Regular Submanifolds in the Conformal Space $\mathbb{Q}_p^n$ *

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Abstract. There is a Lorenzian group acting on the conformal space $\mathbb{Q}_p^n$. We study the regular submanifolds in the conformal space $\mathbb{Q}_p^n$ and construct general submanifold theory in the conformal space $\mathbb{Q}_p^n$. Finally we give the first variation formula of the Willmore volume functional of submanifolds in the conformal space $\mathbb{Q}_p^n$ and classify the conformal isotropic submanifolds in the conformal space $\mathbb{Q}_p^n$.

§ 1. Introduction.

A pseudo-riemannian manifold is a manifold with an indefinite metric of index $p(p \geq 1)$. Such structures arise naturally in relativity theory and, more recently, string theory. Unlike the considerably more familiar Riemannian manifolds (with the metric index 0), Lorentzian manifolds are poorly understood. In this paper we study the conformal submanifold geometry in pseudo-riemannian space forms.

Let $\mathbb{R}_s^N$ be the real vector space $\mathbb{R}^N$ with the Lorenzian inner product $\langle \cdot, \cdot \rangle$ given by
\[
\langle X, Y \rangle = \sum_{i=1}^{N-s} x_i y_i - \sum_{i=N-s+1}^{N} x_i y_i,
\]
where $X = (x_1, \cdots, x_N), Y = (y_1, \cdots, y_N) \in \mathbb{R}^N$. We denote by $C^{n+1}$ the cone in $\mathbb{R}_{p+1}^{n+2}$ and by $\mathbb{Q}_p^n$ the conformal space in $\mathbb{R}^{P^{n+1}}$:
\[
C^{n+1} := \{ X \in \mathbb{R}_{p+1}^{n+2} | \langle X, X \rangle = 0, X \neq 0 \},
\]
\[
\mathbb{Q}_p^n := \{ [X] \in \mathbb{R}^{P^{n+1}} | \langle X, X \rangle = 0 \} = C^{n+1}/(\mathbb{R}\{0\}).
\]

Topologically $\mathbb{Q}_p^n$ is $\mathbb{S}^{n-p} \times \mathbb{S}^p/\mathbb{Z}_2$, which is endowed by a standard Lorentzian metric $h = g_{\mathbb{S}^{n-p}} \oplus (-g_{\mathbb{S}^p})$ and the corresponding conformal structure $[h] :=$
We define the psudo-riemannian sphere space $S^n_p$ and psudo-riemannian hyperbolic space $H^n_p$ by

$$S^n_p = \{ u \in \mathbb{R}^{n+1}_p | (u, u) = 1 \}, \quad H^n_p = \{ u \in \mathbb{R}^{n+1}_p | (u, u) = -1 \}. $$

We call $\mathbb{R}^n_p, S^n_p,$ and $H^n_p$ psudo-riemannian space forms with index $p$. Denote

$$\pi = \{ [x] \in \mathbb{Q}^n | x_1 = x_{n+2} \}, \quad \pi_+ = \{ [x] \in \mathbb{Q}^n_1 | x_{n+2} = 0 \}, \quad \pi_- = \{ [x] \in \mathbb{Q}^n | x_1 = 0 \}. $$

Observe the conformal diffeomorphisms

$$\sigma : \mathbb{R}^n_p \to \mathbb{Q}^n_p \backslash \pi, \quad u \mapsto [(\frac{(u,u)-1}{2}, u, \frac{(u,u)+1}{2})],$$
$$\sigma_+ : S^n_p \to \mathbb{Q}^n_p \backslash \pi_+, \quad u \mapsto [(u, 1)],$$
$$\sigma_- : H^n_p \to \mathbb{Q}^n_p \backslash \pi_-, \quad u \mapsto [(1, u)].$$

We may regard $\mathbb{Q}^n_p$ as the common compactification of $\mathbb{R}^n_p, S^n_p,$ and $H^n_p$, while $\mathbb{R}^n_p, S^n_p,$ and $H^n_p$ as the subsets of $\mathbb{Q}^n_p$. Therefore we research the conformal geometry in the conformal space $\mathbb{Q}^n_p$ with index $p$, while it is not necessary to do so in these three psudo-riemannian space forms respectively.

When $p = 0$, our analysis in this text can be reduced to the Moebius submanifold geometry in the sphere space (see Wang[10]). For more details of Moebius submanifold geometry see refs [3, 4, 5, 11, 12], etc. Some other results about Lorentz conformal geometry see refs.[6-9], etc.

This paper is organized as follows. In Section 2 we prove the conformal group of the conformal space $\mathbb{Q}^n_p$ is $O(n-p+1, p+1)/\{ \pm I \}$. In Section 3 we construct general submanifold theory in the conformal space $\mathbb{Q}^n_p$ and give the relationship between conformal invariants and isometric ones for hypersurfaces in Lorentzian space forms. In Section 4 we give the first variation formula of the Willmore volume functional of regular space-like or time-like submanifolds in the conformal space $\mathbb{Q}^n_p$ In Section 5 we classify the conformal isotropic submanifolds in the conformal space $\mathbb{Q}^n_p$.

§ 2. The conformal group of the conformal space $\mathbb{Q}^n_p$.

First we introduce

**Lemma 2.1.** Let $\varphi : M \to M$ be a conformal transformation on $m(m > 2)$ dimensional psudo-riemannian submanifold $(M, g)$, i.e., $\varphi$ is a diffeomorphism and $\varphi^*g = e^{2\tau}g, \tau \in C^\infty(M)$. If $M$ is connected, then $\varphi$ is determined by the tangent map $\varphi_*$ and 1-form $d\tau$ at one fixed point.

**Proof** For any point $p \in M$, there is a local coordinate $(x^i)$ around $p$. And $(y^i)$ is a local coordinate around $\varphi(p)$.
For pseudo-riemannian metric \( \tilde{g} = e^{2\tau} g = \varphi^* g \) on \( M \), we denote \( \tilde{\nabla} \) the connection of \( \tilde{g} \), and \( \tilde{R} \) the curvature tensor, \( \tilde{Ric} \) the Ricci curvature tensor. Respect to \( g \), the corresponding operators are \( \nabla, R, \tilde{Ric}, \) respectively. The relation of these operators is as the following equations

\[
\tilde{\nabla}_X Y = \nabla_X Y + X(\tau)Y + Y(\tau)X - g(X,Y)\nabla \tau, \tag{2.1}
\]

\[
\tilde{R}(X,Y) Z = R(X,Y) Z + g(X,Z)\nabla_Y \nabla \tau - g(Y,Z)\nabla_X \nabla \tau + [g(X,\nabla \tau)g(Y,Z) - g(Y,\nabla \tau)g(X,Z)]\nabla \tau + [\nabla_Y Z(\tau) + Y(\tau)Z(\tau) - YZ(\tau) - g(Y,Z)g(\nabla \tau, \nabla \tau)]X - [\nabla_X Z(\tau) + X(\tau)Z(\tau) - XZ(\tau) - g(X,Z)g(\nabla \tau, \nabla \tau)]Y, \tag{2.2}
\]

\[
\tilde{R}(X,Y,W,Z) = e^{2\tau}\{R(X,Y,W,Z) + g(X,Z)g(W,\nabla_Y \nabla \tau) - g(Y,Z)g(W,\nabla_X \nabla \tau) + [g(X,\nabla \tau)g(Y,Z) - g(Y,\nabla \tau)g(X,Z)]g(W,\nabla \tau) + [\nabla_Y Z(\tau) + Y(\tau)Z(\tau) - YZ(\tau) - g(Y,Z)g(\nabla \tau, \nabla \tau)]g(W,X) - [\nabla_X Z(\tau) + X(\tau)Z(\tau) - XZ(\tau) - g(X,Z)g(\nabla \tau, \nabla \tau)]g(W,Y)\}, \tag{2.3}
\]

where \( X, Y, Z, W \) are smooth vector fields on \( M \), and \( \nabla \tau \) is the gradient of \( \tau \) respect to \( g \).

