Best possible bounds on the number of distinct differences in intersecting families

Peter Frankl∗, Sergei Kiselev†, Andrey Kupavskii‡

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Abstract

For a family ℱ, let ℰ(ℱ) stand for the family of all sets that can be expressed as 𝐹 \ 𝐺, where 𝐹, 𝐺 ∈ ℱ. A family ℱ is intersecting if any two sets from the family have non-empty intersection. In this paper, we study the following question: what is the maximum of |ℰ(ℱ)| for an intersecting family of 𝑘-element sets? Frankl conjectured that the maximum is attained when ℱ is the family of all sets containing a fixed element. We show that this holds if 𝑛 ⩾ 50 ln 𝑘 and 𝑘 ⩾ 50. At the same time, we provide a counterexample for 𝑛 < 4 𝑘.

1 Introduction

Let 𝑛 ⩾ 𝑘 ⩾ 1 be integers and let [𝑛] = {1, . . . , 𝑛} be the standard 𝑛-element set. Let further \( \binom{[𝑛]}{𝑘} \) denote the collection of all 𝑘-element subsets (𝑘-sets) of [𝑛]. A subset ℱ of \( \binom{[𝑛]}{𝑘} \) is called an intersecting family if 𝐹 ∩ 𝐹′ ̸= ∅ for all 𝐹, 𝐹′ ∈ ℱ. Since for 𝑛 < 2𝑘, 𝐹 ∩ 𝐹′ ̸= ∅ is always true, from now on we always assume that 𝑛 ⩾ 2𝑘. Let us recall the Erdős-Ko-Rado Theorem, one of the fundamental results in extremal set theory.

Theorem 1 ([2]). Suppose that ℱ ⊂ \( \binom{[𝑛]}{𝑘} \) is intersecting and 𝑛 ⩾ 2𝑘. Then

\[
|ℱ| \leq \binom{n - 1}{k - 1}.
\] (1)

For a fixed element 𝑥 ∈ [𝑛] one defines the full star \( ℳ_𝑥 := \{ 𝐹 ∈ \binom{[𝑛]}{𝑘} : 𝑥 ∈ 𝐹 \} \). Subfamilies of \( ℳ_𝑥 \) are called stars.

Full stars provide equality in (1). On the other hand Hilton and Milner [7] proved that for 𝑛 > 2𝑘 no other intersecting family attains equality in (1).

∗ Rényi Institute, Budapest, Hungary and Moscow Institute of Physics and Technology, Russia, Email: peter.frankl@gmail.com
† Moscow Institute of Physics and Technology, Email: kiselev.sg@gmail.com
‡ G-SCOP, CNRS, University Grenoble-Alpes, France and Moscow Institute of Physics and Technology, Russia; Email: kupavskii@yandex.ru. Research of the author is supported by the grant RSF N 21-71-10092.
Theorem 2 ([7]). Suppose that $\mathcal{F} \subset \binom{[n]}{k}$, $\mathcal{F}$ is intersecting but $\mathcal{F}$ is not a star. If $n > 2k$ then

$$|\mathcal{F}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1. \tag{2}$$

For a family $\mathcal{F}$ define the family of (setwise) differences: $\mathcal{D}(\mathcal{F}) \coloneqq \{F \setminus F' : F,F' \in \mathcal{F}\}$. For $0 \leq \ell \leq k$ define also $\mathcal{D}(\ell)(\mathcal{F}) \coloneqq \{D \in \mathcal{D}(\mathcal{F}) : |D| = \ell\}$. Note that $\mathcal{F} \subset \binom{[n]}{k}$ is intersecting if and only if $\mathcal{D}(k)(\mathcal{F}) = \emptyset$. We also note that one of the first correlation inequalities is the Marica–Schönheim Theorem [13], which states that $|\mathcal{D}(\mathcal{F})| \geq |\mathcal{F}|$ for any family $\mathcal{F} \subset 2^{[n]}$.

Observation 3. Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ is intersecting. Then

$$|\mathcal{D}(\mathcal{F})| = \sum_{0 \leq \ell < k} |\mathcal{D}(\ell)(\mathcal{F})| \leq \sum_{0 \leq \ell < k} \binom{n}{\ell}.$$ 

On the other hand,

$$|\mathcal{D}(\mathcal{S}_x)| = \sum_{0 \leq \ell < k} \binom{n-1}{\ell}.$$ 

In [5], the following conjecture is stated.

Conjecture 4 ([5]). Suppose that $n > 2k > 0$, $\mathcal{F} \subset \binom{[n]}{k}$ is intersecting. Then

$$|\mathcal{D}(\mathcal{F})| \leq \sum_{0 \leq \ell < k} \binom{n-1}{\ell}. \tag{3}$$

In [5] the conjecture was proved for $n \geq k(k + 3)$. The main result of this paper is the following theorem.

Theorem 5. Conjecture 4 is true for $n \geq 50k \ln k$, $k \geq 50$. Moreover, equality in (3) is attained only for full stars.

We also provide a counterexample for the conjecture when $n$ is close to $2k$.

Theorem 6. For integers $n,k$, where $n = ck$, $2 < c < 4$ and $k > k(c)$ sufficiently large, there is an intersecting family $\mathcal{F}$ with $|\mathcal{D}(\mathcal{F})|$ greater than the right hand side of (3).

Let us outline the approach to prove Theorem 5. Since every $D \in \mathcal{D}(k-1)(\mathcal{F})$ is a subset of some $F \in \mathcal{F}$, we have

$$|\mathcal{D}(k-1)(\mathcal{F})| \leq k \cdot |\mathcal{F}|.$$ 

One can use this easy bound to deduce the conjecture for $\mathcal{F}$ with $|\mathcal{F}|$ small.

Proposition 7. Let $\mathcal{F} \subset \binom{[n]}{k}$ be intersecting. Suppose that $|\mathcal{F}| \leq \binom{n-1}{k-1}/(2k)$, $n \geq 3k$. Then

$$|\mathcal{D}(\mathcal{F})| \leq \sum_{0 \leq \ell < k} \binom{n-1}{\ell}.$$ 

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We will prove Proposition 7 in the beginning of Section 4.2.

