Tiling Groupoids and Bratteli Diagrams II: Structure of the Orbit Equivalence Relation

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Abstract. In this second paper, we study the case of substitution tilings of $\mathbb{R}^d$. The substitution on tiles induces substitutions on the faces of the tiles of all dimensions $j = 0, \ldots, d - 1$. We reconstruct the tiling’s equivalence relation in a purely combinatorial way using the AF-relations given by the lower dimensional substitutions. We define a Bratteli multi-diagram $B$ which is made of the Bratteli diagrams $B^j, j = 0, \ldots, d$, of all those substitutions. The set of infinite paths in $B^d$ is identified with the canonical transversal $\Xi$ of the tiling. Any such path has a “border”, which is a set of tails in $B^j$ for some $j \leq d$, and this corresponds to a natural notion of border for its associated tiling. We define an étale equivalence relation $R_B$ on $B$ by saying that two infinite paths are equivalent if they have borders which are tail equivalent in $B^j$ for some $j \leq d$. We show that $R_B$ is homeomorphic to the tiling’s equivalence relation $R_\Xi$.

1. Introduction

1.1. Context

In this article, we present a generalized version of Bratteli diagrams and use it to encode, in a purely combinatorial way, the orbit equivalence relation on the transversal of substitution tiling spaces. The structure of the diagram also brings an understanding of the structure of the equivalence relation.

Bratteli diagrams. Bratteli diagrams have been efficiently used to encode $\mathbb{Z}$-actions on the Cantor set. A Bratteli diagram $B$ is given by sets of edges and vertices:

$$E = \bigcup_{n \in \mathbb{N} \cup \{0\}} E_n; \quad V = \bigcup_{n \in \mathbb{N} \cup \{0\}} V_n,$$

and the set of infinite paths on the diagram, $\Pi_\infty$, is a closed subset of $\prod_{n \in \mathbb{N}} E_n$ (a path being a sequence of composable edges).
Under some conditions on the diagram, $\Pi_\infty$ is a Cantor set. An additional structure on $B$ gives a partial order on the set of paths. With respect to this order, the “successor” function is well defined, and defines a minimal action via the Vershik map [19]. Conversely, to any minimal action on the Cantor set, one can associate a Bratteli diagram with order, such that the actions are conjugate. Therefore, ordered Bratteli diagrams provide combinatorial models for minimal $\mathbb{Z}$-actions on the Cantor set [5, 9, 12].

**Tilings.** A particular case of minimal $\mathbb{Z}$-action on a Cantor set is the following. Consider a bi-infinite word $w \in \mathcal{A}^\mathbb{Z}$, with $\mathcal{A}$ a finite set of symbols (alphabet). Then, consider all translates of $w$ (i.e. its orbit by the shift), and take a closure in $\mathcal{A}^\mathbb{Z}$:

$$\Xi_w := \{\sigma^n(w) ; n \in \mathbb{Z}\}.$$

Under suitable conditions on $w$, the $\mathbb{Z}$-dynamical system $(\Xi_w, \sigma)$ is minimal, and $\Xi_w$ is a Cantor set. In the case where $w$ is a word obtained by a symbolic substitution (see Sect. 2), the associated Bratteli diagram can be chosen to be stationary. Figure 1 shows the example of the construction of Definition 2.6 for Fibonacci substitution: $\mathcal{A} = \{a, b\}$ and $\omega(a) = ab, \omega(b) = a$.

Tilings and, in particular, substitution tilings (see Fig. 2 for an example), are higher dimensional analogues. Given a tiling of $\mathbb{R}^d$, let $\Omega$ be its tiling space (a closure of its family of translates, see Sect. 2.1). It is an $\mathbb{R}^d$-dynamical system which is minimal under some conditions, and one can choose a transversal

![Figure 1: A self-similar (or stationary) Bratteli diagram (root on the left): for all $n \geq 1$, $V_n \cong \{a, b\}$, and $E_n$ corresponds to the substitution $a \rightarrow ab, b \rightarrow a$ (see Definition 2.6)](image)

![Figure 2: A process of inflation and substitution: chair (left) and Penrose (right) tilings. Whole tilings can be obtained as a fixed points of these maps)](image)
Ξ to the $\mathbb{R}^d$-flow (Definition 2.11). The transversal in this case is a Cantor set which has the property (amongst others) to meet all orbits. If $d = 1$, there is a first-return map on $\Xi$ which implements a $\mathbb{Z}$-action; the transversal with the $\mathbb{Z}^d$-action can then really be seen as a subshift (as described above). In higher dimension, there is no longer a group action, but a groupoid replaces $\mathbb{Z}$. This is the groupoid of the orbit equivalence relation $\mathcal{R}_\Xi \subset \Xi \times \Xi$ defined by:

$$(T, T') \in \mathcal{R}_\Xi \iff T' = T + a,$$

and with a certain topology (called étale, and which is not induced by the product topology, see Definition 2.13). Therefore, in dimension $d \geq 2$, the problems need then to be rephrased in terms of equivalence relations.

The questions we address in this series of papers are the following:

- Given a tiling space transversal, is it possible to give a combinatorial description of its orbit equivalence relation in terms of Bratteli diagrams?
- Is it possible to describe precisely the structure of this equivalence relation?

### 1.2. Previous Work

In a previous paper [4], we addressed these questions for general tilings. It is a known construction [7,13] that one can associate a Bratteli diagram to any given tiling space transversal $\Xi$. On this diagram, if two paths of $\Pi_\infty$ eventually agree, then the corresponding tilings are in the same orbit. This gives a strict sub-equivalence relation of $\mathcal{R}_\Xi$, called the tail or AF-equivalence relation, and denoted $\mathcal{R}_{\text{AF}}$:

$$\mathcal{R}_{\text{AF}} \subset \mathcal{R}_\Xi.$$  

In [4], we added “horizontal edges” to the diagram, as well as labels on the edges. Using these data, it was possible to reconstruct the whole equivalence relation: adding the missing parts to $\mathcal{R}_{\text{AF}}$ in order to recover $\mathcal{R}_\Xi$. This diagram allows to recover $\mathcal{R}_\Xi$ by defining a generalized tail equivalence relation. However, it fails to be purely combinatorial, as the labels carry geometric information (they are essentially translation vectors).

### 1.3. Present Work

In this paper, we give a construction which holds a priori for substitution tilings (see Sect. 1.4 for a discussion), but gives a much better understanding of what the “missing parts” are. This work is somehow related to the combinatorial representation of minimal $\mathbb{Z}$-actions on the Cantor set by ordered Bratteli diagrams.

We assume that tiles have a good notion of faces in all dimension: see Hypothesis 2.2 and Sect. 2.3. The diagram we use here is a multi-diagram built as follows. Given the transversal $\Xi$ of a tiling space of dimension $d$ associated with a substitution $\omega_d$, build its usual Bratteli diagram $B^d$. Let then $\omega_j$ be the substitution induced by $\omega_d$ on the $j$-dimensional faces of the tiles. For all $0 \leq j \leq d - 1$, build $B^j$ the Bratteli diagram of $\omega_j$. All these diagrams are then linked by horizontal edges, which encode adjacencies (how a face of dimension $j$ contains sub-faces of dimension $j - 1$), see Sect. 3.2.2 and Fig. 6.
As before, there is a homeomorphism between the set $\Pi_\infty^d$ of infinite paths in $B^d$, and the transversal. We define *borders of a path* $x \in \Pi_\infty^d$ (Definition 4.4), as sets of tails in the diagrams of lower dimensions (infinite paths that need not start at depth 1)—the rule on how to derive them being given by horizontal edges. We denote by $b^j(x)$, $j \leq d$, the set of tails in $B^j$ derived from $x$, and call it its $j$th border. The smallest $j$ for which $b^j(x)$ is non empty, is called the *border dimension* of $x$, and written $\text{bdim}(x)$. This is equivalent to the following natural notion of border for tilings (Proposition 4.8): for $T \in \Xi$ define its $j$th border as $b^j(T) = \bigcap_{n \in \mathbb{N}} \lambda^n \omega_n^{-n}(T)^j$, where $T^j$ denote the $j$-skeleton of $T$ (seen as a CW-decomposition of $\mathbb{R}^d$ given by the tiles), and $\lambda$ is the dilation factor of the substitution; this is discussed in detail in Sect. 4.1.

If $\text{bdim}(x) = j$, then any two tails in $b^j(x)$ are tail-equivalent in $B^j$ (Lemma 4.11). We can then define an equivalence relation $\mathcal{R}_B$ on $\Pi_\infty^d$ as follows. We say that two paths $x, y \in \Pi_\infty^d$ are *border equivalent*, and write $x \sim y$, if they have the same border dimension $j$, and their borders are tail equivalent in $B^j$:

(i) $\text{bdim}(x) = \text{bdim}(y) = j$, for some $0 \leq j \leq d$, and
(ii) $b^j(x) \sim b^j(y)$ in $B^j$.

That is, if one writes $x \sim_j y$ to specify the border dimension, and call $\mathcal{R}^j_B$ the corresponding sub-relation, then $\mathcal{R}_B$ is the union of the $\mathcal{R}^j_B$. As a consequence of minimality, for any $j < d$, the set of paths of border dimension $j$ is dense in $\Pi_\infty^d$ (Proposition 4.13), and it has measure zero with respect to any translation invariant measure [16]. So the relation $\mathcal{R}^j_B$, for $j < d$, is defined on a *thin set*. The relation $\mathcal{R}_B^d$ is the standard AF-relation on $B^d$. But for $j < d$, it is important to notice that $\mathcal{R}^j_B$ is *not* an AF-relation (see Remark 5.3). We prove here the following (Theorem 5.8 and Corollary 5.9).

**Theorem.** The equivalence relation $\mathcal{R}_B$ is étale, and it is homeomorphic to $\mathcal{R}_\Xi$.

This gives a decomposition of $\mathcal{R}_\Xi$ in sub-equivalence relations: the AF-equivalence relation homeomorphic to $\mathcal{R}_B^d$, and the “missing parts” which are pairs of tilings of border dimension smaller than $d$.

The definition of the topology of $\mathcal{R}_B$ is technical, and requires a finer analysis of the combinatorics of the substitution, as well as its encoding in the multi-diagram. By minimality, a path $x$ can be the limit of a sequence $(x_n)_{n \in \mathbb{N}}$ in $\Pi_\infty^d$, with $\text{bdim}(x_n) \neq \text{bdim}(x)$ for all $n$. So we loosen up the notion of paths to allow for changes in border dimensions. We introduce *generalized paths* in the multi-diagram, which are infinite paths that have tails in $B^j$ for some $j$ but can start in $B^i$ with $i < j$ (see Sect. 3.2.3). For this purpose, we define *escaping edges* which go from one vertex $v$ in $B^i$ at depth $n$ to a vertex $u$ in $B^j$, $j > i$, at depth $n + 1$, whenever the face corresponding to $v$ lies in the interior of
the substitute of that associated with \( u \) (see Fig. 7, and Sect. 3.2.3). This determines the topology of \( \mathcal{R}_B \) from the combinatorics of the multi-diagram.

1.4. Perspectives

In this paper, we present how to describe in a combinatorial way the equivalence relation on a substitution tiling space. Several questions arise from this work: first, the question of a generalization to groupoids arising from more general tilings; then, the question whether Bratteli multi-diagrams provide a model for a certain class of equivalence relations; finally, the implications of this construction on the \( C^* \)-algebraic level.

As far as the generalization to more general tilings is concerned, it seems easy to make our construction work for substitution tilings with tiles with “wild boundaries”. Our definition of a face of a tile (Definition 2.16) is related to the cell-complex structure of the tiles. However, it is possible to use a more combinatorial definition of faces. We could remove the assumptions that tiles are \( CW \)-complexes, to cover cases where tiles have fractal boundaries for instance.

**Combinatorial faces.** A way to define faces in a more “geometry-independent” way would be the following: define combinatorially a face of dimension \( j \) as being a patch \( \{ p_i \}_{i \in I} \), such that the intersection \( \bigcap_{i \in I} p_i \) is “something” of dimension \( j \). It seems that we need some notion of dimension in order to define a face. However, for simplicial tilings (tilings in which, in dimension \( d \), the intersection of any \( d + 2 \) tiles is empty), there is a combinatorial notion of dimension: the intersection of \( j + 1 \) adjacent tiles has dimension \( d - j \).

In a simplicial tiling, define a face of dimension \( j \) as a patch of \( d - j + 1 \) tiles. Intuitively, the face itself is the intersection of the tiles, but it is possible that it is not a very “clean” geometric object, which makes it actually easier to work purely on the combinatorial side. The boundary of a tile can be defined combinatorially: if \( p \) is a patch defining a face, and \( q \) is its decoration, the faces of \( p \) are all the \( p \cup \{ t \} \) for \( t \in q \).