Locally, let

\[
\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x^k}, \quad \nabla_{\frac{\partial}{\partial y^j}} \frac{\partial}{\partial x^i} = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial y^k},
\]

\[
g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right), \quad (g^{ij}) = (g_{ij})^{-1}, \quad \varphi^* \frac{\partial}{\partial x^i} = \sum \Lambda_j^i \frac{\partial}{\partial y^j}, \quad d\tau = \sum dA d^i.
\]

First we have

\[
g(\varphi^* \frac{\partial}{\partial x^i}, \varphi^* \frac{\partial}{\partial x^i}) \circ \varphi = e^{2\tau} g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^i}\right) \tag{2.4}
\]

Acting with \( \frac{\partial}{\partial x^i} \) on the both sides of (2.4), we get

\[
2B_k g(\varphi^* \frac{\partial}{\partial x^i}, \varphi^* \frac{\partial}{\partial x^j}) = g(\nabla_{\varphi^* \frac{\partial}{\partial x^i}} \varphi^* \frac{\partial}{\partial x^i} - \varphi^* \nabla_{\frac{\partial}{\partial x^i}} \varphi^* \frac{\partial}{\partial x^i}, \varphi^* \frac{\partial}{\partial x^j}) + g(\nabla_{\varphi^* \frac{\partial}{\partial x^j}} \varphi^* \frac{\partial}{\partial x^j} - \varphi^* \nabla_{\frac{\partial}{\partial x^j}} \varphi^* \frac{\partial}{\partial x^j}, \varphi^* \frac{\partial}{\partial x^i}).
\]

Alternating the positions of \( i, j, k \), and by the use of

\[
\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i}, \quad \nabla_{\varphi^* \frac{\partial}{\partial x^i}} \varphi^* \frac{\partial}{\partial x^j} = \nabla_{\varphi^* \frac{\partial}{\partial x^j}} \varphi^* \frac{\partial}{\partial x^i},
\]

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one will obtain
\[ B_i g(\varphi, \frac{\partial}{\partial x^i}, \varphi, \frac{\partial}{\partial x^k}) + B_j g(\varphi, \frac{\partial}{\partial x^j}, \varphi, \frac{\partial}{\partial x^k}) - B_k g(\varphi, \frac{\partial}{\partial x^k}, \varphi, \frac{\partial}{\partial x^l}) \]
\[ = g(\nabla_{\varphi, \frac{\partial}{\partial x^j}} - \varphi \nabla_{\varphi, \frac{\partial}{\partial x^l}}), \]
and
\[ B_k g(\varphi, \frac{\partial}{\partial x^k}) = g(\nabla_{\tau}, \frac{\partial}{\partial x^k}) e^{2\tau} g_{ij} = g_{ij} g(\varphi, \nabla_{\tau}, \varphi, \frac{\partial}{\partial x^k}), \]
where
\[ \nabla_{\tau} = \sum_{ij} g^{ij} B_i \frac{\partial}{\partial x^j}. \]
Therefore
\[ \nabla_{\varphi, \frac{\partial}{\partial x^j}} - \varphi \nabla_{\varphi, \frac{\partial}{\partial x^l}} = B_i \varphi \frac{\partial}{\partial x^j} + B_j \varphi \frac{\partial}{\partial x^k} - g_{ij} \nabla_{\tau}. \]
We collect the terms of \( \frac{\partial}{\partial y^s} \) and get
\[ \frac{\partial A_i^k}{\partial x^s} = B_i A_i^k + B_j A_j^k + \Gamma_{ij}^l A_i^k - g_{ij} \sum_{st} g_{st} A_s^t A_t^k - \sum_{st} A_i^s A_j^t \Gamma_{st}^k. \quad (2.5) \]
Denote
\[ r_{ij} = Ric\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right), \tilde{r}_{ij} = \tilde{Ric}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right). \]
On one hand, form (2.3) we have
\[ \tilde{r}_{ij} = r_{ij} - g_{ij} \Delta \tau + (m - 2)[B_i B_j - \frac{\partial B_i}{\partial x^i} + \sum_t \Gamma_{ij}^t B_t - g_{ij} g(\nabla_{\tau}, \nabla_{\tau})], \quad (2.6) \]
where \( \Delta \) is the Laplacian respect to \( g \). On the other hand, we have
\[ \tilde{Ric}(X, Y) = Ric(\varphi_\ast X, \varphi_\ast Y) \circ \varphi, \quad (2.7) \]
Therefore
\[ \tilde{r}_{ij} = \tilde{Ric}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \sum_{st} A_i^s A_j^t r_{st}^t, r_{st}^t = Ric\left(\frac{\partial}{\partial y^s}, \frac{\partial}{\partial y^t}\right). \quad (2.8) \]
Combining with (2.6) and (2.8), we have
\[ \frac{\partial B_j^i}{\partial x^j} = B_i B_j + \sum_t \Gamma_{ij}^t B_t - g_{ij} \sum_{st} g_{st} B_s B_t \]
\[
+ \frac{1}{m-2}(r_{ij} - g_{ij}\Delta \tau - \sum_{st} A^s_i A^t_j\tau^s_{st}). \tag{2.9}
\]

To the first order ODE (2.5), (2.8), one may notice \( A^k_j = \frac{\partial \varphi}{\partial x^j}, B_j = \frac{\partial \varphi}{\partial \tau}. \) If \( M \) is connected, then \( \varphi \) is determined by the tangent map \( \varphi^* \) and 1-form \( d\tau \) at one fixed point. \( \Box \)

**Theorem 2.1.** Suppose that \( \varphi \) is a conformal transformation on \( \mathbb{Q}_p^n, \varphi^* h = e^{2\tau}h, \) and \( x_0 \) is a fixed point of \( \varphi, \) then there is \( A \in O(n-p+1, p+1), \) such that \( \varphi = \Phi_A \) and \( \Phi_A([X]) = [XA]. \)

**Proof** Let \((U, x^i)\) be a coordinate chart around \( x_0.\) At point \( x_0, \) denote

\[
\frac{\partial \varphi_i}{\partial x^j}|_{x_0} = A^i_j, \quad \frac{\partial \tau}{\partial x^j}|_{x_0} = B_j, \quad h_{ij} = h_{ij}(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})|_{x_0}, \quad (h^{ij}) = (h_{ij})^{-1}.
\]

Suppose that

\[
x_0 = [u_0], \quad u_0 = (u_p, u_2) \in \mathbb{S}^{n-1} \times \mathbb{S}^1 \subset \mathbb{R}^{n+2}_2, \quad Ju_0 = (u_p, -u_2).
\]

And if

\[
e_i \in E_{u_0}^\perp, \quad \pi_\tau e_i = \frac{\partial}{\partial x^j}|_{x_0},
\]

\(\{u_0, Ju_0, e_p, \cdots, e_n\}\) construct a basis of \( \mathbb{R}^{n+2}_2, \) then there is an orthonormal decomposition of \( \mathbb{R}^{n+2}_2:\)

\[
\mathbb{R}^{n+2}_2 = \text{span}\{u_0, Ju_0\} \oplus \text{span}\{e_p, \cdots, e_n\}.
\]

Define linear transformation \( A : \mathbb{R}^{n+2}_2 \to \mathbb{R}^{n+2}_2 \) on this basis:

\[
A(u_0) = e^{-\tau(x_0)}u_0, \quad A(e_i) = e^{-\tau(x_0)}(\sum_j A^j_i e_j - B_i u_0), \tag{2.10}
\]

\[
A(J u_0) = e^{\tau(x_0)Ju_0} + 2e^{-\tau(x_0)}(\sum_{ijk} h_{ik} B_j A^i_k e_i - \sum_{ij} h^{ij} B_i B_j u_0). \tag{2.11}
\]

First, it is easy to know that \( A \in O(n-p+1, p+1). \) In fact, it is guaranteed by \( \sum_{st} A^s_i A^t_j h_{st} = h_{ij} e^{2\tau(x_0)} \) (check it on the basis).

