SLOCC invariants for multipartite mixed states

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Abstract
We construct a nontrivial set of invariants for any multipartite mixed states under the stochastic local operations and classical communication symmetry. These invariants are given by hyperdeterminants and independent of basis change. In particular, a family of \(d^2\) invariants for arbitrary \(d\)-dimensional even partite mixed states are explicitly given.

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1. Introduction

Classification of multipartite states under stochastic local operations and classical communication (SLOCC) has been a central problem in quantum communication and computation. Recently, advances have been made in the classification of pure multipartite states under SLOCC \([1, 2]\) and the dimension of the space of homogeneous SLOCC invariants in a fixed degree is given as a function of the number of qudits. In this work we present a general method for constructing polynomial invariants for mixed states under SLOCC. In particular, we also derive general invariants under local unitary (LU) symmetry for mixed states.
Polynomial invariants have been investigated in [3–6], which allow us in principle to determine all the invariants of LU transformations. However, in practice it is a daunting task to solve the large system of algebraic equations. Moreover, little is known for mixed multipartite states even for $2 \times 2 \times 2$ systems. In [7] a complete set of 18 polynomial invariants is presented for the locally unitary equivalence of two-qubit mixed states. Partial results have been obtained for three-qubit states [8], tripartite pure and mixed states [9], and some generic mixed states [10–12]. Recently the LU equivalence problems for multiqubit [13] and general multipartite [14] pure states have also been solved.

In general there are limited tools available for resolving the SLOCC symmetry for mixed multipartite states. Ideally, it is hoped that a complete set of invariants under LU transformations can be found. However, usually these invariants depend on the detailed expressions for pure state decompositions of a state. For a given state, such pure state decompositions are infinitely many. Particularly when the density matrices are degenerate, the problem becomes more complicated.

In this paper, we present a general method for constructing nontrivial invariants under the SLOCC symmetry group. All invariants found using this method are \textit{a priori} LU invariants, as the LU symmetry group is contained in the SLOCC symmetry group. Therefore SLOCC invariants are generally far fewer than LU invariants. Very little information is known about the SLOCC invariants except for the degree of the invariant subring [1], and even less is known about how to systematically construct the invariants. We will give a general method for constructing four families of SLOCC invariants for any mixed state of even partite cases plus an iterative method for constructing general fundamental invariants in all dimensions.

We show that the Cayley’s two hyperdeterminants [15, 16] can be used to construct such invariants. Cayley’s second hyperdeterminant has been used to study entanglement measures like concurrence [17–21] and 3-tangles [22]. It also played an important role in the classification of multipartite pure states [23–25]. In [26] we proved that the hyperdeterminants give LU invariants that are independent of the pure state decomposition and base transition. One of our current main results is that the generalized characteristic polynomials actually give SLOCC invariants and we further show that in the even partite cases the Cayley first hyperdeterminants also lead to some of the nontrivial SLOCC invariants for mixed states.

2. Hypermatrix representations of mixed states

Let $H$ be the state space of the mixed multipartite quantum state: $H = H_1 \otimes H_2 \otimes \cdots \otimes H_n$, where $H_k$ is a Hilbert space of dimension $d_k$ with inner product $\langle , \rangle_i$. Suppose the $\sigma^{(i)}_k$ $(i = 0, \ldots, d_k^2 - 1)$ are an orthonormal basis of Hermitian operators in $\text{End}(H_i)$. Here we can take the Gell-Mann basis, where $\sigma^{(i)}_0 = I$ is the identity. In particular, they are Pauli spin matrices when $d_k = 2$. Then $\rho$ can be expressed as

$$\rho = \sum_{i_1 \ldots i_n} a_{i_1 \ldots i_n} \sigma^{(1)}_1 \otimes \cdots \otimes \sigma^{(n)}_n,$$

where the $n$-dimensional matrix or hypermatrix $A = (a_{i_1 \ldots i_n})$ is called the matrix representation of $\rho$ with respect to the bases $\{\sigma^{(i)}_k\}$. Here the format of the matrix $A$ is $d_1^2 \times \cdots \times d_n^2$, and usual rectangular matrices of size $m \times n$ are two-dimensional matrices of format $m \times n$. One can use $A = [\rho]_{\sigma}$ to denote the matrix representation of $\rho$ with respect to the orthonormal bases $\sigma = \{\sigma^{(k)}_i\}$. Sometimes the reference to $\sigma$ is omitted if it is clear from the context.
We recall the multiplication of the hypermatrix $A$ of format $f_1 \times \cdots \times f_n$ by an $f_i \times f_i$ matrix $B$ as follows:

$$B \ast_i A = C,$$

where

$$C_{i_1 \ldots i_k} = \sum_{j=1}^{f_i} b_{i_1 \ldots j} a_{i_2 \ldots i_{k+1} \ldots i_k}.$$  

This generalizes the notion of matrix multiplication. When $A$ is a regular rectangular matrix, then $B \ast_1 A = BA, B \ast_2 A = AB^t$ and $(B \ast_1 C \ast_2)A = BAC^t$, where $t$ stands for transpose.

