Coalgebraic Semantics for Nominal Automata

Florian Frank, Stefan Milius, Henning Urbat

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Overview

An exercise in coalgebra:

1. Kleisli style coalgebraic trace semantics
   
ah à Jacobs, Hasuo, and Sokolova revisited

2. The language semantics of nominal automata arises as an instance of
   
   - Kleisli-style coalgebraic trace semantics
   - Eilenberg-Moore-style coalgebraic language semantics
     (generalized determinization)

   The coincidence is an instance of Jacobs, Silva, and Sokolova’s
   extension semantics.

Disclaimer

No surprising results for the cognoscenti.

But some pitfalls and surprises in the proofs.
Part 1:
Coalgebraic Trace Semantics
The Motivating Example

Labelled transition systems (with explicit termination) are coalgebras

\[ X \to \mathcal{P}(1 + \Sigma \times X) = TFX \]

- the initial algebra

\[ \mu F = (1 + \Sigma \times \Sigma^* \xrightarrow{[\text{nil,cons}]} \Sigma^*) \]

yields a terminal coalgebra in \( \text{Kl}(\mathcal{P}) \)

- the unique Kleisli-morphism \( X \to \mathcal{P}(\Sigma^*) \) is \( x \mapsto \{ w \in \Sigma^* : x \xrightarrow{w} y \downarrow \} \)

Reasons:

- \( F \) has an extension \( \bar{F} : \text{Kl}(T) \to \text{Kl}(T) \)
- \( \text{Kl}(\mathcal{P}) \) is enriched in complete partial orders
- \( \bar{F} \) is finitary and locally monotone/continuous
- A Smyth-Plotkin-type limit-colimit-coincidence argument shows that the initial-algebra and terminal coalgebra (chains) coincide.
**DCPO Enriched Categories**

**Definition**

1. A category $\mathcal{C}$ is **left-strictly DCPO$_\perp$-enriched** provided that each hom-set is a dcpo with bottom, and composition is monotone, preserves directed joins and left-strict: for every morphism $f$ and appropriate directed sets of morphisms $g_i$ ($i \in D$) we have

\[
\perp \cdot f = \perp, \quad f \cdot \bigvee_{i \in D} g_i = \bigvee_{i \in D} f \cdot g_i, \quad \left( \bigvee_{i \in D} g_i \right) \cdot f = \bigvee_{i \in D} g_i \cdot f.
\]

2. A functor on $\mathcal{C}$ is **locally monotone** if its restrictions $\mathcal{C}(A, B) \to \mathcal{C}(FA, FB)$ to the hom-sets are monotone.

**Example**

- $\text{Kl}(\mathcal{P})$ is left-strictly DCPO$_\perp$-enriched
- $\bar{F}X = 1 + \Sigma \times X$ is locally monotone
Theorem

Let $C$ be left-strictly DCPO$_\bot$-enriched and $F: C \to C$ locally monotone. Then: $FI \xrightarrow{\iota} I$ initial algebra $\implies I \xrightarrow{\iota^{-1}} FI$ terminal coalgebra.

Freyd proved this for locally continuous functors using Kleene’s theorem.

Proof. **Existence:** given $C \xrightarrow{\gamma} FC$, take

$$C(\mathcal{C}, I) \xrightarrow{\Phi} C(\mathcal{C}, I) \quad (C \xrightarrow{h} I) \quad \iff \quad (C \xrightarrow{\gamma} FC \xrightarrow{Fh} FI \xrightarrow{\iota} I).$$

Since $C(\mathcal{C}, I)$ is a DCPO with $\bot$ and $g$ is monotone, it has a least fixed point $\mu \Phi = h$.

This is a coalgebra homomorphism:

$$h = \iota \cdot Fh \cdot \gamma$$

$$\xymatrix{ C \ar[d]^{\gamma} \ar[r]^{h} & I \ar[d]^{\iota^{-1}} \\
FC \ar[r]^{Fh} & FI}$$
Theorem
Let $\mathcal{C}$ be left-strictly DCPO$_\perp$-enriched and $F: \mathcal{C} \to \mathcal{C}$ locally monotone. Then: $FI \overset{\iota}{\longrightarrow} I$ initial algebra $\implies I \overset{\iota^{-1}}{\longrightarrow} FI$ terminal coalgebra.

