Some Norm Inequalities for Upper Sector Matrices

Dengpeng Zhang, Ning Zhang

Abstract. We generalize some norm inequalities for $2 \times 2$ block accretive-dissipative matrices and positive semi-definite matrices that compare the diagonal blocks with the off-diagonal blocks. Moreover, we partially extend a norm inequality of $n \times n$ block accretive-dissipative matrices.

1. Introduction

Let $M_n(\mathbb{C})$ be the set of all $n \times n$ complex matrices and $I_n$ be the identity matrix in $M_n(\mathbb{C})$. For any $T \in M_n(\mathbb{C})$, $T^*$ stands for the conjugate transpose of $T$. Every matrix $T$ has the Cartesian (or Toeplitz) decomposition,

$$T = A + iB,$$

in which $A = \frac{1}{2}(T + T^*)$, $B = \frac{1}{2i}(T - T^*)$ are Hermitian. We say that $T$ is called accretive-dissipative if $A$, $B$ are positive semidefinite. In this paper, we will always represent the decomposition (1) as follows,

$$
\begin{pmatrix}
T_{11} & T_{12} \\
T_{21} & T_{22}
\end{pmatrix} =
\begin{pmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{pmatrix} + i
\begin{pmatrix}
B_{11} & B_{12} \\
B_{21} & B_{22}
\end{pmatrix},
$$

where $T_{jk} \in M_n(\mathbb{C})$, $j,k=1,2$.

Recall that a norm $\| \cdot \|$ on $M_n$ is unitarily invariant if $\|ULAV\| = \|A\|$ for any $A \in M_n(\mathbb{C})$ and unitarily matrices $U, V \in M_n(\mathbb{C})$. For $p \geq 1$ and $A \in M_n(\mathbb{C})$, let $\|A\|_p = (\sum_{i=1}^n s_i^p(A))^\frac{1}{p}$, where $s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A)$ are the singular values of $A$. This is the Schatten $p$-norm of $A$. If $A$ is Hermitian, then all eigenvalues of $A$ are real and ordered as $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$. We denote $s(A) = (s_1(A), s_2(A), \ldots, s_n(A))$ and $\lambda(A) = (\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A))$.

Let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. We rearrange the components of $x$ and $y$ in nonincreasing order: $x_1^1 \geq \cdots \geq x_n^1$, $y_1^1 \geq \cdots \geq y_n^1$. If $\sum_{i=1}^k x_i^1 \leq \sum_{i=1}^k y_i^1 \left( \prod_{i=1}^k x_i^1 \leq \prod_{i=1}^k y_i^1 \right)$, $k = 1, \ldots, n$. We say that $x$ is
Clearly, $A$ is positive definite if and only if $W$ and let $W$ be weakly (log) majorized by $y$.

Let $A = (a_{ij})$ and $B = (b_{ij})$ be $m \times n$ matrices, the Hadamard product of $A$, $B$ is the entry-wise product: $A \circ B = (a_{ij}b_{ij})$.

The numerical range of $A \in \mathbb{M}_n(C)$ is defined by $W(A) = \{ x'Ax | x \in C^n, x'x = 1 \}$.

For $\alpha \in [0, \frac{\pi}{2})$, $S_\alpha$ denotes the sector in the complex plane given by $S_\alpha = \{ z \in \mathbb{C} | \Re z \geq 0, |3z| \leq (\Re z) \tan(\alpha) \}$

and let $S_\alpha = \{ z \in \mathbb{C} | \Re z \geq 0, 3z \geq 0, 3z \leq (\Re z) \tan(\alpha) \}$.

Clearly, $A$ is positive definite if and only if $W(A) \subseteq S_0$, and if $W(A), W(B) \subseteq S_\alpha$ for some $\alpha \in [0, \frac{\pi}{2})$, then $W(A + B) \subseteq S_\alpha$. As $0 \notin S_\alpha$, then $A$ is nonsingular. Some recent studies of sector matrices can be found in [6, 12, 14–17].

Recent research interest in this class of matrices starts with a resolution of a problem from numerical analysis [3].

