Unification of Gravity and Electromagnetism Revisited

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Abstract

Within the context of a 5D space-time, we construct a unified theory of gravity and electromagnetism from which the Einstein field equations and Maxwell equations emerge, with homogeneous Maxwell equations appearing naturally. We also introduce a well-defined five dimensional energy-momentum tensor consistent with our unification scheme. A correction term appears in Maxwell equations which can be used to explain the recently discovered galactic magnetic fields.

1 Introduction

After Einstein published the general theory of relativity in 1915, numerous attempts were made to generalize the theory in such a way as to encompass both gravitation and electromagnetism in a unique geometrical structure. Three historically important such generalizations which are relevant to our present work here are the Einstein-Schrodinger theory, the Weyl Geometry and the Kaluza-Klein theory, briefly reviewed below.

The Einstein-schrodinger Theory \cite{1} is a generalization of vacuum general relativity which allows for non-symmetric fields. Such a theory, without a cosmological constant, was first proposed by Einstein and Straus \cite{2, 3, 4, 5, 6}. Later on Schrodinger showed that it could be derived from a very simple Lagrangian density if a cosmological constant was included \cite{7, 8, 9}. Einstein and Schrodinger suspected that the theory might include electrodynamics, but no Lorentz force was found \cite{10, 12} when using the Einstein-Infeld-Hoffmann (EIH) method \cite{10, 11}. For detail study of non-symmetric fields take a look at \cite{6}.

The Weyl Geometry \cite{13} came into being after Einstein put forth his general theory of relativity, which provided a geometrical description of gravitation. Later on, Weyl proposed a more general setup which also included a geometrical description of electromagnetism. In the case of general relativity one has a Riemannian geometry with a metric tensor $g_{\mu\nu}$ \cite{14}. If a vector undergoes a parallel displacement in this geometry, its direction may change, but not its length. In Weyl geometry, there is a given vector $k^{\mu}$ which, together with $g_{\mu\nu}$, characterizes the geometry. For any given vector $\xi^{\mu}$ undergoing parallel displacement in this geometry, not only the direction but also the length $\xi$ may change and this change depends on $k_{\mu}$ according to the relation

$$d\xi = \xi k_{\mu}dx^{\mu},$$

(1)
\[ \xi = \xi_0 e^{\int k \mu dx^\mu}, \]  

(2)

where \( \xi_0 \) is the length of the original vector before displacement. The change in the length of \( \xi \) in going from one point to another depends on the path followed, i.e., length is not integrable. Mathematically the Weyl geometry is rich but it seems not to coincide with nature. In general relativity and other main theories of physics, measuring tools are the same for every event. Therefore, We have to use the same clocks and rulers in our experiments. In the Weyl geometry one can arbitrarily regauge the measuring tools for every event. Einstein is among those who have objected the theory. A very detail review on the Weyl geometry can be found in [15].

The Kaluza-Klein (KK) theory [18, 19, 20, 21] is perhaps the most successful of any generalization of general relativity. Historically, the unification of gravity and electromagnetism was first addressed in this model by applying Einstein’s general theory of relativity to a five, rather than four-dimensional space-time manifold. The field equations would logically be expected to be \( \hat{G}_{\alpha\beta} = k \hat{T}_{\alpha\beta} \) in 5D with some appropriate coupling constant \( k \) and a 5D energy-momentum tensor. However, since the latter is unknown, from the time of Kaluza and Klein onward much attention has been paid to the vacuum form of the field equations \( \hat{G}_{\alpha\beta} = 0 \), where \( \hat{G}_{\alpha\beta} = \hat{R}_{\alpha\beta} - \frac{1}{2} \hat{R} \hat{g}_{\alpha\beta} \) is the Einstein tensor, \( \hat{R}_{\alpha\beta} \), \( \hat{R} \) are the five-dimensional Ricci tensor and scalar respectively, and \( \hat{g}_{\alpha\beta} \) is the five-dimensional metric tensor\(^1\). Equivalently, the defining equations are

\[ \hat{R}_{\alpha\beta} = 0, \quad A, B = 0 \cdots 4. \]  

(3)