The difficult case is how to deal with relatively large intersecting families. There are various structural theorems concerning relatively large intersecting families, e.g. [1, 4, 11]. However, the smaller is $n$ w.r.t. $k$, the less we can infer. For $n < k^2$ the bound in (2) is larger than $\binom{n-1}{k-1}/3$. For $n = o(k^2)$ there are various intersecting families that have, say, very large covering number, but still contain $(1 - o(1))\binom{n-1}{k-1}$ members. Even the junta approximation of Dinur and Friedgut gives results that are strong enough to combine with Proposition 7, say, only for $n > Ck^{1+\varepsilon}$, where $\varepsilon > 0$ is some fixed constant and $C = C(\varepsilon)$ is a very large constant depending on $\varepsilon$.

The approach that we found is to apply a powerful concentration inequality of the third author (cf. [6]) to prove that for $n > C k \ln k$ the lower bound $|\mathcal{F}| > \binom{n-1}{k-1}/(2k)$ is sufficient to guarantee that the overwhelming majority of the members of $\mathcal{F}$ contain a fixed vertex. To state this result let us recall the definition of $\gamma(\mathcal{F})$, the diversity of the family $\mathcal{F}$.

$$\gamma(\mathcal{F}) := \min_{x} |\{F \in \mathcal{F} : x \notin F\}|.$$ 

Clearly, $\gamma(\mathcal{F}) = 0$ iff $\mathcal{F}$ is a star. Then Theorem 5 is implied by the following two lemmas.

**Lemma 8.** Let $\mathcal{F} \subset \binom{[n]}{k}$ be intersecting. Suppose that $n \geq 50k \ln k$, $k \geq 50$ and (3) does not hold. Then

$$\gamma(\mathcal{F}) \leq \binom{n - \ln k}{n - k - 1}.$$ 

**Lemma 9.** Let $n, k$ be positive integers satisfying $n \geq 50k \ln k$, $k \geq 50$. Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ is an intersecting family with

$$1 \leq \gamma(\mathcal{F}) \leq \binom{n - \ln k}{n - k - 1},$$

Then

$$|\mathcal{D}(\mathcal{F})| < \sum_{0 \leq i < k} \binom{n - 1}{i}.$$ 

The paper is organised as follows. In Section 2 we introduce the necessary notation and list some standard tools that we use. In Section 3 we give a purely combinatorial proof of Lemma 9. In Section 4 we state the concentration inequality and apply it to prove Lemma 8. In Section 5 we prove Theorem 6. The last section contains some remarks and open problems.

## 2 Notation and tools

For a family $\mathcal{F} \subset 2^{[n]}$ and a pair of subsets $A \subset B \subset [n]$ we use the standard notation

$$\mathcal{F}(A, B) := \{F \setminus A : F \in \mathcal{F}, F \cap B = A\}.$$ 

In the cases $A = B$ or $A = \emptyset$ we use the shorter form $\mathcal{F}(B) := \mathcal{F}(B, B)$, $\mathcal{F}(\emptyset) := \mathcal{F}(\emptyset, B)$. If $B = \{x\}$, then we set $\mathcal{F}(x) := \mathcal{F}(\{x\})$ and $\mathcal{F}(\bar{x}) := \mathcal{F}(\{\bar{x}\})$.

For $0 \leq i \leq k$ and $\mathcal{F} \subset \binom{[n]}{k}$ we define the $i$-th level shadow $\partial^i \mathcal{F} := \{S \in \binom{[n]}{i} : \exists F \in \mathcal{F}, S \subset F\}$. For $k$ and $\mathcal{F}$ fixed Kruskal [10] and Katona [9] determined the minimum of $|\partial^i \mathcal{F}|$ for all $0 \leq i < k$. The following handy version of the Kruskal–Katona theorem is due to Lovász [12].
Kruskal-Katona Theorem (Lovász version). Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ and $x$ is the unique real number satisfying $k \leq x \leq n$, $|\mathcal{F}| = \binom{x}{k}$. Then

$$|\partial^i \mathcal{F}| \geq \binom{x}{i}, \quad 0 \leq i < k. \quad (4)$$

(For a short proof of (4), c.f. [3].)

Two families $\mathcal{F} \subset \binom{[n]}{k}$, $\mathcal{G} \subset \binom{[n]}{\ell}$ are called cross-intersecting if $F \cap G \neq \emptyset$ for all $F \in \mathcal{F}$, $G \in \mathcal{G}$. For a family $\mathcal{F} \subset \binom{[n]}{k}$ let $\mathcal{F}^c$ denote the family of complements, $\mathcal{F}^c := \{[n] \setminus F : F \in \mathcal{F}\}$.

Katona [8] was the first to note that for $n \geq k + \ell$, $\mathcal{F}$ and $\mathcal{G}$ are cross-intersecting iff $\mathcal{F} \cap \partial^k(\mathcal{G}^c) = \emptyset$. Together with the Kruskal-Katona Theorem, this immediately implies

**Theorem 10.** Let $m \geq a + b$. If $\mathcal{A} \subset \binom{[m]}{a}$ and $\mathcal{B} \subset \binom{[m]}{b}$ are cross-intersecting and $|\mathcal{A}| \geq \binom{x}{a}$ for some $m - a \leq x \leq m$, then $|\mathcal{B}| \leq \binom{m}{b} - \binom{x}{b}$.

### 3 Families with small diversity

In this section, we prove Lemma 9. W.l.o.g. we assume that element 1 has maximal degree in $\mathcal{F}$, i.e., that $\gamma(\mathcal{F}) = |\mathcal{F}(1)|$.

