Coulomb drag between a metal and a Wigner crystal.

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We calculated the Coulomb drag contribution to the resistivity and the transresistivity for a double layer system, in which the passive layer is a Wigner crystal pinned by impurities, and the active one is a metal. We found that in quasi one dimensional systems both quantities are suppressed at low temperatures by the constraints of energy and momentum conservation in electron-phonon scattering. In two dimensions both quantities are \( \propto T^4 \), which is consistent with Bloch law for the contribution of electron-phonon interaction to the resistivity. Strong impurities in the Wigner crystal decrease the transresistivity significantly when their number reaches the logarithm of the number of sites in the crystal. In contrast, the drag contribution to the resistivity is found to be independent of the impurities, at least as long as their density is kept zero in the thermodynamic limit.

I. INTRODUCTION

Coupled quantum wells are useful tools for studying electron-electron interaction in low-dimensional systems both experimentally and theoretically. By changing the distance between the layers, one can allow or prohibit interlayer tunneling and also control the interaction between electrons in different layers. Moreover, by changing parameters such as the density of electrons, the temperature and the magnetic field one can examine different electronic phases.

One effect, which has attracted much attention in recent years, is Coulomb drag\(^1\), where current \( I_1 \) in one layer ("the active layer") induces a voltage \( V_2 \) in the other ("passive") layer, which is kept in a zero current state \( (I_2 = 0) \). The measured quantity is the drag resistivity, or trans-resistivity, which for a square sample is defined as \( \rho_D \equiv -\frac{I_1}{V_2} \). Since electron-electron collisions have only indirect consequences for transport properties in a single quantum well, because they conserve momentum, the Coulomb drag is unique in directly measuring effects of electron-electron interaction through a transport measurement.

Coulomb drag has been studied extensively in the regime in which carriers in both layers form Fermi liquids. For low enough density the two-dimensional electron liquid should condense into a Wigner crystal\(^2\). A clean Wigner crystal should have no electrical resistivity. However, it is well appreciated by now, from analogous behavior in charge density wave systems\(^3\), that an arbitrarily small disorder potential should pin the crystal so that there are no charge carriers that can flow in response to an arbitrarily small electric field. The disorder also causes the crystal to deform, so that it has a finite correlation length, or domain size. As a result of pinning, the linear response conductivity vanishes.

In this work we study Coulomb drag in a double layer system in which carriers in the passive layer form a pinned Wigner crystal. The pinning of the crystal is due to its interaction with strong impurities, whose density is assumed to vanish in the thermodynamic limit. The common methods for calculating drag resistivity are not directly applicable to this case: the Boltzmann equation\(^4\) can be used to calculate the flow of momentum from the active to the passive layer, but does not distinguish between momentum flow to the electrons in the Wigner crystal and momentum flow to the impurities that pin the crystal. This distinction is important for separating between two different measurable quantities: the contribution of the passive layer to the resistivity of the active layer and the trans-resistivity, proportional to the voltage developing on the passive layer. The Kubo formalism\(^5\) is useful for calculating the transconductivity. However, the inversion of the conductivity matrix, needed to obtain the transresistivity, is delicate, due to the vanishing zero temperature conductivity of the passive layer. Thus, this problem necessitates a different method.

To that end, we introduce an harmonic potential of frequency \( \omega_0 \), that confines the passive layer and does not allow it to move. When momentum flows from the active to the passive layer, the passive layer is displaced from its equilibrium position. Its displacement, which we call \( u_0 \), is determined by the balance between the momentum flux from the active layer and the restoring force exerted by the harmonic confining potential. We calculate \( u_0 \), from which we deduce the restoring force and the transresistivity. The introduction of the harmonic potential into the problem allows us to calculate the transresistivity directly by using standard linear response formalism.

II. THE MODEL AND THE PROBLEM

The model is defined as follows. There are two conductors (wires in the quasi-1D case and layers in the 2D case). Current flows in the active one, which is in a weakly disordered Fermi liquid state. No current flows in
the passive conductor, which is in a Wigner crystal state pinned by impurities.

