Hamiltonian analysis of the noncommutative Chern-Simons theory

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In this paper the hamiltonian analysis of the pure Chern-Simons theory on the noncommutative plane is performed. We use the techniques of geometric quantization to show that the classical reduced phase space of the theory has nontrivial topology and that quantization of the symplectic structure on this space is possible only if the Chern-Simons coefficient is quantized. Also we show that the physical Hilbert space of the theory is one dimensional and give an explicit expression for the vacuum wavefunction. This vacuum state is found to be related to the noncommutative Wess-Zumino-Witten action.

I. INTRODUCTION

Chern-Simons (CS) theories have been extensively investigated in various contexts since their appearance in physics literature as topological mass terms for odd dimensional gauge theories [1]. On the other hand, recent progress in understanding connection between string theory and noncommutative geometry [2] has brought much attention to studies of the field theories over the noncommutative spaces. In particular, noncommutative version of the Chern-Simons theory have been proposed in both star-product [3] and operator formalism [4]. And although this theory has been discussed by many authors by now [5], much of the previous analysis was done using conventional Lagrangian formalism and path-integral quantization techniques. It is, however, well-known that at least in the commutative case, hamiltonian approach has been much more useful in illuminating various aspects of the CS theory. Therefore it appears to be interesting to extend the canonical formalism to the noncommutative Chern-Simons theory (NCCS) as well.

Also as it was found recently in [6], the noncommutative $U(1)$ Chern-Simons theory can be quite useful in describing fractional quantum Hall effect. The argument of that paper crucially depends on whether Chern-Simons coefficient (also known as level number) is quantized or not. After some initial controversy [7] it was finally shown in [8] that noncommutative CS theory shares the same property as its commutative counterpart, i.e. the quantization of the level number. In fact the result is even stronger in the noncommutative case. CS coefficient is quantized even for the $U(1)$ theory indicating that as the noncommutativity parameter $\theta$ approaches zero, the noncommutative $U(1)$ Chern-Simons theory does not go over smoothly to the commutative one.

In the Lagrangian formalism the reason for level quantization is standard. Like in the commutative case, one can show [9] that NCCS action is not invariant under the gauge transformations belonging to nontrivial homotopy classes of the gauge group $\mathcal{G}$. For transformation with winding number $n$ the action changes by $8\pi^2\lambda n$ and the requirement of single-valuedness of the path-integral measure leads to the following quantization condition on the Chern-Simons coefficient $\lambda$

$$\lambda = \frac{n}{4\pi}, \quad n = \pm 1, \pm 2, \ldots$$

In this paper we would like to consider canonical quantization of the pure $U(N)$ Chern-Simons theory on the noncommutative plane and to show how this quantization condition appears in the Hamiltonian formalism. In Section 2 we give the classical analysis of the phase space of the theory in the framework of geometric quantization. In particular, we show that the reduced phase space is topologically nontrivial and therefore consistent quantization of the symplectic structure on this space is possible only if the level number is quantized. Section 3 describes canonical quantization of the theory in the functional Schrödinger representation. We find that consistent realization of the Gauss law constraint on the wave functionals leads to nontrivial transformation law of physical states under the gauge transformations. And like in the commutative case, this transformation law combined with single-valuedness of the path-integral measure leads to the following quantization condition on the Chern-Simons coefficient $\lambda$

$$\lambda = \frac{n}{4\pi}, \quad n = \pm 1, \pm 2, \ldots$$

In Section 4 we explain how to construct the most general functionals obeying the gauge transformation law mentioned above. It turns out that the physical Hilbert space for our choice of the flat space geometry is one-dimensional. We find an explicit expression

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for the only physical state of the theory and show that it is naturally connected to the noncommutative Wess-Zumino- Witten action. At the end of this section we describe how the results of the ordinary commutative CS theory can be recovered from our expressions in the limit of vanishing noncommutativity parameter $\theta$.

Throughout this paper we use the following conventions. The noncommutative plane is defined in the usual way \[9\] by introducing the coordinate operators $(x_1, x_2)$ which satisfy the commutation relation

\[ [x_i, x_j] = i\theta \delta_{ij} \quad i, j = 1, 2, \] \hspace{1cm} (2)

where $\theta$ is the $c$-number parameter characterizing the noncommutativity of space. The actual space can be thought of as a representation of the operator algebra generated by $(x_1, x_2)$, and the standard realization of the noncommutative plane is given by the Fock oscillator basis $|n\rangle$, $n = 0, 1, 2, \ldots$

\[ \bar{z}|n\rangle = \sqrt{2\theta \sqrt{n} + 1}|n\rangle, \]

\[ z|n\rangle = \sqrt{2\theta \sqrt{n} - 1}|n - 1\rangle, \]

\[ z|0\rangle = 0. \] \hspace{1cm} (3)

Functions on this space are the elements of the enveloping algebra of $(x_1, x_2)$, while derivatives are given by the inner automorphisms of that algebra

\[ \partial_i(\ldots) = [\hat{\partial}_i, \ldots]. \] \hspace{1cm} (5)

These automorphisms are generated by derivative operators

\[ \hat{\partial}_i = \frac{i}{\theta} \epsilon_{ij} \hat{x}_j \] \hspace{1cm} (6)

or

\[ \hat{\partial} = -\frac{\bar{z}}{2\theta} \quad \hat{\bar{\partial}} = \frac{z}{2\theta}. \] \hspace{1cm} (7)

if we use complex coordinates \[44\]. Since we are going to analyze noncommutative Chern-Simons theory in operator formulation, it is convenient to think of the functions as infinite matrices with raw and column indices labeling the oscillator states. In the case of the $U(N)$ theory (as opposed to $U(1)$) we should take the direct sum of $N$ copies of the Fock space (which is certainly isomorphic to a single space) and consider all the relevant quantities (gauge connections, covariant derivative operators, etc) as operator-valued $N \times N$ matrices or as infinite matrices with double index labeling both oscillator states and internal degrees of freedom. The space integral on the commutative plane becomes a trace in the noncommutative case

\[ \int d^2x \rightarrow 2\pi \theta \text{ Tr} \] \hspace{1cm} (8)

and for $U(N)$ theory Tr stands for the integration over the noncommutative plane as well as for the $U(N)$ group trace.

