H-integral and Gaussian integral normal mixed Cayley graphs

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Abstract

If all the eigenvalues of the Hermitian-adjacency matrix of a mixed graph are integers, then the mixed graph is called H-integral. If all the eigenvalues of the (0,1)-adjacency matrix of a mixed graph are Gaussian integers, then the mixed graph is called Gaussian integral. For any finite group Γ, we characterize the set S for which the normal mixed Cayley graph Cay(Γ, S) is H-integral. We further prove that a normal mixed Cayley graph is H-integral if and only if it is Gaussian integral.

Keywords. integral graphs; H-integral mixed graph; Gaussian integral mixed graph; normal mixed Cayley graph.

Mathematics Subject Classifications: 05C50, 20C15.

1 Introduction

A mixed graph G is a pair (V(G), E(G)), where V(G) and E(G) are the vertex set and the edge set of G, respectively. In this case, E(G) ⊆ V(G) × V(G) \ {(u, u): u ∈ V(G)}. If G is a mixed graph, then (u, v) ∈ E(G) need not imply that (v, u) ∈ E(G); for further information see [19]. If both (u, v) and (v, u) are members of E(G), then we call (u, v) to be an undirected edge. If only one of (u, v) and (v, u) is a member of E(G), then we call (u, v) to be a directed edge. It is clear that a mixed graph G can have both undirected and directed edges. If all the edges of G are undirected (resp. directed) then we call G to be a simple graph (resp. oriented graph). Some definitions and results of this paper have similarities with those in the paper [13]. Throughout the paper, we consider i = \sqrt{-1}.

The (0,1)-adjacency matrix and the Hermitian-adjacency matrix of a mixed graph G on n vertices
are denoted by $A(G) = (a_{uv})_{n \times n}$ and $H(G) = (h_{uv})_{n \times n}$, respectively, where

$$a_{uv} = \begin{cases} 1 & \text{if } (u, v) \in E \\
0 & \text{otherwise,} \end{cases} \quad \text{and} \quad h_{uv} = \begin{cases} 1 & \text{if } (u, v) \in E \text{ and } (v, u) \in E \\
i & \text{if } (u, v) \in E \text{ and } (v, u) \notin E \\
-i & \text{if } (u, v) \notin E \text{ and } (v, u) \in E \\
0 & \text{otherwise.} \end{cases}$$

The Hermitian-adjacency matrix of mixed graphs was introduced by Liu and Li [19] in 2015, and later by Guo and Mohar [11] independently. Indeed, Bapat et al. [4] introduced the notion of 3-colored digraph and its adjacency matrix in 2012. The Hermitian-adjacency matrix of mixed graphs is a special case of the adjacency matrix of a 3-colored digraph.

Let $G$ be a mixed graph. We refer to an eigenvalue of $H(G)$ as an $H$-eigenvalue of $G$. An eigenvalue of $A(G)$ is referred to as an eigenvalue of $G$. Similarly, the $H$-spectrum of $G$ is the multi-set of the $H$-eigenvalues of $G$, and the spectrum of $G$ is the multi-set of the eigenvalues of $G$. Since $H(G)$ is a Hermitian matrix, its $H$-eigenvalues are real numbers. However, if $G$ has at least one directed edge then $A(G)$ is not symmetric, and so the eigenvalues of $G$ may or may not be real numbers.

If all of the $H$-eigenvalues of a mixed graph $G$ are integers, it is said to be $H$-integral. If all of the eigenvalues of a mixed graph $G$ are Gaussian integers, then the mixed graph is said to be Gaussian integral. The term integral graph refers to an $H$-integral simple graph. For a simple graph $G$, note that $A(G) = H(G)$. As a result, the terms $H$-eigenvalue, $H$-spectrum and $H$-integrality of a simple graph $G$ have the same meaning with that of the eigenvalue, spectrum and integrality of $G$, respectively.

In 1974, Harary and Schwenk [12] proposed the question of characterization of integral graphs. This problem has inspired a lot of interest over the last half-century. For more information on integral graphs, we refer the reader to [1, 3, 7, 23, 24].

Throughout the paper, we consider $\Gamma$ to be a finite group with identity element $1$. Let $S$ be a subset of $\Gamma$ that does not contain $1$. If $S$ is closed under inverse (resp. $a^{-1} \notin S$ for all $a \in S$), it is said to be symmetric (resp. skew-symmetric). Define $\overline{S} = \{u \in S : u^{-1} \notin S\}$. Clearly, $S \setminus \overline{S}$ is symmetric, while $\overline{S}$ is skew-symmetric. The mixed Cayley graph $Cay(\Gamma, S)$ is a mixed graph, where $V(Cay(\Gamma, S)) = \Gamma$ and $E(Cay(\Gamma, S)) = \{(a, b) : a, b \in \Gamma, ba^{-1} \in S\}$. If $S$ is symmetric (resp. skew-symmetric), we call $Cay(\Gamma, S)$ to be a simple Cayley graph (resp. oriented Cayley graph). A mixed Cayley graph $Cay(\Gamma, S)$ is called normal if $S$ is the union of some conjugacy classes of the group $\Gamma$.

In 1982, Bridge and Mena [5] presented a characterization of integral Cayley graphs over abelian groups. Later on, same characterization was obtained by Wasin So [21] for cyclic groups in 2005. In 2009, Klotz and Sander [16] proved that if $Cay(\Gamma, S)$ over an abelian group $\Gamma$ is integral, then $S$ belongs to the Boolean algebra $\mathbb{B}(\Gamma)$ generated by the subgroups of $\Gamma$. Moreover, they conjectured that the converse is also true, which was proved by Alperin and Peterson [2]. For results on integral Cayley
graphs over non-abelian groups, we refer the reader to [6, 17, 20]. In [14] and [15], we characterized H-integral mixed Cayley graphs over cyclic group and abelian group in terms of their connection set. In 2014, Godsil et al. [10] characterized integral normal Cayley graphs. Xu et al. [25] and Li [18] characterized the set $S$ for which the mixed circulant graph $\text{Cay}(\mathbb{Z}_n, S)$ is Gaussian integral.

The paper is organized as follows. In Section 2, we present some preliminary notions and known results. We also express the H-eigenvalues of a normal mixed Cayley graph $\text{Cay}(\Gamma, S)$ in terms of the irreducible characters of $\Gamma$. In Section 3, we characterize the set $S$ for which the normal mixed Cayley graph $\text{Cay}(\Gamma, S)$ is H-integral. This extends the results of [14, 15] to normal mixed Cayley graphs. In Section 4, we prove that a normal mixed Cayley graph is H-integral if and only if it is Gaussian integral.

2 Preliminaries

In this section, we determine the H-eigenvalues of a normal mixed Cayley graph $\text{Cay}(\Gamma, S)$ in terms of the irreducible characters of $\Gamma$. Finally, we show that the normal mixed Cayley graph is H-integral if and only if each of its directed and undirected portions are H-integral.

For $x \in \Gamma$, let $\text{ord}(x)$ denote the order of $x$. If $g$ and $h$ are elements of the group $\Gamma$, then we call $h$ a conjugate of $g$ if $g = x^{-1}hx$ for some $x \in \Gamma$. The conjugacy class of $g$, denoted $\text{Cl}(g)$, is the set of all conjugates of $g$ in $\Gamma$. Define $C_\Gamma(g)$ to be the set of all elements of $\Gamma$ that commute with $g$. We denote the group algebra of $\Gamma$ over a field $F$ by $F\Gamma$. That is, $F\Gamma$ is the set of all formal sums $\sum_{g \in \Gamma} a_g g$, where $a_g \in F$, and we assume $1.g = g$ to have $\Gamma \subseteq F\Gamma$.

A representation of a finite group $\Gamma$ is a homomorphism $\rho: \Gamma \to \text{GL}_n(\mathbb{C})$, where $\text{GL}_n(\mathbb{C})$ is the set of all $n \times n$ invertible matrices with complex entries. Here, the number $n$ is called the degree of $\rho$. Two representations $\rho_1$ and $\rho_2$ of $\Gamma$ of degree $n$ are equivalent if there is a $T \in \text{GL}_n(\mathbb{C})$ such that $T \rho_1(x) = \rho_2(x)T$ for each $x \in \Gamma$.

Let $\rho: \Gamma \to \text{GL}_n(\mathbb{C})$ be a representation of $\Gamma$. The character $\chi_\rho: \Gamma \to \mathbb{C}$ of $\rho$ is defined by setting $\chi_\rho(x) := \text{Tr}(\rho(x))$ for $x \in \Gamma$, where $\text{Tr}(\rho(x))$ is the trace of $\rho(x)$. By degree of $\chi_\rho$, we mean the degree of $\rho$, which is simply $\chi_\rho(1)$. If $W$ is a $\rho(x)$-invariant subspace of $\mathbb{C}^n$ for each $x \in \Gamma$, then we say that $W$ is a $\rho(\Gamma)$-invariant subspace of $\mathbb{C}^n$. If $\{0\}$ and $\mathbb{C}^n$ are the only $\rho(\Gamma)$-invariant subspaces of $\mathbb{C}^n$, then we say $\rho$ an irreducible representation of $\Gamma$, and the corresponding character $\chi_\rho$ an irreducible character of $\Gamma$.

For a group $\Gamma$, we denote by $\text{IRR}(\Gamma)$ and $\text{Irr}(\Gamma)$ the complete set of non-equivalent irreducible representations of $\Gamma$ and the complete set of non-equivalent irreducible characters of $\Gamma$, respectively. For $z \in \mathbb{C}$, let $\overline{z}$ denote the complex conjugate of $z$ and $\Re(z)$ (resp. $\Im(z)$) denote the real part (resp. imaginary part) of the complex number $z$.

Theorem 2.1 ([22]). Let $\Gamma$ be a finite group and $\rho$ be a representation of $\Gamma$ of degree $k$ with corresponding character $\chi$. If $x \in \Gamma$ and $\text{ord}(x) = m$, then the following assertions hold.
(i) $\rho(x)$ is similar to a diagonal matrix with diagonal entries $\epsilon_1, \ldots, \epsilon_k$, where $\epsilon_i^m = 1$ for each $i \in \{1, \ldots, k\}$.

