On the Minimum Size of Signed Sumsets in Elementary Abelian Groups

Béla Bajnok

Department of Mathematics, Gettysburg College
300 N. Washington Street, Gettysburg, PA 17325-1486 USA
E-mail: bbajnok@gettysburg.edu

and

Ryan Matzke

Department of Mathematics, Gettysburg College
300 N. Washington Street, Gettysburg, PA 17325-1486 USA
E-mail: matzry01@gettysburg.edu

December 2, 2014

Abstract

For a finite abelian group $G$ and positive integers $m$ and $h$, we let

$$
\rho(G, m, h) = \min \{|hA| : A \subseteq G, |A| = m\}
$$

and

$$
\rho_{\pm}(G, m, h) = \min \{|h_{\pm}A| : A \subseteq G, |A| = m\},
$$

where $hA$ and $h_{\pm}A$ denote the $h$-fold sumset and the $h$-fold signed sumset of $A$, respectively. The study of $\rho(G, m, h)$ has a 200-year-old history and is now known for all $G$, $m$, and $h$. In previous work we provided an upper bound for $\rho_{\pm}(G, m, h)$ that we believe is exact, and proved that $\rho_{\pm}(G, m, h)$ agrees with $\rho(G, m, h)$ when $G$ is cyclic. Here we study $\rho_{\pm}(G, m, h)$ for elementary abelian groups $G$: in particular, we determine all values of $m$ for which $\rho_{\pm}(\mathbb{Z}_p^2, m, 2)$ equals $\rho(\mathbb{Z}_p^2, m, 2)$ for a given prime $p$.

*Corresponding author
Let $G$ be a finite abelian group written with additive notation, let $m$ be a positive integer with $m \leq |G|$, and let $h$ be a nonnegative integer. In [3], we introduced the function

$$\rho_{\pm}(G, m, h) = \min\{|h_{\pm}A| : A \subseteq G, |A| = m\},$$

where $h_{\pm}A = \{\sum_{i=1}^{m} \lambda_i a_i : (\lambda_1, \ldots, \lambda_m) \in \mathbb{Z}^m, \sum_{i=1}^{m} |\lambda_i| = h\}$ is the $h$-fold signed sumset of an $m$-subset $A = \{a_1, \ldots, a_m\}$ of $G$ (as usual, $|S|$ denotes the size of the finite set $S$). The function $\rho_{\pm}(G, m, h)$ is the analogue of the well-known $\rho(G, m, h) = \min\{|hA| : A \subseteq G, |A| = m\}$, where $hA = \{\sum_{i=1}^{m} \lambda_i a_i : (\lambda_1, \ldots, \lambda_m) \in \mathbb{N}_0^m, \sum_{i=1}^{m} \lambda_i = h\}$ is the usual $h$-fold sumset of $A$.

Signed sumsets have already been studied in the past: For example, in [4], the first author and Ruzsa investigated the independence number of a subset $A$ of $G$, defined as the maximum value of $t \in \mathbb{N}$ for which $0 \notin \bigcup_{h=1}^{t} h_{\pm}A$ (see also [1] and [2]); and in [14], Klopsch and Lev discussed the diameter of $G$ with respect to $A$, defined as the minimum value of $s \in \mathbb{N}$ for which

$$\bigcup_{h=0}^{s} h_{\pm}A = G$$

(see also [15]). The independence number of $A$ in $G$ quantifies the “degree” to which $A$ is linearly independent in $G$, while the diameter of $G$ with respect to $A$ measures how “effectively” $A$ generates $G$ (if at all). While research on minimum sumset size goes back to the work of Cauchy and is now known for all $G$, $m$, and $h$, to the best of our knowledge, [3] is the first systematic study of the minimum size of signed sumsets. In this paper we continue our work and consider $\rho_{\pm}(G, m, h)$ for elementary abelian groups $G$.

Let us review what we need to know about $\rho(G, m, h)$. It has been over two hundred years since Cauchy [5] found the minimum possible size of

$$A + B = \{a + b : a \in A, b \in B\}$$

among subsets $A$ and $B$ of the cyclic group $\mathbb{Z}_p$ of given sizes. (Here and elsewhere in the paper $p$ denotes a positive prime.) Over a hundred years later, Davenport [6] (cf. [7]) rediscovered Cauchy’s result, which is now known as the Cauchy–Davenport Theorem:
Theorem 1 (Cauchy–Davenport Theorem) If $A$ and $B$ are nonempty subsets of the group $\mathbb{Z}_p$ of prime order $p$, then

$$|A + B| \geq \min\{p, |A| + |B| - 1\}.$$ 

It can easily be seen that the bound is tight for all values of $|A|$ and $|B|$, and thus

$$\rho(\mathbb{Z}_p, m, 2) = \min\{p, 2m - 1\}.$$ 

Relatively recently, $\rho(G, m, h)$ was finally evaluated for all parameters by Plagne [18] (see also [17], [10], and [11]) in 2003. To state the result, we introduce the function

$$u(n, m, h) = \min\{f_d(m, h) : d \in D(n)\},$$

where $n$, $m$, and $h$ are positive integers, $D(n)$ is the set of positive divisors of $n$, and

$$f_d(m, h) = (h \lceil m/h \rceil - h + 1) \cdot d.$$ 

(Here $u(n, m, h)$ is a relative of the Hopf–Stiefel function used also in topology and bilinear algebra; see, for example, [9], [12], [17], and [19].)

