Polyfractal driving for engineering Hamiltonians and symmetries.

Kartiek Agarwal\textsuperscript{1,*} and Ivar Martin\textsuperscript{2}

\textsuperscript{1}Department of Physics, McGill University, Montréal, Québec H3A 2T8, Canada
\textsuperscript{2}Material Science Division, Argonne National Laboratory, Argonne, IL 60439, USA

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We propose a new protocol for engineering quantum many-body Hamiltonians with enhanced symmetries. The protocol is based on repeated pulsed application of a set of unitary operators $X_i$, with $X_i^2 = 1$, (which can be generalized to $X_i^n = 1$, $n > 2$) in a self-similar-in-time (“polyfractal”) manner. For local initial Hamiltonians, the protocol can simultaneously implement multiple global and local symmetries, with the accuracy improving superpolynomially with the fastest drive period. The effective Hamiltonian remains local and avoids heating over time scales that are stretched-exponentially long in the drive frequency. Such Floquet engineering can be used to realize novel quantum models, or in the case when two or more global symmetries $X_i$ anti-commute, engender a degenerate many-body spectrum that can be used to encode topological qubits controlled precisely by the same $X_i$.

\textbf{Introduction.}— Understanding the phase structure of quantum systems in non-equilibrium settings is an important and challenging goal. Contrary to prior expectations, we now know that driven many-body quantum systems can avoid the fate of rapidly heating up to infinite temperature\textsuperscript{1–6}, and can thus exhibit a rich phase diagram akin to their equilibrium counterparts\textsuperscript{3,7–20}. This has been made possible, in large part, due to the theoretical discoveries of many-body localization\textsuperscript{21–23}, and long-lived prethermal states in systems driven at high frequencies\textsuperscript{1,2,5,24}. Additionally, the associated Floquet unitaries of driven systems can exhibit an extensively degenerate spectrum\textsuperscript{25}, give rise to phenomena such as time-crystallinity\textsuperscript{3,10,19}, and host dynamical Majorana modes that can be braided in a single wire\textsuperscript{25–27}.

The novel dynamical phenomena mentioned above arise when the system develops an additional, emergent symmetry—for instance, Refs.\textsuperscript{5,10,16} show how a $\mathbb{Z}_2$-symmetry emerges and can spontaneously break to give rise to time-crystalline behavior. More generally, symmetries and their spontaneous breaking underlie the rich variety of equilibrium and non-equilibrium phases and their dynamics. From the quantum information perspective, a large enough induced symmetry set can cancel the entire original Hamiltonian which can be advantageous for quantum information preservation\textsuperscript{25–31}. It is thus a desirable goal to find ways to engineer symmetries in non-equilibrium settings.

In this Letter, we describe a flexible protocol that can build local and global symmetries into effective Floquet Hamiltonians. Specifically, it involves injecting a finite set, of say, $n_s$ unitary operators $X_i$, at specific times corresponding to a fractal pattern, during the course of time evolution of the system under its physical Hamiltonian $H$. We focus on the case $X_i^2 = 1$, but generalization to the $\mathbb{Z}_n$ case with $n > 2$ is straightforward and will be discussed elsewhere. As we show, such a fractal application of $X_i$ can be optimized in the number of fractal layers $n_f$, to result in an effective Hamiltonian for which $X_i$s are symmetries to an accuracy that is nearly exponential in the drive frequency. This sensitivity to the drive frequency allows for accurately implementing global symmetries while requiring a drive frequency that scales merely logarithmically with system size $N$.