For notational convenience, set $\mathcal{G} := \{[2, n] \setminus F : F \in \mathcal{F}(\bar{1})\}$ and note that $\mathcal{G} \subset \binom{[2, n]}{n-k-1}$. Choose $x$ so that $|\mathcal{G}| = \binom{x}{n-k-1}$ and note that $x \leq n - \ln k$ by the assumption. Put $r := \ln k$ for shorthand. Using (4), we get

$$\frac{|\mathcal{G}|}{|\partial^{k-1}\mathcal{G}|} \leq \frac{\binom{x}{n-k-1}}{\binom{x}{k-1}} = \prod_{i=k}^{n-k-1} \frac{x-i+1}{i} \leq \prod_{i=k}^{n-k-1} \frac{n-r-i+1}{i} = \prod_{i=k}^{n-k-1} \frac{i-r+2}{i}$$

$$= \prod_{i=k}^{n-k-1} \left(1 - \frac{r-2}{i}\right) \leq \exp\left(- (r-2) \sum_{i=k}^{n-k-1} \frac{1}{i}\right) \leq \exp\left(- (r-2) \ln \frac{n-k}{k}\right)$$

$$\leq e^{-(\ln k-2)\ln(50 \ln k-1)} < k^{-5.27} \quad (\text{by } (5))$$

Set $\mathcal{D} := \mathcal{D}(\mathcal{F})$. In order to give an upper bound for $|\mathcal{D}|$, we use $|\mathcal{D}| = |\mathcal{D}(1)| + |\mathcal{D}(\bar{1})|$ and distinguish $D \in \mathcal{D}$ according to their size. We will need the following two inequalities.

$$\sum_{0 \leq i \leq k-2} |\mathcal{D}^{(i)}(\bar{1})| \leq \sum_{0 \leq i \leq k-2} \binom{n-1}{i}, \quad (6)$$

$$|\partial^{k-1}\mathcal{F}(\bar{1})| \leq k|\mathcal{F}(\bar{1})| = k|\mathcal{G}| \quad (5)$$

The inequality (6) is true simply because $\mathcal{D}^{(i)}(\bar{1}) \subset \binom{[2, n]}{i}$. The inequality (7) uses a trivial upper bound on the size of the $(k-1)$-shadow and inequality (5). Next, we prove the following inequality.

$$|\mathcal{D}^{(k-1)}(\mathcal{F}(1))| \leq \binom{n-1}{k-1} - |\partial^{k-1}\mathcal{G}|. \quad (7)$$
Proof. Suppose that $H \in \mathcal{D}^{(k-1)}(\mathcal{F}(1))$. Then $H \in \binom{[2,n]}{k-1}$ and $\{1\} \cup H \in \mathcal{F}$. To prove (8), we show $H \notin \partial^{k-1}\mathcal{G}$. Indeed, the opposite would mean $H \subseteq [2,n] \setminus F$ for some $F \in \mathcal{F}(1)$. However this would imply $(\{1\} \cup H) \cap F = \emptyset$, a contradiction.

Note that the members $D \in \mathcal{D}^{(k-1)}(\bar{1})$ that are not accounted for in the left hand side of (8) are of the form $F \setminus F'$ with $F \in \mathcal{F}(1), F' \in \mathcal{F}$. Consequently, each such $D$ is a member of $\partial^{k-1}\mathcal{F}(1)$. Combining this with (7) and (8), we infer

$$|\mathcal{D}^{(k-1)}(\bar{1})| < \binom{n-1}{k-1} - \frac{k-1}{k} |\partial^{k-1}\mathcal{G}|.$$  

Adding (6) to it:

$$|\mathcal{D}(\bar{1})| < \sum_{0 \leq i < k} \binom{n-1}{i} - \frac{k-1}{k} |\partial^{k-1}\mathcal{G}|.  \quad (9)$$

In order to bound $\mathcal{D}(1)$, first note that $H \in \mathcal{D}(1)$ means that $\{1\} \cup H = F \setminus F'$ for $F, F' \in \mathcal{F}$, where $1 \in F, 1 \notin F'$. Since $H \cap F' = \emptyset, H \in \partial^{H} \mathcal{G}$. Also, $F \cap F' \neq \emptyset$ implies $0 \leq |H| \leq k-2$. Consequently,

$$|\mathcal{D}(1)| \leq \sum_{0 \leq i \leq k-2} |\partial^{i}\mathcal{G}|.$$  

Let us prove that

$$|\partial^{i-1}\mathcal{G}| < \frac{1}{8} |\partial^{i}\mathcal{G}| \quad \text{for } 1 \leq i \leq k-1. \quad (10)$$

Proof. This follows from the local LYM inequality, but for completeness we give a direct proof. To this end, consider the bipartite graph with parts $\partial^{i-1}\mathcal{G}$ and $\partial^{i}\mathcal{G}$, where an edge is $(A, B)$ is drawn iff $A \subseteq B$. Let $e$ be the number of edges in this graph. Then $e = i|\partial^{i}\mathcal{G}|$ is obvious. On the other hand, if $A \in \partial^{i-1}\mathcal{G}$ then for some $(n-k-1)$-set $G \in \mathcal{G}$ we have $A \subseteq G$. Consequently, there are at least $n-k-i$ edges with $A$ as one of the endpoints. Thus, $e \geq (n-k-i)|\partial^{i-1}\mathcal{G}|$. Using $n-k-i > 8k$ and $i < k$, (10) follows.  

From (10) we deduce

$$|\mathcal{D}(1)| \leq \sum_{0 \leq i \leq k-2} |\partial^{i}\mathcal{G}| < \frac{1}{7} |\partial^{k-1}\mathcal{G}|. \quad (11)$$

Combining (9) and (11), the statement of the lemma follows for $k \geq 2$. The case $k = 1$ is obvious, and thus the proof is complete.

4 The diversity is small

We start this section with the aforementioned concentration result. We believe that it may be of good use in many other settings in extremal set theory.
4.1 Concentration

The following result states that, informally, for $k$-uniform families of density $\alpha$ the densities of the sizes of their intersections with large subsets of the ground set are tightly concentrated around $\alpha$.