These 15 relations serve to determine the 15 components of the metric \( \hat{g}_{\alpha\beta} \), at least in principle. In practice, this is impossible without some starting assumption on \( \hat{g}_{\alpha\beta} \). Kaluza was interested in electromagnetism and realized that \( \hat{g}_{\alpha\beta} \) can be expressed in the form of a 4-potential \( A_\alpha \) which appears in Maxwell’s theory. He adopted the cylinder condition, namely the independence of the components of the metric to the fifth coordinate, but also assumed \( \hat{g}_{44} = \text{const} \). Here we look at the more general case where \( \hat{g}_{44} = -\phi^2(x^\alpha) \). The coordinates or gauges are chosen so as to write the 5D metric tensor in the form

\[
\hat{g}_{\alpha\beta} = \begin{pmatrix}
  g_{\alpha\beta} - \kappa^2 \phi^2 A_\alpha A_\beta & -\kappa \phi^2 A_\alpha \\
  -\kappa \phi^2 A_\beta & -\phi^2
\end{pmatrix},
\]  

(4)

where \( \kappa \) is a coupling constant. The field equations then reduce to

\[
G_{\alpha\beta} = \frac{k^2 \phi^2}{2} T_{\alpha\beta} - \frac{1}{\phi} (\nabla_\alpha \nabla_\beta \phi - g_{\alpha\beta} \Box \phi),
\]

(5)

\[
\nabla^\alpha F_{\alpha\beta} = -3 \frac{\nabla^\alpha \phi}{\phi} F_{\alpha\beta},
\]

(6)

and

\[
\Box \phi = -\frac{k^2 \phi^3}{4} F_{\alpha\beta} F^{\alpha\beta}.
\]

(7)

Here, \( G_{\alpha\beta} \) and \( F_{\alpha\beta} \) are the usual 4D Einstein and Faraday tensors and \( T_{\alpha\beta} \) is the energy-momentum tensor for an electromagnetic field given by

\[
T_{\alpha\beta} = \frac{1}{2} \left( \frac{1}{4} g_{\alpha\beta} F_{\gamma\delta} F^{\gamma\delta} - F_{\alpha}^\gamma F_{\beta}^\gamma \right).
\]

(8)

Equation (5) gives back the 10 Einstein equations of 4D general relativity, but with a right-hand side which in some sense represents the energy and momentum tensor. Kaluza’s case \( \hat{g}_{44} = -\phi^2 = -1 \) together with the identification \( \kappa = \left( \frac{16\pi G}{c^4} \right)^{\frac{1}{2}} \) makes (5) and (6) to read

\[
G_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta},
\]

(9)

\(^1\)Capital Latin indices \( A, B, \cdots \) assume the values \( 0 \cdots 4 \), Greek indices run through \( 0 \cdots 3 \) and the five-dimensional Ricci tensor and Christoffel symbols are defined in terms of the metric as in four dimensions.
and
\[ \nabla^\alpha F_{\alpha\beta} = 0. \] 

Note that this theory does not contain the two homogeneous Maxwell equations, \( \nabla (k F_{mn}) = 0. \) Of course, it goes without saying that the homogeneous Maxwell equations are constraints on the Faraday tensor and are implicit in the KK theory. However, in what follows, we show that within the context of the theory presented here, these equations can be independently obtained from the torsion free condition, in contrast to the conventional KK theory.

All the studies carried out in this regard in the past have found a common problem, namely that the Homogeneous Maxwell equations do not appear naturally. Also in most five dimensional theories there is no obvious reason for the cylinder condition. We try to obtain not only the Einstein and non-homogeneous Maxwell equations but also homogeneous Maxwell equations and a solution to the latter by only one assumption. In this paper we investigate a five dimensional space-time and examine its consequences. This is done by forcing the connection to have certain properties, thus enabling one to define the resulting space-time quantities. We show how our assumption leads to cylinder condition and also how the Einstein field equations in five dimensions separate into the 4-dimensional Einstein and Maxwell equations and that the homogeneous Maxwell equations are included in the theory as \( \hat{R}_{(nmk)_4} = 0. \) The corresponding energy-momentum tensor is then defined and presented. The conclusions are drawn in the last section.

