Macroscopic behavior of Lipschitz random surfaces

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Abstract

The motivation for this article is to derive strict convexity of the surface tension for Lipschitz random surfaces, that is, for models of random Lipschitz functions from $\mathbb{Z}^d$ to $\mathbb{Z}$ or $\mathbb{R}$. An essential innovation is that random surface models with long- and infinite-range interactions are included in the analysis. More specifically, we cover at least: uniformly random graph homomorphisms from $\mathbb{Z}^d$ to a $k$-regular tree for any $k \geq 2$ and Lipschitz potentials which satisfy the FKG lattice condition. The latter includes perturbations of dimer- and six-vertex models and of Lipschitz simply attractive potentials introduced by Sheffield. The main result is that we prove strict convexity of the surface tension—which implies uniqueness for the limiting macroscopic profile—if the model of interest is monotone in the boundary conditions. This solves a conjecture of Menz and Tassy, and answers a question posed by Sheffield. Auxiliary to this, we prove several results which may be of independent interest, and which do not rely on the model being monotone. This includes existence and topological properties of the specific free energy, as well as a characterization of its minimizers. We also prove a general large deviations principle which describes both the macroscopic profile and the local statistics of the height functions. This work is inspired by, but independent of, Random Surfaces by Sheffield.

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1 Introduction

1.1 Preface

We study the macroscopic behavior of models of Lipschitz random surfaces, that is, random Lipschitz functions from \( \mathbb{Z}^d \) to \( \mathbb{Z} \) or \( \mathbb{R} \). Examples of such models include height functions of dimer models and six-vertex models and uniformly random \( K \)-Lipschitz functions. One studies in particular the local Gibbs measures, subject to boundary conditions. It is generally expected that the macroscopic limit of a random surface under the influence of boundary conditions is governed by a variational principle. This variational principle asserts that, under suitable boundary conditions on a bounded domain \( D \subset \mathbb{R}^d \), the asymptotic macroscopic profile \( f^* \) must concentrate on any neighborhood of the set of minimizers of the integral

\[
\int_D \sigma(\nabla f(x)) dx
\]

over all those functions \( f \) that match these boundary conditions.

The convex function \( \sigma \), which is called the surface tension, is specific to the model and encodes the free energy density of gradient Gibbs measures which are constrained to a certain slope. Sheffield proves in his seminal work Random Surfaces [She05] that this variational principle can be generalized into a large deviation principle that governs not only the macroscopic profile, but also the local statistics of a random surface over macroscopic regions. These results apply to a significant number of models. The fundamental integral in (1.1) connects the large deviations principle and the variational principle: it appears as the rate function in the large deviations principle, which implies the asserted concentration. When \( \sigma \) is strictly convex, the rate function of the large deviations principle has a unique minimizer \( f^* \) and the random functions concentrate around this unique minimizer (see [DS10] for a proof that strict convexity of \( \sigma \) implies uniqueness of the minimizer of the integral). This also implies that the model is stable under microscopic changes in the boundary conditions. On the other hand, when \( \sigma \) fails to be strictly convex, simulations have suggested that microscopic changes to boundary conditions might have macroscopic effects, and (more generally) that random surfaces might be macroscopically disordered. To illustrate this point, we refer to Figure 1 for two samples from the 5-vertex model, one with parameters which make \( \sigma \) strictly convex, and one with parameters for which \( \sigma \) is not strictly convex. The difference in the macroscopic appearance of these two figures is striking. This dichotomy underlines the pivotal role played by the surface tension in the study of the asymptotic behavior of random surfaces.

In the last thirty years, there have been various models in statistical physics for which strict convexity of the surface tension has been derived. The two most famous are probably the dimer model [CKP01] for \( \mathbb{Z} \)-valued random surfaces and the Ginzburg-Landau \( \nabla \phi \)-interface under suitable conditions [FS97, DF05] for \( \mathbb{R} \)-valued random surfaces. In either case, the strategy employed to demonstrate strict convexity of the surface tension relies heavily on particular properties of the model under consideration. For dimer models, one is able to calculate \( \sigma \) due to exact integrability of the model [CKP01]; for the Ginzburg-Landau \( \nabla \phi \)-interface, the strategy relies on the fact that the potentials considered are almost Gaussian [FS97]. A decisive breakthrough was made in [She05] in the pursuit of a more general approach. In this work, Sheffield proves that statistical physics models associated with simply attractive potentials—that is, convex potentials for which the interactions are exclusively between pairs of points—must have a strictly convex surface tension. Beyond the surprising generality of the result, this work also distinguishes itself by the method that was used to prove strict convexity of the surface tension. Rather than using direct
computational arguments, the author reasons by contradiction: if there is a line segment on which the surface tension is affine, then the minimizing measures corresponding to either endpoint are used to construct a new measure which minimizes the specific free energy, but is not a Gibbs measure. This is then shown to be impossible.

Despite this significant progress, the techniques used in [She05] rely heavily on the interactions being between pairs of points only—they cannot capture what happens for models with interactions involving larger clusters of points. The purpose of this article is to dramatically increase the class of models for which strict convexity of the surface tension can be derived. We do so by providing a new approach which does not rely on a particular formalism of the model in terms of a potential, but instead on stochastic monotonicity. Notably, the new class includes all Lipschitz models for which the interaction potential satisfies the Fortuin-Kasteleyn-Ginibre (FKG) lattice condition. Such potentials are also called submodular, and form a natural generalization of the class of simply attractive potentials. Moreover, the new class also covers interaction potentials which assign a weight to each level set of the height function, in the spirit of the random-cluster model. Such models have infinite-range interactions, and we use them to derive strict convexity of the surface tension for the tree-valued graph homomorphisms studied in [MT20].

There are several ideas which suggest that stochastic monotonicity is a suitable starting point for studying the macroscopic behavior of random surfaces. First, for general percolation models, such as independent percolation and Fortuin-Kasteleyn percolation, the FKG inequality is essential to the understanding of the macroscopic behavior of the model: most, if not all, modern techniques in percolation theory rely on this crucial observation. It appears that stochastic monotonicity is the most general equivalent of the FKG inequality in the context of random height functions. Second, when the height functions of interest are also Lipschitz, the Azuma-Hoeffding inequality implies immediately that the random surface concentrates in some precise sense; the picture on the right in Figure 1 is therefore instantaneously ruled out. Third, it turns out that for this 5-vertex model, stochastic monotonicity (which depends on the choice of parameters), is in fact equivalent to strict convexity of $\sigma$.

Finally, stochastic monotonicity does not depend on any formalism of potentials. This is a significant difference with the class of simply attractive models in [She05], which depends on a particular representation of the model in terms of an underlying interaction potential.
Stochastic monotonicity is thus practical: it suffices to check the Holley criterion. For discrete finite-range models, this is particularly efficient, as it amounts to evaluating a finite number of cases.

1.2 Description of the main results

Let us now broadly describe the main results of this article. Precise statements of the corresponding theorems are to be found in Section 4. Write $\Omega$ for the set of height functions, that is, functions $\phi$ from $\mathbb{Z}^d$ to $E$, where the choice of $d$ and $E \in \{\mathbb{Z}, \mathbb{R}\}$ depends on the model of interest. Write $\Lambda \subset \subset \mathbb{Z}^d$ if the former is a finite subset of the latter; the model of interest is formalized in terms of a specification $\gamma = (\gamma_{\Lambda})_{\Lambda \subset \subset \mathbb{Z}^d}$ which allows one to forget about the values of $\phi$ on $\Lambda$ and resample those values according to the model. The measure $\gamma_{\Lambda}(\cdot, \phi)$ is also called the local Gibbs measure in $\Lambda$ with boundary conditions $\phi$. This model must be invariant by some full-rank sublattice $\mathcal{L}$ of $\mathbb{Z}^d$ if any convergent macroscopic behavior is to be expected. We impose two key restrictions on $\gamma$ for the main results to apply: that $\gamma_{\Lambda}(\cdot, \phi)$ is supported on height functions which are suitably Lipschitz whenever $\phi$ is Lipschitz, and that $\gamma_{\Lambda}(\cdot, \phi) \leq \gamma_{\Lambda}(\cdot, \psi)$ whenever $\phi \leq \psi$. Models satisfying the former condition are called Lipschitz, if they satisfy the latter then they are called stochastically monotone. Finally, for the thermodynamical formalism, we require that the specification $\gamma$ is generated by some interaction potential $\Phi$ which encodes the interactions of the values of $\phi$ at different vertices. We shall see that the heart of the proof does not rely on the formalism of potentials as it is expressed directly in terms of the specification. As a consequence, we are able to incorporate potentials $\Phi$ belonging to a very large class which is described in detail in Section 3. Informally, we allow any potential $\Phi$ which decomposes as the sum of two potentials $\Psi$ and $\Xi$, where $\Psi$ is a potential of finite range which enforces the Lipschitz property (by assigning infinite potential to functions which are not Lipschitz), and where $\Xi$ is potentially an infinite-range potential whose intensity decays fast enough for the specific free energy to be well-defined.

While the finite-range part $\Psi$ of the potential encompasses all common finite-range models in statistical physics, the infinite-range part $\Xi$ is tailored to fit long-range interaction potentials such as those associated with the random-cluster model or the Loop $O(n)$ model. We demonstrate in Subsection 13.3 that this formalism can even be used to prove a conjecture on the limiting behavior of uniformly random graph homomorphisms from $\mathbb{Z}^d$ to a $k$-regular tree for $k \geq 2$.

Let us now introduce a few notions before describing the main results. Write $\mathcal{P}_\mathcal{L}(\Omega, F^\mathcal{V})$ for the collection of $\mathcal{L}$-invariant gradient measures on $\Omega$. Any measure $\mu \in \mathcal{P}_\mathcal{L}(\Omega, F^\mathcal{V})$ has an associated slope $S(\mu)$ which is the unique linear functional $u \in (\mathbb{R}^d)^*$ such that

$$u(x) = \mu(\phi(x) - \phi(0))$$

for all $x \in \mathcal{L}$. The specific free energy of $\mu$ is defined by the limit

$$\mathcal{H}(\mu|\Phi) := \lim_{n \to \infty} n^{-d} \mathcal{H}_{\Pi_n}(\mu|\Phi),$$

where $\Pi_n \subset \subset \mathbb{Z}^d$ denotes a box of sides $n$, and where $\mathcal{H}_{\Lambda}(\mu|\Phi)$ denotes the free energy of $\mu$ over $\Lambda$ with respect to the interior Hamiltonian generated by $\Phi$; this quantity is introduced formally in Section 2. The surface tension is the function $\sigma : (\mathbb{R}^d)^* \to \mathbb{R} \cup \{\infty\}$ defined by

$$\sigma(u) := \inf_{\mu \in \mathcal{P}_\mathcal{L}(\Omega, F^\mathcal{V}) \text{ with } S(\mu) = u} \mathcal{H}(\mu|\Phi).$$
This function is automatically convex as $S(\cdot)$ and $\mathcal{H}(\cdot|\Phi)$ are affine over $\mathcal{P}_\mathcal{L}(\Omega, \mathcal{F}^\wedge)$—as will be shown—and we write $U_\Phi$ for the topological interior of the set $\{\sigma < \infty\} \subset (\mathbb{R}^d)^*$. Finally, call a shift-invariant measure $\mu \in \mathcal{P}_\mathcal{L}(\Omega, \mathcal{F}^\wedge)$ a minimizer if $\mu$ satisfies the equation

$$\mathcal{H}(\mu|\Phi) = \sigma(S(\mu)) < \infty.$$ 

Let us start with the motivating result of this article.

**Theorem** (strict convexity of the surface tension). Let $\Phi$ denote a potential which decomposes as described above, and such that the induced specification $\gamma^\Phi$ is monotone.

1. If $E = \mathbb{R}$, then $\sigma$ is strictly convex on $U_\Phi$.

2. If $E = \mathbb{Z}$, then $\sigma$ is strictly convex on $U_\Phi$ if for any affine map $h : (\mathbb{R}^d)^* \to \mathbb{R}$ with $h \leq \sigma$, the set $\{h = \sigma\} \cap \partial U_\Phi$ is convex. In particular, $\sigma$ is strictly convex on $U_\Phi$ if $E = \mathbb{Z}$ and at least one of the following conditions is satisfied:

   (a) $\sigma$ is affine on $\partial U_\Phi$, but not on $\bar{U}_\Phi$,

   (b) $\sigma$ is not affine on $[u_1, u_2]$ for any distinct $u_1, u_2 \in \partial U_\Phi$ such that $[u_1, u_2] \not\subset \partial U_\Phi$.

See Theorem 4.12 for the formal statement of this theorem. The extra condition for $E = \mathbb{Z}$ is necessary to control the behavior of ergodic measures whose slope is extremal. It is shown in the last part of this article that this condition holds true for all classical models. What happens in general is that measures whose slope lies in $\partial U_\Phi$ have zero combinatorial entropy, which makes it straightforward to derive the inequalities required for satisfying the extra condition. However, it is possible to design exotic models for which it is not known if the condition holds true or not, and consequently we cannot rule out the existence of an affine part of the surface tension for such exotic models.

Our second main result concerns a characterization of minimizers, for potentials which decompose as described above. This generalizes the results of [LT19] to the gradient setting. It is valid even if $\gamma^\Phi$ fails to be monotone, and if $\sigma$ fails to be strictly convex. However, if $\sigma$ is strictly convex, then there exists an ergodic minimizer of slope $u$ for any $u \in U_\Phi$.

**Theorem** (minimizers of the specific free energy). Consider a potential $\Phi$ which decomposes as described, as well as a minimizer $\mu \in \mathcal{P}_\mathcal{L}(\Omega, \mathcal{F}^\wedge)$. Then $\mu$ has finite energy in the sense of Burton and Keane, which means that any local configuration that is Lipschitz, has a positive density (if $E = \mathbb{R}$) or probability (if $E = \mathbb{Z}$) of occurring. Moreover, if the specification $\gamma^\Phi$ is quasilocal, then $\mu$ is a Gibbs measure, and if $\gamma^\Phi$ is not quasilocal but if $\mu$ is supported on its points of quasilocality, then $\mu$ is an almost Gibbs measure—which implies in particular that $\mu = \mu_\gamma^\Phi$ for any $\Lambda \subset \mathbb{Z}^d$. Finally, if $\mu$ is not supported on the points of quasilocality of $\gamma^\Phi$, then we obtain results on the regular conditional probability distributions of $\mu$ which are similar in spirit to those obtained in [LT19].

See Theorem 4.4 for the formal statement of this theorem. The third main result of this paper is a large deviations principle. This large deviations principle concerns both the macroscopic profile of a height function, as well as the local statistics of the height function within a region of macroscopic size. Its formal description requires a significant amount of technical constructions, for which we refer to Sections 4 and 11. One can also consider the large deviations principle on macroscopic profiles only, and the rate function so appearing is given by (1.1) up to an additive constant so that its minimum equals zero. This immediately implies the classical variational principle of [CKP01]. The formal statements are included in Theorem 4.10, Corollary 4.11, and Theorem 11.5.
**Theorem** (variational principle). Consider a potential $\Phi$ which decomposes as above. Let $(D_n, b_n)_{n \in \mathbb{N}}$ denote a sequence of pairs of discrete regions $D_n \subset \subset \mathbb{Z}^d$ and boundary conditions $b_n \in \Omega$ which, after rescaling, suitably approximates some continuous region $D \subset \mathbb{R}^d$ endowed with some boundary function $b : \partial D \to \mathbb{R}$. Then the random function $f_n$ obtained by sampling a configuration from $\gamma_{D_n}^{\Phi}(\cdot, b_n)$ and rescaling, is contained with high probability as $n \to \infty$ in any neighborhood of the set of minimizers $f^*$ of the integral

$$\int_D \sigma(\nabla f(x)) dx$$

over all functions $f : \bar{D} \to \mathbb{R}$ which equal $b$ on $\partial D$. If $\sigma$ is strictly convex, then this minimizer $f^*$ is unique, in which case $f_n \to f^*$ in probability as $n \to \infty$.

In the final part of this article, we provide several applications of our results. Sheffield conjectured that similar results to those obtained in [She05] apply to finite-range submodular potentials, that is, finite-range potentials which satisfy the FKG lattice condition. We prove that our framework applies to submodular Lipschitz potentials, and we prove that the extra condition for $E = \mathbb{Z}$ is automatically satisfied if the model of interest is $\mathcal{L}$-invariant for $\mathcal{L}$ equal to the full lattice $\mathbb{Z}^d$. In fact, we do not even require that the submodular potential of interest has finite range. See Theorem 4.14 for the corresponding formal statements.

We furthermore consider the model of uniformly random graph homomorphisms from $\mathbb{Z}^d$ to a $k$-regular tree. Remark that $k$-regular trees are also Cayley graphs of finitely generated free groups. We confirm the conjecture in [MT20], which asserts that the surface tension associated with this model is strictly convex: see Theorem 4.15. This is remarkable because our theory is phrased in terms of $\mathbb{R}$- or $\mathbb{Z}$-valued functions only.

### 1.3 Ideas and strategy of the proof

The proof of the main results splits into two parts. The first part develops a range of thermodynamical machinery for the class of potentials under consideration. The line of thought motivating these results and proofs was already present in the literature, most notably in the work of Georgii [Geo11], Sheffield [She05], and a previous work of the authors [LT19]. However, it requires significant effort to adapt these existing tools to the generality of our setting. The second part provides a proof of strict convexity of the surface tension, if the potential of interest furthermore induces a specification that is stochastically monotone. This is where we break new ground. Sheffield [She05] proves that the surface tension is strictly convex by employing the following general strategy:

1. Suppose that $\sigma$ is affine on a line segment $[u_1, u_2]$ for $u_1, u_2 \in U_\Phi$ distinct,
2. Construct a shift-invariant gradient measure in $P_\mathcal{L}(\Omega, \mathcal{F}^\mathcal{V})$ of slope $u = (u_1 + u_2)/2$ with minimal specific free energy and which does not have finite energy,
3. Conclude that this contradicts the characterization of the minimizers of the specific free energy, as mentioned earlier in this introduction.

The same strategy is employed here, but the construction of the gradient measure, as well as the heuristic that this construction is based on, are entirely original. The remainder of this subsection gives an overview of this construction.

First, the surface tension $\sigma(u)$ at some slope $u$ can be expressed in terms of the asymptotic behavior of the partition function of $\gamma_{D_n}^{\Phi}(\cdot, \phi^u)$ where $\phi^u$ approximates $u$ in...
some precise sense: this is a consequence of the large deviations principle. We then consider the product measure \( \mu := \gamma_n(\cdot, \phi^n) \times \gamma_n(\cdot, \phi^n) \); write \((\phi_1, \phi_2)\) for the random pair of height functions in \( \mu \), and write \( f \) for the difference \( \phi_1 - \phi_2 \). One can use the fact that \( \sigma \) is affine on the line segment \([u_1, u_2]\) to derive that the function \( f \) deviates macroscopically—that is, at scale \( n \)—from 0 with log probability of order \( o(n^d) \) as \( n \to \infty \). We then use monotonicity of the specification \( \gamma \) to compare the probability of a macroscopic deviation of \( f \) to the probability that the set \( \{ f \in [a, b] \} \subset \Pi_n \subset \mathbb{Z}^d \) has many large connected components for fixed \( 0 < a < b < \infty \). This requires the development of an essential and original geometrical construction. The connected components of \( \{ f \in [a, b] \} \) of interest are called moats. Finally, we randomly shift the functions \( \phi_1 \) and \( \phi_2 \) by a vector in \( \Pi_n \cap \mathcal{L} \) and take limits to produce a shift-invariant measure on the product space, such that each marginal has slope \( u \). The two lower bounds on probabilities imply an upper bound on the specific free energy of this product measure. We show that the moats—the large connected components of \( \{ f \in [a, b] \} \)—grow to be distinct infinite components in this limiting procedure. This contradicts that for a shift-invariant measure with finite energy, the random set \( \{ f \in [a, b] \} \) cannot have more than one infinite component due to the argument of Burton and Keane: the desired contradiction.

Let us finally elaborate briefly on the geometrical construction involving moats. The goal is to find a lower bound on the probability that \( \{ f \in [a, b] \} \) has many large level set, in terms of the probability that \( f \) deviates macroscopically from 0. Write \( c_n := ([n/2], \ldots, [n/2]) \in \Pi_n \) for the center vertex of \( \Pi_n \), and suppose, by means of illustration, that \( f(c_n) > \varepsilon n \) for some \( \varepsilon > 0 \). If \( \phi_1 \) and \( \phi_2 \) are \( K \)-Lipschitz for some \( K \in (0, \infty) \), then \( f \) is \( 2K \)-Lipschitz. Choose \( a = 4K \) and \( b = 8K \). Since \( f(c_n) \) is large and since \( f \) equals 0 on the complement of \( \Pi_n \), we observe that \( \{ f \in [a, b] \} \) must contain a connected component which is contained in \( \Pi_n \) and surrounds the vertex \( c_n \) in some precise sense. This connected component is called a moat. Now fix an arbitrary connected set \( M \subset \Pi_n \), and condition on the event that \( M \) is a moat, and that \( f \) is larger than \( b \) directly inside \( M \). Equipped with monotonicity, it is straightforward to demonstrate that it is more likely (in this conditioned measure) that \( f(c_n) \leq -\varepsilon n + 10K \), than that \( f(c_n) \geq \varepsilon n \). But if \( f(c_n) \leq -\varepsilon n + 10K \) and if \( f \) is larger than \( b \) directly on the inside of \( M \), then \( \{ f \in [a, b] \} \) must have another connected component which surrounds \( c_n \), and which is in turn surrounded by the original moat \( M \). One can continue this procedure to generate a sequence of moats of length \( \varepsilon n / 10K \), such that each moat surrounds the moat that succeeds it. It is important that the union of all moats occupy a uniformly positive proportion of \( \Pi_n \) as \( n \to \infty \), so that they do not disappear in the limiting procedure after rerandomizing the position of the origin; this is indeed the case because of the lower bound on the number of moats.

### 1.4 Open questions

The first natural question which is left open in this work is to decide if it is possible to drop the requirement that random functions are Lipschitz. We believe that it is indeed the case, a significant clue being that this requirement does not appear in [She05]. Finding a way around this restriction would open the main result to a whole new class of interactions. However, the geometrical construction involving the moats relies heavily on the Lipschitz property.

Secondly, it would be interesting to study how the requirement of stochastic monotonicity can be relaxed. Results on strict convexity of the surface tension have been obtained for some non-monotone models for a class of non-convex potentials [CDM09, CD12, AKM16], and for small non-monotone perturbations of dimer models [GMT17, GMT19]. In the simulation on the right in Figure 1, macroscopic disorder is explained by a heuristic. For
this simulation, the parameters of the model are chosen such that straight lines are much preferred over corners. This means that the random surface is able to build momentum: deviations from the mean reinforce each other. This is the exact opposite of stochastic monotonicity. However, there are more subtle (and potentially more local) ways in which stochastic monotonicity might fail. A simple example would be to consider random 1-Lipschitz functions from $\mathbb{Z}^d$ to $\mathbb{Z}$, where the potential discourages neighboring vertices from taking the exact same value. It is easy to show that this model is not monotone, but there is no heuristic of momentum building which would imply macroscopic disorder. Perhaps it would be possible to prove that this model is stochastically monotone in some relaxed sense, in which case the results on moats could be adapted to fit this model.

2 The thermodynamical formalism

The interest is in distributions of the random function $\phi$ which assigns a value $\phi(x)$ from $E$ to each vertex $x \in \mathbb{Z}^d$, where $d \geq 2$ and—depending on the model of interest—$E$ denotes either $\mathbb{Z}$ or $\mathbb{R}$. Such distributions are studied in relation to an underlying model, which encodes the interactions that exist between the function values of $\phi$ at different vertices in $\mathbb{Z}^d$. At the very least, the underlying model must give rise to a functional, which assigns a real number—the specific free energy—to any shift-invariant distribution of $\phi$. In the non-gradient setting there are at least three ways to characterize the model of interest:

1. Through a reference measure on $E$ and an interaction potential,
2. Through a reference distribution of $\phi$,
3. Directly through the specification.

Each formulation has slightly different properties, but they all generate a suitable entropy functional whenever the correct conditions are imposed. See [LT19] for an overview. In the gradient setting of this paper we must be more careful, and it seems that only the first formulation generates a suitable entropy functional. The goal of this section is to efficiently describe the standard objects for the formal framework of gradient models on $\mathbb{Z}^d$.

Subsection 2.1 introduces the necessary objects and symmetries for the shift-invariant gradient setting. The same subsection also introduces the key restrictions on the model: that the specification is monotone, and that it produces Lipschitz functions. Subsection 2.2 describes the formalism of potentials. Subsection 2.3 introduces the specific free energy and the surface tension. The specific free energy is well-defined for all potentials $\Phi$ in the class $S_L + W_L$ which is introduced in Section 3; we prove existence of the specific free energy in Section 7. All definitions in the current section are standard.

2.1 The gradient formalism

2.1.1 Height functions

We are interested in distributions of the random function $\phi$, which assigns values from the measure space $(E, \mathcal{E}, \lambda)$ to the vertices of the square lattice $\mathbb{Z}^d$. Here $E$ refers to either $\mathbb{Z}$ or $\mathbb{R}$, depending on the context, $\mathcal{E}$ is the Borel $\sigma$-algebra, and $\lambda$ denotes the counting measure (if $E = \mathbb{Z}$) or the Lebesgue measure (if $E = \mathbb{R}$). The choice of $E$ is considered fixed throughout the entire work. The set of all functions $\phi$ from $\mathbb{Z}^d$ to $E$ is denoted by $\Omega$. Functions in $\Omega$ are called samples or height functions. For $\Lambda \subset \mathbb{Z}^d$ and $\phi \in \Omega$, write $\phi_\Lambda \in E^\Lambda$ for the restriction $\phi|_\Lambda$. If furthermore $\Delta \subset \mathbb{Z}^d$ and $\psi \in \Omega$ with $\Lambda$ and $\Delta$ disjoint, then write $\phi_\Lambda \psi_\Delta \in E^{\Lambda \cup \Delta}$ for the unique function that restricts to $\phi$ on $\Lambda$ and to $\psi$ on $\Delta$. 

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2.1.2 Subsets of $\mathbb{Z}^d$

Write $\Lambda \subset \subset \mathbb{Z}^d$ if $\Lambda$ is a finite subset of $\mathbb{Z}^d$. Throughout this article, we shall reserve the notation $(\Pi_n)_{n \in \mathbb{N}}$ for the sequence of subsets of $\mathbb{Z}^d$ defined by $\Pi_n := [0, n)^d \subset \subset \mathbb{Z}^d$ for each $n \in \mathbb{N}$. Remark that $|\Pi_n| = n^d$ for any $n \in \mathbb{N}$.

Next, introduce two notions of boundary for subsets $\Lambda$ of $\mathbb{Z}^d$. Write $\partial \Lambda$ for the set of the vertices which are adjacent to $\Lambda$ in the square lattice. Write $\partial^n \Lambda$ for the set of vertices in $\Lambda$ which are at $d_1$-distance at most $n$ from $\mathbb{Z}^d \setminus \Lambda$, for any $n \in \mathbb{Z}_{\geq 0}$; here $d_1$ is the graph metric corresponding to the square lattice. Write also $\Lambda^{-n}$ for $\Lambda \setminus \partial^n \Lambda$. If $D \subset \mathbb{R}^d$, then write $\Lambda(D) := D \cap \mathbb{Z}^d$ and $\Lambda^{-n}(D) := (\Lambda(D))^{-n}$.

Now let $(\Lambda_n)_{n \in \mathbb{N}}$ denote a sequence of subsets of $\mathbb{Z}^d$. If all sets $\Lambda_n$ are finite with $|\Lambda_n| \to \infty$ and $|\partial \Lambda_n|/|\Lambda_n| \to 0$ as $n \to \infty$, then $(\Lambda_n)_{n \in \mathbb{N}}$ is called a Van Hove sequence. We write $(\Lambda_n)_{n \in \mathbb{N}} \uparrow \mathbb{Z}^d$ to mean that $(\Lambda_n)_{n \in \mathbb{N}}$ is a Van Hove sequence. The sequence $(\Pi_n)_{n \in \mathbb{N}}$ is an example of a Van Hove sequence.

2.1.3 $\sigma$-Algebras and random fields

If $(X, \mathcal{X})$ is any measurable space, then write $\mathcal{P}(X, \mathcal{X})$ for the set of probability measures on it, and $\mathcal{M}(X, \mathcal{X})$ for the set of $\sigma$-finite measures. Define the following $\sigma$-algebras on $\Omega$ for any $\Lambda \subset \mathbb{Z}^d$:

$$\mathcal{F} := \sigma(\phi(x) : x \in \mathbb{Z}^d), \quad \mathcal{F}_\Lambda := \sigma(\phi(x) : x \in \Lambda),$$

$$\mathcal{F}^\nabla := \sigma(\phi(y) - \phi(x) : x, y \in \mathbb{Z}^d), \quad \mathcal{F}_\Lambda^\nabla := \sigma(\phi(y) - \phi(x) : x, y \in \Lambda).$$

A random field is a probability measure in $\mathcal{P}(\Omega, \mathcal{A})$ for some $\sigma$-algebra $\mathcal{A} \subset \mathcal{F}$. We introduce the gradient $\sigma$-algebra $\mathcal{F}^\nabla$ because it is often not possible to measure the height $\phi(x)$ directly; only the height differences $\phi(y) - \phi(x)$ are measurable. Note that, with the above definitions, $\mathcal{F}_\Lambda^\nabla = \mathcal{F}^\nabla \cap \mathcal{F}_\Lambda$. For $\Lambda \subset \mathbb{Z}^d$, write $\pi_\Lambda$ for the natural probability kernel from $(\Omega, \mathcal{F})$ to $(E^\Lambda, \mathcal{E}^\Lambda)$ which restricts random fields to $\Lambda$.

A cylinder set is a measurable subset of $\Omega$ which is contained in $\mathcal{F}_\Lambda$ for some $\Lambda \subset \subset \mathbb{Z}^d$; a cylinder function is a function $\Omega \to \mathbb{R}$ which is $\mathcal{F}_\Lambda$-measurable for some $\Lambda \subset \subset \mathbb{Z}^d$. A cylinder function is called continuous if it is continuous with respect to the topology of uniform convergence on $\Omega$. Note that all cylinder functions are continuous whenever $E = \mathbb{Z}$.

Define the further $\sigma$-algebras on $\Omega$ for any $\Lambda \subset \mathbb{Z}^d$:

$$\mathcal{T}_\Lambda := \mathcal{F}^\nabla \setminus \Lambda, \quad \mathcal{T} := \cap_{\Delta \subset \subset \mathbb{Z}^d} \mathcal{T}_\Delta, \quad \mathcal{T}_\Lambda^\nabla := \mathcal{T}_\Lambda \cap \mathcal{F}^\nabla, \quad \mathcal{T}^\nabla := \mathcal{T} \cap \mathcal{F}^\nabla.$$

Sets in $\mathcal{T}$ are called tail-measurable.

2.1.4 The topology of (weak) local convergence

The topology of local convergence is the coarsest topology on $\mathcal{P}(\Omega, \mathcal{F}^\nabla)$ that makes the map $\mu \mapsto \mu(f)$ continuous for any bounded cylinder function $f$. The topology of weak local convergence is the coarsest topology on $\mathcal{P}(\Omega, \mathcal{F}^\nabla)$ that makes the map $\mu \mapsto \mu(f)$ continuous for any bounded continuous cylinder function $f$. Note that the two topologies coincide whenever $E = \mathbb{Z}$. Section 10 uses a particular basis $\mathcal{B}$ for the topology of weak local convergence on $\mathcal{P}(\Omega, \mathcal{F}^\nabla)$. This basis $\mathcal{B}$ is defined such that it contains exactly all sets $B \subset \mathcal{P}(\Omega, \mathcal{F}^\nabla)$ which can be written as finite intersections of open sets of the form $\{\mu : a < \mu(f) < b\}$, where $a, b \in \mathbb{R}$ and where $f$ is a continuous bounded cylinder function.
2.1.5 Shift-invariance and ergodicity

To see convergence of the model at a macroscopic scale it is important that the model exhibits shift-invariance. For \( x \in \mathbb{Z}^d \), write \( \theta_x : \mathbb{Z}^d \to \mathbb{Z}^d \), \( y \mapsto y + x \). Throughout this paper, the letter \( \mathcal{L} \) denotes a fixed full-rank sublattice of \( \mathbb{Z}^d \), and \( \Theta = \Theta(\mathcal{L}) = \{ \theta_x : x \in \mathcal{L} \} \) is the corresponding group of translations of \( \mathbb{Z}^d \). If \( \phi \in \Omega \) and \( \theta \in \Theta \), then \( \theta \phi \) denotes the unique height function satisfying \( (\theta \phi)(x) = \phi(\theta x) \) for all \( x \). Similarly, define

\[
\theta A := \{ \theta \phi : \phi \in A \}, \quad A := \{ \theta A : A \in \mathcal{A} \}, \quad \theta \mu : \theta A \to [0, \infty], \quad \theta \mu(\theta A) \mapsto \mu(A)
\]

for \( A \subset \Omega \), for \( A \) a sub-\( \sigma \)-algebra of \( \mathcal{F} \), and for \( \mu \) a measure on \( A \). Any of these three objects is called \( \mathcal{L} \)-invariant if they are invariant under \( \theta \) for any \( \theta \in \Theta \). If \( A \) is an \( \mathcal{L} \)-invariant \( \sigma \)-algebra on \( \Omega \), then write \( \mathcal{P}_\mathcal{L}(\Omega, \mathcal{A}) \) for the collection of \( \mathcal{L} \)-invariant probability measures on \((\Omega, \mathcal{A})\). Note that \( \mathcal{P}_\mathcal{L}(\Omega, \mathcal{A}) \) is the set of probability measures on \((\Omega, \mathcal{A})\) such that \( \phi \) and \( \theta \phi \) have the same distribution for any \( \theta \in \Theta \).

Define finally

\[
\mathcal{I}_\mathcal{L} := \{ A \in \mathcal{F} : A = \theta A \text{ for all } \theta \in \Theta \}, \quad \mathcal{I}_\mathcal{L}^\gamma := \mathcal{I}_\mathcal{L} \cap \mathcal{F}^\gamma.
\]

A gradient measure \( \mu \in \mathcal{P}(\mathcal{F}, \mathcal{F}^\gamma) \) is called ergodic if \( \mu \) is \( \mathcal{L} \)-invariant and trivial on \( \mathcal{I}_\mathcal{L}^\gamma \). Write \( \text{ex}\mathcal{P}_\mathcal{L}(\mathcal{F}, \mathcal{F}^\gamma) \) for the set of all such ergodic gradient measures. Write \( \text{e}(\text{ex}\mathcal{P}_\mathcal{L}(\mathcal{F}, \mathcal{F}^\gamma)) \) for the smallest \( \sigma \)-algebra that makes the map \( A \mapsto \mu(A) \) measurable for all \( A \in \mathcal{F} \).

2.1.6 Specifications

A specification is a family \( \gamma = (\gamma_\Lambda)_{\Lambda \subset \subset \mathbb{Z}^d} \) of probability kernels, such that

1. \( \gamma_\Lambda \) is a probability kernel from \((\Omega, \mathcal{T}_\Lambda)\) to \((\mathcal{F}, \mathcal{F})\) for each \( \Lambda \subset \subset \mathbb{Z}^d \),

2. \( \mu \gamma_\Lambda(A) = \mu(A) \) for any \( \Lambda \subset \subset \mathbb{Z}^d \), \( A \in \mathcal{T}_\Lambda \), and \( \mu \in \mathcal{P}(\Omega, \mathcal{F}) \),

3. \( \gamma_\Lambda \gamma_\Delta = \gamma_\Lambda \) for any \( \Delta \subset \Lambda \subset \subset \mathbb{Z}^d \).

The specification defines the local behavior of the model, and we think of \( \gamma_\Lambda(\cdot, \phi) \) as the local Gibbs measure in \( \Lambda \subset \subset \mathbb{Z}^d \) with boundary conditions \( \phi \in \Omega \). A specification \( \gamma \) is called \( \mathcal{L} \)-invariant if \( \gamma_\Lambda(\cdot, \theta \phi) = \theta \gamma_\Lambda(\cdot, \phi) \) for any \( \Lambda \subset \subset \mathbb{Z}^d \), \( \phi \in \Omega \), and \( \theta \in \Theta \). Call \( \gamma \) a gradient specification if the distribution of \( \psi + a \) in \( \gamma_\Lambda(\cdot, \phi) \) equals that of \( \psi \) in \( \gamma_\Lambda(\cdot, \phi + a) \) for any \( \Lambda \subset \subset \mathbb{Z}^d \), \( \phi \in \Omega \), and \( a \in e \), where \( \psi \) denotes the random height function in each local Gibbs measure. Note that each kernel \( \gamma_\Lambda \) restricts to a kernel from \((\Omega, \mathcal{T}_\Lambda^\gamma)\) to \((\Omega, \mathcal{F}^\gamma)\) whenever \( \gamma \) is a gradient specification.

2.1.7 Monotonicity

An event \( A \in \mathcal{F} \) is called increasing if \( \phi \in A \) and \( \psi \geq \phi \) implies \( \psi \in A \). Consider two measures \( \mu_1, \mu_2 \in \mathcal{P}(\Omega, \mathcal{F}) \). Say that \( \mu_2 \) stochastically dominates \( \mu_1 \), and write \( \mu_1 \preceq \mu_2 \), if \( \mu_1(A) \leq \mu_2(A) \) for any increasing event \( A \). This is equivalent to asking that there exists a coupling between the two measures such that \( \phi_1 \leq \phi_2 \) almost surely, where the distributions of \( \phi_1 \) and \( \phi_2 \) are prescribed by the measures \( \mu_1 \) and \( \mu_2 \) respectively. A specification \( \gamma \) is called monotone if for each \( \Lambda \subset \subset \mathbb{Z}^d \), the kernel \( \gamma_\Lambda \) preserves the partial order \( \preceq \) on \( \mathcal{P}(\Omega, \mathcal{F}) \). Now consider a fixed measurable set \( A \in \mathcal{F} \), and use—in this definition—the shorthand \( \mathcal{P}_A \) for the set \( \{ \mu \in \mathcal{P}(\Omega, \mathcal{F}) : \mu(A) = 1 \} \). The specification \( \gamma \) is called monotone over \( A \) if \( \mu \gamma_\Lambda \in \mathcal{P}_A \) for any \( \Lambda \subset \subset \mathbb{Z}^d \) and \( \mu \in \mathcal{P}_A \), and if \( \gamma_\Lambda \) preserves the partial order \( \preceq \) on \( \mathcal{P}_A \). The assumption that \( \gamma \) is monotone over a suitable set of Lipschitz functions is crucial to the proof of strict convexity of the surface tension.
2.1.8 The Lipschitz property

Consider some fixed constant $K \in [0, \infty)$. A height function is called $K$-Lipschitz if that height function is $K$-Lipschitz with respect to the graph metric $d_1$ on the square lattice $\mathbb{Z}^d$. A measure is called $K$-Lipschitz if it is supported on $K$-Lipschitz functions. The Lipschitz property is further refined in Subsection 3.1.

2.1.9 The slope

Consider $\mu \in \mathcal{P}_\mathcal{L}(\Omega, \mathcal{F}^\mathcal{V})$. If $\phi(y) - \phi(x)$ is $\mu$-integrable for any $x, y \in \mathbb{Z}^d$, then $\mu$ is said to have finite slope. If $\mu$ has finite slope, then shift-invariance of $\mu$ implies that the function

$$\mathcal{L} \to \mathbb{R}, \ x \mapsto \mu(\phi(x) - \phi(0))$$

is additive. In particular, this means that there is a unique linear functional $u \in (\mathbb{R}^d)^*$ such that

$$u(x) = \mu(\phi(x) - \phi(0))$$

for any $x \in \mathcal{L} \subset \mathbb{R}^d$. This linear functional $u$ is called the slope of $\mu$, and we write $S(\mu)$ for it. The map $S$ is affine: it is clear that $S((1-t)\mu + t\nu) = (1-t)S(\mu) + tS(\nu)$ for any $t \in [0,1]$ and for any $\mu, \nu \in \mathcal{P}_\mathcal{L}(\Omega, \mathcal{F}^\mathcal{V})$ with finite slope.

If we restrict to $K$-Lipschitz measures in $\mathcal{P}_\mathcal{L}(\Omega, \mathcal{F}^\mathcal{V})$ for fixed $K \in [0, \infty)$, then all measures have finite slope, and the map $\mu \mapsto S(\mu)$ is then continuous with respect to the topology of (weak) local convergence.

2.2 Interaction potentials, reference measures, and specifications

2.2.1 Interaction potentials

The model of interest is formalized in terms of an interaction potential $\Phi = (\Phi_\Lambda)_{\Lambda \subset \subset \mathbb{Z}^d}$, which is a family of potential functions $\Phi_\Lambda : \Omega \to \mathbb{R} \cup \{\infty\}$ where each function $\Phi_\Lambda$ is required to be measurable with respect to $\mathcal{F}_\Lambda$. The potential $\Phi$ is called a gradient potential if each function $\Phi_\Lambda$ is in addition $\mathcal{F}^\mathcal{V}_\Lambda$-measurable. The potential $\Phi$ is furthermore called $\mathcal{L}$-invariant or periodic if $\Phi_\Lambda(\theta \phi) = \Phi_\Lambda(\phi)$ for all $\theta \in \Theta$ and for any $\phi \in \Omega$. In the sequel, $\Phi$ shall always denote a fixed periodic gradient potential. It is always conventionally assumed that $\Phi_\Lambda \equiv 0$ whenever $\Lambda$ is a singleton or empty because the $\sigma$-algebra $\mathcal{F}^\mathcal{V}_\Lambda$ is then trivial.

Next, introduce the Hamiltonian. For $\Lambda \subset \subset \mathbb{Z}^d$ and $\Delta \subset \mathbb{Z}^d$ containing $\Lambda$, let $H_{\Lambda,\Delta}$ denote the $\mathcal{F}^\mathcal{V}_\Lambda$-measurable function from $\Omega$ to $\mathbb{R} \cup \{\infty\}$ defined by

$$H_{\Lambda,\Delta} := \sum_{\Gamma \subset \subset \mathbb{Z}^d \text{ with } \Gamma \subset \Delta \text{ and } \Gamma \text{ intersecting } \Lambda} \Phi_\Gamma.$$

In particular, we write $H_{\Lambda} := H_{\Lambda,\mathbb{Z}^d}$ and $H^0_{\Lambda} := H_{\Lambda,\Lambda}$. We shall soon introduce further conditions on $\Phi$ which ensure that the sum in the display is always well-defined and bounded below. The function $H_{\Lambda}$ is called the Hamiltonian of $\Lambda$ and $H^0_{\Lambda}$ is called the interior Hamiltonian of $\Lambda$. We add a superscript $\Phi$ to this notation whenever multiple interaction potentials are considered and confusion might possibly arise.

2.2.2 Reference measures

For any fixed nonempty $\Lambda \subset \subset \mathbb{Z}^d$, there exist natural reference measures on the measurable spaces $(\Omega, \mathcal{F}_\Lambda)$ and $(\Omega, \mathcal{F}^\mathcal{V}_\Lambda)$, in terms of the previously introduced reference measure $\lambda$ on $(E, \mathcal{E})$. In the non-gradient setting this is straightforward: the map $\phi \mapsto \phi_\Lambda$ extends
to a bijection from $\mathcal{F}_\Lambda$ to $\mathcal{E}_\Lambda$, and $\lambda^\Lambda$ is a measure on $(E^\Lambda, \mathcal{E}^\Lambda)$. With only slight abuse of notation, we write also $\lambda^\Lambda$ for the unique measure on $(\Omega, \mathcal{F}_\Lambda)$ that makes the map $\phi \mapsto \phi_\Lambda$ into a measure-preserving projection from $(\Omega, \mathcal{F}_\Lambda, \lambda^\Lambda)$ to $(E^\Lambda, \mathcal{E}^\Lambda, \lambda^\Lambda)$. We must be more subtle in the gradient setting: we cannot measure the height of $\phi$ directly, and so we cannot pullback the measure $\lambda^\Lambda$. Fix therefore some reference point $x \in \Lambda$ and set $\Lambda' := \Lambda \setminus \{x\}$, and consider instead the map $\phi \mapsto \phi_{\Lambda'} - \phi(x)$. This map extends to a bijection from $\mathcal{F}_\Lambda'$ to $\mathcal{E}_\Lambda'$. Abuse notation again by writing $\lambda^{\Lambda'}$ for the unique measure on $(\Omega, \mathcal{F}_\Lambda', \lambda^{\Lambda'})$ that turns the map $\phi \mapsto \phi_{\Lambda'} - \phi(x)$ into a measure-preserving projection from $(\Omega, \mathcal{F}_\Lambda', \lambda^{\Lambda'})$ to $(E^\Lambda, \mathcal{E}^\Lambda, \lambda^\Lambda)$. The notation $\lambda^{\Lambda'}$ bears no reference to the choice of $x \in \Lambda$, as the resulting measure $\lambda^{\Lambda'}$ is indeed independent of this arbitrary choice. The gradient reference measures $\lambda^{\Lambda'}$ are not used in the definition of the specification that $\Phi$ generates; they will first appear in the definition of the specific free energy.

2.2.3  The specification generated by a potential

The potential $\Phi$ generates a specification $\gamma^\Phi = (\gamma^\Phi_\Lambda)_{\Lambda \subset \subset \mathbb{Z}^d}$ defined by

$$\gamma^\Phi_\Lambda(A, \phi) := \frac{1}{Z^\Lambda_\Phi(\phi)} \int_{E^\Lambda} 1_A(\psi \phi_{\mathbb{Z}^d \setminus \Lambda}) e^{-H^\Lambda_\Phi(\psi \phi_{\mathbb{Z}^d \setminus \Lambda})} d\lambda^\Lambda(\psi),$$

for any $\Lambda \subset \subset \mathbb{Z}^d$, $\phi \in \Omega$, and $A \in \mathcal{F}$, where $Z^\Lambda_\Phi(\phi)$ is the normalizing constant

$$Z^\Lambda_\Phi(\phi) := \int_{E^\Lambda} e^{-H^\Lambda_\Phi(\psi \phi_{\mathbb{Z}^d \setminus \Lambda})} d\lambda^\Lambda(\psi).$$

We drop the superscript $\Phi$ in this notation unless the choice of potential is ambiguous. Of course, $\gamma_\Lambda(\cdot, \phi)$ is a well-defined probability measure on $(\Omega, \mathcal{F})$ only if $Z_\Lambda(\phi) \in (0, \infty)$. Say that $\phi$ has finite energy if $\Phi_\Lambda(\phi) < \infty$ for any $\Lambda \subset \subset \mathbb{Z}^d$, and say that $\phi$ is admissible if it has finite energy and $Z_\Lambda(\phi) \in (0, \infty)$ for any $\Lambda \subset \subset \mathbb{Z}^d$. To draw a sample $\psi$ from $\gamma_\Lambda(\cdot, \phi)$, set first $\psi$ equal to $\phi$ on the complement of $\Lambda$, then sample $\psi_\Lambda$ proportional to $e^{-H_\Lambda \lambda^\Lambda}$. Similarly, if $\mu$ is a probability measure on $(\Omega, \mathcal{T}_\Lambda)$ supported on admissible height functions, then $\mu \gamma_\Lambda$ is a probability measure on $(\Omega, \mathcal{F})$; to sample from $\mu \gamma_\Lambda$ one first obtains an auxiliary sample $\phi$ from $\mu$; then one draws the final sample $\psi$ from $\gamma_\Lambda(\cdot, \phi)$.