**Theorem 11.** Fix integers $m, \ell, \ell', t$ such that $m \geq \ell + \ell'$, fix $a > 0$ and set $\varepsilon = \frac{2a + \sqrt{8 \ln 2}}{\sqrt{t}}$. Let $\mathcal{G} \subseteq \binom{[m]}{\ell}$ be a family and set $\alpha := |\mathcal{G}|/\binom{m}{\ell}$. Let $H$ be distributed uniformly on $\binom{[m]}{\ell}$. Then

$$P \left[ |\mathcal{G}(H)| < (\alpha - \varepsilon) \left( \frac{m-\ell'}{\ell} \right) \right] < 2e^{-a^2/2}. \quad (12)$$

We note that the same bound for the probability holds for the upper deviations. The proof of this result relies on the following concentration result by Kupavskii.

**Theorem 12 ([6]).** Fix integers $m, l, t$ such that $m \geq tl$. Let $\mathcal{G} \subseteq \binom{[m]}{l}$ be a family and set $\alpha := |\mathcal{G}|/\binom{m}{l}$. Let $\mathcal{B}$ be chosen uniformly at random out of all $t$-matchings of $l$-sets and let $\eta$ be a random variable $|\mathcal{G} \cap \mathcal{M}|$. Then $E[\eta] = \alpha t$ and, for any positive $a$ and $\delta \in \{-1, 1\}$, we have

$$P[\delta \cdot (\eta - \alpha t) \geq 2a\sqrt{l}] \leq e^{-a^2/2}. \quad (15)$$

**Proof of Theorem 11.** Let $\mathcal{H} \subseteq \binom{[m]}{l'}$ be the family of sets $H$ such that $|\mathcal{G}(H)| < (\alpha - \varepsilon) \left( \frac{m-\ell'}{\ell} \right)$. Denote $\beta := |\mathcal{H}|/\binom{m}{l'}$ and assume that $\beta \geq 2e^{-a^2/2}$. Let $(H, B_1, \ldots, B_t)$, where $|H| = \ell'$ and $|B_i| = \ell$ for $i \in [t]$, be a $(t+1)$-matching chosen uniformly at random out of all matchings of sets with such sizes. Note that $H$ is distributed uniformly on $\binom{[m]}{l'}$ and, in particular, $P[H \in \mathcal{H}] = \beta$. Moreover, the subset of the matching above $\mathcal{B} := (B_1, \ldots, B_t)$ is distributed uniformly on the set of all $t$-matchings of $\ell$-element subsets of $[m]$, and by Theorem 12 we have

$$P[|\mathcal{B} \cap \mathcal{G}| < \alpha t - 2a\sqrt{l}] < e^{-a^2/2} \leq \beta/2. \quad (13)$$

Therefore, we have $P[|\mathcal{B} \cap \mathcal{G}| < \alpha t - 2a\sqrt{l} \mid H \in \mathcal{H}] < 1/2$, and, in particular, there is a set $H' \in \mathcal{H}$ such that

$$P[|\mathcal{B} \cap \mathcal{G}| < \alpha t - 2a\sqrt{l} \mid H = H'] < 1/2. \quad (13)$$

Fix such a set $H'$ and put $\mathcal{G}' := \mathcal{G}(H')$, $\alpha' := |\mathcal{G}'|/\binom{m-\ell'}{\ell} < \alpha - \varepsilon$. Denote a uniformly random $t$-matching of $\ell$-subsets in $[m] \setminus H'$ by $\mathcal{B}'$. Note that for a fixed $H = H'$ the random matching $\mathcal{B}$ is distributed the same way as $\mathcal{B}'$. Then (13) can be written as

$$P[|\mathcal{B}' \cap \mathcal{G}'| < \alpha t - 2a\sqrt{l}] < 1/2. \quad (14)$$

On the other hand, from Theorem 12 we have

$$P[|\mathcal{B}' \cap \mathcal{G}'| > \alpha' t + 2\sqrt{2t \ln 2}] \leq e^{-\ln 2} = 1/2. \quad (15)$$

With positive probability, neither of the events in the left hand sides of (14) and (15) hold. This implies $\alpha t - 2a\sqrt{l} < \alpha' t + 2\sqrt{2t \ln 2} < (\alpha - \varepsilon)t + \sqrt{8t \ln 2}$, which is a contradiction with the definition of $\varepsilon$. \qed

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4.2 Proof of Lemma 8

Let $\mathcal{F} \subset \binom{[n]}{k}$ be an intersecting family that is not a star and that does not satisfy (3). Denote $C := n/k$.

Note that

$$\sum_{i=1}^{k-2} \binom{n-1}{i-1} \leq 2 \binom{n-1}{k-3} = \frac{2(k-1)(k-2)}{(n-k+1)(n-k+2)} \binom{n-1}{k-1} < \frac{2}{(C-1)^2} \binom{n-1}{k-1}.$$ 

Therefore, if (3) does not hold for $\mathcal{F}$, we get that

$$|\mathcal{D}^{(k-1)}(\mathcal{F})| > \sum_{i=0}^{k-1} \binom{n-1}{i} - \sum_{i=0}^{k-2} \binom{n}{i} = \binom{n-1}{k-1} - \sum_{i=1}^{k-2} \binom{n-1}{i-1} \geq \left(1 - \frac{2}{(C-1)^2}\right) \binom{n-1}{k-1}.$$ 

Since $|\mathcal{D}^{(k-1)}(\mathcal{F})| < k|\mathcal{F}|$, we also have $|\mathcal{F}| \geq \frac{1}{2k} \binom{n-1}{k-1}$ for $C \geq 3$, which proves Proposition 7.