2 Five Dimensional Riemannian Space-Time

Let us start with the line-element for a five dimensional Riemannian space-time
\[ ds^2 = \hat{g}_{AB} d\hat{x}^A d\hat{x}^B, \] 
where a hat represents five dimensional objects while objects without it belong to the ordinary 4D space-time. Here \( \hat{x}^\alpha = x^\alpha \) are ordinary 4D coordinates. Capital latin indices also run over 0, 1, 2, 3, and 4. The contravariant components of the metric are defined as inverse of the covariant components
\[ \hat{g}_{AB} \hat{g}^{BC} = \delta^C_A. \]

In such a space-time, the covariant derivatives are defined as
\[ \hat{\nabla}_A \hat{f}^B = \partial_A \hat{f}^B + \hat{\Gamma}^B_{AC} \hat{f}^C. \]

Now we can define a geodesic path as the straightest possible curve, i.e. a curve whose tangent vectors are connected by parallel transport. The tangent vector to a curve is given by \( \hat{x}^A \hat{e}_A \) where \( \hat{e}_A \) are a set of the basis vectors expanding the space and \( \hat{x}^A = \frac{ds^A}{d\tau} \). The equation representing parallel transportation may then be written as
\[ \hat{x}^A \hat{\nabla}_{\hat{e}_A} \hat{x}^B \hat{e}_B = 0, \]
or
\[ \hat{x}^B + \hat{\Gamma}^B_{AC} \hat{x}^A \hat{x}^C = 0. \]

The 5D curvature tensor, \( \hat{R}^D_{ABC} \), can be defined as follows
\[ \hat{\nabla}_A \hat{\nabla}_B \hat{f}_C - \hat{\nabla}_B \hat{\nabla}_A \hat{f}_C = \hat{R}^D_{ABC} \hat{f}_D, \]
where \( \hat{f}_C \) is an arbitrary 5D vector. Therefore, the curvature tensor will be as follows
\[ \hat{R}^D_{CBA} = \partial_B \hat{\Gamma}^D_{AC} - \partial_A \hat{\Gamma}^D_{BC} + \hat{\Gamma}^D_{BE} \hat{\Gamma}^E_{AC} - \hat{\Gamma}^D_{AE} \hat{\Gamma}^E_{BC}. \]
The Ricci tensor in a five dimensional Riemannian space-time can also be defined as follows

\[ \hat{R}_{CA} = \hat{R}^{B}_{CBA}. \]  

(18)

Although the 5D Ricci scalar is defined as \( \hat{R} = \hat{g}^{CA} \hat{R}_{CA} \), we do not need it in our 5D field equations any more. One may now easily write 5D field equations as

\[ \hat{R}_{AB} - \frac{1}{2} \hat{\mathbf{I}} \hat{g}_{AB} = \kappa \hat{T}_{AB}, \]  

(19)

where \( \hat{R}_{AB} \), \( \hat{g}_{AB} \) and \( \hat{T}_{AB} \) are the 5D Ricci, metric, and energy-momentum tensor respectively. \( \hat{\mathbf{I}} \) is a 5D scalar which can be determined using conservation laws and will be discussed later.

### 2.1 The Setup

At this point, it is appropriate to make our main assumption

\[ \hat{\Gamma}^{4}_{AC} = 0, \]  

(20)

in the whole 5D space-time. Substituting this equation into (15) leads to the fact that \( \dot{x}^{4} = 0 \) for particles that move on geodesics, meaning that for these particles we have \( \ddot{x}^{4} = \text{constant} \). The implication is that the fifth component of the position of a particle in our space is the same as any other. We hope our space-time to include both gravitational and electromagnetic forces which means that even charged particles in presence of electromagnetic waves move on geodesics. As a result, in macroscopic scales one can say that particles never see the fifth dimension. It is something like the cylinder condition which means that the derivative with respect to the fifth coordinate on our 4D hypersurface is zero. We are now in a position to investigate the functional form of the Christoffel symbols which we assume are symmetric in lower indices. Let us begin by considering the well known fundamental equation

\[ \hat{\nabla}_{A} \hat{g}_{BC} = 0, \]  

(21)

which is a result of

\[ \partial_{A} \hat{g}_{BC} = \hat{\Gamma}^{D}_{AB} \hat{g}_{DC} + \hat{\Gamma}^{D}_{AC} \hat{g}_{BD}. \]  

(22)

We may then write

\[ \partial_{A} \hat{g}_{BC} + \partial_{B} \hat{g}_{AC} - \partial_{C} \hat{g}_{AB} = 2 \hat{\Gamma}^{D}_{AB} \hat{g}_{DC}. \]  

