SPANNING TREES WITH AT MOST 4 LEAVES IN $K_{1,5}$–FREE GRAPHS

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Abstract. A graph $G$ is said to be $K_{1,5}$-free graph if it contains no $K_{1,5}$ as an induced subgraph. Let $\sigma_5(G)$ denote the minimum degree sum of five independent vertices of a graph $G$. In this article, we will prove that the connected $K_{1,5}$-free graph $G$ has a spanning tree with at most 4 leaves if $\sigma_5(G) \geq |G| - 1$. We also show that the bound $|G| - 1$ is sharp. Beside that, a related result also is introduced.

1. Introduction

In this article, we always consider simple graphs, which have neither loops nor multiple edges. For a graph $G$, let $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of $G$, respectively. We write $|G|$ for the order of $G$ (i.e., $|G| = |V(G)|$). For a vertex $v$ of $G$, we denote by $\deg_G(v)$ the degree of $v$ in $G$. Let $\sigma_k(G)$ be the minimum degree sum of $k$ independent vertices in $G$. Let $T$ be a tree, a vertex of degree one and a vertex of degree at least three is called a leaf and a branch vertex, respectively. Many researchers have investigated the degree sum conditions for the existence of a spanning tree with bounded number of leaves or branch vertices (see the survey article [8] for more details).

On the other hand, for a positive integer $p$, a graph $G$ is said to be $K_{1,p}$–free graph if it contains no $K_{1,p}$ as an induced subgraph. Many results on the degree sum conditions for the existence of a spanning tree with bounded number of leaves or branch vertices are known even if we restrict ourselves to the $K_{1,3}$–free graphs (also called the claw-free graphs) and $K_{1,4}$–free graphs are studied. We list here for some of them.

Theorem 1.1 ([3, L. Gargano et al.]). Let $k$ be a non-negative integer and let $G$ be a connected claw-free graph. If $\sigma_{k+3}(G) \geq |G| - k - 2$, then $G$ has a spanning tree with at most $k$ branch vertices.

Theorem 1.2 ([4, M. Kano et al.]). Let $k$ be a non-negative integer and let $G$ be a connected claw-free graph. If $\sigma_{k+3}(G) \geq |G| - k - 2$, then $G$ has a spanning tree with at most $k + 2$ leaves.

Theorem 1.3 ([7, H. Matsuda, K. Ozeki and T. Yamashita]). Let $G$ be a connected claw-free graph. If $\sigma_5(G) \geq |G| - 2$, then $G$ has a spanning tree with at most one branch vertex.

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Theorem 1.4 ([1, X. Chen, M. Li and M. Xu]). Let $G$ be a $m-$connected claw-free graph. If $\sigma_{m+3}(G) \geq |G| - m$, then $G$ has a spanning tree with at most 3 leaves.

Theorem 1.5 ([5, A. Kyaw]). Every connected $K_{1,4}-$ free graph with $\sigma_4(G) \geq |G| - 1$ contains a spanning with at most 3 leaves.

Theorem 1.6 ([6, A. Kyaw]). Let $G$ be a connected $K_{1,4}$-free graph.

1. If $\sigma_3(G) \geq |G|$, then $G$ has a hamiltonian path.
2. If $\sigma_{k+1}(G) \geq |G| - \frac{k}{2}$ for an integer $k \geq 3$, then $G$ has a spanning with at most $k$ leaves.

Theorem 1.7 ([9, C. Yuan, C. GuanTao and H. ZhiQuan]). Let $G$ be a $m-$connected $K_{1,4}$-free graph with $m \geq 2$. If $\sigma_{m+3}(G) \geq |G| + 2m - 2$ then $G$ contains a spanning with at most 3 leaves.

The main purposes of this article are to give sufficient conditions for a connected $K_{1,5}-$free graph to have a spanning tree with few leaves or few branch vertices. In particular, our first main result is the following.

Theorem 1.8. Let $G$ be a connected $K_{1,5}-$free graph. If $\sigma_5(G) \geq |G| - 1$ then $G$ contains a spanning tree with at most 4 leaves.

In the case the degree of $G$ is bounded by 3, we have a slightly stronger result as the following.

Theorem 1.9. Let $G$ be a connected graph with $\Delta(G) \leq 3$. If $\sigma_5(G) \geq |G| - 2$ then $G$ has a spanning tree with at most 4 leaves.

Moreover, it is easy to see that if a tree has at most $k$ leaves then $T$ has at most $k - 2$ branch vertices. We thus have a condition to show that a connected $K_{1,5}$-free graph has a spanning tree with few branch vertices.

Corollary 1.10. Let $G$ be a connected $K_{1,5}$-free graph. If $\sigma_5(G) \geq |G| - 1$ then $G$ contains a spanning tree with at most 2 branch vertices.

Before proving the main theorems, we first give an example to show that the condition in Theorem 1.8 is sharp.

