Tail probabilities of random linear functions of regularly varying random vectors

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Abstract
We provide a new extension of Breiman’s Theorem on computing tail probabilities of a product of random variables to a multivariate setting. In particular, we give a characterization of regular variation on cones in \([0, \infty)^d\) under random linear transformations. This allows us to compute probabilities of a variety of tail events, which classical multivariate regularly varying models would report to be asymptotically negligible. We illustrate our findings with applications to risk assessment in financial systems and reinsurance markets under a bipartite network structure.

Keywords Bipartite graphs · Heavy-tails · Multivariate regular variation · Networks

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1 Introduction

In this article we study the probability of tail events for random linear functions of regularly varying random vectors. Throughout all random elements are defined on the same probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Suppose \(Z \in [0, \infty)^d =: \mathbb{R}_+^d\) is a random vector with multivariate regularly varying tail distribution on \(\mathbb{R}_+^d \setminus \{0\}\) with index \(-\alpha \leq 0\). For notational convenience, we denote multivariate regular variation on a space \(\mathbb{E}\) with index \(-\alpha \leq 0\) by \(\mathcal{MRV}(\alpha, \mathbb{E})\); hence in this case \(Z \in \mathcal{MRV}(\alpha, \mathbb{R}_+^d \setminus \{0\})\). A precise definition of this notion is given in Sect. 2. Furthermore, let \(A\) be a random matrix in \(\mathbb{R}^{q \times d}\) independent of \(Z\). For \(X = AZ\), our goal is to find \(\mathbb{P}(X \in tC)\) for large values of \(t\) and a wide variety of sets \(C \subset \mathbb{R}_+^d\).

A classical result on the tail behavior of a product of random variables, known as Breiman’s Theorem, states that given independent non-negative random variables \(Z\) and \(A\), where \(Z\) has a univariate regularly varying tail distribution with index \(-\alpha \leq 0\) and \(\mathbb{E}[A^{\alpha+\delta}] < \infty\) for some \(\delta > 0\), the tail distribution of \(X = AZ\) is also regularly varying with index \(-\alpha\), and in particular,

\[
\mathbb{P}(AZ > t) \sim \mathbb{E}[A^\alpha] \mathbb{P}(Z > t), \quad t \to \infty. \tag{1}
\]

This was stated first in Breiman (1965) for \(\alpha \in [0, 1]\) and established for all \(\alpha \geq 0\) in Cline and Samorodnitsky (1991).

A first vector-valued generalization of (1) was obtained in Basrak et al. [2002, Proposition A.1] where a random vector \(Z \in \mathcal{MRV}(\alpha, \mathbb{R}_+^d \setminus \{0\})\) for \(\alpha \geq 0\) is independent of a random matrix \(A \in \mathbb{R}^{q \times d}\) with \(0 < \mathbb{E}\|A\|^{\alpha+\delta} < \infty\) for some arbitrary norm \(\| \cdot \|\) and some \(\delta > 0\). The result states that then \(X = AZ \in \mathcal{MRV}(\alpha, \mathbb{R}_+^d \setminus \{0\})\). In a different set-up, Janssen and Drees [2016, Theorem 2.3] generalized Proposition A.1 in Basrak et al. (2002) for square random matrices \(A \in \mathbb{R}^{d \times d}\) of full rank (and certain other conditions), and computed probabilities of tail sets \(C\) contained in \((0, \infty)^d\). For \(Z \in \mathcal{MRV}(\alpha, (0, \infty)^d)\) they show that \(X = AZ \in \mathcal{MRV}(\alpha, (0, \infty)^d)\).

Consider the following example to combine ideas in the above two settings. Let \(Z = (Z_1, \ldots, Z_d)\) be comprised of iid (independent and identically distributed) Pareto random variables with \(\mathbb{P}(Z_i > z) = z^{-\alpha}\) for \(z > 1\), where \(\alpha > 0\), and let \(\mathbb{A}\) be a \(d \times d\) random matrix independent of \(Z\) satisfying the conditions of both Basrak et al. [2002, Proposition A.1] and Janssen and Drees [2016, Theorem 2.3]. Then \(X = AZ \in \mathcal{MRV}(\alpha, \mathbb{R}_+^d \setminus \{0\})\) and \(X = AZ \in \mathcal{MRV}(\alpha, (0, \infty)^d)\). Hence for sets of the form \([0, x]^c\) and \((x, \infty)\) with \(x > 0\), we compute for \(t \to \infty\),

\[
\mathbb{P}(AZ \in t[0, x]^c) \sim K_1 t^{-\alpha}, \tag{2}
\]

\[
\mathbb{P}(AZ \in t(x, \infty)) \sim K_2 t^{-\alpha x}, \tag{3}
\]

for finite values \(K_1, K_2 \geq 0\) depending on \(x\). Thus (2) allows us to compute probabilities of events described as “at least one of the components of \(X\) is large”, whereas (3) allows us to compute probabilities of events described as “all components of \(X\) are large”. Natural questions to inquire are, what if we want to compute probabilities as in (3) when the matrix \(A\) is not invertible or even when \(q \neq d\). We may also wish...
to compute the probability that “at least three of the components of \(X\) are large” or “exactly two of the components of \(X\) are large”. We can check that, although a probability computation akin to (2) may be possible, it will often render the right hand side of (2) to be zero. On the other hand, (3) will fail to answer such a question, if either \(q \neq d\) or if for the set of concern not all components are large. To the best of our knowledge, (2) and (3) are the only results that compute probabilities of extreme sets for random linear functions of regularly varying random vectors.

In our work, we provide a generalization of Breiman’s Theorem which allows us to compute such probabilities for much more general extreme sets than those in (2) and (3). However, we restrict our attention to random vectors \(Z\) and random matrices \(A\) with non-negative entries. For example, in this particular setting of \(Z\) having iid Pareto margins and \(A\) being a random matrix in \(\mathbb{R}^{d \times d}_+\), our results show that for a general class of sets \(C \subset \mathbb{R}^d_+\),

\[
P(AZ \in tC) \sim K_C t^{-i}u
\]

where the index \(i \in \{1, \ldots, d\}\) depends on the structure of the matrix \(A\) and the set \(C\), and \(K_C > 0\) is a finite constant. The number \(i\) reflects the number of components of \(Z\) to be large such that \(AZ \in tC\) for large \(t\), and finding the correct number \(i\) under a sufficiently general set-up forms the basis of this paper.

We compare our assumptions in detail with those of Basrak et al. [2002, Proposition A.1] (see Theorem 3.1 below) and Janssen and Drees [2016, Theorem 2.3] (see Theorem 3.2 below). Notably, Basrak et al. [2002, Proposition A.1], provides a limit for \(P(AZ \in tC)\) under the appropriate scaling \(P(\|Z\| > t)\), where the Borel set \(C \subset \mathbb{R}^d_+ \setminus \{0\}\) is bounded away from \(0\) satisfying natural continuity assumptions, where both \(Z\) and \(A\) have real-valued entries; however, the limit in (2) is not necessarily non-null. In particular, it might happen that the limit measure has no mass on the non-negative orthant such that the scaled probability \(P(AZ \in tC)/P(\|Z\| > t)\) converges to zero for \(C \subset \mathbb{R}^d_+ \setminus \{0\}\). On the other hand, Janssen and Drees (2016) derive a limit for the properly scaled \(P(AZ \in tC)\) under the constraint that \(C \subset (0, \infty)^d\), the positive quadrant in \(\mathbb{R}^d_+\), when \(A\) is an invertible square matrix whose inverse has all entries non-negative. In our set-up, we restrict \(A\) to have non-negative entries, but allow for \(q \times d\) matrices \(A\), which are not necessarily square or of full rank. We show that different limit measures with different scaling rates can appear depending on the type of the tail set. We also provide sufficient conditions for the limit measures to be non-null, i.e., \(K_C > 0\) in (4) and obtain multivariate regular variation on different subcones. We indicate possible extensions of our results to matrices \(A\) with real-valued entries, but providing sufficient conditions for non-null limits in such cases would require further assumptions; see Remark 8.

Our interest in the computation of probabilities of the form (4) is motivated by a wide range of applications in mind. Regularly varying tail distributions have been used to model power-law tail behavior in stochastic models for applications such as hydrology and climate change, finance and insurance, queueing systems and telecommunication, social and other networks, and many more. A regularly varying random vector like \(Z \in [0, \infty)^d\) often represents extreme risks like storms, hail and heavy rain in the environmental sciences, investment risks from multiple stocks in
finance, losses pertaining to different insurance companies or risk classes, or large jobs in queueing systems or telecommunication.

Such risks are then distributed according to some (random) linear mechanism modeled by a \( q \times d \) random matrix \( A \). Thus a common quantity of interest to compute here is \( P(AZ \in tC) \) for tail sets \( C \) representing a variety of worst case scenarios. Moreover, in many applications it is natural for the underlying risk vector and the linear function given by a (random or fixed) matrix to have non-negative entries only. Examples include dimension reduction in the context of extreme precipitation, extreme losses for financial data, and dietary requirement analysis (Cooley and Thibaud 2019; Chautru 2015; Klüppelberg and Krali 2020), or network models for investments in overlapping portfolios (Kley et al. 2018), insurance risk sharing (Kley et al. 2016), operational risk exposures to business lines (Kley et al. 2020), and job sharing between servers (Adan et al. 2001; Behme and Strietzel 2020). The structure of such applications that we detail in Sect. 4 is that of a bipartite network structure for modeling sharing of losses in insurance markets or investment into overlapping portfolios, where we relate the risk of the underlying objects \( Z \) to that of the risk of agents \( X = AZ \).

The inherent applicability of Breiman’s one- and multi-variate results in finance, insurance, and econometrics, and particularly on portfolio tail risk and stochastic recurrence equations has lead to a number of generalizations in the past decades. A generalization of Breiman’s Theorem by relaxing the assumption of independence of the random variables \( A \) and \( Z \) to asymptotic independence was provided in Maulik et al. (2002). On the other hand, a weakening of the moment condition on \( A \) such that (1) holds can be found in Denisov and Zwart (2007). An extension to componentwise vector multiplication based on several joint regular variation conditions was given in Fougeres and Mercadier (2012).

Other interesting generalizations of Breiman’s Theorem have appeared in the literature, albeit in different contexts. In Jessen and Mikosch (2006), the authors provide partial converses to Breiman’s Theorem: assuming \( A \) and \( Z \) to be non-negative independent random variables, if \( AZ \) has a regularly varying tail distribution, they find conditions when \( Z \) will also have a regularly varying tail distribution. In Tillier and Wintenberger (2017) we find an extension of Breiman’s multivariate result to vectors of random length, determined for instance by a Poisson random variable. A characterization of the tail behavior of homogeneous functions (extending from products) of independent regularly varying random vectors is found in Dyszewski and Mikosch (2019). In a more general setting, Chakraborty and Hazra (2018), extend Breiman’s result for multiplicative Boolean convolution of regularly varying measures. Finally, the monograph Buraczewski et al. (2016) provides many applications of Breiman’s result and its generalizations in the area of stochastic modeling with power-law tails.

Our paper is organized as follows. We conclude the introduction with a summary of notations used in the paper. In Sect. 2, we discuss multivariate regular variation with \( \mathcal{M} \)-convergence in different subcones of \( [0, \infty)^d \), which provides a set-up for the results.
of the paper. Our main results extending Breiman’s Theorem are developed in Sect. 3. In Sect. 4, we provide applications of the model in the context of bipartite networks, where \( q \) agents can be exposed to the risk of \( d \) objects where \( Z \in [0, \infty)^d \) are the risks of the objects. The exposures of the agents are represented by \( X = AZ \). We illustrate the behavior of tail risk of the agents for possible structures of the weighted adjacency matrix \( A \in [0, \infty)^{q \times d} \). We conclude our paper indicating future directions of research in Sect. 5.

Various notations and concepts used in this paper are summarized below. Vector differences are provided wherever applicable.

\[ \mathcal{R}_{\beta} \quad \text{Regularly varying functions with index } \beta \in \mathbb{R}; \text{ that is, functions } f : \mathbb{R}_+ \to \mathbb{R}_+ \text{ satisfying } \lim_{t \to \infty} f(tx)/f(t) = x^\beta, \text{ for } x > 0; \text{ see Bingham 1989; de Haan and Ferreira 2006; Resnick 2008 for further details.} \]

\[ V \in \mathcal{R}_{\alpha} \quad \text{A random variable } V \text{ with distribution function } F_V \text{ is regularly varying (at infinity) if } F_V := 1 - F_V \in \mathcal{R}_{\alpha} \text{ for some } \alpha \geq 0. \]

\[ \mathbb{R}^d_+ \quad [0, \infty)^d \text{ for dimension } d \geq 1. \]

\[ v^{(1)}, \ldots, v^{(d)} \quad \text{Order statistics of } v = (v_1, \ldots, v_d)^\top \in \mathbb{R}^d_+ \text{ such that } v^{(1)} \geq v^{(2)} \geq \ldots \geq v^{(d)}. \]

\[ \mathbb{C}\mathbb{A}_d^{(i)} \quad \{v \in \mathbb{R}^d_+ : v^{(i+1)} = 0\}, \ i = 1, \ldots, d - 1; \text{ also define } C_{d}^{(d)} = (0, \infty)^d. \]

\[ \widetilde{\mathbb{C}}\mathbb{A}_d^{(i)}(j) \quad \text{denotes the } j\text{-th } i\text{-dimensional co-ordinate hyperplane in } \mathbb{R}^d_+ \text{ with } i \text{ positive and } (d - i) \text{ zero co-ordinates in some ordering of the hyperplanes}. \]

\[ \mathbb{E}^{(i)}_d \quad \mathbb{R}^d_+ \setminus \mathbb{C}\mathbb{A}_d^{(i-1)} = \{v \in \mathbb{R}^d_+ : v^{(i)} > 0\} \text{ for } i = 1, \ldots, d. \]

\[ \| \cdot \| \quad \text{For } x \in \mathbb{R}^d, \| x \| \text{ denotes an arbitrary vector norm, and for a matrix } A \in \mathbb{R}^{q \times d}, \| A \| \text{ denotes the corresponding operator norm.} \]

\[ d(x, y) \quad \| x - y \|_{\infty}, \text{ where } \| \cdot \|_{\infty} \text{ denotes the supremum norm.} \]

\[ \tau^{(k)}(x) \quad x^{(k)} \text{ for } x = (x_1, \ldots, x_d)^\top \in \mathbb{R}^d_+. \]

\[ \tau^{(k,j)}(A) \quad \sup_{x \in \mathbb{E}^{(j)}_d} \frac{\tau^{(k,x)}(A)}{\tau^{(j,x)}} \text{ for } A \in \mathbb{R}^{q \times d}; \text{ see Sect. 3 for details.} \]

\[ \mathbb{M}(\mathbb{C} \setminus \mathbb{C}_0) \quad \text{The set of all non-null measures on } \mathbb{C} \setminus \mathbb{C}_0 \text{ which are finite on Borel subsets bounded away from } \mathbb{C}_0. \]

\[ \mu_n \to \mu \quad \text{Convergence in } \mathbb{M}(\mathbb{C} \setminus \mathbb{C}_0); \text{ see Sect. 2.1 and Das, Mitra and Resnick (2013); Hult and Lindskog (2006); Lindskog, Resnick and Roy (2014) for details.} \]

\[ \mathcal{MRV}(\alpha, b, \mu, \mathbb{E}) \quad \text{Multivariate regular variation on the space } \mathbb{E} = \mathbb{C} \setminus \mathbb{C}_0, \text{ where } \mathbb{C} \text{ and } \mathbb{C}_0 \text{ are closed cones in } \mathbb{R}^d_+. \text{ Here } -\alpha \leq 0 \text{ is the index of regular variation, } b \text{ is the scaling function, and } \mu \text{ is the limit measure. We often omit one or more of the arguments. See Definition 2.2 for details.} \]
2 Multivariate regular variation and convergence concepts

We use the notion of $\mathcal{M}$-convergence of measures to define multivariate regular variation on Euclidean spaces and subsets thereof; see Das, Mitra and Resnick (2013); Lindskog, Resnick and Roy (2014) for details. In particular, we investigate regular variation of a random vector $X$, which is given as $X = AZ$, where $Z \in \mathbb{R}^d_+$ is multivariate regularly varying with index $-\alpha \leq 0$ and $A$ is a $q \times d$ random matrix in $\mathbb{R}^{q \times d}$ independent of $Z$.

