New supersymmetric $\sigma$-model duality

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Abstract

We study dualities in off-shell 4D $\mathcal{N} = 2$ supersymmetric $\sigma$-models, using the projective superspace approach. These include (i) duality between the real $O(2n)$ and polar multiplets; and (ii) polar-polar duality. We demonstrate that the dual of any superconformal $\sigma$-model is superconformal. Since $\mathcal{N} = 2$ superconformal $\sigma$-models (for which target spaces are hyperkähler cones) formulated in terms of polar multiplets are naturally associated with Kähler cones (which are target spaces for $\mathcal{N} = 1$ superconformal $\sigma$-models), polar-polar duality generates a transformation between different Kähler cones. In the non-superconformal case, we study implications of polar-polar duality for the $\sigma$-model formulation in terms of $\mathcal{N} = 1$ chiral superfields. In particular, we find the relation between the original hyperkähler potential and its dual. As an application of polar-polar duality, we study self-dual models.

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1 Introduction

Dualities in supersymmetric theories have a long history. In four-dimensional \(\sigma\)-models the duality between scalar and tensor multiplets, e.g., was discussed in \(\mathcal{N} = 1\) superspace both for \(\mathcal{N} = 1\) and \(\mathcal{N} = 2\) models already in [1]. Here we shall be interested in \(\mathcal{N} = 2\) supersymmetric \(\sigma\)-models and their dualities. These are best described in projective superspace [2, 3] where the \(\mathcal{N} = 2\) supersymmetry is manifest.\(^1\) There are several types of dualities (obtained by applying generalized Legendre transformations) for off-shell multiplets in projective superspace. These include (i) duality between so-called real \(O(2n)\) and polar multiplets which was introduced in [3] (see also [7]); and (ii) the duality between polar multiplets which was introduced in [8] and also studied in [9]. Of particular interest to us here is the latter, polar-polar duality.

We pay special attention to off-shell \(\mathcal{N} = 2\) superconformal \(\sigma\)-models formulated in terms of projective superconformal multiplets [10]. As always, superconformal invariance is of interest in itself. What is more important, the \(\sigma\)-models under consideration can be coupled to \(\mathcal{N} = 2\) conformal supergravity [11].

Superconformal \(\sigma\)-model dynamics turns out to require interesting target space geometry. In the component approach, general \(\mathcal{N} = 2\) superconformal \(\sigma\)-models were studied in [12, 13, 14] (see also [15]). Their target spaces are hyperkähler spaces possessing a homothetic conformal Killing vector which is the gradient of a function, and hence an isometric action of SU(2) rotating the complex structures. Such spaces are known as “hyperkähler cones” [14] and they are intimately related to quaternion Kähler manifolds which are

\(^1\)See [4, 5] for alternative approaches. General \(\mathcal{N} = 2\) supersymmetric \(\sigma\)-models in harmonic superspace and their dualities were studied in [6]. Such \(\sigma\)-models do not possess a natural decomposition in terms of standard \(\mathcal{N} = 1\) superfields, a property that is desirable for various applications. The existence of such a decomposition is one of the powerful inborn features of \(\mathcal{N} = 2\) multiplets in projective superspace.
target spaces for $\mathcal{N} = 2$ locally supersymmetric $\sigma$-models [16]. Specifically, there exists a one-to-one correspondence [17] (see also [18]) between $4n$-dimensional quaternion Kähler manifolds and $4(n + 1)$-dimensional hyperkähler cones.

In the projective superspace approach, general off-shell $\mathcal{N} = 2$ superconformal $\sigma$-models were studied in [10, 19, 20], while the superconformal couplings of $\mathcal{N} = 2$ tensor multiplets had appeared already in [2] without a discussion of the conformal properties. The superconformal couplings of $\mathcal{N} = 2$ tensor multiplets were systematically discussed in the component approach in [14].

General off-shell $\mathcal{N} = 2$ superconformal $\sigma$-models are associated with Kähler cones [10, 19, 20]. If the dimension of the $\sigma$-model hyperkähler target space is $4n$, then the associated Kähler cone (following the terminology of [21, 22]) has dimension $2n$. As defined in [21, 22], a Kähler cone $\mathcal{M}$ is a Kähler space possessing a homothetic conformal Killing vector which is the gradient of a function, and therefore holomorphic. If $(\chi^I, \bar{\chi}^J)$ are the components of the homothetic conformal Killing vector, and $g_{IJ}$ is the Kähler metric, then

$$ \nabla_I \chi^J = \delta_I^J, \quad \bar{\nabla}_I \bar{\chi}^J = \bar{\delta}_I^J = 0 \quad (1.1a) $$

$$ \chi_I := g_{IJ} \bar{\chi}^J = \partial_I K, \quad g_{IJ} = \partial_I \partial_J K \quad (1.1b) $$

where $K$ can be chosen to be

$$ K = g_{IJ} \chi^I \bar{\chi}^J. \quad (1.2) $$

We can choose local complex coordinates, $\Phi^I$, on $\mathcal{M}$ in such a way that $\chi^I = \Phi^I$. Then $K(\Phi^I, \bar{\Phi}^J)$ obeys the following homogeneity condition:

$$ \Phi^I \frac{\partial}{\partial \Phi^I} K(\Phi, \bar{\Phi}) = K(\Phi, \bar{\Phi}) \quad (1.3) $$

Any Kähler cone is a cone [21]. If $\mathcal{M}$ in the above discussion is hyperkähler, it is called a hyperkähler cone [14]. For the general properties of hyperkähler cones, see [14, 21]. As shown in [10, 19, 20], the target spaces for general off-shell $\mathcal{N} = 2$ superconformal $\sigma$-models, which are hyperkähler cones, are locally cotangent bundles over Kähler cones.

As is seen from (1.1b), the function $K(\Phi, \bar{\Phi})$ can be identified with the Kähler potential of $\mathcal{M}$. Kähler cones are target spaces for $\mathcal{N} = 1$ superconformal $\sigma$-model, see, e.g., [20] for a detailed discussion. The relationship between the hyperkähler potential in the target space of a $\mathcal{N} = 2$ superconformal $\sigma$-model and the associated Kähler cone was elaborated in some detail in [20].
At the level of $\mathcal{N} = 2$ superfields polar-polar duality amounts to a particular diffeomorphism \[9\]. Here we shall see that the $\mathcal{N} = 1$ interpretation is considerably more interesting. It turns out that polar-polar duality exchanges one Kähler cone with a different (dual) cone. Since any $\mathcal{N} = 1$ superconformal $\sigma$-model has a Kähler cone as its target space, we may interpret polar-polar duality as a transformation in the set of $\mathcal{N} = 1$ superconformal $\sigma$-models.

We further discuss the interpretation of polar-polar duality for the non-superconformal $\sigma$-models in terms of physical $\mathcal{N} = 1$ fields and show that it defines a transformation of certain $n$-dimensional Kähler spaces to other $n$-dimensional Kähler spaces.

Finally, as an important application, polar-polar duality allows us to introduce the family of self-dual models.

The paper is organized as follows: In section 2 we recapitulate some salient features of projective superspace and the definition of superconformal projective multiplets. Section 3 starts our duality discussion by providing a manifestly $\mathcal{N} = 2$ supersymmetric (and, where appropriate, superconformal) description of $\mathcal{O}(2n)/\text{polar}$ and polar/polar dualities in terms of $\mathcal{N} = 2$ projective superfields. In section 4 we examine these dualities when reduced to $\mathcal{N} = 1$ superspace. One of the main results obtained in sections 3 and 4 is the proof of the fact that the dual of any $\mathcal{N} = 2$ superconformal $\sigma$-model is superconformal. Our analysis is deepened and carried out in more detail in section 5 for models with one polar multiplet. This section also contains several important examples. In section 6 we extend the analysis of the previous section to models containing a set of $n$ polar multiplets, again giving several examples. Section 7 contains a discussion of the intriguing possibility of self-dual models in the present setting, while section 8 contains a few concluding comments. We have collected some relevant features of superconformal Killing vectors in Appendix A. Finally, in Appendix B we discuss properties of the tensor multiplet formulation for $\sigma$-models with $\text{U}(1) \times \text{U}(1)$ symmetry \[5.43\]. Review material is collated in section 2 and Appendix A.

\[2\]More specifically, locally it is a symplectomorphism amounting to a change of polarization for the Darboux coordinates that describe the (2,0) holomorphic symplectic form of the hyperkähler manifold as it fibers the $\mathbb{C}P^1$ of complex structures.
2 Superconformal projective multiplets

We start from the algebra of \( \mathcal{N} = 2 \) spinor covariant derivatives:

\[
\{ D^i_\alpha, D^j_\beta \} = 0, \quad \{ \bar{D}^i_\bar{\alpha}, \bar{D}^j_\bar{\beta} \} = 0, \quad \{ D^i_\alpha, \bar{D}^j_\bar{\beta} \} = 2i \varepsilon^{ij} (\sigma^m)_{\alpha\bar{\beta}} \partial_m .
\]  

(2.1)

These relations encode an important structure that can be uncovered by introducing an auxiliary isotwistor \( v^i \in \mathbb{C}^2 \setminus \{0\} \) and defining the following operators: \( D_\alpha := v_i D^i_\alpha \) and \( \bar{D}_{\bar{\alpha}} := v_i \bar{D}^i_{\bar{\alpha}} \). Then, the anti-commutation relations (2.1) imply that

\[
\{ D_\alpha, D_\beta \} = \{ D_\alpha, \bar{D}_{\bar{\beta}} \} = \{ \bar{D}_{\bar{\alpha}}, \bar{D}_{\bar{\beta}} \} = 0 .
\]  

(2.2)

These identities constitute the integrability conditions for existence of certain constrained \( \mathcal{N} = 2 \) superfields that live in \( \mathbb{R}^{4|8} \times \mathbb{C}P^1 \) and are annihilated by \( D_\alpha \) and \( \bar{D}_{\bar{\alpha}} \).

Following [10], a superconformal projective multiplet of weight \( n, Q^{(n)}(z, v) \), is a superfield that lives on \( \mathbb{R}^{4|8} \), is holomorphic with respect to \( v^i \) on an open domain of \( \mathbb{C}P^1 \), and is characterized by the following conditions:

(a) it obeys the analyticity constraints

\[
D_\alpha Q^{(n)} = \bar{D}_{\bar{\alpha}} Q^{(n)} = 0 ;
\]  

(2.3)

(b) it is a homogeneous function of \( v^i \) of degree \( n \), that is

\[
Q^{(n)}(z, c v) = c^n Q^{(n)}(z, v) , \quad c \in \mathbb{C} \setminus \{0\} \equiv \mathbb{C}^* ;
\]  

(2.4)

(c) it obeys the following \( \mathcal{N} = 2 \) superconformal transformation law:

\[
\delta Q^{(n)} = -\left( \xi - \frac{\Lambda^{(2)}}{(v, u)} u^i \partial_{v^i} \right) Q^{(n)} - n \, \Sigma \, Q^{(n)} .
\]  

(2.5)

Here \( \xi = \xi^A(z) D_A \) is a \( \mathcal{N} = 2 \) superconformal Killing vector,

\[
\Lambda^{(2)} := \Lambda_{ij}(z) v^i v^j , \quad \Sigma = \frac{\Lambda_{ij}(z) v^i u^j}{(v, u)} + \sigma(z) + \bar{\sigma}(z) ,
\]  

(2.6)

and \( \Lambda_{ij}(z) \) and \( \sigma(z) \) are related to \( \xi \) as in eqs. (A.4)–(A.7). In the transformation law (2.5), \( u_i \) denotes a fixed isotwistor chosen to be arbitrary modulo the condition \( (v, u) := v^i u_i \neq 0 \). Both \( Q^{(n)} \) and \( \delta Q^{(n)} \) are independent of \( u_i \). The parameters \( \Sigma \) and \( \Lambda^{(2)} \) obey the identities:

\[
\mathcal{D}_\alpha \Lambda^{(2)} = \bar{\mathcal{D}}_{\bar{\alpha}} \Lambda^{(2)} = 0 , \quad \mathcal{D}_\alpha \Sigma = \bar{\mathcal{D}}_{\bar{\alpha}} \Sigma = 0 , \quad u_i \partial_{v^i} \Sigma = \frac{\Lambda^{(2)}}{(v, u)} .
\]  

(2.7)

Internal indices take two values, \( i, j = \underline{1, 2} \). We use underlined symbols to avoid notational confusion (say, between \( D^2 \) and \( D^2 = D \cdot D \)).
Given a superconformal weight-$n$ multiplet $Q^{(n)}(v^i)$, its smile conjugate $\hat{Q}^{(n)}(v^i)$, is defined by

$$Q^{(n)}(v^i) \rightarrow \hat{Q}^{(n)}(\bar{v}_i) \rightarrow \hat{Q}^{(n)}(\bar{v}_i \rightarrow -v_i) =: \hat{Q}^{(n)}(v^i) ,$$

with $\hat{Q}^{(n)}(\bar{v}_i) := \overline{Q^{(n)}(v^i)}$ the complex conjugate of $Q^{(n)}(v^i)$, and $\bar{v}_i$ the complex conjugate of $v^i$. One can show that $\hat{Q}^{(n)}(v)$ is a superconformal weight-$n$ multiplet, unlike the complex conjugate of $Q^{(n)}(v)$. One can also check that

$$\hat{\hat{Q}}^{(n)}(v^i) = (-1)^n Q^{(n)}(v^i) .$$

Therefore, if $n$ is even, one can define real isotwistor superfields, $\hat{Q}^{(2m)}(v^i) = Q^{(2m)}(v^i)$.

Our next goal is to understand how to engineer $\mathcal{N} = 2$ superconformal field theories described by superconformal projective multiplets. Let $\mathcal{L}^{(2)}$ be a real superconformal weight-2 multiplet, which is constructed in terms of the dynamical superfields. Associated with $\mathcal{L}^{(2)}$ is the superconformal action:

$$S := -\frac{1}{2\pi} \gamma v_i dv^i \int d^4 x \Delta^{(-4)} \mathcal{L}^{(2)}_{\theta = \bar{\theta} = 0} .$$

Here $\gamma$ denotes a closed contour in $\mathbb{C}P^1$, $v^i(t)$, parametrized by an evolution parameter $t$. The action makes use of the following fourth-order differential operator:

$$\Delta^{(-4)} := \frac{1}{16} \nabla^\alpha \nabla_\alpha \nabla^\beta \nabla_\beta , \quad \nabla_\alpha := \frac{1}{(v,u)} u_i D^i_\alpha , \quad \nabla^\beta := \frac{1}{(v,u)} \bar{u}_i \bar{D}^i_\beta .$$

Here $u_i$ is defined below eq. (2.7), and it is kept fixed along the integration contour. The action can be shown to be invariant under arbitrary infinitesimal $\mathcal{N} = 2$ superconformal transformations $[10]$.

An important property of the action (2.10) is its invariance under projective transformations of the form:

$$\left( u_i , v_i(t) \right) \rightarrow \left( u'_i , v'_i(t) \right) = \left( u_i , v_i(t) \right) R , \quad R = \begin{pmatrix} a(t) & 0 \\ b(t) & c(t) \end{pmatrix} \in \text{GL}(2, \mathbb{C}) ,$$

where $t$ is the evolution parameter along the contour, and the matrix elements $a(t)$ and $b(t)$ obey the first-order equations:

$$\dot{a} = b \frac{(\dot{v}, v)}{(v,u)} , \quad \dot{b} = -b \frac{(\dot{v}, u)}{(v,u)} ,$$

The smile conjugation is the real structure pioneered by Rosly [4] and re-discovered in [5, 2, 23].
with $\psi$ denoting the derivative of a function $\psi(t)$ with respect to $t$. Equations (2.13) guarantee that the transformed isotwistor $u'_i$ is $t$-independent. This invariance allows one to make $u_i$ arbitrary modulo the constraint $(v, u) \neq 0$, and therefore the action is independent of $u_i$, that is $(\partial/\partial u_i)S = 0$.