Lin and Zhou [13, Theorem 3.3, Theorem 3.11] proved the following unitarily invariant norm inequalities:

**Theorem 1.1.** [13, Theorem 3.3] Let $T \in \mathcal{B}(\mathcal{H})$ be accretive-dissipative and partitioned as in (2). Then

$$||T_{12}|| ||T_{21}|| \leq \max(||T_{12}||^2, ||T_{21}||^2) \leq 4 ||T_{11}|| ||T_{22}||$$

(3)

for any unitarily invariant norm $|| \cdot ||$.

**Theorem 1.2.** [13, Theorem 3.11] Let $T \in \mathcal{B}(\mathcal{H})$ be accretive-dissipative and partitioned as in (2). Then

$$||T|| \leq \sqrt{2} ||T_{11}|| + ||T_{22}||$$

(4)

for any unitarily invariant norm $|| \cdot ||$. Furthermore, if $T_{12} = T_{21}$, then

$$||T|| \leq \sqrt{2} ||T_{11} + T_{22}||$$

(5)

Gumus et al. [7, Theorem 4.2] proved the following Schatten $p$-norm and quasinorm inequalities.

**Theorem 1.3.** [7, Theorem 4.2] Let $T \in \mathbb{M}_n(C)$ be accretive-dissipative partitioned as in (2). Then

$$||T_{12}||^p + ||T_{21}||^p \leq 2^{p-1} ||T_{11}||^{p/2} ||T_{22}||^{p/2}, \quad \text{for } p \geq 2$$

and

$$||T_{12}||^p + ||T_{21}||^p \leq 2^{3-p} ||T_{11}||^{p/2} ||T_{22}||^{p/2}, \quad \text{for } 0 < p \leq 2.$$
is accretive-dissipative. Then
\[
\sum_{i\neq j} \|T_{ij}\|_p^p \leq (n - 1)2^{p-2} \sum_{i=1}^{n} \|T_{ii}\|_p^p, \quad (p \geq 0).
\]

Lin and Fu [14, Theorem 2.9] extended the above Theorem 1.4 to the sector matrices.

**Theorem 1.5.** [14, Theorem 2.9] Suppose that \( T \) is a sector matrix represented as in (6). Then
\[
\sum_{i\neq j} \|T_{ij}\|_p^p \leq (n - 1) \sec^p(\alpha) \sum_{i=1}^{n} \|T_{ii}\|_p^p \quad \text{for} \quad p > 0.
\]

Gumus et al. [7, Definition 3.1] introduced the special class \( C \) of all nonnegative increasing functions \( h \) on \([0, \infty)\) satisfying the following condition: If \( x = (x_1, x_2, \ldots, x_n) \) and \( y = (y_1, y_2, \ldots, y_n) \) are two decreasing sequences of nonnegative real numbers such that \( \prod_{k=1}^{n} x_j \leq \prod_{k=1}^{n} y_j \) \((k = 1, 2, \ldots, n)\), then \( \prod_{j=1}^{n} h(x_j) \leq \prod_{j=1}^{n} h(y_j) \) \((k = 1, 2, \ldots, n)\).

Afraz et al. [1, Theorem 17] extended Theorem 1.4 to the sector matrices involving the functions of class \( C \).

**Theorem 1.6.** Suppose that \( T \) is a sector matrix represented as in (6), \( h \in C \) is submultiplicative and \( \alpha \in \left[0, \frac{\pi}{2}\right)\). If \( p \) is positive real number, then
\[
\sum_{i\neq j} \left\| h\left(\|T_{ij}\|_2^2\right)\right\|_p^p \leq (n - 1) \sum_{i=1}^{n} \left\| h^2(\sec(\alpha)|T_{ii}|)\right\|_p^p
\]
for every unitarily invariant norm \( \| \cdot \| \). In particular, we have
\[
\sum_{i\neq j} \left\| h\left(\|T_{ij}\|_2^2\right)\right\|_p^p \leq (n - 1) \sum_{i=1}^{n} \left\| h^2(\sec(\alpha)|T_{ii}|)\right\|_p^p.
\]

At last, Lee [11, Theorem 2.1] proved the following result which is considered as an extension of the classical Rotfel’d theorem.

