SAMPLE COMPLEXITY OF POLICY-BASED METHODS UNDER OFF-POLICY SAMPLING AND LINEAR FUNCTION APPROXIMATION*

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In this work, we study policy-based methods for solving the reinforcement learning problem, where off-policy sampling and linear function approximation are employed for policy evaluation, and various policy update rules, including natural policy gradient (NPG), are considered for policy update. To solve the policy evaluation sub-problem in the presence of the deadly triad, we propose a generic algorithm framework of multi-step TD-learning with generalized importance sampling ratios, which includes two specific algorithms: the λ-averaged Q-trace and the two-sided Q-trace. The generic algorithm is single time-scale, has provable finite-sample guarantees, and overcomes the high variance issue in off-policy learning.

As for the policy update, we provide a universal analysis using only the contraction property and the monotonicity property of the Bellman operator to establish the geometric convergence under various policy update rules. Importantly, by viewing NPG as an approximate way of implementing policy iteration, we establish the geometric convergence of NPG without introducing regularization, and without using mirror descent type of analysis as in existing literature. Combining the geometric convergence of the policy update with the finite-sample analysis of the policy evaluation, we establish for the first time an overall $\tilde{O}(\epsilon^{-2})$ sample complexity for finding an optimal policy (up to a function approximation error) using policy-based methods under off-policy sampling and linear function approximation.

1. Introduction. Policy-based methods including approximate policy iteration and various policy gradient methods are popular approaches to solve the reinforcement learning (RL) problem (Sutton and Barto, 2018). Two key ideas that are behind these successes are function approximation and off-policy sampling. Since we usually have to deal with extremely large or even continuous state and action spaces, function approximation enables the agent to overcome the curse of dimensionality so that RL is computationally tractable. On the other hand, sampling can be of high risk and/or expensive in many RL problems such as clinical trials (Zhao, Kosorok and Zeng, 2009) and power systems (Glavic, Fonteneau and Ernst, 2017). Off-policy sampling overcomes this challenge since the agent can learn in an off-line manner using historical data.

On the theoretical side, there is an increasing interest in understanding the finite-sample convergence behavior of policy-based methods. However, policy-based methods under off-policy sampling and function approximation are in general not well understood. This leads to our main contributions in the following.

Off-Policy TD-Learning under Linear Function Approximation. To solve the policy evaluation sub-problem in general policy-based approaches, we propose a generic algorithm design of multi-step TD-learning with generalized importance sampling ratios, including two specific algorithms: the λ-averaged Q-trace algorithm and the two-sided Q-trace algorithm.

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Importantly, our proposed algorithm is single time-scale, and successfully overcomes the high variance issue in off-policy learning (albeit at a cost of introducing an asymptotic bias). For our proposed algorithm, we establish its finite-sample convergence guarantees, characterize the limit point as a solution to the generalized multi-step projected Bellman equation (PBE), and provide performance bound on the limit point in terms of the error compared to the true value function.

The $\tilde{O}(\epsilon^{-2})$ Sample Complexity of Policy-Based Methods with Various Policy Update Rules. For the policy improvement, we consider using various policy update rules, including approximate policy iteration with different exploration policies and natural policy gradient (NPG). We present a universal approach to establish the geometric convergence for all the policy update rules we study. The geometric convergence of the policy update and our finite-sample analysis of the policy evaluation together lead to an overall $\tilde{O}(\epsilon^{-2})$ sample complexity for general policy-based methods. This matches with the sample complexity of typical value-based algorithms such as $Q$-learning. To our knowledge, we are the first to establish the $\tilde{O}(\epsilon^{-2})$ sample complexity of policy-based methods under off-policy sampling and linear function approximation.

Geometric Convergence of NPG. An important by-product of our policy update analysis is the geometric convergence of NPG without using regularization. As opposed to existing literature where NPG was commonly viewed as a policy mirror descent algorithm (Lan, 2021), we take a different viewpoint by considering NPG as an approximate policy iteration algorithm, which enables us to use the contraction property and the monotonicity property of the Bellman operator to establish the geometric convergence.

1.1. Related Literature. At a high level, RL algorithms can be divided into two categories: value-based methods and policy-based methods. Popular value-space methods include $Q$-learning (Watkins and Dayan, 1992) and variants of TD-learning (Sutton, 1988), and popular policy-based methods include (natural) actor-critic (Kakade, 2001; Konda and Tsitsiklis, 2000) and approximate policy iteration (Bertsekas, 2011). General policy-based methods usually consist of two phases: policy evaluation and policy update.

TD-Learning. The policy evaluation sub-problem is usually solved with TD-learning and its variants. The asymptotic convergence of TD-learning was established in Bertsekas and Yu (2009); Dayan and Sejnowski (1994); Tsitsiklis (1994). Finite-sample analysis of variants of TD-learning algorithms using on-policy sampling was performed in Chen et al. (2021a), and using off-policy sampling in Chen et al. (2021b); Khodadadian et al. (2021). In the function approximation setting, TD-learning with linear function approximation was studied in Bhandari, Russo and Singal (2018); Lazaric, Ghavamzadeh and Munos (2012); Srikant and Ying (2019); Tsitsiklis and Van Roy (1997) when using on-policy sampling. In the off-policy linear function approximation setting, due to the presence of the deadly triad, TD-learning algorithms can diverge (Sutton and Barto, 2018). Variants of TD-learning algorithms such as TDC (Sutton et al., 2009), GTD (Sutton, Szepesvári and Maei, 2008), emphatic TD (Sutton, Mahmood and White, 2016), and $n$-step TD (with a large enough $n$) (Chen, Khodadadian and Maguluri, 2021) were used to resolve the divergence issue, and the finite-sample bounds were studied in Chen, Khodadadian and Maguluri (2021); Ma, Zhou and Zou (2020); Wang, Zou and Zhou (2021). Note that TDC, GTD, and emphatic TD are two time-scale algorithms, while vanilla $n$-step TD is single time-scale, it suffers from a high variance due to the cumulative product of the importance sampling ratios. See Appendix E of this work for a detailed discussion.

(Natural) Policy Gradient. The policy gradient method was proposed and was shown to converge in Agarwal et al. (2021); Baxter and Bartlett (2001); Sutton et al. (1999). NPG, proposed in Kakade (2001), is a variant of policy gradient method where the inverse of the
fisher information matrix was used as a pre-conditioner. The $O(1/k)$ convergence of NPG was shown in Agarwal et al. (2021). Later, by introducing regularization, geometric convergence of NPG was shown in Cayci, He and Srikant (2021); Cen et al. (2021); Lan (2021). Beyond geometric convergence, Khodadadian et al. (2022) shows that asymptotically NPG converges at a super-linear rate. However, the NPG algorithm studied in Khodadadian et al. (2022) uses adaptive stepsizes that depend on model parameters, and hence cannot be used in an actor-critic framework where the model is unknown to the agent. In this paper, we take a different perspective and view NPG as an approximate policy iteration algorithm, which enables us to show the geometric convergence of NPG using only the properties (i.e., contraction and monotonicity) of the Bellman operator. Importantly, we do not require regularization, and our result can be directly used to show an overall $O(ε^2)$ sample complexity of natural actor-critic.

**Approximate Policy Iteration.** Policy iteration has been a popular method to solve MDPs (Puterman, 1995). In the RL setting, policy iteration has to be implemented in an approximate manner due to the lack of knowledge about the environmental model. The convergence rate and asymptotic error bound of approximate policy iteration have been studied in Munos (2003); Scherrer (2014). A convergent form of approximate policy iteration with Lipschitz continuous policy update was proposed in Perkins and Precup (2002). Other variants of approximate policy iteration such as least square policy iteration and rollout sampling approximate policy iteration schemes were proposed and studied in Lagoudakis and Parr (2003) and Dimitrakakis and Lagoudakis (2008), respectively. See Bertsekas (2011); Powell and Ma (2011) for a detailed survey and review about approximate policy iteration methods.

**(Natural) Actor-Critic.** The asymptotic convergence of on-policy actor-critic was established in Borkar (2009); Borkar and Konda (1997); Williams and Baird (1990) when using a tabular representation, and in Bhatnagar et al. (2009); Konda and Tsitsiklis (2000) when using function approximation. In recent years, there has been an increasing interest in understanding the finite-sample behavior of (natural) actor-critic algorithms. Here is a non-exhaustive list: Cayci, He and Srikant (2021); Khodadadian et al. (2021); Kumar, Koppel and Ribeiro (2019); Lan (2021); Liu et al. (2019, 2020); Qiu et al. (2019); Wang et al. (2019); Wu et al. (2020); Zhang et al. (2019). The state-of-the-art sample complexity of on-policy natural actor-critic is $O(ε^{-2})$ Lan (2021). However, only tabular RL was considered in Lan (2021). In the off-policy setting, finite-sample analysis of natural actor-critic was studied in Khodadadian, Chen and Maguluri (2021) in the tabular setting, and in Chen, Khodadadian and Maguluri (2021) under linear function approximation, and the sample complexity in both cases is $O(ε^{-3})$.

1.2. **Background on Reinforcement Learning.** We consider modeling the RL problem as an infinite horizon Markov decision process (MDP), which is defined in the following.

**DEFINITION 1.1.** An infinite horizon MDP is composed by a 5-tuple $(S, A, P, R, γ)$, where

- $S$ represents the finite state-space,
- $A$ represents the finite action-space,
- $P = \{P_a \in \mathbb{R}^{|S|\times|S|} | a \in A\}$ is a set of transition probability matrices, and $P_a(s, s')$ is the probability of going from state $s$ to state $s'$ under action $a$,
- $R : S \times A \mapsto \mathbb{R}_+$ is the reward function, i.e., $R(s, a)$ is the reward of taking action $a$ at state $s$,
- $γ \in (0, 1)$ is the discount factor, which captures how much weight we assign to future reward.
We assume without loss of generality that \( \max_{s,a} R(s,a) \leq 1 \). The goal is to find an optimal policy \( \pi^* \) of selecting actions so that the long term reward is maximized. Formally, define the state-action value function associated with a policy \( \pi \) at state-action pair \( (s,a) \) by

\[
Q_\pi(s,a) = \mathbb{E}_\pi \left[ \sum_{k=0}^{\infty} \gamma^k R(S_k, A_k) \mid S_0 = s, A_0 = a \right],
\]

where we use the notation \( \mathbb{E}_\pi[\cdot] \) to indicate that the actions are chosen according to the policy \( \pi \). Then the goal is to find an optimal policy \( \pi^* \) such that \( Q^* := Q^{\pi^*} \) is maximized uniformly for all \( (s,a) \). A popular approach to solve the RL problem is to use the policy-based methods. In every iteration of general policy-based algorithms, the agent first performs a policy evaluation step to estimate the value function of the current policy iterate, which is then followed by a policy improvement step to update the policy.

2. Off-Policy TD-Learning with Linear Function Approximation. This section is dedicated to solving the policy evaluation sub-problem within general policy-based methods. Consider estimating the \( Q \)-function \( Q_\pi \) of a given target policy \( \pi \) using TD-learning. Depending on whether the policy \( \pi_b \) used to collect samples (called the behavior policy) is equal to the target policy \( \pi \) or not, there are on-policy TD-learning (i.e., \( \pi_b = \pi \)) and off-policy TD-learning (i.e., \( \pi_b \neq \pi \)). Compared to on-policy sampling, off-policy sampling is often more preferred in practice due to its improved sample efficiency and more flexible sample collection procedure.

TD-learning becomes computationally intractable when the size of the state-action space is large. This motivates the use of function approximation. In linear function approximation, we choose a set of basis vectors \( \varphi_i \in \mathbb{R}^{|S||A|} \), \( 1 \leq i \leq d \), and try to approximate the target value function \( Q^\pi \) using linear combination of the basis vectors. Specifically, let \( \Phi \in \mathbb{R}^{|S||A| \times d} \) be a matrix defined by \( \Phi = [\varphi_1, \ldots, \varphi_d] \), and let \( \phi(s,a) = [\varphi_1(s,a), \varphi_2(s,a), \ldots, \varphi_d(s,a)]^T \in \mathbb{R}^d \) be the \( (s,a) \)-th row of the matrix \( \Phi \), which can be viewed as the feature vector associated with the state-action pair \( (s,a) \). Then, the goal is to find from the linear sub-space \( \hat{Q} = \{ \hat{Q}_w = \Phi w \mid w \in \mathbb{R}^d \} \) the “best” approximation of the \( Q \)-function \( Q^\pi \), where \( w \in \mathbb{R}^d \) is the weight vector.

When TD-learning is used along with off-policy sampling and linear function approximation, the deadly triad is formed and the algorithm can be unstable. We next propose a generic framework of TD-learning algorithms (including two specific algorithms: the \( \lambda \)-averaged \( Q \)-trace and the two-sided \( Q \)-trace), which provably converge in the presence of the deadly triad, and do not suffer from the high variance issue in off-policy learning. Throughout this paper, we impose the following assumption on the basis vectors.

**Assumption 2.1.** The matrix \( \Phi \) has linearly independent columns, and satisfies \( \| \Phi \|_\infty \leq 1 \).

Assumption 2.1 is indeed without loss of generality since disregarding dependent basis vectors or performing constant scaling of the basis vectors does not change the approximation power of the linear sub-space \( \hat{Q} \).

2.1. Algorithm Design. We present in Algorithm 1 a generic TD-learning algorithm using off-policy sampling and linear function approximation. In Algorithm 1, the choice of the generalized importance sampling ratios \( c(\cdot, \cdot) \) and \( \rho(\cdot, \cdot) \) is of vital importance. We next present two specific choices, resulting in two novel algorithms called \( \lambda \)-averaged \( Q \)-trace and two-sided \( Q \)-trace.
Khodadadian et al. reduces to the convergent multi-step off-policy TD-learning algorithm presented in 2021, 2016, 2021. The idea of truncating the importance sampling ratios from above was already employed in existing algorithms such as Retrace(λ) (Munos et al., 2016), V-trace (Espeholt et al., 2018), and Q-trace (Khodadadian et al., 2021), and is used to control the high variance in off-policy learning. However, none of them were shown to converge in the function approximation setting. Introducing the lower truncation level is crucial to ensure the convergence of the two-sided Q-trace algorithm in the presence of the deadly triad. This will be illustrated in detail in Section 2.3.