Theorem 2 (Plagne; cf. [18]) Let $n$, $m$, and $h$ be positive integers with $m \leq n$. For any abelian group $G$ of order $n$ we have

$$\rho(G, m, h) = u(n, m, h).$$

Let us turn now to $\rho_{\pm}(G, m, h)$. It is easy to see that $\rho_{\pm}(G, 1, h)$ and $\rho_{\pm}(G, m, 0)$ both equal 1 and that $\rho_{\pm}(G, m, 1)$ equals $m$ for all $G$, $m$, and $h$. (To see the last equality, it suffices to verify that one can always find a symmetric subset of size $m$ in $G$, that is, an $m$-subset $A$ of $G$ for which $A = -A$.) Therefore, from now on, we assume that $m \geq 2$ and $h \geq 2$.

Perhaps surprisingly, we find that, while the $h$-fold signed sumset of a given set is generally much larger than its sumset, $\rho_{\pm}(G, m, h)$ often agrees with $\rho(G, m, h)$; in particular, this is always the case when $G$ is cyclic:

Theorem 3 (Cf. [3]) For all positive integers $n$, $m$, and $h$, we have

$$\rho_{\pm}(\mathbb{Z}_n, m, h) = \rho(\mathbb{Z}_n, m, h).$$

The situation seems considerably more complicated for noncyclic groups: in contrast to $\rho(G, m, h)$, the value of $\rho_{\pm}(G, m, h)$ depends on the structure of $G$ rather than just the order $n$ of $G$.

Observe that by Theorem 2 we have the lower bound

$$\rho_{\pm}(G, m, h) \geq u(n, m, h) = \min\{f_d(m, h) : d \in D(n)\}.$$ 

In [3], we proved that with a certain subset $D(G, m)$ of $D(n)$, we have

$$\rho_{\pm}(G, m, h) \leq u_{\pm}(G, m, h) = \min\{f_d(m, h) : d \in D(G, m)\};$$

here $D(G, m)$ is defined in terms of the type $(n_1, \ldots, n_r)$ of $G$, that is, via integers $n_1, \ldots, n_r$ such that $n_1 \geq 2$, $n_i$ divides $n_{i+1}$ for each $i \in \{1, \ldots, r-1\}$, and for which $G$ is isomorphic to the invariant product

$$\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_r}.$$ 

Namely, we proved the following result:
Theorem 4 (Cf. [3]) The minimum size of the \( h \)-fold signed sumset of an \( m \)-subset of a group \( G \) of type \((n_1, \ldots, n_r)\) satisfies

\[
\rho_{\pm}(G, m, h) \leq u_{\pm}(G, m, h),
\]

where

\[
u_{\pm}(G, m, h) = \min\{f_d(m, h) : d \in D(G, m)\}
\]

with

\[
D(G, m) = \{d \in D(n) : d = d_1 \cdots d_r, d_1 \in D(n_1), \ldots, d_r \in D(n_r), d_{n_r} \geq d_rm\}.
\]

Observe that, for cyclic groups of order \( n \), \( D(G, m) \) is simply \( D(n) \).

Additionally, we believe that \( u_{\pm}(G, m, h) \) actually yields the exact value of \( \rho_{\pm}(G, m, h) \) in all cases except for one very special situation (which occurs only when \( h = 2 \)). In particular, we made the following conjecture:

Conjecture 5 (Cf. [3]) Suppose that \( G \) is an abelian group of order \( n \) and type \((n_1, \ldots, n_r)\).

If \( h \geq 3 \), then

\[
\rho_{\pm}(G, m, h) = u_{\pm}(G, m, h).
\]

If each odd divisor of \( n \) is less than \( 2m \), then

\[
\rho_{\pm}(G, m, 2) = u_{\pm}(G, m, 2).
\]

If there are odd divisors of \( n \) greater than \( 2m \), let \( d_m \) be the smallest one. We then have

\[
\rho_{\pm}(G, m, 2) = \min\{u_{\pm}(G, m, 2), d_m - 1\}.
\]

We will need to use the following “inverse type” result from [3] regarding subsets that achieve \( \rho_{\pm}(G, m, h) \). Given a group \( G \) and a positive integer \( m \leq |G| \), we define a certain collection \( A(G, m) \) of \( m \)-subsets of \( G \). We let

- Sym\((G, m)\) be the collection of symmetric \( m \)-subsets of \( G \), that is, \( m \)-subsets \( A \) of \( G \) for which \( A = -A \);
- Nsym\((G, m)\) be the collection of near-symmetric \( m \)-subsets of \( G \), that is, \( m \)-subsets \( A \) of \( G \) that are not symmetric, but for which \( A \setminus \{a\} \) is symmetric for some \( a \in A \);
- Asym\((G, m)\) be the collection of asymmetric \( m \)-subsets of \( G \), that is, \( m \)-subsets \( A \) of \( G \) for which \( A \cap (-A) = \emptyset \).

We then let

\[
A(G, m) = \text{Sym}(G, m) \cup \text{Nsym}(G, m) \cup \text{Asym}(G, m).
\]

In other words, \( A(G, m) \) consists of those \( m \)-subsets of \( G \) that have exactly \( m, m - 1, \) or \( 0 \) elements whose inverse is also in the set.

