COHEN-MACAULAY GRAPHS AND FACE VECTORS OF FLAG COMPLEXES

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Abstract. We introduce a construction on a flag complex that, by means of modifying the associated graph, generates a new flag complex whose $h$-factor is the face vector of the original complex. This construction yields a vertex-decomposable, hence Cohen-Macaulay, complex. From this we get a (non-numerical) characterisation of the face vectors of flag complexes and deduce also that the face vector of a flag complex is the $h$-vector of some vertex-decomposable flag complex. We conjecture that the converse of the latter is true and prove this, by means of an explicit construction, for $h$-vectors of Cohen-Macaulay flag complexes arising from bipartite graphs. We also give several new characterisations of bipartite graphs with Cohen-Macaulay or Buchsbaum independence complexes.

1. Introduction

Simplicial complexes are combinatorial objects at the intersection of many fields of mathematics including algebra and topology. Passing from a simplicial complex to its barycentric subdivision yields a flag complex—a simplicial complex whose minimal non-faces are edges—while preserving topological properties. This, among other reasons, lead Stanley to state [21, p. 100] that “Flag complexes are a fascinating class of simplicial complexes which deserve further study.” One simple enumeration of a simplicial complex is the face vector. Face vectors are conveniently expressed as $h$-vectors which admit a more algebraic interpretation (via Hilbert series). An important task is to establish restrictions on the face or, equivalently, $h$-vectors of simplicial flag complexes. In this note we contribute to this problem by exploring the interplay of face vectors and $h$-vectors of flag complexes.

According to [21, p. 100], Kalai conjectured that the face vector of a flag complex $\Delta$ is also the face vector of a balanced complex $\Gamma$. Moreover, if $\Delta$ is Cohen-Macaulay, then $\Gamma$ can be chosen to be Cohen-Macaulay. If this conjecture is true (in its entirety), then it would follow that the $h$-vector of a Cohen-Macaulay flag complex is the face vector of a balanced complex. The first part of the conjecture, which does not assume Cohen-Macaulayness, was also conjectured by Eckhoff [6] and has recently been proven by Frohmader in [9, Theorem 1.1]. However, the second part of the conjecture remains open. Similarly, [2] discusses the relation between $h$-vectors of Cohen-Macaulay complexes and the face vectors of multi-complexes. In this note, we begin studying the question of which Cohen-Macaulay flag complexes have $h$-vectors that are also the face vectors of flag complexes.

In Section [2], we recall some basic concepts used throughout the paper. We introduce clique-whiskering, a generalisation of whiskering graphs (see [5], [8], [13], [20], [23], and [25]),

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in Section 3. An advantage of clique-whiskering is that it produces flag complexes of smaller dimension than whiskering. We show that the independence complex of a clique-whiskered graph is vertex-decomposable (Theorem 3.3) and hence squarefree glicci (so, in particular, in the Gorenstein liaison class of a complete intersection, see [15]). This generalises results by Villarreal [23] and Dochtermann and Engström [5]. Moreover, we prove that the face vector of the independence complex of the base graph is the $h$-vector of the independence complex of the clique-whiskered graph (Proposition 3.8). This provides a characterisation of the face vectors of flag complexes as the $h$-vectors of the independence complexes of clique-whiskered graphs (Theorem 3.9) and also shows that the face vector of every flag complex is the $h$-vector of some vertex-decomposable (hence Cohen-Macaulay) flag complex (Corollary 3.10; compare also the independently-found result in [3, Proposition 4.1]). We conjecture that the converse of the latter is also true (Conjecture 3.11). As evidence, we establish the conjecture in the case of independence complexes of bipartite graphs by means of another explicit construction (Proposition 4.11).

In Section 4, we restrict ourselves to bipartite graphs. We find another classification of bipartite graphs with Cohen-Macaulay independence complexes. From this, we establish that bipartite graphs with Buchsbaum independence complexes are exactly those that are complete or have Cohen-Macaulay independence complexes (Theorem 4.10); this result was found independently in [10]. Moreover, we define the compression of a bipartite graph and show that the $h$-vector of a Cohen-Macaulay flag complex arising from a bipartite graph is the face vector of the independence complex of the associated compression (Proposition 4.11).

After this note had been written, the paper [3] of Constantinescu and Varbaro appeared which treats topics similar to those presented here, though in a greatly different manner. We make more specific references in the main body of the text.

2. Preliminaries

A simplicial complex $\Delta$, on a finite set $V$, is a set of subsets of $V$ closed under inclusion; elements of $\Delta$ are called faces. The dimension of a face $\sigma$ is $\# \sigma - 1$ and of a complex $\Delta$ is the maximum of the dimensions of its faces. A complex whose maximal faces, called facets, are equi-dimensional is called pure and a complex with a unique facet is called a simplex.

