Mesons in Light-Front QCD\(_{2+1}\): Investigation of a Bloch Effective Hamiltonian

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Abstract

We study the meson sector of 2+1 dimensional light-front QCD using a Bloch effective Hamiltonian in the first non-trivial order. The resulting two dimensional integral equation is converted into a matrix equation and solved numerically. We investigate the efficiency of Gaussian quadrature in achieving the cancellation of linear and logarithmic light-front infrared divergences. The vanishing energy denominator problem which leads to severe infrared divergences in 2+1 dimensions is investigated in detail. Our study indicates that in the context of Fock space based effective Hamiltonian methods to tackle gauge theories in 2+1 dimensions, approaches like similarity renormalization renormalization method may be mandatory due to uncanceled infrared divergences caused by the vanishing energy denominator problem. We define and study numerically a reduced model which is relativistic, free from infrared divergences, and exhibits logarithmic confinement. The manifestation and violation of rotational symmetry as a function of the coupling are studied quantitatively.

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I. INTRODUCTION AND MOTIVATION

There are various well-known motivations \cite{1} to study QCD in the light-front Hamiltonian formalism. In fact, there have been many attempts recently to study relativistic bound state problem in the Hamiltonian formalism in a light-front Fock space basis (For a review see, Ref. \cite{2}). It has been realized that a major impediment to a straightforward diagonalization of the Hamiltonian is the rapid growth of the dimension of the Hamiltonian matrix with particle number. An alternative approach will be to use an effective Hamiltonian that operates in

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a few particle basis. A challenging problem here is that, for a successful description of low energy observables, the effective Hamiltonian must incorporate main features of strong interaction dynamics.

One of the first attempts invoked Tamm-Dancoff or the Bloch-Horowitz effective Hamiltonian. Though it was successful in tackling 1+1 dimensional gauge theories, its deficiencies become apparent when attempts were made in 3+1 dimensions. First and foremost is the lack of confinement in the case of QCD. Second is the appearance of the bound state eigenvalue in the energy denominators. This has two undesirable consequences. Firstly, a light-front singularity of the type $\frac{1}{k^+}$, where $k^+$ is the light front longitudinal momentum of the exchanged gluon, remains in the bound state equation, which would have canceled if free energies appeared in the energy denominators. Secondly, from the fermion self energy contribution, in addition to the mass divergence another ultraviolet divergence appears (for an example in the context of 3+1 dimensional Yukawa model see Ref. [4]) which contributes to the renormalization of the coupling. This contribution is also infrared divergent and can be identified as arising from fermion wave function renormalization. It is the Fock space truncation that has produced this unphysical divergence which would otherwise have been canceled by vertex renormalization in a strict order by order perturbative calculation.

It is well-known that various standard formulae for effective Hamiltonian all have drawbacks. Some of the deficiencies of the Bloch-Horowitz formalism are absent in the Bloch effective Hamiltonian which was reinvented in the context of renormalization group by Wilson. Bloch Hamiltonian has two desired properties, namely, the effective Hamiltonian is (1) Hermitian and (2) involves only unperturbed energies in the energy denominator. Use of Bloch effective Hamiltonian eliminates two major problems of the Tamm-Dancoff approach to gauge theories mentioned above. However, Bloch effective Hamiltonian has an undesirable feature, namely, the vanishing energy denominator. To the best of our knowledge, Bloch effective Hamiltonian was never assessed in terms of its strengths and weaknesses in the study of bound state problems in field theory.

In the study of bound states, QCD poses challenging problems. To overcome many pitfalls of standard effective Hamiltonians, similarity renormalization was proposed. It avoids vanishing energy denominators and thus provides an improvement over Bloch effective Hamiltonian. Initial attempts in the similarity renormalization approach worked in either the non-relativistic limit or the heavy quark effective theory context. Only recently, work has begun to address many practical problems, especially the numerical ones one faces in this approach.

A major feature of gauge theories on the light-front is severe light-front infrared divergence of the type $\frac{1}{(k^+)^2}$ where $k^+$ is the exchanged gluon longitudinal momentum which appears in instantaneous four-fermion, two-fermion two-gluon, and four-gluon interactions. In old-fashioned perturbation theory these divergences are canceled by transverse gluon interactions. In similarity perturbation theory these divergences are canceled by transverse gluon interactions. In similarity perturbation theory the cancellation is only partial and singular interactions survive. Before embarking on a detailed study of effective Hamiltonian in the similarity renormalization approach which is a modification of the Bloch effective Hamiltonian, it is quite instructive to study the Bloch effective Hamiltonian itself. The result of such a study can serve as benchmark against which one can evaluate the merits of similarity renormalization scheme. This will also provide us quantitative measures on the strengths and weaknesses of numerical procedures in handling singular interactions (in the context of
light-front field theory) on the computer. It is crucial to have such quantitative measures in order to study the effects of similarity cutoff factors on the nature of the spectrum. This is one of the motivations for the present work.

Just as the Tamm-Dancoff or the Bloch-Horowitz formalism, Bloch effective Hamiltonian of QCD in the first non-trivial order also does not exhibit confinement in 3+1 dimensions. Since one of our major concerns is the study of spectra for confining interactions, we go to 2+1 dimensions. In this case, in the limit of heavy fermion mass, a logarithmic confining potential emerges. There are several other reasons also to study light-front QCD in 2+1 dimensions. They arise from both theoretical and computational issues which we discuss next.

First of all, issues related to ultraviolet divergence become more complicated in the light front approach since power counting is different on the light front. We get products of ultraviolet and infrared divergent factors which complicate the renormalization problem. Going to two space one time dimensions greatly simplifies this issue due to the absence of ultraviolet divergences except in mass corrections. An extra complication is that Fock space truncation introduces extra ultraviolet divergences which complicate the situation in non-perturbative bound state computations. Such special divergences do not occur in 2+1 dimensions. A third complication one faces in 3+1 dimensions is that on enlarging the Fock space in a bound state calculation, one soon faces the running of the coupling constant. At low energy scales, the effective coupling grows resulting in a strongly coupled theory making the weak coupling approach with a perturbatively determined Hamiltonian unsuitable or making it mandatory to invent mechanisms like non-zero gluon mass to stop the drastic growth. In 2+1 dimensional QCD we do not face this problem since the coupling constant is dimensionful in this superrenormalizable field theory and does not run due to ultraviolet divergence. We can keep the coupling arbitrarily small and study the structure of the bound states in a weakly coupled theory.

Secondly, in 1+1 dimensions, in the gauge $A^+ = 0$, dynamical gluons are absent and their effect is felt only through instantaneous interactions between fermions. Further, recall that in light front theory, vacuum is trivial. As a result, the Fock space structure of the bound states are remarkably simple. For example, the ground state meson is just a $q\bar{q}$ pair both at weak and strong couplings. In contrast, in 2+1 dimensions, one component of the gauge field remains dynamical and one can systematically study the effects of dynamical gluons. Also note that 2+1 dimensions are the lowest dimensions where glueball states are possible and offers an opportunity to study their structure in the Fock space language without additional complications of 3+1 dimensions.

A third reason deals with aspects of rotational symmetry. 2+1 dimensions offer the first opportunity to investigate violations of Lorentz invariance introduced by various cutoffs (momenta and/or particle number) in the context of bound state calculations. This is to be contrasted with 1+1 dimensions where the sole Lorentz generator, namely boost, is kinematical in light-front field theory. Since in 2+1 dimensions we have a superrenormalizable field theory, violations introduced by transverse momentum cutoffs are minimal. Thus in contrast to 3+1 dimensions, one can study the violations caused by truncation of particle number alone and longitudinal momentum cutoffs. It is also conceivable that one can enlarge the Fock space sector and investigate their effect on restoring Lorentz invariance. It is expected that such investigations are more viable in 2+1 dimensions compared to 3+1
dimensions due to less severe demand on computational resources.

A fourth reason concerns similarity renormalization approach. In 3+1 dimensions it has been shown that similarity renormalization group approach \[7\] to effective Hamiltonian in QCD leads to logarithmic confining interaction \[12\]. It is of interest to investigate corresponding effective Hamiltonian in 2+1 dimensions especially since the canonical Hamiltonian already leads to logarithmic confinement in the nonrelativistic limit in this case. It is also known that in 3+1 dimensions the confining part of the effective Hamiltonian violates rotational symmetry. Does the violation of rotational symmetry occur also in 2+1 dimensions? If so, how does it manifest itself?

