory of the quantum affine algebras \( U_q(\hat{g}) \) [1]. Given an \( R \)-matrix for \( U_q(\hat{g}) \),

\[ R(z_1/z_2) \in \text{End}_\mathbb{C}(V_{z_1} \otimes V_{z_2}), \]

the corresponding Hamiltonian is given by

\[ H = \sum_{l \in \mathbb{Z}} h_{l+1} h_l, \]  

(1.3)

where \( h_{l+1} h_l \) is the following operator \( h \) acting on the \((l+1)\)-th and the \( l \)-th components:

\[ PR(z_1/z_2) = \text{const}(1 + uh + \cdots), \quad u \to 0, \]

(1.4)

\[ P : \text{the transposition,} \quad z_1/z_2 = e^u. \]

If \( g = \mathfrak{sl}_2 \) and \( V_z \) is the two dimensional \( U_q(\hat{\mathfrak{sl}}_2) \)-module, we have the Hamiltonian (1.1) with \( \Delta = (q + q^{-1})/2 \). An immediate consequence of (1.3) is that the Hamiltonian has the quantum affine symmetry, i.e.,

\[ [\Delta^{(\infty)}(g), H] = 0, \]

where \( g \) is one of the Chevalley generators of \( U_q(\hat{g}) \) and \( \Delta^{(\infty)} \) is the infinite coproduct which formally realizes the action of \( g \) on (1.2).

In [2] a new method was given for the diagonalization of (1.1) in the anti-ferroelectric regime \(-1 < q < 0\). It exploits the quantum affine symmetry of (1.1) and gives all the eigenvectors of (1.1) in the thermodynamic limit. As is well known the Yang-Baxter equation for the \( R \) matrix entails that the row transfer matrices \( T(z) \) of the corresponding vertex model (the six vertex model in the case (1.1)) are mutually commuting. The relation (1.4) says that (1.1) is the first in the hierarchy of commuting Hamiltonians derived from the transfer matrices

\[ H_n = \text{const.} \left( z \frac{d}{dz} \right)^n \log T(z) \big|_{z=1}. \]

In this paper we will formulate a similar method for diagonalizing the transfer matrices of vertex models associated with \( U_q(\hat{\mathfrak{sl}}_2) \).

Consider level \( k \) vertex operators in the sense of [3]:

\[ \Phi_\lambda^{\mu V}(z) : V(\lambda) \to V(\mu) \otimes V_z \]  

(1.5)
where $V(\lambda)$ and $V(\mu)$ are level $k$ $U_q(\hat{\mathfrak{g}})$-modules and $V$ is finite-dimensional. Suppose $V$ has a perfect crystal of level $k$ in the sense of [4]. Then for any level $k$ dominant integral weight $\lambda$, there exists a unique level $k$ dominant integral weight $\mu = \sigma(\lambda)$ such that the vertex operator $\Phi^{\lambda \sigma(\lambda)}_\lambda (z)$ exists, and the latter is unique up to normalization [5]. For example, if $\mathfrak{g} = \mathfrak{sl}_2$, then the $(k + 1)$-dimensional irreducible $U_q(\hat{\mathfrak{sl}}_2)$-module $V^{(k)}(\lambda)$ has a perfect crystal of level $k$. The corresponding vertex model is the spin-$k/2$ integrable generalization of the six vertex model introduced in [1].

The main idea of the diagonalization scheme is to replace the infinite tensor product (1.2) by the level 0 module

$$\mathcal{F} = \oplus_{\lambda, \lambda'} \text{Hom}_{U(\mathfrak{c})}(V(\lambda), V(\lambda')) = \oplus_{\lambda, \lambda'} V(\lambda) \hat{\otimes} V(\lambda')^*,$$

where $V(\lambda')^*$ signifies the dual module defined via the antipode $\hat{a}$. The vertex operator (1.5) enables one to disclose the ‘local component’ $V^{(k)}_l$ hidden in $V(\lambda)$:

$$V(\lambda) \hat{\otimes} V(\lambda')^* \xrightarrow{\Phi^{\lambda \sigma(\lambda)}_\lambda (z) \otimes \text{id}} V(\mu) \hat{\otimes} V^{(k)}_z \hat{\otimes} V(\lambda')^*, \quad \mu = \sigma(\lambda).$$

The tensor structure (1.2) is hidden behind the ‘doors’ $V(\lambda)$ and $V(\lambda')^*$, and the vertex operators open/close these doors to show the tensor components. We postulate that, in the anti-ferroelectric regime of the model, the space spanned by finite excitations over the ground states is given precisely by $\mathcal{F}$. On the basis of this postulate, the transfer matrices, the norm of vectors, the vacuum vectors, the creation and the annihilation operators and local operators can all be translated in the language of representation theory, in much the same way as in [2].

The creation operator is interpreted as a vertex operator $V(\lambda) \rightarrow V^{(1)}_l \hat{\otimes} V(\mu)$ (called type II in [2]). In contrast to the level 1 case, for $k > 1$ there are various candidates for $l = 1, \ldots, k$. The correct choice is dictated by the knowledge of the particle structure of the space of states, i.e. the decomposition of $V(\lambda) \hat{\otimes} V(\lambda')^*$ into the irreducible pieces. In the $q = 0$ limit, the Hamiltonian becomes diagonal and the decomposition can be carried through by the crystal base theory. Namely, we can identify the particles with the connected components of the crystal $B(\lambda) \hat{\otimes} B(\lambda')^*$. In the case of $\mathfrak{g} = \mathfrak{sl}_2$, even for higher $k$, we find only spin-1/2 particles: the connected components of $B(\lambda) \hat{\otimes} B(\lambda')^*$ are isomorphic to the connected components of $\text{Aff}(B^{(1)} \hat{\otimes} \mathfrak{n})$ where $B^{(1)}$ is the crystal of the spin-1/2 (i.e., the two dimensional) $U_q(\hat{\mathfrak{sl}}_2)$-module. This is consistent with the known Bethe Ansatz result [6], and motivates us to formulate the creation operator by using the vertex operator

$$\Phi^{(k)}_{\lambda \lambda^\pm}(z) : V(\lambda) \rightarrow V^{(1)}_l \hat{\otimes} V(\lambda^\pm),$$

where $\lambda = (k - j)\lambda_0 + j\lambda_1$ and $\lambda^\pm = (k - 1\mp 1)\lambda_0 + (j \pm 1)\lambda_1$. This choice leads us to the correct computation of the momentum, the energy and the $S$-matrix of the particles.

The paper is organized as follows. In Sect. 2 we describe the decomposition of the crystal $B(\lambda) \hat{\otimes} B(\lambda')^*$ in the case of $\mathfrak{g} = \mathfrak{sl}_2$. In Sect. 3 we compute the two point correlation functions for various types of vertex operators. In Sect. 4, we introduce the space of states, row transfer matrix and creation/annihilation operators. We calculate their commutation relations and the formula for the energy-momentum using the results of Sect. 3. We also introduce the local operators and derive the $q$-KZ type difference equations for their from factors. They are a lattice analog of those found by Smirnov [7] in the massive integrable field theories. The text is followed by two appendices. In appendix 1 we give a brief summary of the theory of the $q$-KZ equation for the correlation functions of vertex operators [3]. Appendix 2 is a summary of elementary properties of the basic hypergeometric series used in the text.
§2. Decomposition of crystals

An integrable spin-\(k/2\) Hamiltonian is obtained from the \(R\)-matrix, \(\hat{R}(x/y) = PR(x/y) \in \text{Hom}(V_x^{(k)} \otimes V_y^{(k)}, V_y^{(k)} \otimes V_x^{(k)})\). Choosing an appropriate base \(\{u_l\}\) (the crystal base) of \(V^{(k)}\), the Hamiltonian is diagonal in the limit of \(q = 0\). Hence pure tensors
\[
\cdots \otimes u_{p(2)} \otimes u_{p(1)} \otimes u_{p(0)} \otimes u_{p(-1)} \otimes \cdots
\]
are eigenvectors at least formally. Let \(B(\lambda)\) be the crystal of the irreducible level \(k\) highest weight \(U_q(\widehat{\mathfrak{sl}}_2)\)-module \(V(\lambda)\) with highest weight \(\lambda\), and \(B(\mu)^*\) the crystal of the irreducible level \(-k\) lowest weight \(U_q(\widehat{\mathfrak{sl}}_2)\)-module \(V(\mu)^*\) with lowest weight \(-\mu\). We know that the crystal \(B(\lambda) \otimes B(\mu)^*\) of \(V(\lambda) \otimes V(\mu)^*\) is identified with a set of vectors of the form (2.1) with some boundary condition for \(\{\lambda_m\}\). It has a graph structure with respect to the action of \(\tilde{e}_i\) and \(\tilde{f}_i\) and decomposes into connected components. In what follows we shall give a description of these connected components.

2.1 Paths

A path is a map \(p : \mathbb{Z} \rightarrow \{0, 1, \ldots, k\}\). We write it as
\[
p = (\ldots, p(2), p(1), p(0), p(-1), \ldots).
\]
For \(m \in \{0, 1, \ldots, k\}\) there is a path defined by
\[
p(l) = \begin{cases} 
m & \text{for } l \equiv 0 \text{ mod } 2, \\
k - m & \text{for } l \equiv 1 \text{ mod } 2.\end{cases}
\]
We denote it by \(\bar{p}_m\). Here \(\lambda_m = (k - m)\Lambda_0 + m\Lambda_1\) \((m = 0, 1, \ldots, k)\) are the level \(k\) dominant integral weights. A path is called a \((\lambda_m, \lambda_m')\)-path \((m, m' \in \{0, 1, \ldots, k\})\) if it is subject to the boundary condition
\[
p(l) = \begin{cases} 
\bar{p}_m(l) & \text{for } l \gg 0, \\
\bar{p}_{m'}(l) & \text{for } l \ll 0.\end{cases}
\]
The reason why we use \(\lambda_m\)'s to specify the boundary conditions is explained at the end of this subsection. We denote a set of \((\lambda_m, \lambda_m')\)-paths by \(P^{(m, m')}\), and write
\[
P = \bigsqcup_{(m, m')} P^{(m, m')}.
\]
In the following, we consider only paths belonging to \(P\) and identify a path \(p\) with the tensor (2.1).

Let \(B^{(k)} = \{u_l\}\) be the crystal base of the \((k + 1)\)-dimensional (or spin \(k/2\)) irreducible representation \(V^{(k)}\) of \(U_q(\widehat{\mathfrak{sl}}_2)\) \([8]\); see also \([4]\)). The action of the operators \(\tilde{e}_i, \tilde{f}_i (i = 0, 1)\) is given by
\[
\tilde{f}_1 u_l = u_{l+1} \quad \text{for } 0 \leq l < k, \\
= 0 \quad \text{for } l = k; \\
\tilde{f}_0 u_l = u_{l-1} \quad \text{for } 0 < l \leq k, \\
= 0 \quad \text{for } l = 0;
\]
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Let \( B = \{ v_+, v_- \} \) be the \( U_q(\mathfrak{sl}_2) \) crystal such that \( \tilde{f} v_+ = v_- \), \( \tilde{f} v_- = 0 \), \( \tilde{e} v_+ = 0 \), and \( \tilde{e} v_- = v_+ \). When the color \( i = 0, 1 \) is fixed, it is convenient to embed \( B^{(k)} \) (considered as a \( U_q(\mathfrak{sl}_2) \) crystal via \( \tilde{f} = \tilde{f}_i, \tilde{e} = \tilde{e}_i \)) into the \( k \)-fold tensor product \( B \otimes_k \): For \( i = 1 \) (resp. \( 0 \)), \( u_l \) is replaced by \( v_- \otimes \cdots \otimes v_- \otimes v_+ \otimes \cdots \otimes v_+ \) (resp. \( k-l \otimes v_- \otimes \cdots \otimes v_- \otimes v_+ \otimes \cdots \otimes v_+ \)).

Now we describe the rules for the action of \( \tilde{e}_i, \tilde{f}_i \) on \( P \), which is obtained from the knowledge of the action on tensor products (later in Sect. 2.2 we shall give much simpler rules).

1. Using (2.2) replace each component of the path by \( - \)'s and \( + \)'s: for \( i = 1 \) (resp. \( 0 \)), \( p(l) \mapsto \underbrace{- - \cdots -}_{p(l)(\text{resp. } k-p(l))} \underbrace{+ + \cdots +}_{k-p(l)(\text{resp. } p(l))} \).

Then we get an infinite sequence, denoted by \( C(p) \).

2. Remove ‘singlet pairs’ ++ successively from \( C(p) \), and then \( C(p) \) is reduced to a finite sequence of the form \((- \cdots - + \cdots + ) \). We denote it by \( \tilde{C}(p) \).

3. The action of \( \tilde{f}_i \) on \( \tilde{C}(p) \) changes the left most + to -. If there is no + in \( \tilde{C}(p) \), then \( \tilde{f}_i \tilde{C}(p) = 0 \).

4. The action of \( \tilde{f}_i \) on \( \tilde{C}(p) \) changes the +, which corresponds to the one flipped in (iii), to -. If \( \tilde{f}_i \tilde{C}(p) = 0 \) then \( \tilde{f}_i p = 0 \).

5. \( \tilde{f}_i p \) is obtained from \( \tilde{f}_i \tilde{C}(p) \) by reversing the replacement of (i). If \( \tilde{f}_i \tilde{C}(p) = 0 \) then \( \tilde{f}_i p = 0 \).

Example. Let \( k = 2 \) and \( p = (\ldots , 0, 2, 1, 2, 0, 2, \ldots ) \). For \( i = 1 \),

\[
\begin{align*}
C(p) & = \cdots + + - - - + - + - - - \cdots \\
\tilde{C}(p) & = - - \\
\tilde{f}_1 \tilde{C}(p) & = 0 \\
\tilde{f}_1 C(p) & = 0
\end{align*}
\]

i.e., \( \tilde{f}_1 p = 0 \).

