THE GRADIENT FLOW OF THE L² CURVATURE ENERGY NEAR THE ROUND SPHERE

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Abstract. We investigate the low-energy behavior of the gradient flow of the L² norm of the Riemannian curvature on four-manifolds. Specifically, we show long time existence and exponential convergence to a metric of constant sectional curvature when the initial metric has positive Yamabe constant and small initial energy.

1. Introduction

In this paper we study the low-energy behavior of the gradient flow of the L² norm of the curvature tensor on four-dimensional manifolds with positive Yamabe constant. Let us first introduce some notation. Let Rm denote the Riemannian curvature tensor, W the Weyl curvature, r the Ricci tensor, z the traceless Ricci tensor, and s the scalar curvature. Furthermore, let

\[ F(g) := \int_M |Rm|_g^2 dV_g. \]

In what follows we will often drop the explicit reference to g, as all objects in sight will be referencing a given time-dependent metric. A basic calculation ([4] Proposition 4.70) shows that

\[ \text{grad} F = \delta dr - \check{R} + \frac{1}{4} |Rm|^2. \]

where d is the exterior derivative acting on the Ricci tensor treated as a one-form with values in the tangent bundle, and δ is the adjoint of d. Moreover,

\[ \check{R}_{ij} = R_{ipqr} R_{j}^{pqr}. \]

A metric is called critical if

\[ \text{grad} F = 0. \]

Critical points of quadratic curvature functionals on four-manifolds are very natural geometric objects to study. See [10] for a nice overview and many interesting results relating the existence of such metrics to the topology of the underlying manifold.

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Given the importance of critical metrics, it is natural to consider the negative gradient flow of $F$:

$$\frac{\partial}{\partial t} g = - \nabla F,$$

$$g(0) = g_0.$$  

(2)

This is a nonlinear fourth order degenerate parabolic equation. Since the equation is fourth order maximum principle techniques are not available, and the analysis largely relies on integral estimates. In [12] we showed short-time existence of the initial value problem as well as derivative estimates and a long-time existence obstruction. Furthermore, in [13] we showed a convergence result for (2) when the energy is close to zero. In this paper we examine the behavior of (2) when the energy is close to its (topologically determined) minimum and the initial Yamabe constant is positive.

Before stating the main result let us fix some further notation. Given a smooth manifold, $\chi(M)$ will denote the Euler characteristic of $M$. Also, denote the concircular curvature tensor by

$$\circ Rm := Rm - \frac{1}{24} sg \circ g$$

where $\circ$ is the Kulkarni-Nomizu product. Let $(S^4, g_{S^4})$ denote the sphere with sectional curvature equal to 1, and likewise $(\mathbb{RP}^4, g_{\mathbb{RP}^4})$ is the real projective four-space with $g_{\mathbb{RP}^4}$ equal to the $\mathbb{Z}_2$-quotient of $g_{S^4}$. Also, for a tensor $T$ we define

$$||T||_{L^p} := \left( \int_M |T|^p \right)^{\frac{1}{p}}, \quad ||T||_{L^\infty} := \sup_{x \in M} |T|(x)$$

and

$$||T||_{H^k}^2 := \sum_{j=0}^k ||\nabla^j T||_{L^2}^2.$$  

**Theorem 1.** There is a constant $\epsilon > 0$ so that if $(M^4, g)$ is a Riemannian manifold satisfying

$$Y[g] > 0,$$

$$||\circ Rm||_{L^2}^2 \leq \epsilon \chi(M)$$

then then the solution to (2) with initial condition $g$ exists for all time and converges exponentially to either $(S^4, g_{S^4})$ or $(\mathbb{RP}^4, g_{\mathbb{RP}^4})$.

An important remark on the hypotheses is in order. In dimension 4, one has the pointwise equality $||\circ Rm||_{L^2} = |W|^2 + 2 |z|^2$. Therefore the hypothesis includes the statement that

$$||W||_{L^2}^2 \leq \epsilon \chi(M).$$
It follows from [5] Theorem A that once \( \epsilon < \frac{16}{\pi^2} \), \( M \) is diffeomorphic to either \( S^4 \) or \( \mathbb{R}P^4 \). Therefore the theorem is not providing a new topological conclusion. Furthermore, the proof relies on compactness arguments, and so the constant \( \epsilon \) is not computable from the proof. There is a discussion of the conjecturally optimal value of \( \epsilon \) in section 7. Conversely, the constants \( \epsilon \) in the gap theorems below are computable from the proof, though we do not do this here.

One would hope for an analogous result for metrics with negative scalar curvature, however the positive sign is crucial for two main ingredients in the proof. First of all, in section 2 we exploit a well-known relationship between the Yamabe constant, Sobolev constant and the Gauss-Bonnet formula to show that the hypotheses of a lower bound on the Yamabe constant and small \( L^2 \) norm of the traceless curvature tensor imply an a-priori estimate of the Sobolev constant. Next, in section 3, we derive a coercivity estimate for \( \text{grad} \ F \) which holds only for metrics of positive scalar curvature. In particular we show that the \( H^2 \) norm of \( \text{grad} \ F \) dominates the \( L^2 \) norm of \( z \). This estimate is used to show exponential decay of the \( L^2 \) norm of \( z \) along solutions to (2), which is one of the main analytic tools in the proof of the theorem.

Another consequence of this key coercivity estimate is the following “gap theorem” for critical metrics, which plays an important role in the proof of Theorem 1. Recall that Einstein metrics and scalar flat, half-conformally flat metrics are critical for \( F \). However, a complete classification of critical metrics is lacking. What the following corollary says is that when the traceless curvature is small in \( L^2 \) and the Yamabe constant is positive, a critical metric has constant positive sectional curvature.

**Theorem 2. Gap Theorem I** There exists \( \epsilon > 0 \) so that if \((M^4, g)\) is a compact critical Riemannian manifold with \( \|\运算\text{Rm}\|_{L^2}^2 \leq \epsilon \chi(M) \) and \( Y_g > 0 \), then \((M^4, g)\) is isometric to \((S^4, g_{S^4})\) or \((\mathbb{R}P^4, g_{\mathbb{R}P^4})\).

Related estimates allow us to prove an analogous gap theorem for non-compact critical manifolds. This theorem will play a key role in ruling out bubbles in the proof of Theorem 1.

**Theorem 3. Gap Theorem II** Let \((M^4, g)\) be a noncompact complete critical Riemannian four-manifold with zero scalar curvature and \( C_S < \infty \). There is a small constant \( \epsilon = \epsilon(C_S) > 0 \) so that if \( \|\运算\text{Rm}\|_{L^2}^2 \leq \epsilon \) then \((M^4, g)\) is flat.

Here is an outline of the rest of the paper. In section 2 we estimate the Sobolev constant of metrics with positive Yamabe constant and small \( L^2 \) norm of \( \text{Rm} \). Section 3 contains the main coercive estimate for \( \text{grad} \ F \) for metrics of positive scalar curvature. Theorem 2 is a consequence of this
estimate, and we finish section 3 by giving the proof of Theorem 3 using related arguments. In section 4 we give the first main component of the proof of Theorem 1, in particular showing that for \( \epsilon \) chosen small enough solutions to (2) have a definite lower-bound on their existence time. This uses an analysis of bubbles, exploiting Theorem 3 to rule them out. After this lower bound is established one can directly show exponential decay of the energy and hence convergence of the flow, and this is carried out in sections 5 and 6. We conclude in section 7 with some related questions. Section 8 is an appendix wherein we show a multiplicative Sobolev inequality which is used in the proof of the main theorem.

2. Sobolev Constant Estimate

In this section we exhibit an estimate of the Sobolev constant of metrics with positive Yamabe constant and small \( L^2 \) norm of traceless curvature. Estimates of this kind have appeared in many places recently, see [5], [6] for example. We start by recalling the Gauss-Bonnet theorem for smooth compact Riemannian four-manifolds:

\[
\chi(M) = \frac{1}{8\pi^2} \int_M \left( \frac{s^2}{24} + |W|^2 - \frac{|z|^2}{2} \right) dV.
\]

Furthermore note that this formula and the conformal invariance of \( \int_M |W|^2 \) together imply that

\[
\sigma_2(g) := \frac{1}{8\pi^2} \int_M \left( \frac{s^2}{24} - \frac{|z|^2}{2} \right) dV
\]

is also conformally invariant.

Next recall that the Yamabe constant of a conformal class \([g]\) on a compact four-manifold is

\[
Y_{[\bar{g}]} = \inf_{\tilde{g} \in [\bar{\bar{g}}]} \frac{\int_M \tilde{s}d\tilde{V}}{\left( \int_M d\tilde{V} \right)^{\frac{2}{n}}}.
\]

Applying the solution of the Yamabe problem due to Aubin, Trudinger and Schoen ([1], [11]) this infimum is achieved by a metric of constant scalar curvature. Using the expression for the scalar curvature of a conformal metric \( \bar{g} = u^2 g \) we conclude

\[
Y_{[g]} = \inf_{u \neq 0} \frac{\int_M \left( 6 |\nabla u|^2 + su^2 \right) dV}{\left( \int_M u^4 dV \right)^{\frac{2}{3}}}.
\]

In particular it follows that

\[
Y_{[g]} \|u\|_{L^4}^2 \leq 6 \|\nabla u\|_{L^2}^2 + \int_M su^2 dV
\]
holds for all $u \in C^1(M)$. Recall that the Sobolev constant of a metric $g$ on a four-dimensional manifold is the smallest constant $C_S$ such that the inequality

$$||u||_{L^4}^2 \leq C_S \left( ||\nabla u||_{L^2}^2 + V^{-\frac{1}{2}} ||u||_{L^2}^2 \right)$$

holds for all $u \in C^1(M)$. Now let $(M^4, g)$ be a compact Riemannian manifold satisfying

$$\left|\left| Rm \right|\right|_{L^2}^2 \leq \epsilon \chi(M). \quad (5)$$

As we noted in the introduction, once $\left|\left| Rm \right|\right|_{L^2}^2 \leq 16\pi^2 \chi(M)$, $M$ is already diffeomorphic to $S^4$ or $\mathbb{RP}^4$. We assume for the rest of this section that $M$ is oriented and so $M \cong S^4$, and so $\chi(M) = 2$. In particular, using the orthogonal decomposition of the curvature tensor in dimension 4, it follows from (5) that

$$||W||_{L^2}^2 + 2||z||_{L^2}^2 \leq 2\epsilon$$

Furthermore it follows from the Gauss-Bonnet theorem that

$$2 + \frac{\epsilon}{4\pi^2} \geq \frac{1}{8\pi^2} \int_M s^2 \geq 2 - \frac{\epsilon}{4\pi^2}.$$ 

