Abstract

Plasma lensing events can have significant observational consequences, including flux density modulations and perturbations in pulse arrival times. In this paper we develop and apply a formalism that extends geometrical optics to describe the effects of two dimensional plasma lenses of arbitrary shape. We apply insights from catastrophe theory and the study of uniform asymptotic expansions of integrals to describe the lensing amplification close to fold caustics and in shadow regions, and explore the effects of image appearance and disappearance at caustics in the time of arrival (TOA) perturbations due to lensing. The enhanced geometric optics approach successfully reproduces the predictions from wave optics and can be efficiently used to simulate multifrequency TOA residuals during lensing events. Lensing will introduce perturbations both in the way the residuals change as a function of frequency and also in the magnitude and sign of the residuals averaged over a frequency band. The deviations from the expected dispersive $\nu^{-2}$ scaling will be most significant when including observations at low frequencies. We examine the consequences of lensing in the context of precision pulsar timing and touch on its potential relevance to the study of FRBs.

1. INTRODUCTION

The phenomenon of astrophysical plasma lensing has attracted considerable attention ever since the first detections of so-called “extreme scattering events” (ESEs) in the late 1980s (Fiedler et al. 1987) and early 1990s (Cognard et al. 1993), during which the measured flux density of the observed objects (a millisecond pulsar in the latter case, and a quasar in the former) underwent large fluctuations with a frequency dependent structure over a period of time of the order of months. Subsequent works describing observations of ESEs, such as those by Fiedler et al. (1994) and Clegg et al. (1996) mentioned the idea, introduced in Cognard et al. (1993), that these events were the result of plasma overdensities in the interstellar medium that act as lenses as they cross the line of sight between the Earth and the source of radiation, refracting the incoming radio waves and creating observable regions of focusing and defocusing.

Clegg et al. (1998) gave a detailed exposition of the geometric optics of one dimensional Gaussian lenses and performed numerical simulations to find appropriate lens parameters that could match the observed flux fluctuations of specific ESEs, and some subsequent works have also aimed to derive the characteristics of specific lenses deemed to be responsible for particular ESE observations (Pushkarev et al. 2013; Bannister et al. 2016; Tuntsov et al. 2016; Vedantham et al. 2017; Kerr et al. 2017; Main et al. 2018).

More recently, plasma lensing has also been suggested as a possible mechanism to explain certain properties of FRBs (Cordes et al. 2017; Dai & Lu 2017), and other works have examined different kinds of lens models, as well as their possible observational signatures (Pen & King 2012; Er & Rogers 2017), although most of the analysis so far has been done in only one dimension and for a few specific lens shapes.

Plasma lensing events do not only have observable effects in the source’s light curve, they also introduce perturbations in the times of arrival (TOAs) of the radiation, via a combination of geometric and dispersive effects. Thus, plasma lensing events can have potentially important consequences for pulsar timing, as the possible detection of low frequency gravitational waves via this method is dependent on our ability to detect $\lesssim 100$ ns deviations in pulse arrival times. In fact, some plasma lensing events have been inferred by their effects on observed pulsar TOAs (Lam et al. 2018) and dispersion measures (DMs) (Coles et al. 2015); instead of their effects on measured flux density, since in some cases the presence of strong scintillations can effectively mask whatever effects the lensing events have on the source’s light curve.

In contrast to the random fluctuations in the electron column density that are responsible for scintillation, plasma lensing events are produced by larger scale inhomogeneities in the ISM, motivating the use of geometrical optics. Nevertheless, it has been useful for some authors modelling scintillation phenomena to...
study the effect of nonturbulent phase screens, particularly in the transition regime from weak to strong scintillations (Watson & Melrose 2006; Melrose & Watson 2006). Furthermore, the underlying optics based on the Kirchhoff diffraction integral (KDI) is the same for both scintillations and plasma lensing, meaning that a considerable amount of the formalism used in the study of scintillations can be applied in the latter context.

A potentially important effect of plasma lensing is the appearance and disappearance of multiple images of the source as the lens crosses the line of sight. Such multiple imaging has been directly observed in cases in which the angular separation of some of the images has been large enough (Gupta et al. 1999; Pushkarev et al. 2013), and can be inferred from the existence of fringes in the dynamic spectra of pulsars during certain epochs of observation (Cordes & Wolszcan 1986; Gupta et al. 1994; Cordes et al. 2006). The coalescence of images is associated with regions in which a straightforward calculation of the flux using geometric optics diverges; these regions are known as caustics, and the ability to describe these regions is of importance both in the context of plasma lensing and scintillation (Goodman et al. 1987; Melrose & Watson 2006; Cordes et al. 2017). The geometric optics framework, however, is useful because it provides information about the different images, including their amplitudes, phases, and locations, and at the same time provides a relatively simple way of calculating the total flux without the need of finding a full solution to the KDI. Thus, it is desirable to describe the amplification in the caustic regions without having to abandon the geometric optics point of view. Different authors in the astrophysical context have employed a variety of methods to handle the geometrical optics infinities, but so far the problem has not been solved using wave asymptotic methods derived from the geometrical theory of diffraction (Borovikov & Kinber 1994) and catastrophe optics (Poston & Stewart 1978; Berry & Upstill 1980; Stammes 1986; Kravtsov & Orlov 1999; Katsaounis et al. 2001; Kryukovskii et al. 2006), in order to predict the potential observational signatures of two dimensional plasma lenses of arbitrary shape.

Our primary goal in this paper is therefore to use wave asymptotic methods to characterize the effects of astrophysical plasma lensing, develop the resulting formalism that describes the observational effects of two dimensional plasma lenses that cross our line of sight, and present some numerical results based on the application of this formalism. We restrict ourselves to cases in which the source of radiation can be accurately regarded as a point source, and focus on the effects of plasma lensing on pulsar timing. The paper is divided as follows. In §2, we present what we call the zeroth and first order geometric optics of two dimensional lenses, which formally yields infinite flux amplitudes at caustic regions. In §3 we use wave asymptotic methods to construct a second order geometric optics description. In §4 we use the concepts developed in §2 and §3 to examine the TOA and DM perturbations due to a specific plasma lens realization, and we summarize conclusions in §5. We expect to apply the methodology presented here to specific events in subsequent work.

2. ZEROOTH AND FIRST ORDER GEOMETRIC OPTICS

2.1. Geometrical picture

We follow the basics of the treatment given in Clegg et al. 1998 and Cordes et al. 2017 but extend their results to two dimensions. We start by defining planes for the source, the lens, and the observer with coordinates $x_s$, $x$, and $x_{obs}$, respectively, with a source-lens distance $d_{sl}$, a lens-observer distance $d_{lo}$, and a source-observer distance $d_{so} = d_{sl} + d_{lo}$, as depicted in Figure 1. The geometric optics approximation treats the radiation emitted from the source as a cone of rays, and the effects of lensing can be described by the way the lens affects the mapping of the rays from the source plane to the observer plane. From the geometry in the figure, we see that the 2D angle of incidence of a ray into the lens plane $\theta_i$ and its deviation angle are given (in the paraxial approximation) by

![Figure 1. Lensing geometry.](image)
\[ \theta_i = \frac{x_s - x}{d_{so}} \]

\[ \theta_r = \frac{x_{obs} - x}{d_{io}} - \theta_i. \]  

Combining into a single equation in terms of \( \theta_r \) gives the lens equation,

\[ x_s \left( \frac{d_{io}}{d_{so}} \right) + x_{obs} \left( \frac{d_{sl}}{d_{so}} \right) = x + \theta_r \left( \frac{d_{sl}d_{io}}{d_{so}} \right). \]  

We now define a new set of coordinates \( x' \) as a combination of the source and observer coordinates scaled by the distances, namely

\[ x' \equiv x_s \left( \frac{d_{io}}{d_{so}} \right) + x_{obs} \left( \frac{d_{sl}}{d_{so}} \right). \]

and write the lens equation in the simpler form

\[ x' = x + \theta_r \left( \frac{d_{sl}d_{io}}{d_{so}} \right). \]

This expression is perfectly general and not only applies to plasma lensing, but to gravitational lensing as well (Schneider et al. 1992). The nature of the lensing is what determines the formula for the deviation angle \( \theta_r \).

A general expression for this angle can be obtained with the additional assumptions that the lens’s surface slope is small, and that the lens’s medium is uniform. The result of the ray propagating through the lens is that the lens advances or retards the ray’s phase, depending on whether the value of the refractive index \( n_r \) is greater or smaller than unity, because the phase velocity \( v_p \) will be smaller or greater than \( c \). More precisely, we can write this phase difference \( \delta \phi_{lens} \) as

\[ \delta \phi_{lens} = \omega \tau = kct, \]

where \( \tau \) is the propagation time difference between a lensed ray and an unlensed ray, \( k = 2\pi/\lambda \) is the wavenumber, and \( \omega \) is the radiation’s angular frequency. By this definition, \( \tau < 0 \) implies that \( \delta \phi_{lens} < 0 \) and therefore \( v_p > c \). For a lens of length \( l \) parallel to the direction of propagation, this is

\[ \tau = \frac{l}{c} (n_r - 1). \]

For a cold, unmagnetized plasma, the frequency dependent index of refraction is given by

\[ n_r = \sqrt{1 - \left( \frac{\omega^2}{\omega_c^2} \right)^2} \approx 1 - \frac{\lambda^2 r_e n_e}{\pi}, \]

where \( \omega_c^2 = 4\pi n_e e^2/m_e \) corresponds to the square of the electron plasma frequency, \( e \) is the elementary charge, \( m_e \) is the mass of the electron, \( r_e \) is the electron’s classical radius, and \( n_e \) is the electron number density, and the approximate equality comes from the fact that \( \omega_c \ll \omega \) for \( \omega \) within the radio spectrum. According to geometrical optics, rays propagate in the direction normal to the surfaces of constant phase (Born & Wolf 1999, Ch. 3), so the refractive angle \( \theta_r \) is given by

\[ \theta_r = \frac{1}{k} \nabla \delta \phi_{lens}. \]

When the electron column density or dispersion measure perturbation \( DM = n_e l \) at the lens plane varies as a function of transverse position, \( DM \rightarrow DM(x) \), \( \theta_r \neq 0 \), and lensing occurs. Putting everything together, the phase perturbation becomes

\[ \delta \phi_{lens}(x) = -\lambda r_e DM(x), \]

which implies that the refractive angle is

\[ \theta_r = \frac{-\lambda^2 r_e}{2\pi} \nabla DM(x) = -\frac{c^2 r_e}{2\pi\nu^2} \nabla DM(x). \]

For convenience, we write \( DM(x) \) as the product of a maximum perturbation \( DM_l \) and a function with unit maximum \( \psi(x) \), and take the origin of the lens plane’s coordinate system to coincide with the lens’s center. Thus Eq. 11 takes the form

\[ \theta_r = -\frac{c^2 r_e DM_l}{2\pi\nu^2} \nabla \psi(x). \]

We now define the Fresnel scale as \( r_F = \sqrt{cd_{io}d_{so}/2\pi d_{so} \nu} \), the lens phase as \( \phi_0 = -cr_e DM_l/\nu \), and a new parameter \( A = r_F^2/\phi_0 \), and substitute Eq. 12 in terms of these new quantities into the lens equation, which yields a more compact form that is specific to plasma lensing,

\[ x' = x + A \nabla \psi(x). \]

Finally, we define dimensionless coordinates using the characteristic lens scales \( a_x \) and \( a_y \), such that \( u'_x = x/a_x \) and \( u'_y = y/a_y \), and explicitly write Eq. 13 in its adimensionlized component form. Using the notation \( \psi_{ij} = \partial^i \psi/\partial u'_j \partial u'_j \), and defining \( \alpha_{x,y} = A/a_x^2 \),

\[ \begin{bmatrix} u'_x \\ u'_y \end{bmatrix} = \begin{bmatrix} u_x + \frac{A}{a_x^2} \psi_{10}(u_x, u_y) \\ u_y + \frac{A}{a_y^2} \psi_{01}(u_x, u_y) \end{bmatrix} = \begin{bmatrix} u_x + \alpha_x \psi_{10} \\ u_y + \alpha_y \psi_{01} \end{bmatrix}. \]

In general, Eq. 14 must be solved numerically using a root finding algorithm. More details on the numerical techniques used to produce the examples presented throughout the paper can be found in Appendix C. The vector \( u'(t) \) changes as a function of time as the Earth, the lens, and the source move with different velocities, and the nature of this change will partly determine the observational signature of a specific lens realization. The number of solutions of the equation corresponds to the
number of images of the source as seen by the observer, and in general vary as a function of $u'(t)$ and the parameters $\alpha_{x,y}$.