For \( i = 0 \),

\[
\begin{align*}
C(p) & = \cdots - - + + - + + - - + + \cdots \\
\tilde{C}(p) & = + + \\
\tilde{f}_0 \tilde{C}(p) & = - + \\
\tilde{f}_0 C(p) & = \cdots - - - + - + + + - - + + \cdots 
\end{align*}
\]

i.e., \( \tilde{f}_0 p = (\ldots , 0, 1, 1, 2, 0, 2, \ldots ) \).
The action of $\hat{e}_i$ on $p \in \mathcal{P}$ is such that if $\{p' \in \mathcal{P} \mid f_i p' = p\}$ is empty then $\hat{e}_i p = 0$; otherwise $\hat{e}_i p = p'$ where $p'$ is the unique one satisfying $f_i p' = p$. Note that only one component, say $p(l)$ of the path $p$ is affected by an application of one operator: $p(l) \mapsto p(l) \pm 1$ where $+$ is chosen for $f_1, \hat{e}_0$ and $-$ for $f_0, \hat{e}_1$.

Besides the action of operators we define the weight of a path. For a path $p \in \mathcal{P}(m,m')$

$$wt(p) = (m - m')(\Lambda_1 - \Lambda_0) + s(p)\alpha_1 - h(p)\delta$$

We note $2(\Lambda_1 - \Lambda_0) = \alpha_1$ and $\delta = \alpha_0 + \alpha_1$. Here the integer-valued function $h: \mathcal{P}(m,m') \to \mathbb{Z}$ is defined by

$$h(p) = \sum_{l \in \mathbb{Z}} l\left( H(p(l+1), p(l)) - H(\tilde{p}_{m,m'}(l+1), \tilde{p}_{m,m'}(l)) \right),$$

where $\tilde{p}_{m,m'}(l) = \check{p}_m(l)$ for $l > 0$, $= \check{p}_{m'}(l)$ for $l \leq 0$, and

$$H(j, j') = \begin{cases} -j' & \text{for } j + j' \leq k \\ j - k & \text{for } j + j' \geq k, \end{cases}$$

$j, j' = 0, 1, \ldots, k$ ($H$ is called energy function); the ‘spin-deviation’ of the path $s: \mathcal{P}(m,m') \to \mathbb{Z}$ is defined by

$$s(p) = \sum_{l \in \mathbb{Z}} (\tilde{p}_{m,m'}(l) - p(l)).$$

Equipped with the action of operators and the weight, the set of $(\lambda_m, \lambda_{m'})$-paths $\mathcal{P}(m,m')$ is identified with the crystal $B(\lambda_m) \otimes B(\lambda_{m'})^* \otimes V(\lambda_m) \otimes V(\lambda_{m'})^{\ast \ast}$. In particular, $\check{p}_m$ is the unique singlet in $\mathcal{P}(m,m)$, which we call ground state path. Drawing an arrow $p \to p'$ when $p' = f_i p$, we equip $\mathcal{P}$ with a structure of colored oriented graph. It decomposes into a number of (in fact infinite) connected parts.

2.2 Domain walls

An alternative description of paths is obtained when one observes the deviation from the ground state path $\check{p}_m$.

Fix a path $p$ for a while. A segment $(p(l_1), p(l_1 - 1), \ldots, p(l_2))$, $l_1 \geq l_2$, is called $\lambda_m$-domain $(m = 0, 1, \ldots, k)$ if

$$p(l) = \check{p}_m(l) \quad \text{for } l_1 \geq l \geq l_2,$$

$$p(l) \neq \check{p}_m(l) \quad \text{for } l = l_1 + 1 \text{ and } l = l_2 - 1.$$ 

The integer $l_1 - l_2 + 1$ and the weight $\lambda_m$ are called length and type of the domain, respectively. A successive pair $(p(l + 1), p(l))$ belongs to the same domain if and only if $p(l + 1) + p(l) = k$. A path is partitioned into several domains. The boundary between adjacent domains is called (domain) wall. If it occurs at $(p(l + 1), p(l))$, we say the position of the wall is $l$.

A domain wall is an elementary wall if the types of adjacent domains are successive; i.e., if a $\lambda_m$-domain and a $\lambda_{m+1}$-domain are adjacent then the wall between them is elementary. Note that if a wall at $l$ is elementary, $p(l + 1) + p(l) = k + 1$ or $k - 1$. Accordingly we call the wall of type 0 (0-wall) or of type 1 (1-wall) respectively (As
we shall see the $i$-wall may change its position by the action of $\tilde{f}_i$ but never by $\tilde{f}_{1-i}$.

A wall which is not elementary is thought to be a composition of elementary walls: Let $\lambda_m$ and $\lambda_m^\prime (m' > m + 1)$ be the types of domains left and right to the wall at $l$. We consider that, at the position of the wall, there are $m' - m - 1$ domains of length zero (types of which are $\lambda_{m+1}, \lambda_{m+2}, \ldots, \lambda_{m^\prime-1}$ from left to right) and $m' - m$ elementary walls between them (types of which are all 0 (resp. 1) if $p(l+1) + p(l) = k + (m'-m)$ (resp. $= k - (m'-m)$)). In the case $m > m' + 1$ things are similarly defined.

Let $n(p)$ be the number of elementary walls in a path $p$. Let $(a_0, a_1, \ldots, a_{n(p)})$ be a sequence of types of domains of the path, counted from left to right, where $a_0$ (resp. $a_{n(p)}$) is the type of the left (resp. right) half infinite domain. Domains that have zero length are included in the list. Noting that by definition $a_j - a_{j+1} = \pm (\Lambda_0 - \Lambda_1)$, $j = 0, 1, \ldots, n(p) - 1$, we can consider that the list $(a_0, a_1, \ldots, a_{n(p)})$ defines a state (or 'path') of length $n(p)$ of the $(k+1)$-state ABF model.

**Example.** Several paths in the case $k = 2$ (spin 1) are shown. Elementary walls are denoted by $|$. Types of domains $(a_0, a_1, \ldots, a_{n(p)})$ are also given.

$$
\begin{array}{cccccccc}
0 & 2 & 0 & 2 & 1 & 1 & 1 & \ldots \\
1 & 1 & 1 & 1 & 0 & 2 & \ldots \\
0 & 2 & 0 & 1 & 1 & 1 & \ldots \\
0 & 2 & 1 & 1 & 2 & 0 & \ldots \\
0 & 2 & 0 & 2 & 0 & 0 & \ldots \\
\end{array}
\begin{array}{c}
(\lambda_0, \lambda_1) \\
(\lambda_1, \lambda_0) \\
(\lambda_0, \lambda_1) \\
(\lambda_0, \lambda_1) \\
(\lambda_0, \lambda_1) \\
\end{array}
\begin{array}{c}
types of domains \\
\end{array}
\begin{array}{c}
\lambda_0, \lambda_1, \lambda_2 \\
\lambda_0, \lambda_1, \lambda_2 \\
\lambda_0, \lambda_1, \lambda_2 \\
\lambda_0, \lambda_1, \lambda_2 \\
\lambda_0, \lambda_1, \lambda_2 \\
\end{array}

The last one shows an example of a composite wall.

In the language of domain walls, the action of $\tilde{f}_i, \tilde{e}_i$ can be described most simply. The rules are obtained if we notice the following: after the reduction steps (i) and (ii) in Sect. 2.1, a 0-wall is considered to be $v_-$ and a 1-wall to be $v_+$ with respect to color $i = 1$. With respect to color $i = 0$, signs $\pm$ are interchanged.

**Example.** In the case $k = 2$ (spin 1) there are four elementary walls $(2,1), (1,2), (0,1),$ and $(1,0)$, which we can regard as $v_-$ and a 1-wall to be $v_+$ with respect to color $i = 1$. With respect to color $i = 0$, signs $\pm$ are interchanged.

$$
\begin{array}{ccc}
21 & 12 & 01 & 10 \\
- - - + & - + - & + + - & - + - ++ \\
\downarrow & \downarrow & \downarrow & \downarrow \\
- & - & + & + \\
\end{array}
\begin{array}{c}
(\cdots + - - + - - + - - + - - + \cdots) \\
\end{array}

In the case $k = 3$ (spin 3/2), we have six elementary walls: for $i = 1$,

$$
\begin{array}{ccc}
31 & 22 & 13 \\
- - - + + & - + - & - + - \cdots \\
\downarrow & \downarrow & \downarrow \\
- & - & - \\
\end{array}
\begin{array}{c}
(\cdots + - - + - - + - - + \cdots) \\
\end{array}
$$

In the case $k = 3$ (spin 3/2), we have six elementary walls: for $i = 1$,
Rules for the action of $f_1$ (resp. $f_0$) are as follows:
(i) Let $n = \nu(p)$, and let $(i_1, \ldots, i_n)$ be a sequence of wall-types of $p$ (from left to right). Replacing 1 by $+$ (resp. $-$) and 0 by $-$ (resp. $+$) in the sequence, we get a $\pm$ sequence of length $n$, denoted by $\cal C'(p)$.
(ii) Removing singlet pairs $+-$ successively from $\cal C'(p)$, we get a sequence of the form $- \cdot \cdot \cdot - + \cdot \cdot \cdot +$, denoted by $\tilde{\cal C}'(p)$.
(iii) The action of $\tilde{f}_i$ on $\cal C'(p)$ changes the left most $+$ to $-$. If there is no $+$ in $\cal C'(p)$, then $\tilde{f}_i \cal C'(p) = 0$.
(iv) The action of $\tilde{f}_i$ on $\cal C'(p)$ changes the $+$, which corresponds to the one flipped in (iii), to $-$. If $\tilde{f}_i \cal C'(p) = 0$ then $\tilde{f}_i \cal C'(p) = 0$.
(v) $\tilde{f}_i \cal p$ is obtained from $\tilde{f}_i \cal C'(p)$: first, by reversing the replacement of (i) we get a sequence of types of walls; second, we reconstruct the path $\tilde{f}_i \cal p$ from the sequence of wall-types. If $\tilde{f}_i \cal C'(p) = 0$, we get $\tilde{f}_i \cal p = 0$.

Note that one of the walls shifts to the left and changes its type ($0 \leftrightarrow 1$) in the action. The number of walls $\nu(p)$ and the list of domain-types $(a_0, a_1, \ldots, a_{\nu(p)})$ are invariant under the action.

We have a natural decomposition
$$ P = \bigsqcup_{(m,m')} \cal P^{(m,m')}, $$
$$ \cal P^{(m,m')} = \bigsqcup_{n=0}^{\infty} \cal P^{(m,m')}_{(n)}, $$
where $\cal P^{(m,m')}_{(n)}$ signifies the set of $(\lambda_m, \lambda_{m'})$-paths with $n$ elementary walls. A set $\cal P^{(m,m')}_{(0)}$ contains only one element $\bar{p}_m$, the ground state path, if $m = m'$ (otherwise it is empty). $\cal P^{(m,m')}_{(n)}, n \geq 1$, is an infinite set (or empty). Define an equivalence relation in a set of paths by
$$ p \sim p' \iff p' = x_1 \cdot \cdot \cdot x_r p \text{ for some } x_j \text{'s} \in \{ \bar{e}_i, \tilde{f}_i \mid i = 0, 1 \}. $$
In other words, $p \sim p'$ if and only if they belong to the same connected part in $\cal P$. We write $\bar{P} = \cal P / \sim$, and denote its element by $[p]$. Then $\bar{\cal P}^{(m,m')}_{(1)}$ contains only one element (or is empty), and $\bar{\cal P}^{(m,m')}_{(n)} (n \geq 2)$ contains infinite number of elements (or is empty).

2.3 Classification of paths It appears that an elementary wall behaves like a local spin-1/2 state. In fact it is true in the following sense. Let $\text{Aff}(B^{(1) \otimes n})$ be the affinization of $B^{(1) \otimes n}$ ([4]). $\text{Aff}(B^{(1) \otimes n})$ consists of finitely many connected parts. They are all isomorphic as crystals without the weight structure. For convenience, set $v_i = u_{i-1} \in B^{(1)}$. There is an isomorphism between a connected part $[p]$ in $\cal P$ and one of the connected parts of $\text{Aff}(B^{(1) \otimes n})$ where $n = \nu(p)$, induced from the map
$$ \iota : [p] \rightarrow B^{(1) \otimes n}, \ p' \mapsto v_{i_1} \otimes \cdot \cdot \cdot \otimes v_{i_n}, $$
where $(i_1, \ldots, i_n)$ is the sequence of wall-types of $p' \in [p]$. Physically speaking, there are only spin-1/2 particles (magnons), but no higher-spin magnons. This agrees with the result of Reshetikhin [6].
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For a later use we define the ‘depth’ of a path \( \nu(p) \) (\( \in \mathbb{Z} \)) by

\[
\nu(p) = -\lfloor \text{wt}(p)/n(p) \rfloor
\]

where \( \lfloor x \rfloor : \mathbb{P} \times \mathbb{P}^* \rightarrow \mathbb{Z} \), \( \lfloor \delta, d \rfloor = 1 \), \( \langle a_1, d \rangle = 0 \), and \( \lfloor x \rfloor \) denotes the integer part of \( x \). Then \( \{((i_1, \ldots, i_n), \nu) \} \leftrightarrow [p] \) is one to one.

Now we proceed to the classification of connected parts. We shall parametrize the elements in \( \mathcal{P}(\mathbb{P}^m, \mathbb{P}^m)(n) \). To do this it is necessary to define parameters related to ‘lengths of domains’.

Let \( \bar{p} \) be a representative of \( [p] \in \mathcal{P} \) such that its sequence of wall-types is \( (0, \ldots, 0) \). Let \( n = n(\bar{p}) \), and let \( (a_0, a_1, \ldots, a_n) \) be the sequence of domain-types of \( \bar{p} \). For the \( j \)-th domain \( (j = 1, 2, \ldots, n − 1) \) of \( \bar{p} \) (which is an \( a_j \)-domain) we define a non-negative integer \( m_j \in \mathbb{Z}_{\geq 0} \) by

\[
\begin{align*}
2m_j &= \text{length of domain}, \quad \text{if } a_{j−1} − a_j = a_j − a_{j+1}; \\
2m_j + 1 &= \text{length of domain}, \quad \text{otherwise}.
\end{align*}
\]

Then we have

\[
\mathcal{P}(\mathbb{P}^m, \mathbb{P}^m)(n) = \bigsqcup_{a_1, \ldots, a_{n−1}} \mathcal{P}(n; a_0, a_1, \ldots, a_n),
\]

\[
\mathcal{P}(n; a_0, a_1, \ldots, a_n) \simeq \{(m_1, \ldots, m_{n−1}) | m_j \in \mathbb{Z}_{\geq 0}, j = 1, \ldots, n − 1\}.
\]

In the first equation the summation is over all \( a_j \in \{\lambda_0, \lambda_1, \ldots, \lambda_k\} \), \( j = 1, \ldots, n − 1 \), with restriction \( a_j − a_{j+1} = \pm(\lambda_0 − \lambda_1) \), \( j = 0, 1, \ldots, n − 1 \), and \( a_0, a_n \) are fixed to \( \lambda_m, \lambda_{m'} \), respectively. This completes the classification.

