The Dynamics of D-3-brane Dyons
and Toric Hyper-Kähler Manifolds

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ABSTRACT

We find the dyonic worldvolume solitons due to parallel \((p,q)\) strings ending on a D-3-brane. These solutions preserve \(1/4\) of bulk supersymmetry. Then we investigate the scattering of well-separated dyons and find that their moduli space is a toric hyper-Kähler manifold. In addition, we present the worldvolume solitons of the D-3-brane which are related by duality to the M-theory configuration of two orthogonal membranes ending on a M-5-brane. We show that these solitons preserve \(1/8\) of supersymmetry and compute their effective action.
1. Introduction

The low energy scattering of solitons can be investigated using the moduli space approximation. One of the attractive features of this picture is that moduli spaces are equipped with novel geometries. For example, the moduli spaces of BPS monopoles are hyper-Kähler manifolds [1] and the moduli spaces of (4+1)-dimensional black holes which preserve 1/4 of supersymmetry are hyper-Kähler manifolds with torsion [2]. Other examples include the moduli spaces of some eight-dimensional black holes which are octonionic manifolds with torsion.

In the last few years it has emerged that the worldvolume solitons of branes have a bulk interpretation. These solitons are due to brane intersections or branes ending on other branes [3, 4]. From the perspective of one of the branes involved in such configurations, the intersection or the brane boundary is associated with the soliton of its worldvolume theory. The moduli spaces of worldvolume solitons of branes which preserve 1/4 of bulk supersymmetry and involve a non-trivial Born-Infeld field have recently been investigated [5, 6, 7]. It has been found that some of their geometries are similar to those of the moduli spaces of supersymmetric black holes.

In this paper, we shall investigate the worldvolume dyons that arise from (p,q)-strings [8, 9] ending on a D-3-brane. The bulk configuration of a fundamental string along the direction 4 and ending on a D-3-brane, which lies in the directions 1, 2, 3, is

\[
D3 : 0, 1, 2, 3, *, *, *, *, *
\]
\[
F1 : 0, *, *, *, 4, *, *, *, *
\] (1.1)

Acting on the above configuration with the U-duality group \( \text{SL}(2, \mathbb{Z}) \) of type IIB strings, the D-3-brane remains invariant but the fundamental string turns into a (p,q) string. It is known that the above bulk configuration preserves 1/4 of supersymmetry. From the perspective of the D-3-brane, the string boundary is associated with a dyonic worldvolume soliton that carries (p,q) charge and preserves 1/4 of supersymmetry. A related bulk configuration is that of two orthogonal
strings, one of which is fundamental and the other is a D-string, ending possibly at different points on a D-3-brane. The bulk configuration is

\[
\begin{align*}
D3 : & \ 0, 1, 2, 3, *, *, *, *, *, * \\
F1 : & \ 0, *, *, 4, *, *, *, *, * \\
D1 : & \ 0, *, *, *, 5, *, *, *, *
\end{align*}
\]  \tag{1.2}

This can be derived from the M-theory configuration of a M-5-brane and two orthogonal membranes intersecting on a 0-brane by first reducing to ten dimensions and then using IIA/IIB T-duality. This supergravity solution preserves 1/8 of supersymmetry. If the fundamental string and the D-string end at the same point in D-3-brane, the associated worldvolume soliton is a (1, 1)-dyon, i.e. a dyon of unit electric and unit magnetic charges, preserving 1/4 of bulk supersymmetry. This worldvolume soliton is also associated with a (1,1)-string ending on the D-3-brane. More generally, if \( q \) fundamental strings and \( p \) D-strings all end at the same point in D-3-brane, the associated worldvolume solitons are \( N (p', q') \)-dyons, where \( N \) is the maximal common divisor of \( p, q \). This dyon is also associated with \( N (p', q') \)-strings ending on the D-3-brane. The supersymmetry preserved is 1/4. More general IIB configurations can be found by acting on (1.2) with type IIB U-duality.

The effective theory of \( n \) coincident D-3-branes in the linearized limit is N=4 super-Yang Mills with gauge group \( U(n) \). This theory is conjectured to have an \( SL(2, \mathbb{Z}) \) duality. This is required in order to describe the low energy dynamics of the D-3-brane. The worldvolume solitons associated with a \( (p,q) \)-string ending on the D-3-brane are BPS \( (p,q) \)-dyons. The moduli spaces of these dyons are hyper-Kähler manifolds. For well-separated dyons, the moduli space becomes a toric hyper-Kähler manifold [10]. In this approximation, the forces due to massive intermediate gauge bosons are suppressed. More recently a manifestly (non-linear) Dirac-Born-Infeld-type of action has been proposed for a D-3-brane in [11]. This action is kappa-symmetric and its field equations are manifestly \( SL(2, \mathbb{R}) \) invariant. This invariance is achieved by using two \( U(1) \) gauge fields which transform as a
doublet under $SL(2, \mathbb{R})$ and then imposing a self-duality condition to reduce the physical degrees of freedom due to gauge fields to two. This non-linear action can also be used to investigate the (p,q)-dyons on the D-3-brane. Since in this action the intermediate massive gauge bosons are suppressed, the moduli spaces of these dyons are expected to be toric hyper-Kähler manifolds.

In this paper, we shall use the non-linear $SL(2, \mathbb{Z})$-invariant D-3-brane action of [11] modified appropriately by adding source terms to find the worldvolume solutions that are associated with both the above bulk configurations. In particular, we shall find dyonic configurations that have the interpretation of $N$ parallel (p,q)-strings ending on the D-3-brane. We shall also find solutions that have the interpretation of orthogonal fundamental strings and D-strings ending at different points on the D-3-brane. We shall show, using the kappa-symmetry transformation of the D-3-brane and its relation to supersymmetric configurations found in [12], that these dyons preserve 1/4 and 1/8 of the bulk supersymmetry, respectively. Then we shall use the $SL(2, \mathbb{R})$-invariant action to show that the moduli spaces of $N$ worldvolume well-separated dyons that preserve 1/4 of supersymmetry are toric hyper-Kähler manifolds. In addition, we shall give the effective action of some of the worldvolume solutions that preserve 1/8 of spacetime supersymmetry.