**Algorithm 1:** A Generic Multi-Step Off-Policy TD-Learning with Linear Function Approximation

1: **Input:** Integer $K$, bootstrapping parameter $n$, stepsize sequence $\{\alpha_k\}$, initialization $w_0$, target policy $\pi$, behavior policy $\pi_b$, generalized importance sampling ratios $c, \rho : S \times A \mapsto \mathbb{R}_+$, and a single trajectory of samples $\{(S_k, A_k)\}_{0 \leq k \leq K+n-1}$ generated by the behavior policy $\pi_b$.

2: for $k = 0, 1, \ldots, K-1$ do
3: $\Delta_i(w_k) = \mathcal{R}(S_i, A_i) + \gamma \rho(S_{i+1}, A_{i+1}) \varphi(S_{i+1}, A_{i+1})^\top w_k - \varphi(S_i, A_i)^\top w_k$, for all $i \in \{k, k+1, \ldots, k+n-1\}$
4: $w_{k+1} = w_k + \alpha_k \varphi(S_k, A_k) \sum_{i=k}^{k+n-1} \gamma^{i-k} \prod_{j=k+1}^{i} c(S_j, A_j) \Delta_i(w_k)$
5: end for
6: **Output:** $w_K$

**The λ-Averaged Q-Trace Algorithm.** Let $\lambda \in \mathbb{R}^{|S|}$ be a vector-valued tunable parameter satisfying $\lambda \in [0, 1]$. Then the generalized importance sampling ratios are chosen as

$$c(s, a) = \rho(s, a) = \frac{\pi(a|s)}{\pi_b(a|s)} + 1 - \lambda(s)$$

for all $(s, a)$. Observe that when $\lambda = 1$, we have $c(s, a) = \rho(s, a) = \frac{\pi(a|s)}{\pi_b(a|s)}$, and Algorithm 1 reduces to the convergent multi-step off-policy TD-learning algorithm presented in Chen, Khodadadian and Maguluri (2021), which however suffers from an exponential large variance due to the cumulative product of the importance sampling ratios. See Appendix E for more details. On the other hand, when $\lambda = 0$, we have $c(s, a) = \rho(s, a) = 1$, and hence the product of the generalized importance sampling ratios is deterministically equal to one, resulting in no variance at all. However in this case, we are essentially performing policy evaluation of the behavior policy $\pi_b$ instead of the target policy $\pi$, hence there will be an asymptotic bias in the limit of Algorithm 1. More generally, when $\lambda \in (0, 1)$, there is a trade-off between the variance in the stochastic iterates $\{w_k\}$ and the bias in the limit point. Such trade-off will be studied quantitatively in Section 2.3.

**The Two-Sided Q-Trace Algorithm.** To introduce the algorithm, we first define the two-sided truncation function. Given upper and lower truncation levels $a, b \in \mathbb{R}$ satisfying $0 < a < b$, define $g_{a,b} : \mathbb{R} \mapsto \mathbb{R}$ by

$$g_{a,b}(x) = \begin{cases} a, & x < a, \\ x, & a \leq x \leq b, \\ b, & x > b. \end{cases}$$

Let $\ell, u \in \mathbb{R}^{|S|}$ be two vector-valued tunable parameters satisfying $0 \leq \ell \leq 1 \leq u$. Then, for the two-sided Q-trace algorithm, the generalized importance sampling ratios are chosen as

$$c(s, a) = \rho(s, a) = g_{\ell(s), u(s)}\left(\frac{\pi(a|s)}{\pi_b(a|s)}\right)$$

for all $(s, a)$. The idea of truncating the importance sampling ratios from above was already employed in existing algorithms such as Retrace$(\lambda)$ (Munos et al., 2016), V-trace (Espeholt et al., 2018), and Q-trace (Khodadadian et al., 2021), and is used to control the high variance in off-policy learning. However, none of them were shown to converge in the function approximation setting. Introducing the lower truncation level is crucial to ensure the convergence of the two-sided Q-trace algorithm in the presence of the deadly triad. This will be illustrated in detail in Section 2.3.
2.2. The Generalized Projected Bellman Equation. We next theoretically analyze Algorithm 1. Specifically, in this section, we formulate Algorithm 1 as a stochastic approximation algorithm for solving a generalized PBE and study its properties. We begin by stating our assumption.

**Assumption 2.2.** The behavior policy \(\pi_b\) satisfies \(\pi_b(a|s) > 0\) for all \((s, a)\), and induces an irreducible and aperiodic Markov chain \(\{S_k\}\).

Intuitively, Assumption 2.2 is imposed to ensure that the behavior policy has sufficient exploration, which is known to be a necessary component for learning. Under Assumption 2.2, the Markov chain \(\{S_k\}\) induced by \(\pi_b\) has a unique stationary distribution \(\mu \in \Delta^{|S|}\). Moreover, there exist \(C \geq 1\) and \(\sigma \in (0, 1)\) such that \(\max_{s \in S} \|P_{\pi_b}^k(s, \cdot) - \mu(\cdot)\|_{TV} \leq C\sigma^k\) for all \(k \geq 0\) (Levin and Peres, 2017), where \(P_{\pi_b}\) is the transition probability matrix of the Markov chain \(\{S_k\}\) under \(\pi_b\).

For simplicity of notation, denote \(c_{i,j} = \prod_{k=i}^j c(S_k, A_k)\). Algorithm 1 can be viewed as a stochastic iterative algorithm for solving the following system of equations (in terms of \(w\)):

\[
E_{S_0 \sim \mu} \left[ \phi(S_0, A_0) \sum_{i=0}^{n-1} \gamma^i c_{1,i} \Delta_i(w) \right] = 0,
\]

where \(A_i \sim \pi_b(\cdot|S_i)\) and \(S_{i+1} \sim P_{A_i}(S_i, \cdot)\). The following lemma formulates Eq. (1) in the form of a generalized PBE.

To present the lemma, we first introduce some notation. Let \(K_{SA} \in \mathbb{R}^{|S||A| \times |S||A|}\) be a diagonal matrix with diagonal entries \(\{\mu(s)\pi_b(a|s)\}_{(s,a) \in S \times A}\), and let \(K_{SA,\min}\) be the minimal diagonal entry. Let \(\| \cdot \|_{K_{SA}}\) be the weighted \(\ell_2\)-norm with weights \(\{\mu(s)\pi_b(a|s)\}_{(s,a) \in S \times A}\), and denote \(\text{Proj}_Q\) as the projection operator onto the linear sub-space \(Q\) with respect to \(\| \cdot \|_{K_{SA}}\). Let \(\mathcal{T}_c, \mathcal{H}_\rho : \mathbb{R}^{|S||A|} \mapsto \mathbb{R}^{|S||A|}\) be two operators defined by

\[
\mathcal{T}_c(Q)(s, a) = \sum_{i=0}^{n-1} E_{\pi_b} [\gamma^i c_{1,i} Q(S_i, A_i) \mid S_0 = s, A_0 = a]
\]

\[
\mathcal{H}_\rho(Q)(s, a) = \mathcal{R}(s, a) + \gamma E_{\pi_b} [\rho(S_1, A_1)Q(S_1, A_1) \mid S_0 = s, A_0 = a]
\]

for any \(Q \in \mathbb{R}^{|S||A|}\) and state-action pair \((s,a)\).

**Lemma 2.1.** Eq. (1) is equivalent to:

\[
\Phi w = \text{Proj}_Q \mathcal{B}_{c,\rho}(\Phi w),
\]

where \(\mathcal{B}_{c,\rho}(\cdot)\) is the generalized Bellman operator defined by \(\mathcal{B}_{c,\rho}(Q) = \mathcal{T}_c(\mathcal{H}_\rho(Q) - Q) + Q\) for any \(Q \in \mathbb{R}^{|S||A|}\).

The generalized Bellman operator \(\mathcal{B}_{c,\rho}(\cdot)\) was previously introduced in Chen et al. (2021b) to study off-policy TD-learning algorithms in the *tabular* setting (i.e., \(\Phi = I_{SA}\)), where the contraction property of \(\mathcal{B}_{c,\rho}(\cdot)\) (as well as its asynchronous variant) was shown. However, \(\mathcal{B}_{c,\rho}(\cdot)\) alone being a contraction is not enough to guarantee the convergence of Algorithm 1 because of function approximation, which introduces an additional projection operator \(\text{Proj}_Q\). What we truly need for the stability and accuracy of Algorithm 1 is that (1) the composed operator \(\text{Proj}_Q \mathcal{B}_{c,\rho}(\cdot)\) is a contraction mapping, and (2) the solution \(w^n_{c,\rho}\) of Eq. (2) is such that \(\Phi w^n_{c,\rho}\) is an approximation of the \(Q\)-function \(Q^n\). We next provide sufficient conditions on the choices of the generalized importance sampling ratios \(c(\cdot, \cdot)\) and \(\rho(\cdot, \cdot)\), and the bootstrapping parameter \(n\) so that the above two requirements are satisfied.
Let $D_c, D_\rho \in \mathbb{R}^{|S||A| \times |S||A|}$ be two diagonal matrices such that $D_c((s, a), (s, a)) = \sum_{a' \in A} \pi_b(a'|s)c(s, a')$ and $D_\rho((s, a), (s, a)) = \sum_{a' \in A} \pi_b(a'|s)\rho(s, a')$ for all $(s, a)$. Let $D_{c,\max}$ and $D_{\rho,\max}$ ($D_{c,\min}$ and $D_{\rho,\min}$) be the maximum (minimum) diagonal entries of the matrices $D_c$ and $D_\rho$, respectively.

**Condition 2.1.** The generalized importance sampling ratios $c(\cdot, \cdot)$ and $\rho(\cdot, \cdot)$ satisfy

1. $c(s, a) \leq \rho(s, a)$ for all $(s, a)$,
2. $D_{\rho,\max} < 1/\gamma$,
3. $\frac{\gamma (D_{\rho,\max} - D_{\rho,\min})}{(1 - \gamma D_{c,\min})\sqrt{K_{SA,\min}}} < 1$.

Condition 2.1 (1) and (2) were previously introduced in Chen et al. (2021b), and were used to show the contraction property of the operator $B_{c,\rho}(\cdot)$. In particular, it was shown that the generalized Bellman operator $B_{c,\rho}(\cdot)$ is a contraction mapping with respect to $\|\cdot\|_\infty$, with contraction factor $\tilde{\gamma}(n) = 1 - f_n(\gamma D_{c,\min})(1 - \gamma D_{\rho,\max})$, where $f_n : \mathbb{R} \mapsto \mathbb{R}$ is defined as $f_n(x) = \sum_{i=0}^{n-1} x^i$ for any $x \geq 0$. It is clear that $\tilde{\gamma}(n) \in (0, 1)$, and is a decreasing function of $n$.

As illustrated earlier, $B_{c,\rho}(\cdot)$ being a contraction mapping is not sufficient to guarantee the stability of Algorithm 1. We need the composed operator $\text{Proj}_Q B_{c,\rho}(\cdot)$ to be contraction mapping with appropriate choice of $n$. This is guaranteed by Condition 2.1 (3). To see this, first note that we have the following lemma, which is obtained by using the contraction property of the operator $B_{c,\rho}(\cdot)$ and the “equivalence” between norms in finite-dimensional spaces.

**Lemma 2.2.** Under Condition 2.1, it holds for any $Q_1, Q_2 \in \mathbb{R}^{|S||A|}$ that

$$\|\text{Proj}_Q B_{c,\rho}(Q_1) - \text{Proj}_Q B_{c,\rho}(Q_2)\|_{K_{SA}} \leq \frac{\tilde{\gamma}(n)}{\sqrt{K_{SA,\min}}} \|Q_1 - Q_2\|_{K_{SA}}$$

In view of Lemma 2.2, the composed operator $\text{Proj}_Q B_{c,\rho}(\cdot)$ can be made a contraction mapping with an appropriate choice of $n$ as long as $\lim_{n \to \infty} \tilde{\gamma}(n)/\sqrt{K_{SA,\min}} < 1$, which after straightforward algebra is equivalent to Condition 2.1 (3).

To satisfy Condition 2.1 (3), intuitively we should make $D_{\rho,\max}$ and $D_{c,\min}$ arbitrarily close to each other. It is not clear if this is possible for existing off-policy TD-learning algorithms such as Retrace($\lambda$) (Munos et al., 2016), $Q^\pi(\lambda)$ (Harutyunyan et al., 2016), V-trace (Espeholt et al., 2018), and Q-trace (Khodadadian, Chen and Maguluri, 2021). That is the reason why none of them were shown to converge in the function approximation setting. In contrast, consider the $A$-averaged $Q$-trace algorithm. Both $D_c$ and $D_\rho$ are identity matrices (which implies $D_{\rho,\max} = D_{c,\min} = 1$), hence Condition 2.1 (3) is always satisfied. Similarly, in the two-sided $Q$-trace algorithm, for any choice of the upper truncation level $u \geq 1$, we can always choose the lower truncation level $0 \leq \ell \leq 1$ appropriately to satisfy Condition 2.1 (3). Specifically, for any $s \in S$ and $u(s) \geq 1$, choosing $\ell(s) \leq 1$ such that $\sum_{a \in A} \pi_0(a|s)\delta_0(s,u(s)) \pi(a|s)/\pi_0(a|s) = 1$ satisfies Condition 2.1 (3). Therefore, compared to V-trace, Retrace($\lambda$), and Q-trace, where the importance sampling ratios were only truncated above, the primary reason for introducing the lower truncation level is to satisfy Condition 2.1 (3), thereby ensuring the convergence of the resulting two-sided $Q$-trace algorithm.

In the next lemma, we show that under Condition 2.1, with properly chosen $n$, the composed operator $\text{Proj}_Q B_{c,\rho}(\cdot)$ is a contraction mapping, which ensures that Eq. (2) has a unique solution, denoted by $w_{c,\rho}^\pi$. Moreover, we provide performance guarantees on the solution $w_{c,\rho}^\pi$ in terms of an upper bound on the difference between $Q^\pi$ and $\Phi w_{c,\rho}^\pi$. Let $Q_{c,\rho}^\pi$ be the solution
of generalized Bellman equation $Q = B_{c,\rho}(Q)$, which is guaranteed to exist and is unique since $B_{c,\rho}(\cdot)$ itself is a contraction mapping under Condition 2.1 (1) and (2) (Chen et al., 2021b).