Theorem 6 (Cf. [3]) For every \( G, m, \) and \( h \), we have

\[
\rho_{\pm}(G, m, h) = \min\{|h_{\pm}A| : A \in A(G, m)\}.
\]
We should add that each of the three types of sets are essential as can be seen by examples (cf. [3]).

Our goal in this paper is to investigate $\rho_\pm (G, m, h)$ for elementary abelian groups $G$. In particular, we wish to classify all cases for which

$$\rho_\pm (\mathbb{Z}_r^p, m, h) = \rho (\mathbb{Z}_r^p, m, h),$$

where $p$ denotes a positive prime and $r$ is a positive integer. By Theorem 3, we assume that $r \geq 2$, and, since obviously $\rho_\pm (\mathbb{Z}_r^2, m, h) = \rho (\mathbb{Z}_r^2, m, h)$ for all $m, h$, and $r$, we will also assume that $p \geq 3$.

Let us first exhibit a sufficient condition for $\rho_\pm (\mathbb{Z}_r^p, m, h)$ to equal $\rho (\mathbb{Z}_r^p, m, h)$. When $p \leq h$, our result is easy to state; we will prove the following:

**Theorem 7** If $p \leq h$, then for all values of $1 \leq m \leq p^r$ we have

$$\rho_\pm (\mathbb{Z}_r^p, m, h) = \rho (\mathbb{Z}_r^p, m, h).$$

The case $h \leq p - 1$ is more complicated and delicate. In order to state our results, we will need to introduce some notations. Suppose that $m \geq 2$ is a given positive integer. First, we let $k$ be the maximal integer for which

$$p^k + \delta \leq hm - h + 1,$$

where $\delta = 0$ if $p - 1$ is divisible by $h$, and $\delta = 1$ if it is not. Second, we let $c$ be the maximal integer for which

$$(hc + 1) \cdot p^k + \delta \leq hm - h + 1.$$ 

Note that $k$ and $c$ are nonnegative integers and $c \leq p - 1$, since for $c \geq p$ we would have

$$(hc + 1) \cdot p^k \geq p^{k+1} + \delta > hm - h + 1.$$ 

It is also worth noting that

$$f_1 (m, h) = hm - h + 1.$$ 

Our sufficient condition can now be stated as follows:

**Theorem 8** Suppose that $2 \leq h \leq p - 1$, and let $k$ and $c$ be the unique nonnegative integers defined above. If

$$m \leq (c + 1) \cdot p^k,$$

then

$$\rho_\pm (\mathbb{Z}_r^p, m, h) = \rho (\mathbb{Z}_r^p, m, h).$$

In fact, we believe that this condition is also necessary:

**Conjecture 9** The converse of Theorem 8 is true as well; that is, if $2 \leq h \leq p - 1$, $k$ and $c$ are the unique nonnegative integers defined above, and

$$m > (c + 1) \cdot p^k,$$

then

$$\rho_\pm (\mathbb{Z}_r^p, m, h) > \rho (\mathbb{Z}_r^p, m, h).$$
We are able to prove that Conjecture 9 holds in the case of \( \rho_\pm(Z_p^2, m, 2) \):

**Theorem 10** Let \( p \) be an odd prime and \( m \leq p^2 \) be a positive integer. Then

\[
\rho_\pm(Z_p^2, m, 2) = \rho(Z_p^2, m, 2),
\]

if, and only if, one of the following holds:

- \( m \leq p \),
- \( m \geq (p^2 + 1)/2 \), or
- there is a positive integer \( c \leq (p - 1)/2 \) for which
  \[
  c \cdot p + (p + 1)/2 \leq m \leq (c + 1) \cdot p.
  \]

Our proof of Theorem 10 involves some of the deeper methods of additive combinatorics, including Vosper’s Theorem and (Lev’s improvement of) Kemperman’s results on critical pairs.

According to Theorem 10, for a given \( p \), there are exactly \((p - 1)^2/4\) values of \( m \) for which \( \rho_\pm(Z_p^2, m, 2) \) and \( \rho(Z_p^2, m, 2) \) disagree—fewer than 1/4 of all possible values. We have not been able to find any groups where this proportion is higher than 1/4.

## 2 The proofs of Theorems 7 and 8

In this section we establish the two sufficient conditions for the equality

\[
\rho_\pm(Z_p^r, m, h) = \rho(Z_p^r, m, h)
\]

that we stated in Theorems 7 and 8. In order to do so, we first classify all cases with

\[
u_\pm(Z_p^r, m, h) = u(p^r, m, h).
\]

Let \( p \) be an odd positive prime, \( r \geq 2 \) an integer, and \( m \leq p^r \) a positive integer. Via the (unique) base \( p \) representation of \( m - 1 \), we write \( m \) as

\[
m = q_{r-1}p^{r-1} + \cdots + q_1p + q_0 + 1,
\]

where \( q_{r-1}, \ldots, q_0 \) are all integers between 0 and \( p - 1 \), inclusive. We will also need to identify three special indices:

