The Lagrangian and Hamiltonian Aspects of the Electrodynamic Vacuum-Field Theory Models

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Abstract

We review the mathematical backgrounds of the modern Maxwell-Lorentz classical electrodynamic models and the related charged particles interaction-radiation problems, analyze the fundamental least action principles within the canonical Lagrangian and Hamiltonian formalisms. The corresponding electrodynamic vacuum field theory aspects of the classical Maxwell-Lorentz theory are analyzed in detail.

The electrodynamic models of the charged point particle dynamics, based on a Maxwell type vacuum field medium description are described, new field theory concepts related with the mass particle paradigms are discussed. We also revisited and reanalyzed the mathematical structure of the classical Lorentz force expression with respect to arbitrary inertial reference frames and presented new interpretations of some classical special relativity theory relationships.

Based on the Feynman proper time paradigm and a recently devised vacuum field theory approach to the Lagrangian and Hamiltonian, the formulations of alternative classical electrodynamics models are analyzed in detail and their Dirac type quantization is suggested. Problems closely related to the radiation reaction force and electron mass inertia are analyzed. The validity of the Abraham-Lorentz electromagnetic electron mass origin hypothesis is argued. The electromagnetic Dirac-Fock-Podolsky problem of the Maxwell and Yang-Mills type dynamical systems is analyzed within the classical Dirac-Marsden-Weinstein symplectic reduction theory.

Making use of the Gelfand-Vilenkin representation theory of infinite dimensional groups and the Goldin-Menikoff-Sharp theory of generating Bogolubov type functionals the problem of constructing Fock type representations and retrieving their creation-annihilation operator structure an effective approach to study dynamical systems in Hilbert spaces is reviewed. An application of the suitable current algebra representation to describing the non-relativistic Aharonov-Bohm paradox is presented.

Keywords: Lagrangian and Hamiltonian Formalism; Maxwell Equations; Lorentz Constraint; Fock Space; Quantum Current Algebra; Bogolubov Functional Equations; Radiation Theory.

1 Classical Relativistic Electrodynamics Models and Their Least Action Principle Description Revisiting: Lagrangian and Hamiltonian Analysis

1.1 Introductory Setting

Classical Maxwell-Lorentz electrodynamics is nowadays considered [80, 97] as the most fundamental physical theory, largely owing to the depth of its theoretical foundations and wealth of experimental verifications. In this work we present the main mathematical structures lying in foundations of the modern classical electrodynamics, based on a new least action principle approaches to the classical Maxwell-Lorentz electromagnetic theory, taking into account a vacuum field medium model interacting with material charged objects. We reanalyze in detail some of the important modern electrodynamics
problems related with description of a charged point particle dynamics under external electromagnetic field with respect to arbitrary inertial reference frames. We remark here that under "a charged point particle" we as usually understand an elementary material charged particle whose internal spatial structure is assumed to be unimportant and is not taken into account, if the contrary is not specified.

The important physical principles, characterizing the related electrodynamic vacuum field structures, are from the mathematical point of view based on the least action principle, which we discuss subject to different charged point particle dynamics. In particular, the main classical special relativity relationships, characterizing the charge point particle dynamics, we obtain by means of the least action principle within the original Feynman’s approach to the Maxwell electromagnetic equations and the Lorentz type force derivation. Moreover, for each least action principle constructed in the work, we describe the corresponding Hamiltonian pictures and present the related energy conservation laws. Making use of the developed modified least action approach a classical hadronic string model is analyzed in detail.

As the classical Lorentz force expression with respect to an arbitrary inertial reference frame is related with many theoretical and experimental controversies, such as the relativistic potential energy impact into the charged point particle mass, the Aharonov-Bohm effect [3] and the Abraham-Lorentz-Dirac radiation force [69, 34, 80] expression, the analysis of its structure subject to the assumed vacuum field medium structure is a very interesting and important problem, which was discussed by many physicists including E. Fermi, G. Schott, R. Feynman, F. Dyson [44, 121, 46, 42, 54] and many others. To describe the essence of the electrodynamic problems related with the description of a charged point particle dynamics under external electromagnetic field, let us begin with analyzing the classical Lorentz force expression

\[ dp/dt = F_\xi := \xi E + \xi \times B, \]

where \( \xi \in P \) is a particle electric charge, \( u \in T(P^3) \) is its velocity [2, 25] vector, expressed here in the light speed \( c \) units,

\[ E = -\partial A/\partial t - \nabla \varphi \]

is the corresponding external electric field and

\[ B := \nabla \times A \]

is the corresponding external magnetic field, acting on the charged particle, expressed in terms of suitable vector \( A : M^4 \rightarrow E^3 \) and scalar \( \varphi : M^4 \rightarrow P \) potentials. Here "\( \nabla \)" is the standard gradient operator with respect to the spatial variable \( r \in E^3 \), "\( \times \)" is the usual vector product in three-dimensional Euclidean vector space \( E^3 := (P^3, < \cdot, \cdot >) \), which is naturally endowed with the classical scalar product \( < \cdot, \cdot > \). These potentials are defined on the Minkowski space \( M^4 ; \ P \times E^3 \), which models a chosen laboratory reference frame \( K \). Now, it is a well known fact [80, 97, 46, 127] that the force expression (1) does not take into account the dual influence of the charged particle on the electromagnetic field and should be considered valid only if the particle charge \( \xi \rightarrow 0 \). This also means that expression (1) cannot be used for studying the interaction between two different moving charged point particles, as was pedagogically demonstrated in classical manuals [80, 46].

Other questionable inferences, which strongly motivated the analysis in this work, are related both with an alternative interpretation of the well-known Lorentz condition, imposed on the four-vector of electromagnetic potentials \((\varphi, A) : M^4 \rightarrow M^4 \) and the classical Lagrangian formulation [80] of charged particle dynamics under external electromagnetic field. The Lagrangian approach latter is strongly dependent on an important Einsteinian notion of the rest reference frame \( K^\prime \) and the related least action principle, so before explaining it in more detail, we first to analyze the classical Maxwell electromagnetic theory from a strictly dynamical point of view.

Let us consider with respect to a laboratory reference frame \( K \) the additional Lorentz condition

\[ \partial \varphi \partial t + < \nabla, A >= 0, \]

(4)

a priori assumed the Lorentz invariant wave scalar field equation

\[ \partial^2 \varphi / \partial t^2 - \nabla^2 \varphi = \rho \]

(5)

and the charge continuity equation

\[ \partial \rho / \partial t + < \nabla, J >= 0, \]

(6)
where \( \rho : M^4 \rightarrow P \) and \( J : M^4 \rightarrow E^3 \) are, respectively, the charge and current densities of the ambient matter. Then one can derive \([104, 105]\) that the Lorentz invariant wave equation

\[
\frac{\partial^2}{\partial t^2} A - \nabla^2 A = J \tag{7}
\]

and the classical electromagnetic Maxwell field equations \([69, 80, 46, 97, 127]\)

\[
\nabla \times E + \partial B/\partial t = 0, \quad \nabla \cdot E = \rho, \tag{8}
\]

\[
\nabla \times B - \partial E/\partial t = J, \quad \nabla \cdot B = 0,
\]

hold for all \((t, r) \in M^4\) with respect to the chosen laboratory reference frame \(K\).

Notice here that, inversely, Maxwell’s equations (8) do not directly reduce, via definitions (2) and (3), to the wave field equations (5) and (7) without the Lorenz condition (4). This fact is very important and suggests that when it comes to a choice of governing equations, it may be reasonable to replace Maxwell’s equations (8) with the Lorenz condition (4) and the charge continuity equation (6). To make the equivalence statement, claimed above, more transparent we formulate it as the following proposition.

**Proposition 1.1** The Lorentz invariant wave equation (5) together with the Lorenz condition (4) for the observable potentials \((\varphi, A) : M^4 \rightarrow T^* (M^4)\) and the charge continuity relationship (6) are completely equivalent to the Maxwell field equations (8).

Proof. Substituting (4), into (5), one easily obtains

\[
\frac{\partial^2}{\partial t^2} \varphi - \nabla^2 \varphi = \rho
\]

which implies the gradient expression

\[
\nabla \left[ \left( \frac{\partial}{\partial t} \varphi - \nabla \varphi \right) \right] = \rho. \tag{9}
\]

Taking into account the electric field definition (2), expression (10) reduces to

\[
\nabla \cdot E = \rho, \tag{10}
\]

which is the second of the first pair of Maxwell’s equations (8).

Now upon applying \(\nabla \times\) to definition (2), we find, owing to definition (3), that

\[
\nabla \times E + \partial B/\partial t = 0, \tag{12}
\]

which is the first pair of the Maxwell equations (8). Having differentiated with respect to the temporal variable \(t \in P\) the equation (5) and taken into account the charge continuity equation (6), one finds that

\[
\nabla \cdot \left( \frac{\partial^2}{\partial t^2} A - \nabla^2 A - J \right) = 0. \tag{13}
\]

The latter is equivalent to the wave equation (7) if to observe that the vector potential \(A : M^4 \rightarrow E^3\) is defined by means of the Lorenz condition (4) up to a vector function \(\nabla \times S : M^4 \rightarrow E^3\). Now applying operation \(\nabla \times\) to the definition (3), owing to the wave equation (7) one obtains

\[
\nabla \times B = \nabla \times (\nabla \times A) = \nabla \left( \nabla \cdot A \right) - \nabla^2 A =
\]

\[
= -\nabla (\partial \varphi/\partial t) - \frac{\partial^2}{\partial t^2} A + (\varphi - \nabla^2 A) =
\]

\[
= \frac{\partial}{\partial t} \left( \nabla \varphi - \varphi \right) + J = \partial E/\partial t + J, \tag{14}
\]

which leads directly to

\[
\nabla \times B = \partial E/\partial t + J,
\]

which is the first of the second pair of the Maxwell equations (8). The final “no magnetic charge” equation
\[ \langle \nabla, B \rangle = \langle \nabla, \nabla \times A \rangle = 0, \]
in (8) follows directly from the elementary identity \( \langle \nabla, \nabla \times \rangle = 0 \), thereby completing the proof.

This proposition allows us to consider the potential functions \((\varphi, A) : M^4 \to M^4\) as fundamental ingredients of the ambient vacuum field medium, by means of which we can try to describe the related physical behavior of charged point particles imbedded in space-time \(M^4\). The following observation provides strong support for this approach:

**Observation.** The Lorenz condition (4) actually means that the scalar potential field \(\varphi : M^4 \to P\) continuity relationship, whose origin lies in some new field conservation law, characterizes the deep intrinsic structure of the vacuum field medium.

To make this observation more transparent and precise, let us recall the definition [80, 97, 46, 127] of the electric current \(J : M^4 \to \mathbb{E}^3\) in the dynamical form

\[ J := \rho u, \]

where the vector \(u \in T(P^3)\) is the corresponding charge velocity. Thus, the following continuity relationship

\[ \partial \varphi \hat{t} + \langle \nabla, \rho u \rangle = 0 \]

holds, which can easily be rewritten [88] as the integral conservation law

\[ \frac{d}{dt} \int_{\Omega_t} \varphi d^3 r = 0 \]

for the charge inside of any bounded domain \(\Omega_t \subset \mathbb{E}^3\), moving in the space-time \(M^4\) with respect to the natural evolution equation

\[ dr/dt := u. \]

Following the above reasoning, we are led to the following result.

**Proposition 1.2** The Lorenz condition (4) is equivalent to the integral conservation law

\[ \frac{d}{dt} \int_{\Omega_t} \varphi d^3 r = 0, \]

where \(\Omega_t \subset \mathbb{E}^3\) is any bounded domain, moving with respect to the evolution equation

\[ dr/dt := v, \]

which represents the velocity vector of local potential field changes propagating in the Minkowski space-time \(M^4\).

Proof. Consider first the corresponding solutions to potential field equations (5), taking into account condition (15). Owing to the results from [46, 80], one finds that

\[ A = \varphi v, \]

which gives rise to the following form of the Lorenz condition (4):

\[ \partial \varphi \hat{t} + \langle \nabla, \varphi v \rangle = 0. \]

This obviously can be rewritten [88] as the integral conservation law (19), so the proof is complete.

The above proposition suggests a physically motivated interpretation of electrodynamic phenomena in terms of what should naturally be called the vacuum potential field, which determines the observable interactions between charged point particles. More precisely, we can a priori endow the ambient vacuum medium with a scalar potential field function \(W := \varphi : M^4 \to P\), satisfying the governing vacuum field equations

\[ \partial^2 W/\partial t^2 - \nabla^2 W = 0, \quad \partial W/\partial t + \langle \nabla, Wv \rangle = 0, \]

taking into account that there are no external sources besides material particles, which possess only a virtual capability for
disturbing the vacuum field medium. Moreover, this vacuum potential field function \( W : M^4 \rightarrow P \) allows the natural potential energy interpretation, whose origin should be assigned not only to the charged interacting medium, but also to any other medium possessing interaction capabilities, including for instance, material particles, interacting through the gravity. The latter leads naturally to the next important step, consisting in deriving the equation governing the corresponding potential field \( \overrightarrow{W} : M^4 \rightarrow P \), assigned to a charged point particle moving in the vacuum field medium with velocity \( u \in T(P^3) \) and located at point \( r(t) = R(t) \in E^3 \) at time \( t \in P \). As can be readily shown [104, 106, 114], the corresponding evolution equation governing the related potential field function \( \overrightarrow{W} : M^4 \rightarrow P \), assigned to a moving in the space \( E^3 \) charged particle \( \zeta \) has the form

\[
\frac{d}{dt}(-\overrightarrow{W}u) = -\nabla \overrightarrow{W}, \tag{24}
\]

where \( \overrightarrow{W} := W(r, t) \mid_{r \rightarrow R(t)} \), \( u(t) := dR(t)/dt \) at point particle location \( (t, R(t)) \in M^4 \).

Similarly, if there are two interacting charged point particles, located at points \( r(t) = R(t) \) and \( r_j(t) = R_j(t) \in E^3 \) at time \( t \in P \) and moving, respectively, with velocities \( u := dR(t)/dt \) and \( u_j := dR_j(t)/dt \), the corresponding potential field function \( \overrightarrow{W} : M^4 \rightarrow P \), considered with respect to the reference frame \( K_j \) specified by Euclidean coordinates \( (t', r - r_j) \in E^4 \) and moving with the velocity \( u_j \in T(P^3) \) subject to the laboratory reference frame \( K \), should satisfy [104, 106] the dynamical equality

\[
\frac{d}{dt}[-\overrightarrow{W} (u - u_j)] = -\nabla \overrightarrow{W}, \tag{25}
\]

where, by definition, we have denoted the velocity vectors \( u' := dr/dt, u_j' := dr_j/dt \in T(P^3) \). The dynamical potential field equations (24) and (25) appear to have important properties and can be used as means for representing classical electrodynamic phenomena. Consequently, we shall proceed to investigate their physical properties in more detail and compare them with classical results for Lorentz type forces arising in the electrodynamics of moving charged point particles in an external electromagnetic field.

Our investigations were in part inspired by works [32, 33, 132, 69] devoted to solving the classical problem of reconciling gravitational and electrodynamic charges within the Mach-Einstein ether paradigm. First, we will revisit the classical Mach-Einstein relativistic electrodynamics of a moving charged point particle, and second, we study the resulting electrodynamic theories associated with our vacuum potential field dynamical equations (24) and (25), making use of the fundamental Lagrangian and Hamiltonian formalisms which were specially devised in [24, 105].

1.1.1 Classical Relativistic Electrodynamics Revisited

The classical relativistic electrodynamics of a freely moving charged point particle in the Minkowski space-time \( M^4 \); \( P \times E^3 \) is based on the Lagrangian approach [80, 46, 97, 127, 34] with Lagrangian function

\[
\Lambda_0 := -m_0(1 - |u|^2)^{1/2}, \tag{26}
\]

where \( m_0 \in P_+ \) is the so-called particle rest mass parameter and \( u \in T(P^3) \) is its spatial velocity in the Euclidean space \( E^3 \), expressed here and in the sequel in light speed units (with light speed \( c = 1 \)). The least action principle in the form

\[
\delta S = 0, \quad S := -\int_{t_1}^{t_2} m_0 (1 - |u|^2)^{1/2} dt \tag{27}
\]

for any fixed temporal interval \( [t_1, t_2] \subset P \) gives rise to the well-known relativistic relationships for the mass of the particle

\[
m = m_0(1 - |u|^2)^{-1/2}, \tag{28}
\]
the momentum of the particle

\[ p := mu = m_0 u \left( 1 - |u|^2 \right)^{-1/2} \]  

(29)

and the energy of the particle

\[ E_0 = m = m_0 \left( 1 - |u|^2 \right)^{-1/2}. \]  

(30)

It follows from [80, 97], that the origin of the Lagrangian (26) can be extracted from the action

\[ S := -\int_{t_1}^{t_2} m_0 \left( 1 - |u|^2 \right)^{1/2} dt = -\int_{t_1}^{t_2} m_0 \, d\tau, \]  

(31)

on the suitable temporal interval \([t_1, t_2] \subset P\). Here \( m_0 \in P_+ \) is considered as a constant positive parameter a priori attributed to the point particle,

\[ d\tau := dt \left( 1 - |u|^2 \right)^{1/2} \]  

(32)

and \( \tau \in P \) is the so-called, proper temporal parameter assigned to a freely moving particle with respect to the rest reference frame \( K_\tau \). The action (31) is rather questionable from the dynamical point of view, since it is physically defined with respect to the rest reference frame \( K_\tau \), giving rise to the constant action \( S = -m_0 (\tau_2 - \tau_1) \), as the limits of integrations \( \tau_1 < \tau_2 \in P \) were taken to be fixed from the very beginning. Moreover, considering this particle to have a charge \( \xi \in P \) and be moving in the Minkowski space-time \( M^4 \) under action of an external electromagnetic field \( (\varphi, A) \in M^4 \), the corresponding classical (relativistic) action functional is chosen (see [80, 46, 12, 97, 127, 24, 105]) with respect to the rest reference system \( K_\tau \) as follows:

\[ S := \int_{t_1}^{t_2} \left[ -m_0 + \xi < A, \dot{r} > -\xi \varphi (1 + |\dot{r}|^2) \right] d\tau, \]  

(33)

being parameterized by the Euclidean space-time variables \( (\tau, r) \in E^4 \) satisfying the infinitesimal relationship

\[ d\tau^2 + |dr|^2 = dt^2, \]

where we have denoted \( \dot{r} := dr/d\tau \) in contrast to the definition \( u := dr/dt \). The action (33) can be rewritten with respect to the laboratory reference frame \( K \) as

\[ S = \int_{t_1}^{t_2} \Lambda dt, \quad \Lambda := -m_0 \left( 1 - |u|^2 \right)^{1/2} + \xi < A, u > -\xi \varphi, \]  

(34)

defined on the suitable temporal interval \([t_1, t_2] \subset P\). The action function (34) contains two physically incompatible sub-integral parts - the first one \(-m_0 \left( 1 - |u|^2 \right)^{1/2} dt = -m_0 d\tau\), having sense with respect to the rest reference frame \( K_\tau \), and the second one \( \xi < A, dr > -\xi \varphi dt \), having sense with respect to the laboratory reference frame \( K \). Nonetheless, the least action principle applied to the functional (34) gives rise to the following [80, 46, 97, 127] dynamical equation

\[ dP/dt = -\nabla (\xi \varphi - < \xi A, u >), \]  

(35)

where, by definition, the generalized particle-field momentum

\[ P = p + \xi A, \]  

(36)

the own particle momentum

\[ p = mu = m_0 u \left( 1 - |u|^2 \right)^{-1/2} \]  

(37)

and its so called "inertial" mass
The corresponding particle conserved energy equals
\[ E = (m_0^2 + |p|^2)^{1/2} + \tilde{\varphi}, \]
that is
\[ dE/\,dt = 0 = dE/\,d\tau \]
with respect to both the laboratory reference frame \( K \) and the rest reference frame \( K_r \).

The obtained above expression (39) for the particle energy \( E \in P \) appears to be open to question, since the electrical potential energy \( \tilde{\varphi} \), entering additively, has no effect on the relativistic particle mass \( m = m_0(1 - |u|^2)^{-1/2} \), contradicting the experimental facts \([46, 69]\) that some part of the observable charged particle mass is of the electromagnetic origin. This fact was also underlined by L. Brillouin \([31]\), who remarked that the fact that the potential energy has no effect on the particle mass tells us that "... any possibility of existence of a particle mass related with an external potential energy, is completely excluded". Moreover, it is necessary to stress here that the least action principle, based on the action functional (34) and formulated with respect to the laboratory reference frame \( K \) time parameter \( t \in P \), appears to be logically inadequate, for there is a strong physical inconsistency with other time parameters of the Lorentz equivalent laboratory reference frames depending simultaneously both on the spatial and temporal coordinates. This was first mentioned by R. Feynman in \([46]\), in his efforts to rewrite the Lorentz force expression with respect to the rest reference frame \( K_r \). This and other special relativity theory and electrodynamics problems stimulated many prominent physicists of the past \([31, 46, 130, 97, 29]\) and present \([93, 91, 132, 32, 33, 82, 83, 114, 96, 61]\) to try to develop alternative relativity theories based on completely different space-time and matter structure principles.

There also is another controversial inference from the action expression (34) and resulting dynamical equation (35): the force \( F_\xi = dP/dt \), exerted by the external electromagnetic field on the particle-field cluster \([46]\) carrying the momentum \( P = p + \tilde{\varphi}A \), appears to be the standard gradient expression
\[ F_\xi = -\nabla W_\xi, \]
where the generalized "potential energy"
\[ W_\xi = \tilde{\varphi} - \langle \tilde{\varphi}A, u \rangle. \]
Its first part \( \tilde{\varphi} \in P \) equals the classical \([46, 69]\) electrical potential energy, but its second part \(- \langle \tilde{\varphi}A, u \rangle\) is strictly related with magnetic vector potential \( A \in E^3 \) and has now adays no reasonable physical explanation. As one can easily show \([80, 97, 46, 127, 25]\) from (35), the corresponding expression for the classical Lorentz force is given as
\[ dp/dt = F = \tilde{\varphi} + \tilde{\varphi} \times B, \]
where we have defined, as before,
\[ E = -\partial \tilde{\varphi}/\partial t - \nabla \varphi \]
for the corresponding electric field and
\[ B := \nabla \times A \]
for the related magnetic field, acting on the point particle with the electric charge \( \xi \in P \). The expression (43) means, in particular, that the Lorentz force (43) depends linearly on the particle velocity vector \( u \in T(P^\times) \), and so there is a strong dependence on the reference frame with respect to which the charged point particle \( \xi \) moves. Attempts to reconcile this and some related controversies \([31, 46, 114, 71]\) forced Einstein to devise his special relativity theory and proceed further to creating his general relativity theory trying to explain the gravity by means of geometrization of space-time and matter in the Universe.

Here we once more mention that the classical Lagrangian function \( L : T(M^4) \to P \) in (34) is simultaneously written as a combination of terms incompatibly expressed from the physical point of view by means of both the Euclidean rest
reference frame variables \((\tau, r) \in \mathbb{E}^4\), naturally attributed to the charged point particle, and arbitrarily chosen Minkowski reference frame variables \((t, r) \in M^4\).

These problems were recently analyzed using a completely different "no-geometry" approach [104, 106], where new dynamical equations were derived, which were free of the controversial elements mentioned above. Moreover, this approach avoided the introduction of the well known Lorentz transformations of the space-time reference frames with respect to which the action functional (34) is invariant. From this point of view, there are interesting for discussion conclusions in [120, 62, 9, 11, 10], and where some electrodynamic models, possessing intrinsic Galilean and Poincaré-Lorentz symmetries, are reanalyzed from diverse geometrical points of view. Subject to a possible geometric space-type structure and the related vacuum field background, exerting the decisive influence on the particle dynamics, we need to mention here recent works [5, 124] and the closely related with their ideas the classical articles [70, 99]. Next, we shall revisit the results obtained recently in [104, 106, 104, 25] from the classical Lagrangian and Hamiltonian formalisms [24] in order to shed new light on the physical underpinnings of the vacuum field theory approach to the study of combined electromagnetic and, eventually, also gravitational effects.

1.2 The Vacuum Field Theory Electrodynamics Equations: Lagrangian Analysis

1.2.1 A Point Particle Moving in Vacuum - An Alternative Electrodynamic Model

Within the vacuum field theory approach to electromagnetism, devised in [104, 106], the main vacuum potential field function \(\vec{W} : M^4 \to \mathbb{P}\), related to a charged point particle \(\xi\), satisfies the differential evolution equation (24), namely
\[
\frac{d}{dt}(-\vec{W}u) = -\nabla \vec{W},
\]
in the case when all of the external charged particles are at rest, that is \(\partial \vec{W}/\partial t = 0\), and as above, \(u := dr/dt\) is the particle velocity with respect to some laboratory reference system \(\mathbb{K}\), specified by the Minkowski coordinates \((t, r) \in M^4\).

To analyze the dynamical equation (46) from the Lagrangian point of view, we write the corresponding action functional as
\[
S := -\int_{\tau_1}^{\tau_2} \vec{W} dt = -\int_{\tau_1}^{\tau_2} \vec{W} (1 + |\vec{r}|^2)^{1/2} d\tau,
\]
expressed with respect to the rest reference frame \(\mathbb{K}\), specified by the Euclidean coordinates \((\tau, r) \in \mathbb{E}^4\). Fixing the proper temporal parameters \(\tau_1 < \tau_2 \in \mathbb{P}\), one finds from the least action principle \(\delta S = 0\) that
\[
p := \partial N \partial \vec{r} = -\vec{W}(1 + |\vec{r}|^2)^{-1/2} = -\vec{W}u,
\]
\[
\dot{p} := dp/d\tau = \partial N \partial \vec{r} = -\nabla \vec{W}(1 + |\vec{r}|^2)^{1/2},
\]
where, owing to (47), the corresponding Lagrangian function is
\[
\Lambda := -\vec{W}(1 + |\vec{r}|^2)^{1/2}.
\]
Recalling now the definition of the particle "inertial" mass
\[
m := -\vec{W}
\]
and the relationships
\[
d\tau = dt(1 - |u|^2)^{1/2} = dt(1 + |\vec{r}|^2)^{-1/2}, \dot{r} dt = u dt,
\]
from (48) we easily obtain the classical dynamical equation exactly coinciding with (46):
\[
dp/dt = -\nabla \vec{W}.
\]
Moreover, one now readily finds that the corresponding dynamical mass, defined by means of expression (50), is given as
\[ m = m_0(1 - |u|^2)^{-1/2}, \quad m_0 := \overline{W}(R(t_0)), \]  

where \( u(t)|_{t=t_0} = 0 \) at the spatial point \( r = R(t_0) \in \mathbb{R}^3 \), and which completely coincides with expression (28) of the preceding section. Now one can formulate the following proposition using the results obtained above.

**Proposition 1.3** The alternative freely moving point particle electrodynamic model (46) allows the physically reasonable least action formulation based on the action functional (47) with respect to the "rest" reference frame variables, where the Lagrangian function is given by expression (49). The related electrodynamics is completely equivalent to that of a classical relativistic freely moving point particle, described in Subsection 1.1.

### 1.2.2 A Moving in Vacuum Interacting Two Charges System - An Alternative Electrodynamic Model

We proceed now to the case when our charged point particle \( \xi \) moves in the space-time with velocity vector \( u \in T(P^3) \) and interacts with another external charged point particle \( \xi_f \), moving with velocity vector \( u_f \in T(P^3) \) with respect to a common reference frame \( K \). As was shown in [104, 106], the respectively modified dynamical equation for the vacuum potential field function \( \overline{W} : M^4 \rightarrow \mathbb{R} \) subject to the moving reference frame \( K' \) is given by equality (25), or

\[
\frac{d}{dt} [-\overline{W}(u' - u_f')] = -\nabla \overline{W},
\]

where, as before, the velocity vectors 

\[
u := \frac{d\tau}{dt}, \quad u_f := \frac{d\tau_f}{dt} \in T(P^3).
\]

Since the external charged particle \( \xi_f \) moves in the space-time \( M^4 \), it generates the related magnetic field \( B := \nabla \times A \), whose magnetic vector potential \( A : M^4 \rightarrow \mathbb{R}^3 \) is defined, owing to the results of [104, 106, 114], as

\[
\xi A := \overline{W}u_f.
\]

Whence, taking into account that the field potential

\[
\overline{W} = \overline{W}(1 - |u_f|^2)^{-1/2}
\]

and the particle momentum \( p' = -\overline{W}u' = -\overline{W}u \), equality (54) becomes equivalent to

\[
\frac{d}{dt}(p' + \xi A) = -\nabla \overline{W},
\]

if considered with respect to the moving reference frame \( K' \), or to the Lorentz type force equality

\[
\frac{d}{dt}(p + \xi A) = -\nabla \overline{W}(1 - |u_f|^2),
\]

if considered with respect to the laboratory reference frame \( K \) owing to the classical Lorentz invariance relationship (56), as the corresponding magnetic vector potential, generated by the external charged point test particle \( \xi_f \) with respect to the reference frame \( K' \), is identically equal to zero. To imbue the dynamical equation (58) into the classical Lagrangian formalism, we start from the following action functional, which naturally generalizes the functional (47):

\[
S := -\int_{t_1}^{t_2} \overline{W} (1 + |\dot{r} - \dot{r}_f|^2)^{1/2} d\tau.
\]

Here, as before, \( \overline{W} \) is the respectively calculated vacuum field potential \( \overline{W} \) subject to the moving reference frame \( K' \), \( \dot{r} := u \frac{dt}{d\tau}, \dot{r}_f := u_f \frac{dt}{d\tau}, \quad d\tau = dt(1 - |u - u_f|^2)^{1/2} \), which take into account the relative velocity of the charged point particle \( \xi \) subject to the reference frame \( K' \), specified by the Euclidean coordinates.
\((t', r - r_f) \in \mathbb{E}^4\), and moving simultaneously with velocity vector \(\mathbf{u}_f \in T(\mathbb{P}^3)\) with respect to the laboratory reference frame \(\mathbf{K}_r\), specified by the Minkowski coordinates \((t, r) \in \mathbb{M}^4\) and related to those of the reference frame \(\mathbf{K}_f\) and \(\mathbf{K}_r\) by means of the following infinitesimal relationships:

\[
dt^2 = (dt')^2 + |dr_f|^2, \quad (dt)^2 = d\tau^2 + |dr - dr_f|^2.
\]

So, it is clear in this case that our charged point particle \(\xi\) moves with the velocity vector \(\mathbf{u}' - \mathbf{u}_f \in T(\mathbb{E}^3)\) with respect to the reference frame \(\mathbf{K}'\) in which the external charged particle \(\xi_f\) is at rest. Thereby, we have reduced the problem of deriving the charged point particle \(\xi\) dynamical equation to that before solved in Subsection 1.2.1.

Now we can compute the least action variational condition \(\partial \mathcal{S} = 0\), taking into account that, owing to (59), the corresponding Lagrangian function with respect to the rest reference frame \(\mathbf{K}_r\) is given as

\[
\mathcal{L} := -\mathbf{u} \cdot (1 + |\dot{r} - \dot{r}_f|^2)^{1/2}.
\]

As a result of simple calculations, the generalized momentum of the charged particle \(\xi\) equals

\[
P := \partial \mathcal{L}/\partial \dot{r} = -\mathbf{u}' \cdot (1 + |\dot{r} - \dot{r}_f|^2)^{-1/2} =
\]

\[
= -\mathbf{u}' (1 + |\dot{r} - \dot{r}_f|^2)^{-1/2} + \mathbf{u}_f (1 + |\dot{r} - \dot{r}_f|^2)^{-1/2} =
\]

\[
= m\mathbf{u}' + \xi \mathbf{A}' := \mathbf{p}' + \xi \mathbf{A} = \mathbf{p} + \xi \mathbf{A},
\]

where, owing to (56) the vectors \(\mathbf{p}' := -\mathbf{u}' = -\mathbf{u} = p \in \mathbb{E}^3\), \(\mathbf{A}' = \mathbf{u}' \mathbf{u}_f = \mathbf{W} \mathbf{u}_f = \mathbf{A} \in \mathbb{E}^3\), and giving rise to the dynamical equality

\[
\frac{d}{d\tau}(\mathbf{p}' + \xi \mathbf{A}') = -\nabla \mathbf{W},
\]

with respect to the rest reference frame \(\mathbf{K}_r\). As \(dt' = d\tau (1 + |\dot{r} - \dot{r}_f|^2)^{1/2}\) and \(1 + |\dot{r} - \dot{r}_f|^2 = (1 - |\mathbf{u}' - \mathbf{u}_f|^2)^{-1/2}\), we obtain from (63) the equality

\[
\frac{d}{dt}(\mathbf{p}' + \xi \mathbf{A}') = -\nabla \mathbf{W},
\]

exactly coinciding with equality (57) subject to the moving reference frame \(\mathbf{K}'\). Now, making use of expressions (60) and (56), one can rewrite (64) as that with respect to the laboratory reference frame \(\mathbf{K}\):

\[
\frac{d}{dt}(\mathbf{p}' + \xi \mathbf{A}') = -\nabla \mathbf{W} \Rightarrow
\]

\[
\Rightarrow \frac{d}{dt} \left( \frac{-\mathbf{u} \cdot u_f'}{(1 + |u_f'|^2)^{1/2}} + \frac{\xi \mathbf{W} u_f'}{(1 + |u_f'|^2)^{1/2}} \right) = -\frac{\nabla \mathbf{W}}{(1 + |u_f'|^2)^{1/2}} \Rightarrow
\]

\[
\Rightarrow \frac{d}{dt} \left( \frac{-\mathbf{W} dr}{(1 + |u_f'|^2)^{1/2}} + \frac{\xi \mathbf{W} dr_f}{(1 + |u_f'|^2)^{1/2}} \right) = -\frac{\nabla \mathbf{W}}{(1 + |u_f'|^2)^{1/2}} \Rightarrow
\]

\[
\Rightarrow \frac{d}{dt} (-\mathbf{W} \frac{dr}{dt} + \xi \mathbf{W} \frac{dr_f}{dt}) = -\nabla \mathbf{W} (1 - |u_f'|^2),
\]
exactly coinciding with (58):

$$\frac{d}{dt}(p + \xi A) = -\nabla W (1 - |u_f|^2).$$  \tag{66}

**Remark 1.4** The equation (66) allows to infer the following important and physically reasonable phenomenon: if the test charged point particle velocity $u_f \in T(P^3)$ tends to the light velocity $c = 1$, the corresponding acceleration force

$$F_{ac} := -\nabla W (1 - |u_f|^2)$$

is vanishing. Thereby, the electromagnetic fields, generated by such rapidly moving charged point particles, have no influence on the dynamics of charged objects if observed with respect to an arbitrarily chosen laboratory reference frame $K$.

