Quantum Time-evolution in Qubit Readout Process with a Josephson Bifurcation Amplifier

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We analyzed the Josephson bifurcation amplifier (JBA) readout process of a superconducting qubit quantum mechanically. This was achieved by employing numerical analyses of the dynamics of the density operator of a driven nonlinear oscillator and a qubit coupled system during the measurement process. In purely quantum cases, the wavefunction of the JBA is trapped in a quasienergy-state, and bifurcation is impossible. Introducing decoherence enables us to reproduce the bifurcation with a finite hysteresis. Moreover, we discuss in detail the dynamics involved when a qubit is initially in a superposition state. We have observed the qubit-probe (JBA) entangled state and it is divided into two separable states at the moment of the JBA transition begins. This corresponds to “projection”. To readout the measurement result, however, we must wait until the two JBA states are macroscopically well separated. The waiting time is determined by the strength of the decoherence in the JBA.

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The readout of superconducting qubit states with the Josephson bifurcation amplifier (JBA) technique provides non-destructive and high visibility readout. Therefore, now it is widely and successfully used in actual experiments [1]. Mathematically, a JBA is described as a driven Duffing oscillator [2]. It enhances a small difference in operation conditions by utilizing the bifurcation phenomenon. Under an appropriate driving force, a classical nonlinear oscillator becomes bistable [2]. One stable state has a small amplitude (low-amplitude state), and the other has a larger amplitude (high-amplitude state). The critical driving force $f_c$ or the critical detuning $\delta_c$ for the transition between these two states is very sensitive to small changes in the operational parameters of the oscillator. For example, when we increase or decrease the driving force continuously, the amplitude of the oscillation behaves hysteretically as shown in Fig. 1. When using a JBA as a qubit state readout probe, the JBA detects a small change depending on the qubit state. However, the quantum-mechanical behavior of the JBA readout process has not been established theoretically. This is because the bifurcation phenomenon can be discussed only for classical oscillators, and is impossible from the view point of pure quantum mechanics for an isolated system [3]. A classical analysis gives no information on entanglement between the qubit and the probe (JBA) or the decoherence in the composite system, although all the quantum properties (projection, measurement back-action, etc.) in the readout are contained in such information. A quantum mechanical analysis is indispensable if we are to understand the readout process.

In this letter, we analyze the quantum-mechanical time evolution of a JBA, and clarify how a bifurcation appears in an actual situation. Moreover, we investigate what happens during the process of the qubit state readout with a JBA by analyzing dynamics of the qubit-JBA composite system.

In a highly quantum-mechanical JBA case, tunneling between classically stable states destroys the criticality in a classical oscillator. This type of phenomenon has been precisely discussed in [3] and in references therein. That the charging energy of the JBA (\( \sim 2e^2/C \), where $C$ is the effective capacitance in the JBA circuit) is comparable to the energy barrier (\( \sim \) the nonlinearity introduced below) between two stable states, and decoherence is negligibly small. However, actual JBA measurements are made with more classical conditions. Rigo et al. [4] investigated such an oscillator with a semi-classical trajectory analysis. In order to obtain quantum information more directly, here, we analyze the time evolution of a JBA.
A JBA can be modeled as an anharmonic oscillator in a rotating frame approximation with a Hamiltonian:

\[ H_J = (\Omega - \omega)n_a + \alpha n_a^2 - \frac{1}{2} f (a^\dagger + a) \tag{1} \]

where, \(a^\dagger(a)\) is the creation (annihilation) operator of the Josephson plasma oscillation. \(n_a = a^\dagger a\), and \(\Omega\) is the linear resonant frequency of the JBA oscillator. \(\omega\) is the driving frequency, which is slightly smaller than \(\Omega\) by the detuning \(\delta \equiv \Omega - \omega\). \(f\) is the driving strength, and \(\alpha(>0)\) is the nonlinearity. In a classical approximation, this model shows the bifurcation in an appropriate parameter region. However, for a quantum-mechanical junction with \([a, a^\dagger] = 1\), the transition from \(|G\rangle\) (low-amplitude state) to \(|E\rangle\) (high-amplitude state) or, from \(|E\rangle\) to \(|G\rangle\) is impossible.