Our goal is to obtain a complete picture concerning linear functions $X = AZ$ which possess multivariate regular variation on different subcones of $\mathbb{R}^q_+$ (also called hidden regular variation), thus extending results from Basrak, Davis and Mikosch (2002) and Janssen and Drees (2016). The particular choice of subsets where we seek regular variation are natural, depending on the type of extreme sets for which we seek to find probabilities; see Mitra and Resnick (2011) for examples. The necessary definitions and results formulated with respect to $\mathcal{M}$-convergence are discussed below.

Recall that a cone $C \subset \mathbb{R}^d_+$ is a set which is closed under scalar multiplication: if $x \in C$ then $cx \in C$ for $c > 0$. A closed cone of course, is a cone which is a closed set in $\mathbb{R}^d_+$. Now we define multivariate regular variation using convergence of measures on a closed cone $C \subset \mathbb{R}^d_+$ with a closed cone $C_0 \subset C$ deleted. Moreover, we say that a subset $\Lambda \subset C \setminus C_0$ is bounded away from $C_0$ if

$$d(\Lambda, C_0) = \inf \{ d(x, y) : x \in \Lambda, y \in C_0 \} > 0,$$

where $d(x, y) = ||x - y||_\infty$ is the sup-norm on $\mathbb{R}^d_+$. This guarantees that the distance of a point $y \in \mathbb{R}^d_+$ to a specific closed set can be represented as an order statistic of the co-ordinates of $y$; see (12).

The class of Borel measures on $C \setminus C_0$ that assign finite measures to all Borel sets $B \subset C \setminus C_0$, which are bounded away from $C_0$, is denoted by $\mathcal{M}(C \setminus C_0)$.

In this paper, regular variation on cones is defined using $\mathcal{M}$-convergence, which is slightly different from vague convergence which has been traditionally used in multivariate regular variation. Reasons for the preference of $\mathcal{M}$-convergence are presented in Das and Resnick [2015, Remark 1.1]; see also Das et al. (2013); Lindskog et al. (2014). In the space $\mathcal{E}^{(1)}_d = \mathbb{R}^d_+ \setminus \{0\}$ the notions of vague convergence and $\mathcal{M}$-convergence are identical.

**Definition 2.1** Let $C_0 \subset C \subset \mathbb{R}^d_+$ be closed cones containing 0. Let $\mu_n, \mu$ be Borel measures on $\mathcal{M}(C \setminus C_0)$ and $\int f \, d\mu_n \to \int f \, d\mu$ as $n \to \infty$ for any bounded, continuous, real-valued function $f$ whose support is bounded away from $C_0$, then we say $\mu_n$ converges to $\mu$ in $\mathcal{M}(C \setminus C_0)$, and write $\mu_n \to \mu$ in $\mathcal{M}(C \setminus C_0)$.

**Definition 2.2** Let $C_0 \subset C \subset \mathbb{R}^d_+$ be closed cones containing 0. A random vector $V = (V_1, \ldots, V_d)^T \in C$ is regularly varying on $C \setminus C_0$ if there exists a function $b \in \mathcal{R} \mathcal{V}_{1/\alpha}$ for $\alpha \geq 0$, called the scaling function, and a non-null (Borel) measure $\mu \in \mathcal{M}(C \setminus C_0)$ called the limit or tail measure such that

$$t \mathcal{P}(V/b(t) \in \cdot) \to \mu(\cdot), \quad t \to \infty,$$
in $\mathbb{M}(\mathbb{C} \setminus \mathbb{C}_0)$. We write $V \in MRV(\alpha, b, \mu, \mathbb{C} \setminus \mathbb{C}_0)$ or, $V \in MRV(\alpha, \mu, \mathbb{C} \setminus \mathbb{C}_0)$ if the scaling function is contextually irrelevant. If $\mathbb{C} \setminus \mathbb{C}_0 = \mathbb{R}_+^d \setminus \{0\} =: E_d^{(1)}$, we simply write $V \in MRV(\alpha, \mu, E_d^{(1)})$ or $V \in MRV(\alpha, \mu)$.

Since for $\alpha > 0$ the function $b \in \mathcal{R}V_{1/a}$, the limit measure $\mu$ has the scaling property:

$$\mu(c \cdot) = c^{-\alpha} \mu(\cdot), \quad c > 0.$$  

### 2.1 Regular variation on subcones of the positive quadrant

We define regular variation on subcones of $\mathbb{R}_+^d$ following Mitra and Resnick (2011). For $v \in \mathbb{R}_+^d$ write $v = (v_1, \ldots, v_d)^T$. Moreover, the (decreasing) order statistics for any vector $v \in \mathbb{R}_+^d$ is defined as

$$v^{(1)} \geq v^{(2)} \geq \ldots \geq v^{(d)},$$

where $v^{(i)}$ denotes the $i$-th largest component of $v$. First we define closed sets which we think of as a union of co-ordinate hyper-planes of various dimensions in $\mathbb{R}_+^d$. For $0 \leq i \leq d - 1$ define

$$\mathbb{C}A_{d}^{(i)} := \bigcup_{1 \leq j_1 < \ldots < j_{i-1} \leq d} \{ v \in \mathbb{R}_+^d : v_{j_1} = 0, \ldots, v_{j_{i-1}} = 0 \} = \{ v \in \mathbb{R}_+^d : v^{(i+1)} = 0 \}.$$  

Here $\mathbb{C}A_{d}^{(i)}$ represents the union of all $i$-dimensional co-ordinate hyperplanes in $\mathbb{R}_+^d$. Also define $\mathbb{C}A_{d}^{(d)} := \{ v \in \mathbb{R}_+^d : v^{(d)} > 0 \}$. Now define the following sequence of subcones of $\mathbb{R}_+^d$:

$$E_d^{(i)} := \mathbb{R}_+^d \setminus \mathbb{C}A_{d}^{(i-1)} = \{ v \in \mathbb{R}_+^d : v^{(i)} > 0 \}, \quad 1 \leq i \leq d.$$  

Hence $E_d^{(1)}$ is the non-negative orthant with $\{0\} = \mathbb{C}A_{d}^{(0)}$ removed, $E_d^{(2)}$ is the non-negative orthant with all one-dimensional co-ordinate axes removed, $E_d^{(3)}$ is the non-negative orthant with all two-dimensional co-ordinate hyperplanes removed, and so on. Clearly, we have

$$E_d^{(1)} \supset E_d^{(2)} \supset \ldots \supset E_d^{(d)}.$$  

Note that according to our definition $E_d^{(d)} = \mathbb{C}A_{d}^{(d)}$. We also define for $i = 1, \ldots, d$,

$$\mathbb{C}A_{d}^{(i)} \setminus \mathbb{C}A_{d}^{(i-1)} = \{ v \in \mathbb{R}_+^d : \text{exactly } i \text{ co-ordinates of } v \text{ are positive} \} =: \bigcup_{j=1}^{d} \mathbb{C}A_{d}^{(i,j)},$$  

where $\mathbb{C}A_{d}^{(i,j)}$ denotes the $j$-th $i$-dimensional co-ordinate hyperplane in $\mathbb{R}_+^d$ with $i$ positive and $(d - i)$ zero co-ordinates in some ordering of the hyperplanes. We note in passing that
\[ \mathcal{CA}_d^{(d)} = \mathbb{E}_d^{(d)} = \overset{\sim}{\mathcal{CA}}_d^{(d)} \] (1).

A recipe for finding regular variation in the above sequence of cones can be devised as follows. To start with, suppose \( V \in \mathcal{MRV}(\alpha_1, b_1, \mu_1, \mathbb{E}_d^{(1)}) \) with \( \alpha_1 > 0 \).

1. If \( \mu_1(\mathbb{E}_d^{(d)}) > 0 \), we seek no further regular variation on cones of \( \mathbb{R}_d^+ \).
2. If \( \mu_1(\mathbb{E}_d^{(d)}) = 0 \), we may find an \( i \in \{2, \ldots, d\} \) such that
   \[ i = \inf\{j \in \{2, \ldots, d\} : \mu_1(\mathbb{E}_d^{(j-1)}) > 0 \text{ and } \mu_1(\mathbb{E}_d^{(j)}) = 0\} \).

Hence \( \mu_1 \) concentrates on \( \mathcal{CA}_d^{(i-1)} \). So we seek regular variation in \( \mathbb{E}_d^{(i)} = \mathbb{R}_d^+ \setminus \mathcal{CA}_d^{(i-1)} \). Suppose there exists \( b_i(t) \uparrow \infty \) with \( \lim_{t \to \infty} b_i(t)/b_i(t) = \infty \) and \( \mu_i \neq 0 \) on \( \mathbb{E}_d^{(i)} \) such that \( V \in \mathcal{MRV}(a_i, b_i, \mu_i, \mathbb{E}_d^{(i)}) \). Then, \( a_i \geq \alpha_i, b_i \in \mathcal{R} \mathcal{V}_{1/\alpha_i} \) and \( \mu_i(c \cdot) = c^{-\alpha_i} \mu_i(\cdot) \) for \( c > 0 \). Hence \( V \) has regular variation on \( \mathbb{E}_d^{(i)} \) with index \( -\alpha_i \).
3. In the next step, if \( \mu_i(\mathbb{E}_d^{(d)}) > 0 \), we stop looking for regular variation; otherwise we keep seeking regular variation through \( \mathbb{E}_d^{(i+1)}, \ldots, \mathbb{E}_d^{(d)} \) sequentially.

The idea of regular variation on a sequence of cones is easier understood with an example.

**Example 2.1** For \( d \geq 2 \), suppose \( V = (V_1, \ldots, V_d)^T \) and \( V_1, \ldots, V_d \) are iid Pareto(\( \alpha \)) random variables with \( \alpha > 0 \) such that \( \mathbb{P}(V_i > t) = t^{-\alpha}, t \geq 1 \).

(i) First we observe that for all \( i = 1, \ldots, d \), we have \( V \in \mathcal{MRV}(a_i, b_i, \mu_i, \mathbb{E}_d^{(i)}) \) with \( a_i = ia \) and \( b_i(t) = t^{1/(ia)} \) where the limit measure \( \mu_i \) on \( \mathbb{E}_d^{(i)} \) is such that for any \( \mathbf{z} = (z_1, \ldots, z_d)^T \in \mathbb{E}_d^{(i)} \),

\[
\mu_i((v \in \mathbb{E}_d^{(i)} : v_{j_1} > z_{j_1}, \ldots, v_{j_i} > z_{j_i} \text{ for some } 1 \leq j_1 < \ldots < j_i \leq d)) = \frac{1}{\binom{d}{i}} \sum_{1 \leq i_1 < \ldots < i_i \leq d} (z_{i_1}z_{i_2} \ldots z_{i_i})^{-\alpha}. \tag{7}
\]

This follows from Example 5.1 in Maulik and Resnick (2005) and Example 2.2 in Mitra and Resnick (2011). Hence, if

\[
C = \{v \in \mathbb{R}_d^+ : v_1 > z_1, \ldots, v_i > z_i \} \tag{8}
\]

we find

\[
\mathbb{P}(V \in tC) = t^{-ia}(z_1z_2 \ldots z_i)^{-a} + o(t^{-ia}), \quad t \to \infty. \tag{9}
\]

(ii) The measure \( \mu_i \) as defined in (7) concentrates on \( \mathcal{CA}_d^{(i)} \setminus \mathcal{CA}_d^{(i-1)} \).

(iii) In general, from part (i) we conclude that for any Borel set \( C \subset \mathbb{E}_d^{(i)} \) which is bounded away from \( \mathcal{CA}_d^{(i-1)} \) and \( \mu_i(\partial C) = 0 \),
\[
P(V \in tC) = t^{-\alpha} \mu_t(C) + o(t^{-\alpha}), \quad t \to \infty.
\]

So, in case \( \mu_t(C) = 0 \), we get \( P(V \in tC) = o(t^{-\alpha}) \) as \( t \to \infty \). However, if \( C \) is of the form (8), or a finite union of such sets (for fixed \( i \)), from (9) we know that \( \mu_t(C) > 0 \).

**Remark 1** Multivariate regular variation can also be defined for a very general class of sub-cones of \( \mathbb{R}_+^d \) (see Das, Mitra and Resnick (2013); Lindskog, Resnick and Roy (2014); Mitra and Resnick (2011) for examples). In this paper, our interest is in finding probabilities of risk sets which are threshold exceedances for a subset of coordinates of a linear transformation of a regularly varying vector. Any such risk set is necessarily a subset of one or more of the sub-cones \( E^{(i)}_d, \ldots, E^{(d)}_d \) as defined in (5). Hence we restrict our attention to regular variation on such subcones. For an example of regular variation with an infinite sequence of indices on an infinite sequence of cones contained in the space \( \mathbb{R}_+^2 \) we refer to Das et al. 2013, [Example 5.3] and leave an extension of our results to the interested reader.

**Definition 2.3** Suppose \( V = (V_1, \ldots, V_d)^T \in MRV(\alpha, b, \mu, E) \) and \( F_{V^{(i)}}(s) = \inf \{ t \in \mathbb{R} : F_{V^{(i)}}(t) \geq s \} \) is the generalized inverse of the distribution function \( F_{V^{(i)}} \) of \( V^{(i)} \), where \( V^{(i)} \geq \ldots \geq V^{(d)} \) are the order statistics of \( V_1, \ldots, V_d \). If \( b_i(t) = F_{V^{(i)}}(1 - 1/t) \), we call \( b_i \) the canonical choice of the scaling function.

### 3 Breiman’s Theorem and regular variation on Euclidean subspaces

In this section we characterize the multivariate generalization of Breiman’s Theorem addressed in Basrak et al. [2002, Proposition A.1] and its subsequent extension provided in Janssen and Drees [2016, Theorem 2.3]. More precisely, we investigate regular variation properties of the vector \( X = AZ \), where \( A \in \mathbb{R}^{q \times d} \) is a random matrix which is independent of \( Z \in \mathbb{R}_+^q \), and \( Z \) is multivariate regularly varying on subspaces \( E^{(i)}_d \) for \( i = 1, \ldots, d \). We provide asymptotic rates of convergence for tail probabilities like \( P(AZ \in tC) \) for Borel sets \( C \subset E^{(k)}_q \) for \( k = 1, \ldots, q \).

For the sake of convenience, first we present the two available results addressing this issue. Most results quoted from previous papers have appeared with asymptotic properties and definitions in terms of vague convergence, we restate them here with respect to \( \mathbb{M} \)-convergence.

**Theorem 3.1** (Basrak et al. [2002, Proposition A.1]) Let \( Z \in \mathbb{R}^d \) be a random vector such that \( Z \in MRV(\alpha_1, \mu_1, \mathbb{R}^d \setminus \{0\}) \) with \( \alpha_1 \geq 0 \) and \( A \in \mathbb{R}^{q \times d} \) be a random matrix independent of \( Z \) with \( 0 < E[||A||^{\alpha_1 + \delta}] < \infty \) for some \( \delta > 0 \). Then

\[
\frac{P(t^{-1}AZ \in \cdot)}{P(||Z|| > t)} \to E[\mu_1(\{z \in \mathbb{R}^d \setminus \{0\} : Az \in \cdot\})] =: \mu_t(\cdot), \quad t \to \infty, \quad (10)
\]

in \( \mathbb{M}(\mathbb{R}^d \setminus \{0\}) \).
Remark 2  A couple of remarks are in order here.

(i)  If $\mu_1$ is a non-null measure, $AZ \in \mathcal{MRV}(\alpha_1, \mu_1, \mathbb{R}^q \setminus \{0\})$. However, the assumptions of Basrak et al. [2002, Proposition A.1] are not sufficient to guarantee a non-null limit measure.

