Approximate Completely Positive Semidefinite Rank

Paria Abbasi\textsuperscript{1}  Andreas Klingler\textsuperscript{2}  Tim Netzer\textsuperscript{1}

\textsuperscript{1}Department of Mathematics, University of Innsbruck
\textsuperscript{2}Institute for Theoretical Physics, University of Innsbruck

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What Is It About?

It is about a special matrix cone called the cone of completely positive semidefinite matrices ($\text{CPSD}$) and in particular, the related rank. The cone $\text{CPSD}$ is a canonical non-commutative generalization of the intensively-studied completely positive cone ($\text{CP}$) and both together are subcones of doubly non-negative cone ($\text{DNN}$).

\[ \text{CP}^n \subseteq \text{CPSD}^n \subseteq \text{DNN}^n \]
Doubly Non-Negative Cone

$$\mathcal{PSD}^n = \{ A \in S^n \mid \exists x_1, x_2, \cdots, x_n \in \mathbb{R}^m, \text{for some } m \geq 1; A_{ij} = \langle x_i, x_j \rangle \}$$
Doubly Non-Negative Cone

\[ \mathcal{P}_{SD}^n = \{ A \in S^n \mid \exists x_1, x_2, \ldots, x_n \in \mathbb{R}^m, \text{ for some } m \geq 1; A_{ij} = \langle x_i, x_j \rangle \} \]

\[ \mathcal{DNN}^n = \mathcal{P}_{SD}^n \cap \mathbb{R}^{n \times n} \]

\[ = \{ A \in S^n \mid \exists x_1, x_2, \ldots, x_n \in \mathbb{R}^m, \text{ for some } m \geq 1 \text{ with } \angle(x_i, x_j) \leq \pi/2; A_{ij} = \langle x_i, x_j \rangle \} \]

- Proper Cone
- smallest possible \( m = \text{rank}(A) \leq n \)
Completely Positive Cone

\[ A \in DNN_n \quad \xrightarrow{\exists \text{iso.}} \quad A \in CP^n \]

\[ CP_n = \{ A \in S_n | \exists x_1, x_2, \ldots, x_n \in \mathbb{R}_{m+}^n, \text{for some } m \geq 1; A_{ij} = \langle x_i, x_j \rangle \} \]

\[ \text{Proper Cone} \quad \text{CP}-\text{rank}(A) = \min \{ m \geq 1 | \exists x_1, \ldots, x_n \in \mathbb{R}_m^+; A = (\langle x_i, x_j \rangle)_{i,j=1}^n \} \]

\[ \leq n \leq 4 \quad [\text{Shaked-Monderer, Berman, Bo-}] \quad \leq \frac{(n+1)^2}{2} - 4 \quad n \geq 5 \quad [\text{mze, Jarre and Schachinger}] \]
Completely Positive Cone

\[ A \in \mathcal{DNN}^n \rightarrow \exists \text{iso.} \rightarrow A \in \mathcal{CP}^n \]

\[ \mathcal{CP}^n = \{ A \in \mathcal{S}^n \mid \exists x_1, x_2, \ldots, x_n \in \mathbb{R}_+^m, \text{ for some } m \geq 1; A_{ij} = \langle x_i, x_j \rangle \} \]
Completely Positive Cone

\[ \mathcal{CP}^n = \{ A \in S^n \mid \exists x_1, x_2, \cdots, x_n \in \mathbb{R}_+^m, \text{ for some } m \geq 1; A_{ij} = \langle x_i, x_j \rangle \} \]