In this work we initiate a systematic study of light-front QCD in 2+1 dimensions to investigate the various issues discussed above. The plan of the rest of this paper is as follows: In Sec. II we present the canonical Hamiltonian of 2+1 dimensional QCD. The Bloch effective Hamiltonian in the \(q\bar{q}\) sector in the lowest non-trivial order is derived in Sec. III and the bound state equation is derived. The divergence structure is discussed in detail in Sec. IV. In Sec. V we investigate numerically, cancellation of light-front linear infrared divergences and the consequences of the vanishing energy denominator problem which leads to uncanceled infrared divergences in the bound state equation. A model which is relativistic, free from infrared divergences, and exhibits logarithmic confinement is presented in Sec. VI. In Sec. VII we present the numerical investigation of this model in the weak coupling limit. In this section, we also discuss the violation of rotational symmetry in this model at strong coupling. Finally Sec. VIII contains discussion and conclusions. Since Bloch effective Hamiltonian is unfamiliar to most of the readers, we present a detailed derivation in Appendix A. Details of the numerical procedures used in this work are given in Appendix B.

II. CANONICAL HAMILTONIAN

In this section we present the canonical light front Hamiltonian of 2+1 dimensional QCD. The Lagrangian density is given by

\[
\mathcal{L} = \left[ -\frac{1}{4}(F_{\lambda\sigma a})^2 + \bar{\psi}(\gamma^\lambda iD_\lambda - m)\psi \right]
\]

(2.1)

with

\[
iD^\mu = \frac{1}{2} i\partial^\mu + gA^\mu,
\]

\[
F^{\mu\lambda a} = \partial^\mu A^{\lambda a} - \partial^\lambda A^{\mu a} + gf^{abc}A^{\mu b}A^{\lambda c}.
\]

(2.2)

We have the equations of motion,

\[
\left[i\gamma^\mu \partial_\mu + g\gamma^\mu A_\mu - m\right] \psi = 0, 
\]

(2.3)

\[
\partial_\mu F^{\mu a} + gf^{abc}A_{\mu b}F_{\tau}^{\tau a} + g\bar{\psi}\gamma^\nu T^a \psi = 0.
\]

(2.4)

Because we are in 2+1 dimensions, we immediately face an ambiguity since there are no \(\gamma\) matrices in 2+1 dimensions. In the literature both two component \[13\] and four component representation \[14\] have been in use. For simplicity, we pick the two component representation. Explicitly,
\[\gamma^0 = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma^1 = i\sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \gamma^2 = i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \quad (2.5)\]

\[\gamma^\pm = \gamma^0 \pm \gamma^2, \quad \gamma^+ = \begin{pmatrix} 0 & 0 \\ 2i & 0 \end{pmatrix}, \quad \gamma^- = \begin{pmatrix} 0 & -2i \\ 0 & 0 \end{pmatrix}. \quad (2.6)\]

\[\Lambda^\pm = \frac{1}{4}\gamma^\pm \gamma^\pm, \quad \Lambda^+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \Lambda^- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \quad (2.7)\]

Fermion field operator \(\psi^\pm = \Lambda^\pm \psi\). We have

\[\psi^+ = \begin{pmatrix} \xi \\ 0 \end{pmatrix}, \quad \psi^- = \begin{pmatrix} 0 \\ \eta \end{pmatrix} \quad (2.8)\]

where \(\xi\) and \(\eta\) are one component fields. We choose the light front gauge \(A^{+a} = 0\). From the equation of motion, we get the equation of constraint

\[i\partial^+ \psi^- = \left[\alpha^1(i\partial^1 + gA^1) + \gamma^0 m\right]\psi^+. \quad (2.9)\]

Thus the fermion constrained field

\[\eta = \frac{1}{\partial^+}\left[-(i\partial^1 + gA^1) + im\right]\xi. \quad (2.10)\]

From the equation of motion, in the gauge \(A^{+a} = 0\), we have the equation of constraint

\[-\frac{1}{2}(\partial^+)^2 A^{-a} = -\partial^1 \partial^+ A^{1a} - gf^{abc} A^{1b} \partial^+ A^{1c} - 2g\xi^1 T^a\xi. \quad (2.11)\]

Using the equations of constraint, we eliminate \(\psi^-\) and \(A^-\) in favor of dynamical field \(\psi^+\) and \(A^1\) and arrive at the canonical Hamiltonian given by

\[H = H_0 + H_{int} = \int dx^- dx^1 (\mathcal{H}_0 + \mathcal{H}_{int}). \quad (2.12)\]

The free Hamiltonian density is given by

\[\mathcal{H}_0 = \xi^\dagger \frac{-(\partial^1)^2 + m^2}{i\partial^+} \xi + \frac{1}{2}\partial^1 A^{1a} \partial^1 A^{1a}. \quad (2.13)\]

The interaction Hamiltonian density is given by

\[\mathcal{H}_{int} = \mathcal{H}_1 + \mathcal{H}_2 \quad (2.14)\]

with

\[\mathcal{H}_1 = g\xi^\dagger A^1 \frac{\partial^1}{\partial^+} \xi + g\xi^\dagger \frac{\partial^1}{\partial^+} (A^1 \xi) + \frac{1}{2} \partial^1 A^{1a} \partial^1 A^{1a} \quad (2.15)\]
and

\[ \mathcal{H}_2 = -2g^2 \xi^\dagger T^a \xi \left( \frac{1}{\partial^+} \right)^2 \xi^\dagger T^a \xi + g^2 \xi^\dagger A^1 \frac{1}{\partial^+} (A^1 \xi) + 2g^2 f^{abc} \frac{1}{\partial^+} (\xi^\dagger T^a \xi) \frac{1}{\partial^+} (A^{1b} \partial^+ A^{1c}) + \frac{1}{2} g^2 f^{abc} f^{ade} \frac{1}{\partial^+} (A^{1b} \partial^+ A^{1c}) \frac{1}{\partial^+} (A^{1d} \partial^+ A^{1e}). \] (2.16)

The one component fermion field is given by

\[ \xi(x^+, x^-, x^1) = \int \frac{dk^+ dk^1}{2(2\pi)^2 \sqrt{k^+}} \left[ b(k) e^{-ik \cdot x} + d^\dagger(k) e^{ik \cdot x} \right]. \] (2.17)

The Fock operators obey the anti commutation relation

\[ \{ b(k), b^\dagger(q) \} = 2(2\pi)^2 k^+ \delta^2(k - q), \quad \{ d(k), d^\dagger(q) \} = 2(2\pi)^2 k^+ \delta^2(k - q), \] (2.18)

other anti commutators being zero. Note that in two component representation, light front fermions do not carry helicity in 2+1 dimensions.

In free field theory, the equation of motion of the dynamical field \( A^1 \) is the same as that of a free massless scalar field [15] and hence we can write

\[ A^1(x^+, x^-, x^1) = \int \frac{dk^+ dk^1}{2(2\pi)^2 \sqrt{k^+}} \left[ a(k) e^{-ik \cdot x} + a^\dagger(k) e^{ik \cdot x} \right]. \] (2.19)

The Fock operators obey the commutation relation

\[ [a(k), a^\dagger(q)] = 2(2\pi)^2 k^+ \delta^2(k - q), \] (2.20)

other commutators being zero.

We substitute the Fock expansions, Eqs. (2.17) and (2.19) into the Hamiltonian and treat all the terms to be normal ordered. Thus we arrive at the canonical Hamiltonian in the Fock basis.

### III. BLOCH EFFECTIVE HAMILTONIAN IN THE MESON SECTOR AND THE BOUND STATE EQUATION

In this section we evaluate the Block effective Hamiltonian to the lowest non-trivial order for a meson state and derive the effective bound state equation. We define the \( P \) space to be \( q\bar{q} \) sector of the Fock space and \( Q \) space to be the rest of the space. In the lowest non-trivial order, the Bloch effective Hamiltonian is given by (see Appendix A for details)

\[ \langle i \mid H_{\text{eff}} \mid j \rangle = \langle i \mid (H_0 + H_{\text{int}}) \mid j \rangle + \frac{1}{2} \sum_k \langle i \mid v \mid k \rangle \langle k \mid v \mid j \rangle \left[ \frac{1}{\epsilon_i - \epsilon_k} + \frac{1}{\epsilon_j - \epsilon_k} \right]. \] (3.1)

The states \( \mid i \rangle \) and \( \mid j \rangle \) are, explicitly,
The gluon exchange contributions are
\[ \langle a | k | b \rangle = b^\dagger (p_1, \alpha) d^\dagger (p_2, \alpha) | 0 \rangle, \]
\[ | b \rangle = b^\dagger (p_3, \beta) d^\dagger (p_4, \beta) | 0 \rangle, \quad (3.2) \]
where \( p_1, p_2 \) denote momenta and \( \alpha, \beta \) denote color which is summed over. Explicitly, \( p_1 = (p_1^+, p_1^\perp) \) etc., where \( p_1^+ \) is the plus component and \( p_1^\perp \) is the transverse component. For simplicity of notation, we will denote the transverse component of momenta without the superscript 1.