- \( i_1 \) denotes the largest index \( i \) for which \( q_i \geq 1 \); if there is no such index (that is, if \( m = 1 \)), then we let \( i_1 = -1 \).
- \( i_2 \) denotes the largest index \( i \) for which \( q_i \geq p/h \); if there is no such index, we let \( i_2 = -1 \).
- \( i_3 \) denotes the largest index \( i \) for which \( i > i_2 \) and
  \[
  q_i = q_{i-1} = \cdots = q_{i_2+1} = (p - 1)/h;
  \]
  if there is no such index, we let \( i_3 = i_2 \).
We have \( i_1 = \lceil \log_p m \rceil - 1 \), and
\[
 r - 1 \geq i_1 \geq i_3 \geq i_2 \geq -1.
\]

Recall that for positive integers \( n, m, \) and \( h, \)
\[
u(n, m, h) = \min \{ f_d(m, h) \mid d \in D(n) \},
\]
where
\[
f_d(m, h) = (h \lceil m/d \rceil - h + 1) \cdot d.
\]

Our next proposition exhibits all values of \( d \) for which \( f_d(m, h) \) equals \( u(p^r, m, h) \).

**Proposition 11** With our notations as above, for a nonnegative integer \( i \) we have
\[
u(p^r, m, h) = f_{p_i}(m, h)
\]
if, and only if,
\[
i_2 + 1 \leq i \leq i_3 + 1.
\]

Remark: The fact that \( u(p^r, m, h) = f_{p_{i_3+1}}(m, h) \)
was established for \( h = 2 \) by Eliahou and Kervaire in [8].

Proof: Given the representation of \( m \) as above, we find that for every \( 0 \leq i \leq r, \)
\[
f_{p_i} = f_{p_i}(m, h) = h \cdot (q_{r-1}p^{r-1} + \cdots + q_ip^i) + p^i.
\]
Therefore, when \( i \geq i_3 + 2, \) then
\[
f_{p_i} - f_{p_{i_3+1}} = p^i - p_{i_3+1} - h \cdot (q_{i_3+1}p^{i_3+1} + \cdots + q_ip^i)
\]
\[
> p^i - p_{i_3+1} - (p - 1) \cdot (p^{i-1} + \cdots + p_{i_3+1})
\]
\[
= 0.
\]
Similarly, when \( i_2 + 1 \leq i \leq i_3 + 1, \) then
\[
f_{p_{i_3+1}} - f_{p_i} = p_{i_3+1} - p^i - h \cdot (q_{i_3+1}p^{i_3+1} + \cdots + q_ip^i)
\]
\[
= p_{i_3+1} - p^i - (p - 1) \cdot (p^{i_3} + \cdots + p_{i_3+1})
\]
\[
= 0.
\]
Finally, if \( 0 \leq i \leq i_2, \) then \( hq_{i_2} \geq p, \) so we have
\[
f_{p_i} - f_{p_{i_2+1}} = p^i - p_{i_2+1} + h \cdot (q_{i_2}p^{i_2} + \cdots + q_ip^i)
\]
\[
\geq p^i - p_{i_2+1} + hq_{i_2}p^{i_2}
\]
\[
> 0,
\]
completing our proof. \( \Box \)

Recall that for a group of type \((n_1, \ldots, n_r)\) and positive integers \( m, \) and \( h, \) we defined
\[
u_{\pm}(G, m, h) = \min \{ f_d(m, h) \mid d \in D(G, m) \},
\]
where
\[
D(G, m) = \{ d \in D(n) \mid d = d_1 \cdots d_r, d_1 \in D(n_1), \ldots, d_r \in D(n_r), d_n \geq d_rm \}.
\]
Our next result finds all values of \( i \) for which \( f_{p_i}(m, h) \) equals \( u_{\pm}(\mathbb{Z}_p^r, m, h). \)
Proposition 12 Let \( m \geq 2 \). With our notations as above, for a nonnegative integer \( i \) we have
\[
 u_\pm(\mathbb{Z}_p^r, m, h) = f_p(m, h)
\]
if, and only if,
\[
i = \begin{cases} 
 i_1 + 1 & \text{if } hq_i \geq p; \\
 i_1 \text{ or } i_1 + 1 & \text{if } hq_i = p - 1; \\
 i_1 & \text{if } hq_i \leq p - 2.
\end{cases}
\]

Proof: By Theorem 4
\[
 u_\pm(\mathbb{Z}_p^r, m, h) = \min\{f_d(m, h) \mid d \in D(\mathbb{Z}_p^r, m)\}.
\]
By our definition of \( i_1 \),
\[
 D(\mathbb{Z}_p^r, m) = \{p^i \mid i_1 \leq i \leq r\}.
\]
The explicit result now follows via the same considerations as in the proof of Proposition 11 above—we omit the details. \( \square \)

Propositions 11 and 12 then imply the following:

Proposition 13 With our notations as above, we have
\[
 u(p^r, m, h) = u_\pm(\mathbb{Z}_p^r, m, h)
\]
if, and only if, \( i_1 = i_3 \) or \( i_1 = i_3 + 1 \).

We are now ready for the proofs of Theorems 7 and 8.