Let $\Delta$ be a $(d-1)$-dimensional simplicial complex. There are two vectors and two sub-complexes of interest. The face vector (or $f$-vector) of $\Delta$ is the $(d+1)$-tuple $(f_{-1}, \ldots, f_{d-1})$, where $f_i$ is the number of $i$-dimensional faces of $\Delta$. The $h$-vector of $\Delta$ is the $(d+1)$-tuple $(h_0, \ldots, h_d)$ given by $h_j = \sum_{i=0}^{j} (-1)^{j-i} \binom{d-i}{j-i} f_{i-1}$. Notice that given the $h$-vector of a simplicial complex we can recover the face vector; indeed, for $0 \leq j \leq d$, one has

$$f_{j-1} = \sum_{i=0}^{j} \binom{d-i}{j-i} h_i.$$  

(2.1)

Let $\sigma$ be a face of $\Delta$, then the link and deletion of $\sigma$ from $\Delta$ are given by

$$\text{link}_\Delta \sigma := \{ \tau \in \Delta \mid \tau \cap \sigma = \emptyset, \tau \cup \sigma \in \Delta \} \quad \text{and} \quad \text{del}_\Delta \sigma := \{ \tau \in \Delta \mid \sigma \not\subseteq \tau \}.$$  

Moreover, following [18, Definition 2.1], a pure complex $\Delta$ is said to be to be vertex-decomposable if either $\Delta$ is a simplex or there exists a vertex $v \in \Delta$, called a shedding vertex, such that both $\text{link}_\Delta v$ and $\text{del}_\Delta v$ are vertex-decomposable. Checking if a particular simplicial complex is vertex-decomposable can be achieved using a computer program such as Macaulay2 [14]; the package described in [4] provides the appropriate methods.
A graph $G = (V, E)$ is a pair consisting of a finite vertex set $V$ and an edge set $E$ of two-element subsets of $V$. Two vertices $u$ and $v$ are adjacent in $G$ if $uv \in E$, the neighborhood of a vertex $v$ is the set $N_G(v)$ of vertices adjacent to $v$ in $G$, and the degree of a vertex is the cardinality of its neighborhood. A graph $G$ is complete if every vertex is adjacent to every other vertex and a graph is connected if there is a path in $G$ between every pair of vertices of $G$.

Let $G = (V, E)$ be a graph. A subset of vertices $U$ is an independent set if no two elements of $U$ are adjacent. The independence complex of $G$ is the simplicial complex $\text{Ind}_G$ with faces generated by the independent sets of $G$. Hence graphs can be studied by looking at simplicial complexes.

The following result was first observed in [7] and follows directly from the definitions.

**Lemma 2.1.** Let $v$ be a vertex of $G$, then
\[ \text{del}_{\text{Ind}_G} v = \text{Ind}(G \setminus v) \quad \text{and} \quad \text{link}_{\text{Ind}_G} v = \text{Ind}(G \setminus (v \cup N_G(v))). \]

These combinatorial objects are related to squarefree monomial ideals. Let $\Delta$ be a simplicial complex on the set $V$. The Stanley-Reisner ideal of $\Delta$ is the ideal $I(\Delta)$ generated by the minimal non-faces of $\Delta$ and the Stanley-Reisner ring of $\Delta$ is $K[\Delta] = K[V]/I(\Delta)$, for a field $K$. Thus, the Stanley-Reisner ideals of simplicial complexes on some vertex set $V$ are exactly the squarefree monomial ideals in $K[V]$.

The Stanley-Reisner ideals of independence complexes of graphs are exactly the ideals in $K[V]$ generated by quadratic squarefree monomials. Moreover, the generating monomials correspond to the edges of the graphs. When a simplicial complex has a quadratic Stanley-Reisner ideal, that is, it is the independence complex of some graph, then it is called a flag complex.

A simplicial complex is pure if and only if its Stanley-Reisner ring is unmixed, that is, has associated prime ideals that are equi-dimensional. If $(h_0, \ldots, h_d)$ is the $h$-vector of some $(d - 1)$-dimensional complex $\Delta$, then the Hilbert series of $K[\Delta]$ is given by
\[
\frac{h_0 + h_1 t + h_2 t^2 + \cdots + h_d t^d}{(1 - t)^d}.
\]

Moreover, if $\Delta$ and $\Sigma$ are complexes on disjoint vertex sets, then the Hilbert series of $\Delta \cup \Sigma$ is the product of the Hilbert series of $\Delta$ and $\Sigma$. Finally, a simplicial complex is Cohen-Macaulay (resp. Buchsbaum) if and only if the associated Stanley-Reisner ring is Cohen-Macaulay (resp. Buchsbaum).

The following example demonstrates that not every face vector of a simplicial complex is the $h$-vector of a flag complex.

**Example 2.2.** Consider the simplicial complex with facets $\{uv, uw, vw\}$; this complex has face vector $(1, 3, 3)$. Suppose $G$ is a graph whose independence complex has an $h$-vector whose non-zero part is $(1, 3, 3)$ and is of dimension $d - 1$, for some $d \geq 2$. Then the Hilbert series of $K[\text{Ind} G]$ is $\frac{1 + 3t + 3t^2}{(1 - t)^2}$.

Assuming $G$ is on $n$ vertices, then $d = n - 3$ and so $n \geq 5$. Further still, $G$ must have 3 edges. However, $1 + 3t + 3t^2$ is irreducible over $\mathbb{Z}$ so $G$ must be connected; this is impossible when $n \geq 5$. Thus no such $G$ can exist.

Notice however, in Theorem 3.9 we prove that the face vector of every flag complex is indeed the $h$-vector of another flag complex.
3. CLIQUE-WHISKERED GRAPHS

Let \( G = (V, E) \) be a (non-empty) graph with \( V = \{v_1, \ldots, v_n\} \).

Adding a whisker to \( G \) at \( v_i \) means adding a new vertex \( w \) and edge \( v_iw \) to \( G \). It was shown in [23, Proposition 2.2] that if a whisker is added to every vertex of \( G \), then the resulting graph has a Cohen-Macaulay independence complex. Furthermore, in [5, Theorem 4.4] it was shown that the independence complex of a fully-whiskered graph is also pure and vertex-decomposable. We give a generalisation of whiskering in this section and explore its properties.