(ii) $\chi(x) = \sum_{i=1}^{k} \epsilon_i$, where $\epsilon_i^m = 1$ for each $i \in \{1, \ldots, k\}$.

(iii) $\chi(x^{-1}) = \overline{\chi(x)}$.

Proof. Note that $\rho(x)^m$ is an identity matrix. Therefore, $\rho(x)$ is diagonalizable, and that its eigenvalues are $m$-th roots of unity. Thus the proofs of Part (i) and Part (ii) follow.

Again, $xx^{-1} = 1$ gives that $\rho(x^{-1}) = \rho(x)^{-1}$. Therefore if $\chi(x) = \sum_{i=1}^{k} \epsilon_i$, then we have that $\chi(x)^{-1} = \sum_{i=1}^{k} \epsilon_i^{-1} = \sum_{i=1}^{k} \overline{\epsilon_i} = \overline{\chi(x)}$.

For a representation $\rho: \Gamma \to \text{GL}_n(\mathbb{C})$ of $\Gamma$, define $\overline{\rho}: \Gamma \to \text{GL}_n(\mathbb{C})$ by $\overline{\rho}(x) := \overline{\rho(x)}$, where $\overline{\rho(x)}$ is the matrix whose entries are the complex conjugates of the corresponding entries of $\rho(x)$. Note that if $\rho$ is irreducible, then $\overline{\rho}$ is also irreducible. Hence we have the following lemma. See Proposition 9.1.1 and Corollary 9.1.2 in [22] for details.

**Lemma 2.2** ([22]). Let $\Gamma$ be a finite group and $\text{Irr}(\Gamma) = \{\chi_1, \ldots, \chi_h\}$. If $j \in \{1, \ldots, h\}$, then there exists $k \in \{1, \ldots, h\}$ satisfying $\overline{\chi_k} = \chi_j$, where $\overline{\chi_k}: \Gamma \to \mathbb{C}$ such that $\overline{\chi_k(x)} = \overline{\chi_j(x)}$ for each $x \in \Gamma$.

**Theorem 2.3** ([22]). Let $\Gamma$ be a finite group and $x, y \in \Gamma$. If $\text{Irr}(\Gamma) = \{\chi_1, \ldots, \chi_h\}$, then

(i) \[
\sum_{x \in \Gamma} \chi_j(x) \overline{\chi_k(x)} = \begin{cases} |\Gamma| & \text{if } j = k \\ 0 & \text{otherwise,} \end{cases}
\]

(ii) \[
\sum_{j=1}^{h} \chi_j(x) \overline{\chi_j(y)} = \begin{cases} |C_\Gamma(x)| & \text{if } x \text{ and } y \text{ are conjugates to each other} \\ 0 & \text{otherwise.} \end{cases}
\]

For a function $f: \Gamma \to \mathbb{C}$, let $[f(yx^{-1})]_{x,y \in \Gamma}$ be the matrix whose rows and columns are indexed by the elements of $\Gamma$, and for $x, y \in \Gamma$, the $(x, y)$-th entry of the matrix is $f(yx^{-1})$.

**Theorem 2.4** ([9]). Let $\Gamma$ be a finite group and $\text{Irr}(\Gamma) = \{\chi_1, \ldots, \chi_h\}$. If $f: \Gamma \to \mathbb{C}$ is a class function, then the spectrum of the matrix $[f(yx^{-1})]_{x,y \in \Gamma}$ is $\{[\gamma_1]^{d_1^2}, \ldots, [\gamma_h]^{d_h^2}\}$, where

$\gamma_j = \frac{1}{\chi_j(1)} \sum_{x \in \Gamma} f(x) \chi_j(x)$ and $d_j = \chi_j(1)$

for each $j \in \{1, \ldots, h\}$.
Lemma 2.5. Let $\Gamma$ be a finite group. If $\text{Irr}(\Gamma) = \{\chi_1, \ldots, \chi_h\}$, then the H-spectrum of the normal mixed Cayley graph $\text{Cay}(\Gamma, S)$ is $\{[\gamma_1]^{d_1^2}, \ldots, [\gamma_h]^{d_h^2}\}$, where $\gamma_j = \lambda_j + \mu_j$,

\[
\lambda_j = \frac{1}{\chi_j(1)} \sum_{s \in S \setminus \overline{S}} \chi_j(s), \quad \mu_j = \frac{i}{\chi_j(1)} \sum_{s \in S} (\chi_j(s) - \chi_j(s^{-1})),
\]

and $d_j = \chi_j(1)$ for each $j \in \{1, \ldots, h\}$.

Proof. Let $f : \Gamma \to \{0, 1, i, -i\}$ be the function such that

\[
f(s) = \begin{cases} 
1 & \text{if } s \in S \setminus \overline{S} \\
1 & \text{if } s \in \overline{S} \\
i & \text{if } s \in \overline{S}^{-1} \\
0 & \text{otherwise.}
\end{cases}
\]

Since $S$ is a union of some conjugacy classes of $\Gamma$, $f$ is a class function. The Hermitian adjacency matrix of $\text{Cay}(\Gamma, S)$ is equal to $[f(\mu x^{-1})]_{x,y \in \Gamma}$. By Theorem 2.4,

\[
\gamma_j = \frac{1}{\chi_j(1)} \left( \sum_{s \in S \setminus \overline{S}} \chi_j(s) + \sum_{s \in \overline{S}} i \chi_j(s) + \sum_{s \in \overline{S}^{-1}} (-i) \chi_j(s) \right),
\]

and the result follows. \square

As special cases of Lemma 2.5, we have the following two corollaries.

Corollary 2.5.1. Let $\Gamma$ be a finite group. If $\text{Irr}(\Gamma) = \{\chi_1, \ldots, \chi_h\}$, then the H-spectrum (or spectrum) of the normal simple Cayley graph $\text{Cay}(\Gamma, S)$ is $\{[\lambda_1]^{d_1^2}, \ldots, [\lambda_h]^{d_h^2}\}$, where

\[
\lambda_j = \frac{1}{\chi_j(1)} \sum_{s \in S} \chi_j(s) \quad \text{and} \quad d_j = \chi_j(1) \quad \text{for each } j \in \{1, \ldots, h\}.
\]

Corollary 2.5.2. Let $\Gamma$ be a finite group. If $\text{Irr}(\Gamma) = \{\chi_1, \ldots, \chi_h\}$, then the H-spectrum of the normal oriented Cayley graph $\text{Cay}(\Gamma, S)$ is $\{[\mu_1]^{d_1^2}, \ldots, [\mu_h]^{d_h^2}\}$, where

\[
\mu_j = \frac{i}{\chi_j(1)} \sum_{s \in S} (\chi_j(s) - \chi_j(s^{-1})) \quad \text{and} \quad d_j = \chi_j(1) \quad \text{for each } j \in \{1, \ldots, h\}.
\]

Lemma 2.6. If $\Gamma$ is a finite group, then the normal mixed Cayley graph $\text{Cay}(\Gamma, S)$ is H-integral if and only if $\text{Cay}(\Gamma, S \setminus \overline{S})$ is integral (or H-integral) and $\text{Cay}(\Gamma, \overline{S})$ is H-integral.

Proof. Let $\text{Irr}(\Gamma) = \{\chi_1, \ldots, \chi_h\}$ and $\gamma_j$ be an H-eigenvalue of the normal mixed Cayley graph $\text{Cay}(\Gamma, S)$. By Lemma 2.5, we have $\gamma_j = \lambda_j + \mu_j$, where

\[
\lambda_j = \frac{1}{\chi_j(1)} \sum_{s \in \mathbb{S} \setminus \overline{\mathbb{S}}} \chi_j(s) \quad \text{and} \quad \mu_j = \frac{i}{\chi_j(1)} \sum_{s \in \overline{\mathbb{S}}} (\chi_j(s) - \chi_j(s^{-1})).
\]
for each \( j \in \{1, \ldots, h\} \). Assume that Cay(\( \Gamma, S \)) is H-integral and \( j \in \{1, \ldots, h\} \). By Lemma 2.2, there exists \( k \in \{1, \ldots, h\} \) such that \( \overline{\chi}_k = \chi_j \). Thus

\[
\gamma_k = \overline{\gamma}_k = \overline{\lambda}_k + \overline{\mu}_k = \lambda_j - \mu_j.
\]

By assumption, \( \gamma_j \) and \( \gamma_k \) are integers. As \( \gamma_j = \lambda_j + \mu_j \) and \( \gamma_k = \lambda_j - \mu_j \), we get

\[
\lambda_j = \frac{j + \gamma_k}{2} \quad \text{and} \quad \mu_j = \frac{j - \gamma_k}{2}.
\]

Thus \( \lambda_j \) and \( \mu_j \) are rational algebraic integers, and so they are integers. Hence by Corollaries 2.5.1 and 2.5.2, Cay(\( \Gamma, S \)) is integral and Cay(\( \Gamma, \overline{S} \)) is H-integral.

Conversely, assume that Cay(\( \Gamma, S \)) is integral and Cay(\( \Gamma, \overline{S} \)) is H-integral. Using Lemma 2.5, Cay(\( \Gamma, S \)) is H-integral. \( \square \)

Let \( n \geq 2 \) be a positive integer. For a divisor \( d \) of \( n \), define \( G_n(d) = \{ k : 1 \leq k \leq n-1, \gcd(k,n) = d \} \). It is clear that \( G_n(d) = dG_n^\#(1) \).

Let \( \mathbb{B}(\Gamma) \) be the boolean algebra generated by the subgroups of \( \Gamma \). That is, \( \mathbb{B}(\Gamma) \) is the set whose elements are obtained by intersections, unions and complements of subgroups of \( \Gamma \). Define an equivalence relation \( \sim \) on \( \Gamma \) such that \( x \sim y \) if and only if \( y = x^k \) for some \( k \in G_m(1) \), where \( m = \text{ord}(x) \). For \( x \in \Gamma \), let \([x]\) denote the equivalence class of \( x \) with respect to the relation \( \sim \). Note that minimal non-empty sets in a boolean algebra are called its atoms.