**Theorem 1.7.** [11, Theorem 2.1] Let \( f(t) \) be a non-negative concave function on \([0, \infty)\). Then, given an arbitrary partitioned positive semi-definite matrix,
\[
\|f\left(\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}\right)\| \leq \|f(A)\| + \|f(B)\|.
\]
(7)
for all unitarily invariant norms.

What are we interested in the above theorem is whether the right-hand side of the inequality (7) can be placed in one norm. And we give a result under some conditions.

Besides, in this paper, we will extend inequalities (3), (4) and (5) to a larger class matrices, i.e. the upper sector matrices. And on the basis of the extension of (3), we partially generalize Theorem 1.4.

2. Main result

We begin this section with some lemmas which are useful to establish our main results.
Lemma 2.1. [2, p. 54] Let \( x = (x_1, x_2, \cdots), \ y = (y_1, y_2, \cdots), \ \alpha = (\alpha_1, \alpha_2, \cdots) \) be sequence of real numbers with entries arranged in decreasing order. Moreover, we assume the entries of \( \alpha \) are nonnegative. If \( \sum_{j=1}^{k} x_j \leq \sum_{j=1}^{k} y_j \) for all \( k = 1, 2, \cdots \), then
\[
\sum_{j=1}^{k} \alpha_j x_j \leq \sum_{j=1}^{k} \alpha_j y_j
\]
for all \( k = 1, 2, \cdots \).

Lemma 2.2. Let \( A, B \in \mathbb{M}_n, \ W(A + iB) \subseteq S'_\alpha \) for some \( \alpha \in [0, \frac{\pi}{2}) \) and \( A + iB \) be the Cartesian decomposition of the full matrix like (1). Then

\[
s_j(B) \leq \sin \alpha \ s_j(A + iB). \tag{8}
\]

Proof. First, when \( \alpha = 0 \), inequality (8) is trivial.
Label the eigenvectors of \( B \) as \( e_1, \cdots, e_n \) in such a way that
\[
s_j(B) = |\langle e_j, Be_j \rangle|.
\]
For \( W(A + iB) \subseteq S_\alpha \), we get
\[
B \preceq A \tan(\alpha). \tag{9}
\]
\[
csc \alpha s_j(B) = csc \alpha |\langle e_j, Be_j \rangle| = \sqrt{1 + \cot^2 \alpha |\langle e_j, Be_j \rangle|}
\leq \langle e_j, (A + iB)e_j \rangle \quad \text{by (9)}
\leq \|e\| \| (A + iB)e_j \|.
\]
Since \( s_j(A) = \max_{\dim(M) = j} \min_{x \in M} \|Ax\| \) (see, e. g. [2, p.75]), where \( M \) represent a subspace of \( \mathbb{C}^n \) for \( A \in \mathbb{M}_n \), we deduce the inequality (8). \( \square \)

Lemma 2.3. [18, p. 352] Let \( A, B, C \) be \( n \times n \) complex matrices such that \( \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \succeq 0 \). Then
\[
s(B) \prec_{\omega \log} \lambda^\frac{1}{2}(A) \circ \lambda^\frac{1}{2}(C).
\]

Lemma 2.4. [4, 9] Let \( A, B \in \mathbb{M}_n^+ \) and \( W(A + iB) \subseteq S'_\alpha \) for some \( \alpha \in [0, \frac{\pi}{2}) \). Then for any unitarily invariant norm \( \| \cdot \| \),
\[
\|A + iB\| \leq \|A + B\| \leq a \|A + iB\|,
\]
where \( a = \min\{1 + \tan(\alpha), \sqrt{2}\} \).

The first main result can be stated as follows.

