Untangling tradeoffs between recurrence and self-attention in neural networks

Giancarlo Kerg\textsuperscript{1,2, *}, Bhargav Kanuparthi\textsuperscript{1,2, *}, Anirudh Goyal\textsuperscript{1,2}, Kyle Goyette\textsuperscript{1,2,3}

Yoshua Bengio\textsuperscript{1,2,4} Guillaume Lajoie\textsuperscript{1,2,5}

Abstract

Attention and self-attention mechanisms, inspired by cognitive processes, are now central to state-of-the-art deep learning on sequential tasks. However, most recent progress hinges on heuristic approaches with limited understanding of attention’s role in model optimization and computation, and rely on considerable memory and computational resources that scale poorly. In this work, we present a formal analysis of how self-attention affects gradient propagation in recurrent networks, and prove that it mitigates the problem of vanishing gradients when trying to capture long-term dependencies. Building on these results, we propose a relevancy screening mechanism, inspired by the cognitive process of memory consolidation, that allows for a scalable use of sparse self-attention with recurrence. While providing guarantees to avoid vanishing gradients, we use simple numerical experiments to demonstrate the tradeoffs in performance and computational resources by efficiently balancing attention and recurrence. Based on our results, we propose a concrete direction of research to improve scalability of attentive networks.

1 Introduction

We live in a world where most of the information takes a sequential form, largely because it is delivered over time. Performing computations on streams of sequential inputs requires extracting relevant temporal dependencies and learning to recognize patterns across several timescales. Humans can effortlessly make associations relating events stored in memory which are far from each other in time and thus, capture long-term dependencies.

Historically, recurrent neural networks (RNNs) have been the deep network architecture of choice for this type of task since, just like neural circuits in the brain, they enable dynamics that can be shaped to interact with input streams. However, RNNs (including gated RNNs [32, 10]) still struggle with large timescales as their iterative nature leads to unstable information propagation [5, 27, 32, 16]. This is because most standard RNNs rely on their current state $h_t$, a vector of fixed dimension, to represent a summary of relevant past information. Indeed, Bengio et al. [5] showed that without making additional assumptions, storing information in a fixed-size state vector in a stable way necessarily leads to vanishing gradients when back-propagating through time (see also [16]).

\textsuperscript{*}Indicates first authors. Ordering determined by coin flip.
1: Mila - Quebec AI Institute, Canada
2: Université de Montréal, Département d’Informatique et Recherche Opérationnelle, Montreal, Canada
3: Université de Montréal, CIRRELT, Montreal, Canada
4: CIFAR senior fellow
5: Université de Montréal, Département de Mathématiques et Statistiques, Montreal, Canada

Correspondence to: <giancarlo.kerg@gmail.com>

Preprint. Under review.
Several attempts have been made to augment RNN dynamics with external memory to mitigate these issues [34, 12, 31, 13], but it is only recently that access to externally stored information has become effective with the introduction of attention, and more particularly soft attention mechanisms [4]. Attention provides a way by which a system can dynamically access past states and inputs across several timescales, bypassing the need of sequential propagation and ignoring irrelevant information (or distractor information). There is substantial empirical evidence that attention, especially self-attention (Vaswani et al. [35], Ke et al. [19]), is very helpful to improve learning and computations over long-term dependencies.

However, this requires to hold inputs and/or past states in memory in order to be considered for attentive computations, a process that typically scales quadratically with sequence length. As for recurrent networks, a promising solution to this has been the use of sparse attention (see e.g. SAB [19]), leading to models with varying degrees of reliance on both recurrence and self-attention. However most of these models only delay computational issues by a constant factor and are thus still scaling quadratically with sequence length.

In this paper, we are pushing the idea of sparse attention in RNNs further, by asking: how can we sparsify attention in RNNs, in order to maximally reduce computational complexity and memory usage, while simultaneously maintaining good gradient propagation over large time scales?

In Section 2, we give a brief outline of related cognitive processes and neural network mechanisms. In Section 3, we derive asymptotic guarantees for gradient propagation in self-attentive recurrent networks. Building on these results in Section 4, we showcase a simple relevancy screening mechanism that aims to efficiently consolidate relevant memory, reducing the size of the computational graph from quadratic to linear in sequence length. Finally, in Section 5, we compare various recurrent and attention models with our proposed relevancy screening mechanism on a series of simple numerical experiments, while, in Section 6, we analyze their gradient propagation properties together with their GPU usage.

2 Background

To perform complex tasks, our brains rely on mechanisms to encode and retrieve information to and from memory [37, 30]. In contrast, standard RNNs follow rigid sequential dynamics as they are parametric i.e with a fixed-size state vector. Self-attention methods can overcome this limitation by giving access to previous past states for computing the next state. For the sake of the discussion, we call such RNNs, which are augmented by the memory of the past states as semi-parametric RNNs.

The use of soft-attention [4] in such models has improved performance on many tasks such as reading comprehension, abstractive summarization, textual entailment and learning task-independent sentence representations [26, 23, 28, 36] as well as in the self-supervised training of extremely large language models [11, 29] due to their ability to handle long-term dependencies.

Intriguingly, the most notable advances in the use of attention is in purely attention-based systems such as the Transformer [35], which completely foregoes recurrence and inspired some of the work listed above. While the performance of these systems is impressive, their memory and computation requirements grows quadratically with the total sequence length. To address this issue, many variants that aim to “sparsify” the attention matrix have been proposed. Notably, Ke et al. [19] developed the Sparse Attentive Backtracking model (SAB), a self-attentive Long Short-Term Memory network (LSTM) [32] that leverages sparsity by selecting only the top-\(k\) states in memory based on an attention score, propagating gradients only to those chosen hidden states. Recently, Zhao et al. [38] propose to use a similar top-\(k\) attention, and Child et al. [9] introduce sparse masks which attends to roughly \(\sqrt{n}\) locations in memory, implementing explicit selection methods for Transformers. Reformer models [20] replace the dot-product attention by locality-sensitive hashing, changing its complexity from \(O(T^2)\) to \(O(T)\), where \(T\) is the sequence length.

Still, most of these approaches naively sub-sample input streams for memory storage. Our brains on the other hand, seem to select relevant information from the recent past to commit to long term memory based on their relevancy, a process often referred to as memory consolidation [1]. Attempts at mimicking this sparse temporal selectivity process has shown great promise in a variety of contexts [12, 25, 14], and our work aims to formalize this idea for self-attentive recurrent networks.
3 Theoretical analysis of gradient propagation

In this section, we analyze the influence of self-attention onto gradient propagation in recurrent networks with self-attention. In order to do so let us first recall the equations of a recurrent neural network with self-attention. We note that even though we are using "vanilla RNNs" in the formulations of our results, any recurrent network can take its place (see Section 5 where we use LSTMs in the experiments). Let \( x_t \in \mathbb{R}^m \) be the input and \( h_t \in \mathbb{R}^n \) be the hidden state at time step \( t \), satisfying the update equation for all \( t \geq 1 \),

\[
\begin{align*}
  h_{t+1} &= \phi(Vs_t + Ux_{t+1} + b) \\
  s_t &= f(h_t, c_t)
\end{align*}
\]

where \( \phi \) is a non-linearity, \( f : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n, V \in \mathbb{R}^{n \times n}, U \in \mathbb{R}^{n \times n}, b \in \mathbb{R}^n \) and \( c_t = \alpha_{t,1}h_1 + \alpha_{t,2}h_2 + \ldots + \alpha_{t,n}h_n \) with \( \alpha_{t,i} := \frac{\exp(e_{t,i})}{\sum_{j=1}^{n} \exp(e_{t,j})} \) and \( e_{t,i} := a(s_{t-1}, h_i) \), where \( a : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) is the attention alignment function. Throughout, we assume training is done via gradient descent of a cost function \( L \) using back-propagation.

Oftentimes, one uses \( s_t = f(h_t, c_t) = h_t + c_t \) (but concatenation would be more general), and for all \( t > 1 \) and \( 1 \leq j \leq t \), \( a(s_{t-1}, h_j) = v_j^T \cdot \tanh(W_{v_j} \cdot s_{t-1} + U_h \cdot h_j) \), where \( v_j \in \mathbb{R}^n \), and \( W_{v_j}, U_h \in \mathbb{R}^{n \times n} \). The latter choice for alignment function is sometimes referred to as "additive self-attention" and was used in the original paper [4]. Let us emphasize that the results presented in this section hold independently of the choice of the alignment function as we will discuss later in this section.

3.1 Preliminaries

Our goal in this section is to establish formal propagation rules for a system where multiple paths of signal propagation are possible. We would like to understand the relationship between skip connections (those coming from self-attention) and recurrent connections, as well as how the interplay between the two leads to good gradient propagation. In order to achieve this, we seek to analyze the asymptotic behaviour of \( \|\nabla_h L\| = \| \left( \frac{\partial E}{\partial h_t} \right)^T \nabla_s L \| \), as \( T \rightarrow \infty \). We accomplish this by decomposing \( \nabla_h L \) with respect to all possible gradient backpropagation paths, or in other words, by decomposing \( \frac{\partial E}{\partial h_t} \) into sums of products of Jacobian matrices corresponding to those gradient paths, using Proposition 1.

**Proposition 1.** For all \( t \geq 1, k \geq j \geq 0, k' \geq 0 \), let \( E_k^{(t)} \), \( F_{k+1,j}^{(t)} \), and \( J_{t+j}^{(t)} \) be the Jacobian matrices \( \frac{\partial E_k}{\partial h_t} \), \( \frac{\partial F_{k+1,j}}{\partial h_t} \), and \( \frac{\partial J_{t+j}}{\partial h_t} \), with \( h_t \) the hidden state at time step \( t \). Then, we have

\[
\frac{\partial E_k}{\partial h_t} = \sum_{s=k}^{k'} \xi_{s,k}^{(t)}(h_t)
\]

where for all \( s \geq 1 \), \( \xi_{s,k}^{(t)}(h_t) = \sum_{0 \leq i_1 < \ldots < i_s < k} F_{i_1}^{(t)} \cdot F_{i_2,i_1}^{(t)} \cdot \ldots \cdot F_{i_s,i_{s-1}}^{(t)} \cdot E_{i_s}^{(t)} \) and where \( \xi_{0,k}^{(t)}(h_t) = E_k^{(t)} \). (Proof in Appendix A.2 Proposition 1)

Here, each term \( F_{i_1}^{(t)} \cdot F_{i_2,i_1}^{(t)} \cdot \ldots \cdot F_{i_s,i_{s-1}}^{(t)} \cdot E_{i_s}^{(t)} \) corresponds to exactly one gradient path involving exactly \( s + 1 \) skip connections going from \( t \) to \( t + k \), via the \( s \) hidden states \( h_{t+i_1}, \ldots, h_{t+i_s+1} \). In particular, \( \xi_{s,k}^{(t)}(h_t) \) is the sum of all terms containing exactly \( s \) Jacobian matrices \( J \), and thus the larger \( s \) is, the more \( \xi_{s,k}^{(t)}(h_t) \) is prone to vanishing.

**Intuition:** In order to find paths that are not vanishing as \( T \rightarrow \infty \), we want to find gradient paths with: (i) a bounded path length \( s \) so that the number of Jacobian matrices involved in the product is limited. (ii) attention scores that are sufficiently bounded away from 0, so that the resulting product of attention scores is sufficiently bounded away from 0 as well. In order to see how exactly the attention weights come into play via matrices \( E \) and \( F \), we refer to Proposition 2 from Appendix A.2.

**Definitions:** Let us fix an integer \( t \geq 1 \), an integer \( s \in \{1, 2, \ldots, T-t\} \), and an ordered set of indices \( i_1, i_2, \ldots, i_s \in \{0, 1, \ldots, T-t-1\} \), verifying \( i_1 \leq i_2 \leq \ldots \leq i_s \).
• For sequences \( \{g(T)\}_{T \geq 1} \) and \( \{f(T)\}_{T \geq 1} \), we say that \( f(T) = \Omega(g(T)) \) if there exists positive constants \( c \) and \( T_0 \) such that \( f(T) \geq c \cdot g(T) \) for all \( T \geq T_0 \).

• At time \( t \), we call a past hidden state \( h_t \) a **relevant event** if the attention weight \( \alpha_{i,t} \) is sufficiently bounded away from zero.

• We call the \( s \)-tuple \( (i_1, i_2, \ldots, i_s) \) a **dependency chain** \( \gamma \) of depth \( s \), as it induces a gradient backpropagation path going via the \( s \) hidden states \( h_{t+i_1+1}, \ldots, h_{t+i_s+1} \).

• We call **dependency depth** the smallest depth among all dependency chains where the product of the corresponding attention scores is \( \Omega(1) \) as \( T \to \infty \).

The central message is that if the dependency depth is bounded from above and sufficiently small, then we **mitigate gradient vanishing**. As we see below, task structure introduces different ways in which this may happen. We now present a formal treatment for specific cases, and lay the groundwork to take advantage of this structure during learning.

### 3.2 Uniform relevance case

Suppose each state has equal relevance in some task. What can be said about gradient propagation? This translates to having each attention weight \( \alpha_{i,t} = 1/t \) for all \( t \geq i \geq 1 \). We then have dependency chains of depth 1 but with vanishing rate \( \Omega(1/T) \), as formalized in the following theorem (cf. A.3).

**Theorem 1.** Let \( h_t \) be the hidden state at time \( t \) of a vanilla RNN with uniform attention, under mild assumptions on the connectivity matrix \( V \), and trained with respect to a loss \( L \), then if \( T \) is the total sequence length, we have

\[
\| \nabla h_t, L \| = \Omega(1/T)
\]

as \( T \to \infty \). (proof in Appendix A.3, Theorem 7)

This corresponds to the case where all past events contribute equally error signals. We also note that this result is independent of the choice of the alignment function \( a \) (cf. Remark 8 in the Appendix A.3).

**Intuition:** As a "worst case scenario" Theorem 1 reveals the true trade-off of early self-attentive recurrent networks [4]. On one hand, the lower bound obtained on gradient norm is substantially better than in a vanilla RNN without attention, where vanishing happens at an exponential rate, as opposed to a polynomial one here. On the other, this situation fulfills the quadratic scaling bounds for computations, and requires that all events be held in memory. Many inputs and tasks do not call for uniform attention and naturally lend themselves to sparse dependency paths for computation. The next case treats this situation.

### 3.3 Sparse relevance case with bounded dependency depth

Now let us look at a more realistic case where only a sparse subset of past states are relevant for the task at hand, and the gradient needs to access those states efficiently for good learning. Figure 1 illustrates this scenario by showing the attention scores for two input examples computed by a simple self-attentive model [4], trained on Copy and Denoise tasks respectively (see Section 5). This structure introduces the possibility to impose sparsity in the computational graph, and to limit memory use. With these constraints in mind, the goal is to engineer dependency chains that enable best gradient propagation between these relevant events.

**Notation:** We consider a \( \kappa \)-sparse attention mechanism of dependency depth \( d \).

- **Sparsity coefficient:** \( \kappa \geq 1 \). Borrowing from the SAB model [19], at each time step, attention is allowed at most \( \kappa \) relevant events from the past. That is, for any \( t \) there are at most \( \kappa \) indices \( i \) such that \( \alpha_{i,t} \neq 0 \), which gives rise to a sparse temporal segmentation via the most relevant events.

- **Maximal dependency depth:** \( d \). This is the maximal dependency depth across all time steps.

**Theorem 2.** Let \( h_t \) be the hidden state at time \( t \) of a vanilla RNN with \( \kappa \)-sparse uniform attention mechanism of maximal dependency depth \( d \), and under mild assumptions on the connectivity matrix...
Figure 1: Magnitude of attention weights between states in a trained, fully recurrent and fully attentive model (Bahdanau et al. [4]). Each pixel in the lower triangle corresponds to the attention weight of the skip connection departing from the state marked on the y-axis to the state marked on the x-axis. Left shows Copy task, right shows Denoise task. Task details in Section 5.

\[ \| \nabla h_i L \| = \Omega(1/\kappa^d) \]  

as \( T \to \infty \). (proof in Appendix A.4, Theorem 2)

Similarly to uniform case, Theorem 2 is independent of the choice of the alignment function \( a \) (cf. remark 19 in the appendix).

**Intuition:** If \( \kappa \) and \( d \) are assumed to be bounded, we have \( \Omega(1/\kappa^d) = \Omega(1) \), and thus we mitigate gradient vanishing. Furthermore, notice the dependency depth \( d \) affects the lower bound exponentially, while \( \kappa \) affects it polynomially. In other words, the number of relevant events attended to at each time step contributes far less to gradient vanishing than the number of events in the longest dependency chain. Theorem 2 outlines the tradeoff between computational complexity as \( T \to \infty \) and gradient propagation when balancing attention and recurrence. Attending directly to many relevant past events reduces \( d \) and ensures good gradients at the expense of the complexity cost associated with storing past events and computing attention scores (the strategy employed by Transformers [35]). On the other hand, enforcing small sparsity coefficient \( \kappa \) helps keep computational complexity low, but forces the error gradient through recurrent paths, thereby augmenting the dependency depth \( d \) and degrading gradient signal strength. These heuristics were known before, but Theorem 2 explicitly quantifies the distinct contributions to gradient propagation. Equipped with this result, we further develop these heuristics in what follows.