A subset \( C \) of the vertices is a clique if it induces a complete subgraph of \( G \). A clique vertex-partition of \( G \) is a set \( \pi = \{W_1, \ldots, W_t\} \) of disjoint (possibly empty) cliques of \( G \) such that their disjoint union forms \( V \). Notice that \( G \) may permit many different clique vertex-partitions, and every graph has at least one clique vertex-partition, in particular, the trivial partition, \( \tau = \{\{v_1\}, \ldots, \{v_n\}\} \).

Clique-whiskering, which [25, Proposition 22] called clique-starring, a clique \( W \) of \( G \) is done by adding a new vertex \( w \) and connecting \( w \) to every vertex in \( W \), resulting in the graph \( G^W \). We further define fully clique-whiskering \( G \) by a clique vertex-partition \( \pi = \{W_1, \ldots, W_t\} \) to be \( G \) clique-whiskered at every clique of \( \pi \); it produces the graph

\[
G^\pi := (V \cup \{w_1, \ldots, w_t\}, E \cup \{vw_i \mid v \in W_i\}).
\]

Notice that \( G^\tau \) is the fully-whiskered graph when \( \tau \) is the trivial partition and empty cliques produce isolated vertices.

**Example 3.1.** Let \( G \) be the three-cycle on vertices \( \{u, v, w\} \). There are three distinct clique vertex-partitions of \( G \) (without empty cliques): the trivial partition \( \tau = \{\{u\}, \{v\}, \{w\}\} \), \( \pi = \{\{u, v\}, \{w\}\} \), and \( \rho = \{\{u, v, w\}\} \). These are shown in Figure 3.1.

![Figure 3.1. Clique-whiskerings of the three-cycle](image)

Following directly from the definition, we get purity of the independence complex.

**Lemma 3.2.** Let \( \pi = \{W_1, \ldots, W_t\} \) be a clique vertex-partition of \( G \). Then \( \text{Ind } G^\pi \) is pure and \((t - 1)\)-dimensional.

**Proof.** Let \( w_1, \ldots, w_t \) be the new vertices associated to \( W_1, \ldots, W_t \), respectively.

Each clique-whisker \( B_i = W_i \cup \{w_i\} \) is a clique of \( G^\pi \), hence any independent set of \( G^\pi \)

has at most one vertex from each \( B_i \); that is, \( \dim \text{Ind } G^\pi < t \). Moreover, \( \{w_1, \ldots, w_t\} \) is an independent set in \( G^\pi \), as \( N(w_i) = W_i \) are pairwise disjoint. Thus \( \dim \text{Ind } G^\pi = t - 1 \).

Let \( I = \{u_1, \ldots, u_k\} \) be an independent set of \( G^\pi \), and suppose, without loss of generality, that \( u_j \in B_j \). Then \( I \cup \{w_{k+1}, \ldots, w_t\} \) is a maximal independent set in \( G^\pi \) of size \( t \). Thus \( \text{Ind } G^\pi \) is pure. \( \square \)
The following result generalizes [5, Theorem 4.4]. Further, it was shown in [3, Proposition 4.1] that fully-whiskered graphs have balanced independence complexes; the proof easily extends to independence complexes of clique-whiskered graphs.

**Theorem 3.3.** Let \( \pi = \{W_1, \ldots, W_t\} \) be a clique vertex-partition of \( G \). Then \( \text{Ind} G^\pi \) is vertex-decomposable.

**Proof.** Suppose \( \pi = \{W_1, \ldots, W_t\} \) is a clique vertex-partition of \( G \). If, without loss of generality, \( W_t \) is an empty clique, then \( \rho = \{W_1, \ldots, W_{t-1}\} \) is a clique vertex-partition of \( G \) and \( \text{Ind} G^\rho \) is a cone over \( \text{Ind} G^\pi \). Recall that coning over a simplicial complex does not affect vertex-decomposability.

Now, we proceed by induction on \( n \), the number of vertices of \( G \). If \( n = 1 \), then for any \( \pi \) of \( G \), \( \text{Ind} G^\pi \) is a pair of disjoint vertices (or a cone thereof), hence is vertex-decomposable.

Let \( n \geq 2 \) and, without loss of generality, assume \( v \in W_t \). Define \( \rho = \{W_1, \ldots, W_{t-1}, W_t \setminus v\} \), then \( \rho \) is a clique vertex-partition of \( G \setminus v \), a graph on \( n - 1 \) vertices, and

\[
\text{del}_{\text{Ind} G^\pi} v = \text{Ind}(G^\pi \setminus v) = \text{Ind}((G \setminus v)^\rho)
\]

is vertex-decomposable by induction. Define \( \sigma = \{W_1 \setminus N_G(v), \ldots, W_{t-1} \setminus N_G(v)\} \), then \( \sigma \) is a clique vertex-partition of \( G \setminus (v \cup N_G(v)) \), a graph on \( n - 1 - \#N_G(v) \) vertices, and

\[
\text{link}_{\text{Ind} G^\pi} v = \text{Ind}(G^\pi \setminus (v \cup N_G(v))) = \text{Ind}((G \setminus (v \cup N_G(v)))^\sigma)
\]

is vertex-decomposable by induction.

Thus \( v \) is a shedding vertex of \( \text{Ind} G^\pi \) and \( \text{Ind} G^\pi \) is vertex-decomposable. \( \square \)

Notice the proof of Theorem 3.3 shows every vertex of \( G \) is a shedding vertex of \( \text{Ind} G^\pi \).

By [20, Corollary 2.3], a fully-whiskered star graph is licci (in the liaison class of a complete intersection). However, not every fully-whiskered graph has this property.

