ERG AND SCHWINGER-DYSON EQUATIONS
– COMPARISON IN FORMULATIONS AND APPLICATIONS –

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The advantageous points of ERG in applications to non-perturbative analyses of quantum field theories are discussed in comparison with the Schwinger-Dyson equations. First we consider the relation between these two formulations specially by examining the large N field theories. In the second part we study the phase structure of dynamical symmetry breaking in three dimensional QED as a typical example of the practical applications.

1 Introduction
It has been a desire to have practically useful frameworks for analytical studies of non-perturbative dynamics of field theories. So far the Schwinger-Dyson (SD) equations have been mostly used in many aspects (except for some supersymmetric theories). The gap equations and also the loop equations are known as the typical examples of them.

However Exact Renormalization Group (ERG) is expected to offer a equally or more powerful non-perturbative method. These two formulations share the following common features. The correlation functions are given as those solutions. Regularization is necessary to define the equations. The full equations are given in functional forms, therefore approximations are inevitable in practical calculations.

On the other hand the characteristics of the ERG are as follows. ERG gives the RG flows for the effective couplings, while the SD equations give the order parameters in terms of the bare couplings. Therefore the phase diagrams are given in the effective (renormalized) coupling space in the ERG formalism. Also the fixed points, the critical exponents and the renormalized trajectories (continuum limit) are directly evaluated. It should be also noted that the ERG allows systematic improvement of approximations: the derivative expansion, while the systematic treatment has not been known in general for the SD equations.

In this paper we are going to see the interrelations between these formulations and compare the characteristics more closely. In the first part we discuss the relations in formulations by considering large N field theories. In the second part the dynamical symmetry breaking in QED is examined for the comparison in the practical applications.

2 SD and ERG in Large N Field Theories
2.1 The large N vector model
Let us begin with the large N vector models. It is well known that this class of models may be solved by using the so-called auxiliary field method, or the Hubbard-
Stratonovich transformation. Here, however, we restrict ourselves to the ERG and SD approaches. The euclidean bare action is given by

\[ S_b = \int d^4x \left( \frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{g}{8N} (\phi^2)^2 \right), \]  

where \( \phi^2 = \phi^a \phi^a \) \((a = 1, \ldots, N)\). The cutoff effective action \( \Gamma[\phi] \) satisfies the so-called Legendre flow equation given (apart from the canonical scaling part) by

\[ \frac{\partial \Gamma_\Lambda[\phi]}{\partial \Lambda} = \frac{1}{2} \text{Tr} \left( \frac{\partial \Delta^{-1}}{\partial \Lambda} \left[ \Delta^{-1} \delta_{ab} + \frac{\delta^2 \Gamma_\Lambda[\phi]}{\delta \phi^a \delta \phi^b} \right]^{-1} \right), \]  

where \( \Delta(p) \) is the cutoff propagator.

In the large N limit we obtain the ERG equation

\[ \frac{\partial \Gamma_\Lambda[\rho]}{\partial \Lambda} = \frac{1}{2} \text{Tr} \left( \frac{\partial \Delta^{-1}}{\partial \Lambda} \left[ \Delta^{-1} + \frac{\delta \Gamma_\Lambda[\phi]}{\delta \rho} \right]^{-1} \right) \]  

by redefining \( \Gamma_\Lambda \rightarrow N \Gamma_\Lambda \) and \( \rho = \phi^2 / 2N \). It is seen that the flow equation for the potential is exactly extracted in this case as

\[ \frac{\partial V(\rho)}{\partial \Lambda} = \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \frac{\partial \Delta^{-1}(p)}{\partial \Lambda} \frac{1}{\Delta^{-1}(p) + V'(\rho)}. \]  

Below we see explicitly that the solution of the ERG equation indeed satisfies the SD equation also.

