The value of informational arbitrage

Huy N. Chau1 · Andrea Cosso2 · Claudio Fontana3

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Abstract In the context of a general semimartingale model, we aim at determining how much an investor is willing to pay to learn additional information that allows achieving arbitrage. If such a value exists, we call it the value of informational arbitrage. We are interested in the case where the information yields arbitrage opportunities but not unbounded profits with bounded risk. As in Amendinger et al. (Finance Stoch. 7:29–46, 2003), we rely on an indifference valuation approach and study optimal consumption–investment problems under initial information and arbitrage. We establish some new results on models with additional information and characterise when the value of informational arbitrage is universal.

Keywords Value of information · Enlargement of filtration · Arbitrage · Indifference price · Martingale representation

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C. Fontana
fontana@math.unipd.it

H.N. Chau
chau@sigmath.es.osaka-u.ac.jp

A. Cosso
andrea.cosso@unibo.it

1 Center for Mathematical Modeling and Data Science, Osaka University, Osaka, Japan
2 Department of Mathematics, University of Bologna, Bologna, Italy
3 Department of Mathematics “Tullio Levi-Civita”, University of Padova, Padova, Italy
1 Introduction

The notion of information plays a crucial role in the analysis of investment decisions. In line with economic intuition, access to more precise sources of information gives an informational advantage leading to better performing portfolios. The problem of quantifying such an informational advantage represents a central question in finance and has constantly attracted significant attention in financial economics and more recently in mathematical finance.

We develop a general approach for quantifying in monetary terms the informational advantage associated to some additional information, in the context of a general semimartingale model of a complete market, under weak assumptions on the random variable (denoted by $L$) representing the additional information. We adopt an indifference valuation approach and determine a value $\pi(v)$ which makes a risk averse agent with initial capital $v$ indifferent towards the following two alternatives: (i) investing optimally the initial capital $v$ by relying on the publicly available information only; (ii) acquiring additional information $L$ at the price $\pi(v)$ and investing optimally the residual capital $v - \pi(v)$ by relying on the publicly available information enriched by the additional information.

The idea of quantifying information by an indifference valuation approach can be traced back to early contributions in information economics; see in particular La Valle et al. [29], Morris [31] and Willinger [38]. The same approach has been pursued in the context of mathematical finance in Amendinger et al. [4], which represents the main starting point for the present work. In contrast to [4], we assume that the additional information can be potentially exploited to realise arbitrage opportunities, but unbounded profits with bounded risk cannot be achieved (this represents the minimal condition allowing a meaningful solution to optimal portfolio problems; see Karatzas and Kardaras [26], Choulli et al. [13] and Chau et al. [11]). In this framework, we call the indifference value $\pi(v)$ the \textit{value of informational arbitrage}.

As we are going to show, informational arbitrage appears whenever the additional information reveals that some events, which are believed to occur with strictly positive probability by public opinion, are actually impossible. To illustrate the notion of value of informational arbitrage, let us give a simple example which will be analysed in a more general version in Sect. 5.1.

\textbf{Example 1.1} Consider a financial market with a single risky asset with price process

$$S_t = \exp(W_t - t/2) \quad \text{for all } t \in [0,1],$$

where $(W_t)_{t \in [0,1]}$ is a standard Brownian motion. The \textit{ordinary information} (publicly available) is given by the observation of the price process, corresponding to the filtration $\mathcal{F} = (\mathcal{F}_t)_{t \in [0,1]}$. We suppose that the \textit{additional information} is generated by the observation at $t = 0$ of the random variable $L = 1_{\{W_1 \geq 0\}}$. The information flow available to an informed agent is described by the initially enlarged filtration $\mathcal{G} = (\mathcal{G}_t)_{t \in [0,1]}$, where $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(L)$ for all $t \in [0,1]$.

Clearly, the ordinary information does not permit any kind of arbitrage. In contrast, the additional information $L$ yields arbitrage opportunities. In this sense, we say that $L$ yields \textit{informational arbitrage} and we aim at determining the maximal amount
\(\pi(v)\) that an agent with initial wealth \(v > 0\) accepts to pay to learn the realisation of \(L\) before the beginning of trading.

In the context of this example, we show that for any risk averse agent constrained to invest in nonnegative portfolios, the value of informational arbitrage is always given by

\[
\pi(v) = v/2.
\]

Moreover, there exists one arbitrage strategy which is optimal for every risk averse informed agent. We remark that in this example, the value \(\pi(v)\) presents the striking feature of being a universal indifference value which does not depend on the preference structure of the agent.

In the present work, we aim at revealing which features of the additional information are at the origin of arbitrage, and at understanding the indifference value of informational arbitrage in a general setting. Motivated by Example 1.1 and similarly as in Amendinger et al. [4], the problem is naturally framed in the context of an initial enlargement of filtration (see also Danilova et al. [15] in a related setting). In order to allow the possibility of informational arbitrage, we have to depart from the conventional assumption that \(L\) is independent of the ordinary information flow \(F\) under an equivalent probability measure (called decoupling measure in Amendinger et al. [4]). The notion of decoupling measure goes back to early works in the theory of enlargement of filtrations and has been widely employed in the insider trading literature (see e.g. Grorud and Pontier [20, 21] and Amendinger [3]). The existence of a decoupling measure is tantamount to the equivalence between the \(F\)-conditional law of \(L\) and its unconditional law.

We assume the validity of Jacod’s density hypothesis, as introduced in the seminal work by Jacod [23]. This condition is weaker than the existence of a decoupling measure, as it corresponds to the absolute continuity (but not necessarily equivalence) of the \(F\)-conditional law of \(L\) with respect to its unconditional law. While the passage from an equivalence to an absolute continuity relation could appear as a technical generalisation, it turns out to require the development of a new approach. Most importantly, it allows the additional information to generate arbitrage as in Example 1.1, thus covering situations that cannot be addressed by the theory of Amendinger et al. [4]. Models where arbitrage opportunities appear due to the presence of additional information, while preserving the well-posedness of expected utility maximisation problems, have been previously considered in Pikovsky and Karatzas [33], Ankirchner [6], Ankirchner et al. [7], Ankirchner and Imkeller [8], Chau et al. [12] and Choulli and Yansori [14] (see also Remark 3.9 in this regard).

The main results and contributions of the paper can be outlined as follows. First, we show that market completeness can be transferred from \(F\) to \(G\) up to a change of numéraire. By relying on this result, we obtain a complete characterisation of the validity of no free lunch with vanishing risk (NFLVR) and no unbounded profit with bounded risk (NUPBR) in \(G\). This provides the necessary foundations for the solution of optimal consumption–investment problems under additional information and possibly in the presence of arbitrage and leverage. Under natural assumptions, we prove that \(\pi(v)\) is finite and also strictly positive and increasing in the allowable
leverage whenever $L$ generates arbitrage opportunities, regardless of the preference structure. For logarithmic and power utility functions, we obtain explicit expressions for $\pi(v)$. We provide universal bounds for the value of informational arbitrage and characterise when it is a universal value which does not depend on the preference structure, as in Example 1.1. In particular, we show that this can happen in a non-trivial way only in the presence of arbitrage.

The paper is structured as follows. In Sect. 2, we introduce the general setting. We provide a new martingale representation result and study no-arbitrage properties in the presence of additional information. Section 3 deals with optimal consumption–investment problems under non-trivial initial information, leverage and arbitrage. In Sect. 4, we study the indifference value of additional information and characterise its universal properties. Section 5 contains three examples. For better readability, the proofs of some technical results are deferred to the Appendix. Additional comments and side results can be found in the preprint version of this paper.1

Throughout the paper, we adopt the following conventions and notations, referring to He et al. [22] and Jacod and Shiryaev [24] for all unexplained notions related to stochastic calculus. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a generic probability space endowed with some filtration $\mathcal{F} = (\mathcal{F}_t)_{t \in [0,T]}$ satisfying the usual conditions of right-continuity and $\mathbb{P}$-completeness, with $T \in (0, +\infty)$ a fixed time horizon. We denote by $\mathcal{M}(\mathbb{P}, \mathcal{F})$ ($\mathcal{M}_{loc}(\mathbb{P}, \mathcal{F})$, resp.) the set of martingales (local martingales, resp.) on $(\Omega, \mathcal{F}, \mathbb{P})$ and tacitly assume that every local martingale has càdlàg paths. For a given $\mathbb{R}^d$-valued semimartingale $X = (X_t)_{t \in [0,T]}$ on $(\Omega, \mathcal{F}, \mathbb{P})$, we denote by $L(X, \mathcal{F})$ the set of all $\mathcal{F}$-predictable $\mathbb{R}^d$-valued processes $\varphi = (\varphi_t)_{t \in [0,T]}$ which are integrable with respect to $X$ in the filtration $\mathcal{F}$. Recall that the set $L(X, \mathcal{F})$ is invariant under equivalent changes of probability (see e.g. He et al. [22, Theorem 12.22]). The stochastic integral of $\varphi \in L(X, \mathcal{F})$ with respect to $X$ is denoted by $(\varphi \cdot X)_t := \int_{(0,t]} \varphi_u dX_u$ for all $t \in [0, T]$, with $(\varphi \cdot X)_0 = 0$. Finally, we denote by $\mathcal{O}(\mathcal{F})$ and $\mathcal{P}(\mathcal{F})$, respectively, the optional and predictable sigma-fields on $\Omega \times [0, T]$ with respect to the filtration $\mathcal{F}$. For a process $Y = (Y_t)_{t \in [0,T]}$, we write $Y \in \mathcal{O}_+(\mathcal{F})$ to denote that $Y$ is a nonnegative $\mathcal{O}(\mathcal{F})$-measurable process.

2 The ordinary and the informed financial markets

In this section, we first present the ordinary financial market (Sect. 2.1), consisting of a general arbitrage-free complete financial market with respect to a reference filtration $\mathcal{F}$. In Sect. 2.2, we introduce the initially enlarged filtration $\mathcal{G}$ associated to the additional information $L$ and state a new martingale representation result in $\mathcal{G}$. In Sect. 2.3, we characterise the (no-)arbitrage properties of the financial market under additional information.

2.1 The ordinary financial market

We consider a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with a filtration $\mathcal{F} = (\mathcal{F}_t)_{t \in [0,T]}$ satisfying the usual conditions, where $T < +\infty$ represents a fixed time horizon. For simplicity

1The longer preprint version is available online on arXiv at https://arxiv.org/abs/1804.00442v1.
of presentation, we assume that the initial sigma-field $\mathcal{F}_0$ is trivial. On $(\Omega, \mathcal{F}, \mathbb{P})$, we let $S = (S_t)_{t \in [0, T]}$ be a $d$-dimensional nonnegative semimartingale. We assume that the financial market contains $d + 1$ traded assets, with the first asset representing a baseline security with respect to which all prices are discounted. In discounted terms, the first asset is therefore a riskless asset with a constant price equal to one, while the remaining $d$ assets are risky and with price process $S$.

We call ordinary financial market the tuple $(\Omega, \mathcal{F}, \mathbb{P}; S)$, where the filtration $\mathcal{F}$ is supposed to represent the publicly available information. We assume that $S$ satisfies no free lunch with vanishing risk (NFLVR) on $(\Omega, \mathcal{F}, \mathbb{P})$; see Delbaen and Schachermayer [17]. More specifically, we assume the validity of the following condition.

**Assumption 2.1** There exists a unique probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F}_T)$ such that $\mathbb{Q} \approx \mathbb{P}$ and $S \in \mathcal{M}_{loc}(\mathbb{Q}, \mathcal{F})$.

Assumption 2.1 implies that the ordinary financial market $(\Omega, \mathcal{F}, \mathbb{P}; S)$ is arbitrage-free (in the sense of NFLVR) and complete, in the sense that every bounded $\mathcal{F}_T$-measurable random variable $\xi$ can be represented as $\xi = \mathbb{E}^\mathbb{Q}[\xi] + (\varphi \cdot S)_T$ a.s. for some $\varphi \in L(S, \mathcal{F})$. We denote the density process of $\mathbb{Q}$ with respect to $\mathbb{P}$ on $\mathcal{F}$ by $Z = (Z_t)_{t \in [0, T]}$, i.e., $Z_t = \frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t}$ for all $t \in [0, T]$.

**Remark 2.2** Assumption 2.1 can be relaxed by requiring the existence of a unique equivalent local martingale deflator for $S$ on $(\Omega, \mathcal{F}, \mathbb{P})$. This ensures NUPBR in $\mathcal{F}$ and also implies that the financial market $(\Omega, \mathcal{F}, \mathbb{P}; S)$ is complete (see Stricker and Yan [35]). However, since our main goal is to study the value of an additional information generating arbitrage opportunities when the latter are impossible to achieve on the basis of $\mathcal{F}$ alone, we find it more natural to work under Assumption 2.1.

