Inverse problem for a three-parameter space-time fractional diffusion equation

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Abstract
In this article, we consider the space-time Fractional (nonlocal) diffusion equation

$$\partial_t^{\beta} u(t, x) = \mathcal{L}u(t, x), \quad t \geq 0, \quad -1 < x < 1,$$

where $\partial_t^{\beta}$ is the Caputo fractional derivative of order $\beta \in (0, 1)$ and the differential operator $\mathcal{L}$ is the generator of a Lévy process, sum of two symmetric independent $\alpha_1$-stable and $\alpha_2$-stable processes. We consider a nonlocal inverse problem and show that the fractional exponents $\beta$ and $\alpha_i$, $i = 1, 2$ are determined uniquely by the data $u(t; 0) = g(t), 0 < t < T$. The uniqueness result is a theoretical background for determining experimentally the order of many anomalous diffusion phenomena, which are important in physics and in environmental engineering.

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1 Introduction

While the traditional diffusion equation $\partial_t u = \Delta u$ describes a cloud of spreading particles at the macroscopic level, the space-time fractional diffusion equation $\partial^\beta_t u = -(-\Delta)^{\alpha/2} u$ with $0 < \beta < 1$ and $0 < \alpha < 2$ models anomalous diffusions. The fractional derivative in time can be used to describe particle sticking and trapping phenomena. The fractional space derivative models long particle jumps. The combined effect produces a concentration profile with a sharper peak, and heavier tails [6, 15]. Here the fractional Laplacian $(-\Delta)^{\alpha/2}$ is the infinitesimal generator of a symmetric $\alpha-$ stable process $X = \{X_t, t \geq 0, \mathbb{P}_x, x \in \mathbb{R}^d\}$, a typical example of a non-local operator. This process is a Lévy process satisfying

$$\mathbb{E}\left[e^{\xi (X_t - X_0)}\right] = e^{-t|\xi|^\alpha} \quad \text{for every } x, \xi \in \mathbb{R}^d.$$ 

In this paper, we consider the equation

$$\partial^\beta_t u = -(-\Delta)^{\alpha_1/2} u - (-\Delta)^{\alpha_2/2} u \quad \text{with } 0 < \beta < 1 \quad \text{and } 0 < \alpha_1 < \alpha_2 < 2.$$ 

Suppose $X$ is a symmetric $\alpha_1-$ stable process and $Y$ is a symmetric $\alpha_2-$stable process, both defined on $\mathbb{R}^d$, and that $X$ and $Y$ are independent. We define the process $Z = X + Y$. Then the infinitesimal generator of $Z$ is $(-\Delta)^{\alpha_1/2} + (-\Delta)^{\alpha_2/2}$. The Lévy process $Z$ runs on two different scales: on the small spatial scale, the $\alpha_2$ component dominates, while on the large spatial scale the $\alpha_1$ component takes over. Both components play essential roles, and so in general this process can not be regarded as a perturbation of the $\alpha_1-$stable process or of the $\alpha_2-$ stable process. Note that this process can not be obtained from symmetric stable processes through a combination of Girsanov transform and Feynman-Kac transform [4].

The fractional-time derivative considered here is the Caputo fractional derivative of order $0 < \beta < 1$ and is defined as

$$\partial^\beta_t q(t) = \frac{\partial^\beta q(t)}{\partial t^\beta} := \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{\partial q(s)}{\partial s} \frac{ds}{(t-s)^\beta},$$

where $\Gamma(.)$ is the Euler’s gamma function. For example, $\partial^\beta_t (t^p) = \frac{t^{\beta-p} \Gamma(p+1)}{\Gamma(p+1-\beta)}$ for any $p > 0$.

This definition of the Caputo fractional derivative is intended to properly handle initial values [2, 6, 8], since its Laplace transform $s^\beta \tilde{q}(s) - s^{\beta-1} \tilde{q}(0)$ incorporates the initial value in the same way the first derivative does. Here, $\tilde{q}(s) = \int_0^\infty e^{-ts} q(t) dt$ represents the usual Laplace transform of the function $q$.

It is also well known that, if $q \in C^1(0,\infty)$ satisfies $|q(t)| \leq C t^{\nu-1}$ for some $\nu > 0$, then by (1.2), the Caputo derivative of $q$ exists for all $t > 0$ and the derivative is continuous in $t > 0$ [11, 16].