Next it follows from the definition of $\sigma_2$ that

$$\sigma_2(g) > 2 - \frac{\epsilon}{2\pi^2}.$$ 

Moreover, due to the conformal invariance of $\sigma_2$, the above inequality holds for every metric in the conformal class of $g$. In particular, applying it to the constant scalar curvature Yamabe minimizer $\tilde{g}$ we conclude that

$$\frac{1}{192\pi^2} Y_{[g]}^2 = \frac{1}{192\pi^2} \left( \int_M \tilde{s} d\tilde{V} \right)^2 = \frac{1}{192\pi^2} \int_M \tilde{s}^2 d\tilde{V} \geq \sigma_2(g) > 2 - \frac{\epsilon}{2\pi^2}.$$ 

Since $Y_{[g]} > 0$, we conclude $Y_{[g]} > \sqrt{384\pi^2 - 96\epsilon}$. This also allows us to conclude pinching of the Yamabe energy of $g$. In particular we note

$$\int_M s^2 \leq 384\pi^2 + 96\epsilon < Y_{[g]}^2 + 194\epsilon$$

$$\leq \left( \frac{\int_M s dV}{\int_M dV} \right)^2 + 194\epsilon$$

$$\leq \int_M s^2 + 194\epsilon.$$
Let \( \bar{s} = \frac{\int_M s \, dv}{\int_M dv} \). Then this estimate implies
\[
||s - \bar{s}||^2_{L^2} = \int_M s^2 - \bar{s}^2 \leq 194 \varepsilon.
\]

Returning to (1) we may rewrite it as
\[
Y_{[g]} ||u||^2_{L^4} - \int_M (s - \bar{s}) u^2 \leq 6 \|\nabla u\|^2_{L^2} + \int_M \bar{s} u^2 dv
\leq 6 \|\nabla u\|^2_{L^2} + V^{-\frac{1}{2}} \sqrt{384 \pi^2 + 48 \varepsilon \|u\|^2_{L^2}}
\]

Applying the Cauchy-Schwarz inequality and the above estimate we conclude
\[
\int_M (s - \bar{s}) u^2 \leq ||s - \bar{s}||_{L^2} ||u||^2_{L^4}
\leq \sqrt{194 \varepsilon} \|u\|^2_{L^4}.
\]

Collecting the above estimates together we conclude that if \( \varepsilon < \frac{1}{196} \),
\[
||u||^2_{L^4} \leq 768 \pi^2 \left( ||\nabla u||^2_{L^2} + V^{-1} ||u||^2_{L^2} \right).
\]

This completes the proof of the following proposition.

**Proposition 4.** If \((M^4, g)\) is a Riemannian manifold satisfying
\[
Y_{[g]} > 0,
\]
\[
\left\| R^\circ \right\|^2_{L^2} \leq \varepsilon \chi(M),
\]
where \( \varepsilon \leq \frac{1}{196} \), then
\[
C_S \leq 768 \pi^2
\]
and
\[
Y_{[g]} > \sqrt{384 \pi^2 - 96 \varepsilon}.
\]

3. Coercive Estimate and Gap Theorems

In this section we exploit the algebraic structure of the gradient of \( F \) to derive a coercive estimate for \( grad F \) under the assumptions
\[
0 < \mu_1 \leq s \leq \mu_2
\]
\[
C_S \leq A
\]
\[
||W||^2_{L^2} \leq \varepsilon
\]
\[
||z||^2_{L^2} \leq \varepsilon
\]
We will apply this estimate to prove Theorem 2. Finally we give the proof of Theorem 3 which uses related estimates in the noncompact setting. In all estimates below the usage of the constants \( \mu_i, A \), and will always refer to these constants, while \( C \) will denote a generic constant. The constant \( \varepsilon \)
is to be determined by the estimates below. In the end it will depend on 
\( \mu_i \) and \( A \) in a way which is computable in principle, although we do not do this here.

The first step is to derive a partial coercivity estimate from the trace component of \( \text{grad} \mathcal{F} \).

**Lemma 5.** There is a constant \( C \) so that if \( \epsilon \) is chosen small with respect to \( A \) and \( \mu_1 \) we have

\[
C \| \text{grad} \mathcal{F} \|^2_{L^2} \geq \int_M \left| \nabla^2 s \right|^2 + s \left| \nabla s \right|^2.
\]

**Proof.** One can directly compute

\[
\text{tr} \text{grad} \mathcal{F} = -\Delta s.
\]

Therefore

\[
4 \| \text{grad} \mathcal{F} \|^2_{L^2} \geq \int_M (\text{tr} \text{grad} \mathcal{F})^2
\]

\[
= \int_M \nabla^i \nabla_i s \nabla^j \nabla_j s
\]

\[
= -\int_M \nabla_i s \nabla^i \nabla_j s
\]

\[
= -\int_M \nabla_i s \left( \nabla^j \nabla_i \nabla_j s + R^p_{ijj} \nabla_p s \right)
\]

\[
= \int_M \left| \nabla^2 s \right|^2 + r_{ij} \nabla_i s \nabla_j s
\]

\[
= \int_M \left| \nabla^2 s \right|^2 + z_{ij} \nabla_i s \nabla_j s + \frac{1}{4} s \left| \nabla s \right|^2
\]

Next by applying Hölder’s inequality we estimate

\[
\int_M z_{ij} \nabla^i s \nabla^j s \leq \|z\|_{L^2} \left\| \left| \nabla^2 s \right| \right\|_{L^2}
\]

\[
\leq \epsilon \left( \int_M \left| \nabla s \right|^4 \right)^{\frac{1}{2}}
\]

\[
\leq A \epsilon \left( \int_M \left| \nabla^2 s \right|^2 + \int_M \left| \nabla s \right|^2 \right)
\]

\[
\leq A \epsilon \left( \int_M \left| \nabla^2 s \right|^2 + \frac{1}{\mu_1} \int_M s \left| \nabla s \right|^2 \right).
\]

Thus for \( \epsilon \) chosen small enough with respect to \( A \) and \( \mu_1 \), the result follows. \( \square \)

Next we derive a coercivity estimate from the full tensor \( \text{grad} \mathcal{F} \). Before the proof we will record a special expression for \( \text{grad} \mathcal{F} \) in four dimensions.
Lemma 6. Let \((M^4, g)\) be a Riemannian four-manifold. Then

\[
\text{grad} \mathcal{F} = -2\Delta r + \nabla^2 s + \frac{8}{3} z + 4z \circ z - |z|^2 g - 4W \circ z.
\]

Proof. First of all ([4] Proposition 4.70) implies that

\[
\text{grad} \mathcal{F} = \delta dr - \tilde{R} + \frac{1}{4} |\text{Rm}|^2 g.
\]

Next note that, in four dimensions ([4] (4.72)),

\[
\tilde{R} - \frac{1}{4} |\text{Rm}|^2 g = \frac{s}{3} z + 2W \circ z.
\]

Also,

\[
\delta dr = -2\Delta r + \nabla^2 s + 2r \circ r - 2R \circ r.
\]

Combining these yields

\[
\text{grad} \mathcal{F} = -2\Delta r + \nabla^2 s + 2r \circ r - 2R \circ r - \frac{s}{3} z - 2W \circ z.
\]

Now we write

\[
2r \circ r = 2 \left( z + \frac{s}{4} g \right) \circ \left( z + \frac{s}{4} g \right) = 2z \circ z + sz + \frac{1}{8}s^2 g.
\]

Also, recall the four-dimensional curvature decomposition

\[
R_{ijkl} = W_{ijkl} + \frac{1}{2} (z_{ik} g_{jl} - z_{ik} g_{jl} + z_{jk} g_{il} - z_{jk} g_{il}) + \frac{1}{12}s (g_{jk} g_{il} - g_{jl} g_{ik}).
\]

(7)

We conclude that

\[
-2R \circ r = -2W \circ z - |z|^2 g + 2z \circ z - \frac{s}{3} z - \frac{1}{8}s^2 g.
\]

Combining these calculations yields the result. \(\square\)

Proposition 7. Given \(\mu_i, A > 0\), there are constants \(\delta = \delta(\mu_i) > 0\) and \(\epsilon = \epsilon(\mu_i, A)\) such that if \((M^4, g)\) is a compact Riemannian manifold satisfying (6), then we have

\[
\|\text{grad} \mathcal{F}\|^2_{L^2} \geq \delta \left( \|\Delta r\|^2_{L^2} + \|z\|^2_{H^1} \right).
\]
Proof. We start with the result of Lemma 6 and expand the $L^2$ inner product
\[ ||\nabla F||^2_{L^2} = \left| \left| -2\Delta r + \nabla^2 s + \frac{s}{3} z + 4z \circ z - |z|^2 g - 4W \circ z \right| \right|^2_{L^2} \]
\[ = 4 ||\Delta r||^2_{L^2} + ||\nabla^2 s||^2_{L^2} + \left| \left| \frac{s}{3} z \right| \right|^2_{L^2} \]
\[ + \left| \left| 4z \circ z - |z|^2 g \right| \right|^2_{L^2} + 16 ||W \circ z||^2_{L^2} \]
\[ - 4 \left< \Delta r, \nabla^2 s \right>_{L^2} - 4 \left< \Delta r, \frac{s}{3} z \right>_{L^2} - 4 \left< \Delta r, 4z \circ z - |z|^2 g \right>_{L^2} \]
\[ + 16 \left< \Delta r, W \circ z \right> + 2 \left< \nabla^2 s, \frac{s}{3} z \right>_{L^2} + 2 \left< \nabla^2 s, 4z \circ z - |z|^2 g \right>_{L^2} \]
\[ - 8 \left< \nabla^2 s, W \circ z \right> + 2 \left< \frac{s}{3} z, 4z \circ z \right>_{L^2} - 8 \left< \frac{s}{3} z, W \circ z \right> \]
\[ =: \sum_{j=1}^{14} I_j. \]