(ii)  For $||Z||$ to become large, it suffices that one component of $Z$ becomes large. Hence, if $Z \in \mathbb{R}^q_+$ and $\mu_1$ is a non-null measure, then $P(||Z|| > t) \sim c P(Z^{(1)} > t)$ as $t \to \infty$ for some constant $c > 0$, and $P(Z^{(1)} > t)$ provides the rate of convergence of $P(t^{-1}AZ \in \cdot)$ to zero.

(iii)  The observation in (2) is an easy consequence of this theorem.

(iv)  The limit measure $\mu_1$ might have no mass on the non-negative orthant such that for any $\mu_1$-continuous Borel set $C \subset \mathbb{R}^q_+ \setminus \{0\}$ bounded away from 0, an estimation of the tail probability gives zero.

(v)  The limit measure $\mu_1$ might have mass only on a very specific subset of the non-negative orthant, and hence, it is possible that $\mu_1(C) = 0$ on the right hand side of (10) for specific subsets $C$.

Remark 3  As stated in the Introduction, one of our primary interests is in bipartite network structures, in particular in $A$ being an adjacency matrix, having naturally a discrete distribution, or its weighted version where non-zero entries are positive random variables with arbitrary distributions. The following considerations show that indeed such $A$ create the most interesting regular variation scenarios. Because, if we assume absolute continuity for the joint distribution of the entries of $A$, then Theorem 3.1 covers limit measures for the variety of sets that we are interested in. This can be seen by the following example. Let $Z \in \mathcal{MRV}(\alpha_1, \mu_1, \mathbb{E}^{(1)})$ and let $A = (A_{ki}) \in \mathbb{R}^{q \times d}$ be a random matrix independent of $Z$ satisfying $P(\min_{k,i} A_{ki} > 0) > 0$. Then 3.1 entails $AZ \in \mathcal{MRV}(\alpha, \mu_1, \mathbb{E}^{(1)})$ and for $C = (1, \infty)^q$ we have

$$\bar{\mu}_1(C) = E[\mu_1(A^{-1}(C))] \geq E \left[ \mu_1 \left( \left\{ z \in \mathbb{R}^{d}_+ : z^{(1)} \geq \left( \min_{k,i} A_{ki} \right)^{-1} \right\} \right) \right]$$

$$= E \left[ \min_{k,i} A_{ki}^q \right] \mu_1 \left( \left\{ z \in \mathbb{R}^{d}_+ : z^{(1)} \geq 1 \right\} \right) > 0.$$ 

Thus, $AZ \in \mathcal{MRV}(\alpha, \mu_1, \mathbb{E}^{(k)})$ for all $k = 1, \ldots, q$.

The condition $P(\min_{k,i} A_{ki} > 0) > 0$ is satisfied under the weak regularity condition that the vectorised matrix vec($A$) is absolutely continuous with respect to Lebesgue-measure and the Lebesgue measure has positive mass in the interior of the support. Hence, random matrices $A$, where zeroes occur with positive probability, are in the focus of our research.

A partial solution for obtaining a non-null limit in (10) is provided in Janssen and Drees (2016), when $q = d$ and where convergence occurs in the space $(0, \infty)^d = \mathbb{E}^{(d)}$, which means to focus on sets, where all components of $X = AZ$ are large, translated into the event $\{X^{(d)} > t\}$. The formal setting in Janssen and Drees (2016) is as

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follows. Define \( \tau : \mathbb{R}^d_+ \to \mathbb{R}_+ \) to be the distance of a point \( z \in \mathbb{R}^d_+ \) from the space \( \mathcal{C}^d_{\alpha d^{-1}} := \mathbb{R}^d_+ \setminus (0, \infty)^d \), given by \( \tau(z) := d(z, \mathcal{C}^d_{\alpha d^{-1}}) = z^\perp(d) \). For a deterministic matrix \( A \in \mathbb{R}^{d \times d} \) define the analog

\[
\tau(A) := \sup_{z \in \mathbb{E}^{(d)}_+ : \tau(z) = 1} \tau(Az). \tag{11}
\]

**Theorem 3.2** [Janssen and Drees 2016, [Theorem 2.3]] Let \( Z \in \mathbb{R}^d_+ \) be a random vector such that \( Z \in \mathcal{M}RV(\alpha_d, \mu_d, \mathbb{E}^{(d)}_+) \) with \( \alpha_d \geq 0 \) and \( A \in \mathbb{R}^{d \times d} \) be a random matrix independent of \( Z \). Assume \( \tau(A) > 0 \) almost surely and \( \mathbb{E}[\tau(A)^{\alpha_d+\delta}] < \infty \) for some \( \delta > 0 \). Then

\[
\frac{\mathbb{P}(\tau^{-1}AZ \in \cdot)}{\mathbb{P}(\tau(Z) > t)} \to \mathbb{E} \left[ \mu_d(\{z \in \mathbb{E}^{(d)}_+ : Az \in \cdot \}) \right] =: \overline{\mu}_d(\cdot), \quad t \to \infty,
\]

in \( \mathbb{M}(\mathbb{E}^{(d)}_+) \). In particular, we have \( AZ \in \mathcal{M}RV(\alpha_d, \overline{\mu}_d, \mathbb{E}^{(d)}_+) \).

**Remark 4** A couple of remarks to this result are useful.

(i) Note that \( \mathbb{P}(\tau(Z) > t) = \mathbb{P}(Z(d) > t) \) which provides the rate of convergence of \( \mathbb{P}(\tau^{-1}AZ \in \cdot) \) to zero as \( t \to \infty \).

(ii) The observation in (3) is an easy consequence of Theorem 3.2.

(iii) Since \( \overline{\mu}_d \) is a non-null measure by definition, we also get \( \overline{\mu}_d(\mathbb{E}^{(d)}_+) > 0 \).

(iv) Theorem 3.2 was proposed in the context of stochastic volatility models. In such a context, for a square random matrix \( A \), Theorem 3.2 assumes that \( A \) is almost surely invertible and its inverse has almost surely non-negative entries (see Janssen and Drees 2016, [Lemma 2.2]). If moreover we assume that \( A \) itself has non-negative entries then almost all realizations of \( A \) are row permutations of diagonal matrices with positive diagonal entries (cf. Ding and Rhee (2014)). In our results we relax the condition of invertibility and allow for rectangular matrices, but assume that the entries of \( A \) are non-negative. In particular we see that \( AZ \) is not necessarily multivariate regularly varying with index \( \alpha_d \) on \( \mathbb{E}^{(d)}_+ \). The index of multivariate regular variation might be smaller.

### 3.1 Extension of Breiman’s Theorem to Euclidean subspaces

In light of the previous results, we provide a multivariate extension to Breiman’s Theorem which leads to convergence for a multitude of forms of \( A \). Let \( A \in \mathbb{R}^{q \times d}_+ \) be deterministic. We define the analogous subcones of \( \mathbb{R}^q_+ \) as in (5) and proceed as follows. Recall that \( d(x,y) = ||x-y||_\infty \) is the sup-norm on \( \mathbb{R}^d_+ \), which represents the distance of a point \( x \in \mathbb{R}^q_+ \) to a specific closed set as an order statistic of the coordinates. This gives rise to the following definition.

**Definition 3.1** For \( k = 1, \ldots, q \), define \( \tau^{(k)} : \mathbb{R}^q_+ \to \mathbb{R}_+ \) to be the distance of a point \( x \in \mathbb{R}^q_+ \) from the space \( \mathcal{C}A_{(k-1)}^q \), given by
Furthermore, we define in analogy to (11) for \( k = 1, \ldots, q \) and \( i = 1, \ldots, d \) the functions \( \tau^{(k,i)} : \mathbb{R}^q_{+} \to \mathbb{R}_{+} \) given by

\[
\tau^{(k,i)}(A) = \sup_{z \in \mathbb{E}^i_d} \frac{\tau^{(k)}(Az)}{\tau^{(i)}(z)} = \sup_{z \in \mathbb{E}^i_d} \frac{(Az)^{(k)}}{z^{(i)}}.
\]  

Although the functions \( \tau^{(k)} \) and \( \tau^{(k,i)} \) are not necessarily seminorms on the induced vector space (see Horn and Johnson [2013, Section 5.1]), they admit some useful properties as listed below. We call a row of \( A \) trivial, if it is a zero vector. Note that \( \tau^{(q,d)}(A) = \tau(A) \) from (11) if \( q = d \).

**Lemma 3.1** For every deterministic matrix \( A \in \mathbb{R}^q_{+} \times d \) and \( z \in \mathbb{R}^d_{+} \) the following hold for \( i = 1, \ldots, d \) and \( k = 1, \ldots, q \):

(a) \( \tau^{(k)}(Az) \leq \tau^{(k,i)}(A)\tau^{(i)}(z) \).
(b) \( \tau^{(k,i)}(A) \leq \tau^{(k-1,i)}(A) \).
(c) \( \tau^{(k,i)}(A) \leq \tau^{(k,i+1)}(A) \).
(d) \( \tau^{(q,1)}(A) > 0 \) if and only if all rows of \( A \) are non-trivial.
(e) \( \tau^{(k,1)}(A) \leq \tau^{(1,1)}(A) < \infty \).

**Proof**

(a) By definition we have

\[
\tau^{(k)}(Az) = \tau^{(k)}(A - \frac{z}{\tau^{(i)}(z)})\tau^{(i)}(z) \leq \tau^{(k,i)}(A)\tau^{(i)}(z).
\]

(b) and (c) immediately follow from the definition.

(c) If \( A = (A_y) \) has no trivial row, denoting \( e = (1, \ldots, 1)^T \in \mathbb{R}^d_{+} \), we have

\[
\tau^{(q,1)}(A) \geq \frac{\tau^{(q)}(Ae)}{\tau^{(q)}(e)} = \min_{1 \leq i \leq q} \sum_{j=1}^d A_{ij} > 0,
\]

the final domination being a consequence of each row of \( A \) having at least one positive entry. On the other hand, suppose that \( \tau^{(q,1)}(A) > 0 \) and \( A \) has a trivial row. Then for any \( z \in \mathbb{E}^1_d \), we have

\[
\tau^{(q)}(Az) = \min_{1 \leq i \leq q} \sum_{j=1}^d A_{ij}z_j = 0.
\]

This implies
\[ \tau^{(q-1)}(A) = \sup_{z \in \mathbb{E}_q^{(i)}} \frac{\tau^{(q)}(Az)}{\tau^{(1)}(z)} = 0, \]

which is a contradiction. Hence \( A \) cannot have a trivial row.

(d) The first inequality follows from (b). Moreover,

\[ \tau^{(1,1)}(A) = \sup_{z^{(i)}=1} (AZ)^{(i)} = \max_{1 \leq i \leq d} \sum_{1 \leq j \leq q} A_{ij} < \infty. \]

For a deterministic matrix \( A \in \mathbb{R}_+^{qd} \) and \( C \subseteq \mathbb{R}_+^q \), the *pre-image of \( C \) is given by

\[ A^{-1}(C) = \{ z \in \mathbb{R}_+^d : Az \in C \}. \]

The following lemma characterizes the mapping of the subcones of \( \mathbb{R}_+^d \) under the linear map \( A \) and is key to the results to follow.

**Lemma 3.2** Let \( A \in \mathbb{R}_+^{qd} \) be a deterministic matrix with all rows non-trivial. Then for fixed \( i \in \{1, \ldots, d\} \) and fixed \( k \in \{1, \ldots, q\} \), the following are equivalent:

(a) \( \emptyset \neq A^{-1}(\mathbb{E}_q^{(k)}) \subseteq \mathbb{E}_d^{(i)} \).

(b) \( 0 < \tau^{(k,i)}(A) < \infty \).

**Proof** (a)\( \Rightarrow \) (b): Let \( \emptyset \neq A^{-1}(\mathbb{E}_q^{(k)}) \subseteq \mathbb{E}_d^{(i)} \). First suppose that \( \tau^{(k,i)}(A) = 0 \). Hence by definition, from (13) we have that \( \tau^{(k)}(Az) = (Az)^{(k)} = 0 \) for every \( z \in \mathbb{E}_q^{(i)} \). Thus

\[ A^{-1}(\mathbb{E}_q^{(k)}) \cap \mathbb{E}_d^{(i)} = \emptyset \]

contradicting the premise.

Now suppose that \( \tau^{(k,i)}(A) = \infty \). Let \( M = \tau^{(1)}(Ae) \) where \( e = (1, 1, \ldots, 1)^T \in \mathbb{R}_+^d \). Then there exists a \( z \in \mathbb{R}_+^d \) with \( z^{(i)} = 1 \) such that \( \tau^{(k)}(Az) \geq M + d \). Fix such a \( z \) and without loss of generality assume that \( z_1 \geq z_2 \geq \ldots \geq z_d \) (otherwise we may arrange columns of \( A \) accordingly). Hence \( z^{(i)} = z_i = 1 \). Define \( z^* \in \mathbb{R}_+^d \) by converting the last \( d - i \) components of \( z \) to 1. Hence

\[ z^* = (z_1, \ldots, z_{i-1}, 1, \ldots, 1)^T. \]

Since the components of \( z^* \) and \( z \) are ordered and component-wise \( z^* \geq z \), we have \( \tau^{(k)}(Az^*) \geq \tau^{(k)}(Az) \geq M + d \). Now, define

\[ z^*_e := z^* - e = (z_1 - 1, \ldots, z_{i-1} - 1, 0, \ldots, 0)^T. \]

Clearly \( z^*_e \in \mathbb{R}_+^d \) as well as \( z^*_e \notin \mathbb{E}_d^{(i)} \) since \( z^{*(i)}_e = z^*_e = 0 \). Note that \( \tau^{(k)}(Az^*_e) \geq M + d \) means at least \( k \) elements of \( Az^*_e \) are larger than \( M + d \), whereas \( \tau^{(k)}(Ae) \leq \tau^{(1)}(Ae) = M \) by definition. Hence all elements of \( Ae \) are at most \( M \).
Since $A\z^*_e = Az^* - Ae$, at least $k$ elements of $A\z^*_e$ are greater or equal to $d$. Therefore, $\tau^{(k)}(A\z^*_e) \geq d > 0$. Thus $A\z^*_e \in \E_d^k$ which is a contradiction.

(b)⇒(a): Let $x \in \E_d^k$. Then $\tau^{(k)}(x) > 0$. Furthermore, let $A^{-1}(x) := \{z \in \R^d_+ : Az = x\} \subseteq \R^d_+$. Assume that $A^{-1}(x) \neq \emptyset$, since $\tau^{(k,i)}(A) > 0$, such $x \in \E_d^k$ exists; and let $z_x \in A^{-1}(x)$. Then by Lemma 3.1(a),

$$\tau^{(i)}(z_x) = \frac{\tau^{(k,i)}(A z_x)}{\tau^{(k,i)}(A)} = \frac{\tau^{(k,i)}(x)}{\tau^{(k,i)}(A)} > 0,$$

implying $z_x \in \E_d^i$. Hence $\emptyset \neq A^{-1}(\E_d^k) \subseteq \E_d^i$. \hfill $\square$

The following lemma shows that positivity and finiteness of $\tau^{(k,i)}(A)$ are characterized by the non-zero entries in $A$ regardless of their magnitude.

**Lemma 3.3** Let $A = (A_{ki}) \in \R_+^{q \times d}$ be a deterministic matrix with all rows non-trivial and define $I_A = (1_{A_{ki} > 0}) \in \R_+^{q \times d}$. Then for fixed $k \in \{1, \ldots, q\}$ and fixed $i \in \{1, \ldots, d\}$:

$$0 < \tau^{(k,i)}(A) < \infty \quad \text{if and only if} \quad 0 < \tau^{(k,i)}(I_A) < \infty.$$  

**Proof** Define

$$A_{\min} := \min\{A_{mn} > 0 : 1 \leq m \leq q, 1 \leq n \leq d\} > 0,$$

$$A_{\max} := \max\{A_{mn} : 1 \leq m \leq q, 1 \leq n \leq d\} < \infty.$$  

Suppose first that $0 < \tau^{(k,i)}(A) < \infty$.

