Learning Multivariate Hawkes Processes at Scale

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Abstract
Multivariate Hawkes Processes (MHPs) are an important class of temporal point processes that have enabled key advances in understanding and predicting social information systems. However, due to their complex modeling of temporal dependencies, MHPs have proven to be notoriously difficult to scale, what has limited their applications to relatively small domains. In this work, we propose a novel model and computational approach to overcome this important limitation. By exploiting a characteristic sparsity pattern in real-world diffusion processes, we show that our approach allows to compute the exact likelihood and gradients of an MHP – independently of the ambient dimensions of the underlying network. We show on synthetic and real-world datasets that our model does not only achieve state-of-the-art predictive results, but also improves runtime performance by multiple orders of magnitude compared to standard methods on sparse event sequences. In combination with easily interpretable latent variables and influence structures, this allows us to analyze diffusion processes at previously unattainable scale.

1. Introduction
Time, and in particular the magnitude of time intervals between events, carries important information about latent structures of event sequences. Temporal point processes (TPPs) are a flexible and powerful paradigm for modeling such discrete event sequences that are localized in continuous time. Consequently, TPPs have found important applications in diverse fields such as social and information systems (Gomez-Rodriguez et al., 2011; Iwata et al., 2013), human mobility and learning (Jankowiak & Gomez-Rodriguez, 2017; Mavroforakis et al., 2017), finance (Bacry et al., 2015), health (Alaa et al., 2017), and recommender systems (Jing & Smola, 2017; Kumar et al., 2019).

Hawkes Processes (Hawkes, 1971) are an important class of TPPs which are widely used for modeling temporal events with self-exciting behaviour, i.e., processes were the occurrence of an event increases the likelihood of another event happening in the near future. Multivariate Hawkes Processes (MHPs) extend this approach by also accounting for the mutual excitation of events along different entities (or dimensions). Such self- and mutually-exciting processes are central to modeling a wide range of real-world phenomena. For instance, Hawkes Processes have been found to be an excellent approach for modeling the self-exciting and bursty nature of information cascades in social networks (Zhao et al., 2015); for modeling the mutually exciting and retaliatory patterns in gang violence (Linderman & Adams, 2014); for modeling earthquake aftershock sequences (Ogata, 1998); and have been proposed to mitigate the spread of fake news (Farajtabar et al., 2017b).

Moreover, a distinctive advantage of MHPs is that their model parameters are readily interpretable and can provide valuable insights into the latent structure of event sequences (see also Figure 1). This has, for instance, been exploited to infer latent influence structures from diffusion processes...
Despite this diverse range of applications and appealing properties, wider adoption of MHPs has been hindered due to a fundamental limitation: existing models and inference methods for MHPs have proven to be very difficult to scale due to their complex modeling of temporal dependencies. For this reason, MHPs have typically been applied only to relatively small domains, e.g., diffusion networks with a few hundred or thousand nodes (although the number of events can be larger). A common strategy for larger domains is often to limit the number of entities by subsampling the dataset. This not only reduces the coverage of the model, but can also lead to incorrectly estimated influence structures as subsampling affects a process’ latent relational structures. Altogether, this leads to an unfortunate dilemma: many domains where MHPs would enable the most promising applications are exactly those that are intractable with current inference methods.

In this work, we propose a new approach to overcome this issue in the context of large-scale diffusion processes. Our approach is based on the important observation that such processes exhibit a characteristic sparsity pattern in that only a very small fraction of all possible entities participate in any given event sequence. To exploit this property, we develop a combined model and inference method that does not only allow to compute the exact likelihood and gradients on large-scale data, but also improves runtime performance by multiple orders of magnitude – even when compared to state-of-the-art neural methods. In our experiments, we show that this enables interpretable models of diffusion processes at previously unattainable scale.

The remainder of this paper is structured as follows: In Section 2, we discuss related work. In Section 3, we discuss our model and how to train it at scale. In Section 4, we evaluate our method both with regard to model quality and runtime complexity on synthetic and real-world data.