Otherwise, we have

\[
\Phi_A(x_0) = \varphi(x_0) = x_0, \tag{2.12}
\]

\[
\Phi_A^*|_{x_0} (\frac{\partial}{\partial x^i}) = \pi_\tau|_{x_0} \mathbb{Q}_p^n \circ A \circ (\pi_\tau|_{x_0} \mathbb{Q}_p^n)^{-1}(\frac{\partial}{\partial x^i}) = \sum_j A^j_i e_j = \varphi_\tau|_{x_0} (\frac{\partial}{\partial x^i}). \tag{2.13}
\]

Suppose that \([u] \in \mathbb{Q}_p^n, \) for any \( X, Y \in T_{[u]} \mathbb{Q}_p^n, \) there are \( \alpha, \beta \in E_u^\perp \subset T_u \mathbb{Q}_p^{n+1} \) such that

\[
\pi_\tau \alpha = X, \quad \pi_\tau \beta = Y.
\]

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Therefore, from (2.1) we have
\[
(\Phi^* h)_{|u}(X, Y) = (\Phi^* h)_{|u}(\pi_* \alpha, \pi_* \beta) = (\pi^* \Phi^* h)_{|u}(\alpha, \beta)
\]
\[
= (A \circ \pi^* h)_{|u}(\alpha, \beta) = (\pi^* h)_{A(u)}(\alpha A, \beta A) = \frac{2}{|A(u)|^2} \langle \alpha A, \beta A \rangle
\]
\[
= \frac{|u|^2}{|A(u)|^2} \cdot \frac{2}{|u|^2} \langle \alpha, \beta \rangle = \frac{|u|^2}{|A(u)|^2} h_{|u}(X, Y).
\]
Therefore \(\Phi^*_A h = \frac{|u|^2}{|A(u)|} h\). Next we prove that
\[
\frac{\partial}{\partial x^i}|_{x_0} (\frac{|u|^2}{|uA|^2}) = e^{2\tau(x_0)} B_i.
\]
Suppose that there is a local lift of \(Q^n_p\) around \(x_0 \in Q^n_p\) such that \(u : U \subset Q^n_p \rightarrow C^{n+1}\). Then \(\pi \circ u = \text{id}\), and
\[
\frac{\partial u}{\partial x^i}|_{x_0} = u_*(\frac{\partial}{\partial x^i}|_{x_0}) = u_* \circ \pi_*(e_i) = (\pi \circ u)_*(e_i) = e_i.
\]
Suppose that
\[
u = au_0 + bu_0 + \sum_i c^i e_i,
\]
where \(a, b, c^i\) are local smooth functions. Without difference, we let
\[
a(x_0) = 1, \quad b(x_0) = 0, \quad c^i(x_0) = 0.
\]
Using (2.10) and (2.11), we denote \(A(u)\) by
\[
A(u) = (a - 2h(\nabla \tau(x_0), \nabla \tau(x_0))b - \sum_i B_i c^i) u_0 + bc^{\tau(x_0)}Ju_0
\]
\[
+ e^{-\tau(x_0)} \sum_{ik} (2b \sum_j B_j h^{ik} + c^k) A_k^i e_i := a' u_0 + b' Ju_0 + \sum c' e_i.
\]
It is easy to check that
\[
\frac{\partial a}{\partial x^i}|_{x_0} = 0, \quad \frac{\partial b}{\partial x^i}|_{x_0} = 0, \quad \frac{\partial c^i}{\partial x^i}|_{x_0} = \delta^i_i.
\]
Consequently,
\[
\frac{\partial}{\partial x^i}|_{x_0} (|u|^2) = \frac{\partial}{\partial x^i}|_{x_0} (2a^2 + 2b^2 + \sum c^j c^k \langle e_j, e_k \rangle) = 0.
\]
Thereby

$$\frac{\partial}{\partial x^i} |x_o (|A(u)|^2) = 4 \frac{\partial a'}{\partial x^i} |x_o, a'(x_o) = -2e^{-2\tau(x_o)} B_i. \quad (2.21)$$

Thereby

$$\frac{\partial}{\partial x^i} |x_o \left( \frac{|u|^2}{|A(u)|^2} \right) = \frac{|u_o|^2 \frac{\partial}{\partial x^i} |x_o (|A(u)|^2)}{|A(u_o)|^4} = e^{2\tau(x_o)} B_i. \quad (2.22)$$

From Lemma 2.1, we have $\Phi_A = \varphi. \Box$

Suppose that for some fixed point $x_o = [(a, b)] \in \mathbb{Q}^n_p$, a conformal transformation $\varphi: \mathbb{Q}^n_p \rightarrow \mathbb{Q}^n_p$ have

$$\varphi([a, b]) = [(c, d)],$$

where

$$(a, b), (c, d) \in \mathbb{S}^{n-p} \times \mathbb{S}^p. \quad (2.23)$$

We can certainly find $C \in O(n - p + 1), D \in O(p + 1)$, such that $a = cC, b = dD$. That is, $A_p = \text{diag}(C, D) \in O(n - p + 1, p + 1)$ such that $\Phi_{A_p}[(c, d)] = [(a, b)]$. Clearly, the conformal transformation $\Phi_{A_p} \circ \varphi$ of $\mathbb{Q}^n_p$ has fixed point $x_o$. From the above theorem, there is $A \in O(n - p + 1, p + 1)$ such that $\Phi_{A_p} \circ \varphi = \Phi_A$. Thus $\varphi = \Phi_{A_p^{-1}}$. At last, since

$$\Phi: O(n - p + 1, p + 1) \rightarrow \text{the conformal group of } \mathbb{Q}^n_p, A \mapsto \Phi_A$$

is a epimorphism and $\ker(\Phi) = \{\pm I\}$, we have

**Theorem 2.2.** The conformal group of the conformal space $\mathbb{Q}^n_p$ is $O(n - p + 1, p + 1)/\{\pm I\}$.

### § 3. Fundamental equations of submanifolds.

Suppose that $x: \mathbb{M} \rightarrow \mathbb{Q}^n_p$ is an $m$-dimensional psudo-riemannian submanifold. That is, $x_*(TM)$ is non-degenerated subbundle of $(T\mathbb{Q}^n_p, h)$. Let $y: U \rightarrow C^{n+1}$ be a lift of $x: \mathbb{M} \rightarrow \mathbb{Q}^n_p$ defined in an open subset $U$ of $\mathbb{M}$. We denote by $\Delta$ and $\kappa$ the Laplacian operator and the normalized scalar curvature of the local non-degenerated metric $\langle dy, dy \rangle$. Then we have

**Theorem 3.1.** On $\mathbb{M}$ the 2-form $g := \pm (\langle \Delta y, \Delta y \rangle - m^2 \kappa) \langle dy, dy \rangle$ is a globally defined invariant of $x: \mathbb{M} \rightarrow \mathbb{Q}^n_p$ under the Lorentzian group transformations of $\mathbb{Q}^n_p$.