Now we can bound density of $\mathcal{D}^{(k-1)}(\mathcal{F})$:

$$|\mathcal{D}^{(k-1)}(\mathcal{F})|/\binom{n}{k-1} \geq \left(1 - \frac{2}{(C-1)^2}\right) \frac{n-k+1}{n} \geq \left(1 - \frac{2}{(C-1)^2}\right) \left(1 - \frac{1}{C}\right) \geq 1 - \frac{2}{C}.$$ 

We apply Theorem 11 for $\mathcal{D}^{(k-1)}(\mathcal{F})$ with $m = n$, $\ell = k-1$, $\ell' = k$, $t = |C-1|$, $a = \sqrt{2 \ln 8n}$, $\varepsilon = \frac{2a + \sqrt{8 \ln 2}}{\sqrt{t}}$. Note that

$$\varepsilon < 2 \sqrt{\frac{2 \ln (8Ck)}{\sqrt{C-2}}} + 2.4 < 2 \sqrt{\frac{2 \ln (400k \ln k) + 2.4}{50 \ln k - 2}} < 2 \sqrt{\frac{2}{49} \left(1 + \frac{\ln 400}{\ln k} + \frac{\ln \ln k}{\ln k}\right)} + 0.18$$

$$< 2 \sqrt{\frac{2}{49} (1 + 1.54 + 0.35) + 0.18} < 0.87 < 0.9 - \frac{2}{C}.$$ 

We conclude that the probability in the right hand side of (12) is at most $2e^{-a^2/2} = 2e^{-\ln 8n} = \frac{1}{4n}$. Therefore, the property in (12) cannot be satisfied for all $H \in \mathcal{F}$ (indeed, the probability that a randomly chosen $H$ belongs to $\mathcal{F}$ is at least $\frac{1}{4n}$). Therefore, unveiling what (12) states with these parameters, we get that there is $F \in \mathcal{F}$ such that $|\mathcal{D}^{(k-1)}(\mathcal{F}) \cap \binom{\mathcal{F}}{k-1}| \geq \frac{1}{10} \binom{n-k}{k-1}$. W.l.o.g. we assume that $F = \{1, \ldots, k\}$.

By the definition of $\mathcal{D}^{(k-1)}(\mathcal{F})$ and since $\mathcal{F}$ is intersecting, for each $D \in \mathcal{D}^{(k-1)}(\mathcal{F}) \cap \binom{\mathcal{F}}{k-1}$ there is a $x \in F$ such that $\{x\} \cup D \in \mathcal{F}$. Split $\mathcal{D}^{(k-1)}(\mathcal{F}) \cap \binom{\mathcal{F}}{k-1}$ into $\bigcup_{x \in F} \mathcal{E}_x$ such that for any $D \in \mathcal{E}_x$ we have $\{x\} \cup D \in \mathcal{F}$. Note that this partition is not unique and the families $\mathcal{E}_x$ are pairwise cross-intersecting.

Put $m = n - k$, $\ell = k-1$ and $r := \ln k$.

**Claim 13.** For some $i$ we have $|\mathcal{E}_i| \geq \binom{m}{\ell} - \binom{m-r}{\ell}$.

**Proof.** Assume that there is no $i$ such that $|\mathcal{E}_i| > |\mathcal{D}^{(k-1)}(\mathcal{F}) \cap \binom{\mathcal{F}}{k-1}| - \binom{m-r}{\ell-1}$. Then we can partition $F = I_1 \cup I_2$ such that $|\bigcup_{i \in I_1} \mathcal{E}_i| > \binom{m}{\ell-1}$ and $|\bigcup_{i \in I_2} \mathcal{E}_i| > \binom{m}{\ell-1}$. But the families $\bigcup_{i \in I_1} \mathcal{E}_i$ and $\bigcup_{i \in I_2} \mathcal{E}_i$ are cross-intersecting, which contradicts Theorem 10 with $a = b = \ell$ and $x = m - 1$.  

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Therefore, we can assume that $|\mathcal{E}_i| > |\mathcal{D}^{(k-1)}(\mathcal{F}) \cap (\mathcal{F}_{k-1})| - (\binom{m-1}{\ell-1})$. We are only left to verify that this quantity is at least the bound stated in the claim. Note that

$$|\mathcal{E}_i| > |\mathcal{D}^{(k-1)}(\mathcal{F}) \cap (\mathcal{F}_{k-1})| - (\binom{m-1}{\ell-1}) > \left(\frac{1}{10} - \frac{\ell}{m}\right) \left(\frac{m}{\ell}\right) > \frac{1}{12} \left(\frac{m}{\ell}\right).$$

In order to prove the claim, we are thus only left to show that $(\binom{m-r}{\ell}) > \frac{11}{12}(\binom{m}{\ell})$. Indeed, noting that $m - \ell - r + 1 = C\ell + C - 2\ell - r > (C - 2)\ell$, we have

$$\left(\frac{m}{\ell}\right)/\left(\frac{m}{\ell} + 1\right) \leq \exp\left(\frac{\ell r}{m - r - \ell + 1}\right) \leq \exp\left(\frac{r}{C - 2}\right) < \exp\left(\frac{1}{20}\right) < \frac{12}{11}.$$

In what follows, we w.l.o.g. assume that the $i$ guaranteed by Claim 13 is equal to 1.

**Claim 14.** Let $A$ be such that $A \subset F$, $1 \notin A$. Then $|\mathcal{F}(A, F)| \leq \binom{m-r}{m-k+|A|}$.

**Proof.** Note that $\mathcal{F}(A, F) \subseteq \binom{n}{k-A|F}$. Since $\mathcal{E}_i$ and $\mathcal{F}(A, F)$ are cross-intersecting and $|\mathcal{E}_i| > (\binom{m}{\ell}) - (\binom{m-r}{\ell})$, from Theorem 10 we get $|\mathcal{F}(A, F)| \leq \binom{m-r}{m-k+|A|}$. $\square$

**Claim 15.** We have

$$|\mathcal{F}(\bar{1})| = \binom{n-r-1}{n-k-1}.$$

**Proof.** Decomposing and applying Claim 14 we get

$$|\mathcal{F}(\bar{1})| = \sum_{A \subseteq F, 1 \notin A} |\mathcal{F}(A, F)| \leq \sum_{a=0}^{k-1} \binom{k-1}{k-1-a} \binom{m-r}{m-k+a} = \binom{n-r-1}{n-k-1},$$

(in the last inequality we use that $m = n - k$). $\square$

Lemma 8 follows from Claim 15 immediately.