4 Relevancy screening mechanism

Equipped with the results from the previous section, we wish to refine heuristics that strike a balance between good gradient propagation and computational/memory complexity. Building on the SAB model [19], we remark that although sparse attention attends to the top-\( \kappa \) events at any point in time, attention scores must be computed on all events stored in memory to extract the \( \kappa \) best ones. Thus, the resource bottleneck is not controlled by \( \kappa \), but rather by the number of stored events in memory. In SAB, there is a naive attempt to control this number by only recording network states at each 10 time steps. However, this reduces the size of the computational graph only by a constant factor, but retains \( O(T^2) \) complexity. In contrast, Theorem 2 tells us that the only important events to conserve for good gradient propagation are the relevant ones (also see Remark 6 in Appendix A.2). Thus, we propose to reduce complexity while maintaining good gradient propagation by selectively storing events that are predicted to be relevant in the future, using a **relevancy screening mechanism**.

The idea is simple: devise a screening function \( C(i) \) which estimates the future relevance of \( h_i \), and store selected events in a relevant set \( R_t = \{ h_i | i < t \land C(i) = True \} \) for future attention. In principle, one can explicitly control how \( R_t \) grows with \( t \), thus mitigating the complexity scaling outlined above. Here, \( C(i) \) could take many forms, the best of which depends on task structure. In what follows, we present an example screening mechanism meant to showcase the lessons learned from Theorem 2 but we refer the interested reader to Section 7 for further possibilities.
We note that the sets \( S \Relevance RNN /LSTM \) which uses an RNN/LSTM base, as RelRNN/RelLSTM ("Relevancy Screening Algorithm 1") on simple tasks, but use considerably less memory and compute complexity. In what
case? 

The first category specifies tasks with sparse dependency chains, the second one those with dense

hidden state in the argument.

We note that the sets \( S \) and \( R \) give rise to a sparse attention mechanism with sparsity coefficient \( \kappa \) satisfying \( \kappa = \nu + \rho \geq |S| + |R| \). Hence, memory complexity is constant while the \( O(T^2) \)
bottleneck of computational complexity is replaced by \( O((\rho + \nu) \cdot T) = O(T) \). Lastly, applying

Theorem[2] we get the following guarantee for all \( t \geq 0: \|\nabla h_t L\| = \Omega(1/(\rho + \nu)^2) \) as \( T \to \infty \). Thus

the choices of \( \nu \) and \( \rho \) not only directly impact computational complexity and gradient propagation, but also indirectly influence gradient propagation via the implicit effect of \( \kappa = \nu + \rho \) on \( \alpha \) as already discussed in Section[3]

5 Experiments

Before describing experiments, we make a few remarks. First, we stress that Relevancy Screening
can be applied to any semi-parametric attentive model but we refer to the version presented below,
which uses an RNN/LSTM base, as RelRNN/RelLSTM ("Relevancy Screening RNN/LSTM"). Second, our
objective is not to find state-of-the-art performance but to highlight the advantages of event relevancy
and selective sparsity. Finally, we note that relevancy-based sparsity does not necessarily improve
performance over fully attentive models, but rather allows efficient and scalable usage. As we show
below, RelRNN and RelLSTM perform almost identically to other self-attentive recurrent models
(e.g. [4][19]) on simple tasks, but use considerably less memory and compute complexity. In what
follows, we denote MemRNN/MemLSTM for vanilla self-attention RNN/LSTM as defined in [4].

We consider three task categories to highlight distinct impacts of selective attention and recurrence.
The first category specifies tasks with sparse dependency chains, the second one those with dense
temporal dependencies, and the third includes a variety of reinforcement learning (RL) tasks. All
implementation details and hyper-parameters can be found in the Appendix[6]

5.1 Tasks with sparse dependency chains

A good stereotypical task type that captures sparse sequences of important events are memorization
tasks. Here, the network has to memorize a sequence of relevant characters presented among several
non-relevant ones, store it for some given time delay and output them in the same order as they were
read towards the end of the sequence.

Copy task [17]: The characters to be copied are presented in the first 10 time steps, then must be
outputted after a long delay of \( T \) time steps (see full description in Arjovsky et al. [2]). Thus,
all the relevant events occur in the first 10 time steps. This can be corroborated by the attention

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**Algorithm 1 Relevancy Screening**

1: **procedure:** RelRNN\((s_t, x_t)\)
   2: **Require:** Previous macro-state - \( s_{t-1} \)
   3: **Require:** Input - \( x_t, \nu > 0, \rho > 0 \)
   4: **Require:** Short-term buffer \( s_{t-1}^{(i)} \in S_{t-1} \)
   5: **Require:** Relevant set \( \nu > 0 \)

2: \( h_t \leftarrow o(V s_{t-1} + U x_t + b) \)
3: \( S_t = S_{t-1}.add(h_t) \)
4: if \( t - \nu > 0 \) then
5: \( S_t = S_t.remove(h_{t-\nu}) \)
6: if \( t - \nu > 0 \) and \( C(t - \rho) = True \) then
7: \( R_t = R_{t-1}.replaceWith(h_{t-\rho}) \)
8: \( M_t = [S_t, R_t] \)
9: for all \( m^{(i)} \in M_t \) do
10: \( \tilde{z}^{(i)} \leftarrow v^T \cdot \tanh(W a s_{t-1} + U a m^{(i)}) \)
11: \( z \leftarrow \text{softmax}(\tilde{z}) \)
12: \( s_t = h_t + \sum_i \tilde{z}^{(i)} m^{(i)} \)
13: return \( s_t \)

We take inspiration from memory consolidation principles in human cognition [1], which defines the
transfer of events from short-term to long-term memory. We remark that for some tasks such as those
depicted in Figure[1] relevance varies very little across time. To implement relevancy screening for such
tasks, at every time step \( t \) we attend to two subsets of the past hidden states. We call the first subset
a **short-term buffer** \( S_t = \{h_{t-\nu}, h_{t-\nu+1}, ..., h_{t-1} \} \) which consists of the hidden states of the last \( \nu \) time steps, while the second subset is the relevant set \( R_t \). We compute the **relevance score** at time
step \( i, \beta(i) = \sum_{j=i}^{i+\nu-1} a_{i,j} \), measuring the integrated attention scores over our short-term buffer \( S_t \). More precisely, \( C(i) \) is satisfied if \( \beta(i) \) is part of the top \( \rho \) relevance scores when compared to
all previously observed hidden states, where \( \rho \) is a fixed hyper-parameter satisfying \( \rho \geq |R_t| \) for all \( t \). The pseudo-code in Algorithm[1] describes the screening mechanisms and the interaction between
the short-term buffer \( S_t \) and a finite size relevant set \( R_t \). "replaceWith()" is a function replacing the
hidden state with the lowest relevance score by the
Table 1: Results for Transfer Copy task.

| Model       | $T$ 100 | $T$ 200 | $T$ 400 | $T$ 2000 | $T$ 5000 |
|-------------|---------|---------|---------|---------|---------|
| orth-RNN    | 99%     | 4%      | 16%     | 10%     | 0%      |
| expRNN      | 100%    | 86%     | 73%     | 58%     | 50%     |
| MemRNN      | 99%     | 99%     | 99%     | 92%     | OOM     |
| RelRNN      | 100%    | 99%     | 99%     | 99%     | 99%     |
| LSTM        | 99%     | 64%     | 48%     | 19%     | 14%     |
| h-detach    | 100%    | 91%     | 77%     | 51%     | 42%     |
| SAB         | 99%     | 95%     | 95%     | 95%     | 95%     |
| RelLSTM     | 100%    | 99%     | 99%     | 99%     | 99%     |

Table 2: Results for Denoise task.

| Model       | $T$ 100 | $T$ 300 | $T$ 500 | $T$ 1000 | $T$ 2000 |
|-------------|---------|---------|---------|---------|---------|
| orth-RNN    | 90%     | 71%     | 61%     | 29%     | 3%      |
| expRNN      | 34%     | 25%     | 20%     | 16%     | 11%     |
| MemRNN      | 99%     | 99%     | 99%     | 99%     | OOM     |
| RelRNN      | 100%    | 99%     | 99%     | 99%     | 99%     |
| LSTM        | 82%     | 41%     | 33%     | 21%     | 15%     |
| GORU        | 92%     | 93%     | 91%     | 93%     | 73%     |
| SAB         | 99%     | 99%     | 99%     | 99%     | 99%     |
| RelLSTM     | 100%    | 99%     | 99%     | 99%     | 99%     |

Table 3: PTB and pMNIST results.

| Model   | PTB | pMNIST |
|---------|-----|--------|
|         | BPC | Accuracy | BPC | Accuracy |
| RNN     | 1.56 | 66% | 90.4% |
| orth-RNN | 1.53 | 66% | 93.4% |
| expRNN  | 1.49 | 68% | 96.6% |
| RelRNN  | 1.43 | 69% | 92.8% |
| LSTM    | 1.36 | 73% | 91.1% |
| h-detach | -   | -   | 92.3% |
| SAB     | 1.37 | -   | 94.2% |
| RelLSTM | 1.36 | 73% | 94.3% |

In contrast to sparse information found in the tasks above, we now illustrate RelRNN and RelLSTM’s performance on tasks with densely distributed information on long sequences.

Here, we perform tests on pMNIST [21], a variant of MNIST [22] where pixels are fed sequentially in a permuted order to the network, as well as character level Penn Tree Bank corpus (PTB) [24] where the next letter in a text needs to be predicted. See Table 3 for results. Implementation details and further test data found in Appendix C including attention heatmaps such as the ones found in Figure 1 showing dense attention for RelRNN in both tasks. We note that gated RNNs such as LSTMs are known to perform well here, and that orthogonal RNNs such as those tested here are also very good. The full attention model

Transfer Copy task: An important advantage of sparse attentive recurrent models such as RelRNN is that of generalization. This is illustrated by the Transfer Copy scores [17] where models are trained on Copy task for $T = 100$ and evaluated for $T > 100$. Table 1 shows results for the models listed above, in addition to h-detach [3], an LSTM-based model with improved gradient propagation. Importantly, where purely recurrent networks performed well on the original task, all fail to transfer, with discrepancy growing with $T$. As expected, MemRNN and SAB keep good performance but RelRNN outperforms them, with almost perfect performance for all $T$. While both SAB and RelRNN use sparse memory storage and retrieval, the distinguishing factor is RelRNN’s use of relevancy screening, indicating its importance for transfer. The performance of RelLSTM on Transfer Copy is exactly the same as RelRNN.

Denoise task: Jing et al. [18]: This generalizes the Copy task as the symbols that need to be copied are now randomly distributed among the $T$ time steps, requiring the model to selectively pick the inputs that need to be copied. We test our method against all the previously mentioned models in addition to GORU [18] for various values of $T$ (Table 2). RelLSTM performs exactly as RelRNN and again, we see RelRNN maintain complete performance across all $T$ values, outperforming all purely recurrent models. MemRNN performs as RelRNN/RelLSTM but fails to train due to memory overflow beyond $T = 500$.

5.2 Tasks with dense temporal dependencies
(MemRNN) fails to train on the optimization setup used here for both tasks, again due to overflow in memory.

5.3 MiniGrid reinforcement learning tasks

We next consider a few tasks from MiniGrid [8] in the OpenAI gym [6] in which an agent must get to certain goal states. We use a partially observed formulation of the task, where the agent only sees a small number of squares ahead of it.

Table 4: Average Train and Test Rewards for MiniGrid Reinforcement Learning task. The models were trained on the smaller version of the environment and tested on the larger version to test to generalization of the solution learned.

| Environment     | LSTM | MemLSTM | RelLSTM |
|-----------------|------|---------|---------|
| RedBlueDoors-6x6| 0.97 | 0.97    | 0.97    |
| GoToObject-6x6  | 0.81 | 0.85    | 0.84    |
| MemoryS7        | 0.96 | 0.4     | 0.94    |
| GoToDoor-5x5    | 0.28 | 0.17    | 0.25    |
| Fetch-5x5       | 0.48 | 0.42    | 0.5     |
| DoorKey-5x5     | 0.94 | 0.94    | 0.93    |
| RedBlueDoors-8x8| 0.95 | 0.95    | 0.95    |
| GoToObject-8x8  | 0.66 | 0.66    | 0.74    |
| MemoryS13       | 0.25 | 0.24    | 0.30    |
| GoToDoor-8x8    | 0.13 | 0.11    | 0.15    |
| Fetch-8x8       | 0.38 | 0.44    | 0.45    |
| DoorKey-16x16   | 0.09 | 0.31    | 0.44    |

These tasks are difficult to solve with standard RL algorithms, due to (1) the partial observability of the environment and (2) the sparsity of the reward, given that the agent receives a reward only after reaching the goal. We use Proximal Policy Optimization (PPO, [33]) along with LSTM, MemLSTM, and RelLSTM as the recurrent modules. All models were each trained for 5000000 steps on each environment. The hyperparameters used for RelLSTM are \( \nu = 5 \) and \( \rho = 5 \). Our goal is to compare generalization of the solutions learned by each model by training on smaller version of an environment and testing it on a larger version. On the MiniGrid-DoorKey-5x5-v0 environment the average reward for LSTM is 0.94, MemLSTM is 0.94 and RelLSTM is 0.93. On transferring the learned solution to the 16x16 version of that environment the average reward for LSTM is 0.09, MemLSTM is 0.31 and RelLSTM is 0.44. As illustrated in 4, we find that transfer scores for RelLSTM are much higher than for MemLSTM across several environments.

6 Analysis

In this section we analyze the maximal GPU usage and gradient norm of \( \| \nabla h_t L \| \) across time \( t \) for the Denoise Task. All the models were run using a NVIDIA TitanXP GPU and their peak usage was recorded in order to quantify the amount of computational resources used for each of them. We varied sequence length \( T \) from 200 to 2000 in order to measure the trend in the usage. To measure propagating gradients as a function of \( t \), we trained models on \( T = 1000 \) and computed \( \log \| \nabla h_t L \| \).

As illustrated in Figure 6 (center), we confirm MemRNN scales quadratically with \( T \), same as SAB which shows an improvement but only by a constant factor. We also confirm that RelLSTM scales linearly with \( T \) similar to RNN and LSTM. Figure 6 (left) shows that the gradient norms for RNN explode and for LSTM vanish as \( t \) increases. The gradient norms of all attention models were stable, as expected from the results of Section 3. To better visualize the interplay between gradient norm and GPU usage, Figure 6 (right) shows the final averaged log gradient norm against Max GPU usage for different times \( T = \{ 400, 600, 800 \} \). As expected, purely recurrent models (RNN, LSTM) show very little GPU usage differences across distinct \( T \) values, while their performance and gradients degrade with increasing \( t \). Note that the RNN’s gradients explode while the LSTM’s vanish, both
Standard self attentive models (MemRNN, SAB) on the other hand, show opposite trends, with stable gradients but GPU usage quadratically increasing in $T$. As expected from Theorem 2 (Section 3), RelLSTM shows both stable gradients and stable GPU usage. The optimal trade-off between memory usage and good gradient propagation achieved by RelLSTM highlights the importance of a dynamic memory that attempts to predict relevancy in order to only store exactly those events that help with learning. We note the Denoise task has a small number of relevant events and that not all tasks share this structure. Nevertheless, this experiment highlights how important resource gains can be made by shifting efforts from offsetting memory growth by a constant factor, to a relevancy screening method.

7 Conclusion & Discussion

Our main contribution is a formal analysis of gradient propagation in self-attention RNNs, from which we derive two quantities that are governing gradient propagation: sparsity and dependency depth. Meanwhile we identify event relevancy as a key concept to efficiently scale attentive systems to very long sequential computations. This is illustrated via a Relevancy Screening Mechanism, inspired by the cognitive process of memory consolidation in the brain, that efficiently selects network states, called relevant events, to be committed to long-term memory based on a screening heuristic operating on a fixed-size short-term memory buffer. We showcase the benefits of this mechanism in an attentive RNN and LSTM which we call RelRNN and RelLSTM respectively, using simple but illustrative numerical experiments, and demonstrate the optimal trade-off between memory usage and good gradient propagation it achieves.

As outlined in Sections 3 and 4, this trade-off is a reflection of the task-specific balance between sparsity and dependency depth parameters. While our proposed relevancy screening mechanism exploits "local" attention scores (measured while events are in short-term memory buffer), we acknowledge other types of relevancy could be formulated with heuristics better suited to distinct environments. For instance, promising directions include leveraging predictive coding principles to select "surprising events", or neural networks could be used to learn the screening function $C(i)$ in an end-to-end fashion.

References

[1] Cristina Alberini, Sarah Johnson, and Xiaojing Ye. Memory Reconsolidation, pages 81–117. 12 2013. ISBN 9780123868923. doi: 10.1016/B978-0-12-386892-3.00005-6.

[2] Martin Arjovsky, Amar Shah, and Yoshua Bengio. Unitary evolution recurrent neural networks. In Proceedings of the 33rd International Conference on International Conference on Machine Learning - Volume 48, ICML’16, pages 1120–1128. JMLR.org, 2016. URL http://dl.acm.org/citation.cfm?id=3045390.3045509.