The Hamiltonian of the system is given by

\[ H = H_1 + H_2 + H_{1-2}. \]  

(1)

Here \( H_1 \) and \( H_2 \) are the Hamiltonians of the active and passive conductors, and \( H_{1-2} \) is the interaction between them. Thus

\[ H_1 = \sum_p \epsilon_p c_p^\dagger c_p + H_{\text{disorder}}, \]

where \( H_{\text{disorder}} \) is the standard interaction of electrons in a metal with dilute impurities; the Hamiltonian of the passive conductor, \( H_2 \), describes a crystal interacting with impurities:

\[ H_2 = \int \frac{dx}{s_0} \left( \frac{p(x)^2}{2m_2} + \frac{1}{2} m_2 v_s^2 (\nabla_i u_j(x))(\nabla_i u_j(x)) + \frac{1}{2} m_2 \omega_0^2 u(x)^2 - \sum_{i,\text{imp}} \frac{1}{2} A_G G^2 s_0 u(x)^2 \delta(x - R_{\text{imp}}) \right), \]

(2)

where \( u \) is the displacement field, \( p \) - its conjugate momentum, \( s_0 \) is the size of a unit cell of the Wigner crystal, \( m_2 \) - the mass of electrons constituting the Wigner crystal and \( v_s \) - the sound velocity of the crystal (for simplicity we assume it to be equal for the longitudinal and transverse modes). \( G \) is a basis vector of the reciprocal Wigner lattice, \( A_G \) - a parameter, defining the strength of the impurities in the crystal. We expect the induced field (which enters into the definition of trans-resistivity) to be independent of \( \omega_0 \), and at the end of the calculation \( \omega_0 \) is taken to zero. In our analysis of the Hamiltonian \( H_2 \) we assume that the impurities in the passive layer are dilute and strong, in a sense that is defined below.

The derivation of this Hamiltonian is similar to one given in [4]. We start from a discrete harmonic lattice, interacting with impurities:

\[ H_2 = \sum_i \frac{p(x_i)^2}{2m_2} + \frac{1}{2} \sum_{i \neq j} u^0(x_i) D(x_i - x_j) u^0(x_j) + \sum_{i,\text{imp}} V(x_i + u^0(x_i) - R_{\text{imp}}) \]

\[ + \frac{1}{2} \sum_{i \neq j} u^1(x_i) D(x_i - x_j) u^1(x_j) + \sum_{i,\text{imp}} \frac{1}{2} (u^1(x_i) \nabla)(u^1(x_i) \nabla) V(x_i + u^0(x_i) - R_{\text{imp}}), \]

(4)

where there is no linear in \( u^1 \) term, because \( u^0 \) minimizes the potential energy, including the impurities, and the crystal is harmonically pinned at each impurity site. We now assume that the impurity strength is much larger than elastic forces in the crystal, so that the lattice adjusts itself in order to minimize the interaction energy with impurities at each impurity site. This is a strong pinning condition. It is valid when impurities are strong and dilute. As derived in detail below, the impurities are strong when

\[ G^2 A_G >> 2m_2 v_s \omega_0 / a_0 \]

(5)

in quasi 1D, and

\[ G^2 A_G >> \frac{2\pi m_2 v_s^2}{s_0 \ln(\frac{v_s}{a_0 \omega_0})} \]

(6)

in 2D. Here \( a_0 \) is the lattice spacing.

Assuming strong pinning, \( u^0 \) can be found simply by minimizing \( \sum_i V(x_i + u^0(x_i) - R_{\text{imp}}) \) at each impurity site. Assuming now that the range of the impurities is much larger than the lattice spacing, but much smaller than the inter-impurity distance, we can write

\[ \sum_i V(x_i + u^0(x_i) - R_{\text{imp}}) \approx \sum_i V(x_i + u^0(R_{\text{imp}}) - R_{\text{imp}}), \]

which is periodic in \( u^0(R_{\text{imp}}) \) with lattice period. Expanding in a Fourier series and taking only the lowest Fourier components, we may approximate

\[ \approx \sum_G A_G \cos(G(u^0(R_{\text{imp}}) - R_{\text{imp}})), \]

