On invariant analysis and conservation laws for degenerate coupled multi-KdV equations for multiplicity \( l = 3 \)

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Abstract. The degenerate coupled multi-Korteweg–de Vries equations for coupled multiplicity \( l = 3 \) are studied. The equations, also known as three-field Kaup–Boussinesq equations, are considered for invariant analysis and conservation laws. The classical Lie’s symmetry method is used to analyse the symmetries of equations. Based on the Killing’s form, which is invariant of adjoint action, the full classification for Lie algebra is presented. Further, one-dimensional optimal group classification is used to obtain invariant solutions. Besides this, using general theorem proved by Ibragimov, we find several non-local conservation laws for these equations. The conserved currents obtained in this work can be useful for the better understanding of some physical phenomena modelled by the underlying equations.

Keywords. Lie symmetries; optimal system; exact solutions; conservation laws.

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1. Introduction

Recently, degenerate coupled Korteweg–de Vries (KdV) equations for coupled multiplicity \( l = 2, 3, 4 \) have been considered for travelling wave solutions by Gürses and Pekcan [1,2]. The multisystem of Kaup–Boussinesq equations is given by (see [3,4])

\[
\begin{align*}
    u_t & = \frac{3}{2} uu_x + q^2_x, \\
    q^2_t & = q^2 u_x + \frac{1}{2} u q_x^2 + q_x^3, \\
    & \vdots \\
    q^{l-1}_t & = q^{l-1} u_x + \frac{1}{2} u q_x^{l-1} + v_x, \\
    v_t & = -\frac{1}{4} u_{xxx} + v u_x + \frac{1}{2} u v_x,
\end{align*}
\]

(1)

where \( q^1 = u \) and \( q^l = v \). For \( l = 2 \), system (1) reduces to

\[
\begin{align*}
    u_t & = \frac{3}{2} uu_x + v_x, \\
    v_t & = -\frac{1}{4} u_{xxx} + v u_x + \frac{1}{2} u v_x.
\end{align*}
\]

(2)

System (2) has been studied in detail by Gürses and Pekcan [1] for travelling wave solutions, and they have proved that there exists no asymptotically vanishing travelling wave solution of system (2). Wazwaz [5] has studied generalised version of system (2) for multiple-soliton solutions. It is pertinent to mention that system (2), which is also known as Kaup–Boussinesq system, exhibits the same shallow water wave characteristics in the same approximation as the well-known Boussinesq equation in the lowest order in small parameters controlling weak dispersion and nonlinearity effects [6–9]. Moreover, the function \( v(x, t) \) denotes the height of the water surface above a horizontal bottom, whereas the function \( u(x, t) \) denotes its velocity averaged over depth. The Kaup–Boussinesq system (2) corresponds to the case when the gravity force dominates over the capillary one and it is completely integrable [10–12].

For \( l = 3 \), system (1) has the following form:

\[
\begin{align*}
    u_t - \frac{3}{2} uu_x - v_x & = 0, \\
    v_t - vu_x - \frac{1}{2} uu_x - w_x & = 0, \\
    w_t & + \frac{1}{4} u_{xxx} - uu_x - \frac{1}{2} uw_x = 0.
\end{align*}
\]

(3)
The system of eq. (3), which is also known as three-field Kaup–Boussinesq equations [13], has been discussed for travelling wave solutions [2], wherein authors have given a general approach to solve eq. (1) for \( l \geq 3 \). Subsequently, by using the bifurcation analysis, Li and Chen [14] have given complete parametric representations of travelling wave solutions of system (1) for \( l = 2, 3, 4 \), which was missing in the work of Gürses and Pekcan [1,2]. In the literature, we have noticed that eq. (3) has not been completely analysed and so in this work, we propose to investigate eq. (3) for Lie’s symmetry analysis and for conservation laws using the recently proposed new theorem by Ibragimov. 

The paper is organised as follows. In §2, based on the classical Lie symmetry analysis, we have obtained four-dimensional Lie algebra. Starting with a brief discussion about classification techniques, Lie algebra is then classified into mutually conjugate classes by identifying the Killing’s form which is invariant of full adjoint action. Reductions are also presented corresponding to every conjugate class and exact solutions are also obtained. In §3, based on a new theorem proposed by Ibragimov, several non-local conservation laws are also constructed. Finally, in §4, the conclusion is drawn.

2. Lie symmetry analysis of Kaup–Boussinesq equation (3)

In order to identify Lie point symmetries for eq. (3), we follow the standard procedure given in [15–18]. The procedure is so algorithmic that it has been successfully implemented in symbolic languages such as ‘Maple’ and ‘Mathematica’. The Maple package ‘PDEtools’ written by Terrab [19] is quite interactive and efficient. It becomes indispensable for researchers in the field of partial differential equations (PDEs). In the following, we have used this Maple package to find out Lie symmetries for eq. (3). So, we consider one-parameter local Lie group of point transformations:

\[
\begin{align*}
\tilde{x} &= x + \epsilon \xi(x, t, u, v, w) + O(\epsilon^2), \\
\tilde{t} &= t + \epsilon \tau(x, t, u, v, w) + O(\epsilon^2), \\
\tilde{u} &= u + \epsilon \eta_1(x, t, u, v, w) + O(\epsilon^2), \\
\tilde{v} &= v + \epsilon \eta_2(x, t, u, v, w) + O(\epsilon^2), \\
\tilde{w} &= w + \epsilon \eta_3(x, t, u, v, w) + O(\epsilon^2),
\end{align*}
\]