Proof of Theorem 7: Note that when \( p \leq h \), then, for each index \( i \), \( q_i \geq 1 \) is equivalent to \( q_i \geq p/h \). Therefore, we have
\[
 i_1 = i_2 = i_3,
\]
so our result follows from Proposition 13 and Theorem 4. \( \square \)

Proof of Theorem 8: By assumption, we have nonnegative integers \( k \) and \( c \) with \( c \leq p - 1 \) so that
\[
 c \cdot p^k + \frac{p - 1}{h} \cdot p^{k-1} + \frac{p - 1}{h} \cdot p^{k-2} + \cdots + \frac{p - 1}{h} + 1 \leq m \leq (c + 1) \cdot p^k,
\]
where \( \delta = 0 \) if \( p - 1 \) is divisible by \( h \), and \( \delta = 1 \) if it is not. Therefore,
\[
i_1 = \begin{cases} 
 k - 1 & \text{if } c = 0, \\
 k & \text{if } c \geq 1.
\end{cases}
\]

To find \( i_3 \), assume first that \( p - 1 \) is divisible by \( h \). Our bounds for \( m \) above can then be written as
\[
 c \cdot p^k + \frac{p - 1}{h} \cdot p^{k-1} + \frac{p - 1}{h} \cdot p^{k-2} + \cdots + \frac{p - 1}{h} + 1 \leq m \leq (c + 1) \cdot p^k.
\]
Thus we see that, no matter what \( i_2 \) equals, we have
\[
i_3 = \begin{cases} 
 k - 1 & \text{if } c < (p - 1)/h, \\
 k & \text{if } c \geq (p - 1)/h.
\end{cases}
\]
Our result now follows from Proposition 13 and Theorem 4.

The case when \( p - 1 \) is not divisible by \( h \) is similar; this time the bounds for \( m \) are

\[
c \cdot p^k + \frac{p}{h} \cdot p^{k-1} + 1 \leq m \leq (c + 1) \cdot p^k,
\]

so

\[
i_2 = i_3 = \begin{cases} 
  k - 1 & \text{if } c < p/h, \\
  k & \text{if } c \geq p/h;
\end{cases}
\]

implying our claim as before. Our proof is thus complete. \( \square \)

3 The proof of Theorem 10

We now turn to the question of determining all values of \( m \) for which

\[
\rho_{\pm}(\mathbb{Z}_p^2, m, 2) = \rho(\mathbb{Z}_p^2, m, 2).
\]

In order to do so, we will need to discuss some results on the so-called inverse problem in additive combinatorics; in particular, we will review some of what is known about subsets \( A \) and \( B \) of \( G \) when their sumset \( A + B \) is small.

Recall that a subset \( A \) of an abelian group \( G \) is called an arithmetic progression if it is of the form

\[
A = \{a + i \cdot b : 0 \leq i \leq m - 1\}
\]

for some elements \( a, b \in G \) and \( m \in \mathbb{N} \); \( b \) must have order at least \( m \) in \( G \). Here \( m \) is called the length of the arithmetic progression (we allow length 1), and \( b \) is called the common difference of the progression.

The first nontrivial inverse theorem is Vosper’s classic result for groups of prime order:

Theorem 14 (Vosper; cf. [20] and [21]) Suppose that \( A \) and \( B \) are nonempty subsets of \( \mathbb{Z}_p \) satisfying

\[
|A + B| = |A| + |B| - 1.
\]

Then at least one of the following holds:

- \( |A| = 1 \) or \( |B| = 1 \);
- \( |A| + |B| = p + 1 \);
- \( A = \mathbb{Z}_p \setminus (g - B) \) where \( \{g\} = \mathbb{Z}_p \setminus (A + B) \); or
- \( A \) and \( B \) are both arithmetic progressions with the same common difference.

For our use below, the following immediate consequence of Vosper’s Theorem is sufficient:

Corollary 15 Suppose that \( A \) and \( B \) are nonempty subsets of \( \mathbb{Z}_p \) satisfying \( |B| \geq 2 \) and

\[
|A + B| = |A| + |B| - 1 \leq p - 2.
\]

Then \( A \) is an arithmetic progression.
For groups of composite order, the situation is considerably more complicated due to the existence of nontrivial proper subgroups. Nevertheless, Kemperman [13] gave a complete characterization of all critical pairs of finite subsets of an abelian group; that is, all finite subsets \( A \) and \( B \) for which
\[
|A + B| \leq |A| + |B| - 1.
\]
Kemperman’s characterization was rather complicated, but it facilitated several improvements, of which we find Lev’s following result most helpful for our purposes:

**Theorem 16 (Lev; cf. [16] Theorem 4)** Let \( A \) and \( B \) be nonempty finite subsets of an abelian group \( G \) satisfying \(|B| \geq 2\) and
\[
|A + B| \leq \min\{|G| - 2, |A| + |B| - 1\}.
\]

Then at least one of the following holds:

- \( A \) is an arithmetic progression;
- there exists a nonzero subgroup \( H \) of \( G \) with finite index \( t \geq 2 \) so that \( A \) is the disjoint union of an arithmetic progression and \( t - 1 \) cosets of \( H \); or
- there exists a finite, nonzero subgroup \( H \) of \( G \) such that
\[
|A + H| \leq \min\{|G| - 1, |A| + |H| - 1\}.
\]

We are now ready to embark on our proof of Theorem 10.