The following class of functions will play an important role in this article.
Definition 1.1. The Generalized (two-parameter) Mittag-Leffler function is defined by:

\[
E_{\beta,\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta k + \alpha)}, \quad z \in \mathbb{C}, \quad \text{Re}(\alpha) > 0, \quad \text{Re}(\beta) > 0,
\]

where \(\text{Re}(\cdot)\) is the real part of a complex number. When \(\alpha = 1\), this function reduces to \(E_{\beta}(\cdot) := E_{\beta,1}(\cdot)\).

It is well-known that the Caputo derivative has a continuous spectrum \([0,1]\), with eigenfunctions given in terms of the Mittag-Leffler function. In fact, it is not hard to check that the function \(q(t) = E_{\beta}(-\lambda t^\beta)\) is a solution of the eigenvalue equation

\[
\partial_t^\beta q(t) = -\lambda q(t) \quad \text{for any } \lambda > 0.
\]

For \(0 < \alpha_1 < \alpha_2 < 2\), \((-\Delta)^{\alpha_1/2}h - (-\Delta)^{\alpha_2/2}h\) is defined for

\[
h \in \text{Dom}(-(-\Delta)^{\alpha_1/2} - (-\Delta)^{\alpha_2/2}) := \left\{ h \in L^2(\mathbb{R}^d; dx) : \int_{\mathbb{R}^d} (|\xi|^{\alpha_1} + |\xi|^{\alpha_2}) |\hat{h}(\xi)|^2 d\xi < \infty \right\}
\]

as the function with Fourier transform

\[
\mathcal{F}\left[ -(-\Delta)^{\alpha_1/2}h(\xi) - (-\Delta)^{\alpha_2/2}h(\xi) \right] = -\left(|\xi|^{\alpha_1} + |\xi|^{\alpha_2}\right)|\hat{h}(\xi)|^2.
\]

Here, \(\mathcal{F}(h) = \hat{h}\) represents the usual Fourier transform of the function \(h\).

The main purpose of this article is to establish the determination of the unique exponents \(\beta\) and \(\alpha_i\), \(i = 1, 2\) in the fractional time and space derivatives by means of the observed data (also called additional condition) \(u(t,0) = g(t), \ 0 < t < T\). We assume \(g(t) \neq 0\). We show in another article that such inversion algorithm exists and we provide some numerical examples.

Many works have been done recently in inverse problems \([3, 10, 13, 14, 17, 18, 20, 21, 22, 23, 24]\). While most of these works have been dedicated to fractional derivatives only in the time variable \([3, 10, 13, 14, 17, 18, 22, 23, 24]\), space-time fractional derivatives were considered in \([20, 21]\), similarly as in this article. However, a substantial difference is that our work considers diffusion equation involving two independent processes.

The rest of this article is organized as follows: in the next section we provide a review of main properties of the direct problem and introduce the inverse problem. Section 3 is devoted to both the statement and the proof of the main result of this paper. Throughout this article, the letter \(c\), in upper or lower case, with or without a subscript, denotes a constant whose value is not of interest in this article and may stay the same or change from line to line. For simplicity, we will fix \(d = 1\) in the remainder of this paper. The following notation will be used in the sequel: for \(a, b \in \mathbb{R}\), \(a \wedge b := \min(a, b)\); for any two positive functions \(p\) and \(q\), \(p \asymp q\) means that there is a positive constant \(c \geq 1\) so that \(c^{-1}q \leq p \leq cq\) on their common domain of definition. For a given set \(A \subset \mathbb{R}\), \(A^C = \mathbb{R} - A\).
2 Analysis of the direct problem and formulation of the inverse problem

We start by considering the direct problem. The equation we are interested in reads as

\[ \begin{align*}
\partial_t^\beta u(t, x) &= -\varepsilon \Delta_1^{\alpha_1/2} u(t, x) - \varepsilon \Delta_2^{\alpha_2/2} u(t, x), \quad (t, x) \in (0, T) \times (-1, 1), \\
\; u(t, x) &= 0, \quad x \in (-1, 1)^c, \quad 0 < t < T, \\
\; u(0, x) &= f(x), \quad -1 < x < 1.
\end{align*} \tag{2.1} \]

Here \( T > 0 \) is a final time and \( f \) is a given function.