(4)

where \( \epsilon \) is the group parameter. The invariance of eq. (3) under symmetry transformations (4) gives rise to overdetermined system of linear partial differential equations in \( \xi, \tau, \eta_1, \eta_2 \) and \( \eta_3 \). Such overdetermined system may be derived by considering the associated vector field, which may be expressed as

\[ V = \xi \frac{\partial}{\partial x} + \tau \frac{\partial}{\partial t} + \eta_1 \frac{\partial}{\partial u} + \eta_2 \frac{\partial}{\partial v} + \eta_3 \frac{\partial}{\partial w}. \]  

(5)

Third-order prolongation of vector field (5) when applied in the following manner

\[ V^{(3)}(\Delta)|_{(3)} = 0, \quad \text{here } \Delta \text{ is system (3)}, \]  

(6)

will give infinitesimals of symmetry transformation as follows:

\[
\begin{align*}
\xi &= -\frac{5c_3}{6} t + \frac{3c_1}{5} x + c_4, \\
\tau &= c_1 t + c_2, \\
\eta_1 &= -\frac{2c_1}{5} u + c_3, \\
\eta_2 &= -\frac{4c_1}{5} v - \frac{2c_3}{3} u, \\
\eta_3 &= -\frac{c_3}{3} v - \frac{6c_1}{5} w.
\end{align*}
\]  

(7)

Infinitesimals (7) give the following four-dimensional Lie algebra:

\[
\begin{align*}
V_1 &= \frac{\partial}{\partial t}, \\
V_2 &= \frac{\partial}{\partial x} \quad \text{(translation),} \\
V_3 &= -\frac{5t}{6} \frac{\partial}{\partial x} + \frac{\partial}{\partial u} - \frac{2u}{3} \frac{\partial}{\partial v} - \frac{v}{3} \frac{\partial}{\partial w} \quad \text{(Galilean boost),} \\
V_4 &= \frac{3x}{5} \frac{\partial}{\partial x} + \frac{t}{5} \frac{\partial}{\partial t} - \frac{2u}{5} \frac{\partial}{\partial u} - \frac{4v}{5} \frac{\partial}{\partial v} - \frac{6w}{5} \frac{\partial}{\partial w} \quad \text{(dilation).}
\end{align*}
\]  

(8)

The non-zero Lie commutations of Lie algebra (8) are obtained as follows:

\[
\begin{align*}
[V_1, V_3] &= -\frac{5V_2}{6}, \\
[V_1, V_4] &= V_1, \\
[V_2, V_4] &= \frac{3V_2}{5}, \\
[V_3, V_4] &= -\frac{2V_3}{5}.
\end{align*}
\]  

(9)

The non-zero Lie brackets (9) show that the Lie algebra (8) is solvable.

2.1 Construction of optimal system for Lie algebra (8)

In the symmetry analysis, it is well-known that whenever PDEs or system of PDEs admit the symmetry group (or group of invariant transformations), then one can find group invariant solution corresponding to each subgroup by reducing the number of independent variables in the original system. There exist infinitely many such subgroups and hence infinitely many group invariant solutions. But, most of these group invariant solutions would be equivalent by some transformation in the full symmetry group. In order to minimise the search of inequivalent group invariant solutions under transformations in the full symmetry group, the concept of optimal system is introduced. Although the classification of Lie algebras by using adjoint transformations was known to Lie himself, it was Ovsiannikov [15] who first used
the Lie group classification to derive inequivalent group invariant solutions. Ovsienikov used a global adjoint matrix to construct optimal systems and he further extended his technique to derive multidimensional optimal systems. In the construction of the two-dimensional optimal system, Galasi and Richter [20] made some modifications in the technique of Ovsienikov by selecting elements from the normaliser of the one-dimensional optimal system.

Apart from these techniques, the method of classifying the subalgebras proposed by Patera et al [21] is par excellence (see [22] for recent applications) and in their subsequent work [23] they have classified all the real Lie algebras of dim ≤ 4 under the group of inner automorphisms. It is worth mentioning that the Lie algebra of dimension greater than 4 has also been fully classified. For example, Turkowski [24] has classified all six-dimensional solvable Lie algebras containing inner automorphisms. It is worth mentioning that the real Lie algebras of dim ≤ 4 have classified. For example, Turkowski [24] has classified all six-dimensional solvable Lie algebras containing inner automorphisms. It is worth mentioning that the real Lie algebras of dim ≤ 4 have also been fully classified. For example, Turkowski [24] has classified all six-dimensional solvable Lie algebras containing inner automorphisms. It is worth mentioning that the real Lie algebras of dim ≤ 4 have also been fully classified.
connect these solutions by some four parameter group transformation \( \tilde{\psi} = \exp[\sum_{i=1}^{4} a_i V_i] \psi \). In this manner, the group invariant solutions separate into equivalence classes, and the collection of generators corresponding to these classes would constitute an optimal system. In order to find such equivalence classes, we define the following adjoint operator:

\[
\text{Ad}_{\exp(eV)}(W) = \exp(-eV)W \exp(eV) = \tilde{W}(e).
\]