**Proof** First we can check it out that the expression of $g$ is invariant to different local lifts. Suppose that $y: U \rightarrow C^{n+1}, \tilde{y}: \tilde{U} \rightarrow C^{n+1}$ are different lifts of $x: \mathbb{M} \rightarrow \mathbb{Q}^n_p$ defined in open subsets $U$ and $\tilde{U}$ of $\mathbb{M}$. For the local non-degenerated metrics $\langle ., . \rangle_y = \langle dy, dy \rangle$, we denote by $\Delta$ the Laplacian, by $\nabla f$ the gradient of a function $f$, and by $\kappa$ the normalized scalar curvatures. And for $\langle d\tilde{y}, d\tilde{y} \rangle$, we denote by $\tilde{\Delta}$ the Laplacian, and by $\tilde{\kappa}$ the normalized scalar curvatures. On $U \cap \tilde{U}$,
we find that $\tilde{y} = e^\tau y$, where $\tau$ is local smooth function on $U \cap \tilde{U}$. Therefore $\langle d\tilde{y}, d\tilde{y} \rangle = e^{2\tau} \langle dy, dy \rangle$, and they are conformal on $U \cap \tilde{U}$. We have

$$\tilde{\omega}_i = \omega_i^j + \tau_i \omega^j - \tau^j \omega_i + \delta_i^j d\tau,$$

(3.1)

$$e^{2\tau} \tilde{\Delta} f = \Delta f + (m - 2) \langle \nabla \tau, \nabla f \rangle_y,$$

(3.2)

$$e^{2\tau} \tilde{\kappa} = \kappa - \frac{2}{m} \Delta \tau - \frac{m - 2}{m} \langle \nabla \tau, \nabla \tau \rangle_y.$$

(3.3)

It follows that

$$\langle (\Delta y, \Delta y) - m^2 \kappa \rangle \langle dy, dy \rangle = \langle (\tilde{\Delta} \tilde{y}, \tilde{\Delta} \tilde{y}) - m^2 \tilde{\kappa} \rangle \langle d\tilde{y}, d\tilde{y} \rangle.$$

(3.4)

If there is a Lorenzian rotation $T$ acting on $Q^m_p$ and $y : U \rightarrow C^{n+1}$ is a lift of $x : M \rightarrow Q^m_p$ defined in open subsets $U$, then the submanifold $\tilde{x} = x \circ T$ must have a local lift like $\tilde{y} = e^\tau y T$. Since $T$ preserves the Lorentzian inner product and the dilatation of the local lift $y$ will not impact the term $\langle (\Delta y, \Delta y) - m^2 \kappa \rangle \langle dy, dy \rangle$, the 2-form $g$ is conformally invariant. □

**Definition 3.1.** We call an $m$-dimensional submanifold $x : M \rightarrow Q^m_p$ a regular submanifold if the 2-form $g := \pm \langle (\Delta y, \Delta y) - m^2 \kappa \rangle \langle dy, dy \rangle$ is non-degenerated. And $g$ is called the conformal metric of the regular submanifold $x : M \rightarrow Q^m_p$.

In this paper we assume that $x : M \rightarrow Q^m_p$ is a regular submanifold. Since the metric $g$ is non-degenerated (we call it the conformal metric), there exists a unique lift $Y : M \rightarrow C^{n+1}$ such that $g = \langle dY, dY \rangle$ up to a signature. We call $Y$ the canonical lift of $x$. By taking $y := Y$ in (3.1) we get

$$\langle \Delta Y, \Delta Y \rangle = m^2 \kappa \pm 1.$$  

(3.5)

Theorem 3.1 implies that

**Theorem 3.2.** Two submanifolds $x, \tilde{x} : M \rightarrow Q^m_p$ are conformal equivalent if and only if there exists $T \in O(n - p + 1, p + 1)$ such that $\tilde{Y} = YT$, where $Y, \tilde{Y}$ are canonical lifts of $x, \tilde{x}$, respectively.

Let $\{e_1, \ldots, e_m\}$ be a local basis of $M$ with dual basis $\{\omega^1, \ldots, \omega^m\}$. Denote $Y_i = e_i(Y)$. We define

$$N := -\frac{1}{m} \Delta Y - \frac{1}{2 m^2} \langle \Delta Y, \Delta Y \rangle Y,$$

(3.6)

then we have

$$\langle N, Y \rangle = 1, \langle N, N \rangle = 0, \langle N, Y_k \rangle = 0, \quad 1 \leq k \leq m.$$  

(3.7)
And we may decompose $\mathbb{R}^{n+2}_{p+1}$ such that
\[ \mathbb{R}^{n+2}_{p+1} = \text{span}\{Y, N\} \oplus \text{span}\{Y_1, \cdots, Y_m\} \oplus \mathbb{V} \]
(3.8)
where $\mathbb{V} \perp \text{span}\{Y, N, Y_1, \cdots, Y_m\}$. We call $\mathbb{V}$ the conformal normal bundle for $x : M \to \mathbb{Q}_p^n$. Let $\{\xi_{m+1}, \cdots, \xi_n\}$ be a local basis for the bundle $\mathbb{V}$ over $M$. Then $\{Y, N, Y_1, \cdots, Y_m, \xi_{m+1}, \cdots, \xi_n\}$ forms a moving frame in $\mathbb{R}^{n+2}_{p+1}$ along $M$. We adopt the conventions on the ranges of indices in this paper:
\[ 1 \leq i, j, k, l, r, q \leq m; \quad m + 1 \leq \alpha, \beta, \gamma, \nu \leq n. \]
(3.9)
We may write the structure equations as follows
\[ dY = \sum_i \omega^i Y_i; \quad dN = \sum_i \psi^i Y_i + \sum_{\alpha} \phi^\alpha \xi_{\alpha}; \]
(3.10)
\[ dY_i = -\psi_i Y - \omega_i N + \sum_j \omega^j Y_j + \sum_{\alpha} \omega^\alpha \xi_{\alpha}; \]
(3.11)
\[ d\xi_{\alpha} = -\phi_{\alpha} Y + \sum_i \omega^i \xi_{\alpha} + \sum_{\beta} \omega_{\beta} \xi_{\alpha}, \]
(3.12)
where the coefficients of $\{Y, N, Y_i, \xi_{\alpha}\}$ are 1-forms on $M$. It is clear that $A := \sum_i \psi_i \otimes \omega^i, B := \sum_{i, \alpha} \omega_i^\alpha \otimes \omega^i e_{\alpha}, \Phi := \sum_{\alpha} \phi^\alpha \xi_{\alpha}$ are globally defined conformal invariants. If we denote
\[ \psi_i = \sum_j A_{ij} \omega^j, \quad \omega^i_{\alpha} = \sum_j B_{ij}^\alpha \omega^j, \quad \phi^\alpha = \sum_i C_i^\alpha \omega^i, \]
(3.13)
then we can define the covariant derivatives of these tensors and curvature tensor with respect to conformal metric $g$:
\[ \sum_j C_{i,j} \omega^j = dC_i^\alpha - \sum_j C_j^\alpha \omega^j + \sum_{\beta} C_i^\beta \omega_\beta; \]
(3.14)
\[ \sum_k A_{i,j,k} \omega^k = dA_{ij} - \sum_k A_{ik} \omega_j^k - \sum_k A_{kj} \omega_i^k; \]
(3.15)
\[ \sum_k B_{i,j,k}^\alpha \omega^k = dB_{ij}^\alpha - \sum_k B_{ik}^\alpha \omega_j^k - \sum_k B_{kj}^\alpha \omega_i^k + \sum_{\beta} B_{ij}^\beta \omega^\beta; \]
(3.16)
\[ d\omega_j^i + \sum_k \omega_k^i \wedge \omega_j^k = \omega^i \wedge \psi_j + \psi^i \wedge \omega_j - \sum_{\alpha} \omega^i_{\alpha} \wedge \omega_{\alpha}^j = \frac{1}{2} \sum_{kl} R_{ijkl} \omega^k \wedge \omega^l; \]
(3.17)
\[ d\omega_{\alpha}^i + \sum_k \omega_k^\alpha \wedge \omega_{\alpha}^k = - \sum_{\alpha} \omega^i_{\alpha} \wedge \omega_{\alpha}^i = \frac{1}{2} \sum_{kl} R_{\beta kl} \omega^k \wedge \omega^l. \]
(3.18)
Denote
\[ g_{ij} = \langle Y_i, Y_j \rangle, \quad g_{\beta\gamma} = \langle \xi_{\beta}, \xi_{\gamma} \rangle, \quad (g^{ij}) = (g_{ij})^{-1}, \quad (g^{\beta\gamma}) = (g_{\beta\gamma})^{-1}, \]