**Example 3.4.** Let \( G \) be the full-whiskering of the three-cycle (see Figure 3.1, \( G^\tau \)) and \( R = K[u, v, w, x, y, z] \). Then the Stanley-Reisner ring \( K[\text{Ind} G] \) has a free resolution of the form

\[
0 \rightarrow R^3(-4) \rightarrow R^8(-3) \rightarrow R^6(-2) \rightarrow R \rightarrow K[\text{Ind} G] \rightarrow 0
\]

and hence, by [12, Corollary 5.13], is not licci.

Being glicci is a weaker condition than being licci, though it is still true that every glicci ideal is Cohen-Macaulay. It is one of the main open questions in liaison theory if every Cohen-Macaulay ideal is glicci. In this regard, we obtain:

**Corollary 3.5.** Let \( \pi \) be a clique vertex-partition of \( G \). Then \( \text{Ind} G^\pi \) is squarefree glicci and Cohen-Macaulay.

**Proof.** Pure vertex-decomposable simplicial complexes are squarefree glicci [15, Theorem 3.3] and pure shellable [18, Theorem 2.8], hence Cohen-Macaulay. \( \square \)

The graphs that are full clique-whiskerings of graphs can be described in terms of clique vertex-partitions. We note that [3, Lemma 6.1] gives an equivalent statement regarding the flag complexes whose associated graphs are full clique-whiskerings.

**Proposition 3.6.** A graph \( G \) is a full clique-whiskering if and only if there exists a clique vertex-partition of \( G \) such that every clique in the partition contains a vertex whose neighborhood in \( G \) is contained in the clique.
Proof. This follows directly from the definition of full clique-whiskering.

Unfortunately, not every pure vertex-decomposable flag complex comes from the independence complex of a fully clique-whiskered graph. For example, the independence complex of the five-cycle is a pure vertex-decomposable flag complex, but the five-cycle is not the full clique-whiskering of any graph.

A pure simplicial complex $\Delta$ is called partitionable if it can be written as a disjoint union of intervals $[G_1, F_1] \cup \cdots \cup [G_s, F_s]$, called a partitioning of $\Delta$, where $F_1, \ldots, F_s$ are the facets of $\Delta$ and $[G, F] := \{H \mid G \subseteq H \subseteq F\}$. As pure vertex-decomposable simplicial complexes are shellable ([18, Theorem 2.8]), they are also partitionable ([21, Statement before III.2.3]). Further, the $h$-vector of a partitionable simplicial complex can be written in terms of a given partitioning.

**Proposition 3.7.** ([21, Proposition III.2.3]) Let $\Delta$ be a pure partitionable simplicial complex and let $[G_1, F_1] \cup \cdots \cup [G_s, F_s]$ be a partitioning of $\Delta$. Then the $h$-vector of $\Delta$ is given by $h_i = \#\{j \mid \#G_j = i\}$.

An interesting feature of clique-whiskered graphs is that the $h$-vectors of their independence complexes are the same as the face vectors of the independence complexes of their base graphs. This can be seen as a generalisation of [13, Theorem 2.1], which provides the relation for the face vectors with respect to whiskering.

**Proposition 3.8.** Let $\pi$ be a clique vertex-partition of $G$. Then the $h$-vector of Ind $G^\pi$ is the face vector of Ind $G$.

Proof. The result can be obtained from the methods of the proof of [1, Theorem 6.3]; however, we prefer to give a more direct argument.

Let $d = \#\pi = \dim\text{Ind } G^\pi + 1$. There are $f_{i-1}(\text{Ind } G)$ independent sets of size $i$ in $G$; let $I$ be one of these independent sets. Then there are exactly $d - i$ vertices added during clique-whiskering independent from $I$ in $G^\pi$. Hence $I$ can be expanded to the independent set $\hat{I}$ of size $d$ in $G^\pi$ and so $\hat{I}$ is (with abuse of notation) a facet of Ind $G^\pi$. Moreover, we have a partitioning of Ind $G^\pi$ given by the set of intervals $[I, \hat{I}]$, where $I$ runs through the independent sets of $G$. Hence, by Proposition 3.7, the $h$-vector of Ind $G^\pi$ is the face vector of Ind $G$.

A consequence of the previous proposition, along with Proposition 3.6, is a (non-numerical) characterisation of the face vectors of flag complexes.

**Theorem 3.9.** Let $f$ be a finite sequence of positive integers. Then the following conditions are equivalent:

(i) $f$ is the face vector of a flag complex,

(ii) $f$ is the $h$-vector of the independence complex of a clique-whiskered graph, and

(iii) $f$ is the $h$-vector of the independence complex of a fully-whiskered graph.

Another consequence is a (perhaps more useful) condition on the face vectors of flag complexes, which was established independently with the added condition that the vertex-decomposable complex is also balanced (as noted above) in [3, Proposition 4.1].

**Corollary 3.10.** The face vector of every flag complex is the $h$-vector of some vertex-decomposable flag complex.
We believe that the converse of Corollary 3.10 is true as well. If so, this would provide a more complete (non-numerical) characterisation of the face vectors of flag complexes.

**Conjecture 3.11.** The $h$-vector of every vertex-decomposable flag complex is the face vector of some flag complex.

For evidence of this conjecture, see Proposition 4.11 below which shows the conjecture is true for the independence complexes of bipartite graphs.

Conjecture 3.11 was stated independently in [3, Conjecture 1.4] and further expanded in [3, Conjecture 1.5]. A strengthening of Conjecture 3.11 in the case of a Gorenstein flag complex has been proposed in [16, Conjecture 1.4]; instances of this conjecture have been established in [17].