The adjoint transformation (12) can be written through the following adjoint operator:

\[
\text{Ad}_{\exp(eV)}(W) = W - e[V, W] + \frac{e^2}{2} [V, [V, W]] - \cdots,
\]

where \([\cdot, \cdot]\) is the Lie bracket defined by (9). Let \( V = \sum_{i=1}^{4} a_i V_i \), and based on this Lie bracket and formula (13), the straightforward calculations show that

\[
\text{Ad}_{\exp(\epsilon_1 V_1) \text{ Ad}_{\exp(\epsilon_2 V_2)}}(V) = \sum_{i=1}^{4} \tilde{a}_i V_i.
\]

The full adjoint transformation (14) in the matrix notation is

\[
A = \begin{pmatrix}
\epsilon^4 & 0 & 0 & -\epsilon_1 \epsilon^4 \\
-\frac{5 \epsilon_3}{6} \epsilon^4 & \frac{3 \epsilon_4}{5} \epsilon^3 & \frac{5 \epsilon_1}{6} \epsilon^3 & -\frac{3 \epsilon_2}{5} \epsilon^3 + \frac{5 \epsilon_1 \epsilon_3}{6} \epsilon^4 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}.
\]

The construction of adjoint matrix (15) is discussed in [15], where the coefficients \( \tilde{a}_1, \ldots, \tilde{a}_4 \) in (14) are given by the following relations:

\[
\begin{align*}
\tilde{a}_1 &= -a_4 \epsilon_1 \epsilon^4 + a_1 \epsilon^4, \\
\tilde{a}_2 &= \frac{5a_3 \epsilon_1}{6} \epsilon^3 - \frac{3a_4 \epsilon_2}{5} \epsilon^3 + a_2 \epsilon^3 + \frac{5a_4 \epsilon_1 \epsilon_3}{6} \epsilon^4 - \frac{5a_1 \epsilon_3}{6} \epsilon^4, \\
\tilde{a}_3 &= \frac{2a_4 \epsilon_3}{5} + a_3 \epsilon^\frac{2}{3}, \\
\tilde{a}_4 &= a_4.
\end{align*}
\]

The last equation in (16) agrees with the invariance of the Killing’s form under full adjoint transformation (14).

\textbf{Theorem 3.} The one-dimensional optimal system corresponding to Lie algebra (7) is \( \{V_1, V_2, V_3, V_4, aV_1 \pm V_3\} \).

\textbf{Proof.} Let \( V = \sum_{i=1}^{4} a_i V_i \) and \( K = \frac{38}{25} a_4^2 \). We have the following cases for \( K \).

\textbf{Case 1.} For \( K \neq 0 \), we take \( a_4 = 1 \). Choosing \( \epsilon_4 = 0 \), system of eq. (16) becomes

\[
\begin{align*}
\tilde{a}_1 &= -\epsilon_1 + a_1, \\
\tilde{a}_2 &= \frac{5a_3 \epsilon_1}{6} - \frac{3 \epsilon_2}{5} + a_2 + \frac{5 \epsilon_1 \epsilon_3}{6} - \frac{5a_1 \epsilon_3}{6}, \\
\tilde{a}_3 &= \frac{2 \epsilon_3}{5} + a_3, \\
\tilde{a}_4 &= 1.
\end{align*}
\]

The selection

\[
\epsilon_1 = a_1, \quad \epsilon_2 = \frac{25a_3 a_1}{18} + \frac{5a_2}{3}, \quad \epsilon_3 = -\frac{5a_3}{2}
\]

gives \( \tilde{a}_1 = \tilde{a}_2 = \tilde{a}_3 = 0 \), and we obtain the simplification \( V = V_4 \).

\textbf{Case 2.} For \( K = 0 \), we have to take \( a_4 = 0 \).

\[ \begin{align*}
(1) & \ a_3 = 1. \text{ Choosing } \epsilon_4 = \frac{5}{2} \ln(a_3) \text{ gives } \tilde{a}_3 = \pm 1, \\
& \text{ and appropriate selection of } \epsilon_1, \epsilon_3 \text{ gives } \tilde{a}_1 = a_1 a_3^{5/2} \text{ and } \tilde{a}_2 = 0. \text{ We obtain the simplification } V = a V_1 \pm V_3, a = a_1 a_3^{5/2}. \\
(2) & \ a_3 = 0. \text{ System (16) reduces to } \tilde{a}_1 = a_1 \epsilon^4, \tilde{a}_2 = a_2 \epsilon^3 / 5 - (5a_1 \epsilon_3 / 6) \epsilon^4. \text{ By taking } \epsilon_3 = (6a_2 e^{-2(4/5)}) / 5a_1, \text{ we obtain the simplification } V = V_1. \\
(3) & \ a_3 = 0, a_1 = 0. \text{ In this case we obtain the straightforward simplification } V = V_2. \\
(4) & \ a_3 \neq 0, a_1 = 0. \\
\end{align*} \]

\[
\begin{align*}
\tilde{a}_2 &= \frac{5a_3 \epsilon_1}{6} \epsilon^3 + \frac{a_2 \epsilon^3}{5}, \\
\tilde{a}_3 &= a_3 \epsilon^{-2/3}.
\end{align*}
\]