2.2. Zeroth order gain

A large majority of the existing literature on plasma lensing (Clegg et al. 1998; Pen & King 2012; Tuntsov et al. 2016; Cordes et al. 2017; Er & Rogers 2017; Vedantham et al. 2017) derives the gain (or magnification) for an individual image $G_j$ directly from some version of Eq. 14, and the total gain is found by adding together the gains of all $n$ images. More specifically, the image magnification is said to correspond to the absolute value of the inverse of the Jacobian of the mapping between the $u$ and $u'$ planes, evaluated at a solution to the lens equation $u = u'_j$.

$$G_j = |J|^{-1} = \left| (1 + \alpha_x \psi_{20}) (1 + \alpha_y \psi_{02}) - \alpha_x \alpha_y \psi_{11} \right|^{-1}$$

and the total gain is

$$G = \sum_{j=1}^{n} G_j.$$  

We refer to this expression as the “zeroth order” geometrical optics gain. It corresponds to a sum of intensities, and as such it fails to take into account the interference between the images that arises from the phase differences of the corresponding fields. An accurate description of the interference pattern can be obtained by solving the Kirchhoff diffraction integral (KDI), which we introduce below.

2.3. The 2D Kirchhoff diffraction integral

Once we adopt a wave description of the radiation, the scalar wavefield as a function of position with respect to the source is given by the time independent Helmholtz equation. The general form of the KDI is a formal solution to the Helmholtz equation (Born & Wolf 1999, Ch. 8; Thorne & Blandford 2017, Ch. 8). In the paraxial approximation and for the near field, as is the case for AU sized lenses and astronomical distances, the integral can be written in terms of dimensionless coordinates (Goodman et al. 1987; Melrose & Watson 2006; Cordes et al. 2017)

$$\varepsilon(u', \nu) = \frac{a_x a_y}{2 \pi r_F^2} \int d^2u \exp(i \Phi),$$

where the phase $\Phi$ is the sum of a geometric term and the phase perturbation due to the lens, $\delta \phi_{\text{lens}} = \phi_0 \psi(u)$,

$$\Phi(u', u, \nu) = \frac{1}{2r_F^2} \left[ a_x^2 (u_x - u'_x)^2 + a_y^2 (u_y - u'_y)^2 \right] + \phi_0 \psi(u).$$

The integral is normalized such that in the absence of a lens (ie. $\delta \phi_{\text{lens}} = 0$), $\varepsilon(u', \nu) = 1$ for all $u'$ and $\nu$. Analytic solutions to the integral are only available for a few specific forms of $\psi$ (Watson & Melrose 2006). As detailed in Appendix A, this representation of $\Phi$ allows us to write the integral as a convolution of two functions, which can then be solved numerically by employing the convolution theorem and the Fast Fourier Transform (FFT). However, this method is only adequate for lenses that have sizes that are a small fraction of an AU and in cases where $|\phi_0|$ is relatively small, because the required grid size for proper sampling grows prohibitively large as the oscillations of $\exp(i \Phi)$ become more pronounced.

An approximate solution that grows more accurate as the strength of the lens increases follows by applying the method of stationary phase. For a rapidly oscillating two dimensional integral of the form $I(x) = \int \int d^2x g(x) \exp(i f(x))$, the stationary phase lemma (Bleistein & Handelsman 1975) indicates that the principal contributions to the integral’s value come from the points in which the phase is stationary, that is, where the derivatives of the phase vanish, $f_{x0} = f_{y0} = 0$. In the general case where these points $x = x_0$ are complex, each provides a contribution to the integral $I_j$ given by (Connor 1973a)

$$I_j = \frac{2 \pi g_j \exp(i f_j)}{\Delta_j^{1/2}},$$

where $f_j = f(x_0^j)$, $g_j = g(x_0^j)$, $\Delta_j = f_{x0}f_{y0} - f_{x1}f_{y1}$ evaluated at $x_0^j$, and the square root in the denominator is taken to be positive or negative depending on the context. When the stationary point is purely real, the contribution reduces to (Bleistein & Handelsman 1975; Cooke 1982)

$$I_j = \frac{2 \pi g_j \exp(i f_j + i \frac{\pi}{4}(\delta_j + 1)\sigma_j)}{\Delta_j^{1/2}},$$

where $\sigma_j = \text{sgn}(\Delta_j)$, $\delta_j = \text{sgn}(f_{y0})$, and the square root in the denominator is now taken to be positive. In the case of the KDI as given in the form Eq. 17, the points of stationary phase correspond to the points that satisfy the two dimensional equation

$$\Phi_{10} = \frac{a_x^2 (u_x - u'_x)}{r_F^2} + \phi_0 \psi_{10},$$
$$\Phi_{01} = \frac{a_y^2 (u_y - u'_y)}{r_F^2} + \phi_0 \psi_{01}.$$

A quick examination reveals that this is precisely equivalent to the lens equation Eq. 14, given our definitions of the parameters $\alpha_{x,y}$, which therefore implies that solving the KDI by the method of stationary phase
Figure 2. Comparison of the gains obtained from a full numerical solution of the KDI, zeroth order geometrical optics, and first order geometrical optics. The top panel corresponds to a lens with $\phi_0 = -50$ rad and the bottom panel corresponds to one with $\phi_0 = -250$ rad (thus $DM_\ell > 0$ in both cases, and the lenses are diverging). The frequency of observation is $\nu = 0.8$ GHz, $d = 1$ kpc, $d_{\text{sl}} = 0.5$ kpc for both the top and bottom panels. For the top panel, $a_x = a_y = 1.5 \times 10^{-2}$ AU, and for the bottom panel, $a_x = a_y = 1.5 \times 10^{-2}$ AU. The lens shape is described by a two dimensional Gaussian, $\psi(u) = \exp(-u^2_x - u^2_y)$. The left column shows color maps of the gain obtained by solving the KDI via the FFT. The white circles correspond to caustic curves, and the straight white line shows the path of the observer through the $u'$ plane. The right column shows the gain along this path as calculated via the FFT method, zeroth order geometrical optics, and first order geometrical optics. The points of intersection between the caustics and the observer path are marked by white points in the left column and by dashed black lines on the plots in the right column. The geometric optics gain at the caustics is formally infinite, so the GO gains were evaluated up to a short distance away from the caustic.

It is also possible to derive the geometric optics quantities by directly solving the Helmholtz equation via WKB methods (see, e.g. Born & Wolf 1999, Ch. 3; Katsaounis et al. 2001; Poston & Stewart 1978, Ch. 12).

where now we have $\Delta_j = \Phi_{20} - \Phi_{11}$, $\sigma_j = \text{sgn}(\Delta_j)$, $\delta_j = \text{sgn}(\Phi_{02})$, and all quantities are evaluated at the stationary points, $u = u_0^j$. This gives the normalized scalar field due to one image of the source, with a maximum amplitude

$$A_j = |J|^{-1/2} = \frac{a_x a_y}{r_{\ell}^2 |\Delta_j|^{1/2}}$$

and an oscillating component with phase

$$\beta_j = \Phi_j + \frac{\pi}{4} (\delta_j + 1) \sigma_j.$$  (24)
\( n \) real stationary points,
\[
\varepsilon_r(u', \nu) = \sum_{j=1}^{n} \varepsilon_j = \sum_{j=1}^{n} A_j e^{i\beta_j},
\]
(25)

The gain can then be obtained by taking the squared modulus of this last expression, \( G = |\varepsilon_r(u', \nu)|^2 \). This is the “first order” geometrical optics gain. The presence of the oscillatory component in each of the images results in interference. As noted above, this is not correctly captured by Eq. 16.

2.4. Accuracy and regions of applicability

A curious feature of the phasors that emerge from the stationary phase solutions is that they include not only the geometric phase \( \Phi \) but also a potential phase shift related to the signs of the second derivatives at the stationary points. This phase shift is physically associated with the passage of a ray through a caustic. A caustic corresponds to a surface in parameter space that yields a null Jacobian, \( J = 0 \) (Berry & Upstill 1980; Kravtsov & Orlov 1999). As we approach a caustic, \( A_j \to \infty \), and the zeroth and first order geometric optics approximation fail. The reason for this failure is that the approximations do not take into account diffraction effects that occur due to the finite frequency of the waves. Caustics also correspond to boundaries that separate regions in parameter space that contain different numbers of real solutions to the lens equation.

Figure 2 illustrates the difference between the gains obtained from the zeroth and first order approaches in the case of an overdense (DM > 0), two dimensional Gaussian lens described by \( \psi(u) = \exp(-u_x^2 - u_y^2) \) with equal lens scales, \( a_x = a_y \). For a fixed frequency of observation, the wave optics amplification as a function of \( u' \) can be calculated by solving the KDI using the FFT. The left column shows this amplification as a function of \( u' \) for the range \(-5.5 \leq u'_x \leq 5.5 \) and two different values of \( \phi_\alpha \), -50 rad (top) and -250 rad (bottom). The white circles correspond to the caustics, and the straight white line denotes the observer’s path along the plane. The right column shows the zeroth (red) and first (orange) order gains along the path superposed with the wave optics gain (blue) for both cases.

From the figure, we can see that unlike the zeroth order approximation, the first order approach is able to reproduce the wave optics oscillations accurately in bright regions that contain more than one real solution to the lens equation. However, wave optics also predicts that in regions with only one real image of the source, the observer should still see an interference pattern that decays (grows) exponentially as she crosses from the caustic’s bright (dark) side to the dark (bright) side.

For instance, the Gaussian lens from the figure shows two sets of circular caustics in the \( u' \) plane. An observer crossing the \( u' \) plane through its center will pass through three regions in which the form of the gain is qualitatively different. Far away from the two caustic zones, \( G = 1 \), and the intensity shows no modulations due to lensing. This corresponds to the dark side of the outer caustic. As the observer approaches the outer caustic singularity, the intensity starts showing oscillations whose amplitude grows exponentially, even though there is still only one real solution to the lens equation. Crossing into the region between the two caustics, the oscillations’ amplitude reach a peak shortly after the boundary, and the observer sees three images corresponding to three real solutions to the lens equation. After that, the amplitude decays and then recovers, peaking right next to the boundary that separates the bright region from the dark side of the inner caustic. This dark region contains a single, highly demagnified image of the source, but there is still a hint of an exponentially decaying interference pattern that disappears a short distance away from the boundary. After crossing the center, an equivalent pattern is observed in reverse as the observer moves from the inner dark side to the bright region and then to the outer dark side. Clegg et al. (1998), Melrose & Watson (2006), and Cordes et al. (2017) studied an analogous lens shape in one dimension. The short paper by Stinebring et al. (2007) presents similar two dimensional plots without the geometrical optics curves.

