Semi-analytical modelling of the forward and inverse problems in photoacoustic tomography of a femtosecond laser filament in water accounting for refraction and acoustic attenuation

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Keywords: photoacoustic tomography, femtosecond filament, geometrical acoustics, back-projection algorithm.

Abstract. The semi-analytical technique for calculating the acoustic response of a femtosecond filament accounting for acoustic attenuation in water and refraction at a plane boundary is proposed. The technique uses the approximation of geometrical acoustics and an analytical solution of wave equation for 2D Gaussian source in lossless medium modified to account for acoustic attenuation. The response calculated using the proposed approximated formulas agrees with the one calculated using $k$-space pseudospectral method within 1% accuracy and correlates with the waveform obtained in experiments with Cr:Forsterite laser and a wideband lithium niobate piezosensor. The geometrical acoustic approach is used to modify the back-projection algorithm for the corresponding inverse problem of PA tomography. The experimental PA image of a filament is presented.

1. Introduction

Femtosecond filamentation is a complex phenomenon of ultrashort laser pulse self-channeling due to a dynamical balance between Kerr self-focusing, plasma defocusing, diffraction and dispersion [1]. Filamentation attracts attention of scientific society and industry and find applications such as THz generation [2], lighting protection [3], atmosphere sensing [4] and material processing. High (about $10^{13}$ W/cm$^2$) intensity in the core of the filament leads to the formation of plasma column [5]. The plasma starts to expand (temperature $\sim10^5$ K, pressure $\sim1$ GPa) and a shock wave is generated [6]. The shape of the shock wave is completely determined by the initial energy delivery into the medium [5]. The commonly applied methods for filament dynamics characterization are based on the measurements of plasma electron densities and cannot give information about energy reservoir [7] that surrounds the filament core and stores $\sim90\%$ of laser pulse energy [8]. Nevertheless, the mechanical...
post-effects (such as acoustic waves) depend on the energy delivery can be used to characterize the energy distribution remotely using photoacoustic tomography (or PA imaging) [6,7,9,10]. The development of fast semi-analytical acoustic models of femtosecond filament is important for optimization of the parameters of the PA imaging setups as well as for interpretation of the results.

The spatial resolution is determined by the bandwidth and the angular size of the receiver. The bandwidth should exceed \( f_b \approx 0.8 c_0 / \delta \), where \( c_0 \) is the speed of sound in water, to achieve the spatial resolution \( \delta \) [11]. If \( \delta = 12 \mu \text{m} \), then \( f_b \approx 100 \text{ MHz} \). The acoustic frequencies higher than \((\alpha_0 d)^{-1/2}\) are attenuated in viscous media. Here \( \alpha_0 = 25 \text{ mm}^{-1} \text{GHz}^{-2} \) is the attenuation coefficient in water, \( d \) – propagation distance, so the receiver should be closer to the filament. However, it is often technically challenging to manufacture a small wideband piezoelectric transducer with high signal-to-noise ratio. So the transducer should be further away from the filament to decrease its angular size. If a spacer with low acoustic attenuation is placed between the transducer and the filament, the transducer can be far from the filament and the wave travels mostly through the spacer with low attenuation. However, the refraction and partial reflection of acoustic waves can occur at the “water-spacer” boundary.

2. Theory

The ionization, recombination of filament plasma and energy thermalization result in a formation of a heat source take place at a ps timescale. The characteristic times of mechanical relaxation (~ns) and thermal diffusion (~ms) are much greater. The thermal source can be assumed to form instantly and it is immediately transformed into the initial pressure distribution \( p_0(r) = \beta c_0^3 Q(r) \), where \( Q(r) \) is the absorbed energy density distribution, \( \beta \) – volumetric thermal expansion coefficient, \( C_v \) – specific heat capacity at constant volume. If \( Q<1 \) mJ/mm\(^3\), acoustic nonlinearity can also be neglected and the initial value problem in an acoustically homogeneous and lossless medium can be formulated:

\[
\frac{\partial^2 p}{\partial t^2} - c_0^2 \nabla^2 p = 0, \quad p(r,t = 0) = p_0(r), \frac{\partial p}{\partial t} (r,t = 0) = 0
\]  

(1)

Let the space have a plane boundary \( y = 0 \) and a speed of sound \( c_1 \). Let the filament pass through the point \( F(0,a) \) in the \( xy \) plane and the acoustic response be recorded in the point \( D(h,-b) \) (figure 1a). The half-space \( y > 0 \) is filled with water and \( y < 0 \) is filled with the spacer material.

