Pareto Law in a Kinetic Model of Market with Random Saving Propensity

Arnab Chatterjee, Bikas K. Chakrabarti, and S. S. Manna

1 Saha Institute of Nuclear Physics, Block-AF, Sector-I Bidhannagar, Kolkata-700064, India.
2 Satyendra Nath Bose National Centre for Basic Sciences Block-JD, Sector-III, Salt Lake, Kolkata-700098, India.

We have numerically simulated the ideal-gas models of trading markets, where each agent is identified with a gas molecule and each trading as an elastic or money-conserving two-body collision. Unlike in the ideal gas, we introduce (quenched) saving propensity of the agents, distributed widely between the agents (0 ≤ λ < 1). The system remarkably self-organizes to a critical Pareto distribution of money \( P(m) \sim m^{-(\nu+1)} \) with \( \nu \approx 1 \). We analyse the robustness (universality) of the distribution in the model. We also argue that although the fractional saving ingredient is a bit unnatural one in the context of gas models, our model is the simplest so far, showing self-organized criticality, and combines two century-old distributions: Gibbs (1901) and Pareto (1897) distributions.

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Considerable investigations have already been made to study the nature of income or wealth distributions in various economic communities, in particular, in different countries. For more than a hundred years, it is known that the probability distribution \( P(m) \) for income or wealth of the individuals in the market decreases with the wealth \( m \) following a power law, known as Pareto law \( I \):

\[
P(m) \propto m^{-(1+\nu)},
\]

where the value of the exponent \( \nu \) is found to lie between 1 and 2 \( 2 \leq \nu \leq 4 \). It is also known that typically less than 10% of the population in any country possesses about 40% of the wealth and follow the above power law. The rest of the low-income group population, in fact the majority, clearly follows a different law, identified very recently to be the Gibbs distribution \( 5,6,7 \). Studies on real data show that the high income group indeed follows the above Gibbs distribution, with \( \nu \) varying from 1.6 for USA \( 4 \), to 1.8 – 2.2 in Japan \( 5 \). The value of \( \nu \) thus seem to vary a little from economy to economy.

We have studied here numerically a gas model of a trading market. We have considered the effect of saving propensity of the traders. The saving propensity is assumed to have a randomness. Our observations indicate that Gibbs and Pareto distributions fall in the same category and can appear naturally in the century-old and well-established kinetic theory of gas \( 3 \). Gibbs distribution for no saving and Pareto distribution for agents with quenched random saving propensity. Our model study also indicates the appearance of self-organized criticality \( 8 \) in the simplest model so far, namely in the kinetic theory of gas models, when the stability effect of savings \( 10 \) is incorporated.

We consider an ideal-gas model of a closed economic system where total money \( M \) and total number of agents \( N \) is fixed. No production or migration occurs and the only economic activity is confined to trading. Each agent \( i \), individual or corporate, possess money \( m_i(t) \) at time \( t \). In any trading, a pair of traders \( i \) and \( j \) randomly exchange their money \( 3,6,11 \), such that their total money is (locally) conserved and none end up with negative money (\( m_i(t) \geq 0 \), i.e. debt not allowed):

\[
m_i(t) + m_j(t) = m_i(t + 1) + m_j(t + 1);
\]

time \( t \) changes by one unit after each trading. The steady-state \( t \to \infty \) distribution of money is Gibbs one:

\[
P(m) = (1/T) \exp(-m/T); T = M/N.
\]

Hence, no matter how uniform or justified the initial distribution is, the eventual steady state corresponds to Gibbs distribution where most of the people have got very little money. This follows from the conservation of money and additivity of entropy:

\[
P(m_1)P(m_2) = P(m_1 + m_2).
\]

This steady state result is quite robust and realistic too! In fact, several variations of the trading, and of the ‘lattice’ (on which the agents can be put and each agent trade with its ‘lattice neighbors’ only), whether compact, fractal or small-world like \( 2 \), leaves the distribution unchanged. Some other variations like random sharing of an amount \( 2m_2 \) only (not of \( m_1 + m_2 \)) when \( m_1 > m_2 \) (trading at the level of lower economic class in the trade), lead to even drastic situation: all the money in the market drifts to one agent and the rest become truly pauper \( 12,13 \).

