Axisymmetric gravitational MHD equilibria in the presence of plasma rotation

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Abstract

In this paper, extending the investigation developed in an earlier paper (Cremaschini et al., 2008), we pose the problem of the kinetic description of gravitational Hall-MHD equilibria which may arise in accretion disks (AD) plasmas close to compact objects. When intense EM and gravitational fields, generated by the central object, are present, a convenient approach can be achieved in the context of the Vlasov-Maxwell description. In this paper the investigation is focused primarily on the following two aspects:

1) the formulation of the kinetic treatment of G-Hall-MHD equilibria. Based on the identification of the relevant first integrals of motion, we show that an explicit representation can be given for the equilibrium kinetic distribution function. For each species this is represented as a superposition of suitable generalized Maxwellian distributions;

2) the determination of the constraints to be placed on the fluid fields for the existence of the kinetic equilibria. In particular, this permits a unique determination of the functional form of the species number densities and of the fluid partial pressures, in terms of suitably prescribed flux functions.

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I. INTRODUCTION

An open issue in astrophysics is understanding the dynamics of accretion disk (AD) plasmas occurring near compact objects (black holes, neutron stars, etc.) [1, 2, 3, 4, 5, 6]. It is well known that this problem must be generally formulated in the context of kinetic theory for the proper description of macroscopic plasma dynamics. In fact, unless direct information about the fluid fields which characterize the plasma is available (for example from theory, particle numerical simulations or direct observations), a purely fluid treatment of these plasmas may pose serious difficulties. These are related to the notorious impossibility of uniquely defining consistent closure conditions for the fluid equations. A convenient solution can, however, be reached by adopting a kinetic approach. An important motivation for this investigation is, in fact, that of avoiding the closure problem characteristic of fluid equations (see the related discussion in Paper I, Cremaschini et al., 2008). A first step in this direction, realized in this paper, is related to the investigation of kinetic equilibria (in this context, what is meant by an "equilibrium" is stationary-flow solution). The purpose of this paper is to investigate, in particular, kinetic axi-symmetric gravitational Hall-MHD (G-Hall-MHD) equilibria occurring in AD plasmas arising around compact stars, which are locally characterized by the presence of a family of nested axi-symmetric toroidal magnetic surfaces \( \{ \psi(r) = \text{const} \} \), \( \psi \) denoting the so-called poloidal flux. These plasmas are expected to be collisionless and characterized by the presence of intense EM fields, as well as - at the same time - by a strong gravitational field generated by the central object. Extending the investigation developed in an earlier paper (see Paper I) and based on a fluid description, in this paper we intend to analyze, in particular, the kinetic constraints placed on the relevant fluid fields by the kinetic treatment of G-Hall-MHD equilibria, i.e., the requirement that the kinetic distribution function is itself a stationary solution of the relevant kinetic equation. Ignoring possible weakly-dissipative and time-dependent effects as well as assuming the possible presence of a radial flow velocity in the AD, we shall assume - in particular - that the kinetic distribution function and the EM fields associated with the plasma obey the system of Vlasov-Maxwell equations. In addition, the effect of the gravitational field produced by the central object is treated by the introduction of a pseudo-Newtonian gravitational potential which permits to retain weakly relativistic effects on the AD plasma dynamics. In such a case, in general, the form of the equilibrium kinetic distribution remains completely
arbitrary. The only restriction on its form, besides its (strict) positivity and the assumption of its suitable smoothness in the relevant phase-space, is obviously due to the requirement that it must be a function only of the independent first integrals of motion. As such, it is always possible to represent it as superpositions of suitable generalized Maxwellian distributions (GMD). In this paper we wish to analyze the constraints placed on them by the assumption of the existence of kinetic equilibria of this type. We intend to prove, in particular, that:

1. The assumption of kinetic equilibrium determines uniquely the functional form of the number densities, flow velocities and temperatures carried by each GMD, which are found to depend on appropriate flux functions to be suitably prescribed.

2. In particular, we intend to prove that the flow velocities carried by each GMD are species-dependent but coincide, for each GMD belonging to the same species.

3. In addition, the related angular velocities $[\Omega_s]$ are constant on each toroidal magnetic surface $\{\psi(r) = \text{const}\}$, while the form of $\Omega_s$ is uniquely prescribed in terms of normal derivatives of suitable flux-surface averages of the electrostatic and gravitational pseudo-Newtonian potentials.