\[
R_{ijkl} = \sum_p g_{ip} R^p_{jkl}, \quad R_{\alpha\beta kl} = \sum_\nu g_{\alpha\nu} R^\nu_{\beta kl}.
\]

Then the integrable conditions of the structure equations contain
\[ A_{ij,k} - A_{ik,j} = -\sum_{\alpha\beta} g_{\alpha\beta} (B^\alpha_{ij} C^\beta_k - B^\alpha_{ik} C^\beta_j); \quad B^\alpha_{ij,k} - B^\alpha_{ik,j} = g_{ij} C^\alpha_k - g_{ik} C^\alpha_j; \quad (3.19) \]
\[ C^\alpha_{ij} - C^\alpha_{j,i} = \sum_{kl} g_{kl} (B^\alpha_{ik} A_{lj} - B^\alpha_{jk} A_{li}); \quad R_{\alpha\beta ij} = \sum_{kl\gamma\nu} g_{\alpha\gamma} g_{\beta\nu} g_{kl} (B^\gamma_{ik} B^\nu_{lj} - B^\gamma_{ik} B^\nu_{lk}); \]
\[ R_{ijkl} = \sum_{\alpha\beta} g_{\alpha\beta} (B^\alpha_{ik} B^\beta_{jl} - B^\alpha_{il} B^\beta_{jk}) + (g_{ik} A_{jl} - g_{il} A_{jk}) + (A_{ik} g_{jl} - A_{il} g_{jk}). \quad (3.21) \]

Furthermore, we have
\[ \text{tr}(A) = \frac{1}{2m} (m^2 \kappa \pm 1); \quad R_{ij} = \text{tr}(A) g_{ij} + (m - 2) A_{ij} - \sum_{kl\alpha\beta} g^{kl} g_{\alpha\beta} B^\alpha_{ik} B^\beta_{jl}; \quad (3.22) \]
\[ (1 - m) C^\alpha_i = \sum_{jk} g^{jk} B^\alpha_{ij,k}; \quad \sum_{ijkl\alpha\beta} g^{ij} g^{kl} g_{\alpha\beta} B^\alpha_{ik} B^\beta_{jl} = \frac{m - 1}{m}; \quad \sum_{ij} g^{ij} B^\alpha_{ij} = 0, \forall \alpha. \quad (3.23) \]

From above we know that in case \( m \geq 3 \) all coefficients in the PDE system (3.10)-(3.12) are determined by the conformal metric \( g \), the conformal second fundamental form \( \mathbb{B} \) and the normal connection \( \{\omega^\alpha_i\} \) in the conformal normal bundle \( \mathbb{V} \). Then we have

**Theorem 3.3.** Two hypersurfaces \( x : \mathcal{M}^m \to \mathbb{Q}^{m+1}_p \) and \( \tilde{x} : \tilde{\mathcal{M}}^m \to \mathbb{Q}^{m+1}_p (m \geq 3) \) are conformal equivalent if and only if there exists a diffeomorphism \( f : \mathcal{M} \to \tilde{\mathcal{M}} \) which preserves the conformal metric and the conformal second fundamental form. In another word, \( \{g, \mathbb{B}\} \) is a complete invariants system of the hypersurface \( x : \mathcal{M}^m \to \mathbb{Q}^{m+1}_p (m \geq 3) \).

Next we give the relations between the conformal invariants induced above and \( SO(n - p + 1, p+1) \)-invariants of \( u : \mathcal{M} \to \mathbb{R}^n_p \). We give also a conformal fundamental theorem for hypersurfaces in \( \mathbb{R}^n_p \). The pseudo Euclidean space \( \mathbb{R}^n_p \) has an inner product \( \langle ., . \rangle \), whose signature is \( (+, \cdots, +, -, \cdots, -) \). From the conformal map
\[
\sigma : \mathbb{R}^n_p \to \mathbb{Q}^{m+1}_p, \quad u \mapsto \left[ \left( \frac{\langle u, u \rangle - 1}{2}, u, \frac{\langle u, u \rangle + 1}{2} \right) \right],
\]
\[ (3.24) \]
we may recognize that \( \mathbb{R}^n_p \subset \mathbb{Q}^n_p \). Let \( u : M \rightarrow \mathbb{R}^n_p \) be a submanifold. Let \( \{e_1, \ldots, e_m\} \) be an local basis for \( u \) with dual basis \( \{\omega^1, \ldots, \omega^m\} \). Let \( \{e_{m+1}, \ldots, e_n\} \) be a local basis of the normal bundle of \( u \) in \( \mathbb{R}^n_p \). Then we have the first and second fundamental forms \( I, II \) and the mean curvature vector \( \mathbf{H} \). We may write

\[
I = \sum_{ij} I_{ij} \omega^i \otimes \omega^j, \quad II = \sum_{ija} h_{ija} \omega^i \otimes \omega^j e_\alpha
\]

\[
(I^{ij}) = (I_{ij})^{-1}, \quad \mathbf{H} = \frac{1}{m} \sum_{ija} I^{ij} h_{ija} e_\alpha := \sum_\alpha H^\alpha e_\alpha.
\]

Denote \( \Delta_M \) the Laplacian and \( \kappa_M \) the normalized scalar curvature for \( I \). It is easy to see that

\[
\Delta_M u = m\mathbf{H}, \quad \kappa_M = \frac{1}{m(m-1)}(m^2|\mathbf{H}|^2 - |II|^2),
\]

where

\[
|\mathbf{H}|^2 = \sum_{\alpha\beta} I_{\alpha\beta} H^\alpha H^\beta, I_{\alpha\beta} = (e_\alpha, e_\beta); \quad |II|^2 = \sum_{ijkl\alpha\beta} I_{\alpha\beta} I^{ij} h_{ija} h_{i\beta j}.
\]

