A modular invariance property of multivariable trace functions for regular vertex operator algebras

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Abstract
We prove an $SL_2(\mathbb{Z})$-invariance property of multivariable trace functions on modules for a regular VOA. Applying this result, we provide a proof of the inversion transformation formula for Siegel theta series. As another application, we show that if $V$ is a regular VOA containing a regular subVOA $U$ whose commutant $U^c$ is regular and satisfies $(U^c)^c = U$, then all simple $U$-modules appear in some simple $V$-module.

1 Introduction
The concept of a vertex operator algebra (VOA) was introduced by Borcherds [2] to explain a mysterious relation between the Monster simple group and the elliptic modular function $J(\tau)$. In the years since, this connection has been elucidated further and generalized to encompass a wide class of VOAs and elliptic modular forms. In the heart of this developing theory reside trace functions over modules of endomorphisms associated with the VOA. In particular, these functions include an operator formed from a matching of a distinguished element from the VOA, and a single variable in the complex upper half-plane. Meanwhile, the element resulting from this pairing resides in a one-dimensional Jordan subalgebra of the VOA, and begs the question whether trace functions exist which instead incorporate elements from larger Jordan subalgebras. The primary aim of this paper is to study such multivariable trace functions and establish functional equations for them with respect to the group $SL_2(\mathbb{Z})$.

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The development of these equations utilizes a seminal result of Zhu [8], which establishes that the space of trace functions on simple modules of a regular VOA is invariant under the standard action of $\text{SL}_2(\mathbb{Z})$. In particular, Zhu shows that the action of an element of $\text{SL}_2(\mathbb{Z})$ on a single-variable trace function on a simple module is a linear combination of the trace functions for all simple modules of the VOA with coefficients dependent on the representation of the element in $\text{SL}_2(\mathbb{Z})$. As we lift Zhu’s theory to the multivariable case below, we find that we recover these same coefficients. Using Verlinde’s formula, we exploit this fact to show that every simple module of a regular subVOA whose commutant satisfies certain conditions is contained in a simple module of the VOA (see Theorem 2 below).

Beyond considering such regular subVOAs and their commutant, a number of important classes of VOAs are known to contain appropriate Jordan subalgebras and fit the framework presented here to construct multivariable trace functions. We discuss some of these below and look more closely at an application to lattice VOAs, where we formulate another proof of the transformation properties for Siegel theta functions. To explain our results in more detail, we first review the relevant theory and notation pertaining to vertex operator algebras.

A VOA is a quadruple $(V, Y(\cdot, z), 1, \omega)$, which we simply denote by $V$, consisting of a graded vector space $V = \bigoplus_{n \in \mathbb{Z}} V_n$, a linear map $Y(\cdot, z) : V \to \text{End}(V)[[z^{-1}, z]]$, and two notable elements $1 \in V_0$ and $\omega \in V_2$ called the Vacuum and Virasoro elements, respectively. We say $v$ has weight $n$ if $v \in V_n$ and denote the weight of $v$ by $\text{wt}(v)$ if it is not specified. An image $Y(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1}$ of $v \in V$ is called a vertex operator of $v$, and it can be shown that $v_{\text{wt}(v)-1}$ is a weight-preserving operator for a homogeneous element $v$. We denote this unique operator by $o(v)$ and extend it linearly. Meanwhile, the operators $L(n)$ defined by $Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}$ satisfy a Virasoro algebra bracket relation

$$[L(n), L(m)] = (n - m)L(n + m) + \delta_{n+m,0} \frac{n^3 - n}{12} c$$

for some $c \in \mathbb{C}$, called the central charge of $V$. The eigenvalues of $L(0)$ provide the weights on $V$, that is, $V_n = \{ v \in V \mid L(0)v = nv \}$.

In this paper, we assume that $V$ is a regular VOA of CFT-type (i.e. $C_2$-cofinite, rational, and $\mathbb{N}$-graded with $V_0 = \mathbb{C}1$) of central charge $c$. A number of important consequences can be drawn from these assumptions. For one, such a $V$ has only finitely many isomorphism classes of simple $V$-modules $\{W^1, \ldots, W^r\}$, and all of them are $\mathbb{N}$-gradable. Additionally, we have Zhu’s [8] modular-invariance results for single-variable trace functions mentioned above. Specifically, Zhu defines a formal trace function $\widehat{\text{Tr}}_{W^\ell}(\ast : \tau)$ on $W^\ell$ by

$$\widehat{\text{Tr}}_{W^\ell}(v : \tau) := \text{Tr}_{W^\ell} o(v) e^{2\pi i \tau(L(0)) - c/24},$$

and proves that these functions are well-defined as analytic functions on the upper half-plane $\mathcal{H} = \{ \tau \in \mathbb{C} \mid \text{Im}(\tau) > 0 \}$. He then shows that for each $\gamma = (a \ b \ c \ d) \in \text{SL}_2(\mathbb{Z})$ there exists a complex matrix $(A_{ik})$ such that

$$\widehat{\text{Tr}}_{W^\ell}(v : \frac{a\tau + b}{c\tau + d}) = (c\tau + d)^{\text{wt}(v)} \sum_{k=1}^r A_{ik} \widehat{\text{Tr}}_{W^k}(v : \tau)$$

(1.2)
for any \( \tau \in \mathcal{H} \) and \( \text{wt}[-] \)-homogeneous element \( v \in V \). Here \( \text{wt}[-] \) is the weight given by the Virasoro element \((2\pi i)^2 \tilde{\omega} \) of another vertex operator algebra structure \((V, Y[-], 1, (2\pi i)^2 \tilde{\omega})\) on \( V \), which is given by setting

\[
Y[v, z] = Y(v, e^{2\pi iz} - 1)e^{2\pi iz\text{wt}(v)}, \quad \tilde{\omega} = \omega - \frac{c}{24} 1 \in V[2].
\]

We note that the action of \( \text{SL}(2, \mathbb{Z}) \) on \( \mathcal{H} \) is generated by an inversion \( S = (\frac{0}{1} -1) : \tau \to \frac{-1}{\tau} \) and a parallel translation \( T = (\frac{1}{0} 1) : \tau \to \tau + 1 \). The invariance property of trace functions for \( T \) follows easily from the structure of simple modules. Meanwhile, the matrix \( (A^S_{ij}) \) in (1.2) produced by the matrix \( S \) contains interesting and exploitable information about \( V \) and its modules. It is often called the \( S \)-matrix of \( V \) and is denoted by \((s_{ij})\) rather than \((A^S_{ij})\).