(23)

Therefore, knowing that \( \hat{\Gamma}^{4}_{AC} = 0 \), we find

\[ \partial_{A} \hat{g}_{BC} + \partial_{B} \hat{g}_{AC} - \partial_{C} \hat{g}_{AB} = 2 \hat{\Gamma}^{4}_{AB} \hat{g}_{4C}. \]  

(24)

Before going any further we should make it clear what we mean by the 4D metric. If we write the line-element as

\[ ds^{2} = \hat{g}_{\alpha\beta} dx^{\alpha} dx^{\beta} + 2 \hat{g}_{\alpha4} dx^{\alpha} d\hat{x}^{4} + \hat{g}_{44} d\hat{x}^{4} d\hat{x}^{4}, \]  

(25)

it would be obvious that \( \hat{g}_{\alpha\beta} = g_{\alpha\beta} \) are the covariant components of the metric tensor for the four dimensional space-time, and the contravariant components of the 4D metric tensor are defined as

\[ g_{\alpha\beta} g^{\beta\gamma} = \delta_{\alpha}^{\gamma}. \]  

(26)

Now going back to our approach to investigate the functional form of the Christoffel symbols, we replace \( \hat{g}_{\alpha\beta} \) in equation (24) with \( \sigma \) and find

\[ \partial_{A} \hat{g}_{B\sigma} + \partial_{B} \hat{g}_{A\sigma} - \partial_{\sigma} \hat{g}_{AB} = 2 \hat{\Gamma}^{4}_{AB} \hat{g}_{4\sigma}, \]  

(27)

which leads to what we were looking for

\[ \hat{\Gamma}^{4}_{AB} = \frac{1}{2} \hat{g}^{\lambda\sigma}(\partial_{A} \hat{g}_{B\sigma} + \partial_{B} \hat{g}_{A\sigma} - \partial_{\sigma} \hat{g}_{AB}). \]  

(28)
One can simply check that when all capital Latin indices are replaced with Greek indices, then
\[ \hat{\Gamma}^\lambda_{\alpha\beta} = \Gamma^\lambda_{\alpha\beta}, \]
where \( \Gamma^\lambda_{\alpha\beta} \) is the ordinary 4D connection. This is our final result. Alternatively, if \( C \) in equation (24) is replaced with 4, we find
\[ \partial_A \hat{g}_{BA} + \partial_B \hat{g}_{A4} - \partial_4 \hat{g}_{AB} = 2\hat{\Gamma}^\lambda_{AB} \hat{g}_{\lambda 4}, \]  
(29)

or
\[ \partial_A \hat{g}_{BA} + \partial_B \hat{g}_{A4} = 2\hat{\Gamma}^\lambda_{AB} \hat{g}_{\lambda 4} \mid \text{on the hypersurface}. \]
(30)

Now, taking all the possible values of \( A \) and \( B \) will arrive at the following useful results
\[ \partial_\alpha \hat{g}_{\beta 4} + \partial_\beta \hat{g}_{\alpha 4} = 2\hat{\Gamma}^\lambda_{\alpha\beta} \hat{g}_{\lambda 4} \mid \text{on the hypersurface}; \]  
(31)

and
\[ \hat{\Gamma}^\lambda_{44} \hat{g}_{\lambda 4} = 0 \mid \text{on the hypersurface}. \]
(33)

Using equations (20) and (28), the Riemann tensor, (17), will separate into
\[ \hat{R}^\kappa_{CBA} = \partial_B \hat{\Gamma}^\kappa_{AC} - \partial_A \hat{\Gamma}^\kappa_{BC} + \hat{\Gamma}^\kappa_{B\gamma} \hat{\Gamma}^\gamma_{AC} - \hat{\Gamma}^\kappa_{A\gamma} \hat{\Gamma}^\gamma_{BC}, \]  
(34)

and
\[ \hat{R}^4_{CBA} = 0. \]
(35)