It is easy to see that the matrix representation of $\rho$ under basis change from $\sigma$ to $\sigma'$ can be simply described by

$$[\rho]_{\sigma'} = (P_1 \ast_1 \ldots P_n \ast_n)[\rho]_{\sigma},$$

where the unitary matrix $P_k$ is the matrix of transition from the orthonormal basis $\{\sigma_i^{(k)}\}$ to the orthonormal basis $\{\sigma_i^{(k)}\}$ for $k = 1, \ldots, n$.

A function $F(\rho)$ is invariant under SLOCC transformation if $F(\rho) = F(A\rho A^t)$ for any invertible $A = A_1 \otimes \cdots \otimes A_n \in G = SL(d_1) \otimes \cdots \otimes SL(d_n)$, the SLOCC symmetry group, where $t$ denotes transpose conjugate. A true SLOCC invariant should only depend on $\rho$, i.e., an invariant $F(\rho)$ given in terms of the matrix $[\rho]$ should be independent of the matrix representation [26]. Since our matrix representation is given as an unnormalized form, the SLOCC invariants can be taken as a relative invariant under the larger group $G = GL(d_1) \otimes \cdots \otimes GL(d_n)$, or a projective invariant under $\hat{G}$.

Two hypermatrices of the same format can be added together to give rise to a third hypermatrix, via $(a_{i_1 \ldots i_k}) + (b_{i_1 \ldots i_k}) = (a_{i_1 \ldots i_k} + b_{i_1 \ldots i_k})$. The scalar product of a hypermatrix and a number is defined as usual. Then the set of hypermatrices of format $f_1 \times \cdots \times f_n$ forms a vector space of dimension $f_1 \cdots f_n$. We denote the vector space by $\text{Mat}(f_1, \ldots, f_n)$.

We consider invariants under the SLOCC symmetry. Let us first look at the special example of a bipartite case to motivate our later discussion. Suppose $A$ is the (square) matrix representation of the mixed state $\rho$ on $H^{\otimes 2}$. Using the same method as in [26], it is easy to see the following result.

**Proposition 1.** Suppose $A$ is the Bloch matrix representation of the bipartite mixed state $\rho$ on $(\mathbb{C}^d)^{\otimes 2}$. Then the coefficients $F_i(A)$, $i = 1, 2, \ldots, d^2$, of the characteristic polynomials of $A$:

$$\det(\lambda I - A) = \lambda^d - \text{tr}(A)\lambda^{d-1} + \cdots + (-1)^d\det(A)$$

$$= \sum_{i=0}^{d} (-1)^i F_i(A)$$

are LU invariants. In particular, $\text{tr}(A)$ and $\det(A)$ are LU invariants of the two-partite state $\rho$.

We remark that when $\rho$ is a bipartite mixed state in arbitrary dimensional space $H_1 \otimes H_2$, then the Bloch matrix representation $A$ is a rectangular matrix. Then $\det(AA^t)$ and $\det(A^tA)$ are LU invariants. When $m \neq n$, it seems that these are the only known SLOCC invariants according to [1].

With pure states $|\phi\rangle = \sum_{i_1 \ldots i_n} a_{i_1 \ldots i_n}|i_1 \ldots i_n\rangle$, we can associate the hypermatrix $A(|\phi\rangle) = (a_{i_1 \ldots i_n})$; then the format will be simpler. For the bipartite pure state $|\phi\rangle$, the determinant $\det(|\phi\rangle)$ is clearly a SLOCC invariant.