Let $\xi_K$ be a superconformal Killing vector obeying the conditions

$$\Lambda_{ij}(z) = \sigma(z) = 0 ,$$

with $\Lambda_{ij}(z)$ and $\sigma(z)$ defined in eqs. (A.7) and (A.5), respectively. It is called a $\mathcal{N} = 2$ Killing vector, for the set of all such vectors can be seen to form a superalgebra isomorphic to the $\mathcal{N} = 2$ super-Poincaré algebra. In the super-Poincaré case, the transformation law (2.5) reduces to the universal (weight-independent) form:

$$\delta Q^{(n)} = -\xi_K Q^{(n)} .$$

If we are interested in general $\mathcal{N} = 2$ supersymmetric (i.e. super-Poincaré invariant) theories, not necessarily superconformal ones, projective multiplets should be defined by the relations (2.3), (2.4) and (2.15).

Suppose we wish to construct an off-shell $\mathcal{N} = 2$ superconformal theory described by a given set of superconformal projective multiplets $\mathcal{P}_A^{(n_A)}(z, v)$. Then, the corresponding Lagrangian must be an algebraic function of the dynamical superfields,

$$\mathcal{L}^{(2)}_{\text{s-conformal}} = \mathcal{L}(\mathcal{P}_A^{(n_A)}) ,$$

and possess no explicit dependence on the isotwistor $v^i$. Imposing the homogeneity condition

$$\mathcal{L}(c^{n_A} \mathcal{P}_A^{(n_A)}) = c^2 \mathcal{L}(\mathcal{P}_A^{(n_A)}) , \quad c \in \mathbb{C}^* .$$

guarantees that $\mathcal{L}^{(2)}_{\text{s-conformal}}$ is a superconformal weight-two projective multiplet.

In the more general case of super-Poincaré invariant theories, the Lagrangian may depend explicitly on the isotwistor $v^i$,

$$\mathcal{L}^{(2)}_{\text{s-Poincaré}} = \mathcal{L}^{(2)}(\mathcal{P}_A^{(n_A)}; v) ,$$

and must obey the homogeneity condition

$$\mathcal{L}(c^{n_A} \mathcal{P}_A^{(n_A)}; c v) = c^2 \mathcal{L}(\mathcal{P}_A^{(n_A)}; v) , \quad c \in \mathbb{C}^* .$$

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It is easy to show that the action (2.10) generated by $\mathcal{L}^{(2)}_{\text{sp-Poincaré}}$ is $\mathcal{N} = 2$ supersymmetric. In the super-Poincaré case, the action (2.10) can be shown to be equivalent to that proposed originally in [2].

Without loss of generality, we can assume that the integration contour $\gamma$ in (2.10) does not pass through the “north pole” $v^i_{\text{north}} \sim (0,1)$ of $\mathbb{C}P^1$. It is then useful to introduce a complex (inhomogeneous) coordinate $\zeta$ in the north chart, $\mathbb{C}$, of $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$:

$$v^i = v^i(1,\zeta), \quad \zeta := \frac{v^2}{v^1}, \quad i = 1,2$$

and define projective multiplets in this chart. Given a weight-$n$ projective superfield $Q^{(n)}(z,v)$, we can associate with it a new object $Q^{[n]}(z,\zeta)$ defined as

$$Q^{(n)}(z,v) \rightarrow Q^{[n]}(z,\zeta) \propto Q^{(n)}(z,v), \quad \frac{\partial}{\partial \bar{\zeta}} Q^{[n]} = 0.$$ (2.21)

The explicit form of $Q^{[n]}(z,\zeta)$ depends on the multiplet under consideration, and will be specified below. In terms of $Q^{[n]}(z,\zeta)$, the analyticity constraints (2.3) take the form:

$$D_{\alpha}^2 Q^{[n]}(\zeta) = \zeta D_{\bar{\alpha}}^2 Q^{[n]}(\zeta), \quad D_{\dot{\alpha},\dot{2}} Q^{[n]}(\zeta) = -\frac{1}{\zeta} D_{\dot{\alpha},\dot{2}} Q^{[n]}(\zeta).$$ (2.22)

The $Q^{[n]}(z,\zeta)$ can be represented by a Laurent series

$$Q^{[n]}(z,\zeta) = \sum Q_k(z)\zeta^k,$$ (2.23)

with $Q_k(z)$ some ordinary $\mathcal{N} = 2$ superfields. In accordance with (2.8), the smile conjugate of $Q^{[n]}(z,\zeta)$ is defined as follows:

$$\tilde{Q}^{[n]}(z,\zeta) := \sum (-1)^k \tilde{Q}_k(z)\zeta^{-k}.$$ (2.24)

Unlike eq. (2.9), we now have

$$\tilde{Q}^{[n]}(\zeta) = Q^{[n]}(\zeta).$$ (2.25)

A real projective superfield is characterized by the properties:

$$\tilde{Q}^{[n]}(z,\zeta) = Q^{[n]}(z,\zeta) = \sum Q_k(z)\zeta^k, \quad \tilde{Q}_k = (-1)^k Q_{-k}.$$ (2.26)

When switching from $Q^{(n)}(v)$ to $Q^{[n]}(\zeta)$, the information about the degree of homogeneity, $n$, remains encoded only in the superconformal transformation law, eq. (2.5). In the super-Poincaré case, the superscript $[n]$ becomes redundant and is usually omitted.
We conclude this introductory section by listing those projective multiplets which are used for $\sigma$-model constructions. Our first example is the so-called real $O(2n)$ multiplet\(^5\), \(n = 1, 2, \ldots\), which is described by a real weight-2 projective superfield $\eta^{(2n)}(z, v)$ of the form:

$$
\eta^{(2n)}(z, v) = \eta_{i_1\cdots i_{2n}}(z) v^{i_1} \cdots v^{i_{2n}} = \bar{\eta}^{(2n)}(z, v) .
$$

Here $\eta_{i_1\cdots i_{2n}}(z)$ are completely symmetric $N = 2$ superfields obeying the constraints

$$
D_\alpha(j \eta_{i_1\cdots i_{2n}}) = \bar{D}_{\dot{\alpha}}(j \eta_{i_1\cdots i_{2n}}) = 0
$$

which follow from (2.23). It should be pointed out that the reality condition $\bar{\eta}^{(2n)} = \eta^{(2n)}$ is equivalent to

$$
\eta_{i_1\cdots i_{2n}}(z) = \bar{\epsilon}^{i_1j_1} \cdots \bar{\epsilon}^{i_{2n}j_{2n}} \eta_{j_1\cdots j_{2n}} .
$$

Associated with $\eta^{(2n)}(z, v)$ is the superfield $\eta^{[2n]}(z, \zeta)$ defined by

$$
\eta^{[2n]}(z, \zeta) = (v_{\perp}^{\zeta} v^z)^n \eta^{[2n]}(z, \zeta) = (v_{\perp}^{\zeta})^n \eta^{[2n]}(z, \zeta) ,
$$

$$
\eta^{[2n]}(z, \zeta) = \sum_{k=-n}^{n} \eta_k(z) \zeta^k , \quad \bar{\eta}_k = (-1)^k \eta_{-k} .
$$

The superfield $\eta^{[2n]}(z, \zeta)$ is real in the sense of (2.26).

To describe charged hypermultiplets, one uses the so-called \textit{arctic} multiplet $\Upsilon^{(n)}(z, v)$ \[3\], which is defined to be holomorphic in the north chart of $\mathbb{C}P^1$,

$$
\Upsilon^{(n)}(z, v) = (v_{\perp}^{\zeta})^{n} \Upsilon^{[n]}(z, \zeta) , \quad \Upsilon^{[n]}(z, \zeta) = \sum_{k=0}^{\infty} \Upsilon_k(z) \zeta^k ,
$$

and its smile-conjugate \textit{antarctic} multiplet $\bar{\Upsilon}^{(n)}(z, v)$,

$$
\bar{\Upsilon}^{(n)}(z, v) = (v_{\perp}^{\zeta})^{n} \bar{\Upsilon}^{[n]}(z, \zeta) , \quad \bar{\Upsilon}^{[n]}(z, \zeta) = \sum_{k=0}^{\infty} \bar{\Upsilon}_k(z) \frac{(-1)^k}{\zeta^k} .
$$

The pair $\Upsilon^{[n]}(\zeta)$ and $\bar{\Upsilon}^{[n]}(\zeta)$ constitute the so-called polar multiplet. The components $\Upsilon_k(z)$ in (2.31) are constrained $\mathcal{N} = 2$ superfields, in accordance with (2.22).\(^5\)

\[^5\]Here and below, we use the terminology introduced originally in [7] for non-superconformal projective multiplets.
To describe gauge superfields \((n=0)\) and Lagrange multipliers for various duality transformations, one uses the so-called real tropical multiplet \(U^{(2n)}(z,v)\) defined by

\[
U^{(2n)}(z,v) = (i,v \frac{1}{n})^n U^{[2n]}(z,\zeta) = (v \frac{1}{n})^n U^{[2n]}(z,\zeta),
\]

\[
U^{[2n]}(z,\zeta) = \sum_{k=-\infty}^{\infty} U_k(z) \zeta^k, \quad U_k = (-1)^k U_{-k}.
\]

The superfield \(U^{[2n]}(z,\zeta)\) is real in the sense of \((2.26)\).

The \(\mathcal{N}=2\) superconformal transformation laws of the superfields \(\eta^{[2n]}(\zeta), \Upsilon^{[n]}(\zeta), \bar{\Upsilon}^{[n]}(\zeta)\) and \(U^{[2n]}(\zeta)\) are given in \([10]\).

3 Formulation in \(\mathcal{N}=2\) superspace

The formalism presented in the previous section is convenient for the formulation of manifestly \(\mathcal{N}=2\) supersymmetric duality transformations. The main purpose of this section is to show that the dual of any superconformal field theory is superconformal.

3.1 Duality between the real \(\mathcal{O}(2n)\) and polar multiplets

Consider an off-shell \(\mathcal{N}=2\) supersymmetric \(\sigma\)-model described by an \(\mathcal{O}(2n)\)-multiplet \(\eta^{(2n)}(z,v)\) and some other projective multiplets \(\Omega^{(na)}(z,v)\). Let \(\mathcal{L}^{(2)}(\eta^{(2n)}, \Omega^{(na)}; v)\) be the Lagrangian of the theory. Note that, in general, \(\mathcal{L}^{(2)}\) may explicitly depend on the isotwistor \(v^i\). In the superconformal case, however, the Lagrangian must be \(v\)-independent, \(\mathcal{L}^{(2)}(\eta^{(2n)}, \Omega^{(na)})\).

The theory under consideration has a dual formulation given by a different Lagrangian \(\mathcal{L}^{(2)}_D(\Xi^{(2-2n)}, \bar{\Xi}^{(2-2n)}, \Omega^{(na)}; v)\), in which \(\Xi^{(2-2n)}\) is an arctic multiplet, and \(\bar{\Xi}^{(2-2n)}\) its smile conjugate antarctic multiplet. The dual description is obtained by Legendre transformation. One proceeds by replacing the original system by an auxiliary first-order formulation with Lagrangian

\[
\mathcal{L}^{(2)}_{\text{first-order}} = \mathcal{L}^{(2)}(U^{(2n)}, \Omega^{(na)}; v) + U^{(2n)}(\Xi^{(2-2n)} + \bar{\Xi}^{(2-2n)}),
\]

where \(U^{(2n)}\) is a real tropical multiplet. This model is equivalent to the original one. Indeed, varying the first-order action, \(S_{\text{first-order}}\), with respect to \(\Xi^{(2-2n)}\) and \(\bar{\Xi}^{(2-2n)}\) proves to constrain \(U^{(2n)}\) to become a real \(\mathcal{O}(2n)\) multiplet,

\[
\frac{\delta}{\delta \Xi^{(2-2n)}} S_{\text{first-order}} = 0 \implies U^{(2n)} = \eta^{(2n)}.
\]
and then $S_{\text{first-order}}$ reduces to the original action. On the other hand, varying the first-order action with respect to $U^{(2n)}$ gives\footnote{Since $\Xi^{(2-2n)}$ and $\bar{U}^{(2n)}$ are constrained $\mathcal{N} = 2$ superfields, the equations (3.2) and (3.3) are quite nontrivial. They can be derived using a formulation in terms of $\mathcal{N} = 1$ superfields, as was done in the original publications [3, 7]; see also subsection 5.1 below.}

$$
\frac{\partial}{\partial U^{(2n)}} \mathcal{L}^{(2)}(U^{(2n)}, \Omega^{(n_a)}; v) + \Xi^{(2-2n)} + \bar{\Xi}^{(2-2n)} = 0 . \quad (3.3)
$$

Suppose this equation allows us to uniquely express $U^{(2n)}$ as a function of the other variables, that is $U^{(2n)} = U^{(2n)}(\Xi^{(2-2n)}, \bar{\Xi}^{(2-2n)}, \Omega^{(n_a)}; v)$. Then, the dual Lagrangian is defined by

$$
\mathcal{L}^{(2)}(\Xi^{(2-2n)}, \bar{\Xi}^{(2-2n)}, \Omega^{(n_a)}; v) = \left\{ \mathcal{L}^{(2)}(U^{(2n)}, \Omega^{(n_a)}; v) + U^{(2n)}(\Xi^{(2-2n)} + \bar{\Xi}^{(2-2n)}) \right\} , \quad (3.4)
$$

where the vertical stroke on the right indicates that the variable $U^{(2n)}$ should be replaced by its on-shell value $U^{(2n)}(\Xi^{(2-2n)}, \bar{\Xi}^{(2-2n)}, \Omega^{(n_a)}; v)$.

The duality transformation presented is compatible with $\mathcal{N} = 2$ superconformal invariance. Indeed, suppose the original model is superconformal, and hence its Lagrangian has no explicit $v$-dependence, $\mathcal{L}^{(2)} = \mathcal{L}^{(2)}(\eta^{(2n)}, \Omega^{(n_a)})$. It leads to the first-order Lagrangian

$$
\mathcal{L}_{\text{first-order}} = \mathcal{L}^{(2)}(U^{(2n)}, \Omega^{(n_a)}) + U^{(2n)}(\Xi^{(2-2n)} + \bar{\Xi}^{(2-2n)}) , \quad (3.5)
$$

which also has no explicit $v$-dependence, and therefore generates a superconformal theory. Integrating out $U^{(2n)}$ does not generate any explicit $v$-dependence. We conclude that the dual Lagrangian is $v$-independent, $\mathcal{L}^{(2)}_D = \mathcal{L}^{(2)}_D(\Xi^{(2-2n)}, \bar{\Xi}^{(2-2n)}, \Omega^{(n_a)})$, and therefore the dual theory is $\mathcal{N} = 2$ superconformal.

### 3.2 Polar-polar duality

A different type of duality can be defined in the case of a nonlinear $\sigma$-model in which the dynamical variables include a polar multiplet realized in terms of an arctic superfield $\Upsilon^{(n)}(z, v)$ and its smile conjugate antarctic superfield $\tilde{\Upsilon}^{(n)}(z, v)$. Along with this polar multiplet, the theory may also describe the dynamics of some other multiplets $\Omega^{(n_a)}(z, v)$. We denote the corresponding Lagrangian by $\mathcal{L}^{(2)}(\Upsilon^{(n)}, \tilde{\Upsilon}^{(n)}, \Omega^{(n_a)}; v)$.

