A NEAR-OPTIMAL RATE OF PERIODIC HOMOGENIZATION FOR CONVEX HAMILTON-JACOBI EQUATIONS

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ABSTRACT. We consider a Hamilton-Jacobi equation where the Hamiltonian is periodic in space and coercive and convex in momentum. Combining the representation formula from optimal control theory and a theorem of Alexander, originally proved in the context of first-passage percolation, we find a rate of homogenization which is within a log-factor of optimal and holds in all dimensions.

1. INTRODUCTION

Let the Hamiltonian $H: \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be continuous, $\mathbb{Z}^d$-periodic in the first variable, $x$, and coercive in the second variable, $p$. We assume that the coercivity is uniform in $x$; that is,

$$\liminf_{|p| \to \infty} \inf_{x \in \mathbb{R}^d} H(x, p) = +\infty.$$ 

Let $u_0: \mathbb{R}^d \to \mathbb{R}$ be continuous. Our goal is to study, as $\varepsilon \to 0^+$, the behavior of the unique viscosity solution $u^\varepsilon: \mathbb{R}_{\geq 0} \times \mathbb{R}^d \to \mathbb{R}$ to the initial-value problem

$$\begin{cases}
D_t u^\varepsilon(t, x) + H(x, D_x u^\varepsilon(t, x)) = 0 & \text{in } \mathbb{R}_{\geq 0} \times \mathbb{R}^d \\
u^\varepsilon(0, x) = u_0(x) & \text{in } \mathbb{R}^d.
\end{cases}$$

Lions–Papanicolaou–Varadhan [7] proved that $u^\varepsilon \to \overline{u}$ locally uniformly as $\varepsilon \to 0^+$, where $\overline{u}: \mathbb{R}_{\geq 0} \times \mathbb{R}^d \to \mathbb{R}$ is the solution to the effective problem

$$\begin{cases}
D_t \overline{u}(t, x) + \overline{H}(D_x \overline{u}(t, x)) = 0 & \text{in } \mathbb{R}_{\geq 0} \times \mathbb{R}^d \\
\overline{u}(0, x) = u_0(x) & \text{in } \mathbb{R}^d.
\end{cases}$$

Here, $\overline{H}: \mathbb{R}^d \to \mathbb{R}$ is called the effective Hamiltonian; we define $\overline{H}(p)$ as the unique constant such that the cell problem

$$H(x, p + D_x v_p) = \overline{H}(p)$$

has some $\mathbb{Z}^d$-periodic continuous viscosity solution $v_p: \mathbb{R}^d \to \mathbb{R}$, called a corrector.

Our main result is the following rate of convergence, under additional assumptions on $H$ and $u_0$.

Theorem 1. If $H$ is convex in $p$ and $u_0$ is Lipschitz, then there is a constant $C = C(H, \text{Lip}(u_0)) > 0$ such that, for all $t > 0$ and $x \in \mathbb{R}^d$,

$$|u^\varepsilon(t, x) - \overline{u}(t, x)| \leq C\varepsilon \log(C + t\varepsilon^{-1}).$$

Additionally, in the case of dimension $d = 2$, we provide a new proof of a result of Mitake–Tran–Yu.

Theorem 2 (Mitake–Tran–Yu [3]). If $d = 2$, $H$ is convex in $p$, and $u_0$ is Lipschitz, then there is a constant $C = C(H, \text{Lip}(u_0)) > 0$ such that, for all $t > 0$ and $x \in \mathbb{R}^d$,

$$|u^\varepsilon(t, x) - \overline{u}(t, x)| \leq C\varepsilon.$$

The proofs exploit the control formulation of the initial value problem (1), which reduces homogenization to a question about convergence of a subadditive function. In both the $d = 2$ and $d \geq 3$ case, results of Alexander [1] [2] apply to quantify the convergence.