In fact, from the structure equations

\[
\begin{align*}
\mathrm{d}u &= \sum_i \omega^i u_i, \quad \mathrm{d}u_i = \sum_j \theta^i_j u_j + \sum_\alpha \theta^i_\alpha e_\alpha, \\
\mathrm{d}e_\alpha &= \sum_j \theta^i_\alpha u_j + \sum_\beta \theta^\beta_\alpha e_\beta
\end{align*}
\]

we have

\[
\sum_j u_{i,j} \omega^j = \mathrm{d}u_i - \sum_j \theta^i_j u_j = \sum_\alpha \theta^i_\alpha e_\alpha, \quad u_{i,j} = \sum_\alpha h_{ija} e_\alpha.
\]

For \( x = \sigma \circ u : M \rightarrow \mathbb{R}^n_p \), there is a global lift

\[
y : M \rightarrow C^{n+1}, \quad y = \left(\frac{\langle u, u \rangle - 1}{2}, u, \frac{\langle u, u \rangle + 1}{2}\right).
\]

So we will get

\[
\langle \mathrm{d}y, \mathrm{d}y \rangle = \langle \mathrm{d}u, \mathrm{d}u \rangle = I; \quad \Delta = \Delta_M; \quad \kappa = \kappa_M.
\]

It follows from (3.25) that

\[
\langle \Delta Y, \Delta Y \rangle - m^2 \kappa = \frac{m}{m-1}(|II|^2 - m|\mathbf{H}|^2).
\]
Therefore the conformal metric of $x$

$$g = \pm \frac{m}{m-1}(|II|^2 - m|H|^2)(du, dw) := e^{2\tau}I.$$  \hfill (3.30)

Let

$$y_i = e_i(y) = (0, u_i, 0) + (u, u_i)(1, 0, 1), \zeta_\alpha = (0, e_\alpha, 0) + (u, e_\alpha)(1, 0, 1).$$

Through some direct calculation it reaches

$$Y = e^{\tau}y, \quad Y_i = e_i(Y) = e^{\tau}(\tau_i y + y_i), \quad \zeta_\alpha = H_\alpha y + \zeta_\alpha,$$  \hfill (3.31)

$$-e^{\tau}N = \frac{1}{2}(|\nabla \tau|^2 + |H|^2)y + \sum_i \tau_i y_i + \sum_\alpha H_\alpha \zeta_\alpha + (1, 0, 1),$$  \hfill (3.32)

where $\tau_i = \sum_j I^i_j \tau_j, (I^i_j) = (I_{ij})^{-1}$; $|\nabla \tau|^2 = \sum_i \tau_i \tau_i$; $H_\alpha = \sum_\beta I_{\alpha\beta} H^\beta$.

By a direct calculation we get the following expression of the conformal invariants $A, B,$ and $\Phi$:

$$A_{ij} = \tau_i \tau_j - \sum_\alpha h^\alpha_{ij} H_\alpha - \tau_i, j - \frac{1}{2}(|\nabla \tau|^2 + |H|^2)I_{ij},$$  \hfill (3.33)

$$B^\alpha_{ij} = e^{\tau}(h^\alpha_{ij} - H^\alpha I_{ij}), \quad e^{\tau}C^\alpha_i = H^\alpha \tau_i - \sum_j h^\alpha_{ij} \tau_j^i - H^\alpha_{ij},$$  \hfill (3.34)

where $\tau_i, j$ is the Hessian of $\tau$ respect to $I$ and $H^\alpha_{ij}$ is the covariant derivative of the mean curvature vector field of $u$ in the normal bundle $N(M)$ respect to $I$.

Now we consider the case that $u : M \to \mathbb{R}^n_p$ is a hypersurface. Observing the PDE system (3.10)-(3.12), from Theorem 3.3 we have

**Theorem 3.4.** Two hypersurfaces $u, \tilde{u} : M \to \mathbb{R}^n_p(n \geq 4)$ are conformally equivalent if and only if there exists a diffeomorphism $f : M \to M$ which preserves the conformal metric and the conformal second fundamental form $\{g, B\}$.

**Remark 3.1.** For psudo sphere space with index $p$

$$S^n_p = \{u = (u^1, \ldots, u^{n+1}) \in \mathbb{R}^{n+1} \mid \langle u, u \rangle := (u^1)^2 + \cdots + (u^{n+p+1})^2 - (u^{n-p+2})^2 - \cdots - (u^{n+1})^2 = 1\}$$

and psudo hyperbolic space with index $p$

$$H^n_p = \{u = (u^1, \ldots, u^{n+1}) \in \mathbb{R}^{n+1} \mid \langle u, u \rangle := (u^1)^2 + \cdots + (u^{n-p})^2 - (u^{n-p+1})^2 - \cdots - (u^{n+1})^2 = -1\}$$
we obtain analogous conclusion:

\[
A_{ij} = \tau_i \tau_j - \sum_\alpha h^\alpha_{ij} H_\alpha - \tau_{i,j} - \frac{1}{2}(|\nabla \tau|^2 + |H|^2 - \epsilon) I_{ij},
\]

\[
B^\alpha_{ij} = e^\tau (h^\alpha_{ij} - H^\alpha I_{ij}), \quad e^\tau C^\alpha_i = H^\alpha \tau_i - \sum_j h^\alpha_{ij} \tau_j - H^\alpha_{i,j},
\]

where \(\epsilon\) corresponds the sectional curvature of pseudo sphere space or pseudo hyperbolic space with index \(p\).

§ 4. The first variation of the conformal volume functional

Let \(x_0 : M \to \mathbb{Q}_p^n\) be a compact oriented regular submanifold with boundary \(\partial M\). Suppose that local basis \(\{e_1, \cdots, e_m\}\) on \(M\) satisfy the orientation. Denote \(g_{ij} = g(e_i, e_j)\). If the conformal metric \(g\) has \(s\) negative signature and \((g_{ij}) = (-I_s) \oplus (I_{m-s})\), we call \(\{e_1, \cdots, e_m\}\) a local orthonormal basis for \(g\). In the following let \(\{e_1, \cdots, e_m\}\) be a local orthonormal basis for \(g\) with dual basis \(\{\omega^1, \cdots, \omega^m\}\).

We define the generalized Willmore functional \(\mathbb{W}(M)\) as the volume functional of the conformal metric \(g\):

\[
\mathbb{W}(M) = \text{Vol}_g(M) = \int_M dM_g.
\]

The conformal volume element \(dM_g\) is defined by

\[
dM_g = \omega^1 \wedge \cdots \wedge \omega^m,
\]

which is well-defined.