The theory under consideration possesses an equivalent first-order formulation generated by

$$
\mathcal{L}^{(2)}_{\text{first-order}} = \mathcal{L}^{(2)}(W^{(n)}, \tilde{W}^{(n)}, \Omega^{(n_a)}; v) + i W^{(n)} \Xi^{(2-n)} - i \tilde{W}^{(n)} \bar{\Xi}^{(2-n)} , \quad (3.6)
$$
where $W^{(n)}$ is *complex tropical*, and $\Xi^{(2-n)}$ *arctic*. Let $S_{\text{first-order}}$ be the corresponding action. Indeed, it will be shown in subsection 5.1 that the equation of motion for $\Xi^{(2-n)}$ implies that $W^{(n)}$ is a weight-$n$ arctic multiplet,

$$
\frac{\delta}{\delta \Xi^{(2-n)}} S_{\text{first-order}} = 0 \implies W^{(n)} = \Upsilon^{(n)}.
$$

(3.7)

Then, the action $S_{\text{first-order}}$ reduces to that generated by $L^{(2)}(\Upsilon^{(n)}, \bar{\Upsilon}^{(n)}, \Omega^{(n)}; v)$. On the other hand, the equations of motion for $W^{(n)}$ and $\bar{W}^{(n)}$ are:

$$
\frac{\partial}{\partial W^{(n)}} L^{(2)}(W^{(n)}, \bar{W}^{(n)}, \Omega; v) + i \Xi^{(2-n)} = 0 ,
$$

(3.8a)

$$
\frac{\partial}{\partial \bar{W}^{(n)}} L^{(2)}(W^{(n)}, \bar{W}^{(n)}, \Omega; v) - i \bar{\Xi}^{(2-n)} = 0 .
$$

(3.8b)

Under rather general assumptions, these algebraic equations can be used to express $W^{(n)}$ and $\bar{W}^{(n)}$ in terms of the other variables. This leads to the dual Lagrangian:

$$
L^{(2)}_{\text{D}}(\Xi^{(2-n)}, \bar{\Xi}^{(2-n)}, \Omega; v) = \left\{ L^{(2)}(W^{(n)}, \bar{W}^{(n)}, \Omega; v) + i W^{(n)} \Xi^{(2-n)} - i \bar{W}^{(n)} \bar{\Xi}^{(2-n)} \right\},
$$

(3.9)

where the vertical stroke on the right indicates that the variables $W^{(n)}$ and $\bar{W}^{(n)}$ should be replaced by their on-shell values. In the special case $n = 1$, both the original and dual polar multiplets have the same weight.

### 3.3 Polar-polar duality and superconformal $\sigma$-models

We consider a system of interacting weight-one* arctic multiplets, $\Upsilon^{+I}(z, v)$, and their smile-conjugates, $\bar{\Upsilon}^{+I}(z, v)$, described by a Lagrangian of the form [10]:

$$
L^{(2)}(\Upsilon^{+}, \bar{\Upsilon}^{+}) = i K(\Upsilon^{+}, \bar{\Upsilon}^{+}) ,
$$

(3.10)

Here $K(\Phi^{I}, \bar{\Phi}^{J})$ is a real function of $n$ complex variables $\Phi^{I}$, with $I = 1, \ldots, n$, which obeys the homogeneity condition (1.3). The function $K(\Phi^{I}, \bar{\Phi}^{J})$ can be interpreted as the Kähler potential of a Kähler cone (following the terminology of [21, 22]). Of course, this interpretation requires the Kähler metric $g_{I\bar{J}} := K_{I\bar{J}}$ to be non-singular,

$$
\det (K_{I\bar{J}}) \neq 0 ,
$$

(3.11)

\text{To simplify the notation, in this subsection we denote } \Upsilon^{+} = \Upsilon^{(1)}.
where we have used the standard the notation:

\[ K_{I_1 \ldots I_p \bar{J}_1 \ldots \bar{J}_q} := \frac{\partial^{p+q} K}{\partial \Phi_{I_1} \ldots \partial \Phi_{I_p} \bar{\Phi}_{\bar{J}_1} \ldots \bar{\Phi}_{\bar{J}_q}} . \]  

(3.12)

The action

\[ S[\Phi, \bar{\Phi}] = \int d^4x \ d^4\theta K(\Phi^I, \bar{\Phi}^\bar{J}) , \quad \bar{D}_a \Phi^I = 0 , \]  

(3.13)

with \( K(\Phi^I, \bar{\Phi}^\bar{J}) \) obeying the homogeneity condition (1.3), defines the most general \( N = 1 \) superconformal \( \sigma \)-model, see, e.g., [20].

We are interested in the dual formulation for the theory (3.10) which is obtained by performing the polar-polar duality with respect to all the multiplets:

\[ \mathcal{L}^{(2)}_D(\Xi^+, \check{\Xi}^+) = i \left\{ K(W^+, \check{W}^+) + W^+ \Xi^+_I - \check{W}^+ \check{\Xi}^+_I \right\} , \]  

(3.14)

where the vertical stroke on the right indicates that the complex tropical superfields \( W^+ \) and their smile-conjugates \( \check{W}^+ \) should be expressed in terms of weight-one arctic superfields \( \Xi^+_I \) and their smile-conjugates \( \check{\Xi}^+_I \) using the following equations of motion:

\[ \frac{\partial}{\partial W^+_I} K(W^+, \check{W}^+) + \Xi^+_I = 0 , \]  

(3.15a)

\[ \frac{\partial}{\partial \check{W}^+_I} K(W^+, \check{W}^+) - \check{\Xi}^+_I = 0 . \]  

(3.15b)

This requires the Kähler potential \( K(\Phi^I, \bar{\Phi}^\bar{J}) \) to obey the condition

\[ \det \begin{pmatrix} K_{IJ} & K_{I\bar{J}} \\ K_{I\bar{J}} & K_{\bar{J}\bar{J}} \end{pmatrix} \neq 0 . \]  

(3.16)

Making use of the equations (3.15a) and (3.15b), in conjunction with (1.3) and the standard properties of the Legendre transformation, one can show that the dual Lagrangian (3.14) obeys the homogeneity condition:

\[ \Xi^+_I \frac{\partial}{\partial \Xi^+_I} \mathcal{L}^{(2)}_D(\Xi^+, \check{\Xi}^+) = \mathcal{L}^{(2)}_D(\Xi^+, \check{\Xi}^+) . \]  

(3.17)

As a result, we can represent

\[ \mathcal{L}^{(2)}_D(\Xi^+, \check{\Xi}^+) = i K_D(\Xi^+, \check{\Xi}^+) , \]  

(3.18)

where \( K_D(\Psi_I, \bar{\Psi}_J) \) is a real analytic function of \( n \) complex variables \( \Psi_I \), with \( I = 1, \ldots, n \), which obeys the homogeneity condition

\[ \Psi_I \frac{\partial}{\partial \Psi_I} K_D(\Psi, \bar{\Psi}) = K_D(\Psi, \bar{\Psi}) . \]  

(3.19)
This function can be interpreted to be the Kähler potential of a Kähler cone. For such an interpretation to be consistent, the corresponding Kähler metric \( g_{D}^{IJ} := K_{D}^{IJ} \) should be nonsingular,

\[
\det (K_{D}^{IJ}) \neq 0 ,
\]

(3.20)

This indeed follows from eqs. (3.11) and (3.16) of which the latter implies

\[
\det \left( K_{D}^{IJ} K_{D}^{IJ} \right) \neq 0 .
\]

(3.21)

We conclude that the \( \mathcal{N} = 2 \) polar-polar duality transformation induces a transformation in the family of \( \mathcal{N} = 1 \) superconformal \( \sigma \)-models. Specifically, the \( \sigma \)-model (3.13) turns into

\[
S_{D}[\Psi, \bar{\Psi}] = \int d^{4}x \ d^{4}\theta K_{D}(\Psi_{I}, \bar{\Psi}_{J}) , \quad \bar{D}_{\dot{\alpha}} \Psi_{I} = 0 .
\]

(3.22)

It should be emphasized that the above conclusions hold if the duality transformation is applied to all the polar multiplets in the superconformal \( \sigma \)-model (3.10) and (1.3). Had we dualized some of the polar multiplets, we would have ended up with a dual formulation in which the Lagrangian obeys a different homogeneity condition. Specifically, let us split the original set of arctic multiplets, \( \Upsilon^{I} \), into two subsets \( \Upsilon^{I} = (\Upsilon^{i}, \Upsilon^{a}) \), and apply the polar-polar duality to the first subset. Then, we generate a dual Lagrangian

\[
\mathcal{L}_{D}^{(2)}(\Xi_{i}^{+}, \bar{\Upsilon}^{+a}, \bar{\Xi}_{i}^{+}, \tilde{\Upsilon}^{+})
\]

obeying the homogeneity condition

\[
\left( \frac{\partial}{\partial \Xi_{i}^{+}} + \tilde{\Upsilon}^{+a} \frac{\partial}{\partial \tilde{\Upsilon}^{+a}} \right) \mathcal{L}_{D}^{(2)}(\Xi^{+}, \Upsilon^{+}, \bar{\Xi}^{+}, \tilde{\Upsilon}^{+}) = \mathcal{L}_{D}^{(2)}(\Xi^{+}, \Upsilon^{+}, \bar{\Xi}^{+}, \tilde{\Upsilon}^{+}) .
\]

(3.23)

4 Formulation in \( \mathcal{N} = 1 \) superspace

From the point of view of various applications, one of the powerful properties of projective multiplets, \( Q^{[n]}(z, \zeta) \), is that they admit a simple decomposition in terms of standard \( \mathcal{N} = 1 \) superfields. This follows, in particular, from the analyticity constraints (2.22) which can be interpreted as follows. For the component \( \mathcal{N} = 2 \) superfields \( Q_{k}(z) \) of \( Q^{[n]}(z, \zeta) \) appearing in the series (2.23), their dependence on \( \theta^{a}_{2} \) and \( \bar{\theta}_{2}^{\dot{a}} \) is uniquely determined, according to (2.22), in terms of their dependence on the variables \( \theta^{\alpha}_{2} := \theta^{\alpha} \) and \( \bar{\theta}_{2}^{\dot{\alpha}} := \bar{\theta}^{\dot{\alpha}} \), which can be identified with the Grassmann coordinates of \( \mathcal{N} = 1 \) superspace parametrized by (\( x^{m}, \theta^{\alpha}, \bar{\theta}_{\dot{\alpha}} \))\(^{8}\). In other words, all information about the the projective

\[^{8}\text{The } \mathcal{N} = 1 \text{ spinor covariant derivatives are } D_{\alpha} := D^{\alpha}_{\dot{\alpha}} \text{ and } \bar{D}_{\dot{\alpha}} := \bar{D}_{\dot{\alpha}}^{\alpha} .\]
multiplet $Q^{[n]}(z, \zeta)$ is encoded in its $\mathcal{N} = 1$ projection

$$Q^{[n]}(x, \theta_i, \bar{\theta}^i, \zeta)\big|_{\theta_\alpha = \bar{\theta}_\alpha = 0} .$$

(4.1)

What is the structure of the $\mathcal{N} = 1$ superfields $Q_k\big|_{\theta_\alpha = \bar{\theta}_\alpha = 0}$ associated with the $\mathcal{N} = 2$ projective multiplet? If the Laurent series (2.23) terminates from below,

$$Q^{[n]}(z, \zeta) = \sum_p Q_k(z)\zeta^k , \quad -\infty < p ,$$

then the analyticity constraints (2.22) imply that the lowest components $Q_p$ and $Q_{p+1}$ are $\mathcal{N} = 1$ chiral and linear superfields, respectively.

$$\bar{\mathcal{D}}_\alpha Q_p = 0 , \quad \bar{\mathcal{D}}^2 Q_{p+1} = 0 .$$

(4.3)

If the Laurent series (2.23) terminates from above,

$$Q^{[n]}(z, \zeta) = \sum_q Q_k(z)\zeta^k , \quad q < \infty ,$$

then the analyticity constraints (2.22) imply that the highest components $Q_q$ and $Q_{q-1}$ are $\mathcal{N} = 1$ anti-chiral and anti-linear superfields, respectively.

$$D_\alpha Q_q = 0 , \quad D^2 Q_{q-1} = 0 .$$

(4.5)

The other $\mathcal{N} = 1$ superfields $Q_k\big|_{\theta_\alpha = \bar{\theta}_\alpha = 0}$ in (2.23) turn out to be unconstrained, modulo possible reality conditions.

In the $\mathcal{N} = 2$ supersymmetric action (2.10), the Lagrangian $\mathcal{L}^{(2)}$ is a projective multiplet, and therefore it is fully determined by its $\mathcal{N} = 1$ projection $\mathcal{L}^{(2)}\big|_{\theta_\alpha = \bar{\theta}_\alpha = 0}$. Let us express the action (2.10) in terms of this projection. We recall that the integration contour $\gamma$ in (2.10) is chosen to lie outside the “north pole” $v^\text{north}_i \sim (0, 1)$ of $\mathbb{C}P^1$, which allows us to use the inhomogeneous complex coordinate, $\zeta$, defined by $v^i = v^\perp(1, \zeta)$. Since the action is independent of $u_i$, the latter can be chosen to be $u_i = (1, 0)$, such that $(v, u) = v^\perp \neq 0$. We represent the Lagrangian in the form:

$$\mathcal{L}^{(2)}(z, v) = i v^\perp v^2 \mathcal{L}(z, \zeta) = i(v^\perp)^2 \zeta \mathcal{L}(z, \zeta) , \quad \tilde{\mathcal{L}} = \mathcal{L} .$$

(4.6)

It is important to remark that $\mathcal{L}(z, \zeta)$ is a real projective superfield. Now, a short calculation (see, e.g. [37]) allows us to bring the action (2.10) to the form:

$$S = \frac{1}{2\pi i} \oint_\gamma \frac{d\zeta}{\zeta} \int d^4x d^4\theta \mathcal{L}(z, \zeta)\big|_{\theta_\alpha = \bar{\theta}_\alpha = 0} .$$

(4.7)
Here the integration is carried out over the $\mathcal{N} = 1$ superspace. The action is now formulated entirely in terms of $\mathcal{N} = 1$ superfields. At the same time, by construction, it is off-shell $\mathcal{N} = 2$ supersymmetric.

The main goal of this section is to reformulate the duality transformations, which we presented in section 3, in terms of $\mathcal{N} = 1$ superfields. In what follows, the symbol of $\mathcal{N} = 1$ projection in expressions like (4.7) is omitted.

