On non-projective normal surfaces

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Abstract

In this note we construct examples of non-projective normal proper algebraic surfaces and discuss the somewhat pathological behaviour of their Neron-Severi group. Our surfaces are birational to the product of a projective line and a curve of higher genus.

1. Introduction

The aim of this note is to construct some simple examples of non-projective normal surfaces, and discuss the degeneration of the Neron-Severi group and its intersection form. Here the word surface refers to a 2-dimensional proper algebraic scheme.

The criterion of Zariski [3, Cor. 4, p. 328] tells us that a normal surface Z is projective if and only if the set of points $z \in Z$ whose local ring $\mathcal{O}_{Z,z}$ is not $\mathbb{Q}$-factorial allows an affine open neighborhood. In particular, every resolution of singularities $X \to Z$ is projective. In order to construct $Z$, we therefore have to start with a regular surface $X$ and contract at least two suitable connected curves $R_i \subset X$.

Our surfaces will be modifications of $Y = P^1 \times C$, where $C$ is a smooth curve of genus $g > 0$; the modifications will replace some fibres $F_i \subset Y$ over $P^1$ with rational curves, thereby introducing non-rational singularities and turning lots of Cartier divisors into Weil divisors.

Key words: non-projective surface, Neron-Severi group.

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The Neron-Severi group $\text{NS}(\mathbb{Z}) = \text{Pic}(\mathbb{Z})/\text{Pic}^0(\mathbb{Z})$ of a non-projective surface might become rather small, and its intersection form might degenerate. Our first example has $\text{NS}(\mathbb{Z}) = \mathbb{Z}$ and trivial intersection form. Our second example even has $\text{Pic}(\mathbb{Z}) = 0$. Our third example allows a birational morphism $Z \to S$ to a projective surface.

These examples provide answers for two questions concerning surfaces posed by Kleiman [2, XII, rem. 7.2, p. 664]. He has asked whether or not the intersection form on the group $\text{N}(X) = \text{Pic}(X)/\text{Pic}^0(X)$ of numerical classes is always non-degenerate, and the first example shows that the answer is negative. Here $\text{Pic}^0(X)$ is the subgroup of all invertible sheaves $\mathcal{L}$ with $\text{deg}(\mathcal{L}_A) = 0$ for all curves $A \subset X$. He also has asked whether or not a normal surface with an invertible sheaf $\mathcal{L}$ satisfying $c_1^2(\mathcal{L}) > 0$ is necessarily projective, and the third example gives a negative answer. This should be compared with a result on smooth complex analytic surfaces [1, IV, 5.2, p. 126], which says that such a surface allowing an invertible sheaf with $c_1^2(\mathcal{L}) > 0$ is necessarily a projective scheme.

In the following we will work over an arbitrary ground field $k$ with uncountably many elements. It is not difficult to see that a normal surface over a finite ground field is always projective. It would be interesting to extend our constructions to countable fields.

2. A surface without ample divisors

In this section we will construct a normal surface $Z$ which is not embeddable into any projective space. The idea is to choose a suitable smooth curve $C$ of genus $g > 0$ and perform certain modifications on $Y = P^1 \times C$ called mutations, thereby destroying many Cartier divisors.

(2.1) We start by choosing a smooth curve $C$ such that $\text{Pic}(C) \otimes \mathbb{Q}$ contains uncountably many different classes of rational points $c \in C$. For example, let $C$ be an elliptic curve with at least two rational points. We obtain a Galois covering $C \to P^1$ of degree 2 such that the corresponding involution $i : C \to C$ interchanges the two rational points. Considering its graph we conclude that $i$ has at most finitely many fixed points; since there are uncountably many rational points on the projective line, the set $C(k)$ of rational points is also uncountable.
Since the group scheme of $n$-torsion points in the Picard scheme $\text{Pic}_{C/k}$ is finite, the torsion subgroup of $\text{Pic}(C)$ must be countable. Since $C$ is a curve of genus $g > 0$, any two different rational points $c_1, c_2 \in C$ are not linearly equivalent, otherwise there would be a morphism $C \to P^1$ of degree 1. We conclude that $\text{Pic}(C) \otimes \mathbb{Q}$ contains uncountably many classes of rational points.