Let \(x : M \times \mathbb{R} \to \mathbb{Q}_p^n\) be an admissible variation of \(x_0\) such that \(x(\cdot, t) = x_t\) and \(dx_t(T_p M) = dx_0(T_p M)\) on \(\partial M\) for each small \(t\). For each \(t\), \(x_t\) has the conformal metric \(g_t\). As in §3, we have a moving frame \(\{Y, N, Y_i, \xi_\alpha\}\) in \(\mathbb{R}^{n+2}_{p+1}\) along \(M \times \mathbb{R}\) and the conformal volume \(W(t) = \mathbb{W}(x_t)\). Let \(\{\xi_\alpha\}\) be a local orthonormal basis for the conformal normal bundle \(\mathbb{V}_t\) of \(x_t\). Denote \(\tilde{d}\) and \(d\) the differential operators on \(M \times \mathbb{R}\) and \(M\), respectively. Then we have

\[
\tilde{d} = d + dt \wedge \frac{\partial}{\partial t}
\]

on \(T^*(M \times \mathbb{R}) = T^*M \oplus T^*\mathbb{R}\). We also have

\[
d \circ \frac{\partial}{\partial t} = \frac{\partial}{\partial t} \circ d.
\]
Denote \( P = (Y, N, Y_i, \xi_\alpha)^T \). Suppose that \( dP = \Omega P, \frac{\partial}{\partial t} P = LP \), where

\[
\Omega = \begin{pmatrix}
0 & 0 & \omega^j & 0 \\
0 & 0 & \phi^\beta & 0 \\
-\psi_i & -\omega_i & \omega_i^\beta & \phi_i \\
-\phi_\alpha & 0 & \omega^\beta_\alpha & \omega^\beta_\alpha
\end{pmatrix},
L = \begin{pmatrix}
w & 0 & v^j & v^\beta \\
0 & -w & w^j & u^\beta \\
-u_i & -v_i & L^j_i & L^i_\beta \\
-u_\alpha & -v_\alpha & L^i_\alpha & L^j_\alpha
\end{pmatrix}.
\]

From (4.2) it is easy to get

\[
\frac{\partial}{\partial t} \Omega = dL + L\Omega - \Omega L. \tag{4.3}
\]

Therefore we have

\[
\frac{\partial \omega^i}{\partial t} = \sum_j (v^i_j + L^i_j - \sum_{k\alpha\beta} g_{\alpha\beta} v^\alpha B^\beta_{kj} g^{jk}) \omega^j + \sum_{\alpha} v^\alpha \omega^i_\alpha + w \omega^i, \quad L^\alpha_i = v^\alpha_i + \sum_j B^\alpha_{ij} v^j, \tag{4.4}
\]

where \( \{v^i_j\} \) is the covariant derivative of \( \sum v^i e_i \) with respect to \( g \), and \( \{v^\alpha_i\} \) is the covariant derivative of \( \sum v^\alpha \xi_\alpha \). Here we have used the notations of conformal invariants \( \{A_{ij}, B^\alpha_{ij}, C^\alpha_i\} \) for \( x_t \) defined in §3. Furthermore we have

\[
\frac{\partial \omega^\alpha}{\partial t} = \sum_j (L^\alpha_{ij} + \sum_k L^k_i B^\alpha_{kj} - \sum_{\beta} B^\beta_{ij} L^\alpha_{\beta} + A_{ij} v^\alpha - v_i C^\alpha_j) \omega^j + u^\alpha \omega_i, \tag{4.5}
\]

where \( \{L^\alpha_{ij}\} \) is the covariant derivative of \( \sum \omega^i \xi_\alpha \). Using (4.4) and (4.5) we get

\[
\frac{\partial B^\alpha_{ij}}{\partial t} + w B^\alpha_{ij} = v^\alpha_{ij} + A_{ij} v^\alpha + \sum_{kl\gamma} g_{kl} B^\alpha_{ik} B^\gamma_{lj} v_{\gamma} \\
+ u^\alpha g_{ij} + \sum_k L^k_i B^\alpha_{kj} - \sum_{\gamma} B^\gamma_{ij} L^\alpha_{\gamma} + \sum_k v^k B^\alpha_{ik,j} - v_i C^\alpha_j. \tag{4.6}
\]

It follows from (3.19) and (3.23) that

\[
\frac{m-1}{m} w = \sum_{ijkl\alpha\beta} g_{i\beta} g^{jk} B^\beta_{kl} (v^\alpha_{ij} + A_{ij} v^\alpha + \sum_{kl\gamma} g_{kl} B^\alpha_{ik} B^\gamma_{lj} v_{\gamma}). \tag{4.7}
\]

Now we calculate the first variation of the conformal volume functional

\[
W(t) = \text{vol}(g_t) = \int_M \omega^1 \wedge \cdots \wedge \omega^m = \int_M dM_g,
\]

where \( dM_g \) is the volume for \( g_t \). From (4.4) we get

\[
W'(t) = \sum_i \int_M \omega^1 \wedge \cdots \wedge \frac{\partial \omega^i}{\partial t} \wedge \cdots \wedge \omega^m = \int_M \sum_i v^i_j dM_g + m \int_M w dM_g. \tag{4.8}
\]
From the fact that the variation is admissible we know \( v^i = 0, v^\alpha = 0 \) and \( v^\alpha_{,i} = 0 \) on \( \partial M \). It follows from (4.8) and Green’s formula that

\[
W'(t) = \frac{m^2}{m - 1} \int_M \sum_{\alpha} v^\alpha \left[ \sum_{ijkl\beta} g_{\alpha\beta} g^{ik} g^{jl} \cdot (B^{\beta}_{ij,kl} + A_{ij} B^{\beta}_{kl} + \sum_{r\gamma\nu} g^{r\gamma} B^{\beta}_{ir} B^{\nu}_{\gamma j} B^{\rho}_{kl}) \right] dM_g. \quad (4.9)
\]

It follows from (4.9) that

**Theorem 4.1.** The variation of the conformal volume functional depends only on the normal component of the variation field \( \frac{\partial Y}{\partial t} \). A submanifold \( x : M \rightarrow Q^n_p \) is a Willmore submanifold (i.e., a critical submanifold to the conformal volume functional) if and only if

\[
\sum_{ijkl\beta} g_{\alpha\beta} g^{ik} g^{jl} (B^{\beta}_{ij,kl} + A_{ij} B^{\beta}_{kl} + \sum_{r\gamma\nu} g^{r\gamma} B^{\beta}_{ir} B^{\nu}_{\gamma j} B^{\rho}_{kl}) = 0, \quad \forall \alpha. \quad (4.10)
\]

We call the equation (4.10) the Euler-Lagrange equations or Willmore equations. Using (3.22) and (3.23) we can write the Willmore equations (4.10) as

\[
\sum_{\beta} g_{\alpha\beta} \left[ \sum_{ij} g^{ij} C_{i,j}^\beta + \sum_{ijkl} g^{ik} g^{jl} (\frac{1}{m - 1} R_{ij} - A_{ij}) B_{kl}^\beta \right] = 0, \quad \forall \alpha. \quad (4.11)
\]

**Theorem 4.2.** Any stationary (means that whose curvature vector is vanishing) regular surface in pseudo Euclidean space \( \mathbb{R}^n_p \), pseudo sphere space \( S^n_p \) and pseudo hyperbolic space \( H^n_p \) is Willmore.

**Proof** Let \( u : M \rightarrow \mathbb{R}^n_p \) be a regular surface, whether space-like or time-like. Let \( \{e_1, e_2\} \) be a local basis of \( \langle du, du \rangle \) and \( \{e_\alpha\}^n_{\alpha=3} \) a local basis for the normal bundle. If \( x \) is a stationary regular surface, we have \( H^\alpha = 0, \forall \alpha \). From (3.33) and (3.34) we get

\[
\sum_{ijkl} g^{ik} g^{jl} A_{ij} B_{kl}^\beta = \sum_{ijkl} g^{ik} g^{jl} B_{kl}^\beta (\tau_i \tau_j - \tau_{ij}) = e^{-3\tau} \sum_{ijkl} I^{ik} I^{jl} h_{kl}^\beta (\tau_i \tau_j - \tau_{ij}). \quad (4.12)
\]

Now we know from (3.34) that

\[
-e^\tau C_i^\beta = \sum_{kl} I^{kl} h_{ik}^\beta :\tau_i := W_i^\beta. \quad (4.13)
\]

