We present a comprehensive review of the nodal domains and lines of quantum billiards, emphasizing a quantitative comparison of theoretical findings to experiments. The nodal statistics are shown to distinguish not only between regular and chaotic classical dynamics but also between different geometric shapes of the billiard system itself. We discuss, in particular, how a random superposition of plane waves can model chaotic eigenfunctions and highlight the connections of the complex morphology of the nodal lines thereof to percolation theory and Schramm-Loewner evolution. Various approaches to counting the nodal domains—using trace formulae, graph theory, and difference equations—are also illustrated with examples. The nodal patterns addressed pertain to waves on vibrating plates and membranes, acoustic and electromagnetic modes, wavefunctions of a “particle in a box” as well as to percolating clusters, and domains in ferromagnets, thus underlining the diversity—and far-reaching implications—of the problem.

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I. INTRODUCTION

In February 1809, Napoleon Bonaparte invited a certain Dr. Chladni of Wittenberg to his court at the Tuileries Palace in Paris. The Emperor of the French people had been greatly enthused by the prospect of a firsthand demonstration of the “sound patterns” that Ernst Florens Friedrich Chladni (1802) had documented in his book, Die Akustik, some seven years earlier. The path leading up to this meeting had been paved by Chladni’s longstanding scholarly interest in the vibrations of bars and plates. Inspired by Lichtenberg’s experiments on using sulphur and minium powder to visualize electric discharges in insulators, he decided to replicate the procedure with vibrating brass plates (see Fig. 1). The rest,
as they say, is history. Stöckmann (2007) narrates: “He spread sand on the plate, stroked it with the violin bow, and within a few seconds the sand brought about the shape of a star with ten rays.” Thus, Chladni’s sound figures were born. The study of these vibrational patterns soon generated such intense interest that it prompted no less a man than Michael Faraday (1859) to remark, “The beautiful series of forms assumed by sand, filings, or other grains, when lying upon vibrating plates, discovered and developed by Chladni, are so striking as to be recalled to the minds of those who have seen them by the slightest reference. They indicate the quiescent parts of the plates, and visibly figure out what are called the nodal lines.” Over the last quarter of the 18th century, Chladni (1787) systematically probed the sound patterns of circular, quadratic, and rectangular plates, by fixing them with his fingers at different points, thereby enforcing the occurrence of nodal lines (Ullmann, 2007). In fact, his investigations—coupled with pure empirical reasoning—led to the discovery of the relation between the frequency $f$ of a vibrating circular plate and the number of its diametric ($m$) and radial ($n$) nodal lines:

$$f \sim (m + 2n)^2,$$

which, today, is better known as Chladni’s law.

FIG. 1 Experimental approaches for generating (a) classical and (b) modern Chladni figures. The resonant oscillations excited by the bow, in the original scheme, are eigenmodes of the plate. In modern experiments (Jensen, 1955; Waller, 1937, 1938, 1940), the resonant modes are locally excited by driving the plate at varying frequencies with an electronically-controlled mechanical oscillator. From Tuan et al. (2015).

Underlying the patterns traced by the tiny sand particles on an excited plate—and dictating the genesis of the nodal lines—is the fundamental question of symmetry. If a plate has $N$ lines of symmetry (each of which divides its surface into two equal parts that are mirror images of each other), Waller (1957) observed that all the normal modes of vibration supported by the plate can be classified according to the relation $c = 2F$, where $c$ denotes the number of symmetry classes and $F$, the number of factors of $N$. For instance, for a square plate, there are $(N = 4)$ 4 lines of symmetry, the factors being 1, 2, 4 ($F = 3$); the number of classes is then 6. Such taxonomical endeavors were of great personal interest to Chladni. In Die Akustik, he categorizes the patterns observed for rectangular plates according to the number of nodal lines parallel to both sides, meticulously sketching each individual excitation (Fig. 2) at assigned frequencies. These bicentenarian figures bespeak a few general features of note. Suppose a square plate, driven into vibration by a violin bow at one corner, is pinned by, say, a finger, at the midpoint of one of the edges. The resultant nodal pattern partitions the square into four (Fig. 2: 1st row, 1st column), each nodal line being the mean level, having one side of it a rising and on the other a falling surface (Rossiter, 1871). Alternatively, if the positions of driving and pinning are exchanged, the lines of least agitation span across the diagonals, dividing the plate into four equal triangles (Fig. 2: 1st row, 2nd column). The nodal portraits therefore strongly suggest that Chladni figures always conform with the symmetry of the geometrical surfaces upon which they are produced (Waller, 1952). Any disturbance, in the slightest, suffices to change from one standing wave to another among the normal modes with the same frequency, but not to a combination thereof (Waller, 1954). The regions that the nodal contours divide the plate into—where the vibrations are out of phase—are called nodal domains. Understanding these patterns (and counting
the number of such domains) has remained a problem generating great intellectual curiosity since the time it was first treated mathematically by Sophie Germain and Gustav Robert Kirchhoff. This question has today become a part of a more general class of problems which include percolation theory, where percolation clusters appear, ferromagnetism, where magnetic domains appear, and microwave cavities, where domains of electromagnetic modes appear—all sharing certain features, and subtle differences.

Meanwhile, across the English channel, the stage was set for the birth of the physics of billiards on April 27, 1900. Lest we sound facetious in making so sweeping a statement, let us clarify the historical context in which it is made. On that fateful day, Lord Kelvin delivered a lecture entitled “Nineteenth century clouds over the dynamical theory of heat and light” at the Royal Institution of Great Britain. One of the portentous clouds that he had been alluding to was the breakdown of the ergodicity hypothesis—the assumption that the phase-space average of a physical quantity should accord with its time average, taken over sufficiently long times. The setup that Kelvin [1901] considered consisted of a billiard table—“a finite area of plane or curved surface, bounded by . . . walls, from which impinging particles are reflected at angles equal to the angles of incidence”—and an “ideal, perfectly smooth, non-rotating billiard ball, moving in straight lines except when it strikes the boundary.” Examining the motion of a point particle bouncing off the hard walls of a scalene-triangular billiard (now known to be pseudointegrable) and a flower-like billiard (nonintegrable), he was able to explicitly demonstrate a contradiction to the equipartition theorem (Nakamura and Harayama, 2004). Actually, in these experiments, Kelvin approximated the flower-like billiard by a polygonal one, substituting for each arc of the flower (the semicircular corrugations of the circular boundary) its chord. Little did he know that this anodyne approximation would be the reason for the violation of the ergodicity hypothesis. Nonetheless, his investigation set in motion studies on the nonlinear dynamics of classical billiard systems, and in particular, on pseudointegrable and chaotic billiards [Arnold and Avez, 1967; Bunimovich and Sinai, 1981; Krylov, 1979; Sinai, 1970, 1976].

So far, the protagonists in our narrative have been two disparate physical systems—quivering plates and dynamical billiards—that, seemingly, lack any connection. The unifying link between the two was provided by the advent of quantum mechanics, and the subsequent inception of quantum billiards. The quantum analogue to a classical billiard is governed by the stationary Schrödinger equation

$$\Delta \psi_j \equiv \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \psi_j = -k_j^2 \psi_j \tag{2}$$

for a “particle in a box”. Information about the walls of the billiard table appear only through the boundary conditions enforced by, say, a confining potential in the Hamiltonian. Eq. (2) is nothing but a simple time-independent wave equation—the Helmholtz equation. If one identifies the wavefunction $\psi$ as the amplitude of a wave field (sound waves, to be concrete), it is evident that the wavefunctions of quantum billiards allow essentially the same description as the Chladni figures of yore. This opens up new avenues to address questions and theories, which were originally prompted by quantum mechanics, by means of classical wave experiments on acoustics [Ellegaard et al., 1995; Tanner and Søndergaard, 2007; Weaver, 1989], water waves [Blümel et al., 1992], light [Chen et al., 2012, 2006; Huang et al., 2002], and microwave networks [Hul et al., 2004, 2012; Lawniczak et al., 2014]. In the course of the last two decades, quantum billiards have been experimentally realized in gated, mesoscopic GaAs tables [Berry et al., 1994], microwave cavities [Richter, 1999; Stöckmann and Stein, 1990], and ultracold atom traps [Andersen et al., 2006; Friedman et al., 2001; Milner et al., 2001; Mottangero et al., 2009]. The eigenfunctions of these planar quantum billiards once again organize themselves into domains, with positive and negative signs, often in remarkably complicated geometric shapes. Formally, such nodal domains may be defined as the maximally connected regions wherein the wavefunction does not change sign. Unfortunately, quantifying the nodal patterns is a major challenge since it is extremely hard to discern any order when the wavefunctions are arranged in ascending order of energy. In principle, the problem seems (deceptively) straightforward—for each billiard of interest, we need only solve the Schrödinger equation in appropriate coordinates, and count the domains as a function of the quantum numbers. However, in the absence of an exact solution to the counting problem, the statistics of nodal domains and lines have been studied for regular to chaotic billiards. Besides its connections to an array of different subjects such as the seismic response of sedimentary valleys [Flores, 2007], violins [Gough, 2007], floats in surface waves [Lukaschuk et al., 2007], evanescent waves in the brain [Schmabel et al., 2007],

\[ \Delta^2 \psi_j \equiv \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)^2 \psi_j = k_j^4 \psi_j \tag{3} \]

for the $j$th resonance, or, with driving, by the Kirchhoff-Love equation:

$$\left( D \nabla^4 + \rho \frac{\partial^2}{\partial t^2} \right) \psi(x, y, t) = F(x, y, t), \tag{4}$$

where $D$ is the flexural rigidity, $\rho$ the mass density, $h$ the thickness, and $F(x, y, t)$ the effective force function. As opposed to the vibrations of membranes without internal stiffness, it is the square of the Laplace operator that figures in Eq. (2).
and turbulence [Falkovich 2007] to name a few, there are two primary reasons why nodal portraits are of academic interest. Firstly, a host of analytical and numerical evidence [Blum et al. 2002; Toth and Wigman 2009] today suggests that the sequence of nodal counts in a billiard encodes the difference between integrability and chaos, i.e., it can shed light on the quantum signatures of classical chaos. Secondly, and equally interestingly, the nodal count also bears additional information about the dynamics and the geometry of the billiard that are not accessible from the spectral statistics alone [Gnutzmann et al. 2006].

This article is organized as follows. We begin with a brief overview of the classical and quantum dynamics of billiard systems in Sec. II. The nodal properties of Laplacian eigenfunctions of these billiards constitute a subject that has garnered widespread attention from mathematicians and physicists alike. In Sec. III we rummage through the motley mathematical literature, sans rigor and jargon, to demonstrate that far from being l’art pour l’art, some crucial results therefrom find direct applicability to quantum billiards. A lot of work has been devoted to chaotic systems owing to Berry’s random wave hypothesis, which we examine in Sec. IV. This correspondence between nodal domains and random waves also draws us closer to the problems of percolation theory and, relatively more recently, Schramm-Loewner evolution. Proceeding thereafter, in Sec. V we explore the various established results on nodal domain and line statistics. As we show, for separable billiards, it is possible to write down exact expressions for certain limiting distributions as also for fully chaotic systems, whereas for integrable but nonseparable, quasi-integrable, and pseudo-integrable billiards, only partial results exist. Sec. VI brings us to the question of counting nodal domains and, by association, “the shape of a drum”. For separable systems, a trace formula has been developed in this regard. Likewise, for integrable systems, one can formulate a difference-equation formalism. Other geometries where the problem of counting has been well studied include flat tori, surfaces of revolution, and the boundaries of 2D quantum billiards—we also survey the mathematical developments in these directions. In Sec. VII we revisit our previously abstract discussions but now, grounded in actual experiments on microwave billiards, which have been instrumental in the investigation of not only nodal statistics, but also of quantum chaos, in general. Finally, we conclude this Review by enlisting some open questions in Sec. VIII and summarizing our perspectives on the prospective directions of this burgeoning field.

II. BILLIARDS: CLASSICAL AND QUANTUM DYNAMICS

A point particle moving freely inside an enclosure, reflecting from the (piecewise smooth) boundary in accordance with Snell’s law—this is the dynamical system, almost endearing in its simplicity, termed a billiard. Amongst other abstractions, billiards serve as useful models for many-body systems, which can be approximated by a single particle in a mean field, the complexity of the former being simulated by the boundary [Glashow and Mittag 1997; Krishnamurthy et al. 1982; Rabouw and Ruijgrok 1981; Sen 1996; Sinai 1976; Turner 1984]. For instance, the Boltzmann-Gibbs gas in statistical mechanics—elastically colliding hard balls in a box—can be exactly mapped onto a billiard [Bunimovich et al. 2013]. The dynamics of the billiard is completely determined by the shape of its boundary, which also defines its symmetries and therefore, the constants of motion. Classifying along these lines, billiards can, by and large, be divided into a few broad categories such as (i) convex with smooth boundaries, (ii) polyhedral or polygonal, and (iii) dispersive or semi-dispersive.

A trajectory of a particle undergoing reflection from a surface can be straightened by reflecting the domain $D$...
however, for billiards with non-symmetrical shapes, the dynamics become unpredictable in the sense that any initial correlations decay after a few reflections from the boundary. Identifying the periodic trajectories for such systems is also a much more complex problem as the number of these trajectories proliferates exponentially (Ott 2002). Obviously, in quantum mechanics, these notions break down as the very concept of a trajectory becomes undefined due to the uncertainty principle $\Delta x \Delta p \geq \hbar/2$. Contrary to classical mechanics, a quantum system necessarily requires quantum mechanics to reduce to the classical description in the semiclassical limit of large quantum numbers, i.e., $E \to \infty$. This dichotomy is certainly nontrivial and therein lies the tale. However, before we get ahead of ourselves, let us halt to make precise the definitions of integrable and chaotic systems.

A. From integrability to chaos

A dynamical system is termed as integrable in the sense of Liouville-Arnold (2013) if there are $f$ constants of motion for a system with $f$ degrees of freedom. These constants must be functionally independent and in involution. Furthermore, the vector fields have to remain regular everywhere in phase space. A circular billiard ($f = 2$) is an example of this type, where the total energy $E$ and the angular momentum $L$ are conserved. Since the system is integrable, the distance between two “nearby” trajectories increases linearly with time. For any generic system with two degrees of freedom, one can find action-angle variables $(I_1, I_2, \theta_1, \theta_2)$ by employing an appropriate generating function (Lichtenberg and Lieberman 2013). There are two canonical frequencies, $\omega_i = \frac{d\theta_i}{dt}$, $i = 1, 2$, which are functions of phase-space variables for a nonlinear system. In exceptional situations (called nonlinear resonances), the ratio of these two frequencies becomes a rational number and certain regions in phase space are endowed with a chain of islands. Each island has at its center an elliptic point (stable equilibrium), and a separatrix connecting hyperbolic points. The invariant tori are broken, and new intricate structures are formed (see, for example, Berry 1981a). Such systems are said to be quasi-integrable. However, there exists another qualitatively different class of behavior demonstrated by most billiards. Consider the cardioid billiard of Fig. 4, now, the only constant of motion is the total energy $E$. Classically, the motion on the billiard table is irregular and unpredictable, displaying extreme sensitivity to initial conditions. More importantly, the separation between neighboring trajectories grows exponentially with time with a divergence characterized by the Lyapunov exponent. The cardioid billiard is therefore chaotic. A “typical” billiard is expected to be non-integrable (Siegel 1941) and most of these are believed to be chaotic, possessing a positive Kolmogorov-Sinai entropy (Bunimovich 2007).

Clearly, this trajectory-centric distinction between order and chaos does not hold in the quantum regime. Moreover, since the Schrödinger equation is linear, there is no scope for chaos in the conventional sense. However, the latent classical dynamical properties do leave their imprints on the statistical behavior of eigenvalues. The statistics of the energy levels of a generic integrable system can be characterized by a Poissonian random process (Berry and Tabor 1977b). Conversely, the energy-level statistics of fully chaotic systems are described by the eigenvalues of random matrices obeying appropriate symmetries (Bohigas et al. 1984; Mehta 2004; Reichl 1992). For quasi-integrable systems, the statistics can be modeled as a superposition of Poisson and Wigner distributions with weights proportional to the relative fractions of regular and chaotic subregions (Berry and Robnik 1984). Of course, as always, exceptions abound: prominent counterexamples include arithmetic systems (Aurich and Steiner 1988; Bogomolny et al. 1992; Bolte 1993; Bolte et al. 1992; Sarnak 1995) and quantized cat maps (Hannay and Berry 1980; Kéating 1991).

The classical dynamics of the billiard is also betokened by the structure of the eigenfunctions themselves. The semiclassical eigenfunction hypothesis (Berry 1977; 1983; Voros 1976, 1977, 1979) asserts that the eigenstates should concentrate on those regions which a generic orbit explores in the long-time limit. This statement is best formulated in terms of the Wigner function $W(p,q)$, which is a phase-space representation of the wavefunction $\psi(p,q)$. In integrable systems, the motion remains confined to invariant tori in phase space—so $W(p,q)$ should localize on these tori in the semiclassical limit—whereas in ergodic systems, the whole energy surface is filled in a uniform manner and $W(p,q)$ is expected to semiclassically condense on the energy shell as $W(p,q) \sim \left[1/V(\Sigma_E)\right] \delta(H(p,q) - E)$, where $H$ is the Hamiltonian and $V(\Sigma_E)$ is the volume of the shell set by $H(p,q) = E$ (Bäcker et al. 1998b). For the latter
class, this was established to be equivalent to the quantum ergodicity theorem (Gérard and Leichtnam, 1993; Helffer et al., 1987; Shnirel’man, 1974; Colin de Verdière, 1985; Zelditch, 1987; Zelditch and Zworski, 1996)—which states that “most” eigenstates are equidistributed in the semiclassical limit—by Bäcker et al. (1998a,b). Written in position space, this implies

\[
\lim_{j \to \infty} \int_{\mathcal{D}} |\psi_{n_j}(r)|^2 \, dr = \frac{\text{vol}(\mathcal{R})}{\text{vol}(\mathcal{D})} \quad (5)
\]

for a subsequence \(\{\psi_{n_j}\} \subset \{\psi_n\}\) of density one. Thus, as \(E \to \infty\), the probability of finding a particle in a certain region of the billiard \(\mathcal{R} \subset \mathcal{D}\) is identical to that for the classical system, for almost all eigenfunctions. This consonance is pictorially conveyed by Fig. 4.

The phase space representation of eigenfunctions of integrable systems reveals a direct correspondence with the underlying periodic orbits. As the degree of nonintegrability increases, the eigenfunctions become progressively more structured. For chaotic systems, the eigenfunctions often evince an “anomalous enhancement or suppression of eigenstate intensity on or near an unstable periodic orbit and its invariant manifolds” (Kaplan, 1999), over and beyond the statistically expected density. Discovered by McDonald and Kaufman (1979) and christened “scarring” by Heller (1984), this observation is notably distinct from the attraction of eigenstate intensity associated with stable classical orbits, the latter being due to reasons well-understood by a purely semiclassical theory of integrable systems (Berry and Tabor, 1976; Berry and Fishman, 1993, 1994). Although the scars themselves do not disappear in the semiclassical limit (Kaplan and Heller, 1998), the scarred area of phase space (surrounding the orbit), and hence the total weight of the scar, vanishes as \(\hbar \to 0\). One could equally well look for “strong scarring” (Rudnick and Sarnak, 1994) in the semiclassical limit—this necessitates the convergence of the total weight of the wavefunction on the unstable orbit (Kaplan, 1999). However, this property has not been found to hold for any physical system thus far and its existence has even been explicitly disproved for certain arithmetic hyperbolic manifolds.

Incidentally, addressing quasi-integrable or mixed systems, Keating and Prado (2001) found that the probability density for perturbed cat maps is localized strongly around the bifurcated periodic orbits—the phenomenon was homologously styled “superscarring”.

B. Pseudointegrable billiards

Liouville-Arnold integrability is easily broken, when, for instance, the vector fields become singular without disturbing other conditions of integrability. Such dynamical systems are titled pseudointegrable, named thus by Richens and Berry (1981). Nonintegrable polygonal billiards are pseudointegrable; they are also non-chaotic insofar as the Lyapunov exponent is zero. In this regard, we now turn to a billiard in a polygon \(Q \subset \mathbb{R}^2\) where all angles are commensurate with \(\pi\) (Zemlyakov and Katok, 1975). Let us fix a direction \(\hat{e}\), say, along one of the sides of \(Q\). One can write the angles between \(\hat{e}\) and the remaining sides in the form \(m_r \pi / 2n_r\) with \(r = 1,2,\ldots\). Furthermore, let \(N = \text{gcd}(Q)\) denote the least common denominator of the fractions \(m_r/n_r\). It was observed that the function

\[
F(x, y, p_x, p_y) = F(\phi) = |\phi| \mod \frac{\pi}{N} \quad (6)
\]

is well-defined on the energy surface \(\mathcal{M}\) and invariant with the phase-space flow. For a given \(c \in [0, \pi/2N]\),

\[
\psi(x, y, p_x, p_y) = \psi\left(x - \frac{c}{N} p_y, y, x, y + \frac{c}{N} p_x\right)
\]

is well-defined on \(\mathcal{M}\) and invariant with the phase-space flow.

Moreover, the semiclassical defect measures, to adopt mathematical parlance, cannot be a finite sum of delta functions on closed geodesics on compact Riemannian \(C^\infty\) manifolds with Anosov flow and the high-energy eigenfunctions are thus, at least, “half-delocalized” (Anantharaman, 2008; Anantharaman and Nonnenmacher, 2007).

A finite-time exponent can also be defined for polygonal billiards as suggested by Moudgalya et al. (2013).
there is an invariant subset $\mathcal{M}_c$ of $\mathcal{M}$. If $c \notin \{0, \pi/2N\}$, then the set $\mathcal{M}_c$ is obtained by pairwise identification of the sides of $4N$ non-intersecting copies of the polygon labelled by $\phi_s^c = \pm c + s \pi/N$, with $s = 0, 1, \ldots, (2N - 1)$. The identification of the sides is dictated, once again, by Snell’s law. Zemlyakov and Katok (1975) proved that $\mathcal{M}_c$ is a two-dimensional manifold of genus $g > 1$ depending on the shape of the polygon. The fact that the motion in phase space is not restricted to a torus as for integrable systems but rather, to a surface with a more complicated topology, is conveyed by the moniker pseudointegrable. A prototypical example is the $(\pi/3, 2\pi/3)$-rhombus billiard, illustrated in Fig. 5.

![FIG. 5](image)

FIG. 5 Upon three subsequent reflections, the rhombus billiard returns onto itself, albeit with a flipped orientation. This double-valuedness is taken care of by constructing a fundamental domain with the six copies required for identification of the sides; the vertex is a “monkey-saddle” (Eckhardt et al., 1984). The manifold $\mathcal{M}_c$ is thus, topologically, a sphere with two handles or a double torus with genus $g = 2$. Stacking the domains fills the plane, as seen in the lower figure. The bold line segments, although inaccessible, connect two branch points, forming a branch cut. The branch cuts are arranged in a doubly-periodic manner throughout the plane and a classical trajectory is a zig-zag line reflecting from the cuts as the phase changes by $\pi$.

The statistical properties of the eigenvalues of classically pseudointegrable quantum billiards have been numerically established (Biswas and Jain, 1990; Bogomolny et al., 1999; Wiersig, 2002) to be intermediate between those of regular and chaotic systems. The wavefunctions are also equally intriguing, demonstrating pronounced scarring behavior that can be related to families of periodic orbits (Bogomolny and Schmit, 2004). For example, many of the solutions to the Schrödinger equation for the rhombus billiard (Biswas and Jain, 1990) exhibit significantly enhanced intensities in the close neighborhood of a periodic orbit. However, in contrast to chaotic systems, these scar structures persist even at large quantum numbers, thereby earning the epithet “superscars”. Experimentally, superscarring (Fig. 6) has been observed in microwave billiards (Bogomolny et al., 2006a; Richter, 2008) and LiNC $\rightleftharpoons$ LiCN isomerization reactions (Prado et al., 2009). In many-body systems, it is often the case that the amplitude of a collective excitation is spread over states forming the background, well-known examples being the giant dipole resonances in nuclei (Bohr and Mottelson, 1998) and metallic clusters (Brack, 1993). The distinctly created excitation acts as a “doorway” to the background states. Analogously, a superscar of a pseudointegrable billiard also spreads over a large number of non-scarred wavefunctions. This convenient parallel enables the modelling of doorway states in the quantum spectra of nuclei (Åberg et al., 2008).

![FIG. 6](image)

FIG. 6 Superscars in a pseudointegrable barrier billiard, experimentally obtained using the perturbing bead method (Sec. VII.C). Observed is a clear wave function structure connected with the family of classical periodic orbits as well as a distinct localization of excitation strength (Richter, 2008). The bottom row indicates the corresponding classical orbits (dashed lines). From Bogomolny et al. (2006a).

C. Flows and vortices

The nodal lines of Chladni’s vibrating plates are a window to a deeper theory—of vortices—that, paradoxically, neither Chladni nor anyone else ever witnessed in the motion of plates (Courtial and O’Holleran, 2007). A direct (theoretical) route to observe the emergence of vortices is to study the displacement of the plate $A(x, y, t)$ from its equilibrium position, near the nodal lines. As with any wave phenomenon, it is convenient to regard the actual displacement as the real part of a complex amplitude $u(x, y) \exp(i \omega t)$. In the immediate neighborhood of a nodal line along the $x$-axis, the complex field

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is \( u^{(c)}(x, y) = y \); similarly, \( u^{(g)}(x, y) = x \) for a nodal line along \( \hat{y} \). The displacements, pictured in Fig. 7, pivot about these lines as time evolves. Now, consider two frequency-degenerate eigenmodes with nodal lines in the \( \hat{x} \) and \( \hat{y} \) directions, which are locally described as above—obviously, these orthogonal lines cross each other at a point, somewhere on the plate. It is interesting to inspect the ensuing oscillation structures in the vicinity of the intersection point for a linear superposition of the eigenstates. If the complex fields are simply summed up as

\[
u(x, y) = x + y\tag{7}
\]

the resultant eigenmode displays a nodal line, inclined at 45° to the two axes. Alternatively, if the fields are added out of phase by \( \pi/2 \),

\[
u(x, y) = x + i y,\tag{8}
\]

then the amplitude field \( A \) does not merely pivot about a new nodal line but instead, rotates around the crossing point, thus engendering a vortex (at the intersection point). On retracing Chladni’s steps, these vortices should, in principle, be visible as fine islands of sand.

![Fig. 7 Chladni lines and vortices.](image)

As early as 1931, Dirac (1931) demonstrated that nodal points give rise to current vortices. Vortices, in the most general sense, refer to singular points at which the phase of a field becomes undefined. They are commonplace in physical fields of at least two variables, where the physical quantity signified by the field can be naturally represented in a plane \( \mathbb{R}^2 \), such as the Argand plane or \( \mathbb{R}^2 \). In particular, the field could be the wavefunction for an open quantum billiard, or one with broken time-reversal symmetry—in either case, \( \psi \) acquires a finite imaginary part. It could also just as well be the complex order parameter of a superconductor or a scalar optical field. Consider a generic complex scalar wave

\[
\psi(r, t) = \xi(r, t) + i \eta(r, t) = \rho(r, t) \exp \{i \chi(r, t)\}\tag{9}
\]

with \( \xi, \eta, \rho, \chi \in \mathbb{R} \). All is well with this representation, except at the nodal points where \( \psi \) (and consequently, the intensity = \( \rho^2 \)) is zero. Just as the angle of polar coordinates is ill-defined precisely at the origin, along loci of vanishing \( \rho \), the phase \( \chi \) is indeterminate. These phase singularities—also called optical vortices—were first illustrated by Nye and Berry (1974) as dislocations of wavefronts and later, investigated by Berry (1981b, 1998), Nye (1999). The dislocations reside on the contour lines of intersection of the two surfaces

\[
x, t) = 0, \quad \eta(r, t) = 0\tag{10}
\]

as seen in Fig. 8, we call these lines \( \xi NL \) and \( \eta NL \), respectively. To borrow Berry and Dennis (2000) eloquent description, “in light, they are lines of darkness; in sound, threads of silence.” Unfortunately, the nodal lines \( \xi NL \) and \( \eta NL \), being dependent on the overall phase of the wavefunction, are hardly uniquely defined. The nodal points, on the contrary, are invariant under a phase transformation of the wave function and remain unmoved in space on multiplying \( \psi \) by an (arbitrary) constant phase factor \( \exp(i \alpha) \) [Berggren et al. 2002]. This makes them of greater utility when attempting to characterize a complex chaotic wave function.

![Fig. 8 Typical pattern of the nodal lines Im[\( \psi(x, y) \)] = 0 (black lines, \( \eta NL \)) and Re[\( \psi(x, y) \)] = 0 (red lines, \( \xi NL \)) for a random superposition of plane waves. Nodal lines in each set do not cross. Around the nodal points at which the two sets intersect, there is vortical flow in either clockwise (green dots) or anti-clockwise (blue dots) directions. From Saichev et al. 2001.](image)

Around a vortex, the phase assumes all possible values in its \( 2\pi \) range, leading to a circulation of the wave...
energy. Although the change in $\chi$ is usually nonuniform \cite{Mondragon and Berry 1989}, upon traversing a closed circuit $\gamma$ enclosing the phase-singular point, there is a net change of phase

$$Q = \frac{1}{2\pi} \oint \mathbf{d} r \cdot \nabla \chi = \frac{1}{2\pi} \oint \mathbf{d} \chi,$$

(11)

which is quantized in units of $2\pi$. The integer $Q$ is referred to as the strength or topological charge \cite{Halperin 1981} and $\delta = \text{sgn} (Q)$, which is positive if the phase increases in a right-handed sense, is colloquially termed the sign of the singularity. In the language of Eqs. (7, 8), the complex field near the center of a canonical charge-$Q$ vortex located at the origin takes the form \cite{Molina-Terriza et al. 2001}

$$u (x, y) = (x + i y)^{|Q|} = \exp (i Q \phi).$$

(12)

Higher-charge vortices with $Q > 1$ (Fig. 7), however, are non-generic in that they are not stable with respect to perturbations \cite{Freund 1999}, which split a charge-$Q$ vortex into $|Q|$ vortices of charge $\pm 1$ \cite{Courtial and O’Holleran 2007}.

Associated with the field $\psi$ is a current

$$j = \text{Im} (\psi^* \nabla \psi) = \xi \nabla \eta - \eta \nabla \xi = \rho^2 \nabla \chi.$$

(13)

This is the usual quantum-mechanical probability current density. In electrodynamics \cite{Jackson 1999}, $j$ is the familiar Poynting vector if $\psi$ stands for a polarized (scalar) component of the electric field. Since $j$ is aligned along the gradient $\nabla \chi$, i.e., in the direction of change of phase, phase singularities constitute vortices of the optical current flow \cite{Dennis et al. 2000}. The corresponding vorticity is defined as

$$\Omega = \frac{1}{2} \nabla \times j = \frac{1}{2} \text{Im} (\nabla \psi^* \times \nabla \psi) = \nabla \xi \times \nabla \eta.$$

(14)

The vector $\Omega$, being perpendicular to the normals to the two surfaces in Eq. (10), points along the dislocation line. In three dimensions, the total length of dislocation line in a volume $V$ is

$$L(V) = \int_V \mathbf{d} \mathbf{r} \delta (\xi) \delta (\eta) |\Omega(\mathbf{r})|,$$

(15)

whereby we can calculate the dislocation line density. For the time being, however, let us focus on a unit topological charge in one lower dimension. This simplifies life, and Eqs. (13)–(13), a little, whereupon

$$Q = \text{sgn} (\Omega \cdot \hat{z}) = \text{sgn} (\xi_x \eta_y - \xi_y \eta_x),$$

(16)

$$j = (\xi \eta_x - \eta \xi_x, \xi \eta_y - \eta \xi_y),$$

(17)

the subscripts indicating partial derivatives. Note that the strength $Q$ is now identical to the sign $\delta$—also called the winding number (WN)—that defines the sense of circulation of the current swirling about the vortex. Neighboring nodal points on the same nodal line (either $\xi$ or $\eta$) always have opposite WNs \cite{Berggren et al. 2002} and generically, any nearest neighbors are strongly anticorrelated in sign \cite{Shvartsman and Freund 1994a}. In fact, the sign of any single vortex in a random Gaussian wave field determines the sign of all other vortices in the field.

The current is zero at both vortices, where $\psi = \xi = \eta = 0$, and saddle points (stagnation points in the flow), where $\xi/\eta = \xi_x/\eta_x = \xi_y/\eta_y$ \cite{Hohmann et al. 2009}. To tell them apart, we scrutinize the Jacobian

$$J = \partial_x j_x - \partial_y j_y - \partial_y j_x \partial_x j_y,$$

(18)

which is positive at vortices but negative at saddles. As a consequence of $\psi$ satisfying the Helmholtz equation, $J$ neatly separates into two parts

$$J = J_v - J_s,$$

(19)

with the individual contributions

$$J_v \equiv (\xi \eta_y - \xi_y \eta_x)^2,$$

(20)

$$J_s \equiv \frac{1}{2} \left[ (\xi \eta_{xx} - \eta \xi_{xx})^2 + (\xi \eta_{yy} - \eta \xi_{yy})^2 \right] + (\xi \eta_{xy} - \eta \xi_{xy})^2.$$

(21)

From the general density of critical points

$$D_{\text{crit}}(r) = \sum_{\{r': \|r-r'\|=0\}} \delta^2 (r - r') = \delta^2 (j(r)) |J(r)|,$$

(22)

it is now easy to compute the saddle and vortex densities \cite{Berry and Dennis 2000, Dennis 2001, Saichev et al. 2001},

$$D_s(r) = \delta^2 (j(r)) J_s(r),$$

(23)

$$D_v(r) = \delta^2 (j(r)) J_v(r) = \delta (\xi) \delta (\eta) |\xi_x \eta_y - \xi_y \eta_x|,$$

respectively. $D_v$, which can be determined statistically (at least, in the mean), specifies the dislocation point (or vortex) density—the mean number of dislocation lines piercing unit area of a plane \cite{Berry 1978}. These statistics have been thoroughly studied for quasimonochromatic paraxial waves \cite{Baranova et al. 1981}, monochromatic waves in two dimensions \cite{Freund 1994, Freund and Freilikher 1997, Freund and Shvartsman 1994, Freund et al. 1993, Freund and Wilkinson 1998, Shvartsman and Freund 1994a}, isotropic random waves \cite{Berry and Dennis 2000}, and random waves subject to the

\footnote{In conventional mathematical nomenclature, vortices (saddles) are the elliptic (hyperbolic) points in the flow.}
Aharonov-Bohm effect (Houston et al., 2017) among others, with novel extensions to the statistics of knotted nodal lines by Taylor and Dennis (2014, 2016).