2. Related Work

Temporal point processes have long been explored for modeling diffusion processes in social networks. Following the pioneering work of Gomez-Rodriguez et al. (2011) on uncovering latent influence structures from information cascades, TPPs have been applied, for instance, to modeling information pathways in online media, Gomez Rodriguez et al. (2013), to analyze the diffusion of policies in US states (Boehmke et al., 2018), and to topic modeling from text cascades (He et al., 2015). However, none of these methods can scale to a large number of entities that participate in the diffusion process and simultaneously model their interdependencies. Other methods such as SEISMIC (Zhao et al., 2015) or CONTINEST (Du et al., 2013) and COEVOLVE (Farajtabar et al., 2017a) model information cascades using either univariate HPs, or require the underlying network structure to be known. Lemonnier et al. (2017) proposed factorized MHPs which improve runtime and memory complexity significantly with respect to the number of nodes in an influence network. However, even this approach requires more than $3.7 \cdot 10^5$ seconds of training time for a relatively small dataset with 1075 nodes. Hence, no existing method is currently able to model large-scale influence structures even though there has been considerable work in applying TPPs and MHPs to model diffusion processes over the years.

Another important approach that has recently received increased attention is to model TPPs using recurrent neural architectures. This includes, for instance Recurrent Marked Temporal Point Processes (RMTPP; Du et al. 2016), Dynamic Embeddings for Temporal Recommendations (Kumar et al., 2019), and Neural Hawkes Processes (NHPs; Mei & Eisner 2017). An appealing property of neural methods for TPPs is that they scale well with regard to the length of an event sequence and offer a flexible framework to parameterize an intensity function. However, current neural methods do not scale well to datasets that consist of a large number of entities and a large number of event sequences. Moreover, most existing neural methods are essentially black-box methods that are not interpretable.

Another shortcoming of many recurrent neural methods is that they need to resort to uniform sampling to compute the compensator of a TPP. This is clearly suboptimal for application areas with very irregular time intervals such as bursty information cascades. For this reason, another line
of research has recently explored methods based on Neural ODEs (Chen et al., 2018) which allow to compute the exact intensity function using neural methods. This includes, for instance, Latent ODEs for irregularly sampled time series (Rubanova et al., 2019) and Neural Jump Stochastic Differential Equations (Jia & Benson, 2019). However, current Neural ODE-based methods suffer from the same interpretability shortcomings as recurrent architectures and are also difficult to scale and train. Another promising line of research explores intensity-free learning of TPPs to circumvent some of the computational issues associated with parameterizing intensity functions (Shchur et al., 2019; Xiao et al., 2017).

In this work, we pursue a different approach for modeling large-scale diffusion processes: by going back to the original idea of MHPs and by carefully adjusting model and inference method to properties of the data, we show that MHPs can in fact be applied to large-scale domains while retaining all of its appealing properties including interpretability and state-of-the-art performance.

3. Method

In the following, let \( t^x \) denote an event occurring at time \( t \) and at entity \( x \in X \).\(^1\) Furthermore, let \( h = \{t_i^x\}_{i=1}^n \) denote a single event sequence with \( t_i < t_{i+1} \). Furthermore, let \( T_h \) denote the maximum observation time for event sequence \( h \). A temporal point process is then fully characterized through its conditional intensity function

\[
\lambda_x(t|h) = \lim_{\Delta t \downarrow 0} \frac{\Pr(t_i^x \in [t, t + \Delta t] \mid h)}{\Delta t}
\]  

which specifies the infinitesimal probability that an event \( t_i^x \) occurs in the time interval \([t, t + \Delta t]\) given past events \( h \). We follow Daley & Vere-Jones (2008) and use \( \lambda_x(t) \) as shorthand for \( \lambda_x(t|h) \). Given a dataset \( H = \{h_i\}_{i=1}^m \) of event sequences, we can then jointly estimate the parameters for all \( \lambda_x \) by maximizing the log-likelihood

\[
\mathcal{L}(H) = \sum_{h \in H} \sum_x \left( \sum_{t=1}^n \log \lambda_x(t_i^x) - \int_0^{T_h} \lambda_x^*(s)ds \right)
\]