In summary, then, although the inclusion of ray interference dramatically improves the accuracy of the geometric optics approximation, this approach is still unable to reproduce the correct form of the gain close to the caustic singularity (where it blows up), and on the dark side of caustic boundaries (where it fails to account for oscillations).

3. SECOND ORDER GEOMETRIC OPTICS

3.1. Complex rays

So far, we have limited our analysis to the case in which coordinates in the \( u \) plane, and the solutions to the lens equation are purely real. In order to reproduce the oscillations that occur in the caustic shadows, however, it is necessary to extend the analysis to the complex plane. When two or more real roots of the lens equation merge at a caustic, they reemerge at the dark side as a complex conjugate pair of solutions to the lens equation that yield a complex phase \( \Phi = \Phi_r + i\Phi_i \). \( \Phi_i > 0 \) grows as we move farther into the shadow side in parameter space. From Eq. 19, we can write the field \( \varepsilon_\pm \) due to this complex conjugate pair as
\[
\varepsilon_\pm(u', \nu) = Ae^{\mp \Phi_r} e^{i\beta_\pm},
\]
(26)
where \( A = a_x a_y |\Delta_\pm|^{-1/2} / r_F^2 \) is the same as in the case of a purely real stationary point (Eq. 24), and
\[ \beta^c_{\pm} = \Phi_r + \pi/2 - \text{Arg}\left(\Delta_{\pm}\right)/2. \]

This expression implies that \( \varepsilon^c_{\pm} \) decreases exponentially as a function of \( \Phi_t \), whereas \( \varepsilon^c_{\pm} \) increases exponentially. The exponentially increasing solution can be disregarded as unphysical (Kravtsov et al. 1999), but the exponentially decaying contribution can be included as part of the asymptotic approximation to the KDI. Doing so effectively reproduces the shadow side oscillatory pattern predicted by wave optics, as long as we remain far enough away from the caustic. At the caustic, the complex conjugate pair of solutions merge, and \( A \to \infty \).

The idea of looking for complex solutions to the lens equation has surfaced in a variety of contexts. Schramm & Kayser (1995) apply the concept to gravitational lensing, and Budden & Terry (1971) apply it in the context of radio ray tracing in the atmosphere. There is also a direct connection between complex stationary points, the method of steepest descent, and hyperasymptotics of oscillatory integrals (Kaminski 1994; Howls 1997).

### 3.2. Caustic location and extent of the caustic zone

In the language of geometric optics, caustics correspond to envelopes of families of rays, and are formed at the surfaces on which rays cross each other. Determining the parameter values for ray crossings to occur is, in general, a non trivial problem in more than one dimension and for an arbitrary lens shape. For a fixed frequency of observation, the necessary condition is that

\[
(1 + \alpha_x \psi_0) (1 + \alpha_y \psi_0) - \alpha_x \alpha_y \psi_1^2 = 0
\]

for at least some value of \( u \). If this is the case, caustic curves will show up in the \( u' \) plane, and their form in the \( u' \) plane can be determined by mapping these curves via the lens equation. The caustic curves plotted over the colormaps in the left column of Figure 2 were constructed using this method.

On the other hand, for a fixed \( u' \) coordinate, the locations of caustics in the frequency line need to be determined by solving the set of equations

\[
\frac{\psi_{10} \psi_{01}}{\Delta u_x \Delta u_y} + \frac{\psi_{20} \psi_{10}}{\Delta u_y} + \frac{\psi_{02} \psi_{10}}{\Delta u_x} + \psi_{20} \psi_{02} - \psi_{11}^2 = 0
\]

\[
\left(\frac{\alpha_x}{\alpha_y}\right)^2 \frac{\Delta u_x}{\psi_{10}} - \frac{\Delta u_y}{\psi_{01}} = 0
\]

for \( u \), where \( \Delta u_{x,y} = u'_{x,y} - u_{x,y} \). The caustics will be located at frequencies \( \nu_{\text{caus}} \), given by

\[
\nu_{\text{caus}} = \frac{c}{\alpha_x} \left[ \frac{d_{sl} d_{lo} r_e DM}{2 \pi d_{so} \Delta u_x} \psi_{10} \right]^{1/2}
\]

\[
= \frac{c}{\alpha_y} \left[ \frac{d_{sl} d_{lo} r_e DM}{2 \pi d_{so} \Delta u_y} \psi_{01} \right]^{1/2},
\]

evaluated at the solutions of Eq. 28 for which the argument under the square root is positive. Numerical results indicate that the formation of caustics at fixed frequencies, for lenses with Gaussian-like shapes (with a maximum electron column density at the center that falls off relatively quickly) occurs when \( \alpha_{x,y} \approx -1.2 \) for the positive \( \text{DM}_r \) case and \( \alpha_{x,y} \approx 0.5 \) for the negative \( \text{DM}_r \) case. If both \( \alpha_x \) and \( \alpha_y \) satisfy this condition, two sets of caustics form; if only one does, just one set appears.

A consequence of this requirement is that larger lenses require larger magnitudes of \( \text{DM}_r \) in order to form caustics in the \( u' \) plane at a fixed frequency of observation. Thus, small values of \( \text{DM}_r \) will only lead to caustic formation in cases involving small lenses or highly elongated lenses. For example, keeping the relevant distances fixed at \( d_{so} = 1 \) kpc and \( d_{sl} = 0.5 \) kpc, a value of \( \text{DM}_r = \pm 10^{-6} \) pc cm\(^{-3} \), which corresponds to a lens phase of \( \phi_0 \approx \mp 33 \) rad at 0.8 GHz, yields a maximum value of \( \alpha_{x,y} \approx 2.4 \times 10^{-2} \) AU for the overdense case and \( \alpha_{x,y} \approx 3.6 \times 10^{-2} \) AU for the underdense case. Ray crossings for lenses with \( \alpha_{x,y} \approx 1 \) AU would require a minimum value of \( |\text{DM}_r| \approx 2 \times 10^{-3} \) pc cm\(^{-3} \) for the diverging lens and \( |\text{DM}_r| \approx 7 \times 10^{-4} \) pc cm\(^{-3} \) for the converging lens, which correspond to lens phases at 0.8 GHz of \( \phi_0 \approx -5.8 \times 10^4 \) rad and \( \phi_0 \approx 2.4 \times 10^4 \) rad, respectively. Changing the lens-observer distance \( d_{so} \) and source-lens distance \( d_{sl} \) also leads to changes in \( \alpha_{x,y} \), although not in a very simple way because the value of \( d_{lo} = d_{so} - d_{sl} \) also factors into the expression. In general, however, if we keep \( d_{sl} \) fixed at \( d_{so}/2 \), increasing \( d_{so} \) also increases \( \alpha_{x,y} \) and makes caustic formation more likely. The radius of the caustic curves tends to increase linearly with \( |\alpha_{x,y}|^3 \). 

Figure 3 shows the caustics in the dynamic spectra of a lensing event for underdense (left) and overdense (right) Gaussian lenses for multiple paths along the \( u' \) plane, constructed by repeated application of Eq. 28 and Eq. 29 over a range of \( u' \) coordinates. Although the lens parameters are identical in both cases, it can be seen that flipping the sign of \( \text{DM}_r \) generates a completely different set of caustic curves, and that the path of the observer through the \( u' \) plane can also significantly alter the caustic shapes.

Caustics will show up as a function of \( \nu \) at a fixed value of \( u' \) if we search within a range of frequencies that contains a value of \( \nu \) that leads to at least one of the \( \alpha_{x,y} \) parameters having a magnitude larger than the

\[^3\]An important exception is the underdense \((\text{DM}_r < 0)\) Gaussian lens with \( \alpha_x = \alpha_y \), which presents an infinitely small caustic at the center corresponding to a focus, and a single circular caustic surrounding it.
Figure 3. Caustic curves in the dynamic spectra of underdense (left) and overdense (right) two dimensional Gaussian lenses for different paths across the \( u' \) plane. The blue caustics derive from a path with slope \( m = 1 \) and \( y \)-intercept \( n = 0 \), the red caustics have \( m = 0.5 \) and \( n = 1 \), the green caustics correspond to \( m = 0 \), \( n = 1.5 \), and the grey caustics are produced by \( m = 0.3 \) and \( n = 2 \). We use a value of \( DM_\ell = \pm 10^{-3} \) pc cm\(^{-3} \), which corresponds to a lens phase \( \phi_0 \approx \pm3 \times 10^4 \) rad. The source-observer distance \( d_{so} = 1 \) kpc, and the source-lens distance \( d_{sl} = 0.5 \) kpc in both cases. Both lenses have scales \( a_x = 0.5 \) AU and \( a_y = 1 \) AU.

required minimum. Since \( |\alpha_{x,y}| \propto \nu^{-2} \), caustic curves in dynamic spectra, such as the ones depicted in Figure 3, will show up only at low frequencies.

In practical terms, it is useful to be able to locate caustics as functions of both \( u' \) and \( \nu \). Telescope observations made during an observing epoch correspond roughly to observations made at a fixed \( u' \). Observations with a large enough frequency range would in principle allow us to see the effects of caustics (under the right circumstances) in a single epoch of observation if a lensing event is taking place. At the same time, since the coordinates in \( u' \) change as a function of time, we also expect to see caustic effects in observations made within a narrow frequency band over a range of epochs.

At a caustic boundary, two or more images of the source appear or merge, depending on whether the caustic is crossed from one side or the other. In other words, the number of real roots of the lens equation changes by at least two. The first order geometric approximation breaks down in the vicinity of the caustic when two or more images of the source become indistinguishable from each other. As noted by Kravtsov & Orlov (1999), a useful operational definition for the width of the caustic zone is the boundary at which the absolute value of the geometrical phase difference \( |\Delta \Phi_{ij}| \) between two or more roots is less than \( \pi \),

\[
|\Delta \Phi_{ij}| \lesssim \pi, \quad (30)
\]

where \( i, j \) are the labels of each of the roots. The number of coalescing images determines the type of caustic, as it describes the kind of singularity, or catastrophe, that occurs within the caustic zone.

A number of previous works (Chako 1965; Bleistein & Handelsman 1975; Cooke 1982; Wong 2001; Cordes et al. 2017) have dealt with the problem of obtaining the maximum gain within this region by employing an extension of the stationary phase method to approximate the gain at the singularity. Although the derived formulae (some of which are presented in Appendix B) are relatively simple to apply and can be useful for some types of analyses, it is not in general correct because the geometric optics approximation breaks down some distance away from the caustic, close to the boundary defined by Eq. 30.

3.3. Gain inside the caustic zone: catastrophe theory and uniform asymptotics

Catastrophe theory, first developed by the mathematician René Thom (Thom 1972) and subsequently applied to optics by Sir Michael Berry and others (Berry 1976; Nye 1978; Berry & Upstill 1980), provides a useful way of categorizing geometric optics singularities. The basic idea is that close to a caustic, the phase function can be locally mapped into a standard form that is determined by the number of merging images. This standard form is expressed in terms of a fixed number of state and control variables, which are related by the mapping to the physical variables. Solving the KDI for the particular case of
this standard form yields a transitional approximation that describes the gain within the caustic region.