4.1 Duality between the real $\mathcal{O}(2n)$ and polar multiplets

We revisit the duality transformation between the real $\mathcal{O}(2n)$ and polar multiplets considered in subsection 3.1. Associated with the $\mathcal{O}(2n)$ multiplet $\eta^{(2n)}(v)$ is the superfield $\eta^{[2n]}(\zeta)$ defined by eq. (2.30). The two lowest components in the expansion (2.30), $\eta_{-n}$ and $\eta_{-n+1}$, are constrained $\mathcal{N} = 1$ superfields, chiral and linear, respectively,

$$\bar{D}_a \eta_{-n} = 0, \quad \bar{D}^2 \eta_{-n+1} = 0. \quad (4.8)$$

The $\mathcal{N} = 1$ superfields $\eta_{-n+2}, \ldots, \eta_{-1}$ are complex unconstrained, while $\eta_{0}$ is real unconstrained. Finally, the components $\eta_{1}, \ldots, \eta_{n}$ are related to those already considered by complex conjugation, eq. (2.30). For the other projective multiplets, $\Omega^{(na)}_{a}(v)$, entering the Lagrangian $\mathcal{L}^{(2)}(\eta^{(2n)}, \Omega^{(na)}_{a}; v)$, we appropriately replace $\Omega^{(na)}_{a}(v) \to \tilde{\Omega}^{(na)}_{a}(\zeta)$. The supersymmetric action turns into

$$S = \oint_{\gamma} \frac{d\zeta}{2\pi i} \int d^4x d^4\theta \mathcal{L}(\eta^{[2n]}, \Omega^{[na]}_{a}; \zeta). \quad (4.9)$$

Now, let us turn to the dual formulation. In complete analogy with $\eta^{(2n)}$, associated with the real tropical multiplet $U^{(2n)}(v)$ is the superfield $U^{[2n]}(\zeta)$ defined by (2.33). Associated with the arctic multiplet $\Xi^{(2-2n)}(v)$ is $\Xi^{[2-2n]}(\zeta)$ defined by

$$\Xi^{(2-2n)}(v) = (i)^{1-n}(v \zeta)^{2-2n} \Xi^{[2-2n]}(\zeta), \quad \Xi^{[2-2n]}(\zeta) = \sum_{k=0}^{\infty} \Xi_k \zeta^k, \quad \bar{D}_{\dot{a}} \Xi_0 = 0, \quad \bar{D}^2 \Xi_1 = 0. \quad (4.10)$$

For the smile-conjugate antarctic multiplet, $\tilde{\Xi}^{(2-2n)}(z,v)$, we get

$$\tilde{\Xi}^{(2-2n)}(v) = (i)^{n-1}(v \zeta)^{2-2n} \tilde{\Xi}^{[2-2n]}(\zeta), \quad \tilde{\Xi}^{[2-2n]}(\zeta) = \sum_{k=0}^{\infty} \tilde{\Xi}_k \frac{(-1)^k}{\zeta^k}. \quad (4.11)$$

9In the special case $n = 1$, which corresponds to the $\mathcal{N} = 2$ tensor multiplet [2], the component $\eta_0$ is a real linear $\mathcal{N} = 1$ superfield.
The first-order action becomes
\[
S_{\text{first-order}} = \oint \frac{d\zeta}{2\pi i} \int d^4x \, d^4\theta \left\{ \mathcal{L}(U^{[2n]}, \Omega_a^{[na]}; \zeta) + U^{[2n]} \left( \zeta^{n-1} \Xi^{[2-2n]} + (\zeta - 1)^{1-n} \tilde{\Xi}^{[2-2n]} \right) \right\} . \tag{4.12}
\]
This action coincides in form with that introduced in [3] (see also [7]).

4.2 Polar-polar duality

We turn to a \( \mathcal{N} = 1 \) formulation for the theory with Lagrangian \( \mathcal{L}^{(2)}(\Upsilon^{(n)}, \bar{\Upsilon}^{(n)}, \Omega; v) \) and its dual version considered in subsection 3.2.

The arctic multiplet \( \Upsilon^{(n)}(v) \) is represented by the series (2.31), in which the two leading components \( \Upsilon_0 \) and \( \Upsilon_1 \) are, respectively, chiral and complex linear \( \mathcal{N} = 1 \) superfields,
\[
\bar{D}_a \Upsilon_0 = 0, \quad \bar{D}^2 \Upsilon_1 = 0, \tag{4.13}
\]
while the other components \( \Upsilon_2, \Upsilon_3, \ldots \), are complex unconstrained \( \mathcal{N} = 1 \) superfields. Its smile-conjugate antarctic multiplet, \( \bar{\Upsilon}^{(n)}(v) \), is given by eq. (2.32). For the other projective multiplets, \( \Omega_a^{(na)} \), in the Lagrangian \( \mathcal{L}^{(2)}(\Upsilon^{(n)}, \bar{\Upsilon}^{(n)}, \Omega_a^{(na)}; v) \), we appropriately replace \( \Omega_a^{(na)}(z, v) \rightarrow \Omega_a^{[na]}(z, \zeta) \). The supersymmetric action turns into
\[
S = \oint \frac{d\zeta}{2\pi i} \int d^4x \, d^4\theta \mathcal{L}(\Upsilon^{[n]}, \bar{\Upsilon}^{[n]}, \Omega_a^{[na]}; \zeta) . \tag{4.14}
\]
Consider now the dual formulation. Associated with the complex tropical multiplet \( W^{(n)}(z, v) \) is the superfield \( W^{[n]}(z, \zeta) \) defined by
\[
W^{(n)}(v) = (v^1)^n W^{[n]}(\zeta) , \quad W^{[n]}(\zeta) = \sum_{k=-\infty}^{\infty} W_k \zeta^k . \tag{4.15}
\]
For its smile-conjugate antarctic multiplet, \( \bar{W}^{(n)}(z, v) \), we get
\[
\bar{W}^{(n)}(v) = (v^{-1})^n \bar{W}^{[n]}(\zeta) , \quad \bar{W}^{[n]}(\zeta) = \sum_{k=-\infty}^{\infty} \bar{W}_k \frac{(-1)^k}{\zeta^k} . \tag{4.16}
\]
Finally, the arctic superfield \( \Xi^{(2-n)}(z, v) \) and its smile-conjugate \( \tilde{\Xi}^{(2-n)}(z, v) \) will be represented similarly to eqs. (2.31) and (2.32) with the replacement \( n \rightarrow 2 - n \). The first-order action becomes
\[
S_{\text{first-order}} = \oint \frac{d\zeta}{2\pi i} \int d^4x \, d^4\theta \left\{ \mathcal{L}(W^{[n]}, \bar{W}^{[n]}, \Omega_a^{[na]}; \zeta) + \frac{1}{\zeta} W^{[n]} \Xi^{[2-n]} - \zeta \bar{W}^{[n]} \tilde{\Xi}^{[2-n]} \right\} . \tag{4.17}
\]
This formulation of the polar-polar duality coincides with that given in [8, 9].
4.3 Polar-polar duality and superconformal $\sigma$-models

Of special interest for us is the superconformal $\sigma$-model defined by eqs. (3.10) and (1.3), for it can be argued to realize general $N = 2$ superconformal couplings. We represent the weight-one arctic multiplets as $\Upsilon^I(\zeta) = \Phi^I + \zeta \Sigma^I + O(\zeta^2)$, where

$$\Upsilon^I(\zeta) = \sum_{k=0}^{\infty} \Upsilon^I_k \zeta^k = \Phi^I + \zeta \Sigma^I + O(\zeta^2), \quad \bar{D}_\Phi \Phi^I = 0, \quad \bar{D}^2 \Sigma^I = 0. \quad (4.18)$$

We recall that the components $\Upsilon_2, \Upsilon_3, \ldots$, are complex unconstrained $N = 1$ superfields.

The $N = 2$ superconformal action turns into

$$S = \oint d\gamma \frac{d\zeta}{2\pi i} \int d^4x \, d^4\theta \, K(\Upsilon^I, \tilde{\Upsilon}^J). \quad (4.19)$$

The dual formulation is described by the following $N = 2$ superconformal action:

$$S_D = \oint d\gamma \frac{d\zeta}{2\pi i} \int d^4x \, d^4\theta \, K_D(\Xi^I, \tilde{\Xi}^J). \quad (4.20)$$

Here the arctic multiplets $\Xi^I(\zeta)$ is related to $\Xi_I^+(v)$ by the rule $\Xi_I^+(v) = v \frac{\partial}{\partial v} \Xi^I(\zeta)$, and the structure of $\Xi^I(\zeta)$ is completely similar to that given in eq. (4.18). The dual Lagrangian is defined by

$$K_D(\Xi^I, \tilde{\Xi}^J) = \left\{ \right. K(W, \bar{W}) + \frac{1}{\zeta} W^I \Xi^I - \zeta \bar{W}^I \Xi^I \left. \right\}, \quad (4.21)$$

where the tropical superfields $W^I$ and $\bar{W}^I$ must be unique solutions of the algebraic equations:

$$\frac{\partial}{\partial W^I} K(W, \bar{W}) + \frac{1}{\zeta} \Xi^I = 0, \quad \frac{\partial}{\partial \bar{W}^I} K(W, \bar{W}) - \zeta \tilde{\Xi}^I = 0. \quad (4.22)$$

The Kähler potential $K(\Phi, \bar{\Phi})$ and its dual $K_D(\Psi, \bar{\Psi})$ correspond, in general, to different Kähler cones. As will be argued in the remainder of this paper, both potentials are encoded in the hyperkähler potential, $K(\Phi, \bar{\Phi}, \Psi, \bar{\Psi})$, in the target space for the original $\sigma$-model (4.19). It will also be shown that the original hyperkähler potential $K$ and its dual $K_D$ are related to each other by a holomorphic reparametrization.

To conclude this section, we summarize, without proof, the explicit structure of the hyperkähler potential $K(\Phi^I, \bar{\Phi}^J, \Psi_J, \bar{\Psi}_j)$; the technical details can be found in [20]. It has the form:

$$K(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) = K(\Phi, \bar{\Phi}) + H(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) \quad (4.23)$$
where the second term obeys the homogeneity condition

\[
\left( \Phi^I \frac{\partial}{\partial \Phi^I} + \Psi^I \frac{\partial}{\partial \Psi^I} \right) \mathcal{H}(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) = \mathcal{H}(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) ,
\]

as well as the condition

\[
\Psi^I \frac{\partial \mathcal{H}}{\partial \Psi^I} = \bar{\Psi}^I \frac{\partial \mathcal{H}}{\partial \bar{\Psi}^I} ,
\]

and hence

\[
\Phi^I \frac{\partial \mathcal{H}}{\partial \Phi^I} = \bar{\Phi}^I \frac{\partial \mathcal{H}}{\partial \bar{\Phi}^I} .
\]

5 Polar-polar duality with a single hypermultiplet

In this section we carry out a more systematic study of the polar-polar duality. In particular, we provide proofs for several statements, specifically eqs. (3.7), (3.8a) and (3.8b), which were taken for granted in our previous consideration. For the sake of simplicity, our discussion is restricted to the case of \( \mathcal{N} = 2 \) supersymmetric sigma-models described by a single polar multiplet. However, many results can be readily extended to the case of \( n \) polar multiplets. All nontrivial \( \mathcal{N} = 2 \) supersymmetric sigma-models of the type specified (that is, described by one polar multiplet) are non-superconformal, except \( \mathcal{L} = \bar{\Upsilon} \Upsilon \), when \( \Upsilon(\zeta) \) has weight one.

5.1 General analysis

Consider an off-shell \( \mathcal{N} = 2 \) supersymmetric nonlinear \( \sigma \)-model described by a polar multiplet realized in terms of an arctic superfield \( \Upsilon(\zeta) \) and its smile conjugate \( \bar{\Upsilon}(\zeta) \).

\[
S = \oint_{\gamma} \frac{d\zeta}{2\pi i \zeta} \int d^4x d^4\theta \mathcal{L}(\Upsilon, \bar{\Upsilon}; \zeta) ,
\]

where \( \Upsilon(\zeta) \) looks like

\[
\Upsilon(\zeta) = \sum_{n=0}^{\infty} \zeta^n \Upsilon_n = \Phi + \zeta \Sigma + O(\zeta^2) , \quad \bar{D}_a \Phi = 0 , \quad \bar{D}^2 \Sigma = 0 ,
\]

and its smile-conjugate \( \bar{\Upsilon}(\zeta) \) has the form

\[
\bar{\Upsilon}(\zeta) = \sum_{n=0}^{\infty} (-\zeta)^{-n} \bar{\Upsilon}_n .
\]
We recall that the components \( \Upsilon_2, \Upsilon_3, \ldots \), in (5.2) are complex unconstrained \( \mathcal{N} = 1 \) superfields. These superfields appear in the action without derivatives, and therefore they are purely auxiliary.

Before discussing the dual formulation for the theory (5.1), it is worth recalling the explicit structure of the equations of motion, see, e.g., [8, 9]. Since the \( \mathcal{N} = 1 \) superfields \( \Upsilon_2, \Upsilon_3, \ldots \), in (5.2) are complex unconstrained, their equations of motion

\[
\frac{\delta}{\delta \Upsilon_n} S = 0 , \quad n \geq 2
\]  

have the form:

\[
\oint \frac{d \zeta}{\zeta} \zeta^n \frac{\partial \mathcal{L}(\Upsilon, \tilde{\Upsilon} ; \zeta)}{\partial \Upsilon} = 0 , \quad n \geq 2 .
\]  

This infinite set of nonlinear algebraic equations are equivalent to

\[
\frac{\partial}{\partial \Upsilon} \mathcal{L}(\Upsilon, \tilde{\Upsilon} ; \zeta) + \frac{1}{\zeta} \Pi = 0 , \quad \Pi(\zeta) = \sum_{n=0}^{\infty} \zeta^n \Pi_n ,
\]  

for some superfield \( \Pi(\zeta) \). These equations can be used, in principle, to express the auxiliary superfields \( \Upsilon_n \), with \( n \geq 2 \), in terms of the physical superfields \( \Phi \) and \( \Sigma \) and their conjugates; after that, the explicit form of \( \Pi(\zeta) \) can be determined as well. It remains to consider the equations of motion for the physical chiral \( \Phi := \Upsilon_0 \) and complex linear \( \Sigma := \Upsilon_1 \) superfields\(^{10}\). These equations imply that \( \Pi(\zeta) \) is an arctic multiplet. We see that the off-shell constraint \( (\Upsilon(\zeta) \text{ is arctic}) \) and the equation of motion \( (\Pi(\zeta) \text{ is arctic}) \) have the same superfield type. This is characteristic of duality-covariant theories in which the Bianchi identities and equations of motion have the same functional type (see, e.g., [38] for a review).

Now, let us apply the polar-polar duality transformation to the action (5.1). Following subsection 4.2, we replace (5.1) by the first-order action

\[
S_{\text{first-order}} = \oint \frac{d \zeta}{2\pi i \zeta} \int d^4x \, d^4\theta \left\{ \mathcal{L}(W, \tilde{W} ; \zeta) + \frac{1}{\zeta} W \Xi - \zeta \tilde{W} \Xi \right\} ,
\]  

where \( W(\zeta) \) is complex tropical,

\[
W(\zeta) = \sum_{n=-\infty}^{\infty} \zeta^n W_n , \quad \tilde{W}(\zeta) = \sum_{n=-\infty}^{\infty} (-\zeta)^{-n} \tilde{W}_n ,
\]  

\(^{10}\)In deriving the equations of motion for \( \Phi \) and \( \Sigma \), it is useful to represent \( \Phi = \bar{D}^2 \bar{R} \) and \( \Sigma = \bar{D}_\alpha \bar{\xi}^\alpha \), for unconstrained superfields \( \bar{R} \) and \( \bar{\xi}^\alpha \).
and the Lagrange multiplier \( \Xi(\zeta) \) is arctic,

\[
\Xi(\zeta) = \sum_{n=0}^{\infty} \zeta^n \Xi_n = \Psi + \zeta \Gamma + O(\zeta^2) , \quad \bar{D}_a \Psi = 0 , \quad \bar{D}^2 \Gamma = 0 . \tag{5.9}
\]

We would like to show that the theory with action (5.7) is equivalent to the original one, eq. (5.1). To vary (5.7) with respect to \( \Xi(\zeta) \), it is useful (i) to do the contour integrals in the second and third terms on the right of (5.7), as well as (ii) to separate the contributions involving the auxiliary and the physical superfields contained in \( \Xi(\zeta) \):

\[
S_{\text{first-order}} = \oint \frac{d\zeta}{2\pi i \zeta} \int d^4 x d^4 \theta \mathcal{L}(W, \bar{W} ; \zeta) + \sum_{n=2}^{\infty} \int d^4 x d^4 \theta \left\{ \Xi_n W_{-n+1} + \text{c.c.} \right\}
\]
\[
+ \int d^4 x d^4 \theta \left\{ \Psi W_1 + + \Gamma W_0 + \text{c.c.} \right\} . \tag{5.10}
\]

Since the superfields \( \Xi_n \), with \( n = 2, 3, \ldots \), are complex unconstrained, their equations of motion are

\[
W_{-n+1} = 0 , \quad n \geq 2 . \tag{5.11}
\]

Next, the equations of motion for \( \Psi \) and \( \Gamma \) are equivalent to the conditions that \( W_1 \) and \( W_0 \) are complex linear and chiral, respectively. Our conclusion is thus the following:

\[
\delta \delta \Xi S_{\text{first-order}} = 0 \implies W(\zeta) = \Upsilon(\zeta) . \tag{5.12}
\]

As a result, the second and third terms in (5.10) drop out, and \( S_{\text{first-order}} \) reduces to (5.1).