We define the operator \( \mathcal{L} := -\varepsilon \Delta_1^{\alpha_1/2} - \varepsilon \Delta_2^{\alpha_2/2} \) for \( 0 < \alpha_1 < \alpha_2 < 2 \). We will also set \( D := (-1, 1) \). The notation \( \mathcal{L}_D \) will be used to emphasize the underlying domain of interest.

**Definition 2.1** ([6]). A function \( u(t, x) \) is said to be a weak solution of (2.1) if the following conditions are satisfied:

\[ \begin{align*}
u(t, .) &\in W_0^{\alpha_2, 2}(D) \quad \text{for each } t > 0, \\
\lim_{t \downarrow 0} u(t, x) &= f(x) \quad \text{a.e,} \\
\partial_t^\beta u(t, x) &= \mathcal{L}_D u(t, x) \quad \text{in the distributional sense, i.e}
\end{align*} \tag{2.2} \]

\[ \int_{\mathbb{R}} \left( \int_0^\infty u(t, x) \partial_t^\beta \psi(t) \right) \phi(x) dx = \int_0^\infty \varepsilon \mathcal{D}(u(t, .), \phi) \psi(t) dt \]

for every \( \psi \in C_0^1(0, \infty) \) and \( \phi \in C_0^2(D) \). Here, \( W_0^{\alpha_2, 2}(D) \) is the \( \sqrt{\varepsilon} \)-completion of the space \( C_0^\infty(D) \) of smooth functions with compact support in \( D \), where

\[ \varepsilon_1(u, u) = \varepsilon(u, u) + \int_{\mathbb{R}} u^2(x) dx, \]

\[ \varepsilon(u, v) = \varepsilon^D(u, v) \quad \text{for } u, v \in W_0^{\alpha_2, 2}(D), \]

and

\[ \varepsilon^D(u, v) = \frac{1}{2} \int_{D^2} \left( u(x) - u(y) \right) \left( v(x) - v(y) \right) \frac{A(-\alpha_2)}{|x-y|^{1+\alpha_2}} + \frac{b}{|x-y|^{1+\alpha_1}} \; dx dy, \]

where \( A(-\alpha) = \alpha 2^{\alpha-1} \pi^{-1/2} \Gamma((1+\alpha)/2) \Gamma(1-\alpha/2)^{-1} \) and \( b \in \mathbb{R}, \; \text{for } u, v \in \mathcal{F} /[7]. \)

\( \varepsilon^D(u, v) \) comes from variational formulation and symmetry, and

\[ \mathcal{F} := W_0^{\alpha_2, 2}(D) := \left\{ u \in L^2(D; dx) : \int_{D^2} (u(x)-u(y))^2 \left( \frac{A(-\alpha_2)}{|x-y|^{1+\alpha_2}} + \frac{b}{|x-y|^{1+\alpha_1}} \right) \; dx dy < \infty \right\}. \]
Following [6], a weak solution of Problem (2.1) is given by the following formula

\[
\begin{align*}
u(t,x) &= \int_0^\infty \mathbb{E}_x \left[ f(Z_s); s < \tau_D \right] f_t(s) ds \\
&= \int_0^\infty \left( \sum_{n=1}^\infty e^{-\mu_n t} \langle f, \phi_n \rangle \phi_n(x) \right) f_t(s) ds \\
&= \sum_{n=1}^\infty E_{\beta}(-\mu_n t^\beta) \langle f, \phi_n \rangle \phi_n(x),
\end{align*}
\]

(2.3)

where \( f_t(.) \) is defined in [6, (2.1)], \( \tau_D \) is defined later in (2.14), \( (\mu_n)_{n \geq 1} \) is a sequence of positive numbers satisfying \( 0 < \mu_1 \leq \mu_2 \leq \cdots \) and \( (\psi_n)_{n \geq 1} \) is an orthonormal basis of \( L^2(D) \), satisfying the following system of equations

\[
\begin{align*}
\mathcal{L}_D \psi_n &= -\mu_n \psi_n \quad \text{on } D \\
\psi_n &= 0 \quad \text{on } \partial D.
\end{align*}
\]