As eluded to above, our motivation stems from observing that the power of \( \text{exp} \) in (1.1), i.e. \( 2\pi i\tau (L(0) - c/24) \), is the grade-preserving operator \( o(2\pi i\tau \tilde{\omega}) \) of an element in a one-dimensional Jordan subalgebra \( \tilde{\mathcal{H}} \) of \( V[2] \). We therefore treat cases where \( V[2] \) (and also \( V_2 \)) contain a larger Jordan subalgebra \( \mathcal{G} \). Then, for \( u \in \mathcal{G} \) and \( v \in V \), we define a multivariable formal trace function \( \text{Tr}_{W^\alpha} o(v)e^{2\pi i\alpha(u)} \) and establish a new \( \text{SL}_2(\mathbb{Z}) \)-invariance property for these functions.

The first case we consider is that of an associative Jordan subalgebra. Let \( V = (V, Y, 1, \omega) \) be a regular VOA and \( \omega = e^1 + \cdots + e^g \), where \( e^j \) are mutually orthogonal conformal vectors. Here, an element \( e \in V_2 \) is called a conformal vector if \( e \) is a Virasoro element of the subVOA generated by \( e \), which we denote \( \text{VOA}(e) \). In this case, \( \oplus_{j=1}^g \mathbb{C}e^j \) is an associative Jordan subalgebra of \( V_2 \). Set \( \vec{e}^j = e^j - \frac{c_j}{24} 1 \), where \( c_j \) is the central charge of \( e^j \). Under this setting, for a grade-preserving operator \( \alpha \) and \((\tau_1, \ldots, \tau_g) \in \mathcal{H}_g \), we define a multivariable formal trace function by

\[
\text{Tr}_{W^\alpha}(\alpha : \tau_1, \ldots, \tau_g) := \text{Tr}_{W^\alpha}e^{\alpha(\sum_{j=1}^g 2\pi i\tau_j \vec{e}^j)}.
\]  

(1.3)

If an element \( u \in V \) is homogeneous with respect to the grading induced by the operator \( o(\vec{e}^j) \), we denote its weight under this operator by \( \text{wt}_j[u] \). We say an element of \( V \) is multi-\( \prod \text{ht} \cdot \text{wt}_j[-] \)-homogeneous if it is homogeneous with respect to \( o(\vec{e}^j) \) for all \( j \). Then we have the following theorem, which is proved in Section 2.

**Theorem 1** Let \( V \) be a regular VOA and \( \omega = \sum_{j=1}^g e^j \) be a decomposition of the Virasoro element \( \omega \) by mutually orthogonal conformal vectors \( e^j \). Let \( w \in V \) be a multi-\( \prod \text{ht} \cdot \text{wt}_j[-] \)-homogeneous element and assume that

\[
\text{Tr}_{W^\alpha}(o(w) : \tau_1, \ldots, \tau_g)
\]

are well-defined as analytic functions on \( \mathcal{H}_g \). Then for all \( v \in \oplus_{j=1}^g \text{VOA}(e^j)w \), the functions \( \text{Tr}_{W^\alpha}(o(v) : \tau_1, \ldots, \tau_g) \) are analytic on \( \mathcal{H}_g \) and

\[
\text{Tr}_{W^\alpha}(o(v) : \frac{a\tau_1 + b}{c\tau_1 + d}, \ldots, \frac{a\tau_g + b}{c\tau_g + d}) = \prod_{p=1}^q (c\tau_p + d)^{\text{wt}_p[v]} \sum_{h=1}^r A^\gamma_{dh} \text{Tr}_{W^\alpha}(o(v) : \tau_1, \ldots, \tau_g)
\]

(3)
for \((\tau_1, \ldots, \tau_g) \in \mathcal{H}^{\otimes g}\), where \((A^\tau_g)\) is the matrix given in (1.2) and \(\otimes^g_{j=1}\text{VOA}(e^j)w\) denotes a \(\otimes^g_{j=1}\text{VOA}(e^j)\)-submodule generated by \(w\).

One important instance when \(\omega\) decomposes as stated in the previous theorem is when \(V\) contains a simple, regular subVOA \(U = (U, Y, 1, e)\), and we additionally consider the commutant of \(U\) in \(V\) given by \(U^c = (U^c = \text{Com}_V(U), Y, 1, f = \omega - e)\), where \(\text{Com}_V(U) := \{v \in V \mid u_n v \text{ for } u \in U, n \in \mathbb{N}\}\). As an application of Theorem 1, we prove the following theorem in Section 3.

**Theorem 2** Let \(V\) be a regular VOA and \(U\) a regular subVOA of \(V\). If the commutant \(U^c\) of \(U\) is also regular and \((U^c)^c = U\), then all simple \(U\)-modules appear in some simple \(V\)-module.

In the second case, we consider a Jordan algebra of type \(B_g\). That is, a Jordan algebra isomorphic to the space \(\text{Sym}_g(\mathbb{C})\) consisting of all symmetric complex matrices of degree \(g\). More specifically, we have \(V_2\) contains a Griess subalgebra \(G := \oplus_{1 \leq i \leq j \leq g} C\omega^{ij}\) and there exists an algebra isomorphism \(\mu : \text{Sym}_g(\mathbb{C}) \to G\) satisfying \(\mu(E_{ij} + E_{ji}) = \omega^{ij}\) and \(2\mu(I_g) = \omega\), where \(E_{ij}\) denotes an elementary matrix which has 1 in the \((ij)\)-entry and zeros elsewhere. Here we call a subalgebra \(G\) of \((V_2, \times_1)\) a Griess subalgebra if \(v_2u = 0\) for \(v, u \in G\), where a 1-product \(u \times_1 v\) is given by \(u_1 v\).

Under this setting, for \(A = (\tau_{ij}) \in \mathcal{H}_g\) we define a multivariable trace function

\[
\hat{\text{Tr}}_{W^\ell}(\omega(v) : A) = \text{Tr}_{W^\ell} \omega(v) e^{o(2\pi i(2\mu(A) - \frac{\omega(v) - \omega(A)}{2\pi i}))},
\]

where \(\mathcal{H}_g = \{X + Yi \mid X, Y \in \text{Sym}_g(\mathbb{R}), Y\ \text{is positive definite}\}\) is the Siegel upper half-space. The action of \(\text{SL}_2(Z)\) on \(\mathcal{H}_g\) is given by \(T(Z) = Z + E_g\) and \(S(Z) = -Z^{-1}\) for \(Z \in \mathcal{H}_g\), where \(E_g\) is the \(g \times g\) identity matrix. Our next result, which is found in Section 4, establishes the invariance for a Siegel-type inversion.

**Theorem 3** Suppose \(\hat{\text{Tr}}_{W^j}(\omega(1) : A)\) is a well-defined analytic function on \(\mathcal{H}_g\) for \(j = 1, \ldots, r\). Then

\[
\hat{\text{Tr}}_{W^j}(\omega(1) : -A^{-1}) = \sum_{k=1}^r s_{jh} \hat{\text{Tr}}_{W^k}(\omega(1) : A),
\]

where \((s_{jh})\) is the \(S\)-matrix given for \(\gamma = S\) in (1.2).