Let $m \geq 1$ be an integer, and let $D_1, D_2, D_3, C_1, C_2$ be disjoint copies of $K_m$. Let $z, t$ be vertices not contained in $D_1 \cup D_2 \cup D_3 \cup C_1 \cup C_2$. Join $z, t$ together and to all vertices of $D_i (1 \leq i \leq 3), C_j (1 \leq j \leq 2)$ by edges, respectively. Let $G$ denote the resulting graph. Then $G$ is a connected $K_{1,5}$-free graph. We may see that $\sigma_5(G) = 5m = |G| - 2$ but each spanning tree of $G$ has at least 5 leaves. Therefore, the condition of Theorem 1.8 is sharp.
2. Proof of the main theorems

Lemma 2.1. Suppose that $G$ does not have a spanning tree with at most 4 leaves. Let $T$ be a maximal tree of $G$ with 5 leaves. Then, there does not exist a tree $T'$ in $G$ such that it has at most 4 leaves and $V(T') = V(T)$.

Proof. Assume that there exists a tree $T'$ with at most 4 leaves such that $V(T') = V(T)$. Since $G$ has no a spanning tree with at most 4 leaves, $T'$ is not a spanning tree of $G$. Then, there exists a vertex $w \in V(G) - V(T')$. Let $Q$ be a shortest path joining $w$ with $T'$.

If $T' + Q$ has 5 leaves then it contradicts to the maximality of $T$. Otherwise, we repeat the above process till to get a tree $T''$ with 5 leaves such that $|V(T'')| > |V(T)|$. This is a contradiction. \qed

Proof of Theorem 1.8.
Let $G$ be a connected $K_{1,5}$-free graph with $\sigma_5(G) \geq |G| - 1$.
Assume that $G$ has no any spanning tree with at most 4 leaves. So we may choose a maximal tree $T$ which has exactly 5 leaves. Denote by $U = \{u_1, u_2, u_3, u_4, u_5\}$ the set of leaves of $T$.

For a subset $X$ in $V(G)$, set $N(X) = \{x \in V(G) | xy \in E(G) \text{ for some } y \in X\}$. By the maximality of $T$ then we have $N(U) \subset V(T)$.

Now we will prove Theorem 1.8 by giving contradictions in three steps.

Step 1. If there exists a maximal tree $T$ which has exactly two branch vertices $s, t$ such that $\deg_T(s) = 3$, $\deg_T(t) = 2$.

For an integer $k \geq 1$, we denote $N_k(X) = \{x \in V(G) | |N(x) \cap X| = k\}$. Let $B_i$ be a vertex set of components of $T - \{s, t\}$ such that $U \cap B_i = \{u_i\}$ for $1 \leq i \leq 5$ and the only vertex of $N_T(\{s, t\}) \cap B_i$ is denoted $v_i$. For $u, v \in V(T)$, denote by $P_T[u, v]$ the unique path in $T$ connecting $u$ and $v$. We assign an orientation in $P_T[u, v]$ from $u$ to $v$. For each $x \in P_T[u, v]$, its successor $x^+$ and the predecessor $x^-$ are defined, if they exist. Without loss of generality, we may assume $B_i \cap N_T(s) \neq \emptyset (1 \leq i \leq 3), B_j \cap N_T(t) \neq \emptyset (4 \leq j \leq 5)$.

For this case, we choose the maximal tree $T$ with two branch vertices such that:

(C1) Distance between $s$ and $t$ is as small as possible.

(C2) $\sum_{i=1}^{3} |B_i|$ is as large as possible subject to (C1).

Base on the claim 1 in [5], we have the following claim.

Claim 2.2. For all $1 \leq i \neq j \leq 5$, if $x \in B_i \cap N(u_j)$ then $x \neq u_i, x \neq v_i, x^- \notin N(U - \{u_j\})$.

Proof. If $x = u_i$ then we consider the tree

$$T' = \begin{cases} T + u_i u_j - v_i s & \text{if } i \in \{1; 2; 3\}, \\ T + u_i u_j - v_i t & \text{if } i \in \{4; 5\}. \end{cases}$$
The resulting tree $T'$ has 4 leaves and $V(T') = V(T)$. This gives a contradiction with Lemma 2.1.

If $x = v_i$ then we consider the tree $$T' = \begin{cases} T + v_iu_j - v_is & \text{if } i \in \{1; 2; 3\}, \\ T + v_iu_j - v_it & \text{if } i \in \{4; 5\}. \end{cases}$$

Then, the resulting tree $T'$ has 4 leaves and $V(T') = V(T)$, a contradiction with Lemma 2.1.

If $x^- \in N(U - \{u_j\})$ then there exists $k \neq j$ such that $x^-u_k \in E(G)$. Set $$T' = \begin{cases} T + u_jx + x^-u_k - xx^- - v_is & \text{if } i \in \{1; 2; 3\}, \\ T + u_jx + x^-u_k - xx^- - v_it & \text{if } i \in \{4; 5\}. \end{cases}$$

Then, $T'$ has 4 leaves and $V(T') = V(T)$. This contradicts to Lemma 2.1. Hence Claim 2.4 is completed. \qed

By Claim 2.4, $U$ is the independent set. Combining with the fact $G$ is $K_{1,5}$-free, we get $N_5(U) = \emptyset$.