In general, it is very difficult to rigorously construct a mapping that takes the global form of the phase to the standard form. Instead, the mapping is performed by expanding the phase in a Taylor series at the point that satisfies both the lens equation and Eq. 27, in addition to rotating and scaling the coordinate system such that it is possible to match the coefficients present in this form of the phase to the standard form of the catastrophe. This procedure is described in Kravtsov & Orlov (1999), and performed specifically for the case of two dimensional scattering screens in the context of scintillation by Goodman et al. (1987).

Watson & Melrose (2006) rely on an analogous procedure to derive the one dimensional transitional approximation for the case of two merging images, which corresponds to a fold caustic. The fold catastrophe is the first of the seven elementary catastrophes described by Thom in his original work, and it is the simplest to model and describe. In the vicinity of the fold, the phase can be locally mapped to a cubic, and the KDI can be mapped into the canonical integral (Berry & Upstill 1980)

\[
I_{\text{fold}}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \exp \left[ i \left( \frac{t^3}{3} + \xi t \right) \right] = \sqrt{2\pi} \text{Ai}(\xi), \tag{31}
\]

where \( \xi \) denotes a control variable and \( t \) denotes a state variable, and \( \text{Ai}(\xi) \) is the Airy function. The observer sees no real images on the dark side of the caustic, and two images on the bright side, but the intensity at the dark side does not drop to zero instantly as predicted by first order geometric optics. For practical purposes, it is possible to adopt this transitional form within the caustic region, and revert back to the regular geometrical optics description far away from the caustic, as in Watson & Melrose (2006).

A better, more general solution is to employ the method of uniform asymptotics, as initially developed by Chester et al. (1957), Ursell (1965), and Ludwig (1966) for oscillatory integrals, and later explicitly applied to optics and related to catastrophe theory by Kravstov (1968) and Kravtsov & Orlov (1999). This solution enables us to describe the gain in regions both close and far away from the caustics by the application of a single, global expression that employs the integral of the standard form associated with the type of catastrophe involved, the derivatives of this integral, and some combination of the parameters derived from geometrical optics. Close to the caustic, the expression behaves like the transitional approximation, and far away from it, it matches the field given by the regular geometrical optics approximation.

Uniform asymptotic expressions for the fold caustic have been derived by multiple authors starting with Chester et al. (1957), and in general there are slight variations between each of the presented expressions. We derive it here in an intuitive manner.

The general scheme consists in starting with an ansatz with the same number of terms as there are rays involved in the formation of the caustic, one term involving the function corresponding to the canonical caustic integral, and the rest involving its derivatives. Each of these terms is multiplied by an unknown coefficient, and their sum is multiplied by a phasor. For the fold caustic, it is possible to construct the uniform asymptotic simply by starting with the ansatz and matching the relevant parameters to the geometrical optics coefficients far away from the caustic, by employing the asymptotic forms of the Airy function and its derivative for large negative and positive arguments. Thus, for the bright side, we start with an ansatz of the form,

\[
\varepsilon_{\text{bright}}(\mathbf{u}', \nu) = e^{i\gamma} \left[ g_1 I_{\text{fold}}(\xi) + g_2 I'_{\text{fold}}(\xi) \right] = \sqrt{2\pi} e^{i\chi} \left[ g_1 \text{Ai}(\xi) + g_2 \text{Ai}'(\xi) \right], \tag{32}
\]

where \( g_j, \chi, \) and \( \xi \) are all potentially functions of \( \mathbf{u}' \). From Eq. 25, we have that the first order geometrical optics solution in the case of two real rays can be written as

\[
\varepsilon^r(\mathbf{u}', \nu) = A_1 e^{i\beta_1} + A_2 e^{i\beta_2}. \tag{33}
\]

The asymptotic forms of the Airy function and its derivative for large negative argument are the well known formulas,

\[
\text{Ai}(\xi) \approx \frac{1}{\sqrt{\pi}} (-\xi)^{-1/4} \cos \left[ \frac{2}{3} (-\xi)^{3/2} - \frac{\pi}{4} \right], \tag{34}
\]

\[
\text{Ai}'(\xi) \approx \frac{1}{\sqrt{\pi}} (-\xi)^{-1/4} \sin \left[ \frac{2}{3} (-\xi)^{3/2} - \frac{\pi}{4} \right], \tag{35}
\]

which are obtained by applying the one dimensional stationary phase method to the integral in Eq. 31. Defining \( \gamma = 2(-\xi)^{3/2}/3 - \pi/4 \), using Euler’s identity, and substituting into Eq. 32, we get

\[
\varepsilon_{\text{bright}}(\mathbf{u}', \nu) = \frac{e^{i\chi}}{\sqrt{2}} \left\{ e^{i\gamma} \left[ g_1 (-\xi)^{-1/4} + ig_2 (-\xi)^{-1/4} \right] \right\} = +e^{-i\gamma} \left[ g_1 (-\xi)^{-1/4} - ig_2 (-\xi)^{-1/4} \right]. \tag{36}
\]

Matching this to Eq. 33, we obtain two sets of equations that can be used to determine \( g_1, g_2, \chi, \) and \( \xi \) in terms of the geometrical optics amplitudes \( A_j \), and the phases \( \beta_j \). The first set is

\[\text{Also see Ludwig (1966); Connor (1973a); Stamnes (1986); Borovikov & Kinber (1994); Kravtsov & Orlov (1999); Qiu & Wong (2000); Katsaounis et al. (2001)\]
Figure 4. Comparison of the gains obtained from a full numerical solution of the KDI and second order geometric optics. The left column shows color maps of the gain obtained by solving the KDI via the FFT. The white circles correspond to caustic curves, and the straight white line shows the path of the observer through the \( u' \) plane. The right column shows the gain along this path as calculated via the FFT method (blue) and via second order geometric optics (red). The points of intersection between the caustics and the observer path are marked by white points in the left column and by dashed vertical black lines on the plots in the right column. The top panels shows results for an underdense elliptical Gaussian lens with \( \psi(u) = \exp(-u_x^2 - u_y^2) \), lens phase \( \phi_0 = 100 \) rad, and lens scales \( a_x = 2 \times 10^{-2} \) AU and \( a_y = 3 \times 10^{-2} \) AU. The bottom panel corresponds to an overdense ring-like lens with \( \psi(u) = 2.72 (u_x^2 + u_y^2) \exp(-u_x^2 - u_y^2) \), lens phase \( \phi_0 = -30 \) rad, and lens scales \( a_x = 2 \times 10^{-2} \) AU and \( a_y = 3 \times 10^{-2} \) AU. The frequency of observation is \( \nu = 0.8 \) GHz, \( d_{so} = 1 \) kpc, \( d_{sl} = 0.5 \) kpc for both the top and bottom panels. The central caustic at the center of both \( u' \) planes in the left column occur because \( a_x \neq a_y \), and is known as a structurally stable caustic of primary aberration (Berry & Upstill 1980).

The second set of equations is

\[
A_1 = \frac{1}{\sqrt{2}} \left[ g_1(-\xi)^{-1/4} + ig_2(-\xi)^{1/4} \right] \\
A_2 = \frac{1}{\sqrt{2}} \left[ g_1(-\xi)^{-1/4} - ig_2(-\xi)^{1/4} \right]. \tag{37}
\]

Solving for \( g_1 \) and \( g_2 \) gives

\[
\begin{align*}
g_1 &= \frac{1}{\sqrt{2}} (A_1 + A_2)(-\xi)^{1/4} \\
g_2 &= i \frac{1}{\sqrt{2}} (A_1 - A_2)(-\xi)^{-1/4}.
\end{align*} \tag{38}
\]

which leads to

\[
\begin{align*}
\chi + \gamma &= \beta_1^2, \\
\chi - \gamma &= \beta_2^2,
\end{align*} \tag{39}
\]

Putting everything together, we obtain the uniform
asymptotic for the fold caustic’s bright side,
\[ \varepsilon_{\text{bright}}(u', \nu) = \sqrt{2\pi} e^{i\chi} \left( (A_1 + A_2)(-\xi)^{1/4} \text{Ai}(\xi) + i(A_1 - A_2)(-\xi)^{-1/4} \text{Ai}'(\xi) \right). \] (41)

The ambiguity in the labeling is resolved by the condition \( \beta_1 - \beta_2 + \pi/2 > 0 \). The merging rays will in general have opposite parities, with \( \beta_j^+ - \Phi_j = 0 \) for one ray and \( \beta_j^+ - \Phi_j = \pm \pi/2 \) for the other, so this condition is equivalent to \( \Phi_1 - \Phi_2 > 0 \). Note that even though \( A_1 \) and \( A_2 \) diverge as they approach the singularity, the quantity \((A_1 + A_2)(-\xi)^{1/4}\) goes to a finite limit, because \( \xi \to 0 \) at the caustic. By the same token, although \((A_1 - A_2)(-\xi)^{-1/4}\) goes to infinity at the singularity, the quantity \((A_1 - A_2)(-\xi)^{-1/4}\) does not, because \( A_1 - A_2 \to 0 \).

At the caustic’s dark side, we know from §3.1 that far from the singularity, the geometrical optics field reduces to that of a single complex ray, \( \varepsilon_{\text{+}} = Ae^{-\Phi_j e^{i\beta_j^+}} \). Therefore, our ansatz no longer contains \( \text{Ai}'(\xi) \). Instead, we have that
\[ \varepsilon_{\text{dark}}(u', \nu) = \sqrt{2\pi} e^{i\chi} g_0 \text{Ai}(\xi). \] (42)

The asymptotic of \( \text{Ai}(\xi) \) for large positive argument is
\[ \text{Ai}(\xi) \approx \frac{\exp\left(-\frac{2}{3} \xi^{3/2}\right)}{2^{1/4} \sqrt{\pi}}. \] (43)

Matching coefficients as before, we obtain \( \chi = \beta_+^-, \xi = \left[\frac{3}{2} \Phi_j\right]^{3/2} \), and \( g_0 = A \xi^{1/4} \sqrt{2} \). Thus,
\[ \varepsilon_{\text{dark}}(u', \nu) = 2 \sqrt{\pi} e^{i\beta_-^\epsilon} A \xi^{-1/4} \text{Ai}(\xi). \] (44)

Again, even though at the caustic \( A \to \infty \), the expression does not diverge because \( \xi \to 0 \) at the same point.