4. Bipartite graphs

We now restrict ourselves to bipartite graphs and explore both the Buchsbaum and Cohen-Macaulay properties for the associated independence complexes.

Let $G$ be a (non-empty) graph with vertex set $V$. We call $G$ bipartite if $V$ can be partitioned into disjoint sets $V_1$ and $V_2$, such that each is an independent set in $G$. If $G$ is a bipartite graph with $\#V_1 = m$ and $\#V_2 = n$, such that every vertex in $V_1$ is adjacent to every vertex in $V_2$, then $G$ is the complete bipartite graph $K_{m,n}$.

In the case of bipartite graphs, there are known results characterising when the independence complex of the graph is pure or Cohen-Macaulay.

**Theorem 4.1.** [24, Theorem 1.1] Let $G$ be a bipartite graph without isolated vertices. Then Ind $G$ is pure if and only if there is a partition $V_1 = \{x_1, \ldots, x_n\}$ and $V_2 = \{y_1, \ldots, y_n\}$ of the vertices of $G$ such that:

(i) $x_iy_k$ is an edge of $G$, for all $1 \leq i \leq n$, and

(ii) if $x_iy_j$ and $x_jy_k$ are edges in $G$, for $i, j$, and $k$ distinct, then $x_iy_k$ is an edge in $G$.

In this case, we call such a partition and ordering of the vertices a pure order of $G$. Further we will say that a pure order has a cross if, for some $i \neq j$, $x_iy_j$ and $x_jy_i$ are edges of $G$, otherwise we say the order is cross-free.

**Theorem 4.2.** [11, Theorem 3.4] Let $G$ be a bipartite graph on a vertex set $V$ without isolated vertices. Then Ind $G$ is Cohen-Macaulay if and only if there is a pure ordering $V_1 = \{x_1, \ldots, x_n\}$ and $V_2 = \{y_1, \ldots, y_n\}$ of $G$, such that $x_iy_j$ being an edge in $G$ implies $i \leq j$.

Notice that if $G$ has isolated vertices $Z = \{z_1, \ldots, z_m\}$, then Ind $G$ is pure (resp. Cohen-Macaulay) if and only if Ind $(G \setminus Z)$ is pure (resp. Cohen-Macaulay).

A rather direct consequence of Theorem 4.2 is that the Cohen-Macaulayness of the independence complex of a bipartite graph implies vertex-decomposability.

**Corollary 4.3.** [22, Theorem 2.10] Let $G$ be a bipartite graph. Then its independence complex Ind $G$ is Cohen-Macaulay if and only if Ind $G$ is vertex-decomposable.

Thus, the Cohen-Macaulayness of the independence complex of a bipartite graph also implies being squarefree glicci. This provides a nice class of examples of Cohen-Macaulay simplicial complexes that are squarefree glicci.
Corollary 4.4. Let $G$ be a bipartite graph. If $\text{Ind} \, G$ is Cohen-Macaulay, then $\text{Ind} \, G$ is squarefree glicci.

Proof. Since $\text{Ind} \, G$ is vertex-decomposable by Corollary 4.3, then $\text{Ind} \, G$ is squarefree glicci [15, Theorem 3.3].

4.1. Buchsbaum bipartite graphs. In order to classify which bipartite graphs have Buchsbaum independence complexes, we need to find a new classification of bipartite graphs with Cohen-Macaulay independence complexes. First, we see that for a bipartite graph with pure independence complex, we only need to look at one pure order to determine if the graph is cross-free. Hence, cross-free is a property of the graph itself, rather than a property of a particular pure order.

Lemma 4.5. Let $G$ be a bipartite graph with pure independence complex. Then every pure order of $G$ has a cross if and only if some pure order of $G$ has a cross.

Proof. Since we may look at each component individually, it is sufficient to consider connected graphs. Let $G$ be a connected bipartite graph with a pure independence complex and let $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ be a pure order of $G$ which has a cross, say $x_ey_d$ and $x_by_d$.

As $G$ is connected, then the bi-partitioning of the vertex set is unique and so every pure order of $G$ is of the form $\{x_{\alpha(1)}, \ldots, x_{\alpha(n)}\}$ and $\{y_{\beta(1)}, \ldots, y_{\beta(n)}\}$, for some permutations $\alpha$ and $\beta$ in $\mathcal{S}_n$, the symmetric group on $n$ elements. Notice then $x_iy_j$ is in $G$ if and only if $x_{\alpha(i)}y_{\beta(j)}$ is in $G$.

If $\beta^{-1}(\alpha(c))$ is $c$ or $d$, then clearly $x_{d}y_{\beta^{-1}(\alpha(c))}$ is in $G$. On the other hand, suppose $\beta^{-1}(\alpha(c))$ is neither $c$ or $d$. As the order is pure, $x_{\alpha(c)}y_{\alpha(c)}$ is in $G$ and so $x_{c}y_{\beta^{-1}(\alpha(c))}$ is in $G$. Further, since $x_{d}y_{\alpha(c)}$ is in $G$ and the order is pure, then $x_{d}y_{\beta^{-1}(\alpha(c))}$ is in $G$. Thus, regardless of $\alpha$ and $\beta$, we have that $x_{d}y_{\beta^{-1}(\alpha(c))}$ is in $G$ and so $x_{\alpha(d)}y_{\alpha(c)}$ is in $G$.

We can similarly show that $x_{\alpha(c)}y_{\alpha(d)}$ is in $G$. Thus, the cross given by $x_{\alpha(c)}y_{\alpha(d)}$ and $x_{\alpha(d)}y_{\alpha(c)}$ is in $G$. □

Corollary 4.6. Any bipartite graph with a pure order that has a cross has a non-Cohen-Macaulay independence complex.