Closer to the theme of our discussion, vortices were predicted in eigenfunctions of highly symmetric billiards (Chen and Huang, 2003a,b; Chen et al., 2002) and their telltale fingerprints have been spotted in closed, microwave cavities, coherently excited at multiple points (Dembowski et al., 2003). Similar predictions (Berry and Robnik, 1986) and experiments (Barth and Stöckmann, 2002; Berggren et al., 2002; Chibotaru et al., 2001; Oleniuk and Mikhailovska, 2003; Sadreev and Berggren, 2004) were proffered for quantum billiards without time-reversal symmetry, and more recently, for open billiards (Kuhl et al., 2007). On the optical physics front too, phase singularities have generated considerable enthusiasm, in part due to their connections with beams carrying orbital angular momentum (Allen et al., 1992, 1999; Bliokh et al., 2017; García-Chávez et al., 2003; He et al., 1995; O’Neil et al., 2002; Simpson et al., 1996; Turlaub et al., 1996), which fueled remarkable experiments (Beijersbergen, 1996; Karman et al., 1997; O’Holleran et al., 2006a,b) seeking to actively create and annihilate optical dislocations in laser fields. However, as fascinating as the physics thereof may be, we wouldn’t want to let our focus stray too far from nodal domains, to which we now thus return.

III. MATHEMATICAL PRELIMINARIES

The classical eigenvalue problem

$$\Delta u(x) = -\lambda u(x)$$  \hspace{1cm} (24)

for the Laplace operator

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_d^2},$$  \hspace{1cm} (25)

is, to say the least, ubiquitous in all branches of physics. Known as the Helmholtz equation (von Helmholtz, 1865), it naturally arises on separating out the time variable from the wave equation. To begin with, let the Laplacian be defined in $d$ spatial dimensions on an open bounded connected domain $\mathcal{D} \subset \mathbb{R}^d$ with a piecewise smooth boundary $\partial \mathcal{D}$. Although, in the context of quantum billiards, we focus on Euclidean domains in two dimensions ($d = 2$), the discussion can be completely generalized to any-dimensional manifolds.

The actual solutions to Eq. (24) are inseparably intertwined with the boundary conditions on $\partial \mathcal{D}$. Perhaps the most frequently encountered is the Dirichlet boundary condition where the eigenfunctions are constrained to vanish on the boundary, i.e.,

$$u(x) = 0 \quad \text{on} \quad x \in \partial \mathcal{D}. \hspace{1cm} (26)$$

For instance, borrowing from acoustics, the vibrational modes of a thin membrane (or drum) that is clamped along its boundary are exactly the Dirichlet Laplacian eigenfunctions with the drum frequencies proportional to $\sqrt{\lambda_j}$ (Rayleigh, 1945); a particular vibrational eigenmode can thus be selectively excited at the corresponding frequency (Sapoval and Gobron, 1993; Sapoval et al., 1991, 1997). The combination of Eqs. (24) and (26) also describes the propagation of a wave down a waveguide with cutoff frequency $\sim \sqrt{\lambda_j}$; in this case, the eigenfunctions $u_j$ correspond to the so-called TM-mode (Collin, 1960). Other choices of boundary condition include the Neumann condition (which corresponds to the vibration of a free membrane)

$$\frac{\partial u}{\partial n}(x) = 0 \quad \text{on} \quad x \in \partial \mathcal{D}, \hspace{1cm} (27)$$

$\partial/\partial n$ being the normal derivative pointed outwards from the domain, or the Robin boundary condition

$$\frac{\partial u}{\partial n}(x) + h u(x) = 0 \quad \text{on} \quad x \in \partial \mathcal{D}, \hspace{1cm} (28)$$

for some positive constant $h$.

For more than a century, it has been known that the Laplace operator has a discrete spectrum [of infinitely many non-negative eigenvalues (Pockels, 1892)]. The eigenvalues can be labeled (and arranged in ascending order) by an integer index $j = 1, 2, 3, \ldots$ as

$$0 \leq \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \hspace{1cm} (29)$$

with possible multiplicities (degeneracies). The set of eigenfunctions \{uj(x)\} forms a complete basis in the functional space $L_2(\mathcal{D})$ of measurable and square-integrable functions on $\mathcal{D}$ (Courant and Hilbert, 1953; Simon and Reed, 1979). To avoid any ambiguity about multiplicative factors, the eigenfunctions are conventionally normalized to unit $L_2$-norm

$$\|u_j\|_2 = \|u_j\|_{L_2(\mathcal{D})} = \left( \int_{\mathcal{D}} |u_j(x)|^2 \right)^{1/2} = 1 \hspace{1cm} (30)$$

and therefore, constitute an orthonormal set:

Orthonormality: $\int_{\mathcal{D}} dx \ u_i(x) u_j(x) = \delta_{i,j}, \hspace{1cm} (31)$

Completeness: $\sum_j u_j(x) u_j^*(y) = \delta(x - y). \hspace{1cm} (32)$

---

7Here, the word “domain” is loosely used to signify the enclosed region on which the Laplacian eigenfunctions are defined. This is not to be confused with nodal domains (also called subdomains); the difference should be clear from the context of use.

8This discreteness, however, cannot always be taken for granted as warned by Hempel et al. (1991) for the spectrum of a bounded domain with Neumann conditions on an irregular boundary.
The enormity of the literature on the eigenvalue problem is the quintessential paradox of plenty. From the study of elliptic operators to spectral theory and stochastic processes, the properties of eigenvalues and their attendant eigenfunctions have been at the heart of investigations aplenty. Here, we confront and sieve through this vast mathematical scholarship in attempting to bring together various “facts” about Laplacian eigenvalues and eigenfunctions (and eventually, their nodal domains) in one consolidated Section. Besides several books on the subject of terminology, we add that the ratio of dependent continuous and piecewise-differentiable functions is often termed the Sobolev space $H^1$. Neumann boundary conditions. It is easy to recognize simplification immediately follows for both Dirichlet and Rayleigh quotient. Although Eq. (34) looks cumbersome, the minimax principle has other important consequences like the Rayleigh-Ritz method for obtaining upper bounds for eigenvalues. The Rayleigh-Ritz method has also been used to estimate the eigenvalues of the Laplacian for rhombical (Birkhoff and Fix 1970; Weinstein 1966) and parallelogram (Durvasula 1968) regions. For a more detailed discussion about the numerical aspects of this connection, the curious reader is directed to Ch. 12 of (Kuttler and Sigillito 1984).

Another significant implication of the minimax principle is the property of domain monotonicity for Dirichlet boundary conditions, i.e., eigenvalues monotonously decrease when the domain is enlarged. To see this, consider a region $D$ that is properly contained in $D'$; a set of admissible eigenfunctions for $D'$ is then simply $u'_j(x) = u_j$ if $x \in D$ and 0 otherwise. Applying the minimax principle shows that $\lambda_j(D) > \lambda_j(D')$ or in words, the larger the region, the smaller the eigenvalues. This argument breaks down for Neumann or Robin boundary conditions, as illustrated by Fig. 9. A word of caution, however: domain monotonicity, in general, is not equivalent to the (trivially true) statement that on magnifying/shrinking a domain by a factor $\alpha$, all the eigenvalues are rescaled by $1/\alpha^2$, which is a much weaker property instead. Another aspect of monotonicity implied by the minimax principle is that the eigenvalues $\lambda_j$ increase with $h$, namely; if $h < h'$, then $\lambda_j(h) \leq \lambda_j(h')$; hence, the eigenvalues of the Robin problem are always intermediate between those of the corresponding Neumann and Dirichlet problems.

A. Eigenvalues: Basic properties

Even without inspecting the eigenfunctions, important information about the domain $D$ can be gleaned from the eigenvalues alone as we demonstrate in this subsection. For the sake of brevity and continuity, we refrain from presenting detailed proofs; instead, we simply state the relevant results, the emphasis being on their underlying physics.

A good point to start is the variational formulation of the eigenvalue problem which owes itself to the minimax principle (Courant and Hilbert 1953)

$$\lambda_j = \min\max \frac{||\nabla v||^2_{L^2(D)} + h ||v||^2_{H^1(D)}}{||v||^2_{L^2(D)}},$$

$$= \min\max \frac{\int_D \left( \frac{\partial v}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 \, dx \, dy}{\int_D v^2 \, dx \, dy}$$

for $d = 2$, where the maximum is over all linear combinations of the form

$$v = a_1 u_1 + a_2 u_2 + \cdots + a_j u_j,$$

and the minimum is over all choices of $j$ linearly independent continuous and piecewise-differentiable functions $u_1, u_2, \ldots, u_j$ (Henrot 2006); this space of functions is often termed the Sobolev space $H^1(D)$. While on the subject of terminology, we add that the ratio of quadratic forms in the third equality is known as the Rayleigh quotient. Although Eq. (34) looks cumbersome, simplification immediately follows for both Dirichlet and Neumann boundary conditions. It is easy to recognize that the second term in the numerator of Eq. (34) vanishes with these conditions; in the former case, $v$ itself is zero on the boundary $\partial D$ whereas for the latter, $h = 0$. In addition to its mathematical elegance, the minimax principle has other important consequences like the Rayleigh-Ritz method for obtaining upper bounds for eigenvalues. The Rayleigh-Ritz method has also been used to estimate the eigenvalues of the Laplacian for rhombical (Birkhoff and Fix 1970; Weinstein 1966) and parallelogram (Durvasula 1968) regions. For a more detailed discussion about the numerical aspects of this connection, the curious reader is directed to Ch. 12 of (Kuttler and Sigillito 1984).

FIG. 9 A counterexample to the property of domain monotonicity for the Neumann boundary condition. Although $D_1 \subset D_2$, the second eigenvalue $\lambda_2(D_1) = \pi^2/c_1^2 < \lambda_2(D_2) = \pi^2/c_2^2$ (if $a > b$) when $c = \sqrt{(a-\alpha)^2 + (b-\beta)^2} > a$. From Grebenkov and Nguyen (2013); figure by Naoki Saito.

The importance of restructuring the problem in variational terms is that Eq. (33) directly ensures that all eigenvalues are non-negative. In fact, the first eigenvalue $\lambda_1$ is simple (i.e., nondegenerate) and strictly positive for Dirichlet and Robin boundaries but zero with Neumann boundary conditions for which $u_1$ is itself a con-
stant. For the second Dirichlet eigenvalue, Cheng (1976) proved that the multiplicity \( M(\lambda_2) \leq 3 \) and the ineq


te can be inverted and rewritten as a formula for the index \( j \) instead—this naturally leads us to define a counting function (the number of eigenvalues smaller than \( \lambda \)) as

\[
N(\lambda) = \sum_{j=1}^{\infty} \Theta(\lambda - \lambda_j) \asymp \frac{\omega_d \mu_d(D)}{d} \lambda^{d/2} \quad (\lambda \to \infty),
\]

with \( \Theta \) denoting the Heaviside step function. The description of Eq. (36) only accounts for the leading-order term in the counting function and there are higher-order corrections that yield information about the boundary of the domain. These corrections, initially proposed by Weyl (and justified later by Ivrii (1980) and Melrose (1980) for convex \( D \)), are given by

\[
N(\lambda) = \begin{cases} 
\frac{A}{4\pi} \lambda^d \mp \frac{P}{4\pi} \sqrt{\lambda} & \text{for } d = 2 \\
\frac{V}{6\pi^2} \lambda^{3/2} \mp \frac{S}{16\pi} \sqrt{\lambda} & \text{for } d = 3
\end{cases},
\]

where the \(- (+)\) sign is applicable for Dirichlet (Neumann) boundary conditions. Here, and later (unless explicitly stated otherwise), \( A, P, V \) and \( S \) stand for the area, perimeter, volume and surface area of the billiard, respectively, in the appropriate dimensions. Berry (1979, 1980) hypothesized that for irregular boundaries, the correction term should be \( \lambda^{H/2} \), where \( H \) is the Hausdorff dimension of the boundary (rather than \( \lambda^{(d-1)/2} \) as Eq. (37) seems to suggest)—this conjecture was disproved by Brossard and Carmona (1986) who suggested the use of the Minkowski (or box-counting) dimension instead. The thus modified Weyl-Berry conjecture was proved shortly thereafter for \( d = 1 \) (Lapidus, 1991; Lapidus and Pomerance, 1993) and falsified for \( d > 1 \) (Lapidus and Pomerance, 1996). Extensions to domains with fractal (Levitin and Vassiliev, 1996) or rough boundaries (Netrusov and Safarov, 2005) as well as to manifolds and higher-order Laplacians (Desjardins, 1998; Desjardins and Gilkey, 1994) soon followed. However, we digress.

C. Eigenfunctions and nodal lines

After much deliberation on the eigenvalues of the Laplacian in the preceding subsections, we now examine the universal properties of the eigenfunctions themselves as a segue into nodal portraits. Of great import are questions surrounding the smoothness of the eigenfunctions. First of all, the eigenfunctions are infinitely differentiable \((C^\infty)\) at any point inside the region \( D \) (Bernstein, 1950). Observe that this \( C^\infty \) smoothness is preserved upon reflecting \( u_j \) as an odd function across any straight (or more generally, \( C^\infty \)) portion of the boundary—the resultant function still satisfies the Helmholtz equation locally in a neighborhood of that boundary segment. The fact that such odd reflections can be used to extend the eigenfunctions to larger and larger regions was precisely exploited by Lamé (1866) to obtain the eigenfunctions

\[
\lambda_j \propto \frac{4\pi^2}{(\omega_d \mu_d(D))^{2/d} j^{2/d}}; \quad \omega_d = \frac{\pi^{d/2}}{\Gamma(d/2 + 1)},
\]

where \( \mu_d(D) \) is the Lebesgue measure of \( D \) (nothing but the area in two dimensions or the volume in three) and \( \omega_d \) is the volume of a \( d \)-dimensional unit sphere, \( \Gamma \) being the Gamma function. Therefore, the area (volume) in two (three) dimensions can be extracted from the slope of \( \lambda_j \) graphed against \( j^{2/d} \), i.e., by counting how rapidly the eigenvalues grow. Eq. (35) can be inverted and rewritten as a formula for the index \( j \) instead—this naturally leads us to define a counting function (the number of eigenvalues smaller than \( \lambda \)) as

\[
N(\lambda) = \sum_{j=1}^{\infty} \Theta(\lambda - \lambda_j) \asymp \frac{\omega_d \mu_d(D)}{d} \lambda^{d/2} \quad (\lambda \to \infty),
\]

with \( \Theta \) denoting the Heaviside step function. The description of Eq. (36) only accounts for the leading-order term in the counting function and there are higher-order corrections that yield information about the boundary of the domain. These corrections, initially proposed by Weyl (and justified later by Ivrii (1980) and Melrose (1980) for convex \( D \)), are given by

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\end{cases},
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where the \(- (+)\) sign is applicable for Dirichlet (Neumann) boundary conditions. Here, and later (unless explicitly stated otherwise), \( A, P, V \) and \( S \) stand for the area, perimeter, volume and surface area of the billiard, respectively, in the appropriate dimensions. Berry (1979, 1980) hypothesized that for irregular boundaries, the correction term should be \( \lambda^{H/2} \), where \( H \) is the Hausdorff dimension of the boundary (rather than \( \lambda^{(d-1)/2} \) as Eq. (37) seems to suggest)—this conjecture was disproved by Brossard and Carmona (1986) who suggested the use of the Minkowski (or box-counting) dimension instead. The thus modified Weyl-Berry conjecture was proved shortly thereafter for \( d = 1 \) (Lapidus, 1991; Lapidus and Pomerance, 1993) and falsified for \( d > 1 \) (Lapidus and Pomerance, 1996). Extensions to domains with fractal (Levitin and Vassiliev, 1996) or rough boundaries (Netrusov and Safarov, 2005) as well as to manifolds and higher-order Laplacians (Desjardins, 1998; Desjardins and Gilkey, 1994) soon followed. However, we digress.
and eigenvalues of the equilateral triangle by tiling the plane therewith. Moreover, the eigenfunctions are characterized by the so-called unique continuation property (Kuttler and Sigillito [1984]): a function $u_j$ satisfying the Helmholtz equation on a domain $D$ cannot vanish on an open subset $\mathbb{D} \subset D$ without vanishing identically in the region (i.e., $u_j(r)$ must be zero $\forall r \in \mathbb{D}$).

To gain some insight into the geometrical structure of the eigenfunctions, it is rewarding to look at the corresponding nodal lines. The nodal set is formally defined as

$$N_j = \{ r \subset D \mid u_j(r) = 0 \}. \quad (38)$$

Owing to the unique continuation property, the nodal set is also comprised of curves that are $C^\infty$ in $D$. Where nodal lines cross, they do so at equal angles (Courant and Hilbert [1953]). To put it formally, if $u(r) = 0$, then in any neighborhood of $r$, the nodal line is either a smooth curve, or an intersection of $n$ smooth curves at equal angles (Bers [1955]). This equiangularity also extends to when nodal lines intersect a $C^\infty$ portion of the boundary. Thus, as Kuttler and Sigillito [1984] note, “a single nodal line intersects the $C^\infty$ boundary at right angles, two intersect it at 60° angles, and so forth.” These nodal lines branch out across $D$, forming an intricate network that partitions the region into nodal domains. The mathematical corpus on nodal domains is rich and varied with several deep and powerful theorems. Interestingly enough, it is also replete with rather many equally valuable but incorrect assertions and falsified conjectures that strikingly bespeak the circuitous evolution of the subject.

For concreteness, let us illustrate these results in the context of a specific example—the two-dimensional rectangular billiard with (for now) Dirichlet boundary conditions. The solutions for the free vibrations of a thusly-shaped membrane were first studied by Poisson (1829) whose analysis, as has been remarked, “left little to be desired” (Rayleigh [1945]). For the rectangle defined by $D = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x < a, 0 \leq y < b\}$, the Helmholtz equation can be solved quite easily by the method of separation of variables. The eigenfunctions are

$$u_{m,n}(x,y) = \sqrt{\frac{4}{ab}} \sin \left( \frac{m \pi x}{a} \right) \sin \left( \frac{n \pi y}{b} \right), \quad (39)$$

with eigenvalues

$$\lambda_{m,n} = \pi^2 \left[ \left( \frac{m}{a} \right)^2 + \left( \frac{n}{b} \right)^2 \right]; \quad m,n = 1,2,\ldots \quad (40)$$

Since the eigenfunction (39) neatly splits into a product of functions that depend on $x$ and $y$ individually (and not on any combination thereof), the rectangle is the prototypical example of a separable billiard. The nodal set is straightforward to visualize: it consists of a grid formed by vertical and horizontal lines at $x = q_1 a/m$ and $y = q_2 b/n$, respectively, where $q_1,q_2 \in \mathbb{N}$; $q_1 < m$, $q_2 < n$. Evidently, the total number of nodal domains for the eigenstate $(m,n)$ is $\nu_{m,n} = mn$. Instead of the pair of quantum numbers $(m,n)$, the same state can be interchangeably indexed by a label $j$ on the eigenvalues such that

$$0 < \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_j \leq \lambda_{j+1} \leq \ldots \quad (41)$$

To highlight some of the nontrivial features that emerge, even in this simple system, we list a sequence of such eigenstates and their associated number of nodal domains for a square billiard of side $\pi$ in Table I.

| $j$ | $(m,n)$ | $E_{m,n}$ | $\nu_{m,n}$ | $\xi_j$ |
|-----|---------|-----------|-------------|--------|
| 5   | (1,3)   | 10        | 3           | 3/5    |
| 6   | (3,1)   | 10        | 3           | 1/2    |
| 7   | (2,3)   | 13        | 6           | 6/7    |
| 8   | (3,2)   | 13        | 6           | 3/4    |
| 9   | (1,4)   | 17        | 4           | 4/9    |
| 10  | (4,1)   | 17        | 4           | 2/5    |
| 11  | (3,3)   | 18        | 9           | 9/11   |
| 12  | (2,4)   | 20        | 8           | 2/3    |
| 13  | (4,2)   | 20        | 8           | 8/13   |
| 14  | (3,4)   | 25        | 12          | 6/7    |

TABLE I The eigenstates for a square of side $\pi$ are arranged in increasing order of their eigenvalues $E_j = E_{m,n} = m^2 + n^2$, indexed by the label $j$. For various pairs of quantum numbers, the number of domains $\nu_j = \nu_{m,n}$, and the normalized mode number $\xi_j = \nu_{j}/j$ are shown here. This tabulation is a direct way to appreciate the complexity and nonmonotonicity of the sequence $\{\xi_j\}$ even for so tractable a system. This makes the pursuit of finding a statistical description for $\xi$ important and worthwhile.

Two particular attributes become apparent upon perusing Table I for a while. Firstly, for $j > 1$, the eigenvalues can be degenerate—this is characteristic of regions with symmetries. Oftentimes, the degeneracy can be lifted by choosing a rectangle with incommensurate side lengths, thereby breaking the aforesaid symmetry. Whenever $m \neq n$, $E$ necessarily has a multiplicity of at least two as it is symmetric under $i \leftrightarrow j$. Another class of degeneracies stems from purely number-theoretic origins. An example in this category is the fourfold-degenerate eigenvalue $65 = 1^2 + 8^2 = 4^2 + 7^2$. The question therefore is, in how many ways can a given integer $E$ be written as the sum of the squares of two integers? Number theory provides an answer. If $E$ is decomposed into distinct primes as

$$E = 2^a p_1^{s_1} \cdots p_k^{s_k} q_1^{s_1} \cdots q_l^{s_l}, \quad (42)$$

where the $p_i$ are of the form $4t+1$ and the $q_i$ of the form $4t+3$, then $s_i$ is even $\forall i$ and the multiplicity of $E$ is
As expected, a quick check shows that this formula correctly yields the multiplicity of 65 as 4. Accordingly, one can have eigenvalues of arbitrarily large multiplicity. For eigenvalues with multiplicities greater than one, the nodal sets, now formed by a superposition of eigenstates, can be much more exotic as Fig. [10] illustrates. Referring to Béard and Helffer (2015) provides one with ample pictures to argue that the number of nodal domains for a linear combination of two given independent eigenfunctions can be smaller or larger than the number of nodal domains of either Béard and Helffer (2015). However, for a generic region, there are still certain constraints on the eigenfunctions. Uhlenbeck (1972, 1976) proved that for “most” regions, the eigenvalues are all simple (nondegenerate), the nodal lines do not intersect, and the eigenfunctions’ critical points are either maxima or minima. Given a region $D$ that violates one or more of these properties, one can always obtain another $D’$—by arbitrarily small (perhaps symmetry-breaking) perturbations—that indeed satisfies them. Fig. [10] gives a tangible example of how nodal crossings pull apart under such perturbations. Recognizing the instability of Dirichlet nodal domains under various perturbations due to the “avoided crossings”, McDonald and Fulling (2014) proposed a partition of the domain $D$ by trajectories of the gradient linking saddle points to extrema. These lines are (misleadingly enough) designated “Neumann nodal curves” since their tangent vectors are always parallel to $\nabla u$ and resultantly, $u$ satisfies the Neumann boundary condition along them. Such a construction, as artificial as it may seem, largely eliminates the problem of avoided crossings.

The second observation underscored by the data in Table [1] concerns the ratio $\xi_j = \nu_j/j$. The perceptive reader would surely have presaged that it is not mere coincidence that $\xi_j$ always happens to be less than unity. In fact, it is not and that brings us to our next set of theorems.

1. Courant’s nodal domain theorem

Setting aside billiards and membranes for a moment, let us regress to the simplest possible example that permits a description of nodal sets—a one-dimensional, vibrating string. The nodes, which are now points rather than lines, are the spots on the string that remain stationary at all times. As regards the nodal “domains” (more correctly, the nodal intervals that the string is partitioned into), Sturm (1836a,b) proved the following result, made rigorous by Bocher (1889).

**Theorem 1. Sturm oscillation theorem (Simon, 2005):**

$$M(E) = \prod_{i=1}^{k} (r_i + 1). \quad (43)$$

FIG. 10 Consider the family of (linear superpositions of) eigenfunctions of the Laplacian on the square $[0,\pi]^2$

$$\Phi_{j,\nu}(x, y) = \cos \theta \sin x \sin(2\nu y) + \sin \theta \sin(2\nu x) \sin y. \quad (44)$$

associated with the Dirichlet eigenvalue $1 + 4\nu^2$. For $\theta = \pi/4$ (left), the nodal pattern exhibits 12 domains whereas for $\theta \leq \pi/4$ (right), the double points all disappear and the nodal set (a single connected line) divides the square into only two domains Béard and Helffer (2015). In fact, Stern (1924) established that there are actually infinitely many eigenfunctions having exactly two nodal domains. This example also illustrates why counting nodal domains is such a hard problem, requiring one to resort to powerful numerical techniques. From Courant and Hilbert (1953). © 2004 WILEY-VCH Verlag GmbH & Co. KGaA.

1. Let $\lambda_j$ be the eigenvalues of $H = -d^2/dx^2 + V(x)$ with boundary conditions $u_j(0) = u_j(a) = 0$. Then $u_j(x)$ has exactly $j$ zeros in $(0, a)$.

2. The number of eigenvalues of $H$ strictly below $\Lambda$ is exactly the number of zeros of $u_{j(\Lambda)}(x)$ in $(0, a)$.

Sturm, Liouville, and Rayleigh extended this statement to add that a linear combination of $u_m, u_{m+1}, \ldots, u_n$ with constant coefficients has at least $m - 1$ and at most $n - 1$ zeros in the open interval $(0, a)$ spanned by the string Pleijel (1956). Generalizing results of this sort to higher-dimensional regions dovetails into Courant’s theorem. Although the original proof outlined by Courant and Hilbert (1953) was for planar domains, it has since been adapted to compact Riemannian manifolds Béard and Meyer (1982).

**Theorem 2. Courant’s nodal domain theorem:**

1. The first eigenfunction, $u_1(r)$ corresponding to the smallest eigenvalue, $\lambda_1$, on a domain $D$ with arbitrary homogeneous boundary conditions does not have any nodes.

2. For $j \geq 2$, $u_j(r)$, corresponding to the $j$th eigenvalue of the Laplacian counting multiplicity, divides the region $D$ into at least two and no more than $j$ domains.

No assumptions are made about the number of independent variables.
Even without getting into the details of the derivation, which can be found elsewhere [Strauss 1992], we can easily see that assuming Theorem 2.1 is correct, \( u_j(r) \) for \( j > 2 \), must divide \( D \) into two domains at the very least. The proof proceeds as follows [Saito 2007]. Since \( u_1 \) is orthogonal to \( u_j \neq 1 \), we have

\[
\int_{D} u_1(r) u_j(r) \, dr = 0, \quad (45)
\]

and from Theorem 2.1, we also know that \( u_1(r) > 0 \) or \( u_1(r) < 0 \) \( \forall r \in D \). Hence, \( u_j(r) \) must necessarily change its sign somewhere in \( D \) and therefore, appealing to the continuity of \( u_j(r) \), there exist zeros of \( u_j(r) \) in \( D \). These zeros form the nodal set. At the end of Courant’s original proof of Theorem 2 [Courant and Hilbert 1953], there appears a rather innocuous (and now notorious) footnote:

The theorem just proved may be generalized as follows: Any linear combination of the first \( n \) eigenfunctions divides the domain, by means of its nodes, into no more than \( n \) subdomains.

This assertion, attributed to Hermann [1932, 1936], is—without additional qualifiers—egregiously untrue. Evidence to its falsification was first pointed out by V. I. Arnold. He realized that by generalizing Courant’s theorem to include Hermann’s, one could arrive at conclusions about the topology of algebraic curves [Arnold 1973] that outright contradicted the known results of quantum field theory. Eventually, putting the final nail in the coffin, Viro [1979] constructed a real algebraic hypersurface as an explicit counterexample. In general, Hermann’s theorem is valid only under some restrictions on the number of independent variables; in particular, it is false for the Laplacian on \( S^3 \) and higher-dimensional spheres [Kuznetsov 2015].

On the contrary, there do exist other modifications to Courant’s proposition that are actually correct. Pleijel [1956] showed that for planar domains with Dirichlet boundary conditions, Courant’s bound can be asymptotically improved, proving that for an infinitely long sequence,

\[
\limsup_{j \to \infty} \frac{\nu_j}{j} \leq \left( \frac{2}{\beta_{0,1}} \right)^2 \approx 0.691, \quad (46)
\]

where \( \beta_{0,1} \) is the first positive zero of the Bessel function \( J_0(z) \). Let us briefly recall Pleijel’s argument here, not only because the proof is enlightening but also because it gives us the chance to justify our compilation of the much-vaulted isoperimetric inequalities in Appendix A. Denote by \( \omega_1, \omega_2, \ldots, \omega_p \) the nodal domains of the \( j^{th} \) eigenfunction of the Dirichlet Laplacian, \( u_j \). Inside each domain \( \omega_i \), the function \( u_j \) is nonzero; hence, \( \lambda_j \) is the first eigenvalue (with Dirichlet boundary conditions along the perimeter) of the region \( \omega_i \). One can then apply the Faber-Krahn inequality (Eq. A2) individually to each \( \omega_i \) and sum over \( i \) to get

\[
\sum_i A(\omega_i) \frac{\lambda_j}{\pi \beta_{0,1}^2} \geq \sum_i \frac{1}{\lambda_j} \nu_j, \quad (47)
\]

following which, we can decompose \( \nu_j/\lambda_j \) as \( \nu_j/j \times j/\lambda_j \). According to Weyl’s law (Eq. 33), in the limit \( j \to \infty \), \( j/\lambda_j \propto A/4\pi \). Making this substitution in the second term of the product, we obtain Eq. (46). To attain the maximal upper bound of \( \nu_j = j \) in Courant’s inequality, a necessary condition is that

\[
A \frac{\lambda_j}{\pi \beta_{0,1}^2} \geq j, \quad (48)
\]

Suppose \( \nu_j = j \) for infinitely many values of \( j \). Reusing Weyl’s law, the right-hand side above can be replaced as:

\[
A \frac{\lambda_j}{\pi \beta_{0,1}^2} \geq \frac{A}{4 \pi}, \quad (49)
\]

which is blatantly false because \( \beta_{0,1} \approx 2.4048 > 2 \). This led Pleijel [1956] to conclude that for the eigenfunctions \( u_j \) of a membrane with a fixed boundary, the maximum \( j \) of the number of nodal domains is attained only for a finite number of eigenvalues. For the square membrane, we can actually identify the eigenvalues that are Courant sharp (i.e., \( \lambda_j \) such that \( \nu_j = j \)). In this case, Pleijel [1956] derived the identity

\[
j > \frac{\pi}{4} \lambda_j - 2 \sqrt{\lambda_j} + 2, \quad (50)
\]

which, combined with Eq. (48) (now in the form \( j < 0.54323 \lambda_j \) for \( A = \pi^2 \)), can be effectively rearranged to \( \lambda_j < 51 \). Manually examining the spectral sequence (and deferring the missing analysis of \( \lambda_5, \lambda_7, \) and \( \lambda_9 \) in Pleijel’s proof to [Bérard and Helffer 2015]) we find that \( \lambda_j \) is Courant sharp only for \( j = 1, 2, \) and 4. Pleijel’s theorem was generalized to surfaces by Peetre [1957] and therefore, the sphere too has only finitely many Courant-sharp eigenvalues. Additionally, the estimate (46) also holds for a piecewise real analytic domain with Neumann boundary conditions [Polterovich 2009], for which the only Courant-sharp eigenvalues are [Helffer and Sundqvist 2015] \( \lambda_j \) with \( j \in \{1, 2, 4, 5, 9\} \) (Fig. 11).

Further generalizations of Courant’s nodal domain theorem include extensions to nonlinear eigenvalue problems for the \( p \)-Laplacian \( \Delta_p \) [Cuesta et al. 2006; Drábek and...
FIG. 11 Nodal patterns of Neumann eigenfunctions in the square. The nodal sets are depicted for the four nontrivial Courant-sharp cases, $\lambda_2$, $\lambda_4$, $\lambda_5$, and $\lambda_9$, corresponding to wavefunctions $\cos \theta \cos x + \sin \theta \cos y$ ($\theta = 1$ here), $\cos x \cos y$, $\cos 2x + \cos 2y$, and $\cos 2x \cos 2y$, respectively. Regions where the wavefunction is positive (negative) are painted black (white).

Robinson [2002], self-adjoint second-order elliptic operators [Alessandrini 1998], and inequalities on spectral counting functions [Ancona et al. 2004]. Specifically worth mentioning in this list is the important conjecture for the biharmonic eigenvalue problem:

$$\Delta^2 u = \lambda u \quad \text{in } \mathcal{D}$$

$$u = \partial \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial \mathcal{D}. \quad (51)$$

Szegö (1950) hypothesized that if $\mathcal{D} \subseteq \mathbb{R}^2$ is a “nice” domain (i.e., $\partial \mathcal{D}$ is an analytic curve), then $u_1$ for Eq. (51) does not change its sign (Sweers 2001). However, surprisingly, the conjecture is not even true for the first eigenfunction (Coffman 1982; Coffman and Duffin 1980; Duffin 1948; Garabedian 1950; Kozlov et al. 1990; Loewner 1953; Shapiro and Tegmark 1994).

2. Nodal line conjecture

Another richly-debated conjecture that sparked intense discussion in the field was the proposition by Payne (1967) that the second Dirichlet eigenfunction $u_2$ does not have a closed interior nodal line in a bounded planar domain $\mathcal{D} \subseteq \mathbb{R}^2$ of any arbitrary shape. This longstanding question was also concurrently posed by Yang (1982). Payne (1973) himself verified the conjecture under the added condition that $\mathcal{D}$ is symmetric with respect to a line and convex with respect to the direction vertical to it. Thereafter, Lin (1987) ratified it for a smooth, convex domain invariant under rotations by $2\pi p/q$; $p, q \in \mathbb{Z}^+$; a few years later, Jerison (1991) did so for long, thin convex sets. Finally, Melas (1992), furnishing the first fully-general result, certified that “if $\mathcal{D} \subseteq \mathbb{R}^2$ is a bounded, convex domain with $C^\infty$ boundary, then the nodal line $\mathcal{N}$ of any second eigenfunction $u_2$ must intersect the boundary $\partial \mathcal{D}$ at exactly two points” — the analogous statement for simply-connected concave domains was proved by Yang and Guo (2013). Since Courant’s theorem ensures that $u_2$ can have at most two nodal domains, the (only) nodal line must connect these intersection points.\(^{10}\) The impossibility of a closed nodal curve thus follows. This argument was extended upon by Alessandrini (1994) by removing the requirement of $C^\infty$ smoothness on the boundary. The higher-dimensional generalization of this conjecture is due to a theorem by Liboff (1994): the nodal surface of the first excited state of a three-dimensional convex domain intersects its boundary in a single simple closed curve (Grebenkov and Nguyen 2013).

Noteworthily, Payne’s surmise does not pass muster for nonconvex domains; the nodal line of the second eigenfunction of the Laplacian can be closed in $\mathbb{R}^d$ (Fournais 2001) and, in particular, in $\mathbb{R}^2$ with Dirichlet (Hoffmann-Ostenhof et al. 1997) and Robin (Kennedy 2011) boundary conditions. It has also been explicitly disproved for the eigenvalue problem where the Schrödinger operator has a potential $V \neq 0$ in addition to the Laplacian $\Delta$ (Lin and Ni 1988).

