Anisotropic neutrino effect on magnetar spin: constraint on inner toroidal field

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ABSTRACT
The ultra-strong magnetic field of magnetars modifies the neutrino cross section due to the parity violation of the weak interaction and can induce asymmetric propagation of neutrinos. Such an anisotropic neutrino radiation transfers not only the linear momentum of a neutron star but also the angular momentum, if a strong toroidal field is embedded inside the stellar interior. As such, the hidden toroidal field implied by recent observations potentially affects the rotational spin evolution of new-born magnetars. We analytically solve the transport equation for neutrinos and evaluate the degree of anisotropy that causes the magnetar to spin-up or spin-down during the early neutrino cooling phase. Supposing that after the neutrino cooling phase the dominant process causing the magnetar spin-down is the canonical magnetic dipole radiation, we compare the solution with the observed present rotational periods of anomalous X-ray pulsars 1E 1841-045 and 1E 2259+586, whose poloidal (dipole) fields are \( \sim 10^{15} \) G and \( 10^{14} \) G, respectively. Combining with the supernova remnant age associated with these magnetars, the present evaluation implies a rough constraint of global (average) toroidal field strength at \( B_\phi \lesssim 10^{15} \) G.

Key words: magnetic fields — neutrinos — radiative transfer — pulsars: general — stars: neutron

1 INTRODUCTION
Soft Gamma Repeaters (SGRs) and Anomalous X-ray Pulsars (AXPs) are two examples of the astronomical objects collectively known as magnetars. These objects emit a large amount of energy in soft gamma rays and X-rays, and their energy source cannot be explained in terms of the canonical rotation energy of neutron stars (NSs). Magnetic fields inside and outside magnetars are conjectured to be the main source of energy, with very strong magnetic fields required to explain their activity. Magnetars are therefore a special class of NSs that have strong magnetic fields. Based on their periods \( P \) and the time derivative of their periods \( \dot{P} \), this class is thought to have magnetic fields larger than the critical strength \( B_Q \approx 4.4 \times 10^{13} \) G, beyond which the perturbative approach of quantum-electro dynamics breaks down.

Recently, two magnetars with surface dipole magnetic fields smaller than \( B_Q \) were reported [Rea et al. 2010, 2012]. These objects gave us important clues as to the nature of the magnetic field inside magnetars. Since \( P \) and \( \dot{P} \) measurements can only provide information on the dipole (poloidal) component of the field, there is no constraint on the toroidal component. As such, the unknown toroidal fields are often thought to provide the large energy required to account for magnetar activity. The two low-magnetic field SGRs are thought to be explained by hidden internal magnetic fields (e.g., SGR 0418+5729, Tiengo et al. 2013).

It is often discussed in the literature that parity violation in weak interactions can lead to asymmetric neutrino emission in strongly magnetized NSs. Given that neutrinos transfer momentum, asymmetric neutrino emission originating from poloidal fields can therefore impart linear momentum to a NS, which is a possible cause of pulsar kicks [Arras & Las 1999a; Ando 2003; Kotake et al. 2005; Maruyama et al. 2012]. Furthermore, asymmetric neutrino emission could also transfer angular momentum from newborn NSs [Maruyama et al. 2014].

In this paper we investigate the effect of a magnetic field on the opacity of NSs to the neutrinos that carry away the
thermal energy. We specifically focus on the toroidal component and the spin evolution of magnetars. Section 2 opens with the basic picture of this paper. Section 3 is devoted to the derivation of the neutrino transfer equation and its solution. In addition, we give simple relations between the total angular momentum of a NS and the angular momentum emitted by neutrinos. In Section 4 we give the constraint on the magnetar’s internal field. We summarize our results and discuss their implications in Section 5.

2 PHYSICAL SCENARIO

In this section we briefly outline the basic picture studied in this paper. As is well known, NSs are formed by the gravitational collapse of massive stars, leading to core-collapse supernova explosions. At first, just after their formation, NSs are hot (the temperature is typically $O(10^{11})$ K), and in this phase they are referred to as protoneutron stars (PNSs). The stars then proceed to cool down due to neutrino emission (see e.g. Burrows & Lattimer 1986; Fischer et al. 2010; Suwa 2014). The typical timescale of the cooling, referred to as the Kelvin-Helmholtz cooling time and denoted $\tau$ in the following, is $O(1) \times 10^5$ years. In this paper, we are focusing on this early PNS cooling phase. Note that this is different from conventional NS cooling, the timescale of which is typically of $O(10^5)$ years.

During the PNS cooling phase, the strong magnetic field induces anisotropic interactions between neutrinos and polarized nucleons and electrons. These interactions lead to an anisotropic deformation of the neutrino flux, which in turn imparts a linear momentum to the PNS and produces a pulsar kick (Section 4). The emitted neutrinos may also transfer angular momentum, causing the PNS to spin-up/down. These linear and angular momentum transfers are caused by the strong poloidal and toroidal components of magnetic fields, respectively. A quantitative evaluation of the angular momentum allows us to determine the dependency of the NS spin on the toroidal field strength. The optical depth of neutrinos during this period is much higher than unity, so the spin on the toroidal field strength. The optical depth of neutrinos during this period is much higher than unity, so the spin evolution of NSs for $t < \tau_\nu$ is similar independent on the evolution for $t < \tau_\nu$. It is clear that the rotation period of NSs distribute broadly if the neutrinos significantly affect the spin evolution. Therefore, if the neutrino effect dominates the spin evolution of NSs for $t < \tau_\nu$, in order to concentrate the current spin period of NSs in a narrow range, neutrino effect upon the NS spin should be small enough.