It is important to observe that $\gamma$ is a gradient specification. This is due to the fact that $\Phi$ is a gradient potential which makes $H_\Lambda$ measurable with respect to $\mathcal{F}_\Lambda'$, and because the reference measures $\lambda$ and $\lambda^\Lambda$ are invariant under translations.

2.3  The surface tension

2.3.1  Relative entropy

Recall first the relative entropy. If $(X, \mathcal{X}, \nu)$ is an arbitrary $\sigma$-finite measure space and $\mu$ another probability measure on $(X, \mathcal{X})$, then the relative entropy of $\mu$ with respect to $\nu$ is defined by

$$\mathcal{H}(\mu|\nu) := \begin{cases} \mu(\log f) = \nu(f \log f) & \text{if } \mu \ll \nu \text{ where } f = d\mu/d\nu, \\ \infty & \text{otherwise.} \end{cases}$$

Remark that $\mathcal{H}(\mu|\nu) \in \mathbb{R} \cup \{-\infty, \infty\}$ in general, and that $\mathcal{H}(\mu|\nu) \geq -\log \nu(X)$. If $\nu$ is a finite measure, then we have equality if and only if $\mu$ is a scalar multiple of $\nu$. If $\mathcal{A}$ is a sub-$\sigma$-algebra of $\mathcal{X}$, then use the shorthand $\mathcal{H}_{\mathcal{A}}(\mu|\nu)$ for $\mathcal{H}(\mu|\mu_{|\mathcal{A}}|\nu_{|\mathcal{A}})$. 

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2.3.2 The free energy

We are now ready to introduce the free energy. This already requires the presence of some gradient potential $\Phi$, although we do not yet impose any condition on it. Consider also some gradient random field $\mu \in \mathcal{P}(\Omega, \mathcal{F}^\nabla)$, and some finite set $\Lambda \subset \subset \mathbb{Z}^d$. Then the free energy of $\mu$ in $\Lambda$ with respect to $\Phi$ is defined by

$$H^\Lambda(\mu|\Phi) := H^\Lambda_F(\mu|e^{H^0_\Lambda} \lambda^{\Lambda-1}) = H^\Lambda_F(\mu|\lambda^{\Lambda-1}) + \mu(H^0_\Lambda, \Phi).$$

The free energy is sometimes decomposed into the entropy and the energy of $\mu$ in $\Lambda$—the two terms in the rightmost expression in the display respectively. (For the final equality, we adopt the convention that $\infty - \infty = \infty$.)

2.3.3 The specific free energy

The specific free energy of a shift-invariant random field $\mu \in \mathcal{P}(\Omega, \mathcal{F}^\nabla)$ with respect to $\Phi$ is defined by the limit

$$\mathcal{H}(\mu|\Phi) := \lim_{n \to \infty} n^{-d} \mathcal{H}_\Pi_n(\mu|\Phi).$$

The specific free energy thus describes the asymptotic of the normalized free energy of $\mu$ with respect to $\Phi$ over a large box. In Section 7 we prove that the limit converges for all $\Phi$ in the class $\mathcal{S}L + \mathcal{W}_L$ which is described in Section 3. It is also shown in Section 7 that $\mathcal{H}(\cdot|\Phi)$ is affine and bounded below.

2.3.4 The surface tension

Consider a potential $\Phi$ in our class $\mathcal{S}L + \mathcal{W}_L$, which implies that the specific free energy is well-defined, affine and bounded below. The surface tension is the function $\sigma : (\mathbb{R}^d)^* \to \mathbb{R} \cup \{\infty\}$ defined by

$$\sigma(u) := \inf_{\mu \in \mathcal{P}(\Omega, \mathcal{F}^\nabla) \text{ with } S(\mu) = u} \mathcal{H}(\mu|\Phi).$$

The function $\sigma$ must be convex because both $S(\cdot)$ and $\mathcal{H}(\cdot|\Phi)$ are affine. We shall write $U_\Phi$ for the interior of the convex set $\{\sigma < \infty\} \subset (\mathbb{R}^d)^*$. Slopes in $U_\Phi$ are called allowable. The major contribution of this article is that we show that $\sigma$ is strictly convex on $U_\Phi$ whenever $\gamma^\Phi$ is monotone over the set of admissible height functions and if $\Phi$ is in our class $\mathcal{S}L + \mathcal{W}_L$ (and under an additional condition whenever $E = \mathbb{Z}$).

3 The class of models under consideration

In the following four subsections, we describe the conditions which are imposed on the model of interest: these are specific to this article, and this is where we broaden the class of models for which strict convexity of the surface tension can be derived. Subsection 3.1 describes the Lipschitz setting in more detail. We take great care in formulating the Lipschitz condition: this is not necessary for the arguments to work, but it rather minimizes the restrictions imposed on the class of models. Let us now consider the potential which generates the model. The potential of the model of interest must decompose as the sum of two potentials, where the first component is a strong, local potential which—at the very least—enforces the Lipschitz condition (Subsection 3.2), and where the second component is a weak interaction of infinite range (Subsection 3.3). The word weak here is only relative to the word strong that was used to describe the first potential: in particular, we do not mean to imply that the second component demonstrates any sort of decay over long distances. It is the second
potential that allows us to assign energy to large geometric objects, such as level sets. Subsection 3.4 finally gives an overview of the objects describing the model of interest, and which are considered fixed throughout most of the analysis.

3.1 Local Lipschitz constraints

We require that a height function has finite energy if and only if it is Lipschitz with respect to the correct quasimetric. We shall allow quasimetrics (subject to certain necessary constraints) in order to be as general as possible. The Lipschitz constraint must be enforced locally by the potential, due to the nature of the arguments that we use to derive the main result. This means that for each vertex \(x \in \mathbb{Z}^d\) we are allowed to enforce a Lipschitz constraint between \(x\) and only finitely many other vertices \(y \in \mathbb{Z}^d\). In other words, what we have in mind is a set \(A \subset \mathbb{Z}^d \times \mathbb{Z}^d \times \mathbb{R}\), such that a height function \(\phi\) is Lipschitz if and only if \(\phi(y) - \phi(x) \leq a\) for any \((x, y, a) \in A\), and such that \(A\) becomes a finite set once we identify each triple of the form \((x, y, a)\) with all triples of the form \((\theta x, \theta y, a)\) as \(\theta\) ranges over \(\Theta\). The local Lipschitz constraint also enforces that the functions are globally Lipschitz with respect to the correct quasimetric. This is formalized as follows.

**Definition 3.1** (local Lipschitz constraint). Call an edge set \(A\) on \(\mathbb{Z}^d\) an admissible graph if \(A\) is \(L\)-invariant and makes \((\mathbb{Z}^d, A)\) a connected graph of bounded degree. Call a function \(q: \mathbb{Z}^d \times \mathbb{Z}^d \rightarrow \mathbb{R}\) an admissible quasimetric if

1. \(q(x, x) = 0\) for any \(x \in \mathbb{Z}^d\),
2. \(q(x, y) + q(y, x) > 0\) for any \(x, y \in \mathbb{Z}^d\) distinct,
3. \(q(x, z) \leq q(x, y) + q(y, z)\) for any \(x, y, z \in \mathbb{Z}^d\),
4. \(q(\theta x, \theta y) = q(x, y)\) for any \(x, y \in \mathbb{Z}^d\) and \(\theta \in \Theta\).

Such a function is called integral if it takes integral values. A local Lipschitz constraint is a pair \((A, q)\) where

1. \(A\) is an admissible graph,
2. \(q\) is an admissible quasimetric,
3. \(q\) is maximal among all admissible quasimetrics that equal \(q\) on \(A\), in the sense that \(p \leq q\) for any admissible quasimetric \(p\) with \(p(x, y) \leq q(x, y)\) for all \(\{x, y\} \in A\).

If \((A, q)\) is a local Lipschitz constraint and \(\varepsilon \geq 0\) a sufficiently small constant, then write \(q_\varepsilon\) for the largest admissible quasimetric subject to \(q_\varepsilon(x, y) \leq q(x, y) - \varepsilon\) for all \(\{x, y\} \in A\). (It is demonstrated in Proposition 6.5 that this is indeed well-defined for \(\varepsilon > 0\) sufficiently small.) Note that the resulting pair \((A, q_\varepsilon)\) is also a local Lipschitz constraint.

**Remarks.**

1. The last condition in the definition of a local Lipschitz constraint guarantees that \(q\) is fully determined by its values on the edges in \(A\).

2. We shall sometimes omit the reference to \(A\) and simply call \(q\) the local Lipschitz constraint. If \((A, q)\) is a local Lipschitz constraint and \(B\) another admissible graph on \(\mathbb{Z}^d\), then the pair \((A \cup B, q)\) is also a local Lipschitz constraint producing the same quasimetric \(q\). We shall always assume, without loss of generality, that \(A\) contains the edges of the square lattice.

3. If \(q\) is a local Lipschitz constraint, then there is a constant \(K < \infty\) such that \(Kd_1 \geq q\).
4. We do not impose that \( q \) takes values in \([0, \infty)\). This restriction is not necessary to make the arguments work.

From now on, we shall always have in mind a fixed local Lipschitz constraint \((\Lambda, q)\).

**Definition 3.2** \((q\text{-Lipschitz})\). A function \( \phi : \mathbb{Z}^d \to \mathbb{R} \) is called \( q\text{-Lipschitz} \) if, for every \( x, y \in \mathbb{Z}^d \),

\[
\phi(y) - \phi(x) \leq q(x, y).
\]

The function \( \phi \) is called \( q\text{-Lipschitz at } z \in \mathbb{Z}^d \) if this inequality is satisfied for any edge \( \{x, y\} \in \Lambda \) containing \( z \). Naturally extend these definitions to cover the cases that \( \phi : \Lambda \to \mathbb{R} \) for some \( \Lambda \subset \mathbb{Z}^d \). Write \( \Omega_q \) for the collection of \( q\text{-Lipschitz height functions} \). A measure is called \( q\text{-Lipschitz} \) if it is supported on \( \Omega_q \). A specification is called \( q\text{-Lipschitz} \) if it maps \( q\text{-Lipschitz} \) measures to \( q\text{-Lipschitz} \) measures. Finally, a function is called \emph{strictly} \( q\text{-Lipschitz} \) if it is \( q_\varepsilon\text{-Lipschitz} \) for \( \varepsilon > 0 \) sufficiently small.

We now construct a number of objects which derive from \( q \). These are necessary to state the main results, which address the macroscopic behavior of Lipschitz surfaces.

**Definition 3.3** \((U_q, \| \cdot \|_q)\). By a \emph{slope} we simply mean an element \( u \) in the dual space \( (\mathbb{R}^d)^* \) of \( \mathbb{R}^d \). Write \( U_q \) for the interior of the set of slopes \( u \) such that \( u|_\mathbb{C} \) is \( q\text{-Lipschitz} \). The set \( U_q \) is nonempty and convex—this follows from the definition of a local Lipschitz constraint; see Lemma 6.1. Introduce furthermore the function \( \| \cdot \|_q : \mathbb{R}^d \to \mathbb{R} \) defined by

\[
\|x\|_q := \sup\{u(x) : u \in U_q \}.
\]

The function \( \| \cdot \|_q \) is positive homogeneous: we have \( \|ax\|_q = a\|x\|_q \) for \( a \in [0, \infty) \) and \( x \in \mathbb{R}^d \). It also satisfies the triangle inequality, in the sense that \( \|x + y\|_q \leq \|x\|_q + \|y\|_q \) for any \( x, y \in \mathbb{R}^d \).

**Definition 3.4** \((\| \cdot \|_\Lambda\text{-Lipschitz})\). If \( \| \cdot \| : \mathbb{R}^d \to \mathbb{R} \) is any positive homogeneous function satisfying the triangle inequality, then any other function \( f : D \to \mathbb{R} \) defined on a subset \( D \) of \( \mathbb{R}^d \) is called \( \| \cdot \|_\Lambda\text{-Lipschitz} \) if \( f(y) - f(x) \leq \|y - x\| \) for any \( x, y \in D \). The function \( f \) is called \emph{strictly} \( \| \cdot \|_\mathbb{L}\text{-Lipschitz} \) if it is \( \| \cdot \|_\Lambda\text{-Lipschitz} \) for some \( \varepsilon > 0 \). If \( D \) is open, then \( f \) is called \emph{locally strictly} \( \| \cdot \|_\mathbb{L}\text{-Lipschitz} \) if \( f|_K \) is strictly \( \| \cdot \|_\Lambda\text{-Lipschitz} \) for all compact sets \( K \subset D \).

For example, \( U_q \) is the interior of the set of slopes \( u \in (\mathbb{R}^d)^* \) which are \( \| \cdot \|_q\text{-Lipschitz} \).

### 3.2 Strong interactions

Let \( \Psi \) denote an arbitrary periodic gradient potential. The potential \( \Psi \) is called \emph{positive} if \( \Psi_\Lambda \geq 0 \) for any \( \Lambda \subset \mathbb{Z}^d \). The potential \( \Psi \) is said to have \emph{finite range} if \( \Psi_\Lambda \equiv 0 \) whenever the diameter of \( \Lambda \)—in the graph metric \( d_1 \) on the square lattice—exceeds some fixed constant \( R \in \mathbb{N} \); in that case the smallest such \( R \) is called the \emph{range} of \( \Psi \). The potential \( \Psi \) is called \emph{Lipschitz} if there exists a local Lipschitz constraint \((\Lambda, q)\) such that \( \Psi_\Lambda(\phi) = \infty \) if and only if \( \Lambda = \{x, y\} \in \Lambda \) and \( \phi(y) - \phi(x) > q(x, y) \) for some \( x, y \in \mathbb{Z}^d \). If \( E = \mathbb{R} \) and \( \Psi \) Lipschitz with constraint \((\Lambda, q)\), then \( \Psi \) is called \emph{locally bounded} if for any \( \varepsilon > 0 \) sufficiently small, there exists a fixed constant \( C_\varepsilon < \infty \), such that

\[
H_{\Psi|x}(\phi) \leq C_\varepsilon
\]

for any \( x \in \mathbb{Z}^d \) and for any \( \phi \in \Omega \) which is \( q_\varepsilon\text{-Lipschitz} \) at \( x \).
Definition 3.5 (strong interaction, $\mathcal{S}_\mathcal{L}$). A potential $\Psi$ is called a strong interaction if $\Psi$ has all of the above properties, that is, if $\Psi$ is a positive Lipschitz periodic gradient potential of finite range, and if it is locally bounded in the case that $E = \mathbb{R}$. We shall write $\mathcal{S}_\mathcal{L}$ for the collection of strong interactions.

The class $\mathcal{S}_\mathcal{L}$ includes all so-called Lipschitz simply attractive potentials. These are convex Lipschitz nearest-neighbor interactions, see [She05].

3.3 Weak interactions

Let $\Xi$ denote an arbitrary periodic gradient potential.

Definition 3.6 (summability). The potential $\Xi$ is called summable if it has finite norm

$$\|\Xi\| := \sup_{(x,\phi) \in \mathbb{Z}^d \times \Omega} \sum_{\Lambda \subset \subset \mathbb{Z}^d \text{ with } x \in \Lambda} |\Xi_\Lambda(\phi)|.$$

This requirement is significantly weaker than the absolutely summable setting of Georgii [Geo11].

Definition 3.7 (amenability). By an amenable function we mean a function $f$ which assigns a number in $[0, \infty)$ to each finite subset of $\mathbb{Z}^d$, such that:

1. $f(\Lambda) = f(\theta \Lambda)$ for all $\Lambda \subset \subset \mathbb{Z}^d$ and for any $\theta \in \Theta$,
2. $f(\Lambda \cup \Delta) \leq f(\Lambda) + f(\Delta)$ for all $\Lambda, \Delta \subset \subset \mathbb{Z}^d$ disjoint,
3. $f(\Lambda_n) = o(|\Lambda_n|)$ as $n \to \infty$ for any $(\Lambda_n)_{n \in \mathbb{N}} \uparrow \mathbb{Z}^d$.

Definition 3.8 (lower exterior bound). Let us now turn back to the potential $\Xi$ and define, for any $\Lambda \subset \subset \mathbb{Z}^d$,

$$e^-(\Lambda) := \sup_{\phi \in \Omega} \sum_{\Delta \subset \subset \mathbb{Z}^d \text{ with } \Delta \text{ intersecting both } \Lambda \text{ and } \mathbb{Z}^d \setminus \Lambda} |\Xi_\Delta(\phi)|.$$

The function $e^-(\cdot)$ is called the lower exterior bound of $\Xi$.

The key property of the function $e^-(\cdot)$ is that $|H^\Xi_\Lambda - H^{\Xi,0}_\Lambda| \leq e^-(\Lambda)$. The lower exterior bound satisfies Properties 1 and 2 from the definition of an amenable function; this is immediate from the definition.

Definition 3.9 (weak interaction, $\mathcal{W}_\mathcal{L}$). A weak interaction is a summable periodic gradient potential for which the lower exterior bound is amenable. Write $\mathcal{W}_\mathcal{L}$ for the collection of weak interactions.

It is straightforwardly verified that amenability of $e^-(\cdot)$ is equivalent to asking that $e^-(\Pi_n) = o(n^d)$ as $n \to \infty$. Remark that $(\mathcal{W}_\mathcal{L}, \| \cdot \|)$ is a Banach space.

3.4 Overview

Let us fix a number of notations, in order to avoid an excessive number of declarations. We notify the reader of any deviation from this notation. We had already agreed that the choices for $d \geq 2$ and $E \in \{\mathbb{Z}, \mathbb{R}\}$ are fixed, and that $\mathcal{L}$ denotes a fixed full-rank sublattice of $\mathbb{Z}^d$ with corresponding translation group $\Theta$. The letter $\Phi$ denotes a fixed potential in $\mathcal{S}_\mathcal{L} + \mathcal{W}_\mathcal{L}$, and we fix some pair $(\Psi, \Xi) \in \mathcal{S}_\mathcal{L} \times \mathcal{W}_\mathcal{L}$ such that $\Phi = \Psi + \Xi$. This decomposition
is not unique, but this is never a problem. The specification generated by $\Phi$ is denoted $\gamma = \gamma^\Phi$. The pair $(\Lambda, q)$ always denotes the local Lipschitz constraint corresponding to $\Psi$, and the range of $\Psi$ is denoted by $R$. If $E = \mathbb{Z}$, then $q$ is always assumed to be integral. The function $e^{-\cdot}$ denotes the lower exterior bound of $\Xi$. Finally, let $K \in (0, \infty)$ denote the smallest constant such that $Kd_1 \geq q$, and let $N \in \mathbb{N}$ denote the smallest positive integer such that $N \cdot \mathbb{Z}^d \subset \mathcal{L}$.

**Definition 3.10.** The potential $\Phi \in \mathcal{S}_\mathbb{L} + \mathcal{W}_\mathbb{L}$ is called *monotone* if the induced specification $\gamma = \gamma^\Phi$ is monotone over $\Omega_q$.

### 4 Main results

The motivation for writing this article was to demonstrate that the surface tension is strictly convex on $U_\Phi$ if the potential of interest is in the class $\mathcal{S}_\mathbb{L} + \mathcal{W}_\mathbb{L}$ and monotone. If $E = \mathbb{Z}$, then we require an extra condition to be met, but we also demonstrate that this condition is satisfied for many natural models. This section contains an overview of the main results, including several results and applications which are of independent interest. The results are presented roughly in the order in which they appear in the article.

#### 4.1 The specific free energy and its minimizers

The specific free energy functional plays a fundamental role in the analysis. The following result is therefore of independent interest in the study of Lipschitz random surfaces; it is a direct extension of a result of Sheffield [She05] to the setting of this article.

**Theorem 4.1** (specific free energy). If $\Phi \in \mathcal{S}_\mathbb{L} + \mathcal{W}_\mathbb{L}$, then the specific free energy functional

$$
\mathcal{H}(\cdot|\Phi): \mathcal{P}_\mathbb{L}(\Omega, \mathcal{F}^\nabla) \to \mathbb{R} \cup \{\infty\}, \mu \mapsto \lim_{n \to \infty} n^{-d} \mathcal{H}_{\Pi_n}(\mu|\Phi)
$$

is well-defined, affine, bounded below, lower-semicontinuous, and for each $C \in \mathbb{R}$ its lower level set

$$
M_C := \{\mu \in \mathcal{P}_\mathbb{L}(\Omega, \mathcal{F}^\nabla) : \mathcal{H}(\mu|\Phi) \leq C\}
$$

is a compact Polish space, with respect to the topology of (weak) local convergence. In fact, the two topologies coincide on each set $M_C$.

A measure $\mu \in \mathcal{P}_\mathbb{L}(\Omega, \mathcal{F}^\nabla)$ is called a *minimizer of the specific free energy*, or simply a *minimizer*, if it satisfies the equation

$$
\mathcal{H}(\mu|\Phi) = \sigma(S(\mu)) < \infty.
$$

For the purpose of deriving the main result, all that we require is that such minimizers have finite energy, in a sense which is similar to the notion of finite energy in the original paper of Burton and Keane [BK89]. There is a canonical way to translate the concept of finite energy to the gradient Lipschitz setting: we shall see that the following result fits our arguments. Recall that $\Omega_q$ denotes the set of $q$-Lipschitz height functions, and that $\pi_\Lambda$ is the kernel which restrict measures to $\Lambda$, for any $\Lambda \subset \mathbb{Z}^d$.

**Theorem 4.2** (finite energy). Consider $\Phi \in \mathcal{S}_\mathbb{L} + \mathcal{W}_\mathbb{L}$, and suppose that $\mu \in \mathcal{P}_\mathbb{L}(\Omega, \mathcal{F}^\nabla)$ is a minimizer. Then for any $\Lambda \subset \subset \mathbb{Z}^d$, we have

$$
1_{\Omega_q}(\mu \pi_{\mathbb{Z}^d \setminus \Lambda} \times \lambda^\Lambda) \ll \mu.
$$
In [She05], finite energy follows from the variational principle, which asserts that shift-invariant measures $\mu$ which satisfy $\mathcal{H}(\mu|\Phi) = \sigma(S(\mu))$ must also be Gibbs measures with respect to the specification $\gamma = \gamma^\Phi$ induced by the potential $\Phi$—which has finite range. In the infinite-range setting one cannot hope for such a statement, because the specification $\gamma$ is not necessarily quasilocal. This pathology, and its relation to the variational principle, is discussed extensively in [LT19]. One of the key observations in that article is that minimizers of the specific free energy must have finite energy, even if the concept of a Gibbs measure is not well-defined because the specification fails to be quasilocal. There, finite energy is an immediate corollary of a result (Lemma 5.4) which is not quite equivalent to the variational principle, but it is “as close as one expects to get” to it in the non-quasilocal setting. We shall follow the same strategy here: the following theorem states the strongest result on minimizers, which is of independent interest.

**Theorem 4.4** (minimizers of the specific free energy). Consider $\Phi \in \mathcal{S}_\mathcal{L} + \mathcal{W}_\mathcal{L}$, and suppose that $\mu \in \mathcal{P}_\mathcal{L}(\Omega, \mathcal{F})$ is a minimizer. Fix $\Lambda \subseteq \mathbb{Z}^d$, and write $\mu^\phi$ for the regular conditional probability distribution of $\mu$ on $(\Omega, \mathcal{F})$ corresponding to the projection map $\Omega \to E^{2d\setminus \Lambda}$. Then for $\mu$-almost every $\phi \in \Omega$, we have $\mu^\phi|\pi_\Lambda \in \mathcal{A}_{\Lambda, \phi}$. In particular, if $\mu(\Omega_\gamma) = 1$, then $\mu$ is an almost Gibbs measure, and if $\Omega_\gamma = \Omega$, then $\mu$ is a Gibbs measure.

We shall furthermore demonstrate that in each of our applications, all minimizers are indeed (almost) Gibbs measures. We finally derive the following result.

**Theorem 4.5** (existence of ergodic minimizers). Suppose that $\Phi \in \mathcal{S}_\mathcal{L} + \mathcal{W}_\mathcal{L}$. Then for any exposed point $u \in \bar{U}_\Phi$ of $\sigma$, there exists an ergodic gradient measure $\mu$ of slope $u$ which is also a minimizer. In particular, if $\sigma$ is strictly convex on $U_\Phi$, then for each $u \in U_\Phi$, there is an ergodic minimizer of that slope.

Theorem 4.1 is proven in Section 7. Theorems 4.2 and 4.4 are proven in Section 8. Theorem 4.5 is proven in Section 9.

### 4.2 Large deviations principle and variational principle

In Section 11 we prove a large deviations principle (LDP) of similar strength to the one stated in Chapter 7 of [She05], with the noteworthy difference that we express it directly in
terms of the Gibbs specification. This LDP captures both the macroscopic profile of each sample, as well as its local statistics. In this subsection however, we shall state a simpler LDP: one that captures only the macroscopic profile. By doing so we deliver on the premise that limit shapes are characterized by a variational principle, without spending many pages discussing the exact topology for the LDP with local statistics. However, the full LDP is also of independent interest, and we refer the interested reader to Subsection 11.1. Before stating the LDP, we must first describe how a sequence of discrete boundary conditions can approximate a continuous boundary profile, and we must also introduce a topology which captures the macroscopic profile of each sample. Let \( \Phi \) denote a fixed potential throughout this subsection, and adopt the standard notation from Subsection 3.4.

**Definition 4.6** (asymptotic boundary profile). A domain is a nonempty bounded open subset of \( \mathbb{R}^d \) such that its boundary has zero Lebesgue measure. An asymptotic boundary profile is a pair \((D, b)\) where \( D \) is a domain and \( b \) a \( \|\cdot\|_q \)-Lipschitz function on \( \partial D \). If \( E = \mathbb{R} \), then call an asymptotic boundary profile \((D, b)\) good if \( b \) is strictly \( \|\cdot\|_q \)-Lipschitz. If \( E = \mathbb{Z} \), then call an asymptotic boundary profile good if it is non-taut. An asymptotic boundary profile \((D, b)\) is called non-taut if \( b \) has an extension \( \bar{b} \) to \( \bar{D} \) such that \( \bar{b}|_D \) is locally strictly \( \|\cdot\|_q \)-Lipschitz. This is equivalent to asking that the largest and smallest \( \|\cdot\|_q \)-Lipschitz extensions \( b^\pm \) of \( b \) to \( \bar{D} \) satisfy \( b^- < b^+ \) on \( D \).

**Definition 4.7** (discrete approximations). Let \((D, b)\) denote an asymptotic boundary profile. Call a sequence of pairs \((D_n, b_n)_{n \in \mathbb{N}}\) of finite subsets of \( \mathbb{Z}^d \) and height functions an approximation of \((D, b)\) if

1. For all \( n \in \mathbb{N} \), the function \( b_n \) is \( q \)-Lipschitz if \( E = \mathbb{Z} \) or strictly \( q \)-Lipschitz if \( E = \mathbb{R} \),
2. We have \( \frac{1}{n} D_n \to D \) in the Hausdorff metric on \( \mathbb{R}^d \),
3. We have \( \frac{1}{n} \text{Graph}(b_n|_{\partial D_n}) \to \text{Graph}(b) \) in the Hausdorff metric on \( \mathbb{R}^d \times \mathbb{R} \).

Moreover, if \( E = \mathbb{R} \), then an approximation \((D_n, b_n)_{n \in \mathbb{N}}\) is called good if the constant \( \varepsilon > 0 \) which makes each function \( b_n \) a \( q \)-Lipschitz function, is independent of \( n \). If \( E = \mathbb{Z} \), then any approximation is called good.

We have in mind a good approximation \((D_n, b_n)_{n \in \mathbb{N}}\) of some fixed good asymptotic boundary profile \((D, b)\). The sequence of local Gibbs measures which are of interest in the LDP is the sequence \((\gamma_n)_{n \in \mathbb{N}}\) defined by \( \gamma_n := \gamma_{D_n}(\cdot, b_n) \). All samples from the sequence of measures \((\gamma_n)_{n \in \mathbb{N}}\) must be brought to the same topological space, in order for us to formulate the LDP. We will now describe this topology, as well as the map from \( \Omega \) to this topological space.

**Definition 4.8** (topology for macroscopic profiles). For any \( U \subset \mathbb{R}^d \), write \( \text{Lip}(U) \) for the set of real-valued \( K \|\cdot\|_1 \)-Lipschitz functions on \( U \), where we recall that \( K \) is minimal subject to \( K d_1 \geq q \). Suppose given a sample \( \phi \) from \( \gamma_n \). Define the scaled interpolation \( \Phi_n(\phi) \in \text{Lip}(\bar{D}) \) of \( \phi \), which captures the global shape of \( \phi \), as follows. The sample \( \phi \) is almost surely \( q \)-Lipschitz, and therefore also \( K d_1 \)-Lipschitz. First, write \( \bar{\phi} : \mathbb{R}^d \to \mathbb{R} \) for the smallest \( K \|\cdot\|_1 \)-Lipschitz extension of \( \phi \) to \( \mathbb{R}^d \). Next, we simply scale back each sample by \( n \) and restrict it to the set \( \bar{D} \). Formally, this means that we define

\[
\Phi_n(\phi) : \bar{D} \to \mathbb{R}, \ x \mapsto \frac{1}{n}\bar{\phi}(nx).
\]

This function is \( K \|\cdot\|_1 \)-Lipschitz, that is, \( \Phi_n(\phi) \in \text{Lip}(\bar{D}) \). Endow the space \( \text{Lip}(\bar{D}) \) with the topology of uniform convergence, denoted by \( \mathcal{X}^\infty \). The map \( \Phi_n : \Omega \to \text{Lip}(\bar{D}) \) captures the global profile of the height functions in the large deviations principle.
**Definition 4.9** (rate function, pressure). The rate function associated to the profile \((D, b)\) is the function \(I : \text{Lip}(D) \to [0, \infty] \) defined by

\[
I(f) := -P_\Phi(D, b) + \int_D \sigma(\nabla f(x))dx
\]

if \(f|_{\partial D} = b\) and \(I(f) := \infty\) otherwise. Here \(P_\Phi(D, b)\) is the pressure associated to this profile, which is defined precisely such that the minimum of \(I\) is zero.

**Theorem 4.10** (large deviations principle). Let \(\Phi \in \mathcal{S}_L + \mathcal{W}_L\), and let \((D_n, b_n)_{n \in \mathbb{N}}\) denote a good approximation of some good asymptotic profile \((D, b)\). Let \(\gamma^*_n\) denote the pushforward of \(\gamma_n := \gamma_{D_n}(\cdot, b_n)\) along the map \(\Theta_n\), for any \(n \in \mathbb{N}\). Then the sequence of probability measures \((\gamma^*_n)_{n \in \mathbb{N}}\) satisfies a large deviations principle with speed \(n^d\) and rate function \(I\) on the topological space \((\text{Lip}(D), \mathcal{X}_\infty)\). Moreover, the sequence of normalizing constants \((Z_n)_{n \in \mathbb{N}} := (Z_{D_n}(b_n))_{n \in \mathbb{N}}\) satisfies \(-n^{-d} \log Z_n \to P_\Phi(D, g)\) as \(n \to \infty\).

**Corollary 4.11** (variational principle). Let \(\Phi \in \mathcal{S}_L + \mathcal{W}_L\), and let \((D_n, b_n)_{n \in \mathbb{N}}\) denote a good approximation of some good asymptotic profile \((D, b)\). Let \(\gamma^*_n\) denote the pushforward of \(\gamma_n := \gamma_{D_n}(\cdot, b_n)\) along the map \(\Theta_n\), for any \(n \in \mathbb{N}\). Write \(f_n\) for the random function in \(\gamma^*_n\), which—as a random object—takes values in \(\text{Lip}(D)\). If \(\sigma\) is strictly convex on \(U_\Phi\), then the random function \(f_n\) converges to the unique minimizer \(f^*\) of the rate function \(I\), in probability in the topology of uniform convergence as \(n \to \infty\). In other words, \(f^*\) is the unique minimizer of the integral

\[
\int_D \sigma(\nabla f(x))dx
\]

over all Lipschitz functions \(f : \overline{D} \to \mathbb{R}\) which equal \(b\) on the boundary of \(D\). If however \(\sigma\) fails to be strictly convex on \(U_\Phi\), then for any neighborhood \(A\) of the set of minimizers of the integral in the topology of uniform convergence, we have \(f_n \in A\) with high probability as \(n \to \infty\).

## 4.3 The surface tension

Let us now state the motivating result on the surface tension.

**Theorem 4.12** (strict convexity of the surface tension). Let \(\Phi\) denote a potential in \(\mathcal{S}_L + \mathcal{W}_L\) which is monotone.

1. If \(E = \mathbb{R}\), then \(\sigma\) is strictly convex on \(U_\Phi\),
2. If \(E = \mathbb{Z}\), then \(\sigma\) is strictly convex on \(U_\Phi\) if for any affine map \(h : (\mathbb{R}^d)^* \to \mathbb{R}\) with \(h \leq \sigma\), the set \(\{h = \sigma\} \cap \partial U_\Phi\) is convex. In particular, \(\sigma\) is strictly convex on \(U_\Phi\) if at least one of the following conditions is satisfied:
   (a) \(\sigma\) is affine on \(\partial U_\Phi\), but not on \(U_\Phi\),
   (b) \(\sigma\) is not affine on \([u_1, u_2]\) for any distinct \(u_1, u_2 \in \partial U_\Phi\) such that \([u_1, u_2] \not\subset \partial U_\Phi\).

Strict convexity of the surface tension is important because of Theorem 4.5, Theorem 4.10, and Corollary 4.11. Let us also mention some other properties of the surface tension which are useful to keep in mind.

**Theorem 4.13** (general properties of the surface tension). If \(\Phi \in \mathcal{S}_L + \mathcal{W}_L\), then

1. We have \(U_\Phi = U_q\),
2. If $E = \mathbb{R}$, then $\sigma(u)$ tends to $\infty$ as $u$ approaches the boundary of $U_{\Phi}$.

3. If $E = \mathbb{Z}$, then $\sigma$ is bounded and continuous on the closure of $U_{\Phi}$.

Theorem 4.12 is proven in Section 12, and Theorem 4.13 is proven in Section 7.

4.4 Note on the Lipschitz setting

Local Lipschitz constraints are designed to be as flexible as possible. Essential in the argument is that a height function $\phi : \mathbb{Z}^d \to E$ has finite energy if and only if it is Lipschitz with respect to the local Lipschitz constraint. This means that we can rely on the Kirszbraun theorem (Theorem 6.4) to join together Lipschitz functions defined on disjoint parts of the space. However, this formulation is sometimes inconvenient. There are, as we shall see, several natural models in which the admissible height functions are exactly the graph homomorphisms from $\mathbb{Z}^d$ to $\mathbb{Z}$: these are functions $\phi : \mathbb{Z}^d \to \mathbb{Z}$ which satisfy $\phi(0) \in 2\mathbb{Z}$ and $|\phi(y) - \phi(x)| = 1$ for each edge $\{x, y\}$ of the square lattice. For example, the canonical height functions corresponding to the six-vertex model are precisely the graph homomorphisms from $\mathbb{Z}^2$ to $\mathbb{Z}$. Since the zero transition is not allowed, it might appear that this model does not fit the Lipschitz framework: it is the first if in the if and only if that is violated. However, this problem is only cosmetic in nature: by a simple transformation one can move from graph homomorphisms to the Lipschitz framework. Write $h : \mathbb{Z}^d \to \mathbb{Z}$ for the function $h(x) := \sum_i x_i$, and consider the map

$$\phi \mapsto (\phi + h)/2.$$ 

This map is a bijection from the set of graph homomorphisms to the set of functions which are $q$-Lipschitz for $q$ defined by

$$q(x, y) := \sum_i 0 \lor (y - x)_i.$$ 

By applying this transformation, it is thus clear that models of graph homomorphisms do fit into the local Lipschitz setting of this article. In fact, the exact same trick applies to dimer models, and perhaps other models of discrete height functions.

4.5 Application to submodular potentials

A potential $\Phi$ is said to be submodular if for every $\Lambda \subset \subset \mathbb{Z}^d$, $\Phi_\Lambda$ has the property that

$$\Phi_\Lambda(\phi \land \psi) + \Phi_\Lambda(\phi \lor \psi) \leq \Phi_\Lambda(\phi) + \Phi_\Lambda(\psi).$$

Sheffield proposes this family of potentials as a natural generalization of simply attractive potentials, and asks if similar results as the ones proved for simply attractive potentials in [She05] could be proved for finite-range submodular potentials. We provide an answer to this question for the case that the model is also Lipschitz. (In fact, we do not even require the potential to be finite-range.) It is easy to see that submodular potentials generate monotone specifications. If $E = \mathbb{R}$ and $\Phi$ a submodular Lipschitz potential fitting the framework of this article (which is a very mild requirement), then we derive immediately from Theorem 4.12 that the surface tension is strictly convex. If $E = \mathbb{Z}$, then we must also fulfill the extra condition in Theorem 4.12. We show that we can fulfill the extra condition if all shift-invariant measures $\mu$ which are supported on $q$-Lipschitz functions and which have $S(\mu) \in \partial U_{\Phi}$, are frozen, in the sense that for any $\Lambda \subset \subset \mathbb{Z}^d$, the values of $\phi_\Lambda$ depend deterministically on $\phi_{\partial R_\Lambda}$ in $\mu$. This is a property of the local Lipschitz constraint $q$, and such local Lipschitz constraints are called freezing.
Theorem 4.14 (strict convexity for submodular potentials). Suppose that the potential \( \Phi \in S_C + W_C \) is submodular. Then it is monotone. Moreover,

1. If \( E = \mathbb{R} \), then \( \sigma \) is strictly convex on \( U_\Phi \).
2. If \( E = \mathbb{Z} \), then \( \sigma \) is strictly convex on \( U_\Phi \) if the local Lipschitz constraint \( q \) is freezing.

Note that \( q \) is automatically freezing if it is \( \mathbb{Z}^d \)-invariant.

Of course, Theorem 4.5 applies, and if the potential is finite-range, then the specification is quasilocal (\( \Omega = \Omega_\tau \)) so that all minimizers are Gibbs measures (Theorem 4.4).

4.6 Application to tree-valued graph homomorphisms

The flexibility of the main theorem in this article can also be used to prove statements about the behavior of random functions taking values in target spaces other than \( \mathbb{Z} \) and \( \mathbb{R} \). A noteworthy example is the model of tree-valued graph homomorphisms described in \([MT20]\).

The theorem is extensively discussed in \([MT20]\). We confirm the conjecture in \([MT20]\), which asserts that the entropy function describes the macroscopic behavior of the model, as it is equivalent to the number of graph homomorphisms with nearly-linear boundary conditions. This entropy function describes the macroscopic behavior of the model, as is extensively discussed in \([MT20]\). We confirm the conjecture in \([MT20]\), which asserts that this entropy function is strictly convex. We can do so because the model of uniformly random \( T_k \)-valued graph homomorphisms can be translated into a model of \( \mathbb{Z} \)-valued graph homomorphisms after introducing an infinite-range interaction.

Let us now rigorously describe the conjecture which we prove is correct. Figure 2 displays a sample from the model; the limit shape is clearly visible. In this context, tree-valued graph homomorphisms are functions from \( \mathbb{Z}^d \) to a \( k \)-regular tree \( T_k \) which also map the edges of the square lattice to the edges of the tree. Regular trees are natural objects in several fields of mathematics: in group theory, for example, they arise as Cayley graphs of free groups on finitely many generators. As a significant result in \([MT20]\), the authors characterize the surface tension for the model (there named entropy) and show that it is equivalent to the number of graph homomorphisms with nearly-linear boundary conditions. This entropy function describes the macroscopic behavior of the model, as is extensively discussed in \([MT20]\). We confirm the conjecture in \([MT20]\), which asserts that this entropy function is strictly convex. We can do so because the model of uniformly random \( T_k \)-valued graph homomorphisms can be translated into a model of \( \mathbb{Z} \)-valued graph homomorphisms after introducing an infinite-range interaction.

Let us now rigorously describe the conjecture which we prove is correct. Write \( U \) for the set of slopes \( u \in (\mathbb{R}^d)^* \) such that \( |u(e_i)| < 1 \) for each element \( e_i \) in the natural basis of \( \mathbb{R}^d \). For fixed \( u \in U \), write \( \phi^u : \mathbb{Z}^d \to \mathbb{Z} \) for the graph homomorphism defined by

\[
\phi^u(x) := |u(x)| + \begin{cases} 
0 & \text{if } d_1(0, x) \equiv |u(x)| \mod 2, \\
1 & \text{if } d_1(0, x) \equiv |u(x)| + 1 \mod 2.
\end{cases}
\]

Then \( \phi^u \) approximates \( u \) and it thus nearly linear, in the sense that \( \|\phi^u - u\|_{\mathbb{Z}^d} \leq 1 \).

Let \( g \) denote a bi-infinite geodesic through \( T_k \), that is, a \( \mathbb{Z} \)-indexed sequence of vertices \( g = (g_n)_{n \in \mathbb{Z}} \subset T_k \) such that \( d_{T_k}(g_n, g_m) = |m - n| \) for any \( n, m \in \mathbb{Z} \). The geodesic \( g \) is thought of as a copy of \( \mathbb{Z} \) in \( T_k \), and is used as reference frame. Write \( \tilde{\phi}^u : \mathbb{Z}^d \to T_k \) for the graph homomorphism defined by \( \tilde{\phi}^u(x) := g_{\phi^u(x)} \) for every \( x \in \mathbb{Z}^d \). It is shown in \([MT20]\) that the macroscopic behavior of uniformly random \( T_k \)-valued graph homomorphisms is characterized by the function

\[
\text{Ent} : U \to [-\log k, 0], \quad u \mapsto \lim_{n \to \infty} -n^{-d} \log |\{ \tilde{\phi} \in \tilde{\Omega} : \tilde{\phi}_{\mathbb{Z}^d \setminus \Pi_n} = \tilde{\phi}^u_{\mathbb{Z}^d \setminus \Pi_n} \}|,
\]

where \( \tilde{\Omega} \) denotes the set of all graph homomorphisms from \( \mathbb{Z}^d \) to \( T_k \). It is conjectured in \([MT20]\) that \( \text{Ent} \) is strictly convex on \( U \), which we prove is correct. Figure 2 displays a sample from the model; the limit shape is clearly visible.

Theorem 4.15 (strict convexity of the entropy for tree-valued graph homomorphisms). For any \( d, k \geq 2 \), the entropy function \( \text{Ent} : U \to [-\log k, 0] \) associated to uniformly random graph homomorphisms from \( \mathbb{Z}^d \) to a \( k \)-regular tree, is strictly convex on \( U \).
Figure 2: This figure shows the boundaries of the upper level sets of the horocyclic height function (presented in Subsection 13.3) of a random $T_3$-valued graph homomorphism. The boundary conditions resemble the Aztec diamond for domino tilings. The simulation hints at the presence of an arctic circle, alongside the limit shape which we prove appears inside.
5 Moats

The following section is at the heart of this work. Its purpose is to show that for a specification which is stochastically monotone, two configurations sampled independently with the same boundary conditions are, on the scale of the specific free energy, at least as likely to oscillate a large number of times than to deviate from each other macroscopically. Moats are introduced in Definition 5.2 to formalize this statement. Informally, moats are clusters surrounding a given connected set, and on which the height difference between two configurations is prescribed between two fixed bounds. The proof relies crucially on the reflection principle which is stated in Lemma 5.1.

In this section, the implicit graph structure on $\mathbb{Z}^d$ is always the square lattice. As per usual, $(\Lambda, q)$ denotes the local Lipschitz constraint, and $K \in (0, \infty)$ is chosen minimal subject to $Kd_1 \geq q$. We have in mind a gradient specification $\gamma$ which is $q$-Lipschitz and monotone over $\Omega_q$. From this specification we draw two height functions $\phi_1, \phi_2 \in \Omega$, and $f$ shall generally denote the difference function $\phi_1 - \phi_2$, which is thus $2K$-Lipschitz.

5.1 Reflection principle

We first state and prove the reflection principle, which does not rely on the Lipschitz property. Throughout this section only, we shall adopt the following notation. Suppose that $f_1$ and $f_2$ are random functions in $\Omega$, in some probability measures $\mu_1$ and $\mu_2$ respectively. Then write $f_1 \preceq f_2$ if $f_1$ is stochastically dominated by $f_2$, that is, $\mu_1(f_1 \in A) \leq \mu_2(f_2 \in A)$ for any increasing set $A \in \mathcal{F}$. Note that this notation still makes sense if $\mu_1 = \mu_2$, even if $\mu_1$ and $\mu_2$ are finite measures rather than probability measures.

**Lemma 5.1 (Reflection principle).** Let $\gamma = (\gamma_\Lambda)_{\Lambda \subset \subset \mathbb{Z}^d}$ denote a monotone gradient specification. Fix $\Lambda \subset \subset \mathbb{Z}^d$, and consider a probability measure $\mu$ on the product space $(\Omega^2, \mathcal{F}^2)$, writing $(\phi_1, \phi_2)$ for the random pair of height functions, and with $f := \phi_1 - \phi_2$. Suppose that

$$\mu = \mu(\gamma_\Lambda \times \gamma_\Lambda) .$$

If $\mu$-almost surely $f_{\mathbb{Z}^d \setminus \Lambda} \geq a$ for some $a \in \mathbb{R}$, then

$$-f \preceq f - 2a .$$

Similarly, if $\mu$-almost surely $f_{\mathbb{Z}^d \setminus \Lambda} \leq b$ for some $b \in \mathbb{R}$, then

$$f \preceq -f + 2b .$$

The same holds true if $\mu$ is a finite measure rather than a probability measure.