**Proof of Theorem 10**: First, we show how the “if” direction follows from Theorem 8. Keeping the notations introduced there, we see that
\[
(k, c) = \begin{cases} 
(0, m - 1) & \text{if } 1 \leq m \leq (p - 1)/2; \\
(1, 0) & \text{if } (p + 1)/2 \leq m \leq p; \\
(1, q) & \text{if } m = qp + v \text{ with } 1 \leq q \leq (p - 1)/2 \text{ and } (p + 1)/2 \leq v \leq p; \\
(2, 0) & \text{if } (p^2 + 1)/2 \leq m \leq p^2.
\end{cases}
\]

In each case we find that \( m \leq (c + 1) \cdot p^k \).

This leaves us with subset sizes of the form
\[
m = qp + v
\]
with
\[
1 \leq q \leq (p - 1)/2 \quad \text{and} \quad 1 \leq v \leq (p - 1)/2;
\]
we will prove that, in this case,
\[
\rho_\pm(\mathbb{Z}_p^2, m, 2) > \rho(\mathbb{Z}_p^2, m, 2).
\]
Since Proposition 11 yields
\[
\rho(\mathbb{Z}_p^2, m, 2) = f_1(m, 2) = 2m - 1,
\]

10
our goal is to prove that
\[ \rho_{\pm}(\mathbb{Z}_p^2, m, 2) \geq 2m. \]

Let \( A \) be an \( m \)-subset of \( \mathbb{Z}_p^2 \) for which
\[ |2 \pm A| = \rho_{\pm}(\mathbb{Z}_p^2, m, 2); \]
furthermore, by Theorem 6, we may also assume that \( A \) is symmetric, near-symmetric, or asymmetric. We will prove that \( |2 \pm A| \geq 2m \).

First, let us deduce what Theorem 16 says about our situation. Following an indirect approach, let us assume that \( 2 \pm A \), and thus \( 2A \), have size at most \( 2m - 1 \). (They will then have size \( 2m - 1 \).) Note that
\[ 2m - 1 = 2qp + 2v - 1 \leq p^2 - 2, \]
so the conditions of the theorem are met with \( B = A \).

Note also that \( q, v \geq 1 \), so \( m > p \), and thus \( A \) cannot be an arithmetic progression in \( \mathbb{Z}_p^2 \).

Furthermore, a nonzero subgroup \( H \) of index at least 2 must be of order \( p \) and index \( p \); with
\[ |A| = qp + v \leq (p^2 - 1)/2 < (p - 1)p, \]
\( A \) cannot contain the disjoint union of \( p - 1 \) distinct cosets of \( H \).

This leaves only one possibility: there must exist a subgroup \( H \) of \( \mathbb{Z}_p^2 \) of order \( p \) for which
\[ |A + H| \leq \min\{p^2 - 1, m + p - 1\} = m + p - 1. \]

Now \( A \subseteq A + H \), so
\[ qp < m \leq |A + H| \leq m + p - 1 = (q + 1)p + v - 1 < (q + 2)p, \]
and thus \( A + H \) is the union of exactly \( q + 1 \) distinct cosets of \( H \). Let \( A_1, \ldots, A_{q+1} \) be the intersections of these cosets with \( A \); \( A \) is then the union of these \( q + 1 \) components.

Let us see what we can say about the sizes of these components. Since
\[ (q - 1) \cdot p + 2 \cdot (p - 1)/2 < m, \]
and
\[ (q - 1) \cdot p + p < m \]
as well, so any two distinct components have a combined size of at least \( p + 1 \). Therefore, we have two possibilities:

(i) each component has size at least \( (p + 1)/2 \); or

(ii) one component, say \( A_1 \), has size at most \( (p - 1)/2 \), all other components have size at least \( (p + 1)/2 \), and \( |A_1| + |A_i| \geq p + 1 \) for all \( i = 2, 3, \ldots, q + 1 \).

By the Cauchy–Davenport Theorem, applied to the \( q + 1 \) cosets in the group
\[ \mathbb{Z}_p^2/H \cong \mathbb{Z}_p, \]
we can conclude that $2A$ lies in—that is, intersects non-trivially—at least
\[ \min\{p, 2(q + 1) - 1\} = 2q + 1 \]
cosets of $H$.

Citing the Cauchy–Davenport Theorem again, we see that in case (i), $2A$ is actually the union of these cosets, and so we have
\[ |2A| \geq (2q + 1)p > 2m, \]
contradicting our indirect assumption.

Observe that even in case (ii), with the possible exception of the coset containing $2A_1$, all the cosets that $2A$ lies in are entirely contained in $2A$. Therefore, if $2A$ lies in at least $2q + 2$ cosets of $H$, then we still have
\[ |2A| \geq (2q + 1)p > 2m, \]
contradicting our indirect assumption.

Suppose then that we are in case (ii) and that $2A$ lies in exactly $2q + 1$ cosets of $H$. Now if $|A_1| \geq v + 1$, then
\[ |2A| \geq 2qp + \min\{p, 2(v + 1) - 1\} = 2qp + 2v + 1 > 2m, \]
a contradiction again. On the other hand, if $|A_1| \leq v$, then, since
\[ qp + v = |A| = |A_1| + \cdots + |A_{q+1}|, \]
we must have $|A_1| = v$ and $|A_i| = p$ for each $i = 2, 3, \ldots, q + 1$. Thus we are in the situation where there are exactly $q + 1$ cosets intersecting $A$, one component—namely, $A_1$—has size $v$, and the other components all have size $p$. Furthermore, $2A$ lies in exactly $2q + 1$ cosets, and $2q$ of these cosets—all but the one containing $2A_1$—are entirely in $2A$.