Example. We give below the types of domains \( (a_0, a_1, \ldots, a_{n(p)}) \) and the ‘length’ of domains \( (m_1, \ldots, m_{n(p)−1}) \) for several paths in the case \( k = 3 \) (spin 3/2). The asterisk shows the position of \( p(0) \).

* \hspace{1cm} types of domains \hspace{1cm} ‘length’

\[
\begin{array}{ccc}
\cdots & 0 & 3 | 1 2 | 1 2 | 2 1 2 1 \cdots & (\lambda_0, \lambda_1, \lambda_2) & (2) \\
\cdots & 0 & 3 | 1 2 | 1 2 | 1 3 0 3 \cdots & (\lambda_0, \lambda_1, \lambda_0) & (2) \\
\cdots & 2 & 1 2 | 2 1 2 1 | 2 1 \cdots & (\lambda_2, \lambda_1, \lambda_2) & (2) \\
\cdots & 0 & 3 | 1 2 | 1 2 | 2 1 | 3 0 \cdots & (\lambda_0, \lambda_1, \lambda_2, \lambda_3) & (2, 1)
\end{array}
\]

Let us summarize the classification of the paths in \( \mathcal{P} \). The paths are parametrized uniquely by the number of walls in path \( n \in \mathbb{Z}_{\geq 0} \); sequence of types of walls \( (i_1, \ldots, i_n) \), \( i_j = 0, 1 \); depth parameter \( \nu \in \mathbb{Z} \); sequence of types of domains \( (a_0, a_1, \ldots, a_n) \); \( a_j \in \{\lambda_0, \lambda_1, \ldots, \lambda_k\} \) with \( a_j − a_{j+1} = \pm(\lambda_0 − \lambda_1) \), \( j = 0, 1, \ldots, n − 1 \); and ‘length-of-domain’ parameters \( (m_1, \ldots, m_{n−1}) \), \( m_j \in \mathbb{Z}_{\geq 0} \). The boundary condition fixes the values of \( a_0 \) and \( a_n \).
§3. Two point functions of the vertex operators for $\hat{sl}_2$

3.1 Notations We shall follow the notations of [2] unless otherwise stated. The main change is that here we normalize the invariant bilinear form on $P = Z\Lambda_0 \oplus Z\Lambda_1 \oplus Z\delta$ by $\langle \alpha_i, \alpha_j \rangle = 2$, as opposed to the convention $\langle \alpha_i, \alpha_j \rangle = 1$ in [2]. We have $\langle \Lambda_i, \Lambda_j \rangle = \delta_{i,1}\delta_{j,2}/2$, $\langle \Lambda_i, \delta \rangle = 1$ and $\langle \delta, \delta \rangle = 0$. We identify $P^\ast = Z\mathfrak{h}_0 \oplus Z\mathfrak{h}_1 \oplus Zd$ with a subset of $P$ via $(\ , \ )$, so that $\alpha_i = h_i$ and $\rho = \Lambda_0 + \Lambda_1 = h_1/2 + 2d$. We set $P_+ = \mathbb{Z}_{\geq 0}\Lambda_0 \oplus \mathbb{Z}_{\geq 0}\Lambda_1$, $P_0 = \{ \lambda \in P_+ \mid \langle \lambda, c \rangle = k \}$. For $\lambda \in Z\Lambda_0 \oplus Z\Lambda_1$ we define $\sigma(\lambda)$, $\overline{\lambda}$ by

$$\sigma(m\Lambda_0 + n\Lambda_1) = n\Lambda_0 + m\Lambda_1, \quad m\Lambda_0 + n\Lambda_1 = n(\Lambda_1 - \Lambda_0). \quad (3.1)$$

In the rest of this paper we denote by $F$ the base field $Q(q)$. The quantized affine algebra $U$ is the $F$-algebra generated by $e_i, f_i$ ($i = 0, 1$) and $q^h$ ($h \in P^\ast$) with the defining relations as in [8], e.g. $[e_i, f_j] = \delta_{i,j}(t_i - t_i^{-1})/(q - q^{-1})$ ($t_i = q^{h_i}$). We take the coproduct

$$\Delta(e_i) = e_i \otimes 1 + t_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i,$$

$$\Delta(q^h) = q^h \otimes q^h \quad (h \in P^\ast) \quad (3.2)$$

and the antipode

$$a(e_i) = -t_i^{-1}e_i, \quad a(f_i) = -t_i f_i, \quad a(q^h) = q^{-h} \quad (h \in P^\ast). \quad (3.3)$$

$U'$ will denote the subalgebra generated by $e_i, f_i$ and $t_i$ ($i = 0, 1$).

Let $M$ be the left $U'$-module with weight space decomposition $\oplus_\nu M_\nu$. By $M^{*n \pm 1}$ we mean the restricted dual space $M^* = \oplus_\nu M^*_\nu$ endowed with the left module structure via $a^{\pm 1}$:

$$\langle xu, v \rangle = \langle u, a^{\pm 1}(x)v \rangle \quad \text{for } x \in U, u \in M^*, v \in M.$$

The irreducible left (resp. right) highest weight module with highest weight $\lambda \in P_k^0$ will be denoted by $V(\lambda)$ (resp. $V^r(\lambda)$). We fix a highest weight vector $|\lambda\rangle \in V(\lambda)$ (resp. $|\lambda\rangle \in V^r(\lambda)$) once for all. There is a unique non-degenerate symmetric bilinear pairing $V^r(\lambda) \times V(\lambda) \rightarrow F$ such that

$$\langle \lambda | \lambda \rangle = 1, \quad \langle xu | u' \rangle = \langle u | xu' \rangle \quad \text{for any } |u\rangle \in V^r(\lambda), \ |u'\rangle \in V(\lambda).$$

3.2 $R$-matrices For $l \in \mathbb{Z}_{\geq 0}$ let $V(l)$ denote the $(l + 1)$-dimensional $U'$-module with basis $\{ v_j^{(l)} \}_{0 \leq j \leq l}$ given by

$$e_0v_j^{(l)} = [j]v_{j+1}^{(l)}, \quad f_0v_j^{(l)} = [l - j]v_{j-1}^{(l)}, \quad t_1v_j^{(l)} = q^{l-2j}v_j^{(l)},$$

$$e_0 = f_1, \quad f_0 = e_1, \quad t_0 = t_1^{-1} \quad \text{on } V(l). \quad (3.4)$$

Here $v_j^{(l)}$ with $j < 0$ or $j > l$ is understood to be 0. The $\{ v_j^{(l)} \}$ constitute an upper global base of $V(l)$ [9]. In the case $l = 1$ we also write $v_0^{(1)} = v_+ \text{ and } v_1^{(1)} = v_-$. We equip $V_2(l) = V(l) \otimes F[z, z^{-1}]$ with a $U$-module structure via

$$e_{l}v_j^{(l)} \otimes z^n = e_{l}v_j^{(l)} \otimes z^{n+\delta_{l,0}}, \quad f_{l}v_j^{(l)} \otimes z^n = f_{l}v_j^{(l)} \otimes z^{n-\delta_{l,0}},$$

$$\text{wt}(v_j^{(l)} \otimes z^n) = n\delta + (l - 2j)\overline{\Lambda}_1. \quad (3.5)$$
Namely $V_z^{(l)}$ is the affinization of $V^{(l)}$ [4]. Let $\{v_j^{(l)}\}$ be a basis of $V^{(l)*}$ dual to $\{v_j^{(l)}\}$. Then the following are isomorphisms of $U$-modules:

$$C_{\pm}^{(l)} : V_{z, q^\pm}^{(l)} \sim V_z^{(l)* \pm 1}, \quad C_{\pm}^{(l)} v_j^{(l)} = (-1)^j q^{-j(l-j+1)} \left[ \frac{t_1}{j} \right] \pm 1 v_{l-j}^{(l)*}.$$  (3.6)

For $l = 1$ they simplify to $C_+ v_+ = v_+^*$, $C_- v_- = -q^{-1} v_+^*$ where $C_\pm = C_{\pm}^{(1)}$.

Consider now an $R$ matrix $R(z_1, z_2) \in \text{End}(V_z^{(m)} \otimes V_z^{(n)})$ which intertwines the action of $U$ in the sense

$$R(z_1, z_2) \Delta(x) = \Delta'(x) R(z_1, z_2) \quad \forall x \in U.$$  

These equations admit a unique solution up to scalar multiple. We denote by $R_{mn}(z)$ the solution normalized as

$$R_{mn}(z)(v_0^{(m)} \otimes v_0^{(n)}) = v_0^{(m)} \otimes v_0^{(n)}.$$  

The following properties are relevant to the subsequent discussions.

**The inversion relations.**

$$R_{mn}(z) P R_{mn}(z^{-1}) P = 1 \quad \text{where} \quad P(u \otimes v) = v \otimes u, \quad (R_{mn}(z^{-1}))^\dagger = \beta_{mn}(z) (C_+^{(m)} \otimes \text{id}) R_{mn}(z q^{-2}) (C_+^{(m)} \otimes \text{id})^{-1},$$  \quad (3.8)

$$\beta_{mn}(z) = q^{-mn} \prod_{j=0}^{n-1} \frac{1 - q^{m-n+2jz}}{1 - q^{-m-n+2jz}}.$$  

The symbol $^\dagger$ means the transposition in the first component, i.e. we set $(f \otimes g)^\dagger = f^t \otimes g \in \text{End}(V^* \otimes W)$ for $f \in \text{End}(V)$ and $g \in \text{End}(W)$.

**The case $m = 1$ or $n = 1$.** If we write

$$R_{11}(z)(v_\epsilon \otimes v_j^{(n)}) = \sum_{\epsilon' = \pm} v_{\epsilon'} \otimes r_{\epsilon' \epsilon}^{1n}(z) v_j^{(n)}, \quad R_{mn}(z)(v_\epsilon^{(m)} \otimes v_\epsilon) = \sum_{\epsilon' = \pm} r_{\epsilon' \epsilon}^{mn}(z) v_j^{(m)} \otimes v_{\epsilon'},$$  

then we have

$$
\begin{align*}
\left( \begin{array}{cc}
  r_{1+}^{1n}(z) & r_{1-}^{1n}(z) \\
  r_{1+}^{1-}(z) & r_{1-}^{1-}(z)
\end{array} \right) &= \frac{1}{q^{1+n/2} - q^{-1/2}-n/2} \left( \begin{array}{cc}
  q_{1/2}^{1/2} - z^{-1} t_1^{-1/2} & (q - q^{-1}) z^{-1/2} t_1^{-1/2} \\
  (q - q^{-1}) t_1^{-1/2} & q_{1/2}^{-1/2} - z^{-1} t_1^{-1/2}
\end{array} \right), \\
\left( \begin{array}{cc}
  r_{m+}^{m}(z) & r_{m-}^{m}(z) \\
  r_{m+}^{m-}(z) & r_{m-}^{m-}(z)
\end{array} \right) &= \frac{1}{zt_1^{1/2} - q^{-1} t_1^{-1/2}} \left( \begin{array}{cc}
  z t_1^{1/2} - q^{-1} t_1^{-1/2} & (q - q^{-1}) z t_1^{1/2} f_1 \\
  (q - q^{-1}) f_1 & z t_1^{-1/2} - q^{-1} t_1^{-1/2}
\end{array} \right).
\end{align*}
$$

Here $t_1^{1/2}$ is understood to act on $v_j^{(l)}$ as $q^{l/2-j} v_j^{(l)}$. In fact only integral powers of $q$ appear in these formulas.

Let us determine the image of the universal $R$ matrix (A1.17) in $\text{End}(V_z^{(m)} \otimes V_z^{(n)})$

$$R_{mn}^{+}(z) = \pi_{V^{(m)} \otimes V^{(n)}}(R'(z)).$$
From the uniqueness of the $R$ matrix we have
\[ R^\pm_{mn}(z) = \rho_{mn}(z)\overline{R}_{mn}(z) \]  
(3.9)
with some scalar $\rho_{mn}(z) \in F[[z]]$. From (A1.3) it follows that $R^+_{mn}(z)$ satisfies the second inversion relation (3.8) with $\beta_{mn}(z) = 1$. This leads to the relation $\rho(z)\rho(zq^{-2}) = \beta_{mn}(z)$, giving
\[ \rho_{mn}(z) = q^{-mn/2}(q^{m-n+2z};q^4)_\infty(q^{-m+n+2z};q^4)_\infty/(q^{m+n+2z};q^4)_\infty(q^{-m-n+2z};q^4)_\infty. \]  
(3.10)
Here
\[ (z;p)_\infty = \prod_{j=0}^{\infty}(1-zp^j). \]

The argument leading to (3.10) is due to [3].