3. Geometry of nodal sets

Before concluding this Section, we quickly expound on two other results of a purely geometric nature on the size—or volume—of nodal domains. To characterize the asymptotic geometry of the domains, we quantify the size by the inradius. Let $r_\lambda$ be the inradius of a nodal domain for the eigenfunction with eigenvalue $\lambda$. On a closed Riemannian manifold of dimension $d \geq 3$, we then have (Mangoubi 2008)

$$\lambda^{d^2-15d/8+1/4} (\log \lambda)^{2d-4} \leq r_\lambda \leq \frac{C_2}{\sqrt{\lambda}} \quad (52)$$

In two dimensions also, there is a sharp bound (Donnelly and Fefferman 1988, 1990; Nadirashvili 1988; Zelditch 2013):

$$\frac{C_3}{\sqrt{\lambda}} \leq r_\lambda \leq \frac{C_4}{\sqrt{\lambda}} \quad (53)$$

After some juggling and gentle coaxing, this inequality offers a crude but utilitarian estimate for the number of nodal domains:

$$\frac{A}{\pi C_4^2} \leq \nu_j \approx \frac{A}{\pi r_j^2} \leq \frac{A \lambda_j}{\pi C_3^2} \quad (54)$$

These relations reappear in different guises when we talk about arithmetic random waves and difference equations later in this review. In the meantime, the more mathematically-oriented reader can find a delightful survey of results on the geometric properties of eigenfunctions in (Jakobson et al. 2001).

\(^{10}\) In fact, when the eccentricity of $\mathcal{D}$ is sufficiently large, $\mathcal{N}$ is close to a straight line in the sense that the width of the nodal line $< C/r_D$, where $C$ is a constant and $r_D$ is the inradius of $\mathcal{D}$ (the radius of the largest ball that can be inscribed in $\mathcal{D}$) (Grieser and Jerison 1995; Jerison 1995).
IV. NODAL SETS OF CHAOTIC BILLIARDS

A. The random wave model—An introduction

The statistical properties of wavefunctions of classically chaotic systems are well-described by a surprisingly simple model, first conjectured by Berry (1977). Known as the random wave model (RWM), it proposes that the eigenfunctions of strongly chaotic systems “behave” like a random superposition of plane waves of fixed wavelength magnitude. Even today, nearly half a century after Berry’s initial proposition, there is no formal proof of this statement but rather, only heuristic justifications. The starting point for any such argument is the semiclassical eigenfunction hypothesis that we have seen previously in Sec. II.A. Then, the model can be motivated by the underlying (classical) chaotic dynamics of the quantum billiard where a typical trajectory gets arbitrarily close to every point in position space with apparently random directions and random phases (corresponding to the length of trajectory segments) (Bäcker, 2007a). Since the whole energy surface is filled uniformly for ergodic systems, the probability density of finding the particle somewhere has, on average, a uniform distribution over the full billiard (Bäcker, 2007a). In other words, as Urbina and Richter (2007) note, “in the semiclassical regime, the eigenfunctions should appear isotropic, structureless and roughly homogeneous owing to the lack of structure of the classical phase space.” Hence, a random superposition of plane waves suffices for a reasonably good description of the system as Fig. 12 indeed demonstrates.

![Figure 12: Example of a random wave (Eq. 55) in comparison to the 6000th eigenfunction of the chaotic cardioid billiard (of odd symmetry). Locally the appearance of the states is practically indistinguishable. O’Connor and Heller (1988) investigated the properties of the eigenfunctions constructed by such random superpositions and revealed the existence of structures resembling precursors of periodic orbit scar localization (Heller, 1984; Kaplan and Heller, 1999). From Bäcker (2007a).](image)

The most intriguing aspect is the universality of this deceptively simple model, which finds applications in diverse fields ranging from optics (Berry and Dennis, 2000) to wave mechanics in disordered media (Barth and Stöckmann, 2002; Mirlin, 2000). In studies of mesoscopic systems as well, the RWM has proven to be immensely successful in describing conductance fluctuations in quantum dots (Alhassid, 2000; Baranger and Mello, 1994; Beenakker, 1997; Jalabert et al., 1994), especially in the Coulomb blockade (CB) regime (Aleiner et al., 2002; Beenakker, 1991; Patel et al., 1998) where it relates the distribution of tunneling amplitudes to the statistics of CB peak heights (Alhassid and Lewenkopf, 1997; Jalabert et al., 1992; Ulme, 2008). This robustness has prompted observers to regard the RWM as a litmus test for wave signatures of classically chaotic dynamics (Blum et al., 2002). Further applications and theoretical details of the RWM are discussed by Urbina and Richter (2013) in their state-of-the-art review.

That the model works so well is no coincidence as reasoned succinctly by Urbina and Richter (2003). Their reasoning hinges on the two fundamental arguments at play here. First, the RWM wavefunction is, roughly speaking, a function taking random values at each point (formally called a stationary random process) (Goodman, 2015); in addition, it is also Gaussian, which implies that it can be uniquely characterized by a two-point correlation function that encodes the appropriate symmetries. It is this generality of the random wave two-point correlation that accounts for the effectiveness of the theory when neglecting boundary effects, as for bulk properties. A cautionary remark, however: despite its spectacular triumphs, one would do well to remember that the random wave model is precisely just that—namely, a model and that too, of sufficiently excited states. For instance, it must be emphasized that the ground-state eigenfunction for a quantum chaotic billiard, in general, would be unassumingly well-behaved, not resembling anything that looks like a random superposition of plane waves (Jain, 2009; Jain et al., 2002). Nonetheless, for the object of our interest, namely, the nodal set, it works very well, numerically (see Fig. 13).

Now, to the mathematics. In two Euclidean dimensions, the random superposition of plane waves on a region $D \subset \mathbb{R}^2$ may be written as

$$\psi_{N}(\mathbf{r}) = \sqrt{\frac{2}{\text{vol}(D)N}} \sum_{i=1}^{N} a_{i} \cos(\mathbf{k}_{i} \cdot \mathbf{r} + \phi_{i})$$

(55)

where $a_{i} \in \mathbb{R}$ are independent Gaussian random variables with zero mean and unit variance, $\phi_{i}$ are uniformly-distributed random variables on $[0, 2\pi)$, and the momenta $\mathbf{k}_{i} \in \mathbb{R}^2$ are randomly equidistributed, lying on the circle of radius $\sqrt{E}$. Note that the factor of $1/\sqrt{N}$ up front takes care of the normalization in the limit $N \to \infty$. A property of random waves of this type is the universality
of the spatial autocorrelation function
\[ C(\mathbf{r}, \delta \mathbf{r}) = \langle \psi_{\text{RWM}}(\mathbf{r} - \delta \mathbf{r}/2) \psi_{\text{RWM}}(\mathbf{r} + \delta \mathbf{r}/2) \rangle = \frac{1}{\text{vol}(D)} J_0(\sqrt{E} |\delta \mathbf{r}|). \] (56)

One should be wary that Eq. (56) is not a consequence of the randomness but rather, of the dispersion relation $|\mathbf{k}| = \sqrt{E}$, and more directly, of the quantum ergodicity theorem. Although semiclassical techniques cannot be directly used to calculate higher-order correlations, Berry went on to conjecture that all the statistical properties of the spatial fluctuations of the eigenfunctions of a chaotic system are described by a superposition of waves with fixed wavenumber and random phases. This description is reminiscent of the ergodicity hypothesis in statistical physics in its assertion that the spatial average in Eq. (56) is essentially equivalent to averaging over the random phases of Eq. (55). Such universal spatial fluctuations have also been identified in systems with nontrivial spin dynamics such a confined two-dimensional electron gas have also been identified in systems with nontrivial spin phases of Eq. (55). Such universal spatial fluctuations is essentially equivalent to averaging over the random noise on the unit circle. This in context, let $L^2_{\text{sym}}(\mathbb{T})$ denote the Hilbert space of square-integrable ($L^2$) functions on the unit circle with the symmetry $f(-z) = f^*(z)$. The Fourier image of the space $L^2_{\text{sym}}$ is simply the space of real analytic functions, $\mathcal{H}$, satisfying the Helmholtz equation. A random plane wave is then $F = \sum_n C_n \Phi_n$ where the $C_n$ are independent Gaussians and $\{\Phi_n\}$ is any orthonormal basis in $\mathcal{H}$; this has a covariance function $\text{Cov}\{F(x), F(y)\} = J_0(|x - y|)$. It may seem a bit pompous to replace the simple sums over exponentials and Bessel functions with a Gaussian field in its full glory but this is, in many ways, the more natural representation to use since the first two definitions can always be extracted from it. The raison d’être for this excursion into overtly mathematical terrain is that it enables us to link our discussion to the Gaussian spherical harmonics (Fig. 14) and tap into the treasure trove of already well-known results (Wigman, 2009) on nodal sets thereof. The connection is simple: the Gaussian plane wave is a large $n$ limit of the Gaussian spherical harmonic of degree $n$.

To be precise, consider the eigenfunctions of the Laplacian $\Delta$ on the $m$-dimensional unit sphere $S^m$. Given an eigenvalue $E_n (n = 1, 2, \ldots)$, the corresponding eigenspace is the space $\mathcal{H}_n$ of the spherical harmonics of degree $n$, which form a $2n+1$ dimensional space of eigenfunctions of the Laplacian on $S^m$; let $g_n \in \mathcal{H}_n$. Using integral formulae due to Poincaré and Kac-Rice (see, for example, Cramer and Leadbetter (2013)), it is possible to obtain some universal estimates such as the following:

- For each $g_n$, every nodal domain of $g_n$ contains a disk of radius $c n^{-1}$, where $c$ is an absolute constant; this tells us about the minimum size of a domain.

11We use the term in the generic sense of a random signal that has equal intensity at all frequencies and thus, a flat power spectrum.
Let \( Z(g_n) \) be the ‘volume’ of the nodal set (in two dimensions, the total length of the nodal lines). Yau (1982, 1993) conjectured that for any smooth metric on a manifold \( M \), there exist constants \( c(M), C(M) \) such that for every \( g_n \), \( c \sqrt{E_n} < Z(g_n) < C \sqrt{E_n} \); the upper bound was also proposed by Cheng (1976). The lower bound has been proved for the planar case (Brüning, 1978; Brüning and Gromes, 1972) and for smooth metrics (Logunov, 2016a,b). We refer to (Savo, 2000) and (Logunov and Malinnikova, 2016) for estimates of the constants \( c \) and \( C \), respectively. Although Donnelly and Fefferman (1988) proved Yau’s conjecture for real-analytic metrics, the problem, in its full generality, still remains open.

For Gaussian spherical harmonics \( g_n \), the expectation value \( \mathbb{E}[Z(g_n)] = \pi \sqrt{2E_n} = \sqrt{2\pi n + \mathcal{O}(1)} \) (Berard, 1985; Zelditch, 2009b). Subsequently, it was proved (Wigman, 2010, 2012a) that

\[
\text{Var} Z(g_n) = \frac{1}{32} \log n + \mathcal{O}(1),
\]

which is much smaller than the prior \( \mathcal{O}(n) \) estimate (Wigman, 2009) on account of Berry’s cancellation phenomenon (Berry, 2002).

By Courant’s nodal domain theorem, and Pleijel’s refinement, \( \nu(g_n) < 0.69 n^2 \). Since there are spherical harmonics with one or two nodal domains, there is no nontrivial deterministic lower bound (Lewy, 1977). A sharp upper bound for the number of nodal domains of spherical harmonics, for the first six eigenvalues, is given by (Leydold, 1996). On the other side of the coin, it is still an open problem whether—for a general surface (or higher-dimensional Riemannian manifold)—there exists a sequence of eigenfunctions for which the number of nodal domains tends to infinity with the eigenvalue. This question was answered in the affirmative for nodal domains of Maass forms (Ghosh et al., 2013, 2015) and non-positively curved surfaces with concave boundaries, i.e., generalized Sinai or Lorentz billiards (Jung and Zelditch, 2016).

Specializing to two dimensions, \( m = 2 \), we can exploit all of the above results on \( S^2 \) for billiards.

After all this song and dance, it becomes imperative to address the elephant in the room: what are the quantitative predictions (if any) of the random wave model? One (and the simplest) of the many answers to this query concerns the amplitude distribution of the wavefunction. Using the central limit theorem one immediately obtains that random waves exhibit a Gaussian distribution of eigenfunction amplitudes

\[
P(\psi) = \frac{1}{\sqrt{2\pi \sigma}} \exp \left(-\frac{\psi^2}{2\sigma^2}\right),
\]

where \( \sigma^2 = 1/\text{vol}(\Omega) \). This prediction was borne out by several numerical studies (Aufrich and Steiner, 1993; Li and Robnik, 1994; McDonald and Kaufman, 1988; O’Connor and Heller, 1988).

Buoyed by this preliminary success, we now direct our attention to explicitly constructing random wave models for billiards with the intent to eventually study their nodal structure. The first hurdle that we run into is that the theory we have looked at so far is isotropic. For a billiard, with well-defined boundaries, we need a non-isotropic RWM constructed from a random superposition of waves satisfying both the Schrödinger equation and the boundary conditions—this problem, regrettably enough, turns out to be at least as difficult as solving the full quantum mechanical problem employing standard techniques (Urbina and Richter, 2003). The nontrivial deviations from the isotropic case owing to finite-size effects (Ullmo and Baranger, 2001) emphasize the relevance of extending this approach to include arbitrary boundaries.

In order to incorporate boundaries into the model, our desideratum is an ensemble of random functions, constructed so as to respect the boundary conditions of the billiard. This patently calls for a departure from the spatial averaging prescribed in the original RWM, such as in Eq. (56), as it would destroy any information about the boundary. The way out of this conundrum is to substitute the spatial average with a spectral one. Instead of dealing with a single eigenfunction, we now evaluate the average for fixed sets of positions (without any spatial integration) over a set of normalized solutions \( \psi_j(r) \) of the Schrödinger equation with nondegenerate eigenvalues \( E_j \) lying in the interval \( w = [c-e\delta e/2, c+\delta e/2] \). The principal idea is that given a functional \( F[\psi] \equiv F(\psi(r_1), \ldots, \psi(r_N)) \), we can always define the spectral average of the functional \( F \) around
energy \( e \) as (Urbina and Richter 2006)

\[
F = \frac{1}{\rho_w(e)} \sum_j w(e - E_j) F[\psi_n],
\]

where \( w(x) \) represents a normalized window function around \( x = 0 \) and \( \rho_w(e) \) is the density of states smoothed over the window \( w \). The two-point correlation function associated with this spectral average is

\[
R_w(r_1, r_2; e) = \frac{1}{\rho_w(e)} \sum_j w(e - E_j) \psi_n(r_1) \psi_n(r_2),
\]

which exactly satisfies the boundary conditions. This correlation function can actually be calculated using semiclassical expansions of the propagator and represented in a multiple reflection expansion (Hortikar and Srednicki 1998; Urbina and Richter 2004) of the form:

\[
R_w(r_1, r_2; e) = \frac{1}{A(D)} J_0(k e |r_1 - r_2|) + \text{sum over reflections at the boundary}.
\]

Thus, the prescription most amenable to generalization is to work directly with the two-point correlation function that assimilates all nonuniversal effects stemming from boundary constraints.

To make the above notions precise, let us demonstrate this approach in the context of computing the following one-point averages that find use in the nodal counting statistics:

\[
B(x) \equiv \langle \psi(x)^2 \rangle, \quad K_y(x) \equiv \langle \psi(x) \frac{\partial \psi(x)}{\partial y} \rangle,
\]

\[
D_x(x) \equiv \left\langle \left( \frac{\partial \psi(x)}{\partial x} \right)^2 \right\rangle, \quad D_y(x) \equiv \left\langle \left( \frac{\partial \psi(x)}{\partial y} \right)^2 \right\rangle,
\]

written in the notation of Berry (2002). For billiard systems, the isotropic RWM is defined by the ensemble

\[
\psi^i(x) = \sqrt{\frac{2}{J}} \sum_{j=1}^{N} \cos(kx \cos \theta_j + ky \sin \theta_j + \phi_j)
\]

with \( \theta_j = 2\pi j/J \) and the average \( \langle \cdots \rangle \) being determined by integration over a set of independent random phases \( \phi_j \in (0, 2\pi) \) (formally, the limit \( J \to \infty \) is taken after averaging). Explicit calculation yields (Berry and Dennis 2000)

\[
B^i(x) = 1, \quad K_y^i(x) = 0, \quad D_x^i(x) = D_y^i(x) = \frac{k^2}{2}.
\]

Since boundary effects are neglected in this calculation, these results only represent bulk approximations to the system. In a first attempt to overcome this limitation, Berry (2002) introduced the following ensemble of nonisotropic superpositions of random waves for an idealized system with an infinite straight wall at \( y = y_0 \), on which the wavefunction satisfies Dirichlet (D) or Neumann (N) boundary conditions:

\[
\psi^{(D,N)}_i(x) = \sqrt{\frac{1}{J}} \sum_{j=1}^{N} \cos(ky(y-y_0)) \cos(kx \cos \theta_j + \phi_j).
\]

The nonisotropic analogues of Eq. (64) can again be worked out in this ensemble. For a more generic situation, where the confining potential is smooth, one can construct an ensemble of random Airy functions \( A_i(r) \) to locally satisfy the Schrödinger equation, as demonstrated by Bies and Heller (2002) for a linear ramp potential \( V(x, y) = V y \). Another (and, to the best of our knowledge, the only other) boundary to which a nonisotropic RWM has been adapted is the edge between two infinite lines angled at a rational multiple of \( \pi \) (Bies et al. 2003).

The modus operandi for actually calculating the above-defined averages is to evaluate the two-point correlation function defining the nonisotropic and finite-size RWM

\[
R(r_1, r_2) \equiv \langle \psi(r_1) \psi^*(r_2) \rangle = \frac{1}{N} \sum_{j \in w} \psi_j(r_1) \psi_j^*(r_2)
\]

and then take derivatives. The correlation function is more conveniently expressed in terms of the Green’s function of the system

\[
G(r_1, r_2, E + i0^+) = \sum_{j=1}^{\infty} \frac{\psi_j(r_1) \psi_j^*(r_2)}{E - E_j + i0^+},
\]

upon converting the sum to an integral by introducing the (approximately constant) mean level spacing \( \Delta(e) \). This form of the correlation function—in analogy with the partition function in statistical mechanics—can be used to compute the relevant averages by differentiation. For instance,

\[
D_x(r) = \left[ \frac{\partial^2 F(r_1, r_2)}{\partial x_1 \partial x_2} \right]_{r_1 = r_2 = r},
\]

\[
K_y(r) = \left[ \frac{1}{2} \left( \frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2} \right) F(r_1, r_2) \right]_{r_1 = r_2 = r}.
\]

For billiards, the bulk results are obtained by replacing the exact Green’s function with the free propagator given by the Hankel function (Heller and Landry 2007)

\[
G^0(r_2, r_1, E + i0^+) = \frac{i}{4\pi} H^0_{1} \left( \frac{\sqrt{E}}{h} |r_1 - r_2| \right).
\]

The (bulk) contribution to the two-point correlation is

\[
F^b(r_1, r_2) = \frac{1}{A \delta e} \int_{e - \frac{\delta e}{2}}^{e + \frac{\delta e}{2}} J_0 \left( \frac{\sqrt{E}}{h} |r_1 - r_2| \right) dE.
\]
This not only agrees with the previous calculation of Eq. (64) upon differentiation but also reduces to Berry’s result, Eq. (56) (identifying $\text{vol}(D) = A$), when $|r_1 - r_2| \ll \sqrt{4A/\pi}$. In the opposite limit, the correlation decays much faster, as long as $\delta e \geq h\sqrt{\pi e/4A}$. Recognizing $\sqrt{4A/\pi}$ as a length scale that sets the average linear size of the system, $L$, it is apparent that the RWM defined by Eq. (70) subsumes finite size effects when scales of averaging are larger than the ballistic Thouless energy, i.e., $\delta e \geq e_{\text{Th}} = h\sqrt{\pi e/L}$ (Urbina and Richter 2003). Similarly, one can find the two-point correlation in the case of an infinite (smooth or straight) barrier and then take the appropriate limits (either short distances or infinite system size) to recover the known RWM results. The salient attribute worth noting in this elegant calculation is that despite Berry’s original conjecture being an inherently statistical statement, the two-point correlation function defining the RWM, Eq. (70), is derived from purely quantum mechanical expressions without reference to any statistical assumptions about the wavefunction and is thus independent of the character of the classical system. An instructive summary of the different approaches we have discussed here can be espied in the example of the wedge billiard, reviewed by Heller and Landry (2007).

1. The normalization problem

Before moving on, we would like to stress a subtlety about normalization first presented by Mirlin (2000) and explored by Narimanov et al. (2001). The snake lurking in the grass is the assumption of a chaotic wavefunction being a Gaussian process, which apparently contradicts the normalization condition for the wavefunction (Gornyi and Mirlin, 2002); more specifically, the existence of finite boundaries is incompatible with Gaussian statistics. This disagreement can be seen as follows. Consider the functional

$$\eta[\psi] = \int |\psi(\mathbf{r})|^2 \, d\mathbf{r},$$

(71)

where $\psi(\mathbf{r})$ belongs to the ensemble chosen to describe the statistical properties of the wavefunction. The normalization of all $\psi(\mathbf{r})$ in the ensemble imposes the constraint that the ensemble variance $\text{Var}(\eta) = \langle \eta[\psi]\rangle^2 - \langle \eta[\psi]\rangle^2 = 0$ since $\eta[\psi] = 1$ over the set of normalized eigenfunctions. However, with a Gaussian distribution of wavefunction amplitudes, we find

$$\text{Var}(\eta) = 2 \int \int |\langle \psi(\mathbf{r}_1) \psi^*(\mathbf{r}_2)\rangle|^2 \, d\mathbf{r}_1 \, d\mathbf{r}_2 \neq 0.$$  

(72)

Physically, the ensuing implication is that the Gaussian distribution, owing to fluctuations of the normalization integral, would produce spurious contributions to any statistics beyond the two-point correlation function, or, for that matter, to the spectral average of any functional of order higher than two. This ostensible discrepancy was partially resolved by Urbina and Richter (2003), who explicitly proved that $\text{Var}(\eta) \sim O(1/N)$ and hence, goes to zero as $N \to \infty$, thereby avoiding any conflict. Nonetheless, in the presence of boundaries, the Gaussian conjecture for the fluctuations of irregular eigenfunctions must be modified—it turns out that the Principle of Maximum Entropy (Grandy Jr., 1987) selects a particular kind of distribution, known as the “Gaussian Projected Ensemble” (Goldstein et al., 2006). This leads to yet another alternate formulation of Berry’s conjecture: to quote Urbina and Richter (2007), “in systems with classically chaotic dynamics, spectral averages of functionals defined over the set of eigenfunctions are given by the corresponding average over the Gaussian Projected Ensemble with fixed system-dependent covariance matrix.”

2. Systems with mixed phase space

A generic system, in contrast to our discussion heretofore, is neither fully chaotic nor regular; instead typically, systems would have a mixed phase space characterized by the coexistence of both regular and chaotic motions. Perelomov (1973) conjectured that for such mixed systems, like a mushroom billiard (Buninovich, 2001; Gomes, 2015), a full density subsequence of a complete set of eigenfunctions divides into two disjoint subsets, one corresponding to the ergodic and completely integrable regions of phase space each. The transition of the system from integrable to mixed dynamics brought about by small perturbations is described by KAM theory (Arnold, 1963; Kolmogorov, 1954; Moser, 1962). Broadly speaking, the KAM theorem propounds that if the system is subjected to a weak nonlinear perturbation, some of the invariant tori (that satisfy the non-resonance condition of having “sufficiently irrational” frequencies (Casati and Ford, 1979)) are deformed but survive nonetheless, while others are destroyed and become invariant Cantor sets (Percival, 1979). The fingerprints of this behavior can be discerned in a Poincaré section of the billiard flow, which exhibits a prominent irregular component—the chaotic sea—along with regular islands accompanying the stable periodic orbits. The same structure is manifest in the quantum states as well (see Fig. 15).\footnote{To be fair, chaotic states can actually extend into the region of the regular islands when far away from the semiclassical limit (Bécker et al., 2005).}

The organization of the eigenstates prompts one to believe that if we consider irregular eigenfunctions that are concentrated on a region $\mathcal{R}$ in phase space, the statistical properties should once again be described by a superposition of plane waves but with wave vectors of the same lengths and directions distributed uniformly on $\mathcal{R}$.
B. Random wavefunctions and percolation

Building upon our analysis of the nodal structure of chaotic wavefunctions in the previous subsections, we pick up the thread that runs through this entire review, namely, the problem of enumerating the nodal domains. However, analytical results for the actual number of nodal domains, alas, are relatively scarce. The main difficulty in counting stems from non-locality: local observation of the nodal curves alone does not permit one to arrive at conclusions about the number of connected components. A major breakthrough in such enumerative pursuits was the introduction of the percolation model for nodal domains of chaotic wave functions by Fogedby and Schmit (2002). This model not only facilitated the analytical calculation of several physical quantities but also enabled one to exploit the link with percolation theory to gain a foothold on the structure of chaotic wavefunctions. Today, such borrowed ideas regularly find applications in quantum chaos and varied problems where nodal domains of random functions are of importance (see, for example, Berk (1987)). Given the overarching reach of percolation in diverse contexts such as clustering, diffusion, fractals, phase transitions, and disordered systems, as well as its recurrent presence in the rest of this review, this would be a good point to quickly recapitulate the basic ideas behind the phenomenon.

1. Percolation in statistical physics

A simple model for the percolation process can be described as follows. Consider a triangular lattice where every lattice point in the upper half-plane is colored white or black, independently, with probability $p$ and $1-p$, respectively. It is perhaps easiest to envisage a white vertex as being an “open” site, which permits the flow of a liquid through it; alternatively, one can think of the black (white) vertices as being the (un)occupied sites of a given lattice. Consequently, percolation may be reckoned as a model of the permeability of a material, regulated by the value of $p$. The question of interest concerns the existence of an infinite and connected collection of such open sites; in the language of our fluid flow analogy, this asks whether there exists a continuous “pipe” through the extent of the lattice. In statistical physics, we are chiefly concerned with critical phenomena, i.e., the study of systems at or near the point where a phase transition occurs. At a critical concentration $p_c$, one finds that an infinite cluster of white sites, embedded in the black background, extends across the lattice. For site percolation on a triangular lattice, this so-called percolation threshold is known to be $p_c = 1/2$ (Fogedby 2012). The point $p = 1/2$ is “critical” in the sense that for $p > 1/2$, there will always be an infinite connected cluster of white sites whereas this is certainly not the case for $p < 1/2$. A
realization of this system at criticality is drawn in Fig. 16 with the sole difference of a boundary condition on the bottom row that segregates black and white sites on different sides. The rest of the upper half-plane is colored according as the chosen value of $p$. Once all the sites have been filled in, there exists a unique curve commencing at the bottom row such that it has only white vertices on its one side and all black vertices on the other. Thus, by imposing an appropriate boundary condition, we have induced a domain wall that meanders across the system.

$$\begin{align*}
\text{\ldots\ldots\ldots\ldots\ldots}\text{\ldots\ldots\ldots\ldots\ldots}
\end{align*}$$

FIG. 16 The percolation process on the triangular lattice at $p = 1/2$. Figure prepared by Geoffrey Grimmett. First published as Fig. 6 by Lawler (2009). © 2008 American Mathematical Society (AMS); used by permission of the AMS.

Closely related to the problem of site percolation is the Ising model, which is, in all likelihood, the simplest interacting many-particle system in statistical physics (Binney et al., 1992; Stanley, 1987). Each lattice site $i$ is occupied by an Ising “spin” $\sigma_i = \pm 1$ (or more generally, a single degree of freedom), which points either up or down. As a model of a ferromagnet, the spins interact via a short-range exchange interaction $J$ and are described by the nearest-neighbor coupled Hamiltonian $H = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j$, which favors parallel alignment of adjacent spins for $J > 0$. The statistical weight assigned to each spin configuration is given by the conventional Boltzmann factor $\exp(-\beta H)$ and the partition function assumes the form

$$Z = \sum_{\{\sigma_i\}} \exp(-H/k_B T). \quad (74)$$

In the limit of small $\beta \ll 1/J$ (or high “temperature”) the system is disordered: the spin correlations are localized and decay exponentially fast, which means that spins separated by a large distance are almost independent of one another. On the other hand, when $\beta \gg 1/J$ at low temperatures, the system possesses long-range order. The Ising model thus exhibits a phase transition, at a critical temperature, from a disordered paramagnetic phase (at $T > T_c$) to a ferromagnetic phase (for $T < T_c$) with a non-zero order parameter $m \neq 0$ (the magnetization). Near the critical point, the order parameter, the correlation length, and the correlation function scale as

$$\begin{align*}
m &\sim |T - T_c|^{\beta}; \quad (T \to T_c^-) \quad (75) \\
\xi &\sim |T - T_c|^{-\nu}; \quad (T \to T_c^+) \quad (76) \\
\langle \sigma_i \sigma_j \rangle &\sim \frac{1}{|i-j|^{\eta}} \exp (-|i-j|/\xi), \quad (77)
\end{align*}$$

respectively, with critical exponents $\beta$, $\nu$ and $\eta$. It has long been known (Fisher, 1967) that the phase transition at $T = T_c$ is signaled by the divergence of the correlation length $\xi$ as the system becomes scale invariant. The intimate connection of the Ising model to our previous description (in Fig. 16) is evinced by the duality between site percolation and domain wall formation, which can be seen as follows. Instead of selecting a configuration of all the spins $\sigma_i$ first and then identifying the domain wall, the curve can be generated by sequential steps in what is known as an exploration process, as demonstrated by Fig. 17 for a honeycomb lattice. This exploration path is the interface for the statistical mechanics of percolation (Fogedby, 2012). Another interesting parallel between the two descriptions is that the lattice dual to the honeycomb $\text{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots}$

FIG. 17 The exploration process for the Ising model. At each step the walk turns left or right according as the value of the spin in front of it. The relative probabilities are determined by

That is all we have to say about the percolation process for now. We will come back to it shortly in Sec. IV.C when we take a closer look at conformal invariance and scaling limits in the context of Schramm-Loewner Evolution (SLE). At present, however, we return to the description of nodal lines afforded by the original model of Bogomolny and Schmit (2002).
2. The Bogomolny-Schmit percolation model

Consider the nodal portrait of a random function; we came across one in Fig. 13. The mean number of zeroes (or nodes) of a wavefunction of this type along a given straight line (without loss of generality, let’s say the vertical one) can be estimated from the approximate quantization condition $k_y L_y \approx \pi m$, where $m \in \mathbb{Z}$ and $k_y = k^2 / 2$ is the mean-square momentum along the $y$-axis. The mean density of nodal lines is roughly

$$\bar{p}_y = \frac{m}{L_y} = \frac{k}{\pi \sqrt{2}}$$

(78)

Hence, the nodal lines of random functions form a rectangular grid in the mean and the total number of sites in the resultant lattice is asymptotically

$$N_{\text{tot}} = (\bar{p}_y)^2 A = k^2 \frac{A}{2 \pi^2} = \frac{2}{\pi} N(E),$$

(79)

where $N(E) = AE/4\pi$, in accordance with the Weyl formula, is the mean number of levels below energy $E$. In principle, this result can be derived rigorously using the methods of Bogomolny et al. (1996). This back-of-the-envelope calculation, albeit appealing in its simplicity, comes with a potential pitfall. At any point inside the billiard, the actual wavefunction can be written as the sum of its average and a small correction as $\psi(x, y) = \bar{\psi}(x, y) + \delta\psi(x, y)$. Now, even if the average wavefunction forges a checkerboard nodal picture, the addition of the correction term recasts the crossing of nodal lines into one of the two possible avoided crossings in Fig. 18. The sign of the critical point between two maxima or minima determines whether the positive or negative nodal components connect.

![FIG. 18 (a) True nodal crossing. (b) and (c) Avoided nodal crossings. Typically, nodal lines do not intersect as this would imply that $\psi, \psi_x$ and $\psi_y$ are simultaneously zero at some point. These three functions being independent Gaussians, the probability of this event is nearly zero. From Bogomolny and Schmit (2002).](image)

This observation lies at the crux of the Bogomolny-Schmit conjecture, which posits that the distribution of nodal domains for random functions is the same as that for a specific percolation-like process. Starting with a rectangular lattice where each of the $N_{\text{tot}}$ sites represents a saddle point \(^{13}\) (with zero saddle height) akin to Fig. 18 (a), each nodal crossing is amended to one of the permitted avoided intersections. As previously, the bond between two neighboring maxima is set to be “open” if the saddle height is positive and “closed” otherwise. The only stipulation governing this percolation-like process is that the saddle heights are uncorrelated and have equal probabilities of being positive or negative. Hence, the process, although random, is well-defined and corresponds to critical bond percolation. A particular instance is to be found in the inset of Fig. 19. It is not difficult to discern that the original lattice can be decomposed into two dual lattices (of size $a = 2\pi/k$, the de Broglie wavelength) with sites at the centers of the regions where the wavefunction is positive or negative. Two vertices on a dual lattice are connected by an edge if and only if the corresponding cells of the grid belong to the same nodal domain of the random function (Sodin, 2016). Any realization of the aforementioned random process therefore uniquely delineates two graphs on these lattices and conversely, there is a one-to-one correspondence between a graph on a dual lattice and an allowed realization. Bogomolny and Schmit (2002) cognized that the number of connected nodal domains equals the sum of the numbers of different components of both the positive and negative graphs, $n_{+}$, and bond percolation on the square lattice thus provides a good description of the nodal domains.

Utilizing the correspondence between the generating function

$$Z(x) = \sum_{\text{realizations}} x^{n_{+} + n_{-}}$$

(80)

and the partition sum of the Potts model (Wu, 1982), it was shown that the total number of nodal domains for a random function has a universal Gaussian distribution with mean $\bar{n}(E)$ and variance $\sigma^2(E)$ given by

$$\bar{n}(E) = \frac{3 \sqrt{3} - 5}{\pi} \approx 0.0624,$$

(81)

$$\sigma^2(E) = \frac{18}{\pi^2} + \frac{4 \sqrt{3}}{\pi} - \frac{25}{2 \pi} \approx 0.0502.$$  

(82)

Similar formulae were obtained by Ziff et al. (1997) from Monte-Carlo simulations for the two-dimensional percolation problem. The asymptotic predictions of Eqs. (81, 82) are compared with numerical calculations of the mean value and the variance for several random functions in Fig. 19 which demonstrates reasonable agreement, within statistical errors. Deviations from Eq. (81) larger than that sanctioned by Eq. (82) can be attributed

\(^{13}\)Strictly speaking, this picture is not accurate as the function $\psi$ and its gradient $\nabla \psi$ cannot vanish simultaneously.
to the existence of wavefunction scars. Regions of quasi-
integrable behavior have considerably fewer nodal do-
mains than chaotic regions, which accounts for the origin
of large fluctuations \cite{Bogomolny2007}.