Two months after we posted this article, Hung Tran and Yifeng Yu pointed out that, by replacing Step 1 in our proof of Lemma 3 with Lemma 2 of Burago [3], one obtains the optimal $O(\varepsilon)$ rate in all dimensions. In fact, this key lemma is exactly the Hobby–Rice theorem [5], proved in 1965.
2. Prior work

After Lions–Papanicolaou–Varadhan proved qualitative homogenization, there have been two main quantitative results. Under the assumptions that \( u_0 \) is Lipschitz and \( H \) is locally Lipschitz, Capuzzo-Dolcetta–Ishii \( [4] \) proved a rate of \( O(\varepsilon^{1/3}) \), using the perturbed test function method with approximate correctors. Under the additional assumption that \( H \) is convex in \( p \), Mitake–Tran–Yu \( [6] \) proved a rate of \( O(\varepsilon) \) in dimension \( d = 2 \) and a rate of \( O(\varepsilon^{1/2}) \) in dimensions \( d \geq 3 \) using weak KAM theory.

From the definition \( \mathcal{H} \) of \( \mathcal{H} \), we can heuristically hope for the expansion

\[
H^\varepsilon(t, x) \approx \mathcal{H}(t, x) + \varepsilon v_{D, t_0}(x)(\varepsilon^{-1}x),
\]

which suggests a rate of \( O(\varepsilon) \). However, the correctors are not unique, \( u \) is not \( C^1 \) but only Lipschitz, and a continuous selection \( p \mapsto v_p \) of correctors (let alone a Lipschitz selection) does not exist in general (see section 5 of \( [6] \) for an example). The assumptions on the initial data and the Hamiltonian help by giving additional structure to the problem, in the form of the control formulation.

3. Subadditive convergence

We begin by presenting a result of Alexander. In this section, we let \( \Omega \subseteq \mathbb{R}^N \) denote an open convex cone. First, we make a few definitions.

**Definition 1.** A function \( f : \Omega \cap \mathbb{Z}^N \to \mathbb{R}_{\geq 0} \) has approximate geodesics if there is a constant \( K > 0 \) such that, for every \( x \in \Omega \cap \mathbb{Z}^N \), there are \( x_0, x_1, \ldots, x_n \in \Omega \cap \mathbb{Z}^N \) with \( x_0 = 0, x_n = x, x_{i+1} - x_i \in \Omega, |x_{i+1} - x_i| \leq K \), and

\[
|f(x_{i+1} - x_i) - f(x_{i+1} - x_i) - f(x_i - x_i)| \leq K
\]

for all \( i \leq j \leq k \).

**Definition 2.** A function \( f : \Omega \cap \mathbb{Z}^N \to \mathbb{R}_{\geq 0} \) is subadditive if \( f(x + y) \leq f(x) + f(y) \) for all \( x, y \in \Omega \cap \mathbb{Z}^N \).

**Definition 3.** A function \( f : \Omega \cap \mathbb{Z}^N \to \mathbb{R}_{\geq 0} \) has linear growth if there is a constant \( K \geq 1 \) such that \( K^{-1}|x| \leq f(x) \leq K|x| + K \) for all \( x \in \Omega \cap \mathbb{Z}^N \).

**Theorem 3** (Alexander \( [3] \)). If \( f : \Omega \cap \mathbb{Z}^N \to \mathbb{R}_{\geq 0} \) is subadditive, has linear growth, and has approximate geodesics, then there is a constant \( C > 0 \) such that, for all \( x \in \Omega \cap \mathbb{Z}^N \),

\[
|f(x) - \lim_{n \to \infty} n^{-1}f(nx)| \leq C \log(C + |x|).
\]

*Proof.* Without loss of generality, we assume \( K \geq 2 \). Define \( \overline{f} : \Omega \cap \mathbb{Q}^N \to \mathbb{R} \) by

\[
\overline{f}(x) := \lim_{n \to \infty} n^{-1}f([nx]),
\]

where \([\cdot]\) denotes coordinate-wise rounding to integers. Then \( \overline{f} \) is also subadditive with linear growth. From the scaling, it is immediate that \( t\overline{f}(x) = \overline{f}(tx) \) for all \( t \geq 0 \). From subadditivity of \( f \), we see that \( \overline{f} \leq f \).

For the rest of the argument, we let \( C > 1 > c > 0 \) be constants which depend only on \( K \) and \( N \) and may differ from line to line.