Figure 1. Schematic view of the time evolution of angular velocity, $\Omega$. For $t < \tau_\nu$, the neutrino emission changes the NS spin and for $\tau_\nu < t < \tau_0$ the NS rotation is decelerated by the usual dipole radiation. Depending on the direction of magnetic fields, the NS spin evolution can be classified as following. In the case (a), since the toroidal field is absent, for $t < \tau_\nu$ the rotation velocity is not altered by neutrino emission; In the case (b), the neutrinos decelerate the NS spin; In the case (c), the neutrinos accelerate the NS spin; In the case (d), the neutrinos first decelerate the NS spin and eventually the NS rotation is stopped. Since the neutrinos transfer the angular momentum even after the NS rotation stops, then the NS starts counterrotating (dotted line). The spin deceleration by dipole radiation does not depend on the rotation direction, so that the spin evolution for $t < \tau_\nu$ is similar. Independent of the evolution for $t < \tau_\nu$. It is clear that the rotation period of NSs distribute broadly if the neutrinos significantly affect the spin evolution. Therefore, if the neutrino effect dominates the spin evolution of NSs for $t < \tau_\nu$, in order to concentrate the current spin period of NSs in a narrow range, neutrino effect upon the NS spin should be small enough.

3 ANISOTROPIC NEUTRINO FLUX AND MOMENTUM TRANSFER

3.1 Neutrino transfer equation

Following Arras & Lai (1999), we solve the transfer equation for neutrinos. The Boltzmann equation for neutrinos is

\[ \frac{\rho \gamma \alpha}{L_\nu} \]
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given by
\[ \frac{1}{c} \frac{\partial f_\nu(\vec{p}_\nu)}{\partial t} + \vec{\Omega} \cdot \nabla f_\nu(\vec{p}_\nu) = S, \]
where \( c \) is the speed of light, \( f_\nu(\vec{p}_\nu) \) is the distribution function for neutrinos with momentum \( \vec{p}_\nu \), \( t \) is time, \( \vec{\Omega} \) is the propagation direction of neutrinos, and \( S \) is the source term, in which scattering and absorption are included.

Since we are considering the neutrino transfer inside a PNS, where the neutrinos propagate diffusely, we employ the following diffusion approximation for the neutrino distribution function,
\[ f_\nu(\vec{p}_\nu) = f_\nu^{(0)}(\epsilon_\nu) + g(\epsilon_\nu) + 3 \vec{\Omega} \cdot \vec{h}(\epsilon_\nu), \]
where \( f_\nu^{(0)} \) is the Fermi-Dirac distribution function for neutrinos, \( \epsilon_\nu \) is the energy, \( g(\epsilon_\nu) \) is the deviation from thermal equilibrium and \( \vec{h}(\epsilon_\nu) \) is the dipole component that is connected to the neutrino flux.

By averaging Eq. (1) over the whole solid angle and omitting the time derivative term, we get the following moment equation for steady state \(^{3} \text{Arras & Lai 1999a}\)
\[ \nabla \left[ \vec{f}_\nu^{(0)} + g \right] + \epsilon_{\text{abs}} k_{\text{abs}} \vec{g} \vec{B} = -3 k_{\text{tot}} \vec{r} \vec{h}, \]
where \( \epsilon_{\text{abs}} \) is a coefficient related to absorption and originates from the existence of strong magnetic fields (if there are no magnetic fields \( \epsilon_{\text{abs}} \) is zero). \( k_{\text{abs}} \) is the inverse of the mean free path for neutrino emission and absorption \( (p + e^- \rightarrow n + \nu_e) \) and \( k_{\text{tot}} \) is the inverse of the mean free path for all interactions, including isoelectronic scattering by nucleons without magnetic fields. Lastly, \( \vec{B} \equiv \vec{B}/|\vec{B}| \).

Similarly, we obtain the first order moment equation by integrating Eq. (1) multiplied by \( \mu = \vec{\Omega} \cdot \vec{r}/|\vec{r}| \) as
\[ \nabla \cdot \vec{h} = -k_{\text{abs}} \epsilon_{\text{abs}} \vec{h} - k_{\text{tot}} \vec{r} \vec{h}. \]
Note that to obtain Eqs. (3) and (4) we omitted source terms relating to the scattering originating from the existence of magnetic fields (denoted \( \epsilon_{\text{sc}} \) in \(^{4} \text{Arras & Lai 1999a}\)). This is because this contribution is much smaller that from the terms proportional to \( \epsilon_{\text{abs}} \).

Combining Eqs. (3) and (4), we get the following diffusion equation
\[ \frac{1}{3 \epsilon^2} \frac{\partial}{\partial r} \left[ \epsilon_{\text{tot}} \frac{\partial (f_\nu + g)}{\partial r} \right] = \epsilon_{\text{abs}} g. \]
Note that we omitted the higher-order term proportional to \( \epsilon_{\text{abs}}^2 \). Using the specified opacities for \( k_{\text{abs}} \) and \( k_{\text{tot}} \), we can solve this diffusion equation.

