BEHAVIOR OF ZEROS OF $X_1$-JACOBI AND $X_1$-LAGUERRE EXCEPTIONAL POLYNOMIALS

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Abstract. The $X_1$-Jacobi and the $X_1$-Laguerre exceptional orthogonal polynomials have been introduced and studied by Gómez-Ullate, Kamran and Milson in a series of papers. In this note, we establish some properties about the called regular and exceptional zeros of these two classes of polynomials such as interlacing, monotonicity with respect to the parameters and asymptotic behavior.

1. Introduction and statement of the results

In [7] is initially introduced the exceptional orthogonal polynomials $X_1$-Jacobi and $X_1$-Laguerre and many properties of them is defined as well. Since then, a series of results, generalizations and conjectures about the sequence of exceptional orthogonal polynomials sequence(XOPS, in short) and about the zeros were published [6,8,9,11–13]. In this paper, we will comment briefly on these two classes of polynomials and list some crucial properties in [7] to help the development of this paper.

Given $\alpha > 0$, the $X_1$-Laguerre XOPS, denoted by $\hat{L}_n^{(\alpha)}(x)$, is obtained by the Gram-Schmidt orthogonalization process from the monic polynomials given by $x + \alpha + 1, (x + \alpha)^2, (x + \alpha)^3, \ldots,$ where the inner product is given by

$$\langle P, Q \rangle_\alpha := \int_0^\infty P(x)Q(x)e^{-x(\alpha+1)}dx.$$ 

By $x_{n,k}^{(\alpha)}$, $k = 1, \ldots, n$, we denote the zeros in an increasing order. In the same work (see [7]), there are established that the $\hat{L}_n^{(\alpha)}(x)$ has $n - 1$ distinct zeros in $(0, \infty)$ and the remaining in $(-\infty, -\alpha)$. The first ones are called the regular zeros and the last one the exceptional zero. Moreover, there are the existence of the following relationship with the Laguerre orthogonal polynomials sequence(ORPS, in short) and the XOPS $X_1$-Laguerre

$$\hat{L}_n^{(\alpha)}(x) = -(x + \alpha + 1)L_{n-1}^{(\alpha)}(x) + L_n^{(\alpha)}(x),$$

where $L_n^{(\alpha)}(x)$ is $n$-th Laguerre polynomial. Another important property is the $X_1$-Laguerre XPOS be a solution of the following differential equation

$$x(x + \alpha)y'' - (x - \alpha)(x + \alpha + 1)y' + [nx + (n - 2)\alpha]y = 0.$$
Given $\alpha, \beta > -1$ with $\text{sign}(\alpha) = \text{sign}(\beta)$ and $\alpha \neq \beta$, let

$$a = \frac{1}{2}(\beta - \alpha), \ b = \frac{\beta + \alpha}{\beta - \alpha} \quad \text{and} \quad c = b + \frac{1}{a}.$$ 

The $X_1$-Jacobi XOPS denoted by $\hat{P}_n^{(\alpha,\beta)}(x)$ is obtained by the Gram-Schmidt orthogonalization process from the monic polynomials

$$(x-c), (x-b)^2, (x-b)^3, \ldots,$$

where the inner product is given by

$$\langle P, Q \rangle_{\alpha,\beta} := \int_{-1}^{1} P(x)Q(x)\frac{(1-x)^{\alpha}(1+x)^{\beta}}{(x-b)^2}dx.$$ 

By $x_{n,k}^{(\alpha,\beta)}$, $k = 1, \ldots, n$, we denote zeros of $\hat{P}_n^{(\alpha,\beta)}(x)$ in an increasing order. In [8] is established that the $\hat{P}_n^{(\alpha,\beta)}(x)$ has $n-1$ distinct zeros in $(-1,1)$ and the remaining in $(-\infty,b)$, or $(b,\infty)$, depending on the parameters $\alpha$ and $\beta$. The first ones are called regular zeros and the last one the exceptional zero. Moreover, there are the relationship with the usual polynomials of Jacobi and XOPS $X_1$-Jacobi

$$\hat{P}_n^{(\alpha,\beta)}(x) = -\frac{1}{2}(x-b)P_{n-1}^{(\alpha,\beta)}(x) + \frac{bP_{n-1}^{(\alpha,\beta)}(x) - P_{n-2}^{(\alpha,\beta)}(x)}{2n-2+\alpha+\beta},$$

