Algebraic model for single-particle energies of Λ hypernuclei

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Abstract

A model is proposed for the spectrum of Λ hypernuclei based on the $u(3) \times u(2)$ Lie algebra, in which the internal degrees of freedom of the spin-1/2 Λ particle are treated in the Fermionic $u(2)$ scheme, while the motion of the hyperon inside a nucleus is described in the Bosonic $u(3)$ harmonic oscillator scheme. Within this model, a simple formula for single-particle energies of the Λ particle is obtained from the natural dynamical symmetry. The formula is applied to the experimental data on the reaction spectroscopy for the $^{89}_\Lambda Y$ and $^{51}_\Lambda V$ hypernuclei, providing a clear theoretical interpretation of the observed structures.

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1. Introduction

Hypernuclear physics has rapidly been developed in the past few decades [1], deepening our knowledge of how hyperons, i.e., particles containing at least one $s$ quark, interact with nuclear matter. A characteristic feature of hyperons is that they are free from the Pauli blocking due to the nucleons, and thus they can probe the deep interior of atomic nuclei. Furthermore, they may attract surrounding nucleons, leading to important modifications in the nuclear structure [2,3,4,5,6].

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One of the most remarkable findings in hypernuclear physics is the spectral signature of a clear single-particle structure of \( \Lambda \) particle in single-\( \Lambda \) hypernuclei \([1, 8, 9, 10]\). In particular, single-particle \( \Lambda \) states with orbital angular momentum ranging from \( l = 0 \) to \( l = 3 \) have been clearly identified in medium-heavy \( ^{89}\Lambda Y \) hypernucleus using the \((\pi^+, K^+)\) reaction spectroscopy \([1, 8]\). These single-particle levels have been theoretically analyzed with the distorted-wave impulse approximation (DWIA) based on shell model calculations \([11]\).

In this paper, we re-analyze the experimental data for the \((\pi^+, K^+)\) reaction by introducing an algebraic model to describe single-particle levels of single-\( \Lambda \) hypernuclei. In this approach, single-particles levels are classified according to the underlying symmetries. The energy of each level is then given in terms of expectation values of Casimir operators associated with the dynamical symmetries. An advantage of the algebraic method is that the spectrum can be predicted with a minimal set of requirements associated with symmetry even when the exact shape of mean-field potential experienced by the hyperon is not known. In this sense, the model is applicable for a whole class of potentials that share the same asymptotic behavior. In fact, the formalism which we present in this paper is general enough, and we expect it to be universally valid, even beyond the applications presented in this paper.

We mention that a mass formula for \( \Lambda \) hypernuclei has been constructed in Ref. \([12]\) based on a similar algebraic approach. In contrast to Ref. \([12]\), our interest in this paper is in the excitation spectra of single-\( \Lambda \) hypernuclei, rather than the ground state mass.

2. Algebraic model for hypernuclei

Our aim in this paper is to describe the spectrum of hypernuclei with an algebraic model. To this end, one can model the total Hamiltonian in two terms:

\[
\hat{H} = \hat{H}_{\text{Nucl}} + \hat{H}_{\text{Hyp}},
\]

where \( \hat{H}_{\text{Nucl}} \) is for the core nucleus without hyperons, and \( \hat{H}_{\text{Hyp}} \) for the hyperons interacting with non-strange nucleons. In this paper, we are not concerned with the specific model for the core nucleus, \( \hat{H}_{\text{Nucl}}\): it might be e.g., a shell-model, an interacting boson model (IBM) based on \( U(6) \) symmetry, a collective model, or any other suitable model, provided that it gives eigenstates with good total angular momentum \( J \). As a matter of fact, for simplicity, we assume that the core nucleus remains in the ground state with \( J = 0 \), and discard \( \hat{H}_{\text{Nucl}} \) from Eq. \((1)\) in the following discussion.

We model the hyperon part of the Hamiltonian, \( \hat{H}_{\text{Hyp}}\), as follows. First we notice that, despite that the nuclear medium is so dense, the quantized motion of the hyperon through it proceeds with an almost constant attractive interaction within the nuclear volume, because there is no Pauli blocking effect. The \( \Lambda \) particle then bounces back at the surface, being attracted inward by a restoring force. The bosonic algebra of the three-dimensional harmonic oscillator \( u(3) \) can be used to model this situation, in which the motion of any other strange
or charmed chargeless particle would behave similarly. We then couple a $u_F(2)$ fermionic Lie algebra scheme for Λ spin-1/2 particles to the bosonic $u_B(3)$ Lie algebra for the harmonic motion of the hyperon in the nuclear interior.