**Lemma 2.3.** Under Condition 2.1, suppose that the parameter $n$ is chosen such that $\gamma_c := \gamma(n)/\sqrt{K_{SA,\min}} < 1$. Then the composed operator $\text{Proj}_Q B_{c,\rho}(\cdot)$ is a $\gamma_c$-contraction mapping with respect to $\| \cdot \|_{K_{SA}}$. In this case, the unique solution $w^c_{c,\rho}$ of the generalized PBE (cf. Eq. (2)) satisfies

$$
\|Q^c - \Phi w^c_{c,\rho}\|_{K_{SA}} \leq \frac{\|Q^c_{c,\rho} - \text{Proj}_Q Q^c_{c,\rho}\|_{K_{SA}}}{\sqrt{1 - \gamma_c^2}} + \frac{\gamma \max_{s\in S} \|\pi(s) - \pi_b(s)\rho(s, \cdot)\|_1}{(1 - \gamma)(1 - \gamma D_{\rho,\max})}.
$$

(3)

The first term on the RHS of Eq. (3) captures the error due to function approximation, which is in the same spirit to Theorem 1 (4) of the seminal paper Tsitsiklis and Van Roy (1997), and vanishes in the tabular setting. The second term on the RHS of Eq. (3) arises because of the use of generalized importance sampling ratios, which are introduced to overcome the high variance in off-policy learning. Note that the second term vanishes when $\rho(s, a) = \pi(a|s) / \pi_b(a|s)$ for all $(s, a)$, which corresponds to choosing $\lambda = 1$ in $\lambda$-averaged $Q$-trace or choosing $\ell(s) \leq \min_{s,a} \pi(a|s) / \pi_b(a|s)$ and $u(s) \geq \max_{s,a} \pi(a|s) / \pi_b(a|s)$ for all $s$ in two-sided $Q$-trace. However, in these cases, the cumulative product of importance sampling ratios leads to a high variance in Algorithm 1. The trade-off between the variance and the bias in $w^c_{c,\rho}$ (i.e., second term on the RHS of Eq. (3)) will be illustrated in detail in the next subsection.

2.3. Finite-Sample Analysis. With the contraction property of the generalized PBE established, when the stepsize sequence is nonsummable but squared-summable, the almost sure convergence of Algorithm 1 directly follows from standard stochastic approximation results in the literature (Bertsekas and Tsitsiklis, 1996; Borkar, 2009). In this section, we perform finite-sample analysis of Algorithm 1. For ease of exposition, we here only present the finite-sample bounds of $\lambda$-averaged $Q$-trace and two-sided $Q$-trace, where $c(\cdot, \cdot)$ and $\rho(\cdot, \cdot)$ are explicitly specified. For the finite-sample guarantees of Algorithm 1, please see Section A.

For any $\delta > 0$, let $t_\delta = \min\{k \geq 0 : \max_{s\in S} \|P^k_{\pi_b}(s, \cdot) - \mu(\cdot)\|_{TV} \leq \delta\}$ be the mixing time of the Markov chain $\{S_k\}$ induced by $\pi_b$. With precision $\delta$, note that Assumption 2.2 implies that $t_\delta = O(\log(1/\delta))$. Let $\lambda_{\min}$ be the minimum eigenvalue of the positive definite matrix $\Phi^T K_{SA} \Phi$. Let $L = 1 + (\gamma \rho_{\max})^n$, where $\rho_{\max} = \max_{s,a} \rho(s, a)$. We next present finite-sample guarantees of the $\lambda$-averaged $Q$-trace algorithm.

**Theorem 2.1.** Consider $\{w_k\}$ of the $\lambda$-averaged $Q$-trace Algorithm. Suppose that (1) Assumptions 2.1 and 2.2 are satisfied, (2) $\lambda \in [0, 1]$, (3) the parameter $n$ is chosen such that $\gamma_c := \gamma^n / \sqrt{K_{SA,\min}} < 1$. Then, when using constant stepsize $\alpha$ satisfying $\alpha(t_\alpha + n + 1) \leq (1 - \gamma_c)\lambda_{\min}$, we have for all $k \geq t_\alpha + n + 1$ that

$$
\mathbb{E}[\|w_k - w^c_{c,\rho}\|_2^2] \leq c_1 (1 - (1 - \gamma_c)\lambda_{\min}\alpha)^{k - (t_\alpha + n + 1)} + c_2 \frac{\alpha L^2(t_\alpha + n + 1)}{(1 - \gamma_c)\lambda_{\min}},
$$

(4)

where $c_1 = (\|w_0\|_2 + \|w_0 - w^c_{c,\rho}\|_2 + 1)^2$ and $c_2 = 130(\|w^c_{c,\rho}\|_2 + 1)^2$. Moreover, the limit point $w^c_{c,\rho}$ satisfy

$$
\|Q^c - \Phi w^c_{c,\rho}\|_{K_{SA}} \leq \frac{\|Q^c_{c,\rho} - \text{Proj}_Q Q^c_{c,\rho}\|_{K_{SA}}}{\sqrt{1 - \gamma_c^2}} + \frac{\gamma \max_{s\in S} \|\pi(s) - \pi_b(s)\rho(s, \cdot)\|_1}{(1 - \gamma)^2}.
$$

(5)
Using the common terminology in stochastic approximation literature, we call the first term on the RHS of Eq. (4) convergence bias, and the second term variance. When constant stepsize is used, the convergence bias goes to zero at a geometric rate while the variance is a constant roughly proportional to $\alpha t$. Since $\lim_{\alpha \to 0} \alpha t = 0$ under Assumption 2.2, the variance can be made arbitrarily small by using small $\alpha$.

The parameter $L = 1 + (\gamma \rho_{\text{max}})^n$ plays an important role in the finite-sample bound. In fact, $L$ appears quadratically in the variance term of Eq. (4), and captures the impact of the cumulative product of the importance sampling ratios. To overcome the high variance in off-policy learning (i.e., to make sure that the parameter $L = 1 + (\gamma \rho_{\text{max}})^n$ does not grow exponentially fast with respect to $n$), we choose $\lambda \in \mathbb{R}_{>0}$ such that $ho_{\text{max}} = \max_s \lambda(s)(\max_a \pi(a|s)/\pi_b(a|s) - 1) + 1 \leq 1/\gamma$. However, as long as $\lambda \neq 1$, the limit point of the $\lambda$-averaged Q-trace algorithm involves an additional bias term (i.e., the second term on the RHS of Eq. (5)) that does not vanish even in the tabular setting.

In light of the discussion above, it is clear that there is a trade-off between the variance (cf. second term on the RHS of Eq. (4)) and the bias in the limit point (cf. the second term on the RHS of Eq. (3)) in choosing the parameter $\lambda$. Specifically, large $\lambda$ leads to large $\rho_{\text{max}}$ and hence large $L$ and large variance, but in this case the second term on the RHS of Eq. (3) is smaller, implying that we have a smaller bias in the limit point.

Next, we present the finite-sample bounds of the two-sided Q-trace algorithm.

**Theorem 2.2.** Consider $\{w_k\}$ of the two-sided Q-trace Algorithm. Suppose that (1) Assumptions 2.1 and 2.2 are satisfied, (2) the upper and lower truncation levels $\ell, u \in \mathbb{R}_{>0}$ are chosen such that $\sum_{a \in A} \pi_b(a|s)g_{\ell(s),u(s)}(\pi(a|s)/\pi_b(a|s)) = 1$ for all $s$, (3) the parameter $n$ is chosen such that $\gamma_c := \gamma^n / \sqrt{K_{SA\min}} < 1$, and (4) the stepsize $\alpha$ is chosen such that $\alpha(t + n + 1) \leq (1 - \gamma_c)\lambda_{\text{min}}$. Then, we have for all $k \geq t + n + 1$

$$E[\|w_k - w_{\pi,\rho,\ell,u}\|_2^2] \leq c_1(1 - (1 - \gamma_c)\lambda_{\text{min}}\alpha)^{k-(t+\alpha+n+1)} + c_2 \frac{\alpha L^2(t + n + 1)}{(1 - \gamma_c)\lambda_{\text{min}}},$$

where $c_1 = (\|w_0\|_2 + \|w_0 - w_{\pi,\rho}\|_2 + 1)^2$ and $c_2 = 130(\|w_{\pi,\rho}\|_2 + 1)^2$. Moreover, we have

$$\|Q^\pi - \Phi w_{\pi,\rho,\ell,u}\|_{K_{SA}} \leq \frac{1}{1 - \gamma_c^2}\|Q_{\pi,\rho}^\pi - \Phi w_{\pi,\rho,\ell,u}\|_{K_{SA}}$$

$$+ \frac{\gamma \max_{s \in S} \sum_{a \in A} (u_{\pi,\rho}(s, a) - \ell_{\pi,\rho}(s, a))}{(1 - \gamma)^2},$$

where

$$u_{\pi,\rho}(s, a) = \max(\pi(a|s) - \pi_b(a|s)u(s), 0),$$

$$\ell_{\pi,\rho}(s, a) = \min(\pi(a|s) - \pi_b(a|s)\ell(s), 0).$$

The finite-sample bound of the two-sided Q-trace algorithm is qualitatively similar to that of the $\lambda$-averaged Q-trace algorithm. To overcome the high variance issue in off-policy learning, we choose the upper truncation level such that $\gamma u(s) \leq 1$ for all $s$, which ensures that the parameter $L = 1 + (\gamma \rho_{\text{max}})^n \leq 1 + (\gamma \max_s u(s))n$ does not grow exponentially with respect to $n$. Then we choose the lower truncation level accordingly to satisfy requirement (2) stated in Theorem 2.2. However, as long as there exists $s \in S$ such that $u(s) < \max_{s,a} \pi(a|s)/\pi_b(a|s)$ or $\ell(s) > \min_{s,a} \pi(a|s)/\pi_b(a|s)$, the second term on the RHS of Eq. (7) is in general non-zero, hence adding an additional bias term to the limit point even in the tabular setting. As a result, the trade-off between the variance and the bias in the limit point is also present in the two-sided Q-trace algorithm.
In view of Theorems 2.1 and 2.2, one limitation of this work is that the choice of $n$ to make $\gamma_c < 1$ depends on the unknown parameter $K_{S,A,\min}$ of the problem. In practice, one can start with a specific choice of $n$ and then gradually tune $n$ to achieve the convergence of the $\lambda$-averaged $Q$-trace algorithm or the two-sided $Q$-trace algorithm.

3. Policy-Based Methods. In this section we study various policy-based algorithms and establish their finite-sample convergence guarantees. The policy evaluation sub-problem is solved with Algorithm 1.

3.1. Policy Update Rules. We begin by presenting a generic policy-based algorithm in the following. For simplicity of notation, for a given target policy $\pi$, behavior policy $\pi_b$, constant stepsize $\alpha$, initialization $w_0$, and samples $\{(S_k, A_k)\}_{0 \leq k \leq K+n-1}$, we denote the output of Algorithm 1 after $K$ iterations by $w = \text{ALG}(w_0, \pi, \pi_b, \alpha, K, \{(S_k, A_k)\}_{0 \leq k \leq K+n-1})$.

\begin{algorithm}
\caption{A Generic Policy-Based Algorithm}
\begin{algorithmic}[1]
  \State \textbf{Input:} Integers $T$, $K$, initial policy $\pi_0$, sample trajectory $\{(S_t, A_t)\}_{0 \leq t \leq T(K+n)}$ collected under the behavior policy $\pi_b$.
  \For{$t = 0, 1, \ldots, T-1$}
  \State dataset = $\{(S_k, A_k)\}_{t(K+n) \leq k \leq (t+1)(K+n)-1}$
  \State $w_t = \text{ALG}(0, \pi_t, \pi_b, \alpha, K, \text{dataset})$
  \State $\pi_{t+1} = G(\Phi w_t, \pi_t)$
  \EndFor
  \State \textbf{Output:} $\pi_T$.
\end{algorithmic}
\end{algorithm}

Although Algorithm 2 is presented with a fixed behavior policy $\pi_b$, our results can be generalized to the case where the behavior policy is updated across $t$. The only requirement on the behavior policy is that it should enable the agent to sufficiently explore the state-action space. In Algorithm 2 line 5, the function $G(\cdot, \cdot)$ represents the policy update rule, which takes the current policy iterate $\pi_t$ and the $Q$-function estimate $\Phi w_t$ as inputs. Many existing policy update rules fit into this framework, as elaborated below.

$1/\beta_1$-Greedy Update. Let $\beta_1 \in [1, \infty]$ be a tunable parameter. For any $t \geq 0$ and state-action pair $(s, a)$, we update the policy as

$$
\pi_{t+1}(a|s) = \begin{cases} 
\frac{1}{\beta_1|A|}, & a \neq \arg \max_{a' \in A} \phi(s, a')^\top w_t, \\
\frac{1}{\beta_1|A|} + 1 - \frac{1}{\beta_1}, & a = \arg \max_{a' \in A} \phi(s, a')^\top w_t.
\end{cases}
$$

Since $\arg \max_{a' \in A} \phi(s, a')^\top w_t$ is strictly speaking a set, we adopt the convention that whenever the arg max is not unique, we break tie arbitrarily. More generally, we allow the tunable parameter $\beta_1$ to be time-dependent (i.e., $\beta_1$ is a function of the iteration index $t$) and/or state-dependent (i.e., $\beta_1$ is a function of the state $s$).