The approach is motivated by the following intuition. The periodic application of $X_i$ flips the sign of the terms in the Hamiltonian that anti-commute with $X_i$; thus, frequent application of $X_i$ leads to their cancellation after even number of drive periods. In particular, the resulting effective Hamiltonian commutes with $X_i$ up to $\sim O(T_0)$ corrections, where $T_0$ is the drive period. As we show, these corrections can be further suppressed by applying $X_i$ periodically at intervals of $2T_0, 4T_0, ...$ (see Fig. 1). (Note since $X_i^2 = 1$, $X_i$ will effectively not be applied at some time instances.) Alternatively, one may apply other generators $X_j$s at self-similar intervals to generate additional symmetries. This work follows through with the above intuition, and shows that for local Hamiltonians, there exists an optimal number of fractal layers $n_f$ which leads to superpolynomial (in drive-frequency) suppression of symmetry-violating terms, while the system heats up only on a stretched-exponentially long timescale.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Illustration of the protocol for $n_s = 2$ unitaries $X_1, X_2$. The two unitaries are applied in a self-similar fashion in time. The Floquet unitary $U_F(T) \equiv X_1^{n_f} X_2^{n_f} X_1^{n_f} X_2^{n_f} \cdots$ corresponds to a fractal pattern, during the course of time evolution of the system under its physical Hamiltonian $H$. As we show, this results in a cancellation of symmetry-violating terms, while the system heats up only on a stretched-exponentially long timescale.}
\end{figure}
This work has implications for engineering novel Floquet Hamiltonians, symmetry-protected topological phases, and stabilization of quantum memories, among others. We explore some of these ideas in a companion paper. When the engineered symmetries anti-commute, they give rise to a robust degeneracy structure in the entire spectrum. This potentially could be exploited to engineer topologically protected qubits. These qubits would be manipulated by the same $X_i$, to near-exponential precision in the drive frequency.

Finally, note that there is a precedent for fractal pulsing in finite-sized systems. However, as opposed to the more general scheme we identify, these are system specific. Moreover, they rely on operator expansions that have null convergence in the thermodynamic limit. One may thus view this work as a formal extension of dynamical decoupling techniques to many-body systems. Below we describe our results, before providing more detailed proofs and numerical validation.

**Description of protocol and main results.**— We assume a physical system described by a Hamiltonian $H$ comprising of a sum of local terms, with a local norm $\| h \|$. We also assume a set of $n_f$ unitaries $X_i$ that further satisfy the condition $X_i^2 = 1$. These unitaries may themselves either commute or anti-commute with one another. The protocol we study involves applying $X_i$ at times

$$t_i = m 2^{i+n_s(j-1)} T_0, \ i \in [1, n_s], j \in [1, n_f], m \in \mathbb{Z}^+ \tag{1}$$

amidst the regular Heisenberg evolution, where $n_f$ is the number of “fractal layers” in the composite Floquet unitary; see Fig. 1 for illustration. To illustrate by example, suppose we have $n_s = 2$ unitaries, and apply these $n_f = 1$ times. Then, $U(T_0) = e^{-iHT_0}; U(2T_0) = X_1 U(T_0) X_1 U(T_0); U(4T_0) = X_2 U(2T_0) X_2 U(2T_0).$ Subsequent time-evolution at periods of $T = 2^{n_f+n_s} T_0 = 4T_0$ is given by the repeated application of the Floquet unitary $U(4T_0)$. For $n_f = 2$, the above recursion relations would be repeated for another fractal layer: $U(8T_0) = X_1 U(4T_0) X_1 U(4T_0); U(16T_0) = X_2 U(8T_0) X_2 U(8T_0),$ and $U(16T_0)$ would subsequently serve as the Floquet unitary.

In what follows, we decompose $H$ into terms which transform differently under $X_i$:

$$H = \sum_\epsilon A_\epsilon \quad \text{where } \epsilon = (\epsilon_1, \ldots, \epsilon_{n_s}); \epsilon_i \in \{0, 1\},$$

$$X_j A_\epsilon X_j = (-1)^{\epsilon_j} A_\epsilon. \tag{2}$$

This decomposition is unique if $X_i$s commute or anti-commute with one another, which we assume. With this terminology, one may represent the Floquet unitary in time-ordered notation as

$$U(T) = 2^{n_f+n_s} T_0 = T \left\{ e^{-i f_\epsilon(t)} dt \right\} \sum_\epsilon A_\epsilon f_\epsilon(t) ,$$

where $f_\epsilon(t)$ simply tracks the times at which $X_i$ is applied: this corresponds to a sign change of terms $A_\epsilon$ for which $\epsilon_i = 1$. The integral over a complete period is guaranteed to be zero except for $f_0$. Thus, in a time-averaged sense, the effective Hamiltonian is $A_0$ comprising of only the terms even under all $X_i$.