### 2.2 The initially enlarged filtration $\mathcal{G}$

The additional information is generated by an $\mathcal{A}$-measurable random variable $L$ taking values in a Lusin space $(E, \mathcal{B}_E)$, where $\mathcal{B}_E$ is the Borel sigma-field of $E$. The **initially enlarged filtration** $\mathcal{G} = (\mathcal{G}_t)_{t \in [0, T]}$ is defined as the smallest filtration containing $\mathcal{F}$ and such that $L$ is $\mathcal{G}_0$-measurable, i.e., $\mathcal{G}_t := \mathcal{F}_t \vee \sigma(L)$ for all $t \in [0, T]$ (right-continuity of $\mathcal{G}$ follows from Lemma 2.4). We denote by $\lambda : \mathcal{B}_E \to [0, 1]$ the unconditional law of $L$ so that $\lambda(B) = \mathbb{P}[L \in B]$ holds for all $B \in \mathcal{B}_E$. For each $t \in [0, T]$, let $\nu_t : \Omega \times \mathcal{B}_E \to [0, 1]$ be a regular version of the $\mathcal{F}_t$-conditional law of $L$.

We assume the validity of the following condition which is known as Jacod’s density hypothesis in the enlargement of filtrations theory.

**Assumption 2.3** For all $t \in [0, T]$, $\nu_t \ll \lambda$ holds in the a.s. sense.

Assumption 2.3 was introduced in the seminal work by Jacod [23] to prove the $H’$-hypothesis (i.e., every $\mathcal{F}$-semimartingale is also a $\mathcal{G}$-semimartingale). In a frictionless financial market, the failure of the semimartingale property is incompatible
with NUPBR (see Kardaras and Platen [28]), which is in turn a necessary condition for the solution of portfolio optimisation problems (see Karatzas and Kardaras [26, Proposition 4.19]). Therefore, the validity of the \( H' \)-hypothesis represents a necessary requirement in our framework.

A central feature of our work is that Assumption 2.3 is only required to hold as an absolute continuity relation and not as an equivalence. This fact turns out to be intimately linked to the existence of arbitrage opportunities in \( G \) (see Theorem 2.6). The following lemma presents some first consequences of Assumption 2.3 (see Jacod [23] as well as Fontana [19, Lemma 4.2]).

**Lemma 2.4** Suppose Assumption 2.3 holds. Then the filtration \( G \) is right-continuous and every semimartingale on \((\Omega, F, P)\) is also a semimartingale on \((\Omega, G, P)\). Moreover, there exists a function \( E \times \Omega \times [0, T] \ni (x, \omega, t) \mapsto q_t^x(\omega) \in \mathbb{R}_+ \) which is \((B_E \otimes \mathcal{O}(F))\)-measurable, càdlàg in \( t \in [0, T] \) and such that

(i) for every \( t \in [0, T] \), \( \nu_t(dx) = q_t^x \lambda(dx) \) holds a.s.;
(ii) for every \( x \in E \), the process \( q_t^x = (q_t^x)_{t \in [0, T]} \) is a martingale on \((\Omega, F, P)\).

Furthermore, it holds that \( P[q_t^L > 0] = 1 \) for all \( t \in [0, T] \).

The following implication of Lemma 2.4 is used repeatedly: for every \( t \in [0, T] \) and \((B_E \otimes \mathcal{F}_t)\)-measurable function \( E \times \Omega \ni (x, \omega) \mapsto f_t(x)(\omega) \in \mathbb{R}_+ \), it holds that

\[
\mathbb{E}[f_t^L] = \mathbb{E}\left[ \int_E f_t^x q_t^x \lambda(dx) \right] = \int_E \mathbb{E}[f_t^x q_t^x] \lambda(dx). \tag{2.1}
\]

Under the present assumptions, we can prove the following proposition which shows that the martingale representation property of \( S \) on \((\Omega, F, Q)\) can be transferred to the initially enlarged filtration \( G \) under \( P \) up to a suitable “change of numéraire”.

**Proposition 2.5** Suppose that Assumptions 2.1 and 2.3 hold and let the process \( M = (M_t)_{t \in [0, T]} \) be a local martingale on \((\Omega, G, P)\). Then there exists a process \( K = (K_t)_{t \in [0, T]} \in L(S, G) \) such that

\[
M_t = \frac{Z_t^L}{q_t^L} (M_0 + (K \cdot S)_t) \quad \text{a.s. for all } t \in [0, T].
\]

**Proof** Define the \( \mathbb{R}^{d+1} \)-valued semimartingale \( X := (1, S) \). Due to Assumption 2.1, one can verify that \( ZX \) has the martingale representation property on \((\Omega, F, P)\). So by Fontana [19, Proposition 4.10], there exists a process \( H \in L(ZX, G) \) such that

\[
M_t = \frac{1}{q_t^L} \left( M_0 + (H \cdot (ZX))_t \right) = \frac{Z_t}{q_t^L} \frac{M_0 + (H \cdot (ZX)_t)}{Z_t} \quad \text{a.s.} \tag{2.2}
\]

for all \( t \in [0, T] \). Furthermore, due to the martingale representation property of \( S \) on \((\Omega, F, Q)\), there exists a process \( \theta \in L(S, F) \) such that \( 1/Z = 1 + \theta \cdot S \). For each
\( n \in \mathbb{N} \), define \( H^n := H^1_{\|H\| \leq n} \). Using integration by parts and the associativity of the stochastic integral, we have that

\[
\frac{M_0 + H^n \cdot (ZX)}{Z} = M_0 + \left( M_0 + (H^n \cdot (ZX))_\theta \right) \cdot \frac{1}{Z} + \frac{H^n}{Z} \cdot (ZX) + H^n \cdot \left[ ZX, \frac{1}{Z} \right] \\
= M_0 + \left( \left( M_0 + (H^n \cdot (ZX))_\theta \right) \cdot S + H^n \cdot X - (H^n)^T X \cdot Z \cdot \frac{1}{Z} \right) \\
= M_0 + K^n \cdot S,
\]

where the \( \mathbb{R}^d \)-valued process \( K^n = (K^n_i)_{i \in [0,T]} \) is defined by

\[
K_i^{n,i} := \left( M_0 + (H^n \cdot (ZX))_\theta \right)_i - (H^n)^T X_i \cdot Z_i \cdot \frac{1}{Z} + H_i^{n,i+1}
\]

for all \( i = 1, \ldots, d \) and \( t \in [0, T] \). The fact that \( H \in L(ZX, G) \) implies that the stochastic integrals \( H^n \cdot (ZX) \) converge to \( H \cdot (ZX) \) in the semimartingale topology as \( n \to +\infty \) (compare with Rheinländer and Schweizer \([34, Proposition 8]\)). Hence in view of Jacod and Shiryaev \([24, Proposition III.6.26]\), the processes \( K^n \cdot S = (M_0 + H^n \cdot (ZX))/Z - M_0 \) also converge in the semimartingale topology to \( K \cdot S \) for some \( K \in L(S, G) \), thus proving that \( (M_0 + H \cdot (ZX))/Z = M_0 + K \cdot S \). Together with (2.2), this completes the proof. \( \square \)

2.3 Market viability under additional information

An informed agent is supposed to have access to the information generated by \( L \), i.e., to the enlarged filtration \( G \). Such an agent can trade in the same set of securities available in the ordinary financial market, but is allowed to rely on the information flow \( G \) when constructing portfolios. We call the tuple \((\Omega, G, \mathbb{P}; S)\) the informed financial market, recalling that Assumption 2.3 ensures that \( S \) is a semimartingale on \((\Omega, G, \mathbb{P})\) (see Lemma 2.4).

We are especially interested in the situation where the additional information generated by \( L \) yields arbitrage opportunities, so that NFLVR does not hold in the informed financial market \((\Omega, G, \mathbb{P}; S)\). However, we need to ensure that \((\Omega, G, \mathbb{P}; S)\) still represents a viable financial market. For this, the minimal requirement is the no unbounded profit with bounded risk (NUPBR) condition. By Takaoka and Schweizer \([37, Theorem 2.6]\), \( S \) satisfies NUPBR on \((\Omega, G, \mathbb{P})\) if and only if

\[
\mathcal{Z} := \{ Z \in \mathcal{M}_{loc}(\mathbb{P}, G) : Z > 0, \ Z_0 = 1 \ \text{and} \ ZS \in \mathcal{M}_{loc}(\mathbb{P}, G) \} \neq \emptyset.
\]

The set \( \mathcal{Z} \) is the set of all equivalent local martingale deflators (ELMDs) for \( S \) on \((\Omega, G, \mathbb{P})\).

The following result gives a complete characterisation of the no-arbitrage properties of the informed financial market \((\Omega, G, \mathbb{P}; S)\) in the sense of NUPBR and

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NFLVR. Our assumption of completeness of \((\Omega, \mathcal{F}, \mathbb{P}; S)\) (Assumption 2.1) enables us to derive necessary and sufficient conditions for NUPBR and NFLVR to hold on \((\Omega, \mathcal{G}, \mathbb{P})\), while existing results only provide sufficient conditions (see Amendinger et al. [5, Theorem 2.5], Aksamit et al. [2, Theorem 6] and Acciaio et al. [1, Theorem 1.12]).

**Theorem 2.6** Suppose that Assumptions 2.1 and 2.3 hold and that \(L^1(\Omega, \mathcal{F}_T, \mathbb{P})\) is separable.\(^2\) Then NUPBR holds on \((\Omega, \mathcal{G}, \mathbb{P})\) if and only if the set \(\{q^L_x = 0 < q^x\}\) is evanescent for \(\lambda\)-a.e. \(x \in E\). In this case, it holds that \(Z = \{Z/q^L\}\). Moreover, the following properties are equivalent:

(i) \(S\) satisfies NFLVR on \((\Omega, \mathcal{G}, \mathbb{P})\).

(ii) For all \(t \in [0, T]\), \(\lambda \ll v_t\) holds in the a.s. sense.

(iii) \(\mathbb{P}[q^L_x > 0] = 1\) for \(\lambda\)-a.e. \(x \in E\).

(iv) \(\mathbb{E}[1/q^L_x] = 1\).

(v) \(\mathbb{E}[Z_T/q^L_T] = 1\).

(vi) The process \(1/q^L = (1/q^L_t)_{t \in [0, T]}\) is a martingale on \((\Omega, \mathcal{G}, \mathbb{P})\).

(vii) The process \(N/q^L = (N_t/q^L_t)_{t \in [0, T]}\) is a martingale on \((\Omega, \mathcal{G}, \mathbb{P})\) for every \(N \in \mathcal{M}(\mathbb{P}, \mathcal{F})\).

**Proof** The sufficiency part of the first assertion follows directly from Acciaio et al. [1, Theorem 1.12]. To prove the necessity, define for \(x \in E\) the \(\mathcal{F}\)-stopping times

\[
\zeta^x := \inf\{t \in [0, T] : q^x_t = 0\}, \quad \eta^x := \zeta^x 1_{\{q^x_{\zeta^x} = 0\}} + (+\infty) 1_{\{q^x_{\zeta^x} = 0\}}
\]

and suppose there exists a set \(B \in \mathcal{B}_E\) with \(\lambda(B) > 0\) such that \(\mathbb{P}[\eta^x < +\infty] > 0\) for all \(x \in B\). By Lemma 2.4, we have \(\eta^x = \zeta^x = +\infty\) a.s. For each \(x \in B\), we define the \(\mathcal{F}\)-martingale \(M^x := -(\mathbf{1}_{[\eta^x, T]} - (\mathbf{1}_{[\eta^x, T]})^p)\), where \(A^p\) denotes the dual \(\mathcal{F}\)-predictable projection of \(A\). Since \(L^1(\Omega, \mathcal{F}_T, \mathbb{P})\) is separable, Stricker and Yor [36, Proposition 4] ensure the existence of a \((\mathcal{P}(\mathcal{F}) \otimes \mathcal{B}_E)\)-measurable version of \((\mathbf{1}_{[\eta^x, T]})^p\). As a consequence of Assumption 2.1 together with Fontana [19, Proposition 4.9], there exists a \((\mathcal{P}(\mathcal{F}) \otimes \mathcal{B}_E)\)-measurable process \(H^x\) such that for each \(x \in E\) it holds that \(H^x \in L(S, \mathcal{F})\) and \(M^x = H^x \cdot S\). Moreover, the same arguments as in the proof of Fontana [19, Proposition 4.10] allow showing that the process \(H^L\) belongs to \(L(S, \mathcal{G})\) and that \(H^L \cdot S = M^L = (\mathbf{1}_{[\eta^L, T]})^p|_{x=L}\). The process \(H^L \cdot S\) is nonnegative, nondecreasing and by an application of (2.1),

\[
\mathbb{E}[(H^L \cdot S)_T] = \mathbb{E}[(\mathbf{1}_{[\eta^L, T]})^p_T|_{x=L}] = \int_E \mathbb{E}[q^x_T(\mathbf{1}_{[\eta^x, T]})^p_T]\lambda(dx) = \int_E \mathbb{E}[q^x_{\eta^x} 1_{\{\eta^x \leq T\}}]\lambda(dx) > 0,
\]

\(^2\)As mentioned by Stricker and Yor [36], separability of \(L^1(\Omega, \mathcal{F}_T, \mathbb{P})\) is always satisfied in practice; necessary and sufficient conditions are given in Auerhan et al. [9, Lemma on page 200].
where the third equality follows from He et al. [22, Theorems 5.32 and 5.33]. This contradicts the validity of NUPBR on \((\Omega, \mathcal{G}, \mathbb{P})\), thus proving the first assertion of the theorem. The fact that \(\mathcal{Z} = \{Z/q^L\} \) follows by Proposition 2.5 together with Corollary 2.1 in Stricker and Yan [35].