Here again it is clear that \( \hat{R}^\kappa_{\alpha\beta\gamma} = R^\kappa_{\alpha\beta\gamma} \), where \( R^\kappa_{\alpha\beta\gamma} \) is an ordinary 4D Riemann tensor. It is now interesting to study the symmetry properties of the curvature tensor. For this purpose, let us adopt the geodesic coordinate system in which the curvature tensor can be written as
\[ \hat{R}^D_{CBA} = \partial_B \hat{\Gamma}^D_{AC} - \partial_A \hat{\Gamma}^D_{BC}, \]  
(36)

from which we immediately see that
\[ \hat{R}^D_{CBA} = -\hat{R}^D_{CAB}. \]
(37)

Now, to see if other symmetry properties are satisfied, we write our equation as
\[ \hat{R}_{ABCD} = \hat{g}_{AE} \hat{R}^E_{BCD} = \partial_C (\hat{g}_{AE} \hat{\Gamma}^E_{BD}) - \partial_D (\hat{g}_{AE} \hat{\Gamma}^E_{BC}), \]  
(38)

showing that unlike the ordinary 4D space-time, because of the unusual characteristics of the connections, the other common symmetries are not satisfied. In general we have
\[ \hat{R}_{DCBA} \neq -\hat{R}^D_{CDBA} \neq \hat{R}^D_{BADC}. \]
(39)

In addition, the first and second Bianchi identities can be written as
\[ \hat{R}^A_{BCD} + \hat{R}^A_{DBC} + \hat{R}^A_{CDB} = 0, \]  
(40)

and
\[ \hat{\nabla}_E \hat{R}^D_{ABC} + \hat{\nabla}_C \hat{R}^D_{AEB} + \hat{\nabla}_B \hat{R}^D_{ACE} = 0, \]  
(41)

where both can be verified easily in geodesic coordinate. Since not all the symmetry properties of ordinary 4D space-time appear in our theory, naturally not all the known Bianchi identity forms in 4D are correct here. However, there is another form of the First Bianchi identity which is of much importance to us, namely
\[ \hat{R}_{\alpha\beta\gamma4} + \hat{R}_{\gamma\alpha\beta4} + \hat{R}_{\beta\gamma\alpha4} = 0 \mid \text{on the hypersurface}, \]
(42)
which can be easily proved in geodesic coordinates using $\partial_4 = 0$. The Ricci tensor in this space-time can be obtained using equations (18), (34), and (35) as

$$\hat{R}_{CA} = \partial_4 \hat{\Gamma}_{AC} - \partial_A \hat{\Gamma}_{\kappa C}^\kappa + \hat{\Gamma}_{\kappa \gamma}^\kappa \hat{\Gamma}_{AC}^\gamma - \hat{\Gamma}_{A\gamma}^\kappa \hat{\Gamma}_{\kappa C}^\gamma. \quad (43)$$

Again

$$\hat{R}_{\alpha \gamma} = R_{\alpha \gamma}, \quad (44)$$

where $R_{\alpha \gamma}$ is the ordinary 4D Ricci tensor. It is very simple to show that the Ricci tensor is symmetric.

Now we can investigate the field equations, (19), and determine the scalar appearing in it. We know that, as mentioned before, all the matter is confined to our 4D hypersurface and therefore energy-momentum is a conservative quantity in our 4D volume. We can interpret $\hat{J}_A = \hat{T}_{AB} \dot{x}^B$ as current density. Hence the conservation law can be written as

$$\oint_{\partial \Omega} \hat{J}_\alpha d f^\alpha = \oint_{\partial \Omega} \hat{T}_{AB} \dot{x}^B d f^\alpha = 0, \quad (45)$$

where $df^\alpha$ is the vector representing the area element perpendicular to the surface $\dot{x}^\alpha = \text{const}$. Using Gauss’s law, we will have the following equation

$$\int_{\Omega} \nabla^\alpha \hat{J}_\alpha \sqrt{-g} d^4 x = 0, \quad (46)$$

where $\nabla^\alpha$ is the 4D covariant derivative. This integral implies

$$\nabla^\alpha \hat{J}_\alpha = \nabla^\alpha (\hat{T}_{AB} \dot{x}^B) = 0, \quad (47)$$

or

$$\nabla^\alpha (\hat{T}_{\alpha \beta} \dot{x}^\beta - \hat{T}_{\alpha 4}) = 0, \quad (48)$$

where we use $\dot{x}^4 = -1$ on the hypersurface. For any arbitrary but small region we can find vector fields with $\nabla^\alpha \dot{x}^\beta \simeq 0$ [22]. As a result

$$\nabla^\alpha (\hat{T}_{\alpha \beta} \dot{x}^\beta) \dot{x}^\beta - \nabla^\alpha \hat{T}_{\alpha 4} = 0. \quad (49)$$