### 3.4. Uniform asymptotics in plasma lensing

For the present case of plasma lenses and astrophysical distances, the idealized situation presented above involving two real images on the caustic’s bright side and no real images on its dark side does not actually occur, as the lens is not opaque and the immense distances allow the initial cone of emitted radiation to grow to a size much larger than that of the lens by the time the two encounter each other. Thus, Eqs. 41 and 44 cannot be applied directly as given: there will always be at least one real ray involved in the description of the field, and the total number of rays will always be odd. The more general way of dealing with such a situation would be to implement the uniform asymptotic for the next catastrophe in the series, the cusp. The canonical integral in that case is

\[ I_{\text{cusp}}(\xi_1, \xi_2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \exp \left[ i \left( \xi_1 t + \frac{\xi_2 t^2}{2} - \frac{t^4}{4} \right) \right], \] (45)

which is related to the Pearcey integral \( P(\xi_1, \xi_2) \) (Pearcey 1946) by the relationship \( I_{\text{cusp}}(\xi_1, \xi_2) = P^*(-\sqrt{2}\xi_1, -\xi_2)/\sqrt{\pi} \). The observer sees three images in the bright side and one image in the dark side, which is exactly what happens for the Gaussian lens analyzed in Figure 2. The corresponding ansatz for the bright side of the caustic would then be

\[ \varepsilon(u', \nu) = e^{i\chi} \left[ g_1 I_{\text{cusp}}(\xi_1, \xi_2) + g_2 \frac{\partial I_{\text{cusp}}}{\partial \xi_1} + g_3 \frac{\partial I_{\text{cusp}}}{\partial \xi_2} \right]. \] (46)

Finding the unknown parameters \( g_j, \xi_j, \chi \), however, is not possible via implementation of the same matching procedure we used above for the fold caustic. One reason for this is that the asymptotic forms of the Pearcey integral are much more complicated than those of the Airy integral (Paris 1991). Instead, the correct strategy involves obtaining systems of equations for the relevant quantities by exploiting the correspondence between the phase function of the canonical integral and the phase function of the KDI at the stationary points, as described in detail by Connor (1973b) and Katsanis et al. (2001). Unfortunately, in the case of cusps and higher order catastrophes, it is not possible to express all the unknown parameters as a function of the geometrical optics quantities in a simple form.

For practical purposes, however, this is rarely necessary. Cusps correspond to points in which three solutions of the lens equation merge. These points are connected to each other by curves which correspond to fold catastrophes, where only two images merge. Far from these cusp points, Eq. 46 can be written as the sum of the uniform asymptotic for the fold caustic and the regular geometrical optics contribution from each of the \( n \) images not involved in the formation of the fold lines. This also holds for higher order catastrophes. Thus, as long as we are not too close to catastrophes of higher order, the total field can be written as

\[ \varepsilon(u', \nu) = \varepsilon_{\text{fold}} + \sum_{j=1}^{n} A_j e^{i\beta_j^\epsilon}, \] (47)

where \( \varepsilon_{\text{fold}} \) is given by Eq. 41 or Eq. 44 depending on whether we are at the caustic’s dark side or bright side.

Figure 4 shows a comparison between the gain obtained from the FFT and that obtained using the uniform asymptotic formulas for a slice across the \( u' \) plane, for two different lens shapes \( \psi \) and lens phases \( \phi_0 \). Unlike the case of the circular Gaussian lens with positive \( DM_\ell \) depicted in Figure 2, the lenses in these figures show cusps as well as folds. In Figure 4, both the elliptical Gaussian with \( DM_\ell < 0 \) (top panels) and the ring-like lens with \( DM_\ell > 0 \) (bottom panels) show fold lines interrupted by cusp points at which three roots merge and the curvature of the fold lines is reversed. The num-
Figure 5. Sections of dynamic spectra and slices across them for overdense and underdense perturbed Gaussian lenses with \( \psi(u) = \exp(-u_x^2-u_y^2) \{ 1 - A [\sin(Bu_x) + \sin(Bu_y)] \} \) and different DM\(_L\) magnitudes. The left column shows the two dimensional spectrum for both lenses, with the top row corresponding to the overdense lens and the bottom row to the underdense lens. The vertical and horizontal lines correspond to the slices across the spectra plotted in the right column. Caustic intersections are marked by white dots in the left column plots and by dashed black lines in the right column plots. The overdense lens has a maximum column density of DM\(_L\) = 10\(^{-4}\) pc cm\(^{-3}\) and lens scales of \( a_x = 0.1\) AU and \( a_y = 0.2\) AU, whereas the underdense lens has DM\(_L\) = 10\(^{-5}\) pc cm\(^{-3}\), and \( a_x = a_y = 0.04\) AU. Both lenses have perturbation parameters \( A = 1.5 \times 10^{-2} \) and \( B = 5 \), source-observer distance \( d_{so} = 1\) kpc, and source-lens distance \( d_{sl} = 0.5\) kpc. The path through the \( u' \) plane in both cases is a straight line with slope \( m = 0.5 \) and y-intercept \( n = 2.5 \).

The number of images that can be seen varies depending on the position in the \( u' \) plane and the type of lens. For the negative DM\(_L\) elliptical Gaussian, the observer sees one image in the dark side of the outer caustic zone, three images after crossing the outer caustic boundary, and five images in the central caustic. For the ring-like lens in the bottom panels, the number of images is equal to one outside the caustic zones, three inside the mirrored crescent shaped caustics and in between the two central caustic curves, and five at the center. Other lens shapes can show larger numbers of images and catastrophes of higher order. Some examples are shown in Appendix D.

3.5. Advantages of second order geometric optics

As long as \( |\phi_0| \gg 1 \), second order geometric optics is able to produce remarkably accurate results. Unlike the FFT method, it can be implemented for essentially arbitrary values of \( a_x, a_y \), and \( \phi_0 \) without difficulty. We have applied the second order approach only to the case of slices across the \( u' \) plane at a fixed frequency of observation, but the equations for the field hold identically if we were to vary any of the parameters present in the phase function Eq. 18. Thus, we can use second order geometric optics to produce accurate plots of the gain as a function of \( \nu \) at a fixed position in the \( u' \) plane. Even for small values of the lens scales, constructing such a plot using the FFT would be extremely computationally expensive, as it would require performing two dimensional FFTs at each frequency of observation.

Using the concepts developed so far, we can construct sections of the dynamic spectrum of a lens event, at least
for the case in which these show no cusps. Plots of the gain as a function of position along a line in the $u'$ plane at a single frequency will then correspond to horizontal slices of the dynamic spectrum, whereas plots of the gain as a function of $\nu$ at fixed $u'$ coordinates will correspond to vertical slices. This is illustrated further in Figure 5. From the figure, it is also apparent that larger magnitudes of the maximum column density $|DM|$ induce faster oscillations in the gain, and the contributions from complex rays in the shadow sides of caustics become less important.

4. ASTROPHYSICAL APPLICATIONS

4.1. TOA perturbations

One of the important potential effects of plasma lensing, in particular with regards to its consequences to pulsar timing, is the issue of perturbations in pulse arrival times. The importance of these potential perturbations has been clear for a long time (see, e.g. Cordes & Wolszcan 1986; Cordes et al. 1986) and has resurfaced more recently given the potential of PTAs to detect low frequency gravitational waves (Cordes & Shannon 2010; Cordes et al. 2016) and in the context of FRBs (Cordes et al. 2017; Dai & Lu 2017). Our analysis will rely on examples that use parameters that are more likely to be relevant for pulsar timing, where the resulting perturbations are in the order of microseconds, and the distances place the source and lens inside the Milky Way galaxy. Nevertheless, the same concepts can be applied to the FRB case by increasing the distances, the lens sizes, and the magnitude of the maximum dispersion measure perturbations.

4.1.1. Geometry and dispersion

Refractive due to plasma lensing invariably introduces a geometric delay into the time of arrival of radiation, independently of whether the lensing effect is produced by an underdensity or an overdensity in the interstellar medium. By Fermat’s principle, an unlensed ray will travel in a straight line from the source to the observer, and lensing introduces a deviation from this straight path. Referring to the geometry of Figure 1, we can write the magnitude of the geometric delay $\Delta t_{\text{geo}}$ as

$$\Delta t_{\text{geo}} = t_{g_u}(u_x - u'_x)^2 + t_{g_y}(u_y - u'_y)^2,$$

(48)

where the $t_{g_u}$ and $t_{g_y}$ are the geometrical delay coefficients along the $u_x$ and $u_y$ axes. The location of images in the $u$ plane is determined by the coordinates in the $u'$ plane and the lens equation, so for a given image located at $u = u^0_j$, we can express the geometric delay as

$$\Delta t_{\text{geo}} = t_{g_u}x_j^2 + t_{g_y}y_j^2 \psi_j^2.$$

(49)

Independently of the geometric delay, the lens will also introduce a dispersive perturbation in pulse arrival time due to the plasma’s effect on the radiation’s group velocity $v_g$. For a cold plasma, $v_g = c^2$, which means that the dispersive perturbation in the TOA is given by

$$\Delta t_{DM} = \frac{cr_DDM}{2\pi v^2} \psi(u)$$

$$= 4.149 \times \frac{DM}{v^2} \psi(u),$$

(50)

where the second equality applies for a $DM$ in units of pc cm$^{-3}$ and $v$ in GHz. If $DM > 0$, the dispersive perturbation will introduce a TOA delay, as the column density of electrons along the line of sight will increase. On the other hand, if $DM < 0$, the lens will constitute a “pinhole” in the interstellar medium, and radiation passing through the lens will experience less of a dispersive delay than radiation traveling outside of it.

The total TOA perturbation for each image $\Delta t_j$ is simply the sum of the geometric and dispersive perturbations,

$$\Delta t_j = \Delta t_{geo} + \Delta t_{DM}.$$  

(51)

When $DM > 0$, both perturbations are positive, and the total TOA perturbation will be positive for any combination of parameters, frequency of observation, and position in the $u'$ plane. On the other hand, when $DM < 0$, $\Delta t_j$ can be either positive or negative depending on the relative magnitudes of $\Delta t_{geo}^j$ and $\Delta t_{DM}^j$. For an observer close to the origin of the $u'$ plane and a lens with a maximum dispersion measure perturbation at the center of the lens plane, the maximum TOA advance will occur for solutions to the lens equation that are within the $u'$ plane’s central region, since at these points the geometric delay will be minimum and the dispersive advance will be maximum. For a fixed position in the $u'$ plane, the magnitude of the geometric perturbation for an individual image will decrease as $\nu^{-4}$ (Rickett 1990), whereas the dispersive delay will decrease as $\nu^{-2}$, which means that dispersive delays will dominate geometric perturbations at large frequencies. Geometric delays will grow as a function of the lens size, but larger lenses do not necessarily increase the maximum dispersion measure perturbation, so geometric delays acquire more significance as the lens size grows and DM stays constant. In general, the magnitude of the total TOA perturbation per image decreases as a function of frequency.

Figure 6 shows a sequence of plots of $\Delta t$ along a
Figure 6. Timing perturbations of for pulses corresponding to different images as a function of observer position for overdense (top row) and underdense (bottom row) lenses with $\text{DM}_\ell = \pm 5 \times 10^{-4}$ pc cm$^{-3}$. Different frames in each row correspond to different frequencies of observation. Both overdense and underdense lenses have a Lorentzian shape with $\psi(u) = \frac{1}{[\left(u^2 + u_x^2\right)^{\frac{1}{2}}]^2 + 1}$ and lens scales $a_x = 0.25$ AU, $a_y = 0.4$ AU. The distances used were $d_{\text{so}} = 1$ kpc and $d_{\text{sl}} = 0.5$ kpc, and the path through the $u'$ plane has slope $m = 0.2$ and $y$-intercept $n = 0.5$. The subplot in the top corner of each subpanel shows the (blue) caustic curves in the $u'$ plane for the corresponding frequency of observation, together with the (green) path of the observer through the $u'$ plane. The different colors in the $\Delta t$ vs $u_x'$ plots trace the timing perturbation for each individual image.

path through the $u'$ plane for frequencies of observation 0.8, 1.0, and 1.2 GHz for overdense and underdense Lorentzian lenses with $\text{DM}_\ell = \pm 5 \times 10^{-4}$ pc cm$^{-3}$, which gives a lens phase $\phi_o$ for each of the frequencies of $\mp 1.6 \times 10^4$ rad, $\mp 1.3 \times 10^4$ rad, and $\mp 1.1 \times 10^4$ rad, respectively. For the overdense lens sequence in the top panels, both the geometric and dispersive perturbations are positive. At $\nu = 0.8$ GHz, the geometric contribution dominates over the dispersive contribution, as is apparent by the facts that one, the maximum TOA delay occurs far from the origin of the $u'$ coordinate system, where the geometric perturbation is larger than the dispersive perturbation, and two, the minimum delay in the caustic zone occurs at the origin, where the dispersive delay is maximum and the geometric delay is minimum. Outside of the caustic region, the delay is negligible. As we increase the frequency, it can be seen that the difference between the minimum delay at the center and the maximum delay at the edges of the caustic zone becomes less noticeable, as the magnitude of the geometric delay decreases faster than that of the dispersive delay. The maximum number of images produced in the case of the overdense lens is three, and the caustic pattern is very similar to that of an overdense two dimensional Gaussian like the one depicted in Figure 2.