Theorem 2.5. Let \( T \in \mathbb{M}_{2n}(\mathbb{C}) \) be partitioned as in (2) and assume \( W(T) \subseteq S'_\alpha \) for some \( \alpha \in [0, \frac{\pi}{2}) \). Then
\[
\max\{\|T_{12}\|^2, \|T_{21}\|^2\} \leq (1 + \sin \alpha)^2 \|T_{11}\| \|T_{22}\| \tag{10}
\]
for any unitarily invariant norm \( \| \cdot \| \).
Proof. Let \( \nu = (\nu_1, \nu_2, \cdots, \nu_n) \) be a sequence with nonnegative entries and \( \nu_1 \geq \nu_2 \geq \cdots \geq \nu_n \). Define \( \|X\|_\nu = \sum_{k=1}^n \nu_k \|s_k(X)\| \) for \( X \in M_n \).

\[
\|T_{12}\|_\nu = \left\| \sum_{k=1}^n \nu_k s_k(T_{12}) \right\|
\leq \sum_{k=1}^n \nu_k [s_k(A_{12}) + s_k(B_{12})] \quad \text{(by Lemma 2.1)}
\leq \sum_{k=1}^n \nu_k [s_k(A_{11})^{1/2} s_k(A_{22})^{1/2} + s_k(A_{11})^{1/2} s_k(B_{22})^{1/2}] \quad \text{(by Lemma 2.3)}
\leq \sum_{k=1}^n \nu_k [s_k(A_{11}) + s_k(B_{11})]^{1/2} [s_k(A_{22}) + s_k(B_{22})]^{1/2} \quad \text{(by Cauchy – Schwarz)}
\leq \sum_{k=1}^n \nu_k [(1 + \sin \alpha) s_k(A_{11} + iB_{11})]^{1/2} [(1 + \sin \alpha) s_k(A_{22} + iB_{22})]^{1/2} \quad \text{(by [2, Proposition III.5.1] and Lemma 2.2)}
\leq (1 + \sin \alpha) \sum_{k=1}^n \nu_k [s_k(T_{11})]^{1/2} [s_k(T_{22})]^{1/2}
\leq (1 + \sin \alpha) (\sum_{k=1}^n \nu_k [s_k(T_{11})])^{1/2} (\sum_{k=1}^n \nu_k [s_k(T_{22})])^{1/2} \quad \text{(by Cauchy – Schwarz)}
= (1 + \sin \alpha) \|T_{11}\|_\nu^{1/2} \|T_{22}\|_\nu^{1/2}.
\]

Similarly, we can get

\[
\|T_{21}\|_\nu \leq (1 + \sin \alpha) \|T_{11}\|_\nu^{1/2} \|T_{22}\|_\nu^{1/2}.
\]

As \( \nu \) is arbitrarily chosen, the alleged inequality follows form [8, Corollary 3.5.9]. \( \square \)

Seeing this result, we naturally want to make a comparison between the result of the above Theorem 2.5 and that of Lemma 2.6 in [14] (i.e. [17, Theorem 3.2]). Whether \( (1 + \sin \alpha)^2 \) can be less than \( \sec^2 \alpha \) when \( (1 + \sin \alpha)^2 \) is less than \( \sec^2 \alpha \)? Now we define a function

\[
f(\alpha) = \cos \alpha (1 + \sin \alpha) - 1 \quad \alpha \in (0, \frac{\pi}{2}),
\]

so

\[
(1 + \sin \alpha)^2 \leq \sec^2 \alpha \iff f(\alpha) \leq 0,
\]

By the calculation of matlab, we get \( f(\alpha) \leq 0 \) on \( (0.9960, \frac{\pi}{2}) \), i.e. \( 1 + \sin \alpha \leq \sec \alpha, \alpha \in (0.9960, \frac{\pi}{2}) \) and \( 1 + \sin \alpha > \sec \alpha, \alpha \in (0, 0.9960) \).

For \( p \geq 1 \), since the Schatten p-norms are the examples of the unitarily invariant norms, we could get the following two results.

