Lectures on Geometry and Topology of Polynomials—Surrounding the Jacobian Conjecture—

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Introduction

A polynomial ring over a field is very simple to define but yet very difficult to handle in the case with two or more variables. Although polynomial rings modulo ideals give rise to affine algebras, which are fundamental in algebraic geometry, complexity of polynomial computations bars trials to looking into the properties of polynomials. For example, one would like to know the relations between the form of a polynomial standardized after some operations and geometric or topological properties of the hypersurface defined by the given polynomial. Some scattered results are obtained through various investigations around this direction, e.g., polynomial mappings defined by the given polynomial and other topics.

Recent developments of affine algebraic geometry, especially the theory of open algebraic surfaces, provide means to systematically explore geometric and topological properties of polynomials in two variables. Nevertheless, there is one unsurmountable problem remained even in the case of two variables, that is the Jacobian conjecture. It has been unsolved for more than sixty years since O.H. Keller [Ganze Cremona-Transformationen, Monats. Math. Physik 47 (1939), 299–306] proved erroneously that a polynomial endomorphism \( \varphi \) of the complex affine plane \( \mathbb{C}^2 \) (which we denote also by \( \mathbb{A}^2 \)) with everywhere non-vanishing Jacobian determinant is an automorphism.

Difficulty of the conjecture lies probably in the point that the affine plane \( \mathbb{C}^2 \) is so simple and there are no clues to tackle the problem geometrically. The following four approaches or extra conditions to be put additionally will explain what we need or expect to have as clues in dimension two. In order
to explain these approaches, we note first that the condition on the Jacobian determinant to be nowhere vanishing is equivalent to the condition that \( \varphi \) is étale (or unramified if there are singular points on varieties concerned).

(1) Let \( G \) be a finite subgroup of \( \text{Aut}(\mathbb{C}^2) \). Suppose that the endomorphism \( \varphi \) commutes with the \( G \)-action. Then \( \varphi \) induces an endomorphism \( \overline{\varphi} \) of the quotient surface \( \mathbb{C}^2/G \) such that \( \overline{\varphi} \) is unramified. Since \( G \) is regarded as a linear subgroup, the quotient surface \( \mathbb{C}^2/G \) has a nice structure of a Platonic \( \mathbb{C}^* \)-fiber space if the unique singular point removed, which is, by definition, a smooth algebraic surface with a \( \mathbb{C}^* \)-fibration over the projective line \( \mathbb{P}^1 \) and with three multiple fibers whose multiplicity sequence is a Platonic triplet. If one can show that \( \overline{\varphi} \) (or its restriction onto \( \mathbb{C}^2/G - \{ \text{a singular point} \} \)) is an automorphism, then \( \varphi \) itself is an automorphism. This is the so-called \textit{equivariant version} of the Jacobian conjecture in dimension two.

(2) Let \( C \) be a curve on \( \mathbb{C}^2 \). Suppose that \( \varphi \) satisfies the condition \( \varphi^{-1}(C) \subseteq C \). Then \( \varphi \) induces an étale endomorphism of \( \mathbb{C}^2 \setminus C \). It is shown in [3, 35] that if \( C \) is irreducible the conjecture is affirmative with this extra condition. But the case where \( C \) is reducible is not worked out.

(3) Let \( X \) be a smooth affine surface and let \( \varphi \) is an étale endomorphism of \( X \). One can ask as an analogy of the Jacobian conjecture if \( \varphi \) is an automorphism. But this is not quite so as the \( n \)-th power mapping of the algebraic torus group \( G_m \) is an étale finite endomorphism of degree \( n \). So we ask instead if \( \varphi \) is a finite morphism, and we call this \textit{the generalized Jacobian conjecture}. It will be reviewed later that the generalized Jacobian conjecture is again false in general as there are many counterexamples. But one can
expect that the conjecture holds for surfaces which are quite similar to $\mathbb{C}^2$. One of the candidate surfaces is the complement $\mathbb{P}^2 \setminus C$ of a plane curve $C$ defined by $X_0X_1^{d-1} = X_2^d$ for $d \geq 2$. This surface is a kind of $\mathbb{Q}$-homology plane and has a structure which we generalize as an affine pseudo-plane. Namely it has an $\mathbb{A}^1$-fibration over the affine line $\mathbb{A}^1$ which has a unique multiple fiber. We shall discuss in the subsection 1.4 some results on affine pseudo-planes concerning the cancellation problem. See also [10, 30]. It is strongly desired to verify if the generalized Jacobian conjecture holds for affine pseudo-planes.

(4) Suppose that there exists a non-isomorphic, étale endomorphism $\varphi$ of $\mathbb{C}^2$. We are interested in algebraic surfaces $Y$ which might factorize $\varphi$ in the sense that $\varphi : \mathbb{C}^2 \xrightarrow{\varphi_1} Y \xrightarrow{\varphi_2} \mathbb{C}^2$. Note that $\varphi_1$ and $\varphi_2$ are étale and we may assume that they are almost surjective in the sense that the image includes all codimension one points. In general, given a morphism $f : X \rightarrow Y$ of smooth affine varieties, we say that $X$ is an affine pseudo-covering of $Y$ if $f$ is almost surjective and étale. So, we are interested in what kind of surfaces are affine pseudo-coverings of $\mathbb{C}^2$ or what kind of surfaces have $\mathbb{C}^2$ as affine pseudo-coverings. These points of view are taken up in the subsection 2.4.

The author started to write the present lecture notes for the lecture(s) to be delivered at the Mathematics Department of l’Université de Bordeaux I and, in fact, talked on the beginning parts 1.1 and 1.2 in January, 2003. The most of the first section was lectured at the Department of Mathematics, Osaka University during the first semester of the year 2003. There are some influences in the notes by lecturing in the classes. For example, the
subsections 1.1 and 1.2 as well as the other parts are elementary, well-known or short of full explanations. But we left these parts in the notes as they are because the notes grew as the author teaches in the classes and thinks the Jacobian problem once and again. So, the readers might as well skip 1.1, 1.2 and 1.3. The author gradually began to have the idea of putting together various results of the author and his collaborators which were published in journals and proceedings [3, 19, 20, 28, 33, 35]. But these notes are not simple reproductions of the past results. There are some new results contained in the subsections 1.4 and 2.5. Meanwhile, most results in the subsections 2.2, 2.3 and 2.4 are the reproductions from [35]. We added the subsection 2.1 for the readers convenience to have a quick overview on the present (not necessarily updated) status of the Jacobian conjecture.

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Chapter 1

Characterization of polynomial rings

§1. General properties of polynomial rings

§2. Squeezing-out one variable

§3. Fibrations of the affine plane by polynomials

§4. Plane-like affine surfaces
1.1 General properties of polynomial rings

Let $k$ be a field. A polynomial ring $k[x_1, \ldots, x_n]$ in $n$ variables over $k$ is a commutative $k$-algebra consisting of finite sums of monomials $a_{d_1 \ldots d_n} x_1^{d_1} \cdots x_n^{d_n}$, where $a_{d_1 \ldots d_n} \in k$ and monomials are multiplied by the rule

$$x_1^{d_1} \cdots x_n^{d_n} \times x_1^{e_1} \cdots x_n^{e_n} = x_1^{d_1+e_1} \cdots x_n^{d_n+e_n}.$$

Hence two polynomials $f = \sum_d a_{d_1 \ldots d_n} x_1^{d_1} \cdots x_n^{d_n}$ and $g = \sum_d b_{d_1 \ldots d_n} x_1^{d_1} \cdots x_n^{d_n}$ are added and multiplied as follows:

$$f + g = \sum_d (a_{d_1 \ldots d_n} + b_{d_1 \ldots d_n}) x_1^{d_1} \cdots x_n^{d_n},$$

$$fg = \sum_c \sum_{c=d+e} a_{d_1 \ldots d_n} b_{e_1 \ldots e_n} x_1^{e_1} \cdots x_n^{e_n}.$$

For a ring $R$, we denote by $R^*$ the set of invertible elements of $R$. Note that $R^*$ is a group under the multiplication of $R$. For an element $a \in R^*$, the group inverse is the inverse $a^{-1}$ of $a$ in $R$. For an integral domain $R$, an element $a$ is irreducible if $a = bc$ with $b, c \in R$ implies either $b \in R^*$ or $c \in R^*$. $R$ is said to be decomposable if for any nonzero element $a \in R$, there is at least one irreducible decomposition $a = ub_1 \cdots b_n$, where $u \in R^*$ and $b_1, \ldots, b_n$ are irreducible elements.

**Lemma 1.1.1** If $R$ is a noetherian domain then $R$ is decomposable.

**Proof.** Suppose that an element $a$ has no irreducible decomposition. Then $a$ is not irreducible. Hence $a = a_1 b_1$ with non-invertible elements $a_1, b_1 \in R$. Then either $a_1$ or $b_1$ has no irreducible decomposition, say $a_1$. Repeating this argument, we find an infinite series of non-invertible elements $a, a_1, a_2, \cdots$
such that \( a_i = a_{i+1}b_{i+1} \) and \( a_i \) has no irreducible decomposition. This implies that there is an ascending chain of ideals \( aR \subset a_1R \subset \cdots \subset a_nR \subset \cdots \). This contradicts the hypothesis that \( R \) is noetherian. Q.E.D.

Finally, \( R \) is said to be factorial or a unique factorization domain (UFD) if \( R \) is decomposable and if whenever there exist two expressions \( a = ub_1 \cdots b_n = vc_1 \cdots c_m \) with \( u, v \in R^* \) then \( n = m \) and \( b_{\sigma(i)} = u_ic_i \ (1 \leq i \leq n) \) for a suitable permutation \( \sigma \) of \( \{1, 2, \ldots, n\} \) and \( u_i \in R^* \).

Let \( R \) be an integral domain. Let \( a, b \in R \). Then \( a \) divides \( b \) if \( b = ac \) for some \( c \in R \). An element \( p \in R \) is called a prime element if \( p \) divides \( ab \) with \( a, b \in R \) then \( p \) divides either \( a \) or \( b \). This definition is equivalent to saying that the ideal \( aR \) is a prime ideal, i.e., the ring \( R/(a) \) is an integral domain. A prime element \( p \) is irreducible, for if \( p = ab \) then \( a = pu \) or \( b = pv \) with \( u, v \in R \), and \( ub = 1 \) or \( av = 1 \), i.e., either \( u \) or \( v \) is invertible.

**Exercise 1.1.1.** Let \( R \) be an integral domain. Show that if \( R \) is factorial then every irreducible element is a prime element. Suppose further that \( R \) is decomposable. If every irreducible element is a prime element, then \( R \) is factorial.