The free part of the Hamiltonian leads to the matrix element
\[ \langle a \mid H \mid b \rangle = \left( \frac{m^2 + p_1^2}{p_1^+} + \frac{m^2 + p_2^2}{p_2^+} \right) 2(2\pi)^2 p_1^+ \delta^2 (p_1 - p_3) 2(2\pi)^2 p_2^+ \delta^2 (p_2 - p_4) \delta_{\alpha \beta}. \quad (3.3) \]
From the four fermion interaction, we get the contribution
\[ -4 g^2 (T^a T^a)_{\alpha \alpha} \frac{1}{(p_1^+ - p_2^+)^2} 2(2\pi)^2 \sqrt{p_1^+ p_2^+ p_3^+ p_4^+} \delta^2 (p_1 + p_2 - p_3 - p_4) \delta_{\alpha \beta}. \quad (3.4) \]
Next we evaluate the contribution from the second order term. The intermediate state \( | k \rangle \) is a quark, anti-quark, gluon state. This intermediate state gives rise to both self energy and gluon exchange contributions.

The self energy contributions are
\[ g^2 C_f \delta_{\alpha \beta} p_1^+ 2(2\pi)^2 \delta^2 (p_1 - p_3) p_2^+ 2(2\pi)^2 \delta^2 (p_2 - p_4) \]
\[ \int \frac{dk_1^+ dk_2}{2(2\pi)^2 (p_1^+ - k_1^+)} \left\{ -2 \frac{(p_1 - k_1)}{(p_1^+ - k_1^+)} + \frac{k_1}{k_1^+} + \frac{p_1}{p_1^+} - i \frac{m}{p_1^+} \right\} \frac{1}{ED_1} \]
\[ \left\{ -2 \frac{(p_1 - k_1)}{(p_1^+ - k_1^+)} + \frac{k_1}{k_1^+} + \frac{p_1}{p_1^+} + i \frac{m}{p_1^+} \right\} \frac{1}{ED_1} \]
\[ + g^2 C_f \delta_{\alpha \beta} p_1^+ 2(2\pi)^2 \delta^2 (p_1 - p_3) p_2^+ 2(2\pi)^2 \delta^2 (p_2 - p_4) \]
\[ \int \frac{dk_1^+ dk_2}{2(2\pi)^2 (p_2^+ - k_2^+)} \left\{ -2 \frac{(p_2 - k_2)}{(p_2^+ - k_2^+)} + \frac{k_2}{k_2^+} + \frac{p_2}{p_2^+} - i \frac{m}{p_2^+} \right\} \frac{1}{ED_2} \]
\[ \left\{ -2 \frac{(p_2 - k_2)}{(p_2^+ - k_2^+)} + \frac{k_2}{k_2^+} + \frac{p_2}{p_2^+} + i \frac{m}{p_2^+} \right\} \frac{1}{ED_2} \], \quad (3.5) \]
with
\[ ED_1 = \frac{p_1^2 + m^2}{p_1^+} - \frac{m^2 + k_1^2}{k_1^+} - \frac{(p_1 - k_1)^2}{(p_1^+ - k_1^+)^2}, \]
\[ ED_2 = \frac{p_2^2 + m^2}{p_2^+} - \frac{m^2 + k_2^2}{k_2^+} - \frac{(p_2 - k_2)^2}{(p_2^+ - k_2^+)^2}. \quad (3.6) \]
The gluon exchange contributions are
\[ -g^2 (T^a T^a)_{\alpha \alpha} 2(2\pi)^2 \delta^2 (p_1 + p_2 - p_3 - p_4) \sqrt{p_1^+ p_2^+ p_3^+ p_4^+} \]
\[ \left\{ -2 \frac{(p_1 - p_3)}{(p_1^+ - p_3^+)} + \frac{p_3}{p_3^+} + \frac{p_1}{p_1^+} - i \frac{m}{p_3^+} + i \frac{m}{p_1^+} \right\} \frac{1}{ED_1} \]
\[ \left\{ -2 \frac{(p_1 - p_3)}{(p_1^+ - p_3^+)} + \frac{p_3}{p_3^+} + \frac{p_4}{p_4^+} + i \frac{m}{p_3^+} - i \frac{m}{p_4^+} \right\} \]
\begin{align}
\frac{1}{2} \frac{(p_2^- - p_3^-)}{p_1^- - p_3^-} \left\{ \begin{array}{c}
\frac{1}{m^2 + p_3^2} - \frac{(p_1 - p_3)^2}{p_1^2} + \frac{1}{m^2 + p_3^2} - \frac{(p_1 - p_3)^2}{p_1^2} \\
\frac{1}{m^2 + p_3^2} - \frac{(p_1 - p_3)^2}{p_1^2} + \frac{1}{m^2 + p_3^2} - \frac{(p_1 - p_3)^2}{p_1^2}
\end{array} \right\} \\
- \frac{g^2}{(T^a T^a)} \alpha \beta 2(2\pi)^2 \delta^2(p_1 + p_2 - p_3 - p_4) \sqrt{p_1^+ p_2^+ p_3^+ p_4^+} \\
\left\{ \begin{array}{c}
- \frac{2}{(p_3 - p_1)} + \frac{p_3}{p_3} - \frac{p_1}{p_1} - i \frac{m}{p_3} + i \frac{m}{p_1} \\
- \frac{2}{(p_3 - p_1)} + \frac{p_3}{p_2} + \frac{p_4}{p_4} + \frac{m}{p_2} - \frac{m}{p_4}
\end{array} \right\} \\
\frac{1}{2} \frac{(p_2^+ - p_1^+)}{p_3^+ - p_1^+} \left\{ \begin{array}{c}
\frac{1}{m^2 + p_3^2} - \frac{(p_1 - p_3)^2}{p_1^2} + \frac{1}{m^2 + p_3^2} - \frac{(p_1 - p_3)^2}{p_1^2} \\
\frac{1}{m^2 + p_3^2} - \frac{(p_1 - p_3)^2}{p_1^2} + \frac{1}{m^2 + p_3^2} - \frac{(p_1 - p_3)^2}{p_1^2}
\end{array} \right\}. 
\end{align}

After the construction of $H_{\text{eff}}$ in the two particle space, we proceed as follows. Consider the bound state equation

$$
H_{\text{eff}} | \Psi \rangle = \frac{M^2 + P^2}{P^+} | \Psi \rangle
$$

where $P^+$, $P$, and $M$ are the longitudinal momentum, the transverse momentum and the invariant mass of the state respectively. The state $| \Psi \rangle$ is given by

$$
| \Psi \rangle = \sum_{\beta} \int \frac{dp_3^+ dp_3}{\sqrt{2(2\pi)^2 p_3^2}} \int \frac{dp_4^+ dp_4}{\sqrt{2(2\pi)^2 p_4^2}} \phi_2(P; p_3, p_4) b^\dagger(p_3, \beta) d^\dagger(p_4, \beta) | 0 \rangle \\
\sqrt{2(2\pi)^2 P^+ \delta^2(P - p_3 - p_4)}
$$

which we symbolically represent as

$$
| \Psi \rangle = \sum_j \phi_{2j} | j \rangle.
$$

Taking projection with the state $\langle i | = \langle 0 | d(p_2, \alpha) b(p_1, \alpha)$, we get the effective bound state equation,

$$
\frac{M^2 + P^2}{P^+} \phi_{2i} = H_{\text{eff}} \phi_{2i} + \sum_j \langle i | H_{\text{eff}} | j \rangle \phi_{2j}.
$$

Introduce the internal momentum variables $(x, k)$ and $(y, q)$ via $p_1^+ = x P^+$, $p_1 = x P + k$, $p_2^+ = (1 - x) P^+$, $p_2 = (1 - x) P - k$, $p_3^+ = y P^+$, $p_3 = y P + q$, $p_4^+ = (1 - y) P^+$, $p_4 = (1 - y) P - q$ and the amplitude $\phi_2(P; p_1, p_2) = \frac{1}{\sqrt{P^+}} \psi_2(x, k)$.

The fermion momentum fractions $x$ and $y$ range from 0 to 1. To handle end point singularities, we introduce the cutoff $\eta \leq x, y \leq 1$. This does not prevent the gluon longitudinal momentum fraction $x - y$ from becoming zero and we introduce the regulator $\delta$ such that $| x - y | \geq \delta$. To regulate ultraviolet divergences, we introduce the cutoff $\Lambda$ on the relative transverse momenta $k$ and $q$. We remind the reader that in the superrenormalizable field theory under study, only ultraviolet divergence is in the fermion self energy contribution which we remove by a counterterm before discretization.