(7)

which for attractive impurities \( (A_G < 0) \) is minimized when \( \cos(G(u^0(R_{\text{imp}}) - R_{\text{imp}})) = 1 \). Substituting this approximation in the Hamiltonian \( H_2 \), we get:
\[
\sum_{i,imp} \frac{1}{2} (u^1(x_i) \nabla)(u^1(x_i) \nabla)V(x_i + u^0(x_i) - R_{imp}) \\
\approx -\frac{1}{2} A_G \sum_{i,imp} (u^1(x_i)G)(u^1(x_i)G) \delta_{x_i - R_{imp}}.
\] 

We now omit the non-dynamical parts of \( \mathcal{H} \) (i.e., the parts that depend only on \( u^0 \)), add a term \( \frac{1}{2} \sum_{x} m_{2} \omega_{0}^{2}(u(x))^{2} \) to represent a harmonic confining potential for the passive layer, and go to the continuum limit. We obtain the Hamiltonian \( \mathcal{H} \). Since this Hamiltonian does not depend on \( u^0 \), we can simplify the notation by omitting the superscript from \( u^1 \) and calling it \( u \). It should be remembered, however, that from now on \( u \) is not the deviation of the electron position from a lattice site, but its deviation from the position it holds in equilibrium (which is shifted from the lattice site by the impurities).

The interaction between the conductors is given by

\[
H_{1-2} = \left( \delta q + G + i \delta k + G - q \right) \mathcal{U}_{q} \delta k \left( 1 + \frac{1}{2} \delta k_{1} + \delta k_{2} \right) n_{2} U_{q} c_{p}^{\dagger} c_{p} -
\]

where \( n_{2} \) is the density of lattice sites in the Wigner crystal. This is just the inter-layer Coulomb interaction expanded to second order in terms of the crystal displacement field. We omitted here a term arising from the static deformation of the Wigner crystal due to impurities. \( U_{q} \) is the Fourier component of the screened interlayer Coulomb potential. We take it to be constant at wave length smaller than the interlayer spacing, while at larger wavelengths it decreases exponentially.

The problem is posed as follows: given a current density \( j_{1} \) in the active conductor and \( j_{2} = 0 \) in the passive conductor, what are the electric fields \( E_{1} + \delta E_{1} \) and \( E_{2} \) in the conductors? (Here \( E_{1} \) is the field which would be in the active conductor in the absence of the passive one). Using the resistivity matrix, we can write this as

\[
\begin{pmatrix}
E_{1} + \delta E_{1} \\
E_{2}
\end{pmatrix} =
\begin{pmatrix}
\rho + \delta \rho & - \rho_D \\
- \rho_D & \rho_w
\end{pmatrix}
\begin{pmatrix}
j_{1} \\
0
\end{pmatrix}.
\] 

where \( \rho \) is the resistivity of the active layer in the absence of drag. We want to find are \( \delta \rho \) and \( \rho_D \).

We find the field \( E_{2} \) by calculating \( u_{0} \), since the two are related by

\[
E_{2} = \frac{m_{2} \omega_{0}^{2} u_{0}}{e},
\]

while \( \delta E_{1} \) is found by calculating \( \delta j_{1} - \) the negative contribution to the current in the active conductor at a given field \( E_{1} \) due to the drag. Then, for weak drag,

\[
\delta j_{1} = - \frac{\delta E_{1}}{\rho}.
\]

We consider both quasi 1D coupled conductors (by which we mean wires of several conduction channels, in which electrons form either a Fermi liquid or a Wigner crystal), and two dimensional conductors. We focus mostly on the latter.

### III. RESULTS

In quasi 1D conductors the transresistivity and the drag contribution to the resistivity are suppressed at low temperatures in the limit of vanishing \( \omega_{0} \) and infinite \( L \), since in that limit momentum and energy cannot be conserved simultaneously for electron-phonon scattering.

In the 2D case the transresistivity is found to be

\[
\rho_{D} = Z \frac{m_{1}}{m_{2}} \frac{U_{0}^{2} T^{4}}{e^{2} h^{5} v_{f s} v_{s}^{2} n_{1}} \left( 1 - \frac{N_{imp}}{2 \pi \ln(L/a_{0})} \right),
\]

where \( U_{0} \) is the zero wave vector component of the interlayer interaction, \( Z \) - a numerical factor given by

\[
Z = \frac{2}{\pi^{2}} \int_{0}^{\infty} \frac{z^{4} dz}{\sin^{4}(z)},
\]