By taking \( \epsilon_1 = -(6a_2 / 5a_3) \) we obtain the simplification \( V = V_3 \). \( \square \)
2.2 Symmetry reductions and invariant solutions

By virtue of vector fields V₁ and V₂, we can see that eq. (3) admits symmetry in space and time translation. So by letting $\xi = x - ct$, we have similarity transformations $u = F(\xi)$, $v = G(\xi)$ and $w = H(\xi)$. Substituting into system (3), we obtain

\[-cF_\xi - \frac{3}{2} FF_\xi - G_\xi = 0, \tag{17a}\]
\[-cG_\xi - G F_\xi - \frac{1}{2} FG_\xi - H_\xi = 0, \tag{17b}\]
\[-cH_\xi + \frac{1}{4} F_\xi,\xi,\xi - HF_\xi - \frac{1}{2} FH_\xi = 0. \tag{17c}\]

Integrating (17a) with respect to $\xi$ gives

\[G = -cf - \frac{3}{4} F^2 + d_1, \tag{18}\]

here $d$ is a constant of integration.

Again substituting this $G$ into (17b) gives

\[H_\xi = (c^2 - d_1)F_\xi + 3cFF_\xi + \frac{3}{2} F^2 F_\xi, \tag{19}\]

followed by integrating once with respect to $\xi$

\[H = (c^2 - d_1)F + \frac{3c}{2} F^2 + \frac{1}{2} F^3 + d_2. \tag{20}\]

Substituting (18) and (19) into (17c) gives

\[-c^3 F_\xi + cd_1 F_\xi - \frac{9}{2} c^2 FF_\xi - \frac{9}{2} cF_\xi F^2 + \frac{1}{4} F_\xi,\xi,\xi + \frac{3}{2} d_1 F_\xi F - \frac{5}{4} F_\xi F^3 - d_2 F_\xi = 0. \tag{21}\]

Integrating (21) with respect to $\xi$ and then second integration after using the integrating factor $F_\xi$ gives

\[(F_\xi)^2 = \frac{1}{2} F^5 + 3cF^4 + (6c^2 - 2d_1)F^3 + 4(c^3 - cd_1 + d_2)F^2 + 8d_3F + 8d_4, \tag{22}\]

where $c$, $d_1$, $d_2$, $d_3$, $d_4$ are constants of integration. Detailed discussion about the solution of (22) can be seen in [2]. The reductions corresponding to the rest of the vector fields have been classified in the following cases.

Case 3. Reduction under the subalgebra $V_3$.

- Similarity variables:
  \[\xi = t, \quad u = F(t) - \frac{6x}{5t}, \quad v = G(t) + \frac{4ux}{5t} + \frac{12x^2}{25t^2}, \quad w = H(t) + \frac{2vx}{5t} - \frac{4ux^2}{25t^2} - \frac{8x^3}{125t^3}. \tag{23}\]

- Reduced system: Substituting (22) into (3), the reduced system is obtained as follows:
  \[t F_\xi + F = 0, \quad 2tF_\xi^2 - 5t^2 G_\xi - 4tx F_\xi - 4xF - 4tG = 0, \quad 4tx F_\xi^2 + 5t^2 F_\xi F - 25t^3 H_\xi - 10t^2 xG_\xi - 4tx^2 F_\xi - 4x^2 F - 8txG - 30t^2 H = 0. \tag{24}\]

- Similarity solutions:
  \[u = \frac{c_1}{t} + \frac{6x}{5t}, \quad v = -\frac{c_1^2}{3t} + \frac{c_2}{t^{3/5}} + \frac{c_3}{t^{1/5}} + \frac{2c_1x}{15t^3}, \quad w = -\frac{c_1c_2}{3t^{9/5}} + \frac{c_1^3}{27t^3} + \frac{c_3}{t^{6/5}} + \frac{2c_1x^2}{5t^{15/5}} + \frac{4c_1x^2}{25t^3} - \frac{8x^3}{125t^3}. \tag{25}\]

Case 4. Reduction under subalgebra $V_4$.

- Similarity variables:
  \[\xi = \frac{t}{x^{5/3}}, \quad u = \frac{1}{x^{2/3}} F(\xi), \quad v = \frac{1}{x^{4/3}} G(\xi), \quad w = \frac{1}{x^2} H(\xi). \tag{26}\]

- Reduced system: Substituting (28) into (3), the reduced system is obtained as follows:
  \[15F_\xi F_\xi + 6F + 10G \xi G_\xi + 8G + 6F_\xi = 0, \quad 5F_\xi G_\xi + 10G_\xi F_\xi + 8FG + 10G H_\xi + 12H + 6G F_\xi = 0, \quad 125F_\xi F_\xi,\xi,\xi - 90F_\xi H_\xi - 180H_\xi F_\xi + 750F_\xi F + 830F_\xi F + 80F - 108H_\xi = 0. \tag{27}\]

- Similarity solutions:
  \[u = \frac{1}{x^{2/3}} \sum_{n=0}^{\infty} P_n \left( \frac{t}{x^{5/3}} \right)^n, \quad v = \frac{1}{x^{4/3}} \sum_{n=0}^{\infty} Q_n \left( \frac{t}{x^{5/3}} \right)^n, \quad w = \frac{1}{x^2} \sum_{n=0}^{\infty} R_n \left( \frac{t}{x^{5/3}} \right)^n, \tag{28}\]

where the coefficients $P_n$, $Q_n$ and $R_n$ are obtained in Theorem 4.
Case 5. Reduction under subalgebra $aV_1 + V_3$.