The SD equations are nothing but the identities between correlation functions followed by the trivial equation

\[ 0 = \int \mathcal{D}\phi \frac{\delta}{\delta \phi^a(x)} O[\phi] e^{-S[\phi]} = \int \mathcal{D}\phi \left( \frac{\delta O[\phi]}{\delta \phi^a(x)} - \frac{\delta S[\phi]}{\delta \phi^a(x)} O[\phi] \right) e^{-S[\phi]} . \]  

However it is necessary to perform regularization in order to make the equations meaningful. Here we introduce cutoff to the propagator just as done for the Legendre flow equations:

\[ S[\phi] = -\frac{1}{2} \phi^a \cdot \Delta^{-1} \cdot \phi^a - S_b^{\text{int}}[\phi] \]  

If we set \( O = \phi^b(y) \) for example, then we obtain

\[ (\Delta^{-1} + m^2)\langle \phi^a(x) \phi^b(y) \rangle + \frac{g}{2N} \langle \phi^2(x) \phi^a(x) \phi^b(y) \rangle = \delta^{ab} \delta^{(d)}(x - y). \]  

These equations are not closed, therefore we need to handle the infinite set of the SD equations in general. The factorization property in the large N limit reduces the above SD equation to the one for the two point functions:

\[ \left( \Delta^{-1} + m^2 + g \rho + \frac{g}{2N} \langle \phi^2(x) \rangle_c \right) \langle \phi^a(x) \phi^b(y) \rangle = \delta^{ab} \delta^{(d)}(x - y), \]  

where the subscript c indicates the connected part. It is seen from this equation that the mass function defined by \( \Sigma = m^2 + g \rho + (g/2N) \langle \phi^2(x) \rangle_c \) satisfies the so-called gap equation

\[ \Sigma = m^2 + g \rho + \frac{g}{2} \int \frac{d^d p}{(2\pi)^d} \frac{1}{\Delta^{-1} + \Sigma}. \]
Here it should be noted that the mass function is related to the cutoff effective potential as

\[ \Sigma(\rho) = \frac{\partial^2 V(\phi)}{\partial \phi^2} = \frac{\partial V(\rho)}{\partial \rho}. \]  

(10)

By solving the gap equation the mass function is given in terms of \( \rho \) and the cutoff scale \( \Lambda \). The derivatives \( \partial \Sigma / \partial \rho \) and \( \partial \Sigma / \partial \Lambda \) satisfies the following equations,

\[ \frac{\partial \Sigma}{\partial \rho} = g - \frac{g}{2} \int \frac{d^4p}{(2\pi)^d} \left( \frac{1}{(\Delta^{-1} + \Sigma)^2} \right) \frac{\partial \Sigma}{\partial \rho}, \]  

(11)

\[ \frac{\partial \Sigma}{\partial \Lambda} = -\frac{g}{2} \int \frac{d^4p}{(2\pi)^d} \left( \frac{1}{(\Delta^{-1} + \Sigma)^2} \right) \left( \frac{\partial \Delta^{-1}}{\partial \Lambda} + \frac{\partial \Sigma}{\partial \Lambda} \right). \]  

(12)

Further we may rewritten the scale dependence of the mass function as

\[ \frac{\partial \Sigma}{\partial \Lambda} = \frac{g}{2} \int \frac{d^4p}{(2\pi)^d} \frac{\partial \Delta^{-1}}{\partial \Lambda} \left( \frac{1}{\Delta^{-1} + \Sigma} \right) \frac{\partial \Sigma}{\partial \rho}, \]  

(13)

by using Eq. (9) and (11). This differential equation is found to be just equivalent to the ERG equation for the effective potential (4). Thus the equivalence in the large N limit is shown.

2.2 The Gross-Neveu model

In this case as well the equivalence of ERG and SD may be shown by repeating the similar argument given above. However we should pay attention to that the effective potential treated by the ERG formulation is not the potential in terms of the order parameter given by the fermion composite. Let us mention the ERG analysis briefly also for the later conveniences.

We start with the bare action given by

\[ S_b = \int d^4x \bar{\psi}_i \Delta^{-1} \psi^i - \frac{G}{2N} (\bar{\psi}_i \psi^i)^2, \quad (i = 1, \cdots, N), \]  

(14)

where we introduced the cutoff propagator \( \Delta(p) = C(p^2/\Lambda^2)/ig \). Note that chiral symmetry is preserved by this regularization. In the large N limit the ERG equation for the cutoff effective potential \( V(\psi, \bar{\psi}; \Lambda) \) may be exactly derived again. By redefining \( V \to NV \) it is found to be

\[ \frac{\partial V(\sigma)}{\partial \Lambda} = \int \frac{d^4p}{(2\pi)^d} \text{Tr} \left( \frac{\partial \Delta^{-1}}{\partial \Lambda} \left( \frac{1}{\Delta^{-1} + V'(\sigma)} \right) \right), \]  

(15)

where \( \sigma \) denotes a product of the classical fields, \( \bar{\psi} \psi \). It should not be confused with \( \langle \bar{\psi} \psi \rangle \).