**Proof.** We focus on the first statement; the second statement then follows by symmetry. Fix $a \in \mathbb{R}$. Suppose first that $\mu$ restricted to $\mathbb{Z}^d \setminus \Lambda$ is a Dirac measure, that is,

$$\mu = \gamma_\Lambda(\cdot, \psi_1) \times \gamma_\Lambda(\cdot, \psi_2)$$

for some $\psi_1, \psi_2 \in \Omega$ with $\psi_1 - \psi_2 \geq a$. As $\gamma$ is a monotone gradient specification, we have $\phi_1 \succeq \phi_2 + a$.

But $\phi_1$ and $\phi_2$ are independent, and therefore

$$-f = \phi_2 - \phi_1 \preceq (\phi_1 - a) - (\phi_2 + a) = f - 2a .$$

This inequality is generalized to the case that $\mu$ restricted to $\mathbb{Z}^d \setminus \Lambda$ is not a Dirac measure, simply by averaging the inequality over all possible values of $\phi_1$ and $\phi_2$ on $\mathbb{Z}^d \setminus \Lambda$ with respect to $\mu$. □
5.2 Definition of moats

Definition 5.2 (Moats). Let \( f : \mathbb{Z}^d \to \mathbb{R} \) be a \( 2K \)-Lipschitz function and \( \Lambda \subset \subset \mathbb{Z}^d \) connected. Consider two real numbers \( a \) and \( b \) with \( b - a \geq 4K \).

1. A set \( M \subset \mathbb{Z}^d \) is called an \( a,b \)-moat of \((f,\Lambda)\) or simply a moat if \( M \) is a finite connected component of the set \( \{a \leq f < b\} = \{x \in \mathbb{Z}^d : a \leq f(x) < b\} \subset \mathbb{Z}^d \) such that \( \Lambda \) is contained in a bounded connected component of \( \mathbb{Z}^d \setminus M \).

2. The boundary of \( M \), that is, the set of vertices \( x \in \mathbb{Z}^d \setminus M \) adjacent to \( M \), is denoted by \( \partial M \). Write \( \bar{M} \) for the closure of \( M \), that is, \( M \cup \partial M \).

3. The connected component of \( \mathbb{Z}^d \setminus M \) containing \( \Lambda \) is called the inside of \( M \), and the inside boundary is the intersection of the inside with \( \partial M \). Write \( M^\Lambda \) and \( \partial_\Lambda M \) for the inside and the inside boundary respectively.

4. The unbounded connected component of \( \mathbb{Z}^d \setminus M \) is called the outside of \( M \), and the outside boundary is the intersection of the outside with \( \partial M \). Write \( M^\infty \) and \( \partial_\infty M \) for the outside and the outside boundary respectively.

5. A moat \( M \) is said to surround another moat \( N \), if \( N \subset \bar{M} \Lambda \).

6. A moat \( M \) is called a climbing moat if \( f_{\partial_\infty M} < a \) and \( f_{\partial_\Lambda M} \geq b \) and it is called a descending moat if \( f_{\partial_\infty M} \geq b \) and \( f_{\partial_\Lambda M} < a \). From now on, we shall only consider moats which are either climbing or descending; when speaking of a moat, it is implicit that it belongs to one of these categories.

7. A finite sequence of moats \((M_k)_{1 \leq k \leq n}\) is called nested if \( M_k \) surrounds \( M_{k+1} \) for all \( 1 \leq k < n \), and if the moats are alternatingly climbing and descending, with \( M_1 \) climbing.

We immediately collect a number of important properties.

Proposition 5.3. Work in the context of the previous definition.

1. There exists at most one moat \( M \) with \( x \in M \), for any fixed \( x \in \mathbb{Z}^d \).

2. If \( M \) is a moat, then \( a - 2K \leq f < b + 2K \) on \( \bar{M} \).

3. Suppose that \( M \) is a moat, and that \( p = (p_k)_{0 \leq k \leq n} \subset \mathbb{Z}^d \) is a path through the square lattice from \( M^\Lambda \) to \( M^\infty \). Then \( p_k \in M \) for at least \( \lfloor (b - a)/2K \rfloor \geq 2 \) consecutive integers \( k \).

4. If \( \Delta \subset \subset \mathbb{Z}^d \) contains \( \Lambda \), then the number of moats \( M \) of \((f,\Lambda)\) for which \( M \cup M^\Lambda \subset \Delta \), is bounded by the \( d_1 \)-distance from \( \Lambda \) to \( \mathbb{Z}^d \setminus \Delta \).

5. Suppose that \( M \) is a moat of \((f,\Lambda)\), and that \( g \) is another \( 2K \)-Lipschitz function with \( g = f \) on \( \bar{M} \). Then \( M \) is also a moat of \((g,\Lambda)\). If \( M \) was climbing (resp. descending) w.r.t. \((f,\Lambda)\) then it is climbing (resp. descending) w.r.t. \((g,\Lambda)\). In other words, for \( M \subset \mathbb{Z}^d \), the event
\[
\{M \text{ is a (climbing or descending) moat of } (f,\Lambda)\}
\]
is \( \mathcal{F}_F^2 \)-measurable.
6. Suppose that $A \subset \mathbb{Z}^d$ such that $\Lambda$ is contained in a finite connected component of $\mathbb{Z}^d \setminus A$, and write $A^\Lambda$ for this connected component. If $f < a$ on $A$ and $f \geq b$ on $\Lambda$, then $A^\Lambda$ contains a climbing moat. If $f \geq b$ on $A$ and $f < a$ on $\Lambda$, then $A^\Lambda$ contains a descending moat.

7. If $a'$ and $b'$ are real numbers with $b' - a' \geq 4K$ and $[a',b'] \subset [a,b]$, then any $a,b$-moat contains an $a',b'$-moat.

Proof. The first three statements follow from the definitions, where it is important that $f$ is $2K$-Lipschitz and that any moat is either climbing or descending. For the fourth statement, observe that a path of minimal length from $\Lambda$ to $\mathbb{Z}^d \setminus \Delta$ through the square lattice must intersect any moat $M$ for which $M \cup M^c \subseteq \Delta$. The fifth statement is immediate from the definition. The sixth statement follows from the connectivity properties of the square lattice, as well as the fact that $f$ is $2K$-Lipschitz. The final statement is a corollary of the sixth. \hfill \Box

5.3 Moats and macroscopic deviations

Theorem 5.4. Let $\gamma = (\gamma_A)_{A \subset \mathbb{Z}^d}$ denote a $q$-Lipschitz gradient specification which is monotone over $\Omega_q$. Fix $\Delta \subset \mathbb{Z}^d$, and consider a $q$-Lipschitz probability measure $\mu$ on the product space $(\Omega^2, \mathcal{F}^2)$, writing $(\phi_1, \phi_2)$ for the random pair of height functions, and with $f := \phi_1 - \phi_2$. Suppose that

$$\mu = \mu(\gamma \times \gamma) \quad \text{and} \quad \mu\text{-almost surely } |f_{\mathbb{Z}^d \setminus \Delta}| \leq 2K.$$ 

Fix a connected set $\Lambda \subset \Delta$, and write $E(n)$ for the event that there exists a sequence of $n$ nested $a,b$-moats of $(f, \Lambda)$, where $a = 4K$ and $b \geq 8K$. Then

$$m^{2n} \mu(E(2n)) \geq \mu(f_\Lambda \geq 3bn) \quad (5.5)$$

for all $n \in \mathbb{N}$, where $m = d_1(\Lambda, \mathbb{Z}^d \setminus \Delta)$.

The idea of the proof is as follows. If $f \geq 3bn$ on $\Lambda$ and $f \leq 2K$ on $\mathbb{Z}^d \setminus \Delta$, then $\Delta$ must contain a climbing $a,b$-moat. Suppose now that we fix a subset $M$ of $\Delta$, and condition on the event

$$A := \{M \text{ is a climbing } a,b\text{-moat of } (f, \Lambda)\} \in \mathcal{F}^2_M.$$ 

If we write $\Gamma$ for the set $\Delta \setminus M$, then the conditioned measure $\mu(\cdot | A)$ satisfies

$$\mu(\cdot | A) = \mu(\cdot | A)(\gamma_\Gamma \times \gamma_{\Gamma}) \quad \text{and} \quad \mu(\cdot | A)\text{-almost surely } -2K \leq f_{\mathbb{Z}^d \setminus \Gamma} \leq b + 2K.$$ 

By the reflection principle, we thus have

$$\mu(f_\Lambda \leq -3bn + 2b + 4K | A) \geq \mu(f_\Lambda \geq 3bn | A).$$

In other words, this means that it is as least as likely to observe the set $M$ as a climbing moat and a large negative deviation on $\Lambda$, than to see the set $M$ as a climbing moat and a slightly larger positive deviation on $\Lambda$. But if $f$ is negative on $\Lambda$ then we can find a descending moat in the inside $M^\Lambda$ of $M$. One repeats this reflection procedure to generate a full nested sequence of moats, while retaining a sufficiently large probability. The formalism is slightly more convoluted because one needs to choose the set $M$ appropriately. This produces the extra factor $m^{2n}$ in (5.5).
Proof of Theorem 5.4. We proceed along the same spirit. Write \( \Delta_k := \{ \Lambda \subset \Delta \}^k \), and define
\[
A(M) := \{ M \text{ is a nested sequence of a, b-moats of } (f, \Lambda) \} \in \mathcal{F}_{\cup_i \tilde{M}_i}^2
\]
for \( M \in \Delta_k \). We also write \( \Gamma(M) := \Delta \setminus \cup_i \tilde{M}_i \). Define \( \mu_B := \mu(\cdot \cap B) \) for any \( B \in \mathcal{F}^2 \).

For any \( k \in \mathbb{N} \) and \( M \in \Delta_k \), we have
\[
\mu_{A(M)}(\gamma_{\Gamma(M)} \times \gamma_{\Gamma(M)}) \quad \text{and} \quad \mu_{A(M)} \text{-a.e. } -2K \leq f_{\mathbb{Z}^d \setminus \Gamma(M)} \leq b + 2K,
\]
which means that the reflection principle applies to this measure. Claim that
\[
\mu(f_{\Lambda} \geq 3bn) \leq \sum_{M \in \Delta_1} \mu_{A(M)}(f_{\Lambda} \geq 3bn) \quad (5.6)
\]
\[
\leq \sum_{M \in \Delta_2} \mu_{A(M)}(f_{\Lambda} \leq -3bn + 2b + 4K) \quad (5.7)
\]
\[
\leq \sum_{M \in \Delta_2} \mu_{A(M)}(f_{\Lambda} \leq -3bn + 2b + 4K) \quad (5.8)
\]
\[
\leq \sum_{M \in \Delta_2} \mu_{A(M)}(f_{\Lambda} \geq 3bn - 2b - 8K) \quad (5.9)
\]
\[
\leq \sum_{M \in \Delta_2} \mu_{A(M)}(f_{\Lambda} \geq 3b(n - 1)). \quad (5.10)
\]

Here (5.6) follows from the fact that \( \Delta \) contains a moat whenever \( f \leq 2K \) on the complement of \( \Delta \) and \( f \geq 3bn \) on \( \Lambda \), and (5.7) follows from the reflection principle applied to each measure in the finite sum. Now isolate one set \( M \in \Delta_1 \) and consider the measure \( \mu_{A(M)} \). If \( f_{\Lambda} \leq -3bn + 2b + 4K \), then there must be a descending moat in the inside of \( M_1 \)—recall that \( M_1 \) is a climbing moat, by definition of a nested sequence of moats. In particular, this proves (5.8). Inequality (5.9) follows again from the reflection principle applied to each separate measure, and (5.10) follows from the fact that \( 3bn - 2b - 8K \geq 3b(n - 1) \). A continuation of this series of inequalities leads to the equation
\[
\mu(f_{\Lambda} \geq 3bn) \leq \sum_{M \in \Delta_2n} \mu_{A(M)}(f_{\Lambda} \geq 0).
\]

The proof is nearly done. Note that \( \mu_{A(M)}(\Omega^2) = \mu_{A(M)}(E(2n)) \) for \( M \in \Delta_{2n} \) by definition of \( A(M) \) and \( E(2n) \), and therefore
\[
\mu(f_{\Lambda} \geq 3bn) \leq \sum_{M \in \Delta_{2n}} \mu_{A(M)}(f_{\Lambda} \geq 0) \leq \sum_{M \in \Delta_{2n}} \mu_{A(M)}(E(2n)).
\]

To deduce (5.5), it suffices to demonstrate that, as measures,
\[
\sum_{M \in \Delta_{2n}} \mu_{A(M)} \leq m^{2n} \mu.
\]

The measure on the left equals \( X \mu \), where \( X \) is the number of ways to choose a nested sequence of \( 2n \) moats contained in \( \Delta \). Since \( \Delta \) contains at most \( m \) moats, we have \( X \leq \binom{m}{2n} \leq m^{2n} \).

We state an immediate corollary, which is an adaptation of the previous result to the case that \( \Delta \) and \( \Lambda \) are not connected.

Proposition 5.11. Assume the setting of the previous theorem, only suppose now that \( \Delta \) and \( \Lambda \) each decompose into \( k \) connected components denoted by \( (\Delta_i)_i \) and \( (\Lambda_i)_i \) respectively with \( \Lambda_i \subset \Delta_i \), and write \( E(n) \) for the event that each \( \Delta_i \) contains a sequence of \( n \) nested a, b-moats of \( (f, \Lambda_i) \). Then (5.5) holds true once we replace \( m \) by
\[
m = \prod_{i=1}^{k} d_i(\Lambda_i, \mathbb{Z}^d \setminus \Delta_i).
\]
6 Analysis of local Lipschitz constraints

This section contains several results on local Lipschitz constraints—most are deduced directly from Definition 3.1. Fix, throughout this section, a local Lipschitz constraint \( (A,q) \), and let \( R \in \mathbb{N} \) denote a fixed constant such that \( d_1(x,y) \leq R \) for all \( \{x,y\} \in A \). For example, one can take \( (A,q) \) to be the local Lipschitz constraint of \( \Psi \), and \( R \) its range. These results are near-trivial for most commonly studied models; they require some work in the generality of Definition 3.1.

Throughout this section, we adopt the following notation. If \( p = (p_k)_{0 \leq k \leq n} \) is a path through \( (\mathbb{Z}^d, A) \), then we write \( q(p) \) for \( \sum_{k=1}^n q(p_{k-1}, p_k) \). If \( q(p) = q(p_0, p_n) \), then \( p \) is called an optimal path.

6.1 Homogenization of local Lipschitz constraints

The following lemma characterizes \( U_q \) in terms of \( q \). It also provides a relation between the local Lipschitz constraint \( q \) and the map \( \| \cdot \|_q \) that it generates. The proof is similar to the proof in [She05], although the formulation of the lemma is different.

**Lemma 6.1.** The set \( U_q \) is nonempty. Its closure \( \overline{U}_q \) can be written as the intersection of finitely many half-spaces. For each contributing half-space \( H \), there exists a path \( p = (p_k)_{0 \leq k \leq n} \) through \( (\mathbb{Z}^d, A) \) with \( p_n - p_0 \in L \) such that

\[
H = H(p) := \{ u \in (\mathbb{R}^d)^* : u(p_n - p_0) \leq q(p) \}.
\]

Moreover, there exists a constant \( C < \infty \) such that

\[
\|y - x\|_q - C \leq q(x, y) \leq \|y - x\|_q + C
\]

for any \( x, y \in \mathbb{Z}^d \).

**Proof.** Call some path \( p = (p_k)_{0 \leq k \leq n} \) through \( (\mathbb{Z}^d, A) \) a cycle lift if the projection of \( p \) onto \( \mathbb{Z}^d/L \) is a cycle. Since \( \mathbb{Z}^d/L \) is finite and \( (\mathbb{Z}^d, A) \) of bounded degree, there exist only finitely many cycle lifts once we identify paths which differ by a shift by a vector in \( L \).

Claim that

\[
\{ u \in (\mathbb{R}^d)^* : u|_L \text{ is } q\text{-Lipschitz} \} = \cap_{p : p \text{ is a cycle lift}} H(p).
\]

(6.2)

It is clear that the left set is contained in the right set. Focus now on the other containment. Fix a slope \( u \) in the set on the right. Suppose, for the sake of contradiction, that \( u \) is not in the set on the left, that is, that \( u|_L \) is not \( q\)-Lipschitz. Then there is some vertex \( x \in L \) and a path \( p \) from 0 to \( x \) through \( (\mathbb{Z}^d, A) \) such that \( u(x) > q(0, x) = q(p) \). But \( p \) decomposes into a finite collection of cycle lifts \( (p^i) \). By choice of \( u \), we have \( u(x) \leq \sum_i q(p^i) = q(p) \), a contradiction. This proves the claim.

The set \( U_q \) equals the interior of the left and right in (6.2). Suppose that \( U_q \) is empty. Select a minimal family of cycle lifts \( (p^k)_{1 \leq k \leq m} \) such that the corresponding intersection of interiors of half-spaces \( \cap_k \overline{H}(p^k) \) is empty—by minimal we simply mean that \( m \) is as small as possible. For each \( 1 \leq k \leq m \), write \( x^k \in L \) for the endpoint of \( p^k \) minus \( p^k_0 \). Then each vector \( x^k \) is orthogonal to the affine hyperplane \( \partial H(p^k) \). By Helly’s theorem, we observe that \( m \leq d + 1 \). In fact, it is easy to see that, regardless of the value of \( m \), the set \( \{x^k : 1 \leq k \leq m \} \) is linearly dependent, with any strict subset linearly independent. It is a simple exercise in linear algebra to derive from the fact that the intersection of half-spaces \( \cap_k \overline{H}(p^k) \) is empty, that there is some slope \( u \) which is contained in the complement of
\( \hat{H}(p^k) \) for any \( k \), and that there exists a family of positive integers \((a^k)_{1 \leq k \leq m} \subset \mathbb{N} \) such that \( \sum_k a^k x^k = 0 \). Since \( u(x^k) \geq q(0, x^k) \) for each \( k \) by choice of \( u \), we have
\[
\sum_k q(0, a^k x^k) \leq \sum_k a^k q(0, x^k) \leq \sum_k a^k u(x^k) = u(\sum_k a^k x^k) = 0.
\]

However, the triangle inequality and the inequality \( q(x, y) + q(y, x) > 0 \) for \( x \neq y \) from the definition of an admissible quasimetric imply that
\[
\sum_{k=1}^m q(0, a^k x^k) = q(0, a^1 x^1) + \sum_{k=2}^m q(0, a^k x^k) \\
\geq q(0, a^1 x^1) + q(0, -a^1 x^1) = q(0, a^1 x^1) + q(a^1 x^1, 0) > 0,
\]
a contradiction. This proves that \( U_q \) is nonempty.

Now let \( x, y \in \mathbb{Z}^d \) arbitrary, and let \( p \) denote an optimal path from \( x \) to \( y \). Then \( p \) decomposes into cycle lifts and at most \( |\mathbb{Z}^d / \mathcal{L}| - 1 \) remaining edges. It is straightforward to derive from this decomposition that the difference between \( q(x, y) \) and \( \|y - x\|_q \) is bounded uniformly over the choice of \( x \) and \( y \).

Let us also state the following result, which follows immediately from the definition of \( \| \cdot \|_q \) in terms of \( q \).

**Proposition 6.3.** If \( f : D \to \mathbb{R} \) is \( \| \cdot \|_q \)-Lipschitz for \( D \subset \mathbb{R}^d \), then \( f \mid_{D \cap \mathcal{L}} \) is \( q \)-Lipschitz. If furthermore \( q \) is integral, then \( \| f \| \mid_{D \cap \mathcal{L}} \) is also \( q \)-Lipschitz.

### 6.2 General observations

First state the Kirszbraun theorem: this is an elementary result in the theory of Lipschitz functions. It asserts that a Lipschitz function defined on part of the space can be extended to a Lipschitz function on the entire space, with the same Lipschitz constant.

**Proposition 6.4** (Kirszbraun theorem). If \( \Lambda \subset \mathbb{Z}^d \) is nonempty and if \( \phi : \Lambda \to \mathbb{R} \) is \( q \)-Lipschitz, then the function
\[
\phi^* : \mathbb{Z}^d \to \mathbb{R}, x \mapsto \sup_{y \in \Lambda} \phi(y) - q(x, y)
\]
is the unique smallest \( q \)-Lipschitz extension of \( \phi \) to \( \mathbb{Z}^d \). If \( \phi \) and \( q \) are integral, then so is \( \phi^* \). Suppose that \( \| \cdot \| : \mathbb{R}^d \to \mathbb{R} \) is any positive homogeneous function satisfying the triangle inequality. If \( D \subset \mathbb{R}^d \) is nonempty and if \( f : D \to \mathbb{R} \) is \( \| \cdot \| \)-Lipschitz, then the function
\[
f^* : \mathbb{R}^d \to \mathbb{R}, x \mapsto \sup_{y \in D} f(y) - \| y - x \|
\]
is the unique smallest \( \| \cdot \| \)-Lipschitz extension of \( f \) to \( \mathbb{R}^d \).

Next, we discuss the derived local Lipschitz constraint \( q_\varepsilon \) for \( \varepsilon \) sufficiently small. For example, if \( \Lambda \) is the edge set of the square lattice and \( q = K d_1 \) for \( K \in [0, \infty) \), then \( q_\varepsilon \) is well-defined for \( \varepsilon \in [0, K) \), and \( q_{\varepsilon} = (K - \varepsilon) d_1 \) for such \( \varepsilon \). For the more general case, we use a technical construction to understand the derived local Lipschitz constraint \( q_\varepsilon \).

**Proposition 6.5.** There exist constants \( \eta > 0 \) and \( C < \infty \) such that for any \( 0 \leq \varepsilon \leq \eta \),

1. We have \( \varepsilon d_1 / R \leq q - q_\varepsilon \leq C \varepsilon d_1 \),
2. We have \( q_{\varepsilon + \varepsilon'} = (q_\varepsilon)_{\varepsilon'} = (q_{\varepsilon'})_\varepsilon \) for any \( \varepsilon' \geq 0 \) with \( \varepsilon + \varepsilon' \leq \eta \),
3. For any $\varepsilon' \geq 0$ with $\varepsilon + 2\varepsilon' \leq \eta$, if $\phi, \psi : \Lambda \to \mathbb{R}$ are functions for some $\Lambda \subset \mathbb{Z}^d$ where $\phi$ is $q_\varepsilon + 2\varepsilon'$-Lipschitz and $\|\phi - \psi\|_\infty \leq \varepsilon'$, then $\psi$ is $q_\varepsilon$-Lipschitz.

Proof outline. Claim that there exists a uniform constant $C < \infty$ such that $n(p) \leq Cd_1(x, y)$ for any optimal path $p$ from $x$ to $y$, where $n(p)$ denotes the length of that path. To see that the claim is true, observe that $U_\varepsilon$ is nonempty and open, and therefore there exists a constant $\alpha > 0$ such that $\|x\|_q + \| - x\|_q \geq \alpha\|x\|_1$ for any $x \in \mathbb{R}^d$. Moreover, the difference between $q(x, y)$ and $\|y - x\|_q$ is bounded uniformly over $x, y \in \mathbb{Z}^d$ (Lemma 6.1). It is straightforward to deduce the claim from these two facts.

One now defines the map $X_q : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{Z}_{\geq 0}$ by

$$X_q(x, y) := \max\{n(p) : p \text{ is an optimal path from } x \text{ to } y\}.$$ Then $d_1/R \leq X_q \leq Cd_1$ by the previous discussion. It is straightforward, but slightly technical, to see that $q_\varepsilon = q - \varepsilon X_q$ for $\varepsilon$ sufficiently small. This implies the three statements of the proposition. \qed

Proposition 6.6. We have $U_q = \cup_{\varepsilon \geq 0} U_{q_\varepsilon}$.

6.3 Approximation of continuous profiles

Recall that $\Lambda^{-m}(D) := (\mathbb{Z}^d \cap D) \setminus \partial^m(\mathbb{Z}^d \cap D)$ for any $m \in \mathbb{Z}_{\geq 0}$ and $D \subset \mathbb{R}^d$.

Theorem 6.7. Consider $\varepsilon > 0$ sufficiently small so that $q_\varepsilon$ is well-defined, and fix $C < \infty$. Then there is a constant $m \in \mathbb{Z}_{\geq 0}$ such that the following statement holds true. Suppose given a collection $(D_i)_i$ of disjoint subsets of $\mathbb{R}^d$, and write $D := \cup_i D_i$. Let $f : D \to \mathbb{R}$ denote a $\| \cdot \|_q$-Lipschitz function such that $f(y) - f(x) \leq \|y - x\|_q$, for $x \in D_i$ and $y \in D_j$ with $i \neq j$. Define $\Lambda_i := \Lambda^{-m}(D_i)$ and $\Lambda := \cup_i \Lambda_i$. Let $\phi : \Lambda \to E$ denote a function such that $\phi_{\Lambda_i}$ is $q$-Lipschitz for all $i$ and with $|\phi - f|_E \leq C$. Then $\phi$ is $q$-Lipschitz, and has a $q$-Lipschitz extension to $\mathbb{Z}^d$.

Proof. This follows from Lemma 6.1 and Proposition 6.5. \qed

In the remainder of this section, we specialize to the case that $(\Lambda, q)$ is the local Lipschitz constraint associated to the strong interaction $\Psi$ as described in Subsection 3.4. The previous theorem is particularly useful in the case that the function $f$ is affine on each set $D_i$, say with slope $u_i \in U_q$. In that case, we want the height function $\phi$ to approximate the slope $u_i$ on each set $\Lambda_i$. To this end we will choose for each $u \in U_q$ a canonical Lipschitz height function $\phi^u$ to represent that slope $u$. This is the purpose of the following definition.

Definition 6.8. Consider some fixed slope $u \in U_q$. If $E = \mathbb{Z}$, then write $\phi^u \in \Omega$ for the unique smallest $q$-Lipschitz extension of the function $|u|_E$ to $\mathbb{Z}^d$. If $E = \mathbb{R}$, then write $\phi^u \in \Omega$ for the unique smallest $q_\varepsilon$-Lipschitz extension of $|u|_E$ to $\mathbb{Z}^d$, where $\varepsilon$ is the largest positive real number such that $u|_E$ is $q_\varepsilon$-Lipschitz (subject to $\varepsilon \leq \eta$, where $\eta$ is as in Proposition 6.5).

If $E = \mathbb{Z}$, then $q$ is integral, and therefore the smallest $q$-Lipschitz extension of $|u|_E$ to $\mathbb{Z}^d$ is also integer-valued. The rounding procedure in the discrete setting makes that the gradient of $\phi^u$ is not $\mathcal{L}$-invariant. In the continuous setting $E = \mathbb{R}$ there is no rounding, and therefore the gradient of $\phi^u$ is $\mathcal{L}$-invariant. Finally, we want to remark that, in both the discrete and the continuous setting, there exists a constant $C < \infty$ such that $|\phi^u - u|_{\mathbb{Z}^d} \leq C$ for any $u \in U_q$. This is due to Lemma 6.1. This observation, combined with the previous theorem, implies the following result.

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Theorem 6.9. Let $C < \infty$ denote the smallest constant such that $|\phi^u - u|_{Z^d} + 1 \leq C$ for all $u \in U_q$. Consider $\varepsilon > 0$ so small that $q_\varepsilon$ is well-defined. Then there exists a constant $m \in \mathbb{Z}_{\geq 0}$ such that the following holds true. Suppose given a collection $(D_i)_i$ of disjoint subsets of $\mathbb{R}^d$, and write $D := \cup_i D_i$, $\Lambda_i := \Lambda^{-m}(D_i)$, and $\Lambda := \cup_i \Lambda_i$. Let $f : D \to \mathbb{R}$ denote a $\|\cdot\|_{q_\varepsilon}$-Lipschitz function which is affine with slope $u_i \in U_q$ whenever restricted to $D_i$. Then there exists a $q$-Lipschitz function $\phi : \Lambda \to E$ which satisfies $|\phi - f|_{\Lambda} \leq C$ and $\nabla \phi|_{\Lambda_i} = \nabla \phi^u|_{\Lambda}$ for all $i$. If $E = \mathbb{R}$ then we may furthermore impose that $\phi$ is $q_\varepsilon$-Lipschitz for fixed $0 < \varepsilon' < \varepsilon$ (that $m$ is allowed to depend upon).

For this result, the notation $\nabla \phi = \nabla \psi$ means that the difference $\phi - \psi$ is constant.

7 The specific free energy

7.1 The attachment lemmas

The letter $\Phi$ denotes a potential in $\mathcal{S}_c + W_c$ throughout this section. For the thermodynamical formalism, it is crucial that we are able to attach height functions defined on disjoint subsets of $\mathbb{Z}^d$ without losing or gaining too much energy. More precisely, if $\Lambda_1, \Lambda_2 \subset \subset \mathbb{Z}^d$ are disjoint with $\Lambda := \Lambda_1 \cup \Lambda_2$, then we want to find bounds on the difference between $H^0_\Lambda(\phi)$ and $H^0_{\Lambda_1}(\phi) + H^0_{\Lambda_2}(\phi)$. Similarly, we will require bounds on the difference between $H^0_{\Lambda_1}(\phi)$ and $H_{\Lambda}(\phi)$. In this section, we present simple tools for doing this: the attachment lemmas. We first state and prove the lower attachment lemma, which is easier.

Lemma 7.1 (Lower attachment lemma). Let $\Lambda_1, \Lambda_2 \subset \subset \mathbb{Z}^d$ disjoint, and write $\Lambda := \Lambda_1 \cup \Lambda_2$. Then

$$H^0_{\Lambda_1} \geq H^0_{\Lambda_1} + H^0_{\Lambda_2} - \min_{i \in \{1,2\}} e^{-}(\Lambda_i),$$

where $e^{-}$ is the lower exterior bound of $\Xi$. We also have $H_{\Lambda} \geq H^0_{\Lambda} - e^{-}(\Lambda)$ for any $\Lambda \subset \subset \mathbb{Z}^d$.

Proof. The inequality $H^0_{\Lambda} \geq H^0_{\Lambda_1} + H^0_{\Lambda_2}$ is obvious because $\Psi$ is positive. The inequality $H_{\Lambda} \geq H_{\Lambda_1} + H_{\Lambda_2} - \min_{i \in \{1,2\}} e^{-}(\Lambda_i)$ is immediate from the definition of $e^{-}$ in terms of $\Xi$. This proves the inequality in the display. The other inequality follows from a similar decomposition.

More care is required for the upper bound. There is a difference between the discrete case $E = \mathbb{Z}$ and the continuous case $E = \mathbb{R}$. If $E = \mathbb{Z}$ then the strong interaction $\Psi$ can be described by finite information. The effect of this is that there exists a uniform bound $C < \infty$ such that

$$H_{\{x\}}^{\Psi}(\phi) \leq C \quad (7.2)$$

for any $x \in \mathbb{Z}^d$ and any $q$-Lipschitz function $\phi \in \Omega$. If $E = \mathbb{R}$ then there exists no such a priori bound, and it is this specific reason reason that we introduce the locally bounded property in Subsection 3.2, so that at least

$$H_{\{x\}}^{\Psi}(\phi) \leq C_\varepsilon \quad (7.3)$$

whenever $\phi$ is $q_\varepsilon$-Lipschitz at $x$.

For the upper bound, one requires control especially over the potential $\Psi$ which enforces the Lipschitz constraint. The height function $\phi$ must therefore be sufficiently well-behaved for the lemma to work, at least on the boundary where $\Lambda_1$ meets $\Lambda_2$. 

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Lemma 7.4 (Upper attachment lemma). Let $\phi \in \Omega$ and $\Lambda_1, \Lambda_2 \subset \subset \mathbb{Z}^d$ disjoint, and write $\Lambda := \Lambda_1 \cup \Lambda_2$. If $E = \mathbb{Z}$, then there exists an amenable function $e^+$, dependent only on $\Phi$, such that

$$H^0_\Lambda(\phi) \leq H^0_{\Lambda_1}(\phi) + H^0_{\Lambda_2}(\phi) + \min_{i \in \{1, 2\}} e^+(\Lambda_i)$$

(7.5)

whenever $\phi_{\partial R \Lambda_1 \cup \partial R \Lambda_2}$ is $q$-Lipschitz, and such that

$$H_\Lambda(\phi) \leq H^0_\Lambda(\phi) + e^+(\Lambda)$$

(7.6)

whenever $\phi_{\partial R \Lambda \cup \partial R (\mathbb{Z}^d \setminus \Lambda)}$ is $q$-Lipschitz. If $E = \mathbb{R}$ and $\varepsilon > 0$, then there exists an amenable function $e^+_\infty$, dependent only on $\Phi$ and $\varepsilon$, such that (7.5) and (7.6) hold true whenever the restrictions of $\phi$ are $q_\varepsilon$-Lipschitz, and with $e^+$ replaced by $e^+_\infty$.

Definition 7.7. The functions $e^+$ and $e^+_\infty$ are called upper exterior bounds.

Proof of Lemma 7.4. It suffices to consider the contributions of the potentials $\Psi$ and $\Xi$ to each Hamiltonian separately; one can simply sum the two upper exterior bounds $e^{-, \Psi}$ and $e^{+, \Xi}$ so obtained. In fact, the upper exterior bound $e^{+, \Xi} := e^-$ suffices for the long-range interaction $\Xi$. Let us therefore focus on the contribution from the potential $\Psi$.

We shall simultaneously consider the discrete case and the continuous case. In this proof we shall reserve the name Lipschitz for $q$-Lipschitz whenever $E = \mathbb{Z}$ and for $q_\varepsilon$-Lipschitz whenever $E = \mathbb{R}$. Write $C$ for a fixed constant such that $H^\Psi_{\{x\}}(\psi) \leq C$ for any $x \in \mathbb{Z}^d$ and for any Lipschitz height function $\psi$. Because $\Xi$ is positive and of range $R$ and because the restriction of $\phi$ to $\partial^R \Lambda_1 \cup \partial^R \Lambda_2$ is Lipschitz, we have

$$H^\Psi_{\Lambda_1}(\phi) - H^0_{\Lambda_1}(\phi) - H^\Psi_{\Lambda_2}(\phi) = \sum_{\Delta \subset \partial^R \Lambda_1 \cup \partial^R \Lambda_2} \Psi_{\Delta}(\phi)$$

$$= \sum_{\Delta \subset \partial^R \Lambda_1 \cup \partial^R \Lambda_2} \Psi_{\Delta}(\phi) \leq \min_{i \in \{1, 2\}} H^\Psi_{\partial^R \Lambda_i \cup \partial^R \Lambda_1 \cup \partial^R \Lambda_2} + \Psi_{\Delta}(\phi)$$

$$\leq \min_{i \in \{1, 2\}} C \cdot |\partial^R \Lambda_i| \leq \min_{i \in \{1, 2\}} e^{+\Psi}(\Lambda_i)$$

if we define $e^{+\Psi}(\Lambda) := C(2R + 1)^d \cdot |\partial \Lambda|$; this function satisfies the desired constraints. It is clear that this choice for $e^{+\Psi}$ also implies that

$$H^\Psi_{\Lambda}(\phi) \leq H^0_{\Lambda}(\phi) + e^{+\Psi}(\Lambda)$$

whenever the restriction of $\phi$ to $\partial^R \Lambda \cup \partial^R (\mathbb{Z}^d \setminus \Lambda)$ is Lipschitz.

\[ \blacksquare \]

7.2 Density limits of functions on finite subsets of $\mathbb{Z}^d$

Proposition 7.8. Consider two $\mathcal{L}$-invariant real-valued functions $f$ and $b$ on the finite subsets of $\mathbb{Z}^d$, with $b$ amenable and

$$f(\Lambda_1 \cup \Lambda_2) \leq f(\Lambda_1) + f(\Lambda_2) + \min_{i \in \{1, 2\}} b(\Lambda_i)$$

(7.9)

for disjoint $\Lambda_1, \Lambda_2 \subset \subset \mathbb{Z}^d$. Then $(n^{-d} f(\Pi_n))_{n \in \mathbb{N}}$ tends to a limit in $[-\infty, \infty]$ as $n \to \infty$, and

$$\lim_{n \to \infty} n^{-d} f(\Pi_n) = \inf_{n \in \mathbb{N} \setminus N} n^{-d} (f(\Pi_n) + b(\Pi_n))$$

where $N \in \mathbb{N}$ is minimal subject to $N \cdot \mathbb{Z}^d \subset \mathcal{L}$. Finally, if $(\Lambda_n)_{n \in \mathbb{N}} \uparrow \mathbb{Z}^d$, then

$$\limsup_{n \to \infty} |\Lambda_n|^{-1} f(\Lambda_n) \leq \lim_{n \to \infty} n^{-d} f(\Pi_n).$$

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If we weaken the assumptions, and suppose only that (7.9) holds true whenever $\Lambda_1$ contains some vertex $x$ adjacent to some vertex $y$ in $\Lambda_2$, then each statement in this proposition remains valid, except that, for the final assertion, we also require that each set $\Lambda_n$ is connected.

Definition 7.10. Write $\langle \cdot | \Phi \rangle : \mathcal{P}_\mathcal{L}(\Omega, \mathcal{F}^\Lambda) \rightarrow [-\|\Xi\|, \infty]$ for the unique functional which satisfies

$$
\langle \mu | \Phi \rangle := \mu(\Phi) := \lim_{n \to \infty} n^{-d} \mu(H_{1,n}^0).
$$

The limit on the right converges due to the lower attachment lemma and the previous proposition. This quantity is called the specific energy of $\mu$ with respect to $\Phi$.

7.3 Free energy attachment lemma

Definition 7.11. Define $e^* := e^- + \log(2K + 1)$, where $K$ is minimal subject to $Kd_1 \geq q$. Call the amenable function $e^*$ the free energy exterior bound.

Lemma 7.12 (Free energy attachment lemma). Fix $\mu \in \mathcal{P}(\Omega, \mathcal{F}^\Lambda)$, and consider some disjoint sets $\Lambda_1, \Lambda_2 \subset \mathbb{Z}^d$ with some vertex $x$ of $\Lambda_1$ adjacent to some vertex $y$ of $\Lambda_2$ in the square lattice. Write $\Lambda : = \Lambda_1 \cup \Lambda_2$. Then

$$
\mathcal{H}_\Lambda(\mu | \Phi) \geq \mathcal{H}_{\Lambda_1}(\mu | \Phi) + \mathcal{H}_{\Lambda_2}(\mu | \Phi) - \min_{i \in \{1,2\}} e^*(\Lambda_i).
$$

Moreover, for $\Lambda \subset \subset \mathbb{Z}^d$ connected and nonempty, we have

$$
\mathcal{H}_\Lambda(\mu | \Phi) \geq -((|\Lambda| - 1) \max_{x \in \mathbb{Z}^d/\mathcal{L}} e^* \{x\}.
$$

Proof. Fix $K$ minimal subject to $Kd_1 \geq q$. Recall that $\mu_\Pi$ is the restriction of $\mu$ to $\Pi$. We assume that $\mu_\Pi$ is supported on $Kd_1$-Lipschitz functions; if this is not the case, then $\mathcal{H}_\Lambda(\mu | \Phi)$ is infinite, and we are done. For any $\Delta \subset \subset \Lambda$, we have

$$
\mathcal{H}_\Delta(\mu | \Phi) = \mathcal{H}_{\mathcal{F}^\Lambda}(\mu | \lambda^{\Delta^{-1}}) + \mu(H_{\Delta}^0).
$$

By the lower attachment lemma, we have $\mu(H_{\Lambda_1}^0) \geq \mu(H_{\Lambda_1}^0) + \mu(H_{\Lambda_2}^0) - \min_{i \in \{1,2\}} e^- (\Lambda_i)$. Therefore it suffices to show that

$$
\mathcal{H}_{\mathcal{F}^\Lambda}(\mu | \lambda^{\Delta^{-1}}) \geq \mathcal{H}_{\mathcal{F}^\Lambda}(\mu | \lambda^{\Delta_1^{-1}}) + \mathcal{H}_{\mathcal{F}^\Lambda}(\mu | \lambda^{\Delta_2^{-1}}) - \log(2K + 1)
$$

whenever $\mu_\Pi$ is supported on $Kd_1$-Lipschitz functions. This follows from the following two facts:

1. We have $\mathcal{H}_{\mathcal{F}^\Lambda_{\{x,y\}}}(\mu | \lambda^{\{x,y\}^{-1}}) \geq -\log(2K + 1)$,

2. If $\Delta_1, \Delta_2 \subset \Lambda$ share a single vertex $z$ and $\Delta := \Delta_1 \cup \Delta_2$, then

$$
\mathcal{H}_{\mathcal{F}^\Lambda}(\mu | \lambda^{\Delta^{-1}}) \geq \mathcal{H}_{\mathcal{F}^\Lambda_1}(\mu | \lambda^{\Delta_1^{-1}}) + \mathcal{H}_{\mathcal{F}^\Lambda_2}(\mu | \lambda^{\Delta_2^{-1}}).
$$

Note that (7.13) then follows by applying the second fact twice, first to the sets $\Lambda_1$ and $\{x,y\}$, then to the sets $\Lambda_1 \cup \{y\}$ and $\Lambda_2$. Let us first prove the first fact. Since $\mu$ is supported on $Kd_1$-Lipschitz functions, we have

$$
\mathcal{H}_{\mathcal{F}^\Lambda_{\{x,y\}}}(\mu | \lambda^{\{x,y\}^{-1}}) \geq -\log \lambda^{\{x,y\}^{-1}} \{||\phi(y) - \phi(x)|| \leq K\} \geq -\log(2K + 1).
$$
For the second fact, we can simply choose the point \( z \) as a reference point for all gradient measures, such that the measurable space \((\Omega, \mathcal{F}_\nabla)\) becomes effectively a product space; the measure \( \lambda^{A-1} \) is then the product measure of \( \lambda^{A1-1} \) and \( \lambda^{A2-1} \). The second fact now follows; the inequality in the display is well-known for product spaces.

The final assertion of the lemma is a direct consequence of the first assertion and the fact that \( H_\Lambda(\mu|\Phi) = 0 \) whenever \( \Lambda \) is a singleton. \( \square \)

### 7.4 Convergence and properties of the specific free energy

The two results in this subsection jointly imply Theorem 4.1.

#### Theorem 7.14

If \( \Phi \in S_L + W_L \), then the functional \( H(\cdot|\Phi) : \mathcal{P}_L(\Omega, \mathcal{F}_\nabla) \to \mathbb{R} \cup \{\infty\} \) is well-defined and satisfies

\[
H(\mu|\Phi) := \lim_{n \to \infty} n^{-d} H_{\Pi_n}(\mu|\Phi) = \sup_{n \in N \cdot N} n^{-d} (H_{\Pi_n}(\mu|\Phi) - e^*(\Pi_n)) \geq - \max_{x \in \mathbb{Z}^d/L} e^*(\{x\}),
\]

where \( N \) is minimal subject to \( N \cdot \mathbb{Z}^d \subset L \). Moreover, \( H(\cdot|\Phi) \) is lower-semicontinuous, and for each \( C \in \mathbb{R} \) the lower level set

\[
M_C := \{ \mu \in \mathcal{P}_L(\Omega, \mathcal{F}_\nabla) : H(\mu|\Phi) \leq C \}
\]

is a compact Polish space, with respect to the topology of (weak) local convergence. In fact, the two topologies coincide on each set \( M_C \).

**Proof.** The statements in the first display follow from Lemma 7.12 and Proposition 7.8. For the remainder of the theorem, observe that

\[
\mathcal{M}_C = \mathcal{P}_L(\Omega, \mathcal{F}_\nabla) \cap \bigcap_{n \in N \cdot N} \{ \mu \in \mathcal{P}(\Omega, \mathcal{F}_\nabla) : H_{\Pi_n}(\mu|\Phi) \leq n^d C + e^*(\Pi_n) \}.
\]

Each of these sets is closed (in the topology of weak local convergence), and therefore \( \mathcal{M}_C \) is closed; the functional \( H(\cdot|\Phi) \) must be lower-semicontinuous (in either topology). Moreover, for each \( n \in N \cdot \mathbb{N} \), the set

\[
\{ \mu \in \mathcal{P}(\Omega, \mathcal{F}_{\Pi_n}) : H_{\Pi_n}(\mu|\Phi) \leq n^d C + e^*(\Pi_n) \}
\]

is a compact Polish space with respect to both the weak and strong topologies, which coincide on this set. Write \( \delta_n \) for the corresponding metric. Then \( \mathcal{M}_C \) is a compact Polish space with metric \( \delta(\mu, \nu) := \sum_{n \in N \cdot \mathbb{N}} e^{-n}(\delta_n(\mu, \nu) \wedge 1) \).

**Theorem 7.15** If \( \Phi \in S_L + W_L \), then the functional \( H(\cdot|\Phi) \) is affine, in the sense that

\[
H((1-t)\mu + t\nu|\Phi) = (1-t)H(\mu|\Phi) + tH(\nu|\Phi)
\]

for \( \mu, \nu \in \mathcal{P}_L(\Omega, \mathcal{F}_\nabla) \) and \( 0 \leq t \leq 1 \).

**Proof.** It follows from a direct entropy calculation that for fixed \( \Lambda \subset \subset \mathbb{Z}^d \),

\[
0 \leq (1-t)H_\Lambda(\mu|\Phi) + tH_\Lambda(\nu|\Phi) - H_\Lambda((1-t)\mu + t\nu|\Phi) \leq 2 \log 2.
\]

This error term vanishes in the normalization of the specific free energy. \( \square \)
7.5 The surface tension

Recall that the surface tension $\sigma : (\mathbb{R}^d)^* \rightarrow \mathbb{R} \cup \{\infty\}$ is defined by

$$\sigma(u) := \inf_{\mu \in \mathcal{P}_L(\Omega, \mathcal{F}^\phi)} \mathcal{H}(\mu|\Phi).$$

The function $\sigma$ must be convex because both $S(\cdot)$ and $\mathcal{H}(\cdot|\Phi)$ are affine. It is also bounded from below because $\mathcal{H}(\cdot|\Phi)$ is bounded from below by $-\max_{x \in \mathbb{Z}^d/L} e^+(\{x\})$. Recall that $U_\Phi$ is defined to be the interior of the set $\{\sigma < \infty\} \subset (\mathbb{R}^d)^*$. The set $U_\Phi$ is convex, and $\sigma$ is continuous on $U_\Phi$. Moreover, $\sigma$ must equal $\infty$ on the complement of the closure of $U_\Phi$. Recall the statement of Theorem 4.13, for which we now provide a proof.

**Proof of Theorem 4.13.** Observe that $\sigma$ is lower-semicontinuous, because $S(\cdot)$ is continuous and because $\mathcal{H}(\cdot|\Phi)$ is lower-semicontinuous with compact lower level sets.