Theorem 2.7 ([2]). The atoms of the boolean algebra \( \mathbb{B}(\Gamma) \) are the sets \([x]\) for each \( x \in \Gamma \).

By Theorem 2.7, we observe that each element of \( \mathbb{B}(\Gamma) \) can be expressed as a disjoint union of the equivalence classes of the relation \( \sim \) on \( \Gamma \). Thus

\[
\mathbb{B}(\Gamma) = \{ [x_1] \cup \cdots \cup [x_k] : x_1, \ldots, x_k \in \Gamma, k \in \mathbb{N} \}.
\]

Theorem 2.8 ([10]). Let \( \Gamma \) be a finite group and Cay(\( \Gamma, S \)) be a normal simple Cayley graph. Then Cay(\( \Gamma, S \)) is integral if and only if \( S \in \mathbb{B}(\Gamma) \).

Let \( n \equiv 0 \pmod{4} \). For a divisor \( d \) of \( \frac{n}{r} \) and \( r \in \{1, 3\} \), define

\[
G_n^r(d) = \{ dk : k \equiv r \pmod{4}, \gcd(dk,n) = d \}.
\]

It is easy to see that \( G_n(d) = G_n^1(d) \cup G_n^3(d), G_n^1(d) \cap G_n^3(d) = \emptyset \) and \( G_n^r(d) = dG_n^r(1) \) for \( r \in \{1, 3\} \).

Let \( \Gamma(4) \) be the set of all \( x \in \Gamma \) satisfying \( \text{ord}(x) \equiv 0 \pmod{4} \). That is, \( \Gamma(4) := \{ x \in \Gamma : \text{ord}(x) \equiv 0 \pmod{4} \} \). Define an equivalence relation \( \approx \) on \( \Gamma(4) \) such that \( x \approx y \) if and only if \( y = x^k \) for some \( k \in G_m^1(1) \), where \( m = \text{ord}(x) \). Observe that if \( x, y \in \Gamma(4) \) and \( x \approx y \) then \( x \sim y \), but the converse need not be true. For example, consider \( x = 5 \pmod{12} \), \( y = 11 \pmod{12} \) in \( \mathbb{Z}_{12} \). Here \( x, y \in \mathbb{Z}_{12}(4) \) and \( x \sim y \), but \( x \not\approx y \). For \( x \in \Gamma(4) \), we denote the equivalence class of \( x \) with respect to the relation \( \approx \) by
[x]. For \( \Gamma(4) \neq \emptyset \), define \( \mathbb{D}(\Gamma) \) to be the class of all skew-symmetric subsets \( S \), where \( S = \{x_1 \cup \cdots \cup x_k\} \) for some \( x_1, \ldots, x_k \in \Gamma(4) \). For \( \Gamma(4) = \emptyset \), define \( \mathbb{D}(\Gamma) := \{\emptyset\} \). Thus

\[
\mathbb{D}(\Gamma) = \begin{cases} 
\{\{x_1 \cup \cdots \cup x_k\} : x_1, \ldots, x_k \in \Gamma(4), k \in \mathbb{N}\} & \text{if } \Gamma(4) \neq \emptyset \\
\{\emptyset\} & \text{if } \Gamma(4) = \emptyset.
\end{cases}
\]

3 H-integral normal mixed Cayley graphs

Let the order of the group \( \Gamma \) be \( n \) and \( \chi \in \text{Irr}(\Gamma) \) be of degree \( d \). Let \( x \in \mathbb{Q}(i)\Gamma \) be such that \( x = \sum_{g \in \Gamma} ic_gg \), where \( c_g \in \mathbb{Z} \) for all \( g \in \Gamma \). Define \( \chi(x) := \sum_{g \in \Gamma} ic_g \chi(g) \). Note that \( \chi(g)^n = \chi(g^n) = \chi(1) = d \), and so \( \chi(g) \) is an algebraic integer for each \( g \in \Gamma \). Therefore \( ic_g \chi(g) \) is an algebraic integer for each \( g \in \Gamma \), and hence \( \chi(x) \) is an algebraic integer.

Let \( \text{Irr}(\Gamma) = \{\chi_1, \ldots, \chi_h\} \). Let \( E \) be the matrix \( [E_{ig}] \) of size \( h \times n \), whose rows are indexed by \( 1, \ldots, h \) and columns are indexed by the elements of \( \Gamma \) such that \( E_{ig} = \chi_j(g) \). Note that \( EE^* = nI_h \) and the rank of \( E \) is \( h \), where \( E^* \) is the conjugate transpose of \( E \) and \( I_h \) is the \( h \times h \) identity matrix.

Let \( \text{Gal}(\mathbb{K}/\mathbb{F}) \) denote the Galois group of an extension \( \mathbb{K} \) over the field \( \mathbb{F} \). It is well known that \( \text{Gal}(\mathbb{Q}(\omega_m)/\mathbb{Q}) = \{\sigma_r : r \in G_m(1), \sigma_r(\omega_m) = \omega_m^r\} \). For example, see Section 14.5 in [8]. If \( m \equiv 0 \pmod{4} \), then \( \mathbb{Q}(i, \omega_m) = \mathbb{Q}(\omega_m) \). Therefore, \( \text{Gal}(\mathbb{Q}(i, \omega_m)/\mathbb{Q}(i)) \) is a subgroup of \( \text{Gal}(\mathbb{Q}(\omega_m)/\mathbb{Q}) \).

Thus \( \text{Gal}(\mathbb{Q}(i, \omega_m)/\mathbb{Q}(i)) \) contains those automorphisms in \( \text{Gal}(\mathbb{Q}(\omega_m)/\mathbb{Q}) \) that fix \( i \). Note that \( G_m(1) = G_m^1(1) \cup G_m^3(1) \) and \( G_m^1(1) \cap G_m^3(1) = \emptyset \). If \( r \in G_m^1(1) \) then \( \sigma_r(i) = i \), and if \( r \in G_m^3(1) \) then \( \sigma_r(i) = -i \). Thus

\[
\text{Gal}(\mathbb{Q}(i, \omega_m)/\mathbb{Q}(i)) = \{\sigma_r : r \in G_m^1(1), \sigma_r(\omega_m) = \omega_m^r\}.
\]

If \( m \equiv 0 \pmod{4} \), then \( [\mathbb{Q}(i, \omega_m) : \mathbb{Q}(i)] = \varphi(m) \). Thus the field \( \mathbb{Q}(i, \omega_m) \) is a Galois extension of \( \mathbb{Q}(i) \) of degree \( \varphi(m) \). Any automorphism of the field \( \mathbb{Q}(i, \omega_m) \) is uniquely determined by its action on \( \omega_m \). Hence \( \text{Gal}(\mathbb{Q}(i, \omega_m)/\mathbb{Q}(i)) = \{\tau_r : r \in G_m(1), \tau_r(\omega_m) = \omega_m^r \text{ and } \tau_r(i) = i\} \).

Let \( g \in \Gamma, m = \text{ord}(g) \), and \( \chi \) be a character of \( \Gamma \). By Theorem 2.1, \( \chi(g) = \sum_{i=1}^k \epsilon_i \), where \( \epsilon_1, \ldots, \epsilon_k \) are some \( m \)-th roots of unity. If \( m \equiv 0 \pmod{4} \) and \( \sigma_r \in \text{Gal}(\mathbb{Q}(i, \omega_m)/\mathbb{Q}(i)) \), then

\[
\sigma_r(\chi(g)) = \sigma_r \left( \sum_{i=1}^k \epsilon_i \right) = \sum_{i=1}^k \sigma_r(\epsilon_i) = \sum_{i=1}^k \epsilon_i^r = \chi(g^r).
\]

Similarly, if \( m \equiv 0 \pmod{4} \) and \( \tau_r \in \text{Gal}(\mathbb{Q}(i, \omega_m)/\mathbb{Q}(i)) \), then also \( \tau_r(\chi(g)) = \chi(g^r) \).

**Theorem 3.1.** Let \( \Gamma \) be a finite group and \( \text{Irr}(\Gamma) = \{\chi_1, \ldots, \chi_h\} \). If \( x = \sum_{g \in \Gamma} ic_gg \), where \( c_g \in \mathbb{Z} \) for all \( g \in \Gamma \), then \( \chi_j(x) \) is an integer for each \( j \in \{1, \ldots, h\} \) if and only if the following conditions hold:

(i) \( \sum_{s \in \text{Cl}(g_1)} c_s = \sum_{s \in \text{Cl}(g_2)} c_s \) for each \( g_1, g_2 \in \Gamma(4) \) and \( g_1 \approx g_2 \);
(ii) \( \sum_{s \in \text{Cl}(g)} c_s = - \sum_{s \in \text{Cl}(g^{-1})} c_s \) for each \( g \in \Gamma \);

(iii) \( \sum_{s \in \text{Cl}(g)} c_s = 0 \) for all \( g \in \Gamma \setminus \Gamma(4) \).

**Proof.** Let \( L \) be a set of representatives of the conjugacy classes in \( \Gamma \). Since characters are class functions, we have

\[
\chi_j(x) = \sum_{g \in L} \left( \sum_{s \in \text{Cl}(g)} i c_s \right) \chi_j(g) \text{ for each } j \in \{1, \ldots, h\}.
\]  

(1)

Assume that \( \chi_j(x) \) is an integer for each \( j \in \{1, \ldots, h\} \). Let \( g_1, g_2 \in \Gamma(4) \), \( g_1 \approx g_2 \) and \( m = \text{ord}(g_1) \).