2 The measurements for both GPU usage and gradient norm are identical for both RelLSTM and RelRNN.
[3] Devansh Arpit, Bhargav Kanuparthi, Giancarlo Kerg, Nan Rosemary Ke, Ioannis Mitliagkas, and Yoshua Bengio. h-detach: Modifying the lstm gradient towards better optimization. arXiv preprint arXiv:1810.03023, 2018.

[4] Dzmitry Bahdanau, Kyunghyun Cho, and Yoshua Bengio. Neural Machine Translation by Jointly Learning to Align and Translate. arXiv e-prints, art. arXiv:1409.0473, Sep 2014.

[5] Y Bengio, P Simard, and P Frasconi. Learning long-term dependencies with gradient descent is difficult. IEEE Transactions on Neural Networks, 5(2):157–166, 1994.

[6] Greg Brockman, Vicki Cheung, Ludwig Pettersson, Jonas Schneider, John Schulman, Jie Tang, and Wojciech Zaremba. Openai gym. 2016. URL http://arxiv.org/abs/1606.01540.

[7] Mario Lezcano Casado and David Martínez-Rubio. Cheap orthogonal constraints in neural networks: A simple parametrization of the orthogonal and unitary group. CoRR, abs/1901.08428, 2019. URL http://arxiv.org/abs/1901.08428.

[8] Maxime Chevalier-Boisvert and Lucas Willems. Minimalistic gridworld environment for openai gym. https://github.com/maximecb/gym-minigrid, 2018.

[9] Rewon Child, Scott Gray, Alec Radford, and Ilya Sutskever. Generating long sequences with sparse transformers. arXiv preprint arXiv:1904.10509, 2019.

[10] Kyunghyun Cho, B van Merrienboer, Caglar Gulcehre, F Bougares, H Schwenk, and Yoshua Bengio. Learning phrase representations using rnn encoder-decoder for statistical machine translation. In Conference on Empirical Methods in Natural Language Processing (EMNLP 2014), 2014.

[11] Jacob Devlin, Ming-Wei Chang, Kenton Lee, and Kristina Toutanova. Bert: Pre-training of deep bidirectional transformers for language understanding. arXiv preprint arXiv:1810.04805, 2018.

[12] Alex Graves, Greg Wayne, and Ivo Danihelka. Neural turing machines. arXiv preprint arXiv:1410.5401, 2014.

[13] Alex Graves, Greg Wayne, Malcolm Reynolds, Tim Harley, Ivo Danihelka, Agnieszka Grabska-Barwińska, Sergio Gómez Colmenarejo, Edward Grefenstette, Tiago Ramalho, John Agapiou, et al. Hybrid computing using a neural network with dynamic external memory. Nature, 538 (7626):471, 2016.

[14] Anna Harutyunyan, Will Dabney, Thomas Mesnard, Mohammad Gheshlaghi Azar, Bilal Piot, Nicolas Heess, Hado P van Hasselt, Gregory Wayne, Satinder Singh, Doina Precup, and Remi Munos. Hindsight Credit Assignment. pages 12467–12476, 2019.

[15] Mikael Henaff, Arthur Szlam, and Yann LeCun. Recurrent orthogonal networks and long-memory tasks. arXiv preprint arXiv:1602.06662, 2016.

[16] Sepp Hochreiter. Untersuchungen zu dynamischen neuronalen netzen [in german] diploma thesis. TU Münich, 1991.

[17] Sepp Hochreiter and Jürgen Schmidhuber. Long short-term memory. Neural computation, 9(8):1735–1780, 1997.

[18] Li Jing, Caglar Gulcehre, John Peurifoy, Yichen Shen, Max Tegmark, Marin Soljacic, and Yoshua Bengio. Gated orthogonal recurrent units: On learning to forget. Neural computation, 31(4):765–783, 2019.

[19] Nan Rosemary Ke, Anirudh Goyal, Olexa Bilaniuk, Jonathan Binas, Michael C Mozer, Chris Pal, and Yoshua Bengio. Sparse attentive backtracking: Temporal credit assignment through reminding. In Advances in Neural Information Processing Systems, pages 7640–7651, 2018.

[20] Nikita Kitaev, Łukasz Kaiser, and Anselm Levskaya. Reformer: The efficient transformer. arXiv preprint arXiv:2001.04451, 2020.
[21] Quoc V Le, Navdeep Jaitly, and Geoffrey E Hinton. A simple way to initialize recurrent networks of rectified linear units. *arXiv preprint arXiv:1504.00941*, 2015.

[22] Yann LeCun and Corinna Cortes. MNIST handwritten digit database. 2010. URL [http://yann.lecun.com/exdb/mnist/](http://yann.lecun.com/exdb/mnist/).

[23] Zhouhan Lin, Minwei Feng, Cicero Nogueira dos Santos, Mo Yu, Bing Xiang, Bowen Zhou, and Yoshua Bengio. A structured self-attentive sentence embedding. *arXiv preprint arXiv:1703.03130*, 2017.

[24] Mitchell P. Marcus, Mary Ann Marcinkiewicz, and Beatrice Santorini. Building a large annotated corpus of english: The penn treebank. *Comput. Linguist.*, 19(2):313–330, June 1993. ISSN 0891-2017. URL [http://dl.acm.org/citation.cfm?id=972470.972475](http://dl.acm.org/citation.cfm?id=972470.972475).

[25] Tsendsuren Munkhdalai, Alessandro Sordoni, TONG WANG, and Adam Trischler. Metalearned Neural Memory. pages 1331–13342, 2019.

[26] Ankur P Parikh, Oscar Täckström, Dipanjan Das, and Jakob Uszkoreit. A decomposable attention model for natural language inference. *arXiv preprint arXiv:1606.01933*, 2016.

[27] R Pascanu, T Mikolov, and Y Bengio. On the difficulty of training recurrent neural networks. *ICML*, 2013.

[28] Romain Paulus, Caiming Xiong, and Richard Socher. A deep reinforced model for abstractive summarization. *arXiv preprint arXiv:1705.04304*, 2017.

[29] Alec Radford, Jeffrey Wu, Rewon Child, David Luan, Dario Amodei, and Ilya Sutskever. Language models are unsupervised multitask learners. *OpenAI Blog*, 1(8):9, 2019.

[30] Gabriel A Radvansky and Jeffrey M Zacks. Event boundaries in memory and cognition. *Current opinion in behavioral sciences*, 17:133–140, 2017.

[31] Adam Santoro, Ryan Faulkner, David Raposo, Jack Rae, Mike Chrzanowski, Theophane Weber, Daan Wierstra, Oriol Vinyals, Razvan Pascanu, and Timothy Lillicrap. Relational recurrent neural networks. In *Advances in Neural Information Processing Systems 31*. 2018.

[32] J Schmidhuber and S Hochreiter. Long short-term memory. *Neural Computation*, 1997.

[33] John Schulman, Filip Wolski, Prafulla Dhariwal, Alec Radford, and Oleg Klimov. Proximal policy optimization algorithms. 2017. URL [http://arxiv.org/abs/1707.06347](http://arxiv.org/abs/1707.06347).

[34] Sainbayar Sukhbaatar, arthur szlam, Jason Weston, and Rob Fergus. End-to-end memory networks. In *Advances in Neural Information Processing Systems 28*, pages 2440–2448. 2015. URL [http://papers.nips.cc/paper/5846-end-to-end-memory-networks.pdf](http://papers.nips.cc/paper/5846-end-to-end-memory-networks.pdf).

[35] Ashish Vaswani, Noam Shazeer, Niki Parmar, Jakob Uszkoreit, Llion Jones, Aidan N Gomez, Lukasz Kaiser, and Illia Polosukhin. Attention is all you need. *NeurIPS*, 2017.

[36] Zhilin Yang, Zihang Dai, Yiming Yang, Jaime Carbonell, Russ R Salakhutdinov, and Quoc V Le. Xlnet: Generalized autoregressive pretraining for language understanding. In *Advances in neural information processing systems*, pages 5754–5764, 2019.

[37] Jeffrey M Zacks, Nicole K Speer, Khena M Swallow, Todd S Braver, and Jeremy R Reynolds. Event perception: a mind-brain perspective. *Psychological bulletin*, 133(2):273, 2007.

[38] Guangxiang Zhao, Junyang Lin, Zhiyuan Zhang, Xuancheng Ren, Qi Su, and Xu Sun. Explicit sparse transformer: Concentrated attention through explicit selection. *arXiv preprint arXiv:1912.11637*, 2019.
Supplemental Material for:
Untangling tradeoffs between recurrence and self-attention in artificial neural networks

A Theoretical analysis of gradient propagation

A.1 Notational convention

In this paper, we use the notation \( \frac{df}{dx} \) to denote the total derivative of \( f \) with respect to \( x \), and \( \frac{\partial f}{\partial x} \) to denote the partial derivative of \( f \) with respect to \( x \).

If we assume \( f : \mathbb{R}^n \rightarrow \mathbb{R}^m \), and \( x \in \mathbb{R}^n \), then \( \frac{df}{dx} \) denotes the Jacobian matrix \( J_f \) such that

\[
(J_f)_{ij} = \frac{df_i}{dx_j}
\]  

(6)

In particular, with this notation, we have that if a function \( L : \mathbb{R}^m \rightarrow \mathbb{R} \), and \( y \in \mathbb{R}^m \) then \( \frac{dL}{dy} \) is a row vector, while the conventional notation for \( \nabla_y L \) indicates a column vector. In other words, \((\nabla_y L)^T = \frac{dL}{dy}\). Hence if \( L \) is a function of \( f(x) \), then

\[
\frac{dL}{dx} = \frac{dL}{df} \cdot \frac{df}{dx}
\]  

(7)

while

\[
\nabla_x L = \left( \frac{df}{dx} \right)^T \cdot \nabla f(x) L = J_f^T \cdot \nabla f(x) L
\]  

(8)

Similarly, we have that \( \frac{\partial L}{\partial y} \) is a row vector.

A.2 Preliminary results

Let

\[
s_t = \psi_t(h_1, h_2, \ldots, h_t, s_{t-1})
\]  

(9)

where

\[
h_{t+1} = \phi(Vs_t + Ux_{t+1} + b)
\]  

(10)

Lemma 1. For all \( t, k \geq 0 \), we have

\[
\frac{ds_{t+k+1}}{dh_t} = \frac{\partial s_{t+k+1}}{\partial h_t} + \sum_{j=0}^{k} \frac{\partial s_{t+k+1}}{\partial h_{t+j+1}} \cdot \frac{dh_{t+j+1}}{dh_t} + \frac{\partial s_{t+k+1}}{\partial s_t} \cdot \frac{ds_t}{dh_t}
\]  

(11)

Proof. Follows directly from the following multivariable chain rule: if

\[
g(t) = f(g_1(t), g_2(t), \ldots, g_n(t))
\]  

(12)

then

\[
\frac{dg}{dt} = \sum_{i=1}^{n} \frac{\partial f}{\partial g_i} \frac{dg_i}{dt}
\]  

(13)

Lemma 2. If we further denote the Jacobian matrix \( J_k = \frac{\partial s_{t+k+1}}{\partial h_t} \), then we get that for all \( t, k \geq 0 \), we have

\[
\frac{ds_{t+k+1}}{dh_t} = \frac{\partial s_{t+k+1}}{\partial h_t} + \sum_{j=0}^{k} \left( \frac{\partial s_{t+k+1}}{\partial h_{t+j+1}} \cdot J_{t+j} + 1_{j=k} \cdot \frac{\partial s_{t+k+1}}{\partial s_t} \right) \cdot \frac{ds_t}{dh_t}
\]  

(14)
Proof. Follows directly from the observation that
\[
\frac{dh_{t+j+1}}{dh_t} = \frac{\partial h_{t+j+1}}{\partial s_{t+j}} \frac{ds_{t+j}}{dh_t} = J_{t+j} \cdot \frac{ds_{t+j}}{dh_t}
\] (15)
\[\square\]

**Remark 1.** Let us denote
\[
C_{k+1}^{(t)} = \frac{ds_{t+k+1}}{dh_t}
\] (16)
\[
E_{k+1}^{(t)} = \frac{\partial s_{t+k+1}}{\partial h_t}
\] (17)
and
\[
F_{k+1,j}^{(t)} = \frac{\partial s_{t+k+1}}{\partial h_{t+j+1}} \cdot J_{t+j+1} \cdot \frac{\partial s_{t+k+1}}{\partial s_{t+k}}
\] (18)
and thus the recursion formula in Lemma 2 rewrites as
\[
C_{k+1}^{(t)} = E_{k+1}^{(t)} + \sum_{j=0}^{k} F_{k+1,j}^{(t)} \cdot C_j^{(t)}
\] (19)

The next two results highlight how to solve this recursion.

**Lemma 3.** Let \(C_i, E_i, F_{i,j} \in \mathbb{R}^{n \times n}\) such that for all \(k \geq 0\), we have
\[
C_{k+1} = E_{k+1} + \sum_{j=0}^{k} F_{k+1,j} \cdot C_j
\] (20)
Then for all \(k \geq 1\), we have
\[
C_k = \xi_{0:k} C_0 + \sum_{r=1}^{k} \xi_{r:k} E_r
\] (21)
where
\[
\xi_{r:k} = \sum_{s=1}^{k-r} \xi_{r:k}(s)
\] (22)
with
\[
\xi_{r:k}(s) = \sum_{r_1 \leq \ldots \leq r_s = k} F_{i_{s+1},i_s} \cdot F_{i_{s-1},i_{s-2}} \cdot \ldots \cdot F_{i_2,i_1}
\] (23)
and \(\xi_{k:k} = \text{Id}\).

Proof. Let us prove the statement by induction on \(k \geq 1\).
For \(k = 1\), we have
\[
C_1 = E_1 + F_{1,0} C_0 = \xi_{1:1} E_1 + \xi_{0:1} C_0
\] (24)
Now let us assume the statement to be true for $k$, then we get

$$C_{k+1} = E_{k+1} + \sum_{j=0}^{k} F_{k+1,j} \cdot \left( \xi_{0:j} C_0 + \sum_{r=1}^{j} \xi_{r,j} E_r \right)$$  \hspace{1cm} (25)

$$= E_{k+1} + \left( \sum_{j=0}^{k} F_{k+1,j} \cdot \xi_{0:j} \right) \cdot C_0 + \sum_{j=0}^{k} \sum_{r=1}^{j} F_{k+1,j} \xi_{r,j} E_r$$  \hspace{1cm} (26)

$$= E_{k+1} + \xi_{0:k+1} C_0 + \sum_{r=1}^{k} \left( \sum_{j=r}^{k} F_{k+1,j} \xi_{r,j} \right) \cdot E_r$$  \hspace{1cm} (27)

$$= \xi_{k+1:k+1} E_{k+1} + \xi_{0:k+1} C_0 + \sum_{r=1}^{k} \xi_{r:k+1} E_r$$  \hspace{1cm} (28)

$$= \xi_{0:k+1} C_0 + \sum_{r=1}^{k+1} \xi_{r:k+1} E_r$$  \hspace{1cm} (29)

\[ \square \]

**Lemma 4.** If we further assume that $C_0 = E_0$, then we have for all $k \geq 1$

$$C_k = E_k + \sum_{s=1}^{k} \sum_{q=s}^{k} \xi_{k-q,k}(s) E_{k-q}$$  \hspace{1cm} (31)

**Proof.** Using the previous lemma, we get

$$C_k = E_k + \sum_{s'=1}^{k} \xi_{0:k}(s') C_0 + \sum_{r=1}^{k-1} \sum_{s=1}^{k-r} \xi_{r,k}(s) E_r$$  \hspace{1cm} (32)

Using the assumption $C_0 = E_0$, we get

$$C_k = E_k + \sum_{s'=1}^{k} \xi_{0:k}(s') E_0 + \sum_{r=1}^{k-1} \sum_{s=1}^{k-r} \xi_{r,k}(s) E_r$$  \hspace{1cm} (33)

$$= E_k + \sum_{r=0}^{k-1} \sum_{s=1}^{k-r} \xi_{r,k}(s) E_r$$  \hspace{1cm} (34)

Now let us put $q = k - r$, we get

$$C_k = E_k + \sum_{q=1}^{k} \sum_{s=1}^{q} \xi_{k-q,k}(s) E_{k-q}$$  \hspace{1cm} (36)

$$= E_k + \sum_{s=1}^{k} \sum_{q=s}^{k} \xi_{k-q,k}(s) E_{k-q}$$  \hspace{1cm} (37)

\[ \square \]
Remark 2. First, note that Lemma 4 applies here, since $C_0^{(t)} = E_0^{(t)}$, and thus

$$C_k^{(t)} = E_k^{(t)} + \sum_{s=1}^{k} \sum_{q=s}^{k} \xi_{k-q:k}^{(t)}(s) E_{k-q}^{(t)}$$

The idea of Lemma 4 was to regroup all terms with the same number of $F$ factors (where each $F$ contains a Jacobian matrix $J_k$ which contains the connectivity matrix $V$ of the recurrent net). One could roughly perceive the term

$$\sum_{q=s}^{k} \xi_{k-q:k}^{(t)}(s) E_{k-q}^{(t)}$$

as being the term of degree $s$ for $s = 1, 2, \ldots, k$ and $E_k^{(t)}$ the term of degree 0. This will allow us to consider the terms $C$ roughly as a polynomial in $V$ and we can look the asymptotic behaviour of each of the coefficients of this polynomial individually. This will then give us a very good understanding on how the distribution of the attention weights are affecting the magnitude of total gradient.