Proof. Let $G$ be a bipartite graph with a pure order that has a cross. Then every pure order of $G$ has a cross by Lemma 4.5, hence has an edge $x_iy_j$ such that $j < i$. Thus, by Theorem 4.2, $\text{Ind} \, G$ is not Cohen-Macaulay. □

Next, we see that a cross-free bipartite graph has two vertices of degree one.

Lemma 4.7. Any cross-free bipartite graph with at least two vertices has, for any pure order, two vertices both of degree one that are in separate components of the bi-partition.

Proof. It is sufficient to consider connected graphs; otherwise, we may look at each component individually. Let $G$ be a cross-free connected bipartite graph and let $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ be a pure order of $G$. If $n = 1$, then clearly $\text{deg}_G \, x_1 = \text{deg}_G \, y_1 = 1$.

Suppose $n \geq 2$. Let $H = G \setminus \{x_1, y_1\}$, then $H$ is also cross-free, and by induction has two vertices, say $x_i$ and $y_j$ of degree one.

Assume $G$ has no vertices of degree one. As $G$ has no vertices of degree one, if $i = j$, then $x_1y_i$ and $x_iy_1$ are edges in $G$; this contradicts $G$ being cross-free. Suppose then $i \neq j$. Then $x_iy_1$ and $x_1y_j$ are edges in $G$. As the order is pure, then $x_iy_j$ is in $G$ hence in $H$, but this contradicts $\text{deg}_H \, x_i = 1$. □
Assume $G$ has vertices $\{x_{i_1}, \ldots, x_{i_m}\}$ of degree one, $m \geq 1$, but all vertices $\{y_1, \ldots, y_n\}$ have degree at least two. If $m = n$, then the $y_i$ must be connected, contradicting $G$ bipartite. Let $J = G \setminus \{x_{i_1}, \ldots, x_{i_m}, y_{i_1}, \ldots, y_{i_m}\}$, then $J$ is also cross-free, and by induction has two vertices, say $x_i$ and $y_j$ of degree one. But then $y_j$ must be connected to one of the $x_{i_k}$ which contradicts their having degree one. \hfill $\square$

We summarise the above results to get a characterisation of bipartite graphs with Cohen-Macaulay independence complexes.

**Proposition 4.8.** Let $G$ be a bipartite graph. Then $G$ has a cross-free pure order if and only if its independence complex $\operatorname{Ind} G$ is Cohen-Macaulay.

**Proof.** It is sufficient to consider connected graphs; otherwise, we may look at each component individually. Let $G$ be a cross-free connected bipartite graph and let $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ be a pure order of $G$. If $n = 1$, then clearly $\operatorname{Ind} G$ is Cohen-Macaulay.

Assume $n \geq 2$. Then by Lemma 4.7 we may assume $y_1$ has degree one. As $H = G \setminus \{x_1, y_1\}$ has a Cohen-Macaulay independence complex by induction, and all edges in $G$ not in $H$ are of the form $x_1 y_i$, for some $1 \leq i \leq n$, then $\operatorname{Ind} G$ is also Cohen-Macaulay.

If $G$ is a cross, then by Corollary 1.6 $\operatorname{Ind} G$ is not Cohen-Macaulay. \hfill $\square$

We will use the following classification of Buchsbaum complexes. Note that the following theorem, [10, Theorem 3.2], has been rewritten using Reisner’s Theorem.

**Theorem 4.9.** A simplicial complex is Buchsbaum if and only if it is pure and the link of each vertex is Cohen-Macaulay.

Finally, we can classify all bipartite graphs with Buchsbaum independence complexes, and surprisingly, find yet another classification of bipartite graphs with Cohen-Macaulay independence complexes. Also note, that this theorem was proven independently, and in a different manner, in [10, Theorem 1.3].

**Theorem 4.10.** Let $G$ be a bipartite graph. Then its independence complex $\operatorname{Ind} G$ is Buchsbaum if and only if $G$ is a complete bipartite graph $K_{n,n}$, for some $n$, or $\operatorname{Ind} G$ is Cohen-Macaulay.

**Proof.** If $\operatorname{Ind} G$ is Cohen-Macaulay, then $\operatorname{Ind} G$ is Buchsbaum (this holds in general). If $G = K_{n,n}$, then for all vertices $v$ of $G$, $\operatorname{link}_{\operatorname{Ind} G} v = \operatorname{Ind}(G \setminus (v \cup N_G(v)))$ is a simplex on $n - 1$ vertices, which is Cohen-Macaulay. Thus, by Theorem 4.9, $\operatorname{Ind} G$ is Buchsbaum.

Suppose $G$ is not $K_{n,n}$ and $\operatorname{Ind} G$ is not Cohen-Macaulay. Then by Proposition 4.8, every pure order of $G$ has a cross, say $x_1 y_2, x_2 y_1$. Notice then $N_G(x_1) = N_G(x_2)$ and $N_G(y_1) = N_G(y_2)$, as $\operatorname{Ind} G$ is pure. Furthermore, if $N_G(y_1) = \{x_1, \ldots, x_n\}$ and $N_G(x_1) = \{y_1, \ldots, y_n\}$, then $G$ is $K_{n,n}$. Hence we may assume there exists an $x \notin N_G(y_1)$. Then $G \setminus (x \cup N_G(x))$ still contains $x_1 y_2, x_2 y_1$, hence does not have a Cohen-Macaulay independence complex by Proposition 4.8. Therefore, $\operatorname{Ind} G$ is not Buchsbaum, by Theorem 4.9. \hfill $\square$

We note that the independence complex of $K_{n,n}$ is Cohen-Macaulay if and only if $n = 1$.