The latter equation (66) can be easily rewritten as

$$\frac{dp}{dt} = -\nabla W - \xi dA/dt + \nabla W |u_f|^2 =$$

$$= \xi (\varepsilon^{-1} \nabla W - \partial A/\partial t) - \xi <u, \nabla> A + \xi \nabla <A, u_f>,$$

or, using the well-known [80] identity

$$\nabla <a, b> = <a, \nabla> b + <b, \nabla> a + b \times (\nabla \times a) + a \times (\nabla \times b),$$  \tag{68}

where $a, b \in \mathbb{E}^3$ are arbitrary vector functions, in the standard Lorentz type form

$$\frac{dp}{dt} = \xi E + \xi u \times B - \nabla <\xi A, u - u_f>.$$  \tag{69}

The result (69), being before found with respect to the moving reference frame $K$, in [104, 106, 114], makes it possible to formulate the next important proposition.

**Proposition 1.5** The alternative classical relativistic electrodynamic model (57) allows the least action formulation based on the action functional (59) with respect to the rest reference frame $K_r$, where the Lagrangian function is given by expression (61). The resulting Lorentz type force expression equals (69), being modified by the additional force component $F_\xi := -\nabla <\xi A, u - u_f>$, important for explanation [3, 28, 128] of the well known Aharonov-Bohm effect.

1.2.3 A Moving Charged Point Particle Formulation Dual to the Classical Alternative Electrodynamic Model

It is easy to see that the action functional (59) is written utilizing the standard classical Lorentz transformations of reference frames. If we now consider the action functional (47) for a charged point particle moving with respect to the rest reference frame $K_r$, and take into account its interaction with an external magnetic field generated by the vector potential $A : M^4 \rightarrow \mathbb{E}^3$, it can be naturally generalized as

$$S := \int_{t_1}^{t_2} (-\nabla W dt + \xi <A, dr>) = \int_{t_1}^{t_2} (-\nabla W (1 + |\dot{r}|^2)^{1/2} + \xi <A, \dot{r}>) dt,$$  \tag{70}

where $dt = dt(1 - |u|^2)^{1/2}$. The chosen form of functional (70) can be explained by means of the following physically motivated reasonings. Consider an action functional like (70) and calculate its value along any smooth arbitrarily chosen and dynamically admissible closed path $l \subset M^4$, which should be naturally put to be zero:

$$0 = \int_l (-\nabla W dt + \xi <A, dr>).$$  \tag{71}

Having applied to the right-hand side of (71) the standard Stokes theorem [2], one easily obtains that

$$\int_l (-\nabla W dt + \xi <A, dr>) = \int_{\delta(l)} (- <\nabla W, dr \wedge dt > -$$  \tag{72}
if and only if the charged point particle energy $E \in P$ is conserved along this arbitrarily chosen and admissible path $I \subset M^4$. As a simple consequence of (71) the work performed by the electromagnetic force $F_\xi$ depends only on the electric field $E \in \mathbb{E}^3$, not depending on the related magnetic field $B = \nabla \times A \in \mathbb{E}^3$. Thus, having assumed that the corresponding charged point particle dynamical equations conform the energy conservation condition mentioned above, the action functional (70) can be accepted as reasonable from physical point of view.

**Remark 1.6** It is also interesting to remark that a condition $\int_I \lambda dt = 0$, similar to (71), calculated for the Lagrangian function $\Lambda = \frac{m|\dot{r}|^2}{2} - \bar{W}$ in the classical mechanics of a point particle with mass $m \in P_+$, moving under an external potential $\bar{W} : \mathbb{P}^3 \to P$, gives rise to the true classical Newton’s mechanics:

$$\int_{\mathcal{S}(I)} \left( \frac{m}{2} |\dot{r}|^2 - \bar{W} \right) dt = \int_{\mathcal{S}(I)} \left( \frac{m}{2} \langle \dot{r}, \dot{r} \rangle - \bar{W} \right) dt =$$

$$= \int_{\mathcal{S}(I)} \left( -m \dot{r} \cdot \dot{r} - \langle \nabla \bar{W}, \dot{r} \rangle \right) dt =$$

$$= \int_{\mathcal{S}(I)} \left( -m \ddot{r} \cdot \ddot{r} - \langle \nabla \bar{W}, \ddot{r} \rangle \right) dt =$$

$$= \int_{\mathcal{S}(I)} \left( -m \dddot{r} \cdot \dddot{r} + \langle \nabla \bar{W}, \dddot{r} \rangle \right) dt =$$

$$= -\int_{\mathcal{S}(I)} \left( m \dddot{r} + \nabla \bar{W}, \dddot{r} \right) dt = 0,$$

if and only if the Newton’s equation

$$m\dddot{r} = -\nabla \bar{W}$$

(74)

holds.

The least action condition $\partial \mathcal{S} = 0$, as calculated with respect to the rest reference frame $K_\tau$, states in the Feynman’s spirit [46] of reasonings that the charged point particle $\xi$ chooses in the Minkowski space-time $M^4$ such a trajectory of its motion, which realizes the least action value of the functional (70), calculated namely with respect to its own rest reference time parameter $\tau \in P$, being a unique physically sensible quantity attributed to the charged point particle dynamics. Really, as it was stressed by R. Feynman [46], the least action principle, as applied to the functional (70) with respect to the laboratory reference frame time parameter $t \in P$, gives rise to a senseless expression, whose value is both ambiguous and physically not well defined. Thus, the corresponding common generalized particle-field momentum takes the form

$$P := \partial N \partial \dot{r} = -\bar{W}r(1 + |\dot{r}|^2)^{-1/2} + \xi A =$$

(75)
where

\[ \Lambda := -\hat{W}(1+|\hat{r}|^2)^{1/2} + \xi < A, \hat{r}> \]

is the corresponding Lagrangian function. Since \( d\tau = dt(1-|u|^2)^{1/2} \), one easily finds from (76) that

\[ dP/dt = -\nabla(\hat{W} - <\xi A, u>). \]

Upon substituting (75) into (78) and making use of the identity (68), we obtain the classical expression for the Lorentz force \( F_{\xi} \), acting on the moving charged point particle \( \xi \):

\[ d\tau = \hat{F}_{\xi} = \xi E + \xi u \times B, \]

where, by definition,

\[ E := -\xi^{-1}\nabla\hat{W} - \partial A/\partial t \]

is its associated electric field and

\[ B := \nabla \times A \]

is the corresponding magnetic field. This wondering result can be summarized as follows.

**Proposition 1.7** The classical relativistic Lorentz force (79) allows the least action formulation based on the action functional (70) with respect to the rest reference frame \( K_{\tau} \), where the Lagrangian function is given by formula (77).

Concerning the related electrodynamics of a charged point particle \( \xi_f \), described by the dual classical Lorentz force (79), we need to state that it is not equivalent to that described by means of the classical Lorentz force (43). Moreover, one can easily observe that the classical Lorentz force \( F_{\xi_f} = \xi F + \xi u \times B \), exerted on the charged point particle \( \xi \) by an external charged point test particle \( \xi_f \) is not a priori vanishing as it should follow from the relativistic physics point of view. The details of these aspects will be analyzed in more details in the next Section to follow.

Comparing the obtained above Lorentz type forces expressions (79) and (69), differing by the gradient term \( \xi \), which reconciles the dual Lorentz force acting on a moving charged point particle \( \xi \) with respect to an arbitrarily chosen laboratory reference frames \( K \), and, as it was mentioned below, is responsible for the Aharonov-Bohm effect. This fact is important for our vacuum field theory approach since it uses no special geometry and makes it possible to analyze electromagnetic and, under some conditions, also gravitational fields simultaneously by employing the new definition of the dynamical mass by means of expression (50).

### 1.3 The Vacuum Field Theory Electrodynamics Equations: Hamiltonian Analysis

Any Lagrangian theory has an equivalent canonical Hamiltonian representation via the classical Legendre transformation [8, 127, 2, 103, 25]. As we have already formulated our vacuum field theory of a moving particle with a charge \( \xi \in P \) in Lagrangian form, we proceed now to its Hamiltonian analysis making use of the action functionals (47), (61) and (70).

Take, first, the Lagrangian function (49) and the momentum expression (48) for defining the corresponding Hamiltonian function

\[ H := < p, \hat{r} > - \Lambda = -|p|^2 \hat{W}^{-1}(1-|p|^2/\hat{W}^2)^{-1/2} + \hat{W}(1-|p|^2/\hat{W}^2)^{-1/2} = \]
\begin{equation}
-|P|^2 \frac{W^{-1}}{W^2}(1-|P|^2/W^2)^{-1/2} + \frac{W^2}{W^4} \frac{W^{-1}}{W^2}(1-|P|^2/W^2)^{-1/2} = \ \tag{82}
\end{equation}

Consequently, it is easy to show [2, 8, 127, 103] that the Hamiltonian function (82) is a conservation law of the dynamical field equation (46); that is, for all \( r, t \in P \)

\[
dH/dt = 0 = dH/d\tau,
\]

which naturally leads to an energy interpretation of \( H \). Thus, we can represent the particle energy as

\[
E = (\frac{W^2}{W^2} - |P|^2)^{1/2}.
\]  

The corresponding Hamiltonian system equivalent to the vacuum field equation (46) can be written as

\[
\dot{r} := dP/d\tau = \partial H/\partial P = P(\frac{W^2}{W^2} - |P|^2)^{-1/2}
\]

\[
\dot{p} := dP/d\tau = -\partial H/\partial r = \frac{W^2}{W^2}(\frac{W^2}{W^2} - |P|^2)^{-1/2},
\]

and we have the following result.

**Proposition 1.8** The alternative freely moving point particle electrodynamic model, based on the action functional (47), allows the canonical Hamiltonian formulation (85) with respect to the rest reference frame \( K \), where the Hamiltonian function is given by expression (82).

Concerning the charged point particle electrodynamics, based on the dynamical equations (85), we state completely equivalent to the classical relativistic freely moving point particle electrodynamics described above in Subsection 1.1.

In an analogous manner, one can now use the Lagrangian (61) and equation (76) to construct the Hamiltonian function for the dynamical field equation (58), describing the motion of a charged point particle \( \xi \) in an external electromagnetic field as

\[
\dot{r} := dP/d\tau = \partial H/\partial P, \quad \dot{p} := dP/d\tau = -\partial H/\partial r,
\]  

where, by definition,

\[
H := \langle P, \dot{r} \rangle - A = \langle P, \dot{r}_f + \ddot{P}W^{-1}(1-|P|^2/W^2)^{-1/2} + W[\frac{W^2}{W^2} - |P|^2]^{-1/2} \rangle = \langle P, \dot{r}_f + |P|^2 (\frac{W^2}{W^2} - |P|^2)^{-1/2} - \frac{W^2}{W^2} (\frac{W^2}{W^2} - |P|^2)^{-1/2} \rangle = -\langle \frac{W^2}{W^2} - |P|^2 \rceil (\frac{W^2}{W^2} - |P|^2)^{-1/2} + \langle P, \dot{r}_f \rangle = -\langle \frac{W^2}{W^2} - |P|^2 \rangle^{1/2} - \langle \xi \cdot A, P \rangle (\frac{W^2}{W^2} - |P|^2)^{-1/2}.
\]

Here we took into account that, owing to definitions (55) and (62),

\[
\xi_f A := \frac{W}{W} \frac{du_f}{dt} = \frac{W}{W} \frac{dr_f}{dt} = W u_f = \xi_f A = \ \tag{88}
\]

\[
= \frac{W}{W} \frac{dr_f}{dt} \frac{d\tau}{dt} = \frac{W}{W} \frac{dr_f}{dt} (1-|u^\prime - u_f^\prime|)^{1/2} = \frac{W}{W} \dot{r}_f (1+|\dot{r} - \dot{r}_f|^2)^{-1/2} = -\frac{W}{W} \dot{r}_f (\frac{W^2}{W^2} - |P|^2)^{1/2} W^{-1} = -\frac{W}{W} \dot{r}_f (\frac{W^2}{W^2} - |P|^2)^{1/2},
\]  
or
\[ \dot{r}_f = -\xi_f A \left( \overline{W}^2 - |P|^2 \right)^{-1/2}, \]  

(89)

where \( A : M^4 \to \mathbb{P}^3 \) is the related magnetic vector potential generated by the moving external charged particle \( \xi_f \) with respect to the laboratory reference frame \( K \). Equations (86) can be easily rewritten with respect to the laboratory reference frame \( K \) in the form

\[ \begin{align*}
    dl/dt &= u, \\
    dp/dt &= \xi E + \xi u \times B - \nabla <\xi A, u - u_f>,
\end{align*} \]

(90)

which coincide with the result (69).

Whence, we see that the Hamiltonian function (87) satisfies the energy conservation conditions

\[ dH/dt = 0 = dHld\tau, \]

(91)

for all \( \tau, t \in \mathbb{P} \), and the suitable energy expression, owing to (56), is

\[ E = \left( \overline{W}^2 - |\xi A|^2 - |P|^2 \right)^{1/2} + <\xi A, P> \left( \overline{W}^2 - |\xi A|^2 - |P|^2 \right)^{1/2}, \]

(92)

where the generalized momentum \( P = p + \xi A \). The result (92) differs essentially from that obtained in [80], which is strongly based on the Einsteinian Lagrangian for a moving charged point particle \( \xi \) in the external electromagnetic fields, generated by a charged point test particle \( \xi_f \), moving with velocity \( u_f \in T(P^3) \) with respect to a laboratory reference frame \( K \). Thus, we obtained the following proposition.

**Proposition 1.9** The alternative classical relativistic electrodynamic model (90), which is intrinsically compatible with the classical Maxwell equations (6), allows the Hamiltonian formulation (86) with respect to the rest reference frame \( K_0 \). The latter gives rise to the following crucial relationship between the particle energy \( E_0 \) and its rest mass \( m_0 = -\overline{W}_0 \) (for the velocity \( u = 0 \) at the initial time moment \( t = 0 \)):

\[ E_0 = m_0 \left( 1 - \frac{|\xi A_0/m_0|^2}{(1 - 2 |\xi A_0/m_0|^2)^{1/2}} \right), \]

(93)

or, equivalently, at the condition \( |\xi A_0/m_0|^2 < 1/2 \)

\[ m_0 = E_0 \left( \frac{1}{2} + |\xi A_0/E_0|^2 \pm \frac{1}{2} \sqrt{1 - 4 |\xi A_0/E_0|^2} \right)^{1/2}, \]

(94)

where \( A_0 := A|_{t=0} \in \mathbb{P}^3 \), which strongly differs from the classical expression \( m_0 = E_0 - \xi \phi_0 \), following from (39) and is not depending a priori on the external potential energy \( \xi \phi_0 \). As the quantity \( |\xi A_0/E_0| \to 0 \), the following asymptotical mass values follow from (94):

\[ m_0^{(+)}, \ E_0, \ m_0^{(-)}; \ E_0, \pm \sqrt{2} |\xi A_0|. \]

(95)

The first mass value \( m_0^{(+)} \); \( E_0 \) is physically correct, giving rise to the bounded charged particle energy \( E_0 \), but the second mass value \( m_0^{(-)}; \pm \sqrt{2} |\xi A_0| \) is not physical, as it gives rise to the vanishing denominator \((1 - 2 |\xi A_0/m_0^{(-)}|^2)^{1/2} \to 0 \) in (93), being equivalent to the unboundedness of the charged particle energy \( E_0 \).
To make this difference more clear, we now will analyze the dual classical Lorentz force (79) from the Hamiltonian point of view, based on the Lagrangian function (77). Thus, we can easily obtain that the corresponding Hamiltonian function

$$ H := \langle P, \dot{r} \rangle - A = \langle P, \dot{r} \rangle + \overline{W} (1 + | \dot{r} |^2)^{1/2} - \xi < A, \dot{r} > $$

(96)

$$ = \langle P - \xi A, \dot{r} \rangle + \overline{W} (1 + | \dot{r} |^2)^{1/2} $$

$$ = - | p |^2 \overline{W}^{-1} (1 - | p |^2 / \overline{W}^2)^{-1/2} + \overline{W} (1 - | p |^2 / \overline{W}^2)^{-1/2} $$

$$ = -(\overline{W}^2 - | p |^2)(\overline{W}^2 - | p |^2)^{-1/2} = -(\overline{W}^2 - | p |^2)^{1/2}. $$

Since $ p = P - \xi A $, expression (96) assumes the final "no interaction" [80, 97, 79, 108] form

$$ H = -(\overline{W}^2 - | P - \xi A |^2)^{1/2}, $$

(97)

which is conserved with respect to the evolution equations (75) and (76), that is

$$ dH/dt = 0 = dH/d\tau $$

(98)

for all $ \tau, t \in \mathbb{P} $ . These equations are equivalent to the following Hamiltonian system

$$ \dot{r} = \partial H / \partial P = (P - \xi A)(\overline{W}^2 - | P - \xi A |^2)^{-1/2}, $$

(99)

$$ \dot{P} = -\partial H / \partial \dot{r} = (\overline{W} \nabla \overline{W} - \nabla < \xi A, (P - \xi A) >)(\overline{W}^2 - | P - \xi A |^2)^{-1/2}, $$

as one can readily check by direct calculations. Actually, the first equation

$$ \dot{r} = (P - \xi A)(\overline{W}^2 - | P - \xi A |^2)^{-1/2} = p(\overline{W}^2 - | p |^2)^{-1/2} $$

(100)

$$ = mu(\overline{W}^2 - | p |^2)^{-1/2} = -\overline{W}u(\overline{W}^2 - | p |^2)^{-1/2} = u(1 - | u |^2)^{-1/2}, $$

holds, owing to the condition $ d\tau = dt(1 - | u |^2)^{1/2} $ and definitions $ p := mu \ , \ m = -\overline{W} $, postulated from the very beginning. Similarly we obtain that

$$ \dot{P} = -\nabla \overline{W} (1 - | p |^2 / \overline{W}^2)^{-1/2} + \nabla < \xi A, u > (1 - | p |^2 / \overline{W}^2)^{-1/2} $$

(101)

$$ = -\nabla \overline{W} (1 - | u |^2)^{-1/2} + \nabla < \xi A, u > (1 - | u |^2)^{-1/2}, $$

or equivalently, the dual Lorentz dynamical expression

$$ dp/dt = \xi E + \xi u \times B, $$

(102)

exactly coinciding with that of equation (79). This result can be reformulated as the next proposition.

**Proposition 1.11** The dual to the classical relativistic electrodynamic model (79) allows the canonical Hamiltonian formulation (99) with respect to the rest reference frame $ K_r $, where the Hamiltonian function is given by expression (97). Moreover, this formulation circumvents the "mass-potential" energy controversy attached to the classical electrodynamic model, based the classical action functional (34).

The classical Lorentz force expression (102) and the related conserved energy relationship

$$ E = (\overline{W}^2 - | P - \xi A |^2)^{1/2} $$

(103)

are characterized by the following remark.

**Remark 1.12** If we make use of the modified relativistic Lorentz force expression (102) as an alternative to the classical one of (43), the corresponding charged particle energy relationship (103) gives rise to a different energy expression (for the velocity $ u = 0 $ at the initial time moment $ t = 0 $). Namely, one naturally obtains the physically reasonable Einsteinian mass-energy relationship $ E_0 = m_0 $ instead of the senseless classical expression $ E_0 = m_0 + \xi \rho_0 $, following from (39), where $ \phi_0 := \phi |_{t=0} $ and where the mass parameter $ m_0 $ is a constant parameter not depending on the external electromagnetic field.


1.4 Comments

All of dynamical field equations discussed above are canonical Hamiltonian systems with respect to the corresponding physically proper rest reference frames \( K_r \), parameterized by the Euclidean coordinates \((\tau, r) \in \mathbb{R}^4\). Upon passing to the basic laboratory reference frame \( K \), naturally parameterized by the Minkowski coordinates \((t, r) \in M^4\), the related Hamiltonian structure is lost, giving rise to a suitably altered interpretation of the real particle motion. Namely, as it was demonstrated above, a least action principle for a charged point particle dynamics makes sense only with respect to the proper rest reference frame \( K_r \), as, otherwise, it becomes completely senseless with respect to all other laboratory reference frames. As for the Hamiltonian expressions (82), (87) and (97), one observes that they all depend strongly on the vacuum potential field function \( W : M^4 \to \mathbb{R}, \) thereby avoiding the mass problem related with the well known classical energy expression and pointed out by L. Brillouin [31].

Some comments are also on order concerning the classical relativity principle and a way of its application to real physical phenomena. We have obtained our results with using the standard Lorentz transformations of reference frames - relying only on the natural notion of the rest reference frame \( K_r \) and its suitable parametrization with respect to any other laboratory reference frame \( K \). It seems physically reasonable that the true state changing of a moving charged particle \( \xi \) is exactly realized only with respect to its proper rest reference system \( K_r \).

Thus, the only remaining question would be about the physical justification of the corresponding relationship between time parameters of the corresponding laboratory and rest reference frames. The relationship between these reference frames that we have used through is simply expressed as

\[
d\tau = dt(1-|u|^2)^{1/2},
\]

(104)

where \( u := dr/dt \in \mathbb{E}^3 \) is the velocity vector with which the rest reference frame \( K_r \) moves with respect to another arbitrarily chosen reference frame \( K \). Expression (104) implies, in particular, that

\[
dt^2 - |dr|^2 = d\tau^2,
\]

(105)

which is evidently identical to the classical infinitesimal Lorentz invariant. This is not a coincidence, since all our dynamical vacuum field equations were derived in turn [104, 106] from the governing equations of the vacuum potential field function \( W : M^4 \to \mathbb{P} \) in the form

\[
\partial^2W/\partial\tau^2 - \nabla^2W = \rho, \partial W/\partial\tau + \nabla, vW > 0, \partial \rho/\partial\tau + \nabla, \nu > 0,
\]

(106)

which is a priori Lorentz invariant. Here \( \rho \in \mathbb{P} \) is the charge density and \( v := dr/dt \) the associated local velocity of the vacuum field potential evolution. Consequently, the dynamical infinitesimal Lorentz invariant (105) reflects this intrinsic structure of equations (106). If it is rewritten in the following slightly nonstandard Euclidean form:

\[
dt^2 = d\tau^2 + |dr|^2
\]

(107)

it gives rise to a completely different relationship between the reference frames \( K \) and \( K_r \), namely

\[
dt = d\tau(1+ |\dot{r}|^2)^{1/2},
\]

(108)

where \( \dot{r} := dr/d\tau \) is the related particle velocity with respect to the rest reference system \( K_r \). Thus, we observe that all our Lagrangian analysis is strongly related to the functional expressions written in these "Euclidean" space-time coordinates and with respect to which the least action principle was applied. So we see that there are two alternatives - the first one is to apply the least action principle to the corresponding Lagrangian functions, expressed in the Minkowski space-time variables with respect to an arbitrarily chosen reference frame \( K \) and the second one is to apply the least action principle to the corresponding Lagrangian functions expressed in the Euclidean space-time variables with respect to the rest reference frame \( K_r \). But, as it was demonstrated above, the second alternative appeared to be physically reasonable in contrast to the first one, which gives rise to different physically senseless controversies.

All that above entails also a next slightly amusing but thought-provoking observation: "It follows that all of the results of classical special relativity related with the electrodynamics of charged point particles can be obtained (but not in a one-to-one correspondence) using our new reasonable definitions of the dynamical particle mass and the physically
motivated least action principle, calculated with respect to the related Euclidean space-time variables specifying the rest reference frame $K_r$.

2 The Electromagnetic Dirac-Fock-Podolsky Problem and Symplectic Properties of the Maxwell and Yang-Mills Type Dynamical Systems

2.1 Introduction

When investigating different dynamical systems on canonical symplectic manifolds, invariant under action of certain symmetry groups, additional mathematical structures often appear, the analysis of which shows their importance for understanding many related problems under study. Amongst them we here mention the Cartan type connection on an associated principal fiber bundle, which enables one to study in more detail the properties of the investigated dynamical system in the case of its reduction upon the corresponding invariant submanifolds and quotient spaces, associated with them.

Problems related to the investigation of properties of reduced dynamical systems on symplectic manifolds were studied, e.g., in [2, 103, 111, 25], where the relationship between a symplectic structure on the reduced space and the available connection on a principal fiber bundle was formulated in explicit form. Other aspects of dynamical systems related to properties of reduced symplectic structures were studied in [78, 113, 65, 66], where, in particular, the reduced symplectic structure was explicitly described within the framework of the classical Dirac scheme, and several applications to nonlinear (including celestial) dynamics were given.

It is well-known [26, 127, 37, 104, 106, 97] that the Hamiltonian theory of electromagnetic Maxwell equations faces a very important classical problem of introducing into the unique formalism the well known Lorentz conditions, ensuring both the wave structure of propagating quanta and the positivity of energy. Regrettfully, in spite of classical studies on this problem given by Dirac, Fock and Podolsky [38], the problem remains open, and the Lorentz condition is imposed within the modern electrodynamics as the external constraint not entering a priori the initial Hamiltonian (or Lagrangian) theory. Moreover, when trying to quantize the electromagnetic theory, as it was shown by Pauli, Dirac, Bogolubov and Shirkov and others [26, 97, 37], within the existing approaches the quantum Lorentz condition could not be satisfied, except in the average sense, since it becomes not compatible with the related quantum dynamics. This problem stimulated us to study this problem from the so called symplectic reduction theory [89, 25, 111], which allows the systematic introduction into the Hamiltonian formalism the external charge and current conditions, giving rise to a partial solution to the Lorentz condition problem mentioned above. Some applications of the method to Yang-Mills type equations interacting with a point charged particle, are presented in detail. In particular, based on analysis of reduced geometric structures on fibered manifolds, invariant under the action of a symmetry group, we construct the symplectic structures associated with connection forms on suitable principal fiber bundles. We present suitable mathematical preliminaries of the related Poissonian structures on the corresponding reduced symplectic manifolds, which are often used [2, 89, 79] in various problems of dynamics in modern mathematical physics, and apply them to study the non-standard Hamiltonian properties of the Maxwell and Yang-Mills type dynamical systems. We formulate a symplectic analysis of the important Lorentz type constraints, which describe the electrodynamics真空 properties.

We formulate a symplectic reduction theory of the classical Maxwell electromagnetic field equations and prove [104] that the important Lorentz condition, ensuring the existence of electromagnetic waves [26, 46, 80], can be naturally included into the Hamiltonian picture, thereby solving the Dirac, Fock and Podolsky problem [38] mentioned above. We also study from the symplectic reduction theory the Poissonian structures and the classical minimal interaction principle related with Yang-Mills type equations.

2.2 The Symplectic Reduction on Cotangent Fiber Bundles with Symmetry

Consider an $m$-dimensional smooth manifold $M$ and the cotangent vector fiber bundle $T^*(M)$. We equip (see [56], Chapter VII; [40]) the cotangent space $T^*(M)$ with the canonical Liouville 1-form $\lambda^{(1)} := pr_M^*\alpha^{(1)} \in \Lambda^1(T^*(M))$, where $pr_M : T^*(M) \to M$ is the canonical projection, $pr_M^* := (d \circ pr_M)^*: T^*(M) \to T^*(T^*(M))$ is the adjoint to the standard tangent mapping $d \circ pr_M := pr_M*: T(T^*(M)) \to T(M)$ with respect to the natural convolution on the product $T^*(M)) \otimes T(M)$. Then for a general one-form

$$\alpha^{(1)}(u) = \sum_{j=1}^m y_j du^j,$$

(109)
where \((u,v)\in T^*(M)\) are the corresponding canonical local coordinates on \(T^*(M)\), the canonical symplectic structure on \(T^*(M)\) will be equal to \(\omega^{(2)}(\alpha^{(1)}) := d\lambda(\alpha^{(1)}) = \sum_{j=1}^{m} dy_j \wedge du^i \in \Lambda^2(T^*(M))\). The any group of diffeomorphisms of the manifold \(M\), naturally lifted to the fiber bundle \(T^*(M)\), preserves the invariance of the canonical 1-form \(\hat{\lambda}(\alpha^{(1)}) \in \Lambda^1(T^*(M))\). In particular, if a smooth action of a Lie group \(G\) is given on the manifold \(M\), then every element \(a \in \Gamma\), where \(\Gamma\) is the Lie algebra of the Lie group \(G\), generates the vector field \(k_a: M \to T(M)\) in a natural manner. Furthermore, since the group action on \(M\), i.e.,

\[
\varphi: G \times M \to M,
\]

(110)
generates a diffeomorphism \(\varphi^*_g \in \text{Diff} M\) for every element \(g \in G\), this diffeomorphism is naturally lifted to the corresponding diffeomorphism \(\varphi^*_g \in \text{Diff} T^*(M)\) of the cotangent fiber bundle \(T^*(M)\), which also leaves the canonical 1-form \(pr^*_M \alpha^{(1)} \in \Lambda^1(T^*(M))\) invariant. Namely, the equality

\[
\varphi^*_g \hat{\lambda}(\alpha^{(1)}) = \hat{\lambda}(\alpha^{(1)})
\]

(111)
holds \([2, 56, 103]\) for every 1-form \(\alpha^{(1)} \in \Lambda^1(M)\). Thus, we can define on \(T^*(M)\) the corresponding vector field \(K_a : T^*(M) \to T(T^*(M))\) for every element \(a \in \Gamma\). Then condition (111) can be rewritten in the following form for all \(a \in \Gamma\):

\[
L_{K_a} : pr^*_M \alpha^{(1)} = pr^*_M \cdot L_{k_a} \alpha^{(1)} = 0,
\]

where \(L_{K_a}\) and \(L_{k_a}\) are the usual Lie derivatives on \(\Lambda^1(T^*(M))\) and \(\Lambda^1(M)\), respectively.

The canonical symplectic structure on \(T^*(M)\) defined above as

\[
\omega^{(2)}(\alpha^{(1)}) := d\lambda(\alpha^{(1)})
\]

(112)
is also invariant, i.e., \(L_{K_a} \omega^{(2)} = 0\) for all \(a \in \Gamma\).

For any smooth function \(H \in D(T^*(M))\), a Hamiltonian vector field \(K_H : T^*(M) \to T(T^*(M))\) such that

\[
i_{K_H} \omega^{(2)} = -dH
\]

(113)
is defined, and vice versa, because the symplectic 2-form (112) is non-degenerate. Using (113) and (112), we easily establish that the Hamiltonian function \(H := H_K \in D(T^*(M))\) is given by the expression

\[
H_K = pr^*_M \alpha^{(1)}(K_H) = \alpha^{(1)}(pr^*_MK_H) = \alpha^{(1)}(k_H),
\]

where \(k_H \in T(M)\) is the corresponding vector field on the manifold \(M\), whose lifting to the fiber bundle \(T^*(M)\) coincides with the vector field \(K_H : T^*(M) \to T(T^*(M))\).

For \(K_a : T^*(M) \to T(T^*(M))\), \(a \in \Gamma\), it is easy to establish that the corresponding Hamiltonian function

\[
H_a = \alpha^{(1)}(k_a) = pr^*_MK^a(\alpha^{(1)}(K_a))
\]

for \(a \in \Gamma\) defines \([2, 103, 63]\) a linear momentum mapping \(l : T^*(M) \to \Gamma^*\) according to the rule

\[
H_a := <l, a>,
\]

(114)
where \(<\cdot, \cdot>\) is the corresponding convolution on \(\Gamma^* \times \Gamma\). By virtue of definition (114), the momentum mapping \(l : T^*(M) \to \Gamma^*\) is invariant under the action of any invariant Hamiltonian vector field \(K_b : T^*(M) \to T(T^*(M))\) for any \(b \in \Gamma\). Indeed, \(L_{K_b} <l, a> = L_{k_b} H_a = -L_{k_a} H_b = 0\), because, by definition, the Hamiltonian function \(H_b \in D(T^*(M))\) is invariant under the action of any vector field \(K_a : T^*(M) \to T(T^*(M))\), \(a \in \Gamma\).
We now fix a regular value of the momentum mapping $l(u, v) = \xi \in \Gamma^*$ and consider the corresponding submanifold $M_\xi := \{ (u, v) \in T^*(M) : l(u, v) = \xi \in \Gamma^* \}$. On the basis of definition (109) and the invariance of the 1-form $pr_M^*\alpha \in \Lambda^1(T^*(M))$ under the action of the Lie group $\Gamma$ on $T^*(M)$, we can write the equalities
\[ <l(g \circ (u, v)), a > = pr_M^*(K_{\alpha})(g \circ (u, v)) = \]
\[=pr_M^*(K_{\alpha})(u,v) = \]
\[ <l(u,v), Ad_{g^{-1}}a > = Ad_{g^{-1}}l(u,v), a > \]
for any $g \in \Gamma$ and all $a \in \Gamma$ and $(u,v) \in T^*(M)$. Using (115) we establish that, for every $g \in \Gamma$ and all $(u, v) \in T^*(M)$, the following relation is true: $l(g \circ (u, v)) = Ad_{g^{-1}}l(u,v)$. This means that the diagram
\[ T^*(M) \rightarrow \Gamma^* \]
\[ g \downarrow \downarrow Ad_{g^{-1}} \]
\[ T^*(M) \rightarrow \Gamma^* \]
is commutative for all elements $g \in \Gamma$; the corresponding action $g : T^*(M) \rightarrow T^*(M)$ is called equivariant [2, 103].

