Note on group distance magic graphs $G[C_4]$

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Abstract

A *group distance magic labeling* or a $G$-distance magic labeling of a graph $G(V, E)$ with $|V| = n$ is an injection $f$ from $V$ to an Abelian group $G$ of order $n$ such that the weight $w(x) = \sum_{y \in N_G(x)} f(y)$ of every vertex $x \in V$ is equal to the same element $\mu \in G$, called the magic constant. In this paper we will show that if $G$ is a graph of order $n = 2^p(2k + 1)$ for some natural numbers $p, k$ such that $\deg(v) \equiv c \pmod{2^p + 1}$ for some constant $c$ for any $v \in V(G)$, then there exists an $G$-distance magic labeling for any abelian group $G$ for the graph $G[C_4]$. Moreover we prove that if $G$ is an arbitrary abelian group of order $4n$ such that $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times A$ for some abelian group $A$ of order $n$, then exists a $G$-distance magic labeling for any graph $G[C_4]$.

Keywords: distance magic labeling, magic constant, sigma labeling, graph labeling, abelian group
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1 Introduction

All graphs considered in this paper are simple finite graphs. Consider a simple graph $G$ whose order we denote by $|G| = n$. Write $V(G)$ for the vertex set and $E(G)$ for the edge set of a graph $G$. The *neighborhood* $N(x)$

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of a vertex \( x \) is the set of vertices adjacent to \( x \), and the degree \( \text{deg}(x) \) of \( x \) is \( |N(x)| \), the size of the neighborhood of \( x \).

Let \( w(x) = \sum_{y \in N_G(x)} l(y) \) for every \( x \in V(G) \).

*Distance magic labeling* (also called *sigma labeling*) of a graph \( G = (V, E) \) of order \( n \) is a bijection \( l: V \rightarrow \{1, 2, \ldots, n\} \) with the property that there is a positive integer \( k \) such that \( w(x) = k \) for every \( x \in V \). If a graph \( G \) admits a distance magic labeling, then we say that \( G \) is *distance magic graph* (12). The concept of distance magic labeling has been motivated by the construction of magic squares.

The following observations were independently proved:

**Observation 1.1** (8, 9, 10, 12). Let \( G \) be a \( r \)-regular distance magic graph on \( n \) vertices. Then \( k = \frac{r(n+1)}{2} \).

**Observation 1.2** (8, 9, 10, 12). No \( r \)-regular graph with \( r \)-odd can be a distance magic graph.

Problem of distance magic labeling of \( r \)-regular graphs was studied recently (see 2, 3, 6, 9, 11). It is interesting that if you blow up a \( r \)-regular \( G \) graph into some specific \( p \)-regular graph (like \( C_4 \) or \( K_{2n} \)), then the obtained graph \( H \) is distance magic. More formally, we have the following definition.

**Definition 1.3.** Let \( G \) and \( H \) be two graphs where \( \{x^1, x^2, \ldots, x^p\} \) are vertices of \( G \). Based upon the graph \( G \), an isomorphic copy \( H^1 \) of \( H \) replaces every vertex \( x^j \), for \( j = 1, 2, \ldots, p \) in such a way that a vertex in \( H^1 \) is adjacent to a vertex in \( H^1 \) if and only if \( x^j x^i \) was an edge in \( G \). Let \( G[H] \) denote the resulting graph.

Miller at al. [9] proved the following results.

**Theorem 1.4** ([9]). The cycle \( C_n \) of length \( n \) is a distance magic graph if and only if \( n = 4 \).

**Theorem 1.5** ([9]). If \( r \geq 1 \), \( n \geq 3 \), \( G \) is an \( r \)-regular graph and \( C_n \) the cycle of length \( n \). Then \( G[C_n] \) admits a distance magic labeling if and only if \( n = 4 \).

**Theorem 1.6** ([9]). Let \( G \) be an arbitrary regular graph. Then \( G[K_n] \) is distance magic for any even \( n \).
The following problem was posted in [2].

**Problem 1.7** ([2]). *If G is non-regular graph, determine if there is a distance magic labeling of G[C₄].*

It seems to be very hard to characterize such graphs. For example there were considered all graphs $K_{m,n}[C_4]$ for $1 \leq m < n \leq 2700$ and only $K_{9,21}[C_4]$, $K_{20,32}[C_4]$, $K_{428,548}[C_4]$ are distance magic (see [1]).

Froncek in [5] defined the notion of *group distance magic graphs*, i.e. the graphs allowing the bijective labeling of vertices with elements of an abelian group resulting in constant sums of neighbor labels.