- Similarity variables:
  \[ \xi = \frac{12\alpha x}{5} + t^2, \]
  \[ u = -F(\xi) - \frac{t}{\alpha}, \]
  \[ v = G(\xi) - \frac{2t \xi}{3\alpha} + \frac{4x}{5\alpha}, \]
  \[ w = H(\xi) - \frac{t G(\xi)}{3\alpha} - \frac{4x F(\xi)}{15\alpha} - \frac{4tx}{45\alpha^2} - \frac{2t \xi}{27\alpha^3}. \]  

- Reduced system: Substituting (31) into (3), the reduced system is obtained as follows:
  \[ 18F \alpha^2 F_\xi + 12\alpha^2 G_\xi - 1 = 0, \]
  \[ 9F \alpha^2 G_\xi + 18G\alpha^2 F_\xi + 18\alpha^2 H_\xi + 4\alpha \xi F_\xi + 6\alpha = 0, \]
  \[ 3888\alpha^6 F_{\xi, \xi} - 1350 F\alpha^4 H_\xi - 2700 H\alpha^4 F_\xi + 450 F\alpha^2 \xi F_\xi + 150 F^2 \alpha^2 - 375 G \alpha^2 - 125 \xi = 0. \]  

- Similarity solutions:
  \[ u = -\sum_{n=0}^{\infty} P_n \left( \frac{12\alpha x}{5} + t^2 \right)^n - \frac{t}{\alpha}, \]
  \[ v = \sum_{n=0}^{\infty} Q_n \left( \frac{12\alpha x}{5} + t^2 \right)^n - \frac{2t}{3\alpha} \sum_{n=0}^{\infty} P_n \left( \frac{12\alpha x}{5} + t^2 \right)^n + \frac{4x}{5\alpha}, \]
  \[ w = \sum_{n=0}^{\infty} R_n \left( \frac{12\alpha x}{5} + t^2 \right)^n - \frac{t}{3\alpha} \sum_{n=0}^{\infty} Q_n \left( \frac{12\alpha x}{5} + t^2 \right)^n - \frac{4x}{15\alpha} \sum_{n=0}^{\infty} P_n \left( \frac{12\alpha x}{5} + t^2 \right)^n - \frac{4tx}{45\alpha^2} - \frac{2t \left( \frac{12\alpha x}{5} + t^2 \right)}{27\alpha^3}. \]  

where the coefficients $P_n$, $Q_n$ and $R_n$ are obtained in Theorem 5.

For similarity solutions of reductions corresponding to vector $V_4$ and $aV_1 + V_3$, we seek power series solution of the form

\[ F = \sum_{n=0}^{\infty} P_n \xi^n, \quad G = \sum_{n=0}^{\infty} Q_n \xi^n, \quad H = \sum_{n=0}^{\infty} R_n \xi^n, \]  

where $P_n$, $Q_n$ and $R_n$ are unknown coefficients of power series that need to be determined later. On substituting (34) into reductions corresponding to respective vector fields we have the following theorems:

Theorem 4. Substitution of power series (34) into reductions corresponding to vector field $V_4$ gives the following recurrence relations:

\[ P_{n+1} = -\frac{1}{6(n+1)} \left( 10n Q_n + 15 \sum_{k=0}^{n} (n-k) P_k P_{n-k} + 6 \sum_{k=0}^{n} P_k P_{n-k} + 8 Q_n \right), \]
\[ Q_{n+1} = \frac{1}{6(n+1)} \left( 10n R_n + 5 \sum_{k=0}^{n} (n-k) P_k Q_{n-k} + 10 \sum_{k=0}^{n} (n-k) Q_k P_{n-k} + 8 \sum_{k=0}^{n} P_k Q_{n-k} + 12 R_n \right), \]
\[ R_{n+1} = \frac{-1}{108(n+1)} \left( -125n^3 P_n - 375 n^2 P_n - 330 n P_n + 180 \sum_{k=0}^{n} P_k R_{n-k} + 90 \sum_{k=0}^{n} (n-k) P_k R_{n-k} + 180 \sum_{k=0}^{n} (n-k) R_k P_{n-k} - 80 P_n \right), \]  

where $P_0$, $Q_0$, $R_0$ ought to be taken as arbitrary, and

\[ P_1 = -P_0^2 - \frac{4Q_0}{3}, \quad Q_1 = -\frac{4P_0 Q_0}{3} - 2R_0, \]
\[ R_1 = -\frac{5P_0 R_0}{3} + \frac{20P_0}{27}. \]  