There are many known VOAs containing a Jordan algebra of type \(B_g\). For example, a VOA \(M(1)^{\otimes g}\) of free boson type constructed from a \(g\)-dimensional vector space \(\mathbb{C}^g\) and its fixed point subVOA \((M(1)^{\otimes g})^+\) by an automorphism \(-1\) on \(\mathbb{C}^g\) contain a Griess subalgebra isomorphic to a Jordan algebra of type \(B_g\) (that is, a Jordan algebra consisting of all symmetric complex matrices of degree \(g\)). The famous moonshine VOA \(V^2\) also contains a Griess subalgebra isomorphic to a Jordan algebra of type \(B_{24}\). Moreover, \(V^2\) has only one simple module and its \(S\)-matrix is \((s_{ij}) = I_4\). We also note that the second author and
Ashihara have shown in [1] that for any $c \in \mathbb{C}$ and $g \in \mathbb{N}$, there is a VOA $AM(g, c)$ with central charge $c$ whose Griess algebra is a Jordan algebra of type $B_g$.

We conclude this paper with Section 5, where we apply the above results to prove the inversion transformation property and convergence for ordinary Siegel theta series. See Proposition 7 below for a detailed statement of this result.

## 2 Preliminaries and simultaneous transformations

We first recall the following notation and results from [2].]

2 Preliminaries and simultaneous transformations

We first recall the following notation and results from [2]. Since we will treat power series of $e^{2\pi i\tau_j}$ for various $\tau_j \in \mathcal{H}$, we denote the $q$-power expansion of Eisenstein series $G_{2k}(\tau)$ by $\tilde{G}_{2k}(\tau)$, where $q = e^{2\pi i}$. Namely,

$$\tilde{G}_{2k}(\tau) = 2\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k - 1)!} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n. \quad (2.1)$$

Under the action of a matrix $\gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}_2(\mathbb{Z})$, these transform as

$$\tilde{G}_2 \left( \frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^2 \tilde{G}_2(\tau) - 2\pi i c(c\tau + d) \quad \text{and}$$

$$\tilde{G}_{2k} \left( \frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^{2k} \tilde{G}_{2k}(\tau) \quad \text{for } k > 1. \quad (2.2)$$

One of the most important results in [8] is the following, which we will often use.

**Lemma 4** For any vertex operator algebra $V$ and any $L(0)$-gradable module $M$, we have

$$\widetilde{\text{Tr}}_M (o(a[0]b) : \tau) = 0, \quad \text{and} \quad (2.3)$$

$$\widetilde{\text{Tr}}_M (o(a)[0]b) : \tau) = \widetilde{\text{Tr}}_M (o(a[-1]b) : \tau) - \sum_{k=1}^{\infty} \tilde{G}_{2k}(\tau) \widetilde{\text{Tr}}_M (o(a[2k-1]b) : \tau). \quad (2.4)$$

as formal complex power series of $e^{2\pi i\tau}$ for $a, b \in V$.

In this section, we let $V = (V, Y(\cdot, \cdot), 1, \omega)$ be a regular VOA of CFT-type and assume that $V$ has a set $\{e^1, \ldots, e^g\}$ of mutually orthogonal conformal vectors such that $\omega = e^1 + \cdots + e^g$. Then $(2\pi i)^2\tilde{c}^j$ are mutually orthogonal conformal vectors of the deformed VOA $V = (V, Y[\cdot, \cdot], 1, (2\pi i)^2 \tilde{\omega})$ and $\tilde{\omega} = \sum_{j=1}^{g} \tilde{c}^j$, where $c_j$ is the corresponding central charges of $e^j$ and $\tilde{e}^j = e^j - \frac{c_j}{24}1$. Let $M$ be a $V$-module and recall the multivariable functions (1.3). Clearly, we have

$$\frac{\partial}{\partial \tau_j} \widetilde{\text{Tr}}_M (o(v) : \tau_1, \ldots, \tau_g) = 2\pi i \widetilde{\text{Tr}}_M \left( o(\tilde{e}^j) o(v) : \tau_1, \ldots, \tau_g \right). \quad (2.5)$$

Since $e^j$ are mutually orthogonal, $[o(\tilde{e}^j), o(\tilde{e}^h)] = 0$ and so we have the commutativity of partial differentials,

$$\frac{\partial}{\partial \tau_j} \frac{\partial}{\partial \tau_h} \widetilde{\text{Tr}}_M (o(v) : \tau_1, \ldots, \tau_g) = \frac{\partial}{\partial \tau_h} \frac{\partial}{\partial \tau_j} \widetilde{\text{Tr}}_M (o(v) : \tau_1, \ldots, \tau_g), \quad (2.6)$$

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for any \( j \) and \( h \). We also note that
\[
\lim_{\forall \tau \to \tau} \tilde{\text{Tr}}_M(o(v) : \tau_1, \ldots, \tau_g) = \tilde{\text{Tr}}_M(v : \tau).
\]

Viewing a \( V \)-module \( M \) as a \( \text{VOA}(e^j) \)-module, (2.3) in Lemma 4 becomes
\[
\tilde{\text{Tr}}_M(o(e^j[0]b) : \tau_1, \ldots, \tau_g) = 0,
\]
while (2.4) becomes
\[
\tilde{\text{Tr}}_M(o(e^j)o(b) : \tau_1, \ldots, \tau_g) = \tilde{\text{Tr}}_M(o(e^j[-1]b) : \tau_1, \ldots, \tau_g) - \sum_{k=1}^{\infty} G_{2k}(\tau_j) \tilde{\text{Tr}}_M( o(e^j[2k - 1]b) : \tau_1, \ldots, \tau_g).
\]

We are now in position to prove Theorem 1.

[Proof of Theorem 1] In this proof, \( L_k[m] \) denotes \((2\pi i)^2 e_k[m + 1]\). To simplify the notation, we will write the proof for the case \( g = 2 \), but there is no difference for \( g \geq 3 \).

We first prove the statement that \( \tilde{\text{Tr}}_{W^h}(o(v) : \tau_1, \ldots, \tau_g) \) is a well-defined analytic function on \( \mathcal{H}^{\otimes 2} \). We do so by induction on \( \text{wt}_1[v] + \text{wt}_2[v] \), after assuming this is true for the base case \( v = w \). More generally, any \( v \in \otimes_{j=1}^{2} \text{VOA}(e^j)w \) is of the form \( v = \otimes_{j=1}^{2} L_j[-m_{ij}] \cdot \cdots \cdot L_j[-m_{dj}]w \) for \( m_{ij} \geq 1 \). Since \( L_j[-n] \) is generated by \( L_j[-1] \) and \( L_j[-2] \), we may take \( m_{ij} = 1, 2 \). Moreover, by (2.7) we may assume \( m_{1j} = 2 \). Since \( \tilde{\text{Tr}}_{W^h}(o(\otimes_{j=1}^{2} [L_j][m_{ij}] \cdots \cdot L_j[-m_{dj}]w) : \tau_1, \tau_2) \), where \( L_j[-m_{ij}] \) denotes the omission of one or both of these terms, is analytic by our induction hypothesis, then (2.5) and (2.8) imply \( \tilde{\text{Tr}}_{W^h}(o(v) : \tau_1, \tau_2) \) is also analytic.