Let us first prove that $U_\Phi \subset U_q$. Suppose that the slope $u := S(\mu)$ of $\mu \in \mathcal{P}_L(\Omega, \mathcal{F}^\phi)$ is not in $U_q$. It suffices to demonstrate that $\mathcal{H}(\mu|\Phi) = \infty$. Since $u \notin U_q$, we know that $u|_L$ is not $q$-Lipschitz, and therefore with positive $\mu$-probability, $\phi|_L$ is not $q$-Lipschitz. In particular, this means that $\mu(H^n_{\Pi_n}) = \infty$ for $n$ sufficiently large. This proves that $\mathcal{H}(\mu|\Phi) = \infty$.

For the remainder of the proof, we distinguish between the discrete and the continuous setting. Consider first the case that $E = \mathbb{Z}$. For the lemma, it suffices to demonstrate that $\sigma$ is bounded on $U_q$. If $\mu \in \mathcal{P}_L(\Omega, \mathcal{F}^\phi)$ is supported on $q$-Lipschitz functions, then

$$\mathcal{H}_{\Pi_n}(\mu|\Phi) = \mu(H^n_{\Pi_n}) + \mathcal{H}_{\mathcal{F}^\phi_{\Pi_n}}(\mu|\lambda^{\Pi_n} - 1) \leq C n^d \quad \text{where} \quad C := \max_{x \in \mathbb{Z}^d/L} e^+(\{x\});$$

the energy term is bounded by $C n^d$ because $\phi$ is $q$-Lipschitz $\mu$-almost surely, and the entropy term is nonpositive because $\lambda^{\Pi_n} - 1$ is a counting measure. In particular, $\mathcal{H}(\mu|\Phi) \leq C$. Fix $u \in U_q$, and consider a subsequential limit $\mu$ of the sequence

$$\mu_n := \frac{1}{|\Pi_n \cap L|} \sum_{x \in \Pi_n \cap L} \delta_{x, \phi^n}.$$

This limit $\mu$ is clearly supported on $q$-Lipschitz functions and is automatically shift-invariant and satisfies $S(\mu) = u$; in particular, $\sigma(u) \leq C < \infty$. This proves that $\sigma$ is bounded by $C$ on $U_q$.

Consider now the continuous case $E = \mathbb{R}$. For the lemma, we must show that $\sigma$ is finite on $U_q$, and infinite on $\partial U_q$. Fix $u \in U_q$. Then $\phi^n$ is $q_\varepsilon$-Lipschitz for $\varepsilon > 0$ sufficiently small.

Let $X = (X_x)_{x \in \mathbb{Z}^d}$ denote an i.i.d. family of random variables which are uniformly random in the interval $[0, \varepsilon]$. Write $\mu \in \mathcal{P}_L(\Omega, \mathcal{F}^\phi)$ for the measure in which $\phi$ has the distribution of $\phi^n + X$. Then $\phi$ is $q_\varepsilon$-Lipschitz almost surely. It is straightforward to see that

$$\mathcal{H}_{\Pi_n}(\mu|\Phi) = \mu(H^n_{\Pi_n}) + \mathcal{H}_{\mathcal{F}^\phi_{\Pi_n}}(\mu|\lambda^{\Pi_n} - 1) \leq (C - \log \varepsilon) n^d \quad \text{where} \quad C := \max_{x \in \mathbb{Z}^d/L} e^+(\{x\});$$

in particular, $\mathcal{H}(\mu|\Phi) \leq C - \log \varepsilon < \infty$. Clearly $S(\mu) = u$, and so $\sigma(u) < \infty$. Finally, consider $u \in \partial U_q$. Suppose that $\mu \in \mathcal{P}_L(\Omega, \mathcal{F}^\phi)$ has slope $u$. Then at least one of the following two must hold true:

1. $\phi$ is not $q$-Lipschitz, with positive $\mu$-probability,
2. $\phi(y) - \phi(x)$ is deterministic in $\mu$ for some distinct vertices $x$ and $y$. 

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This follows from Lemma 6.1 which gives a characterization of $U_q$. In the former case we have $\mathcal{H}(\mu|\Phi) = \infty$ as was shown at the beginning of this proof. In the latter case, we observe that

$$\mathcal{H}_{\pi_n}^{\nabla}(\mu|\lambda^{n-1}) = \infty$$

for $n$ sufficiently large, because $\mu\pi_n$ is not absolutely continuous with respect to $\lambda^{n-1}$. This also implies that $\mathcal{H}(\mu|\Phi) = \infty$. We have now shown that $\sigma = \infty$ on $\partial U_q$. \hfill \square

# 8 Minimizers of the specific free energy

Recall that a minimizer is a shift-invariant measure $\mu \in \mathcal{P}_L(\Omega, \mathcal{F}^\nabla)$ which satisfies

$$\mathcal{H}(\mu|\Phi) = \sigma(S(\mu)) < \infty,$$

and recall the discussion of minimizers in Subsection 4.1, in particular Definition 4.3. The purpose of this section is to prove the following theorem, which provides us with several properties of minimizers, and is equivalent to the conjunction of Theorem 4.2 and Theorem 4.4.

**Theorem 8.1.** Let $\Phi \in S_L + W_L$, and consider a minimizer $\mu \in \mathcal{P}_L(\Omega, \mathcal{F}^\nabla)$. Fix $\Lambda \subset \subset \mathbb{Z}^d$, and write $\mu^\phi$ for the regular conditional probability distribution of $\mu$ on $(\Omega, \mathcal{F})$ corresponding to the projection map $\Omega \to E^{2d-\Lambda}$. Then for $\mu$-almost every $\phi \in \Omega$, we have $\mu^\phi\pi_\Lambda \in \mathcal{A}_{\Lambda,\phi}$. In particular, if $\mu(\Omega_{\gamma}) = 1$, then $\mu$ is an almost Gibbs measure. In general, the former implies that $\mu$ has finite energy, in the sense that

$$1_{\Omega_q}(\mu\pi_{Z_d \setminus \Lambda} \times \lambda^\Lambda) \ll \mu,$$

where $\Omega_q$ is the set of $q$-Lipschitz height functions.

We first introduce the definition of the max-entropy, which is due to Datta [Dat09].

**Definition 8.2.** Let $(X, \mathcal{X})$ denote a measurable space, endowed with some finite measures $\mu$ and $\nu$. Then the max-entropy of $\mu$ with respect to $\nu$ is defined by

$$\mathcal{H}^\infty(\mu|\nu) := \log \inf \{ \lambda \geq 0 : \mu \leq \lambda \nu \} = \begin{cases} \text{ess sup log } f & \text{if } \mu \ll \nu \text{ where } f = d\mu/d\nu, \\ \infty & \text{otherwise}. \end{cases}$$

The max-diameter of a non-empty set $\mathcal{A}$ of finite measures on $(X, \mathcal{X})$ is defined by

$$\text{Diam}^{\infty} \mathcal{A} := \sup_{\mu, \nu \in \mathcal{A}} \mathcal{H}^\infty(\mu|\nu).$$

If $\text{Diam}^{\infty} \mathcal{A} < \infty$, then all measures in $\mathcal{A}$ are absolutely continuous with respect to one another, with uniform lower and upper bounds on the Radon-Nikodym derivatives.

**Proposition 8.3.** Suppose that $\Lambda \subset \Delta \subset \subset \mathbb{Z}^d$ with $\Lambda \subset \Delta^{-R}$. Then $\text{Diam}^{\infty} \mathcal{C}(\mathcal{A}_{\Lambda,\Lambda,\phi}) \leq 4e^{-\Lambda} < \infty$. In particular, $\text{Diam}^{\infty} \mathcal{A}_{\Lambda,\phi} \leq 4e^{-\Lambda} < \infty$.

**Proof.** Claim first that $\text{Diam}^{\infty} \mathcal{A}_{\Lambda,\Lambda,\phi} \leq 4e^{-\Lambda}$. Consider two random fields $\nu_1, \nu_2 \in \mathcal{P}(\Omega, \mathcal{F})$ with $\nu_1\pi_\Lambda = \nu_2\pi_\Lambda = \delta_{\phi_\Lambda}$. Then

$$\nu_1\gamma_\Lambda \pi_\Lambda = \int \frac{1}{Z_\Lambda(\psi)} e^{-H(\cdot,\psi_{Z_d \setminus \Lambda}) \lambda^\Lambda} d\nu_1(\psi).$$
But since $\psi_\Delta = \phi_\Delta$ almost surely in both $\nu_1$ and $\nu_2$, the dependence of $H(\cdot, \psi_{\mathbb{Z}^d, \Delta})$ on $\psi$ is bounded by $e^{-\gamma}(\Lambda)$. This error term appears twice in each measure $\nu_1 \gamma_\Lambda \pi_{\Lambda}$; directly in the Hamiltonian, and indirectly in the normalization constant. Thus, in calculating the Radon-Nikodym derivative between the two measures, the term appears four times. This proves the claim. By Lemma 5.1 in [LT19], this also implies that $\text{Diam}^\infty C(\mathcal{A}_\Delta, \phi) \leq 4e^{-\gamma}(\Lambda)$. □

We also need the following lemma, which is an adaptation of an intermediate result in [LT19] to the gradient setting.

**Lemma 8.4.** Fix $\mu \in \mathcal{P}(\Omega, \mathcal{F}^\nabla)$, and define, for $\Lambda \subset \Delta \subset \mathbb{Z}^d$,

$$K_\mu(\Lambda, \Delta) := \inf_{\nu \in \mathcal{P}(\Omega, \mathcal{F}^\nabla) \text{ with } \nu \pi_\Delta = \mu \pi_\Delta} \mathcal{H}_{\mathcal{F}_\nabla^\Lambda}(\mu|\nu \gamma_\Lambda) \geq 0.$$  

Then $K_\mu(\cdot, \cdot)$ is superadditive in the first argument, and increasing in the second argument.

**Proof.** It is straightforward to see that $K_\mu(\Lambda, \Delta)$ is increasing in $\Delta$: increasing $\Delta$ restricts the set of measures $\nu$ for the infimum, while increasing the $\sigma$-algebra $\mathcal{F}_\nabla^\Lambda$ for the entropy. Both operations increase the value of $K_\mu(\Lambda, \Delta)$. For superadditivity in $\Lambda$, it suffices to prove that

$$\inf_{\nu \in \mathcal{P}(\Omega, \mathcal{F}^\nabla) \text{ with } \nu \pi_\Delta = \mu \pi_\Delta} \mathcal{H}_{\mathcal{F}_\nabla^\Lambda}(\mu|\nu \gamma_\Lambda) \geq \sum_{i \in \{1, 2\}} \inf_{\nu \in \mathcal{P}(\Omega, \mathcal{F}^\nabla) \text{ with } \nu \pi_\Delta = \mu \pi_\Delta} \mathcal{H}_{\mathcal{F}_\nabla^\Lambda}(\mu|\nu \gamma_{\Lambda_i})$$

for $\Lambda_1$ and $\Lambda_2$ disjoint with $\Lambda := \Lambda_1 \cup \Lambda_2 \subset \Delta$. This follows from Lemma 4.1 in [LT19]. Observe that that lemma does not concern the gradient setting, which provides us with a slight complication. However, since we choose $\Lambda$ to be a strict subset of $\Delta$, we can fix a vertex $x \in \Delta \setminus \Lambda$ to serve as a reference vertex for the gradient setting for all three entropy calculations in the display, thus translating the inequality to the non-gradient setting. □

**Lemma 8.5.** If $\mu$ is a minimizer, then $K_\mu \equiv 0$.

**Proof.** Fix $\mu \in \mathcal{P}_\mathcal{L}(\Omega, \mathcal{F}^\nabla)$. Then $K_\mu$ is shift-invariant, in the sense that $K_\mu(\Lambda, \Delta) = K_\mu(\theta \Lambda, \theta \Delta)$ for $\Lambda \subset \Delta \subset \mathbb{Z}^d$ and $\theta \in \Theta$. Using also the properties of $K_\mu$ in the previous defining lemma, it is immediate that $K_\mu \equiv 0$ if and only if $K_\mu(\Pi_n^R, \Pi_n) = o(n^d)$ as $n \to \infty$. Moreover, by definition of $K_\mu$, it is immediate that

$$K_\mu(\Pi_n^R, \Pi_n) \leq \mathcal{H}_{\mathcal{F}_\nabla^{\Pi_n}}(\mu|\nu \gamma_{\Pi_n}).$$

We must therefore prove that $\mathcal{H}(\mu|\Phi) = \sigma(S(\mu)) < \infty$ implies that the expression on the right in this display is of order $o(n^d)$ as $n \to \infty$. If this expression is not of order $o(n^d)$, then there is an $n \in \mathbb{N}$ and an $\varepsilon > 0$ such that

$$\mathcal{H}_{\mathcal{F}_\nabla^{\Pi_n}}(\mu|\nu \gamma_{\Pi_n}) \geq 2e^{-\gamma}(\Pi_n) + \varepsilon.$$  \hspace{1cm} (8.6)

We will use this inequality to construct another $\mathcal{L}$-invariant measure $\mu''$ of the same slope as $\mu$ and with a strictly smaller specific free energy. This proves that $\mathcal{H}(\mu|\Phi) \neq \sigma(S(\mu))$.

For $\Lambda \subset \mathbb{Z}^d$, we denote by $\gamma^*_\Lambda$ the kernel $\gamma_{\Lambda^c}$, only now with respect to the partial Hamiltonian $H_{\Lambda^c, \Lambda}$ rather than the full Hamiltonian $H_{\Lambda^c}$. With a straightforward entropy calculation one can demonstrate that

$$\mathcal{H}_\Lambda(\nu \gamma_{\Pi_n}) \leq \mathcal{H}_\Lambda(\nu|\Phi) - \varepsilon$$

for any $\Lambda \subset \mathbb{Z}^d$ containing $\Pi_n$, and for any $\nu \in \mathcal{P}(\Omega, \mathcal{F}^\nabla)$ with $\nu \pi_{\Pi_n} = \mu \pi_{\Pi_n}$. This can be done by calculating each free energy term first over the $\sigma$-algebra generated by the

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vertices in $\Lambda \setminus \Pi^{-R}_n$, then over the remaining vertices. The first term is the same for $\nu \gamma^*_n$ and $\nu$ since the kernel modifies the values of $\phi$ in $\Pi^{-R}_n$ only; the difference between the two measures for the second term is at least $\varepsilon$ due to (8.6) and because

$$|H_{\Pi^{-R}_n, \Lambda} - H_{\Pi^{-R}_n, \Pi_n}| \leq \varepsilon (\Pi_n).$$

If $\Lambda$ and $\Delta$ are disjoint, then clearly $\gamma^*_\Lambda$ and $\gamma^*_\Delta$ commute. Let $M$ denote the smallest multiple of $N$ which exceeds $n$, and write

$$\mu' := \mu \prod_{x \in M \cdot \mathbb{Z}^d} \gamma^*_\Pi_n+x;$$

this measure is $M \cdot \mathbb{Z}^d$-invariant, but not necessarily $\mathcal{L}$-invariant. By the inequality in the previous paragraph, we have $H_{\Pi, M}(\mu' | \Phi) \leq H_{\Pi, M}(\mu | \Phi) - k^d \varepsilon$ for any $k \in \mathbb{N}$. As $M \cdot \mathbb{Z}^d$-invariant measures, we have $\mathcal{S}(\mu') = \mathcal{S}(\mu)$ and $H(\mu' | \Phi) \leq H(\mu | \Phi) - \varepsilon / M^d < H(\mu | \Phi)$. To make $\mu'$ also $\mathcal{L}$-invariant, simply define

$$\mu'' := \frac{1}{|\mathcal{L}/(M \cdot \mathbb{Z}^d)|} \sum_{x \in \mathcal{L}/(M \cdot \mathbb{Z}^d)} \theta_x \mu'.$$

The averaging procedure does not change the slope or the specific free energy. This is the desired measure. \hfill $\Box$

**Proof of Theorem 8.1.** The theorem contains three claims. The second claim follows directly from the first claim and the definition of an almost Gibbs measure. We shall quickly demonstrate that the third claim also follows from the first claim, before focusing on that first claim. Assume that the first claim holds true. Observe first that, by assumption, for $\mu$-almost every $\phi$,

$$1_{\Omega_n}(\delta_{\phi \mu \Delta} \times \lambda^\Lambda) \pi_\Lambda \ll \gamma_\Lambda(\cdot, \phi) \pi_\Lambda \in A_{\Lambda, \phi} \ni \mu^\phi \pi_\Lambda.$$ 

But all measures in $A_{\Lambda, \phi}$ are absolutely continuous with respect to one another, by Proposition 8.3 and the comment preceding it. Therefore

$$1_{\Omega_n}(\delta_{\phi \mu \Delta} \times \lambda^\Lambda) \ll \mu^\phi$$

for $\mu$-almost every $\phi$, which implies that $1_{\Omega_n}(\mu \pi_{\mathbb{Z}^d \setminus \Lambda} \times \lambda^\Lambda) \ll \mu$.

Focus finally on the first claim. By the previous lemma, it suffices to prove that $K_\mu \equiv 0$ implies that $\mu^\phi \pi_\Lambda \in A_{\Lambda, \phi}$ for $\mu$-almost every $\phi$. The proof is nearly identical to the proof of Lemma 5.4 in [LT19]. Fix $\Delta \subset \subset \mathbb{Z}^d$ with $\Lambda \subset \Delta^{-R}$; it suffices to demonstrate that $K_\mu \equiv 0$ implies that $\mu^\phi \pi_\Lambda \in C(A_{\Lambda, \Delta, \phi})$ for $\mu$-almost every $\phi$. By choice of $\Delta$, we have $\text{Diam}^\Lambda C(A_{\Lambda, \Delta, \phi}) < \infty$. Write $\Delta_n$ for $\{ -n, \ldots, n \}^d \subset \subset \mathbb{Z}^d$, and write $\mu_n^\phi$ for the regular conditional probability distribution of $\mu$ on $(\Omega, \mathcal{F})$ corresponding to the projection map $\Omega \rightarrow E^{\Delta_n \setminus \Lambda}$. We only consider $n \in \mathbb{N}$ so large that $\Lambda \subset \Delta \subset \Delta_n$. As in the proof of Lemma 5.4 in [LT19], we observe that $K_\mu(\Lambda, \Delta_n) = 0$ implies that for $\mu$-almost every $\phi$,

1. $\mu_n^\phi \pi_\Lambda \in C(A_{\Lambda, \Delta_n, \phi}) \subset C(A_{\Lambda, \Delta, \phi})$ for fixed $n$—this follows from Lemma 5.1 in [LT19],
2. $\mu_n^\phi(A) \rightarrow \mu^\phi(A)$ for fixed $A \in \mathcal{F}_\Lambda$, by the bounded martingale convergence theorem,
3. $\mu_n^\phi \pi_\Lambda \rightarrow \mu^\phi \pi_\Lambda \in C(A_{\Lambda, \Delta, \phi})$ by compactness of $C(A_{\Lambda, \Delta, \phi})$ in the strong topology.

Compactness of $C(A_{\Lambda, \Delta, \phi})$ follows from Lemma 5.1 in [LT19] and the fact that $A_{\Lambda, \Delta, \phi}$ has finite max-diameter. \hfill $\Box$
9 Ergodic decomposition of shift-invariant measures

In this section we cite some standard results on ergodic decompositions of shift-invariant random fields from the work of Georgii [Geo11]. Recall that $\mathcal{L}_c^\nabla$ is the $\sigma$-algebra of shift-invariant gradient events, and that $\text{exP}_c(\Omega, F^\nabla)$ is the set of ergodic gradient measures, endowed with the $\sigma$-algebra $e(\text{exP}_c(\Omega, F^\nabla))$.

The following result is a direct adaptation of Theorem 14.10 in [Geo11] to the gradient setting of this article. Informally, the theorem asserts that if $\mu$ is a shift-invariant gradient random field, then the regular conditional probability distribution of $\mu$ given the information in $\mathcal{L}_c^\nabla$ is well-defined.

**Theorem 9.1.** There is a unique affine bijection

$$ w : \mathcal{P}_c(\Omega, F^\nabla) \to \mathcal{P}(\text{exP}_c(\Omega, F^\nabla), e(\text{exP}_c(\Omega, F^\nabla))), \mu \mapsto w_\mu $$

such that

$$ \mu = \int \nu d w_\mu(\nu) $$

for all $\mu \in \mathcal{P}_c(\Omega, F^\nabla)$. For any $A \in F^\nabla$ and $c \in \mathbb{R}$, this bijection satisfies

$$ w_\mu(\nu(A) \leq c) = \mu(\mu(A|\mathcal{L}_c^\nabla) \leq c). $$

**Definition 9.2.** The measure $w_\mu$ is called the **ergodic decomposition** of $\mu$.

**Proof of Theorem 9.1.** Let $(e_1, \ldots, e_d)$ denote the standard basis of $\mathbb{R}^d$. The measure $\mu$ can be considered a non-gradient measure, by associating to each vertex $x \in \mathbb{Z}^d$ the tuple $(\phi(x + e_1) - \phi(x), \ldots, \phi(x + e_d) - \phi(x)) \in \mathbb{R}^d$. Theorem 14.10 in [Geo11] applies to this non-gradient measure, which immediately implies the current theorem.

It was shown in previous sections that the slope and specific free energy are affine. In fact, these functionals are also strongly affine. This is the subject of the following two results.

**Proposition 9.3.** The functional $S$ is strongly affine, that is,

$$ S(\mu) = \int S(\nu) d w_\mu(\nu) $$

for any $\mu \in \mathcal{P}_c(\Omega, F^\nabla)$ with finite slope.

This proposition is immediate from the definition of $S$.

**Theorem 9.4.** If $\Phi \in \mathcal{S}_c + \mathcal{W}_c$, then the functional $\mathcal{H}(\cdot|\Phi)$ is strongly affine, that is,

$$ \mathcal{H}(\mu|\Phi) = \int \mathcal{H}(\nu|\Phi) d w_\mu(\nu) \quad (9.5) $$

for any $\mu \in \mathcal{P}_c(\Omega, F^\nabla)$.

**Proof.** If $\mu$ is not supported on $q$-Lipschitz functions, then the left and right of (9.5) equal $\infty$; recall that $\mathcal{H}(\cdot|\Phi)$ is bounded below by Theorem 4.1 so that the integral on the right in (9.5) is always well-defined.

Consider now the case that $\mu$ is supported on $q$-Lipschitz functions, which means in particular that $\mu$ is $K$-Lipschitz for $K$ minimal subject to $Kd_1 \geq q$. In that case we have

$$ \mathcal{H}(\mu|\Phi) = \langle \mu|\Phi \rangle + \lim_{n \to \infty} n^{-d} \mathcal{H}_{\mu|\Lambda^n}(\mu|\Lambda^{n-1}), \quad (9.6) $$

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once it is established that the sequence on the right tends to some limit in \((-\infty, \infty]\). The functional \((\cdot | \Phi)\) is clearly strongly affine on \(\mathcal{P}_\mathbb{L}(\Omega, \mathcal{F}^\mathbb{V})\). Let us therefore focus on the limit on the right in the display. It suffices to demonstrate that the second limit in the display is well-defined, bounded below, and strongly affine in its dependence on \(\mu\), once restricted to \(K\)-Lipschitz measures. The idea is to use Theorem 15.20 in [Geo11], which concerns the non-gradient setting. The measure \(\mu\) can be made into a shift-invariant, non-gradient measure by considering the values of \(\phi\) modulo \(4K\). It is clear that the gradient of \(\phi\) can be reconstructed from this reduced height function, if we use the extra information that \(\phi\) is \(K\)-Lipschitz. This is formalized as follows. Write \(\hat{E}\) for the set \(E/4K\mathbb{Z}\), and endow it with the Borel \(\sigma\)-algebra \(\hat{\mathcal{E}}\) and the Lebesgue measure \(\hat{\lambda}\) which satisfies \(\hat{\lambda}(\hat{E}) = 4K\). Write \(\hat{\Omega}\) for the set of functions from \(\mathbb{Z}^d\) to \(\hat{E}\), and \(\hat{\mathcal{F}}\) for the product \(\sigma\)-algebra on \(\hat{\Omega}\). Define the measure \(\hat{\mu}\) on \((\hat{\Omega}, \hat{\mathcal{F}})\) as follows: first sample a pair \((\phi, a)\) from \(\mu \times (\lambda/4K)\), the final sample \(\phi\) is then obtained by setting \(\hat{\phi}(x) = \phi(x) - \phi(0) + a \in \hat{E}\). The measure \(\hat{\mu}\) is clearly \(\mathcal{L}\)-invariant. Note that, for \(\Lambda \subset \subset \mathbb{Z}^d\) nonempty,
\[
\mathcal{H}_{\hat{\mathcal{F}}_\Lambda}(\mu|\lambda^{\Lambda^{-1}}) = \mathcal{H}_{\hat{\mathcal{F}}_\Lambda}(\hat{\mu}^\Lambda) + \log 4K,
\]
where \(\hat{\mathcal{F}}_\Lambda := \sigma(\hat{\phi}(x) : x \in \Lambda)\). By Theorem 15.20 in [Geo11], the limit
\[
\lim_{n \to \infty} n^{-d} \mathcal{H}_{\hat{\mathcal{F}}_{\Pi_n}}(\mu|\lambda^{\Pi_n^{-1}}) = \lim_{n \to \infty} n^{-d} \mathcal{H}_{\hat{\mathcal{F}}_{\Pi_n}}(\hat{\mu}^\Pi)\n\]
is well-defined, bounded below by \(-\log 4K\), and strongly affine over \(\mu\).

**Definition 9.7.** For \(\mu \in \mathcal{P}_\mathbb{L}(\Omega, \mathcal{F}^\mathbb{V})\) a \(K\)-Lipschitz measure, define
\[
\mathcal{H}(\mu|\lambda) := \lim_{n \to \infty} n^{-d} \mathcal{H}_{\hat{\mathcal{F}}_{\Pi_n}}(\mu|\lambda^{\Pi_n^{-1}}) \in [-\log 4K, \infty],
\]
the specific entropy of \(\mu\). This quantity is well-defined and strongly affine over \(\mu\) due to the proof of the previous theorem. Remark that \(\mathcal{H}(\mu|\lambda) \leq 0\) whenever \(E = \mathbb{Z}\).

**Lemma 9.8.** Consider a potential \(\Phi \in \mathcal{S}_\mathbb{L} + \mathcal{W}_\mathbb{L}\) and a measure \(\mu \in \mathcal{P}_\mathbb{L}(\Omega, \mathcal{F}^\mathbb{V})\). Fix \(K\) minimal subject to \(Kd_1 \geq q\). If \(\mu\) is not \(K\)-Lipschitz, then \(\mathcal{H}(\mu|\Phi) = \langle \mu|\Phi \rangle = \infty\), and if \(\mu\) is \(K\)-Lipschitz, then \(\mathcal{H}(\mu|\Phi) = \langle \mu|\Phi \rangle + \mathcal{H}(\mu|\lambda)\).

**Proof.** This also follows from the proof of the previous theorem.

We are now able to prove Theorem 4.5.

**Proof of Theorem 4.5.** Suppose that \(u \in \hat{U}_\Phi\) is an exposed point of \(\sigma\). By compactness of the lower level sets of \(\mathcal{H}(\cdot|\Phi)\) (Theorem 4.1) and by continuity of \(S(\cdot)\), there exists a minimizer \(\mu \in \mathcal{P}_\mathbb{L}(\Omega, \mathcal{F}^\mathbb{V})\) of slope \(u\). Write \(w_\mu\) for the ergodic decomposition of \(\mu\). Since both \(S(\cdot)\) and \(\mathcal{H}(\cdot|\Phi)\) are strongly affine (due to Proposition 9.3 and Theorem 9.4) and because \(u\) is an exposed point, we observe that \(w_\mu\)-almost every component \(\nu\) is an ergodic minimizer of slope \(u\).

\[\square\]

## 10 Limit equalities

This section provides the fundamental building blocks for the large deviations principle in the next section. The motivating thesis for this section is that \(\sigma(u)\) can be approximated by integrals of \(\exp -\mathcal{H}^{\Pi_n}_0\) after restricting to height functions which are close to the slope \(u\) on \(\partial^R\Pi_n\). It is possible to be more subtle: if one considers a measure \(\mu \in \mathcal{P}_\mathbb{L}(\Omega, \mathcal{F})\) with \(\mathcal{H}(\mu|\Phi) < \infty\) and \(S(\mu) \in U_\Phi\), then one can approximate \(\mathcal{H}(\mu|\Phi)\) by integrals of \(\exp -\mathcal{H}^{\Pi_n}_0\).
after restricting to height functions which are close to the slope \(S(\mu)\) on \(\partial^R\Pi_n\), and after restricting further to height functions \(\phi\) whose empirical measure in \(\Pi_n\) approximates \(\mu\). The empirical measure of \(\phi\) in \(\Pi_n\) is obtained by randomly shifting \(\phi\) by a vertex in \(\mathcal{L} \cap \Pi_n\). Analogous results for finite-range non-Lipschitz potentials can be found in Chapter 6 in [She05]. However, the proof presented here differs from the proof in [She05] to account for the generality of our setting, and the specificity of the discrete Lipschitz case.

## 10.1 Formal statement

Let us first introduce some simple notation for fixing boundary conditions.

**Definition 10.1.** Write \(0_\Lambda\) for the smallest element in \(\Lambda\) in the dictionary order on \(\mathbb{Z}^d\) whenever \(\Lambda \subset\subset \mathbb{Z}^d\). Let \(u \in U_\Phi\). If \(E = \mathbb{Z}\), then write \(C^u_\Lambda\) for the set of height functions

\[
\{\phi \in \Omega : \phi_{\partial^R \Lambda} - \phi(0_\Lambda) = \phi^u_{\partial^R \Lambda} - \phi^u(0_\Lambda)\} \in \mathcal{F}_{\partial^R \Lambda}^\Lambda.
\]

Now consider \(E = \mathbb{R}\), and fix \(\varepsilon > 0\). Write \(C^u_{\Lambda,\varepsilon}\) for the set

\[
\{\phi \in \Omega : |(\phi_{\partial^R \Lambda} - \phi(0_\Lambda)) - (\phi^u_{\partial^R \Lambda} - \phi^u(0_\Lambda))| \leq \varepsilon\} \in \mathcal{F}_{\partial^R \Lambda}^\Lambda.
\]

Abbreviate \(C^u_{\Pi_n}\) and \(C^u_{\Pi_n,\varepsilon}\) to \(C^u_{n}\) and \(C^u_{n,\varepsilon}\) respectively.

Next, we formally define the empirical measure of a height function \(\phi\) in \(\Lambda\). Recall the definition of the basis \(B\) of the topology of weak local convergence on \(\mathcal{P}(\Omega, \mathcal{F}_\Lambda)\) from Subsection 2.1.4.

**Definition 10.2.** In this definition, we adopt the following notation: if \(\phi\) is a height function and \(\Lambda \subset\subset \mathbb{Z}^d\), then write \(\partial_\Lambda\) for unique extension of \(\phi_\Lambda\) to \(\mathbb{Z}^d\) which equals \(\phi(0_\Lambda)\) on the complement of \(\Lambda\). For \(\Lambda \subset\subset \mathbb{Z}^d\) and \(\phi \in \Omega\), we define the measure \(L_{\Lambda}(\phi)\) by

\[
L_{\Lambda}(\phi) := \frac{1}{|\mathcal{L} \cap \Lambda|} \sum_{x \in \mathcal{L} \cap \Lambda} \delta_{\partial_\Lambda \phi}.
\]

This is called the empirical measure of \(\phi\) in \(\Lambda\). The kernel \(L_{\Lambda}\) is thus a probability kernel from \((\Omega, \mathcal{F}_\Lambda)\) to \((\Omega, \mathcal{F})\) which restricts to a kernel from \((\Omega, \mathcal{F}_\Lambda)\) to \((\Omega, \mathcal{F}_\mathbb{R})\). Now consider \(B \in \mathcal{B}\). Write \(B_{\Lambda}\) for the event \(B_{\Lambda} := \{\phi \in \Omega : L_{\Lambda}(\phi) \in B\}\); this event is \(\mathcal{F}_\Lambda^\mathbb{R}\)-measurable. We shall also write \(L_n\) and \(B_n\) for \(L_{\Pi_n}\) and \(B_{\Pi_n}\) respectively.

We start with the introduction of free boundary limits, which is slightly easier than the definition of pinned boundary limits. For free boundary limits, we integrate over all height functions having the appropriate empirical measure, irrespective of boundary conditions. It will be useful to define free boundary limits also for measures \(\mu \in \mathcal{P}(\Omega, \mathcal{F}_\mathbb{R})\) which are not shift-invariant.

**Definition 10.3.** Let \(\Lambda \subset\subset \mathbb{Z}^d\) and \(B \in \mathcal{B}\). The free boundary estimate of \(B\) over \(\Lambda\) is given by

\[
FB_{\Lambda}(B) := -\log \int_{B_{\Lambda}} e^{-H^\Lambda_{\Lambda}} d\Lambda^{\Lambda - 1}.
\]

Let \(\mu \in \mathcal{P}(\Omega, \mathcal{F}_\mathbb{R})\). The free boundary limits of \(B\) and \(\mu\) respectively are given by

\[
FB(B) := \liminf_{n \to \infty} n^{-d} FB_{\Pi_n}(B) \quad \text{and} \quad FB(\mu) := \sup_{A \in \mathcal{B} \text{ with } \mu \in A} FB(A).
\]
Free boundary limits should be thought of as an asymptotic upper bound on the integral in the display, and this is why we take the limit inferior in the definition of FB(B)—taking into account the minus sign which appears in the definition of FBΛ(B). Indeed, the free boundary estimates are useful in proving the upper bound on probabilities in the large deviations principle in the next section. Remark that it is immediate from the definition of FB(µ) that FB(·) is lower-semicontinuous on the set of gradient measures in the topology of weak local convergence for which B forms a basis.

Finally, we introduce pinned boundary limits, which take into consideration also the value of φ on the boundary of Πn. In this case, it is the lower bound on the integral of interest that matters to us; pinned boundary limits play a crucial role in the proof of the lower bound on probabilities in the large deviations principle.

**Definition 10.4.** Fix u ∈ Uφ and ε > 0, and let Λ ⊂⊂ Zd and B ∈ B. If E = R, then define

\[ PB_{Λ,u,ε}(B) := -\log \int_{C_σ^u \cap B_Λ} e^{-H^B_λ} dλ^{Λ-1}. \]

If E = Z, then define

\[ PB_{Λ,u}(B) := -\log \int_{C_σ^u \cap B_Λ} e^{-H^B_λ} dλ^{Λ-1}. \]

These are called the pinned boundary estimates of B over Λ. In either case, we set PB_{Λ,u,ε}(B) := ∞ and PB_{Λ,u}(B) := ∞ whenever u ∉ Uφ. Consider now also some random field µ ∈ PC(Ω, FV). The pinned boundary limits of B and µ are defined as follows:

\[ PB_u(B) := \limsup_{n → ∞} n^{-d} PB_{Π_n,u,ε}(B), \quad PB_\mu(A) := \sup_{\varepsilon > 0} \sup_{A ∈ B with \mu ∈ A} PB_{S(\mu),ε}(A) \]

whenever E = R, and if E = Z, then

\[ PB_u(B) := \limsup_{n → ∞} n^{-d} PB_{Π_n,u}(B), \quad PB_\mu(A) := \sup_{A ∈ B with \mu ∈ A} PB_{S(\mu)}(A). \]

It is again immediate from these definitions that for fixed u ∈ Uφ, the functional PB(·) is lower-semicontinuous on the set \{S(·) = u\} ⊂ PC(Ω, FV).

For the proof of the large deviations principle in the next section, we require the following equalities and inequalities.

**Theorem 10.5.** If Φ ∈ SL + WL and µ ∈ PC(Ω, FV), then

\[ \mathcal{H}(µ|Φ) = FB(µ) = PB(µ), \]

unless E = Z and S(µ) ∈ ∂Uφ. If however E = Z and S(µ) ∈ ∂Uφ, then

\[ FB(µ) ≥ \mathcal{H}(µ|Φ). \]

Finally, if µ ∈ PC(Ω, FV) \ PC(Ω, FV), then FB(µ) = ∞.

Free and pinned boundary limits are calculated along the sequence (Πn)n∈N. This choice is convenient, but by no means necessary. In the following sections, we do not only prove the inequalities presented in the theorem: we also prove some generalizations thereof where these quantities are calculated over sequences of the form (Λn)n∈N with Λn := Λ−m(nD), where D is a bounded convex subset of R^d of positive Lebesgue measure, and where m ∈ Z_≥0. Observe that in this notation, Π_n = Λ_n for m = 0 and D = [0, 1)^d ⊂ R^d.
Definition 10.6. Write $C$ for the set of bounded convex subsets of $\mathbb{R}^d$ of positive Lebesgue measure.

The definitions imply that $\text{PB}(\mu) \geq \text{FB}(\mu)$ for $\mu$ shift-invariant. In Subsection 10.2 we discuss free boundary limits. In particular, we show that $\text{FB}(\mu) \geq \mathcal{H}(\mu|\Phi)$ whenever $\mu$ is shift-invariant, and that $\text{FB}(\mu) = \infty$ whenever $\mu$ is not shift-invariant. In Subsection 10.3 we prove that $\text{PB}(\mu) \leq \mathcal{H}(\mu|\Phi)$ whenever $\mu$ is ergodic with $S(\mu) \in U_{\Phi}$. In Subsection 10.4 we extend this inequality to shift-invariant measures $\mu$ which are not ergodic.

10.2 Free boundary limits: empirical measure argument

The idea in this subsection is always to use the set $B$, the empirical measures $L_n(\phi)$ for $\phi \in B_n$, as well as the subsequential limits thereof as $n \to \infty$, to derive the desired inequalities which were mentioned in the previous subsection. Let us first cover the case that $\mu$ is not shift-invariant.

Lemma 10.7. If $\mu \in \mathcal{P}(\Omega, \mathcal{F}^\Lambda) \setminus \mathcal{P}_C(\Omega, \mathcal{F}^\Lambda)$, then $\text{FB}(\mu) = \infty$.

Proof. If $\mu$ is not shift-invariant, then there is a shift $\theta \in \Theta$ and a continuous cylinder function $g : \Omega \to [0, 1]$ such that $\mu(g - \theta g) \neq 0$. Define $f := g - \theta g$; this is a bounded continuous cylinder function such that $\mu(f) \neq 0$. Define $\varepsilon := |\mu(f)|/2$ and $B := \{\nu : |\nu(f) - \mu(f)| < \varepsilon\} \in \mathcal{B}$. For $\Lambda \subset \subset \mathbb{Z}^d$ fixed and for $n$ large, the measure $L_n(\phi) = L_{\Pi_n}(\phi)$ restricted to $\mathcal{F}^\Lambda$ looks almost shift-invariant. More precisely, the sequence of functions

$$
\Omega \to [-1, 1], \phi \mapsto L_n(\phi)(f)
$$

converges to 0 uniformly over $\phi \in \Omega$ as $n \to \infty$. This proves that $B_n = B_{\Pi_n}$ is empty for $n$ sufficiently large, that is, $\text{FB}(\mu) \geq \text{FB}(B) = \infty$.

Next, we consider shift-invariant gradient random fields.

Lemma 10.8. For any $\mu \in \mathcal{P}_C(\Omega, \mathcal{F}^\Lambda)$, we have $\text{FB}(\mu) \geq \mathcal{H}(\mu|\Phi)$.

We start with the following auxiliary lemma.

Lemma 10.9. Suppose that $B \in \mathcal{B}$ satisfies $\text{FB}(B) < \infty$. Then $B$ contains a shift-invariant measure $\mu$ with $\mathcal{H}(\mu|\Phi) \leq \text{FB}(B)$.

Proof. Write $\nu_n^B$ for the normalized version of the measure $1_{B_n} e^{-H_{\Pi_n}^{B_n}} \Lambda_n^{-1}$ for each $n \in \mathbb{N}$, and observe that

$$
\text{FB}(B) = \liminf_{n \to \infty} n^{-d} \mathcal{H}_{\Pi_n}(\nu_n^B|\Phi).
$$

We focus on good subsequences of $n$, that is, subsequences along which the limit inferior is reached.

Write $m : \mathbb{N} \to \mathbb{N}$ for a sequence of integers with $m(n) \to \infty$ and $m(n)/n \to 0$ as $n \to \infty$, and set $\Pi_n := \Pi_n^{-m(n)} = \{m(n), \ldots, n - m(n) - 1\} \subset \Pi_n$. Fix $N \in \mathbb{N}$ minimal subject to $N \cdot \mathbb{Z}^d \subset \mathcal{L}$, and let $k$ denote an integer multiple of $N$. Let $n$ denote another integer, which is so large that $m(n) > k$. The idea is now to apply Lemma 7.12 to translates of $\Pi_k$. In particular, if we write $\Pi_n^{x}$ for the set $\Pi_n$ with the sets $\Pi_k + x$ removed for all $x$ in $\Pi_n \cap (k \cdot \mathbb{Z}^d)$, then that lemma asserts that

$$
\mathcal{H}_{\Pi_n}(\nu_n^B|\Phi) \geq \mathcal{H}_{\Pi_n^{x}}(\nu_n^B|\Phi) + \sum_{x \in \Pi_n \cap (k \cdot \mathbb{Z}^d)} \mathcal{H}_{\Pi_k + x}(\nu_n^B|\Phi) - e^*(\Pi_k + x).
$$

(10.10)
The set $\Pi_{n,k}^{D}$ is always connected and, as $n \to \infty$, we have $|\Pi_{n,k}^{D}| = o(n^d)$. Therefore the first term on the right in (10.10) has a lower bound of order $o(n^d)$. Moreover, the value of $e^*(\Pi_k + x)$ is independent of $x$ as long as $x$ lies in $\mathcal{L}$, and therefore we obtain the asymptotic bound

$$\frac{1}{|\Pi_n |} \sum_{x \in \Pi_n \cap \mathbb{Z}^d} \mathcal{H}_{\Pi_k + x}(\nu_n^B|\Phi) \leq k^d \text{FB}(B) + e^*(\Pi_k) + o(1)$$

(10.11)
as $n \to \infty$ along a good subsequence. Moreover, if we write $\mu_{n,k}$ for the measure

$$\frac{1}{|\Pi_n |} \sum_{x \in \Pi_n \cap \mathbb{Z}^d} \theta_x \nu_n^B,$$

then the previous inequality and convexity of relative entropy imply that

$$\mathcal{H}_{\Pi_k}(\mu_{n,k}|\Phi) \leq k^d \text{FB}(B) + e^*(\Pi_k) + o(1)$$
as $n \to \infty$ along a good subsequence. We may replace the sublattice $k \cdot \mathbb{Z}^d$ by another set $k \cdot \mathbb{Z}^d + y$ for $y \in \mathcal{L}/(k \cdot \mathbb{Z}^d)$ in the previous discussion, and by doing so and averaging further, it is immediate that the sequence of measures $\mu^n$ defined by

$$\frac{1}{|\Pi_n | \cdot \mathcal{L}} \sum_{x \in \Pi_n \cap \mathcal{L}} \theta_x \nu_n^B,$$

also satisfies

$$\mathcal{H}_{\Pi_k}(\mu^n|\Phi) \leq k^d \text{FB}(B) + e^*(\Pi_k) + o(1)$$
as $n \to \infty$ along a good subsequence. Compactness of the lower level sets of relative entropy implies that the sequence $\mu^n$ has a subsequential limit—at least when restricted to $\mathcal{F}_{\Pi_k}^\infty$. Using a standard diagonal argument for convergence for all integers $k \in \mathbb{N}$, one obtains a subsequential limit $\mu$ which is shift-invariant and satisfies

$$\mathcal{H}_{\Pi_k}(\mu|\Phi) \leq k^d \text{FB}(B) + e^*(\Pi_k)$$
for all $k$, that is, $\mathcal{H}(\mu|\Phi) \leq \text{FB}(B)$. This measure must clearly lie in $\mathcal{B}$ by construction. \(\square\)

Proof of Lemma 10.8. Fix $\mu \in \mathcal{P}_\mathcal{L}(\Omega, \mathcal{F}^\infty)$, and suppose that $\text{FB}(\mu) < \infty$. Then the lower level set of the specific free energy $M_{\text{FB}(\mu)}$ endowed with the topology of weak local convergence is metrizable, and therefore we may choose for each $n$ an open set $B^n \in \mathcal{B}$, containing $\mu$ and with $B^n \cap M_{\text{FB}(\mu)}$ of diameter at most $1/n$ in this metric. Then each set $\bar{B}^n$ contains a measure $\mu^n$ with $\mathcal{H}(\mu^n|\Phi) \leq \text{FB}(B^n) \leq \text{FB}(\mu)$. By choice of $B^n$ we must have $\mu^n \to \mu$, and lower-semicontinuity implies that

$$\mathcal{H}(\mu|\Phi) \leq \liminf_{n \to \infty} \mathcal{H}(\mu^n|\Phi) \leq \liminf_{n \to \infty} \text{FB}(B^n) \leq \text{FB}(\mu).$$ \(\square\)

Finally, we discuss how to extend this result to other shapes.

Definition 10.12. For fixed $D \in \mathcal{C}$ and $\mu \in \mathcal{P}(\Omega, \mathcal{F}^\infty)$, we write

$$\text{FB}(\mu : D) := \sup_{B \in \mathcal{B} \text{ with } \mu \in B} \liminf_{n \to \infty} n^{-d} \text{FB}_{\Lambda_n}(B),$$

where we write $\Lambda_n$ for $\Lambda(nD) = nD \cap \mathbb{Z}^d$.

The previous results extend immediately as follows—the Lebesgue measure $\text{Leb}(D)$ would first appear as a factor on the left in (10.11) in the generalized argument.

Lemma 10.13. Consider $D \in \mathcal{C}$ and $\mu \in \mathcal{P}(\Omega, \mathcal{F}^\infty)$. If $\mu$ is not shift-invariant, then $\text{FB}(\mu : D) = \infty$, and if $\mu$ is shift-invariant, then $\text{FB}(\mu : D) \geq \text{Leb}(D) \cdot \mathcal{H}(\mu|\Phi)$. 

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10.3 Pinned boundary limits for $\mu$ ergodic: truncation argument

The goal of this section is to derive the following lemma. The proof starts with a simple reduction, and is then intermitted to state an auxiliary result and to give an overview of the remainder of the proof. The proof extends the random truncation argument in [She05] to the infinite-range Lipschitz setting.