The paper is organized as follows. In section two, we present the appropriate static solutions of the $SL(2, \mathbb{Z})$-invariant action. In section three, we show that our classical solutions preserve either $\frac{1}{4}$ or $\frac{1}{8}$ of bulk supersymmetry using $\kappa$-symmetry. In section four, we review and improve certain aspects of the computation of the metric on moduli space of 0-brane worldvolume solitons of D-p-branes that we shall use here. In section five, we find that the moduli space of solutions preserving 1/4 of supersymmetry is a toric hyper-Kähler manifold. In section six, we find the metric on the moduli space of some of the configurations that preserve 1/8 of supersymmetry. Finally, in section seven we give our conclusions.
2. The $SL(2, \mathbb{Z})$ invariant action and static solutions.

To find the worldvolume dyons of D-3-brane, we shall use the $SL(2, \mathbb{Z})$ covariant action of [11] and choose as a IIB supergravity background the ten-dimensional Minkowski spacetime. We also introduce the two-form field strength

$$F_{\mu \nu} \equiv U^r (F_r)_{\mu \nu} = U^r (\partial_\mu A_{r\nu} - \partial_\nu A_{r\mu})$$

(2.1)

where $U^r$ are complex constants satisfying

$$\frac{i}{2} \epsilon_{rs} U^r U^s = 1 ,$$

(2.2)

and $A_r$ are one-form gauge potentials; $r = 1, 2$ and $\mu, \nu = 0, \ldots, 3$. Under $A \in SL(2, \mathbb{R})$, the constants

$$U^T = (U^1, U^2)$$

(2.3)

and field strengths

$$F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$$

(2.4)

transform as $U^T \rightarrow U^T A^{-1}$ and $F \rightarrow AF$. So the two-form field strength $\mathcal{F} = U^T F$ is $SL(2, \mathbb{R})$ invariant. The constant IIB supergravity axion $\rho$ and dilaton $\phi$ are given in terms of $\{U^r\}$ as

$$\tau \equiv \rho + i e^{-\phi} = \frac{U^2}{U^1} .$$

(2.5)

Under $SL(2, \mathbb{R})$, $\tau$ transforms as usual with fractional linear transformations. The bosonic part of the D-3-brane action [11] is

$$S_{D3} = \int d^4 x \sqrt{-\text{det}(g)} (1 + \frac{1}{2} \mathcal{F} \cdot \bar{\mathcal{F}} - \frac{1}{16} (\mathcal{F} \cdot \ast \mathcal{F})(\bar{\mathcal{F}} \cdot \ast \bar{\mathcal{F}}) + \frac{1}{4} F \cdot F)$$

(2.6)

where here $\lambda$ is a Lagrange multiplier, $F = dW$ is a 4-form field strength and
$g$ is the induced worldvolume metric. This action is supplemented with a ‘self-duality’-like relation
\[ \frac{i}{2} (\ast F)^{\ast} F = F - \frac{1}{4} (F \cdot \ast F)^{\ast} \hat{F}. \]  
(2.7)

The duality relation has the status of a field equation and should not be substituted into the action. We remark that in the static gauge the action (2.6) has six scalars which describe the position of the D-3-brane in the ten-dimensional Minkowski spacetime. Both that action and the self-duality condition are invariant under the $U(1)$ transformation
\[ F \rightarrow e^{i\theta} F, \]  
(2.8)

where $\theta$ is a angle.

There are four in total electric and magnetic charges associated with $F_1$ and $F_2$. However due to the self-duality condition, only two are independent. We shall choose
\[ q = \frac{1}{4\pi} \int_{S^2} F_1 \]  
\[ p = \frac{1}{4\pi} \int_{S^2} F_2. \]  
(2.9)

We shall investigate dyon configurations for which the non-vanishing fields are $F, F, \lambda$ and two transverse scalars $X, Y$. In addition, since $F$ is $SL(2, \mathbb{R})$ invariant, we can set without loss of generality
\[ U^1 = 1 \]  
\[ U^2 = i, \]  
(2.10)

after possibly using a $SL(2, \mathbb{R})$ transformation and so $F = F_1 + iF_2$. This choice corresponds to the $\tau = i$ vacuum of supergravity theory. A $U(1)$ subgroup of $SL(2, \mathbb{R})$ preserves (2.10).

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* We are using the conventions for the inner product $\phi \cdot \psi = \frac{1}{g^{\mu_1 \cdots \mu_p}} \phi_{\mu_1 \cdots \mu_p} \psi_{\mu_1 \cdots \mu_p}$ and the dual form $\ast \phi^{\mu_0 \cdots \mu_3} = \frac{1}{g^{\mu_0 \cdots \mu_3 \mu_1 \cdots \mu_p}} \epsilon^{\mu_1 \cdots \mu_3 \mu_4 \cdots \mu_p} \phi_{\mu_0 \cdots \mu_3 \mu_4 \cdots \mu_p}$, where $\epsilon^{0123} = 1$. 