The quasienergy-states (eigenstates of the Hamiltonian Eq. (1) in the rotating approximation) are easily calculated and it is found that eigenstates never cross when the driving strength \(f\) is changed adiabatically. This means that if the JBA is initially in the ground state without driving, it never moves to the high-amplitude resonant state even if we increase the driving field because the JBA state only moves along the initial quasienergy-state and never jumps to the quasienergy-state which the high-amplitude resonant state belongs.

Therefore, we expect that when a transition between quasienergy-states is caused by perturbation from outside the system the bifurcation phenomenon is reproduced. This is the case when decoherence is introduced into the present model. Here, we only take into account the decoherence caused by a bath coupled to the JBA because decoherence that directly attacks the qubit is not limited to the readout process. Even for this model, indirect decoherence via the JBA occurs in the qubit.

For example, we introduce linear loss in the oscillator (JBA). The time evolution of the system (qubit-JBA) is governed by a Liouville equation:

\[ \frac{d\rho}{dt} = \frac{1}{i}[\rho, H] + \frac{\Gamma}{2} (2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a), \tag{2} \]

where \(\rho\) is the density operator of the system, and \(\Gamma\) is the relaxation rate due to the linear loss in the JBA. The \(Q\)-value is given by \(\Omega/\Gamma\).

First, we show a numerical example of JBA dynamics without a qubit in Figs. 2 and 3. Here, the parameters used are \(\delta = 0.007\Omega, \alpha = 8 \times 10^{-5}\Omega, Q = 2500\), and \(f\) is operated as \(0 \rightarrow 0.025\Omega \rightarrow 0\). These parameters are similar to those used in actual experiments 1. However, \(\delta\) and \(\Gamma\) are a factor of \(10^{-2}\) times smaller than real cases in order to emphasize quantunmness and discuss the influences of decoherence. Even if we use different parameters we obtain qualitatively same behaviors for a JBA with similar \(\delta/\Gamma\) ratio value.

Figure 2(a) approximately corresponds to the square of the JBA amplitude shown in Fig. 1. So, we can see that our calculation with decoherence reproduces the bifurcation phenomenon well. We find that the critical driving \(f_c\) is approximately 0.011\(\Omega\). Once the driving exceeds this \(f_c\), the behaviors of the JBA is the same not depending on the maximum \((f = 0.025\Omega)\) for Fig. 3 driving strength. Moreover, our calculation provides a lot of quantum-mechanical information about the JBA transition. Figure 3(b) shows the time variation of the purity of the JBA state. Increasing the driving force \(f\), we found that the purity decreases abruptly \((t = \tau_a)\). This cor-
corresponds to the beginning of the transition from \(|G\rangle_J\) to \(|E\rangle_J\) of the JBA. This is a manifestation of the fact that the transition needs an intense emission/absorption of energy to/from an external energy bath. This energy transfer is incoherent. After the rapid decrease, the purity recovers to some extent and the JBA approaches the classically stable state \(|E\rangle_J\) (\(t = \tau_0\)). Since \(|E\rangle_J\) is a meta-stable state (stationary point of the classical Hamiltonian), dragging JBA into the state by decoherence (linear loss) leads to the recovery of the purity. However, the purity does not reach unity because it is not a true ground state. The fluctuation in the JBA amplitude is plotted in Fig. 3(c). We can see a divergence of the fluctuation at the moment of the rapid decrease in the purity (\(t = \tau_0\)). This suggests that this JBA transition between \(|G\rangle_J\) and \(|E\rangle_J\) is one of a phase transitions in bosonic systems with many degrees of freedom.

Now we discuss the criterion of decoherence that determines whether a bifurcation is observed or not. From the above analyses we know that there is no critical value. When the decoherence is very small (\(\Gamma < \delta\)), the speed of the transition from \(|G\rangle_J\) to \(|E\rangle_J\) becomes exponentially slower as (schematically) \(\exp[-\eta \delta / \Gamma]\), where \(\eta\) is a numerical factor of the order of unity.

Information about the qubit state is transferred to the probe (JBA) through the formation of an entanglement between the qubit and the probe. What we actually observe is the macroscopic state of the JBA, and merely postulate the qubit state. Therefore, the process by which the entanglement is formed and split into separable states due to decoherence (“projection”) is very important for understanding the readout process.