- **Proper Cone**

- **\( CP\)-rank(\( A \)) = \min \{ m \geq 1 \mid \exists x_1, \cdots, x_n \in \mathbb{R}_+^m; A = (\langle x_i, x_j \rangle)_{i,j=1}^n \} \]
  \[
  \leq n \quad n \leq 4 \quad \text{[Shaked-Monderer, Berman, Bomze, Jarre and Schachinger]}
  \]
  \[
  \leq \left( \frac{n+1}{2} \right) - 4 \quad n \geq 5
  \]
Completely Positive Semidefinite Cone

\[ x_i = (x_{i1}, \ldots, x_{im})^T \in \mathbb{R}^m \rightarrow D_i = \text{Diag}(x_{i1}, \ldots, x_{im})\] and \[ x_i^T x_j = \text{tr}(D_i D_j)\]

\[ A = (\langle x_i, x_j \rangle)_{i,j=1}^n \in \mathcal{CP}^n \rightarrow A = (\langle D_i, D_j \rangle)_{i,j=1}^n\] and \[ D_i \in \mathcal{PSD}^m\]
Completely Positive Semidefinite Cone

\[ x_i = (x^i_1, \ldots, x^i_m)^T \in \mathbb{R}^m \rightarrow D_i = \text{Diag}(x^i_1, \ldots, x^i_m) \text{ and } x_i^T x_j = \text{tr}(D_i D_j) \]

\[ A = (\langle x_i, x_j \rangle)_{i,j=1}^n \in \mathcal{CP}^n \rightarrow A = (\langle D_i, D_j \rangle)_{i,j=1}^n \text{ and } D_i \in \mathcal{PSD}^m \]

\( \mathcal{CPSD}^n = \{ A \in S^n \mid \exists X_1, \ldots, X_n \in \mathcal{PSD}^m, \text{ for some } m \geq 1; A_{ij} = \underbrace{\langle X_i, X_j \rangle}_{\text{tr}(X_i X_j)} \} \)
The set of completely positive semidefinite matrices is a proper subset of the cone of positive semidefinite matrices, which in turn is a proper subset of the cone of positive matrices. It is closed for $n \in \{4\}$ (J.E. Maxfield and H. Minc), but not closed for $n \geq 10$ as shown by the non-closure of a certain affine section of the cone of completely positive semidefinite matrices (K. Dykema, V. I. Paulsen, and J. Prakash). It is still unknown for $n \in \{5, \ldots, 9\}$.
Set of $CPSD$ matrices is pointed, full-dimensional convex cone, but closed?
Set of $\mathcal{CPSD}$ matrices is pointed, full-dimensional convex cone, but closed? 

It is closed for $n \in [4]$ ($\mathcal{CP}^n = \mathcal{DN}^n$ [J.E. Maxfield and H. Minc]), not closed for $n \geq 10$ shown by non-closure of a certain affine section of $\mathcal{CPSD}^n$ [K. Dykema, V.I. Paulsen and J. Prakash] 

Still unknown for $n \in \{5, \cdots, 9\}!$
Completely Positive Semidefinite Cone

| Lemma |
|-------|
| For each $n \geq 10$, $CPSD^n$ is not a semialgebraic set. |
Lemma

For each $n \geq 10$, $\mathcal{CPSD}^n$ is not a semialgebraic set.

$$\mathcal{CPSD}^n = \bigcup_{r \in \mathbb{N}} \mathcal{CPSD}_{\leq r}^n = \bigcup_{r \in \mathbb{N}} \{ A = (\langle X_i, X_j \rangle)_{i,j}^n \mid X_1, \ldots, X_n \in \mathcal{PSD}^r \},$$

$$\mathcal{CPSD}_{\leq r}^n \subseteq \mathcal{CPSD}_{\leq r+1}^n \quad \forall r \geq 1$$

Lemma

For each $n, r \geq 1$, the set $\mathcal{CPSD}_{\leq r}^n$ is closed and semialgebraic.
Completely Positive Semidefinite Rank

For $A \in \text{CP}SD^n$,

$$\text{CP}SD\text{-rank}(A) = \min\{m \geq 1 | \exists \{X_i\}_{i=1}^n \subseteq \text{PSD}^m; A = (\langle X_i, X_j \rangle)_{i,j=1}^n\}$$
Completely Positive Semidefinite Rank