We now estimate the individual terms $I_j$. First using Lemma 5 we conclude that
\[ I_6 \geq -\theta ||\Delta r||^2_{L^2} - \frac{C}{\theta} ||\nabla F||^2_{L^2} \]
where $\theta$ is a small constant to be determined later. Next consider
\[ I_7 = -\frac{4}{3} \int_M \left< \Delta r, sz \right> \]
\[ = \frac{4}{3} \left< \nabla r, z \nabla s + s \nabla z \right> \]
\[ = \frac{4}{3} \int_M s |\nabla z|^2 + \int_M \nabla z \ast \nabla s \ast z. \]

Now using that $\nabla s$ may be expressed in terms of $\nabla z$ by the Bianchi identity we estimate
\[ \left| \int_M \nabla z \ast \nabla s \ast z \right| \leq C ||z||_{L^2} \left| \left| |\nabla z|^2 \right| \right|_{L^2} \]
\[ \leq C \epsilon^\frac{1}{2} \left( \int_M |\nabla z|^4 \right)^\frac{1}{2} \]
\[ \leq C A \epsilon^\frac{1}{2} \left( \int_M |\nabla^2 z|^2 + \int_M |\nabla z|^2 \right). \]
Thus for $\epsilon$ chosen small with respect to $A$ we conclude
\[ I_7 \geq \frac{4}{3} \int_M s |\nabla z|^2 - CA \epsilon^\frac{1}{2} ||z||^2_{H^2}. \]

Similar estimates yield
\[ I_8 + I_{11} \geq -CA \epsilon^\frac{1}{2} ||z||^2_{H^2}. \]
Next consider
\[ I_9 \geq - \theta \| \Delta r \|_{L^2}^2 - \frac{C}{\theta} \| W \circ z \|_{L^2}^2. \]

Next we estimate, again using the Bianchi identity,
\[ |I_{10}| = 2 \left| \int_M \left\langle \nabla^2 s, \frac{s}{3} z \right\rangle \right| \]
\[ = \frac{2}{3} \left| \int_M \nabla^i \nabla^j s (sz_{ij}) \right| \]
\[ = \left| \int_M z * \nabla s^2 + s * \nabla s^2 \right| \]
\[ \leq C \int_M |z| |\nabla s|^2 + |s| |\nabla s|^2. \]

From (8) and Lemma 5 we conclude
\[ I_{10} \geq - CA \epsilon \| z \|_{L^2}^2 - C \| \text{grad } F \|_{L^2}^2. \]

Next applying Lemma 5 we conclude
\[ I_{12} \geq - \theta \| W \circ z \|_{L^2}^2 - \frac{C}{\theta} \| \text{grad } F \|_{L^2}^2. \]

Next we have
\[ |I_{13}| = \left| \int_M sz^3 \right| \]
\[ \leq C \| sz \|_{L^2} \left( \int_M |z|^4 \right)^{1/2} \]
\[ \leq CA \| sz \|_{L^2} \left( \int_M |\nabla z|^2 + \int_M |z|^2 \right). \]

Next we estimate
\[ CA \| sz \|_{L^2} \| z \|_{L^2}^2 = CA \epsilon^{1/2} (\| sz \|_{L^2} \| z \|_{L^2}) \]
\[ \leq CA \epsilon^{1/2} (\| sz \|_{L^2}^2 + \| z \|_{L^2}^2). \]

Also,
\[ CA \| sz \|_{L^2} \int_M |\nabla z|^2 = - CA \| sz \|_{L^2} \int_M \langle z, \nabla^2 z \rangle \]
\[ \leq CA \| sz \|_{L^2} \| z \|_{L^2} \| \nabla^2 z \|_{L^2} \]
\[ \leq CA \epsilon^{1/2} \left( \| sz \|_{L^2}^2 + \| z \|_{H^2}^2 \right). \]

Combining these we conclude
\[ I_{13} \geq - CA \epsilon^{1/2} \left( \| sz \|_{L^2}^2 + \| z \|_{H^2}^2 \right). \]
Finally we estimate
\[ I_{14} \geq -\theta \|sz\|_{L^2}^2 - \frac{C}{\theta} \|W \circ z\|_{L^2}^2. \]

Collecting these individual estimates, choosing \( \epsilon \) small with respect to \( A \), and choosing \( \theta \) small yields
\[
C \|\text{grad} F\|_{L^2}^2 \geq \frac{1}{10} \left( \|\Delta r\|_{L^2}^2 + \|\nabla^2 s\|_{L^2}^2 + \left\|s \nabla z\right\|_{L^2}^2 + \int_M s |\nabla z|^2 \right) 
- CA\epsilon^2 \left\|z\right\|_{H^2}^2 - C \|W \circ z\|_{L^2}^2.
\]

It remains to estimate the last two terms, which we do in the next two lemmas. First consider

**Lemma 8.** We may choose \( \epsilon \) small with respect to \( A \) so that
\[
\left\|W\right\|_{L^2}^2 \leq C(\mu_i, A)\epsilon \left\|z\right\|_{H^2}^2.
\]

**Proof.** Begin by applying Hölder’s inequality and the Sobolev inequality to yield
\[
\int_M |W|^2 |z|^2 \leq \left( \int_M |W|^4 \right)^{\frac{1}{2}} \left( \int_M |z|^4 \right)^{\frac{1}{2}} 
\leq A^2 \left( \int_M |\nabla W|^2 + \int_M |W|^2 \right) \left( \int_M |\nabla z|^2 + \int M |z|^2 \right) 
= A^2 \left( \|\nabla W\|_{L^2}^2 + \|W\|_{L^2}^2 \right) \left( \|z\|_{H^1}^2 \right).
\]

Before continuing we need a nice expression for \( \Delta W \). First apply the second Bianchi identity and commute derivatives to yield
\[
\nabla_i \nabla_i W_{mklm} = \nabla_i (\nabla_m W_{iklm} + \nabla_k W_{milm}) + \nabla^2 z 
= \nabla_m \nabla_l W_{iklm} + \nabla_k \nabla_l W_{milm} + \text{Rm} * W + \nabla^2 z 
= \nabla_m \nabla_k W_{iilm} + \nabla_k \nabla_m W_{iilm} + \text{Rm} * W + \nabla^2 z 
= \nabla^2 z + \text{Rm} * W
\]

We conclude
\[
\|\nabla W\|_{L^2}^2 = -\int_M \langle W, \Delta W \rangle 
= \int_M \text{Rm} * W^2 + \int_M W \nabla^2 z.
\]

To estimate the first term in the line above, note
\[
\int_M s * W^2 \leq C \mu_2 \left\|W\right\|_{L^2}^2,
\]
Also we have
\[
\int_M W^3 \leq C \|W\|_{L^2} \left( \int_M |W|^4 \right)^{1/2} \\
\leq CAe^{\frac{\epsilon}{2}} \left( \|\nabla W\|_{L^2}^2 + \|W\|_{L^2}^2 \right).
\]

There is also the curvature term
\[
\int_M z \ast W^2 \leq C \|z\|_{L^2} \left( \int_M |W|^4 \right)^{1/2} \\
\leq CAe^{\frac{\epsilon}{2}} \left( \|\nabla W\|_{L^2}^2 + \|W\|_{L^2}^2 \right).
\]

Collecting these calculations and choosing \(\epsilon\) small with respect to \(A\) we conclude
\[
\|\nabla W\|_{L^2}^2 \leq C(\mu_2) \|W\|_{L^2}^2 + \int_M W \ast \nabla^2 z.
\]

Plugging this into (10) yields
\[
\int_M |W|^2 |z|^2 \leq C(\mu_2)A^2 \epsilon \|z\|_{H^1}^2 + A^2 \left( \int_M W \ast \nabla^2 z \right) \|z\|_{H^1}^2.
\]

To estimate the final term we first consider
\[
A^2 \left( \int_M W \ast \nabla^2 z \right) \|\nabla z\|_{L^2}^2 \leq CA^2 \|W\|_{L^2} \|\nabla^2 z\|_{L^2} \|\nabla z\|_{L^2}^2 \\
\leq CA^2 \|W\|_{L^2} \|z\|_{L^2} \|\nabla^2 z\|_{L^2}^2 \\
\leq CA^2 \epsilon \|\nabla^2 z\|_{L^2}^2.
\]

Lastly estimate
\[
A^2 \left( \int_M W \ast \nabla^2 z \right) \|z\|_{L^2}^2 \leq CA^2 \|W\|_{L^2} \|\nabla^2 z\|_{L^2} \|z\|_{L^2}^2 \\
\leq CA^2 \epsilon \|\nabla^2 z\|_{L^2} \|z\|_{L^2} \\
\leq CA^2 \epsilon \left( \|\nabla^2 z\|_{L^2}^2 + \|z\|_{L^2}^2 \right).
\]

The result follows.

\[\Box\]

Lemma 9. There is a constant \(C\) so that if we choose \(\epsilon\) small with respect to \(A\),
\[
\|\nabla^2 z\|_{L^2}^2 \leq C \left( \|\Delta z\|_{L^2}^2 + \|z\|_{H^1}^2 + \|s z\|_{L^2}^2 + \int_M s |\nabla z|^2 + |||W| z|||_{L^2}^2 \right).
\]
Proof. We integrate by parts and estimate
\[
\|\nabla^2 z\|^2_{L^2} = \int_M \nabla_i \nabla_j z_{kl} \nabla_i \nabla_j z_{kl} + \text{Rm} \ast z \ast \nabla^2 z
\]
\[
\leq - \int_M \nabla_i z_{kl} \nabla_i \nabla_j z_{kl} + C \int_M \|\text{Rm}\|^2 |z|^2 + \frac{1}{2} \int_M \|\nabla^2 z\|^2
\]
\[
= - \int_M \nabla_i z_{kl} \nabla_i \Delta z_{kl} + \int_M \text{Rm} \ast \nabla z \ast \nabla z + C \int_M \|\text{Rm}\|^2 |z|^2 + \frac{1}{2} \int_M \|\nabla^2 z\|^2
\]
\[
\leq \|\Delta z\|^2_{L^2} + C \int_M \|\text{Rm}\| |\nabla z|^2 + C \int_M \|\text{Rm}\|^2 |z|^2 + \frac{1}{2} \int_M \|\nabla^2 z\|^2
\]
\[
\leq C \left( \|\Delta z\|^2_{L^2} + \int_M \|\text{Rm}\| |\nabla z|^2 + \int_M \|\text{Rm}\|^2 |z|^2 \right).
\]