We now turn to proving the functional equation. Set \( \gamma = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \text{SL}_2(\mathbb{Z}) \). We will prove
\[
\tilde{\text{Tr}}_{W^\ell}(o(v) : \begin{pmatrix} \alpha_1 + b & \alpha_2 + b \\ c \alpha_1 + d & c \alpha_2 + d \end{pmatrix}) = (c \alpha_1 + d)^{\text{wt}_1[v]}(c \alpha_2 + d)^{\text{wt}_2[v]} \sum_{h=1}^{r} A_{\gamma,h} \tilde{\text{Tr}}_{W^h}(o(v) : \gamma \tau_1, \gamma \tau_2).
\]

(2.9)

For ease of notation, we set \( \gamma \tau_i := \frac{\alpha_i + b}{c \alpha_i + d} \) and \( j(\gamma, \tau) := (c \tau_i + d) \). To begin, we consider \( (\tau_1, \tau_2) = (\tau, \tau) \) as a base point. Since \( \lim_{\forall \tau \to \tau} \tilde{\text{Tr}}_{W^\ell}(o(v) : \tau_1, \tau_2) = \tilde{\text{Tr}}_{W^\ell}(v : \tau) \), Zhu’s theorem (cf. (1.2)) implies
\[
\lim_{\forall \gamma \to \tau} \tilde{\text{Tr}}_{W^\ell}(o(v) : \gamma \tau_1, \gamma \tau_2) = \sum_{h=1}^{r} A_{\gamma,h} \tilde{\text{Tr}}_{W^h}(o(v) : \tau_1, \tau_2).
\]

(2.10)

Namely, (2.9) is true for the base point \( (\tau, \tau) \). We will next show that higher order partial differentials on both sides of (2.10) by \( \tau_1 \) and \( \tau_2 \) still coincide with each other when evaluated at \( (\tau, \tau) \). This in turn implies the Taylor series expansions about \( (\tau, \tau) \) of the analytic left
and right hand sides of (2.9) are equal on a neighborhood about \((\tau, \tau)\), and thus on all of \(H^{\mathbb{R}^2}\).

To simplify the arguments, we will prove the equality for higher partial differentials by \(\tau_1\). Namely, we will prove

\[
\lim_{\nu \tau_1 \rightarrow \tau_1} \frac{\partial^p}{\partial \tau_1^p} \mathcal{W}(o(v) : \gamma \tau_1, \gamma \tau_2) = \lim_{\nu \tau_1 \rightarrow \tau_1} \frac{\partial^p}{\partial \tau_1^p} j(\gamma, \tau_1)^{wt_1[v]} j(\gamma, \tau_2)^{wt_2[v]} \sum_{h=1}^{r} A^h_{\theta} \mathcal{W}(o(v), \tau_1, \tau_2)
\]

\[(2.11)\]

for any \(p \in \mathbb{N}\) and \(v \in \otimes_{j=1}^{2} \text{VOA}(e^j)w\) by induction. For the combinations with \(\frac{\partial}{\partial \tau_1}\), we can prove the assertion by using (2.5),(2.6), and (2.8). We note that (2.11) is true for \(p = 0\), and we next assume that it holds for all \(v \in \otimes_{j=1}^{2} \text{VOA}(e^j)w\) and \(p \leq m\). In particular, we have

\[
\lim_{\nu \tau_1 \rightarrow \tau_1} \frac{\partial^m}{\partial \tau_1^m} j(\gamma, \tau_1)^{wt_1[v]} j(\gamma, \tau_2)^{wt_2[v]} \sum_{h=1}^{r} A^h_{\theta} \mathcal{W}(o(L_1[-2]v) : \tau_1, \tau_2)
\]

\[(2.12)\]

Moreover, since \(\lim_{\nu \tau_1 \rightarrow \tau_1} (ct_j + d)^n G_{2k}(\tau_j) = (ct + d)^n G_{2k}(\tau)\) for \(k, n \geq 0\) and \(j = 1, 2\), the induction hypothesis also implies

\[
\lim_{\nu \tau_1 \rightarrow \tau_1} (ct_j + d)^n G_{2k}(\tau_j) \frac{\partial^m}{\partial \tau_1^m} j(\gamma, \tau_1)^{wt_1[v]} j(\gamma, \tau_2)^{wt_2[v]} \sum_{h=1}^{r} A^h_{\theta} \mathcal{W}(o(L_1[-2]v) : \tau_1, \tau_2)
\]

\[(2.13)\]

for such \(k, n, \) and \(j\), where we set \(G_0(\tau_j) := 1\).

Since \(L_1[-2] = (2\pi i)^2 \varepsilon^2[-1]\), (2.8) and direct calculation gives

\[
\text{RHS of (2.12)} = \lim_{\nu \tau_1 \rightarrow \tau_1} (2\pi i)^2 \frac{\partial^m}{\partial \tau_1^m} \mathcal{W}(o(\varepsilon^2) o(v) : \gamma \tau_1, \gamma \tau_2) + \lim_{\nu \tau_1 \rightarrow \tau_1} \frac{\partial^m}{\partial \tau_1^m} G_2(\gamma \tau_1)^{wt_1[v]} \mathcal{W}(o(v) : \gamma \tau_1, \gamma \tau_2) + \lim_{\nu \tau_1 \rightarrow \tau_1} \frac{\partial^m}{\partial \tau_1^m} \sum_{k=2}^{\infty} G_{2k}(\gamma \tau_1) \mathcal{W}(o(L_1[2k-2]v) : \gamma \tau_1, \gamma \tau_2).
\]

Furthermore, from (2.5) we have

\[
(2\pi i)^2 \frac{\partial^m}{\partial \tau_1^m} \mathcal{W}(o(\varepsilon^2) o(v) : \gamma \tau_1, \gamma \tau_2) = \frac{\partial^m}{\partial \tau_1^m} 2\pi i \frac{\partial}{\partial (\gamma \tau_1)} \mathcal{W}(o(v) : \gamma \tau_1, \gamma \tau_2) = 2\pi i \frac{\partial^m}{\partial \tau_1^m} j(\gamma, \tau_1)^2 \frac{\partial}{\partial \tau_1} \mathcal{W}(o(v) : \gamma \tau_1, \gamma \tau_2).
\]
On the other hand, by (2.5) and (2.8) we find