Prior work on scaling MHPs has mainly focused on the integral (compensator) in Equation (2) which is intractable to compute in general. However, a second important aspect regarding scalability is the number of entities \(|X|\) and the number of event sequences \(|H|\): It can be seen from Equation (2) that the computation of the log-likelihood for a single sequence \( h \) involves all entities of the TPP. Hence, the overall runtime complexity for a single evaluation of

\[\mathcal{O}(|X| \cdot |H|).\]^2

This becomes prohibitively expensive in domains such as social networks and information systems, where both \(|X|\) and \(|H|\) can easily be in the millions.

To overcome this issue, our approach is based on the important observation that sequences in large-scale domains are typically sparse, i.e., that only a small fraction of all possible entities participate in a given episode. For instance, only very few (out of all possible) users in a social network typically interact with (e.g., like, share) a given meme. Figure 2 illustrates this property on two larger real-world datasets: Memetracker (Leskovec et al., 2009) which consists of \( \sim 75 M \) meme cascades collected from \( \sim 1.3M \) websites; and Reddit Hyperlinks (Kumar et al., 2018) which consists of \( \sim 860 K \) hyperlinks between \( \sim 55 K \) Reddit communities (subreddits). For Memetracker, it can be seen that even for the most dense sequences, less than 10% of all entities are active and for the vast majority of sequences it is even less than 0.001%. The smaller Reddit Hyperlinks dataset exhibits a similar distribution. Based on these observations, we will now propose a combined model and inference method to compute the exact likelihood and gradients at scale.

Model To develop our model, we build on prior work on Exponential Hawkes Processes which are defined through the conditional intensity function

\[
\lambda_x^*(t) = \mu_x + \sum_{t^y_i < t} \alpha_{xy} e^{-\beta(t-t^y_i)}
\]

where \( \mu_x, \alpha_{xy}, \beta > 0 \) and \( t^y_i \in h \). For Equation (3), it is well known (Ozaki, 1979) that the compensator can be computed in closed form via

\[
\int_0^{T_h} \lambda_x^*(s)ds = \mu_x T_h + \sum_{t^y_i \in h} \frac{\alpha_{xy}}{\beta} \left(1 - e^{-\beta(T_h-t^y_i)}\right)
\]

Moreover, the Markovian nature of Exponential Hawkes Processes allows to compute their log-likelihood in \(\mathcal{O}(|H|)\), using a recursive algorithm (Ogata, 1981):

\[
\mathcal{L}(h) = \sum_x \sum_{i=1}^n \log \left(\mu_x + A(i)\right) - \int_0^{T_h} \lambda_x^*(s)ds
\]

where

\[
A(i) = \begin{cases} 
\sum_{t_j < t_i} \alpha_{xy} e^{-\beta(t_i-t_j)} & i > 1 \\
0 & i = 1
\end{cases}
\]

However, while this solves some of the scalability issues associated with MHPs, it does not address the core concern of our work, i.e., scalability with regard to \(|X|\) and

\(^1\)For notational convenience, we will simply write \( t_i^x \) if the identity of an event’s entity is not relevant in a given context.

\(^2\)In our runtime analysis, we absorb the cost of \( \lambda_x^*(t|h) \) into \(|H|\) since it does not differ between the model classes we consider.
A crucial difference to prior work on Exponential Hawkes Processes is that we factorize the influence structure as 

\[ \alpha_{xy} = \begin{cases} \phi(\theta^v_y) & \text{if } x = y \\ u_x^\top v_y & \text{otherwise} \end{cases} \]

This parameterization ensures that \( \lambda_x \) stays always positive while simultaneously being differentiable such that we can employ gradient-based optimization.

A crucial difference to prior work on Exponential Hawkes Processes is that we factorize the influence structure as \( \alpha_{xy} = \langle \phi(\theta^v_x), \phi(\theta^v_y) \rangle \) as opposed to \( \alpha_{xy} = \phi(\theta^v_x, \theta^v_y) \). This non-negative factorization is key to our scalable inference method as we will show in the following.