By way of contradiction assume $\tau^{(k,i)}(I_A) = 0$. Then for any $z \in \E_d^i$ with $z(i) = 1$ we have $(I_A z)^{(k)} = 0$. Since $A$ and $z$ have all non-negative entries, term by term, we have $Az \leq A_{\max} I_A z$. Therefore

$$0 \leq (Az)^{(k)} \leq (A_{\max} I_A z)^{(k)} = A_{\max} \cdot (I_A z)^{(k)} = 0.$$  

Hence, $(Az)^{(k)} = 0$, which implies $\tau^{(k,i)}(A) = 0$. But this is a contradiction to $\tau^{(k,i)}(A) > 0$.

Now assume $\tau^{(k,i)}(I_A) = \infty$. Hence for every large $M > 0$, there exists $z_M \in \E_d^i$ with $z(i) = 1$ such that $(I_A z_M)^{(k)} > M / A_{\min}$. Again notice that term by term, we have $Az_M \geq A_{\min} I_A z_M$ and hence

$$(Az_M)^{(k)} \geq (A_{\min} I_A z_M)^{(k)} = A_{\min} \cdot (I_A z_M)^{(k)} > A_{\min} \frac{M}{A_{\min}} = M.$$  

Hence, $(Az_M)^{(k)} > M$ for every $M$, which is a contradiction to $\tau^{(k,i)}(A) < \infty$.

The reverse can be proved using similar arguments. \hfill $\square$

Because of Remark 3 and Lemma 3.3 in what follows we restrict ourselves to matrices, which have only entries 0 and 1.
Example 3.1 The following example illustrates the equivalence shown in Lemma 3.2. Suppose that

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

and $z = (z_1, z_2, z_3, z_4)^T$. Then

$$x = Az = (z_1 + z_2 + z_3 + z_4, z_1 + z_3 + z_4, z_2 + z_3 + z_4)^T.$$ 

For $k = q = 4$ we find

$$\tau^{(4,1)}(A) = \sup_{z \in E_4^{(1)}} \frac{x(4)}{z(1)} = 3 < \infty, \quad \tau^{(4,2)}(A) = \sup_{z \in E_4^{(2)}} \frac{x(4)}{z(2)} = 3 < \infty,$$

$$\tau^{(4,3)}(A) = \sup_{z \in E_4^{(3)}} \frac{x(4)}{z(3)} = \infty.$$

The supremum value of 3 in the first two cases is attained at $z = (z, z, z, z)^T$ for $z > 0$. The final equality is attained by using $z^* = (z^4, z^3, z^2, z)^T$ for $z > 0$, where $z^* \in E_4^{(3)}$. Hence according to Lemma 3.2 we have

$$A^{-1}(E_4^{(4)}) \subseteq E_4^{(2)}$$ (and by inclusion also $E_4^{(1)}$).

This means that the pre-image $A^{-1}(E_4^{(4)})$ contains vectors $z \in \mathbb{R}_+^4$, whose largest two components are positive, and the other two components can be either zero or positive. The results of Janssen and Drees (2016) (cf. Theorem 3.2) can not be applied to this example since $\tau(A) = \tau^{(4,4)}(A)$ is infinite.

3.2 Main Results

The key results extending Theorems 3.1 and 3.2, incorporating general random matrices $A \in \mathbb{R}_+^{q \times d}$ and a wide variety of tail sets is provided in this section. If $Z \in \mathcal{MRV}(\alpha, \mu, E_4^{(1)})$ satisfying $\mu(\mathbb{R}_+^d \setminus CA_d^{(d-1)}) = \mu([z \in \mathbb{R}_+^d : z(\ell) > 0]) = 0$, we may seek and find multivariate regular variation in subcones $E_d^{(i)}$ for $i = 1, \ldots, d$ as seen in Sect. 2.1. Theorem 3.4 provides the appropriate non-null limit and its rate of convergence in the presence of regular variation for $AZ$.

Definition 3.2 Let $A \in \mathbb{R}_+^{q \times d}$ be a random matrix. For $k = 1, \ldots, q$ and $\omega \in \Omega$, define $A_\omega := A(\omega)$ and

$$i_k(A_\omega) := \max\{j \in \{1, \ldots, d\} : \tau^{(k,j)}(A_\omega) < \infty\},$$

which creates a partition $\Omega^{(k)}(A) = (\Omega_i^{(k)}(A))_{i=1,\ldots,d}$ of $\Omega$ given by
\[ \Omega^{(k)}_i := \Omega^{(k)}_i(A) := \{ \omega \in \Omega : i_k(A_\omega) = i \}, \quad i = 1, \ldots, d. \]

We write \( P^{(k)}_i(\cdot) := P(\cdot \cap \Omega^{(k)}_i) \) and \( E^{(k)}_i[\cdot] := E[\cdot 1_{\Omega^{(k)}_i}] \).

Note, that due to Lemma 3.3 for a random matrix \( A \in \mathbb{R}^{q \times d}_+ \), the partition \( \Omega^{(k)}(A) \) and the partition \( \Omega^{(k)}(I_A) \) are a.s. the same and hence, \( P^{(k)}_i \) and \( E^{(k)}_i \) depend only on \( I_A \).

Our main results are formulated for fixed \( k \in \{1, \ldots, q\} \) on measure spaces \((\Omega^{(k)}_i, \mathcal{F} \cap \Omega^{(k)}_i, P^{(k)}_i)\) indexed by \( i = 1, \ldots, d \). In Theorem 3.3 we investigate the asymptotic behavior of \( AZ \) restricted to such a subspace \( \Omega^{(k)}_i \). In Theorem 3.4 below we use the asymptotic behavior of \( AZ \) on \( \Omega^{(k)}_i \) to derive the asymptotic of \( AZ \) on \( \mathbb{E}^q \) using that \( \Omega \) is the disjoint union \( \bigcup_{i=1}^d \Omega^{(k)}_i \).

**Theorem 3.3** Let \( k \in \{1, \ldots, q\} \) and \( i \in \{1, \ldots, d\} \) be fixed and \( Z \in \mathbb{R}^d \) be a random vector such that \( Z \in \mathcal{MRV}(\alpha_0, b_0, z_0) \) with canonical choice of \( b_i \) as in Definition 2.3. Also let \( A \in \mathbb{R}^{q \times d}_+ \) be a random matrix with almost surely no trivial rows and independent of \( Z \). Furthermore, assume that for some \( \delta = \delta(i, k) > 0 \) we have

\[
E^{(k)}_i\left[ (\tau^{(k,j)}(A))^{q_0+\delta} \right] := \int_{\Omega^{(k)}_i} \tau^{(k,j)}(A)^{q_0+\delta} \, dP < \infty. \tag{14}
\]

Then

\[
\frac{P^{(k)}_i(AZ \in t \cdot)}{P(\tau^{(i)}(Z) > t)} \to E^{(k)}_i\left[ \mu_i([z \in \mathbb{E}_d^{(i)} : AZ \in \cdot]) \right] =: \mu^{i,k}(\cdot), \quad t \to \infty, \tag{15}
\]

in \( \mathcal{M}(\mathbb{E}^q) \).

**Proof** If \( P(\Omega^{(k)}_i) = 0 \) then (15) is trivially satisfied because the left and the right hand side are zero. Thus we assume that \( P(\Omega^{(k)}_i) > 0 \). Let \( C \subset \mathbb{E}^q \) be a Borel set which is bounded away from \( CA^{(k-1)} \) and satisfies \( E^{(k)}_i[\mu_i(\partial A^{-1}(C))] = 0 \). Then there exists a constant \( \delta_C > 0 \) such that \( \tau^{(k)}(x) = \tau^{(k)}_d > \delta_C \) for all \( x \in C \). Using Lemma 3.1(a), we have for all \( t > 0, M > 0 \)

\[
P^{(k)}_i(AZ \in tC, \tau^{(k,j)}(A) > M) \leq P^{(k)}_i(\tau^{(k,j)}(A) > t\delta_C, \tau^{(k,j)}(A) > M) \leq P(\tau^{(k,j)}(A) \tau^{(i)}(Z) > t\delta_C, \tau^{(k,j)}(A) > M, \Omega^{(k)}_i).
\]

Since \( \tau^{(i)}(Z) = Z^{(i)} \in \mathcal{R} \cup_{-q} \), and \( A \) and \( Z \) are assumed to be independent, the univariate version of Breiman's Theorem in combination with \( E^{(k)}_i[\tau^{(k,j)}(A)^{q_0+\delta}] < \infty \) yields

\[
\limsup_{t \to \infty} \frac{P^{(k)}_i(AZ \in tC, \tau^{(k,j)}(A) > M)}{P(\tau^{(i)}(Z) > t)} \leq \limsup_{t \to \infty} \frac{P(1_{\tau^{(k,j)}(A) > M} \Omega^{(k)}_i) \tau^{(k,j)}(A) \tau^{(i)}(Z) > t\delta_C)}{P(\tau^{(i)}(Z) > t)} \leq \delta_C^{-q_0} E^{(k)}_i[\tau^{(k,j)}(A)^{q_0} 1_{\tau^{(k,j)}(A) > M} \tau^{(i)}(Z)].
\]

\[ \]
Note that \( A^{-1}(C) := \{ z \in \mathbb{R}^d : Az \in C \} \) is again a.s. bounded away from \( \mathbb{C} \mathbb{A}^{(i-1)} \), since for \( x \in C, \omega \in \Omega_i^{(k)} \), and \( z_x \in A^{-1}(C) \subseteq \mathbb{R}_+^d \) with \( A_{\omega}z_x = x \) we have by Lemma 3.1(a),

\[
\tau^{(i)}(z_x) \geq \frac{\tau^{(k)}(A_{\omega}z_x)}{\tau^{(k,i)}(A_{\omega})} = \frac{\tau^{(k)}(x)}{\tau^{(k,i)}(A_{\omega})} > \frac{\delta_C}{\tau^{(k,i)}(A_{\omega})} > 0
\]

and, thus, \( P_i^{(k)}(A^{-1}(C) \subseteq E_d^{(i)}) = 1 \). Hence abbreviating \( a := A_{\omega} \) and conditioning on \( A \), by independence of \( Z \) and \( A \), we obtain

\[
\lim_{t \to \infty} \frac{P_i^{(k)}(AZ \in tC, \tau^{(k,i)}(A) \leq M)}{P(\tau^{(i)}(Z) > t)} = \lim_{t \to \infty} \int_{\{\tau^{(k,i)}(a) \leq M\}} \frac{P(Z \in ta^{-1}(C))}{P(\tau^{(i)}(Z) > t)} dP_i^{(k)}(a) = \int_{\{\tau^{(k,i)}(a) \leq M\}} \mu_i(a^{-1}(C)) dP_i^{(k)}(a) = E_i^{(k)}\left[\mu_i(A^{-1}(C)1_{\{\tau^{(k,i)}(A) \leq M\}})\right],
\]

where we used for the third equality that \( E_i^{(k)}[\mu_i(\partial A^{-1}(C))] = 0 \) in combination with Pratt’s lemma (Pratt, 1960), since for \( \tau^{(k,i)}(A_{\omega}) \leq M \) we have for the integrand

\[
P(Z \in ta^{-1}(C)) \leq \frac{P(\tau^{(k,i)}(A_{\omega}) \tau^{(i)}(Z) > t\delta_C)}{P(\tau^{(i)}(Z) > t)} \leq \frac{P(M \tau^{(i)}(Z) > t\delta_C)}{P(\tau^{(i)}(Z) > t)} \to M^{a_k}\delta_C^{-a_k}, \quad t \to \infty.
\]

Now, we need to show that \( E_i^{(k)}[\mu_i(A^{-1}(C))] < \infty \). Define \( B_d^{(i)}(\delta) := \{ z \in \mathbb{R}^d_+ : \tau^{(i)}(z) \leq \delta \} \). By (16) and the homogeneity of \( \mu_i \) we obtain

\[
E_i^{(k)}\left[\mu_i(A^{-1}(C))\right] \leq E_i^{(k)}\left[\mu_i\left(\left(B_d^{(i)}(\delta_C/\tau^{(k,i)}(A))\right)^c\right)\right] = \mu_i\left(\left(B_d^{(i)}(\delta_C)\right)^c\right) \left(E_i^{(k)}\left[\tau^{(k,i)}(A)^{a_k}\right]\right) < \infty,
\]

which is the final piece of the proof.

\( \square \)

**Remark 5** The condition (14) and the limit \( \overline{\mu}_{i,k} \) in (15) need some explanation.

(i) The moment condition in (14) is necessary for obtaining multivariate regular variation of \( X \) and is similar to moment conditions assumed in various versions of Breiman’s Theorem.

(ii) Although the limit measure \( \overline{\mu}_{i,k} \) is finite, our result does not say that it is necessarily non-null. This is in accordance with Basrak, Davis and Mikosch [2002, Proposition A.1] (Theorem 3.1 and Remark 2). In the following proposition, we provide a sufficient condition for \( \overline{\mu}_{i,k} \) to be non-null.

**Proposition 3.1** Let the assumptions and notations of Theorem 3.3 hold. Moreover, assume that \( P(\Omega_i^{(k)}) > 0 \) and for the \( i \)-dimensional hyperplanes we have
\( \mu_t(\overline{C\ell}_d(j)) > 0 \) for all \( j = 1, \ldots, \binom{d}{i} \). Then the limit measure \( \overline{\mu}_{i,k} \) in (15) is a non-null measure. Moreover, \( AZ \in \mathcal{M}\mathcal{R}\mathcal{V}(\alpha_i, \overline{\mu}_{i,k}, \mathbb{E}_q^{(k)}) \) under the probability measure \( \mathbb{P}_i^{(k)} \).

**Proof** We show that \( \overline{\mu}_{i,k}(\mathbb{E}_q^{(k)}) = \mathbb{E}_i^{(k)}[\mu_t(A^{-1}(\mathbb{E}_q^{(k)}))] > 0. \)

**Case 1:** Suppose \( 1 \leq i < d \). Let \( \omega \in \Omega_d^{(d)} \). We know from Lemma 3.2 that \( A_{0,1}^{-1}(\mathbb{E}_q^{(k)}) \subseteq \mathbb{E}_d^{(i)} \). By definition, \( C\ell_d^{(i)} \setminus C\ell_d^{(i-1)} \subseteq \mathbb{E}_d^{(i)} \). We claim that

\[
(C\ell_d^{(i)} \setminus C\ell_d^{(i-1)}) \cap A_{0,1}^{-1}(\mathbb{E}_q^{(k)}) \neq \emptyset.
\]

If not, then we have \( A_{0,1}^{-1}(\mathbb{E}_q^{(k)}) \subseteq \mathbb{E}_d^{(i)} \setminus (C\ell_d^{(i)} \setminus C\ell_d^{(i-1)}) = \mathbb{R}_+^d \setminus C\ell_d^{(i)} = \mathbb{E}_d^{(i+1)} \). Therefore by Lemma 3.2, \( \tau^{(k,j+1)}(A_{0,1}) < \infty \). But this is a contradiction to the definition of \( \Omega_d^{(k)} \) since \( \omega \in \Omega_d^{(k)} \).

So let \( z \in (C\ell_d^{(i)} \setminus C\ell_d^{(i-1)}) \cap A_{0,1}^{-1}(\mathbb{E}_q^{(k)}) \). Then by (6), we have \( z \in C\ell_d^{(i)}(j^*) \) for some \( 1 \leq j^* \leq \binom{d}{i} \). Let \( I_z := \{ j \in \{1, \ldots, d\} : z_j > 0 \} \). Clearly,

\[
C\ell_d^{(i)}(j^*) = \{ z \in \mathbb{R}_+^d : z_j > 0 \text{ for } j \in I_z \text{ and } z_j = 0 \text{ for } j \in \{1, \ldots, d\} \setminus I_z \}.
\]

Hence, for every \( z^* \in C\ell_d^{(i)}(j^*) \) we have that some component of \( A_{0,1}z^* \) is positive if and only if the corresponding component of \( A_{0,1}z \) is positive, since \( A_{0,1} \) has only non-negative entries. Thus \( A_{0,1}z^* \in \mathbb{E}_q^{(k)} \), i.e., \( z^* \in A_{0,1}^{-1}(\mathbb{E}_q^{(k)}) \). Hence, we get that

\[
C\ell_d^{(i)}(j^*) \subseteq A_{0,1}^{-1}(\mathbb{E}_q^{(k)}) \subseteq \mathbb{E}_d^{(i)}.
\]

Since \( \mu_t \) has positive mass on each of the \( \binom{d}{i} \) hyperplanes \( C\ell_d^{(i)}(j) \), this results in

\[
\mu_t(A_{0,1}^{-1}(\mathbb{E}_q^{(k)})) \geq \mu_t(C\ell_d^{(i)}(j^*)) \geq \min_j \mu_t(C\ell_d^{(i)}(j)) > 0,
\]

and

\[
\mathbb{E}_i^{(k)}[\mu_t(A^{-1}(\mathbb{E}_q^{(k)}))] \geq \min_j \mu_t(C\ell_d^{(i)}(j)) \mathbb{P}^{(k)}(\Omega_i^{(k)}) > 0,
\]

which proves the claim for \( 1 \leq i < d \).