We further note that one may represent this unitary $U(T)$ as an expansion in the exponent

$$U(T) = e^{-i T \sum_{n=0} T^n \Omega_n} \tag{4}$$

with operators $\Omega_n$ that can be arrived at using the Magnus expansion, or, in this case, a repeated application of the BCH formula; the first term is simply the time-averaged Hamiltonian $\Omega_0 = A_0$. In general, the operator $\Omega_n$ involves $n$ nested commutators of the local operators $A_\epsilon$. Thus, if the local terms comprising $A_\epsilon$ involve at most $k$ sites, $\Omega_n$ can be represented as a sum of terms comprising at most $nk$ sites. Finally, the series expansion is only useful if we can truncate it at some order and effectively approximate the unitary dynamics; we define the approximate Hamiltonian

$$H_F^{(n)} = \sum_{0 \leq n \leq n_0} T^n \Omega_n. \tag{5}$$

Our main results concern the properties of the Floquet unitary $U(T)$, and the associated effective Floquet Hamiltonian $H_F^{(n)}$. The first part of our results are directly adapted from the results of Refs. which state that the difference between the exact reduced density matrix of a region of size $N_\rho$ and that obtained by evolving it with $H_F^{(n)}$ is bounded in norm by $c N_\rho 2^{-n_0}$, for some finite constant $c$. Here $n_0 \sim 1/(T \| h \|) \equiv \omega/\| h \| \gg 1$ scales linearly with the effective drive frequency $\omega = 1/(2^{n_f+n_s} T_0)$. The norm of this error sets the inverse of the time scale up to which $H_F^{(n)}$ provides a good description of the dynamics of local operators (alternatively, the time scale for heating in the system)—the crucial result is that this time scale grows exponentially with the drive frequency. For global operators $X_i$ with finite norm, but also for $H_F^{(n)}$ itself, $N_\rho$ is the system size $N$, but the error can still be made small by scaling $\omega$ merely logarithmically with the system size. Hereon, we will assume such frequency scaling.

Having established the conditions under which $H_F^{(n)}$ faithfully describes the time evolution of $X_i$, we seek to establish a bound on the norm of terms in $H_F^{(n)}$ that do not commute with $X_i$. Defining the time scale $\tau_X \approx \min \left\{ 1/ \left\| H_F^{(n)} X_i \right\| \right\}$ which sets the shortest timescale at which unitaries $X_i$ relax, we find

$$\tau_X \geq 1/N \left\| C_{n_f} \right\| \right\| |T_0|/\| h \| n_f |T_0|$$

$$\tau_H \geq 1/N \left\| C \right\| \right\| \| h \| /\| h \| n_f |T_0| \tag{6}$$

where $C_{1,2}$ are $O(1)$ combinatorial constants. $\tau_H$ is a bound on the time for which global operators such as energy are accurately described by $H_F^{(n)}$. Note that $\tau_X$...
initially increases with the number of fractal layers \( n_f \) but eventually begins to decrease again. There is therefore an optimal \( n_f \) for which \( X_i \) become effective symmetries. Note also that \( n_f \) cannot be made arbitrary large since its increase rapidly decreases the thermalization time scale \( \tau_H \).

We now describe how to optimize \( n_f \) to maximize \( \tau_X, \tau_H \). First, note that to maintain exponential dependence on the reference drive frequency \( \omega_0 \equiv 1/T_0 \), \( n_f \) must scale at most logarithmically in the small parameter \( T_0 \| h \| \). This implies \( n_f = \frac{x}{\pi} \log_2 \left( \frac{1}{\tau_0 \| h \|} \right) \) with \( 0 < x < 1 \). Plugging this into the result for \( \tau_X \), we find

\[
\tau_X \geq \frac{1}{N} \left( c_2 \| \log_2 (T_0 \| h \|) \right)^{(1-x)} \frac{\log_2 \left( \frac{1}{\tau_0 \| h \|} \right)}{N},
\]

\[
\tau_H \geq \frac{1}{N} e^{-\frac{\omega_0}{\beta}} \frac{1}{\| h \|^{1-x}},
\]

for some \( 0 < x < 1 \). (7)

where \( c_2 \) is an \( O(1) \) constant. Thus, we can vary \( \omega_0 \) to control \( \tau_H \) with (stretched-) exponential sensitivity, and \( \tau_X \) as a power-law that can be made arbitrarily large. Consequently, a very slow increase of \( \omega_0 \) with system size \( N \) is sufficient to cancel the prefactor of \( 1/N \) in both \( \tau_X \) and \( \tau_H \).

