ON PSEUDO-RIEMANNIAN MANIFOLDS
WITH RECURRENT CONCIRCULAR CURVATURE TENSOR

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Abstract. It is proved that every concircularly recurrent manifold must be necessarily a recurrent manifold with the same recurrence form.

1. Introduction

For a pseudo-Riemannian manifold \((M, g)\), by \(T, U, V, W, X, Y, Z\) will be denoted arbitrary smooth vector fields on \(M\), and for the Riemann curvature operator \(R\) and the Riemann curvature \((0,4)\)-tensor \(R\), we assume the following conventions

\[
R(X,Y) = [\nabla X, \nabla Y] - \nabla [X,Y] = \nabla^2 X,Y - \nabla^2 Y,X,
\]
\[
R(W,X,Y,Z) = g(R(W,X)Y,Z),
\]
where \(\nabla\) indicates the covariant derivative with respect to the Levi-Civita connection, and \(\nabla^2\) is the second covariant derivative,

\[
\nabla^2 X,Y = \nabla X \nabla Y - \nabla \nabla X Y.
\]

Pseudo-Riemannian manifolds are assumed to be connected.

A pseudo-Riemannian manifold \((M, g)\) is said to be recurrent \([13, 10]\) if its Riemann curvature operator \(R\) is recurrent, that is, \(R\) is non-zero and its covariant derivative \(\nabla R\) satisfies the condition

\[
\nabla R = \lambda \otimes R
\]
for a certain 1-form \(\lambda\) (the recurrence form).

For a pseudo-Riemannian manifold \((M, g)\), the concircular curvature tensor field \(\mathcal{C}\) is defined as

\[
\mathcal{C} = R - \frac{r}{n(n-1)} \mathcal{G},
\]
where \(n = \dim M\), \(r\) is the scalar curvature and \(\mathcal{G}\) is the curvature like operator defined as

\[
\mathcal{G}(X,Y)Z = g(Y,Z)X - g(X,Z)Y.
\]

The tensor \(\mathcal{C}\) is an invariant of the concircular transformations which have many important geometric and algebraic applications; see \([13, 15, 8, 7]\), etc. For our purpose, we recall only two facts: (1) when \(\dim M = 2\), then \(\mathcal{C} = 0\) and such a manifold realizes the condition \(\nabla R = \lambda \otimes R\) with \(\lambda = \nabla (\ln |r|)\) at each point at which \(R \neq 0\); (2) when \(\dim M \geq 3\), \(\mathcal{C} = 0\) if and only if the manifold is of constant sectional curvature.

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A pseudo-Riemannian manifold \((M, g)\) is said to be concircularly recurrent if its concircular curvature tensor \(\mathcal{C}\) is recurrent, that is, \(\mathcal{C}\) is non-zero and its covariant derivative \(\nabla \mathcal{C}\) satisfies the condition

\[
\nabla \mathcal{C} = \lambda \otimes \mathcal{C}
\]

for a certain 1-form \(\lambda\). For a concircularly recurrent manifold, \(n = \dim M \geq 3\) and \(\mathcal{C} \neq 0\) at every point of \(M\).

It is obvious that a recurrent manifold is concircularly flat \((\mathcal{C} = 0)\) or concircularly recurrent with the same recurrence form. The purpose of the presented paper is to prove the following theorem:

**Theorem.** Every concircularly recurrent manifold is necessarily a recurrent manifold with the same recurrence form.

The above theorem seems to be very important since concircularly recurrent manifolds were studied by many authors, see \([2, 3, 4, 5, 6, 10, 11, 14]\), etc. In view of our theorem, many results and proofs occurred in some of the listed papers can be simplified, sometimes radically.

**Hint.** With the recurrence notion used in the present paper, we follow Y.-C. Wong \([17, 18]\); cf. also \([9]\). Due to Theorem 3.8 from \([17]\), on a connected differentiable manifold endowed with an affine connection, any recurrent tensor field has no zeros.

### 2. Proof of the theorem

For a concircularly recurrent manifold, as a consequence of (2) and (3), we claim that the Riemann curvature operator satisfies the following condition

\[
\nabla \mathcal{R} = \lambda \otimes \mathcal{R} + \mu \otimes \mathfrak{S},
\]

where the 1-form \(\lambda\) is the same as in (3) and the 1-form \(\mu\) is given by

\[
\mu = \frac{1}{n(n-1)}(dr - r\lambda).
\]

Conversely, if a pseudo-Riemannian manifold satisfies (4) with certain 1-forms \(\lambda\) and \(\mu\), then \(\mu\) must be of the form (5), and (3) must be realized. The equivalence of (3) and (4) has already been noticed in \([2]\). It is worth to remark that if \(\mu = 0\), then obviously (1) holds so that the concircular recurrence reduces to the recurrence.