and

$$-\frac{1}{4}(x-b)^2P_n^{(\alpha,\beta)}(x) = f_{n+1}\hat{P}_{n+2}^{(\alpha,\beta)}(x) - 2b\hat{g}_n\hat{P}_{n+1}^{(\alpha,\beta)}(x) + \hat{h}_n\hat{P}_n^{(\alpha,\beta)}(x),$$

where

$$f_n = \frac{n(n+\alpha+\beta)}{(2n-1+\alpha+\beta)(2n+\alpha+\beta)},$$

$$\hat{g}_n = \frac{(n+\alpha)(n+\alpha)}{(2n+2+\alpha+\beta)(2n+\alpha+\beta)},$$

$$\hat{h}_n = \frac{(n-1+\alpha)(n-1+\alpha)}{(2n+1+\alpha+\beta)(2n+1+\alpha+\beta)}.$$ 

There are established in [6] that the exceptional zeros of $X_1$-Jacobi XOPS belong in $(b,c)$ and the interlacing property of the regular zeros of the consecutive degrees polynomials in this sequence. Furthermore, the regular zeros are increasing functions of $\beta$ and decreasing functions of $\alpha$ on certain conditions in this parameters. One of our contributions is to establish the same properties about zeros of $X_1$-Laguerre XOPS

**Theorem 1.** Let $\{\hat{P}_n^{(\alpha)}(x)\}_{n=1}^{\infty}$ $X_1$-Laguerre XOPS. Then, the regular zeros of consecutive degrees polynomials interlace. In the other words,

$$x_{n+1,2}^{(\alpha)} < x_{n,2}^{(\alpha)} < \ldots < x_{n,n}^{(\alpha)} < x_{n+1,n+1}^{(\alpha)}.$$ 

Moreover, the regular zeros are increasing functions of $\alpha$.

It is known that all families of exceptional polynomials have the weight function as follows

$$W(x) = \frac{W_{0}(x)}{\eta(x)^2},$$

where

$$\eta(x) = (1-x)^{\alpha}(1+x)^{\beta}.$$
where $W_0(x)$ is a classical orthogonal polynomials weight and the $\eta(x)$ is a certain polynomial, whose degree is equal to the number of exceptional zeros or the number of gaps in the degree of XPOS, and which does not vanish on the domain of orthogonality. In [11] is postulated and partially proved the following conjecture for the class of exceptional Hermite polynomials introduced in [10].

**Conjecture 1.** The regular zeros of XOPS have the same asymptotic behaviour as the zeros of their classical counterpart. The exceptional zeros converge to the zeros of the denominator polynomial $\eta(x)$.

In the case of $X_1$-Jacobi and $X_1$-Laguerre XOPS, there are established in [9] the asymptotic behavior of them (see Proposition 3.4 and Proposition 5.6) which imply immediately the convergence of the exceptional zeros of $\hat{L}_n^{(\alpha)}(x)$ and $\hat{P}_n^{(\alpha,\beta)}(x)$ to $-\alpha$ and $b$, respectively. The main results of our work are establish the order of the exceptional zeros sequence. In mathematical words,

**Theorem 2.** Given $\hat{P}_n^{(\alpha,\beta)}(x)$ the $n$-th $X_1$-Jacobi XOPS and denote by $\hat{x}_{n,k}^{(\alpha,\beta)}$, $k = 1, \ldots, n$, the zeros in increasing order. If $0 \leq \alpha < \beta$, then

$$b < \hat{x}_{n,n}^{(\alpha,\beta)} < \frac{2n + \alpha + \beta}{2n - 2 + \alpha + \beta} b.$$  

Moreover, the sequence of exceptional zeros $\{\hat{x}_{n,n}^{(\alpha,\beta)}\}_{n=1}^\infty$ is strictly increasing to $b$. In the other words,

$$b < \ldots < \hat{x}_{n,n}^{(\alpha,\beta)} < \ldots < \hat{x}_{2,2}^{(\alpha,\beta)} < \hat{x}_{1,1}^{(\alpha,\beta)} = c.$$  

Note that the convergence of exceptional zeros sequence of $X_1$-Jacobi XOPS can be established by the inequality (5).