Let us now prove the second part of the theorem. The equivalence between properties (ii)–(vii) as well as the implication (ii) \(\Rightarrow\) (i) easily follows from Amendinger et al. [5, Proposition 2.3 and Theorem 2.5]. Therefore, we only need to show that under Assumption 2.1, any of the properties (ii)–(vii) is necessary for NFLVR to hold on \((\Omega, \mathcal{G}, \mathbb{P})\). To this end, we prove that (i) \(\Rightarrow\) (v). Arguing by contradiction, suppose that \(\mathbb{E}[Z_T/q^L_T] \neq 1\). Since \(Z/q^L\) is a supermartingale on \((\Omega, \mathcal{G}, \mathbb{P})\) (being a nonnegative local martingale, see Acciaio et al. [1, Proposition 3.4]), we must have \(\mathbb{E}[Z_T/q^L_T] < 1\). Define \(M_t = \mathbb{E}[Z_T/q^L_T|\mathcal{G}_t]\), for all \(t \in [0, T]\). By Proposition 2.5, there exists \(K \in L(S, \mathcal{G})\) such that \(M_t = Z_t/q^L_t(M_0 + (K \cdot S)_t)\) a.s. for all \(t \in [0, T]\). Note that

\[(K \cdot S)_t = \frac{q_t^L}{Z_t} M_t - M_0 \geq -M_0 \geq -1 \quad \text{a.s. for all } t \in [0, T],\]

where the last inequality follows from the \(\mathcal{G}\)-supermartingale property of \(Z/q^L\). Therefore, the strategy \(K\) is 1-admissible in the sense of Delbaen and Schachermayer [16]. Moreover, it holds that \((K \cdot S)_T = 1 - M_0 \geq 0\) a.s. and \(\mathbb{P}[(K \cdot S)_T > 0] > 0\) since \(\mathbb{E}[M_0] < 1\), thus showing that \(K\) is an arbitrage opportunity. \(\square\)

We remark that in Theorem 2.6, the separability assumption is only needed for proving the necessity of the evanescence of the set \(\{q^x = 0 < q^x_\lambda\}\), for \(\lambda\)-a.e. \(x \in E\), for NUPBR to hold on \((\Omega, \mathcal{G}, \mathbb{P})\). Motivated by Theorem 2.6, let us now introduce the following assumption.

**Assumption 2.7** The set \(\{q^x = 0 < q^x_\lambda\}\) is evanescent for \(\lambda\)-a.e. \(x \in E\).

We are especially interested in the case where the densities \(q^x\) can reach zero as this corresponds to the existence of arbitrage opportunities in the informed financial market \((\Omega, \mathcal{G}, \mathbb{P}; S)\). In general, the densities \(q^x\) can reach zero either in a continuous way or due to a jump to zero. Assumption 2.7 excludes a jump to zero behaviour. As shown in Theorem 2.6, under the present assumptions, the set of ELMDs for \(S\) on \((\Omega, \mathcal{G}, \mathbb{P})\) is nonempty and consists of the unique element \(Z/q^L\).

In view of Theorem 2.6, the additional information generates arbitrage opportunities if and only if the \(\mathcal{F}_T\)-conditional law \(\nu_T\) of \(L\) fails to be equivalent with respect to the unconditional law \(\lambda\). The failure of the equivalence means that there exist some scenarios that from the point of view of an ordinary agent are a priori possible (i.e., they have a strictly positive mass for the measure \(\lambda\)), but can be later revealed to be impossible (i.e., they can be assigned zero mass by the measure \(\nu_T\)). For an informed agent, such scenarios would be excluded already before the beginning of trading, thus providing a clear informational advantage. This phenomenon is clarified by the examples in Sect. 5.
3 Optimal consumption–investment problems under additional information

In this section, we study general optimal consumption–investment problems, allowing state-dependent utilities and intermediate consumption. Similarly as in Amendinger et al. [4], we allow a non-trivial initial information, represented by \(L\), with the additional feature of the possibility of arbitrage. For better readability, the technical proofs of the results in this section are deferred to the Appendix.

We fix a stochastic clock \(\kappa = (\kappa_t)_{t \in [0,T]}\), which is a nondecreasing càdlàg \(\mathcal{F}\)-adapted bounded process with \(\kappa_0 = 0\) and \(\mathbb{P}[\kappa_T > 0|\mathcal{G}_0] > 0\) a.s. The process \(\kappa\) represents the notion of time according to which consumption is assumed to occur.

A portfolio is defined as a triplet \(\Pi = (v, \vartheta, c)\), where \(v \in \mathbb{R}\) represents an initial capital, \(\vartheta = (\vartheta_t)_{t \in [0,T]}\) is an \(\mathbb{R}^d\)-valued \(S\)-integrable process representing the holdings in the \(d\) risky assets and \(c = (c_t)_{t \in [0,T]}\) is a nonnegative process representing the consumption rate. For an ordinary agent, the strategy \(\vartheta\) and the consumption process \(c\) are required to be measurable with respect to \(\mathcal{P}(\mathcal{F})\) and \(\mathcal{O}(\mathcal{F})\), respectively. On the other hand, an informed agent is allowed to construct portfolios by choosing \(\mathcal{P}(\mathcal{G})\)-measurable strategies \(\vartheta\) together with \(\mathcal{O}(\mathcal{G})\)-measurable consumption processes \(c\). The value process \(V^{v,\vartheta,c}_t = (V^{v,\vartheta,c}_t)_{t \in [0,T]}\) of \(\Pi = (v, \vartheta, c)\) is defined as

\[
V^{v,\vartheta,c}_t := v + \int_0^t \vartheta_u dS_u - \int_0^t c_u d\kappa_u \quad \text{for all } t \in [0, T].
\]

**Definition 3.1** Let \(H \in \{F, G\}\), \(k \in \mathbb{R}_+\) and \(v \in \mathbb{R}\). The set of \(H\)-admissible portfolios with initial capital \(v\) and allowable credit line \(k\), denoted by \(A^{H,k}(v)\), is defined as

\[
A^{H,k}(v) := \{(\vartheta, c) \in L(S, H) \times O_+(H) : V^{v,\vartheta,c}_t \geq -k \text{ a.s. for all } t \in [0, T] \quad \text{and} \quad V^{v,\vartheta,c}_T \geq 0 \text{ a.s.}\}.\]

According to Definition 3.1, we assume that investors have access to a finite and fixed credit line \(k\) over the investment horizon \([0, T]\) and are required to fully repay their debts by the terminal date \(T\). Observe that in the absence of arbitrage opportunities, the requirement \(V^{v,\vartheta,c}_T \geq 0\) a.s. automatically implies that \(V^{v,\vartheta,c}_T \geq 0\) a.s. for all \(t \leq T\) (see Delbaen and Schachermayer [16, Proposition 3.5]), so that \(A^{H,k}(v) = A^{H,0}(v)\) for all \(k \in \mathbb{R}_+\). However, this is no longer true in the presence of arbitrage opportunities. For \(k = 0\), we recover the usual notion of admissibility via nonnegative portfolios. Note that \(A^{F,k}(v) \subseteq A^{G,k}(v)\), meaning that every portfolio which is admissible for an ordinary agent is also admissible for an informed agent. This follows from the fact that \(L(S, F) \subseteq L(S, G)\), as a consequence of Jeulin [25, Proposition 2.1] together with Lemma 2.4.

We assume that preferences are defined with respect to intermediate consumption over \([0, T]\) and/or wealth at the terminal date \(T\). More specifically, we introduce a *utility stochastic field* \(U = U(\omega, t, x) : \Omega \times [0, T] \times \mathbb{R}_+ \rightarrow \mathbb{R} \cup \{-\infty\}\) satisfying the following requirements.

**Assumption 3.2** For every \((\omega, t) \in \Omega \times [0, T]\), the function \(x \mapsto U(\omega, t, x)\) is strictly concave, strictly increasing, continuously differentiable on \((0, +\infty)\) and, with
$U'$ denoting the derivative of $U$ with respect to $x$, satisfies the Inada condition

$$\lim_{x \to +\infty} U' (\omega, t, x) = 0.$$ 

By continuity, we assume that $U (\omega, t, 0) = \lim_{x \downarrow 0} U (\omega, t, x)$. Finally, for every $x \geq 0$, the stochastic process $U (\cdot, \cdot, x)$ is $\mathcal{O} (\mathbb{F})$-measurable.

**In the following**, we always assume that a utility stochastic field satisfies Assumption 3.2. For $H \in \{F, G\}$, we define the set of consumption processes

$$\mathcal{C}^{H,k} (v) := \{ c \in \mathcal{O}_+ (H) : \exists \vartheta \in L (S, H) \text{ such that } (\vartheta, c) \in A^{H,k} (v) \}$$

consisting of all consumption plans that can be financed by portfolios with initial capital $v$ respecting the allowable credit line $k$. The optimal consumption–investment problem of an agent having access to the information flow $H$ and with initial capital $v$ is defined as

$$u^{H,k} (v) := \sup_{c \in \mathcal{C}^{H,k} (v)} \mathbb{E} \left[ \int_0^T U (u, c_u) d\kappa_u \right],$$

with the convention $\mathbb{E} [\int_0^T U (u, c_u) d\kappa_u] = -\infty$ if $\mathbb{E} [\int_0^T U^- (u, c_u) d\kappa_u] = +\infty$. We also define the related $\mathcal{H}_0$-conditional optimisation problem

$$\text{ess sup}_{c \in \mathcal{C}^{H,k} (v)} \mathbb{E} \left[ \int_0^T U (u, c_u) d\kappa_u \bigg| \mathcal{H}_0 \right],$$

with the analogous convention that we set $\mathbb{E} [\int_0^T U (u, c_u) d\kappa_u] = -\infty$ on the set $\{ \mathbb{E} [\int_0^T U^- (u, c_u) d\kappa_u | \mathcal{H}_0] = +\infty \}$. Note that an element $c \in \mathcal{C}^{H,k} (v)$ attains the supremum in (3.1) if it attains the supremum in (3.2) (see e.g. Amendinger et al. [4, Sect. 4]). We also remark that the set $\mathcal{C}^{H,k} (v)$ is closed in the topology of convergence in measure ($d\kappa \otimes \mathbb{P}$) (see Chau et al. [11] and compare also with Lemma 3.3 below).

The following lemma provides a characterisation of the set of financeable consumption plans. For notational convenience, we define the processes $Z^F = (Z^F_t)_{t \in [0, T]}$ and $Z^G = (Z^G_t)_{t \in [0, T]}$ by

$$Z^F_t := Z_t \quad \text{and} \quad Z^G_t := Z_t / q^L_t \quad \text{for all } t \in [0, T].$$

**Lemma 3.3** Suppose that Assumptions 2.1, 2.3 and 2.7 hold. Let $H \in \{F, G\}$, $k \in \mathbb{R}_+$ and $v \in \mathbb{R}$. Then for every consumption process $c \in \mathcal{O}_+ (H)$, it holds that $c \in \mathcal{C}^{H,k} (v)$ if and only if $\mathbb{E} [\int_0^T Z^H_u c_u d\kappa_u | \mathcal{H}_0] \leq v + k (1 - \mathbb{E} [Z^H_T | \mathcal{H}_0])$ a.s.

**Remark 3.4** It is important to observe that the credit line (or allowable leverage) $k$ plays no role in the characterisation of financeable consumption plans if and only if $Z^H \in \mathcal{M} (\mathbb{P}, H)$. In turn, this can be easily shown to imply that $u^{H,k} (v) = u^{H,0} (v)$.

---

3For ease of notation, we omit to indicate explicitly the dependence on $\omega$ in the utility stochastic field $U$. 

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Proposition 3.5
Suppose that Assumptions 2.1, 2.3, 2.7 and 3.2 hold. Let \( H \in \{ F, G \} \), \( k \in \mathbb{R}_+ \) and \( v \geq -k(1 - \| \mathbb{E}[Z_T^H|\mathcal{H}_0] \|_{\infty}) =: v_k^H \). Suppose that there exists an \( \mathcal{H}_0 \)-measurable random variable \( \Lambda^H,k(v) : \Omega \rightarrow (0, +\infty) \) such that
\[
\mathbb{E}
\left[
\int_0^T Z_u^H I(u, \Lambda^H,k(v)Z_u^H) d\kappa_u \mid \mathcal{H}_0
\right] = v + k(1 - \mathbb{E}[Z_T^H|\mathcal{H}_0]) \quad \text{a.s.} \quad (3.3)
\]
and such that the process \( (I(t, \Lambda^H,k(v)Z_t^H))_{t \in [0,T]} \) satisfies the integrability condition
\[
\int_0^T U^{-}(-u, I(u, \Lambda^H,k(v)Z_u^H)) d\kappa_u \in L^1(\mathbb{P}).
\]
Then the optimal consumption process \( c^H = (c_t^H)_{t \in [0,T]} \) solving (3.2) with initial capital \( v \) and allowable leverage \( k \) is given by
\[
c_t^H = I(t, \Lambda^H,k(v)Z_t^H) \quad \text{for all } t \in [0, T].
\]
If \( u^H,k(v) < +\infty \), the strict concavity of \( U \) implies that the optimal consumption process \( c^H = (c_t^H)_{t \in [0,T]} \) is unique up to a \( (d\kappa \otimes \mathbb{P}) \)-nullset. The associated optimal strategy \( \vartheta^H \in L(S, H) \) is given by the integrand appearing in the representation of the local martingale \( M = (M_t)_{t \in [0,T]} \) defined by
\[
M_t := \mathbb{E}
\left[
\int_t^T Z_s^H c_s^H d\kappa_s \mid \mathcal{H}_t
\right] + Z_t^H \int_0^t c_u^H d\kappa_u + k(\mathbb{E}[Z_T^H|\mathcal{H}_t] - Z_t^H)
\]

\footnote{Similarly as above, for simplicity of notation, we omit to write explicitly the dependence on \( \omega \) in the stochastic field \( I \).}
for \( t \in [0, T] \). Note that the optimal consumption does not depend on the allowable leverage \( k \) if NFLVR holds on \((\Omega, \mathbf{H}, \mathbb{P}; S)\). The quantity \( v^H_k \) introduced in Proposition 3.5 represents the maximum amount of liabilities with which an agent can start at \( t = 0 \). For \( v < v^H_k \), there does not exist any strategy which can fully insure the agent against his liabilities at \( T \), so that \( C^{H,k}(v) = \emptyset \).