Next we estimate
\[
\int_M |\text{Rm}| |\nabla z|^2 \leq \int_M s |\nabla z|^2 + \int_M (|W| + |z|) |\nabla z|^2
\]
\[
\leq \int_M s |\nabla z|^2 + \left( \|W\|_{L^2} + \|z\|_{L^2} \right) \left( \int_M |\nabla z|^4 \right)^{\frac{1}{4}}
\]
\[
\leq \int_M s |\nabla z|^2 + A \left( \|W\|_{L^2} + \|z\|_{L^2} \right) \left( \int_M |\nabla^2 z|^2 + \int_M |\nabla z|^2 \right)
\]
\[
\leq \int_M s |\nabla z|^2 + A e^{\frac{1}{4}} \left( \|\nabla^2 z\|^2_{L^2} + \|\nabla z\|^2_{L^2} \right)
\]

Also we estimate
\[
\int_M |\text{Rm}|^2 |z|^2 \leq \|sz\|^2_{L^2} + \int_M |z|^4 + \|W\| |z|^2_{L^2}
\]
\[
\leq \|sz\|^2_{L^2} + 2 \left( \int_M |\nabla z|^2 \right)^2 + 2 \left( \int_M |z|^2 \right)^2 + \|W\| |z|^2_{L^2}
\]
\[
\leq \|sz\|^2_{L^2} + 2 \epsilon \|\nabla^2 z\|_{L^2}^2 + 2 \epsilon \|z\|^2_{L^2} + \|W\| |z|^2_{L^2}.
\]

Combining these estimates yields the result. \(\square\)

Applying Lemmas 8 and 9 and the fact that \(s > \mu_1\), we conclude from \(9\) that if \(\epsilon\) is chosen small we have
\[
C \|\nabla \mathcal{F}\|^2_{L^2} \geq \frac{1}{20} \|\Delta r\|^2_{L^2} + \left( \frac{\mu_1}{20} - C(A, \mu_1) \epsilon \right) \|\nabla z\|^2_{L^2}
\]
\[
+ \left( \frac{\mu_1^2}{20} - C(A, \mu_1) \epsilon \right) \|z\|^2_{L^2}.
\]

The proposition follows. \(\square\)
**Theorem 10. Gap Theorem I** There exists $\epsilon > 0$ so that if $(M^4, g)$ is a compact critical Riemannian manifold with $\left\| \overset{\circ}{Rm} \right\|_{L^2}^2 \leq \epsilon \chi (M)$ and $Y_{[g]} > 0$, then $(M^4, g)$ is isometric to $(S^4, g_{S^4})$ or $(\mathbb{RP}^4, g_{\mathbb{RP}^4})$.

**Proof.** As pointed out in the introduction, the hypotheses already imply that $M$ is diffeomorphic to $S^4$ or $\mathbb{RP}^4$, and by passing to the double cover we may as well assume $M \cong S^4$. Since $\text{tr}_g \text{grad} F = \Delta s$ and $g$ is critical it follows that $g$ has constant scalar curvature. Scale $g$ so that it has unit volume, then we have $s \equiv Y_{[g]}$. Apply Proposition \[3\] to conclude that the Sobolev constant of $g$ is bounded above and the Yamabe constant is bounded below, so the scalar curvature is bounded above and below. We now may apply Proposition \[7\] to conclude that if $\left\| \overset{\circ}{Rm} \right\|_{L^2}$ is chosen small enough with respect to $Y_{[g]}$ then $g$ is Einstein. It follows from the arguments of section 2 that in fact for $\epsilon$ small the Yamabe constant of $(M^4, g)$ is close to that of $S^4$. It now follows from Theorem C of \[7\] that $(M^4, g)$ is isometric to $(S^4, g_{S^4})$. We also sketch another argument below to finish the theorem which is more in line with the type of arguments we have been using.

Since $g$ is now Einstein, it follows that the traceless part of the curvature tensor satisfies the elliptic equation

$$\Delta \overset{\circ}{Rm} = \overset{\circ}{Rm} \ast \overset{\circ}{Rm} + s \ast \overset{\circ}{Rm}.$$ 

It follows that the curvature satisfies the local elliptic estimate

$$\sup_{B_r} \left\| \overset{\circ}{Rm} \right\| \leq \frac{C}{r} \left\| \overset{\circ}{Rm} \right\|_{L^2(B_r)}$$

for balls satisfying

$$\left\| \overset{\circ}{Rm} \right\|_{L^2(B_r)} \leq \epsilon_0.$$ 

The constants $\epsilon_0$ and $C$ here depend on a bound for $s$ and a bound on the Sobolev constant, both of which are bounded by $Y_{[g]}$. Therefore, for $\epsilon$ chosen small enough, $\left\| \overset{\circ}{Rm} \right\|_\infty \leq C(Y_{[g]}) \epsilon$. In particular, for $\epsilon$ chosen small enough we can conclude that $g$ has positive curvature operator. It now follows from the main theorem of \[8\] that in fact $g$ is isometric to the round metric on $S^4$. \qed

**Theorem 11. Gap Theorem II** Let $(M^4, g)$ be a noncompact complete critical Riemannian four-manifold with zero scalar curvature and $C_S < \infty$. There is a small constant $\epsilon = \epsilon(C_S) > 0$ so that if $\left\| \overset{\circ}{Rm} \right\|_{L^2}^2 \leq \epsilon$ then $(M^4, g)$ is flat.
Proof. Since $s \equiv 0$, let us write the critical equation in the simple form

$$0 = \Delta r + \text{Rm}^2$$

Let $\phi$ be some compactly supported function. First observe the inequality

$$- |\text{Rm}| \Delta |\text{Rm}| = - \frac{1}{2} \Delta |\text{Rm}|^2 + |\nabla |\text{Rm}| = - \langle \Delta \text{Rm}, \text{Rm} \rangle - |\nabla \text{Rm}|^2 + |\nabla |\text{Rm}| \leq - \langle \Delta \text{Rm}, \text{Rm} \rangle$$

using the Kato inequality $|\nabla |\text{Rm}| |\leq| \nabla |\text{Rm}|$. By the Bianchi identity, one can show that $\Delta \text{Rm} = \mathcal{L}(\nabla \text{div Rm}) + \text{Rm} \ast \text{Rm}$ for some universal linear operator $\mathcal{L}$. Therefore we may estimate for any Riemannian metric

$$- \int_M \phi^2 |\text{Rm}| \Delta |\text{Rm}| \, dV \leq \int_M \phi^2 (- \langle \Delta \text{Rm}, \text{Rm} \rangle) \, dV$$

\begin{align*}
&= \int_M \phi^2 (- \langle \mathcal{L}(\nabla \text{div Rm}) + \text{Rm}^2, \text{Rm} \rangle) \, dV \\
&= \int_M (\phi \ast \nabla \phi \ast \text{div Rm} \ast \text{Rm} + \phi^2 \text{div Rm}^2 + \phi^2 \ast \text{Rm}^3) \, dV \\
&\leq C \int_M |\nabla \phi|^2 |\text{Rm}|^2 + \phi^2 |\text{div Rm}|^2 + |\text{Rm}|^3 \phi^2 \, dV.
\end{align*}

Next we use the critical equation to estimate

$$0 = \int_M \phi^2 \langle \Delta r + \text{Rm}^2, r \rangle \, dV$$

\begin{align*}
&= - \int_M \phi^2 |\nabla r|^2 + \phi r \ast \nabla r \ast \nabla \phi + \phi^2 \text{Rm}^3 \, dV \\
&\leq - \frac{1}{2} \int_M \phi^2 |\nabla r|^2 \, dV + C \int_M \left( |\nabla \phi|^2 |\text{Rm}|^2 + \phi^2 |\text{Rm}|^3 \right) \, dV.
\end{align*}

Since $\text{div Rm} = \nabla^i \text{Rm}_{ijkl} = \nabla_k r_{jl} - \nabla_j r_{jk}$ by the Bianchi identity, we conclude that

$$- \int_M \phi^2 |\text{Rm}| \Delta |\text{Rm}| \, dV \leq C \int_M \left( |\nabla \phi|^2 |\text{Rm}|^2 + \phi^2 |\text{Rm}|^3 \right) \, dV.$$ 

Applying the Sobolev inequality we conclude using the above estimate that

$$||\phi ||\text{Rm}||^2_{L^4} \leq C \int_M \left( |\nabla \phi|^2 |\text{Rm}|^2 + |\nabla |\text{Rm}|^2 \phi^2 \right) \, dV$$

\begin{align*}
&\leq C \int_M \left( |\nabla \phi|^2 |\text{Rm}|^2 + \phi^2 |\text{Rm}|^3 \right) \, dV \\
&\leq C \int_M |\nabla \phi|^2 |\text{Rm}|^2 \, dV + C ||\phi ||\text{Rm}||^2_{L^4} ||\text{Rm}||_{L^2}.
\end{align*}
Therefore for \( \varepsilon \) chosen small with respect to the Sobolev constant we conclude

\[
||\phi |\text{Rm}||_{L^4}^2 \leq C \int_M |\nabla \phi|^2 |\text{Rm}|^2 dV.
\]

Fix some point \( x \in M \), and let \( \phi \) be a cutoff function for the ball of radius \( \rho \). In particular choose \( \phi \) such that

\[
0 \leq \phi \leq 1,
\]

\[
\phi = 1 \text{ on } B_\frac{\rho}{2}(x)
\]

\[
\phi = 0 \text{ on } M \setminus B_{\rho}(x)
\]

\[
|\nabla \phi| \leq \frac{4}{\rho}.
\]

It follows that

\[
||\phi |\text{Rm}||_{L^4}^2 \leq \frac{4}{\rho^2} \int_{B_\rho - B_{\frac{\rho}{2}}} |\text{Rm}|^2
\]

\[
\leq \frac{C}{\rho^2}.
\]

Letting \( \rho \to \infty \) we conclude \( |\text{Rm}| \equiv 0 \), and the result follows.

\[\square\]

4. PROOF OF THEOREM 1

\textbf{Proof.} We proceed by contradiction. If the statement is false, then we may choose \( \varepsilon_i \to 0 \) and metrics \( g_i \) such that \( ||\circ \text{Rm}_{g_i}||_{L^2} \leq \varepsilon_i \chi(M) \), and the solution to (2) with initial condition \( g_i \) exists on a finite time interval. As noted in the introduction, once \( \varepsilon_i < 16\pi^2 \) it follows that \( M \) is diffeomorphic to either \( S^4 \) or \( \mathbb{RP}^4 \), so we can conclude that \( \chi(M) = 2, 1 \). By lifting to the double cover, we may assume without loss of generality that \( M \cong S^4 \), and by redefining \( \varepsilon_i \) that

\[
||\circ \text{Rm}_{g_i}||_{L^2} \leq \varepsilon_i.
\]

The first major step is to use a blowup argument to show that the existence time is bounded below for \( i \) sufficiently large. It is important to note here that the small energy condition above is not a priori preserved in general for solutions to (2). Indeed, it follows from the Gauss-Bonnet theorem (3) that

\[
\mathcal{F}(g) = 8\pi^2 \chi(M) + \int_M |z|^2 dV.
\]

Therefore an upper bound on \( ||z||_{L^2}^2 \) is automatically preserved, but it is possible that the balance between the scalar curvature and Weyl curvature contributions to \( \mathcal{F} \) could change along the flow. This important technical issue is discussed in some more detail in section 7. To control the balance...
between scalar and Weyl curvatures we first need a lemma which bounds the decay of the Yamabe energy under a solution to (2).