Is well known the existence of a connection between the zeros of Jacobi and Laguerre OPRS (see [15]). The following theorem establishes the same statement about the zeros of $X_1$-Jacobi and $X_1$-Laguerre XOPS which is essential to establish the order of exceptional zeros sequence of the $X_1$-Laguerre XOPS.

**Proposition 1.** Given $0 \leq \alpha < \beta$, let $\hat{x}_{n,k}^{(\alpha)}$ and $\hat{x}_{n,n+1-k}^{(\alpha)}$ be the zeros of $n$-th $X_1$-Jacobi and $X_1$-Laguerre XOPS, respectively. Then,

$$\lim_{\beta \to \infty} \frac{\beta (1 - \hat{x}_{n,k}^{(\alpha)})}{2} = \hat{x}_{n,n+1-k}^{(\alpha)},$$

for $k = 1, \ldots, n$.

**Corollary 1.** Given $\hat{L}_n^{(\alpha,\beta)}(x)$ the $n$-th $X_1$-Laguerre XOPS and denote by $\hat{x}_{n,k}^{(\alpha)}$, for $k = 1, \ldots, n$, the zeros in increasing order. If $0 \leq \alpha$, then $\hat{x}_{n,1}^{(\alpha)}$ belongs to the interval $[-\alpha - 1, -\alpha)$, $n \geq 1$. Moreover, the exceptional zeros sequence $\{\hat{x}_{n,1}^{(\alpha)}\}_{n=1}^\infty$ is strictly increasing to $-\alpha$. In the other words,

$$-(\alpha + 1) = \hat{x}_{1,1}^{(\alpha)} < \ldots < \hat{x}_{n-1,1}^{(\alpha)} < \hat{x}_{n,1}^{(\alpha)} < \ldots < -\alpha.$$  

Another well known connection is between Hermite and Laguerre OPRS in [1] due to F. Calogero,

$$n! \lim_{\alpha \to \infty} \left( \frac{2}{\alpha} \right)^{n/2} L_n^{(\alpha)}(x) = \lim_{\alpha \to \infty} H_n\left( \frac{x - \alpha}{\sqrt{2\alpha}} \right).$$
in the other words, the zeros of Laguerre OPRS, by a change of variable, converges to zeros of Hermite OPRS. The similar statement can be established about the regular zeros of $X_1$-Laguerre XOPS

**Proposition 2.** Given $\hat{L}^{(\alpha, \beta)}_n(x)$ and $H_n(x)$ the $n$-th polynomials of $X_1$-Laguerre XOPS and Hermite OPRS, respectively, and denoted by $\hat{x}^{(\alpha)}_{n,k}$ and $h_{n,k}$, $k = 1, \ldots, n$, the zeros in increasing order. Then

$$\lim_{\alpha \to \infty} \frac{\hat{x}^{(\alpha)}_{n,k} - \alpha}{\sqrt{2\alpha}} = h_{n-1,k-1},$$

for $k = 2, \ldots, n$.

2. Preliminaries

The tools to establish the results of our work are the Descartes’s sign rule and the Sturm comparison theorem in the classic and refined version.

**Theorem A** (Descartes’s Sign Rule). Let $P(x)$ a polynomial with real coefficients. The number of positive zeros of $P(x)$ is equal the change of signs of coefficients or is less than it by an even number.

**Theorem B** (Sturm Comparison Theorem). Let $y(x)$ and $Y(x)$ be solutions of the differential equations

$$y''(x) + f(x)y(x) = 0$$

and

$$Y''(x) + F(x)Y(x) = 0,$$

where $f, F \in C(r, s)$ and $f(x) \leq F(x), f \not\equiv F$, in $(r, s)$. Let $x_1$ and $x_2$, with $r < x_1 < x_2 < s$ be two consecutive zeros of $y(x)$. Then the function $Y(x)$ has at least one variation of sign in the interval $(x_1, x_2)$ provided $f(x) \not\equiv F(x)$ there. The statement holds also:

- for $x_1 = r$ if

$$y(r + 0) = 0 \text{ or } \lim_{x \to r^+} \{y'(x)Y(x) - y(x)Y'(x)\} = 0;$$

- for $x_2 = s$ if

$$y(s - 0) = 0 \text{ or } \lim_{x \to s^-} \{y'(x)Y(x) - y(x)Y'(x)\} = 0.$$

Note that in the above mentioned Sturm comparison theorem, denoted by classical version, the function $F - f$ does not change of sign in $(r, s)$. In [2,3], there are established the refined version, in which the difference $F - f$ changes of sign only once in the interval $(r, s)$. This modification allows a huge application(see [3,6]).