\section{II. CANONICAL FORMALISM}

The starting point of our analysis is the matrix form of the noncommutative Chern-Simons action proposed in \[4\]

\[ S_{NCCS} = 2\pi i \theta \lambda \int dt \text{ Tr}(\frac{2}{3} D_\mu D_\nu D_\rho \epsilon^{\mu\nu\rho}) + 4\pi \lambda \int dt \text{ Tr}D_0. \] \hspace{1cm} (9)

Here $D_\mu, \mu = 0, 1, 2$ - are hermitian matrix-valued covariant derivative operators, transforming adjointly under the $U(N)$ gauge transformations

\[ D_\mu \rightarrow D^U_\mu = UD_\mu U^{-1}. \] \hspace{1cm} (10)

These operators are related to ordinary noncommutative gauge connections $A_\mu, \mu = 0, 1, 2$ via

\[ D_i = -i\hat{\partial} + A_i \quad i = 1, 2 \]

\[ D_0 = -i\hat{\partial}_t + A_0. \] \hspace{1cm} (11)
For canonical quantization purposes it is most convenient to choose the time-axial gauge $A_0 = 0$. In this gauge, the action (9) is quadratic in $D_1, D_2$
\[ S_{NCCS} = 2\pi\theta\lambda \int dt \text{Tr}(\partial_t D_1 D_2 - \partial_t D_2 D_1) \] (12)
and since it is first order in time derivatives, we can immediately write the symplectic 2-form $\Omega$
\[ \Omega = 8\pi\theta\lambda \text{Tr}(\delta Z \delta \bar{Z}) \] (13)
as well as Poisson brackets
\[ \{Z_{ij}, \bar{Z}_{kl}\} = \frac{i}{8\pi\theta\lambda} \delta_{il} \delta_{jk} \] (14)
on the space of all covariant derivatives $Z$. Here we introduced the complex coordinates $Z = \frac{1}{2}(D_1 - iD_2)$, $\bar{Z} = \frac{1}{2}(D_1 + iD_2)$ on $Z$, and $\delta$ in (13) is to be interpreted as denoting exterior derivative on this space (we do not write the wedge sign for exterior products on $Z$ since it is clear from the context). With this choice of coordinates $Z$ can be considered as a Kähler manifold with $\Omega$ being the Kähler form and
\[ K = 8\pi\theta\lambda \text{Tr}(Z \bar{Z}) \] (15)
being the Kähler potential.
Moreover, the Hamiltonian is obviously zero meaning that there is no time evolution in this theory. Equivalently, we can say that equations of motion for $Z, \bar{Z}$
\[ \partial_t Z = 0 \quad \partial_t \bar{Z} = 0 \] (16)
are satisfied trivially with time-independent matrices. However, these matrices have to satisfy an extra constraint (the Gauss law constraint)
\[ \theta[Z, \bar{Z}] + \frac{1}{2} = 0 \] (17)
which appears from (9) as an equation of motion for $A_0$. Equation (17) is a matrix identity, although matrix indices are suppressed for simplicity. It is also easy to see that this equation can not be satisfied with finite matrices, meaning that we consider an essentially infinite-dimensional matrix model.
At this point we have the two alternatives. One may impose the Gauss law constraint at the classical level. This yields the reduced phase space, the set of matrices $Z, \bar{Z}$ satisfying (17) up to gauge transformations, endowed with a symplectic structure inherited from (13). This reduced phase space may be quantized using the holomorphic polarization induced by complex structure on the noncommutative plane. In this paper, however, we will follow an alternative procedure of quantizing the Poisson brackets (14) first. In this case $Z$ may be considered as the phase space of the theory before reduction by the action of the gauge symmetries. Reduction is done by requiring that Gauss law constraint acts on the Hilbert space thus selecting the subspace of physical states. As will be shown shortly, the quantization of the level number appears in this case as a consistency condition for performing such reduction.

But before we can proceed a few words about gauge transformations are in order. The $A_0 = 0$ condition does not fix the gauge completely: one can still make time-independent gauge transformations. However, we have to be careful about what the allowed gauge transformations are. We want to show now that the group $\mathcal{G}$ of the valid gauge transformations is given by those unitary matrices only, which act as identity on the oscillator basis $|n\rangle$ as $n \to \infty$.
This property is the noncommutative version of the requirement that gauge transformations go to identity at spatial infinity.