![Image](https://example.com/image.png)

**FIG. 19** Mean values of the nodal domains (dots) and their
variances (squares) for random functions, normalized by and
plotted against \( N \equiv N(E) \). The solid and dashed horizontal
lines represent theoretical predictions of Eqs. (81) and (82),
respectively. [Inset]: A realization of a random percolation-
like process. The centers of the positive (negative) regions,
denoted by + (−), constitute two dual lattices. Dashed (solid)
lines indicate graphs for the positive (negative) dual lattice.
Adapted from \cite{Bogomolny2002}.

Additionally, percolation theory prognosticates that the
distribution of the areas \( s \) of clusters (the connected
nodal domains), \( n(s) \), should follow a power-law behavior

\[
P_A(s) \propto \left( \frac{s}{s_{\text{min}}} \right)^{-\tau} \tag{83}
\]

where \( \tau = 187/91 \) is the Fisher exponent \cite{Stauffer1994, Aharony1994}. The constant \( s_{\text{min}} = \pi (J_{0,1}/k)^2 \) is the
smallest possible area for a fixed wavenumber \( k \) (Eq. A2).
The nodal domain perimeters \( l \) should also exhibit a
similar dependence \( n_l \propto l^{-\tau'} \) with the scaling exponent
\( \tau' = 15/7 \) \cite{Ziff1986}. Another result that was originally
derived in the context of percolation and can be directly
carried over is the fractal dimension of the nodal domains.
Within the framework of the percolation model, this is
given by the fractal dimension of critical percolation clusters,
\( D = 91/48 \) \cite{Stauffer1994}. Numerically,
the fractal dimension of a domain can be extracted by
juxtaposing it onto a grid of squares (say, of side \( R \))
and counting the number of crossings of the region with
the grid, which is expected to scale as

\[
n_{\text{crossing}} \propto R^{-D} \tag{84}
\]

whenever \( a = 2\pi/k \ll R \ll l \) (the size of the domain).

Thus began the steady influx of ideas from percolation
into the study of nodal domains, which soon proved to be
a rather fruitful intellectual exercise. However, this
celebrated conjecture has not been without controversy
and has received its fair share of criticism and defense
alike. At first sight, the manifest connection between
critical percolation and the random plane wave model
should itself come as a surprise \cite{Foltin2004a}. In the
standard percolation scheme, each site/edge of a lat-
tice is set to be occupied/positive with probability \( p \); this
assignment has the important property that the occupa-
 tion fraction (or concentrations) at different points are
independent random variables. In contrast, for a Gaus-
sian random function, the probability of its values having
the same sign at two points, far from being uncorrelated,
is

\[
P(|r - r'|) = \frac{1}{2} + \frac{1}{\pi} \arcsin G(|r - r'|), \tag{85}
\]

which, assuming that the two-point correlation function
is normalized as \( G(0) = 1 \), is nearly one for nearby pairs
of points. It is only in the limit of large distances when
\( G(r) \to 0 \) that the probabilities of finding a point positive
and negative equalize. For a random superposition of
plane waves of the form \eqref{eq:superposition}, it follows that\footnote{This correlator holds whenever the assumption that wavefunctions have quantum chaotic correlations does. For instance, applied to fluctuations of the transmission phase in interacting quantum dots, Eq. \eqref{eq:transmission_phase} predicts large universal sequences of resonances and transmission zeros \cite{Molina2012}.}

\[
\langle \psi(r) \psi(r') \rangle = J_0(k|r - r'|) \sim \frac{\cos(k|r - r'| - \pi/4)}{\sqrt{|r - r'|}} \tag{86}
\]

and the random wavefunction correlation \( G(r) \) decays
rather slowly (\( \sim r^{-1/2} \)). In fact, even for the density
of the random field,

\[
\chi = -\text{sgn}(\psi) \Theta(-\det(\partial_x \partial_y \psi)) \det(\partial_x \partial_y \psi) \delta^2(\nabla \psi), \tag{87}
\]

the correlations decay just as slowly:

\[
\langle \chi(r) \chi(0) \rangle = -\frac{1}{72 \pi^3} J_0(r) + O \left( J_0^3(r) \right) \sim \frac{1}{\sqrt{r}} \cos(2\pi r) + O \left( r^{-3/2} \right), \tag{88}
\]

and consequently, are long-ranged \cite{Foltin2003a}. It is
not unreasonable to expect such long-range correlations
to undermine the validity of percolation theory\footnote{Otherwise, if one constructs a Gaussian ensemble of random functions characterized by the correlation function \( G_0(r) = \exp(-k^2 r^2/4) \), the applicability of the critical (short-range) percolation picture is almost self-evident \cite{Weinrib1982, Zallen1971}.}.
This contention was addressed by Bogomolny and Schmit [2007] by invoking what is known as Harris’ criterion [Harris 1974] to argue that the spurious problem of the slow decay of Eq. (86) is effectively circumvented. To see this, consider a critical percolation problem where the concentrations at different sites, denoted by \( \psi(x) \), are correlated. The mean concentration at each site, of course, equals the critical value \( p_c \). Let \( p_v \) represent the average concentration in a (finite) volume \( V \):

\[
p_v = \frac{1}{V} \sum_{r \in V} \psi(r);
\]

obviously, \( \langle p_v \rangle = p_c \). Furthermore, we assume, on grounds of translational invariance, that the connected correlation function depends not on \( r \) and \( r' \) individually, but rather only on the difference between them, i.e.,

\[
\langle (\psi(r) - p_c) (\psi(r') - p_c) \rangle = G(|r - r'|). 
\]

Briefly, Harris’ eponymous criterion, later extended by Weinrib [1984] to correlated percolation, can be formulated as follows. If the variance of \( p_v \),

\[
\Delta \equiv \langle (p_v - p_c)^2 \rangle = \frac{1}{V^2} \sum_{r,r' \in V} \langle (\psi(r) - p_c)(\psi(r') - p_c) \rangle 
\]

is small in the sense that \( \Delta \ll |p_v - p_c|^2 \), then correlations are unessential and all critical quantities are the same as for the standard uncorrelated percolation (Bogomolny and Schmit [2007]). For random wavefunctions, \( \Delta \sim |x|^3 \) and \( \xi \sim |p_v - p_c|^{-\nu} \), where \( \nu = 4/3 \) is a critical index for two-dimensional percolation. Hence, Harris’ criterion is satisfied [16] and the system belongs to the short-range percolation universality class. For the sake of completeness, let us mention that the Bogomolny-Schmit conjecture can be generalized along these lines to level domains, i.e., regions where \( \psi(x,y) > \varepsilon \) for some fixed \( \varepsilon \neq 0 \). Level domains are also described by percolation theory but by noncritical percolation wherein the deviation from criticality \( p - p_c \sim \varepsilon \) (Bogomolny and Schmit [2007]).

Disconcertingly enough, more serious objections to the conjecture have come to the forefront in recent times, fueled by several high-precision numerical studies. The first questions were raised by Nastasescu [2011], who computed the density of the mean number of nodal domains for random spherical harmonics to be \( 0.0598 \pm 0.0003 \), nearly \( 5\sigma \) away from the theorized value. Initially, this discrepancy was brushed away by attribution to finite size and curvature effects. Adding to the growing unease, Konrad [2012] repeated the calculation for plane waves to obtain a density of \( 0.0589 \pm 0.000142 \), which is 6% below the prediction. The general belief is that the normalized number of nodal domains should behave like \( a + b/k \), for constants \( a \) and \( b \). While Konrad [2012] found a best fit of \( 0.0589 + 4.6209/k \), simulations at higher energies (Beliaev and Kereta [2013]) determined the fitting parameters to be \( a = 0.0589 \) and \( b = 4.717 \). Puzzlingly, the latter group also observed that the crossing probabilities for a nodal line of a random plane wave to connect the sides of a box \( D \) converge to their percolation counterparts (Cardy [1992] Watts [1996]). That the probabilities are macroscopic observables and hence, universal from a percolation perspective, reaffirms the suspicion that this is no coincidence. Seeking better agreement with the numerics, Beliaev and Kereta [2013] proposed an alternative normalization scheme in which the number of vertices of the square lattice chosen is the same as the number of local maxima of the random plane wave (see Fig. 20). Pursuant to this prescription, the technical details of which are to be found in Kereta [2012], the average density of the critical points can be computed using Gaussian integrals to be 0.0919\( k^2 \) (a quarter of which are maxima) and the number of domains is estimated as \( \bar{n}(E)/\bar{N}(E) = 0.0566 \). It is important to emphasize that the choice of normalization condition should not affect the density of nodal domains, which is believed to be a universal quantity in the following sense. Given Laplacian eigenfunctions \( \psi_n \), we can define

\[
f_n = \sum_{k=n}^{n+c\sqrt{n}} c_k \psi_k, 
\]

where \( c_k \) are i.i.d. normal variables and \( C \) is a (large) constant. Then, the properly rescaled number of nodal domains of \( f_n \) (asymptotically) has the same density as the random plane wave (Beliaev and Kereta [2013]). The same cannot be said, however, for the number of clusters per vertex, which, being a nonuniversal quantity in percolation theory, is strongly dependent on the lattice structure.

All things considered, at present, we are far from understanding “a hidden universality law” of sorts that would provide a rigorous foundation for the Bogomolny-Schmit conjecture. One of the strongest known results in this direction pertains to the number of nodal domains \( \nu(g_n) \) for a two-dimensional Gaussian spherical harmonic of degree \( n \). Nazarov and Sodin [2009] proved that for every \( \varepsilon > 0 \), there exist positive constants \( C(\varepsilon) \) and \( \varepsilon(\varepsilon) \) such that the probability tail

\[
P \left\{ \left| \frac{\nu(g_n)}{n^2} - \nu_{BS} \right| > \varepsilon \right\} \leq C(\varepsilon) e^{-c(\varepsilon)n}. 
\]
In addition, the expected number of nodal domains is asymptotic to

\[ \mathbb{E} [\nu(g_n)] = \nu_{ns} n^2 + O(n^2). \] (94)

This theorem, which is based on the Gaussian isoperimetric inequality [Borell 1975; Sudakov and Tsirel’son 1978], implies that the normalized number of nodal domains is exponentially concentrated around some strictly positive constant, which, in this case, happens to be \( \nu_{ns} = (3 \sqrt{3} - 5)/\pi \). The constant \( \nu_{ns} \) is certainly not universal and genuinely depends on the underlying random Gaussian field [Kurlberg and Wigman 2015]. Recently, Rozenshein (2016) extended these results to random eigenfunctions of the Laplacian on the torus \( \mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2 \), showing that the number of nodal domains \( \nu_n \) localizes around its median, mean, and limiting mean, exponentially, with an optimal lower bound [Buckley and Wigman 2016]. The asymptotic law for the expectation of \( \nu(g_n) \), Eq. (94), is actually much more general as proved by Nazarov and Sodin (2016) in a setting of ensembles of Gaussian functions on Riemannian manifolds. These and other methods have been systematically exploited to obtain distributions of nodal volumes [Beliaev and Wigman 2016; Rudnick and Wigman 2008; Wigman 2009] and topological invariants [Canzani and Sarnak 2014; Gayet and Welschinger 2014a,b; 2015].

C. Morphology of nodal lines: Schramm-Loewner Evolution

One of the oldest tricks in the playbook of critical phenomena is to define a model on a finite subset of a lattice and then ask what happens as the lattice size is allowed to grow. Ceteris paribus, one could also work in a bounded region and increase the resolution of the grid thereon by considering finer and finer lattices. Either way, the objective of such endeavors is to determine the scaling (continuum) limit of the system, if it exists, and understand its geometric and fractal properties. Many years ago, Belavin et al. [1984a,b] postulated that several two-dimensional systems had scaling limits at criticality that were, loosely speaking, conformally invariant. Although not rigorous, their predictions were consistent with numerical simulations and hinted at some deeper physics beneath the surface.

1. Scaling and conformal invariance

As is probably evident from our preceding remarks, the recurrent motifs of this narrative are the conjoint ideas of scaling and conformal invariance. Scale invariance is ubiquitous in nature; its most notable application in physics is perhaps the renormalization group flow [Wilson and Kogut 1974]. Mathematically, a function [or scaling operator] \( \Phi(\mathbf{r}) \) is said to be scale invariant if

\[ \Phi(\lambda \mathbf{r}) = \lambda^\Delta \Phi(\mathbf{r}) \] (95)

holds with the same exponent \( \Delta \) for all rescaling factors \( \lambda \) (which generate the dilatation \( \mathbf{r} \mapsto \mathbf{r}' = \lambda \mathbf{r} \)); in other words, if \( \Phi(\mathbf{r}) \) is a generalized homogeneous function. Discrete scale invariance is better recognized as self-similarity: for instance, the famous Koch curve scales with \( \Delta = 1 \), but only for values of \( \lambda = 1/3^n; n \in \mathbb{Z} \). If the rescaling factors are permitted to be space-dependent \( \lambda = \lambda(\mathbf{r}) \), a natural generalization of global scale-invariance, Eq. (95), is

\[ \Phi(\mathbf{r}) \mapsto \Phi'(\mathbf{r}) = J(\mathbf{r})^{x/d} \Phi(\mathbf{r}/\lambda(\mathbf{r})) \] (96)

where \( J(\mathbf{r}) \) is the Jacobian of the transformation \( \mathbf{r} \mapsto \mathbf{r}' = \mathbf{r}/\lambda(\mathbf{r}) \) in \( d \) spatial dimensions and \( x \) the scaling dimension of \( \Phi \). Restricting to those coordinate transforms that additionally conserve angles, one arrives at conformal transformations [Henkel and Karevski 2012]. Visualized on an elastic medium, conformal transformations represent deformations without shear [Landau and Lifshitz 1959]. In two dimensions, these concepts can easily be extended to the complex plane by bartering a vector \( \mathbf{r} = (r_1, r_2) \) for a complex number \( z = r_1 + i r_2 \). Then, Riemann’s mapping theorem [Ahlfors 2010] ensures that we can map any simply connected domain \( \mathcal{D} \) (topologically equivalent to a disk) to another, \( \mathcal{D}' \), i.e., there exists an invertible holomorphic (complex-analytic) map \( g \) between them. In quotidian dealings, both \( \mathcal{D} \) and \( \mathcal{D}' \) are customarily the upper half-plane \( \mathbb{H} \). Moreover, any analytic or anti-analytic coordinate transformation, \( z \mapsto g(z) \) or \( \bar{z} \mapsto g(\bar{z}) \), is always conformal. The bourne of the mathematicians to whom SLE owes its present form was to make these notions of conformal invariance precise and introduce the rigor that field theorists had previously glossed over.

Before diving into the details, we must concede that it is nigh impossible to do justice to this incredibly rich field...
in just a few pages and hence it is obligatory for us to point the interested reader to further references. Besides the original set of articles by Lawler et al. (2001a,b, 2002) relating SLE to various aspects of Brownian motion, mention must be made of the excellent albeit mathematically intensive reviews by Kager and Nienhuis (2004); Lawler (2001, 2008); Werner (2004). For the reader dissuaded towards the jargon of mathematicians, there are also a few semi-pedagogical (and more accessible) introductions written for physicists (Cardy, 2005; Fogedby, 2012; Lawler, 2009), from which we mainly source our discussion. An exhaustive bibliography up to 2003 can be found in Gruzberg and Kadanoff (2004).

2. Critical interfaces in 2D: Lattice models

Our spotlight now falls on models that describe random non-intersecting paths, which define the boundaries of clusters on a lattice, with the hope that a picture of their continuum limits might be rendered by SLE. Motivated by analogous ideas in two-dimensional critical behavior such as the Coulomb gas approach (Nienhuis, 1987), the central question that arises is, what are the properties of the measure on such curves as the lattice spacing tends to zero (Cardy, 2005)? The simplest, if not the oldest, example is the random walk, which has been extensively studied over the decades (Ash and Doleans-Dade, 2000; Feder, 1988; Reichl, 2016). We begin with an unbiased simple random walk (SRW) consisting of exactly $n$ steps on a two-dimensional plane. The $i$th step, represented by the vector $s_i$ (of magnitude equal to a constant step size $S$), is random, isotropic and uncorrelated, i.e., $\langle s_i \rangle = 0$, and $\langle s_i s_j \rangle \propto \delta_{i,j} \delta_{\alpha,\beta}$. The net displacement during the drunkard’s excursion is just $x = \sum_{i=1}^{n} s_i$ and the corresponding end-to-end (Euclidean) distance traversed is $R = \sqrt{(x^2)} \sim \sqrt{n} \sim n^{1/2}$, characteristic of diffusive motion. Calculating the fractal dimension\(^\text{17}\) of a typical path using the box-counting procedure (Feder, 1988; Mandelbrot, 1983) yields $D = 2$, implying that the walk fills the plane modulo the lattice spacing. As a consequence, the scaling exponent, defined by the relation $R \sim n^\nu$, is found to be $\nu = D^{-1} = 1/2$.

To formally characterize this process, let us represent a walk by a sequence of vertices $\omega = [\omega_1, \omega_2, \ldots, \omega_n]$ with $||\omega_j - \omega_{j-1}|| = S$ for each $j$. Each such walk $\omega$ is assigned a probability $P_n(n) = \exp (-\beta n)$ resembling the familiar Boltzmann factor, whereby all walks of the same length have the same weight. The sum of the weights of all possible paths is recognized as the partition function

$$Z_\beta = \sum_\omega e^{-\beta |\omega|},$$

where the length (number of edges) $|\omega|$ can be arbitrarily large. At, and only at, a critical value of $\beta = \beta_c = \log 4$ (: the number of SRWs of length $n$ is $4^n$ in 2D), this sum neither grows nor decays exponentially with the number of lattice sites (or equivalently, the fineness of the mesh), scaling as a power law instead. Said otherwise, it becomes scale invariant! The scaling limit of an unbiased random walk is actually Brownian motion (Ash and Doléans-Dade, 2000), denoted $B_t$. This limit is obtained as $S \to 0$ but with the number of steps $n$ simultaneously scaled up, so as to keep the size $R \sim \sqrt{n} S$ constant. The resultant Brownian path is a continuous non-differentiable random curve, once again plane-filling with fractal dimension $D = 2$. The most significant departure from our previous discourse is that the notion of the probability density of a walk no longer makes sense. Indeed, what do the probabilities $P_n(n)$ that we allotted so insouciantly even mean when $n \to \infty$? Nevertheless, this is not an insurmountable hurdle as we can make the appropriate replacements with the more general concept of a measure. The partition function in Eq. (97) is now identified with what mathematicians would call the total mass of this measure. For concreteness, consider a Brownian excursion on a domain $D$, traced by a curve $\gamma$ that commences at $r_1$ and concludes at $r_2$. This acquires a measure $\mu (\gamma; D, r_1, r_2)$, which is conformally invariant (Lévy, 1965). The proof, which follows directly from Itô’s lemma (Itô, 1944), hinges on the premise that if $B_t$ is a complex Brownian motion and $g$ a conformal mapping

\(^{17}\)Crudely, a subset of $\mathbb{Z}^d$ is said to have fractal dimension $D$ if the number of points in a disk of radius $R$ grows as $R^D$.\n
FIG. 21 Comparison between scale and conformal transformations. The latter are also angle-preserving and basically correspond to a combination of a local rotation, local translation, and local dilatation. There is no a priori reason why a scale-invariant system must also be conformally invariant although it is often found to be so. From Nakayama (2015), with permission from Elsevier.
between simply connected domains \( D \to \mathcal{D} \), then \( g(B_t) \), modulo a reparametrization of time, is also a Brownian motion. Consequently, we have

**Property 1. Conformal Invariance:**

\[
(g \circ \mu) (\gamma; D, r_1, r_2) = \mu (g(\gamma); D', r'_1, r'_2). 
\] (98)

Although Brownian motion does not fall within the purview of SLE owing to self-crossings that annul Riemann’s mapping theorem, its perimeter (which has a fractal dimension of \( D = 4/3 \)) does \[Lawler et al. (2000)], as do several other variations upon it.

A common variant is the loop-erased random walk (LERW), which was studied by \[Schramm (2000)\]. This process, a subset of a broader (and more intractable) class of self-avoiding random walks, describes systems like polymers that possess strong tendencies to be self-avoiding. The prescription, outlined in Fig. 22 is to sample an unconstrained random walk and then, chronologically erase the loops along the way, in the order in which they are encountered; the resultant path, by construction, does not cross itself. The bad news is that LERWs forfeit a salient property of the SRW—Markovian (memoryless) time evolution. Since the LERW curve cannot cross itself, the future path, at any point, is always highly correlated with and dependent on its past. To make things explicit, consider Fig. 22 (b), which depicts a LERW, specifically conditioned to exit \( D \) at a fixed point on the boundary, \( y \in \partial D \). We call this law \[LERW_S (\gamma; D, z, y)\]. Let us trace out this path in reverse: having observed the first \( k \) steps of the walk (Fig. 22 d), it turns out that the law of the remaining path is nothing but \( LERW_S (\gamma; D \setminus [y, y_k], z, y_k) \). Thus we recover the original law itself but this time, on a truncated slit domain, \( D_k \equiv D \setminus [y, y_k] \). One can therefore still salvage a Markov property in the LERW process, albeit in the evolution of the domain rather than the curve. Formally \[Cardy (2005)\].

**Property 2. Domain Markovian condition:** If the curve \( \gamma \) is divided into two disjoint parts: \( \gamma_1 \) from \( r_1 \) to \( r \), and \( \gamma_2 \) from \( r \) to \( r_2 \), then the conditional measure satisfies

\[
\mu (\gamma_2 | \gamma_1; D, r_1, r_2) = \mu (\gamma_2; D \setminus \gamma_1, r_1, r_2). 
\] (99)

The LERW measure of each self-avoiding path \( \omega \) is the total measure of all the different SRWs that can be loop erased to \( \omega \). This weight can be written as

\[
4^{-|\omega|} e^{\Lambda (\omega)},
\] (100)

where \( \Lambda (\omega) \) is a measure of the number of loops in the domain that intersect \( \omega \) \[Lawler (2009)\]. Properties 1 and 2 in combination with Loewner evolution, are sufficient to determine the measures in the scaling limit \[Fogedby (2012)\]. LERW, which satisfies both, was proved to have a conformally-invariant scaling limit that can be accessed by SLE \[Lawler et al. (2004)\]. By the way, the growth of the curve \( \gamma \) is also an example of an exploration process. This should immediately remind us of our old friend, percolation. It is easily seen that the exploration curve in Fig. 17 exhibits the same domain Markov property as LERW. This correspondence can be taken a step further. Indeed, in the scaling limit, which does manifest conformal invariance, the critical percolation cluster has a fractal boundary described by SLE. Interestingly, the model also bears an additional conformal invariant, which was predicted by \[Cardy (1984, 1992)\].

**FIG. 22** (a) An unbiased random walk on a domain \( D \) (a 20 \( \times \) 20 square lattice) that starts in the interior and terminates when it encounters the boundary. (b) The same, after loop erasure. (c) LERW on a 80 \( \times \) 80 grid (in red) together with its shadow. (d) Construction of the domain Markov property. Figures courtesy Tom Alberts (2008).

3. The Loewner differential equation

Armed with this background, we can now begin the story of Loewner evolution, which originally unfolded in the context of the Bieberbach conjecture. Proposed in
1916, the latter surmised that if \( f(z) = z + \sum_{n \geq 2} a_n z^n \) is a holomorphic, injective (one-to-one) function on the unit disc \( \mathbb{D} = \{ z \in \mathbb{C} ; |z| < 1 \} \), then \( |a_n| \leq n \) for \( n \geq 2 \) (Bieberbach 1916, Gong 1999). In 1912, Bieberbach proved \( |a_2| \leq 2 \) and later, Löwner (1923) considered the dynamics of the coefficients \( a_n \) to prove \( |a_3| \leq 3 \). It then took a further 62 years for conclusive progress before the conjecture was finally proved by De Branges (1985). Löwner’s ideas in this regard can be stated quite simply: we closely follow the discussion by Alberts (2008). Consider a self-avoiding curve \( \gamma : [0, \infty) \to \mathbb{H} \) such that \( \gamma(0) = 0 \) and \( \gamma(\infty) = \infty \). It follows that as the curve grows, \( \forall t \geq 0, \mathbb{H} \setminus \gamma([0,t]) \) is a simply connected domain. SLE catalogs (local) growth processes of this type for which the resulting set \( \gamma \) is eventually a continuous curve (Henkel and Karevski 2012). Riemann’s mapping theorem averts that there exists a ‘time-dependent’ conformal transformation \( g_t : \mathbb{H} \setminus \gamma([0,t]) \to \mathbb{H} \) as Fig. 23 limns. The map \( g_t \) is certainly not unique and most generally, will have three real degrees of freedom. For instance, \( \mathbb{H} \) is transformed to itself by a three-parameter group of fractional transformations

\[
g_t(z) = \frac{a z + b}{cz + d}, \tag{101}
\]

for \( a, b, c, d \in \mathbb{R} \) (Bogomolny et al. 2006b, Siegel et al. 1969). Firstly, two degrees of freedom can be absorbed by imposing the hydrodynamic normalization which constrains the behavior out at infinity: \( g_t(\infty) = \infty, g_t'(\infty) = 1 \). The Laurent expansion for \( |z| \to \infty \) must therefore resemble

\[
g_t(z) = z + a_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \ldots. \tag{102}
\]

Specifying \( a_0 = 0 \) fixes all the other coefficients uniquely. The coefficient \( b_1 \) (which is also a function of \( t \)) is, for reasons historical and arcane, called the half-plane capacity of \( \gamma([0,t]) \), and notationally denoted by \( a(\gamma([0,t])) \). The fact that the capacity is additive, i.e.,

\[
a(\gamma|0,t + s]) = a(\gamma|0,t]) + a(g_t(\gamma[t,t+s])) \tag{103}
\]

lends credence to our interpretation of \( t \) as a “conformal time”. Furthermore, since \( a(\gamma[0,t]) \) is continuous and increasing, one can reparametrize the curve such that \( a(\gamma[0,t]) = 2t \) (again, by convention) and Eq. (102) is recast as

\[
g_t(z) = z + a(\gamma[0,t]) \frac{t}{z} + \ldots = z + \frac{2t}{z} + \ldots \tag{104}
\]

An instructive example is when \( \gamma \) is a straight line, growing vertically upwards in the upper half-plane. If \( (\zeta, h) \) are the coordinates of the tip of the slit, then \( z = \zeta + i h \) and the required transformation is

\[
g_t(z) = \zeta + \sqrt{(z - \zeta)^2 + 4t}, \tag{105}
\]

with \( h = \sqrt{4t} \) being the parametric equation of the growing slit. In fact, \( g_t(\gamma(t)) \propto \sqrt{t} \) is a signature of a straight line growing at a fixed angle to the real axis (Cardy 2005). More involved deterministic examples can be found in (Kager et al. 2004).

This nifty calculation enables us to derive the Loewner differential equation (LDE) that describes the evolution of the maps \( g_t \) with the growth of the curve \( \gamma([0,t]) \). One starts by asking if the map \( g_{t+dt} \) could be determined just from knowledge of the one infinitesimally earlier, \( g_t \). The procedure to do so is sketched in Fig. 23. In particular, we can explicitly compute the map \( h_{t,dt} \), which, by Eq. (105), is

\[
h_{t,dt}(w) = U_{t+dt} + \sqrt{(w - U_t)^2 + 2dt} \approx w + \frac{2dt}{w - U_t}, \tag{106}
\]

where \( U_t = g_t(\gamma(t)) \), called the “driving” or “forcing” function, encodes all the topological properties of the curve \( \gamma([0,t]) \). Recognizing that

\[
g_{t+dt}(z) = h_{t,dt}(g_t(z)) \approx g_t(z) + \frac{2}{g_t(z) - U_t}, \tag{107}
\]

we are led to the celebrated (and till now, elusive) LDE:

\[
\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}; g_0(z) = z. \tag{108}
\]
Given a curve $\gamma$, the maps $g_t$ must necessarily satisfy Eq. (108). The converse is also possible: given a driving function $U_t: [0, \infty) \rightarrow \mathbb{R}$, the LDE can be solved for $g_t$, which then determines $\gamma([0, t])$. When $U_t$ itself is a continuous random function, Eq. (108) yields a stochastic equation of motion and consequently, a stochastic map $g_t(z)$. The growing tip then traces out a random curve, determined by \cite{Bauer2003} and \cite{Kennedy2007}:

$$\gamma(t) = g_t^{-1}(U_t). \quad (109)$$

A sufficient condition for generating such a curve is that the driving function $U_t$ be Hölder continuous with exponent $\alpha > 1/2$. By a deep general result \cite{Henkel2012}, any random process $\zeta(t)$ with continuous samples and independent identically distributed increments must necessarily be of the form $\zeta(t) = \sigma B_t + \rho t$ for some $\sigma > 0$ and $\rho \in \mathbb{R}$. In his seminal work, Oded Schramm \cite{2000} connected the dots to show that conformal invariance and the domain Markov property together conspire to impose this very form on $U_t$, which is thereby restricted to standard Brownian motions with drift. In addition, invariance under reflections about the imaginary axis requires $\rho = 0$ with the outcome

$$U_t = \sqrt{\kappa} B_t; \text{ with } \langle B_t \rangle = 0, \langle B_t B_s \rangle = \min(t, s), \quad (110)$$

where $\kappa$ is the diffusion constant. To be precise, $U_t$ is distributed as a Gaussian

$$P(U, t) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{U^2}{2t}\right) \quad (111)$$

with correlations $\langle (U_t - U_s)^2 \rangle = \kappa |t - s|$ \cite{Fogedby2012}. We define (chordal) $\text{SLE}_{\kappa}$ (from 0 to $\infty$ in $\mathbb{H}$) as the random collection of conformal maps $g_t$ obtained by solving Eq. (108) with $U_t = \sqrt{\kappa} B_t$; the curve itself is called the $\text{SLE}_{\kappa}$ trace. On another simply connected domain $\mathbb{D}$ with distinct boundary points $w, z$ (such as in the percolation exploration process), $\text{SLE}_{\kappa}$ from $z$ to $w$ is directly obtained by a conformal transformation \cite{Lawler2009} mapping $z$ to the origin and $w$ to infinity. Likewise, one can define radial $\text{SLE}_{\kappa}$ for paths from a boundary point to one in the interior (as for LERW)—this is the “law” that we had blithely anticipated earlier.

The only caveat is that $B_t$ is not Hölder-1/2 continuous, which insinuates that the credibility of the entire approach might be resting on tenuous grounds. Fortunately, \cite{Rohde2005} proved that although it is not immediately obvious, and perhaps, even questionable, that a curve should always be produced for $\text{SLE}_{\kappa}$, in practice, it is indeed the case. The value of $\kappa$ determines the universality class of critical behavior, thereby establishing a well-defined classification into different phases as shown in Fig. 24. This innocuous parameter has yet another significance: it is related to the Hausdorff dimension of the $\text{SLE}_{\kappa}$ trace \cite{Bauer2003} and \cite{2008} as

$$D_{\kappa} \equiv \dim(\gamma([0, t])) = \min\left(2, 1 + \frac{\kappa}{8}\right). \quad (112)$$

Thus a typical LERW path, which corresponds to $\text{SLE}_2$, has a fractal dimension of $5/4$ in the continuum limit and accordingly, the average number of steps in a LERW across a $N \times N$ grid is $N^{5/4}$ \cite{Kenyon2000}. Other familiar lattice models that are known to have a $\text{SLE}$ scaling limit include the self-avoiding walk $[\kappa = 8/3] \cite{Lawler2004a}$ as well as cluster boundaries in the Ising model $[\kappa = 3]$ and in percolation $[\kappa = 6] \cite{Smirnov2001}$.

4. \text{SLE}_6 description of nodal lines

After having perhaps tested the patience of the reader who would not be unreasonable to wonder what the preceding mathematical palaver might possibly have to do with nodal lines, we are finally ready to explore the connections between the two. We know that the boundaries of percolation clusters are generated by $\text{SLE}_6$ traces, as proved by \cite{Smirnov2001} for critical percolation on the triangular lattice. We have also seen that the nodal portraits of random wavefunctions are adequately captured by a critical percolation model. Putting two and two together, we conclude that the nodal lines of chaotic billiards should be described by $\text{SLE}_6$ curves. This correspondence was examined in detail by \cite{Bogomolny2006b} with the aid of numerical calculations on a semi-circular region of area $4\pi$. For each of $N = 2248$ realizations of the random wavefunction $\left(\text{57}\right)$, they inspected the longest nodal line stretching from the origin to the boundary and the statistical properties of its forcing func-

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Footnote: A real or complex-valued function $f$ is said to be Hölder continuous if $3$ non-negative real constants $M$, $\alpha$, such that $|f(x) - f(y)| \leq M ||x - y||^\alpha \forall x, y \in$ the domain of $f$. 
tion $U_t$, determined using the geodesic algorithm [Marchall and Rohde 2006]. Strictly speaking, SLE predicts that

$$U_t = \frac{1}{N} \sum_{j=1}^{N} U_t(j) = 0, \quad (113)$$

$$\sigma^2(t) = \frac{1}{N} \sum_{j=1}^{N} (U_t(j) - U_t)^2 \approx \left( \kappa \pm \kappa \sqrt{\frac{2}{N}} \right) t. \quad (114)$$

The reality, however, differs from this simple picture. Contrary to expectations, the dependence of the calculated variance $\sigma^2$ on $t$ is not linear. At best, the initial (approximately) straight segment, which constitutes only $\sim 1/4$ of the total curve, can be fit to a quadratic equation as

$$\sigma^2(t) = -0.003 + 6.05 t - 10.0 t^2. \quad (115)$$

Noteworthily, the slope of the linear term, which yields $\kappa = 6.05$ (as opposed to 6 for pure percolation) is within the confidence interval $6 \pm 0.18$ set by Eq. (114).

These aberrations can be ascribed to the inexorable finite-size effects concomitant with numerical computations. Whilst the parallel between percolation boundaries and SLE$_6$, technically, only holds for infinite curves, realistically, one is always restricted to curves of finite size. Bogomolny et al. (2006b) reasoned that ideally, by dint of the locality of SLE$_6$ (Lawler et al. 2001a, 2003), the trace should not “feel” the boundary until it actually encounters a point thereon. This argument breaks down because even for infinitesimally small capacities (or time scales), there exist traces that can go arbitrarily far from the origin (Bogomolny et al. 2006b). The statistical properties of such curves are bound to be affected by the presence of boundaries, which therefore, must be taken into account.