**Lemma 12.** Let \((M^4, g(t))\) be a solution to (2). Then

\[
\left( \int_M s dV \right)(t) \geq \left( \int_M s dV \right)(0) - Ct^\frac{1}{2} \mathcal{F}(0) (\mathcal{F}(0) - \mathcal{F}(t)) .
\]

**Proof.** Recall that if \(g(t)\) is a one-parameter family of metrics with \(\frac{\partial}{\partial t} g = h\), then

\[
\frac{\partial}{\partial t} s = - \Delta \text{tr} h + \text{div div} h - \langle h, \text{Rc} \rangle.
\]

Note that \(\text{div grad} \mathcal{F} = 0\) as a consequence of diffeomorphism invariance of \(\mathcal{F}\). Thus for a solution to (2) we conclude

\[
\frac{\partial}{\partial t} \int_M s = \int_M \left( \langle \text{grad} \mathcal{F}, \text{Rc} \rangle - \frac{1}{2} s \text{tr} \text{grad} \mathcal{F} \right) dV
\]

We directly estimate

\[
\left| \frac{\partial}{\partial t} \int_M s dV \right| \leq \left| \int_M \left( \langle \text{grad} \mathcal{F}, \text{Rc} \rangle - \frac{1}{2} s \text{tr} \text{grad} \mathcal{F} \right) dV \right|
\]

\[
\leq C \|\text{Rc}\|_{L^2} \|\text{grad} \mathcal{F}\|_{L^2} .
\]

Thus we may integrate in time to yield

\[
\left( \int_M s dV \right)(t) - \left( \int_M s dV \right)(0) \geq - C \int_0^t \|\text{Rc}\|_{L^2} \|\text{grad} \mathcal{F}\|_{L^2} dt
\]

\[
\geq - C \mathcal{F}(0) \left( \int_0^t dt \right)^{\frac{1}{2}} \left( \int_0^t \|\text{grad} \mathcal{F}\|_{L^2}^2 dt \right)
\]

\[
= - Ct^{\frac{1}{2}} \mathcal{F}(0) (\mathcal{F}(0) - \mathcal{F}(t)) .
\]

\( \square \)

So, consider \(g = g_i\) some element of the above sequence. Suppose \(T \leq 1\) is the maximal existence time of the flow \(g(t)\). By the gradient flow property and equation (12) we conclude that

\[
\|z\|_{L^2}^2(T) \leq \epsilon_i
\]

for all \(t \in [0, T]\). Also, applying Lemma 12 we conclude, since \(T \leq 1\), that

\[
\int_M s dV(T) \geq \int_M s dV(0) - C \epsilon_i
\]

\[
\geq Y_{[g(0)]} - C \epsilon_i .
\]
Since $Y_{g(0)} \geq \sqrt{384\pi^2 - 96\epsilon_i}$ by Proposition 4 it follows from Hölder’s inequality that

\[
(\sqrt{384\pi^2 - C\epsilon_i})^2 \leq \left[ \int_M sdV(T) \right]^2 \leq \int_M s^2dV(T).
\]

It now follows from the Gauss-Bonnet formula that

$$\|W\|_{L^2}^2 (T) \leq C\epsilon_i.$$

In particular, we have now shown that there is a universal constant $C$ so that

$$\left\| \sqrt{384\pi^2 - C\epsilon_i} \right\|_{L^2}^2 (T) \leq C\epsilon_i. \tag{13}$$

Given this, we return to Proposition 4 to conclude that the Sobolev constant is bounded on $[0, T]$. Suppose

$$\limsup_{t \to T} \|\text{Rm}\|_{\infty} \leq C.$$

Since the curvature and Sobolev constants are bounded, it follows from Theorem 6.2 that the flow exists smoothly up to time $T$, and hence past it, contradicting maximality of $T$. Therefore we conclude that

$$\limsup_{t \to T} \|\text{Rm}\|_{\infty} = \infty.$$ 

Let $(x_j, t_j)$ be a sequence of points such that $t_j \to T$ and

$$\limsup_{t \to T} \|\text{Rm}\|_{\infty} = \lim_{j \to \infty} \|\text{Rm}\|_{\infty} (x_j, t_j) =: \lambda_j.$$

Let

$$g_j(t, x) := \lambda_j \left( t_j + \frac{t}{\lambda_j^2}, x \right).$$

Consider the sequence of pointed Riemannian manifolds $(M, g_j(t), x_j)$. They have uniformly bounded curvatures on the time interval $[-t_j\lambda_j^2, 0]$ and uniformly bounded Sobolev constants, and hence by Theorem 7.1 of Theorem 6.2 we conclude subsequential convergence to a solution $(M_\infty, g_\infty(t), x_\infty)$ of on the time interval $[-\infty, 0]$. Note that quadratic curvature functionals are scaling invariant on $M$, so by Fatou’s lemma upper bounds on such integrals pass to the limit $g_\infty$.
we conclude that
\[ \int_{-1}^{0} \int_{M} |\text{grad } \mathcal{F}_\infty|^2 dV_\infty dt \leq \lim_{j \to \infty} \int_{\frac{1}{\lambda_j}}^{t_j} \int_{M} |\text{grad } \mathcal{F}_j|^2 dV_j dt \]
\[ = \lim_{j \to \infty} \mathcal{F} \left( g \left( \frac{-1}{\lambda_j^2} + t_j \right) \right) - \mathcal{F}(g(t_j)) \]
\[ = 0. \]

Therefore \( g_\infty(t) = g_\infty(0) \) is a critical metric for all \( t \) with \( \| \text{Rm}_\infty \|_\infty = 1 \).
Furthermore, we conclude that \( \text{tr } \text{grad } \mathcal{F}_\infty = \Delta s = 0 \). Since \( \int_M s_\infty^2 dV_\infty \leq C \), it follows by the maximum principle that \( s \) is constant, and this constant must be zero. Moreover, the limiting manifold is noncompact, and satisfies the Sobolev inequality
\[ \| u \|_{L^4} \leq C \| \nabla u \|_{L^2} \]
where the constant \( C \) is bounded uniformly the Sobolev constants of the metrics \( g_i \). Therefore we may apply Theorem 3 to conclude that for \( \epsilon_i \) small enough, \( \| \text{Rm}_\infty \|_\infty = 0 \), a contradiction. Thus
\[ \limsup_{t \to T} \| \text{Rm} \|_\infty \leq C \]

Thus \( T \) is not the maximal existence time, and we have shown that for sufficiently large \( i \) the solution to (2) with initial condition \( g_i \) exists at least on \( [0, 1] \). Note that it follows from the above argument that there is a constant \( C > 0 \) so that
\[ \sup_{M \times [\frac{1}{2}, 1]} \| \text{Rm}(g_i) \|_\infty \leq K. \]

Indeed, if this were not the case, one could choose a sequence \( i \to \infty \) and points \((x_i, t_i)\), \( t_i \in [\frac{1}{2}, 1] \) and repeat the blowup process. The resulting blow-up metric will be critical since \( \epsilon_i \to 0 \), and then another application of Theorem 3 provides the contradiction. It is important to note that we do not have any a priori control over this constant \( K \), we merely know it exists.

Since the curvatures and Sobolev constant are bounded on \([\frac{1}{2}, 1]\), it follows from Theorem 5.4 and the Sobolev inequality that there exist constants \( C_m \) such that
\[ \sup_{M \times [\frac{1}{2}, 1]} \| \nabla^m \text{Rm}(g_i) \|_\infty \leq C_m C S K^{m+5} \]

To finish the first step we show that for \( i \) sufficiently large the scalar curvature of \( g_i(1) \) is bounded away from zero. Using the above estimates, it is clear that if we fix \( x \in M \) the sequence of pointed Riemannian manifolds \( \{ M, g_i(1), x \} \) has a subsequence which converges, up to diffeomorphisms, to a new smooth metric \( g_\infty \). Since \( \epsilon_i \to 0 \), it follows from the above estimates that \( g_\infty \) satisfies \( \text{Rm}_\infty \equiv 0 \), and it then follows from Schur’s lemma that \( s_\infty \) is constant, and in particular \( g_\infty \) is isometric to \( g_{S^4} \). This metric has constant
positive scalar curvature, and since a lower bound on scalar curvature is
diffeomorphism invariant, we conclude that given \( \delta > 0 \), for \( i \) sufficiently
large one has

\[
s_{g_i(1)} \geq s_{g^4} - \delta. \tag{16}
\]

The second main step is completed in Proposition \([18]\), where it is shown that
for \( \epsilon \) sufficiently small with respect to \( K \), metrics satisfying \([13]\), \([14]\), \([15]\)
and \([16]\), the solution to \((2)\) exists for all time and converges exponentially
to \( g_{S^4} \). This contradicts the initial hypothesis, and finishes the proof of the
theorem. \( \square \)

5. A-priori \( L^2 \) Growth Estimate for \( \nabla F \)

In this section we give a bound on the growth of \( \| \nabla F \|_{L^2} \) over time intervals of small energy decay. This is the key input in showing exponential
convergence of long-time solutions of \((2)\) near round metrics. To simplify
notation we will set

\[
E := \nabla F.
\]