In the following we extend the result of proposition 1 to show that the $F_i(A)$ are in fact SLOCC invariants.
3. Hyperdeterminants and SLOCC invariants

Given any hypermatrix $A$, one defines the associated multilinear form $f_A : H_1 \otimes \cdots \otimes H_n \mapsto \mathbb{C}$ given by

$$f_A(x^{(1)}, \ldots, x^{(n)}) = \sum_{i_1, \ldots, i_n} a_{i_1, \ldots, i_n} x_{i_1}^{(1)} \cdots x_{i_n}^{(n)}.$$  \hspace{1cm} (5)

The multilinear form $f_A$ can also be written as a tensor in $H_1^* \otimes \cdots \otimes H_n^*$:

$$f_A = \sum_{i_1, \ldots, i_n} a_{i_1, \ldots, i_n} f_i^{(1)} \otimes \cdots \otimes f_i^{(n)},$$  \hspace{1cm} (6)

where the $f_i^{(k)}$ are standard 1-forms on $H_k$. For any rectangular matrix $B = [b_{mn}]$, the column vector realignment is given by $v(B) = [b_{11}, \ldots, b_{m1}, \ldots, b_{1n}, \ldots, b_{mn}]$. The vector realignment satisfies the following fundamental property [27]:

$$v(ABC) = (C \otimes A)v(B),$$  \hspace{1cm} (7)

where $C \otimes A$ is the Kronecker tensor product. It is also clear that $v(B_1), \ldots, v(B_{mn})$ form a basis of $\mathbb{C}^{mn}$. Therefore the map $\phi$ induces an isomorphic linear operator on $M_{mn} \simeq \mathbb{C}^{mn}$ via $\phi(v(B)) = v(ABC)$ as $v(\cdot)$ is apparently linear. We will denote the isomorphic map by the same symbol $\phi$. It then follows that

$$\phi[v(B_1), \ldots, v(B_{mn})] = [v(AB_1C), \ldots, v(AB_{mn}C)]$$

$$= (C \otimes A)[v(B_1), \ldots, v(B_{mn})],$$

which means that $C \otimes A$ is the matrix of the linear operator $\phi \in \text{End}(\mathbb{C}^{mn})$. Hence $\det(\phi) = \det(C \otimes A) = \det(C)^m \det(A)^n$. \hfill $\square$

Consider a SLOCC symmetry $g = A_1 \otimes \cdots \otimes A_n \in G = \text{SL}(d_1) \otimes \cdots \otimes \text{SL}(d_n)$ on the state space $H = H_1 \otimes \cdots \otimes H_n$. This action is given by $\rho \mapsto g \rho g^\dagger = (A_1 \otimes \cdots \otimes A_n) \rho (A_1^\dagger \otimes \cdots \otimes A_n^\dagger)$. Let $B_k$ be the matrix of the action of $A_k$ on the orthonormal basis $\{\sigma_j^{(k)}\}$, then the multilinear form $g \rho g^\dagger$ under the action of $g \in G$ is seen to be

$$[g \rho g^\dagger] = (B_1 \ast_1 \cdots \ast_{n-1} B_n \ast_n)[\rho].$$  \hspace{1cm} (9)

One usually writes it as $(B_1 \otimes \cdots \otimes B_n)[\rho]$ when the hypermatrix $[\rho]$ is identified with its associated tensor via (6). This shows that the action of the SLOCC symmetry on the mixed state $\rho$ is represented by similar hypermatrix multiplications when one introduces the matrix $B_k$ through (8) for each individual action of $A_k$ of the local SLOCC symmetry $g = A_1 \otimes \cdots \otimes A_n$. The great advantage of (9) is the similarity with the basis transition (3). The following result is immediate from lemma 2 and the action (9).
Theorem 3. The action of the SLOCC group $\text{SL}(d_1) \otimes \cdots \otimes \text{SL}(d_n)$ on $\rho$ induces an action of $\text{SL}(d_1^2) \times \cdots \times \text{SL}(d_n^2)$ on its hypermatrix representation $[\rho]$.

We comment that a similar result also holds for a larger symmetry group $\hat{\text{SLOCC}} = \{g = A_1 \otimes \cdots \otimes A_n | \det(g) = 1, A_k \in \text{GL}(d_k)\}$ when all $d_i$ are equal.

Theorem 3 translates the problem of SLOCC invariants of $\rho$ into a specific problem of invariants of its hypermatrix representation $[\rho]$ under the group $\text{SL}(d_1^2) \times \cdots \times \text{SL}(d_n^2)$. This is exactly the classical problem studied long ago by Cayley.

Cayley [15] developed the theory of hyperdeterminants as the theory of homogeneous polynomials in matrix elements $A_{i,j,k}$. A hyperdeterminant is a special homogeneous polynomial invariant under the action of $\text{SL}(n_1) \times \cdots \times \text{SL}(n_1)$. One modern account of Cayley’s theory is the beautiful monograph [16]. Cayley studied two kinds of hyperdeterminants, and each is used in our current work. Cayley’s (second) hyperdeterminant $\text{Det}(A)$ is defined as the resultant of the multilinear form $f_A$, that is, $\text{Det}(A)$ is a certain polynomial in components of the tensor $f_A$ which is zero if and only if if the map $f_A$ has a nontrivial point where all partial derivatives with respect to the components of its vector arguments vanish (a nontrivial point means that none of the vector arguments are zero). For example, the resultant of $a_1 x_1^2 + a_2 x_1 x_2 + a_3 x_2^2 = 0$ is the polynomial $a_2^2 - 4a_1 a_3$.