Following \(^{5} \text{Arras & Lai 1999b}\), the opacities are estimated as:
\[ \kappa_{0}^{(\text{abs})}(\nu) = \frac{(G_F \hbar c)^2}{\pi} (\epsilon_\nu + Q)^2 n_e (\epsilon_\nu^2 + 3 c_A^2) \left[ 1 - f_\nu (\epsilon_\nu + Q) \right] \]
\[ = 3.66 \times 10^{-9} \text{ cm}^{-1} \left( \frac{\epsilon_\nu + Q}{2.29 \text{ MeV}} \right)^2 \frac{(\epsilon_\nu + Q)^2}{10^{11} \text{ g cm}^{-2}} \times \left[ 1 - f_\nu (\epsilon_\nu + Q) \right], \]
\[ \kappa_{0}^{(\text{sc})}(\nu) = \frac{2}{3\pi} (G_F \hbar c)^2 \nu_\nu (\epsilon_\nu^2 + 5 c_A^2) n \]
\[ = 3.38 \times 10^{-10} \text{ cm}^{-1} \left( \frac{\epsilon_\nu}{1 \text{ MeV}} \right)^2 \frac{(\epsilon_\nu + Q)^2}{10^{11} \text{ g cm}^{-2}} \]
\[ \kappa_{0}^{(\text{tot})}(\nu) = \kappa_{0}^{(\text{abs})}(\nu) + \kappa_{0}^{(\text{sc})}(\nu). \]

Here, \( G_F = 1.166 \times 10^{-5} \text{ GeV}^{-2} \) is Fermi's constant, \( h = 1.054 \times 10^{-27} \text{ cm}^2 \text{ g}^{-1} \) is the reduced Planck constant. \( Q = 1.29 \text{ MeV} \) is the difference in mass between a neutron and proton, \( n_e \) is the number density of neutrons, \( c_\nu \) and \( c_A \) are weak interaction constants \(^{6} \text{Arras & Lai 1999b}\). This is because this contribution is much smaller that from the terms proportional to \( \epsilon_{\text{abs}} \).

The absorption coefficient, as given by \(^{6} \text{Arras & Lai 1999b}\), is
\[ \epsilon_{\text{abs}} = \frac{1}{2} \left( \frac{\hbar c}{(\epsilon_\nu + Q)^2} \frac{c_v}{c_A} - \frac{c_A}{3c_A} \right) \]
\[ = -0.0575 \left( \frac{B}{10^{15} \text{ G}} \right) \left( \frac{\epsilon_\nu + Q}{2.29 \text{ MeV}} \right)^{-2}, \]
where \( c_v = 1 \) and \( c_A = 1.26 \) for absorption.

The density profile employed in this study, which mimics the structure of the protoneutron star, is
\[ \rho(r) = \rho_c \left( \frac{r}{R_c} \right)^{-3}, \]
where \( \rho_c \) is the density of the PNS surface and \( R_c \) is the radius of the protoneutron star. Here we take \( \rho_c = 10^{11} \text{ g cm}^{-3} \) and \( R_c = 100 \text{ km} \). Although the density diverges at the center, it does not matter in this study because neutrinos are tightly coupled with matter and \( f_\nu = f_\nu^{(0)} \) there.

By assuming that the matter temperature is constant and neutrinos are not degenerated (i.e. taking the chemical potential of neutrinos to be vanishing) \(^{6} \text{Arras & Lai 1999b}\) we obtain the following steady state equation for \( G \equiv g/f_\nu^{(0)} \)
\[ G'' + \frac{5}{r} G' - \alpha \left( \frac{R_c}{r} \right)^6 G = 0, \]
\[ \frac{1}{c} \frac{\partial f_\nu(\vec{p}_\nu)}{\partial t} + \vec{\Omega} \cdot \nabla f_\nu(\vec{p}_\nu) = S, \]
where \( e_{\text{sc}} \sim 10^{-2} \epsilon_{\text{abs}}(T) (KT/1 \text{ MeV})^{-1} (e_\nu/1 \text{ MeV})^2 \) (see equations 7.1 and 7.2 in their paper), where \( \epsilon_{\text{abs}}(T) \) is the asymmetry coefficient for neutrino absorption by electrons, \( k \) is Boltzmann's constant and \( T \) is the matter temperature. Since we are interested in the region where \( kT \sim e_\nu \sim O(1) \text{ MeV} \), omitting \( e_{\text{sc}} \) is a reasonable approximation.

\[ \epsilon_{\text{sc}}(T) = -1/2 \text{ and } c_A = -1.23/2. \]
\[ c_\nu = 1/2 - 2 \sin^2 \theta_w = 0.035 \text{ and } c_A = 1.23/2, \]
where \( \theta_w \) is the Weinberg angle.