Suppose that \( R \) is noetherian and hence decomposable. Given two elements \( a, b \) of \( R \), we say that \( c \in R \) is a common divisor of \( a, b \) if \( c \) divides \( a \) and \( b \), \( c \mid a \) and \( c \mid b \) by notation. An element \( d \) of \( R \) is a greatest common divisor of \( a, b \) if \( d \) is a common divisor of \( a, b \) and any common divisor \( e \) of \( a, b \) divides \( d \). If \( R \) is factorial, such an element \( d \) exists and is uniquely determined up to multiplication of an invertible element. So, we denote \( d \) by \( \gcd(a, b) \). For elements \( a_1, \ldots, a_m \) of \( R \), we can also define the greatest
common divisor \( \gcd(a_1, \ldots, a_m) \) if \( R \) is factorial.

**Exercise 1.1.2.** Suppose \( R \) is noetherian and factorial. For elements \( a_1, \ldots, a_m \) of \( R \), write

\[
a_i = u_i p_1^{\alpha_i(1)} p_2^{\alpha_i(2)} \cdots p_n^{\alpha_i(n)},
\]

where \( u_i \in R^* \), \( \alpha_i(j) \geq 0 \) (\( 1 \leq j \leq n \)) and \( p_1, p_2, \ldots, p_n \) are mutually non-divisible prime elements. Show then that \( \gcd(a_1, \ldots, a_m) \) is given by

\[
p_1^{\beta_1} p_2^{\beta_2} \cdots p_n^{\beta_n},
\]

where \( \beta_j = \min(\alpha_1^{(j)}, \ldots, \alpha_m^{(j)}) \).

**Theorem 1.1.2** The polynomial ring \( k[x_1, \ldots, x_n] \) has the following properties.

1. \( k[x_1, \ldots, x_n]^* = k^* \).
2. \( k[x_1, \ldots, x_n] \) is factorial.

**Proof.** The proof consists of showing that (1) a polynomial ring \( k[x] \) is factorial and (2) \( R[x] \) is noetherian and factorial provided so is \( R \). The assertion (1) is Exercise 1.1.3 below. We shall prove the assertion (2). Let \( K \) be the quotient field of \( R \). Note that \( R[x] \subset K[x] \). Let \( f(x) = a_0 x^d + a_1 x^{d-1} + \cdots + a_{d-1} x + a_d \) be a polynomial in \( R[x] \). We say that \( f \) is primitive if the coefficients \( a_0, a_1, \ldots, a_d \) has no common divisor. If \( f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n \) and \( g(x) = b_0 x^m + b_1 x^{m-1} + \cdots + b_m \) are primitive polynomials with \( a_0 b_0 \neq 0 \), then the product \( f(x)g(x) = \sum_{i=0}^{n+m} (\sum_{j=0}^{i} a_j b_{i-j}) x^{n+m-i} \) is primitive as well. In fact, if \( f(x)g(x) \) is not primitive, then there is a prime element \( p \) of \( R \) such that \( p \) divides all the coefficients of \( f(x)g(x) \). Suppose
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that \( p \mid a_j \) for \( 0 \leq j < \ell_1, p \nmid a_{\ell_1} \) and \( p \mid b_j \) for \( 0 \leq j < \ell_2 \) and \( p \nmid b_{\ell_2} \). Then \( p \) does not divide the coefficient \( a_{\ell_1}b_{\ell_2} + (\sum_{i=0}^{\ell_1+\ell_2} a_ib_{\ell_1+i-1} - a_{\ell_1}b_{\ell_2}) \) of the term of degree \( \ell_1 + \ell_2 \). This is a contradiction.

Given a polynomial \( h(x) \) of \( K[x] \), we find elements \( c, d \in R \) and a primitive polynomial \( f(x) \in R[x] \) such that \( \gcd(c, d) = 1 \) and \( ch(x) = df(x) \). Note that \( c = 1 \) if \( h(x) \in R[x] \). Furthermore, if \( h(x) \) is an irreducible element of \( R[x] \), then \( d = 1 \). Hence \( h(x) \) is primitive. Let \( h(x) \) be an irreducible element of \( R[x] \) again. If \( h(x) \) splits as \( h(x) = f'(x)g'(x) \) in \( K[x] \) with \( \deg f'(x) > 0 \) and \( \deg g'(x) > 0 \), then \( ch(x) = df(x)g(x) \), where \( c, d \in R \) with \( \gcd(c, d) = 1 \) and \( f(x), g(x) \in R[x] \) which are primitive and differ from \( f'(x), g'(x) \) by elements of \( K \). Since \( f(x)g(x) \) is primitive, it follows that \( c = 1 \). Hence \( h(x) \) is not irreducible. This implies that an irreducible element of \( R[x] \) is irreducible in \( K[x] \) as well. We shall show that if \( f(x) \) is an irreducible element of \( R[x] \) then \( f(x) \) is a prime element. In fact, if \( f(x) \) is constant, i.e., \( f(x) = a \in R \), then \( a \) is irreducible in \( R \) and hence prime. So, the ring \( R[x]/(f(x)) \) is isomorphic to \( (R/(a))[x] \), which is an integral domain. So, \( f(x) \) is a prime element in \( R[x] \). Assume that \( f(x) \not\in R \). Suppose that \( g(x)h(x) = f(x)q(x) \), where \( g(x), h(x), q(x) \in R[x] \). Since \( f(x) \) is irreducible, \( f(x) \) divides either \( g(x) \) or \( h(x) \) in \( K[x] \). Suppose \( f(x) \mid g(x) \) in \( K[x] \). Write \( g(x) = f(x)q'(x) \) with \( q'(x) \in K[x] \) and write \( cq'(x) = dq(x) \), where \( c, d \in R \) with \( \gcd(c, d) = 1 \) and \( q(x) \in R[x] \) is primitive. Then \( cg(x) = df(x)q(x) \), where \( f(x)q(x) \) is primitive. Hence \( c = 1 \), and \( g(x) = df(x)q(x) \), i.e., \( f(x) \mid g(x) \). So, \( f(x) \) is a prime element of \( R[x] \). Since \( R[x] \) is noetherian, it follows that \( R[x] \) is factorial.

**Exercise 1.1.3.** Show that \( k[x] \) is a principal ideal domain (PID) and that
a PID is factorial.

**Exercise 1.1.4.** Let $R$ be an integral domain. Show that if a polynomial ring $R[x]$ is factorial so is $R$.

### 1.2 Squeezing-out one variable

Let $R$ be an integral domain which we assume to be finitely generated over a field $k$ of characteristic zero. We are interested in a question asking which conditions make $R$ a polynomial ring over $k$. One approach is to squeeze out one variable from $R$ via an algebraic group action. Let $\delta$ be a $k$-linear endomorphism of $R$. We call $\delta$ a $k$-derivation if $\delta(c) = 0$ for $\forall c \in k$ and $\delta(ab) = a\delta(b) + b\delta(a)$ for $a, b \in R$. Then we can show that

$$
\delta^n(ab) = \delta^n(a)b + n\delta^{n-1}(a)\delta(b) + \cdots + \binom{n}{i} \delta^{n-i}(a)\delta^i(b) + \cdots + a\delta^n(b).
$$

Furthermore, we say that $\delta$ is locally nilpotent if for every $a \in R$, $\delta^N(a) = 0$ for $N \gg 0$. Define a subset $\text{Ker} \, \delta$ by $\{a \in R \mid \delta(a) = 0\}$. Then $\text{Ker} \, \delta$ is a $k$-subalgebra of $R$. It is straightforward to prove the following result by making use of the above formula.

**Lemma 1.2.1** Let $t$ be a variable and define a $k$-linear mapping $\varphi : R \to R[t]$ by

$$
\varphi(a) = \sum_{n=0}^{\infty} \frac{1}{n!} \delta^n(a)t^n.
$$

Then $\varphi$ is a $k$-algebra homomorphism.

In terms of schemes, the homomorphism $\varphi$ gives rise to a morphism $\sigma : G_a \times X \to X$, where $G_a = \text{Spec} \, k[t]$ endowed with a group law and
called the *additive group scheme* and $X = \text{Spec } R$. The group law on $G_a$ is given in such a way that for a $k$-algebra $A$ and for elements $g, g' \in G_a(A) := \text{Hom}_{k-\text{alg}}(k[t], A) = A$, the product of $g, g'$ under the group law is the addition of $g, g'$ in $A$. The morphism $\sigma$ defines a group action on $X$ in the sense that for a $k$-algebra $A$ and for $g, g' \in G_a(A)$ and $x \in X(A) := \text{Hom}_{\text{Sch}}(\text{Spec } A, X) = \text{Hom}_{k-\text{alg}}(R, A)$, we have $\sigma(g, \sigma(g', x)) = \sigma(g \cdot g', x)$ and $\sigma(e, x) = x$, where $e$ is the identity of the group $G_a(A)$ that is 0 in $A$. If we identify $g \in G_a(A)$ with an element of $A$ and $x \in X(A)$ with a $k$-homomorphism $R \to A$ defined by $a \mapsto \sum_{n=0}^{\infty} \frac{1}{n!}x(\delta^n(a))g^n$.

**Lemma 1.2.2** Let $\delta$ be a non-trivial, locally nilpotent derivation on $R$ and let $R_0 = \text{Ker } \delta$. Suppose there exists an element $u$ of $R$ such that $\delta(u) = 1$. Then $R = R_0[u]$ and $u$ is considered to be a variable over $R_0$.

**Proof.** For $a \in R$, define the $\delta$-*length* $\ell(a)$ by $\ell(a) = \min\{n \mid \delta^{n+1}(a) = 0\}$. It is equal to the degree of a $t$-polynomial $\varphi(a)$. We proceed by induction on $\ell(a)$ to show that $a \in R_0[u]$. If $\ell(a) = 0$, then $a \in R_0$. Let $n = \ell(a)$. Then $\delta^n(a) \neq 0$ but $\delta^{n+1}(a) = 0$. Hence $\delta^n(a) \in R_0$. Set $a_1 = a - \delta^n(a)u^n/n!$. Then $\delta^n(a_1) = 0$. Hence $\ell(a_1) < n$. So, $a_1 \in R_0[u]$ by the hypothesis of induction. Then $a = a_1 + (\delta^n(a)/n!)u^n \in R_0[u]$. So, $R = R_0[u]$. We have to show that $u$ is a variable over $R_0$, i.e., $u$ is algebraically independent over $R_0$. Suppose to the contrary that there exists an algebraic relation

$$c_0u^n + c_1u^{n-1} + \cdots + c_n = 0,$$
where $c_0, \ldots, c_n \in R_0$ with $c_0c_n \neq 0$. We assume that $n$ is the smallest among all such relations. Apply $\delta$ to the left hand side of the equation to get

$$nc_0u^{n-1} + (n-1)c_1u^{n-2} + \cdots + c_{n-1} = 0.$$ 

Since $c_0 \neq 0$, this is a non-trivial relation. Hence this contradicts the choice of $n$. Q.E.D.