II. KINETIC G-HALL-MHD EQUILIBRIA

As a starting point, we require that AD plasma admits fluid equilibrium, i.e., a G-Hall-MHD equilibrium in the sense defined in Paper I. In particular, we assume that the equilibrium magnetic field $B$ admits, at least locally, a family of magnetic surfaces $\{\psi(r) = \text{const}\}$, all mutually nested and bounded, which are represented by smooth toroidal surfaces, $\psi$ denoting the poloidal magnetic flux of $B$ (see Paper I). By assumption, these surfaces and all equilibrium fluid fields are axi-symmetric. This means that the relevant dynamical variables characterizing the plasma are required to be independent of the toroidal angle $\varphi$, when referred to a set of cylindrical coordinates $(R, \varphi, z)$. In particular, the axis $R = 0$ can be identified with the principal axis of the toroidal surfaces. The assumption of the existence of a family of nested magnetic surfaces $\{\psi(r)\}$ implies that locally a set of magnetic coordinates $(\psi, \varphi, \vartheta)$ can be defined, with $\vartheta$ denoting, in particular, a curvilinear angle-like coordinate on a magnetic surface $\psi(r) = \text{const}$. For definiteness we shall assume,
furthermore, that the plasma is: a) *non-relativistic*, in the sense that - for an inertial observer at rest with respect to the central object and in the considered region of the AD - the species flow velocities are much smaller, in magnitude, than the speed of light in vacuum; b) *collisionless*, so that the mean free path of the particles of the plasma is assumed to be much larger than the largest characteristic scale length of the plasma; c) the plasma can be treated in the *pseudo-Newtonian approximation*, whereby the gravitational force produced by the central object is treated by means of an appropriate pseudo-Newtonian gravitational potential acting on the plasma.

Let us now introduce the key assumption that the G-Hall-MHD equilibrium also corresponds to a kinetic equilibrium, namely that the kinetic distribution functions characterizing the plasma species are stationary solutions of the Vlasov kinetic equation. We stress that the condition of kinetic equilibrium is here assumed to apply perhaps only in a local sense, i.e., in an appropriate sub-domain of the AD where the plasma can be treated, in particular, according to the previous assumptions a)-c). In particular, for greater generality and in contrast with respect to Paper I, here we shall allow that the equilibrium magnetic field admits a non-vanishing toroidal component \((B_T)\), possibly produced by currents (of the AD plasma) located far from the considered region. This means that the kinetic distribution functions must be an exact first integral of motion. For definiteness we shall consider here a plasma formed by at least two species of charged particles: one species of ions and one of electrons. Regarding the specific form of the species-equilibrium kinetic distribution functions \(f_k\) (where \(k = i, e\) in the case of a two-species plasma formed by one species of ions and the electrons), we shall assume that they can be approximated by a superposition of suitable GMD’s \(f_{ss}\), namely for \(k = i, e\)

\[
f_k = \sum_{s \in I_k} f_{ss}
\]

where \(I_i\) and \(I_e\) are suitable sets of indices. Thus, in this sense for the kinetic description of the plasma, several sub-species \(s = 1, \ldots, n\) can in principle be distinguished, each one by assumption defined in such a way that:

I) the kinetic distribution \(f_{ss}\) is a first integral of motion, in the sense that:

\[
\frac{d}{dt} \ln f_{ss} = 0.
\] (2)

II) for each \(s = 1, n\) the equilibrium kinetic distribution function, denoted by \(f_{ss}\), is - in a suitable asymptotic sense to be defined later - a drifted local Maxwellian distribution of
the form:

\[
f_{Ms} = \frac{n_s}{\pi^{3/2}v_{ths}^3} \exp \left\{ -\frac{M_s (v - V_s)^2}{2T_s} \right\}.
\]  

(3)

This means that there exists an infinitesimal dimensionless parameter \( \varepsilon \) such that

\[
f_{ss} = f_{Ms} \left[ 1 + o(\varepsilon) \right].
\]  

(4)

Here the notation is standard, thus \( n_s, T_s \) and \( V_s \) are respectively the sub-species number density, temperature and flow velocity. Moreover, \( v_{ths} = \left( 2T_s/M_s \right)^{1/2} \) is the thermal velocity and \( \mathbf{b} \) is the unit vector of the magnetic field \( \mathbf{B}(\mathbf{r},t) \).

