The experimental and theoretical evidence relating spin dependence and long range confinement is reviewed. One of the simplest confinement ideas, the dynamic electric flux tube picture of Buchmüller, can be exploited to yield a complete effective Hamiltonian. In addition to pure Thomas spin-orbit splitting the other relativistic corrections are calculated. It is shown that our effective Hamiltonian is identical to the mechanical relativistic flux tube model and gives relativistic corrections in agreement with QCD predictions.

1 Introduction

The linear growth in energy with increasing quark and anti-quark separation has long been accepted as a natural prediction of QCD. On the other hand, the quark's orientational energy has been a matter of continuing discussion. It is clear that the long range spin interaction cannot be similar to the short range gluon exchange since this would imply a large hyperfine splitting in the $p$-wave charmonium states which is at least an order of magnitude larger than observed. About twenty years ago Schnitzer considered the ratio of $p$-wave $\chi$-state masses,

$$R = \frac{\chi_2 - \chi_1}{\chi_1 - \chi_0} \simeq 0.5 \text{ (exp.)}$$

If the interaction potential were of the Lorentz vector type $R > 0.8$ but it was observed that a long range Lorentz scalar interaction would decrease the ratio $R$ and successful phenomenological models could be constructed. The improvement of course is due to the negative spin-orbit energy from scalar confinement whereas the vector case is positive. This observation ignited a long tradition of assuming scalar confinement. The $R$ argument assumes the spin dependence is given by the Breit-Fermi Hamiltonian in the ladder approximation. The argument also assumes that virtual coupling to open flavor channels can be ignored. It should also be pointed out that the justification of the hypothesis of scalar confinement from a fundamental QCD point of view has never been very satisfactory.
The discussion of long range spin dependence was placed on a firm foundation by the low velocity expansion of the Wilson loop minimal area law by Eichten and Feinberg. For one static and one slowly moving quark the spin-orbit energy is given in this formalism by

\[ H_{\text{s.o.}} \simeq \frac{1}{2m^2r} \left( \frac{dV}{dr} + \frac{2dV_1}{dr} \right) \mathbf{S} \cdot \mathbf{L}, \]  

(2)

where \( V \) is the static potential and \( V_1 \) is the long range potential. It was then observed by Gromes that there is a further constraint on these potentials,

\[ \frac{dV}{dr} = \frac{dV_2}{dr} - \frac{dV_1}{dr}, \]  

(3)

where \( V_2 \) is the short range interaction. At large quark separations \( V_2 \ll V_1 \) and with linear confinement \( V \to ar \) the spin-orbit energy (2) becomes

\[ H_{\text{s.o.}} \simeq -\frac{a}{2m^2r} \mathbf{S} \cdot \mathbf{L}, \]  

(4)

which is exactly the Thomas spin-orbit energy term. The “Gromes relation” thus provides further evidence for a pure Thomas spin-orbit interaction at large separations. Some doubt has been cast on the original derivation in recent work by Williams. We also await a careful study of the \( B_c \) p-wave potential by lattice simulation.

A simple and attractive picture relating confinement and spin-dependence was proposed by Buchmüller. In this picture the quarks carry the color field around as they move and for large separations the the field collapses to a purely electric flux tube. There are two immediate conclusions one can draw in this limit. First, since the color magnetic moments of the quarks are decoupled there can be no long-range hyperfine or tensor interactions. Second, since there is no magnetic field the only spin-orbit interaction is kinematic, known as the “Thomas” spin-orbit energy. In this talk I will extend the Buchmüller picture considerably to show that this simple concept leads to a complete theory and that it is consistent with other results based in more formal applications of QCD.

2 A Review of Buchmüller’s Argument

Although QCD is the proper theory of quark interaction, it is not unreasonable to discuss some questions within the context of electrodynamics. The flux-tube field configuration certainly depends crucially on the nonabelian nature of QCD while the interactions of the quarks with the fields does not in our case. This
is because we assume a flux-tube field in which only one component of the field strength $F_{\mu\nu}$ (i.e., the radial color electric field) is non-zero in a co-rotating frame. We may thus gauge transform $F_{\mu\nu}$ to an abelian subgroup of SU(3).