Set $P = V(P_T[s, t] - \{s, t\})$.

**Claim 2.3**. $N(u_4) \cap P = N(u_5) \cap P = \emptyset$.

**Proof**. Because of the same role of $u_4$ and $u_5$, we just need to prove $N(u_4) \cap P = \emptyset$. Indeed, suppose that $N(u_4) \cap P \neq \emptyset$. Let $y \in N(u_4) \cap P$. Consider the tree $$T' = T + yu_4 - v_4t.$$ We have $d_{T'}(s, y) < d_T(s, t)$, a contradiction with the condition (C1). So $N(u_4) \cap P = \emptyset$. Claim 2.3 is proved. \qed

**Claim 2.4**. If $P \neq \emptyset$, for all $y \in P$ then we have $|N(y) \cap \{u_1; u_2; u_3\}| \leq 1$.

**Proof**. Otherwise, suppose that there exists $y \in P$ such that $|N(y) \cap \{u_1; u_2; u_3\}| \geq 2$. Then, there are two distinct vertices $u_i, u_j$ (i, j $\in \{1; 2; 3\}$) satisfying $yu_i \in E(G), yu_j \in E(G)$. We consider the tree $$T' = T + yu_i + yu_j - sv_i - sv_j.$$ We have $d_{T'}(y, t) < d_T(s, t)$. This contradicts to the choice $T$.

So $|N(y) \cap \{u_1; u_2; u_3\}| \leq 1$. Claim 2.4 is completed. \qed

**Claim 2.5**. If $P \neq \emptyset$, then we have $s^+ \notin N(U)$.

**Proof**. Since Claim 2.3 we have $s^+ \notin (N(u_4) \cup N(u_5))$.

Now, suppose that there exists $i \in \{1, 2, 3\}$ such that $s^+u_i \in E(G)$. Consider the tree $$T' = T + s^+u_i - ss^+.$$
Then the resulting tree $T'$ has 4 leaves and $V(T') = V(T)$. This contradicts to Lemma 2.1. Hence, $s^+ \notin N(U)$. This completes Claim 2.5.

**Claim 2.6.** If $\sum_{i=1}^{5} |N(u_i) \cap \{t\}| \geq 3$ then $P \neq \emptyset$.

**Proof.** Suppose that $\sum_{i=1}^{5} |N(u_i) \cap \{t\}| \geq 3$. Then, there exists $i \in \{1, 2, 3\}$ such that $u_it \in E(G)$. If $P = \emptyset$ then $st \in E(G)$. We consider the tree $T' = T + u_it - st,$

then $T'$ has 4 leaves and $V(T') = V(T)$. This is a contradiction with Lemma 2.1. Hence, $st \notin E(G)$. Therefore Claim 2.6 is proved.

**Claim 2.7.** $|N(u_j) \cap \{s\}| = 0$ for all $4 \leq j \leq 5$.

**Proof.** Suppose that $u_js \in E(G), (4 \leq j \leq 5)$. If $P = \emptyset$ we consider the tree $T' = T + u_js - st,$

then the resulting tree $T'$ has 4 leaves. This is a contradiction with Lemma 2.1. If $P \neq \emptyset$ then $s^+ \in P$. Since Claim 2.2, Claim 2.5 and $G$ is a $K_{1,5}$-free graph then $v_is^+ \in E(G)$ for some $i \in \{1, 2, 3\}$ or $v_iv_k \in E(G)$ for some $1 \leq i \neq k \leq 3$. We consider the tree $T' = \begin{cases} T + u_js - ss^+ - sv_i + v_is^+ & \text{if } v_is^+ \in E(G), \\ T + u_js + v_iv_k - sv_i - v_jt & \text{if } v_iv_k \in E(G). \end{cases}$

Then the resulting tree $T'$ gives a contradiction with Lemma 2.1 for first case and a contradiction with the condition (C1) for last case. Claim 2.7 is proved.

**Claim 2.8.** $N_4(U) = \emptyset$.

**Proof.** Suppose that there exists a vertex $z \in N_4(U)$. Set $N(z) \cap U = \{u_{j_1}, u_{j_2}, u_{j_3}, u_{j_4}\}$. By Claim 2.3 we have $z \notin P$. So it remains following three cases.

*Case 1. $z \in B_i$ for some $i \in \{1, 2, 3, 4, 5\}$.*

Since Claim 2.2 we have $z^- \notin N(U)$. Then, $G$ has a $K_{1,5}$ subgraph with vertices $z, z^-, u_{j_1}, u_{j_2}, u_{j_3}, u_{j_4}$, a contradiction with the assumption of Theorem 1.8.

*Case 2. $z = s$.*

If $P \neq \emptyset$ then using Claim 2.3 we get $s^+ \notin N(U)$. Then, $G$ has a $K_{1,5}$ subgraph with vertices $s, s^+, u_{j_1}, u_{j_2}, u_{j_3}, u_{j_4}$. This gives a contradiction.

