Transporting positive definiteness

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Abstract. This is a rough guide to the topic which might be worthy to develop further on. Expanding positive definiteness beyond its presupposed scope is intriguing due to a number of possible applications. The paper though looking at the first glance a little bit sketchy may provide a basis for further research.

A little piece of history

1. The outstanding Béla Szőkefalvi–Nagy appendix [17] concerns Hilbert space operator valued functions defined and positive definite on *-semigroups. More precisely and with a somehow neutral notation, let $\mathcal{G}$ be a *-semigroup (an involution semigroup alternatively); a map $\varphi: \mathcal{G} \to B(H)$ is said to be positive definite if

$$\sum_{i,j=1}^{n} \langle \varphi(s^*js_i)f_j, f_i \rangle \geq 0, \quad s_1, \ldots, s_n \in \mathcal{G}, \quad f_1, \ldots, f_n \in H. \quad (1)$$

It is said to satisfy the boundedness condition if for every $u \in \mathcal{G}$ there is $c(u) > 0$ such that

$$\sum_{i,j=1}^{n} \langle \varphi(s^*ju^*us_i)f_j, f_i \rangle \leq c(u) \sum_{i,j=1}^{n} \langle \varphi(s^*js_i)f_j, f_i \rangle, \quad s_1, \ldots, s_n \in \mathcal{G}, \quad f_1, \ldots, f_n \in H. \quad (2)$$

Suppose $\mathcal{G}$ is a *-semigroup with unit 1. The mapping $\varphi: \mathcal{G} \to B(H)$ is positive definite (cf. (1)) and satisfies the boundedness condition (2) if and only if there is a Hilbert space $K$, a bounded linear operator $V: H \to K$ and a *-representation $\Phi: \mathcal{G} \to B(K)$ such that

$$\varphi(s) = V^*\Phi(s)V, \quad s \in \mathcal{G}. \quad (3)$$

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1 For equivalent definitions of the boundedness condition look at [10] or [9]
The above makes immediately
\[
\langle \varphi(s)f, g \rangle_{\mathcal{H}} = \langle \Phi(s)Vf, Vg \rangle_{\mathcal{K}}, \ s \in \mathcal{G}, \ f, g \in \mathcal{H}.
\]
which is sometimes referred to as a weak dilation; the other described above is just a dilation.

If \( \varphi(1) = I \) then \( V \) is an isometric embedding of \( \mathcal{H} \) into \( \mathcal{K} \). Identifying \( \mathcal{H} \) with its image \( \mathcal{VH} \) in \( \mathcal{K} \) allows us to think of \( V^* \) in (3) as an orthogonal projection\(^2\); this is in the flavour [17].

Notice properly understood minimality forces uniqueness to hold.

2. The content of [3,4] encouraged us to proclaim in [8] the following result \(^3\)

**Theorem 1.** Suppose \( \mathcal{G} \) is a \(*\)-semigroup and \( \mathcal{I} \) is a \(*\)-ideal in it. A mapping \( \varphi: \mathcal{I} \to \mathcal{B}(\mathcal{H}) \) extends to a positive definite \( \varphi_{\text{ext}}: \mathcal{G} \to \mathcal{B}(\mathcal{H}) \) if every \( s \in \mathcal{G} \setminus \mathcal{I} \) there is \( c(s) \geq 0 \) such that
\[
\sum_{i,j=1}^{n} \langle \varphi(u_j^*u_i) f_j, f_i \rangle \geq c(s) \left\| \sum_{i=1}^{n} \varphi(u_i s) f_i \right\|^2,
\]
\[
u_1, \ldots, u_n \in \mathcal{I}, \ f_1, \ldots, f_n \in \mathcal{H}
\]
and only if (4) holds for \( s \in \mathcal{G} \).