3.3 Vertex operators Let $\lambda$, $\mu$ be dominant integral weights of level $k$. We set $\Delta_\lambda = (\lambda, \lambda + 2\rho)/(2k + 4)$. By vertex operators (VOs) we mean the following types of formal series
\[ \Phi^{(V)}(\lambda)(z) = z^{\Delta_\mu - \Delta_\lambda} \Phi^{(V)}(\mu)(z), \quad \Phi^{(V)}(\lambda) = \sum \Phi^{(V)}(\lambda)_{j,n} \otimes v_j^{(l)} z^{-n}, \]  
(3.11)
\[ \Phi^{(V)}(\mu)(z) = z^{\Delta_\mu - \Delta_\lambda} \Phi^{(V)}(\mu)(z), \quad \Phi^{(V)}(\mu) = \sum v_j^{(l)} z^{-n} \otimes (\Phi^{(V)}(\mu))_{j,n}. \]  
(3.12)
whose coefficients are the maps
\[ \hat{\Phi}^{(V)}_{\lambda}(\mu)_{j,n} : V(\mu)_{\xi} \to V(\mu)_{\xi-(2l)k+n}, \]  
(3.13)
such that they are intertwiners of $U$-modules
\[ \hat{\Phi}^{(V)}_{\lambda}(\lambda) : V(\lambda) \to V(\mu) \otimes V_{l}^{(l)}, \]  
(3.14)
\[ \hat{\Phi}^{(V)}_{\lambda}(\mu) : V(\lambda) \to V_{l}^{(l)} \otimes V(\mu). \]  
(3.15)
This means that, as a formal series, e.g. (3.14) satisfy
\[ \hat{\Phi}^{(V)}_{\lambda}(\lambda) \circ x = \Delta(x) \circ \hat{\Phi}^{(V)}_{\lambda}(\lambda), \quad \forall x \in U \]
where the action on $V_{l}^{(l)}$ is according to (3.5). In (3.14, 3.15) we set $M \otimes N = \oplus_\xi \prod_\mu M_\mu \otimes N_{\xi-\mu}$. Henceforth we will not bother writing $\otimes$. Note that (3.13) implies
\[ (d \otimes \text{id})\hat{\Phi}^{(V)}_{\lambda}(\mu)(z) = z\frac{d}{dz}\hat{\Phi}^{(V)}_{\lambda}(\mu)(z) \]  
(3.16)
and likewise for $\hat{\Phi}^{(V)}_{\lambda}(\mu)(z)$. By abuse of notation we shall also write $\Phi^{(V)}_{\lambda}(\mu)$ (1) to mean the intertwiner of $U'$-modules $\sum (\hat{\Phi}^{(V)}_{\lambda}(\mu))_{j,n} \otimes v_j^{(l)} : V(\lambda) \to \hat{V}(\mu) \otimes V_{l}^{(l)}$ with $\hat{V}(\mu) = \prod_\nu V(\mu)_\nu$. In [2] we called (3.14) a VO of type I, (3.15) of type II.

For a VO (3.11) we define its ‘leading term’ $v_\ell$ by
\[ \hat{\Phi}^{(V)}_{\lambda}(\mu)(\lambda) = |\mu\rangle \otimes v_\ell + \cdots, \]  
(3.17)
where $\cdots$ means terms of the form $|u\rangle \otimes v$ with $|u\rangle \in \bigoplus_{\xi \neq \mu} V(\mu)_\xi$. Set
\[ V^{(l)}(\mu) = \{ v \in V^{(l)} \mid \lambda \equiv \mu + \text{wt } v \mod \delta, \, e_i^{(\lambda,\mu) + 1} v = 0 \quad i = 0, 1 \}. \]
We have the following criterion about the existence of VOs.
Proposition 3.1 [5]. Associating $\Phi^{V^{(i)}_\lambda}(z)$ with its leading term we have an isomorphism of vector spaces
\[
\{ \text{VOs } V(\lambda) \rightarrow V(\mu) \otimes V^{(i)}_\lambda \} \sim V^{(i)\mu}_\lambda.
\]
Similar statement holds for the VOs (3.12).

Let $\lambda = m\Lambda_0 + n\Lambda_1$ ($m + n = k$), and take $l = k$ for (3.11), $l = 1$ for (3.12). Then up to scalar the nontrivial VOs are the following ones:
\[
\Phi^{V^{(k)}_\lambda}|\lambda\rangle = |\mu\rangle \otimes v^{(k)}_m + \cdots, \quad \text{if } \mu = \sigma(\lambda) = n\Lambda_0 + m\Lambda_1, \quad (3.18)
\]
\[
\Phi^{V^{(i)}_\lambda}|\lambda\rangle = v^\pm \otimes |\mu\rangle + \cdots, \quad \text{if } \mu = \lambda_{\pm} = (m \mp 1)\Lambda_0 + (n \pm 1)\Lambda_1 \in P_+ . \quad (3.19)
\]
Note that in (3.18) both $\mu$ and $v^{(k)}_m$ are uniquely determined from $\lambda$; this is a general feature of perfect modules [4]. Henceforth we shall fix the normalization (3.18), (3.19) for the VOs.

The isomorphism (3.6) gives rise to the VOs
\[
\Phi^{(\lambda)}_\lambda^{(\sigma)(V^{(k)}) = \pm 1}(z) = \alpha_\pm^{(k)}(\lambda) (\text{id} \otimes C^{(k)}_{\pm}) \Phi^{V^{(i)}_\lambda}(zq^{\mp 2}),
\]
\[
\Phi^{(\lambda)}_\lambda^{(\sigma)(V^{(i)}) = \pm 1}(z) = \alpha_\pm^{(1)}(\lambda, \mu)(C^{(1)}_{\pm} \otimes \text{id}) \Phi^{(\lambda)}_\lambda^{(\sigma)(V^{(i)})}(zq^{\mp 2}),
\]
where the normalization constants $\alpha_\pm^{(k)}(\lambda), \alpha_\pm^{(1)}(\lambda, \mu)$ are so chosen that
\[
\Phi^{(\lambda)}_\lambda^{(\sigma)(V^{(k)}) = \pm 1}(z)|\lambda\rangle = |\mu\rangle \otimes v^{(k)}_m + \cdots \quad \text{if } \mu = \sigma(\lambda) = n\Lambda_0 + m\Lambda_1,
\]
\[
\Phi^{(\lambda)}_\lambda^{(\sigma)(V^{(i)}) = \pm 1}(z)|\lambda\rangle = v^\pm \otimes |\mu\rangle + \cdots \quad \text{if } \mu = (m \mp 1)\Lambda_0 + (n \pm 1)\Lambda_1 \in P_+ .
\]
For $k = 1$ we have
\[
\alpha_\pm^{(1)}(\lambda, \mu) = \begin{cases} -q^{\mp 1} & \text{if } \mu = (m - 1)\Lambda_0 + (n + 1)\Lambda_1, \\ 1 & \text{if } \mu = (m + 1)\Lambda_0 + (n - 1)\Lambda_1. \end{cases}
\]
The fractional powers are included by the same rule, e.g.
\[
\Phi^{(\lambda)}_\lambda^{(\sigma)(V^{(i)}) = \pm 1}(z) = z^{\Delta_\mu - \Delta_\lambda} \times \Phi^{(\lambda)}_\lambda^{(\sigma)(V^{(i)}) = \pm 1}(z).
\]

3.4 Explicit formulas for two point functions For our subsequent discussions, we need the following types of correlation functions:
\[
\langle \nu|\Phi^{V^{(k)}_\mu}(z_2)\Phi^{V^{(k)}_\lambda}(z_1)|\lambda\rangle, \quad (3.22)
\]
\[
\langle \nu|\Phi^{V^{(i)}_\mu}(z_2)\Phi^{V^{(i)}_\lambda}(z_1)|\lambda\rangle, \quad (3.23)
\]
\[
\langle \nu|\Phi^{V^{(k)}_\mu}(z_2)\Phi^{V^{(i)}_\lambda}(z_1)|\lambda\rangle, \quad (3.24)
\]
\[
\langle \nu|\Phi^{V^{(i)}_\mu}(z_2)\Phi^{V^{(i)}_\lambda}(z_1)|\lambda\rangle. \quad (3.25)
\]
See Appendix 1, (A1.13)–(A1.16) as for the convention. Let
\[
z_1^{\Delta_\mu - \Delta_\lambda} z_2^{\Delta_\nu - \Delta_\sigma} \Psi(z_1/z_2)
\]
be one of (3.22)–(3.25). Thus, for instance, \( \Psi(z_1/z_2) = \langle \Phi^{\lambda; \nu}_{\sigma; \lambda}(z_2) \Phi^{\nu; \lambda\nu}_{\lambda; \nu}(z_1) \rangle \) for (3.22) (we omit writing \((\nu, |\lambda|)\)). Then \( \Psi(z) \) is a formal power series in \( z \) taking values in \( \Phi^{(k)}_{1} \otimes V^{(l_1)}_{2} \) with appropriate \( l_1, l_2 \), and satisfies the \( q \)-KZ equation (A1.18) where \( R^+(z) = R_+^{\nu\lambda}(z) \) is given in (3.9, 3.10). Thanks to the perfectness of \( V^{(k)} \), in the cases (3.22)–(3.24) \( \Psi(z) \) can be determined up to scalar by using Proposition A1.4. Hence the \( q \)-KZ equation reduces to a scalar equation. In the case (3.25) the \( q \)-KZ equation can be solved by using basic hypergeometric functions (see Appendix 2).

We list the formulas for \( \Psi(z) \) below. Fix

\[
\lambda = m\Lambda_0 + n\Lambda_1, \quad m, n \in \mathbb{Z}_{\geq 0}, \quad m + n = k
\]

and set

\[
\lambda_\pm = \lambda \pm (\Lambda_1 - \Lambda_0), \quad \lambda_\pm = \lambda \pm 2(\Lambda_1 - \Lambda_0), \quad s = \frac{1}{2(k+2)}, \quad p = q^{2k+4},
\]

\[
\xi(z; a, b) = \frac{(az; p, q^4)_\infty(a^{-1}bz; p, q^4)_\infty}{(q^2az; p, q^4)_\infty(q^{-2}a^{-1}bz; p, q^4)_\infty}, \quad (z; p, q)_\infty = \prod_{i,j \geq 0} (1 - zp^i q^j).
\]

(3.26)

The case (3.22). In this case \( \mu = \sigma(\lambda) \) and \( \nu = \lambda \). We have

\[
\langle \Phi^{\lambda; \nu}_{\sigma(\lambda)}(z_2) \Phi^{\sigma(\lambda)\nu}_{\lambda}(z_1) \rangle = \psi(z_1/z_2) \sum_{j=0}^{k} a_j(z_1/z_2) v_{k-j}^{(k)} \otimes v_{j}^{(k)},
\]

(3.27)

\[
\psi(z) = \frac{(q^{2(k+2)}z; q^4)\infty}{(q^4z; q^4)\infty},
\]

\[
a_j(z) = (-1)^{n+j} q^{j(k+n+1-j)} \sum_{\max(0, j-m) \leq i \leq \min(n, j)} \binom{n}{i} \binom{m}{j-i} q^{n-(k+2)i} z^{n-i}.
\]

The case (3.23). \( \Psi(z) \) is non-trivial when \( \mu \) and \( \nu \) are the following.

\[
\mu = \sigma(\lambda) = n\Lambda_0 + m\Lambda_1, \quad \nu = \sigma(\lambda_\pm) = (n \pm 1)\Lambda_0 + (m \mp 1)\Lambda_1.
\]

The solutions are

\[
\langle \Phi^{\lambda; \nu}_{\sigma(\lambda)}(z_2) \Phi^{\sigma(\lambda)\nu}_{\lambda}(z_1) \rangle = \frac{(q^5 z_1/z_2; q^4)\infty}{(q^4 z_1/z_2; q^4)\infty} \times \begin{cases}
    (v_{m}^{(k)} \otimes v_+ - q v_{m-1}^{(k)} \otimes v_-) & \text{for +,} \\
    (v_{m}^{(k)} \otimes v_- - q z_1/z_2 v_{m+1}^{(k)} \otimes v_+) & \text{for -}.
\end{cases}
\]

(3.28)

The case (3.24). \( \Psi(z) \) is non-trivial when \( \mu \) and \( \nu \) are the following.

\[
\mu = \lambda_\pm = (m \mp 1)\Lambda_0 + (n \pm 1)\Lambda_1, \quad \nu = \sigma(\lambda_\pm) = (n \pm 1)\Lambda_0 + (m \mp 1)\Lambda_1, \quad (3.29)\]

\[
\mu = \lambda_\pm = (m \mp 1)\Lambda_0 + (n \pm 1)\Lambda_1, \quad \nu = \sigma(\lambda_\pm) = (n \pm 1)\Lambda_0 + (m \mp 1)\Lambda_1, \quad (3.29)\]
The solutions are

\[
\langle \Phi_{\lambda \pm}^{(\lambda \pm)} V_i^{(k)} (z_2) \Phi_{\lambda}^{V_1^{(1)} \lambda \pm} (z_1) \rangle \\
= \frac{(qz_1/z_2; q^4)_\infty}{(q^{-1}z_1/z_2; q^4)_\infty} \times \left\{ \begin{array}{ll} 
(v_- \otimes v_{m-1}^{(k)} - q^{-1}v_+ \otimes v_m^{(k)}) & \text{for } +, \\
(v_+ \otimes v_{m+1}^{(k)} - q^{-1}(z_1/z_2)v_- \otimes v_m^{(k)}) & \text{for } -. 
\end{array} \right.
\]

The case (3.25). \( \Psi(z) \) is non-trivial when \( \mu \) and \( \nu \) are the following.

\[
\mu = \lambda_\pm = (m \pm 1)\Lambda_0 + (n \pm 1)\Lambda_1, \quad \nu = \lambda, \quad (3.31) \\
\mu = \lambda_\pm = (m \pm 1)\Lambda_0 + (n \pm 1)\Lambda_1, \quad \nu = \lambda^\pm = (m \pm 2)\Lambda_0 + (n \pm 2)\Lambda_1, \quad (3.32)
\]

The solutions are as follows.

\[
\langle \Phi_{\mu}^{V_1^{(1)} \nu} (z_2) \Phi_{\lambda}^{V_1^{(1)} \mu} (z_1) \rangle / \xi(z_1/z_2; 1, q^4) \\
= \phi \left( \frac{(2n + 4)s}{2n + 2s}; \frac{q^{-2}z_1/z_2}{(2n + 2)s} \right) v_- \otimes v_+ \\
- \frac{1 - q^2}{q(1 - q^{2n+2})} \phi \left( \frac{1 + 2s}{1 + (2n + 2)s}; \frac{q^{-2}z_1/z_2}{2n + 2s} \right) v_+ \otimes v_- \quad \text{for (3.31) with } +, \\
= \phi \left( \frac{1 - 2ns}{1 - (2n + 2)s}; \frac{q^{-2}z_1/z_2}{2n + 2s} \right) v_+ \otimes v_- \\
- \frac{1 - q^2}{q(1 - q^{2n+2})} \frac{z_1}{z_2} \times \phi \left( \frac{1 + 2s}{2 - (2n + 2)s}; \frac{q^{-2}z_1/z_2}{2n + 2s} \right) v_- \otimes v_+ \quad \text{for (3.31) with } -, \\
= v_+ \otimes v_+ \quad \text{for (3.32),}
\]

where \( \phi \left( a, b; c ; z \right) = 2\phi_1 \left( a^p, b^p; c, z \right) \) (see Appendix 2.).