If $P = \emptyset$ then $st \in E(G)$. Since $s \in N_4(U)$ there exists $i \in \{4, 5\}$ such that $su_i \in E(G)$. Consider the tree $T' = T + su_i - st$ then $T'$ has 4 leaves and $V(T') = V(T)$. It is a contradiction with Lemma 2.1.

*Case 3. $z = t$.*
Since \( t \in N_4(U) \) and Claim 2.6, we have \( P \neq \emptyset \) and \( |N\{t\} \cap \{u_1,u_2,u_3\}| \geq 2 \). On the other hand, for each \( i,j \in \{1;2;3\} \), \( i \neq j \) such that \( tu_j \in E(G) \) and \( t-u_i \in E(G) \), we consider a new tree \( T' = T + tu_j + t-u_i - vs - tt^- \) then \( T' \) has 4 leaves and \( V(T') = V(T) \). This implies a contradiction with Lemma 2.1. Hence, \( t-u_i \notin E(G) \) for all \( i \in \{1,2,3\} \). Therefore, combining with Claim 2.3 the graph \( G \) has a \( K_{1,5} \) subgraph with vertices \( t, t^- , u_{j_1}, u_{j_2}, u_{j_3}, u_{j_4} \). This implies a contradiction.

Hence, we get \( N_4(U) = \emptyset \). Claim 2.8 is proved.

Claim 2.9. \( (N_3(U) - N(u_i)) \cap B_i = \emptyset \) for all \( 1 \leq i \leq 5 \).

Proof. Suppose that there exists a vertex \( z \in N_3(U) \cap B_i \) for some \( i \). Set \( N(z) \cap U = \{u_{j_1}, u_{j_2}, u_{j_3}\}(i \notin \{j_1,j_2,j_3\}) \). By Claim 2.2 we have \( z \neq u_i, z \neq v_i \) and \( z^- , z^+ \notin N(U-u_i) \).

Then, \( G \) has a \( K_{1,5} \) subgraph with vertices \( z, z^- , z^+, u_{j_1}, u_{j_2}, u_{j_3} \). This gives a contradiction. Claim 2.9 is proved.

By the condition (C2) we have the following claim.

Claim 2.10. \( N(u_j) \cap B_i = \emptyset \) for all \( 1 \leq j \leq 3, 4 \leq i \leq 5 \). In particular, \( N_3(U) \cap N(u_i) \cap B_i = \emptyset \) for all \( 4 \leq i \leq 5 \).

Claim 2.11. \( |N_3(U) \cap N(u_i) \cap B_i| \leq 1 \) for all \( 1 \leq i \leq 3 \).

Proof. For convenience, we fix \( i = 1 \). Suppose that there exist two vertices \( x,y \in N_3(U) \cap N(u_1) \cap B_1 \). Without loss of generality, we may assume that \( y \in P_7[x, u_1] \). By Claim 2.2, we have \( x^+ \neq y \). Assume that \( u_{j_1} \in N(x)(j_1 \neq 1), u_{i_1} \in N(y)(i_1 \neq 1,j_1) \). If \( x^- x^+ \in E(G) \), we consider a new tree

\[
T' = T + xu_1 + x^- x^+ + xu_{j_1} + yu_{i_1} - xx^- - xx^+ - yy^+ - sv_1
\]

then the resulting tree has 4 leaves. This contradicts to Lemma 2.1. Otherwise, since \( G \) is \( K_{1,5} \)-free graph then \( u_1 x^+ \in E(G) \). Now we consider the tree

\[
T' = T + u_1 y - yy^- + u_1 x^+ - xx^+ + u_{j_1} x - sv_1
\]

then the resulting tree has 4 leaves. This contradicts to Lemma 2.1. Claim 2.11 is proved.

Claim 2.12. If \( u_iv_i \in E(G) \) then \( |N_3(U) \cap N(u_i) \cap B_i| = 0 \) for all \( 1 \leq i \leq 5 \).

Proof. Suppose that there exists a vertex \( x \in N_3(U) \cap N(u_i) \cap B_i \) and \( u_j x \in E(G) \) with \( i \neq j \). Set the tree \( T' = T + u_j x + u_iv_i - xx^- - sv_i \) then the resulting tree has 4 leaves. This contradicts to Lemma 2.1. Claim 2.12 is proved.

Claim 2.13. If \( u_is \in E(G) \) then \( |N_3(U) \cap N(u_i) \cap B_i| = 0 \) for all \( 1 \leq i \leq 3 \).
PROOF. For convenience, we fix \( i = 1 \). Suppose that \( su_1 \in E(G) \) and there exists a vertex \( x \in N_3(U) \cap N(u_1) \cap B_1 \).

**Case 1.** If \( v_1 v_2 \in E(G) \), we consider the tree \( T' = T + su_1 + v_1 v_2 - sv_1 - sv_2 \) then the resulting tree has 4 leaves. This contradicts to Lemma 2.1. So \( v_1 v_2 \notin E(G) \). By the same arguments we also give \( su_1 \notin E(G) \).