Now our interest is to see dynamical mass generation in this model. In the SD approach the mass function \( \Sigma \) defined by

\[ \langle \psi^i_\alpha(p) \bar{\psi}^j_\beta(-p) \rangle = \delta_{\alpha\beta} \delta_j^i \frac{1}{\Delta^{-1}(p) + \Sigma(p)}, \]  

(16)
is examined. Here it should be noted that we first assume the order parameter apriori to see the critical phenomena in the SD approach. In the large N limit the gap equation is found to be

\[ \Sigma = -G\sigma - G \int \frac{d^dp}{(2\pi)^d} \text{Tr} \frac{1}{\Delta^{-1} + \Sigma}, \]  

(17)

where we kept the classical fields \( \sigma \). The critical (bare) coupling for the dynamical chiral symmetry breaking is found by (non-)existence of non-trivial solutions for this gap equation.

We may derive the scale dependence of the mass function by considering \( \partial \Sigma / \partial \sigma \) and \( \partial \Sigma / \partial \Lambda \) to the gap equation. The resultant ERG for the mass function turns out to be

\[ \frac{\partial \Sigma}{\partial \Lambda} = - \int \frac{d^dp}{(2\pi)^d} \text{Tr} \left( \frac{\partial \Delta^{-1}}{\partial \Lambda} \frac{1}{(\Delta^{-1} + \Sigma)^2} \right) \frac{\partial \Sigma}{\partial \sigma}. \]  

(18)

Once we note that the mass function is related with the effective potential as \( \Sigma(\sigma) = dV/d\sigma \), then the equivalence of these formulations is readily seen. However the ERG has a great advantage to find the phase structures and also the critical exponents compared with the SD approaches. If we perform the operator expansion of the effective potential into

\[ V(\sigma; \Lambda) = -\frac{1}{2\Lambda^{d-2}} G(\Lambda)\sigma^2 + \frac{1}{8\Lambda^{d-4}} G_8(\Lambda)\sigma^4 + \cdots, \]  

(19)

then the beta function for each coupling is derived by substituting into Eq. (15). It is found that the effective 4-fermi coupling \( G \) is subject to the ERG equation isolated from other couplings:

\[ \beta_G = \Lambda \frac{dG}{d\Lambda} = (d-2)G - AG^2, \]  

(20)

where \( A \) is a cutoff scheme dependent constant. This beta function has two fixed points: \( G^* = 0 \) (IR attractive) and \( G^* = (d-2)/A \) (IR repulsive). The IR repulsive fixed point gives the critical coupling of the chiral symmetry breaking and there are found to be broken phase and unbroken one. It is also quite easy to see the anomalous dimensions of the operators \( \bar{\psi}\psi \), \( (\bar{\psi}\psi)^2 \) and so on.

However it may seem curious how the dynamical mass can be generated in the broken phase in the ERG formulation. Indeed the chiral symmetry prohibits the mass term and any symmetry breaking operators to appear in the effective potential \( V(\sigma, \Lambda) \). It is found that the operator expansion given by Eq. (19) is not proper in order to see that composite order parameter.

The dynamical mass is rather evaluated as the mass function at \( \sigma = 0 \):

\[ m_{\text{eff}} = \lim_{\Lambda \to 0} \Sigma(\sigma, \Lambda)|_{\sigma = 0} = \lim_{\Lambda \to 0} V'(\sigma, \Lambda)|_{\sigma = 0}. \]  

(21)

If we solve the ERG equation for the effective potential \( V(\sigma; \Lambda) \), then it is found that the potential is evolved to be non-analytic at the origin due to the IR singularity of massless fermion loops. On the other hand the fermion condensate \( \langle \bar{\psi}\psi \rangle \) is also

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\( ^a \) Actually the gap equation leads to also the solutions corresponding to unstable vacua. Further study is needed to understand these solutions in the ERG point of view.
obtained by introducing the bare mass $m_0$ in the original action. Since $m_0$ plays a role of source for the fermion composite, $\langle \bar{\psi}\psi \rangle$ is evaluated by

$$\langle \bar{\psi}\psi \rangle = \lim_{\Lambda \to 0} \frac{\partial V(\sigma, m_0, \Lambda)}{\partial m_0} \bigg|_{\sigma = 0^+}. \quad (22)$$

Thus we need to analyze the whole potential, not only the four fermi coupling, to see generation of the order parameters of the dynamical symmetry breaking. However the phase structure of dynamical symmetry breaking is immediately found out from the RG flows of couplings.