In any trading, savings come naturally \( 10 \). A saving propensity factor \( \lambda \) is therefore introduced in the same model \( 3 \) (see \( 11 \) for model without savings), where each trader at time \( t \) saves a fraction \( \lambda \) of its money \( m_i(t) \) and trades randomly with the rest:

\[
m_i(t + 1) = m_i(t) + \Delta m; \quad m_j(t + 1) = m_j(t) - \Delta m
\]
\[ \Delta m = (1 - \lambda)[\epsilon \{ m_i(t) + m_j(t)\} - m_i(t)], \quad (6) \]

\( \epsilon \) being a random fraction, coming from the stochastic nature of the trading.

The market (non-interacting at \( \lambda = 0 \) and 1) becomes 'interacting' for any non-vanishing \( \lambda (<1) \): For fixed \( \lambda \) (same for all agents), the steady state distribution \( P_f(m) \) of money is exponentially decaying on both sides with the most-probable money per agent shifting away from \( m = 0 \) (for \( \lambda = 0 \)) to \( M/N \) as \( \lambda \to 1 \) (Fig. 1(a)). This self-organizing feature of the market, induced by sheer self-interest of saving by each agent without any global perspective, is quite significant as the fraction of paupers decrease with saving fraction \( \lambda \) and most people end up with some fraction of the average money in the market (for \( \lambda \to 1 \), the socialists' dream is achieved with just people's self-interest of saving!). Interestingly, self-organisation also occurs in such market models when there is restriction in the commodity market [14]. Although this fixed saving propensity does not give yet the Pareto-like power-law distribution, the Markovian nature of the scattering or trading processes (eqn. (1)) is lost and the system becomes co-operative. Indirectly through \( \lambda \), the agents get to know (start interacting with) each other and the system co-operatively self-organises towards a most-probable distribution \( (m_p \neq 0) \).

\[ P(m) \sim |\lambda_0 - \lambda|^{\alpha}, \quad \lambda_0 \neq 1, \quad 0 < \lambda < 1, \quad (8) \]

of quenched \( \lambda \) values among the agents. The Pareto law with \( \nu = 1 \) is universal for all \( \alpha \). The data in Fig. 2 corresponds to \( \lambda_0 = 0, \alpha = 0 \). For negative \( \alpha \) values, however, we get an initial (small \( m \)) Gibbs-like decay in \( P(m) \) (see Fig. 3).

In a real society or economy, \( \lambda \) is a very inhomogeneous parameter: the interest of saving varies from person to person. We move a step closer to the real situation where saving factor \( \lambda \) is widely distributed within the population. One again follows the same trading rules as before, except that

\[ \Delta m = \epsilon (1 - \lambda_i) m_j(t) - (1 - \lambda_i)(1 - \epsilon)m_i(t) \quad (7) \]

here; \( \lambda_i \) and \( \lambda_j \) being the saving propensities of agents \( i \) and \( j \). The agents have fixed (over time) saving propensities, distributed independently, randomly and uniformly (white) within an interval 0 to 1 (see [15] for preliminary results): agent \( i \) saves a random fraction \( \lambda_i \) (\( 0 \leq \lambda_i < 1 \)) and this \( \lambda_i \) value is quenched for each agent (\( \lambda_i \) are independent of trading or \( t \)). Starting with an arbitrary initial (uniform or random) distribution of money among the agents, the market evolves with the tradings. At each time, two agents are randomly selected and the money exchange among them occurs, following the above mentioned scheme. We check for the steady state, by looking at the stability of the money distribution in successive Monte Carlo steps \( t \). Eventually, after a typical relaxation time (\( \sim 10^5 \) for \( N = 200 \) and uniformly distributed \( \lambda \)) dependent on \( N \) and the distribution of \( \lambda \), the money distribution becomes stationary. After this, we average the money distribution over \( \sim 10^3 \) time steps. Finally we take configurational average over \( \sim 10^5 \) realizations of the \( \lambda \) distribution to get the money distribution \( P(m) \).