Finally, we note that \( H_{F}^{(n)} \) is quasi-local in the sense that the amplitude of terms decays exponentially with the spatial range\(^1,2\). In general, this operator may be hard to evaluate exactly, but it can be approximated by

\[
H_{F}^{(n)} \approx H_{F}^{(0)} = A_0
\]

where \( A_0 \) commutes with all \( X_i \) by construction. Since it captures the time-evolution of local operators and, importantly, also reflects the global symmetry properties of \( H_{F}^{(n)} \), it is a good approximation to the effective Floquet Hamiltonian for times \( t \leq \min (\tau_X, \tau_H) \).

The design of the protocol which is crucial to the bound obtained in the first part of Eq. (6), the proof of the bound, and the result of Eqs. (7) that the times \( \tau_X \) and \( \tau_H \) are almost exponentially sensitive to the base drive frequency are the central results of this paper.

**Fractal driving with a single unitary \( X \).—** We now derive the bound in the first part of Eqs. (6) for the case \( n_s = 1 \) of a single unitary \( X_1 \equiv X \). The derivation of the result will also help the reader intuit the logic behind fractal driving.

Using the terminology introduced above, the Hamiltonian is composed of two (kinds of) terms: \( H = A_0 + A_1 \), where \( A_{0(1)} \) is even (odd) under \( X \). In this case, fractal driving can be described by the simple recurrence relations

\[
U(2^n T_0) = X \cdot U(2^{n-1} T_0) \cdot X \cdot U(2^{n-1} T_0), \forall n \geq 2
\]

with \( U(T_0) = e^{-i H T_0} \). At the first stage, this implies

\[
U(2T_0) = e^{-i T_0(A_0 - A_1)} e^{-i T_0(A_0 + A_1)} = e^{-i T_0^1(A_0^{(1)} + A_1^{(1)})},
\]

where we define \( T_0^1 = 2T_0 \), and \( A_0^{(1)} \) and \( A_1^{(1)} \) are the new effective terms that are even and odd, respectively, under \( X \). The BCH formula then yields

\[
A_0^{(1)} = A_0 + O \left( T_0^2 \right),
\]

\[
A_1^{(1)} = -i \frac{T_0}{2} [A_0, A_1] + O \left( T_0^2 \right).
\]

After \( n_f \) fractal layers, this implies

\[
T^{(n_f)} = 2^n T_0, \quad A_0^{(n_f)} = A_0;
\]

\[
A_1^{(n_f)} = \left( -i 2^n \frac{T_0}{2} A_0^{(1)} \right)^{n_f} [A_0, \ldots, [A_0, A_1] \ldots] + O \left( T_0^{n_f+1} \right),
\]

(12)

Importantly, terms in \( H_{F}^{(n_f)} \) that anti-commute with \( X \) appear first at \( O \left( T_0^{n_f} \right) \). These terms are a subset of all terms that appear at \( O \left( T_0 \right) \) in the expansion of the Floquet Hamiltonian, Eq. (5). Their norm is therefore bounded by \( T_0^{n_f} \| \Omega_{n_f} \| \). Further, the norm of all terms that may anti-commute with \( X \) can be bounded by \( \sum_{n=0}^{n_f} T_0^{n_f} \| \Omega_n \| \). We note from Ref.\(^2\), that

\[
\| \Omega_n \| T_0^n \leq N \left( c T \| h \| \right)^n 1 \leq N(cT \| h \| n)\]

for some \( O(1) \) constant \( c \). Using the above, we can bound the ratio \( \| \Omega_{n_f+1} \| / \| \Omega_n \| < 1/2 \forall n \leq n_f \), if we set \( n_f = 1/(2cT \| h \|) \). This finally implies

\[
\| \left[ H_{F}^{(n_f)}, X \right] \| \leq 2 T_0^{n_f} \| \Omega_{n_f} \| .
\]

Eqs. (14) and (13) give the result in Eq. (6) for \( n_s = 1 \).
Note that terms which are odd under just one symmetry are canceled to \( \mathcal{O}(T_0) \), while terms odd under both \( X_1, X_2 \) are canceled to higher order. Similar conclusions apply for the general case of \( n_f \geq 2 \).