For each \( x \in \Omega \cap \mathbb{Q}^N \), define \( \overline{f}_x \) to be a supporting affine functional of \( \overline{f} \) at \( x \), chosen consistently so that \( \overline{f}_x(x) = \overline{f}_x(x) \) for all \( t > 0 \). We think of \( \overline{f}_x(v) \) as the amount of progress that a step \( v \) makes in the direction \( x \). Given \( x \in \Omega \cap \mathbb{Q}^N \), define the set of “good” increments

\[
Q_x = \{ v \in \Omega \cap \mathbb{Z}^N \mid f(v) - 5K^2 \leq \overline{f}_x(v) \leq \overline{f}(x) \}.
\]

We think of \( f(v) - \overline{f}_x(v) \) as the amount of inefficiency in the increment \( v \) on a path toward \( x \), so a good increment is one which has inefficiency at most \( 5K^2 \). The second part of the inequality means that good increments don’t “overshoot” in the direction of \( x \), which implies (from linear growth) that good increments have length at most \( C|x| \).

**Step 1.** We show that if \( x \in \Omega \cap \mathbb{Q}^N \) with \( |x| \geq C \), then there is \( \alpha \in [c, 1] \) such that \( \alpha x \) lies in the convex hull of \( Q_x \). Let \( n \in \mathbb{N} \) be large enough so that

\[
|n^{-1}f(nx) - \overline{f}(x)| \leq 1.
\]
Let $x_0, x_1, \ldots, x_m$ be an approximate geodesic for $nx$. We iteratively define a subsequence $y_k = x_{j_k}$ by letting $j_0 = 0$ and, as long as $j_k < m$, we define $j_{k+1} \in [j_k + 1, \ldots, m]$ to be maximal such that $y_{k+1} - y_k \in Q_x$. By linear growth of $f$ and $\overline{f}$, we have
\[
\overline{f}_x(x_{j_k+1} - x_{j_k} - 5K^2 \leq K^2 + K - 5K^2 \leq f(x_{j_k+1} - x_{j_k}) \leq K^2 + K \leq K - C \leq \overline{f}(x),
\]
as long as $C$ was chosen large enough, so $j_k + 1$ is admissible and therefore the subsequence exists, and we let $p \in \mathbb{N}$ be the index where $j_p = m$. If $k$ is such that $j_k+1 < m$ and $\overline{f}_x(x_{j_k+1} - x_{j_k}) > \overline{f}(x)$, then the choice of $x \geq C$, linear growth, and the approximate geodesic property yields
\[
f(y_{k+1} - y_k) \geq (K^2 - K) - (K^2 + 2K).
\]
Choosing $C$ large enough and summing over $k$ (using the approximate geodesic property again) shows that there are $O(n)$ many such $k$.

On the other hand, let $\ell$ be the number of $k$ such that $j_k+1 < m$ and
\[
f(x_{j_k+1} - x_{j_k}) - 5K^2 > \overline{f}_x(x_{j_k+1} - x_{j_k}).
\]
For such $k$, we have
\[
\overline{f}_x(y_{k+1} - y_k) \leq f(y_{k+1} - y_k) - 5K^2 + 2(K^2 + K) \leq f(y_{k+1} - y_k) - K^2.
\]
Linearity of $\overline{f}_x$ and the approximate geodesic property shows that
\[
\overline{f}_x(nx) = n\overline{f}(x) \leq f(nx) + pK - \ell K^2,
\]
so the choice of $n$ implies that $\ell K^2 - pK \leq n$ and therefore $\ell \leq \frac{1}{4}n + \frac{1}{4}p$. All together, we have shown that $p \leq Cn$. We conclude this step by noting that
\[
x = \frac{1}{n} \sum_{k=1}^{p} (y_k - y_{k-1}),
\]
and $n \leq p \leq Cn$ (the first part of the inequality follows from applying $\overline{f}_x$ to both sides of the equation).