For infinitesimal gauge transformation $U = 1 + \phi + \ldots$ ($\phi$ - antihermitian matrix) we obtain from (10)
\[ \delta Z = [\phi, Z] \] \[ \delta \bar{Z} = [\phi, \bar{Z}] \] (18)
The vector field on $Z$ generating such transformation is
\[ \xi = [\phi, Z]_{ij} \frac{\delta}{\delta Z_{ij}} + [\phi, \bar{Z}]_{ij} \frac{\delta}{\delta \bar{Z}_{ij}}. \] (19)
By contracting this with Ω we get

\[ i\xi\Omega = 8\pi\theta\lambda i\text{Tr}([\phi, Z]\delta\bar{Z} - [\phi, \bar{Z}]\delta\bar{Z}) = 8\pi\theta\lambda i\text{Tr}(\phi[Z, \delta\bar{Z}] + \phi[\delta Z, \bar{Z}]) = 8\pi\theta\lambda i\text{Tr}(\phi\delta[Z, \bar{Z}]). \]  

(20)

This identifies the generator \( G \) of infinitesimal gauge transformation \( (15) \) up to an arbitrary constant as

\[ G_{ij} = 8\pi\lambda i\text{Tr}(\phi[Z, \bar{Z}]) + \text{const}\delta_{ij}. \]  

(21)

We can fix this constant by requiring that \( G(\phi) = 0 \) condition is equivalent to the Gauss law constraint \( (17) \). Therefore, the fixed expression for \( G(\phi) \) is

\[ G(\phi) = 8\pi\lambda i\text{Tr}(\phi[\theta, Z] + \frac{1}{2}). \]  

(22)

As a consistency check we can evaluate, using the canonical Poisson brackets \( (16) \), the commutator of two such transformations with infinitesimal parameters \( \phi \) and \( \rho \)

\[ [G(\phi), G(\rho)] = -iG([\phi, \rho]) - 4\pi\lambda\text{Tr}[\phi, \rho]. \]  

(23)

From this expression we see that the algebra of these generators gives the representation of the Lie algebra of the gauge group \( G \) provided

\[ \text{Tr}[\phi, \rho] = 0. \]  

(24)

This last condition is satisfied only by functions \( \phi, \rho \) which act as zero on the oscillator basis states \( |n⟩ \) for large \( n \). In this case, \( \phi \) and \( \rho \) are essentially finite matrices and we can use the cyclicity of trace to prove \( (24) \). This also validates the statement we have made above that closure of the algebra of gauge transformations restricts \( G \) only to those unitary matrices which go to identity at spatial infinity.

Once we have identified the group \( G \) we would like to analyze the reduced phase space \( Z/G \) of covariant derivatives modulo gauge transformations in more detail. We want to show now that this space has nontrivial topology, in particular, that there are closed noncontractible two-surfaces in \( Z/G \).

Let \((Z_0, \bar{Z}_0)\) denote a specific set of matrices corresponding to a point in \( Z \). Consider now the 2-surface in this space parameterized by \( \sigma \) and \( \tau \)

\[ Z = (1 - \sigma)Z_0 + \sigma UZ_0U^{-1} \]

\[ \bar{Z} = (1 - \sigma)\bar{Z}_0 + \sigma U\bar{Z}_0U^{-1} \]  

(25)

where \( 0 \leq \sigma \leq 1 \) and \( U(\tau) \) is a one-parameter family of gauge transformations with \( U(0) = U(1) \equiv 1, 0 \leq \tau \leq 1 \).

Easy to see that in the reduced phase space \( Z/G \), the boundary of this surface corresponds to the single point \((Z_0, \bar{Z}_0)\) and so we have a closed 2-surface in \( Z/G \). This closed surface is not contractible if \( U(\tau) \) traces a noncontractible path in \( G \). We can now integrate the symplectic 2-form \( \Omega \) over this surface:

\[ \delta Z = \delta\sigma(UZ_0U^{-1} - Z_0) - \sigma U[Z_0, U^{-1}\delta U]U^{-1} \]

\[ \delta\bar{Z} = \delta\sigma(U\bar{Z}_0U^{-1} - \bar{Z}_0) - \sigma U[\bar{Z}_0, U^{-1}\delta U]U^{-1} \]  

(26)

so

\[ \int \Omega = 8\pi\theta\lambda i\int \delta\sigma\sigma Tr[(\bar{Z}_0 - U^{-1}\bar{Z}_0)U^{-1}\delta U] - (Z_0 - U^{-1}Z_0U)\bar{Z}_0, U^{-1}\delta U] \]  

(27)

The last term integrates to \( \text{Tr}(\bar{Z}_0UZ_0U^{-1} - Z_0U\bar{Z}_0U^{-1}) \) at \( \tau = 0 \) and \( \tau = 1 \). Since \( U \equiv 1 \) at these points, this term should give zero. Therefore,

\[ \int \Omega = -16\pi\theta\lambda i\int \delta\sigma\sigma Tr[U^{-1}\bar{Z}_0U^{-1}\delta U] \]

\[ = -8\pi\theta\lambda i\int \text{Tr}([\bar{Z}_0, U^{-1}]U^{-1}\delta U] + 4\pi\lambda i\int_0^1 d\tau U^{-1}dU \]  

(28)
and we see that if the Gauss law constraint (17) is satisfied, then the first term disappears and we are left with

$$\int \Omega = 4\pi \lambda i \int_{0}^{1} d\tau \text{Tr} U^{-1} d\tau U.$$  \hfill (29)

only. As it was shown previously, \( \Pi_{1}(G) = \mathbb{Z} \) and

$$Q[U] = \frac{i}{2\pi} \int_{0}^{1} d\tau \text{Tr} U^{-1} d\tau U$$  \hfill (30)

is an integer equal to the winding number of the class in \( \Pi_{1}(G) \) to which \( U(\tau) \) belongs. Also from the general principles of geometric quantization we know that the reduced phase space can be quantized only if the integral of \( \Omega/2\pi \) over any closed noncontractible surface is an integer. Therefore we can write the following quantization condition

$$4\pi \lambda Q[U] = \text{integer}$$  \hfill (31)

which can be satisfied for arbitrary \( Q[U] \in \mathbb{Z} \) only if the level number is quantized as

$$4\pi \lambda = k \quad k = 0, \pm 1, \ldots$$  \hfill (32)

This is exactly the same quantization condition as was found in [8] using Lagrangian approach to the noncommutative Chern-Simons theory.