\( m_{1} \) is the electron mass in the active (metallic) layer, \( m_{2} \) is the electron mass in the Wigner crystal, \( v_{s} \) is the sound velocity in the Wigner crystal, \( v_{f s} \equiv \sqrt{v_{f}^{2} - v_{s}^{2}} \), with \( v_{f} \) being the Fermi velocity in the metallic layer (we assume \( v_{f} > v_{s} \) and that \( v_{f}, v_{s} \) and \( v_{f s} \) are of the same order of magnitude); \( n_{1} \) is the density of electrons, \( N_{imp} \) is the number of impurities, and \( L/a_{0} \) - the ratio between the size of the system and that of a unit cell. The impurities are supposed to be dilute enough, so that a condition \( N_{imp} \ll \ln(L/a_{0}) \) is satisfied.

We see that impurities in the passive conductor decrease the transresistivity. Due to the force they exert on the crystal, part of the momentum flux from the active layer is transferred to the impurities. This part does not lead to drag. Stretching the approximation to its limit, we find that the influence of the impurities becomes significant once their number reaches the logarithm of the number of electrons in the Wigner crystal. The impurity strength does not appear in the final expression as long as it is much larger than elastic forces in the Wigner crystal. The temperature dependence is consistent with the 2D extension of the Bloch law for the resistance due to electron-phonon scattering.

The drag contribution to the resistivity in 2D is, in the limit of dilute impurities,

\[
\delta \rho = \frac{n_{2}}{n_{1}} \rho_{D} \left( 1 - \frac{N_{imp}}{2 \pi \ln(L/a_{0})} \right)^{-1}
\]

\[
= Z \frac{n_{2} m_{1}}{n_{1} m_{2} e^{2} h^{5} v_{f s} v_{s}^{2} n_{1}} \frac{U_{0}^{2} T^{4}}{1 - \frac{N_{imp}}{2 \pi \ln(L/a_{0})}}
\]

This contribution is not suppressed by the impurity induced factor \( 1 - \frac{N_{imp}}{2 \pi \ln(L/a_{0})} \). The momentum, that
flows to the impurities, does increase the resistivity in the active layer, although it does not induce voltage on the electrons in the passive layer. In the absence of impurities, the ratio between the trans-resistivity and the drag contribution to the resistivity is $n_1/n_2$, as required by the Galilean invariance. Due to this invariance the two electronic systems can flow at equal speeds without any friction resulting from their interaction. Thus when $j_1/j_2 = n_1/n_2$ we must have

$$\delta E = j_1\delta \rho - j_2\rho_D = 0.$$  \hspace{1cm} (15)

IV. CALCULATION

As explained in the introduction, the calculation is done using the linear response formalism. We first consider the trans-resistivity of the passive conductor and then the contribution of inter-layer interaction to the resistance of the active conductor.

A. The trans-resistivity

From the Kubo formula we find the displacement of the Wigner crystal:

$$\langle u_0^0(\omega) \rangle = \lim_{\omega \to 0} \frac{1}{\omega \hbar} \int_0^\infty dt \exp(i\omega t) \langle \{ u_0^0(t), j_0^\beta \} \rangle E_1^\beta,$$  \hspace{1cm} (16)

where $j_0 = e/m \sum_q q c_q^\dagger c_q$ is the current operator for the active layer and $E$ is the applied electric field (in the same layer). Proceeding as usual, we define the correlator $\Pi_n$ by

$$\langle u_0(\omega) \rangle = \frac{i}{\omega} \Pi_n(\omega) \frac{e}{m_1} E,$$  \hspace{1cm} (17)

where we have assumed isotropy. Then in imaginary time the correlator is given by

$$\Pi_u(i\omega) = \int_0^\beta \tau \exp(i\omega \tau) (T u_0(\tau) p_f^E).$$  \hspace{1cm} (18)

Here $\beta$ is the inverse temperature and $p_f^E$ is the component of the total momentum of electrons in the active conductor in the direction of the applied field. This correlator is then calculated to lowest order of inter-layer interaction. The corresponding diagrams are shown in Fig. 1. Note that the last pair of diagrams vanishes, since the transferred momentum is zero.