Step 2. We show that if $x \in \Omega \cap Q^N, |x| \geq C$, $t \geq 1$, and $tx \in \mathbb{Z}^N$, then there is a $z \in \Omega \cap \mathbb{Z}^N$ with
\[
f(tx) - \overline{f}(tx) \leq f(z) - \overline{f}(z) + Ct.
\]
Using the previous step, write $tx = z + \sum_{k=1}^{m} v_k$, where $|z| \leq C|x|, \overline{f}(z) \leq \overline{f}_x(z) + C, v_k \in Q_x$, and $m \leq Ct$. Indeed, for some $\alpha \in [c, 1]$ we first write
\[
\alpha x = \sum_{i=1}^{N+1} p_i v_i,
\]
where $v_i \in Q_x$ and $p_i \geq 0, \sum_i p_i = 1$. Note that the sum only requires $N + 1$ terms by Caratheodory’s theorem on convex hulls, since we are working in $\mathbb{R}^N$. To decompose $tx$, we write
\[
tx = \sum_{i=1}^{N+1} (t\alpha^{-1} p_i - |t\alpha^{-1} p_i|) v_i + \sum_{i=1}^{N+1} |t\alpha^{-1} p_i| v_i =: z + (tx - z),
\]
so $z$ satisfies the required properties. By subadditivity of $f$ and linearity of $\overline{f}_x$,
\[
f(tx) \leq f(z) + \sum_{k=1}^{m} f(v_k)
\]
\[
\leq f(z) + \sum_{k=1}^{m} (\overline{f}_x(v_k) + 5K^2)
\]
\[
= f(z) + \overline{f}_x(tx - z) + 5CK^2 t
\]
\[
\leq f(z) + \overline{f}_x(tx - z) + Ct.
\]
Finally, we write $\overline{f}(tx) = \overline{f}_x(z) + \overline{f}_x(tx - z)$ and subtract from both sides of the inequality above to get
\[
f(tx) - \overline{f}(tx) \leq f(z) - \overline{f}(z) + Ct,
\]
where we used the fact that $\overline{f}(z) \leq \overline{f}_x(z) + C$.
Step 3. For some large $M > 1$, the previous step yields
\[
\sup_{|x| \leq M^k+1C} f(x) - \overline{f}(x) \leq \sup_{|x| \leq M^kC} f(x) - \overline{f}(x) + CM.
\]
We conclude by induction on $k$. \(\square\)

4. Homogenization via the metric problem

Let $C > 1 > c > 0$ denote constants which depend on $H$ and Lip$(u_0)$ and may differ from line to line. If $a \in \mathbb{R}$, then replacing $H$ by $H - a$ replaces solutions $u^\varepsilon$ by $u^\varepsilon + ta$, so we lose no generality in assuming that $H(x,0) \leq -1$ for all $x \in \mathbb{R}^d$. It is well-known (see, e.g. Theorem 1.34 from [1]) that the solutions $u^\varepsilon$ are Lipschitz, with bound Lip$(u^\varepsilon) \leq C$ independent of $\varepsilon$. In particular, only the values of $H(x,p)$ for $|p| \leq C$ are needed to solve the initial-value problem (1). Therefore, we lose no generality in assuming that $H(x,p) = |p|^2$ for $|p| \geq C$. We write $L(x,v)$ to denote the Lagrangian
\[
L(x,v) := \sup_{p \in \mathbb{R}^d} p \cdot v - H(x,p),
\]
which we use to define the metric
\[
m(t,x,y) := \inf_{\Gamma(t,x,y)} \int_0^t L(\gamma(s), \gamma'(s)) \, ds,
\]
where $\Gamma(t,x,y)$ is the set of paths $\gamma \in W^{1,1}([0,t]; \mathbb{R}^d)$ with $\gamma(0) = x$ and $\gamma(t) = y$. We also define the homogeneous metric
\[
\overline{m}(t,x,y) := \lim_{n \to \infty} n^{-1} m(nt,nx,ny).
\]
Given a path $\gamma \in \Gamma(t,x,y)$, we refer to $\int_0^t L(\gamma(s), \gamma'(s)) \, ds$ as the cost of $\gamma$. Noting that the assumption on $H$ implies that $L(x,v) = |v|^2$ for $|v| \geq C$, it is a standard fact that a minimizer $\gamma \in \Gamma(t,x,y)$ exists for the infimum in equation (4) which satisfies
\[
\text{Lip}(\gamma) \leq C + C t^{-1} |x - y|.
\]
The optimal control formulation of (1) is
\[
u^\varepsilon(t,y) = \inf_{|x-y| \leq Ct} u_0(x) + \varepsilon m(\varepsilon^{-1} t, \varepsilon^{-1} x, \varepsilon^{-1} y).
\]
Define the cone $\Omega := \{(t,x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^d \mid |x| \leq Ct\}$. For any $(t,y-x) \in \Omega$, we have
\[
|m(t,x,y) - m([t],|x|,[y])| \leq C,
\]
where $\lfloor \cdot \rfloor$ denotes coordinate-wise rounding to integers in a way that stays inside $\Omega$. By $\mathbb{Z}^d$-periodicity,
\[
m([t],|x|,[y]) = m([t],0,[y]-[x]).
\]
The Lipschitz estimate (5) for minimizers shows that $f(t,x) := m(t,0,x)$ has approximate geodesics. Indeed, we find an approximate geodesic by chopping up a minimizing path, and the Lipschitz estimate (5) shows that the pieces lie in $\Omega$. Since $L(x,v) \geq 1$ for all $x,v \in \mathbb{R}^d$, it is clear that $f$ has linear growth (when restricted to $\Omega$) and is subadditive and nonnegative. We finish by applying Alexander’s theorem, which yields
\[
|\varepsilon m(\varepsilon^{-1} t, \varepsilon^{-1} x, \varepsilon^{-1} y) - \overline{m}(t,x,y) \leq C \varepsilon \log(C + \varepsilon^{-1} t + \varepsilon^{-1} |x - y|),
\]
for all $(t,x,y)$ with $(t,y-x) \in \Omega$. The main result follows.
5. The case \( d = 2 \)