The above derivation of eq. (5.12) can be readily extended to justify the equation (3.7) in the general case.

On the other hand, instead of varying (5.7) with respect to \( \Xi(\zeta) \), we can first vary \( S_{\text{first-order}} \) with respect to \( W(\zeta) \). Since all the components \( W_n \) in (5.8) are complex unconstrained superfields, we immediately obtain

\[
\delta \delta W S_{\text{first-order}} = 0 \implies \frac{\partial}{\partial W} \mathcal{L}(W, \bar{W} ; \zeta) + \frac{1}{\zeta} \Xi = 0 . \tag{5.13}
\]

This equation and its smile-conjugate can be used to express \( W(\zeta) \) in terms of \( \Xi(\zeta) \), \( \tilde{\Xi}(\zeta) \) and \( \zeta \). As a result, \( S_{\text{first-order}} \) turns into the dual action

\[
S_D = \oint \frac{d\zeta}{2\pi i \zeta} \int d^4 x d^4 \theta \mathcal{L}_D(\Xi, \tilde{\Xi} ; \zeta) . \tag{5.14}
\]

Our derivation of eq. (5.13) can be readily generalized to justify the equation (3.8a).
5.2 Chiral-linear duality

The $\mathcal{N} = 2$ supersymmetric nonlinear $\sigma$-model (5.1) can be formulated solely in terms of the physical superfields $\Phi$, $\Sigma$ and their conjugates. The equations (5.6) can be used to express all the auxiliary superfields $\Upsilon_2, \Upsilon_3, \ldots$ (as well as the components $\Pi_n$ in (5.6)) in terms of the physical ones. Then, the action (5.1) turns into the chiral-linear (CL) one

$$S^{(\text{CL})} = \int d^4x \, d^4\theta \, L^{(\text{CL})}(\Phi, \bar{\Phi}; \Sigma, \bar{\Sigma}).$$

(5.15)

The chiral-linear formulation can also be obtained for the dual theory (5.14) following the same rules. This leads to

$$S^{(\text{CL})}_D = \int d^4x \, d^4\theta \, L^{(\text{CL})}_D(\Psi, \bar{\Psi}; \Gamma, \bar{\Gamma}).$$

(5.16)

We now demonstrate that $L^{(\text{CL})}_D(\Psi, \bar{\Psi}; \Gamma, \bar{\Gamma})$ is a Legendre transform of $L^{(\text{CL})}(\Phi, \bar{\Phi}; \Sigma, \bar{\Sigma})$.

Let us return to the first-order formulation (5.7) for the theory (5.1). This first-order action is equivalent to (5.10). Consider the equations of motion for the auxiliary superfields $\Xi_n$ and $W_n$, where $n \geq 2$. The equations of motion for $\Xi_n$, with $n \geq 2$, are given by (5.11). The equations of motion for $W_n$, with $n \geq 2$, are

$$\frac{\partial}{\partial W} L(W, \bar{W}; \zeta) + \frac{1}{\zeta} \Lambda = 0, \quad \Lambda(\zeta) := \sum_{n=0}^{\infty} \zeta^n A_n,$$

(5.17)

with $\Lambda(\zeta)$ some superfield. Eq. (5.11) tells us that $W(\zeta)$ is now represented by a Taylor series. Eq. (5.17) has the same functional form as the auxiliary field equation of motion, eq. (5.6), in the theory (5.1). Therefore, making use of eqs. (5.11) and (5.17) allows us to transform (5.10) to the form:

$$S'_{\text{first-order}} = \int d^4x \, d^4\theta \, L^{(\text{CL})}(U, \bar{U}; V, \bar{V}) + \int d^4x \, d^4\theta \left\{ \Gamma U - \Psi V + \text{c.c.} \right\},$$

(5.18)

where we have denoted

$$U := W_0, \quad V := -W_1.$$

(5.19)

It is clear that the first-order model (5.18) is equivalent to (5.1). The latter is also equivalent to (5.16). This indeed shows that $L^{(\text{CL})}_D(\Psi, \bar{\Psi}; \Gamma, \bar{\Gamma})$ is a Legendre transform of $L^{(\text{CL})}(\Phi, \bar{\Phi}; \Sigma, \bar{\Sigma})$.

As is clear from the above consideration, the transformation

$$L^{(\text{CL})}(\Phi, \bar{\Phi}; \Sigma, \bar{\Sigma}) \quad \longrightarrow \quad L^{(\text{CL})}_D(\Psi, \bar{\Psi}; \Gamma, \bar{\Gamma})$$
actually involves two independent Legendre transformations:

(a) dualization of the (anti) chiral variables $\Phi$ and $\bar{\Phi}$ into (anti) linear ones $\Gamma$ and $\bar{\Gamma}$;

(b) dualization of the (anti) linear variables $\Sigma$ and $\bar{\Sigma}$ into (anti) chiral ones $\Psi$ and $\bar{\Psi}$.

It is easy to see the order in which these Legendre transformations is performed (say, first carry out (a) and then (b), or vice versa) does not matter. We can also apply single Legendre transformations, specifically:

\[
L^{(CL)}(\Phi, \bar{\Phi}; \Sigma, \bar{\Sigma}) \rightarrow L^{(CC)}(\Phi, \bar{\Phi}; \Psi, \bar{\Psi}) ,
\]

\[
L^{(CL)}(\Phi, \bar{\Phi}; \Sigma, \bar{\Sigma}) \rightarrow L^{(LL)}(\Gamma, \bar{\Gamma}; \Sigma, \bar{\Sigma}) .
\]

The Lagrangians obtained can be further Legendre-transformed

\[
L^{(CC)}(\Phi, \bar{\Phi}; \Psi, \bar{\Psi}) \rightarrow L^{(LC)}(\Gamma, \bar{\Gamma}; \Psi, \bar{\Psi}) ,
\]

\[
L^{(LL)}(\Gamma, \bar{\Gamma}; \Sigma, \bar{\Sigma}) \rightarrow L^{(LC)}(\Gamma, \bar{\Gamma}; \Psi, \bar{\Psi}) ,
\]

where the notation introduced should be quite transparent. One has

\[
L^{(LC)}(\Gamma, \bar{\Gamma}; \Psi, \bar{\Psi}) = L^{(CL)}(\Psi, \bar{\Psi}; \Gamma, \bar{\Gamma}) .
\]

It should be pointed out that

\[
\mathbb{K}(\Phi, \bar{\Phi}; \Psi, \bar{\Psi}) := L^{(CC)}(\Phi, \bar{\Phi}; \Psi, \bar{\Psi})
\]

coincides with the hyperkähler potential in the target space of the $\mathcal{N} = 2$ supersymmetric $\sigma$-model (5.1). Let $\mathbb{K}_D(\Psi, \bar{\Psi}, \Phi, \bar{\Phi})$ be the hyperkähler potential in the target space of the dual model (5.14). It follows from (5.24) that

\[
\mathbb{K}_D(\Psi, \bar{\Psi}, \Phi, \bar{\Phi}) = \mathbb{K}(\Phi, \bar{\Phi}; \Psi, \bar{\Psi}) .
\]

5.3 $\mathcal{N} = 2$ $\sigma$-models on cotangent bundles of Kähler manifolds

Let us now consider those $\sigma$-models (5.1) in which the Lagrangian has no explicit dependence on $\zeta$,

\[
\mathcal{L}(\Upsilon, \bar{\Upsilon}; \zeta) \rightarrow \mathcal{L}(\Upsilon, \bar{\Upsilon}) = K(\Upsilon, \bar{\Upsilon}) .
\]

Here $K(\Phi, \bar{\Phi})$ is real analytic function that can be consistently interpreted as the Kähler potential of a two-dimensional Kähler manifold $M$. The action (5.1) associated with the Lagrangian (5.27) is invariant under U(1) transformations of the form:

\[
\Upsilon(\zeta) \rightarrow \Upsilon'(\zeta) = \Upsilon(e^{i\beta} \zeta) , \quad \beta \in \mathbb{R} .
\]
This invariance follows from the fact that the contour integration measure in (5.1) is invariant under transformations $\zeta \to e^{i\beta} \zeta$ \(^{11}\) and thus
\[
\oint_\gamma \frac{d\zeta}{2\pi i} L\left(\gamma'(\zeta), \dot{\gamma}'(\zeta); \zeta\right) = \oint_\gamma \frac{d\zeta}{2\pi i} L\left(\gamma(\zeta), \dot{\gamma}(\zeta); e^{-i\beta}\zeta\right), \tag{5.29}
\]
with $\gamma'(\zeta)$ defined in (5.28). The hyperkähler potential in the target space turns out to have the form:
\[
K(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) = K(\Phi, \bar{\Phi}) + H(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}), \tag{5.30}
\]
where the complex variables $(\Phi, \Psi)$ parametrize (an open domain of the zero section of) the holomorphic cotangent bundle $T^*M$ of the Kähler manifold $M$ \(^{12}\). The second term in (5.30) must be invariant under arbitrary phase transformations of the one-form $\Psi$,
\[
H(\Phi, \bar{\Phi}, e^{i\beta}\Psi, e^{-i\beta}\bar{\Psi}) = H(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}), \quad \beta \in \mathbb{R}. \tag{5.31}
\]

The hyperkähler potential must obey the Monge-Ampère equation (see, e.g., [39])
\[
\det \begin{pmatrix}
\frac{\partial^2 K}{\partial \Phi \partial \bar{\Phi}} & \frac{\partial^2 K}{\partial \Phi \partial \Psi} \\
\frac{\partial^2 K}{\partial \bar{\Phi} \partial \Psi} & \frac{\partial^2 K}{\partial \bar{\Phi} \partial \bar{\Psi}}
\end{pmatrix} = 1. \tag{5.32}
\]
It can be represented in the form:
\[
H(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) = \sum_{n=1}^{\infty} H_n(\Phi, \bar{\Phi}) \left[ \frac{\Psi \bar{\Psi}}{g_{\Phi\Phi}(\Phi, \bar{\Phi})} \right]^n, \quad H_1 = 1 \tag{5.33}
\]
where $g_{\Phi\Phi}(\Phi, \bar{\Phi}) = \partial_\Phi \partial_{\bar{\Phi}} K(\Phi, \bar{\Phi})$ is the Kähler metric on $M$, and $H_n(\Phi, \bar{\Phi})$ are real analytic scalar fields on $M$. We would like to rewrite the Monge-Ampère equation in terms of objects intrinsic to the Kähler base manifold. From the point of view of the Kähler base, $H$ can be thought of as a linear combination of tensor fields where $\Psi$ play

\[^{11}\text{Transformations } \zeta \to e^{i\beta} \zeta \text{ can be interpreted as time translations along } \gamma. \text{ This becomes manifest if the integration contour } \gamma \text{ in (5.1) is chosen to be } \zeta(t) = Re^{it}. \text{ Thus, if } \zeta \text{ is viewed as a complex evolution parameter, the Lagrangian (5.27) is a generalization of mechanical systems with conserved energy (for such a system, its Lagrangian } L(q, \dot{q}) \text{ has no explicit time dependence).}

\[^{12}\text{More generally, in the case of } n \text{ self-interacting polar multiplets described by a } \zeta \text{-independent Lagrangian, } L = K(\gamma^I, \dot{\gamma}^J), \text{ it was shown in [3, 25] that the } \sigma \text{-model target space is (an open domain of the zero section of) the cotangent bundle } T^*M \text{ of a Kähler manifold } M \text{ for which } K(\Phi^I, \bar{\Phi}^J) \text{ is the Kähler potential. This supersymmetric } \sigma \text{-model result implies the existence of a hyperkähler structure on } T^*M, \text{ for an arbitrary real-analytic Kähler manifold [3, 25]. The latter result was independently established by purely mathematical means [20, 27].}

the role of base co-vectors. By the definition of the covariant derivative on the Kähler base manifold we realize that
\[ \nabla_{\Phi} \mathcal{H} = \sum_{n=1}^{\infty} \nabla_{\Phi} H_n(\Phi, \bar{\Phi}) \left[ \frac{\Psi \bar{\Psi}}{g_{\Phi \bar{\Phi}}(\Phi, \bar{\Phi})} \right]^n. \] (5.34)

Using this, the Monge-Ampère equation is equivalent to
\[ g_{\Phi \bar{\Phi}} \frac{\partial^2 \mathcal{H}}{\partial \Psi \partial \bar{\Psi}} - 1 = \left( \nabla_{\Phi} \frac{\partial \mathcal{H}}{\partial \Psi} \right) \nabla_{\Phi} \frac{\partial \mathcal{H}}{\partial \bar{\Psi}} - \left[ \nabla_{\Phi} \nabla_{\bar{\Phi}} \mathcal{H} + \frac{1}{2} g_{\Phi \bar{\Phi}} R \frac{\partial \mathcal{H}}{\partial \Psi} \frac{\partial^2 \mathcal{H}}{\partial \Psi \partial \bar{\Psi}} \right], \] (5.35)

with \( R \) denoting the scalar curvature of \( \mathcal{M} \), that is \( g_{\Phi \bar{\Phi}} R = -\frac{1}{2} \partial_{\Phi} \Gamma_{\Phi \Phi}^{\Phi} \). In the right-hand side of (5.35), the covariant derivatives \( \nabla_{\Phi} \) and \( \nabla_{\bar{\Phi}} \) act only on the scalar fields \( H_n \) appearing in (5.33), that is
\[ (\nabla_{\bar{\Phi}})^a (\nabla_{\Phi})^b \mathcal{H}(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) := \sum_{n=1}^{\infty} \left[ (\nabla_{\bar{\Phi}})^a (\nabla_{\Phi})^b H_n(\Phi, \bar{\Phi}) \right] \left[ \frac{\Psi \bar{\Psi}}{g_{\Phi \bar{\Phi}}(\Phi, \bar{\Phi})} \right]^n. \] (5.36)

Eq. (5.35) is equivalent to a recursion relation to uniquely compute the coefficients \( H_n \). The recursion relation is as follows:
\[ H_m = \frac{1}{m^2} \sum_{n=1}^{m-1} \frac{m-n}{g_{\Phi \bar{\Phi}}} \left( n(\nabla_{\Phi} H_n) \nabla_{\bar{\Phi}} H_{m-n} - (m-n)(\nabla_{\bar{\Phi}} \nabla_{\Phi} H_n) H_{m-n} \right. \]
\[ \left. \quad - \frac{1}{2} n(m-n) g_{\Phi \bar{\Phi}} R H_n H_{m-n} \right), \quad m \geq 2. \] (5.37)

To construct the dual formulation, we should consider the first-order action
\[ S_{\text{first-order}} = \oint_{\gamma} \frac{d\zeta}{2\pi i \zeta} \int d^4 x \, d^4 \theta \left\{ K(W, \bar{W}) + \frac{1}{\zeta} W \Xi - \zeta \bar{W} \bar{\Xi} \right\}, \] (5.38)

with \( W(\zeta) \) a complex tropical multiplet, and \( \Xi(\zeta) \) an arctic superfield. The U(1) symmetry of the original model, eq. (5.28), turns into
\[ W(\zeta) \longrightarrow W(e^{i\beta} \zeta), \quad \Xi(\zeta) \longrightarrow e^{-i\beta} \Xi(e^{i\beta} \zeta). \] (5.39)

From (5.38) we read off the dual Lagrangian
\[ \mathcal{L}_D(\Xi, \bar{\Xi}; \zeta) = -K_D(-\zeta^{-1} \Xi, \zeta \bar{\Xi}) \] (5.40)

for some real function \( K_D(\Psi, \bar{\Psi}) \). Unlike the original Lagrangian, eq. (5.27), the dual Lagrangian depends, in general, on \( \zeta \). Such a dependence disappears in special cases which will be discussed below.
5.4 $\mathcal{N} = 2 \sigma$-models with U(1) × U(1) symmetry

Here we would like to consider a subclass of hypermultiplet models (5.27) which are invariant under two rigid U(1) symmetries: phase transformations

$$\Upsilon(\zeta) \rightarrow e^{i\alpha} \Upsilon(\zeta), \quad \alpha \in \mathbb{R}$$

(5.41)

and shadow chiral rotations\(^\text{13}\)

$$\Upsilon(\zeta) \rightarrow \Upsilon'(\zeta) = e^{-(i/2)\beta} \Upsilon(e^{i\beta} \zeta), \quad \beta \in \mathbb{R}.$$ 

(5.42)

The most general Lagrangian compatible with such symmetries is

$$\mathcal{L}(\Upsilon, \bar{\Upsilon}; \zeta) = \mathcal{L}(\Upsilon \bar{\Upsilon}),$$

(5.43)

with $\mathcal{L}(x)$ a real analytic function of one real variable. We can interpret $K(\Phi, \bar{\Phi}) := \mathcal{L}(\Phi \bar{\Phi})$ as the Kähler potential of a two-dimensional space $\mathcal{M}$ in canonical (or Kähler normal) complex coordinates, see subsection 6.1.