**Proposition 1.** For all $t \geq 1$, and all $k \geq 0$, we have that

$$\frac{ds_{t+k}}{dh_t} = \sum_{s=0}^{k} \tilde{\xi}_{s:k}^{(t)}(s)$$

where for all $s \geq 1$,

$$\tilde{\xi}_{s:k}^{(t)}(s) = \sum_{0 \leq i_1 < \ldots < i_s < k} F_{k,i_1}^{(t)} \cdot F_{i_2,i_1}^{(t)} \cdot \ldots \cdot F_{i_s,i_1}^{(t)} \cdot E_{i_1}^{(t)}$$

and where $\tilde{\xi}_{s:k}^{(t)}(0) = E_k^{(t)}$. With for all $k \geq 0$ we have

$$E_k^{(t)} = \frac{\partial s_{t+k}}{\partial h_t}$$

and for all $k \geq j$ we have

$$F_{k+1,j}^{(t)} = \frac{\partial s_{t+k+1}}{\partial h_{t+j+1}} \cdot J_{t+j} + 1_{j=k} \frac{\partial s_{t+k+1}}{\partial s_{t+k}}$$

**Proof.** Let $t \geq 1$, and recall that we defined $C_k^{(t)} = \frac{ds_{t+k}}{dh_t}$, for all $k \geq 0$.

As already pointed out, we know that $C_0^{(t)} = E_0^{(t)}$ (thus the claim holds for $k = 0$).

Then by Lemma 4 we know that for all $k \geq 1$ we have

$$C_k^{(t)} = E_k^{(t)} + \sum_{s=1}^{k} \sum_{q=s}^{k} \xi_{k-q:k}^{(t)}(s) E_{k-q}^{(t)}$$

$$= \tilde{\xi}_{s:k}^{(t)}(0) + \sum_{s=1}^{k} \sum_{q=s}^{k} \sum_{0 \leq i_1 < \ldots < i_s+1 = k} F_{k,i_1}^{(t)} \cdot F_{i_2,i_1}^{(t)} \cdot \ldots \cdot F_{i_s,i_1}^{(t)} \cdot E_{i_1}^{(t)}$$

$$= \tilde{\xi}_{s:k}^{(t)}(0) + \sum_{s=1}^{k} \sum_{0 \leq i_1 < \ldots < i_s < k} F_{k,i_1}^{(t)} \cdot F_{i_2,i_1}^{(t)} \cdot \ldots \cdot F_{i_s,i_1}^{(t)} \cdot E_{i_1}^{(t)}$$

$$= \tilde{\xi}_{s:k}^{(t)}(0) + \sum_{s=1}^{k} \tilde{\xi}_{s:k}^{(t)}(s)$$

$$= \sum_{s=0}^{k} \tilde{\xi}_{s:k}^{(t)}(s)$$

$\square$
Remark 3. In what follows the main emphasis will be to calculate the $F_{t,i,j}$ and $E_{t,i}$ terms explicitly, since they are the building blocks of the mentioned polynomials in $2$.

We will assume that
\[
s_t = f(h_t, c_t)
\]
with
\[
c_t = \alpha_{1,t}h_1 + \alpha_{2,t}h_2 + \ldots + \alpha_{t,t}h_t
\]
and
\[
\alpha_{j,t} = \exp (e_{j,t}) \sum_{i=1}^{t} \exp (e_{i,t})
\]
where
\[
e_{i,t} = a(s_{t-1}, h_i)
\]
Let us recall that for all $k \geq 0$ we have
\[
E_{t,k} = \frac{\partial s_{t+k}}{\partial h_t}
\]
and for all $k \geq j$ we have
\[
F_{t,k+1,j} = \frac{\partial s_{t+k+1}}{\partial h_{t+j+1}} \cdot J_{t+j} + 1_{j=k} \cdot \frac{\partial s_{t+k+1}}{\partial s_{t+k}}
\]

Lemma 5. With the assumption of Remark 3, we have that for all $t \geq 2$
\[
\frac{\partial s_t}{\partial s_{t-1}} = \partial^2 f(h_t, c_t) \cdot \left( \sum_{i=1}^{t} \alpha_{i,t} Y_{i,t} \right)
\]
where $\partial^2 f$ is the the partial derivative of $f$ with respect to the second variable, and where we define
\[
Y_{i,t} = h_i \cdot \left( \frac{\partial e_{i,t}}{\partial s_{t-1}} - \sum_{j=1}^{t} \alpha_{j,t} \cdot \frac{\partial e_{j,t}}{\partial s_{t-1}} \right)
\]

Proof.
\[
\frac{\partial s_t}{\partial s_{t-1}} = \partial^2 f(h_t, c_t) \cdot \frac{\partial c_t}{\partial s_{t-1}}
\]
\[
= \partial^2 f(h_t, c_t) \cdot \left[ \sum_{i=1}^{t} h_i \cdot \left( \frac{\partial \alpha_{i,t}}{\partial s_{t-1}} \right) \right]
\]
\[
= \partial^2 f(h_t, c_t) \cdot \left[ \sum_{i=1}^{t} h_i \cdot \left( \sum_{j=1}^{t} \frac{\partial \alpha_{i,t}}{\partial e_{j,t}} \cdot \frac{\partial e_{j,t}}{\partial s_{t-1}} \right) \right]
\]
\[
= \partial^2 f(h_t, c_t) \cdot \left[ \sum_{i=1}^{t} h_i \cdot \left( \sum_{j=1}^{t} \alpha_{i,t} (1_{i=j} - \alpha_{j,t}) \cdot \frac{\partial e_{j,t}}{\partial s_{t-1}} \right) \right]
\]
\[
= \partial^2 f(h_t, c_t) \cdot \left[ \sum_{i=1}^{t} \alpha_{i,t} h_i \left( \frac{\partial e_{i,t}}{\partial s_{t-1}} - \sum_{j=1}^{t} \alpha_{j,t} \frac{\partial e_{j,t}}{\partial s_{t-1}} \right) \right]
\]
\[
= \partial^2 f(h_t, c_t) \cdot \left( \sum_{i=1}^{t} \alpha_{i,t} Y_{i,t} \right)
\]
\[
\square
\]
Lemma 6. With the assumption of Remark 3 we have that for all $k \geq j$:

$$\frac{\partial s_k}{\partial h_j} = 1_{k=j} \cdot \partial_1 f(h_k, c_k) + \alpha_{j,k} \partial_2 f(h_k, c_k) \cdot (I + X_{j,k}) \tag{64}$$

where $\partial_1 f$ and $\partial_2 f$ are the partial derivatives of $f$ with respect to the first and second variable, respectively, and where we define

$$X_{j,k} = \left( h_j - \sum_{i=1}^{k} h_i \alpha_{i,k} \right) \cdot \frac{\partial e_{j,k}}{\partial h_j} \tag{65}$$

Proof.

$$\frac{\partial s_k}{\partial h_j} = 1_{k=j} \cdot \partial_1 f(h_k, c_k) \cdot \frac{\partial h_k}{\partial h_j} + \partial_2 f(h_k, c_k) \cdot \frac{\partial c_k}{\partial h_j} \tag{66}$$

$$= 1_{k=j} \cdot \partial_1 f(h_k, c_k) + \partial_2 f(h_k, c_k) \cdot \left[ \alpha_{j,k} \cdot I + \sum_{i=1}^{k} h_i \cdot \frac{\partial \alpha_{i,k}}{\partial h_j} \right] \tag{67}$$

$$= 1_{k=j} \cdot \partial_1 f(h_k, c_k) + \partial_2 f(h_k, c_k) \cdot \left[ \alpha_{j,k} \cdot I + \sum_{i=1}^{k} h_i \cdot \alpha_{i,k} \left( 1_{i=j} - \alpha_{j,k} \right) \frac{\partial e_{j,k}}{\partial h_j} \right] \tag{68}$$

$$= 1_{k=j} \cdot \partial_1 f(h_k, c_k) + \partial_2 f(h_k, c_k) \cdot \left[ \alpha_{j,k} \cdot I + \left( h_j \alpha_{j,k} - \alpha_{j,k} \sum_{i=1}^{k} h_i \cdot \alpha_{i,k} \right) \frac{\partial e_{j,k}}{\partial h_j} \right] \tag{69}$$

$$= 1_{k=j} \cdot \partial_1 f(h_k, c_k) + \alpha_{j,k} \partial_2 f(h_k, c_k) \cdot \left[ \alpha_{j,k} \cdot I + \left( h_j - \sum_{i=1}^{k} h_i \cdot \alpha_{i,k} \right) \frac{\partial e_{j,k}}{\partial h_j} \right] \tag{70}$$

$$= 1_{k=j} \cdot \partial_1 f(h_k, c_k) + \alpha_{j,k} \partial_2 f(h_k, c_k) \cdot \left( I + X_{j,k} \right) \tag{71}$$

$$E^{(t)}_{k'} = 1_{k'=0} \partial_1 f(h_t, c_t) + \alpha_{t,t+k'} \partial_2 f(h_{t+k'}, c_{t+k'}) \cdot [I + X_{t,t+k'}] \tag{73}$$

and for all $k \geq j$,

$$E^{(t)}_{k+1,j} = \alpha_{t+j+1,t+k+1} \cdot \partial_2 f(h_{t+k+1}, c_{t+k+1}) \cdot [I + X_{t+j+1,t+k+1}] \cdot J_{t+j} \tag{74}$$

$$+ 1_{k=j} \cdot \left( \partial_1 f(h_{t+k+1}, c_{t+k+1}) \cdot J_{t+j} + \partial_2 f(h_{t+k+1}, c_{t+k+1}) \cdot \left[ \sum_{i=1}^{t+k+1} \alpha_{t+k+1} Y_{i,t+k+1} \right] \right) \tag{75}$$

Proof. Applying lemma 6 we get that for all $k \geq 0$,

$$E^{(t)}_{k'} = \frac{\partial s_{t+k'}}{\partial h_t} \tag{76}$$

$$= 1_{k'=0} \cdot \partial_1 f(h_t, c_t) + \alpha_{t,t+k'} \cdot \partial_2 f(h_{t+k'}, c_{t+k'}) \cdot [I + X_{t,t+k'}] \tag{77}$$

and then by applying lemma 5 and 6 we get that for all $k \geq j$,
Proof. Follows straight from Corollary 1.

\[ F_{k+1,j}^{(t)} = \frac{\partial s_{t+k+1}}{\partial t_{t+j+1}} \cdot J_{t+j} + 1_{j=k} \cdot \frac{\partial s_{t+k+1}}{\partial s_{t+k}} \]

\[ = [1_{k=j} \partial_t f(h_{t+k+1}, c_{t+k+1}) + \alpha_{t+j+1, t+k+1} \partial_2 f(h_{t+k+1}, c_{t+k+1}) \cdot (I + X_{t+j+1, t+k+1})] \cdot J_{t+j} + 1_{k=j} \cdot \partial_2 f(h_{t+k+1}, c_{t+k+1}) \cdot \left( \sum_{i=1}^{t+k+1} \alpha_{i, t+k+1} Y_{i, t+k+1} \right) \]

Proposition 2. We can rewrite for all \( k' \geq 0 \) and all \( k \geq j \geq 0 \)

\[ E_{k'}^{(t)} = \alpha_{t, t+k'} \cdot \tilde{D}_{k',0}^{(t)} + 1_{k'=0} \tilde{R}_{0}^{(t)} \]

\[ F_{k+1,j}^{(t)} = \alpha_{t+j+1, t+k+1} \cdot D_{k+1,j}^{(t)} + 1_{k=j} \cdot R_{k+1}^{(t)} \]

where

\[ D_{k+1,j+1}^{(t)} = \partial_2 f(h_{t+k+1}, c_{t+k+1}) \cdot [I + X_{t+j+1, t+k+1}] \cdot J_{t+j} \]

\[ R_{k+1}^{(t)} = \partial_1 f(h_{t+k+1}, c_{t+k+1}) \cdot [I + X_{t+t+k}] \cdot \left( \sum_{i=1}^{t+k+1} \alpha_{i, t+k+1} Y_{i, t+k+1} \right) \]

\[ \tilde{D}_{k'}^{(t)} = \partial_2 f(h_{t+k'}, c_{t+k'}) \cdot [I + X_{t+t+k'}] \]

\[ \tilde{R}_{0}^{(t)} = \partial_1 f(h_{t}, c_{t}) \]

while \( X_{t,i} \) and \( Y_{t,i} \) are defined as in lemma 5 and 6.

Proof. Follows straight from Corollary 1.

Remark 4. If we are further assuming that

\[ s_{t} = f(h_{t}, c_{t}) = h_{t} + c_{t} \]

then for all \( k \geq 0 \), we have

\[ E_{k}^{(t)} = 1_{k=0} \cdot I + \alpha_{t, t+k} \cdot [I + X_{t+t+k}] \]

and for all \( k \geq j \), we have

\[ F_{k+1,j}^{(t)} = \alpha_{t+j+1, t+k+1} \cdot [I + X_{t+j+1, t+k+1}] \cdot J_{t+j} + 1_{k=j} \cdot \left( J_{t+j} + \sum_{i=1}^{t+k+1} \alpha_{i, t+k+1} Y_{i, t+k+1} \right) \]

Proof. This follows directly from corollary 1 and the observation that

\[ \partial_1 f(h_{t}, c_{t}) = \partial_2 f(h_{t}, c_{t}) = I \]
Remark 5. If we are further assuming that 
\[ e_{j,t} = a(s_{t-1}, h_j) = v_a^T \cdot \tanh (W_a s_{t-1} + U_a h_j) \] 
(95) 
as done in [3], we get that 
\[ \frac{\partial e_{j,t}}{\partial h_{j}} = v_a^T \cdot \text{diag}[1 - \tanh^2 (W_a s_{t-1} + U_a h_j)] \cdot U_a \] 
(96) 
and 
\[ \frac{\partial e_{j,t}}{\partial s_{t-1}} = v_a^T \cdot \text{diag}[1 - \tanh^2 (W_a s_{t-1} + U_a h_j)] \cdot W_a \] 
(97) 
which we can plug into the definitions of \( X_{j,k} \) and \( Y_{j,k} \) to get explicit expressions for matrices \( E_k(t) \) and \( F_{k+1,j} \).

Lemma 7. If, with the assumptions Remark 3, we assume that for all \( i, t \geq 1 \), we have \( e_{i,t} = a(s_{t-1}, h_i, \theta) \) depending on some parameter \( \theta \in \mathbb{R}^{N \times M} \), then we have 
\[ \frac{dL}{d\theta} = \sum_{i,t} \alpha_{j,t} \cdot \frac{dL}{ds_t} \cdot \partial_2 f(h_t, c_t) \cdot h_j \cdot \left[ \sum_i (1_{i=j} - \alpha_{i,t}) \cdot \frac{\partial e_{i,t}}{\partial \theta} \right] \] 
(98) 

Proof. If we denote \( \theta^{(i,t)} \) to be the parameter for \( e_{i,t} \), then we have 
\[ \frac{dL}{d\theta} = \sum_{i,t} \frac{dL}{d\theta^{(i,t)}} \] 
(99) 
\[ = \sum_{i,j,t} \frac{dL}{d\alpha_{j,t}} \cdot \frac{\partial e_{i,t}}{\partial \alpha_{j,t}} \cdot \frac{\partial e_{i,t}}{\partial \theta^{(i,t)}} \] 
(100) 
\[ = \sum_{i,j,t} \alpha_{j,t} (1_{i=j} - \alpha_{i,t}) \cdot \frac{dL}{d\alpha_{j,t}} \cdot \frac{\partial e_{i,t}}{\partial \theta} \] 
(101) 
where 
\[ \frac{dL}{d\alpha_{j,t}} = \frac{dL}{ds_t} \cdot \frac{\partial s_t}{\partial c_t} \cdot \frac{\partial c_t}{\partial \alpha_{j,t}} \] 
(102) 
Hence 
\[ \frac{dL}{d\theta} = \sum_{i,j,t} \alpha_{j,t} (1_{i=j} - \alpha_{i,t}) \cdot \frac{dL}{ds_t} \cdot \partial_2 f(h_t, c_t) \cdot h_j \cdot \frac{\partial e_{i,t}}{\partial \theta} \] 
(103) 
\[ = \sum_{i,j,t} \alpha_{j,t} \cdot \frac{dL}{ds_t} \cdot \partial_2 f(h_t, c_t) \cdot h_j \cdot \left[ \sum_i (1_{i=j} - \alpha_{i,t}) \cdot \frac{\partial e_{i,t}}{\partial \theta} \right] \] 
(104) 
\[ \square \]

Lemma 8. Let us recall that for all \( t \geq 0 \), we have 
\[ h_{t+1} = \phi(V s_t + U x_{t+1} + b) \] 
(105) 
where \( \phi \) is a non-linear activation function, \( V \in \mathbb{R}^{n \times n}, U \in \mathbb{R}^{n \times m} \) and \( b \in \mathbb{R}^n \). Then we have that 
\[ \begin{bmatrix} \frac{dL}{dV} & \frac{dL}{dU} & \frac{dL}{db} \end{bmatrix} = \sum_{t=1}^{T} [s_{t-1}, x_t, 1] \cdot \frac{dL}{dh_t} \cdot \text{diag}(\phi'(\alpha_t)) \] 
(106)
Proof. Let us denote $V^{(t)}$, $U^{(t)}$, $b^{(t)}$ the matrices $V$, $U$, $b$ of $a_{t-1}$ respectively, then
\[
\begin{bmatrix}
\frac{dL}{dV}, \frac{dL}{dU}, \frac{dL}{db}
\end{bmatrix} = \sum_t \begin{bmatrix}
\frac{dL}{dV^{(t)}}, \frac{dL}{dU^{(t)}}, \frac{dL}{db^{(t)}}
\end{bmatrix}
\]
(107)
\[
= \sum_t [s_{t-1}, x_t, 1] \cdot \frac{dL}{da_{t-1}}
\]
(108)
\[
= \sum_t [s_{t-1}, x_t, 1] \cdot \frac{dL}{dh_t} \cdot \frac{dh_t}{da_{t-1}}
\]
(109)
\[
= \sum_t [s_{t-1}, x_t, 1] \cdot \frac{dL}{dh_t} \cdot \text{diag}(\phi'(a_{t-1}))
\]
(110)

Remark 6. Combining the fact that $\frac{dL}{dh_t} = \frac{dL}{ds_t} \frac{ds_t}{dh_t}$, the results from propositions 1 and 2, with lemma 8, we see that attention weights $\alpha_{i,t}$ which are very close to 0, do not contribute to the gradient and the learning of $V$, $U$ and $b$.