**Definition 1.8.** A *group distance magic labeling* or a $G$-*distance magic* labeling of a graph $G(V,E)$ with $|V| = n$ is an injection $f$ from $V$ to an abelian group $G$ of order $n$ such that the weight $w(x) = \sum_{y \in N_G(x)} f(y)$ of every vertex $x \in V$ is equal to the same element $\mu \in G$, called the magic constant.

Obviously, every graph with $n$ vertices and a distance magic labeling also admits a $\mathbb{Z}_n$-distance magic labeling. The converse is not necessarily true.

It was proved that Observation 1.2 is also true for $G$-distance magic labeling ([4]) in case $n \equiv 2(\mod 4)$.

**Observation 1.9** ([4]). *Let G be a $r$-regular distance magic graph on $n \equiv 2(\mod 4)$ vertices, where $r$ is odd. There does not exists an abelian group $G$ of order $n$ such that $G$ is $G$-distance magic.*

In this paper we will prove that if $G$ is a graph of order $n = 2^p(2k + 1)$ for some natural numbers $p, k$ such that $\deg(v) \equiv c \pmod{2^p+1}$ for some constant $c$ for any $v \in V(G)$, then there exists an $G$-distance magic labeling for any abelian group $G$ for the graph $G[C_4]$. Moreover we show that if $G$ is an abelian group of order $4n$ such that $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times A$ for some abelian group $A$ of order $n$, then there exists a $G$-distance magic labeling for any graph $G[C_4]$.

## 2 Main results

We start with the following lemma.
Lemma 2.1. Let $G$ be a graph of order $n$ and $\mathcal{G}$ be an arbitrary abelian group of order $4n$ such that $\mathcal{G} \cong \mathbb{Z}_{2^p} \times A$ for $p \geq 2$ and some abelian group $A$ of order $\frac{n}{2^p}$. If $\deg(v) \equiv c \mod 2^{p-1}$ for some constant $c$ and any $v \in V(G)$, then there exists a $\mathcal{G}$-distance magic labeling for the graph $G[C_4]$.

**Proof.** Let $G$ has the vertex set $V(G) = \{x^0, x^1, \ldots, x^{n-1}\}$, $C_4 = v_0v_1v_2v_3v_0$ and $H = G[C_4]$.

For $0 \leq i \leq n - 1$ and $j = 0, 1, 2, 3$, let $v^i_j$ be the vertices of $H$ that replace $x^i$, $0 \leq i \leq n - 1$ in $G$.

If $g \in \mathcal{G}$, then we can write that $g = (w, a_i)$ for $w \in \mathbb{Z}_{2^p}$ and $a_i \in A$ for $i = 0, 1, \ldots, n - 1$.

Label the vertices of $H$ in the following way

$$f(v^i_j) = \begin{cases} (2i + j \mod 2^{p-1}, a_{i2^{-p+2}}) & \text{for } j = 0, 1 \\ (2^p - 1, 0) - f(v^i_{j-2}) & \text{for } j = 2, 3 \end{cases}$$

for $i = 0, 1, \ldots, n - 1$.

Notice that for every $i$

$$f(v^i_0) + f(v^i_1) = f(v^i_2) + f(v^i_3) = (2^p - 1, 0).$$

So the sum of the labels in the $i$th part is

$$f(v^i_0) + f(v^i_1) + f(v^i_2) + f(v^i_3) = (2^p - 2, 0),$$

which is independent of $i$. Since $c \equiv \deg(v) \mod 2^{p-1}$ for any $v \in V(G)$, therefore, for every $x \in V(H)$

$$w(x) = (-2c - 1, 0).$$

Theorem 2.2. Let $G$ be a graph of order $n$ and $\mathcal{G}$ be an arbitrary abelian group of order $4n$ such that $\mathcal{G} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times A$ for some abelian group $A$ of order $n$. There exists a $\mathcal{G}$-distance magic labeling for the graph $G[C_4]$.

**Proof.** Let $G$ has the vertex set $V(G) = \{x^0, x^1, \ldots, x^{n-1}\}$ and $C_4 = v_0v_1v_2v_3v_0$. For $0 \leq i \leq n - 1$ and $j = 0, 1, 2, 3$, let $v^i_j$ be the vertices of $H$ that replace $x^i$, $0 \leq i \leq n - 1$ in $G$.