Proof. For brevity, we have omitted detailed calculations and results in the form of power series solutions for system (3) corresponding to reductions under vector field $V_4$ are interpreted in Case 4. \[\square\]

Theorem 5. Substitution of power series (34) into reductions corresponding to vector field $aV_1 + V_3$ gives the following recurrence relations:
3. Conservation laws

In physics, the conservation laws are fundamental laws which ensure that certain physical quantity will not change with time during the course of physical process [29]. Some of the well-known conservation laws in physics are conservation of mass, momentum, energy, electric charge, etc. It is a well-known fact that the Noether’s theorem gives conservation laws for a system only when it has variational principle. To establish conservation laws for a system without variational structure, Ibragimov [30] has given a new theorem based on the concept of adjoint equations for nonlinear equations. In the recent literature, many authors have applied the theorem of Ibragimov to derive conservation laws. For instance, in [31] it was proved that the Camassa–Holm is strictly self-adjoint and conservation laws were also obtained without classical Lagrangians. Freire and Sampaio [32] have constructed some conservation laws for the nonlinear self-adjoint class of the generalised fifth-order equation, such as a general Kawahara equation, modified Kawahara equation and simplified modified Kawahara equation. Johnpillai and Khalique [33] have applied the same theorem to derive conservation laws for the generalised KdV equation of time-dependent variable coefficients. For further details about the application of theorem by Ibragimov, see [22,34–40].

Based on the theory developed in [30] and notations adopted therein, we define formal Lagrangian for eq. (3) in the following manner:

\[ I = \phi(x,t) \left( u_t - \frac{3}{2} uu_x - v_x \right) + \psi(x,t) \left( v_t - uu_x - \frac{1}{2} uu_x - w_x \right) + \theta(x,t) \left( w_t + \frac{1}{4} u_{xxx} - uu_x - \frac{1}{2} uu_x \right), \quad (38) \]

where \( \phi(x,t), \psi(x,t) \) and \( \theta(x,t) \) are new dependent variables. The adjoint equations for (3) can be written as

\[ F^* = \frac{\delta I}{\delta u} = 0, \quad G^* = \frac{\delta I}{\delta v} = 0, \quad H^* = \frac{\delta I}{\delta w} = 0, \quad (39) \]

where we have used the variational derivative \( \delta / \delta u^\alpha \) defined by the relation

\[ \frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s=1}^{\infty} (-1)^s D_{i_1} \ldots D_{i_s} \frac{\partial}{\partial u_{i_1} \ldots i_s}. \quad (40) \]
Substituting Lagrangian (38) into (39) and using relation (40), we obtain the adjoint equations:
\[
F^* = \frac{1}{2} \psi v_x + \frac{1}{2} \theta w_x + \frac{3}{2} \phi_x u + \psi_x v + \theta_s w \eta + \psi_x v
\]
\[
geq \frac{1}{4} \theta_{xx} = 0,
\]
\[
G^* = -\frac{1}{2} \psi u_x + \phi_x + \frac{1}{2} \psi_x u - \psi_t = 0,
\]
\[
H^* = -\frac{1}{2} \theta u_x + \psi_x + \frac{1}{2} \theta_s u - \theta_t = 0.
\]
For conservation laws we shall use the following theorem proved in [30].

**Theorem 6.** Any infinitesimal symmetry (Lie point, Lie Bäcklund, non-local)
\[
V = \xi^i(x, u, u(1), \ldots) \frac{\partial}{\partial x^i} + \eta^a(x, u, u(1), \ldots) \frac{\partial}{\partial u^a}
\]
leads to conservation laws \(D_i(C^i) = 0\) constructed by the formula
\[
C^i = \xi^i + W^a \left[ \frac{\partial I}{\partial u^a_i} - D_j \left( \frac{\partial I}{\partial u^a_{ij}} \right) \right]
\]
\[
+ D_j D_k \left( \frac{\partial I}{\partial u^a_{ijk}} - \cdots \right)
\]
\[
+ D_j (W^a) \left[ \frac{\partial I}{\partial u^a_{ij}} - D_k \left( \frac{\partial I}{\partial u^a_{ijk}} \right) + \cdots \right]
\]
\[
+ D_j D_k (W^a) \left[ \frac{\partial I}{\partial u^a_{ijk}} - \cdots \right],
\]
where \(W^a = \eta^a - \xi^i u^a_i\) and I is the Lagrangian defined by (38).

Relation (42) can be simplified by writing Lagrangian I with respect to all mixed derivative \(u^a_{ij}, u^a_{ijk}, \ldots\) in a symmetric manner. We obtain
\[
C^x = \xi^x + W^{(1)} \left[ \frac{\partial I}{\partial u^x} - D_x \left( \frac{\partial I}{\partial u^{xx}} \right) \right]
\]
\[
+ W^{(2)} \frac{\partial I}{\partial v_x} + W^{(3)} \frac{\partial I}{\partial w_x} + D_x (W^{(1)})
\]
\[
\times \left[ \frac{\partial I}{\partial u^{xx}} - D_x \left( \frac{\partial I}{\partial u^{xxx}} \right) \right]
\]
\[
+ D_x^2 (W^{(1)}) \frac{\partial I}{\partial u^{xxx}},
\]
\[
C^t = \tau I + W^{(1)} \frac{\partial I}{\partial t} + W^{(2)} \frac{\partial I}{\partial v_t} + W^{(3)} \frac{\partial I}{\partial w_t},
\]
where \(D_i\) denotes the operator of total differentiation:
\[
D_i = \frac{\partial}{\partial x^i} + u^a_i \frac{\partial}{\partial u^a} + u^a_{ij} \frac{\partial}{\partial u^a_j} + \cdots,
\]
and rest of the details about notations can be seen in [30].