It is known that the hyperdeterminant $\text{Det}(A)$ exists for a given format and is unique up to a scalar factor provided that any factor in the format is less than or equal to the sum of the other factors in the format. Cayley’s two hyperdeterminants are all invariant under the group $\text{SL}(f_1) \otimes \cdots \otimes \text{SL}(f_n)$. In fact the hyperdeterminant $\text{Det}$ satisfies the following multiplicative property. Suppose $A$ is a hypermatrix of format $f_1 \times \cdots \times f_n$ and $B$ is any $f_i \times f_j$ matrix; then

$$\text{Det}(B \ast_i A) = \text{Det}(A) \text{ det}(B)^{N/f_i},$$

where $N$ is the degree of $\text{Det}(A)$. Therefore we have the following result.

Theorem 4. The hyperdeterminant of the matrix of $\rho$ is a relative invariant under the action of $\text{GL}(d_1) \otimes \cdots \otimes \text{GL}(d_n)$, and thus invariant under the SLOCC symmetry group $G = \text{SL}(d_1) \otimes \cdots \otimes \text{SL}(d_n)$.

For an even dimensional matrix $A = (A_{i,j,k})$, where $1 \leq i_1, \ldots, i_{2n} \leq N$, Cayley’s first hyperdeterminant $\text{hdet}(A)$ is the following polynomial:

$$\text{hdet}(A) = \frac{1}{N!} \sum_{\sigma_1, \ldots, \sigma_{2n} \in S_N} \text{sgn}(\sigma) \prod_{i=1}^N A_{\sigma_1(i), \ldots, \sigma_{2n}(i)},$$

where $\text{sgn}(\sigma) = \prod_{i=1}^n \text{sgn}((i))$. The hyperdeterminant $\text{hdet}$ reduces to the usual determinant when $A$ is a square matrix.

Cayley’s second hyperdeterminant [15] of the format $2 \times 2 \times 2$ for the hypermatrix $A$ with components $a_{ijk}, i, j, k \in \{0, 1\}$, is given by

$$\text{Det}(A) = a_{000}^2 a_{111}^2 + a_{001}^2 a_{110}^2 + a_{010}^2 a_{101}^2 + a_{100}^2 a_{011}^2 - 2a_{000} a_{001} a_{110} a_{111} - 2a_{000} a_{110} a_{010} a_{111} - 2a_{001} a_{010} a_{101} a_{111} - 2a_{000} a_{010} a_{110} a_{101} + 4a_{000} a_{111} a_{110} + 4a_{001} a_{101} a_{100}.$$

This hyperdeterminant can be written in a more compact form by using the Einstein convention and the Levi-Civita symbol $\epsilon^{ijk}$, with $\epsilon^{00} = \epsilon^{11} = 0$, $\epsilon^{01} = -\epsilon^{10} = 1$; and $b_{iln} = (1/2)\epsilon^{ijk} a_{ijkl} a_{iln}$. $\text{Det}(A) = (1/2)\epsilon^{ijk} b_{iln} b_{jln}$. The four-dimensional hyperdeterminant of the format $2 \times 2 \times 2 \times 2$ has been given in [23].
In order to generate more invariants from the hyperdeterminants, observe that the tensor product $I_n = I_d \otimes \cdots \otimes I_d$ is a special hypermatrix of format $d_1^2 \times \cdots \times d_d^2$ provided that one views it as a tensor. Moreover, for any $A = A_1 \otimes \cdots \otimes A_n \in G$, one has that
\[(A_1 \ast_1 \ldots \ast_{n-1} A_n \ast_n) I_n = A_1 \otimes \cdots \otimes A_n\] which is also an element of $\text{Mat}(d_1^2, \ldots, d_n^2)$.