Let $\delta$ be a non-trivial locally nilpotent derivation on $R$. Then there exists an element $z$ such that $\delta(z) \neq 0$. Let $n = \ell(z)$. Then $n \geq 1$. So, set $x = \delta^{n-1}(z)$ and $a = \delta^n(z)$. Then $\delta(x) = a$ and $a \in R_0$. Denote by $R[a^{-1}]$ the ring of fractions of the form $b/a^m$, where $b \in R$ and $m \geq 0$. It is then easy to see that $\delta$ extends to a locally nilpotent derivation (denoted by the same letter $\delta$) defined by $\delta(b/a^m) = \delta(b)/a^m$. Since $\delta(x/a) = 1$, Lemma 1.2.2 implies that $R[a^{-1}] = R_0[a^{-1}][x/a] = R[a^{-1}][x]$. Thus we have shown the following result.

**Lemma 1.2.3** Let $\delta$ be a non-trivial locally nilpotent derivation on $R$. Then there exist elements $a, x$ such that $a \in R_0$, $\delta(x) = a$ and $R[a^{-1}] = R_0[a^{-1}][x]$.

An affine $k$-domain is by definition an integral domain which is finitely generated over a field $k$. Let $K$ be the quotient field of $R$. We define transcendence degree of $K$ over $K$ (denoted by $\text{tr.deg}_k K$) as follows and then define dimension of $R$ over $k$ (denoted by $\text{dim} R$) by $\text{tr.deg}_k K$. Note that $K$ is finitely generated over $k$ as a field. Namely there exists a system of elements $\{\xi_1, \ldots, \xi_n\}$ such that every element of $K$ is written as a fraction $f(\xi_1, \ldots, \xi_n)/g(\xi_1, \ldots, \xi_n)$ of two polynomials $f(x_1, \ldots, x_n), g(x_1, \ldots, x_n)$ with coefficients in $k$ and with $x_1, \ldots, x_n$ replaced by $\xi_1, \ldots, \xi_n$. We apply the fol-
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Following process of taking in and throwing away the elements $\xi_1, \ldots, \xi_n$. Consider $\xi_1$. If $\xi_1$ is algebraically independent over $k$, then take it in. Otherwise throw it away. Consider next $\xi_2$ over $k(\xi_1)$. If $\xi_2$ is algebraically independent over $k(\xi_1)$ then take it in and throw it away otherwise. Continue this process. At the $i$-th step, consider $\xi_i$ over $k(\xi_1, \ldots, \xi_{i-1})$ and take it in if it is algebraically independent over $k(\xi_1, \ldots, \xi_{i-1})$. Otherwise throw it away. After considering all elements $\xi_1, \ldots, \xi_n$, let $\eta_1, \ldots, \eta_d$ be the elements we took in. Then $K$ is algebraic over $k(\eta_1, \ldots, \eta_d)$. The above process depends on the choice of generators $\{\xi_1, \ldots, \xi_n\}$ and the order of arranging these $n$ elements. But the number $d$ of the elements we take in is independent of these choices. So, it is an invariant which is proper to $K$. This number $d$ is called *transcendence degree* of $K$ over $k$ and denoted by $\text{tr.deg}_k K$.

**Theorem 1.2.4** Let $R$ be an affine domain of $\dim R = 1$. If $R$ is given a non-trivial locally nilpotent derivation $\delta$, then $R$ is a polynomial ring in one variable over a field $K_0$ which is a finite algebraic extension of $k$.

**Proof.** By Lemma 1.2.3, $R[a^{-1}] = R_0[a^{-1}][x]$, where $R_0[a^{-1}]$ is finitely generated over $k$. Since $x$ is algebraically independent over $R_0[a^{-1}]$ and since $\dim R = 1$, it follows that every element of $R_0[a^{-1}]$ is algebraic over $k$. Hence $R_0[a^{-1}]$ is a finite algebraic extension of $k$ and coincides with the quotient field $K_0$ of $R_0$. Q.E.D.

**Remark.** It is known [31] that an affine domain $R$ of $\dim R = 1$ defined over an algebraically closed field $k$ is a polynomial ring over $k$ if and only if $R$ is factorial and $R^* = k^*$. 

Let \( R \) be an affine domain and let \( R_0 \) be a \( k \)-subalgebra of \( R \) such that \( R[a^{-1}] = R_0[a^{-1}][x] \) with \( a \in R_0 \) and with \( x \) algebraically independent over \( R_0 \). Define a locally nilpotent derivation \( \partial \) on \( R[a^{-1}] \) by setting \( \partial(x) = 1 \) and \( \partial(b) = 0 \) for \( \forall b \in R_0 \). Write \( R = k[u_1, \ldots, u_n] \) with a system of generators \( \{u_1, \ldots, u_n\} \) of \( R \). Then \( a^m \partial(u_i) \in R \) for \( m_i \geq 0 \). Let \( m = \max\{m_1, \ldots, m_n\} \) and let \( \delta = a^m \partial \). Then it is easy to see that \( \delta \) is a locally nilpotent derivation on \( R \) with \( \delta|_{R_0} = 0 \). We shall show that \( \text{Ker} \delta = R_0 \) provided \( R_0 \) satisfies the property that \( a^rb \in R_0 \) with \( r > 0 \) and \( b \in R_0 \) implies \( b \in R_0 \). In fact, we have \( R_0 \subset \text{Ker} \delta \). Suppose \( \delta(b) = 0 \) for \( b \in R \). Then \( b \in R_0[a^{-1}] \). Hence \( a^rb \in R_0 \). By the assumed property, we have \( b \in R_0 \). Hence \( \text{Ker} \delta = R_0 \).

**Theorem 1.2.5** Let \( R \) be a factorial affine domain defined over an algebraically closed field \( k \) of characteristic zero. Suppose that there is a non-trivial locally nilpotent derivation \( \delta \) on \( R \) and that \( R^* = k^* \). Then \( R \) is isomorphic to a polynomial ring in two variables over \( k \).

**Proof.** (I) Let \( R_0 = \text{Ker} \delta \). Let \( a \neq 0 \) be an element of \( R_0 \). Suppose \( a = bc \) with \( b, c \in R \). By applying \( \varphi : R \to R[t] \) associated to \( \delta \) (cf. Lemma 1.2.1), we have \( a = \varphi(b)\varphi(c) \). Since \( R \) is an integral domain, both \( \varphi(b) \) and \( \varphi(c) \) are constant polynomials in \( t \). Namely, \( b, c \in R_0 \). A \( k \)-subalgebra \( R_0 \) of \( R \) having this property that \( a = bc \) with \( b, c \in R \) implies \( b, c \in R_0 \) is said to be inert. Then it is easy to show that an inert \( k \)-subalgebra of a factorial domain is factorial as well. Since \( R_0^* = k^* \), \( R_0 \) is a polynomial ring \( k[f] \) by the remark after Theorem 1.2.4. Since \( R_0 \) is inert, it is easy to show that \( f - \alpha \) is irreducible in \( R \) for every \( \alpha \in k \).
1.2. SQUEEZING-OUT ONE VARIABLE

(II) By Lemma 1.2.3 there exist an element \( a = \prod_{i=1}^{n} (f - \alpha_i) \) of \( R_0 \) and an element \( g \in R - R_0 \) such that \( R[a^{-1}] = R_0[a^{-1}][g] \). If \( f - \alpha_i \) divides \( g - \beta \) for some \( 1 \leq i \leq n \) and \( \beta \in k \), then replace \( g \) by \( g' = (g - \beta) / (f - \alpha_i) \). If \( g' - \beta' \) is divisible by some \( f - \alpha_i \) for some \( \beta' \), we do the same replacement. Hence we may assume that \( g - \beta \) is not divisible by \( f - \alpha_i \) for \( \forall \beta \in k \) and \( \forall i \). Consider an \( R_0 \)-algebra homomorphism \( \pi : k[f, g] \to R \). For any \( \alpha \in k \), let \( \pi_\alpha : k[g] \to R_\alpha \) be the reduction of \( \pi \) modulo \( (f - \alpha) \). Since \( g - \beta \) is not divisible by \( f - \alpha \), the homomorphism \( \pi_\alpha \) is injective. Note that \( R_\alpha \) is an affine domain of dimension one, for \( f \) is algebraically independent over \( k \). Under this condition, Zariski’s main theorem (cf. [23]) says that \( \pi \) is an isomorphism. Namely, \( R \cong k[f, g] \). Q.E.D.

As a corollary of Theorem 1.2.5, we have the following result of Rentschler [46].

**Corollary 1.2.6** Let \( k \) be an algebraically closed field of characteristic zero and let \( k[x, y] \) be a polynomial ring in two variables. Let \( \delta \) be a locally nilpotent derivation of \( k[x, y] \). Then \( \delta \) is written in the form \( \delta = f(x) \partial / \partial y \) after a suitable change of variables.

**Proof.** By Theorem 1.2.5 there are two polynomials \( f, g \) such that \( \text{Ker} \delta = k[f], k[x, y] = k[f, g] \) and \( \delta = a \partial / \partial g \), where \( a \in k[f] \). Then make the following change of variables \( (x, y) \mapsto (f, g) \). Then \( \text{Ker} \delta = k[x] \) and \( \delta \) has the form \( f(x) \partial / \partial y \). Q.E.D.
1.3 Fibrations of the affine plane by polynomials

We shall consider mostly a polynomial ring $k[x, y]$ in two variables, where $k$ is an algebraically closed field of characteristic zero. We may consider $k$ to be the complex field $\mathbb{C}$ if necessary. Our objective is to look into the geometric properties of a given polynomial $f \in k[x, y]$. One way is to consider a totality of the polynomials $\{f - \alpha \mid \alpha \in k\}$.

Hereafter, we use $a, b, c, \cdots$ instead of $\alpha, \beta, \gamma, \cdots$ to denote the elements of $k$. For the said purpose, we consider the affine plane $\mathbb{A}^2$ which is the set of pairs $\{(a, b) \mid a, b \in k\}$. A curve on $\mathbb{A}^2$ defined by an equation $f = 0$ is a subset $\{(a, b) \mid f(a, b) = 0\}$, where $f \in k[x, y]$. We use the notations like $V(f), C(f)$ to denote this subset. The residue ring $k[x, y]/(f)$ is called the coordinate ring of the curve defined by $f = 0$. Sometimes, we consider a topology called the Zariski topology on $\mathbb{A}^2$ (and the induced topology on $V(f)$ as well) such that the open sets are generated by the complements $D(g)$ in $\mathbb{A}^2$ of the curves $V(g)$ with $g \in k[x, y]$. If $g = 0$, $V(f)$ is not the curve but the total space $\mathbb{A}^2$, while $V(1)$ is the empty set. Furthermore, $D(fg) = D(f) \cap D(g)$ and $D(f) \cup D(g) = \mathbb{A}^2 - V(\mathfrak{a})$, where $\mathfrak{a}$ is the ideal $(f, g)$ and $V(\mathfrak{a})$ is the set of points $P$ such that $h(P) = 0$ for all $h \in \mathfrak{a}$. For example, $D(x) \cap D(y) = \mathbb{A}^2 - \{x\text{-axis}\} \cup \{y\text{-axis}\}$ and $D(x) \cup D(y) = \mathbb{A}^2 - \{(0, 0)\}$. So, all the points $(a, b)$ are closed points.