The length of the filament along \( z \) axis is usually much greater than its diameter in \( xy \) plane, so we can assume that \( p_0(x,y,z) = p_0(x,y) \). If the initial pressure distribution has a 2D Gaussian profile: \( p_0(r) = P_0 \exp(-r^2/w_0^2) \), where \( r = \{x,y\}, r = |r| \) and \( w_0 \) is the radius, the analytical solution can be found in [12]. The dispersion relation is \( k(\omega) = \omega/c_0 \) in a lossless medium and \( k(\omega) = \omega/c_0 - i\alpha_0 \omega^2 \) in a viscous medium. If \( P_{\text{ideal}} \) is the solution to the problem (1), then in the medium with arbitrary \( k(\omega) \) [13]:

\[
\tilde{p}_{\text{ot}}(r,\omega) = \mathcal{F}[k(\omega)c_0]^{-1} \tilde{p}_{\text{ideal}}(r,k(\omega)c_0), \quad \text{where} \quad \mathcal{F}[p(r,t)] = \tilde{p}(r,\omega)
\]

(1)

The refraction and partial reflection at the plane “water-spacer” boundary can be accounted for in the approximation of geometrical acoustics where the acoustic ray that connects the points \( F \) and \( D \) is introduced (figure 1a). This ray intersects the “water-spacer” boundary at point of refraction \( V \), which can be found using the Snell’s law: \( \sin \theta_F/c_0 = \sin \theta_V/c_1 \). The acoustic pulse travels across this ray, so the time of flight is \( T_{FVD} = r_F/c_0 + r_D/c_1 \). The amplitude of the wave at the point \( D \) can be found from the energy flux conservation inside the ray tube between rays \( FVD \) and \( FVD' \) (\( d\theta \) is small) [14]. The energy flux through the rectangles formed by translation of \( VV' \) and \( DD' \) along the \( z \) axis has to be the same and \( p(r_F,t + r_F/c_1)/p(r_D,t) = e = (VV'/DD')^{1/2} \). Neglecting \( \alpha_0 c_1 \omega \ll 1 \) for \( \omega \ll 1 \) GHz the resulting pressure distribution can be found as:

\[
p(r_D,t) \approx e W(\theta) \frac{P_0}{2} \left( \frac{w_0}{2\pi r_a} \right)^{1/2} \left( \frac{w_0}{w_0'} \right)^{1/2} f \left( \frac{r_F - c_0(t - r_F/c_1)}{w_0'} \right)
\]

(2)
where \( w_0^* = \left(w_0^2 + 4\alpha_0\varepsilon_0 \varepsilon^2 \right)^{1/2} \), \( f(x) = \left[ \Gamma \left( \frac{1}{2} \right), F_1 \left( -\frac{1}{2}, \frac{1}{2}; x^2 \right) + 2x \Gamma \left( \frac{1}{2} \right), F_1 \left( \frac{1}{2}, \frac{1}{2}; x^2 \right) \right] \exp (-x^2) \), \( \Gamma(x) \) is the Euler’s Gamma function, \( F_1(a; b; x) \) is the confluent hypergeometric function, \( W(\theta) \) – transmission coefficient for plane waves given in [14], \( \varepsilon = \left[ 1 + (bc\cos^3 \theta)/(ac\cos^3 \theta) \right]^{-1/2} \).

Figure 1. (a) – the refraction of an acoustic ray from a filament and an experimental PA image in the inset. (b) – the numerical (black and blue) and experimental (red) acoustic signals from a filament. Dashed black – a single Gaussian PA source; \( w_0=28 \) \( \mu \)m. Solid blue – two Gaussian PA sources; \( w_0=28 \) \( \mu \)m, \( w_1=65 \) \( \mu \)m, \( P_0; P_1=43:7, u=52 \) \( \mu \)m.

The inverse problem of PA imaging is finding \( p_0(r) \) using the signals \( p(r_s, t) \) recorded on a 2D surface \( r_s \in S_0 \) surrounding the PA source. The back-projection algorithm for homogeneous media and arbitrary \( S_0 \) was derived in [15] using the far-field approximation. This formula expresses \( p_0(r) \) as an integral over the surface \( S_0 \). When \( p_0(x, y, z) = p_0(x, y) \) and \( S_0 \) is an infinite cylindrical surface parallel to \( z \) axis the integration can be done over a 1D directrix \( C_0 \) of the surface \( S_0 \) [16].

The approximation of geometrical acoustics can be used to derive an approximate inverse formula accounting for refraction at a plane “water-spacer” boundary [7,17]. Since a single point of refraction \( V \) exists for each source-receiver pair (points \( F \) and \( D \) respectively) this point can be a “virtual” receiver. The actual detection curve \( C_0 \) can be replaced with a curve \( C_V \) that consists of the points \( V \) for all pixel-receiver pairs. The “virtual” receivers are located in water and the inverse problem can be solved as a homogeneous one. The modified formula is:

\[
p_0(x, y) \approx -2 \int_{c_v} \int_{t_a}^{t_a + t_b} \frac{1}{c_0 t_a} \frac{\hat{p}(r_x, t = t_a + t_y)}{\partial t} \frac{dt_y}{\sqrt{c_0^2 t_a^2 - d_y^2}} \frac{(n_y \cdot d_y)}{\partial W} \frac{dC_V}{\Omega_v}
\]

where \( d_y = \{x-x_y, y-y_y\} \), \( n_y \) is the unit normal to curve \( C_V \) at the point \( r_y \) pointing inwards, \( \Omega_v \) is a full angular aperture of the curve \( C_V \) as seen from the point \( \{x, y\} \).