Proof. **Unicity:** given $h': (\mathcal{C}, \gamma) \to (\mu F, \iota^{-1})$ form

\[
\begin{align*}
\mathcal{C}(I, I) &\xrightarrow{\Psi} \mathcal{C}(I, I) & (I \xrightarrow{k} I) &\implies (I \xrightarrow{\iota^{-1}} FI \xrightarrow{Fk} FI \xrightarrow{\iota} I) \\
\mathcal{C}(h', I) &\downarrow \downarrow \mathcal{C}(h', I) & (I \xrightarrow{k} I) &\implies (C \xrightarrow{h'} I \xrightarrow{k} I) \\
\mathcal{C}(C, I) &\xrightarrow{\Phi} \mathcal{C}(C, I) & (C \xrightarrow{h} I) &\implies (C \xrightarrow{\gamma} FC \xrightarrow{Fh} FI \xrightarrow{\iota} I)
\end{align*}
\]

- $\mathcal{C}(h', I)$ is strict and monotone, whence preserves least fixed points:
  
  \[C(h', I)(\mu \Psi) = \mu \Phi\] (uniformity)

- $\mu \Psi = id_I$ unique fixed point since $F(\mu F) \overset{\iota}{\longrightarrow} \mu F$ initial algebra

- Thus: $\mu \Phi = C(h', I)(id_I) = id_I \cdot h' = h'$. 
\[\square\]
Instance: Kleisli Categories

**Theorem (Hermida and Jacobs)**

Let $T : C \to C$ be a monad and let $F : C \to C$ have an extension $\bar{F} : \text{Kl}(T) \to \text{Kl}(T)$. Then $J : C \to \text{Kl}(T)$

$FI \xrightarrow{\iota} I$ initial algebra $\iff J \iota = (FI \xrightarrow{\iota} I \xrightarrow{\eta} TI)$ initial $\bar{F}$-algebra.

**Theorem**

Let $T$ be a monad and $F$ a functor on $C$. Assume that

- $\text{Kl}(T)$ is left-strictly DCPO$_\bot$-enriched,
- $F$ has a locally monotone extension $\bar{F} : \text{Kl}(T) \to \text{Kl}(T)$.

Then $(I, \iota)$ initial $F$-algebra $\implies (I, J\iota^{-1})$ terminal $\bar{F}$-coalgebra.

**Proof.** Immediate from the previous two theorems.
Coalgebraic Trace Semantics

Definition

Given $T$ and $F$ as before and a coalgebra $X \xrightarrow{c} TFX$.

The coalgebraic trace map is given by final semantics in $\text{Kl}(T)$:

$$
\begin{array}{ccc}
X & \xrightarrow{\text{tr}_c} & \mu F \\
\downarrow c & & \downarrow \text{J}_t^{-1} \\
\bar{F}X & \xrightarrow{\bar{F}\text{tr}_c} & \bar{F}(\mu F)
\end{array}
$$

Example: Traces of labelled transition systems

Given $X \xrightarrow{c} \mathcal{P}(1 + \Sigma \times X)$, we have $X \xrightarrow{\text{tr}_c} \mathcal{P}(\Sigma^*)$ given by

$$
\text{tr}_c(x) = \{w \in \Sigma^* : x \xrightarrow{w} y\downarrow\} \quad \text{for all } x \in X.
$$
Part 2: Nominal Automata and Their Language Semantics
Nominal Sets (aka Sets with Atoms)

- **\( \mathbb{A} \)** = infinite set of **atoms** or **names**
- **\( \text{Perm}(\mathbb{A}) \)** = \( \{ \pi \mid \pi : \mathbb{A} \to \mathbb{A} \text{ finite permutation} \} \)
- **Nominal set** = set \( X \) with **\( \text{Perm}(\mathbb{A}) \)-action** \( (\pi, x) \mapsto \pi \cdot x \) such that every \( x \in X \) has a finite **support** \( S \) if \( \forall s \in S. \pi(s) = s \), then \( \pi \cdot x = x \)
- **\( \text{supp}(X) \)** = least support of \( X \)
- **\( \text{Nom} \)** category of nominal set and **equivariant** maps \( f : X \to Y \):
  \[
  f(\pi \cdot x) = \pi \cdot f(x) \quad \text{for every } \pi \in \text{Perm}(\mathbb{A}), x \in X.
  \]
Functors on Nom

- Coproducts and finite products as in sets: $X \times Y$, $X + Y$
- $\mathcal{P}_{fs}X = \{ Y \subseteq X \mid Y \text{ finitely supported} \}$, $\pi \cdot Y = \{ \pi \cdot y \mid y \in Y \}$
- $\mathcal{P}_{ufs}X = \{ Y \subseteq X \mid Y \text{ uniformly finitely supported} \}$