Similarly, it follows directly from lemma 7 that attention weights $\alpha_{i,t}$ which are very close to 0, do not contribute to the gradient and the learning of any parameters $\theta$ of the alignment function $e_{i,t} = \alpha(s_{t-1}, h_i, \theta)$. In case we have an alignment function as in remark 5, these parameters are $W_a$, $U_a$ and $v_a$.

If we have the case where one state $h_i$ is such that all attention weights $\alpha_{i,t} \approx 0$ for all $t \geq i$, then we can see that $h_i$ does not contribute to the gradient and learning to any parameters be it parameters from the recurrence or the alignment function.

In practice we have observed that in the majority of tasks, most states $h_i$ fall in either of two categories:

- $\alpha_{i,t}$ is sufficiently bounded away from 0 for most $t \geq i$, and thus contributes to learning. This is what we call a "relevant state”.
- $\alpha_{i,t} \approx 0$ for almost all $t \geq i$, and thus doesn’t contribute much to learning, and the gradient can be approximated by assuming $\alpha_{i,t} = 0$ for all $t \geq i$. This is what he call a "non-relevant state”.

This observation is what lead us to the intuition that we can approximate the gradient, by decomposing it via proposition 7 into gradient paths involving only skip connections between "relevant states".

A.3 Uniform attention case

Remark 7. In this subsection, we are going to assume:

- no non-linearity in the hidden-to-hidden connection: $J_t = V$ for all $t$.
- all assumptions from Remark 3.
- uniform attention: $\alpha_{i,t} = 1/t$ for all $t \geq 1$.

A.3.1 Overview

Remark 8. Recalling corollary 7 together the main proposition 7 form last section, we can hope to simplify these expressions using the new assumptions from the previous remark 7. Recalling
Hence, for our calculations we are going to assume that \( h_j - \frac{1}{t} \sum_{i=1}^{t} h_i \approx 0 \), and thus \( X_{j,t} \approx 0 \) for all \( 1 \leq j \leq t \). Similarly,

\[
\sum_{i=1}^{t} \alpha_{i,t} Y_{i,t} = \sum_{i=1}^{t} \alpha_{i,t} h_i \cdot \left( \frac{\partial e_{i,t}}{\partial s_{t-1}} - \sum_{j=1}^{t} \alpha_{j,t} \cdot \frac{\partial e_{j,t}}{\partial s_{t-1}} \right)
\]

(113)

\[
= \frac{1}{t} \sum_{i=1}^{t} h_i \cdot \left( \frac{\partial e_{i,t}}{\partial s_{t-1}} - \sum_{j=1}^{t} \frac{1}{t} \cdot \frac{\partial e_{j,t}}{\partial s_{t-1}} \right)
\]

(114)

\[
= \frac{1}{t} \sum_{i=1}^{t} h_i \cdot \frac{\partial e_{i,t}}{\partial s_{t-1}} - \frac{1}{t} \sum_{j=1}^{t} \left( \frac{1}{t} \sum_{i=1}^{t} h_i \right) \cdot \frac{\partial e_{j,t}}{\partial s_{t-1}}
\]

(115)

\[
= \frac{1}{t} \sum_{i=1}^{t} h_i \cdot \frac{\partial e_{i,t}}{\partial s_{t-1}} - \frac{1}{t} \sum_{i=1}^{t} \left( \frac{1}{t} \sum_{j=1}^{t} h_j \right) \cdot \frac{\partial e_{i,t}}{\partial s_{t-1}}
\]

(116)

\[
= \frac{1}{t} \sum_{i=1}^{t} \left( h_i - \frac{1}{t} \sum_{j=1}^{t} h_j \right) \cdot \frac{\partial e_{i,t}}{\partial s_{t-1}}
\]

(117)

\[
\approx 0
\]

(118)

Recalling the expression from corollary 1 and that \( f(h_t, c_t) = h_t + c_t \) by remark 3 and that \( J_t = V \) for all \( t \), this will give for all \( k' \geq 0 \)

\[
E_{k'}^{(t)} = \left( \frac{1}{t+k'} + 1_{k'=0} \right) \cdot I
\]

(119)

and for all \( k \geq j \), we get

\[
F_{k+1,j}^{(t)} = \left( \frac{1}{t+k+1} + 1_{k=j} \right) \cdot V
\]

(120)

Hence by recalling proposition 5, the main expression of interest becomes

\[
\frac{ds_{t+k}}{dh_t} = \sum_{s=0}^{k} \xi_{0:k}^{(t)}(s) = \sum_{s=0}^{k} V^s \cdot \chi_{0:k}^{(t)}(s)
\]

(121)

where

\[
\chi_{0:k}^{(t)}(s) \text{ def } \sum_{0 \leq i_1 < \ldots < i_s < k} \left( \frac{1}{t+k} + 1_{k-i_s=1} \right) \cdot \left( \frac{1}{t+i_s} + 1_{i_s-i_{s-1}=1} \right) \cdot \ldots
\]

(122)

\[
\ldots \left( \frac{1}{t+i_2} + 1_{i_2-i_1=1} \right) \cdot \left( \frac{1}{t+i_1} + 1_{i_1=0} \right)
\]

(123)

**Remark 9.** The goal is thus to have a good estimation of the terms

\[
\chi_{0:k}^{(t)}(s)
\]

(124)

in order to then find an asymptotic estimation for

\[
\frac{ds_{t+k}}{dh_t} = \sum_{s=0}^{k} V^s \cdot \chi_{0:k}^{(t)}(s)
\]

(125)

as \( k \to \infty \). In order to do so, we will adopt the following strategy:
Step 1. Estimate the expression
\[
\omega_{t:k}(s) \overset{\text{def}}{=} \sum_{l \leq i_1 < \ldots < i_s < k} \frac{1}{t + i_s} \cdot \frac{1}{t + i_{s-1}} \cdot \ldots \cdot \frac{1}{t + i_2} \cdot \frac{1}{t + i_1}
\] (126)
for all \(s \geq 1\). This will be done in sub-subsection A.3.2.

Step 2. Estimate the expression
\[
\theta_{t:k}(s) \overset{\text{def}}{=} \sum_{l \leq i_1 < \ldots < i_s < k} \left( \frac{1}{t + i_s} + 1_{i_s-1=1} \right) \cdot \left( \frac{1}{t + i_{s-1}} + 1_{i_{s-1}-i_s-2=1} \right) \cdot \ldots
\]
\[
\ldots \cdot \left( \frac{1}{t + i_2} + 1_{i_2-i_1=1} \right) \cdot \left( \frac{1}{t + i_1} + 1_{i_1=0} \right)
\] (127)
for all \(s \geq 1\), because as we will see the expression \(\theta_{t:k}(s)\) can be decomposed into \(\omega_{t:k}(s')\) expressions for \(s \geq s' \geq 1\). This will be done in sub-subsection A.3.3.

Step 3. The final step will consist in putting the results from the two previous sub-subsections together, and getting a final asymptotic estimate for \(ds_t \cdot dh_t\) as \(k \to \infty\), by noting that
\[
\chi_{0:k}(s) = \frac{1}{t + k} \cdot \theta_{0:k}(s) + \frac{1}{t + k - 1} \cdot \theta_{0:k-1}(s - 1) + \ldots
\]
\[
\ldots + \frac{1}{t + k - s + 1} \cdot \theta_{0:k-s+1}(1) + \frac{1}{t + k - s} + 1_{k=s}
\] (129)
This will be treated in sub-subsection A.3.4.

A.3.2 Estimating \(\omega\)

Remark 10. In this sub-subsection we are going to estimate \(\omega_{0:k}(s)\), which is a sum of products of \(s\) distinct factors. The idea will be to start from the expression
\[
\left( \frac{1}{t} + \frac{1}{t + 1} + \ldots + \frac{1}{t + k - 1} \right)^s
\] (131)
and subtract all products containing at least two identical factors, followed by a division by \(s!\).

This approach will be similar in spirit to the inclusion-exclusion principle, with the only difference that the desired term will not computed directly, but instead one first establishes a recursive formula using \(\omega_{0:k}(s')\) with \(s' \leq s\).

Solving this recursive formula will enable us to express \(\omega_{0:k}(s)\) only in terms of \(\left( \frac{1}{t} + \frac{1}{t+1} + \ldots + \frac{1}{t+k-1} \right)^s\). In fact, \(\omega_{0:k}(s)\) will be a polynomial of degree \(s\) in \(\left( \frac{1}{t} + \frac{1}{t+1} + \ldots + \frac{1}{t+k-1} \right)^s\).

We adopt this approach, because we have a very good estimate for
\[
\frac{1}{t} + \frac{1}{t + k - 1} + \ldots + \frac{1}{t + k - 1} = \ln n + \gamma + \varepsilon_n \leq \ln n + 1
\] (132)
Namely, we know that for all \(n\), we have
\[
1 + \frac{1}{2} + \ldots + \frac{1}{n-1} + \frac{1}{n} = \ln n + \gamma + \varepsilon_n \leq \ln n + 1
\] (133)
where $\gamma > \frac{1}{2}$ is the Euler-Mascheroni constant and $\varepsilon_n$ behaves asymptotically as $\frac{1}{2^n}$. In other words,

$$
\frac{1}{t} + \frac{1}{t+1} + \ldots + \frac{1}{t+k-1} = \ln \left( \frac{t+k-1}{t-1} \right) + \varepsilon_{t+k-1} - \varepsilon_{t-1}
$$

(134)

which is equivalent to

$$
\ln \left( \frac{t+k-1}{t-1} \right) \cdot \exp (\varepsilon_{t+k-1} - \varepsilon_{t-1})
$$

(135)

and

$$
= \ln \beta_{t-1,t+k-1}
$$

(136)

where $\beta_{l,l'} = e^{\varepsilon_{l'} - \varepsilon_l}$. In order to reinforce the intuition here, let us imagine that $T = t + k$, then

$$
\ln \beta_{t-1,t+k-1} \sim \ln T
$$

(137)

as $T \to \infty$. Hence we should expect $\omega_{0,k}^{(i)}(s)$ to behave asymptotically as a polynomial of degree $s$ in $\ln T$.

Let us emphasize that we would like to express $\omega_{0,k}^{(i)}(s)$ with as much precision as possible (i.e. not omitting the monomials in $\ln T$ of degree less than $s$), since we would like to later on use this estimate in subsequent steps when summing multiple $\omega_{0,k}^{(i)}(s)$ terms over $s$.

In order to further ease notation, we will simply write $\omega(s)$ for $\omega_{0,k}^{(i)}(s)$, whenever there is no ambiguity.

Finally, for this sub-subsection only we will use the following notation

$$
S_l = \frac{1}{l!} + \frac{1}{(t+1)^l} + \ldots + \frac{1}{(t+k-1)^l}
$$

(138)

for all $l \geq 1$, and keeping in mind that $S_l$ converges as $k \to \infty$, for all $l \geq 2$.

---

**Remark 11.** Let us now build a first intuition on how to apply an inclusion-exclusion-like principle in order to calculate $\omega(s)$ for small $s$.

For $s = 1$:

$$
\omega(1) = S_1
$$

(139)

For $s = 2$:

$$
\omega(2) = \frac{1}{2!} (S_1^2 - S_2)
$$

(140)

Here we expand $S_1^2$, then substract the sum of products of doubles $S_2$, followed by a division of $2! = 2$ to divide out the number of permutations.

For $s = 3$: first we need to substract the sum of products of triples $S_3$, and then the sum products where exactly two factors are identical $S_2 \cdot \omega(1) - S_3$. The latter appears $(\frac{3}{2!}) = \frac{3!}{2!} = 3$ times in the expansion of $S_3^3$. Similarly, we need to divide out the number of permutations $3!$. Hence

$$
\omega(3) = \frac{1}{3!} [S_1^3 - S_3 - 3 \cdot (S_2 \cdot \omega(1) - S_3)] = \frac{S_1^3}{3!} - \frac{1}{2} S_2 \cdot \omega(1) + \frac{1}{3} S_3
$$

(141)

Let us form now on denote $(3)$ for the sum of products of triples, and $(2, 1)$ the sum of products where exactly two factors are the same.
More generally we would denote

\[(j_1, j_2, \ldots, j_k)\]  \hspace{1cm} (142)

with \(j_1 \geq j_2 \geq \ldots \geq j_k \geq 1\), to denote the sum of products where one factor appears exactly \(j_1\) times, another factor (distinct from the previous one!) appears exactly \(j_2\) times, and another factor (distinct from the previous two!) appears exactly \(j_3\) times, etc. This leaves us with exactly \(k\) distinct factors each having multiplicity \(j_1, j_2, \ldots, j_k\) respectively. This sum appears with

\[
\binom{s}{j_1, j_2, \ldots, j_k} = \frac{s!}{j_1! \cdot j_2! \cdot \cdots \cdot j_k!}
\]  \hspace{1cm} (143)

repetitions in the expansion of \(S_1^s\), where \(s = j_1 + j_2 + \ldots + j_k\).

For \(s = 4\): when expanding \(S_1^4\), we need to take into account

- \((4) = S_4\) with \(\binom{4}{1} = \frac{4!}{1!} = 1\) repetition.
- \((3, 1) = S_3 \cdot \omega(1) - S_4\) with \(\binom{4}{3, 1} = \frac{4!}{3!1!} = 4\) repetitions.
- \((2, 2) = S_2^2 - S_4\) with \(\binom{4}{2, 2} = \frac{4!}{2!2!} = 6\) repetitions.
- \((2, 1, 1) = S_2 \cdot \omega(2) - (3, 1) = S_2 \cdot \omega(2) - S_3 \cdot \omega(1) + S_4\) with \(\binom{4}{2, 1, 1} = \frac{4!}{2!1!1!} = 12\) repetitions.

Hence we get

\[
\omega(4) = \frac{1}{4!} \left[ S_1^4 - S_4 - 4 \cdot (S_3 \cdot \omega(1) - S_4) - 6 \cdot (S_2^2 - S_4) \right] 
- 12 \cdot (S_2 \cdot \omega(2) - S_3 \cdot \omega(1) + S_4) 
= \frac{1}{4!} \left[ S_1^4 - 4 \cdot S_3 \cdot \omega(1) + 4 \cdot S_4 - 6 \cdot S_2^2 + 6 \cdot S_4 - 12 \cdot S_2 \omega(2) 
+ 12 \cdot S_3 \cdot \omega(1) - 12 \cdot S_4 
= \frac{1}{4!} \left[ S_1^4 - 12 \cdot S_2 \cdot \omega(2) + 8 \cdot S_3 \cdot \omega(1) - 3 \cdot (S_4 + S_2^2) \right] 
= \frac{S_1^4}{4!} - \frac{S_2}{2} \omega(2) + \frac{S_3}{3} \omega(1) - \frac{(S_4 + S_2^2)}{8} \right] 
\]  \hspace{1cm} (144-149)

Notice how, as we progress with higher values of \(s\), we build a recursive formula in \(\omega(s')\) with \(s' \leq s\).

**Intuition.** Note that the coefficient of \(\omega(2)\) for \(s = 4\), is the same as the coefficient for \(\omega(1)\) for \(s = 3\), and is the same as the 'constant term' for \(s = 2\). Similarly, the coefficient of \(\omega(1)\) for \(s = 4\) is the same as the 'constant term' for \(s = 3\). (By convention here, we don’t consider the terms \(\frac{S_1^4}{4!}\) to not be part of the 'constant term'.)

Hence, in the recursive formula for \(\omega(s)\), we would expect the coefficient of \(\omega(s')\) with \(s' < s\) to be equal to the 'constant term' in the formula for \(\omega(s - s')\).