In the following cases, we shall find conserved currents (43) corresponding to every symmetry generator of optimal system obtained in Theorem 3.

**Case 1.** For the generator \(V_1 = \partial/\partial t\), the Lie’s characteristic functions are obtained as follows:
\[
W^{(1)} = -u_t, \quad W^{(2)} = -v_t, \quad W^{(3)} = -w_t.
\]
Substituting (44) into (43) yields the following conserved currents:
\[
C^x = \frac{3}{2} u_t \phi u + u_t \psi v + u_t \theta w
\]
\[
- \frac{1}{4} u_t \psi_{xx} + v_t \phi + \frac{1}{2} v_t \psi u
\]
\[
+ w_t \psi + \frac{1}{2} w_t \theta u + \frac{1}{4} u_t \psi w - \frac{1}{4} u_{xxx} \theta,
\]
\[
C^t = -\frac{3}{2} \phi u u_x - \phi v_x - \psi v u_x - \frac{1}{2} \psi u v_x - \psi w_x
\]
\[
+ \frac{1}{4} \theta u_{xxx} - \theta w u_x - \frac{1}{2} \theta w w_x,
\]
where \(\phi(x,t), \psi(x,t)\) and \(\theta(x,t)\) are arbitrary solutions of adjoint equation (41).

**Case 2.** For the generator \(V_2 = \partial/\partial x\), the Lie’s characteristic functions are obtained as follows:
\[
W^{(1)} = -u_x, \quad W^{(2)} = -v_x, \quad W^{(3)} = -w_x.
\]
Substituting (46) into (43) yields the following conserved currents:
\[
C^x = u_t \phi + v_t \psi + w_t \theta - \frac{1}{4} u_x \psi_{xx} + \frac{1}{4} u_{xxx} \theta_x,
\]
\[
C^t = -\psi u_x - \phi v_x - \theta w_x,
\]
where \(\phi(x,t), \psi(x,t)\) and \(\theta(x,t)\) are the arbitrary solutions of adjoint equation (41).

**Case 3.** For the generator
\[
V_3 = -\frac{5 t}{6} \frac{\partial}{\partial x} + \frac{\partial}{\partial u} - \frac{2 u_t}{3} \frac{\partial}{\partial v} - \frac{u_v}{3} \frac{\partial}{\partial w},
\]
the Lie’s characteristic functions are obtained as follows:
\[
W^{(1)} = 1 + \frac{5 t}{6} u_x, \quad W^{(2)} = -\frac{2 u}{3} + \frac{5 t}{6} v_x,
\]
\[
W^{(3)} = -\frac{v}{3} + \frac{5 t}{6} w_x.
\]
Substituting (48) into (43) yields the following conserved currents:

\[
C^x = \frac{1}{3} \psi u^2 - \frac{5}{6} t u_t \phi - \frac{5}{6} t v_t \psi
- \frac{5}{6} t w_t \phi + \frac{5 t u_x \theta x x}{24}
+ \frac{1}{6} v \theta u - \frac{5 t u_x \theta x}{24} + \frac{1}{4} \theta x x - \frac{2}{3} \psi v
- \theta w - \frac{5}{6} \phi u,
\]

\[
C^t = \phi + \frac{5}{6} \phi t u_x - \frac{2}{3} \psi u + \frac{5}{6} \psi t v_x
- \frac{1}{3} \psi \theta + \frac{5}{6} \theta t w_x,
\]  

(49)

where \( \phi(x, t), \psi(x, t) \) and \( \theta(x, t) \) are arbitrary solutions of the adjoint equation (41).

**Case 4.** For the generator

\[
V_4 = \frac{3 x \frac{\partial}{\partial x} + t \frac{\partial}{\partial t} - \frac{2 u}{5} \frac{\partial}{\partial u} - \frac{2 v}{5} \frac{\partial}{\partial v} - \frac{6 w}{5} \frac{\partial}{\partial w}},
\]

the Lie’s characteristic functions are obtained as follows:

\[
W^{(1)} = - \frac{2 u}{5} + \frac{3 x}{5} u_x - t u_t,
\]

\[
W^{(2)} = - \frac{4 v}{5} - \frac{3 x}{5} v_x - t v_t,
\]

\[
W^{(3)} = - \frac{6 w}{5} - \frac{3 x}{5} w_x - t w_t.
\]  

(50)