- Abstraction functor: $[\mathbb{A}]X = \mathbb{A} \times X / \sim$ where $c \notin \{a, b\} \cup \text{supp}(x) \cup \text{supp}(y)$

\[(a, x) \sim (b, y) : \iff (a \cdot c) \cdot x = (b \cdot c) \cdot y \text{ for some fresh } c\]

Notation: $\langle a \rangle x = [(a, x)]_\sim$. For an equivariant $f : X \rightarrow Y$

$[\mathbb{A}]f : [\mathbb{A}]X \rightarrow [\mathbb{A}]Y$ \quad $\langle a \rangle x \mapsto \langle a \rangle f(x)$.

Intuition: $[\mathbb{A}]X \rightarrow X$ is a name-binding operation e.g. $\lambda$-abstraction
Nominal Automata (which) are Coalgebras

**NOFAs (Bojańczyk, Klin, Lasota 2014)**

- Non-deterministic orbit-finite automata (NOFAs) are coalgebras
  \[ Q \rightarrow \{0, 1\} \times \mathcal{P}_{fs}(A \times Q). \]
- Every state \( q \in Q \) accepts its data language \( L_q \subseteq A^*. \)

**RNNAs (Schröder, Kozen, M, Wißmann 2017)**

Regular nominal non-deterministic automata (RNNAs) are coalgebras
\[ Q \rightarrow \{0, 1\} \times \mathcal{P}_{ufs}(A \times Q) \times \mathcal{P}_{ufs}([A]Q). \]

Concretely: \( \overline{A} = A \cup \{|a : a \in A\}; \)
ordinary and **binding** transitions

Every state accepts its bar language
\[ L_q \subseteq \overline{A}^*/\equiv_{\alpha}. \]

Equivalent generated by \( x|av =_{\alpha} x|bw \)
\( \forall a, b \in A \) and \( x, v, w \in \overline{A}^* \) s.th. \( \langle a \rangle v = \langle b \rangle w. \)
Problem: \( \mathcal{P}_{fs}X \) and \( \mathcal{P}_{ufs}X \) are no complete lattices (not even \( \omega \)-cpos)!

Theorem 😊

The categories \( \text{Kl}(\mathcal{P}_{fs}) \) and \( \text{Kl}(\mathcal{P}_{ufs}) \) are left strictly DCPO\(_{\perp}\)-enriched.

Which functors extend to \( \text{Kl}(\mathcal{P}_{fs}) \) and \( \text{Kl}(\mathcal{P}_{ufs}) \)?
Extending Functors to $\text{Kl}(\mathcal{P}_{fs})$ and $\text{Kl}(\mathcal{P}_{ufs})$

**Definition**

Binding polynomial functors on $\text{Nom}$ are given by the grammar

$$F ::= C \mid \text{Id} \mid [\mathbb{A}](\_ \_) \mid F \times F \mid \bigsqcup_{i \in I} F_i \mid FF,$$

where $C$ ranges over constant functors and $I$ is an index set.

**Theorem**

1. Every binding polynomial functor $F$ has a canonical locally monotone extension $\bar{F}$ on $\text{Kl}(\mathcal{P}_{ufs})$.

   $\leadsto \mu F = \text{terminal } \bar{F} \text{-coalgebra}$

2. Every polynomial functor $F$ has a canonical locally monotone extension $\bar{F}$ on $\text{Kl}(\mathcal{P}_{fs})$.

   $\leadsto \mu F = \text{terminal } \bar{F} \text{-coalgebra}$
There is a canonical natural isomorphism $\psi_X : \mathcal{P}_{fs}(\mathcal{A}X) \to \mathcal{A}\mathcal{P}_{fs}X$. Its inverse is not a distributive law $[\mathcal{A}](\cdot)$ over $\mathcal{P}_{fs}$.