3.5 Commutation relations \( \) In this subsection we shall give the commutation relations of the various vertex operators.

The case (3.22). \( \) We modify the scalar factor of the \( R \) matrix and define

\[
R_{kk}(z) = r_{kk}(z) \mathcal{T}_{kk}(z), \quad r_{kk}(z) = z^{-k/2}(q^2 z; q^4)_\infty (q^{2k+2} z^{-1}; q^4)_\infty. 
\]

Then we have

\[
R_{kk}(z_1/z_2) \Phi_{\sigma(\lambda)}^{\alpha \gamma(k)} (z_1) \Phi_{\lambda}^{\sigma \gamma(k)} (z_2) = \Phi_{\sigma(\lambda)}^{\alpha \gamma(k)} (z_2) \Phi_{\lambda}^{\sigma \gamma(k)} (z_1). \quad (3.35)
\]

In fact, this equality can be verified for the matrix element \( \langle \lambda | \cdots | \lambda \rangle \) by rewriting the \( q \)-KZ equation. The equality for arbitrary matrix elements follows automatically since both sides are intertwiners (the argument is the same as in the proof of Proposition 6.1 of [2]).
The case (3.23), (3.24). Using the formulas (3.28), (3.30) and arguing similarly as above, we obtain the following commutation relations.

\[
\Phi_{\sigma(\lambda)}^{V(1)\sigma(\lambda^{\pm})}(z_1)\Phi_{\lambda}^{(\sigma(\lambda)VV^t)}(z_2) = \tau(z_1/z_2)^{-1}\Phi_{\lambda^{\pm}}^{\sigma(\lambda^{\pm})V^t}(z_2)\Phi_{\lambda}^{(1)\lambda^{\pm}}(z_1),
\]

(3.36)

\[
\Phi_{\sigma(\lambda)}^{V(1)\sigma(\lambda^{\pm})}(z_1)\Phi_{\lambda}^{(\sigma(\lambda)VV^t)}(z_2) = \tau(z_1/z_2)\Phi_{\lambda^{\pm}}^{\sigma(\lambda)VV^t}(z_2)\Phi_{\lambda}^{(1)\lambda^{\pm}}(z_1).
\]

(3.37)

Here

\[
\tau(z) = z^{-1/2}\frac{\Theta_{\mu}(qz)}{\Theta_{\mu^t}(qz^{-1})}
\]

(3.38)

with \(\Theta_{\mu}(z)\) being defined in Appendix 2, (A2.2).

The case (3.25). This case is more complicated compared to the previous ones. To derive the commutation relations we invoke the connection formula (A2.8) for the basic hypergeometric functions.

Let \(R_{VV}(z) = -R_{11}(z)\) be defined in (3.34) with \(k = 1\) and set

\[
R_{VV^t}(z) = (id \otimes C_-)R_{VV}(zq^{-2})(id \otimes C_-)^{-1},
\]

\[
R_{V^tV}(z) = (C_- \otimes C_-)R_{VV}(z)(C_- \otimes C_-)^{-1}.
\]

(Our \(R_{VV}(z)\) and \(r_{11}(z)\) are \(R_{VV}(z)\) and \(-r_0(z)\) in [2] respectively). The inversion relations take the form

\[
R_{VV}(z)PR_{VV}(z^{-1})P = 1,
\]

\[
(R_{VV}(z^{-1})^{-1} = -R_{VV^t}(z).
\]

Retaining the notations of (3.33) we have the following commutation relations.

\[
\Phi_{\mu}^{V(1)\nu}(z_1)\Phi_{\lambda}^{V(1)\mu}(z_2) = R_{VV}(z_1/z_2)\sum_{\mu'}\Phi_{\mu'}^{V(1)\nu}(z_2)\Phi_{\lambda}^{V(1)\mu'}(z_1)W\left(\frac{\lambda}{\mu'}\frac{\mu}{\nu}\frac{z_1}{z_2}\right),
\]

(3.39)

Here

\[
W\left(\frac{\lambda}{\mu'}\frac{\mu}{\nu}\frac{z}{\lambda}\right) = -z^{\Delta + \Delta_v - \Delta_v - \Delta_v - 1/2}s^2(z; 1, pq^2)\times W\left(\frac{\lambda}{\mu'}\frac{\mu}{\nu}\frac{z}{\lambda}\right),
\]

\[
\hat{W}\left(\frac{\lambda}{\mu'}\frac{\mu}{\nu}\frac{z}{\lambda}\right) = \frac{\Theta_{\mu}(pq^2)}{\Theta_{\mu}(pq^{-2n+2})},
\]

\[
\hat{W}\left(\frac{\lambda}{\mu'}\frac{\mu}{\nu}\frac{z}{\lambda}\right) = q^{-1} \times \frac{\Gamma_p(2n+2s)\Gamma_p(2n+2s)}{\Gamma_p(2n+4s)\Gamma_p(2n+2s)} \times \Theta_{\mu}(pq^2),
\]

\[
\hat{W}\left(\frac{\lambda}{\mu'}\frac{\mu}{\nu}\frac{z}{\lambda}\right) = q^{-1} \times \frac{\Gamma_p(1-2n+2s)\Gamma_p(1-2n+2s)}{\Gamma_p(1-2n+4s)\Gamma_p(1-2n+2s)} \times \Theta_{\mu}(pq^2),
\]

\[
\hat{W}\left(\frac{\lambda}{\mu'}\frac{\mu}{\nu}\frac{z}{\lambda}\right) = 1,
\]

\[
\hat{W}\left(\frac{\lambda}{\mu'}\frac{\mu}{\nu}\frac{z}{\lambda}\right) = 0 \text{ otherwise,}
\]
and $\Gamma_\mu(z)$ denotes the $q$-gamma function (A2.2).

The same relations (3.39) hold if we replace $\Phi_{\lambda}^{V\mu}$ by $\Phi_{\lambda}^{V*a-1\mu}$ and $R_{VV}(z)$ by $R_{V\ast V\ast}(z)$.

We have further

$$
\Phi_{\lambda}^{V(1)}(z_1)\Phi_{\lambda}^{V(1)* a-1\mu}(z_2) = R_{V\ast V\ast}(z_1/z_2) \sum_{\mu'} \Phi_{\lambda}^{V(1)* a-1\mu'}(z_2)\Phi_{\lambda}^{V(1)\mu'}(z_1)
$$

$$
\times (-q)^{\pm\delta_{\lambda,\mu}\delta_{\mu',\nu}} W \left( \lambda, \mu' \left| \frac{z_1}{q^2 z_2} \right. \right).
$$

(3.40)

In the neighborhood of $|z_1/z_2| = 1$ both side of (3.39) are holomorphic, while those of (3.40) with $\lambda = \nu$ have a simple pole at $z_1 = z_2$. The residues of them are given by the following formulas.

$$
\text{Res}_{z_1=z_2} \Phi_{\lambda \pm}^{V(1)}(z_1)\Phi_{\lambda \pm}^{V(1)* a-1}\lambda \pm(z_2) d \left( \frac{z_1}{z_2} \right) = (v_+ \otimes v_+^* + v_- \otimes v_-^*) \otimes g^{\pm}_\lambda \text{id}_{V(\lambda)}.
$$

(3.41)

where

$$
g^{+}_\lambda = q^{-1}\xi(q^2; 1, q^4)(q^2; p)_{\infty}(q^{2n+4}; p)_{\infty},
$$

(3.42)

$$
g^{-}_\lambda = \xi(q^2; 1, q^4)(q^2; p)_{\infty}(pq^{-2n}; p)_{\infty},
$$

(3.43)

and $\xi(z; a, b)$ is defined in (3.26).
§4. Vertex models

4.1 Space of states  
We now turn to the discussion of the vertex model associated with the spin $k/2$ representation $V^{(k)}$ of $U'$.  
Let $\mathcal{H}_k = \bigoplus_{\lambda \in P_0} V(\lambda)$. We set  
\begin{align}
\mathcal{F} &= \text{End}(\mathcal{H}_k) = \bigoplus_{\lambda, \mu \in P_0} \mathcal{F}_{\lambda\mu}, \quad (4.1) \\
\mathcal{F}_{\lambda\mu} &= \text{Hom}(V(\mu), V(\lambda)) \cong V(\lambda) \otimes V(\mu)^* a. \quad (4.2)
\end{align}

Strictly speaking we consider the completion of these spaces in the topology of formal power series in $q$. This is necessary in order to accommodate the action of VO's which produce infinite sums. This point is discussed in Sect. 7 of [2]. We shall not go into the details here. The symbols $\otimes$, etc. are to be understood appropriately.  
We take (4.1) to be the mathematical definition of the space of states of the vertex model. As discussed in the Introduction it should replace the usual naïve picture of the infinite tensor product space $\cdots \otimes V^{(k)} \otimes V^{(k)} \otimes V^{(k)} \otimes \cdots$, taking into account the boundary conditions for the states. We shall come to such an interpretation shortly.  
The left action of $U$ on $\mathcal{F}_{\lambda\mu}$ is given by  
$$xf = \sum x(1) \circ f \circ a(x(2))$$
where $\Delta(x) = \sum x(1) \otimes x(2)$ for $x \in U$. One can also endow the same underlying space with a structure of right $U$-module by setting  
$$fx = \sum a^{-1}(x(2)) \circ g \circ x(1).$$
We use the letter $\mathcal{F}_{\lambda\mu}^r$ to denote this right $U$-module. In the case $\mu = \lambda$, $\mathcal{F}_{\lambda\lambda}$ has the unique canonical element, i.e. the identity $\text{id}_{V(\lambda)}$. We call it the vacuum and denote it by $|\text{vac}\rangle_{\lambda} \in \mathcal{F}_{\lambda\lambda}, \langle \text{vac}| \in \mathcal{F}_{\lambda\lambda}^r$. There is a natural inner product [2]  
$$\langle f | g \rangle = \frac{\text{tr}_{V(\lambda)}(q^{-2\rho}fg)}{\text{tr}_{V(\lambda)}(q^{-2\rho})} \quad f \in \mathcal{F}_{\lambda\mu}^r, \quad g \in \mathcal{F}_{\mu\lambda}. \quad (4.3)$$
It enjoys the property  
$$\langle fx | g \rangle = \langle f | xg \rangle \quad \forall x \in U.$$

4.2 Local structure  
Consider now the VO of type I, $\tilde{\Phi}_\lambda^{\sigma(\lambda)V^{(k)}}(z)$ and $\tilde{\Phi}_\lambda^{\sigma(\lambda)V^{(k)}}(z)$.  
We normalize the latter as  
$$\tilde{\Phi}_\lambda^{\sigma(\lambda)V^{(k)}}(z) = gn \times \text{id}_{V^{(k)}}, \quad \lambda = m\Lambda_0 + n\Lambda_1. \quad (4.4)$$

Proposition 4.1.  
\begin{align}
\tilde{\Phi}_\lambda^{\sigma(\lambda)V^{(k)}}(z) \circ \tilde{\Phi}_\lambda^{\sigma(\lambda)V^{(k)}}(z) &= gn \times \text{id}_{V^{(k)}} \quad (4.4) \\
\tilde{\Phi}_\lambda^{\sigma(\lambda)V^{(k)}}(z) \circ \tilde{\Phi}_\lambda^{\sigma(\lambda)V^{(k)}}(z) &= gn \times \text{id}_{V^{(k)}}, \quad (4.4)
\end{align}
where
\[ g_{mL_0+nL_1} = q^{mn} \left[ k \right] \frac{(q^{2(k+1)}; q^4)^\infty}{(q^2; q^4)^\infty} \, \, \, k = m + n. \] (4.6)

Proof. It is known [5] that (4.4) holds with some scalar \( g_\lambda \). This scalar can be calculated using the two point function (3.27) and (3.6). Calculating similarly we find
\[ \langle \Phi_\lambda^{\sigma(\lambda)V(z)}(z) \Phi_{\sigma(\lambda)}^{\mu\lambda}(z) \rangle = g_\lambda \sum_{j=0}^{k} v_j^{(k)} \otimes v_j^{(k)*}. \]

Noting that the right hand side belongs to the trivial representation, we easily see that
\[ g_\lambda \# \Phi_{\sigma(\lambda)}^{\mu\lambda}(z) = g_\lambda \# \Phi_{\sigma(\lambda)}^{\mu\lambda}(z) \]

This is equivalent to (4.5). \( \Box \)

The results (4.4, 4.5) tell that \( \Phi_\lambda^{\sigma(\lambda)V(z)} \) gives an isomorphism (on the spaces being completed properly). Iterating this we have for any \( n \) an isomorphism
\[ \Phi_\lambda^{(n)} : V(\lambda) \sim V(\sigma^n(\lambda)) \otimes (V(k))^{\otimes n}. \] (4.7)

This makes it possible to realize the space \( \mathcal{F}_{\lambda\mu} \) in a way which incorporates the ‘local structure’ \( V(k) \otimes \cdots \otimes V(k) \) in between:
\[ \mathcal{F}_{\lambda\mu} \simeq V(\lambda) \otimes V(\mu)^{\ast a} \sim V(\sigma^n(\lambda)) \otimes (V(k))^{\otimes n} \otimes V(\mu)^{\ast a}. \] (4.8)

One may do an analogous construction for the right half \( V(\mu)^{\ast a} \). The weight \( \lambda \) (resp. \( \mu \)) in \( \mathcal{F}_{\lambda\mu} \) signifies the boundary condition to the left (resp. right) spatial infinity of the lattice.