Lemma 10.14. If $\mu \in \mathcal{P}_C(\Omega, \mathcal{F}_\infty)$ is ergodic and $u := S(\mu) \in U_\Phi$, then $PB(\mu) \leq \mathcal{H}(\mu|\Phi)$.

Proof. It suffices to consider the case that $\mathcal{H}(\mu|\Phi) < \infty$, which implies in particular that $\mu$ is $K$-Lipschitz. We first focus on the discrete case $E = \mathbb{Z}$, then generalize to the continuous case $E = \mathbb{R}$; the latter comes with some additional technical complications.

The discrete case. Pick $B \in \mathcal{B}$ with $\mu \in B$. It suffices to show that

$$\mathcal{H}(\mu|\Phi) = \lim_{n \to \infty} n^{-d} \mathcal{H}_{\Pi_n}(\mu|\Phi) \geq \limsup_{n \to \infty} n^{-d} \mathcal{P}B_{\Pi_n,u}(B) = \limsup_{n \to \infty} n^{-d} \mathcal{H}_{\Pi_n}(\nu_n^B|\Phi),$$

where $\nu_n^B$ is the normalized measure

$$\nu_n^B := \frac{1}{Z} e^{-\mathcal{H}_{\Pi_n}^B}.$$

Observe that $\nu_n^B$ minimizes $\mathcal{H}_{\Pi_n}(\cdot|\Phi)$ over all measures which are supported on $\mathcal{C}_n^u \cap B_n$. Therefore it suffices to construct a sequence of measures $(\mu_n)_{n \in \mathbb{N}}$, with each $\mu_n$ supported on $\mathcal{C}_n^u \cap B_n$, and such that $\mathcal{H}_{\Pi_n}(\mu_n|\Phi) \leq \mathcal{H}_{\Pi_n}(\mu|\Phi) + o(n^d)$ as $n \to \infty$. Let us now intermit the proof to give an overview of the remainder of the proof, before continuing.

One continues roughly as follows. Always take 0 as a reference point for all gradient measures. This means that $\mu$-almost surely $\phi(0) = 0$. Write $\phi_n^\pm$ for the largest and smallest $q$-Lipschitz extensions of $\phi_{\partial^q \Pi_n}$ to $\Pi_n$ respectively, for each $n \in \mathbb{N}$. Define the random sets

$$A_n^- \equiv \{ x \in \Pi_n : \phi(x) < \phi_n^-(x) \} \quad \text{and} \quad A_n^+ := \{ x \in \Pi_n : \phi(x) > \phi_n^+(x) \}.$$

Note that $\phi_n^+(0) = 0$ by definition; 0 is $\mu$-almost surely not contained in $A_n^\pm$.

Since $\mu$ is ergodic and $K$-Lipschitz, almost every sample $\phi$ from $\mu$ is asymptotically close to $u$, in the sense of Theorem 10.15. As $u$ belongs to $U_\eta$, the interior of the set of Lipschitz slopes, the function $\phi_n^+$ is substantially larger than $u$ on most vertices in $\Pi_n$. This means that $\mu(|A_n^-|) = o(n^d)$ as $n \to \infty$, and similarly $\mu(|A_n^+|) = o(n^d)$. For each $n \in \mathbb{N}$, define the measure $\mu_n^\pm$ as follows: to draw a sample from $\mu_n^\pm$, sample first a height function $\phi$ from $\mu$, then replace this sample by $\psi := \phi_n^- \lor \phi_{\Pi_n} \land \phi_n^+$. Note that $\phi$ and $\psi$ differ at at most $o(n^d)$ vertices in $\Pi_n$ on average as $n \to \infty$. Moreover, the modified height function $\psi$ is $q$-Lipschitz if the original height function $\phi$ was $q$-Lipschitz. In particular, we deduce that

$$\mathcal{H}_{\Pi_n}(\mu_n^\pm|\Phi) = \mathcal{H}_{\Pi_n}(\mu|\Phi) + o(n^d).$$

The measure $\mu_n^\pm$ is clearly supported on $\mathcal{C}_n^u$, because $\phi_n^-, \phi_n^+$, and $\phi$ are equal on $\partial^q \Pi_n$. Using again the ergodicity of $\mu$ through Theorem 10.15, one can show that $\mu(B_n) \to 1$ as $n \to \infty$, and consequently $\mu_n^\pm(B_n) \to 1$ because $\phi$ and $\psi$ agree on most vertices of $\Pi_n$. This proves that the sequence $(\mu_n)_{n \in \mathbb{N}}$ defined by $\mu_n := \mu_n^\pm(\cdot|B_n)$ is the desired sequence of measures. This concludes the proof overview for $E = \mathbb{Z}$. In the real case $E = \mathbb{R}$, the details are more involved, owing to the following two difficulties:

1. We cannot simply replace $\phi$ by $\phi_n^- \lor \phi_{\Pi_n} \land \phi_n^+$, because the measure so produced would not be absolutely continuous with respect to Lebesgue measure,
2. We only have a bound on \( H_{\{x\}}(\phi) \) if \( \phi \) is \( q \)-Lipschitz at \( x \); it is not sufficient to make modifications which are \( q \)-Lipschitz.

Let us now state Theorem 10.15 before continuing the proof of Lemma 10.14.

**Theorem 10.15.** Consider \( \mu \in \mathcal{P}_c(\Omega, \mathcal{F}^\infty) \) ergodic. Then \( L_n(\phi)(f) \rightarrow \mu(f) \) as \( n \rightarrow \infty \) for \( \mu \)-almost every \( \phi \), for any bounded cylinder function \( f \). Suppose now that \( \mu \) is also \( K \)-Lipschitz with slope \( u := S(\mu) \). Then \( \mu \)-almost surely \( \|\phi_{\Pi_n} - \phi(0) - u|_{\Pi_n}\|_\infty \leq \varepsilon n \) for \( n \) sufficiently large, for any fixed constant \( \varepsilon > 0 \).

The first assertion is the ergodic theorem. The second assertion is straightforward: in the Lipschitz setting, the height difference \( (\phi(x) - \phi(0))/\|x\|_1 \) is approximately equal to the average of the gradient—which is bounded in magnitude—of \( \phi \) over a large set \( \Lambda \subset\subset \mathbb{Z}^d \).

**Continuation of the proof of Lemma 10.14.** Recall that \( E = \mathbb{Z} \). By taking a smaller set \( B \in \mathcal{B} \) if necessary, we suppose that \( B \) is of the form

\[
B = \{ \nu \in \mathcal{P}(\Omega, \mathcal{F}^\infty) : |\nu(f) - \mu(f)| < 2\eta \text{ for all } f \}
\]

for a finite collection \((f_i)\) of continuous cylinder functions \( f_i : \Omega \rightarrow [0, 1] \) and for some \( \eta > 0 \), and we write \( B^* \) for the same set with \( 2\eta \) replaced by \( \eta \). The ergodic theorem asserts that \( \mu(B_n^*) \rightarrow 1 \) as \( n \rightarrow \infty \). Consider \( \mu \) a non-gradient measure on \((\Omega, \mathcal{F})\) by taking \( 0 \in \Pi_n \) as a reference point: this means that \( \phi(0) = 0 \) almost surely in \( \mu \).

Recall the definitions of \( \phi_n^+ \) and \( A_n^+ \) from the proof overview, and claim that \( \mu(|A_n^+|) = o(n^d) \) as \( n \rightarrow \infty \). The function \( \phi_n^+ \) is pyramid-shaped, as in Figure 3—that figure concerns the more complicated continuous setting \( E = \mathbb{R} \), but the shape of \( \phi_n^+ \) is the same. Formally, this means that there exist constants \( C' > 0 \) and \( \varepsilon' > 0 \) such that for any \( n \in \mathbb{N} \) and for any \( x \in \Pi_n \),

\[
\phi_n^+(x) \geq u(x) + \varepsilon'd_1(x, \partial B^\Pi_n) - C'.
\]

(10.16)

This is a consequence of Lemma 6.1 and the fact that \( u \) is in \( U_\phi \), the interior of the set of slopes \( u \) for which \( u'_{\mathcal{L}} \) is \( q \)-Lipschitz. Now fix \( \varepsilon'' > 0 \). By (10.16), the number of points \( x \in \Pi_n \) at which \( \phi_n^+(x) \leq u(x) + \varepsilon''n \) is bounded from above by \( (2d\varepsilon''/\varepsilon')n^d + o(n^d) \) as \( n \rightarrow \infty \). Theorem 10.15 tells us that

\[
\mu(\{ x \in \Pi_n : \phi(x) > u(x) + \varepsilon''n \}) = o(n^d).
\]

Combining the two bounds gives \( \mu(|A_n^+|) \leq (2d\varepsilon''/\varepsilon')n^d + o(n^d) \). The constant \( \varepsilon'' \) may be chosen arbitrarily small, and therefore we obtain \( \mu(|A_n^+|) = o(n^d) \). In the same spirit, one obtains \( \mu(|A_n^-|) = o(n^d) \). This proves the claim.

Next, we construct for each \( n \in \mathbb{N} \) a new measure \( \mu_n^+ \), the upper truncation of \( \mu \). To sample from \( \mu_n^+ \), first sample \( \phi \) from \( \mu \), then replace \( \phi(x) \) by \( \phi_n^+(x) \) for any \( x \in A_n^+ \). This means that the distribution of \( \phi_{\Pi_n} \) in \( \mu_n^+ \) is the same as the distribution of \( \phi_{\Pi_n} \wedge \phi_n^+ \) in \( \mu \). Assert that

\[
\mathcal{H}_{\Pi_n}(\mu_n^+|\Phi) = \mathcal{H}_{\Pi_n}(\mu|\Phi) + o(n^d).
\]

We present an alternative three-stage construction of \( \mu_n^+ \), and demonstrate that the free energy changes by no more than \( o(n^d) \) at every stage. Write \( S \) for the set of finite subsets of \( \mathbb{Z}^d \), which is countable, and write \( \alpha \) for the counting measure on \( S \). Write \( \mathcal{G}_n \) for the smallest \( \sigma \)-algebra on \( \Omega \times S \) containing \( A \times \{ A \} \) for any \( A \in \mathcal{F}_{\Pi_n}^\infty \) and any \( A \subset \Pi_n \).

For the first stage, write \( \tilde{\mu}_n \) for the measure \( \mu \) with the set \( A_n^+ \) attached to every sample \( \phi \in \Omega \). The measure \( \tilde{\mu}_n \) is thus a probability measure on the measurable space \((\Omega \times S, \mathcal{G}_n)\).
Moreover, the distribution of $\phi|\Pi_n$ is the same in $\mu$ as it is in $\tilde{\mu}_n$, and the set $A_n^+$ depends deterministically on $\phi|\Pi_n$. Therefore

$$
\mathcal{H}_{\Pi_n}(\mu|\Phi) = \mathcal{H}_{\mathcal{F}\Xi_n}\left(\mu \left| e^{-H_{\text{ref}}_{\Pi_n} \lambda_{\Pi_n}^{-1}} \right\right) = \mathcal{H}_{\mathcal{G}_n}\left(\tilde{\mu}_n \left| e^{-H_{\text{ref}}_{\Pi_n} \lambda_{\Pi_n}^{-1}} \right\right) \times \alpha \right) .
\tag{10.17}
$$

For the second stage, introduce a new measure $\tilde{\mu}_n$ on $(\Omega \times \mathcal{S}, \mathcal{G}_n)$. To sample from $\tilde{\mu}_n$, sample first a pair $(\phi, A)$ from $\tilde{\mu}_n$, then replace $\phi(x)$ by $\phi_n(x)$ for every $x \in A$. Remark that the distribution of $\phi|\Pi_n$ is the same in $\tilde{\mu}_n$ as it is in $\mu_n^+$. Write $A'$ for the set $\Pi_n \setminus A$. The entropies of $\tilde{\mu}_n$ and $\mu_n^+$ (relative to the reference measure in the final term of (10.17)) can be calculated in three steps. First, calculate the entropy of the choice of the set $A$. Second, calculate the entropy of the choice of the values of $\phi$ on $A'$. Third, calculate the entropy of the choice of the values of $\phi$ on $A$. In the construction of $\tilde{\mu}_n$ we only change the values of $\phi$ on $A$, and therefore the third step is the only step that produces a different entropy term. We have

$$
\mathcal{H}_{\mathcal{G}_n}\left(\tilde{\mu}_n \left| e^{-H_{\text{ref}}_{\Pi_n} \lambda_{\Pi_n}^{-1}} \right\right) - \mathcal{H}_{\mathcal{G}_n}\left(\tilde{\mu}_n \left| e^{-H_{\text{ref}}_{\Pi_n} \lambda_{\Pi_n}^{-1}} \right\right) \times \alpha
\tag{10.18}
$$

In these equations, $\delta$ denotes the Dirac measure, $\pi_A$ is the projection kernel onto $A$, and $\mu(A|\phi,A')$ denotes the original measure $\mu$ conditioned on seeing $A_\Pi^n = A$ and on the values of $\phi$ on the set $A'$. For the first term in (10.18) we observe that

$$
|\tilde{\mu}_n \left(\Pi_n \left(\phi \wedge \phi_n^+ \right) - \Pi_n \left(\phi \right)\right)| = O(\mu(A_n^+)) = o(n^d);
$$

this follows from the claim and (7.2)—noting that $\phi$ and $\phi \wedge \phi_n^+$ are $q$-Lipschitz. For $\tilde{\mu}_n$-a.e. $(\phi, A)$, we observe that the measure $\mu(A|\phi,A')$ produces $K$-Lipschitz height functions almost surely, and consequently the same measure—restricted to $A$—is supported on a set of cardinality at most $(2K + 1)^{|A|}$. Conclude that the second term in (10.18) is bounded absolutely by

$$
\mu(A_n^+) \log(2K + 1) = o(n^d).
$$

To sample from $\mu_n^+$, sample a pair $(\phi, A)$ from $\tilde{\mu}_n$, then simply forget about the set $A$. This is the third stage. Write $\nu_n$ for marginal of $\tilde{\mu}_n$ on $\mathcal{S}$. Then

$$
\mathcal{H}_{\Pi_n}(\mu_n^+|\Phi) + \mathcal{H}(\nu_n|\alpha) \leq \mathcal{H}_{\mathcal{G}_n}\left(\tilde{\mu}_n \left| e^{-H_{\text{ref}}_{\Pi_n} \lambda_{\Pi_n}^{-1}} \right\right) \times \alpha \right) \leq \mathcal{H}_{\Pi_n}(\mu_n^+|\Phi).
$$

Evidently $\mathcal{H}(\nu_n|\alpha) \leq 0$; the goal is to find a lower bound on $\mathcal{H}(\nu_n|\alpha)$. The measure $\nu_n$ is a probability measure on the set of subsets of $\Pi_n$ and we also know that $\nu_n(|A|) = \mu(|A_n^+|)$. The entropy of $\nu_n$ is minimized (among all probability measures with these two properties) if one samples from $\nu_n$ by flipping a coin independently for every vertex $x \in \Pi_n$ to determine if $x \in A$. The Bernoulli parameter of the coin is $\mu(|A_n^+|)/n^d$ so that $\nu_n(|A|) = \mu(|A_n^+|)$. Write $f(p) = p \log p + (1 - p) \log(1 - p)$, the entropy of a Bernoulli trial with parameter $p$. Then the entropy of the entropy-minimizing measure is $n^d f(\mu(|A_n^+|)/n^d)$. Now $\lim_{n \to \infty} f(p) = 0$ and therefore $\mathcal{H}(\nu_n|\alpha) = o(n^d)$. Conclude that the assertion holds true, that is,

$$
\mathcal{H}_{\Pi_n}(\mu_n^+|\Phi) = \mathcal{H}_{\Pi_n}(\mu|\Phi) + o(n^d).
$$

The measure $\mu_n^\pm$ is now obtained from $\mu_n^+$ by applying a lower truncation. To sample from $\mu_n^\pm$, sample first a height function $\phi$ from $\mu_n^+$, then replace $\phi(x)$ by $\phi_n^-(x)$ for any
Again, we present an alternative three-stage construction of \( \phi \) value the entropy does not increase by more than and the randomly truncated function \( \phi \) for each

\[(10.16)\]

Let the extension pyramid-shaped in the sense of(10.16) because

\[\mu \]

Note that this means that the measures \( \mu \) and because \( \varphi \). The measure \( \mu \) and \( \varphi \) is supported on \( C_{\varphi}^u \) and satisfy \( H_{\Pi_n}(\mu_n|\Phi) \leq H_{\Pi_n}(\mu|\Phi) + o(n^d) \) as \( n \to \infty \). This concludes the proof for \( E = \mathbb{Z} \).

The continuous case. Fix \( \varepsilon > 0 \) so small that \( \varphi \) is \( q_{L} \)-Lipschitz, and pick \( B \in \mathbb{B} \) with \( \mu \in B \). Assume a choice of \( B \) and \( B^* \) as for the discrete case. It suffices to find a sequence of measures \( (\mu_n)_{n \in \mathbb{N}} \) with \( \mu_n \) supported on \( C_{\varphi}^u \) and with \( H_{\Pi_n}(\mu_n|\Phi) \leq H_{\Pi_n}(\mu|\Phi) + o(n^d) \). Write \( \varphi \) for the largest and smallest \( q_{L} \)-Lipschitz extensions of \( \varphi_{\mu|\Pi_n} \) to \( \Pi_n \) respectively, for each \( n \in \mathbb{N} \). Take again 0 as reference vertex for the gradient setting (as for the discrete case), and define the random sets

\[A_n^- := \{ x \in \Pi_n : \varphi(x) < \varphi_n^-(x) - 2\varepsilon \} \quad \text{and} \quad A_n^+ := \{ x \in \Pi_n : \varphi(x) > \varphi_n^+(x) + 2\varepsilon \}.\]

Note that \( \varphi_n^+(0) = 0 \) by definition, and therefore \( \mu \)-almost surely \( 0 \notin A_n^+ \). Observe that \( \mu(|A_n^+|) = o(n^d) \) by arguments identical to the case \( E = \mathbb{Z} \); one can show that \( \varphi_n^+ \) is pyramid-shaped in the sense of(10.16) because \( \varphi \) is \( q_{L} \)-Lipschitz and because we chose the extension \( \varphi_n^+ \) to be the largest \( q_{L} \)-Lipschitz extension.

For each \( n \in \mathbb{N} \) we construct a new measure \( \mu_n^+ \) on \( (\Omega, \mathcal{F}_{\Pi_n}) \), the upper truncation of \( \mu \). Let \( (X(x))_{x \in \mathbb{Z}^d} \) be a process of i.i.d. random variables, uniformly random in the interval \([0,\varepsilon]\), in some new measure \( \nu \). To sample from \( \mu_n^+ \), first sample \( (\varphi, X) \) from \( \mu \times \nu \). Then, for each \( x \in A_n^+ \), replace \( \varphi(x) \) by \( \varphi_n^+(x) + X(x) \). Figure 3 displays the original function \( \varphi \) and the randomly truncated function \( \psi \); the upper truncation is located on the right hand side, a lower truncation (which is defined at a later stage) occurs on the left. The new measure \( \mu_n^+ \) is absolutely continuous with respect to \( \lambda_{\Pi_n}^{-1} \) because we replaced each value \( \varphi(x) \) by a continuously distributed random variable. Assert that

\[H_{\Pi_n}(\mu_n^+|\Phi) \leq H_{\Pi_n}(\mu|\Phi) + o(n^d). \quad (10.19)\]

Again, we present an alternative three-stage construction of \( \mu_n^+ \), and we demonstrate that the entropy does not increase by more than \( o(n^d) \) at every stage.

Figure 3: The random truncation for \( E = \mathbb{R} \). The randomly truncated sample \( \psi \) remains between \( \phi_n^- - 2\varepsilon \) and \( \phi_n^+ + 2\varepsilon \).
For the first stage, write $\tilde{\mu}_n$ for the measure $\mu$ with the set $A^+_n$ attached to every sample $\phi \in \Omega$. Define $\alpha$ and $G_n$ as before. The measure $\tilde{\mu}_n$ is a probability measure on $(\Omega \times S, G_n)$. Note that (10.17) holds for this measure as $A^+_n$ depends deterministically on $\phi_{\Pi_n}$.

For the second stage, introduce a new measure $\tilde{\mu}_n^+$ on $(\Omega \times S, G_n)$. To sample from $\tilde{\mu}_n^+$, sample first a triple $(\phi, A, X)$ from $\tilde{\mu}_n \times \nu$, then replace $\phi(x)$ by $\phi^+_n(x) + X(x)$ for every $x \in A$. Write $A'$ for $\Pi_n \setminus A$. Write $\psi$ for the function on $\Pi_n$ defined by $\psi_A = \phi^+_n|_A + X_A$ and $\tilde{\psi}_{A'} = \tilde{\phi}_{A'}$. One calculates that $\tilde{\mu}_n$ and $\tilde{\mu}_n^+$ as in the discrete case to deduce that

$$\mathcal{H}_{G_n} \left( \tilde{\mu}_n^+ \left| e^{-H^{0}_{\tilde{\mu}_n} \lambda^{\Pi_n-1}} \right) \times \alpha \right) - \mathcal{H}_{G_n} \left( \tilde{\mu}_n \left| e^{-H^{0}_{\tilde{\mu}_n} \lambda^{\Pi_n-1}} \right) \times \alpha \right)$$

$$= \int \left( \mathcal{H} \left( \left( \nu + \phi_n^+ \right) \tau_A \right) - \mathcal{H} \left( \mu^{(A,\phi^+)} \tau_A \right) \right) d\tilde{\mu}_n(\phi, A)$$

$$= \tilde{\mu}_n(\psi_{A,\Pi_n}) - \tilde{\mu}_n(\Pi_n) \log \varepsilon - \int \mathcal{H} \left( \mu^{(A,\phi^+)} \tau_A \right) d\tilde{\mu}_n(\phi, A).$$

In these equations, $\mu^{(A,\phi^+)}$ denotes the original measure $\mu$ conditioned on seeing $A^+_n = A$ and on the values of $\phi$ on the set $A'$. By $\nu + \phi_n^+$ we simply mean the measure obtained by shifting each sample $X$ from $\nu$ by $\phi_n^+$. As in the discrete setting, the last two terms have an upper bound of order $o(n^d)$ as $n \to \infty$. It suffices to find an appropriate upper bound for the first term in the final expression.

Let $(\mathbb{A}, q)$ denote the local Lipschitz constraint. By Proposition 6.5, it is possible to find a constant $0 < \varepsilon' \leq \varepsilon$, such that for any $\{x, y\} \in \mathbb{A}$, we have

$$q_{\varepsilon'}(x, y) \geq q(x, y) - \varepsilon.$$

Claim that $\tilde{\mu}_n \times \nu$-almost surely, $\psi$ is $q_{\varepsilon'}$-Lipschitz at every $x \in A$. In other words, we claim that

$$-q_{\varepsilon'}(y, x) \leq \psi(y) - \psi(x) \leq q_{\varepsilon'}(x, y)$$

(10.20)

whenever $x \in A$, $y \in \Pi_n$, and $\{x, y\} \in \mathbb{A}$. Suppose first that $y \in A$. The function $\phi_n^+$ is $q_{3\varepsilon}$-Lipschitz and $0 \leq (\psi - \phi_n^+)(x, y) = X(x, y) \leq \varepsilon$ for $x, y \in A$, and therefore (10.20) holds true with $\varepsilon'$ replaced by $\varepsilon$. But $q_{\varepsilon} \leq q_{\varepsilon'}$, which implies (10.20) without said replacement. Now suppose that $y \not\in A$, so that $\psi(y) - \psi(x) = \phi(y) - \phi_n^+(x) - X(x)$. For the righthand inequality of (10.20) we have (almost surely)

$$\phi(y) - \phi_n^+(x) - X(x) \leq (\phi_n^+(y) + 2\varepsilon) - \phi_n^+(x) \leq q_{3\varepsilon}(x, y) + 2\varepsilon \leq q_{\varepsilon}(x, y) \leq q_{\varepsilon'}(x, y).$$

For the inequality on the left we see that (using $\phi_n^+(x) + X(x) \leq \phi(x) - \varepsilon$ for the first inequality)

$$\phi(y) - \phi_n^+(x) - X(x) \geq \phi(y) - \phi(x) + \varepsilon \geq -q(y, x) + \varepsilon \geq -q_{\varepsilon'}(y, x).$$

The middle inequality in this equation is due to the fact that $\phi$ is $\mu$-almost surely $q$-Lipschitz. This proves the claim.

By the claim and (7.3), we have

$$\tilde{\mu}_n \times \nu \left( H_{A,\Pi_n}(\psi) \right) \leq O(\mu(|A^+_n|)) = o(n^d).$$

For the other Hamiltonian we simply observe that

$$\tilde{\mu}_n \times \nu \left( H_{A,\Pi_n}(\phi) \right) = \tilde{\mu}_n \left( H_{A,\Pi_n}(\phi) \right) \geq -\|\Xi\|\mu(|A^+_n|)) = o(n^d).$$
Putting all estimates together, we see that
\[
\mathcal{H}_n \left( \tilde{\mu}_n^+ \left( \left( e^{-H_n^0 \lambda^{n-1}} \right) \times \alpha \right) \right) \leq \mathcal{H}_n(\mu|\Phi) + o(n^d).
\]
To prove the original assertion, simply observe that, as in the discrete case, forgetting about the information encoded in the set \( A \) changes the entropy of \( \tilde{\mu}_n^+ \) by no more than \( o(n^d) \):
\[
\mathcal{H}_n(\mu_n^+|\Phi) = \mathcal{H}_n \left( \tilde{\mu}_n^+ \left( \left( e^{-H_n^0 \lambda^{n-1}} \right) \times \alpha \right) \right) + o(n^d).
\]
This proves the assertion (10.19).

Finally one constructs a lower truncation \( \mu_n^+ \) from \( \mu_n^+ \). To sample from \( \mu_n \), one first samples \( \phi \) from \( \mu_n^+ \). Then, for every \( x \in A_n^+ \), one resamples \( \phi(x) \) independently and uniformly at random from the interval \([\phi_n^-(x) - \varepsilon, \phi_n^+(x)]\). As before, we have
\[
\mathcal{H}_n(\mu_n^+|\Phi) \leq \mathcal{H}_n(\mu_n^+|\Phi) + o(n^d) \leq \mathcal{H}_n(\mu|\Phi) + o(n^d).
\]
Now \( \phi_n^+ - 2\varepsilon \leq \phi_n^+ \leq \phi_n^+ + 2\varepsilon \) almost surely in the measure \( \mu_n^\pm \) and this implies in particular that
\[
\phi_n^\pm \sim \phi_n^\pm - 2\varepsilon \leq \phi_n^\pm \leq \phi_n^\pm + 2\varepsilon,
\]
that is, \( \mu_n^\pm \) is supported on \( C_n \). Since \( \mu(B_n^+) \to 1 \) and \( \mu(|A_n^\pm|) = o(n^d) \) as \( n \to \infty \), we have \( \mu_n^\pm(B_n) \to 1 \) as \( n \to \infty \). This proves that the sequence \( \mu_n := \mu_n^\pm(\cdot|B_n) \) has the desired properties.

We now proceed as for free boundary limits, and define pinned boundary limits over other Van Hove sequences.

**Definition 10.21.** Fix \( D \in \mathcal{C} \) and \( m \in \mathbb{Z}_{\geq 0} \), and write \( \Lambda_n \) for \( \Lambda^{-m}(nD) = (nD \cap \mathbb{Z}^d)^{-m} \). Consider \( \mu \in \mathcal{P}_n(\Omega, \mathcal{F}) \). If \( E = \mathbb{Z} \), then define
\[
PB(\mu : D, m) := \sup_{B \in \mathcal{B} \text{ with } \mu \in B} \limsup_{n \to \infty} n^{-d} PB_{\Lambda_n, S(\mu)}(B),
\]
and if \( E = \mathbb{R} \), then define
\[
PB(\mu : D, m) := \sup_{\varepsilon > 0} \limsup_{n \to \infty} n^{-d} PB_{\Lambda_n, S(\mu), \varepsilon}(B).
\]
Finally, write \( PB^*(\mu) := \sup_{(D, m) \in \mathcal{C} \times \mathbb{Z}_{\geq 0}} PB(\mu : D, m) / \text{Leb}(D) \).

It is immediate that \( PB^*(\mu) \geq PB(\mu) \) because one can take \( D = [0, 1]^d \) and \( m = 0 \) in the supremum in this new definition. By reordering the suprema in the definitions, it is also clear that \( PB^* \) is lower-semicontinuous on the set \( \{ S(\cdot) = u \} \subset \mathcal{P}_n(\Omega, \mathcal{F}) \) for any \( u \).

**Lemma 10.22.** Consider \( D \in \mathcal{C}, m \in \mathbb{Z}_{\geq 0} \), and \( \mu \in \mathcal{P}_n(\Omega, \mathcal{F}) \) ergodic with \( S(\mu) \in U_\Phi \). Then
\[
PB(\mu : D, m) \leq \text{Leb}(D) \cdot \mathcal{H}(\mu|\Phi).
\]
In other words, \( PB^*(\mu) \leq \mathcal{H}(\mu|\Phi) \).

**Proof.** Write \( u := S(\mu) \) and \( \Lambda_n := \Lambda^{-m}(nD) \), and fix \( B \in \mathcal{B} \) with \( \mu \in B \). The truncation argument in the proof of Lemma 10.14 implies that \( PB_{\Lambda_n, u}(B) \leq \mathcal{H}_{\Lambda_n}(\mu|\Phi) + o(n^d) \) as \( n \to \infty \) if \( E = \mathbb{Z} \), and \( PB_{\Lambda_n, u, \varepsilon}(B) \leq \mathcal{H}_{\Lambda_n}(\mu|\Phi) + o(n^d) \) as \( n \to \infty \) for any \( \varepsilon > 0 \) if \( E = \mathbb{R} \). Therefore it suffices to demonstrate that
\[
\limsup_{n \to \infty} n^{-d} \mathcal{H}_{\Lambda_n}(\mu|\Phi) \leq \text{Leb}(D) \cdot \mathcal{H}(\mu|\Phi).
\] (10.23)
Consider also some finite family $D$ without loss of generality, we suppose that when constructing the contradiction which leads to a proof of strict convexity of the surface.

The purpose of this subsection is to demonstrate that one appeals to the following result.

Proposition 10.25.

Consider some set $\phi \in C(\Omega, F_N)$ with $S(\phi) \in U_\Phi$. The previous subsection proved this for $\phi$ ergodic. First, we demonstrate that $\text{PB}^*(\mu) \leq \mathcal{H}(\mu|\Phi)$ for any shift-invariant random field $\mu$ with $S(\mu) \in U_\Phi$. The idea is then to use lower-semicontinuity of $\text{PB}^*$ in the topology of weak local convergence to derive the inequality for all non-ergodic measures (Lemma 10.33). Extra care must be taken whenever $E = \mathbb{Z}$, because in that case there exist ergodic measures with finite specific free energy which have their slope in $\partial U_\Phi$ rather than $U_\Phi$. This pathology is dealt with in Lemma 10.32.

Lemma 10.24. The functional $\text{PB}^*$ is convex.

Consider $\nu_1, \nu_2 \in \mathcal{P}_\mathbb{C}(\Omega, F_N)$ with $S(\nu_1), S(\nu_2) \in U_\Phi$, and define $\mu := (1-t)\nu_1 + t\nu_2$ for some $t \in (0, 1)$. If we take the value of $\text{PB}^*(\nu_1)$ and $\text{PB}^*(\nu_2)$ for granted, then we look for an upper bound on $\text{PB}^*(\mu)$. This means that we look for asymptotic lower bounds on the integrals defining the pinned boundary estimates of $\mu$.

The proof of the lemma uses a general strategy which produces an asymptotic lower bound on this particular integral, and which is used again twice in this article: in the lower bound on probabilities in the proof of the large deviations principle in Subsection 11.4, and when constructing the contradiction which leads to a proof of strict convexity of the surface tension in Subsection 12.2. The general idea is as follows: Lemma 10.14, and later (once it is proven) Lemma 10.33, provide the fundamental building blocks for the lower bounds. One then shows that these building blocks can be put together without gaining too much energy, that is, without decreasing the value of the integral of interest by too much. For this, one appeals to Theorem 6.9, which allows one to find suitable discrete approximations of continuous Lipschitz profiles, and the upper attachment lemma (Lemma 7.4), which allows one to bound the energy increase due to combining height functions defined on different parts of $\mathbb{Z}^d$. This is already sufficient to understand the macroscopic shape of the height functions. In the context of boundary limits, this is expressed through the pinning of the height functions on the boundary $\partial \Lambda_\Phi$ of the set $\Lambda$ of interest—essentially by restricting to the set $C^\Lambda_\Phi$ or $C^\Lambda_{\Lambda_\Phi}$. It is, however, also necessary to understand the behavior of the local statistics of the height functions—expressed in the boundary limits through the sets $B_\Lambda$—under the operation of putting together the fundamental building blocks. For this, one appeals to the following result.

Proposition 10.25. Consider some set $D \in \mathcal{C}$ and a nonnegative integer $m \in \mathbb{Z}_{\geq 0}$. Consider also some finite family $(D_i, m_i)_i \subset \mathcal{C} \times \mathbb{Z}_{\geq m}$ with the sets $D_i$ disjoint and contained in $D$. Write $\Lambda_n := \Lambda^{-m}(nD)$ and $\Lambda_n^i := \Lambda^{-m}(nD_i)$. Then for any cylinder function $f : \Omega \to [0, 1]$, we have

$$
\limsup_{n \to \infty} \sup_{\phi \in \Omega} \left| \int_{\Lambda_n^i} \phi - \sum_i \frac{\text{Leb}(D_i)}{\text{Leb}(D)} \int_{\Lambda_n^i} \phi \right| \leq \frac{\text{Leb}(D \setminus \bigcup_i D_i)}{\text{Leb}(D)}.
$$
Proof. Note that
\[
\frac{|\mathcal{L} \cap \Lambda^i_n|}{|\mathcal{L} \cap \Lambda_n|} \rightarrow \text{Leb}(D_1) / \text{Leb}(D)
\]
as \(n \rightarrow \infty\), and therefore it suffices to prove the proposition for the latter fraction replaced by the former. Suppose that \(f\) is \(\mathcal{F}_\Delta\)-measurable for some \(\Delta \subset \mathbb{Z}^d\) which contains 0. Write \(\mathbb{P}_n\) for the uniform probability measure on \(\{\theta_x : x \in \mathcal{L} \cap \Lambda_n\}\). By coupling the measures in the obvious way, we observe that
\[
L_{\Lambda_n}(\phi)(f) - \sum_i \frac{|\mathcal{L} \cap \Lambda^i_n|}{|\mathcal{L} \cap \Lambda_n|} L_{\Lambda^i_n}(\phi)(f) = \mathbb{E}_n(g_n)
\]
where \(g_n\) is defined by
\[
g_n(\theta) = \begin{cases} 
 0 & \text{if } \theta \Delta \subset \Lambda^i_n \text{ for some } i, \\
 2 & \text{if } \theta \in \Lambda^i_n \text{ for some } i \text{ but } \theta \Delta \not\subset \Lambda^i_n, \\
 f(\theta) & \text{otherwise.}
\end{cases}
\]
Now \(|g_n| \leq 1\) and \(\mathbb{P}_n(\theta \Delta \not\subset \Lambda^i_n \text{ for any } i) = \text{Leb}(D \setminus \cup_i D_i) / \text{Leb}(D) + o(1)\) as \(n \rightarrow \infty\), which implies the proposition. \(\square\)

The particular proof of Lemma 10.24 utilizes the so-called washboard construction (see Figure 4), which appears in the work of Sheffield [She05], and is adapted here to the particular Lipschitz setting.

Proof of Lemma 10.24. Consider \(\mu := s\nu_1 + t\nu_2\) for \(s, t \in (0, 1)\) with \(s + t = 1\) and for some measures \(\nu_1, \nu_2 \in \mathcal{P}_C(\Omega, \mathcal{F}_\Omega)\) which have their slope in \(U_{\Phi}\). The goal is to prove that \(\text{PB}(\mu) \leq s \text{PB}(\nu_1) + t \text{PB}(\nu_2)\).

Write \(u := S(\mu), u_1 := S(\nu_1),\) and \(u_2 := S(\nu_2)\). Consider \(D \in \mathcal{C}, m \in \mathbb{Z}_{\geq 0},\) and \(B \in \mathcal{B}\) with \(\mu \in B\). Write \(\Lambda_n := \Lambda^{-m}(nD)\). Fix also some \(\varepsilon > 0\). If \(E = \mathbb{Z}\), then we must show that
\[
\limsup_{n \rightarrow \infty} n^{-d} \text{PB}_{\Lambda_n, u}(B) \leq \text{Leb}(D)(s \text{PB}(\nu_1) + t \text{PB}(\nu_2)),
\]
and if \(E = \mathbb{R}\), then we must show that
\[
\limsup_{n \rightarrow \infty} n^{-d} \text{PB}_{\Lambda_n, u, \varepsilon}(B) \leq \text{Leb}(D)(s \text{PB}(\nu_1) + t \text{PB}(\nu_2)).
\]
By choosing \(\varepsilon > 0\) smaller if necessary, we suppose that \(u, u_1, u_2 \in U_{q\varepsilon}\). By choosing \(B\) smaller if necessary, we suppose that \(B\) is of the form
\[
B = \{\pi : |\mu(f_i) - \pi(f_i)| < 2\eta \text{ for all } i \in \mathcal{B}\}
\]
for some finite family \((f_i)_{i}\) of continuous cylinder functions \(f_i : \Omega \rightarrow [0, 1]\) and for some \(\eta > 0\), and we write
\[
B^j := \{\pi : |\nu_j(f_i) - \pi(f_i)| < \eta \text{ for all } i \in \mathcal{B}\}
\]
for \(j \in \{1, 2\}\).

The idea of the proof is roughly as follows. First, we partition a large subset of \(D\) into finitely many convex shapes. Second, we find a continuous Lipschitz function \(f\) which equals \(u\) on \(\partial D\), and which is affine on each convex shape in this partition, with slope either \(u_1\) or \(u_2\). This function is chosen such that the Lebesgue measure of the convex shapes with slope \(u_j\) is roughly \(s \text{Leb}(D)\) for \(j = 1\) and roughly \(t \text{Leb}(D)\) for \(j = 2\). Informally, the
function $f$ looks like a “washboard”. Next, we define $f_n := nf(\cdot/n)$, and use the existence of the function $f_n$ and Theorem 6.9 to find for each $n \in \mathbb{N}$ a corresponding height function $\phi_n$. The existence of the height function $\phi_n$ and the previously described general strategy allow us to build a direct comparison between $\text{PB}_\Lambda(A, u)(\mathcal{B})$ or $\text{PB}_\Lambda(A, u, \varepsilon)(\mathcal{B})$ and the numbers $\text{PB}^*(\nu_1)$ and $\text{PB}^*(\nu_2)$.

We start by constructing the continuous “washboard”—see Figure 4. Set $v := u_1 - u_2$ if $u_1 \neq u_2$, and choose $v \in (\mathbb{R}^d)^* \setminus \{0\}$ arbitrary otherwise. Define $w : \mathbb{R}^d \to \mathbb{Z}$, $x \mapsto \begin{cases} 2 [v(x)] & \text{if } v(x) - \lfloor v(x) \rfloor \in [0, t), \\ 2 [v(x)] + 1 & \text{if } v(x) - \lfloor v(x) \rfloor \in [t, 1). \end{cases}$

Write $p : \mathbb{R}^d \to \mathbb{R}$ for the unique continuous function that maps 0 to 0, and which has gradient $u_j$ on the interior of $\{w \in 2\mathbb{Z} + j\} \subset \mathbb{R}^d$ for $j \in \{1, 2\}$. For $\alpha > 0$, write $w_\alpha$ for the map $w(\cdot/\alpha)$, and write $p_\alpha$ for the map $p_\alpha(\cdot) := \alpha p(\cdot/\alpha)$. It is straightforward to see that $p_\alpha$ has gradient $u_j$ on $\{w_\alpha \in 2\mathbb{Z} + j\}$ for $j \in \{1, 2\}$, and that $\|p_\alpha - u\|_\infty \propto \alpha$. Observe also that $p$ and $p_\alpha$ are $\|\cdot\|_{\text{lip}}$-Lipschitz.

In the remainder of the proof, we shall work with three limits. First we take $n \to \infty$, then $\varepsilon_2 \to 0$, then $\varepsilon_1 \to 0$. Reference to these variables is sometimes omitted for brevity. Define $\partial' D := \{x \in \mathbb{R}^d : d_2(x, \partial D) < \varepsilon_1\}$, $D' := \{x \in D : d_2(x, \partial D) > 2\varepsilon_1\}$, $D'_k := D' \cap \{w_{\varepsilon_2} = k\}$, where $d_2$ denotes Euclidean distance. Write $f : \partial' D \cup D' \to \mathbb{R}$ for the function defined by $f(x) := \begin{cases} u(x) & \text{if } x \in \partial' D, \\ p_{\varepsilon_2}(x) & \text{if } x \in D'. \end{cases}$

This function is $\|\cdot\|_{\text{lip}_{\varepsilon_2}}$-Lipschitz for $\varepsilon_2$ sufficiently small, depending on $\varepsilon_1$. Note that $f$ is affine with gradient $u_1$ on $D'_k$ for $k$ odd and with gradient $u_2$ on $D'_k$ for $k$ even. Moreover,
the family $\{D'_k\}_{k \in \mathbb{Z}}$ is a partition of $D'$. Only finitely many members are nonempty, and the nonempty members are convex, bounded, and have positive Lebesgue measure. The merit of this construction is that
\begin{align}
\text{Leb}(\bigcup_{k \in 2\mathbb{Z}+1} D'_k) &\to_{\varepsilon_2 \to 0} s \text{Leb}(D') \to_{\varepsilon_1 \to 0} s \text{Leb}(D), \quad (10.26) \\
\text{Leb}(\bigcup_{k \in 2\mathbb{Z}} D'_k) &\to_{\varepsilon_2 \to 0} t \text{Leb}(D') \to_{\varepsilon_1 \to 0} t \text{Leb}(D). \quad (10.27)
\end{align}

For $n \in \mathbb{N}$, define $f_n : n(\partial' D \cup D') \to \mathbb{R}$ by $f_n(\cdot) := n f(\cdot/n)$—this function is also $\| \cdot \|_{q_u}$-Lipschitz. In particular, Theorem 6.9 implies that for some $M \in \mathbb{Z}_{\geq m}$ depending only on $\varepsilon$, there exists a $q$-Lipschitz height function $\phi_n \in \Omega$ such that
\begin{enumerate}
  \item $\nabla \phi_n|_{\Lambda^{-M}(n\partial'D)} = \nabla \phi^u|_{\Lambda^{-M}(n\partial'D)}$,
  \item $\nabla \phi_n|_{\Lambda^{-M}(n\partial'_k)} = \nabla \phi^u|_{\Lambda^{-M}(n\partial'_k)}$ for all $k$ odd,
  \item $\nabla \phi_n|_{\Lambda^{-M}(n\partial'_k)} = \nabla \phi^{u2}|_{\Lambda^{-M}(n\partial'_k)}$ for all $k$ even,
  \item $\phi_n$ is $q_u$-Lipschitz if $E = \mathbb{R}$.
\end{enumerate}

Recall the definition of $\Lambda_n$, and define
\[ \Lambda_{n,k} := \Lambda^{-M}(nD'_k), \quad \Lambda_n^0 := \Lambda_n \setminus \cup_k (\Lambda_{n,k} \setminus \{0\Lambda_{n,k}\}), \quad \Lambda_n^* := \Lambda_n \setminus \cup_k \Lambda_{n,k}^{-R}. \]

Note that $\partial R\Lambda_n \subset \Lambda^{-M}(n\partial'D)$ for $n$ sufficiently large, and consequently $\nabla \phi_n|_{\partial R\Lambda_n} = \nabla \phi^u|_{\partial R\Lambda_n}$. This also implies that the sets $\partial R\Lambda_n$ and $\Lambda_{n,k}$ are all disjoint for fixed $n$ as $k$ ranges over $\mathbb{Z}$. Finally, $\Lambda_{n,k} \subset \Lambda_n$ for all $k$.

The idea is now to use the existence of the function $\phi_n$ to derive the inequalities. We distinguish two cases, depending on whether $E = \mathbb{Z}$ or $E = \mathbb{R}$. Start with the former, which is easier. Write $A_n$ for the set of height functions $\phi$ such that
\begin{enumerate}
  \item $\nabla \phi$ equals $\nabla \phi_n$ on $\Lambda_n^*$,
  \item $\phi \in B_{\Lambda_{n,k}}^1$ for all $k$ odd,
  \item $\phi \in B_{\Lambda_{n,k}}^2$ for all $k$ even.
\end{enumerate}

Note that $A_n \subset C_{\Lambda_n}^u$ because $\partial R\Lambda_n \subset \Lambda_n^*$ and because $\nabla \phi_n = \nabla \phi^u$ on $\partial R\Lambda_n$. It is straightforward to work out that $A_n \subset B_{\Lambda_n}$ for $n$ sufficiently large and $\varepsilon_1, \varepsilon_2$ sufficiently small, by application of Proposition 10.25 combined with (10.26) and (10.27).