The only possible dynamical symmetry arising in this simple scheme is given by the subalgebra chain, where for our purposes we can discard the lower $so(2)$ symmetries as they will affect only magnetic substates. In this paper, we consider the following model Hamiltonian:

$$H_{\text{Hyp}} = H_{u(3)} + H_{u(2)} + \hat{V}_{\text{int}},$$

(3)

where the $u(3)$ part provide a global mean-field dynamics and the second term, $H_{u(2)}$, is a constant energy that depends on the number of Λ particle ($n = 0, 1, \text{ and } 2$). The last term, $\hat{V}_{\text{int}}$, describes any additional coupling interaction such as a spin-orbit coupling, that we will discuss later. For simplicity, we will neglect higher order terms in the present analysis.

2.1. $u(3)$ harmonic oscillator

Let us first discuss the $u(3)$ part in Eq. (3). Up to two body, the simplest and yet already general Hamiltonian with dynamical symmetry reads:

$$H_{u(3)} = \alpha \hat{C}_1(u(3)) + \beta \hat{C}_2(u(3)) + \gamma \hat{C}_2(so_L(3))$$

(4)

where $\alpha, \beta,$ and $\gamma$ are free parameters and the Casimir operators are given by $\hat{C}_1(u(3)) = \hat{N}$, $\hat{C}_2(u(3)) = \hat{N}^2$ and $C_2(so_L(3)) = L^2$. The spectrum of this Hamiltonian is given in term of the eigenvalues of the Casimir operators by

$$E_{u(3)} = \alpha N + \beta N(N + 2) + \gamma L(L + 1)$$

(5)

where $N$ is the number of quanta and $L$ is the orbital angular momentum of the confined Λ particle inside the nuclear volume. This clearly gives an (an)harmonic spectrum with rotational-vibrational levels. The allowed symmetric representations of $u(3)$ are labeled by integers $N = 0, 1, 2, \ldots$ and, for each value of $N$, possible values of $L$ are given by

$$L = N, N - 2, \ldots, 1 \text{ or 0 } \quad (N = \text{ odd or even})$$

(6)

(see Sect.7.5.1 in Ref. [13]).

2.2. $u(2)$ algebra for Λ fermion with $s = 1/2$

Let us next consider the $u(2)$ part. We associate a fermionic creation and annihilation operators to each substate of the $s = 1/2$ state as,

$$a_{1/2,+1/2}^\dagger \quad a_{1/2,-1/2}^\dagger$$
such that their anticommutator reads
\[
\left\{ a_{1/2,m}^{\dagger}, a_{1/2,m'}^{\dagger} \right\} = \delta_{m,m'}.
\] (7)

With the bilinear products of $a$ and $a^{\dagger}$, one can construct the $u(2)$ Lie algebra. The four elements for this algebra are the total spin operator, $\hat{S}_\mu$, with $\mu = 0, \pm 1$ and the number operator for fermions, $\hat{N}_F$ (see Sect. 8.4.2 in Ref. [13]). These 4 elements are related to a vector operator defined as
\[
A^{(1)}_\mu = \left[ a_{1/2}^{\dagger} \times \tilde{a}_{1/2} \right]^{(1)}_\mu = -\sqrt{2} \hat{S}_\mu,
\] (8)
and a scalar operator defined as
\[
A^{(0)}_0 = \left[ a_{1/2}^{\dagger} \times \tilde{a}_{1/2} \right]^{(0)}_0 = -\sqrt{\frac{1}{2}} \hat{N}_F.
\] (9)

Here, we have used the definition $\tilde{a}_{j,m} = (-1)^{j-m}a_{j,-m}$. In our case with $s = 1/2$, $a_{1/2,+1/2} = a_{1/2,-1/2}$ and $\tilde{a}_{1/2,-1/2} = -a_{1/2,+1/2}$.

The relevant linear combination of Casimir operators for the fermionic part of the Hamiltonian is:
\[
\hat{H}_{u(2)} = A\hat{C}_1(u(2)) + B\hat{C}_2(u(2)),
\] (10)
and the corresponding energy formula reads,
\[
E_{u(2)} = A\langle C_1 \rangle + B\langle C_2 \rangle.
\] (11)

The representations of $u(2)$ are given in general by a pair of numbers $[\lambda_1, \lambda_2]$. Using the algorithm in Sect. 5.4.1 in Ref. [13], one can calculate the eigenvalues of the linear and quadratic Casimir operators as:
\[
\langle C_1 \rangle = \lambda_1 + \lambda_2, \quad \langle C_2 \rangle = \lambda_1^2 + \lambda_1 + \lambda_2^2 - \lambda_2.
\] (12)

Their eigenvalues in fermionic (antisymmetric) representations are thus given by,
\[
\begin{array}{ccc}
[0] & [1] & [1,1] \\
\langle C_1 \rangle & 0 & 1 & 2 \\
\langle C_2 \rangle & 0 & 2 & 2
\end{array}
\] (13)
where the notation $[0]$, $[1]$ and $[1,1]$ means zero fermions, one fermion (in either spin state) and two fermions, respectively.