2.3 Formal equivalence

It is also shown formally that the solutions of the ERG equation necessarily satisfy the SD equation in generic field theories. First we define the cutoff generating functional

$$Z_{\Lambda}[J] = e^{W_{\Lambda}[J]} = \int \mathcal{D}\phi e^{-S_{\Lambda} + J \cdot \phi} \quad (23)$$

by using the regularized action

$$S_{\Lambda}[\phi] = \int \int d^d x d^d y \frac{1}{2} \phi(x) \Delta(x - y; \Lambda) \phi(y) + S[\phi]. \quad (24)$$

The variation of the generating functional under shift of the cutoff $\Lambda$ is given as

$$\frac{\partial}{\partial \Lambda} Z_{\Lambda}[J] = \frac{1}{2} \int \int d^d x d^d y \frac{\delta}{\delta J(x)} \frac{\delta}{\delta J(y)} \Delta \frac{\partial}{\partial \Lambda} \frac{\delta}{\delta J(y)} Z_{\Lambda}[J], \quad (25)$$

which may be rewritten also into the Polchinski equation. On the other hand the general SD equations are represented in terms of the source function as

$$0 = \left( J(x) - \frac{\delta S_{\Lambda}}{\delta \phi(x)} \left[ \frac{\delta}{\delta J} \right] \right) Z_{\Lambda}[J]. \quad (26)$$

Now suppose that the generating functional $Z_{\Lambda}[J]$ satisfies the SD equation derived for the action $S_{\Lambda}$. Under variation of the scale $\Lambda \to \Lambda + \delta \Lambda$, the generating functional is shifted by

$$Z_{\Lambda + \delta \Lambda}[J] = Z_{\Lambda}[J] + \frac{1}{2} \int \int d^d x d^d y \frac{\delta}{\delta J(x)} \frac{\delta}{\delta J(y)} Z_{\Lambda}[J] \delta \Lambda. \quad (27)$$

Then we may show that the generating functional at scale $\Lambda + \delta \Lambda$ indeed satisfies the SD equation deduced from the action $S_{\Lambda + \delta \Lambda}$:

$$\left( J(x) - \frac{\delta S_{\Lambda + \delta \Lambda}}{\delta \phi(x)} \left[ \frac{\delta}{\delta J} \right] \right) Z_{\Lambda + \delta \Lambda}[J] = 0, \quad (28)$$

by noting that $Z_{\Lambda}[J]$ satisfies Eq. (26). If we perform both UV and IR cutoff to define the generating functional, then the solutions of the SD equation are obtained by removing the IR cutoff. Therefore it is seen generally that the solutions of the ERG equation at the IR limit should satisfies the SD equation.

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* It has been found that if we introduce the composite operators corresponding to the fermion condensate by extending the theory space, then the operator expansion scheme works quite well. The author became aware after the conference that the same argument has been already given by Ellwanger et al.
2.4 The large N matrix model

In this subsection we consider to apply the ERG method to the large N matrix model given by the action

$$S = N \int d^2x \left( \frac{1}{2} \text{Tr}[(\partial_\mu \phi)^2] + \frac{1}{2} m^2 \text{Tr}[\phi^2] + \frac{g}{4} \text{Tr}[\phi^4], \right)$$

(29)

where $\phi$ is a $N \times N$ hermitian matrix. The SD equations for the matrix model are known as the loop equations. For example we obtain in the large N limit

$$(-\partial_x + m^2) \langle \text{Tr}\phi(x)\phi(y) \rangle + g \langle \text{Tr}\phi^3(x)\phi(y) \rangle = \frac{1}{N} \sum_{p=0}^{n-1} \langle \text{Tr}\phi^p(x) \rangle \langle \text{Tr}\phi^{n-p-1}(x) \rangle.$$

(30)

However it has not been known how to treat such equations.