Since $\dot{x}^\beta$ can be chosen arbitrarily, this equation gives us two distinct equations

$$\nabla^\alpha (\hat{T}_{\alpha \beta}) = 0, \quad (50)$$

and

$$\nabla^\alpha \hat{T}_{\alpha 4} = 0. \quad (51)$$

Now, it is appropriate to take $\hat{T}_{\alpha \beta}$ as the four dimensional energy-momentum tensor, $T_{\alpha \beta}$, and $\hat{T}_{\alpha 4}$ as the four dimensional electromagnetic current vector, $j_\alpha$, and write the five dimensional energy-momentum tensor in the following form

$$\hat{T}_{AB} = \begin{pmatrix} T_{\alpha \beta} & \frac{q}{m} \epsilon j_0 \\ \frac{q}{m} \epsilon j_1 & \frac{q}{m} \epsilon j_2 \\ \frac{q}{m} \epsilon j_3 & \frac{q}{m} \epsilon j_4 \end{pmatrix}, \quad (52)$$

where $\epsilon$ is a coupling constant. If we replace $A$ by $\alpha$ and $B$ by $\beta$, equation (19) will reduce to

$$\hat{R}_{\alpha \beta} - \frac{1}{2} \hat{g}_{\alpha \beta} \kappa \hat{T}_{\alpha \beta} = 0. \quad (53)$$
or equivalently
\[ R_{\alpha\beta} - \frac{1}{2} \mathring{\hat{\text{g}}} g_{\alpha\beta} = \kappa T_{\alpha\beta}. \]  
(54)

Since \( \nabla^\alpha T_{\alpha\beta} = 0 \), we have to have \( \nabla^\alpha (R_{\alpha\beta} - \frac{1}{2} \mathring{\hat{\text{g}}} g_{\alpha\beta}) = 0 \) which leads to the following equation
\[ \mathring{\hat{\text{g}}} = R. \]  
(55)

Here \( R \) is 4D Ricci scalar. It is easy to show that the 4D Ricci scalar is also a scalar in five dimensions. Therefore, we can write the field equations (19) as follows
\[ \hat{R}_{AB} - \frac{1}{2} R \hat{g}_{AB} = \kappa \hat{T}_{AB}. \]  
(56)

### 3 Electromagnetism

As the first step let us derive the Lorentz force which is a part of the geodesic equation. Knowing that on the geodesic equation \( \dot{x}^4 = -1 \), we can write equation (15) as
\[ \ddot{\xi}^\kappa + \Gamma^\kappa_{\alpha\beta} \dot{\xi}^\alpha \dot{x}^\beta - 2 \Gamma^\kappa_{\alpha 4} \dot{\xi}^\alpha + \Gamma^\kappa_{4 4} = 0. \]  
(57)

The geodesic equation is then given by
\[ \ddot{\xi}^\kappa + \frac{1}{2} g^{\lambda\kappa} (\hat{g}_{\lambda\beta,\alpha} + \hat{g}_{\lambda\alpha,\beta} - \hat{g}_{\alpha\beta,\lambda}) \dot{\xi}^\alpha \dot{\xi}^\beta - g^{\lambda\kappa} (\hat{g}_{4\lambda,\alpha} - \hat{g}_{4\alpha,\lambda}) \dot{\xi}^\alpha - \frac{1}{2} g^{\lambda\kappa} \hat{g}_{4 4,\lambda} = 0, \]  
(58)

noting that partial derivatives with respect to \( \dot{x}^4 \) are zero, and a comma indicates partial derivative.

We may now define the vector potential, \( A_\mu \), as
\[ A_\mu = \frac{m}{q} \hat{g}_{4\mu}, \]  
(59)

where \( m \) and \( q \) are the mass and electric charge of our test particle respectively. Therefore, we define the Faraday tensor as
\[ F_{\mu\nu} = \hat{g}_{4\mu,\nu} - \hat{g}_{4\nu,\mu}, \]  
(60)

where by means of this equation, one can simply find the Lorentz force in equation (58).