**Case 2.** If \( v_2 v_3 \in E(G) \) then

**Subcase 2a.** If \( u_2 x \in E(G) \) or \( u_3 x \in E(G) \) we consider the tree \( T' = T + u_2 x - sv_2 - sv_3 \) or \( T' = T + u_3 x - sv_2 - sv_3 \) respectively, then the resulting tree has 4 leaves. This is a contradiction with Lemma 2.1.

**Subcase 2b.** If \( u_4 x \in E(G) \) and \( u_5 x \in E(G) \) we consider the tree \( T' = T - sv_2 - tt^- + su_1 + u_4 x + v_2 v_3 - xx^- \), then the resulting tree has 4 leaves if \( P = \emptyset \), a contradiction with Lemma 2.1. Otherwise, this contradicts with the condition (C1).

So \( v_2 v_3 \notin E(G) \).

**Case 3.** If \( s^+ v_1 \in E(G) \) we consider the tree \( T' = T + s^+ v_1 + su_1 - sv_1 - ss^+ \) then the resulting tree has 4 leaves. This contradicts to Lemma 2.1. So \( s^+ v_1 \notin E(G) \).

**Case 4.** If \( s^+ v_2 \in E(G) \) then

**Subcase 4a.** If \( u_4 x \in E(G) \) or \( u_5 x \in E(G) \) we consider the tree \( T' = T + s^+ v_2 + u_4 x - ss^+ - sv_2 \) or \( T' = T + s^+ v_2 + u_5 x - ss^+ - sv_2 \) respectively, then the resulting tree has 4 leaves. This is a contradiction with Lemma 2.1.

**Subcase 4b.** If \( u_4 x \notin E(G) \) and \( u_5 x \notin E(G) \) then \( u_2 x \in E(G) \) and \( u_3 x \in E(G) \) we consider the tree \( T' = T - ss^+ - sv_2 - xx^- + su_1 + xu_2 + xu_3 + v_2 s^+ \), then the resulting tree has 4 leaves. This is a contradiction with Lemma 2.1.

So \( s^+ v_2 \notin E(G) \). By the same arguments we also have \( s^+ v_3 \notin E(G) \).

**Summery,** we have \( u_j v_j \notin E(G)(1 \leq i \neq j \leq 3) \) and \( s^+ v_i \notin E(G)(1 \leq i \leq 3) \). Now, since \( G \) is \( K_{1.5} \)-free graph then \( u_1 v_1 \in E(G) \). This contradicts to Claim 2.12.

Claim 2.13 is proved. \( \square \)

Since Claim 2.2[2.13] for \( 1 \leq i \leq 5 \), \( \{u_i\}, N(u_i) \cap B_i, (N(U - \{u_i\}))^- \cap B_i \) and \( (N_2(U) - N(u_i)) \cap B_i \) are pair-wise disjoint subsets of \( B_i \), where \( (N(U - \{u_i\}))^- \cap B_i = \{x^-: x \in N(U - \{u_i\}) \cap B_i\} \) and \( N_4(U) = N_3(U) = (N_2(U) - N(u_i)) \cap B_i = \emptyset \).

For \( 1 \leq i \leq 3 \), we have

\[
|B_i| \geq 1 + |N(u_i) \cap B_i| + |(N(U - \{u_i\}))^- \cap B_i| + |(N_2(U) - N(u_i)) \cap B_i| \\
- |N_3(U) \cap N(u_i) \cap B_i| \\
= 1 + |N(u_i) \cap B_i| + |N(U - \{u_i\}) \cap B_i| + |(N_2(U) - N(u_i)) \cap B_i| \\
- |N_3(U) \cap N(u_i) \cap B_i|
\]
\[ \geq 1 + \sum_{j=1}^{5} |N(u_j) \cap B_i| - |N_3(U) \cap N(u_i) \cap B_i| \]
\[ \geq \sum_{j=1}^{5} |N(u_j) \cap B_i| + |N(u_i) \cap \{s\}|. \]

Hence,
\begin{equation}
\sum_{i=1}^{3} |B_i| \geq 5 \sum_{i=1}^{5} \sum_{j=1}^{5} |N(u_j) \cap B_i| + 3 \sum_{i=1}^{3} |N(u_i) \cap \{s\}|. \tag{1}
\end{equation}

For \(4 \leq i \leq 5\), we also have
\[ |B_i| \geq 1 + |N(u_i) \cap B_i| + |(N(U - \{u_i\}))^- \cap B_i| + |(N_2(U) - N(u_i)) \cap B_i| \]
\[ = 1 + |N(u_i) \cap B_i| + |N(U - \{u_i\}) \cap B_i| + |(N_2(U) - N(u_i)) \cap B_i| \]
\[ \geq 1 + \sum_{j=1}^{5} |N(u_j) \cap B_i|. \]

Hence,
\[ \sum_{i=4}^{5} |B_i| \geq 2 + \sum_{i=4}^{5} \sum_{j=1}^{5} |N(u_j) \cap B_i| \]
\[ \Rightarrow \sum_{i=4}^{5} |B_i| \geq 2 + \sum_{i=4}^{5} \sum_{j=1}^{5} |N(u_j) \cap B_i| + \sum_{i=4}^{5} |N(u_i) \cap \{s\}|. \tag{2} \]