**Case 2:** Suppose \( i = d \). Let \( \omega \in \Omega_d^{(d)} \). Take \( z \in A_{0,1}^{-1}(\mathbb{E}_q^{(k)}) \subseteq \mathbb{E}_d^{(d)} \), then all components of \( z \) and \( A_{0,1}z \in \mathbb{E}_q^{(k)} \) are positive. Thus \( A_{0,1} \) has no trivial row and we get that for every \( z^* \in \mathbb{E}_d^{(d)} \) also \( A_{0,1}z^* \) has only positive components, i.e., \( A_{0,1}z^* \in \mathbb{E}_q^{(k)} \). This results in \( \mathbb{E}_d^{(d)} = A_{0,1}^{-1}(\mathbb{E}_q^{(k)}) \) and \( \mathbb{E}_d^{(d)}[\mu_t(A^{-1}(\mathbb{E}_q^{(k)}))] = \mu_d(\mathbb{E}_q^{(k)}) \mathbb{P}^{(k)}(\Omega_d^{(k)}) > 0 \).

As a consequence of (15) and \( \overline{\mu}_{i,k}(\mathbb{E}_q^{(k)}) > 0 \), we have \( AZ \in \mathcal{M}\mathcal{R}\mathcal{V}(\alpha_i, \overline{\mu}_{i,k}, \mathbb{E}_q^{(k)}) \) under the probability measure \( \mathbb{P}_i^{(k)} \). \( \square \)
Remark 6

(i) Proposition 3.1 shows how regular variation of $Z$ on the space $E_q(i)$ transforms to regular variation of $X = AZ$ on the space $E_q(k)$ under the probability measure $\mu_i(k)$ for fixed $i \in \{1, \ldots, d\}$ and fixed $k \in \{1, \ldots, q\}$.

(ii) Proposition 3.1 only provides a sufficient condition for the limit measure to be non-null. In many examples, the measures $\mu_i$ turn out to be exchangeable with respect to their co-ordinates and the assumption being true for all $j = 1, \ldots, (d/i)$ is not uncommon. In particular, Example 2.1 where $Z_i$ are iid Pareto$(a)$ for $a > 0$ is an easy and often used example of $Z$. In fact, we have $\mu_i(C^{(i)}_{A_d}(j)) > 0$ for all $j = 1, \ldots, (d/i)$ for all examples in this paper. Popular dependence structures, say, defined using Gaussian or Archimedean copulas will often have this property. A possible relaxation of this condition will assume that $\mu_i(C^{(i)}_{A_d}(j)) > 0$ for at least one $j \in \{1, \ldots, (d/i)\}$ or for some subset of $j$’s, which will put additional non-trivial assumptions on $A$.

In the following theorem, we apply Theorem 3.3 to identify the appropriate subcones of $\mathbb{R}_+^q$ where $AZ$ has regular variation with a non-null limit measure. To be more precise, for fixed $k \in \{1, \ldots, q\}$ we derive the asymptotic behavior of $AZ$ on $E_q(k)$.

Theorem 3.4 Let $Z \in \mathbb{R}_+^d$ be a random vector such that for all $i = 1, \ldots, d$, we have $Z \in \text{MRV}((\alpha_i, b_i, \mu_i, E_q(i))$ with canonical choices of $b_i$ as in Definition 2.3. Moreover, for all $l = 1, \ldots, d - 1$ we assume $b_l(t)/b_{l+1}(t) \to \infty$ as $t \to \infty$. Let $A \in \mathbb{R}_{+}^{q \times d}$ be a random matrix with almost surely no trivial rows and independent of $Z$. For fixed $k \in \{1, \ldots, q\}$ let $C \subset E_q(k)$ be a Borel set bounded away from $C^{(k-1)}_{A_q}$ with $E_{i}^{(k)}[\mu_i(A^{-1}(C))] = 0$ for all $i = 1, \ldots, d$. Suppose further that for all $i = 1, \ldots, d$ we have $E_{i}^{(k)}[(\sigma^{(k)}(A))^{q_i+\delta}] < \infty$ for some $\delta = \delta(i, k) > 0$. Then the following results hold.

(a) We have

$$P(AZ \in tC) = \sum_{i=1}^{d} P(Z^{(i)} > t) \left[ E_{i}^{(k)}[\mu_i(A^{-1}(C))] + o(1) \right], \quad t \to \infty. \quad (17)$$

(b) Define

$$i^*_k := \arg\min\{i \in \{1, \ldots, d\} : P(\Omega_i^{(k)} > 0)\}. \quad (18)$$

Then we have

$$\frac{P(AZ \in tC)}{P(Z^{(i^*_k)} > t)} \to \frac{E_{i^*_k}^{(k)}[\mu_{i^*_k}(A^{-1}(C))]}{\mu_{i^*_k}(C)} = \tilde{\mu}_{i^*_k}(C) =: \mu_k(C), \quad t \to \infty, \quad (19)$$
in $\mathbb{M}(E^{(k)}_q)$. Moreover, if $\mu_k$ is a non-null measure then $AZ \in \mathcal{MRV}(\alpha, \mu_k, E^{(k)}_q)$.

**Proof**

(a) Since $\{\Omega^{(k)}_i, 1 \leq i \leq d\}$ forms a partition of $\Omega$, $P(AZ \in tC) = \sum_{i=1}^d P_i^{(k)}(AZ \in tC)$. Hence using (15) and observing that

$$P(t^{(i)}(Z) > t) = P(Z^{(i)} > t) \sim 1/b_i^-(t), \quad t \to \infty,$$

we have

$$P(AZ \in tC) = \sum_{i=1}^d \left[ P(Z^{(i)} > t) \left[E_i^{(k)}[\mu_i(A^{-1}(C))] + o(1)\right]\right], \quad t \to \infty.$$

(b) Now, since $\{\Omega^{(k)}_i, 1 \leq i \leq d\}$ forms a partition of $\Omega$, there exists a $j \in \{1, \ldots, d\}$ with $P(\Omega^{(k)}_i) > 0$ and hence, $i^*_k$ is well-defined. Note that using (17), we have for any Borel set $C \subset E^{(k)}_q$ bounded away from $C_{A_{d-1}^{(k-1)}}$ with $E_i^{(k)}[\mu_i(\partial A^{-1}(C))] = 0$ for $i = 1, \ldots, d$, the following asymptotic behavior:

$$P(AZ \in tC) = \frac{P(Z^{(i)} > t)}{P(Z^{(i)}_b > t)} = \left[ E_i^{(k)}[\mu_i(A^{-1}(C))] + o(1) \right]$$

$$+ \sum_{i=i^*_k+1}^d \left[ \frac{P(Z^{(i)} > t)}{P(Z^{(i)}_b > t)} E_i^{(k)}[\mu_i(A^{-1}(C))] + \frac{o(P(Z^{(i)} > t))}{P(Z^{(i)}_b > t)} \right]$$

$$\sim E_i^{(k)}[\mu_i(A^{-1}(C))] = \overline{\mu}(C), \quad t \to \infty,$$

since for all $i = i^*_k + 1, \ldots, d$ we have $b_i^- / b_i^-(t) \to \infty$, and hence

$$\frac{P(Z^{(i)} > t)}{P(Z^{(i)}_b > t)} \sim \frac{b_i^-}{b_i^-} \to 0, \quad t \to \infty.$$

Now, if $\overline{\mu} \neq 0$ then $AZ \in \mathcal{MRV}(\alpha, \mu_k, E^{(k)}_q)$. □

**Remark 7** The assumption $Z \in \mathcal{MRV}(\alpha, b, \mu, E^{(i)}_d)$ for all $i = 1, \ldots, d$ with $b_i(t) / b_{i+1}(t) \to \infty$ as $t \to \infty$ for all $i = 1, \ldots, d - 1$ restricts the supports of the measures $\mu_i$ to $C_{A_{d-1}^{(i)}} \setminus C_{A_{d-1}^{(i-1)}}$; a specific case is discussed in Example 2.1.

**Remark 8** We are now able to indicate extensions of our work from matrices $A \in \mathbb{R}^{q,d}$ with non-negative entries to matrices $A \in \mathbb{R}^{q,d}$ by comparing our previous results to Janssen and Drees [2016, Theorem 2.3] (Theorem 3.2 of this paper). Namely, if for any general matrix $A \in \mathbb{R}^{q,d}$ we impose the condition that it has full rank and that its left inverse $A^{-} = (A^TA)^{-1}A^T$ has non-negative entries, then we can find regular variation of $X = AZ$ on the space $E^{(q)}_q$. This follows from an analog of
Janssen and Drees [2016, Lemma 2.1] and we obtain the analogous result of 3.2. We leave further exploration in this direction for future research.

**Remark 9** If \( Z \) has asymptotically independent components, and each component has distribution tail \( P(Z_j > t) \sim \kappa_j t^{-\alpha} \) as \( t \to \infty \) for some \( \alpha, \kappa_j > 0 \), then we get a generalization of Theorem 3.2 of Kley et al. (2016). We investigate such structures further in the next section.

The following example illustrates the image and pre-image of sets under the map \( A : z \mapsto Az = x \) as well as the regions, where the limit measure is positive in a 3-dimensional setting. This example is not covered by Janssen and Drees (2016) (see Theorem 3.2), since \( AZ \) has rate \( t^{-2\alpha} \) on \( \mathbb{E}^{(3)}_3 \), and Basrak et al. (2002) (see Theorem 3.1) give a zero estimate in the limit. We emphasize that for this example our theory is not really necessary, the calculations can be done by hand, but it helps clarifying the ideas and the notation; more complex examples follow in Sect. 4.

**Example 3.2** Let \( Z = (Z_1, Z_2, Z_3)^T \) have iid Pareto(\( \alpha \)) components with \( P(Z_i > z) = z^{-\alpha}, z > 1 \) for some \( \alpha > 0 \) as in Example 2.1. Then \( Z \in \mathcal{MRV}(\alpha, b_i, \mathbb{E}^{(i)}_3) \) for \( i = 1, 2, 3 \) where \( b_1(t) = (3t)^{1/\alpha}, b_2(t) = (3t)^{1/(2\alpha)} \) and \( b_3(t) = t^{1/(3\alpha)} \) are the canonical choices and

\[
\mu_1\left( \bigcup_{i=1}^{3} \{ v \in \mathbb{R}^3_+ : v_i > z_j \} \right) = \frac{1}{3}(z_1^{-\alpha} + z_2^{-\alpha} + z_3^{-\alpha}),
\]

\[
\mu_2\left( \bigcup_{1 \leq i \neq j \leq 3} \{ v \in \mathbb{R}^3_+ : v_i > z_i, v_j > z_j \} \right) = \frac{1}{3}\{(z_1z_2)^{-\alpha} + (z_2z_3)^{-\alpha} + (z_3z_1)^{-\alpha}\},
\]

\[
\mu_3((z_1, \infty) \times (z_2, \infty) \times (z_3, \infty)) = (z_1z_2z_3)^{-\alpha}
\]

for \( z_1, z_2, z_3 > 0 \). Consider the matrix

\[
A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.
\]

Then, under the map \( A : z \mapsto Az = x \), the region \( C = (1, \infty)^3 \subset \mathbb{E}^{(3)}_3 \) has pre-image given by

\[
A^{-1}(C) = \{ z \in \mathbb{R}^3_+ : z_1 > 1, z_2 > 1 \} \cup \{ z \in \mathbb{R}^3_+ : z_2 > 1, z_3 > 1 \} \\
\cup \{ z \in \mathbb{R}^3_+ : z_3 > 1, z_1 > 1 \}.
\]

It is easy to check that \( i_3^* = 2 \), as defined in (18). Hence, with \( \mu_2(A^{-1}(C)) = 1 \) we obtain

\[
P(AZ \in tC) \sim P(Z^{(2)} > t)\mu_2(A^{-1}(C)) \sim 3t^{-2\alpha}, \quad t \to \infty.
\]
Figure 1 gives a plot of the region $C$ and the transformed region $A^{-1}(C)$ colored in blue. The red region on the right plot shows the support of the measure $\mu_2$.

Remark 10 In Theorem 3.4, we ascertain the asymptotic behavior of $P(AZ \in tC)$ for certain sets $C \subset \mathbb{E}^{(k)}_q$. Specifically, from Theorem 3.4 (b) we have

$$P(AZ \in tC) = P(Z^{(\bar{t})} > t) E^{(k)}_{t,\bar{t}} [\mu_{t,\bar{t}}(A^{-1}(C))] + o(P(Z^{(\bar{t})} > t), \quad t \to \infty,$$

where $i^*_k$ is defined in (18). If $E^{(k)}_{t,\bar{t}} [\mu_{t,\bar{t}}(A^{-1}(C))] = 0$, we only get

$$P(AZ \in tC) = o(P(Z^{(\bar{t})} > t), \quad t \to \infty.$$

However, under certain assumptions on $A$ and $C$ we can say more about the precise rates, illustrated in the following results.

Proposition 3.2 Let the assumptions and notations of Theorem 3.4 hold. Define

$$\bar{t} = \bar{t}_C := \min \{d, \inf \{i \in \{i^*_k, \ldots, d\} : E^{(k)}_{i,\bar{t}} [\mu_{i,\bar{t}}(A^{-1}(C))] > 0\}\}.$$  

(20)

Suppose for all $i = i^*_k, \ldots, \bar{t} - 1$ and $\omega \in \Omega^{(k)}_i$ that $A^{-1}_\omega(C) = \emptyset$. Then

$$P(AZ \in tC) = P(Z^{(\bar{t})} > t) E^{(k)}_{\bar{t},\bar{t}} [\mu_{\bar{t},\bar{t}}(A^{-1}(C))] + o(P(Z^{(\bar{t})} > t)), \quad t \to \infty.$$  

(21)

Proof Since by assumption, for all $i = i^*_k, \ldots, \bar{t} - 1$ and $\omega \in \Omega^{(k)}_i$, we have $A^{-1}_\omega(C) = \emptyset$, we obtain

$$P^{(k)}_{i,\bar{t}}(AZ \in tC) = 0.$$

Therefore due to the definition of $i^*_k$,
\[
P(AZ \in tC) = \sum_{i=1}^{d} P_{i}^{(k)}(AZ \in tC)
\]
\[
= \sum_{i=1}^{d} P_{i}^{(k)}(AZ \in tC)
\]
\[
= P(Z^{(i)}> t) E_{i}^{(k)}[\mu_{i}(A^{-1}(C))] + o(P(Z^{(i)}> t)), \quad t \to \infty.
\]

using Theorem 3.3. \hfill \Box

The additional assumption made in Proposition 3.2 is often satisfied by random matrix structures. One such example is a random matrix with only one positive entry in each row. Such matrices are for instance proposed in the examples of Sect. 4.2. Moreover, if \( A \in \mathbb{R}^{d \times d} \) and we follow the assumptions of Janssen and Drees [2016, Theorem 2.3], we also obtain such matrices; cf. Remark 4(iii). The following proposition formalizes the result in this case.