4.3 Transfer matrix Let us define the row transfer matrix of the vertex model in our framework
\[ T(z) = T_{\lambda\mu}^{\sigma(\lambda)\sigma(\mu)}(z) : \mathcal{F}_{\lambda\mu} \longrightarrow \mathcal{F}_{\sigma(\lambda)\sigma(\mu)}. \] (4.9)

By definition it is the composition of
\[ V(\lambda) \otimes V(\mu)^{\ast a} \xrightarrow{\Phi(z) \otimes \text{id}} V(\sigma(\lambda)) \otimes V(\mu)^{\ast a} \xrightarrow{\text{id} \otimes \Phi^{\ast}(z)} V(\sigma(\lambda)) \otimes V(\sigma(\mu))^{\ast a} \]

where \( \Phi(z) = \Phi_\lambda^{\sigma(\lambda)V(z)} \) and \( \Phi^{\ast}(z) \) is the transpose of \( \Phi_{\sigma(\mu)}^{\muV^{\ast-1}}(z) \). In terms of the weight components
\[ \Phi_\lambda^{\sigma(\lambda)V}(z) = \sum_j \Phi_j(z) \otimes v_j^{(k)}, \]
\[ \Phi_{\sigma(\mu)}^{\muV^{\ast-1}}(z) = \sum_j \Phi^*_j(z) \otimes v_j^{(k)*}, \]
we have
\[ T(z)(f) = \sum_j \Phi_j(z) \otimes f \otimes \Phi^*_j(z) \quad \text{for} \quad f \in \text{Hom}(V(\mu), V(\lambda)). \] (4.10)

We call \( T = g_\lambda^{-1} T(1) \) the translation operator where \( g_\lambda \) is given in (4.6) (see the proposition below). From the property (3.16) we find that \( [d, T(z)] = -z d \# T(z) \), and hence
\[ T(z) = z^{-d} \circ T(1) \circ z^d. \]
The following states that the vacuum is invariant under the transfer matrix and the translation in particular.
Proposition 4.2.

\[ T^{\sigma(\lambda)\sigma(\lambda)}(z)\langle \text{vac} \rangle_\lambda = g_\lambda \langle \text{vac} \rangle_{\sigma(\lambda)}. \]

**Proof.** From the definition of the vacuum and the transfer matrix the assertion is equivalent to

\[ \sum_j \Phi_j(z) \circ \Phi^*_j(z) = g_\lambda \times \text{id} \]

in the notation of (4.10). This can be shown similarly as in the proof of Proposition 4.1.

Let us consider the action of \( T(z) \) in the picture (4.8). We set

\[ T^{(n)}(z) = (\Phi^{(n)}_{\sigma(\lambda)} \otimes \text{id}) \circ T(z) \circ (\Phi^{(n)}_{\sigma(\lambda)} \otimes \text{id})^{-1}, \]

\[ T^{(n)}(z) : V(\sigma^n(\lambda)) \otimes V^{(k)\otimes n} \otimes V(\mu)^a \rightarrow V(\sigma^{n+1}(\lambda)) \otimes V^{(k)\otimes n} \otimes V(\mu)^a \] (4.11)

Denote by \( T(z) \) the monodromy matrix on a lattice of horizontal length \( n \)

\[ T(z) = R^{0n}(z) \cdots R^{02}(z)R^{01}(z) \in \text{End}(V_0^{(k)} \otimes V_1^{(k)} \otimes \cdots \otimes V_n^{(k)}) \]

where \( R^{ij}(z) \) signifies the \( R \) matrix \( R(z) = R_{kk}(z) \in \text{End}(V^{(k)} \otimes V^{(k)}) \) (3.34) acting on the \( i \)-th and \( j \)-th components. Let \( T(z)^{j'}_j \) be its matrix elements given by

\[ T(z)(v_j^{(k)} \otimes w) = \sum_j v_j^{(k)} \otimes T(z)^{j'}_j w. \]

Usual definition of the row transfer matrix (in the periodic boundary condition) is

\[ \sum_j T(z)^{j'}_j. \] The following motivates our definition of the row transfer matrix (4.9).

**Proposition 4.3.** In the picture (4.8) the row transfer matrix (4.9) acts as follows:

\[ T^{(n)}(z)(u \otimes v \otimes w) = \sum_j \Phi_j(z)u \otimes T(z)^{j'}_j v \otimes \Phi^*_j(z)w \]

\[ u \otimes v \otimes w \in V(\sigma^n(\lambda)) \otimes V^{(k)\otimes n} \otimes V(\mu)^a. \]

Here \( \Phi^*_j(z) \) signifies the transpose of \( \Phi^*_j(z) \).

**Proof.** Using the commutation relation (3.35) repeatedly we have

\[ T(z) \Phi_{\sigma^n(\lambda)}^{\sigma^{n+1}V_0}(z) \Phi_{\lambda}^{(n)} = \Phi_{\sigma(\lambda)}^{(n)} \Phi^\sigma_{\lambda}V_0(z), \]

or in components

\[ \Phi_{\sigma(\lambda)}^{(n)} \Phi_j(z)(\Phi_{\lambda}^{(n)})^{-1} = \sum_j \Phi_j(z)T(z)^{j'}_j. \]

The assertion follows from this, (4.10) and (4.11).
4.4 Creation and annihilation operators  In the same way as [2], the particles are created over the vacuum by the type II VOs. Let us define their components by

\[ \Phi^{(1)}_{\lambda}^\mu(z) = v_+ \otimes \Phi_{\lambda,+}^\mu(z) + v_- \otimes \Phi_{\lambda,-}^\mu(z), \]

(4.12)

\[ \Phi^{(1)+1-\mu}(z) = v_+^* \otimes \Phi_{\lambda,+}^{*\mu}(z) + v_-^* \otimes \Phi_{\lambda,-}^{*\mu}(z). \]

(4.13)

From (3.21) we have the relation

\[ \Phi_{\lambda,-}^\mu(z) = (-q)^{-1} \Phi_{\lambda,+}^{*-\mu}(z q^{-2}), \quad \Phi_{\lambda,+}^\mu(z) = \Phi_{\lambda,-}^{*\mu}(z q^{-2}), \]

\[ \Phi_{\lambda,-}^{*\mu}(z) = \Phi_{\lambda,+}^{(-1)}(z q^{-2}), \quad \Phi_{\lambda,+}^{*\mu}(z) = (-q) \Phi_{\lambda,-}^{*\mu}(z q^{-2}). \]

Define the creation operator \( \varphi_{\lambda,\nu}^\mu(z) \) on \( \mathcal{F}_{\lambda \mu} = \text{Hom}(\mathcal{V}(\mu), \mathcal{V}(\nu)) \) by

\[ \varphi_{\lambda,\nu}^\mu(z) : \mathcal{F}_{\lambda \mu} \to \mathcal{F}_{\lambda \nu}, \quad f \mapsto \Phi_{\lambda,\nu}^\mu(z) \circ f. \]

Likewise define the annihilation operator \( \varphi_{\lambda,\nu}^{*\mu}(z) \) to be the adjoint (with respect to the inner product (4.3)) of

\[ \varphi_{\lambda,\nu}^{*\mu}(z) : \mathcal{F}_{\lambda \nu} \to \mathcal{F}_{\lambda \mu}, \quad f \mapsto f \circ \Phi_{\lambda,\nu}^{*\mu}(z). \]

In both cases the quasi-momentum \( z \) is supposed to be on the unit circle \( |z| = 1 \).

From (3.39) and (3.40) we obtain the following commutation relations of creation and annihilation operators.

\[ \varphi_{\mu,\nu,1}(z_1) \varphi_{\lambda,\nu,2}(z_2) = \sum_{\mu',\nu',1} R_{V \nu,1}(z_1/z_2) \varphi_{\mu',\nu',2}(z_2) \varphi_{\lambda,\nu,1}(z_1) W \left( \lambda_{\mu',\nu'}, \frac{z_1}{z_2} \right), \]

(4.14)

\[ \varphi_{\mu,\nu,1}(z_1) \varphi_{\lambda,\nu,2}(z_2) = \sum_{\mu',\nu',1} R_{V \nu,1}(z_1/z_2) \varphi_{\mu',\nu',2}(z_2) \varphi_{\lambda,\nu,1}(z_1) W \left( \lambda_{\mu',\nu'}, \frac{z_1}{z_2} \right), \]

(4.15)

\[ \varphi_{\mu,\nu,1}(z_1) \varphi_{\lambda,\nu,2}(z_2) = \sum_{\mu',\nu',1} R_{V \nu,1}(z_1/z_2) \varphi_{\mu',\nu',2}(z_2) \varphi_{\lambda,\nu,1}(z_1) \]

\[ \times (-q)^{\pm} \delta_{\lambda,\nu,\mu,\nu'} W \left( \lambda_{\mu',\nu'}, \frac{z_1}{q^2 z_2} \right) + g_{\mu,\nu,\delta_{1,\nu}} \delta_{1,\nu} \delta(z_1/z_2). \]

(4.16)

Here we have set \( R_{V \nu,1}(z_1) = \sum_{\nu,1} v_{\nu,1} \otimes v_{\nu,1} R_{V \nu,1}(z_1) \), \( \delta(z) = \sum_{n \in \mathbb{Z}} z^n \) in (4.16) arises because of the pole of (3.40) at \( z_1 = z_2 \).

To find the commutation relation with the transfer matrix \( T(z) \) we use (3.37) to get

\[ T_{\lambda,\mu}^{\sigma(\lambda) \sigma(\mu)}(z_2) \Phi_{\lambda,\nu,1}^\mu(z_1) = \tau(z_1/z_2)^{-1} \Phi_{\sigma(\lambda),\nu}^{\sigma(\lambda) \sigma(\mu)}(z_1) T_{\lambda,\mu}^{\sigma(\lambda) \sigma(\mu)}(z_2), \]

(4.17)

where \( \tau(z) \) is given in (3.38). Let \( \tilde{T}(z) = T_{\lambda,\mu}^{\sigma(\lambda) \sigma(\mu)}(z) T_{\lambda,\mu}^{\sigma(\lambda) \sigma(\mu)}(z) \) be the transfer matrix for two lattice steps which does not change the boundary conditions. Then (4.17) implies

\[ \tilde{T}(z_2) \varphi_{\lambda,\nu}^\mu(z_1) \tilde{T}(z_2)^{-1} = \tau(z_1/z_2)^2 \varphi_{\lambda,\nu}^\mu(z_1), \]

\[ \tilde{T}(z_2) \varphi_{\lambda,\nu}^{*\mu}(z_1) \tilde{T}(z_2)^{-1} = \tau(z_1/z_2)^{-2} \varphi_{\lambda,\nu}^{*\mu}(z_1). \]
Note that the function $\tau(z)$ is independent of the level $k$ and therefore is the same as (7.11) in [2]. The energy can be obtained by differentiating $\log\tau(z)$. Hence our formulas agree with Sogo's result [10] found through the Bethe Ansatz method.

4.5 Local operators The local operators can be defined making use of the realization (4.8). For a linear map $L \in \text{End}(V^{(k)\otimes n})$ we define its action on $F_{\lambda\mu}$ by $f \mapsto L_\lambda \circ f$ where we set [11]

$$L_\lambda = \Phi_\lambda^{(n)-1}(\text{id} \otimes L)\Phi_\lambda^{(n)}$$

in the notation of (4.7). Using (3.36, 3.37) we find that the components of the creation operator (4.13) commutes with $L_\lambda$ in the sense

$$L_\mu \Phi_\mu^{(n)}(z) = \Phi_\mu^{(n)}(z)L_\lambda.$$

For simplicity of notation, we shall write in the sequel

$$V^* = V^{(1)*a^{-1}}, \quad R(z) = R_{kk}(z), \quad R^*(z) = R_{V^*V^*}(z),$$

$$\text{Tr}(\cdot) = \text{tr}(\cdot)/\text{tr}(q^{-2\rho}).$$

Consider the $n$-particle states created from the vacuum

$$\varphi^{a_{n}}_{\sigma_{n},(\lambda),\epsilon_{n}}(z_{n}) = \varphi^{a_{n}}_{\sigma_{n^{-1}}(\lambda),\epsilon_{n}}(z_{n}) \Phi^{a_{n}}_{\lambda,\epsilon_{n}}(z_{1})$$

$$= \Phi_{\sigma_{n^{-1}}(\lambda)}(z_{n}) \cdots \Phi^{a_{n}}_{\lambda,\epsilon_{n}}(z_{1})$$

$$\in \text{Hom}(V(\lambda), V(\sigma_{n}(\lambda)))$$

(4.19)

where $\epsilon_{1}, \cdot \cdot \cdot, \epsilon_{n} \in \{\pm\}$. A form factor of $L \in \text{End}(V^{(k)\otimes n})$ is defined by (we put $\lambda_{n} = \lambda_{0}$)

$$F_{\lambda_{n},\cdot \cdot \cdot,\lambda_{0}}(z_{n}, \cdot \cdot \cdot, z_{1}) = \text{Tr}_{V(\lambda_{0})}(q^{-2\rho}L_{\lambda_{0}}\Phi^{V^{*}\lambda_{0}}_{\lambda_{n^{-1}}}(z_{n}) \cdots \Phi^{V^{*}\lambda_{1}}_{\lambda_{0}}(z_{1})).$$

(4.20)

It is a $V^*_{1} \otimes \cdots \otimes V^*_{n}$-valued function where $V^*_{j}$ is a copy of $V^*$, the suffix $j$ indicating that it is put at the $j$-th place in the tensor product. More generally let us consider the following trace

$$F_{\lambda_{n},\cdot \cdot \cdot,\lambda_{0}}(\zeta_{1}, \cdot \cdot \cdot, \zeta_{m}, z_{n}, \cdot \cdot \cdot, z_{1})$$

$$= \text{Tr}_{V(\lambda)}(q^{-2\rho}\Phi^{V^{*}}_{\lambda}(\zeta_{1}) \cdots \Phi^{a_{m^{-1}}(\lambda)}\Phi^{V^{*}\lambda_{m}}_{\lambda_{n^{-1}}}(z_{n}) \cdots \Phi^{V^{*}\lambda_{1}}_{\lambda_{0}}(z_{1})).$$

(4.21)

where $\lambda_{0} = \lambda$ and $\lambda_{n} = \sigma^{m}(\lambda)$. It is a function with values in $V^*_{1} \otimes \cdots \otimes V^*_{n} \otimes V^{(k)}_{1} \otimes \cdots \otimes V^{(k)}_{m}$.