Since there are only three possible fermionic states, the formula can take the values of
\[
E_0 = 0, \quad E_\Lambda = A + 2B, \quad E_{\Lambda \Lambda} = 2A + 2B.
\] (14)

Together with Eq. (13), the energies of hypernuclei then read
\[
E_{\text{Hyp}} = \alpha N + \beta N(N + 2) + \gamma L(L + 1) + E_{n\Lambda},
\] (15)
where the last term is given by Eq. (14), depending on the number of $\Lambda$ particles in the system.
3. Reanalysis of the experimental data

We now apply the energy formula introduced in the previous section to single-Λ hypernuclei and reanalyze the experimental data obtained by Hotchi et al. [8] for $^{89}\Lambda Y$ and $^{51}\Lambda V$ hypernuclei. The measured cross-sections (integrated in the $2^\circ$-$14^\circ$ range) for the $(\pi^+,K^+)$ reaction leading to the formation of the hypernuclei show several peaks as a function of energy [8], that are interpreted as corresponding to different angular momentum states of the Λ particle.

In Ref. [8], the experimental data for $^{89}\Lambda Y$ have been empirically fitted with a combination of 10 Gaussian functions (with a total of 18 parameters, 8 of which are energy centroids and the remaining 10 are connected to the height of each peak). With this procedure, the authors of Ref. [8] have concluded that the observed broad bumps contain at least two sub-peaks. The width of the bumps in the spectrum has been attributed to i) the experimental energy resolution, that was estimated to be 1.65 MeV for this hypernucleus [8] and ii) the spreading width due to presence of several low-lying excited states of the core nucleus.

The fit obtained in Ref. [8] well reproduces the experimental spectra, but it lacks a theoretical understanding, even though it would be essentially correct that the peaks are associated with growing angular momenta of Λ particle. We therefore re-fit here each major peak using a mathematically complete formalism with all quantum numbers attributed to the $u(3) \times u(2)$ chain. That is, each major peak is reassigned to the different harmonic oscillator shells with increasing $N$, whereas the lower component within each peak is assigned to the largest possible $L$ according to the rule given by Eq. (6). For example, the first peak has $N = 0$ and therefore only $L = 0$, while the third peak with $N = 2$ has $L = 0$ and 2 components.

To this end, we have undertaken a new fit with a Gaussian function given by

$$G(E; b_{N,L}, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\left(\frac{(E-b_{N,L})^2}{2\sigma^2}\right)} ,$$

where $b_{N,L}$ is the centroid energy given by Eq. (15) with $E_{n\Lambda} = E_\Lambda$ (see Eq. (14)). We superpose 8 Gaussian functions as,

$$G(E) = \Delta E_{\text{bin}} \sum_{N=0}^{4} \sum_{L} a_{N,L} G(E; b_{N,L}, \sigma) ,$$

where $\Delta E_{\text{bin}}$ is the bin width. The value of $L$ is determined for each $N$ according to the rule, Eq. (6), except for $N = 4$, for which we have found that the $L = 0$ component provides only a negligibly small contribution, at least in the energy region where the experimental data were taken. Following Ref. [8], we have used $\Delta E_{\text{bin}} = 0.25$ MeV and $\sigma = 1.65$ MeV. In this way, the fit contains 12 parameters in total, 8 of which are for heights, $a_{N,L}$, and the remaining 4 parameters are for the energy formula, Eq. (15).

The resultant fit for the $^{89}\Lambda Y$ hypernucleus is shown in Fig. 1 together with the parameters in Table 1. The quantum number assignments to each Gaussian
are also shown in the figure, where the components with \( n = 0 \) and \( 1 \), defined as \( N = 2n + L \), are shown by the dashed and the dotted lines, respectively. One can see that the fit is as good as the previous empirical fit. While we do not want to put stress on the statistical comparison of the fitting procedures or on the fact that we use only 12 parameters with respect to 18, we consider that this is a considerable gain in the theoretical interpretation of the experimental data, because one can now assign quantum numbers and determine the splitting of some of the sub-peaks on the basis of the algebraic theory.