A curve $V(f)$ is called irreducible if $f$ is irreducible. Then the residue ring $\Gamma(f) := k[x, y]/(f)$ is an integral domain, and one can consider its quotient field which we denote by $Q(f)$ or $k(C)$ if the curve is denoted by $C$. We call
1.3. FIBRATIONS OF THE AFFINE PLANE BY POLYNOMIALS

\( Q(f) \) the function field of the curve \( V(f) \).

**Example 1.3.1.** Let \( f = y + \lambda(x) \) with \( \lambda(x) \in k[x] \). Then \( \Gamma(f) \cong k[y], \Gamma(f)^* = k^* \) and \( Q(f) \cong k(y) \). Any curve whose coordinate ring is isomorphic to a polynomial ring in one variable is said to be isomorphic to the affine line \( \mathbb{A}^1 \). Hence \( V(f) \) for the above \( f \) is isomorphic to \( \mathbb{A}^1 \). Any irreducible curve \( V(f) \) is called rational if \( Q(f) \) is generated by one element over the field \( k \).

A curve isomorphic to \( \mathbb{A}^1 \) is classified by the following result of Abhyankar-Moh-Suzuki [2, 50].

**Theorem 1.3.1** Let \( f \in k[x, y] \) satisfy the condition that \( V(f) \cong \mathbb{A}^1 \). Then there exists an automorphism \( \sigma \) of \( k[x, y] \) such that \( \sigma(f) = x \). Hence the curves \( V(f - a) \) are isomorphic to \( \mathbb{A}^1 \) for all \( a \in k \).

We can consider a mapping of the sets \( \mathbb{A}^2 \to \mathbb{A}^1 \) defined by \( (a, b) \mapsto f(a, b) \). Then the inverse image (called the fiber) of a point \( c \) of \( \mathbb{A}^1 \) is the curve \( V(f - c) \). We denote this mapping by \( \Lambda(f) \). It is called a linear pencil defined by \( f \). In the case of Theorem [1.3.1] all the fibers of \( \Lambda(f) \) are isomorphic to \( \mathbb{A}^1 \).

**Example 1.3.2.** A rational curve which is not isomorphic to \( \mathbb{A}^1 \) is exemplified by \( V(f) \) with \( f = xy - 1 \). Then \( \Gamma(f) \cong k[x, x^{-1}] \). Hence \( \Gamma(f)^* = k^* \times \mathbb{Z} \), where \( \mathbb{Z} \cong \{x^n \mid m \in \mathbb{Z}\} \), and \( Q(f) \cong k(x) \). So, \( V(xy - 1) \) is not isomorphic to \( \mathbb{A}^1 \). A curve isomorphic to \( V(xy - 1) \) is called an algebraic torus of dimension 1 and denoted by \( \mathbb{A}^1_\ast \).

**Example 1.3.3.** It is known that a curve defined by \( f_a = y^2 - x^3 - a \) with
a ∈ \k^* is not rational. Let C be a curve defined by \( f = y^2 - x^3 \) and let \( \Lambda(f) : \mathbb{A}^2 \to \mathbb{A}^1 \) be a linear pencil defined by \( f = y^2 - x^3 \). Then the fiber \( V(f_a) \) with \( a \neq 0 \) is not rational, while the fiber \( V(f_0) \) is rational.

**Exercise 1.3.1.** Show that the curve \( V(f) \) with \( f = y^2 - x^3 - 1 \) is not rational. In fact, if it were rational, there would exist polynomials \( g(t), h(t), \ell(t) \in k[t] \) such that \( \gcd(g(t), h(t), \ell(t)) = 1 \) and \( \ell(t)h(t)^2 = g(t)^3 + \ell(t)^3 \). Show that this leads to a contradiction.

In general, if the total degree of \( f \) becomes bigger, then the curve \( V(f) \) becomes far from being rational. In 1.2, we observed that a non-trivial locally nilpotent derivation \( \delta \) on an affine \( k \)-domain gives a variable \( x \) such that \( R[a^{-1}] = R_0[a^{-1}][x] \), where \( R_0 = \text{Ker} \delta \) and \( a \in R_0 \). The subalgebra \( R_0 \) needs not to be an affine \( k \)-domain as seen by many counterexamples to the Fourteenth Problem of Hilbert \[41, 47, 35, 16, 9\].\footnote{Kuroda of Tōhoku University has recently shown that a counterexample exists even in case \( \dim R_0 = 3 \).} It is known by Zariski \[42\] that \( R_0 \) is finitely generated if \( \dim R_0 \leq 2 \). Suppose that \( R_0 \) is an affine domain. We denote by \( \text{Spec} \ R \) an affine variety (or an affine scheme, in general) with coordinate ring \( R \). Then the inclusion \( R_0 \hookrightarrow R \) defines a morphism \( f : X \to B \), where \( X = \text{Spec} \ R \) and \( B = \text{Spec} \ R_0 \). Let \( U \) be an open set of \( B \) such that \( a \neq 0 \) on \( U \), i.e., \( U = \text{Spec} R_0[a^{-1}] \). Then \( f^{-1}(U) \cong U \times \mathbb{A}^1 \). So, the fibers \( f^{-1}(P) \) over a point \( P \in U \) is isomorphic to \( \mathbb{A}^1 \). We define, in general, an \( \mathbb{A}^1 \)-fibration to be a morphism of algebraic varieties \( f : X \to B \) such that \( f^{-1}(U) \cong U \times \mathbb{A}^1 \) for an non-empty open set \( U \) of \( B \). Hence the morphism \( f : \text{Spec} \ R \to \text{Spec} \ R_0 \) is an \( \mathbb{A}^1 \)-fibration, which we call the quotient morphism under \( \delta \) or the associated \( G_\alpha \)-action.
Conversely, if given an $A^1$-fibration $f : X \to B$ with affine varieties $X = \text{Spec } R$ and $B = \text{Spec } R_0$, we find an open set $U = D(a) \subset U$ such that $R[a^{-1}] = R_0[a^{-1}][x]$. As explained after Theorem 1.2.4, there exists a locally nilpotent derivation $\delta$ on $R$ such that $R_0 = \text{Ker } \delta$ and $f$ coincides with the quotient morphism under $\delta$. This is the case only if $B$ is an affine variety. If $B$ is not affine, $f$ does not come from a $G_a$-action.

For an affine variety $X = \text{Spec } R$, we define the Makar-Limanov invariant $\text{ML } (X)$ as the intersection of Ker $\delta$ in the coordinate ring $R$ of $X$, where $\delta$ ranges over all locally nilpotent derivations of $R$. Hence $\text{ML } (X)$ is a subalgebra of $R$. If $\dim X = 1$ and there is a non-trivial locally nilpotent derivation $\delta$, then $\text{Ker } \delta = k$. Hence $\text{ML } (X) = k$. When $\dim X = 2$, we have the following result.

**Lemma 1.3.2** Let $X = \text{Spec } R$ be an affine surface. Then one of the following cases takes place.

1. $\text{ML } (X) = R$ and there are no locally nilpotent derivations on $R$.
2. $\text{ML } (X) = \text{Ker } \delta$ and there is a nontrivial locally nilpotent derivation $\delta$. If $\delta'$ is another locally nilpotent derivation on $R$, then $\delta'$ is conjugate to $\delta$, which signifies, by definition, that $\text{Ker } \delta' = \text{Ker } \delta$ and $c\delta = d\delta'$ for some $c, d \in \text{Ker } \delta$.
3. $\text{ML } (X) = k$ and there are two algebraically independent locally nilpotent derivations $\delta$ and $\delta'$, which signifies, by definition, that there are elements $\xi \in \text{Ker } \delta$ (resp. $\xi' \in \text{Ker } \delta'$) such that $\xi$ (resp. $\xi'$) is algebraically independent over $\text{Ker } \delta'$ (resp. $\text{Ker } \delta$).
CHAPTER 1. CHARACTERIZATION OF POLYNOMIAL RINGS

Proof. (1) If there are no locally nilpotent derivations on \( R \), then \( ML(X) = R \).

(2) If \( \delta \) is a nontrivial locally nilpotent derivation, then \( R[a^{-1}] = R_0[a^{-1}][x] \) for \( a \in R_0 = \text{Ker} \delta \) and \( x \in R \). Let \( \delta' \) be another locally nilpotent derivation on \( R \). If \( \delta' \) is conjugate to \( \delta \) (\( \delta \sim \delta' \) by notation), then it is obvious that \( \text{Ker} \delta' = R_0 \). Suppose \( \dim ML(X) = 1 \). Then \( R_0 \supseteq ML(X) \) and \( R_0 \) is algebraic over \( ML(X) \). Since \( \text{Ker} \delta' \supseteq ML(X) \) and \( R_0 \) is algebraic over \( ML(X) \), it follows that \( \text{Ker} \delta' \supseteq R_0 \). This implies that \( \text{Ker} \delta' = R_0 = ML(X) \).

Considering the derivation \( \delta' \), we can write \( R[b^{-1}] = R_0[b^{-1}][y] \) with \( b \in R_0 \) and \( y \in R \). In particular, \( R \otimes_{R_0} K_0 = K_0[x] = K_0[y] \), where \( K_0 = Q(R_0) \). So, \( y = \alpha x + \beta \) with \( \alpha, \beta \in K_0 \). Hence we may assume \( y = x \) by replacing \( b \) with a suitable element of \( R_0 \). Since \( \delta = d \partial / \partial x \) and \( \delta' = c \partial / \partial x \) with \( c, d \in R_0 \), we have \( c \delta = d \delta' \). Namely \( \delta \sim \delta' \).