Set \(d_i = |N(u_i) \cap P|\), for \(1 \leq i \leq 3\). By Claim 2.4, \(N(u_1) \cap P, N(u_2) \cap P, N(u_3) \cap P\) are distinct. Moreover, since \(N(u_4) \cap P = N(u_5) \cap P = \emptyset\) we have
\[ |N(U) \cap P| = d_1 + d_2 + d_3. \]

Now, if \(\sum_{i=1}^{5} |N(u_i) \cap \{t\}| \leq 2\) then we have
\[ |P_T[s, t]| \geq 2 + d_1 + d_2 + d_3 \]
\[ \geq \sum_{i=1}^{5} |N(u_i) \cap \{t\}| + |N(U) \cap P|. \]

If \(\sum_{i=1}^{5} |N(u_i) \cap \{t\}| = 3\) then since Claim 2.6 we have \(P \neq \emptyset\). Combining with Claim 2.5 we get \(s^+ \notin N(U)\). Hence,
\[ |P_T[s, t]| \geq 3 + d_1 + d_2 + d_3 = \sum_{i=1}^{5} |N(u_i) \cap \{t\}| + |N(U) \cap P|. \]
So we get

\[(3) \quad |P_T[s,t]| \geq \sum_{i=1}^{5} |N(u_i) \cap \{t\}| + |N(U) \cap P|.
\]

By (1), (2) and (3), we have

\[
|T| \geq \sum_{i=1}^{5} |B_i| + |P_T[s,t]|
\geq 2 + \sum_{i=1}^{5} \sum_{j=1}^{5} |N(u_j) \cap B_i| + \sum_{i=1}^{5} |N(u_i) \cap \{s,t\}| + |N(U) \cap P|
\geq 2 + \sum_{i=1}^{5} \sum_{j=1}^{5} |N(u_j) \cap B_i| + |N(U) \cap P_T[s,t]|
= 2 + \deg(U).
\]

This implies that

\[|G| \geq |T| \geq 2 + \deg(U).
\]

Hence,

\[\sigma_5(G) \leq \deg(U) \leq |G| - 2.
\]

This contradicts to the assumption of Theorem 1.8.

**Step 2.** The maximal tree \(T\) has a branch vertex \(r\) with \(\deg_T(r) = 5\). Setting \(N_T(r) = \{v_1, v_2, v_3, v_4, v_5\}\). Because \(G\) is \(K_{1,5}\)-free, there exist two distinct vertices \(v_i, v_j (1 \leq i \neq j \leq 5)\) such that \(v_i v_j \in E(G)\).

If \(v_i\) is a leaf of \(T\) then we consider the tree \(T' = T + v_i v_j - rv_j\). The resulting tree has 4 leaves and \(V(T') = V(T)\). This is a contradiction with Lemma 2.1. By the same argument we also get a contradiction if \(v_j\) is a leaf of \(T\). Otherwise, we consider the tree \(T' = T + v_i v_j - rv_j\) then \(T'\) has exactly two branch vertices. This also gives a contradiction by the same arguments as in Step 1.

**Step 3.** The maximal tree \(T\) has three branch vertices \(s, w, t\) such that \(w \in P_T[s,t]\).

Now we recall the same notations as in Step 1. Let \(B_i\) be a vertex sets of components of \(T - \{s, w, t\}\) such that \(U \cap B_i = \{u_i\}\) for \(1 \leq i \leq 5\) and the only vertex of \(N_T\{s, w, t\} \cap B_i\) is denoted by \(v_i\). For each \(x \in B_i\), the vertex that precedes \(x\) on \(P_T[s, x]\) or \(P_T[w, x]\) or \(P_T[t, x]\) is denoted by \(x^-\). Without loss of generality, we may assume \(B_i \cap N_T(s) \neq \emptyset (1 \leq i \leq 2), B_j \cap N_T(t) \neq \emptyset (3 \leq j \leq 4)\) and \(B_5 \cap N_T(w) \neq \emptyset\).

For this case, we choose \(T\) such that:

(C3) Distance between \(s\) and \(t\) is as small as possible.

(C4) \(\sum_{i=1}^{2} |B_i|\) is as large as possible subject to (C3).

By the same arguments as in Step 1 we also have the following claim.
Claim 2.14. For all $1 \leq i \neq j \leq 5$, if $x \in B_i \cap N(u_j)$ then $x \neq u_i, x \neq v_i, x \notin N(U - \{u_j\})$.

Set $P_1 = V(P_T[s, w] - \{s, w\}), P_2 = V(P_T[w, t] - \{w, t\})$ and $P = P_1 \cup P_2$.

Claim 2.15. $N(u_i) \cap P = \emptyset$ for all $1 \leq i \leq 4$.