The bottom panels, corresponding to the lens with $\text{DM}_\ell < 0$, show a different sequence. This time the maximum number of images (five) is seen along the section of the observer’s path through $u'$ that is closer to the center of the caustic region, and the caustic curves form cusps as well as folds. The dispersive perturbation is now negative, and is able to overpower the geometric delay only in regions close to the origin, where the geometric delay is at a minimum. Nevertheless, only one of the five images actually shows a TOA advance.

In both the overdense and the underdense case, we see that the total magnitudes of the perturbations decrease as a function of frequency, and almost no lensing effects are apparent at 1.2 GHz, although this is more dramatic for the underdense lens than for the overdense one. Both the size of the caustic zone and the distance between each of the caustic curves decrease as as a function of frequency, because of the weakening of the lens's refractive power.

4.1.2. Telescope observations of TOA perturbations during a lensing event

The examples from the previous section apply only to the unrealistic case of measurements performed at an infinitely narrow frequency band, and ignore the fact that in general a telescope will be unable to resolve individual images. In reality, the incident electric field $E(t)$ is
sampled as a function of time by the telescope’s receiver, and individual pulse shapes are constructed by taking the Fourier transform of $E(t)$, $\hat{E}(\nu) = \int dt \, E(t)e^{-2\pi i \nu t}$ and transforming back after filtering $\hat{E}(\nu)$ with a band-pass of bandwidth $\Delta \nu_r$ centered on frequency $\nu_0$. Then, the electric field measured by the telescope across a single band $E_{\text{band}}$ can be written as (Cordes & Wasserman 2016)

$$E_{\text{band}}(t, \nu_0; \Delta \nu_r) = \int_{\nu_0 - \Delta \nu_r / 2}^{\nu_0 + \Delta \nu_r / 2} d\nu \, \hat{E}(\nu)e^{2\pi i \nu t}. \quad (52)$$

The pulse profile for the band can then be constructed by taking the square modulus of Eq. 52. The effects of lensing can be quantified as follows. Let an unlensed pulse be described by a normalized electric field $V_0(t)$ and Fourier transform $\hat{V}_0(\nu)$. Then, the lensed pulse over a band $V_{\text{band}}$ will be given by

$$V_{\text{band}}(u', t, \nu_0; \Delta \nu_r) = \int_{\nu_0 - \Delta \nu_r / 2}^{\nu_0 + \Delta \nu_r / 2} d\nu \, \hat{V}_0(\nu)\varepsilon(u', \nu)e^{2\pi i \nu t}, \quad (53)$$

where $\varepsilon(u', \nu)$ is the normalized scalar field from the monochromatic KDI, Eq. 17. As we showed in §3, we can accurately and efficiently solve the KDI using second order geometric optics, by expressing $\varepsilon(u', \nu)$ as a sum of terms of the same form as Eq. 47. In practice, $V_0(t)$ looks like modulated white noise, and the processing of the data captured by the telescope will involve heterodyning to baseband, coherently dedispersing, and the folding of multiple pulses to obtain a better signal to noise ratio. Nevertheless, in the context of numerical simulations, we can get an idea about how lensing events will show up in our data by regarding the unlensed pulse as a unit impulse at $t = 0$, $V_0(t) = \delta(t)$. Then, $\hat{V}_0(\nu) = 1$, and the deviation of $V_{\text{band}}$ from $V_0$ will be exclusively due to the characteristics of the lens and the observing position $u'$. We can mimic how the perturbation will look in real data by convolving $I = |V_{\text{band}}|^2$ with a suitable pulse template and adding white noise. The TOA perturbation for each band can be calculated afterwards using PyPulse\(^6\).

Figure 7 shows numerically simulated pulse dynamic spectra, individual image gains and TOA perturbations, and combined TOA perturbations as a function of frequency for a single epoch of observation (fixed $u'$). In the top panel, lensing occurs as a result of an under-density with Gaussian shape, $\psi(u) = \exp(-u_2^2 - u_y^2)$, whereas in the bottom panel, the lens is overdense with shape $\psi(u) = \exp[-(u_x^2 + u_y^2)^2]$. The parameters used, listed in the figure’s caption, lead to caustic formation in both cases. The effects of the lenses in pulse TOAs vary dramatically as a function of frequency, especially close to the caustics, where the signal to noise ratio can be observed to increase as the image magnification becomes large, and sharp discontinuities arise as images appear or disappear.

The case of the underdense lens is especially complex due to the fact that different images can have either positive or negative TOA perturbations, meaning that the overall pulse TOA can be delayed or advanced depending on the frequency and epochs of observation, the lens parameters, and the lens shape. It is a generic feature of underdense lenses that the number of images tends to decrease as a function of frequency, and thus we expect to be able to observe multiple imaging events more often at low frequencies than at large frequencies. The size of the multiple images regions, as well as the distance between caustics, will in general change as we vary the coordinates in the $u'$ plane, the value of $\Delta M_\ell$, and the size of the lens.

For the overdense lens in the figure, the region of multiple imaging follows a region of very large demagnification of a single image, a behavior that can also be observed in the case of the stochastic Gaussian lens shown in Figure 5, and appears to be generic for the case of Gaussian-like overdense lenses, although more complicated lens shapes can lead to other types of behavior. The signal to noise ratio in the first region is therefore extremely low, and the perturbations are dominated by white noise. The region of multiple imaging shows a gradual increase in the TOA delay as a function of frequency, with the signal to noise ratio increasing as we move closer to the caustic, after which the lensing effects are minimal. Again, we can see sharp discontinuities in the behavior of the perturbations at both caustic points.

4.2. Dispersion measure perturbations

Modern pulsar timing models and pulsar timing packages like TEMPO and TEMPO2 operate on the assumption that the frequency dependent delay of incoming radiation is purely dispersive, with the total delay being given by

$$\Delta t = 4.149 \times \frac{\Delta M_\ell}{\nu^2}, \quad (54)$$

where $\Delta M_\ell$ is in standard units of pc cm$^{-3}$ and $\nu$ is in GHz. Physically, $\Delta M_\ell$ corresponds to the total integrated column density of electrons along the line of sight between the Earth and the pulsar. As discussed in the previous section, a lens changes the dispersive contribution depending on its characteristic shape and the parameter $\Delta M_\ell$, but also introduces a geometric perturbation in the TOAs due to refraction. These perturbations will be different for each image of the source.

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\(^6\)Lam, M. T., 2017. PyPulse, Astrophysics Source Code Library, record ascl:1706.011
Figure 7. Pulse dynamic spectra, individual image gains and TOA perturbations, and combined TOA perturbation as a function of frequency for a single epoch of observation for an underdense lens (top panel) and an overdense lens (bottom panel). The lens in the top panel has Gaussian shape $\psi(u) = \exp(-u_x^2 - u_y^2)$, $DM_{\ell} = -7 \times 10^{-4}$ pc cm$^{-3}$, $d_{so} = 1$ kpc, $d_{sl} = 0.5$ kpc, $a_x = 0.5$ AU and $a_y = 1.1$ AU, and the epoch corresponds to a position in the $u'$ plane with coordinates $u' = (0.1, 0.1)$. The lens in the bottom panel has shape $\psi(u) = \exp \left[-(u_x^2 + u_y^2)\right]$, $DM_{\ell} = 10^{-3}$ pc cm$^{-3}$, $d_{so} = 2$ kpc, $d_{sl} = 1.5$ kpc, $a_x = 0.8$ AU, $a_y = 1.1$ AU, and $u' = (-1.5, -0.55)$. The pulse profile contains 2048 bins, and the pulse repetition period is $T = 5$ ms, giving an integration time $\Delta t_{\text{int}} \approx 2.44 \mu$s. The channel bandwidth is $\Delta \nu_r = 1.5$ MHz. We use a Gaussian template to model the pulse shape.
for the cases in which the lensing is strong enough for ray crossings to occur. Thus, during a strong lensing event like the ones we have analyzed in this work, the expected $\nu^{-2}$ relationship for the group delay will not in general hold. Furthermore, we would expect that attempts at finding the best value of DM according to Eq. 54 will yield different best fit values and different deviations from the expected $\nu^{-2}$ scaling depending on the frequency band. This follows from the fact that the nature of the frequency dependence of the perturbations due to the lens can change drastically as a caustic is crossed, as illustrated in Figure 7. This also means that a lensing event will not necessarily show up in the data as an increase in the $\nu^{-4}$ dependence of the residuals, except in cases in which the frequency band across which the data is being analyzed contains only a single image. A more sophisticated analysis, taking into account the details involved in the operational determination of DM and the way it changes in time, as described in Keith et al. (2013), is outside the scope of this work.

5. SUMMARY AND CONCLUSIONS

We have built on previous works that have studied the phenomenon of astrophysical plasma lensing in the context of ESEs, scintillations, and FRBs by developing a more general formalism that applies to two dimensional plasma lenses formed by both underdensities and overdensities in the ISM, and that can be used to study and predict the many possible ways in which lensing can affect observational quantities such as pulse intensities and TOAs. We showed that the geometrical optics method commonly employed in previous works to construct lensed light curves is unable to properly describe the fluctuations in the gain due to the interference between multiple source images, and is also unable to properly describe the gain within caustic zones.

By incorporating elements of catastrophe theory and the study of uniform asymptotic approximations of highly oscillatory integrals, we have developed an enhanced version of geometric optics that is able to account for such oscillatory features, and that does not break down at caustic curves in which two geometric optic images merge. We showed how this type of geometric optics can be successfully leveraged to construct the flux perturbations due to a variety of lens shapes and sizes, overcoming some of the limitations of other numerical approaches. We also apply some elements of this approach to characterize the possible form of TOA perturbations due to lensing events.

Our results indicate that there are many ways in which lensing effects can present themselves to an observer, depending on the lens shape, the magnitude of the electron density’s departure from the surrounding ISM, whether this departure acquires the form of an overdensity or underdensity, and a series of other parameters such as the lens size, distances, and the frequencies of observation. The two dimensional model also adds an important degree of freedom in the form of the observer’s path through the $u'$ plane, something that cannot be correctly accounted for by one dimensional models. This extra degree of freedom also leads to the appearance of higher order diffraction catastrophes in parameter space that our approach is presently unable to accurately model. We expect to solve this problem in future work, as the successful implementation of uniform asymptotic methods for catastrophes like the cusp can greatly expand the the volume of parameter space that can be explored accurately in simulations.