To this end, one employs an amended version of the traditional formalism, known as dipolar SLE [Bauer et al. 2005]. Consider a region with two boundary arcs capped by points $z_+$ and $z_-$. One of the arcs is assumed to be an absorbing (Dirichlet) boundary and the other reflecting (Neumann). The origin of the random curves, which meander all the way only to be rudely stopped by the absorbing arc, is a point $z_0$ on the reflecting boundary. The guile lies in that the reflecting boundary, such as the real axis in Fig. 25, now thwarts the ambitions of the more adventurous curves desirous of a quick stroll outside the domain. This setup can be conformally mapped to the strip

$$S = \{ z \in \mathbb{C}, 0 < \text{Im} z < \pi \} \quad (116)$$

such that the points $z_-$, $z_+$, and $z_0$ are mapped to $-\infty$, $\infty$, and 0, respectively. For the semicircle of radius $L$, this is achieved by the map

$$F(z) = \log \left( \frac{(L + z)^2}{(L - z)^2} \right), \quad (117)$$

whereas for a rectangle $[-L/2, L/2] \times [0, l]$, the same is accomplished by

$$F(z) = \log \left( \frac{\wp(z + L/2) - \wp(L)}{\wp(L/2) - \wp(L)} \right), \quad (118)$$

where $\wp(z)$ is the Weierstrass elliptic function with periods $2L$ and $2il$. The dipolar SLE$_\kappa$ process is effected by the Loewner-type equation:

$$\partial_t g_t(z) = \frac{2}{\tanh (g_t(z) - U_t)}; \quad g_0(z) = z, \quad (119)$$

whereupon subsequent calculations refine the quadratic fitting to

$$\sigma^2(t) = 5.92 t - 0.103 t^2 \quad (120)$$

Eq. (120), if not a better estimate of $\kappa$, at least minimizes the coefficient of the quadratic term compared to our previous estimate in Eq. (114). The remaining mismatches are vestiges of discretization errors that fade away upon including more points along a trace and do not bear any new physical information. In the same vein, Keating et al. (2006, 2008) corroborated that the nodal lines for a perturbed quantum cat map are described by SLE$_6$. The nodal lines of the vorticity field in two-dimensional turbulence are also known to bear resemblance to SLE$_6$ curves [Bernard et al. 2006, Cardy 2006].

From a broader viewpoint, the universally accredited importance of SLE is due in no small part to its intimate and profound connections to conformal field theories [Bauer and Bernard 2002, 2003, Cardy 2003 and...
the flavor it lends to general ideas such as criticality and dualities (Beffara [2004] Duplantier [2000]). In all fairness, one could go on all day about SLE and its myriad applications. However, to prevent this from devolving into a full-blown review about SLE alone, we have to draw the line somewhere and this seems to be as good a place as any.

V. STATISTICAL MEASURES

A. Nodal domain statistics

The nodal domain statistics of quantum billiards, as promised, can be used to distinguish between quantum systems with integrable and chaotic classical dynamics; the limiting distributions of the same are believed to be universal (system-independent). This yields a criterion for quantum chaos, which is complementary to that established based on spectral statistics. In this subsection, we examine each of these statistical measures individually.

1. Limiting distributions of nodal counts

Bearing Courant’s nodal domain theorem and Eq. (146) in mind, Blum et al. (2002) defined the normalized number of nodal domains as

\[ \xi_j = \frac{\nu_j}{j}, \quad 0 < \xi_j \leq 1. \]  

(121)

To extract the universal features thereof, it was proposed that a limiting distribution be constructed as

\[ P(\xi) = \lim_{E \to \infty} P(\xi, I_g(E)) \]  

by considering the energy levels in an interval \( I_g(E) = [E, E + gE] \), \( g > 0 \). With the number of eigenvalues in \( I_g(E) \) given by the Weyl formula, the distribution of \( \xi \) associated with \( I_g(E) \) is

\[ P(\xi, I_g(E)) = \frac{1}{N_1} \sum_{E_j \in I_g(E)} \delta \left( \xi - \frac{\nu_j}{j} \right). \]  

(123)

a. Separable, integrable billiards For integrable (separable) systems, the Hamiltonian can be expressed in terms of action-angle variables, \( H (I_1, I_2) \)—this is a homogeneous function of degree two. The Einstein-Brillouin-Keller quantization [Bleher 1994] of such systems gives the energy levels \( E_{n_1, n_2} = H(n_1 + \alpha_1, n_2 + \alpha_2) + O(H) \), \( n_1, n_2 \in \mathbb{Z} \); \( \alpha_1, \alpha_2 \) being Maslov indices. Separability of the system implies that the number of domains of the eigenfunction with quantum numbers, \( n_1, n_2 \) is \( \nu_{n_1, n_2} = n_1 n_2 + O(1) \). The nodal domain number is thus \( \xi_{n_1, n_2} = n_1 n_2 / N(E_{n_1, n_2}) \), where \( N(E) = A E (1 + O(E^{-1/2})) \). Eq. (123) can be rewritten as

\[ P(\xi, I_g(E)) = \frac{1}{g E A} \int_{H \to I_g} dI_1 dI_2 \delta \left( \xi - \frac{I_1 I_2}{AH(I_1, I_2)} \right) + O(E^{-1/2}). \]  

(124)

The homogeneity of \( H \) leads to an expression for \( P \) that is independent of \( g \) and \( E \). Employing a change of variables \( (I_1, I_2) \to (\mathcal{E}, s) \), where \( \mathcal{E} = H(I_1, I_2) \) and \( s \) is the constant-energy curve, Eq. (124) becomes an integral

\[ P(\xi) = \frac{1}{A} \int_{\Gamma} d\xi \delta \left( \xi - \frac{I_1(s) I_2(s)}{A} \right). \]  

(125)

over the line \( \Gamma \) defined by \( H(I_1, I_2) = 1 \). The limiting distributions for rectangular and circular billiards, calculated numerically according to Eq. (125), are seen in Fig. 26.

![FIG. 26 Numerically calculated limiting distributions of the nodal domain number for rectangular (blue) and circular (red) billiards.](image)

For the rectangular billiard, we elucidate the calculations in more detail now, following Smilansky and Sankaranarayanan (2005). Let the lengths of the sides of the rectangle be \( L_x \) (set equal to \( \pi \)) and \( L_y = L_x / \alpha \) where \( 0 < \alpha < 1 \). The energy spectrum is proportional to \( n^2 + \alpha^2 m^2 \). Weyl’s law gives us the cumulative density of energy levels:

\[ N(E) \simeq \frac{\pi E}{4\alpha} \left( 1 - \frac{2}{\pi} \frac{1 + \alpha}{E} \right). \]  

(126)

The simple form of the energy level sequence admits easy parametrization in terms of the continuous variables

\[ n(E, \theta) = \sqrt{E} \cos \theta, \quad m(E, \theta) = \sqrt{E} \sin \theta / \alpha, \]  

(127)
whereupon the normalized nodal domain number for the \( j \)th state is

\[
\xi_j = \frac{2}{\pi} \sin 2\theta \left[ 1 - \frac{2}{\pi} \frac{1 + \alpha}{\sqrt{E}} \right]^{-1}.
\]  
\( \text{(128)} \)

The distribution \( P(\xi, I) \) can be expressed in terms of \( E, \theta \):

\[
P(\xi, I) \approx \frac{1}{2\alpha N_I} \int_{E_0}^{E_1} \int_0^{\pi/2} \delta[\xi - \xi_j(E, \theta)] \, dE \, d\theta.
\]  
\( \text{(129)} \)

\( N_I \) gives the number of energy levels between \( E_0 \) and \( E_1 \). We first perform the \( \theta \)-integral by taking advantage of the Dirac delta function; with some convenient definitions, this can be easily achieved so that

\[
P(\xi, I) \approx \frac{E_0}{\alpha N_I} \int_1^\ell x \left[ \frac{2 \cos 2\theta_0}{\pi (1 - \varepsilon/x)} \right]^{-1} \, dx,
\]  
\( \text{(130)} \)

where \( x = \sqrt{E/E_0}, \varepsilon(\alpha) = 2 (1 + \alpha)/(\pi \sqrt{E_0}), \sin 2\theta_0 = (\pi \xi/2)(1 - \varepsilon/x) \), and

\[
\ell = \begin{cases} 
G = E_1/E_0, & \text{if } \xi < 2/\pi, \\
\min \left[ G, \frac{\varepsilon \pi \xi}{2} - 1 \right] & \text{if } \frac{2}{\pi} < \xi \leq \frac{2}{\pi(1 - \varepsilon)}. 
\end{cases}
\]

\( P(\xi, I) \) is zero for \( \xi > 2/[\pi (1 - \varepsilon)] \). In this manner, the aspect ratio \( \alpha \) of a rectangular domain (with Dirichlet boundary conditions) can be retrieved by counting its nodal domains. The limiting distribution is

\[
P(\xi) = \begin{cases} 
\frac{1}{(1 - (\pi \xi/2))^2} & \text{for } \xi < 2/\pi, \\
0 & \text{for } \xi > \xi_{\text{max}} = 2/\pi.
\end{cases}
\]  
\( \text{(131)} \)

In Fig. 26, the peak around \( \xi \approx 0.64 \) exactly coincides with the value of \( 2/\pi \) in Eq. \( \text{(131)} \).

b. Surfaces of revolution The Schrödinger equation can be solved for certain surfaces of revolution in \( \mathbb{R}^3 \). Karabassis, George and Smilansky (2008) presented the first results for ellipsoids with different eccentricities and proved that “the nodal sequence of a mirror-symmetric surface is sufficient to uniquely determine its shape (modulo scaling).” The form of the limiting distribution is nearly identical to that for separable billiards (Fig. 26) but, of course, with differing values of \( \xi_{\text{max}} \) (that actually decrease with the eccentricity \( \varepsilon \)); specifically, for the sphere, \( \varepsilon = 1 \) and \( \xi_{\text{max}} = 0.5 \). In general, for separable systems in \( d \) dimensions,

\[
P(\xi) \approx (1 - \xi/\xi_{\text{max}})^{(d-3)/2}.
\]  
\( \text{(132)} \)

c. Nonseparable, integrable billiards The two chief billiards of interest in this category are the right-angled isosceles triangle and the equilateral triangle, the nodal domain distributions for both of which have been studied numerically.

First, let us consider the right-isosceles triangle. For simplicity, let the length of the equal legs be \( \pi \); the area is just \( \pi^2/2 \). The unnormalized solutions of the Schrödinger problem in the interior of the triangle, with Dirichlet conditions on the boundary, have the form

\[
\psi_{mn}(x,y) = \sin(mx) \sin(ny) - \sin(nx) \sin(my),
\]  
\( \text{(133)} \)

and the spectrum of eigenvalues is given by

\[
E_{m,n} = m^2 + n^2, \quad m, n \in \mathbb{N}, \ m > n.
\]  
\( \text{(134)} \)

When \( \gcd(m,n) = d > 1 \) for \( m > n \), the nodal set is composed of \( d^2 \) identical nodal patterns (“tiles”), each contained within a subtriangle. The nodal domains of the billiard were counted using the Euler formula for graphs by Aronovitch et al. (2012) (refer to Sec. VI.C for details). Fig. 27 shows the corresponding distribution of the nodal domain counts. One observes a large number of peaks, each seemingly converging to a Dirac delta function.

![Fig. 27 The nodal count distribution for the right-isosceles triangle, considering the energy interval \([9000^2, 2 \times 9000^2]\).](image-url)

The analysis for the equilateral triangle proceeds analogously. An equilateral-triangular domain of side \( L = \pi \) and area \( A = \sqrt{3} \pi^2/4 \) is

\[
\mathcal{D} = \left\{ (x, y) \in \left[0, \frac{\pi}{2}\right] \times \left[0, \frac{\sqrt{3}\pi}{2}\right] : y \leq \sqrt{3}x \right\}
\]  
\( \text{(135)} \)

\[
\cup \left\{ (x, y) \in \left[\frac{\pi}{2}, \pi\right] \times \left[0, \frac{\sqrt{3}\pi}{2}\right] : y \leq \sqrt{3}(\pi - x) \right\}.
\]
The Dirichlet eigenfunctions, which form a complete orthogonal basis, are (Brack and Bhaduri 2003)

\[
\psi_{m,n}^\xi(x, y) = \sin \left( \frac{n \pi}{\sqrt{3} L} y \right) \left( \cos \sin \left( \frac{(2m-n) \pi}{3L} x \right) - \sin \left( \frac{m \pi}{\sqrt{3} L} y \right) \cos \sin \left( \frac{(2n-m) \pi}{3L} x \right) + \sin \left( (m-n) \frac{\pi}{\sqrt{3} L} y \right) \cos \sin \left( -(m+n) \frac{\pi}{3L} x \right) \right),
\]

(136)

where \(m\) and \(n\) are integer quantum numbers with the restriction \(m \geq 2n\) and \(m, n > 0\). The eigenfunctions \(\psi_{m,n}^\xi\) and \(\psi_{m,n}^\eta\) correspond to the symmetric and antisymmetric modes respectively (McCartin 2003). The eigenenergies of the Hamiltonian for the system is

\[
E_{m,n} = \frac{16 \pi^2 \hbar^2}{9 m L^2} (m^2 + n^2 - mn)
\]

(137)

for a particle of mass \(m\). This spectrum possesses interesting and deep number-theoretic properties as shown by Itzykson and Luck (1986). The nodal patterns of the eigenfunctions of the equilateral triangle also exhibit certain symmetry relations, including, once again, a tiling structure of the nodal lines: for \(m \geq 2n\) and \(\gcd(m, n) = d > 1\), \(\psi_{m,n}\) is tiled by \(\psi_{m',n'}\) with \(m' = m/d\) and \(n' = n/d\). This arrangement, which follows directly from Eq. (136), is illustrated by the examples in Fig. 28. Counting the nodal domains for this billiard with the Hoshen-Kopelman algorithm (Hoshen and Kopelman 1976), one arrives at the distribution \(P(\xi)\), displayed for two spectral intervals in Fig. 29.

FIG. 28 (a) Nodal domains for the cosine combination of the eigenfunctions of the equilateral triangle for \((m, n) = (10, 3)\). (b) The tiling pattern of the domains (with 4 tiles) is seen for \((m, n) = (10, 2)\). (c) The antisymmetric (sine) combination for \((m, n) = (10, 3)\). The eigenfunctions are positive (negative) in the white (black) regions.

2. Distribution of boundary intersections

For bounded domains, one can also study the statistics of the number of nodal intersections with the boundary of the billiard, \(\tilde{\nu}\)—this is exactly the number of times the normal derivative at the boundary vanishes. The appropriate normalized parameter is now \(\eta_j = \tilde{\nu}_j \sqrt{\lambda_j}\). Even with Neumann boundary conditions, \(\tilde{\nu}_j \sim \mathcal{O}(\lambda_j^{1/2})\) (Toth and Zelditch 2009), so \(\eta_j\) is correctly normalized either way. The distribution of \(\eta\) for the interval \(I_\delta(E)\), which is a characteristic of the system, is

\[
P[\eta, I_\delta(E)] = \frac{1}{N_I} \sum_{E_I \in I_\delta(E)} \delta(\eta - \eta_I).
\]

(138)

Hence, the limiting distribution for the system is defined, in exact correspondence to Eq. (122), as

\[
P(\eta) = \lim_{E \to \infty} P[\eta, I_\delta(E)].
\]

(139)

a. Separable, integrable billiards In terms of the action variables \(I_{1,2}\), the analogue of Eq. (131) for the boundary intersections was calculated by Blum et al. (2002) to be

\[
P(\eta) = \begin{cases} \frac{1}{4 \sqrt{\lambda}} (I_2(I_1) - I'_2(I_1) I_1) \bigg|_{I_1 = \frac{2 \sqrt{\lambda}}{\eta}}, & \eta < \eta_m, \\ 0, & \eta \geq \eta_m, \end{cases}
\]

where the maximum value of \(\eta\), \(\eta_m = 2 I_{1,m}/\sqrt{\lambda}\), is determined by the intersection point, \(I_{1,m}\), of the \(I_1\)-axis with the contour line \(\Gamma\). The numerics (Fig. 30) are consistent with this prediction.

b. Nonseparable, integrable billiards We revert to our old example of the equilateral triangle as a prototype for this class of systems. As seen in Fig. 31, \(P(\eta)\) consists of multiple dominant peaks of different strengths at certain characteristic values of \(\eta\) (Samajdar and Jain 2014b).
obtained an analytical form for this distribution for non-tiling wavefunctions. The number of eigenfunctions in a generic spectral interval \( I = [E_0, E_1] \) is set by Weyl’s law:

\[
N_I = \frac{\sqrt{3}\pi}{16} \left( E_1 - E_0 \right) - \frac{4\sqrt{3}}{\pi} \left( \sqrt{E_1} - \sqrt{E_0} \right),
\]

Introducing the variable \( \varepsilon = 4\sqrt{3}/(\pi\sqrt{E_0}) \) and retaining \( x, G \) from Eq. (130), the distribution can be expressed as

\[
P(\eta, I) = \frac{E_0 \sqrt{\pi \varepsilon}}{4 N_I} \int_1^\ell x \left( \frac{1 - \varepsilon/x}{1 - \{f(x)/2\}} \right) \, dx,
\]

\[
f(x) = \frac{16 + \sqrt{\pi E_0 \sqrt{3}\eta} \left( x^2 - \varepsilon^2 \right)^{1/2}}{8 x \sqrt{E_0}},
\]

\[
\ell = \begin{cases} G, & \text{if } \eta < \varphi_1, \\ \min \left[ G, \mathcal{X}_{\max} \right], & \text{if } \eta > \varphi_1, \end{cases}
\]

\[
\varphi_1 = \begin{cases} \frac{8}{3} \sqrt{6 - \sqrt{3}\pi}, & \text{if } 0 < \sqrt{E_0} \leq \frac{2\sqrt{3}}{\pi - \sqrt{3}}, \\ \frac{8}{3^{1/4}} \frac{\sqrt{E_0} - 2}{\pi E_0 - 4\sqrt{3}E_0}, & \text{otherwise}. \end{cases}
\]

Furthermore, \( P(\eta, I) = 0 \) for all \( \eta > \varphi_2 \) specified by

\[
\varphi_2 = \begin{cases} \frac{8}{3^{1/4}} \frac{\sqrt{E_0} - 2}{\pi E_0 - 4\sqrt{3}E_0}, & \text{if } 4\sqrt{3}/\pi < \sqrt{E_0} \leq \frac{\pi}{\pi - \sqrt{3}}, \\ \frac{8}{3^{1/4}} \frac{\sqrt{E_0} - 2}{\pi \sqrt{3}}, & \text{if } \sqrt{E_0} > \frac{\pi}{\pi - \sqrt{3}}. \end{cases}
\]

However, there does not exist any such upper bound when \( \sqrt{E_0} \leq 4\sqrt{3}/\pi \). \( \mathcal{X}_{\max} \), appearing in the definition of \( \ell \) above, is the maximum permissible value of \( x \) as regulated by the inequality

\[
0 < \varphi_1 \leq \frac{8 x \sqrt{E_0} - 16}{\sqrt{\pi E_0 \sqrt{3}(x^2 - \varepsilon^2)}} \leq \varphi_2.
\]

Evaluating \( X_{\max} \) for the most general case (considering sufficiently excited states such that \( E_0 > [2\sqrt{3}/(\pi - \sqrt{3})]^2 \)), the integral for \( P(\eta, I) \) can be easily computed numerically for any \( E_0 \). The theoretical estimate of \( \varphi_2 = 3.43 \) for \( E_0 = 2000^2 \) is in close agreement with the numerical result (Fig. 31), which suggests that the distribution \( P(\eta) \) is zero beyond, approximately, 3.35.

c. Chaotic billiards For the chaotic Sinai and stadium billiards, the nodal domain statistics are usually analyzed in the random wave approximation. Given a wavefunction of the type (57), the number of zeroes of its normal derivative \( u(\theta) = \partial_r \psi(r, \theta)|_{r = R} \), on a circle of radius \( R \), is (Kac 1959)

\[
\tilde{\nu}_u = \int_0^{2\pi} \frac{d\theta}{2} \int_{-\infty}^{\infty} \frac{d\xi}{2(\pi\eta)^2} e^{i\xi u(\theta)} \left( 1 - e^{i\eta \tilde{u}(\theta)} \right),
\]

where \( \tilde{u}(\theta) = d_0 u(\theta) \). The mean and variance of \( \tilde{\nu}_j \) are thus (Blum et al. 2002)

\[
\langle \tilde{\nu} \rangle = k R \sqrt{1 + \left( \frac{2}{k R} \right)^2} \approx k L \approx \frac{k P}{2\pi}, \quad \text{Var}(\tilde{\nu}) \approx 0.0769 k P.
\]

This implies that the scaled number of nodal intersections tends to a Dirac delta function, centered at \( \eta \sqrt{\pi A/P} = 1 \) (seen in the inset of Fig. 32).
3. Geometric characterization of nodal domains

a. Area-to-perimeter ratio  A nodal domain of an eigenfunction, being a bounded region in space itself, has obvious geometric characteristics like a well-defined perimeter and area. The ratio of these two quantities turns out to be another statistically significant tool to sniff out the underlying dynamics of the system. To interrogate the morphology of the nodal lines, Elon et al. (2007) considered the set of nodal domains of the \( j \)-th eigenfunction of a billiard in a domain \( D \); this can be represented as the sequence \( \{ \omega_j^{(m)} \} \), \( m = 1, 2, \ldots, \nu_j \). One can then define the ratio

\[
\rho_j^{(m)} = \frac{A_j^{(m)}}{L_j^{(m)}} \sqrt{E_j},
\]

where \( A_j^{(m)}, L_j^{(m)} \) denote the area and perimeter of the nodal domain; the factor of the energy eigenvalue ensures the correct scaling. As in Eq. (122), we inspect the probability measure

\[
P_D(\rho, E, g) = \frac{1}{N_I} \sum_{E_j \in I} \frac{1}{\nu_j} \sum_{m=1}^{\nu_j} \delta \left( \rho - \rho_j^{(m)} \right),
\]

which again tends to a limiting distribution in the same fashion as previously. For a rectangular billiard, the distribution is of the form

\[
P_{\text{rectangle}}(\rho) = \begin{cases} 
\frac{4}{\rho^2 \sqrt{8\rho^2 - \pi^2}}, & \frac{\pi}{\sqrt{8}} \leq \rho \leq \frac{\pi}{2}, \\
0, & \text{otherwise},
\end{cases}
\]

This function (Fig. 33), in addition to being independent of the aspect ratio of the billiard, is analytic and monotonically decreasing in the compact interval \([\pi/\sqrt{8}, \pi/2]\) but is discontinuous at the endpoints. All of these properties, including the support, are believed to be universal features for all two-dimensional separable surfaces. Explicit derivations of the limiting distributions for the family of simple surfaces of revolution and the disc billiard (Elon et al., 2007) further lend weight to this hypothesis. However, for integrable but non-separable and pseudointegrable billiards, the form of \( P(\rho) \) is unknown. For chaotic Sinai and stadium billiards, numerics suggest a universal limiting distribution \( P(\rho) \), which converges to that for the random-wave ensemble. This agreement can also be demonstrated for finite energies, as in Fig. 33, by considering only the inner nodal domains (away from the billiard’s boundary).
b. Signed area distribution  Instead of scrutinizing the areas of individual nodal domains, one may alternatively peruse the collective statistics of the total area where the wavefunction is positive (negative), denoted hereafter by \(|A|_+; \text{clearly, } \langle |A|_+ \rangle = A/2\). Exploiting the identity

\[
|A|_\pm = \frac{1}{2\pi} \lim_{\epsilon \to 0^+} \int_A \int_{-\infty}^{\infty} d\xi \frac{e^{\pm i \xi \psi(r)}}{\xi - i \epsilon},
\]

Blum et al. (2002) reckoned the signed area variance

\[
\frac{\langle (|A|_+ - (\langle |A|_+ \rangle)^2 \rangle}{A^2} \approx 0.0386 \frac{0.0386}{(R/2)^2} \equiv 0.0386 N^{-1}
\]

for the random wave model on a circle of radius \(R\). Fig. 34 affirms the convergence of the variance to this asymptotic limit for the stadium and the Sinai billiards but also proclaims the qualitatively different—non-oscillatory—behavior of the data for the equilateral triangle billiard.

![FIG. 34 The normalized signed area variance for the equilateral triangle. (Inset): The same for the stadium and Sinai billiards but plotted as a function of \(N\). The analytical expression and numerical simulation for random waves are depicted by smooth curves and triangles, respectively. Adapted from Blum et al. (2002).](image)

4. Nodal volume statistics

The nodal volume is the hypersurface volume of the nodal set of the \(j\)th eigenfunction, denoted by \(H_j\). In order to facilitate comparison of nodal volumes of eigenfunctions at different energies, it is prudent to scale the volumes with the typical wavelength, \(\sqrt{\ell_j} \), \(\ell_j > 0\), and then define the rescaled dimensionless variable, \(\sigma_j = H_j/V \sqrt{\ell_j} \), where \(V\) is the volume of the manifold \(M\). It is this rescaled quantity on which Yau’s conjecture places the bounds \(c_1 \leq \sigma_j \leq c_2\) for \(j \geq 2\), where \(c_1, c_2\) depend only on the manifold and the metric. We now present some recent results (Gnutzmann and Lois, 2014) on the nodal volume statistics for an \(s\)-dimensional cuboid—a paradigm of regular classical dynamics—and for boundary-adapted planar random waves—an established model for chaotic wave functions—in irregular shapes (Berry, 2002).

a. \(s\)-dimensional cuboid  The normalized eigenfunctions of an \(s\)-dimensional cuboidal Dirichlet billiard with sides of lengths \(\{\alpha_\ell, \ell = 1, 2, \ldots, s\}\) and volume \(V = \prod_{\ell=1}^s a_\ell\) are

\[
\psi_n(q) = \frac{(2\pi)^{s/2}}{V^{1/2}} \prod_{\ell=1}^s \sin \left( \frac{\pi n_\ell q_\ell}{a_\ell} \right),
\]

\((n_\ell)\) being positive integers. The corresponding energies and rescaled nodal volumes are

\[
E_n = \pi^2 \sum_{\ell=1}^s \frac{n_\ell^2}{a_\ell^2}, \quad \sigma_n = \frac{1}{\sqrt{E_n}} \sum_{\ell=1}^s \frac{n_\ell - 1}{a_\ell}. \quad (152)
\]

Let us define an asymptotic mean value of \(\sigma_n\) in a spectral interval \(E_n \in [E, E + \Delta E]\) of width \(\Delta E\) near \(E\),

\[
\langle \sigma_n \rangle_{[E, E + \Delta E]} = \frac{1}{N_{[E, E + \Delta E]}} \sum_{n \in [E, E + \Delta E]} \sigma_n \chi_{[E, E + \Delta E]}(E_n),
\]

where \(\chi\) is the characteristic function on the interval and \(N_{[E, E + \Delta E]}\) is the number of eigenfunctions with energies in \([E, E + \Delta E]\). For the asymptotic behavior, \(\Delta E\) can be chosen to be \(g E^{1/4}\), \(g > 0\), without loss of generality. Weyl’s law for the cumulative level density, adapted for an \(s\)-dimensional cuboid, is repackaged to

\[
N_{[E, E + \Delta E]} = \frac{2 \zeta_s V}{\pi \zeta_s} E^{s/2} - \frac{\zeta_s - 1}{2} S E^{(s-1)/2} + O(E^{(s-2)/2}),
\]

\((153)\)

where \(\zeta_s = \pi^{s/2}/\Gamma(1 + s/2)\) is the volume of an \(s\)-dimensional unit sphere and \(S = 2V \sum_{\ell=1}^s a_\ell^{-1}\) is the \((s-1)\)-dimensional volume of the surface of the \(s\)-cuboid. Using Eq. \(153\) to obtain \(N_{[E, E + \Delta E]}\) and employing the Poisson summation formula, the mean value is found to be (Gnutzmann and Lois, 2014)

\[
\langle \sigma_n \rangle_{[E, E + \Delta E]} = \frac{2 \zeta_{s-1}}{\pi \zeta_s} \left( 1 - \beta_s \frac{S}{2} E^{-1/2} + O(E^{-3/4}) \right),
\]

\[
\beta_s = \frac{\pi (s - 1) \zeta_{s-2}}{2 s \zeta_{s-1}} + \frac{\pi \zeta_s}{4 \zeta_{s-1}} - \frac{\pi (s - 1) \zeta_{s-1}}{2 s \zeta_s}. \quad (154)
\]

Similarly, the variance of \(\sigma_n\) can be expressed in an asymptotic series for large \(E\) as

\[
\text{Var}(\sigma_n) = \frac{1}{\pi^2} + \frac{4}{\pi^2} \frac{(s - 1) \zeta_{s-2}}{2 \pi^2 \zeta_s} - \frac{4 \zeta_{s-2}}{\pi^2 \zeta_s} + O(E^{-1/2}).
\]

Utilizing the higher moments, the limiting distribution

\[
P_s(\sigma) = \lim_{E \to \infty} \delta(\sigma - \langle \sigma \rangle_{[E, E + \Delta E]})
\]

\((155)\)

can be calculated for any \(s\). The limiting distributions thus evaluated by Gnutzmann and Lois (2014) are non-zero only over a finite interval. For instance, \(P_2(\sigma)\) is non-zero only over \([1/\pi, \sqrt{2}/\pi]\) wherein it varies as \(4/\sqrt{2} - \pi^2 \sigma^2\), bringing to mind Eq. \(131\). Importantly, this observation is also in line with Yau’s conjecture.
b. Random wave model

On a different note from the integrable cuboid, we can repeat the above-defined procedure for the eigenfunctions of a chaotic billiard modeled according to boundary-adapted RWM. In \( s \) dimensions, the mean of \( \sigma \) is \( \langle \sigma \rangle_G = \rho_{\text{bulk}} \left( 1 - \frac{S}{\sqrt{32\pi k}} + \mathcal{O}(k^{-1}) \right) \), \( s = 2 \),

\[
= \rho_{\text{bulk}} \left( 1 - \frac{S}{\sqrt{32\pi k}} + \mathcal{O}(k^{-1}) \right), \quad s \geq 3,
\]

where \( \rho_{\text{bulk}} = \Gamma((s+1)/2)/[\sqrt{\pi s} \Gamma(s/2)] \) is the constant nodal density of the standard RWM without boundaries and \( I_s \) are constants \( (I_3 \approx 0.758, I_4 = 0.645) \). The limiting distribution of nodal volumes is now sharply peaked for a finite energy interval and converges to \( P(\sigma) = \delta(\sigma - \rho_{\text{bulk}}) \). This is to be contrasted with the finite support for the cuboid’s distribution—the distinct characters of \( P(\sigma) \) therefore differentiate between chaotic and regular manifolds. Moreover, the variance decreases with increasing energies for irregularly-shaped billiards whereas for separable shapes, it remains finite and bounded. Additionally, the boundary corrections to expected rescaled nodal volumes—\( 1/k \) for the cuboid and \( k^{-1} \log k \) for chaotic shapes—also herald this distinction.

B. Nodal line statistics

The sinuous nature of the nodal curves for classically chaotic systems, particularly for the excited states, renders their study challenging. However, the wealth of statistical information borne by the nodal lines also makes the problem equally rewarding.

1. Length fluctuations

The total length of the nodal curve of a real wavefunction \( u(r) \), \( r = (x, y) \), is proffered by elementary calculus:

\[
L = \int \int_A \mathrm{d}r \frac{\delta(u(r))}{|\nabla u(r)|},
\]

where the integral runs over the area of the billiard enclosure \( A \). Although, on paper, Eq. (156) offers a deterministic expression for the nodal length, the practical drawback is that exact analytical results for eigenfunctions of chaotic billiards are extremely rare and only known for a few low-lying states (Jain et al. 2002). For the excited states of these systems, the eigenfunctions display intricate scaling properties, reminiscent of multifractal objects, but the exact forms thereof remain an open question. Thus, foiled, we resort to the random wave model, consoling ourselves that since the RWM describes the high-energy eigenvalues, the nodal lines of random waves should also model the nodal lines of honest eigenfunctions (Wigman 2012b). To adapt Eq. (63) to the boundary conditions, we switch to coordinates in which, at any point on the billiard’s boundary, \( \hat{x} \) is along \( \partial \mathcal{D} \) (traversing counterclockwise) and \( \hat{y} \) points along the normal to it, inwards (much like \( \hat{\theta} \) and \(-\hat{r}\), respectively, in polar coordinates). For real eigenfunctions, the superpositions can be written as (cf. Eq. (63))

\[
u_D(\mathbf{R}) = \sqrt{\frac{2}{N}} \sum_{j=1}^{N} \sin(Y \sin \theta_j) \cos(X \cos \theta_j + \phi_j);
\]

\[
u_N(\mathbf{R}) = \sqrt{\frac{2}{N}} \sum_{j=1}^{N} \cos(Y \sin \theta_j) \cos(X \cos \theta_j + \phi_j);
\]

\[
(\partial \mathbf{u}/\partial \mathbf{y} = 0 \text{ for } y = 0),
\]

where \( \mathbf{R} = (k x, k y) = (X, Y) \) introduces a convenient set of dimensionless coordinates. On replacing cosines by sines in these equations, we obtain the corresponding expressions for complex waves. To perform the averages, we integrate over the random phases \( \phi_j \) and directions \( \theta_j \), both of which are uniformly distributed in \([0, 2\pi] \).

The average total length of nodal lines \( \langle L \rangle \) can be calculated on knowing the mean density of nodal length as a function of \( y \) (Berry 2002). From Eq. (156), this is

\[
\langle L(k) \rangle = \frac{k}{2\sqrt{2}} \int_A \mathrm{d}r \rho_L(Y);
\]

the prefactor ensures that \( \rho_L(Y) \) tends to unity as \( Y \to \infty \). The mean density is

\[
\rho_L(Y) = 2\sqrt{2} \langle \delta(u) | \nabla \mathbf{u} \rangle,
\]

which requires information about the distribution functions of \( u_X \) and \( u_Y \). These distributions are Gaussian,

\[
P(u = 0, u_Y) = \frac{1}{(2\pi B D_Y - K^2)^{3/2}} \exp \left[-\frac{B u_Y^2}{2(B D_Y - K^2)}\right],
\]

with the parameters expressible in terms of various cylindrical Bessel functions as \( D_X = (1\mp J_0(2Y)\mp J_2(2Y))/2, D_Y = (1\pm J_0(2Y)\mp J_2(2Y))/2, B = 1 \mp J_0(2Y), \) and \( K = \pm J_1(2Y) \) for Dirichlet and Neumann boundary conditions, respectively. Employing these distributions, the density reduces to

\[
\rho_L(Y) = \frac{2\sqrt{2}}{\pi} D_X (B D_Y - K^2)
\]

\[
\times \int_0^{\pi/2} \frac{d\theta}{B D_X \cos^2 \theta + (B D_Y - K^2) \sin^2 \theta}^{1/2},
\]

\[
= \frac{2\sqrt{2}}{\pi B} \frac{B D_Y - K^2}{B D_Y - B D_X} \sqrt{1 + \frac{B D_X}{K^2 - B D_Y}},
\]

(162)
\( E \) being the complete elliptic integral \[ \text{[Whittaker and Watson, 1996].} \]

The behavior of \( \rho_L(Y) \) for Dirichlet (Neumann) boundary conditions in Fig. 35 shows a stronger (weaker) repulsion from the boundary, along with an oscillatory asymptotic behaviour for large \( Y \), eventually settling at unity. Eqs. (159, 160) now tell us that

\[
\langle L \rangle = \sqrt{\frac{\pi J}{2}} - \frac{P}{128 \pi \sqrt{2} A} \log j + O(1). \tag{163}
\]

The first piece is an “area term”, originating from the \( kA/2\sqrt{2} \) factor, whereas the second term stems from the boundary. More generally, it is conjectured \[ \text{(Nonnenmacher, 2013)} \] that on any chaotic surface \( M \), such as an ergodic billiard, the nodal lines are asymptotically equidistributed in \( M \) \[ \text{[Wigman, 2012b].} \] In particular, for a discrete spectrum of eigenvalues \( \{E_j\} \), the nodal length of the \( j^{th} \) eigenfunction is asymptotically shaped to obey

\[
\langle L_j \rangle \sim c_M \cdot \sqrt{E_j} \tag{164}
\]

for some constant \( c_M > 0 \).