LHS of (2.12)
\[
= (2\pi i)^2 \lim_{\gamma \to \tau} \frac{\partial}{\partial \gamma_1} j(\gamma, \tau_1)\Gamma_{\gamma_1+2} j(\gamma, \tau_2)\Gamma_{\tau_2} \sum_{h=1}^{r} A_{\gamma_h}^2 \Gamma_{\tau_h} \left( o(e^1 o(v) : \tau_1, \tau_2) + \lim_{\gamma \to \tau} \frac{\partial}{\partial \gamma_1} j(\gamma, \tau_1)\Gamma_{\gamma_1+2} j(\gamma, \tau_2) G_2(\tau_1) \Gamma_{\gamma_1} \sum_{h=1}^{r} A_{\gamma_h}^2 \Gamma_{\tau_h} \left( o(v) : \tau_1, \tau_2 \right) + \lim_{\gamma \to \tau} \frac{\partial}{\partial \gamma_1} j(\gamma, \tau_1)\Gamma_{\gamma_1+2} j(\gamma, \tau_2) \sum_{k=2}^{\infty} G_{2k}(\tau_1) \sum_{h=1}^{r} A_{\gamma_h}^2 \Gamma_{\tau_h} \left( o(L_1[2k-2]v) : \tau_1, \tau_2 \right) \right)
\]
\[
= 2\pi i \lim_{\gamma \to \tau} \frac{\partial}{\partial \gamma_1} \left( j(\gamma, \tau_1)\Gamma_{\gamma_1+2} j(\gamma, \tau_2)\Gamma_{\gamma_1} \sum_{h=1}^{r} A_{\gamma_h}^2 \Gamma_{\tau_h} \left( o(v) : \tau_1, \tau_2 \right) + \lim_{\gamma \to \tau} \frac{\partial}{\partial \gamma_1} j(\gamma, \tau_1)\Gamma_{\gamma_1+2} j(\gamma, \tau_2) G_2(\tau_1) \Gamma_{\gamma_1} \sum_{h=1}^{r} A_{\gamma_h}^2 \Gamma_{\tau_h} \left( o(v) : \tau_1, \tau_2 \right) + \lim_{\gamma \to \tau} \frac{\partial}{\partial \gamma_1} j(\gamma, \tau_1)\Gamma_{\gamma_1+2} j(\gamma, \tau_2) \sum_{k=2}^{\infty} G_{2k}(\tau_1) \sum_{h=1}^{r} A_{\gamma_h}^2 \Gamma_{\tau_h} \left( o(L_1[2k-2]v) : \tau_1, \tau_2 \right) \right)
\]
where we also used the transformations (2.2). Therefore, by the derivation property of \( \frac{\partial}{\partial \tau_i} \) along with (2.13), we have

\[
\text{LHS of (2.12)} \\
= 2 \pi i \lim_{\gamma \tau_i \rightarrow \tau_r} \frac{\partial^m}{\partial \tau_1^m} \left( j(\gamma, \tau_1)^2 \frac{\partial}{\partial \tau_1} j(\gamma, \tau_1) \right) j(\gamma, \tau_2)^{wt_1[v]} \sum_{h=1}^r A_{\gamma h} \text{Tr}_{W_h} (o(v) : \tau_1, \tau_2) \\
+ \lim_{\gamma \tau_i \rightarrow \tau_r} \frac{\partial^m}{\partial \tau_1^m} G_2(\gamma \tau_1)^{wt_1[v]} \text{Tr}_{W'} (o(v) : \gamma \tau_1, \gamma \tau_2) \\
+ \lim_{\gamma \tau_i \rightarrow \tau_r} \frac{\partial^m}{\partial \tau_1^m} \sum_{k=2}^\infty G_{2k}(\gamma \tau_1) \text{Tr}_{W'} (o(L_1[2k - 2][v]) : \gamma \tau_1, \gamma \tau_2).
\]

Equating our calculations for the left and right hand sides of (2.12), we obtain

\[
\lim_{\gamma \tau_i \rightarrow \tau_r} \frac{\partial^m}{\partial \tau_1^m} j(\gamma, \tau_1)^2 \frac{\partial}{\partial \tau_1} j(\gamma, \tau_1) j(\gamma, \tau_2)^{wt_1[v]} \sum_{h=1}^r A_{\gamma h} \text{Tr}_{W_h} (o(v) : \tau_1, \tau_2) \\
= \lim_{\gamma \tau_i \rightarrow \tau_r} \frac{\partial^m}{\partial \tau_1^m} \text{Tr}_{W'} (o(v) : \gamma \tau_1, \gamma \tau_2).
\]

Using the derivation property of the partial derivatives and (2.13) again with \( k = 0 \), we deduce

\[
\lim_{\gamma \tau_i \rightarrow \tau_r} \frac{\partial^{m+1}}{\partial \tau_1^{m+1}} j(\gamma, \tau_1)^{wt_1[v]} j(\gamma, \tau_2)^{wt_2[v]} \sum_{h=1}^r A_{\gamma h} \text{Tr}_{W_h} (o(v) : \tau_1, \tau_2) \\
= \lim_{\gamma \tau_i \rightarrow \tau_r} \frac{\partial^{m+1}}{\partial \tau_1^{m+1}} \text{Tr}_{W'} (o(v) : \gamma \tau_1, \gamma \tau_2),
\]

as desired. This completes the proof of Theorem 1.

\[\blacksquare\]

3 Commutant decomposition

In this section we prove Theorem 2.

[Proof of Theorem 2] Set \( W = U^c \). Let \( \{V = V^1, \ldots, V^p\} \) denote the set of simple \( V \)-modules and \( \{U = U^1, \ldots, U^q, \ldots, U^s\} \) and \( \{W = W^1, \ldots, W^h, \ldots, W^l\} \) be the sets of simple \( U \)-modules and simple \( W \)-modules, respectively, where the sets \( \{U^1, \ldots, U^q\} \) and \( \{W^1, \ldots, W^h\} \) denote the modules which appear in some simple \( V \)-module. We will prove the theorem by contradiction, and assume that \( s > g \).

Set \( \sigma = (0 \ 1 \ -1 \ 0) \), and let \( S = (s_{ij}), S^U, \) and \( S^W \) denote the matrices \( (A_{ij}^e) \) given in (1.2) of the trace functions on \( V \)-modules, \( U \)-modules, and \( W \)-modules, respectively. Moreover, let \( M_{r,s}(F) \) denote the set of \( r \times s \)-matrices with entries in \( F \).