Consistent with the results of previous works (Goodman et al. 1987; Melrose & Watson 2006; Watson & Melrose 2006; Stinebring et al. 2007), we find that lensing effects tend to be stronger at lower frequencies since the refractive power of plasma is more pronounced at large wavelengths. We also find our results for the overdense Gaussian lens to be consistent with results presented in previous works (Clegg et al. 1998; Stinebring et al. 2007; Cordes et al. 2017; Er & Rogers 2017). Unlike these studies, however, we also analyze underdense Gaussian lenses, and find that their observational consequences are dramatically different from the overdense case. We also apply the uniform asymptotics approach to other types of lens shapes that have not been explored in the past.

The increasing accuracy of pulsar timing methods and procedures, as well as the growing population of pulsars under observation, imply that relatively rare phenomena like lensing events will be observed more often, and that their impact on the timing residuals will be more noticeable. Thus, being able to model such events will become increasingly more important. We expect to apply the methodology outlined in this work to establish whether chromatic aberrations such as the ones reported recently by Coles et al. (2015) and Lam et al. (2018) are indeed the results of lensing phenomena and, if so, develop a model of the lensing structures responsible for such occurrences. The concepts developed here also have direct application to the modelling of ESEs for sources other than pulsars, and it is possible that lensing could be part of the explanation for some of the mysteries surrounding FRBs, which makes future work on this topic all the more important.

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APPENDIX

A. SOLVING THE KDI USING THE FFT

The two dimensional Kirchoff diffraction integral (KDI) introduced in §2.3, gives the normalized wave optics field $\varepsilon$ as a function of the observer coordinates $u'$ by integrating over an angular spectrum of plane waves,

$$\varepsilon(u', \nu) = \frac{a_x a_y}{2\pi r_F^2} \int d^2u \exp(i \Phi),$$

where $\Phi$ is the geometric phase,

$$\Phi(u', u, \nu) = \frac{1}{2r_F} [a_x^2 (u_x - u'_x)^2 + a_y^2 (u_y - u'_y)^2] + \phi \psi(u),$$

with $r_F$ the Fresnel scale, $a_x$ and $a_y$ the lens scales, $\phi_o$ the lens strength parameter, and $\psi$ the lens shape. Given the form of the phase function, the KDI can be written as a two dimensional convolution integral,

$$\varepsilon(u', \nu) = \iint d^2u G(u - u', \nu)H(u, \nu),$$

where

$$G(u, \nu) = \frac{a_x a_y}{2\pi r_F^2} \exp \left( \frac{i}{2r_F} (a_x^2 u_x^2 + a_y^2 u_y^2) \right),$$

$$H(u, \nu) = \exp [i \phi \psi(u)].$$

From the discrete version of the convolution theorem (Schmidt 2010), we have that

$$\varepsilon(u', \nu) = \mathcal{F}^{-1} \{ \mathcal{F} [G(u, \nu)] \cdot \mathcal{F} [H(u, \nu)] \},$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ correspond to the discrete Fourier transform and its inverse, respectively, and $\cdot$ denotes element by element multiplication. Thus, it is in principle possible to solve the KDI numerically for arbitrary lens shapes using the Fast Fourier Transform (FFT). The technique is applied for plasma lenses in one dimension by Watson & Melrose (2006) and Melrose & Watson (2006), and in two dimensions by Stinebring et al. (2007) using code developed by Coles et al. (1995), and we use it in the main text to show that it is possible to use an enhanced version of geometric optics to reproduce the intensity fluctuations predicted by wave optics.

Although useful, this approach suffers from serious limitations. First, it does not give information about the number of images of the source that can potentially be seen by the observer or the respective amplifications, phases, and TOAs of each of these images. Second, in practice the method can only be applied for a restricted range of lens scales $a_{xy}$ and relatively small values of $\phi_o$. The issue is the grid size necessary to properly sample the oscillations of the functions $G(u, \nu)$ and $H(u, \nu)$. We illustrate this for the former case. Consider a lens with characteristic scales $a_x = a_y = a$. By Nyquist’s sampling theorem, the maximum array index $n_{max}$ that can be sampled along a given axis is given by (Schmidt 2010)

$$n_{max} = \frac{\pi r_F^2}{(\Delta x)^2},$$

where $\Delta x$ is the grid spacing in physical units. Now, let $u'_{max}$ be the half-width of the $u'$ plane along either of the axes, and $N$ be the size of the array along that axis. Then, the sampling interval can be written as

$$\Delta x = \frac{2 a u'_{max}}{N}.$$  \hspace{1cm} (A8)

Setting $N = n_{max}$ and rearranging, we have that the size of the grid along one axis required to ensure proper sampling is

$$N = \frac{4 a^2 (u'_{max})^2}{\pi r_F^2}.$$  \hspace{1cm} (A9)

This means that if we want to properly calculate the field for a lens with size $a = 1$ AU up to $u'_{max} = 5$ and with distances $d_{sl} = 0.5$ kpc, $d_{ao} = 1$ kpc, and frequency of observation $\nu = 0.8$ GHz, we need $N \approx 1.5 \times 10^6$. This might be acceptable for the one dimensional case, but a two dimensional grid with side of size $N$ is too big for even a modern desktop computer to handle. A more detailed analysis of sampling constraints and the numerical simulation of wave propagation using Fourier optics can be found in Schmidt (2010).
Figure A1. Left: Colormap of the gain in the \( u' \) plane overlaid with the caustic curves in white and slices along the plane in different colors for two different lens shapes. The points of intersection between the slices and the caustic lines are marked by points. The top panel corresponds to a lens with shape \( \psi(u) = 0.74(u_x^2 + u_y^2) \exp(-u_x^2 - u_y^2) \), and parameters \( a_x = a_y = 0.02 \) AU, \( DM_L = -1.5 \times 10^{-6} \) pc cm\(^{-3} \), \( \nu = 0.8 \) GHz, \( d_{so} = 1 \) kpc, and \( d_{sl} = 0.5 \) kpc. The bottom lens has \( \psi(u) = \exp(-u_x^4 - u_y^4) \), \( a_x = 0.04 \) AU, \( a_y = 0.05 \) AU, \( DM_L = -2 \times 10^{-6} \) pc cm\(^{-3} \), \( \nu = 1.0 \) GHz, \( d_{so} = 5 \) kpc, \( d_{sl} = 2.5 \) kpc. In both cases, \( \phi_0 \approx 50 \) rad. Right: Plots of the gain along the paths shown in the left panel for each lens. Both kinds of lens show folds, cusps, and higher order catastrophes. The top lens can generate up to nine images of the source, whereas the bottom lens can produce up to seventeen.

Perhaps the primary advantage of this numerical strategy is that it does not have any problem calculating the field at caustic regions for any kind of catastrophe, even the higher order ones. Figure A1 shows the gain obtained using this method for different paths through the \( u' \) plane for a lens that shows higher order catastrophes than the ones in the main text.

B. ESTIMATION OF THE VALUE OF THE GAIN AT THE CAUSTIC

For very large values of \( \phi_0 \), it might be desirable in some cases to find the gain due to a lens using zeroth order geometric optics (Eq. 16), since the oscillations due to multiple imaging will give a value of the flux consistent with the prediction from that equation once we take into account the frequency resolution of the observations. Close to the caustics, however, the gain diverges. When \( \phi_0 \) is large and the lens has strong refractive power, the gain can diverge in such a way that the maximum value occurs extremely close to the caustic, and this value can be estimated by the an extension of the method of stationary phase.

This estimate has been derived in more than one dimension by just a handful of authors in the context of asymptotic expansions of integrals, and their results do not necessarily agree with each other. Here we give two of the published formulas, specifically applied to the KDI, although we do not derive them. According to Chako (1965) and Wong
(2001), the gain at the singularity is

\[ G_{\text{max}} = \frac{a_x^2 a_y^2}{12 \pi r^2} \frac{\Gamma^2 (1/3)}{|\Phi_{20}| |\Phi_{03}|^{2/3}}. \]  

(B10)

Bleistein & Handelsman (1975) and Cooke (1982) give a more complicated expression,

\[ G_{\text{max}} = \frac{a_x^2 a_y^2}{4 \pi^2 r^2} |\Phi_{20}| \Gamma^2 (1/3) \left( \frac{32 \pi^2}{3 |B|^2} \right)^{1/3}, \]  

(B11)

where \( B = \Phi_{20}^3 \Phi_{03} - 3 \Phi_{20}^2 \Phi_{11} + 3 \Phi_{20} \Phi_{11}^2 \Phi_{21} - \Phi_{11}^3 \Phi_{30} \). All derivatives of the phase in both equations are evaluated at the degenerate stationary phase point for which \( \Phi_{10} = \Phi_{01} = \Phi_{20} \Phi_{02} - \Phi_{11}^2 = 0 \). Some numerical experimentation has determined that both formulas give similar but not the same results.

C. NUMERICS

The key behind successful application of geometric optics as presented in the main text is the ability to numerically solve the lens equation, Eq. 14. This is essentially a two dimensional nonlinear root finding problem, with the added difficulties that the number of roots can be more than one, and that roots can appear or disappear as a function of the input parameters. A general method for two dimensional root finding consists in rewriting the system of equations in the form

\[
\begin{bmatrix}
  f(x, y) \\
  g(x, y)
\end{bmatrix} = \begin{bmatrix}
  0 \\
  0
\end{bmatrix},
\]

(C12)

where \( f(x, y) \) and \( g(x, y) \) are the two equations that must be solved simultaneously. Once this is done, we can produce contour plots of both equations in order to find the sets of curves that satisfy \( f(x, y) = 0 \) and \( g(x, y) = 0 \). The roots of the two dimensional system will then correspond to the points of intersection of these sets of curves. When implemented properly, this method allows one to find all the roots of a two dimensional system within a range of values for \( x \) and \( y \). A similar idea was pursued by Schramm & Kayser (1987) to solve the lens equation for gravitational lensing. The disadvantage of this scheme is that it requires the evaluation of both \( f(x, y) \) and \( g(x, y) \) in a two dimensional grid that spans the area in which we are looking for solutions, which can be very computationally expensive if done repeatedly.

Since we are interested in solving the equation at many different points in parameter space, it is desirable to find a way to solve the lens equation that does not require us to apply the above algorithm at every single point of the independent variable. We can do this by combining it with other, more efficient numerical techniques that have been developed for numerical root finding in an arbitrary number of dimensions. These have existed for a long time, and are available for a variety of programming languages. In Python, some of these routines are available via the SciPy library’s optimization package. Although more efficient, these algorithms have the limitation that they rely on the user to input a guess solution that must be close enough to the actual solution. Furthermore, if there are multiple roots, they will only find the one closest to the input guess. This means that there is no way to find out exactly how many roots there are for a particular set of parameters.

Our strategy consists in combining the contour plotting method with the optimization algorithms in SciPy. First, we find the caustic locations for the range of parameters that we want to find the solutions of the lens equation for. If we are looking for solutions as a function of \( u' \), we apply the contour plotting algorithm to find the intersections between the curves in the \( u' \) plane that satisfy Eq. 25 with the line \( u'_n = mu'_0 + n \), where \( m \) and \( n \) parameterize the path through the \( u' \) plane. If we are looking for the solutions as a function of \( \nu \), we apply the contour plotting algorithm to simultaneously solve the system of equations given in Eq. 26.