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A solution to the field equations of (2.6) is

\[ F_{on} = \partial_n Z \]
\[ F_{pq} = i \epsilon_{pqr} \delta^{rl} \partial_l Z \]
\[ X = H_1 \]
\[ Y = H_2, \]

(2.11)

where \( H_1, H_2 \) are harmonic functions in \( \mathbb{R}^3 \), i.e.

\[ H_1 = m_1 + \sum_{i=1}^{N} \frac{\alpha_i}{|x - y_i|} \]
\[ H_2 = m_2 + \sum_{i'=1}^{N'} \frac{\beta_{i'}}{|x - y_{i'}'|} \]
\[ Z = H_1 - iH_2, \]

(2.12)

and \( m, n, p, q, r, l = 1, 2, 3 \). The rest of the fields are easily determined. For example,

\[ *F = (2 + \delta^{mn} \partial_m Z \partial_n \bar{Z}) \]

(2.13)

where the Hodge star operation is taken with respect to the flat metric on \( \mathbb{E}^3 \). The macroscopic bulk interpretation of this solution is as \( N \) fundamental- and \( N' \) D-strings ending at the points \( \{ y_i; i = 1, \ldots, N \} \) and \( \{ y_{i'}'; i' = 1, \ldots, N' \} \) of the D-3-brane, respectively. The charges of this solution are \( p_i = \alpha_i \) and \( q_{i'} = \beta_{i'} \).

Next let us suppose that \( N = N' \) and \( y_i = y_{i'}' \). Such solution describes \( N \) world-volume dyons and has the macroscopic bulk interpretation of \( N \) dyonic strings with charges \( \{(p_i, q_i) = (\alpha_i, \beta_i); i = 1, \ldots, N\} \) ending at the points \( \{y_i; i = 1, \ldots, N\} \) of the D-3-brane. These strings are not parallel in the \( X, Y \) plane but end orthogonally on the D-3-brane. The angle that these strings lie relative to \( X, Y \) directions is

\[ \tan \phi_i = \frac{\beta_i}{\alpha_i}. \]

(2.14)

We shall show in the next section that when all the strings are parallel the solution preserves 1/4 of supersymmetry. The mass of the dyons that we have found is
infinite since the length of the associated (p,q)-strings is infinite (for a similar discussion see [13]). Therefore to compute the metric on the moduli space of such solutions, we have to ‘regularize’ the above solutions. This will be explained in detail in section four.

To explore the $SL(2, \mathbb{R})$ invariance of the field equations of our D-3-brane solution and demonstrate further their dyonic nature, let us take $H_1 = 0$ and

$$H = H_2 = 1 + \frac{\beta}{|x|}.$$  \hspace{1cm} (2.15)

For such solution, the electric part of $F_1$ and the magnetic part of $F_2$ vanish. So it describes a particle of charge $(0, \beta)$ in the $\tau = i$ vacuum. We can now use the method in [9], to construct a dyonic solution with charge $(p, q)$ at any vacuum $\tau$. For this we take $H$ as above for some $\beta$ parameter and set $f = F_1$ for the non-vanishing two-form field strength. Then we act on the magnetic part of $(F_1, F_2) = (f, 0)$ with the transformation

$$A = \frac{1}{\sqrt{p^2 + q^2}} \begin{pmatrix} q & -p \\ p & q \end{pmatrix}.$$  \hspace{1cm} (2.16)

of $U(1) \subset SL(2, \mathbb{R})$ that stabilizes the vacuum $\tau = i$ to give

$$f_1 = \frac{q}{\sqrt{p^2 + q^2}} f$$
$$f_2 = \frac{p}{\sqrt{p^2 + q^2}} f.$$  \hspace{1cm} (2.17)

Now if we choose $\beta = \sqrt{p^2 + q^2}$, then the solution with magnetic part $(F_1, F_2) = (f_1, f_2)$ describes a dyon with charges $(p, q)$ in the $\tau = i$ vacuum. The electric part of $(F_1, F_2)$ can be easily determined in a similar way. To give the solution of a $(p, q)$-dyon at a generic vacuum $\lambda_0$, we first write for the magnetic part of
\[(F_1, F_2) = (f_1, f_2) = (\cos \theta f, \sin \theta f), \text{ where}\]

\[e^{i\theta} = \frac{q + ip}{\sqrt{p^2 + q^2}}. \quad (2.18)\]

Then we act with the transformation

\[A = \begin{pmatrix} e^{-\frac{\phi_0}{2}} & -\rho_0 e^{\frac{\phi_0}{2}} \\ 0 & e^{\frac{\phi_0}{2}} \end{pmatrix}. \quad (2.19)\]

The vacuum \(\tau = i\) is transformed to \(\tau_0 = \rho_0 + ie^{-\phi_0}\). Moreover the magnetic part \((B_1, B_2)\) of \((F_1, F_2)\) is

\[B_1 = e^{-\frac{\phi_0}{2}} \cos \theta f - \rho_0 e^{\frac{\phi_0}{2}} \sin \theta f \quad (2.20)\]

\[B_2 = e^{\frac{\phi_0}{2}} \sin \theta f.\]

For this dyon to have charge \((p, q)\), we set

\[e^{i\theta} = \frac{(p\lambda_0 + q)e^{\frac{\phi_0}{2}}}{\sqrt{(p\rho_0 + q)^2e^{\phi_0} + p^2e^{-\phi_0}}} \quad (2.21)\]

and

\[\beta = \sqrt{(p\rho_0 + q)^2e^{\phi_0} + p^2e^{-\phi_0}}. \quad (2.22)\]

The electric parts of \((F_1, F_2)\) can be easily determined in a similar way.