3. Let us supply with a resume of the RKHS story (the long established reference is [1], the assuming is [16]). Given a set \( X \), a function of two variables \( K: X \times X \to \mathbb{C} \) is customarily called a kernel on \( X \). A kernel \( K \) is said to be positive definite if
\[
\sum_{i,j=1}^{N} K(x_i, x_j) \lambda_i \bar{\lambda}_j \geq 0, \ x_1, \ldots, x_N \in X, \ \lambda_1, \ldots, \lambda_N \in \mathbb{C}.
\]

Denote by \( \mathcal{H} \) a Hilbert space of complex valued functions on \( X \). We will say that the couple \( (K, \mathcal{H}) \) enjoys the reproducing property, if the sections
\[
K_x \overset{\text{def}}{=} K(\cdot, x), \quad x \in X,
\]
which have to be called kernel functions, enjoy the following two properties
\[
\mathcal{D} \overset{\text{def}}{=} \text{lin}\{K_x\}_{x \in X} \subset \mathcal{H},
\]
\[
f(x) = \langle f, K_x \rangle_{\mathcal{H}}, \quad f \in \mathcal{H}, \ x \in X.
\]
Condition (5) ensures the reproducing property (6) to be executable.

\(^2\)If someone does not want to go along with idea considering \( VV^* \) is another option.

\(^3\) Notice there is a change of fonts comparing to [8].
Formulae (5) and (6) are trivially equivalent to

\[ K(x, y) = \langle K_y, K_x \rangle_{\mathcal{H}}, \quad x, y \in X \quad (7) \]

The members of the couple \((K, \mathcal{H})\) are tied one to the other:
- \(\mathcal{H}\) can be obtained from \(K\) via the Moore–Aronszajn completion procedure of \(\mathcal{D}\), this can be made possible because due to (6) \(\mathcal{D}\) is dense in \(\mathcal{H}\);
- (6) gives us immediately continuity of the functional

\[ f(x) = \Phi_x(f) = \langle f, K_x \rangle, \quad f \in \mathcal{H}, \ x \in X \]

determining the kernel functions \(K_x\) by the Riesz representation theorem.

Now (7) turns out into the definition of \(K\).

Notice by the way (5) implies positive definiteness of \(K\) and the above makes these two counterparts equivalent which rounds the whole story up. In other words, we have

**Remark 2.** The formula (7) linking both \(K\) and \(\mathcal{H}\) may serve as a definition of the other side depending on what one is starting with.

**Focal results**

4. The introductory stuff in 1 sets our main point forth.

Suppose we are given \(k: X \rightarrow \mathbb{C}\). Denote by \(A_k\) a set of mappings \(\alpha: X \rightarrow X\) such that there is a \(c_\alpha\) for which

\[ \sum_{i, j=1}^{N} K(x_i, x_j)\lambda_i\bar{\lambda}_j \geq c_\alpha \left| \sum_{i=1}^{N} k(\alpha(x_i))\lambda_i \right|^2, \quad x_1, \ldots, x_N \in X, \ \lambda_1, \ldots, \lambda_N \in \mathbb{C}. \quad (8) \]

Notice \(k\) is a fixed parameter the whole construction depends on.

It is clear that from (8) we infer that \(K\) is a positive definite kernel on \(X\). Denote by \(\mathcal{H}_K\) the corresponding reproducing kernel Hilbert space.

Due to (7) and (8) the formula

\[ \Phi_\alpha \left( \sum_{i=1}^{N} K_{x_i} \lambda_i \right) = \sum_{i=1}^{N} k(\alpha(x_i))\lambda_i, \quad x_1, \ldots, x_N \in X, \ \lambda_1, \ldots, \lambda_N \in \mathbb{C} \quad (9) \]

defines a linear functional on \(\mathcal{H}_K\), which is continuous with the norm not exceeding \(c_\alpha^2\).