III) each distribution function \( f_{Ms} \) (for \( s = 1, n \)) is an approximate stationary solution of the Vlasov equation, in the sense that, consistent with the requirement (4), it must give

\[
\frac{d}{dt} \ln f_{Ms} = 0 + o(\varepsilon).
\]  

(5)

It can be proved, as a consequence, that the following kinetic constraints must be imposed on the fluid fields:

1. \( \mathbf{V}_s(\mathbf{r}) \) can be be identified with a toroidal flow velocity

\[
\mathbf{V}_s = e_\phi R \Omega_s,
\]  

(6)

where the toroidal angular frequency takes the form \( \Omega_s = \Omega_s(\psi) \), i.e., it is constant on magnetic surfaces \( \psi(\mathbf{r}) = \text{const} \);

2. the number density \( n_s \) is:

\[
n_s = n_{os} \exp \left\{ -\frac{\bar{S}_s}{T_s} \right\},
\]  

(7)

where

\[
\bar{S}_s = Z_s e \phi + U_{Grav} - \frac{M_s^2 V_s^2}{2}
\]  

(8)

and \( \bar{A} = A - \langle A \rangle \). Here the notation is standard (see Paper I). In particular, \( \phi \) and \( U_{Grav} \) are respectively the equilibrium electrostatic potential and an effective Pseudo-Newtonian gravitational potential. Moreover, \( \langle A \rangle \) is the \( \psi \)-surface average (defined on a flux surface \( \psi(\mathbf{r}) = \text{const} \)) of function \( A(\mathbf{r}) \), by assumption independent of the toroidal angle \( \varphi \), which is defined as \( \langle A \rangle = \xi^{-1} \oint d\vartheta A(\mathbf{r}) / |\mathbf{B} \cdot \nabla \vartheta| \), with \( \xi \) denoting

\[
\xi \equiv \oint d\vartheta / |\mathbf{B} \cdot \nabla \vartheta|;
\]
3. ignoring possible slow spatial dependencies, the pseudo-density $n_{os}$ and the temperature $T_s$ are flux functions, namely they depend only on the poloidal flux $\psi$ so that:

\begin{align}
    n_{os} &= n_{os}(\psi), \\
    T_s &= T_s(\psi);
\end{align}

4. the toroidal angular velocity $\Omega_s(\psi)$ can always be identified with the flux function

\begin{equation}
    \Omega_s(\psi) = \Omega(\psi) + \Delta \Omega_s(\psi),
\end{equation}

where $\Omega(\psi) = c \frac{\partial}{\partial \psi} \langle \Phi \rangle$ and $\Delta \Omega_s(\psi) = \frac{1}{Z_s e} \langle U_{grav} \rangle$ are the toroidal angular velocities driven, respectively, by the electrostatic and effective gravitational potentials.

Let us first determine the equilibrium distribution function $f_s$. For this purpose, we first notice that it must be necessarily a function only of the independent first integrals of motion and/or of the relevant independent adiabatic invariants (at least for the leading orders in $\varepsilon$). In axisymmetry these are well known in particular the canonical momentum $p_{\varphi s}$ and the total particle energy, $E_s = \frac{M_s v^2}{2} + \frac{Z_s e}{\varepsilon} \Phi + \frac{1}{\varepsilon} U_{grav}$. In the following we assume that the plasma is strongly magnetized in the sense that for all species $s = e, i$

\begin{equation}
    \frac{r_{L_s}}{L} \sim \varepsilon,
\end{equation}

where $r_{L_s} = v_{\perp ths}/\Omega_{cs}$ is the species average Larmor radius and $\Omega_{cs} = Z_s e B/M_s c$ is the species Larmor frequency. Here we shall assume that the plasma is subject to intense electrostatic and gravitational fields (defined respectively by the electrostatic potential $\Phi$ and the effective gravitational potential $U_{grav}$), thus requiring

\begin{equation}
    \frac{T_s}{Z_s e \Phi} \sim \frac{T_s}{U_{grav}} \sim o(\varepsilon).
\end{equation}