Suppose now that $q \leq (p - 3)/2$. Then $2q + 1 \leq p - 2$, so by Corollary 15 of Vosper’s Theorem, the $q + 1$ cosets that $A$ lies in must form an arithmetic progression. Therefore, we can write $A$ in the form
\[ A = \bigcup_{i=0}^{q}(a + ig + H_i), \]
where $a$ and $g$ are group elements, $H_i$ is a subset of $H$ for each $i$, and $H_i = H$ for all but one $i$. Consequently, for distinct $i_1$ and $i_2$, at least one of $H_{i_1}$ or $H_{i_2}$ equals $H$, so we have
\[ (a + ig + H_{i_1}) - (a + ig + H_{i_2}) = (i_1 - i_2)g + H \subseteq 2A. \]
Furthermore, since $q + 1 \geq 2$, there is an $i$ such that $H_i = H$, so for this $i$ we have
\[ (a + ig + H_i) - (a + ig + H_i) = H, \]
and thus
\[ H \setminus \{0\} \subseteq 2A. \]
This implies that
\[ \bigcup_{i=-q}^{q}(ig + H) \setminus \{0\} \subseteq 2A, \]
and so
\[ |2A| \geq (2q + 1)p - 1 \geq 2m, \]
a contradiction with our indirect assumption.

We are left with the case when $q = (p - 1)/2$, in which case
• \( A \) has size \((p^2 - p)/2 + v\) and is the union of \(A_1\) and \((p - 1)/2\) cosets of \(H\), and
• \(2A\) has size \(p^2 - p + 2v - 1\) and is the union of \(2A_1\) and \(p - 1\) cosets of \(H\).

Recall that we are assuming that \(A\) is symmetric, near-symmetric, or asymmetric—we attend to each of these cases separately.

Suppose first that \(A\) is near-symmetric. Then, by definition, we can find an element \(a \in \mathbb{Z}_p^2 \setminus A\) for which

\[
A' = A \cup \{a\}
\]

is symmetric. We can easily check that we then have

\[
2 \pm A' = 2 \pm A
\]

(note that \(0 \in 2 \pm A\) since \(m \geq 2\)). Therefore,

\[
2m - 1 \geq |2 \pm A'| = |2 \pm A| \geq \rho(\mathbb{Z}_p^2, m + 1, 2) \geq \rho(\mathbb{Z}_p^2, m + 1, 2).
\]

However, this is a contradiction, since by Proposition 11, we get

\[
\rho(\mathbb{Z}_p^2, m + 1, 2) = \begin{cases} 
  f_1(m + 1, 2) = 2(m + 1) - 1 = 2m + 1 & \text{if } v \leq (p - 3)/2; \\
  f_p(m + 1, 2) = p^2 = 2m + 1 & \text{if } v = (p - 1)/2.
\end{cases}
\]

Next, assume that \(A\) is symmetric. Observe that we then must have \(A_1 \subseteq H\), since otherwise \(A\) would contain fewer than \((p - 1)/2\) full cosets of \(H\).

Now suppose that \(a + H\) is one of the cosets that make up \(A\). Then, because it is symmetric, \(A\) also contains \(-a + H\); and since these are two disjoint cosets, \(2A\) will contain their sum, which is \(H\). But this then implies that \(2A = \mathbb{Z}_p^2\), which again contradicts \(|2A| \leq 2m - 1\).

Finally, suppose that \(A\) is asymmetric. Consider the partition

\[
H \cup (\pm a_1 + H) \cup \cdots \cup (\pm a_{(p-1)/2} + H)
\]

of \(\mathbb{Z}_p^2\) (here \(a_1, \ldots, a_{(p-1)/2}\) are appropriately chosen group elements). Recall that \((p - 1)/2\) of these \(p\) cosets must lie entirely in \(A\), so the fact that \(A\) is asymmetric implies that \(A_1 \subseteq H\). With \(a + H\) being one of the cosets in \(A\),

\[
(a + H) - (a + H) = H,
\]

and thus

\[
H \setminus \{0\} \subseteq 2 \pm A.
\]

Therefore,

\[
|2 \pm A| \geq (p - 1)p + (p - 1) = p^2 - 1 \geq 2m,
\]

a contradiction.

Our proof is now complete. \(\square\)

We have thus identified each value of \(m\) for which \(\rho(\mathbb{Z}_p^2, m, 2)\) equals \(\rho(\mathbb{Z}_p^2, m, 2)\)—and thus also equals \(u(p^2, m, 2)\)—and those for which it does not, but how about the exact value of \(\rho(\mathbb{Z}_p^2, m, 2)\)? We believe that \(\rho(\mathbb{Z}_p^2, m, 2)\) equals \(u(\mathbb{Z}_p^2, m, 2)\) for all values of \(m\), except for when

\[
\frac{p^2 - p + 2}{2} \leq m \leq \frac{p^2 - 1}{2}.
\]
Indeed, in this case we have
\[ u_\pm(\mathbb{Z}_p^2, m, 2) = p^2 \]
(cf. Proposition 12), but
\[ \rho_\pm(\mathbb{Z}_p^2, m, 2) \leq p^2 - 1, \]
as demonstrated by any asymmetric \( m \)-subset \( A \) of \( \mathbb{Z}_p^2 \) (we then have \( 0 \not\in 2 \pm A \)).