**Proposition 3.3** Let the assumptions and notations of Theorem 3.4 hold. Moreover, let \( C \subset E_{q}^{(k)} \) be such that \( C = \bigcup_{i=1}^{l} \Gamma_{i} \) for some \( N \in \mathbb{N} \), where each \( \Gamma_{i} \) is of the form:
\[
\Gamma = \{ x \in \mathbb{R}_{+}^{q} : x_{j_{1}} > \gamma_{1}, \ldots, x_{j_{l}} > \gamma_{l} \}
\]
and \( E_{i}^{(k)}[\mu_{i}(\partial A^{-1}(C))] = 0 \) for all \( i = 1, \ldots, d \). Suppose \( i = i_{C} \) is defined as in (20) and \( \mu_{i}(\bar{CA}_{d}^{(j)}(j)) > 0 \) for all \( j = 1, \ldots, \left( {d \atop i} \right) \) and \( i = i_{k}, \ldots, i_{-1} \). If the random matrix \( A \) has a discrete distribution and has exactly one positive entry in each row, then (21) holds.

**Proof** If we show that for all \( i = i_{k}, \ldots, i - 1 \) and \( \omega \in \Omega_{i}^{(k)} \), we have \( A_{-1}^{\omega}(C) = \emptyset \), then applying Proposition 3.2 we get the result. Fix \( i \in \{ i_{k}, \ldots, i_{-1} \} \). By definition, we have \( E_{i}^{(k)}[\mu_{i}(A^{-1}(C))] = 0 \). Suppose there exists \( \omega \in \Omega_{i}^{(k)} \) with
\[
A_{-1}^{\omega}(C) \cap (CA_{d}^{(i)} \setminus CA_{d}^{(i-1)}) \neq \emptyset.
\]
Then there exists \( x^{*} \in C \subset E_{q}^{(k)} \) and \( z^{*} \in CA_{d}^{(i)} \setminus CA_{d}^{(i-1)} \subset E^{(i)} \) with \( A_{\omega}z^{*} = x^{*} \in C \). Since \( z^{*} \in CA_{d}^{(i)} \setminus CA_{d}^{(i-1)} \), exactly \( i \) components of \( z^{*} \) are positive. Without loss of generality let \( z_{j_{1}}, \ldots, z_{j_{i}} > 0 \). Now for any \( z = (z_{1}, \ldots, z_{i}, 0, \ldots, 0) \) with \( z_{j} \geq z_{j}^{*} \), we have \( Az \geq Az^{*} = x^{*} \) and by the structure of \( C \), we have \( Az \in C \). Hence
\[
\{ z \in \mathbb{R}_{+}^{d} : z_{j} \geq z_{j}^{*}, j = 1, \ldots, i \text{ and } z_{i+1} = \ldots = z_{d} = 0 \} \subseteq A_{-1}^{\omega}(C) \cap (CA_{d}^{(i)} \setminus CA_{d}^{(i-1)}).
\]
Now, due to \( \mu_{i}(\bar{CA}_{d}^{(j)}(j)) > 0 \) for all \( j = 1, \ldots, \left( {d \atop i} \right) \) and the homogeneity of the measure \( \mu_{i} \), we have
0 < \mu_i \left( \{ z \in \mathbb{R}_+^d : z_j \geq z_j^*, j = 1, \ldots, i \text{ and } z_{i+1} = \ldots = z_d = 0 \} \right) \\
\leq \mu_i(A_{\omega}^{-1}(C)) \leq \frac{E_i^{(k)}[\mu_i(A^{-1}(C))]}{P(A = A_{\omega})},
\text{since } A \text{ has a discrete distribution and } P(A = A_{\omega}) > 0. \text{ Hence } E_i^{(k)}[\mu_i(A^{-1}(C))] > 0, \text{ which is a contradiction. Thus}
A_{\omega}^{-1}(C) \cap (\mathbb{C}A_d^{(i)} \setminus \mathbb{C}A_d^{(i-1)}) = \emptyset. \quad (22)
\text{Now suppose that } A_{\omega}^{-1}(C) \neq \emptyset. \text{ Then there exist } x \in C \subset \mathbb{E}^{(k)}_q \text{ and } z \in \mathbb{E}^{(i)}_d \text{ with } A_{\omega}z = x \in C. \text{ Since } \omega \in \Omega_i^{(k)}, \text{ exactly } i \text{ columns of } A_{\omega} \text{ have at least one positive entry. W.l.o.g. assume these are the first } i \text{ columns of } A_{\omega}. \text{ Then for}

z^* := (z_1, \ldots, z_i, 0, \ldots, 0) \in \mathbb{C}A_d^{(i)} \setminus \mathbb{C}A_d^{(i-1)}

\text{we have}
A_{\omega}z^* = A_{\omega}x = x \in C
\text{since columns } i+1, \ldots, d \text{ of } A_{\omega} \text{ have all entries zero and hence the last } d-i \text{ entries of } z \text{ or } z^* \text{ do not count towards the computation of } x. \text{ Hence } z^* \in A_{\omega}^{-1}(C) \cap (\mathbb{C}A_d^{(i)} \setminus \mathbb{C}A_d^{(i-1)}), \text{ which is a contradiction to } (22). \text{ This gives the statement.} \square

4 Bipartite networks

Risk-sharing in complex systems is often modeled using a graphical network model, one such example being the bipartite network structure for modeling sharing of losses in insurance markets or investment into overlapping portfolios as proposed in Kley et al. (2016, 2018). In these papers, only first order asymptotics of risk measures based on the agents’ and market’s tail risks are derived. In the same spirit, but going beyond first order approximations, we consider a vertex set of agents \( A = \{1, \ldots, q\} \) and a vertex set of objects \( O = \{1, \ldots, d\} \).

Each agent \( k \in A \) chooses a number of objects \( i \in O \) to connect with. Figure 2 provides an example of such a network. This choice can be random according to some probability distribution. A basic model assumes \( k \) and \( i \) connect with probability \( \mu_i \).
Let $Z_i$ denote the risk attributed to the $i$-th object and $Z = (Z_1, \ldots, Z_d)^T$ forms the risk vector. Assume that the graph creation process is independent of $Z$. The proportion of loss of object $i$ affecting agent $k$ is denoted by $f_k(Z_i) = 1(k \sim i)W_{ki}Z_i$, where $W_{ki} > 0$ denotes the effect of the $i$-th object on the $k$-th agent. Now define the weighted adjacency matrix $A = (A_{ki}) \in \mathbb{R}^{q \times d}_+$ by $A_{ki} = 1(k \sim i)W_{ki}$.

The total exposure of the agents is given by $X = (X_1, \ldots, X_q)^T$, where $X_k = \sum_{i=1}^{d} f_k(Z_i)$ can be represented as $X = AZ$.

Our goal is to find the probability of tail risks of some or all agents in terms of $X$.

In Kley et al. (2016, 2018), proportional weights are used to distribute the insurance loss of object $i$ affecting agent $k$ or to diversify the investment risk of an agent. From Lemma 3.3 we know that $0 < \tau^{(k,i)}(A) < \infty$ a.s. is independent of the weights $W_{ki}$, hence the partitions $\Omega^{(k)}(A)$ and $\Omega^{(k)}(I_A)$ (where $I_A = (1(k \sim i)) \in \mathbb{R}^{q \times d}$) of $\Omega$ from Definition 3.2 are a.s the same. This implies that the value $i_k^*$ of (18) is the same for the weighted adjacency matrix $A$ and the unweighted matrix $I_A$. The weights only affect the values of the limit measures, resulting in different positive constants, whereas the rate of convergence remains the same.

Consequently, we work in Sect. 4.1 with unweighted adjacency matrices $A = I_A$ with focus on the random structures of the adjacency matrices. Weights can be incorporated in the calculations by appropriate multiplications using the independence of $A$ and $Z$. We consider a single weighted adjacency matrix in Sect. 4.2, when we investigate dependent objects in contrast to independent ones; see Examples 4.2 and 4.3 below.

Throughout we formulate our results and examples in terms of investment risk.
4.1 Independent objects

For illustration we start with an example, which shows how regular variation of $X = AZ$ for independent Pareto-tailed components of $Z$ and random adjacency matrices transforms into regular variation of $X$. The choice of the set $C$ specifies the tail risk, and in this example we calculate the asymptotic tail risk explicitly for two different risk sets leading to two different asymptotic rates. This example is beyond the scope of Janssen and Drees (2016) since $A$ is not a square matrix, and Basrak et al. (2002) give zeros estimates.

**Example 4.1** Suppose there are two investment possibilities (objects) $O_1$ and $O_2$ in the market with associated risks $Z_1$ and $Z_2$ respectively, which are independent and $P(Z_i > z) \sim \kappa_i z^{-\alpha}$ for $\alpha > 0$ as $z \to \infty$ with constants $\kappa_i > 0$ for $i = 1, 2$. Note that $Z \in \mathcal{MRV}(\alpha, b_1, \mu_1, \mathbb{E}_1^{(1)})$ and $Z \in \mathcal{MRV}(2\alpha, b_2, \mu_2, \mathbb{E}_2^{(2)})$ where $b_1(t) = ((\kappa_1 + \kappa_2)t)^{1/\alpha}, b_2(t) = (\kappa_1\kappa_2 t)^{1/(2\alpha)}$ and for $(z_1, z_2) \in (0, \infty)^2$,

$$
\mu_1(\{v \in \mathbb{E}_1^{(1)} : v_1 > z_1 \text{ or } v_2 > z_2\}) = \frac{\kappa_1}{\kappa_1 + \kappa_2} z_1^{-\alpha} + \frac{\kappa_2}{\kappa_1 + \kappa_2} z_2^{-\alpha},
$$

$$
\mu_2(\{v \in \mathbb{E}_2^{(2)} : v_1 > z_1, v_2 > z_2\}) = (z_1 z_2)^{-\alpha}.
$$

Assume there are three investors (agents), each of whom may either invest in $O_1$ or in $O_2$ or in both (we assume that they always invest). Hence there are $3 \times 3 \times 3 = 27$ possible market investment strategies, which are represented by the adjacency matrix $A = (\mathbf{1}(k \sim i)) \in \mathbb{R}^{q \times d}$. Assume $Z$ is independent of $A$.

Our first aim is to derive the partition of $\Omega$ from Definition 3.2 and find $i^*_k$ of (18), and we present the 27 realisations of the random matrix $A$:

$$
A_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix},
$$

$$
A_4 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad A_5 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad A_6 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \quad A_7 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad A_8 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad A_9 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix},
$$

$$
A_{10} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad A_{11} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad A_{13} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad A_{14} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad A_{15} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},
$$

$$
A_{16} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_{17} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad A_{18} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad A_{19} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad A_{20} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad A_{21} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix},
$$

$$
A_{22} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad A_{24} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad A_{25} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_{26} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_{27} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.
$$

We can check that for $m = 1, \ldots, 15,$
\[
\tau^{(k,1)}(A_m) < \infty, \quad \tau^{(k,2)}(A_m) = \infty, \quad k = 1, 2, 3.
\]

Hence, for \( m = 1, \ldots, 15 \), we get \( i_k(A_m) = 1 \) for \( k = 1, 2, 3 \).

On the other hand for \( m = 16, \ldots, 27 \), we observe that

\[
\tau^{(k,1)}(A_m) < \infty, \quad k = 1, 2, 3, \quad \tau^{(3,2)}(A_m) < \infty, \quad \tau^{(2,2)}(A_m) = \tau^{(1,2)}(A_m) = \infty.
\]

Therefore, for \( m = 16, \ldots, 27 \), we get \( i_3(A_m) = 2 \) and \( i_2(A_m) = i_1(A_m) = 1 \).

For each \( k = 1, 2, 3 \) we obtain partitions of \( \Omega \) as follows. For \( k = 1, 2 \) we obtain \( \Omega_1^{(1)} = \Omega_1^{(2)} = \Omega \). For \( k = 3 \) we obtain \( \Omega_1^{(3)} = \{1, \ldots, 15\} \) and \( \Omega_2^{(3)} = \{16, \ldots, 27\} \). Hence, Theorem 3.4 applies and by (18), we have \( i_1^* = i_2^* = i_3^* = 1 \) such that part (b) may yield a limit result with rate \( t^{-\alpha} \).

We consider the non-weighted risk \( X = AZ \). Suppose we want to assess the risk of all investors being above a high threshold \( t > 0 \). Moreover, we also want to find the probability that, when all risks of the investors are above \( t \), the risk of the first investor is larger than that of the second which is larger than the one of the third, i.e., \( X_1 > X_2 > X_3 > t \). Hence given \( t > 0 \) and

\[
C_1 = \{x \in E_3^{(3)} : x_i > 1, i = 1, 2, 3\} = (1, \infty),
\]

\[
C_2 = \{x \in E_3^{(3)} : x_1 > x_2 > x_3 > 1\},
\]

we want to compute \( P(X \in tC_i) \) for \( i = 1, 2 \).

Note that both \( C_1, C_2 \subset E_3^{(3)} \), and since \( i_3^* = 1 \), we apply Theorem 3.4(b) and obtain \( X \in \mathcal{M}(\mathcal{R})\mathcal{V}(\alpha, b_1(t) = ((\kappa_1 + \kappa_2)t)^{1/\alpha}, \mu_3, E_3^{(3)}) \), where

\[
\overline{\mu}_3(\cdot) = E_1^{(3)}[\mu_1(A^{-1}(\cdot))].
\]

In order to compute the constants \( \overline{\mu}_3(C_1) \) and \( \overline{\mu}_3(C_2) \) as in (19) we need the following for \( q_m := P(A = A_m) \geq 0 \):

\[
P(\Omega_1^{(3)}) = \sum_{m=1}^{15} q_m = q_1 + Q_1 + Q_2 > 0 \quad \text{and} \quad q_{16} + q_{19} > 0,
\]

respectively, where

\[
Q_1 = q_2 + q_4 + q_5 + q_6 + q_{10} + q_{11} + q_{12}, \quad Q_2 = q_3 + q_7 + q_8 + q_9 + q_{13} + q_{14} + q_{15}.
\]

(i) We first consider the set \( C_1 \) as defined in (25). Since

\[
E_1^{(3)}[\mu_1(A^{-1}(C_1))] = \sum_{m=1}^{15} q_m \mu_1(\{z \in E_2^{(1)} : A_mz \in C_1\}) = \frac{(q_1 + Q_1)\kappa_1 + (q_1 + Q_2)\kappa_2}{\kappa_1 + \kappa_2} > 0,
\]

Theorem 3.4(b) gives

\[\heartsuit\] Springer
\[P(X \in tC_1) \sim P(Z^{(1)}> t) E_1^{(3)}[\mu_1(A^{-1}(C_1))]\\sim [(q_1 + Q_1)\kappa_1 + (q_1 + Q_2)\kappa_2] t^{-\alpha}, \quad t \to \infty.\]

(ii) Now consider the set \(C_2\) as in (26). Note that in this case we have \(A_m^{-1}(C_2) = \emptyset\) for \(m = \{1, \ldots, 27\} \setminus \{16, 19\}\), and hence

\[E_1^{(3)}[\mu_1(A^{-1}(C_2))] = \sum_{m=1}^{15} \mu_1(A_m^{-1}(C_2))q_m = 0, \quad \text{but}\]

\[E_2^{(3)}[\mu_2(A^{-1}(C_2))] = q_{16}\mu_2([z \in E_2^{(2)} : A_{16}z \in C_2]) + q_{19}\mu_2([z \in E_2^{(2)} : A_{19}z \in C_2]) = \frac{1}{2}(q_{16} + q_{19}) > 0.\]

Now using Theorem 3.4(b) or Basrak et al. [2002, Proposition A.1], we get \(\lim_{t \to \infty} tP(X \in tC_1) = 0\). But Proposition 3.2 with \(i = i_{C_2} = 2\) yields

\[P(X \in tC_2) \sim P(Z^{(2)}> t) E_2^{(3)}[\mu_2(A^{-1}(C_2))] \sim \frac{1}{2}\kappa_1\kappa_2(q_{16} + q_{19}) t^{-2\alpha}, \quad t \to \infty.\]

(iii) Finally, suppose that \(q_1 + Q_1 + Q_2 = 0\) and we consider the set \(C_1\). From (27), we have \(E_1^{(3)}[\mu_1(A^{-1}(C_1))] = 0\) and hence Theorem 3.4(b) (as well as Basrak et al. [2002, Proposition A.1]) gives \(\lim_{t \to \infty} tP(X \in tC_1) = 0\). But note that

\[E_2^{(3)}[\mu_2(A^{-1}(C_1))] = \sum_{m=16}^{27} q_m\mu_2([z \in E_2^{(1)} : A_mz \in C_1]) = \sum_{m=16}^{27} q_m = 1. \quad (28)\]

Hence we can use Theorem 3.4(a) to get as \(t \to \infty\),

\[P(X \in tC_1) = \sum_{i=1}^{3} P(Z^{(i)}> t) \left[ E_i^{(3)}[\mu_i(A^{-1}(C))] + o(1) \right] \sim P(Z^{(2)}> t) = \kappa_1\kappa_2 t^{-2\alpha},\]

the asymptotic convergence rate.