As a consequence, the magnetic flux $\psi$, as well as $\Phi$ and the effective gravitational potential $U_{grav}$ are all considered to be of order $1/\varepsilon$, while the species thermal velocities, i.e., $v_{\parallel ths}$ and $v_{\perp ths}$ (for $s = e, i$), are assumed to be of order $o(\varepsilon^0)$. Constructing the adiabatic invariants in the following we shall consider in particular "thermal" test-particles for which $|v|$ is of the same order as $v_{th}$ (for $s = 1, n$). It follows that, thanks to axisymmetry, the canonical momentum conjugate to the ignorable (toroidal) angle $\varphi$ is by assumption a first integral. Taking into account the previous asymptotic orderings, it is given by

\begin{equation}
    p_{\varphi s} = M_s R \hat{v} \cdot \hat{e}_\varphi + \frac{Z_s e}{\varepsilon c} \psi \equiv \frac{Z_s e}{\varepsilon c} \psi_{ss}
\end{equation}
Similarly at equilibrium the test particle total energy, \( E_s = \frac{M_s v^2}{2} + \frac{Z_s e}{\varepsilon} \Phi + \frac{1}{\varepsilon} U_{grav} \) is by assumption a first integral of motion. It follows that the dynamical variable

\[
H_{ss} = E_s - \frac{Z_s e}{\varepsilon c} \psi_{ss} \Omega_s(\psi_{ss})
\]

is manifestly a first integral of motion too. We notice that introducing the velocity vector \( \mathbf{V}_{ss} = e_\varphi R \Omega_s(\psi_{ss}) \), it follows that \( H_{ss} \) is also given by

\[
H_{ss} = E_{Rs}(\mathbf{V}_s) - \frac{Z_s e}{\varepsilon} \psi \Omega_s(\psi_{ss}) - \frac{M_s V^2}{2},
\]

where

\[
E_{Rs}(\mathbf{V}_s) = \frac{M_s (\mathbf{v} - \mathbf{V}_{ss})^2}{2} + \frac{Z_s e}{\varepsilon} \Phi + \frac{1}{\varepsilon} U_{grav},
\]

so that \( E_{Rs}(\mathbf{V}_s) \) can be interpreted as the total energy relative to the reference frame moving with velocity \( \mathbf{V}_{ss} \). Next, we note that invoking the orderings (12), (13) and assuming \( |\mathbf{v}| / v_{th} \sim o(\varepsilon^0) \), the variables \( \psi_{ss} \) and \( H_{ss} \) can be conveniently approximated as

\[
\psi_{ss} \cong \psi [1 + o(\varepsilon)]
\]

\[
H_{ss} = H_s [1 + o(\varepsilon^2)]
\]

where

\[
H_s = E_{Rs}(\mathbf{V}_s) - \frac{M_s V^2}{2} - \frac{Z_s e}{\varepsilon} \psi \Omega_s(\psi),
\]

and \( E_{Rs}(\mathbf{V}_s) \) is the total energy relative to the reference frame moving with the (fluid) velocity \( \mathbf{V}_s = e_\varphi R \Omega_s(\psi) \) [see Eq.(6)].

Thus, a possible definition for the equilibrium distributions \( f_{ss} \) is provided by the distribution function

\[
f_{ss} = \frac{\hat{n}_{ss}}{\pi^{3/2} (2T_{ss}/M_s)^{3/2}} \exp \left\{ -\frac{H_{ss}}{T_{ss}} \right\}
\]

*(generalized Maxwellian distribution)*, where \( \hat{n}_{ss} \) and \( T_{ss} \) are defined as \( \hat{n}_{ss} = \hat{n}_s(\psi_{ss}) \) and \( T_{ss} = T_s(\psi_{ss}) \). It clearly follows that:

- for all \( s = 1, n, f_{ss} \) is a function of first integrals and hence, by construction, is itself a first integral of motion. This result holds, in principle, for an arbitrary function \( \Omega_s(\psi_{ss}) \) (assumed to be suitably smooth);
- the kinetic constraints (6)-(11) are necessarily identically satisfied, by suitable definition of the functions \( \hat{n}_s(\psi_{ss}) \) and \( \Omega_s(\psi_{ss}) \);
- invoking the asymptotic orderings (12), (13) and \( |\mathbf{v}| / v_{th} \sim o(\varepsilon^0) \), Eq.(17) is satisfied for \( f_{Ms} \) defined by Eq.(3).
To prove these statements, we invoke Eqs. (16) and (17) so that $f_{ss}$ can be written

$$f_{ss} = \frac{\hat{n}_s(\psi)}{\pi^{3/2} (2T_s(\psi)/M_s)^{3/2}} \exp \left\{ -\frac{H_s}{T_s(\psi)} \right\} \left[ 1 + o(\varepsilon) \right].$$