It is quite obvious that the nodal lines’ length depends on the state; correspondingly, there are fluctuations over different eigenstates. These fluctuations are quantified by the variance

\[
(\delta L)^2 \equiv \left( \sqrt{\langle L^2 \rangle - \langle L \rangle^2} \right)^2 \approx \frac{A}{256\pi} \log (k \sqrt{A}). \tag{165}
\]

Written in the dimensionless form (for level number \( j \))

\[
\frac{\delta L}{\sqrt{A}} \approx \frac{1}{10} \sqrt{\frac{\log j}{2\pi}}, \tag{166}
\]

this estimate \[ \text{[Berry, 2002]} \] is palpably smaller than the boundary term, thereby hailing the expectation that the boundary corrections could be gauged without the necessity of averaging over long sequences of states. Although, these results were obtained for a straight-line boundary, the average nodal line length has the same leading-order logarithmic boundary term for a convex circular boundary as well \[ \text{[Wheeler, 2005].} \]

While chaotic billiards were the cynosure of the preceding exercise, studying the nodal lines of the purportedly “simpler” integrable systems also provides unique insights. An example that has recently spurred great inquiry in mathematical circles is the square, but with periodic boundary conditions, which reduces it to the standard 2-torus \( T^2 = \mathbb{R}^2/\mathbb{Z}^2 \) by sewing together opposite sides. The energy spectrum is \( E_j = 4\pi^2 j \) with \( j \in \mathbb{N} \), as we have seen earlier, and the cardinality \( N_j \) of the set of frequencies

\[
\Lambda_j = \{ \lambda = (m, n) \in \mathbb{Z}^2 : m^2 + n^2 = j \} \tag{167}
\]

is just the degeneracy of the level, which grows on average as \( \sqrt{\log j} \) \[ \text{[Landau, 1909].} \] The eigenspace is spanned by the complex exponentials \( e_j(\mathbf{r}) = \exp (2\pi i \lambda \cdot \mathbf{r}) \) or their trigonometric equivalents. For this system (or any other with a checkerboard nodal pattern), the total nodal length of a “pure” (non-superposed) eigenstate is trivial; in a side of square \( L \), it equals \( (m-1)(n-1) L \). It is much more interesting to look at the nodal lengths and their fluctuations when the state is instead a linear superposition of such individual eigenfunctions. In this context, we consider arithmetic random waves (also called random Gaussian toral Laplacian eigenfunctions), which are the random fields

\[
T_j(\mathbf{r}) = \frac{1}{N_j} \sum_{\lambda \in \Lambda_j} a_{\lambda} e_{\lambda}(\mathbf{r}); \quad \mathbf{r} \in \mathbb{T}, \tag{168}
\]

where the coefficients \( a_{\lambda} \) are independent standard complex-Gaussian random variables save for the relations \( a_{-\lambda} = a_{\lambda} \). The expected total nodal length of the random eigenfunctions is \[ \text{[Rudnick and Wigman, 2008]} \]

\[
\mathbb{E} \left[ L_j \right] = \frac{1}{2\sqrt{2}} \sqrt{E_j}, \tag{169}
\]

in consistence with Yau’s conjecture \[ \text{[Donnelly and Fefferman, 1988; Yau, 1982].} \] The corresponding variance was calculated by Krishnapur \textit{et al.} \[ \text{[2013]} \] as

\[
\text{Var} \left( L_j \right) = c_j \frac{E_j}{N_j^2} \left[ 1 + O_{N_j \rightarrow \infty}(1) \right], \tag{170}
\]

where \( c_j \in [1/512, 1/256] \). This expression is to be juxtaposed with Eq. \[ \text{[58]} \] for the Gaussian spherical harmonic, which, in addition, is garnished by the peculiar factor of 1/32 arising from the nontrivial local geometry of the sphere. In this fashion, Eqs. \[ \text{(169) and (170)} \] define the integrable analogues to the chaotic versions,
Eqs. [163] and [164], respectively. The fine asymptotic behavior can be extracted by examining the distributions of the sequence of normalized random variables

\[ \hat{L}_j \equiv \frac{L_j - \mathbb{E}[L_j]}{\sqrt{\text{Var}(L_j)}}, \quad j \in \mathbb{N}. \quad (171) \]

For \( \eta \in [0, 1] \), we also introduce the random variable

\[ M_\eta = \frac{1}{2 \sqrt{1 + \eta^2}} (2 - (1 + \eta) \chi_1^2 - (1 - \eta) \chi_2^2), \quad (172) \]

\( \chi_1 \) and \( \chi_2 \) being standard independent Gaussians. As proved by Marinucci et al. (2016), the sequence \( \hat{L}_j \) converges to \( M_\eta \) under certain technical measure-theoretic conditions determining \( \eta \), which we will not get into except to note that \( \hat{L}_j \) does not converge in distribution for \( N_j \to \infty \) (Kurlberg and Wigman 2016).

2. Curvature distributions

The avoidance of the labyrinthine nodal curves for nonseparable and chaotic systems suggests studying their curvature. Separable billiards can hold eigenfunctions with nodal curves of zero curvature. Let us denote the local curvature by \( \kappa = 1/r \) (\( r \) being the radius of curvature) and the length of nodal curves with curvature \( \leq \kappa \) by \( \ell(\kappa) \). A differential measure of curvature can be defined as (Simmel and Eckert 1996)

\[ C(\kappa) = \frac{1}{\ell(\infty)} \frac{d\ell(\kappa)}{d\kappa} \quad (173) \]

with \( \int_0^\infty C(\kappa) \, d\kappa = 1 \). For separable billiards, due to the checkerboard arrangement of the nodal lines, \( C(\kappa) \) is trivially \( \delta(\kappa) \). Numerical studies on pseudointegrable and chaotic billiards reveal that the curvature distribution, averaged over the eigenstates, is similar (Fig. 36).

The nodal lines of the eigenfunctions of nonseparable systems avoid intersections. To illustrate this, let us expand a plane wave in cylindrical coordinates, following Courant and Hilbert (1953)

\[ \psi(r) = \sum_{l=-\infty}^{\infty} \alpha_l J_l(k r) e^{i l \theta}, \quad (174) \]

where \( \alpha_l^* = (-1)^l \alpha_{-l} \). The origin can be made arbitrary by shifting it by \( \rho \), whereupon the wavefunction can be rewritten using the addition theorem of Bessel functions (Watson 1995):

\[ \psi(r + \rho) = \sum_m \alpha_m(r) J_m(k r) e^{i m \phi}, \quad (175) \]

where \( \phi \) is measured from the direction defined by \( r \). Here, \( \alpha_m = \beta_m + i \gamma_m \) is given by

\[ \alpha_m(r) = \sum_{l=-\infty}^{\infty} \alpha_l(0) J_{l-m}(k r) e^{i l \phi}. \quad (176) \]

The real and imaginary parts of \( \alpha_m \) are related to the wavefunction and its derivatives at \( r \); for instance, (Monastra et al. 2003)

\[ \beta_0(r) = \psi(r), \quad \beta_1(r) = \frac{1}{k} \frac{\partial \psi}{\partial r}, \quad \beta_2(r) = \psi(r) + \frac{2}{k^2} \frac{\partial^2 \psi}{\partial r^2}, \quad \gamma_1(r) = -\frac{1}{k r} \frac{\partial \psi}{\partial \theta}, \quad \gamma_2(r) = \frac{2}{(k r)^2} \left( \frac{\partial \psi}{\partial \theta} - r \frac{\partial^2 \psi}{\partial r \partial \theta} \right), \]

\[ \frac{\partial^2 \psi}{\partial \theta^2} = -k r \beta_1(r) - \frac{(k r)^2}{2} (\beta_2(r) + \beta_0(r)). \quad (177) \]

For \( k \rho < 1 \), the wavefunction to second order in \( k \rho \) is

\[ \psi(r + \rho) = \beta_0(r) \left[ 1 - \left( \frac{(k \rho)^2}{2} \right) \right] + |\alpha_1(r)| \left( \frac{(k \rho)^2}{2} \right) \cos(\phi + \phi_1) + \frac{1}{2} |\alpha_2(r)| \left( \frac{(k \rho)^2}{2} \right) \cos[2(\phi + \phi_2)], \quad (178) \]

taking \( \phi_{1,2} \) to be the phases of \( \alpha_{1,2} \). Two nodal lines intersect at \( r \) if \( \beta_0 = 0, \alpha_1 = 0, \) and \( \alpha_2 \neq 0 \). The intersection of the nodal lines is at right angles since \( \cos(2(\phi + \phi_2)) = 0 \) along two perpendicular lines intersecting at \( r \). If more than two nodal lines intersect such that the first non-vanishing coefficient at \( r \) is \( \alpha_q \), then \( r \) is a nodal point of order \( q \). At this point, \( q \) nodal lines intersect at angles \( \pi/q \). For higher-order \( q \), more conditions need to be satisfied by the coefficients. Thus the intersections become rarer, which is the essential result of Uhlenbeck’s theorem.

An avoided crossing of two nodal lines occurs at \( r \) if \( r \) is a saddle point of the wavefunction. In terms of the

![FIG. 36 Nodal line curvature distribution, C(κ), over 201–600 eigenfunctions for pseudointegrable (two-step) and chaotic (Sinai, deformed Sinai, Stadium) billiards show comparable trends. From Simmel and Eckert (1996), with permission from Elsevier.](image-url)
local coordinates of \( \rho = (\xi, \eta) \), Eq. \[178\] takes the form
\[
(1 - \frac{\alpha_2(\mathbf{r})}{\beta_0(\mathbf{r})}) \xi^2 + (1 + \frac{\alpha_2(\mathbf{r})}{\beta_0(\mathbf{r})}) \eta^2 = 1. \tag{179}
\]
This is the equation of a hyperbola or an ellipse according as whether \( |\alpha_2(\mathbf{r})| \) is larger or smaller than \( |\beta_0(\mathbf{r})| \). The coefficients, \( \alpha_2(\mathbf{r}) \) and \( \beta_0(\mathbf{r}) \), are, of course, related to the wavefunction and its derivatives. One can quantify the scaled distance between the two branches by defining an avoidance range associated with the avoided crossing at \( \mathbf{r} \) (Monastra et al. 2003):
\[
z(\mathbf{r}) = \left[ \frac{16 k^2 |\psi|}{k^2 |\psi| + \sqrt{4 \psi_x^2 + (\psi_{xx} - \psi_{yy})^2}} \right]^{1/2} \tag{180}
\]
the subscripts denoting partial derivatives. Clearly, \( z \) is zero at an intersection of nodal lines. If the intersection is at a saddle point, then \( |\alpha_2(\mathbf{r})| > |\beta_0(\mathbf{r})| \), which puts a bound on \( z \) so that it can range only between 0 and \( 2\sqrt{2} \).

\[D\] to obtain
\[
N_S = \frac{k^2}{4} \int_{D} r \, d\, r \, d\theta \, \delta(\beta_1(\mathbf{r})) \delta(\gamma_1(\mathbf{r})) \left( |\alpha_2(\mathbf{r})|^2 - \beta_0^2(\mathbf{r}) \right) \times \Theta(|\alpha_2(\mathbf{r})|^2 - \beta_0^2(\mathbf{r})). \tag{181}
\]
Similarly,
\[
\tilde{I}(z) = \frac{k^2}{4} \int_{D} r \, d\, r \, d\theta \, \delta(\beta_1(\mathbf{r})) \delta(\gamma_1(\mathbf{r})) \left( |\alpha_2(\mathbf{r})|^2 - \beta_0^2(\mathbf{r}) \right) \times \Theta(|\alpha_2(\mathbf{r})|^2 - \beta_0^2(\mathbf{r})) \Theta(z - r(\mathbf{r})). \tag{182}
\]
Equipped with the above, \( I(z) = \tilde{I}(z)/N_S \) can be studied for different billiards.

Closed-form expressions for \( I(z) \) can be found if the eigenfunction is assumed to be drawn from a random wave ensemble. In this case, Gaussian integrations over the parameters \( \beta_0, \beta_1, \beta_2, \gamma_1, \gamma_2 \) give the averaged values, \( \langle N_S \rangle \) and \( \langle \tilde{I}(z) \rangle \), leading to
\[
I_{\text{rand}}(z) = \frac{3 \sqrt{3} \, z \, (16 - z^2)^2}{(512 - 64 \, z^2 + 3 \, z^4)^{3/2}}; \quad 0 < z < 2\sqrt{2}. \tag{183}
\]
The trend displayed by the Sinai billiard in Fig. \[37\] is closely mimicked by Eq. \[183\]. The probability distribution of the avoidance \( P(z) = d\, I(z)/dz \) exhibits linear repulsion. However, the proportionality constant is different from the case of the Gaussian Orthogonal Ensemble of random matrices (Mehta 2004). The analogy with random matrix theory (RMT) is unsurprising given the results of Johansson (2002) who considered the trajectories of \( N \) non-intersecting Brownian particles on a line starting from \( t = 0 \) and returning at \( t = 2T \) to show that the distribution function of their nearest-neighbor spacings is the same as that for the eigenvalues of random matrices.

More formally, the curvature of \( u(r) \) can be defined (Struik 2012) as the rate of turning of the tangent of a contour line of \( u \) at a point \( r \) (Berry 2002), i.e.,
\[
\kappa(r) = \frac{u_x^2 u_{xx} + u_y^2 u_{yy} - 2 u_x u_y u_{xy}}{|\nabla u|^3}. \tag{184}
\]
The probability distribution of \( \kappa \) is procured by appropriately weighting it over the nodal line length:
\[
P(\kappa) = \frac{\langle \delta(u) |\nabla u| \delta(\kappa - \kappa(r)) \rangle}{\langle \delta(u) |\nabla u| \rangle}. \tag{185}
\]
On the nodal curves, since the wavefunction is zero, \( u_{xx} + u_{yy} = 0 \). This fact, in combination with the polar representation of \( \nabla u \) for Gaussian random waves, Eq. \[63\], yields the distribution (Berry 2002)
\[
P\left( \frac{\kappa}{\kappa} \right) = \frac{4}{\pi \left[ 1 + \left( \frac{\kappa}{\kappa} \right)^2 \right]^2}. \tag{186}
\]
This implies
\[
\langle | \kappa | \rangle = \frac{k}{\pi} = \frac{2}{\lambda}; \quad \sqrt{\langle \kappa^2 \rangle} = \frac{k}{2} = \frac{\pi}{\lambda},
\]
which is a physically appealing result—the radii of curvature of the nodal lines are of the same order as the wavelength. Higher moments of \( \kappa \) are divergent and correspond to regions where the nodal lines are strongly curved.

3. Complexity of the network of nodal lines

Nodal lines form a serpentine network in nonseparable and nonintegrable billiards. One of the ways of quantifying such a network is to examine its intersection with a reference curve [Aronovitch and Smilansky 2007]. Interestingly, the first efforts to do so were made in the context of studying a moving surface of the sea in terms of the curvature of the nodal lines along a straight line [Longuet-Higgins, 1957]. In order to characterize the complex wave pattern belonging to a random wave ensemble, consider the points generated by the intersection of a reference curve with the nodal lines. The nearest-neighbor spacing statistics for these points can be found numerically and correspond to regions where the nodal lines are strongly curved.

We can write the density of the points of intersection as the density of zeros of a function \( f \):
\[
\rho(t) = \sum_i \delta(t - t_i) = \sum_i \delta[f(t)]/|\partial f/\partial t|, \tag{188}
\]
where \( f(t_i) = 0 \). Kac’s Fourier representation [Kac 1959] gives the mean density function:
\[
\langle \rho(t) \rangle = \frac{1}{2\pi^2} \int_{-\infty}^{\infty} d\xi d\eta \left\langle e^{i\xi f(t)}(1 - e^{i\eta \partial_t f(t)}) \right\rangle. \tag{189}
\]
This can be easily evaluated for Gaussian \( f(t) \) and used to calculate the two-point correlation function:
\[
R(t, t') = \left\langle \sum_{i \neq j} \delta(t - t_i) \delta(t' - t_j) \right\rangle = \langle \rho(t) \rho(t') \rangle - \delta(t - t') \langle \rho(t) \rangle. \tag{190}
\]

The random wave model is appropriate for excited states satisfying \( k \gg \kappa \), where \( \kappa \) is now the curvature of the reference curve. In this limit, the first such reference curve of relevance is a straight line. Restricting our discussion to this example, the average density is \( \langle \rho \rangle = k/(\sqrt{2} \pi) \). In terms of \( s = (t' - t) \langle \rho \rangle \), the normalized correlation function, \( R(s) = R/\langle \rho \rangle^2 - 1 \) exhibits the following behaviour [Aronovitch and Smilansky 2007]:
\[
R(s) \sim 1 + \frac{\pi^2}{16}s + \frac{37\pi^4}{2304}s^3 + \frac{\pi^4}{1296\sqrt{2}}s^4 + \ldots; s \to 0,
\]
\[
\sim \frac{1}{2\sqrt{2}\pi^2} \left[ 1 + 9 \sin(2\sqrt{2}\pi s) \right]; s \to \infty. \tag{191}
\]

Fig. 38 presents a comparison of these asymptotics along with the results expected from the three random matrix ensembles—the agreement is only on average.

Fig. 38 Density \( p(s) \) of the nearest-neighbour spacings \( s \) between the points where a reference curve intersects a network of nodal lines created by a random wave (RW), the normal derivative of a random wave (NRW), or the Wigner surmise of RMT. For small distances, \( p(s) \) grows linearly. On closer inspection, one can observe persistent oscillations relative to the mean decaying curve for the RW and the NRW. From [Aronovitch and Smilansky 2006].

Fig. 39 Normalized correlation: nodal intersections of RW versus RMT level spacings—GOE (\( \beta = 1 \)), GUE (\( \beta = 2 \)), and GSE (\( \beta = 4 \)). From [Aronovitch and Smilansky 2006].
When working with chaotic billiards such as the (desymmetrized) Sinai and stadium billiards, one can choose a reference curve inside the domain and inspect the nodal intersections with this curve. The near-perfect agreement of the spacing statistics of nodal intersections between RWM and chaotic billiards\(^{21}\) demonstrates the success of Berry’s conjecture. In fact, for random Gaussian toral eigenfunctions in two and three dimensions, the expected intersection number (against any smooth curve) is universally proportional to the length of the reference curve times the wavenumber, but independent of the geometry (Maffucci, 2016; Rudnick and Wigman, 2015).

4. Density of line shapes

One of the ways to find the distribution of the shapes of nodal lines is to compute the probability that a nodal line matches a given reference curve \(r(s)\) (parametrized by the arclength \(s\) in the plane) within a certain precision \(\varepsilon\) \cite{Foltin et al., 2004a}. Of interest is the integral of the square of the amplitude of a random function \(\psi(r)\),

\[
X = \frac{1}{2} \int ds \, \psi^2(r(s)); \quad (192)
\]

\(X\) is also a random variable itself. When \(\psi\) has a nodal line close to \(r(s)\), \(X\) is small. Foltin et al. (2004a) calculated the distribution of \(X\) and its cumulants for a circular reference curve, \(C\), for the random wave and short-range ensembles (corresponding to critical percolation). The hope was that the scaling properties of these moments as a function of the radius (size) of \(r(s)\) would detect the long-range correlations in \(\psi\). An approximate expression for the probability that a nodal line falls inside a tube of width \(\varepsilon\) is thus considered.

Assuming that \(\psi(r)\) has a nodal line close to \(C\), the normed distance \(\zeta(s)\) of the nodal line from the reference curve can be obtained by linearization

\[
\psi(r + \zeta n) \simeq \zeta \frac{\partial \psi(r)}{\partial n} + \psi(r) = 0; \quad \text{wherefore,} \quad \zeta = -\frac{\psi(r)}{\partial \psi(r)/\partial n}. \quad (193)
\]

\(\hat{n}(s)\) being the unit vector normal to \(C\). Finding the probability that a nodal line lies within a tube \(|\zeta(s)| < \vartheta\) by analytical means alone is difficult. We thus diffuse the boundary of the tube and consider instead the expectation value

\[
P_\varepsilon = \left\langle \exp \left( -\frac{1}{2\varepsilon} \int ds \, \zeta^2 \right) \right\rangle \quad (195)
\]

where the line integral is along \(C\) and \(\varepsilon \sim \vartheta^2\). However, we need to simplify even further as the numerator and denominator of \(\zeta\) are not independent, implying that \(\eta\) itself need not be Gaussian distributed. Thus, we consider an isotropic, mean-field expression for the shape probability which can be evaluated:

\[
P_\varepsilon = \left\langle \exp \left( -\frac{1}{\varepsilon} \int ds \frac{\psi^2}{\langle (\nabla \psi)^2 \rangle} \right) \right\rangle = \det \left( 1 + \frac{\hat{B}}{\varepsilon \langle (\nabla \psi)^2 \rangle / 2} \right)^{-1/2}. \quad (196)
\]

\(\hat{B}\) is an integral operator with symmetric kernel, which gives the correlation function of \(\psi(r(s))\),

\[
B(s, s') = \langle \psi(r(s)) \psi(r(s')) \rangle = G(|r(s) - r(s')|). \quad (197)
\]

Since the operator is positive semi-definite with a finite trace, \(\int ds \, B(s, s) = L\), its eigenvalues \(\beta_\mu \geq 0\) have an accumulation point at zero. Thus we arrive at the final form of the generating function,

\[
F(\varepsilon) \equiv \log P_\varepsilon = -\frac{1}{2} \sum_\mu \log \left( 1 + \frac{\beta_\mu}{\varepsilon \langle (\nabla \psi)^2 \rangle / 2} \right),
\]

\[
= \log \langle \exp(-X/\varepsilon) \rangle, \quad (198)
\]

which entails all the cumulants upon expansion in powers of \(\tilde{\varepsilon} = \varepsilon \langle (\nabla \psi)^2 \rangle / 2\).
It is well-known \cite{Berry1977, HorticarSrednicki1998} that for a random wave ensemble (RWE), the correlation function is \( G_{RW}(r) = \langle \psi(r) \psi(0) \rangle = J_0(kr) \) whereas for the short-range ensemble (SRE), it is \( \exp(-k^2 r^2/4) \) \cite{Foltinetal2004}. The probability \( \beta \) can now be calculated for a circle of radius \( R \) for both. The kernel of the integral operator corresponding to the correlation function \( G_{RW}(r) \) is \( B(\theta' - \theta) = J_0(2kr \sin(\theta - \theta'))/2 \) for two positions \( \theta, \theta' \) on the circle. The eigenfunctions of \( B \) are \( \exp(i m \theta) \); \( m = 0, \pm 1, \pm 2, \ldots \), with eigenvalues

\[
\beta_m = R \int_0^{2\pi} d\theta J_0(2kr \sin(\theta/2)) e^{i m \theta} = 2\pi R J_m^2(kR).
\]

This long-range character stands in contrast to the situation with SRE where the eigenvalues are

\[
\beta_m = 2\pi R e^{-k^2 r^2/2} I_m(k^2 R^2/2) \approx \frac{2\sqrt{\pi}}{k} e^{-m^2/(kR)^2}.
\]

The spectrum is seen to behave smoothly for SRE whereas there are strong fluctuations for RWE.

The generating function \( F(z) \) can now be expressed for different regions where \( m < kR \) or \( m \approx kR \). Let us scrutinize RWE first: the asymptotic expansions of the Bessel functions in both regimes can be combined into a single scaling law with a universal scaling function \( f(x) \) \cite{Foltinetal2004} as

\[
|J_m(kR)| \sim (kR)^{-1/3} f \left( \frac{m^2 - (kR)^2}{(kR)^{4/3}} \right).
\]

Thus, the eigenvalues of the operator scale as

\[
\beta_m = \frac{2\pi}{k} (kR)^{1/3} f \left( \frac{m^2 - (kR)^2}{(kR)^{4/3}} \right)^2.
\]

The leading behavior of the \( v \)th cumulant of \( X \), which is proportional to the trace of the \( v \)th power of \( B \), can be simplified as a function of \( kR \) (to first-order therein):

\[
\langle X^v \rangle_c \sim k^{1-v} \begin{cases} kR, & v < 2, \\ kR \log(kR), & v = 2, \\ (kR)^{(1+v)/3}, & v > 2. \end{cases}
\]

For the SRE, the scaling behavior is \cite{Foltinetal2004}

\[
\langle X^v \rangle_c \sim k^{-v} kR; \quad \forall \ v > 0.
\]

Clearly, below the critical exponent \( v = 2 \), the two results agree for large \( kR \). For \( v > 2 \), Eq. \eqref{eq:201} suggests a rather nontrivial scaling of the cumulants for the RWE. Numerical computations for the case of \( v = 3 \) also corroborate the scaling relation, with an exponent \( 4/3 \).

**VI. COUNTING NODAL DOMAINS**

A. Can one count the shape of a drum?

By now, we can all unanimously agree that the shape of a domain (the geometry of its boundary) is wedded to and determines its Laplacian eigenspectrum. One can turn the tables to pose the question in reverse: does the set of eigenvalues (or emitted frequencies, if we are talking about a vibrating drum) uniquely identify the domain? Or as \textit{Kac} \cite{Kac1966} put it more colorfully, “can one hear the shape of a drum?” A short answer is yes; a shorter answer is no.

The connection between isometry and isospectrality is a question that has both perplexed and fascinated physicists and mathematicians alike. Upon a moment’s thought, the nontriviality of the problem becomes apparent. Most definitely, certain geometrical and topological constants associated with the domain, say, \( D \), can indeed be drawn from the spectrum. For instance, \textit{Kac} \cite{Kac1966} himself conjectured the asymptotic relation for the heat trace, also called the spectral function,

\[
H(t) = \text{Tr} (e^{t \Delta_D}) = \sum_{j=1}^{\infty} e^{-\lambda_j t}
\]

\[
\sim \frac{A}{4\pi t} - \frac{\mathcal{P}}{8\sqrt{\pi} t} + \frac{1}{6} (1 - h) \quad \text{as } t \to 0,
\]

where \( h \) is the number of holes (genus) in \( D \). The heat trace is thus a spectral invariant. The first term in Eq. \eqref{eq:200} is essentially Weyl’s law \cite{Vanaletal2005} whereas the second and third components follows from the results of \textit{Pleijel} \cite{Pleijel1954} and McKean and Singer \cite{McKeanSinger1967}, respectively. Similarly, Van den Berg and Srisatkumarajah \cite{Srisatkumarajah1988} found that for any polygon in \( \mathbb{R}^2 \) with angles \( \alpha_i \), the heat trace can be written as

\[
\sum_{j=1}^{\infty} e^{-\lambda_j t} \sim \frac{A}{4\pi t} - \frac{\mathcal{P}}{8\sqrt{\pi} t} + \frac{1}{24} \sum_i \left( \pi \frac{\alpha_i}{\alpha_i - \pi} \right) + O \left( e^{-\frac{t}{2}} \right).
\]

However, besides information about \( \mathcal{P} \) and \( A \), both Eq. \eqref{eq:35} and Eq. \eqref{eq:189} are silent about retrieving the shape itself. Moreover, unlike with isoperimetric inequalities, here, we are grappling with the whole spectrum rather than individual eigenvalues, which renders the problem fundamentally different from any we have encountered up to this point.

Without further ado, let us illustrate why, formally, the answer to Kac’s question is in the negative. After a string of early counterexamples \cite{Ikeda1980, Milnor1964, Protter1987, Urakawa1982, Vigneras1980} in \( \mathbb{R}^n, n \geq 4 \), the fate of the conjecture for planar domains was finally sealed by the discovery of a pair of two-dimensional polygonal billiards \cite{Gordonetal1992a,b} having exactly the same eigenspectrum (Fig. \ref{fig:41}), for both Dirichlet and Neumann boundary conditions. This
seemingly serendipitous finding was actually based on Sunada's (1985) systematic “paper-folding” construction. Many other instances of isospectral domains were soon provided by Buser et al. (1994); Chapman (1995), including experimental realizations by Sridhar and Kudrolli (1994) (see also Cipra (1992)). Nonetheless, only 17 families of examples that say no to Kac’s question were found in a 40-year period (Giraud and Thas, 2010). All these specimens, however, were either disjoint or non-convex. For \( d \geq 4 \), Gordon and Webb (1994) put this objection to rest by showing the existence of isospectral convex connected domains but the question remains open for billiards in \( d = 2 \).

![FIG. 41 Two domains that are isospectral but not isometric in the sense of geometric congruence. The 19th (top) and 20th (bottom) eigenfunctions are shown. Isospectrality can be proved by the transplantation technique (Bérard 1992). Note that the isospectral billiards have the same area and perimeter. Figure generated from code by Moler (2012).](image)

At the very onset, we had, somewhat tantalizingly, hinted that Kac’s question might also have an affirmative answer. This is indeed true for certain classes of domains. Zelditch (2000) proved that for domains possessing elliptical symmetry, the spectrum of the Dirichlet Laplacian uniquely determines the region, subject to the satisfaction of some generic conditions on the boundary. Analogous results hold for real analytic, planar domains with only one symmetry (Zelditch 2004, 2009). Moreover, for rectangular or triangular billiards, a finite number of eigenvalues suffice to completely specify the shape (Chang and DeTurck 1989). In fact, two isospectral triangles must necessarily be isometric (Grieser and Maronna 2013). The prevailing belief is that a smooth boundary on the domain \( D \) is a sufficient condition for a positive answer to Kac’s question; unfortunately this is yet to be proved.

Even if, disappointingly enough, one cannot “hear” the shape of a drum, one can still “count” the shape to address the question of isospectrality. This scheme was first developed by Gnutzmann et al. (2006, 2005) who provided a combination of heuristic arguments and numerical simulations to support the proposition that sequences of nodal counts store information on the geometry (metric) of the domain where the wave equation is considered. To put it explicitly, the information contained in the nodal count sequence is different from that borne by the spectral sequence. Thus the spectral ambiguity—and its associations with isometry—can be resolved by comparing nodal sequences, as we discuss in the following subsection.

B. Trace formula approach

The nodal count sequence for separable Laplacians can be described by a semiclassical trace formula (Gnutzmann et al. 2006, 2007, 2005). This is analogous to the Gutzwiller spectral trace formula, which relates the quantum density of states, \( g(E) \), to its semiclassical counterpart, \( g_{sc} \),

\[
g(E) = \sum_{j=0}^{\infty} \delta(E-E_j) \leftrightarrow g_{sc}(E) = g_0(E) + \sum_{\alpha \in po} A_{\alpha} e^{ikL_\alpha},
\]

as a sum over all the periodic orbits (of length \( L_\alpha \)) of the corresponding classical system. Likewise, the trace formula for the nodal counts can be expressed as sums over closed ray trajectories on the manifold where each term bears geometric information about the orbit (Aronovitch et al. 2012). The approach is particularly simple for flat tori (in \( \mathbb{R}^2 \) and \( \mathbb{R}^4 \)) and surfaces of revolution.

1. Flat tori

The simple 2-dimensional torus can be represented as a rectangular billiard in \( \mathbb{R}^2 \) with length \( a \) and breadth \( b = a/b \notin \mathbb{Q} \) with periodic boundary conditions. The number of nodal domains is, trivially,

\[
\nu_{m,n} = (2|m| + \delta_{m,0})(2|n| + \delta_{n,0}). \tag{205}
\]

The cumulative density of energy levels affords an asymptotic expression in terms of classical periodic orbits, obtained from saddle-point approximations of all oscillatory integrals:

\[
\mathcal{N}(E) = \sum_{m,n = -\infty}^{\infty} \Theta \left( E - E_{m,n} \right) \tag{206}
\]

\[
= AE + \sqrt{\frac{8}{\pi}} A E^{1/4} \sum_{po} \sin(L_{po} \sqrt{E - \pi/4}) L_{po} \frac{\sqrt{E - \pi/4}}{L_{po}} + O(E^{-3/4}), \tag{207}
\]
where \( A = ab/4\pi \) and \( L_{po} = \sqrt{(Na)^2 + (Mb)^2} \); \( N, M \in \mathbb{Z} \), is the length of a periodic orbit. Similarly, the cumulative nodal count

\[
C(K) = \sum_{j=1}^{[K]} v_j, \quad \text{for } K > 0, \tag{208}
\]

after modification to set a unique order within the degenerate states, can be brought to the form

\[
c(E) = \sum_{j=1}^{\infty} v_j \Theta(E - E_j)
= \frac{2A^2}{\pi^2} E^2 + \frac{211/2}{\pi^{1/2}} A^{5/4} \sum_{po} \frac{|MN|}{L_{po}^{7/2}} \sin \left( \frac{L_{po} \sqrt{E - \frac{\pi}{4}}}{A} \right) + \mathcal{O}(E). \tag{209}
\]

The next step is to invert \( \mathcal{N}(E) = K \) with \( \mathcal{N}(E) \) from Eq. (206), thereby yielding, to leading order,

\[
E(K) = \frac{K}{A} - K^{1/4} \frac{2^{3/2}}{\mathcal{A} A^{3/2}} \sum_{po} \sin \left( \frac{L_{po} \sqrt{K - \frac{\pi}{4}}}{\sqrt{A}} \right) \frac{M^2}{|L_{po}^{3/2} / (|A|^{3/2}} + \mathcal{O}(K). \tag{210}
\]

which we can now substitute in Eq. (209). Thus, the cumulative counting function \( c(K) \equiv \tilde{c}(E(K)) \) is formulated as the sum of an average part, \( c_{av}(K) \), and an oscillating part, \( c_{osc}(K) \), where

\[
c_{av}(K) = \frac{2}{\pi^2} K^2 + \mathcal{O}(K),
\]

\[
c_{osc}(K) = \sum_{po} \frac{1}{L_{po}^{3/2}} \left( \frac{4\pi^2 |MN|}{L_{po}^3 / A} - 1 \right) \sin \left( \frac{L_{po} \sqrt{K - \frac{\pi}{4}}}{A} \right) \times K^{5/4} \sqrt{\frac{27A^{3/2}}{\pi^5}} + \mathcal{O}(K). \tag{211}
\]

The oscillatory part of \( c(K) \) is especially interesting as its Fourier transform with respect to \( \sqrt{K} \) entails the length spectrum of the periodic orbits in contrast to the smooth part, which is independent of the geometry of the torus. In fact, \( c_{osc}(K) \) explicitly depends on the aspect ratio \( \tau = a/b \), through \( L_{po} \), and can therefore resolve different geometries. A kindred calculation in \( \mathbb{R}^4 \) insinuates that “one can count the shape of a drum (if it is designed as a flat torus in four dimensions).”