### III. SCHröDINGER REPRESENTATION

After the preliminary analysis of the classical phase space in the previous section, we would like now to explicitly quantize our theory and to show how does the Chern-Simons coefficient quantization condition (32) appear as a requirement of consistency in implementing the Gauss law (17) on physical states.

Canonical quantization of the Poisson structure (14) leads to the following quantum commutation relations

$$[Z_{ij}, \bar{Z}_{kl}] = -\frac{1}{8\pi \theta \lambda} \delta_{i\mu} \delta_{jk}.$$  \hfill (33)

In order to construct a unitary representation of this canonical algebra, we have to choose polarization on the phase space of the theory. In this paper we use a holomorphic polarization condition. The wave functionals \( \Psi[Z] \) are functionals of \( Z \) only; \( Z \) is represented trivially as multiplication by \( Z \), while \( \bar{Z} \) acts as a functional derivative with respect to \( Z 

$$\bar{Z}_{ij} \Psi[Z] = \frac{1}{8\pi \theta \lambda} \frac{\delta}{\delta Z_{ij}} \Psi[Z].$$  \hfill (34)

In this representation generator (22) of infinitesimal gauge transformations becomes

\[ G(\phi) = i[\phi, Z]_{ij} \frac{\delta}{\delta Z_{ij}} + 4\pi \lambda i \text{Tr} \phi. \]  \hfill (35)

It is easy to verify that the algebra of these generators closes

$$[G(\phi), G(\rho)] = -iG([\phi, \rho])$$  \hfill (36)

provided that we choose \( \phi \) and \( \rho \) to satisfy (24). Closure of this algebra means that there is no apparent obstruction to demanding that the Gauss law constraint (17) be met by requiring that \( G(\phi) \) annihilates physical states

$$G(\phi)\Psi[Z] = 0.$$  \hfill (37)

However, as it is known from the ordinary commutative Chern-Simons theory [10, 12] such condition does not necessarily mean gauge-invariance of the physical wave-functionals. In fact, in the commutative case consistent implementation of (37) requires that the action of the gauge group on states is realized with a 1-cocycle which leads to multivalued wave-functionals unless the level number \( \lambda \) is quantized. We want to show now that similar arguments apply in the noncommutative case as well.
For an arbitrary gauge transformation $g = e^{i\phi}$ its realization on states $\Psi[Z]$ is given by the unitary operator

$$U(g) = e^{-iG(\phi)}. \quad (38)$$

As it follows from the definition (35), we can split $G(\phi)$ as

$$G(\phi) = G_Z(\phi) + 4\pi\lambda i \text{Tr} \phi, \quad (39)$$

where

$$G_Z(\phi) = i[\phi, Z]_{ij} \frac{\delta}{\delta Z_{ij}} \quad (40)$$

is the generator of infinitesimal gauge transformations on $Z$. Therefore,

$$\Psi[Z] \rightarrow U(g)\Psi[Z] = e^{-iG(\phi)} e^{iG_Z(\phi)} \Psi[Z^g] \quad (41)$$

$Z^g = gZg^{-1}$.

The prefactor $e^{-iG(\phi)} e^{iG_Z(\phi)}$ can easily be evaluated since $[G_Z(\phi), 4\pi\lambda i \text{Tr} \phi] = 0$ and the result is just $e^{4\pi\lambda \text{Tr} \phi}$. The Gauss law constraint (37) requires that physical states $\Psi_{\text{phys}}[Z]$ be left unchanged by the action of $U(g)$, since the generator $G(\phi)$ annihilates them (37)

$$U(g)\Psi_{\text{phys}}[Z] = \Psi_{\text{phys}}[Z]. \quad (42)$$

Therefore, in the noncommutative Chern-Simons theory, functionals describing physical states are not gauge invariant; rather, according to (41), they satisfy

$$\Psi_{\text{phys}}[Z^g] = e^{-4\pi\lambda \text{Tr} \phi} \Psi_{\text{phys}}[Z]. \quad (43)$$

However, the above expression can not be met with single-valued functionals unless $\lambda$ is quantized. To see this it is useful to rewrite (43) in the following equivalent way

$$\Psi_{\text{phys}}[Z^g] = (\det g)^{-4\pi\lambda} \Psi_{\text{phys}}[Z]. \quad (44)$$

As was argued in the previous section, the valid gauge transformations are given by essentially finite (although they can be very large) unitary matrices $g$. For such matrices $\det g$ is well-defined (it is basically a complex number with unit modulus). However, $(\det g)^{-4\pi\lambda}$ is multivalued unless the exponent $-4\pi\lambda$ is an integer. Therefore, for (44) to make sense, $\lambda$ has to be quantized in units of $\frac{1}{4\pi}$

$$4\pi\lambda = k, \quad k = 0, \pm 1, \ldots \quad (45)$$

so again we obtain the same quantization condition as (32).