Let us consider the ERG in the LPA approximation, in which radiative corrections including the derivatives are discarded. The generic Legendre flow equation leads to the ERG for the effective potential $V(\phi)$ as

$$\frac{\partial V}{\partial \Lambda} = \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \frac{\partial \Delta^{-1}}{\partial \Lambda} \text{Tr} \left[ \Delta^{-1} \delta_{ijkl} + \frac{\delta^2 V}{\delta \phi_i \delta \phi_k} \right]^{-1},$$

(31)

apart from the canonical scaling terms. As is seen later on the corrections are limited to the single trace operators in the large N limit. Therefore we may suppose that the potential is consist of single trace operators:

$$V(\phi) = \sum_{n=1}^{\infty} a_n(\Lambda) \text{Tr}\phi^{2n}.$$  

(32)

Then in the second derivatives in Eq. (31),

$$\frac{\delta^2 V}{\delta \phi_i \delta \phi_k} = \sum_{n=1}^{\infty} 2na_n(\Lambda) \left\{ \delta_{il}(\phi^{2n-2})_{jk} + (\phi^{2n-2})_{li} \delta_{jk} + \sum_{p=1}^{2n-3} (\phi^p)_{li}(\phi^{2n-p-2})_{jk} \right\},$$

(33)

only the first and the second terms can contribute to the radiative corrections in the large N limit. Therefore the trace in the ERG may be evaluated as

$$\text{Tr} \left[ \Delta^{-1} + \frac{\delta^2 V}{\delta \phi \delta \phi} \right]^{-1} = 2N \text{Tr} \left( \Delta^{-1} + \sum_{n=1}^{\infty} 2na_n \phi^{2n-2} \right)^{-1}. $$

(34)

Thus only the single trace operators are found to appear through the corrections. In Fig. 1 the large N leading corrections are shown schematically. It is realized also that solving the ERG equation generates planar diagrams.

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* In the large N limit the multi trace operators are not generated from the single trace ones through radiative corrections. Therefore let us start with this type of actions. General cases containing the multi trace operators may be analyzed similarly.

* Except for the 0 dimensional model.
We may deduce the beta functions for the couplings $a_n$ as follows. If we define a function depending on $a_n$ as

$$v(y) = \sum_{n=1}^{\infty} a_n y^n,$$

then the beta functions are obtained by expanding the equation,

$$\Lambda \frac{\partial v}{\partial \Lambda} = N \int \frac{d^4p}{(2\pi)^4} \Lambda \frac{\partial \Delta^{-1}}{\partial \Lambda} \left[ \Delta^{-1} + 2v'(y) \right]^{-1},$$

which is found to be identical to the flow equation for the effective potential of the large N vector model. Therefore there are found a critical surface dividing two phases. Contrary to the vector model, however, the LPA is not exact, since there are additional corrections from the derivative interactions. Namely the large N matrix model and the vector model belong to different universality classes. Here we would like to remark that ERG is able to handle the matrix models as well by applying the derivative expansion.

3 Dynamical Symmetry Breaking in QED$_3$

Now we discuss the feature of ERG in applications to the dynamical symmetry breaking by gauge interactions. Here we consider especially the application to QED$_3$, which serves a typical example.

Dynamical symmetry breaking by gauge interaction is one of the most interesting non-perturbative phenomena not only in condensed matter physics but also in particle physics. In practice many works using the SD equations has been done mostly in the applications to such problems, for example, chiral symmetry breaking in 4 dimensional gauge theories, color superconductivity in high density QCD and so on.

In the SD approach we have to assume the order parameter essential for the dynamical symmetry breaking. For the chiral symmetry breaking we examine the equation for the mass function. In general, however, we do not know apriori which symmetry should be broken dynamically. This problem becomes important in the case that there are more than two phases. Indeed we will face with such a situation in considering QED$_3$. It should be mentioned also that the ladder approximation
scheme, which has been frequently used in the SD approach, suffers from large gauge parameter dependence. Moreover, if we try to improve the approximation so as to obtain gauge independent results, we are obliged to treat much more complicated equations. In this section we examine QED$_3$ by the ERG equations in comparing with such features in the SD approach.

Let us consider QED$_3$ with $N$ flavors of 4 component spinors $\psi^i$ ($i = 1, \cdots, N$) without the Chern-Simons term. The bare lagrangian is given by

\[ S_b = \sum_{i=1}^{N} \bar{\psi}_i (i \partial x + e A) \psi^i + \frac{1}{4} F^2_{\mu \nu} - \frac{1}{2 \xi} (\partial_\mu A_\mu)^2. \]  

(37)

Here we use the 4 by 4 $\gamma$ matrices given by

\[ \gamma^0 = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \]  

(38)