Calculation of the topmost pair of diagrams yields an expression for $\Pi_n$:
where \( \tau_i \) is the electron transport scattering time for the active conductor and \( \epsilon_p \) - electron energy for the active conductor. Now we sum over \( p \), assuming \( T \ll \epsilon_F \) and \( 1/d \ll k_f \) (where \( d \) is the spacing between the conductors and \( k_f \) the Fermi wave vector). Then

\[
\Pi_u(\omega \rightarrow 0) = \sum_{k_1, k_2} \frac{1}{2} N(0) D_{k_10}(0) \left( \frac{\hbar}{2m_2} \right)^2 \frac{\hbar \omega}{\beta} k^2 k_2^2 f_{1f}^2 \frac{m_{HF}^2 \cos \theta}{S^2 \sqrt{\omega \omega_{k_1} \omega_{k_2}}} \sinh^2(\beta \hbar v_F k \cos \theta / 2) (D_{k_2 k}(\hbar v_F k \cos \theta) - D_{k_2 k}^*(\hbar v_F k \cos \theta)),
\]

(21)

where \( N(0) \) is the extensive density of states for the metallic conductor. To proceed further, we have to calculate the phonon propagator \( D \) in the presence of impurities. Here we calculate the phonon propagator in different dimensions, assuming that strength of the impurities is much larger than typical elastic forces in the crystal, i.e., strong pinning. To formulate quantitatively this condition, we consider the static deformation of the Wigner lattice due to all impurities. In our derivation of \( H_2 \) above, we assumed that the fragment of the lattice, that interacts with each impurity, is positioned in a way, that minimizes this interaction. In doing so, we neglected elastic forces on the scale of inter-impurity distance and found out that, up to a lattice constant, the displacement of a fragment interacting with an impurity at \( \mathbf{R}_i \), satisfies \( u_i = 0 \). The elastic forces between fragments will modify this displacement. We now express this modification in terms of phonon propagator. In coordinate representation the propagator is

\[
\tilde{D}(\omega, x_1, x_2) = \int \frac{dt}{\hbar} e^{i \omega t} (-i \theta(t) | [u_{x_1}(t), u_{x_2}(0)] |),
\]

(22)

and it can be written in momentum representation as:

\[
\tilde{D}(\omega, x_1, x_2) = \sum_{k_1, k_2} \frac{\hbar D_{k_1 k_2}(\omega) e^{i k_1 x_1 - i k_2 x_2}}{2N_2 m_2 \sqrt{\omega_{k_1} \omega_{k_2}}},
\]

(23)

where \( N_2 \) is the number of sites in the Wigner lattice. The propagator \( \tilde{D} \) relates a point force \( f \) applied at \( x_2 \) to the displacement \( u_{x_1} \) it generates at \( x_1 \)

\[
u_{x_1} = -\tilde{D}(\omega = 0, x_1, x_2) f.
\]

(24)

The force exerted on the lattice by an impurity at \( \mathbf{R}_i \) is (assuming \( G(u^0(\mathbf{R}_i) - u^0_{x_i}) \ll 1 \))

\[G^2 A_G(\mathbf{R}_i - u^0_{x_i}).\]

Consequently, the displacement \( u^0_{x_i} \) satisfies the equation

\[
u_{x_i}^0 = -\sum_i \tilde{D}^f(\omega = 0, x_k, x_i) G^2 A_G(\mathbf{R}_i - u^0_{x_i}),
\]

(25)

where \( \tilde{D}^f(\omega = 0, x_1, x_2) \) is the propagator of a clean crystal in the coordinate representation. When the impurities are dilute, \( -\tilde{D}^f(\omega = 0, x_k, x_k) \gg -\tilde{D}^f(\omega = 0, x_k, x_i) \) for \( i \neq k \). Under this condition we may retain only the term with \( i = k \) in the sum, and the strong pinning condition, requiring that \( u^0 \gg \delta_{imp} - u^0 \) becomes:

\[-\tilde{D}^f(\omega = 0, x_k, x_k) G^2 A_G \gg 1.\]

(26)

Physically, the neglect of the \( i \neq k \) terms in the sum \( (25) \) implies that the local displacement \( u_{x_k}^0 \) is determined by a balance between the potential exerted by the closest impurity and the confining harmonic potential characterized by \( \omega_0 \).