**Remark 3.6** Let \( 0 \leq k_1 < k_2 \) and suppose that for some \( v > v^H_{k_1} \) there exist \( \mathcal{H}_0 \)-measurable random variables \( \Lambda^{H,k_1}(v) \) and \( \Lambda^{H,k_2}(v) \) satisfying (3.3). Because \( \mathbb{P}[\kappa_T > 0|\mathcal{H}_0] > 0 \) a.s., it can be shown that \( \Lambda^{H,k_1}(v) \geq \Lambda^{H,k_2}(v) \) a.s., with strict inequality holding on \( \{\mathbb{E}[Z^H_T|\mathcal{H}_0] < 1\} \). This means that in the presence of arbitrage, a deeper credit line yields a higher consumption rate. In turn, this implies that \( u^{H,k}(v) \) is strictly increasing with respect to \( k \) if \( \mathbb{E}[Z^H_T] < 1 \).

**Remark 3.7** The existence of an \( \mathcal{H}_0 \)-measurable random variable \( \Lambda^{H,k}(v) \) solving (3.3) is ensured if \( \int_0^T Z^H_u I(u, yZ^H_u) d\kappa_u \in L^1(\mathbb{P}) \) for all \( y > 0 \). This corresponds to a classical condition in the theory of expected utility maximisation (see Amendinger [3, Lemma 5.2] and Karatzas and Shreve [27, Chap. 3]).

On the basis of the previous general results, we now derive explicit solutions to optimal consumption–investment problems in the case of logarithmic, power and exponential utility functions. Besides allowing intermediate consumption, this generalises Amendinger et al. [4, Corollary 4.7] to the case where the additional information can generate arbitrage opportunities. The following result will be used in Sect. 4 for the explicit computation of the value of informational arbitrage.

**Corollary 3.8** Suppose Assumptions 2.1, 2.3 and 2.7 hold. Let \( k \in \mathbb{R}_+, \mathbf{H} \in \{\mathbf{F}, \mathbf{G}\} \) and \( v > v^H_k \). Then the optimal expected utilities in problem (3.1) for logarithmic, power and exponential utility functions are explicitly given as follows:

(i) Let \( U(\omega, t, x) = \log x \) for \((\omega, t, x) \in \Omega \times [0, T] \times (0, +\infty) \). If we have \( \int_0^T \log(1/Z^H_u) d\kappa_u \in L^1(\mathbb{P}) \), then

\[
\begin{align*}
u^{H,k}(v) &= \mathbb{E}\left[ \log \left( v + k(1 - \mathbb{E}[Z^H_T|\mathcal{H}_0])\kappa_T \right) - \mathbb{E}[\kappa_T \log \mathbb{E}[\kappa_T|\mathcal{H}_0]] \right] \\
&\quad + \mathbb{E}\left[ \int_0^T \log \frac{1}{Z^H_u} d\kappa_u \right].
\end{align*}
\]

(ii) Let \( U(\omega, t, x) = x^p/p \) with \( p \in (0, 1) \) for \((\omega, t, x) \in \Omega \times [0, T] \times (0, +\infty) \). If \( \mathbb{E}[\int_0^T (Z^H_u)^{p/(p-1)} d\kappa_u|\mathcal{H}_0] < +\infty \) a.s., then

\[
\begin{align*}
u^{H,k}(v) &= \frac{1}{p} \mathbb{E}\left[ (v + k(1 - \mathbb{E}[Z^H_T|\mathcal{H}_0]))^p \mathbb{E}\left[ \int_0^T (Z^H_u)^{\frac{p}{p-1}} d\kappa_u \bigg| \mathcal{H}_0 \right]^{1-p} \right].
\end{align*}
\]

and \( u^{H,k}(v) < +\infty \) if \( \mathbb{E}[\int_0^T (Z^H_u)^{\frac{p}{p-1}} d\kappa_u|\mathcal{H}_0]^{1-p} \in L^1(\mathbb{P}) \).

(iii) Let \( U(\omega, t, x) = -e^{-\alpha x} \) with \( \alpha > 0 \) for \((\omega, t, x) \in \Omega \times [0, T] \times (0, +\infty) \). Then

\[
\begin{align*}\nu^{H,k}(v) &= -\frac{1}{\alpha} \mathbb{E}\left[ \int_0^T \left( \Lambda^{H,k}(v)Z^H_u \wedge \alpha \right) d\kappa_u \right].
\end{align*}
\]
where the $\mathcal{H}_0$-measurable random variable $\Lambda^{H,k}(v)$ is the a.s. unique solution to the equation

$$
\frac{1}{\alpha} \mathbb{E} \left[ \int_0^T Z_u^H \left( \log \frac{\alpha}{\Lambda^{H,k}(v) Z_u^H} \right) \, d\kappa_u \bigg| \mathcal{H}_0 \right] = v + k (1 - \mathbb{E}[Z_T^H|\mathcal{H}_0]). \tag{3.7}
$$

Observe that the optimal expected utilities do not depend on $k$ if and only if there are no arbitrage opportunities in $(\Omega, H, \mathbb{P}; S)$, in line with Remark 3.4. On the other hand, in the presence of arbitrage, the optimal expected utilities are strictly increasing in $k$, reflecting the fact that higher levels of consumption can be financed by taking leveraged positions in an arbitrage strategy. Equation (3.7) can be explicitly solved in some simple models. In particular, if $d\kappa_u = \delta_T (du)$ and $k = 0$, a sufficient condition is that $\log Z_T^H \leq \mathbb{E}[Z_T^H \log Z_T^H|\mathcal{H}_0]/\mathbb{E}[Z_T^H|\mathcal{H}_0]$ a.s. This last condition is always satisfied if $Q = \mathbb{P}$ and $L$ is a discrete $\mathcal{F}_T$-measurable random variable generating arbitrage opportunities in $(\Omega, G, \mathbb{P}; S)$ (see the examples in Sects. 5.1 and 5.2).

**Remark 3.9** Consider the classical setting where $d\kappa_u = \delta_T (du)$ and the utility is given by $U(\omega, t, x) = \log x$, corresponding to maximisation of expected logarithmic utility from terminal wealth. Suppose that $u^{G,k}(v) < +\infty$ for some $k \in \mathbb{R}_+$ and $v > 0$. Part (i) of Corollary 3.8 implies that

$$
u^{G,k}(v) - \nu^{F,k}(v) = \mathbb{E} \left[ \log \left( v + k (1 - \mathbb{E}[Z_T^G|G_0]) \right) \right]
+ \mathbb{E}[\log(1/Z_T^G)] - \log v - \mathbb{E}[\log(1/Z_T^F)]
= \mathbb{E} \left[ \log \left( 1 + \frac{k}{v} (1 - Q[q_T > 0]|_{x=L}) \right) \right] + \mathbb{E}[\log q_T^L]. \tag{3.8}
$$

which represents the utility gain of an informed agent with allowable leverage $k$. This generalises Amendinger et al. [5, Theorem 3.7], where (3.8) has been obtained in the case $k = 0$ under the additional assumption that the densities $q^x$ are a.s. strictly positive and continuous. Note also that

$$
u^{G,k}(v) - \nu^{F,k}(v) \geq \nu^{G,0}(v) - \nu^{F,0}(v) \geq - \log \mathbb{E}[1/q_T^L] \geq 0.
$$

In particular, these inequalities imply that the utility gain is always strictly positive whenever the additional information $L$ yields arbitrage opportunities in $(\Omega, G, \mathbb{P}; S)$ (see Theorem 2.6).

For continuous $S$, the logarithmic utility gain of an informed agent has been studied in detail in Ankirchner et al. [7] and Ankirchner and Imkeller [8]. In particular, Ankirchner and Imkeller [8, Theorem 2.13] show that the utility gain $\mathbb{E}[\log q_T^L]$ can be expressed in terms of the information drift of $G$ with respect to $F$, even when $L$ generates arbitrage in $(\Omega, G, \mathbb{P}; S)$. Moreover, as a consequence of Ankirchner et al. [7, Theorem 5.13], the quantity $\mathbb{E}[\log q_T^L]$ corresponds to the Shannon information between $L$ and $\mathcal{F}_T$. In the case where $L$ is a discrete $\mathcal{F}_T$-measurable random variable, as in Sects. 5.1 and 5.2, and $k = 0$, the utility gain equals the entropy of the random variable $L$, i.e., $\mathbb{E}[\log q_T^L] = - \sum_{x \in E} \mathbb{P}[L = x] \log \mathbb{P}[L = x]$ (see also Springer).
The value of informational arbitrage

Ankirchner et al. [7, Remark 5.14] and Ankirchner [6, Theorem 12.6.1]. We point out that the results of [6, 7, 8] are not limited to initial filtration enlargements, but can be applied to more general situations. On the other hand, [6, 7, 8] work under the assumption that $S$ is continuous, while we allow a general (possibly discontinuous) semimartingale.

For utility functions other than the logarithmic one, the utility gain of an informed agent admits a representation in terms of an $f$-divergence, as shown in Ankirchner [6, Chap. 12] under the assumption that $S$ has continuous paths. In particular, for a power utility function, it can be verified that the optimal expected utility $u^{G,0}(v)$ given in (3.5) coincides with the expression given in Ankirchner [6, Proposition 12.5.1].

4 The utility indifference value of additional information

By relying on the results established in the previous section, we are now in a position to study and compute the value of an additional information which potentially enables an informed agent to achieve arbitrage opportunities. Inspired by Amendinger et al. [4], we introduce the following definition.

**Definition 4.1** For $k \in \mathbb{R}_+$ and $v > 0$, the utility indifference value of the additional information $L$ is defined as a solution $\pi = \pi^{U,k}(v) \in \mathbb{R}_+$ to the equation

$$u^{F,k}(v) = u^{G,k}(v - \pi).$$

As explained in the introduction, the value $\pi^{U,k}(v)$ is such that an investor is indifferent between two alternatives: (i) invest optimally the total initial capital $v$ on the basis of the publicly available information $F$; (ii) acquire the additional information $L$ at the price $\pi^{U,k}(v)$ and invest optimally the residual capital $v - \pi^{U,k}(v)$, possibly exploiting the arbitrage opportunities generated by the knowledge of $L$. If the additional information $L$ allows an investor to achieve arbitrage, then we call the quantity $\pi^{U,k}(v)$ the indifference value of informational arbitrage.

Under natural assumptions, the utility indifference value $\pi^{U,k}(v)$ exists and is unique as long as the optimal consumption–investment problem of an informed agent is well posed.

**Theorem 4.2** Suppose that Assumptions 2.1, 2.3, 2.7 and 3.2 hold. Suppose furthermore that $u^{F,0}(v) > -\infty$ for every $v > 0$ and that the assumptions of Proposition 3.5 are satisfied for every $k \in \mathbb{R}_+$, $v > v^H_k$ and $H \in \{F, G\}$. If $u^{G,k}(v_0) < +\infty$ for some $v_0 > v^G_k$, then for every $v > 0$, the following hold:

(i) If $\lim_{w \downarrow v^G_k} u^{G,k}(w) < u^{F,0}(v)$, the utility indifference value $\pi^{U,k}(v)$ exists and is unique.

(ii) The map $k \mapsto \pi^{U,k}(v)$ is strictly increasing if and only if $\mathbb{E}[1/q^T] < 1$.

(iii) If $\int_E (\mathbb{E}[\int_0^T 1_{\{q^T_\omega = 0\}} \mathbf{d} \kappa_t] + k \mathbb{P}(q^T_\omega = 0)) \lambda(dx) > 0$ and $U'(\omega, t, 0) = +\infty$ for all $(\omega, t) \in \Omega \times [0, T]$, then it always holds that $\pi^{U,k}(v) > 0$. 

\[ Springer \]
Proof (i) By concavity of \( U \), the assumption that \( u^{G,k}(v_0) < +\infty \) for some \( v_0 > v_k^G \) implies that the function \( u^{G,k} \) is concave and \( u^{G,k}(v) < +\infty \) for all \( v \geq v_k^G \). Moreover, it holds that \( u^{G,k}(v) \geq u^{F,k}(v) = u^{F,0}(v) > -\infty \). Therefore, for every \( v > 0 \), (4.1) admits a unique nonnegative solution \( \pi^{U,k}(v) \) if the function \( u^{G,k} \) is continuous, strictly increasing and satisfies \( \lim_{w \downarrow 0} u^{G,k}(w) < u^{F,0}(v) \). Under the present assumptions, these properties hold. Indeed, by concavity, the function \( u^{G,k} \) is continuous on \( (v_k^G, +\infty) \). As a consequence of (3.3) and since \( I(\omega, t, \cdot) \) is decreasing for all \( (\omega, t) \in \Omega \times [0, T] \), it holds that \( \Lambda^{G,k}(v + \delta) < \Lambda^{G,k}(v) \) a.s. for every \( v > v_k^G \) and \( \delta > 0 \). In turn, by Proposition 3.5, this implies that \( u^{G,k} \) is strictly increasing.