The estimate applies in a more general situation which we describe now. Let \( \epsilon_0 \) be a small constant which will be fixed later and fix a time interval
\( [t_0, t_1] \) such that some solution to \((2)\) exists on \( [t_0, t_1] \), has unit volume, and satisfies

\[
\int_{t_0}^{t_1} \int_M |E|^2 dV dt \leq \epsilon \leq 1. \tag{17}
\]

Note that this condition is satisfied for arbitrary time intervals if the initial
condition satisfies

\[
\|z\|^2_{L^2} \leq \epsilon.
\]

Furthermore assume that for any \( t \in [t_0, t_1] \) one has

\[
C_S(g_t) \leq A. \tag{18}
\]

Without loss of generality we assume \( A \geq 1 \). In this setting we derive an
estimate for the \( L^2 \) norm of \( E \). A direct calculation (see \([13]\) Lemma 13)
yields

\[
\frac{\partial}{\partial t} \|E\|^2_{L^2} = -\|\Delta E\|^2_{L^2} + \int_M E \ast \nabla^2 E \ast Rm
\]
\[
+ \int_M E \ast \nabla E \ast \nabla Rm + E^2 \ast Rm^2 + E^2 \ast \nabla^2 Rm. \tag{19}
\]
Integrating by parts and commuting derivatives yields

\[
\|\Delta E\|_{L^2}^2 = \int_M \nabla_i \nabla_j E \nabla_i \nabla_j E \\
= \int_M |\nabla^2 E|^2 + Rm \cdot \nabla E^2 \\
\geq \int_M |\nabla^2 E|^2 - C \int_M |E| |\nabla^2 E| |Rm| \\
- C \int_M |E| |\nabla E| |\nabla Rm|.
\]

Combining this with (19) and integrating over the time interval \([t_0, t_1]\) yields

\[
\|E\|_{L^2(g_{t_1})}^2 + \int_{t_0}^{t_1} \int_M |\nabla^2 E|^2 \\
\leq \|E\|_{L^2(g_{t_0})}^2 + C \int_{t_0}^{t_1} \int_M |E| |\nabla^2 E| |Rm| \\
+ |E| |\nabla E| |\nabla Rm| + |E|^2 |Rm|^2 + |E|^2 |\nabla^2 Rm|.
\]

We now proceed to bound the terms on the right hand side of the above inequality in a series of lemmas.

**Lemma 13.** Given \((M^4, g(t))\) a solution to (2) satisfying (17) and (18), there is a constant \(C\) depending on \(\mathcal{F}(g(t_1))\) such that

\[
\int_{t_0}^{t_1} \int_M |E|^2 |Rm|^2 \leq CA^2 \epsilon^2 \left[1 + \sup_{t_0 \leq t \leq t_1} \int_M |\nabla^2 Rm|^2 + \int_{t_0}^{t_1} \int_M |\nabla^2 E|^2\right].
\]

**Proof.** First we apply Hölder’s inequality and the Sobolev inequality to yield

\[
\int_M |E|^2 |Rm|^2 \leq \left(\int_M |E|^4\right)^{\frac{1}{2}} \left(\int_M |Rm|^4\right)^{\frac{1}{2}} \\
\leq CA^2 \left(\int_M |\nabla E|^2 + \int_M |E|^2\right) \left(\int_M |\nabla Rm|^2 + \int_M |Rm|^2\right) \\
= I + II + III + IV,
\]

where

\[
I = \int_M |\nabla E|^2, \\
II = \int_M |E|^2, \\
III = \int_M |\nabla Rm|^2, \\
IV = \int_M |Rm|^2.
\]
where the Roman numerals refer to the four different terms in the expanded product. First we bound the main term, integrating by parts

\[ I = CA^2 \int_M |\nabla E|^2 \int_M |\nabla Rm|^2 \]

\[ = CA^2 \left( \int_M \langle E, \Delta E \rangle \right) \left( \int_M \langle Rm, \Delta Rm \rangle \right) \]

\[ \leq CA^2 \left( \int_M |E|^2 \right)^{\frac{1}{2}} \left( \int_M |\Delta E|^2 \right)^{\frac{1}{2}} \left( \int_M |Rm|^2 \right)^{\frac{1}{2}} \left( \int_M |\Delta Rm|^2 \right)^{\frac{1}{2}} \]

\[ \leq CA^2 \left( \int_M |E|^2 \right)^{\frac{1}{2}} \left( \int_M |\nabla^2 E|^2 \right)^{\frac{1}{2}} \left( \int_M |\nabla^2 Rm|^2 \right)^{\frac{1}{2}} \].

Integrating this bound in time and applying Hölder’s inequality to the time integral yields

(22)

\[ \int_{t_0}^{t_1} I \leq CA^2 \sup_{t_0 \leq t \leq t_1} \left( \int_M |\nabla^2 Rm|^2 \right)^{\frac{1}{2}} \left( \int_{t_0}^{t_1} \int_M |E|^2 \right)^{\frac{1}{2}} \left( \int_{t_0}^{t_1} \int_M |\nabla^2 E|^2 \right)^{\frac{1}{2}} \]

\[ \leq CA^2 \epsilon^{\frac{1}{2}} \left[ \sup_{t_0 \leq t \leq t_1} \int_M |\nabla^2 Rm|^2 + \int_{t_0}^{t_1} \int_M |\nabla^2 E|^2 \right]. \]

The lower order terms are easier to bound. We bound by interpolation

\[ \int_{t_0}^{t_1} II = \int_{t_0}^{t_1} \int_M |\nabla E|^2 \int_M |Rm|^2 \]

\[ \leq C \int_{t_0}^{t_1} \left( \int_M |E|^2 \right)^{\frac{1}{2}} \left( \int_M |\nabla^2 E|^2 \right)^{\frac{1}{2}} \]

\[ \leq C \left( \int_{t_0}^{t_1} \int_M |E|^2 \right)^{\frac{1}{2}} \left( \int_{t_0}^{t_1} \int_M |\nabla^2 E|^2 \right)^{\frac{1}{2}} \]

\[ \leq C \epsilon^{\frac{1}{2}} \left[ 1 + \int_{t_0}^{t_1} \int_M |\nabla^2 E|^2 \right]. \]

For the third term we again interpolate

\[ \int_{t_0}^{t_1} III = C \int_{t_0}^{t_1} \int_M |E|^2 \int_M |\nabla Rm|^2 \]

\[ \leq C \int_{t_0}^{t_1} \int_M |E|^2 \left( \int_M |Rm|^2 \right)^{\frac{1}{2}} \left( \int_M |\nabla^2 Rm|^2 \right)^{\frac{1}{2}} \]

\[ \leq C \sup_{t_0 \leq t \leq t_1} \left( \int_M |\nabla^2 Rm|^2 \right)^{\frac{1}{2}} \int_{t_0}^{t_1} \int_M |E|^2 \]

\[ \leq C \epsilon^{\frac{1}{2}} \left[ 1 + \sup_{t_0 \leq t \leq t_1} \int_M |\nabla^2 Rm|^2 \right]. \]
Finally we make the bound
\[
\int_{t_0}^{t_1} IV = C \int_{t_0}^{t_1} \int_M |E|^2 \int_M |\mathsf{Rm}|^2 \\
\leq C \int_{t_0}^{t_1} \int_M |E|^2 \\
\leq C\varepsilon.
\]
Combining these bounds gives the result. \(\square\)

**Lemma 14.** Given \((M^4, g(t))\) a solution to (2) satisfying (17) and (18), there is a constant \(C\) depending on \(\mathcal{F}(g(t_1))\) such that
\[
\int_{t_0}^{t_1} \int_M |E|^2 |\nabla^2 \mathsf{Rm}| \leq CA^2 \varepsilon^2 \left[ 1 + \sup_{t_0 \leq t \leq t_1} \int_M |\nabla^2 \mathsf{Rm}|^2 + \int_{t_0}^{t_1} \int_M |\nabla^2 E|^2 \right].
\]

**Proof.** First we apply Hölder’s inequality and the Sobolev inequality to bound
\[
\int_M |E|^2 |\nabla^2 \mathsf{Rm}| \leq \int_M |E|^2 |\nabla^2 \mathsf{Rm}|
\]
\[
\leq \left( \int_M |E|^4 \right)^{1/2} \left( \int_M |\nabla^2 \mathsf{Rm}|^2 \right)^{1/2}
\]
\[
\leq A \left( \int_M |\nabla E|^2 + \int_M |E|^2 \right) \left( \int_M |\nabla^2 \mathsf{Rm}|^2 \right)^{1/2}
\]
\[
\leq CA \left[ \left( \int_M |E|^2 \right)^{1/2} \left( \int_M |\nabla^2 E|^2 \right)^{1/2} + \int_M |E|^2 \right] \left( \int_M |\nabla^2 \mathsf{Rm}|^2 \right)^{1/2}.
\]
In the last line we applied interpolation to the integral \(\int_M |\nabla E|^2\). The second term above may be integrated in time to yield
\[
CA \int_{t_0}^{t_1} \int_M |E|^2 \left( \int_M |\nabla^2 \mathsf{Rm}|^2 \right)^{1/2} \leq CA\varepsilon \left( \sup_{t_0 \leq t \leq t_1} \int_M |\nabla^2 \mathsf{Rm}|^2 \right)^{1/2}
\]
\[
\leq CA\varepsilon \left[ 1 + \sup_{t_0 \leq t \leq t_1} \int_M |\nabla^2 \mathsf{Rm}|^2 \right].
\]
The first term above is integrated in time and bounded as in line (22), yielding the result. \(\square\)

**Lemma 15.** Given \((M^4, g(t))\) a solution to (2) satisfying (17) and (18), there is a constant \(C\) depending on \(\mathcal{F}(g(t_1))\) such that
\[
\int_{t_0}^{t_1} \int_M |E| |\nabla E| |\nabla \mathsf{Rm}| \leq CA^2 \varepsilon^2 \left[ 1 + \sup_{t_0 \leq t \leq t_1} \int_M |\nabla^2 \mathsf{Rm}|^2 + \int_{t_0}^{t_1} \int_M |\nabla^2 E|^2 \right].
\]
Proof. We apply Hölder’s inequality, the Sobolev inequality and interpolation to bound

\[
\int_M |E| |\nabla E| |\nabla Rm| \leq \left( \int_M |E|^2 \right)^{\frac{1}{4}} \left( \int_M |\nabla E|^4 \right)^{\frac{1}{4}} \left( \int_M |\nabla Rm|^4 \right)^{\frac{1}{4}} \]

\[
\leq CA^2 \left( \int_M |E|^2 \right)^{\frac{1}{2}} \left( \int_M |\nabla^2 E|^2 + \int_M |\nabla E|^2 \right)^{\frac{1}{2}} \cdot \left( \int_M |\nabla^2 Rm|^2 + \int_M |Rm|^2 \right)^{\frac{1}{2}}.
\]