We consider a charged particle of spin $S$ and gyromagnetic ratio $g$. Its classical motion in electric and magnetic fields $E$ and $B$ is given by the Thomas equation. If the particle is viewed from a frame in which its motion is only radial (the co-rotating frame) the fields in this frame ($E'$ and $B'$) are given by

$$
\begin{align*}
E &= \gamma_\perp (E' - \mathbf{v}_\perp \times \mathbf{B}') , \\
B &= \gamma_\perp (B'_\perp + \mathbf{v}_\perp \times \mathbf{E}') .
\end{align*}
$$

The spin-orbit energy in terms of co-rotating fields is

$$
H_{s.o.} = -\frac{1}{m} \mathbf{S} \cdot \left( \frac{g}{2\gamma_\perp} \mathbf{B}' + \frac{1}{\gamma_\perp + 1} \mathbf{v}_\perp \times \mathbf{E}' \right) ,
$$

where $\gamma_\perp^{-2} = 1 - \mathbf{v}_\perp^2$ and the charge has been absorbed into the fields. For an electric flux tube $B'_\perp \equiv 0$ at every point along the flux tube (and also in the quark rest frame). The co-rotating electric field is given by

$$
E' = -a\hat{r}
$$

in the flux tube limit. For a slowly moving quark ($v \ll 1$) the spin-orbit energy becomes

$$
H_{s.o.} \simeq -\frac{a}{2m^2r} \mathbf{S} \cdot \mathbf{L} ,
$$

where $\mathbf{L} \simeq m\mathbf{r} \times \mathbf{v}$. We note this is the same as the Gromes result \[\text{[4]}\]. In what follows, the validity of the Buchmüller picture is assumed as is the consequent Thomas-type spin-orbit interaction.

### 3 Equivalent Laboratory Fields

For simplicity we will assume here that the anti-quark is infinitely heavy and fixed at the origin. A straightforward extension to a general meson would consider two collinear flux-tube segments in the CM system with CM at the coordinate origin. As in Fig. 1 an arbitrary point on the tube is $\mathbf{r}$, the quark’s position is $\mathbf{r}_Q$, and the momentum conjugate to $\mathbf{r}_Q$ is $\mathbf{p}_Q = -i\nabla_Q$.

As we have discussed, the fields in the co-rotating frame in Buchmüller’s picture are

$$
\begin{align*}
E' &= -a\hat{r} , \\
B' &= 0 .
\end{align*}
$$
Figure 1: Flux tube with static $\bar{Q}$ at the coordinate origin. The moving quark is located at $r_Q$. In a non-rotating frame the fields are $E$ and $B$ at the point $r$ on the flux tube.

We now use the Lorentz transformations (5) to find the lab fields $E$ and $B$ which (at the same location on the flux tube) are equivalent to the co-rotating fields (9). They are

\[
E = \gamma_\perp E' = -a \gamma_\perp \hat{r},
\]
\[
B = a \gamma_\perp v_\perp (\hat{r} \times \hat{v}_\perp).
\]

For a straight flux tube the magnitude of the perpendicular velocity is

\[
v_\perp = \frac{r}{r_Q} v_{Q,\perp},
\]

where $v_{Q,\perp}$ is the component of quark velocity perpendicular to the flux tube.

We next proceed to define a four vector potential at points along the tube in the lab system. The field vector potential is defined by

\[
B(r, r_Q, p_Q) = \nabla_r \times A(r, r_Q, p_Q)
\]
\[
= a \frac{r}{r_Q} v_{Q,\perp} \gamma_\perp \left( \frac{r}{r_Q} v_{Q,\perp} \right) (\hat{r} \times \hat{v}_\perp),
\]

where we explicitly denote the spatial dependence of $\gamma_\perp(v_\perp)$ using (11). We choose the gauge $A = A_\perp(r) \hat{v}_\perp$, which implies $\nabla \cdot A = 0$ and $A$ parallel to
the direction of motion of the tube. The solution in spherical coordinates is then

\[ A(r, r_Q, p_Q) = \hat{v}_\perp \left( \frac{\partial v_{Q\perp}}{\partial r_Q} \right) \int_0^r dr' r'^2 \gamma_\perp \left( r'_q v_{Q\perp} \right) . \] (13)

It is easy to verify that \( B = (\hat{r} \times \hat{v}_\perp)\left( \frac{1}{r^2} \frac{\partial}{\partial r} \right) r A_\perp \) is the same as in (12).