Viewing \( V^k \) as a \( U \otimes W \)-module, we have the existence of a matrix \( R^k = (r_{ij}^k) \in M_{s,t}(\mathbb{N}) \) such that \( V^k \) decomposes as

\[
V^k = \bigoplus_{i,j} r_{ij}^k (U^i \otimes W^j),
\]
where $r_{ij}^k (U^i \otimes W^j)$ denotes a direct product of $r_{ij}^k$ copies of $U^i \otimes W^j$. By the transformation property of Theorem 1, we have

$$ (S^U) R^k (t^{SW}) = \sum_{i=1}^p s_{ki} R^i. \quad (3.1) $$

For variables $x_1, \ldots, x_p$, let $\underline{x}$ denote the set of variables $\{x_1, \ldots, x_p\}$. Set $R(\underline{x}) = R(x_1, \ldots, x_p) := \sum_{i=1}^p x_i R^i$, and view $R(\underline{x})$ as a matrix over $K := \mathbb{C}(x_1, \ldots, x_p)$. Then replacing $R^k$ in (3.1) with $R(\underline{x})$, we find

$$ (S^U) R(\underline{x}) (t^{SW}) = \sum_{\ell=1}^p x_\ell \left[ (S^U) R^\ell (t^{SW}) \right] = \sum_{\ell=1}^p x_\ell \left( \sum_{i=1}^p s_{\ell i} R^i \right) = \sum_{i=1}^p \left( \sum_{\ell=1}^p s_{\ell i} x_\ell \right) R^i = \sum_{i=1}^p \hat{x}_i R^i = R(\underline{x}), \quad (3.2) $$

where we set $\hat{x}_j = \sum_{\ell=1}^p s_{\ell j} x_\ell$. Moreover, we extend the transformation $(s_{ij})$: $x_j \rightarrow \sum_{\ell=1}^p s_{\ell j} x_\ell = \hat{x}_j$ to a $\mathbb{C}$-automorphism $\phi$ over $K$. We also let $\underline{x}^\phi$ denote the application of $\phi$ on each $x_i$. For example,

$$ R(\underline{x}^\phi) = R(\phi(x_1), \ldots, \phi(x_p)) = R \left( \sum_{\ell=1}^p s_{\ell 1} x_\ell, \ldots, \sum_{\ell=1}^p s_{\ell p} x_\ell \right). $$

We can then rewrite (3.2) as

$$ (S^U) R(\underline{x}) = R(\underline{x}^\phi) (t^{SW})^{-1}. \quad (3.3) $$

It follows from the assumption $s > g$ that $r_{jk}^k = 0$ for all $j, k$. Hence, the $s$-th row of $R(\underline{x})$ is zero, as is the $s$-th row of $R(\underline{x}^\phi)$. Therefore the $s$-th row of $R(\underline{x}^\phi) (t^{SW})^{-1}$ on the right hand side of (3.3) is zero, and thus so is the $s$-th row of $(S^U) R(\underline{x})$. In particular, the $(s, 1)$-entry of $(S^U) R(\underline{x})$ is zero. Explicitly, we have

$$ \sum_{j=1}^s S^U_{ij} R(\underline{x})_{j1} = 0, \quad (3.4) $$

where $S^U_{ij}$ and $R(\underline{x})_{ij}$ denote the $(i, j)$-entries of the matrices $S^U$ and $R(\underline{x})$, respectively. Meanwhile, $S^U_{11}$ is nonzero by the Verlinde formula. Additionally, since $U^c = W$ and $W = UC$, we have $r_{1j}^j = 0$ for $j > 1$. This implies $R(\underline{x})_{j1} \in \mathbb{N}[x_2, \ldots, x_p]$ for $j > 1$. Finally, noting that $r_{11}^1 = 1$, we have $R(\underline{x})_{11} \in x_1 + \mathbb{N}[x_2, \ldots, x_p]$. It follows that

$$ \sum_{j=1}^s S^U_{sj} R(\underline{x})_{j1} \in S^U_{s1} x_1 + \mathbb{C}[x_2, \ldots, x_p], $$

which cannot equal zero. This contradicts (3.4), and the proof is complete.
4 Jordan algebra of type $B_g$

Let $\text{Sym}_g(\mathbb{C})$ denote the set of symmetric matrices of degree $g$ and $\mathcal{H}_g$ be the Siegel upper half-space $\{X + Y i \mid X, Y \in \text{Sym}_g(\mathbb{R}), Y \text{ is positive definite}\}$. We note that $\text{Sym}_g(\mathbb{C})$ is a Jordan algebra of type $B_g$.

In this section, we assume that there is a Griess subalgebra $\mathcal{G} \subseteq V_2$ isomorphic to a Jordan algebra $\text{Sym}_g(\mathbb{C})$ such that the identity matrix corresponds to $\omega/2$, and a primitive idempotent corresponds to one half of a conformal element with a central charge $c/g$. We denote its ring isomorphism by $\mu : \text{Sym}_g(\mathbb{C}) \to \mathcal{G}$.

For $A \in \text{Mat}_g(\mathbb{C})$, let $A^{ss}$ and $A^{nil}$ denote the semisimple and nilpotent parts, respectively. We note that $A^{ss}$ is also a symmetric matrix.

**Lemma 5** If $A \in \mathcal{H}_g$, then the eigenvalues of $A^{ss}$ are all in $\mathcal{H}$ and there is a complex orthogonal matrix $R$ such that $R^{-1}A^{ss}R$ is a diagonal matrix.

**[Proof]** We define an inner product $(u, v)$ by $^t uv$ for $u, v \in \mathbb{C}^g$, and view $A$ as an endomorphism of $\mathbb{C}^g$. If an eigenvalue $\lambda$ of $A$ is not in $\mathcal{H}$, then $A - \lambda I_g$ is again in $\mathcal{H}_g$ and thus nonsingular. However, the determinant $\text{det}(A - \lambda I_g)$ is zero, which is a contradiction. This proves the claim on the eigenvalues, and we now assume $A$ is semisimple. If $u, v \in \mathbb{C}^g$ are eigenvectors with different eigenvalues, say $\lambda$ and $\mu$, respectively, then since $\lambda^t uv = \mu^t uv$, we have $t uv = 0$. Therefore, $\mathbb{C}^g$ is an orthogonal sum of eigenspaces as desired.

We also note that

$$\mathcal{H}_g^{ss} = \{R^{-1} DR \in \mathcal{H}_g \mid R \in O_g(\mathbb{C}), D \text{ is diagonal}\}$$

is a dense subset of $\mathcal{H}_g$. Recall that every idempotent matrix in $\mathcal{G}$ is one half of a conformal vector. Therefore, for each $A \in \mathcal{H}_g^{ss}$ there are mutually orthogonal conformal vectors $e^1, \ldots, e^g$ with central charges $c_1, \ldots, c_g$, respectively, and scalars $\tau_1, \ldots, \tau_g$ such that $\sum_{j=1}^g e^j = \omega$ and $2\mu(A) = \tau_1 e^1 + \cdots + \tau_g e^g$. We again note that $\tau_j \in \mathcal{H}$. Then $\{(2\pi i)^2 \bar{e}^j = (2\pi i)^2 (e^j - \frac{\tau_j}{2\pi} \mathbf{1}) \mid j = 1, \ldots, g\}$ are mutually orthogonal conformal vectors for $(V, Y[\cdot], 1, (2\pi i)^2 \bar{\omega})$. Let $\text{wt}_j[A]$ denote the weight given by $(2\pi i)^2 \bar{\omega}$. Clearly, $2\mu(-A^{-1}) = \frac{\tau_1}{2\pi} e^1 + \cdots + \frac{\tau_g}{2\pi} e^g$ for $A \in \mathcal{H}_g^{ss}$. Therefore, by Theorem 1, we have the following result.