**Theorem C** ([2,3]). Let $y(x; \tau)$ be solution of the differential equation

$$y''(x; \tau) + f(x; \tau)y(x; \tau) = 0$$

which depends on a parameter $\tau$, where the differentiation is with respect to the variable $x$ and $f \in C[(r, s) \times (c, d)]$. Given $\tau_1, \tau_2$ in $(c, d)$, let $\{x_j(\tau_1)\}_{j=1}^n$ and $\{x_j(\tau_2)\}_{j=1}^n$ be the distinct zeros of $y(x; \tau_1)$ and $y(x; \tau_2)$ in $(r, s)$, respectively. Suppose further that

$$\lim_{x \to r^+} \{y'(x; \tau_1) \cdot y(x; \tau_2) - y'(x; \tau_2) \cdot y(x; \tau_1)\} = 0$$
ZEROS OF $X_1$-JACOBI AND $X_1$-LAGUERRE EXCEPTIONAL POLYNOMIALS

and

$$\lim_{x \to \infty} \{y'(x; \tau_1) \cdot y(x; \tau_2) - y'(x; \tau_2) \cdot y(x; \tau_1)\} = 0.$$  

If exists $\eta \in (r, s)$ such that $f(\eta; \tau_1) = f(\eta; \tau_2)$ and

- $f(x; \tau_2) - f(x; \tau_1) < 0$ for $x \in (0, \eta)$, $f(x; \tau_2) - f(x; \tau_1) > 0$ for $x \in (\eta, s)$, then $x_k(\tau_1) < x_k(\tau_2)$ for every $k = 1, \ldots, n$;
- $f(x; \tau_2) - f(x; \tau_1) > 0$ for $x \in (0, \eta)$, $f(x; \tau_2) - f(x; \tau_1) < 0$ for $x \in (\eta, s)$, then $x_k(\tau_1) > x_k(\tau_2)$ for every $k = 1, \ldots, n$.

The $X_1$-Laguerre XOPS satisfy the differential equation (2) which is equivalent to the Sturm-Liouville form

$$u''(x) + \lambda_{n,4}(x)u_n(x) = 0,$$

where

$$\lambda_{n,4}(x) = \frac{-x^4 + A_3(n,\alpha)x^3 + A_2(n,\alpha)x^2 + A_1(n,\alpha)x + A_0(n,\alpha)}{4x^2(x + \alpha)^2},$$

for $n = 1, 2, \ldots$, the zeros of $P_n^{(\alpha, \beta)}(x)$ have $n + 1$ zeros in $(-1, 1)$. Suppose there is $\xi > 1$ such that

$$P_n^{(\alpha, \beta)}(\xi) \geq P_{n+1}^{(\alpha, \beta)}(\xi),$$

then $Q_{n+1}(x)$ would have one more zero in $(1, \infty)$ which is an obvious contradiction.

\textbf{Lemma 1.} Let $\{P_n^{(\alpha, \beta)}(x)\}_{n=0}^{\infty}$ be Jacobi OPRS. Then, $P_n^{(\alpha, \beta)}(x) < P_{n+1}^{(\alpha, \beta)}(x)$ for every $x > 1$.

\textbf{Proof.} Given $n \geq 1$, denote by $x_n^{(\alpha, \beta)}$, $k = 1, \ldots, n$, the zeros of $P_n^{(\alpha, \beta)}(x) = a_{n, n}^{(\alpha, \beta)} x^n + \ldots$ in increasing order. By the interlacing property and $a_{n, n}^{(\alpha, \beta)} > 0$, the polynomial $Q_{n+1}(x) := P_{n+1}^{(\alpha, \beta)}(x) - P_n^{(\alpha, \beta)}(x)$ has $n$ zeros in $(x_1^{(\alpha, \beta)}, x_{n+1}^{(\alpha, \beta)})$ and

$$P_{n+1}^{(\alpha, \beta)}(x_{n+1}^{(\alpha, \beta)} + \epsilon) < P_{n+1}^{(\alpha, \beta)}(x_{n+1}^{(\alpha, \beta)} + \epsilon),$$