In 1D the condition \( (24) \) is

\[G^2 A_G \gg 2m_2 v_{\omega_0}/a_0,\]

(27)

while in 2D:

\[G^2 A_G \gg \frac{2\pi m_2 v_{\omega_0}^2}{s_0 \ln(\pi m_2 \omega_0)},\]

(28)

where \( s_0 \) is the area of the unit cell.

The calculation of the phonon propagator in the presence of the impurities goes as follows. We start from a differential equation satisfied by the propagator in the coordinate representation:

\[
\left( \omega^2 + v_{\omega_0}^2 \nabla^2 - \omega_0^2 + \sum_i \frac{s_0 G^2 A_G}{m_2} \delta(\mathbf{x} - \mathbf{x}_i) \right) \tilde{D}(\omega, \mathbf{x}, \mathbf{x}') = \frac{s_0 \delta(\mathbf{x} - \mathbf{x}')}{m_2},
\]

(29)

where the sum is over the impurities. This equation can be solved in terms of the free propagator

\[
\tilde{D}(\omega, \mathbf{x}, \mathbf{x}') = \tilde{D}^f(\omega, \mathbf{x} - \mathbf{x}') - \sum_i G^2 A_G \tilde{D}^f(\omega, \mathbf{x} - \mathbf{x}_i) \tilde{D}(\omega, \mathbf{x}_i, \mathbf{x}'),
\]

(30)

which gives

\[
\tilde{D}(\omega, \mathbf{x}_k, \mathbf{x}') = \sum_j \left( \mathcal{A}^{-1} \right)_{kj} \tilde{D}^f(\omega, \mathbf{x}_j - \mathbf{x}'),
\]

(31)

with matrix \( \mathcal{A} \) defined by

\[\mathcal{A}_{ji} = \delta_{ji} + G^2 A_G \tilde{D}^f(\omega, \mathbf{x}_j - \mathbf{x}_i).\]

(32)

Substituting this back into \( (30) \), we obtain
\[ \tilde{D}(\omega, \mathbf{x}, \mathbf{x}') = \tilde{D}^f(\omega, \mathbf{x}, \mathbf{x}') - \sum_{ij} G^2 A_{ij} \tilde{D}^f(\omega, \mathbf{x} - \mathbf{x}_i)(A^{-1})_{ij} \tilde{D}^f(\omega, \mathbf{x}_j - \mathbf{x}'). \]  

(33)

Using (29), we can obtain an expression for the propagator in the momentum representation:

\[ D_{k_1 k_2}(\omega) = D_{k_1}^f(\omega)\delta_{k_1 k_2} - \sum_{ij} \frac{\hbar}{2N_2m_2} G^2 A_{ij}(A^{-1})_{ij} D_{k_1}^f(\omega)D_{k_2}^f(\omega)e^{-i\mathbf{k}_1\mathbf{x}_i + i\mathbf{k}_2\mathbf{x}_j} \]

\[ = \frac{2\omega_{k_1}\delta_{k_1 k_2}}{\hbar((\omega + i\delta)^2 - \omega_{k_1}^2)} - \sum_{ij} \frac{2\sqrt{\omega_{k_1}\omega_{k_2}}G^2 A_{ij}(A^{-1})_{ij}}{\hbar m_2 N_2} \frac{e^{-i\mathbf{k}_1\mathbf{x}_i + i\mathbf{k}_2\mathbf{x}_j}}{((\omega + i\delta)^2 - \omega_{k_1}^2)((\omega + i\delta)^2 - \omega_{k_2}^2)}, \]  

(34)

where the free propagator in the momentum representation was used:

\[ D_{k_1 k_2}^f(\omega) = \delta_{k_1 k_2} D_{k_1}^f(\omega) = \frac{2\omega_{k_1}}{\hbar((\omega + i\delta)^2 - \omega_{k_1}^2)}. \]  

(35)

In the limit of strong pinning, the propagator satisfies an important relationship:

\[ \sum_k e^{i\mathbf{k} \cdot \mathbf{x}_i} D_{k,0}(\omega = 0) \propto O \left( \frac{1}{G^2 A_G \tilde{D}^f(\omega = 0, \mathbf{x} = 0)} \right) \approx 0, \]  

(36)

where \( \mathbf{x}_i \) is an impurity site. This relationship means that a force, when applied at an impurity site, causes a much smaller deformation, than when applied anywhere else, since it is opposed by the impurity. The proof of this relationship follows from the equation satisfied by the propagator \( \tilde{D} \), namely \( \tilde{D} = A^{-1}\tilde{D}^f \). In the strong pinning limit \( A_{ij} \approx G^2 A_G \tilde{D}^f(\omega, \mathbf{x}_j - \mathbf{x}_i) \) and hence (36).