### Scalable Likelihood and Gradient Computation

First, note that the log-likelihood can be decomposed into active and inactive entities in an event sequence:

\[
\ell(\mathcal{H}) = \sum_{h \in \mathcal{H}} \left( \sum_{x \in h} \ell_x(h) + \sum_{x \not\in h} \ell_x(h) \right) = \sum_{h \in \mathcal{H}} \frac{\lambda^v_h(s)ds}{h \mathcal{H}^+} 
\]

Second, for inactive entities \( x \not\in h \), \( \ell_x(h) \) simplifies to

\[
\ell_x(h) = \int_0^{t_h} \lambda^v_x(s)ds 
\]

Furthermore, due to our non-negative factorization, \( \alpha_{xy} \) is now a linear function of \( u_x \) and \( v_y \) and we can therefore rewrite \( \ell_x(h) \) for inactive entities as

\[
\sum_{x \not\in h} \ell_x(h) = - \sum_{x \not\in h} \mu_x T_h - \frac{1}{\beta} \left( \sum_{x \not\in h} u_x \right) \sum_{t \not\in h} v_y \left( 1 - e^{-\beta(T_h - t_i^h)} \right)
\]

Combining these three properties, our approach is then based on the following idea: for sparse event sequences where \( |x| < |h| \) it holds in terms of runtime complexity that

\[
O\left( \sum_{x \not\in h} u_x \right) \ll O\left( \tilde{u} - \sum_{x \in h} u_x \right)
\]

where \( \tilde{u} = \sum_x u_x \). This is the case because \( \tilde{u} \) is identical for all \( x \in \mathcal{X} \) and \( h \in \mathcal{H} \). We can therefore regard it as a constant that has to be computed only once.

Using this reparameterization trick, we can then compute the full likelihood via

\[
\ell(\mathcal{H}) = \sum_{h \in \mathcal{H}} \left( \sum_{x \in h} \ell_x(h) - \frac{1}{\beta} C_x - \mu_x D_x \right)
\]

where \( \tilde{u} = \tilde{u}/(x \in h) \) and where \( \mathcal{H}_x = \{ h : x \in h \} \) and \( \mathcal{H}_x = \{ h : x \not\in h \} \) are the sets of all sequences in which \( x \) does and does not occur, respectively. Although \( D_x \) depends on \( \mathcal{H}_x \), it depends on no parameter of the model. It can therefore be precomputed and regarded as a constant. Furthermore, let \( \bar{E} \) denote the average number of active entities per event sequence. It can then easily be verified that Equation (5) has a runtime complexity of \( O(|\mathcal{X}| + |\mathcal{H}| \cdot \bar{E}) \) since processing each sequence \( h \) depends only on precomputed quantities for entities \( x \not\in h \). For sparse sequences where \( \bar{E} \ll |\mathcal{X}| \), Equation (5) can therefore yield substantial runtime improvements compared to existing inference methods.

To compute the gradients of Equation (2), we can apply similar ideas. In particular, the gradient for post-activation weights \( \mu_x \) can be computed as

\[
\frac{\partial}{\partial \mu_x} \ell(\mathcal{H}) = \sum_{h \in \mathcal{H}_x} \left( \frac{\partial}{\partial \mu_x} \ell_x(h) - G^v_x \right)
\]

where \( G^v_x = \sum_{h \not\in \mathcal{H}_x} T_h / |\mathcal{H}_x| \).

For \( \beta \), the gradient can be computed via

\[
\frac{\partial}{\partial \beta} \ell(\mathcal{H}) = \sum_{h \in \mathcal{H}_x} \left( \frac{\partial}{\partial \beta} \ell_x(h) - C^v_x \right)
\]

where \( C^v_x = (\tilde{u} - u_x) \sum_{t \not\in h} v_y \frac{1}{\beta} \left( 1 - e^{-\beta(T_h - t_i^h)} \right) \).