With the above Lagrangian, the first-order action (5.7) takes the form

$$S_{\text{first-order}} = \oint_{\gamma} \frac{d\zeta}{2\pi i} \int d^4 x \ d^4 \theta \left\{ \mathcal{L}(W \bar{W}) + \frac{1}{\zeta} W \Xi - \zeta \bar{W} \bar{\Xi} \right\},$$

(5.44)

with $W(\zeta)$ a complex tropical multiplet, and $\Xi(\zeta)$ an arctic multiplet. This action is invariant under arbitrary phase transformations

$$W(\zeta) \rightarrow e^{i\alpha} W(\zeta), \quad \Xi(\zeta) \rightarrow e^{-i\alpha} \Xi(\zeta)$$

(5.45)

and shadow chiral rotations

$$W(\zeta) \rightarrow e^{-i(1/2)\beta} W(e^{i\beta} \zeta), \quad \Xi(\zeta) \rightarrow e^{-(i/2)\beta} \Xi(e^{i\beta} \zeta).$$

(5.46)

These symmetries are therefore present in the dual theory (5.14). As a result, the corresponding Lagrangian has the form

$$\mathcal{L}_D(\Xi, \bar{\Xi}; \zeta) = \mathcal{L}_D(\Xi \bar{\Xi}).$$

(5.47)

\(^{13}\text{Such transformations naturally originate in } \mathcal{N} = 2 \text{ superspace parametrized by } z^A = (x^a, \theta^i, \bar{\theta}^\dot{i}), \text{ with } i = 1, 2, \text{ as part of the } R\text{-symmetry group } SU(2) \times U(1), \text{ see } [10] \text{ for more details. A shadow chiral rotation is a phase transformation of } \theta^2 \text{ only, with } \bar{\theta}^2 \text{ kept unchanged. The } \mathcal{N} = 1 \text{ superspace is identified with the surface } \theta^2 = 0.\)
The dual Lagrangian, $L_D$, and the original one, $L$, are related to each other as follows

$$L_D(\Xi \breve{\Xi}) = L(W \breve{W}) - 2L'(W \breve{W}) W \breve{W}, \quad (5.48)$$

where $W$ and its smile-conjugate $\breve{W}$ are to be expressed via $\Xi$ and $\breve{\Xi}$ using the equations

$$L'(W \breve{W}) \breve{W} + \frac{1}{\zeta} \Xi = 0, \quad L'(W \breve{W}) W - \zeta \breve{\Xi} = 0. \quad (5.49)$$

The dual Lagrangian $L_D$ can be given a geometric interpretation of the Kähler potential, $K_D(\Psi, \bar{\Psi}) := L_D(\Psi \bar{\Psi})$, of a two-dimensional space $\mathcal{M}$ with $U(1) \times U(1)$ isometry. By construction, the Kähler potential is given in canonical complex coordinates.

It turns out that both the original Kähler potential $K(\Phi, \bar{\Phi})$ and its dual $K_D(\Psi, \bar{\Psi})$ are encoded in the hyperkähler potential $(5.30)$. Before turning to a detailed justification of this claim, it is worth considering an example.

### 5.5 The Eguchi-Hanson metric and polar-polar duality

As an instructive example of the sigma-models studied in the previous subsection, consider the Kähler potential corresponding to $\mathbb{CP}^1$:

$$K(\Phi, \bar{\Phi}) = \ln (1 + \Phi \bar{\Phi}). \quad (5.50)$$

Associated with this potential is the polar multiplet Lagrangian:

$$L(\Upsilon, \bar{\Upsilon}) = \ln (1 + \Upsilon \bar{\Upsilon}). \quad (5.51)$$

A short calculation gives for the dual Lagrangian:

$$L_D(\Xi, \breve{\Xi}) = \sqrt{1 + 4\Upsilon \bar{\Upsilon}} - 1 - \ln \frac{1 + \sqrt{1 + 4\Upsilon \bar{\Upsilon}}}{2}. \quad (5.52)$$

The dual Lagrangian is associated with the Kähler potential

$$K_D(\Psi, \bar{\Psi}) = \sqrt{1 + 4\Psi \bar{\Psi}} - 1 - \ln \frac{1 + \sqrt{1 + 4\Psi \bar{\Psi}}}{2}. \quad (5.53)$$

which corresponds to a new Kähler manifold that differs from the two-sphere. This follows from the fact that the Kähler metric for $\mathbb{CP}^1$,

$$g_{\Phi \bar{\Phi}} = (1 + \Phi \bar{\Phi})^{-2}, \quad (5.54)$$
is characterized by a constant curvature, while the Kähler metric generated by the dual
Kähler potential \( (5.53) \),
\[
g_{\bar{\Psi}\Psi} = (1 + \bar{\Psi}\Psi)^{-1/2},
\]
is no longer a metric of constant curvature. In addition, the dual Kähler manifold is
non-compact, unlike \( \mathbb{C}P^1 \).

To get a better understanding of the relationship between the Kähler potential \( (5.50) \)
and its dual \( (5.53) \), consider the hyperkähler potential generated by the \( \mathcal{N} = 2 \) super-symmetric Lagrangian \( (5.51) \).\(^{14}\) It is
\[
\mathbb{K}(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) = K(\Phi, \bar{\Phi}) + \sqrt{1 + 4|\Psi|^2} - 1 - \ln \frac{1 + \sqrt{1 + 4|\Psi|^2}}{2},
\]
\[
|\Psi|^2 := \frac{\Psi\bar{\Psi}}{g_{\Phi\bar{\Phi}}} = (1 + \Phi\bar{\Phi})^2\Psi\bar{\Psi}.
\]
The Kähler potential \( (5.50) \) and its dual \( (5.53) \) can be seen to correspond to two different
limits one can define in terms of the hyperkähler potential \( \mathbb{K}(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) \):
\[
\mathbb{K}(\Phi, \bar{\Phi}, 0, 0) = K(\Phi, \bar{\Phi}),
\]
\[
\mathbb{K}(0, 0, \Psi, \bar{\Psi}) = K_D(\Psi, \bar{\Psi}).
\]
Since these limits are defined in terms of local complex coordinates on \( T^*\mathbb{C}P^1 \), one might
think they are non-geometric. This is not quite true, for the coordinate system used in
\( (5.56) \) is canonical, and canonical coordinates for Kähler manifolds \(^{28}\) are intrinsic (they
are defined modulo linear holomorphic reparametrizations), see subsection 6.1.

6 Polar-polar duality with \( n \) hypermultiplets

This section generalizes the analysis given in subsections 5.4 and 5.5 to the case of \( n \)
interacting hypermultiplets.

6.1 Canonical coordinates for Kähler manifolds

We start by recalling the concept of canonical coordinates for Kähler manifolds \(^{28}\).\(^{14}\)
Given a Kähler manifold \( \mathcal{M} \), for any point \( p_0 \in \mathcal{M} \) there exists a neighborhood of \( p_0 \) such
\(^{14}\)Within the projective superspace approach, the hyperkähler potential \( (5.56) \) was computed in \(^{8, 34}\).
that holomorphic reparametrizations and Kähler transformations can be used to choose coordinates with origin at $p_0$ in which the Kähler potential is

$$K(\Phi, \bar{\Phi}) = g_{IJ} \Phi^I \bar{\Phi}^J + \sum_{m,n \geq 2} K^{(m,n)}(\Phi, \bar{\Phi}),$$

$$K^{(m,n)}(\Phi, \bar{\Phi}) := \frac{1}{m! n!} K_{I_1 \ldots I_m, J_1 \ldots J_n} | \Phi^{I_1} \ldots \Phi^{I_m} \bar{\Phi}^{J_1} \ldots \bar{\Phi}^{J_n}. \quad (6.1)$$

Such a coordinate system in the Kähler manifold is called canonical. It was first introduced by Bochner [28] and extensively used by Calabi in the 1950s [30]. There still remains freedom to perform linear holomorphic reparametrizations which can be used to set the metric at the origin, $p_0 \in \mathcal{M}$, to be $g_{IJ} = \delta_{IJ}$. The resulting frame is defined modulo linear holomorphic $U(n)$ transformations.

It turns out that the coefficients $K_{I_1 \ldots I_m, J_1 \ldots J_n}$ in (6.1) are tensor functions of the Kähler metric $g_{IJ}$, the Riemann curvature $R_{IJKL}$ and its covariant derivatives, all evaluated at the origin. In particular, one finds

$$K^{(2,2)} = \frac{1}{4} R_{I_1, I_2 J_1, J_2} | \Phi^{I_1} \Phi^{I_2} \bar{\Phi}^{J_1} \bar{\Phi}^{J_2},$$

$$K^{(3,2)} = \frac{1}{12} \nabla_{I_3} R_{I_1, I_2 J_1, J_2} | \Phi^{I_1} \ldots \Phi^{I_3} \bar{\Phi}^{J_1} \bar{\Phi}^{J_2},$$

$$K^{(4,2)} = \frac{1}{48} \nabla_{I_3} \nabla_{I_4} R_{I_1, I_2, I_3 J_1, J_2} | \Phi^{I_1} \ldots \Phi^{I_4} \bar{\Phi}^{J_1} \bar{\Phi}^{J_2},$$

$$K^{(3,3)} = \frac{1}{12} \left\{ \frac{1}{6} \nabla_{I_3, I_4} R_{I_1, I_2 J_1 J_2} | + R_{I_1, I_3, I_4 J_1, J_2} | R_{I_3, J_1 J_2} \bar{R}_{I_1 J_1 J_2} \bar{R}_{I_3 J_1 J_2} \right\} \times \Phi^{I_1} \ldots \Phi^{I_3} \bar{\Phi}^{J_1} \ldots \bar{\Phi}^{J_3}. \quad (6.2d)$$

The functions $K^{(4,3)}$ and $K^{(4,4)}$ are given in [10].

### 6.2 $\mathcal{N} = 2$ σ-models with $U(1) \times U(1)$ symmetry

Here we consider a family of Kähler manifolds $\mathcal{M}$ with holomorphic $U(1)$ isometry. In canonical coordinates, the corresponding Kähler potential is

$$K(\Phi^I, \bar{\Phi}^\bar{J}) = g_{IJ} | \Phi^I \bar{\Phi}^\bar{J} + \sum_{n \geq 2} K^{(n,n)}(\Phi, \bar{\Phi}),$$

$$K^{(n,n)}(\Phi, \bar{\Phi}) := \frac{1}{(n!)^2} K_{I_1 \ldots I_n, J_1 \ldots J_n} | \Phi^{I_1} \ldots \Phi^{I_n} \bar{\Phi}^{J_1} \ldots \bar{\Phi}^{J_n}. \quad (6.3)$$

\footnote{In the modern literature on supersymmetric σ-models, some authors, unaware of the work of [28], refer to the canonical coordinates as a normal gauge [31] or Kähler normal coordinates [32].}
The relevant isometry acts as a phase transformation
\[ \Phi^I \rightarrow e^{i\alpha\Phi^I}, \quad \alpha \in \mathbb{R} \] (6.4)
which leaves the Kähler potential invariant. The condition (6.3) is equivalent to
\[ \Phi^I \frac{\partial}{\partial \Phi^I} K(\Phi, \bar{\Phi}) = \bar{\Phi}^J \frac{\partial}{\partial \bar{\Phi}^J} K(\Phi, \bar{\Phi}). \] (6.5)
All Hermitian symmetric spaces belong to this family of manifolds. Note that eq. (6.5) is weaker than the homogeneity condition (1.3) which corresponds to the superconformal action (4.19).

Associated with such a Kähler manifold \( \mathcal{M} \) is the \( \mathcal{N} = 2 \) supersymmetric \( \sigma \)-model
\[ S = \oint \frac{d\zeta}{2\pi i} \int d^4x d^4\theta K(\Upsilon^I, \bar{\Upsilon}^J). \] (6.6)
This model possesses \( U(1) \times U(1) \) symmetry. The relevant symmetry transformations are derived from (5.41) and (5.42) simply by replacing \( \Upsilon(\zeta) \rightarrow \Upsilon'(\zeta) \). The model (6.6) is non-supersymmetry except for the trivial case of a quadratic Kähler potential.

We apply the polar-polar duality transformation to all the arctic multiplets \( \Upsilon^I \) and their conjugates in (6.6). The dual Lagrangian \( K_D(\Xi_I, \bar{\Xi}_J) \), which is defined as in eqs. (4.21) and (4.22), has the same functional form as the Kähler potential in eq. (6.3). We conclude that polar-polar duality generates a transformations between Kähler spaces, \( \mathcal{M} \rightarrow \mathcal{M}_D \), described (in canonical coordinates) by Kähler potentials of the form (6.3).

There is a simple relationship between the Kähler potential \( K(\Phi^I, \bar{\Phi}^J) \) and its dual \( K_D(\Psi_I, \bar{\Psi}_J) \). Let us first discuss the structure of the hyperkähler target space for the \( \mathcal{N} = 2 \) supersymmetric \( \sigma \)-model (6.6). As argued in [8, 25], the \( \sigma \)-model target space is an open domain of the zero section of the cotangent bundle \( T^* \mathcal{M} \) parametrized by complex variables \( (\Phi^I, \Psi_I) \) and their conjugates, with \( \Psi_I \) a holomorphic one-form at the point \( (\Phi, \bar{\Phi}) \) of the base space \( \mathcal{M} \). The hyperkähler potential for \( T^* \mathcal{M} \) can be chosen as
\[ K(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) = K(\Phi, \bar{\Phi}) + \mathcal{H}(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}), \] (6.7)
where the function \( \mathcal{H}(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) \) can be represented by a Taylor series
\[ \mathcal{H}(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) = \sum_{n=1}^{\infty} \mathcal{H}^{I_1\ldots I_n\bar{J}_1\ldots \bar{J}_n}(\Phi, \bar{\Phi}) \Psi_{I_1} \cdots \Psi_{I_n} \bar{\Psi}_{\bar{J}_1} \cdots \bar{\Psi}_{\bar{J}_n}, \]
\[ \mathcal{H}^{IJ}(\Phi, \bar{\Phi}) = g^{IJ}(\Phi, \bar{\Phi}). \] (6.8)
Here the Taylor coefficients \( \mathcal{H}_{I_1 \ldots I_n \bar{J}_1 \ldots \bar{J}_n}(\Phi, \bar{\Phi}) \) are some tensor functions of the Kähler metric \( g_{IJ}(\Phi, \bar{\Phi}) = \partial_I \partial_J K(\Phi, \bar{\Phi}) \), the Riemann curvature \( R_{IJKL}(\Phi, \bar{\Phi}) \) and its covariant derivatives. One can see from (6.7) that the complex coordinate system \((\Phi^I, \bar{\Psi}_I)\) in \( T^* \mathcal{M} \) is canonical.