## 5 Counterexamples

For $2 \leq p \leq k$, $n > 2k$, define the family $\mathcal{A}_p(n, k)$ by

$$\mathcal{A}_p(n, k) := \left\{ A \in \binom{n}{k} : 1 \in A, A \cap [2, p+1] \neq \emptyset \right\} \cup \left\{ A \in \binom{n}{k} : [2, p+1] \subset A \right\}.$$

Note that $\mathcal{A}_k(n, k)$ is the only family that attains equality in the Hilton–Milner Theorem (2) for $k > 3$. In [4] it was shown that families $\mathcal{A}_p(n, k)$ are extremal among intersecting families with fixed maximum degree. In [11] the stronger result showing that they are extremal among intersecting families with fixed minimal diversity is proven. Also let us note that $|\mathcal{A}_2(n, k)| = |\mathcal{A}_3(n, k)|$.

In the following, we show that both $\mathcal{A}_3(n, k)$ and $\mathcal{A}_k(n, k)$ are counterexamples to Conjecture 4 for $n = ck$ and small $c$. We have $|\mathcal{D}(\mathcal{A}_3(n, k))| > |\mathcal{D}(\mathcal{S}_x)|$ for $2 < c < \frac{1}{2}(3 + \sqrt{5})$ and $|\mathcal{D}(\mathcal{A}_k(n, k))| > |\mathcal{D}(\mathcal{S}_x)|$ for $2 < c < 4$. At the same time, it is not difficult to see that for $2 < c < \frac{1}{2}(3 + \sqrt{5})$ we have $|\mathcal{D}(\mathcal{A}_3(n, k))| > |\mathcal{D}(\mathcal{A}_k(n, k))|$. We believe that each $\mathcal{A}_t(n, k)$ for $4 \leq t \leq k$ provides a counterexample to the conjecture for some values of $c > 2$, but it requires some tedious computations, so we have not checked it.
5.1 $\mathcal{A}_k(n, k)$

Let us put $\mathcal{B} := \mathcal{A}_k(n, k)$ for shorthand. Comparing it with the full star $\mathcal{S}_1 = \{A \in \binom{[n]}{k} : 1 \in A\}$ we have $\mathcal{B} = \left(\mathcal{S}_1 \setminus \left\{F \cup \{1\} : F \in \binom{[k+2,n]}{k-1}\right\}\right) \cup \{[2, k + 1]\}$. This easily implies

$$\mathcal{D}(\mathcal{S}_1) \setminus \mathcal{D}(\mathcal{B}) = \binom{[k + 2, n]}{k-1},$$

i.e., we “lose” $(n-k-1)$ sets.

On the other hand for all $D \subset [k + 2, n]$, $0 \leq |D| \leq k - 2$, the sets $\{1\} \cup D$ is in $\mathcal{D}(\mathcal{B}) \setminus \mathcal{D}(\mathcal{S}_1)$. Indeed, fix $E \subset [2, k + 1]$, $|E| = k - 1 - |D|$. Then $\{1\} \cup D \cup E \in \mathcal{B}$ and $((\{1\} \cup D) \cup E) \setminus [2, k + 1] = \{1\} \cup D$. Thus

$$|\mathcal{D}(\mathcal{B}) \setminus \mathcal{D}(\mathcal{S}_1)| \geq \sum_{\ell=0}^{k-2} \binom{n-k-1}{\ell}.$$  \hspace{1cm} (17)

As long as the RHS of (16) is smaller than the RHS of (17), $|\mathcal{D}(\mathcal{B})| > |\mathcal{D}(\mathcal{S}_1)|$, i.e., we get a counterexample.

**Claim 16.** If $2 < c < 4$ and $n = ck$, $k > k_0(c)$, then

$$\sum_{\ell=0}^{k-2} \binom{n-k-1}{\ell} > \binom{n-k-1}{k-1}.$$  \hspace{1cm} (18)

**Proof.** Take $d = c/2$, $1 < d < 2$. Define $t$ as the minimum integer satisfying

$$\frac{1}{d} + \frac{1}{d^2} + \ldots + \frac{1}{d^t} \geq 1.$$  \hspace{1cm} (19)

We want to choose $k_0 = k_0(c)$ to satisfy

$$\binom{n-k-1}{k-1-j} > d^{-j} \binom{n-k-1}{k-1} \mbox{ for } 1 \leq j \leq t.$$  \hspace{1cm} (20)

Combined with (19) this would clearly imply (18). For (20) it is sufficient that

$$\binom{n-k-1}{k-1-j} / \binom{n-k-1}{k-j} = \frac{k-j}{n-2k+j} \geq \frac{1}{d} = \frac{2}{c} \mbox{ for } 1 \leq j \leq t.$$  

Equivalently, $(4+c)k \geq 2n + (2+c)j$ or $k \geq \frac{2+c}{4+c} \cdot t$. Thus, for sufficiently large $k$ the inequalities (20) hold and therefore the claim is proved. \hfill $\square$

5.2 $\mathcal{A}_3(n, k)$

Put $\mathcal{C} := \mathcal{A}_3(n, k)$. For a set $F \in \binom{[n]}{k}$ its membership in $\mathcal{C}$ is decided by the intersection $F \cap [4]$, it is a so-called “junta”. Define

$$\mathcal{J}^* := \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}.$$
then \( F \in \mathcal{C} \) iff \( F \cap [4] \in \mathcal{J}^* \). (We say that \( \mathcal{J}^* \) is the defining family of \( \mathcal{J} \).) Note that \( \mathcal{J}^* \) is intersecting. We need to show that

\[
|\mathcal{D}(\mathcal{C})| > \sum_{i=1}^{k} \binom{n-1}{k-i}.
\]

First put \( \mathcal{L}_i := \{ F \setminus F' : \# \varnothing, \# F' \in \mathcal{J}^*, |F \cap F'| \leq i \} \). Note that by the definition \( \varnothing = \mathcal{L}_0 \subset \mathcal{L}_1 \subset \mathcal{L}_2 \subset \mathcal{L}_3 \subset \mathcal{L}_4 = \mathcal{L}_5 = \ldots = \mathcal{L}_k \).