Finally, to have a well-defined quantum theory we need to define the inner product on the Hilbert space of physical states. The inner product of two wave functionals is given by

$$\langle \Phi | \Psi \rangle = \int [dZ, d\bar{Z}] e^{-8\pi\theta \lambda \text{Tr} \bar{Z}Z} \Phi^* \bar{Z} | \Psi[Z] \quad (46)$$

where the exponential prefactor is just the Kähler potential (15), as is standard in holomorphic quantization. This prefactor ensures that $\bar{Z}$ is the hermitian conjugate of $Z$, the quantum version of the classical relation $\bar{Z} = (Z)^\dagger$.

Also it can be easily verified that this inner product is insensitive to the gauge noninvariance of states meaning that physical expectation values do not depend on the gauge choice as they should.

### IV. PHYSICAL STATES

Given that we know the gauge transformation properties of the physical states, we now want to explicitly construct functionals that obey (43). But before we proceed to the details of this construction, we would like to briefly outline our strategy. Equation (44) tells us, that under the gauge transformation $g$ the wave functionals are multiplied by some power of $\det g$. Therefore, if $h$ is some noncommutative matrix field parametrizing covariant derivative $Z$ and transforming as

$$h \longrightarrow h^g = gh, \quad (47)$$
then \((\text{det } h)^{-4\pi \lambda}\) transforms exactly as \((43)\), and the most general functional with the correct transformation properties is given by

\[
\Psi_{\text{phys}}[Z] = (\text{det } h)^{-4\pi \lambda} \psi[Z].
\]  

(48)

Here \(\psi[Z]\) is an arbitrary gauge-invariant functional of \(Z\) only. In the case of pure Chern-Simons theory on the noncommutative plane the only such functional is \(\psi[Z] \sim 1\) so the physical Hilbert space of the theory is one-dimensional with the only vacuum state given (up to normalization) by

\[
\Psi_{VAC}[Z] = (\text{det } h)^{-4\pi \lambda}.
\]  

(49)

To make these heuristic arguments precise we need, first of all, parametrization of the covariant derivative \(Z\) in terms of the matrix field \(h\) obeying \((47)\) and then, we need to give an exact meaning to \(\text{det } h\) since, unlike the gauge transformations \(g\), \(h\) is an infinite matrix and it is not clear \textit{a priori} what \(\text{det } h\) is in this case and whether the usual property of determinants \(\text{det } h^g = (\text{det } h)(\text{det } g)\), which is crucial in deriving \((49)\), still holds in the noncommutative case.

In the commutative field theory in two space dimensions, the following parametrization of the gauge potential \(A_z\) is frequently used

\[
A_z = -\partial_z h h^{-1}.
\]  

(50)

The reason why such parametrization is possible is that in two dimensional operator \(\partial_z\) is invertible and, for any field configuration \(A_z\) we can invert \((50)\) and find corresponding \(h\) at least perturbatively as a series in powers of \(A_z\). It is also easy to verify, that under the gauge transformations \(h\) transforms as in \((47)\).

With these ideas in mind, we introduce the following parametrization of the noncommutative covariant derivative

\[
Z = \frac{i}{2\theta} \hat{h} \hat{z} h^{-1}.
\]  

(51)

One can use the relationship \((11)\) between covariant derivative \(Z\) and noncommutative gauge potential \(A_z\) as well as the definition of the noncommutative derivatives \((9)\) to see that \((51)\) is the noncommutative analogue of \((50)\). It is shown in the Appendix that \((51)\) gives the valid parametrization in the sense that it can be perturbatively solved for \(h\).

Now we have to clarify the meaning of \(\text{det } h\) in \((49)\) and to do that we start with the well-known expression

\[
\text{det } h = e^{\text{Tr} \log h}
\]  

(52)

which we use to put \(\text{det } h\) into an exponential form. Then \(\text{Tr} \log h\) can be written as

\[
\text{Tr} \log h = \int_0^1 d\tau \text{Tr} (\hat{h}^{-1} \partial_\tau \hat{h})
\]  

(53)

where \(\hat{h}(\tau)\) is the smooth extension of \(h\) onto the line segment \(\tau \in [0,1]\) with the following values at the boundary

\[
\hat{h}(0) \equiv 1 \quad \hat{h}(1) \equiv h.
\]  

(54)

One can use a power series expansion of \(\hat{h}(\tau) = e^{\chi(\tau)}\) to verify \((55)\). Under the gauge transformation \(U = e^\phi\) the value of \(\chi(\tau)\) on the \(\tau = 1\) boundary transforms as \(\chi^U(1) = \chi(1) + \phi + [\phi, \chi(1)] + \ldots\) and if \(\phi\) goes sufficiently fast to zero at infinity (as required by \((24)\)) then under the trace all the commutator terms vanish and we get

\[
e^{\text{Tr} \log h^U} = e^{\text{Tr} \log h + \text{Tr} \phi}.
\]  

(55)

This is exactly the transformation we need for the physical states. As a result we see that \((55)\) has all the required transformation properties and therefore can be used to write the vacuum state as

\[
\Psi_{VAC}[Z] = e^{-4\pi \lambda \int_0^1 d\tau (h^{-1} \partial_\tau h)}.
\]  

(56)

Although this expression can already be used as the definition of \(\Psi_{VAC}[Z]\), we would like to explore it a bit further. In particular, we want to show how it is related to the noncommutative WZW action

\[
S_{NCWZW} = S_KIN + S_{NCWZ} = 2\pi \theta \text{Tr} (h^{-1} \partial_z h)(h^{-1} \partial_z h) + 2\pi \theta \int d\tau \text{Tr} (h^{-1} \partial_\tau h [h^{-1} \partial_z h, h^{-1} \partial_z h])
\]  