$\beta_2$-Softmax Update. Let $\beta_2 > 0$ be a tunable parameter, which is allowed to be time varying and state-dependent. Then the policy is updated by

$$
\pi_{t+1}(a|s) = \frac{\exp(\beta_2 \phi(s, a)^\top w_t)}{\sum_{a' \in A} \exp(\beta_2 \phi(s, a')^\top w_t)}, \quad \forall (s, a).
$$

In $1/\beta_1$-greedy update or $\beta_2$-softmax update, there is no need to parametrize the policy because it is uniquely determined by the estimate of the $Q$-function, which already uses linear function approximation.
At a first glance of Algorithm 2 line 5, it seems that we need to work with $|S||A|$-dimensional objects to update the policy at each state-action pair, which contradicts to the motivation of using function approximation. However, there is an equivalent way of implementing Algorithm 2 without explicitly executing line 5. To see this, first note that the target policy $\pi_t$ in each iteration is only used in the policy evaluation step (Algorithm 2 line 4). In view of our policy evaluation algorithm (cf. Algorithm 1), we only need to compute the policy value of $\pi_t$ at state-action pairs that are visited by the sample trajectory $\{(S_k, A_k)\}$. When using $1/\beta_1$-greedy update or $\beta_2$-softmax update, Algorithm 2 subsumes the popular value-based method SARSA (Bertsekas and Tsitsiklis, 1996) as its special case. To see this, suppose that we are in the on-policy setting (i.e., $\pi = \pi_b$), and the inner-loop iteration number $K$ is set to 1. Then Algorithm 2 corresponds to SARSA with $1/\beta_1$-greedy exploration policy or softmax exploration policy. However, we need to point out that our result does NOT imply finite-sample bounds for SARSA since we need a relatively large $K$ to provide a sufficiently accurate estimate of the value function before using it in the policy improvement.

**$\beta_3$-NPG Update.** Unlike $1/\beta_1$-greedy update or $\beta_2$-softmax update, where we need only the estimate of the $Q$-function to perform to update, in NPG, to update the policy, we need both the current policy and the estimate of its $Q$-function. Therefore, to keep track of the policy, in this case we also need to parametrize the policy using softmax parametrization and compatible linear function approximation. Specifically, with parameter $\theta \in \mathbb{R}^d$, the policy $\pi$ associated with parameter $\theta$ is given by $\pi_\theta(a|s) = \frac{\exp(\phi(s,a)\top \theta)}{\sum_{a' \in A} \exp(\phi(s,a')\top \theta)}$.

Let $\beta_3 > 0$ be a tunable parameter, which is allowed to be time varying. Then NPG updates the parameter $\theta_t$ of the policy according to the formula

$$\theta_{t+1} = \theta_t + \beta_3 w_t.$$  

(8)

See Agarwal et al. (2021) for more details about this update rule. Denote $\pi_t$ as $\pi_{\theta_t}$ for simplicity of notation. Then the update equation (8) can be equivalently written in terms of the policy update (and also in the form of Algorithm 2 line 5) as

$$\pi_{t+1}(a|s) = \frac{\pi_t(a|s) \exp(\beta_3 \phi(s,a)\top w_t)}{\sum_{a' \in A} \pi_t(a'|s) \exp(\beta_3 \phi(s,a')\top w_t)}, \quad \forall (s,a).$$

This enables us to use the previous equation for our analysis of NPG while using Eq. (8) for the implementation of Algorithm 2.

### 3.2. Finite-Sample Analysis.

In this section, we present the finite-sample guarantees of Algorithm 2. For ease of exposition, we implement line 4 of Algorithm 2 with the $\lambda$-averaged $Q$-trace algorithm. The results for using either two-sided $Q$-trace algorithm or Algorithm 1 with more general choices of $c(\cdot,\cdot)$ and $\rho(\cdot,\cdot)$ (as long as Condition 2.1 is satisfied) are straightforward extensions. As for the policy improvement (cf. line 5 of Algorithm 2), we use either $1/\beta_1$-greedy policy update, or $\beta_2$-softmax policy update, or $\beta_3$-NPG update, with the corresponding parameters satisfying the following condition. Denote $a_{t,s} = \arg \max_{a' \in A} \phi(s,a')\top w_t$. Recall that we break tie arbitrarily when the $\arg \max$ is not unique. Let $\beta > 0$ be a tunable parameter.

**CONDITION 3.1.** The parameters $\beta_1$, $\beta_2$, and $\beta_3$ satisfy the following requirements.

1. The parameter $\beta_1$ is time-varying and state-dependent, and satisfies

$$\beta_1(t,s) \geq \frac{2\gamma}{\beta} \max_{a \in A} |\phi(s,a)\top w_t|, \quad \forall s,t.$$  

2. The parameter $\beta_2$ is chosen such that $\beta_2 \geq \frac{2}{\beta} \log(|A|)$. 


The parameter $\beta_3$ is time-varying, and satisfies $\beta_3(t) \geq \frac{\gamma}{\beta} \log(1/\min_{s \in S} \pi_t(a_t, s) )$ for all $t$.

**Theorem 3.1.** Consider $\pi_t$ of Algorithm 2. Suppose that the assumptions for applying Theorem 2.1 are satisfied, and the choices of $\beta_1$, $\beta_2$, and $\beta_3$ satisfy Condition 3.1. Then we have for any $T \geq 0$:

$$
\mathbb{E}[(Q^* - Q_{\pi^t})_{\infty}] \leq \frac{2\gamma E_{\text{approx}}}{(1 - \gamma)^2} + \frac{2\gamma^2 E_{\text{bias}}}{(1 - \gamma)^4} + \frac{\gamma^T (Q^* - Q_{\pi^0})_{\infty}}{N_2} \text{ Convergence bias in the actor}
$$

$$
+ \frac{6(1 - (1 - \gamma_c)\lambda_{\min}(\alpha))(K - (t_\alpha + n + 1))}{(1 - \gamma)^3(1 - \gamma_c)\lambda_{\min}^2} \text{ Convergence bias in the critic}
$$

$$
+ \frac{2\gamma^2 \beta}{(1 - \gamma)^2} \text{ N_6: Critic variance}
$$

where $E_{\text{approx}} = \sup_{Q_{\pi}} \|Q_{\pi,c,\rho} - \Phi u_{\pi,c,\rho}\|_{\infty}$ and $E_{\text{bias}} = \max_{0 \leq t \leq T} \max_{s \in S} (1 - \lambda(s))\|\pi_t(\cdot|s) - \pi_0(\cdot|s)\|_1$.

Notably on the LHS, our finite-sample guarantees are stated for the last policy iterate $\pi_T$, while in many existing literature it was stated for the best policy among $\{\pi_t\}_{0 \leq t \leq T}$ (Agarwal et al., 2021), or a uniform sample from $\{\pi_t\}_{0 \leq t \leq T-1}$ (Khodadadian, Chen and Maguluri, 2021; Xu, Wang and Liang, 2020).

**The Terms $N_1$ and $N_2$.** The term $N_1$ represents the function approximation bias, and is present in all existing literature that study policy-based methods under function approximation. Note that $N_1 = 0$ when we use a complete basis. The term $N_2$ represents the bias introduced to the algorithm by using generalized importance sampling ratios $c(\cdot, \cdot)$ and $\rho(\cdot, \cdot)$. Note that we have $N_2 = 0$ when $c(s,a) = \rho(s,a) = \pi(a|s)/\pi_0(a|s)$, which corresponds to using $\lambda = 1$ in the $\lambda$-averaged $Q$-trace algorithm, and using $u(s) \geq \max_{s,a} \pi(a|s)/\pi_0(a|s)$ and $\ell(s) \leq \min_{s,a} \pi(a|s)/\pi_0(a|s)$ for all $s$ in the two-sided $Q$-trace algorithm. However, this choice of $\lambda$ (or $u$ and $\ell$) might lead to a high variance. In particular, the parameter $L$ within the term $N_5$ could be large. Such bias-variance trade-off was illustrated in detail in Section 2.

**The terms $N_3$ and $N_4$.** The term $N_3$ represents the convergence bias in the actor, and goes to zero geometrically fast as the outer loop iteration number $T$ goes to infinity. Such geometric convergence is the main reason why we obtain improved sample complexity of natural actor-critic compared to Chen, Khodadadian and Maguluri (2021), where the convergence rate of the actor is $O(1/T)$. The term $N_4$ represents the convergence bias in the critic, and goes to zero geometrically fast as the inner loop iteration number $K$ goes to infinity.

**The terms $N_5$ and $N_6$.** The term $N_5$ represents the variance in the critic, and is proportional to $\sqrt{\alpha T_\alpha} = O(\sqrt{\alpha \log(1/\alpha)})$. Therefore, $N_5$ can be made arbitrarily small by using small enough stepsize $\alpha$. The term $N_6$ captures the error introduced to the algorithm by the policy update rule $G(\cdot, \cdot)$. To elaborate, consider the following example. Suppose that the underlying MDP model has a unique optimal policy, and suppose we use $1/\beta_1$-greedy update (with a fixed $\beta_1$) in Algorithm 2 line 5. Then as long as $\beta_1$ is finite, we can never truly find the optimal policy $\pi^*$ because of the deterministic nature of $\pi^*$ and the stochastic nature of our policy iterates $\{\pi_t\}$. As a result, the difference between $Q^*$ and $Q_{\pi^t}$ will always be above some threshold, which depends on the choice of $\beta_1$, and is captured by $N_6$. Observe that
$N_6$ can be made arbitrarily small by using small enough $\beta$. Alternatively, we can gradually increase the parameter $\beta$ to eliminate $N_6$, which leads to the following result.

**Theorem 3.2.** Under the same assumptions as in Theorem 3.1, suppose that the tunable parameter $\beta$ is chosen as $\beta = \beta_t = \frac{t}{T}$. Then we have for all $T \geq 0$

$$\mathbb{E} [\|Q^* - Q^{\pi_T}\|_\infty] \leq \sum_{i=0}^{5} N_i + \frac{2\gamma^T}{1 - \gamma}.$$ 

Based on Theorem 3.1 (or Theorem 3.2), we next derive the sample complexity of Algorithm 2. To enable fair comparison with existing literature, we choose $\lambda = 1$ to eliminate the error due to using generalized importance sampling ratios. Note that $\lambda = 1$ implies $\mathcal{E}_{\text{bias}} = 0$ (and hence $N_2 = 0$) in Theorem 3.1 (and Theorem 3.2).

**Corollary 3.2.1.** For a given accuracy level $\epsilon > 0$, to achieve $\mathbb{E} [\|Q^* - Q^{\pi_T}\|_\infty] \leq \epsilon + N_1$, the number of samples (e.g. the integer $TK$) required is of the size

$$\mathcal{O} \left( \frac{\log^3 (1/\epsilon)}{\epsilon^2} \right) \tilde{\mathcal{O}} \left( \frac{L^2 n}{(1 - \gamma)^7 (1 - \gamma_c)^3 \lambda_{\min}^2} \right).$$

Notably, we obtain $\tilde{\mathcal{O}}(\epsilon^{-2})$ sample complexity for policy-based methods, which matches with the sample complexity of value-based algorithms such as $Q$-learning (Li et al., 2020). In the case of $\beta_3$-NPG update, to our knowledge, Cayci, He and Srikant (2021); Cen et al. (2021); Lan (2021) establish the $\tilde{\mathcal{O}}(\epsilon^{-2})$ sample complexity of on-policy NAC under regularization, and Chen, Khodadadian and Maguluri (2021) establishes the $\tilde{\mathcal{O}}(\epsilon^{-3})$ sample complexity of a variant of off-policy NAC (where the infamous deadly triad is present). We improve the sample complexity in Chen, Khodadadian and Maguluri (2021) by a factor of $\epsilon^{-1}$, and we do not use regularization.

In addition to the dependence on $\epsilon$, the dependence on $1/(1 - \gamma)$ (which is usually called the effective horizon) is also improved by a factor of $1/(1 - \gamma)$ compared to existing work (Agarwal et al., 2021; Chen, Khodadadian and Maguluri, 2021). The bootstrapping parameter $n$ appears linearly in our sample complexity bound. This matches with the results for $n$-step TD-learning in the on-policy tabular setting (Chen et al., 2021a).

### 3.3. Geometric Convergence of NPG

Importantly, Theorem 3.1 applies to natural actor-critic. The main reason that we achieve $\tilde{\mathcal{O}}(\epsilon^{-2})$ sample complexity of natural actor-critic is that we establish geometric convergence of NPG. Due to the popularity of NPG, we next present the result separately in the following.

While NPG is a special case of Algorithm 2, we next explicitly describe the NPG algorithm first in the tabular setting and then under linear function approximation. See Agarwal et al. (2021) for more details about the algorithm.

**Algorithm 3 NPG: Tabular Setting**

1: **Input:** Integer $T$, initial policy $\pi_0$
2: **for** $t = 0, 1, \ldots, T - 1$ **do**
3: \[ \pi_{t+1}(a|s) = \frac{\pi_t(a|s) \exp(\beta_3(t)Q^\pi_t(s,a))}{\sum_{a' \in A} \pi_t(a'|s) \exp(\beta_3(t)Q^\pi_t(s,a'))} \] \text{for all } (s,a). 
4: **end for**
5: **Output:** $\pi_T$
THEOREM 3.3. Consider $\pi_T$ generated by Algorithm 3. Suppose that $\beta_3(t) \geq T^{-1} \log(1/\min_{s \in S} \pi_{t}(a_t,s \mid s))$. Then we have for all $T \geq 0$ that

$$
\mathbb{E}[\|Q^* - Q^\pi_T\|_\infty] \leq \gamma^T \left(\|Q^* - Q^\pi_0\|_\infty + \frac{2}{1 - \gamma}\right).
$$

Geometric convergence of NPG has been previously established in Cen et al. (2021); Khodadadian et al. (2022); Lan (2021). The result in Khodadadian et al. (2022) requires complete knowledge about the environmental model, and hence cannot be applied to the RL setting. The authors of Cen et al. (2021) employ entropy regularization in the algorithm design to achieve geometric convergence. Lan (2021) also employs a time-varying regularization, and shows that NPG achieves geometric convergence when using geometrically increasing stepsizes. This is also suggested by Theorem 3.3. Intuitively, the reason is that NPG becomes more and more like policy iteration when using large enough stepsizes, and this is the perspective we are taking in this paper.

Similarly, we next show that NPG with linear function approximation achieves geometric convergence up to a function approximation error. Recall that we use softmax parametrization $\pi_\theta(a \mid s) = \frac{\exp(\phi(s,a)\top \theta)}{\sum_{a' \in A} \exp(\phi(s,a')\top \theta)}$, and we denote $w_{c,\rho}^\pi$ as the solution to the generalized PBE (2).

