“Good Propagation” Constraints on Dual Invariant Actions in Electrodynamics and on Massless Fields

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We present some consequences of non-anomalous propagation requirements on various massless fields. Among the models of nonlinear electrodynamics we show that only Maxwell and Born–Infeld also obey duality invariance. Separately we show that, for actions depending only on the $F^2_{\mu\nu}$ invariant, the permitted models have $L \sim \sqrt{1 + F^2}$.

We also characterize acceptable vector–scalar systems. Finally we find that wide classes of gravity models share with Einstein the null nature of their characteristic surfaces.

The nonlinear electrodynamics of Born–Infeld (BI) \hspace{1em} has been the subject of frequent revivals, not least because it enjoys two quite separate properties, shared with Maxwell theory. The first is that its excitations propagate without the shocks common to generic nonlinear models \hspace{1em}, the second is duality invariance \hspace{1em}. Here, we want to complete this subject in several ways. Our primary result will be that imposition of both duality and good propagation singles out BI and Maxwell, without even requiring the solutions to reduce to Maxwell in the weak field limit (something that is used to select BI in the derivations of each separate demand). We will

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then show separately that for actions depending only on the Maxwell invariant \( \alpha \equiv \frac{1}{2} F_{\mu \nu}^2 \) (rather than on both \( \alpha \) and the other invariant, \( \beta \equiv \frac{1}{8} \epsilon^{\mu \nu \sigma \tau} F_{\mu \nu} F_{\sigma \tau} \), allowed in D=4) correct propagation implies the form \( L \sim \sqrt{1 + \alpha} \), which shares a square root (but not a determinant) form with BI.

In this connection, we will also discuss the scalar analogs of BI and characterize systems involving both scalars and vectors. Criteria for physical propagation of hyperbolic systems can be derived in many ways; in a separate work \([6]\) we will consider the “complete exceptional” (CE) approach \([7]\) to these questions, in more detail. Since this method was useful in some of our derivations below, we will use “CE” to mean propagation without emerging shocks, with characteristic flow remaining parallel along the waves (but waves need not travel with the same speed in all directions.)

In the following we will be concerned with systems of partial differential equations which arise as the variational equations of a relativistic Lagrangian theory. Thus they will be quasilinear (linear in highest derivatives) and the coefficient functions will not explicitly depend on the coordinates. Without loss of generality, they can be reduced to a set of equations first order in derivatives. For definiteness then, with \( u \) an \( N \)-vector of fields, \( A \) an \( N \times N \) matrix and \( B \) an \( N \)-vector (both arbitrary (smooth enough) functions of \( u \)), the equations of interest are

\[
A^\mu(u) \partial_\mu u + B(u) = 0. \tag{1}
\]

Of course, the theory of such equations in arbitrary dimensions is quite difficult, but we will be interested in the evolution of the spatial boundary of a wave propagating into some given vacuum state. So, with \( \bar{u} \) some smooth (say at least \( C^1 \)) solution, at some initial time we have some spatial region outside of which the “state” is the “vacuum solution” \( \bar{u} \), and across the boundary surface the full solution \( u \) is continuous but its first derivative may not be. We now consider the evolution of this initial “wavefront”.

One discussion of this situation has been developed (\([3]\) and references therein) which we will
follow here. Let the hypersurface $S$, specified by

$$\varphi(x) = 0,$$

(2)
denote the surface of evolution of the initial wavefront; i.e., the initial wavefront is the spatial surface $\varphi(x, 0) = 0$. Since the field $u$ is continuous, only the normal derivative will be allowed to be discontinuous. Choosing a local coordinate system $x^\mu = (\varphi, \psi^i)$, we can define the “first order discontinuity” in a given quantity $f$ to be

$$\delta_1 f \equiv \left[ \frac{\partial f}{\partial \varphi} \right],$$

(3)

where

$$[X] \equiv X|_{\varphi=0^+} - X|_{\varphi=0^-}$$

(4)

and we will sometimes write $[X] \equiv \delta_0 X$.

Taking the discontinuity of (1) we obtain ($\varphi_\mu = \partial_\mu \varphi$)

$$(\mathcal{A}_\mu \varphi_\mu) \delta_1 u = 0,$$

(5)

where the matrix $\mathcal{A} = \mathcal{A}(\bar{u})$. Hence, since $\delta_1 u \neq 0$, we see that $S$ must be a characteristic surface; i.e., the characteristic equation

$$H(x, \varphi_\mu) = \det (\mathcal{A}^\mu \varphi_\mu) = 0$$

(6)

must hold on $S$, where $H$ is homogeneous of order $N$ in $\varphi_\mu$. The characteristic curves, which solve

$$\frac{dx^\mu}{ds} = \frac{\partial H}{\partial \varphi_\mu},$$

are clearly tangential to the characteristic surface.