Proof. If $u_i$ is adjacent to $x \in P$ for some $1 \leq i \leq 4$, we consider the tree

$$T_1 = \begin{cases} T + u_i x - v_i s & \text{if } 1 \leq i \leq 2, \\ T + u_i x - v_i t & \text{if } 3 \leq i \leq 4. \end{cases}$$

This implies a contradiction with the condition (C3). Then, this completes Claim 2.15. □

Claim 2.16. $N(u_i) \cap \{w\} = \emptyset$ for all $1 \leq i \leq 4$.

Proof. If $u_i$ is adjacent to $w$ for some $1 \leq i \leq 4$, we consider the tree

$$T_1 = \begin{cases} T + u_i w - v_i s & \text{if } 1 \leq i \leq 2, \\ T + u_i w - v_i t & \text{if } 3 \leq i \leq 4. \end{cases}$$

Then the resulting tree has exactly two branch vertices. Using Step 1 we get a contradiction. Claim 2.16 is proved. □

Claim 2.17. $N(u_i) \cap \{t\} = \emptyset$ for all $1 \leq i \leq 2$ and $N(u_i) \cap \{s\} = \emptyset$ for all $3 \leq i \leq 4$.

Proof. If $u_i$ is adjacent to $t$ for some $1 \leq i \leq 2$, we consider the tree $T_1 = T + u_i t - v_i s$. Then the resulting tree has exactly two branch vertices. Using Step 1 we get a contradiction. So $N(u_i) \cap \{t\} = \emptyset$ for all $1 \leq i \leq 2$. Repeating the same arguments we get $N(u_i) \cap \{s\} = \emptyset$ for all $3 \leq i \leq 4$. Claim 2.17 is proved. □

Claim 2.18. $N(u_5) \cap \{s, t\} = \emptyset$.

Proof. Assume that $N(u_5) \cap \{s, t\} \neq \emptyset$. Without loss of generality we may assume $u_5$ is adjacent to $t$ in $G$. We consider the tree $T_1 = T + u_5 t - v_5 w$. Then the resulting tree has exactly two branch vertices. Using Step 1 we get a contradiction. Claim 2.18 is proved. □

Using Claim 2.14-2.18 we have

$$\sum_{i=1}^{5} |N(u_i) \cap \{w\}| \leq 1, \sum_{i=1}^{5} |N(u_i) \cap \{s\}| \leq 2, \sum_{i=1}^{5} |N(u_i) \cap \{t\}| \leq 2.$$

Then $\sum_{i=1}^{5} |N(u_i) \cap \{s, w, t\}| \leq 5$.

On the other hand, since Claim 2.15 we have $|N(U) \cap P| \leq |P|$. So we get

$$|P_T[s, t]| = 3 + |P| \geq 3 + |N(U) \cap P|$$

(4) \[\geq \sum_{i=1}^{5} |N(u_i) \cap \{s, w, t\}| + |N(U) \cap P| - 2 = \sum_{i=1}^{5} |N(u_i) \cap P_T[s, t]| - 2.\]
Claim 2.19. $N_2(U - u_5) \cap B_5 = \emptyset$.

Proof. Since the condition (C4), $N(u_i) \cap B_5 = \emptyset$ for all $1 \leq i \leq 2$. So if there exists a vertex $x \in N_2(U - u_5) \cap B_5$ then $u_3x, u_4x \in E(G)$. We consider the tree $T_1 = T + u_3x + u_4x - v_5w - v_3t$. Then the resulting tree has exactly two branch vertices. Using Step 1 we get a contradiction. Claim 2.19 is proved. \qed

Claim 2.20. $N_2(U - u_i) \cap B_i = \emptyset$ for all $1 \leq i \leq 4$.

Proof. Assume that there exists a vertex $x \in N(U - u_i) \cap B_i$ and $xu_j, xu_k \in E(G)$ for some $i \notin \{j, k\}$.

If $\{j, k\} = \{1, 2\}$ or $\{j, k\} = \{3, 4\}$ we consider the tree

$$T_1 = \begin{cases} T + xu_j + xu_k - v_js - w^-w & \text{if } \{j, k\} = \{1, 2\}, \\ T + xu_j + xu_k - v_jt - w^+w & \text{if } \{j, k\} = \{3, 4\}, \end{cases}$$

where $w^-, w^+ \in P_T[s, t]$. Then the resulting tree has exactly two branch vertices. Using Step 1 we get a contradiction.

Otherwise, remove the edges in $T$ joining $v_j, v_k$ to $s$ (or $t$, or $w$) and add the edges $xu_j, xu_k$. Denote the resulting tree of $G$ by $T_1$. Then $T_1$ has exactly two branch vertices. Using Step 1 we get a contradiction.

Claim 2.20 is proved. \qed

By Claim 2.14 Claim 2.19 and 2.20 we have

$$|B_i| \geq 1 + \sum_{j=1}^{5} |N(u_j) \cap B_i|, 1 \leq i \leq 5. \quad (5)$$

Hence, combining (1) and (5) we get

$$|T| = \sum_{i=1}^{5} |B_i| + |P_T[s, t]| \geq 3 + \sum_{i=1}^{5} \sum_{j=1}^{5} |N(u_j) \cap B_i| + \sum_{i=1}^{5} |N(u_i) \cap P_T[s, t]| = 3 + \deg(U) \geq 1 + |G|.$$ 

This implies a contradiction. This completes step 3.