Before we start with the proof, it will be useful to recall the famous Walker identity (see \([16\), Lemma 1]) stating that for any pseudo-Riemannian manifold it holds

\[
(\nabla^2_{U,V} R - \nabla^2_{V,U} R)(W, X, Y, Z) + (\nabla^2_{W,X} R - \nabla^2_{X,W} R)(Y, Z, U, V) + (\nabla^2_{Y,Z} R - \nabla^2_{Z,Y} R)(U, V, W, X) = 0.
\]

Rewrite the Walker identity in the following form, which will be more convenient for us

\[
(\mathcal{R}(U, V) R)(W, X, Y, Z) + (\mathcal{R}(W, X) R)(Y, Z, U, V) + (\mathcal{R}(Y, Z) R)(U, V, W, X) = 0.
\]

We are going to reach the assertion of the Theorem in the following three steps.
However, we would like to add that the closedness of the form \( \lambda \) (see the first step below) has already been proved in [2]. We have included it only for the completeness of the whole of the proof.

**Step 1.** _For a concircularly recurrent manifold, the recurrence form \( \lambda \) is closed._

**Proof.** Let \( C \) denotes the \((0, 4)\)-tensor related to \( \mathcal{C} \) by
\[
C(W, X, Y, Z) = g(\mathcal{C}(W, X)Y, Z).
\]
Using (3), we have \( \nabla_V C = \lambda(V)C \), and next
\[
\nabla^2_{U,V} C = ((\nabla_U \lambda)(V) + \lambda(U)\lambda(V))C.
\]
Therefore,
\[
\mathcal{R}(U, V)C = \nabla^2_{U,V} C - \nabla^2_{V,U} C = 2d\lambda(U, V)C,
\]
where we have used the formula
\[
d\lambda(U, V) = \frac{1}{2}((\nabla_U \lambda)(V) - (\nabla_V \lambda)(U)).
\]
Consequently, we obtain
\[
(\mathcal{R}(U, V)C)(W, X, Y, Z) + (\mathcal{R}(W, X)C)(Y, Z, U, V) + (\mathcal{R}(V, U)C)(Y, Z, W, X) = 2d\lambda(U, V)C(W, X, Y, Z),
\]
On the other hand, note that from (2) it follows that
\[
C = R - \frac{r}{n(n-1)} G,
\]
where \( G \) is the curvature like \((0, 4)\)-tensor related to \( \mathcal{S} \) by
\[
G(W, X, Y, Z) = g(\mathcal{S}(W, X)Y, Z).
\]
Using (5) and the identities \( \mathcal{R}(U, V)r = 0, \mathcal{R}(U, V)G = 0 \), we find
\[
(\mathcal{R}(U, V)C)(W, X, Y, Z) + (\mathcal{R}(W, X)C)(Y, Z, U, V) + (\mathcal{R}(V, U)C)(Y, Z, W, X) = 2d\lambda(U, V)C(W, X, Y, Z),
\]
Therefore, from (6), (9) and (7), we have
\[
d\lambda(U, V)C(W, X, Y, Z) + d\lambda(W, X)C(Y, Z, U, V) + d\lambda(Y, Z)C(U, V, W, X) = 0.
\]
Since \( C \neq 0 \), by applying the famous Walker lemma ([16, Lemma 2]), we obtain from the last formula
\[
d\lambda = 0,
\]
completing the proof of Step 1. \( \square \)

**Step 2.** _A concircularly recurrent manifold is semisymmetric._

**Proof.** Using (4), for the first covariant derivative of the Riemann curvature \((0, 4)\)-tensor \( R \), we find
\[
\nabla_V R = \lambda(V)R + \mu(V)G,
\]
and for the second covariant derivative,
\[
\nabla^2_{U,V} R = ((\nabla_U \lambda)(V) + \lambda(U)\lambda(V))R + ((\nabla_U \mu)(V) + \mu(U)\lambda(V))G.
\]
Hence, using also (10), we obtain
\begin{equation}
\mathcal{R}(U, V)R = \nabla^2_{U,V} R - \nabla^2_{V,U} R = 2(d\mu + \mu \wedge \lambda)(U, V)G.
\end{equation}

Applying the above identity into the Walker identity (6), we obtain
\[ (d\mu + \mu \wedge \lambda)(U, V)G(W, X, Y, Z) + (d\mu + \mu \wedge \lambda)(W, X)G(Y, Z, U, V) + (d\mu + \mu \wedge \lambda)(Y, Z)G(U, V, W, X) = 0. \]

Since \( G \neq 0 \), by applying the famous Walker lemma, we obtain
\[ d\mu + \mu \wedge \lambda = 0. \]

The last relation reduces (12) to
\begin{equation}
\mathcal{R}(U, V)R = 0,
\end{equation}
which is just the semisymmetry (cf. [1]). \(\square\)