From (5), it further follows that $\delta_1 u$ may be expanded in a basis of $\text{Ker} (\mathcal{A}^\mu \varphi_\mu)$, the coefficients of this expansion being called the “coefficients of discontinuity”. In general, the coefficients of
discontinuity evolve according to a nonlinear differential equation. Following the discussion in [3], based on [8], it can be shown that when we ensure that the characteristic curves do not intersect locally (and thus “shock” singularities do not develop), the evolution of the equations of discontinuity is correspondingly linear. We will impose this condition as our specification of “nonanomalous evolution”. It is the CE condition mentioned earlier, and can be imposed as the condition that, on the characteristic surface $H = 0$, we have [3]

$$\delta_0 H = 0. \quad (7)$$

We now apply these ideas to the systems of interest, without however showing any details of the underlying CE derivation. For general nonlinear gauge invariant actions in $D = 4$ that depend only on field strengths but not explicit derivatives, the Lagrangian $L(\alpha, \beta)$ must obey the two equations (subscripts indicate partial differentiation)

$$-L_\alpha \left(4 L_{\alpha\alpha} - L_{\beta\beta}\right) + 2 \alpha \left[L_{\alpha\alpha} L_{\beta\beta} - (L_{\alpha\beta})^2\right] = 0, \quad (7a)$$

and

$$-L_\alpha L_{\alpha\beta} + \beta \left[L_{\alpha\alpha} L_{\beta\beta} - (L_{\alpha\beta})^2\right] = 0. \quad (7b)$$

in order for its excitations to be CE and for light to travel according to only one dispersion law, i.e. no birefringence [3, 4]. To obtain them, however, it was necessary to assume nontrivial $\beta$ dependence; see below. Quite independently the duality invariance requirement is given by [3]

$$\left(L_\alpha\right)^2 - \frac{\alpha}{2\beta} L_\alpha L_\beta - \frac{1}{4} (L_\beta)^2 = \frac{1}{4}. \quad (8)$$

The simultaneous solution of (7) and (8) can be obtained as follows.

We first change to variables $(a, b)$ with $\alpha = a + b$, $\beta^2 = -ab$ (these new variables have the fundamental advantage that they factorize $L_{BI}$); then (7a), (7b) and (8) transform into

$$-4ab(a + b)(a - b)^2[L_{\alpha\alpha} L_{\beta\beta} - (L_{\alpha\beta})^2] + 4ab(a^2 - b^2)L_{\alpha\beta}(L_\alpha - L_\beta) + 8ab(b - a)L_{\alpha\beta}(bL_\beta - aL_\alpha)$$
\[-2(a - b)^2(a + b)\left[aL_bL_{aa} + bL_aL_{bb}\right] + (4ab + (a + b)^2)(L_a - L_b)(aL_a - bL_b) = 0, \tag{9a}\]

\[2ab(a - b)^2[L_{aa}L_{bb} - (L_{ab})^2] + (a + b)(L_a - L_b)(bL_b - aL_a) + (a - b)^2[aL_bL_{aa} + bL_aL_{bb}] - (a^2 - b^2)L_{ab}(aL_a - bL_b) + 2ab(b - a)L_{ab}(L_a - L_b) = 0, \tag{9b}\]

\[L_aL_b = \frac{1}{4}, \tag{10}\]

respectively. Note the symmetry in \((a, b)\) of each equation. Multiplying (9b) by \(2(a + b)\) and adding the result to (9a), one finds

\[(a - b)^2(aL_a - bL_b)[2(b - a)L_{ab} - (L_a - L_b)] = 0 . \tag{11}\]

Vanishing of the \((aL_a - bL_b)\) factor is not a useful solution of (11) because it would imply \(L = L(ab) \equiv L(\beta^2),\) for which (8) implies \(L \sim \beta,\) a total divergence. Hence we must impose

\[2(b - a)L_{ab} - (L_a - L_b) = 0 . \tag{12}\]

Substituting this back into (9a) gives the simplified form

\[2ab[L_{aa}L_{bb} - (L_{ab})^2] + [a(L_aL_b)_a + b(L_aL_b)_b] = 0 . \tag{13}\]

So now instead of (9) and (10), we can study (12), (13) and (10). But actually it is easy to see that (13) follows from (10), since (10) is a first integral of the Monge–Ampère factor \([L_{aa}L_{bb} - (L_{ab})^2],\)

\(i.e.\) this factor vanishes as a consequence of (10). Thus we are left with the system (10), (12) \footnote{An alternative derivation inserts the general parametric solution \([9]\) of (10) into (12).}.