(ii) If \( \mathbb{E}[1/q_T^F] < 1 \), then \( \mathbb{E}[Z_T^G] < 1 \) (see Theorem 2.6). As explained in Remark 3.6, this entails that \( k \mapsto u^{G,k}(v) \) is strictly increasing. In turn, in view of Definition 4.1, this implies that \( k \mapsto \pi^{U,k}(v) \) is strictly increasing for every \( v > 0 \). Conversely, if the map \( k \mapsto \pi^{U,k}(v) \) is strictly increasing, we necessarily have \( u^{G,k}(v) > u^{F,0}(v) \). In view of Remark 3.4 together with Theorem 2.6, this implies that \( \mathbb{E}[1/q_T^F] < 1 \).

(iii) It suffices to show that if \( \int_E (\mathbb{E}[\int_0^T 1_{|q^F_T| = 0} d\kappa])) + k\mathbb{P}[q_T^F = 0]) \lambda(dx) > 0 \), then \( u^{G,k}(v) > u^{F,0}(v) \). Under the present assumptions and in view of Lemma 3.3, there exists a pair \((\tilde{d}^F, c^F) \in \mathcal{A}^{F,0}(v) \) such that \( c^F \) solves problem (3.1) in \( F \). Hence

\[
M_0 := \mathbb{E} \left[ \int_0^T Z_u^G c_u^F d\kappa_u + k Z_T^G \right] \leq v + k \quad \text{a.s.}
\]

By (2.1), the random variable \( M_0 \) can be computed explicitly. Indeed, let \( h : E \to \mathbb{R} \) be an arbitrary \( B_E \)-measurable bounded function. Then

\[
\mathbb{E} \left[ h(L) \left( \int_0^T Z_u^G c_u^F d\kappa_u + k Z_T^G \right) \right] = \int_E h(x) \mathbb{E} \left[ \int_0^T \frac{Z_u^F}{q_u^F} 1_{|q_u^F| > 0} c_u^F d\kappa_u + k Z_T^F 1_{|q_T^F| > 0} \right] \lambda(dx)
\]

where the second equality follows from He et al. [22, Theorem 5.32]. We have thus shown that

\[
M_0 = \mathbb{E} \left[ \int_0^T Z_u^F 1_{|q_u^F| > 0} c_u^F d\kappa_u + k Z_T^F 1_{|q_T^F| > 0} \right] |_{x=L} \quad \text{a.s.}
\]

Since \( U'(\omega, t, 0) = +\infty \) for all \( (\omega, t) \in \Omega \times [0, T] \), the process \( c^F \) is strictly positive \((d\kappa \otimes \mathbb{P})\)-a.e., and hence \( \int_E \mathbb{E}[\int_0^T 1_{|q^F_T| = 0} d\kappa]) + k\mathbb{P}[q_T^F = 0]) \lambda(dx) > 0 \) implies that \( \mathbb{P}[M_0 < v + k] > 0 \). Define then an \( \mathcal{O}_+(G) \)-measurable process \( \hat{c} = (\hat{c}_t)_{t \in [0, T]} \) by

\[
\hat{c}_t := c^F_t + \frac{v + k - M_0}{Z_t^G \mathbb{E}[kT | G_0]} \quad \text{for all } t \in [0, T].
\]
By Lemma 3.3, \( \hat{c} \in C^{G,k}(v) \). Furthermore, since \( \mathbb{P}[\hat{c}_t > c^F_t] > 0 \) for all \( t \in [0, T] \), it holds that
\[
u^G(v) = \mathbb{E} \left[ \int_0^T U(u, \hat{c}_u) d\kappa_u \right] > \mathbb{E} \left[ \int_0^T U(u, c^F_u) d\kappa_u \right] = \nu^F(v),
\]
which completes the proof. \( \square \)

Note that \( \nu^F(v) > -\infty \) for every \( v > 0 \) always holds if \( U \) is bounded from below by a real-valued function (in particular, if \( U \) is deterministic). The condition in part (i) of Theorem 4.2 is always satisfied in the absence of leverage (i.e., if \( k = 0 \)).

Part (ii) of Theorem 4.2 shows that whenever the additional information \( L \) yields arbitrage, the indifference value of informational arbitrage is strictly increasing in the credit line \( k \). Having access to a deeper line of credit, an informed agent can take more leveraged positions, yielding arbitrage profits which can be scaled up to the limit of the allowable leverage.

The condition
\[
\int_E \left( \mathbb{E} \left[ \int_0^T 1_{\{q^L_t = 0\}} d\kappa_t \right] + k \mathbb{P}[q^T_T = 0] \right) \lambda(dx) > 0
\]
in part (iii) of Theorem 4.2 implies that an informed agent can finance any consumption plan \( c \in C^{F,k}(v) \) at a cost smaller than \( v \), using the remaining resources to increase consumption. This is possible since an informed agent does not need to finance consumption in states of the world incompatible with the realisation of \( L \) observed at \( t = 0 \). In this case, an investor is always willing to pay a strictly positive price to learn the additional information, regardless of the specific preference structure.

The conclusions of Theorem 4.2 always hold for the utility functions considered in Corollary 3.8, under suitable integrability conditions. This enables us to obtain explicit expressions for the utility indifference value of the additional information \( L \) as shown in the next proposition, which generalises Theorem 5.3 of Amendinger et al. [4] and follows as a direct consequence of Corollary 3.8.

Proposition 4.3 Suppose that Assumptions 2.1, 2.3 and 2.7 hold. Assume furthermore that \( k = 0 \). Then the utility indifference value of the additional information \( L \) is explicitly given as follows:

(i) Let \( U(\omega, t, x) = \log x \) for \( (\omega, t, x) \in \Omega \times [0, T] \times (0, +\infty) \). If we have \( \int_0^T \log(q^L_u / Z_u) d\kappa_u \in L^1(\mathbb{P}) \), then for every \( v > 0 \),
\[
\pi^{\log}(v) = v \left( 1 - \exp \left( \frac{1}{\mathbb{E}[\kappa_T]} \left( \chi^G - \chi^F - \mathbb{E} \left[ \int_0^T \log q^L_u d\kappa_u \right] \right) \right) \right), \tag{4.2}
\]
where \( \chi^H := \mathbb{E}[\kappa_T \log \mathbb{E}[\kappa_T | H_0]] \) for \( H \in \{ F, G \} \).

(ii) Let \( U(\omega, t, x) = x^p / p \) with \( p \in (0, 1) \) for \( (\omega, t, x) \in \Omega \times [0, T] \times (0, +\infty) \). If \( \mathbb{E}[\int_0^T (Z_u / q^L_u)^{p/p} d\kappa_u | G_0]^{1-p} \in L^1(\mathbb{P}) \), then for every \( v > 0 \),
\[
\pi^{pwt}(v) = v \left( 1 - \frac{1}{\mathbb{E}[\int_0^T Z_u^{p/p} d\kappa_u | G_0]^{1-p} / \mathbb{E}[\mathbb{E}[\int_0^T (Z_u / q^L_u)^{p/p} d\kappa_u | G_0]^{1-p}]^{1/p}} \right). \tag{4.3}
\]
In general, the utility indifference value of the additional information cannot be computed in explicit form for exponential utility, and for \( k > 0 \) also not for logarithmic and power utilities, as can be seen from (3.4) and (3.6). However, in view of part (ii) of Theorem 4.2, (4.2) and (4.3) represent lower bounds for the logarithmic and power indifference values when \( k > 0 \).

Proposition 4.3 reveals several features of the value of informational arbitrage in the case of CRRA utility functions. First, the relative indifference value \( \pi^{U,0}(v)/v \) is constant. Furthermore:

- If \( \int E[\int_0^T 1_{\{q_T^x=0\}}d\kappa]\lambda(dx) > 0 \), then \( \pi^{\log}(v) \) and \( \pi^{\text{pwr}}(v) \) are always strictly increasing with respect to \( v \). In other words, the value of informational arbitrage is strictly increasing with respect to initial wealth, in line with the analysis of Liu et al. [30].

- In the case of logarithmic utility, the indifference value \( \pi^{\log}(v) \) is lower when preferences are defined over intermediate consumption rather than terminal wealth only, confirming some empirical findings of Liu et al. [30]. This follows from the observation that by Jensen’s inequality together with the \( G \)-supermartingale property of \( 1/q^L_T \), we have

\[
\mathbb{E} \left[ \int_0^T \log q^L_T d\kappa_u \right] \leq \mathbb{E} \left[ \int_0^T \log q^F_T d\kappa_u \right] = \mathbb{E} [\kappa_T \log q^F_T].
\]

- Jensen’s inequality applied to the convex function \( x \mapsto x \log x \) implies that the term \( \chi^G - \chi^F \) appearing in (4.2) is nonnegative, with \( \chi^G = \chi^F \) if and only if \( \mathbb{E} [\kappa_T | G_0] = \mathbb{E} [\kappa_T] \) a.s. In turn, this means that if the additional information \( L \) has predictive power on \( \kappa_T \), then the indifference value \( \pi^{\log}(v) \) is lower than in the case where \( L \) has no predictive power on \( \kappa_T \). Note that \( \chi^G = \chi^F \) if \( \kappa_T \) is deterministic, as in the case of utility from terminal wealth.

**Remark 4.4** In the case of utility from terminal wealth (so that \( d\kappa_u = \delta_T (du) \)), it can be easily verified that (4.2) and (4.3) reduce to the expressions in Theorem 5.3 of Amendinger et al. [4] whenever one of the equivalent conditions of the second part of Theorem 2.6 holds, i.e., whenever the additional information does not lead to arbitrage in \( (\Omega, G, \mathbb{P}; S) \). For \( d\kappa_u = \delta_T (du) \), (4.2) reduces to

\[
\pi^{\log}(v) = v \left( 1 - \exp(-\mathbb{E} [\log q^F_T]) \right).
\]

In line with Theorem 4.2 (see also Remark 3.9), this confirms that the indifference value is always strictly positive if \( L \) generates arbitrage in \( (\Omega, G, \mathbb{P}; S) \). The quantity \( \pi^{\log}(v) \) can be expressed in terms of the Shannon information between \( L \) and \( F_T \), which reduces to the entropy of \( L \) whenever \( L \) is a discrete \( F_T \)-measurable random variable. For a power utility function, it can be shown that the indifference value computed in (4.3) can be expressed in terms of an \( f \)-divergence, along the lines of Ankirchner [6, Chap. 12].
In general, the indifference value of the additional information depends on the considered stochastic utility field. However, in some special cases (for instance in Example 1.1), the indifference value is a universal value which does not depend on the preference structure. This situation is clarified by the next theorem. We denote by $\mathcal{U}$ the class of all strictly increasing and strictly concave deterministic utility functions $U : \mathbb{R}_+ \rightarrow \mathbb{R} \cup \{-\infty\}$. In the statement of the following theorem, we denote by $u^{H,k}(v)$ the value function associated to problem (3.1) in the case of expected utility from consumption only at date $T$ (i.e., terminal wealth) with utility function $U$.

**Theorem 4.5** Suppose that Assumptions 2.3 and 2.7 hold. Suppose furthermore that Assumption 2.1 holds with $Q = P$ and that $d\kappa_u = \delta_T(du)$. Then the following three conditions are equivalent:

(i) $P[q_L^T = q] = 1$ for some constant $q \geq 1$.

(ii) For every $k \in \mathbb{R}_+$ and $v > 0$, there exists a universal value $\pi^k(v) \in [0, v + k)$ such that

$$u^{G,k}(v - \pi^k(v)) = u^{F,k}(v) \quad \text{for all } U \in \mathcal{U}.$$

(iii) For every $v > 0$, there exists a universal value $\pi^0(v) \in [0, v)$ such that

$$u^{G,0}(v - \pi^0(v)) = u^{F,0}(v) \quad \text{for all } U \in \mathcal{U}.$$

In those cases, for every $U \in \mathcal{U}$, $k \in \mathbb{R}_+$ and $v > 0$, the indifference value $\pi^k(v)$ is always given by

$$\pi^k(v) = (v + k)\left(1 - \frac{1}{q}\right), \quad (4.4)$$

and the optimal wealth process $V^G = (V^G_t)_{t \in [0,T]}$ in problem (3.1) for $H = G$ is always given by

$$V^G_t = (v + k)\frac{q_L^T - k}{q_L^T} \quad \text{for all } t \in [0, T]. \quad (4.5)$$

**Proof** As a preliminary, observe that Jensen’s inequality and the assumption that $S \in \mathcal{M}_{\text{loc}}(\mathbb{P}, \mathbb{F})$ imply that $u^{F,k}(v) = U(v)$ for every $U \in \mathcal{U}$.