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4 Distinguish them by the name of \(\textit{actions}\) on \(X\). For the time being we do not impose any structure on \(A_k\)'s.
5. Applying the Riesz representation theorem to (9) we get existence of a function
\[ \kappa_{\alpha}^{(k)} \in \mathcal{H}, \quad \alpha \in A_k. \tag{10} \]
such that
\[ k(\alpha(x)) = \Phi_{\alpha}(K_x) = \langle K_x, \kappa_{\alpha}^{(k)} \rangle_{\mathcal{H}} = \langle \kappa_{\alpha}^{(k)}, K_x \rangle_{\mathcal{H}} = \kappa_{\alpha}^{(k)}(x), \quad x \in X, \alpha \in A_k; \]
the latter results in \(^5\)
\[ \kappa_{\alpha}^{(k)} = k \circ \alpha, \quad \alpha \in A_k \tag{11} \]
We can declare \(^6\) now
\[ K^{(k)}(\alpha, \beta) \overset{\text{def}}{=} \langle \kappa_{\beta}^{(k)}, \kappa_{\alpha}^{(k)} \rangle_{\mathcal{H}}, \quad \alpha, \beta \in A_k \tag{12} \]
getting a kernel \(K^{(k)}\) which is positive definite on \(A_k\). Denoting by \(\mathcal{H}^{(k)}\) the corresponding reproducing kernel Hilbert space, composed of complex functions on \(A_k\), we arrive at what we mean here by transporting the positive definite structure of the reproducing couple \((K, \mathcal{H})\) to \((K^{(k)}, \mathcal{H}^{(k)})\).

Summing the above up we state the following

**Theorem 3.** Given the above data satisfying in particular (8) we conclude in getting a positive definite kernel \(K^{(k)}\) defined by (12) which in turn generates the inner product in \(\mathcal{H}^{(k)}\), the corresponding reproducing kernel Hilbert space. For the functions \(\kappa_{\alpha}^{(k)}, \alpha \in A_k\) the composition formula (11) is a kind of bonus.

With \(K_{\alpha}^{(k)}, \alpha \in A_k\) standing for kernel functions of \((K^{(k)}, \mathcal{H}^{(k)})\), after employing Remark 2 and formula (12) we reach
\[ \langle K_{\alpha}^{(k)}, K_{\beta}^{(k)} \rangle_{\mathcal{H}^{(k)}} = \langle \kappa_{\alpha}^{(k)}, \kappa_{\beta}^{(k)} \rangle_{\mathcal{H}}, \quad \alpha, \beta \in A_k. \tag{13} \]

**Example 4.** Suppose \(A_k\) is a singleton \(\{\text{id}_X\}\). Then \(K^{(k)} = K, \mathcal{H}^{(k)} = \mathcal{H}\), the condition (8) reads as
\[ \sum_{i,j=1}^{N} K(x_i, x_j)\lambda_i \bar{\lambda}_j \geq \frac{1}{c} \sum_{i=1}^{N} k(x_i)\lambda_i^2, \quad x_1, \ldots, x_N \in X, \lambda_1, \ldots, \lambda_N \in \mathbb{C} \]
and Theorem 3 becomes, what we call in (the first appearance is in [11]) the **RKHS test.** It settles when a complex function on \(X\), in this case \(k\) belongs to \(\mathcal{H}\). This can be deduced directly using (10) and (11).

It may be interesting to notify

\(^5\) The notation \(\circ\) stands apparently for composition of functions.
\(^6\) The right hand side of (12) makes sense because of (10).
Corollary 5. Formula (13) makes the linear mapping
\[ \mathcal{H}^{(k)} \ni K^{(k)}_\alpha \mapsto \kappa^{(k)}_\alpha \in \mathcal{H} \]
an isometry of \( \mathcal{H}^{(k)} \) into \( \mathcal{H} \). It becomes a unitary operator if we restrict the target space to \( \text{clolin}_H\{\kappa_\alpha: \alpha \in A^{(k)}\} \).

Create \( K^{(k)} \overset{\text{def}}{=} \text{clolin}_H\{\kappa_\alpha: \alpha \in A^{(k)}\} \oplus \mathcal{H} \) (14) and suppose the kernel \( K \) is diagonal, that is
\[ K(x, y) = \mu_x \delta_{x,y}, \quad x, y \in X. \]
This implies \( \mathcal{H} = \ell^2(X, \mu) \) with the orthonormal basis \( K_x = \mu^{\frac{1}{2}} \epsilon_x, x \in X \), and the measure \( \mu \) designated as \( \mu(\{x\}) = \mu_x^{-1} \).