Let $g_\xi \subset \Gamma$ denote the stabilizer of a regular element $\xi \in \Gamma^*$ with respect to the related co-adjoint action. It is obvious that in this case the action of the Lie subgroup $g_\xi$ on the submanifold $M_\xi \subset M := T^*(M)$ is naturally defined; we assume that it is free and proper. According to this action on $M_\xi$, we can define [2, 113, 65, 79] a so-called reduced space $\tilde{M}_\xi$ by taking the factor with respect to the action of the subgroup $g_\xi$ on $M_\xi$, i.e.,
\[ \tilde{M}_\xi := M_\xi / G_\xi. \]  
(116)
The quotient space (116) induces a symplectic structure $\tilde{\omega}_{\xi}^{(2)} \in \Lambda^2(\tilde{M}_\xi)$ on itself, which is defined as follows:
\[ \tilde{\omega}_{\xi}^{(2)}(\tilde{\eta}_1, \tilde{\eta}_2) = \omega_{\xi}^{(2)}(\eta_1, \eta_2), \]  
(117)
where $\tilde{\eta}_1, \tilde{\eta}_2 \in T(\tilde{M}_\xi)$ are arbitrary vectors onto which vectors $\eta_1, \eta_2 \in T(M_\xi)$ are projected, taken at any point $(u_\xi, v_\xi) \in M_\xi$, being uniquely projected onto the point $\tilde{\mu}_\xi \in \tilde{M}_\xi$, according to (116).

Let $\pi_\xi : M_\xi \rightarrow M$ denote the corresponding imbedding mapping into $M$ and let $r_\xi : M_\xi \rightarrow \tilde{M}_\xi$ denote the corresponding reduction to the space $\tilde{M}_\xi$.

Then relation (117) can be rewritten equivalently in the form of the equality
\[ r_\xi^*\tilde{\omega}_{\xi}^{(2)} = \pi_\xi^*\omega_{\xi}^{(2)}, \]  
(118)
defined on vectors on the cotangent space $T^*(M_\xi)$. To establish the symplecticity of the 2-form $\omega_{\xi}^{(2)} \in \Lambda^2(\tilde{M}_\xi)$, we use the corresponding non-degeneracy of the Poisson bracket $\{ \cdot, \cdot \}_\xi^\prime$ on $\tilde{\omega}_{\xi}^{(2)}$. To calculate it, we use a Dirac type construction, defining functions on $\tilde{M}_\xi$ as certain $G_\xi$-invariant functions on the submanifold $M_\xi$. Then one can calculate the Poisson bracket $\{ \cdot, \cdot \}_\xi^\prime$ of such functions that corresponds to symplectic structure (112) as an ordinary Poisson bracket on $M_\xi$. Arbitrarily extending these functions from the submanifold $M_\xi \subset M$ to a certain neighborhood $U(M_\xi) \subset M$, it is obvious that two extensions of a given function to the neighborhood $U(M_\xi)$ of
this type differ by a function that vanishes on the submanifold $M_\xi \subset M$. The difference between the corresponding Hamiltonian fields of these two different extensions to $U(M_\xi)$ is completely controlled by the conditions of the following lemma (see also [2, 103, 65, 113, 111]).

**Lemma 2.1** Suppose that a function $f : U(M_\xi) \to \mathbb{P}$ is smooth and vanishes on $M_\xi \subset T^*(M)$, i.e., $f|_{h_\xi} = 0$. Then, at every point $(u_\xi, v_\xi) \in M_\xi$ the corresponding Hamiltonian vector field $K_f \in T(U(M_\xi))$ is tangent to the orbit $Or(G; (u_\xi, v_\xi))$.

Proof. It is obvious that the submanifold $M_\xi \subset T^*(M)$ is defined by a certain collection of relations of the type

$$H_{a_s} = \xi_s, \quad \xi_s := <\xi_s, a_s >,$$

(119)

where $a_s \in \Gamma, s = 1, \dim G$, is a certain basis of the Lie algebra $\Gamma$, which follows from definition (114). Since a function $f : U(M_\xi) \to \mathbb{P}$ vanishes on $M_\xi$, we can write the following equality:

$$f = \sum_{s=1}^{\dim G} (H_{a_s} - \xi_s) f_s,$$

where $f_s : U(M_\xi) \to \mathbb{P}, \quad s = 1, \dim G$, is a certain collection of functions in the neighborhood $U(M_\xi)$. We take an arbitrary tangent vector $\eta \in T(U(M_\xi))$ at the point $(u_\xi, v_\xi) \in M_\xi$ and calculate the expression

$$< df(u_\xi, v_\xi), \eta(u_\xi, v_\xi) > = \sum_{s=1}^{\dim G} < dH_{a_s}(u_\xi, v_\xi), \eta(u_\xi, v_\xi) > f_s(u_\xi, v_\xi) =$$

$$= -\sum_{s=1}^{\dim G} \omega^{(2)}(K_{a_s}(u_\xi, v_\xi), \eta(u_\xi, v_\xi)) f_s(u_\xi, v_\xi) =$$

$$= -\omega^{(2)}(\sum_{s=1}^{\dim G} K_{a_s}(u_\xi, v_\xi) f_s(u_\xi, v_\xi), \eta(u_\xi, v_\xi)) =$$

$$= -< \sum_{s=1}^{\dim G} K_{a_s}(u_\xi, v_\xi) f_s(u_\xi, v_\xi), \eta(u_\xi, v_\xi) >.$$ (120)

It follows from the arbitrariness of the vector $\eta \in T(M_\xi)$ at the point $(u_\xi, v_\xi) \in M_\xi$ and relation (120) that

$$K_f = \sum_{s=1}^{\dim G} K_{a_s} f_s,$$

i.e., $K_f : M_\xi \to T(Or(G))$, which was to be proved.

As a corollary of Lemma 2.1, we obtain an algorithm for the determination of the reduced Poisson bracket $\{ . , . \}_r^f$ on the space $\overline{M}_\xi$ according to definition (118). Namely, we choose two functions defined on $M_\xi$ and invariant under the action of the subgroup $G_\xi$ and arbitrarily smoothly extend them to a certain open domain $U(M_\xi) \subset M$. Then we determine the corresponding Hamiltonian vector fields on $M$ and project them onto the space tangent to $M_\xi$, adding, if necessary, the corresponding vectors tangent to the orbit $Or(G)$. It is obvious that the projections obtained depend on the chosen extensions to the domain $U(M_\xi) \subset M$. As a result, we establish that the reduced Poisson bracket $\{ . , . \}_r^f$ is uniquely defined via the restriction of the initial Poisson bracket upon $M_\xi \subset M$. By virtue of the non-degeneracy of
the latter and the functional independence of the basis functions (119) on the submanifold $U(M_{\xi}) \subset M$, the reduced Poisson bracket $\{ \cdot , \cdot \}_{\xi}$ appears to be [2, 103, 113] non-degenerate on $\overline{M}_{\xi}$. As a consequence of the non-degeneracy, we establish that the dimension of the reduced space $\overline{M}_{\xi}$ is even. Taking into account that the element $\xi \in \Gamma^*$ is regular and the dimension of the Lie algebra of the stabilizer $\Gamma_{\xi}$ is equal to $\dim G_{\xi}$, we easily establish that $\dim \overline{M}_{\xi} = \dim M - 2\dim \Gamma_{\xi}$. Since, by construction, $\dim M = 2m$, we conclude that the dimension of the reduced space $\overline{M}_{\xi}$ is necessarily even.

For the correctness of the algorithm, it is necessary to establish the existence of the corresponding projections of Hamiltonian vector fields onto the tangent space $T(M_{\xi})$. The following statement is true.

**Theorem 2.2** At every point $(u_{\xi}, v_{\xi}) \in M_{\xi} \subset M$, one can choose a vector $V_f \in T(Or(G))$ such that $K_f(u_{\xi}, v_{\xi}) + V_f(u_{\xi}, v_{\xi}) \in T(u_{\xi}, v_{\xi})(M_{\xi})$. Furthermore, the vector $V_f \in T(Or(G))$ is determined uniquely up to a vector tangent to the orbit $Or(G_{\xi})$.

Proof. Note that the orbit $Or(G; (u_{\xi}, v_{\xi}))$ passing through the point $(u_{\xi}, v_{\xi}) \in M_{\xi}$ is always symplectically orthogonal to the tangent space $T(u_{\xi}, v_{\xi})(M_{\xi})$. Indeed, for any vector $\eta \in T(M_{\xi})$ and $a \in \Gamma$, we have $\omega^{(2)}(\eta, K_{\eta}) = -i_{K_{\eta}} \omega^{(2)}(\eta) = dH_a(\eta) = 0$, because the submanifold $M_{\xi} \subset M$ is defined by the equality $\langle \xi, a \rangle = H_a$ for all $a \in \Gamma$, i.e., $dH_a = 0$ on $M_{\xi}$. Thus, $T(M_{\xi}) \cap T(Or(G)) = T(Or(G))$ because $H_a \circ g_{\xi} = H_a$ for all $g_{\xi} \in G_{\xi}$, which follows from the invariance of the element $\xi \in \Gamma^*$ under the action of the Lie group $G_{\xi}$. We now solve the imbedding condition $K_f + V_f \in T(M_{\xi})$, or the equation

$$\omega^{(2)}(K_f + V_f, K_a) = 0$$

on the manifold $M_{\xi} \subset T^*(M)$ for all $a \in G$. We rewrite equality (121) in the form

$$K_a f = \omega^{(2)}(V_f, K_a)$$

(122)

on $M_{\xi}$ for all $a \in G$; it is obvious that the 2-form on the right-hand side of (122) depends only on the element $\xi \in G^*$. Taking into account the equivariance of the group action on $M$ and the obvious equality

$$\omega^{(2)}(K_a, K_b) = pr_M^* \alpha^{(1)}([K_a, K_b]) = -pr_M^* \alpha^{(1)}(K_{[a, b]})$$

for all $a, b \in \Gamma$, we establish that there exists an element $a_f \in \Gamma$ such that $V_f = K_{a_f} \in T(Or(G))$ and

$$\omega^{(2)}(V_f, K_a) = \omega^{(2)}(K_{a_f}, K_a) = pr_M^* \alpha^{(1)}([K_{a_f}, K_a]) =$$

$$= pr_M^* \alpha^{(1)}(K_{[a_f, a]}) = H_{[a_f, a]} = \langle l_{[a_f, a]} \rangle =$$

$$= \langle \xi, [a_f, a] \rangle = \langle ad_{a_f}^* \xi, a \rangle$$

(123)

on $M_{\xi}$ for all $a \in \Gamma$. Since $ad_{a_f}^* \xi = 0$ for any $a_f \in \Gamma_{\xi}$, we conclude that, on the quotient space $\Gamma/\Gamma_{\xi}$, the right-hand side of (123) defines a non-degenerate skew-symmetric form associated with the canonical isomorphism $\xi: \Gamma/\Gamma_{\xi} \rightarrow (\Gamma/\Gamma_{\xi})^*$, where, by definition,

$$\langle \hat{\xi}(a), \hat{b} \rangle = \langle \xi, [a, b] \rangle$$

(124)
for any \( \tilde{a} \) and \( \tilde{b} \in \Gamma/\Gamma_\xi \) with the corresponding representatives \( a \) and \( b \in \Gamma \). Further, since the function \( f : M_\xi \to P \) is \( G_\xi \)-invariant on \( M_\xi \subset M \), the right-hand side of (122) defines an element \( \mu_f \in (\Gamma/\Gamma_\xi)\) by the equality

\[
\mu_f : \tilde{a} := -K_uf
\]

for all \( a \in G \). Using relations (123) and (124), we establish that there exists the element

\[
\tilde{a}_f = \hat{\xi}^{-1} \circ \mu_f \in \Gamma/\Gamma_\xi.
\]

Since the element \( \tilde{a}_f \in \Gamma/\Gamma_\xi \) is associated with the element \( a_f \) (mod \( \Gamma_\xi \)) \( \in \Gamma \), which uniquely generates a locally defined vector field \( K_{a_f} : Or(G) \to T(Or(G)) \), using the fact that \( V_f = K_{a_f} \) on \( M_\xi \subset M \), we complete the proof of the theorem.

Now assume that two functions \( f_1, f_2 \in D(M_\xi) \) are \( G_\xi \)-invariant. Then their reduced Poisson bracket \( \{f_1, f_2\}_\xi \) on \( \overline{M}_\xi \) is defined according to the rule:

\[
\{f_1, f_2\}_\xi := -\omega(2)(K_{a_1} + V_{f_1}, K_{a_2} + V_{f_2}) = \{f_1, f_2\} + \omega(2)(V_{f_1}, V_{f_2}),
\]

where we have used the following identities on \( M_\xi \subset T^*(M) \):

\[
\omega(2)(K_{a_1} + V_{f_1}, V_{f_2}) = 0 = \omega(2)(K_{a_2} + V_{f_2}, V_{f_1}),
\]

being simple consequences of equality (121) on \( M_\xi \). Regarding (123), relation (125) takes the form

\[
\{f_1, f_2\}_\xi = \{f_1, f_2\} + \frac{1}{2}(V_{f_1} f_2 - V_{f_2} f_1),
\]

where \( f_1, f_2 \in D(M_\xi) \) are arbitrary smooth extensions of the \( G_\xi \)-invariant functions defined earlier on the domain \( U(M_\xi) \). Thus, the following theorem holds.

**Theorem 2.3** The reduced Poisson bracket of two functions on the quotient space \( \overline{M}_\xi = M_\xi/G_\xi \) is determined with the use of their arbitrary smooth extensions to functions on an open neighborhood \( U(M_\xi) \) according to the Dirac-type formula (126).

### 2.2.1 The Symplectic Reduction on Principal Fiber Bundles with Connection

We begin by reviewing the backgrounds of the reduction theory subject to Hamiltonian systems with symmetry on principle fiber bundles. The material is partly available in [53, 78], so here it will be only sketched in notations suitable for us.

Let \( G \) denote a given Lie group with the unity element \( e \in G \) and the corresponding Lie algebra \( \mathfrak{g} = T_e(G) \). Consider a principal fiber bundle \( p : (\mathcal{M}, \phi) \to \mathcal{N} \) with the structure group \( G \) and base manifold \( \mathcal{N} \), on which the Lie group \( G \) acts by means of a mapping \( \phi : \mathcal{M} \times G \to \mathcal{M} \). Namely, for each \( g \in G \) there is a group diffeomorphism \( \phi_g : \mathcal{M} \to \mathcal{M} \), generating for any fixed \( u \in \mathcal{M} \) the following induced mapping: \( \hat{u} : G \to \mathcal{M} \), where

\[
\hat{u}(g) = \phi_g(u).
\]

On the principal fiber bundle \( p : (\mathcal{M}, \phi) \to \mathcal{N} \) a connection \( \Gamma(\mathcal{M}) \) is assigned by means of such a morphism \( A : (T(M), \phi_{\ast\ast}) \to (\Gamma, \mathrm{Ad}_{\phi^{-1}}) \) that for each \( u \in \mathcal{M} \) a mapping \( A(u) : T_u(M) \to \Gamma \) is a left inverse.
one to the tangent mapping \( d\hat{u}(e) := \hat{u}_*(e) : \Gamma \rightarrow T_u(M) \) at unity element \( e \in G \), that is
\[
A(u)\hat{u}_*(\xi) = 1. \tag{128}
\]

As usual, denote by \( \varphi^\ast_g : T^\ast(M) \rightarrow T^\ast(M) \) the corresponding cotangent lift of the mapping \( \varphi^\ast : M \rightarrow M \) at any \( g \in G \). If \( \alpha^{(1)} \in \Lambda^1(M) \) is the canonical \( G \)-invariant 1-form on \( M \), the canonical symplectic structure \( \omega^{(2)} \in \Lambda^2(T^\ast(M)) \) given by
\[
\omega^{(2)} := dp^\ast\alpha^{(1)} \tag{129}
\]
generates the corresponding momentum mapping \( l : T^\ast(M) \rightarrow \Gamma^\ast \), where
\[
l(\alpha^{(1)})(u) = \hat{u}^\ast(e)\alpha^{(1)}(u) \tag{130}
\]
for all \( u \in M \). Remark here that the principal fiber bundle structure \( p : (M, \varphi) \rightarrow N \) means in part the exactness of the following sequences of mappings:
\[
0 \rightarrow \Gamma \rightarrow T_u(M) \rightarrow T_{p(u)}(N) \rightarrow 0, \tag{131}
\]
that is
\[
p_* (u)\hat{u}_*(e) = 0 = \hat{u}^\ast(e)p^\ast(u) \tag{132}
\]
for all \( u \in M \). Combining (132) with (128) and (130), one obtains such an embedding:
\[
1 - A^\ast(u)\hat{u}^\ast(e)[\alpha^{(1)}(u)] \in \text{range} p^\ast(u) \tag{133}
\]
for the canonical 1-form \( \alpha^{(1)} \in \Lambda^1(M) \) at \( u \in M \). The expression (133) means of course, that
\[
\hat{u}^\ast(e)[1 - A^\ast(u)\hat{u}^\ast(e)]\alpha^{(1)}(u) = 0 \tag{134}
\]
for all \( u \in M \). Now taking into account that the mapping \( p^\ast(u) : T^\ast(N) \rightarrow T^\ast(M) \) is for each \( u \in M \) injective, it has the unique inverse mapping \( (p^\ast(u))^{-1} \) upon its image \( p^\ast(u)T^\ast_{p(u)}(N) \subset T^\ast_u(M) \). Thereby for each \( u \in M \) one can define a morphism \( p^\Lambda : (T^\ast(M), \varphi^\ast_g) \rightarrow T^\ast(N) \) as
\[
p^\Lambda (u) : \alpha^{(1)}(u) \rightarrow (p^\ast(u))^{-1}[1 - A^\ast(u)\hat{u}^\ast(e)]\alpha^{(1)}(u). \tag{135}
\]
Based on the definition (135) one can easily check that the diagram
\[
\begin{array}{ccc}
T^\ast(M) & \xrightarrow{p^\Lambda} & T^\ast(N) \\
p_{\mathcal{M}} & \downarrow & \downarrow p_{\mathcal{N}} \\
M & \xrightarrow{p} & N
\end{array} \tag{136}
\]
is commutative.

Let an element \( \xi \in \Gamma^\ast \) be \( G \)-invariant, that is \( Ad_{g^{-1}}^\ast \xi = \xi \) for all \( g \in G \). Denote also by \( p^\xi_{\mathcal{A}} \) the restriction of the mapping (135) upon the subset \( \mathcal{M}_\xi := l^{-1}(\xi) \subset T^\ast(M) \), that is \( p^\xi_{\mathcal{A}} : \mathcal{M}_\xi \rightarrow T^\ast(N) \), where for all \( u \in M \)
\[
p^\xi_{\mathcal{A}}(u) : l^{-1}(\xi) \rightarrow (p^\ast(u))^{-1}[1 - A^\ast(u)\hat{u}^\ast(e)]l^{-1}(\xi). \tag{137}
\]
Now one can characterize the structure of the reduced phase space \( \overline{\mathcal{M}}_\xi := l^{-1}(\xi)/G \) by means of the following lemma.
Lemma 2.4 The mapping $p^\xi_A(u) : M_{\xi} \to T^*(N)$, where $M_{\xi} := l^{-1}(\xi)$, is a principal fiber $G$-bundle with the reduced space $\overline{M}_{\xi}$, being diffeomorphic to $T^*(N)$.

Denote by $< \ldots >_G$ the standard $Ad$-invariant non-degenerate scalar product on $\Gamma \times \Gamma$. Based on Lemma 2.4 one derives the following characteristic theorem.

Theorem 2.5 Given a principal fiber $G$-bundle with a connection $\Gamma(A)$ and a $G$-invariant element $\xi \in \Gamma^*$, then every such connection $\Gamma(A)$ defines a symplectomorphism $\nu_{\xi} : \overline{M}_{\xi} \to T^*(N)$ between the reduced phase space $\overline{M}_{\xi}$ and cotangent bundle $T^*(N)$, where $l : T^*(M) \to \Gamma^*$ is the naturally associated momentum mapping for the group $G$-action on $M$. Moreover, the following equality

$$
(p^\xi_A(dpr^*_N\beta^{(1)} + pr^*_N\Omega^{(2)}_{\xi})) = dpr^*_M\alpha^{(1)}
$$

holds for the canonical 1-forms $\beta^{(1)} \in \Lambda^1(N)$ and $\alpha^{(1)} \in \Lambda^1(M)$, where $\Omega^{(2)}_{\xi} := < \xi, \Omega^{(2)} >_\Gamma$ is the $\xi$-component of the corresponding curvature form $\Omega^{(2)} \in \Lambda^2(N) \otimes \Gamma$.

Proof. One has that on $l^{-1}(\xi) \subset M$ the following expression, due to (135), holds:

$$
p^*(u)p^\xi_A(\alpha^{(1)}(u)) = p^*(u)\beta^{(1)}(pr_N(u)) = \alpha^{(1)}(u) - \Lambda^*(u)u^*(e)\alpha^{(1)}(u)
$$

for any $\beta^{(1)} \in T^*(N)$ and all $u \in M_{\xi} := p_Ml^{-1}(\xi) \subset M$. Thus we easily get that

$$
\alpha^{(1)}(u) = (p^\xi_A)^{-1}\beta^{(1)}(p_N(u)) = p^*(u)\beta^{(1)}(pr_N(u)) + < \xi, \Lambda(u) >_\Gamma
$$

for all $u \in M_{\xi}$. Recall now that in virtue of (136) on the manifold $M_{\xi}$ there hold relationships:

$$
p \circ pr^*_M_{\xi} = pr_N \circ p^\xi_A, \quad pr^*_M_{\xi} \circ p^* = (p^\xi_A)^* \circ pr^*_N.
$$

Therefore we can now write down that

$$
pr^*_M_{\xi}\alpha^{(1)}(u) = pr^*_M_{\xi}\beta^{(1)}(p_N(u)) + pr^*_M_{\xi} < \xi, \Lambda(u) >_\Gamma

= (p^\xi_A)^*(pr^*_M\beta^{(1)}(u)) + pr^*_M_{\xi} < \xi, \Lambda(u) >_\Gamma,
$$

whence taking the external differential, one arrives at the following equalities:

$$
n\circ pr^*_M_{\xi}\alpha^{(1)}(u) = (p^\xi_A)^d(pr^*_M\beta^{(1)}(u)) + pr^*_M_{\xi} < \xi, d\Lambda(u) >_\Gamma =

= (p^\xi_A)^*d(pr^*_M\beta^{(1)}(u)) + pr^*_M_{\xi} < \xi, \Omega(p(u)) >_\Gamma =

= (p^\xi_A)^*d(pr^*_M\beta^{(1)}(u)) + pr^*_M_{\xi} p^* < \xi, \Omega >_\Gamma (u) =

= (p^\xi_A)^*d(pr^*_M\beta^{(1)}(u)) + (p^\xi_A)^*pr^*_N < \xi, \Omega >_\Gamma (u) =

= (p^\xi_A)^*[(d(pr^*_N\beta^{(1)}(u)) + pr^*_N < \xi, \Omega >_\Gamma (u)].
$$

When deriving the above expression we made use of the following property satisfied by the curvature 2-form $\Omega \in \Lambda^2(M) \otimes \Gamma$:

$$
< \xi, d\Lambda(u) >_\Gamma = < \xi, d\Lambda(u) + \Lambda(u) \wedge \Lambda(u) >_\Gamma - < \xi, \Lambda(u) \wedge \Lambda(u) >_\Gamma
$$
\[ \langle \xi, \Omega(p_N(u)) \rangle \rangle = \langle \xi, p_N^* \Omega \rangle \ (u) \]

at any \( u \in M_{\xi} \), since for any \( A, B \in \Gamma \) there holds \( \langle \xi, [A, B] \rangle \rangle = \langle \xi, p_N^* A, B \rangle \rangle = 0 \) in virtue of the invariance condition \( Ad_\gamma^* \xi = \xi \). Thereby the proof is finished.

**Remark 2.6** As the canonical 2-form \( d \ pr_N^* \alpha^{(1)} \in \mathcal{N}^2(T^* (M)) \) is \( G \)-invariant on \( T^* (M) \) due to construction, it is evident that its restriction upon the \( G \)-invariant submanifold \( M_{\xi} \subset T^* (M) \) will be effectively defined only on the reduced space \( \overline{M}_{\xi} \), that ensures the validity of the equality sign in (138).

As a consequence of Theorem 2.5 one can formulate the following useful for applications theorems.

**Theorem 2.7** Let an element \( \xi \in \Gamma^* \) have the isotropy group \( G_{\xi} \) acting on the subset \( M_{\xi} \subset T^* (M) \) freely and properly, so that the reduced phase space \( (\overline{M}_{\xi}, \overline{\sigma}_{\xi}^{(2)}) \) where, by definition, \( \overline{M}_{\xi} := \text{ker} (\xi)/G_{\xi} \), is symplectic whose symplectic structure is defined as

\[ \overline{\sigma}_{\xi}^{(2)} := \text{dpr}_{N}^* \alpha^{(1)} \bigg|_{\overline{M}_{\xi}} \] (139)

If a principal fiber bundle \( p : (M, \varphi) \to N \) has a structure group coinciding with \( G_{\xi} \), then the reduced symplectic space \( (\overline{M}_{\xi}, \overline{\sigma}_{\xi}^{(2)}) \) is symplectomorphic to the cotangent symplectic space \( (T^* (N), \overline{\omega}_{\xi}^{(2)}) \), where

\[ \overline{\omega}_{\xi}^{(2)} = \text{dpr}_{N}^* \beta^{(1)} + pr_N^* \Omega_{\xi}^{(2)} \] (140)

and the corresponding symplectomorphism is given by a relation like (138).

**Theorem 2.8** In order that two symplectic spaces \( (\overline{M}_{\xi}, \overline{\sigma}_{\xi}^{(2)}) \) and \( (T^* (N), \text{dpr}_{N}^* \beta^{(1)}) \) were symplectomorphic, it is necessary and sufficient that the element \( \xi \in \ker h \), where for \( G \)-invariant element \( \xi \in \Gamma^* \) the mapping \( h : \xi \to \Omega_{\xi}^{(2)} \in H^2 (N; \mathbb{Z}) \), with \( H^2 (N; \mathbb{Z}) \) being the cohomology class of 2-forms on the manifold \( N \).

### 2.3 The Hamiltonian Analysis of the Maxwell Electromagnetic Dynamical Systems

We take the Maxwell electromagnetic equations to be

\[ \partial E / \partial t = \nabla \times B - j, \quad \partial B / \partial t = - \nabla \times E, \] (141)

on the cotangent phase space \( T^* (N) \) to \( N \subset T \) (\( D; \mathbb{E}^3 \)), being the smooth manifold of smooth vector fields on an open domain \( D \subset \mathbb{P}^3 \), all expressed in the light speed units. Here \( (E, B) \in T^* (N) \) is a vector of electric and magnetic fields, \( \rho : D \to \mathbb{P} \) and \( j : D \to \mathbb{E}^3 \) are, simultaneously, fixed charge and current densities in the domain \( D \), satisfying the equation of continuity

\[ \partial \rho / \partial t + \langle \nabla, j \rangle = 0, \] (142)

holding for all \( t \in \mathbb{P} \), where we denoted by the sign "\( \nabla \)" the gradient operation with respect to a variable \( x \in D \), by the sign "\( \times \)" the usual vector product in \( \mathbb{E}^3 := (\mathbb{P}^3, < \cdot, >) \), being the standard three-dimensional Euclidean vector space \( \mathbb{P}^3 \) endowed with the usual scalar product "\( < \cdot, > \)."

Aiming to represent equations (141) as those on reduced symplectic space, we define an appropriate configuration space \( M \subset T \) (\( D; \mathbb{E}^3 \)) with a vector potential field coordinate \( A \in M \). The cotangent space \( T^* (M) \) may be identified with pairs \( (A; Y) \in T^* (M) \), where \( Y \in T^* (D; \mathbb{E}^3) \) is a suitable vector field density in \( D \). On the space \( T^* (M) \) there exists the weak canonical symplectic form \( \omega^{(2)} \in \wedge^2 (T^* (M)) \), allowing, owing to the definition of the
Liouville from
\[ \lambda(\alpha^{(1)})(A;Y) = \int_D d^3 x < Y, dA > := (Y, dA), \]  
(143)
the canonical expression
\[ \omega^{(2)}(\alpha^{(1)}) := d\lambda(\alpha^{(1)}) = (dY, \wedge dA). \]  
(144)
Here we denoted by "^\wedge" the usual external differentiation, by \( d^3 x \), \( x \in D \), the Lebesgue measure in the domain \( D \) and by \( pr : T^*(M) \to M \) the standard projection upon the base space \( M \). Define now a Hamiltonian function \( \tilde{\mathcal{H}} \in \Delta(T^*(M)) \) as
\[ H(A,Y) = 1/2[(Y,Y) + (\nabla \times A, \nabla \times A) + (\nabla, \nabla A >, A >)], \]  
(145)
describing the well-known Maxwell equations in vacuum, if the densities \( \rho = 0 \) and \( j = 0 \). Really, owing to (144) one easily obtains from (145) that
\[ \partial A / \partial t := \partial \mathcal{H} / \partial Y = Y, \]  
(146)
\[ \partial Y / \partial t := -\mathcal{H} / \partial A = -\nabla \times B + \nabla < \nabla, A >, \]  
(147)
being true wave equations in vacuum, where we put, by definition,
\[ B := \nabla \times A, \]  
(148)
being the corresponding magnetic field. Now defining
\[ E := -Y - \nabla W \]  
(149)
for some function \( W : M \to P \) as the corresponding electric field, the system of equations (146) will become, owing to definition (147),
\[ \partial B / \partial t = -\nabla \times E, \quad \partial E / \partial t = \nabla \times B, \]  
(149)
exactly coinciding with the Maxwell equations in vacuum, if the Lorentz condition
\[ \partial W / \partial t + < \nabla, A > = 0 \]  
(150)
is involved.

Since definition (148) was essentially imposed rather than arising naturally from the Hamiltonian approach and our equations are valid only for a vacuum, we shall try to improve upon these matters by employing the reduction approach devised in Section 2. Namely, we start with the Hamiltonian (145) and observe that it is invariant with respect to the following abelian symmetry group \( G := \exp \Gamma \), where \( \Gamma ; \mathcal{C}^{(1)}(D;P) \) acting on the base manifold \( M \) naturally lifted to \( T^*(M) \); for any \( \psi \in \Gamma \) and \( (A,Y) \in T^*(M) \)
\[ \phi_\psi(A) := A + \nabla \psi, \quad \phi_\psi(Y) = Y. \]  
(151)
The 1-form (143) under transformation (151) also is invariant since
\[ \phi_\psi^* \lambda(\alpha^{(1)})(A,Y) = (Y, dA + \nabla d\psi) = \]  
(152)
\[ = (Y, dA) - (\nabla, Y >, d\psi) = \lambda(\alpha^{(1)})(A,Y), \]
where we made use of the condition \( d\psi ; 0 \in \Lambda^1(T^*(M)) \) for any \( \psi \in \Gamma \). Thus, the corresponding momentum mapping (130) is given as
\[ l(A,Y) = -< \nabla, Y > \]  
(153)
for all \( (A,Y) \in T^*(M) \). If \( \rho \in \Gamma^* \) is fixed, one can define the reduced phase space \( \overline{M}_\rho := \Lambda^1(\rho)/G \), since evidently, the isotropy group \( G_\rho = G \), owing to its commutativity and the condition (151). Consider now a principal
fiber bundle \( p : M \rightarrow N \) with the abelian structure group \( G \) and a base manifold \( N \) taken as
\[
N := \{ B \in T (D; E^3) : < \nabla, B >= 0, < \nabla, E(S) >= \rho \},
\]
(154)
where, by definition,
\[
p(A) = B = \nabla \times A.
\]
(155)
We can construct a connection 1-form \( A \in \Lambda^1(M) \otimes \Gamma \) on this bundle, where for all \( A \in M \)
\[
A(A) \cdot \hat{A}(l) = 1, \ d < \rho, A(A) >_{\Gamma} = \Omega^{(2)}_{\rho}(A) \in H^2(M; Z),
\]
(156)
where \( A(A) \in \Lambda^1(M) \) is some differential 1-form, which we choose in the following form:
\[
A(A) := -(W, d < \nabla, A >),
\]
(157)
where \( W \in C^{(1)}(D; P) \) is some scalar function, still not defined. As a result, the Liouville form (143) transforms into
\[
\lambda(\tilde{\alpha}^{(2)}_{\rho}) := (Y, dA) - (W, d < \nabla, A >) = (Y + \nabla W, dA) := (\tilde{Y}, dA), \tilde{Y} := Y + \nabla W,
\]
(158)
giving rise to the corresponding canonical symplectic structure on \( T^*(M) \) as
\[
\tilde{\alpha}^{(2)}_{\rho} := d\lambda(\tilde{\alpha}^{(1)}_{\rho}) = (d\tilde{Y}, \wedge dA).
\]
(159)
Respectively, the Hamiltonian function (145), as a function on \( T^*(M) \), transforms into
\[
\tilde{H}_{\rho}(A, \tilde{Y}) = 1/2[(\tilde{Y}, \tilde{Y}) + (\nabla \times A, \nabla \times A) + (\nabla < \nabla, A > < \nabla, A >)],
\]
(160)
coinciding with the well-known Dirac-Fock-Podolsky [26, 38] Hamiltonian expression. The corresponding Hamilton equations on the cotangent space \( T^*(M) \)
\[
\partial A/\partial t := \tilde{\alpha}^{(1)}_{\rho} / \tilde{\alpha}^{(1)}_{\rho} = \tilde{Y}, \tilde{Y} := -E - \nabla W,
\]
\[
\partial \tilde{Y}/\partial t := -\tilde{\alpha}^{(1)}_{\rho} / \tilde{\alpha}^{(1)}_{\rho} = -\nabla \times (\nabla \times A) + \nabla < \nabla, A >
\]
describe true wave processes related to the Maxwell equations in vacuum, which do not take into account boundary charge and current densities conditions. Really, from (160) we obtain that
\[
\partial^2 A/\partial t^2 - \nabla^2 A = 0 \Rightarrow \partial E/\partial t + \nabla(\partial W/\partial t + < \nabla, A >) = -\nabla \times B,
\]
(161)
giving rise to the true vector potential wave equation, but the electromagnetic Faraday induction law is satisfied if one to impose additionally the Lorentz condition (150).