For \( u_x \), the gradient can be computed via

\[
\frac{\partial}{\partial u_x} \ell(\mathcal{H}) = \sum_{h \not\in \mathcal{H}_x} \left( \frac{\partial}{\partial u_x} \ell_x(h) + \frac{1}{\beta} G^u_x \right)
\]

where \( G^u_x = \sum_{t \not\in h} v_y \left( 1 - e^{-\beta(T - t_i^h)} \right) - \hat{z} \), and

\[
\hat{z} = \sum_{h \not\in \mathcal{H}_x} T_h / |\mathcal{H}_x|\left( 1 - e^{-\beta(T - t_i^h)} \right) \cdot \tilde{u}
\]

Similar to \( \tilde{u} \), \( \hat{z} \) is identical for all \( x \) and \( h \) and, therefore, needs to be computed only once per epoch.
For \( v_y \), the gradient can be computed via
\[
\frac{\partial}{\partial v_y} \mathcal{L}(H) = \sum_{h \in H} \left( \sum_{x \in h} \frac{\partial}{\partial v_y} L_x(h) - \frac{1}{\beta} G_y \right)
\]
\[
G_y^{\prime} = (\hat{u} - u_x) \sum_{t_i = y} (1 - e^{-\beta(T_h - t_i^\prime)})
\]  

Finally, self-excitation parameters \( \alpha_{xx} \) receive gradients only for sequences where \( x \in h \) and require no special treatment. For the full derivation of all calculations, we refer the reader to the supplementary material.

By using this approach, we can now compute the exact log-likelihood and gradients of the model by processing only the active entities in an event sequence and with an overall runtime complexity of \( O(|\mathcal{X}| + |H| \cdot E) \). We will refer to this combined model and inference method as Lazy MHPs.

**Stochastic Training** The likelihood and gradients in Section 3 can directly be used for parameter estimation using batch methods such as L-BFGS (Byrd et al., 1995). However, batch optimization can be difficult to scale and sensitive to initialization and local minima. For this reason, we discuss in the following how to apply stochastic optimization methods in our setting.

Naive stochastic optimization using the gradients in Section 3 would again lead to bad runtime performance as it would require to compute \( \hat{u} = \sum_x u_x \) for each minibatch. However, due to its linear structure, we can update \( \hat{u} \) on-the-fly during the optimization process. In particular, let \( \Theta_{\hat{u}}^{-} \), \( \Theta_{\hat{u}}^{+} \) denote parameters before and after an update. We can then simply compute the new state of \( \hat{u} \) via
\[
\hat{u} \leftarrow \hat{u} - \phi(\Theta_{\hat{u}}^{-}) + \phi(\Theta_{\hat{u}}^{+})
\]

As such, it is sufficient to compute \( \hat{u} \) only once prior to training. The second is \( \bar{z} \). Since \( \bar{z} \) is not a linear function of either \( v_y \) or \( \beta \), we employ a different strategy: First, note that \( \bar{z} \) can be computed in one pass through the data and by only considering active entities. Hence, we can compute \( \bar{z} \) on-the-fly while processing an epoch and use it in the next epoch to compute gradients. While this leads to lagging updates for \( u_x \), we did not observe any deterioration in terms of convergence or model quality. The full algorithm for stochastic training of Lazy MHPs is listed in Algorithm 1.

**Implementation and Training Details** Since the efficient computation of recursive gradients is not supported by standard frameworks such as PyTorch or Tensorflow, we implemented our model as a custom C++ extension to PyTorch. In our implementation, we exploit that a large number of sequences and entities allows for massively parallel training using Hogwild (Recht et al., 2011) as we can dispatch each \( L_x(h) \) to a separate thread. This has the advantage over mini-batching that we do not need to resort to padding or to group sequences by length (both would be problematic as sequence length can differ by multiple orders of magnitude). For optimization we employ Adam (Kingma & Ba, 2015).