(2.2) We will examine the product ruled surface $Y = P^1 \times C$, and the corresponding projections $pr_1 : Y \to P^1$ and $pr_2 : Y \to C$. Let $y \in Y$ be a rational point, $f : X \to Y$ the blow-up of this point, $E \subset X$ the exceptional divisor, and $R \subset X$ the strict transform of $F = pr_1^{-1}(pr_1(y))$. Then we can view $f$ as the contraction of the curve $E \subset X$, and I claim that there is also a contraction of the curve $R \subset X$. Let $D \subset X$ be the strict transform of $pr_2^{-1}(pr_2(y))$ and $L = \mathcal{O}_X(D)$ the corresponding invertible sheaf. Obviously, the restriction $L | D$ is relatively ample with respect to the projection $pr_1 \circ f : X \to P^1$; according to [4] some $L^{\otimes n}$ with $n > 0$ is relatively base point free, hence the homogeneous spectrum of $(pr_1 \circ f)_\ast(\text{Sym } L)$ is a normal projective surface $Z$, and the canonical morphism $g : X \to Z$ is the contraction of $R$, which is the only relative curve disjoint to $D$. We call $Z$ the mutation of $Y$ with respect to the center $y \in Y$.

(2.3) We observe that the existence of the contraction $g : X \to Z$ is local over $P^1$; hence we can do the same thing simultaneously for finitely many rational points $y_1, \ldots, y_n$ in pairwise different closed fibres $F_i = pr_1^{-1}(pr_1(y_i))$. If $f : X \to Y$ is the blow-up of the points $y_i$, and $E_i \subset X, R_i \subset X$ are the corresponding exceptional curves and strict transforms respectively, we can construct a normal proper surface $Z$ and a contraction $g : X \to Z$ of the union $R = R_1 \cup \ldots \cup R_n$ by patching together quasi-affine pieces over $P^1$. Since $Z$ is obtained by patching, there is no reason that the resulting proper surface should be projective. We also will call $Z$ the mutation of $Y$ with respect to the centers $y_1, \ldots, y_n$.

(2.4) Let us determine the effect of mutations on the Picard group. One easily sees that the maps

$$H^1(C, \mathcal{O}_C) \to H^1(Y, \mathcal{O}_Y) \to H^1(X, \mathcal{O}_X)$$
are bijective. Let \( \mathfrak{X} \) be the formal completion of \( X \) along \( R = \cup R_i \); since the composition
\[
H^1(C, \mathcal{O}_C) \to H^1(\mathfrak{X}, \mathcal{O}_\mathfrak{X}) \to H^1(R, \mathcal{O}_R)
\]
is injective, the same holds for the map on the left. Hence the right-hand map in the exact sequence
\[
0 \to H^1(Z, \mathcal{O}_Z) \to H^1(X, \mathcal{O}_X) \to H^1(\mathfrak{X}, \mathcal{O}_\mathfrak{X})
\]
injective, and \( H^1(Z, \mathcal{O}_Z) \) must vanish. We deduce that the group scheme \( \text{Pic}^0_{Z/k} \), the connected component of the Picard scheme, is zero. Since the Neron-Severi group of \( Y \) is torsion free, the same holds true for \( Z \), and we conclude \( \text{Pic}^\tau(Z) = 0 \).

(2.5) Now let \( F_1, F_2 \subset Y \) be two different closed fibres over rational points of \( P^1 \) and \( y_1 \in F_1 \) a rational point. The idea is to choose a second rational point \( y_2 \in F_2 \) in a generic fashion in order to eliminate all ample divisors on the resulting mutation. Let \( Z' \) be the mutation with respect to \( y_1 \). By finiteness of the base number, \( \text{Pic}(Z') \) is a countable group, in fact isomorphic to \( \mathbb{Z}^2 \). On the other hand, \( \text{Pic}(F_2) \) is uncountable, and there is a rational point \( y_2 \in F_2 \) such that the classes of the divisors \( ny_2 \) in \( \text{Pic}(F_2) \) for \( n \neq 0 \) are not contained in the image of \( \text{Pic}(Z') \). Let \( Z \) be the mutation of \( Y \) with respect to the centers \( y_1, y_2 \).

I claim that there is no ample Cartier divisor on \( Z \). Assuming the contrary, we find an ample effective divisor \( D \subset Z \) disjoint to the two singular points \( z_1 = g(R_1) \) and \( z_2 = g(R_2) \) of the surface. Hence the strict transform \( D' \subset Z' \) is a divisor with
\[
D' \cap F_2 = \{y_2\},
\]
contrary to the choice of \( y_2 \in F_2 \). We conclude that \( Z \) is a non-projective normal surface. More precisely, there is no divisor \( D \in \text{Div}(Z) \) with \( D \cdot F > 0 \), where \( F \subset Z \) is a fibre over \( P^1 \), since otherwise \( D + nF \) would be ample for \( n \) sufficiently large. Hence the canonical map \( \text{Pic}(P^1) \to \text{Pic}(Z) \) is bijective, \( \text{Pic}(Z)/\text{Pic}^\tau(Z) = \mathbb{Z} \) holds, and the intersection form on \( N(Z) \) is zero.