For $A \in \mathcal{CP}^n$,

$$\mathcal{CPSD}\text{-rank}(A) = \min \{ m \geq 1 \mid \exists \{X_i\}_{i=1}^n \subseteq \mathcal{PSD}^m; A = (\langle X_i, X_j \rangle)_{i,j=1}^n \}$$

$A \in \mathcal{CP}^n$, \quad $\mathcal{CPSD}\text{-rank}(A) \leq \mathcal{CP}\text{-rank}(A) \leq n^2/2 + O(n)$
Completely Positive Semidefinite Rank

For $A \in \mathcal{CPSD}^n$,

$$\text{CPSD-rank}(A) = \min\{m \geq 1| \exists \{X_i\}_{i=1}^n \subseteq \mathcal{PSD}^m; A = (\langle X_i, X_j \rangle)_{i,j=1}^n\}$$

$$A \in \mathcal{CP}^n, \quad \text{CPSD-rank}(A) \leq \text{CP-rank}(A) \leq \frac{n^2}{2} + O(n)$$

Is there any upper bound on the $\text{CPSD}$-rank of general $\mathcal{CPSD}^n$-matrices in terms of matrix size?

$$\left\{ \begin{array}{ll}
\leq n & 1 \leq n \leq 4 \\
\text{Unknown} & 5 \leq n \leq 9 \\
\text{No} & n \geq 10 
\end{array} \right.$$
Completely Positive Semidefinite Rank

For $A \in \mathbb{CPSD}^n$,

$$\mathbb{CPSD}\text{-rank}(A) = \min\{m \geq 1 | \exists \{X_i\}_{i=1}^n \subseteq \mathbb{PSD}^m; A = (\langle X_i, X_j \rangle)_{i,j=1}^n\}$$

$$A \in \mathbb{CP}^n, \quad \mathbb{CPSD}\text{-rank}(A) \leq \mathbb{CP}\text{-rank}(A) \leq n^2/2 + O(n)$$

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$$\begin{cases} \leq n & 1 \leq n \leq 4 \\ \text{Unknown} & 5 \leq n \leq 9 \\ \text{No} & n \geq 10 \end{cases}$$

For $n \geq 10$ the cpsd-rank of elements from $\mathbb{CPSD}^n$ is unbounded.
Completely Positive Semidefinite Rank

For $A \in \mathcal{CP}_{SD}^n$,

\[
\text{CP}_{SD}\text{-rank}(A) = \min \{ m \geq 1 | \exists \{X_i\}_{i=1}^n \subseteq \mathcal{P}SD^m; A = (\langle X_i, X_j \rangle)_{i,j=1}^n \}
\]

\[
A \in \mathcal{CP}^n, \quad \text{CP}_{SD}\text{-rank}(A) \leq \text{CP}\text{-rank}(A) \leq \frac{n^2}{2} + O(n)
\]

Is there any upper bound on the $\text{CP}_{SD}$-rank of general $\mathcal{CP}_{SD}^n$-matrices in terms of matrix size?

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\begin{cases}
\leq n & 1 \leq n \leq 4 \\
\text{Unknown} & 5 \leq n \leq 9 \\
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\end{cases}
\]

For $n \geq 10$ the cpsd-rank of elements from $\mathcal{CP}_{SD}^n$ is unbounded.