The bound state equation is
\[ M^2 - \frac{m^2 + k^2}{x(1-x)} \psi_2(x, k) = SE \psi_2(x, k) - 4 \frac{g^2}{2(2\pi)^2} C_f \int dq dy \, \psi_2(y, q) \frac{1}{(x - y)^2} \]
\[ -\frac{g^2}{2(2\pi)^2} C_f \int dq dy \, \psi_2(y, q) \frac{1}{2 ED}. \]
\[ \text{(3.12)} \]

The self energy contribution
\[ SE = -\frac{g^2}{2(2\pi)^2} C_f \int_0^x dq \int dy \, xy \frac{\left[\left(\frac{q}{y} + \frac{k}{x} - \frac{2(k-q)}{x-y}\right)^2 + \frac{m^2(x-y)^2}{x^2y^2}\right]}{(ky-qx)^2 + m^2(x-y)^2} \]
\[ -\frac{g^2}{2(2\pi)^2} C_f \int_1^x dy \int dq \, (1-x)(1-y) \frac{\left[\left(\frac{q}{1-y} + \frac{k}{1-x} + \frac{2(q-k)}{y-x}\right)^2 + \frac{m^2(x-y)^2}{(1-y)^2(1+y)^2}\right]}{[k(1-y) - q(1-x)]^2 + m^2(x-y)^2}. \]
\[ \text{(3.13)} \]

The boson exchange contribution
\[ \frac{V}{ED} = \frac{\theta(x-y)}{(x-y)} \left[ \frac{1}{m^2 + \frac{(k-q)^2}{(x-y)}} - \frac{m^2 + k^2}{x} + \frac{1}{(1-x) - \frac{(k-q)^2}{y} - \frac{m^2 + q^2}{1-y}} \right] \]
\[ \times \left[ K(k, x, q, y) + iV_f \right] \]
\[ + \frac{\theta(y-x)}{(y-x)} \left[ \frac{1}{m^2 + \frac{(k-q)^2}{(y-x)}} - \frac{m^2 + q^2}{y} + \frac{1}{1-y - \frac{(k-q)^2}{x} - \frac{m^2 + k^2}{1-x}} \right] \]
\[ \times \left[ K(q, y, k, x) + iV_f \right], \]
\[ \text{(3.14)} \]

where
\[ K(k, x, q, y) = \left(\frac{q}{y} + \frac{k}{x} - \frac{2(k-q)}{x-y}\right) \left(\frac{q}{1-y} + \frac{k}{1-x} + \frac{2(k-q)}{(x-y)}\right) - \frac{m^2(x-y)^2}{xy(1-x)(1-y)}. \]
\[ \text{(3.15)} \]
\[ V_f = -\frac{m}{xy(1-x)(1-y)} \left[q(2-y-3x) + k(3y+x-2)\right]. \]
\[ \text{(3.16)} \]

**IV. DIVERGENCE STRUCTURE**

In this subsection we carry out a detailed analysis of the divergence structure of the effective bound state equation. We encounter both infrared and ultraviolet divergences.

**A. Ultraviolet Divergences**

First consider ultraviolet divergences. In the super renormalizable field theory under consideration, with the terms appearing in the canonical Hamiltonian as normal ordered, ultraviolet divergence is encountered only in the self energy contributions. To isolate the ultraviolet divergence, we rewrite the self energy integrals as
\[ SE = -\frac{g^2}{2(2\pi)^2} C_f \int_0^x dy \int_{-\Lambda}^{+\Lambda} dq \left[ \frac{(x + y)^2}{xy(x-y)^2} - \frac{4m^2}{(ky - qx)^2 + m^2(x-y)^2} \right] \]
\[ -\frac{g^2}{2(2\pi)^2} C_f \int_0^1 dy \int_{-\Lambda}^{+\Lambda} dq \left[ \frac{(2 - x - y)^2}{(y-x)^2(1-x)(1-y)} - \frac{4m^2}{k(1-y) - q(1-x)^2 + m^2(x-y)^2} \right]. \]  

The first term inside the square brackets in the above equation is ultraviolet divergent, which we cancel by adding an ultraviolet counterterm given by

\[ CT = +\frac{g^2}{2(2\pi)^2} C_f \int_{-\Lambda}^{+\Lambda} dq \left[ \int_0^x dy \frac{(x + y)^2}{xy(x-y)^2} + \int_x^1 dy \frac{(2 - x - y)^2}{(y-x)^2(1-x)(1-y)} \right]. \]  

After the addition of this counterterm, the bound state equation is ultraviolet finite.

**B. Infrared Divergences**

The infrared divergences that appear in the bound state equation are of two types: (1) light front infrared divergences that arise from the gluon longitudinal momentum fraction \( x_g = 0 \), (2) true infrared divergences that arise from gluon transverse momentum \( k_g = 0 \) and gluon longitudinal momentum fraction \( x_g = 0 \).

1. **Cancellation of Light-front Infrared Divergences in the Effective Bound State Equation**

First consider light front infrared divergences. The effective bound state equation Eq. (3.12) explicitly has a linear light front infrared divergent term \( \frac{1}{(x-y)^2} \) coming from instantaneous gluon exchange. The most divergent part of the numerator of the transverse gluon exchange term in this equation is \(-4\frac{(k-q)^2}{(x-y)^2}\). After combining the terms, the linear infrared divergent term is completely canceled and the resultant effective bound state equation takes the form

\[ \left[ M^2 - \frac{m^2 + k^2}{x(1-x)} \right] \psi_2(x, k) = SE1 \psi_2(x, k) - \frac{g^2}{2(2\pi)^2} C_f \int dy dq \, \psi_2(y, q) \times \frac{1}{2} \left[ \frac{\tilde{V}_1}{E_1} + \frac{\tilde{V}_2}{E_2} + iV_I \left( \frac{1}{E_1} + \frac{1}{E_2} \right) \right]. \]  

The self energy contribution, made ultraviolet finite by the addition of the counterterm is

\[ SE1 = +\frac{g^2}{2(2\pi)^2} C_f \int_0^x dy \int_{-\Lambda}^{+\Lambda} dq \frac{4m^2}{(ky - qx)^2 + m^2(x-y)^2} \]
\[ + \frac{g^2}{2(2\pi)^2} C_f \int_x^1 dy \int_{-\Lambda}^{+\Lambda} dq \frac{4m^2}{k(1-y) - q(1-x)^2 + m^2(x-y)^2}. \]  

The energy denominator factors are
\[
\frac{1}{E_1} = \frac{xy}{[ky-qx]^2 + m^2(x-y)^2}, \quad \frac{1}{E_2} = \frac{(1-x)(1-y)}{[k(1-y) - q(1-x)]^2 + m^2(x-y)^2}. \tag{4.5}
\]

The vertex terms are
\[
\tilde{V}_1 = \theta(x-y) \tilde{U}(k, x, q, y) + \theta(y-x) \tilde{U}(q, y, k, x), \tag{4.6}
\]
\[
\tilde{V}_2 = \theta(x-y) \tilde{U}(k, 1-x, q, 1-y) + \theta(y-x) \tilde{U}(q, 1-y, k, 1-x), \tag{4.7}
\]

with
\[
\tilde{U}(k, x, q, y) = 4 \frac{m^2}{xy} - \frac{m^2(x-y)^2}{xy(1-x)(1-y)} \\
+ \frac{k^2}{y(1-y)} + \frac{k}{x(1-x)} - 2 \frac{k^2}{x-y} \frac{1}{x(1-x)} + 2 \frac{q^2}{x-y} \frac{1}{y(1-y)} \\
+ \frac{kq}{x(1-y)} + \frac{kq}{y(1-x)} + 2 \frac{kq}{x-y} \left[ \frac{1-2y}{y(1-y)} - \frac{1-2x}{x(1-x)} \right]. \tag{4.8}
\]

In addition to the $\frac{1}{x_3}$ singularity which is canceled, transverse gluon exchange contributions also contain $\frac{1}{x_3}$ singularity which is removed by the principal value prescription. Cancellation of this singularity is an appealing feature of the Bloch effective Hamiltonian in contrast to the Tamm-Dancoff effective Hamiltonian where the singularity cancellation does not occur because of the presence of invariant mass in the energy denominator [16].

2. “True” infrared divergences

Next we consider true infrared divergences. Consider the self energy integrals. The energy denominators in these expressions vanish when $k = q$ and $x = y$ which correspond to vanishing gluon momentum. By carrying out the integrals explicitly, in the limit $\Lambda \to \infty$ we get,
\[
SE_1 = \frac{mg^2}{2\pi} C_f \left[ \frac{1}{x} \ln \frac{x}{\delta} + \frac{1}{1-x} \ln \frac{1-x}{\delta} \right]. \tag{4.9}
\]

Thus the singular part of self energy is
\[
SE_{1\text{, singular}} = -\frac{mg^2}{2\pi} C_f \frac{1}{x(1-x)} \ln \delta. \tag{4.10}
\]

The infrared divergent contribution from self energy gives a positive contribution to the fermion mass. It is important to note that the vanishing of energy denominator is possible also in 3+1 dimensions, but in that case we do not encounter any divergence. It is the peculiarity of 2+1 dimensions that the vanishing energy denominators cause a severe infrared divergence problem.