(i) $\Rightarrow$ (ii) Let $U$ be an arbitrary element of $\mathcal{U}$, $k \in \mathbb{R}_+$ and $v > 0$. Consider the consumption process $c^G = (c^G_t)_{t \in [0,T]}$ given by $c^G_t = v1_{t=T}$ for $t \in [0,T]$. Since $d\kappa_u = \delta_T(du)$ and $P[q_L^T = q] = 1$ with $q \geq 1$, Lemma 3.3 implies that $c^G \in C^{G,k}((v + k)/q - k)$. As a consequence, we have that

$$u^{G,k}((v + k)/q - k) \geq \mathbb{E}[U(c^G_T)] = U(v) \quad \text{for every } v > 0.$$

On the other hand, for any consumption process $c \in C^{G,k}((v + k)/q - k)$, we have

$$\mathbb{E}[U(c_T)] \leq U(\mathbb{E}[c_T]) = U(q \mathbb{E}[c_T/q_L^T]) \leq U\left(q((v + k)/q - k + k - k/q)\right) = U(v).$$
where the two inequalities follow respectively from Jensen’s inequality and from Lemma 3.3, since \( Q = P \) and \( d\kappa_u = \delta_T (du) \). We have therefore shown that we have \( u^G_k((v + k)/q - k) = \Pi U(v) = u^{F,k} U(v) \) for every \( U \in U \), thus proving that (ii) holds with the indifference value \( \pi^k(v) \) given as in (4.4).

(ii) \( \Rightarrow \) (iii) This trivially follows by taking \( k = 0 \) in (ii).

(iii) \( \Rightarrow \) (i) Consider the utility functions \( U_1(x) = \log x \) and \( U_2(x) = x^p/p \) for \( p \in (0, 1) \). For \( H \in \{ F, G \} \) and \( i \in \{ 1, 2 \} \), denote by \( u_i^{H,0}(v) \) the value function of the corresponding expected utility maximisation problem (3.1) for \( v > 0 \) and \( k = 0 \). Suppose that for every \( v > 0 \), there exists a value \( \pi^0(v) \) such that \( u_i^{G,0}(v - \pi^0(v)) = u_i^{F,0}(v) \) for every \( i \in \{ 1, 2 \} \) and all \( p \in (0, 1) \). In particular, this implies \( u_i^{G,0}(v - \pi^0(v)) < +\infty \) for \( i \in \{ 1, 2 \} \), and \( \pi^0(v) = \pi^{\log}(v) = \pi^{\text{pwr}}(v) \) for all \( p \in (0, 1) \), using the notation introduced in Proposition 4.3. The assumptions of Proposition 4.3 are therefore satisfied, and in view of (4.2) and (4.3), we get

\[
\exp(\mathbb{E} [\log q_T^L]) = \mathbb{E} [\mathbb{E} [(q_T^L)^{1/p} | G_0]^{1-p}]^{1/p}
\]

for all \( p \in (0, 1) \). By Jensen’s inequality, we have \( \exp(\mathbb{E} [\log q_T^L]) \leq \mathbb{E} [q_T^L] \). On the other hand, the function \( x \mapsto x^{1/(1-p)} \) is convex, and again by Jensen’s inequality,

\[
\mathbb{E} [\mathbb{E} [(q_T^L)^{1/p} | G_0]^{1-p}]^{1/p} \geq \mathbb{E} [\mathbb{E} [(q_T^L)^p | G_0]]^{1/p} = \mathbb{E} [(q_T^L)^p]^{1/p}.
\]

We have thus shown that

\[
\mathbb{E} [(q_T^L)^p]^{1/p} \leq \mathbb{E} [\mathbb{E} [(q_T^L)^{1/p} | G_0]^{1-p}]^{1/p} \leq \mathbb{E} [q_T^L]
\]

and \( \mathbb{E} [(q_T^L)^p]^{1/p} < +\infty \) for all \( p \in (0, 1) \). Therefore, \( \mathbb{E} [(q_T^L)^{1/p} | G_0]^{1-p} \) converges to \( \mathbb{E} [q_T^L] \) as \( p \to 1 \). In turn, this implies that for every \( v > 0 \),

\[
v(1 - e^{-\mathbb{E} [\log q_T^L]}) = \pi^{\log}(v) = \pi^{\text{pwr}}(v) = v(1 - \mathbb{E} [\mathbb{E} [(q_T^L)^{1/p} | G_0]^{1-p}]^{-1/p})
\]

as \( p \to 1 \). As a consequence, we obtain \( \mathbb{E} [\log q_T^L] = \log \mathbb{E} [q_T^L] \). Since the function \( x \mapsto \log x \) is strictly concave, this implies that there exists a strictly positive constant \( q \) such that \( \mathbb{P} [q_T^L = q] = 1 \). The fact that \( q \geq 1 \) follows since \( \mathbb{E} [1/q_T^L] \leq 1 \) by the supermartingale property of \( 1/q^L \) on \( (\Omega, G, \mathbb{P}) \).

It remains to show that the wealth process associated to the optimal consumption plan \( e^G \) constructed in the first part of the proof is given as in (4.5). This follows since by optimality, it holds that \( (V^G + k)/q^L \in \mathcal{M}(\mathbb{P}, G) \). \( \square \)

In the setting of the above theorem, the optimal strategy for an informed agent is given by a multiple of the process \( \phi \in L(S, G) \) appearing in the stochastic integral representation \( q^L = 1 + \phi \cdot S \). Under the conditions of Theorem 4.5, the constant payoff \( v = v - \pi^k(v) + (v + k - \pi^k(v))(\phi \cdot S)_T \) dominates, according to the second order stochastic dominance criterion, all possible outcomes of admissible portfolios for an informed agent.
Remark 4.6 The random variable $q^L_T$ is always a.s. constant whenever $L$ is an $\mathcal{F}_T$-measurable discrete random variable with uniform distribution on a finite set $E$ so that $\mathbb{P}[L = x] = 1/|E|$ for all $x \in E$ and the entropy of the random variable $L$ is given by $\log |E|$. Indeed, in this case, we have $q^L_T = 1_{\{L = x\}}/E$ for all $x \in E$ so that $q^L_T = |E|$. This also happens in Example 1.1, as we explain in detail in Sect. 5.1.

Remark 4.7 If there are no arbitrage opportunities in $(\Omega, \mathcal{G}, \mathbb{P}; S)$, the only case in which condition (i) of Theorem 4.5 holds is when the random variable $L$ is independent of $\mathcal{F}_T$. In this case, it is never attractive to buy the informational content of the random variable $L$ because it does not provide any useful information on the financial market.

The assumptions of Theorem 4.5 cannot be easily relaxed. Indeed, if $d\mathbb{P}_u = \delta_T(du)$ but $\mathbb{Q} \neq \mathbb{P}$, condition (i) does not suffice to ensure the existence of a universal indifference value, as can be shown by a simple modification of the example in Sect. 5.1. Similarly, even if $\mathbb{Q} = \mathbb{P}$, in the presence of intermediate consumption, the utility indifference value can depend on the preference structure even if $q^L_T$ is a.s. constant (apart from the trivial case where $L$ is independent of $\mathcal{F}_T$).

Under the assumptions of Theorem 4.5, we can establish some universal bounds for the indifference value of informational arbitrage, as shown in the following result.

**Proposition 4.8** Suppose that Assumptions 2.3 and 2.7 hold. Suppose furthermore that Assumption 2.1 holds with $\mathbb{Q} = \mathbb{P}$, $d\mathbb{P}_u = \delta_T(du)$ and there exist constants $q_{\min}$ and $q_{\max}$ with $q_{\min} \leq q_{\max}$ such that $\mathbb{P}[q^L_T \in [q_{\min}, q_{\max}]] = 1$. Then for every utility function $U \in \mathcal{U}$, $k \in \mathbb{R}_+$ and $v > 0$, it holds that

\[
(v + k) \left(1 - \frac{1}{q_{\min}}\right) \leq \pi^{U,k}(v) \leq (v + k) \left(1 - \frac{1}{q_{\max}}\right).
\]  

**Proof** Similarly as in the proof of Theorem 4.5, we have $u^{F,k}(v) = U(v)$ for every $U \in \mathcal{U}$. The process $c^G = (c^G_t)_{t \in [0,T]}$ defined by $c^G_t = v_1\mathbb{1}_{\{t = T\}}$ for $t \in [0, T]$ belongs to $C^{G,k}(v + k)/q_{\min} - k)$. Indeed, under the present assumptions, it holds that $\mathbb{E}[(v + k)/q^L_T|G_0] \leq (v + k)/q_{\min}$ a.s. Therefore for all $k \in \mathbb{R}_+$ and $v > 0$, we have that

\[
u^{F,k}(v) = U(v) = \mathbb{E}[U(c^G_T)] \leq u^{G,k}(v + k)/q_{\min} - k),
\]

which implies that $v - \pi^{U,k}(v) \leq (v + k)/q_{\min} - k$, proving the first inequality in (4.6). To prove the second, consider an arbitrary consumption process $c = (c_t)_{t \in [0,T]}$ in $C^{G,k}(v + k)/q_{\max} - k)$. By Jensen’s inequality, it holds that

\[
\mathbb{E}[U(c_T)] \leq U(\mathbb{E}[c_T]) \leq U\left(q_{\max} \mathbb{E}\left[\frac{c^L_T}{q_T}ight]\right) \leq U(v) = u^{F,k}(v),
\]

where the third inequality follows from Lemma 3.3. By the arbitrariness of $c$, this implies that $u^{G,k}(v + k)/q_{\max} - k) \leq u^{F,k}(v)$, thus showing that

\[
v - \pi^{U,k}(v) \geq (v + k)/q_{\max} - k.
\]

This completes the proof of the universal bounds (4.6).
5 Examples

In this section, we illustrate some of the main concepts and results in the context of three examples. The first (Sect. 5.1) is a general version of Example 1.1, and the second (Sect. 5.2) considers a two-dimensional discontinuous financial market. In these two examples, the random variable \( L \) is discrete. In the third example (Sect. 5.3), we consider a continuous random variable \( L \) generating informational arbitrage.

5.1 One-dimensional geometric Brownian motion

Let \( W = (W_t)_{t \in [0,T]} \) be a one-dimensional Brownian motion on the filtered probability space \( (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}) \), where \( \mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]} \) is the \( \mathbb{P} \)-augmentation of the natural filtration of \( W \). We consider a financial market where besides a riskless asset with a constant price of one, a single risky asset is traded with discounted price process \( S = (S_t)_{t \in [0,T]} \) satisfying

\[
dS_t = S_t \sigma_t \, dW_t, \quad S_0 > 0, \tag{5.1}
\]

where \( \sigma = (\sigma_t)_{t \in [0,T]} \) is a strictly positive \( \mathbb{F} \)-predictable process with \( \int_0^T \sigma_t^2 \, dt < +\infty \) a.s. According to the notation introduced in Sect. 2.1, the tuple \( (\Omega, \mathbb{F}, \mathbb{P}; S) \) represents the ordinary financial market and Assumption 2.1 is satisfied with \( \mathbb{Q} = \mathbb{P} \).

Similarly as in Pikovsky and Karatzas [33, Example 4.6] (see also Ankirchner and Imkeller [8, Example 2.12]), we suppose that the additional information is generated by the random variable \( L := I_{\{WT \geq c\}} \), where \( c \) is a constant such that \( \mathbb{P}[WT \geq c] = r \in (0, 1) \). In this setting, \( E = \{0, 1\} \) and the unconditional law of \( L \) is given by \( \lambda(\{0\}) = 1 - r \) and \( \lambda(\{1\}) = r \). Since \( L \) is discrete, Assumption 2.3 is automatically satisfied. In particular, we have

\[
q^0_t = \frac{\mathbb{P}[L = 0|\mathcal{F}_t]}{\mathbb{P}[L = 0]} = \frac{1}{1 - r} \Phi\left(\frac{c - W_t}{\sqrt{T - t}}\right),
\]

\[
q^1_t = \frac{\mathbb{P}[L = 1|\mathcal{F}_t]}{\mathbb{P}[L = 1]} = \frac{1}{r} \Phi\left(\frac{W_t - c}{\sqrt{T - t}}\right)
\]

for all \( t \in [0, T) \), where \( \Phi(x) := \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \, dz \). For \( t = T \), we have

\[
q^0_T = \frac{1}{1 - r} I_{\{W_T < c\}}, \quad q^1_T = \frac{1}{r} I_{\{W_T \geq c\}}.
\]

Since \( q^0 \) and \( q^1 \) have continuous paths, Assumption 2.7 is satisfied. Moreover, it holds that

\[
q^L_T = \frac{1}{1 - r} I_{\{W_T < c\}} + \frac{1}{r} I_{\{W_T \geq c\}}.
\]

By Theorem 2.6, NUPBR holds in the informed financial market \( (\Omega, \mathbb{G}, \mathbb{P}; S) \) and \( 1/q^L_T \) is the associated ELMD. However, since \( \mathbb{E}[1/q^L_T] < 1 \), the additional information leads to arbitrage and NFLVR does not hold. The boundedness of \( q^L_T \) ensures...
that the assumptions of Proposition 4.3 are satisfied, and so we can compute explicitly the indifference value of informational arbitrage. For simplicity of presentation, let us consider the problem of maximising expected utility of terminal wealth (i.e., \(d\kappa_u = \delta_T(du)\)) for \(k = 0\). In this case, for every \(v > 0\), it holds that

\[
\pi^{\log}(v) = v \left(1 - (1 - r)^{1-r} r^r\right),
\]

\[
\pi^{\text{powr}}(v) = v \left(1 - \left((1 - r)^{1-p} + r^{1-p}\right)^{-1/p}\right).
\]