**Algorithm 4** NPG: Linear Function Approximation

1. **Input:** Integer $T$, initial policy $\pi_0$
2. **for** $t = 0, 1, \ldots, T - 1$ **do**
   3. $\theta_{t+1} = \theta_t + \beta_3(t)w_{c,\rho}^\pi$
4. **end for**
5. **Output:** $\pi_{\theta_T}$

THEOREM 3.4. Consider $\pi_{\theta_T}$ generated by Algorithm 3. Suppose that $\beta_3(t) \geq T^{-1} \log(1/\min_{s \in S} \pi_{\theta_t}(a_t,s \mid s))$. Then we have for all $T \geq 0$ that

$$
\mathbb{E}[\|Q^* - Q^\pi_T\|_\infty] \leq \gamma^T \left(\|Q^* - Q^\pi_0\|_\infty + \frac{2}{1 - \gamma}\right) + \frac{2\gamma\mathcal{E}_{\text{approx}}}{(1 - \gamma)^2}.
$$

4. Conclusion. In this work, we study finite-sample guarantees of general policy-based algorithms under off-policy sampling and linear function approximation. To overcome the deadly triad and the high variance in policy evaluation, we design a convergent framework of TD-learning algorithms, including two specific algorithms called $\lambda$-averaged $Q$-trace and two-sided $Q$-trace. The resulting overall sample complexity bound is $\mathcal{O}(\epsilon^{-2})$, which matches with typical value-based algorithms such as $Q$-learning. In the case of natural actor-critic with function approximation, this advances the existing state-of-the-art result by a factor of $\epsilon^{-1}$.

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APPENDIX A: PROOFS OF THEOREM 2.1 AND THEOREM 2.2

Instead of proving Theorems 2.1 and 2.2, we will state and prove finite-sample bounds for Algorithm 1 with $c(\cdot, \cdot)$ and $\rho(\cdot, \cdot)$ satisfying Condition 2.1, which subsumes Theorems
2.1 and 2.2 as its special cases. In this more general setup where we do not necessarily have 
\( c(\cdot, \cdot) = \rho(\cdot, \cdot) \), we define the constant parameter \( L \) as
\[
L = \begin{cases} 
(1 + (\gamma \rho_{\text{max}})^n), & c(\cdot, \cdot) = \rho(\cdot, \cdot), \\
(1 + \gamma \rho_{\text{max}}) f_n(\gamma c_{\text{max}}), & c(\cdot, \cdot) \neq \rho(\cdot, \cdot),
\end{cases}
\]
(10)
where \( c_{\text{max}} = \max_{s, a} c(s, a) \) and \( \rho_{\text{max}} = \max_{s, a} \rho(s, a) \). For simplicity of notation, we write \( t_k \) for \( t_{\alpha_k} \), which is the mixing time of the Markov chain \( \{S_k\} \) (induced by \( \pi_0 \)) with precision \( \alpha_k \).

**Theorem A.1.** Consider \( \{w_k\} \) of Algorithm 1. Suppose that (1) Assumptions 2.1 and 2.2 are satisfied, (2) the generalized importance sampling ratios satisfy Condition 2.1, (3) the parameter \( n \) is chosen such that \( \gamma_c := \frac{\gamma(n)}{\sqrt{K}\nu_{\text{min}}} < 1 \). Then, when using constant stepsize \( \alpha_k \equiv \alpha \), where \( \alpha \) is chosen such that \( \alpha(t_\alpha + n + 1) \leq \frac{(1-\gamma_c)\lambda_{\text{min}}}{1-\gamma_c} \), we have for all \( k \geq t_\alpha + n + 1 \):
\[
\mathbb{E}[\|w_k - w^\pi_{c,\rho}\|_2^2] \leq c_1 (1 - (1-\gamma_c)\lambda_{\text{min}}\alpha)^{k-t_\alpha-n+1} + c_2 L^2 \alpha (t_\alpha + n + 1) / (1-\gamma_c)\lambda_{\text{min}},
\]
(11)
where \( c_1 = (\|w_0\|_2 + \|w_0 - w^\pi_{c,\rho}\|_2 + 1)^2 \) and \( c_2 = 130(\|w^\pi_{c,\rho}\|_2 + 1)^2 \). When using diminishing stepsizes of the form \( \alpha_k = \frac{\alpha}{k+h} \) with \( \alpha > \frac{1}{(1-\gamma_c)\lambda_{\text{min}}} \) and \( h \) chosen such that
\[
\sum_{i=k-(t_\alpha+n+1)}^{k-1} \alpha_i \leq \frac{(1-\gamma_c)\lambda_{\text{min}}}{130L},
\]
for all \( k \geq k_0 \), where \( k_0 := \min\{k : k \geq t_\alpha + n + 1\} \).

To prove Theorem A.1, we first rewrite Algorithm 1 as a stochastic approximation algorithm. Let \( \{X_k\} \) be a finite-state Markov chain defined as \( X_k = (S_k, A_k, \ldots, S_{k+n}, A_{k+n}) \) for any \( k \geq 0 \). Denote the state-space of \( \{X_k\} \) by \( \mathcal{X} \). It is clear that under Assumption 2.2, the Markov chain \( \{X_k\} \) also admits a unique stationary distribution, which we denote by \( \nu \in \Delta^{\mathcal{X}} \). Let \( F : \mathbb{R}^d \times \mathcal{X} \mapsto \mathbb{R}^d \) be an operator defined by
\[
F(w, x) = \phi(s_0, a_0) \sum_{i=0}^{n-1} \gamma_i c_{1,i} \Delta_i(w)
\]
for any \( w \in \mathbb{R}^d \) and \( x = (s_0, a_0, \ldots, s_n, a_n) \in \mathcal{X} \). Let \( F : \mathbb{R}^d \mapsto \mathbb{R}^d \) be the “expected” operator of \( F(\cdot, \cdot) \) defined by \( \bar{F}(w) = \mathbb{E}_{X \sim \nu}[F(w, X)] \). Using the notation above, the update equation (line 4) of Algorithm 1 can be compactly written as
\[
w_{k+1} = w_k + \alpha_k F(w_k, X_k),
\]
(12)
which is a stochastic approximation algorithm for solving the equation \( \bar{F}(w) = 0 \) with Markovian noise. Note that \( \bar{F}(w) = 0 \) is equivalent to the generalized PBE (2) (cf. Lemma 2.1). We next establish the properties of the operators \( F(\cdot, \cdot), F(\cdot), \) and the Markov chain \( \{X_k\} \) in the following proposition, which enables us to use standard stochastic approximation results in the literature to derive finite-sample bounds of Algorithm 1.

**Proposition A.1.** The following statements hold:
(1) \( \|F(w_1, x) - F(w_2, x)\|_2 \leq L \|w_1 - w_2\|_2 \) for any \( w_1, w_2 \in \mathbb{R}^d \) and \( x \in \mathcal{X} \), and
(2) \( \|F(0, x)\|_2 \leq f_n(\gamma c_{\text{max}}) \) for any \( x \in \mathcal{X} \).
We first rewrite the operator $F(\cdot, \cdot)$ in the following equivalent way. For any $w \in \mathbb{R}^d$ and $x = (s_0, a_0, ..., s_n, a_n) \in \mathcal{X}$, we have

$$F(w, x) = \phi(s_0, a_0) \sum_{i=0}^{n-1} \gamma^i \prod_{j=1}^{i} c(s_j, a_j) \times$$

$$\left( R(s_i, a_i) + \gamma \rho(s_{i+1}, a_{i+1}) \phi(s_{i+1}, a_{i+1})^\top w - \phi(s_i, a_i)^\top w \right)$$

$$= \phi(s_0, a_0) \sum_{i=0}^{n-1} \gamma^i \prod_{j=1}^{i} c(s_j, a_j) R(s_i, a_i) - \phi(s_0, a_0) \sum_{i=0}^{n-1} \gamma^i \prod_{j=1}^{i} c(s_j, a_j) \phi(s_i, a_i)^\top w$$

$$+ \phi(s_0, a_0) \sum_{i=0}^{n-1} \gamma^{i+1} \prod_{j=1}^{i} c(s_j, a_j) \rho(s_{i+1}, a_{i+1}) \phi(s_{i+1}, a_{i+1})^\top w$$

$$= \phi(s_0, a_0) \sum_{i=0}^{n-1} \gamma^i \prod_{j=1}^{i} c(s_j, a_j) R(s_i, a_i) - \phi(s_0, a_0) \sum_{i=0}^{n-1} \gamma^i \prod_{j=1}^{i} c(s_j, a_j) \phi(s_i, a_i)^\top w$$

$$+ \phi(s_0, a_0) \sum_{i=1}^{n} \gamma^i \prod_{j=1}^{i-1} c(s_j, a_j) \rho(s_i, a_i) \phi(s_i, a_i)^\top w$$

$$= \phi(s_0, a_0) \sum_{i=0}^{n-1} \gamma^i \prod_{j=1}^{i} c(s_j, a_j) R(s_i, a_i) - \phi(s_0, a_0) \phi(s_0, a_0)^\top w$$

$$+ \phi(s_0, a_0) \sum_{i=1}^{n-1} \gamma^i \prod_{j=1}^{i-1} c(s_j, a_j) \rho(s_i, a_i) - c(s_i, a_i) \phi(s_i, a_i)^\top w$$

$$+ \phi(s_0, a_0) \gamma^n \prod_{j=1}^{n-1} c(s_j, a_j) \rho(s_n, a_n) \phi(s_n, a_n)^\top w.$$

We now proceed and show the Lipschitz property. For any $w_1, w_2 \in \mathbb{R}^d$ and $x = (s_0, a_0, ..., s_n, a_n) \in \mathcal{X}$, using the fact that $\|\phi(s, a)\|_2 \leq \|\phi(s, a)\|_1 \leq \|\Phi\|_\infty \leq 1$, we have

$$\|F(w_1, x) - F(w_2, x)\|_2$$

$$\leq \|\phi(s_0, a_0) \phi(s_0, a_0)^\top (w_1 - w_2)\|_2$$

$$+ \left\| \phi(s_0, a_0) \sum_{i=1}^{n-1} \gamma^i \prod_{j=1}^{i-1} c(s_j, a_j) (\rho(s_i, a_i) - c(s_i, a_i)) \phi(s_i, a_i)^\top (w_1 - w_2) \right\|_2$$

$$+ \left\| \phi(s_0, a_0) \gamma^n \prod_{j=1}^{n-1} c(s_j, a_j) \rho(s_n, a_n) \phi(s_n, a_n)^\top (w_1 - w_2) \right\|_2.$$
Using the fact that

\[ \text{Assumption } k \]

for all \( (x, \cdot) \). Similarly, for any \( x = (s_0, a_0, \ldots, s_n, a_n) \in \mathcal{X} \), we have

\[ \| F(0, x) \|_2 = \left\| \sum_{i=0}^{n-1} \gamma^i c^i \right\|_2 \leq \sum_{i=0}^{n-1} \gamma^i c^i \leq f_n(\gamma c_{\max}). \]

(2) Under Assumption 2.2, it is clear that the stationary distribution \( \nu \) of the Markov chain \( \{X_k\} \) is given by

\[ \nu(s_0, a_0, \ldots, s_n, a_n) = \mu(s_0) \prod_{i=0}^{n-1} \pi_b(a_i | s_i) P_{a_i}(s_i, s_{i+1}) \pi_b(a_n | s_n), \]

for all \( (s_0, a_0, \ldots, s_n, a_n) \in \mathcal{X} \). Moreover, for any \( x = (s_0, a_0, \ldots, s_n, a_n) \in \mathcal{X} \), we have for any \( k \geq 0 \) that

\[ \text{TV} \left( P_{X_n}^k(x, \cdot) - \nu(\cdot) \right) \leq C \sigma^k. \]

Since the RHS of the previous inequality does not depend on \( x \), we in fact have

\[ \max_{x \in \mathcal{X}} \text{TV} \left( P_{X_n}^k(x, \cdot) - \nu(\cdot) \right) \leq C \sigma^k, \quad \forall k \geq 0. \]

(3) Using the fact that \( \mathcal{B}_{c, \rho}(\cdot) \) is a linear operator, we have for any \( w \in \mathbb{R}^d \) that

\[ (w - w_{c, \rho})^\top \tilde{F}(w) \]
\( (w - w_{c,\rho}^\pi)^\top K_{SA} (B_{c,\rho}(\Phi w) - \Phi w) \)
\( = (w - w_{c,\rho}^\pi)^\top K_{SA} (B_{c,\rho}(\Phi w) - B_{c,\rho}(\Phi w_{c,\rho}^\pi)) - (w - w_{c,\rho}^\pi)^\top K_{SA} \Phi (w - w_{c,\rho}^\pi) \)
\( = (w - w_{c,\rho}^\pi)^\top K_{SA} \Phi (w - w_{c,\rho}^\pi) \)
\( \leq \| \Phi (w - w_{c,\rho}^\pi) \|_{K_{SA}} \| \Phi (w - w_{c,\rho}^\pi) \|_{K_{SA}} - \| \Phi (w - w_{c,\rho}^\pi) \|_{K_{SA}}^2 \)
\( \leq (1 - \gamma_c) \| \Phi (w - w_{c,\rho}^\pi) \|_{K_{SA}} \quad \text{for all} \quad \gamma_c > 0 \).

\( = (1 - \gamma_c) \lambda_{\min} \| w - w_{c,\rho}^\pi \|_2^2. \)

Proposition A.1 (1) establishes the Lipschitz continuity of the operator \( F(\cdot, \cdot) \), Proposition A.1 (2) establishes the geometric mixing of the auxiliary Markov chain \( \{X_k\} \), and Proposition A.1 (3) essentially guarantees that the ODE \( \dot{x}(t) = F(x(t)) \) associated with stochastic approximation algorithm (12) is globally geometrically stable. The rest of the proof follows by applying Theorem 2.1 of Chen et al. (2019) to Algorithm 1. In particular, when using constant stepsize (i.e., \( \alpha_k \equiv \alpha \)) with \( \alpha \) chosen such that \( (1 - \gamma_c) \lambda_{\min} \leq \frac{1}{130L^2} \), we have for all \( k \geq t_\alpha + n + 1 \)

\[ \mathbb{E}[\| w_k - w_{c,\rho}^\pi \|_2^2] \leq c_1 (1 - (1 - \gamma_c) \lambda_{\min} \alpha)^{k - (t_\alpha + n + 1)} + c_2 \frac{\alpha L^2 (t_\alpha + n + 1)}{(1 - \gamma_c) \lambda_{\min}}, \]

where \( c_1 = (\| w_0 \|_2 + \| w_0 - w_{c,\rho}^\pi \|_2 + 1)^2 \) and \( c_2 = 130(\| w_{c,\rho}^\pi \|_2 + 1)^2 \).