The same vanishing energy denominators occur also in the one gluon exchange contributions. Let us now consider various terms in the numerator separately. The terms
proportional to $4m^2$ arose from the denominator of the transverse gluon exchange. A straightforward calculation shows that this term leads to both finite and infrared divergent contributions. The infrared divergent contribution is given by

$$\frac{mg^2}{2\pi} C_f \frac{1}{x(1-x)} \ln \delta$$

which exactly cancels the infrared divergent contribution from self energy. The finite part, in the nonrelativistic limit, can be shown to give rise to the logarithmically confining potential. Next we have to consider the remaining terms in the numerator. Rest of the terms proportional to $m^2$ are multiplied by $(x - y)^2$ so that they do not lead to an infrared divergence problem. The numerator of the imaginary part vanishes at $k = q$, and $x = y$ and hence is also infrared finite. It is easy to verify that the rest of the (transverse momentum dependent) terms in the numerator does not vanish when the denominator vanishes and hence the resulting bound state equation is inflicted with infrared divergences arising from the vanishing energy denominator. This problem was first noted in the context of QED in 2+1 dimensions by Tam, Hamer, and Yung [18] but was not investigated by these authors. We remind the reader that this is a peculiarity of 2+1 dimensions which provides us a unique opportunity to explore the consequences of the vanishing energy denominator problem.

V. NUMERICAL STUDY OF THE BOUND STATE EQUATION

We convert the integral equation into a matrix equation with the use of Gaussian Quadrature. (For details of the numerical procedure see Appendix B.) $C_f$ is set to 1 for all the numerical calculations presented. As mentioned before, an important feature of gauge theories on the light-front is the presence of linear infrared divergences. They appear in the canonical Hamiltonian in instantaneous four fermion interaction term. When the $q\bar{q}g$ states are integrated out completely in perturbation theory, they also appear in the effective four fermion interaction and cancel against each other. Non-cancellation of this divergence is a major feature of similarity renormalization approach. We first address the issue of how linear divergences manifest in the non-uniform grid of the Gaussian Quadrature and how well it can handle linear light front infrared divergence. We have studied numerically discretized versions of Eq. (3.12) where the divergences are present separately in the discretized version together with the counterterm given in Eq. (4.2). For $g = .2$, we have calculated the eigenvalues with and without the instantaneous interaction. The results presented in Fig. 1(a) for the lowest eigenvalue shows that the Gaussian Quadrature can handle the cancellation very efficiently.

After the cancellation of linear light-front infrared divergence, a logarithmic infrared divergence which arises from the vanishing energy denominator survives in the bound state equation. Here we have to distinguish two types of terms. First type, where the coefficient of the logarithmic infrared divergence is independent of the fermion transverse momentum and the second type where the coefficient is dependent. Self energy and Coulomb interaction are of the first type. In the weak coupling limit, since the wavefunction is dominated by very low transverse momentum, we anticipate that contributions of the second type will be dynamically suppressed even though both are multiplied by the same coupling constant. This is especially true of any discrete grid which automatically imposes a lower limit on the
smallest longitudinal momentum fraction allowed. Thus at weak coupling, even if there are 
uncanceled infrared divergences (divergences of the second type), they may not be significant 
numerically whereas divergences of the second type are significant. By switching the self 
energy contribution off and on, we have studied this interplay. The lowest eigenvalue with 
and without self energy contribution is plotted in Fig. 1(b). This shows the cancellation 
of the dominant logarithmic infrared divergence. Since there are still uncanceled infrared 
divergences in the bound state equation (with coefficient proportional to fermion transverse 
momenta) this figure further illustrate the fact that such divergences are not numerically 
significant at weak coupling.

As the strength of the interaction grows, wavefunction develops medium to large trans-
verse momentum components and the infrared catastrophe triggered by the vanishing energy 
denominator becomes manifest numerically. This is illustrated in Table I where we present 
the variation with $\delta$ of the first five eigenvalues for two different choices of the coupling $g$.
The table clearly shows that on a discrete grid, the uncanceled infrared divergences due to 
the vanishing energy denominator problem are not numerically significant at weak coupling 
but their effect is readily felt at a stronger coupling.

VI. REDUCED MODEL

In this section we consider a model Hamiltonian free from infrared divergences con-
structed by dropping the transverse momentum dependent terms from the numerator of the 
effective Hamiltonian. For convenience, we further drop the terms proportional to $(x - y)^2$ 
and the imaginary part. This defines our reduced model which is also ultraviolet finite. The 
equation governing the model is given by

$$\left[M^2 - \frac{m^2 + k^2}{x(1-x)}\right] \psi_2(x,k) = SE1 \psi_2(x,k) + BE. \quad (6.1)$$

The self energy contribution $SE1$ is the same as given in Eq. (4.4). The boson exchange 
contribution $BE$ is given by

$$BE = -\frac{1}{2} \frac{g^2}{2(2\pi)^2} C_f \int_0^1 dy \int_{-\Lambda}^{+\Lambda} dq \frac{4m^2}{(ky-qx)^2 + m^2(x-y)^2} \psi_2(y,q)$$ 
$$- \frac{1}{2} \frac{g^2}{2(2\pi)^2} C_f \int_0^1 dy \int_{-\Lambda}^{+\Lambda} dq \frac{4m^2}{[k(1-y) - q(1-x)]^2 + m^2(x-y)^2} \psi_2(y,q). \quad (6.2)$$

Again we discretize the Eq. (6.1) by Gaussian Quadrature. The convergence of the 
eigenvalues as a function of the number of grid points is presented in Table II. In this table 
we also present the (in)dependence of eigenvalues on the momentum cutoff. 2+1 dimensions 
provide an opportunity to study the manifestation and violation of rotational symmetry in 
light front field theory in a simpler setting compared to 3+1 dimensions. The absence of spin 
further facilitates this study. Rotational symmetry in this case simply implies degeneracy 
with respect to the sign of the azimuthal quantum number $l$. Thus we expect all $l \neq 0$ states 
to be two fold degenerate.

By a suitable change of variables, one can easily show that our reduced model, in the 
nonrelativistic limit reduces to Schrödinger equation in two space dimensions with a loga-
rithmic confining potential. In the weak coupling limit, since $C_f$ is set to 1, we can compare...
our results of the reduced model (where we do not make any nonrelativistic approximation) with the spectra obtained in nonrelativistic $QED_{2+1}$. Tam et al. solved the radial Schrödinger equation in momentum space for $l = 0$ states and Koures solved the coordinate space radial Schrödinger equation for general $l$. Since we are solving the light front bound state equation, rotational symmetry is not at all manifest. However, at weak coupling we expect that the spectra exhibit rotational symmetry to a very good approximation. Our numerical results are compared with those of Koures in Table II for two values of the coupling. At $g = 0.2$ we find reasonable agreement with the degeneracy in the spectrum. Even at $g = 0.6$ the violation of rotational symmetry is very small. Splitting of levels which are supposed to be degenerate become more visible at very strong coupling as can be seen from Table IV for $g = 5$.

Along with the eigenvalues, the diagonalization process also yields wavefunctions. We have plotted the wavefunctions corresponding to the first four eigenvalues in Fig. 2 as a function of $x$ and $k$. All wavefunctions are normalized to be $\int_0^1 dx \int dk \psi^2(x, k) = 1$. The lowest state is nodeless and corresponds to $l = 0$. The next two states correspond to $l = 1$ and have one node. It is interesting to note the way the node appears in the wavefunctions which correspond to degenerate levels. Since the rotational symmetry cannot be manifest in the variables $x$ and $k$, how can the wavefunctions still indicate this? From Fig. 2, it is clear that the way this problem is resolved is by one wavefunction having a node in $k$ and the other wavefunction having a node in $x$. Thus even if we did not know about the underlying symmetry from other means, the light-front wavefunctions have a subtle way of indicating the symmetry.

VII. SUMMARY, DISCUSSION AND CONCLUSIONS

In light-front Hamiltonian approach to the bound state problem in gauge theories, the Bloch effective Hamiltonian has certain advantages compared to the Tamm-Dancoff or the Bloch-Horowitz formalism. Furthermore, the recently proposed similarity renormalization approach is a modification of the Bloch approach. In order to quantitatively estimate the impact of similarity form factors in the similarity renormalization approach, it is extremely useful to have a quantitative study of the bound state problem in Bloch formalism. As far as we know, Bloch effective Hamiltonian has never been investigated in the context of the bound state problem in light-front field theory.