**Notation.** For all \(s \geq l \geq 0\), let us denote \(\delta_{s,l}\) to be the coefficient of the term \(\omega(l)\) in the recursive formula for \(\omega(s)\). By convention, we denote \(\delta_{s,0}\) for the 'constant term' in the recursive formula for \(\omega(s)\). Hence for all \(s \geq 1\), we have

\[
\omega(s) = \frac{S_1^s}{s!} + \delta_{s,s-1} \cdot \omega(s-1) + \delta_{s,s-2} \cdot \omega(s-2) + \ldots + \delta_{s,1} \cdot \omega(1) + \delta_{s,0} 
\]  \hspace{1cm} (150)

**Hypothesis.** The hypothesis will thus rewrite as

\[
\delta_{s,l} = \delta_{s-l,0} 
\]  \hspace{1cm} (151)
for all $s > l \geq 0$, which will prove by induction on $s$ in the next lemma.

**Lemma 9.** Let $s \geq 1$. Then

$$
\omega(s) = \frac{S^s}{s!} + \delta_{1,0} \cdot \omega(s-1) + \delta_{2,0} \cdot \omega(s-2) + \ldots + \delta_{s-1,0} \cdot \omega(1) + \delta_{s,0}
$$

(152)

**Proof.** Let us prove by induction on $s$ that for all $s > l \geq 0$, we have

$$
\delta_{s,l} = \delta_{s-l,0}
$$

(153)

. We already verified the cases $s = 1, 2, 3, 4$ in the previous remark. Thus let us suppose the induction hypothesis is true for $s$, and consider the mapping

$$
\Upsilon : (j_1, j_2, \ldots, j_k) \mapsto (j_1, j_2, \ldots, j_k, 1)
$$

(154)

where $j_1 \geq j_2 \geq \ldots \geq j_k \geq 1$ and $s = j_1 + j_2 + \ldots + j_k$, mapping a partition of $s$ onto a partition of $s + 1$.

If we suppose that $(j_1, j_2, \ldots, j_k)$ consists of exactly $r$ 1’s, then we can write

$$
(j_1, j_2, \ldots, j_k) = c_r \cdot \omega(r) + c_{r-1} \cdot \omega(r-1) + \ldots + c_1 \cdot \omega(1) + c_0
$$

(155)

for some coefficients $c_r, c_{r-1}, \ldots, c_1, c_0$, and with

$$
\frac{s}{j_1, j_2, \ldots, j_k} = \frac{s!}{j_1! \cdot j_2! \cdot \ldots \cdot j_k!}
$$

(156)

repetitions in the expansion of $S^s_1$.

The contribution of $(j_1, j_2, \ldots, j_k)$ to the coefficient $\delta_{s,r'}$ of $\omega(r')$ with $r' \leq r < s$, in the final recursive formula of $\omega(s)$ will be

$$
\frac{c_{r'}}{j_1! \cdot j_2! \cdot \ldots \cdot j_k!}
$$

(157)

(keeping in mind that we are dividing by $s!$ after having done all the substractions from $S^s_1$).

Meanwhile,

$$
(j_1, j_2, \ldots, j_k, 1) = c_r \cdot \omega(r+1) + c_{r-1} \cdot \omega(r) + \ldots + c_1 \cdot \omega(2) + c_0 \cdot \omega(1) + c_0
$$

(158)

for some coefficient $c_0$, with

$$
\frac{s+1}{j_1, j_2, \ldots, j_k, 1} = \frac{(s+1)!}{j_1! \cdot j_2! \cdot \ldots \cdot j_k!}
$$

(159)

repetitions in the expansion of $S^{s+1}_1$.

The contribution of $(j_1, j_2, \ldots, j_k, 1)$ to the coefficient $\delta_{s+1,r'+1}$ of $\omega(r'+1)$ with $r' \leq r < s$, in the final recursive formula of $\omega(s+1)$ will be

$$
\frac{c_{r'}}{j_1! \cdot j_2! \cdot \ldots \cdot j_k!}
$$

(160)

(keeping in mind that we are dividing by $(s+1)!$ after having done all the substractions from $S^{s+1}_1$).

Conversely, the coefficient $\delta_{s+1,r'+1}$ only receives contributions from partitions of $(s+1)$ having at least $(r'+1)$ 1’s, which correspond exactly to the contributions from the partitions of $s$ having at least $r'$ 1’s. Hence

$$
\delta_{s+1,r'+1} = \delta_{s,r'}
$$

(161)

Then by the induction hypothesis, we have $\delta_{s,r'} = \delta_{s-r',0}$. In other words

$$
\delta_{s+1,r'+1} = \delta_{s-r',0}
$$

(162)

which completes the proof by induction. \( \square \)
Remark 12. Note that all the coefficients $\delta_{s,l}$ consist of linear combination of products with factors equal to $S_j$ with $j \geq 2$, which are known to converge as $T \to \infty$. Thus those can be considered constants when doing an asymptotic analysis in the subsequent sub-subsections. Also note that $\delta_{s,s-1} = \delta_{1,0} = 0$.

Proposition 3. For all $s \geq 1$, we have

$$\omega(s) = \sum_{r=0}^{s} \psi_{s-r} \frac{S_r^r}{r!}$$

(163)

where for $l \geq 2$

$$\psi_1 \overset{\text{def}}{=} \sum_{k=1}^{l-1} \sum_{(j_1,j_2,...,j_k) \in \Psi_{1,k}} \delta_{j_1,0} \cdot \ldots \cdot \delta_{j_k,0}$$

(164)

with

$$\Psi_{l,k} \overset{\text{def}}{=} \{(j_1,j_2,...,j_k) \text{ with } j_1 \geq \ldots \geq j_k > 1 \text{ and } j_1 + \ldots + j_k = l\}$$

(165)

and where we define $\psi_0 = 1$ and $\psi_1 = 0$.

Proof. For $l \geq 2$, we have

$$\psi_l = \sum_{k=1}^{l-1} \sum_{(j_1,j_2,...,j_k) \in \Psi_{l,k}} \delta_{j_1,0} \cdot \ldots \cdot \delta_{j_k,0}$$

(166)

$$= \delta_{l,0} + \sum_{k=1}^{l-1} \left( \sum_{j=2}^{l-2} \sum_{(j_2,...,j_k) \in \Psi_{l-1,k-1}} \delta_{j_2,0} \cdot \ldots \cdot \delta_{j_k,0} \cdot \delta_{j_1,0} \right)$$

(167)

$$= \delta_{l,0} + \sum_{j=2}^{l-2} \left( \sum_{k=1}^{l-j} \sum_{(j_2,...,j_k) \in \Psi_{l-1,j-1}} \delta_{j_2,0} \cdot \ldots \cdot \delta_{j_k,0} \cdot \delta_{j_1,0} \right)$$

(168)

$$= \delta_{l,0} + \sum_{j=2}^{l-2} \delta_{j,0} \cdot \left( \sum_{k=1}^{l-j} \sum_{(j_2,...,j_k) \in \Psi_{l-1,j-1}} \delta_{j_2,0} \cdot \ldots \cdot \delta_{j_k,0} \right)$$

(169)

$$= \delta_{l,0} + \sum_{j=2}^{l-2} \delta_{j,0} \cdot \psi_{l-j}$$

(170)

$$= \sum_{j=1}^{l} \delta_{j,0} \cdot \psi_{l-j}$$

(171)

In other words, we have shown that for all $l \geq 2$,

$$\psi_l = \sum_{j=0}^{l-1} \delta_{l-j} \psi_j$$

(173)

Let us now prove the proposition by induction on $s$.

The case $s = 1$ is trivial by the definition of $\psi_0$ and $\psi_1$. 



26
Let us now assume the formula is true for $s$, and let us prove it for $s+1$. By the previous lemma, we know that

$$
\omega(s + 1) = \frac{S^{s+1}_1}{(s+1)!} + \sum_{l=1}^s \delta_{s+1-l,0} \cdot \omega(l) + \delta_{s+1,0}
$$

(174)

$$
= \frac{S^{s+1}_1}{(s+1)!} + \sum_{l=1}^s \delta_{s+1-l,0} \cdot \left( \sum_{r=0}^l \psi_{l-r} \cdot \frac{S^r_1}{r!} \right) + \delta_{s+1,0}
$$

(175)

$$
= \frac{S^{s+1}_1}{(s+1)!} + \sum_{l=1}^s \sum_{r=0}^l \delta_{s+1-l,0} \cdot \psi_{l-r} \cdot \frac{S^r_1}{r!} + \delta_{s+1,0}
$$

(176)

$$
= \frac{S^{s+1}_1}{(s+1)!} + \sum_{r=0}^s \sum_{l=r}^s \delta_{s+1-l,0} \cdot \psi_{l-r} \cdot \frac{S^r_1}{r!}
$$

(177)

$$
= \frac{S^{s+1}_1}{(s+1)!} + \sum_{r=0}^s \sum_{l'=0}^{s-r} \delta_{s+1-r-l',0} \cdot \psi_{l'+r} \cdot \frac{S^r_1}{r!}
$$

(178)

$$
= \frac{S^{s+1}_1}{(s+1)!} + \sum_{r=0}^{s+1} \psi_{s+1-r} \cdot \frac{S^r_1}{r!}
$$

(179)

$$
= \sum_{r=0}^{s+1} \psi_{s+1-r} \cdot \frac{S^r_1}{r!}
$$

(180)

completing the proof by induction.

Remark 13. Hence we have shown that for all $s \geq 1$

$$
\omega(s) = \sum_{r=0}^s \psi_{s-r} \frac{S^r_1}{r!} = \frac{S^s_1}{s!} + \sum_{r=0}^{s-2} \psi_{s-r} \frac{S^r_1}{r!}
$$

(182)

or in other words

$$
\omega(s) = \frac{(\ln \beta_{t-1,t+k-1})^s}{s!} + \sum_{r=0}^{s-2} \psi_{s-r} \left( \frac{(\ln \beta_{t-1,t+k-1})^r}{r!} \right) \sim \frac{(\ln T)^s}{s!} + \sum_{r=0}^{s-2} \psi_{s-r} \left( \frac{(\ln T)^r}{r!} \right)
$$

(183)

as $t + k = T \to \infty$, which is roughly the polynomial in $\ln T$ of degree $s$ we were anticipating.

A.3.3 Estimating $\theta$

Remark 14. Let us now recall the definition for all $s \geq 1$,

$$
\theta_{t,k}^{(s)}(s) \text{ def } \sum_{l \leq i_0 < \ldots < i_s < k} \left( \frac{1}{t+i_s} + 1_{i_s-i_{s-1}=1} \right) \cdot \left( \frac{1}{t+i_{s-1}} + 1_{i_{s-1}-i_{s-2}=1} \right) \cdot \ldots
$$

(184)

$$
\ldots \cdot \left( \frac{1}{t+i_0} + 1_{i_0-i_1=1} \right) \cdot \left( \frac{1}{t+i_1} + 1_{i_1=0} \right)
$$

(185)

which we would like to estimate using $\omega_{t,k}^{(1)}(s)$.

In order to build a first intuition, let us look at how it plays out for small values for $s$. 
Notation. In this subsection we omit the superscript \((t)\) notation because there is no ambiguity. We will also occasionally do the abuse of notation and assume \(\omega_{l,k}(0) = 1\) for all \(l < k\).

For \(s = 1\), we get

\[
\theta_{0;k}(1) = 1 + \omega_{0;k}(1)
\]  

(186)

For \(s = 2\), we get

\[
\theta_{0;k}(2) = 1 + \omega_{1;k}(1) + \omega_{0;k-1}(1) + \omega_{0;k}(2)
\]  

(187)

In what follows, we will use the following recursive formula quite frequently

\[
\theta_{0;k}(s + 1) = \theta_{0;k-1}(s) + \sum_{j=s}^{k-1} \frac{1}{t+j} \theta_{0;j}(s)
\]

(188)

Hence for \(s = 3\), we get

\[
\theta_{0;k}(3) = 1 + \omega_{1;k-1}(1) + \omega_{0;k-2}(1) + \omega_{2;k}(1) + \omega_{0;k-1}(2)
\]  

(189)

\[
+ \omega_{1;k}(2) + \sum_{j=2}^{k-1} \frac{\omega_{0;j-1}(1)}{t+j} + \omega_{0;k}(3)
\]

(190)

Now let us further observe that for all \(s \geq 1\) and \(0 \leq r \leq l\), we have

\[
\omega_{l+r;k+r}(s) \leq \omega_{l;k}(s) \leq \omega_{l-r;k-r}(s)
\]

(191)

This implies that

\[
1 + 2 \cdot \omega_{1;k}(1) + \omega_{0;k}(2) \leq \theta_{0;k}(2) \leq 1 + 2 \cdot \omega_{0;k-1}(1) + \omega_{0;k}(2)
\]  

(192)

and, similarly,

\[
1 + 3 \cdot \omega_{2;k}(1) + 3 \cdot \omega_{1;k}(2) + \omega_{0;k}(3) \leq \theta_{0;k}(3) \leq 1 + 3 \cdot \omega_{0;k-2}(1) + 3 \cdot \omega_{0;k-1}(2) + \omega_{0;k}(3)
\]  

(193)

Hypothesis. We can thus see the binomial coefficients arising, and we would expect that in general, we have

\[
\sum_{r=0}^{s} \binom{s}{r} \cdot \omega_{0;k-s+r}(r) \geq \theta_{0;k}(s) \geq \sum_{r=0}^{s} \binom{s}{r} \cdot \omega_{s-r;k}(r)
\]

(194)

Lemma 10. For all \(k \geq s \geq 1\), we have

\[
\sum_{r=0}^{s} \binom{s}{r} \cdot \omega_{0;k-s+r}^{(t)}(r) \geq \theta_{0;k}^{(t)}(s) \geq \sum_{r=0}^{s} \binom{s}{r} \cdot \omega_{s-r;k}^{(t)}(r)
\]

(195)

Proof. Let us prove this lemma by induction on \(s\). The cases \(s = 1, 2, 3\) have already been treated in the previous remark.

Let us now assume that the claim holds for \(s\), and prove it for \(s + 1\) using the recursive formula

\[
\theta_{0;k}(s + 1) = \theta_{0;k-1}(s) + \sum_{j=s}^{k-1} \frac{1}{t+j} \theta_{0;j}(s)
\]

(196)
For the lower bound, using the induction hypothesis, we get

\[
\theta_{0:k}(s + 1) \geq \sum_{r=0}^{s} \binom{s}{r} \cdot \omega_{s-r:k-1}(r) + \sum_{j=s}^{k-1} \sum_{r=0}^{s} \frac{1}{l + j} \cdot \omega_{s-r:j}(r)
\]

(197)

\[
= \sum_{r=0}^{s} \binom{s}{r} \cdot \omega_{s-r:k-1}(r) + \sum_{r=0}^{s} \binom{s}{r} \cdot \sum_{j=s}^{k-1} \frac{1}{l + j} \cdot \omega_{s-r:j}(r)
\]

(198)

\[
= \sum_{r=0}^{s} \binom{s}{r} \cdot \omega_{s-r:k-1}(r) + \sum_{r=0}^{s} \binom{s}{r} \cdot \omega_{s-r:k}(r + 1)
\]

(199)

\[
= \sum_{r=0}^{s} \binom{s}{r} \cdot \omega_{s-r:k-1}(r) + \sum_{r=1}^{s+1} \binom{s}{r} \cdot \omega_{s-r+1:k}(r)
\]

(200)

\[
= 1 + \omega_{0:k}(s + 1) + \sum_{r=1}^{s+1} \binom{s+1}{r} \cdot \omega_{s-r+1:k}(r)
\]

(201)

\[
= 1 + \omega_{0:k}(s + 1) + \sum_{r=1}^{s+1} \binom{s+1}{r} \cdot \omega_{s-r+1:k}(r)
\]

(202)

\[
= \sum_{r=0}^{s+1} \binom{s+1}{r} \cdot \omega_{s-r+1:k}(r)
\]

(203)

For the upper bound, using the induction hypothesis, we get

\[
\theta_{0:k}(s + 1) \leq \sum_{r=0}^{s} \binom{s}{r} \cdot \omega_{0:k-1-(s-r)}(r) + \sum_{j=s}^{k-1} \sum_{r=0}^{s} \frac{1}{l + j} \cdot \omega_{0:j-(s-r)}(r)
\]

(205)

\[
= \sum_{r=0}^{s} \binom{s}{r} \cdot \omega_{0:k-1-(s-r)}(r) + \sum_{r=0}^{s} \binom{s}{r} \cdot \sum_{j=s}^{k-1} \frac{1}{l + j} \cdot \omega_{0:j-(s-r)}(r)
\]

(206)

\[
\leq \sum_{r=0}^{s} \binom{s}{r} \cdot \omega_{0:k-1-(s-r)}(r) + \sum_{r=0}^{s} \binom{s}{r} \cdot \sum_{j=s}^{k-1} \frac{1}{l + j - (s - r)} \cdot \omega_{0:j-(s-r)}(r)
\]

(207)

\[
= \sum_{r=0}^{s} \binom{s}{r} \cdot \omega_{0:k-1-(s-r)}(r) + \sum_{r=0}^{s} \binom{s}{r} \cdot \sum_{j=r}^{k-1} \frac{1}{l + j'} \cdot \omega_{0:j'}(r)
\]

(208)

\[
= \sum_{r=0}^{s} \binom{s}{r} \cdot \omega_{0:k-1-(s-r)}(r) + \sum_{r=0}^{s} \binom{s}{r} \cdot \omega_{0:k-(s-r)}(r + 1)
\]

(209)

\[
= \sum_{r=0}^{s} \binom{s}{r} \cdot \omega_{0:k-1-(s-r)}(r) + \sum_{r=1}^{s+1} \binom{s}{r} \cdot \omega_{0:k-(s+1-r)}(r)
\]

(210)

\[
= 1 + \omega_{0:k}(s + 1) + \sum_{r=1}^{s+1} \left[ \binom{s}{r} + \binom{s}{r-1} \right] \cdot \omega_{0:k-(s+1-r)}(r)
\]

(211)

\[
= 1 + \omega_{0:k}(s + 1) + \sum_{r=1}^{s+1} \binom{s+1}{r} \cdot \omega_{0:k-(s+1-r)}(r)
\]

(212)

\[
= \sum_{r=0}^{s+1} \binom{s+1}{r} \cdot \omega_{0:k-(s+1-r)}(r)
\]

(213)

completing the proof by induction. □
Remark 15. Let us recall that
\[ \omega_{l,k}(r) = \sum_{q=0}^{r} \psi_{r-q} \frac{(\ln \beta_{t+i-t+k-1})^q}{q!} \]  
(215)

Thus the difference between the upper-bound and the lower-bound becomes
\[ \sum_{r=0}^{s} \binom{s}{r} [\omega_{l,k-(s-r)}(r) - \omega_{s-r,k}(r)] = \sum_{r=0}^{s} \binom{s}{r} \sum_{q=0}^{r} \psi_{r-q} \frac{(\ln \beta_{t+i-t+k-(s-r)-1})^q - (\ln \beta_{t+i+s-r-1,t+k-1})^q}{q!} \]  
(216)

which converges to zero as \( T = t + k \to \infty \).