3. A surface without invertible sheaves

In this section we will construct a normal surface \( S \) with \( \text{Pic}(S) = 0 \). We start with \( Y = P^1 \times C \), pass to a suitable mutation \( Z \), and obtain the desired surface as a contraction of \( Z \).
Let $y_1, y_2 \in Y$ be two closed points in two different closed fibres $F_1, F_2 \subset Y$ as in (2.5) such that the mutation with respect to the centers $y_1, y_2$ is non-projective. Let $y_0 \in Y$ be another rational point in $\text{pr}_2^{-1}(\text{pr}_1(y_1))$, and consider the mutation

$$Y \xleftarrow{f} X \xrightarrow{g} Z$$

with respect to the centers $y_0, y_1, y_2$. We obtain a configuration of curves on $X$ with the following intersection graph:

Here $A$ is the strict transform of $\text{pr}_2^{-1}(\text{pr}_1(y_1))$ and $B$ is the strict transform of $\text{pr}_2^{-1}(\text{pr}_1(y_2))$. Consider the effective divisor $D = 3B + 2R_0 + 2R_1$; one easily calculates

$$D \cdot B = 1, \quad D \cdot R_0 = 1, \quad \text{and} \quad D \cdot R_1 = 1,$$

hence the associated invertible sheaf $\mathcal{L} = \mathcal{O}_X(D)$ is ample on $D \subset X$. According to [4], the homogeneous spectrum of $\Gamma(X, \text{Sym} \mathcal{L})$ yields a normal projective surface and a contraction of $A \cup R_2$. On the other hand, the curves $R_0$ and $R_1$ are also contractible. Since the curves $R_0, R_1$ and $A \cup R_2$ are disjoint, we obtain a normal surface $S$ and a contraction $h : Z \to S$ of $A$ by patching.

Let $\mathcal{L}$ be an invertible $\mathcal{O}_S$-module; then $\mathcal{M} = h^*(\mathcal{L})$ is an invertible $\mathcal{O}_Z$-module which is trivial in a neighborhood of $A \subset Z$. Since the maps in

$$\text{Pic}(P^1) \longrightarrow \text{Pic}(Z) \longrightarrow \text{Pic}(A)$$

are injective, we conclude that $\mathcal{M}$ is trivial. Hence $S$ is a normal surface such that $\text{Pic}(S) = 0$ holds.
4. A counterexample to a question of Kleiman

In this section we construct a non-projective normal surface $Z$ containing an integral Cartier divisor $D \subset Z$ with $D^2 > 0$. We obtain such a surface by constructing a non-projective normal surface $Z$ which allows a birational morphism $h : Z \to S$ to a projective surface $S$; then we can find an integral ample divisor $D \subset S$ disjoint to the image of the exceptional curves $E \subset Z$.

(4.1) Again we start with $Y = P^1 \times C$ and choose two closed points $y_1, y_2 \in Y$ as in [2.5] such that the resulting mutation is non-projective. Let $y'_2$ be the intersection of $F_2 = \text{pr}_1^{-1}(\text{pr}_1(y_2))$ with $\text{pr}_2^{-1}(\text{pr}_2(y_1))$, and $f : X \to Y$ the blow up of $y_1, y_2$ and $y'_2$. We obtain a configuration of curves on $X$ with intersection graph

Here $A$ is the strict transform of $\text{pr}_2^{-1}(\text{pr}_2(y_2))$, and $A'$ is the strict transform of $\text{pr}_2^{-1}(\text{pr}_2(y'_2))$. One easily sees that there is a contraction $X \to S$ of the curve $R_1 \cup R_2 \cup E_2$ and another contraction $X \to Z$ of the curve $R_1 \cup R_2$. The divisor $A'$ is relatively ample on $S$ and shows that this surface is projective. On the other hand, I claim that there is no ample divisor on $Z$. Assuming the contrary, we can pick an integral divisor $E \subset Z$ disjoint to the singularities; its strict transform $D \subset Y$ satisfies

$$D \cap F_1 = \{y_1\} \quad \text{and} \quad D \cap F_2 = \{y_2, y'_2\},$$

where $F_i$ are the fibres containing $y_i$. Since $A' \cap F_2 = \{y'_2\}$ holds, the class of some multiple $ny_2 \in \text{Div}(F_2)$ is the restriction of an invertible $\mathcal{O}_Y$-module,
contrary to the choice of \( y_2 \). Hence the surface \( Z \) and the morphism \( Z \to S \) are non-projective.

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