\[ \text{Arras & Lai 1999a}, \text{ Arras & Lai 1999b}\]

\[ \text{Janka 2001}\]

\[ \text{Januska 2001}\]
where a prime denotes the derivative with respect to $r$ and
\[
\alpha = 4.01 \times 10^{-17} \text{ cm}^{-2} \left( \frac{\epsilon_{\nu} + Q}{2.29 \text{ MeV}} \right)^4 (1 - f_{\nu})^2 + 3.71 \times 10^{-18} \text{ cm}^{-2} \left( \frac{\epsilon_{\nu} + Q}{2.29 \text{ MeV}} \right)^2 \left( \frac{\epsilon_{\nu}}{1 \text{ MeV}} \right)^2 \times (1 - f_{\nu}).
\]
(13)
The solution to Eq. (12) is given by
\[
g = C_1 \frac{I_1(\sqrt{B} \rho_0/2r^2)}{r^2} + C_2 \frac{K_1(\sqrt{B} \rho_0/2r^2)}{r^2},
\]
(14)
where $I$ and $K$ denote modified Bessel functions of the first and second kind, respectively, and $C_1$ and $C_2$ are constants. At the center, neutrinos are tightly coupled with matter so that $f_{\nu} = f_{\nu}^{(0)}$ and $g = 0$, meaning that $C_1 = 0$. From Eq. (14), the flux is given as
\[
\tilde{h} = -\frac{1}{3 \rho_0 c} \left( G' f_{\nu}^{(0)} \hat{r} + \epsilon_{\abs{a} \abs{b}} G f_{\nu}^{(0)} \hat{B} \right),
\]
(15)
where $\hat{r}$ denotes the unit vector in the radial direction. Since the specific neutrino flux is given by $\tilde{F}_{\nu} = (\epsilon_{\nu}/2\pi \hbar c)^{1/2} \text{ch}$, $r$- and $\phi$-components are given as
\[
F_{\nu}^r = -\frac{c}{3 \rho_0 c} \left( \frac{\epsilon_{\nu}}{2\pi \hbar c} \right)^{3/2} \left( G' + \frac{\epsilon_{\abs{x} \abs{y}} G}{B} \right) f_{\nu}^{(0)},
\]
(16)
\[
F_{\nu}^\phi = -\frac{c}{3 \rho_0 c} \left( \frac{\epsilon_{\nu}}{2\pi \hbar c} \right)^{3/2} \frac{\epsilon_{\abs{x} \abs{y}} G}{B} B f_{\nu}^{(0)}.
\]
(17)
Here, $B^r$ and $B^\phi$ correspond to the $r$- and $\phi$-components of the magnetic field, respectively. $F_{\nu}^r$ should be positive at $R_{\nu}$ so that $C_2 < 0$.

By integrating over energy, using the matter temperature $kT = 4 \text{ MeV}$ and vanishing chemical potentials for $f_{\nu}^{(0)}$ and $f_{\nu}$, the ratio between fluxes in the radial and orthogonal directions at the neutrinosphere surface is given by
\[
\frac{\int d\epsilon_{\nu} \tilde{F}_{\nu}^r}{\int d\epsilon_{\nu} \tilde{F}_{\nu}^r} \approx -0.013 \left( \frac{B^\phi}{10^{15} \text{ G}} \right) \left( \frac{R_{\nu}}{100 \text{ km}} \right)^{1/2}.
\]
(18)
The second term in Eq. (16) is neglected in this estimation.

The total neutrino luminosity is given by
\[
L_{\nu} = \int d\epsilon_{\nu} d\Omega F_{\nu}^r R_{\nu}^2,
\]
(19)
and the rate of angular momentum transfer by neutrinos is given by
\[
J_{\nu} = \frac{1}{c} \int d\epsilon_{\nu} d\Omega F_{\nu}^\phi R_{\nu}^2 \sin \theta.
\]
(20)
The factor $R_{\nu} \sin \theta$ comes from the distance from the symmetry axis. By combining Eqs. (15), (16) and (20), and assuming that $F_{\nu}^r$ is independent of the angle, we obtain
\[
J_{\nu} = -0.013 \left( \frac{\langle B^\phi \rangle}{10^{15} \text{ G}} \right) \left( \frac{R_{\nu}}{100 \text{ km}} \right)^{1/2} \frac{R_{\nu} L_{\nu}}{c},
\]
(21)
\[
= -4.3 \times 10^{-17} \frac{\text{g cm}^2 \text{s}^{-2}}{\text{s}} \times \left( \frac{\langle B^\phi \rangle}{10^{15} \text{ G}} \right) \left( \frac{R_{\nu}}{100 \text{ km}} \right)^{3/2} \frac{L_{\nu}}{10^{33} \text{ erg s}^{-1}},
\]
(22)
where $\langle B^\phi \rangle \equiv \int d\Omega B^\phi \sin \theta/4\pi$, which is the angle-averaged strength.

### 3.2 Angular momentum transfer by neutrinos

In this subsection we evaluate the angular momentum transferred by the anisotropic neutrino radiation that interacts with the toroidal magnetic field. This process occurs during the PNS cooling phase when the neutrino diffusion approximation is valid in the stellar interior (Section 4). By comparing it with the total angular momentum of a rotating NS, we are able to determine an expression for the critical magnetic field strength at which the NS rotation period is drastically affected by the anisotropic neutrino radiation. To compare with present observations, here we employ the NS angular momentum at a stellar radius of 10 km after the PNS cooling phase. This assumption is valid if the angular momentum is conserved when the PNS (i.e. hot NS) contracts to a cold NS, where the radius shrinks from $\sim 100$ km to $\sim 10$ km.