**Corollary 2.6.** Let \( T \in M_{2n}(\mathbb{C}) \) be partitioned as in (2) and assume \( W(T) \subseteq S'_n \). Then

\[
\max\{\|T_{12}\|_p, \|T_{21}\|_p\} \leq (1 + \sin \alpha)^p \|T_{11}\|^{p/2}_p \|T_{22}\|^{p/2}_p, \quad \text{for } p \geq 1.
\]
**Theorem 2.7.** Let $T \in \mathcal{M}_{2n}(\mathbb{C})$ be partitioned as in (2) and $W(T) \subseteq S'_\alpha$. Then
\[\|T_{12}\|_p^p + \|T_{21}\|_p^p \leq 2(1 + \sin \alpha)^p \|T_{11}\|_p^{p/2} \|T_{22}\|_p^{p/2}, \quad \text{for } p \geq 1.\] (13)

**Proof.**
\[\|T_{12}\|_p^p + \|T_{21}\|_p^p \leq (1 + \sin \alpha)^p \|T_{11}\|_p^{p/2} \|T_{22}\|_p^{p/2} + (1 + \sin \alpha)^p \|T_{11}\|_p^{p/2} \|T_{22}\|_p^{p/2},\] (12)

\[= 2(1 + \sin \alpha)^p \|T_{11}\|_p^{p/2} \|T_{22}\|_p^{p/2}.\]

□

In view of the above results, we give a generalization of the Theorem 1.4 in the case $p \geq 1$.

**Theorem 2.8.** For $i,j = 1,2,\ldots,n$, let $T_{ij}$ be square matrices of the same size such that
\[
T = \begin{pmatrix}
T_{11} & T_{12} & \cdots & T_{1n} \\
T_{21} & T_{22} & \cdots & T_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
T_{n1} & T_{n2} & \cdots & T_{nn}
\end{pmatrix}
\]
and assume $W(T) \subseteq S'_\alpha$. Then
\[
\sum_{i\neq j} \|T_{ij}\|_p^p \leq (n-1)(1 + \sin \alpha)^p \sum_{i=1}^n \|T_{ii}\|_p^p \quad \text{for } p \geq 1.
\]

**Proof.** It is easy to obtain that a principal submatrix \(\begin{pmatrix} T_{ii} & T_{ij} \\ T_{ji} & T_{jj} \end{pmatrix}\) of $T$ is also accretive-dissipative and its numerical range is contained in $S'_\alpha$. Now, applying (13) to \(\begin{pmatrix} T_{ii} & T_{ij} \\ T_{ji} & T_{jj} \end{pmatrix}\), we get
\[
\|T_{ij}\|_p^p + \|T_{ji}\|_p^p \leq 2(1 + \sin \alpha)^p \|T_{ii}\|_p^{p/2} \|T_{jj}\|_p^{p/2}
\]
for $i \neq j$ and $p \geq 1$.

Consequently, using the arithmetic-geometric mean inequality, we have
\[
\|T_{ij}\|_p^p + \|T_{ji}\|_p^p \leq (1 + \sin \alpha)^p(\|T_{ii}\|_p^p + \|T_{jj}\|_p^p)
\]
for $i \neq j$ and $p \geq 1$.

Adding up the previous inequalities for $i,j = 1,2,\ldots,n$, we get
\[
\sum_{i\neq j} \|T_{ij}\|_p^p \leq (n-1)(1 + \sin \alpha)^p \sum_{i=1}^n \|T_{ii}\|_p^p,
\]
which proves the inequality. □

**Remark 2.9.** From inequality (11), we know that the results of Theorem 2.7 and Theorem 2.8 are tighter than that of [14, Theorem 2.8, 2.9], correspondingly, when $\alpha \in (0.9960, \frac{\pi}{2})$, for $p \geq 1$.

Next, we extend Theorem 1.2 to the upper sector matrices.
Theorem 2.10. Let $T \in M_{2n}(\mathbb{C})$ be partitioned as in (2) and assume $W(T) \subseteq S_1'$. Then
\[
\|T\| \leq a(\|T_{11}\| + \|T_{22}\|),
\]
for any unitarily invariant norm $\| \cdot \|$. Furthermore, if the off diagonal blocks of $RT$ and $ST$ are Hermitian or skew-Hermitian, then
\[
\|T\| \leq a(\|T_{11} + T_{22}\|),
\]
where $a = \min\{1 + \tan(\alpha), \sqrt{2}\}$.