We also introduce $\gamma^5 = \gamma^0 \gamma^1 \gamma^2 \gamma^3$ and $\tau = -i \gamma^5 \gamma^3$. This action is invariant under the global U(2N) and also parity symmetry, which is made transparent by reformulating in terms of 2n 2-component spinors $\chi^I$, ($I = 1, \cdots, 2N$):

\[ \psi^i = \begin{pmatrix} \chi^i \\ i \chi^{i+N} \end{pmatrix}, \quad \bar{\psi}_i = (\chi^i \sigma_3, -\chi^{i+N} \sigma_3) = (\chi^i, \bar{\chi}^{i+N}) \tau. \]  

(39)

The 2-component field are transformed by the U(2N) matrix $U$ as $\chi^I \rightarrow \chi'^I = U^I_J \chi^J$. The parity transformation is defined by $\psi \rightarrow \psi' = i \gamma^5 \gamma^3 \psi$. Therefore $\bar{\psi}_i \gamma^\mu \psi^j = \bar{\chi}_I \gamma^\mu \chi^J$ is invariant under the both symmetry. The ordinary mass operator, $\bar{\psi}_i \psi^i = \bar{\chi}_I \chi^I$, is parity even but not invariant under U(2N) transformation. If this operator acquires a non-vanishing vacuum expectation value, then U(2N) is spontaneously broken to U(N)$\times$U(N). Thus we may regards this U(2N) symmetry as a sort of chiral transformation. While we find a U(2N) invariant operator, $\bar{\psi}_I \tau^I \psi^i = \bar{\chi}_I \chi^I$, which is parity odd in turn. Therefore non-vanishing expectation value of this operator leads to spontaneous breakdown of the parity symmetry. However it is expected from Vafa-Witten’s theorem that parity is never broken in QED$_3$.

In section 1 we saw that the RG flows of the effective four fermi interactions are important to distinguish the phases. All the local four-fermi operators invariant under U(2N) and parity transformations are listed up as follows:

\[ \mathcal{O}_P = (\bar{\psi}_I \tau^I \psi^i)^2 = (\bar{\chi}_I \chi^I)^2 \]

\[ \mathcal{O}_V = (\bar{\psi}_I \gamma^\mu \psi^i)^2 = (\bar{\chi}_I \gamma^\mu \chi^J)^2 \]

\[ \mathcal{O}_S = \bar{\psi}_i \psi^j \bar{\psi}_j \psi^i - \bar{\psi}_i \gamma^3 \psi^j \bar{\psi}_j \gamma^3 \psi^i - \bar{\psi}_i \gamma^5 \psi^j \bar{\psi}_j \gamma^5 \psi^i + \bar{\psi}_i \tau^I \psi^j \bar{\psi}_j \tau^I \psi^i \]

\[ = 2 \bar{\chi}_I \chi^I \bar{\chi}_J \chi^J \]

\[ \mathcal{O}_{V'} = \bar{\psi}_i \gamma^\mu \psi^j \bar{\psi}_j \gamma^\mu \psi^i - \bar{\psi}_i \gamma^3 \gamma^\mu \psi^j \bar{\psi}_j \gamma^3 \gamma^\mu \psi^i \]

\[ - \bar{\psi}_i \gamma^5 \gamma^\mu \psi^j \bar{\psi}_j \gamma^5 \gamma^\mu \psi^i + \bar{\psi}_i \tau^I \psi^j \bar{\psi}_j \tau^I \psi^i \]

\[ = 2 \bar{\chi}_I \gamma^\mu \chi^J \bar{\chi}_J \gamma^\mu \chi^I \]  

(40)

These operators are induced by radiative corrections. However it is found by the Fierz transformation that two of them are independent. We choose $\mathcal{O}_S$ and $\mathcal{O}_P$ as
the independent ones and always rewrite others by using the Fierz transformation, whenever these are induced.

Before going into the ERG analysis of QED\textsubscript{3}, let us briefly summarize the results obtained by other methods. Appelquist et al\textsuperscript{15} examined the SD equations for the chiral symmetry breaking mass in the ladder approximation and found the novel phase transition. They claimed that there are two phases depending only on the number of flavors \( N \) and if \( N \) is less than the critical value \( N_c = 32/\pi^2 \sim 3.2 \), then the chiral symmetry is spontaneously broken. In any cases there are a single phase chirally broken or unbroken. In this analysis they approximated the photon self-energy by its large \( N \) leading part. Also it is assumed that parity is not broken a la Vafa-Witten’s theorem. After this work many works have been devoted to improvement of the approximations and the similar results have been obtained\textsuperscript{16}. On the other hand the MC simulation of the noncompact lattice QED also has been examined. E.Dagotto et al\textsuperscript{17} reported the qualitatively same results as the above. The critical number was estimated as \( N_c = 3.5 \pm 0.5 \).