(2.4)

Hence, any function \( f \in L^2(D; dx) \) has the representation

\[
f(x) = \sum_{n=1}^\infty \langle f, \phi_n \rangle \phi_n(x).
\]

(2.5)

Using the spectral representation, one has

\[
\text{Dom}(\mathcal{L}_D) = \left\{ f \in L^2(D) : \| \mathcal{L}_D f \|_{L^2(D)}^2 = \sum_{n=1}^\infty \mu_n^2 \langle f, \phi_n \rangle^2 < \infty \right\}
\]

(2.6)

and

\[
\mathcal{L}_D f(x) = -\sum_{n=1}^\infty \mu_n \langle f, \phi_n \rangle \phi_n(x).
\]

For any real-valued function \( \phi : \mathbb{R} \to \mathbb{R} \), one can also define the operator \( \phi(\mathcal{L}_D) \) as follows:

\[
\text{Dom}(\phi(\mathcal{L}_D)) = \left\{ f \in L^2(D) : \| \phi(\mathcal{L}_D) f \|_{L^2(D)}^2 = \sum_{n=1}^\infty \phi(\mu_n)^2 \langle f, \phi_n \rangle^2 < \infty \right\}
\]

(2.7)

and

\[
\phi(\mathcal{L}_D) f = \sum_{n=1}^\infty \phi(\mu_n) \langle f, \phi_n \rangle \phi_n.
\]

(2.8)

For the remainder of this article, we will use \( \phi(t) = t^k \) for some \( k > 0 \). For technical reasons (cf. proof of main Theorem), we also restrict \( f \) to the class of functions satisfying

\[
\langle f, \phi_n \rangle > 0, \ n \geq 1 \quad \text{or} \quad \langle f, \phi_n \rangle < 0, \ n \geq 1.
\]

(2.9)

The following lemma indicates an important property of the Mittag-Leffler function. It will be used frequently in the sequel.
Lemma 2.2. For each $0 < \alpha < 2$ and $\pi/2 < \mu < \min(\pi, \pi\alpha)$, there exists a constant $C_0 > 0$ such that

$$|E_\beta(z)| \leq \frac{C_0}{1 + |z|}, \quad \mu \leq |\arg(z)| \leq \pi. \quad (2.10)$$

Theorem 2.3. The eigenvalues of the spectral problem for the one-dimensional double fractional Laplace operator, i.e. $(-\Delta)^{\alpha_1}u(x) + (-\Delta)^{\alpha_2}u(x) = \mu_n u(x)$ in the interval $D \subset \mathbb{R}$ satisfy the following bounds

$$c_1(n^{\alpha_1} + n^{\alpha_2}) \leq \mu_n \leq c_2(n^{\alpha_1} + n^{\alpha_2}), \quad \text{for all } n \geq 1 \text{ and } c_1, c_2 > 0. \quad (2.11)$$

Proof. This follows easily from [5, Theorem 4.4] by taking $\phi(s) = s^{\alpha_1} + s^{\alpha_2}$. \hfill \Box

For the existence of a solution to (2.1), we now show that the series given in (2.3) is uniformly convergent for $(t,x) \in (0,T] \times (-1,1)$. To this aim, we use the following Lemma giving bounds for the eigenvalues and eigenfunctions:

Lemma 2.4. Suppose that the initial value $f$ in (2.1) is such that $f \in \text{Dom}(\phi(L_D^k))$ for $k > -1 + \frac{3}{2\alpha_2}$. Let $(\mu_n, \varphi_n)$ be the eigenpair from (2.4), then

$$\begin{align*}
|\langle f, \varphi_n \rangle| &\leq \sqrt{M\mu_n^{-k}} \\
|\varphi_n(x)| &\leq c_3 \left( \mu_n^{1/2\alpha_1} \wedge \mu_n^{1/2\alpha_2} \right),
\end{align*} \quad (2.12)$$

where

$$M := \sum_{n=1}^{\infty} \mu_n^{2k} |\langle f, \varphi_n \rangle|^2 < \infty \quad \text{and} \quad c_3 > 0.$$

Proof. The first bound in (2.12) follows directly from the definition of $M$. So we only show the second bound.