Therefore, there is \( r \in G_m^1(1) \) and \( \sigma_r \in \text{Gal}(\mathbb{Q}(\omega_m) / \mathbb{Q}(i)) \) such that \( g_2 = g_1^r \) and \( \sigma_r(\omega_m) = \omega_m^r \). Note that \( \sigma_r(\chi_j(g_1)) = \chi_j(g_1^r) \) for each \( j \in \{1, \ldots, h\} \). For \( t \in \Gamma \), let \( \theta_t = \sum_{j=1}^{h} \chi_j(t) \overline{\chi_j} \), where \( \overline{\chi_j(g)} = \chi_j(g^{-1}) \) for each \( g \in \Gamma \). By Theorem 2.3, we have

\[
\theta_t(u) = \begin{cases} |C_T(t)| & \text{if } u \text{ and } t \text{ are conjugates to each other} \\ 0 & \text{otherwise}. \end{cases}
\]

So \( \theta_t(x) = |C_T(t)| \sum_{s \in \text{Cl}(t)} i c_s \in \mathbb{Q}(i) \), and it gives that \( \sigma_r(\theta_t(x)) = \theta_t(x) \). Since \( \chi_j(x) \) is assumed to be an integer, we have \( \sigma_r(\chi_j(x)) = \chi_j(x) \) for each \( j \in \{1, \ldots, h\} \). Thus

\[
|C_T(g_1)| \sum_{s \in \text{Cl}(g_1)} i c_s = \theta_{g_1}(x) = \sigma_r(\theta_{g_1}(x)) = \sum_{j=1}^{h} \sigma_r(\chi_j(g_1)) \sigma_r(\overline{\chi_j}(x)) = \sum_{j=1}^{h} \chi_j(g_1^r) \overline{\chi_j}(x) = \theta_{g_1^r}(x) = \theta_{g_2}(x) = |C_T(g_2)| \sum_{s \in \text{Cl}(g_2)} i c_s.
\]  

(2)

Since \( g_1 \approx g_2 \), we have \( C_T(g_1) = C_T(g_2) \). So Equation (2) implies that \( \sum_{s \in \text{Cl}(g_1)} c_s = \sum_{s \in \text{Cl}(g_2)} c_s \). Hence condition (i) holds. Again

\[
0 = \chi_j(x) - \overline{\chi_j(x)} = \sum_{g \in L} \left( \sum_{s \in \text{Cl}(g)} i c_s \right) \chi_j(g) - \sum_{g \in L} \left( \sum_{s \in \text{Cl}(g)} -i c_s \right) \overline{\chi_j(g)} = \sum_{g \in L} \left( \sum_{s \in \text{Cl}(g)} i c_s \right) \chi_j(g) + \sum_{g \in L} \left( \sum_{s \in \text{Cl}(g)} i c_s \right) \chi_j(g^{-1}) = \sum_{g \in L} i \left( \sum_{s \in \text{Cl}(g)} c_s + \sum_{s \in \text{Cl}(g^{-1})} c_s \right) \chi_j(g),
\]

and so

\[
\sum_{g \in L} i \left( \sum_{s \in \text{Cl}(g)} c_s + \sum_{s \in \text{Cl}(g^{-1})} c_s \right) \begin{bmatrix} \chi_1(g) \\ \vdots \\ \chi_h(g) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.
\]

(3)
Note that the number of irreducible characters of \( \Gamma \) is equal to the number of conjugacy classes of \( \Gamma \), that is, \(|L| = h\). Since characters are class functions and the rank of \( E \) is \( h \), the columns of \( E \) corresponding to the elements of \( L \) are linearly independent. Thus by Equation (3),

\[
\sum_{s \in \text{Cl}(g)} c_s + \sum_{s \in \text{Cl}(g^{-1})} c_s = 0
\]

for all \( g \in L \), and so condition (ii) holds.

Let \( g \in \Gamma \setminus \Gamma(4) \) and \( m = \text{ord}(g) \). Then there exists \( \tau_{m-1} \in \text{Gal}(\mathbb{Q}(i, \omega_m)/\mathbb{Q}(i)) \) such that \( \tau_{m-1}(\omega_m) = \omega_m^{m-1} \). Note that \( \tau_{m-1}(\chi_j(g)) = \chi_j(g^{m-1}) \) for each \( j \in \{1, \ldots, h\} \). Now

\[
|C_\Gamma(g)| \sum_{s \in \text{Cl}(g)} ic_s = \theta_g(x) = \tau_{m-1}(\theta_g(x))
\]

\[
= \sum_{j=1}^h \tau_{m-1}(\chi_j(g))\tau_{m-1}(\chi_j(x))
\]

\[
= \sum_{j=1}^h \chi_j(g^{m-1})\chi_j(x)
\]

\[
= \theta_{g^{m-1}}(x) = \theta_{g^{-1}}(x) = |C_\Gamma(g^{-1})| \sum_{s \in \text{Cl}(g^{-1})} ic_s.
\]

Since \( C_\Gamma(g) = C_\Gamma(g^{-1}) \), Equation (4) implies that \( \sum_{s \in \text{Cl}(g)} c_s = \sum_{s \in \text{Cl}(g^{-1})} c_s \). This, together with condition (ii), gives \( \sum_{s \in \text{Cl}(g)} c_s = 0 \) for all \( g \in \Gamma \setminus \Gamma(4) \). Hence condition (iii) also holds.

Conversely, assume that all the three conditions of the theorem hold. Let \( n \) be the order of \( \Gamma \). If \( n \not\equiv 0 \pmod{4} \) then \( \Gamma(4) = \emptyset \). Therefore by condition (iii) and Equation (1), we have \( \chi_j(x) = 0 \). Thus, \( \chi_j(x) \) is an integer for each \( j \in \{1, \ldots, h\} \). Now assume that \( n \equiv 0 \pmod{4} \). Let \( L(4) \) be a set of representatives of the conjugacy classes of \( \Gamma(4) \). Since characters are class functions, using condition (iii) we have

\[
\chi_j(x) = \sum_{g \in L(4)} \left( \sum_{s \in \text{Cl}(g)} ic_s \right) \chi_j(g) \text{ for each } j \in \{1, \ldots, h\}.
\]

Let \( \sigma_k \in \text{Gal}(\mathbb{Q}(i, \omega_n)/\mathbb{Q}(i)) \). Therefore \( \sigma_k(\omega_n) = \omega_n^k \) and \( k \in G_n^1(1) \). Thus

\[
\sigma_k(\chi_j(x)) = \sum_{g \in L(4)} \left( \sum_{s \in \text{Cl}(g)} ic_s \right) \sigma_k(\chi_j(g))
\]

\[
= \sum_{g \in L(4)} \left( \sum_{s \in \text{Cl}(g)} ic_s \right) \chi_j(g^k).
\]

Since \( g \approx g^k \), by condition (i) we have \( \sum_{s \in \text{Cl}(g)} c_s = \sum_{s \in \text{Cl}(g^k)} c_s \). From Equation (6), we get

\[
\sigma_k(\chi_j(x)) = \sum_{g \in L(4)} \left( \sum_{s \in \text{Cl}(g^k)} ic_s \right) \chi_j(g^k) = \chi_j(x).
\]
The second equality in Equation (7) holds, because \( \{g^k : g \in L(4)\} \) is also a set of representatives of conjugacy classes of \( \Gamma(4) \). Now \( \sigma_k(\chi_j(x)) = \chi_j(x) \) for each \( k \in G_n(1) \), and so \( \chi_j(x) \in \mathbb{Q}(i) \). Taking complex conjugates in Equation (5), we have

\[
\overline{\chi_j(x)} = \sum_{g \in L(4)} \left( \sum_{s \in \text{Cl}(g)} -ic_s \right) \overline{\chi_j(g)} = \sum_{g \in L(4)} \left( \sum_{s \in \text{Cl}(g)} -ic_s \right) \chi_j(g^{-1})
\]

\[
= \sum_{g \in L(4)} \left( \sum_{s \in \text{Cl}(g^{-1})} ic_s \right) \chi_j(g^{-1})
\]

\[
= \chi_j(x).
\]

Thus Equation (8) implies that \( \chi_j(x) \in \mathbb{Q} \). As \( \chi_j(x) \) is a rational algebraic integer, it must be an integer for each \( j \in \{1, \ldots, h\} \).

\[\square\]

Indeed, we can replace condition (i) of Theorem 3.1 by \( \sum_{s \in \text{Cl}(x)} c_s = \sum_{s \in \text{Cl}(y)} c_s \) for all \( x, y \in [g] \) and \( g \in \Gamma(4) \).

**Theorem 3.2.** Let \( \Gamma \) be a finite group and \( \text{Cay}(\Gamma, S) \) be a normal oriented Cayley graph. Then \( \text{Cay}(\Gamma, S) \) is \( H \)-integral if and only if \( S \in \mathbb{D}(\Gamma) \).

**Proof.** Let Irr(\( \Gamma \)) = \( \{\chi_1, \ldots, \chi_h\} \) and \( x = \sum_{g \in \Gamma} ic_g g \), where

\[
c_g = \begin{cases} 
1 & \text{if } g \in S \\
-1 & \text{if } g \in S^{-1} \\
0 & \text{otherwise.}
\end{cases}
\]

Observe that \( \chi_j(x) = \sum_{s \in S} i(\chi_j(s) - \chi_j(s^{-1})) \), and so \( \frac{\chi_j(x)}{\chi_j(1)} \) is an \( H \)-eigenvalue of \( \text{Cay}(\Gamma, S) \). Assume that the normal oriented Cayley graph \( \text{Cay}(\Gamma, S) \) is \( H \)-integral. Thus \( \chi_j(x) \) is an integer for each \( j \in \{1, \ldots, h\} \), and therefore all the three conditions of Theorem 3.1 are satisfied for \( x \). By the third condition of Theorem 3.1, we get \( \sum_{s \in \text{Cl}(g)} c_s = 0 \) for all \( g \in \Gamma \setminus \Gamma(4) \). Note that \( S \) is a union of some conjugacy classes of \( \Gamma \). Therefore, if \( g \in S \) then \( \text{Cl}(g) \subseteq S \), and so by the definition of \( c_g \) we get \( \sum_{s \in \text{Cl}(g)} c_s = |\text{Cl}(g)| \neq 0 \). Thus \( S \cap (\Gamma \setminus \Gamma(4)) = \emptyset \), that is, \( S \subseteq \Gamma(4) \). Again, let \( g_1 \in S \), \( g_2 \in \Gamma(4) \) and \( g_1 \approx g_2 \). By the first condition of Theorem 3.1, we get \( 0 < \sum_{s \in \text{Cl}(g_1)} c_s = \sum_{s \in \text{Cl}(g_2)} c_s \), which implies that \( g_2 \in S \). Thus \( g_1 \in S \) gives \( [g_1] \subseteq S \). Hence \( S \in \mathbb{D}(\Gamma) \).