Examples for different choices of risk sets \(C\) are plenty considering the numerous risk situations for a group of agents. In what follows, we address tail risks for extreme events where the portfolio risk of all agents are above a high threshold in a more systematic way. Such events are represented by sets of the form \(t(x, \infty)\). In case we want to study the problem for a specific set of agents, we need only to consider a reduced set of rows of the adjacency matrix \(A\).

In the following we suppose that each agent takes investment decisions according to a probability distribution, where the agents’ choices are independent of each other. Hence, we may assume for each agent \(k \in A = \{1, \ldots, q\}\) that there exist subsets \(J_{k1}, \ldots, J_{km_k}\) of investments \(O = \{1, \ldots, d\}\) such that

\[P(A_{ki} = 1 \text{ for } i \in J_{ki}, \text{ and } A_{ki} = 0 \text{ for } i \in J_{ki}^c) = p_{kl} \quad (29)\]
for \( p_{kl} \in [0, 1] \) and \( \sum_{i=1}^{m_k} p_{kl} = 1 \).

The following two propositions describe specific scenarios of (29), where agents choose exactly one investment uniformly out of a possible set, but some agents disregard one or more specific investments right away. To invest into one investment possibility is a risk averse strategy for small \( \alpha \). According to Remark 13.3(b) of Rüschendorf (2013), for \( \alpha \leq 1 \) portfolio diversification does not reduce the danger of extreme losses, but typically increases extreme risks. Moreover, the investment possibilities are independent of each other and all agents take their investment decisions independently. Because of the structure of the resulting adjacency matrix, the tail probabilities for the risk exposure of all agents can be computed explicitly using the multivariate regular variation on each relevant cone.

From a mathematical point of view, the two propositions below also show exactly how our results extend those of Janssen and Drees [2016, Theorem 2.3] (Theorem 3.2 of this paper) in multiple directions. As already discussed in the Introduction and Remark 4 we allow for a non-square matrix \( A \). For computational ease we restrict to independent components of \( Z \), resulting in \( \mathbb{Z} \in \mathbb{M} \mathbb{R} \mathbb{V}(\mathbb{d} \alpha, \mu_d, \mathbb{E}_d^{(d)}) \). We obtain \( AZ \) to be multivariate regularly varying with different indices in different spaces; whereas in the aforementioned paper the indices of regular variation of \( Z \) and \( AZ \) on \( \mathbb{E}_d^{(d)} \) remain identical.

**Proposition 4.1** Let \( Z_1, \ldots, Z_d \) be independent random variables such that \( \mathbb{P}(Z_i > t) \sim \kappa_i t^{-\alpha} \) for \( \alpha > 0 \) as \( t \to \infty \) with constants \( \kappa_i > 0 \) for \( i \in \mathcal{O} = \{1, \ldots, d\} \). Let \( A \in \{0, 1\}^{q\times d} \) for \( q \geq d \) be a random adjacency matrix, where for all \( k \in A = \{1, \ldots, q\} \) independently,

\[
\begin{align*}
\mathbb{P}(A_{ki} = 1 \text{ and } A_{kj} = 0 \text{ for } j \neq i) &= \frac{1}{d^{1-i}} , \quad i \in \{1, \ldots, d\} \setminus \{k\} , \quad k \in \{1, \ldots, d\} , \\
\mathbb{P}(A_{ki} = 1 \text{ and } A_{kj} = 0 \text{ for } j \neq i) &= \frac{1}{d} , \quad i \in \{1, \ldots, d\} , \quad k \in \{d + 1, \ldots, q\} .
\end{align*}
\]

(a) For \( 1 \leq k \leq q - 1 \) we have \( AZ \in \mathbb{M} \mathbb{R} \mathbb{V}(\mathbb{a}, \mathbf{b}_1, \mathbf{\mu}_k, \mathbb{E}_q^{(k)}) \) with

\[
\bar{\mu}_k(\cdot) = \mathbf{E}_1^{(k)} \left[ \mu_1(\{z \in \mathbb{E}_d^{(1)} : AZ \in \cdot \}) \right],
\]

where \( \mu_1([0, z]^t) = K_1^{-1} \sum_{i=1}^{d} \kappa_i z_i^{-\alpha} \) for \( z \in \mathbb{E}_d^{(1)} \), \( b_1(t) \sim (K_1 t)^{1/\alpha} \) as \( t \to \infty \) and \( K_1 = \sum_{i=1}^{d} \kappa_i \).

(b) We have \( \mathbb{P}(\Omega_1^{(q)} = 0) = 0 \) and for \( 2 \leq i \leq d \) and \( x = (x_1, \ldots, x_q)^T \in \mathbb{E}_q^{(q)} \) we have as \( t \to \infty \),

\[
\mathbb{P}_i^{(q)}(AZ \in t(x, \infty)) = \Delta_i \sum_{1 \leq j_1 < \cdots < j_i \leq d} \left\{ \prod_{l=1}^{i} \kappa_{j_l} x_{j_l} \right\} t^{-i \alpha} + o(t^{-i \alpha}),
\]

where

\[
\Delta_i = \left( \frac{i-1}{d-1} \right) \left( \frac{i}{d-1} \right)^{d-i} \left( \frac{i}{d} \right)^{q-d} - i \left( \frac{i-2}{d-1} \right)^{d+1-i} \left( \frac{i-1}{d-1} \right)^{d+1-i} \left( \frac{i-1}{d} \right)^{q-d}.
\]
(c) For $k = q$ we have $AZ \in \mathcal{MRV}(2\alpha, b_2, \mu_q, \mathcal{E}_q^{(q)})$ with $b_2(t) \sim (K_2t^{1/(2\alpha)})$ as $t \to \infty$ where $K_2 = \sum_{1 \leq i < j \leq d} \kappa_i \kappa_j$ and

$$\bar{\mu}_q((x, \infty)) = \frac{2^{q-2}}{(d-1)^d \alpha^{d-1}} K_2^{-1} \sum_{1 \leq i < j \leq d} \kappa_i \kappa_j (x_i x_j)^{-\alpha},$$

such that for $x \in \mathcal{E}_q^{(q)}$ as $t \to \infty$,

$$\mathbf{P}(AZ \in t(x, \infty)) = K_2 \bar{\mu}_q((x, \infty)) t^{-2\alpha} + o(t^{-2\alpha}).$$

**Proof** First note that using similar arguments as in Example 2.1(a), we have for any $i = 1, \ldots, d$ that $Z \in \mathcal{MRV}(i\alpha, b_i, \mu_i, \mathcal{E}_d^{(i)})$ with canonical choices $b_i(t) \sim (K_i t)^{1/(i\alpha)}$ and

$$K_i = \sum_{1 \leq i_1 < \cdots < i_d \leq d} \prod_{l=1}^i K_{i_l}.$$  \hspace{1cm} (30)

Moreover, for $z = (z_1, \ldots, z_d)^T \in \mathcal{E}_d^{(d)}$ we have

$$\mu_i(\{ v \in \mathcal{E}_d^{(i)} : v_{i_1} > z_{j_1}, \ldots, v_{i_d} > z_{j_d} \text{ for some } 1 \leq j_1 < \cdots < j_d \leq d \}) = \frac{1}{K_i} \sum_{1 \leq i_1 < \cdots < i_d \leq d} \prod_{l=1}^i K_{i_l} x_{i_l}^{-\alpha},$$  \hspace{1cm} (31)

and

$$\mathbf{P}(x^{(i)}(Z) > t) = \mathbf{P}(Z^{(i)} > t) \sim 1/b_i^{x \alpha} \sim K_i t^{-i\alpha}, \quad t \to \infty.$$  \hspace{1cm} (32)

The structure of $A$ guarantees that $\mathbf{E}_i^{(q)}[(x^{(k,i)}(A))^{i\alpha}] = 1$. Now we show the various parts of the result.

(a) Let $1 \leq k \leq q - 1$. Then $\mathbf{P}(\Omega_1^{(k)}) > 0$. Also, referring to Remark 6 and (31), we have $\mu_i^i(\langle \bar{C}_A_d^{(i)}(j) \rangle) > 0$ for all $j = 1, \ldots, \begin{pmatrix} d \\ i \end{pmatrix}$ so that the assumptions of Proposition 3.1 are satisfied and Theorem 3.4(b) gives the statement.

(b) Since each row of $A$ has exactly one entry 1 and all others zero, we have for $i = 1, \ldots, d$,

$$\Omega_i^{(q)} = \{ \omega \in \Omega : \text{exactly } i \text{ columns of } A_\omega \text{ have at least one entry 1} \},$$

because $x^{(k,i)}(A_\omega) < \infty$ if and only if there are not more than $i$ columns of $A_\omega$ with positive entries. Clearly $\Omega_i^{(q)} = \emptyset$ and we have $\mathbf{P}(\Omega_i^{(q)}) = 0$. Now for $1 \leq j_1 < \cdots < j_i \leq d$ and $i = 1, \ldots, d$ define

$$\Omega_{j_1,\ldots,j_i}^{(q)} := \{ \omega \in \Omega : \text{exactly columns } j_1, \ldots, j_i \text{ of } A_\omega \text{ have at least one entry 1} \}.$$  \hspace{1cm} (33)

Hence for $2 \leq i \leq d$,
\( \mathbf{P}(\Omega_{j_1, \ldots, j_d}^{(q)}) = \left( \frac{i - 1}{d - 1} \right)^i \left( \frac{i}{d - 1} \right)^{d - i} \left( \frac{q}{d} \right)^{q-d} = \Delta_i. \)

Now an application of Theorem 3.3 yields
\[
\mathbf{P}_i^q(AZ \in t(x, \infty)) = \mathbf{P}(t(i)(Z) > t) = \sum_{1 \leq j_1 < \ldots < j_d \leq d} \mu_i(z \in \mathbb{R}_+^d : z_j > x_j, \ldots, z_j > x_j) \mathbf{P}(\Omega_{j_1, \ldots, j_d}^{(q)}) \kappa^\alpha t^{-\alpha} + o(t^{-\alpha})
\]

which is the result in (b).

(c) Using notation from Theorem 3.4, we have \( i^*_d = 2. \) Also note that for \( i < j, \)
\[
\mathbf{P}(\Omega_{i,j}^{(q)}) = \Delta_2 = \frac{2q^2}{(d - 1)!d^{q-d}}.
\]

Therefore, using Proposition 3.1 and Theorem 3.4(b) we have \( AZ \in \mathcal{MRV}(2\alpha, b_2, \overline{u}_q, \mathbb{E}_q) \) and as \( t \to \infty, \)
\[
\mathbf{P}(AZ \in t(x, \infty)) = \mathbb{E}_2^q [\mu_2(A^{-1}((x, \infty))))] \mathbf{P}(Z(2) > t) + o(\mathbf{P}(Z(2) > t))
\]
\[
= \left\{ \sum_{1 \leq j_1 < \ldots < j_d \leq d} \mu_j(z \in \mathbb{R}_+^d : z_i > x_i, z_j > x_j) \mathbf{P}(\Omega_{i,j}^{(q)}) \right\} K_j t^{-2\alpha} + o(t^{-2\alpha})
\]
\[
= \left\{ \sum_{1 \leq j_1 < \ldots < j_d \leq d} \kappa_j \kappa_j(x_j)^{-\alpha} \right\} \frac{2q^2}{(d - 1)!d^{q-d}} t^{-2\alpha} + o(t^{-2\alpha})
\]

which shows (c).

Proposition 4.1 shows that for continuous sets \( C \subseteq \mathbb{E}_q^{(k)} \) for \( k \in \{1, \ldots, q - 1\}, \)
\( \mathbf{P}(AZ \in tC) \) is of the order \( t^{-\alpha}. \) But for sets of the form \( t(x, \infty) \) which belong to \( \mathbb{E}_q^{(k)}, \) we observe a tail probability of the order \( t^{-2\alpha}. \) However, if we restrict \( A \)
\( \Omega_{i,j}^{(k)} \) as in part (b), we may observe tail probabilities of the order \( t^{-i\alpha} \) for all \( i = 2, \ldots, d. \)

In the next example we show that tail probabilities of other orders can also be observed for risk sets of the form \( t(x, \infty). \) Here we fix \( q = d \) and consider the same investment scenario as in Proposition 4.1; i.e., each agent invests in exactly one investment possibility and agents take their investment decisions independently. As before each row is a unit vector, but the distribution of \( A \) changes. Given \( m \in \{1, \ldots, d - 1\}, \) the single 1 in each row is chosen uniformly on a subset of size \( d - m, \) the subset changing across each row.

From a mathematical point of view, we obtain multivariate regular variation with different indices on \( \mathbb{E}_q^{(d)} \) depending on the choice of \( m. \) Such a model leads to explicit expressions for the asymptotic tail probabilities \( \mathbf{P}(AZ \in t(x, \infty)). \)
Proposition 4.2 Let $Z_1, \ldots, Z_d$ be independent random variables such that $P(Z_i > t) \sim \kappa_i t^{-\alpha}$ for $\alpha > 0$ as $t \to \infty$ with constants $\kappa_i > 0$ for $i \in \mathcal{O} = \{1, \ldots, d\}$. Let $1 \leq m \leq d - 1$, $m \in \mathbb{N}$ and $A \in \{0, 1\}^{d \times d}$ be a random adjacency matrix, where for all $k \in \mathcal{A} = \{1, \ldots, d\}$ independently

$$P(A_{ki} = 1 \text{ and } A_{kj} = 0 \text{ for } j \neq i) = \frac{1}{d - m}, \quad i \in I_k,$$

(32)

where $I_k$ is defined as

$$I_k = \begin{cases} \{1, \ldots, d\} \setminus \{k, \ldots, k + m - 1\} & \text{if } k + m - 1 \leq d, \\ \{1, \ldots, d\} \setminus \{\{k, \ldots, d\} \cup \{1, \ldots, k + m - 1 - d\}\} & \text{if } k + m - 1 > d. \end{cases}$$

Also define for $m + 1 \leq i \leq d$,

$$q^{(m)}_i = \frac{\rho^{d-(i+m-1)}(i-m)^{i-m+1}}{(d-m)^d} \prod_{l=1}^{m-1}(i-l)^2.$$

Then the following assertions hold:

(a) For $1 \leq k \leq d - m$ we have $AZ \in \mathcal{M}\mathcal{R}\mathcal{V}(\alpha, b_1, \bar{\mu}_k, \mathbb{E}^{(k)}_d)$ with

$$\bar{\mu}_k(\cdot) = \mathbb{E}^{(k)}_1 \mu_1([z \in \mathbb{E}^{(1)}_d : Az \in \cdot]),$$

and for $d - m < k \leq d$ we have $AZ \in \mathcal{M}\mathcal{R}\mathcal{V}(ja, b_j, \bar{\mu}_k, \mathbb{E}^{(k)}_d)$ with

$$\bar{\mu}_k(\cdot) = \mathbb{E}^{(k)}_j \mu_j([z \in \mathbb{E}^{(j)}_d : Az \in \cdot]),$$

where $j = k + m + 1 - d$, $\mu_j$ is defined as in (31), and $b_j(t) \sim (K_j t)^{1/ja}$ as $t \to \infty$ with $K_j$ as in (30).