(57)
Using the definition of derivatives on the noncommutative plane and formally expanding the commutators we can transform the Wess-Zumino term as

\[ S_{NCWZ} = 2\pi \theta \int \! d\tau \text{Tr} \left( h^{-1} \partial_\tau h \left[ h^{-1} \partial_{\tau} h, h^{-1} \partial_\tau h \right] \right) \]

\[ = \frac{\pi}{\theta} \int \! d\tau \text{Tr} \left( h^{-1} \partial_\tau h \left[ h^{-1} [z, h], h^{-1} [\bar{z}, h] \right] \right) \]

\[ = 2\pi \int \! d\tau \text{Tr} \left( h^{-1} \partial_\tau h \right) - \frac{\pi}{\theta} \int \! d\tau \text{Tr} \left( h^{-1} \partial_\tau \left( [h^{-1} zh, \bar{z}] + [z, h^{-1} \bar{z}h] \right) \right) \]

\[ = 2\pi \int \! d\tau \text{Tr} \left( h^{-1} \partial_\tau h \right) + \frac{\pi}{\theta} \int \! d\tau \partial_\tau \text{Tr} \left( h^{-1} zh\bar{z} - h^{-1} \bar{z}zh \right) \]

\[ = 2\pi \int \! d\tau \text{Tr} \left( h^{-1} \partial_\tau h \right) + \frac{\pi}{\theta} \text{Tr} (h^{-1} zh\bar{z} - z\bar{z}) - \frac{1}{4\theta^2} \text{Tr}(hzh^{-1} \bar{z} - z\bar{z}). \]

Similarly the kinetic term becomes

\[ S_{KIN} = 2\pi \theta \text{Tr} (h^{-1} \partial_\tau h (h^{-1} \partial_\tau h)) \]

\[ = -\frac{\pi}{\theta} \text{Tr} (h^{-1} zh - z)(h^{-1} \bar{z}h - \bar{z}) \]

\[ = \frac{\pi}{\theta} \text{Tr} (h^{-1} zh\bar{z} - z\bar{z}) + \frac{\pi}{\theta} \text{Tr} (hzh^{-1} \bar{z} - z\bar{z}). \]

Note that we did not use the cyclicity of trace while performing these transformations. Also, although each of the expressions like \( \text{Tr} z\bar{z} \) or \( \text{Tr} h^{-1} zh\bar{z} \) is divergent, their difference, as it appears in the last lines of (58) and (59), is, in fact, a well-defined finite quantity. These two simple observations in certain sense validate the formal manipulations that we have done.

Now, if we put the two terms together, we get the following identity

\[ 2\pi \int \! d\tau \text{Tr} h^{-1} \partial_\tau h = S_{NCWZW} + 2\pi i \text{Tr}(ZA) \]

which can be used to rewrite the vacuum state as

\[ \Psi_{VAC}[Z] = e^{-2\lambda S_{NCWZW} - 4\pi \lambda^2 \text{Tr}(ZA)}. \]

This expression is much more convenient for the analysis of the transformation properties of \( \Psi_{VAC}[Z] \) since now we can use the well-known Polyakov-Wiegmann identity (which still holds in the noncommutative theory)

\[ S_{NCWZW}(gh) = S_{NCWZW}(g) + S_{NCWZW}(h) + 2 \int \! \text{Tr} (g^{-1} \partial_g \partial_h h^{-1}) \]

(62)

to prove that (61) transforms properly under the unitary gauge transformations. Also it is easy to see that our parametrization of \( Z \) in terms of \( h \) is somewhat ambiguous. Really, if \( h \) is some solution of (61), then \( h f(\bar{z}) \), where \( f(\bar{z}) \) is some antiholomorphic function, gives an equivalent solution of that equation and can be used to define the vacuum state of the theory as well. \( \Psi_{VAC}[Z] \) should certainly be invariant with respect to such ambiguity in the choice of parametrization and again we can use the Polyakov-Wiegmann identity to see that this is indeed the case.

Finally the above expression for the ground state looks very similar to the well-known vacuum wave functional of the commutative Chern-Simons theory

\[ \Psi_{VAC_{comm}}[A] = e^{-2\lambda S_{WZW}(h)} \]

(63)
except for the last term in the exponential of (61). The reason for appearance of such term can be easily tracked down to our choice of the covariant derivative operators \( Z, \bar{Z} \) as the fundamental set of variables of the theory. Really, the change in the phase space parametrization from \( Z, \bar{Z} \) to \( A, \bar{A} \) is essentially the canonical shift of variables according to (14). Upon such transformation, the path-integral measure in the inner product (66) becomes

\[ \int [dZ][d\bar{Z}] e^{-8\pi \theta \lambda \text{Tr}(\bar{Z}Z)} = \int [dA][d\bar{A}] e^{-8\pi \theta \lambda \text{Tr}(\bar{A}A) + 4\pi \lambda^2 \text{Tr}(zA) - 4\pi \lambda \text{Tr}(\bar{z}A) - \frac{1}{4\theta^2} \text{Tr}(z\bar{z})} \]