In light of this, we make the following conjecture:

**Conjecture 17** Let us write \( m \) as
\[ m = cp + v \]
with
\[ 0 \leq c \leq p - 1 \quad \text{and} \quad 1 \leq v \leq p. \]
We then have:

| \( c \) | \( v \) | \( \rho(\mathbb{Z}_p^2, m, 2) \) | \( \rho_\pm(\mathbb{Z}_p^2, m, 2) \) | \( u_\pm(\mathbb{Z}_p^2, m, 2) \) |
|---|---|---|---|---|
| 0 | \( v \leq (p - 1)/2 \) | \( 2m - 1 \) | \( 2m - 1 \) | \( 2m - 1 \) |
| 0 | \( v \geq (p + 1)/2 \) | \( p \) | \( p \) | \( p \) |
| \( 1 \leq c \leq (p - 3)/2 \) | \( v \leq (p - 1)/2 \) | \( 2m - 1 \) | \( (2c + 1)p \) | \( 2c + 1 \) |
| \( 1 \leq c \leq (p - 3)/2 \) | \( v \geq (p + 1)/2 \) | \( (2c + 1)p \) | \( (2c + 1)p \) | \( (2c + 1)p \) |
| \( c = (p - 1)/2 \) | \( v \leq (p - 1)/2 \) | \( 2m - 1 \) | \( \frac{p^2 - 1}{p^2} \) | \( p^2 \) |
| \( c = (p - 1)/2 \) | \( v \geq (p + 1)/2 \) | \( p^2 \) | \( p^2 \) | \( p^2 \) |
| \( c \geq (p + 1)/2 \) | any \( v \) | \( p^2 \) | \( p^2 \) | \( p^2 \) |

The two boxed entries in this table remain unproven.

**References**

[1] B. Bajnok, Spherical Designs and Generalized Sum-Free Sets in Abelian Groups. Special issue dedicated to Dr. Jaap Seidel on the occasion of his 80th birthday (Oisterwijk, 1999). *Des. Codes Cryptogr.* 21(1-3) (2000), 11-18.

[2] B. Bajnok, The Spanning Number and the Independence Number of a Subset of an Abelian Group. In *Number Theory*, D. Chudnovsky, G. Chudnovsky, and M. Nathanson (Ed.), Springer-Verlag (2004), 1-16.

[3] B. Bajnok and R. Matzke, The Minimum Size of Signed Sumsets, www.arxiv.org (2014).

[4] B. Bajnok and I. Ruzsa, The Independence Number of a Subset of an Abelian Group. *Integers* 3(A2) (2003), 23 pp.

[5] A.-L. Cauchy, Recherches sur les nombres, *J. École Polytechnique* 9 (1813) 99–123.

[6] H. Davenport, On the addition of residue classes, *J. London Math. Soc.* 10 (1935) 30–32.
7] H. Davenport, A historical note, *J. London Math. Soc.* 22 (1947) 100–101.

[8] S. Eliahou and M. Kervaire, Restricted Sumsets in Finite Vector Spaces: The Case $p = 3$, *Integers*, 1 (2001) #A02.

[9] S. Eliahou and M. Kervaire, Old and new formulas for the Hopf–Stiefel and related functions, *Expo. Math.*, 23:2 (2005) 127–145.

[10] S. Eliahou and M. Kervaire, Some extensions of the Cauchy–Davenport Theorem, *Electronic Notes in Discrete Math.*, 28 (2007) 557–564.

[11] S. Eliahou, M. Kervaire, and A. Plagne, Optimally small sumsets in finite abelian groups, *J. Number Theory*, 101 (2003) 338–348.

[12] Gy. Károlyi, A note on the Hopf–Stiefel function. *European J. Combin.*, 27 (2006) 1135–1137.

[13] J. H. B. Kemperman, On small sumsets in an abelian group, *Acta Mathematica*, 103 (1960) 63–88.

[14] B. Klopsch and V. F. Lev, How long does it take to generate a group? *J. Algebra*, 261, (2003) 145–171.

[15] B. Klopsch and V. F. Lev, Generating abelian groups by addition only. *Forum Math.*, 21:1, (2009) 23–41.

[16] V. F. Lev, Critical pairs in abelian groups and Kemperman’s structure theorem, *Int. J. Number Theory* 3 (2006) 379–396.

[17] A. Plagne, Additive number theory sheds extra light on the Hopf–Stiefel ◦ function, *Enseign. Math., II Sér.*, 49:1–2 (2003) 109–116.

[18] A. Plagne, Optimally small sumsets in groups, I. The supersmall set property, the $\mu_G^{(k)}$ and the $\nu_G^{(k)}$ functions, *Unif. Distrib. Theory*, 1:1 (2006) 27–44.

[19] D. Shapiro, Products of sums of squares, *Expo. Math.*, 2 (1984) 235–261.

[20] A. G. Vosper, The critical pairs of subsets of a group of prime order. *J. London Math. Soc.* 31 (1956) 200–205.

[21] A. G. Vosper, Addendum to “The critical pairs of subsets of a group of prime order.” *J. London Math. Soc.* 31 (1956) 280–282.