To remedy this situation, we will apply to this symplectic space the reduction technique devised in Subsection (2.2). Namely, owing to Theorem 2.7, the constructed above cotangent manifold \( T^*(N) \) is symplectomorphic to the corresponding reduced phase space \( \overline{M}_{\rho} \), that is
\[
\overline{M}_{\rho}; \{ (B; S) \in T^*(N): < \nabla, E(S) >= \rho, < \nabla, B >= 0 \}
\]
(162)
with the reduced canonical symplectic 2-form
\[
\tilde{\omega}^{(2)}_{\rho}(B, S) = (dB, \wedge dS = d\lambda(\alpha^{(1)}_{\rho})(B, S), \lambda(\alpha^{(1)}_{\rho})(B, S) := -(S, dB),
\]
(163)
where we put, by definition,
\[
\nabla \times S + F + \nabla W = -\tilde{Y} := E + \nabla W, < \nabla, F >= \rho,
\]
(164)
for some fixed vector mapping \( F \in C^{(1)}(D; E^3) \), depending on the imposed boundary conditions. The result (163)
follows right away upon substituting the expression for the electric field \( E = \nabla \times S + F \) into the symplectic structure (159), and taking into account that \( dF = 0 \) in \( A(M) \). The Hamiltonian function (160) reduces, respectively, to the following symbolic form:

\[
H_p(B, S) = \frac{1}{2}[(B, B) + (\nabla \times S + F + \nabla W, \nabla \times S + F + \nabla W) + \\
+ (\langle \nabla, (\nabla \times)^{-1} B \rangle, \langle \nabla, (\nabla \times)^{-1} B \rangle)],
\]

(165)

where \( (\nabla \times)^{-1} \) means, by definition, the corresponding inverse curl-operation, mapping [89] the divergence-free subspace \( C_{div}^{(1)}(D; \mathbb{E}^3) \subset C^{(1)}(D; \mathbb{E}^3) \) into itself. As a result from (165), the Maxwell equations (141) become a canonical Hamiltonian system upon the reduced phase space \( T^*(N) \), endowed with the canonical symplectic structure (163) and the modified Hamiltonian function (165). Really, one easily obtains that

\[
\partial S/\partial t = \partial H/\partial B = B - (\nabla \times)^{-1} \nabla \times \langle \nabla, (\nabla \times)^{-1} B \rangle,
\]

(166)

where we make use of the definition \( E = \nabla \times S + F \) and the elementary identity \( \nabla \times \nabla = 0 \). Thus, the second equation of (166) coincides with the second Maxwell equation of (141) in the classical form

\[
\partial B/\partial t = -\nabla \times E.
\]

Moreover, from (164), owing to (166), one obtains via the differentiation with respect to \( t \in \mathbb{R} \) that

\[
\partial E/\partial t = \partial F/\partial t + \nabla \times \partial S/\partial t = \\
\partial F/\partial t + \nabla \times B,
\]

(167)

(168)

as well as, owing to (142),

\[
\nabla \times (\partial E/\partial t) = \nabla \times (\partial F/\partial t) = -\nabla \times j.
\]

So, we can find from (168) that, up to non-essential curl-terms \( \nabla \times (\cdot) \), the following relationship

\[
\nabla \times (\partial F/\partial t) = -j
\]

(169)

holds. Really, the current density vector \( j \in C^{(1)}(D; \mathbb{E}^3) \), owing to the equation of continuity (142), is defined up to curl-terms \( \nabla \times (\cdot) \) which can be included into the right-hand side of (169). Having now substituted (169) into (167), we obtain exactly the first Maxwell equation of (141):

\[
\partial E/\partial t = \nabla \times B - j,
\]

(170)

being supplemented, naturally, with the external boundary constraint conditions

\[
\langle \nabla, B \rangle = 0, \quad \langle \nabla, E \rangle = \rho,
\]

(171)

\[
\partial \rho/\partial t + \langle \nabla, j \rangle = 0,
\]

owing to the continuity relationship (142) and definition (162).

Concerning the wave equations, related to the Hamiltonian system (166), we obtain the following: the electric field \( E \) is recovered from the second equation as

\[
E := -\nabla \cdot (\partial A/\partial t) - \nabla W,
\]

(172)

where \( W \in C^{(1)}(D; \mathbb{P}) \) is some smooth function, depending on the vector field \( A \in M \). To retrieve this dependence, we substitute (169) into equation (170), having taken into account that \( B = \nabla \times A \):

\[
\nabla^2 A/\partial t^2 - \nabla (\partial W/\partial t + \langle \nabla, A \rangle) = \nabla^2 A + j.
\]

(173)

With the above, if we now impose the Lorentz condition (150), we obtain from (173) the corresponding true wave
equations in the space-time, taking into account the external charge and current density conditions (171).

Notwithstanding our progress so far, the problem of fulfilling the Lorentz constraint (150) naturally within the canonical Hamiltonian formalism still remains to be completely solved. To this end, we are compelled to analyze the structure of the Liouville 1-form (158) for Maxwell equations in vacuum on a slightly extended functional manifold $M \times L$. As a first step, we rewrite 1-form (158) as

$$\lambda(\vec{\alpha}_\rho^{(1)}) := (\tilde{Y}, dA) = (Y + \nabla W, dA) = (Y, dA) + (W, -d < \nabla, A >) := (Y, dA) + (W, d\chi),$$

(174)

where we put, by definition,

$$\chi := -< \nabla, A >.$$  

(175)

Considering now the elements $(Y, A; \chi, W) \in T^*(M \times L)$ as new canonical variables on the extended cotangent phase space $T^*(M \times L)$, where $L := C^{(1)}(D; P)$, we can rewrite the symplectic structure (159) in the following canonical form

$$\tilde{\omega}_\rho^{(2)} := d\lambda(\vec{\alpha}_\rho^{(1)}) = (dY, \wedge dA) + (dW, \wedge d\chi).$$

(176)

Subject to the Hamiltonian function (160) we obtain the expression

$$H(A, Y; \chi, W) = 1/2[(Y - \nabla W, Y - \nabla W) + (\nabla \times A, \nabla \times A) + (\chi, \chi)],$$

(177)

with respect to which the corresponding Hamiltonian equations take the form:

$$\partial A/\partial t := \partial H/\partial Y = Y - \nabla W, \ Y := -E,$$

$$\partial Y/\partial t := -\partial H/\partial A = -\nabla \times (\nabla \times A),$$

$$\partial \chi/\partial t := \partial H/\partial W = <\nabla, Y - \nabla W >,$$

$$\partial W/\partial t := -\partial H/\partial \chi = -\chi.$$  

(178)

From (178) we obtain, owing to external boundary conditions (171), successively that

$$\partial B/\partial t + \nabla \times E = 0, \ \partial^2 W/\partial t^2 - \nabla^2 W = \rho,$$

(179)

$$\partial E/\partial t - \nabla \times B = 0, \ \partial^2 A/\partial t^2 - \nabla^2 A = -\nabla(\partial W/\partial t + < \nabla, A >).$$

As is seen, these equations describe electromagnetic Maxwell equations in vacuum, but without the Lorentz condition (150). Thereby, as above, we will apply to the symplectic structure (176) the reduction technique devised in Section 2. We obtain that under transformations (164) the corresponding reduced manifold $\tilde{M}_\rho$ becomes endowed with the symplectic structure

$$\tilde{\omega}_\rho^{(2)} := (dB, \wedge dS) + (dW, \wedge d\chi),$$

(180)

and the Hamiltonian (177) assumes the form

$$H(S, B; \chi, W) = 1/2[(\nabla \times S + F + \nabla W, \nabla \times S + F + \nabla W) + (B, B) + (\chi, \chi)],$$

(181)

whose Hamiltonian equations

$$\partial S/\partial t := \partial H/\partial B = B, \ \partial W/\partial t := -\partial H/\partial \chi = -\chi,$$

(182)

$$\partial B/\partial t := -\partial H/\partial S = -\nabla \times (\nabla \times S + F + \nabla W) = -\nabla \times E,$$

$$\partial \chi/\partial t := \partial H/\partial W = -< \nabla, \nabla \times S + F + \nabla W > = -< \nabla, E > -\Delta W,$$

coincide completely with Maxwell equations (141) under conditions (164), describing true wave processes in vacuum, as well as the electromagnetic Maxwell equations, taking into account a priori both the imposed external boundary conditions (171) and the Lorentz condition (150), solving the problem mentioned in [26, 38]. Really, it is easy to obtain from (182) that
Based now on (183) and (171) one can easily calculate [106, 104] the magnetic wave equation

\[ \frac{\partial^2 A}{\partial t^2} - \Delta A = j, \]  

supplementing the suitable wave equation on the scalar potential \( W \in \mathbb{L} \), finishing the calculations. Thus, we can formulate the following proposition.

**Proposition 2.9** The electromagnetic Maxwell equations (141) jointly with Lorentz condition (150) are equivalent to the Hamiltonian system (182) with respect to the canonical symplectic structure (180) and Hamiltonian function (181), which correspondingly reduce to electromagnetic equations (183) and (184) under external boundary conditions (171).

The obtained above result can be, eventually, used for developing an alternative quantization procedure of Maxwell electromagnetic equations, being free of some quantum operator problems, discussed in detail in [26]. We hope to consider this aspect of quantization problem in a specially devoted study.

**Remark 2.10** If one considers a motion of a charged point particle under a Maxwell field, it is convenient to introduce a trivial fiber bundle structure \( \pi : M \rightarrow N \), such that \( M = N \times G \), \( N = D \subset \mathbb{P}^3 \), with \( G = \mathbb{P} \backslash \{0\} \) being the corresponding (abelian) structure Lie group. An analysis similar to the above gives rise to the reduced (on the space \( \mathbb{M}_\xi = l^{-1}(\xi)G \); \( T^*(N) \), \( \xi \in \mathbb{G}^+ \)) symplectic structure

\[ \omega^{(2)}_\xi(q, p) = \langle dp, \wedge dq \rangle + d < \xi, A(q, g) > _G, \]

where \( A(q, g) := g^{-1}(d + \xi < A(q), dq >) \) \( g \in \Gamma \) is a suitable connection 1-form on phase space \( M \), with \( (q, p) \in T^*(N) \) and \( g \in G \). The corresponding canonical Poisson brackets on \( T^*(N) \) are easily found to be

\[ \{ q^i, q'^j \} = 0, \quad \{ p_j, q^i \} = \delta^i_j, \quad \{ p_j, p_j \} = F_{ij}(q) \]  

for all \( (q, p) \in T^*(N), i, j = 1, 3 \). If to introduce a new momentum variable \( \tilde{p} := p + \xi A(q) \) on \( T^*(N) \), \( \omega^{(2)}_\xi(q, p) \), it is easy to verify that \( \omega^{(2)}_\xi \rightarrow \omega^{(2)}_{\tilde{p}} := \langle d\tilde{p}, \wedge dq \rangle \), giving rise to the following Poisson brackets [79, 111, 25]:

\[ \{ q^i, q'^j \} = 0, \quad \{ \tilde{p}_j, q^i \} = \delta^i_j, \quad \{ \tilde{p}_j, \tilde{p}_j \} = 0, \]

where \( i, j = 1, 3 \), iff the standard Maxwell field equations

\[ \partial F_{ij}/\partial q_k + \partial F_{ik}/\partial q_j + \partial F_{jk}/\partial q_i = 0 \]

are satisfied on \( N \) for all \( i, j, k = 1, 3 \) with the curvature tensor \( F_{ij}(q) := \partial A_j/\partial q^i - \partial A_i/\partial q^j, \quad i, j = 1, 3 \), \( q \in N \).

Such a construction permits a natural generalization to the case of non-abelian structure Lie group yielding a description of Yang-Mills field equations within the reduction approach, to which we proceed below.

### 2.4 The Hamiltonian Analysis of the Yang-Mills Type Dynamical Systems

As above, we start with defining a phase space \( M \) of a particle under a Yang-Mills field in a region \( D \subset \mathbb{P}^3 \) as \( M := D \times G \), where \( G \) is a (not in general semi-simple) Lie group, acting on \( M \) from the right. Over the space \( M \) one can define quite naturally a connection \( \Gamma(A) \) if we consider the following trivial principal fiber bundle \( p : M \rightarrow N \), where \( N := D \), with the structure group \( G \). Namely, if \( g \in G \), \( q \in N \), then a connection 1-form on \( M \rightarrow (q, g) \) can be expressed [53, 103, 63] as

\[ A(q; g) := g^{-1}(d + \sum_{i=1}^n q_i A^{(i)}(q))g, \]  

where \( A^{(i)}(q) \) represents the field strength 1-form on \( M \) corresponding to the \( i \)-th field component of the vector potential.
where \( \{a_i \in \Gamma : i = 1, n \} \) is a basis of the Lie algebra \( \Gamma \) of the Lie group \( G \), and \( A_i : D \to \Lambda^1(D), \ i = 1, n, \) are the Yang-Mills fields on the physical space \( D \subset \mathbb{R}^3 \).

Now one defines the natural left invariant Liouville form on \( M \) as
\[
\alpha^{(1)}(q; g) := < p, dq > + < y, g^{-1} dg >, \tag{189}
\]
where \( y \in T^*(G) \) and \( < \cdot, \cdot > \) denotes, as before, the usual Ad-invariant non-degenerate bilinear form on \( \Gamma^* \times \Gamma \), as evidently, \( g^{-1} dg \in \Lambda^1(G) \otimes \Gamma \). The main assumption we need to proceed is that the connection 1-form is compatible with the Lie group \( G \) action on \( M \). The latter means that the condition
\[
R_h^* \alpha(q; g) = Ad_h^{-1} \alpha(q; g) \tag{190}
\]
is satisfied for all \( (q, g) \in M \) and \( h \in G \), where \( R_h : G \to G \) means the right translation by an element \( h \in G \) on the Lie group \( G \).

Having stated all preliminary conditions needed for the reduction Theorem 2.7 to be applied to our model, suppose that the Lie group \( G \) canonical action on \( M \) is naturally lifted to that on the cotangent space \( T^*(M) \) endowed due to (endowed owing to (134)) with the following \( G \)-invariant canonical symplectic structure:
\[
\omega^{(2)}(q, p; g, y) := dp^* \alpha^{(1)}(q, p; g, y) = < dp, \wedge dq > + < dy, g^{-1} dg >, \tag{191}
\]
for all \( (q, p; g, y) \in T^*(M) \). Take now an element \( \xi \in \Gamma^* \) and assume that its isotropy subgroup \( G_\xi = G \), that is \( Ad^*_h \xi = \xi \) for all \( h \in G \). In the general case such an element \( \xi \in \Gamma^* \) cannot exist but trivial \( \xi = 0 \) as it happens, for instance, in the case of the Lie group \( G = SL_2(\mathbb{P}) \). Then one can construct the reduced phase space \( l^{-1}(\xi)/G \) symplectomorphic to \( (T^*(N), \omega^{(2)}_\xi) \), where owing to (138) for any \( (q, p) \in T^*(N) \)
\[
\omega^{(2)}_\xi(q, p) := < dp, \wedge dq > + \sum_{j=1}^{n-3} e_j F^{(s)}_{ij}(q) dq^i \wedge dq^j. \tag{192}
\]
In the above we have expanded the element \( \xi = \sum_{i=1}^{n} e_a a^i \in \Gamma^* \) with respect to the bi-orthogonal basis \( \{a^i \in \Gamma^*, a_j \in \Gamma : < a^i, a_j > = \delta^i_j, \ i, j = 1, n\} \), with \( e_i \in \mathbb{P}, \ i = 1, 3, \) being some constants, and we, as well, denoted by \( F^{(s)}_{ij}(q), \ i, j = 1, 3, \) \( s = 1, n \), the corresponding curvature 2-form \( \Omega^{(2)}(q) \in \Lambda^2(N) \otimes \Gamma \) components, that is
\[
\Omega^{(2)}(q) := \sum_{s=1}^{n} \sum_{j=1}^{3} a_s F^{(s)}_{ij}(q) dq^i \wedge dq^j, \tag{193}
\]
for any point \( q \in N \). Summarizing the calculations accomplished above, we can formulate the following result.

**Theorem 2.11** Suppose the Yang-Mills field (281) on the fiber bundle \( p : M \to N \) with \( M = D \times G \) is invariant with respect to the Lie group \( G \) action \( G \times M \to M \). Suppose also that an element \( \xi \in \Gamma^* \) is chosen so that \( Ad^*_G \xi = \xi \). Then for the naturally constructed momentum mapping \( l : T^*(M) \to G^* \) (being equivariant) the reduced phase space \( l^{-1}(\xi)/G; T^*(N) \) is endowed with the symplectic structure (287), having the following component-wise Poisson brackets form:
\[ \{ p_i, q^j \}_\xi = \delta^j_i, \quad \{ q^i, q^j \}_\xi = 0, \quad \{ p_i, p_j \}_\xi = \sum_{s=1}^{n} e_{ij}^{(s)}(q) \]

for all \( i, j = 1, 3 \) and \( (q, p) \in T^*(N) \).

The respectively extended Poisson bracket on the whole cotangent space \( T^*(M) \) amounts owing to (151) into the following set of Poisson relationships:

\[ \{ y_i, y_j \}_\xi = \sum_{r=1}^{n} c_{ir}^r y_r, \quad \{ p_i, q^j \}_\xi = \delta^j_i, \quad \{ y_i, p_j \}_\xi = 0 = \{ q^i, q^j \}_\xi, \quad \{ p_i, p_j \}_\xi = \sum_{s=1}^{n} y_s F_{ij}^{(s)}(q), \]

where \( i, j = 1, n, \quad c_{ir}^r \in \mathbb{P}, \quad s, k, r = 1, m, \) are the structure constants of the Lie algebra \( \Gamma \), and we made use of the expansion \( A_j^{(s)}(q) = \sum_{j=1}^{n} A_j^{(s)}(q) dq^j \) as well we introduced alternative fixed values \( e_i := y_i, \quad i = 1, n \). The result (292) can be easily seen if one one makes a shift within the expression (284) as \( \sigma^{(2)} \rightarrow \sigma^{(2)}_{ext} \), where \( \sigma^{(2)}_{ext} := \sigma^{(2)}|_{A_0 = A} \), \( A_0(g) := g^{-1} dq, \quad g \in G \). Thereby one can obtain in virtue of the invariance properties of the connection \( \Gamma(A) \) that

\[ \sigma^{(2)}_{ext}(q, p; u, y) = \langle dp, \wedge dq \rangle + d < y(g), Ad_{g^{-1}} A(q; e) >_\Gamma = \langle dp, \wedge dq \rangle + d < Ad^{* g^{-1}} y(g) , \wedge A(q; e) >_\Gamma + \sum_{j=1}^{m} dy_j \wedge du_j + + \sum_{j=1}^{m} \sum_{s=1}^{n} A_j^{(s)}(q) dy_s \wedge dq^j < Ad_{g^{-1}} y(g), A(q, e) \wedge A(q; e) >_\Gamma + + \sum_{k=1}^{m} \sum_{s=1}^{n} y_s c_{sk}^s du^s \wedge du^k + \sum_{j=1}^{m} \sum_{s=1}^{n} y_s F_{ij}^{(s)}(q) dq^i \wedge dq^j, \]

where coordinate points \( (q, p; u, y) \in T^*(M) \) are defined as follows: \( \Lambda_0(e) := \sum_{i=1}^{m} du_i \wedge a_i \), \( Ad_{g^{-1}} y(g) = y(e) := \sum_{i=1}^{m} y_s \wedge a_i \) for any element \( g \in G \). Hence one gets straightaway the Poisson brackets (2.8) plus additional brackets connected with conjugated sets of variables \( \{ u^i \in \mathbb{P} : s = 1, m \} \in \Gamma^* \) and \( \{ y_s \in \mathbb{P} : s = 1, m \} \in \Gamma \):

\[ \{ y_s, u^k \}_\xi = \delta^k_s, \quad \{ u^k, q^j \}_\xi = 0, \quad \{ p_j, u^i \}_\xi = A_j^{(s)}(q), \quad \{ u^i, u^k \}_\xi = 0, \]

where \( j = 1, n, \quad k, s = 1, m, \) and \( q \in N \).

Note here that the transition suggested above from the symplectic structure \( \sigma^{(2)} \) on \( T^*(N) \) to its extension \( \sigma^{(2)}_{ext} \) on \( T^*(M) \) just consists formally in adding to the symplectic structure \( \sigma^{(2)} \) an exact part, which transforms it into an equivalent one. Looking now at the expressions (293), one can infer immediately that an element \( \xi := \sum_{i=1}^{m} e_i a_i \in \Gamma^* \) will be invariant with respect to the \( Ad^* \)-action of the Lie group \( G \) iff

\[ \{ y_s, y_k \}_\xi |_{y_s \neq y_k} = \sum_{j=1}^{m} e_j e_r = 0 \]
identically for all \( s, k = 1, m, \) \( j = 1, n \) and \( q \in N \). In this, and only this case, the reduction scheme elaborated above will go through.

Returning our attention to expression (294), one can easily write the following exact expression:

\[
\omega^{(2)}_{\text{cut}}(q, p; u, y) = \omega^{(2)}(q, p + \sum_{i=1}^{n} y_{s} A^{(i)}(y); u, y),
\]

(199)
on the phase space \( T^{*}(M) \ni (q, p; u, y) \), where we abbreviated \( < A^{(i)}(q), dq > = \sum_{j=1}^{n} A^{(i)}_{j}(q) \ dq^{j} \). The transformation like (296) was discussed within somewhat different contexts in articles [79, 111] containing also a good background for the infinite dimensional generalization of symplectic structure techniques. Having observed from (296) that the simple change of variable

\[
\tilde{p} := p + \sum_{i=1}^{m} y_{s} A^{(i)}(q)
\]

(200)
of the cotangent space \( T^{*}(N) \) recasts our symplectic structure (293) into the old canonical form (284), one obtains that the following new set of canonical Poisson brackets on \( T^{*}(M) \ni (q, \tilde{p}; u, y) \):

\[
\{ y_{s}, y_{k} \}_{\xi} = \sum_{i=1}^{n} c_{ij}^{s} y_{r}, \quad \{ \tilde{p}_{i}, \tilde{p}_{j} \}_{\xi} = 0, \quad \{ \tilde{p}_{i}, q^{j} \}_{\xi} = \delta_{ij},
\]

(201)
\[
\{ y_{s}, q^{j} \}_{\xi} = 0 = \{ q^{i}, q^{j} \}_{\xi}, \quad \{ u^{s}, u^{k} \}_{\xi} = 0, \quad \{ y_{s}, \tilde{p}_{j} \}_{\xi} = 0,
\]

\[
\{ u^{s}, \tilde{p}_{j} \}_{\xi} = 0, \quad \{ y_{s}, u^{k} \}_{\xi} = \delta^{k}_{s}, \quad \{ u^{s}, \tilde{p}_{j} \}_{\xi} = 0,
\]

where \( k, s = 1, m \) and \( i, j = 1, n \), holds iff the non-abelian Yang-Mills type field equations

\[
\partial F_{ij}^{(s)}/\partial q^{j} + \partial F_{lj}^{(s)}/\partial q^{j} + \partial F_{li}^{(s)}/\partial q^{j} + \sum_{k,r=1}^{m} c_{kj}^{s}(F_{ij}^{(r)} A_{l}^{(r)} + F_{lj}^{(r)} A_{i}^{(r)} + F_{li}^{(r)} A_{j}^{(r)}) = 0
\]

(202)
are fulfilled for all \( s = 1, m \) and \( i, j, l = 1, n \) on the base manifold \( N \). This effect of complete reduction of gauge Yang-Mills type variables from the symplectic structure (293) is known in literature [79] as the principle of minimal interaction and appeared to be useful enough for studying different interacting systems as in [89, 110]. We plan to continue further the study of the geometric properties of reduced symplectic structures connected with such interesting infinite-dimensional coupled dynamical systems of Yang-Mills-Vlasov, Yang-Mills-Bogolubov and Yang-Mills-Josephson types [89, 110] as well as their relationships with associated principal fiber bundles endowed with canonical connection structures.

3 The Maxwell Electromagnetic Equations and the Lorentz Type Force Derivation - The Feynman’s Approach Legacy

3.1 Introduction

Still in 1948 R. Feynman presented but not published [41, 42] a very interesting, in some aspect “ heretical”, quantum-mechanical derivation of the classical Lorentz force acting on a charged particle under influence of an external electromagnetic field. His result was analyzed by many authors [81, 30, 39, 122, 6, 43, 129, 68, 64] from different points of view, including its relativistic generalization [125]. As this problem is completely classical, we reanalyze the Feynman’s derivation from the classical Hamiltonian dynamics point of view on the coadjoint space \( T^{*}(N), N \subset P^{3} \), and construct its nontrivial generalization compatible with results [104, 24, 101] of Section 1, based on a recently devised vacuum field theory approach [105, 106]. Having further obtained the classical Maxwell electromagnetic equations we supply the complete legacy of the Feynman’s approach to the Lorentz force derivation and demonstrate its compatibility with the relativistic generalization, presented in Section 1.

Consider a motion of a charged point particle under a electromagnetic field. For its description, following Section
3, it is convenient to introduce a trivial fiber bundle structure \( \pi : M \rightarrow N, M = N \times G \), with the abelian structure group \( G := P \setminus \{0\} \), equivariantly acting on the canonically symplectic coadjoint space \( \mathcal{T}^*(M) \), and to endow it with some connection one-form \( A : M \rightarrow \mathcal{T}^*(M) \times \Gamma \) as

\[
A(q) := g^{-1}(d + \alpha^{(1)}(q)) \ g
\]

on the phase space \( M \), where \( d : \Lambda(M) \rightarrow \Lambda(M) \) is the usual exterior differentiation, \( \alpha^{(1)} : M \rightarrow \Lambda^1(N) \otimes G \) is some smooth mapping, \( q \in N \) and \( g \in G \). If \( l : T^*(M) \rightarrow \mathcal{G}^* \) is the related momentum mapping, one can respectively construct the reduced phase space \( \overline{M}_\xi := l^{-1}(\xi)/G; T^*(N) \), where \( \xi \in \mathcal{G}^*; \mathcal{R} \) is taken to be fixed, possessing the reduced symplectic structure

\[
\overline{\omega}^{(2)}_\xi(q, p) = \langle dp, \wedge dq \rangle + d\xi, \alpha^{(1)}(q) >_G.
\]

From (227) one finds easily the corresponding Poisson brackets on \( T^*(N) \):

\[
\{q^i, q^j\}_{\omega^{(2)}_\xi} = 0, \quad \{p_j, q^i\}_{\omega^{(2)}_\xi} = \delta^i_j, \quad \{p_i, p_j\}_{\omega^{(2)}_\xi} = \xi F_{ji}(q)
\]

for \( i, j = \overline{1,3} \) with respect to the reference frame \( K(t, q) \), characterized by the phase space coordinates \((q, p) \in T^*(N)\). If one introduces a new momentum variable \( \tilde{p} := p + \xi A(q) \) on \( T^*(N) \triangleright (q, p) \), where \( \alpha^{(1)}(q) := A(q), dq \rangle \in T^*_q(N) \), it is easy to verify that \( \overline{\omega}^{(2)}_\xi \rightarrow \overline{\omega}^{(2)'\xi} := \langle d\tilde{p}, \wedge dq \rangle \rangle \), giving rise to the following "minimal interaction" canonical Poisson brackets [79, 111, 25]:

\[
\{q^i, q^j\}_{\omega^{(2)'\xi}} = 0, \quad \{p_j, q^i\}_{\omega^{(2)'\xi}} = \delta^i_j, \quad \{p_i, p_j\}_{\omega^{(2)'\xi}} = 0
\]

for \( i, j = \overline{1,3} \) with respect to the reference frame \( K_f(t, q - q_f) \), characterized by the phase space coordinates \((q, \tilde{p}) \in T^*(N) \), iff the Maxwell field equations

\[
\partial F_{ij}/\partial q_k + \partial F_{jk}/\partial q_i + \partial F_{ik}/\partial q_j = 0
\]

are satisfied on \( N \) for all \( i, j, k = \overline{1,3} \) with the curvature tensor \( F_{ij}(q) := \partial A_i/\partial q^j - \partial A_j/\partial q^i \), \( i, j = \overline{1,3} \)

\( q \in N \).

### 3.2 The Lorentz Type Force and Maxwell Electromagnetic Field Equations - The Lagrangian Analysis

The Poisson structure (229) makes it possible to describe a charged particle \( \xi \in \mathbb{R} \), located at point \( q \in N \subset \mathbb{R}^3 \), moving with a velocity \( dq/dt = u \in T_q(N) \) with respect to the laboratory reference frame \( K(t, q) \), specified by coordinates \((t, q) \in M^4 \), being under the electromagnetic influence of an external charged particle \( \xi_f \in \mathbb{R} \) located at point \( q_f \in N \subset \mathbb{R}^3 \) and moving with respect to the same reference frame \( K(t, q) \) with a velocity \( dq_f/dt = u_f \in T_{q_f}(N) \). Really, consider a new shifted reference frame \( K_f(t', q - q_f) \) moving with respect to the reference frame \( K(t, q) \) with the velocity \( u_f \). With respect to the reference frame \( K_f(t', q - q_f) \), specified by coordinates \((t', q - q_f) \in M^4 \), the charged point particle \( \xi_f \) moves with the velocity \( u' - u_f := dr/dt' - dr_f/dt \) \((N) \) and, respectively, the charged particle \( \xi_f \) stays in rest. Then one can write down the standard classical Lagrangian function of the charged particle \( \xi \) with a mass \( m' \in \mathbb{R}_+ \) subject to the reference frame \( K_f(t', q - q_f) \):

Volume 1, Issue 1 available at http://scitecresearch.com/journals/index.php/bjmp/index
\[ L_f(q, u') = \frac{m}{2} |u - u'|^2 - \varphi', \]  
(208)

and the suitably Lorentz transformed scalar potential \( \varphi' = \varphi(1 + |u'|^2) \in C^2(N; \mathbb{R}) \) is the corresponding potential energy with respect to the reference frame \( K_f(t', q - q_f) \). On the other hand, owing to (231) and the Poisson brackets (229) the following equality for the charged particle \( \xi \) canonical momentum with respect to the reference frame \( K_f(t', q - q_f) \) holds:

\[ \bar{p}' := p' + \xi A(q) = \partial L_f(q, u')/\partial u', \]  
(209)

or, equivalently,

\[ p' + \xi A(q) = m'(u' - u_f'), \]  
(210)

expressed in the units when the light speed \( c = 1 \). Taking into account that the charged particle \( \xi \) momentum with respect to the reference frame \( K(t, q) \) equals \( p' := m u \in T_q(N) \), one can easily obtain from (233) the important relationship

\[ \xi A(q) = -m' u_f' \]  
(211)

for the vector potential \( A \in C^2(N; \mathbb{E}^3) \), which was before obtained in [105, 106, 114] and described before in Section 1. Taking now into account (231) and (234) one finds the following Lagrangian equation:

\[ \frac{d}{dt}[p' + \xi A(q)] = \partial L_f(q, u')/\partial u = -\xi \nabla \varphi', \]  
(212)

obtained before with respect to the shifted reference frame \( K_f(t', q - q_f) \) in [105, 106] and giving rise, as the result of obvious relationships \( p' = p, A' = A \), to the following charged point particle \( \xi \) dynamics:

\[ \frac{dp}{dt} = -\xi \partial A/\partial t - \xi \nabla \varphi(1 - |u'|^2) - \xi < u, \nabla \times A = \]  
(213)

\[ = -\xi \partial A/\partial t - \xi \nabla \varphi - \xi < u, \nabla \times A > + \xi \nabla < u, A > - \xi < u - u_f, A >/\]  

\[ = -\xi (\partial A/\partial t + \nabla \varphi) + \xi u \times (\nabla \times A) - \xi < u - u_f, A > \]  

with respect to the laboratory reference frame \( K(t, q) \). Based now on (236) we obtain the modified Lorentz type force

\[ \frac{dp}{dt} = \xi E + \xi u \times B = \xi < u - u_f, A >, \]  
(214)

where we put, as usually by definition,

\[ E := -\partial A/\partial t - \nabla \varphi, \quad B := \nabla \times A, \]  
(215)

and slightly differing from the classical [69, 34, 46, 80] Lorentz force expression

\[ \frac{dp}{dt} = \xi E + \xi u \times B \]  
(216)

by the gradient component.
Remark now that the modified Lorentz type force expression (237) can be naturally generalized to the relativistic case if to take into account that the standard Lorenz condition

$$0 \geq \frac{\rho}{At} \nabla + \frac{\partial}{\partial t} \nabla $$

is imposed on the electromagnetic potential $$(\varphi, A) \in C^2(\mathbb{N}; M^4)$$. Really, from (238) one obtains that the Lorentz invariant field equation

$$\partial^2 \varphi / \partial t^2 - \Delta \varphi = \rho_f,$$

where $\rho_f : \mathbb{N} \to \mathbb{D}'(\mathbb{N})$ is a generalized density function of the external charge distribution $\zeta_j$. Following now the calculations from [105, 106] we can easily find from (242) and the charge conservation law

$$\partial \rho_f / \partial t + \varphi_j = 0$$

the next Lorentz invariant equation on the vector potential $A \in C^2(\mathbb{N}; \mathbb{E}^3)$$

$$\partial^2 A / \partial t^2 - \Delta A = j_f.$$

Moreover, relationships (238),(242) and (244) easily entail the true classical Maxwell equations

$$\nabla \times E = -\partial B / \partial t, \nabla \times B = \partial E / \partial t + j_f,$$

$$\nabla \cdot E = \rho_f, \nabla \cdot B = 0$$

on the electromagnetic field $$(E, B) \in C^2(\mathbb{N}; \mathbb{E}^3 \times \mathbb{E}^3)$$.