Substituting for \(L_b\) in (12), it can be written as \((a - b)^2 \left(\frac{dL_a}{a-b}\right)_b = 1,\) with first integral

\[L_a = \frac{1}{2} \sqrt{1 + (a - b)f(a)} . \tag{14}\]
But by (10), $4(L_a)^2 = L_a/L_b = 1 + (a - b)f(a)$ which gives $L_b = \frac{1}{2}[1 + (a - b)f(a)]^{-1/2}$, whose integral is

$$L = -\frac{1}{f(a)}\sqrt{1 + (a - b)f(a) + h(a)} . \quad (15a)$$

Given the $(a, b)$ symmetry of the equations, the above procedure based on $L_b$ instead of $L_a$ gives the corresponding form

$$L = -\frac{1}{k(b)}\sqrt{1 + (b - a)k(b) + m(b)} . \quad (15b)$$

Consistency with (10) demands that $\sqrt{1 + (a - b)f(a)}/\sqrt{1 + (b - a)k(b)} = 1$, which implies $k(b) = f(a)/(1 + (a - b)f(a))$. Substituting for $k(b)$ in (15b) then shows that

$$L = -\frac{1}{f(a)}\sqrt{1 + (a - b)f(a) + m(b)} \quad (16)$$

which implies $h(a) = m(b) = d = \text{const.}$ Differentiating (16) with respect to $a$ and comparing the result with (14) one finally finds that $(2 + (a - b)f(a))(f' - f^2) = 0$, which when integrated gives (for $s$ an integration constant) $f(a) = \frac{1}{s-a}$ (or $f = 0$ trivially, the Maxwell case.) Finally then, renaming constants, $L = -\frac{1}{\epsilon}\sqrt{(1 + ac)(1 + bc) + d}$ (and $L = -\frac{1}{2}(a + b)$ for $f = 0$). Rewriting these using $(\alpha, \beta)$, we find using allowed rescalings that $L = -\frac{1}{\epsilon}\alpha$, Maxwell, and $L = 1 - \sqrt{1 + \alpha - \beta^2} = 1 - \sqrt{-\det[\eta_{\mu\nu} + F_{\mu\nu}]}$, Born-Infeld, are the only possible solutions. This demonstrates the uniqueness of BI and Maxwell as the simultaneously CE and duality invariant electrodynamics. The CE requirement (7) alone permits additional solutions, such as $L = \alpha/\beta$, that are not duality invariant. [Conceivably, requiring power series expandability in $(\alpha, \beta)$ might restrict the solutions of (7) to Maxwell and BI.]

As was mentioned earlier, the CE criteria (7) are only correct for nontrivial $\beta$-dependence; for pure $L(\alpha)$ they just imply $L' = \text{const.}$, namely Maxwell. Instead, if one goes back to the complete CE requirements, they imply, for pure $L(\alpha)$

$$L'L'' - 3(L'')^2 = 0 . \quad (17)$$
The solution, apart from Maxwell ($L'' = 0$) is

$$L(\alpha) = k + (d + c\alpha)^{\frac{1}{2}}. \quad (18)$$

[We remark that in $D = 3$, where $\alpha$ is the only invariant, this is also the CE result, there also

$$\sqrt{1 + \alpha} = \sqrt{-\det[\eta_{\mu\nu} + F_{\mu\nu}]}$$

with the BI determinant form. In $D = 2$ there is of course no

propagation for any $L(\alpha)$ and correspondingly no restrictions are imposed.] The above BI form

is very analogous to that obtained for a scalar field [3]. There, (in any dimension) for $L(z),

$$z \equiv \frac{1}{2}(\partial_{\mu}\phi)(\partial^{\mu}\phi)$$

being the only invariant (in first derivatives), we find the same requirement (17). Hence we obtain, apart from the free scalar solution $L = -\frac{1}{2}z$, the same square root solution.