Remark 6. Supposing \( A_k \) and \( X \) are disjoint by \( f_1 \cup f_2 \) we mean the function (call it union) defined on \( A_k \cup X \) which agree with \( f_1 \) and \( f_2 \) on \( A_k \) and \( X \) respectively.

For kernels, the procedure goes as follows. Let \( K_1 \) be a kernel on \( A_k \) and \( K_2 \) that on \( X \), the union \( K_1 \cup K_2 \) of these kernel is defined as
\[ (K_1 \cup K_2)(x, y) = \begin{cases} 
K_1(x, y) & (x, y) \in A_k \times A_k, \\
K_2(x, y) & (x, y) \in X \times X, \\
0 & \text{otherwise}.
\end{cases} \]
If \( K_1 \) and \( K_2 \) are positive definite, then so is the kernel \( K_1 \cup K_2 \) and \( \mathcal{H}_{K_1 \cup K_2} = \mathcal{H}_{K_1} \cup \mathcal{H}_{K_2} \).

This is another way than (14) of glueing the spaces \( \text{clolin}_H\{\kappa_\alpha: \alpha \in A^{(k)}\} \) and \( \mathcal{H} \).

6. Come back to the case when the function \( \varphi \) is operator valued like in (4). We appeal here to the environment provided by Pedrick’s approach (see [6] and even better [15]; here we follow the exposition in [15].)

Duality as it appears in Functional Analysis, cf. [5, Chapter 3], [2, p. 155], [7, p. 59] declares three objects to be given: linear spaces \( E \) and \( F \), and a bilinear form
\[ B: E \times F \ni (f, g) \mapsto B(x, y) \in \mathbb{C} \]
which is separating in a sense that
\[ B(f, g) = 0 \text{ for all } g \in F \implies f = 0, \]

---

7 This is a RKHS due to the criterion involving evaluation functionals.
8 \( \{\epsilon_x\}_{x \in X} \) is the “zero-one” orthonormal basis in \( \ell^2(X) \).
9 This does not happen in the case of Theorem 1.
$B(f, g) = 0$ for all $f \in \mathcal{E} \implies g = 0$.

The spaces $\mathcal{E}$ and $\mathcal{F}$ when accompanied by $B$ are referred to as being in duality. As the spaces are complex and the Hilbert space is highlighted we prefer to impose $B$ to be Hermitian bilinear.$^{10}$

The most recognised examples are

- $\mathcal{E}$ is a linear space and $\mathcal{F} = \mathcal{E}^*$, where $\mathcal{E}^*$ is the algebraic dual of $\mathcal{E}$, that is the space of all linear functionals on $\mathcal{E}$;
- $\mathcal{E}$ is a locally convex space and $\mathcal{F} = \mathcal{E}'$, where $\mathcal{E}'$ is the topological dual of $\mathcal{E}$, that is the space of all continuous linear functionals on $\mathcal{E}$.

The bilinear form in both these cases is just the standard pairing, that is $B = \langle \cdot, - \rangle$; from now on we use the latter instead of $B$. Even more, we replace “duals” by “antiduals”, the latter comes from the previous by taking the complex conjugates of the values of functionals in question which results in $B$ supposed to be Hermitian bilinear. This is synchronised with the conclusion the classical F. Riesz representation theorem which in fact determines such kind of “duality” when establishing isomorphism between functionals (the topological dual of a Hilbert space) and the elements represented by them.

Suggestive enough $\langle \mathcal{E}, \mathcal{F} \rangle$ is a shorthand notation of the triplet $(\mathcal{E}, \mathcal{F}, \langle \cdot, - \rangle)$. Incidentally we abridge antiduality to duality hoping no confusion arises.