The angular momentum of a NS is written as
\[
M_{\text{NS}}^0 = \Omega \cdot \left( 7.0 \times 10^{35} \frac{\text{g cm}^2 \text{s}^{-1}}{\text{s}} \left( \frac{P}{1 \text{ s}} \right)^{-1} \left( \frac{M}{1.4 M_\odot} \right) \left( \frac{R_{\text{NS}}}{10 \text{ km}} \right)^2 \right),
\]
(23)
where $I = \frac{2}{5} M R_{\text{NS}}^2$ is the moment of inertia, $\Omega$ is the angular velocity, $P$ is the rotation period, $P = 2\pi/\Omega$, $M$ is the NS mass and $R_{\text{NS}}$ is the NS radius.

The angular momentum transferred by neutrino radiation is given by
\[
M_{\nu}^0 = \frac{\beta R_{\nu} E_{\nu}}{c},
\]
(24)
where $\beta$ is the asymmetry parameter for neutrino emission and $E_{\nu}$ is the total energy emitted by the neutrinos responsible for the change in spin, which is related to the luminosity as $E_{\nu} = \int dt L_{\nu}$. Note that a PNS has larger radius than an ordinary NS due to the existence of thermal pressure (see e.g. [Janka 2012, Suwa et al. 2013]). Although the total amount of energy that can be released by the neutrinos is $\sim 3 \times 10^{53}$ erg, the contributions from $\nu_e$ ($\bar{\nu}_e$) and $\nu_x$ ($\bar{\nu}_x$) to the change in spin cancel each other [Arras & Lai 1999]. As such, we only consider the energy released due to the $\nu_e$ emitted in electron capture ($p + e^{-} \rightarrow n + \nu_e$) just after the core bounce of supernova shock, which is $\sim O(10^{52})$ erg. The total number of $\nu_e$ emitted due to electron capture is estimated as
\[
N_{\nu_e} = N_p = \frac{M Y_p}{m_p} = 8.3 \times 10^{56} \left( \frac{M}{1.4 M_\odot} \right) \left( \frac{Y_p}{0.5} \right),
\]
(25)
where $N_p$ is the total number of protons in the neutron star, $m_p$ is the proton mass and $Y_p$ is the proton fraction. By taking the average energy of emitted $\nu_e$ to be $3.15kT = 12.6 \text{ MeV}$ ($kT/4 \text{ MeV}$), the total energy released due to $\nu_e$ emission in the neutronization process is given as $E_{\nu_e} = 1.7 \times 10^{57} \text{ erg} (M/1.4 M_\odot) (Y_p/0.5) (kT/4 \text{ MeV})$.

Comparing Eqs. (23) and (24), one recognizes that the
slowly rotating \((P \sim 1 \text{ s})\) PNS’s rotation can be significantly affected if \(\beta \sim 10^{-3}\). This condition can be used to put a constraint on the strength of internal toroidal magnetic fields. From Eqs. (22) and (24), \(\beta\) is given as

\[
\beta \approx -0.013 \left(\frac{\langle B^0 \rangle}{10^{15} \text{ G}}\right) \left(\frac{R_{\text{NS}}}{10 \text{ km}}\right)^{1/2},
\]

where we have used \(\int dt L_\nu = E_\nu\). Using these relations, in the next section we will constrain the internal toroidal field.

4 CONSTRAINT ON INTERNAL TOROIDAL FIELDS

It is natural to expect that the angular momentum transferred by neutrinos should be smaller than the total angular momentum of the PNS at \(t = \tau_\nu\) (see Section 2). As such, using Eqs. (23) and (24) we get the following constraint:

\[
|\beta| \lesssim 1 \times 10^{-3} \left(\frac{P}{1 \text{ s}}\right)^{-1} \left(\frac{M}{1.4M_\odot}\right) \left(\frac{R_{\text{NS}}}{10 \text{ km}}\right)^2 \times \left(\frac{R_\nu}{100 \text{ km}}\right)^{-1} \left(\frac{E_\nu}{2 \times 10^{52} \text{ erg}}\right)^{-1},
\]

which can be rewritten as a constraint on the magnetic fields using Eq. (26) as

\[
\left|\langle B^0 \rangle\right| \lesssim 8.1 \times 10^{13} \text{ G} \left(\frac{P}{1 \text{ s}}\right)^{-1} \left(\frac{M}{1.4M_\odot}\right) \left(\frac{R_{\text{NS}}}{10 \text{ km}}\right)^2 \times \left(\frac{R_\nu}{100 \text{ km}}\right)^{-3/2} \left(\frac{E_\nu}{2 \times 10^{52} \text{ erg}}\right)^{-1}.
\]

By exploiting the fact that the magnetic field is conserved during the PNS cooling phase, i.e. \(\left\langle B^0_{\text{NS}} \right\rangle R_{\text{NS}}^2 = \left\langle B^0 \right\rangle R_\nu^2\), we can evaluate the field strength inside a cold NS whose radius is \(R_{\text{NS}}\) as

\[
\left\langle B_{\text{NS}}^0 \right\rangle \approx 8.1 \times 10^{15} \text{ G} \left(\frac{P}{1 \text{ s}}\right)^{-1} \left(\frac{M}{1.4M_\odot}\right) \left(\frac{R_{\text{NS}}}{10 \text{ km}}\right)^2 \times \left(\frac{R_\nu}{100 \text{ km}}\right)^{1/2} \left(\frac{E_\nu}{2 \times 10^{52} \text{ erg}}\right)^{-1}.
\]

We therefore see that the constraint on the magnetic field strength depends on the rotation period \(P\) at \(t = \tau_\nu\). The typical spin period of magnetars at \(t = \tau_\nu\) is unclear due to the lack of knowledge on magnetar formation. However, if we take \(P = 10 \text{ ms}\) at \(t = \tau_\nu\), we obtain \(\left\langle B_{\text{NS}}^0 \right\rangle \lesssim 10^{18} \text{ G}\).