Recall that the fundamental solution $p(t,x,y)$, also referred to as the heat kernel of $L$, is the unique solution to

$$\partial_t u = Lu. \quad (2.13)$$

It represents the transition density function of $Z$. Denote the first exit time of the process $Z$ by

$$\tau_D := \inf\{t \geq 0 : Z_t \notin D\}. \quad (2.14)$$

Let $Z_D^\tau$ denote the process $Z$ "killed" upon exiting $D$, i.e.

$$Z_D^\tau := \begin{cases} Z_t, & t < \tau_D \\ \partial, & t \geq \tau_D \end{cases} \quad (2.15)$$
Here, $\partial$ is a cemetery point added to $D$. Throughout this paper, we use the convention that any real-valued function $f$ can be extended by taking $f(\partial) = 0$. Then $Z^D$ has a jointly continuous transition density function $p_D(t, x, y)$. Moreover, by the strong Markov property of $Z$, one has for $t > 0$ and $x, y \in D$,

\begin{equation}
(2.16) \quad p_D(t, x, y) = p(t, x, y) - E[p(t - \tau_D, X_{\tau_D}, y); t < \tau_D] \leq p(t, x, y).
\end{equation}

By [4, (1.4)],

\begin{equation}
(2.17) \quad p(t, x, y) \asymp \left( t^{-1/\alpha_1} \wedge t^{-1/\alpha_2} \right) \wedge \left( \frac{t}{|x - y|^{1+\alpha_1}} + \frac{t}{|x - y|^{1+\alpha_2}} \right).
\end{equation}

In particular, one has $\sup_{x \in D} \int_D p(t, x, y)^2 dy \leq \infty$ for all $t > 0$. Denote by $\{p^D_t, t \geq 0\}$ the transition semigroup of $Z^D$, i.e

\begin{equation*}
p^D_t f(x) = \int_D p_D(t, x, y) f(y) dy.
\end{equation*}

It is well known (cf. [9]) that $u(t, x) = p^D_t f(x)$ is the unique weak solution to

\begin{equation*}
\partial_t u = \mathcal{L}_D u
\end{equation*}

with initial condition $u(0, x) = f(x)$ on the Hilbert space $L^2(D; dx)$. Therefore, for each $t > 0$, $p^D_t$ is a Hilbert-Schmidt operator in $L^2(D; dx)$ so it is compact [6]. Consequently, for the eigenpair defined in (2.4), we have $p^D_t \varphi_n = e^{-\mu_n t} \varphi_n$ in $L^2(D; dx)$ for $n \geq 1$ and $t > 0$. Combining this with (2.5), it follows that

\begin{equation*}
p^D_t f(x) = \sum_{n=1}^{\infty} \langle f, \varphi_n \rangle p^D_t \varphi_n = \sum_{n=1}^{\infty} e^{-\mu_n t} \langle f, \varphi_n \rangle \varphi_n.
\end{equation*}

In particular, the transition density $p_D(t, x, y)$ is given by

\begin{equation}
(2.18) \quad p_D(t, x, y) = \sum_{n=1}^{\infty} e^{-\mu_n t} \varphi_n(x) \varphi_n(y).
\end{equation}

Next,

\begin{equation*}
e^{-\mu_n t} |\varphi_n(x)|^2 \leq \sum_{m=1}^{\infty} e^{-\mu_n t} |\varphi_m(x)|^2 = p_D(t, x, x) \leq p(t, x, x) \leq C_1 \left( t^{-1/\alpha_1} \wedge t^{-1/\alpha_2} \right).
\end{equation*}

Hence, taking the square root of both sides, we get

\begin{equation}
(2.19) \quad |\varphi_n(x)| \leq C_2 e^{\mu_n t/2} \sqrt{t^{-1/\alpha_1} \wedge t^{-1/\alpha_2}}.
\end{equation}
Finally, taking $t = \mu_n^{-1}$ concludes the proof.

With everything set, we can now proceed to show the uniform convergence of the series given in (2.3). In fact, using (2.10), (2.11) and (2.12), we have

\[
\sum_{n=1}^{\infty} \max_{x \in D} \left| E_{\beta}(-\mu_n t^\beta) \langle f, \varphi_n(x) \rangle \right| \leq \sqrt{MC} \sum_{n=1}^{\infty} \frac{1}{1 + |\mu_n t^\beta|} \mu_n^{-k} \left( \mu_n^{1/2\alpha_1} \wedge \mu_n^{1/2\alpha_2} \right)
\]

by our choice of $k$ in Lemma 2.4. This shows that the series in (2.3) is uniformly convergent.