**Non substitutive tilings.** Is there hope to generalize the results of this papers to equivalence relations arising from general tilings, or from \( \mathbb{Z}^d \)-actions? To any transversal of a minimal tiling space with finite local complexity, it is possible to associate a Bratteli diagram. The construction of the diagram relies on the construction of \emph{refined tessellations} (expanding-flattening sequences in the sense of [2]). At step \( n \), build a tiling space \( \Omega_n \) whose prototiles get bigger and bigger with \( n \), and can be tiled by tiles of \( \Omega \). However, the topology of these tiles becomes increasingly complicated: there is \emph{a priori} no good geometrical notion of a \emph{face} of such tiles. With a combinatorial definition of faces (along the lines developed above), it seems, however, possible to tackle such a problem. It requires to build refined tessellations which are all simplicial. It turns out that in their work, Giordano et al. [10] were able to describe a tiling space as an inverse limit of simplicial complexes (which corresponds exactly to building a refined tessellation by simplicial tilings). So it seems reasonable to define faces (and maps between faces of \( \Omega_{n+1} \) to faces of \( \Omega_n \)) using their formalism of \emph{well-separated tessellations}. For these reasons, it could be expected to extend the
results of the present article to general tilings and, in particular, to groupoids arising from minimal $\mathbb{Z}^d$-actions on a Cantor set.

**$C^*$-algebras.** On the $C^*$-algebraic level, Bratteli diagrams were originally used to classify AF-algebras [3]. It would be interesting to see whether our multi-diagram and the structure of the equivalence relation $R_B$ that it encodes could shed some light on the structure of the tilings $C^*$-algebras.

2. **Substitution Tilings**

In this section, we briefly define the notions of tile, tiling, tiling space, and canonical transversal arising from a substitution rule. Given a tiling, there is a natural action on the associated tiling space, which is given by translation. The orbit equivalence relation induced by the action restricts to the transversal. We give some details on the topology of these equivalence relations. Finally, we define *decorations* of tiles and faces of tiles.

2.1. **Some Notions on Substitution Tilings**

We work in the $d$-dimensional Euclidean space $\mathbb{R}^d$. Let us first define some vocabulary. We refer the reader to [8,18] for a complete exposition.

**Definition 2.1.**

- A **tile** is a compact subset of $\mathbb{R}^d$, which is homeomorphic to a ball.
- A **partial tiling** is a set of tiles $p = \{t_i\}_{i \in I}$ which have pairwise disjoint interiors. We set $\text{Supp}(p) := \bigcup_{i \in I} t_i$ the *support* of the patch $p$.
- A **patch** is a finite partial tiling.
- A **tiling** is a partial tiling with support $\mathbb{R}^d$.

Note that the support of a single tile is the tile seen as a subset of $\mathbb{R}^d$, forgetting the cell structure, and any label or decoration it could have. One can think of tiles as being polyhedra meeting face-to-face. More generally, we ask that the tiles satisfy the following hypothesis.

**Hypothesis 2.2.** The tiles and tilings have a *cellular* structure, that is:

(i) tiles are assumed to be finite CW-complexes,\(^1\) which have a unique $d$-dimensional cell;
(ii) in a (partial) tiling or a patch, the intersection of any number of tiles is either empty or a sub-complex of each of them.

The cellular structure means that tiles are made of cells of various dimensions. A $k$-cell is homeomorphic to an open ball in $\mathbb{R}^k$. This allows to define faces and, in particular, to define faces of a certain dimension. See Definition 2.16 in Sect. 2.3. If tiles were polyhedra, 0-cells would be vertices, 1-cells would be edges without their endpoints, $\ldots$, and the $d$-cell would be the interior of the tile. Assumption (iii) above would ensure that tiles meet face-to-face. When we introduce a substitution, we require an additional assumption on the cellular structure (see Hypothesis 2.9).

\(^1\) See [11, Chapter 0] for definitions of cells, CW-complexes and sub-complexes.
There is a natural action of $\mathbb{R}^d$ on the set of tiles by translation. This action extends to patches, partial tilings and tilings:

$$T + x := \{ t + x ; \, t \in T \} \quad \text{for} \, \, x \in \mathbb{R}^d.$$ 

Note that the tilings are not regarded up to translation: if $T$ is a tiling, then $T$ and $T + x$ are different for $x \neq 0$. We do not consider tiles and patches up to translation either, but as subsets of $\mathbb{R}^d$.

**Definition 2.3.** A puncturing of the tiles is a function $\text{punc}$, which associates to a tile $t$ a point in its interior, such that:

$$\forall x \in \mathbb{R}^d \quad \text{punc}(t + x) = \text{punc}(t) + x.$$ 

Then, the tile $t$ is said to be punctured, and $\text{punc}(t)$ is called the puncture of $t$. A set of punctured tiles is a set of tiles with a puncturing function defined on it.

**Notation 2.4.** Given a tiling $T$, we set

$$T^{\text{punc}} = \{\text{punc}(t); \, t \in T\},$$

the set of punctures of its tiles. If $t$ is a tile and $p$ a patch, the notations $t \in T$, and $p \subset T$, respectively mean “$t$ is a tile and $p$ is a patch of the tiling $T$, at the positions they have as subsets of $\mathbb{R}^d$”. We will use the following notation

$t$ appears in $p$, $t$ appears in $T$, and $p$ appears in $T$, if there exists $a \in \mathbb{R}^d$ such that we respectively have $t + a \in p$, $t + a \in T$, and $p + a \subset T$.

Let us now define substitution tilings. Start with a set of prototiles, then define a substitution rule on it.

**Definition 2.5.**

- A prototile set $\mathcal{A}$ is a finite family of equivalence classes of tiles of $\mathbb{R}^d$ under translation.
- A set of punctured prototiles is a set of prototiles $\mathcal{A}$, together with a puncturing function $\text{punc}$ defined on the set of all tiles with class in $\mathcal{A}$.

By abuse of notation, we may identify an element $t \in \mathcal{A}$ with its unique representative $t_0$ which satisfies $\text{punc}(t_0) = 0_{\mathbb{R}^d}$. We also say that a “patch with tiles in $\mathcal{A}$” is a patch whose tiles have their translational classes in $\mathcal{A}$. We define similarly a (partial) tiling with tiles in $\mathcal{A}$.

Note that in the definition of tiles and prototiles, we allow “labels”: it is possible that two elements of $\mathcal{A}$ have the same shape, but a label indicates that they should be regarded as different elements (a typical example being a tiling of the plane by squares of different colors).

**Definition 2.6.** A substitution rule $\omega$ with inflation factor $\lambda$ on the prototile set $\mathcal{A}$ is a map which, to a tile $t$ of $\mathcal{A}$, associates a patch with tiles in $\mathcal{A}$, such that:

$$\text{Supp}(\omega(t)) = \lambda \text{Supp}(t),$$
and for all $x \in \mathbb{R}^d$,
\[ \omega(t + x) = \omega(t) + \lambda x. \]

See Fig. 2 for an example of substitution rule with four prototiles.

**Definition 2.7.** Given a substitution $\omega$, define the Abelianization matrix of $\omega$ as the matrix $A = (A_{ij})_{i,j \in I}$, where $I$ is in bijection with the set of prototiles via a map $i \mapsto t_i$, and such that:
\[ A_{ij} = \text{number of times } t_i \text{ appears in } \omega(t_j). \]
Each time, the tile $t_i$ appears in $\omega(t_j)$ is called an occurrence of $t_i$ in $\omega(t_j)$.

The substitution allows to define what “acceptable tilings” (with respect to $\omega$) are, and to define $\Omega$, the set of all acceptable tilings.

**Definition 2.8.** The tiling space $\Omega$ associated to $\omega$ is the set of all tilings $T$ such that for all patch $p \subset T$, there exists $t \in A$ and $n \in \mathbb{N}$, such that $p$ appears in $\omega^n(t)$.

It is clear that for any $T \in \Omega$ and $x \in \mathbb{R}^d$, $T + x \in \Omega$. Therefore, there is a natural $\mathbb{R}^d$-action on $\Omega$. It is classical that $\Omega$ is not empty: it is possible to build a fixed point of some power of the substitution $\omega$; such a fixed point then belongs to $\Omega$ (see Fig. 2 for an example).

We now make some assumptions on the substitution.

**Hypothesis 2.9.**
(i) The substitution is primitive, which means the associated Abelianization matrix is primitive;
(ii) the substitution is strongly aperiodic: for all tiling $T \in \Omega$, $(T + x = T) \Rightarrow (x = 0)$;
(iii) the tiling space has finite local complexity (FLC): there are finitely many patches of a given size, up to translation.
(iv) the substitution is well behaved with respect to the cellular structure of the tiles: for any tile $t$ and any $t' \in \omega(t)$, any cell $c'$ of $t'$ is included in a unique cell $c$ of $\lambda t$, with $\dim(c') \leq \dim(c)$.

Primitivity for a matrix $A$ means that there is some integer $n > 0$ such that all entries of $A^n$ are strictly positive. Hypothesis (iv) is interpreted as follows: we know that the substitution, when applied to a tile, gives a patch of tiles. This assumption ensures that the substitution changes a face in a set of faces (see Definition 2.16).

The tiling space $\Omega$ can be given a topology. It is defined by the following basis. For a patch $p$, and $r > 0$, let
\[ \Omega(p, r) := \{ T \in \Omega \mid \exists x \in \mathbb{R}^d, \| x \| < r, p \subset (T - x) \}. \]

The sets $\Omega(p, r)$ form a basis for a topology on $\Omega$. With this topology, the $\mathbb{R}^d$-action by translations is continuous.

**Proposition 2.10** (Anderson and Putnam [1]). With Hypothesis 2.9, $(\Omega, \mathbb{R}^d)$ is a compact and minimal dynamical system.
Minimality means that all orbits are dense. Minimality is actually equivalent to the combinatorial condition of repetitivity on the tiling: all patches repeat “often” in a certain sense. The critical condition to ensure minimality is the primitivity of the substitution.

We now define a transversal \( \Xi \) for the action of \( \mathbb{R}^d \) in \( \Omega \).

**Definition 2.11.** The canonical transversal of \( \Omega \) (with respect to the \( \mathbb{R}^d \)-action) is:

\[
\Xi := \{ T \in \Omega ; \ 0_{\mathbb{R}^d} \in T^{\text{punc}} \}.
\]

It is easily shown that the relative topology of \( \Omega \) restricted to \( \Xi \) is given by the following basis of open sets: given a patch \( p \) such that \( 0_{\mathbb{R}^d} \in p^{\text{punc}} \), consider

\[
\Xi(p) = \{ T \in \Xi ; \ p \subset T \}.
\]

**Proposition 2.12.** The canonical transversal \( \Xi \) is a Cantor set, that is a compact Hausdorff, totally disconnected set, with no isolated points. Furthermore, the sets defined in (3) form a basis of clopen sets (sets which are both open and closed).

### 2.2. Tiling Equivalence Relations and Groupoids

Let \( \Omega \) be a tiling space, and let \( \Xi \) be its canonical transversal. We define two equivalence relations associated with \( \Omega \) and \( \Xi \).

**Definition 2.13.** The equivalence relation of the tiling space is the set

\[
R_{\Omega} = \{ (T, T') \in \Omega \times \Omega ; \ \exists a \in \mathbb{R}^d, \ T' = T + a \}
\]

with the following topology: a sequence \( (T_n, T'_n = T_n + a_n) \) converges to \( (T, T' = T + a) \) if \( T_n \to T \) in \( \Omega \) and \( a_n \to a \) in \( \mathbb{R}^d \).

The equivalence relation of the transversal is the restriction of \( R_{\Omega} \) to \( \Xi \):

\[
R_{\Xi} = \{ (T, T') \in \Xi \times \Xi ; \ \exists a \in \mathbb{R}^d, \ T' = T + a \}
\]

Note that the equivalence relations are not endowed with the relative topology of \( R_{\Omega} \subset \Omega \times \Omega \) and \( R_{\Xi} \subset \Xi \times \Xi \). For example, by minimality, for \( a \) large, \( T \) and \( T + a \) might be close in \( \Omega \), so that \( (T, T + a) \) is close to \( (T, T') \) for the relative topology, but not for that from \( \Omega \times \mathbb{R}^d \). The map \( (T, a) \mapsto (T, T + a) \) from \( \Omega \times \mathbb{R}^d \) to \( \Omega \times \Omega \) is injective because \( \Omega \) is strongly aperiodic (contains no periodic points), and its image is \( R_{\Omega} \). It is, therefore, a bijection onto its image, and the topology on \( R_{\Omega} \) is the topology which makes this map a homeomorphism.

**Definition 2.14.** An topological equivalence relation \( \mathcal{R} \) on a compact metrizable space \( X \) is called \( \text{étale} \) when the following holds.

(i) The set \( R^2 = \{ ((x, y), (y, z)) \in \mathcal{R} \times \mathcal{R} \} \) is closed in \( \mathcal{R} \times \mathcal{R} \) and the maps sending \( ((x, y), (y, z)) \) in \( \mathcal{R} \times \mathcal{R} \) to \( (x, z) \) in \( \mathcal{R} \), and \( (x, y) \) in \( \mathcal{R} \) to \( (y, z) \) in \( \mathcal{R} \) are continuous.