Next, let us turn to the dual \( \sigma \)-model. Its target space is an open domain of the zero section of the cotangent bundle \( T^* \mathcal{M}_D \) parametrized by complex variables \((\Psi_I, \Phi^I)\) and their conjugates. The hyperkähler potential for \( T^* \mathcal{M}_D \) is

\[
\mathbb{K}_D(\Psi, \Phi, \bar{\Phi}, \bar{\Psi}) = K_D(\Psi, \bar{\Psi}) + \mathcal{H}_D(\Psi, \Phi, \bar{\Phi}, \bar{\Psi}) ,
\]

where \( \mathcal{H}_D \) has a series representation which is similar to that given in (6.8) and is obtained from the latter by the replacement \( \Phi \leftrightarrow \Psi \) (including the replacement of all relevant geometric objects).

Finally, we can apply the chiral-linear duality of subsection 5.2 to show that

\[
\mathbb{K}_D(\Psi, \Phi, \bar{\Phi}) = \mathbb{K}(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) .
\]

We can immediately conclude that

\[
\mathbb{K}(\Phi, \bar{\Phi}, 0, 0) = K(\Phi, \bar{\Phi}) , \quad \mathbb{K}(0, 0, \Psi, \bar{\Psi}) = K_D(\Psi, \bar{\Psi}) .
\]

As a result, we see that the Kähler potential \( K(\Phi, \bar{\Phi}) \) and its dual \( K_D(\Psi, \bar{\Psi}) \) are encoded in the hyperkähler potential \( \mathbb{K}(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) \).

### 6.3 \( \mathcal{N} = 2 \) \( \sigma \)-models on cotangent bundles of Hermitian symmetric spaces

Hermitian symmetric spaces form a subclass in the family of Kähler manifolds introduced in the previous subsection. If \( \mathcal{M} \) is Hermitian symmetric, then

\[
\nabla_L R_{I_1 \bar{J}_1 I_2 \bar{J}_2} = \nabla_L R_{\bar{J}_1 I_1 \bar{J}_2 I_2} = 0 \implies K^{(m,n)} = 0 , \quad m \neq n .
\]

This follows from the fact that, for Hermitian symmetric spaces, there exists a closed-form expression for the Kähler potential in the canonical coordinates [36):

\[
K(\Phi, \bar{\Phi}) = -\frac{1}{2} \Phi^T g \frac{\ln (1 - R_{\Phi, \Phi})}{R_{\Phi, \Phi}} \Phi , \quad \Phi := \left( \begin{array}{c} \Phi^I \\ \bar{\Phi}^\bar{I} \end{array} \right) .
\]
Here we have introduced
\begin{equation}
g := \begin{pmatrix} 0 & g_{I\bar{J}} \\ g_{I\bar{J}} & 0 \end{pmatrix}, \quad R_{\Phi, \bar{\Phi}} := \begin{pmatrix} 0 & (R_{\Phi})_{I\bar{J}} \\ (R_{\Phi})_{I\bar{J}} & 0 \end{pmatrix},
\end{equation}
\begin{equation}
(R_{\Phi})_{I\bar{J}} := \frac{1}{2} R_{K\bar{L}I\bar{J}} \Phi^K \Phi^L, \quad (R_{\bar{\Phi}})_{I\bar{J}} := \overline{(R_{\Phi})_{I\bar{J}}}.
\end{equation}

The program of deriving the hyperkähler potential on $T^*M$ from the $\sigma$-model (6.6), for various Hermitian symmetric spaces, has been carried in a series of papers [8, 25, 33, 34, 35]. It has resulted in a universal expression for the hyperkähler potential derived in [36] using the results of [35]. It is given by eq. (6.7), where
\begin{equation}
\mathcal{H}(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) = \frac{1}{2} \Psi^T g^{-1} \mathcal{F} \left( -R_{\Psi, \bar{\Psi}} \right) \Psi, \quad \Psi := \begin{pmatrix} \Psi_I \\ \bar{\Psi}_I \end{pmatrix}.
\end{equation}

Here the function $\mathcal{F}(x)$ is defined as
\begin{equation}
\mathcal{F}(x) := \frac{1}{x} \left\{ \sqrt{1 + 4x} - 1 - \ln \frac{1 + \sqrt{1 + 4x}}{2} \right\}, \quad \mathcal{F}(0) = 1,
\end{equation}
and the operator $R_{\Psi, \bar{\Psi}}$ has the form:
\begin{equation}
R_{\Psi, \bar{\Psi}} := \begin{pmatrix} 0 & (R_{\Psi})_{I\bar{J}} \\ (R_{\Psi})_{I\bar{J}} & 0 \end{pmatrix},
\end{equation}
\begin{equation}
(R_{\Psi})_{I\bar{J}} = (R_{\Psi})_{IK} g^{K\bar{J}}, \quad (R_{\bar{\Psi}})_{KL} := \frac{1}{2} R_{K\bar{L}I\bar{J}} \Psi_I \Psi_J.
\end{equation}

In the canonical coordinates, the curvature $R_{K\bar{L}I\bar{J}}$ is a constant tensor, see, e.g., [36] for more details.

Using eq. (6.11), from (6.15) we can immediately read off the polar-polar dual of the $\sigma$-model (6.6) if $K(\Phi, \bar{\Phi})$ is the Kähler potential of a Hermitian symmetric space.

### 7 Self-dual hypermultiplet models

The concept of polar-polar duality allows us to introduce self-dual hypermultiplet models. For simplicity, here we consider the case of a single polar hypermultiplet.

The theory with action (5.1) is said to be self-dual if the dual Lagrangian coincides with the original one,
\begin{equation}
\mathcal{L}_D(\Upsilon, \bar{\Upsilon}; \zeta) = \mathcal{L}(\Upsilon, \bar{\Upsilon}; \zeta).
\end{equation}
The simplest example of self-dual systems was given in [8]. It is the free hypermultiplet model
\[ \mathcal{L}_{\text{free}}(\Upsilon, \bar{\Upsilon}) = \Upsilon \bar{\Upsilon} \quad (7.2) \]
Below we construct an infinite family of self-dual nonlinear hypermultiplet models.

### 7.1 The meaning of self-duality
If the off-shell \( \sigma \)-model is self-dual, eq. (7.1), then we also have
\[ L_{D}^{(\text{CL})}(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) = L_{D}^{(\text{CL})}(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) \quad (7.3) \]
This is equivalent to the condition
\[ L^{(\text{CL})}(\Psi, \bar{\Psi}; \Gamma, \bar{\Gamma}) = L^{(\text{CL})}(U, \bar{U}; V, \bar{V}) + \Gamma U + \bar{\Gamma} \bar{U} - \Psi V - \bar{\Psi} \bar{V} \quad (7.4) \]
where \( U \) and \( V \) are functions of \( \Psi, \Gamma \) and their conjugates which have to be determined by solving the equations
\[ \frac{\partial}{\partial U} L_{D}^{(\text{CL})}(U, \bar{U}; V, \bar{V}) + \Gamma = 0 \quad (7.5a) \]
\[ \frac{\partial}{\partial V} L_{D}^{(\text{CL})}(U, \bar{U}; V, \bar{V}) - \Psi = 0 \quad (7.5b) \]
In the case of a self-dual model, one can readily see that the Lagrangians \( L^{(\text{CC})}(\Phi, \bar{\Phi}; \Psi, \bar{\Psi}) \) and \( L^{(\text{LL})}(\Gamma, \bar{\Gamma}; \Sigma, \bar{\Sigma}) \) must be symmetric functions.
\[ L^{(\text{CC})}(\Phi, \bar{\Phi}; \Psi, \bar{\Psi}) = L^{(\text{CC})}(\Psi, \bar{\Psi}; \Phi, \bar{\Phi}) \quad (7.6) \]
\[ L^{(\text{LL})}(\Gamma, \bar{\Gamma}; \Sigma, \bar{\Sigma}) = L^{(\text{LL})}(\Sigma, \bar{\Sigma}; \Gamma, \bar{\Gamma}) \quad (7.7) \]
In accordance with the above consideration, the nonlinear \( \sigma \)-model
\[ S = \int d^4 x \, d^4 \theta \, L^{(\text{CC})}(\Phi, \bar{\Phi}; \Psi, \bar{\Psi}) \quad (7.8) \]
is \( \mathcal{N} = 2 \) supersymmetric, and therefore \( L^{(\text{CC})}(\Phi, \bar{\Phi}; \Psi, \bar{\Psi}) \) should be the hyperkähler potential of a hyperkähler manifold \( \mathcal{M} \) [29]. If the original off-shell action (5.1) is self-dual, then the hyperkähler potential \( L^{(\text{CC})}(\Phi, \bar{\Phi}; \Psi, \bar{\Psi}) \) must be symmetric, eq. (7.6). This means that the target space \( \mathcal{M} \) must possess a \( \mathbb{Z}_2 \) symmetry.


7.2 Self-duality equation

In this section we consider a simple special class of self-dual models. We assume that the dependence on the polar multiplet is through the combination \( x = \Upsilon \bar{\Upsilon} \) with no explicit \( \zeta \)-dependence so that the Lagrangian is given by an ordinary function \( \mathcal{L}(x) \).

Suppose the theory under consideration is self-dual, \( \mathcal{L}_D = \mathcal{L} \). Then the Lagrangian \( \mathcal{L}(x) \) can be seen to obey the algebraic equation

\[
\mathcal{L}'(x) \mathcal{L}'(y) = 1 , \quad (7.9a)
\]

where the variables \( x \) and \( y \) are related to each other as follows

\[
y = -x [\mathcal{L}'(x)]^2 . \quad (7.9b)
\]

It follows from eqs. (7.9a) and (7.9b) that

\[
x = -y [\mathcal{L}'(y)]^2 . \quad (7.9c)
\]

In particular, any function \( f(x) \) with the property that \( x = f(f(x)) \) gives a self-dual Lagrangian through

\[
\mathcal{L}(x) = \int^x dx' \sqrt{-f(x')} . \quad (7.10)
\]

Self-dual Lagrangians are not rare. Given any function \( h(x) \) we can construct a function satisfying the above properties as \( f(x) = h^{-1}(-h(x)) \) (compare with [40]). If the function \( h(x) \) is even or odd, the solution is trivial \( f(x) = \pm x \), so for nontrivial solutions we need functions \( h(-x) \neq \pm h(x) \).

A simple partial solution of the equations (7.9a) and (7.9b) is

\[
\mathcal{L}(x) = \frac{-2}{g^2} \left\{ 1 - \sqrt{1 + g^2 x} \right\} , \quad (7.11)
\]

with \( g \) a coupling constant.

To work out the geometry of this self-dual model one could use that models with this particular dependence on the polar multiplet can be dualized to an \( \mathcal{O}(2) \) multiplet \( \eta \) as

\[
\mathcal{L}_2(\eta) = -\frac{2}{g^2} + \eta + \sqrt{\eta^2 + \frac{4}{g^4}} - \eta \ln \left( \frac{g^2 \eta^2}{2} + \sqrt{\eta^2 + \frac{g^4 \eta^4}{4}} \right) , \quad (7.12)
\]

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although there may appear contour ambiguities [9]. After working out the component content of the dual model one would have to dualize the $\mathcal{N} = 1$ linear superfield of the $\mathcal{O}(2)$ multiplet to get the hyperkähler potential. We leave the explicit solution of this model for a future publication.

In theories with more polar multiplets there are other ways to construct self-dual models. For instance if we have two polar multiplets we may start with an action $\mathcal{L}(\Upsilon_1 \tilde{\Upsilon}_1 + \Upsilon_2 \tilde{\Upsilon}_2)$. Making a polar-polar duality on only one of the polar multiplets we create a model which will be self-dual with respect to a duality transformation of all polar multiplets.

8 Final comments

In this paper, we demonstrated that polar-polar duality generates a transformation between different Kähler cones. This is a new type of duality. It relates Kähler manifolds that are target space geometries for $\mathcal{N} = 1$ $\sigma$-models, although the duality is intrinsically $\mathcal{N} = 2$ supersymmetric. These dual Kähler manifolds are both embedded in the hyperkähler target space of the original $\mathcal{N} = 2$ supersymmetric $\sigma$-model.

For non-superconformal $\sigma$-models, we derived a simple relationship between the hyperkähler potential $\mathcal{K}$ and its dual $\mathcal{K}_D$, eq. (6.10). It can naturally be extended to the superconformal case.

In order to find a hyperkähler potential corresponding to a $\mathcal{N} = 2$ supersymmetric $\sigma$-model formulated in terms of polar multiplets, one has to eliminate the auxiliary $\mathcal{N} = 1$ superfields. This technical problem has been solved for the $\sigma$-models on cotangent bundles of Hermitian symmetric spaces. The full solution is given in [35, 36]. Polar-polar duality allows us to extend the class of $\sigma$-models for which this is possible. As an example, we looked at the Eguchi-Hanson geometry whose known solution allowed us to solve the auxiliary field problem for the dual model (5.52). The duality thus allows for a treatment of more complicated geometries.

We have discussed a family of self-dual models and identified some interesting features. The intrinsic meaning of self-duality as well as the relation between the geometry of the dual models remain to be understood, however. Here an example with the geometry worked out would be of great help.

For more than one polar multiplet, there are more possibilities to construct self-dual models. In particular, we can also consider self-dual superconformal models. Their properties remain to be investigated.
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Projective superspace and its various implications have been frequently discussed with M. Roček over the years. His inspiration and insights are gratefully acknowledged.

A Superconformal Killing vectors

In this appendix we recall salient properties of the $\mathcal{N} = 2$ superconformal Killing vectors, following [41, 42].