(Anti)duality is encoded in

$$\langle e, f \rangle_{\langle \mathcal{E}, \mathcal{F} \rangle} = \langle f, e \rangle_{\langle \mathcal{F}, \mathcal{E} \rangle}, \quad e \in \mathcal{E}, \; f \in \mathcal{F}.$$  

The first of these two itemised cases can always be directed in the second by introducing the so called $\sigma(\mathcal{E}, \mathcal{F})$ topology in $\mathcal{E}$ or reversing the role the spaces play a $\sigma(\mathcal{F}, \mathcal{E})$ topology in $\mathcal{F}$; in general they are not the only possibilities.

7. Pedrick’s device is to turn a kernel of (general or abstract) values into a scalar kernel like those in subsections 4 and 5 (this is what he calls “tilde correspondence”). Then after performing specific tasks in this friendly scalar territory the way back (that is the reciprocal of tilde correspondence, which is always possible) establishes the wanted properties. Let us scrutinise this now.

The following items are the main objects in [15, cf. p.16]

- two complex linear spaces $\mathcal{E}$ and $\mathcal{F}$ being in antiduality, the Hilbert spaces $\mathcal{H}_\mathcal{E} \subset \mathcal{E}^X$ and $\mathcal{H}_\mathcal{F} \subset \mathcal{F}^X$ which are composed of $\mathcal{E}$-valued or $\mathcal{F}$-valued resp. functions on $X$,
- a family of linear operators $K(x, y): \mathcal{F} \to \mathcal{E}$, $x, y \in X$, which may be also recollected as$^{11}$ $K: X \times X \to L(\mathcal{F}, \mathcal{E})$ and which in turn let $K$ be thought of as $L(\mathcal{F}, \mathcal{E})$-valued kernel on $X$.

$^{10}$ Instead of being linear in the second variable it is linear conjugate; call $B$ to be Hermitian bilinear and the spaces $\mathcal{E}$ and $\mathcal{F}$ being in antiduality.

$^{11}$ $L(X, \mathcal{Y})$ denotes the totality of all linear operators from $X$ into $\mathcal{Y}$, which by the way is a complex linear space; as always shorten $L(X, \mathcal{X}')$ to $L(X)$. 

\[ Springer \]
Consider $\mathcal{G} \subset \mathcal{E}^X$ and assume the couple $(K, \mathcal{G})$ enjoys the reproducing property on $X$, that is

the functions $K_{x,f} \overset{\text{def}}{=} K(\cdot, x)f$ are in $\mathcal{G}$ for any $x \in X, f \in \mathcal{F}$  \hspace{1cm} (15)

and

$$
\langle \varphi(x), f \rangle_{(\mathcal{E}, \mathcal{F})} = \langle \varphi, K_{x,f} \rangle_{\mathcal{G}}, \quad \varphi \in \mathcal{G}, x \in X, f \in \mathcal{F}.
$$

This intensionally turns out $K$ to be an $L(\mathcal{F}, \mathcal{E})$-valued positive definite kernel. Positive definiteness means here

$$
\sum_{i,j=1}^{N} \langle K(x_i, x_j) f_i, f_j \rangle_{(\mathcal{E}, \mathcal{F})} \geq 0, \quad x_1, \ldots, x_N \in X, \ f_1, \ldots, f_N \in \mathcal{F}.
$$

(17)

Notice the formula (16) is just the reproducing property as adjusted to the current circumstances.

8. An important consequence of the reproducing formula (16), when combined with (15), is

$$
\langle K(\cdot, x)f, K(-, y)f' \rangle_{\mathcal{G}} = \langle K(y, x)f, f' \rangle_{(\mathcal{E}, \mathcal{F})}
$$

which is a counterpart of (7). Now we can offer the following definitions

$$
\tilde{K}(x, f, y, f') \overset{\text{def}}{=} \langle K(x, y)f, f' \rangle_{(\mathcal{E}, \mathcal{F})}, \quad x, y \in X, f, f' \in \mathcal{F}
$$

and for $\varphi \in \mathcal{G} \subset \mathcal{E}^X$ encode

$$
\tilde{\varphi}(x, f) \overset{\text{def}}{=} \langle \varphi(x), f \rangle_{(\mathcal{E}, \mathcal{F})}, \quad (x, f) \in \tilde{X} \overset{\text{def}}{=} X \times \mathcal{F};
$$