(3) Suppose \( \dim ML(X) = 0 \). Then there exist locally nilpotent derivations \( \delta, \delta' \) of \( R \) such that \( \delta' \) is nontrivial on \( \text{Ker} \delta \). Hence there exists an element \( \xi \in \text{Ker} \delta \) such that \( \delta'(\xi) \neq 0 \). If one writes \( R[b^{-1}] = \text{Ker} \delta'[b^{-1}][y] \) with \( b \in \text{Ker} \delta' \) and \( y \in R \), the element \( \xi \) is a polynomial in \( y \) of positive degree with coefficients in \( \text{Ker} \delta'[b^{-1}] \). Since \( y \) is algebraically independent over \( \text{Ker} \delta' \), so is the element \( \xi \). Since \( \delta \) is nontrivial on \( \text{Ker} \delta' \), we find similarly an element \( \xi' \in \text{Ker} \delta' \) which is algebraically independent over \( \text{Ker} \delta \). This implies that \( \dim(\text{Ker} \delta \cap \text{Ker} \delta') = 0 \) and \( \text{Ker} \delta \cap \text{Ker} \delta' \) is algebraic over \( k \). So, \( \text{Ker} \delta \cap \text{Ker} \delta' = ML(X) = k \). Q.E.D.
1.4 Plane-like affine surfaces

We say that a morphism \( f : X \to B \) of algebraic varieties is an \( G \)-fibration if almost all fibers \( f^{-1}(Q) \) with \( Q \in B \) are isomorphic to one and the same algebraic variety \( G \). We mostly consider the case where \( \dim B = 1 \) and \( G \) is isomorphic to \( \mathbb{A}^1, \mathbb{A}^1_* \) and \( \mathbb{P}^1 \). Any fiber \( F \) which is not isomorphic to \( G \) is called a singular fiber of \( f \).

By Theorem 1.2.5, the affine plane \( \mathbb{A}^2 \) is an affine surface such that its coordinate ring \( R \) is factorial, \( R^* = k^* \) and there exists an \( \mathbb{A}^1 \)-fibration \( \rho : X \to \mathbb{A}^1 \). In the following, we consider an affine surface \( X = \text{Spec} R \) with an \( \mathbb{A}^1 \)-fibration \( \rho : X \to \mathbb{C} := \mathbb{A}^1 \) such that \( R^* = k^* \). In order to look into the properties of \( X \), we need to consider a smooth projective surface \( V \) which contains \( X \) as an open set and has a \( \mathbb{P}^1 \)-fibration \( p : V \to \overline{C} \cong \mathbb{P}^1 \) extending the \( \mathbb{A}^1 \)-fibration \( \rho \). This implies that \( p \mid_X = \rho \). Such a surface \( V \) is called a smooth compactification or a smooth completion of \( X \).

The existence of \( V \) and \( p \) is shown as follows. Since \( X \) is a closed subvariety of a certain affine space \( \mathbb{A}^N \), we take the closure \( \overline{X} \) of \( X \) in the projective space \( \mathbb{P}^N \) to which \( \mathbb{A}^N \) is naturally embedded as the complement of a hyperplane. Though \( \overline{X} \) might have singular points, we can resolve singularities of \( \overline{X} \) which are centered on \( \overline{X} \setminus X \). Thus there exists a smooth projective surface \( V' \) containing \( X \) as an open set. Then \( \rho : X \to C \) extends to a rational mapping \( p' : V' \to \overline{C} \), where \( \overline{C} \) is a smooth completion of \( C \), which means that \( \overline{C} \) is a smooth projective curve containing \( C \) as an open set. In our case \( \overline{C} \) is isomorphic to \( \mathbb{P}^1 \). If there exists points of \( V' \) where \( p' \) is not defined, we can eliminate indeterminacy by applying a sequence of blowing-ups \( \sigma : V \to V' \) with centers outside \( X \). Namely, the rational mapping \( p := p' \cdot \sigma : V \to \overline{C} \) is
a morphism extending \( \rho \). Note that there a general fiber of \( \rho \) is isomorphic to \( \mathbb{A}^1 \). Hence a general fiber of \( \rho \) is a smooth projective curve containing \( \mathbb{A}^1 \) as an open set. Hence the general fiber of \( p \) is isomorphic to \( \mathbb{P}^1 \). Namely \( p \) is a \( \mathbb{P}^1 \)-fibration.

Here is another reasoning of finding an open embedding \((X, \rho) \hookrightarrow (V, p)\) with \( \rho = p|_X \). Note that \( \rho^{-1}(U) \cong U \times \mathbb{A}^1 \) for an open set \( U \) of \( C \), where \( \rho|_U \) is identified with the first projection of \( U \times \mathbb{A}^1 \) onto \( U \). Since \( \mathbb{A}^1 \) is embedded into \( \mathbb{P}^1 \) as the complement of the point at infinity, \( U \times \mathbb{A}^1 \) is embedded as an open set into \( V_0 := \overline{C} \times \mathbb{P}^1 \). Hence there exists a birational mapping \( \phi_0 : X \to V_0 \) such that \( p_0 \cdot \phi_0 = \rho \), where \( p_0 \) denotes the first projection \( \overline{C} \times \mathbb{P}^1 \to \overline{C} \). If there are points on \( V_0 \) (resp. \( X \)) corresponding to irreducible curves on \( X \) (resp. \( V_0 \)) under the above birational mapping, we blow up the surfaces \( X \) and \( V_0 \) to obtain \( X_1 \) and \( V_1 \) so that there are no points on \( X_1 \) (resp. \( V_1 \)) corresponding to irreducible curves on \( V_1 \) (resp. \( X_1 \)). Then \( X_1 \) is identified with an open set of \( V_1 \) by virtue of Zariski’s Main Theorem. Let \( \sigma : X_1 \to X \) and \( \tau : V_1 \to V_0 \) be the sequences of blowing-ups. Since the exceptional curves in \( X_1 \) arising from \( \sigma \) is contained in \( V_1 \) as well, we may contract these exceptional curves in \( V_1 \) to obtain a smooth projective surface \( V \). Then \( X \) is identified with an open set of \( V \). It is then clear that there is a \( \mathbb{P}^1 \)-fibration \( p : V \to \overline{C} \) such that \( p|_X = \rho \).

We shall recall some of the basic properties of a \( \mathbb{P}^1 \)-fibration on a smooth projective surface.

**Lemma 1.4.1** Let \( p : V \to B \) be a \( \mathbb{P}^1 \)-fibration on a smooth projective surface \( V \) with base a smooth curve \( B \). Let \( F \) be a singular fiber of \( p \). Then the following assertions hold.
(1) Every irreducible component of $F$ is isomorphic to $\mathbb{P}^1$. If two irreducible components meet each other, they meet transversally. Furthermore, no three components meet in one point. Hence $F$ is a tree of smooth rational curves.

(2) $F$ contains a $(-1)$-curve, called a $(-1)$-component, and any $(-1)$-curve in $F$ meets at most two other components of $F$. If a $(-1)$-curve $E$ occurs with multiplicity 1 in the scheme-theoretic fiber $F$ then $F$ contains another $(-1)$-curve.

(3) By successively contracting $(-1)$-curves in $F$ and its images we can reduce $F$ to a regular fiber.

For the proof, see Gizatullin [18].

A smooth projective surface $V$ with a $\mathbb{P}^1$-fibration is called a minimal rational ruled surface or simply a Hirzebruch surface if every fiber is irreducible and the base curve is rational. Minimal rational ruled surfaces are classified in the following result.

**Lemma 1.4.2** The following assertions hold true.

(1) Every minimal rational ruled surface is isomorphic to $\Sigma_n := \text{Proj}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$, where $n \geq 0$.

(2) There is a section $M$ (called minimal section) on $\Sigma_n$ such that $(M^2) = -n$. If $n > 0$ then $M$ is unique. For other sections $S$, $(S^2) \geq n$. Hence $S \sim M + a\ell$ with $a \geq n$.

A smooth projective surface $V$ with a $\mathbb{P}^1$-fibration is obtained from a Hirzebruch surface $\Sigma_n$ by applying several blowing-ups. A singular fiber,
which is then a reducible fiber of the $\mathbb{P}^1$-fibration, is described in Lemma 1.4.1. Write such a singular fiber as $F = \sum_{i=1}^r a_i F_i$, where the $F_i$ are irreducible components and the $a_i$ are their coefficients (hence $a_i > 0$). As explained above, any $\mathbb{P}^1$-fibration has a cross-section, say $S$, which comes from the (a) minimal section of a Hirzebruch surface and for which we have $(F \cdot S) = 1$. So, one of the $a_i$ must be equal to 1. Hence any $\mathbb{P}^1$-fibration has no multiple fibers.

As a consequence of the above observations, we have the following result which describes the singular fibers of an $\mathbb{A}^1$-fibration on a smooth affine surface. Let $F = \sum_{i=1}^r a_i F_i$ denote anew a fiber of such $\mathbb{A}^1$-fibration. Then $F$ is singular if either $r \geq 2$, or $r = 1$ and $a_1 \geq 2$. If $F$ is singular, we call $m := \gcd(a_1, \ldots, a_r)$ the multiplicity of $F$. We have the following result.

**Lemma 1.4.3** Let $X$ be a smooth affine surface with an $\mathbb{A}^1$-fibration $\rho : X \to C$, where $C \cong \mathbb{A}^1$ or $C \cong \mathbb{P}^1$. Then the following assertions hold.

1. Any fiber of $\rho$ is a disjoint union of the affine lines with suitable multiplicities.

2. If $C \cong \mathbb{A}^1$, then $\text{rank Pic}(X) = \sum_{P \in C} (r_P - 1)$, where $r_P$ is the number of irreducible components of the fiber $\rho^{-1}(P)$. If $C \cong \mathbb{P}^1$, then $\text{rank Pic}(X) = 1 + \sum_{P \in C} (r_P - 1)$.

3. Suppose that $C \cong \mathbb{A}^1$ and $r_P = 1$ for every $P \in C$. Then $\text{Pic}(X) \cong \prod_{P \in C} \mathbb{Z}/m_P \mathbb{Z}$, where $m_P$ is the multiplicity of the fiber $\rho^{-1}(P)$.

4. With the same assumptions as in the assertion (3), $X$ is a $\mathbb{Q}$-homology plane. Namely, $X$ is a smooth affine surface with $H_1(X; \mathbb{Q}) = (0)$ for
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every $i > 0$.

Proof. (1) Embed $X$ into a smooth projective surface $V$ with a $\mathbb{P}^1$-fibration $p : V \to \overline{C}$. Let $D := V - X$, which we consider as a reduced effective divisor. Since $X$ is affine, there is a very ample divisor whose support is equal to $D$. In particular, $D$ is connected (cf. [23]). Since $D$ contains a (unique) cross-section, say $S$, we know that $D$ consists of $S$ and some connected parts of the fibers of $\rho$ which meet $S$. By Lemma 1.4.1, the fibers of $\rho$ are trees of rational curves. Since $X$ contains no complete curves, it follows that any fiber of $\rho$ with the part in $D$ removed off consists of irreducible components which are located at the tips of the tree (the ends of tree branches) and each of which is deprived of one point where it meets the part in $D$. This means that any fiber of $\rho$ is a disjoint union of the affine lines.