(ii) The diagonal \( \Delta(\mathcal{R}) = \{ (x, x) ; x \in X \} \) is open in \( \mathcal{R} \).
(iii) The range and source maps \( r, s : \mathcal{R} \to X \) given by \( r(x, y) = x, s(x, y) = y \), are open and are local homeomorphisms.

Note that the equivalence relation being topological, it comes with a topology which is in general not induced by the product topology on \( X \times X \).

A set \( O \subset \mathcal{R} \) is called an \( \mathcal{R} \)-set, if \( O \) is open, and \( r|_O \) and \( s|_O \) are homeomorphisms.

The collection of \( \mathcal{R} \)-sets forms a base of open sets for the topology of \( \mathcal{R} \). For this topology, it is proven in [13] that \( \mathcal{R}_\Omega \) and \( \mathcal{R}_\Xi \) are \( \acute{e}tale \) equivalence relations.

The tiling space \( \Omega \) has a foliated space structure with leaves identified to \( \mathbb{R}^d \) and Cantorian transversals [2]. We now define the groupoid of tiling space \( \Gamma_\Xi \) which encodes the “first return map” to the transversal \( \Xi \) in \( \Omega \) under the flow of \( \mathbb{R}^d \). This groupoid encodes essential dynamical and topological properties of \( \Omega \).

A groupoid [17] is a small category (the collections of objects and morphisms are sets) whose morphisms are all invertible. A topological groupoid, is a groupoid \( G \) whose sets of objects \( G^0 \) and morphisms \( G \) are topological spaces, and such that the composition of morphisms \( G \times G \to G \), the inverse of morphisms \( G \to G \), and the source and range maps \( G \to G^0 \) are all continuous maps.

Given an equivalence relation \( R \) on a topological space \( X \), there is a natural topological groupoid \( G \) associated with \( R \), with objects \( G^0 = X \), and morphisms \( G = \{(x, x'); x \sim R x'\} \). The topology of \( G \) is then inherited from that of \( \mathcal{R} \).

**Definition 2.15.** The groupoid of the tiling space is the groupoid of \( \mathcal{R}_\Xi \), with set of objects \( \Gamma^0_\Xi = \Xi \) and morphisms

\[
\Gamma_\Xi = \{(T, a) \in \Xi \times \mathbb{R}^d ; T + a \in \Xi\}. \quad (6)
\]

There is also a notion of \( \acute{e}tale \) groupoids [17]. Essentially, this means that the range and source maps are local homeomorphisms. It can be shown that \( \Gamma_\Xi \) is an \( \acute{e}tale \) groupoid [13].

### 2.3. Faces and Induced Substitutions

In this section, we define faces of tiles, and describe how \( \omega \) induces substitutions on faces of any dimension. A key point is that we need decorated faces (or collared faces). The use of decorations is related to the notion of border forcing introduced by Kellendonk [13]. In their paper [1], Anderson and Putnam used collared tiles to build approximants of the tiling space, and describe the tiling space as an inverse limit. Bratteli diagrams can also be seen as an inverse limit construction which describes the transversal. Therefore, decoration is also an essential feature.

Consider a set of prototiles \( A \) of dimension \( d \), and a substitution \( \omega \) on it. Let \( \Omega \) be the associated tiling space. Remember (Hypothesis 2.2) that tiles are CW-complexes. Furthermore, the tilings in \( \Omega \) are cellular in the sense that the
intersection of two adjacent tiles in a tiling is a sub-complex of both (this is the analogue of meeting face-to-face for a tiling by polygons).

**Definition 2.16.** Let \( T \in \Omega \), and \( t \in T \). A \( j \)-dimensional face \( f_0 \) of a tile \( t \) is the closure of one of the \( j \)-cells of \( t \), which satisfies the following condition:

\[
\bigcap_{t' \in T : f_0 \subset t'} t' = f_0.
\]  

(7)

A \( j \)-dimensional decorated face of \( t \) (with respect to its position in \( T \)) is a pair 

\[
f = (f_0, \text{Col}(f))
\]

where \( f_0 \) is a \( j \)-dimensional face of \( t \), and \( \text{Col}(f) \) is the patch defined as the set of all tiles in \( T \) which intersect \( f_0 \). It is called the collar of \( f_0 \) in \( T \). It is also called the collar of \( f \).

Condition (7) ensures that a face is on the boundary of a tile. More generally, a \( j \)-faces is a boundary between at least two \((j - 1)\)-faces. Furthermore, since a face is an intersection of tiles, it has itself a cell complex structure (Hypothesis 2.2).

From now on, when the term “face” will be used, it will denote a decorated face. Note that as a particular case, a \( d \)-dimensional face is actually a tile \( t \in \mathcal{A} \), together with an additional decoration (its collar). Remark that any tiling \( T \in \Omega \) can be seen as a tiling by decorated tiles.

We really want to consider decorated tiles and faces as geometric objects. In particular, the terminology is the same for tiles and for faces: a \( j \)-dimensional face \( f \) appears in a tiling \( T \) if there is some \( a \in \mathbb{R}^n \) such that \( f + a \) is included in \( T \) (together with its decoration). The face \( f \) is in \( T \) (noted abusively \( f \in T \)) if \( f \) is included in \( T \), with its decoration, at the same position.

We extend the puncturing function to faces, such that \( \text{punc}(f + x) = \text{punc}(f) + x \).

**Definition 2.17.** For \( 0 \leq j \leq d \), define \( \mathcal{A}^j \) as the set of all equivalence classes of \( j \)-dimensional (punctured) faces up to translation.

All these sets are finite, by the finite local complexity property. By abuse of notation, we may consider \( f \in \mathcal{A}^j \) as a specific element in such an equivalence class.

If \( f, f' \) are two faces in \( T \), it is clear that \( f \cap f' \) is well defined (and possibly empty), and is made of faces (by Hypothesis 2.9). Unless \( f \subset f' \) or \( f' \subset f \), the dimension of the intersection is strictly smaller than the dimension of \( f \) and \( f' \). Similarly, given a decorated tile, it is possible to define its faces in various dimensions independently of its position in a tiling. The reason for this is that the collar of a face is included in the collar of the tile.

Since we can define what it means for a face to be on the boundary of a tile, we can give the following definition.

**Definition 2.18.** Let \( f \) be a \( j \)-dimensional face. We define \( \partial f \) (the boundary of \( f \)), as the set of all \((j - 1)\)-dimensional faces which are on its boundary in the usual (geometrical) sense. Orientation is not taken into account.
Figure 3. The substitution of the marked face (thick line) depends on its neighborhood in the tiling: with the substitution pictured in Fig. 2, the induced substitution on the left face would be two long faces, and the induced substitution on the right face would be two short and one long face.

The point in the definition above is that if $f$ is decorated, its boundary is made of decorated faces.

Definition 2.19. For all $0 \leq j \leq d$, we define $\omega_j$ as the substitution induced by $\omega$ on (decorated) faces. Given $f \in A^j$, $\omega_j(f)$ is defined as the set of all $j$-dimensional faces which intersect $\lambda f$ in $\omega(\text{Col}(f))$.

By point (iii) of Hypothesis 2.9, $\omega_j(f)$ is a set of faces, the union of which is $\lambda f$. The faces in $\omega_j(f)$ come with a decoration, because the collar of any element $f' \in \omega_j(f)$ is included in $\omega(\text{Col}(f))$.

It is essential here to consider tiles and faces with decorations. Indeed, two different decorated versions of the same undecorated face may have different substitutions (see Fig. 3). The definition above of $\omega_j$ is non-ambiguous because of the choice of decorations. Note that in the special case $j = d$, we get the substitution induced by $\omega$ on decorated faces: $\omega_d(t)$ is a patch of decorated tiles. We may write $\omega$ instead of $\omega_d$ when there is no risk of confusion.

We conclude this section by the following lemma, which is geometrically intuitive and is actually a consequence of finite local complexity.

Lemma 2.20. There exists a $\rho > 0$ which depends only on the prototile set and the substitution rule, such that:

- for any face $f$, the collar of $f$ contains a $\rho$-tubular neighborhood of $f$. In other words, the distance of $f$ to the outside of its collar is at least $\rho$.
- if any two faces in a tiling $T$ are closer than a distance $\rho$, then they intersect.

3. Multi-Diagram

We define here a Bratteli multi-diagram associated with a substitution tiling of $\mathbb{R}^d$. We first recall the construction of the usual Bratteli diagram of substitution.

3.1. Usual Bratteli Diagram and AF-Equivalence Relations

Let us remind the reader how a Bratteli diagram is defined for a primitive substitution [7,13].
Construction of the diagram. An example of a Bratteli diagram is given in the introduction (Fig. 1). The formal definition is the following.

Definition 3.1. Let $\omega$ be a primitive and totally aperiodic substitution (defined on decorated tiles), with prototile set $A$, and let $A = (A_{ij})_{i,j \in I}$ be it associated Abelianization matrix (Definition 2.7). The stationary Bratteli diagram associated with $\omega$ is the graph $B = (V, E)$, with

$$V = \left( \bigcup_{n \geq 1} V_n \right) \cup \{ \circ \}, \quad E = \bigcup_{n \geq 0} E_n,$$

where all the $V_n$ are copies of the index set of the matrix $A$, and there are exactly $A_{ij}$ edges in $E_n$ ($n \geq 1$) between $v \in V_n$ and $v' \in V_{n+1}$ if $v$ is the vertex corresponding to $i$ and $v'$ is the vertex corresponding to $j$. Finally, there is a single edge in $E_0$ between the root $\circ$ and each vertex of $V_1$.

The adjacency of edges and vertices is given by two maps $r$ and $s$ (range and source maps), such that $r : E_n \rightarrow V_{n+1}$ and $s : E_n \rightarrow V_n$.

Remark 3.2. Each vertex corresponds to an element of the index set of $A$. By definition of the Abelianization matrix, it means that each vertex corresponds to a tile, and an edge from $v$ to $v'$ corresponds to an occurrence of $t_v$ in the substitution of $t_{v'}$ (where $t_v, t_{v'}$ are the tiles corresponding to $v$ and $v'$ respectively).

Definition 3.3. A path in the Bratteli diagram $B$ is a sequence of edges $\gamma = (e_1, \ldots, e_m)$, for $n < m$ and $m \in \mathbb{N} \cup \{\infty\}$, satisfying $e_i \in E_i$ and $r(e_i) = s(e_{i+1})$ for all $i$. We denote by $\Pi_{n,m}$ the set of such paths. If $m < \infty$ we extend the functions $r$ and $s$ to $\Pi_{n,m}$, so that $r(\gamma) \in V_{m+1}$ and $s(\gamma) \in V_n$.

We will use the shorthand notations $\Pi_n$ and $\Pi_\infty$ for $\Pi_{0,n}$ and $\Pi_{0,\infty}$ respectively. We endow each $E_i$ with the discrete topology, and $\Pi_\infty$ with the relative topology of the product topology on $\Pi_{n \geq 0} E_n$. Since the relation $r(e_i) = s(e_{i+1})$ is closed, it is clear that $\Pi_\infty$ is a compact and totally disconnected set (as a closed subset of a Cantor set). The primitivity of the substitution ensures that it is itself a Cantor set.

Notation 3.4. For $x \in \Pi_{n,m}$ and $n \leq k < l \leq m$ we denote by $x_{[k,l]}$, $x_{[k,l]}$, $x_{(k,l)}$, and $x_{(k,l)}$ the restrictions of $x$ from depths $k$ through $l$ with end points included or excluded. For instance, if $x \in \Pi_\infty$ we shall denote by $x_{[n,\infty)}$ the tail of $x$ from depth $n$ on, and by $x_{[0,n)}$ its head from the root down to depth $n$ (excluded).

If $\gamma, \eta$ are two paths with $s(\eta) = r(\gamma)$, we denote by $\gamma \cdot \eta$ the concatenated path.

A family of clopen sets which generates the topology can be given explicitly.

Notation 3.5. Given $\gamma \in \Pi_n$, with $n < \infty$, define:

$$[\gamma] := \{ x \in \Pi_\infty ; x_{[0,n]} = \gamma \}.$$
AF-equivalence relation.

**Definition 3.6.** Let $\mathcal{B}$ be a Bratteli diagram and let

$$\mathcal{R}_m = \{ (x, \gamma) \in \Pi_\infty \times \Pi_m ; r(x_{[0,m]}) = r(\gamma) \}.$$ 

with the product topology (discrete topology on $\Pi_m$).

The AF-equivalence relation is the direct limit of the $\mathcal{R}_m$ given by

$$\mathcal{R}_{AF} = \lim_{m \to \infty} \mathcal{R}_m = \{ (x, y) \in \Pi_\infty \times \Pi_\infty ; \exists m, \ x_{[m, \infty]} = y_{[m, \infty]} \},$$

with the direct limit topology. For $(x, y) \in \mathcal{R}_{AF}$ we write $x \sim y$, or $x \sim_{tail} y$, and say that the paths are tail equivalent.