Amusingly, this one can be put into BI-like form, since (rescaling $\phi$)

$$\sqrt{1 + 2z} \equiv \sqrt{1 + \phi_{\mu}^{2} = \sqrt{-\det[\eta_{\mu\nu} + \phi_{\mu}\phi_{\nu}]} \quad (19)$$

where $\phi_{\mu} \equiv \partial_{\mu}\phi$ denotes the field strength. If one combines Maxwell and the (neutral) scalar into an

action $L(\alpha, \beta, z)$, then the CE conditions further require $L_{z\alpha} = 0 = L_{z\beta}$, reducing the Lagrangian
to the noninteracting $L(\alpha, \beta) + L(z)$ form. Having the “fully” BI form $\sqrt{-\det[\eta_{\mu\nu} + F_{\mu\nu} + \phi_{\mu}\phi_{\nu}]}$ mind, one can actually show more generally that there are no CE actions with nontrivial dependence

on the other possible variable $y \equiv \frac{1}{2}(F_{\mu\nu}\phi^{\nu})^{2}$.

Finally, we turn to gravitation. For Einstein’s gravity in vacuum, as well as the linearized

theory, the gravitational waves are CE, the characteristic surfaces describing discontinuities being

null (see e.g. [10]). It can be shown that this result holds for any $D > 4$. [For $D = 3$, there is of
course no propagation and no restrictions are imposed.] One can further look at pure gravitational

actions of the form $pR_{\mu\nu}^{2} - qR^{2}$ in $D = 4$ and $f(R)$ in $D > 3$ and show that the same conclusion

remains unchanged.

To reduce these theories to a first order system would be inconvenient, but is fortunately

made unnecessary by a simple extension of the previous discussion. Clearly, if we rebuilt the
original higher order equations from the set \( \mathbb{I} \), we would simply have the situation that all the
derivatives of the field are assumed continuous except the highest one. Thus for quasilinear systems
higher order, say \( q \), in derivatives, we define
\[
\delta_r f = \frac{\partial^r f}{\partial \varphi^r},
\]
and will consider the case that
\[
\delta_q u \neq 0; \quad \delta_r u = 0, 0 \leq r < q.
\]
Notice that
\[
\delta_r \partial_\mu = \partial_\mu \varphi \delta_{r+1}.
\]

Let us first sketch the Einstein case to establish notation. Considering a second order dis-
continuity in the metric across some characteristic surface \( \varphi = 0 \), \( \delta_2 g_{\mu \nu} = \pi_{\mu \nu} \), we have \( (\varphi_\mu \equiv \partial_\mu \varphi) \)
\[
\delta_1 \Gamma^\lambda_{\mu \nu} = \frac{1}{2} (\varphi_\mu \varphi^\lambda\nu + \varphi_\nu \varphi^\lambda\mu - \varphi^\lambda \pi_{\mu \nu}) ,
\]
\[
\delta_0 R_{\mu \nu} = \varphi_\lambda (\delta_1 \Gamma^\lambda_{\mu \nu}) - \varphi_\nu (\delta_1 \Gamma^\lambda_{\lambda \mu})
= \frac{1}{2} (\varphi_\mu \varphi_\lambda \pi^\lambda\nu + \varphi_\nu \varphi_\lambda \pi^\lambda\mu - \varphi_\mu \varphi_\nu \pi^\lambda \lambda - \varphi_\lambda \varphi_\mu \pi_{\mu \nu})
\]
and
\[
\delta_0 R = g^{\mu \nu} (\delta_0 R_{\mu \nu}) = \varphi^\mu \varphi^\nu \pi_{\mu \nu} - \varphi_\mu \varphi^\mu \pi^\nu
\]
which implies for
\[
\delta_0 G_{\mu \nu} = \delta_0 (R_{\mu \nu} - \frac{1}{2} \pi_{\mu \nu} R) = \delta_0 R_{\mu \nu} - \frac{1}{2} \pi_{\mu \nu} \delta_0 R = 0
\]
\[
\delta_0 G_{\mu \nu} = \frac{1}{2} \left[ \varphi_\mu \varphi_\lambda \pi^\lambda\nu + \varphi_\nu \varphi_\lambda \pi^\lambda\mu - \varphi_\mu \varphi_\nu \pi^\lambda \lambda - \varphi_\lambda \varphi_\mu \pi_{\mu \nu} - g_{\mu \nu} \left( \varphi_\sigma \varphi^\sigma \pi_{\sigma \tau} - \varphi_\sigma \varphi_\sigma \pi^\tau \tau \right) \right] = 0.
\]

(20)
In the harmonic gauge $g^{\mu\nu}\Gamma^\sigma_{\mu\nu} = 0$, one finds that its first discontinuity implies

$$2\pi^{\mu\nu}\varphi_\mu - \pi^{\mu}_{\phantom{\mu}\mu}\varphi_\nu = 0 \quad (21)$$