When using diminishing stepsizes of the form \( \alpha_k = \frac{\alpha}{k + h} \) with \( \alpha > \frac{1}{(1 - \gamma_c) \lambda_{\min}} \) and \( h \) chosen such that \( \sum_{i=k-(t_\alpha+n+1)}^{k-1} \alpha_i \leq \frac{(1 - \gamma_c) \lambda_{\min}}{130L^2} \) for all \( k \geq t_k + n + 1 \), we have

\[ \mathbb{E}[\| w_k - w_{c,\rho}^\pi \|_2^2] \leq c_1 \frac{k_0 + h}{k + h} + c_2 \frac{8\alpha^2}{(1 - \gamma_c) \lambda_{\min} \alpha - 1} \frac{t_k + n + 1}{k + h} \]

for all \( k \geq k_0 \), where \( k_0 := \min\{k : k \geq t_k + n + 1\} \).

The finite-sample bounds of \( \lambda \)-averaged Q-trace and two-sided Q-trace directly follow from Theorem A.1. To show the performance bound (i.e. Eqs. (5) and (7)) on the limit point \( w_{c,\rho}^\pi \), we apply Lemma 2.3. Note that when \( c(s, a) = \rho(s, a) = \lambda(s) \frac{\pi(a|s)}{\pi(s|a)} + 1 - \lambda(s) \) for all \( (s, a) \), we have for any \( s \in S \)

\[ \sum_{a \in A} |\pi(a|s) - \pi_b(a|s)| \rho(s, a) |(1 - \lambda(s)) \sum_{a \in A} |\pi(a|s) - \pi_b(a|s)| |(1 - \lambda(s)) \| \pi(\cdot|s) - \pi_b(\cdot|s) \|_1. \]
This proves Theorem 2.1. When \( c(s, a) = \rho(s, a) = g_{\ell(u)}(\pi(a|s)/\pi_b(a|s)) \) for all \((s, a)\), we have

\[
\sum_{a \in A} |\pi(a|s) - \pi_b(a|s)\rho(s, a)| = \sum_{a \in A} |(\pi(a|s) - \pi_b(a|s)\ell(s))I\{\pi(a|s) < \ell(s)\pi_b(a|s)}| \\
+ (\pi(a|s) - \pi_b(a|s)u(s))I\{\pi(a|s) > u(s)\pi_b(a|s)}| \\
\leq \sum_{a \in A} |(\pi(a|s) - \pi_b(a|s)\ell(s))I\{\pi(a|s) < \ell(s)\pi_b(a|s)}| \\
+ \sum_{a \in A} |(\pi(a|s) - \pi_b(a|s)u(s))I\{\pi(a|s) > u(s)\pi_b(a|s)}| \\
= \sum_{a \in A} \max(\pi(a|s) - \pi_b(a|s)u(s), 0) - \min(\pi(a|s) - \pi_b(a|s)\ell(s), 0) \\
= \sum_{a \in A} (u_{\pi,\pi_b}(s, a) - \ell_{\pi,\pi_b}(s, a)).
\]

This proves Theorem 2.2.

APPENDIX B: PROOF OF THEOREM 3.1 AND THEOREM 3.2

We first introduce some notation. Let \( \mathcal{H} : \mathbb{R}^{|S||A|} \rightarrow \mathbb{R}^{|S||A|} \) be the Bellman optimality operator defined by

\[
[\mathcal{H}(Q)](s, a) = R(s, a) + \gamma \mathbb{E}\left[ \max_{a' \in A} Q(S_{k+1}, a') | S_k = s, A_k = a \right], \forall Q \in \mathbb{R}^{|S||A|}, \forall (s, a),
\]

and let \( \mathcal{H}_\pi : \mathbb{R}^{|S||A|} \rightarrow \mathbb{R}^{|S||A|} \) be the Bellman operator associated with policy \( \pi \) defined by

\[
[\mathcal{H}_\pi(Q)](s, a) = R(s, a) + \gamma \mathbb{E}_\pi\left[ Q(S_{k+1}, A_{k+1}) | S_k = s, A_k = a \right], \forall Q \in \mathbb{R}^{|S||A|}, \forall (s, a).
\]

The key to prove Theorem 3.1 and Theorem 3.2 is the following proposition.

PROPOSITION B.1. Consider \( \{\pi_T\} \) of Algorithm 2. The following inequality holds for any \( T \geq 0 \):

\[
\mathbb{E}[\|Q^* - Q^{\pi_T}\|_\infty] \leq \frac{2\gamma}{1 - \gamma} \left[ \frac{1}{T} \sum_{t=0}^{T-1} \gamma^{T-1-t}\mathbb{E}[\|Q^{\pi_t} - \Phi w_t\|_\infty] \right]
\]

Before we present the proof of Proposition B.1, we note that in most of the existing literature, for policy-based type of algorithms, the analysis is usually based on the mirror descent analysis in optimization (Lan, 2020), where the KL-divergence was chosen as a potential/Lyapunov function, and the performance difference lemma was extensively used (Agarwal et al., 2021; Cayci, He and Srikant, 2021). To establish Proposition B.1, we
use a completely different approach, where we only exploit the contraction and the monotonicity of the Bellman operators $H \pi(s, a)$ and $H(s)$. Such proof technique was inspired by Bertsekas and Tsitsiklis (1996) Section 6.2. However, only asymptotic error bound of approximate policy iteration was established in Bertsekas and Tsitsiklis (1996), while we establish finite-sample bounds for various policy update rules.

**Proof of Proposition B.1.** For simplicity, denote $\delta_t = \max_{s,a} (Q^\pi_t(s, a) - Q^{\pi_{t+1}}(s, a))$, $\zeta_t = \max_{s,a} (Q^*(s, a) - Q^{\pi_{t}}(s, a)) = \|Q^* - Q^{\pi_{t}}\|_{\infty}$, and $\psi_t = \Phi w_t$ for all $t = 0, 1, \ldots, T$. Note that we have by definition that $Q^{\pi_{t+1}} \geq Q^* - \delta_t \mathbf{1}$ and $Q^\pi_t \geq Q^* - \zeta_t \mathbf{1}$. Using the monotonicity of the Bellman operator (Bertsekas and Tsitsiklis, 1996, Lemma 2.1 and Lemma 2.2) and we have

$$Q^{\pi_{t+1}} = H_{\pi} (Q^{\pi_{t+1}}) \geq H_{\pi} (Q^\pi_t - \delta_t \mathbf{1}) = H_{\pi} (Q^\pi_t - \gamma \delta_t \mathbf{1}).$$

It follows that

$$Q^\pi_t - Q^{\pi_{t+1}} = Q^\pi_t - H_{\pi} (Q^\pi_t - \delta_t \mathbf{1}) \leq Q^\pi_t - H_{\pi} (Q^\pi_t) + \gamma \delta_t \mathbf{1}$$

$$= Q^\pi_t - H_{\pi} (Q^\pi_t) + H_{\pi} (Q^\pi_t) - H_{\pi} (Q^\pi_t) + \gamma \delta_t \mathbf{1}$$

$$\leq H_{\pi} (Q^\pi_t) - H_{\pi} (Q^\pi_t + H_{\pi} (Q^\pi_t) - H_{\pi} (Q^\pi_t) + \gamma \delta_t \mathbf{1}$$

$$\leq 2\gamma \|Q^\pi_t - Q^\pi_t \|_{\infty} + 2\|H_{\pi} (Q^\pi_t) - H_{\pi} (Q^\pi_t)\|_{\infty} + \gamma \delta_t \mathbf{1}.$$ 

Therefore, we have

$$\delta_t \leq 2\gamma \|Q^\pi_t - Q_t \|_{\infty} + \|H_{\pi} (Q^\pi_t) - H_{\pi} (Q^\pi_t)\|_{\infty} + \gamma \delta_t ,$$

which implies

$$\delta_t \leq \frac{2\gamma \|Q^\pi_t - Q_t \|_{\infty} + \|H_{\pi} (Q^\pi_t) - H_{\pi} (Q^\pi_t)\|_{\infty}}{1 - \gamma} . \quad (13)$$

Using the monotonicity of the Bellman operator and we have

$$Q^{\pi_{t+1}} = H_{\pi} (Q^{\pi_{t+1}})$$

$$\geq H_{\pi} (Q^\pi_t - \max_{s,a} (Q^\pi_t(s, a) - Q^{\pi_{t+1}}(s, a)) \mathbf{1})$$

$$= H_{\pi} (Q^\pi_t) - \gamma \max_{s,a} (Q^\pi_t(s, a) - Q^{\pi_{t+1}}(s, a)) \mathbf{1}$$

$$\geq H_{\pi} (Q^\pi_t) - \frac{2\gamma \|Q^\pi_t - Q_t \|_{\infty} + \|H_{\pi} (Q^\pi_t) - H_{\pi} (Q^\pi_t)\|_{\infty}}{1 - \gamma} \mathbf{1} . \quad (14)$$

where the last line follows from Eq. (13). We next control $H_{\pi} (Q^\pi_t)$ from below in the following. Again by monotonicity of the Bellman operator we have

$$H_{\pi} (Q^\pi_t) \geq H_{\pi} (Q^\pi_t) - \|Q^\pi_t - Q^\pi_t \|_{\infty} \mathbf{1}$$

$$= H_{\pi} (Q^\pi_t) - \gamma \|Q^\pi_t - Q^\pi_t \|_{\infty} \mathbf{1}$$

$$= H_{\pi} (Q^\pi_t) - H_{\pi} (Q^\pi_t)^{+} + H_{\pi} (Q^\pi_t) - \gamma \|Q^\pi_t - Q^\pi_t \|_{\infty} \mathbf{1}$$

$$\geq H_{\pi} (Q^\pi_t) - H_{\pi} (Q^\pi_t)^{+} + H_{\pi} (Q^\pi_t - \|Q^\pi_t - Q^\pi_t \|_{\infty} \mathbf{1}) - \gamma \|Q^\pi_t - Q^\pi_t \|_{\infty} \mathbf{1}$$

$$= H_{\pi} (Q^\pi_t) - H_{\pi} (Q^\pi_t)^{+} + H_{\pi} (Q^\pi_t - 2\gamma \|Q^\pi_t - Q^\pi_t \|_{\infty} \mathbf{1}$$

$$\geq H_{\pi} (Q^\pi_t) - H_{\pi} (Q^\pi_t)^{+} + H_{\pi} (Q^\pi_t - \zeta_t \mathbf{1}) - 2\gamma \|Q^\pi_t - Q^\pi_t \|_{\infty} \mathbf{1}$$

$$= H_{\pi} (Q^\pi_t) - H_{\pi} (Q^\pi_t)^{+} + H_{\pi} (Q^\pi_t - \gamma \zeta_t \mathbf{1}) - 2\gamma \|Q^\pi_t - Q^\pi_t \|_{\infty} \mathbf{1}$$

$$\geq -\|H_{\pi} (Q^\pi_t) - H_{\pi} (Q^\pi_t)\|_{\infty} \mathbf{1} + Q^\pi_t - \gamma \zeta_t \mathbf{1} - 2\gamma \|Q^\pi_t - Q^\pi_t \|_{\infty} \mathbf{1}.$$
Using the previous inequality in Eq. (14) and we have

\[ Q^{\pi_{t+1}} - Q^* \geq -\gamma \zeta_t 1 - \frac{2\gamma \|Q^{\pi_t} - Q_t\|_\infty + \|H_{\pi_{t+1}}(Q_t) - H(Q_t)\|_\infty}{1 - \gamma}, \]

which implies

\[ \zeta_{t+1} \leq \gamma \zeta_t + \frac{2\gamma \|Q^{\pi_t} - Q_t\|_\infty + \|H_{\pi_{t+1}}(Q_t) - H(Q_t)\|_\infty}{1 - \gamma}. \]

Repeatedly using the previous inequality and we obtain

\[ \zeta_T \leq \gamma^T \zeta_0 + \frac{2\gamma}{1 - \gamma} \sum_{t=0}^{T-1} \gamma^{T-1-t} \|Q^{\pi_t} - Q_t\|_\infty \\
+ \frac{2\gamma}{1 - \gamma} \sum_{t=0}^{T-1} \gamma^{T-1-t} \|H_{\pi_{t+1}}(Q_t) - H(Q_t)\|_\infty. \]

The result follows by taking expectation on both sides of the previous inequality.

In light of Proposition B.1, to proceed and establish finite-sample bound of Algorithm 2, it remains to control the terms \( A_2 \) and \( A_3 \) when the policy evaluation algorithm and the policy update rule are specified. Specifically, we control \( A_2 \) by using Theorem 2.1, and control \( A_3 \) by using Condition 3.1 on the parameters \( \beta_1, \beta_2, \) and \( \beta_3 \) for various policy update rules.