for $\epsilon > 0$ sufficient small. On the other hand, we have

$$P_n^{(\alpha, \beta)}(1) = \binom{n + \alpha}{n} < \binom{n + 1 + \alpha}{n + 1} = P_{n+1}^{(\alpha, \beta)}(1),$$

which implies $Q_{n+1}(x)$ has $n + 1$ zeros in $(-1, 1)$. Suppose there is $\xi > 1$ such that $P_n^{(\alpha, \beta)}(\xi) \geq P_{n+1}^{(\alpha, \beta)}(\xi)$, then $Q_{n+1}(x)$ would have one more zero in $(1, \infty)$ which is an obvious contradiction.

\section{3. Proofs of main results}

\textbf{Proof of Theorem 1.} The first statement is a consequence of the direct application of Theorem 2. For this purpose, note that

$$\lambda_{n+1,4}(x) - \lambda_{n,4}(x) = \frac{1}{x}$$

and the solution $u_{n+1}(x)$ satisfy the conditions (11) and (12) for $r = 0$ and $s = \infty$. 


The monotonicity of zeros is established by the application of Theorem C. Note that
\[
\frac{\partial \lambda_{n,k}(x)}{\partial \alpha} = \frac{p_{n,k}(x;\alpha)}{2x^2(\alpha+x)^3} = \frac{x^4 + B_3(\alpha)x^3 + B_2(\alpha)x^2 + B_1(\alpha)x + B_0(\alpha)}{2x^2(x+\alpha)^3},
\]
with
\[
B_3(\alpha) = 2\alpha, \\
B_2(\alpha) = 6, \\
B_1(\alpha) = -2\alpha(1+\alpha^2), \\
B_0(\alpha) = -\alpha^4.
\]
Obviously, the sequence \( \{B_k(\alpha)\}_{k=0}^3 \) has just one sign change for every \( \alpha > 0 \). Therefore, by the Descartes sign rule, \( p_{n,k}(x;\alpha) \) has just one positive zero and the conclusion follows by the classical Sturm comparison theorem. \( \square \)

**Proof of Theorem Z** Firstly, we show that there is a partition in \((b, c)\] which divide the sequence of exceptional zeros of \( \{\hat{P}_n(\alpha,\beta)(x)\}_{n=1}^\infty \) in \((b, c)\]. This zero belongs to \((b, \gamma_n b)\), where \( \gamma_n := (2n + \alpha + \beta)/(2n - 2 + \alpha + \beta) \), for \( n \geq 2 \). In fact, by \( \{3\} \) we have
\[
\hat{P}_n(\alpha,\beta)(x) = f_1(x)P_{n-1}^{(\alpha,\beta)}(x) - \frac{P_{n-2}^{(\alpha,\beta)}(x)}{2n - 2 + \alpha + \beta},
\]
where
\[
f_1(x) := -\frac{1}{2}(x-b) + \frac{b}{2n - 2 + \alpha + \beta},
\]
and \( f_1(x) = 0 \) if and only if \( x = (2n + \alpha + \beta)b/(2n - 2 + \alpha + \beta) \). Since the leading coefficient of the Jacobi polynomial is positive and the exceptional zeros of \(X_1\)-Jacobi XOPS belong in \((b, c)\], we have \( f_1(x)P_{n-1}^{(\alpha,\beta)}(x) \) and \( -P_{n-2}^{(\alpha,\beta)}(x)/(2n - 2 + \alpha + \beta) \) have opposite signs in \((b, \gamma_n b)\) and the same sign in \( (\gamma_n b, c)\]. Therefore, the exceptional zero of \( \hat{P}_n(\alpha,\beta)(x) \) belongs to \((b, \gamma_n b)\).