We are now ready to state and prove our main result.

3 Statement and proof of the main result

We open this section straight with our main result. We then provide its proof.

**Theorem 3.1.** Let $u$ be the weak solution of (2.1) and let $v$ be the weak solution of the following problem

\[
\begin{align*}
\frac{\partial^\gamma v(t,x)}{\partial t^\gamma} & = -(-\Delta)^{n/2} v(t,x) - (-\Delta)^{\eta_2/2} v(t,x), \quad x \in D, \quad 0 < t < T, \\
v(t,x) & = 0, \quad x \in D^c, \quad 0 < t < T, \\
v(0,x) & = f(x), \quad x \in D.
\end{align*}
\]

If $u(t,0) = v(t,0)$, $0 < t < T$ and (2.9) holds, then

$\beta = \gamma$ and $\alpha_i = \eta_i$, $i = 1, 2$.

**Proof.** The proof follows a similar argument as in [21]. Using the explicit formula (2.3), the weak solutions $u$ and $v$ can be written as

\[
u(t,x) = \sum_{n=1}^{\infty} E_{\beta}(-\mu_n t^\beta) \langle f, \varphi_n(x) \rangle \varphi_n(x)
\]

and

\[
v(t,x) = \sum_{n=1}^{\infty} E_{\gamma}(-\lambda_n t^\gamma) \langle f, \psi_n(x) \rangle \psi_n(x),
\]

where the eigenpairs $\left( \mu_n, \varphi_n \right)$ and $\left( \lambda_n, \psi_n \right)$ satisfy
\[
\begin{aligned}
\mathcal{L}_D \varphi_n &= -\mu_n \varphi_n \quad \text{on } D, \\
\varphi_n &= 0 \quad \text{on } D^c
\end{aligned}
\]

and
\[
\begin{aligned}
\mathcal{L}_D^{\eta_1; \eta_2} \psi_n &= -\lambda_n \psi_n \quad \text{on } D, \\
\psi_n &= 0 \quad \text{on } D^c,
\end{aligned}
\]

where \( \mathcal{L}_D^{\eta_1; \eta_2} \) is the operator \( \mathcal{L}_D \) with \( \eta_1 \) and \( \eta_2 \) replacing the fractional exponents. Without loss of generality, we can normalize the eigenfunctions such that \( \varphi_n(0) = \psi_n(0) = 1 \) for all \( n \geq 1 \). This implies that

\[
\sum_{n=1}^{\infty} E_\beta(-\mu_n t^\beta) \langle f, \varphi_n \rangle = \sum_{n=1}^{\infty} E_\gamma(-\lambda_n t^\gamma) \langle f, \psi_n \rangle
\]

if we assume that \( u(t, 0) = v(t, 0) \).

Next, we use the following asymptotic property of the Mittag-Leffler function \([11, 16]\)

\[
E_l(-t) = \frac{1}{t^{1-l}} + O(|t|^{-2}), \quad 0 < l < 1.
\]

Combining (2.11) and (3.5), we get

\[
\sum_{n=1}^{\infty} \left| E_\beta(-\mu_n t^\beta) - \frac{1}{\Gamma(1-\beta)} \frac{1}{\mu_n t^\beta} \right| \leq C t^{-2\beta}.
\]

By adding and subtracting the term \( \frac{1}{\Gamma(1-\beta)} \frac{1}{\mu_n t^\beta} \) in the left side term in (3.4), we get the following asymptotic equation

\[
\sum_{n=1}^{\infty} E_\beta(-\mu_n t^\beta) \langle f, \varphi_n \rangle = \sum_{n=1}^{\infty} \langle f, \varphi_n \rangle \left[ \frac{1}{\Gamma(1-\beta)} \frac{1}{\mu_n t^\beta} + E_\beta(-\mu_n t^\beta) - \frac{1}{\Gamma(1-\beta)} \frac{1}{\mu_n t^\beta} \right]
\]

\[
= \sum_{n=1}^{\infty} \langle f, \varphi_n \rangle \frac{1}{\Gamma(1-\beta)} \frac{1}{\mu_n t^\beta} + O(|t|^{-2\beta}).
\]