2. Surfaces of revolution

A surface of revolution \( \mathcal{M} \) is the collection of points traced out on rotating a curve \( y = f(x) \) for \( x \in [-1, 1] \) about the \( x \)-axis. We concern ourselves only with surfaces which are smooth and convex and, in particular, focus on mild deformations of ellipsoids of revolution, in essence, imposing that \( \mathcal{M} \) has no boundary and \( f(x) \) possesses a single maximum at \( x_{\text{max}} \).

The Euclidean metric in \( \mathbb{R}^3 \) induces a metric on the surface

\[
d s^2 = \left[ 1 + \left( \frac{\partial f^2}{\partial x^2} \right) \right] dx^2 + [f(x)]^2 d\theta^2; \tag{212}
\]

\( \theta \) is the azimuthal angle. The wave equation on the surface of revolution is simply \( \nabla^2_M + E \psi(x, \theta) = 0 \) where

\[
\nabla^2_M = \frac{1}{f(x) \sigma(x) \sigma(x)} \frac{\partial}{\partial x} \left( \frac{1}{f(x)^2} \frac{\partial}{\partial \theta} \right)
\]

with \( \sigma(x) = \sqrt{1 + (\partial f / \partial x)^2} \). The wave equation is separable with solutions

\[
\psi(x, \theta) = \begin{cases} 
\cos(m \theta) \phi_m(x), & m \geq 0, \\
\sin(m \theta) \phi_m(x), & m < 0,
\end{cases} \tag{214}
\]

in which \( \phi_m(x) \) satisfies the Sturm-Liouville equation:

\[
-\frac{1}{f(x) \sigma(x)} \frac{d}{dx} \left( f(x) \frac{d \phi_m(x)}{dx} \right) + \frac{m^2}{f(x)^2} \phi_m(x) = E \phi_m(x). \tag{215}
\]

For each \( m \), one has a sequence of solutions \( \phi_{n,m} \), \( n = 0, 1, 2, \ldots, \) with eigenvalues \( E_{n,m} \). The nodal domains, of which there are

\[
\nu_{n,m} = (n + 1) (2|m| + \delta_{m,0}), \tag{215}
\]

are arranged in a checkerboard pattern. A semiclassical treatment leads to an asymptotic expression (large-wavenumber expansion) for the cumulative density of energy levels in terms of the classical periodic orbits formed by a rational ratio of windings \( (M, N) \) along the \( (\theta, x) \) directions (Gnutzmann et al. 2007):

\[
\mathcal{N}(E) = \mathcal{A} E + E^{1/4} \sum_{(M,N)\neq(0,0)} \mathcal{N}_{M,N}(E); \tag{216}
\]

note that the area of the surface is now \( 4\pi \mathcal{A} \). In the stationary phase approximation (Bleher 1994)

\[
\mathcal{N}_{M,N}(E) = (-1)^N \frac{\sin(L_{M,N} \sqrt{E} + \sigma \pi / 4)}{2\pi [N^3 n_{M,N}''(m=m_{M,N})]^{1/2}} + \mathcal{O}(E^{-1/2}). \tag{217}
\]

Here, \( \sigma = \text{sgn}(n_{M,N}'') \) and the function \( n(m) \equiv n(E = 1, m) \) is itself defined by the action variable

\[
n(E, m) = \frac{1}{2\pi} \int p_{x,E}(x) \, dx. \tag{218}
\]

Formally, for Eq. (217) to remain convergent, \( n_{M,N} \) should follow the twist condition, \( n_{M,N}'' \neq 0 \) for \( 0 < m \leq f_{\text{max}} \). Inverting \( \mathcal{N}(E) = K \) now gives

\[
E(K) = \frac{K}{A} - \left( \frac{K}{A} \right)^{1/4} \sum_{M,N} \mathcal{N}_{M,N}(K/A) A + \mathcal{O}(1). \tag{219}
\]
Accordingly, the cumulative nodal count is
\[
c(K) = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \nu_{m,n} \Theta [E(K) - E_{m,n}]
\]
\[
= \overline{c}(K) + c_{\text{osc}}(K)
\]
with
\[
\overline{c}(K) = 2 \frac{m \overline{n}}{A} K^2 + \frac{m}{\sqrt{A}} K^{3/2} + \mathcal{O}(K),
\]
\[
c_{\text{osc}}(K) = K^{-5/4} \sum_{\text{po}} a_{\text{po}} \sin \left( \frac{L_{\text{po}}}{A} \sqrt{\frac{K}{A}} + \sigma \pi \right) + \mathcal{O}(K),
\]
where the amplitudes are given by
\[
a_{\text{po}} = (-1)^N \frac{m_{M,N} n_{m,M,N} - 2 \overline{m} \overline{n}}{\pi A^{5/4} |N^3 n_{M,N}^{\prime}|^{1/2}}
\]
and the sum runs over periodic geodesics respecting \((-M/N) \in \text{Range}(a'(m))\), corresponding to the classically accessible domain. The overbars are evaluated as
\[
\overline{m} \overline{n} = \left\{ \frac{1}{A} \int_{E_{m,n} < 0} d\,m\,d\,n \, |m|^p \, n^q \right\}.
\]

That there exists a relation between the nodal count and the periodic geodesics is nothing short of remarkable, even more so given that the cumulative nodal count does not bear any spectral information except the ordering it inherits from the spectrum \([\text{Gnutzmann et al., 2007}]\). Fig. 42 presents the length spectra of periodic geodesics for some examples of flat tori and surfaces of revolution. The length spectrum \(S(l)\) is extracted from the Fourier transform of \(c_{\text{osc}}(K)\) with respect to \(\kappa = \sqrt{K}\).

3. Periodic orbits of nonseparable, integrable billiards

It is perhaps worth mentioning that the manifestations of periodic orbits in nodal data were also observed for the right-angled isosceles \([\text{Aronovitch et al., 2012}]\) and equilateral \([\text{Samajdar and Jain, 2014b}]\) triangles, which although not separable are at least integrable. For these billiards, the power spectrum of the cumulative count of nodal loops fluctuates prominent peaks at the lengths of the periodic orbits (Fig. [43]), suggesting forthright the possible existence of a comparable (but yet undetermined) trace formula.

a. Inverse nodal problems: Shape analysis The trace formula approach, although physically appealing, is a little too roundabout when it comes to actually determining the geometry of a billiard. However, there is a more direct way to do so, which is tied to the general inverse nodal problem of determining the metric of a Riemannian manifold using its nodal sequence. Formally, a nodal sequence is defined to be \(S = \{\nu_1, \nu_2, \ldots\}\), arranged in increasing order of \(\lambda_j\). To convey the main idea, we concentrate on rectangular billiards \([0, a] \times [0, b]\) with Dirichlet boundary conditions (in short, Dirichlet rectangles), which are parametrized by the aspect ratio \(\tau = 1\) such that \(\tau b = a = 1\). \([\text{Klawonn, 2009}]\) proved that the parameter \(\tau\) is uniquely determined by the nodal sequence \(S\). Similar results hold for flat tori and Klein bottles, which too can be correspondingly parametrized. Let us now unequivocally outline the methodology for Dirichlet rectangles, \(R_{\tau}\), following closely \([\text{Klawonn, 2009}]\). We need to prove that \(\tau \neq \tau'\) implies that \(S(R_{\tau}) \neq S(R_{\tau'})\) by constructing a sequence limiting to \(\tau\). We denote by \(P\) the set of primes and choose \(p \in P \cup \{1\}\). For the spectrum of \(R_{\tau}\)s, there are exactly two positions \(i, j (i < j)\) with \(p \in \nu(\lambda_i), \nu(\lambda_j)\); note that \(i\) stands for the pair of quantum numbers \((m, n)\). Then, it is easy to see that
\[
\lambda_i = \lambda_{i_p} = 1^2 + \nu^2 p^2 \leq \nu^2 + \tau^2 = \lambda_{p1} = \lambda_j.
\]
In general, let us consider a natural number \(N\) with prime decomposition, \(N = p_1 p_2 \ldots p_k; p_i < p_{i+1}, p_i \in P\). For any such decomposition, one can rewrite \(N\) using the permutation \(\pi\{1, 2, \ldots, k\} = \{i_1, i_2, \ldots, i_k\}\) as the product \(p_{i_1} \cdots p_{i_k} p_{i_{k+1}} \cdots p_{i_k}\) with \(1 < l < k\) such that
\[
N^2 + \tau^2 > (p_{i_1} \cdots p_{i_k})^2 + \tau^2 (p_{i_{k+1}} \cdots p_{i_k})^2.
\]
Thus, we can detect the position of every eigenvalue of the forms $N^{2} + \tau^{2}$ for $N \in \mathbb{N}$ and $i^2 + \tau^2 p^2$ for $p \in \mathbb{P} \cup \{1\}$. To extract $\tau$ from this data, we construct two sequences, for every $h \in \mathbb{P} \cup \{1\}$,

$$h_{+} = \min \{ i \in \mathbb{N} : 1 + h^{2} \tau^{2} < i^{2} + \tau^{2} \},$$

$$h_{-} = \max \{ i \in \mathbb{N} : i^{2} + \tau^{2} < 1 + h^{2} \tau^{2} \}, \quad (226)$$

and use the more dense sequence to put bounds on the other. The inequality

$$H^h_{+} = \frac{h^2 - 1}{h^2 - 1} < \tau^2 < \frac{h^2 - 1}{h^2 - 1} = H^h_{+}, \quad \forall \ h \in \mathbb{P} \quad (227)$$

is easily verified. Finally, Klawonn (2009) illustrates that $\lim_{h \to \infty} H^h_{+} - H^h_{-} \to 0$, which implies $H^h_{+} \to \tau^2$, thereby completing the proof.

The Laplace-Beltrami nodal counts have also been advertised to provide a new signature for 3D shape analysis. Specifically, it was used by Lai et al. (2009) to experimentally resolve ambiguities left unaddressed by the “shape DNA” (the distribution of eigenvalues). Their method was based on a distance function defined between nodal count sequences $S$ and $S'$:

$$D(S, \tilde{S}) = \frac{1}{\alpha} \sqrt{\sum_{n=1}^{\infty} \left(\frac{1}{n^\alpha}\right)^2 [\tilde{N}(\lambda_n) - \tilde{N}(\lambda_n)]^2}, \quad \alpha > 0. \quad (228)$$

This spectral distance function meets all the conditions of a metric and, hence, can be reliably used in shape analysis (Lai et al., 2010). Pairwise distances, as used in Eq. (228), are stored in a distance matrix. Further details concerning the usage of this matrix and the multi-dimensional scaling technique to embed the surfaces into Euclidean space can be found in (Lai et al., 2009). Recently, this method has been paired with landmark-based morphometric studies (Shi et al., 2017) using the neuroimaging data from works on Alzheimer’s disease (Wang and Wang, 2017).

### C. Graph-theoretic analysis

The wavefunction for a right-angled isosceles triangle, Eq. (133), is easily decomposed into the difference of $\psi_{m,n}^{(1)} = \sin(mx) \sin(ny)$ and $\psi_{m,n}^{(2)} = \sin(nx) \sin(my)$, which, individually, possess a lattice structure (the length of the legs have been assumed to be $\pi$). Let us denote the nodal sets for the two functions by $\mathcal{N}^{(1)}_{m,n}$ and $\mathcal{N}^{(2)}_{m,n}$. Their intersection, $\mathcal{N}^{(1)}_{m,n} \cap \mathcal{N}^{(2)}_{m,n}$, is the set of points

$$V_{m,n} = \left\{ \left\lfloor \frac{m}{n} (i,j) \right\rfloor 0 < j < i < m \right\} \cup \left\{ \left\lfloor \frac{n}{m} (i,j) \right\rfloor 0 < j < i < n \right\}$$

Since $\psi_{m,n}$ necessarily vanishes at these points, the nodal lines pass through them. The union $\mathcal{N}^{(1)}_{m,n} \cup \mathcal{N}^{(2)}_{m,n}$ divides the triangle $D$ into cells. Aronovitch et al. (2012) translated this to a pictorial representation by constructing a graph $G_{m,n}$ with vertices $V_{m,n}$ and an additional vertex $v_0$ for the boundary of the triangle. The edges of the graph stand for the nodal lines, connecting the vertices. The graph $G_{m,n}$ is endowed, according to certain specified rules, with one, two, or three edge(s) for each (shaded) cell that a nodal line runs over. The number of vertices in a cell determines its connectivity. Once $G_{m,n}$ has been constructed, the number of nodal domains can be counted by the Euler formula for planar graphs:

$$\nu_{m,n} = 1 + E(G_{m,n}) - |V_{m,n}| + c(G_{m,n}), \quad (229)$$

where $E(G_{m,n})$, $|V_{m,n}|$, and $c(G_{m,n})$ denote the number of edges, vertices, and connected components of $G_{m,n}$, respectively. An example of such a construction is to be seen in Fig. 44.

The nodal domain count can be related to the number of intersections of the nodal lines with the boundary, $\eta_{m,n}$; and the number of nodal loops, $I_{mn}$. For instance, in Fig. 44 $\eta_{9,4} = 10$ and $I_{9,4} = 4$. In the generic-
where (230) \( \eta_{m,n} = m + n - 3 \).

Extensive analysis of the data on the number of domains, obtained numerically using the graph-theoretic algorithm, led Aronovitch et al. (2012) to propose that the loop count is given by

\[
I_{m,n} = \tilde{I}(n, (m - n - 1)/2, 0)
\]

where

\[
\tilde{I}(n, k, l) = \begin{cases} 
0; & n = 1 \text{ or } k = 0, \\
\frac{n}{2k + 1} (l k + (2l + 1)k^2) + \tilde{I}(n \mod (2k + 1), k, l); & 2k + 1 < n, \\
\frac{1}{2} \left\lfloor \frac{k}{n} \right\rfloor (2l + 1)(n^2 - n) + \tilde{I}(n, k \mod n, l); & 2k + 1 > 2n, \\
(l + 1/2) \left\lfloor \frac{2k^2 + n^2 - n - 2nk + k}{2} \right\rfloor + \tilde{I}(2k - n + 1, n - k - 1, l + 1); & n < (2k + 1) < 2n.
\end{cases}
\]

This recursive relation was used, subsequent to verification of its predictions for the first 100,000 non-tiling loop counts, to produce the \( P(\xi) \) distribution of Fig. 27.

### D. Difference-equation formalism

For integrable billiards, such as the right-isosceles triangle above, Samajdar and Jain (2014) put forth an alternative method to count the number of domains. Firstly, the eigenfunctions of all integrable billiards can be classified by an index uniquely specified by the quantum numbers. Within each congruence class, the wavefunctions display exceptional structural similarities and their nodal counts obey certain difference equations. A trivial example is the rectangular billiard; its eigenfunctions are classified by \( k = m \mod n \) and we can set up the relation \( \nu_{m+n,n} - \nu_{m,n} = n^2 \). Fortunately, the equations for other billiards too turn out to have constant coefficients and can be solved readily given a “boundary condition”, namely, the number of domains for a low-lying state (which can be counted manually if need be). The solutions thereto explicitly yield the number of nodal domains for integrable (and importantly, nonseparable) billiards, thus partially solving an otherwise intractable problem. For completeness, we list the analytically-known difference equations below.

1. Circles (and annuli/sectors thereof)

Let \( \Delta_{a,b} = \nu_{a+b,b} - \nu_{a,b} \). With this notation, the difference equation for the circle (or a circular annulus) is simply \( \Delta_{m,n} = 2n^2 \) for \( m \neq 0 \), where \( n \) and \( m \) are radial and angular quantum numbers, respectively (Manjunath et al., 2010); this is solved by \( \nu_{m,n} = 2m \). When \( m = 0 \), the number of domains is just \( n \). For a sector, the domain \( D \) is restricted in radial and angular variables and, like for the rectangular billiard, we find \( \nu_{m+n,n} - \nu_{m,n} = n^2 \), leading to \( \nu_{m,n} = mn \).

2. Ellipses and elliptic annuli

The Helmholtz equation for an elliptical billiard separates into Mathieu equations for the radial and angular components. They can be categorized into symmetry classes by noting whether they are symmetric (+) or antisymmetric (−) with respect to the \( x \)- and \( y \)-axes. So, a state designated as \((++)\) is (anti)symmetric about the \( x \) (\( y \)) axis. The difference equations are:

\[
\Delta_{r,l} = \begin{cases} 
4l^2; & (++), \\
4l^2 + 2l; & (++) \\
4l^2 + 2l; & (+-) \\
4l^2 + 4l; & (-) \\
\end{cases}, \quad \nu_{r,l} = \begin{cases} 
2l(2r + 1) + 1; & (++) \\
2(2l + 1)(r + 1); & (++) \\
(2l + 1)(2r + 1) + 1; & (+-) \\
4l + 1)(r + 1); & (-)
\end{cases}
\]

where \( l \) (\( r \)) is the angular (radial) quantum number. For elliptic annuli, the difference equations and number of domains are almost identical:

\[
\Delta_{r,l} = \begin{cases} 
4l^2; & (++), \\
4l^2 + 8l; & (++) \\
4l^2 + 8l; & (+-) \\
4l^2 + 4l; & (-) \\
\end{cases}, \quad \nu_{r,l} = \begin{cases} 
4l(r + l); & (++) \\
2(2l + 1)(r + 1); & (++) \\
2(2l + 1)(r + 1); & (+-) \\
4(r + l); & (-)
\end{cases}
\]
3. Confocal parabolae

The detailed solutions of the confocal parabolic billiard can be found in [Manjunath et al. 2016]. The Helmholtz equation separates in parabolic coordinates, \((\tau, \sigma)\), related to \((x, y)\) by \(x = \tau \sigma\) and \(y = (\tau^2 - \sigma^2)/2\) with \(\sigma \geq 0\). The wavefunctions \(\psi(\tau, \sigma)\) can be written as a product \(S(\sigma)T(\tau)\). For Dirichlet boundary conditions, the functions \(T\) and \(S\) ought to be either both even or both odd. In the former case,

\[
\Delta_{m,n} = 2n^2 - n, \quad \text{and} \quad \nu_{m,n} = 2 mn - m - n + 1. \tag{233}
\]

When \(S\) and \(T\) are odd, \(\Delta_{m,n} = 2n^2\) and \(\nu_{m,n} = 2 mn\). The parabolic annuli and sectors can be analyzed along the same lines as previously. For all the separable billiards above, the difference equations are but restatements of the exactly attainable nodal counts, which might make the utility of the approach seem dubious. The nonseparable billiards below, hopefully, assuage such fears.

4. Right-angled isosceles triangle

The eigenfunctions of the right-angled isosceles triangle are classified by the index \(C_{2n} = m \mod 2n\). It is sufficient to consider the general non-tiling case as the number of domains with tiling can always be recovered on knowing that for each tile. From extensive numerical analysis, [Samajdar and Jain 2014a] found that a simple set of equations hold for the number of nodal domains and loops:

\[
\Delta_{m,2n} = \nu_{m+2n,n} - \nu_{m,n} = \frac{n(n+1)}{2}, \quad I_{m+2n,n} - I_{m,n} = \frac{n(n-1)}{2}. \tag{234}
\]

These equations can be retroactively motivated by successively inserting \(\nu_{m+2n,n}\) and \(\nu_{m,n}\) into Eq. (54), which thus predicts \(\Delta_{m,2n} \propto n^2 + n\). Let \(\zeta_1 = n \mod C_{2n}\) and \(\zeta_2 = n \mod 2C_{2n}\). Taking advantage of Eqs. (232) and (234), we arrive at an exact solution for the number of domains for even \(C_{2n}:

\[
\nu_{m,n} = \frac{m(n+1) + n - 2}{4} + \left[ -\frac{n^2}{4} + \left(\frac{C_{2n}}{2}\right) \right] n - \left(\frac{C_{2n} - C_{2n} - 1}{2} \pm \frac{1}{4} (\zeta_2 - 1) \right), \tag{235}
\]

with the \((-\) sign applicable when \(C_{2n} < \zeta_2\) (otherwise).

For odd \(C_{2n}:

\[
\nu_{m,n} = \frac{m(n+1) + n - 2}{4} + \left[ -\frac{n^2}{4} + \left(\frac{C_{2n}}{2}\right) \right] n - \left(\frac{2C_{2n} - C_{2n} - 2}{4} + \gamma \right). \tag{236}
\]

The precise form of \(\gamma\) is uncertain, but asymptotically \(\lim_{k \to \infty}(\nu_{m+k,n,n}) = 0\). Since the fluctuations die out as \(E \to \infty\), Eq. (236) lends itself to studying the limiting distributions without problem.

5. Equilateral triangle

The eigenfunctions (136) enable classification by \(C_{3n} = m \mod 3n\) [Samajdar and Jain 2014b] as Table II explicates. Restricting ourselves to non-tiling patterns, the number of domains and loops satisfy the equations:

\[
\nu_{m+6n,n} - 2\nu_{m+3n,n} + \nu_{m,n} = 3n^2,
\]

\[
I_{m+6n,n} - 2I_{m+3n,n} + I_{m,n} = 3n^2. \tag{237}
\]

The solution to Eq. (257), along with detailed numerical observations, lead to the following formulæ for the nodal domain counts:

\[
\nu_{m,n} = \frac{m^2}{6} - \frac{(4n-3)m}{6} + n^2 - \frac{C_{3n}n - \lambda_1(C_{3n},n)}{3} \tag{0 < C_{3n} < n),
\]

\[
\nu_{m,n} = \frac{m^2}{6} - \frac{(4n-3)m}{6} + n^2 - \frac{2(C_{3n} - n)n - \lambda_2(C_{3n},n)}{3} \tag{n < C_{3n} < 3n). \tag{238}
\]

\(\lambda_1\) and \(\lambda_2\) are parameters that contribute to small variations in the count. Their explicit forms are not known.

| \(m\) | \(n\) | \(m \mod 3n\) | \(\nu_{m,n}\) | \(\Delta\nu_{m,n}\) | \(\Delta^2\nu_{m,n}\) |
|---|---|---|---|---|---|
| 7 | 2 | 1 | 6 | – | – |
| 13 | 2 | 1 | 21 | 15 | – |
| 19 | 2 | 1 | 48 | 27 | 12 |
| 25 | 2 | 1 | 87 | 39 | 12 |
| 31 | 2 | 1 | 138 | 51 | 12 |
| 37 | 2 | 1 | 201 | 63 | 12 |
| 9 | 2 | 3 | 10 | – | – |
| 15 | 2 | 3 | 29 | 19 | – |
| 21 | 2 | 3 | 60 | 31 | 12 |
| 27 | 2 | 3 | 103 | 43 | 12 |
| 33 | 2 | 3 | 158 | 55 | 12 |
| 39 | 2 | 3 | 225 | 67 | 12 |

TABLE II  The second difference of the number of nodal domains for an equilateral-triangular billiard is seen to remain constant if one considers a sequence of wavefunctions differing in \(m\) by steps of \(3n\), for a fixed value of \(n\). This defines an equivalence relation, and correspondingly, congruence classes, indexed by \(k = m \mod 3n\). The second difference \(\Delta^2\nu_{m,n} = \nu_{m+6n,n} - 2\nu_{m+3n,n} + \nu_{m,n}\) is \(12 = 3n^2\) for all the eigenfunctions tabulated.
analytically although a number of relations satisfied by
them are (Samajdar and Jain 2014a,b).

For the closely related hemi-equilateral-triangular bili-
diard (a $30^\circ - 60^\circ - 90^\circ$ scalene triangle), the difference
equation turns out to be

$$\nu_{m+6n,n} - 2\nu_{m+3n,n} + \nu_{m,n} = 0,$$

(239)
i.e., it is the first rather than the second difference that
remains constant. This brings to a close our compila-
tion of the difference equations and domain counts for
all planar integrable billiards—both separable and non-
separable. The main advantage of this formalism is that
it empowers one to determine $\nu_{m,n}$ for a whole hierar-
chy of states by starting with the domain count for a
simpler wavefunction, in the same congruence class, and
ascending the ladder (Mandwal and Jain 2017).

E. Counting with Potts spins

Exact counting of the nodal domains of a chaotic bili-
diard is still an outstanding open problem. Within the
random wave model, certain correlation functions can be
computed, as we have already seen. Here, we discuss
a method for calculating the moments of the number
of nodal domains with the help of auxiliary Potts spins
(Foltin 2003a) that sheds light on the percolative struc-
ture of the wavefunction. Suppose the nodal pattern is
placed on top of a square lattice of side $a \ll 2\pi/k$. To
each node $r_i$, we assign the variable $\sigma_i = \text{sgn}(\psi(r_i))$ and
a Potts spin $s_i$ (Baxter 2007). The spins can take val-
ues $s_i = 1, 2, \ldots, q$ but are constrained to have the same
value if they belong to the same nodal domain. Prac-
tically, this constraint is implemented as follows. The
product over adjacent lattice bonds $(ij)$ given by

$$\mathcal{N}(s_i) = \prod_{(ij)} \left(1 - \frac{\sigma_i \sigma_j}{2}\delta_{s_i,s_j} + \frac{1 + \sigma_i \sigma_j}{2}\delta_{s_i,s_j}\right)$$

$$= \prod_{(ij)} \left(1 - \frac{1 + \sigma_i \sigma_j}{2}(1 - \delta_{s_i,s_j})\right),$$

(240)
is one if and only if $\sigma_i$ and $\sigma_j$ are different, or if $\sigma_i = \sigma_j$
and $s_i = s_j$. Summing over all possible spin configura-
tions, we get (Baxter 2007)

$$Z(\{\sigma_i\}) = \sum_{s_i} \mathcal{N}(s_i) = q^C,$$

(241)

where $C$ is the number of nodal domains. To find the
mean number of domains and higher moments, it is ex-
pedient to construct a partition function. This is readily
defined by averaging over the Gaussian random fields $\psi$
with the correlation function from Eq. (86)

$$Z = \sum_{\{s_i\}} \left\langle \exp \left( -\beta \sum_{(ij)} \frac{1 + \sigma_i \sigma_j}{2}(1 - \delta_{s_i,s_j}) \right) \right\rangle_\psi$$

$$\to \langle q^C \rangle_\psi, \quad \text{as } \beta \to \infty.$$

(242)
The mean number of nodal domains is then

$$\langle C \rangle_\psi = \frac{\partial Z}{\partial q}_{|q=1}$$

(243)

for $\beta \to \infty$. As we have endowed each node with a spin, we can also
formulate an order parameter $o_i = \delta_{s_i,1} - 1/q$. The cor-
relation function of $o$ is

$$G_{k,l} = \sum_{\{s_i\}} \delta_{o_k,o_l} \prod_{(ij)} \left(1 - \frac{1 + \sigma_i \sigma_j}{2}(1 - \delta_{s_i,s_j})\right),$$

and since spins on different domains are independent,

$$G_{k,l} = \begin{cases}
0, & \text{if } k, l \text{ are disconnected,} \\
q^C - q^{C-1}, & \text{if } k, l \text{ are connected.}
\end{cases}$$

(244)
The partial derivative of $\langle G_{k,l} \rangle_\psi$ with respect to $q$ gives
the probability that the nodes $k$ and $l$ are connected.

VII. EXPERIMENTAL REALIZATIONS

For a monograph on a subject that traces its roots to
purely experimental origins, it is almost heretical to have
a colloquy as unabashedly theoretical as ours thus far. In
this section, we make amends and correct course by delv-
ing into some of the pioneering studies on nodal patterns
in the laboratory. As Stockmann (2006) cedes, prior to
the onset of the 1990s, experiments on the quantum me-
chanics of chaotic systems were few and far between. The
initial impetus was borrowed from analyses of nuclear
spectra (Porter 1965), which, more than half a century
later, still continues to reveal new aspects of quantum
chaos (Dietz et al. 2017). Contemporarily, intriguing in-
vestigations were underway on hydrogen atoms in strong
microwave (Bayfield and Koch 1974) and magnetic fields
(Holle et al. 1986; Main et al. 1986). A completely new
direction was lent to the subject by Stockmann and Stein
(1990) with their experiments on irregularly shaped mi-
crowave cavities—the microwave billiards, which we turn
to discussing first. In Sec. VII D immediately thereafter,
we also detail a few experiments probing the statistics of
cavities.

A. Microwave billiards: The physicist’s pool table

Playing billiards with microwaves is made feasible by
the mathematical analogy between electromagnetism and
quantum mechanics in that both are described by linear second-order differential equations. Maxwell’s equations, subject to everyday manipulations (Jackson, 1999), can be massaged into the Helmholtz equations for the electric and magnetic fields:

\[(\Delta + k^2)\mathbf{E} = 0; \quad (\Delta + k^2)\mathbf{B} = 0,\]  \hspace{1cm} (245)

where \(k = \omega/c\) is the wavenumber and \(\omega\) is the angular frequency. The solutions are the familiar transverse magnetic (TM) and transverse electric (TE) modes. The fields belonging to the former category are, generically,

\[\hat{n} \times \mathbf{E} = 0; \quad \hat{n} \cdot \mathbf{B} = 0\]  \hspace{1cm} (246)

where \(\hat{n}\) is the unit vector normal to the surface. In a sufficiently flat cylindrical resonator \(D\) (of depth \(d\)), where all the walls are parallel or perpendicular to \(\hat{z}\), these assume the form

\[\mathbf{E}_z|_{\partial D} = 0; \quad \nabla \perp \mathbf{B}_z|_{\partial D} = 0.\]  \hspace{1cm} (247)

The solutions are the familiar transverse magnetic (TM) and transverse electric (TE) modes. The fields belonging to the former category are, generically,

\[\mathbf{E}_z(x, y, z) = \mathcal{E}(x, y) \cos\left(j \frac{\pi z}{d}\right); \quad j = 0, 1, 2, \ldots,\]  \[\mathbf{B}_z(x, y, z) = 0,\]  \hspace{1cm} (248)

where \(\mathcal{E}(x, y)\), which is just a multivariate scalar function, is characterized by the stationary two-dimensional Helmholtz equation

\[\left[\Delta + k^2 - \left(j \frac{\pi}{d}\right)^2\right] \mathcal{E} = 0; \quad \mathcal{E}(x, y)|_{\partial D} = 0.\]  \hspace{1cm} (249)

For frequencies below the cutoff \(c/2d\) (or for wavenumbers \(k < \pi/d\)), only TM modes with \(j = 0\) are permitted and Eq. (249) reduces to a more recognizable one, \((\Delta + k^2)\mathcal{E} = 0\). Endowed with appropriate boundary conditions, this has a discrete spectrum

\[-\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \mathcal{E}_n = k^2 \mathcal{E}_n,\]  \hspace{1cm} (250)

which, fortuitously, bears a striking resemblance to the time-independent Schrödinger equation (cf. Eq. 3)

\[-\frac{\hbar^2}{2m} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \psi_n = \mathcal{E}_n \psi_n,\]  \hspace{1cm} (251)

with \(n\) now labelling the energy eigenstates. In fact, there is an exact one-to-one correspondence with the two related as \(E \propto k^2\) and \(\psi_n \propto \mathcal{E}_n\). The analogy can be pushed further. Quantum mechanically, a particle must be trapped within an infinite potential well in order to mimic the hard boundaries of the classical billiard. This infinite barrier exactly translates to the Dirichlet boundary conditions on the surface of the cylinder. Therefore, in a cavity with height \(d \leq \lambda_{\min}/2 = c/(2f_{\max})\), with \(\lambda_{\min}\) being the minimum de Broglie wavelength accessible to experiments, the quantum billiard can be simulated with electromagnetic waves (Richter, 1999), as Fig. 45 outlines. To get a sense of the actual numbers involved, consider a typical “quasi-two-dimensional” cavity: with \(d = 0.8\ cm\), we find \(\lambda_{\min} = 1.6\ cm\) and \(f_{\max} = 18.75\ GHz\).

![FIG. 45 Schematic illustration of a realization of the quantum analogue of a classical billiard through a flat electromagnetic cavity, taking a quarter of the Bunimovich stadium as an example. From Richter (1999). With permission of Springer Nature.](image)

Such lines of reasoning are not exactly new. Even earlier, the parallels between microwaves and sound waves were used to simulate the acoustics of enclosed spaces (Schroeder, 1987). Hence, it was no surprise that the first experiments by Stockmann and Stein (1990) precipitated a flurry of intense investigation into both two- (So et al., 1995; Sridhar, 1991; Stein and Stöckmann, 1992) and three-dimensional (Alt et al., 1997; 1996; Deus et al., 1995) microwave cavities. A time-honored protocol for measurement of the eigenmodes thereof would be as follows. One proceeds by first pumping in microwaves into the resonator with an antenna—this is usually a minuscule wire, no more than a few millimeters in diameter. The antennae are cased in wider semi-rigid leads that are attached to the billiard. To eliminate the possibility of the former influencing the field distributions, they are introduced into the walls of the cavity through small orifices with an abundance of caution to ensure negligible penetration into the resonance chamber itself. Depending on the specifications of the experiment, one could then either record the reflected microwave power at the same antenna as a function of frequency (taking care to separate the incoming and outgoing waves with a microwave bridge) or measure the transmission between two or more antennae. The entire procedure is repeated with different placements of the antennae so that no electromagnetic mode goes undetected simply due to the un-
eigenmodes could evidently wreak havoc on the spacing when working with experimental data, is the uncertainty in counting resonances: if two maxima are separated by a distance smaller than the experimental line width, they are registered as one (Stöckmann, 2006). Such missed eigenmodes could evidently wreak havoc on the spacing distribution. The origin of this nuisance is a straightforward consequence of elementary electromagnetic theory. The microwaves inevitably have a finite skin depth, wherefore they penetrate the walls up to distances on the scale of \( \delta = \sqrt{2/(\mu_0 \omega \sigma)} \sim 1 \mu m \), determined by the conductivity \( \sigma \). Accordingly, the electric fields are damped as \( E(t) = E_0 \exp(-t/2\tau) \cos(\omega_0 t) \) and the power spectrum

\[
\hat{S}(\omega) = \left| \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \ E(t) \exp(i\omega t) \right|^2
\]

becomes Lorentzian: \( [(\omega - \omega_0)^2 + (1/2 \tau)^2]^{-1} \). Careful calculations by Balian and Bloch (1970, 1971) show that the spectral broadening of the resonances is inversely correlated with the maximum number of resolvable resonances as

\[
N_{\text{max}} = \frac{1}{3} Q = \frac{1}{3} \frac{\omega_{\text{max}}}{\Delta \omega},
\]

where \( \Delta \omega = 1/\tau \) is the full width at half maximum (FWHM) of a resonance peak and \( Q \) is the quality factor of the resonator. Typical qualities of normally conducting cavities are in the range of \( 10^3 - 10^4 \) (Stöckmann, 2006) and one can distinctly identify about 1000 resonances through the smudging. The workaround to this unavoidable impediment eventually came from the cryostats of a linear accelerator at Darmstadt, the S-DALINAC (Auerhammer et al., 1993). Graf et al. (1992) discovered that a superconducting niobium billiard, immersed in a liquid He bath at 4 K, could yield \( Q \sim 10^5 \) to even \( 10^7 \)—a remarkable thousandfold improvement upon the experiments of old! This enhancement of the quality factor is not just academic but very much visible to the naked eye as Fig. 46 seeks to convince. The resultant extraordinarily sharp spectral peaks meant that one could resolve resonances to an unprecedented accuracy of \( \Delta f < 100 \text{ kHz} \), nearly two orders of magnitude below the mean level spacing of 17 MHz.