If $g \in \mathcal{G}$, then we can write that $g = (j_1, j_2, a_i)$ for $j_1, j_2 \in \mathbb{Z}_2$ and $a_i \in A$ for $i = 0, 1, \ldots, n - 1$.

Label the vertices of $H$ in the following way
\[ f(v_j^i) = \begin{cases} 
(0, 0, a_i) & \text{for } j = 0, \\
(1, 0, a_i) & \text{for } j = 1, \\
(1, 1, -a_i) & \text{for } j = 2, \\
(0, 1, -a_i) & \text{for } j = 3 
\end{cases} \]

for \( i = 0, 1, \ldots, n - 1 \).

Notice that for every \( i = 0, \ldots, n - 1 \)
\[ f(v_0^i) + f(v_2^i) = f(v_1^i) + f(v_3^i) = (1, 1, 0). \]

So the sum of the labels in the \( i \)th part is
\[ f(v_0^i) + f(v_1^i) + f(v_2^i) + f(v_3^i) = (0, 0, 0), \]
which is independent of \( i \). Therefore, for every \( x \in V(H) \),
\[ w(x) = (1, 1, 0). \]

\[ \square \]

**Theorem 2.3.** Let \( G \) be a graph of order \( n \) and \( \mathcal{G} \) be an abelian group of order \( 4n \). If \( n = 2^p(2k + 1) \) for some natural numbers \( p, k \) and \( \deg(v) \equiv c \pmod{2^{p+1}} \) for some constant \( c \) for any \( v \in V(G) \), then there exists a \( \mathcal{G} \)-distance magic labeling for the graph \( G[C_4] \).

**Proof.**
The fundamental theorem of finite abelian groups states that the finite abelian group \( \mathcal{G} \) can be expressed as the direct sum of cyclic subgroups of prime-power order. This implies that \( \mathcal{G} \cong \mathbb{Z}_{2^{\alpha_0}} \times \mathbb{Z}_{p_1^{\alpha_1}} \times \mathbb{Z}_{p_2^{\alpha_2}} \times \ldots \times \mathbb{Z}_{p_m^{\alpha_m}} \)
for some \( \alpha_0 > 0 \), where \( 4n = 2^{\alpha_0} \prod_{i=1}^m p_i^{\alpha_i} \) and \( p_i \) for \( i = 1, \ldots, m \) are not necessarily distinct primes.

Suppose first that \( \mathcal{G} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathcal{A} \) for some abelian group \( \mathcal{A} \) of order \( n \), then we are done by Theorem 2.2. Observe now that the assumption \( \deg(v) \equiv c \pmod{2^{p+1}} \) and unique decomposition of any natural number \( c \) into powers of 2 apply that there exist constants \( c_1, c_2, \ldots, c_p \) such that \( \deg(v) \equiv c_i \pmod{2^i} \) for \( i = 1, 2, \ldots, p \) for any \( v \in V(G) \). Hence if \( \mathcal{G} \cong \mathbb{Z}_{2^{\alpha_0}} \times \mathcal{A} \) for some \( 2 \leq \alpha_0 \leq p + 2 \) and some abelian group \( \mathcal{A} \) of order \( \frac{4n}{2^{p+1}} \), then we obtain by Lemma 2.1 that there exists a \( \mathcal{G} \)-distance magic labeling for the graph \( G[C_4] \). \[ \square \]
The observation follows easily from the above Theorem 2.3 however the below Observation 2.4 shows an infinite family of Eulerian graphs with odd order such that none of graphs was distance magic.

**Observation 2.4.** Let $G$ be a graph of odd order $n$ and $\mathcal{G}$ be an abelian group of order $4n$. If $G$ is an Eulerian graph (i.e. all vertices of the graph $G$ have even degrees), then there exists a $\mathcal{G}$-distance magic labeling for the graph $G[\mathcal{C}_4]$.

Before we proof the Observation 2.5 we need the following definition. The Dutch windmill graph $D^t_m$ is the graph obtained by taking $t > 1$ copies of the cycle $C_m$ with a vertex $c$ in common ([7]). Thus for $t$ being even a graph $D^t_4$ is an Eulerian graph of odd order $3t + 1$. Let $i$th copy of a cycle in $C^i_4$ is $cy^ix^iz^jc$ for $i = 0, \ldots, t - 1$, $C_4 = v_0v_1v_2v_3v_0$ and $H = C^i_4[C_4]$. For $0 \leq i \leq t - 1$ and $j = 0, 1, 2, 3$, let $x^i_j$ $(y^i_j, z^i_j, c_j$ resp.) be the vertices of $H$ that replace $x^i$ $(y^i, z^i, c$, resp.) $0 \leq i \leq t - 1$ in $C^i_4$.