**Scope and Limitations** Our approach is explicitly designed to train MHPs in large-scale domains with possibly millions of entities and sparse event sequences. In smaller domains or on data with dense sequences, our approach will likely not provide significant advantages over traditional methods. Furthermore, our method relies on a non-negative factorization of \( \alpha \) to achieve scalability. While this has the additional positive effect of interpretable latent factors, it will also lead to larger embedding dimensions compared to standard factorizations, which can have negative effects on runtime performance. Due to the non-negative factorization, our method is also limited to modeling mutual excitation (e.g., it cannot model inhibition). Overall, we believe these are acceptable tradeoffs for learning large-scale MHPs.

**4. Experiments**

In the following, we evaluate our model on various synthetic and real-world datasets to demonstrate three key properties of our method: a) Lazy MHPs improve runtime by multiple orders of magnitude compared to existing methods b) since Lazy MHPs compute exact gradients, we are able to infer accurate model parameters c) interpretable model parameters allow us to analyze diffusion processes at scale.

We compare Lazy MHPs to typical factorized MHPs where \( \alpha_{xy} = \phi(\Theta_x, \Theta_y) \) and which are trained with standard maximum-likelihood (MLE). This allows us to compare both the performance of lazy training as well as the effect of our non-negative factorization. If applicable, we also compare to ADM4 (Zhou et al., 2013) as implemented in
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Figure 3: Runtime experiments on Memetracker subsets (log-log scale).

tick (Bacry et al., 2017). Furthermore, we compare to Neural Hawkes Processes (Mei & Eisner, 2017) as a scalable neural method. Due to space constraints, we report full hyperparameter settings in the supplementary material.

Runtime Analysis on Real-World Cascades First, we focus on the runtime performance of our model on real-world cascades. For this purpose, we employ various subsets of Memetracker (Leskovec et al., 2009) as extracted by Gomez Rodriguez et al. (2013). Each subset consists of information cascades that are related to a specific topic in the news cycle from March 2011 to February 2012. For comparison, we selected four subsets whose sizes are still tractable with standard MHPs (see Table 1 for data statistics). Unfortunately, these datasets were already too large for ADM4 and EM as implemented in tick, causing out-of-memory errors on 500GB memory machines.

In our experiments, we compared the runtime for a single epoch on these datasets for: Lazy MHPs, standard MHPs (MLE), and the Neural Hawkes Process (NHP). Both MHP models were trained using 40 threads while the NHP was trained on a GPU. It can be seen from Figure 3 (upper row) that Lazy MHPs improve the runtime compared to standard MHPs by approximately four orders of magnitude on all datasets. Even for Neural Hawkes Processes – which are one of the most scalable available methods – our method improves the runtime by approximately two orders of magnitude. This is remarkable as NHPs are a black-box model that makes various approximations to achieve scalability while Lazy MHPs compute an exact, interpretable MHP.

Log-likelihood

Amy Winehouse
Arab Spring
Bail out
Miami Heat

Parameter Estimation on Synthetic Data Next, we evaluate our model on synthetic datasets that have been generated from standard MHPs with known parameters $\mu$, $\beta$, and $\alpha$. This allows us to not only test our stochastic training method, but also evaluate the ability of our non-negative factorization to recover the correct model parameters $\alpha$. In detail, the datasets have been generated as follows: We first sampled random directed scale-free graphs as proposed by Bollobás et al. (2003) to generate synthetic influence structures. We varied the size of the graph between 50 and 250 nodes. Furthermore, we set the global timescale parameter

| Topic          | Nodes | Episodes | Events   |
|----------------|-------|----------|----------|
| Amy Winehouse  | 1,561 | 109,650  | 226,247  |
| Arab Spring    | 1,377 | 179,681  | 400,199  |
| Bail Out       | 835   | 44,130   | 64,138   |
| Miami Heat     | 1,173 | 61,280   | 133,451  |
Table 2: Model quality on synthetic data and example of generated sequences.