It is well known that $\mathcal{R}_{AF}$ is an AF-equivalence relation, as the direct limit of the compact étale relations $\mathcal{R}_m$, see [15]. A motivation for this name is that the $C^*$-algebra associated with $\mathcal{R}_{AF}$ (with the construction of [17]) is an AF-algebra, also the AF-algebra is associated with the Bratteli diagram (see [6]).

**The Robinson map.** The Robinson map is a map from $\Pi_\infty$ to $\Xi$, which relates the Bratteli diagram with the tiling space.

Let us describe the construction of the Robinson map. First, remember (Remark 3.2) each vertex in the Bratteli diagram corresponds to a proto-tile. The proto-tile associated to $v$ is determined by the sequence of inclusions above lies at the origin. Since $e_i \in \mathcal{E}_n$ between $v \in \mathcal{V}_n$ and $v' \in \mathcal{V}_{n+1}$ corresponds to the inclusion of $t_v$ in $\omega(t_{v'})$, then by applying $\omega^{-1}$, it corresponds to the inclusion of $\omega^{-1}(t_v)$ in $\omega(t_{v'})$.

Therefore, a path $\gamma = (e_0, e_1, e_2, \ldots, e_n)$ (with $e_i \in \mathcal{E}_i$) corresponds to a sequence of inclusions of $\omega^{i-1}(t_{s(e_i)})$ in $\omega^{i}(t_{r(e_i)})$, at a position prescribed by edge $e_i$ for $1 \leq i \leq n$. Since $r(e_i) = s(e_{i+1})$ for all $i$, one has a sequence of inclusions:

$$t_{r(e_0)} \subset \omega(t_{r(e_1)}) \subset \cdots \subset \omega^n(t_{r(e_n)}) .$$

$\varphi_n(\gamma)$ is then defined as the translate of the patch $\omega^n(t_{r(e_n)})$ such that the puncture of the tile $t_{r(e_0)}$ determined by the sequence of inclusions above lies at the origin.
The properties of the $\varphi_n$ make it possible to define $\varphi(x)$ as the union of the patches $\varphi_n(x_{[0,n]})$. It can happen that this union is only a partial tiling. However, since we used decorated tiles, for all $x$, there is a unique tiling in $\Omega$ which contains $\bigcup_{n \in \mathbb{N}} \varphi_n(x)$.

Indeed, $\varphi_n(\gamma)$ is a patch of decorated tiles: it is some translate of $\omega^n(t_{r(\gamma)})$, which contains the origin in its support. Using the collar of $\omega^n(t_{r(\gamma)})$, one sees that $\varphi_n(\gamma)$ determines a bigger undecorated tiling, which is some translate of $\omega^n(\text{Col}(t_{r(\gamma)}))$. Since the distance of $t_{r(\gamma)}$ to the edge of its collar is uniformly bounded below by $\rho > 0$ (Lemma 2.20), then $\varphi_n(\gamma)$ determines an undecorated tiling up to a distance $\lambda^n \rho$ around the origin. Letting $n$ tend to infinity, we see that $\varphi(x)$ may not be a full (decorated) tiling, but determines uniquely an undecorated tiling of all $\mathbb{R}^d$. Finally, one simply derives a decorated tiling from this undecorated tiling (Fig. 4).

We still call $\varphi(x)$ the resulting tiling.

**Definition 3.7.** Let $\varphi : \Pi_\infty \rightarrow \Xi$ be the map defined inductively from the $\varphi_n$. This map is called the Robinson map.

**Theorem 3.8** (Kellendonk [13]). The map $\varphi$ is a homeomorphism between the set of infinite rooted paths in $B$ and the canonical transversal of the tiling space.

### 3.2. Definition of the Multi-Diagram

We now turn to the definition of a generalized Bratteli diagram. The basic idea is the following: some of the paths in a usual Bratteli diagram only define partial tilings. How much of these paths are there, and what is the structure of these special tilings? For one-dimensional tilings, this is known. There are
finely many such tilings, and they correspond to fixed points of some power of the substitution. In higher dimension, however, a *nested structure* appears. A half-tiling of the plane, e.g., has a boundary which is a sequence of one-dimensional faces. In other words, it looks very much like a one-dimensional substitution tiling. The generalized Bratteli diagrams contains the usual Bratteli diagram of the $d$-dimensional substitution. Also, for each dimension $0 \leq j \leq d - 1$, it contains a Bratteli diagram given by the substitution induced on $j$-faces. It also has a horizontal structure: edges linking these different diagrams. These edges encode the informations about faces being boundaries of tiles. Altogether, the information added to the original Bratteli diagram is purely combinatorial and allows to define an equivalence relation on the set of infinite paths. This equivalence relation contains the tail-equivalence relation, and is mapped homeomorphically via the Robinson map to the translational equivalence relation on the transversal of the tiling space, see Sect. 5.

3.2.1. First Step: Dual Diagram. For technical reasons, it is more natural to start from a dual diagram rather that from the usual Bratteli diagram. The construction is done as follows. Consider $B_0$ the usual Bratteli diagram associated to $\omega$, as defined in Sect. 3.1. Let us construct $B^d = (V^d, E^d)$ as follows.

- For all $n \geq 1$, the set $V^d_n$ of vertices at depth $n$ in $B^d$ is isomorphic to the set of edges in $B_0$ at depth $n$;
- For all $n \geq 1$, there is an edge $e \in E^d_n$ between $s(e) \in V^d_n$ and $r(e) \in V^d_{n+1}$ if the corresponding edges are composable in $B_0$;
- Add a root and a set $E^d_0$ of edges from the root to elements of $V^d_1$.

This new diagram is *simple*: there is at most one edge between two given vertices. Therefore, a path in $B^d$ (which is a sequence of composable edges) is entirely given by the sequence of vertices it goes through. Since vertices in $B^d$ correspond to edges in $B_0$, the trivial map

$$(e_0, e_1, \ldots, e_n) \mapsto (e_1, \ldots, e_n),$$

provides a canonical identification between paths in $B_0$ and paths in $B^d$ (where the $e_i$ on the left are edges in $B_0$, and the $e_i$ on the right are vertices in $B^d$). It makes, therefore, sense to define a Robinson map in this context: a finite path still corresponds to a partial tiling, and an infinite path to a tiling.

What do vertices correspond to, via the identification made by the Robinson map? Since a vertex of $B^d$ corresponds to an edge in $B_0$, it corresponds to some tile, sitting inside a patch which is itself the substitution of a tile. It is also possible to consider a vertex of $B^d$ simply as a (decorated) tile, with an additional label. This additional label corresponds to the fact that this tile lies inside of a given super-tile in a predetermined position: it is not only information about the neighborhood of the tile, but about its position in the hierarchical structure of a tiling. This is shown on the left of Fig. 5.

**Notation 3.9.** Since a vertex in $B_d$ corresponds to a pair of tiles $t_1, t_2$, such that $t_1 \subset \omega(t_2)$, we will refer to it as the vertex "$t_1 \subset t_2$". This notation is a slight abuse of notation, since $t_1$ is not actually a subset of $t_2$, and furthermore
there could be several “$t_1 \subset t_2$” vertices. However, this notation is convenient when no confusion can occur.

3.2.2. Bratteli Diagrams in Lower Dimensions and Horizontal Structure. For all $j < d$, let us build a Bratteli diagram associated to $\omega_j$. We proceed exactly as before (Definition 3.1), except that we do not include a root. Then, we take the dual diagram. The resulting diagram is called $B^j$.

To sum up given $0 \leq j \leq d - 1$, a vertex of $B^j$, say $v \in V^j_n$, corresponds to one occurrence of some $j$-face $f$ in the substitution of some other $j$-face $f'$ under $\omega_j$. Again, by simplicity and when there is no possible confusion, we may note “$v = f \subset f'$”. There is an edge between $v$ and $v'$ if the inclusions are compatible (i.e., $v = f \subset f'$ and $v' = f' \subset f''$). This defines the diagram $B^j = (V^j, E^j)$.

Each diagram is related to the other thanks to the horizontal structure, which we define now. This definition strongly depends on the fact that, for a $j$-dimensional decorated face $f$, we are able to define $\partial f$ as a set of $(j - 1)$-dimensional decorated faces.

We shall denote by $V$ the union of the sets $V^j$ and $E$ the union of the sets $E^j$ over $j = 0, \ldots, d$.

We now define horizontal edges, which links the diagrams $B^j$ together, see Fig. 5 for an illustration of this definition.

**Figure 5.** Two vertices in the dual diagram $B^2$ linked by horizontal edges (dotted lines) to a common face in $B^1$.

**Definition 3.10.** For all $n \in \mathbb{N}$, define the set of horizontal edges $H_n$. We have $H_n = \bigcup_{j=1}^d H^j_n$, and there is an edge $h \in H^j_n$ from $v = (g \subset g') \in V^j$ to $u = (f \subset f') \in V^{j-1}$ if:

(a) $f \in \partial g$ and $f' \in \partial g'$;  
(b) the occurrence of $g$ in $\omega_j(g')$ encoded by $v$ actually lies on the boundary of $\omega_j(g')$, and induces the inclusion encoded by $u$.

We also set $H$ to be the union of the sets $H^j$ over $j = 0, \ldots, d$. We extend the range and source maps to $H$ as follows $r : H_n \rightarrow V^{j-1}_n$ and $s : H^j_n \rightarrow V^j_n$.

How horizontal structure corresponds to adjacency is represented on Figs. 6 and 5.

**Paths.** Paths in the diagrams of lower dimensions are defined in the same way as paths in the diagram of maximum dimension. We, however, accept non-rooted paths. Let us fix the notations here.
Figure 6. A tile (in gray in the center, with decoration shown around it) has here three 1-dimensional faces, and three vertices. The arrows shown here encode adjacencies. The induced substitution on tiles is determined by the gray patches. The white patches represent the decoration of the faces.

Notation 3.11. We use the following notations: $\Pi_{n,m}^j$ is the set of all paths in $B^j$ starting at depth $n$ and ending at depth $m$ (with possibly $m = \infty$).

Any $x \in \Pi_{n,m}^j$ is of the form

$$x = (v_n, v_{n+1}, \ldots, v_m),$$

such that for all $i \in \{n, \ldots, m\}, v_i \in V_i^j$, and there is an edge $e \in E_i^j$, with $s(e) = v_i$ and $r(e) = v_{i+1}$.

If $j = d$, then it could be that $n = 0$, in which case it means that the path starts from the root, and we write $\Pi_{0,\infty}^d := \Pi_{0,\infty}^j$ for short.

Finally, $\Pi_{n,m}^j$ with $m \in \mathbb{N} \cup \{+\infty\}$ denotes the union of $\Pi_{n,m}^j$ for $n \leq m$.

Note that paths do not go through horizontal edges. Horizontal edges are not used to define paths, but to define relations between paths (a path in $B^j$ can be on the boundary of a path in $B^{j+1}$, see Sect. 4 for a definition).

3.2.3. Escaping Edges and Vanishing Faces. We complete the construction of the multi-diagram by constructing a new type of edges, which we call escaping edges. This construction is used to define the topology on the equivalence relation extending $R_{AF}$, and can be skipped for now. The reader is advised to come back to these definitions before reading Sect. 5.2.
Figure 7. On the first level, the two tiles have a common face determined by their decorations (not shown here). On the next level, this common face lies in the interior of a super-tile, and is no longer visible on the diagram $B^1$. There is an escaping edge.

Figure 8. A situation in which two escaping edges follow one another.

The set of escaping edges keeps track combinatorially of the fact that, in the substitution process, $\omega(t)$ contains faces in its interior, which are “created” by $\omega$, and do not come from the substitution of faces of $t$ (see Fig. 7 and Fig. 8). Any tile comes from the substitution of some other tile (which we call a super-tile in this definition in order to keep track of the hierarchy), but not any face appears as a sub-face of a super-face.

**Definition 3.12.** Define the set of edges $S$ on the multi-diagram as follows. For all $n \geq 1$, there are edges $e \in S$ in the following cases.

- From a vertex $v \in V_n^{(j)}$ to a pair of distinct vertices $\{w, w'\} \subset V_{n+1}^{(k)}$ in the case pictured in Fig. 7, that is if:
  - $k > j$;
  - $w$ and $w'$ correspond to $g \subset g''$ and $g' \subset g''$ for the same $k$-face $g''$;
  - $v$ corresponds to $f \subset f'$ where $f, f'$ are $j$-faces, and $f' \subset g \cap g'$.
• From a pair of vertices \( \{v, v'\} \subset \mathcal{V}_n^j \) to a pair of vertices \( \{w, w'\} \subset \mathcal{V}_{n+1}^k \) in the case pictured in Fig. 8, that is if:
  - \( k > j \);
  - \( v \) and \( v' \) correspond to \( f \subset f'' \) and \( f' \subset f'' \) respectively (these are \( j \)-faces);
  - \( w \) and \( w' \) correspond to \( g \subset g'' \) and \( g' \subset g'' \) respectively (these are \( k \)-faces);
  - \( f'' \subset g \cap g' \).