3. Results and Discussion
The solution (2) was tested in \( k \)-wave MATLAB toolbox that solves wave equation directly using \( k \)-space pseudospectral method [18]. For \( w_0=80 \) \( \mu \)m and the grid step size of 10 \( \mu \)m the error did not exceed 2% for \( a=3 \) mm, \( b>3 \) mm. However, the acoustic response of the filament recorded in the experiments with Cr:Forsterite laser (\( \lambda = 1240 \) nm, pulse duration \( \approx 150 \) fs, pulse energy up to 2 mJ) cannot be described completely by the PA signal from a single 2D Gaussian source (figure 1b). The sum of two 2D Gaussian sources with different radii \( w_0 < w_1 \) and amplitudes \( P_0 > P_1 \):
\[ p_0(r) = P_0 \exp(-r^2/w_0^2) + P_1 \exp\left[-(r - u)^2/w_1^2\right], \]
describe the waveform of the experimental signal better. The second source can be interpreted as the reservoir of the filament.

The acoustic response of the filament was recorded experimentally at the points of the 220° circular arc by rotating the wideband lithium niobate piezoelectric sensor around the axis parallel to the \( z \) axis with a 1° angular step. The spacer was made of fused quartz (\( b=32.5 \) mm). The transverse size of the sensor was 6 mm. The experimental signals were bandpass filtered from 1 MHz to 150 MHz to reduce noise. The inset in figure 1a shows the normalized PA image (101 by 101 pixels, 0.25 mm by 0.25 mm) reconstructed using (2). The halo around the filament core can be interpreted as the energy reservoir. The position of the filament can be determined from the maximum value of the image. The translations of the filament along \( xy \) plane as small as 10 \( \mu \)m were detected in the experiments.

The reconstruction of PA images for single 2D Gaussian sources with various radii \( w_0 \) from 5 \( \mu \)m to 70 \( \mu \)m showed that acoustic attenuation added 5–10 \( \mu \)m and integration over the surface of the sensor added ~15 \( \mu \)m to the FWHM of the image. The details finer than 20 \( \mu \)m could not be resolved. The estimated size of the core of the filament was ~50 \( \mu \)m.

In conclusion, the novel semi-analytical framework for fast modelling of femtosecond filaments was developed. Within this framework the inverse formula accounting for refraction at a plane “water-spacer” boundary was derived. The numerical modelling helped interpret the experimental acoustic signals and quantify the effects that cause blurring of the PA images.

Acknowledgments
The reported study was funded by RFBR according to the research project No. 18-32-00696. The theory was developed under the funding of Russian Science Foundation (grant No. 16-17-10181). The experimental results were obtained under the funding of Russian Science Foundation (grant No. 17-72-20130). E.I. Mareev has a scholarship of BASIS foundation (Russia).

References
[1] Couairon A and Mysyrowicz A 2007 Phys. Rep. 441 47–189
[2] Amico C D, Houard A, Akturk S, Liu Y, Le Bloas J, Franco M, Prade B, Couairon A, Tikhonchuk V T and Mysyrowicz A 2008 New J. Phys. 10 13015
[3] Kasparian J et al. 2003 Science 301 61–4
[4] Bergé L, Skupin S, Nuter R, Kasparian J and Wolf J-P 2007 Rep. Prog. Phys. 70 1633
[5] Potemkin F V, Mareev E I, Podshivalov A A and Gordinenko V M 2015 New J. Phys. 17 53010
[6] Potemkin F V, Mareev E I, Podshivalov A A and Gordinenko V M 2014 Laser Phys. Lett. 11 106001
[7] Potemkin F V, Mareev E I, Rumiantsev B V, Bychkov A S, Karabutov A A, Cherepetskaya E B and Makarov V A 2018 Laser Phys. Lett. 15 075403
[8] Liu W, Théberge F, Arévalo E, Gravel J F, Becker A and Chin S L 2005 Opt. Lett. 30 2602–4
[9] Potemkin F V and Mareev E I 2015 Laser Phys. Lett. 12 15405
[10] Bychkov A S, Cherepetskaya E B, Karabutov A A and Makarov V A 2016 Laser Phys. Lett. 13 085401
[11] Xu M and Wang L V 2003 Phys. Rev. E 67 056605
[12] Heritier J M 1983 Opt. Commun. 44 267–72
[13] Treeby B E 2013 J. Biomed. Opt. 18 036008
[14] Brekhovskikh L M 1980 Waves in Layered Media (New York: Academic Press) pp. 276–9
[15] Burgholzer P, Matt G J, Haltmeier M and Paltauf G 2007 Phys. Rev. E 75 046706
[16] Burgholzer P, Bauer-Marschallinger J, Grün H, Haltmeier M and Paltauf G 2007 Inverse Probl. 23 S65
[17] Zarubin V, Bychkov A, Simonova V, Zhigarkov V, Karabutov A, Cherepetskaya E 2018 Appl. Phys. Lett. 112 214102
[18] Treeby B E and Cox B T 2010 J. Biomed. Opt. 15 021314