4.2. **Compress and extrude.** We now return to the question of whether the $h$-vector of a Cohen-Macaulay flag complex is a face vector of a flag complex (see Conjecture 3.11).

Let $G$ be a bipartite graph with a Cohen-Macaulay independence complex. Then there exists some tri-partitioning $X = \{x_1, \ldots, x_n\}, Y = \{y_1, \ldots, y_n\}$, and $Z = \{z_1, \ldots, z_m\}$ such that:
(i) the vertices of $Z$ are exactly the isolated vertices of $G$,
(ii) $X$ and $Y$ are a pure order of $G \setminus Z$, and
(iii) $x_iy_j$ in $G$ implies $i \leq j$.

Define the compression of $G$, denoted by $\tilde{G}$, by “compressing” all the edges $x_iy_i$ to the vertex $x_i$ and removing the vertices of $Z$ altogether. That is,

$$\tilde{G} := (X, \{x_i x_j \mid i < j \text{ and } x_i y_j \in G\}).$$

It turns out that compressing is the right action to find the desired simplicial complex.

**Proposition 4.11.** Let $G$ be a bipartite graph. If $\text{Ind } G$ is Cohen-Macaulay, then $h(\text{Ind } G) = f(\text{Ind } \tilde{G})$.

**Proof.** Let $X,Y,Z$ be as above. Notice that $\dim \text{Ind } G = n + m - 1$, as, e.g., $X \cup Z$ is a maximal independent set in $G$.

If $n = 0$, then $G$ is a graph of $m$ disjoint vertices so $\tilde{G}$ is the empty graph and $\text{Ind } \tilde{G}$ is the empty complex, hence $h(\text{Ind } G) = (1) = f(\text{Ind } \tilde{G})$.

Suppose $n \geq 1$ and let $H = G \setminus (x_n \cup N_G(x_n))$; that is, $\text{Ind } H = \text{link}_{\text{Ind } G} x_n$. Then $\text{Ind } H$ is Cohen-Macaulay and, further, $\dim \text{Ind } H = n + m - 2$, as $(X \setminus x_n) \cup Z$ is a maximal independent set in $H$. Thus, by induction, $h(\text{Ind } H) = f(\text{Ind } \bar{H})$. Using Equation (2.1), we obtain

$$f_{j-1}(\text{Ind } H) = \sum_{i=0}^{j} \binom{n + m - 1 - i}{j - i} h_i(\text{Ind } H) = \sum_{i=0}^{j} \binom{n + m - 1 - i}{j - i} f_{i-1}(\text{Ind } \bar{H}).$$

Further, we have

$$f_{j-1}(\text{Ind } H) + f_{j-2}(\text{Ind } H) = \sum_{i=0}^{j} \binom{n + m - i}{j - i} f_{i-1}(\text{Ind } \bar{H}),$$

where $f_{j-1}(\text{Ind } H)$ counts the number of independent sets of size $j$ in $G$ without $x_n$ and $y_n$ and $f_{j-2}(\text{Ind } H)$ counts the number of independent sets of size $j$ in $G$ with $x_n$.

Let $J = G \setminus (y_n \cup N_G(y_n))$; that is, $\text{Ind } J = \text{link}_{\text{Ind } G} y_n$. Then $\text{Ind } J$ is Cohen-Macaulay and, further, $\dim \text{Ind } J = n + m - 2$, as $(Y \setminus y_n) \cup Z$ is a maximal independent set in $J$. Thus, by induction, $h(\text{Ind } J) = f(\text{Ind } \bar{J})$ and so

$$f_{j-2}(\text{Ind } J) = \sum_{i=0}^{j-1} \binom{n + m - 1 - i}{j - 1 - i} h_i(\text{Ind } J) = \sum_{i=0}^{j-1} \binom{n + m - 1 - i}{j - 1 - i} f_{i-1}(\text{Ind } \bar{J}),$$

which counts the number of independent sets of size $j$ in $G$ with $y_n$, as the vertices of $J$ are exactly those independent of $y_n$.

Now, $f_{i-1}(\text{Ind } \bar{H}) + f_{i-2}(\text{Ind } \bar{J}) = f_{i-1}(\text{Ind } \tilde{G})$ as the first counts the independent sets of size $i$ in $\tilde{G}$ without $y_n$ and the second counts the independent sets of size $i$ in $\tilde{G}$ with $y_n$. 


Hence, for $0 \leq j \leq n + m$,
\[
 f_{j-1}(\text{Ind } G) = f_{j-1}(\text{Ind } H) + f_{j-2}(\text{Ind } H) + f_{j-2}(\text{Ind } J)
 = \sum_{i=0}^{j} \binom{n+m-i}{j-i} f_{i-1}(\text{Ind } H) + \sum_{i=0}^{j-1} \binom{n+m-1-i}{j-1-i} f_{i-1}(\text{Ind } J)
 = \sum_{i=0}^{j} \binom{n+m-i}{j-i} \left( f_{i-1}(\text{Ind } H) + f_{i-2}(\text{Ind } J) \right)
 = \sum_{i=0}^{j} \binom{n+m-i}{j-i} f_{i-1}(\text{Ind } \hat{G}).
\]
Solving this system for $f_{j-1}(\text{Ind } \hat{G})$ yields
\[
 f_{j-1}(\text{Ind } \hat{G}) = \sum_{i=0}^{j} (-1)^{j-i} \binom{n+m-i}{j-i} f_{i-1}(\text{Ind } G) = h_{j}(\text{Ind } G),
\]
for $0 \leq j \leq n + m$. □

The following consequence was independently, and in a different manor, shown in [3, Corollary 5.4].