Similarly,

\[
\sum_{n=1}^{\infty} E_\gamma(-\lambda_n t^\gamma) \langle f, \psi_n \rangle = \sum_{n=1}^{\infty} \langle f, \psi_n \rangle \left[ \frac{1}{\Gamma(1-\gamma)} \frac{1}{\lambda_n t^\gamma} + E_\gamma(-\lambda_n t^\gamma) - \frac{1}{\Gamma(1-\gamma)} \frac{1}{\lambda_n t^\gamma} \right]
\]

\[
= \sum_{n=1}^{\infty} \langle f, \psi_n \rangle \frac{1}{\Gamma(1-\gamma)} \frac{1}{\lambda_n t^\gamma} + O(|t|^{-2\gamma}).
\]
Now combining (3.4), (3.7) and (3.8), we get, as $t \to \infty$

$$\sum_{n=1}^{\infty} \langle f, \varphi_n \rangle \frac{1}{\Gamma(1-\beta) \mu_n t^\beta} + O(|t|^{-2\beta}) = \sum_{n=1}^{\infty} \langle f, \psi_n \rangle \frac{1}{\Gamma(1-\gamma) \lambda_n t^\gamma} + O(|t|^{-2\gamma}). \quad (3.9)$$

Now assume, for example, that $\beta > \gamma$. Then multiply (3.9) by $t^\gamma$ to get

$$-t^{\gamma-\beta} \sum_{n=1}^{\infty} \langle f, \varphi_n \rangle \frac{1}{\Gamma(1-\beta) \mu_n} + O(|t|^{\gamma-2\beta}) + \sum_{n=1}^{\infty} \langle f, \psi_n \rangle \frac{1}{\Gamma(1-\gamma) \lambda_n} + O(|t|^{-\gamma}) = 0. \quad (3.10)$$

Letting $t \to \infty$ in (3.10) yields

$$\sum_{n=1}^{\infty} \langle f, \psi_n \rangle \frac{1}{\Gamma(1-\gamma) \lambda_n} = 0: \ a \ contradiction \ to \ (2.9)! \quad (3.11)$$

Similarly, assuming $\gamma > \beta$ also leads to a contradiction. Thus $\beta = \gamma$.

We now prove the second part of the Theorem, i.e $\alpha_i = \eta_i$, $i = 1, 2$. To this aim, we will show that $\mu_n = \lambda_n$ for all $n \geq 1$.

Since $\beta = \gamma$, (3.4) becomes

$$\sum_{n=1}^{\infty} E_\beta(-\mu_n t^\beta) \langle f, \varphi_n \rangle = \sum_{n=1}^{\infty} E_\beta(-\lambda_n t^\beta) \langle f, \psi_n \rangle. \quad (3.12)$$

Taking the Laplace transform of $E_\beta(-\mu_n t^\beta)$ yields

$$\int_0^\infty e^{-zt} E_\beta(-\mu_n t^\beta) dt = \frac{z^{\beta-1}}{z^{\beta} + \mu_n}, \ Re \ z > 0. \quad (3.13)$$

Furthermore, taking the Laplace transform of the Mittag-Leffler function term by term, we get

$$\int_0^\infty e^{-zt} E_\beta(-\lambda_n t^\beta) dt = \frac{z^{\beta-1}}{z^{\beta} + \lambda_n}, \ Re \ z > \mu_n^{1/\beta}. \quad (3.14)$$

It follows that $\sup_{t \geq 0} |E_\beta(-\mu_n t^\beta)| < \infty$ by (2.10). This implies that $\int_0^\infty e^{-zt} E_\beta(-\mu_n t^\beta) dt$ is analytic in the domain $Re \ z > \mu_n^{1/\beta}$. Then by analytic continuity, $\int_0^\infty e^{-zt} E_\beta(-\mu_n t^\beta) dt$ is analytic in the domain $Re \ z > 0$.