(64)

and we see, that \( e^{4\pi \lambda \text{Tr}(zA)} \) can be absorbed into \( \Psi[Z] \) thus cancelling the extra term in the wave function \( e^{-4\pi \lambda \text{Tr}(\bar{A}z)} \) correspondingly is absorbed by \( \Phi[Z] \) in (61). Therefore, the canonically transformed wave functional of variable \( A \) is

\[ \Psi_{VAC}[A] = e^{-2\lambda S_{NCWZW}(h)}. \]
In the commutative limit $\theta \to 0$ the noncommutative WZW action $S_{NCWZW}(h)$ goes to the commutative one $S_{WZW}(h)$ and we trivially recover the ground state of the commutative Chern-Simons theory (63). Similarly, the only remaining term $1 - 8\pi \theta \lambda \text{Tr}(\bar{A}A)$ in the Hilbert space measure (46) becomes just $-4\lambda \int d^2 z \text{Tr}(\bar{A}A)$ in the limit of vanishing noncommutativity and again we obtain the standard expression for the inner product of states in the commutative case

$$\langle \Phi | \Psi \rangle = \int [dA][d\bar{A}] e^{-4\lambda \text{Tr}(\bar{A}A)} \Phi^* \bar{A} \Psi [A].$$ (66)

V. SUMMARY AND CONCLUSIONS

In this paper the hamiltonian analysis of the pure $U(N)$ Chern-Simons theory on the noncommutative plane was performed. It was found that quantization of the level number in the canonical formalism is a consequence of existence of the closed noncontractible surfaces in the reduced phase space of the theory. The quantization condition (32) is exactly the same as was previously obtained in [8] using Lagrangian approach. Also like its commutative counterpart, pure noncommutative CS theory turns out to be exactly solvable. We use the techniques of holomorphic quantization to construct an explicit representation of the quantum commutator algebra (33). Furthermore, it is shown that the Gauss law constraint (17), which selects the subspace of physical states of our theory, can be solved exactly. The physical Hilbert space for our choice of flat space geometry is found to be one-dimensional and we give an explicit expression for the only physical state of the theory.

Although pure Chern-Simons theory appears to be trivial, it is well-known that in the commutative case it leads to highly nontrivial results when coupled to external sources. Therefore, it appears to be interesting to include external charges into the noncommutative theory as well. In particular, one might address the question of how the presence of such charges affect the quantum holonomy of physical states. This is currently under investigation.

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APPENDIX A

In this Appendix we would like to show that parametrization (51) of the covariant derivative operator $Z$ in terms of an infinite matrix $h$ is well-defined in the sense that for any given matrix $Z$ it is possible, at least perturbatively, to find corresponding matrix $h$. It is useful to rewrite (51) in an equivalent form

$$A = -\frac{i}{2\theta} [\bar{z}, h] h^{-1},$$ (A1)

where $A$ is the noncommutative gauge potential as defined in (11). To be able to solve this equation we have to prove that operator $-\frac{i}{2\theta} [\bar{z}, \ldots]$ is invertible, i.e. that there exists a map (we call it $D(\ldots)$)

$$B \xrightarrow{D} D(B)$$ (A2)

which associates to any given noncommutative function $B$ another function $D(B)$ such that

$$-\frac{i}{2\theta} [\bar{z}, D(B)] = B.$$ (A3)

This map is the noncommutative analogue of $\int dw G(z, w) \ldots$ with $G(z, w)$ being the Green’s function of the ordinary commutative derivative operator $\partial_z$. In terms of $D(\ldots)$ we can write then the solution of (A1) as

$$h = 1 + D(Ah) = 1 + D(A) + D(AD(A)) + \ldots.$$ (A4)

1 The other term $-\frac{2\lambda}{\pi} \text{Tr}(z \bar{z})$, being independent of $A$, can be absorbed into the wavefunction normalization constant.
However, the validity of this expression crucially depends on the existence of map $D(\ldots)$, so we give now the proof that $D(\ldots)$ is indeed a well-defined operation on the noncommutative plane.

Since $B$ and $D(B)$ are both infinite-dimensional matrices, we can represent them in the oscillator basis as

$$B = \sum_{i,j=0}^{\infty} B_{ij} |i\rangle \langle j|$$

$$D(B) = i\sqrt{2\theta} \sum_{i,j=0}^{\infty} C_{ij} |i\rangle \langle j|$$

With this expansion, eq.(A3) now gives the following set of recursion relations for matrix elements of $D(B)$

$$C_{i-1,j} \sqrt{i} - C_{ij} \sqrt{j+1} = B_{ij} \quad i, j = 0, 1, 2, \ldots$$

From these we find

$$B_{00} = -C_{01}$$

$$B_{01} = -C_{02}\sqrt{2}$$

$$\ldots$$

$$B_{0l} = -C_{0l+1}\sqrt{l+1},$$

which means that we can immediately find all $C_{0l}$ coefficients with $l \geq 1$. Next we consider the following set of equations

$$B_{11} = C_{01} - C_{12}\sqrt{2}$$

$$B_{12} = C_{02} - C_{13}\sqrt{3}$$

$$\ldots$$

$$B_{1l} = C_{0l} - C_{1l+1}\sqrt{l+1}$$

and obtain $C_{1l}, l \geq 2$. Now we can proceed iteratively and see that given that we have already found $C_{i-1l}, l \geq i$ for some $i$, we can always find $C_{il}, l \geq i + 1$ from

$$B_{ii} = C_{i-1i} \sqrt{i} - C_{ii+1}\sqrt{i+1}$$

$$\ldots$$

$$B_{il} = C_{i-1l} \sqrt{i} - C_{il+1}\sqrt{l+1}.$$

Therefore, all the matrix elements $C_{ij}$ of $D(B)$ with $i < j$ can be uniquely determined from the above equations.