Therefore it suffices to demonstrate that
\begin{equation}
\liminf n^{-d} \log \int_{A_n} e^{-H^0_{\Lambda_n}} d\lambda^{\Lambda_n-1} \geq - \text{Leb}(D)(s \text{PB}^*(\nu_1) + t \text{PB}^*(\nu_2)) \quad (10.28)
\end{equation}

where the limit is in the variables $n, \varepsilon_2$, and $\varepsilon_1$. Moreover, since $\nabla \phi$ equals $\nabla \phi_n$ on $\Lambda_n^*$ for any $\phi \in A_n$, this restriction to $\Lambda_n^*$ is $q$-Lipschitz, and the upper attachment lemma (Lemma 7.4) implies that
\begin{equation}
H^0_{\Lambda_n} \leq \sum_k H^0_{\Lambda_{n,k}} + \sum_k e^+(\Lambda_{n,k}) + |\Lambda_n \setminus \cup_k \Lambda_{n,k}| \max_{x \in \mathbb{R}/L} e^+(\{x\}) \quad (10.29)
\end{equation}
on $A_n$. For the third term we have $n^{-d}|\Lambda_n \setminus \cup_k \Lambda_{n,k}| \to_{n \to \infty} \text{Leb}(D \setminus D') \to_{\varepsilon_1 \to 0} 0$, and the second term is of order $o(n^d)$ as $n \to \infty$. This implies that
\begin{equation}
\liminf n^{-d} \log \int_{A_n} e^{-H^0_{\Lambda_n}} d\lambda^{\Lambda_n-1} \geq \liminf n^{-d} \log \int_{A_n} e^{-\sum_k H^0_{\Lambda_{n,k}}} d\lambda^{\Lambda_n-1}. \quad (10.30)
\end{equation}
Recall the definition of $\Lambda_n^0$, and consider $\Lambda_n^{\lambda n-1}$ a product measure, by writing
\[
\Lambda_n^{\lambda n-1} := \Lambda_n^{\lambda n-1} \times \prod_k \Lambda_n^{\lambda n,k-1}.
\]

Note that $A_n$ contains exactly all height functions $\phi$ such that
1. $\nabla \phi$ equals $\nabla \phi_n$ on $A_n^0$,
2. $\phi \in C_{\Lambda_n,k}^{\Lambda_n,k} \cap B_{\Lambda_n,k}^1$ for all $k$ odd,
3. $\phi \in C_{\Lambda_n,k}^{\Lambda_n,k} \cap B_{\Lambda_n,k}^2$ for all $k$ even,
and therefore
\[
\int_{A_n} e^{-\sum_k H_{n,k}^0} d\Lambda_n^{\lambda n-1} = \int \{\nabla \phi \text{ equals } \nabla \phi_n \text{ on } A_n^0\} d\Lambda_n^{\lambda n-1}(\phi) \cdot \prod_{k \in 2Z+1} \int_{C_{\Lambda_n,k}^{\Lambda_n,k} \cap B_{\Lambda_n,k}^1} e^{-H_{n,k}^0} d\Lambda_n^{\lambda n,k-1} \cdot \prod_{k \in 2Z} \int_{C_{\Lambda_n,k}^{\Lambda_n,k} \cap B_{\Lambda_n,k}^2} e^{-H_{n,k}^0} d\Lambda_n^{\lambda n,k-1}.
\]

The first factor equals one since we are dealing with the counting measure, and therefore
\[
\log \int_{A_n} e^{-\sum_k H_{n,k}^0} d\Lambda_n^{\lambda n-1} = - \sum_{k \in 2Z+1} \text{PB}_{\Lambda_n,k} u_1(\mathcal{B}^1) - \sum_{k \in 2Z} \text{PB}_{\Lambda_n,k} u_2(\mathcal{B}^2).
\]

For fixed $\varepsilon_1, \varepsilon_2$ only finitely many terms are possibly nonzero—those corresponding to nonempty sets $D_k^j$—and for each term we have (for $j \in \{1, 2\}$)
\[
\limsup_{n \to \infty} n^{-d} \text{PB}_{\Lambda_n,k} u_j(\mathcal{B}^j) \leq \text{PB}(\nu_j : D_k^j, M) \leq \text{Leb}(D_k^j) \text{PB}^*(\nu_j).
\]

Therefore (10.26) and (10.27) imply
\[
\liminf_{n \to \infty} n^{-d} \log \int_{A_n} e^{-H_{\Lambda,n}^0} d\Lambda_n^{\lambda n-1} \geq - \text{Leb}(D)(s \text{PB}^*(\nu_1) + t \text{PB}^*(\nu_2)),
\]

the desired inequality.

Let us now discuss what changes for $E = \mathbb{R}$. Write $A_n$ for the set of samples $\phi$ such that
1. $|\phi_{\Lambda_n} - \phi(0_{\Lambda_n}) - (\phi_n|_{\Lambda_n} - \phi_n(0_{\Lambda_n}))| \leq \varepsilon$,
2. $\phi \in C_{\Lambda_n,k}^{\Lambda_n,k} \cap B_{\Lambda_n,k}^1$ for all $k$ odd,
3. $\phi \in C_{\Lambda_n,k}^{\Lambda_n,k} \cap B_{\Lambda_n,k}^2$ for all $k$ even.

Note that $A_n \subset C_{\Lambda_n,k}^{\Lambda_n,k}$. The proof that $A_n \subset B_{\Lambda_n}^1$ is the same as before. We must again prove (10.28). The definition of $A_n$ implies that $|\phi_{\Lambda_n} - \phi(0_{\Lambda_n}) - (\phi_n|_{\Lambda_n} - \phi_n(0_{\Lambda_n}))| \leq 2\varepsilon$ for any $\phi \in A_n$, which in turn implies that $\phi_{\Lambda_n}^\ast$ is $q_\varepsilon$-Lipschitz as $\phi_n$ was $q_{5\varepsilon}$-Lipschitz—see Proposition 6.5. Therefore (10.29) holds true with $e^+ (\cdot)$ replaced by $e_\varepsilon^+ (\cdot)$, which implies (10.30). We now have
\[
\int_{A_n} e^{-\sum_k H_{n,k}^0} d\Lambda_n^{\lambda n-1} = \int \{(\phi_{\Lambda_n}^0 - \phi(0_{\Lambda_n})) - (\phi_n|_{\Lambda_n}^0 - \phi_n(0_{\Lambda_n}))| \leq \varepsilon\} d\Lambda_n^{\lambda n-1} \cdot \prod_{k \in 2Z+1} \int_{C_{\Lambda_n,k}^{\Lambda_n,k} \cap B_{\Lambda_n,k}^1} e^{-H_{n,k}^0} d\Lambda_n^{\lambda n,k-1} \cdot \prod_{k \in 2Z} \int_{C_{\Lambda_n,k}^{\Lambda_n,k} \cap B_{\Lambda_n,k}^2} e^{-H_{n,k}^0} d\Lambda_n^{\lambda n,k-1}.
\]
The first integral equals \((2\varepsilon)^{|\Lambda_n^0| - 1}\), and therefore
\[
\log \int_{A_n} e^{-\sum_k H_{n,k}^0 d\lambda_{n-1}} = (|\Lambda_n^0| - 1) \log(2\varepsilon) - \sum_{k \in 2\mathbb{Z} + 1} \text{PBA}_{n,k,u_1,\varepsilon}(B^1) - \sum_{k \in 2\mathbb{Z}} \text{PBA}_{n,k,u_2,\varepsilon}(B^2).
\]

The first term vanishes in the limit in the three variables after normalizing by \(n^{-d}\). The remainder of the proof is the same as before. \(\square\)

Let us now discuss briefly how to deal with ergodic measures with finite specific free energy which have their slope in \(\partial U_\Phi\), before proving that \(\text{PB}^*(\mu) \leq \mathcal{H}(\mu|\Phi)\) for any shift-invariant random field \(\mu\) with \(S(\mu) \in U_\Phi\).

**Definition 10.31.** Consider a measure \(\mu \in \mathcal{P}_\mathcal{L}(\Omega, F^\mathbb{V})\) with finite specific free energy. Classify \(\mu\) as **taut** if \(w_{\mu^-}\)-almost surely \(S(\nu) \in \partial U_\Phi\), and as **non-taut** if \(w_{\mu^-}\)-almost surely \(S(\nu) \in U_\Phi\). A **non-taut approximation** of \(\mu\) is a sequence \((\mu_n)_{n \in \mathbb{N}} \subset \mathcal{P}_\mathcal{L}(\Omega, F^\mathbb{V})\) of non-taut measures such that \(\mathcal{H}(\mu_n|\Phi) \to \mathcal{H}(\mu|\Phi)\) and \(\mu_n \to \mu\) in the topology of weak local convergence as \(n \to \infty\).

If \(E = \mathbb{R}\) and \(\mu\) a shift-invariant random field with finite specific free energy, then \(w_{\mu^-}\)-almost surely \(S(\nu) \in U_\Phi\), due to Theorem 4.13 and because \(\mathcal{H}(\cdot|\Phi)\) is strongly affine. In other words, \(\mu\) is automatically non-taut. The following lemma is therefore meaningful for \(E = \mathbb{Z}\) only.

**Lemma 10.32.** Any ergodic gradient random field with finite specific free energy has a non-taut approximation.

**Proof.** Let \(E = \mathbb{Z}\), and let \(\mu\) denote an ergodic random field with \(\mathcal{H}(\mu|\Phi) < \infty\) and \(S(\mu) \in \partial U_\Phi\). In this pathological case, we must modify \(\mu\) slightly, so that the modified measure is non-taut, and without changing the specific free energy too much. Let \(\xi\) denote another ergodic measure in \(\mathcal{P}_\mathcal{L}(\Omega, F^\mathbb{V})\) with \(\mathcal{H}(\xi|\Phi) < \infty\) and with \(S(\xi) \in U_\Phi\)—such measures exist, due to the proof of Theorem 4.13 on Page 37. Write \(\rho_n\) for the uniform probability measure on the set \(\{0, \ldots, n - 1\}\). Fix \(n \in \mathbb{N}\); we are going to define a new measure \(\mu_n\). To sample a height function \(\phi\) from \(\mu_n\), sample first a triple \((\phi^\mu, \phi^\xi, a)\) from the measure \(\mu \times \xi \times \rho_n\). The final sample \(\phi\) is then given by the equation
\[
\phi := \phi^\mu - \left[\frac{(\phi^\mu - \phi^\mu(0)) - (\phi^\xi - \phi^\xi(0)) + a}{n}\right].
\]

The random choice of \(a\) makes the rounding operation shift-invariant. Note that the numerator in this fraction is \(2K\)-Lipschitz almost surely for \(K\) minimal subject to \(Kd_1 \geq q\), and therefore the rounded function is 1-Lipschitz for \(n\) sufficiently large. In fact, the density of edges on which the rounded function is not constant, has a bound of order \(O(1/n)\) as \(n \to \infty\). In particular, this implies that \(\mu_n \to \mu\) in the topology of (weak) local convergence. Recall (9.6) from the proof of Theorem 9.4, and observe that the specific free energy of \(\mu\) and \(\mu_n\) can be calculated as in this equation because either measure is \(K\)-Lipschitz. If \(f(p)\) denotes the entropy function of a Bernoulli trial with parameter \(p\) as in the proof of Lemma 10.14, then by arguments similar to those used in that proof, we can bound the difference in the specific entropy between \(\mu\) and \(\mu_n\):
\[
|\mathcal{H}(\mu|\lambda) - \mathcal{H}(\mu_n|\lambda)| = O(f(O(1/n))) = o(1)
\]
as \( n \to \infty \). For \( E = \mathbb{Z} \), we have a lower and upper bound on \( H_{(x)}(\phi) \) for \( q \)-Lipschitz \( \phi \), and this and amenability of the weak interaction \( \Xi \) imply that the specific energy functional

\[
\mu \mapsto \mu(\Phi)
\]

is continuous with respect to the topology of local convergence whenever restricted to shift-invariant random fields which are supported on \( q \)-Lipschitz functions. Jointly these two observations imply that \( \mathcal{H}(\mu_n|\Phi) \to \mathcal{H}(\mu|\Phi) \). It suffices to demonstrate that each measure \( \mu_n \) is non-taut. Claim that \( w_{\mu_n} \)-almost every ergodic component \( \nu \) satisfies \( S(\nu) = (1 - \frac{1}{n})S(\mu) + \frac{1}{n}S(\xi) \in U_{\Phi} \). Recall Theorem 10.15. The final assertion of that theorem tells us that the slope \( S(\nu) \) of each ergodic component can be read off from almost every sample \( \phi \) from \( \nu \), since the slope \( u := S(\nu) \) is almost surely the unique slope such that for any fixed \( \varepsilon > 0 \),

\[
\|\phi_{\Pi_m} - \phi(0) - u|_{\Pi_m}\|_\infty \leq \varepsilon m
\]

for \( m \) sufficiently large. The slope \( (1 - \frac{1}{n})S(\mu) + \frac{1}{n}S(\xi) \) makes this inequality work for samples \( \phi \) from the original measure \( \mu_n \), because \( \mu \) and \( \xi \) are ergodic, and because \( \phi \) equals \( (1 - \frac{1}{n})\phi^{(k)} + \frac{1}{n}\phi^{(\xi)} \) up to bounded differences.

**Lemma 10.33.** For any \( \mu \in \mathcal{P}_L(\Omega, \mathcal{F}^\text{V}) \) with \( S(\mu) \in U_{\Phi} \), we have \( \text{PB}^*(\mu) \leq \mathcal{H}(\mu|\Phi) \).

**Proof.** Let \( \mu \) denote an arbitrary shift-invariant random field with \( H := \mathcal{H}(\mu|\Phi) + 1 < \infty \) and \( u := S(\mu) \in U_{\Phi} \). If \( \mu \) is non-taut and a convex combination of finitely many ergodic random fields, then the lemma follows immediately from Theorem 4.1 and Lemmas 10.22 and 10.24. Let us now consider the case that \( \mu \) is non-taut, but not a convex combination of finitely many ergodic random fields. The lower level set of the specific free energy \( M_H \) is a compact Polish space, and therefore there exists a sequence of continuous cylinder functions \( (f_k)_{k \in \mathbb{N}} \) with \( f_k : \Omega \to [0, 1] \) such that some sequence \( (\mu_n)_{n \in \mathbb{N}} \subset M_H \) satisfies \( \mu_n \to \mu \) in the topology of weak local convergence if and only if \( \mu_n(f_k) \to \mu(f_k) \) as \( n \to \infty \) for every \( k \in \mathbb{N} \). Write \( w_{\mu} \) for the ergodic decomposition of \( \mu \). Let \( (\nu_i)_{i \in \mathbb{N}} \) denote an i.i.d. sequence of samples from \( w_{\mu} \). Define

\[
\mu_n := \sum_{i=1}^n \frac{1}{n} \nu_i.
\]

Then \( w_{\mu} \)-almost surely, \( \mathcal{H}(\mu_n|\Phi) \to \mathcal{H}(\mu|\Phi) \) and \( \mu_n(f_k) \to \mu(f_k) \) as \( n \to \infty \) for all \( k \in \mathbb{N} \). This implies that \( \mu_n \to \mu \) in the topology of weak local convergence. Finally, we have \( S(\mu_n) \to u \). By altering the coefficients in the definition of each measure \( \mu_n \) slightly, we can make sure that \( S(\mu_n) = u \) for \( n \) sufficiently large, while retaining the other properties of this sequence. For each measure \( \mu_n \) we have \( \text{PB}^*(\mu_n) \leq \mathcal{H}(\mu_n|\Phi) \) by the first part of this proof, and \( \text{PB}^*(\mu) \leq \mathcal{H}(\mu|\Phi) \) because \( \text{PB}^*(\cdot) \) is lower-semicontinuous when restricted to \( \{S(\cdot) = u\} \), while \( \mathcal{H}(\mu_n|\Phi) \to \mathcal{H}(\mu|\Phi) \) as \( n \to \infty \).

We now prove the lemma for the case that \( \mu \) is a convex combination of finitely many ergodic measures, but without imposing that \( \mu \) is non-taut. Write

\[
\mu = \sum_{i=1}^n a_i \nu^i
\]

for the decomposition of \( \mu \) into ergodic components. Since each \( \nu^i \) is ergodic, it has a non-taut approximation \( (\nu^i_k)_{k \in \mathbb{N}} \). Define \( \mu_k := \sum_{i=1}^n a_i \nu^i_k \), so that \( \mu_k \to \mu \) in the topology of weak local convergence with \( \mathcal{H}(\mu_k|\Phi) \to \mathcal{H}(\mu|\Phi) \) as \( k \to \infty \). This implies also that \( S(\mu_k) \to S(\mu) \), and by altering the coefficients in the definition of each measure \( \mu_k \) slightly, we may ensure that \( S(\mu_k) = u \) for \( k \) sufficiently large, while retaining the previously mentioned properties. By arguing as before, we have \( \text{PB}^*(\mu_k) \leq \mathcal{H}(\mu_k|\Phi) \) and therefore \( \text{PB}^*(\mu) \leq \mathcal{H}(\mu|\Phi) \). The generalization to those measures \( \mu \) which are not a convex combination of finitely many ergodic measures and not non-taut is the same as before. \( \square \)
11 Large deviations principle

Large deviations are the subject of a vast literature within statistical physics [DS89, DZ10, RS15]. In the context of gradient models, the pioneering result was derived by Sheffield in [She05]. In this section we prove a large deviations principle (LDP) of similar strength to the one contained in Chapter 7 of [She05], with the noteworthy difference that we express it directly in terms of the Gibbs specification. The large deviations principle applies to all models described in the introduction, including for example perturbed dimer models [GMT17, GMT19] which are not monotone, even if the perturbation has infinite range. This LDP captures both the macroscopic profile of each sample, as well as its local statistics. We will be using some notations and ideas from [She05] and [KMT19].

Recall Subsection 4.2 for a description of good asymptotic boundary profiles and good approximations. That subsection also contains a description of the topology for the macroscopic profile of each function. The letter \( \Phi \) denotes a fixed potential belonging to the class \( S_L + W_L \) throughout this section.

11.1 Formal description of the LDP

Recall Subsection 4.2, which gave an overview of the large deviations principle without local statistics. Throughout this section, the sequence \((D_n, b_n)_{n \in \mathbb{N}}\) denotes a good approximation of some fixed good asymptotic boundary profile \((D, b)\). The sequence of local Gibbs measures which are of interest in the LDP is the sequence \((\gamma_n)_{n \in \mathbb{N}}\) defined by \(\gamma_n := \gamma_{D_n}(\cdot, b_n)\). We shall also write \(Z_n\) for \(Z_{D_n}(b_n)\); the normalizing constant in the definition of the measure \(\gamma_{D_n}(\cdot, b_n)\). Finally, \(\tilde{\gamma}_n\) shall denote the non-normalized version of \(\gamma_n\), that is, \(\tilde{\gamma}_n := Z_n \gamma_n\).

11.1.1 The topological space

All samples from the sequence of measures \((\gamma_n)_{n \in \mathbb{N}}\) must be brought to the same topological space, in order to formulate the large deviations principle. We want our large deviations principle to describe both the global profile of each sample as well as its local statistics, and this is reflected in the choice of topological space. More concretely, the topological space that we have in mind decomposes as the product of two topological spaces, each describing one of the two aspects of each sample. Recall from Definition 4.8 that \((\operatorname{Lip}(\bar{D}), \mathcal{X}^\infty)\) is space of \(K\|\cdot\|_1\)-Lipschitz functions functions on \(\bar{D}\) endowed with the topology of uniform convergence. Recall also the definition of \(\mathfrak{G}_n\); each map \(\mathfrak{G}_n\) is used to map samples from \(\gamma_n\) to \(\operatorname{Lip}(\bar{D})\). This map characterizes the macroscopic profile of each sample.

Next, we define the empirical measure profile \(\mathfrak{L}_n(\phi)\) of the sample \(\phi\) from \(\gamma_n\). The empirical measure profile captures the local statistics of the height function \(\phi\) in the large deviations principle.

Definition 11.1 (topology for local statistics). Write \(\mathcal{D}\) for the Borel \(\sigma\)-algebra on \(D\), and recall that \(\mathcal{M}(X, X')\) denotes the set of \(\sigma\)-finite measures on the measurable space \((X, X')\). Throughout this article, we shall write \(\mathcal{M}^D\) for the set of measures \(\mu \in \mathcal{M}(D \times \Omega, \mathcal{D} \times \mathcal{F}^\vee)\) which have the property that the first marginal \(\mu_D = \mu(\cdot, \Omega)\) equals the Lebesgue measure on \(D\). The empirical profile \(\mathfrak{L}_n(\phi) \in \mathcal{M}^D\) of \(\phi\) is now defined by the equation

\[
\mathfrak{L}_n(\phi) := \int_D \delta_{(x, \theta_{[nx]_L}(\phi))} dx,
\]

where \(\delta\) denotes the Dirac measure and \([nx]_L\) is the vertex in \(L\) closest to \(nx\) in the Euclidean metric—this is well-defined for almost every \(x\) with respect to the Lebesgue measure. Thus,
to “sample” from $\mathcal{L}_n(\bar{\phi})$—this language is abusive because the size of the measure $\mathcal{L}_n(\bar{\phi})$ is $\text{Leb}(D)$ and therefore not generally a probability measure—one first samples $x$ from $D$ uniformly at random; then one shifts the sample $\bar{\phi}$ by $[nx]$. The map $\mathcal{L}_n : \Omega \to \mathcal{M}^D$ thus captures the local statistics of the height functions in the large deviations principle. For the statement of the large deviations principle, we endow the space $\mathcal{M}^D$ with the topology $\mathcal{X}^\mathcal{L}$.

This is defined to be the weakest topology which makes the map $\mu \to \mu(R, f)$ continuous for any rectangular subset $R$ of $D$, and for any continuous cylinder function $f : \Omega \to [0, 1]$.

**Remark.** If $\phi$ is a height function, $R \subset D$ a bounded convex set of positive Lebesgue measure, and $n$ large, then

$$\text{Leb}(R)^{-1}\mathcal{L}_n(\phi)(R, \cdot) \approx L_{\mathcal{L}(n,R)}(\phi).$$

More precisely, the total variation distance between the two measures goes to zero as $n \to \infty$, uniformly over the choice of $\phi$.

**Definition 11.2** (Product topology for the large deviation principle). The large deviations principle is formulated on the space $X^\mathcal{L} := \text{Lip}(\overline{D}) \times \mathcal{M}^D$ endowed with the topology $\mathcal{X}^\mathcal{L} := \mathcal{X}^\infty \times \mathcal{X}^\mathcal{L}$, and we map each sample $\phi$ from $\gamma_n$ to this space by applying the map $\mathcal{P}_n := \mathcal{G}_n \times \mathcal{L}_n$.

11.1.2 The rate function

Before proceeding, a few definitions for measures $\mu \in \mathcal{M}^D$ are introduced. The measure $\mu$ is called $\mathcal{L}$-invariant if $\text{Leb}(U)^{-1}\mu(U, \cdot) \in \mathcal{P}_\mathcal{L}(\Omega, \mathcal{F}^\mathcal{L})$ for any $U \in \mathcal{D}$ of positive Lebesgue measure. Write $\mathcal{M}^D_{\mathcal{L}}$ for the set of all such shift-invariant measures. If $\mu$ is $\mathcal{L}$-invariant and $U \in \mathcal{D}$ has positive Lebesgue measure, then we write $\mathcal{S}(\mu(U, \cdot))$ for the slope of $\text{Leb}(U)^{-1}\mu(U, \cdot)$. Call a pair $(g, \mu) \in X^\mathcal{L}$ compatible, and write $g \sim \mu$, if $\mu$ is $\mathcal{L}$-invariant with $\nabla g(x) = \mathcal{S}(\mu(x, \cdot))$ as a distribution on $D$. Finally, write $w_\mu$ for the ergodic decomposition of the shift-invariant non-normalized measure $\mu(D, \cdot)$, and define

$$\mathcal{H}(\mu|\Phi) := \mathcal{H}(\mu(D, \cdot)|\Phi) : = \int \mathcal{H}(\nu|\Phi)dw_\mu(\nu) = \text{Leb}(D)\mathcal{H}(\text{Leb}(D)^{-1}\mu(D, \cdot)|\Phi).$$

**Definition 11.3.** Consider a good asymptotic boundary profile $(D, b)$. The rate function associated to this profile is the function $I : X^\mathcal{L} \to \mathbb{R} \cup \{\infty\}$ defined by

$$I(g, \mu) := \bar{I}(g, \mu) - P_\Phi(D, b) \quad \text{where} \quad \bar{I}(g, \mu) := \begin{cases} \mathcal{H}(\mu|\Phi) & \text{if } g|_{\partial D} = b \text{ and } g \sim \mu, \\ \infty & \text{otherwise}. \end{cases}$$

Here $P_\Phi(D, b)$ denotes the pressure of $(D, b)$, which is given by

$$P_\Phi(D, b) := \min_{g \in \text{Lip}(D) \text{ with } g|_{\partial D} = b} \int_D \sigma(\nabla g(x))dx.$$

The function $\bar{I}$ is useful because its definition does not appeal to the pressure. It will later appear as the rate function of the LDP corresponding to the sequence of measures $(\tilde{\gamma}_n)_{n \in \mathbb{N}}$ defined by $\tilde{\gamma}_n := Z_n \gamma_n$, the non-normalized versions of the local Gibbs measures $\gamma_D(\cdot, b_n)$.

**Lemma 11.4.** The following hold true:

1. The rate functions $I$ and $\bar{I}$ are convex.
2. The rate functions $I$ and $\tilde{I}$ are lower-semicontinuous,

3. The lower level sets $\{I \leq C\}$ and $\{\tilde{I} \leq C\}$ are compact Polish spaces for $C < \infty$,

4. There is a probability kernel $u \mapsto \mu_u$ such that for any $u \in \{\sigma < \infty\}$, we have $\mu_u \in \mathcal{P}_\mathcal{L}(\Omega, \mathcal{F}^\nabla)$ with $S(\mu_u) = u$ and $\mathcal{H}(\mu_u(\Phi)) = \sigma(u)$,

5. For fixed $g \in \text{Lip}(\tilde{D})$ with $g|_{\partial D} = b$, we have

$$\min_{\mu \in \mathcal{M}^D} \tilde{I}(g, \mu) = \int_D \sigma(\nabla g(x))dx,$$

6. The minimum of $I$ is 0, and the minimum of $\tilde{I}$ is $P_\Phi(D, b)$.

We provide a proof in the next subsection.

### 11.1.3 Statement of the LDP

**Theorem 11.5** (Large deviations principle). Let $\Phi \in \mathcal{S}_\mathcal{L} + \mathcal{W}_\mathcal{L}$, and let $(D_n, b_n)_{n \in \mathbb{N}}$ denote a good approximation of some good asymptotic profile $(D, b)$. Let $\gamma_n^\ast$ denote the pushforward of $\gamma_n := \gamma_{D_n}(\cdot, b_n)$ along the map $\Psi_n$, for any $n \in \mathbb{N}$. Then the sequence of probability measures $(\gamma_n^\ast)_{n \in \mathbb{N}}$ satisfies a large deviations principle with speed $n^d$ and rate function $I$ on the topological space $(\mathcal{X}^\Psi, \mathcal{X}^\Omega)$. Moreover, the sequence of normalizing constants $(Z_n)_{n \in \mathbb{N}} := (Z_{D_n}(b_n))_{n \in \mathbb{N}}$ satisfies $-n^{-d} \log Z_n \to P_\Phi(D, g)$ as $n \to \infty$.

Remark that Theorem 4.10 follows immediately from this theorem in combination with Lemma 11.4, Statement 5.

### 11.2 Proof overview

Let us start with a proof of some key properties of the rate functions $I$ and $\tilde{I}$.

**Proof of Lemma 11.4.** Note that $\mathcal{M}^D_\mathcal{L}$ is closed in $(\mathcal{M}^D, \mathcal{X}^\Omega)$. It follows immediately from the properties of the original specific free energy functional (Theorem 4.1) that the map $\mathcal{H}(|\cdot|\Phi) : \mathcal{M}^D_\mathcal{L} \to \mathbb{R} \cup \{\infty\}$ is affine and lower-semicontinuous, and that its lower level sets are compact Polish spaces with respect to the $\mathcal{X}^\Omega$-topology.

Observe that the set $\{g \sim \mu\} \subset \mathcal{X}^\Psi$ is convex. This implies that $I$ and $\tilde{I}$ are convex, since the map $(g, \mu) \mapsto \mathcal{H}(\mu(\Phi))$ is affine on $\{g \sim \mu\}$. Observe that the set $\{g \sim \mu\}$ is also closed in $\mathcal{X}^\Psi$. The lower level sets of $I$ and $\tilde{I}$ are compact Polish spaces because

$$\{\tilde{I} \leq C\} = \{(g \in \text{Lip}(\tilde{D}) : g|_{\partial D} = b) \times \{\mu \in \mathcal{M}^D_\mathcal{L} : \mathcal{H}(\mu(\Phi) \leq C)\}\} \cap \{g \sim \mu\},$$

that is, $\{\tilde{I} \leq C\}$ is as a closed subset of a product of two compact Polish spaces. This also implies that $I$ and $\tilde{I}$ are lower-semicontinuous.

The fourth statement is a simple exercise in measure theory; it follows from the topological properties of the specific free energy stated in Theorem 4.1. If $g \sim \mu$, then it is clear that

$$\tilde{I}(g, \mu) = \int_D \mathcal{H}(\mu(x, \cdot)|\Phi)dx \geq \int_D \sigma(S(\mu(x, \cdot)))dx = \int_D \sigma(\nabla g(x))dx.$$

For fixed $g$, this inequality can be turned into an equality, by constructing $\mu$ in terms of $\nabla g$ and the kernel from the fourth statement. This proves the fifth statement. The final statement is now obvious. \qed
Theorem 11.5 states the LDP for the sequence of normalized measures \((\gamma_n)_{n \in \mathbb{N}}\). For the proof, however, it will be beneficial to consider also the sequence of non-normalized measures \((\tilde{\gamma}_n)_{n \in \mathbb{N}}\). Write \(\tilde{\gamma}_n^*\) for the pushforward of \(\tilde{\gamma}_n\) along \(\mathcal{P}_n\). Theorem 11.5 is equivalent to the conjunction of the following two statements:

1. The minimum of \(I\) is \(P_{\Phi}(D, b)\),
2. The sequence \((\tilde{\gamma}_n^*)_{n \in \mathbb{N}}\) satisfies an LDP with speed \(n^d\) and rate function \(\tilde{I}\) in \((X^\Psi, \mathcal{X}^\Psi)\).

The first statement was proven in Lemma 11.4. The second statement is somewhat easier to prove than the original LDP, because it appeals to non-normalized measures only.

Let us first describe a particular basis for the topological space \((X^{\Psi}, \mathcal{X}^{\Psi}) = (\operatorname{Lip}(\bar{D}), \mathcal{X}^{\infty}) \times (\mathcal{M}^D, \mathcal{X}^\mathcal{L})\).

As a basis \(B^\infty\) for \(\mathcal{X}^{\infty}\), we take the sets of the form
\[
B^\infty_\varepsilon (g) := \{ h \in \operatorname{Lip}(\bar{D}) : \| h - g \|_\infty < \varepsilon \}
\]
where \(g \in \operatorname{Lip}(\bar{D})\) and \(\varepsilon > 0\). Write
\[
B^\mathcal{L}_\varepsilon (\mu, (R_i)_i, (f_j)_j) := \{ \nu : |\mu(R_i, f_j) - \nu(R_i, f_j)| < \operatorname{Leb}(R_i)\varepsilon \text{ for all } i, j \} \subset \mathcal{M}^D,
\]
where \(\varepsilon > 0\), \(\mu\) is a measure in \(\mathcal{M}^D\), \((R_i)_i\) is a finite collection of closed rectangular subsets of \(D\), and \((f_j)_j\) is a finite collection of continuous cylinder functions \(f_j : \Omega \rightarrow [0, 1]\). The collection \(B^\mathcal{L}\) of such sets forms a basis of \(\mathcal{X}^\mathcal{L}\). As a basis \(B^\Psi\) for \(\mathcal{X}^\Psi\), we choose the collection of open sets of the form \(B^\Psi_\varepsilon (\cdot, \cdot, \cdot, \cdot) := B^\infty_\varepsilon (\cdot, \cdot) \times B^\mathcal{L}_\varepsilon (\cdot, \cdot, \cdot, \cdot)\).

To prove a large deviations principle, it must first be checked that the rate function is lower-semicontinuous. For this refer again to Lemma 11.4. The large deviations principle (with non-normalized measures) is now a corollary of the following three claims:

1. **Lower bound on probabilities.** For any \((g, \mu) \in A \in B^\Psi\), we have
   \[
   \liminf_{n \to \infty} n^{-d} \log \tilde{\gamma}_n^*(A) \geq -\tilde{I}(g, \mu).
   \]

2. **Upper bound on probabilities.** For any \((g, \mu) \in X^\Psi\), we have
   \[
   \inf_{A \in B^\Psi \text{ with } (g, \mu) \in A} \limsup_{n \to \infty} n^{-d} \log \tilde{\gamma}_n^*(A) \leq -\tilde{I}(g, \mu).
   \]

3. **Exponential tightness.** For all \(\alpha > -\infty\), there is a compact set \(K_\alpha \subset X^\Psi\) such that
   \[
   \limsup_{n \to \infty} n^{-d} \log \tilde{\gamma}_n^*(X^\Psi \setminus K_\alpha) \leq \alpha
   \]
   The next subsection contains an auxiliary result on approximations of Lipschitz functions which is useful for proving the lower bound. Each of the three subsequent sections addresses one of the three claims formulated above.
11.3 Simplicial approximations of Lipschitz function

This subsection is dedicated to providing some results on affine approximations of Lipschitz functions necessary to prove the lower bound on probabilities. For \( x \in \mathbb{R}^d \), the point \( \lfloor x \rfloor \in \mathbb{Z}^d \) is obtained by rounding down each coordinate.

**Definition 11.6.** Let \( S_d \) denote the group of permutations on \( \{1, \ldots, d\} \). For \( x \in \mathbb{R} \), we write \( s(x) \in S_d \) for the permutation which rank-orders the coordinate indices of \( x - \lfloor x \rfloor \). For \( x \in \mathbb{Z}^d \) and \( s \in S_d \), we define the simplex \( C(x, s) \) to be the closure of the set \( \{ y \in \mathbb{R}^d : \lfloor y \rfloor = x, s(y) = s \} \).

By a simplex of scale \( \varepsilon \), we simply mean a scaled simplex of the form \( \varepsilon C(x, s) \). A simplex domain of scale \( \varepsilon \) is a union of finitely many simplices of scale \( \varepsilon \).

**Definition 11.7.** Let \( D \) denote a domain, and \( g \) a real-valued function on \( D \). Consider \( \varepsilon > 0 \). Write \( F_\varepsilon = F_\varepsilon(g) \) for the unique real-valued function on \( D_\varepsilon \) which equals \( g \) on \( D_\varepsilon \cap \varepsilon \mathbb{Z}^d \), interpolated linearly on each simplex.

We will make use of the simplicial Rademacher theorem proven in [KMT19] for which we recall a statement here.

**Lemma 11.8** (Lemma 6.1 from [KMT19]). Consider a positive homogeneous function \( \| \cdot \| : \mathbb{R}^d \to \mathbb{R} \) satisfying the triangle inequality. Let \( D \subset \mathbb{R}^d \) be a domain and \( g : D \to \mathbb{R} \) a \( \| \cdot \| \)-Lipschitz function. For any \( \delta > 0 \) and any \( \varepsilon > 0 \) sufficiently small (depending on \( \delta \)), we have

1. \( \text{Leb}(D \setminus D_\varepsilon) \leq \delta \),
2. \( \| F_\varepsilon - g \|_{D_\varepsilon} \leq \delta \varepsilon \),
3. \( \text{Leb}(\{ x \in D_\varepsilon : \| \nabla F_\varepsilon(x) - \nabla g(x) \|_2 \geq \delta \}) \leq \delta \).

Moreover, \( F_\varepsilon \) is \( \| \cdot \| \)-Lipschitz for any \( \varepsilon > 0 \).

The first property is obvious, and the proof of the second and third property is identical to the proof in [KMT19].

11.4 The lower bound on probabilities

Fix \( (g, \mu) \in A \in \mathcal{B}^\Omega \) and \( \beta > 0 \); the goal of this subsection is to prove that

\[
\liminf_{n \to \infty} n^{-d} \log \tilde{\gamma}_n^*(A) \geq -\tilde{I}(g, \mu) - \beta.
\]

We suppose of course that \( \tilde{I}(g, \mu) \) is finite. For the proof, we require the following result.

**Lemma 11.9.** Consider some fixed \( \varepsilon > 0 \). Then there exists a sufficiently small constant \( \alpha > 0 \) such that the following statement holds true. Suppose that \( \mu \in \mathcal{P}_\mathcal{L}(\Omega, \mathcal{F}) \) satisfies \( u := S(\mu) \in \bar{U}_q \subset U_q \), and that \( v \in U_q \) is another slope with \( \| u - v \|_2 \leq \alpha \). Then there is another measure \( \nu \in \mathcal{P}_\mathcal{L}(\Omega, \mathcal{F}) \) such that \( S(\nu) = v \) and \( \mathcal{H}(\nu|\Phi) \leq \mathcal{H}(\mu|\Phi) + \varepsilon \) and \( \| \mu - \nu \|_{TV} := \| \mu - \nu \|_\infty < \varepsilon \). In particular, if \( f : \Omega \to [0, 1] \) is measurable, then \( |\mu(f) - \nu(f)| < \varepsilon \).
Proof. Note that \( \sigma \) is bounded uniformly on a neighborhood \( A \) of \( U_q \). The proof of the lemma is straightforward: one simply defines \( \nu := (1-t)\mu + t\mu' \) for \( t \) small and \( \mu' \) some minimizer with \( S(\mu') \in A \) in order to adjust the slope of the measure of interest. \( \square \)

Proof of the lower bound on probabilities. It suffices to consider the case that \( \tilde{I}(g, \mu) \) is finite. We claim that it is sufficient to consider the case that \( g \) is strictly \( \| \cdot \|_q \)-Lipschitz (if \( E = \mathbb{R} \)) or that \( g \mid D \) is locally strictly \( \| \cdot \|_q \)-Lipschitz (if \( E = \mathbb{Z} \)). If \( g \) were not \( \| \cdot \|_q \)-Lipschitz and \( g \sim \mu \), then \( \mu \) cannot be supported on \( q \)-Lipschitz functions, and consequently \( \tilde{I}(g, \mu) = \infty \). Therefore \( g \) must be \( \| \cdot \|_q \)-Lipschitz. There is some pair \( (h, \nu) \in X^\Psi \) such that \( h \) is strictly \( \| \cdot \|_q \)-Lipschitz (if \( E = \mathbb{R} \)) or such that \( h \mid D \) is locally strictly \( \| \cdot \|_q \)-Lipschitz (if \( E = \mathbb{Z} \)), and such that \( \tilde{I}(h, \nu) < \infty \)—this follows from the definition of a good asymptotic boundary profile and from Lemma 11.4. Define \( g_t := (1-t)g + th \) and \( \mu_t := (1-t)\mu + t\nu \). Then \( (g_t, \mu_t) \in A \) for \( t \) sufficiently small and \( \limsup_{t \to 0} \tilde{I}(g_t, \mu_t) \leq \tilde{I}(g, \mu) \) if \( \tilde{I} \) is convex. Moreover, for any \( t > 0 \), the function \( g_t \) has the desired properties. Thus, we may replace \( (g, \mu) \) by \( (g_t, \mu_t) \) for small \( t \), by choosing \( \beta \) smaller if necessary. This proves the claim.

The proof follows the general strategy that was outlined after the statement of Lemma 10.24. Let us first consider the case that \( E = \mathbb{R} \). We find an appropriate approximation of \( g \) using the simplicial Rademacher theorem, and then apply Lemma 11.9 and the limit equalities to obtain the desired lower bound on probabilities. For the approximations, it is necessary to take limits in three variables: first we take \( n \to \infty \), then \( \varepsilon_2 \to 0 \), and finally \( \varepsilon_1 \to 0 \). There is also another variable \( \varepsilon \); it is not necessary to take a limit in this variable, but it must be small for the arguments to work.

First fix \( \varepsilon > 0 \) so small that \( b \) and \( g \) are \( \| \cdot \|_{q_{6e}} \)-Lipschitz, and such that all functions \( b_n \) are \( q_{6e} \)-Lipschitz. We also suppose that \( A = B^\Psi_{q_{6e}}(g, \mu, (R_t)_i, (f_j)_j) \), by choosing \( \varepsilon \) and \( A \) smaller if necessary, where \( (R_t)_i \) is a finite family of rectangular subsets of \( D \), and \( (f_j)_j \) a finite family of continuous cylinder functions \( f_j : \Omega \to [0,1] \).

Consider some \( \varepsilon_1 > 0 \), and write \( D' \) for the points in \( D \) at distance more than \( \varepsilon_1 \) from the complement of \( D \). Consider additionally some \( \varepsilon_2 > 0 \), and write \( D'' \) for \( D'_{\varepsilon_2} \): the largest simplex domain of scale \( \varepsilon_2 \) contained in \( D' \). See Figure 5 for a drawing of this construction. Write \( F = \tilde{F}(g) \) for the unique \( \| \cdot \|_{q_{6e}} \)-Lipschitz function on \( D'' \) which equals \( g \) on \( \varepsilon_2 \mathbb{Z}^d \cap D'' \), and which is affine on each simplex of \( D'' \). For \( \varepsilon_2 \) sufficiently small, this function has a \( \| \cdot \|_{q_{6e}} \)-Lipschitz extension \( \tilde{F} \) to \( D \) which equals \( b \) on \( \partial D \). It is clear that any such extension \( \tilde{F} \) is contained in \( B^\infty_{\varepsilon}(g) \), that is, \( \| \tilde{F} - g \|_\infty < \varepsilon \), for \( \varepsilon_1 \) and \( \varepsilon_2 \) sufficiently small.

Write \( \Sigma \) for the set of simplices of scale \( \varepsilon_2 \) in \( D'' \)—this is a finite set. The slope \( \nabla F \) of \( F \) is constant on any \( \Delta \in \Sigma \); write \( S(\Delta) \in U_{q_{6e}} \) for this slope. Write \( \Sigma' \) for the set of simplices \( \Delta \in \Sigma \) for which \( \| S(\Delta) - S(\mu(\Delta, \cdot)) \|_2 \leq \varepsilon_1 \); Lemma 11.8 asserts that \( \| \Sigma' \| / \| \Sigma \| \geq 1 - \varepsilon_1 \) for \( \varepsilon_2 \) sufficiently small. See again Figure 5 for an example of the sets \( \Sigma \) and \( \Sigma' \).

Choose \( C \) minimal subject to \( \| \phi - u \|_{2+} + 1 \leq C \) for all \( u \in U_{6e} \). Let \( M \) denote a constant which makes Theorem 6.9 work for the local Lipschitz constraint \( q_{5e} \), and for the constants \( \varepsilon \) and \( C \)—this constant \( M \) depends on \( \varepsilon \) only. We shall also suppose that \( M \geq R \), by choosing \( M \) larger if necessary. For \( \Delta \in \Sigma \) and \( n \in \mathbb{N} \), define \( \Delta_n := \Lambda^{-M(n)}(\Delta) \). Write also \( D''_n := \cup_{\Delta \in \Sigma} \Delta_n \) and \( D_n := \cup_{\Delta \in \Sigma} \Delta_n \). It follows from the definition of an approximation that \( D_n \subset D_n \) for \( n \) sufficiently large. By Theorem 6.9 there exists, for any \( n \in \mathbb{N} \), a \( q_{5e} \)-Lipschitz function \( F_n : D''_n \to E \) such that:

1. \( |F_n(x) - nF(x/n)| \leq C \) for all \( x \in D''_n \),
2. \( \nabla F_n|_{\Delta_n} = \nabla \phi \mid_{\Delta_n} \) for all \( \Delta \in \Sigma \).

It is straightforward to see that for \( n \) sufficiently large, the function \( F_n \) extends to a \( q_{5e} \)-Lipschitz height function \( \tilde{F}_n \) which equals \( b_n \) on the complement of \( D_n \).
We now use the existence of the function $\bar{F}_n$ to demonstrate that there exists a set $A_n \in \mathcal{F}$ such that $\mathfrak{R}(A_n) \subset A$, and for which we show that $\tilde{\gamma}_n(A_n)$ is sufficiently large as $n \to \infty$. Define $A_n$ to be the set of height functions $\phi$ which are $q$-Lipschitz, and which satisfy the following criteria:

1. If $x \in \mathbb{Z}^d \setminus D_n$, then $\phi(x) = \bar{F}_n(x) = b_n(x)$,
2. If $x \in D_n \setminus D^*_n$, then $|\phi(x) - \bar{F}_n(x)| \leq \varepsilon$,
3. If $x = 0 \Delta_n$ for some $\Delta \in \Sigma^*$, then $|\phi(x) - \bar{F}_n(x)| \leq \varepsilon$,
4. For each $\Delta \in \Sigma^*$, we have $\phi \in C_{\Delta_n,\varepsilon}$,
5. For each $\Delta \in \Sigma^*$, we have $\phi \in B_{\Delta_n}$, that is, $L_{\Delta_n}(\phi) \in B_{\Delta}$, where

$$B_{\Delta} := \{ \nu \in \mathcal{P}(\Omega, \mathcal{F}^\nabla) : |\text{Leb}(\Delta)\nu(f_j) - \mu(\Delta, f_j)| < \mathrm{Leb}(\Delta)\varepsilon \text{ for all } j \} \in \mathcal{B}.$$ 

It suffices to demonstrate that for $\varepsilon$, $\varepsilon_1$, and $\varepsilon_2$ sufficiently small, and for $n$ sufficiently large, we have $\mathfrak{R}(A_n) \subset A$ and

$$\liminf_{n \to \infty} n^{-d} \log \tilde{\gamma}_n(A_n) \geq -I(g, \mu) - \beta.$$ 

Claim first that, in the limit, $\mathfrak{R}(A_n) \subset A$. This is equivalent to asking that $\mathfrak{S}_n(A_n) \subset B_{\varepsilon}(g)$ and $\mathfrak{L}_n(A_n) \subset B_{\varepsilon}(\mu, (R_i)_i, (f_j)_j)$. The former of the two holds true because $\|F - g\|_\infty < \varepsilon$ and because $\|\bar{F} - \mathfrak{S}_n(\phi)\|_\infty$ is small in the described limit, uniformly over the choice of $\phi \in A_n$. The proof that $\mathfrak{L}_n(A_n) \subset B_{\varepsilon}(\mu, (R_i)_i, (f_j)_j)$ in the limit relies again on Proposition 10.25; observe in particular that in the limit most of the volume of each fixed rectangle $R_i$ is covered by simplices in $\Sigma^*$ which are entirely contained in $R_i$.