**Step 3.** A concircularly recurrent manifold is recurrent.

**Proof.** Since \( \mathcal{R}(U, V) \) is a derivation of the tensor algebra on \( M \) (cf. e.g. [12]), we have
\[
(\mathcal{R}(U, V)R)(W, X, Y, Z) = -R(\mathcal{R}(U, V)W, X, Y, Z) - R(W, \mathcal{R}(U, V)X, Y, Z) - R(W, X, \mathcal{R}(U, V)Y, Z) - R(W, X, Y, \mathcal{R}(U, V)Z).
\]

Hence, having the semisymmetry condition (13), we obtain
\begin{equation}
R(\mathcal{R}(U, V)W, X, Y, Z) + R(W, \mathcal{R}(U, V)X, Y, Z) + R(W, X, \mathcal{R}(U, V)Y, Z) + R(W, X, Y, \mathcal{R}(U, V)Z) = 0.
\end{equation}

Now, differentiating the above equality covariantly, we get
\[
(\nabla_T R)(\mathcal{R}(U, V)W, X, Y, Z) + R(\nabla_T \mathcal{R}(U, V)W, X, Y, Z) + R(W, \nabla_T \mathcal{R}(U, V)W, X, Y, Z) + R(W, X, \nabla_T \mathcal{R}(U, V)W, X, Y, Z) \]
\[
+ (\nabla_T R)(W, X, \mathcal{R}(U, V)Y, Z) + R(W, X, \nabla_T \mathcal{R}(U, V)Y, Z) + R(W, X, Y, \nabla_T \mathcal{R}(U, V)Z) = 0.
\]

Hence, by applying (11) and (11), we find
\begin{equation}
(\lambda(T)R + \mu(T)G)(\mathcal{R}(U, V)W, X, Y, Z) + R(\lambda(T)\mathcal{R} + \mu(T)\mathcal{G})(U, V)W, X, Y, Z)
\end{equation}
\[
+ (\lambda(T)R + \mu(T)G)(W, \mathcal{R}(U, V)X, Y, Z) + R(W, \lambda(T)\mathcal{R} + \mu(T)\mathcal{G})(U, V)X, Y, Z)
\]
\[
+ (\lambda(T)R + \mu(T)G)(W, X, \mathcal{R}(U, V)Y, Z) + R(W, X, \lambda(T)\mathcal{R} + \mu(T)\mathcal{G})(U, V)Y, Z)
\]
\[
+ (\lambda(T)R + \mu(T)G)(W, X, Y, \mathcal{R}(U, V)Z) + R(W, X, Y, \lambda(T)\mathcal{R} + \mu(T)\mathcal{G})(U, V)Z) = 0.
\]

Let us assume that \( \mu \neq 0 \) at a certain point of \( M \). At this point, using (14), the equality (15) can be reduced to
\[
G(\mathcal{R}(U, V)W, X, Y, Z) + R(\mathcal{G}(U, V)W, X, Y, Z)
\]
\[
+ G(W, \mathcal{R}(U, V)X, Y, Z) + R(\mathcal{G}(U, V)X, Y, Z)
\]
\[
+ G(W, X, \mathcal{R}(U, V)Y, Z) + R(\mathcal{G}(U, V)Y, Z)
\]
\[
+ G(W, X, Y, \mathcal{R}(U, V)Z) + R(W, X, Y, \mathcal{G}(U, V)Z) = 0.
\]
When using the definitions of the tensor $G$ and the operator $\mathcal{S}$, the last equality takes the following form
\[
g(V, W)R(U, X, Y, Z) - g(U, W)R(V, X, Y, Z) \\
+ g(V, X)R(W, U, Y, Z) - g(U, X)R(W, V, Y, Z) \\
+ g(V, Y)R(W, X, U, Z) - g(U, Y)R(W, V, X, Z) \\
+ g(V, Z)R(W, X, Y, U) - g(U, Z)R(W, X, Y, V) = 0.
\]
Contracting the above with respect to the pair of arguments $V, W$ (this means that we take the trace $\text{Trace}_g\{ (V, W) \mapsto \ldots \}$, where the dots stand for the relation to be traced), we obtain
\[
(n - 2)R(U, X, Y, Z) + R(Y, X, U, Z) + R(Z, X, Y, U) \\
+ g(U, Y)S(X, Z) - g(U, Z)S(X, Y) = 0,
\]
which with the help of the first Bianchi identity becomes
\[
(n - 1)R(U, X, Y, Z) + g(U, Y)S(X, Z) - g(U, Z)S(X, Y) = 0,
\]
$S$ being the Ricci curvature tensor. Contracting the obtained relation with respect to the pair of arguments $X, Z$, we get the Einstein condition $S = (r/n)g$, which applied to (16), gives us $R = (r/(n(n - 1)))G$, or equivalently $C = 0$, contradicting our assumption. Therefore, $\mu = 0$ at every point of $M$, and consequently, the concircular recurrence reduces to the recurrence. □

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