A.3.4 Putting it all together

Remark 16. Now it is time to turn to \( \chi_{0:k}^{(t)}(s) \) and finally put it all together, so that we can finally estimate
\[ \frac{d s_{t+k}}{d h_t} = \sum_{s=0}^{k} V^s \cdot \chi_{0:k}^{(t)}(s) \]  
(217)

and get the asymptotic estimate when \( T = t + k \to \infty \).

Let us recall that
\[ \chi_{0:k}^{(t)}(s) = \frac{1}{t+k} \cdot \theta_{0:k}^{(t)}(s) + \frac{1}{t+k-1} \cdot \theta_{0:k-1}^{(t)}(s - 1) + \ldots \]  
(218)
\[ \ldots + \frac{1}{t+k-s+1} \cdot \theta_{0:k-s+1}^{(t)}(1) + \frac{1}{t+k-s} \cdot \theta_{0:k-s}^{(t)}(0) \]  
(219)

Using the abuse of notation \( \theta_{l:k}^{(t)}(0) = 1 \) for \( l < k \), we can rewrite it as follows
\[ \chi_{0:k}^{(t)}(s) = 1_{k=s} + \sum_{i=0}^{s} \frac{1}{t+k-i} \cdot \theta_{0:k-i}(s - i) \]  
(220)

The idea is to use the inequality from lemma 10 and get a similar result for \( \chi_{0:k}^{(t)}(s) \), then show that the lower and upper bound are no more than \( O(1/T) \) apart, thus enabling us to eventually get an asymptotic estimate for \( \frac{d s_{t+k}}{d h_t} \).

We are also omitting the superscript \( (t) \) notation here because of lack of ambiguity.

Lemma 11. For all \( s \geq 0 \) and \( k \geq 1 \), we have
\[ 1_{k=s} + \frac{1}{t+k} \cdot \sum_{r=0}^{s} \binom{s+1}{r+1} \cdot \omega_{s-r,k}(r) \leq \chi_{0:k}^{(t)}(s) \leq 1_{k=s} + \frac{1}{t+k-s} \cdot \sum_{r=0}^{s} \binom{s+1}{r+1} \cdot \omega_{0,k-(s-r)}(r) \]  
(221)
Proof. Using the upper-bound of lemma 10, we get

\[
\chi_{0,k}(s) = 1_{k=s} + \sum_{i=0}^{s} \frac{1}{t + k - i} \cdot \theta_{0,k-i}(s-i)
\]

(222)

\[
\leq 1_{k=s} + \sum_{i=0}^{s} \frac{1}{t + k - i} \cdot \sum_{r=0}^{s-i} \binom{s-i}{r} \cdot \omega_{0,k-s+r}(r)
\]

(223)

\[
\leq 1_{k=s} + \frac{1}{t+k-s} \sum_{i=0}^{s} \sum_{r=0}^{s-i} \binom{s-i}{r} \cdot \omega_{0,k-s+r}(r)
\]

(224)

\[
= 1_{k=s} + \frac{1}{t+k-s} \sum_{r=0}^{s} \sum_{i=0}^{s-r} \binom{s-r}{r} \cdot \omega_{0,k-s+r}(r)
\]

(225)

\[
= 1_{k=s} + \frac{1}{t+k-s} \sum_{r=0}^{s} \binom{s+1}{r+1} \cdot \omega_{0,k-s+r}(r)
\]

(226)

Similarly, using the lower-bound of lemma 10, we get

\[
\chi_{0,k}(s) = 1_{k=s} + \sum_{i=0}^{s} \frac{1}{t + k - i} \cdot \theta_{0,k-i}(s-i)
\]

(227)

\[
\geq 1_{k=s} + \sum_{i=0}^{s} \frac{1}{t + k - i} \cdot \sum_{r=0}^{s-i} \binom{s-i}{r} \cdot \omega_{s-r,k}(r)
\]

(228)

\[
\geq 1_{k=s} + \frac{1}{t+k} \sum_{i=0}^{s} \sum_{r=0}^{s-i} \binom{s-i}{r} \cdot \omega_{s-r,k}(r)
\]

(229)

\[
= 1_{k=s} + \frac{1}{t+k} \sum_{r=0}^{s} \sum_{i=0}^{s-r} \binom{s-r}{r} \cdot \omega_{s-r,k}(r)
\]

(230)

\[
= 1_{k=s} + \frac{1}{t+k} \sum_{r=0}^{s} \binom{s+1}{r+1} \cdot \omega_{s-r,k}(r)
\]

(231)

Lemma 12. For all \( s \geq 0 \), we have

\[
\chi_{0,k}(s) = 1_{k=s} + \frac{1}{t+k} \left[ \sum_{r=0}^{s} \binom{s+1}{r+1} \cdot \omega_{s-r,k}(r) \right] + \Theta \left( \frac{1}{t+k} \right)
\]

(232)

for all large enough \( k > 1 \), and where the implicit constants from the \( \Theta(\cdot) \) notation are dependent on \( s \).

Proof. Building on the previous lemma 11 and substracting the lower bound from the upper bound, we get

\[
\sum_{r=0}^{s} \binom{s+1}{r+1} \frac{\omega_{0,k-s+r}(r) - \omega_{s-r,k}(r)}{t+k-s} = \sum_{r=0}^{s} \sum_{q=0}^{r} \frac{(s+1)}{(r+1)} \cdot \frac{(ln(\beta_{t-1,t+k-s+r-1})^q - (ln(\beta_{t+s-r-1,t+k-1})^q)}{t+k}
\]

(233)

When assuming that for large \( k \), we have

\[
(ln(\beta_{t-1,t+k-s+r-1})^q \approx (ln(\beta_{t+s-r-1,t+k-1})^q
\]

(234)
then
\[
\frac{(\ln \beta_{t-1,t+k-s+r-1})^q}{t+k-s} - \frac{(\ln \beta_{t+s-r+1,t+k})^q}{t+k} \approx \frac{1}{t+k} \cdot \left[ \frac{s}{t+k-s} \cdot (\ln \beta_{t-1,t+k-s+r-1})^q \right]
\]
\[
\leq \frac{1}{t+k} \cdot \left[ \frac{s}{t+k-s} \cdot (\ln \beta_{t-1,t+k-s+r-1})^s \right]
\]
\[
\leq \frac{\tau_s}{t+k}
\]
for some \( \tau_s > 0 \) depending on \( s \), for all sufficiently large \( k \).

In other words, we have
\[
\sum_{r=0}^{s} \left( \frac{s+1}{r+1} \right) \cdot \left[ \frac{\omega_{0,k-s+1}(r)}{t+k-s} - \frac{\omega_{s-r,k}(r)}{t+k} \right] \leq \frac{\tau_s}{t+k}
\]
for for some \( \tau_s > 0 \) depending on \( s \), for all sufficiently large \( k \).

Meanwhile, for all large enough \( k \), we have
\[
\sum_{r=0}^{s} \frac{s+1}{r+1} \cdot \left[ \frac{\omega_{0,k-s+1}(r)}{t+k-s} - \frac{\omega_{s-r,k}(r)}{t+k} \right] \approx \frac{s}{(t+k)(t+k-s)} \cdot \sum_{r=0}^{s} \sum_{q=0}^{r} \left( \frac{s+1}{q+1} \right) \psi_{r-q} \cdot (\ln \beta_{t-1,t+k-s+r-1})^q
\]
\[
\geq \frac{\tau'_s}{(t+k)^2} \cdot \sum_{r=0}^{s} \sum_{q=0}^{r} \left( \frac{s+1}{q+1} \right) \psi_{r-q} \cdot (\ln \beta_{t-1,t+k-s+r-1})^q
\]
\[
\geq \frac{\tau'_s}{(t+k)^2} \cdot \sum_{r=0}^{s} \sum_{q=0}^{r} \psi_{r-q} \cdot (\ln \beta_{t-1,t+k-s+r+q-1})^q
\]
\[
= \frac{\tau'_s}{(t+k)^2} \cdot \sum_{q=0}^{s-q} \sum_{r=0}^{s-q} \psi_{r'} \cdot (\ln (t+k))^{q}\frac{(\ln (t+k))^q}{q!}
\]
\[
\geq \frac{\tau''_s}{(t+k)^2} \cdot \sum_{q=0}^{s-q} \frac{(\ln (t+k))^q}{q!}
\]
\[
\geq \frac{\tau''_s \cdot \exp[\ln (t+k)]}{(t+k)^2}
\]
\[
= \frac{\tau''_s}{t+k}
\]
for some \( \tau'_s, \tau''_s > 0 \) depending on \( s \).

**Proposition 4.** If \( V \) is a normal matrix with eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_n \) of modulus smaller than 1, then
\[
\frac{d\tau_T}{dh_t} = P\Lambda_T P^*
\]
where \( P^* \) is the conjugate transpose of the unitary matrix \( P \) (independent of \( T \)) and where \( \Lambda_T \) is a diagonal matrix satisfying
\[
(\Lambda_T)_{ii} \sim T^{-1} \cdot c + T^{\lambda_i-1} \cdot c'
\]
for some positive real constants \( c, c' \), as \( T \to \infty \).

32
Proof. Let \( V = \Lambda P^* \) be the Schur decomposition of \( V \), with \( \Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \). Note that since we supposed that \( V \) is normal, we thus have that the Schur matrix \( \Lambda \) is indeed diagonal and is composed of the eigenvalues on the diagonal.

Based on lemma 12 one can show that there exists a function \( g : \mathbb{N} \to \mathbb{R}_0^+ \) such that

\[
\chi_{0,k}(s) = 1_{k=s} + \frac{1}{t + k} \left[ \sum_{r=0}^{s} \binom{s+1}{r+1} \cdot \omega_{s-r+1}(r) + g(s) \right]
\]

Thus

\[
\frac{d \chi_{s+k}}{dh_t} = \sum_{s=0}^{k} V^s \cdot \chi_{0,k}(s)
\]

\[
= \frac{1}{t + k} \left[ \sum_{s=0}^{k} g(s) \cdot V^s + \sum_{s=0}^{k} \sum_{r=0}^{s} \binom{s+1}{r+1} \cdot \omega_{s-r+1}(r) \cdot V^s \right]
\]

\[
= \frac{1}{t + k} \left[ \sum_{s=0}^{k} g(s) \cdot V^s + \sum_{s=0}^{k} \sum_{r=0}^{s} \sum_{q=0}^{r} \binom{s+1}{r+1} \cdot \psi_{s-r} \left( \frac{\ln \beta_{s-r+1,t+k-1} q}{q!} \right) \cdot V^s \right]
\]

Since the eigenvalues of \( V \) are of modulus smaller than 1, we can assume that there exists a constant \( d > 0 \) (dependent on the choice of eigenvalues of \( V \)) such that for all \( k > d \) we have \( V^k \approx 0 \).

Furthermore since \( V^m = (\Lambda P^*)^m = \Lambda^m P^* \) for all \( m \in \mathbb{N}_0 \), while keeping in mind that we pick \( T = t + k \), we can write

\[
\Lambda_T = \frac{1}{T} \left[ \sum_{s=0}^{d} g(s) \cdot \Lambda^s + \sum_{s=0}^{d} \sum_{q=0}^{s} \sum_{r=0}^{s} \binom{s+1}{r+1} \cdot \psi_{s-r} \left( \frac{\ln \beta_{s-r+1,t+k-1} q}{q!} \right) \cdot \Lambda^s \right]
\]

\[
= \frac{1}{T} \left[ \sum_{s=0}^{d} g(s) \cdot \Lambda^s + \sum_{q=0}^{d} \sum_{s=0}^{d-q} \sum_{r=0}^{s} \binom{s+1}{r+1} \cdot \psi_{s-r} \left( \frac{\ln \beta_{s-r+1,t+k-1} q}{q!} \right) \cdot \Lambda^s \right]
\]

\[
= \frac{1}{T} \left[ \sum_{s=0}^{d} g(s) \cdot \Lambda^s + \sum_{q=0}^{d} \sum_{s'=0}^{d-q} \sum_{r'=0}^{s'} \binom{s'+q+1}{r'+q+1} \cdot \psi_{r'} \left( \frac{\Lambda \ln \beta_{s'-r'+1,t+k-1} q}{q!} \right) \cdot \Lambda^{s'} \right]
\]

\[
\approx \frac{1}{T} \left[ \sum_{s=0}^{d} g(s) \cdot \Lambda^s + \sum_{q=0}^{d} \sum_{s'=0}^{d-q} \sum_{r'=0}^{s'} \binom{s'+q+1}{r'+q+1} \cdot \psi_{r'} \left( \frac{\Lambda \ln T q}{q!} \right) \cdot \Lambda^{s'} \right]
\]

\[
= \frac{1}{T} \left[ \sum_{s=0}^{d} g(s) \cdot \Lambda^s \right] + \frac{1}{T} \left[ \sum_{q=0}^{d} \left( \frac{\Lambda \ln T q}{q!} \right) \cdot \left( \sum_{s'=0}^{d-q} \sum_{r'=0}^{s'} \binom{s'+q+1}{r'+q+1} \cdot \psi_{r'} \cdot \Lambda^{s'} \right) \right]
\]

\[
\approx \frac{1}{T} \left[ \sum_{s=0}^{d} g(s) \cdot \Lambda^s \right] + \frac{1}{T} \exp (\Lambda \cdot \ln T) \cdot (c_0 + c_1 \cdot \Lambda + \ldots + c_d \cdot \Lambda^d)
\]

\[
\sim \frac{c'}{T} + c' \exp (\Lambda \cdot \ln T)
\]

for some positive constants \( c', c, c_0, c_1, \ldots, c_d \).
Hence

\[(\Lambda_T)_{ii} \sim c \cdot T^{-1} + c' \cdot T^\lambda_{i-1}\]  

(263)  

**Theorem 1.** If \(V\) is a normal matrix with eigenvalues of modulus smaller than 1, then

\[\| \frac{d s_T}{d h_t} \| = \Omega(1/T)\]  

(264)  
as \(T \to \infty\). (here \(\| . \|\) is the Frobenius norm.)

**Proof.** Let us start off with the observation that

\[T^{-1} \cdot c + T^\lambda_{i-1} \cdot c' = \Omega \left( T^{-\min\{1,1-\Re(\lambda_i)\}} \right)\]  

(265)  
as \(T \to \infty\). And thus, by using proposition 4, we get

\[\| \frac{d s_T}{d h_t} \| = \Omega(T^{-\eta})\]  

(266)  
where

\[\eta = \min_{i=1,...,n} \{ \min(1, 1 - \Re(\lambda_i)) \} \leq 1\]  

(267)  

**Remark 17.** Note that \(V\) being normal is not a necessary condition for the generality of the theorem to hold. We simply chose \(V\) to be normal in order to make the calculations less cumbersome.

In case \(V\) is non-normal, its Schur matrix \(\Lambda\) becomes triangular instead of diagonal. In fact, if \(t_{i,j}\) are the off-diagonal elements of Schur matrix of \(V\) (with \(i < j\), then

\[\|V\| = \sqrt{\text{Tr}(V^*V)} = \sqrt{\text{Tr}(\Lambda^*\Lambda)} = \sqrt{\sum_{i=1}^n |\lambda_i|^2 + \sum_{i<j} |t_{i,j}|^2} \geq \sqrt{\sum_{i=1}^n |\lambda_i|^2}\]  

(268)  

Thus every lower bound on \(\sqrt{\sum_{i=1}^n |\lambda_i|^2}\) induces a lower bound on \(\|V\|\), and in particular an asymptotic lower bound on the modulus of one of the eigenvalues of \(\frac{d s_T}{d h_t}\) induces an asymptotic lower bound on \(\| \frac{d s_T}{d h_t} \|\).