The fit also shows two other interesting facts. Firstly, \( \beta \) is about one tenth of \( \alpha \), that is, the anharmonicity is indeed small but non-negligible for the first few states. Secondly, \( \gamma \), the coefficient of the \( L^2 \) term, is negative, which implies that the state with higher \( L \) comes lower and it is usually stronger. This confirms that the intuition in Ref. [8] of assigning the main peaks to increasing values of \( L \) was indeed correct. At the same time it gives a natural explanation for most of the observed features.

We have repeated a similar analysis for the \( ^{51}_{\Lambda}V \) hypernucleus, and have

\[
\begin{array}{cccccc}
\text{a}_{00} & 1.03458 & a_{11} & 4.774 & a_{20} & 4.26848 & a_{22} & 8.93652 \\
a_{31} & 9.57975 & a_{33} & 14.6199 & a_{42} & 21.4563 & a_{44} & 22.7715 \\
\alpha & 5.39547 & \beta & 0.506972 & \gamma & -0.321663 & E_{\Lambda} & -22.6373
\end{array}
\]

Table 1: The parameters in Eq. (17) (see also Eq. (15)) for the best fit of the empirical mass spectra of \( ^{89}_{\Lambda}Y \). The parameters \( \alpha, \beta, \gamma \) and \( E_{\Lambda} \) are in MeV, while \( a_{nL} \) are in \( \mu \text{b MeV} \).
again achieved a good agreement with the experimental data, as shown in Fig. 2. Since both the experimental error bars and the energy resolution are larger for this hypernucleus, the fit is less accurate as compared to that for $^{89}$Y shown in Fig. 1. Nevertheless, the present algebraic model predicts six peaks in the mass spectra with four major peaks with $N = 0, \ldots, 3$. The number of parameters which we employ is 10, in which 4 parameters are for the energy formula as before and 6 for the peak heights (see Table 2).

Higher precision experimental data would help constraining even more the parameters of the fit. It should be noticed, however, that the energy resolution of the detection apparatus may not be the only origin for the width of the peaks, that is, the excitations of the core nucleus may also contribute to the width. In this case, one would have to perform the fitting procedures by taking this effect into account, as the true lowest state will be found at the lower end (left in the pictures) of each peak.

### 4. Higher order interactions

The energy resolution of the experiment reported in Ref. 8 was sufficiently good to appreciate the bumps corresponding to each major shell $N$ and also to
some extent the splitting of the states with different $L$ within a given $N$. On the other hand, if one looks at finer details, some discrepancies can be seen, that sometimes exceeds a confidence level of 90%. A further insight of the fine structure can be gained from the algebraic model, because the operators that form the $u(2)$ and $u(3)$ algebras naturally provide a way to classify higher order interactions and perturbations starting from the two-body level. For example, the simplest interaction term is the scalar operator obtained by coupling the angular momenta of the two algebras, that is, a spin-orbit operator of the form:

$$V_{\text{int}} \propto \hat{L} \cdot \hat{S},$$

(18)

that gives an energy splitting into two components proportional to $2L + 1$ for each $L \neq 0$ state. With this interaction, the $0p$ state, for example, will separate into a $J = 3/2$ and a $J = 1/2$ peaks.

One should always remember, however, that, at these energies, the core nucleus may also get excited in the reaction process [11] (see also the discussion in the previous section) and therefore there is an even finer structure in each peak that cannot be presently resolved. For this reason, one would obtain a too high value (about 1-2 MeV) of the spin-orbit splitting if one tried to obtain information on the magnitude directly from the fit. Notice that shell-model [11, 14, 15] as well as other experiments on lighter hypernuclei [1] indicate that the spin-orbit splitting is much smaller, of the order of 0.05-0.2 MeV. It would be an interesting future study to investigate how the angular momentum of the nuclear excited states and that of the $\Lambda$ particle give rise to higher-order interactions in the context of the algebraic model.

5. Conclusions

We have introduced a simple algebraic model that accounts for the major features observed in spectra of a $\Lambda$ particle in medium-heavy nuclei. This has allowed us to re-fit the experimental data of the $(\pi^+, K^+)$ reaction with a theoretical model in which the quantum numbers in each state are arranged into an energy formula according to symmetries. We have achieved a good agreement with the measured spectra, to within the limitations of the experimental energy resolution.

We expect that this algebraic model is universally applicable to describe the states of hyperons and other hadrons in the nuclear medium. The model also provides a way to classify higher order interaction terms and would become useful when a finer experimental energy resolution will eventually be attained.

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