|       | Log-Likelihood | RMSE |       |       |       |
|-------|----------------|------|-------|-------|-------|
|       |                | µ    | β     | α     |       |
|       |                | 50   | 250   | 50   | 250   | 50   | 250   |
| Low Rank |                |      |       |       |       |       |       |
| ADM4  | -6.88          | 2.8e-5 | 1.0e-5 | -     | -     | 0.009 | 0.003 |
| NHP   | -7.34          | -     | -     | -     | -     | -     | -     |
| MLE   | -6.93          | 3.0e-5 | 1.2e-5 | 0.017 | 0.006 | 0.009 | 0.003 |
| Lazy MHP | -6.96        | 2.8e-5 | 1.2e-5 | 0.001 | 0.038 | 0.015 | 0.007 |
| Full Rank |                |      |       |       |       |       |       |
| ADM4  | -7.27          | 2.6e-5 | 1.1e-5 | -     | -     | 0.011 | 0.003 |
| NHP   | -8.02          | -     | -     | -     | -     | -     | -     |
| MLE   | -7.28          | 2.8e-5 | 1.3e-5 | 0.022 | 0.060 | 0.015 | 0.004 |
| Lazy MHP | -7.28        | 2.7e-5 | 1.1e-5 | 0.015 | 0.037 | 0.037 | 0.008 |

Table 2: Model quality on synthetic data and example of generated sequences.

In Table 2, we report the RMSE of the inferred parameters as well as the log-likelihood of the observed sequences. Since ADM4 requires β as a hyperparameter, we provide it with the ground-truth value during training. As such ADM4 has a distinct advantage in our evaluation. For the Lazy MHP and MLE models we used a maximum factorization dimension of d = 20. It can be seen that Lazy MHPs show strong results in this task and accurately recover the model parameters as compared to ADM4 and MLE. The log-likelihood results of Lazy MHPs show similarly strong results. The non-negative factorization has only a limited effect on model quality when compared to MLE. Furthermore, even for the full-rank setting a d = 20 factorization was sufficient to achieve strong results. All MHP based methods outperform NHPs on these datasets, which is likely due to the mismatch of bursty event sequences and uniform sampling in NHPs (see suppl. material for a detailed evaluation of this aspect).

SPID In the following, we examine the interpretability of our model with respect to the inferred influence structure and latent factors. For this purpose, we employ the State Policy Innovation and Diffusion Database (SPID; Boehmke et al. 2019) which has been collected to study innovations in public policy and the spread of policies across US states.
The database consists of 728 policies established between the years 1691 to 2017. For each policy, the SPID records the year of its eventual adoption in a state (e.g., “Alcoholic Beverage Control, 1926, Pennsylvania”), (Concealed Carry, 1926, Pennsylvania), (Concealed Carry, 1975, Alabama). We consider each policy as a separate event sequence and embed the entire database using a 4-dimensional Lazy MHP. The influence structure α_{xy} in the MHP reflects the increased likelihood that a state x adopts a policy if state y has adopted the same policy earlier.

Since SPID is a smaller database, it allows us to visually inspect and analyze the inferred model in its entirety. Figure 4 shows the full influence matrix α as well as the inferred latent factors u_x, v_y for each state. To aid visual analysis, we also clustered both latent factors using k-means. For the full influence matrix, we computed a bi-clustering based on the k-means clusters of the latent factors. It can be seen that the inferred model reveals interesting structures of the diffusion process: First, α as well as u_x and v_y exhibit clear regional structures. For instance, New York is influential for its direct neighbors (esp. New Jersey, Massachusetts) and north-eastern states in general. Tennessee on the other hand is mainly influential for states in the south-east such as South Carolina and Mississippi; while California is influential for states such as Oregon and Washington. However, not all influence is purely regional. For instance, although on different coasts, New York is also influential for California, since both states share similar policy adoptions. State-specific plots in Figure 4 further illustrate these patterns on the examples of New York and Tennessee. Overall, New York and California are two of the most influential states in our model, what matches previous analyses (Boehmke et al., 2018).