Observe that \(\pi^{\text{powr}}(v)\) is increasing with respect to \(p\), meaning that the indifference value of informational arbitrage is decreasing with respect to risk aversion. The indifference value of informational arbitrage in the case of exponential utility with risk aversion \(\alpha > 0\) is given by the unique solution \(\pi = \pi^{\exp}(v)\) to the equation

\[
e^{-\alpha v} = (1 - r)e^{-\alpha(v-\pi)} + re^{-\frac{\alpha}{r}(v-\pi)}.
\]

Note also that in the present example, for every strictly increasing and concave utility function \(U: \mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\}\) and every \(k \in \mathbb{R}_+\), it follows from Proposition 4.8 that the indifference value \(\pi_{U,k}(v)\) of informational arbitrage satisfies the bounds

\[
\min\{r, 1 - r\} \leq \frac{\pi_{U,k}(v)}{v + k} \leq \max\{r, 1 - r\} \quad \text{for all } v > 0.
\]

**Analysis of Example 1.1.** If \(c = 0\) (and hence \(r = 1/2\)), the random variable \(q^L_T\) reduces to the constant \(q^L_T = 2\). In this case, in line with Theorem 4.5 (see also Remark 4.6), the value of informational arbitrage for \(k = 0\) is equal to the universal value \(\pi(v) = v/2\). By (4.5), the corresponding optimal wealth process \(V^G = (V^G_t)_{t \in [0,T]}\) is given by

\[
V^G_t = v \frac{q^L_t}{q^L_T} = v \left(\Phi \left(\frac{-W_t}{\sqrt{T-t}}\right)1_{\{W_T < 0\}} + \Phi \left(\frac{W_t}{\sqrt{T-t}}\right)1_{\{W_T \geq 0\}}\right)
\]

for all \(t \in [0,T)\). An application of Itô’s formula yields that the optimal strategy \(\vartheta^G = (\vartheta^G_t)_{t \in [0,T)}\) for the informed agent is given by

\[
\vartheta^G_t = \left(1_{\{W_T \geq 0\}} - 1_{\{W_T < 0\}}\right) v \frac{1}{\sigma_t S_t} \frac{W_t^2}{2\pi(T-t)} \exp \left(-\frac{W_t^2}{2(T-t)}\right)
\]

for all \(t \in [0,T)\), regardless of the utility function being considered. In particular, the strategy \(\vartheta^G\) is an arbitrage strategy for an informed agent. Indeed, it holds that \((\vartheta^G \cdot S)_t = V^G_t - v/2 > -v/2\) for all \(t \in [0,T)\) and \((\vartheta^G \cdot S)_T = v/2 > 0\). This shows that by acquiring the additional information \(L\) at price \(\pi(v) = v/2\) and trading according to the strategy \(\vartheta^G\), an informed agent can achieve exactly the terminal wealth \(v\), which also corresponds to the optimal terminal wealth for an ordinary agent.

### 5.2 Two-dimensional geometric Poisson process

Let \(N^i = (N^i_t)_{t \in [0,T]}, \ i = 1, 2\), be two independent Poisson processes with intensity 1 on a filtered probability space \((\Omega, \mathcal{A}, \mathcal{F}, \mathbb{P})\), where \(\mathcal{F} = (\mathcal{F}_t)_{t \in [0,T]}\) is assumed to
be the \( \mathbb{P} \)-augmentation of the natural filtration of \( (N^1, N^2) \). We consider a financial market where besides a riskless asset with a constant price of one, two risky assets are traded with discounted price processes \( S^i = (S^i_t)_{t \in [0,T]} \), \( i = 1, 2 \), satisfying
\[
dS^i_t = S^i_t (dN^i_t - dt), \quad S^i_0 > 0,
\]
with explicit solutions \( S^i_t = S^i_0 e^{-t} 2^{N^i_t} \) for \( i \in \{1, 2\} \) and \( t \in [0,T] \). The tuple \( (\Omega, F, \mathbb{P}; (S^1, S^2)) \) represents the ordinary financial market. Since \( (S^1, S^2) \) has the martingale representation property on \( (\Omega, F, \mathbb{P}) \), Assumption 2.1 holds with \( \mathbb{Q} = \mathbb{P} \).

Let us define the process \( N = (N_t)_{t \in [0,T]} \) by \( N_t := N^1_t - N^2_t \) for all \( t \in [0,T] \). We suppose that \( L := N_T \), corresponding to the observation of the ratio \( S^1_T / S^2_T \). The distribution of \( L \) is given by
\[
\mathbb{P}[L = x] = e^{-2T} I_{|x|}(2T) = e^{-2T} \sum_{k \in \mathbb{N}} \frac{T^{2k+|x|}}{k!(k + |x|)!} I_{|x|}(2T) \quad \text{for all } x \in \mathbb{Z},
\]
where \( I_{|x|}(2T) \) denotes the modified Bessel function of the first kind. Since \( L \) is discrete, Assumption 2.3 is automatically satisfied and one can compute that
\[
q^x_t = \frac{\mathbb{P}[L = x | F_t]}{\mathbb{P}[L = x]} = \sum_{k \in \mathbb{N}} e^{-\frac{T-t}{k!}} e^{\frac{T-t}{k!}} \frac{T^{2k+|x|}}{k!(k + |x|)!} \quad \text{for all } x \in \mathbb{Z}
\]
for all \( x \in \mathbb{Z} \) and \( t \in [0,T) \); see Chau et al. [12, Example 3.3]. For \( t = T \), we have
\[
q^x_T = \frac{1_{\{L=x\}}}{\mathbb{P}[L = x]} = e^{-2T} \sum_{k \in \mathbb{N}} \frac{T^{2k+|x|}}{k!(k + |x|)!} I_{|x|}(2T) \quad \text{for all } x \in \mathbb{Z}.
\]
Note that \( q^x_t > 0 \) for all \( t \in [0,T) \). Moreover, \( q^x_T \) does not jump to zero due to the quasi-left-continuity of the filtration \( F \). Assumption 2.7 is therefore satisfied and the informed financial market \( (\Omega, G, \mathbb{P}; (S^1, S^2)) \) satisfies NUPBR (see Theorem 2.6). The additional information \( L \) generates arbitrage opportunities for an informed agent, since \( \mathbb{E}[1/q^x_T] = \sum_{x \in \mathbb{Z}} \mathbb{P}[L = x]^2 < 1 \).

The indifference value of informational arbitrage can be explicitly computed for logarithmic and power utility functions by Proposition 4.3 (for \( d\kappa_u = \delta_T (du) \) and \( k = 0 \)); it is given, for every \( v > 0 \), by
\[
\pi^{\log}(v) = v \left( 1 - \exp \left( - \sum_{x \in \mathbb{Z}} \mathbb{P}[L = x] \log \mathbb{P}[L = x] \right) \right),
\]
\[
\pi^{\text{pwr}}(v) = v \left( 1 - \mathbb{E} \left[ \left( \sum_{x \in \mathbb{Z}} 1_{\{L=x\}} \mathbb{P}[L = x]^{p/(p-1)} \right)^{1-p} \right]^{-1/p} \right)
\]
\[
= v \left( 1 - \left( \sum_{x \in \mathbb{Z}} \mathbb{P}[L = x]^{1-p} \right)^{-1/p} \right).
\]
In particular, note that for a logarithmic utility function, the value of informational arbitrage is determined by the entropy of the random variable \( L \) (compare with Remark 3.9). For an exponential utility function with risk aversion \( \alpha > 0 \), the indifference value is given by the unique solution \( \pi = \pi^{\exp}(v) \) to the equation

\[
e^{-\alpha v} = \mathbb{E}[\exp(-\alpha q_T^*(v - \pi))] = \sum_{x \in \mathbb{Z}} \mathbb{P}[L = x] e^{-\frac{q_T^*(v - \pi)}{\mathbb{E}[L = x]}}.
\]

5.3 Informational arbitrage induced by a continuous random variable

We now present an example of a filtration initially enlarged by a continuous random variable \( L \) satisfying the absolute continuity relation of Assumption 2.3 and generating arbitrage opportunities.

Let \( W = (W_t)_{t \in [0, T]} \) be a one-dimensional Brownian motion on \((\Omega, \mathcal{A}, \mathcal{F}, \mathbb{P})\), where \( \mathcal{F} = (\mathcal{F}_t)_{t \in [0, T]} \) is the \( \mathbb{P} \)-augmentation of the natural filtration of \( W \). Let \( U \) be independent of \( W \) with uniform distribution on \([0, 1]\) and \( \mathcal{A} = \mathcal{F}_T \vee \sigma(U) \). We consider a financial market with a riskless asset with a constant price of one and a single risky asset with discounted price process \( S = (S_t)_{t \in [0, T]} \) given as in (5.1). The tuple \((\Omega, \mathcal{F}, \mathbb{P}; S)\) represents the ordinary financial market, and Assumption 2.1 is satisfied with \( \mathbb{Q} = \mathbb{P} \).

We set \( \tilde{W}_T^* := \sup_{t \in [0, T]} W_t \) and define the random variable \( L \) by

\[
L := \frac{\tilde{W}_T^*}{2(1 + \tilde{W}_T^*)} + \frac{U}{1 + \tilde{W}_T^*}.
\]

Then \( L \) takes values in \([0, 1]\) and its conditional law \( \nu_T \) with respect to \( \mathcal{F}_T \) is a uniform distribution on the interval \([a(y), b(y)]\), where \( a(y) := y/(2 + 2y) \) and \( b(y) := (2 + y)/(2 + 2y) \) for \( y \in \mathbb{R}_+ \). The unconditional law \( \lambda \) of \( L \) can be computed as

\[
\lambda([0, x]) = \mathbb{P}[L \leq x] = \mathbb{E}[f(x, \tilde{W}_T^*)] = \sqrt{\frac{2}{\pi T}} \int_0^{+\infty} f(x, z) e^{-\frac{z^2}{2T}} \, dz
\]

for \( x \in [0, 1] \), where \( f(x, z) := (z(x - 1/2) + x)^+ \wedge 1 \) for \((x, z) \in [0, 1] \times \mathbb{R}_+ \). Defining \( \gamma(x) := 2x/(1 - 2x) \) for \( x \in [0, 1/2] \) and \( \gamma(x) := (2 - 2x)/(2x - 1) \) for \( x \in (1/2, 1] \), the conditional density \( q_T^x \) of \( \nu_T \) with respect to \( \lambda \) can be computed as

\[
q_T^x = \begin{cases} 1 & \text{if } x \neq 1/2, \\ \frac{1}{1 + \sqrt{\frac{2T}{\pi}}} & \text{if } x = 1/2, \end{cases}
\]

for all \( x \neq 1/2 \), and for \( x = 1/2 \),

\[
q_T^x = \frac{1 + \tilde{W}_T^*}{1 + \sqrt{\frac{2T}{\pi}}}.
\]
Therefore, we have
\[ q_L^T = \frac{1 + W^*_T}{2\Phi\left(\frac{\gamma(L)}{\sqrt{T}}\right) - 1} + \sqrt{\frac{2T}{\pi}} \left(1 - e^{-\frac{\gamma(L)^2}{2T}}\right) \quad \text{a.s., with } \gamma(L) = \frac{1 + W^*_T}{1 - 2U} - 1. \]

In this example, \( \nu_t \ll \lambda \) holds a.s. for all \( t \in [0, T] \) so that Assumption 2.3 is satisfied. However, \( \nu_t \) and \( \lambda \) fail to be equivalent for every \( t \in (0, T] \). This simply follows from the observation that \( \nu_t \) is null outside of the interval \([a(W^*_t), b(W^*_t)]\), together with the fact that the process \((W^*_t)_{t \in [0, T]}\) is increasing and the functions \( a(\cdot) \) and \( b(\cdot) \) are increasing and decreasing, respectively. Moreover, the continuity of the filtration \( \mathcal{F} \) implies that Assumption 2.7 is satisfied. By Theorem 2.6, the informed financial market \((\Omega_1, \mathcal{G}, \mathbb{P}; S)\) satisfies NUPBR, but arbitrage opportunities exist.

6 Conclusions

In this paper, we have presented a general study of the value of informational arbitrage in the context of a semimartingale model of a complete financial market with additional initial information. In our analysis, the assumption of market completeness is used to obtain necessary and sufficient conditions for the validity of NUPBR and NFLVR in the informed financial market \((\Omega_1, \mathcal{G}, \mathbb{P}; S)\). Furthermore, for typical utility functions, market completeness leads to explicit solutions which reveal interesting features of the value of informational arbitrage.