**Proposition 4.4.** The following difference equations hold for (4.21):

$$F_{\lambda_{n},\cdot \cdot \cdot,\lambda_{0}}(\zeta_{1}, \cdot \cdot \cdot, q^{k}\zeta_{j}, \cdot \cdot \cdot, \zeta_{m}, z_{n}, \cdot \cdot \cdot, z_{1})/\prod_{k=1}^{n} \tau(q^{k}\zeta_{j}/z_{i})$$

$$= R^{j+1}(q^{m}\zeta_{j}/\zeta_{j+1})^{-1} \cdots R^{m}(q^{m}\zeta_{j}/\zeta_{m})^{-1}$$

$$\times \pi_{\lambda_{j}}(q^{2\rho})R^{1j}(\zeta_{1}/\zeta_{j}) \cdots R^{1j}(\zeta_{j-1}/\zeta_{j})F_{\lambda_{n},\cdot \cdot \cdot,\lambda_{0}}(\zeta_{1}, \cdot \cdot \cdot, \zeta_{m}, z_{n}, \cdot \cdot \cdot, z_{1}),$$

where $\lambda_{n} = \lambda$ and $\lambda_{0} = \sigma^{m}(\lambda)$. It is a function with values in $V^*_{1} \otimes \cdots \otimes V^*_{n} \otimes V^{(k)}_{1} \otimes \cdots \otimes V^{(k)}_{m}$. 

**Proposition 4.4.** The following difference equations hold for (4.21):
\[ F_{\lambda_n, \cdots, \lambda_0}(\zeta_1, \cdots, \zeta_m, z_n, \cdots, q^4 z_i, \cdots, z_1)/\prod_{j=1}^{m} \tau(z_i/\zeta_j) \]
\[ = R^{*i-ii}(z_{i-1}/q^4 z_i)^{-1} \cdots R^{*i1}(z_1/q^4 z_i)^{-1} \times \pi_{V*}^i(q^{-2\tilde{\rho}}) R^{*in}(z_i/z_n) \cdots R^{*ii+1}(z_i/z_{i+1}) \]
\[ \sum_{\lambda'_0, \cdots, \lambda'_{i+1}, \lambda'_i, \cdots, \lambda'_1} F_{\sigma^m(\lambda'_1), \lambda'_n, \cdots, \lambda'_1}(\zeta_1, \cdots, \zeta_m, z_n, \cdots, z_1) \]
\[ \times W_{\lambda_0 \lambda_1 \lambda_2} q^4 z_i/z_1 \cdots W_{\lambda_i-2 \lambda_{i-1} \lambda_i} q^4 z_i/z_i-1 \times W_{\lambda_{i+1} \lambda_{i+2} \lambda_i} z_i/z_{i+1} \cdots W_{\lambda_{n-1} \lambda_{n} \lambda'_1} z_i/z_n. \]

Here \( \pi_{V*}^i(q^{-2\tilde{\rho}}) \) (resp. \( \pi_{V*}^i(q^{-2\tilde{\rho}}) \)) signifies the operator \( q^{-2\tilde{\rho}} \) acting on the component \( V_j \) (resp. \( V^*_j \)). The notation \( R^{*j}(z) \) means the \( R(z) \) acting on \( V_i \otimes V_j \) and likewise for \( R^{*i}(z) \) (not to be confused with the \( R_{kk}(z) \) in (4.18)).

Proof. Note that
\[ \Phi^V_{\lambda \mu}(q^4 z) = (q^{-2\tilde{\rho}} \otimes q^{-2\tilde{\rho}}) \Phi^V_{\lambda \mu}(z)q^{2\rho}, \]
\[ \Phi^\sigma(\lambda)V^{(k)}(q^4 z) = (q^{-2\tilde{\rho}} \otimes q^{-2\tilde{\rho}}) \Phi^\sigma(\lambda)V^{(k)}(z)q^{2\rho}. \]

The assertions follow from this, the commutation relations (3.35, 3.36, 3.37, 3.39) and the cyclic property of the trace. 

In particular, replacing the \( \zeta_i \) by \( q^{-2\zeta_1}, \cdots, q^{-2\zeta_m}, \zeta_m, \cdots, \zeta_1 \) and using (3.20) we obtain the following relation for arbitrary form factors
\[ F_{\lambda_n, \cdots, \lambda_0}(z_n, \cdots, q^4 z_i, \cdots, z_0) \]
\[ = R^{*i-ii}(z_{i-1}/q^4 z_i)^{-1} \cdots R^{*i1}(z_1/q^4 z_i)^{-1} \times \pi_{V*}^i(q^{-2\tilde{\rho}}) R^{*in}(z_i/z_n) \cdots R^{*ii+1}(z_i/z_{i+1}) \]
\[ \sum_{\lambda'_0, \cdots, \lambda'_{i+1}, \lambda'_i, \cdots, \lambda'_1} F_{\sigma^m(\lambda'_1), \lambda'_n, \cdots, \lambda'_1}(z_n, \cdots, z_1) \]
\[ \times W_{\lambda_0 \lambda_1 \lambda_2} q^4 z_i/z_1 \cdots W_{\lambda_i-2 \lambda_{i-1} \lambda_i} q^4 z_i/z_i-1 \times W_{\lambda_{i+1} \lambda_{i+2} \lambda_i} z_i/z_{i+1} \cdots W_{\lambda_{n-1} \lambda_{n} \lambda'_1} z_i/z_n. \]

In the case of level \( k = 1 \) there is only one term in the sum. This is then a lattice analog of the qKZ equation found by Smirnov [7] for form factors in massive integrable field theories.

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Appendix 1. The $q$-KZ equation

In this appendix we briefly outline the derivation of the $q$-KZ equation for the correlation functions of vertex operators [3]. We shall not go into the discussions about the topology here.

A1.1 Universal $R$ and Drinfeld’s central element Let $U = U_q(\mathfrak{g})$ be the quantized enveloping algebra associated with an affine Lie algebra $\mathfrak{g}$. Concerning the notations of affine Lie algebras we follow [12], except that we reverse the ordering of vertices for the type $A^{(2)}_2$.

Recall that $U$ has the universal $R$ matrix $\mathcal{R}$ that enjoys the following properties [13,14].

\[
\begin{align*}
\mathcal{R}\Delta(x) &= \Delta'(x)\mathcal{R} \quad \text{for any } x \in U, \\
(\Delta \otimes \text{id})\mathcal{R} &= \mathcal{R}_{13}\mathcal{R}_{23}, \quad (\text{id} \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12}, \\
(\varepsilon \otimes \text{id})\mathcal{R} &= 1 = (\text{id} \otimes \varepsilon)\mathcal{R}, \quad (a \otimes \text{id})\mathcal{R} = \mathcal{R}^{-1} = (\text{id} \otimes a^{-1})\mathcal{R}.
\end{align*}
\]

(A1.1) (A1.2) (A1.3)

Here $\varepsilon$ is the counit, $a$ is the antipode, $\Delta' = \sigma \circ \Delta$ with $\sigma(a \otimes b) = b \otimes a$, and if we write $\mathcal{R} = \sum a_i \otimes b_i$ then $\mathcal{R}_{12} = \sum a_i \otimes b_i \otimes 1$, $\mathcal{R}_{23} = \sum 1 \otimes a_i \otimes b_i$, $\mathcal{R}_{13} = \sum a_i \otimes 1 \otimes b_i$.

Let $\{\mathcal{R}_0\}, \{\mathcal{R}\}$ be the dual bases of the classical part of the Cartan subalgebra of $\mathfrak{g}$, and let $c, d$ be the canonical central element and the scaling element respectively. Let further $u_{\beta i}, u_{-\beta i}$ be dual bases of the weight $\pm \beta$ component of the ‘nilpotent’ subalgebras (see [14]). We set

\[
\mathcal{R}'(z) = q^{-\sum \mathcal{R}_i \otimes \mathcal{R}_j} \sum_{\beta,i} z^{(d,\beta)} u_{\beta i} \otimes u_{-\beta i}.
\]

Note that if $\beta = \sum_{i=0}^n m_i \alpha_i$, then $\langle d, \beta \rangle = m_0$. The universal $R$ matrix itself is given by

\[
\mathcal{R} = q^{-c \otimes d - d \otimes c}\mathcal{R}', \quad \mathcal{R}'(1) = q^{-\sum \mathcal{R}_i \otimes \mathcal{R}_j} \sum u_{\beta i} \otimes u_{-\beta i}.
\]

In terms of $\mathcal{R}'$ the property (A1.2) is translated as follows.

\[
(\Delta \otimes \text{id})(\mathcal{R}') = \mathcal{R}'_{13}(q^{-c})\mathcal{R}'_{23}, \quad (\text{id} \otimes \Delta)(\mathcal{R}') = \mathcal{R}'_{13}(q^{2c})\mathcal{R}'_{12},
\]

(A1.4)

where $c^2 = 1 \otimes c \otimes 1$.

The following are shown in [15].

(i) If $\mathcal{R} = \sum a_i \otimes b_i$, then $C = \sum a_i b_i a_i$ is invertible and $C^{-1} = \sum a_i^{-1} b_i a_i$.

(ii) $C x C^{-1} = a^2(x) \forall x \in U$.

(iii) The element $\mathcal{D} = q^{2b} \mathcal{R}'(\sum A_i)$ is central in certain completion of $U$.

(iv) $\mathcal{D}$ acts on $V(\lambda)$ as a scalar $q^{\lambda,\lambda+2\rho}$. Here $V(\lambda)$ denotes the irreducible $U$-module with highest weight $\lambda$.

A1.2 Difference equations for VO We shall study how the central element $\mathcal{D}$ behaves with respect to intertwiners.

**Proposition A1.1.** Let $V_1$, $V_2$ and $V_3$ be $U$-modules, and let $\Phi : V_1 \rightarrow V_2 \otimes V_3$ be a $U$-linear map. Then we have

\[
(\mathcal{D} \otimes \text{id})\Phi\mathcal{D}^{-1} = \sigma(\mathcal{R}) \sum (1 \otimes b_i q^{-2\rho}) \Phi a_i,
\]

(A1.5)

\[
(\text{id} \otimes \mathcal{D})\Phi\mathcal{D}^{-1} = \sum (b_i q^{-2\rho} \otimes 1) \mathcal{R} \Phi a_i.
\]

(A1.6)
Proof. The derivation being similar, we show (A1.5). We shall use the sigma notation
\[ \Delta(x) = \sum x_{(1)} \otimes x_{(2)}, \quad (\Delta \otimes 1) \Delta(x) = \sum x_{(1)} \otimes x_{(2)} \otimes x_{(3)}, \] etc. for \( x \in U \).

First note that the intertwining property of \( \Phi \) is equivalently stated as follows.
\[ \sum (x_{(1)} \otimes x_{(2)}) \Phi = \Phi x \quad \iff \quad \sum (a(x_{(1)}) \otimes 1) \Phi x_{(2)} = (1 \otimes x) \Phi \] (A1.7)
\[ \iff \sum (1 \otimes a^{-1}(x_{(2)})) \Phi x_{(1)} = (x \otimes 1) \Phi. \] (A1.8)

Using these we have
\[ (C \otimes 1) \Phi = \sum (a(b_i) a_i \otimes 1) \Phi \]
\[ = \sum (a(b_i) \otimes a^{-1}(a_i(2))) \Phi a_i \]
\[ = \sum (a(b_i b_j) \otimes a^{-1}(a_j)) \Phi a_i. \]

In the last step we used the fact that \( \sum a_i \otimes a_i(2) \otimes b_i = \sum a_i \otimes a_j \otimes b_i b_j \), which follows from the first of (A1.2). Similarly
\[ \Phi C^{-1} = \Phi \sum a^{-1}(b_r) a_r \]
\[ = \sum (a^{-1}(b_r(2)) \otimes a^{-1}(b_r(1))) \Phi a_r \]
\[ = \sum (a^{-1}(b_r) \otimes a^{-1}(b_s)) \Phi a(a_r a_s). \]

Combining them we have
\[ C \otimes 1 \Phi C^{-1} = \sum (a(b_i b_j) \otimes a^{-1}(a_j)) \Phi a_i C^{-1} \]
\[ = \sum (a(b_i b_j) \otimes a^{-1}(a_j)(a^{-1}(b_r) \otimes a^{-1}(b_s)) \Phi a(a_r a_s) C a_i C^{-1} \]
\[ = \sum (a(b_i a_j a^{-1}(b_r) \otimes a^{-1}(a_j) a^{-1}(b_i)) \Phi a(a_i a_s) a^2(a_i). \]

On the other hand, (A1.3) implies \( \sum i a^2(a_i) \otimes a(b_i) = \sum i a(a_i) \otimes b_i \) and \( \sum i a_i a_r \otimes b_i a^{-1}(b_r) = 1 \), hence the right hand side can be simplified to
\[ \sum (a(b_i) \otimes a^{-1}(a_i a_s)) \Phi a(a_s) = \sum (a^2(b_j) \otimes a_j b_s) \Phi a^2(a_s). \]

The assertion follows easily from this. \( \square \)