To avoid complexities due to ultraviolet divergences we turn to $2+1$ dimensions. This allows us to investigate light-front infrared divergences in the bound state problem in the presence of transverse dynamics without the additional complication arising from the mixing of ultraviolet and light-front infrared divergences. Further, $2+1$ dimensions allow us to quantitatively study the manifestation and possible violation of rotational symmetry in light-front theory in a simpler setting. The emergence of a logarithmic confining interaction in the limit of heavy fermion masses is an added impetus to study gauge theories in $2+1$ dimensions.

Only very recently, study of various issues that arise in the numerical computations in the similarity approach has begun. Since similarity renormalization approach is a modification of Bloch effective Hamiltonian approach, a detailed numerical study of the latter can serve as benchmark against which one can evaluate the merits of the similarity approach. It is also
important to quantitatively evaluate the strengths and weaknesses of numerical procedures in handling singular interactions in the context of light-front dynamics on the computer.

In this work we have focused on the Gaussian Quadrature (GQ) which is one straightforward procedure to solve the integral equation by converting it into a matrix equation. We have demonstrated the efficiency of the GQ method in handling linear and logarithmic light-front infrared divergences.

A major advantage of the similarity approach is that it avoids the vanishing energy denominator problem that is present in the Bloch formalism. In 2+1 dimensions the vanishing energy denominator leads to severe infrared divergences and hence we are presented with a unique opportunity to study its consequences. We encounter two types of infrared divergences: (1) with a coefficient proportional to fermion mass and (2) with a coefficient proportional to fermion transverse momentum. The former type gets canceled in the bound state equation between fermion dressing by gluon and gluon exchange between fermions. The latter type is uncanceled which however can be dynamically suppressed at very weak coupling on a finite grid. We have demonstrated that on a discrete grid provided by GQ, the uncanceled divergences are numerically insignificant at weak coupling whereas the catastrophe due to their presence is readily felt at stronger coupling.

We proceed to study a reduced model that is free from infrared divergences and which reduces to the Schroedinger equation with a logarithmic potential in the nonrelativistic limit. This model provides us an opportunity to study the simplest manifestation and possible violation of rotational symmetry in the context of light-front field theory. Even though the Hamiltonian does not exhibit rotational symmetry we have shown that at weak coupling spectra exhibit rotational symmetry to a very good approximation. We have also shown that even though the rotational symmetry is not manifest in the variables $x$ and $k$, light-front wavefunctions have a subtler way of indicating the underlying symmetry.

Our study indicates that in the context of Fock space based effective Hamiltonian methods to tackle gauge theories in 2+1 dimensions, approaches like similarity renormalization method are mandatory due to uncanceled infrared divergences caused by the vanishing energy denominator problem. It is important to recall that Bloch effective Hamiltonian is generated by completely integrating out the intermediate gluons irrespective of whether they are low energy or high energy. Is this justified in a confining theory?

Now that we have obtained quantitative measures of the vanishing energy denominator problem and the nature of the spectra at weak coupling of the Bloch effective Hamiltonian, the next step is to study $\text{QCD}_{2+1}$ in the similarity renormalization approach which avoids the vanishing energy denominator problem. An important issue here is the nature of new effective interactions generated by the similarity approach. It has been shown that in 3+1 dimensions, similarity approach generates logarithmic confining interactions which however breaks rotational symmetry. It is interesting to investigate the corresponding situation in 2+1 dimensions.

**APPENDIX A: BLOCH PERTURBATION THEORY FOR EFFECTIVE HAMILTONIAN**

Bloch perturbation theory was introduced in Ref. [5]. Here we follow the treatment in Ref. [17] where the reader can find many examples of perturbative calculations.
Consider a Hamiltonian $H$ defined at a cutoff $\Lambda$. Let us try to lower the cutoff to $\lambda$. In general, the cutoff could be in energy and/or particle number. Let us denote by $Q$ the operator that projects on to all of the states removed when the cutoff is lowered. Let $P = I - Q$. We have

$$Q^2 = Q, \quad P^2 = P, \quad PQ = QP = 0. \quad (A1)$$

Our purpose is to find an effective Hamiltonian $H_{\text{eff}}$ that produces the same eigenvalues in the subspace $P$ as the original Hamiltonian $H$.

Introduce an operator $R$ that satisfies

$$Q | \psi \rangle = RP | \psi \rangle$$

for all eigenstates of the Hamiltonian that have support in the subspace $P$. $R$ gives the part of $| \psi \rangle$ outside the space projected by $P$ in terms of the part of $| \psi \rangle$ inside the space. Require that $R$ gives zero acting on states outside the subspace. This means $R = RP$, $R = QR$, $R^2 = 0$. From $R = QR$, we have, $PR = 0$. Note also that $R^\dagger \neq R$.

Start from the set of equations

$$PHP | \psi \rangle + PHQ | \psi \rangle = EP | \psi \rangle, \quad (A3)$$

$$QHP | \psi \rangle + QHQ | \psi \rangle = EQ | \psi \rangle. \quad (A4)$$

From Eq. (A3),

$$RPHP | \psi \rangle + RPHQRP | \psi \rangle = ERP | \psi \rangle. \quad (A5)$$

From Eq. (A4),

$$QHP | \psi \rangle + QHQRP | \psi \rangle = ERP | \psi \rangle. \quad (A6)$$

Subtracting,

$$RH_{PP} - H_{QQ}R + RH_{PQ}R - H_{QP} = 0. \quad (A7)$$

We have introduced the notations, $PHP = H_{PP}$ and so on. Put $H = h + v$ with $[h, Q] = 0$. Then

$$Rh_{PP} - h_{QQ}R - v_{QP} + Rv_{PP} - v_{QQ}R + Rv_{PQ}R = 0 \quad (A8)$$

which shows that $R$ starts first order in $v$.

We start from the eigenvalue equation,

$$H(P + Q) | \psi \rangle = E(P + Q) | \psi \rangle. \quad (A9)$$

i.e.,

$$H(P + R)P | \psi \rangle = E(P + R)P | \psi \rangle. \quad (A10)$$

Multiplying from the left by $(P + R^\dagger)$ we have,
\[(P + R^\dagger)H(P + R)P | \psi \rangle = E(P + R^\dagger)(P + R)P | \psi \rangle. \]  

(A11)

Using \( PR = 0, R^\dagger P = 0 \), \((P + R^\dagger)(P + R) = P + R^\dagger R \). Thus we can rewrite the eigenvalue equation as

\[
\left[ \frac{1}{1 + R^\dagger R} \right]^{\frac{1}{2}} (P + R^\dagger)H(P + R) \left[ \frac{1}{1 + R^\dagger R} \right]^{\frac{1}{2}} [1 + R^\dagger R]^{\frac{1}{2}} P | \psi \rangle = E[1 + R^\dagger R]^{\frac{1}{2}} P | \psi \rangle. \]  

(A12)

i.e.,

\[
H_{eff} | \phi \rangle = E | \phi \rangle \]  

(A13)

where

\[
| \phi \rangle = [1 + R^\dagger R]^{\frac{1}{2}} P | \psi \rangle \]  

(A14)

and

\[
H_{eff} = \left[ \frac{1}{1 + R^\dagger R} \right]^{\frac{1}{2}} (P + R^\dagger)H(P + R) \left[ \frac{1}{1 + R^\dagger R} \right]^{\frac{1}{2}}. \]  

(A15)

Our next task is to generate a perturbative expansion. Denote free eigenstates in \( P \) by \(| a \rangle, | b \rangle \), etc. Denote free eigenstates in \( Q \) by \(| i \rangle, | j \rangle \), etc. Then

\[
h_{PP} | a \rangle = \epsilon_a | a \rangle, \]  

\[
h_{QQ} | i \rangle = \epsilon_i | i \rangle. \]  

(A16)

Let us compute \( R \) to lowest orders in the perturbation theory. Let us write \( R = R_1 + R_2 + \ldots \) where the subscript denotes orders in \( v \). A straightforward calculation leads to

\[
\langle i | R_1 | a \rangle = \frac{\langle i | v_{QP} | a \rangle}{\epsilon_a - \epsilon_i}, \]  

(A17)

\[
\langle i | R_2 | a \rangle = -\sum_b \frac{\langle b | v | a \rangle \langle i | v | b \rangle}{(\epsilon_a - \epsilon_i)(\epsilon_b - \epsilon_i)} + \sum_j \frac{\langle i | v | j \rangle \langle j | v | a \rangle}{(\epsilon_a - \epsilon_i)(\epsilon_a - \epsilon_j)}. \]  

(A18)

Our next task is to develop a perturbation theory expansion for the effective Hamiltonian to a given order.