Now we want to rewrite \( |\mathcal{D}(k-i)(\mathcal{C})| \) using \( \mathcal{L}_i \). Note that for each \( F \setminus F' \in \mathcal{D}(k-i)(\mathcal{C}) \) we have \( |F \setminus F'| = i \), thus for \( J := F \cap [4] \) and \( J' := F' \cap [4] \) we have \( |J \cap J'| \leq i \) and \( J \setminus J' \in \mathcal{L}_i \). This implies that \( \mathcal{D}(k-i)(\mathcal{C}) \subset \{ F \in \binom{[n]}{k-i} : F \cap [4] \in \mathcal{L}_i \} \). Actually, it is not difficult to see that equality holds in the previous inclusion. Grouping the size of the family in the right hand side by its intersections with \([4]\), we get

\[
|\mathcal{D}(k-i)(\mathcal{C})| = \sum_{b=0}^{4} |\mathcal{L}_i| \left( \binom{n-4}{k-i-b} \right).
\]

Summing this over \( i = 1, \ldots, k \) we get

\[
|\mathcal{D}(\mathcal{C})| = \sum_{i=1}^{k} |\mathcal{D}(k-i)(\mathcal{C})| = \sum_{i=1}^{k} \sum_{b=0}^{4} |\mathcal{L}_i| \left( \binom{n-4}{k-i-b} \right).
\]

(21)

For our \( \mathcal{J}^* \) we have \( \mathcal{L}_1 = \{ \{1\}, \{2\}, \{3\}, \{4\}, \{2, 3\}, \{2, 4\}, \{3, 4\} \} \) and \( \mathcal{L}_b = \mathcal{L}_1 \cup \{\varnothing\} \) for \( b > 1 \). Then we can regroup terms and rewrite

\[
|\mathcal{D}(\mathcal{C})| = \sum_{i=1}^{k} \left( |\mathcal{L}_{i+1}^{(0)}| \left( \binom{n-4}{k-i-1} \right) + \sum_{b=1}^{2} |\mathcal{L}_i| \left( \binom{n-4}{k-i-b} \right) \right)
\]

\[
= \sum_{i=1}^{k} \left( \binom{n-4}{k-i-1} + \sum_{b=1}^{2} \binom{n-4}{k-i-b} \right)
\]

\[
= \sum_{i=0}^{k-2} \binom{n-4}{i} + \sum_{i=0}^{k-3} \binom{n-4}{i}
\]

Note that we can use Pascal’s triangle identity several times and rewrite

\[
|\mathcal{D}(\mathcal{S}_1)| = \sum_{i=0}^{k-1} \binom{n-1}{i} = \sum_{i=0}^{k-1} \left( \binom{n-4}{k-1} - \binom{n-4}{k-2} - \binom{n-4}{k-3} \right) = \binom{n-4}{k-1} - \binom{n-3}{k-2}.
\]

Therefore, we have

\[
|\mathcal{D}(\mathcal{S}_1)| - |\mathcal{D}(\mathcal{C})| = \binom{n-4}{k-1} - \binom{n-4}{k-2} - \binom{n-4}{k-3} = \binom{n-4}{k-1} - \binom{n-3}{k-2}.
\]

The latter expression is negative iff \( \frac{1}{k+1} < \frac{n-k-1}{(n-k-1)(n-k-2)} \), which is equivalent to \( (n-k-1)(n-k-2) < (n-3)(k-1) \). Simplifying, we get \( n^2 - (3k+2)n + k^2 + 6k - 1 < 0 \), which holds for \( n < \frac{1}{2}(3k + 2 + \sqrt{5k^2 - 12k + 8}) \sim \frac{1}{2}(3 + \sqrt{5})k \) for \( k \to \infty \).
Remark. Put $\alpha := 1/c$ and $\mu_\alpha(F) := \alpha^{|F|} (1 - \alpha)^{|\mathcal{J}^*| - |F|}$, $\mu_\alpha(\mathcal{C}) := \sum_{F \in \mathcal{C}} \mu_\alpha(F)$, and $\mathcal{L}_i := \mathcal{L}_i - \mathcal{L}_{i-1}$. Then we can rewrite (21) as follows.

$$|\mathcal{D}(\mathcal{C})| \sim \sum_{i=1}^{k} \binom{n}{k-i} \mu_\alpha(\mathcal{L}_i) = \sum_{i=1}^{k} \sum_{s=1}^{|\mathcal{J}^*|} \binom{n}{k-i+1-s} \mu_\alpha(\mathcal{L}_{s}) \sim \sum_{i=1}^{k} \left( \frac{n}{k-i} \right)^{|\mathcal{J}^*|} \sum_{s=1}^{s-1} \left( \frac{\alpha}{1 - \alpha} \right)^{s-1} \mu_\alpha(\mathcal{L}_{s}).$$

Thus the problem of maximizing $|\mathcal{D}(\mathcal{C})|$ for an intersecting junta can be reduced to a problem of maximizing $\sum_{s=1}^{|\mathcal{J}^*|} \left( \frac{\alpha}{1 - \alpha} \right)^{s-1} \mu_\alpha(\mathcal{L}_{s})$ for the defining family of the junta.