(b) For $m + 1 \leq i \leq d$, and $x \in \mathbb{E}^{(d)}_d$ we have as $t \to \infty$,

$$(c) \quad P^{(d)}_i(AZ \in t(x, \infty)) = \left\{ \sum_{1 \leq l_1 < \cdots < l_i \leq d} \prod_{l=1}^{i} \kappa_{l_{l_i}} x_{j_i}^{-a} \right\} \left\{ q^{(m)}_i - q^{(m)}_{i-1} \right\} t^{-ia} + o(t^{-ia}).$$

(d) For $k = d$, part (a) applies with

$$\mu_d((x, \infty)) = K^{-1}_{m+1} q^{(m)}_{m+1} \left\{ \sum_{1 \leq l_1 < \cdots < l_i \leq d} \prod_{l=1}^{i} \kappa_{l_{l_i}} x_{j_i} \right\}$$

with $K_{m+1}$ as in (30), and we have as $t \to \infty$,

$$P(AZ \in t(x, \infty)) = K_{m+1} \mu_d((x, \infty)) t^{-(m+1)a} + o(t^{-(m+1)a}).$$
Linear functions of regularly varying vectors

**Proof**  For \( m \leq i + 1 \leq d \),

\[
q^{(m)}_i = \mathbf{P} (\{ \omega \in \Omega : \text{only in columns } j_1, \ldots, j_i \text{ of } A_{\omega} \text{ appears } 1 \})
\]

such that \( \mathbf{P}(\Omega_{j_1, \ldots, j_i}^{(d)}) = q^{(m)}_i - q^{(m)}_{i-1} \) and \( \mathbf{P}(\Omega_{j_1, \ldots, j_{m+1}}^{(d)}) = q^{(m)}_{m+1} \). The proposition can then be proved in a similar manner as Proposition 4.1. Hence the proof is omitted here.  

\( \square \)

**Remark 11**

(i) The uniform probability in (32) is not necessary to obtain multivariate regular variation of \( X = AZ \) of that order on the different subcones \( \mathbb{E}^{(k)}_d \). However, the probabilities in (32) allow for the explicit representations of the limit measures in (b) and (c).

(ii) Note again that for \( d - m < k < d - 1 \) Basrak et al. [2002, Proposition A.1] and Janssen and Drees [2016, Theorem 2.3] are not applicable. Even when \( k = d \) the assumptions are not satisfied and we get regular variation with index \((m + 1)\alpha\) on \( \mathbb{E}^{(d)}_d \).

### 4.2 Dependent objects

In this section we contrast independent objects as we have considered previously with a specific dependence structure of the components of \( Z = (Z_1, Z_2, Z_3) \top \) given by

\[
\mathbf{P}(Z_1 \leq z_1, Z_2 \leq z_2, Z_3 \leq z_3) = (1 + \theta(\kappa_1 \kappa_2 \kappa_3)\rho(\kappa_1 z_2 z_3)^{-\rho \alpha}) \prod_{i=1}^{3} (1 - \kappa_i z_i^{-\alpha}), \quad (33)
\]

for \( z_i \geq \kappa_i^{1/\alpha} \), where \( \kappa_i > 0, i = 1, 2, 3, \alpha > 0, \rho \geq 1, 0 \leq \theta \leq 1 \). In the two examples below we contrast two sets of parameters: for \( \theta = 0 \) the components of \( Z \) are independent Pareto (cf. Example 4.2) and for \( \rho = 1, \theta = 1 \) they are dependent (cf. Example 4.3). Such dependence in terms of copulas has been discussed in Rodríguez-Lallena et al. (2004). Hence, the underlying distribution of \( Z \) has either independent marginals or at least it has a tractable form.

Moreover, we also investigate the influence of weights in a numerical example, where the adjacency matrix \( A \) is relatively simple, in order to provide an interpretable illustration. We use a weighted adjacency matrix, which in both examples is given by

\[
A = \begin{bmatrix}
A_{11} & 0 & 0 \\
0 & A_{22} & 0 \\
A_{41} & A_{42} & 0 \\
A_{41} & A_{42} & 0 \\
A_{52} & A_{53}
\end{bmatrix}
\]

with random weights satisfying \( \mathbf{P}(\min(A_{11}, A_{22}, A_{33}, A_{41}, A_{42}, A_{52}, A_{53}) > 0) = 1 \) and \( \mathbf{E} \| A \|^{4 \alpha + 6} < \infty \) for some \( \delta > 0 \). Also for the convenience of computing
the limit measures of the sets $D_k$ we assume $\mathbf{P}(A_{41}A_{11}^{-1} + A_{42}A_{22}^{-1} > 1) = 1$ and $\mathbf{P}(A_{52}A_{22}^{-1} + A_{53}A_{33}^{-1} > 1) = 1$.

Let $X = AZ$ be the investment portfolios of five agents, each of whom connects to a subset of three objects whose risks are given by $Z$. We estimate the tail risks for $k = 1, \ldots, 5$:

$$
\mathbf{P}(\text{risk of at least } k \text{ of the portfolios } > t) =: \mathbf{P}(X \in tD_k). \quad (34)
$$

**Example 4.2** Suppose $Z_1, Z_2, Z_3$ are independent random variables such that $\mathbf{P}(Z_i > z) = \kappa_i z^{-\alpha}, z > \kappa_i^{1/\alpha}$ with constants $\kappa_i > 0$ for $i = 1, 2, 3$.

We calculate all relevant quantities. First, the tails of the order statistics are given for $t \to \infty$,

$$
\begin{align*}
\mathbf{P}(Z^{(1)} > t) &= (\kappa_1 + \kappa_2 + \kappa_3)^{-\alpha}t^{-\alpha} + o(t^{-\alpha}), \\
\mathbf{P}(Z^{(2)} > t) &= (\kappa_1^2 + \kappa_2^2 + \kappa_3^2 \kappa_1^1) t^{-2\alpha} + o(t^{-2\alpha}), \\
\mathbf{P}(Z^{(3)} > t) &= (\kappa_1^2 \kappa_3^2 \kappa_1^1) t^{-3\alpha}.
\end{align*}
\quad (35)
$$

We have $Z \in \mathcal{MRV}(ia, b, \mu, \mathcal{B})$ with canonical $b$ given by

$$
\begin{align*}
b_1(t) &= (\kappa_1 + \kappa_2 + \kappa_3)^{1/\alpha} t^{1/\alpha}, \\
b_2(t) &= (\kappa_1^2 + \kappa_2^2 + \kappa_3^2 \kappa_1^1)^{1/(2\alpha)} t^{1/(2\alpha)}, \\
b_3(t) &= (\kappa_1^2 \kappa_3^2 \kappa_1^1)^{1/(3\alpha)} t^{1/(3\alpha)},
\end{align*}
\quad (36)
$$

and limit measures

$$
\begin{align*}
\mu_1 \left( \bigcup_{i=1}^3 \{ v \in \mathbb{R}_+^3 : v_j > z_j \} \right) &= (\kappa_1 + \kappa_2 + \kappa_3)^{-1} \sum_{i=1}^3 \kappa_i z_i^{-\alpha}, \\
\mu_2 \left( \bigcup_{1 \leq i \neq j \leq 3} \{ v \in \mathbb{R}_+^3 : v_j > z_j, v_j > z_j \} \right) &= (\kappa_1^2 + \kappa_2^2 + \kappa_3^2 \kappa_1^1)^{-1} \sum_{1 \leq i \neq j \leq 3} \kappa_i \kappa_j (z_i z_j)^{-\alpha}, \\
\mu_3 \left( (z_1, \infty) \times (z_2, \infty) \times (z_3, \infty) \right) &= (z_1 z_2 z_3)^{-\alpha}.
\end{align*}
\quad (37)
$$

Note that from its definition in (34), $D_k \subset \mathcal{E}_{\mathcal{F}}^{(k)}$ for $k = 1, \ldots, 5$.

We can check from the form of $A$ and the fact that $I_A$ is a.s. constant in combination with 3.3 that  

$$
i_1^1 = 1, \quad i_2^2 = 1, \quad i_3^3 = 1, \quad i_4^4 = 2, \quad i_5^5 = 3,
$$

$$
\mathbf{P}(\mathbb{O}^{(1)}_1) = \mathbf{P}(\mathbb{O}^{(2)}_1) = \mathbf{P}(\mathbb{O}^{(3)}_1) = \mathbf{P}(\mathbb{O}^{(4)}_2) = \mathbf{P}(\mathbb{O}^{(5)}_3) = 1.
$$

Hence using Proposition 3.1 and Theorem 3.4, along with (35) and (37), we have as $t \to \infty$,
\[ P(AZ \in tD_1) \sim P(Z^{(1)} > t) E[\mu_1(A^{-1}(D_1))] \]
\[ \sim [k_1E[\max(A_{11}^a, A_{41}^a)] + k_2E[\max(A_{22}^a, A_{42}^a, A_{52}^a)] + k_3E[(\max(A_{33}^a, A_{53}^a))] t^{-a}. \]
\[ P(AZ \in tD_2) \sim P(Z^{(1)} > t) E[\mu_1(A^{-1}(D_2))] \]
\[ \sim [k_1E[\min(A_{11}^a, A_{41}^a)] + k_2E[\min(A_{22}^a, A_{42}^a, A_{52}^a)] + k_3E[\min(A_{33}^a, A_{53}^a))] t^{-a}. \]
\[ P(AZ \in tD_3) \sim P(Z^{(1)} > t) E[\mu_1(A^{-1}(D_3))] \]
\[ \sim k_2E[\min(A_{22}^a, A_{33}^a, A_{42}^a)] t^{-a}, \]
\[ P(AZ \in tD_4) \sim P(Z^{(2)} > t) E[\mu_2(A^{-1}(D_4))] \]
\[ \sim [k_1k_2E[A_{11}^a \min(A_{22}^a, A_{42}^a)] + k_2k_3E[\min(A_{22}^a, A_{42}^a)] A_{33}^a)] + k_3k_4E[\min(A_{33}^a, A_{53}^a)) \min(A_{11}^a, A_{41}^a))] t^{-2a}. \]
\[ P(AZ \in tD_5) \sim P(Z^{(3)} > t) E[\mu_3(A^{-1}(D_5))] \]
\[ \sim k_1k_2k_3E[A_{11}^a A_{22}^a A_{33}^a)] t^{-3a}. \]

The forms for \( P(AZ \in tD_4) \) and \( P(AZ \in tD_5) \) become more complicated if we do not assume \( P(\min(A_{11}, A_{22}, A_{33}, A_{41}, A_{42}, A_{52}, A_{53}) > 0) = 1 \). Furthermore, for the unweighted adjacency matrix, if all weights are equal to one, the above formulas hold true with all expectations reducing to 1.

As an illustration, we fix \( k_1 = 1, k_2 = 2, k_3 = 3 \). Moreover, let \( A_{11} = A_{22} = A_{33} = 1, A_{41} = 2, A_{42} = 2, \) and \( A_{52} = A_{53} = 3 \). The five probabilities obtained above are plotted for the case \( \alpha = 1, 2 \) in Fig. 3 for \( 20 \leq t \leq 100 \). We also compare the probability for the event \( tD_5 \) in this example and in Example 4.3; see the right two panels of Fig. 3.

**Example 4.3** Suppose \( Z_1, Z_2, Z_3 \) are dependent with joint distribution (33) for \( \rho = \theta = 1 \). Otherwise assume the setting as in Example 4.2. We calculate again all relevant quantities. First, the tails of the order statistics are given for \( t \to \infty \),
\[ P(Z^{(1)} > t) = (\kappa_1 + \kappa_2 + \kappa_3)t^{-a} + o(t^{-a}), \]
\[ P(Z^{(2)} > t) = (\kappa_1k_2 + \kappa_2k_3 + \kappa_3k_1)t^{-2a} + o(t^{-2a}), \]
\[ P(Z^{(3)} > t) = k_1k_2k_3(\kappa_1 + \kappa_2 + \kappa_3)t^{-4a} + o(t^{-4a}). \]

Notice that the only difference from (35) is in the term \( P(Z^{(3)} > t) \). Hence, for \( i = 1, 2 \) we have \( Z \in \mathcal{MR}(ia, b_i, \mu_i, E^{(i)}) \) with canonical choice for \( b_i \) as in (36) and \( \mu_i \) as in (37). On the other hand, we have \( Z \in \mathcal{MR}(4a, b_3, \mu_3, E^{(3)}) \) with
\[ b_3(t) = (\kappa_1k_2k_3(\kappa_1 + \kappa_2 + \kappa_3))^{1/(4a)} t^{1/(4a)}, \]
and
\[ \mu_3((z_1, \infty) \times (z_2, \infty) \times (z_3, \infty)) = (\kappa_1 + \kappa_2 + \kappa_3)^{-1} \sum_{i=1}^{3} \kappa_i z_i^{-a}(z_1z_2z_3)^{-a}. \]

As in Example 4.2, we have \( i^*_1 = 1, i^*_2 = 1, i^*_3 = 1, i^*_4 = 2, i^*_5 = 3 \) and \( P(\Omega_1^{(1)}) = P(\Omega_2^{(2)}) = P(\Omega_3^{(3)}) = P(\Omega_4^{(4)}) = P(\Omega_5^{(5)}) = 1 \). Using Theorem 3.4, along with (35) and (37), we have the same limits for \( P(AZ \in tD_k) \) for \( k = 1, \ldots, 4 \). The only difference occurs for \( k = 5 \), where we have for \( t \to \infty \),
Again we fix $\mu_1 = 1$, $\mu_2 = 2$, and let $A_{11} = A_{22} = A_{33} = 1$, $A_{41} = 2$, $A_{42} = 2$, and $A_{52} = A_{53} = 3$, as in Example 4.2. The probabilities for events $tD_1, tD_2, tD_3, tD_4$ asymptotically remain the same as in Example 4.2 (matching the plots in the left two panels of Fig. 3 for the case $\mu = 1, 2$). In the right panels of Fig. 3 we plot the values for $tD_5$ when $\alpha = 1, 2$; clearly these values differ in the two examples.

\[
P(AZ \in tD_5) \sim P(Z^{(3)} > t)E[\mu_3(A^{-1}(D_5))] \\
\sim \kappa_1 \kappa_2 \kappa_3 E[A_{11}^{\alpha} A_{22}^{\alpha} A_{33}^{\alpha}(\kappa_1 A_{11}^{\alpha} + \kappa_2 A_{22}^{\alpha} + \kappa_3 A_{33}^{\alpha})] t^{-4\alpha}.
\]

Fig. 3 The probabilities of the tail events $tD_1, tD_2, tD_3, tD_4$ are asymptotically equal in both Examples 4.2 and 4.3 and are plotted in the left panels with $\alpha = 1$ in the top and $\alpha = 2$ in the bottom. The probabilities of the tail events $tD_5$ are asymptotically different in the two examples and are plotted in the right panels ($\alpha = 1$ in the top and $\alpha = 2$ in the bottom). Values are plotted for $20 \leq t \leq 100$. 

\[\]
5 Conclusion

This work is motivated by the need to find probabilities of a variety of extreme events under a linear transformation of regularly varying random vectors with emphasis on bipartite network structures. By an extension of Breiman’s Theorem we have shown that probabilities of many such events can be calculated, if we have information on the regular variation property of the underlying random vector on specific subcones of the non-negative orthant. Most of the subsets $C$ of such cones have linear boundaries and hence form a polytope, whose pre-image under linear transformation also turns out to be a polytope in $\mathbb{R}^d_+$. Computing the limit measures $E_1^{(k)}(\mu_i(A^{-1}(C)))$ in such cases means finding the appropriate boundaries of the polytope which can become quite complicated. For moderate dimensions of the matrix $A$, numerical solutions can be obtained even when the distributional forms of $Z$ and $A$ are more complicated.

We envisage wide application of such results in areas of risk management. There are clear implications for computing a variety of conditional risk measures. We also believe that an alternative characterization of the rate of decay of tail probabilities can be provided via connectivity of the row components (in the bipartite network model, the agents); this work is under current investigation.

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