Proof. Consider the Cartesian decomposition $T = A + iB$, where $A$ and $B$ are positive semi-definite. Compute
\[
\|T\| = \|A + iB\| \\
\leq \|A + B\| \quad \text{(by Lemma 2.4)} \\
\leq \|A_{11} + B_{11} + A_{22} + B_{22}\| \quad \text{(by (7))} \\
\leq a(\|A_{11} + iB_{11}\| + \|A_{22} + iB_{22}\|) \quad \text{(by Lemma 2.4)} \\
= a(\|T_{11}\| + \|T_{22}\|),
\]
which proves the first inequality.

Now we prove the second inequality. We assume that $A + B = \begin{pmatrix} A_{11} + B_{11} & A_{12} + B_{12} \\ A_{21} + B_{21} & A_{22} + B_{22} \end{pmatrix}$ is positive with Hermitian off diagonal blocks and using the simple fact that $T^*T = T \circ T$ (unitarily congruent) we then deduce
\[
A + B \equiv J(A + B)J^* = \begin{pmatrix} A_{11} + B_{11} + A_{22} + B_{22} & \star \\ \star & \frac{1}{2}A_{11} + B_{11} + A_{22} + B_{22} \end{pmatrix},
\]
where $J = \frac{1}{\sqrt{2}} \begin{pmatrix} I & -I \\ I & I \end{pmatrix}$ is a unitary matrix, $I$ is an identity matrix in $I_n$ and $\star$ stands for the unspecified matrices. Then
\[
\|T\| = \|A + iB\| \\
\leq \|A + B\| \quad \text{(by Lemma 2.4)} \\
\leq \left\| \begin{pmatrix} A_{11} + B_{11} + A_{22} + B_{22} & \star \\ \star & \frac{1}{2}A_{11} + B_{11} + A_{22} + B_{22} \end{pmatrix} \right\| \\
\leq \|A_{11} + B_{11} + A_{22} + B_{22}\| + \|\frac{1}{2}(A_{11} + B_{11} + A_{22} + B_{22})\| \quad \text{(by (7))} \\
= \|A_{11} + B_{11} + A_{22} + B_{22}\| \\
\leq a(\|A_{11} + A_{22} + i(B_{11} + B_{22})\|) \quad \text{(by Lemma 2.4)} \\
= a(\|A_{11} + iB_{11} + A_{22} + iB_{22}\|) \\
= a(\|T_{11} + T_{22}\|).
\]

Similarly, if $A + B = \begin{pmatrix} A_{11} + B_{11} & A_{12} + B_{12} \\ A_{21} + B_{21} & A_{22} + B_{22} \end{pmatrix}$ is positive with skew Hermitian off diagonal blocks and still using the simple fact that $T^*T = T \circ T$ (unitarily congruent), $T = \frac{1}{\sqrt{2}} \begin{pmatrix} I & -I \\ I & I \end{pmatrix}$ we then deduce the same result. □

Remark 2.11. It is clear, when $a \leq \sqrt{2}$, i.e. $0 \leq \alpha \leq \arctan(\sqrt{2} - 1)$, the result in Theorem 2.10 is higher than that of Theorem 1.2.

For example, we take $\alpha = 15^\circ$, i.e. $W(T) \subseteq S_{15}'$, we can get
\[
\|T\| \leq 1.268(\|T_{11}\| + \|T_{22}\|).
\]
Inequality (15) correspondingly becomes

$$||T|| \leq 1.268(||T_{11} + T_{22}||).$$

At the end, we generalize the Theorem 1.7, as follows.

**Theorem 2.12.** Let $A, B \in M_n$, and $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \succeq 0$ with Hermitian or skew-Hermitian off diagonal blocks. If $f(t)$ is a non-negative concave function on $[0, \infty)$, then

$$\|f\left(\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}\right)\| \leq 2\|f\left(\frac{1}{2}A\right) + f\left(\frac{1}{2}B\right)\|,$$

(16)

for all unitarily invariant norm.