In $\mathcal{N} = 2$ superspace $\mathbb{R}^{4|8}$ parametrized by coordinates $z^A = (x^a, \theta^\alpha_i, \bar{\theta}^\dot{\alpha}_i)$, with $i = 1, 2$, a first-order differential operator

$$\xi = \bar{\xi} = \xi^A(z)D_A = \xi^a(z)\partial_a + \xi^a_\alpha(z)D^i_{\dot{\alpha}} + \bar{\xi}^i_\dot{\alpha}(z)\bar{D}^i_{\dot{\alpha}}$$

is called a $\mathcal{N} = 2$ superconformal Killing vector if it obeys the condition

$$[\xi, \bar{D}^i_{\dot{\alpha}}] \propto \bar{D}^j_{\dot{\beta}},$$

which implies

$$\bar{D}^i_{\dot{\alpha}}\xi^\beta_j = 0, \quad \bar{D}^i_{\dot{\alpha}}\xi^{\dot{\beta}}_j = 4i\varepsilon^{\dot{\alpha}\dot{\beta}}\xi^i.$$  \hspace{1cm} (A.3)

A short calculation gives

$$[\xi, D^i_{\dot{\alpha}}] = -(D^i_{\dot{\alpha}}\xi^\beta_j)D^j_{\dot{\beta}} = \omega^\beta_\alpha D^i_{\dot{\beta}} - \bar{\sigma} D^i_{\dot{\alpha}} - \Lambda^i_{\dot{\alpha}} D^j_{\dot{\beta}},$$

where the parameters of Lorentz ($\omega$) and scale-chiral ($\sigma$) transformations are

$$\omega^\beta_\alpha(z) = -\frac{1}{2}D^i_{\dot{\alpha}}\xi^\beta_i, \quad \sigma(z) = \frac{1}{4}\bar{D}^i_{\dot{\alpha}}\xi_i.$$  \hspace{1cm} (A.5)

These parameters can be seen to be chiral

$$\bar{D}^i_{\dot{\alpha}}\omega^\beta_\alpha = 0, \quad \bar{D}^i_{\dot{\alpha}}\sigma = 0.$$  \hspace{1cm} (A.6)
The parameters $\Lambda^i_j$ defined by
\[
\Lambda^i_j(z) = \frac{1}{2} \left( D^i_\alpha \xi^j_\alpha - \frac{1}{2} \delta^i_j D^k_\alpha \xi^k_\alpha \right) = -\frac{1}{2} \left( \bar{D}^i_\alpha \bar{\xi}^j_\alpha - \frac{1}{2} \delta^i_j \bar{D}^k_\alpha \bar{\xi}^k_\alpha \right),
\]
\[
\Lambda^i_j = \Lambda^j_i, \quad \bar{\Lambda}^{ij} = \Lambda_{ij}
\]
correspond to SU(2) transformations. One can readily check the identity
\[
D^k_\alpha \Lambda^i_j = -\frac{1}{2} \left( \delta^k_j D^i_\alpha - \frac{1}{2} \delta^i_j D^k_\alpha \right) \sigma,
\]
and therefore
\[
D^i_\alpha \Lambda^{jk} = \bar{D}^i_\alpha \Lambda^{jk} = 0.
\]
Comparing this with eq. (2.28), we see that $\Lambda^{(2)}(v) := \Lambda_{ij} v^i v^j$ is an $O(2)$ multiplet.

\section*{B Tensor multiplet formulation for $\mathcal{N} = 2$ $\sigma$-models with $U(1) \times U(1)$ symmetry}

In subsection 5.4 we discussed the polar-polar duality of $\mathcal{N} = 2$ $\sigma$-models (5.43). There is a different dual formulation for such theories which is given in terms of an $O(2)$ multiplet $\eta = \bar{\xi} + G - \varphi \zeta$ (also known as an $\mathcal{N} = 2$ tensor multiplet) and a $\mathcal{N} = 2$ Lagrangian $\mathcal{L}_2(\eta)$. Here we will elaborate on the structure of such $\sigma$-models.\footnote{Four-dimensional quaternion Kähler metrics with torus symmetry were studied in \cite{43}.} We will assume that the contour integral in the corresponding action has been done,
\[
S = \oint \frac{d\zeta}{2\pi i \zeta} \int d^4x d^4\theta \mathcal{L}_2(\eta) = \int d^4x d^4\theta H(\varphi \bar{\varphi}, G),
\]
where $H(\varphi \bar{\varphi}, G)$ is the resulting $\mathcal{N} = 1$ Lagrangian.

In \cite{1} it is shown that the Lagrangian $H(\varphi \bar{\varphi}, G)$ must satisfy the Laplace equation
\[
\partial_\varphi \partial_{\bar{\varphi}} H + \partial_\zeta^2 H = 0.
\]
This can be used to find the form of $H(\varphi \bar{\varphi}, G)$ from the knowledge of only part of it. For instance, using the ansatz
\[
H = \sum_{n=0}^{\infty} H_n(G)(\varphi \bar{\varphi})^n,
\]
where $H_0(G)$ is the $\varphi$ independent part of the Lagrangian, the Laplace equation gives us the recursion relation
\[
H_n''(G) = -(n + 1)^2 H_{n+1}(G),
\]
which can be solved given the initial data $H_0(G)$. The solution is

$$H_n = \frac{(-1)^n}{(n!)^2} \frac{d^{2n}}{dG^{2n}} H_0(G). \quad (B.4)$$

Then the full Lagrangian can be written compactly as

$$H = J_0 \left( \sqrt{4 \phi \bar{\phi}} \frac{d}{dG} \right) H_0(G), \quad (B.5)$$

where $J_0$ is a Bessel function. The program can be tested on the $\mathcal{N} = 2$ improved tensor multiplet model where $H_0 = -G \ln G$. Applying the differential operator defined above one indeed gets

$$\sqrt{G^2 + 4 \phi \bar{\phi}} - G \ln G + \frac{\sqrt{G^2 + 4 \phi \bar{\phi}}}{2}, \quad (B.6)$$

which agrees (up to terms annihilated by the superspace measure) with the Lagrangian given in [1].

One may also use this scheme if one knows instead the dependence on $\phi$ but not on $G$. Making the ansatz

$$H = \sum_n \tilde{H}_n(4 \phi \bar{\phi}) G^n, \quad (B.7)$$

leads via the Laplace equation to the recursion relation

$$\tilde{H}_{n+2} = -4 \frac{4 \phi \bar{\phi} \tilde{H}_n'' + \tilde{H}_n'}{(n+1)(n+2)}. \quad (B.8)$$

We see that the recursion relation does not mix $H_n$ with odd and even $n$. It means that to find the most general solution it is not enough to know $H_0$ but we also need to know $H_1$. Thus the general solution is

$$\tilde{H}_{2n}(x) = \frac{(-4)^n}{(2n)!} \left( \frac{d}{dx} x \frac{d}{dx} \right)^n \tilde{H}_0(x), \quad (B.9a)$$

$$\tilde{H}_{2n+1} = \frac{(-4)^n}{(2n)!} \left( \frac{d}{dx} x \frac{d}{dx} \right)^n \tilde{H}_1(x), \quad (B.9b)$$

where $x = 4 \phi \bar{\phi}$. Then the full Lagrangian can be written as follows:

$$H = \cos \left( \sqrt{4G \frac{d}{dx} x \frac{d}{dx}} \right) \left( \tilde{H}_0(x) + G \tilde{H}_1(x) \right). \quad (B.10)$$

Applying this to the improved tensor multiplet with $\tilde{H}_0 = \sqrt{x}$ and $\tilde{H}_1 = \frac{1}{2} \ln \frac{x}{4}$ we find the correct solution. It is interesting that since $G \tilde{H}_1$ is in fact a solution to the Laplace equation in itself, the higher odd powers are missing.
References

[1] U. Lindström and M. Roček, “Scalar tensor duality and N=1, N=2 nonlinear sigma models,” Nucl. Phys. B 222, 285 (1983).

[2] A. Karlhede, U. Lindström and M. Roček, “Self-interacting tensor multiplets in N = 2 superspace,” Phys. Lett. B 147, 297 (1984).

[3] U. Lindström and M. Roček, “New hyperkähler metrics and new supermultiplets,” Commun. Math. Phys. 115, 21 (1988).

[4] A. A. Rosly, “Super Yang-Mills constraints as integrability conditions,” in Proceedings of the International Seminar on Group Theoretical Methods in Physics,” (Zvenigorod, USSR, 1982), M. A. Markov (Ed.), Nauka, Moscow, 1983, Vol. 1, p. 263 (in Russian); A. A. Rosly and A. S. Schwarz, “Supersymmetry in a space with auxiliary dimensions,” Commun. Math. Phys. 105, 645 (1986).

[5] A. Galperin, E. Ivanov, S. Kalitsyn, V. Ogievetsky and E. Sokatchev, “Unconstrained N = 2 matter, Yang-Mills and supergravity theories in harmonic superspace,” Class. Quant. Grav. 1, 469 (1984).

[6] A. S. Galperin, E. A. Ivanov and V. I. Ogievetsky, “Duality transformations and most general matter self-couplings in N = 2 supersymmetry,” Nucl. Phys. B 282, 74 (1987).

[7] F. Gonzalez-Rey, M. Roček, S. Wiles, U. Lindström and R. von Unge, “Feynman rules in N = 2 projective superspace. I: Massless hypermultiplets,” Nucl. Phys. B 516, 426 (1998) [arXiv:hep-th/9710250].

[8] S. J. Gates Jr. and S. M. Kuzenko, “The CNM-hypemultiplet nexus,” Nucl. Phys. B 543, 122 (1999) [arXiv:hep-th/9810137].

[9] U. Lindström and M. Roček, “Properties of hyperkähler manifolds and their twistor spaces,” Commun. Math. Phys. 293, 257 (2010) [arXiv:0807.1366 [hep-th]].

[10] S. M. Kuzenko, “On superconformal projective hypermultiplets,” JHEP 0712, 010 (2007) [arXiv:0710.1479[hep-th]].

[11] S. M. Kuzenko, U. Lindström, M. Roček and G. Tartaglino-Mazzucchelli, “4D N=2 supergravity and projective superspace,” JHEP 0809, 051 (2008) [arXiv:0805.4683]; “On conformal supergravity and projective superspace,” JHEP 0908, 023 (2009) [arXiv:0905.0063 [hep-th]].
[12] B. de Wit, B. Kleijn and S. Vandoren, “Rigid N = 2 superconformal hypermultiplets,” in Supersymmetries and quantum symmetries, J. Wess and E. A. Ivanov (Eds.), Springer-Verlag, 1999, p. 37 (Lectures Notes in Physics, Vol. 524) arXiv:hep-th/9808160.

[13] B. de Wit, B. Kleijn and S. Vandoren, “Superconformal hypermultiplets,” Nucl. Phys. B 568, 475 (2000) [arXiv:hep-th/9909228].

[14] B. de Wit, M. Roček and S. Vandoren, “Hypermultiplets, hyperkähler cones and quaternion-Kähler geometry,” JHEP 0102, 039 (2001) [arXiv:hep-th/0101161].

[15] E. Sezgin and Y. Tanii, “Superconformal sigma models in higher than two dimensions,” Nucl. Phys. B 443, 70 (1995) [arXiv:hep-th/9412163].

[16] J. Bagger and E. Witten, “Matter couplings in N = 2 supergravity,” Nucl. Phys. B 222, 1 (1983).

[17] A. Swann, “HyperKähler and quaternion Kähler geometry,” Math. Ann. 289, 421 (1991).

[18] K. Galicki, “Geometry of the scalar couplings in N = 2 supergravity models,” Class. Quant. Grav. 9, 27 (1992).

[19] S. M. Kuzenko, U. Lindström and R. von Unge, “New extended superconformal sigma models and quaternion Kähler manifolds,” JHEP 0909, 119 (2009) [arXiv:0906.4393 [hep-th]].

[20] S. M. Kuzenko, “N = 2 supersymmetric sigma-models and duality,” JHEP 1001, 115 (2010) [arXiv:0910.5771 [hep-th]].

[21] G. W. Gibbons and P. Rychenkova, “Cones, tri-Sasakian structures and superconformal invariance,” Phys. Lett. B 443, 138 (1998) [arXiv:hep-th/9809158].

[22] E. Bergshoeff, S. Cecotti, H. Samtleben and E. Sezgin, “Superconformal sigma models in three dimensions,” Nucl. Phys. B 838, 266 (2010) [arXiv:1002.4411 [hep-th]].

[23] N. J. Hitchin, A. Karlhede, U. Lindström and M. Roček, “Hyperkähler metrics and supersymmetry,” Commun. Math. Phys. 108, 535 (1987).

[24] S. V. Ketov, B. B. Lokhvitsky and I. V. Tyutin, “Hyperkähler sigma models in extended superspace,” Theor. Math. Phys. 71, 496 (1987) [Teor. Mat. Fiz. 71, 226 (1987)].

[25] S. J. Gates Jr. and S. M. Kuzenko, “4D N = 2 supersymmetric off-shell sigma models on the cotangent bundles of Kähler manifolds,” Fortsch. Phys. 48, 115 (2000) [arXiv:hep-th/9903013].
[26] D. Kaledin, “Hyperkähler structures on total spaces of holomorphic cotangent bundles,” in D. Kaledin and M. Verbitsky, Hyperkähler Manifolds, International Press, Cambridge MA, 1999 [alg-geom/9710026]; “A canonical hyperkähler metric on the total space of a cotangent bundle,” in Quaternionic Structures in Mathematics and Physics, S. Marchiafava, P. Piccinni and M. Pontecorvo (Eds.), World Scientific, 2001 [alg-geom/0011256].

[27] B. Feix, “Hyperkähler metrics on cotangent bundles,” Cambridge PhD thesis, 1999; “Hyperkähler metrics on cotangent bundles,” J. Reine Angew. Math. 532, 33 (2001).

[28] S. Bochner, “Curvature in Hermitian metric,” Bull. Amer. Math. Soc. 53, 179 (1947).

[29] C. M. Hull, A. Karlhede, U. Lindström and M. Roček, “Nonlinear sigma models and their gauging in and out of superspace,” Nucl. Phys. B 266, 1 (1986).

[30] E. Calabi, “Isometric imbedding of complex manifolds,” Ann. of Math., 58, 1 (1953); “On compact, locally symmetric Kähler manifolds,” Ann. of Math., 71, 472 (1960).

[31] S. J. Gates, Jr., M. T. Grisaru, M. Roček and W. Siegel, “Superspace, or one thousand and one lessons in supersymmetry,” Front. Phys. 58, 1 (1983) [arXiv:hep-th/0108200].

[32] K. Higashijima, E. Itou and M. Nitta, “Normal coordinates in Kähler manifolds and the background field method,” Prog. Theor. Phys. 108, 185 (2002) [arXiv:hep-th/0203081].

[33] M. Arai and M. Nitta, “Hyper-Kähler sigma models on (co)tangent bundles with SO(n) isometry,” Nucl. Phys. B 745, 208 (2006) [arXiv:hep-th/0602277].

[34] M. Arai, S. M. Kuzenko and U. Lindström, “Hyperkähler sigma models on cotangent bundles of Hermitian symmetric spaces using projective superspace,” JHEP 0702, 100 (2007) [arXiv:hep-th/0612174].

[35] M. Arai, S. M. Kuzenko and U. Lindström, “Polar supermultiplets, Hermitian symmetric spaces and hyperkähler metrics,” JHEP 0712, 008 (2007) [arXiv:0709.2633 [hep-th]].

[36] S. M. Kuzenko and J. Novak, “Chiral formulation for hyperkähler sigma-models on cotangent bundles of symmetric spaces,” JHEP 0812, 072 (2008) [arXiv:0811.0218 [hep-th]].

[37] S. M. Kuzenko, “Lectures on nonlinear sigma-models in projective superspace,” J. Phys. A 43, 443001 (2010) [arXiv:1004.0880 [hep-th]].

[38] S. E. Hjelmeland and U. Lindström, “Duality for the non-specialist,” arXiv:hep-th/9705122.

[39] A. L. Besse, Einstein Manifolds, Springer, Berlin, 1987.

[40] F. Gonzalez-Rey, B. Kulik, I. Y. Park and M. Roček, “Self-dual effective action of N = 4 super-Yang-Mills,” Nucl. Phys. B 544, 218 (1999) [arXiv:hep-th/9810152].
[41] J. H. Park, “Superconformal symmetry and correlation functions,” Nucl. Phys. B 559, 455 (1999) [arXiv:hep-th/9903230].

[42] S. M. Kuzenko and S. Theisen, “Correlation functions of conserved currents in N = 2 superconformal theory,” Class. Quant. Grav. 17, 665 (2000) [arXiv:hep-th/9907107].

[43] D. M. J. Calderbank and H. Pedersen, “Selfdual Einstein metrics with torus symmetry,” J. Differential Geometry 60, 485 (2002).