If we assume that magnetic dipole radiation is the dominant process affecting magnetar spin evolution for \(t > \tau_\nu\), the spin period of 1E 1841-045 at \(t = \tau_\nu\) can be estimated as \(\approx 8-11 \text{ s}\) (see Appendix A). Therefore, using Eq. (29), we can obtain the following constraint on the field strength:

\[
\left|\left\langle B_{\text{NS}}^0 \right\rangle\right| \lesssim 10^{15} \text{ G} \left(\frac{R_\nu}{100 \text{ km}}\right)^{1/2},
\]

where we have employed canonical values for \(M\) and \(E_\nu\). A similar value is obtained for the case of 1E 2259+586\(^{8}\). Thus, the toroidal magnetic fields of these magnetars can be comparable to the dipole component at least at the moment of birth. Note that this constraint only applies to the global toroidal field, i.e. the angle averaged value near the NS surface, since the angular momenta transferred by turbulent components on small scales cancel each other out.

5 SUMMARY AND DISCUSSION

In this paper we studied the spin evolution of magnetars resulting from the anisotropic neutrino emission induced by strong magnetic fields. We solved the diffusion equation for neutrinos and estimated the degree of anisotropy. By considering the toroidal component of the magnetic fields we were able to constrain the unseen internal fields using the current rotation period of magnetars. Supposing that the associated SNR age is the real magnetar age, we found the constraint \(\left|\left\langle B_{\text{NS}}^0 \right\rangle\right| \lesssim 10^{15} \text{ G}\) for 1E1841-045 and 1E 2259+586, whose dipole fields are thought to be \(\sim 10^{15} \text{ G}\) and \(10^{14} \text{ G}\), respectively.

In addition to the spin evolution, we can also estimate the pulsar kick velocity of magnetars using Eq. (25). When we consider the split monopole poloidal field at the PNS surface, the degree of asymmetry \(\gamma\) is \(O(10^{-2})\). The kick velocity can thus be estimated as

\[
\nu_{\text{kick}} = \frac{E_\nu}{Mc} \approx 24.0 \text{ km s}^{-1} \left(\frac{\gamma}{10^{-2}}\right) \left(\frac{E_\nu}{2 \times 10^{52} \text{ erg}}\right) \left(\frac{M}{1.4M_\odot}\right)^{-1}.
\]

We therefore see that the magnetar kick resulting from this mechanism is expected to be very small.

In this paper we focused on magnetars (SGRs and AXPs). However, there are other classes of stars that also have strong dipole fields (see Dall’Osso et al. 2012 for a list). These objects exhibit a similar spin period to magnetars (3 \(s \sim P \leq 11 \text{ s}\), but their magnetic fields are typically weaker. Even though they do not have associated SNR, we can apply the same analysis as discussed in this paper taking \(P_1 \sim O(1) \text{ s}\). Thus, the constraint obtained in this study is applicable for these objects as well as magnetars.

To finish we comment on the assumptions made in this study. First, we employed the diffusion approximation for the neutrino radiative transfer equation. This assumption is essentially valid for the region of the magnetar considered in this work, but near the surface, where the mean free path

\(^8\)Interestingly, this value is similar to the recent observational suggestion by Makishima et al. (2018), which is based on the pulse modulation analysis implying the precession. Note that their employed magnetar is different one from ours so that this coincidence might be just a product of chance.
of neutrinos is comparable to the scale size, this approximation starts to break down. However, since we are considering the region inside the PNS, the effect of the break-down of this assumption is not significant. Secondly, for simplicity we have assumed that the PNS radius is constant during the cooling phase. However, this assumption does not change our discussion drastically because the constraints on the internal toroidal magnetic field given by Eqs. (29) and (30) imply very weak dependence on the PNS radius. In addition, since a smaller PNS radius gives a tighter upper limit for the toroidal field, our assumption of constant radius will tend to give more conservative upper limits. Thirdly, since the real age of a magnetar is unknown, we assumed it to be the same as that of the SNR. Because the SNR age contains systematic errors, this approximation might affect the derived constraint. However, we expect that the corrections to the age do not change it by orders of magnitude, meaning that our discussion in the previous section should not change very much even if we include this systematic error. Finally, we have assumed that after neutrino emission the sole mechanism behind the magnetar spin-down is dipole radiation. There are several other mechanisms that can decelerate a NS’s spin (see e.g. [Thompson et al. 2004]), which will tend to lead to looser constraints on the internal fields. This is because these mechanisms usually act later than the neutrinos so that a smaller $P_i$ is possible. More detailed studies that include the effects of other deceleration mechanisms are necessary. A fundamental limit can be obtained using the fastest rotation of a NS (i.e. the rotational breakup speed), which gives $\langle B_{NS}^p \rangle \lesssim 10^{19}$ G.