**Proposition**

The abstraction functor admits a (functor over monad) distributive law

$$[\mathcal{A}](\mathcal{P}_{ufs}X) \xrightarrow{\varrho_X} \mathcal{P}_{ufs}([\mathcal{A}]X) \quad \langle a \rangle S \longmapsto \{ \langle a \rangle s : s \in S \}$$

Quotient of the canonical distributive law

$$\xymatrix{ \mathcal{A} \times \mathcal{P}_{ufs}X \ar[r]^{\lambda_X} \ar[d]_{q_{\mathcal{P}_{ufs}X}} & \mathcal{P}_{ufs}(\mathcal{A} \times X) \ar[d]^{\mathcal{P}_{ufs}q_X} \\ [\mathcal{A}](\mathcal{P}_{ufs}X) \ar[r]^{\varrho_X} & \mathcal{P}_{ufs}([\mathcal{A}]X) }$$

**Note:** $\varrho_X$ is not well-defined for $\mathcal{P}_{fs}$; the inverse of $\psi_X$ is more involved.
Instance 1: NOFAs

- NOFAs are coalgebras $Q \xrightarrow{c} \{0, 1\} \times \mathcal{P}_{fs}(A \times Q) \cong \mathcal{P}_{fs}(1 + A \times X)$

- Take $FX = 1 + A \times X$ and $T = \mathcal{P}_{fs}$.

- We have a locally monotone extension $\bar{F} : \text{Kl} (\mathcal{P}_{fs}) \to \text{Kl} (\mathcal{P}_{fs})$.

- Its terminal coalgebra is $\mu F = A^*$.

- The coalgebraic trace map $Q \xrightarrow{\text{tr}_c} \mathcal{P}_{fs}(A^*)$ satisfies

  \[\text{tr}_c(q) = \text{data language of state } q.\]
RNNAs are coalgebras

\[ Q \xrightarrow{c} \{0, 1\} \times \mathcal{P}_{ufs}(\mathbb{A} \times X) \times \mathcal{P}_{ufs}([A]X) \cong \mathcal{P}_{ufs}(1 + \mathbb{A} \times X + [A]X) \]

Take \( FX = 1 + \mathbb{A} \times X + [A]X \) and \( T = \mathcal{P}_{ufs} \).

We have a locally monotone extension \( \bar{F} : \mathcal{Kl}(\mathcal{P}_{ufs}) \to \mathcal{Kl}(\mathcal{P}_{ufs}) \).

Its terminal coalgebra is \( \mu F = \bar{A}^* / =^\alpha \).

The coalgebraic trace map \( Q \xrightarrow{tr_c} \mathcal{P}_{ufs}(\bar{A}^* / =^\alpha) \) satisfies

\[ tr_c(q) = \text{bar language of state } q. \]
Summary

- Revisited Kleisli style coalgebraic trace semantics:
  initial algebra $\mu F$ extends to terminal $\tilde{F}$-coalgebra on $\text{KI}(T)$
  (finitariness or convergence of initial-algebra chain are not needed)

- Language semantics of NOFAs and RNNAs are instances of
  Kleisli style coalgebraic trace semantics
Further Work (in the paper):

- EM-style coalgebraic language semantics of NOFAs and RNNAs using extension semantics by Jacobs, Silva, Sokolova

**Subtleties:** Lifting $[\mathbb{A}](\cdot)$ to $\text{EM}(\mathcal{P}_{ufs})$.

(Once again our proof does not work for $\mathcal{P}_{fs}$.)

$$\mathcal{P}_{fs}(\mathbb{A} \times X) \cong (\mathcal{P}_{fs} X)^{\mathbb{A}} \quad \text{but} \quad \mathcal{P}_{ufs}(\mathbb{A} \times X) \not\cong (\mathcal{P}_{ufs} X)^{\mathbb{A}}$$

**Result:** The coalgebraic language semantics arising from generalized determinization yields the data languages of NOFAs and the bar language of RNNAs, respectively.

**Proof:** use extension natural transformations $\varepsilon : TF \to GT$

$$\mathcal{P}_{fs}(1 + \mathbb{A} \times X) \cong \{0, 1\} \times (\mathcal{P}_{fs} X)^{\mathbb{A}}$$

$$\mathcal{P}_{ufs}(1 + \mathbb{A} \times X + [\mathbb{A}]X) \not\cong \{0, 1\} \times (\mathcal{P}_{ufs} X)^{\mathbb{A}} \times [\mathbb{A}] (\mathcal{P}_{ufs} X)$$
Further Work

Further Work (in the paper):

▶ EM-style coalgebraic language semantics of NOFAs and RNNAs using extension semantics by Jacobs, Silva, Sokolova

Further work (in the future):

▶ Coalgebraic methods/techniques applied to NOFAs, RNNAs, and other nominal systems, e.g.
▶ coalgebraic \( \varepsilon \)-elimination
▶ coalgebraic up-to-techniques might lead to new proof principles and algorithms
▶ semantics for nominal systems based on graded monads should lead to a nominal spectrum of system equivalences (generalizing the van Glabbeek linear time – branching time spectrum)