In the sequel, we shall pretend that $A_n \in \mathcal{E}^{D_n}$ by restricting each height function $A_n$ to $D_n$. If $\phi \in \mathcal{E}^{D_n}$, then we write $\psi$ for the height function which restricts to $\phi$ on $D_n$ and to $b_n$ on the complement of $D_n$. We aim to find an asymptotic lower bound on

$$n^{-d} \log \tilde{\gamma}_n(A_n) = n^{-d} \log \int_{A_n} e^{-H_{D_n}(\psi)} d\lambda^{D_n}(\phi).$$
If $\phi \in A$, then $\psi$ is $q_\varepsilon$-Lipschitz whenever restricted to $\mathbb{Z}^d \setminus \cup_{\Delta \in \Sigma^*} \Delta_n^R$, because $\bar{F}_n$ is $q_{5\varepsilon}$-Lipschitz and because $\psi$ and $\bar{F}_n$ differ by at most $2\varepsilon$ at each vertex in this set. Therefore the upper attachment lemma (Lemma 7.4) implies

$$H_{D_n}(\psi) \leq H_{D_n}(\psi) + e_n^+ + \sum_{\Delta \in \Sigma^*} H_{\Delta_n}(\psi) + e_n^+(\Delta_n)$$

for any $\phi \in A$. For fixed $\varepsilon, \varepsilon_1$, and $\varepsilon_2$, the terms of the form $e_n^+(\cdot)$ in this expression are of order $o(n^d)$ as $n \to \infty$, and therefore we may omit them in calculating the limit inferior. Moreover, since $\psi$ is $q_\varepsilon$-Lipschitz on $D_n \setminus D_n^*$, the term $H_{D_n}^0(\psi)$ has an upper bound $C'|D_n \setminus D_n^*|$, where $C'$ depends on $\varepsilon$ only. In particular,

$$\liminf_{n \to \infty} n^{-d} \log \tilde{\gamma}_n(A_n) \geq \liminf_{n \to \infty} n^{-d} \left[ -C'|D_n \setminus D_n^*| + \log \int_{A_n} e^{-\sum_{\Delta \in \Sigma^*} H_{\Delta_n}(\psi)} d\lambda_{D_n}(\phi) \right].$$

It follows from the definition of $A_n$, that the integral decomposes as follows:

$$\int_{A_n} e^{-\sum_{\Delta \in \Sigma^*} H_{\Delta_n}(\psi)} d\lambda_{D_n}(\phi) = \left[ \prod_{x \in D_n \setminus D_n^*} \int_{D_n} F_n(x) + \varepsilon d\lambda \right] \left[ \prod_{\Delta \in \Sigma^*} \int_{D_n} F_n(0_{\Delta_n}) + \varepsilon d\lambda \right] \left[ \prod_{\Delta \in \Sigma^*} \int_{C_\Delta(S(\Delta), 0_{\Delta_n}) \setminus B_\Delta_n} e^{-H_{\Delta_n}(\psi)} d\lambda_{D_n} \right],$$

and therefore the logarithm of this integral equals

$$\left( |D_n \setminus D_n^*| + |\Sigma^*| \right) - \sum_{\Delta \in \Sigma^*} \log 2 \varepsilon - \sum_{\Delta \in \Sigma^*} \int_{D_n} F_n(x) - \varepsilon d\lambda_{D_n}(\phi) = \log 2 \varepsilon - \sum_{\Delta \in \Sigma^*} \int_{D_n} F_n(x) - \varepsilon d\lambda_{D_n}(\phi) + \sum_{\Delta \in \Sigma^*} \int_{D_n} F_n(x) + \varepsilon d\lambda_{D_n}(\phi).$$

But $|\Sigma^*|$ does not depend on $n$, and by choosing $C'$ larger, we obtain

$$\liminf_{n \to \infty} n^{-d} \log \tilde{\gamma}_n(A_n) \geq \liminf_{n \to \infty} n^{-d} \left[ -C'|D_n \setminus D_n^*| - \sum_{\Delta \in \Sigma^*} \int_{D_n} F_n(x) - \varepsilon d\lambda_{D_n}(\phi) \right].$$

It is easy to see that $n^{-d}|D_n \setminus D_n^*| \to \text{Leb}(D \setminus \cup \Sigma^*)$ as $n \to \infty$. Fix $\Delta \in \Sigma^*$. By definition of $\Sigma^*$, we have $\|S(\Delta) - S(\mu(\Delta, \cdot))\|_2 \leq \varepsilon_1$. Then Lemma 11.9 tells us that for $\varepsilon_1$ sufficiently small, the set $B_{\Delta}$ contains another shift-invariant measure $\nu$ of slope $S(\Delta)$ such that $\mathcal{H}(\nu(\Phi)) \leq \mathcal{H}(\text{Leb}(\Delta)^{-1}\mu(\Delta, \cdot)\Phi) + \varepsilon$. In particular, this means that

$$\limsup_{n \to \infty} n^{-d} \int_{D_n} F_n(x) + \varepsilon d\lambda_{D_n}(\phi) = \mathcal{H}(\text{Leb}(\Delta)^{-1}\mu(\Delta, \cdot)\Phi) + \varepsilon$$

and therefore

$$\liminf_{n \to \infty} n^{-d} \log \tilde{\gamma}_n(A_n) \geq -C' \text{Leb}(D \setminus \cup \Sigma^*) - C' \mathcal{H}(\mu(\cup \Sigma^*, \cdot)) \Phi).$$

As $\varepsilon_2 \to 0$ and then $\varepsilon_1 \to 0$, we have

$$\text{Leb}(\cup \Sigma^*) \to \text{Leb}(D), \quad \text{Leb}(D \setminus \cup \Sigma^*) \to 0, \quad \mathcal{H}(\mu(\cup \Sigma^*, \cdot)) \Phi) \to \mathcal{H}(\mu(\Phi).$$

The desired lower bound is thus obtained by setting $\varepsilon$ so small that $\varepsilon \text{Leb}(D) < \beta$. Let us finally describe what changes for $E = \mathbb{Z}$. The first part of the proof is the same, except that the functions $F_n$ and $\bar{F}_n$ are $q$-Lipschitz, and not $q_{6\varepsilon}$-Lipschitz or $q_{6\varepsilon}$-Lipschitz. The $q$-Lipschitz extension $\bar{F}_n$ exists for $n$ sufficiently large, because $g_{1D}$ is locally strictly $q$-Lipschitz. The only thing that changes in the remainder of the proof is that $\lambda$ is now the counting measure rather than the Lebesgue measure. This makes the remainder of the proof easier, exactly as in the proof of Lemma 10.24. \[\hfill \Box\]
11.5 The upper bound on probabilities

Proof of the upper bound on probabilities. Let us first consider the case \( \tilde{I}(g, \mu) < \infty \), in which case \( \tilde{I}(g, \mu) = \mathcal{H}(\mu|\Phi) \). Let \( \Sigma \) denote a finite set of closed disjoint rectangles, contained in \( D \). Define \( R_n := \Lambda(nR) \) for \( R \in \Sigma \) and \( \Sigma_n := \cup_{R \in \Sigma \cap \Lambda(nR)} R_n \), and note that \( \Sigma_n \subset D_n \) for \( n \) sufficiently large. Now choose for each \( R \in \Sigma \) an open set \( B^R \subset \mathcal{B} \) with \( \mu(R, \cdot)/\text{Leb}(R) \in \mathcal{B} \), and define

\[ A_n := \cap_{R \in \Sigma} B^R_{\Lambda(nR)}. \]

It is straightforward to show that \( (g, \mu) \) has a fixed neighborhood which is contained in all sets \( \Psi_n(A_n) \) for \( n \) sufficiently large. Fix \( \beta > 0 \). It suffices to find an appropriate choice for the set of rectangles \( \Sigma \) and the collection of balls \( (B^R_{\Lambda(nR)})_{R \in \Sigma} \), such that

\[ \limsup_{n \to \infty} n^{-d} \log \hat{\gamma}_n(A_n) \leq -\mathcal{H}(\mu|\Phi) + \beta. \]

Remove all height functions from \( A_n \) which do not equal \( b_n \) on \( \mathbb{Z}^d \setminus D_n \) or which are not \( Kd_1 \)-Lipschitz; this obviously does not change the value of \( \hat{\gamma}_n(A_n) \). As in the proof of the lower bound, we shall sometimes pretend that \( A_n \in \mathcal{E}D_n \) by restricting each height function in \( A_n \) to \( D_n \). If \( \phi \in E^D_n \), then we write \( \psi \) for the height function which restricts to \( \phi \) on \( D_n \) and to \( b_n \) on the complement of \( D_n \). We are thus interested in the asymptotic behavior of

\[ \hat{\gamma}_n(A_n) = \int_{A_n} e^{-H_{D_n}^{n}(\psi)} d\lambda^{D_n}(\phi). \]

The lower attachment lemma (Lemma 7.1) asserts that

\[
H_{D_n} \geq H^0_{D_n \setminus \Sigma_n} - e^{-\mathcal{D}_n} + \sum_{R \in \Sigma} H^0_{R_n} - e^{-\mathcal{R}_n} \\
\geq -\|\Xi\| \cdot |D_n \setminus \Sigma_n| - e^{-\mathcal{D}_n} + \sum_{R \in \Sigma} H^0_{R_n} - e^{-\mathcal{R}_n}.
\]

The terms of the form \( e^{-\cdot} \) are of order \( o(n^d) \) as \( n \to \infty \). Moreover, \( n^{-d}|D_n \setminus \Sigma_n| \to \text{Leb}(D \setminus \cup \Sigma) \) as \( n \to \infty \), and therefore

\[
\limsup_{n \to \infty} n^{-d} \log \hat{\gamma}_n(A_n) \leq \|\Xi\| \text{Leb}(D \setminus \cup \Sigma) + \limsup_{n \to \infty} n^{-d} \log \int_{A_n} e^{-\sum_{R \in \Xi} H^0_{R_n}(\psi)} d\lambda^{D_n}(\phi).
\]

Write \( D^0_n := D_n \setminus \cup_{R \in \Sigma} \{R_n \setminus \{0_{R_n}\}\} \), so that \( \lambda^{D^0_n} = \lambda^{D^0_n} \times \prod_{R \in \Sigma} \lambda^{R_n-1} \). Then

\[
\int_{A_n} e^{-\sum_{R \in \Xi} H^0_{R_n}(\psi)} d\lambda^{D_n}(\phi) \leq \left[ \int_{W_n} d\lambda^{D^0_n} \right] \left[ \prod_{R \in \Sigma} \int_{B^R_{\Lambda(nR)}} e^{-H^0_{R_n}} d\lambda^{R_n-1} \right],
\]

where \( W_n \) is the set of \( Kd_1 \)-Lipschitz functions \( \phi : D^0_n \to E \) such that \( \phi b_n |_{\mathbb{Z}^d \setminus D_n} \) is also \( Kd_1 \)-Lipschitz. Remark that

\[
\log \int_{W_n} d\lambda^{D^0_n} \leq |D^0_n| \log(2K + 1)
\]

and that

\[
\log \int_{B^R_{\Lambda(nR)}} e^{-H^0_{R_n}} d\lambda^{R_n-1} = -FB_{\Lambda(nR)}(B^R).
\]
If we write \( m := \|\Xi\| + \log(2K + 1), \) then we have now shown that
\[
\limsup_{n \to \infty} n^{-d} \log \tilde{\gamma}_n(A_n) \leq m \Leb(D \setminus \cup \Sigma) - \sum_{R \in \Sigma} \Leb(R) \FB(B^R).
\]

It now suffices to show that the expression on the right is at most \(-\cH(\mu|\Phi) + \beta\) for an appropriate choice of the set of rectangles in \( \Sigma \) and for the collection \((B^R)_{R \in \Sigma} \subset \cB\). By choosing the rectangles in \( \Sigma \) such that they exhaust most of the space, we can ensure that \( \Leb(D \setminus \cup \Sigma) \leq \beta/2m \), and by also taking each ball \( B^R \) sufficiently small, we can ensure that \( \sum_{R \in \Sigma} \Leb(R) \FB(B^R) \) is at least \( \cH(\mu|\Phi) - \beta/2 \). This proves the upper bound on probabilities.

Consider now the case that \( \tilde{I}(g,\mu) = \infty \). We distinguish several reasons which may cause \( \tilde{I}(g,\mu) \) to be infinite. If \( \mu \) is shift-invariant but \( \cH(\mu|\Phi) = \infty \), then the proof is the same as before. If \( \mu \) is not shift-invariant, then there is a closed rectangle \( R \subset D \) such that \( \mu(R,\cdot) \) is not shift-invariant, and by including \( R \) in \( \Sigma \) and using the free boundary limits for non-shift-invariant measures, we obtain the same result. In fact, in that case it is readily seen that \( A_n \) is empty for \( n \) sufficiently large. (See also the proof of Lemma 10.7).

The remaining cases are: either \( g|_{\partial D} \) does not equal \( b \), or it is not true that \( \nabla g(x) = S(\mu(x,\cdot)) \) as a distribution on \( D \). Consider first the case that \( g|_{\partial D} \) does not equal \( b \). Choose \( \varepsilon := \|g|_{\partial D} - b\|_{\infty}/2 \). In that case, it is readily seen that \( \tilde{\gamma}_n(\Theta_{n}^{-1}(\cB_{\infty}(g))) = 0 \) for \( n \) sufficiently large. Finally consider the case that it is not true that \( \nabla g(x) = S(\mu(x,\cdot)) \) as a distribution on \( D \). In that case, there is a closed rectangle \( R \subset D \) such that the average of \( \nabla g \) over \( R \) does not equal \( S(\mu(R,\cdot)) \). But if \( \Theta_{n}(\phi) \) is close to \( g \), then \( \Sigma_n(\phi)(R,\cdot) \) must have its approximate slope close to the average of \( \nabla g \) over \( R \). Note that we use the words \textit{approximate slope} here rather than the word \textit{slope}, because \( \Sigma_n(\phi)(R,\cdot) \) is not shift-invariant, but it is almost shift-invariant in the sense that \( \Sigma_n(\phi)(R,f-\theta f) \) goes to zero uniformly over \( \phi \) as \( n \to \infty \) for \( f \) a bounded continuous cylinder function and \( \theta \in \Theta(\cL) \); see the proof of Lemma 10.7. In particular, by including \( R \) in \( \Sigma \) in the previous discussion and choosing \( B^R \) sufficiently small, it can again be seen that \( A_n \) is empty for \( n \) sufficiently large, which leads to the desired bound. \( \square \)

### 11.6 Exponential tightness

**Proof of exponential tightness.** The proof is easy. Fix a positive constant \( \varepsilon > 0 \), and let \( K \) denote the smallest constant such that \( Kd_1 \geq q \). Define
\[
K_{\varepsilon} := \{ g \in \Lip(D) : \|g|_{\partial D} - b\|_{\infty} \leq \varepsilon \}, \quad K^\varepsilon := \{ \mu \in \cM^D : \mu(D,\cdot) \text{ is } K\text{-Lipschitz} \}.
\]

It is clear that \( K_{\varepsilon} \) is compact in \( (\Lip(D),\cX^\infty) \), and that \( K^\varepsilon \) is compact in \( (\cM^D,\cX^\varepsilon) \). This means that \( K_{\varepsilon} \times K^\varepsilon \) is compact in \( (\cX^\Phi,\cX^\Phi) \). As in the proof of the upper bound of probabilities, we observe that \( \tilde{\gamma}_n^\varepsilon \) is supported on \( K_{\varepsilon} \times K^\varepsilon \) for \( n \) sufficiently large. This completes the proof; the compact set that we have found is independent of the choice of \( \alpha \) that appeared in the original formulation of exponential tightness. \( \square \)

### 12 Proof of strict convexity

#### 12.1 The product setting

For the proof of strict convexity of \( \sigma \), it is useful to work in the product setting \( \Omega \times \Omega \), because one is then able to study the difference \( \phi_1 - \phi_2 \) of a pair of height functions \( (\phi_1,\phi_2) \) and apply the theory of moats from Section 5. Almost all constructions and results in
the previous sections generalize to the product setting. An alternative way of viewing the product setting is by considering a height function to take values in the two-dimensional space \( \mathbb{E}^2 \) rather than \( E \). This section gives an overview of the definitions and results for the product setting as required for the proof of strict convexity of \( \sigma \).

Write \( \mathcal{P}^2(\Omega, \mathcal{F}^\nabla) \) for the set of probability measures on \((\Omega, \mathcal{F})^2\) whenever \((\Omega, \mathcal{F})\) is a measurable space. If \( \mu \in \mathcal{P}^2(\Omega, \mathcal{F}) \), then write \( \mu_1 \) and \( \mu_2 \) for the marginals of \( \mu \) on the first and second space respectively.

**Definition 12.1.** The *topology of weak local convergence* is the coarsest topology on \( \mathcal{P}^2(\Omega, \mathcal{F}^\nabla) \) that makes the evaluation map \( \mu \mapsto \mu(f) \) continuous for any bounded continuous cylinder function \( f \) on \( \Omega^2 \), that is, a bounded function \( f : \Omega^2 \to \mathbb{R} \) which is \( \mathcal{F}_\Lambda^\nabla \times \mathcal{F}_\Lambda^\nabla \)-measurable for some \( \Lambda \subset \subset \mathbb{Z}^d \), and continuous with respect to the topology of uniform convergence on \( \Omega^2 \)—the set of functions from \( \mathbb{Z}^d \) to \( \mathbb{E}^2 \).

**Definition 12.2.** Write \( \mathcal{P}^2_2(\Omega, \mathcal{F}^\nabla) \) for the set of \( \mathcal{L} \)-invariant probability measures in \( \mathcal{P}^2(\Omega, \mathcal{F}^\nabla) \); a measure \( \mu \in \mathcal{P}^2_2(\Omega, \mathcal{F}^\nabla) \) is called \( \mathcal{L} \)-invariant if \( \mu(A \times B) = \mu(\theta A \times \theta B) \) for any \( A, B \in \mathcal{F}^\nabla \) and \( \theta \in \Theta \). This is equivalent to asking that \((\phi_1, \phi_2)\) and \((\theta \phi_1, \theta \phi_2)\) have the same distribution under \( \mu \).

**Definition 12.3.** By the *slope* of \( \mu \in \mathcal{P}^2_2(\Omega, \mathcal{F}^\nabla) \) we simply mean the pair of slopes of the two marginals of \( \mu ; \) \( S^2(\mu) := (S(\mu_1), S(\mu_2)) \). The slope functional \( S^2 \) is clearly strongly affine, as in the non-product setting.

**Definition 12.4.** For \( \mu \in \mathcal{P}^2(\Omega, \mathcal{F}^\nabla) \) and \( \Lambda \subset \subset \mathbb{Z}^d \), define the *free energy* of \( \mu \) in \( \Lambda \) by

\[
\mathcal{H}^2_\Lambda(\mu|\Phi) := \mathcal{H}_{\mathcal{F}_\Lambda^\nabla \times \mathcal{F}_\Lambda^\nabla}(\mu|\Lambda^{-1} \times \Lambda^{-1}) + \mu(H^0_\Lambda(\phi_1) + H^0_\Lambda(\phi_2)).
\]

Note that we immediately have

\[
\mathcal{H}^2_\Lambda(\mu|\Phi) \geq \mathcal{H}(\mu_1|\Phi) + \mathcal{H}(\mu_2|\Phi), \tag{12.5}
\]

with equality if and only if the restriction of \( \mu \) to \( \mathcal{F}_\Lambda^\nabla \times \mathcal{F}_\Lambda^\nabla \) decomposes as the product of \( \mu_1 \) and \( \mu_2 \), or if either side equals \( \infty \). If \( \mu \) is \( \mathcal{L} \)-invariant, then define the *specific free energy* of \( \mu \) by

\[
\mathcal{H}^2(\mu|\Phi) = \lim_{n \to \infty} n^{-d} \mathcal{H}^2_{\mathcal{F}_\Lambda^\nabla}(\mu|\Phi).
\]

It follows immediately from (12.5) that \( \mathcal{H}^2(\mu|\Phi) \geq \mathcal{H}(\mu_1|\Phi) + \mathcal{H}(\mu_2|\Phi) \). In particular, this implies that \( \mathcal{H}^2(\mu|\Phi) \geq \sigma(S(\mu_1)) + \sigma(S(\mu_2)) \). For convenience, we shall write \( \sigma^2(u,v) := \sigma(u) + \sigma(v) \). Note that

\[
\sigma^2(u,v) := \inf_{\mu \in \mathcal{P}^2_2(\Omega, \mathcal{F}^\nabla) \text{ with } S^2(\mu) = (u,v)} \mathcal{H}^2(\mu|\Phi).
\]

With these definitions, the following results generalize naturally to the product setting:

1. Theorem 4.1 for existence of the specific free energy,
2. Theorem 4.2 for finite energy, where the result applies if

\[
\mathcal{H}^2(\mu|\Phi) = \sigma^2(S^2(\mu)) < \infty,
\]
3. Theorem 9.1, Proposition 9.3, and Theorem 9.4 for ergodic decompositions,
4. Theorem 10.5 for limit equalities and Theorem 11.5 for the large deviations principle.

Rather than repeating each result here, we state clearly the generalized result that is used whenever referring to it.
12.2 Moats in the empirical limit

In this section, we suppose that $\sigma$ is not strictly convex, and construct the pathological measure which derives from this assumption. Let $K$ denote the smallest real number such that $Kd_1 \geq q$, and write $\rho$ for the uniform probability measure on the set $E \cap [0,4K]$, with random variable $U$. For fixed $(\phi_1, \phi_2, U) \in \Omega \times \Omega \times E$, we shall write $\xi = \xi(\phi_1, \phi_2, U)$ for the function

$$\xi := \frac{1}{4K}(\phi_1 - \phi_1(0) - \phi_2 + \phi_2(0) - U) : \mathbb{Z}^d \to \mathbb{Z}.$$

We will refer to $\xi$ as the difference function associated to the triplet $(\phi_1, \phi_2, U)$. Remark that the law of $\nabla \xi$ is $L$-invariant in $\mu \times \rho$ for any $\mu \in \mathcal{P}_L^2(\Omega, F)$; the random variable $U$ makes the rounding operation shift-invariant as in the proof of Lemma 10.32.

**Theorem 12.6.** Let $\Phi$ denote a potential which is monotone and in $\mathcal{S}_L + W_L$. Assume that $\sigma$ is affine on the line segment $[u_1, u_2]$ connecting two distinct slopes $u_1, u_2 \in U_\Phi$, and set $u = (u_1 + u_2)/2$. Select two vertices $x \in \mathcal{L}$ and $y \in \mathbb{Z}^d$ subject only to $(u_1 - u_2)(x) \neq 0$. Then there exists a product measure $\mu \in \mathcal{P}_L^2(\Omega, F^\nabla)$ such that $S^2(\mu) = (u, u)$ and $\mathcal{H}^2(\mu|\Phi) = \sigma^2(u, u) = 2\sigma(u)$, and such that with positive $\mu \times \rho$-probability, the following two events occur simultaneously:

1. The function $\xi$ is not constant on the set $y + \mathbb{Z}x$,

2. The set $\{\xi = 0\} \subset \mathbb{Z}^d$ has at least three distinct infinite connected components.

In the next section, we discuss rigorously how to derive a contradiction from this theorem (under the additional condition whenever $E = \mathbb{Z}$), using Theorem 4.2 and the argument for uniqueness of the infinite cluster of Burton and Keane [BK89]. The purpose of the remainder of this section is to prove Theorem 12.6.

Let us assume the setting of Theorem 12.6: $\Phi$ is a monotone potential in $\mathcal{S}_L + W_L$, $u_1$ and $u_2$ are distinct slopes in $U_\Phi$ such that $\sigma$ is affine on $[u_1, u_2]$, and $u := (u_1 + u_2)/2$. In the proof of the theorem, we shall suppose that $y = 0$, without loss of generality. Fix $0 < \varepsilon < K$ so small that $u_1, u_2 \in U_\Phi$. We shall use the large deviations principle with the good asymptotic profile $(D, b)$ where $D := (0,1)^d \subset \mathbb{R}^d$ and $b := u|_{\partial D}$, and with the good approximation $(D_n, b_n)$ of $(D, b)$ defined by $D_n := \Pi_n$ and $b_n := \delta^\nu$ for all $n \in \mathbb{N}$. As per usual, we write $\gamma_n := \gamma_{D_n}(\cdot, b_n)$, and we shall also write $\gamma_n^2 := \gamma_n \times \gamma_n$.

Set $t = 1/2$, and recall the definitions of $v, p$, and $p_o$ from the proof of Lemma 10.24 (Page 55). Fix $\varepsilon_1, \varepsilon_2,$ and $\varepsilon_3$ strictly positive and consider $n \in \mathbb{N}$. When taking limits we shall take first $n \to \infty$, then $\varepsilon_3 \to 0$, then $\varepsilon_2 \to 0$, and finally $\varepsilon_1 \to 0$; it is again convenient to work on different scales. Define $D' := (\varepsilon_1, 1 - \varepsilon_1)^d \subset D$ and $D'_n := nD' \cap \mathbb{Z}^d$. Write $H_k$ for the affine hyperplane $\{2v = k\varepsilon_2\} \subset \mathbb{R}^d$. Note that the sets $(H_k)_{k \in \mathbb{Z}}$ correspond to the hyperplanes where the gradient of $p_{\varepsilon_2}$ changes. For $k$ even, $p_{\varepsilon_2}$ equals $u$ on $H_k$. For $k$ odd, $p_{\varepsilon_2}$ equals $u + \varepsilon_2/4$ on $H_k$. Finally, write

$$H_{n,k} := \{x \in \mathbb{Z}^d : d_2(x, nH_k) \leq n\varepsilon_3\},$$

$$D'_n := (\bigcup_{k \in \mathbb{Z}} H_{n,k}) \cap D_n,$$

$$D'_n := (\bigcup_{k \in \mathbb{Z}+1} H_{n,k}) \cap D'_n.$$

See Figure 6 for an overview of this construction.

**Proposition 12.7.** Assume the setting of Theorem 12.6. If $E = \mathbb{Z}$, then there is a $\delta > 0$ such that

$$n^{-d} \log \gamma_n(\phi_{D'_n}) = \phi^u_{D'_n} \text{ and } \phi^u_{D'_n} \geq \phi^u_{D'_n} + n\delta\varepsilon_2) = o(1)$$

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in the limit of \( n, \varepsilon_3, \varepsilon_2, \) and \( \varepsilon_1 \). If \( E = \mathbb{R} \) and \( \varepsilon > 0 \), then there is a \( \delta > 0 \) such that

\[
n^{-d} \log \gamma_n(\|\phi_{D_n^0} - \phi_{D_n^0}^u\| \leq \varepsilon \text{ and } \phi_{D_n^+} \geq \phi_{D_n^+}^u + n\delta\varepsilon_2) = o(1)
\]

in the limit of \( n, \varepsilon_3, \varepsilon_2, \) and \( \varepsilon_1 \).

**Proof.** In fact, we shall demonstrate that any \( \delta < 1/4 \) works. Write \( f \) for the smallest \( \|\cdot\|_q \)-Lipschitz function which satisfies \( f \geq u \) and which equals \( p_{\varepsilon_2} \) on \( D' \). This function is well-defined and equals \( u \) on \( \mathbb{R}^d \setminus D \) for \( \varepsilon_2 \) sufficiently small (depending only on \( \varepsilon_1 \)).

The pressure \( P_\Phi(D, b) \) is equal to \( \sigma(u) \) because \( \sigma \) is convex, \( b = u|_{\partial D} \), and \( \text{Vol}(D) = 1 \). Moreover, \( \int_D \sigma(\nabla f(x)) \, dx \) tends to \( \sigma(u) \) in the limit of \( \varepsilon_1 \) and \( \varepsilon_2 \), because \( \sigma \) is affine on the line segment connecting \( u_1 \) and \( u_2 \), and because the gradient of \( f \) equals \( u_1 \) on roughly half of \( D \), and \( u_2 \) on roughly the other half of \( D \) with respect to Lebesgue measure. Note that \( \sigma(\nabla f) \) is bounded uniformly as \( f \) is \( \|\cdot\|_q \)-Lipschitz. This means that for any \( \varepsilon' > 0 \), which is allowed to depend arbitrarily on \( \varepsilon_1 \) and \( \varepsilon_2 \), we have

\[
n^{-d} \log \gamma_n(\mathcal{G}_n^{-1}(B_\varepsilon^\infty(f))) = o(1)
\]

in the limit of \( n, \varepsilon_2, \) and \( \varepsilon_1 \).

Note that for \( \varepsilon_3 \) and \( \varepsilon' \) sufficiently small depending on \( \varepsilon_1 \) and \( \varepsilon_2 \), all height functions \( \phi \in \mathcal{G}_n^{-1}(B_\varepsilon^\infty(f)) \) satisfy \( \phi_{D_n^+} \geq \phi_{D_n^+}^u + n\delta\varepsilon_2 \) (by virtue of the choice of \( f \)). Moreover, \( \phi_{D_n^0} \) and \( \phi_{D_n^0}^u \) must be close for such \( \phi \). By repeating arguments of the proof of the lower bound on probabilities in the large deviations principle, it is straightforward to see that conditioning further on the exact values of \( \phi_{D_n^0} \) (up to \( \varepsilon \) in the continuous case) does not decrease the value of the limit of the normalized probabilities. In particular, this implies the proposition. 

By interchanging the role of \( u_1 \) and \( u_2 \), one obtains the same result as in Proposition 12.7, now with the inequality sign \( \geq \) replaced by \( \leq \), and with \( n\delta\varepsilon_3 \) replaced by \( -n\delta\varepsilon_2 \). By appealing to both the original proposition and the version with replacements, one deduces immediately the following proposition.
Proposition 12.8. Assume the setting of Theorem 12.6. If $E = \mathbb{Z}$, then there exists a $\delta > 0$ such that
\[
 n^{-d} \log \gamma_n^2((\phi_1 - \phi_2)_{D_n}) = 0 \quad \text{and} \quad (\phi_1 - \phi_2)_{D_n^+} \geq n\delta \epsilon_2 = o(1)
\]
in the limit of $n, \epsilon_3, \epsilon_2,$ and $\epsilon_1$. If $E = \mathbb{R}$ and $\epsilon > 0$, then there is a $\delta > 0$ such that
\[
 n^{-d} \log \gamma_n^2((\phi_1 - \phi_2)_{D_n}) \leq 2\epsilon \quad \text{and} \quad (\phi_1 - \phi_2)_{D_n^+} \geq n\delta \epsilon_2 = o(1)
\]
in the limit of $n, \epsilon_3, \epsilon_2,$ and $\epsilon_1$.

Recall Section 5 on moats; we are now ready to apply the theory developed there. If $k$ is odd with $H_{n,k} \cap D_n'$ nonempty, then write $\Lambda_{n,k} := H_{n,k} \cap D_n'$. Note that $D_n^+ = \cup_k \Lambda_{n,k}$. Write also $\Delta_{n,k}$ for the connected component of $D_n \setminus D_n^0$ containing $\Lambda_{n,k}$; see Figure 6 for an example of the sets $\Lambda_{n,k}$ and $\Delta_{n,k}$. Write $E_n^a(m)$ for the event that each connected component $\Delta_{n,k}$ contains a sequence of $\lceil m \rceil$ nested $4K, 4K + a$-moats of $(\phi_1 - \phi_2, \Lambda_{n,k})$.

Lemma 12.9. Assume the setting of Theorem 12.6. For any $a \geq 4K$, there is a $\delta > 0$ such that
\[
 n^{-d} \log \gamma_n^2(E_n^{a}(n\delta \epsilon_2)) = o(1)
\]
in the limit of $n, \epsilon_3, \epsilon_2,$ and $\epsilon_1$.

Proof. This follows immediately from the previous proposition and from Proposition 5.11. Note that the prefactor which appears on the left in (5.5) is of order $n^{O(1/\epsilon_2)} - O(n^2 \delta \epsilon_2)$, because distances are bounded by $n$, there are at most $O(1/\epsilon_2)$ sets $\Lambda_{n,k}$, and because we enforce $n\delta \epsilon_2$ moats around each set $\Lambda_{n,k}$. In particular, keeping all constants other than $n$ fixed, the logarithm of this term is of order $O(n \log n)$, which disappears in the normalization because we normalize by $n^{-d}$ with $d \geq 2$.

Proof of Theorem 12.6. Let us consider a configuration $(\phi_1, \phi_2) \in E_n^a(n\delta \epsilon_2)$, and focus on the collection of moats of $f := \phi_1 - \phi_2$. Fix $x \in \mathcal{L}$ with $u_1(x) - u_2(x) \neq 0$, and define $L := \mathbb{Z}x$ and $L_N := \{-N, \ldots, N\}x$. Write $\bar{L}_N$ for a path through the square lattice of minimal length traversing all the vertices in $L_N$. Draw some vertex $z$ from $L \cap D_n$ uniformly at random, and write $L^z := L + z$, $L_N^z := L_N + z$, and $\bar{L}_N^z := \bar{L}_N + z$. We are interested in the line $L^z$, and the way this line intersects the moats of $f$. We make a series of important geometrical observations. By saying that a quantity is uniformly positive, we mean that it has a strictly positive lower bound which is independent of the four parameters, for $n$ sufficiently large and for $\epsilon_3, \epsilon_2,$ and $\epsilon_1$ sufficiently small.

1. If $a$ is at least $(4\sqrt{2}m)K$, then the $d_1$-distance from the inside to the outside of a fixed climbing or descending $4K, 4K + a$-moat is at least $\lceil m \rceil + 1$, as $f$ is $2K$-Lipschitz (See Proposition 5.3, Statement 3). If $m \geq d_1(0, x)$ and if $L_N^z$ intersects both the inside and outside of some moat, then $L_N^z$ must also intersect that moat. In particular, if $L^z$ intersects $\Lambda_{n,k}$, then $L^z$ must necessarily also intersect all moats surrounding $\Lambda_{n,k}$. In the sequel, we choose $a' := (4\sqrt{2}d_1(0, x))K$ and $a = 3a'$.

2. With uniformly positive probability, $z$ lies in $\Delta_{n,k}$ with $L^z$ intersecting $\Lambda_{n,k}$, for some odd integer $k$. This is illustrated by Figure 6; it is important here that $(u_1 - u_2)(x) \neq 0$ so that $x$ does not lie in the hyperplane $\{u_1 - u_2 = 0\}$. Let us suppose that such an odd integer $k$ indeed exists. Write $m^\pm$ for the smallest and largest integer respectively such that $z + m^\pm x \in \Delta_{n,k}$. Then $m^+ - m^- \leq O(n \epsilon_2)$, where the constant is independent of all four parameters. But $\Delta_{n,k}$ contains a sequence of $\lceil n\delta \epsilon_2 \rceil$ nested $4K, 4K + a$-moats.
of $\Lambda_{n,k}$; $L^z$ intersects each one of them. These moats thus have a uniformly positive density in the set $z + \{m^-, \ldots, m^+\}x$. But $z$ was chosen uniformly random from $\mathcal{L} \cap D_n$ and therefore we may rerandomize its position within $z + \{m^-, \ldots, m^+\}x$. Since the moats are disjoint from one another and have a positive density within this set, we observe there exists a fixed constant $N \in \mathbb{N}$ such that $L^z_N$ intersects at least five distinct nested moats with uniformly positive probability. In fact, each $4K, 4K + a$-moat contains a $4K, 4K + a'$-moat (Proposition 5.3, Statement 7), and $z$ is contained in such a moat with uniformly positive probability. Therefore, the event that $L^z_N$ intersects at least five distinct nested $4K, 4K + a$-moats, and simultaneously $f(z) \in [4K, 4K + a')$, has uniformly positive probability.

3. Let us mention a first consequence of the event described above. Since $L^z_N$ intersects more than three distinct $4K, 4K + a$-moats, it must intersect both the inside and outside of the middle moat. This moat contains both a $4K, 4K + a'$-moat, as well as a $4K + 2a', 4K + 3a'$-moat, which $L^z_N$ must both intersect. The value of $f$ differs by at least $a' \geq 4K$ on these two moats. In particular, $\xi = \xi(\phi_1, \phi_2, U)$ cannot be constant on $L^z_N$, regardless of the value of $U$. Similarly, $\xi(\theta_x\phi_1, \theta_z\phi_2, U)$ cannot be constant on $L^z_N$.

4. Let us mention a second consequence. Since the set $L^z_N$ intersects five distinct nested $4K, 4K + a$-moats, it must intersect both the inside and outside of the three middle moats. Fix $U \in [0, 4K]$, and write $a'' := f(z) + U \in [4K, 4K + 2a')$. The set $L^z_N$ must intersect three $a'', a'' + 4K$-moats: each of the three middle $4K, 4K + a$-moats contains a $a'', a'' + 4K$-moats which $L^z_N$ must also intersect. But these three moats correspond exactly to connected components of $\{\xi = 0\}$ for $\xi := \xi(\theta_x\phi_1, \theta_z\phi_2, U)$, which are intersected by $L^z_N$. We must however limit ourselves to local observations, as we always work in the topology of (weak) local convergence. Write therefore $\Sigma_m := \{-m, \ldots, m\}^d \subset \mathbb{Z}^d$; we only consider $m$ so large that $L^z_N \subset \Sigma_m$. The previous observation means that for any $m \in \mathbb{N}$, $\{\xi = 0\} \cap \Sigma_m$ has three connected components which intersect both $L^z_N$ and $\partial^1 \Sigma_m$, at least if $n$ is sufficiently large—this is because each moat must surround some set $\Lambda_{n,k}$, which grows large whenever $n$ is large.

Let us summarize what we have done so far. We proved that there exist constants $N \in \mathbb{N}$ and $\delta' > 0$ with the following properties. Choose $(\phi_1, \phi_2) \in F^0_n(n\delta \varepsilon_2)$, and choose $z \in \mathcal{L} \cap D_n$ uniformly at random. Then for fixed $m \in \mathbb{N}$, the probability that for any $U \in [0, 4K)$,

1. $\xi := \xi(\theta_x\phi_1, \theta_z\phi_2, U)$ is not constant on $L^z_N$,

2. $\{\xi = 0\} \cap \Sigma_m$ has three connected component which intersect both $L^z_N$ and $\partial^1 \Sigma_m$,

is at least $\delta'$, for $n$ sufficiently large depending on $m$, and for $\varepsilon_3, \varepsilon_2$, and $\varepsilon_1$ small.

In the final part of the proof, we use this intermediate result, as well as the large deviations principle and compactness of the lower level sets $M_C$ of the specific free energy, to construct the desired measure for Theorem 12.6.

Let us first consider the case $E = \mathbb{Z}$. Consider $m \in \mathbb{N}$ so large that $L^z_N \subset \Sigma_m$, and write $A_m \in F^\Sigma_{\Sigma_m} \times F^\Sigma_{\Sigma_m}$ for the event that for any $U \in [0, 4K)$, the function $\xi := \xi(\phi_1, \phi_2, U)$ is not constant on $L^z_N$, and that $\{\xi = 0\} \cap \Sigma_m$ has three connected components intersecting both $\partial^1 \Sigma_m$ and $L^z_N$. Write $B_m$ for the set of measures $\mu \in \mathcal{P}^2(\Omega, F^\Sigma)$ such that $\mu(A_m) > \delta'/2$. Note that $B_m$ is in the basis for the topology of weak local convergence on the space of
product measures $\mathcal{P}^2(\Omega, \mathcal{F}^\nabla)$. Recall the definition of $\mathcal{L}_n(\phi)$ in Subsection 11.1.1, and define, for the product setting,

$$\mathcal{L}_n^2(\phi_1, \phi_2) := \int_D \delta_{(x, \theta_n x, \ldots, \theta_n x)}(\phi_1, \phi_2) dx \in \mathcal{M}_D^\nabla,$$

where $\mathcal{M}_D^\nabla$ we mean the set of measures in $\mathcal{M}(D \times \Omega \times \Omega, D \times \mathcal{F}^{\nabla} \times \mathcal{F}^{\nabla})$ for which the first marginal equals the Lebesgue measure on $D$. By Lemma 12.9 and the intermediate result, we know that

$$n^{-d} \log \gamma_n^2(\mathcal{L}_n^2(D, \cdot) \in B_m) = o(1)$$

as $n \to \infty$. It therefore follows from the large deviations principle that $\bar{B}_m$ contains a shift-invariant measure $\mu_m \in \mathcal{P}^2_2(\Omega, \mathcal{F}^{\nabla})$ with $S^2(\mu_m) = (u, u)$ and $\mathcal{H}^2(\mu_m) \leq 2\sigma(u)$. In particular, this means that $\mu_m(A_m) \geq \delta'/2$, and in fact $\mu_m(A_{m'}) \geq \delta'/2$ for all $m' \leq m$ because $A_m \subset A_{m'}$ for $m' \leq m$. By compactness of the lower level sets of the specific free energy, the sequence $(\mu_m)_{m \in \mathbb{N}}$ has a subsequential limit $\mu \in \mathcal{P}^2_2(\Omega, \mathcal{F}^{\nabla})$ in the topology of local convergence which satisfies $S^2(\mu) = (u, u)$ and $\mathcal{H}^2(\mu) \leq 2\sigma(u)$. In particular, $\mu(A_m) \geq \delta'/2$ for all $m$, which means that $\mu$ satisfies all the requirements of Theorem 12.6; the intersection $\cap_m A_m$ of the decreasing sequence $(A_m)_{m \in \mathbb{N}}$ is precisely the event that $\{\xi = 0\}$ has three infinite level sets which intersect $\mathcal{L}_N$, regardless of the value of $U$.

In the case that $E = \mathbb{R}$, there is a slight complication. If $E = \mathbb{R}$, then the indicator $1_{A_m}$ is not continuous with respect to the topology of uniform convergence on $\Omega^2$, and therefore the sets $B_m$ as defined above are not in the basis of the topology of weak local convergence. Introduce therefore the sequence of functions $(f_{m,k})_{k \in \mathbb{N}}$ where each function $f_{m,k} : \Omega^2 \to [0, 1]$ is defined by $f_{m,k}(\phi_1, \phi_2) := 0 \vee (1 - kd_\infty(A_m, (\phi_1, \phi_2)))$; here $d_\infty$ denotes the metric corresponding to the norm $\| \cdot \|_\infty$ on $\Omega^2$. Write $B_{m,k}$ for the set of product measures $\mu$ such that $\mu(f_{m,k}) > \delta'/2$. Then $B_{m,k}$ is in the basis of the topology of weak local convergence, and we have

$$n^{-d} \log \gamma_n^2(\mathcal{L}_n^2(D, \cdot) \in B_{m,k}) = o(1)$$

as $n \to \infty$. Therefore $\bar{B}_{m,k}$ contains a measure $\mu_{m,k}$ with $S^2(\mu_{m,k}) = (u, u)$ and $\mathcal{H}^2(\mu_{m,k}) \leq 2\sigma(u)$. Moreover, the sequence of measures $(\mu_{m,k})_{k \in \mathbb{N}}$ must have a subsequential limit $\mu_m$ in the topology of local convergence, and this limit must satisfy $\mathcal{H}^2(\mu) \leq 2\sigma(u)$, $S^2(\mu) = (u, u)$, and $\mu_m(f_{m,k}) \geq \delta'/2$ for all $k$. The dominated convergence theorem says that $\mu_m(A_m) = \mu_m(\lim_k f_{m,k}) \geq \delta'/2$. But $\mu_m(\partial A_m) = 0$, since $\mu_m$ has finite specific free energy and is therefore locally absolutely continuous with respect to the Lebesgue measure. In particular, $\mu_m(A_m) \geq \delta'/2$. Take now a subsequential limit of the sequence $(\mu_m)_{m \in \mathbb{N}}$ for the desired measure. For this last step, it is important that the topology of local convergence and the topology of weak local convergence coincide on the lower level sets of the specific free energy.

\[ \square \]

12.3 **Application of the argument of Burton and Keane**

In this subsection we prove Theorem 4.12, which is equivalent to the conjunction of Theorem 12.13 and Theorem 12.14. Recall the definition of $\rho$ and $\xi$ in the previous subsection.

**Lemma 12.10.** Let $\Phi$ denote any potential in $\mathcal{S}_\mathcal{L} + \mathcal{W}_\mathcal{L}$, and consider a measure $\mu \in \mathcal{P}^2_2(\Omega, \mathcal{F}^{\nabla})$. Then one of the following properties must fail:

1. $\mu$ is ergodic and at least one of $S(\mu_1)$ and $S(\mu_2)$ lies in $U_\Phi$,
2. \( \mu \) is a minimizer in the sense that \( \mathcal{H}^2(\mu|\Phi) = \sigma^2(S^2(\mu)) < \infty \).

3. With positive \( \mu \times \rho \)-probability, \( \{\xi = 0\} \) has at least three infinite components.

The proof uses a construction from the part in [Lam19] on strict convexity.

Proof of Lemma 12.10. For a fixed configuration \((\phi_1, \phi_2, U)\), a trifurcation box is a finite set \( \Lambda \subset \subset \mathbb{Z}^d \) such that for some \( a \in \mathbb{Z} \), the set \( \{\xi = a\} \cap \Lambda \) has three infinite connected components, which are contained in a single connected component of \( \{\xi = a\} \). If \( \mu \) is shift-invariant then almost surely \( \mu \times \rho \) has no trifurcation boxes, due to the argument of Burton and Keane [BK89]. Note that it is important for this statement that the gradient is shift-invariant in \( \mu \times \rho \). To arrive at the desired contradiction, we aim to prove that trifurcation boxes occur with positive probability for the measure \( \mu \) described in the statement of the lemma.

Write \( \Omega_q^2 \) for the set of pairs of \( q \)-Lipschitz height functions. The natural adaptation of Theorem 4.2 to the product setting asserts that

\[
1_{\Omega_q^2}(\lambda^A \times \lambda^A \times \mu_{\mathbb{Z}^d \setminus \Lambda}) \times \rho \ll \mu \times \rho
\]  

(12.11)

for any \( \Lambda \subset \subset \mathbb{Z}^d \), where by \( \mu_{\mathbb{Z}^d \setminus \Lambda} \) we mean the product measure \( \mu \) restricted to the vertices in the complement of \( \Lambda \), as in the non-product setting. Therefore it suffices to demonstrate that trifurcation boxes occur with positive measure in the measure on the left in the display, for some \( \Lambda \subset \subset \mathbb{Z}^d \).

Suppose, without loss of generality, that \( S(\mu_1) \in U_\Phi \). Write \( \Sigma_n := \{-n, \ldots, n\}^d \subset \subset \mathbb{Z}^d \) for \( n \in \mathbb{N} \). Then for some fixed \( n \in \mathbb{N} \), three infinite components of \( \{\xi = 0\} \) intersect \( \Sigma_n \) with positive \( \mu \times \rho \)-probability. Moreover, as \( \mu \) is ergodic with \( S(\mu_1) \in U_\Phi \), we observe that the two functions

\[
(\phi_1 \pm 8nK)|_{\Sigma_n} \phi_1|_{\mathbb{Z}^d \setminus \Sigma_N}
\]  

(12.12)

are \( q \)-Lipschitz with high \( \mu \)-probability as \( N \to \infty \). This is due to Lemma 6.1, Theorem 10.15 and because \( S(\mu_1) \in U_\Phi \)—recall for comparison the pyramid construction from the proof of Lemma 10.14. In particular, for \( N \geq n \) sufficiently large, the \( \mu \times \rho \)-probability that three infinite components of \( \{\xi = 0\} \) intersect \( \Sigma_n \) and simultaneously the two functions in (10.14) are \( q \)-Lipschitz, is positive. Now choose \( x \in \mathbb{L} \) such that \( 0 \not\in \Sigma_N + x \), and write \( \Sigma_n := \Sigma_n + x \) and \( \Sigma'_n := \Sigma_N + x \). Due to shift-invariance, have now proven that with positive \( \mu \times \rho \)-probability, \( \Sigma_n \) intersects three connected components of \( \{\xi = a\} \) for some \( a \in \mathbb{Z} \), and the two functions in (12.12) are \( q \)-Lipschitz for \( \Sigma_n \) and \( \Sigma_N \) replaced by \( \Sigma'_n \) and \( \Sigma'_N \) respectively. Let us write \( A \) for this event.