In this section, we assume \( d = 2 \). Rather than working with approximate geodesics as before, it will be more convenient to work directly with the minimizers for the metric \( m \). We follow the same method as Alexander \([2]\), who proved an analogous result in the context of Bernoulli percolation. We first show that \( m \) is approximately superadditive.

**Lemma 4.** If \((t, x) \in \Omega \cap \mathbb{Z}^{d+1}\), then \( 2m(t, 0, x) \leq m(2t, 2x) + C \).

**Proof.** Let \( \gamma \in \Gamma(2t, 0, 2x) \) be a minimizing path.

**Step 1.** We show that we can form a path from \( 0 \) to \( x \) as the concatenation of at most 4 non-overlapping segments of \( \gamma \). Let \( \gamma^1, \gamma^2 : [0, t] \to \mathbb{R}^d \) be the first and second halves of \( \gamma \) respectively, given by

\[
\gamma^1(s) := \gamma(s)
\]

and

\[
\gamma^2(s) := \gamma(t + s) - \gamma(t)
\]

respectively. Then \( \gamma^1(t) + \gamma^2(t) = 2x \), so \( \gamma^1(t) = x - y \) and \( \gamma^2(t) = x + y \) for some \( y \in \mathbb{R}^d \). By a linear transformation of \( \mathbb{R} \times \mathbb{R}^2 \), we lose no generality in assuming that \( x = 0 \) and \( y = (A, 0) \) for some \( A > 0 \). Consider the paths

\[
\eta^1 : s \mapsto (s, \gamma^1(s) + (A, 0))
\]

and

\[
\eta^2 : s \mapsto (s, \gamma^2(s)),
\]

so \( \eta^1(0) = (0, A, 0) \), \( \eta^1(t) = (t, 0, 0) \), \( \eta^2(0) = (0, 0, 0) \), and \( \eta^2(t) = (t, A, 0) \).

For \( k \in \{1, 2\} \) and \( c \in [0, t] \), we define the cyclic shift

\[
\eta^{k, c}(s) := \begin{cases} 
\eta^k(c + s) - \eta^k(c) & \text{if } c + s \leq t \\
\eta^k(s - (t - c)) + \eta^k(t) - \eta^k(c) & \text{otherwise.}
\end{cases}
\]

We claim that some cyclic shifts of \( \eta^1 \) and \( \eta^2 \) intersect. Indeed, we can cyclically shift either path so that it is contained in the half-space \( H^\pm := \{ x \in \mathbb{R}^3 \mid \pm x \cdot (0, 0, 1) \geq 0 \} \). The claim then follows from continuity, starting with \( \eta^1 \) in \( H^- \) and \( \eta^2 \) in \( H^+ \), and cyclically shifting them into \( H^+ \) and \( H^- \) respectively, as we will now explain in detail.