The time integral of each of the terms above has been bounded in the previous two lemmas, and so the result follows.  \(\square\)

**Lemma 16.** Given \((M^4, g(t))\) a solution to (2) satisfying (17) and (18), there is a constant \(C\) depending on \(\mathcal{F}(g(t_1))\) such that

\[
\int_{t_0}^{t_1} \int_M |E| |\nabla^2 E| |Rm| \leq CA^2 \epsilon^\frac{1}{4} \left[ 1 + \sup_{t_0 \leq t \leq t_1} \int_M |\nabla^2 Rm|^2 + \int_{t_0}^{t_1} \int_M |\nabla^2 E|^2 \right].
\]

Proof. We start by applying Hölder’s inequality and the Sobolev inequality to bound

\[
\int_M |E| |\nabla^2 E| |Rm| \leq \left( \int_M |E|^4 \right)^{\frac{1}{4}} \left( \int_M |\nabla^2 E|^2 \right)^{\frac{1}{4}} \left( \int_M |Rm|^4 \right)^{\frac{1}{4}} \leq A^2 \left( \int_M |\nabla E|^2 + \int_M |E|^2 \right)^{\frac{1}{2}} \left( \int_M |\nabla^2 E|^2 \right)^{\frac{1}{2}}.\]

\[
= A^2 \left( I + II + III + IV \right).
\]

where the Roman numerals denote the four terms in the expanded product above after applying the inequality \(\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}\). First we bound the
For the third term we bound
\[
\int_{t_0}^{t_1} III = \int_{t_0}^{t_1} \left( \int_M |E|^2 \right)^{\frac{1}{4}} \left( \int_M |\nabla E|^2 \right)^{\frac{1}{4}} \left( \int_M |\nabla Rm|^2 \right)^{\frac{1}{4}} \\
\leq \int_{t_0}^{t_1} \left( \int_M |E|^2 \right)^{\frac{1}{4}} \left( \int_M |\nabla E|^2 \right)^{\frac{1}{4}} \left( \int_M |\nabla Rm|^2 \right)^{\frac{1}{4}} \left( \int_M |\nabla^2 Rm|^2 \right)^{\frac{1}{4}} \\
\leq C \left( \sup_{t_0 \leq t \leq t_1} \int_M |\nabla Rm|^2 \right)^{\frac{1}{4}} \left( \int_{t_0}^{t_1} \int_M |E|^2 \right)^{\frac{1}{4}} \left( \int_{t_0}^{t_1} \int_M |\nabla E|^2 \right)^{\frac{1}{4}} \\
\leq C \epsilon^{\frac{1}{4}} \left[ 1 + \sup_{t_0 \leq t \leq t_1} \int_M |\nabla^2 Rm|^2 + \int_{t_0}^{t_1} \int_M |\nabla^2 E|^2 \right].
\]

Finally we estimate
\[
\int_{t_0}^{t_1} IV = \int_{t_0}^{t_1} \left( \int_M |E|^2 \right)^{\frac{1}{2}} \left( \int_M |\nabla E|^2 \right)^{\frac{1}{2}} \left( \int_M |Rm|^2 \right)^{\frac{1}{2}} \\
\leq C \left( \int_{t_0}^{t_1} \int_M |E|^2 \right)^{\frac{1}{2}} \left( \int_{t_0}^{t_1} \int_M |\nabla E|^2 \right)^{\frac{1}{2}} \left( \int_{t_0}^{t_1} \int_M |Rm|^2 \right)^{\frac{1}{2}} \\
\leq C \epsilon^{\frac{1}{2}} \left[ 1 + \int_{t_0}^{t_1} \int_M |\nabla^2 E|^2 \right].
\]
Combining these four estimates and using that \( \epsilon \leq 1 \) gives the result. \( \square \)

**Proposition 17.** Given \((M^4, g(t))\) a solution to (2) satisfying (17) and (18), there is a constant \( C > 0 \) depending on \( F(g(t_1)) \) so that if \( \epsilon \) is chosen small with respect to \( A \) and \( F(g(t_1)) \) one has

\[
\sup_{t_0 \leq t \leq t_1} \| E \|_{L^2}^2 + \int_{t_0}^{t_1} \int_M |\nabla^2 E|^2 \leq C A^2 \epsilon + \left[ 1 + \sup_{t_0 \leq t \leq t_1} \int_M |\nabla^2 Rm|^2 \right].
\]

**Proof.** Combining Lemmas 13 - 16 and plugging into (19) yields

\[
\sup_{t_0 \leq t \leq t_1} \| E \|_{L^2}^2 + \int_{t_0}^{t_1} \int_M |\nabla^2 E|^2 \leq C A^2 \epsilon + \left[ 1 + \sup_{t_0 \leq t \leq t_1} \int_M |\nabla^2 Rm|^2 \right]
\]

Therefore for \( \epsilon \) chosen small enough with respect to \( A \) and the constants of the lemmas, which depend on \( F(g(t_1)) \), we conclude the result. \( \square \)

6. **Exponential Convergence**

**Proposition 18.** Given \( K > 0, 0 < \delta << 1 \), there exists \( \epsilon > 0 \) so that if \((M^4, g(t))\) is a solution to (2) which exists on \([0,1]\) and satisfies \( Y[g(0)] > 0 \), then the solution exists for all time and converges to either \( g_{S^4} \) or \( g_{RP^4} \).

**Proof.** The strategy is to use the key coercivity estimate of Proposition 7 to show exponential decay of \( \nabla^2 F \). With this decay in hand, an argument exploiting a multiplicative Sobolev inequality and Moser iteration can be applied to conclude exponential convergence of the flow.

Let \((M^4, g(t))\) be a solution to (2) satisfying the hypotheses of the proposition. As in section 4, assume by passing to the double cover that \( M \) is oriented. Observe that \( g(1) \) trivially satisfies by hypothesis

\[
s > s_{g_{S^4}} - 2\delta
\]

\( \| Rm \|_\infty < 2K \)

(25)

\( \| Rm \|_{L^2}^2 < 2\epsilon \)

Let \( \Omega = \{ t \in [1,\infty) | (25) \text{ is satisfied} \} \). \( \Omega \) is certainly open, and we aim to show that \( \Omega \) is closed. Let \( T \in \Omega \). If \( \epsilon \) is small enough, we may apply Proposition 4 to conclude that there is a uniform constant \( A \) such that

\[
\sup_{t \in [0,T]} C_S(g(t)) \leq A.
\]
Likewise, using (14), we have that \( \sup_{M \times [T-\frac{1}{2}, T]} |Rm| \leq 2K \). Using this and the curvature bound of (25), we may argue as in section 4 using the derivative estimates for solutions to (2) to conclude that there are constants \( C_m \) such that
\[
||\nabla^m Rm(g(T))||_\infty \leq C_m C_S K^{m+5}.
\]
Thus if condition (25) holds on \([0, T)\), the solution to (2) exists smoothly up to and past time \( T \).

We now derive exponential decay of \( ||\text{grad } F||_{L^2} \). First note that, using (12), we have that
\[
\frac{\partial}{\partial t} ||z||^2_{L^2} = \frac{\partial}{\partial t} \left( 8\pi^2 \chi(M) + ||z||^2_{L^2} \right) = \frac{\partial}{\partial t} F = - ||\text{grad } F||^2_{L^2}.
\]
Provided say \( \delta < \frac{1}{100} \), by Proposition 7 we conclude that there is a constant \( \eta > 0 \) so that if \( \epsilon \) is chosen small with respect to \( A \) and \( K \), then given \( t \geq 1 \in \Omega \), we have
\[
||\text{grad } F||^2_{L^2} \geq \eta ||z||^2_{L^2}.
\]
Combining this with the line above we conclude that for \( t \geq 1 \),
\[
||z||^2_{L^2} (t) \leq \epsilon e^{-\eta t}.
\]
Given exponential decay of the energy, it is natural to expect exponential decay of its time derivative. We claim that there exists a constant \( P = P(A, K) \) such that for \( t \geq 1 \),
\[
||\text{grad } F||^2_{L^2} (t) < P \epsilon^{\frac{1}{4}} e^{-\frac{\eta}{4} t}.
\]
We first need to show this estimate on the time interval \([1, \frac{5}{4}]\). Note that
\[
\int_{\frac{3}{4}}^1 ||\text{grad } F||^2_{L^2} = F(\frac{3}{4}) - F(1) \leq \epsilon.
\]
Thus there exists \( s, \frac{3}{4} \leq s \leq 1 \) such that \( ||\text{grad } F||^2_{L^2} (s) \leq \epsilon \). Using Proposition 17 and (15) we conclude that if \( \epsilon \) is chosen small enough we have
\[
\sup_{t \in [1, \frac{5}{4}]} ||\text{grad } F||^2_{L^2} \leq C \epsilon^{\frac{1}{4}}
\]
which proves (28) on \([1, \frac{5}{4}]\) for \( P = C \). Next we show (28) for arbitrary times \( t \geq \frac{5}{4} \). Observe for any \( 1 \leq t_1 \leq t_2 \) the estimate
\[
\int_{t_1}^{t_2} ||\text{grad } F||^2_{L^2} = F(t_1) - F(t_2)
\]
\[
= \left( 8\pi^2 \chi(M) + ||z||^2_{L^2} (t_1) \right) - \left( 8\pi^2 \chi(M) + ||z||^2_{L^2} (t_2) \right)
\]
\[
\leq ||z||^2_{L^2} (t_1)
\]
\[
\leq \epsilon e^{-\eta t_1}.
\]
Now fix some $t \geq \frac{5}{4}$. Applying (29) for $t_1 = t - \frac{1}{4}, t_2 = t$ we conclude that there exists $s \in \left[ t - \frac{1}{4}, t \right]$ such that $\| \text{grad } F \|^2_{L^2}(s) \leq e^{-\eta(t - \frac{1}{4})}$. Next we apply Proposition 17 with $t_0 = s, t_1 = t$ and apply (26), to conclude
\[
\| \text{grad } F \|^2_{L^2}(t) \leq 2 \| \text{grad } F \|^2_{L^2}(s) + C(K)A^2 e^{\frac{1}{4}t} e^{-\eta(t - \frac{1}{4})} + C(K)A^2 e^{-\frac{3}{4}t}.
\]
This finishes the proof of (28). We now use this estimate to show that $[0, \infty) \subset \Omega$. Apply Theorem 19 with $p = 8, m = 2$ and $\alpha = \frac{4}{5}$ and use (26) to estimate
\[
\int_1^T \| \nabla^2 \text{grad } F \|_{\infty} \leq CA \int_1^T \| \nabla^2 \text{grad } F \|^\frac{1}{2}_{L^2} \left( \| \nabla \text{grad } F \|_{L^8} + \| \text{grad } F \|_{L^8} \right) \frac{5}{8}
\]
\[
\leq C(A, K) e^{\frac{1}{40t}} \int_1^T e^{-\frac{3}{40t}}
\]
\[
\leq C(A, K) e^{\frac{1}{40t}}.
\]
Likewise another application of Theorem 19 yields
\[
\int_1^T \| \nabla^3 \text{grad } F \|_{\infty} \leq CA \int_1^T \| \nabla^3 \text{grad } F \|^\frac{1}{2}_{L^2} \left( \| \nabla^3 \text{grad } F \|_{L^8} + \| \text{grad } F \|_{L^8} \right) \frac{5}{8}
\]
Integrating by parts and applying Hölder’s inequality and (26) we conclude that for $t \geq 1$
\[
\| \nabla^2 \text{grad } F \|_{L^2} \leq \| \text{grad } F \|^\frac{1}{2}_{L^2} \| \nabla^4 \text{grad } F \|^\frac{1}{2}_{L^2} \leq C(K) \| \text{grad } F \|^\frac{1}{2}_{L^2}. \]