Conversely, assume that \( S \in \mathbb{D}(\Gamma) \). Let \( \text{Cay}(\Gamma, S) \) be a normal oriented Cayley graph, so that \( S \) is a union of some conjugacy classes of \( \Gamma \). Let

\[
S = [x_1] \cup \cdots \cup [x_r] = \text{Cl}(y_1) \cup \cdots \cup \text{Cl}(y_k) \subseteq \Gamma(4)
\]

for some \( x_1, \ldots, x_r, y_1, \ldots, y_k \in \Gamma(4) \). Therefore

\[
S^{-1} = [x_1^{-1}] \cup \cdots \cup [x_r^{-1}] = \text{Cl}(y_1^{-1}) \cup \cdots \cup \text{Cl}(y_k^{-1}) \subseteq \Gamma(4).
\]
Now for \( g_1, g_2 \in \Gamma(4) \), if \( g_1 \approx g_2 \) then \( \text{Cl}(g_1), \text{Cl}(g_2) \subseteq S \) or \( \text{Cl}(g_1), \text{Cl}(g_2) \subseteq S^{-1} \) or \( \text{Cl}(g_1), \text{Cl}(g_2) \subseteq (S \cup S^{-1})^c \). Here \((S \cup S^{-1})^c\) is the complement of \( S \cup S^{-1} \) in \( \Gamma \). Note that \( |\text{Cl}(g_i)| = |\text{Cl}(g_2)| \). For all the cases, using the definition of \( c_g \), we have

\[
\sum_{s \in \text{Cl}(g_1)} c_s = \sum_{s \in \text{Cl}(g_2)} c_s.
\]

Again, we see that \( \text{Cl}(g) \subseteq S \) if and only if \( \text{Cl}(g^{-1}) \subseteq S^{-1} \). Therefore

\[
\sum_{s \in \text{Cl}(g)} c_s = -\sum_{s \in \text{Cl}(g^{-1})} c_s = 0 \quad \text{or} \quad \sum_{s \in \text{Cl}(g)} c_s = \pm |\text{Cl}(g)| = -\sum_{s \in \text{Cl}(g^{-1})} c_s.
\]

Further, if \( g \not\in \Gamma(4) \) then \( \text{Cl}(g) \cap (S \cup S^{-1}) = \emptyset \), and so \( \sum_{s \in \text{Cl}(g)} c_s = 0 \). Thus the three conditions of Theorem 3.1 are satisfied, and therefore \( \chi_j(x) \) is an integer for each \( j \in \{1, \ldots, h\} \). Consequently, the H-eigenvalue \( \mu_j := \frac{\chi_j(x)}{\chi_j(1)} \) of Cay(\( \Gamma, S \)) is a rational algebraic integer, and hence it an integer for each \( j \in \{1, \ldots, h\} \). \( \Box \)

We give the following example to illustrate Theorem 3.2.

**Example 3.1.** Consider the group \( M_{16} := \langle a, x \mid a^8 = x^2 = 1, xax^{-1} = a^5 \rangle \), and let \( S = \{a, a^5, a^9, a^7x \} \). The conjugacy classes of \( M_{16} \) are \( \{1\}, \{a^4\}, \{a^2\}, \{a^6\}, \{a, a^5\}, \{a^3, a^7\}, \{ax, a^5x\}, \{a^3x, a^7x\}, \{x, a^4x\} \) and \( \{a^2x, a^9x\} \). The normal oriented Cayley graph Cay(\( M_{16}, S \)) is shown in Figure 1a. We see that \( S = [a] \cup [a^3x] = \text{Cl}(a) \cup \text{Cl}(a^3x) \). Thus \( S \in \mathbb{D}(M_{16}) \), and hence Cay(\( M_{16}, S \)) is H-integral. We can also confirm this by finding its H-eigenvalues. Using the GAP software, the character table of \( M_{16} \) is obtained and given in Table 1, where Irr(\( M_{16} \)) = \( \{\chi_1, \ldots, \chi_{10}\} \). Further, using Corollary 2.5.2, the H-spectrum of Cay(\( M_{16}, S \)) is obtained as \( \{[\mu_j] \mid 1 \leq j \leq 8\} \cup \{[\mu_9]^4, [\mu_{10}]^4\} \), where \( \mu_j = 0 \) for \( j \not\in \{5, 6\} \), \( \mu_5 = -8 \) and \( \mu_6 = 8 \). Thus all the H-eigenvalues of Cay(\( M_{16}, S \)) are integers.

**Theorem 3.3.** Let \( \Gamma \) be a finite group and Cay(\( \Gamma, S \)) be a normal mixed Cayley graph. Then Cay(\( \Gamma, S \)) is H-integral if and only if \( S \setminus \overline{S} \in \mathbb{B}(\Gamma) \) and \( \overline{S} \in \mathbb{D}(\Gamma) \).

**Proof.** By Lemma 2.6, Cay(\( \Gamma, S \)) is H-integral if and only if Cay(\( \Gamma, S \setminus \overline{S} \)) is integral and Cay(\( \Gamma, \overline{S} \)) is H-integral. Now the proof follows from Theorem 2.8 and Theorem 3.2. \( \Box \)

The following example uses Theorem 3.3 to check H-integrality of a normal mixed Cayley graph.

**Example 3.2.** Consider the group \( M_{16} \) of Example 3.1 and \( S = \{a, a^3, a^5, a^7, a^3x, a^7x\} \). The normal mixed Cayley graph Cay(\( M_{16}, S \)) is shown in Figure 1b. We have

\[
S = [a] \cup [a^3x] = \text{Cl}(a) \cup \text{Cl}(a^3x).
\]

Therefore \( S \setminus \overline{S} \in \mathbb{B}(M_{16}) \) and \( \overline{S} \in \mathbb{D}(M_{16}) \). Further, using Lemma 2.5, the H-spectrum of Cay(\( M_{16}, S \)) is obtained as \( \{[\gamma_j] \mid 1 \leq j \leq 8\} \cup \{[\gamma_9]^4, [\gamma_{10}]^4\} \), where \( \gamma_1 = \gamma_3 = \gamma_6 = \gamma_7 = 4 \), \( \gamma_2 = \gamma_4 = \gamma_5 = \gamma_8 = -4 \) and \( \gamma_9 = \gamma_{10} = 0 \). Thus Cay(\( M_{16}, S \)) is H-integral.
Table 1: Character table of $M_{16}$

|    | $[1]$ | $[a^4]$ | $[a^2]$ | $[a^6]$ | $[a, a^3]$ | $[a, a^7]$ | $[ax, a^5x]$ | $[a^3x, a^7x]$ | $[x, a^4x]$ | $[a^2x, a^6x]$ |
|----|-------|---------|---------|---------|-----------|-----------|-------------|--------------|-------------|--------------|
| $\chi_1$ | 1     | 1       | 1       | 1       | 1         | 1         | 1           | 1            | 1           | 1            |
| $\chi_2$ | 1     | 1       | 1       | 1       | -1        | -1        | -1          | -1           | 1           | 1            |
| $\chi_3$ | 1     | 1       | 1       | 1       | 1         | 1         | -1          | -1           | -1          | -1           |
| $\chi_4$ | 1     | 1       | 1       | 1       | -1        | -1        | 1           | 1            | -1          | -1           |
| $\chi_5$ | 1     | 1       | -1      | -1      | i         | -i        | -i          | i            | i           | 1            |
| $\chi_6$ | 1     | 1       | -1      | -1      | -i        | i         | i           | -i           | -1          | 1            |
| $\chi_7$ | 1     | 1       | -1      | -1      | i         | -i        | i           | -i           | 1           | 1            |
| $\chi_8$ | 1     | 1       | -1      | -1      | -i        | i         | -i          | i            | 1           | -1           |
| $\chi_9$ | 2     | -2      | 2i      | -2i     | 0         | 0         | 0           | 0            | 0           | 0            |
| $\chi_{10}$ | 2     | -2      | -2i     | 2i      | 0         | 0         | 0           | 0            | 0           | 0            |

Figure 1: The mixed graph Cay($M_{16}, S$)

(a) $S = \{a, a^7, a^3x, a^7x\}$

(b) $S = \{a, a^3, a^5, a^7, a^3x, a^7x\}$
4 Gaussian integral normal mixed Cayley graphs

In [15], the authors proved that if \( \Gamma \) is an abelian group, then \([x] \cup [x^{-1}] = [x]\) for each \( x \in \Gamma(4) \). Note that this result and its proof also hold for non-abelian group. In the subsequent discussion, we use this fact for non-abelian group.

Let \( S \) be a union of some conjugacy classes of a finite group \( \Gamma \) that does not contain \( 1 \), and let \( \text{Irr}(\Gamma) = \{ \chi_1, \ldots, \chi_h \} \). Consider the function \( f: \Gamma \to \{0, 1\} \) defined by

\[
    f(s) = \begin{cases} 
        1 & \text{if } s \in S \\
        0 & \text{otherwise}
    \end{cases}
\]

in Theorem 2.4. We see that \( \frac{1}{\chi_j(1)} \sum_{s \in S} \chi_j(s) \) is an eigenvalue of the normal mixed Cayley graph \( \text{Cay}(\Gamma, S) \) for each \( j \in \{1, \ldots, h\} \). Indeed, all the eigenvalues of \( \text{Cay}(\Gamma, S) \) are of this form. For each \( j \in \{1, \ldots, h\} \), define

\[
    f_j(S) := \frac{1}{\chi_j(1)} \sum_{s \in S} \chi_j(s) + \frac{1}{2\chi_j(1)} \sum_{s \in S, s^{-1}} \chi_j(s)
\]

and

\[
    g_j(S) := \frac{i}{2\chi_j(1)} \sum_{s \in S} (\chi_j(s) - \chi_j(s^{-1})).
\]

We have

\[
    \frac{1}{\chi_j(1)} \sum_{s \in S} \chi_j(s) = f_j(S) - ig_j(S) \quad \text{for each } j \in \{1, \ldots, h\}.
\]

Therefore, the normal mixed Cayley graph \( \text{Cay}(\Gamma, S) \) is Gaussian integral if and only if \( f_j(S) \) and \( g_j(S) \) are integers for each \( j \in \{1, \ldots, h\} \).