As was mentioned above and is well known, the homogeneous Maxwell equations appear as constraints on the Faraday tensor in the KK theory and are implicitly assumed to hold. They do do not appear in an independent manner. However, the situation is different in the theory presented here. It is therefore appropriate at this point to show that this is indeed the case. To begin with we note that in classical electrodynamics, the Jacobi identities lead to the homogenous Maxwell equations if we define our connections as \( D_\mu = \partial_\mu + A_\mu \), where \( A_\mu \) is the four vector potential, so that the Faraday tensor is written as \( F_{\mu\nu} = [D_\mu, D_\nu] \). Now, use of the second Jacobi identity leads to the Bianchi identity in the form
\[ [D_\mu, F_{\nu\lambda}] + [D_{\nu}, F_{\lambda\mu}] + [D_{\lambda}, F_{\mu\nu}] = 0, \]  
(61)

which is equivalent to
\[ D_\mu F_{\nu\lambda} + D_{\nu} F_{\lambda\mu} + D_{\lambda} F_{\mu\nu} = 0. \]  
(62)

This is, of course, the homogenous Maxwell equations [23]. In General Relativity on the other hand, we replace \( D_\mu \) by \( \nabla_\mu \), so that the first Jacobi identity results in
\[ [\nabla_\mu, R(\partial_\nu, \partial_\lambda)] + [\nabla_\nu, R(\partial_\lambda, \partial_\mu)] + [\nabla_\lambda, R(\partial_\mu, \partial_\nu)] = 0, \]  
(63)
or
\[ R_{\alpha\beta;\mu\nu;\lambda} = 0. \]  
(64)
The torsion free condition also provides the first Bianchi identity

$$R_{\lambda[\beta\gamma\delta]} = 0. \quad (65)$$

It is now clear that because of the definition of connections in general relativity, neither the first Bianchi identity, equation (65), nor the second Bianchi identity, equation (64), lead to homogenous Maxwell equations. Let us now show that in the model presented here, the first Bianchi identity leads to the homogenous Maxwell equations. To write the first Bianchi identity in our model we have to show that

$$\hat{R}^\alpha_{\beta\lambda\kappa} = \frac{1}{2} \nabla_\kappa F^\alpha_\beta, \quad (66)$$

where $F_{\alpha\beta}$ is the Faraday tensor in four dimensional space which is defined by $g_{\mu\nu}$. We write

$$\frac{1}{2} F^\kappa_\sigma = \frac{1}{2} g^{\kappa\lambda} (\hat{g}_{4\lambda\sigma} - \hat{g}_{4\sigma\lambda}) = \hat{\Gamma}^\kappa_4\sigma, \quad (67)$$

and

$$\frac{1}{2} \nabla_\kappa F^\alpha_\beta = \partial_\kappa \hat{\Gamma}^\alpha_4\beta + \hat{\Gamma}^\alpha_\kappa\lambda \hat{\Gamma}^\lambda_4\beta - \hat{\Gamma}^\lambda_{\kappa\beta} \hat{\Gamma}^\alpha_4\lambda. \quad (68)$$

On the other hand we have

$$\hat{R}^\alpha_{\beta\lambda\kappa} = \partial_\kappa \hat{\Gamma}^\alpha_4\beta + \hat{\Gamma}^\alpha_\kappa\lambda \hat{\Gamma}^\lambda_4\beta - \hat{\Gamma}^\lambda_{\kappa\beta} \hat{\Gamma}^\alpha_4\lambda, \quad (69)$$

$$\hat{R}^\alpha_{\beta\lambda\kappa} = \partial_\kappa \hat{\Gamma}^\alpha_4\beta + \hat{\Gamma}^\alpha_\kappa\lambda \hat{\Gamma}^\lambda_4\beta - \hat{\Gamma}^\lambda_{\kappa\beta} \hat{\Gamma}^\alpha_4\lambda. \quad (70)$$

Hence

$$\hat{R}^\alpha_{\beta\lambda\kappa} = \frac{1}{2} \nabla_\kappa F^\alpha_\beta. \quad (71)$$

Now using equation (71), the first Bianchi identity, equation (42), leads to

$$\frac{1}{2} \nabla_\kappa F^\alpha_\beta + \frac{1}{2} \nabla_\beta F^\alpha_\kappa + \frac{1}{2} \nabla_\alpha F^\beta_\kappa = 0, \quad (72)$$

showing that, as expected, the first Bianchi identity results in the homogeneous Maxwell equations. It is worth mentioning that in other 5D theories, e.g. KK, equations (40) or (42) do not lead to equation (72).