Consider now the Lorenz condition (241) and observe that it is equivalent to the following local conservation law:

$$\frac{d}{dt} \int_{\Omega} w d^3 q = 0,$$

giving rise to the important relationship for the magnetic potential $A \in C^2(\mathbb{N}; \mathbb{E}^3)$

$$A = u_f \varphi$$

with respect to the laboratory reference frame $K(t, q)$, where $\Omega \subset \mathbb{N}$ is any open domain with the smooth boundary $\partial \Omega$, moving jointly with the charge distribution $\zeta_j$ in the domain $\mathbb{N} \subset \mathbb{R}^3$ with the corresponding velocity $u_f$. Taking into account relationship (234) one can find the expression for our charged particle $\zeta$ "inertial" mass:

$$m = -\bar{W}, \bar{W} := \zeta \varphi,$$

coinciding with that obtained before in [105, 106, 114], where we denoted by $\bar{W} \in C^2(\mathbb{N}; \mathbb{R})$ the corresponding potential energy of the charged point particle $\zeta$.

4 The Maxwell Electromagnetic Equations and the Lorentz Type Force Derivation - The Feynman’s Approach Legacy

4.1 Problem Setting

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view, including its relativistic generalization [125]. As this problem is completely classical, we reanalyze the Feynman’s derivation from the classical Hamiltonian dynamics point of view on the coadjoint space \( T^*(N), N \subset \mathbb{R}^1 \), and construct its nontrivial generalization compatible with results \([104, 24, 101]\) of Section 1, based on a recently devised vacuum field theory approach \([105, 106]\). Having further obtained the classical Maxwell electromagnetic equations we supply the complete legacy of the Feynman’s approach to the Lorentz force derivation and demonstrate its compatibility with the relativistic generalization, presented in Section 1.

Consider a motion of a charged point particle under a electromagnetic field. For its description, following Section 3, it is convenient to introduce a trivial fiber bundle structure \( \pi : M \rightarrow N, M = N \times G, \ N \subset \mathbb{R}^3 \), with the abelian structure group \( G := \mathbb{R} \setminus \{0\} \), equivariantly acting on the canonically symplectic coadjoint space \( T^*(M) \), and to endow it with some connection one-form \( \mathbf{A} : M \rightarrow T^*(M) \times G \) as

\[
\mathbf{A}(q, g) := g^{-1}(d + \alpha^{(1)}(q)) g
\]

on the phase space \( M \), where \( d : \Lambda(M) \rightarrow \Lambda(M) \) is the usual exterior differentiation, \( \alpha^{(1)} : M \rightarrow \Lambda'(N) \otimes G \) is some smooth mapping, \( q \in N \) and \( g \in G \). If \( l : T^*(M) \rightarrow G^* \) is the related momentum mapping, one can respectively construct the reduced phase space \( \mathbb{M}_\xi := l^{-1}(\xi) / G; T^*(N) \), where \( \xi \in G^*; \mathbb{R} \) is taken to be fixed, possessing the reduced symplectic structure

\[
\omega^{(2)}_\xi(q, p) = \langle dp, \wedge dq \rangle + d < \xi, \alpha^{(1)}(q) >_G.
\]

From (227) one finds easily the corresponding Poisson brackets on \( T^*(N) \):

\[
\{q^i, q^j\} = 0, \quad \{p_j, q^i\} = \delta^i_j, \quad \{p_i, p_j\} = \omega^{(2)}_\xi(q) \]

for \( i, j = 1, 3 \) with respect to the reference frame \( K(t, q) \), characterized by the phase space coordinates \( \{q, p\} \in T^*(N) \). If one introduces a new momentum variable \( \tilde{p} := p + \xi \mathbf{A}(q) \) on \( T^*(N) \) \( \triangleright \) \( \{q, p\} \), where \( \alpha^{(1)}(q) := < \mathbf{A}(q), dq > \in T_q^*(N) \), it is easy to verify that \( \omega^{(2)}_\xi(q) \rightarrow \tilde{\omega}^{(2)}_\xi := < d\tilde{p}, \wedge dq > \), giving rise to the following "minimal interaction" canonical Poisson brackets \([79, 111, 108]\):

\[
\{q^i, q^j\} = 0, \quad \{\tilde{p}_j, q^i\} = \delta^i_j, \quad \{\tilde{p}_i, \tilde{p}_j\} = \omega^{(2)}_\xi(q) = 0
\]

for \( i, j = 1, 3 \) with respect to the reference frame \( K_f(t, q - q_f) \), characterized by the phase space coordinates \( \{q, \tilde{p}\} \in T^*(N) \), iff the Maxwell field equations

\[
\partial F_{ij} / \partial q_k + \partial F_{jk} / \partial q_i + \partial F_{ik} / \partial q_j = 0
\]

are satisfied on \( N \) for all \( i, j, k = 1, 3 \) with the curvature tensor \( F_{ij}(q) := \partial A_j / \partial q^i - \partial A_i / \partial q^j \), \( i, j = 1, 3 \) \( q \in N \).

4.2 The Lorentz Type Force and Maxwell Electromagnetic Field Equations - The Lagrangian Analysis

The Poisson structure (229) makes it possible to describe a charged particle \( \xi \in \mathbb{R} \), located at point \( q \in N \subset \mathbb{R}^3 \), moving with a velocity \( dq/\text{d}t := u \in T_q^*(N) \) with respect to the laboratory reference frame \( K(t, q) \), specified by coordinates \( (t, q) \in M^4 \), being under the electromagnetic influence of an external charged particle \( \xi_f \in \mathbb{R} \) located at point \( q_f \in N \subset \mathbb{R}^3 \) and moving with respect to the same reference frame \( K(t, q) \) with a velocity \( dq_f / \text{d}t := u_f \in T_{q_f}^*(N) \). Really, consider a new shifted reference frame \( K_f(t', q - q_f) \) moving with respect to the reference frame \( K(t, q) \) with the velocity \( u_f \). With respect to the reference frame \( K_f(t', q - q_f) \), specified by
coordinates \((t, q - q_f) \in M^4\), the charged point particle \(\xi\) moves with the velocity \(u' - u'_f := d\mathbf{r}/dt - d\mathbf{r}_f/dt \in T_{q-q_f}(N)\) and, respectively, the charged particle \(\xi_f\) stays in rest. Then one can write down the standard classical Lagrangian function of the charged particle \(\xi\) with a mass \(m' \in \mathbb{R}_+\) subject to the reference frame \(K_f(t', q - q_f)\):

\[
L_f(q, u') = \frac{m'}{2} \left| u' - u'_f \right|^2 - \xi \phi',
\]

and the suitably Lorentz transformed scalar potential \(\phi' = g(1 + \left| u'_f \right|^2) \in C^2(N, \mathbb{R})\) is the corresponding potential energy with respect to the reference frame \(K_f(t', q - q_f)\). On the other hand, owing to (231) and the Poisson brackets (229) the following equality for the charged particle \(\xi\) canonical momentum with respect to the reference frame \(K_f(t', q - q_f)\) holds:

\[
\tilde{p} := p' + \xi A(q) = \partial L_f(q, u')/\partial \dot{u}',
\]

or, equivalently,

\[
p' + \xi A(q) = m'(u' - u'_f),
\]

expressed in the units when the light speed \(c = 1\). Taking into account that the charged particle \(\xi\) momentum with respect to the reference frame \(K(t, q)\) equals \(p' := m' u' \in T_q(N)\), one can easily obtain from (233) the important relationship

\[
\xi A(q) = -m' u'_f
\]

for the vector potential \(A \in C^2(N; \mathbb{R}^3)\), which was before obtained in [105, 106, 114] and described before in Section 1. Taking now into account (231) and (234) one finds the following Lagrangian equation:

\[
\frac{d}{dt} [p' + \xi A(q)] = \partial L_f(q, u')/\partial q = -\xi \nabla \phi',
\]

obtained before with respect to the shifted reference frame \(K_f(t', q - q_f)\) in [105, 106] and giving rise, as the result of obvious relationships \(p' = p, A' = A\), to the following charged point particle \(\xi\) dynamics:

\[
dp/dt = -\xi \nabla A \partial_t - \xi \nabla \phi (1 - |u'_f|^2) - \xi <u, \nabla > A =
\]

\[
= -\xi \partial_t A - \xi \nabla \phi - \xi <u, \nabla > A +
\]

\[
+ \xi \nabla <u, A > -\xi \nabla <u - u'_f, A > =
\]

\[
= -\xi (\partial_t A \partial_t + \nabla \phi) + \xi u \times (\nabla \times A) - \xi \nabla <u - u'_f, A >
\]

with respect to the laboratory reference frame \(K(t, q)\). Based now on (236) we obtain the modified Lorentz type force

\[
dp/dt = \xi E + \xi u \times B - \xi \nabla <u - u'_f, A >,
\]

where we put, as usually by definition,

\[
E := -\partial_t A - \nabla \phi, \quad B := \nabla \times A,
\]

and slightly differing from the classical [69, 34, 46, 80] Lorentz force expression

\[
dp/dt = \xi E + \xi u \times B
\]
by the gradient component

\[ F_{j} := -\xi \nabla < u_{j}, A >. \]  

(240)

Remark now that the modified Lorentz type force expression (237) can be naturally generalized to the relativistic case if to take into account that the standard Lorenz condition

\[ \partial \varphi \partial t + < \nabla, A >= 0 \]  

(241)

is imposed on the electromagnetic potential \( (\varphi, A) \in C^2(N;M^4) \).

Really, from (238) one obtains that the Lorentz invariant field equation

\[ \partial^2 \varphi \partial t^2 - \Delta \varphi = \rho_f, \]  

(242)

where \( \rho_f : N \rightarrow \mathbb{D}'(N) \) is a generalized density function of the external charge distribution \( \xi_f \). Following now by the calculations from [105, 106] we can easily find from (242) and the charge conservation law

\[ \partial \rho_f \partial t + < \nabla, J_f >= 0 \]  

(243)

the next Lorentz invariant equation on the vector potential \( A \in C^2(N;E^3) \):

\[ \partial^2 A \partial t^2 - \Delta A = J_f. \]  

(244)

Moreover, relationships (238),(242) and (244) easily entail the true classical Maxwell equations

\[ \nabla \times E = -\partial B/\partial t, \nabla \times B = \partial E/\partial t + J_f, \]  

(245)

\[ < \nabla, E >= \rho_f, < \nabla, B >= 0 \]

on the electromagnetic field \( (E, B) \in C^2(N;E^3 \times E^3) \).

Consider now the Lorenz condition (241) and observe that it is equivalent to the following local conservation law:

\[ \frac{d}{dt} \int_{\Omega} W d^3 q = 0, \]  

(246)

giving rise to the important relationship for the magnetic potential \( A \in C^2(N;E^3) \)

\[ A = u_f \varphi \]  

(247)

with respect to the laboratory reference frame \( K(t,q) \), where \( \Omega \subseteq N \) is any open domain with the smooth boundary \( \partial \Omega \), moving jointly with the charge distribution \( \xi_f \) in the domain \( N \subseteq \mathbb{R}^3 \) with the corresponding velocity \( u_f \).

Taking into account relationship (234) one can find the expression for our charged particle \( \xi \) “inertial” mass:

\[ m = -\bar{W}, \bar{W} := \xi \varphi, \]  

(248)

coinciding with that obtained before in [105, 106, 114], where we denoted by \( \bar{W} \in C^2(N;\mathbb{R}) \) the corresponding potential energy of the charged point particle \( \xi \).

### 4.3 The Modified Least Action Principle and the Hamiltonian Analysis

Based on the representations (247) and (248) one can rewrite the determining Lagrangian equation (235) with respect to the shifted reference frame \( K_f(t',q_f) \) as follows:

\[ \frac{d}{dt} [-\bar{W}(u' - u_f)] = -\nabla \bar{W}, \]  

(249)

which is reduced to the Lorentz type force expression (237) calculated with respect to the laboratory reference frame.
\[ K(t, q) : \]
\[
dp/dt = \xi E + \xi u \times B - \xi \nabla < u - u_f, A >, \tag{250} \]
where we put, as before,
\[
E := -\partial A/\partial t - \nabla \varphi, \quad B := \nabla \times A. \tag{251} \]

**Remark 4.1** It is interesting to remark here that equation (250) does not allow the Lagrangian representation with respect to the reference frame \( K(t, q) \) in contrast to that of equation (249) which is equivalent to (235).

The remark above is a challenging source of our further analysis concerning the direct relativistic generalization of the modified Lorentz type force (237). Namely, the following proposition holds.

**Proposition 4.2** The Lorentz type force (237) in the case when the charged point particle \( \xi \) momentum is defined, owing to (248), as \( p = -\overline{W} u \) is the exact relativistic expression allowing the Lagrangian representation of the charged particle \( \xi \) dynamics with respect to the rest reference frame \( K_r(\tau, q - q_f) \), related to the shifted reference frame \( K_f(\tau', q - q_f) \) by means of the classical relativistic proper time infinitesimal transformation:
\[
d\tau' = d\tau (1 + |u' - u_f|^2)^{1/2}, \tag{252} \]
where \( \tau \in \mathbb{R} \) is the proper time parameter in the rest reference frame \( K_r(\tau, q - q_f) \).

**Proof.** Take the following action functional with respect to the charged point particle \( \xi \) rest reference frame \( K_r(\tau, q - q_f) \):
\[
S^{(\tau)} := -\int_{t_1(\tau_1)}^{t_2(\tau_2)} \overline{W} \, dt' = \int_{t_1}^{t_2} \overline{W} (1 + |u' - u_f|^2)^{1/2} \, d\tau, \tag{253} \]
where the proper temporal values \( \tau_1, \tau_2 \in \mathbb{R} \) are considered, in a Feynman spirit [46], to be fixed in contrast to the temporal parameters \( t_2(\tau_2), t_2(\tau_2) \in \mathbb{R} \) depending, owing to (252), on the charged particle \( \xi \) trajectory in the phase space \( M^4 \). The least action condition
\[
\delta S^{(\tau)} = 0, \quad \delta q(\tau_1) = 0 = \delta q(\tau_2), \tag{254} \]
applied to (253), entails exactly the dynamical equation (249), being simultaneously equivalent to the relativistic Lorentz type force expression (237) with respect to the laboratory reference frame \( K(t, q) \). The latter proves the proposition.

Making use of the relationships between the reference frames \( K(t, q) \) and \( K_r(\tau, q - q_f) \) in the case when the external charge particle velocity \( u_f = 0 \), we can easily derive the following corollary.

Let the external charge point \( e_f \) be in rest, that is the velocity \( u_f = 0 \). Then equation (249) reduces to
\[
(d/dt)(-\overline{W} u) = -\nabla \overline{W}, \tag{255} \]
allowing the following conservation law:
\[
H_0 = \overline{W} (1 - |u|^2)^{1/2} = \overline{W} (2 - |p|^2)^{1/2} \tag{256} \]
Moreover, equation (255) is Hamiltonian with respect to the canonical Poisson structure (229), Hamiltonian function (256) and the rest reference frame \( K_r(\tau, q) \):
\[
\begin{align*}
\frac{dq}{d\tau} &:= \partial H_0/\partial p = \overline{W} (p - |p|^2)^{-1/2} \\
\frac{dp}{d\tau} &:= -\partial H_0/\partial q = \overline{W} (2 - |p|^2)^{-1/2} \nabla \overline{W} \implies \frac{dq}{d\tau} = -p \overline{W}^{-1}, \\
\frac{dp}{d\tau} &:= -\nabla \overline{W} \end{align*} \tag{257} \]
In addition, if to define the rest particle mass \( m_0 := -H_0 \bigg|_{u=0} \), the "inertial" particle mass quantity \( m \in \mathbb{R} \) obtains the well known classical relativistic form

\[
m = -\bar{W} = m_0 (1 - |u|^2)^{-1/2},
\]

depending on the particle velocity \( u \in \mathbb{R}^3 \).

Concerning the general case of equation (249) analogous ones to the above results hold, which were also described in part in Section 1. We need only to mention that the induced Hamiltonian structure of the general equation (249) results naturally from its least action representation (253) and (254) with respect to the rest reference frame \( K_\Sigma (\tau, q) \).

4.3.1 Comments

Within Section 2 we have demonstrated the complete legacy of the Feynman’s approach to the Lorentz force based derivation of the Maxwell electromagnetic field equations. Moreover, we have succeeded in finding the exact relationship between the Feynman’s approach and the vacuum field approach of Section 1, devised before in [105, 106]. Thus, the results obtained firmly argue for the deep physical backgrounds lying in the vacuum field theory approach, based on which one can simultaneously describe the physical phenomena both of electromagnetic and gravity origins. The latter is physically based on the particle “inertial” mass expression (248), naturally following both from the Feynman’s approach to the Lorentz force derivation and from the vacuum field approach.

5 The Modified Lorentz Force, Radiation Theory And The Abraham–Lorentz Electron Inertia Problem

5.1 Introductory Setting

It is well known that Maxwell equations, which are fundamental in modern physics, allow two main forms of representations: either by means of the electric and magnetic fields or by the electric and magnetic potentials. The latter were mainly considered as a mathematically motivated representation useful for different applications but having no physical significance.

That the situation is not so simple and the evidence that the magnetic potential demonstrates the physical properties was doubtless, the physics community understood when Y. Aharonov and D. Bohm [3] formulated their “paradox” concerning the measurement of magnetic field outside a separated region where it is completely vanishing. Later, similar effects were also revealed in the superconductivity theory of Josephson media. As the existence of any electromagnetic field in the ambient space can be tested only owing to its interaction with electric charges, their dynamical behavior, being of great importance, was deeply studied by M. Faraday, A. Ampère and H. Lorentz subject to its classical Newton’s second law form. Namely, the classical Lorentz force

\[
dp/dt = \mathbf{F} + \frac{\mathbf{\xi} u}{c} \times \mathbf{B}
\]

was derived, where \( E \) and \( B \in \mathbb{R}^3 \), acting on a point charged particle \( \mathbf{\xi} \in \mathbb{R}, \) possessing the momentum \( p = mu \), where \( m \in \mathbb{R} \), is the observed particle mass and \( u \in T(\mathbb{R}^3) \) is its velocity, measured with respect to a suitably chosen laboratory reference frame \( K \).

That the Lorentz force (259) is not a completely satisfactory expression was well known by Lorentz himself, as the nonuniform Maxwell equations also describe the electromagnetic fields, radiated by any accelerated charged particle. This follows directly from well-known expressions for the Liénard–Wiechert electromagnetic four-potential \( (\varphi, A) : M^4 \to T^*(M^4) \), related to the electromagnetic fields by means of the well-known [80, 69, 34] relationships

\[
E := -\nabla \varphi - \frac{1}{c} \frac{\partial A}{\partial t}, \quad B := \nabla \times A.
\]

This fact had inspired many physicists to “improve” the classical Lorentz force expression (259) and its modification was then suggested by G.A. Schott [121] and later by M. Abraham and P.A.M. Dirac (see [69, 34]), who found that the so called classical “radiation reaction” force, owing to the self-interaction of a charged particle with charge \( \mathbf{\xi} \in \mathbb{R} \), equals
\[ \frac{dp}{dt} = \xi E + \xi \frac{u}{c} \times B + \frac{2\xi^2}{3c^3} d^2 u / dt^2. \]  

(261)

The additional self-reaction force expression

\[ F_r := \frac{2\xi^2}{3c^3} \frac{d^2 u}{dt^2}, \]

(262)

depending on the particle acceleration immediately begged questions concerning its physical meaning, since for instance, a uniformly accelerated charged particle, owing to the expression (261), feels no radiation reaction, contradicting the fact that any accelerated charged particle radiates electromagnetic waves. This “paradox” was a challenging problem during the twentieth century [121, 35, 69, 26, 98, 69] and still remains to be explained [115, 93, 90]. As there exist different approaches to explaining this reaction radiation phenomenon, we mention here only the most popular ones such as the Wheeler–Feynman [131] “absorber radiation” theory, based on a very sophisticated elaboration of the retarded and advanced solutions to the nonuniform Maxwell equations, the vacuum Casimir effect approach devised in [95, 112], and the construction of Teitelbom [126] which extensively exploits the intrinsic structure of the electromagnetic energy tensor subject to the advanced and retarded solutions to the nonuniform Maxwell equations.

It is also worth mentioning here very the nontrivial development of the Teitelbom’s theory devised recently in [73, 123] and applied to the non-abelian Yang–Mills equations, which are natural generalizations of the Maxwell equations. Nonetheless, all of these explanations do not prove to be satisfactory from the modern physics of view. Taking this state of art into account, we will reanalyze the structure of the “radiative” Lorentz type force (261) using the vacuum field theory approach of Section 1 and find that this force allows some natural slight modification.

5.2 The Radiation Reaction Force: Vacuum Field Theory Approach

In this section, we will develop further our vacuum field theory approach, devised in [104, 101], to the electromagnetic Maxwell and Lorentz electron theories and show that it is in complete agreement with the classical results and even more: it allows some nontrivial generalizations, which may have some important physical applications. It will also be shown that the closely related electron mass problem can be satisfactorily explained via the devised vacuum field theory approach and the spatial electron structure assumption.

The modified Lorentz force, acting on a particle of charge \( \xi \in \mathbb{R} \) and exerted by a moving with velocity \( u_j \in T(R^3) \) charged particle \( \xi_j \in \mathbb{R} \), was derived in Section 1 and is

\[ \frac{dp}{dt} := F_s = \xi E + \xi \frac{u}{c} \times B - \nabla < \xi A, (u - u_j) / c >, \]  

(263)

where \( (\varphi, A) \in T^* (M^4) \) is the external electromagnetic potential calculated with respect to a fixed laboratory reference frame \( \mathbb{K} \). To take into account the self-interaction of this particle we will make use of a spatially distributed charge density \( \rho : M^4 \rightarrow \mathbb{R} \), satisfying the condition

\[ \xi = \int_{R^3} \rho(t, r) d^3 r \]  

(264)

for all \( t \in \mathbb{R} \) subject to this laboratory reference frame \( \mathbb{K} \) with coordinates \( (t, r) \in M^4 \). Then, owing to (263) and results from Section 1, the self-interacting force of this spatially structured charge \( \xi \in \mathbb{R} \) can be expressed with respect to this laboratory reference frame \( \mathbb{K} \) in the following equivalent form:

\[ \frac{dp}{dt} = - \frac{1}{c} \int_{R^3} d^3 r \varphi(t, r) \frac{d}{dt} A_j(t, r) - \]  

(265)

\[ - \int_{R^3} d^3 r \varphi(t, r) \nabla \varphi_j(t, r) (1 - \frac{|u / c|^2}) = \]

where

\[ \varphi_j(t, r) = \int_{R^3} \rho(t', r') \frac{1}{|r - r'|} d^3 r' \]

(266)

\[ A_j(t, r) = \frac{1}{c} \int_{R^3} \frac{J(t', r')}{|r - r'|} d^3 r', \]

the well-known retarded Lienard–Wiechert potentials, which should be calculated at the retarded time parameter.
\[ i' = t - \frac{|r - r'|}{c} \quad (c \in \mathbb{R}). \]

Taking into account the continuity relationship

\[ \partial \rho / \partial t + \nabla \cdot J = 0 \]  \hfill (267)

for the spatially distributed charge density \( \rho : M^4 \rightarrow \mathbb{R} \) and current \( J = \rho u : M^4 \rightarrow \mathbb{E}^3 \) and the Taylor expansions for retarded potentials \( \phi, A \)

\[ \phi_i(t, r) = \sum_{n=Z_p} \frac{\partial^n}{\partial t^n} \int_{\mathbb{R}^3} \frac{(-1)^n |r - r'|^n \rho(t, r') d^3 r'}{|r - r'|} \]  \hfill (268)

\[ A_i(t, r) = \sum_{n=Z_p} \frac{\partial^n}{\partial t^n} \int_{\mathbb{R}^3} \frac{(-1)^n |r - r'|^n J(t, r') d^3 r'}{|r - r'|} \]  \hfill (269)

from (265) and (268). Assuming for brevity the spherical charge distribution is small \( |u/c| = 1 \) and, respectively, slow acceleration, followed by calculations similar to those of [69, 90], one can obtain that

\[ F_s = \sum_{n=Z_p} \frac{(-1)^n}{n! c^n} \left( 1 - |u/c|^2 \right) \int_{\mathbb{R}^3} d^3 r \rho(t, r) \int_{\mathbb{R}^3} d^3 r' \frac{\partial^n}{\partial t^n} \rho(t, r') \nabla \frac{1}{|r - r'|^{n+1}} + \]

\[ + \sum_{n=Z_p} \frac{(-1)^n}{n! c^{n+2}} \int_{\mathbb{R}^3} d^3 r \rho(t, r) \int_{\mathbb{R}^3} d^3 r' \frac{\partial^{n+2}}{\partial t^{n+2}} J(t, r') = \]

\[ = \sum_{n=Z_p} \frac{(-1)^n}{n! c^{n+2}} \left( 1 - |u/c|^2 \right) \int_{\mathbb{R}^3} d^3 r \rho(t, r) \int_{\mathbb{R}^3} d^3 r' \frac{1}{|r - r'|^{n+1}} \frac{\partial^{n+1}}{\partial t^{n+1}} J(t, r') \]  \hfill (270)

The relationship above can be rewritten, owing to the charge continuity equation \( \partial \rho / \partial t + \nabla \cdot J = 0 \), and gives rise to the radiation force expression

\[ F_s = \sum_{n=Z_p} \frac{(-1)^n}{n! c^{n+2}} \left( 1 - |u/c|^2 \right) \int_{\mathbb{R}^3} d^3 r \rho(t, r) \int_{\mathbb{R}^3} d^3 r' \left| r - r' \right|^{n+1} \frac{\partial^{n+1}}{\partial t^{n+1}} \left( \frac{J(t, r')}{n+2} + \frac{n-1}{n+2} \frac{|r - r'|}{|r - r'|^2} \right) \]

\[ + \sum_{n=Z_p} \frac{(-1)^n}{n! c^{n+2}} \int_{\mathbb{R}^3} d^3 r \rho(t, r) \int_{\mathbb{R}^3} d^3 r' \left| r - r' \right|^{n+1} \frac{\partial^{n+1}}{\partial t^{n+1}} J(t, r') = \]

\[ = \sum_{n=Z_p} \frac{(-1)^n}{n! c^{n+2}} \left( 1 - |u/c|^2 \right) \int_{\mathbb{R}^3} d^3 r \rho(t, r) \int_{\mathbb{R}^3} d^3 r' \left| r - r' \right|^{n+1} \frac{\partial^{n+1}}{\partial t^{n+1}} \left( \frac{J(t, r')}{n+2} + \frac{n-1}{n+2} \frac{|r - r'|}{|r - r'|^2} u^2 \right) \]

\[ + \sum_{n=Z_p} \frac{(-1)^n}{n! c^{n+2}} \int_{\mathbb{R}^3} d^3 r \rho(t, r) \int_{\mathbb{R}^3} d^3 r' \left| r - r' \right|^{n+1} \frac{\partial^{n+1}}{\partial t^{n+1}} J(t, r') \]

Now, having applied to \( \partial \rho / \partial t + \nabla \cdot J = 0 \) the rotational symmetry property for calculation of the internal integral, one easily obtains that
\[ F_s = \sum_{n \in \mathbb{Z}_+} \frac{(-1)^n}{n!c^{n+2}} \int_{\mathbb{R}^3} d^3r \rho(t, r) \int_{\mathbb{R}^3} d^3r' \left| r - r' \right|^{-n} \frac{\partial^{n+1}}{\partial t^{n+1}} \left( \frac{J(t, r')}{n+2} + \frac{(n-1)J(t, r')}{3(n+2)} \right) + \]

\[ + \sum_{n \in \mathbb{Z}_+} \frac{(-1)^{n+1}}{n!c^n} \int_{\mathbb{R}^3} d^3r \rho(t, r) \int_{\mathbb{R}^3} d^3r' \left| r - r' \right|^{-n} \frac{\partial^n}{\partial t^n} J(t, r') = \]

\[ = \frac{d}{dt} \sum_{n \in \mathbb{Z}_+} \frac{2(-1)^{n+1}}{3n!c^{n+3}} \int_{\mathbb{R}^3} d^3r \rho(t, r) \int_{\mathbb{R}^3} d^3r' \left| r - r' \right|^{-n} \frac{\partial^n}{\partial t^n} J(t, r') - \]

\[ - \sum_{n \in \mathbb{Z}_+} \frac{(-1)^n |u|^2}{3n!c^{n+4}} \int_{\mathbb{R}^3} d^3r \rho(t, r) \int_{\mathbb{R}^3} d^3r' \left| r - r' \right|^{-n} \frac{\partial^n}{\partial t^n} J(t, r'), \] \hspace{1cm} (271)

where we took into account [69] that in case of the spherical charge distribution the following equalities

\[ \int_{\mathbb{R}^3} d^3r \int_{\mathbb{R}^3} d^3r' \rho(t, r) \rho(t, r') \left| \frac{\nabla \cdot J(t, r)}{n} \right| = \frac{1}{3} \varepsilon^2, \]

\[ \int_{\mathbb{R}^3} d^3r \nabla \cdot J(t, r) > \int_{\mathbb{R}^3} d^3r' \left| r - r' \right|^{-n} \frac{\partial^n}{\partial t^n} J(t, r') = 0, \] \hspace{1cm} (272)

hold for all \( n \in \mathbb{Z}_+ \). Thus, from (272) one easily finds up to the \( O(1/c^4) \) accuracy the following radiation reaction force expression:

\[ \frac{dp}{dt} = F_s = -\frac{d}{dt} \left( \frac{4E}{3c^2} u(t) \right) - \frac{d}{dt} \left( \frac{2E}{3c^2} |u(t)|^2 u(t) \right) + \frac{2\varepsilon^2}{3c^3} \frac{d^2u}{dt^2} + O(1/c^4) = \]

\[ = -\frac{d}{dt} \left( \frac{4}{3} m_0 c^2 (1 + \frac{1}{2} |u(t)|^2) u(t) \right) + \frac{2\varepsilon^2}{3c^3} \frac{d^2u}{dt^2} + O(1/c^4) = \]

\[ = -\frac{d}{dt} \left( \frac{4}{3} m_0 c^2 u(t) \right) + \frac{2\varepsilon^2}{3c^3} \frac{d^2u}{dt^2} + O(1/c^5) = \]

\[ = -\frac{d}{dt} \left( \frac{4}{3} m_0 u(t) \right) + \frac{2\varepsilon^2}{3c^3} \frac{d^2u}{dt^2} + O(1/c^4), \]

where we defined, respectively, the electrostatic self-interaction repulsive energy as
\[ E_{es} := \frac{1}{2} \int_{\mathbb{R}^3} d^3r \int_{\mathbb{R}^3} d^3r' \frac{\rho(t, r) \rho(t, r')}{|r - r'|}, \]  \hspace{1cm} (274)

the electromagnetic charged particle rest and inertial masses as

\[ m_{0,es} := \frac{E_{es}}{c^2}, ~ m_{es} := \frac{m_{0,es}}{(1 - |u| c^2)^{1/2}}. \]  \hspace{1cm} (275)

Now from (263) one obtains that

\[ \frac{d}{dt} \left[ (m_g + \frac{4}{3} m_{es}) u \right] = \frac{2 \varepsilon^2}{3c^3} \frac{d^2 u}{dt^2} + O(1/c^4), \]  \hspace{1cm} (276)

where we made use of the inertial mass definition

\[ m_g := -\frac{\overline{W}_g}{c^2}, ~ \nabla \overline{W}_g \not= 0, \]  \hspace{1cm} (277)

following from the vacuum field theory approach, where the \( m_g \in \mathbb{R} \) is the corresponding gravitational mass of the charged particle \( \xi \), generated by the vacuum field potential \( \overline{W}_g \). The corresponding radiation force

\[ F_r = \frac{2 \varepsilon^2}{3c^3} \frac{d^2 u}{dt^2} + O(1/c^4), \]  \hspace{1cm} (278)

coincides exactly with the classical Abraham–Lorentz–Dirac results. From (276) it follows that the observable physical charged particle mass \( m_{ph} : m_g + \frac{4}{3} m_{es} \) consists of two impacts: the electromagnetic and gravitational components, giving rise to the final force expression

\[ \frac{d}{dt} (m_{ph} u) = \frac{2 \varepsilon^2}{3c^3} \frac{d^2 u}{dt^2} + O(1/c^4). \]  \hspace{1cm} (279)

This means, in particular, that the real physically observed “inertial” mass \( m_{ph} \) of an electron strongly depends on the external physical interaction with the ambient vacuum medium, as was recently demonstrated using completely different approaches in [112, 95], based on the vacuum Casimir effect considerations. Moreover, the assumed above boundedness of the electrostatic self-energy \( E_{es} \) appears to be completely equivalent to the existence of so-called intrinsic Poincaré type “tensions”, analyzed in [26, 95], and to the existence of a special compensating Coulomb “pressure”, suggested in [112], guaranteeing the observable electron stability.