Substituting (50) into (43) yields the following conserved currents:

\[
C^x = \frac{3}{5} \phi \psi u^2 - \frac{1}{10} u \theta x x + \frac{4}{5} \psi \phi + \frac{6}{5} \psi \psi + \frac{1}{4} \theta x u_x
- \frac{2}{5} \theta u_x x + \frac{3}{2} t u_t \phi u + t u_t \psi v + t u_t \theta w
+ \frac{1}{2} t v_t \psi u + \frac{1}{2} t w_t \theta u + t v_t \phi + u \theta w
- \frac{1}{4} t u_t \theta x x + \frac{3 \theta x u_x x x}{20} + \frac{1}{4} \theta x t u x x - \frac{1}{4} \theta t u x x x
+ \frac{4}{5} \psi v + \frac{3}{5} x \psi v + \frac{3}{5} x \phi u_t
- \frac{3 u_x \theta x x}{20} + \frac{3}{5} x \theta w_t + t w_t \psi,
\]

\[
C^t = - \frac{3}{2} t \phi \psi u_x - t \phi v_x - t \psi v u_x - \frac{1}{2} t \psi u v_x
- t \psi w_x + \frac{1}{4} t u_x x x \theta - t \theta w u x - \frac{1}{2} t \theta w w_x
- \frac{2}{5} \phi u - \frac{3}{5} \phi x u_x - \frac{4}{5} \psi v - \frac{3}{5} \psi x v_x
+ \frac{6}{5} \theta w - \frac{3}{5} \theta x w_x,
\]  

(51)

where \( \phi(x, t), \psi(x, t) \) and \( \theta(x, t) \) are arbitrary solutions of the adjoint equation (41).

**Case 5.** For the generator

\[
\alpha V_1 + V_3 = - \frac{5 t}{6} \frac{\partial}{\partial x} + \alpha \frac{\partial}{\partial t} + \frac{\partial}{\partial u} - \frac{2 u}{3} \frac{\partial}{\partial v} - \frac{v}{3} \frac{\partial}{\partial w},
\]

the Lie’s characteristic functions are obtained as follows:

\[
W^{(1)} = 1 + \frac{5 t}{6} u_x - \alpha u_t,
\]

\[
W^{(2)} = - \frac{2 u}{3} + \frac{5 t}{6} v_x - \alpha v_t,
\]

\[
W^{(3)} = - \frac{v}{3} + \frac{5 t}{6} w_x - \alpha w_t.
\]  

(52)

Substituting (52) into (43) yields the following conserved currents:

\[
C^x = \frac{3}{2} \alpha u_t \phi u + \alpha u_t \psi v + \alpha u_t \theta w + \frac{1}{2} \alpha v_t \psi u
+ \frac{1}{2} \alpha w_t \theta u + \frac{1}{6} v \theta u + \frac{1}{4} \theta x \alpha u x x + \frac{5 t u_x \theta x x}{24}
- \frac{5}{6} t w_t \theta + \alpha v_t \phi - \frac{1}{4} \alpha u_t \theta x x - \frac{1}{4} \theta \alpha u x x
- \frac{5}{6} t v_t \psi - \frac{5}{6} t u_t \phi - \frac{5 t u_x \theta x}{24} + \alpha w_t \psi + \frac{1}{4} \theta x x
- \frac{5}{6} \phi u - \frac{2}{3} \psi v - \theta w + \frac{1}{3} \psi u^2,
\]

\[
C^t = \frac{3}{2} \alpha \phi \theta u_x - \alpha \phi v_x - \alpha \psi \psi u_x
- \frac{1}{2} \alpha \psi \psi u_x - \alpha \psi \psi w_x + \frac{1}{4} \alpha \theta u x x x
- \alpha \alpha \psi w_x - \frac{1}{2} \alpha \theta u w_x + \phi + \frac{5}{6} \phi t u x
- \frac{2}{3} \psi u + \frac{5}{6} \psi v x - \frac{1}{3} \psi w + \frac{5}{6} \theta t w_x,
\]  

(53)

where \( \phi(x, t), \psi(x, t) \) and \( \theta(x, t) \) are arbitrary solutions of the adjoint equation (41). In a similar manner, the conserved currents corresponding to the generator

\[
\alpha V_1 - V_3 = \frac{5 t}{6} \frac{\partial}{\partial x} + \alpha \frac{\partial}{\partial t} - \frac{\partial}{\partial u} + \frac{2 u}{3} \frac{\partial}{\partial v} + \frac{v}{3} \frac{\partial}{\partial w},
\]

can also be calculated.

**Remark 1.** Despite the huge success of new conservation theorem of Ibragimov, the recent comments from Anco [41] confirm the incompleteness of the theorem. In particular, the formulation proposed by Ibragimov can generate trivial conservation laws and does not always yield non-trivial conservation laws. But fortunately, in the present case, all the conservation laws given in (45), (47), (49), (51) and (53) are not trivial. Rather, these conservation laws are non-local.
4. Conclusion

Using the classical Lie symmetry analysis we have analysed three-field Kaup–Boussinesq system (3) in a comprehensive manner. Based on the Killing’s form derived in Lemma 2, the complete classification of Lie algebra (8) is obtained in Theorem 3. Similarity reductions and invariant solutions using the power series method are also presented. Apart from this usual symmetry analysis, we have demonstrated the construction of several non-local conservation laws based on the theory of a new conservation theorem [30]. The work presented here emphasised the relevance of new conservation theorem by Ibragimov for the construction of conservation from Lie symmetries without the formulation of classical Lagrangian.

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