6 Conclusion

It is natural to ask similar questions for intersecting families, where set difference operation is replaced with another binary set operation. E.g. define $\mathcal{SD}(\mathcal{F}) := \{F \Delta F' : F, F' \in \mathcal{F}\}$, where $\Delta$ stands for symmetric difference. It is clear that for a full star $\mathcal{S}_x$ we have

$$|\mathcal{SD}(\mathcal{S}_x)| = \sum_{0 \leq \ell < k} \binom{n-1}{2\ell}.$$

Conjecture 17. For any intersecting family $\mathcal{F}$, $n > 10k$, we have

$$|\mathcal{SD}(\mathcal{F})| \leq \sum_{0 \leq \ell < k} \binom{n-1}{2\ell}.$$  \tag{22}

Using an argument analogous to [5], one can show that (22) holds for, say, $n > 3k^2$. Repeating the argument from Section 4.2 one can show that for intersecting $\mathcal{F} \subset \binom{[n]}{k}$ with $|\mathcal{F}| > n^{-100} \binom{n}{k}$ that maximizes $|\mathcal{SD}(\mathcal{F})|$ we have $\gamma(\mathcal{F}) < n^{-100} \binom{n}{k}$ for $n > 3k \ln k$.

However, the general case is elusive. It is even unclear how to show that $|\mathcal{F}|$ should be large if $\mathcal{F}$ is extremal. It is not difficult to see that a $p$-random subset of a full star $\mathcal{S}_x$ in a wide range of parameters has almost exactly the same number of symmetric differences as $\mathcal{S}_x$. We believe that proving Conjecture 17 even for $n > 100k \ln k$ would already be very interesting.

Another question one may ask is as follows: what is the maximum of $\mu_p(\mathcal{D}(\mathcal{F}))$ for intersecting $\mathcal{F} \subset 2^{[n]}$ and $0 < p < 1/2$? In [5] the first author proved that $\mu_{1/2}(\mathcal{D}(\mathcal{F})) \leq 1/2$. To solve this problem we can essentially repeat the argument from [5]. It was proved in [2] that for any intersecting family $\mathcal{F} \subset 2^{[n]}$ there is an intersecting family $\mathcal{G}$, such that $\mathcal{F} \subset \mathcal{G} \subset 2^{[n]}$, $|\mathcal{G}| = 2^{n-1}$. Using the notation $\overline{X} := [n] \setminus X$, it is clear that for any $X \subset [n]$, either $X$ or $\overline{X}$ belongs to $\mathcal{G}$ and $\mathcal{G}$ is upwards closed. Next, if $X = F \setminus F'$, where $F, F' \in \mathcal{G}$, then $F' \subset \overline{X}$, and so $\overline{X} \in \mathcal{G}$. We immediately get that $X \notin \mathcal{G}$. Conversely, if $X \in \mathcal{G}$ then $[n] \in \mathcal{G}$ implies $\overline{X} \in \mathcal{D}(\mathcal{G})$. Therefore, we have $\mathcal{G} \cup \mathcal{D}(\mathcal{G}) = 2^{[n]}$ and thus $\mu_p(\mathcal{G}) + \mu_p(\mathcal{D}(\mathcal{G})) = 1$. This shows that $\mu_p(\mathcal{D}(\mathcal{G}))$ is maximized whenever $\mu_p(\mathcal{G})$ is minimized. Since $|\{X, \overline{X}\} \cap \mathcal{G}| = 1$, in order to minimize the measure of $\mathcal{G}$ it is better to include the larger set in $\mathcal{G}$. The optimal family also happens to be the intersecting family, corresponding to the majority function (if $n$ is even, then out of all $n/2$-element sets we can take those that contain 1, say). An interesting consequence is that the maximum of $\mu_p(\mathcal{D}(\mathcal{F})) = 1 - e^{-cn}$, where $c > 0$ for any $p < 1/2$. This is in contrast with the uniform case, in which the answer more or less corresponds to the $p$-measure $1 - p$.  

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References

[1] I. Dinur and E. Friedgut, *Intersecting families are essentially contained in juntas*, Combinatorics, Probability and Computing 18 (2009), 107–122.

[2] P. Erdős, C. Ko and R. Rado, *Intersection theorems for systems of finite sets*, The Quarterly Journal of Mathematics 12 (1961), N1, 313–320.

[3] P. Frankl, *A new short proof for the Kruskal-Katona theorem*, Discrete Mathematics 48.2-3 (1984): 327–329.

[4] P. Frankl, *Erdős-Ko-Rado theorem with conditions on the maximal degree*, Journal of Combinatorial Theory, Series A 46.2 (1987): 252–263.

[5] P. Frankl *On the number of distinct differences in an intersecting family*, Discrete Mathematics 344.2 (2021): 112210.

[6] P. Frankl and A. Kupavskii, *The Erdős Matching Conjecture and concentration inequalities*, arXiv preprint arXiv:1806.08855 (2018).

[7] A.J.W. Hilton and E.C. Milner, *Some intersection theorems for systems of finite sets*, Quart. J. Math. Oxford 18 (1967), 369–384.

[8] G. O. H. Katona, *Intersection theorems for systems of finite sets*, Acta Math. Acad. Sci. Hungar. 15 (1964), 329–337.

[9] G. O. H. Katona, *A theorem of finite sets*, Theory of Graphs, Proc. Colloq. Tihany, 1966, pp. 187–207, Akad. Kiadó, Budapest, 1968.

[10] J. B. Kruskal, *The number of simplices in a complex*, Math. Optimization Techniques, pp. 251–278, Univ. of Calif. Press, Berkeley, 1963.

[11] A. Kupavskii, D. Zakharov. *Regular bipartite graphs and intersecting families*, Journal of Combinatorial Theory, Series A 155 (2018): 180–189.

[12] L. Lovász, *Combinatorial Problems and Exercises*, 13.31, Akad. Kiadó, Budapest; North-Holland, Amsterdam, 1979.

[13] J. Marica and J. Schönheim, *Differences of sets and a problem of Graham*, Canad. Math. Bull. 12 (1969), 635–637.

[14] E. Sperner, *Ein Satz über Untermengen einer endlichen Menge*, Math. Z. 27 (1928), 544–548.