**A.4 Sparse relevance case with bounded dependency depth**

**Remark 18.** Similarly to remark 7, we are going to assume for this subsection:

- no non-linearity in the hidden-to-hidden connection: \(J_t = V\) for all \(t\).
- all assumptions from Remark 3.
- \(\kappa\)-sparse attention: for each \(t \geq 1\), there are at most \(\kappa \leq t\) values for \(i\) such that \(\alpha_{i,t} \neq 0\).
  (Let us define \(\kappa_t \overset{\text{def}}{=} |\{i \text{ such that } \alpha_{i,t} \neq 0\}|\) )
- uniform attention across attended states: for all \(t \geq 1\), and all \(i \leq t\) such that \(\alpha_{i,t} \neq 0\), we have \(\alpha_{i,t} = 1/\kappa_i \geq 1/\kappa\).
Remark 19. Similarly to remark 8, let us recall that

\[ X_{i,t} = \left( h_i - \sum_{j=1}^{t} \alpha_{j,t} h_j \right) \cdot \frac{\partial e_{i,t}}{\partial h_i} \quad (269) \]

and that

\[ \sum_{i=1}^{t} \alpha_{i,t} Y_{i,t} = \sum_{i=1}^{t} \alpha_{i,t} \cdot h_i \cdot \frac{\partial e_{i,t}}{\partial s_{t-1}} - \sum_{i=1}^{t} \alpha_{i,t} \left( \sum_{j=1}^{t} \alpha_{j,t} \cdot h_j \right) \cdot \frac{\partial e_{i,t}}{\partial s_{t-1}} \quad (270) \]

Thus we can see that both expressions have the common factor \( \left( h_i - \sum_{j=1}^{t} \alpha_{j,t} h_j \right) \).

By defining

\[ A_t \overset{\text{def}}{=} \{ i \text{ such that } \alpha_{i,t} \neq 0 \} \quad (273) \]

we see that

\[ h_i - \sum_{j=1}^{t} \alpha_{j,t} h_j = h_i - \frac{1}{\kappa_t} \sum_{j \in A_t} h_j \quad (274) \]

and we are going to assume for the sake of simplicity that

\[ h_i \approx \frac{1}{\kappa_t} \sum_{j \in A_t} h_j \quad (275) \]

and thus \( X_{i,t} \approx 0 \) and \( \sum_{i=1}^{t} \alpha_{i,t} Y_{i,t} \approx 0 \).

Recalling the expression from corollary 7 and that \( f(h_t, c_t) = h_t + c_t \) by remark 3 and that \( J_t = V \) for all \( t \), this will give for all \( k' \geq 0 \)

\[ E_{k'}^{(t)} = \left( \frac{1_{i \in A_t} + k'}{\kappa_t + k'} + 1_{k'=0} \right) \cdot I \quad (276) \]

and for all \( k \geq j \), we get

\[ F_{k+1,j}^{(t)} = \left( \frac{1_{i+j+1 \in A_{t+k+1}}}{\kappa_{t+k+1} + 1_{k=j}} + 1_{k'=0} \right) \cdot V \quad (277) \]

Hence by recalling proposition 2, the main expression of interest becomes

\[ \frac{ds_{t+k}}{dh_t} = \sum_{s=0}^{k} \xi_{0:k}(s) = \sum_{s=0}^{k} V^s \cdot \chi_{0:k}(s) \quad (278) \]

where

\[ \chi_{0:k}(s) \overset{\text{def}}{=} \sum_{0 \leq i_1 < \ldots < i_s < k} \left( \frac{1_{i_1+i_2+1 \in A_{t+k}}}{\kappa_{t+k}} + 1_{k-i_s=1} \right) \cdot \left( \frac{1_{i_s-i_{s-1}=1}}{\kappa_{t+i_s} + 1_{i_s-i_{s-1}=1}} \right) \cdot \ldots \cdot \left( \frac{1_{i_t+i_1+1 \in A_{t+i_2}}}{\kappa_{t+i_2}} + 1_{i_2-i_1=1} \right) \cdot \left( \frac{1_{i_t \in A_{t+i_1}}}{\kappa_{t+i_1} + 1_{i_t=0}} \right) \quad (279) \]

\[ \ldots \cdot \left( \frac{1_{i_t+i_1+1 \in A_{t+i_2}}}{\kappa_{t+i_2}} + 1_{i_2-i_1=1} \right) \cdot \left( \frac{1_{i_t \in A_{t+i_1}}}{\kappa_{t+i_1} + 1_{i_t=0}} \right) \quad (280) \]

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Remark 20. Let us now have a look at how we could potentially simplify the analysis of \( \chi_{0:k}^{(t)}(s) \).

If we further assume \( V \) to be normal we can write

\[
V = P \Lambda P^* \tag{281}
\]

where \( \Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \) is the diagonal matrix consisting of the eigenvalues of \( V \), and \( P^* \) is the conjugate transpose of \( P \).

Hence, we can rewrite

\[
\frac{ds_{t+k}}{dh_t} = \sum_{s=0}^{k} V^s \cdot \chi_{0:k}^{(t)}(s) = P \cdot \left( \sum_{s=0}^{k} \Lambda^s \cdot \chi_{0:k}^{(t)}(s) \right) \cdot P^* \tag{282}
\]

We can therefore see that the asymptotic behaviour of \( \frac{ds_{t+k}}{dh_t} \) depends largely on the asymptotic behaviour of the modulus of the complex-valued polynomial

\[
p_{0:k}(\lambda) \overset{\text{def}}{=} \sum_{s=0}^{k} \lambda^s \cdot \chi_{0:k}^{(t)}(s) \tag{283}
\]

and thus

\[
\left\| \frac{ds_{t+k}}{dh_t} \right\| = \sqrt{\sum_{i=1}^{n} |p_{0:k}(\lambda_i)|^2} \tag{284}
\]

where \( \| \cdot \| \) is the Frobenius norm. Hence in order to prove that

\[
\left\| \frac{ds_{t+k}}{dh_t} \right\| = \Omega(1/\kappa^d) \tag{285}
\]

for all large enough \( k \) (note that \( k \) and \( \kappa \) here are two different symbols), it would suffice to show that there exists \( \lambda \in \{ \lambda_1, \ldots, \lambda_n \} \) such that, for all large enough \( k \), we have

\[
|p_{0:k}(\lambda)| = \Omega(1/\kappa^d) \tag{286}
\]

For simplicity we are going to assume that for all \( t \), we have \( \kappa_t = \kappa \).

Let us further define for all \( s \geq 1, \)

\[
f_{0:k}^{(s)}(i_1, \ldots, i_s) \overset{\text{def}}{=} \begin{cases} \left( \frac{1_{t+i_1+1 \in A_{t+k}}}{\kappa_{t+i_1}} + 1_{k-i_1=1} \right) \cdot \left( \frac{1_{t+i_2+1 \in A_{t+i_2}}}{\kappa_{t+i_2}} + 1_{i_2-i_1=1} \right) \cdot \ldots \\ \ldots \cdot \left( \frac{1_{t+i_s+1 \in A_{t+i_s}}}{\kappa_{t+i_s}} + 1_{i_s-i_{s-1}=1} \right) \cdot \left( \frac{1_{t+i_{s-1}+1 \in A_{t+i_{s-1}}}}{\kappa_{t+i_{s-1}}} + 1_{i_{s-1}=1} \right) \end{cases} \tag{287}
\]

\[
\text{whenever } (i_1, \ldots, i_n) \text{ satisfies } 0 \leq i_1 < i_2 < \ldots < i_s < k, \text{ and } \]

\[
f_{0:k}^{(s)}(i_1, \ldots, i_s) \overset{\text{def}}{=} 0 \tag{288}
\]

otherwise.

Theorem 2. Given the \( \kappa \)-sparsity assumption and the dependency depth \( d \), we have that if \( V \) is normal and has one positive real eigenvalue, then

\[
\left\| \frac{ds_{t+k}}{dh_t} \right\| = \Omega(1/\kappa^d) \tag{290}
\]

for all large enough \( k \).
Proof. By the hypothesis on the dependency depth $d$, we know that for each $k$, there exists $s' \leq d$ and $(i_1, i_2, \ldots, i_{s'})$ such that

$$f_{0;k}^{(s')} (i_1, \ldots, i_{s'}) \geq \left( \frac{1}{\kappa} \right)^{s'+1} \geq \left( \frac{1}{\kappa} \right)^{d+1}$$

(291)

Hence if $\lambda$ is real and positive, then for all large enough $k$, we have

$$|p_{0;k}(\lambda)| = \Omega(1/\kappa^d)$$

(292)

Let us recall that, since $V$ is normal we can write

$$\| ds_{t+k} \| = \sum_{i=1}^n |p_{0;k}(\lambda_i)|^2$$

(293)

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $V$.

Hence, if $V$ has at least one positive real eigenvalue then

$$\| ds_{t+k} \| = \Omega(1/\kappa^d)$$

(294)

for all large enough $k$.

\[ \square \]

**Remark 21.** As already mentioned, since $\kappa$ and $d$ are assumed to be constant, the theorem states that

$$\| ds_{t+k} \| = \Omega(1)$$

(295)

The dependence on $\kappa$ and $d$ was simply given in order to get an intuition on how $\kappa$ and $d$ are influencing the lower bound, and that $d$ has more leverage on the lower bound than $\kappa$.

Regarding the normality of $V$, the same remark can be made as in remark 17.

Then note that if $V$ is a (real) $n \times n$ matrix, with $n$ odd, then we have at least one real eigenvalue. Thus the restriction of having at least one positive real eigenvalue is not that severe.

Further, one can show that the theorem holds in a slightly more general setting where one might not have at least one positive real eigenvalue.

Let us consider the case where $\kappa = 1$, $|\lambda| < 1$ such that we could consider $\lambda^c \approx 0$ for some large enough positive integer $c$, and that all states between $T$ and $T - c$ have dependency depth of exactly $d$ (where $T = t + k$), then

$$p_{0;k}(\lambda) = \frac{\lambda^d}{\kappa^d} \cdot (1 + \lambda + \ldots + \lambda^{c-d}) = \frac{\lambda^d}{\kappa^d} \cdot \left( \frac{1 - \lambda^{c-d+1}}{1 - \lambda} \right)$$

(296)

Hence we can see that if we can show that $|1 - \lambda^d| \approx 0$, then $|p_{0;k}(\lambda)| = \Omega(1)$.

We also see that we would like $\lambda$ to be sufficiently bounded away from a small set of critical values such as the $(c - d)$-th roots of unity.

In a more general setting, we can rewrite

$$p_{0;k}(\lambda) = \frac{\lambda^d}{\kappa^d} \cdot q_{0;k}(\lambda)$$

(297)
for some polynomial $g_{0,k}$ with positive real coefficients, and we would like $\lambda$ to be such that $|g_{0,k}(\lambda)| = \Omega(1)$ for all sufficiently large $k$.

Our hypothesis is that the theorem holds as long as $\lambda$ is sufficiently bounded away from a small set of critical values in $\mathbb{C} \setminus \mathbb{R}^+$, or in other words, we would need only at least one eigenvalue to satisfy this condition. This set of critical values is dependent on $\kappa$, $d$ and the overall configuration of the attention weights.
B Tradeoff analysis between sparsity and gradient propagation

As already discussed in Section 4, the sparsity coefficient $\kappa$ verifies $\kappa = \nu + \rho \geq |S_t| + |R_t|$ for all time step $t$, where we denote $\nu$ for the size of the short-term buffer, and $\rho$ for the maximal size of the relevant sets $R_t$. In this section we would like to see how gradient propagation varies when changing sparsity. As already discussed at the end of Section 3 as well as at the end of Section 4, decreasing $\kappa$, would increasingly force gradients to backpropagate through the recurrent connections, thus degrading gradient stability. Meanwhile, increasing $\kappa$ would increase the size of the computational graph. Thus we would like to find the optimal trade-off between sparsity and gradient propagation. This trade-off is clearly task-specific and needs to be determined experimentally. The only way to do so is by either changing $\nu$ or changing $\rho$ (or both). Hence we are going to analyze the effects on gradient propagation by separately changing $\nu$ and $\rho$.

![Gradient norms plot on the Denoise Task](image1)

![Gradient norms plot on the Denoise Task](image2)

Figure 3: Both sides show gradient norm plots of $\|\nabla_h L\|$ in log scale after training for Denoise Task with $t$ ranging from 0 (latest time step) to 1000 (furthest time step). (Left) We took four MemLSTM models for $\rho = 3, 8, 18, 25$ while keeping $\nu = 15$ fixed. (Right) We took four MemLSTM models for $\nu = 3, 8, 18, 25$ while keeping $\rho = 15$ fixed. (Note that the $y$-axis of the two plots have different scales, as indicated in the plots.)

For Figure 3 (left), we can see that when choosing $\rho$ too small (here for instance $\rho = 3$), gradient propagation becomes unstable, while larger values for $\rho$ all show stable gradient propagation. This confirms our initial intuition that we can decrease $\rho$ until a task-specific threshold and maintain stable gradient propagation, while decreasing $\rho$ beyond this threshold would cause gradient propagation to become unstable.

For Figure 3 (right), we can see that changing $\nu$ has much less leverage on gradient propagation than changing $\rho$. Gradient propagation stays relatively stable regardless of the values for $\nu$. The only difference is that for the extreme value of $\nu = 3$, we can see that gradient propagation became slightly less stable, because with smaller $\nu$ predictions for future relevancy might become less accurate.
C Additional Results

Figure 4: Cross-entropy vs training updates for Copy (top) and Denoise (bottom) tasks for $T = \{100, 200, 300, 500, 1000, 2000\}$. 1 unit of the x-axis is equal to 100 iterations of training with the exception of expRNN where 1 unit on the x-axis is 10 iterations of training.

Table 5: Results for Copy Task

| $T$ | LSTM | orth-RNN | expRNN | MemRNN | SAB | RelRNN | RelLSTM |
|-----|------|----------|--------|--------|-----|--------|---------|
| 100 | 100% | 100%     | 100%   | 100%   | 100%| 100%   | 100%    |
| 200 | 100% | 100%     | 100%   | 100%   | 100%| 100%   | 100%    |
| 300 | 100% | 100%     | 100%   | 100%   | 100%| 100%   | 100%    |
| 500 | 12%  | 100%     | 100%   | 100%   | 100%| 100%   | 100%    |
| 1000| 12%  | 80%      | 100%   | 100%   | 100%| 100%   | 100%    |
| 2000| 12%  | 11%      | 100%   | OOM    | 100%| 100%   | 100%    |

Table 6: Hyperparameters used for Copy task

| Model   | lr   | optimizer | non-linearity | $\nu$ | $\rho$ |
|---------|------|-----------|---------------|-------|--------|
| orthRNN | 0.0002 | RMSprop | modrelu       | -     | -      |
| expRNN  | 0.0002 | RMSprop | modrelu       | -     | -      |
| LSTM    | 0.0002 | Adam     | -             | -     | -      |
| RelRNN  | 0.0002 | Adam     | tanh          | 10    | 10     |
Figure 5: Training curves for LSTM on Denoise task

Figure 6: Training curves for GORU on Denoise task

Figure 7: Heatmap of attention scores on PTB task training with full attention and BPTT of 125
Figure 8: Validation accuracy curves for pMNIST

Figure 9: Heatmap of attention scores on MNIST digit classification. 7 pixels were grouped at each time step to make visualization of heatmap easier.

Figure 10: Heatmap of attention scores on Copy task when only doing attention over the Short Term Buffer.
Table 7: Hyperparameters used for Denoise task

| Model   | lr     | optimizer | non-linearity | ν   | ρ   |
|---------|--------|-----------|---------------|-----|-----|
| orthRNN | 0.0002 | RMSprop   | modrelu       | -   | -   |
| expRNN  | 0.0002 | RMSprop   | modrelu       | -   | -   |
| LSTM    | 0.0002 | Adam      | -             | -   | -   |
| GORU    | 0.001  | RMSprop   | -             | -   | -   |
| RelRNN  | 0.0002 | RMSprop   | modrelu       | 10  | 10  |

Table 8: Hyperparameters used for sequential MNIST

| Model   | lr (lr orth) | optimizer | non-linearity | ν   | ρ   |
|---------|--------------|-----------|---------------|-----|-----|
| orthRNN | 0.0001       | Adam      | modrelu       | -   | -   |
| expRNN  | 0.0001(0.00001) | Adam   | modrelu       | -   | -   |
| LSTM    | 0.0002       | -         | -             | -   | -   |
| GORU    | -            | -         | -             | -   | -   |
| RelRNN  | 0.0003       | Adam      | modrelu       | 10  | 10  |

Figure 11: Heatmap of attention scores on Denoise task when only doing attention over the Short Term Buffer.

Table 9: Hyperparameters used for PTB

| Model   | lr (lr orth) | optimizer | non-linearity | ν   | ρ   |
|---------|--------------|-----------|---------------|-----|-----|
| orthRNN | 0.001        | Adam      | tanh          | -   | -   |
| expRNN  | 0.003(0.0003) | Adam   | tanh          | -   | -   |
| LSTM    | 0.0002       | -         | -             | -   | -   |
| GORU    | -            | -         | -             | -   | -   |
| RelRNN  | 0.0003       | Adam      | tanh          | 10  | 5   |