Multiplying this by $g_{\nu\sigma}\varphi_\tau + g_{\nu\tau}\varphi_\sigma$, one gets

$$\varphi_\mu\varphi_\lambda\pi^\lambda_{\phantom{\lambda}\nu} + \varphi_\nu\varphi_\lambda\pi^\lambda_{\phantom{\lambda}\mu} - \varphi_\mu\varphi_\nu\pi^\lambda_{\phantom{\lambda}\lambda} = 0 \quad (22)$$

whereas contracting by $\varphi_\nu$, one finds

$$\varphi^\mu\varphi_\nu\pi^{\mu\nu} = \frac{1}{2}\varphi^\mu\varphi_\mu\pi^{\nu\nu} \quad (23)$$

Using (22) and (23) in (20), one ends up with

$$\delta_0 G_{\mu\nu} = \frac{1}{2}(\varphi_\lambda\varphi^\lambda_{\phantom{\lambda}\mu\nu} + \frac{1}{2}g_{\mu\nu}\varphi_\lambda\varphi^\lambda_{\phantom{\lambda}\sigma\sigma} = 0 \quad (24)$$

Hence taking the trace

$$\delta_0 G^\mu_{\phantom{\mu}\mu} = \frac{(D + 2)}{4}\varphi_\lambda\varphi^\lambda_{\phantom{\lambda}\sigma\sigma} = 0 \quad (25)$$

The discontinuity in $g_{\mu\nu}$ is arbitrary, hence $\pi^\sigma_{\phantom{\sigma}\sigma} \neq 0$, which implies that $\varphi_\lambda\varphi^\lambda = 0$. This tells that the characteristic surfaces are null: the discontinuities travel with the speed of light in all directions. The same holds for the linearized version of the theory as well of course.

For generic quadratic Lagrangians $(pR_{\mu\nu}R^{\mu\nu} - qR^2)\sqrt{-g}$ in $D = 4$, using similar steps (writing the field equations, choosing harmonic gauge as before and utilizing the identities (22), (23)) one finds that ($Q \equiv \varphi^\lambda\varphi_\lambda$, $\pi \equiv \pi^\lambda_{\phantom{\lambda}\lambda}$)

$$Q\left(\frac{1}{2}(p - 2q)\varphi_\mu\varphi_\nu\pi - \frac{p}{2}Q\pi_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(\frac{p}{2} - 2q)Q\pi\right) = 0 \quad (24)$$

Taking the trace, one gets $Q^2\pi(p - 3q) = 0$. (The choice $p = 3q$ corresponds to Weyl–tensor squared; the scalar degree of freedom is absent.) For $p = 3q$, (24) becomes

$$qQ\left(\frac{1}{2}\varphi_\mu\varphi_\nu\pi - \frac{3}{2}Q\pi_{\mu\nu} + \frac{1}{4}g_{\mu\nu}Q\pi\right) = 0 \quad (25)$$
Since $\pi_{\mu\nu}$ is arbitrary, we see that again $Q = 0$, as in Einstein, so $Q = 0$ characterizes both Einstein and the quadratic action.

For actions $f(R)\sqrt{-g}$ in $D \geq 4$, the field equations are

$$E_{\mu\nu} \equiv R_{\mu\nu}f' - \frac{1}{2}g_{\mu\nu}f + (g_{\mu\nu}\nabla_\sigma\nabla^\sigma - \nabla_\mu\nabla_\nu)f' = 0 .$$

Hence the order of highest derivatives is four. Following similar steps by taking $\delta_4 g_{\mu\nu} = \pi_{\mu\nu}$, we find the same expressions for $\delta_3 \Gamma^\lambda_{\mu\nu}$ and $\delta_2 R_{\mu\nu}$ as for $\delta_1 \Gamma^\lambda_{\mu\nu}$ and $\delta_0 R_{\mu\nu}$ in the Einstein case. Using these, we get

$$\delta_0 E_{\mu\nu} = (Qg_{\mu\nu} - \varphi_\mu \varphi_\nu)(\varphi^\sigma \varphi^\tau \pi_{\sigma\tau} - Q\pi)f'' = 0 .$$

Finally, in the harmonic gauge with identity (23) and taking the trace, one gets

$$\delta_0 E^\mu_{\mu} = (1 - D)Q^2 \pi f'' = 0 .$$

Here too $Q = 0$ is the only solution, and so for a wide class of gravitational actions the propagation obeys the Einstein behavior.

This work was supported in part by NSF, under grant no PHY-9315811.

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