Note that \( \gamma_1 b = c = \hat{x}_{1,1}^{(\alpha,\beta)} \) and \( \gamma_n \) decreases strictly in \( n \). Then, the exceptional zeros \( \hat{x}_{2,2}^{(\alpha,\beta)} \) and \( \hat{x}_{1,1}^{(\alpha,\beta)} \) are distinct for any \( \alpha \) and \( \beta \). On the other hand, \( \hat{P}_n^{(\alpha,\beta)}(\gamma_{n+1} b) > 0 \) for \( \alpha \) and \( \beta \) sufficiently close. In fact, by formula \( \{3\} \) and Lemma \( \{4\} \) we have
\[
\hat{P}_n^{(\alpha,\beta)}(\gamma_{n+1} b) = \frac{b}{2n - 2 + \alpha + \beta} \left[ \frac{P_{n-1}^{(\alpha,\beta)}(\gamma_{n+1} b)}{2n + \alpha + \beta} - \frac{\beta - \alpha}{\beta + \alpha} P_{n-2}^{(\alpha,\beta)}(\gamma_{n+1} b) \right] > 0.
\]
Then, subject to this condition for \( \alpha \) and \( \beta \) sufficiently close, we have \( \gamma_{n+1} b < \hat{x}_{n,n}^{(\alpha,\beta)} < \gamma_n b \). Therefore
\[
b < \ldots < \hat{x}_{n,n}^{(\alpha,\beta)} < \ldots < \hat{x}_{2,2}^{(\alpha,\beta)} < \hat{x}_{1,1}^{(\alpha,\beta)} = c.
\]
Since the zeros of \( \hat{P}_n^{(\alpha,\beta)}(x) \) are continuous functions of the parameters \( \alpha \) and \( \beta \), we have the inequality \( \{5\} \) is initially satisfied for them sufficiently close. If we change the parameters, the inequality \( \{5\} \) still remains valid. In fact, suppose that there are the pair \( (\alpha, \beta) \) and \( k \) such that
\[
\hat{x}_{k+2,k+2}^{(\alpha,\beta)} = \hat{x}_{k+1,k+1}^{(\alpha,\beta)} = \hat{x}_{k,k}^{(\alpha,\beta)} < \hat{x}_{k-1,k-1}^{(\alpha,\beta)}.
\]
and denote by \( \xi := x_{k+1,k+1}^{(a,b)} \). Since \( x_{2,2}^{(a,b)} \neq x_{1,1}^{(a,b)} \) for every values of \( a \) and \( b \), then we have \( k \geq 2 \) and, by \([1]\),

\[
0 > -\frac{1}{4} (\xi - b)^2 P_{k-1}^{(a,b)} (\xi) = \hat{h}_{k-1}^{(a,b)} (\xi) > 0,
\]

which is a contradiction. \( \square \)

**Proof of Proposition** [7] The proof is based on the Gaussian hypergeometric representation of the Jacobi polynomials

\[
P_n^{(a,b)} \left( 1 - \frac{2x}{\beta} \right) = \frac{(\alpha + 1)_n}{n!} {\,}_2F_1 \left( -n, \alpha + \beta + n + 1; \alpha + 1; \frac{x}{\beta} \right)
\]

(15)

\[
= \sum_{k=0}^{n} \frac{R_n^{(a,b)}(x)}{\beta^k},
\]

where \( R_n^{(a,b)}(x) = L_n^{(a,b)}(x) \) (see [15, section 5.3]).

For \( \alpha < \beta \), we express

\[
b = \beta + \alpha \frac{\beta - \alpha}{\beta - \alpha} = \left( 1 + \frac{\alpha}{\beta} \right) \sum_{k=0}^{\infty} \left( \frac{\alpha}{\beta} \right)^k = 1 + 2 \sum_{k=1}^{\infty} \left( \frac{\alpha}{\beta} \right)^k.
\]

We apply \([15]\) in \([5]\), we have \((2n - 2 + 2ab) \hat{L}_n^{(a,b)}(1 - 2\beta^{-1}x) = \)

\[
= \left\{ (2n - 2 + \beta + \alpha) \left[ \frac{x + \alpha}{\beta} + \sum_{k=2}^{\infty} \left( \frac{\alpha}{\beta} \right)^k \right] + 1 + 2 \sum_{k=1}^{\infty} \left( \frac{\alpha}{\beta} \right)^k \right\} \left\{ \sum_{k=0}^{n-1} \frac{R_{n-1,k}^{(a,b)}(x)}{\beta^k} \right\}
\]

\[
- \sum_{k=0}^{n-2} \frac{R_{n-2,k}^{(a,b)}(x)}{\beta^k}
\]

\[
= \left[ (2n - 2 + \alpha + \beta) \frac{x + \alpha}{\beta} + 1 \right] R_n^{(a,b)}(x) - R_{n-2,0}^{(a,b)}(x) + O(\beta^{-1})
\]

\[
= (x + \alpha + 1) L_{n-1}^{(a,b)}(x) - L_{n-2}^{(a,b)}(x) + O(\beta^{-1})
\]

\[
= -\hat{L}_n^{(a,b)}(x) + O(\beta^{-1}).
\]