We extend the source and range maps to \( S \) as follows: for \( e \in S \), \( s(e) \) is a vertex or a pair of vertices in some \( \mathcal{V}_n^j \), and \( r(e) \) is a vertex of a pair of vertices in some \( \mathcal{V}_{n+1}^k \), \( k > j \).

**Generalized paths.** Using escaping edges, we can define paths “jumping up in dimension”: generalized paths on the multi-diagram which use the escaping edges we defined above.

**Definition 3.13.** A generalized path \( x \in \tilde{\Pi}_{n,\infty} \) on the Bratteli multi-diagram is a sequence \( (x_n, x_{n+1}, \ldots) \), where each \( x_n \) is either a vertex or a pair of vertices of \( \mathcal{V} \), such that for all \( i \geq n \), one of these situations occurs:

- there is an edge \( e \in E \cup S \) such that \( x_i = s(e) \) and \( x_{i+1} = r(e) \);
- if \( x_i = (v, v') \) and \( x_{i+1} = v'' \), there is a pair of edges \( (e, e') \in (E) \), such that \( s(e) = v, s(e') = v' \) and \( r(e) = r(e') = v'' \).

We finally define the Bratteli multi-diagram, by putting altogether the sets of edges, horizontal edges and escaping edges. We see this diagram as the union of the diagrams \( \mathcal{B}^j \) over \( j = 0, \ldots d \), linked together by the sets \( H \) and \( S \) of horizontal and escaping edges.

**Definition 3.14.** The Bratteli multi-diagram associated with the substitution \( \omega_d \) on \( A^d \) and its induced substitutions \( \omega_j \) on \( A_j \) is the diagram \( \mathcal{B} = (V, E, H, S) \) with range and source maps \( r, s \), as defined previously on each sets of edges.

### 3.3. Robinson Map for the Multi-Diagram

The usual Robinson map (see Definition 3.7) gives a map

\[
\varphi : \Pi_{\infty}^d \to \Xi.
\]

When we extend this map to a map (still called \( \varphi \)) defined on paths of lower dimension (paths in \( \mathcal{B}^j \), with \( j < d \), and to generalized paths, we don’t get a \( \Xi \)-valued map. Actually, when \( x \) is a path which is not in \( \mathcal{B}^d \), we are really interested in the class of \( \varphi(x) \) under translation. We explain now the construction of \( \varphi \), which may involve some arbitrary choices. It is very similar to the construction of the usual Robinson map.

**On paths.** As in Sect. 3.1, the generalized Robinson map is defined inductively: by defining \( \varphi_n, n \in \mathbb{N} \) which satisfy good properties. Let \( \gamma \in \Pi_{n_0,n} \). By similarity with the definition of the usual Robinson map, one wants to define \( \varphi_n \) as follows:
• if the vertex \( r(\gamma) \) corresponds to the inclusion \( f \subset f' \), then \( \varphi_n(\gamma) \) is some translate of \( \omega_j^n(f') \);
• two different paths ending at the same vertex correspond to two (possibly different) translates of the same patch;
• if \( \gamma \in \Pi^{j}_{\eta_0,n} \) is a prefix of a path \( \eta \in \Pi^{j}_{\eta_0,n+m} \), then \( \varphi_n(\gamma) \subset \varphi_{n+m}(\eta) \).

Note also that the second condition implies the following: if \( \gamma \) and \( \eta \cdot \gamma \) are two paths (which start at different depths), then \( \varphi_n(\gamma) \) and \( \varphi_n(\eta \cdot \gamma) \) correspond to the same patch, up to translation.

These definitions need, however, to be explained, since strictly speaking, \( \varphi_n(\gamma) \) is a set of edges, not a patch: \( \varphi_n(x) \) a priori can’t be made into a tiling just by taking a union.

The way to actually define \( \varphi_n(\gamma) \) as a patch is the following. Remember that faces are decorated. Therefore, a \( j \)-dimensional face \( f \) is
\[
\begin{align*}
    f &= (f_0, \text{Col}(f)),
\end{align*}
\]
where \( f_0 \) is a \( j \)-dimensional compact set (dimension defined using the cellular structure of tiles), and \( \text{Col}(f) \) is a patch of tiles which intersect \( f_0 \). Just for this section, define \( p(f) \) as the patch of all tiles in \( \text{Col}(f) \) which have \( f_0 \) as a \( j \)-dimensional face.

Now, we can define \( \varphi_n(\gamma) \) as the patch:
\[
\varphi_n(\gamma) = \omega^n(p(f)) + x,
\]
for some \( x \in \mathbb{R}^d \) determined by the path \( \gamma \) (minus the last edge).

Now, if a sequence of \( \varphi_n \) satisfying the conditions above exist, we just define \( \varphi(x) \) as the union of the \( \varphi_n(x_{[0,n]}) \). To prove that such a sequence of \( \varphi_n \) actually exist: we just need to explain how to:

1. extend \( \varphi_n(\gamma) \) to \( \varphi_n(\gamma \cdot e) \) when \( e \) is a single edge,
2. define \( \varphi_n(\gamma) \) when \( \gamma \) is a path of length 0 (is a single vertex).

The first point is pretty clear, the edge \( e \) encodes the following chain of inclusions:
\[
    f \subset \omega(f') \subset \omega^2(f''),
\]
so (by applying the substitution \( n - 1 \) times), it encodes a specific occurrence of \( \varphi_n(\gamma) \) in the patch \( \varphi_{n+1}(\gamma \cdot e) \). Therefore, if \( \varphi_n(\gamma) \) is well defined, for any edge \( e \) extending \( \gamma \), we can define \( \varphi_{n+1}(\gamma \cdot e) \). For the second point, consider \( \gamma = (v) \) a single vertex \( (v \in \nu^{j}_{\eta}) \), with \( v \) corresponding to the inclusion of faces “\( f \subset f' \)”. Consider the translate of \( \omega(p(f')) \), such that the puncture of \( f \subset \omega(f') \) lies at the origin. Then apply \( \omega^{n-1} \) to this patch, and define it as \( \varphi_n(\gamma) \).

Now that \( \varphi_n \) is defined for all \( n \), define \( \varphi(x_m, x_2, \ldots) \) as the union of the patches \( \varphi_n(x_m, \ldots, x_n) \). As for the usual Robinson map, it could happen that \( \varphi(x) \) is just a partial tiling. However, using the decorations, it is always possible to define \( \varphi(x) \) as being a tiling.
Proposition 3.15. There is a continuous mapping, called Robinson map,
\[ \varphi : \bigcup_j \Pi_j^\infty \longrightarrow \Omega, \]  
(8)
such that for all \( n \in \mathbb{N} \), \( \varphi_n(x) \subset \varphi(x) \).

Let us prove this proposition (the only thing left to prove is that the decoration always allow to associate a tiling of all \( \mathbb{R}^d \) to a path). To this purpose, we introduce a notation to denote the decorated version of the \( \varphi_n \).

Notation 3.16. For \( n \in \mathbb{N} \), define \( \varphi_n^c(\gamma) \) in the same way as \( \varphi_n(\gamma) \), just requiring that if \( r(\gamma) \) is the vertex \( v \) corresponding to \( f \subset f' \), then
\[ \varphi_n^c(\gamma) = \omega_n(\text{Col}(f')). \]

Remark that since \( p(f) \subset \text{Col}(f) \), then \( \varphi_n(\gamma) \subset \varphi_n^c(\gamma) \). The map \( \varphi_n^c \) is the collared version of the map \( \varphi_n \).

Lemma 3.17. Given \( \gamma \in \Pi_n \), \( \varphi_n^c(\gamma) \) is a patch which contains a ball of radius \( \rho \lambda^n \), for a certain constant \( \rho \).

Proof. It is a straightforward consequence of Lemma 2.20: apply the substitution \( n \) times to the first bullet-point of Lemma 2.20. \( \square \)

Now, it is clear that \( \bigcup_{n \in \mathbb{N}} \varphi_n^c(\gamma(x_{[0,n]})) \) is a tiling of all \( \mathbb{R}^d \). We take this as the definition of \( \varphi(x) \), and it proves the proposition. We could have defined directly \( \varphi_n^c \) and not \( \varphi_n \), but it is interesting to see on which paths \( \varphi_n \) fails to be a tiling of all \( \mathbb{R}^d \).

Remark that \( \varphi \) is \textit{a priori} not one-to-one, and never onto.

Notation 3.18. We use the following convention
\[ T_x := \varphi(x), \text{ for } x \in \Pi_j^\infty, \quad \text{and} \quad x_T := \varphi^{-1}(T), \text{ for } T \in \Xi, \]  
(9)
where \( \varphi \) is the homeomorphism of Theorem 3.15.

On generalized paths. The definition of the Robinson map to generalized paths (see Definition 3.13) is very similar (and straightforward). Just note that if \( z = (z_n, \ldots, z_m) \) with \( z_m = (v_1, v_2) \) a pair of vertices, then \( v_1 \) corresponds to \( f_1 \subset f' \) and \( v_2 \) corresponds to \( f_2 \subset f' \) with the same \( f' \).

Furthermore, an escaping edge encodes the occurrence of a face \( f \) in the substitution of a face of greater dimension, but it still allows to define \( \tilde{\varphi}_{n+1} \) in a compatible way with respect to \( \tilde{\varphi}_n \).

For these reasons, it is possible to define \( \tilde{\varphi} \) for generalized paths.

Proposition 3.19. There is a continuous mapping, called Robinson map,
\[ \tilde{\varphi} : \tilde\Pi^\infty \longrightarrow \Omega, \]  
(10)
such that for all \( n \in \mathbb{N} \), \( \tilde{\varphi}_n(z) \subset \tilde{\varphi}(z) \).

Note that the restriction of \( \tilde{\varphi} \) to \( \Pi^d_\infty \) is the usual Robinson map, and is a homeomorphism onto \( \Xi \).
Figure 9. A tiling of border dimension 0 can be obtained as a fixed point of this map.

Figure 10. A tiling of border dimension 1 can be obtained as a fixed point of this map.

Notation 3.20.

\[ \tilde{T}_z := \varphi(z), \text{ for } z \in \tilde{\Pi}^d_{*,\infty}. \] (11)

where \( \varphi \) is the map of Proposition 3.19.

4. Borders

We show here the geometrical meaning of Bratteli multi-diagrams for tilings. Generalizing a construction of Matui [14], we introduce the notion of border of a tiling in Sect. 4.1. We then show how the multi-diagram naturally encodes this in Sect. 4.2. We introduce there the border of a path and show in Proposition 4.8 that the two notions are equivalent. We end with important properties of such borders and their translational orbits.

4.1. Borders of a Tiling

If we iterate the substitution shown in Figs. 9 and 10 (forgetting the border forcing) we obtain only a partial tiling of the space: a half-plane in the second case (with “border” a line), and a quadrant (with “border” two intersecting half-lines) in the second case. Figures 9 and 10 illustrate with the chair tiling the notion of tilings of border dimension 0 and 1 respectively.
This notion of border of a tiling was first introduced by Matui [14] for substitution tilings of $\mathbb{R}^2$.

**Definition 4.1.** For a tiling $T$ in $\Omega$, and $0 \leq j \leq d$, let $T^j$ denote the union of the supports of the $j$-faces of $T$. The $j$th border of a tiling $T$ is defined as

$$b^j(T) = \bigcap_{n \in \mathbb{N}} \lambda^j \omega_n^{-n}(T)^j. \quad (12)$$

The border dimension of a tiling $T \in \Omega$ is the minimum of all $j$ such that $b^j(T)$ is not empty

$$\text{bdim}(T) = \min\{j \leq d \mid b^j(T) \neq \emptyset\}.$$ 

Remark that $b^j(T)$ is the “support” of the union of faces $\bigcap_{n \in \mathbb{N}} \omega_n^{j} (\omega_n^{-n}(T))$, see Fig. 11.

We have the elementary following properties.

**Lemma 4.2.** (i) For any $a \in \mathbb{R}^d$, one has $b^j(T + a) = b^j(T) + a$.

(ii) Let $T \in \Xi$. For $n \in \mathbb{N}$ let $t_n$ be the tile of $\omega_n^{-n}(T)$ that contains the origin, and $p_n^j$ the union of the supports of the $j$-faces that intersect it. One has

$$b^j(T) = \bigcup_{m \in \mathbb{N}} \bigcap_{n \geq m} \lambda^n p_n^j.$$ 

**Proof.** By definition of the substitution one has $\omega_n^a(T + a) = \omega_n^a(T) + \lambda^na$ and (i) follows immediately.