From (3.14) we have

\[
\sum_j e^\tau C_{ij}^\beta \omega^j = d(e^\tau C_i^\beta) - e^\tau C_i^\beta d\tau + \sum_\gamma e^\tau C_i^\gamma \theta^\gamma - \sum_k e^\tau C_k^\beta \omega_i^k
\]
\( = -dW^\beta_i + W^\beta_i d\tau - \sum_{\gamma} W^\gamma_i \theta^\beta + \sum_k W^\beta_k \omega^k_i. \quad (4.14) \)

Combining with

\[ \omega^k_i = \theta^k_i + \tau^k \sum_j I_{ij} \omega^j - \tau^k \omega^k_i + \delta^k_i d\tau \]

and (4.14) we get

\[ e^\tau C^\beta_{i,j} = 2W^\beta_i \tau_j + W^\beta_j \tau_i - \sum_k W^\beta_k \tau^k I_{ij} - W^\beta_{i,j}, \quad (4.15) \]

where \( W^\beta_{i,j} \) is the covariant differential of \( W^\beta_i \) with respect to the first fundamental form \( I \) of \( u \). Therefore

\[ \sum_{ijkl} g^{ij} C^\beta_{i,j} = e^{-3\tau} \sum_{ijkl} I^{ik} I^{jl} h^\beta_{kl} (\tau_i \tau_j - \tau_{ij}). \quad (4.16) \]

Whether the regular surface \( u \) is space-like or time-like, if we choose \( \{e_1, e_2\} \) orthonormal, then a direct calculation leads to

\[ \sum_{ijkl} g^{ik} g^{jl} R_{ij} B^\beta_{kl} = 0. \quad (4.17) \]

Thus we have (4.11) from (4.12), (4.16) and (4.17), which implies that \( u \) is Willmore.

One can verify that stationary regular surfaces in \( \mathbb{S}^n_p \) and \( \mathbb{H}^n_p \) are also Willmore.

\( \square \)

**Remark 4.1.** In some conferences, a surface in pseudo Riemannian space forms with vanishing mean curvature vector is also called maximal or minimal. But in this time the volume functional of the surface is not really maximal or minimal. So we take the place of the above two terms by stationary (also see [1]).

§ 5. Conformal isotropic submanifolds in \( \mathbb{Q}^n_p \)

**Definition 5.1.** We call an m-dimensional submanifold \( x : M \to \mathbb{Q}^n_p \) is conformal isotropic if there exists a smooth function \( \lambda \) on \( M \) such that

\[ A + \lambda g \equiv 0 \quad \text{and} \quad \Phi \equiv 0. \quad (5.1) \]

From precious discuss in §3 we can easily verify

**Proposition 5.1.** If \( u : M \to \mathbb{R}^n_p \) is a stationary regular submanifold with constant scalar curvature, then \( x = \sigma \circ u \) is a conformal isotropic submanifold in \( \mathbb{Q}^n_p \).
Remark 5.1. The same conclusion holds on $\mathbb{S}_p^n$ or $\mathbb{H}_p^n$.

Suppose that $x : M \to \mathbb{Q}_p^n$ is a conformal isotropic submanifold. Then we get

$$dN + \lambda dY = 0, \quad d\lambda \wedge dY = \sum_{i=1}^m (d\lambda \wedge \omega^i)Y_i = 0. \quad (5.2)$$

Since $\{Y_1, \cdots, Y_m\}$ are linearly independent,

$$d\lambda \wedge \omega^i = \sum_{j=1}^m E_j(\lambda)\omega^j \wedge \omega^i = 0. \quad (5.3)$$

If $M$ is connected, we get

$$\lambda = \text{constant}, \quad (5.4)$$

which concludes from (2.3) that

$$\kappa = \text{constant}. \quad (5.5)$$

By (5.2) we can find a constant vector $c \in \mathbb{R}^{n+2}_{p+1}$ such that

$$N + \lambda Y = c. \quad (5.6)$$

It follows that

$$\langle Y, c \rangle = 1, \quad \langle c, c \rangle = 2\lambda = \text{constant}. \quad (5.7)$$

Then we look into three cases.

Case 1: $\langle c, c \rangle = 0$. By making use of a Lorenzian rotation in $\mathbb{R}^{n+2}_{p+1}$ when necessary, we may assume that

$$c = (-1, 0, -1). \quad (5.8)$$

Letting

$$Y = (x_p, u, x_{n+2}), \quad (5.9)$$

by (5.7) and $Y \in C^{n+1}$ we have

$$Y = \left(\frac{\langle u, u \rangle - 1}{2}, u, \frac{\langle u, u \rangle + 1}{2}\right). \quad (5.10)$$

Then $x$ determines a submanifold $u : M \to \mathbb{R}^n_p$ with $I = \langle du, du \rangle = \langle dy, dy \rangle = g$, which implies that

$$\kappa_{M} = \kappa = \text{constant}. \quad (5.11)$$
From (5.7) and (2.13) we have $H^a = 0$, i.e., $u$ is a stationary submanifold in $\mathbb{R}_p^n$. In this case $x$ is conformal equivalent to the image of $\sigma$ of a stationary submanifold with constant scalar curvature in $\mathbb{R}_p^n$.

Case 2: $\langle c, c \rangle = -r^2, r > 0$. By making use of a Lorenzian rotation in $\mathbb{R}_{p+1}$ when necessary, we may assume that

$$c = (0, r). \quad (5.12)$$

Letting

$$Y = (u/r, x_{n+2}), \quad (5.13)$$

by (5.7) we have

$$x_{n+2} = 1/r. \quad (5.14)$$

So

$$Y = (u, 1)/r, \langle u, u \rangle = 1. \quad (5.15)$$

Then $x$ determines a submanifold $u : M \to \mathbb{S}_p^n$ with $I/r^2 = \langle du, du \rangle/r^2 = \langle dy, dy \rangle = g$, which implies that $\kappa_M = \kappa/r^2 = \text{constant}$. From (5.7) and (2.21) we have $H^a = 0$, i.e., $u$ is a stationary submanifold in $\mathbb{S}_p^n$. In this case $x$ is conformal equivalent to the image of $\sigma_+$ of a stationary submanifold with constant scalar curvature in $\mathbb{S}_p^n$.

Case 3: $\langle c, c \rangle = r^2, r > 0$. By making use of a Lorenzian rotation in $\mathbb{R}_{p+1}$ when necessary, we may assume that

$$c = (-r, 0). \quad (5.16)$$

Letting

$$Y = (x_p, u/r), \quad (5.17)$$

by (5.7) we have

$$x_p = 1/r. \quad (5.18)$$

So

$$Y = (1, u)/r, \langle u, u \rangle = -1. \quad (5.19)$$

Then $x$ determines a submanifold $u : M \to \mathbb{H}_p^n$ with $I/r^2 = \langle du, du \rangle/r^2 = \langle dy, dy \rangle = g$, which implies that $\kappa_M = \kappa/r^2 = \text{constant}$. From (5.7) and (2.26) we have $H^a = 0$, i.e., $u$ is a stationary submanifold in $\mathbb{H}_p^n$. In this case $x$ is conformal equivalent to the image of $\sigma_-$ of a stationary submanifold with constant scalar curvature in $\mathbb{H}_p^n$.

So combining with Proposition 5.1 and Remark 5.1 we get

**Theorem 5.2.** Any conformal isotropic submanifold in $\mathbb{Q}_p^n$ is conformal equivalent to a stationary submanifold with constant scalar curvature in $\mathbb{R}_p^n, \mathbb{S}_p^n$, or $\mathbb{H}_p^n$.

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