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**APPENDIX A: SPIN EVOLUTION OF MAGNETARS**

**A1 Case without magnetic field decay**

Since the real age of a magnetar, $\tau_0$, is unknown, the characteristic spin-down time, $\tau_c \equiv P/2\dot{P}$, is conventionally used as an approximation. We also know that some magnetars can be associated with SNRs, for which alternative, better age estimations are possible via X-ray plasma diagnostics. Here we assume that the SNR age is a better estimator of $\tau_0$, and extrapolate the current rotation period to the initial period at $\tau_c$ using the dipole radiation model. In the following discussion we give expressions for the initial rotation period $P_i$ at $\tau_c$ and its evolution. In this subsection we neglect the magnetic field decay, which will be discussed in the next subsection.

When dipole radiation is the leading cause of spin-down, the rotation period as a function of time, $t$, can be written as (Shapiro & Teukolsky 1983)

$$P = P_i \left(1 + \frac{2P_i^2 t}{P_i^2 T} \right)^{1/2},$$  \hspace{1cm} (A1)

where the initial period, $P_i$, at $t = \tau_c$ is given at the time when dipole radiation becomes the dominant process for spin down and

$$T = \frac{P}{\dot{P}} = \frac{3\mu_0^3 P^2}{2\pi^2 B_p^2 R^6 \sin^2 \alpha} \hspace{1cm} (A2)$$

$$= 145 \text{ years} \left(\frac{B_p}{10^{15} \text{ G}}\right)^{-2} \left(\frac{R}{10 \text{ km}}\right)^{-4} \left(\frac{M}{1.4 M_\odot}\right) \left(\frac{P}{1 \text{ s}}\right)^2, $$ \hspace{1cm} (A3)

where $B_p$ is the surface dipole field at the pole. Here we employ $\sin^2 \alpha = 1$ for simplicity. Using this relation we find $B_p = \left(\frac{3\mu_0^3}{2\pi^2 R^6 P^2}\right)^{1/2}$.

$$= 6.75 \times 10^{19} \text{ G} \left(\frac{M}{1.4 M_\odot}\right) \left(\frac{R}{10 \text{ km}}\right)^{-4} \left(\frac{P}{1 \text{ s}}\right)^{1/2} \left(\frac{\dot{P}}{1 \text{ s/s}}\right)^{1/2}.$$ \hspace{1cm} (A4)

Although this result looks different by a factor of two to the more frequently used $B = \frac{3\mu_0}{2\pi R^2 P^2}$, this difference just comes from a difference in notation. By substituting Eq. (A2) into (A1), we get the following simple form as

$$P^2 = P_i^2 + \frac{4\pi^2 B_p^2 R^6 \sin^2 \alpha}{3\mu_0^3} t.$$  \hspace{1cm} (A5)

In Figure A1 we show the evolution of the spin period of neutron stars with various strengths of the constant dipole field. The red crosses correspond to observed magnetars for which the characteristic age is used ($\tau_c \equiv P/2\dot{P}$), whilst the blue points correspond to magnetars that can be associated with SNRs, so that the SNR age is used. For $B_p = 10^{15}$ G we plot the evolution for two different initial periods ($P_i = 1$ s for the top line and 1 ms for the bottom line). One finds that the evolutions coincide after > 1000 years, from which we conclude that $P_i$ does not affect the late time evolution.

As can be seen in Table A1 there are two magnetars for which the SNR age is younger than the characteristic age. For example, 1E 2259+586 and associated SNR CTB 109 exhibit a large discrepancy between the two ages. Here we treat the SNR age as the true age and use this to estimate the spin periods of the magnetars at birth. In Figure A2 we show the time evolution of the spin period for values of $P$...
and $P$ equal to those of 1E 1841-045. We find that $P_i$ should be $\approx 8-11$ s in order to explain the current observation with the age of $\sim 1$ kyr. The same analysis also gives the initial period of 1E 2259+586 as $P_i \approx 7$ s, which is almost the same as the current period. Note that these values would be smaller if decay of the poloidal magnetic field were included, which will be discussed in the next subsection.

A2 Case with magnetic field decay

In this subsection we study spin evolution including phenomenologically the effect of magnetic field decay. It is important to consider the effect of the decaying magnetic field because there is no isolated NS with $P \gtrsim 12$ s, meaning that the dipole radiation can be assumed to become small enough so as to not affect the spin period for slowly rotating NSs. There are several studies that investigate the long-term evolution of magnetic fields including their decay (e.g., Colpi et al. 2000; Dall’Osso et al. 2012; Nakano et al. 2014; Pons et al. 2013).

Using the model of Colpi et al. (2000) and Dall’Osso et al. (2012), after several algebraic steps we get the following expressions for the time evolution of the spin period and the dipole magnetic field strength:

$$P^2(t) = P_\infty^2 - (P_\infty^2 - P_\infty^2) \left(1 + \frac{t}{\tau_d}\right)^{\alpha_B-2}/\alpha_B,$$  \hspace{1cm} (A6)

$$B_p(t) = \frac{B_i}{(1 + t/\tau_d)^{1/\alpha_B}},$$ \hspace{1cm} (A7)

where $P_\infty$ is the final spin period, $\tau_d$ is the decay timescale of the magnetic fields, $\alpha_B$ is a parameter describing the magnetic field decay and $B_i$ is the initial magnetic field strength. In Dall’Osso et al. (2012) it was found that models with $1.5 \leq \alpha_B \leq 1.8$ can explain most of the observational evidence for isolated neutron stars with strong magnetic fields (not only magnetars but also X-ray dim isolated NSs). Although $P_\infty$ is unknown, Dall’Osso et al. (2012) and Pons et al. (2013) suggested that $P_\infty \approx 12$ s, because there is no observed NS with $P > 12$ s. Thus, we employ $P_\infty = 12$ s as a fiducial value here. In addition, Dall’Osso et al. (2012) showed that taking $10^{15}$ G $\leq B_i \leq 10^{16}$ G gives good agreement with the distribution of observed NSs with strong magnetic fields in the $\tau_d$-B$\_p$ plane. We thus use $B_i = 10^{16}$ G in the following. In order to explain observed features, Dall’Osso et al. (2012) suggested that $\tau_d = 1$ kyr/($B_i/10^{15}$ G)$^{\alpha_B}$.