Similarly, the time component \( A_0(r, r_Q, p_Q) \) is defined by

\[ E = -\nabla_r A_0 = -a\gamma_\perp \left( \frac{r}{r_Q} v_{Q\perp} \right) \hat{r}, \] (14)

giving

\[ A_0(r, r_Q, p_Q) = a \int_0^r dr' r'^2 \gamma_\perp \left( \frac{r'}{r_Q} v_{Q\perp} \right) . \] (15)

Both (13) and (15) can be explicitly integrated.

4 Interpretation of the Laboratory Four Potential

We first observe from (13) and (15) that for a pure flux tube the “invariant” potential is

\[ U = A_0 - v \cdot A, \] (16)

\[ U = a \int_0^r dr' r'^2 \gamma_\perp \left( 1 - v_{Q\perp}^2 \right) = a \int_0^r dr' \sqrt{1 - v_{Q\perp}^2}. \] (17)

The resulting Lagrangian \( L_{\text{string}} = -U \) is exactly that of the string action and the dynamics of a straight string with a massive spinless quark at one end (the other end being fixed) follows from the Lagrangian

\[ L = -m \sqrt{1 - v_Q^2} - U(r = r_Q, r_Q, p_Q). \] (18)

The above Lagrangian reminds us that in a first quantized effective Hamiltonian all fields are to be evaluated at the quark and all derivatives refer to quark coordinates. The spinless quark case is straightforward since no derivatives of fields appear and \( U \) is well defined. The numerical methods of solving this problem are by now well established. An example of the global spectroscopy is shown in Fig. 2. In this example the quarks at the ends of the flux tube are assumed massless.

The potentials we have defined in (13) and (15) are unusual in that they depend on the velocity of the quark. This introduces an apparent uncertainty...
between field derivatives and quark derivatives. We therefore define the required quark derivative as
\[
\lim_{r \to r_Q} \nabla_r A_{\mu}(r, r_Q, p_Q) \equiv \lim_{r \to r_Q} \nabla_r A_{\mu}(r, r_Q, p_Q).
\] (19)

This assumption ensures a smooth transition from field point to quark and in particular it guarantees that the Buchm"uller electric tube will yield a pure Thomas spin-orbit energy when analyzed in the lab frame. The assumption (19) is satisfied if spatial derivatives are taken at a constant instantaneous angular velocity.

5 Effective Hamiltonian and Reduction

The one-fermion Bethe-Salpeter equation can, without approximation, be expressed as the one-particle Salpeter equation
\[
\Lambda_+ [\alpha \cdot (p - A) + \beta m + A_0 - E] \Lambda_+ \psi = 0,
\] (20)
where the usual energy projection operators are defined by

\begin{align*}
\Lambda_{\pm} &= E_0 \pm H_0/2E_0, \\
E_0 &= \sqrt{p^2 + m^2}, \\
H_0 &= \alpha \cdot p + \beta m. 
\end{align*}

We suppress the subscript on \( r_Q \) and \( p_Q \) since it is clear that such equations must involve quark coordinates. The effective Hamiltonian is then

\[ H_{\text{eff}} = H_0 + \Lambda_+ (A_0 - \alpha \cdot A) \Lambda_+ . \quad (22) \]

We have assumed that the potentials have been promoted to Hermitian operations by suitable symmetrization. The use of the Salpeter equation avoids possible inconsistencies inherent in the Dirac equation due to the Klein paradox. For the purpose of determining relativistic corrections either equation gives the same result. The proof of this statement follows from the expectation identity

\[ \langle [E_0^\pm, F(r)] \rangle \simeq 0 \quad (23) \]

valid to leading order in \( p^2/m^2 \). From this and the standard reduction of the Dirac equation we obtain the reduced (semi-relativistic) approximation,

\begin{align*}
H_{\text{eff}} &\simeq \left( m + \frac{p^2}{2m} - \frac{p^4}{8m^3} \right) + A_0 + \frac{\nabla^2 A_0}{8m^2} - \frac{A : p}{m} + H_{\text{s.o.}}, \\
H_{\text{s.o.}} &\simeq \frac{1}{2m^2 r} \frac{d A_0}{dr} \mathbf{L} \cdot \mathbf{S} - \frac{1}{m} (\nabla \times A) \cdot \mathbf{S}. \quad (24)
\end{align*}