(18)

$$
\tilde{\mathcal{G}} \overset{\text{def}}{=} \{ \tilde{\varphi} : \varphi \in \mathcal{G} \text{ and } \tilde{\varphi} \text{ defined in (18)} \}.
$$

(19)

Remark 7. The deal is now with the complex valued kernel $\tilde{K}$. Because $\tilde{K}$ is linear in the second variable and antilinear in the fourth, formula (17) ensures its positive definiteness getting now the form

$$
\sum_{i,j=1}^{N} \tilde{K}(x_i, f_i, x_j f_j) \geq 0, \quad x_1, \ldots, x_N \in X, \ f_1, \ldots, f_N \in \mathcal{F}.
$$

Therefore $\tilde{K}$ has its reproducing kernel Hilbert space $\tilde{H}$ composed of complex functions on $\tilde{X} \overset{\text{def}}{=} X \times \mathcal{F}$.

If there is any need to consider other duality than the standard topological one $(\mathcal{E}, \mathcal{E}')$ mentioned in subSection 6, it can be done without any further complication.

The main point is to check whether $\tilde{\mathcal{G}}$ is just the RKHS partner of $\tilde{K}$.

\[ \text{ Springer} \]
Theorem 8. ([15] Theorem 3) For \( \varphi \in \mathcal{G} \)

\[ \| \varphi \|_{\mathcal{G}} = \| \tilde{\varphi} \|_{\tilde{\mathcal{G}}}, \]

where \( \| \cdot \|_{\tilde{\mathcal{G}}} \) stands for the norm in \( \tilde{\mathcal{G}} \). Consequently, \( (\tilde{K}, \tilde{\mathcal{G}}) \) is a reproducing couple.

This pave the way between \( (K, \mathcal{G}) \) and \( (\tilde{K}, \tilde{\mathcal{G}}) \) endowing Pedrick’s “tilde correspondence” with the reputation of a unitary map. Notice that the inverse of Pedrick’s “tilde correspondence” can be determined by applying formula (18) due to the fact that the duality \( \langle \mathcal{E}, \mathcal{F} \rangle \) is separating.

Remark 9. Thinking of \( \mathcal{E} = \mathcal{F} \) = Hilbert space we jump into the situation considered in [17] and [9]. More general context is when a Banach space appears instead; this what some people may like.

9. An instructive example is on hand, it refers to the so called operator valued moment problem, cf. [17, Section 8] for instance. Let us consider it in the Hamburger drawing.

For a given \( a: \mathbb{N} \overset{\text{def}}{=} \{0, 1, \ldots \} \to \mathcal{L}(\mathcal{F}, \mathcal{E}) \) set

\[ K(m, n) \overset{\text{def}}{=} a_{m+n} \quad m, n \in \mathbb{N}. \quad (20) \]

Supposing the sequence \( \{a_k\}_{k \in \mathbb{N}} \) is positive definite, that is

\[ \sum_{m,n}^{\text{finite}} \langle a_{m+n}f_m, f_n \rangle_{\langle \mathcal{E}, \mathcal{F} \rangle} \geq 0, \quad \{f_k\}_{k \in \mathcal{F}}. \quad (21) \]

we get an \( \mathcal{L}(\mathcal{F}, \mathcal{E}) \)-valued kernel on \( \mathbb{N} \) which due to (21) is positive definite in the sense of (17). Pedrick’s tilde correspondence procedure makes \( \tilde{a}_{m+n} = \tilde{K}(m, n) \) a scalar valued kernel (a sequence in fact) matured to be a Hamburger moment sequence having an integral representation in the usual sense. Reversing the tilde correspondence we come from this classical stuff (after some work) to the representation

\[ a_n = \int_{\mathbb{R}} t^n M(dt), \]

where \( M \) is an \( \mathcal{L}(\mathcal{F}, \mathcal{E}) \)-valued measure (its \( \sigma \)-additivity being in the strong operator topology in \( \mathcal{F} \)) with the appropriate meaning of the integral.