(2) Note that rank Pic $(\Sigma_n) = 2$ and each blowing-up increases the rank of the Picard group by one. Then rank Pic $(V) = 2 + \sum_{P \in \overline{C}} (s_P - 1)$, where $s_P$ is the number of irreducible components of the fiber $p^{-1}(P)$. Since $D = S + \sum_{P \in \overline{C}} (D \cap p^{-1}(P))$ and $D \cap p^{-1}(P)$ consists of $(s_P - r_P)$ components, we obtain the asserted result.

(3) If $C \cong \mathbb{A}^1$, one point of $\overline{C}$, say $P_\infty$, is not included in $C$. Let $F_\infty$ be the fiber of $p$ over $P_\infty$. Then every fiber $p^{-1}(P)$ is linearly equivalent to $F_\infty$. Hence if $m_P F_P$ is the component of $p^{-1}(P)$ left in $X$ with multiplicity $m_P$, then we have a relation $m_P F_P \sim 0$. In fact, there are no other relations on the $F_P$. Hence we have the asserted result.

(4) We just indicate the key ingredients of the proof without going too much to details. First of all, $V$ is a smooth rational surface, whence $H^1(V, \mathcal{O}_V) = H^2(V, \mathcal{O}_V) = (0)$. Then $H^1(V; \mathbb{Q}) = (0)$ because the first Betti number $b_1$
equals to $2h^{0,1}$, where $h^{0,1} = \dim H^1(V, \mathcal{O}_V) = 0$. From the well-known long exact sequence

$$\cdots \to H^1(V, \mathcal{O}_V) \to H^1(V, \mathcal{O}_V^*) \to H^2(V; \mathbb{Z}) \to H^2(V, \mathcal{O}_V) \to \cdots,$$

it follows that $\text{Pic} (V) \otimes \mathbb{Q} \cong H^2(V; \mathbb{Q})$, where $\text{Pic}(V) \cong H^1(V, \mathcal{O}_V)^*$. Since the irreducible components of $D$ are independent in $H^2(V; \mathbb{Q})$ and $\text{Pic}(X) \otimes \mathbb{Q} = (0)$, it follows that $H^2(V; \mathbb{Q}) \cong H^2(D; \mathbb{Q})$. Now consider the cohomology exact sequence of the pair $(V, D)$ with $\mathbb{Q}$-coefficients

$$\cdots \to H^1(V) \to H^1(D) \to H^2(V, D) \to H^2(V) \to H^2(D) \to H^3(V, D) \to H^3(V),$$

whence we obtain $H^1(D; \mathbb{Q}) \cong H^2(V, D; \mathbb{Q}) \cong H_2(X; \mathbb{Q})$ and $H_1(X; \mathbb{Q}) \cong H^3(V, D; \mathbb{Q}) = (0)$. Since $D$ is a tree of smooth rational curves, we know that $H^1(D; \mathbb{Q}) = (0)$. Hence $H_2(X; \mathbb{Q}) = (0)$. Since $\dim X = 2$ it is clear that $H_i(X; \mathbb{Q}) = (0)$ if $i > 2$.

Q.E.D.

**Corollary 1.4.4** Let $X$ be as in Lemma 1.4.3. Assume that $C \cong \mathbb{A}^1$ and that every fiber of $\rho$ is irreducible. Then the Picard group of $X$ is a torsion group. In particular, if there is only one multiple fiber $mF$, then $\text{Pic} (X)$ is a cyclic group of order $m$.

There are many kinds of affine surfaces satisfying the same conditions as in the above corollary. We define below just one class of such surfaces which include the surfaces $\mathbb{P}^2 - C$, where $C$ is a curve $X_0X_1^{d-1} = X_2^d$ with $d \geq 2$ (cf. Lemma 1.4.6 below). The remaining part of this subsection will be published elsewhere in our forthcoming paper [30].
**Definition 1.4.5** A smooth affine surface $X$ is an affine pseudo-plane if $X$ satisfies the following conditions:

1. $X$ has an $\mathbb{A}^1$-fibration $\rho : X \to A$, where $A \cong \mathbb{A}^1$.

2. The $\mathbb{A}^1$-fibration $\rho$ has a unique multiple fiber with multiplicity $d \geq 2$.

We say that $X$ has type $(d, n, r)$ if $X$ further satisfies the next condition:

3. $X$ has a smooth completion $(V, D)$ such that the dual graph of $D$ is as given in Figure 1 below, where $n \geq 1$ and $r \geq 2$. Furthermore, $\overline{F}$ is the closure of $F$ in $V$ and $S'$ is the unique cross-section contained in $D$.

If the curves $\overline{F} = E_{d+r}, E_{d+r-1}, \ldots, E_{d+1}, E_d, E_{d-1}, \ldots, E_2, E_1$ are contracted in this order, the resulting surface is a Hirzebruch surface $\Sigma_n$ with the minimal cross-section $S'$.

Choose a point $P$ on the fiber $\ell'_\infty$. Blow up the point $P$ to obtain a $(-1)$ curve $E$. Then the proper transform $L$ of $\ell'_\infty$ is a $(-1)$ curve. Contract $L$ to obtain the same figure as before with $\ell'_\infty$ replaced by the image of $E$ and with $(S'^2)$ is $-n + 1$ if $P \neq S' \cap \ell'_\infty$ and $-n - 1$ if $P = S' \cap \ell'_\infty$. This operation is called the *elementary transformation* with center $P$. After several elementary
transformations, we may and shall assume unless otherwise specified that \( n = -(S'^2) = 1 \). So, we say that an affine pseudo-plane has type \((d, r)\) instead of type \((d, 1, r)\).

**Lemma 1.4.6** Let \( X \) be an affine pseudo-plane of type \((d, r)\). Then \( X \) is isomorphic to the complement of \( M_0 \cup C_d \) if \( r < d \) and \( M_1 \cup C_d \) if \( r \geq d \) in the Hirzebruch surface \( \Sigma_n \) with \( n = |r - d| \), where \( M_0 \) is the minimal section and where \( C_d \) and \( M_1 \) are specified as follows. In the case \( r < d \), \( C_d \) is an irreducible member of the linear system \(|M_0 + d\ell_0|\) which meets \( M_0 \) in the point \( M_0 \cap \ell_0 \) with multiplicity \( r \). In the case \( r \geq d \), \( M_1 \) is a section of \( \Sigma_n \) with \((M_1^2) = n\), and \( C_d \) is an irreducible member of the linear system \(|M_1 + d\ell_0|\) which meets \( M_1 \) in the point \( M_1 \cap \ell_0 \) with multiplicity \( r \). In both cases, \( \ell_0 \cap X = \overline{F} \cap X \) for the fiber \( \ell_0 \) of \( \Sigma_n \).

**Proof.** Contract \( S', \ell'_0, E_2, \ldots, E_d, E_{d+1}, \ldots, E_{d+r-1} \) in this order. Then the resulting surface is the Hirzebruch surface \( \Sigma_n \) with \( n = |r - d| \) and the image of \( \ell'_\infty \) provides \( C_d \). The image of \( E_1 \) provides \( M_0 \) or \( M_1 \) according as \( r - d < 0 \) or \( r - d \geq 0 \), while the image of \( \overline{F} \) is the fiber \( \ell_0 \). Q.E.D.

Lemma 1.4.6 gives a construction of affine pseudo-planes from the Hirzebruch surfaces. Whenever we are conscious of this construction from the Hirzebruch surfaces, we denote the affine pseudo-plane \( X \) in Lemma 1.4.6 by \( X(d, r) \).

**Lemma 1.4.7** Let \( X \) be an affine pseudo-plane. Then \( X \) is isomorphic to \( \mathbb{P}^2 - C \), where \( C \) is a curve on \( \mathbb{P}^2 \) defined by \( X_0X_1^{d-1} = X_2^d \) with \( d \geq 2 \), if and only if \( X \) has type \((d, d - 1)\).
Proof. It is straightforward to see that $\mathbb{P}^2 - C$ is an affine pseudo-plane of type $(d, d - 1)$. Conversely, if $X$ is an affine pseudo-plane of type $(d, d - 1)$, then contract in the Figure 1 of Definition 1.4.5 the curves $S', \ell'_0, E_2, \ldots, E_d, E_{d+1}, \ldots, E_{2d-2}, E_1$ in this order to obtain the plane curve $X_0X_1^{d-1} = X_2^d$ which is the image of $\ell'_\infty$. Q.E.D.

Let $X$ be a smooth (complex) affine surface. A smooth projective surface $V$ is a normal compactification of $X$ if $V \setminus X$ consists of smooth irreducible curves such that they meet each other transversally and that there are no three or more curves meeting in one point. Further, it is a minimal normal compactification if the contraction of any possible $(-1)$ curve in $V \setminus X$ breaks the above normality condition. We consider $V \setminus X$ the boundary divisor and denote it simply by $D := V - X$. So, $D$ is a reduced effective divisor.

Suppose that the divisor $D$ is a tree of rational curves. Extracting the definition from [38] and [45], we shall define a group associated to a weighted graph. Let $\Gamma$ be a weighted connected tree which consists of vertices with weights and edges, each of edges connecting exactly two vertices. The group $\pi_1(\Gamma)$ associated to $\Gamma$ is generated by as many generators as the vertices $\{v_n\}$ of $\Gamma$ which are subject to the following relations. First, order the vertices in some fixed manner as $v_1, v_2, \ldots$. For each vertex $v$ with weight $d$, let $v_{i_1}, v_{i_2}, \ldots, v_{i_r}$ be all the vertices connected to $v$ such that $i_1 < i_2, \ldots < i_r$ (then $v$ occurs among these vertices, say $v = v_{i_a}$). For any vertex $v$, we consider a relation $v_{i_1} \cdots v_{i_{a-1}} \cdot v_{i_a}^d \cdots v_{i_r} = e$. Next, if $v, w$ are any two vertices which are connected by an edge, then we impose the relation $v \cdot w = w \cdot v$ (i.e., the generators corresponding to $v, w$ commute). The quotient of the free group generated by the vertices of $\Gamma$ modulo these relations is the fun-
damental group at infinity $\pi_{1,\infty}(X)$. It is known that $\pi_{1,\infty}(X)$ is independent of the choice of a minimal normal compactification. This group modulo the commutator subgroup is the first homology group at infinity $H_{1,\infty}(X)$.