It is found to follow a strict power-law decay. This decay fits to Pareto law [1] with \( \nu = 1.02 \pm 0.02 \) (Fig. 2). Note, for finite size \( N \) of the market, the distribution has a narrow initial growth up to a most-probable value \( m_p \) after which it falls off with a power-law tail for several decades. As can be seen from the inset of Fig. 2, this Pareto law (with \( \nu \simeq 1 \)) covers the entire range in \( m \) of the distribution \( P(m) \) in the limit \( N \to \infty \). We checked that this power law is extremely robust: apart from the uniform \( \lambda \) distribution used in the simulations in Fig. 2, we also checked the results for a distribution

FIG. 1: Steady state money distribution (a) \( P_f(m) \) for the fixed (same for all agents) \( \lambda \) model, and (b) \( P_f(m) \) for some typical values of \( \lambda \) in the distributed \( \lambda \) model. The data is collected from the ensembles with \( N = 200 \) agents. The inset in (b) shows the scaling behavior of \( P_f(m) \). For all cases, agents start with average money per agent \( M/N = 1 \).

FIG. 2: Steady state money distribution \( P(m) \) in the model for distributed \( \lambda \) (\( 0 \leq \lambda < 1 \)) for \( N = 1000 \) agents. Inset shows that the most probable peak \( m_p \) shifts towards 0 (indicating the same power law for the entire range of \( m \)) as \( N \to \infty \); results for four typical system sizes \( N = 100, 200, 500, 1000 \) are shown. For all cases, agents play with average money per agent \( M/N = 1 \).
Larger and can approach to the order of $N = 100$ agents with $\lambda \rightarrow 1$. Few subtle points may be noted though: while for fixed $\lambda$ (1), for distributed $\lambda$ values of $\alpha$ with $1 + \nu \leq 0$, the trend gets reversed (see Fig. 4). In the distributed $\lambda$ case $m_{p}(\lambda)$ can be considerably larger and can approach to the order of $N$ for large $\lambda$ (see Fig. 1(b)). The other important difference is in the scaling behavior of $\tilde{P}_{f}(m)$, as shown in the inset of Fig. 1(b). In the distributed $\lambda$ ensemble, $\tilde{P}_{f}(m)$ appears to have a very simple scaling:

$$\tilde{P}_{f}(m) \sim (1 - \lambda) F(m(1 - \lambda)), \quad (9)$$

for $\lambda \rightarrow 1$, where the scaling function $F(x)$ has non-monotonic variation in $x$. The fixed (same for all agents) $\lambda$ income distribution $P_{f}(m)$ do not have any such comparative scaling property. It may be noted that a small difference exists between the ensembles considered in Fig 1(a) and 1(b): while $\int m P_{f}(m) dm = M$ (independent of $\lambda$), $\int m \tilde{P}_{f}(m) dm$ is not a constant and infact approaches to order of $M$ as $\lambda \rightarrow 1$. There is also a marked qualitative difference in fluctuations (see Fig. 4): while for fixed $\lambda$, the fluctuations in time (around the most-probable value) in the individuals’ money $m_{i}(t)$ gradually decreases with increasing $\lambda$, for quenched distribution of $\lambda$, the trend gets reversed (see Fig. 4).

We now investigate on the range of distribution of the saving propensities in a certain interval $a < \lambda < b$, where, $0 < a < b < 1$. For uniform distribution within the range, we observe the appearance of the same power law in the distribution but for a narrower region. As may be seen from Fig. 5, as $a \rightarrow b$, the power-law behavior is seen for values $a$ or $b$ approaching more and more towards...
unity: For the same width of the interval \( |b-a| \), one gets power-law (with same \( \nu \)) when \( b \to 1 \). This indicates, for fixed \( \lambda, \lambda = 0 \) corresponds to Gibbs distribution, and one gets Pareto law when \( \lambda \) has got non-zero width of its distribution extending upto \( \lambda = 1 \). This of course indicates a crucial role of these high saving propensity agents: the power law behavior is truely valid upto the asymptotic limit if \( \lambda = 1 \) is included. Indeed, had we assumed \( \lambda_0 = 1 \) in \( \mathbb{R} \), the Pareto exponent \( \nu \) immediately switches over to \( \nu = 1 + \alpha \). Of course, \( \lambda_0 \neq 1 \) in \( \mathbb{R} \) leads to the universality of the Pareto distribution with \( \nu = 1 \) (independent of \( \lambda_0 \) and \( \alpha \)). Indeed this can be easily rationalised from the scaling behavior \( \mathbb{R} \): 

\[
P(m) \sim \int_0^1 \tilde{P}_f(m) \rho(\lambda) \, d\lambda \sim m^{-2} \text{ for } \rho(\lambda) \text{ given by } \mathbb{R} \text{ and } m^{-(2+\alpha)} \text{ if } \lambda_0 = 1 \text{ in } \mathbb{R} \text{ (for large } m \text{ values}).
\]