The qubit-JBA composite system is approximated expressed by the Hamiltonian

\[
H = H_J + k \sigma_z n_a + H_Q, \quad H_Q = \frac{1}{2} (\varepsilon \sigma_z + \Delta \sigma_x) \tag{3}
\]

where \(H_Q\) is the Pauli operator representation of the qubit. \(k\) is the interaction constant between the qubit and the JBA. The qubit state \(\sigma_z\) slightly changes the effective detuning \(\delta + k \sigma_z\), resulting in a change in the critical value \(f_c\). By detecting the change in \(f_c\), we can distinguish the qubit state, i.e., whether \(\sigma_z\) is 1 or -1. For a flux qubit, the eigenstates of \(\sigma_z\) are the two flux states. \(\epsilon\) is the bias provided by an external applied magnetic field, and \(\Delta\) corresponds to the tunneling energy between two flux states.

For the qubit-JBA coupled system, we carried out calculations similar to those without a qubit shown above. The qubit readout process is well understood by employing knowledge of the quantum behavior in the time evolution of the JBA without a qubit that we have already discussed.

We show a numerical example of the dynamics during the qubit readout process in Fig. 4. JBA parameters are the same as for the above example. The initial state is a separable state: \(\frac{1}{\sqrt{2}} |g\rangle_q + \frac{1}{\sqrt{2}} |e\rangle_q \otimes |G\rangle_J\), that is, the qubit is in a superposition. Here, \(|g\rangle_q\) and \(|e\rangle_q\) are the ground and excited states of the qubit, respectively. Qubit parameters are \(\epsilon = 0.2 \Omega, \Delta / \epsilon = 1 / 2\). The coupling between the qubit and the JBA is set at \(k = 0.001 \Omega\). The driving force \(f\) is increased from 0 to 0.012fΩ (slightly larger than \(f_c\) of the JBA) and maintained. This parameter set gives a typical behavior of successful qubit readout.

The \(Q\)-representations of the JBA state \(\text{Tr}_q[\rho]\) are shown in Fig. 4, where \(\rho\) is the density operator of the qubit-JBA coupled system, and \(\text{Tr}_q[\cdots]\) denotes taking partial trace about qubit degrees of freedom. In the readout we can distinguish two peaks appearing in Fig. 4(d), which is the final stage of the readout. These peaks constitute an incoherent mixture, so they correspond to two possibilities in the measurement result.

To discuss the entanglement between the JBA and the qubit, we adopt \(E \equiv \text{Tr} [\rho^2] - \text{Tr} [\text{Tr}_q[\rho]^2]\), as a measure of the entanglement. The reduction in \(\text{Tr} [\text{Tr}_q[\rho]^2]\) is the purity decrease in the reduced density operator of the JBA, that contains the decrease due to both decoherence and the entanglement formation. The reduction in \(\text{Tr} [\rho^2]\) of the total system corresponds to the decrease due to decoherence. Therefore, \(E\) defined above shows the strength of entanglement.

The time variation of the entanglement measure \(E\) is shown in Fig. 5. This process can be schematically expressed as

\[
\rho(0) = |G\rangle_{J1} \otimes \left( \frac{1}{\sqrt{2}} |g\rangle_q + \frac{1}{\sqrt{2}} |e\rangle_q \right) \left( \frac{1}{\sqrt{2}} |g\rangle_e + \frac{1}{\sqrt{2}} |e\rangle_e \right) \\
\rightarrow \rho(\tau_1) = \frac{1}{2} (|G\rangle_{J1} |e\rangle_q + |G\rangle_{J1} |g\rangle_q) + (G |e\rangle_q |g\rangle_q) \\
\rightarrow \rho(\tau_2) = \frac{1}{2} (|G\rangle_{J1} |e\rangle_q |g\rangle_e + |G\rangle_{J1} |g\rangle_q |e\rangle_e) \\
\rightarrow \rho(\tau_3) = \frac{1}{2} (|G\rangle_{J1} |e\rangle_q |g\rangle_e + \frac{1}{2} |E\rangle_J |g\rangle_q |e\rangle_e). \tag{4}
\]