After \( n_f \) fractal layers, symmetry-violating terms appear at order \( \mathcal{O}(T_0^{n_f}) \) or higher. (The terms that are odd under just one symmetry appear at the lowest order.) To estimate the norm of these terms, we can apply the same arguments for the case \( n_f = 1 \), arriving at the results of Eqs. (6). This completes the proof.

**Numerical Results.**— We perform numerical simulations to illustrate the above findings. We consider a generic spin-1/2 chain of length \( L \), treated under open boundary conditions:

\[
H = \mathcal{O}(\sigma_i^z, \sigma_i^x/\sqrt{2}, \sigma_i^{x/y}, \sigma_i^{x/y} \sigma_{i+1}^z, \sigma_i^z \sigma_{i+1}^z, \sigma_i^z \sigma_{i+2}^z) \quad (16)
\]

\( H \) only has a parity symmetry \( P_2 = \prod_i \sigma_i^z \); we choose to work in the sector \( P_2 = 1 \). The system is driven with \( P_X = \prod_i \sigma_i^z \), or both \( P_X \) and \( P_{Z_2} = \prod_i \sigma_i^{z/2} \). For even (odd) \( L \), operators \( P_X, P_{Z_2} \) (anti-) commute.

Time-evolution is obtained by calculating unitaries \( U(mT) \) and using these to evaluate \( x(t) = 1 - 2^{-L} \text{Tr} [P_X(t)P_X] \) and \( z(t) = 1 - 2^{-L} \text{Tr} [P_{Z_2}(t)P_{Z_2}] \) which measure the preservation in time of \( P_X \), and \( P_{Z_2} \) (note that \( x(0) = z(0) = 1 \)), and \( \epsilon(t) = \langle \psi_0(t) | A_0 | \psi_0(t) \rangle - \langle \psi_0 | A_0 | \psi_0 \rangle \) that characterizes heating assuming we start from the ground state \( | \psi_0 \rangle \) of \( A_0 \), the lowest order approximation to \( H_F^{(no)} \). The results generically exhibit rapid initial heating on microscopic timescales, before transitioning to an extremely long-lived prethermal state. The values of \( x(t), z(t), \epsilon(t) \) remain approximately constant in this state, and signify the extent of “symmetry-breaking” and heating; the results correspond well with the norm of terms predicted in Eqs. (6) and (7). See Fig. 2 for more explanation.

**Summary and Outlook.**— We have introduced a novel strong-driving protocol for engineering Floquet Hamiltonians. From a theoretical point-of-view, this work describes a general protocol for controllably introducing structure into the eigenspectrum of Floquet unitaries by creating new local and global symmetries in the Floquet Hamiltonian. It may also be viewed as an extension of dynamical decoupling techniques to local many-body Hamiltonians.

The symmetries can be used to engender a variety of novel Hamiltonians and dynamical phenomena. We leave such exploration to future work but outline some directions. (i) If the set of unitaries \( X_i \) is exhaustive such that \( A_0 \) is identically zero, then the full Hamiltonian is suppressed super-exponentially in the number of fractal layers \( n_f \). This may represent a viable way of stabilizing quantum information in networks of spins or Majoranas. (ii) When \( X_i \) anti-commute; the Floquet Hamiltonian has non-trivial degeneracies at all energies. This degeneracy may be used to engineer topological qubits controlled by precisely \( X_i \). (iii) One can generalize these protocols and results to \( \mathbb{Z}_{n>2} \) by applying the individual symmetry generators in sets of \( n \) instead of twice, as in Eq. (9). We can even combine \( \mathbb{Z}_n \) symmetries with different \( n \), leading to even richer structure. (iv) In the case that \( X_i \)s are themselves products of local unitary transformations, the lowest-order effective Floquet Hamiltonian \( A_0 \) may have an extensive set of local symmetries \( X_i(r) \). \( \mathcal{O}(T_0) \) corrections would violate these local symmetries but it is plausible there exist \( X_i(r) \) related to \( X_i(r) \) by quasi-local basis rotations that remain conserved for long times. (v) Finally, more careful evaluation of symmetry-violating terms may lead to considerably tighter bounds than derived in this work.

Experiments probing non-equilibrium phenomena in driven systems in a variety of setups including Nitrogen-vacancy centers,\(^{35}\) ion traps,\(^{36}\) cold atoms\(^{37,38}\) among others would be the natural setup to explore these ideas.

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* agarwal@physics.mcgill.ca

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