**Lemma 6** Let $A \in \mathcal{H}_g^{ss}$ and $w \in V$ be a multi-$\text{wt}_j^{A[\cdot]}$-homogeneous element. If $\overline{\text{Tr}}(w : A)$ is an analytic function on $\mathcal{H}_g$, then for any $v \in \otimes_{j=1}^g \text{VOA}(e^j) w$, we have

$$\overline{\text{Tr}}_{W_j} (o(v) : -A^{-1}) = \prod_{p=1}^g (\tau_p)^{\text{wt}_p[v]} \sum_{h=1}^r s_{jh} \overline{\text{Tr}}_{W^{ss}} (o(v) : A) .$$

Since $\text{wt}_j^{A[\cdot]}$ is zero for all $A \in \mathcal{H}_g^{ss}$, Theorem 3 is an immediate consequence of Lemma 6 for such $A$. Meanwhile, because $\mathcal{H}_g^{ss}$ is a dense set of $\mathcal{H}_g$ and $\overline{\text{Tr}}_{W_j}(o(1), A)$ and $\overline{\text{Tr}}_{W_j}(o(1), -A^{-1})$ are analytic for all $A \in \mathcal{H}_g$ and $j = 1, \ldots, r$, Theorem 3 holds as claimed.
5 Applications

As a corollary to Theorem 3, we will prove the inversion transformation formula of matrices $A \in \mathcal{H}_g$ for Siegel theta series. To do so we must introduce the lattice VOA $V_L$ for an even positive lattice $L$ of rank $g$.

We begin by first defining the VOA $M(1)$ of free boson type. Viewing $\mathbb{C}L$ as a $g$-dimensional vector space with inner product $\langle \cdot, \cdot \rangle$, we consider an affine Lie algebra

$$\widehat{\mathbb{C}L} := \left( \bigoplus_{j=1}^{g} \bigoplus_{n \in \mathbb{Z}} \mathbb{C}a_j(n) \right) \oplus \mathbb{C},$$

where $\{a_j \mid j = 1, \ldots, g\}$ is an orthonormal basis of $\mathbb{C}L$ and the Lie products are given by $[a(n), b(m)] = \delta_{n+m,0} n(a, b)$. We note $\widehat{\mathbb{C}L}$ does not depend on the choice of the orthonormal basis. Clearly, $\widehat{\mathbb{C}L}_{\geq 0} := \left( \bigoplus_{j=1}^{g} \bigoplus_{n \geq 0} \mathbb{C}a_j(n) \right) \oplus \mathbb{C}$ is a subring. For every $\alpha \in \mathbb{C}L$, we define a one-dimensional $\widehat{\mathbb{C}L}_{\geq 0}$-module $\mathbb{C}e^\alpha$ by

$$a(n)e^\alpha = 0 \text{ for } n > 0, \text{ and } a(0)e^\alpha = \langle \alpha, \alpha \rangle e^\alpha. \quad (5.1)$$

We also consider the induced module

$$M^g(1)e^\alpha := U(\widehat{\mathbb{C}L}) \otimes_{U(\mathbb{C}L_{\geq 0})} \mathbb{C}e^\alpha,$$

where $U(R)$ denotes the universal enveloping algebra of $R$. Among these modules, $M^g(1)e^0$ has a VOA structure of central charge $g$ which we denote by $M^g(1)$. Furthermore, $M^g(1)e^\alpha$ is an $M^g(1)$-module for each $\alpha$. Often $M^g(1)$ is called the VOA of $g$ bosons. Then

$$V_L = \bigoplus_{\alpha \in \mathbb{C}L} M^g(1)e^\alpha$$

becomes a vertex operator algebra of central charge $g$, which is called a lattice VOA. (See [4] for more details on lattice VOAs.) We note that $1 := 1 \otimes e^0$ and $\omega := \frac{1}{2} \sum_{i=1}^{g} a_i(-1)a_i(-1)1$ are the Vacuum and Virasoro elements, respectively, of both $V_L$ and $M^g(1)$.

It is known that $V_L$ is a regular VOA, and its simple modules are given by $V_{L+\beta} = \bigoplus_{\alpha \in L} M^g(1)e^{\alpha+\beta}$ for some $\beta \in \mathbb{Q}L$ with $\langle \beta, L \rangle \subseteq \mathbb{Z}$ (see [3]). We will use the vertex operators

$$Y(a(-1)1, z) = \sum_{n \in \mathbb{Z}} a(n)z^{-n-1} \text{ and }$$

$$Y(a(-1)b(-1)1, z) = \sum_{m \in \mathbb{Z}} \left( \sum_{n \in \mathbb{N}} a(-1-n)b(m+n) + b(-1+m-n)a(n) \right)z^{-m-1}.$$ 

From (5.1), we have $o(a(-1)b(-1)1)e^\beta = \langle a, \beta \rangle \langle b, \beta \rangle e^\beta$ and $\text{wt}(a(-i_k) \cdots a(-i_1)e^\alpha) = i_1 + \cdots + i_k + \frac{(a, \alpha)}{2}$. Therefore, the character $\widehat{\text{Tr}}_{M^g(1)}(o(1): \tau)$ of $M^g(1)$ is $1/\eta(\tau)^g$ and the character of $V_L$ is $\theta_L(\tau)/\eta(\tau)^g$, where $\theta_L(\tau)$ is the theta series associated to the lattice $L$ and $\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$ is the eta-function, where $q = e^{2\pi i \tau}$. 

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By using an orthonormal basis \{a_i \mid i = 1, \ldots, g\} of \mathbb{R}L, we define \omega^{ij} = \frac{1}{2}a_i(-1)a_j(-1)1 for \(i, j = 1, \ldots, g\). We note \(\omega^{ij} = \omega^{ji}\), \{\omega^{ii} \mid i = 1, \ldots, g\} is a set of mutually orthogonal conformal vectors of central charge 1, and \(\omega = \sum_{i=1}^g \omega^{ii}\) is a Virasoro element of \(M^g(1)\).