Let \( V \) be a finite dimensional module over the subalgebra \( U' \) 'with \( d \) being dropped', and let \( V_z \) denote its affinization. We shall apply Proposition A1.1 to the \( U \)-linear maps of the form \( \Phi_1(z) : V(\lambda) \rightarrow V(\mu) \otimes V_z \) and \( \Phi_2(z) : V(\lambda) \rightarrow V_z \otimes V(\mu). \) (Here \( M \otimes N = \sum_{\xi} \prod_{\nu} M_{\nu} \otimes N_{\xi-\nu} \)) Set
\[ p = q^{2(k+\ell)}, \quad \Delta_\lambda = \frac{(\lambda, \lambda + 2\rho)}{2(k + \ell)}, \]
where \( k \) is the level of \( \lambda \) and \( \ell \) denotes the dual Coxeter number. Writing \( \mathcal{R}' = \sum a'_i \otimes b'_i \) we rephrase Proposition A1.1 as
\[ (D \otimes p^d) \Phi_1(z) D^{-1} = \sum \left( b'_i \otimes \text{Ad}(q^{(k+2\ell)d}) a'_i \right) \left( 1 \otimes q^{-2\ell} a^{-2}(b'_j) \right) \Phi_1(z) a'_j, \] (A1.9)
\[ (p^d \otimes D) \Phi_2(z) D^{-1} = \sum \left( q^{-2\ell} (\text{Ad}(q^{kd}) a^{-2}(b'_i)) a'_i \otimes b'_j \right) \Phi_2(z) a'_j, \] (A1.10)
where \( \text{Ad}(q^h)x = q^h x q^{-h} \) for \( x \in U \). Note that \( D \) acts on \( V(\lambda) \) as a scalar \( p^{\Delta_\lambda} \) while \( p^d \) acts on \( V_z \) as the difference operator \( p^d v(z) = v(pz) \). Setting \( \Phi_i(z) = z^{\Delta_\lambda - \Delta_\lambda} \Phi_i(z) \) \( (i = 1, 2) \) we get
Proof. We prove the case (A1.1). We have

\[ \Phi_1(pz) = \sigma(\mathcal{R}'(q^{k+2h^+})) (\text{id} \otimes q^{-2T}) \sum (\text{id} \otimes a^{-2}(b')) \Phi_1(z)a', \]  

(A1.11)

\[ \Phi_2(pz) = (q^{-2T} \otimes \text{id}) \sum (\text{Ad}(q^{kd}) a^{-2}(b') \otimes \text{id}) \mathcal{R}' \Phi_2(z)a', \]  

(A1.12)

A1.2 Two point functions Let \( \Psi(z_1, z_2) \) be one of the following two point correlation functions:

\[ \langle \nu | \Phi^V_{\mu} (z_2) \Phi^V_{\lambda} (z_1) | \lambda \rangle \]  

(A1.13)

\[ \langle \nu | \Phi^V_{\mu} (z_2) \Phi^V_{\lambda} (z_1) | \lambda \rangle \]  

(A1.14)

\[ \langle \nu | \Phi^V_{\mu} (z_2) \Phi^V_{\lambda} (z_1) | \lambda \rangle \]  

(A1.15)

\[ \langle \nu | \Phi^V_{\mu} (z_2) \Phi^V_{\lambda} (z_1) | \lambda \rangle \]  

(A1.16)

Here we use the convention that the space suffixed by 1 (resp. 2) always comes in the first (resp. second) component of the tensor product. For instance, (A1.13) is the expectation value of the composition

\[ V(\lambda) \xrightarrow{\Phi^V_{\mu}(z_2)} V(\mu) \otimes V_{z_2} \xrightarrow{\Phi^V_{\lambda}(z_1)} 1 \otimes V(\nu) \otimes V_{z_2} \]

while (A1.15) corresponds to

\[ V(\lambda) \xrightarrow{\Phi^V_{\mu}(z_1)} V_{z_1} \otimes V(\mu) \xrightarrow{1 \otimes \Phi^V_{\lambda}(z_2)} V_{z_1} \otimes V(\nu) \otimes V_{z_2}. \]

We shall derive the \( q \)-KZ equation for the two point functions. Given finite dimensional representations \( \pi_V : U' \rightarrow \text{End}(V) \), \( \pi_W : U' \rightarrow \text{End}(W) \), we set

\[ R^+_{VW}(z) = \pi_V \otimes \pi_W(\mathcal{R}'(z)). \]  

(A1.17)

Proposition A1.3. We have

\[ \Psi(pz_1, z_2) = A(z_1/z_2) \Psi(z_1, z_2), \quad \Psi(pz_1, pz_2) = (q^{-\phi} \otimes q^{-\phi}) \Psi(z_1, z_2), \]  

(A1.18)

\[ \phi = \tilde{\lambda} + \tilde{\nu} + 2\tilde{\rho}, \]

where

\[ A(z) = R^+(pz)(q^{-\phi} \otimes 1) \text{ for (A1.13)}, \]  

(A1.19)

\[ = (q^{-\tilde{\nu}} \otimes 1) R^+(pq^{-k}z)(q^{-\phi+\tilde{\nu}} \otimes 1) \text{ for (A1.14)}, \]  

(A1.20)

\[ = (q^{-\phi+\tilde{\nu}} \otimes 1) R^+(q^{k}z)(q^{-\tilde{\nu}} \otimes 1) \text{ for (A1.15)}, \]  

(A1.21)

\[ = (q^{-\phi} \otimes 1) R^+(z) \text{ for (A1.16)}, \]  

(A1.22)

and \( R^+(z) = R^+_{V_1V_2}(z) \) is given in (A1.17). In particular we can write

\[ \Psi(z_1, z_2) = \zeta_{z_1}^{\Delta_{\mu} - \Delta_{\lambda}} \zeta_{z_2}^{\Delta_{\nu} - \Delta_{\rho}} \overline{\Psi}(z_1/z_2), \quad \overline{\Psi}(z) \in V_1 \otimes V_2 \otimes F[[z]]. \]

Proof. We prove the case (A1.15). The other cases are similar.
From (A1.12) and (A1.4) we have

\[
\langle \nu | \Phi^V_\mu (z_2) \Phi^V_\lambda (p_{z_1}) | \lambda \rangle \\
= \langle \nu | \Phi^V_\mu \Phi^V_\lambda (z_2) (q^{-2} \otimes \text{id}) \sum (\text{Ad}(q^{-d}) a^{-2}(b'_i) \otimes \text{id}) R'_V \Phi^V_\lambda (z_1) a'_i | \lambda \rangle \\
= \sum \pi_{V_1} (q^{-2} \text{Ad}(q^{-d}) a^{-2}(b'_i) a'_j) \langle \nu | \Phi^V_\mu (z_2) b'_j \Phi^V_\lambda (z_1) a'_i | \lambda \rangle \\
= \sum \pi_{V_1} (q^{-2} \text{Ad}(q^{-d}) a^{-2}(b'_i) a'_j) \sum \pi_{V_2} (b'_j) \langle \nu | \Phi^V_\mu (z_2) \Phi^V_\lambda (z_1) a'_i | \lambda \rangle \\
= \sum \pi_{V_1} (q^{-2} \text{Ad}(q^{-d}) a^{-2}(b'_i) a'_j) \sum \pi_{V_2} (b'_j) \langle \nu | \Phi^V_\mu (z_2) \Phi^V_\lambda (z_1) a'_i | \lambda \rangle \\
= \langle q^{-2} \otimes \text{id} \rangle \pi_{V_1} \otimes \pi_{V_2} (\sum \langle \nu | \Phi^V_\mu (z_2) | \lambda \rangle).
\]

\[ \square \]

Similar argument leads to the \(q\)-KZ equation for general \(n\) point correlation functions, see [3].

The following observations are sometimes useful. We owe the second one to M. Okado.

**Proposition A1.4.**

(i) Suppose \(\lambda, \mu, \nu\) are dominant integral, and consider \(\Psi(z_1, z_2)\) in the case (A1.13). Then for any \(i\) we have

\[ \pi_{V_{z_1}} \otimes \pi_{V_{z_2}} (\Delta'(e_i (h_i, \nu) + 1)) \Psi(z_1, z_2) = 0, \quad \text{wt}(z_1, z_2) = \bar{\lambda} - \bar{\nu}, \]

where \(\Delta' = \sigma \circ \Delta\).

(ii) Let \(\Psi(z_1, z_2)\) be a solution for (A1.19). Then the following give solutions for the other cases:

\[
\langle q^{-\rho} \otimes 1 \rangle \Psi(q^{-k} z_1, z_2) \quad \text{for (A1.20)}, \\\n\langle q^{-\phi + \rho} \otimes 1 \rangle \Psi(p^{-1} q^k z_1, z_2) \quad \text{for (A1.21)}, \\\n\langle q^{-\phi} \otimes 1 \rangle \Psi(p^{-1} z_1, z_2) \quad \text{for (A1.22)}.
\]

**Proof.** Let \(\Phi = \Phi^{V_1}_\mu (z_1) \circ \Phi^{V_2}_\lambda (z_2)\). Then this is an intertwiner \(V(\lambda) \rightarrow V(\nu) \otimes W\) where \(W = V_{z_1} \otimes V_{z_2}\). Clearly \(\text{wt} \langle \nu | \Phi | \lambda \rangle = \bar{\lambda} - \bar{\nu}\). Since \(V^r(\nu)\) is integrable we have \(\langle \nu | e_i (h_i, \nu) + 1 \rangle = 0\). Setting \(l = (h_i, \nu) + 1, x = e_i^l\) and using (A1.8) we get

\[
0 = \langle \nu | x \Phi | \lambda \rangle = \sum \pi_W (a^{-1}(x(z_2))) \langle \nu | \Phi x | \lambda \rangle \\
= \pi_W (a^{-1}(e_i^l)) \langle \nu | \Phi e_i^l | \lambda \rangle.
\]

Here we have used \(e_i | \lambda \rangle = 0\). This implies \(\pi_W (e_i^l) \langle \nu | \Phi | \lambda \rangle = 0\). Multiplying the transposition \(P\) from the left and interchanging \(z_1\) and \(z_2\) we arrive at the assertion (i). Assertion (ii) can be verified directly. \[ \square \]
Appendix 2. Basic hypergeometric series

We recall several facts on the basic hypergeometric series (see e.g. [16]).

The basic hypergeometric series is defined by

$$2\phi_1\left(\begin{array}{c}a \ b \\ c\end{array}; q, z\right) = \sum_{n=0}^{\infty} \frac{(a; q)_n(b; q)_n}{(c; q)_n(q; q)_n} z^n,$$  \hspace{1cm} (A2.1)

where

$$(a; q)_n = \prod_{j=0}^{n-1} (1 - aq^j), \quad (a; q)_0 = 1.$$  

The series (A2.1) converges absolutely for $|z| < 1$ when $|q| < 1$ and can be analytically continued to the domain $|\text{arg}(z)| < \pi$. We shall also use the $q$-Gamma function and the theta function

$$\Gamma_q(z) = \frac{\theta_q(z)}{\theta_q(q^z)} (1-q)^{1-z}, \quad \Theta_q(z) = (z; q)\infty (qz^{-1}; q)\infty (q; q)\infty.$$  \hspace{1cm} (A2.2)

They enjoy the properties

$$\Gamma_q(z+1) = \frac{1-q^2}{1-q} \Gamma_q(z), \quad \Gamma_q(1) = 1,$$  

$$\Theta_q(z^{-1}) = \Theta_q(qz) = -z^{-1}\Theta_q(z), \quad \Gamma_q(z)\Gamma_q(1-z) = \frac{(q; q)\infty^2 (1-q)}{\Theta_q(q^z)}.$$  \hspace{1cm} (A2.3)

Consider the difference equation

$$(q^e - q^{a+b+1})u(q^2z) - \{(q + q^e) + (q^a + q^b)qz\}u(qz) + q(1-z)u(z) = 0.$$  \hspace{1cm} (A2.4)

If we supplement (A2.4) with the condition that $u(z)$ has an expansion of the form $u(z) = z^n \sum_{n=0}^{\infty} a_n z^n$ around $z = 0$, then the solution space of (A2.4) is two-dimensional over $C$, and we can take

$$2\phi_1\left(\begin{array}{c}a \ b \\ c\end{array}; q, z\right) \quad \text{and} \quad z^{1-c}2\phi_1\left(\begin{array}{c}a-c+1 \ b-c+1 \\ c\end{array}; q, z\right)$$

as a basis.

We list some useful formulas below.

$$2\phi_1\left(\begin{array}{c}a \ c \\ b\end{array}; q, z\right) = \sum_{n=0}^{\infty} \frac{(q^a; q)_n q^c}{(q^b; q)_n} z^n = \frac{(q^a z; q)\infty}{(z; q)\infty}, \quad (A2.5)$$

$$\lim_{q \rightarrow 1} \frac{(q^a; q)_\infty}{(z; q)_\infty} = (1-z)^{-a},$$

$$2\phi_1\left(\begin{array}{c}a \ b \\ c\end{array}; q, z\right) = \frac{(q^{a+b-c}; q)_\infty}{(z; q)_\infty} 2\phi_1\left(\begin{array}{c}c-a \ c-b \\ c\end{array}; q, q^{a+b-c}z\right), \quad (A2.6)$$

$$2\phi_1\left(\begin{array}{c}a \ b \\ c\end{array}; q, q^{a+b-c}z\right) = \frac{(q^{c-a}; q)_\infty (q; q)_\infty}{(q^c; q)_\infty (q^{c-a-b}; q)_\infty}, \quad (A2.7)$$

$$2\phi_1\left(\begin{array}{c}a \ b \\ c\end{array}; q, z\right) = \frac{\Gamma_q(c)}{\Gamma_q(b)\Gamma_q(c-b)} \int_0^1 t^{b-1} (tz^q; q)_\infty (tq^c; q)_\infty d_qt.$$  \hspace{1cm} (A2.8)
The last formula involves the Jackson integral defined by
\[
\int_0^1 f(t)d_q t = (1 - q) \sum_{n=0}^{\infty} f(q^n) q^n.
\]
The following connection formula holds
\[
2\phi_1 \left( \frac{q^a}{q^c}; \frac{q}{q^c}, \frac{1}{z} \right) = \frac{\Gamma_q(c)\Gamma_q(b - a)\Theta_q(q^{1 - a}z)}{\Gamma_q(b)\Gamma_q(c - a)\Theta_q(qz)} 2\phi_1 \left( \frac{q^a}{q^{a+1}} \frac{q^{a-c+1}}{q^{a-b+1}z}; q, q^{c-a-b+1}z \right)
+ \frac{\Gamma_q(c)\Gamma_q(a - b)\Theta_q(q^{1 - b}z)}{\Gamma_q(a)\Gamma_q(c - b)\Theta_q(qz)} 2\phi_1 \left( \frac{q^b}{q^{b+1}} \frac{q^{b-c+1}}{q^{b-a+1}z}; q, q^{c-a-b+1}z \right),
\]
provided that \(|\text{arg}(-z)| < \pi\), \(c\) and \(a - b\) are non-integers, and \(z \neq 0\).

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