(20)

By making use of the expressions

$$n_{os}(\psi) = \hat{n}_s(\psi) \exp \left\{ -\frac{\langle S \rangle_s}{T_s} \right\},$$

(21)

it follows that Eq. (20) leads to (4), while Eq. (21) implies (7). Finally, we notice that Eq. (11) can also be obtained in an equivalent way from the momentum balance equation for the $s$-th species [see Eqs. (1) and (2) in Paper I]. In the present case, this gives

$$\frac{q_i n_s}{\varepsilon} \left( -\nabla \phi + E_s^{(d)} \right) - \frac{q_i}{\varepsilon} n_s \mathbf{V}_s \times \mathbf{B} = 0,$$

(22)

where $E_s^{(d)}$ is a suitable diamagnetic electric field (see Eq. (3), Paper I). It follows $\mathbf{V}_s = \Omega \mathbf{e}_\varphi = V_\parallel \mathbf{b} + \frac{\mathbf{b}}{B} \left( -\nabla \phi + E_s^{(d)} \right) \times \mathbf{b}$. As a consequence, taking the scalar products term by term with $\mathbf{b}$, $\mathbf{e}_\varphi$ and $\nabla \psi$ it follows that

$$V_\parallel = \Omega R \frac{B_T}{B},$$

(23)

$$\Omega R \frac{B_p}{B^2} = \nabla \psi \cdot \left( \nabla \phi - E_s^{(d)} \right),$$

(24)

$$\left( -\nabla \phi + E_s^{(d)} \right) \cdot \mathbf{B}_p = 0,$$

(25)

which proves Eq. (11). This result shows that in such a case, i.e., for kinetic G-Hall-MHD equilibria which hold under the validity of the previous asymptotic orderings, no bifurcation arises (in contrast to the general case of fluid equilibria, considered in Paper I). This is due to the functional form of the number density provided by (7). As a consequence, as indicated above, in the present case the fluid velocity ($\mathbf{V}_s$) is uniquely determined and is identical for each sub-species belonging, respectively, to the ion or electron species. In addition, the presence of a finite (equilibrium) toroidal magnetic field is in principle permitted. Although its possible origin has not been explicitly discussed here, we can always assume that it is produced by "external" currents (i.e., located outside the domain of local existence of the kinetic equilibrium). Finally, another interesting consequence concerns the form of the generalized Grad-Shafranov equation, which in the present case becomes

$$\frac{1}{R} \frac{\partial^2 \psi}{\partial z^2} + \frac{1}{R} \frac{\partial^2 \psi}{\partial R^2} = -\frac{4\pi R}{c} \left[ \rho \Omega + \sum_s Z_s e n_s \Delta \Omega_s \right].$$

(26)
This shows that in kinetic G-Hall-MHD equilibria, the electric current density, producing the self-generated poloidal magnetic field, contains two terms: a) the first one driven by the $\psi$-derivative of the average electrostatic potential is proportional to the local charge density of the plasma $\rho$ (Hall effect); b) the second one due to the gravitational potential, occurs even in the case of a quasi-neutral plasma ($\rho \cong 0$).

III. CONCLUSIONS

In this paper, kinetic G-Hall-MHD equilibria have been investigated. In particular, the kinetic equilibrium has been determined imposing three main assumptions:

1) that a fluid G-Hall-MHD equilibrium exists which is characterized locally by a family of nested magnetic surfaces $\{\psi(r) = \text{const}\}$ represented by axi-symmetric nested tori;

2) requiring that the species equilibrium kinetic distribution function can be represented by a superposition of suitable local equilibrium distributions $f_{ss}$ (for $s = 1, n$);

3) imposing that the equilibrium distribution function can be expressed in terms of first integrals of the motion.