We start from the expression for the effective Hamiltonian. Remember that \( R_1 \sim O(v) \), \( R_2 \sim O(v^2) \).

To order \( v \), \( H_{eff} = PHP \) and hence

\[
\langle a | H_{eff} | b \rangle = \langle a | (h + v) | b \rangle. \]  

(A19)

To second order in \( v \), we have

\[
H_{eff} = [1 - \frac{1}{2} R^\dagger R][PHP + PHR + R^\dagger HP + R^\dagger HR][1 - \frac{1}{2} R^\dagger R]. \]  

(A20)

From \( R^\dagger HR \) we get,
\[ \langle a \mid R^\dagger HR \mid b \rangle = \sum_i \epsilon_i \frac{\langle a \mid v \mid i \rangle \langle i \mid v \mid b \rangle}{(\epsilon_a - \epsilon_i)(\epsilon_b - \epsilon_i)}. \] (A21)

From \(PHR\) and \(R^\dagger HP\) terms we get

\[ \sum_i \langle a \mid H \mid i \rangle \langle i \mid R_1 \mid b \rangle + \sum_i \langle a \mid R_1^\dagger \mid i \rangle \langle i \mid H \mid b \rangle = \sum_i \left[ \frac{\langle a \mid v \mid i \rangle \langle i \mid v \mid b \rangle}{\epsilon_a - \epsilon_i} + \frac{\langle a \mid v \mid i \rangle \langle i \mid v \mid b \rangle}{\epsilon_b - \epsilon_i} \right]. \] (A22)

From the normalization factors we get

\[ - \frac{1}{2} R^\dagger RHPH - \frac{1}{2} PHPR^\dagger R = - \frac{1}{2} (\epsilon_a + \epsilon_b) \sum_i \frac{\langle a \mid v \mid i \rangle \langle i \mid v \mid b \rangle}{(\epsilon_a - \epsilon_i)(\epsilon_b - \epsilon_i)} \] (A23)

Adding everything, to second order, we have,

\[ \langle a \mid H_{\text{eff}} \mid b \rangle = \frac{1}{2} \sum_i \langle a \mid v \mid i \rangle \langle i \mid v \mid b \rangle \left[ \frac{1}{\epsilon_a - \epsilon_i} + \frac{1}{\epsilon_b - \epsilon_i} \right]. \] (A24)

If \(a = b\), this expression reduces to the familiar second order energy shift.

Why Bloch formalism is preferred over Bloch-Horowitz formalism?

In the former, eigenstates of the effective Hamiltonian are ortho normalized projections of the original eigenstates. In the latter, they are not.

Consider two ortho normalized eigenstates of the original Hamiltonian \(\mid \psi_1 \rangle\) and \(\mid \psi_2 \rangle\) with \(\langle \psi_1 \mid \psi_2 \rangle = 0\). However, \(P \mid \psi_1 \rangle\) and \(P \mid \psi_2 \rangle\) need not be orthogonal, i.e., \(\langle \psi_1 \mid PP \mid \psi_2 \rangle = \langle \psi_1 \mid P \mid \psi_2 \rangle \neq 0\). Consider

\[ \langle \psi_1 \mid \psi_2 \rangle = \langle \psi_1 \mid P \mid \psi_2 \rangle + \langle \psi_1 \mid Q^2 \mid \psi_2 \rangle = \langle \psi_1 \mid P \mid \psi_2 \rangle + \langle \psi_1 \mid P^\dagger R^\dagger RP \mid \psi_1 \rangle. \] (A25)

Construct \(\tilde{\psi}_1 = [1 + R^\dagger R^\dagger]P \mid \psi_1 \rangle, \tilde{\psi}_2 = [1 + R^\dagger R^\dagger]P \mid \psi_2 \rangle\). Then

\[ \langle \tilde{\psi}_1 \mid \tilde{\psi}_2 \rangle = \langle \psi_1 \mid P \mid \psi_2 \rangle + \langle \psi_1 \mid P^\dagger R^\dagger R P \mid \psi_2 \rangle = \langle \psi_1 \mid \psi_2 \rangle. \] (A26)

**APPENDIX B: DETAILS OF NUMERICAL PROCEDURE**

**Parametrization:** The light-front variables are parametrized in the following ways in our numerical calculations. The full \(k\)-interval is divided into \(n_1\) quadrature points. \(k\) is defined by two different ways. One definition is

\[ k = \frac{u\Lambda m}{(1 - u^2)\Lambda + m}, \] (B1)

where \(\Lambda\) is the ultraviolet cutoff and \(u\)'s are the quadrature points lying between \(-1\) and \(+1\), so that \(k\) goes from \(-\Lambda\) to \(+\Lambda\). The other definition is

\[ k = \frac{u\Lambda m}{(1 - u^2)\Lambda + m}, \] (B1)
\[ k = \frac{1}{\kappa} \tan\left(\frac{u\pi}{2}\right), \]  

(B2)

here \( \kappa \) is a parameter that can be tuned to adjust the ultraviolet cutoff. The second definition (\( B2 \)) of \( k \) is very suitable for weak coupling calculations where we need maximum points to be concentrated near \( k = 0 \) and get better convergence than the first definition (\( B1 \)).

The longitudinal momentum fraction \( x \) ranges from 0 to 1. We divide all \( x \)- integrations in our calculations into two parts, \( x \) ranging from 0 to 0.5 and \( x \) ranging from 0.5 to 1 and discretize each \( x \)-interval into \( n \) 2 quadrature points with the parametrization

\[
x = \frac{1 + v + 2\eta(1 - v)}{4}, \quad \eta \leq x \leq 0.5, \tag{B3}
\]

\[
x = \frac{3 + v - 2\eta(1 + v)}{4}, \quad 0.5 \leq x \leq 1 - \eta, \tag{B4}
\]

where \( v \)'s are the Gauss-quadrature points lying between \(-1\) and \(+1\) and \( \eta(\rightarrow 0) \) is introduced to handle end-point singularities in \( x \) as mentioned before.

To handle the infrared diverging terms we put the cutoff \( |x - y| \geq \delta \) and at the end we take the limit \( \delta \rightarrow 0 \). Numerically, it means that the result should converge as one decreases \( \delta \) if there is no net infrared divergence in the theory.

Diagonalization: After discretisation, solving the integral equation becomes a matrix diagonalization problem. The diagonalization has been performed by using the packed storage LAPACK [20] routines DSPEVX for the reduced model (real symmetric matrix) and ZHPEVX for the full Hamiltonian (Hermitian matrix).
REFERENCES

[1] K. G. Wilson, T. S. Walhout, A. Harindranath, W. Zhang, R. J. Perry and S. D. Glazek, Phys. Rev. D 49, 6720 (1994) [hep-th/9401153].
[2] S. J. Brodsky, H. Pauli and S. S. Pinsky, Phys. Rept. 301, 299 (1998) [hep-ph/9705477]. For most recent work in Discrete Light Cone Quantization see J. Hiller, Application of Discrete Light Cone Quantization to Yukawa Theory in Four Dimensions, [hep-ph/0010061] and references therein.
[3] R. J. Perry, A. Harindranath and K. G. Wilson, Phys. Rev. Lett. 65, 2959 (1990).
[4] S. Glazek, A. Harindranath, S. Pinsky, J. Shigemitsu and K. Wilson, Phys. Rev. D 47, 1599 (1993).
[5] C. Bloch, Nucl. Phys. 6, 329 (1958).
[6] K. G. Wilson, Phys. Rev. D 2, 1438 (1970).
[7] S. D. Glazek and K. G. Wilson, Phys. Rev. D 48, 5863 (1993); 49, 4214 (1994); Similar flow equations were proposed by F. Wegner, Ann. Phys. (Leipzig) 3, 77 (1994).
[8] M. Brisudova and R. Perry, Phys. Rev. D 54, 1831 (1996) [hep-ph/9511443]; M. Brisudova, R. J. Perry and K. G. Wilson, Phys. Rev. Lett. 78, 1227 (1997) [hep-ph/9607280].
[9] W. -M. Zhang, Phys. Rev. D 56, 1528 (1997) [hep-ph/9705220].
[10] B. H. Allen and R. J. Perry, Phys. Rev. D 62, 025005 (2000) [hep-th/9908124]; R. D. Kylin, [hep-ph/01083129].
[11] S. D. Glazek and K. G. Wilson, Phys. Rev. D 57, 3558 (1998) [hep-th/9707028].
[12] R. J. Perry, Hamiltonian Light Front Field Theory and Quantum Chromodynamics, in Hadron Physics '94: Topics on the Structure and Interaction of Hadron Systems, Proceedings, Gramado, Brazil, edited by V. Hercovitz et al., (World Scientific, Singapore 1995) [hep-th/9407056].
[13] K. M. Bitar, Phys. Rev. D 7, 1184 (1973).
[14] M. Burkardt and A. Langnau, Phys. Rev. D 44, 1187 (1991).
[15] B. Binegar, J. Math. Phys. 23, 1511 (1982).
[16] M. Krautgartner, H. C. Pauli and F. Wolz, Phys. Rev. D 45, 3755 (1992).
[17] R. J. Perry, Annals Phys. 232 (1994) 116 [hep-th/9402013]; B. D. Jones and R. J. Perry, Phys. Rev. D 55, 7715 (1997).
[18] A. Tam, C. J. Hamer and C. M. Yung, J. Phys. G21, 1463 (1995).
[19] V. G. Kouris, J. Comp. Phys. 128, 1 (1996). [quant-ph/9510006].
[20] E. Anderson et al., LAPACK Users’ Guide, third edition (Society for Industrial and Applied Mathematics, Philadelphia, 1999). Available on the internet at the URL: http://www.netlib.org/lapack/lug/index.html.
FIGURES