After quantization, the charges \((p, q)\) and \((p', q')\) of two worldvolume dyons are quantized according to the Schwinger-Zwanziger quantization condition

\[pq' - qp' \in \mathbb{Z}. \quad (2.23)\]

Then the bulk interpretation of our dyonic solution is that of \(\{k_i; i = 1, \ldots, N\}\) \((\tilde{\alpha}_i, \tilde{\beta}_i)\)-strings ending on the D-3-brane, where \(k_i\) is the maximal common divisor of \((\alpha_i, \beta_i)\).
3. $\kappa$-symmetry of D-3-brane dyons

To find the supersymmetry preserved by a worldvolume solution of a brane, we make use of the kappa-symmetry transformations

$$\delta \theta = P_+ \kappa \equiv (1 + \Gamma) \kappa ,$$

(3.1)

where $\kappa$ is the kappa-symmetry parameter, $\Gamma$ satisfies $\Gamma^2 = 1$ and $\text{Tr} \Gamma = 0$. Then $P_\pm = (1/2)(1 \pm \Gamma)$ are projection operators. Then it has been shown in [12] that the supersymmetry condition is

$$(1 - \Gamma) \epsilon = 0 ,$$

(3.2)

where $\epsilon$ is the spacetime supersymmetry parameter.

To describe the kappa-symmetry transformations of D-3-brane, we introduce

$$\gamma(k) = \frac{1}{k!} dX^{A_1} \wedge \ldots \wedge dX^{A_k} \Gamma_{A_1} \ldots \Gamma_{A_k} ,$$

(3.3)

where $X$ are the embedding maps of the D-3-brane worldvolume into spacetime and $\{\Gamma_A; A = 0, \ldots , 9\}$ are the spacetime gamma-matrices in the Majorana-Weyl representation. Then for the D-3-brane, the projections $P_\pm$ are

$$(*F) P_\pm \eta = \frac{1}{2} (*F) \eta \mp \frac{i}{2} (*F \wedge \gamma(2)) \bar{\eta} \pm i(*\gamma(4)) \eta ,$$

(3.4)

where $\eta = \eta_1 + i\eta_2$; $\eta_1, \eta_2$ are 16-component Majorana-Weyl spinors of the same chirality. The supersymmetry condition is

$$\frac{1}{2} (*F) \epsilon - \frac{i}{2} (*F \wedge \gamma(2)) \bar{\epsilon} - i(*\gamma(4)) \epsilon = 0 .$$

(3.5)

For our solutions, this condition can be written as

$$(*\text{Im}(F) \wedge \gamma(2)) \epsilon_1 - (2(*\gamma(4)) + (*\text{Re}(F) \wedge \gamma(2))) \epsilon_2 = (1 + \frac{1}{2} |\nabla X|^2 + \frac{1}{2} |\nabla Y|^2) \epsilon_1 ,$$

(3.6)

where $\epsilon = \epsilon_1 + i\epsilon_2$. If the exterior derivatives $dX, dY$ of the fields $X, Y$ of our
solution are linearly independent, then (3.6) implies that

\begin{align*}
\epsilon_1 &= -\Gamma_0 \Gamma_1 \Gamma_2 \epsilon_2 \\
\epsilon_1 &= \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_5 \epsilon_1 \\
\epsilon_1 &= \Gamma_0 \Gamma_4 \epsilon_1,
\end{align*}

and so the solution preserves 1/8 of bulk supersymmetry. If on the other hand

\[ dX - rdY = 0, \]

where \( r \) is a real number, then (3.6) implies that

\begin{align*}
\epsilon_1 &= -\Gamma_0 \Gamma_1 \Gamma_2 \epsilon_2 \\
-\epsilon_1 &= (r(\Gamma_0 + \Gamma_4) + \Gamma_5) \Gamma_1 \Gamma_2 \Gamma_3 \epsilon_1,
\end{align*}

and the solution preserves 1/4 of bulk supersymmetry. It is clear that for such solutions \( N = N', \ y_i = y'_i \) and \( \alpha_i \beta_j = \alpha_j \beta_i \) for every \((i, j)\). The supersymmetry preserved by our dyons is in agreement with that preserved by the bulk configurations in the introduction for which all strings are parallel. We can identify \( r \) with \( \cot \phi \), where \( \phi \) is the angle that the strings lie in the \( X, Y \) plane. We can always take \( r = 0 \) or \( r = \infty \) using possibly a field redefinition, i.e. a rotation in the \( X, Y \) plane.

4. The Moduli Spaces of 0-brane Worldvolume Solitons Revisited

Before we compute the metric on the moduli space of the dyons we have found in the previous sections, we shall briefly review and improve on some aspects of the computation we have done for the metric on the moduli space of 0-brane worldvolume solitons of D-p-branes in [5]. The non-vanishing fields for such solution is a transverse scalar \( Y \) and the electric part of the Born-Infeld field \( F \) of the D-4-brane.
The solution [13, 14] is
\[ F = dt \wedge dH, \]
\[ Y = H, \]
where \( H \) is a harmonic function in \( \mathbb{R}^4 \). We may take
\[ H = 1 + \sum_{i=1}^{N} \frac{\mu_i}{|x - y_i|^{p-2}}, \]
where \( \{x^a; a = 1, \ldots, p\} \) are the spatial worldvolume coordinates of a D-4-brane and \( \{(y^a_i; a = 1, \ldots, p); i = 1, \ldots, N\} \) are the locations on \( N \) 0-brane worldvolume solitons. The location of the D-p-brane at \( |x| \to \infty \) is at \( Y = 1 \).

Next let us consider the case of one such 0-brane, i.e. \( N = 1 \). We can choose without loss of generality \( y_1 = 0 \) and so the 0-brane is located at \( |x| = 0 \). It has been demonstrated in [13] that this solution has infinite mass. For this, the energy \( E \) of the configuration is computed in a region \( |x| > R \) and it is found that \( E(R) \sim |Y(R) - 1| \). This is precisely the mass \( M(R) \) of a string with constant tension and length \( |Y(R) - 1| \). Now it can be easily seen from (4.1) that as \( R \to 0 \), \( Y(R) \to \infty \) and so \( E \to \infty \). Therefore this infinity has the physical interpretation as the mass of a string with constant tension and infinite length. The introduction of the characteristic size \( R \) for the 0-brane can be thought as a cut off for distances close to its location.