The value of informational arbitrage can be studied in general incomplete financial markets. In particular, the existence and uniqueness result of Theorem 4.2 still holds in incomplete markets as long as the optimal consumption–investment problem in \( \mathcal{G} \) is well posed. More precisely, if the primal and dual value functions in \( \mathcal{G} \) are finite and \((\Omega, \mathcal{G}, \mathbb{P}; S)\) satisfies NUPBR (but not necessarily NFLVR), the results of Chau et al. [11] imply that the value function is sufficiently regular to prove existence and uniqueness of the value of informational arbitrage. However, except for specific models, one cannot obtain an explicit description of that value. Furthermore, in general incomplete markets, there does not exist a simple criterion for determining whether the additional information generates arbitrage in \( \mathcal{G} \) (compare with Theorem 2.6).

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Appendix: Proofs of the results in Sect. 3

Proof of Lemma 3.3 Let \((\vartheta, c) \in A_{H,k}^{\vartheta, c}(v)\). For ease of notation, we set \( V := V^{v+k, \vartheta, c}, C := \int_0^t c_u d\kappa_u \) and \( \tilde{C} := \int_0^t Z_u^H dC_u \). Integration by parts gives for all \( t \in [0, T] \)
that

\[ Z_t^H V_t + \tilde{C}_t = Z_t^H(v + k + (\vartheta \cdot S)_t) - Z_t^H C_t + \int_0^t Z_u^H dC_u \]

\[ = Z_t^H(v + k + (\vartheta \cdot S)_t) - (C_\cdot - Z^H)_t. \]

Since \( Z^H \in M_{\text{loc}}(\mathbb{P}, H) \) and \( Z^H S \in M_{\text{loc}}(\mathbb{P}, H) \), this implies that \( Z^H V + \tilde{C} \) is a sigma-martingale on \((\Omega, H, \mathbb{P})\) (see e.g. Fontana [18, Lemma 4.2]). Being nonnegative, it is also a supermartingale. Therefore, since \( V^v,\vartheta,c \geq 0 \) a.s., we get

\[ v + k \geq \mathbb{E}[Z_T^H V_T + \tilde{C}_T | \mathcal{H}_0] \geq \mathbb{E}[k Z_T^H + \tilde{C}_T | \mathcal{H}_0] \]

so that \( \mathbb{E}[\tilde{C}_T | \mathcal{H}_0] \leq v + k(1 - \mathbb{E}[Z_T^H | \mathcal{H}_0]) \) a.s. Conversely, let \( C := \int_0^t c_u d\kappa_u \) and suppose that

\[ \mathbb{E} \left[ \int_0^T Z_u^H dC_u \mid \mathcal{H}_0 \right] \leq v + k(1 - \mathbb{E}[Z_T^H | \mathcal{H}_0]) \quad \text{a.s.} \]

Consider the process \( \hat{V} = (\hat{V}_t)_{t \in [0, T]} \) defined for \( t \in [0, T] \) by

\[ \hat{V}_t := v + Z_t^H C_t - \int_0^t Z_u^H dC_u + \mathbb{E} \left[ \int_0^T Z_u^H dC_u \mid \mathcal{H}_t \right] - \mathbb{E} \left[ \int_0^T Z_u^H dC_u \mid \mathcal{H}_0 \right] \]

\[ + k(1 - \mathbb{E}[Z_T^H | \mathcal{H}_0] + \mathbb{E}[Z_T^H | \mathcal{H}_t] - Z_t^H). \]

The process \( \hat{V} \) is well defined as an element of \( M_{\text{loc}}(\mathbb{P}, H) \). Due to Assumption 2.1 (and Proposition 2.5, in the case \( H = G \)), there exists \( \psi \in L(S, H) \) such that

\[ \hat{V}_t = Z_t^H(v + (\psi \cdot S)_t) \quad \text{a.s. for all } t \in [0, T]. \]

The process \( V^{v+k,\psi,c} = (V_t^{v+k,\psi,c})_{t \in [0, T]} \) associated to the pair \((\psi, c)\) satisfies

\[ Z_t^H V_t^{v+k,\psi,c} + \int_0^t Z_u^H dC_u = v + k(\mathbb{E}[Z_T^H | \mathcal{H}_t] - \mathbb{E}[Z_T^H | \mathcal{H}_0]) \]

\[ + \mathbb{E} \left[ \int_0^T Z_u^H dC_u \mid \mathcal{H}_t \right] - \mathbb{E} \left[ \int_0^T Z_u^H dC_u \mid \mathcal{H}_0 \right] \]

a.s. for all \( t \in [0, T] \). By construction, we have \( Z_t^H V_t^{v+k,\psi,c} \geq 0 \) a.s. for all \( t \in [0, T] \) and \( Z_T^H V_T^{v,\psi,c} \geq 0 \) a.s. Therefore \((\psi, c) \in \mathcal{A}^{\psi,c}(v)\), proving that \( c \in \mathcal{C}^{\psi,c}(v) \). \( \square \)

**Proof of Proposition 3.5** Under the given assumptions, the process \( c^H = (c_t^H)_{t \in [0, T]} \) satisfies

\[ \mathbb{E} \left[ \int_0^T Z_u^H c_u^H d\kappa_u \mid \mathcal{H}_0 \right] = v + k(1 - \mathbb{E}[Z_T^H | \mathcal{H}_0]) \quad \text{a.s.} \]
so that \( c^H \in C^{H,k}(v) \) by Lemma 3.3. Consider an arbitrary consumption process \( c \in C^{H,k}(v) \). By the Fenchel–Legendre duality (see e.g. Karatzas and Shreve [27, Lemma 3.4.3]), the definitions of the stochastic field \( I \) and of the process \( c^H \) imply that

\[
U(t, c^H_t) - \Lambda^{H,k}(v) Z^H_t c^H_t \geq U(t, c_t) - \Lambda^{H,k}(v) Z^H_t c_t \quad \text{for all } t \in [0, T].
\]

Therefore, it holds that

\[
\mathbb{E}
\left[
\int_0^T U(u, c^H_u) d\kappa_u \bigg| \mathcal{H}_0
\right] \geq \mathbb{E}
\left[
\int_0^T U(u, c_u) d\kappa_u \bigg| \mathcal{H}_0
\right] + \Lambda^{H,k}(v) \mathbb{E}
\left[
\int_0^T Z^H_u c^H_u d\kappa_u \bigg| \mathcal{H}_0
\right] - \Lambda^{H,k}(v) \mathbb{E}
\left[
\int_0^T Z^H_u c_u d\kappa_u \bigg| \mathcal{H}_0
\right]
\]

where the last inequality follows from the fact that in view of Lemma 3.3,

\[
\mathbb{E}
\left[
\int_0^T Z^H_u c^H_u d\kappa_u \bigg| \mathcal{H}_0
\right] = v + k \left( 1 - \mathbb{E}[Z^H_T | \mathcal{H}_0] \right) \geq \mathbb{E}
\left[
\int_0^T Z^H_u c_u d\kappa_u \bigg| \mathcal{H}_0
\right] \quad \text{a.s.}
\]

The result then follows by the arbitrariness of \( c \in C^{H,k}(v) \). \( \square \)

**Proof of Corollary 3.8** In view of Proposition 3.5, to compute \( u^{H,k}(v) \), it suffices to find the \( \mathcal{H}_0 \)-measurable random variable \( \Lambda^{H,k}(v) \) satisfying (3.3).

(i) If \( U(\omega, t, x) = \log x \), then \( I(\omega, t, y) = 1/y \) for all \( y \in (0, +\infty) \). Therefore (3.3) can be explicitly solved and we find

\[
\Lambda^{H,k}(v) = \frac{\mathbb{E}[\kappa_T | \mathcal{H}_0]}{v + k \left( 1 - \mathbb{E}[Z^H_T | \mathcal{H}_0] \right)}.
\]

By Proposition 3.5, the solution \( c^H = (c^H_t)_{t \in [0, T]} \) is therefore given by

\[
c^H_t = \frac{1}{\Lambda^{H,k}(v) Z^H_t} = \frac{v + k \left( 1 - \mathbb{E}[Z^H_T | \mathcal{H}_0] \right)}{Z^H_t \mathbb{E}[\kappa_T | \mathcal{H}_0]} \quad \text{for all } t \in [0, T].
\]

Under the integrability assumption in the corollary, the optimal expected utility \( u^{H,k}(v) \) in (3.4) can be obtained by means of a straightforward computation.

(ii) If \( U(\omega, t, x) = x^p/p \), then \( I(\omega, t, y) = y^{1/(p-1)} \) for all \( y \in (0, +\infty) \). By Proposition 3.5, the \( \mathcal{H}_0 \)-measurable random variable \( \Lambda^{H,k}(v) \) must solve

\[
\mathbb{E}
\left[
\int_0^T \left( Z^H_u \right)^{p/(p-1)} \left( \Lambda^{H,k}(v) \right)^{1/(p-1)} d\kappa_u \bigg| \mathcal{H}_0
\right] = v + k \left( 1 - \mathbb{E}[Z^H_T | \mathcal{H}_0] \right).
\]
Therefore if \( \mathbb{E} \left[ \int_0^T (Z_u^H)^{p/(p-1)} \, d\kappa_u \big| \mathcal{H}_0 \right] < +\infty \) a.s., it holds that

\[
\Lambda^{H,k}(v) = (v + k(1 - \mathbb{E}[Z_T^H | \mathcal{H}_0]))^{p-1} \mathbb{E} \left[ \int_0^T (Z_u^H)^{p/(p-1)} \, d\kappa_u \big| \mathcal{H}_0 \right]^{1-p}.
\]

By Proposition 3.5, the optimal consumption process \( c^H = (c_t^H)_{t \in [0,T]} \) is given by

\[
c_t^H = (\Lambda^{H,k}(v) Z_T^H)^{1/(p-1)}
\]

for all \( t \in [0, T] \). If

\[
\mathbb{E} \left[ \int_0^T (Z_u^H)^{p/(p-1)} \, d\kappa_u \big| \mathcal{H}_0 \right]^{1-p} \in L^1(\mathbb{P}),
\]

the optimal expected utility \( u^{H,k}(v) \) is finite and can be explicitly computed as in (3.5).

(iii) We first show that (3.7) admits an a.s. unique solution for every \( v > v^H_0 \). Arguing similarly as in Muraviev [32, Theorem 3.2], define the \( \mathcal{H}_0 \)-measurable function \( g : \Omega \times (0, +\infty) \to \mathbb{R}_+ \) by

\[
g(\lambda) := \frac{1}{\alpha} \mathbb{E} \left[ \int_0^T Z_u^H \left( \log \frac{\alpha}{\lambda Z_u^H} \right)^+ \, d\kappa_u \big| \mathcal{H}_0 \right] \quad \text{for } \lambda \in (0, +\infty).
\]

Note that \( g \) is well defined since

\[
g(\lambda) = \frac{1}{\alpha} \mathbb{E} \left[ \int_0^T Z_u^H \log \left( \frac{\alpha}{\lambda Z_u^H} \right) 1_{\{Z_u^H \leq \alpha/\lambda\}} \, d\kappa_u \big| \mathcal{H}_0 \right] \leq \frac{\mathbb{E}[\kappa_T | \mathcal{H}_0]}{\lambda} < +\infty \text{ a.s.}
\]

Clearly, \( g \) is a decreasing function. Furthermore, dominated convergence implies that \( g \) is continuous. Again by dominated convergence, we get \( \lim_{\lambda \to +\infty} g(\lambda) = 0 \) a.s., and a straightforward application of Fatou’s lemma yields that \( \lim_{\lambda \downarrow 0} g(\lambda) = +\infty \) a.s. Moreover, for all \( 0 < \lambda' < \lambda < +\infty \), it holds that \( g(\lambda') > g(\lambda) \) a.s. on \( \{g(\lambda) > 0\} \). Indeed, if the \( \mathcal{H}_0 \)-measurable set \( G_{\lambda,\lambda'} := \{g(\lambda) = g(\lambda'), g(\lambda) > 0\} \) has strictly positive probability, then it holds on that set that

\[
\mathbb{E} \left[ \int_0^T Z_u^H \left( \log \frac{\alpha}{\lambda Z_u^H} \right)^+ - \left( \log \frac{\alpha}{\lambda' Z_u^H} \right)^+ \right] \, d\kappa_u \big| \mathcal{H}_0 \right] = 0.
\]

However, since \( \log(\alpha/\lambda' Z_u^H) > \log(\alpha/(\lambda Z_u^H)) \) for all \( u \in [0, T] \), this contradicts the assumption that \( g(\lambda) > 0 \). In view of these observations,

\[
v + k \left( 1 - \mathbb{E}[Z_T^H | \mathcal{G}_0](\omega) \right) \in \{g(\lambda)(\omega) : \lambda \in (0, +\infty)\}
\]

for a.a. \( \omega \in \Omega \). Therefore, by Beneš [10, Lemma 1], (3.7) admits a unique strictly positive \( \mathcal{H}_0 \)-measurable solution \( \Lambda^{H,k}(v) \) for every \( v > v^H_0 \). If \( U(\omega, t, x) = -e^{-\alpha x} \), then \( I(\omega, t, y) = (1/\alpha)(\log(\alpha/y))^+ \) for all \( y \in (0, +\infty) \). By Proposition 3.5, the optimal consumption process \( c_t^H = (c_t^H)_{t \in [0,T]} \) is given by

\[
c_t^H = \frac{1}{\alpha} \left( \log \frac{\alpha}{\Lambda^{H,k}(v) Z_t^H} \right)^+
\]

for all \( t \in [0, T] \), thus proving (3.6). □
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