For those $C_{ij}$ with $i \geq j$ we may consider the following set of equations

$$B_{10} = C_{00} - C_{11}$$

$$B_{21} = C_{11}\sqrt{2} - C_{22}\sqrt{2}$$

$$\ldots$$

$$B_{i+1i} = C_{ii} \sqrt{i} + 1 - C_{i+1i+1}\sqrt{i+1}.$$ 

From these equations we can find all diagonal matrix elements $C_{ii}, i \geq 0$ provided that we fix arbitrarily the value of $C_{00}$. Easy to see that this freedom in choosing $C_{00}$ translates into the following ambiguity of $D(B)$

$$C_{00} \sum_{0}^{\infty} |i\rangle \langle i| = C_{00} \mathbf{1}$$

so we can add an arbitrary constant function to $D(B)$. Similarly, from

$$B_{20} = C_{10}\sqrt{2} - C_{21}$$

$$B_{31} = C_{21}\sqrt{3} - C_{32}\sqrt{2}$$

$$\ldots$$

$$B_{i+2} = C_{i+1i} \sqrt{i+2} - C_{i+2i+1}\sqrt{i+1}$$
we can find all $C_{i+1}$, $i \geq 0$, however, solution is not unique again; we can add
\[
C_{10} \sum_{i=0}^{\infty} \sqrt{i+1} |i+1\rangle\langle i| = C_{10} \bar{z}
\] (A13)
to $D(B)$. In exactly the same way one can show that all the remaining matrix elements $C_{ij}$ with $i \geq j$ can be found from $C_{10}$ and this completes the proof of existence of the map $D(A)$. This map, however, is not unique; $D(B)$ is defined up to
\[
C_{0i} + C_{10} \bar{z} + C_{20} \bar{z}^2 + C_{30} \bar{z}^3 + \ldots
\] (A14)
with arbitrary coefficients $C_{0i}, C_{10}, \ldots$, i.e. we can add any noncommutative antiholomorphic function $f(\bar{z})$ to $D(B)$ and still satisfy $D(A)$. One can also see that solution $D(A)$ of equation (A1) is not unique as well. Really, because of the ambiguity in the definition of $D(A)$ we can write an alternative solution of (A1) as
\[
h = f(\bar{z}) + D(A h) = f(\bar{z}) + D(A f(\bar{z})) + D(A D(A f(\bar{z}))) + \ldots
\] (A15)
However, this can also be written as
\[
h = \left[ 1 + D(A f(\bar{z})) f^{-1}(\bar{z}) + \ldots \right] f(\bar{z}) = \left[ 1 + D^f(A) + D^f(A D^f(A)) + \ldots \right] f^{-1}(\bar{z})
\] (A16)
where $D^f(A) = D(\ldots f(\bar{z})) f^{-1}(\bar{z})$ satisfies $D(A)$ and expression in brackets
\[
h^f = 1 + D^f(A) + D^f(A D^f(A)) + \ldots
\] (A17)
is the solution of $D(A)$ as well. Therefore, we see that if $h$ is some solution of $D(A)$ then
\[
h^f = h f(\bar{z})
\] (A18)
is another solution of that equation. This means that our parametrization of the covariant derivative $Z$ in terms of the matrix field $h$ is defined up to right multiplication by an arbitrary antiholomorphic function only. In fact, this can be seen directly from (51) since any such function obviously commutes with the antiholomorphic coordinate operator $\bar{z}$.

[1] S. Deser, R. Jackiw and S. Templeton, Phys. Rev. Lett. 48 (1982) 975; Ann. Phys. 140 (1982) 372.
[2] A. Connes, M. R. Douglas and A. S. Schwarz, JHEP 02 (1998) 003; M. R. Douglas and C. Hull, JHEP 02 (1998) 008; N. Seiberg and E. Witten, JHEP 99 (1999) 032.
[3] A. H. Chamseddine and J. Frolich, J. Math. Phys. 35 (1994) 5195; T. Krajewski, math-phys/9810015; S. Mukhi and N. V. Suryanarayana, JHEP 0011 (2000) 006.
[4] A. P. Polychronakos, JHEP 0011 (2000) 008.
[5] G.-H. Chen and Y.-S. Wu, Nucl. Phys. B593 (2001) 562; N. Grandi and G. A. Silva, Phys. Lett. B507 (2001) 345; G. S. Lozano, E. F. Moreno and F. Schaposnik, JHEP 0102 (2001) 036; D. Bak, S. K. Kim, K.-S. Soh and J. H. Jee, Phys. Rev. D64 (2001) 025018; G. Alexanian, D. Arnaudon and M. B. Paranjape, JHEP 0311 (2003) 011; K. Kaminsky, Y. Okawa and H. Ooguri, Nucl. Phys. B633 (2003) 33.
[6] S. Bachall, L. Susskind, Int. J. Mod. Phys. B5 (1991) 2735; L. Susskind, hep-th/0101029.
[7] M. M. Sheikh-Jabbari, hep-th/0102092.
[8] V. P. Nair, A. P. Polychronakos, Phys. Rev. Lett. 87 (2001) 030403; D. Bak, K. Lee and J.-H. Park, Phys. Rev. Lett. 87 (2001) 030402.
[9] for a recent review see, for example, M. R. Douglas and N. A. Nekrasov, Rev. Mod. Phys. 73 (2001) 997.
[10] M. Bos and V. P. Nair, Int. J. Mod. Phys. A5 (1990) 959.
[11] E. Witten, Comm. Math. Phys. 121 (1989) 351.
[12] G. V. Dunne, R. Jackiw and C. A. Trugenberger, Ann. Phys. 194 (1989) 197.
[13] A. M. Polyakov, P. B. Wiegmann, Phys. Lett. B141 (1984) 223.