Let us first discuss the discrete setting \( E = \mathbb{Z} \). If \( (\phi_1, \phi_2, U) \in A \), then there exists another \( q \)-Lipschitz function \( \phi'_1 \in \Omega \) which equals \( \phi_1 \) on the complement of \( \Sigma'_N \), and such that \( \{\xi = a\} \cup \Sigma'_n \subset \{\xi' = a\} \) where \( \xi' := (\phi'_1, \phi_2, U) \). In particular, this means that \( \Sigma'_N \) is a trifurcation box for \( \xi' \). For example, one can take \( \phi'_1 \) to be the smallest \( q \)-Lipschitz extension of \( \phi_1|_{\mathbb{Z}^d \setminus \Sigma_N} \) to \( \mathbb{Z}^d \) which equals at least

\[
\phi_2 + 4Ka + U + (\phi_1(0) - \phi_2(0))
\]

on \( \{\xi = a\} \cup \Sigma'_n \). This proves that the event that \( \Sigma'_N \) is a trifurcation box has positive measure in the measure on the left in (12.11) if we choose \( \Lambda = \Sigma'_n \). If \( E = \mathbb{R} \), then we must show that not only such a \( q \)-Lipschitz function \( \phi'_1 \) exists, but also that the set of such functions \( \phi'_1 \) has positive Lebesgue measure. The original measure \( \mu \) has finite specific free energy and therefore almost surely the height functions \( \phi_1 \) and \( \phi_2 \) are not taut, that is, for every \( \Lambda \subset \subset \mathbb{Z}^d \) there almost surely exists a positive constant \( \varepsilon > 0 \) such that the restriction
of \( \phi_1 \) and \( \phi_2 \) to \( \Lambda \) are \( q_\varepsilon \)-Lipschitz. Now choose \( \Lambda \) so large that \( \Sigma_N^f \subset \Lambda^{-R} \), choose \( \varepsilon \) at least so small that \( S(\mu_1) \in U_{q_\varepsilon} \), and construct the initial height function \( \phi'_1 \) such that it is also \( q_\varepsilon \)-Lipschitz. It is easy to see that one can employ the remaining flexibility granted by Proposition 6.5, Statement 3 to demonstrate that the set of of suitable height functions has positive Lebesgue measure.

Theorem 12.13. Let \( \Phi \) denote a potential which is monotone and in \( S_{L} + W_{L} \). If \( E = \mathbb{R} \), then \( \sigma \) is strictly convex on \( U_{\Phi} \).

Proof. Let \( \mu \) denote the measure from Theorem 12.6, and write \( w_\mu \) for its ergodic decomposition. The measure \( \mu \) satisfies \( H^2(\mu; \Phi) = \sigma^2(S^2(\mu)) < \infty \), and both \( H^2(\cdot; \Phi) \) and \( S^2(\cdot) \) are strongly affine. This implies that \( w_\mu \)-almost every measure \( \nu \) satisfies \( H^2(\nu; \Phi) = \sigma^2(S^2(\nu)) < \infty \). Since \( E = \mathbb{R} \), this implies also that \( S(\nu_1), S(\nu_2) \in U_{\Phi} \).

With positive \( w_\mu \)-probability, the \( \nu \times \rho \)-probability that \( \{ \xi = 0 \} \) has at least three distinct infinite connected components, is positive. We have now proven the existence of a measure which satisfies all criteria of Lemma 12.10. This is the desired contradiction. \( \square \)

Theorem 12.14. Let \( \Phi \) denote a potential which is monotone and in \( S_{L} + W_{L} \). Consider now the discrete case \( E = \mathbb{Z} \). Suppose that \( \sigma \) satisfies the following property: for any affine map \( h : (\mathbb{R}^d)^* \to \mathbb{R} \) such that \( h \leq \sigma \), the set \( \{ h = \sigma \} \cap \partial U_\Phi \) is convex. Then \( \sigma \) is strictly convex on \( U_{\Phi} \). In particular, \( \sigma \) is convex on \( U_{\Phi} \) if at least one of the following conditions is satisfied:

1. \( \sigma \) is affine on \( \partial U_{\Phi} \), but not on \( U_{\Phi} \).

2. \( \sigma \) is not affine on \( [u_1, u_2] \) for any distinct \( u_1, u_2 \in \partial U_{\Phi} \) such that \( [u_1, u_2] \notin \partial U_{\Phi} \).

Proof. Suppose that \( \sigma \) satisfies the property in the statement. Let \( h : (\mathbb{R}^d)^* \to \mathbb{R} \) denote an affine map such that \( h \leq \sigma \), and such that the set \( \{ h = \sigma \} \cap \partial U_{\Phi} \) contains at least two slopes. We aim to derive a contradiction.

Let us first cover the case that \( \{ h = \sigma \} \subset U_{\Phi} \). Let \( \mu \) denote the measure from Theorem 12.6, with slope \( S(\mu) = (u, u) \) for some \( u \in \{ h = \sigma \} \). Write \( w_\mu \) for the ergodic decomposition of \( \mu \). Then \( w_\mu \)-almost surely \( S(\nu_1), S(\nu_2) \in \{ h = \sigma \} \subset U_{\Phi} \), and therefore the proof is the same as for the real case.

Let us now discuss the case that \( \{ h = \sigma \} \) intersects \( \partial U_{\Phi} \). Recall Lemma 6.1. Since \( \{ h = \sigma \} \cap \partial U_{\Phi} \) is convex, this intersection must be contained in the boundary of one of the half-spaces \( H = H(p) \) contributing to the intersection in Lemma 6.1, where \( p = (p_k)_{0 \leq k \leq n} \) is a path of finite length through \( (\mathbb{Z}^d, A) \) with \( p_n - p_0 \in L \). Set \( y := p_0 \) and \( x := p_n - p_0 \). If a shift-invariant measure in \( \mathcal{P}_{L}(\Omega, \mathcal{F}^V) \) has finite specific free energy and its slope in \( \partial H(p) \), then the random function \( \phi \) must satisfy

\[
\phi(y + kx) - \phi(y) = kq(p) := k \sum_{k=1}^{n} q(p_{k-1}, p_k)
\]  

(12.15)

for any \( k \in \mathbb{Z} \) almost surely. As \( x \) is orthogonal to \( \partial H(p) \), it is straightforward to find two distinct slopes \( u_1, u_2 \in \{ h = \sigma \} \cap \partial U_\Phi \) such that \( (u_1 - u_2)(x) \neq 0 \).

Let \( \mu \) denote the measure from Theorem 12.6, and write \( w_\mu \) for its ergodic decomposition. The measure \( \mu \) satisfies \( H^2(\mu; \Phi) = \sigma^2(S^2(\mu)) \), and both \( H^2(\cdot; \Phi) \) and \( S^2(\cdot) \) are strongly affine. This implies that \( w_\mu \)-almost every measure \( \nu \) satisfies \( H^2(\nu; \Phi) = \sigma^2(S^2(\nu)) < \infty \). We know that \( w_\mu \)-almost surely \( S(\nu_1) \) and \( S(\nu_2) \) lie in \( \{ h = \sigma \} \subset U_{\Phi} \), but it is not guaranteed that these slopes lie in \( U_{\Phi} \).
With positive $w_\mu$-probability, the $\nu \times \rho$-probability that $\xi$ is not constant on $y + \mathbb{Z}x$ and that $\{\xi = 0\}$ has at least three distinct infinite connected components, is positive. But if $\xi$ is not constant on $y + \mathbb{Z}x$, then (12.15) is false for $\phi$ having the distribution of either $\nu_1$ or $\nu_2$, or both, and consequently at least one of $S(\nu_1)$ and $S(\nu_2)$ does not lie in $\partial H(p)$. Conclude that with positive $w_\mu$-probability, at least one of $S(\nu_1)$ and $S(\nu_2)$ lies in $U_\Phi$, and the $\nu \times \rho$-probability that $\{\xi = 0\}$ has three or more infinite connected components, is positive. We have now proven the existence of a measure which satisfies all criteria of Lemma 12.10. This is the desired contradiction.

13 Applications

13.1 The Holley criterion

Each time we apply the theory, we must verify that the specification associated to the model of interest is monotone. An interesting property of stochastic monotonicity is that it does not depend on any formalism and can be checked through the Holley criterion. This criterion is usually stated in the context of the Ising model or Fortuin-Kasteleyn percolation (see for example [Gri18]) but can be extended to random surfaces in a straightforward way. Throughout this section, we will use this criterion in combination with Theorem 4.12 to prove the strict convexity of the surface tension for various interesting models.

**Theorem 13.1** (Holley criterion). The potential $\Phi \in \mathcal{S}_\mathcal{L} + \mathcal{W}_\mathcal{L}$ is monotone if and only if for any two $q$-Lipschitz functions $\phi, \psi \in \Omega$ with $\phi \leq \psi$ and for any $x \in \mathbb{Z}^d$, we have

$$\gamma_{\{x\}}(\cdot, \phi) \leq \gamma_{\{x\}}(\cdot, \psi).$$

**Proof.** Choose $\phi$ and $\psi$ as in the statement of the theorem, and consider $\Lambda \subset \subset \mathbb{Z}^d$. We aim to demonstrate that

$$\gamma_{\Lambda}(\cdot, \phi) \preceq \gamma_{\Lambda}(\cdot, \psi).$$

Write $\kappa_{\Lambda}$ for the probability kernel associated with Glauber dynamics, that is,

$$\kappa_{\Lambda} := |\Lambda|^{-1} \sum_{x \in \Lambda} \gamma_{\{x\}}.$$

It is clear under the assumption of the theorem that $\kappa_{\Lambda}$ preserves the partial order $\preceq$ on $q$-Lipschitz measures. Claim now that

$$\mu \kappa_{n,\Lambda} \to \mu \gamma_{\Lambda}$$

in the strong topology as $n \to \infty$ for any $q$-Lipschitz probability measure $\mu$; this would indeed imply the theorem. This is a standard fact in probability theory. The only detail requiring attention is that it is necessary for any $q$-Lipschitz function $\phi$, that $\gamma_{\Lambda}(\cdot, \phi)$-almost every height function $\psi$ is accessible from $\phi$ by local moves, that is, by updating the value of $\phi$ by one vertex in $\Lambda$ at a time, and such that all intermediate functions are also $q$-Lipschitz. This is straightforward to check from the definition of $q$—in particular, it is important that $q(x, y) + q(y, x) > 0$ for any $x, y \in \mathbb{Z}^d$ distinct.

13.2 Submodular potentials

A potential $\Phi$ is said to be *submodular* if for every $\Lambda \subset \subset \mathbb{Z}^d$, $\Phi_{\Lambda}$ has the property that

$$\Phi_{\Lambda}(\phi \land \psi) + \Phi_{\Lambda}(\phi \lor \psi) \leq \Phi_{\Lambda}(\phi) + \Phi_{\Lambda}(\psi).$$
Sheffield proposes this family of potentials as a natural generalization of simply attractive potentials, and asks if similar results as the ones proved for simply attractive potentials in [She05] could be proved for finite-range submodular potentials. It is easy to see that submodular potentials generate monotone specifications.

**Lemma 13.2.** A submodular potential is monotone.

**Proof.** Let \( \phi_1, \phi_2 \in \Omega \) denote \( q \)-Lipschitz functions with \( \phi_1 \leq \phi_2 \). It suffices to check the Holley criterion (Theorem 13.1). Write \( f_i \) for the Radon-Nikodym derivative of \( \gamma_{\{x\}}(\cdot)\pi_{\{x\}} \) with respect to \( \lambda \), for \( i \in \{1, 2\} \). It suffices to demonstrate that \( f_1 \lambda \leq f_2 \lambda \) as measures on \((E, \mathcal{E})\). Submodularity of \( \Phi \) implies that \( f_1(b)f_2(a) \leq f_1(a)f_2(b) \) for \( \lambda \times \lambda \)-almost every \( a, b \in E \) with \( a \leq b \). It is a simple exercise to see that this implies the desired stochastic domination. \( \square \)

If \( E = \mathbb{R} \) and \( \Phi \) a submodular Lipschitz potential fitting the framework of this article (which is a very mild requirement), then we derive immediately from Theorem 4.12 that the surface tension is strictly convex.

**Corollary 13.3.** Suppose that \( E = \mathbb{R} \) and consider a submodular Lipschitz potential \( \Phi \in \mathcal{S}_L + \mathcal{W}_L \). Then \( \sigma \) is strictly convex on \( U_\Phi \).

In the remainder of this section, we focus on the case \( E = \mathbb{Z} \). If \( E = \mathbb{Z} \), then we cannot immediately conclude that the surface tension is strictly convex, because we must fulfill the additional condition in Theorem 4.12. We demonstrate how to derive this extra condition for many natural discrete models. Let \((\Lambda, q)\) denote the local Lipschitz constraint associated with the potential of interest and fix \( R \in \mathbb{N} \) minimal subject to \( d_1(x, y) \leq R \) for all \( \{x, y\} \in \Lambda \).

A measure \( \mu \in \mathcal{P}_L(\Omega, \mathcal{F}^\mathbb{V}) \) is called frozen if for any \( \Lambda \subset \mathbb{Z}^d \), the values of the random function \( \phi_\Lambda \) in \( \mu \) depend deterministically on the boundary values \( \phi_{\partial^R\Lambda} \). Call a local Lipschitz constraint freezing if any measure \( \mu \in \mathcal{P}_L(\Omega, \mathcal{F}^\mathbb{V}) \) which is supported on \( q \)-Lipschitz functions, and which has \( S(\mu) \in \partial U_\Phi \), is frozen. This condition on the local Lipschitz constraint implies that any such measure has zero specific entropy, that is, \( \mathcal{H}(\mu|\lambda) = 0 \). Indeed, deterministic dependence implies that

\[
\mathcal{H}_{\mathcal{F}^\mathbb{V}}(\mu|\lambda^{\Pi_n}) = \mathcal{H}_{\mathcal{F}^\mathbb{V}}(\mu|\lambda^{\partial^R\Pi_n}) = O(n^{d-1}) = o(n^d)
\]

as \( n \to \infty \).

**Lemma 13.4.** If the local Lipschitz constraint \((\Lambda, q)\) is invariant by the full lattice \( \mathcal{L} = \mathbb{Z}^d \), then it is freezing. In particular, the local Lipschitz constraints corresponding to dimer models, the six-vertex model, and \( Kd_1 \)-Lipschitz functions for \( K \in \mathbb{N} \), are freezing.

**Proof.** Fix \( \mu \in \mathcal{P}_L(\Omega, \mathcal{F}^\mathbb{V}) \) with \( S(\mu) \in \partial U_\Phi \) and supported on \( q \)-Lipschitz functions. As in the proof of Theorem 12.14, there is a path \( p = (p_k)_{0 \leq k \leq n} \) of finite length through \((\mathbb{Z}^d, \Lambda)\) with \( x := p_n - p_0 \in \mathcal{L} \setminus \{0\} \), such that

\[
\phi(p_0 + y + kx) - \phi(p_0 + y)
\]

is deterministic in \( \mu \) for any \( y \in \mathcal{L} \) and \( k \in \mathbb{Z} \). Moreover, this path is a cycle lift as defined in the proof of Lemma 6.1. Since \( \mathcal{L} = \mathbb{Z}^d \), this means that \( \phi(y + kx) - \phi(y) \) is deterministic for any \( y \in \mathbb{Z}^d \), and that \( d_1(0, x) \leq R \). In particular, \( \phi_\Lambda \) depends deterministically on \( \phi_{\partial^R\Lambda} \) in \( \mu \) for any \( \Lambda \subset \subset \mathbb{Z}^d \). \( \square \)

The final goal of this section is to prove the following theorem.
Theorem 13.5. Suppose that \( E = \mathbb{Z} \), and that \( \Phi \in \mathcal{S}_L + \mathcal{W}_L \) is a submodular Lipschitz potential with a freezing local Lipschitz constraint. Then the associated surface tension \( \sigma \) is strictly convex on \( U_\Phi \).

We first prove two auxiliary lemmas.

Lemma 13.6. If \( E = \mathbb{Z} \) and \( \Phi \) a submodular gradient potential, then

\[
\Phi_\Lambda([\frac{\phi_1+\phi_2}{2}]) + \Phi_\Lambda([\frac{\phi_1-\phi_2}{2}]) \leq \Phi_\Lambda(\phi_1) + \Phi_\Lambda(\phi_2)
\]

for any \( \phi_1, \phi_2 \in \Omega \) and \( \Lambda \subseteq \mathbb{Z}^d \).

Proof. Write \( \xi^\pm := \phi_1 \pm \phi_2 \), so that \( \phi_1 = (\xi^+ + \xi^-)/2 \) and \( \phi_2 = (\xi^+ - \xi^-)/2 \). Write

\[
F(\psi^+, \psi^-) := \Phi_\Lambda(\frac{\psi^+ + \psi^-}{2}) + \Phi_\Lambda(\frac{\psi^+ - \psi^-}{2})
\]

for any \( \psi^+, \psi^- \in \Omega \) with \( \psi^+ + \psi^- \equiv 0 \mod 2 \). For example, the right hand side of the display in the statement of the lemma equals \( F(\xi^+, \xi^-) \), and the left hand side equals\( F(\xi^+, p \circ \xi^-) \), where \( p : \mathbb{Z} \to \{0, 1\} \) is the parity function which maps even integers to 0 and odd integers to 1. Therefore it suffices to demonstrate that

\[
F(\psi^+, p \circ \psi^-) \leq F(\psi^+, \psi^-)
\]

for any \( \psi^+, \psi^- \in \Omega \) with \( \psi^+ + \psi^- \equiv 0 \mod 2 \).

Observe that \( F \) has the following four properties:

1. Translation invariance: \( F(\psi^+ + a_1, \psi^- + a_2) = F(\psi^+, \psi^-) \) for any \( a_1, a_2 \in \mathbb{Z} \) with \( a_1 + a_2 \) even, because \( \Phi \) is a gradient specification,

2. Inversion invariance: \( F(\psi^+, -\psi^-) = F(\psi^+, \psi^-) \); replacing \( \psi^- \) by \( -\psi^- \) corresponds to interchanging the sum and difference of \( \psi^+ \) and \( \psi^- \),

3. Submodularity: \( F(\psi^+, |\psi^-|) \leq F(\psi^+, \psi^-) \); equivalent to submodularity of \( \Phi \),

4. Locally measurable: \( F(\psi^+, \psi^-) \) depends on \( \psi^\Lambda \) only.

By applying the three operations on the pair \( (\psi^+, \psi^-) \) finitely many times, one can turn the original pair into a new pair \( (\psi^+, \psi^\Lambda) \), where \( \psi^\Lambda = (p \circ \psi^-) \Lambda \). In particular, since each operation can only decrease the value of \( F \), we have

\[
F(\psi^+, p \circ \psi^-) = F(\psi^+, \psi^\Lambda) \leq F(\psi^+, \psi^-)
\]

as desired. \( \square \)

Corollary 13.7. Suppose that \( E = \mathbb{Z} \) and that \( \Phi \in \mathcal{S}_L + \mathcal{W}_L \) is submodular. If \( \mu_1, \mu_2 \in \mathcal{P}_L(\Omega, \mathcal{F}^\Lambda) \) are ergodic, then there exists an ergodic measure \( \nu \in \mathcal{P}_L(\Omega, \mathcal{F}^\Lambda) \) with

\[
S(\nu) = \frac{S(\mu_1) + S(\mu_2)}{2} \quad \text{and} \quad \langle \nu | \Phi \rangle \leq \frac{\langle \mu_1 | \Phi \rangle + \langle \mu_2 | \Phi \rangle}{2}.
\]

Proof. Write \( \hat{\mu} \in \mathcal{P}_L(\Omega, \mathcal{F}^\Lambda) \) for the following measure: to sample from \( \hat{\mu} \), sample first a pair \( (\phi_1, \phi_2) \) from \( \mu_1 \times \mu_2 \), and sample \( X \) from \( \{0, 1\} \) independently and uniformly at random; the final sample \( \psi \) from \( \hat{\mu} \) is now defined by

\[
\psi := \begin{cases} 
\frac{\phi_1 - \phi_1(0) + \phi_2 - \phi_2(0)}{2} & \text{if } X = 0, \\
\frac{\phi_1 - \phi_1(0) + \phi_2 - \phi_2(0)}{2} & \text{if } X = 1.
\end{cases}
\]
Since $\phi_1 - \phi_1(0)$ and $\phi_2 - \phi_2(0)$ are asymptotically close to $S(\mu_1)$ and $S(\mu_2)$ respectively in the measure $\mu_1 \times \mu_2$ in the sense of Theorem 10.15, it is clear that $\psi$ is asymptotically close to $(S(\mu_1) + S(\mu_2))/2$ in $\hat{\mu}$ (see also the proof of Lemma 10.32). In particular, $S(\nu) = (S(\mu_1) + S(\mu_2))/2$ for $w_{\hat{\mu}}$-almost every $\nu$ in the ergodic decomposition of $\hat{\mu}$. By the previous lemma, we have
\[
\langle \hat{\mu} | \Phi \rangle \leq \frac{\langle \mu_1 | \Phi \rangle + \langle \mu_2 | \Phi \rangle}{2}.
\]
As $\langle \hat{\mu} | \Phi \rangle$ is strongly affine, we have $\langle \nu | \Phi \rangle \leq \langle \hat{\mu} | \Phi \rangle$ with positive $w_{\hat{\mu}}$-probability. This proves the existence of the desired measure $\nu$.

**Lemma 13.8.** Consider the case that $E = \mathbb{Z}$, $\Phi$ a potential in $\mathcal{S}_\mathcal{C} + \mathcal{W}_\mathcal{C}$, and $\mu$ an ergodic minimizer with $S(\mu) \in U_\Phi$. Then $\mathcal{H}(\mu|\lambda) < 0$.

**Proof.** Suppose that $\mu$ does have zero combinatorial entropy; we aim to derive a contradiction. Write $u := S(\mu)$, and write $\hat{\mu} \in \mathcal{P}_\mathcal{C}^2(\Omega, \mathcal{F}^\mathcal{V})$ for the unique measure which has $\mu$ as its first marginal, and in which $\phi_1$ and $\phi_2$ are equal almost surely. Then $S^2(\hat{\mu}) = (u, u)$ and $\mathcal{H}^2(\hat{\mu} | \Phi) = 2\langle u | \Phi \rangle = 2\mathcal{H}(\mu | \Phi) = \sigma^2(S^2(\hat{\mu})) < \infty$, that is, $\hat{\mu}$ is a minimizer in the product setting. The adaptation of Theorem 4.2 to the product setting implies that
\[
\text{1}_{\Omega_\mathcal{V}^2} \langle \hat{\mu} \pi_{\mathcal{Z}^d \times \Lambda} | \lambda^A \times \lambda^A \rangle \ll \hat{\mu}
\]
for any $\Lambda \subset \subset \mathbb{Z}^d$, where $\Omega_\mathcal{V}^2$ is the set of pairs of $q$-Lipschitz height functions. However, since $\mu$ is ergodic with slope in $U_\Phi$, we can find some $\Lambda \subset \subset \mathbb{Z}^d$ such that with positive $\hat{\mu}$-probability $\phi_1|_{\mathcal{Z}^d \times \Lambda}$ has more than a single $q$-Lipschitz extension to $\mathbb{Z}^d$. This contradicts that $\phi_1$ and $\phi_2$ are almost surely equal in $\hat{\mu}$. \qed

We are now ready to prove the second main theorem of this section.

**Proof of Theorem 13.5.** Recall Theorem 4.12. If $\sigma$ is not strictly convex, then there is an affine map $h : (\mathbb{R}^d)^* \to \mathbb{R}$ with $h \leq \sigma$ and such that $\{h = \sigma\} \cap \partial U_\Phi$ is not convex. Write $H$ for the exposed points of $\{h = \sigma\} \subset (\mathbb{R}^d)^*$ which are also in $\partial U_\Phi$. Then the convex envelope of $H$ intersects $U_\Phi$.

Note that each slope in $H$ is also an exposed point of $\sigma$. This means that for each slope in $H$, there is an ergodic minimizer $\mu$ of that slope. Moreover, since $\mathcal{H}(\mu|\lambda) = 0$ for any $\mu$ with $S(\mu) \in H \subset \partial U_\Phi$, we must have $\langle \mu | \Phi \rangle = h(S(\mu)) = \sigma(S(\mu))$ for any such measure $\mu$. The fact that the convex envelope of $H$ intersects $U_\Phi$, together with Corollary 13.7, implies that there exists an ergodic measure $\mu \in \mathcal{P}_\mathcal{C}(\Omega, \mathcal{F}^\mathcal{V})$ with $S(\mu) \in U_\Phi$ and $\langle \mu | \Phi \rangle \leq h(S(\mu)) \leq \sigma(S(\mu))$. But it is only possible that $\langle \mu | \Phi \rangle \leq \sigma(S(\mu))$ if $\langle \mu | \Phi \rangle = \sigma(S(\mu))$ and if $\mu$ is a minimizer with $\mathcal{H}(\mu|\lambda) = 0$. This contradicts Lemma 13.8. \qed

### 13.3 Tree-valued graph homomorphisms

The flexibility of the main theorem in this article can also be used to prove statements about the behavior of random functions taking values in target spaces other than $\mathbb{Z}$ and $\mathbb{R}$. A noteworthy example is the model of tree-valued graph homomorphisms described in [MT20]. Let $k \geq 2$ denote a fixed integer, and let $T_k$ denote the $k$-regular tree, that is, a tree in which every vertex has exactly $k$ neighbors. In this context, tree-valued graph homomorphisms are functions from $\mathbb{Z}^d$ to the vertices of $T_k$ which also map the edges of the square lattice to the edges of the tree. Regular trees are natural objects in several fields of mathematics: in group theory, for example, they arise as Cayley graphs of free groups on finitely many generators. As a significant result in [MT20], the authors characterize the
The gradient of the graph homomorphism

The boundaries of the upper level sets

Figure 7: A random $T_3$-valued graph homomorphism

surface tension for the model (there named \textit{entropy}) and show that it is equivalent to the number of graph homomorphisms with nearly-linear boundary conditions. In this section we will confirm the conjecture from [MT20], which states that this entropy function is strictly convex. We must first show how the model and the corresponding surface tension fit into the framework of this paper. A tree-valued graph homomorphism can be represented by an integer-valued graph homomorphism after introducing an infinite-range potential to compensate for the “loss of information”.

Let us first introduce some definitions. Write $d_{T_k}$ for the graph metric on $T_k$. Let $g$ denote a fixed bi-infinite geodesic through $T_k$, that is, a $\mathbb{Z}$-indexed sequence of vertices $g = (g_n)_{n \in \mathbb{Z}} \subset T_k$ such that $d_{T_k}(g_n, g_m) = |m - n|$ for any $n, m \in \mathbb{Z}$. Let $p : T_k \to \mathbb{Z}$ denote the projection of the tree onto $g$, defined such that $p(x)$ minimizes $d_{T_k}(x, g_p(x))$ for any $x \in T_k$. Write $h$ for the horocyclic height function on $T_k$; this is the function $h : T_k \to \mathbb{Z}$ defined by $h(x) := p(x) + d_{T_k}(x, g_p(x))$ (see also [GL18]). In other words, if $x = g_n$ for some $n \in \mathbb{Z}$, then $h(x) = n$, and $h$ increases by one every time one moves away from the geodesic $g$. The function $h$ can also be characterized as follows: each vertex $x \in T_k$ has a unique neighbor $y$ such that $h(y) = h(x) - 1$, and $h(z) = h(x) + 1$ for every other neighbor $z$ of $x$.

The graphs $\mathbb{Z}^d$, $\mathbb{Z}$, and $T_k$ are bipartite, we shall call the two parts the even vertices and odd vertices respectively; the set of even vertices is the part containing 0 if the graph is $\mathbb{Z}^d$ or $\mathbb{Z}$, and the part containing $g_0$ if the graph is $T_k$. By a graph homomorphism we mean a map from $\mathbb{Z}^d$ to $\mathbb{Z}$ or $T_k$ which preserves the parity of the vertices, and which maps edges to edges. Write $\Omega$ and $\tilde{\Omega}$ respectively for the set of graph homomorphisms from $\mathbb{Z}^d$ to either $\mathbb{Z}$ or $T_k$. For fixed $\phi \in \Omega$ and $n \in \mathbb{Z}$, we call some set $\Lambda \subset \mathbb{Z}^d$ an $n$-upper level set if $\Lambda$ is a connected component of $\{ \phi \geq n \} \subset \mathbb{Z}^d$ in the square lattice graph. An $n$-upper level set is also called an $n$-level set or simply a level set.

Write $U$ for the set of slopes $u \in (\mathbb{R}^d)^*$ such that $|u(e_i)| < 1$ for each element $e_i$ in the natural basis of $\mathbb{R}^d$. For fixed $u \in U$, write $\phi^u \in \Omega$ for the graph homomorphism defined by

$$
\phi^u(x) := |u(x)| + \begin{cases} 
0 & \text{if } d_1(0, x) \equiv |u(x)| \mod 2, \\
1 & \text{if } d_1(0, x) \equiv |u(x)| + 1 \mod 2,
\end{cases}
$$

and write $\tilde{\phi}^u \in \tilde{\Omega}$ for the graph homomorphism defined by $\tilde{\phi}^u(x) = g_{\phi^u(x)}$. 

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It is shown in Section 3 of [MT20] that the entropy function $\text{Ent} : \bar{U} \to [-\log k, 0]$ associated to the model of graph homomorphisms from $\mathbb{Z}^d$ to $\mathcal{T}_k$ can be estimated by counting for each slope $u \in \bar{U}$ the number of graph homomorphisms $\phi : \mathbb{Z}^d \to \mathcal{T}_k$ which equal $\tilde{\phi}^u$ on the complement of $\Pi_n$. More precisely, for $u \in \bar{U}$, we have

$$\text{Ent}(u) = \lim_{n \to \infty} -n^{-d} \log |\{ \tilde{\phi} \in \tilde{\Omega} : \tilde{\phi}_T \downarrow \Pi_n = \tilde{\phi}_T^{u} \downarrow \Pi_n \}|.$$  

Notice that counting the number of functions in this set is similar to considering the normalizing constant in the definition of the specification, as we frequently do in this paper. Before proceeding, let us already remark that $\text{Ent}(u) = 0$ for $u \in \partial U$. Indeed, for such $u$, the set in the display contains only a single element: the original function $\tilde{\phi}^u$. It is also easy to see that $\text{Ent}$ is not identically zero on $\bar{U}$. Consider, for example, the slope $u = 0$, and consider the set of all graph homomorphisms $\tilde{\phi}$ which equal $\tilde{\phi}^u$ on the complement of $\Pi_n$ and which map all the even vertices of the square lattice to $g_0 \in \mathcal{T}_k$. Then this set contains at least $k^{\lfloor n^{d/2} \rfloor}$ functions, proving that $\text{Ent}(u) \leq -\frac{1}{2} \log k < 0$.

We now get to the heart of the case. Let us use the horocyclic height function to count the set in the previous display in a different way. Suppose that some graph homomorphism $\phi \in \Omega$ equals $\tilde{\phi}^u$ on the complement of $\Pi_n$. How many graph homomorphisms $\phi \in \tilde{\Omega}$ do there exist which satisfy $h \circ \tilde{\phi} = \phi$ and equal $\tilde{\phi}^u$ on the complement of $\Pi_n$? It turns out that this number must be precisely $(k-1)^{\text{Fin}_n(\phi)}$, where $F_\Lambda(\phi)$ denotes the number of level sets of $\phi$ which are entirely contained in $\Lambda$, for any $\Lambda \subset \subset \mathbb{Z}^d$. Indeed, each time we see an $n$-level set of $\phi$, the function $\tilde{\phi}$ must be constant on the outer boundary of that $n$-level set—say with value $x \in \mathcal{T}_k$—and there are $k-1$ neighbors of $x$ which lead to an increase of the horocyclic height function by exactly one. In particular, we have

$$\text{Ent}(u) = \lim_{n \to \infty} -n^{-d} \log \sum_{\phi \in \Omega, \phi_T \downarrow \Pi_n = \phi_T^{u} \downarrow \Pi_n} (k-1)^{\text{Fin}_n(\phi)}.$$  

See Figure 7 for a sample of the model, with the gradient of the graph homomorphism on the left, and with the boundaries of the level sets of the horocyclic height function on the right. We have now reduced to a problem expressed entirely in terms of integer-valued functions. In fact, we do no longer require $k$ to be an integer, although we do require that $k \geq 2$. In the remainder of this section, we construct a potential $\Phi$ which fits into our class $S_L + W_L$ and which is monotone, and such that $U_\Phi = U$ and $\sigma = \text{Ent}$. This proves that $\sigma$ and $\text{Ent}$ are strictly convex on $U_\Phi = U$. In fact, the specification induced by the potential that we construct is not perfectly monotone, but we shall demonstrate that it is sufficiently monotone for us to deduce that $\sigma$ is strictly convex.

Unfortunately, we cannot hope to use a potential that counts the level sets directly. The reason is that there is no upper bound on the number of level sets containing a single point; such a potential would always sum to infinity. However, each finite level set has a uniquely defined outer boundary, and each vertex is contained in only finitely many outer boundaries. This means that counting outer boundaries of finite level sets is equivalent to counting finite level sets, and the potential that does so is well-defined and fits our framework, as we will show. It is not possible through this method to count infinite level sets, but we shall demonstrate how to work around this apparent difficulty.

We shall now describe how to characterize the outer boundary of a finite level set. This is not entirely straightforward due to the connectivity properties of the square lattice. By the $*$-graph on $\mathbb{Z}^d$, we mean the graph in which two vertices $x$ and $y$ are neighbors if and only if $\|x - y\|_\infty = 1$. For example, each vertex has $3^d - 1$ distinct $*$-neighbors. On every single occasion that we mention a graph-related notion, we mean the usual square lattice.
Proposition 13.10. Suppose that $\Lambda \subset \subset \mathbb{Z}^d$ is finite and connected, and that its complement $\Delta := \mathbb{Z}^d \setminus \Lambda$ is $*$-connected. Define $\partial^* \Delta$ to be the set of vertices $x \in \mathbb{Z}^d$ such that:

1. Either $x \in \Lambda = \mathbb{Z}^d \setminus \Delta$ and $*$-adjacent to $\Delta$,
2. Or $x \in \Delta$ and adjacent to $\Lambda = \mathbb{Z}^d \setminus \Delta$.

Then $\partial^* \Delta \cap \Lambda = \partial^* \Delta \cap (\mathbb{Z}^d \setminus \Delta)$ is connected, and so is $\partial^* \Delta$.

Consider a finite nonempty connected set $\Lambda \subset \subset \mathbb{Z}^d$. Write $\Lambda^\infty$ for the outside of $\Lambda$, that is, the unique unbounded $*$-connected component of the complement of $\Lambda$. Write also $\bar{\Lambda}$ for the complement of $\Lambda^\infty$: this set is finite and connected, and contains $\Lambda$. The pair $(\bar{\Lambda}, \Lambda^\infty)$ will play the role of $(\Lambda, \Delta)$ in the previous proposition. The set $\partial^* \Lambda^\infty$ can obviously be written as the disjoint union of $\partial^* \Lambda^\infty \cap \Lambda^\infty$ and $\partial^* \Lambda^\infty \cap \bar{\Lambda}$. Claim that $\partial^* \Lambda^\infty \cap \bar{\Lambda} = \partial^* \Lambda^\infty \cap \Lambda$. Indeed, if $x \in \partial^* \Lambda^\infty \cap \bar{\Lambda}$ is not in $\Lambda$, then it should be in $\Lambda^\infty$ as it is $*$-adjacent to $\Lambda^\infty$; this proves the claim. This also means that all vertices in $\partial^* \Lambda^\infty \cap \Lambda^\infty$ are adjacent to $\Lambda$.

Suppose now that $\Lambda$ is also an $n$-level set of some graph homomorphism $\phi \in \Omega$. Then $\phi$ must equal exactly $n-1$ on $\partial^* \Lambda^\infty \cap \Lambda^\infty$, and $\phi$ must be at least $n$ on $\partial^* \Lambda^\infty \cap \bar{\Lambda} = \partial^* \Lambda^\infty \cap \Lambda$. We have now proven the following lemma.

Lemma 13.11. Suppose that $\Delta \subset \subset \mathbb{Z}^d$ is $*$-connected and cofinite, with its complement connected. Then

$$\{ \phi \in \Omega : \Delta \text{ is the outside of a } n \text{-level set of } \phi \text{ for some } n \in \mathbb{Z} \}$$

$$= \{ \phi \in \Omega : \phi_{\partial^* \Delta \cap \Delta} = n - 1 \text{ and } \phi_{\partial^* \Delta \setminus \Delta} \geq n \text{ for some } n \in \mathbb{Z} \} \in \mathcal{F}_{\partial^* \Delta}^\nabla.$$

Moreover, no two level sets of $\phi$ produce the same outside boundary $\partial^* \Delta$.

Define the potential $\Xi = (\Xi_\Lambda)_{\Lambda \subset \subset \mathbb{Z}^d}$ by

$$\Xi_\Lambda(\phi) = -\log(k-1)$$

if $\Lambda = \partial^* \Delta^\infty$ for some finite level set $\Delta$ of $\phi$, and $\Xi_\Lambda(\phi) = 0$ otherwise. For fixed $x \in \mathbb{Z}^d$ and $\phi \in \Omega$, there are at most $3^d$ finite level sets $\Delta$ of $\phi$ such that $x \in \partial^* \Delta^\infty$. In particular, this means that $\|\Xi\| \leq 3^d \log(k-1)$. Moreover, since $\Xi_\Lambda \equiv 0$ whenever $\Lambda \subset \subset \mathbb{Z}^d$ is not connected, it is clear that $e^- \Lambda \leq |\partial \Lambda| \cdot \|\Xi\|$. In particular, $e^-$ is an amenable function, which means that $\Xi \in \mathcal{W}_L$. Remark that $H^\Xi_L(\phi)$ equals $-\log(k-1)$ times the number of finite level sets $\Delta$ of $\phi$ for which $\partial^* \Delta^\infty$ intersects $\Lambda$. Unfortunately, it is not possible to count infinite level sets with this construction; this is a small inconvenience that we must circumvent.

Write $\Psi$ for the potential which forces graph homomorphisms, that is, $\Psi_\Lambda(\phi) = \infty$ if $\Lambda = \{x, y\}$ is an edge of the square lattice and $|\phi(y) - \phi(x)| \neq 1$, and $\Psi_\Lambda(\phi) = 0$ otherwise. This potential belongs to $\mathcal{S}_L$, modulo the detail explained in Subsection 4.4, which we shall simply ignore here.

Lemma 13.12. For any integer $k \geq 2$, the surface tension $\sigma$ associated to the potential $\Phi := \Psi + \Xi$ equals the entropy function $\text{Ent}$. 

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Proof. We prove that $\sigma(u) = \text{Ent}(u)$ for $u \in U_\Phi$, the result extends to all $u \in \bar{U}_\Phi$ because both $\sigma$ and $\text{Ent}$ are continuous on $\bar{U}_\Phi$. Due to Theorem 4.10, we know that

$$\sigma(u) = P_\Phi((0, 1)^d, u|_{\partial(0, 1)^d}) = \lim_{n \to \infty} -n^{-d} \log \int_{E^{\Omega_n}} e^{-H_n^\Phi(\psi_{\phi^m|\partial n^n})} d\Pi_n(\psi) = \lim_{n \to \infty} -n^{-d} \log \sum_{\phi \in \Omega, \phi_{\partial n^n} = \phi^m_{\partial n^n}} e^{-H_n^\Phi(\phi)}.$$ 

But the logarithm of the ratio of $e^{-H_n^\Phi(\phi)}$ with $(k - 1)^{F_n(\phi)}$ is of order $O(n^{d-1}) = o(n^d)$ uniformly over $\phi$ as $n \to \infty$, so that the equality $\sigma(u) = \text{Ent}(u)$ follows from (13.9). \qed

Definition 13.13. Write $\Omega_-$ for the set of graph homomorphisms $\phi \in \Omega$ which have no infinite level sets.

Lemma 13.14. The specification induced by the potential $\Phi := \Psi + \Xi$ is stochastically monotone over $\Omega_-$ for any $k \geq 2$.

Proof. We use the Holley criterion (Theorem 13.1) to prove that $\gamma_\Lambda$ preserves $\preceq$; we suppose that $\Lambda = \{0\}$ without loss of generality. Let $\phi_1, \phi_2 \in \Omega_-$ denote graph homomorphisms without infinite level sets, and which satisfy $\phi_1 \preceq \phi_2$. Notice that the only case where the local Gibbs measure $\gamma_\Lambda(\cdot, \phi)$ is not a Dirac measure, is if there exist a $n \in \mathbb{Z}$ such that $\phi(x) = n$ for any neighbor $x$ of $0$. If this is not the case for $\phi_1$ or $\phi_2$ then the proof is trivial; we reduce to the case that $\phi_1(x) = \phi_2(x) = 1$ for any neighbor $x$ of $0$ in $\mathbb{Z}^d$. It remains to show that $\gamma_\Lambda(\cdot, \phi_1) \preceq \gamma_\Lambda(\cdot, \phi_2)$. Without loss of generality, $\phi_1(0) = \phi_2(0) = 0$.

Write $\psi$ for the random function in either local Gibbs measure. Since $\phi_i(x) = 1$ for any neighbor $x$ of $0$ and for $i \in \{1, 2\}$, the function $\psi$ can only take two values with positive probability: they are $0$ and $2$. What we thus must show is that the quantity

$$a_i := \frac{\gamma_\Lambda(\psi(0) = 2, \phi_i)}{\gamma_\Lambda(\psi(0) = 0, \phi_i)}$$

satisfies $a_1 \leq a_2$. Claim that $a_i = (k - 1)^{2 - X_i}$, where $X_i$ is the number of $1$-level sets of $\phi_i$ which are adjacent to $0$. If $\psi(0) = 0$, then all $1$-level sets adjacent to $0$ are counted separately, and $\{0\}$ is not a level set. If $\psi(0) = 2$, then we count two level sets: the set $\{0\}$ is a $2$-level set, and all neighbors of $0$ are contained in the same $1$-level set. All other level sets remain unaffected. This proves the claim. We must therefore prove that $X_1 \geq X_2$. This is clear: increasing the values of $\phi$ can only increase the size of the $1$-level set containing a fixed vertex $x$, and potentially merge several $1$-level sets. In particular, it can only decrease the number of $1$-level sets adjacent to $0$. \qed

Theorem 13.15. The surface tension $\sigma$ associated to the potential $\Phi$ defined above, is strictly convex on $U_\Phi$ whenever $k \geq 2$.

Proof. We must circumvent the problem that the specification $\gamma$ induced by $\Phi$ is monotone only after restricting it to the set $\Omega_-$. Remark that Theorem 5.4 and Proposition 5.11 remain true in this context if the measure $\mu$ in the statement of Theorem 5.4 is supported on $\Omega_-$. The only time that monotonicity is used in the proof for strict convexity of $\sigma$, is in the application of these two results in Lemma 12.9. Recall that the local Gibbs measure $\gamma_n$ in the statement of Lemma 12.9 was defined to be $\gamma_{\Pi_n}(\cdot, \phi^n)$; this is now problematic because $\phi^n$ does have infinite level sets. This can be easily solved by the following modification. Define $\phi^m_n$ to be the smallest graph homomorphism which equals $\phi^n$ on the set $\Pi_n \cup \partial \Pi_n$. It is easy to check that $\{\phi^m_n \geq m\}$ is finite for any $n \in \mathbb{N}$ and $m \in \mathbb{Z}$; in particular, $\phi^m_n \in \Omega_-$. 85
Moreover, the sequence \((\Pi, \phi_n)_{n \in \mathbb{N}}\) is as much an approximation of \(((0, 1)^d, u|_{\partial(0, 1)^d})\) as the original sequence \((\Pi_n, \phi^u)_{n \in \mathbb{N}}\). In particular, all of the same arguments apply if we simply replace each local Gibbs measure \(\gamma_n = \gamma_{\Pi_n}(\cdot, \phi^u)\) by \(\gamma_{\Pi_n}(\cdot, \phi^u_n)\). We had already seen that \(\sigma = 0\) on \(\partial U_{\Phi}\) and \(\sigma(0) < 0\), which proves that \(\sigma\) is strictly convex.

### 13.4 Stochastic monotonicity in the six-vertex model

Consider the two-dimensional square lattice. An arrow configuration is an orientation of each edge of the square lattice, in such a way that each vertex has exactly two incoming edges and two outgoing edges. This means that there are six configurations for the four edges incident to a fixed vertex; see Figure 8. Each of these six types receives a weight, and one studies the probability measure where the probability of observing an arrow configuration is proportional to the product of the weights over the vertices in that configuration. This is the six-vertex model, which is the subject of an extensive literature. Each arrow configuration has an associated height function, which assigns integers to the faces of the square lattice, and is defined as follows: the height of the face to the right of an arrow is always exactly one more than the height of the face to the left of it, and the height of a fixed reference face is set to zero. It is straightforward to see that this uniquely defines the height functions associated to an arrow configuration. The six-vertex model can thus be considered a Lipschitz random surface. Our main theorem asserts that the surface tension of this random surface model is strictly convex, if the specification is monotone. It is a straightforward exercise to demonstrate that the specification is monotone if and only if

\[
c_+ c_- \geq \max\{a_+ a_-, b_+ b_-\};
\]

this is verified through checking the Holley criterion (Theorem 13.1). Informally, this means that the specification is monotone if the model prefers vertices for which the four values of the adjacent faces are as close to each other as possible. Finally, we should mention that from the perspective of the specification, there is some gauge equivalence in the choice of the six weights; for details we refer to the work of Sridhar [Sri16, Section 2.2].

**Theorem 13.16.** The potential \(\Phi \in \mathcal{S}_L\) corresponding to the six-vertex model is monotone if and only if \(c_+ c_- \geq \max\{a_+ a_-, b_+ b_-\}\), in which case \(\sigma\) is strictly convex on \(U_{\Phi}\).

Although it is not directly stated in *Random Surfaces* [She05], the potential \(\Phi\) can be written as a simply attractive potential whenever \(c_+ c_- \geq \max\{a_+ a_-, b_+ b_-\}\). Therefore this theorem should be considered an alternative proof rather than a novel result.

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