Now, let us see if our field equations, (56), reduce to Maxwell equations. If one, in the field equations replaces $A$ by 4 and $B$ by $\lambda$, one finds

$$\hat{R}^\alpha_4\lambda - \frac{1}{2} \hat{g}_{4\lambda} R = \kappa \hat{T}^\lambda_4. \quad (73)$$

We can also see from equation (66) that

$$\hat{R}^\alpha_4\lambda = \hat{\rho}^\lambda_4 = \hat{R}^\alpha_{\lambda\alpha 4} = \frac{1}{2} \nabla_\alpha F^\alpha_\lambda. \quad (74)$$

Substituting this equation into (73) yields the Maxwell equations with a correction term

$$\nabla_\alpha F^\alpha_\lambda - \hat{g}_{4\lambda} R = 2\kappa \hat{T}^\lambda_4, \quad (75)$$

Note that in cases where the 4D scalar curvature is small, we may neglect the correction term in (75). We can move $\hat{g}_{4\lambda} R$ in (75) to the right hand side and look at it as a new source. In the case of a large scalar curvature, this term, under certain conditions, results in an electromagnetic field which is stronger than that expected from the usual form of the Maxwell equations. Recent observations may provide sufficient evidence towards such a prediction. Indeed, it is possible to take account of the unusually large galactic magnetic fields discovered recently [24, 25, 26] for which no satisfactory explanation as yet exists.
4 Coordinate transformation

If we change our coordinate system so that we again observe the space-time from the 4D hypersurface, namely our universe, the main assumption must be satisfied in order to yield the desired equations. It means that we must have

$$\hat{\Gamma}^4_{AC'} = 0.$$  \hspace{1cm} (76)

Therefore we need to examine the change of the above equation under the transformation introduced as follows

$$\frac{\partial \hat{x}^4}{\partial x^4} = \text{const.},$$ \hspace{1cm} (77)

$$\frac{\partial \hat{x}^4}{\partial x^a} = 0.$$

On the other hand we know that under transformations, the connections change as

$$\hat{\Gamma}^B_{A'E'} = \frac{\partial \hat{x}^B}{\partial x^C} \left( \frac{\partial}{\partial \hat{x}^A'} \frac{\partial \hat{x}^C}{\partial \hat{x}^{E'}} \right) + \frac{\partial \hat{x}^B}{\partial x^C} \frac{\partial \hat{x}^D}{\partial \hat{x}^{A'}} \frac{\partial \hat{x}^H}{\partial \hat{x}^{E'}} \hat{\Gamma}^C_{DH}. \hspace{1cm} (78)$$

If we now replace $B'$ by $A'$ in this equation

$$\hat{\Gamma}^A_{A'E'} = \frac{\partial \hat{x}^A}{\partial x^C} \left( \frac{\partial}{\partial \hat{x}^{A'}} \frac{\partial \hat{x}^C}{\partial \hat{x}^{E'}} \right) + \frac{\partial \hat{x}^A}{\partial x^C} \frac{\partial \hat{x}^D}{\partial \hat{x}^{A'}} \frac{\partial \hat{x}^H}{\partial \hat{x}^{E'}} \hat{\Gamma}^C_{DH}, \hspace{1cm} (79)$$

and substitute (77) into the above equation we find

$$\hat{\Gamma}^A_{A'E'} = \frac{\partial \hat{x}^D}{\partial \hat{x}^{A'}} \frac{\partial \hat{x}^H}{\partial \hat{x}^{E'}} \hat{\Gamma}^4_{DH}. \hspace{1cm} (80)$$

Hence, our assumption is invariant under the introduced transformation.

5 Conclusions

In this paper we have presented a unified theory of gravity and electromagnetism where the resulting inhomogeneous Maxwell equations are modified by a term involving the curvature which, in certain cases, leads to electromagnetic fields which are stronger than those obtained from the usual Maxwell equations in cases where a large scalar curvature is present, contrary to other conventional theories, e.g. Kaluza-Klein. This could have interesting consequences and may be used to describe the recently observed unusually strong galactic magnetic fields. This is currently the focus of a work in progress.

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