### B. The S-matrix and transmission measurements

The unassuming microwave billiard quickly became the workhorse for several ingenious experiments on varied facets of quantum chaos, spanning from tests of random matrix and periodic orbit theory (Kudrolli et al., 1993, 1994; Lewenkopf et al., 1992) to spectral level dynamics (Kollmann et al., 1994) and scattering matrix approaches (Alt et al., 1995; Doron et al., 1990). However, our present considerations are of a slightly different nature. The transmission spectrum, invaluable as it is, does not directly tell us about the nodal structure, which must instead be accessed from the electric field distribution inside the cavity. The formalism best equipped to deal with such open systems is that of the S-matrix. The scattering (whence, S) matrix encapsulates the relation between the

![Figure 46: Measured eigenmode spectrum between 17 and 18 GHz for a cavity in the shape of a quarter stadium billiard. Each maximum in the transmitted microwave power corresponds to an eigenfrequency of the resonator. The upper part is taken at room temperature (normally conducting), the lower part at 2 K (superconducting).](Image)
components of the amplitudes of the waves entering \(a_i\) and departing \(b_i\) through the \(i\)th channel in a single matrix equation \(b = Sa\). The total number of open channels actually depends on the frequency \(f\) because each lead can support \(M = \lfloor 2f/d/c \rfloor\) modes. Using Green’s function techniques (Stöckmann 2006), the \(S\)-matrix can be knoedd to the form

\[
S = 1 - 2iW^\dagger \frac{1}{k^2 - H + iW^\dagger W}, \tag{256}
\]

where \(H = -\Delta\) is the Hamiltonian of the unperturbed system (without channels) and the matrix elements \(W_{i,j}\) describe the coupling of the \(i\)th eigenfunction to the \(j\)th channel. In the presence of time-reversal symmetry, the scattering matrix is unitary and its off-diagonal terms are not independent: \(S_{ij} = S_{ji}\). Assuming non-overlapping resonances and point-like coupling (Kuhl 2007), Eq. (256) simplifies to

\[
S_{ij}(k) = \delta_{i,j} - 2i\gamma \mathcal{G}(r_i, r_j, k), \tag{257}
\]

with the modified Green’s function (Stein et al. 1995)

\[
\mathcal{G}(r_i, r_j, k) = \sum_n \frac{\psi_n(r_i) \psi_n(r_j)}{k^2 - k_n^2 + i\gamma \sum_i |\psi_n(r_i)|^2}, \tag{258}
\]

reminiscent of the Breit-Wigner formula in nuclear physics (Blatt and Weisskopf 1979). It might be helpful to pause for a moment and survey the barrage of notation introduced by Eq. (258) in one fell swoop. Here, \(\psi_n\) are the real eigenfunctions of the closed systems. The couplings to the antennae (captured by the parameter \(\gamma\)), in collusion with the summation \(\sum\) over all open channels, define an effective broadening \(\Gamma_n = \gamma \sum_i |\psi_n(r_i)|^2\) that could also subsume additional effects of absorption in the system (Evodorov et al. 2005; Kuhl et al. 2005a; Schäfer et al. 2003). In words, Eq. (258) claims that \(\mathcal{G}(r_i, r_j, k)\) can be obtained from transmission measurements between two antennae of variable position (Kuhl 2007). The transmission, in turn, solely depends on the field distributions at the locations of the antennae. Consequently, a two-dimensional scan by a probe/exit antenna (indexed 1), keeping the other entrance antenna(s) stationary, maps out the field \(E(x, y)\) inside the resonator (Stein et al. 1995; Stöckmann 2006) and with it, a visual image of the wavefunction \(\psi(x, y)\).

This stratagem was employed to good effect by Kuhl et al. (2007) in investigating a certain class of open cylindrical microwave billiards. Stepping in increments of 1 MHz, they measured the transmission \(S_{12}\) from antenna 1 to 2, and the reflection \(S_{11}\) at antenna 1, over a frequency range of 1–18.6 GHz. The experimental minu-

tiae have been meticulously chronicled by Kuhl et al. (2000); Veble et al. (2000). A particularly noteworthy detail is that with the aid of a vector network analyzer (VNA), the phase of the transmission \(S_{12}\), \(\phi\), can also be measured (Kuhl et al. 2003). Since \(\mathcal{E}(x, y)\) is a real quantity, this corresponds to determining both the real and imaginary parts of the complex wave function \(\psi(x, y) = |\psi(x, y)| \exp(i\phi)\). More often than not, the real \(\langle \psi_\eta \rangle\) and imaginary \(\langle \psi_i \rangle\) parts are correlated due to unwelcome global phase shifts \(\phi_g\), originating primarily from the leads and the antennae. This global phase can always be removed by an overall rotation (Ishio et al. 2001; Saichev et al. 2002).

\[
\psi_\eta + i\psi_i = e^{-i\phi_g} (\psi'_\eta + i\psi'_i), \tag{259}
\]

It does, however, have physical implications. Fig. 17 displays the nodal domains and their number for the real part of a wavefunction, at \(n_{\text{Weyl}} \approx 223\), as \(\phi_g\) is varied. To be pedantic, the Weyl law is not strictly valid for open systems (Lu et al. 2003; Nonnenmacher and Zworski 2005) in which the resonances are shifted to the complex plane or are, occasionally, even removed from the spectrum altogether (Lehmann et al. 1995). Nonetheless, in the absence of a viable alternative at our disposal, it is still the best approximation—a one-eyed king in the land of the blind. The wavefunction under scrutiny is additionally characterized by a phase rigidity (van Langen et al. 1997) that quantifies the extent to which the wavefunction is open

\[
|\rho|^2 = \left| \frac{\int d\mathbf{r} \psi(\mathbf{r})^2}{\int d\mathbf{r} |\psi(\mathbf{r})|^2} \right|^2 = \left| \frac{\langle \psi_\eta^2 \rangle_A - \langle \psi_i^2 \rangle_A}{\langle \psi_\eta^2 \rangle_A + \langle \psi_i^2 \rangle_A} \right|^2 \approx 0.81,
\]

derived under the assumption that the real and imaginary parts, \(\psi_\eta\) and \(\psi_i\), are not correlated. Indeed, the two are uncorrelated for \(\phi_g = 0\) and remain so at \(\phi_g = \pi/2\), switching identities (and getting correlated) in between. While the phase is changing, the nodal lines are shifted and permanently dissolved and reconnected (Kuhl 2007) and such rearrangements continually alter the number of nodal domains. Hence, it is of utmost importance that the effects of the phase are corrected for before any statistical calculations. Respecting this obligation, Kuhl et al. (2007) fit the experimentally observed number of nodal domains to \(\nu_n = an + b \sqrt{n}\) (Blum et al. 2002), taking \(n = n_{\text{Weyl}}\), with \(a = 0.059(0.060)\) and \(b = 1.23(1.30)\) for the real (imaginary) part of the wavefunction. The second term above takes boundary effects

\[22\]For instance, it would be just as difficult, qualitatively speaking, to excite a resonance in the vicinity of a nodal line, as it would be easy in close proximity to a maximum.

\[23\]A popular candidate is the resonance counting function \(n_r(k) = \sum_{i=1}^n \Phi_i(k) - c/2\), \(\Phi_i\) being the eigenphases of the \(S\)-matrix and \(c\) the number of open channels (Doron and Smilansky 1992a,b). Sadly, absorption disallows the calculation of \(n_r\) from data.
The cardinal premise is that the wave functions \( \psi \) of a closed system are localized, and the phases \( \phi \) are uniformly distributed in \([0, 2\pi]\). From this, the density of states can be calculated. A small perturbation \( \eta \) of Eq. (260) only concerns itself with \( |\epsilon_n| \). We will still need to somehow recover the signs. To do this, first, we identify all the minima of \( |\psi_n(R_c, \theta)| \) that are close to zero. Then, we assign the signs “minus” and “plus” in alternating fashion to the region between consecutive minima, starting (arbitrarily) with the negative, generating our trial wavefunction \( \psi_n(R_c, \theta) \) in the process. It cannot be ascertained a priori that this assignment of the signs is correct. An a posteriori sanity check is that if it indeed is, the reconstructed wave function should automatically vanish on the boundary, i.e.,

\[
\psi_n(r_{aD}, \theta_{aD}) = 0.
\]

Formally, the wavefunctions of the

field distribution. This, of course, is easier said than done and actually reconstructing the wavefunction proves to be a formidable challenge. A systematic method to do so using “trial functions” was first developed by Savvytskyy and Sirko (2002), which we outline in the context of experiments on chaotic rough billiards. These, and associated systems, have received considerable attention from motley quarters such as in relation to dynamic localization (Sirko et al., 2000), localization in discontinuous quantum systems (Borgonovi, 1998), microdisk lasers (Nöckel and Stone, 1997; Yamamoto and Slusher, 1993) and ballistic electron transport in microstructures (Blanter et al., 1998). Savvytskyy et al. (2004) constructed a billiard of this type out of an aluminum microwave cavity in the shape of a rough half-circle. The rough segment is described by the radius function

\[
R = R_0 + \sum_{m=2}^{M} a_m \sin (m \theta + \phi_m)
\]

with a mean radius \( R_0 \) and the phases \( \phi_m \) uniformly distributed in \([0, 2\pi]\). On loading a semicircular Teflon insert of radius \( R_c < R_0 \) (Hul et al., 2005), the dielectric constant (or the Schödinger potential) becomes discontinuous inside the resonator and the billiard turns ray-splitting (Blümel et al., 2001; Couchman et al., 1992). The qualifier “ray-splitting” refers to a class of chaotic systems with non-Newtonian and nondeterministic classical dynamics (e.g., Bauch et al., 1998; Hlushchuk et al., 2000; Savvytskyy et al., 2001; Sirko et al., 1997) in which rays, as the name suggests, split upon reflection from sharp boundaries. Both these sets of experiments, with and without ray-splitting, allow one to encroach upon the regime of Shirelman ergodicity in which the wave functions are expected to be homogeneously distributed on the energy surface, abiding by the quantum ergodicity theorem (Shirelman, 1974).

Let us work out the details for the simple rough billiard, for which the calculations are a little less involved. The cardinal premise is that the wave functions \( \psi_n(r, \theta) \) can be determined from the electric field \( \epsilon_n(R_c, \theta) \), evaluated on a semicircle of fixed radius \( R_c < R_0 \). Since the perturbation of Eq. (260) only concerns itself with \( |\epsilon_n| \), we still need to somehow recover the signs. To do this, first, we identify all the minima of \( |\psi_n(R_c, \theta)| \) that are close to zero. Then, we assign the signs “minus” and “plus” in alternating fashion to the region between consecutive minima, starting (arbitrarily) with the negative, generating our trial wavefunction \( \psi_n(R_c, \theta) \) in the process. It cannot be ascertained a priori that this assignment of the signs is correct. An a posteriori sanity check is that if it indeed is, the reconstructed wave function should automatically vanish on the boundary, i.e.,

\[
\psi_n(r_{aD}, \theta_{aD}) = 0.
\]

Formally, the wavefunctions of the

24Of course, \( |\beta_n| \) can also come into play, depending on the mode excited, but its influence on \( \Delta f_n \) can be minimized by using a small, vertically-positioned piece of a metallic pin as a perturber.
FIG. 48 [Left to right]: The experimentally measured number of nodal domains, its scaled variance and the normalized area distribution for two different billiards (inset). The predictions of the Bogomolny-Schmit percolation model (or the linear fits thereto) are drawn as solid/dashed lines for comparison. [Top]: The billiard is a quantum dot-like structure of a rectangular shape with rounded corners and two attached leads. Data points for the real (imaginary) part are represented by crosses/solid histogram (diamonds/red dotted histogram). [Bottom]: The corresponding data for the chaotic half-circular microwave ray-splitting rough billiard, now plotted as solid circles. Adapted from Kuhl et al. (2007) and Hul et al. (2005).

A rough half-circular billiard can be expanded in a basis of circular waves (only odd states are considered here) as

$$\psi_n(r, \theta) = \sum_{s=1}^{L} a_s C_s J_s(k_n r) \sin(s \theta), \quad (261)$$

where the number of basis functions $L = k_n r_{\text{max}}$ grows with the maximum radius of the cavity $r_{\text{max}}$. The accompanying coefficients are

$$C_s = \left[ \frac{\pi}{2} \int_0^{r_{\text{max}}} |J_s(k_n r)|^2 r \, dr \right]^{-1/2}, \quad (262)$$
$$a_s = \left[ \frac{\pi}{2} C_s J_s(k_n R_c) \right]^{-1} \int_0^{\pi} \psi_n(R_c, \theta) \sin(s \theta) \, d\theta.$$ 

The utility of the trial wavefunction is now patent. Putting this technique to use, Savitsky et al. (2004) were able to reconstruct 156 experimental wavefunctions for the rough microwave billiard; Hul et al. (2005), extracted 30 with ray-splitting. The computed statistics (Fig. 48) were consistent with the percolation model with Bogomolny-Schmit parameters $a = 0.058 \pm 0.006$, $\tau = 1.99 \pm 0.14$ and $a = 0.063 \pm 0.023$, $\tau = 2.14 \pm 0.12$, in the absence and presence of ray-splitting, respectively.

D. Current and vortex statistics

Over the years, another broad class of experiments on the nodal structure of quantum billiards have gained prominence. Picking up where we left off in Sec. II.C, we now sketch some of these experimental studies probing the distributions and correlations of wavefunctions, currents flows, and vortices.

Following up on the correspondence between the Schrödinger and Helmholtz equations, we can relate the Poynting vector (Seba et al., 1999)

$$\mathbf{S} = \frac{c}{8\pi k} \mathbf{E} \times \mathbf{H} = \frac{c}{8\pi k} \Im [\mathbf{E}^* \mathbf{E}] , \quad (263)$$

in two dimensions, to the probability current density

$$\mathbf{j}(\mathbf{r}) \equiv (j_x, j_y) = \frac{A}{k} \Im [\psi^*(\mathbf{r}) \nabla \psi(\mathbf{r})]. \quad (264)$$

Using the random plane wave ansatz, it is not difficult to calculate the distributions of the current components (Saichev et al., 2002)

$$P(|j|) = \frac{4j}{(j^2)^{3/2}} K_0 \left( \frac{2j}{\sqrt{j^2}} \right), \quad (265)$$
$$P(j_x) = \frac{1}{\sqrt{2} j_x^2} \exp \left( - \sqrt{\frac{2}{j_x^2}} |j_x| \right), \quad (266)$$
where the parameter

\[ \langle j^2 \rangle = \frac{1}{2} \langle j^2 \rangle = k^2 \langle \psi_R^2 \rangle \langle \psi_I^2 \rangle \]  

(267)

is directly accessible from experiments. Similarly, one can compute the corresponding distribution function for the vorticity (Eq. 14)

\[ P(\Omega) = \frac{1}{\sqrt{2} \sqrt{\Omega}} \exp \left( -\sqrt{2 \left| \frac{\Omega}{\Omega} \right|} \right), \]  

(268)

where, once again,

\[ \langle \Omega^2 \rangle = \frac{1}{2} k^4 \langle \psi_R^2 \rangle \langle \psi_I^2 \rangle \]  

(269)

can be taken straight from the experiment (Barth and Stöckmann 2002). The major stumbling block in experimentally determining the current distributions, however, is that the probe antenna, perforce, gives rise to a leakage current, which tampers with the statistical properties (Seba et al. 1999). The only way to guarantee that this influence is minimal is to either ensure a strong flow through the system or to choose frequencies such that the overall amplitudes are moderate (Barth and Stöckmann, 1999). The only way to guarantee that this influence is minimal is to either ensure a strong flow through the system or to choose frequencies such that the overall amplitudes are moderate (Barth and Stöckmann, 2002). The statistical distributions at one such choice of frequency are shown in Fig. 49 for a limaçon billiard (Robnik 1983, 1984) with a TRS-breaking ferris ring. In particular, the distributions for \( j_x \) and \( j_y \) are found to be identical but it need not always be so. For instance, in an open billiard the maximum of the \( j_x \) distribution could be shifted significantly to negative (positive) values due to transport from the right (left) to the left (right) through the billiard (Barth and Stöckmann, 2002).

Of related interest are the vortices of the current flow (the nodal points of the complex wavefunctions). In the plane, their positions are not independent but rather correlated. To characterize this, Berry and Dennis (2000) presented two types of vortex spatial autocorrelation functions. The first is the pair correlation function \( g(r) \), which quantifies the mean density of vortices at position \( \mathbf{R} + \mathbf{r} \), given that there is a vortex at \( \mathbf{R} \). This is defined by

\[ g(r) = g_0 \langle \delta \left( \psi_R(\mathbf{R} + \mathbf{r}) \right) \delta \left( \psi_I(\mathbf{R} + \mathbf{r}) \right) \rangle \times |\Omega(\mathbf{R} + \mathbf{r})| |\Omega(\mathbf{R})| \]  

(270)

\[ = g_0 \langle D_x(\mathbf{R} + \mathbf{r}) D_y(\mathbf{R}) \rangle_{\mathbf{R} + \mathbf{r}}, \]

\[ = \frac{2}{\pi} \frac{(E^2 - F_0(1 - C^2)^2)}{F_0^2 (E^2 - F_0(1 - C^2)^2)^2} \times \int_0^\infty dt \int_0^{\infty} dt' \frac{3 - Z + 2Y + (3 + Z - 2Y)t^2 + 2Zt^4}{(1 + t^2)^3 \sqrt{1 + (1 + Z - Y)t^2 + Zt^4}}, \]  

(271)

where \( g_0 \) is a normalization factor such that \( g(r) \to 1 \) for \( r \to \infty \); ordinarily, \( g_0 = \rho^{-2} \) for a dislocation point density of \( \rho \) (Eq. 28). Let us go through the dictionary for the remaining symbols. \( C \), as previously (see Eq. 56), is the non-local autocorrelation function

\[ C(r) = \langle \psi_{R,1}(1) \psi_{R,1}(2) \rangle = \langle \psi_{(0)}(k \mathbf{r}) \rangle = \left\langle \sin (K \mathbf{r}) \right\rangle \]  

with 1, 2 denoting different positions and \( k = |(k_x, k_y)| \). Building on this, we define

\[ E \equiv C'(r); F \equiv -C''(r); H \equiv -C'(r)/r, \]

\[ D_1 = [E^2 - (1 + C)(F_0 - F)] \quad [E^2 - (1 - C)(F_0 + F)], \]

\[ D_2 = F_0^2 - H^2, \]  

with \( F_0 = -C''(0) \), and finally,

\[ Y = H^2 \left( CE^2 - F(1 - C^2)^2 \right)^2, \]

\[ Z = \frac{D_1 D_2 (1 - C^2)}{D_0^2 (E^2 - F_0(1 - C^2)^2)^2}. \]

Knowing \( g(r) \), one can calculate the nearest neighbor distribution of vortices. This distribution is dependent on whether the billiard is nominally either regular or irregular (Berggren et al. 1999) and could thus potentially serve as yet another signature of quantum chaos. For ease of statement, we introduce the dimensionless pair correlation function

\[ G(\ell) = \left( \frac{\ell}{\sqrt{\rho}} \right), \]

(272)

where \( \rho = k^2/4\pi \) is the bulk mean density of vortices for a homogeneous Gaussian field and \( \ell = \sqrt{\rho} r \). Under the Poisson approximation, i.e., ignoring n-point correlations beyond \( n = 2 \), Saichev et al. (2002) calculated the distribution function for the nearest distances between the nodal points as

\[ f(\ell) \approx 2\pi \ell G(\ell) \exp \left( -2\pi \int_0^\ell z G(z) \, dz \right) \]

(273)

\[ \sim \frac{\pi}{2} \ell \quad (\ell \to 0). \]  

(274)

Although the Poisson approximation implicitly assumes that all nodal points around a given one are statistically independent, the end result, Eq. (273), is still an extremely useful reference point for experimental data nonetheless (e.g., Kim et al. 2003; Kuhl 2007).

A slightly simpler quantity to consider is the charge correlation function \( q_0(r) \) (Halperin 1981), which gives the normalized density of vortices separated by \( r \), but now weighted by their strengths so that vortices of opposite sign (and sense of rotation) contribute antagonistically. Formally, we have

\[ q_0(r) = \langle \delta \left( \psi_R(\mathbf{R} + \mathbf{r}) \right) \delta \left( \psi_I(\mathbf{R} + \mathbf{r}) \right) \delta \left( \psi_R(\mathbf{R}) \right) \delta \left( \psi_I(\mathbf{R}) \right) \rangle \times |\Omega(\mathbf{R} + \mathbf{r})| |\Omega(\mathbf{R})| \]  

(275)

\[ = \frac{2}{\pi F_0^2 (1 - C^2)^2} \times \int_0^\infty dt \int_0^{\infty} dt' \frac{3 - Z + 2Y + (3 + Z - 2Y)t^2 + 2Zt^4}{(1 + t^2)^3 \sqrt{1 + (1 + Z - Y)t^2 + Zt^4}}, \]  

(276)
Eq. (275) is essentially Eq. (270) but without the modulus signs on the vorticity, wherefore \( g_Q(r) \to 0 \) as \( r \to \infty \).

At the origin, \( g_Q \) and \( g \) are related as

\[
g(0) = -g_Q(0). \tag{276}
\]

Both the functions \( g \) and \( g_Q \) are plotted in Fig. 50 for an open microwave billiard. Assuming isotropy, \( g_Q(r) \) depends only on the scaled distance \( R \equiv k|r_2 - r_1| \) and can actually be kneaded into a surprisingly simple form (Dennis, 2003; Foltin, 2003a, b; Freund and Wilkinson, 1998; Wilkinson, 2004):

\[
g_Q(R) = 4 \frac{d}{dR} \left[ \frac{1}{R} \frac{d \arcsin (J_0(R))}{dR} \right]^2. \tag{277}
\]

Moreover, since the distribution of the phases of the field \( \psi \) is isotropic, \( g_Q \) satisfies the “topological charge screening relation”:

\[
\frac{1}{2} \int_0^\infty dR \ R g_Q(R) = -1, \tag{278}
\]

neglecting the self-interaction at \( R = 0 \). Physically, this means that the integral of the topological charge over all \( r \) necessarily compensates the charge associated with the vortex at \( r = 0 \) (Berry and Dennis, 2000). This polyonymous local neutrality condition appears in many guises such as the Stillinger-Lovett sum rule (Stillinger Jr. and Lovett, 1968a, b) in the theory of ionic liquids, and “critical-point screening” in the context of dislocations (Freund and Wilkinson, 1998).

For \( R \gg 1 \), \( g_Q(R) \sim 8 \cos(2R)/\pi R^2 \); the period of oscillation of \( g_Q \) is thus twice that of the correlation function \( C \). Contrarily, unsigned correlation functions, like those for saddle points, do not reduce to such simple forms. The asymptotic approximations to order \( O(R^{-1}) \) of the unsigned RWM vortex-vortex, vortex-saddle, and saddle-saddle pair correlations were derived by H"ohmann et al. (2009) to be

\[
g_{vv}(R), g_{ss}(R) \sim 1 + \frac{4 \sin 2R}{\pi R}; \quad g_{vs}(R) \sim 1 - \frac{4 \sin 2R}{\pi R}.
\]

Finally, the most demanding asymptotics (short distances from the boundaries, for example) were addressed using supersymmetric techniques. Klein and Agam.
established that (unnormalized) pair correlations of both the unsigned density of critical points and the density of minima points of the Gaussian random field are given by

\[ \tilde{g}(r) = \begin{cases} (D)^2 + \alpha_1 \nabla^4 C(r) + \alpha_2 \text{Tr} [H_C(r)]^2 ; r \gg \sigma \\ \alpha_3 r^{2-d+\kappa}, & 0 < r \ll \sigma \end{cases} \]

with \( \kappa = 0 \) and \( \kappa = 3 \), respectively. The elements involved in computing this correlation function are \( D \), the average density of critical points, \( \alpha_i \), constants that depend on the type of critical-point density considered and the dimensionality of the system \( d \), \( [H_C(r)]_{ij} = \partial^2 C(r)/\partial r_i \partial r_j \), the Hessian matrix of the correlation function of the field, and \( \sigma \), its typical length scale.

A fitting end to this Section and effectively, the Review, is perhaps provided by Carlo Beenakker’s Synopsis for the Seventh Annual Symposium on Frontiers of Science at Irvine, California, 1995. Addressing the then nascent field, he remarks, “Quantum billiards is a game played by physicists at a few academic and industrial laboratories in various parts of the world. It’s a serious game: we are actually getting paid for it. It’s also fun and exciting.” We hope that the experiments described above—and our larger discussion on nodal portraits—have been successful in sharing some of this excitement.

**VIII. CONCLUDING REMARKS**

One of the key features of mathematical description—portability—allows us to reach out under one shade to nodal domains, percolating clusters, spin domains, electromagnetic modes, water waves, and others. From Chladni plates to gallium arsenide tables, the essence of the fundamental questions stays the same even if the details do not. What is clear by now is that nodal portraits distinguish between integrable and chaotic systems and afford details of their geometrical features. However, with a great number of new results comes an equally large number of new puzzles, many of which remain markedly unclear. It is to some of these that we turn now, before closing.

1. The area distribution for the pseudointegrable barrier billiard is known (Dietz et al. 2008) to follow the same scaling as Eq. (83), suggesting that the excited states are about as random as those for chaotic billiards. This is somewhat reminiscent of the time when “linear level repulsion” in spacing distributions was (fallaciously) believed to be an indicator of quantum chaos. A counterexample was the pseudointegrable rhombus billiard (Biswas and Jain 1990), which belongs to a different universality class altogether (Auberson et al. 2001; Bogomolny et al. 1999). It is perhaps not too unreasonable to speculate that these sums of large number of trigonometric products, with random coefficients, might serve as good models for chaotic wavefunctions.

2. There is still a lot to be understood about the nodal statistical features of nonseparable but integrable, and quasi-integrable billiards, the wavefunctions of which have been spurned by both checkerboards and the random wave model. Some noteworthy results have been obtained for special systems (Prado et al. 2009) but the evolution of the domains with quasi-integrable perturbations is an open problem.

3. As we have seen, the statistics of nodal volumes are in tune with Yau’s conjecture and thus, when appropriately scaled, are non-zero only over a finite interval. In light of this conjecture, it would certainly be interesting to seek exact limiting distributions for systems other than the cuboid and the RWM, which lie at opposite ends of the spectrum from order to chaos.

4. For the nonseparable, integrable billiards, it is fascinating to be able to set up difference equations for \( \nu_{m,n} \) but there are important questions that this gives birth to. Firstly, how does one connect the topology of the eigenfunctions to the algebra of the difference equations? Moreover, even though one can count the domains, analytical forms for the limiting distributions \( P(\xi) \) and the trace formulæ portended by the statistics of nodal loops continue to elude us. In this vein, it might be worthwhile to study the sums of trigonometric products and explore the possibility of constructing difference equations for the domains of these functions. It is perhaps not too unreasonable to speculate that these sums of large number of trigonometric products, with random coefficients, might serve as good models for chaotic wavefunctions.

5. The idea of counting nodal domains of chaotic billiards with Potts spins is, by all means, innovative. However, at this stage, it seems far removed from actual quantitative counts and any developments in this direction would be a welcome addition.

6. Although there exist isospectral, convex, connected domains in dimensions larger than three (Gordon and Webb 1994), no known examples thereof have been found for planar billiards. Of course, absence of evidence is not evidence of absence and one would ideally hope to see this conjecture conclusively settled, once and for all.

7. Shortly after Kac’s famous question, Fisher (1966) posed the first instance of this query in the context of graphs. Half a century thence, the nodal domains of quantum graphs (Kuchment 2008) have
Expressed differently, we have the relation
\[ \frac{A}{\pi} \geq \frac{\omega_d(D)}{\mu_d(D)} \left( \frac{\sqrt{2}}{d-1} \right)^2, \]
and later, to regions inside circular sectors by Payne and Weinberger (1960a). It is possible to further improve this result if we now restrict the class of domains under consideration (Antunes and Freitas 2006). One such possibility is to consider the \( n \)-polygons, for which Pólya and Szegő (1951) conjectured that “of all \( n \)-polygons with the same area, the regular \( n \)-polygon has the smallest first Dirichlet eigenvalue.” Utilizing this hypothesis for triangles and quadrilaterals (see, for example, Freitas 2006; Siudeja 2007) yields
\[ \lambda_1^n(\triangle) \geq \frac{4\sqrt{3}\pi^2}{3A}, \quad \text{and} \quad \lambda_1^n(\square) \geq \frac{2\pi^2}{A}. \]

Analogously, the second Dirichlet eigenvalue \( \lambda_2^n \) satisfies
\[ \lambda_2^n \geq 2^{2/d} \left( \frac{\omega_d(D)}{\mu_d(D)} \right)^{2/d} \left( \frac{\sqrt{2}}{d-1} \right)^2, \]
which is minimized by the union of two identical disks (Krahn 1926, Pólya 1955). A related pair of bounds for \( \lambda_1^n \) in a simply connected planar domain is given by
\[ \frac{\alpha}{R^2} \leq \lambda_1^n \leq \frac{1}{R^2} \left( \frac{\sqrt{2}}{d-1} \right)^2, \]
where \( R \) is the inradius of \( D \). The lower bound is due to Makai (1965) and Hayman (1978). Subsequent to several iterative refinements of their original estimates, the best value of the constant \( \alpha \) known today is \( \alpha = \pi^2/4 \approx 3.1416 \).
2.4674 for convex domains (Hersch [1960]) and α = 0.6197 for a general D (Bañuelos and Carroll [1994]). The upper bound, on the other hand, follows directly from domain monotonicity and has been further tightened for planar (Pólya and Szegő [1951] and higher-dimensional (Freitas and Krejčíř [2008]) star-shaped domains. The final result concerning \( \lambda_0^n \) values given Neumann boundary conditions. The first Dirichlet eigenvalues—equally interesting are the eigenvalues for a general \( D \)-strained as

\[
\inf_{R} \left( -\frac{\Delta \phi}{\phi} \right) \leq \lambda_0^n \leq \sup_{R} \left( -\frac{\Delta \phi}{\phi} \right).
\]  

(A7)

2. Payne-Pólya-Weinberger inequality

The preceding inequalities provide individual estimates for the first and second eigenvalues but stop short of relating them. The obvious question that follows is: how do the two compare? For an answer, we turn to the conjecture by Payne et al. (1956), which posits that

\[
\frac{\lambda_{2,n+1}^0}{\lambda_1^n} \leq 3 \lambda_0^n.
\]  

(A8)

and, more generally,

\[
\lambda_{n+1}^0 \leq 3 \lambda_0^n.
\]

(A9)

Although Payne et al. (1956) originally proved Eq. (A8) for a weaker upper bound of \( 1 + 4/d \) in \( d = 2 \) dimensions, it was not until much later that the rigorous proof of the inequality, as it stands today, was provided by Ashbaugh and Benguria (1991, 1992, 1993a,b). Alternatively, instead of looking at the ratios of the two eigenvalues, one can consider their differences. In this case, we know (Singer et al. [1985])

\[
\frac{\pi^2}{4 \delta^2} \leq \lambda_2^0 - \lambda_1^n \leq d \frac{\pi^2}{R^2},
\]

where \( \delta = \max \{|x - y|: x, y \in D\} \) is the diameter of \( D \in \mathbb{R}^d \). As suggested by Donnelly (2011) and shown by Andrews and Clutterbuck (2011), the lower bound can actually be sharpened to \( 3 \pi^2/\delta^2 \), thereby further narrowing the range.

3. Szegő-Weinberger inequality

So far we have only talked about the properties of the Dirichlet eigenvalues—equally interesting are the eigenvalues given Neumann boundary conditions. The first nontrivial Neumann eigenvalue \( \lambda_2^n \) (since \( \lambda_1^n = 0 \)) is constrained as

\[
\frac{\pi^2}{\delta^2} \leq \lambda_2^n \leq \left( \frac{\omega_d}{\mu_d(D)} \right)^{2/d} \left( \frac{\tilde{\lambda}_{d,1}}{\tilde{\lambda}_{1,1}} \right)^2,
\]

where \( \tilde{\lambda}_{v,1} \) is the first positive zero of the function

\[
\frac{d}{dz} \left( z^{1-d/2} J_{\frac{d}{2}+1}(z) \right) = J_{\frac{d}{2}}(z) \quad \text{(for } d = 2\).
\]

This relation actually consists of two independent isoperimetric inequalities compactified into one for ease of presentation. The supremum, proven for simply-connected, planar domains by Szegő (1954) and Weinberger (1956), is usually identified as the inequality bearing the name of its proponents. The other component of Eq. (A11)—the infimum—was obtained by Payne and Weinberger (1960b). For Neumann eigenvalues in planar, bounded, regular domains (domains with a discrete Neumann eigenspectrum), Pólya (1954) hypothesized that

\[
\lambda_j^n \leq \frac{4(j - 1)\pi}{A} \quad (j = 2, 3, 4, \ldots),
\]

(A12)

for simply-connected, regular, planar domains (Girouard et al. [2009]) with the equality attained (in the limit) by a family of domains degenerating to a disjoint union of two identical disks (Grebenkov and Nguyen [2013]). Incidentally, it can be shown that the harmonic mean of the first two nontrivial Neumann eigenvalues is also minimized for a disk (Szegő [1954], Weinberger [1956]), i.e.,

\[
\frac{1}{\lambda_2^n} + \frac{1}{\lambda_3^n} \geq \frac{2A}{\pi \left( \tilde{\lambda}_{1,1} \right)^2}.
\]

(A14)

which permits generalization to the longer sequence (Ashbaugh and Benguria [1993b])

\[
\frac{1}{\lambda_2^n} + \cdots + \frac{1}{\lambda_{m+1}^n} \geq \frac{d}{m + 2} \left( \frac{\mu_d(D)}{\omega_d} \right)^{2/d}.
\]

(A15)

To the reader perplexed as to the point of this compendium of abstruse relations (which might rightfully seem an archival exercise in mathematical stamp collecting), we can only offer solace in the reassurance that isoperimetric inequalities prove to be extremely useful in shape optimization problems, an overview of which is presented in Sec. VI.B.3.a.

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