Using (2.10), (2.11), (2.12) and Lebesgue’s convergence Theorem, we get that $e^{-t Re z t^\beta}$ is integrable for $t \in (0, \infty)$ with fixed $z$ such that $Re \ z > 0$. 

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and

\[ |e^{-t\text{Re}z} \sum_{n=1}^{\infty} E_{\beta}(-\mu_n t^\beta) \langle f, \varphi_n \rangle| \leq C_0 e^{-t\text{Re}z} \left( \sum_{n=1}^{\infty} \langle f, \varphi_n \rangle \frac{1}{\mu_n t^\beta} \right) \]

\[ \leq C_0' e^{-t\text{Re}z} t^{-\beta} \sum_{n=1}^{\infty} n^{-\alpha_2(k+1)} < \infty \]

by the choice of \( k \) in (2.12).

Next, for \( \text{Re} z > 0 \), we have

(3.15)

\[ \int_0^\infty e^{-t\text{Re}z} \sum_{n=1}^{\infty} E_{\beta}(-\mu_n t^\beta) \langle f, \varphi_n \rangle dt = \sum_{n=1}^{\infty} \langle f, \varphi_n \rangle \frac{z^{\beta-1}}{z^\beta + \mu_n} \]

Similarly,

(3.16)

\[ \int_0^\infty e^{-t\text{Re}z} \sum_{n=1}^{\infty} E_{\beta}(-\lambda_n t^\beta) \langle f, \psi_n \rangle dt = \sum_{n=1}^{\infty} \langle f, \psi_n \rangle \frac{z^{\beta-1}}{z^\beta + \lambda_n} \]

This means, by (3.12), (3.15) and (3.16),

(3.17)

\[ \sum_{n=1}^{\infty} \frac{\langle f, \varphi_n \rangle}{\rho + \mu_n} = \sum_{n=1}^{\infty} \frac{\langle f, \psi_n \rangle}{\rho + \lambda_n}, \quad \text{Re} \rho > 0. \]

Since we can continue analytically (in \( \rho \)) both series in (3.17), this equality actually holds for \( \rho \in \mathbb{C} - \left( \{\mu_n\}_{n \geq 1} \cup \{\lambda_n\}_{n \geq 1} \right) \).

We are now ready to show that \( \mu_n = \lambda_n \) for all \( n \geq 1 \). We proceed by induction:

Without loss of generally, assume \( \mu_1 < \lambda_1 \). Thus we can find a suitable disk containing \(-\mu_1\) but not \( \{-\mu_n\}_{n \geq 2} \cup \{-\lambda_n\}_{n \geq 1} \). Then integrating (3.17) over this disk, by the Cauchy’s integral formula, we get

\[ 2\pi i \langle f, \varphi_1 \rangle = 0 : \] this is a clear contradiction to (2.9).

This means that \( \mu_1 = \lambda_1 \) since the reverse inequality would also lead to a contradiction.

A similar argument yields \( \mu_2 = \lambda_2 \). Inductively, we deduce that

(3.18)

\[ \mu_n = \lambda_n \quad \text{for all } n \geq 1. \]

This also means that

(3.19)

\[ c_1(n^{\alpha_1} + n^{\alpha_2}) \leq \mu_n \leq c_2(n^{\alpha_1} + n^{\alpha_2}) \]

and

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(3.20) \[ c_3(n^{\eta_1} + n^{\eta_2}) \leq \mu_n \leq c_4(n^{\eta_1} + n^{\eta_2}), \] where \( c_i > 0, i = 1, 2, 3, 4. \)

Assume for example that \( \alpha_2 < \eta_2 \), then combining (3.19) and (3.20) yields
\[
c_3' n^{\eta_2} \leq \mu_n \leq c_4' n^{\alpha_2}, \text{ for all } n \geq 1: \text{ a contradiction!}
\]

Therefore \( \alpha_2 = \eta_2 \) since the reverse inequality would also lead to a contradiction.

Similarly, assuming \( \alpha_1 > \eta_1 \) and combining (3.19) and (3.20) gives
\[
c_1(n^{\alpha_1} + n^{\alpha_2}) \leq c_4(n^{\eta_1} + n^{\eta_2}), \text{ for all } n \geq 1: \text{ a contradiction!}
\]

Thus \( \alpha_1 = \eta_1 \) and this concludes the proof. \( \square \)
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