For the computation of the moduli space metric, we should keep the masses of the 0-branes finite. This, as we have seen, requires a ‘regularization’ of the solution (4.1). However, the cut off regularization described above is not convenient. A more instructive way to ‘regularize’ the solution is to assume that the 0-brane solitons are balls of radius \( R \) with centres at \( \{y_i; i = 1, \ldots, N\} \) and with constant charge density \( \{\rho_i; i = 1, \ldots, N\} \), where
\[ \rho_i = \begin{cases} 0 & : |x - y_i| > R \\ \frac{\mu_i}{R^p} & : |x - y_i| < R \end{cases}. \]

The regularized 0-brane solution is as in (4.1) but now \( H = H(R) = 1 + \sum_{i=1}^{N} H_i \),
where
\[
H_i = \begin{cases} 
\frac{\mu_i}{|x-y_i|^2} : |x-y_i| > R \\
\frac{p\mu_i}{2R^{p+2}} + \frac{(2-p)\mu_i}{2R^p}|x-y_i|^2 : |x-y_i| < R
\end{cases}.
\] (4.3)

One of the advantages of this regularization is that the fields are continuous at 
\(|x - y_i| = R\) and so certain surface terms due to partial integrations that appear 
in the calculation of the moduli metric can be easily handled. The energy of a single 
0-brane is
\[
E = \frac{2p\Omega_{p-1}\mu}{p+2}(Y(R) - 1)
\] (4.4)
and so again it has the interpretation of the mass of a string with constant tension 
\(T_f = \frac{2p\Omega_{p-1}\mu}{p+2}\) of length \(Y(R) - 1\), where \(\Omega_{p-1}\) is the volume of a \((p-1)\)-sphere of 
unit radius and \(\mu = \mu_1\).

The moduli calculation can proceed as in [5] by using the linearized\(^*\) \(p\)-brane action
\[
S = S_0 + S_{\text{Source}} + S_{\text{free}},
\] (4.5)
where
\[
S_0 = \frac{1}{2} \int d^{p+1}x \left( \eta^{\mu\nu} \partial_\mu Y \partial_\nu Y + \frac{1}{2} F_{\mu\nu} F^{\mu\nu} \right),
\] (4.6)
\[
S_{\text{Source}} = (2-p)\Omega_{p-1} \sum_{i=1}^{N} d\tau_i \rho_i \left( Y + A_\mu \frac{\partial y_i^\mu}{\partial \tau_i} \right)
\] (4.7)
and
\[
S_{\text{free}} = -\sum_{i=1}^{N} M_i \int d\tau_i + (p-2)\Omega_{p-1} \sum_{i=1}^{N} \mu_i \int d\tau_i ;
\] (4.8)
\(\{M_i; i = 1, \ldots, N\}\) are the masses of the worldvolume 0-branes. The sources 
\(S_{\text{Source}}\) that we have added in the action are those of the classical solution (4.1). 
These sources should have a bulk interpretation as a fundamental string and a

\(^*\) This calculation can also be done with the full non-linear Born-Infeld Lagrangian.
D-p-brane; the former is the source of the Born-Infeld field $F$ while the latter is the source of the scalar $Y$. The first part in $S_{\text{free}}$ is the action of $N$ free relativistic particles with masses $\{M_i; i = 1, \ldots, N\}$. The second part in $S_{\text{free}}$ has been added in order to make the effective action of the worldvolume 0-branes independent from the (asymptotic) position of the D-p-brane.

Repeating the computation as in [5] but now with the ‘regularized’ solution, we find that the metric on the moduli space is

$$ds^2 = \sum_{i=1}^{N} M_i |dy_i|^2 + (p-2)\Omega_{p-1} \sum_{i<j}^{N} \mu_i \mu_j k(y_i - y_j) |dy_i - dy_j|^2 , \quad (4.9)$$

where

$$k(y_i - y_j) = \frac{p}{R^p} \int_{Sp^{-1}}^{R} \int_{0}^{R} \frac{r^{p-1}}{|r + y_i - y_j|^{p-2}} dr d\Omega_{p-1} . \quad (4.10)$$

All the properties of the moduli metric found in [5] are still valid provided that $|y_i - y_j| > R$. For example, the moduli space of 0-brane solitons on the D-4-brane is a hyper-Kähler manifold with torsion. The same applied for the moduli space of the worldvolume self-dual string soliton of M-5-brane. Now if the separation distance of worldvolume 0-branes is much larger than the size of the balls, $|y_i - y_j| >> R$, then

$$k(y_i - y_j) = \frac{1}{|y_i - y_j|^{p-2}} + O(\frac{R}{|y_i - y_j|}) . \quad (4.11)$$

Substituting this into (4.9), we recover the result found in [5]. It is clear then that the quadratic in the velocities interaction terms of well-separated 0-brane solitons are independent of the regularization of their masses. Therefore, the moduli metric found above is the moduli metric of well-separated 0-brane solitons. The order parameter is the ratio of the size of the 0-branes with their separation distance. This approximation is the same as that employed in [10] to find that the asymptotic metric of the BPS monopoles moduli space is toric hyper-Kähler. This way of understanding the moduli metric (4.9) will be further enforced by the result of the
next section. Incidentally, the large separation distance approximation is consistent with suppressing the interactions of the 0-branes due the massive intermediate gauge bosons of the Coloumb branch of the full non-abelian effective theory of D-p-branes. This is because the forces of such interactions are short range and so in the above limit do not contribute. However for the full treatment of the moduli space of D-0-branes, they should be taken into account.