**B.1. Bounding the Term \( A_2 \).** Using triangle inequality and we have for any \( 0 \leq t \leq T - 1 \):

\[
\mathbb{E}[\|Q^{\pi_t} - \Phi w_t\|_\infty] \\
\leq \mathbb{E}[\|Q^{\pi_t} - \Phi w_{c,\rho}^{\pi_t}\|_\infty] + \mathbb{E}[\|\Phi(w_{c,\rho}^{\pi_t} - w_t)\|_\infty] \\
\leq \mathbb{E}[\|Q^{\pi_t} - \Phi w_{c,\rho}^{\pi_t}\|_\infty] + \Phi \mathbb{E}[\|w_{c,\rho}^{\pi_t} - w_t\|_\infty] \\
\leq \mathbb{E}[\|Q^{\pi_t} - \Phi w_{c,\rho}^{\pi_t}\|_\infty] + \mathbb{E}[\|w_{c,\rho}^{\pi_t} - w_t\|_\infty] \quad (\|\Phi\|_\infty \leq 1) \\
\leq \mathbb{E}[\|Q_{c,\rho}^{\pi_t} - \Phi w_{c,\rho}^{\pi_t}\|_\infty] + \mathbb{E}[\|Q_{c,\rho}^{\pi_t} - Q^{\pi_t}\|_\infty] + \mathbb{E}[\|w_{c,\rho}^{\pi_t} - w_t\|_\infty] \\
\leq \mathcal{E}_{\text{approx}} + \frac{\gamma}{(1 - \gamma)^2} \max_{s \in S} (1 - \lambda(s)) \|\pi_t(\cdot|s) - \pi_{\rho}(\cdot|s)\|_1 + \mathbb{E}[\|w_{c,\rho}^{\pi_t} - w_t\|_\infty] \\
\leq \mathcal{E}_{\text{approx}} + \frac{\gamma}{(1 - \gamma)^2} \mathcal{E}_{\text{bias}} + \mathbb{E}[\|w_{c,\rho}^{\pi_t} - w_t\|_\infty]. \quad (\text{Apply Eq. (21)})
\]

It remains to control \( \mathbb{E}[\|w_{c,\rho}^{\pi_t} - w_t\|_\infty] \), and this is done by applying Theorem 2.1.

Observe that Theorem 2.1 is stated for a fixed target policy \( \pi \), but here the policy iterate \( \pi_t \) is in fact random. However, note that \( \pi_t \) is completely determined by \( \{(S_k, A_k)\}_{0 \leq k \leq t(K+n)} \), while \( w_t \) is determined by \( \{(S_k, A_k)\}_{t(K+n) \leq k \leq t(t+1)(K+n)} \), and Theorem 2.1 holds for any initialization. Therefore, for any \( 0 \leq t \leq T - 1 \), we have by the tower property of conditional expectation, the Markov property, and Theorem 2.1 that

\[
\mathbb{E}[\|w_{c,\rho}^{\pi_t} - w_t\|_\infty] = \mathbb{E}[\|w_{c,\rho}^{\pi_t} - w_t\|_\infty | \{(S_k, A_k)\}_{0 \leq k \leq t(K+n)}] \\
\leq \mathbb{E}[\|w_{c,\rho}^{\pi_t} - w_t\|_2 | \{(S_k, A_k)\}_{0 \leq k \leq t(K+n)}] \\
\leq \left( \mathbb{E}[\|w_{c,\rho}^{\pi_t} - w_t\|_2^2 | \{(S_k, A_k)\}_{0 \leq k \leq t(K+n)}] \right)^{1/2} \quad (\text{Conditional Jensen’s Inequality})
\]
Substituting the upper bound we obtained for $c$ on which policy update rule we use.

$$\arg \max \quad 0 \leq c$$

Therefore we have $c \leq T - t$ and state-action pair $(s, a)$. Note that we have for any policy $\pi$ that

$$\|w^\pi_{c, \rho}\|_2 \leq \frac{1}{\sqrt{\lambda_{\min}}} \|\Phi w^\pi_{c, \rho}\|_{K_{SA}}$$

$$\leq \frac{1}{\sqrt{\lambda_{\min}}} \left( \|Q^\pi_{c, \rho}\|_{K_{SA}} + \frac{1}{\sqrt{1 - \gamma^2(1 - \gamma)}} \right)$$

$$\leq \frac{1}{\sqrt{\lambda_{\min}(1 - \gamma)} \sqrt{1 - \gamma_c}}$$

Therefore we have $c_1, t \leq \frac{3}{\sqrt{\lambda_{\min}(1 - \gamma)} \sqrt{1 - \gamma_c}}$ and $c_2, t \leq \frac{35L}{\sqrt{\lambda_{\min}(1 - \gamma)} \sqrt{1 - \gamma_c}}$ for any $0 \leq t \leq T - 1$.

Substituting the upper bound we obtained for $\mathbb{E}[\|w^\pi_{c, \rho} - w_t\|_\infty]$ into Eq. (15) and we have for any $0 \leq t \leq T - 1$:

$$\mathbb{E}[|Q^\pi_t - \Phi w_t|_\infty] \leq \mathcal{E}_{\text{approx}} + \frac{\gamma \mathcal{E}_{\text{bias}}}{(1 - \gamma)^2} \frac{3(1 - (1 - \gamma_c)\lambda_{\min} \alpha) \frac{1}{2}[K - (t_o + n + 1)]}{\sqrt{\lambda_{\min}(1 - \gamma)} \sqrt{1 - \gamma_c}}$$

$$\leq \frac{35L}{\sqrt{\lambda_{\min}(1 - \gamma)} \sqrt{1 - \gamma_c}} \frac{[\alpha(t_o + n + 1)]^{1/2}}{(1 - \gamma)(1 - \gamma_c)} \lambda_{\min}$$

Finally, using the previous inequality and we obtain the following bound on the term $A_2$:

$$A_2 = \frac{2\gamma}{1 - \gamma} \sum_{t=0}^{T-1} \gamma^{T-1-t} \mathbb{E}[|Q^\pi_t - \Phi w_t|_\infty]$$

$$\leq \frac{2\gamma \mathcal{E}_{\text{approx}}}{(1 - \gamma)^2} + \frac{2\gamma^2 \mathcal{E}_{\text{bias}}}{(1 - \gamma)^3} + \frac{6(1 - (1 - \gamma_c)\lambda_{\min} \alpha) \frac{1}{2}[K - (t_o + n + 1)]}{(1 - \gamma)^3(1 - \gamma_c) \lambda_{\min}^{1/2}}$$

$$+ \frac{70L}{\lambda_{\min}(1 - \gamma)(1 - \gamma_c)^3} \frac{[\alpha(t_o + n + 1)]^{1/2}}{(1 - \gamma)(1 - \gamma_c)} \lambda_{\min}$$

**B.2. Bounding the Term $A_3$.** Now consider the term $A_3$, whose upper bound depends on which policy update rule we use.

**B.2.1. $1/\beta_1$-Greedy Update.** Recall our notation that $Q_t = \Phi w_t$ and $a_{t,s} = \arg \max_{a \in A} Q_t(s, a)$. Using the definition of $\mathcal{H}(\cdot)$ and $\mathcal{H}_\pi(\cdot)$ and we have for any $0 \leq t \leq T - 1$ and state-action pair $(s, a)$ that

$$0 \leq [\mathcal{H}(Q_t)](s, a) - [\mathcal{H}_{\pi_t+1}(Q_t)](s, a)$$

$$= \gamma \mathbb{E}_{a' \in A} [Q_{t+1}(s, a') - Q_t(s, A_1) \mid S_0 = s, A_0 = a] \quad (A_1 \sim \pi_{t+1}(\cdot|S_1))$$

$$= \gamma \sum_{s'} P_a(s, s') \times$$
\[
\left(\frac{1}{\beta_1(t, s') - |A|\beta_1(t, s')} - \frac{1}{|A|\beta_1(t, s')}\right)Q_t(s', a_{t,s'}) - \sum_{a' \neq a_{t,s'}} \frac{1}{|A|\beta_1(t, s')}Q_t(s', a')
\]
\[
\leq \gamma \sum_{s'} P_a(s, s') \frac{2}{\beta_1(t, s')} \max_{a' \in A}|Q_t(s', a')|.
\]

When using \(\beta_1(t, s) \geq \frac{2\gamma}{\beta} \max_{a \in A}|Q_t(s, a)|\) for all \(s \in S\) (cf. Condition 3.1 (1)), we have
\[
A_3 \leq \frac{2\gamma}{1 - \gamma} \sum_{t=0}^{T-1} \gamma^{T-1-t} \beta \leq \frac{2\gamma \beta}{(1 - \gamma)^2}.
\]

When using \(\beta_1(t, s) \geq \frac{2\gamma}{T} \max_{a \in A}|Q_t(s, a)|\), we have
\[
A_3 \leq \frac{2\gamma}{T(1 - \gamma)} \sum_{t=0}^{T-1} \gamma^{T-1} = \frac{2\gamma T}{1 - \gamma}.
\]

**B.2.2. \(\beta_2\)-Softmax Update.** The following lemma is needed for us to control the term \(A_3\) when using softmax policy update.

**Lemma B.1.** For any \(x \in \mathbb{R}^d\) and \(y \in \Delta^d\) satisfying \(y_i > 0\) for all \(i\), denote \(i_{\max} = \arg \max_{1 \leq i \leq d} x_i\), then the following inequality holds for any \(\beta > 0\):
\[
\max_{1 \leq i \leq d} x_i - \frac{\sum_{i=1}^d x_i y_i e^{\beta x_i}}{\sum_{j=1}^d y_j e^{\beta x_j}} \leq \frac{1}{\beta} \log \left( \frac{1}{y_{i_{\max}}} \right).
\]

**Proof of Lemma B.1.** For any \(\beta > 0\), consider the function \(h_\beta : \mathbb{R}^d \rightarrow \mathbb{R}\) defined by
\[
h_\beta(x) = \frac{1}{\beta} \log \left( \sum_{i=1}^d y_i e^{\beta x_i} \right).
\]
Assume without loss of generality that \(i_{\max} = 1\). Then it is clear that \(h_\beta(x) \leq x_1\). On the other hand, we have
\[
x_1 \leq \frac{1}{\beta} \log \left( \sum_{i=1}^d y_i e^{\beta x_i} \right) = h_\beta(x) + \frac{1}{\beta} \log \left( \frac{1}{y_1} \right). \quad (16)
\]
Since it is well-known that \(h_\beta(x)\) is a convex differentiable function, we have for any \(x \in \mathbb{R}^d\) that \(h_\beta(0) - h_\beta(x) \geq \langle \nabla h_\beta(x), -x \rangle\), which implies
\[
\langle \nabla h_\beta(x), x \rangle = \frac{\sum_{i=1}^d x_i y_i e^{\beta x_i}}{\sum_{j=1}^d y_j e^{\beta x_j}} \geq h_\beta(x) - h_\beta(0) = h_\beta(x). \quad (17)
\]
Using Eqs. (16) and (17) and we finally obtain
\[
\max_{1 \leq i \leq d} x_i - \frac{\sum_{i=1}^d x_i y_i e^{\beta x_i}}{\sum_{j=1}^d y_j e^{\beta x_j}} \leq x_1 - h_\beta(x) \leq \frac{1}{\beta} \log \left( \frac{1}{y_1} \right). \]

\(\square\)
We now proceed to control the term $A_3$ when using the $\beta_2$-softmax update. For any $0 \leq t \leq T - 1$ and state-action pair $(s, a)$, we have

$$0 \leq [\mathcal{H}(Q_t)](s, a) - [\mathcal{H}_{\pi_{t+1}}(Q_t)](s, a)$$

$$= \sum_{s'} P_a(s, s') \left( \max_{a' \in A} Q_t(s', a') - \sum_{a' \in A} \frac{\exp(\beta_2 Q_t(s', a'))}{\sum_{a'' \in A} \exp(\beta_2 Q_t(s', a''))} Q_t(s', a') \right)$$

$$= \sum_{s'} P_a(s, s') \left( \max_{a' \in A} Q_t(s', a') - \frac{\exp(\beta_2 Q_t(s', a'))}{|A|} \sum_{a'' \in A} \exp(\beta_2 Q_t(s', a'')) \right)$$

$$\leq \frac{\gamma}{\beta_2} \log(|A|). \quad \text{(Lemma B.1)}$$

When $\beta_2 \geq \frac{\gamma}{\beta} \log(|A|)$, we have

$$A_3 \leq \frac{2\gamma}{1 - \gamma} \sum_{t=0}^{T-1} \gamma^{T-1-t} \beta \leq \frac{2\gamma \beta}{(1 - \gamma)^2}.$$ 

When $\beta_2 \geq \frac{T \log |A|}{\gamma}$, we have

$$A_3 \leq \frac{2\gamma}{(1 - \gamma)T} \sum_{t=0}^{T-1} \gamma^{T-1} = \frac{2\gamma T}{1 - \gamma}.$$ 

B.2.3. $\beta_3$-NPG Update. Recall that $\beta_3$-NPG updates the policy according to

$$\pi_{t+1}(a|s) = \frac{\pi_t(a|s) \exp(\beta_3(t) Q_t(s, a))}{\sum_{a' \in A} \pi_t(a'|s) \exp(\beta_3(t) Q_t(s, a'))}, \quad \forall (s, a).$$

Therefore, for any $0 \leq t \leq T - 1$ and state-action pair $(s, a)$, we have

$$0 \leq [\mathcal{H}(Q_t)](s, a) - [\mathcal{H}_{\pi_{t+1}}(Q_t)](s, a)$$

$$= \sum_{s'} P_a(s, s') \times$$

$$\left( \max_{a' \in A} Q_t(s', a') - \sum_{a' \in A} \frac{\pi_t(a'|s') \exp(\beta_3(t) Q_t(s', a'))}{\sum_{a'' \in A} \pi_t(a''|s') \exp(\beta_3(t) Q_t(s', a''))} Q_t(s', a') \right)$$

$$\leq \frac{\gamma}{\beta_3(t)} \log \left( \frac{1}{\pi_t(a_{t,s}|s')} \right). \quad \text{(Lemma B.1)}$$

When using $\beta_3(t) \geq \frac{\gamma}{\beta} \log(1/ \min_{s \in S} \pi_t(a_{t,s}|s))$, we have

$$A_3 \leq \frac{2\gamma}{1 - \gamma} \sum_{t=0}^{T-1} \gamma^{T-1-t} \beta \leq \frac{2\gamma \beta}{(1 - \gamma)^2}.$$ 