Let $c^j(T)$ be the right hand side of the equation in (ii). For any $T$ one has $\lambda \omega_n^{-1}(T)^j \subset T^j$, so one can rewrite $b^j(T)$ as $\bigcup_{m \in \mathbb{N}} \bigcap_{n \geq m} \lambda^n \omega_n^{-n}(T)^j$. Since $p_n^j \subset \omega_n^{-n}(T)^j$ one has $c^j(T) \subset b^j(T)$. To prove the other inclusion consider $a \in b^j(T)$, so $a \in \lambda^n \omega_n^{-n}(T)^j$ for all $n$. Let $p_n$ denote the union of faces whose support is $p_n^j$. Then $p_n$ contains $t_n$ in the interior of its support, so as $n \to \infty$, $\lambda^n p_n$ eventually covers $\mathbb{R}^d$. Hence for all $n$ large enough $a \in \lambda^n p_n^j$, and this proves the other inclusion $b^j(T) \subset c^j(T)$ and completes the proof. \hfill $\square$
4.2. Borders of a Path

We will here build borders of paths using the horizontal structure.

Via the Robinson map, a tiling $T$ is encoded by a sequence of vertices $x_T = (x_n)_{n \in \mathbb{N}} \in \Pi^d_{\infty}$ in the multi-diagram. If $\text{bdim}(T) < d$, as shown in Lemma 4.2, the $j$-faces of the tile corresponding to $x_n$ can be used for all $n$ to build the border of $T$. In the multi-diagram, there are horizontal edges between $x_n$ and the vertices encoding those $j$-faces. We show here how to associate to the path $x_T$ sequences of such vertices which represent those $j$-faces for each $n$ and keep track of how they match together from $n$ to $n + 1$: this will define derived paths from $x_T$ and in turns the border of $x_T$.

4.2.1. Derived Paths. There is a natural concatenation on paths. Given two paths $\gamma$ and $\eta$ with $r(\gamma) = s(\eta)$ we denote by $\gamma \cdot \eta$ their concatenation.

Remember that a horizontal edge occurs when a tile is included in a supertile, and this inclusion induces an inclusion of faces.

Definition 4.3. Let $F$ be a set of vertices in some $\mathcal{V}^j$. We define $F'$ as the set of all vertices $v \in \mathcal{V}^{j-1}$ such that there is an edge in $H$ from an element of $F$ to $v$:

$$F' = r\left((s_{|H})^{-1}(F)\right).$$

For all $0 \leq p \leq j$, we define $F^{(p)} := (F^{(p-1)})'$.

In particular, if $x \in \Pi^d_{\infty}$ is a path, it is possible to define all paths which lie in its boundary. We adopt the following notation: if $x$ is a path, $\{x\}$ is the set of all vertices it goes through, and so it makes sense to define $\{x\}'$.

Definition 4.4. Given $x \in \Pi^d_{\infty}$ and $j \in \{0, \ldots, d\}$, define:

$$b^j(x) = \{y = (y_n, y_{n+1}, \ldots) \in \Pi^j_{n, \infty}; \ n \in \mathbb{N}; \ \forall i \geq n, \ y_i \in \{x\}^{(d-j)}\}.$$

We define $b(x) = \bigcup_{0 \leq j \leq d} b^j(x)$. We call $b^j(x)$ the set of boundary paths of $x$ of dimension $j$.

In particular, $b^j(x)$ is a set of paths in $\mathcal{B}^j$. It could very well be empty, and actually, we will see that it is generically empty for $j < d$ (see Remark 4.14).

Definition 4.5. The border dimension of a path $x \in \Pi^d_{\infty}$ is the minimum of all $j$ such that $b^j(x)$ is not empty

$$\text{bdim}(x) = \min\{j \leq d ; \ b^j(x) \neq \emptyset\}$$

The rest of this Sect. 4.2.1 can be skipped on first reading. We can also derive generalized paths from a given path $x = (x_1, x_2, \ldots) \in \Pi^d_{\infty}$ (so that each $x_i \in \mathcal{V}_i$). Let

$$\mathcal{F}_x = \{v \in \mathcal{V}; \ \exists j \leq d \ v \in \{x\}^{(j)}\}.$$

In other words, $\mathcal{F}_x$ is the set of all vertices of the path $x$, or in any vertex set derived from $x$.

Definition 4.6. Let $x \in \Pi^d_{\infty}$. A path $y = (y_n, y_{n+1}, \ldots) \in \tilde{\Pi}_{n, \infty}$ is said to be derived from $x$ if for all $k \geq n$
• either \( y_n \) is a single vertex and belongs to \( \mathcal{F}_x \),
• or \( y_n \) is a pair of vertices, at least one of which belongs to \( \mathcal{F}_x \).

The set of all generalized paths derived for \( x \) is:
\[
\tilde{b}(x) = \{ y \in \tilde{\Pi}_{n,\infty}; \ n \in \mathbb{N} \text{ and } y \text{ is derived from } x \}.
\]

**Remark 4.7.** Any generalized path has a tail in \( \Pi_{j,\infty} \) for some \( 0 \leq j \leq d \). Hence \( x \in \Pi_{d,\infty} \) has border dimension \( j \) if and only if there exists a generalized path in \( \tilde{b}(x) \) with tail in \( \Pi_{j,\infty} \) and none with tail in \( \Pi_{i,\infty} \) for any \( i < j \).

### 4.2.2. Orbit Properties of Borders.

We first show the equivalence between the two notions of borders.

**Proposition 4.8.** For any \( x \in \Pi_{d,\infty} \) one has
\[
\text{bdim}(x) = j \iff \text{bdim}(T_x) = j.
\]

**Proof.** One writes \( t_n \) for the tile of \( \omega^{-n}(T) \) that contains the origin (and corresponds to the vertex of \( x_T \) at depth \( n \)).

Consider the case \( j < d \) first. Assume \( \text{bdim}(x) = j \) and let \( z \in b^j(x) \). Let \( f_n \) be the \( j \)-face on the boundary of the tile \( t_n \), for \( n \) larger than some \( m \) so that it is defined. For all \( n \geq m \) we have \( \text{Col}(f_n) \in p_n^j \), and by Lemma 4.2 (ii) \( b^j(T) \neq \emptyset \). Now if \( b^i(T) \neq \emptyset \) for some \( i < j \), by Lemma 4.2 (ii), for all \( n \) larger than some \( n_1 \) there exists \( i_n \)-faces \( f_n \) such that \( f_n \in \text{Col}(t_n)^i \) and with \( f_n \in \omega(f_{n+1}) \), for some non decreasing integers \( i_1 \) taking values in the finite set \( S = \{ i, i + 1, \cdots, j - 1 \} \). So for all \( n \) large enough the \( f_n \) have eventually a fixed dimension \( i' \in S \), and this proves that \( \text{bdim}(x) = i' < j \) which is a contradiction.

Conversely, if \( b^j(T_x) \neq \emptyset \) and \( b^i(T_x) = \emptyset \) for all \( i < j \), then by Lemma 4.2 (ii) there exists and infinite path in \( b(x) \cap \Pi_{i,\infty} \), and none in \( b(x) \cap \Pi_{i,\infty} \).

This proves that \( \text{bdim}(b) = j \).

Now consider the case \( j = d \). Assume \( \text{bdim}(x) = d \). For all \( n, t_n \subset p_n^d = \text{Col}(t_n)^d \) and, therefore, \( b^d(T_x) \neq \emptyset \). If for some \( i < d \), \( b^i(T_x) \neq \emptyset \) then by the same argument given for the case \( j < d \) above, one can build a generalized path in \( \tilde{b}(x) \) with tail in \( \Pi_{i,\infty} \) and this contradicts \( \text{bdim}(x) = d \). For the converse, the proof is the same as for the case \( j < d \) above. \( \Box \)

We now turn on to orbit properties of borders. We state an important lemma, which justifies our choice for the combinatorial definition of boundary paths.

We need, however, the following hypothesis (which is satisfied for many tilings, such as the Penrose tiling) in order for the lemma to be true. This hypothesis can be lifted by adapting slightly the definition of the diagram. However, this would add a layer of complexity to the construction. We give an informal definition of why this hypothesis can be lifted in the last section.

**Hypothesis 4.9.** For any tile \( t \in A^d \) and any face \( f \in A^j \), there is at most one occurrence of \( f \) in the boundary of \( t \).
Lemma 4.10. Let \( x \in \Pi^d_\infty \) be a path in the Bratteli diagram, and \( y \in \mathcal{B}^j(x) \) be a boundary path of \( x \) of dimension \( j \). Then the tilings \( T_x \) and \( T_y \) are in the same orbit under translation.

Proof. Let \( x = (x_1, x_2, \ldots) \) be the path \( x \) coded by vertices in \( \mathcal{B}^d \). We first assume that \( y = (y_1, y_2, \ldots) \), with \( y_i \in \mathcal{V}^j_i \). We built jointly \( \varphi_n(x_1, \ldots, x_n) \) and \( \varphi_n(y_1, \ldots, y_n) \), and prove that they define the same tiling (up to translation).

It is clear that \( \varphi_1(x_1) \) and \( \varphi_1(y_1) \) fit into the same picture: by definition of the chain of horizontal edges between \( x_1 \) and \( y_1 \), \( x_1 \) corresponds to a tile \( t \) included in the substitution of another tile \( t' \), and this inclusion induces the inclusion of faces corresponding to \( y_1 \).

Now, assume that the definitions of \( \varphi_k \) and \( \varphi_k \) are consistent for \( x \) and \( y \) up to \( k = n \). One has the following square in the diagram.

\[
\begin{array}{ccc}
  x_n & & y_n \\
  \downarrow & & \downarrow \\
  x_{n+1} & & y_{n+1}
\end{array}
\]

To fix the notations, assume \( x_n \) corresponds to \( t \subset t' \), \( x_{n+1} \) corresponds to \( t' \subset t'' \), \( y_n \) corresponds to \( f \subset f' \) and \( y_{n+1} \) corresponds to \( f' \subset f'' \) (using Notations 3.9). Because of Hypothesis 4.9, there is only one occurrence of \( f' \) on the boundary of \( t' \). Therefore, since the occurrence \( x_n \) of \( t \subset t' \) occurs on the face \( f' \) of \( t' \) and \( y_n \) corresponds to \( f' \subset f'' \), then the occurrence of \( t \) in \( t'' \) encoded by the left vertical edge induces the occurrence \( f \subset f'' \) encoded by the right vertical edge.

Therefore, the inclusion \( f \subset f' \subset f'' \) appears in \( \varphi_{n+1}(x_1, \ldots, x_{n+1}) \).

Inductively, the sentence above is true for all \( n \). When applied to \( \varphi \), one gets that “the limit of patches \( \varphi_n(y) \) appears in \( \varphi(x) = T_x \)”, that is the tiling \( \varphi(y) = T_y \) and \( \varphi(x) = T_x \) are translates of each other.

If \( y \) starts at depth \( n \), then the path coding \( \omega^{-n}(\varphi(x)) \) and \( \omega^{-n}(\varphi(y)) \) start from the depth 1, and are on the boundary of one another. Therefore, the result for \( y \) starting at depth \( n \) reduces to the one we proved above. \( \square \)

We further study the properties of borders of the same path. Note that, if not empty, the set \( \mathcal{B}^j(x) \) contains infinitely many paths in \( \Pi^j_{\infty} \). We show now that any two paths in \( \mathcal{B}^j(x) \) are tail-equivalent in \( \mathcal{B}^j \).

Lemma 4.11. Let \( x \in \Pi^d_\infty \) with \( \text{bdim}(x) = j \), then any two infinite paths in \( \mathcal{B}^j(x) \) are tail-equivalent in \( \mathcal{B}^j \).

Proof. Let \( z, z' \in \mathcal{B}^j(x) \). Let \( t_n \) be the tile corresponding to the vertex in \( x \) at depth \( n \), and \( f_n, f'_n \in \partial t_n \) the \( j \)-faces corresponding to the vertices in \( z, z' \), respectively (at depth \( n \) larger than some \( m \) large enough so that they both are defined). For all \( n \geq n_0 \) we have the inclusions

\[
\lambda^{-n+m}\text{Col}(f_m) \subset \text{Col}(f_n) \subset \partial t_n, \quad \text{and} \quad \lambda^{-n+m}\text{Col}(f'_m) \subset \text{Col}(f'_n) \subset \partial t_n.
\]
The faces \( f_m \) and \( f'_m \) belongs to \( \partial t_m \), and \( \lambda^{-n+m}t_m \) shrinks to a point as \( n \) tends to infinity, therefore \( \text{dist}(f_n, f'_n) \rightarrow 0 \) as \( n \rightarrow \infty \). Hence by Lemma 2.20 the faces must eventually intersect: \( f_n \cap f'_n \neq \emptyset \) for all \( n \) greater than some \( n_1 \).

We now show that \( f_n = f'_n \) for all \( n \) greater than some \( n_2 \geq n_1 \), which proves \( z \overset{\text{tail}}{\sim} z' \). It suffices to show that \( f_{n_2} = f'_{n_2} \). Indeed if this holds, then both \( f_{n_2+1} \) and \( f'_{n_2+1} \) contain \( f_{n_2} \) in their substitute, and so must be the same face of \( t_{n_2+1} \). And by immediate induction we get \( f_n = f'_n, n \geq n_2 \).