In Figure A3 we show the period evolution of magnetars as determined using the decaying magnetic field model. In this figure the top axis gives the strength of poloidal field (decreasing from the initial value of $10^{15}$ G). The blank square shows the current position of 1E1841-045 in the $P$-$B_p$ plane, as estimated from $P$ and $P$. We see that the square overlaps with the left-hand cross, which corresponds to the lower limit on the SNR age. As such, this model can be used to consistently explain all three observed quantities $P$, $B_p$, and the SNR age. One can see that $P_i \gtrsim 11$ s is still required in order to explain observations using the decaying magnetic field model with fiducial model parameters (case (a)). As such, the discussion in the previous subsection is still valid in this case. We do note, however, that with a fine tuning of the parameters it is possible to explain observational data with $P_\infty > 12$ s and $P_i < 1$ s (see case (b)). On the other hand, 1E 2259+586 has $P = 6.9789484460$ s. We find that $P_i \sim 5$ s by the same discussion with fiducial parameters, which is similar value as 1E 1841-045. Therefore, even with the decaying magnetic field model, we find that $P_i$ should be $O(1)$ s.

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Figure A3. The same as Fig. A2 but for the decaying magnetic field model (see Eqs. A6 and A7) with the initial magnetic field $B_i = 10^{16}$ G. The top axis corresponds to the strength of poloidal dipole magnetic field (see Eq. A7). The left panel is for $P_\infty = 12$ s and the right panel for $P_\infty = 15$ s. The blank square marks the current observed $B_p$ and $P$, and is almost coincident with the left-hand cross that marks the lower limit on the SNR age.

Table A1. Observational properties of SGRs and AXPs

| SGR/AXP name† | $P$ [s] | $\dot{P}$ [$10^{-11}$ s/s] | $B_p$ [$10^{14}$ G]‡ | $\tau_c$ [kyr]§ | SNR age [kyr] |
|---------------|--------|-----------------|-----------------|-------------|-------------|
| SGR 0418+5729 | 9.07838827(4) | <0.0006 | <0.16 | 2.4×10^4 | — |
| SGR 0501+4516 | 5.76209653(3) | 0.582(3) | 3.9 | 16 | — |
| SGR 0526-66 | 8.054(2) | 3.8(1) | 12 | 3.4 | 4.8† |
| SGR 1627-41 | 2.5945786(6) | 1.9(4) | 4.7 | 2.2 | — |
| SGR 1806-20 | 7.6022(7) | 75(4) | 51 | 0.16 | — |
| Swift J1822.3-1606 | 8.43771977(4) | 0.0254(22) | 0.99 | 530 | — |
| SGR 1833-0832 | 7.5654084(4) | 0.35(3) | 3.5 | 34 | — |
| Swift J1834.9-0846 | 2.4823018(1) | 0.796(12) | 3.0 | 4.9 | 60–200# |
| SGR 1900+14 | 5.19987(7) | 9.2(4) | 15 | 0.90 | — |
| CXOU J010043.1-721134 | 8.020392(9) | 1.88(8) | 8.3 | 6.8 | — |
| 4U 0142+61 | 8.68832877(2) | 0.20332(7) | 2.8 | 68 | — |
| 1E 1048.1-5937 | 6.457875(3) | ~2.25 | 8.1 | 4.5 | — |
| 1E 1547.0-5408 | 2.0721255(1) | ~4.7 | 6.7 | 0.70 N/A |
| PSR J1622-4950 | 4.3261(1) | 1.7(1) | 5.8 | 4.0 | — |
| CXO J164710.2-455216 | 10.6160663(1) | ~0.073 | 1.9 | 230 | — |
| IGR J170849.0-400910 | 11.003027(1) | 1.91(4) | 9.8 | 9.1 | — |
| CXOU J171405.7-381031 | 3.82535(5) | 6.40(14) | 11 | 0.95 | 4.9% |
| XTE J1810-197 | 5.5403537(2) | 0.777(3) | 4.4 | 11 | — |
| 1E 1841-045 | 11.7828977(10) | 3.93(1) | 15 | 4.8 | 0.5–2.6$^c$ |
| 1E 2259+586 | 6.9789484460(39) | 0.048430(8) | 1.2 | 230 | 14$^d$ |

† Data taken from McGill SGR/AXP Online Catalog [Olausen & Kaspi 2014] (see also Viganò et al. 2013).
‡ The estimation is based on Eq. (A4).
§ Characteristic ages estimated as $P/2\dot{P}$.
$^a$ Park et al. (2012).
$^b$ Tian et al. (2007).
$^c$ Aharonian et al. (2008).
$^d$ Tian & Leahy (2008).
$^e$ Sasaki et al. (2013).
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