**Observation 2.5.** There does not exist a distance magic graph $C^t_4[C_4]$.

**Proof.** Suppose that $C^t_4[C_4]$ is a distance magic graph. It is easy to observe that:

- $l(c_0) + l(c_2) = l(c_1) + l(c_3) = a_c$.
- $l(x^0_i) + l(x^1_i) = l(x^1_i) + l(x^1_i) = a_y$ for $0 \leq i \leq t - 1$.
- $l(y^0_i) + l(y^1_i) = l(y^1_i) + l(y^1_i) = a_y$ for $0 \leq i \leq t - 1$.
- $l(z^0_i) + l(z^1_i) = l(z^1_i) + l(z^1_i) = a_z$ for $0 \leq i \leq t - 1$.

Since $w(z^i_j) = a^i_y + 2a^i_x + 2a_c = w(y^i_j) = a^i_x + 2a^i_y + 2a_c$ we obtain that $a^i_y = a^i_x = a^i_z = a$ for $0 \leq i \leq t - 1$. Moreover $w(x^i_j) = a^i_x + 4a^i = w(y^i_j) = a^i + 2a^i_x + 2a_c$, hence $a^i_x = 3a_c - 2a_c$. Furthermore $w(z^i_j) = 7a^i - 2a_c = w(y^i_j) = 7a - 2a_c$ implies that $a^i = a^i = a$.

Since $7a - 2a_c = a_c + 4ta = k$ it has to be that $3a_c = (7 - 4t)a$ and therefore $t = 1$ (because $a_c, a > 0$), a contradiction.

The following observation shows that inverse of the Theorem 2.3 is not true:

**Observation 2.6.** Let $K_{p,q}$ be such complete bipartite graph that $p$ is even and $q$ is odd and $\mathcal{G}$ be an abelian group of order $4(p + q)$. There exists a $\mathcal{G}$-distance magic labeling for the graph $G[\mathcal{C}_4]$. 

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Proof.
If $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times A$ for some abelian group $A$ of order $p+q$, then there exists a $G$-distance magic labeling for the graph $K_{p,q}[C_4]$ by Theorem 2.2. Suppose now that $G \cong \mathbb{Z}_4 \times A$ for some abelian group $A$ of order $p+q$. Let $K_{p,q}$ has the partition vertex sets $A = \{x^0, x^1, \ldots, x^{p-1}\}$, $B = \{y^0, y^1, \ldots, y^{q-1}\}$ and $C_4 = v_0v_1v_2v_3v_0$. For $0 \leq i \leq n-1$ and $j = 0, 1, 2, 3$, let $x^i_j$ (y^l_j respectively) be the vertices of $K_{p,q}[C_4]$ that replace $x^i 0 \leq i \leq p-1$ (y^l 0 \leq l \leq q-1 respectively) in $K_{p,q}$. If $g \in G$, then we can write that $g = (j, a_i)$ for $j \in \mathbb{Z}_4$ and $a_i \in A$ for $i = 0, 1, \ldots, p+q-1$.

Label the vertices of $K_{p,q}[C_4]$ in the following way

$$f(x^i_j) = \begin{cases} (2j, a_i) & \text{for } j = 0, 1 \\ (1, 0) - f(x^i_{j-2}) & \text{for } j = 2, 3 \end{cases}$$

for $i = 0, 1, \ldots, p-1$.

$$f(y^l_j) = \begin{cases} (2j, a_{p+i}) & \text{for } j = 0, 1 \\ (3, 0) - f(y^l_{j-2}) & \text{for } j = 2, 3 \end{cases}$$

for $l = 0, 1, \ldots, q-1$.

Notice that

$$f(x^i_0) + f(x^i_2) = f(x^i_1) + f(x^i_3) = (1, 0),$$

$$f(y^l_0) + f(y^l_2) = f(y^l_1) + f(y^l_3) = (3, 0),$$

for every $i = 0, \ldots, p-1$ and for every $l = 0, \ldots, q-1$. This implies:

$$\sum_{i=0}^{p-1} \left( \sum_{j=0}^{3} f(x^i_j) \right) = p(2, 0) = (0, 0)$$

$$\sum_{l=0}^{q-1} \left( \sum_{j=0}^{3} f(y^l_j) \right) = q(2, 0) = (2, 0)$$

Hence for every $x \in V(K_{p,q}[C_4])$, $w(x) = (3, 0)$.

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