Assume that it is not the case: for each \( n \geq n_1 \), \( f_n \cap f'_n \neq \emptyset \) but \( f_n \neq f'_n \). Then there exists a \( k_n \)-face \( g_n \subset \partial f_n \cap \partial f'_n \), with \( k_n < j \). Since \( f_n \in \omega(f_{n+1}) \) and \( f'_n \in \omega(f'_{n+1}) \) we have \( g_n \in \omega(g_{n+1}) \) for all \( n \geq n_1 \). If \( g_{n_1} \) is a \( k_{n_1}-\text{cell} \), then \( g_n \) is a \( k_n\)-cell with \( k_n \geq k_{n_1} \). The sequence of cells dimensions \((k_n)_{n \geq n_1}\) is thus non decreasing and takes on values in the finite set \( S = \{k_{n_1}, k_{n_1}+1, \ldots, j-1\} \). Therefore, it is eventually constant: there exists \( k \in S \) and \( n_2 \geq n_1 \), such that \( k_n = k \) for all \( n \geq n_2 \). In other words, the sequence of faces \((g_n)_{n \geq n_1}\) defines a generalized path in \( \tilde{b}(x) \) with tail in \( \Pi^k_{\infty} \), for some \( k \leq j-1 \).

We can now characterize the notion of border dimension in terms of orbit properties. The AF-equivalence relation \( R_{\text{AF}} \) on \( \Xi^d \) induces an AF-equivalence relation \( R'_{\text{AF}} \) on \( \Xi \) via the Robinson map (Definition 3.7 and Theorem 3.8): for \( T \in \Xi \) we set

\[
[T]_{\text{AF}} = \left\{ T - a \in \Xi \mid x_{T-a} \overset{\text{AF}}{\sim} x_T \right\}.
\]

**Lemma 4.12.** Let \( T \) be a tiling in \( \Xi \), then

\[
[T]_{\text{AF}} = [T]_{R_{\Xi}} \iff \text{bdim}(T) = d,
\]

and equivalently

\[
[T]_{\text{AF}} \subsetneq [T]_{R_{\Xi}} \iff \text{bdim}(T) < d.
\]

**Proof.** Using the Robinson map and Proposition 4.8, we see that the above statements are equivalent to the following: let \( x \in \Pi^d_{\infty} \) be a rooted infinite path in \( \mathcal{B}^d \), then

\[
\text{bdim}(x) = d \iff \lim_{n \rightarrow +\infty} \text{dist}(\varphi_1(x), \partial \varphi_n(x)) = +\infty,
\]

or equivalently

\[
\text{bdim}(x) < d \iff \lim_{n \rightarrow +\infty} \text{dist}(\varphi_1(x), \partial \varphi_n(x)) < +\infty,
\]

where \( \partial \varphi_n(x) \) is a shorthand notation for the boundary of the support of \( \varphi_n(x) \).

Since the sequence of distances has always a limit in \( \mathbb{R}_+ \cup \{+\infty\} \), the equivalence between the two statements is clear. We prove the second statement. Write \( d_n = \text{dist}(\varphi_1(x), \partial \varphi_n(x)) \). Let \( t_n \) denote the tile corresponding to the vertex in \( x \) at depth \( n \).

If \( \text{bdim}(x) = j < d \), there exists an infinite path \( z \) in \( b(x) \cap \Pi^j_{\infty} \) for some \( m \). Let \( f_n \) denote the \( j \)-cell corresponding to the vertex of \( z \) at depth \( n \geq m \). For all \( n \geq m \), \( f_n \in t_n \), so \( \lambda^n \text{Col}(f_n) \) appears on the boundary of
Col(φₙ(x)), thus dist (φₙ(x), ∂φₙ(x)) = 0 and, therefore, dₙ = dₘ for all n ≥ m.

Conversely, by Lemma 2.20, if φₙ(x) belongs to the interior of φₙ₊₁(x) then dₙ₊₁ > dₙ + ρ. Otherwise if ∂φₙ(x) ∩ ∂φₙ₊₁(x) ̸= ∅, then dₙ = dₙ₊₁. Therefore, if the sequence (dₙ)ₙ∈ℕ converges, it must be eventually constant.

Now assume that the sequence converges, say to a ∈ ℝᵈ, and let n₁ ≥ m be such that dₙ = a for all n ≥ n₁. We thus have ∂φₙ₊₁(x) ∩ ∂φₙ(x) ̸= ∅ for n ≥ n₁. Let then f'ₙ be a cell that appears in tₙ and such that λⁿf'ₙ ⊂ ∂φₙ₊₁(x) ∩ ∂φₙ(x), n ≥ n₁. We, therefore, have f'ₙ ∈ ω(f'ₙ₊₁). If f'ₙ is a k₁-cell, then fₙ is a kₙ-cell with kₙ ≥ k₁. The sequence of cells dimensions (kₙ)ₙ≥n₁ is non decreasing and takes on values in the finite set S = {k₁, k₂ + 1, · · · , d − 1}. So for all n large enough the f'ₙ have eventually a fixed dimension k' ∈ S, and this proves that bdim(x) = k' < d which is a contradiction.

An easy consequence of minimality gives the following.

**Proposition 4.13.** The set \{x ∈ Πᵈ₀; bdim(x) = j\} is dense in Πᵈ₀, for any j ≤ d.

**Proof.** Call the above set Bₗ. Pick x ∈ Πᵈ₀, and y ∈ Bₗ. For n ∈ ℕ, by minimality, there exists m ≥ n such that the tile corresponding to the vertex in x at depth n appears in the (m − n)th substitute of the tile corresponding to the vertex in y at depth m. This means that there exists a path γ ∈ Πᵈ₀ with s(γ) = r(x[0,n]) and r(γ) = s(y[m,∞)). Define xₙ = x[0,n] · γ · y[m,∞]. For all n, xₙ belongs to Bₗ, and the sequence (xₙ)ₙ∈ℕ converges to x in Πᵈ₀. This proves that Bₗ is dense in Πᵈ₀.

**Remark 4.14.** We see from Lemma 4.12 and Proposition 4.8 that the border of a tiling cannot be “crossed” by the AF-equivalence relation R’AF. For example, the tilings shown in Figs. 9 and 10 give rise to only partial tilings of the plane (upper right quadrant, and upper half plane respectively), but extend uniquely to tilings of the whole plane via the Robinson map φ, provided that the Bratteli diagram is built using decorated tiles. However, the R’AF orbits of those tilings do not match their RΞ orbits: the AF-relation cannot identify two tilings whose punctures lie on two different sides of the borders. The authors explained this in detail in the first paper [4], Sect. 3.3 (Remark 3.14 in particular), in a more general setting (without a substitution).

By Proposition 4.13, the set of tilings of border dimension less than or equal to d − 1 is dense in Ξ. A result of Radin and Sadun ("Property F" in [16]) shows, however, that this set has measure zero with respect to any invariant measure on Ξ. Such a set is called thin in the literature. So one sees that RΞ differs from the AF-relation R’AF on thin set.

**5. Equivalence Relation in a Bratteli Multi-Diagram**

In this section, we reconstruct the equivalence relation RΞ from the multi-diagram. In the first section, Sect. 5.1, we define an equivalence relation RΞ
on $\Pi^d_{\infty}$ from the multi-diagram and prove that the Robinson map induces an isomorphism $\mathcal{R}_B \cong \mathcal{R}_\Xi$ (Proposition 5.4). In the last section, Sect. 5.2, we show that this isomorphism is actually a homeomorphism (Theorem 5.8) and deduce that $\mathcal{R}_B$ is an étale equivalence relation. We then describe this étale topology of $\mathcal{R}_B$, and give a base of open sets using the multi-diagram. This last section uses the technicalities of generalized paths in the multi-diagram introduced in Sect. 3.2.3; it is more involved and can be skipped on a first reading.

5.1. The Equivalence Relation on the Multi-Diagram

As a consequence of lemma 4.11, we can associate to a path $x \in \Pi^d_{\infty}$ of border dimension $\text{bdim}(x) = j$, the tail-equivalence class of its $j$th border in $B^j$.

**Definition 5.1.** We say that two infinite paths $x$ and $y$ in $\Pi^d_{\infty}$ are border equivalent, and we write $x \sim y$, if

(i) $\text{bdim}(x) = \text{bdim}(y) = j$ for some $0 \leq j \leq d$, and

(ii) $b^j(x) \sim b^j(y)$ in $B^j$.

This defines an equivalence relation on $\Pi^d_{\infty}$: $\sim$ is clearly symmetric and reflexive, and transitivity follows from that of $\sim^{\text{tail}}$.

**Definition 5.2.** Define the equivalence relation on $\Pi^d_{\infty}$

$$\mathcal{R}_B = \{(x, y) \in \Pi^d_{\infty} \times \Pi^d_{\infty} ; x \sim y\}.$$ 

**Remark 5.3.** We can write $x \sim_j y$, in the case where $\text{bdim}(x) = \text{bdim}(y) = j$ to specify the dimension of the borders, and call $\mathcal{R}_B^j$ the corresponding sub-relation. We thus view the equivalence relation $\mathcal{R}_B$ as the union of the $\mathcal{R}_B^j$ over $j = 0, \cdots d$:

$$\mathcal{R}_B = \bigcup_{j=0}^d \mathcal{R}_B^j.$$ 

By definition, if $\text{bdim}(x) = \text{bdim}(y) = d$ we see that $x \sim y$ if and only if $x \sim^{\text{AF}} y$, so that $\mathcal{R}_B^d = \mathcal{R}_\text{AF}$. So the equivalence relation $\mathcal{R}_B$ contains $\mathcal{R}_\text{AF}$ as a natural sub-relation. However, by Remark 4.14, the inclusion is not an equality:

$$\mathcal{R}_\text{AF} = \mathcal{R}_B^d \subsetneq \mathcal{R}_B.$$ 

Indeed the two relations only coincide on the set of paths or border dimension $d$. They differ on the dense set of paths of border dimension less than $d - 1$ (Proposition 4.13).

Also, it is important to notice that $\mathcal{R}_B^j$ is not an AF-relation for $j < d$. Indeed, its base (namely the set of paths of border dimension $j$) is neither compact, nor locally compact in $\Pi^d_{\infty}$.

**Proposition 5.4.** The Robinson map induces an isomorphism $\mathcal{R}_B \cong \mathcal{R}_\Xi$. 

Proof. We show that $x \sim y$ if and only if $T_x$ and $T_y$ are translate of each other.

Consider $(x, y) \in \mathcal{R}_B$. By definition, $x$ and $y$ have a common derived path $z \in b^j(x) \cap b^j(y)$ for some $j$. By Lemma 4.10, $T_z$ is in the translational orbit of both $T_x$ and $T_y$ and, therefore, $T_x$ and $T_y$ are translate of each other.

Conversely, consider $T, T' \in \Xi$ with $T = T' + a$ for some $a \in \mathbb{R}^d$. There exists $m$ such that for all $n \geq m$, the intersection $\varphi_n(x_T) \cap (a + \varphi_n(x_{T'}))$ is non empty, and thus contains the substitute $\omega^n(f_n)$ of a face. This defines a sequence of faces $(f_n)_{n \geq m}$. For $n \geq m$, $f_n$ appears on the “common boundary” of $t_n$ and $t_n'$ and thus $f_n \subset \omega(f_{n+1})$. The dimensions of those faces can only increase and, therefore, stabilize to a fixed value $j \in \{0, 1, \cdots d\}$ for $n$ large enough, say $n \geq m_1$. The path $z = (f_n)_{n \geq m_1} \in \Pi_{m_1, \infty}$ is, therefore, in the common borders of $x_T$ and $x_{T'}$. Now by Lemma 4.11, this implies $b^j(x_T) \sim b^j(x_{T'})$ and so proves $x_T \sim x_{T'}$. □

5.2. The étale Topology

We now turn on to the topology of $\mathcal{R}_\Xi$, and we use now generalized paths as introduced in Sect. 3.2.3.

First notice that any generalized path has a tail in $\Pi_{\infty}^d$ for some $j$. Hence two infinite paths $x$ and $y$ in $\Pi_{\infty}^d$ are border equivalent if and only if they have a common generalized derived path: $b(x) \cap b(y) \cap \tilde{\Pi}_{\infty} \neq \emptyset$. So one can rewrite the equivalence relation as

$$\mathcal{R}_B = \left\{ (x, y) \in \Pi_{\infty}^d \times \Pi_{\infty}^d ; \ b(x) \cap b(y) \cap \tilde{\Pi}_{\infty} \neq \emptyset \right\}.$$ 

We topologize $\mathcal{R}_B$ as follows.