FIG. 1. Cancellation of infrared divergence. Full line denotes the full Hamiltonian. (a) shows the cancellation of light-front infrared divergence by switching on and off the instantaneous interaction. Filled circles - without instantaneous interaction. (b) shows the cancellation of logarithmic infrared divergence by switching on and off the self energy term. Filled circles - without self energy. The parameters are $g = .2$, $\eta = .00001$, $m = 1$, $\kappa = 20$, $n_1 = 40$, $n_2 = 50$.

FIG. 2. The wavefunctions corresponding to the lowest four eigenvalues of the reduced model as a function of $x$ and $k$. The parameters are $g = .2$, $\eta = .00001$, $m = 1$, $\kappa = 10$, $n_1 = 46$, $n_2 = 74$. (a) Lowest state, (b) first excited state, (c) second excited state, (d) third excited state. The first and second excited states should be degenerate in the absence of violation of rotational symmetry.
### Table I: Variation with $\delta$ of the full Hamiltonian

The parameters are $n_1=40$, $n_2=50$, $\eta=0.00001$, $\kappa=20.0$ in $k = \frac{1}{\kappa} tan(\frac{\eta}{2})$.

| $g$ | $\delta$ | Eigenvalues ($M^2$) |
|-----|-----------|---------------------|
| 0.00001 | 4.0913 | 4.1113 | 4.1122 | 4.1181 | 4.1209 |
| 0.001 | 4.0913 | 4.1113 | 4.1122 | 4.1181 | 4.1209 |
| 0.2 | 0.001 | 4.0913 | 4.1113 | 4.1122 | 4.1181 | 4.1209 |
| 0.005 | 4.0901 | 4.1066 | 4.1099 | 4.1100 | 4.1112 |
| 0.01 | 4.0870 | 4.0972 | 4.0972 | 4.0973 | 4.0973 |
| 0.0001 | -187230.4 | -187225.4 | -186664.9 | -186664.8 | -31506.9 |
| 0.6 | 0.001 | -187230.4 | -187225.4 | -186664.9 | -186664.8 | -31506.9 |
| 0.005 | 1.9094 | 1.9415 | 3.1393 | 3.1399 | 4.5697 |
| 0.01 | 4.5735 | 4.7337 | 4.7667 | 4.7832 | 4.8277 |
| n1 | n2 | eigenvalues (lowest five) ($\kappa = 10.0$) |
|----|----|------------------------------------------|
| 20 | 20 | 4.08926 4.10605 4.10768 4.11061 4.11085 |
| 30 | 30 | 4.09045 4.10909 4.11038 4.11516 4.11699 |
| 40 | 30 | 4.09045 4.10913 4.11035 4.11524 4.11697 |
| 40 | 40 | 4.09102 4.11052 4.11154 4.11711 4.11951 |
| 40 | 50 | 4.09136 4.11133 4.11222 4.11811 4.12096 |
| 50 | 50 | 4.09136 4.11135 4.11219 4.11816 4.12095 |
| 50 | 60 | 4.09158 4.11188 4.11263 4.12189 4.12290 |
| 46 | 60 | 4.09158 4.11187 4.11264 4.11877 4.12189 |
| 46 | 66 | 4.09168 4.11212 4.11284 4.11905 4.12231 |
| 46 | 74 | 4.09179 4.11237 4.11305 4.11934 4.12276 |

| n1 | n2 | eigenvalues (lowest five) ($\Lambda = 20.0$) |
|----|----|------------------------------------------|
| 46 | 74 | 4.09179 4.11240 4.11301 4.11940 4.12273 |

TABLE II: Convergence of eigenvalue with $n_1$ and $n_2$ (reduced model). The parameters are $m=1.0$, $g=0.2$, $\eta = 0.00001$. 

23
| $g$   | eigenvalues          | 0.2   | 4.0918 (4.1227, 4.1235) | (4.1124, 4.1130) | 4.1194 (4.1298, 4.1303) | 4.1144 (4.1268, 4.1273) | 4.1214 (4.1227, 4.1235) | 4.1260 (4.1268, 4.1273) |
|-------|----------------------|-------|------------------------|------------------|------------------------|------------------------|------------------------|------------------------|
|       | This work            |       | 4.0925 ($l = 0$)      | 4.1144 ($l = 1$) | 4.1214 ($l = 0$)      | 4.1303 ($l = 1$)      | 4.1340 ($l = 3$)      |
|       | Koures (Ref. [19])   |       | 4.0925 ($l = 0$)      | 4.1144 ($l = 1$) | 4.1214 ($l = 0$)      | 4.1303 ($l = 1$)      | 4.1340 ($l = 3$)      |
|       |                      |       | 4.1260 ($l = 2$)      | 4.1303 ($l = 1$) | 4.1340 ($l = 3$)      |                        |                        |                        |
| 0.6   | This work            |       | 4.5856 (4.8767, 4.8816) | (4.7741, 4.7821) | 4.8390 (4.9094, 4.9184) | (4.9458, 4.9481)      |                        |                        |
|       | Koures (Ref. [19])   |       | 4.5806 ($l = 0$)      | 4.7777 ($l = 1$) | 4.8409 ($l = 0$)      | 4.9205 ($l = 1$)      | 4.9545 ($l = 3$)      |
|       |                      |       | 4.8827 ($l = 2$)      | 4.9205 ($l = 1$) | 4.9545 ($l = 3$)      |                        |                        |                        |

TABLE III: Reduced model. The parameters are $n_1=46$, $n_2=74$, $\eta = 0.00001$, $m=1.0$. $k = tan(q\pi/2)/\kappa$, $\kappa = 20.0$. Eigenvalues within () are $\pm l$ degenerate (broken) states.
| n1 | n2 | eigenvalues          |
|----|----|----------------------|
|    |    |                      |
| I  | 40 | 18.217 (30.702, 33.499) | 35.206 (39.955, 41.159) | (41.332, 43.271) | (44.134, 45.272) |
| 46 | 70 | 18.276 (30.774, 33.616) | 35.318 (40.106, 41.331) | (41.483, 43.477) | (44.375, 45.503) |
|    |    |                      |
| II | 40 | 18.980 (31.507, 34.219) | 35.826 (40.406, 41.888) | (41.921, 43.788) | (44.345, 45.163) |
| 46 | 70 | 19.008 (31.542, 34.319) | 35.935 (40.626, 42.031) | (42.088, 44.010) | (44.647, 45.780) |

TABLE IV: First few eigenvalues in the reduced model. The parameters are $g=5.0$, $m=1.0$, $\eta=0.00001$. (I) for the parametrization $k = u\Lambda m/((1 - u^2)\Lambda + m)$, $\Lambda = 40.0$. (II) for the parametrization $k = \tan(u\pi/2)/\kappa$, $\kappa = 10.0$. Eigenvalues within ( ) are $\pm l$ degenerate (broken) states.
FIG. 1. Cancellation of infrared divergence. Full line denotes the full Hamiltonian. (a) shows the cancellation of light-front infrared divergence by switching on and off the instantaneous interaction. Filled circles - without instantaneous interaction. (b) shows the cancellation of logarithmic infrared divergence by switching on and off the self energy term. Filled circles - without self energy. The parameters are $g = .2$, $\eta = .00001$, $m = 1$, $\kappa = 20$, $n_1 = 40$, $n_2 = 50$. 
FIG. 2. The wavefunctions corresponding to the lowest four eigenvalues of the reduced model as a function of $x$ and $k$. The parameters are $g = .2$, $\eta = .00001$, $m = 1$, $\kappa = 10$, $n_1 = 46$, $n_2 = 74$. (a) Lowest state, (b) first excited state, (c) second excited state, (d) third excited state. The first and second excited states should be degenerate in the absence of violation of rotational symmetry.