10. The current developments can be perfectly illustrated by the choice of (20) to be scalar valued. This was done successfully in [12] and [13]; it is a need to mention that an attempt made some time ago is in [18].
Concluding remark

As mentioned in Abstract this is a starting point for the matters. As the reproducing kernel technique is deeply engaged in these investigations the lack of appropriate sources (except those not accessible for this or another reason like [14] or [16]) limit them for the time being. Nevertheless, we hope to be able to continue the issue in the desired direction.

References

[1] N. Aronszajn, Theory of reproducing kernels, *Trans. Amer. Math. Soc.*, 68 (1950), 337–404.
[2] S. K. Berberian, *Lectures in Functional Analysis and Operator Theory*, Springer-Verlag, New York – Heidelberg, 1974.
[3] C. H. FitzGerald and R. A. Horn, *On Quadratic and Bilinear Forms in Function Theory*, Technical Report No. 305, The John Hopkins University, June 1979.
[4] C. H. FitzGerald and R. A. Horn, On quadratic and bilinear forms in function theory, *Proc. London Math. Soc.*, 44 (1982), 554–576.
[5] J. Horváth, *Topological Vector Spaces and Distributions. Vol. I*, Addison-Wesley Publishing Co., Reading, Mass.– London –Don Mills, Ont., 1966,
[6] G. Pedrick, *Theory of Reproducing Kernels for Hilbert Spaces of Vector Valued Functions*, Thesis (Ph.D.), University of Kansas, 1958, pp. 59.
[7] G. K. Pedersen, *Analysis Now*, Springer-Verlag, New York, 1989.
[8] F. H. Szafraniec, Sur les fonctions admettant une extension de type positif, *C. R. Acad. Sci. Paris, Sér. I*, 292 (1981), 431–432.
[9] F. H. Szafraniec, Boundedness of the shift operator related to positive definite forms: an application to moment problems, *Ark. Mat.*, 19 (1981), 251–259.
[10] F. H. Szafraniec, Boundedness in dilation theory, *Spectral Theory*, Proceedings Banach Center Publications, vol. 8, September 23 - December 16, 1977, ed. W. Żelazko, PWN - Polish Scientific Publishers, Warszawa, 1982, pp.449–453.
[11] F. H. Szafraniec, Interpolation and domination by positive definite kernels, *Complex Analysis - Fifth Romanian-Finish Seminar*, Part 2, Proceedings, Bucharest (Romania), 1981, eds. C. Andrean Cazacu, N. Boboc, M. Jurchescu and I. Suciu, Lecture Notes in Math., vol. 1014, Springer, Berlin – Heidelberg, 1983, pp. 291-295.
[12] F. H. Szafraniec, On extending backwards positive definite sequences, *Numer. Algorithms*, 3 (1992), 419–425.
[13] F. H. Szafraniec, Detecting mass points of representing measures, *Numer. Algorithms*, 33 (2003), 475–483.
[14] F. H. Szafraniec, *Przestrzenie Hilberta z Jądrem Reprodukującym* (Reproducing Kernel Hilbert Spaces), Wydawnictwo Uniwersytetu Jagiellońskiego, Kraków, 2004 (in Polish).
[15] F. H. Szafraniec, Revitalising Pedrick’s approach to reproducing kernel Hilbert spaces, *Complex Anal. Oper. Theory*, **15** (2021), no. 4, Paper No. 66, 12 pp.

[16] F. H. Szafraniec, *The Reproducing Property: Spaces and Operators*, Cambridge University Press, in progress.

[17] B. Sz.-Nagy, *Prolongement des transformations de l’espace de Hilbert qui sortent de cet espace*, Appendix to F. Riesz and B. Sz.-Nagy, *Leçons d’Analyse Fonctionelle*, Akadémiai Kiadó, Budapest, 1955.

[18] F. M. Wright, On the backward extension of positive definite Hamburger moment sequences, *Proc. Amer. Math. Soc.*, **7** (1956), 413–422.

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