**Theorem 4.1.** Let \( \Gamma \) be a finite group. If the normal mixed Cayley graph \( \text{Cay}(\Gamma, S) \) is Gaussian integral, then it is H-integral.

**Proof.** Let the normal mixed Cayley graph \( \text{Cay}(\Gamma, S) \) be Gaussian integral. Therefore, \( f_j(S) \) and \( g_j(S) \) are integers for each \( j \in \{1, \ldots, h\} \). Note that \( 2g_j(S) \) is an integer H-eigenvalue of the normal oriented Cayley graph \( \text{Cay}(\Gamma, S) \) for each \( j \in \{1, \ldots, h\} \). By Theorem 3.2, we get \( B = \bigcup_{j=1}^h [x_j] = \bigcup_{j=1}^h \text{Cl}(y_j) \) for some \( x_1, \ldots, x_k, y_1, \ldots, y_s \in \Gamma(4) \). Thus

\[
    S \cup S^{-1} = \bigcup_{j=1}^k ([x_j] \cup [x_j^{-1}]) = \bigcup_{j=1}^k [x_j] = \bigcup_{j=1}^s (\text{Cl}(y_j) \cup \text{Cl}(y_j^{-1})) \in B(\Gamma).
\]

Note that \( \frac{1}{\chi_j(1)} \sum_{s \in S \cup S^{-1}} \chi_j(s) \) is an eigenvalue of the normal simple Cayley graph \( \text{Cay}(\Gamma, S \cup S^{-1}) \) for each \( j \in \{1, \ldots, h\} \). By Theorem 2.8, \( \frac{1}{\chi_j(1)} \sum_{s \in S \cup S^{-1}} \chi_j(s) \) is an integer for each \( j \in \{1, \ldots, h\} \). Therefore

\[
    \frac{1}{\chi_j(1)} \sum_{s \in S \cup S^{-1}} \chi_j(s) = f_j(S) - \frac{1}{2\chi_j(1)} \sum_{s \in S \cup S^{-1}} \chi_j(s),
\]
and hence \( \frac{1}{\chi_j(1)} \sum_{s \in S \setminus \overline{S}} \chi_j(s) \) is a rational number. Thus, the eigenvalue \( \frac{1}{\chi_j(1)} \sum_{s \in S \setminus \overline{S}} \chi_j(s) \) of the normal simple Cayley graph \( \text{Cay}(\Gamma, S \setminus \overline{S}) \) is a rational algebraic integer, and hence it is an integer for each \( j \in \{1, \ldots, h\} \). Hence by Theorem 2.8, we get \( S \setminus \overline{S} \in \mathcal{B}(\Gamma) \). Now the result follows from Theorem 3.3. \( \square \)

**Lemma 4.2.** Let \( x \in \Gamma \) and \( \text{ord}(x) = 2^t m \). If \( t \geq 2 \) and \( m \) is odd, then the following assertions hold.

\[
(i) \ [x] = \begin{cases} \ x^{3m}[x^2] & \text{if } m \equiv 1 \ (\mod \ 4) \\ \ x^{m}[x^2] & \text{if } m \equiv 3 \ (\mod \ 4). \end{cases} 
\]

\[
(ii) \ [x^{-1}] = \begin{cases} \ x^{m}[x^2] & \text{if } m \equiv 1 \ (\mod \ 4) \\ \ x^{3m}[x^2] & \text{if } m \equiv 3 \ (\mod \ 4). \end{cases} 
\]

\[
(iii) \ [x] = x^{m}[x^2] \cup x^{3m}[x^2]. 
\]

**Proof.**

(i) Assume that \( t \geq 2 \) and \( k = 2^t m \). Let \( m \equiv 1 \ (\mod \ 4) \) and \( x^{3m+2r} \in x^{3m}[x^2] \) for some \( r \in G_{\frac{k}{2}}(1) \). Now \( \gcd(r, \frac{k}{2}) = 1 \), and it implies that \( \gcd(3m + 2r, k) = 1 \) and \( 3m + 2r \equiv 1 \ (\mod \ 4) \). We have \( x^{3m+2r} \in [x] \). Thus \( x^{3m}[x^2] \subseteq [x] \). Now since the sizes of \( [x] \) and \( x^{3m}[x^2] \) are equal, we have \( [x] = x^{3m}[x^2] \). Similarly, if \( m \equiv 3 \ (\mod \ 4) \) then \( [x] = x^{m}[x^2] \).

(ii) The proof of this part is similar to the proof of Part (i). For the sake of completeness, we provide the proof. Assume that \( t \geq 2 \) and \( k = 2^t m \). Let \( m \equiv 1 \ (\mod \ 4) \) and \( x^{m+2r} \in x^{m}[x^2] \) for some \( r \in G_{\frac{k}{2}}(1) \). Now \( \gcd(r, \frac{k}{2}) = 1 \) implies that \( \gcd(m + 2r, k) = 1 \) and \( m + 2r \equiv 3 \ (\mod \ 4) \). We have \( x^{m+2r} \in [x^{-1}] \). Thus \( x^{m}[x^2] \subseteq [x^{-1}] \). Now since the sizes of \( [x^{-1}] \) and \( x^{m}[x^2] \) are equal, we have \( [x^{-1}] = x^{m}[x^2] \). Similarly, if \( m \equiv 3 \ (\mod \ 4) \) then \( [x^{-1}] = x^{3m}[x^2] \).

(iii) Using Part (i), Part (ii) and \( [x] = [x] \cup [x^{-1}] \), we get the result in desired form. \( \square \)

For \( x \in \Gamma \), define \( S_x^1 := \bigcup_{s \in \text{Cl}(x)} [s] \). We see that if \( m = \text{ord}(x) \), then

\[
S_x^1 = \{ g^{-1}x^rg : g \in \Gamma, r \in \text{G}_m(1) \} = \bigcup_{s \in [x]} \text{Cl}(s).
\]

The set \( S_x^1 \) is also known as the rational conjugacy class of \( x \). See [9] for details. For each \( y \in S_x^1 \), it is clear that \( \text{Cl}(y), [y] \subseteq S_x^1 \). Now let \( A \) be a symmetric subset of \( \Gamma \) such that \( x \in A, \text{ and } \text{Cl}(a), [a] \subseteq A \) for each \( a \in A \). Let \( g^{-1}x^{r}g \in S_x^1 \), where \( g \in \Gamma, r \in \text{G}_m(1) \) and \( m = \text{ord}(x) \). As \( [x] \subseteq A \), we have \( x^{r} \in A \). Now \( \text{Cl}(x^{r}) \subseteq A \), and so \( g^{-1}x^{r}g \in A \). Thus \( S_x^1 \subseteq A \), and therefore \( S_x^1 \) is the smallest symmetric subset of \( \Gamma \) containing \( x \) that is closed under both conjugacy and the equivalence relation \( \sim \). Considering each of the repeated equivalence classes, if any, only once in \( \bigcup_{s \in \text{Cl}(x)} [s] \), we can write \( S_x^1 = \bigcup_{i=1}^{t} [x_i] \), where the equivalence classes \( [x_1], \ldots, [x_t] \) are distinct. We state this fact in the next lemma.

**Lemma 4.3.** If \( x \in \Gamma \), then there exist distinct equivalence classes \( [x_1], \ldots, [x_t] \) such that \( S_x^1 = \bigcup_{i=1}^{t} [x_i] \), where \( x_1, \ldots, x_t \in \text{Cl}(x) \).
Lemma 4.4. If $y \in S^1_x$, then $S^1_y = S^1_x$.

Proof. Let $y \in S^1_x$, so that $y = g^{-1}x^r g$ for some $g \in \Gamma$ and $r \in G_m(1)$, where $m = \text{ord}(x)$. We see that $\text{ord}(y) = \text{ord}(x) = m$. Now let $z \in S^1_y$. Then $z = h^{-1}y^r h$ for some $h \in \Gamma$ and $t \in G_m(1)$. This gives $z = h^{-1}y^r h = h^{-1}x^r g h \in S^1_x$. Conversely, let $w \in S^1_x$ so that $w = h^{-1}x^r h$ for some $h \in \Gamma$ and $t \in G_m(1)$. Therefore

$$w = h^{-1}x^r h = (h^{-1}g)g^{-1}(x^r)^{-1}g(g^{-1}h) = (h^{-1}g)g^{-1}(g^{-1}h) \in S^1_y.$$ 

Here $r^{-1}$ is the multiplicative inverse of $r$ in the group $G_m(1)$. Hence we conclude that $S^1_y = S^1_x$. \qed

Due to Lemma 4.4, the sets $S^1_x$ and $S^1_y$ are either disjoint or equal. Hence the class of distinct subsets of $\Gamma$ of the form $S^1_x$ is a partition of $\Gamma$.

Lemma 4.5. Let $x \in \Gamma(4)$. If $S^1_x = [x_1] \cup \cdots \cup [x_\ell]$ for some $x_1, \ldots, x_\ell \in \text{Cl}(x)$, then $S^1_{x_1} = [x_1^2] \cup \cdots \cup [x_\ell^2]$.