This step allows us to separate the regions in parameter space that contain different numbers of solutions to the lens equation. Now, we can apply the contour plotting method again to find the number of roots at the center of each region. This results in the method being more reliable, because close to region boundaries, at least two roots will be very close to each other, whereas they will be maximally separated at the region’s center. After having found the roots at the center of each of these regions, we find the other roots by iterating forward and backward in parameter space, using the root finding algorithm from SciPy with the previously found roots as the input guess solutions. As long as the distance between neighboring values of the independent variable is small enough, this strategy tends to work. It has the advantage of being much more efficient than applying the contour plotting method repeatedly, and also allows

\footnote{Jones E, Oliphant E, Peterson P, et al. SciPy: Open Source Scientific Tools for Python, 2001-, http://www.scipy.org/}
us to find the roots up to a very close distance to the caustic. This method has been tried for a wide variety of lens shapes and parameters, and has been found to be very reliable, especially for finding the real roots of the lens equation.

In order to find the complex rays, we need to apply a modified version of the above procedure that does not rely on contour plotting. The reason is that extending the search of solutions to the complex plane transforms the two-dimensional lens equation into a four-dimensional equation, and evaluating four different equations in four-dimensional space is not practically feasible. Instead, we exploit the fact that, as discussed in the main text, very close to the shadow side of a caustic, the only important set of complex conjugate solutions to the lens equation is the one that has the smallest magnitude of its imaginary part. The real part of this complex conjugate set will be almost the same as that of the solution to the lens equation that intersects with the singularity. Thus, we use SciPy’s root finding algorithm with a value of the independent variable that falls in the caustic’s shadow side but at the same time is very close to the singularity, and input the value of $u$ at the caustic as the guess solution. From there, we can recursively look for complex solutions that are farther away from the caustic in the same manner as we did for the case of the real solutions.

D. MORE NUMERICAL EXAMPLES AND LENS COLORMAPS

**Figure D2.** Comparison of the gains obtained from a full numerical solution of the KDI and second order geometric optics. The left column shows color maps of the gain obtained by solving the KDI via the FFT. The white circles correspond to caustic curves, and the straight white line shows the path of the observer through the $u'$ plane. The right column shows the gain along this path as calculated via the FFT method and second order geometric optics. The points of intersection between the caustics and the observer path are marked by white points in the left column and by dashed vertical black lines on the plots in the right column. The top panel shows an underdense rectangular Gaussian lens with $\psi(u) = \exp(-u_x^2 - u_y^2)$, lens phase $\phi_0 = 80$ rad, and lens scales $a_x = 1.5 \times 10^{-2}$ AU and $a_y = 3 \times 10^{-2}$ AU. The bottom panel corresponds to an underdense super-Gaussian lens with $\psi(u) = \exp \left(-\left(u_x^2 + u_y^2\right)^3\right)$, lens phase $\phi_0 = 120$ rad, and lens scales $a_x = 2.5 \times 10^{-2}$ AU and $a_y = 4 \times 10^{-2}$ AU. The frequency of observation is $\nu = 1.4$ GHz, $d_{so} = 1$ kpc, $d_{sl} = 0.5$ kpc for both the top and bottom panels.
Figure D3. Individual image TOAs for two different lenses and different paths through the $u'$ plane. The left panel corresponds to a square Gaussian lens with $\psi(u) = \exp\left(-u_x^4 - u_y^4\right)$, $DM_\ell = -5 \times 10^{-4}$ pc cm$^{-3}$, $d_{so} = 1$ kpc, $d_{sl} = 0.5$ kpc, $a_x = 0.5$ AU and $a_y = 0.6$ AU. The frequency of observation is $\nu = 0.8$ GHz, which gives a lens phase of $\phi_0 \approx 1.63 \times 10^4$ rad. The right panel corresponds to a super-Gaussian lens with $\psi(u) = \exp\left[-(u_x^2 + u_y^2)^3\right]$, $DM_\ell = -1 \times 10^{-3}$ pc cm$^{-3}$, $d_{so} = 5$ kpc, $d_{sl} = 2.5$ kpc, $a_x = 0.7$ AU and $a_y = 1$ AU, with $\nu = 1.4$ GHz, and thus $\phi_0 \approx 1.86 \times 10^4$ rad. Different colors denote different images, and the top right subplots show the path of the observer through the $u'$ plane and the caustic curves. The maximum number of images in each plot is seventeen and nine, respectively.

Figure D4. Colormaps of the different types of lensing structures used in the text and appendices. Top row, from left to right: Gaussian lens, $\psi(u) = \exp\left(-u_x^2 - u_y^2\right)$, Lorentzian lens, $\psi(u) = \frac{1}{(u_x^2 + u_y^2)^2 + 1}$, and super-Gaussian lenses $\psi(u) = \exp\left[-(u_x^2 + u_y^2)^2\right]$ and $\psi(u) = \exp\left[-(u_x^2 + u_y^2)^3\right]$. Bottom row, from left to right: Rectangular Gaussian lens, $\psi(u) = \exp\left(-u_x^2 - u_y^4\right)$, square Gaussian lens, $\psi(u) = \exp\left(-u_x^4 - u_y^4\right)$, ring-like lens $\psi(u) = 2.72 (u_x^2 + u_y^2) \exp\left(-u_x^4 - u_y^4\right)$, and double lens $\psi(u) = 0.74 (u_x^2 + u_y^4) \exp\left(-u_x^4 - u_y^4\right)$. 
REFERENCES

Bannister, K. W., Stevens, J., Tuntsov, A. V., et al. 2016, Science, 351, 354
Berry, M. V. 1976, AdPhy, 25, 1
Berry, M. V., & Upstill, C. 1980, in Progress in Optics, ed. E. Wolf, Vol. 18 (Elsevier), 257–346
Bleistein, N., & Handelsman, R. A. 1975, Asymptotic expansions of integrals (Dover Publications)
Born, M., & Wolf, E. 1999, Principles of Optics, seventh edn. (Cambridge University Press)
Borovikov, V. A., & Kinber, B. E. 1994, Geometrical theory of diffraction, Electromagnetic Wave Series No. 37 (The Institution of Electrical Engineers)
Budden, K. G., & Terry, P. D. 1971, RSPSA, 321, 275
Chako, N. 1965, JApMa, 1, 372
Chester, C., Friedman, B., & Ursell, F. 1957, 53, 599
Clegg, A. W., Fey, A. L., & Fiedler, R. L. 1994, ApJ, 430, 581
Clegg, A. W., Fey, A. L., & Lazio, T. J. W. 1998, ApJ, 496, 253
Cognard, I., Bourgois, G., Lestrade, J.-F., et al. 1993, Nature, 366, 320
Connor, J. 1973a, MolPh, 25, 181
Connor, J. N. L. 1973b, MolPh, 26, 1217
Cooke, J. C. 1982, JApMa, 29, 25
Cordes, J. M., & Wasserman, I. 2016, ApJS, 637, 19
Cordes, J. M., Rickett, B. J., Stinebring, D. R., & Coles, W. A. 2006, ApJ, 637, 346
Cordes, J. M., & Shannon, R. M. 2010, https://arxiv.org/abs/1010.3785
Cordes, J. M., Shannon, R. M., & Stinebring, D. R. 2016, ApJ, 817, 16
Cordes, J. M., & Wasserman, I. 2016, MNRAS, 457, 232
Cordes, J. M., Wasserman, I., Hessels, J. W. T., et al. 2017, ApJ, 842, 10
Cordes, J. M., & Woloszcan, A. 1986, ApJ, 307, L27
Dai, L., & Lu, W. 2017, ApJ, 847, 19
Er, X., & Rogers, A. 2017, MNRAS, 475, 867
Fiedler, R., Dennison, B., Johnston, K. J., et al. 1994, ApJ, 430, 581
Fiedler, R. L., Dennison, B., Johnston, K. J., & Hewish, A. 1987, Nature, 326, 675
Goodman, J. J., Romani, R. W., Blandford, R. D., & Narayan, R. 1987, MNRAS, 229, 73
Gupta, Y., Ghat, N. D. R., & Rao, A. P. 1999, ApJ, 520, 173
Gupta, Y., Rickett, B. J., & Lyne, A. G. 1994, MNRAS, 269, 1035
Howls, C. J. 1997, RSPSA, 453, 2271
Kaminski, D. 1994, MApAn, 1, 44
Katsaounis, T., Kossioris, G. T., & Makrakis, G. N. 2001, Math. Models Meth. Appl. Sci., 11, 199
Keith, M. J., Coles, W. A., Shannon, R. M., et al. 2013, MNRAS, 429, 2161
Kerr, M., Coles, W. A., Ward, C. A., et al. 2017, MNRAS, 474, 4637
Krafft, Y. A. 1968, SPhAc, 14, 1
Krafft, Y. A., Forbes, G. W., & Asatryan, A. A. 1999, in Progress in optics, ed. E. Wolf, Vol. 39 (Elsevier), 1–62
Kravtsov, Y. A., & Orlov, Y. I. 1999, Springer Series on Wave Phenomena, Vol. 15, Caustics, catastrophes and wave fields, 2nd edn. (Springer)
Kryukovskii, A. S., Lukin, D. S., Palkin, E. A., & Rastygaev, D. S. 2006, Journal of Communications Technology and Electronics, 51, 1087
Lam, M. T., Ellis, J. A., Grillo, G., et al. 2018, ApJ, 861, 132
Ludwig, D. 1966, Commun. Pure Appl. Math., 19, 215
Main, R., Yang, I. S., Chan, V., et al. 2018, Nature, 557, 522
Melrose, D. B., & Watson, P. G. 2006, ApJ, 647, 1131
Nye, J. F. 1978, RPSA, 361, 21
Paris, R. B. 1991, RPSA, 432, 391
Pearcey, T. 1946, PMag, 37, 311
Pen, U. L., & King, L. 2012, MNRAS, 421, L132
Poston, T., & Stewart, I. 1978, Catastrophe Theory and its Applications (Dover Publications)
Puskarhev, A. B., Kovalev, Y. Y., Lister, M. L., et al. 2013, A&A, 553, A80
Qiu, W.-Y., & Wong, R. 2000, RPSA, 456, 407
Rickett, B. J. 1990, ARA&A, 28, 561
Schmidt, J. D. 2010, Numerical simulation of optical wave propagation with examples in MATLAB
Schneider, P., Ehlers, J., & Falco, E. E. 1992, Gravitational Lenses (Springer)
Schramm, T., & Kayser, R. 1987, A&A, 174, 361
Schramm, T., & Kayser, R. 1995, A&A, 299, 1
Stamnes, J. J. 1986, Waves in Focal Regions: Propagation, Diffraction and Focusing of Light, Sound and Water Waves (Adam Hilger)
Stinebring, D., Matters, J., & Hemberger, D. 2007, in Astronomical Society of the Pacific Conference Series, Vol. 365, SINS - Small Ionized and Neutral Structures in the Diffuse Interstellar Medium, ed. M. Haverkorn & W. M. Goss, 275
Thom, R. 1972, Stabilite structurelle et morphogenese (Benjamin)
Thorne, K. S., & Blandford, R. D. 2017, Modern Classical Physics (Princeton University Press)
Tuntsov, A. V., Walker, M. A., Koopmans, L. V. E., et al. 2016, ApJ, 817, 176
Ursell, F. 1965, PCPS, 61, 113
Vedantham, H. K., Readhead, A. C. S., Hovatta, T., et al. 2017, ApJ, 845, 90
Watson, P. G., & Melrose, D. B. 2006, ApJ, 647, 1142
Wong, R. 2001, Asymptotic approximations of integrals, Vol. 34 (SIAM)