Thus we conclude
\[
\int_1^T \| \nabla^2 \text{grad } F \|_{\infty} \leq C(A, K) \int_1^T \| \text{grad } F \|^\frac{1}{2}_{L^2}
\]
\[
\leq C(A, K) e^{\frac{1}{40t}} \left( \int_1^T e^{-\frac{3}{40t}} \right)
\]
\[
\leq C(A, K) e^{\frac{1}{40t}}
\]
Using these two estimates we can finish the proof. Recall the evolution equation computed above,
\[
\frac{\partial}{\partial t} s = -\Delta^2 s - \langle r, \text{grad } F \rangle.
\]
Therefore, for times \( t \in \Omega \), we conclude using (30) and (31), for any \( x \in M \),

\[
s(x, t) - s(x, \tau) \geq -\int_{\tau}^{t} \left| \nabla^2 \text{grad} F \right|_{\infty} + B \left| \text{grad} F \right|_{\infty} \\
\geq -C(A, K) \epsilon^{\frac{1}{80}}.
\]

It follows that if \( \epsilon \) is chosen initially small enough, then we may conclude \( s > s_{g_{S^4}} - 2\delta \) for all times \( t \leq T \). A completely analogous argument shows that

\[
\| Rm \|_{\infty}(T) \leq \| Rm \|_{\infty}(1) + C(A, K) \epsilon^{\frac{1}{80}}.
\]

Thus again for \( \epsilon \) chosen small with respect to \( A \) and \( K \) we conclude

\[
\| Rm \|_{\infty}(T) < 2K
\]

The final bound of (25) follows in an analogous fashion. Since \( T \) was arbitrary, we conclude \([0, \infty) \subset \Omega \). The estimates we have shown already imply uniform \( C^k \) convergence \( g(t) \to g_{\infty} \) for any \( k \). The decay estimate (28) and the bound \( s > s_{g_{S^4}} - 2\delta \) together imply that \( g_{\infty} \) is a critical metric with small energy and positive Yamabe constant, which is isometric to \((S^4, g_{S^4})\) by Theorem 2. The proposition follows. \( \square \)

7. Related Questions

It is tempting to ask what the optimal value of \( \epsilon \) is in the statement of the three main theorems. At least for Theorem 1, it seems natural, given the main theorem of [5], that \( 16\pi^2 \) is the optimal value. However, this is not completely clear, since solutions to (2) do not necessarily preserve upper bounds on the Weyl tensor. Indeed, it was exactly this problem which forced us to use Lemma 12 to ensure that the \( L^2 \) norm of the Weyl curvature was staying small for a fixed time.

However, if instead of (2), one considered the Bach flow, i.e. the negative gradient flow of the squared \( L^2 \) norm of the Weyl curvature, then the hypothesis \( \| W \|_{L^2} < 16\pi^2 \chi(M) \) becomes quite natural. It is furthermore natural to conjecture in this setting that solutions to the Bach flow with initial condition satisfying this hypothesis exist for all time and converge to round metrics. Many of the techniques used here can likely be adapted to this setting, but new challenges will certainly arise. Indeed, to even define the Bach flow requires adding a certain conformal term to the flow to overcome the nonparabolicity of the Bach flow which arises due to the conformal invariance of the Bach tensor. The existence of this flow with small energy remains an interesting open question.

8. Appendix: Sobolev Inequalities

In this appendix we record a multiplicative Sobolev inequality for Riemannian manifolds. The proof is as adaptation of techniques used in [9].
Theorem 19. Let \((M^4, g)\) be a Riemannian manifold of unit volume. For \(u \in C^1_0(M)\), \(4 < p \leq \infty\), \(0 \leq m \leq \infty\) we have

\[
||u||_{\infty} \leq C_S \cdot C(n, m, p) ||u||^{1-\alpha}_m (||\nabla u||_p + ||u||_p)^\alpha
\]

where \(0 < \alpha \leq 1\) satisfies

\[
\frac{1}{\alpha} = \left(\frac{1}{4} - \frac{1}{p}\right) m + 1
\]

Proof. Let \(A\) denote the Sobolev constant of \((M, g)\). Fix \(p > 4\), and rescale \(u\) such that

\[
A (||\nabla u||_{L^p} + ||u||_{L^p}) = 1
\]

Let \(q = \frac{2p}{p-2}\) and note that for any \(w \geq 0\),

\[
||u^{1+w}||_{L^4} \leq A (||\nabla (u^{1+w})||_{L^2} + ||u^{1+w}||_{L^2}) \\
\leq A (1+w) ||u^w||_{L^q} (||\nabla u||_{L^p} + ||u||_{L^p}) \\
\leq (1+w) ||u^w||_{L^q}
\]

Let \(j = \frac{q}{4} \in (2, 4]\). Then we can rewrite the above estimate as

\[
||u||_{j(1+w)q} \leq (1+w)^{\frac{1}{q+1}} ||u||_{\frac{w+1}{wq}}^{\frac{w}{wq}}
\]

We want to apply this estimate inductively. To that end let \(w_0 = \frac{m}{q}\), \(w_{i+1} = j(1+w_i)\), \(\delta_i = \frac{w_i}{w_{i+1}}\), \(C_i = (1+w_i)^{\frac{1}{q+w_i}}\). Using this notation the above estimate reads

\[
||u||_{w_{i+1}q} \leq C_i ||u||^{\delta_i}_{w_iq}.
\]

Applying this estimate inductively yields

\[
||u||_{w_1q} \leq \left(\prod_{l=0}^{i-1} C_l^{\delta_l+1} \cdot \delta_{l-1}\right) ||u||^{\delta_{0} \cdots \delta_{i-1}}_m.
\]

Now observe the formula

\[
1 + w_i = j^i w_0 + \sum_{l=0}^{i} j^l
\]

This implies that there exists a constant \(C\) depending on \(m\) and \(p\) such that

\[
\frac{1}{C j^i} \leq 1 + w_i \leq C j^i
\]

Since each \(\delta_i \leq 1\) this implies the estimate

\[
\log \prod_{l=0}^{i} C_l^{\delta_l+1} \cdot \delta_l \leq \sum_{l=0}^{i} \frac{1}{1 + w_l} \log(1 + w_l) \\
\leq \sum_{l=0}^{\infty} C j^{-l} (l \log j) \\
\leq C.
\]
Furthermore we compute
\[
\prod_{l=0}^{\infty} \delta_l = \lim_{i \to \infty} \sum_{j=0}^{i} \frac{w_0}{1 + w_j} = \frac{w_0}{w_0 + \frac{1}{j-1}} = 1 - \alpha.
\]
\[\square\]

References

[1] Aubin, T. Équations différentielles non linéaires et problème de Yamabe concertant la courbure scalaire, J. Math. Pures Appl. (9), 55 (1976), 269-296.
[2] Bando, S. Real analyticity of solutions of Hamilton’s equation, Math. Zeit. 195 (1987), 93-97.
[3] Berger, M. Quelques formules de variation pour une structure riemannienne. Ann. Sci. École Norm. Sup. (4) 3 1970, 285-294.
[4] Besse, A. Einstein Manifolds, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 10, Springer-Verlag, Berlin, 1987.
[5] Chang, S.Y.; Gursky, M.; Yang, P. A conformally invariant sphere theorem in four dimensions Publ. Math. Inst. Hautes Études Sci., No. 98 (2003) 105-143.
[6] Chen, X.X.; LeBrun, C.; Weber, B. On conformally Kähler, Einstein manifolds J. Amer. Math. Soc. 21 (2008), no. 4, 1137-1168.
[7] Gursky, M. Four-manifolds with \( \delta W^+ \) and Einstein constants of the sphere, Math. Ann. 318, 417-431 (2000).
[8] Hamilton, R. Four-manifolds with positive curvature operator J. Diff. Geom. 24 (1986), no. 2, 153-179.
[9] Ladyzhenskaya, O.A.; Solonnikov, V.A.; Uraltseva, N.N. Linear and quasilinear equations of parabolic type, Amer. Math. Soc, 1968.
[10] LeBrun, C. Curvature functionals, optimal metrics, and the differential topology of 4-manifolds. Different faces of geometry, 199-256, Int. Math. Ser. (N.Y.),3, New York, 2004.
[11] Lee, J. Parker, T. The Yamabe problem Bull. Am. Math. Soc. 17 (1987).
[12] Streets, J. The gradient flow of \( \int_M |Rm|^2 \), J. Geom. Anal. 18 (2008), no. 1, 249-271.
[13] Streets, J. The gradient flow of \( \int_M |Rm|^2 \) with small initial energy, preprint.

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