If such a bound exists $\rightarrow \exists r; \mathcal{CP}_{SD}^n = \mathcal{CP}_{SD}^{n \leq r}$ $\rightarrow$ $\mathcal{CP}_{SD}^n = \text{cl}(\mathcal{CP}_{SD}^{n})$ \underline{fails for $n \geq 10$}
Completely Positive Semidefinite Rank

For \( A \in \text{CPSD}^n \),

\[
\text{CPSD-rank}(A) = \min\{m \geq 1 \mid \exists \{X_i\}_{i=1}^n \subseteq \text{PSD}^m; A = (\langle X_i, X_j \rangle)_{i,j=1}^n\}
\]

\[
A \in \text{CP}^n, \quad \text{CPSD-rank}(A) \leq \text{CP-rank}(A) \leq n^2/2 + O(n)
\]

Is there any upper bound in terms of matrix size on the \( \text{CPSD} \)-rank of general \( \text{CPSD}^n \)-matrices?

\[
\begin{cases}
\leq n & 1 \leq n \leq 4 \\
\text{Unknown} & 5 \leq n \leq 9 \\
\text{No} & n \geq 10
\end{cases}
\]

For \( n \geq 10 \) the cpsd-rank of elements from \( \text{CPSD}^n \) is unbounded.

How about the approximate case? Can we find an approximation of the \( \text{CPSD} \)-matrices of relatively small \( \text{CPSD} \)-rank?
Theorem. Let $M = (\langle A_i, A_j \rangle)_{i,j=1}^n \in \mathcal{CPSD}^n$, set $\ell := \max_i \text{tr}(A_i)$ and $L := \max_i M_{ii}$. Then for every $0 < \varepsilon < \frac{1}{2} \min\{\ell^2, L\}$ there exists some $N \in \mathcal{CPSD}^n$ with

$$\text{cpsd-rank}(N) \leq \min \left\{ n \left\lfloor \frac{9L\ell^2}{2\varepsilon^2} \right\rfloor, \frac{(6\ell)^4 \log \left( n \left\lfloor \frac{18L\ell^2}{\varepsilon^2} \right\rfloor + 1 \right)}{\varepsilon^2} \right\}$$

and

$$|M_{ij} - N_{ij}| < \varepsilon \quad \text{for all } i, j \in [n].$$
Theorem. Let $M = (\langle A_i, A_j \rangle)_{i,j=1}^n \in \mathbb{C}^{n \times n}$, set $\ell := \max_i \text{tr}(A_i)$ and $L := \max_i M_{ii}$. Then for every $0 < \varepsilon < \frac{1}{2} \min\{\ell^2, L\}$ there exists some $N \in \mathbb{C}^{n \times n}$ with

$$\text{cpsd-rank}(N) \leq \min \left\{ n \left[ \frac{9L\ell^2}{2\varepsilon^2} \right], \frac{(6\ell)^4 \log \left( n \left[ \frac{18L\ell^2}{\varepsilon^2} \right] + 1 \right)}{\varepsilon^2} \right\}$$

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- For the first upper bound we make use of the Approximate Carathéodory Theorem
Theorem. Let \( M = (\langle A_i, A_j \rangle)_{i,j=1}^{n} \in \mathcal{CPSD}^n \), set \( \ell := \max_i \text{tr}(A_i) \) and \( L := \max_i M_{ii} \). Then for every \( 0 < \varepsilon < \frac{1}{2} \min\{ \ell^2, L \} \) there exists some \( N \in \mathcal{CPSD}^n \) with

\[
\text{cpsd-rank}(N) \leq \min \left\{ n \left[ \frac{9L\ell^2}{2\varepsilon^2} \right], \frac{(6\ell)^4 \log \left( n \left[ \frac{18L\ell^2}{\varepsilon^2} \right] + 1 \right)}{\varepsilon^2} \right\}
\]

and

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|M_{ij} - N_{ij}| < \varepsilon \quad \text{for all } i, j \in [n].
\]

- For the first upper bound we make use of the *Approximate Carathéodory* Theorem
- Further, the second bound is obtained by applying the JL-Lemma
Theorem. Let $M = (\langle A_i, A_j \rangle)_{i,j=1}^n \in \mathcal{CPSD}^n$, set $\ell := \max_i \text{tr}(A_i)$ and $L := \max_i M_{ii}$. Then for every $0 < \varepsilon < \frac{1}{2} \min \{ \ell^2, L \}$ there exists some $N \in \mathcal{CPSD}^n$ with