**Theorem 4.12.** The $h$-vector of an independence complex of a bipartite graph that is Cohen-Macaulay is a face vector of a flag complex.

A natural question is, which graphs are compressions of bipartite graphs with Cohen-Macaulay independence complexes? To study this, we define an *extrusion* of any graph $G$ on vertices $\{v_1, \ldots, v_n\}$ to be a new bipartite graph $\hat{G}$ on vertices $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ with edge set given by the edges $x_i y_i$, for $1 \leq i \leq n$, and either $x_i y_j$ or $x_j y_i$, for each edge $v_i v_j$ in $G$. Note extrusion is not unique and if $\text{Ind } \hat{G}$ is pure, then $\hat{G}$ is cross-free by construction and thus $\text{Ind}(\hat{G})$ is Cohen-Macaulay by Proposition 4.8.

We call a graph *Cohen-Macaulay extrudable* if there is some extrusion of the graph with a Cohen-Macaulay independence complex.

**Proposition 4.13.** Bipartite graphs are Cohen-Macaulay extrudable.

*Proof.* Let $G$ be a bipartite graph with bi-partition $U = \{u_1, \ldots, u_m\}, V = \{v_{m+1}, \ldots, v_n\}$. Then $G$ can be extruded to the graph $\hat{G}$ with vertex set $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ and edge set given by $x_i y_i$, for $1 \leq i \leq n$, and $x_i y_j$ or $x_j y_i$, for all edges $u_i v_j$ in $G$.

Hence, as $y_i$, for $1 \leq i \leq m$, and $x_i$, for $m + 1 \leq i \leq n$, are all degree one, we have that $\text{Ind } \hat{G}$ is pure and thus Cohen-Macaulay.

The extrusion described in the proof of Proposition 4.13 is just whiskering the graph at every vertex.

**Example 4.14.** Figure 4.1 demonstrates the extrusion described in the proof of Proposition 4.13. It illustrates that the chosen extrusion $\hat{G}$ is simply a whiskering of the graph $G$ at every vertex.

Hence by Proposition 3.8 we have the following consequence.
Figure 4.1. The extrusion $\hat{G}$ of the bipartite graph $G$ as described in the proof of Proposition 4.13; the dark grey lines in the graph $\hat{G}$ correspond to the lines originating from the graph $G$.

Corollary 4.15. The face vector of the independence complex of every bipartite graph is the $h$-vector of the independence of some vertex-decomposable bipartite graph.

Unfortunately, the converse is not true (see Example 4.16 below). However, by Proposition 4.11 the $h$-vector of the independence complex of a bipartite graph that is vertex-decomposable is the face vector of the independence complex of some (not necessarily bipartite) graph.

Example 4.16. Let $G$ be the Ferrers graph given by the edges $x_1y_1, x_1y_2, x_1y_3, x_2y_2, x_2y_3, x_3y_3$; this graph is shown in figure 4.2. Then Ind$G$ is Cohen-Macaulay and hence vertex-decomposable, moreover, the $h$-vector of Ind$G$ is $(1, 3)$. However, the only graph with independence complex having face vector $(1, 3)$ is the three-cycle, which is not bipartite.

Figure 4.2. The Ferrers graph $G$

A graph is said to have odd-holes if it has an induced subgraph that is an odd-cycle of length at least five.

Proposition 4.17. A graph with odd-holes is not Cohen-Macaulay extrudable.

Proof. Let $G$ be a graph with odd-holes. In particular, let $v_1, \ldots, v_{2m+1}$ be an odd-hole of $G$ and suppose that $v_i$ extrudes to the pair $x_i, y_i$. Without loss of generality, assume the edge $v_1v_2$ is extruded to $x_1y_2$. Then we must extrude the edge $v_2v_3$ to $x_3y_2$, otherwise we introduce an obstruction to purity as $v_1v_3$ is not an edge in $G$. Continuing in this manner, edges $v_{2k-1}v_{2k}$ extrude to $x_{2k-1}y_{2k}$ and edges $v_{2k}v_{2k+1}$ extrude to $x_{2k+1}y_{2k}$.

The remaining edge $v_1v_{2m+1}$ can either be extruded as $x_1y_{2m+1}$ or $x_{2m+1}y_1$. In the first case, then $x_1y_{2m+1}$ and $x_{2m+1}y_2$ would require $x_1y_{2m}$ for purity, but this contradicts our choice of an odd-hole, as this would require $v_1v_{2m}$ to be in $G$. Similarly for the second case. Thus, odd-holes obstruct pure extrusions.

Notice that odd-hole-free graphs without triangles are bipartite, hence Cohen-Macaulay extrudable. However, not all odd-hole-free graphs with triangles are Cohen-Macaulay extrudable.
Example 4.18. Let $G$ be the three-cycle on vertices $\{u, v, w\}$. Then the extrusion $\hat{G} = (\{u, v, w, x, y, z\}, \{ux, vy, wz, uy, vz, uz\})$ has a Cohen-Macaulay independence complex.

Let $H = (\{u, v, w, x, y, z\}, \{uv, uw, vw, xy, xz, yz, ux, uy, vz\})$, then one can use a program such as Macaulay2 [14] to check the $2^9 = 512$ possible extrusions of $H$ to see that it is not Cohen-Macaulay extrudable.

Hence, we close with a question: which odd-hole-free graphs with triangles are Cohen-Macaulay extrudable?

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