Reddit Hyperlinks At last, we apply our model also on a larger-scale dataset, i.e., Reddit Hyperlinks (Kumar et al., 2018) which consists of timestamped hyperlinks between Reddit communities. In total, the dataset consists of over 55K entities (subreddits) and 860K hyperlinks between them. We consider each link target as a separate sequence and embed the entire dataset using a 50-dimensional Lazy MHP. The influence structure α_{xy} reflects then the increased likelihood that a subreddit x links to a target if subreddit y has linked to it earlier.

Due to the non-negative factorization of Lazy MHPs, it is relatively easy to analyze such larger datasets. Since we can interpret each value u_{xk} (and v_{yk}) as the participation of entity x in component (or cluster) k, we can directly inspect the latent factors to analyze the latent structure of the diffusion process. In Table 3, we list subreddits with the highest activations (as measured by u_{xk}) for various latent factors that have been inferred by our model. It can be seen that the model is able to reveal topicaly consistent features that explain the temporal interdependencies within hyperlink cascades – ranging from relatively broad topics such as music and sports to more focused communities about specific games and electronics. Training a Lazy MHP requires ~ 650s per epoch on this dataset and achieves a log-likelihood of ca. −9.1 after 100 epochs. To the best of our knowledge, this is the first time that a MHP has been successfully applied to a dataset of this size.

5. Conclusion

In this work, we developed a new method for modeling diffusion processes at previously unattainable scale. By revisiting the original idea of MHPs and by carefully adjusting model and inference to the sparsity of real-world event sequences, we show that MHPs can in fact be applied to large-scale domains while retaining all of its appealing properties including interpretability and state-of-the-art performance. Our approach reduces runtime complexity from $O(|X| \cdot |H|)$ to $O(|X| + |H| \cdot E)$ what improves runtime performance by multiple orders of magnitude on sparse sequences. Since scalability was an important limitation in existing methods, we believe that our approach opens up promising new opportunities for modeling and understanding social information systems at scale.

| Topic       | Subreddits                                                                 |
|-------------|-----------------------------------------------------------------------------|
| Music       | music, hiphopheads, metal, indieheads, metaljerk                            |
|             | electricforest, electronicmusic, punk, listentothis                        |
| Music Prod. | wearthemusicmakers, edproduction, besked, mcmosaic, audioengineering        |
|             | djs, music, synthesizers                                                   |
| Movies      | movies, starwars, marvel, arrow, marvelstudios, starrek, fantheories        |
|             | comicbooks, titlegore, dccomics                                            |
| Fashion     | malefashionadvice, diy, streetwear, frugalmalesfashionadvice                |
|             | sneakers, charlieputch, watches, octoberstveryown                           |
| Computers   | techsupport, linux, sysadmin, buildapc, windows10                          |
|             | linuxquestions, linux4noobs, datahoarder, titlegore                        |
| Electronics | arduino, electricians, askelectronics, askengineers                        |
|             | raspberry, pi, electronics, thebutton, homeimprovement                      |

Table 3: Latent factors of Reddit Hyperlink model.

| Topic       | Subreddits                                                                 |
|-------------|-----------------------------------------------------------------------------|
| Football    | nfl, greenbaypackers, seahawks, panthers, detroitlions, falcons, kansas city|
|             | chiefs, steelers, miamidolphins                                             |
| Basketball  | nba, bostonceltics, aflahawks, sixers, abaspurs, warriors, pacers,         |
|             | chicagobulls, lakers, denvernuggets                                        |
| Baseball    | baseball, sfgiants, kerryos, braves, azdiamondbacks, phillies,             |
|             | keepwriting, minnesotatwins, pads, brewers                                  |
| Role Playing| nhl, nba, worldbuilding, llg, ndbbehindthestreec, dndnext,                  |
|             | boardgames, gameofthrones, roleplay                                         |
| Pokemon     | pokemon, postpreview, pokemongo, test, thesilphroad                       |
|             | pokemongiveaway, casualpokemontrades, pokemomplaza                         |
| Elite       | elitedangerous, astringduval, kumocrew, elitemahon                         |
|             | elitepatreus, eliterwinters, elitetorval, elitelavniry                      |
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