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- For the first upper bound we make use of the *Approximate Carathéodory* Theorem
- Further, the second bound is obtained by applying the JL-Lemma
- Which of the bounds is better depends on our setup
  - fix $n$ and $\varepsilon$ getting smaller, first upper bound better
  - fix $\varepsilon$ and $n \to \infty$, second upper bound significantly smaller
Some Remarks and Examples

- First approximation procedure can be used to generate a completely positive approximation: for \( M = (\langle D_i, D_j \rangle)_{i,j=1}^n \in \mathcal{CP}^n \subseteq \mathcal{CPSD}^n \), the approximation \( N \) is completely positive and

\[
\mathcal{CPSD}-\text{rank}(N) \leq \mathcal{CP}-\text{rank}(N) \leq n \left[ \frac{9L\ell^2}{2\varepsilon^2} \right]
\]
Some Remarks and Examples

First approximation procedure can be used to generate a completely positive approximation: for \( M = (\langle D_i, D_j \rangle)_{i,j=1}^n \in \mathcal{CP}^n \subseteq \mathcal{CPSD}^n \), the approximation \( N \) is completely positive and

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\]

- \( M \in \mathcal{CP}^n \) with \( M_{ii} = 1 \) → Gram representation by non-negative unit vectors \( \{x_i\}_{i=1}^n \) with \( L = 1 \) and \( \ell = \max_i \|x_i\|_1 \)

\[
\mathcal{CP}\text{-rank}(N) \leq n \left\lfloor \frac{9 \max_i \|x_i\|_1^2}{2\varepsilon^2} \right\rfloor < \binom{n+1}{2} - 4
\]
Some Remarks and Examples

- First approximation procedure can be used to generate a completely positive approximation: for \( M = (\langle D_i, D_j \rangle)_{i,j=1}^n \in \mathcal{CP}^n \subseteq \mathcal{CP}^{SD}^n \), the approximation \( N \) is completely positive and

\[
\mathcal{CP}^{SD}\text{-rank}(N) \leq \mathcal{CP}\text{-rank}(N) \leq n \left\lfloor \frac{9L\ell^2}{2\varepsilon^2} \right\rfloor
\]

- \( M \in \mathcal{CP}^n \) with \( M_{ii} = 1 \rightarrow \) Gram representation by non-negative unit vectors \( \{x_i\}_{i=1}^n \) with \( L = 1 \) and \( \ell = \max_i \|x_i\|_1 \)

\[
\mathcal{CP}\text{-rank}(N) \leq n \left\lfloor \frac{9 \max_i \|x_i\|_1^2}{2\varepsilon^2} \right\rfloor < \left( \frac{n + 1}{2} \right) - 4
\]

- \( \mathcal{I} = (\langle E_{ii}, E_{jj} \rangle) \in \mathcal{CP}^{SD}^n \), \( \mathcal{CP}^{SD}\text{-rank}(\mathcal{I}) = n, \ell = L = 1 \)

\[
\mathcal{CP}^{SD}\text{-rank}(N) \leq \frac{6^4 \log \left( n \left\lfloor \frac{18}{\varepsilon^2} \right\rfloor + 1 \right)}{\varepsilon^2} < n
\]
Let $M \in \mathcal{CPSD}^n$ with Gram representation consisting of orthogonal projections $P_1, \ldots, P_n \in \mathcal{PSD}^m$. Further set $L := \max_i M_{ii}$. Then for all $0 < \varepsilon < \frac{1}{2} L^2$ there exists some $N \in \mathcal{CPSD}^n$ with

$$\text{CPSD-rank}(N) \leq \min \left\{ n \left\lceil \frac{9L^3}{2\varepsilon^2} \right\rceil, \frac{(6L)^4 \log \left( n \left\lceil \frac{18L^3}{\varepsilon^2} \right\rceil + 1 \right)}{\varepsilon^2} \right\}$$

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