5.3 Comments

The charged particle radiation problem, revisited in this section, allows to conceive the following explanation of the point charged particle mass as that of a compact and stable object which should possess the vacuum interaction potential \( \overline{W} \in \mathbb{R}^3 \) of negative sign as follows from (277). The latter can be satisfied iff the equality (277) holds, thereby imposing on the intrinsic charged particle structure [91] some nontrivial geometrical constraints. Moreover, as follows from the physically observed particle mass expressions (277) the electrostatic potential energy, being of repulsive force origin, does contribute to the full mass as its main component.

There exist different relativistic generalizations of the force expression (276), which suffer the same common physical inconsistency related with the no radiation effect of a charged point particle at uniform motion.

Another problem closely related to the radiation reaction force analyzed above is the search for an explanation to the Wheeler and Feynman reaction radiation mechanisms, called the absorption radiation theory, based on the Mach type interaction of a charged point particle with the ambient vacuum electromagnetic medium. Concerning this problem, one can also observe some of its relationships with the one devised here via the vacuum field theory approach, but this question needs a more detailed and extended analysis.
6 Electron Inertia Via the Feynman Proper Time Paradigm and Vacuum Field Theory Approach

6.1 Introduction

As was reported by F. Dyson [41, 42], the original Feynman approach derivation of the electromagnetic Maxwell

equations was based on an a priori general form of the classical Newton type force, acting on a charged point particle

moving in three-dimensional space \( \mathbb{R}^3 \) endowed with the canonical Poisson brackets on the phase variables, defined on

the associated tangent space \( T(\mathbb{R}^3) \). As a result of this approach there only the first part of the Maxwell equations were

derived, as the second part, owing to F. Dyson [41], is related with the charged matter nature, which appeared to be hidden.

Trying to complete this Feynman approach to the derivation of Maxwell’s equations more systematically we have observed

[101] that the original Feynman’s calculations, based on Poisson brackets analysis, were performed on the tangent space

\( T(\mathbb{R}^3) \) which is, subject to the problem posed, not physically proper. The true Poisson brackets can be correctly defined

only on the coadjoint phase space \( T^*(\mathbb{R}^3) \), as seen from the classical Lagrangian equations and the related Legendre

transformation \([2, 8, 56, 25]\) from \( T(\mathbb{R}^3) \) to \( T^*(\mathbb{R}^3) \). Moreover, within this observation, the corresponding dynamical

Lorentz type equation for a charged point particle should be written for the particle momentum, not for the particle velocity,

whose value is well defined only with respect to the proper relativistic reference frame, associated with the charged point

particle owing to the fact that the Maxwell equations are Lorentz invariant.

Thus, from the very beginning, we shall reanalyze the structure of the Lorentz force exerted on a moving charged point particle

with a charge \( \xi \in \mathbb{R} \) by another point charged particle with a charge \( \xi_f \in \mathbb{R} \), making use of the classical

Lagrangian approach, and rederive the corresponding electromagnetic Maxwell equations. The latter appears to be strongly

related to the charged point mass structure of the electromagnetic origin as was suggested by R. Feynman and F. Dyson.

6.2 Feynman Proper Time Paradigm Analysis

Consider a charged point particle moving in an electromagnetic field. For its description, it is convenient to

introduce a trivial fiber bundle structure \( \pi : M \to \mathbb{R}^3, M = \mathbb{R}^3 \times G \), with the abelian structure group \( G := \mathbb{R} \setminus \{0\} \),
equivariantly acting on the canonically symplectic coadjoint space \( T^*(M) \) endowed both with the canonical symplectic

structure

\[
\omega^{(2)}(p, y; r, g) := dp^* \alpha^{(1)}(r, g) = \langle dp, \wedge dr \rangle + \langle dy, \wedge g^{-1} dg \rangle \tag{280}
\]

for all \( (p, y; r, g) \in T^*(M) \), where \( \alpha^{(1)}(r, g) := \langle p, dr \rangle + \langle y, g^{-1} dg \rangle \in T^*(M) \) is the corresponding

Liouville form on \( M \), and with a connection one-form \( A : M \to T^*(M) \times G \) as

\[
A(r, g) := g^{-1} \xi A(r), dr \rangle > g + \langle g^{-1} dg \rangle \tag{281}
\]

with \( \xi \in \mathbb{G}^* \), \( (r, g) \in \mathbb{R}^3 \times G \), \( \cdot ; \cdot \) being the scalar product in \( \mathbb{E}^3 \). The corresponding curvature 2-form

\( \Sigma^{(2)} \in \Lambda^2(\mathbb{R}^3) \otimes G \) is

\[
\Sigma^{(2)}(r) := dA(r, g) + A(r, g) \wedge A(r, g) = \xi \sum_{i,j=1}^{3} F_{ij}(r) dr^i \wedge dr^j, \tag{282}
\]

where

\[
F_{ij}(r) := \frac{\partial A_{j}}{\partial r^i} - \frac{\partial A_{i}}{\partial r^j} \tag{283}
\]

for \( i, j = 1,3 \) with respect to the reference frame \( K_i \), characterized by the phase space coordinates \( (r, p) \in T^*(\mathbb{R}^3) \).

As an element \( \xi \in \mathbb{G}^* \) is still not fixed, it is natural to apply the standard \([2, 8, 25]\) invariant Marsden–Weinstein–Meyer

reduction to the orbit factor space \( \widetilde{P}_\xi := P/\mathbb{G}_\xi \) subject to the related momentum mapping \( I : T^*(M) \to \mathbb{G}^* \),

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constructed with respect to the canonical symplectic structure \( (280) \) on \( T^\ast(M) \), where, by definition, \( \xi \in G^\ast \) is constant, \( P_\xi := I^{-1}(\xi) \subset T^\ast(M) \) and \( G_\xi = \{ g \in G : \text{Ad}_g^\ast \xi \} \) is the isotropy group of the element \( \xi \in G^\ast \). 

As a result of the Marsden–Weinstein–Meyer reduction, one finds that \( G_\xi ; G \), the factor-space \( \tilde{P}_\xi ; T^\ast(R^3) \) is endowed with a suitably reduced symplectic structure \( \tilde{\omega}_\xi^{(2)} \in T^\ast(\tilde{P}_\xi) \) and the corresponding Poisson brackets on the reduced manifold \( \tilde{P}_\xi \) are

\[
\{ r^i , r^j \}_\xi = 0 , \{ p_j , r^i \}_\xi = \delta^i_j , \\
\{ p_j , p_j \}_\xi = \xi F_{ij}(r) 
\]

for \( i , j = 1,3 \), considered with respect to the reference frame \( K(t;r) \). Introducing a new momentum variable

\[
\tilde{p} := p + \xi A(r) 
\]

on \( \tilde{P}_\xi \), it is easy to verify that \( \tilde{\omega}_\xi^{(2)} \rightarrow \tilde{\omega}_\xi^{(2)} := \{ d\tilde{p} , \wedge dr \} \), giving rise to the following “minimal interaction” canonical Poisson brackets:

\[
\{ r^i , r^j \}_{\tilde{\omega}_\xi^{(2)}} = 0 , \{ p_j , r^i \}_{\tilde{\omega}_\xi^{(2)}} = \delta^i_j , \{ p_j , p_j \}_{\tilde{\omega}_\xi^{(2)}} = 0 
\]

for \( i , j = 1,3 \) with respect to some new reference frame \( \tilde{K}_j \), characterized by the phase space coordinates \( (r, \tilde{p}) \in \tilde{P}_\xi \) and an evolution parameter \( t' \in R \) if and only if the Maxwell field equations

\[
\frac{\partial F_{ij}}{\partial r^j} + \frac{\partial F_{jk}}{\partial r^i} + \frac{\partial F_{kj}}{\partial r^i} = 0 
\]

are satisfied on \( R^3 \) for all \( i , j , k = 1,3 \) with the curvature tensor \( F_{ij}(r) := \partial A_i / \partial r^j - \partial A_j / \partial r^i \), \( i , j = 1,3 \) \( r \in R^3 \).

Now we proceed to a dynamic description of the interaction between two moving charged point particles \( \xi \) and \( \xi_j \) moving respectively, with the velocities \( u := dr/dt \) and \( u_j := dr_j/dt \) subject to the reference frame \( K_j \). Unfortunately, there is a fundamental problem in correctly formulating a physically suitable action functional and the related least action condition. There are clearly possibilities such as

\[
S_p^{(t)} := \int_{t_1}^{t_2} dt L_p^{(t)}[r ; dr/dt] 
\]

on a temporal interval \( [t_1 , t_2 ] \subset R \) with respect to the laboratory reference frame \( K(t;r) \),

\[
S_p^{(t)} := \int_{t_1}^{t_2} dt L_p^{(t)}[r ; dr/dt] 
\]

on a temporal interval \( [t_1 , t_2 ] \subset R \) with respect to the moving reference frame \( K_j \) and

\[
S_p^{(r)} := \int_{\tau_1}^{\tau_2} d\tau L_p^{(r)}[r ; dr/d\tau] 
\]

on a temporal interval \( [\tau_1 , \tau_2 ] \subset R \) with respect to the proper time reference frame \( K_r \), naturally related to the moving charged point particle \( \xi \).

It was first observed by Poincaré and Minkowski [97] that the temporal differentials \( dt \) and \( dt' \) are not closed differential one-forms, which physically means that a particle can traverse many different paths in space \( R^3 \) during any
given proper time interval \( d\tau \), naturally related to its motion. This fact was stressed [49, 50, 94, 97, 100] by Einstein, Minkowski and Poincaré, and later exhaustively analyzed by R. Feynman, who argued [46] that the dynamical equation of a moving point charged particle is physically sensible only with respect to its proper time reference frame. This is Feynman’s proper time reference frame paradigm, which was recently further elaborated and applied both to the electromagnetic Maxwell equations in [47, 48] and to the Lorentz type equation for a moving charged point particle under external electromagnetic field in [101, 104, 106, 25]. As it was there argued from a physical point of view, the least action principle should be applied only to the expression (290) written with respect to the proper time reference frame \( K, \) whose temporal parameter \( \tau \in \mathbb{R} \) is independent of an observer and is a closed differential one-form. Consequently, this action functional is also mathematically sensible, which in part reflects the Poincaré’s and Minkowski’s observation that the infinitesimal quadratic interval

\[
d\tau^2 = (dt')^2 - \left| d\tau - d\tau_f \right|^2,
\]

relating the reference frames \( K, \) and \( K, \) can be invariantly used for the four-dimensional relativistic geometry. The most natural way to contend with this problem is to first consider the quasi-relativistic dynamics of the charged point particle \( \xi \) with respect to the moving reference frame \( K, \) subject to which the charged point particle \( \xi_f \) is at rest. Therefore, it possible to write down a suitable action functional (289), up to \( O(1/c^4), \) as the light velocity \( c \to \infty, \) where the Lagrangian function \( L^{(\xi)}[r; dr/dt] \) can be naturally chosen as

\[
L^{(\xi)}[r; dr/dt] := m'(r)dr/dt - dr_f/dt^2/2 - \xi\varphi'(r).
\]

As a result, it is easy to verify that the least action condition \( \delta L^{(\varphi)}[\varphi] = 0 \) is equivalent to the dynamical equation

\[
d\pi/dt = \nabla L^{(\xi)}[r; dr/dt] = \nabla m\left(\frac{1}{2}(dr/dt - dr_f/dt^2)^2\right) - \xi\varphi'(r),
\]

where we have defined the generalized canonical momentum as

\[
\pi := \partial L^{(\xi)}[r; dr/dt]/\partial(\lambda^2) \nabla (dr/dt) = m(d\lambda/dt - dr_f/dt),
\]

with the dash signs dropped and denoted by “\( \nabla \)” the usual gradient operator in \( E^3. \) Equating the canonical momentum expression (296) with respect to the reference frame \( K, \) to that of (285) with respect to the reference frame \( \tilde{K}, \) and identifying the reference frame \( \tilde{K}, \) with \( K, \) one obtains the important particle mass determining expression

\[
m = -\xi\varphi(r),
\]

which follows from the relationship.*
\[ \varphi(r)dr/\rho = A(r). \]  

(298)

This is well known in the classical electromagnetic theory \([69]\) for potentials \( (\varphi, A) \in T^*(M^4) \) satisfying the Lorentz condition

\[ \partial \varphi(r)/\partial t + <\nabla, A(r)> = 0, \]  

(299)

yet the expression (297) looks very nontrivial in relating the "inertial" mass of the charged point particle \( \xi \) to the electric potential, both generated by the ambient charged point particles \( \xi_f \). As was argued in articles \([24, 101, 109]\), the above mass phenomenon is closely related and from a physical perspective shows its deep relationship to the classical electromagnetic mass problem.

Before further analysis of the completely relativistic charge \( \xi \) motion under consideration, we substitute the mass expression (297) into the quasi-relativistic action functional (289) with the Lagrangian (292). As a result, we obtain two possible action functional expressions, taking into account two main temporal parameters choices:

\[ S_{p}^{(\tau)} = -\int_{t_1}^{t_2} \xi \varphi^\prime (r)(1 + 1/2 [d\varphi/d\rho - dr_j/\rho \frac{d}{dt}]^2) d\rho \]  

(300)

on an interval \([t_1, t_2] \subset \mathbb{R}\) or

\[ S_{p}^{(\tau)} = -\int_{t_1}^{t_2} \xi \varphi^\prime (r)(1 + 1/2 [d\varphi/d\rho - dr_j/\rho \frac{d}{dt}]^2) d\tau \]  

(301)

on an \([\tau_1, \tau_2] \subset \mathbb{R}\). It is easy to see that the first expression (295) is unsatisfactory upon transforming to the proper time relativistic representation form the suitable quasi-relativistic limit for the Lagrangian function (292). On the other hand, the direct relativistic generalization of (301) follows:

\[ S_{p}^{(\tau)} = -\int_{t_1}^{t_2} \xi \varphi^\prime (r)(1 + 1/2 [d\varphi/d\rho - dr_j/\rho \frac{d}{dt}]^2) d\tau; \]  

(302)

\[ -\int_{t_1}^{t_2} \xi \varphi^\prime (r)(1 + [d\varphi/d\rho - dr_j/\rho \frac{d}{dt}]^2)^{1/2} d\tau = \]

\[ = -\int_{t_1}^{t_2} \xi \varphi^\prime (r)(1 - [d\varphi/d\rho - dr_j/\rho \frac{d}{dt}]^2)^{1/2} d\tau = -\int_{t_1}^{t_2} \xi \varphi^\prime (r)dt^\prime, \]

giving rise to the correct (from the physical point of view) relativistic action functional form (289), suitably transformed to the proper time reference frame representation (290) via the Feynman proper time paradigm. Thus, we have shown that the true action functional procedure consists in a physically motivated choice of either the action functional expression form (288) or (289). Then, it is transformed to the proper time action functional representation form (290) in the Feynman paradigm, and the least action principle is applied.

Concerning the above problem of describing the motion of a charged point particle \( \xi \) in the electromagnetic field generated by another moving charged point particle \( \xi \), it must be mentioned that we have chosen the quasi-relativistic functional expression (292) in the form (289) with respect to the moving reference frame \( K_r \), because its form is more physically acceptable, since the charged point particle \( \xi_f \) is then at rest.

Based on the above relativistic action functional expression

\[ S_{p}^{(\tau)} := -\int_{t_1}^{t_2} \xi \varphi^\prime (r)(1 + [d\varphi/d\rho - dr_j/\rho \frac{d}{dt}]^2)^{1/2} d\tau \]  

(303)

written with respect to the proper reference from \( K(\tau; r - r_j) \), one finds the following evolution equation:

\[ d\pi_{p}/d\tau = -\xi \nabla \varphi (r)(1 + [d\varphi/d\rho - dr_j/\rho \frac{d}{dt}]^2)^{1/2}, \]  

(304)
where the generalized momentum is given by the relationship (296):
\[ \pi_p = m(dr/dt - dr_j/dt). \] (305)
Making use of the relativistic transformation (293) and the next one (294), the equation (304) is easily transformed to
\[ \frac{d}{dt} (p + \xi A) = -\nabla \varphi(r)(1 - \left|u_j\right|^2), \] (306)
where we took into account the definitions: (297) for the charged particle \( \xi \) mass, (298) for the magnetic vector potential and \( \varphi(r) = \varphi'(r)/(1 - \left|u_j\right|^2)^{1/2} \) for the scalar electric potential with respect to the laboratory reference frame \( K_i \).

Equation (306) can be further transformed, using elementary vector algebra, to the classical Lorentz type form:
\[ \frac{dp}{dt} = \xi E + \xi u \times B - \xi \nabla u - u_j, A >, \] (307)
where
\[ E := -\partial A/\partial t - \nabla \varphi \] (308)
is the related electric field and
\[ B := \nabla \times A \] (309)
is the related magnetic field, exerted by the moving charged point particle \( \xi_j \) on the charged point particle \( \xi \) with respect to the laboratory reference frame \( K(t; r) \). The Lorentz type force equation (307) was obtained in [101, 104] in terms of the moving reference frame \( K_i \), and recently reanalyzed in [109, 106, 118]. The obtained results follow in part [116, 117] from Ampère’s classical works on constructing the magnetic force between two neutral conductors with stationary currents.

For the Lorentz force equation (307) it is a natural problem to analyze its form in the case of many external charged point particles \( \xi_j \in \mathbb{R}, \ j \in \mathbb{Z}_+ \), moving with velocities \( dr_j/dt, j \in \mathbb{Z}_+ \), with respect to the laboratory reference frame \( K_i \). In this case there is no possibility of choosing a common moving reference frame \( K_i \) with respect to which all of the charged particles \( \xi_j, j \in \mathbb{Z}_+ \), are at rest. However, we do have the unique proper time parameter \( \tau \in \mathbb{R} \) related to each charged point particle \( \xi_j, j \in \mathbb{Z}_+ \), via the infinitesimal relativistic transformation expressions
\[ dt_j = d\tau(1 - \left|dr_j/dt - dr_j/dt\right|^2)^{1/2} \] (310)
to the moving reference frames \( K_{i_j}, \ j \in \mathbb{Z}_+ \), fixing the \( \tau \)-clock for all the charged particles. Thus, making use of the same scheme as demonstrated above, we can express together with the superposition principle, the net Lorentz type force expression for the charged point particle \( \xi \) as
\[ \frac{dp}{dt} = \xi \bar{E} + \xi u \times \bar{B} - \xi \nabla \sum_{j \in \mathbb{Z}_+} <u - u_j, A_j >, \] (311)
where
\[ \bar{E} := \sum_{j \in \mathbb{Z}_+} E_j, \bar{B} = \sum_{j \in \mathbb{Z}_+} B_j, \] (312)
and \( A_j \in T^*(\mathbb{R}^3), j \in \mathbb{Z}_+ \), are magnetic vector potentials generated by the set of distant charged point particles \( \xi_j, j \in \mathbb{Z}_+ \). As this system of external charges is on average neutral, that is \( \sum_{j \in \mathbb{Z}_+} \xi_j = 0 \), and their spatial distribution is on average symmetric with respect to the charge signs and velocities, one obtains from (311) that
\[ \frac{dp}{dt} = \xi \bar{E} + \xi u \times \bar{B}, \] (313)
which the classical Lorentz type expression for the charged point particle \( \xi \) moving under the influence of an external electromagnetic field with respect to the laboratory reference frame \( K \).

Equation (313) can naturally be physically interpreted as the Lorentz type force exerted by a virtual net charge \( \bar{\xi} \) at rest and located at the centroid of the charges with respect to \( K \). Consequently, one can write the corresponding effective relativistic invariant action functional in the form

\[
\bar{S}^{(t)} := \int_{t_1}^{t_2} dt(m_{\bar{\xi}} + \langle \bar{A}, dr/dt \rangle - \xi \bar{\varphi})
\]

on an interval \([t_1, t_2] \subset \mathbb{R}\) with respect to \( K \). Here \( m_{\bar{\xi}} \in \mathbb{R} \) is a possible internal charged particle mass energy value and as before, \( \bar{\varphi} := \sum_{j \in Z_+} \varphi_j \), \( \bar{A} := \sum_{j \in Z_+} A_j \), and we also took into account took the suitable relativistic electric potentials transformations from the moving reference frames \( K_{i_j}, j \in Z_+ \), to the laboratory reference frame \( K \) with respect to which the averaged set of charges \( \bar{\xi} \) is assumed to be virtually at rest so that

\[
-\varphi_j dt_j = \varphi_j dt + \langle A_j, dr \rangle,
\]

holds for all \( j \in Z_+ \) and gives rise, upon summing over \( j \in Z_+ \), to

\[
-\sum_{j \in Z_+} \varphi_j dt_j = -\bar{\varphi} dt + \langle \bar{A}, dr \rangle,
\]

used for construction of the action functional (314). As this is considered to be written for the averaged set of charges \( \bar{\xi} \), whose virtual location is now assumed to be at rest, we can apply to this action functional (314) the Feynman proper time paradigm and construct the corresponding physically reasonable action functional

\[
\bar{S}_\rho^{(t)} = \int_{\tau_1}^{\tau_2} d\tau(-\xi \bar{\varphi} + \bar{\xi} \langle \bar{A}, dr/d\tau \rangle)(1 + |dr/d\tau|^2)^{1/2},
\]

defined on an independent time interval \([\tau_1, \tau_2] \subset \mathbb{R}\) with respect to the proper time reference frame \( K \), whose time parameter \( \tau \in \mathbb{R} \) is infinitesimally related to the laboratory time parameter \( t \in \mathbb{R} \) as

\[
d\tau = dt(1 - |dr/dt|^2)^{-1/2}.
\]

Applying the least action principle to the functional (317) one easily obtains the evolution equation

\[
\frac{d}{dt}(p + \xi \bar{A}) = -\bar{\xi} \nabla \bar{\varphi} + \xi \nabla \langle \bar{A}, u \rangle,
\]

where, as before, the charged particle \( \xi \) momentum is defined classically as

\[
p := md\rho dt,
\]

and its mass parameter is defined as

\[
m := -\bar{\xi} \bar{\varphi}(r).
\]

As the four-vector potentials \((\varphi_j, A_j) \in T^*(M^4), j \in Z_+\), where \( M^4 := \mathbb{R} \times \mathbb{E}^3 \) is the standard Minkowski pseudo-Euclidean metric space, satisfy the Lorentz conditions

\[
\partial \varphi_j /\partial t + \langle \nabla, A_j \rangle = 0
\]

for any \( j \in Z_+ \), it is evident that the same condition

\[
\partial \bar{\varphi} /\partial t + \langle \nabla, \bar{A} \rangle = 0
\]
holds also for the averaged potentials \((\overline{\varphi}, \overline{A}) \in T^*(M^4)\). The same standard calculations applied to the expression (319) yield the (same as (313)) Lorentz force equation

\[
dp/dt = \overline{\varphi E} + \overline{\varphi} \times \overline{B},
\]

(324)

thereby demonstrating the mathematical agreement between two physically different approaches to its derivation, based on the classical averaging procedure and the superposition principle.

6.3 Analysis of the Maxwell and Lorentz force Equations

6.3.1 The Maxwell Equations

As a moving charged particle \(\xi_f\) generates the suitable electric field (308) and magnetic field (309) via their electromagnetic potential \((\varphi, A) \in T^*(M^4)\) with respect a laboratory reference frame \(K(t; r)\), we will supplement them naturally by means of the external material equations describing the relativistic charge conservation law:

\[
\partial_t \varphi + \nabla \cdot J = 0
\]

(325)

where \((\rho, J) \in T^*(M^4)\) is a related four-vector for the charge and current distribution in the space \(\mathbb{R}^3\). Moreover, one can augment the equation (325) with the experimentally well established the Gauss law

\[
\nabla \cdot E = \rho
\]

(326)

to calculate the quantity \(\Delta \varphi := \nabla \cdot \varphi\) from the expression (308):

\[
\Delta \varphi = -\frac{\partial}{\partial t} \left( \nabla \cdot A \right) - \nabla \cdot E.
\]

(327)

Having taken into account the relativistic Lorentz condition (299) and the expression (326) one easily finds that the wave equation

\[
\partial^2 \varphi/\partial t^2 - \Delta \varphi = \rho
\]

(328)

holds with respect to the laboratory reference frame \(K(t; r)\). Applying the rot-operation \(\nabla \times\) to the expression (308) we obtain, owing to the expression (309), the equation

\[
\nabla \times E + \partial B/\partial t = 0,
\]

(329)

giving rise, together with (326), to the first pair of the classical Maxwell equations. To obtain the second pair of the Maxwell equations, it is first necessary to apply the rot-operation \(\nabla \times\) to the expression (309):

\[
\nabla \times B = \partial E/\partial t + (\partial^2 A/\partial t^2 - \Delta A)
\]

(330)

and then apply \(-\partial/\partial t\) to the wave equation (328) to obtain

\[
-\frac{\partial^2}{\partial t^2} \left( \frac{\partial \varphi}{\partial t} \right) + \nabla \cdot \nabla \varphi = \frac{\partial^2}{\partial t^2} \left( \nabla \cdot A \right) - \nabla \cdot (\nabla \times A) - \Delta \varphi =
\]

\[
- \nabla \cdot \nabla \left( \nabla \cdot A \right) = \nabla \cdot (\nabla \times A) - \Delta \varphi = \nabla \cdot \frac{\partial^2 A}{\partial t^2} - \Delta A = \nabla \cdot \frac{\partial^2 A}{\partial t^2} - \Delta A = \nabla \cdot J.
\]

(331)

The result (331) leads to the relationship

\[
\frac{\partial^2 A}{\partial t^2} - \Delta A = J,
\]

(332)

if we take into account that both the vector potential \(A \in \mathbb{E}^3\) and the vector of current \(J \in \mathbb{E}\) are determined up to a

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rot-vector expression $\nabla \times S$ for some smooth vector-function $S : M^4 \rightarrow \mathbb{R}^3$. Inserting the relationship (332) into (330), we obtain (329) and the second pair of the Maxwell equations:

$$\nabla \times B = \partial E/\partial t + J, \nabla \times E = \partial B/\partial t. \quad (333)$$

It is important that the system of equations (333) can be represented by means of the least action principle $\delta\mathcal{S}^{(t)}_{f-p} = 0$, where the action functional

$$\mathcal{S}^{(t)}_{f-p} := \int^2_{t_1} dt \mathcal{L}^{(t)}_{f-p} \quad (334)$$

is defined on an interval $[t_1, t_2] \subset \mathbb{R}$ by the Lagrangian function

$$\mathcal{L}^{(t)}_{f-p} = \int_{\mathbb{R}^3} d^3r \left( \frac{1}{2} \left| \frac{\partial J}{\partial t} \right|^2 - |B|^2 \right) / 2 + \left< J, A \right> - \rho \phi \quad (335)$$

with respect to the laboratory reference frame $K(t; r)$. From (335) we deduce that the generalized field momentum

$$\pi_{f} := \partial \mathcal{L}^{(t)}_{f-p} / \partial \partial A/\partial t = -E \quad (336)$$

and its evolution is given as

$$\partial \pi_{f} / \partial t := \delta \mathcal{A}^{(t)}_{f-p} / \partial \mathcal{A} = J - \nabla \times B, \quad (337)$$

which is equivalent to the first Maxwell equation of (333). As the Maxwell equations allow the least action representation, it is easy to derive [2, 8, 25, 104, 109] their dual Hamiltonian formulation with the Hamiltonian function

$$H_{f-p} := \int_{\mathbb{R}^3} d^3r \left< \pi_{f}, \partial A/\partial t \right> - \mathcal{L}^{(t)}_{f-p} = \int_{\mathbb{R}^3} d^3r \left( \left| \frac{\partial J}{\partial t} \right|^2 - |B|^2 \right) / 2 - \left< J, A \right>, \quad (338)$$

satisfying the invariant condition

$$dH_{f-p} / dt = 0 \quad (339)$$

for all $t \in \mathbb{R}$.

It is worth noting here that the Maxwell equations were derived under the important condition that the charged system $(\rho, J) \in T(M^4)$ exerts no influence on the ambient electromagnetic field potentials $(\phi, A) \in T^*(M^4)$. As this is not actually the case owing to the damping radiation reaction on accelerated charged particles, one can try to describe this self-interacting influence by means of the modified least action principle, making use of the Lagrangian expression (335) in the case of a separate charged particle $\zeta$. Following the well-known approach from [80] this Lagrangian can be recast (in the Gauss units) as

$$\mathcal{L}^{(t)}_{f-p} = \int_{\mathbb{R}^3} d^3r \left( \frac{1}{2c} \left< \nabla \phi, \nabla \phi \right> - \left< \partial A/\partial t, \nabla \phi \right> - \frac{1}{c} \left< \partial A/\partial t, \partial A/\partial t \right> + \frac{1}{2} \left< \nabla \times (\nabla \times A), A \right> + \frac{1}{c} \left< J, A \right> - \rho \phi \right) + \left< k(t), dr/dt \right> = \int_{\mathbb{R}^3} d^3r \left( \frac{1}{2} \left< \nabla \phi, E \right> - \frac{1}{2c} \left< \partial A/\partial t, E \right> - \frac{1}{2} \left< A, \nabla \times B \right> + \frac{1}{c} \left< J, A \right> - \rho \phi \right) + \left< k(t), dr/dt \right> \quad (340)$$
\[ \begin{align*}
&= \int_{\mathbb{R}^3} d^3 r \left( \frac{1}{2} \varphi < \nabla, E > + \frac{1}{2c} A, \partial E \partial t > - \frac{1}{2c} A, J + \partial E \partial t > + \frac{1}{c} J, A > - \rho \varphi \right) - \\
&- \frac{1}{2c} \frac{d}{dt} \int_{\mathbb{R}^3} d^3 r < A, E > - \frac{1}{2} \lim \int_{\mathbb{S}^2_r} \varphi dS < A \times B, dS^2 > - < k(t), \partial r/\partial t > = \\
&= \frac{1}{2} \int_{\mathbb{R}^3} d^3 r \left( \frac{1}{c} < J, A > - \rho \varphi \right) - \frac{1}{2c} \frac{d}{dt} \int_{\mathbb{R}^3} d^3 r < A, E > - \\
&- \frac{1}{2} \lim \int_{\mathbb{S}^2_r} \varphi dS < A \times B, dS^2 > - < k(t), \partial r/\partial t > -,
\end{align*} \]

where we have introduced an as yet undetermined internal charged particle stability energy impact \( m_v c^2 \) and radiation damping force \( k(t) \in \mathbb{E}^3 \), as well as a two-dimensional sphere \( \mathbb{S}^2_r \) of radius \( r \to \infty \). If we also assume that the radiated charged particle energy at infinity is negligible, the Lagrangian function (340) becomes equivalent to

\[ L^{(t)}_{r-p} = \frac{1}{2} \int_{\mathbb{R}^3} d^3 r \left( \frac{1}{c} < J, A > - \rho \varphi \right) - < k(t), \partial r/\partial t >, \tag{341} \]

which we now need to calculate taking into account that the electromagnetic potentials \( (\varphi, A) \in T^* (M^4) \) are retarded and given as \( 1/c \to 0 \) in the following Lienard–Wiechert form:

\[ \begin{align*}
\varphi &= \int_{\mathbb{R}^3} d^3 r \frac{\rho(t, r')}{|r - r'|} = \lim \int_{\mathbb{R}^3} d^3 r \frac{\rho(t - \varepsilon, r')}{|r - r'|} + \\
&+ \frac{1}{2c^2} \int_{\mathbb{R}^3} d^3 r \frac{r - r'}{|r - r'|} \partial^2 \rho(t, r')/\partial t^2 + \frac{1}{6c^2} \int_{\mathbb{R}^3} d^3 r \frac{r - r'}{|r - r'|^3} \partial \rho(t, r')/\partial t + O(1/c^4), \\
A &= \frac{1}{c} \int_{\mathbb{R}^3} d^3 r \frac{J(t, r')}{|r - r'|} = \lim \frac{1}{c} \int_{\mathbb{R}^3} d^3 r \frac{J(t - \varepsilon, r')}{|r - r'|} - \\
&- \frac{1}{c^2} \int_{\mathbb{R}^3} d^3 r \frac{\partial J(t, r')}{\partial t} + \frac{1}{2c^3} \int_{\mathbb{R}^3} d^3 r \frac{r - r'}{|r - r'|^2} \partial^2 J(t, r')/\partial t^2 + O(1/c^4). \tag{342}
\end{align*} \]

Here the current density \( J(t, r) = \rho(t, r) dr/\partial t \) for all \( t \in \mathbb{R} \), \( r \in \Omega(\xi) := \text{supp} \ \rho(t; r) \subseteq \mathbb{R}^3 \) is the compact support of the charged particle density distribution. Moreover, the limit as \( \varepsilon \to +0 \) takes into account that the potentials (342) are both retarded and singular at the charged particle \( \xi \) center, moving in space with the velocity \( u \in T(\mathbb{R}^3) \) with respect to the laboratory reference frame \( K(t; r) \). As a result of simple calculations of the kind in [69] and the suitable regularization procedure one finds that, up to \( O(1/c^4) \), the electric potential integral in (341), equals
\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^3} d^3 r \rho(t, r) \int_{\mathbb{R}^3} d^3 r' \frac{\rho(t, r')}{|r - r'|} = \\
= \int_{\mathbb{R}^3} d^3 r \rho(t, r) \int_{\mathbb{R}^3} d^3 r' \frac{\rho(t, r')}{|r - r'|} - \lim_{\varepsilon \to 0} \int_{\mathbb{R}^3} d^3 r \rho(t, r) \int_{\mathbb{R}^3} d^3 r' \frac{\varepsilon \nabla \cdot J(t, r')}{|r - r'|} = \\
= \int_{\mathbb{R}^3} d^3 r \rho(t, r) \int_{\mathbb{R}^3} d^3 r' \frac{\rho(t, r')}{|r - r'|} - \lim_{\varepsilon \to 0} \int_{\mathbb{R}^3} d^3 r \rho(t, r) \int_{\mathbb{R}^3} d^3 r' \frac{\rho(t, r')}{|r - r'|} \\
- \lim_{\varepsilon \to 0} \int_{\mathbb{R}^3} d^3 r \rho(t, r) \int_{\mathbb{R}^3} d^3 r' < \frac{\varepsilon}{|r - r'|} \frac{r - r'}{|r - r'|^2} > \rho(t; r') := 2E_{es} + m_\xi |u|^2, \\
\text{(343)}
\]

where we denoted the averaged, as \( \varepsilon \downarrow 0 \), limiting integral expression by

\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^3} d^3 r \rho(t, r) \int_{\mathbb{R}^3} d^3 r' < \frac{\varepsilon}{|r - r'|} \frac{r - r'}{|r - r'|^2} > \rho(t; r') := m_\xi |u|^2. \\
\text{(344)}
\]

This expression depends strongly on the internal electron structure, thus ensuring its stability. The same regularization scheme applied to the expression \( \lim_{\varepsilon \to 0} \int_{\mathbb{R}^3} d^3 r \varepsilon \nabla \cdot J(t, r') \frac{1}{|r - r'|} \) does not change its value.