Entanglement formation and “projection” correspond to the second (\(t = \tau_1\)) and third (\(t = \tau_2\)) lines of Eq. 4, respectively. At this moment \(\tau_2\), however, it is impossible to obtain any information about the qubit from the observed probe (JBA) state because \(|G\rangle_J\) closely resembles \(|G\rangle|G\rangle\rangle\) in a classical mechanical sense (Fig. 4(b)) although quantum mechanically \(|G\rangle\rangle\) moves to \(|E\rangle_J\rangle\). In contrast, the other \(|G\rangle\rangle\) does not move significantly. (see, Figs. 4(c), (d)) Then \(\tau_3\), we can easily distinguish \(|E\rangle_J\rangle\) or \(|G\rangle\rangle\). This leads to a good postulation of the qubit state \(|g\rangle_q\) or \(e\rangle_q\), which brings us to the end of the readout.

The measure \(E\) is sufficiently quantitative for us to discuss the time variation of the entanglement but it does not show the absolute strength of the entanglement. To estimate the absolute strength we can calculate the entanglement of formation for every eigenstate consisting the total system density operator \(\rho(t)\). For example, the time variation of the value of the most dominant eigenstate is quantitatively proportional to the behavior of \(E\). However, it almost becomes unity when it meets its maximum value. The values for less dominant states also
The backaction on the qubit caused by the measurement is induced as a result of the non-commutation relation between the qubit Hamiltonian and the interaction Hamiltonian. When the qubit gap $\Delta$ is much smaller than other energies, the interaction commutes $\hat{H}_q$. Therefore, the JBA readout causes only pure dephasing on the qubit. This does not pollute the measurement result because the measurement itself requires the projection onto the $\sigma_z$ basis. This is simply the condition of the “non-demolition measurement”. However, when $\Delta$ is not negligible compared with $\varepsilon$, the measurement simultaneously causes qubit relaxation. The non-commuting part induces coherent transition between $|g\rangle_q$ and $|e\rangle_q$ in the qubit. This coherent transition itself is not harmful, but when such a transition is accompanied by decoherence (linear loss), stochastic energy relaxation in the qubit accumulate and a finite error remains. In fact, in the numerical example shown above, the average $\langle \sigma_z \rangle$ of the qubit deviates slightly (0.1%) from the initial value 0 because of qubit relaxation. Stronger decoherence causes larger deformation in the readout result although it is often much smaller than the deformation caused by other factors not discussed here, such as qubit relaxation as a result of decoherence directly attacking the qubit, even if we use $10^5$ times strong decoherence of JBA as in actual experiments.

As described above, the state of the total system is already divided into separable states $|e\rangle_q |G\rangle_1$ and $|g\rangle_q |G'\rangle_1$ just after the transition $|G\rangle_1 \rightarrow |E\rangle_1$ starts. Therefore, “projection” itself is successful even if the transition takes much longer in the absence of sufficiently strong decoherence (here, linear loss $\Gamma$). However, we cannot distinguish $|G\rangle_1$ and $|E\rangle_1$ until the transition finishes. Then, the readout fails unless we can wait and maintain the JBA state until the transition is complete.

In summary, we analyzed the quantum dynamics of the density operator of a system composed of a qubit and a JBA as the probe of the qubit state readout. From the analysis results, we have succeeded in extracting the essential feature of the JBA readout process of a superconducting qubit.

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[1] I. Siddiqi et al., Phys. Rev. Lett. 93, 207002 (2004); I. Siddiqi et al., ibid. 94, 027005 (2005). A. Lupascu et al., ibid. 96, 127003 (2006); I. Siddiqi et al., Phys. Rev. B 73, 054510 (2006); A. Lupascu et al., Nature Physics 3, 119 (2007); N. Boulant et al., Phys. Rev. B 76, 014525 (2007).
[2] W. Jordan and P. Smith, ‘Nonlinear Ordinary Differential Equations’, third ed. (Oxford Univ. Press, 1999).
[3] M. I. Dykman and M. V. Fistul, Phys. Rev. B 71, 140508 (R) (2005); M. I. Dykman, Phys. Rev. E 75, 011101 (2007); V. Peano and M. Thorwart, Chem. Phys. 322, 135 (2006); V. Peano and M. Thorwart, New J. Phys. 8, 21 (2006).
[4] M. Rigo et al., Phys. Rev. A 55, 1665 (1997).
[5] Y. Makhlin, A. Shnirman and G. Schön, Rev. Mod. Phys., 73, 357 (2001).