5. Moduli Space metrics

To determine the metric on the moduli space of our dyons, we have to compute the quadratic term in the velocities of their effective action. For this we should again ‘regularize’ our solution in section two using one of the methods that we have described in the previous section for the other 0-brane worldvolume solitons. To simplify matters, we shall perform our computations assuming that the dyons are well-separated. We shall investigate the moduli space for those solutions for which \( N = N' \) and \( y_i = y'_i \). For this using the method of [15, 16, 17, 5], we add to the action (2.6) the source term

\[
S_{\text{source}} = -4\pi\sigma \sum_{i=1}^{N} \int d\tau_i \delta(x - y_i) (\hat{p}_i X + \hat{q}_i Y + \frac{1}{2}\hat{p}_i (A_1)_\mu \frac{dy_i^\mu}{d\tau_i} - \frac{1}{2}\hat{q}_i (A_2)_\mu \frac{dy_i^\mu}{d\tau_i}) \\
+ \frac{\sigma}{2} \int d^4x \left( \frac{1}{96} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu\rho\sigma} - 2 \right),
\]

(5.1)

where \( \sigma \) is a constant and \( \{\tau_i; i = 1, \ldots, N\} \) are the proper times of dyons. The scalar charges \( \{\hat{p}_i, \hat{q}_i\} \) can be chosen to be independent from \( \{p_i, q_i\} \). Though they can be related for particular solutions. For the purpose of this paper, we take

\[
\hat{p}_i = \sqrt{p_i^2 + q_i^2} \cos \phi_i \\
\hat{q}_i = \sqrt{p_i^2 + q_i^2} \sin \phi_i,
\]

(5.2)

where \( \{\phi_i; i = 1, \ldots, N\} \) are some phase angles. Note that the last two terms in (5.1) do not affect the field equations.
The parameters of the source terms and those of the solution (2.11) are related as

\[ p_i = \hat{p}_i = \alpha_i \]
\[ q_i = \hat{q}_i = \beta_i . \]  
(5.3)

These are the charges of the dyons. The expression (5.3) for the scalar charges is consistent with that of (5.2), if we choose the phase angles for this solution as

\[ \tan \phi_i = \frac{\beta_i}{\alpha_i} . \]  
(5.4)

The field equations of (2.6) with source terms (5.1) for the fields \( \lambda, W, A \) are

\[ *F = 2\sqrt{1 + \frac{1}{2} F \cdot \bar{F} - \frac{1}{16} (F \cdot *F)(\bar{F} \cdot *\bar{F})} \]
\[ \lambda *F = \sigma \]  
(5.5)

\[ \partial_\nu (\sqrt{-g} *F^{\mu \nu}) = 4\pi i \sum_{i=1}^{N} (p_i - iq_i) \delta(x - y_i) \frac{dy_i^\nu}{d\tau_i} , \]
respectively. The field equations for the two transverse scalars \( X, Y \) are

\[ \partial_\nu (\lambda \sqrt{-g} (F^{(\mu}_\alpha F^{[\alpha \mu])}) + \frac{1}{2} (F \cdot \bar{F}) g^{\mu \nu} - \]
\[ \frac{1}{12} F_{\mu \sigma \lambda \tau} F^{\nu \sigma \lambda \tau}) \partial_\mu X) = -4\pi \sigma \sum_{i=1}^{N} \hat{p}_i \delta(x - y_i) \]
\[ \partial_\nu (\lambda \sqrt{-g} (F^{(\mu}_\alpha F^{[\alpha \mu])}) + \frac{1}{2} (F \cdot \bar{F}) g^{\mu \nu} - \]
\[ \frac{1}{12} F_{\mu \sigma \lambda \tau} F^{\nu \sigma \lambda \tau}) \partial_\mu Y) = -4\pi \sigma \sum_{i=1}^{N} \hat{q}_i \delta(x - y_i) . \]  
(5.6)

For the Hodge star operation and for the raising and lowering of indices we have used the induced metric on the D-3-brane. The constant \( \sigma \) is related to the tension of the D-3-brane.
We shall consider perturbations of our solution that involve the fields and the various charges. To illustrate how such a perturbation can be done, let $\mathcal{L}(\phi, s)$ be a Lagrangian of a field $\phi$ and depending on some parameters $s$. Suppose that we have a solution $(\phi_0, s_0)$ of this system. Next consider a perturbation $(\phi(u), s(u))$ of this theory such that $(\phi(0), s(0)) = (\phi_0, s_0)$. It is easy to see that if

$$\frac{\partial \mathcal{L}(\phi, s)}{\partial s}\bigg|_{s=s_0} = 0,$$

then the linear terms in the perturbation of the Lagrangian vanish subject to the field equations* of $\phi$. In addition, it turns out that only the linear terms in the perturbation of the fields and of the parameter $s$ contribute in the quadratic perturbation of the Lagrangian.

Next, we consider the following perturbation:

(i) We allow

$$y_i \rightarrow y_i(t).$$

(ii) The charges are perturbed as

$$\alpha_i \rightarrow p_i = \alpha_i + \beta_i \chi_i$$

$$\beta_i \rightarrow q_i = \beta_i - \alpha_i \chi_i,$$

where $\chi_i$ are the perturbations of the charges.

(iii) The perturbation of the transverse scalars $X, Y$ is induced only by that in $y_i$, i.e.

$$X(\alpha_i, y_i) \rightarrow X(\alpha_i, y_i(t))$$

$$Y(\beta_i, y_i) \rightarrow Y(\beta_i, y_i(t))$$

and so $X, Y$ are not perturbed linear in the velocities.