When using $\beta_3(t) \geq \frac{T \log (1/ \min_{s \in S} \pi_t(a_{t,s}|s))}{\gamma}$, we have

$$A_3 \leq \frac{2\gamma}{(1 - \gamma)T} \sum_{t=0}^{T-1} \gamma^{T-1} = \frac{2\gamma T}{1 - \gamma}.$$
B.3. Putting Together. Using the upper bounds we obtained for the terms $A_2$ and $A_3$ in Proposition B.1, when using constant $\beta$, we have for any $K \geq t_\alpha + n + 1$ and $T \geq 0$ that

$$
\mathbb{E}[\|Q^* - Q^{\pi^*_T}\|_\infty] \leq \gamma^T \|Q^* - Q^{\pi_0}\|_\infty + \frac{2\gamma \mathcal{E}_{\text{approx}}}{(1-\gamma)^2} + \frac{2\gamma^2 \mathcal{E}_{\text{bias}}}{(1-\gamma)^4} + \frac{6(1 - (1 - \gamma_c)\lambda_{\min}(\alpha))^{1/2}|K-(t_\alpha + n + 1)|}{(1-\gamma)^3(1-\gamma)^{1/2}\lambda_{\min}^{1/2}} + \frac{70L[\alpha(t_\alpha + n + 1)]^{1/2}}{\lambda_{\min}(1 - \gamma_c)(1-\gamma)^3} + \frac{2\gamma^T}{1-\gamma}.
$$

When using time-varying $\beta$ such that $\beta = \beta_t = \frac{T}{\gamma}$, we have for any $K \geq t_\alpha + n + 1$ and $T \geq 0$ that

$$
\mathbb{E}[\|Q^* - Q^{\pi^*_T}\|_\infty] \leq \gamma^T \|Q^* - Q^{\pi_0}\|_\infty + \frac{2\gamma \mathcal{E}_{\text{approx}}}{(1-\gamma)^2} + \frac{2\gamma^2 \mathcal{E}_{\text{bias}}}{(1-\gamma)^4} + \frac{6(1 - (1 - \gamma_c)\lambda_{\min}(\alpha))^{1/2}|K-(t_\alpha + n + 1)|}{(1-\gamma)^3(1-\gamma)^{1/2}\lambda_{\min}^{1/2}} + \frac{70L[\alpha(t_\alpha + n + 1)]^{1/2}}{\lambda_{\min}(1 - \gamma_c)(1-\gamma)^3} + \frac{2\gamma^T}{1-\gamma}.
$$

APPENDIX C: PROOF OF THEOREMS 3.3 AND 3.4

We start from the result of Proposition B.1 presented in the following:

$$
\mathbb{E}[\|Q^* - Q^{\pi^*_T}\|_\infty] \leq \gamma^T \|Q^* - Q^{\pi_0}\|_\infty + \frac{2\gamma}{1-\gamma} \sum_{t=0}^{T-1} \gamma^{T-1-t}\mathbb{E}[\|Q^{\pi^*_t} - \Phi w_t\|_\infty] + \frac{2\gamma}{1-\gamma} \sum_{t=0}^{T-1} \gamma^{T-1-t}\mathbb{E}[\|\mathcal{H}_{\pi_{t+1}}(\Phi w_t) - \mathcal{H}(\Phi w_t)\|_\infty].
$$

In tabular NPG, since we do not require a critic to perform policy evaluation and $\Phi = I$, we have $A_2 = 0$. In addition, we have $A_3 \leq \frac{2\gamma^T}{1-\gamma}$ (see Subsection B.2.3). This proves Theorem 3.3.

When using linear function approximation, we have

$$
A_2 \leq \frac{2\gamma}{1-\gamma} \sum_{t=0}^{T-1} \gamma^{T-1-t}\mathcal{E}_{\text{approx}} \leq \frac{2\gamma}{(1-\gamma)^2}\mathcal{E}_{\text{approx}}
$$

and similarly $A_3 \leq \frac{2\gamma^T}{1-\gamma}$. This proves Theorem 3.4.

APPENDIX D: PROOF OF ALL TECHNICAL RESULTS IN SECTION 2

D.1. Proof of Lemma 2.1. We begin by introducing some notation. Let $\pi_c$ and $\pi_\rho$ be two policies defined by

$$
\pi_c(a|s) = \frac{\pi_b(a|s)c(s,a)}{\sum_{a' \in A}\pi_b(a'|s)c(s,a')}, \quad \text{and} \quad \pi_\rho(a|s) = \frac{\pi_b(a|s)\rho(s,a)}{\sum_{a' \in A}\pi_b(a'|s)\rho(s,a')}, \quad \forall (s,a).
$$
Let \( P_\pi \) and \( P_\rho \) be the transition probability matrices of the Markov chain \( \{ S_k \} \) induced by the policies \( \pi_c \) and \( \pi_\rho \), respectively. Then, Eq. (1) can be compactly written in vector form as

\[
\Phi^\top \mathcal{K}_{SA} \sum_{i=0}^{n-1} (\gamma P_\pi D_c)^i (R + \gamma P_\rho D_\rho \Phi w - \Phi w) = 0,
\]

where \( R \in \mathbb{R}^{|S||A|} \) is defined by \( R(s,a) = \mathcal{R}(s,a) \) for all \((s,a)\). Observe that the above equation is further equivalent to

\[
\Phi(\Phi^\top \mathcal{K}_{SA})^{-1} \Phi^\top \mathcal{K}_{SA} \sum_{i=0}^{n-1} (\gamma P_\pi D_c)^i (R + \gamma P_\rho D_\rho \Phi w - \Phi w) = 0. \tag{18}
\]

To see this, note that the matrix \( \Phi \) has full column-rank, and the matrix \( \Phi^\top \mathcal{K}_{SA} \Phi \) is positive definite and hence invertible. Therefore, we have \( x = 0 \) if and only if \( \Phi(\Phi^\top \mathcal{K}_{SA})^{-1} x = 0 \).

To rewrite Eq. (18) in the desired form of the generalized PBE (2), we use the following three observations.

(1) The projection operator \( \text{Proj}_Q(\cdot) \) is explicitly given by

\[
\text{Proj}_Q(\cdot) = \Phi(\Phi^\top \mathcal{K}_{SA} \Phi)^{-1} \Phi^\top \mathcal{K}_{SA}(\cdot).
\]

(2) The operator \( \mathcal{T}_c(\cdot) \) is explicitly given by \( \mathcal{T}_c(\cdot) = \sum_{i=0}^{n-1} \gamma (\gamma P_\pi D_c)^i (\cdot) \).

(3) The operator \( \mathcal{H}_\rho(\cdot) \) is explicitly given by \( \mathcal{H}_\rho(\cdot) = R + \gamma P_\rho D_\rho (\cdot) \).

Therefore, Eq. (18) is equivalent to

\[
\text{Proj}_Q[\mathcal{T}_c(\mathcal{H}_\rho(\Phi w) - \Phi w)] = 0. \tag{19}
\]

Finally, adding and subtracting \( \Phi w \) on both sides of the previous inequality and we obtain the desired generalized PBE:

\[
\Phi w = \text{Proj}_Q[\mathcal{T}_c(\mathcal{H}_\rho(\Phi w) - \Phi w)] + \Phi w
= \text{Proj}_Q[\mathcal{T}_c(\mathcal{H}_\rho(\Phi w) - \Phi w) + \Phi w]
= \text{Proj}_Q B_{c,\rho}(\Phi w),
\]

where the second equality follows from (1) \( \Phi w \in Q \) and (2) \( \text{Proj}_Q(\cdot) \) is a linear operator.

**D.2. Proof of Lemma 2.2.** For any \( Q_1, Q_2 \in \mathbb{R}^{|S||A|} \), using the fact that \( \text{Proj}_Q \) is non-expansive with respect to \( \| \cdot \|_{\mathcal{K}_{SA}} \), we have

\[
\| \text{Proj}_Q B_{c,\rho}(Q_1) - \text{Proj}_Q B_{c,\rho}(Q_2) \|_{\mathcal{K}_{SA}} \leq \| B_{c,\rho}(Q_1) - B_{c,\rho}(Q_2) \|_{\mathcal{K}_{SA}}
\leq \| B_{c,\rho}(Q_1) - B_{c,\rho}(Q_2) \|_\infty \quad (\| \cdot \|_{\mathcal{K}_{SA}} \leq \| \cdot \|_\infty)
\leq \tilde{\gamma}(n) \| Q_1 - Q_2 \|_\infty
\leq \frac{\tilde{\gamma}(n)}{\sqrt{\mathcal{K}_{SA,\min}}} \| Q_1 - Q_2 \|_{\mathcal{K}_{SA}},
\]

where the third inequality follows from \( B_{c,\rho}(\cdot) \) being a \( \tilde{\gamma}(n) \)-contraction operator with respect to \( \| \cdot \|_\infty \) (Chen et al., 2021b) \footnote{Chen et al. (2021b) works with an asynchronous variant of the generalized Bellman operator, which is shown to be a contraction mapping with respect to \( \| \cdot \|_\infty \) with contraction factor \( 1 - \mathcal{K}_{SA,\min} \pi(n) (\gamma D_{c,\min}) (1 - \gamma D_{\rho,\max}) \). In this paper we work with the synchronous generalized Bellman operator \( B_{c,\rho}(\cdot) \). In this case, one can easily verify that the corresponding contraction factor can be obtained by simply dropping the factor \( \mathcal{K}_{SA,\min} \).}.


D.3. Proof of Lemma 2.3. We first show that under Condition 2.1 (3), we have
\[ \lim_{n \to \infty} \frac{\tilde{\gamma}(n)}{\sqrt{K_{SA,\min}}} < 1. \]
Using the explicit expression of \( \tilde{\gamma}(n) \), we have
\[
\begin{align*}
\lim_{n \to \infty} \frac{\tilde{\gamma}(n)}{\sqrt{K_{SA,\min}}} &= \lim_{n \to \infty} \frac{1 - f_n(\gamma D_{c,\min})(1 - \gamma D_{\rho,\max})}{\sqrt{K_{SA,\min}}} \\
&= \lim_{n \to \infty} \frac{1 - \frac{1}{1 - \gamma D_{c,\min}}(1 - \gamma D_{\rho,\max})}{\sqrt{K_{SA,\min}}} \\
&= \frac{\gamma(D_{\rho,\max} - D_{c,\min})}{(1 - \gamma D_{c,\min})\sqrt{K_{SA,\min}}} < 1. \quad \text{(Condition 2.1 (3))}
\end{align*}
\]
Therefore, when \( n \) is chosen such that \( \gamma_c = \frac{\tilde{\gamma}(n)}{\sqrt{K_{SA,\min}}} < 1 \), we have by Lemma 2.2 that
\[
\|\text{Proj}_{\mathcal{Q}}\mathcal{B}_{c,\rho}(Q_1) - \text{Proj}_{\mathcal{Q}}\| \leq \gamma_c\|Q_1 - Q_2\|_{K_{SA}}, \quad \forall \, Q_1, Q_2 \in \mathbb{R}^{\mathcal{S}||\mathcal{A}||}.
\]
It follows that the composed operator \( \text{Proj}_{\mathcal{Q}}\mathcal{B}_{c,\rho}(\cdot) \) is a contraction mapping with respect to \( \| \cdot \|_{K_{SA}} \), with contraction factor \( \gamma_c \).

Next consider the difference between \( Q^\pi \) and \( \Phi w_{c,\rho}^\pi \). First of all, we have by triangle inequality that
\[
\|Q^\pi - \Phi w_{c,\rho}^\pi\|_{K_{SA}} \leq \|Q^\pi - Q^\pi_{c,\rho}\|_{K_{SA}} + \|Q^\pi_{c,\rho} - \Phi w_{c,\rho}^\pi\|_{K_{SA}}.
\]
We next bound each term on the RHS of the previous inequality. For the first term, it was already established in Proposition 2.1 of Chen et al. (2021b) that
\[
\|Q^\pi - Q^\pi_{c,\rho}\|_{K_{SA}} \leq \|Q^\pi - Q^\pi_{c,\rho}\|_{\infty} \leq \frac{\gamma_{\max_{s \in \mathcal{S}}} \sum_{a \in \mathcal{A}} |\pi(a\mid s) - \pi(b\mid s)\rho(s, a)|}{(1 - \gamma)(1 - \gamma D_{\rho,\max})}.
\]  \hspace{1cm} (21)

Now consider the second term on the RHS of Eq. (20). First note that
\[
\|Q^\pi_{c,\rho} - \Phi w_{c,\rho}^\pi\|_{K_{SA}} \leq \frac{1}{\sqrt{1 - \gamma_c^2}}\|Q^\pi_{c,\rho} - \text{Proj}_{\mathcal{Q}}Q^\pi_{c,\rho}\|_{K_{SA}}.
\]  \hspace{1cm} (22)

Substituting Eqs. (21) and (22) into the RHS of Eq. (20) and we finally obtain
\[
\|Q^\pi - \Phi w_{c,\rho}^\pi\|_{K_{SA}} \leq \frac{\gamma_{\max_{s \in \mathcal{S}}} \sum_{a \in \mathcal{A}} |\pi(a\mid s) - \pi(b\mid s)\rho(s, a)|}{(1 - \gamma)(1 - \gamma D_{\rho,\max})} + \frac{1}{\sqrt{1 - \gamma_c^2}}\|Q^\pi_{c,\rho} - \text{Proj}_{\mathcal{Q}}Q^\pi_{c,\rho}\|_{K_{SA}}.
\]
APPENDIX E: THE HIGH VARIANCE IN Chen, Khodadadian and Maguluri (2021)

Consider Theorem 2.1 of Chen, Khodadadian and Maguluri (2021). The constant $c_2$ on the second term is proportional to $\sum_{i=0}^{n-1} (\gamma \max_{s,a} \frac{\pi(a|s)}{\pi_b(a|s)})^i$ (which appears as $f(\gamma \zeta \pi)$ using the notation of Chen, Khodadadian and Maguluri (2021)). When $\frac{\pi(a|s)}{\pi_b(a|s)} > 1/\gamma$ (which can usually happen in practice where $\gamma$ is chosen to be close to 1), the parameter $c_2$ grows exponentially fast with respect to the bootstrapping parameter $n$. Moreover, since $n$ needs to be chosen large enough for the results in Chen, Khodadadian and Maguluri (2021) to hold, the variance term on the finite-sample bound of the $n$-step off-policy TD-learning algorithm with linear function approximation is exponentially large.