By definition, a $\mathbb{Q}$-homology plane $Y$ is a smooth affine surface such that $H_i(Y;\mathbb{Q}) = (0)$ for $i > 0$. It is well-known that the divisor at infinity of a $\mathbb{Q}$-homology plane $Y$ in a normal compactification is a (connected) tree of rational curves (see [36]). Hence we can use Mumford-Ramanujam’s presentation of $\pi_{1,\infty}(Y)$. An appendix of a weighted graph $\Gamma$ is a pair of vertices $\{u, v\}$ such that $u$ is a vertex of $\Gamma$ with weight 0 which is connected only to the vertex $v$ and $v$ is a vertex linked to $u$ and at most one other vertex of $\Gamma$. Removing an appendix $\{u, v\}$ from $\Gamma$ is an operation of removing vertices $u, v$, the edge connecting $u, v$ and the edge which connects $v$ to a third vertex, if it exists.

**Lemma 1.4.8** Let $\Gamma$ be a weighted connected tree. Then the following assertions hold.

1. Let $\{u, v\}$ be an appendix of $\Gamma$ such that the weight of $v$ is $d$ and let $\Gamma'$ be obtained from $\Gamma$ by removing the appendix $\{u, v\}$. Then $\pi_1(\Gamma) = \pi_1(\Gamma')$.

2. Suppose that $\Gamma$ is a linear chain. Then $\pi_1(\Gamma)$ is a cyclic group. If further the determinant of the intersection form on the free abelian group on the vertices of $\Gamma$ is non-zero (in particular, if this form is negative definite) then this group is finite cyclic. In the remaining case, $\pi_1(\Gamma)$ is an infinite cyclic group.

**Proof.** (1) Let $\{u, v\}$ be an appendix of $\Gamma$ with the weight of $v$ being $d$. If $\Gamma$ consists only of vertices $u, v$ and an edge connecting them, it is easy to see
that $\pi_1(\Gamma) = e$. Otherwise, let $w$ be a third vertex connected to $v$. Let the ordering on the vertices of $\Gamma$ be $u, v, w, \ldots$. Since $u^0v = e$ and $uv^dw = e$, we have $v = e$ and $u = w^{-1}$. Hence it follows that $\pi_1(\Gamma) = \pi_1(\Gamma')$.

(2) Let the ordering on $\Gamma$ be $v_1, v_2, \ldots$. From Mumford-Ramanujam’s presentation we see easily that the generator corresponding to $v_1$ generates $\pi_1(\Gamma)$. If the intersection form is negative definite, then the order of this group is the absolute value of the determinant of the intersection form, hence finite. Suppose that the form has at least one positive eigenvalue. It is proved in [21, Lemma 5], that there is a connected linear weighted tree $\Gamma_1$ with the following properties:

(a) $\Gamma_1$ has a connected subtree $T$ with vertices $w_1, w_2, \ldots, w_{2n}$ such that weights of $w_1, w_2, \ldots, w_{2n}$ are all 0.

(b) The subgraph of $\Gamma_1$ obtained by removing $T$, say $\Gamma_1 - T$, is connected (it may be empty) and either consists of a single vertex of weight 0 or has negative definite intersection form.

(c) $\pi_1(\Gamma) \cong \pi_1(\Gamma_1) \cong \pi_1(\Gamma_1 - T)$.

Now if $\Gamma_1 = T$ then by the argument in part (1) we see that $\pi_1(\Gamma) = (e)$. If $\Gamma_1 - T$ is non-empty, then we see by Mumford-Ramanujam’s presentation above that $\pi_1(\Gamma_1 - T)$ is a cyclic group. It is finite if the intersection form on $\Gamma_1 - T$ is negative definite. Q.E.D.

**Lemma 1.4.9** Let $X$ be an affine pseudo-plane of type $(d, r)$. Then $\pi_{1, \infty}(X)$ is a group generated by $x, y$ with relations $x^r = y^d = (xy)^d$. If $r = 1$ then $\pi_{1, \infty}(X)$ is a finite cyclic group of order $d^2$. 
Proof. Let $e_i (1 \leq i \leq d + r - 1), \ell'_0, s', \ell'_\infty$ be the generators corresponding to $E_i (1 \leq i \leq d + r - 1), \ell'_0, S', \ell'_\infty$, respectively. Let $x = e_{d+r-1}, y = \ell'_0$ and $z = e_1$. Then we have $e_i = x^{d+i} - i$ for $d \leq i \leq d + r - 1, s' = 1, \ell'_\infty = y^{-1}, e_i = y^i$ for $2 \leq i \leq d$, and $e_d = z^d$. Furthermore, $x^r = y^d = z^d$ and $zy^{d-1}x^{r-1} = z^{2d}$. Thence we obtain $z = xy$. So, $\pi_{1,\infty}(X)$ has generators and relations as described in the statement. If $r = 1$ then it is easy to see that $\pi_{1,\infty}(X)$ is an abelian group generated by $y$. So, $\pi_{1,\infty}(X) = H_{1,\infty}(X)$. A direct computation shows that $H_{1,\infty}(X)$ is a finite group for $r$ general, and isomorphic to $\mathbb{Z}/d^{2}\mathbb{Z}$ provided $r = 1$. Q.E.D.

Our next result is the following:

**Theorem 1.4.10** Let $X$ be an affine pseudo-plane of type $(d, r)$ with $r \geq 2$. Then $\rho$ is a unique $\mathbb{A}^1$-fibration on $X$.

**Proof.** Suppose that there exists another $\mathbb{A}^1$-fibration $\sigma : X \to B$ which is different from the fixed $\mathbb{A}^1$-fibration $\rho : X \to A$. Then $B \cong \mathbb{A}^1$ because $\text{Pic } (X)_\mathbb{Q} = (0)$ and every fiber of $\sigma$ is isomorphic to $\mathbb{A}^1$ if taken with the reduced structure. Let $M$ be a linear pencil on $V$ spanned by the closures of general fibers of $\sigma$, where the notations $V, D$, etc. are the same as in Definition 1.4.5. Then a general member of $M$ meets the curve $\ell'_\infty$, for otherwise the $\mathbb{A}^1$-fibrations $\rho$ and $\sigma$ coincide with each other. Suppose that $M$ has no base points. Then the curve $\ell'_\infty$ is a cross-section of $M$ and $S' + \ell'_0 + E_1 + \cdots + E_{d+r-1}$ supports a reducible fiber of $M$. Then $r = d = 1$. Since $d \geq 2$ by the hypothesis, this case does not take place. Hence $M$ has a base point, say $P$, on $\ell'_\infty$. Let $Q := \ell'_\infty \cap S'$. We consider two cases separately.

**Case** $P \neq Q$. Then $\ell'_\infty + S' + \ell'_0 + E_1 + E_2 + \cdots + E_{d+r-1}$ will support a
reducible member \( G_0 \) of the pencil \( M \). Let \( s = (F \cdot G) \), where \( G \) is a general member of \( M \). By comparing the intersection numbers of \( G \) with two fibers of \( \rho, \ell'_\infty \) and the one containing \( dF \), it follows that \((\ell'_\infty \cdot G) = ds\). Let \( \mu \) be the multiplicity of \( G \) at \( P \), where \( P \) is a one-place point of \( G \). We have \( ds \geq \mu \). Consider first the case \( n = 1 \). The contraction of \( S', \ell'_0, E_2, \ldots, E_{d-1} \) makes \( E_d \) a \((-1)\) curve meeting three components \( \ell'_\infty, E_1, E_{d+1} \), and this is impossible. So, suppose \( n \geq 2 \). We need an argument for this case too, which we make use of in the case \( P = Q \). The elimination of the base points of \( M \) will be achieved by blowing up the point \( P \) and its infinitely near points. After the elimination of the base points of \( M \), the proper transform \( \tilde{M} \) gives rise to a \( \mathbb{P}^1 \)-fibration, and the proper transform of \( \ell'_\infty \) is a unique \((-1)\) component. If \( gs > \mu \) then the point \( P \) and its infinitely near point of the first order lying on \( \ell'_\infty \) are blown up. Hence the proper transform of \( \ell'_\infty \) is not a \((-1)\) curve. This implies that \( ds = \mu \). Let \( E \) be the exceptional curve arising from the blowing-up of \( P \) and let \( M' \) be the proper transform of \( M \). Then \( E \) is contained in the member \( G'_0 \) of \( M' \) corresponding to \( G_0 \) of \( M \). In fact, we have otherwise \( ds = 1 \), which is impossible because \( d \geq 2 \). Now contract \( \ell'_\infty \) and take the image of \( E \) instead of \( \ell'_\infty \). Then we have the same dual graph as Figure 1 with \((S'^2) = -(n - 1)\). By repeating this argument, we reach to a contradiction.

**Case** \( P = Q \). As above, let \( G_0 \) be a reducible member of \( M \) containing \( S' + \ell'_0 + E_1 + E_2 + \cdots + E_{d+r-1} \). If \( \ell'_\infty \) is not contained in \( G_0 \), the elimination of the base points of \( M \), which is achieved by blowing up the point \( P = Q \) and its infinitely near points, yields a \( \mathbb{P}^1 \)-fibration in which the fiber corresponding to \( G_0 \) is a reducible fiber not containing any \((-1)\) curve. This is a contradiction.
Hence $\ell'_\infty$ is contained in $G$. So, $G$ is supported by $S' + \ell'_0 + E_1 + E_2 + \cdots + E_{d+r-1} + \ell'_\infty$. Now apply the elementary transformation with center $P$. Then we obtain the same dual graph as Figure 1, where $(S'^2) = -(n+1)$ and $\ell'_\infty$ is replaced by the image of $E$. After repeating the elementary transformations several times, we are reduced to the case where $P \neq Q$. So, we reach to a contradiction in the present case as well. Q.E.D.

On the other hand, an affine pseudo-plane of type $(d,1)$ has two algebraically independent $G_a$-actions. It follows from a more general result in [20] when we note that the boundary divisor $D$ is then a linear chain for the normal compactification in Definition 1.4.5 and that $\pi_{1,\infty}(X)$ is a finite cyclic group by Lemma 1.4.9.

**Theorem 1.4.11** Let $X$ be a smooth affine surface. Then $\text{ML}(X)$ is trivial if and only if $X$ has a minimal normal compactification $V$ such that the dual graph of $D := V - X$ is a linear chain of rational curves and $\pi_{1,\infty}(X)$ is a finite group.

One of the possible applications of Theorem 1.4.10 is about determining the automorphism groups of affine pseudo-planes of type $(d, n, r)$ with $r \geq 2$.

**Theorem 1.4.12** Let $X$ be an affine pseudo-plane of type $(d, r)$ with $r \geq 2$. Let $\alpha$ be an automorphism of $X$. Then the following assertions hold true.