Now we consider to apply ERG to this system. We adopt the following scheme of approximation. First we truncate the set of induced operators and restrict the effective lagrangian to

\[
\mathcal{L}_{\text{eff}} = \sum_{i=1}^{N} \bar{\psi}_i (\partial + eA) \psi^i + \frac{1}{4} F_{\mu\nu}^2 - \frac{1}{2\xi} (\partial_{\mu} A_{\mu})^2 - \frac{G_S}{2} \mathcal{O}_S - \frac{G_P}{2} \mathcal{O}_P. \tag{41}
\]

Therefore the RG flows are given in the three dimensional coupling space spanned by \((e^2, G_S, G_P)\). In the RG approach we may naturally incorporate all the theories with the same symmetries, namely QED\textsubscript{3} with four-fermi interactions\textsuperscript{18}, simultaneously.

Since our purpose is to see the chiral phase structure of QED\textsubscript{3}, we adopt sharp momentum cutoff preserving the chiral U(2\( N \)) and parity symmetry, at the cost of the gauge invariance. Here we simply discard the gauge non-invariant corrections, e.g. photon mass induced in such regularization scheme. Of course the gauge invariant scheme\textsuperscript{12} is preferable to see non-perturbative dynamics by gauge interactions. Here we would postpone the gauge invariant analysis to the future studies. Rather we use one-loop perturbative results for the gauge beta function and the fermion anomalous dimension as the first step of approximation.

Thus we may simply solve the RG equations for the four-fermi couplings \((G_S, G_P)\) coupled to the gauge beta function,

\[
\frac{de^2}{dt} = e^2 - \frac{N}{8} e^4, \tag{42}
\]

where \( t = \ln(\Lambda_0/\Lambda) \). The first term represents the canonical scaling of the gauge coupling with dimension one half. It should be noted that there appears an IR stable fixed point (FP) at \( e^2 = e^{2*} = 8/N \). As is seen later on this FP plays an essential role for the novel phase transition. Indeed the FP obtained by the perturbative beta function may not be reliable for \( N \) not large. Therefore our analysis as well as the SD analyses is not totally confidential. In this point also the ERG applicable for the non-perturbative RG equations for the gauge couplings\textsuperscript{12} is strongly desired.
fixed point is assumed, our results are supposed to be qualitatively correct.

The beta functions for the four fermi couplings $G_S$ and $G_P$ are evaluated by summing up the corrections described in Fig. 2. In each four-fermi vertex of the diagram the operator $O_S$ or $O_P$ is inserted. It has been found also that the ladder approximation frequently used in the SD approach can be reproduced by restricting them to the corrections in the first two lines.

The beta functions for the four-fermi couplings in the ladder approximation turn out to be

$$\dot{G}_S = -G_S + \frac{1}{\pi^2} \left[ G_S^2 - G_S G_P + \frac{1}{3} G_P^2 + 2e^2 G_S - \frac{4}{3} e^2 G_P + \frac{2}{3} e^4 \right],$$

$$\dot{G}_P = -G_P + \frac{1}{\pi^2} \left[ -\frac{1}{6} G_S^2 - \frac{2}{3} e^2 G_P - \frac{2}{3} e^4 \right], \tag{43}$$

in the Landau gauge $\xi = 0$. It is found that the flow equations for $G'_S = G_S - (1/2)G_P$ and $G'_P$ are completely decoupled. Therefore we may solve the coupled equations for $G'_S$:

$$\dot{G}'_S = -G'_S + \frac{1}{\pi^2} \left( G'_S + e^2 \right)^2, \tag{44}$$

and Eq. (42). The flow diagrams in the $(G'_S, e^2)$ plane are shown in Fig. 3 and in Fig. 4 for $N = 4$ and for $N = 2$ respectively. It is seen that there appears a critical surface dividing into two phases for $N = 4$, but not for $N = 2$. We may evaluate the critical value of the flavor number as $N_{\text{cr}} = 32/\pi^2$, where the two phase structure collapses to the single phase. This critical number coincides with the value obtained by solving the SD equations in the ladder approximation with Landau gauge. The two phases appearing for $N \geq N_{\text{cr}}$ are supposed as chiral symmetry broken and (un)broken phase. The theories in the unbroken phase turns out to be scale invariant since they are subject to the IR fixed point. For $N < N_{\text{cr}}$ the fixed point disappears and there remains only the broken phase.