**Theorem 5.** (1) The hypercharacteristic polynomial $\text{Det}(\lambda I_n - \rho) = \text{Det}(\lambda I_n - [\rho])$ of the multipartite mixed state $\rho$, with $[\rho] = (A_1, \ldots, n) \in \text{Mat}(d_1^2, \ldots, d_n^2)$, is a SLOCC invariant polynomial in $\lambda$. The coefficients of the characteristic polynomial are all SLOCC invariants. In particular, the hypertrace $\text{tr}(A)$ and the hyperdeterminant $\text{Det}(A)$ of $A$ are two distinguished polynomial SLOCC invariants. (2) The same results hold for even partite quantum states when $\text{Det}$ is replaced with $\text{hdet}$.

**Proof.** The proofs for the two hyperdeterminants are similar, so we only consider $\text{Det}$. By previous discussion either the transition matrix or the action of the SLOCC symmetry is represented by the hypermatrix multiplication; see (3) and (9):
\[ [\rho]_{\sigma'} = (P_1 \ast_1 \ldots \ast_{n-1} \ast P_n \ast_n) [\rho]_{\sigma}, \]
\[ [gpg^\dagger] = (B_1 \ast_1 \ldots \ast_{n-1} \ast B_n \ast_n) [\rho]. \]
Therefore the hypercharacteristic polynomial of $gpg^\dagger$ is
\[ \text{Det} ((B_1 \ast_1 \ldots \ast_{n-1} \ast B_n \ast_n)(\lambda I_n - [\rho])) = \det(B_1)^{\lambda/d_1} \cdots \det(B_1)^{\lambda/d_1} \text{Det}(\lambda I_n - \rho) = \text{Det}(\lambda I_n - \rho). \]
This means that $f(\lambda) = \text{Det}(\lambda I_n - \rho)$ is a SLOCC invariant polynomial and every $\lambda$ coefficient of $f(\lambda)$ is a SLOCC invariant. Note that the above calculation also proves the independence of base change. \(\Box\)

We emphasize that in general the state $I_n$ is not represented by a tensor product of the identity operators except for the special bipartite case. This means that although as regards form our result resembles the usual theory of the characteristic polynomial, the result in fact contains substantially more information in general and constitutes a theoretical method for generating nontrivial invariants for the mixed states.

As an application of our general method, we give a few examples.

**Example 1.** For the two-qubit case, $A = (a_{ij}) \in M_4$, the following are SLOCC invariants:
\[ \text{tr}(A); \quad \det(A); \]
\[ m_{12} + m_{13} + m_{14} + m_{23} + m_{24} + m_{34}; \]
\[ m_{23} + m_{13} + m_{12} + m_{34}, \]
where the $m_{ij}$ etc are the principal minors, e.g. $m_{12}$ is the $(1, 2)$-principal minor of $A$.

**Example 2.** When $\rho$ is a mixed state on $H = \bigotimes_{i=1}^n \mathbb{C}^d$, we will get $d^3$ SLOCC invariants from $\text{hdet}(\lambda I_n - \rho)$. The four nontrivial coefficients of the following polynomial provide SLOCC invariants for the mixed four-qubit state:
\[ \text{hdet}(A - \lambda I) = \frac{1}{2^d} \sum_{\sigma \in S_4} \text{sgn}(\sigma) \prod_{i=1}^4 (A_{\sigma_1(i)\sigma_2(i)\sigma_3(i)\sigma_4(i)} - \lambda \delta_{\sigma_1(i)\sigma_2(i)} \delta_{\sigma_3(i)\sigma_4(i)}). \]
4. Conclusion and discussion

We have introduced a general set of SLOCC invariants for mixed multipartite states by using hyperdeterminants of the matrix representation in terms of orthonormal bases. We have shown that these invariants are independent of matrix representations.

Our method is fundamentally different from previous studies on LU invariants of pure states, and enjoys the favorable property of independence of base change. In fact, our method shows that as long as the transition matrix belongs to $SL(n)$, then the hypermatrix representation of the mixed state $\rho$ can deliver the same result. Even in the case of pure quantum states, our method is also different. Suppose that a pure state is given by $|\phi\rangle = \sum I a_I |I\rangle$, and assume that the matrix of transition from $|I\rangle \langle J|$ to the $\lambda$ basis is $(p_{IJ}, \sigma)$; then the hypermatrix of $\rho = |\phi\rangle \langle \phi|$ is $(\sum_{IJ} a_I a^*_J p_{IJ}, \sigma)$, whose hyperdeterminant is quite different from that of $(a_I)$, even in format.

The method of hyperdeterminants can be used for the classification of multipartite quantum states in addition to the theoretical classification of SLOCC symmetry. Since the invariants involve a large amount of computation, in practice they should be useful only with the help of computer codes. In our opinion, one of the useful messages is that many LU invariants obtained from the hypermatrix representation are actually SLOCC invariants.

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