**Proof.** If $X = X^*$

$$\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} = J\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}J^* = \begin{pmatrix} \frac{A + B}{2} & \star \\ \star & \frac{A + B}{2} \end{pmatrix},$$

where $J = \frac{1}{\sqrt{2}}\begin{pmatrix} I & -I \\ I & I \end{pmatrix}$ is a unitary matrix, I is an identity matrix in $I_n$ and $\star$ stands for the unspecified matrices. Then

$$\|f\left(\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}\right)\| = f(J\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}J^*)$$

$$\leq 2\|f\left(\frac{A + B}{2}\right)\| \quad \text{by (7)}$$

$$\leq 2\|f\left(\frac{1}{2}A\right) + f\left(\frac{1}{2}B\right)\|.$$  

The last inequality is by [5, theorem 1.1]. Similarly, if $X^* = -X$, let $T = \frac{1}{\sqrt{2}}\begin{pmatrix} I & -I \\ I & I \end{pmatrix}$, we still deduce the same result. \qed

**Corollary 2.13.** Let $A, B \in M_n$, and $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \succeq 0$ with Hermitian or skew-Hermitian off diagonal blocks. Then for all unitarily invariant norm $\| \cdot \|$ \n
$$\left\|\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}\right\|^p \leq 2^{1-p}\|A^p + B^p\| \quad (0 < p \leq 1),$$

$$\|\log(I + \begin{pmatrix} A & X \\ X^* & B \end{pmatrix})\| \leq 2\|\log(I + A/2) + \log(I + B/2)\|.$$  

**Acknowledgements**

The authors are grateful to the referee for many valuable remarks and suggestions.
References

[1] D. Afraz, R. Lashkaripour and M. Bakherad, Norm inequalities involving a special class of functions for sector matrices, J. Inequal. Appl. (2020) 122.
[2] R. Bhatia, Matrix Analysis, GTM 169, New York, Springer-Verlag, 1997.
[3] R. Bhatia and JAR. Holbrook, On the Clarkson-McCarthy inequalities, Math. Ann. 281 (1988) 7–12.
[4] R. Bhatia and F. Kittaneh, Norm inequalities for positive operators, Lett. Math. Phys. 43 (1998) 225-231.
[5] J.-C. Bourin and M. Uchiyam, A matrix subadditivity inequality for f(A+B) and f(A)+f(B), Linear Algebra Appl. 423 (2007) 512–518.
[6] S. Drury and M. Lin, Singular value inequalities for matrices with numerical ranges in a sector, Oper. Matrices 8 (2014) 1143–1148.
[7] I. H. Gumus, O. Hirzallah and F. Kittaneh, Norm inequalities involving accretive-dissipative 2 × 2 block matrices, Linear Algebra Appl. 528 (2017) 76–93.
[8] R. A. Horn and C. R. Johnson, Topics in Matrix Analysis, Cambridge University Press, 1991.
[9] L. Hou and D. Zhang, Concave functions of partitioned matrices with numerical ranges in a sector, Math. Inequal. Appl. 20 (2017) 583–589.
[10] F. Kittaneh and M. Sakkijha, Inequalities for accretive-dissipative matrices, Linear Multilinear Algebra 67 (2019) 1037–1042.
[11] EY. Lee, Extension of Rotfel’d theorem, Linear Algebra Appl. 435 (2011) 735–741.
[12] M. Lin, Some inequalities for sector matrices, Operators and Matrices 10 (2016) 915–921.
[13] M. Lin and D. Zhou, Norm inequalities for accretive-dissipative operator matrices, J. Math. Anal. Appl. 407 (2013) 436–442.
[14] S. Lin and X. Fu, On some inequalities for sector matrices, Linear Multilinear Algebra DOI:10.1080/03081087.2019.1600466.
[15] F. Tan and A. Xie, On the logarithmic mean of accretive matrices, Filomat 33 (2019) 4747–4752.
[16] D. Zhang, L. Hou, L. Ma, Properties of matrices with numerical ranges in a sector, Bull. Iranian Math. Soc. 43 (2017) 1699–1707.
[17] F. Zhang, A matrix decomposition and its application, Linear Multilinear Algebra 63 (2015) 2033–2042.
[18] F. Zhang, Matrix Theory: Basic Results and Techniques, Universitext, New York, Springer, 1999.