Regarding the second assumption we remark that: a) experimental observations of collisionless astrophysical plasmas (for example, the solar wind) are consistent with this type of representation; b) it is always possible to represent a kinetic distribution function by an appropriate superposition of drifted local Maxwellian distributions (here replaced by the equilibrium distributions $f_{ss}$). Nevertheless, as shown in this paper, the assumption of kinetic equilibrium invoked here has been obtained by imposing the validity of suitable kinetic constraints on these distributions. As in Paper I, no assumption of local quasi-neutrality has been invoked, thus permitting the consistent treatment of Hall effects. In addition, allowance has been made for the inclusion of relativistic effects, taken into account by means of an effective gravitational potential ($U_{\text{grav}}$), and the presence of a finite toroidal magnetic field. Finally, we remark that the third assumption actually rules out the treatment of possible more general kinetic equilibria, which may be expressed in terms of adiabatic invariants to be established based on the so-called gyrokinetic theory of particle dynamics in strong EM fields \[11\]. The main results of the present theory include: 1) the treatment of kinetic equilibria in the presence of strong EM and gravitational fields, i.e., in particular within the validity of the asymptotic orderings \[12\] \[13\]; 2) the evaluation of the species-dependent toroidal
angular velocity ($\Omega_i$), which - to leading order in the asymptotic parameter $\varepsilon$, is proved to depend on the gradient of the $\psi$—average electrostatic and effective gravitational potentials; 3) the proof that for this type of kinetic equilibrium, no bifurcation occurs in the ion toroidal angular velocity ($\Omega_\psi$). The theory developed here applies to plasmas subject to the action of an intense magnetic field, partly self-generated by the plasma itself. The theory permits also the treatment of intense electric and gravitational fields, including the possible action of relativistic effects produced by the presence of a massive central object. In particular, the equilibrium distribution $f_{ss}$ has been determined by adopting a gyrokinetic approach. This permits $f_{ss}$ to be constructed in such a way that it is - at the same time - a first integral of the motion and is also close, in an appropriate asymptotic sense, to a drifted local Maxwellian distribution. Based on the present theory, G-Hall-MHD equilibria can be investigated, in principle, utilizing either standard numerical solution methods or by a perturbative solution method based on a power series expansion near to the equatorial plane ($z = 0$). Detailed results will be presented elsewhere [10]. The results presented in this paper appear to be potentially significant both for the numerical evaluation of AD plasma equilibria and for the physical interpretation of the theory. We believe, in fact, that applications of the present theory can be useful for the theoretical investigation of AD phenomenology and for gaining a better understanding of the related basic physical processes.

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Notice

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[1] S. Chandrasekhar, ApJ **124**, 232 (1956).
[2] A.I. Morozov and L.S. Solov’ev, Sov.Phys.- Dokl. **8**, 243 (1963).
[3] S. I. Krashennikov and P. J. Catto, Phys. Lett. A **260**, 502 (1999).
[4] G. N. Throumoulopoulos, H. Tasso, Geophys. Astroph. Fluid Dynamics **94**, 249 (2001).
[5] K.G. McClements and A. Thyagaraya, Mon. Not. Roy.Astr.Soc., 323(3), 733 (2001).
[6] J. Ghanbari and S. Abbassi, Mon. Not. Roy.Astr.Soc., 350(4), 1437 (2004).
[7] P. J. Catto, I. B. Bernstein and M. Tessarotto, Phys. Fluids **30**, 2784 (1987).
[8] M. Tessarotto and R. B. White, Phys. Fluids B **4**, 859 (1992).
[9] C. Cremaschini, A. Beklemishev, J.C. Miller and M. Tessarotto, *Generalized Grad-Shafranov equation for gravitational Hall-MHD equilibria*, contributed paper at RGD26 (Kyoto, Japan, July 2008); [arXiv:0806.4522](https://arxiv.org/abs/0806.4522) (2008).
[10] C. Cremaschini, A. Beklemishev, J.C. Miller and M. Tessarotto, *Gravitational MHD equilibria in the presence of differential rotation*, contributed paper to be presented at 7th PAMIR International Conference on Fundamental and Applied MHD (Presqu’ile de Giens, Toulon, France, September 8 - 12, 2008).
[11] M. Tessarotto, C. Cremaschini, P. Nicolini and A. Beklemishev, Proc. 25th RGD (International Symposium on Rarefied gas Dynamics, St. Petersburg, Russia, July 21-28, 2006), Ed. M.S. Ivanov and A.K. Rebrov (Novosibirsk Publ. House of the Siberian Branch of the Russian Academy of Sciences), p.1001 (2007); [arXiv:physics/0611114](https://arxiv.org/abs/physics/0611114) (2007).