Thus, making use of the expressions (342), (343), the Lagrangian function (341) yields

\[
L_{f-p}^{(\xi)} = \frac{E_{es}^a}{6c^2} |d\tau| - <k(t), d\tau/dt> - E_{es}(1 - |u|^2/c^2) - m_\xi |u|^2/2, \\
\text{(345)}
\]

where

\[
E_{es} := \frac{1}{2} \int_{\mathbb{R}^3} d^3 r \int_{\mathbb{R}^3} d^3 r' \frac{\rho(t, r') \rho(t, r')}{|r - r'|} \\
\text{(346)}
\]

is the electrostatic energy of the charged particle \( \xi \).

To obtain the corresponding evolution equation for our charged particle \( \xi \) we need, within the Feynman proper time paradigm, to transform the Lagrangian function (345) to one with respect to the charged particle proper time reference frame \( K_\xi \):\[
L_{f-p}^{(\xi)} = (m_{\xi}/6) |\dot{r}|^2 (1 + |\ddot{r}|^2/c^2)^{-1/2} - m_{\xi} c^2 (1 + |\ddot{r}|^2/c^2)^{-1/2} - \\
- <k(t), \dot{r}> - m_\xi |\dot{r}|^2/2(1 + |\ddot{r}|^2/c^2)^{-1/2}, \\
\text{(347)}
\]

where \( \dot{r} := d\tau/dt \) is the charged particle \( \xi \) velocity with respect to the proper reference frame \( K_\xi \) and by \( m_{\xi} := E_{es}/c^2 \) is its so called electrostatic mass.

As a result, the generalized charged particle \( \xi \) momentum up to \( O(1/c^4) \) equals...
\[
\pi_p := \partial L^{(r)}_{f-p}/\partial r = \frac{1}{3} \frac{m_e \dot{r}}{(1 + |\dot{r}|^2/c^2)^{1/2}} - \frac{m_e}{6c^2} \frac{|\ddot{r}|^2}{(1 + |\dot{r}|^2/c^2)^{3/2}} + \frac{m_e \ddot{r}}{(1 + |\dot{r}|^2/c^2)^{1/2}} - \\
-k(t) - \frac{m_e \ddot{r}}{(1 + |\dot{r}|^2/c^2)^{1/2}} + \frac{m_e}{2(1 + |\dot{r}|^2/c^2)^{3/2}} \cdot \frac{1}{3} \frac{m_e u(1 - |u|^2/c^2)^{1/2}}{1 - \frac{1}{3} m_e u(1 - |u|^2/c^2)^{1/2}} \right) + \\
+ m_e u(1 - |u|^2/c^2)^{1/2} - k(t) - m_e u; (-m_e + \frac{4}{3} m_e) u - k(t),
\]

where \( u := dr/dt \) is the charged particle \( \xi \) velocity with respect to the laboratory reference frame \( K_r \). The generalized momentum (348) satisfies the following evolution equation with respect to \( K_r \)

\[
d\pi_p/d\tau := \partial L^{(r)}_{f-p}/\partial r = 0,
\]

which is equivalent to, with respect to the laboratory reference frame \( K(t; r) \), the Lorentz type equation

\[
\frac{d}{dt} (-m_e u + \frac{4}{3} m_e u) = -dk(t)/dt.
\]

The evolution equation (349) allows the corresponding canonical Hamiltonian formulation on the phase space \( T^*(\mathbb{R}^3) \) with the Hamiltonian function

\[
H_{f-p} := \pi_p \cdot r - L^{(r)}_{f-p} = \frac{1}{3} \frac{m_e \dot{r}}{(1 + |\dot{r}|^2/c^2)^{1/2}} - \frac{m_e}{6c^2} \frac{|\ddot{r}|^2}{(1 + |\dot{r}|^2/c^2)^{3/2}} + \frac{m_e \ddot{r}}{(1 + |\dot{r}|^2/c^2)^{1/2}} - \\
-k(t) - \frac{m_e \ddot{r}}{(1 + |\dot{r}|^2/c^2)^{1/2}} + \frac{m_e}{2(1 + |\dot{r}|^2/c^2)^{3/2}} \cdot \frac{1}{3} \frac{m_e u(1 - |u|^2/c^2)^{1/2}}{1 - \frac{1}{3} m_e u(1 - |u|^2/c^2)^{1/2}} \right) + \\
+ m_e c^2 (1 + |\dot{r}|^2/c^2)^{-1/2} - k(t), \dot{r} > -(m_e/6) |\ddot{r}|^2 (1 + |\dot{r}|^2/c^2)^{-1/2} + \\
+ m_e c^2 (1 + |\dot{r}|^2/c^2)^{-1/2} - k(t), \dot{r} > + (m_e/2) |\ddot{r}|^2 (1 + |\dot{r}|^2/c^2)^{-1/2}
\]

\[
= \frac{1}{3} m_e |\ddot{r}|^2 (1 + |\dot{r}|^2/c^2)^{-1/2} + m_e |\ddot{r}|^2 (1 + |\dot{r}|^2/c^2)^{-1/2} - k(t), \dot{r} > \\
- m_e |\ddot{r}|^2 (1 + |\dot{r}|^2/c^2)^{-1/2} - (m_e/6) |\ddot{r}|^2 (1 + |\dot{r}|^2/c^2)^{-1/2} + \\
+ m_e c^2 (1 + |\dot{r}|^2/c^2)^{-1/2} - k(t), \dot{r} > + (m_e/2) |\ddot{r}|^2 (1 + |\dot{r}|^2/c^2)^{-1/2}
\]

\[
; \frac{m_e}{2(-m_e + 4m_e/3)} \cdot \frac{|\pi_p + k(t)|^2}{(-m_e + 4m_e/3)^2 c^2} \right) + m_e c^2] \left(1 - \frac{|\pi_p + k(t)|^2}{(-m_e + 4m_e/3)^2 c^2} \right)^{1/2},
\]

satisfying for all \( \tau, t \in \mathbb{R} \) the conservation conditions

\[
\frac{d}{d\tau} H_{f-p} = 0 = \frac{d}{dt} H_{f-p}.
\]

To determine the damping radiation force \( k(t) \in \mathbb{E}^1 \), we can make use of the Lorentz type force expression (307) in the proper case \( u = u_f \in T(\mathbb{R}^1) \) and obtain, as in [69], up to \( O(1/c^4) \) accuracy, the resulting
Abraham–Lorentz force as
\[
d\left(-m_\nu u + \frac{4}{3} m_\nu u\right) = \frac{2\varepsilon^2}{3c^3} \rho dt^2.
\] (353)

Comparing the force expressions (350) and (353), one finds that, up to \(O(1/c^4)\) accuracy,
\[
k(t) = \frac{2\varepsilon^2}{3c^3} \rho dt,
\] (354)

which should be understood as a smooth function of the temporal parameter \(t \in \mathbb{R}\). Moreover, looking at the equation (353) one can define the physical observable charged particle \(\xi\) mass parameter as
\[
m_{ph} := -m_\xi + \frac{4}{3} m_\nu.
\] (355)

For the mass parameter \(m_{ph} \in \mathbb{R}\) in the expression (355) to be determined, we need to analyze in detail the charged particle \(\xi\) stability condition and try to understand its relationship to the additional momentum production. Before proceeding to this analysis, we review some important results devoted to the stability problem of a charged particle such as an electron and try to determine a related additional momentum generation mechanism.

**Remark 6.1** Some years ago in [90] a “solution” to the above “4/3-electron mass” problem was suggested that was expressed by the physical mass relationship (355) and formulated more than one hundred years ago by H. Lorentz and M. Abraham. Unfortunately, the “solution” appeared to be erroneous as one can easily see from the incorrect assumptions on which the work in [90] was based. The first one concerns the particle-field momentum conservation, taken in the form
\[
\frac{d}{dt}(p + \xi A) = 0,
\] (356)

and the second one is a speculation about the \(1/2\)-coefficient imbedded into the calculation of the Lorentz type self-interaction force
\[
F := -\frac{1}{2c} \int d^3 r \rho(t; r) \partial A(t; r) / \partial t.
\] (357)

There it was incorrectly argued by the reasoning that the expression (357) represents “... the interaction of a given element of charge with all other parts, otherwise we count twice that reciprocal action” (cited from [90], page 2710). This claim is fallacious as it was not taken into account the important fact that the interaction in the integral (357) is, in reality, retarded and should be considered as that calculated for two virtually different charged particles, as in the classical works of H. Lorentz and M. Abraham. As for the first assumption (356), it suffices to recall that a vector of the field momentum \(\xi A \in \mathbb{R}^3\) is not independent and is, in the charged particle model considered, strongly related to the local flow of the electromagnetic energy in the Lorentz constraint form:
\[
\partial(\xi \rho) / \partial(t c) + \langle \nabla, \xi A \rangle = 0.
\] (358)

The constraint implies the validity of the Lienard–Wiechert expressions (341) for potentials for calculation of the integral (357), which was exploited in [90]. Thus, the equation (356), following the classical Newton second law, should be replaced by
\[
\frac{d}{dt}(p + \xi A) = -\nabla(\xi \varphi),
\] (359)

written with respect to the reference frame \(K(t; r)\) with respect to which the charged particle \(\xi\) is at rest. Taking into account that the relativistic relationships \(dt = dt' (1 - u^2 / c^2)^{-1/2}\) and \(\varphi' = \varphi(1 - u^2 / c^2)^{1/2}\) hold with respect to the laboratory reference frame \(K\), it follows from (359) that
Here we made use of the well-known relationship \( A = u \varphi c \) for the vector potential generated by this charged particle \( \xi \) moving in space with the velocity \( u \in T(\mathbb{R}^3) \) with respect to \( K \). Now, from the equation (360) one can derive the final expression for the evolution of the charged particle \( \xi \) momentum:

\[
\frac{dp}{dt} = -\xi \nabla \varphi - \frac{\xi}{c} dA/dt - \frac{\xi}{c} <u, \nabla > A + \frac{\xi}{c} \nabla <u, A> = \xi E + \frac{\xi}{c} u \times (\nabla \times A) = \xi E + \frac{\xi}{c} u \times B,
\]

which is exactly the well-known Lorentz force expression, used in the works of H. Lorentz and M. Abraham.

Recently there been other interesting research devoted to this “4/3-electron mass” problem, amongst which we would like to mention [95, 112], whose arguments are based on the charged shell electron model and are quite similar - each assumes a virtual interaction of the electron with the ambient “dark” radiation energy. This interaction was first clearly demonstrated in [112], where a suitable compensation mechanism for the related singular electrostatic Coulomb electron energy and the wide band vacuum electromagnetic radiation energy fluctuations deficit inside the electron shell was shown to be harmonically realized as the electron shell radius \( a \to 0 \). Moreover, this compensation occurs when the induced outward directed electrostatic Coulomb pressure on the whole electron coincides, up to the sign, with that induced by the above vacuum electromagnetic energy fluctuations outside the electron shell, as was manifested by their absence inside the electron shell.

Actually, the outward directed electrostatic spatial Coulomb pressure on the electron is

\[
\eta_{\text{coul}} := \lim_{a \to 0} \frac{\varepsilon}{2} \left. \frac{|E|^2}{r^3} \right|_{r=a} = \lim_{a \to 0} \frac{\xi^2}{32 \varepsilon_0 \pi^2 a^4},
\]

where \( E = \frac{\xi}{4 \pi \varepsilon_0} \mathbf{r} r \mathbf{r} \in \mathbb{R}^3 \) is the electrostatic field at point \( r \in \mathbb{R} \) with respect to the electron center at the point \( r = 0 \in \mathbb{R} \). The related inward directed vacuum electromagnetic fluctuations spatial pressure is

\[
\eta_{\text{vac}} := \lim_{\Omega \to 0} \frac{1}{2 \Omega} \int_0^\Omega dE(\omega),
\]

where \( dE(\omega) \) is the electromagnetic energy fluctuations density for a frequency \( \omega \in \mathbb{R} \), and \( \Omega \in \mathbb{R} \) is the corresponding electromagnetic frequency cutoff. The integral (363) can be calculated by taking into account the quantum statistical recipe [45, 67, 20] that

\[
dE(\omega) := \hbar \omega \frac{d^3 p(\omega)}{h^3},
\]

where the Planck constant \( \hbar := 2\pi\hbar \) and the electromagnetic wave momentum \( p(\omega) \in \mathbb{R}^3 \) satisfy the relativistic relationship

\[
|p(\omega)| = \hbar \omega c.
\]

Whence, by substituting (365) into (364) one obtains
\[ dE(\omega) = \frac{\hbar \omega^3}{2 \pi^2 c^3} d\omega, \quad (366) \]

which implies, in view of (363), the following vacuum electromagnetic energy fluctuations spatial pressure

\[ \eta_{\text{vac}} = \lim_{\omega \to \infty} \frac{\hbar \Omega^4}{24 \pi^2 c^3}, \quad (367) \]

For the charged electron shell model to be stable at rest it is necessary to equate the inward (367) and outward (362) spatial pressures:

\[ \lim_{\Omega \to \infty} \frac{\hbar \Omega^4}{24 \pi^2 c^3} = \lim_{\omega \to 0} \frac{\varepsilon^2}{32 \varepsilon_0 \pi^2 a^2}, \quad (368) \]

giving rise to the balance electron shell radius \( a_b \to 0 \) limiting condition:

\[ a_b = \lim_{\Omega \to \infty} \left[ \Omega^{-1} \left( \frac{3 \varepsilon^2 c^2}{2 \hbar} \right)^{1/4} \right], \quad (369) \]

Simultaneously we can calculate the corresponding Coulomb and electromagnetic fluctuations energy deficit inside the electron shell:

\[ \Delta W_b := \frac{1}{2} \int_{\varepsilon_0}^{\varepsilon_b} \int_0^\Omega dE(\omega) = \frac{\varepsilon^2}{8 \pi \varepsilon_0 a_b} - \frac{\hbar \Omega^4 a_b^3}{6 \pi c^3} = 0, \quad (370) \]

additionally ensuring the electron shell model stability.

Another important consequence of this pressure-energy compensation mechanism can be derived concerning the electron mass component \( m_\xi \in \mathbb{R} \), entering the momentum expression (348) in the case of the electron movement. Namely, following the reasoning in [95], one can observe that during the electron movement there arises an additional hidden and not compensated for, velocity \( u \in T(\mathbb{R}^3) \) directed electrostatic Coulomb surface self-pressure acting only on the front half part of the electron shell and equal to

\[ \eta_{\text{surf}} := \frac{E \xi}{4 \pi \varepsilon_0} \left[ \frac{1}{2} \int_{\varepsilon_0}^{\varepsilon_b} \int_0^\Omega dE(\omega) = \frac{\varepsilon^2}{32 \pi \varepsilon_0 a_b^3}, \quad (371) \right. \]

apparently coinciding with the already compensated for outward directed electrostatic Coulomb spatial pressure (362). As it is evident that during the electron motion in space its surface electric current energy flow does not vanish [95], it follows that the electron momentum gains an additional mechanical impact, which can be expressed as

\[ \pi_\xi := -\eta_{\text{surf}} \frac{4 \pi \varepsilon_0^3 a_b^3}{3 \varepsilon c^2} u = - \frac{1}{3} \frac{\varepsilon^2}{8 \pi \varepsilon_0 a_b c^2} u = - \frac{1}{3} m_\xi u, \quad (372) \]

where we took into account that in this electron shell model the corresponding electrostatic electron mass equals its electrostatic energy

\[ m_\xi = \frac{\varepsilon^2}{8 \pi \varepsilon_0 a_b c^2}, \quad (373) \]

Thus, one can claim that, owing to the structural stability of the electron shell model, its generalized self-interaction momentum \( \pi_\rho \in T^*(\mathbb{R}^3) \) gains during the movement with velocity \( u = dr/dt \in T(\mathbb{R}^3) \) the additional backward directed hidden impact (372), which can be identified with the momentum component

\[ \pi_\xi = -m_\xi u, \quad (374) \]

entering the momentum expression (348). Owing to (354), this becomes
The result above makes it possible to reanalyze the calculation of the Lagrangian function \( (345) \), based on the averaged limiting integral expression \( (344) \), taking into account the electron shell model and its dynamical stability. In particular, the averaged limiting integral expression \( (344) \) can be calculated in the above dynamically stable electron shell model as follows:

\[
\lim_{\epsilon \to 0} \frac{1}{2} \int_{r_0}^{r_1} d^3 r \rho(t; r) \int_{r_0}^{r_1} d^3 r' < \frac{\alpha u}{|r' - r|} \frac{r' - r}{|r' - r|^2} > \rho(t; r); \\
= \frac{1}{2} \lim_{\epsilon \to 0} \frac{1}{3} \int_{r_0}^{r_1} d^3 r \rho(t; r) \int_{r_0}^{r_1} d^3 r' < \frac{\alpha u}{|r' - r|} \frac{\alpha u}{|r' - r|^2 c^2} > \rho(t; r) = \\
= 2E_{\alpha} \frac{2E_{\alpha}}{3} |u|^2 = \frac{1}{3} m_{es} |u|^2 := m_{e} |u|^2,
\]

Here, we took into account that, owing to the retarded electron self-interaction, only one half of the charged electron shell, separated by the distance \( |r' - r| = \alpha c \), generates an additional impact in the Lagrangian function \( (345) \), as the second half is shadowed by the electron shell interior with the absent electric field. Thus, upon substituting \( m_{e} = \frac{1}{3} m_{es} \) into the final electron physical mass expression \( (355) \), one obtains

\[
m_{ph} := -\frac{1}{3} m_{es} + \frac{4}{3} m_{es} = m_{e},
\]

which also supports the Abraham–Lorentz suggestion about the origin of the electromagnetic electron mass.

6.3.2 Comments

The electromagnetic mass origin problem was reanalyzed in details within the Feynman proper time paradigm and related vacuum field theory approach by means of the fundamental least action principle and the Lagrangian and Hamiltonian formalisms. The resulting electron inertia appeared to coincide in part, in the quasi-relativistic limit, with the momentum expression obtained more than one hundred years ago by M. Abraham and H. Lorentz \[1, 84, 85, 86\], yet it proved to contain an additional hidden impact owing to the imposed electron stability constraint, which was taken into account in the original action functional as some preliminarily undetermined constant component. As it was demonstrated in \[112, 95\], this stability constraint can be successfully realized within the charged shell model of electron at rest, if to take into account the existing ambient electromagnetic “dark” energy fluctuations, whose inward directed spatial pressure on the electron shell is compensated by the related outward directed electrostatic Coulomb spatial pressure as the electron shell radius satisfies some limiting compatibility condition. The latter also allows to compensate simultaneously the corresponding electromagnetic energy fluctuations deficit inside the electron shell, thereby forbidding the external energy to flow into the electron. In contrary to the lack of energy flow inside the electron shell, during the electron movement the corresponding internal momentum flow is not vanishing owing to the nonvanishing hidden electron momentum flow caused by the surface pressure flow and compensated by the suitably generated surface electric current flow. As it was shown, this backward directed hidden momentum flow makes it possible to justify the corresponding self-interaction electron mass expression and to state, within the electron shell model, the fully electromagnetic electron mass origin, as it has been conceived by H. Lorentz and M. Abraham and strongly supported by R. Feynman in his Lectures \[46\]. This consequence is also independently supported by means of the least action approach, based on the Feynman proper time paradigm and the suitably calculated regularized retarded electric potential impact into the charged particle Lagrangian function.

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The charged particle radiation problem, revisited in this Section, allowed to conceive the explanation of the charged particle mass as that of a compact and stable object which should be exerted by a vacuum field interaction energy potential \( \tilde{W}: M^4 \rightarrow \mathbb{R} \) of negative sign as follows from (277). The latter can be satisfied iff the expression (276) holds, thereby imposing on the intrinsic charged particle structure [91] some nontrivial geometrical constraints. Moreover, as follows from the physically observed particle mass expressions (277) the electrostatic potential energy, being of the repulsive force origin, does contribute into the full mass as its main energy component.

There exist different relativistic generalizations of the force expression (276), which suffer the same common physical inconsistency related to the no radiation effect of a charged particle in uniform motion.

Another deeply related problem to the radiation reaction force analyzed above is the search for an explanation to the Wheeler and Feynman reaction radiation mechanism, called the absorption radiation theory, strongly based on the Mach type interaction of a charged particle with the ambient vacuum electromagnetic medium. Concerning this problem, one can also observe some of its relationships with the one devised here within the vacuum field theory approach, but this question needs a more detailed and extended analysis.

7 The Generalized Fock Spaces, Quantum Currents Algebra Representations and Electrodynamics

7.1 Preliminaries: Fock Space and its Realizations

Let \( \Phi \) be a separable Hilbert space, \( F \) be a topological real linear space and \( A := \{A(f) \colon f \in F\} \) a family of commuting self-adjoint operators in \( \Phi \) (i.e. these operators commute in the sense of their resolutions of the identity). Consider the Gelfand rigging [13] of the Hilbert space \( \Phi \), i.e., a chain

\[
D \subseteq \Phi_+ \subseteq \Phi \subseteq \Phi_- \subseteq D^\prime
\]

(378)

in which \( \Phi_+ \) and \( \Phi_- \) are further Hilbert spaces, and the inclusions are dense and continuous, i.e. \( \Phi_+ \) is topologically (densely and continuously) and quasi-nucleus (the inclusion operator \( i: \Phi_+ \rightarrow \Phi \) is of the Hilbert - Schmidt type) embedded into \( \Phi \), the space \( \Phi_- \) is the dual of \( \Phi_+ \) with respect to the scalar product \( <.,.>_\Phi \) in \( \Phi \), and \( D \) is a separable projective limit of Hilbert spaces, topologically embedded into \( \Phi_+ \). Then, the following structural theorem [13, 15] holds:

**Theorem 7.1** Assume that the family of operators \( A \) satisfies the following conditions:

1. \( D \subseteq \text{Dom}A(f), f \in F, \) and the closure of the operator \( A(f) \uparrow D \) coincides with \( A(f) \) for any \( f \in F \), that is \( A(f) \uparrow D = A(f) \) in \( \Phi \);
2. the Range \( A(f) \uparrow D \subseteq \Phi_+ \) for any \( f \in F \);
3. for every \( \psi \in D \) the mapping \( F \ni f \rightarrow A(f)\psi \in \Phi_+ \) is linear and continuous;
4. there exists a strong cyclic (vacuum) vector \( |\Omega\rangle \in \bigcap_{f \in F} \text{Dom}A(f) \), such that the set of all vectors \( |\Omega\rangle \), \( \prod_{j=1}^n A(f_j)|\Omega\rangle, n \in \mathbb{Z}_{+} \) is total in \( \Phi_+ \) (i.e. their linear hull is dense in \( \Phi_+ \)).

Then there exists a probability measure \( \mu \) on \( (\tilde{F}^\prime, C_\sigma(\tilde{F}^\prime)) \), where \( \tilde{F}^\prime \) is the dual of \( F \) and \( C_\sigma(\tilde{F}^\prime) \) is the \( \sigma \) - algebra generated by cylinder sets in \( \tilde{F}^\prime \) such that, for \( \mu \) - almost every \( \eta \in \tilde{F}^\prime \) there is a generalized joint eigenvector \( \omega(\eta) \in \Phi_- \) of the family \( A \), corresponding to the joint eigenvalue \( \eta \in \tilde{F}^\prime \), that is

\[
< \omega(\eta), A(f)\psi >_\Phi = \eta(f) < \omega(\eta), \psi >_\Phi
\]

(379)

with \( \eta(f) \in \mathbb{R} \) denoting the pairing between \( F \) and \( \tilde{F}^\prime \).

The mapping

\[
\Phi_+ \ni \psi \rightarrow < \omega(\eta), \psi >_\Phi := \psi(\eta) \in \mathbb{C}
\]

(380)
for any \( \eta \in F' \) can be continuously extended to a unitary surjective operator \( F : \Phi \rightarrow L^2_2(F' ; \mathbb{C}) \), where
\[
F \psi(\eta) := \psi(\eta)
\]
(381)
for any \( \eta \in F' \) is a generalized Fourier transform, corresponding to the family \( A \). Moreover, the image of the operator \( A(f) , \ f \in F' \), under the \( F \) – mapping is the operator of multiplication by the function \( F' \ni \eta \rightarrow \eta(f) \in \mathbb{C} \).

We assume additionally that the main Hilbert space \( \Phi \) possesses the standard Fock space (bose)-structure [20, 13, 109], that is
\[
\Phi = \bigoplus_{n \in \mathbb{Z}_+} \Phi^\otimes_n,
\]
(382)
where subspaces \( \Phi^\otimes_n , \ n \in \mathbb{Z}_+ \), are the symmetrized tensor products of a Hilbert space \( H := L^2_2(\mathbb{R}^m ; \mathbb{C}) \). If a vector \( g := (g_0, g_1,..., g_n,...) \in \Phi \), its norm
\[
P g P_\Phi := \left( \sum_{n \in \mathbb{Z}_+} P g^2_n P^2_n \right)^{1/2},
\]
(383)
where \( g_n \in \Phi^\otimes_n ; L^2_2()((\mathbb{R}^m)^\otimes_n ; \mathbb{C}) \) and \( P_\Phi \) is the corresponding norm in \( \Phi^\otimes_n \) for all \( n \in \mathbb{Z}_+ \). Note here that concerning the rigging structure (378), there holds the corresponding rigging for the Hilbert spaces \( \Phi^\otimes_n , \ n \in \mathbb{Z}_+ \), that is
\[
D^\Phi_{\otimes n} \subset \Phi^\otimes_{n-1} \subset \Phi^\otimes_n \subset \Phi^\otimes_{n+1}
\]
(384)
with some suitably chosen dense and separable topological spaces of symmetric functions \( D_{\otimes n} , n \in \mathbb{Z}_+ \). Concerning expansion (378) we obtain by means of projective and inductive limits [16, 18, 13, 15] the quasi-nucleus rigging of the Fock space \( \Phi \) in the form (378):
\[
D \subset \Phi_+ \subset \Phi \subset \Phi_- \subset D.
\]
Consider now any vector \( |(\alpha)_n\rangle \in \Phi^\otimes_n , \ n \in \mathbb{Z}_+ \), which can be written [16, 20, 76] in the following canonical Dirac ket-form:
\[
| (\alpha)_n \rangle := | \alpha_1, \alpha_2,..., \alpha_n \rangle,
\]
(385)
where, by definition,
\[
| \alpha_1, \alpha_2,..., \alpha_n \rangle := \frac{1}{\sqrt{n!}} \sum_{\sigma \in S_n} | \alpha_{\sigma(1)} \rangle \otimes | \alpha_{\sigma(2)} \rangle ... | \alpha_{\sigma(n)} \rangle
\]
(386)
and \( | \alpha_j \rangle \in \Phi^\otimes_1(\mathbb{R}^m ; \mathbb{C}) := H \) for any fixed \( j \in \mathbb{Z}_+ \). The corresponding scalar product of base vectors as (386) is given as follows:
\[
\langle (\beta )_n \ | (\alpha)_n \rangle := \langle \beta_1, \beta_2,..., \beta_j, \beta_j | \alpha_1, \alpha_2,..., \alpha_{n-j}, \alpha_n \rangle = \sum_{\sigma \in S_n} \langle \beta_1 | \alpha_{\sigma(1)} \rangle ... \langle \beta_j | \alpha_{\sigma(j)} \rangle := \text{per}(\langle \beta | \alpha_j \rangle ; i, j = 1,n),
\]
(387)
where “\( \text{per} \)” denotes the permanent of matrix and \( \langle . | . \rangle \) is the corresponding product in the Hilbert space \( H \). Based now on representation (385) one can define an operator \( a^+(\alpha) : \Phi^\otimes_n \rightarrow \Phi^\otimes_{n+1} \) for any \( | \alpha \rangle \in H \) as follows:
\[
a^+(\alpha) | \alpha_1, \alpha_2,..., \alpha_n \rangle := | \alpha, \alpha_1, \alpha_2,..., \alpha_n \rangle,
\]
(388)
which is called the "creation" operator in the Fock space \( \Phi \). The adjoint operator \( a(\beta) := (a^+ (\beta))^* : \Phi^{\otimes (n+1)}_{(s)} \to \Phi^{\otimes n}_{(s)} \) with respect to the Fock space \( \Phi \) (378) for any \( | \beta \rangle \in \mathcal{H} \), called the "annihilation" operator, acts as follows:

\[
a(\beta) | \alpha_1, \alpha_2, \ldots, \alpha_{n+1} \rangle := \sum_{\sigma \in S_n} \langle \beta, \alpha_j | \alpha_1, \alpha_2, \ldots, \alpha_{j-1}, \hat{\alpha}_j, \alpha_{j+1}, \ldots, \alpha_{n+1} \rangle,
\]

(389)

where the "hat" over a vector denotes that it should be omitted from the sequence.

It is easy to check that the commutator relationship

\[
a(\alpha), a^+(\beta) = \langle \alpha, \beta \rangle
\]

holds for any vectors \( | \alpha \rangle \in \mathcal{H} \) and \( | \beta \rangle \in \mathcal{H} \). Expression (390), owing to the rigged structure (378), can be naturally extended to the general case, when vectors \( | \alpha \rangle \) and \( | \beta \rangle \in \mathcal{H}_- \), conserving its form. In particular, taking

\[
| \alpha \rangle := | \alpha(x) \rangle = \frac{1}{\sqrt{2\pi}} e^{i L \cdot \chi} \in \mathcal{H}_- := L^2_{-}(\mathbb{R}^m, \mathbb{C})
\]

for any \( x \in \mathbb{R}^m \), one easily gets from (390) that

\[
a(x), a^+(y) = \delta(x-y),
\]

(391)

where we put, by definition, \( a^+(x) := a^+ (\alpha(x)) \) and \( a(y) := a(\alpha(y)) \) for all \( x, y \in \mathbb{R}^m \) and denoted by \( \delta(\cdot) \) the classical Dirac delta-function.