From the above relationship it follows that for all \( \Omega \gg \omega_0 \),

\[ \sum_{k_1} D_{k_1,0}(0)(D_{k_1 k_2}^f(\Omega) - (D_{k_2 k_1}^f(\Omega))^*) \approx 0, \]  

(37)

where \( D^i \equiv D - D^f \) is a part of the propagator due to the impurities. To see this, we note using Eq. (34) that \( D_{k,0}(0)/\sqrt{\omega_{k}} \) is a sharply peaked function of \( k \).\( k_1 = 0 \): its free part is proportional to \( \delta_{k_1,0} \), and the part due to impurities is peaked around zero with width \( \omega_0/\nu_\theta \), because of the denominator \( 1/\omega_{k}^2 \). In contrast, \( D_{k_2 k_1}^f(\Omega) \) varies slowly in that region, except for the exponential factor \( e^{-i\mathbf{k}_2\mathbf{x}_i} = e^{-i\mathbf{k}_1\mathbf{x}_i + i\mathbf{k}_2\mathbf{x}_i} \). Hence the summation is actually of the form (36).

Using the last relationship in (29), and taking into account that the typical energy transfer in an inter-layer scattering event, \( \nu_F k \cos \theta \), is of the order of the temperature, and thus much larger than \( \omega_0 \), we see that the effect of impurities enters Eq. (29) only in the first phonon propagator, where the frequency is zero, while for the second phonon propagator, that carries a non-zero frequency, only the free part of \( D \) contributes. Thus,
where \( S(i, j) = 1 \) for \( i = j \) and \(-1\) for \( i \neq j \). Substituting this into (30), we find \( D_{00}(0) \) in 2D:

\[
D_{00}(0) = -\frac{2}{\hbar \omega_0} + \frac{4\pi v^2 N_{\text{imp}}}{\hbar^2 L^2 \omega_0^2 \ln\left(\frac{\omega_0}{n_0 \omega_0}\right)},
\]

where \( N_{\text{imp}} \) is the number of impurities. The \( \ln\left(\frac{\omega_0}{n_0 \omega_0}\right) \) originates from the expression for \( \tilde{D}_f(\omega = 0, x_1 = 0, x_2 = 0) \) in 2D.

Substituting expression (42) for the propagator into Eq. (48) and choosing

\[
\omega_0 \sim \frac{v_s}{L},
\]

we obtain the transresistivity Eq. (33).

As evident in Eq. (33), the impurities decrease the trans-resistivity as they absorb part of the momentum transferred from the active layer. That part of the transferred momentum does not generate a drag voltage, and therefore does not lead to trans-resistivity. Under the strong pinning approximation, the influence of the impurities becomes significant when their number is comparable to \( \ln\left(\frac{L}{a_0}\right) \), even though their density is still zero in the thermodynamic limit.

**B. Drag contribution to resistivity**

We now turn to the way electron-electron scattering between different layers affects the resistivity of the active layer. We again apply the Kubo formula to find, to second order in screened inter-layer interaction, the current density in the active layer as a response to an electric field in that same layer (we expect it to be negative as the drag should decrease the current):

\[
j^\alpha = Re \left[ \frac{1}{\hbar \omega S} \int_0^\infty dt \, e^{i\omega t} \langle [j^0_\alpha(t), j^\beta_\beta] \rangle E^\beta \right].
\]

Introducing the correlator from

\[
j^\alpha = \frac{1}{\omega} \lim_{\omega \to 0} \Im \Pi^\alpha_\beta(\omega) E^\beta,
\]

we have in imaginary time:

\[
\Pi(i\omega) = -\frac{1}{S} \int_0^\beta d\tau \, e^{i\omega \tau} \langle [j^0_\alpha(\tau), j^\beta_\beta(0)] \rangle,
\]

where we again assumed isotropy. The corresponding diagrams are shown in Fig. 2.