To explicitly evaluate this reduction we need to expand \( A_\mu \) in powers of \( v^2 \) and evaluate the derivative at the quark coordinate using (19). For low quark velocities we find from the general expressions (13) and (15) that

\begin{align*}
A_0(r, r_Q, p_Q) &\simeq a r + \frac{av_Q^2}{6r_Q^3} r^3, \quad (25) \\
A(r, r_Q, p_Q) &\simeq \frac{a v_Q}{3} r_Q^2 \hat{r} \cdot \hat{v}_\perp. \quad (26)
\end{align*}

Using consistently the prescription (13) to evaluate the field derivatives we obtain

\begin{align*}
\nabla^2 A_0 &\simeq 2a/r_Q, \\
\frac{dA_0}{dr} &\simeq a, \\
\nabla \times A &\simeq \frac{av_Q}{3} (\hat{r} \times \hat{v}_\perp). \quad (27)
\end{align*}
Finally, using the fact that a heavy quark carries most of the orbital angular momentum
\[ L \simeq m r_Q v_Q \perp, \]
(28)
(24) becomes
\[ H_{\text{eff}} \simeq \left( m + \frac{p^2}{m} - \frac{p^4}{8m^3} \right) + a r + \frac{a L^2}{4m^2r} - \frac{a}{2m^2r} L \cdot S. \]
(29)

6 The Mechanical Relativistic Flux Tube Model

The relativistic flux tube (RFT) model is based on the energy-momentum Lorentz transformation. In the co-rotating frame the electric flux tube is assumed to have constant energy per unit length equal to the tension \( a \). In a frame where the tube rotates with angular velocity \( \omega \equiv v_Q \perp / r_Q \) the energy of an infinitesimal element is then \( dE = dE(r) \) and the total energy of the tube is
\[ H_t(r) = a \int_0^r \frac{dr'}{\sqrt{1 - \omega^2 r'^2}}, \]
(30)
which is the same as \( A_0(r) \) in (15). Similarly, the angular momentum of the tube \( L_t \) is
\[ p_t \equiv L_t = \frac{a \omega}{r} \int_0^r \frac{dr' r'^2}{\sqrt{1 - \omega^2 r'^2}} (\hat{r} \times \hat{v}_\perp), \]
(31)
which is the same as \( A(r) \) from (13).

In the free Salpeter equation
\[ \not{\Lambda} + [ \alpha \cdot p + \beta m - E \not{1}] \Lambda \psi = 0, \]
(32)
the confining interaction is introduced by the “covariant tube substitution”
\[ p^\mu = (E, p) \to (E - H_t, p - p_t), \]
(33)
giving
\[ \not{\Lambda} + [ \alpha \cdot (p - p_t) + \beta m + H_t \not{1} - E \not{1}] \Lambda \psi = 0. \]
(34)
This is identical to (23) since \( H_t = A_0 \) and \( p_t = A \). We see that the assumption of pure chromo-electric field in the co-rotating frame is equivalent to the mechanical RFT model.
7 Conclusions

Our main result is contained in the effective Hamiltonian (20) and its reduced form (29). The starting point was Buchmüller's observation that a moving flux tube should be pure electric in the co-rotating frame. Although this makes very plausible the pure Thomas spin-orbit term and the lack of spin-spin interactions at large distance, it did not directly constitute a complete effective Hamiltonian. The present analysis, done in a non-rotating frame, extends Buchmüller's original conclusions to achieve a complete dynamical theory. In addition to the usual kinetic energy terms and the static linearly confining energy, there are three relativistic corrections which are dependent on the type of interaction. These are the last terms in (29): the Darwin, the $L^2$, and the Thomas terms. The Darwin term depends on the method of symmetrizing the Hamiltonian and thus is somewhat ambiguous. The $L^2$ and the Thomas spin-orbit terms are exactly what one expects from a more fundamental QCD approach.