\( \square \)

**Proof of Corollary** [7] We know \( \hat{L}_1^{(a)}(x) = -(x + \alpha + 1) \) and the **exceptional** zeros of \( \hat{L}_n^{(a)}(x) \) are smaller than \(-\alpha\) (see [7]). To establish the lower bound of these zeros is sufficiently to analyze the formula \([11]\). Note that \(-\hat{L}_1^{(a)}(x)L_n^{(a)}(x)\) and \( L_n^{(a)}(x) \) have opposite signs in \((-\alpha - 1, 0)\) and same signs in \((-\infty, -\alpha - 1)\) for \( n \geq 3 \). Then the **exceptional** zeros belong to \([-\alpha - 1, -\alpha]\). The inequality \([8]\) is followed by Theorem \([2]\) and Proposition \([1]\) once

\[
\hat{x}_{n+1,n+1}^{(a,b)} < \hat{x}_{n,n}^{(a,b)} \iff \frac{\beta(1 - \hat{x}_{n,n}^{(a,b)})}{2} < \frac{\beta(1 - \hat{x}_{n+1,n+1}^{(a,b)})}{2},
\]

for \( n \geq 1 \). Taking \( \beta \) sufficiently large, we obtain the inequality \([8]\). \( \square \)
Proof of Proposition 2 By (1), we have

\[(n-1)! \frac{\left(\frac{2}{\alpha}\right)^{(n-1)/2} \tilde{L}_n^{(\alpha)}(x)}{x - \tilde{x}_{n,1}^{(\alpha)}} = -\frac{x + \alpha + 1}{x - \tilde{x}_{n,1}^{(\alpha)}} (n-1)! \frac{\left(\frac{2}{\alpha}\right)^{(n-1)/2} L_{n-1}^{(\alpha)}(x)}{x - \tilde{x}_{n,1}^{(\alpha)}} + \frac{(n-1)\sqrt{2}}{(x - \tilde{x}_{n,1}^{(\alpha)})\sqrt{\alpha}} (n-2)! \frac{\left(\frac{2}{\alpha}\right)^{(n-2)/2} L_{n-2}^{(\alpha)}(x)}{x - \tilde{x}_{n,1}^{(\alpha)}}.\]

Theorem 1 assures that \(-\tilde{x}_{n,1}^{(\alpha)} \in (\alpha, \alpha + 1]\) for \(n \geq 1\). Therefore, applying a limit in the above equation, we obtain

\[(n-1)! \lim_{\alpha \to \infty} \left(\frac{2}{\alpha}\right)^{(n-1)/2} \frac{\tilde{L}_n^{(\alpha)}(x)}{x - \tilde{x}_{n,1}^{(\alpha)}} = -\lim_{\alpha \to \infty} H_{n-1} \left(\frac{x - \alpha}{\sqrt{2\alpha}}\right).\]

4. Final Remarks

Another interesting question involving the identities (7) and (10) is about a monotonicity. Is known by [4, 5] that the product

\[f_n^{(\alpha, \beta)}(1 - x_{n,k}^{(\alpha, \beta)})\]

is strictly increasing to \(x_{n,n-k}^{(\alpha)}\) and

\[x_{n,k}^{(\alpha)} - \frac{(2n + \alpha - 1)}{\sqrt{2(n + \alpha - 1)}}\]

is strictly increasing to \(h_{n,k}\), for each \(k = 1, \ldots, n\), where \(f_n^{(\alpha, \beta)} = 2n^2 + 2n(\alpha + \beta + 1) + (\alpha + 1)(\beta + 1)\) and \(x_{n,k}^{(\alpha)}, x_{n,k}^{(\alpha, \beta)}, h_{n,k}\) denote zeros of Laguerre, Jacobi and Hermite polynomials, respectively. The numerical tests confirm the same behavior in the zeros of \(X_1\)-Jacobi and \(X_1\)-Laguerre XOPS.

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