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* In the supergravity context, (5.7) should be thought as part of the supergravity field equations for the IIB dilaton and axion.
The perturbation in the vector potentials \( A_1, A_2 \) of \( (F_1, F_2) \) is as

\[
A_1(\alpha_i, \beta_i, y_i) \rightarrow A_1(p_i, q_i, y_i(t)) + B_1
\]

\[
A_2(\alpha_i, \beta_i, y_i) \rightarrow A_2(p_i, q_i, y_i(t)) + B_2 ,
\]

where \( B_1, B_2 \) are determined by the field equations.

The perturbations of the scalar charges \( \hat{p_i}, \hat{q_i} \) are determined by (5.2). But in order to perturb the scalar fields as in (5.10) in a way consistent with the field equations, the phase angles \( \phi_i \) remain unperturbed. Their values are those of the original solution. The perturbations of \( F \) and \( \lambda \) are determined by those of the other fields above using the field equations.

To find \( B_1, B_2 \), we consider the fields equations of \( \mathcal{F} \) both linearized in the fields and the velocities. These are

\[
\partial_0(A_1)_n - \partial_n(B_1)_0 = \epsilon^l_s \partial_l(B_2)_s
\]

\[
\partial_0(A_2)_n - \partial_n(B_2)_0 = -\epsilon^l_s \partial_l(B_1)_s .
\]

The solution of these equations linear in velocities is

\[
(B_1)_0 = \sum_{i=1}^{N} \beta_i v_i \cdot w(x - y_i)
\]

\[
(B_2)_0 = \sum_{i=1}^{N} \alpha_i v_i \cdot w(x - y_i)
\]

\[
B_1 = \sum_{i=1}^{N} \frac{\alpha_i}{|x - y_i|} v_i
\]

\[
B_2 = -\sum_{i=1}^{N} \frac{\beta_i}{|x - y_i|} v_i ,
\]

where \( w \) is the Dirac vector potential defined so that

\[
\nabla \wedge w(x) = \frac{1}{|x|^3} x .
\]

So far our analysis applies to all solutions that we have found in section two. In what follows, we shall present the moduli space metric only for those solutions that
preserve 1/4 of spacetime supersymmetry. The moduli space metric for the rest of the solutions will be given in the next section. For the solutions that preserve 1/4 of supersymmetry, the transverse scalars $X, Y$ obey the linear relation (3.8).

Without loss of generality, we can take $r = 0$ to find that $\alpha_i = 0$. So $X = m_1$ constant and we choose $X = 0$. It turns out that in this case the solution (5.13) of (5.12) also solves the field equations of the non-linear theory up to terms linear in velocities. Moreover, the deformation of the charges (5.9) also obeys (5.7) and therefore there are no contributions in the effective action linear in the velocities. Substituting our solution back into the action

$$S = S_{D3} + S_{Source} + S_{free},$$

(5.15)

where

$$S_{free} = -\sum_{i=1}^{N} M_i(R) \int d\tau_i + 4\pi \sigma m \sum_{i=1}^{N} \beta_i \int d\tau_i$$

(5.16)

and collecting the terms quadratic in the velocities, we find

$$S_{Eff} = \int dt \left( \frac{1}{2} \sum_{i=1}^{N} M_i(R) |v_i|^2 - 2\sigma \left[ \sum_{i=1}^{N} m_i \chi_i^2 + \sum_{i\neq j}^{N} \frac{\beta_i \beta_j}{r_{ij}} (\chi_i^2 - \chi_i \chi_j) + \sum_{i\neq j}^{N} \frac{\beta_i \beta_j}{r_{ij}} (v_i, v_j - |v_i|^2) + 2\beta_i \beta_j \chi_i (v_j - v_i) \cdot w_{ij} \right] \right),$$

(5.17)

where $m = m_2$, 

$$r_{ij} = |y_i - y_j|,$$

(5.18)

$\{M_i(R); i = 1, \ldots, N\}$ is the rest mass of the dyons*, and $w_{ij} = w(y_i - y_j)$. To compute (5.17), we remark that both the Lagrangian density (2.6) and the source

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* We have computed the masses of dyons in the linearized theory. Unless the solution is appropriately ‘regularized’, the masses diverge in the same way as those of the rest of 0-brane solitons.
terms (5.1) contribute. Moreover the second term in $S_{\text{free}}$ has been added in order that the effective action of the dyons is independent of the asymptotic position of the D-3-brane as for the 0-branes in the previous section. Because $m$ appears in the effective Lagrangian above, it seems that the moduli metric of the dyons depends on the position of the D-3-brane. However this is not the case since the $m$ dependent terms do not contribute in the field equations and can be tuned appropriately to write the Lagrangian in a more symmetric way.

The effective Lagrangian that we have obtained above is similar, up to adjusting some parameters, to that derived in [10] for well-separated BPS monopoles. To continue, we shall make use of the reasoning in that paper to find that the moduli space of our solutions is a toric hyper-Kähler manifold. For this, we first rewrite the Lagrangian of (5.17) as

$$L = \frac{1}{2}g_{ij}v_i v_j + \frac{1}{\kappa}\chi_i W_{ij} v_j - \frac{1}{2\kappa^2}h^{ij}\chi_i\chi_j$$