From the construction, \(M^g(1)\) has an automorphism \(\sigma\) induced from \(-1\) on \(\mathbb{C}L\), that is, \(\sigma(a_{jk}(-i_k) \cdots a_{ji}(-i_1)1) = (-1)^k a_{jk}(-i_k) \cdots a_{ji}(-i_1)1\). Let \(M^g(1)^+\) denote the fixed point subVOA of \(M^g(1)\) by \(\sigma\). Then by direct calculations, we have \((M^g(1)^+)_0 = \mathbb{C}1^\otimes g\), \((M^g(1)^+)_1 = 0\), and \((M^g(1)^+)_2 = \prod_{1 \leq i < j \leq g} \mathbb{C}\omega^{ij}\) is isomorphic to a Jordan algebra of type \(B_g\) by the 1-products.

We now introduce a multivariable trace function on the Siegel upper half-space \(\mathcal{H}_g\). For \(A = (\tau_{ij}) \in \mathcal{H}_g\) and a \(V_L\)-module \(M\), we recall the function (1.4), and in particular

\[
\widehat{\text{Tr}}_M(o(1) : A) = \text{Tr}_Me^{\alpha(2\pi i(\mu(A) - \frac{\omega(A)}{4}))},
\]

where in this case, \(\mu(A) = \sum_{i=1}^g \sum_{j=1}^g \tau_{ij} \omega^{ij} \in M^g(1)^+_2\).

In order to pick out the lattice parts, we define

\[
\gamma_M(A) = \widehat{\text{Tr}}_M(o(1) : A) \prod_{i=1}^g \eta(\mu_i)
\]

for \(A \in \mathcal{H}_g\), where the \(\mu_i\) are the numbers satisfying \(\text{det}(xE_g - A) = \prod_{i=1}^g (x - \mu_i) = 0\). We note \(\mu_i \in \mathcal{H}\).

We now prove the following result.

**Proposition 7** For a lattice VOA \(V_L\) with inequivalent simple \(V_L\)-modules \(V_L = W_1, \ldots, W_r\), we have \(\widehat{\text{Tr}}_{W_h}(o(1) : A)\) is an analytic function on \(\mathcal{H}_g\) for all \(h = 1, \ldots, r\). Furthermore, \(\gamma_{W_h}(A)\) are ordinary Siegel theta series and

\[
\left(-i \frac{1}{\text{det}(A)}\right)^{g/2} \gamma_{W^1}(-A^{-1}) = \sum_{h=1}^r s_{j_h} \gamma_{W_h}(A).
\]

**Proof** As discussed above, a simple \(V_L\)-module \(M\) is of the form \(M = V_{L+\kappa}\) for some \(\kappa \in \mathbb{Q}L\). If \(A = (\tau_{ij})\) is semisimple, then there is an orthogonal complex matrix \(P \in O_g(\mathbb{C})\) and scalars \(\mu_1, \ldots, \mu_g\) such that \(P^{-1}(\tau_{ij})P = \text{diag}(\mu_1, \ldots, \mu_g)\). Set \((b_1, \ldots, b_g) = (a_1, \ldots, a_g)P\) and \(e^i = \frac{1}{2}b^i(-1)b^i(-1)1\). Then \(b_1, \ldots, b_g\) is an orthonormal basis of \(\mathbb{C}L\), and \(\{e^i \mid i = 1, \ldots, g\}\) is a set of mutually orthogonal conformal vectors of \(V_L\) such that \(\mu(A) = \sum_{i=1}^g \mu_i e^i\). Since

\[
\widehat{\text{Tr}}_M(o(1), A) = \frac{1}{\prod_{i=1}^g \eta(\mu_i)} \sum_{\beta \in L+\kappa} e^{\pi i \sum_{j=1}^g \mu_j(\beta, \beta')}^2,
\]

it follows that

\[
\gamma_M(A) = \sum_{\beta \in L+\kappa} e^{\pi i \sum_{j=1}^g \mu_j(\beta, \beta')}^2.
\]
Moreover, because $\pi i \sum_{j=1}^{g} \mu_j \langle \beta, b_j \rangle^2$ is an eigenvalue of $o(\pi i \sum_{j=1}^{g} \mu_j b_j (-1)b_j (-1)1)$ for $\epsilon^\beta$, it is equal to an eigenvalue of $o(\sum_{j=1}^{g} \tau_{jk} a_j (-1)a_j)$ for $\epsilon^\beta$, that is,

$$\pi i \sum_{j=1}^{g} \mu_j \langle \beta, b_j \rangle^2 = \pi i \sum_{j=1}^{g} \sum_{h=1}^{g} \tau_{jk} \langle a^j, \beta \rangle \langle a^h, \beta \rangle.$$ 

Therefore $\gamma_M(A)$ is an ordinary Siegel theta series of $L + \kappa$. Explicitly, we have

$$\gamma_M(A) = \sum_{\beta \in L + \kappa} e^{\pi i \sum_{j=1}^{g} \sum_{h=1}^{g} \tau_{jk} \langle a^j, \beta \rangle \langle a^h, \beta \rangle} = \sum_{\beta \in L + \kappa} e^{\pi \tilde{\beta} A \tilde{\beta}}, \quad (5.2)$$

where $\tilde{\beta} = (\langle a^1, \beta \rangle, \ldots, \langle a^g, \beta \rangle) \in \mathbb{R}^g$. Since $\text{Im}(A)$ is positive definite, there is a number $\epsilon(A) > 0$ such that $\langle \beta \rangle \geq \epsilon(A) \langle \beta, \beta \rangle$ for all $\beta \in L + \kappa$. It follows that

$$|\gamma_M(A)| \leq \sum_{\beta \in L + \kappa} |e^{-\pi \epsilon(A) \langle \beta, \beta \rangle}| < \infty.$$ 

This implies $\gamma_M(A)$ is an analytic function for any symmetric matrix $A \in H_g$. Furthermore, $(5.2)$ is well-defined for any $A \in H_g$, and so $\gamma_M(A)$ is an analytic function on $H_g$.

**Remark 8** (i) Although we have been treating the cases where the rank $g$ of a lattice coincides with the genus of the Siegel upper half-space, by viewing $H_h \otimes I_{g/h} \subseteq H_g$ for $h | g$, we may treat a Siegel upper half-space of genus $h < g$.

(ii) Even if $V$ has a set $\{W^1, \ldots, W^r\}$ of simple $V$-modules, as Huang has proved in [5], $\sum_{i=1}^{r} \bar{\text{Tr}}_{W_i \otimes W^i} (v : \tau)$ is invariant for an inversion $\tau \mapsto -\frac{1}{\tau}$, where $W^i = \oplus_{p \in \mathbb{C}} \text{Hom}(W_p^i, \mathbb{C})$ denotes a restricted dual of $W^i$. Therefore, $\sum_{i=1}^{r} \bar{\text{Tr}}_{W^i \otimes W^i} (o(1) : A)$ is invariant for an inversion $A \mapsto -A^{-1}$ by viewing $H_g \otimes I_2 \subseteq H_{2g}$.

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