Finally, Theorem 1.8 is proved.

Proof of Theorem 1.9.

Assume $G$ is a connected graph such that $\Delta(G) \leq 3$ and $\sigma_5(G) \geq |G| - 2$.

Case 1. $\Delta(G) = 2$. Then $G$ is a path or a cycle. So $G$ has a spanning tree with 2 leaves.

Case 2. $\Delta(G) = 3$. Assume that $G$ has no any spanning tree with at most 4 leaves. Let $T$ be a maximal tree with 5 leaves in $G$. Since the tree $T$ has 5 leaves and $\Delta(T) \leq 3$, then
$T$ has exactly 3 branch vertices $s, w, t$ whose degrees are 3. Let $U = \{u_1, u_2, u_3, u_4, u_5\}$ be a set of leaves of $T$. By the maximality of $T$ we have $N(U) \subset V(T)$.

Let $B_i$ be vertex sets of components of $T - P_T[s, t]$ such that $U \cap B_i = u_i$ for $1 \leq i \leq 5$. Setting $C_i = B_i - \{u_i\}$ for $1 \leq i \leq 5$ and $P = V(P_T[s, t] - \{s, t, w\}) = \{y_1, y_2, \ldots, y_q\}$. Without loss of generality, we may assume that $w$ is in a path $P_T[s, t]$ in $T$ joining $s$ and $t$ and $B_i \cap N_G(s) \neq \emptyset$ $(1 \leq i \leq 2), B_j \cap N_G(t) \neq \emptyset$ $(4 \leq j \leq 5)$.

For each $y_i \in P$ we have $\deg_T(y_i) = 2$ and $\deg_G(y_i) \leq 3$, then $|N(y_i) \cap U| \leq 1$. This implies that

$$|N(U) \cap P| \leq q. \tag{6}$$

We now prove the following.

$$\sum_{i=1}^{5} |N(u_i) \cap (C_1 \cup C_2 \cup \{s\})| \leq |C_1| + |C_2| + 2. \tag{7}$$

If $C_1 = C_2 = \emptyset$ then $u_1s, u_2s \in E(G)$. Then $\sum_{i=1}^{5} |N(u_i) \cap \{s\}| = 2$ (by $\deg_G(s) \leq 3$). Therefore

$$\sum_{i=1}^{5} |N(u_i) \cap (C_1 \cup C_2 \cup \{s\})| \leq |C_1| + |C_2| + 2.$$

If $C_1 \neq \emptyset$, $C_2 \neq \emptyset$ then $u_1s, u_2s \notin E(G)$.

Since $\deg_T(s) = 3$ and $\deg_G(s) \leq 3$ we get $N(s) \cap U = \emptyset$.

Setting $C_1 = \{z_{11}; z_{12}; \ldots; z_{1k}\}$ such that $z_{11} \in N(u_1)$. We have $|N(z_{11}) \cap U| \leq 2$ and $|N(z_{1i}) \cap U| \leq 1$ for $2 \leq i \leq k$. Then

$$\sum_{i=1}^{5} |N(u_i) \cap C_1| \leq 2 + (k - 1) = k + 1 = |C_1| + 1.$$

By similar arguments, we have

$$\sum_{i=1}^{5} |N(u_i) \cap C_2| \leq |C_2| + 1.$$

So we get

$$\sum_{i=1}^{5} |N(u_i) \cap (C_1 \cup C_2 \cup \{s\})| \leq |C_1| + 1 + |C_2| + 1 = |C_1| + |C_2| + 2.$$
Hence (7) holds.

Using the same arguments, we get

\[(8) \quad \sum_{i=1}^{5} |N(u_i) \cap (C_3 \cup C_4 \cup \{t\})| \leq |C_3| + |C_4| + 2.\]

\[(9) \quad \sum_{i=1}^{5} |N(u_i) \cap (C_5 \cup \{w\})| \leq |C_5| + 1.\]

Now, combining (6), (7), (8) and (9) we have

\[
\deg(U) = \sum_{i=1}^{5} |N(u_i) \cap (C_1 \cup C_2 \cup \{s\})| + \sum_{i=1}^{5} |N(u_i) \cap (C_3 \cup C_4 \cup \{t\})| \\
+ \sum_{i=1}^{5} |N(u_i) \cap (C_5 \cup \{w\})| + \sum_{i=1}^{5} |N(u_i) \cap P| \\
\leq |C_1| + |C_2| + 2 + |C_3| + |C_4| + 2 + |C_5| + 1 + q \\
= 5 + |C_1| + 2 + |C_2| + |C_3| + |C_4| + |C_5| + q \\
\leq 5 + |T| - 8 = |T| - 3.
\]

This implies that

\[|T| - 2 \leq |G| - 2 \leq \sigma_5(G) \leq \deg(U) \leq |T| - 3.\]

This is a contradiction. Therem 1.9 is proved.

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