An additional significant observation is that the original effective Hamiltonian (22) before any semi-relativistic approximations were made is exactly that of the “mechanical relativistic flux tube model”. This model was constructed by considerations of mechanical momentum and energy conservation of a flux tube with quarks at its ends. The relativistic treatment of the present case will also then have the desired Regge behavior (for massless quarks) of a Nambu string.

Finally, we might mention that one can also start with the direct assumption that $B' = 0$ on the quark. This is another way to build in the Thomas spin-orbit interaction and by a suitable Lorentz transform method obtain a complete dynamical model. In this case the resulting model is relatively easy to solve because the Hamiltonian can be explicitly written in terms of canonical momenta and coordinates. Although it promises to be a useful model it does not contain direct reference to the flux tube properties. The relativistic corrections turn out to be different than those in (29) and the Regge slope is the same as from scalar confinement.

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References

1. Kenneth G. Wilson, Phys. Rev. D 10, 2445 (1974).
2. M.G. Olsson, Proceedings of the International Conference on Quark Confinement and the Hadron Spectrum, June 1994, p. 76, ed. by N. Brambilla and G.M. Prosperi (World Scientific, Singapore, 1996).
3. H.J. Schnitzer, Phys. Rev. Lett. 35, 1540 (1975); A.B. Henriques, B.H. Kellett, and R.G. Moorhouse, Phys. Lett. B 64, 95 (1976).
4. E. Eichten and F. Feinberg, Phys. Rev. D 23, 2724 (1981).
5. D. Gromes, Z. Phys. C 22, 265 (1984); 26, 401 (1984).
6. K. Williams, “Revisiting the Eichten-Feinberg-Gromes Q Anti-Q Spin Orbit Interaction”, hep-ph/9607211.
7. W. Buchmüller, Phys. Lett. B 112, 479 (1982).
8. L.T. Thomas, Phil. Mag. 3, 1 (1927); V. Bargmann, L. Michel and V.L. Telegdi, Phys. Rev. Lett 2, 435 (1959).
9. J.D. Jackson, Classical Electrodynamics (John Wiley and Sons, second edition, 1975).
10. A. Chodos and C.B. Thomas, Nucl. Phys. B 72, 509 (1974); I. Bars, Nucl. Phys. B 111, 413 (1974); I. Bars and A. Hanson, Phys. Rev. D 13, 1744; K. Kikkawa and M. Sato, Phys. Rev. Lett. 38, 1309 (1977); M. Ida, Prog. Theor. Phys. 59, 1661 (1978); K. Johnson and C. Nohl, Phys. Rev. D 19, 291 (1979).
11. A. Yu. Dubin, A.B. Kaidalov, and Yu. A. Simonov, Phys. Lett. B 323, 41 (1994); E.L. Guban Kova and A. Yu. Dubin, ibid. 334, 180 (1994).
12. D. LaCourse and M.G. Olsson, Phys. Rev. D39, 2751 (1989); C. Olson, M.G. Olsson, and K. Williams, Phys. Rev. D 45 4307 (1992); M.G. Olsson and S. Veseli, Phys. Rev. D 51, 3578 (1995).
13. C. Semay and B. Silvestre-Brac, Phys. Rev. D 51, 1258 (1995).
14. C. Long and D. Robson, Phys. Rev. D 27, 644 (1983); W. Lucha, F.F. Schöberl, and D. Gromes, Phys. Rep. 200, 127 (1991).
15. M.G. Olsson, S. Veseli, and K. Williams, Phys. Rev. D 51, 5079 (1995).
16. M.G. Olsson and K. Williams, Phys. Rev. D 48, 417 (1993); M.G. Olsson, S. Veseli, and K. Williams, Phys. Rev. D 53, 4006 (1996).
17. A. Barchielli, E. Montaldi, and G.M. Prosperi, Nucl. Phys. B 296, 625 (1988); 303, 752(E) (1988); A. Barchielli, N. Brambilla, and G.M. Prosperi, Nuovo Cimento A 103, 59 (1989); N. Brambilla and G.M. Prosperi, Phys. Lett. B 236, 69 (1990); Phys. Rev. D 46, 1096 (1992); N. Brambilla, P. Consoli, and G.M. Prosperi, Phys. Rev. D 50, 5878 (1994).
18. T.J. Allen, M.G. Olsson, S. Veseli, and K. Williams, University of Wisconsin-Madison report MADPH-96-934 (1996).