Proof. Let $m = \text{ord}(x)$ and $S^1_x = [x_1] \cup \cdots \cup [x_\ell] \in \text{Cl}(x)$. Assume that the sets $[x_1], \ldots, [x_\ell]$ are all distinct. We see that

$$S^1_{x_1} = \{g^{-1}x_1^r g \mid g \in \Gamma, r \in G^m(1)\} = \{g^{-1}x_1^rg \mid g \in \Gamma, r \in G^m(1)\} \cup \{g^{-1}x_1^2 r g \mid g \in \Gamma, r \in G^m(1), r < \frac{m}{2}\} \cup \{g^{-1}x_1^2 g \mid g \in \Gamma, t \in G^m(1), t > \frac{m}{2}\}$$

$$= \{g^{-1}x_1^2 g \mid g \in \Gamma, r \in G^m(1)\} = \{g^{-1}x_1^2 g \mid g \in \Gamma, r \in G^m(1)\} = \{g^{-1}x_1^2 g \mid g \in \Gamma, r \in G^m(1)\} = \{y^2 \mid y \in S^1_x\}.$$ 

Now noting that $\{s^2 \mid s \in [x]\} = [x^2]$ and $S^1_x = [x_1] \cup \cdots \cup [x_\ell]$, we have $S^1_{x_1} = [x_1^2] \cup \cdots \cup [x_\ell^2]$. \qed

Let $x \in \Gamma(4)$ be an element of order $m$. The element $x$ is said to be admissible if $x^r \not\in \text{Cl}(x)$ for all $r \in G^m(1)$. The following lemma characterizes admissible elements in terms of skew-symmetric sets.

Lemma 4.6. If $x \in \Gamma(4)$, then $x$ is admissible if and only if the set $\bigcup_{s \in \text{Cl}(x)} [s]$ is skew-symmetric.

Proof. We see that if $m = \text{ord}(x)$, then

$$\bigcup_{s \in \text{Cl}(x)} [s] = \{g^{-1}x^r g \mid g \in \Gamma, r \in G^m(1)\} = \bigcup_{s \in [x]} \text{Cl}(s).$$

Assume that $x$ is not admissible, so that $x^r \in \text{Cl}(x)$ for some $r \in G^m(1)$. As $m - r \in G^m(1)$ and $\text{Cl}(x) \subseteq \bigcup_{s \in \text{Cl}(x)} [s]$, we find that $x^r, x^{m-r} \in \bigcup_{s \in \text{Cl}(x)} [s]$. Hence $\bigcup_{s \in \text{Cl}(x)} [s]$ is not skew-symmetric.

Now assume that $\bigcup_{s \in \text{Cl}(x)} [s]$ is not skew-symmetric. Then there is an $y = g^{-1}x^r g \in \bigcup_{s \in \text{Cl}(x)} [s]$ for some $r \in G^m(1)$ such that $y^{-1} \not\in \bigcup_{s \in \text{Cl}(x)} [s]$. Therefore $g^{-1}x^{m-r} g = y^{-1} = h^{-1}x^k h$ for some $h \in \Gamma, k \in G^m(1)$. Let $t \in G^m(1)$ be the multiplicative inverse of $m - r$. We have $g^{-1}x^{(m-r)t} g = h^{-1}x^{kt} h$, and it gives
Lemma 4.9. If \( x^{kt} = hgh^{-1} xgh^{-1} \in \text{Cl}(x) \). Since \((m - r)t \equiv 1 \pmod{4}\) and \(m - r \in G_m^3(1)\), we have that \( t \in G_m^3(1)\). Thus \( kt \in G_m^3(1)\) with \( x^{kt} \in \text{Cl}(x)\), giving that \( x \) is not admissible. \( \Box \)

Let \( x \in \Gamma(4) \) be admissible, and define \( S_x^4 := \bigcup_{s \in \text{Cl}(x)} \llbracket s \rrbracket \). The structure and properties of the set \( S_x^4 \) are similar to those of \( S_x^4 \). If \( \Gamma \) is abelian, then \( S_x^4 = \llbracket x \rrbracket \) for each \( x \in \Gamma(4) \).

For each \( y \in S_x^4 \), it is clear that \( \text{Cl}(y), \llbracket y \rrbracket \subseteq S_x^4 \). Now let \( A \) be a skew-symmetric subset of \( \Gamma \) containing an admissible element \( x \), and \( \text{Cl}(a), \llbracket a \rrbracket \subseteq A \) for each \( a \in A \). It is easy to see that \( S_x^4 \subseteq A \). Thus, \( S_x^4 \) is the smallest skew-symmetric subset of \( \Gamma \) containing \( x \) that is closed under both conjugacy and the equivalence relation \( \equiv \). Considering each of the repeated equivalence classes, if any, only once in \( \bigcup_{s \in \text{Cl}(x)} \llbracket s \rrbracket \), we can write \( S_x^4 = \bigcup_{i=1}^r \llbracket y_i \rrbracket \), where the equivalence classes \( \llbracket y_1 \rrbracket, \ldots, \llbracket y_r \rrbracket \) are distinct. We state this fact in the next lemma.

**Lemma 4.7.** If \( x \) is an admissible element in \( \Gamma(4) \), then there are distinct equivalence classes \( \llbracket y_1 \rrbracket, \ldots, \llbracket y_r \rrbracket \) such that \( S_x^4 = \bigcup_{i=1}^r \llbracket y_i \rrbracket \), where \( y_1, \ldots, y_r \in \text{Cl}(x) \).

**Lemma 4.8.** If \( y \in S_x^4 \), then \( S_y^4 = S_x^4 \).

**Proof.** Let \( y \in S_x^4 \), so that \( y = g^{-1}x^{r}g \) for some \( g \in \Gamma \) and \( r \in G_m^1(1) \), where \( m = \text{ord}(x) \). We see that \( \text{ord}(y) = \text{ord}(x) = m \). Now let \( z \in S_x^4 \). Then \( z = h^{-1}y^{-1}h \) for some \( h \in \Gamma \) and \( t \in G_m^1(1) \). This gives \( z = h^{-1}y^{-1}h = h^{-1}g^{-1}x^{r}gh \in S_x^4 \). Conversely, let \( w \in S_x^4 \) so that \( w = h^{-1}x^{t}h \) for some \( h \in \Gamma \) and \( t \in G_m^1(1) \). Therefore

\[
\begin{align*}
w &= h^{-1}x^{t}h = (h^{-1}g)g^{-1}(x^{r})^{-1}g(g^{-1}h) = (h^{-1}g)g^{-r}g^{-1}(g^{-1}h) \in S_y^4.
\end{align*}
\]

Here \( r^{-1} \) is the multiplicative inverse of \( r \) in the subgroup \( G_m^1(1) \). Thus we conclude that \( S_y^4 = S_x^4 \). \( \Box \)

Due to Lemma 4.8, the sets \( S_x^4 \) and \( S_y^4 \) are either disjoint or equal.

**Lemma 4.9.** If \( x \in \Gamma(4) \) is admissible, then \( S_x^4 \cup S_{x-1}^4 = S_x^4 \).

**Proof.** Let \( m = \text{ord}(x) \). We have

\[
\begin{align*}
S_x^4 \cup S_{x-1}^4 &= \{ g^{-1}x^{r}g : g \in \Gamma, r \in G_m^1(1) \} \cup \{ g^{-1}x^{-r}g : g \in \Gamma, r \in G_m^1(1) \} \\
&= \{ g^{-1}x^{r}g : g \in \Gamma, r \in G_m^1(1) \} \cup \{ g^{-1}x^{-r}g : g \in \Gamma, r \in G_m^1(1) \} \\
&= \{ g^{-1}x^{r}g : g \in \Gamma, r \in G_m^1(1) \} \\
&= S_x^4.
\end{align*}
\]

\( \Box \)

**Lemma 4.10.** Let \( x \in \Gamma(4) \) be an admissible element. If \( S_x^4 = \llbracket x_1 \rrbracket \cup \cdots \cup \llbracket x_r \rrbracket \) for some \( x_1, \ldots, x_r \in \text{Cl}(x) \), then \( S_{x^2}^4 = \llbracket x_1^2 \rrbracket \cup \cdots \cup \llbracket x_r^2 \rrbracket \).

16
Proof. Let \( S^x_t = [x_1] \cup \cdots \cup [x_r] \), where \( x_1, \ldots, x_r \in \text{Cl}(x) \). Then \( S^x_{t-1} = [x_1^{-1}] \cup \cdots \cup [x_r^{-1}] \). Therefore
\[
S^x_t = S^x_t \cup S^x_{t-1} = ([x_1] \cup [x_1^{-1}]) \cup \cdots \cup ([x_r] \cup [x_r^{-1}]) = [x_1] \cup \cdots \cup [x_r].
\]
Now the result follows from Lemma 4.5.

For \( x \in \Gamma \) and \( j \in \{1, \ldots, h\} \), define
\[
C_x(j) := \frac{1}{\chi_j(1)} \sum_{s \in S^x_t} \chi_j(s).
\]
Note that \( S^x_t \in \mathbb{B}(\Gamma) \) and \( C_x(j) \) is an eigenvalue of the normal simple Cayley graph \( \text{Cay}(\Gamma, S^x_t) \). As a consequence of Theorem 2.8, \( C_x(j) \) is an integer for each \( x \in \Gamma \) and \( j \in \{1, \ldots, h\} \).

**Lemma 4.11.** Let \( x \in \Gamma \) and \( \text{ord}(x) = 2^t m \). If \( m \) is odd and \( t \geq 2 \), then
\[
C_x(j) = (\chi_j(x^m) + \chi_j(x^{3m}))C_{x^2}(j).
\]
Moreover, \( C_x(j) \) is an even integer for each \( j \in \{1, \ldots, h\} \).

**Proof.** Let \( S^x_{1} = [x_1] \cup \cdots \cup [x_k] \) for some \( x_1, \ldots, x_k \in \text{Cl}(x) \). For \( j \in \{1, \ldots, h\} \), we have
\[
C_x(j) = \frac{1}{\chi_j(1)} \sum_{r=1}^{k} \sum_{s \in [x_r]} \chi_j(s)
= \frac{1}{\chi_j(1)} \sum_{r=1}^{k} \left( \sum_{s \in [x_r]} \chi_j(x_r^m)\chi_j(s) + \sum_{s \in [x_r]} \chi_j(x_r^{3m})\chi_j(s) \right)
= (\chi_j(x^m) + \chi_j(x^{3m})) \frac{1}{\chi_j(1)} \sum_{r=1}^{k} \sum_{s \in [x_r]} \chi_j(s)
= (\chi_j(x^m) + \chi_j(x^{3m}))C_{x^2}(j). \quad (10)
\]
The second equality in the preceding equations follows from Part (iii) of Lemma 4.2 and the fourth equality follows from Lemma 4.5.