Here we should note that our results are slightly different from them claimed by the MC simulation and by the SD analyses, since they tell that no symmetry breaking occurs for $N \geq N_{\text{cr}}$. Actually the continuum limit of the theory is studied in these analyses. In the RG point of view we may take the continuum limit only.
when the gauge coupling is less than the IR fixed point value. If we restrict to such cases, the theories always lie in the unbroken phase for $N \geq N_{cr}$. Thus our results are not conflicting with them. However we may well regards QED$_3$ as an effective theory with a certain underlying cutoff. Then the chiral symmetry is always spontaneously broken when the gauge coupling is strong enough.

It is a great advantage of the ERG approach to enable us to incorporate all the corrections shown in Fig. 2 quite easily. This benefit is not solely the matter of improvement from the ladder approximation, but also saving from the large gauge dependence. The reason is similar with that the gauge independence of the on-shell S-matrix is achieved by summing up all the diagram appearing in each loop order. In our RG equations the one-loop corrections to the four-fermi interactions are completed diagramatically by adding the corrections to the external fermion legs to the full set of the diagrams shown in Fig. 2. This effect may be incorporated by taking the anomalous dimension of fermions into account. If we evaluate the anomalous dimension also by one-loop perturbation, then we may obtain the gauge independent beta functions, which are found out to be

\begin{align}
\dot{G}_S &= -G_S + \frac{1}{\pi^2} \left[ \frac{N+2}{3} G_S^2 - G_S G_P + \frac{4}{3} \epsilon^2 G_S - \frac{8}{3} \epsilon^2 G_P \right], \\
\dot{G}_P &= -G_P + \frac{1}{\pi^2} \left[ (2N-1)G_P^2 - 2(N-1)G_S G_P - \frac{4N-7}{6} G_S^2 - \frac{8}{3} \epsilon^2 G_S + \frac{4}{3} \epsilon^2 G_S - 2 \epsilon^4 \right].
\end{align}

(45)

Numerical analysis tells us that there remains the critical number of flavors at $3 < N_{cr} < 4$. In Fig. 5 and Fig. 6 the RG flows of $(G_S, G_P)$ couplings on the plane of $\epsilon^2 = \epsilon^{2*}$ are shown for $N = 4$ and $N = 2$ respectively. It is seen that there are three phases, which are supposed to be symmetric, chiral symmetry broken and parity broken phases, for $N$ larger than the critical value. As $N$ becomes smaller than the critical value the symmetric phase disappears. It is also seen that the RG flows of QED$_3$ always run outside of the parity broken phase, which is consistent with Vafa-Witten's theorem. However parity can be broken for QED$_3$ with the general
four-fermi interactions. Besides it is seen that the tricritical fixed point appears at the edge of the boundary between the chiral broken phase and the parity broken phase. This implies that the phase transition turns to first order beyond this edge. Thus we are able to grasp the phase structure of dynamical symmetry breakings easily by means of ERG. This is a remarkable point of ERG, though we cannot assert which symmetry is broken in each phase only from the RG flows of the four-fermi couplings. Therefore we may well expect that ERG method will be effective also to other interesting cases, e.g. color superconductivity.

4 Conclusions

First we considered mainly the formal aspects of the ERG in comparison with the SD equations. Their explicit relations are given in large N vector models and in the Gross-Neveu model. It has been shown formally that the ERG equations and the SD equations give the identical generating functional as their solution. It was also shown that the ERG method is applicable to the large N matrix models, where radiative corrections given by the planar diagrams are easily taken by solving the approximated RG equations.

In the second part, we analyzed the phase structure of QED$_3$ by applying the ERG in the primitive level of approximations. By considering the RG flow equations for the four-fermi interactions, we could understand the novel phase transition advocated by the SD approach and also by the lattice simulations after quite simple calculations. The resultant RG flows show the phase structure of chiral and parity symmetries immediately. It is also noted that we do not need to assume the symmetry to be broken apriori contrary to the SD approach. These observations demonstrate that ERG predominates over the SD methods in clarifying complicated phase diagrams of dynamical symmetry breaking. It is indeed true that our analysis is not confidential due to poor treatment of the RG equation for the gauge coupling. We would like to expect future development in this direction.
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