Indeed, suppose that the cyclic shifts \( \eta^{1, c_1} \) and \( \eta^{2, c_2} \) do not intersect for any \( c_1, c_2 \in [0, t] \). Then form the map \( \varphi^{c_1, c_2} : [0, t] \to S^1 \), where \( S^1 \) is the unit circle (identified in \( \mathbb{C} = \mathbb{R}^2 \) for concreteness) by

\[
\varphi^{c_1, c_2}(s) := P \left( \frac{\eta^{1, c_1}(s) - \eta^{2, c_2}(s)}{\|\eta^{1, c_1}(s) - \eta^{2, c_2}(s)\|} \right),
\]

where \( P(x, y, z) := (y, z) \) denotes projection onto the last two coordinates. Since the denominator is always nonzero, shifting \( c_1 \) and \( c_2 \) continuously produces a homotopy. As a homotopy invariant, the winding number of \( \varphi^{c_1, c_2} \) is constant with respect to \( c_1, c_2 \). Choosing

\[
c_1 := \arg \max_c \eta^{1}(c) \cdot (0, 0, 1) \quad \text{and} \quad c_2 := \arg \min_c \eta^{2}(c) \cdot (0, 0, 1)
\]

ensures \( \eta^{1, c_1}(s) \in H^- \) and \( \eta^{2, c_2}(s) \in H^+ \) for all \( s \). Since \( (\eta^{1, c_1}(s) - \eta^{2, c_2}(s)) \cdot (0, 0, 1) \leq 0 \), the map \( \varphi^{c_1, c_2} \) is homotopic to \( s \mapsto e^{-i\pi s/t} \), which has winding number \(-1/2\). On the other hand, choosing

\[
c_1 := \arg \min_c \eta^{1}(c) \cdot (0, 0, 1) \quad \text{and} \quad c_2 := \arg \max_c \eta^{2}(c) \cdot (0, 0, 1)
\]

makes \( \varphi^{c_1, c_2} \) homotopic to \( s \mapsto e^{i\pi s/t} \), which has winding number \( 1/2 \), a contradiction.

Finally, we form a new path following (a cyclic shift of) \( \eta^2 \) from \((0, 0, 0)\) to the point of intersection, and following \( \eta^1 \) the rest of the way to \((t, 0, 0)\).

To summarize, we found a path from \( 0 \) to \( x \) which is composed of a segment of a cyclic shift of \( \gamma^1 \) and a segment of a cyclic shift of \( \gamma^2 \), so the segments don’t overlap. Since we took cyclic shifts, this equates to at most 4 segments from \( \gamma \).
Step 2. Use Step 1 to find an approximate geodesic with subsequence

\[0 = (t_0, x_0), (t_1, x_1), \ldots, (t_9, x_9) = (2t, 2x)\]

for \(m\) along \(\gamma\), such that there are indices \(i_1, \ldots, i_4\) with \(\sum_{k=1}^{4} (t_{i_k} - t_{i_k-1}, x_{i_k} - x_{i_k-1}) = (t, x)\). Rearranging the indices, we find a path \(\tilde{\gamma} \in \Gamma(2t, 0, 2x)\) with \(\tilde{\gamma}(t) = x\) and cost at most \(C\) more than the cost of \(\gamma\). The conclusion follows.

The previous lemma and subadditivity show that

\[m(2t, 0, 2x) \leq 2m(t, 0, x) \leq m(2t, 0, 2x) + C\]

for all \((t, x) \in \Omega \cap \mathbb{Z}^{d+1}\). Then \(m(t, 0, x) - C \leq 2^{-k}(m(2^k t, 0, 2^k x) - C)\) for all \(k \in \mathbb{N}\) by induction, so letting \(k \to \infty\) shows

\[|m(t, x, y) - \overline{m}(t, x, y)| \leq C\]

for all \((t, x) \in \Omega \cap \mathbb{Z}^{d+1}\), so the same holds for all \((t, x) \in \Omega\) since \(m\) is Lipschitz. The result in dimension \(d = 2\) follows.

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