The construction above makes it possible to observe easily that there exists a unique vacuum vector \( | \Omega \rangle \in \mathcal{H}_- \), such that for any \( x \in \mathbb{R}^m \)

\[
a(x) | \Omega \rangle = 0,
\]

(392)

and the set of vectors

\[
\left( \prod_{j=1}^n a^+(x_j) \right) | \Omega \rangle \in \Phi^{\otimes n}_{(s)}
\]

(393)

is total in \( \Phi^{\otimes n}_{(s)} \), that is their linear integral hull over the dual functional spaces \( \hat{\Phi}^{\otimes n}_{(s)} \) is dense in the Hilbert space \( \Phi^{\otimes n}_{(s)} \) for every \( n \in \mathbb{Z}_+ \). This means that for any vector \( g \in \Phi \) the following representation

\[
g = \bigoplus_{n \in \mathbb{Z}_+} \int_{\mathbb{R}^m} \hat{g}_n(x_1, \ldots, x_n) a^+(x_1) a^+(x_2) \ldots a^+(x_n) | \Omega \rangle \]

(394)

holds with the Fourier type coefficients \( \hat{g}_n \in \hat{\Phi}^{\otimes n}_{(s)} \) for all \( n \in \mathbb{Z}_+ \), with \( \hat{\Phi}^{(1)}_{(s)} := \mathcal{H}; L^2_{-}(\mathbb{R}^m; \mathbb{C}) \). The latter is naturally endowed with the Gelfand type quasi-nucleus rigging dual to

\[
\mathcal{H}_- \subset \mathcal{H}_- \subset \mathcal{H}_-,
\]

(395)

making it possible to construct a quasi-nucleolus rigging of the dual Fock space \( \hat{\Phi} := \bigoplus_{n \in \mathbb{Z}_+} \hat{\Phi}^{\otimes n}_{(s)} \). Thereby, chain (395) generates the dual Fock space quasi-nucleolus rigging

\[
\hat{\mathcal{D}} \subset \hat{\Phi}_+ \subset \hat{\Phi} \subset \hat{\Phi}_- \subset \hat{\mathcal{D}}
\]

(396)

with respect to the central Fock type Hilbert space \( \hat{\Phi}_+ \), where \( \hat{\mathcal{D}}; \hat{\mathcal{D}}_+ \) easily following from (378) and (395).

Construct now the following self-adjoint operator \( \rho(x) : \Phi \to \Phi \) as

\[
\rho(x) := a^+(x) a(x),
\]

(397)

called the density operator at a point \( x \in \mathbb{R}^m \), satisfying the commutation properties:

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\[
\rho(x), \rho(y) = 0,
\]
\[
\rho(x), a(y) = -a(y)\delta(x - y),
\]
\[
\rho(x), a^+(y) = a^+(y)\delta(x - y)
\]
for all \( y \in \mathbb{R}^m \).

Now, if to construct the following self-adjoint family \( A := \left\{ \rho(x)f(x)dx : f \in F \right\} \) of linear operators in the Fock space \( \Phi \), where \( F := \mathcal{S}({\mathbb{R}}^m;\mathbb{R}) \) is the Schwartz functional space, one can derive, making use of Theorem 7.1, that there exists the generalized Fourier transform (381), such that

\[
\Phi(\mathcal{H}) = L^2_2(\mathcal{S};\mathbb{C}) \int_\mathcal{S} \Phi_\eta d\mu(\eta)
\]
for some Hilbert space sets \( \Phi_\eta \), \( \eta \in F \), and a suitable measure \( \mu \) on \( \mathcal{S} \), with respect to which the corresponding joint eigenvector \( \varphi(\eta) \in \Phi_+ \) for any \( \eta \in F \) generates the Fourier transformed family \( \hat{\mathcal{U}} = \{ \varphi(f) \in \mathbb{R} : f \in F \} \).

Moreover, if \( \dim \Phi_\eta = 1 \) for all \( \eta \in F \), the Fourier transformed eigenvector \( \hat{\varphi}(\eta) := \Omega(\eta) = 1 \) for all \( \eta \in F \).

Now we will consider the family of self-adjoint operators \( \hat{\mathcal{U}} \) as generating a unitary family \( \mathcal{U} := \{ \mathcal{U}(f) : f \in F \} = \exp(i\hat{\mu}) \), where for any \( \rho(f) \in \hat{\mathcal{U}} \), \( f \in F \), the operator

\[
\mathcal{U}(f) := \exp[i\rho(f)]
\]
is unitary, satisfying the abelian commutation condition

\[
\mathcal{U}(f_1)\mathcal{U}(f_2) = \mathcal{U}(f_1 + f_2)
\]
for any \( f_1, f_2 \in F \).

Since, in general, the unitary family \( \mathcal{U} = \exp(i\hat{\mu}) \) is defined in some Hilbert space \( \Phi \), not necessarily being of Fock type, the important problem of describing its Hilbertian cyclic representation spaces arises, within which the factorization

\[
\rho(f) = \int_{\mathbb{R}^m} a^+(x)a(x)f(x)dx
\]
jointly with relationships (398) hold for any \( f \in F \). This problem can be treated using mathematical tools devised both within the representation theory of \( C^\ast \)-algebras [37, 36] and the Gelfand–Vilenkin [52] approach. Below we will describe the main features of the Gelfand–Vilenkin formalism, being much more suitable for the task, providing a reasonably unified framework of constructing the corresponding representations.

**Definition 7.2** Let \( F \) be a locally convex topological vector space, \( F_0 \subset F \) be a finite dimensional subspace of \( F \). Let \( F^0 \subset F \) be defined by

\[
F^0 := \left\{ \xi \in F : \xi|_{F_0} = 0 \right\}
\]
and called the annihilator of \( F_0 \).

The quotient space \( F^0 := F/F^0 \) may be identified with \( F_0 \subset F \), the adjoint space of \( F_0 \).

**Definition 7.3** Let \( A \subset F \); then the subset

\[
X^{(A)}_{F_0} := \left\{ \xi \in F : \xi + F^0 \subset A \right\}
\]

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is the cylinder set with base \( A \) and generating subspace \( F^0 \).

**Definition 7.4** Let \( n = \dim F_0 = \dim F'_0 = \dim F^0 \). One says that a cylinder set \( X^{(A)} \) has Borel base, if \( A \) is Borel, when regarded as a subset of \( R^n \).

The family of cylinder sets with Borel base forms an algebra of sets.

**Definition 7.5** The measurable sets in \( F' \) are the elements of the \( \sigma \) - algebra generated by the cylinder sets with Borel base.

**Definition 7.6** A cylindrical measure in \( F' \) is a real-valued \( \sigma \) - pre-additive function \( \mu \) defined on the algebra of cylinder sets with Borel base and satisfying the conditions \( 0 \leq \mu(X) \leq 1 \) for any \( X \), \( \mu(F^0) = 1 \) and

\[
\mu\left(\bigcup_{j \in Z_+} X_j\right) = \sum_{j \in Z_+} \mu(X_j),
\]

if all sets \( X_j \subset F' \), \( j \in Z_+ \), have a common generating subspace \( F'_0 \subset F \).

**Definition 7.7** A cylindrical measure \( \mu \) satisfies the commutativity condition if and only if for any bounded continuous function \( \alpha : F^n \to R \) of \( n \in Z_+ \) real variables the function

\[
a_{f_1, f_2, \ldots, f_n} := \int_F \alpha(\eta(f_1), \eta(f_2), \ldots, \eta(f_n))d\mu(\eta)
\]

is sequentially continuous in \( f_j \in F \), \( j = 1, \ldots, n \). (It is well known [52, 57] that in countably normalized spaces the properties of sequential and ordinary continuity are equivalent).

**Definition 7.8** A cylindrical measure \( \mu \) is countably additive if and only if for any cylinder set \( X = \bigcup_{j \in Z_+} X_j \), which is the union of countably many mutually disjoints cylinder sets \( X_j \subset F' \), \( j \in Z_+ \), \( \mu(X) = \sum_{j \in Z_+} \mu(X_j) \).

The following propositions hold.

**Proposition 7.9** A countably additive cylindrical measure \( \mu \) can be extended to a countably additive measure on the \( \sigma \) - algebra, generated by the cylinder sets with Borel base. Such a measure will also be called a cylindrical measure.

**Proposition 7.10** Let \( F \) be a nuclear space. Then any cylindrical measure \( \mu \) on \( F' \), satisfying the continuity condition, is countably additive.

**Definition 7.11** Let \( \mu \) be a cylindrical measure in \( F' \). The Fourier transform of \( \mu \) is the nonlinear functional

\[
\mathcal{L}(f) := \int_F \exp[i \eta(f)]d\mu(\eta).
\]  

**Definition 7.12** The nonlinear functional \( \mathcal{L} : F \to C \) on \( F \), defined by (406), is called positive definite, if and only if for all \( f_j \in F \) and \( \lambda_j \in C \), \( j = 1, \ldots, n \), the condition

\[
\sum_{j, k=1}^n \lambda_j \lambda_k \mathcal{L}(f_k - f_j) \geq 0
\]

holds for any \( n \in Z_+ \).

**Proposition 7.13** The functional \( \mathcal{L} : F \to C \) on \( F \), defined by (406), is the Fourier transform of a cylindrical measure on \( F' \), if and only if it is positive definite, sequentially continuous and satisfying the condition \( \mathcal{L}(0) = 1 \).

Suppose now that we have a continuous unitary representation of the unitary family \( U \) in a Hilbert space \( \Phi \) with a cyclic vector \( |\Omega\rangle \in \Phi \). Then we can put

\[
\mathcal{L}(f) := \langle \Omega | U(f) | \Omega \rangle
\]
for any $f \in F := S$, being the Schwartz space on $\mathbb{R}^m$, and observe that functional (406) is continuous on $F$ owing to the continuity of the representation. Therefore, this functional is the generalized Fourier transform of a cylindrical measure $\mu$ on $S$:

$$
\langle \Omega | U(f) | \Omega \rangle = \int_S \exp[i\eta(f)]d\mu(\eta).
$$

(409)

From the spectral point of view, based on Theorem 7.1, there is an isomorphism between the Hilbert spaces $\Phi$ and $L^2(\mu_k^\psi(S';\mathbb{C}))$, defined by $|\Omega\rangle \mapsto \Omega(\eta) = 1$ and $U(f)|\Omega\rangle \mapsto \exp[i\eta(f)]|\Omega\rangle$ and next extended by linearity upon the whole Hilbert space $\Phi$.

In the case of the non-cyclic case there exists a finite or countably infinite family of measures $\{\mu_k : k \in \mathbb{Z}_+\}$ on $S'$ with $\Phi \cong \bigoplus_{k \in \mathbb{Z}_+} L^2(\mu_k^\psi(S';\mathbb{C}))$ and the unitary operator $U(f) : \Phi \rightarrow \Phi$ for any $f \in S'$ corresponds in all $L^2(\mu_k^\psi(S';\mathbb{C}))$, $k \in \mathbb{Z}_+$, to $\exp[i\eta(f)]$. This means that there exists a single cylindrical measure $\mu$ on $S$ and a $\mu$-measurable field of Hilbert spaces $\Phi_n$ on $S$, such that

$$
\Phi_n = \int_S \Phi_\eta d\mu(\eta),
$$

(410)

with $U(f) : \Phi \rightarrow \Phi$, corresponding [52] to the operator of multiplication by $\exp[i\eta(f)]$ for any $f \in S$ and $\eta \in S'$. Thereby, having constructed the nonlinear functional (406) in an exact analytical form, one can retrieve the representation of the unitary family $\bigcup$ in the corresponding Hilbert space $\Phi$ of the Fock type, making use of the suitable factorization (402) as follows: $\Phi = \bigoplus_{n \in \mathbb{Z}_+} \Phi_n$, where

$$
\Phi_n = \text{span} \left\{ \prod_{j=1}^n a^+(x_j) | \Omega \right\},
$$

(411)

for all $n \in \mathbb{Z}_+$. The cyclic vector $|\Omega\rangle \in \Phi$ can be, in particular, obtained as the ground state vector of some unbounded self-adjoint positive definite Hamilton operator $H : \Phi \rightarrow \Phi$, commuting with the self-adjoint particles number operator

$$
N := \int_{\mathbb{R}^m} \rho(x) dx,
$$

(412)

that is $[H, N] = 0$. Moreover, the conditions

$$
H | \Omega \rangle = 0
$$

(413)

and

$$
\inf_{g \in \text{dom } H} \langle g, Hg \rangle = \langle \Omega | H | \Omega \rangle = 0
$$

(414)

hold for the operator $H : \Phi \rightarrow \Phi$, where $\text{dom } H$ denotes its domain of definition.

To find the functional (408), which is called the generating Bogolubov type functional for moment distribution functions

$$
F_n(x_1, x_2, \ldots, x_n) := \langle \Omega | : \rho(x_1) \rho(x_2) \cdots \rho(x_n) : | \Omega \rangle,
$$

(415)

where $x_j \in \mathbb{R}^m$, $j = 1, n$, and the normal ordering operation $:\cdots:\$ is defined as

$$
:\rho(x_1) \rho(x_2) \cdots \rho(x_n) : = \prod_{j=1}^n \left( \rho(x_j) - \sum_{k=1}^n \delta(x_j - x_k) \right),
$$

(416)

it is convenient to choose the Hamilton operator $H : \Phi \rightarrow \Phi$ in the following [58, 56, 23] algebraic form:
being equivalent in the Hilbert space \( \Phi \) to the positive definite operator expression

\[
H := \frac{1}{2} \int_{\mathbb{R}^m} (K^+(x)\rho^{-1}(x)K(x)dx + V(\rho),
\]

(417)

satisfying conditions (413) and (414), where \( A(x;\rho) : \Phi \to \Phi, \ x \in \mathbb{R}^m, \) is some specially chosen linear self-adjoint operator. The “potential” operator \( V(\rho) : \Phi \to \Phi \) is, in general, a polynomial (or analytical) functional of the density operator \( \rho(x) : \Phi \to \Phi \) and the operator is given as

\[
K(x) := \nabla_x \rho(x)\partial + i J(x),
\]

(419)

where the self-adjoint “current” operator \( J(x) : \Phi \to \Phi \) can be defined (but non-uniquely) from the equality

\[
\partial \rho(\partial t) = \frac{1}{i} [H, \rho(x)] = -< \nabla_x J(x)>.
\]

(420)

holding for all \( x \in \mathbb{R}^m \). Such an operator \( J(x) : \Phi \to \Phi, \ x \in \mathbb{R}^m, \) can exist owing to the commutation condition \([H,N] = 0\), giving rise to the continuity relationship (420), if taking into account that supports \( \text{supp} \ \rho \) of the density operator \( \rho(x) : \Phi \to \Phi, \ x \in \mathbb{R}^m, \) can be chosen arbitrarily owing to the independence of (420) on the potential operator \( V(\rho) : \Phi \to \Phi \), but its strict dependence on the corresponding representation (410). In particular, based on the Fock space \( \Phi, \) defined by (378) and generated by the creation-annihilation operators (388) and (389), the current operator \( J(x) : \Phi \to \Phi, \ x \in \mathbb{R}^m, \) can be constructed as follows:

\[
J(x) = \frac{1}{2i} [a^+(x)\nabla a(x) - \nabla a^-(x)a(x)],
\]

(421)

satisfying jointly with the density operator \( \rho(x) : \Phi \to \Phi, \ x \in \mathbb{R}^m, \) defined by (397), the following quantum current Lie algebra [59, 23, 103] relationships:

\[
[ J(g_1), J(g_2) ] = iJ([g_1,g_2]),
\]

(422)

\[
J(g_1), \rho(f_1)) = i\rho(< g_1, \nabla f_1 >),
\]

\[
\rho(f_1), \rho(f_2)) = 0,
\]

(423)

holding for all \( f_1, f_2 \in F \) and \( g_1, g_2 \in F^3, \) where we put, by definition,

\[
g_1,g_2 := < g_1, \nabla > g_2 - < g_2, \nabla > g_1,
\]

being the usual commutator of vector fields in the Euclidean space \( E^m. \) It is easy to observe that the current algebra (422) is the Lie algebra \( G, \) corresponding to the Banach Lie group \( G = \text{Diff} \ E^3 \hat{a} F, \) the semidirect product of the Banach Lie group of diffeomorphisms \( \text{Diff} \ E^3 \) of the three-dimensional Euclidean space \( E^3 \) and the abelian subject to the multiplicative operation Banach group of smooth functions \( F. \) We note also that representation (418) holds only under the condition that there exists such a self-adjoint operator \( A(x;\rho) : \Phi \to \Phi, \ x \in \mathbb{R}^m, \) that

\[
K(x) \mid \Omega = A(x;\rho) \mid \Omega
\]

(424)

for all ground states \( \mid \Omega \) \( \in \Phi, \) correspond to suitably chosen potential operators \( V(\rho) : \Phi \to \Phi. \)

The self-adjointness of the operator \( A(x;\rho) : \Phi \to \Phi, \ x \in \mathbb{R}^m, \) can be stated following schemes from works [58, 23], under the additional condition of the existence of such a linear anti-unitary mapping \( T : \Phi \to \Phi \) that the
following invariance conditions hold:

$$T \rho(x)T^{-1} = \rho(x), \quad T J(x)T^{-1} = -J(x), \quad T \Omega = \Omega$$

(425)

for any $x \in \mathbb{R}^m$. Thereby, owing to conditions (425), the following expressions

$$K^*(x) \Omega = A(x; \rho) \Omega = K(x) \Omega$$

(426)

hold for any $x \in \mathbb{R}^m$, giving rise to the self-adjointness of the operator $A(x; \rho) : \Phi \rightarrow \Phi, \ x \in \mathbb{R}^m$.

Based now on the construction above one easily deduces from expression (420) that the generating Bogolubov type functional (408) obeys for all $x \in \mathbb{R}^m$ the following functional-differential equation:

$$\left[ \nabla_s - i \nabla_x A \right] \frac{1}{2i} \frac{\delta L(f)}{\delta f} = A \left( x, \frac{1}{i} \frac{\delta}{\delta f} \right) \frac{\delta L(f)}{\delta f},$$

(427)

whose solutions should satisfy the Fourier transform representation (409). In particular, a wide class of special so-called Poissonian white noise type solutions to the functional-differential equation (427) was obtained in [58, 23] by means of functional-operator methods in the following generalized form:

$$L(f) = \exp \left\{ 2A \left( \frac{1}{i} \frac{\delta}{\delta f} \right) \right\} \exp \left\{ \bar{\rho} \int_{\mathbb{R}^m} \left( \exp \{if(x)\} - 1 \right) dx \right\}.$$  

(428)

where $\bar{\rho} := \langle \Omega | \rho | \Omega \rangle \in \mathbb{R}_+$ is a Poisson distribution density parameter.

It is worth to remark here that solutions to equation (427) realize the suitable physically motivated representations of the abelian Banach subgroup $F$ of the Banach group $G = \text{Diff} \ E^3 \hat{A} F$, mentioned above. In the general case of the Banach group $G = \text{Diff} \ E^3 \hat{A} F$ one can also construct [23, 107] a generalized Bogolubov type functional equation, whose solutions give rise to suitable physically motivated representations of the corresponding current Lie algebra $G$.

Consider now the case, when the basic Fock space $\Phi = \bigotimes_{j=1}^{s} \Phi^{(j)}$, where $\Phi^{(j)}, \ j = 1, s$, are Fock spaces corresponding to the different types of independent cyclic vectors $\{ \Omega_j, \} \in \Phi^{(j)}, \ j = 1, s$. This, in particular, means that the suitably constructed creation and annihilation operators $a_j(x), a^*_k(y) : \Phi \rightarrow \Phi, \ j, k = 1, s$, satisfy the following commutation relations:

$$a_j(x), a^*_k(y) = 0,$$

$$a_j(x), a^*_k(y) = \delta_{jk} \delta(x - y)$$

(429)

for any $x, y \in \mathbb{R}^m$.

**Definition 7.14** A vector $|u\rangle \in \Phi, \ x \in \mathbb{R}^m$, is called coherent [54, 102] with respect to a mapping $u \in L_2(\mathbb{R}^m; \mathbb{R}^r) := M$, if it satisfies the eigenfunction condition

$$a_j(x) |u\rangle = u_j(x) |u\rangle$$

(430)

for each $j = 1, s$ and all $x \in \mathbb{R}^m$.

It is easy to check that the coherent vectors $|u\rangle \in \Phi$ exist. Really, the following vector expression

$$|u\rangle := \exp \{ (u, a^*) \} |\Omega\rangle,$$  

(431)

where $(\ , \ )$ is the standard scalar product in the Hilbert space $M$, satisfies the defining condition (430), and moreover, the norm
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\[ \mathcal{P}_\phi := \langle u \mid u \rangle^{1/2} = \exp\left(\frac{1}{2} \mathcal{P}^2\right) < \infty, \]

(432)

since \( u \in \mathcal{M} \) and its norm \( \mathcal{P}_\phi := (u, u)^{1/2} \) is bounded.

### 7.2 The Fock Space Embedding Method, Nonlinear Dynamical Systems and their Complete Linearization

Consider any function \( u \in \mathcal{M} := L_2(\mathbb{R}^m; \mathbb{R}^r) \) and observe that the Fock space embedding mapping

\[ \phi : \mathcal{M} \ni u \rightarrow \langle u \rangle \in \Phi, \]

(433)

defined by means of the coherent vector expression (431) realizes a smooth isomorphism between Hilbert spaces \( \mathcal{M} \) and \( \Phi \). The inverse mapping \( \phi^{-1} : \Phi \rightarrow \mathcal{M} \) is given by the following exact expression:

\[ u(x) = \langle \Omega \mid a(x) \mid u \rangle, \]

(434)

holding for almost all \( x \in \mathbb{R}^m \). Owing to condition (432), one finds from (434) that, the corresponding function \( u \in \mathcal{M} \).

In the Hilbert space \( \mathcal{M} \), let now define a nonlinear dynamical system (which can, in general, be non-autonomous) in partial derivatives

\[ \frac{du}{dt} = K[u], \]

(435)

where \( t \in \mathbb{R}_+ \) is the corresponding evolution parameter, \( [u] := (t, x; u, u_x, \ldots, u_r) \), \( r \in \mathbb{Z}_+ \), and a mapping \( K : \mathcal{M} \rightarrow T(\mathcal{M}) \) is Frechet smooth. Assume also that the Cauchy problem

\[ u \big|_{t=0} = u_0 \]

(436)

is solvable for any \( u_0 \in \mathcal{M} \) in an interval \( [0, T) \subset \mathbb{R}_+ \) for some \( T > 0 \). Thereby, the smooth evolution mapping is defined

\[ T_t : \mathcal{M} \ni u_0 \rightarrow u(t \mid u_0) \in \mathcal{M}, \]

(437)

for all \( t \in [0, T) \).

It is now natural to consider the following commuting diagram

\[ \begin{array}{ccc}
\mathcal{M} & \xrightarrow{\phi} & \Phi \\
T_t \downarrow & & \downarrow T_t \\
\mathcal{M} & \xrightarrow{\phi} & \Phi,
\end{array} \]

(438)

where the mapping \( T_t : \Phi \rightarrow \Phi, \ t \in (0, T), \) is defined from the conjugation relationship

\[ \phi \circ T_t = T_t \circ \phi. \]

(439)

Now take coherent vector \( \langle u_0 \rangle \in \Phi \), corresponding to \( u_0 \in \mathcal{M} \), and construct the vector

\[ \langle u \rangle := T_t \langle u_0 \rangle \]

(440)

for all \( t \in (0, T) \). Since vector (440) is, by construction, coherent, that is

\[ a_j(x) \langle u \rangle := a_j(x, t \mid u_0) \langle u \rangle \]

(441)

for each \( j = 1, \ldots, r \), \( t \in (0, T) \) and almost all \( x \in \mathbb{R}^m \), owing to the smoothness of the mapping \( \xi : \mathcal{M} \rightarrow \Phi \) with respect to the corresponding norms in the Hilbert spaces \( \mathcal{M} \) and \( \Phi \), we derive that coherent vector (440) is differentiable.
with respect to the evolution parameter $t \in (0, T)$. Thus, one can easily find [76, 75, 102, 109] that

$$\frac{d}{dt} |u| = \hat{K}[a^+, a] |u|, \hspace{1cm} (442)$$

where

$$|u| \big|_{t=0} = |u_0\rangle \hspace{1cm} (443)$$

and a mapping $\hat{K}[a^+, a] : \Phi \to \Phi$ is defined by the exact analytical expression

$$\hat{K}[a^+, a] := (a^+, K[a]). \hspace{1cm} (444)$$

As a result of the consideration above we obtain the following theorem.

**Theorem 7.15** Any smooth nonlinear dynamical system (435) in Hilbert space $M := L_2(\mathbb{R}^m; \mathbb{R}^s)$ is representable by means of the Fock space embedding isomorphism $\phi : M \to \Phi$ in the completely linear form (442).

We now make some comments concerning the solution to the linear equation (442) under the Cauchy condition (443). Since any vector $|u\rangle \in \Phi$ allows the series representation

$$|u\rangle = \sum_{n=\sum_{j=1}^s n_j \in \mathbb{Z}_r} \frac{1}{(n_1! \cdots n_s!)^{1/2}} \int_{\mathbb{R}^m_{n_1, \cdots, n_s}} f_{n_1, \cdots, n_s}^{(n)} (x^{(1)}_1, \ldots, x^{(1)}_{n_1}; \ldots; x^{(s)}_1, \ldots, x^{(s)}_{n_s}) \prod_{j=1}^s \prod_{k=1}^{n_j} dx_k^{(j)} (x^{(j)}_k) |\Omega\rangle, \hspace{1cm} (445)$$

where for any $n = \sum_{j=1}^s n_j \in \mathbb{Z}_r$ functions

$$f_{n_1, \cdots, n_s}^{(n)} \in \bigotimes_{j=1}^s L_{2, \sigma}((\mathbb{R}^m_{n_j}; \mathbb{C}) : L_{2, \sigma}(\mathbb{R}^m_{n_1} \times \cdots \times \mathbb{R}^m_{n_s}; \mathbb{C}), \hspace{1cm} (446)$$

and the norm

$$P \mu P^2 = \sum_{n=\sum_{j=1}^s n_j \in \mathbb{Z}_r} P f_{n_1, \cdots, n_s}^{(n)} \mu = \exp (P \mu P^2). \hspace{1cm} (447)$$

By substituting (445) into equation (442), reduces (442) to an infinite recurrent set of linear evolution equations in partial derivatives on coefficient functions (446). The latter can often be solved [75, 102] step by step analytically in exact form, thereby, making it possible to obtain, owing to representation (434), the exact solution $u \in M$ to the Cauchy problem (436) for our nonlinear dynamical system in partial derivatives (435).

**Remark 7.16** Concerning some applications of nonlinear dynamical systems like (433) in mathematical physics problems, it is very important to construct their so called conservation laws or smooth invariant functionals $\gamma : M \to \mathbb{R}$ on $M$.

Making use of the quantum mathematics technique described above one can suggest an effective algorithm for constructing these conservation laws in exact form.

Indeed, consider a vector $|\gamma\rangle \in \Phi$, satisfying the linear equation:

$$\frac{\partial}{\partial t} |\gamma\rangle + \hat{K}^*[a^+, a] |\gamma\rangle = 0. \hspace{1cm} (448)$$

Then, the following proposition [75, 102] holds.

**Proposition 7.17** The functional
\( \gamma := \langle u \mid y \rangle \) (449)

is a conservation law for dynamical system (433), that is

\[ dy/dt \bigg|_k = 0 \] (450)

along any orbit of the evolution mapping (437).

### 7.3 The Quantum Current Lie Algebra and the Magnetic Aharonow-Bohm Effect

In the Section above we could get convinced that different representations of the equal-time current algebra (422)

\[ \rho(f_1), \rho(f_2) \rangle = 0, \]

\[ J(g_1), \rho(f_1) \rangle = i \rho\langle g_1, \nabla f_1 \rangle, \]

where \( f_1, f_2 \in F \) and \( g_2, g_1 \in F^3 \), acting in the Fock space \( \Phi \) and describing a non-relativistic quasi-stationary system consisting of a test charged particle \( q \), imbedded into a cylindrical region \( \Gamma \subset \mathbb{E}^3 \), being under influence of the magnetic field \( B = \nabla \times A \). Here \( A \in \mathbb{E}^3 \) is a magnetic vector potential, the sign \( \times \) means the vector product in \( \mathbb{E}^3 \) and the current \( J(x) : \Phi \rightarrow \Phi, \ x \in \mathbb{R}^m \), is defined, owing to the minimal interaction principle (394), as

\[ J(x) := \frac{1}{2} a^i(x) (\frac{1}{i} \nabla - A) a(x) - [(\frac{1}{i} \nabla + A) a^i(x)] a(x). \] (452)

In particular, it is assumed that \( \text{supp} \ B \subset \Gamma \) that gives rise to the equality \( \rho\langle B, g_1 \times g_2 \rangle = 0 \) for all points \( x \in \mathbb{E}^3 \setminus \Gamma \).

As the suitable representations of the current algebra \( \mathcal{G} \), defined by (452), describe the physical quantum states of the system \( \Gamma \) under regards, we consider them following [59, 60] as those realized in the Hilbert space \( L_2(\mathbb{E}^3; \mathbb{C}) \) under the condition that the charged test particle \( q \) can penetrate the boundary \( \partial \Gamma \) of the region \( \Gamma \). Namely, let for \( \psi \in L_2(\Gamma; \mathbb{C}) \)

\[ \rho(f) \psi(x) := f(x) \psi(x), \] (453)

\[ J(g) \psi(x) = \frac{1}{2i} \left[ \langle g(x), \nabla \rangle + \langle \nabla, g(x) \rangle \right] - \langle g(x), \int d^3 y \frac{\nabla \times B(y)}{4\pi |x-y|} \rangle \psi(x), \]

for all \( x \in \mathbb{E}^3 \), where the sign \( \cdot \) means that the natural operator composition. When deriving (453) there was imposed the invariant Coulomb gauge constraint \( \langle \nabla, A \rangle = 0 \) allowing to determine the vector potential \( A \in \mathbb{E}^3 \) as

\[ A = \int d^3 y \frac{\nabla \times B(y)}{4\pi |x-y|} \] (454)

using the classical Maxwell equations (6), since the electric displacement current component \( \partial E/\partial t = 0 \). The wave function \( \psi \in L_2(\Gamma; \mathbb{C}) \) satisfies in the cylindrical coordinates \( x(r, \theta, z) \in \Gamma \) the natural quasi-periodical condition

\[ \psi(r, \theta + 2\pi n, z) = \exp(i\lambda n) \psi(r, \theta, z) \] (455)

for some \( \lambda \in \mathbb{R} \) and any \( n \in \mathbb{Z} \), which should be determined from the physically realizable representation (453). To do this, we need preliminarily to define the following [60] unitary operator in the Hilbert space \( L_2(\mathbb{E}^3; \mathbb{C}) : \)

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where the path \( l_y \subset \mathbb{E}^3 \setminus \Gamma \) connects an infinite point \( \infty \in \mathbb{E}^3 \) with the chosen point \( x \in \mathbb{E}^3 \setminus \Gamma \). Making use now of the unitary transformation \( \tilde{J}(g) := QJ(g)Q^{-1} \) and the fact that the magnetic field

\[
B(x) = \nabla \times \int_{\Gamma} \frac{\nabla \times B(y) d^3 y}{4 \pi |x - y|}
\]

for any \( x \in \mathbb{E}^3 \), one obtains easily that the current operator \( \tilde{J}(g) : L_2(\mathbb{E}^3; \mathbb{C}) \rightarrow L_2(\mathbb{E}^3; \mathbb{C}) \) is self-adjoint for any \( g \in F^3 \) and

\[
\tilde{J}(g)\psi(x) = \frac{1}{2i} \left< g(x), \nabla \cdot (\psi(x)) \right>
\]

and whose domain of definition \( \text{dom} \tilde{J}(g) \subset L_2(\mathbb{E}^3; \mathbb{C}) \) is constrained by the functions \( \psi \in L_2(\mathbb{E}^3; \mathbb{C}) \), satisfying the condition

\[
\psi(r, \theta + 2\pi, z) = \exp[i \lambda(B)] \psi(r, \theta, z),
\]

where, owing to (456),

\[
\lambda(B) = -\int_{\Gamma} < B, dS >.
\]

Thus, the found above representation (458) of the current Lie algebra (451) in the Hilbert space \( L_2(\mathbb{E}^3; \mathbb{C}) \), in the case when \( \text{supp} B \subset \Gamma \), describes the complete set of observables if the charged particle \( q \) is not excluded from the region \( \Gamma \). In contrast, if the region \( \Gamma \) possesses a potential barrier at the boundary \( \partial \Gamma \), such that the charged particle \( q \) can not penetrate it and enter the region \( \Gamma \), the value of \( \lambda(B) \in \mathbb{R} \), defined by (460), remains constant. This entails that a suitable outside the region \( \Gamma \) measurement can certainly indicate the presence of the magnetic field inside \( \Gamma \). So, as it was mentioned in [59], the constructed above current algebra representation completely describes our non-relativistic quasi-stationary system not giving rise to the Aharonov-Bohm [3] paradox. Moreover, the outside measurements results simply depend on the representation of the current algebra (451), which in turn depends on the history of the system and the topology of the space outside the barrier. As a related physical aspect of the explanation above it is necessary to stress that the vanishing of the magnetic field outside the region \( \Gamma \), possessing a nontrivial topology, does not imply the simultaneous vanishing of the corresponding magnetic potential outside the region \( \Gamma \). Namely the latter in somewhat obscured form was used in the analysis of the current algebra representation, suitable for describing the complete set of physical observables both inside and outside the region \( \Gamma \).

### 7.4 Comments

Within the scope of this Section we have described the main mathematical preliminaries and properties of the quantum mathematics techniques suitable for analytical studying of the important linearization problem for a wide class of nonlinear dynamical systems in partial derivatives in Hilbert spaces. This problem was analyzed in much detail using the Gelfand-Vilenkin representation theory [52, 21] of infinite dimensional groups and the Goldin-Menikoff-Sharp theory [58, 57, 59] of generating Bogolubov type functionals, classifying these representations. The related problem of constructing Fock type space representations and retrieving their creation-annihilation generating structure still needs a deeper investigation within the approach devised. Here we mention only that some aspects of this problem within the so-called Poissonian White noise analysis were studied in a series of works [15, 14, 4, 72, 87], based on some generalizations of the Delsarte type characters technique. It is also necessary to mention the related results obtained in [74, 75, 76, 104, 106, 119], devoted to the application of the Fock space embedding method to studying solutions to a wide nonlinear dynamical systems and to constructing quantum computing algorithms.

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