![Diagrams contributing to reduction of conductivity of the active conductor](image.png)

FIG. 2. Diagrams contributing to reduction of conductivity of the active conductor

Calculating them, we obtain

\[
\Pi = -\left(\frac{e}{m_1}\right)^2 \sum_{p, k} k^2 \left(\frac{U_0}{S}\right)^2 \frac{\hbar}{2 m_2 \omega_k} \omega \beta \frac{n_F(\epsilon_p) - n_F(-\epsilon_{p+k})}{2} \sinh^2\left(\frac{\beta(\epsilon_{p+k} - \epsilon_p)}{2}\right) \Im D_{kk}(\epsilon_{p+k} - \epsilon_p) \hbar \tau^2 k^2.
\]

from which the drag contribution to the resistivity is easily found:

\[
\delta \rho = -\rho^2 \delta \sigma = \frac{n_2^2 U_0^2 T^4 m_1}{e^2 \hbar^5 m_2 v_f s v_s^2} Z.
\]

In the presence of impurities the phonon propagator is composed of a free part and an impurity-induced part. The contribution of the former remains Eq. (48), while the contribution of the latter is again inversely proportional to the system size. Hence in the macroscopic limit the impurities do not affect Eq. (48).

When the passive layer is clean, the ratio between the trans-resistivity and the drag contribution to resistivity of the active layer is \( n_1/n_2 \). This is consistent with the Galilean invariance requirement.
V. SUMMARY

We considered a bi-layer system in which the passive layer is a Wigner crystal pinned by impurities and the active layer is a conductor, and calculated the transresistivity between two layers and the drag contribution to the resistivity of the active layer. We focused on the case of a small number of strong impurities, whose density vanishes in the thermodynamic limit. We found that in quasi 1D case both quantities are exponentially small at low temperature due to constraints of energy and momentum conservation.

In two dimensions we found a $T^4$–dependence of both quantities, consistent with the Bloch law for the electron-phonon interaction contribution to resistivity. Impurities in the Wigner crystal decrease the transresistivity significantly when their number reaches the logarithm of the number of sites in the crystal. In the limit of strong pinning the transresistivity and the active layer resistivity are independent of the impurity strength.

While a minute density of impurities is sufficient to affect the transresistivity, we find that it does not significantly affect the contribution of the Wigner crystal to the resistivity of the active layer.

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1 T. J. Gramila et al., Phys. Rev. Lett. 66, 1216 (1991).
2 T. J. Gramila et al., Phys. Rev. B 47, 12957 (1993).
3 T. J. Gramila et al., Physica B 197, 442 (1994).
4 U. Sivan, P. M. Solomon and H. Shtrikman, Phys. Rev. Lett. 68, 1196 (1992).
5 G. Gruner, Rev. Mod. Phys. 60, 1129 (1988).
6 A.-P. Jauho and H. Smith, Phys. Rev. B 47, 4420 (1993).
7 L. Zheng and A. H. MacDonald, Phys. Rev. B 48, 8203 (1993).
8 A. Kamenev and Y. Oreg, Phys. Rev. B 52, 7516 (1995).
9 K. Flensberg, B. Y.-K. Hu, A.-P. Jauho and J. M. Kinaret, Phys. Rev. B 52, 14761 (1995).
10 H. Fukuyama and P. Lee, Phys. Rev. B17, 335 (1978).
11 L. I. Glazman, I. M. Ruzin and B. I. Shklovskii, Phys. Rev. B 45, 8454 (1993).
12 J. Callaway, Quantum Theory of Solid State (Academic Press, New York, 1974).
13 G. D. Mahan, Many-Particle Physics, 2nd ed. (Plenum Press, New York, 1990).
14 A. A. Abrikosov, L. P. Gorkov and I. E. Dzyaloshinski, Methods of Quantum Field Theory in Statistical Physics, edited by R. A. Silverman (Prentice-Hall, Englewood Cliffs, New Jersey, 1963).
15 L. Bonsall and A. A. Maradudin, Phys. Rev. B 15, 1959 (1977).
16 M. Born and K. Huang, Dynamical Theory of Crystal Lattices (Oxford U. P., Oxford, 1954).