Definition 5.5. The equivalence relation $\mathcal{R}_B$ is endowed with the following topology: $(x_n, y_n)_{n \in \mathbb{N}}$ converges to $(x, y)$, if and only if

(i) $x_n \rightarrow x$ and $y_n \rightarrow y$ in $\Pi_{\infty}^d$, and
(ii) there exists $m \in \mathbb{N}$ such that $b(x_n) \cap b(y_n) \cap \tilde{\Pi}_{m, \infty} \neq \emptyset$ for all $n$ large enough.

We state two technical results that we shall need for Theorem 5.8.

Corollary 5.6. If $\tilde{b}(x) \cap \tilde{b}(y) \cap \tilde{\Pi}_{m, \infty} \neq \emptyset$, then $T_x = T_y + a(x, y)$ with $|a(x, y)| \leq c\lambda^m$, where $c > 0$ is a constant that does not depend on $x, y, m$.

Proof. By the remark preceding Definition 5.5, $x$ and $y$ are border equivalent and, therefore, by Proposition 5.4 there exist $a(x, y) \in \mathbb{R}^d$ such that $T_x = T_y + a(x, y)$. Pick $z \in b(x) \cap b(y)$. There is a vector $u_m$ that links the puncture of $\tilde{x} S_m^c(z)$ to the puncture of a copy of $\varphi_m(x)$ that it contains. Hence $u_m = \lambda^m u$, where $u$ is the vector linking the puncture of the face $f_m$ associated with $z_m$ to the puncture of a copy of the tile associated with $x_m$. The same argument shows that there is a vector $v_m = \lambda^m v$, where $v$ is the vector linking the puncture of $\operatorname{col}(f_m)$ to the puncture of one of its tiles. Now the distance from the puncture of a tile to the puncture of any of its faces is bounded above by the outer radius $R$ of the tiles, so we have $|u|, |v| \leq R$. Hence $|a(x, y)| = |u_m - v_m| \leq |u_m| + |v_m| \leq 2R\lambda^m$. □

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Lemma 5.7. If $T = T' + a$, then $\tilde{b}(x_T) \cap \tilde{b}(x_{T'}) \cap \tilde{\Pi}_{m_a,\infty} \neq \emptyset$ for some $m_a \in \mathbb{N}$ that only depends on $a$.

Proof. For each $n$, the origin and the point $-\lambda^{-n}a$ are punctures of tiles in $\omega^{-n}(T)$. For all $n$ large enough those tiles must then intersect, so they have at least a common face $f_n$. The distance between the punctures of two neighboring tiles is bounded below by $2r$, where $r$ is the inner radius of the tiles. Let $m_a$ be the smallest integer $n$ for which $\lambda^{-n}a \leq 2r$. The sequence of faces $(f_n)_{n \geq m_a}$ defines a generalized path in $\tilde{b}(x_T) \cap \tilde{b}(x_{T'})$, and this completes the proof. \qed

We can now state our main theorem.

Theorem 5.8. The two equivalence relations $\mathcal{R}_B$ and $\mathcal{R}_\Xi$ are homeomorphic:

$$\mathcal{R}_B \cong \mathcal{R}_\Xi.$$  

Proof. Consider the map $\varphi^*: \mathcal{R}_B \to \mathcal{R}_\Xi$, given by

$$\varphi^*(x, y) = (\varphi(x), \varphi(y)) = (T_x, T_y) = (T_x, T_x + a(x, y)).$$

By Proposition 5.4, $\varphi^*$ is a bijection. We now show that $\varphi^*$ and its inverse are continuous.

Consider a sequence $(x_n, y_n)_{n \in \mathbb{N}}$ that converges to $(x, y)$ in $\mathcal{R}_B$, and let $a_n = a(x_n, y_n)$. By definition of the convergence in $\mathcal{R}_B$, there is an $m \in \mathbb{N}$ such that $\tilde{b}(x_n) \cap \tilde{b}(y_n) \cap \tilde{\Pi}_{m,\infty} \neq \emptyset$ for $n$ large enough. Hence by Corollary 5.6, the sequence $(a_n)_{n \in \mathbb{N}}$ is bounded. By finite local complexity, it can only take finitely many values. Any convergent subsequence must, therefore, be eventually stationary; and its limit must be $a(x, y)$. Hence the sequence $(a_n)_{n \in \mathbb{N}}$ has a unique accumulation point, namely $a(x, y)$ and, therefore, converges to $a(x, y)$ in $\mathbb{R}^d$. Since $\varphi$ is a homeomorphism, $T_{x_n} \to T_x$ and $T_{y_n} \to T_y$ in $\Xi$. Hence $(T_{x_n}, T_{y_n}) \to (T_x, T_y)$ in $\mathcal{R}_\Xi$, and this proves that $\varphi^*$ is continuous.

Conversely, if $((T_n, T'_{n} = T_n + a_n))_{n \in \mathbb{N}}$ converges to $(T, T' = T + a)$ in $\mathcal{R}_\Xi$, then $a_n = a$ for all $n$ large enough. By Lemma 5.7, there exists $m_a \in \mathbb{N}$ such that $\tilde{b}(x_{T_n}) \cap \tilde{b}(x_{T'_n}) \cap \tilde{\Pi}_{m_a,\infty} \neq \emptyset$ for all $n$ large enough. Since $\varphi$ is a homeomorphism, $x_{T_n} \to x_T$ and $x_{T'_n} \to x_{T'}$ in $\Pi_{\infty}$ and, therefore, $(x_{T_n}, x_{T'_n}) \to (x_T, x_{T'})$ in $\mathcal{R}_B$. This proves that $(\varphi^*)^{-1}$ is continuous. \qed

Since $\mathcal{R}_\Xi$ is an étale equivalence relation (Definition 2.14), we have the immediate corollary.

Corollary 5.9. The equivalence relation $\mathcal{R}_B$ is étale.

Let $\overline{k}$ be the smallest integer such that for any tile $t$ any face $f$ of $t$, $\text{Col}(\omega^k(f))$ contains $\text{Col}(t)$. As a consequence of Corollary 5.6, one can check that a base of $\mathcal{R}_B$-sets for the étale topology of $\mathcal{R}_B$ is given by the following

$$O_{\gamma, \gamma', \eta, m} = \left\{(x, y) \in \mathcal{R}_B; \ x \in [\gamma], \ y \in [\gamma'], \ \tilde{b}(x) \cap \tilde{b}(y) \cap \tilde{\Pi}_{m-k,\infty} \neq \emptyset, \ \tilde{b}(x) \cap \tilde{b}(y) \cap \tilde{\Pi}_{m-k, m} = \eta\right\}.$$  (13)
where \( \gamma, \gamma' \in \Pi_{d,0}^{m}, m > \bar{k} \), and where \( \eta \) is a generalized path of length \( \bar{k} \) in \( \tilde{b}(\gamma_{[m-k,k]}) \cap \tilde{b}(\gamma'_{[m-k,k]}) \). One can check that those sets are compatible with the topology of \( R_{B} \). Also, one sees that the restrictions of the range and source maps to those sets are homeomorphic as follows.

The condition in equation (13) says that the first \( \bar{k} + 1 \) vertices of any generalized path in \( \tilde{b}(x) \cap \tilde{b}(y) \cap \tilde{\Pi}_{m-k,\infty} \) are determined by \( \eta = (z_{m-k}, z_{m-k+1}, \ldots, z_{m}) \). Hence for any \( (x, y) \in O_{\gamma\gamma',\eta,m} \), the patches \( \varphi_{m-k}(x) = \varphi_{m-k}^{c}(\gamma) \) and \( \varphi_{m-k}(y) = \varphi_{m-k}^{c}(\gamma') \) appear in \( \tilde{\varphi}_{m}(\eta) \), hence in both \( \varphi_{m}(x) = \varphi_{m}(\gamma) \) and \( \varphi_{m}(y) = \varphi_{m}(\gamma') \), and at respective positions which are uniquely determined by \( \eta \). Therefore, the distance between the tilings \( T_{x} \) and \( T_{y} \) is uniquely determined by the paths \( \gamma, \gamma' \) and the generalized path \( \eta \). In other words, given any compatible \( x \in [\gamma] \), there exists a unique \( y \in [\gamma'] \) such that \( (x, y) \in O_{\gamma\gamma',\eta,m} \), see Fig. 12 for an illustration. Notice that for fixed \( \gamma, \gamma', m \), two sets \( O_{\gamma\gamma',\eta_{1},m} \) and \( O_{\gamma\gamma',\eta_{2},m} \) are either equal or disjoint (this is because \( \tilde{\varphi}_{m}^{c} \) is not one-to-one: two generalized paths \( \eta_{1} \) and \( \eta_{2} \) might determine the same faces on the boundary of \( \varphi_{m}(\gamma) \) and \( \varphi_{m}(\gamma') \)).

Another immediate corollary of Theorem 5.8 is the following.

**Corollary 5.10.** The groupoid of the equivalence relation \( R_{B} \) is homeomorphic to \( \Gamma_{\Xi} \).

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6. Appendix

This paper presents in details what the structure of the orbit equivalence relation on a substitution tiling space is. The equivalence relation is made of an
Note that the right and left vertical sides of the gray tile define only one face. The gray tile appears in its own substitute eight times. The center-left tile and the center-right tile appear on the boundary of the patch $\omega$ (gray). These two occurrences induce the same inclusion of faces.

AF-equality relation, whose orbits have been glued together. This gluing process is described by a sequence of equivalence relations $R_{d-1}, \ldots, R_1, R_0$. This is a “cell-complex-like” description of the orbit equivalence relation.

This specificity (substitution tiling spaces) is the price to pay for this level of details. However, there are reasons to think that the results in this paper may hold for in more general cases.

6.1. Lifting Technical Hypothesis

First, let us explain how it is possible to get rid of Hypothesis 4.9. An example in which this hypothesis does not hold is given by the substitution of Fig. 13.

This hypothesis is used to prove Lemma 4.10. One needs to show that if there is a “square” in the diagram such as the one shown in the proof of the lemma, then the inclusion of tiles given by the left edge of the square induces the inclusion of faces given by the right edge of the square.

A solution to get rid of Hypothesis 4.9 is to make a list of “good” squares for each level. Then, adapt the definition to say that a path $x'$ is derived from $x$ if each vertex of $x'$ is linked to a vertex of $x$ by a horizontal edge, and furthermore, all “squares” formed by edges of $x$, of $x'$, and by horizontal edges are “good”. Equivalently, it would have been possible to use the dual diagram of our dual diagram (the bi-dual Bratteli diagram). Then, a vertex $v$ in the bi-dual diagram corresponds to a vertical edge $e_v$ in our multi-diagram, and we put an horizontal edge between $v$ and $v'$ if in the multi-diagram, there is a good square between edges $e_v$ and $e_{v'}$.

Using the bi-dual diagram would, however, have made things significantly more confusing, so Hypothesis 4.9 was used instead.

It is legitimate to wonder whether the problem illustrated by the example above is actually solved by taking the bi-dual. Does the bi-dual diagram solve it, or does it change it into something so complicated that we can’t see the problem anymore? We bring an answer to this question in the next paragraph.

6.2. Description of Boundary Paths in Terms of Shifts of Finite Type

The set of infinite paths $\Pi_\infty$ on a stationary Bratteli diagram can be seen as a (one sided) shift of finite type. A path of $\Pi_\infty$ is a sequence of edges, with compatibility conditions. The edge $e$ may follow the edge $e'$ only if $r(e) = s(e')$. Since there is a finite number of such conditions (by self-similarity), the set of
paths is a subshift of finite type of the set $\mathcal{E}^\mathbb{N}$. The “shift” is here the map

$$(e_1, e_2, \ldots) \mapsto (e_2, e_3, \ldots).$$

It defines an action of $\mathbb{N}$, which is not directly related to the action by translations on tilings (actually, shifting a sequence $(e_n)_{n \in \mathbb{N}}$ would correspond to applying $\omega^{-1}$ to the corresponding tiling).

Consider now the following subshift of finite type: it is made of sequences of “authorized” edges $e \in \mathcal{E}$ which encode the inclusion of a tile $t$ in $\omega(t')$, with $t$ on the boundary of $\omega(t')$. Furthermore, $ee'$ is permitted only if the edges encode compatible inclusions of tiles, so that $ee'$ encodes the inclusion of $t$ in $\omega^2(t'')$ (here $t$ corresponds to $s(e)$ and $t''$ to $r(e')$).

Such a subshift of finite type is made of boundary path in our formalism. An edge in the diagram corresponds to a vertex in the dual diagram. It is authorized if it is linked by an horizontal edge to a vertex in the (dual) diagram of lower dimension. Furthermore, $ee'$ is authorized if the square

```
/e\       f
|   |
|   |
ev'-------f'
```

is authorized in the sense of Sect. 6.1.

This “shift of finite type” characterization of boundary tilings can be captured by the multi-diagram, with horizontal edges, and an additional ingredient:

- either Hypothesis 4.9;
- or with a list of “authorized squares”;
- or using horizontal edges on a “bi-dual” diagram.

These last two possibilities were discussed above. We made the first choice in this paper, for the sake of simplicity.

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