(5.19)

where $\kappa$ is a constant, $h^{ij} = \kappa^2 g_{ij}$,

$$g_{jj} = M_j(R) + 4\pi\sigma\sum_{i \neq j}^N \frac{\beta_i\beta_j}{r_{ij}}$$

(5.20)

$$g_{ij} = -4\pi\sigma\frac{\beta_i\beta_j}{r_{ij}}, \quad i \neq j,$$

and

$$W_{jj} = 4\pi\kappa\sigma\sum_{i \neq j}^N \beta_i\beta_j w_{ij}$$

(5.21)

$$W_{ij} = -4\pi\sigma\beta_i\beta_j\kappa w_{ij}, \quad i \neq j.$$

We have also tuned appropriately the $m$-dependent terms in (5.17). The above Lagrangian has equations of motion which are identical to those of

$$\mathcal{L} = \frac{1}{2}[g_{ij}v_i v_j + g^{ij}(\dot{\theta}_i + \frac{1}{\kappa}W_{ik} v_k)(\dot{\theta}_j + \frac{1}{\kappa}W_{jl} v_l)]$$

(5.22)
up to a rescaling of $\theta_i$. To see this, we remark that

$$Q^i = g^{ij}(\dot{\theta}_j + \frac{1}{\kappa} W_{jl} \nu_l)$$  \hspace{1cm} (5.23)

are conserved subject to field equations. To relate the field equations we simply set $\chi_i = Q^i$. The coordinates $\theta$ parameterize the deformations of the charges.

The metric

$$ds^2 = g_{ij} dx_i dx_j + g^{ij}(d\theta_i + \frac{1}{\kappa} W_{ik} dx_k)(d\theta_j + \frac{1}{\kappa} W_{jl} dx_l)$$  \hspace{1cm} (5.24)

on the moduli space of dyons given by the effective Lagrangian (5.22) is toric hyper-Kähler. In fact it is a special case of the toric hyper-Kähler metrics of [18]. In the context of non-abelian effective theory of D-3-branes, our calculation describes the moduli space of well-separated (non-abelian) dyons. The effective action we have found is expected to be that of a one-dimensional sigma model with eight supersymmetries. The geometry we have found is consistent with the target space geometries of such sigma models (see for example [19]). The dyons we have investigated are T-dual to the worldvolume 0-branes of the D-4-brane [20]. Our results above suggest that this duality is extended to their dynamics. This is because the toric hyper-Kähler moduli spaces that we have found here are, under sigma model duality, dual to the hyper-Kähler with torsion moduli spaces found for the worldvolume 0-brane solitons of D-4-brane in [5]. Note that if we neglect the charge moduli parameters $\chi_i$, then we recover the moduli metric of the 0-brane worldvolume solitons on the D-3-brane of [5] (see also [6]).
6. Other Dyons

The computation of the moduli metric for the configurations that preserve 1/8 of spacetime supersymmetry is similar to that of the dyons that preserve 1/4 of supersymmetry. However there are some differences. First the solution (5.13) of the linearized field equations is no longer a solution of the non-linear theory up to terms linear in velocities. In addition, the parameters do not satisfy the equation (5.7). Consequently, one expects that the effective action should contain linear terms in the velocities. Moreover in order to obtain the full effective action quadratic in the velocities, quadratic velocity deformations in the charges should be taken into account.

We have not been able to obtain the solution for the deformation of the one-form gauge potential within the non-linear theory. So we shall use the solution (5.13) to compute the moduli metric for the linearized theory. The effective action is

\[
\mathcal{L}_{\text{eff}} = \int dt \left( \frac{1}{2} M_i(R) |v_i|^2 - 2 \pi \sigma \left[ \sum_{i=1}^{N} -(m_1 \alpha_i + m_2 \beta_i) |v_i|^2 \\
+ \sum_{i \neq j} \frac{(\alpha_i \alpha_j + \beta_i \beta_j)}{r_{ij}} (x_i^2 - \chi_i \chi_j) + \frac{(\alpha_i \alpha_j + \beta_i \beta_j)}{r_{ij}} (v_i \cdot v_j - |v_i|^2) \\
+ 2(\alpha_i \alpha_j + \beta_i \beta_j) \chi_i (v_j - v_i) \cdot w_{ij} + 2(\alpha_i \beta_j - \alpha_j \beta_i) v_j \cdot w_{ij} \\
+ 2(\alpha_i \beta_j - \alpha_j \beta_i) \chi_i \right] \right)
\]

(6.1)

The second order part of this Lagrangian is very similar to that in (5.17), and it defines a toric hyper-Kähler moduli space metric. However, there are additional first order terms. This effective Lagrangian may have additional nonlinear contributions added to it when the full effect of the nonlinear equations of motion is taken into account. The part of our effective action which does not involve contributions from the charge deformation agrees with the one computed in [6] for solutions of Dirac-Born-Infeld actions that preserve 1/8 of supersymmetry.
7. Conclusions

We have found the static solutions of the field equations of a $SL(2,\mathbb{R})$ covariant D-3-brane action which have the interpretation of dyons on the D-3-brane worldvolume. We have identified them with the dyons generated by a $(p,q)$-string ending on the D-3-brane. We have found using $\kappa$-symmetry that these dyons preserve $1/4$ of spacetime supersymmetry. In addition, we have shown that the moduli space of well-separated dyons is a toric hyper-Kähler manifold. Other worldvolume dyons are also found. We argue that these are associated using duality to the M-theory configuration of two orthogonal membranes ending on a M-5-brane. We have shown that these dyons preserve $1/8$ of supersymmetry and we have computed their effective action.

It would be of interest to have a better description of the moduli space of all worldvolume brane solitons, specially those that preserve less than $1/4$ of bulk supersymmetry. In many cases such description is known. For example, the moduli spaces of worldvolume solitons that are associated with calibrations have been extensively studied in the mathematics literature. The same applies, at least in the linearized limit, for the moduli spaces of the worldvolume solitons that can be identified with BPS solitons or instantons. It would be of interest to investigate the moduli spaces of worldvolume solitons that are associated with both calibration-like surfaces and non-abelian Born-Infeld field configurations.

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