![Graph showing cumulative probability distribution for income in USA and Japan](image)

**FIG. 6:** Cumulative distribution \( Q(m) = \int_{-\infty}^{m} P(m) \, dm \) of wealth \( m \) in USA \( \mathbb{R} \) in 1997 and Japan \( \mathbb{R} \) in 2000. Low-income group follow Gibbs law (shaded region) and the rest (about 5%) of the rich population follow Pareto law. The inset shows the cumulative distribution for a model market where the saving propensity of the agents is distributed following \( \mathbb{R} \) with \( \lambda_0 = 0 \) and \( \alpha = -0.7 \). The dotted line (for large \( m \) values) corresponds to \( \nu = 1.0 \).

These model income distributions \( P(m) \) compare very well with the wealth distributions of various countries: Data suggests Gibbs like distribution in the low-income range \( \mathbb{R} \) (more than 90% of the population) and Pareto-like in the high-income range \( \mathbb{R} \) (less than 10% of the population) of various countries (Fig. 6). In fact, we have compared one model simulation of the market with saving propensity of the agents distributed following \( \mathbb{R} \), with \( \lambda_0 = 0 \) and \( \alpha = -0.7 \). This model result is shown in the inset of Fig. 6. The qualitative resemblance of the model income distribution with the real data for Japan and USA in recent years is quite intriguing. In fact, for negative \( \alpha \) values in \( \mathbb{R} \), the density of traders with low saving propensity is higher and since \( \lambda = 0 \) ensemble yields Gibbs-like income distribution \( \mathbb{R} \), we see an initial Gibbs-like distribution which crosses over to Pareto distribution \( \mathbb{R} \) with \( \nu = 1.0 \) for large \( m \) values. The position of the crossover point depends on the magnitude of \( \alpha \). The important point to note is that any distribution of \( \lambda \) near \( \lambda = 1 \), of finite width, eventually gives Pareto law for large \( m \) limit. The same kind of crossover behavior (from Gibbs to Pareto) can also be reproduced in a model market of mixed agents where \( \lambda = 0 \) for a finite fraction of population and \( \lambda \) is distributed uniformly over a finite range near \( \lambda = 1 \) for the rest of the population.

We also considered annealed randomness in the saving propensity \( \lambda \); here \( \lambda_i \) for any agent \( i \) changes from one value to another within the range \( 0 \leq \lambda_i < 1 \), after each trading. Numerical studies for this annealed model did not show any power law behavior for \( P(m) \); rather it again becomes exponentially decaying on both sides of a most-probable value.

We have numerically simulated here ideal-gas like models of trading markets, where each agent is identified with a gas molecule and each trading as an elastic or money-conserving two-body collision. Unlike in the ideal gas, we introduce (quenched) saving propensity of the agents, distributed widely between the agents (\( 0 \leq \lambda < 1 \)). For quenched random variation of \( \lambda \) among the agents the system remarkably self-organizes to a critical Pareto distribution \( \mathbb{R} \) of money with \( \nu \simeq 1.0 \) (Fig. 2). The exponent is quite robust: for savings distribution \( \rho(\lambda) \sim (|\lambda_0 - \lambda|^\alpha, \lambda_0 \neq 1, \) one gets the same Pareto law with \( \nu = 1 \) (independent of \( \lambda_0 \) or \( \alpha \)). It may be noted that the trading market model we have talked about here has got some apparent limitations. The stochastic nature of trading assumed here in the trading market, through the random fraction \( \epsilon \) in \( \mathbb{R} \), is of course not very straightforward as agents apparently go for trading with some definite purpose (utility maximization of both money and commodity). We are however, looking only at the money transactions between the traders. In this sense, the income distribution we study here essentially corresponds to ‘paper money’, and not the ‘real wealth’. However, even taking money and commodity together, one can argue (see \( \mathbb{R} \)) for the same stochastic nature of the transactions, due to the absence of ‘just pricing’ and the effects of bargains in the market.

Apart from the intriguing observation that Gibbs (1901) and Pareto (1897) distributions fall in the same category and can appear naturally in the century-old and well-established kinetic theory of gas, that this model study indicates the appearance of self-organized criticality in the simplest (gas) model so far, when the stability effect of savings incorporated, is remarkable.

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