We apply induction on \( t \) to prove that \( C_x(j) \) is an even integer. Let \( \rho_j \) be a representation corresponding to \( \chi_j \). If \( t = 2 \) then \( \rho_j(x^m)^4 \) is the identity matrix, and so each eigenvalue of \( \rho_j(x^m) \) is a 4-th root of unity. Thus, \( \chi_j(x^m) \) is the trace of a matrix whose eigenvalues are 4-th roots of unity. Therefore \( \chi_j(x^m) + \chi_j(x^{3m}) = 2\Re(\chi_j(x^m)) \), an even integer. Hence \( C_x(j) \) is an even integer. Assume that the statement holds for each \( z \in \Gamma \) with \( \text{ord}(z) = 2^{t-1} m \), where \( m \) is odd and \( t - 1 \geq 2 \). Let \( x \in \Gamma \) with \( \text{ord}(x) = 2^t m \), where \( m \) is odd and \( t \geq 3 \). Since the order of \( x^2 \) is \( 2^{t-1} m \), by induction hypothesis \( C_{x^2}(j) \) is an even integer. If \( C_{x^2}(j) = 0 \), then clearly \( C_x(j) = 0 \), an even integer. By Equation (10), \( \chi_j(x^m) + \chi_j(x^{3m}) \) is a rational algebraic integer whenever \( C_{x^2}(j) \neq 0 \). Thus if \( C_{x^2}(j) \neq 0 \), then \( \chi_j(x^m) + \chi_j(x^{3m}) \) is an integer. Hence by Equation (10) and induction hypothesis, \( C_x(j) \) is an even integer for each \( j \in \{1, \ldots, h\} \). Thus the proof is complete by induction.
Let \( x \in \Gamma(4) \) be admissible. For \( j \in \{1, \ldots, h\} \), define

\[
S_x(j) := \frac{i}{\chi_j(1)} \sum_{s \in S_x^j} (\chi_j(s) - \chi_j(s^{-1})).
\]

Note that \( S_x(j) \) is an H-eigenvalue of the normal oriented Cayley graph \( \text{Cay}(\Gamma, S_x^1) \) for each \( j \in \{1, \ldots, h\} \). Since \( S_x^1 \in \mathbb{D}(\Gamma) \), by Theorem 3.2 \( S_x(j) \) is an integer for each \( j \in \{1, \ldots, h\} \).

**Lemma 4.12.** Let \( x \in \Gamma(4) \) be admissible and \( \text{ord}(x) = 2^t m \). If \( m \) is odd and \( t \geq 2 \), then

\[
S_x(j) = \begin{cases} 
-23(\chi_j(x^{3m}))C_x^2(j) & \text{if } m \equiv 1 \pmod{4} \\
-23(\chi_j(x^{m}))C_x^2(j) & \text{if } m \equiv 3 \pmod{4}.
\end{cases}
\]

Moreover, \( S_x(j) \) is an even integer for each \( j \in \{1, \ldots, h\} \).

**Proof.** Let \( S_x^1 = \llbracket x_1 \rrbracket \cup \cdots \cup \llbracket x_k \rrbracket \) for some \( x_1, \ldots, x_k \in \text{Cl}(x) \). For \( j \in \{1, \ldots, h\} \), we have

\[
S_x(j) = \frac{i}{\chi_j(1)} \sum_{r=1}^{k} \sum_{s \in [x_r]} (\chi_j(s) - \chi_j(s^{-1}))
\]

\[
= \frac{i}{\chi_j(1)} \sum_{r=1}^{k} \sum_{s \in \llbracket x_r \rrbracket} (\chi_j(s) - \chi_j(s^{-1})) \quad \text{if } m \equiv 1 \pmod{4}
\]

\[
= \frac{i}{\chi_j(1)} \sum_{r=1}^{k} \sum_{s \in \llbracket x_r \rrbracket} (\chi_j(x^{3m})\chi_j(s) - \chi_j(x^{-3m})\chi_j(s^{-1})) \quad \text{if } m \equiv 1 \pmod{4}
\]

\[
= \frac{i}{\chi_j(1)} \sum_{r=1}^{k} \sum_{s \in \llbracket x_r \rrbracket} (\chi_j(x^{m})\chi_j(s) - \chi_j(x^{-m})\chi_j(s^{-1})) \quad \text{if } m \equiv 3 \pmod{4}
\]

\[
= -23(\chi_j(x^{3m}))\frac{i}{\chi_j(1)} \sum_{r=1}^{k} \sum_{s \in \llbracket x_r \rrbracket} \chi_j(s) \quad \text{if } m \equiv 1 \pmod{4}
\]

\[
-23(\chi_j(x^{m}))\frac{i}{\chi_j(1)} \sum_{r=1}^{k} \sum_{s \in \llbracket x_r \rrbracket} \chi_j(s) \quad \text{if } m \equiv 3 \pmod{4}
\]

The third equality in the preceding equations follows from Part (i) of Lemma 4.2 and the fifth equality follows from Lemma 4.10. If \( t = 2 \), then \( \chi_j(x^{3m}) \) and \( \chi_j(x^{m}) \) are traces of matrices whose eigenvalues are 4-th roots of unity. Therefore, \( \Im(\chi_j(x^{3m})) \) and \( \Im(\chi_j(x^{m})) \) are integers. Thus \( S_x(j) \) is an even integer. Now assume that \( t \geq 3 \). If \( C_x^2(j) = 0 \), then clearly \( S_x(j) = 0 \), an even integer. Note that \( 23(\chi_j(x^{3m})) \) and \( 23(\chi_j(x^{m})) \) are rational algebraic integers whenever \( C_x^2(j) \neq 0 \). Thus if \( C_x^2(j) \neq 0 \), then \( 23(\chi_j(x^{3m})) \) and \( 23(\chi_j(x^{m})) \) are integers. Since the order of \( x^2 \) is \( 2t^{-1}m \), by Lemma 4.11 \( S_x(j) \) is an even integer. \( \square \)
Let $S$ be a nonempty set in $\mathbb{D}(\Gamma)$ and that $S$ be expressible as a union of some conjugacy classes of $\Gamma$. Then $S$ is a skew-symmetric subset of $\Gamma$ that is closed under both conjugacy and the equivalence relation $\approx$. Let $S = \text{Cl}(x_1) \cup \cdots \cup \text{Cl}(x_k) = [y_1] \cup \cdots \cup [y_r]$ for some $x_1, \ldots, x_k, y_1, \ldots, y_r \in \Gamma(4)$. We see that

$$S = \text{Cl}(x_1) \cup \cdots \cup \text{Cl}(x_k) = \left( \bigcup_{s \in \text{Cl}(x_1)} [s] \right) \cup \cdots \cup \left( \bigcup_{s \in \text{Cl}(x_k)} [s] \right) = S^1_{x_1} \cup \cdots \cup S^4_{x_k}.$$ 

Due to Lemma 4.8, we can assume that the sets $S^1_{x_1}, \ldots, S^4_{x_k}$ are all distinct. In the next result, we prove the converse of Theorem 4.1.

**Theorem 4.13.** If $\Gamma$ is a finite group, then the normal mixed Cayley graph $\text{Cay}(\Gamma, S)$ is Gaussian integral if and only if it is H-integral.

**Proof.** Assume that the normal mixed Cayley graph $\text{Cay}(\Gamma, S)$ is H-integral. It is enough to show that $f_j(S)$ and $g_j(S)$ are integers for $j \in \{1, \ldots, h\}$. By Theorem 3.3, we get $S \setminus S \in \mathbb{B}(\Gamma)$ and $S \in \mathbb{D}(\Gamma)$. Therefore $S = S^1_{x_1} \cup \cdots \cup S^4_{x_r}$ for some $x_1, \ldots, x_r \in \Gamma(4)$, where the sets $S^1_{x_1}, \ldots, S^4_{x_r}$ are all distinct. By Lemma 4.12, $S_{x_1}(j) + \cdots + S_{x_r}(j)$ is an even integer. As $g_j(S) = \frac{1}{2}(S_{x_1}(j) + \cdots + S_{x_r}(j))$, we find that $g_j(S)$ is an integer. Observe that $S^1_{x_1} \cup S^1_{x_r} = S^1_{x_1}$, and so $S \cup S^{-1} = S^1_{x_1} \cup \cdots \cup S^4_{x_r}$. Therefore $f_j(S) = \frac{1}{\chi_j(1)} \sum_{s \in S \setminus S} \chi_j(s) + \frac{1}{2}(C_{x_1}(j) + \cdots + C_{x_r}(j))$. By Theorem 2.8, $\frac{1}{\chi_j(1)} \sum_{s \in S \setminus S} \chi_j(s)$ is an integer. Also, by Lemma 4.11, $C_{x_i}(j)$ is an even integer for each $i \in \{1, \ldots, r\}$. Hence we find that $f_j(S)$ is an integer. The other part of the theorem is already proved in Theorem 4.1. 

We give the following example to illustrate Theorem 4.13.

**Example 4.1.** Consider the normal mixed Cayley graph $\text{Cay}(M_{16}, S)$ of Example 3.2. We have already seen that it is H-integral, and hence it must be Gaussian integral. Indeed, using Theorem 2.4, the spectrum of $\text{Cay}(M_{16}, S)$ is obtained as

$$\{[\gamma_j]^1: 1 \leq j \leq 8\} \cup \{[\gamma_9]^4, [\gamma_{10}]^4\},$$

where

$$\gamma_j = \frac{1}{\chi_j(1)} \left[ \chi_j(a) + \chi_j(a^3) + \chi_j(a^5) + \chi_j(a^7) + \chi_j(a^9) x \right] \text{ for each } j \in \{1, \ldots, 10\}.$$ 

We find that $\gamma_1 = 6, \gamma_2 = -6, \gamma_3 = 2, \gamma_4 = -2, \gamma_5 = \gamma_7 = 2i, \gamma_5 = \gamma_8 = -2i, \text{ and } \gamma_9 = \gamma_{10} = 0$. Thus $\text{Cay}(M_{16}, S)$ is Gaussian integral.

**Conflict of Interest:** We declare that we have no conflict of interest to this work.

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