1. $\alpha$ preserves the $\mathbb{A}^1$-fibration $\rho : X \to A$. Namely, there exists an automorphism $\beta$ of $A$ such that $\rho \cdot \alpha = \beta \cdot \rho$, where $\beta$ fixes the point $P_0$ of $A$ with $\rho^*(P)$ the multiple fiber $dF_0$ of $\rho$. 
By assigning $\beta$ to $\alpha$, we have a group homomorphism $\pi : \text{Aut} \,(X) \to \text{Aut} \,(A - \{P_0\}) \cong \mathbb{G}_m$, which is surjective if $(d,r) = (d,d - 1)$.

Let $G$ be the kernel of $\pi$. Then $G$ contains the additive group $\mathbb{G}_a$ as a subgroup whose action gives rise to the $\mathbb{A}^1$-fibration $\rho$. Furthermore, $G$ does not contain a torus as a subgroup.

**Proof.** The assertion (1) and the first part of the assertion (3) follow readily from Theorem 1.4.10. The assertion (2) follows from Lemma 1.4.13 below. The second part of the assertion (3) follows from Lemma 1.4.14 below.

**Lemma 1.4.13** Let $C$ be a rational curve on $\mathbb{P}^2$ defined by $X_0X_1^{d-1} = X_2^d$ with $d > 2$ and let $X = \mathbb{P}^2 - C$. Let $x = X_0/X_1, y = X_2/X_1$ and $f = x - y^d$. Then the following assertions hold true.

1. Let $\Lambda$ be a linear pencil on $\mathbb{P}^2$ generated by the curves $C$ and $d\ell_1$, where $\ell_1$ is defined by $X_1 = 0$. Then $\Lambda$ defines an $\mathbb{A}^1$-fibration $\rho : X \to A$, where $A \cong \mathbb{A}^1$. This is a unique $\mathbb{A}^1$-fibration on $X$. Hence there is a canonical projection $\pi : \text{Aut} \,(X) \to \text{Aut} \,(\mathbb{A}^1 - \{0\}) \cong \mathbb{G}_m$, where 0 corresponds to the point of $A$ over which lies the unique multiple fiber $d(\ell_1 \cap X)$.

2. Define a $\mathbb{G}_m$-action $\tau : \mathbb{G}_m \times X \to X$ on $X$ by $(X_0, X_1, X_2) \mapsto (\lambda^d X_0, X_1, \lambda X_2)$, where $\lambda \in k$. Then $\tau$ defines a torus subgroup $T$ of $\text{Aut} \,(X)$ such that $\pi_T : T \to \text{Aut} \,(X - \{0\})$ is the $d$-th power mapping.

3. Let $\delta$ be a locally nilpotent derivation on $\Gamma(X, \mathcal{O}_X)$ such that $\delta(x) = dy^{d-1}f^{-1}$ and $\delta(y) = f^{-1}$. Then $\delta$ defines a $\mathbb{G}_a$-action $\sigma : \mathbb{G}_a \times X \to X$ associated with $\rho$. Furthermore, $\tau_\lambda^{-1}\delta \tau_\lambda = \lambda^{d+1}\delta$ for $\lambda \in k^*$. 

(4) Let $G$ be the kernel of $\pi$. Then $G$ consists of automorphisms $\alpha$ such that

$$\begin{align*}
\alpha(y) &= cy + f^{-m}(a_0 f^r + a_1 f^{r-1} + \cdots + a_r) \\
\alpha(x) &= x - y^d + \alpha(y)^d,
\end{align*}$$

where $c^d = 1$, $a_i \in k$ ($0 \leq i \leq r$) with $a_0 a_r \neq 0$ and $m > r \geq 0$. Hence $\text{Aut}(X)$ is not an algebraic group.

**Proof.** Let $Y = \mathbb{P}^2 - (C + \ell_1)$. Then $Y$ is an affine surface with $\Gamma(Y, \mathcal{O}_Y) = \mathbb{C}[x, y, f^{-1}]$, where $f = x - y^d$. Furthermore, $Y$ has an $\mathbb{A}^1$-fibration $\{f = \lambda | \lambda \in \mathbb{C}^*\}$ which extends to a unique $\mathbb{A}^1$-fibration $\rho : X \to A$. In fact, the $\mathbb{A}^1$-fibration on $Y$ is obtained by removing a unique multiple fiber $d\ell_1$ from $\rho$.

Let $\alpha$ be an automorphism of $X$. Since $\rho$ is unique, the fibers of $\rho$ are transformed by $\alpha$ to the fibers of $\rho$. Since the multiple fiber $d\ell_1$ is unique, $\alpha$ induces an automorphism $\alpha_A$ which fixes the point $\rho(\ell_1)$. Thence follows the assertion (1). It is clear that $\tau_\lambda(f) = \lambda^d f$. Since $A - \{0\} = \text{Spec} k[f, f^{-1}]$, the assertion (2) follows from this remark.

We shall show the assertion (3). It is clear that $\delta$ is a locally nilpotent derivation of $k[x, y, f^{-1}] = \Gamma(Y, \mathcal{O}_Y)$ because $\delta(f) = 0$. The automorphism $\sigma_\lambda$ is given by $\sigma_\lambda(y) = y + f^{-1}\lambda$ and $\sigma_\lambda(x) = x - y^d + \sigma_\lambda(y)^d$, where $\lambda \in k$.

Hence the assertion (4) shows that $\sigma_\lambda$ is an automorphism of $X$.

We shall show the assertion (4). Let $\alpha$ be an automorphism in $G$. Let $t = X_1/X_2$ and $u = X_0/X_2$. Since $\alpha$ induces the identity on $A$, it acts along the fibers of $\rho$. Since $Y = X - d\ell_1 = \text{Spec} k[y, f, f^{-1}]$ and $\alpha(f) = f$, $\alpha$ is
written as

\[ \alpha(y) = cf^n y + f^s (a_0 f^r + a_1 f^{r-1} + \cdots + a_r) \]

\[ \alpha(x) = x - y^d + \alpha(y)^d, \]

where \( c \in k^*, a_i \in k \ (0 \leq i \leq r) \) with \( a_0 a_r \neq 0 \) and \( n, s, r \in \mathbb{Z} \) with \( r \geq 0 \). Conversely, given such an automorphism \( \alpha \) of \( Y \) as written above, we shall consider when \( \alpha \) extends to an automorphism of \( X \). Note that \( C \cap \ell_1 = (1, 0, 0) \) and \( \ell_1 \cap X \) is contained in the open set \( \mathbb{P}^2 - \ell_2 \), where \( \ell_2 = \{ X_2 = 0 \} \). Let \( Z = \mathbb{P}^2 - (C \cup \ell_2) \) be an open set of \( X \) containing \( \ell_1 \cap X \). Since \( t = 1/y, u = x/y \) and \( f = (ut^{d-1} - 1)/t^d \), it follows that \( t^df \) is a nowhere vanishing function on \( Z - \ell_1 \). We write

\[ \alpha(y) = c^* t^{-dn-1} + t^{-(s+r)d} \left( a_0^* + a_1^* t^d + \cdots + a_r^* t^{rd} \right), \]

where \( c^*, a_i^* \ (0 \leq i \leq r) \) are nowhere vanishing functions on \( Z \). Since \( \alpha(t) \) is divisible by \( t \) and not divisible by \( t^2 \) along the curve \( \ell_1 \cap Z \) and since \( d > 2 \), it follows that \( n = 0 \) and \( s + r \leq 0 \). Hence \( c^* = c \). Then we can write

\[ \alpha(u) = \frac{x - y^d + \alpha(y)^d}{\alpha(y)} = \frac{u + (c^d - 1)t^{-(d-1)} + dc^{d-1}g^* t^{-(s+r) - (d-2)} + \cdots + g^d t^{-d(s+r)+1}}{c + t^{1-d(s+r)} g^*}, \]

where \( g^* = (a_0^* + a_1^* t^d + \cdots + a_r^* t^{rd}) \). Since \( 1 - d(s + r) \geq 1 \) in the denominator, the numerator of \( \alpha(u) \) must be regular along the curve \( \ell_1 \cap Z \). Namely, we have \( c^d = 1 \) and \( s + r < 0 \). Set \( s = -m \). Then we obtain a formula for \( \alpha \) as stated in the assertion (4).

Q.E.D.
Lemma 1.4.14 Let $X$ be a smooth affine surface with an $\mathbb{A}^1$-fibration $\rho : X \to B$. Suppose that $\rho$ has a multiple fiber $mA$. Then there are no non-trivial torus actions $\sigma$ which stabilize the fibers of the $\mathbb{A}^1$-fibration $\rho$ in the sense that $\rho \cdot \sigma = \rho$.

Proof. Suppose that $\sigma : G_m \times X \to X$ be a non-trivial action which stabilizes the fibers of the $\mathbb{A}^1$-fibration $\rho$. Then there exists an equivariant smooth compactification $V$ of $X$ such that the boundary divisor $D := V - X$ has smooth normal crossings (cf. [49]) and that the fibration $\rho$ extends to a $\mathbb{P}^1$-fibration $p : V \to \overline{B}$. We denote by $\sigma$ the extended torus action on $V$ which stabilizes the fibers of the fibration $p$. Since each fiber and its irreducible components of $\rho$ are stable under $\sigma$, we may and shall assume that the multiple fiber $mA$ is irreducible, i.e., $A$ is isomorphic to $\mathbb{A}^1$. Let $F$ be the fiber $p^{-1}(\rho(A))$ and let $\overline{A}$ be the closure of $A$. Since $\sigma$ induces a trivial action on $\overline{B}$, it follows that each irreducible component of $F$ is $\sigma$-stable and the induced torus action on each component is non-trivial. In fact, if $\sigma$ is trivial along an irreducible component, say $G$, of $F$. Then $\sigma$ is trivial on an open neighborhood of $G$, hence on $V$ itself. This implies that we can contract the components of $F$ to a smooth fiber by repeating the equivariant blowing-downs. Conversely, in order to obtain the fiber $F$ from a smooth fiber, we always blow up the fixed points which must be either the fixed points of the starting smooth fiber or the intersection points of adjacent components. But, since $A \cong \mathbb{A}^1$ and $mA$ is a multiple fiber, we must blow up a non-fixed point to obtain the component $\overline{A}$. This is a contradiction. Q.E.D.

Let $p : V \to C$ be a $\mathbb{P}^1$-fibration from a smooth projective surface $V$ onto a smooth projective curve $C$, and let $G$ be a singular fiber of $p$. An
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irreducible component $B$ of $G$ is called a branching component if $B$ meets three other components of $G$. For example, in Figure 1 of Definition 1.4.5, the component $E_d$ is a branching component. In connection with Theorem 1.4.12 we have the following result.

**Theorem 1.4.15** Let $X$ be an affine pseudo-plane of type $(d, n, r)$ with $n > 0$ and $r \geq 3$, where we may assume $n = 1$. Let $V$ be a smooth normal compactification as described in Definition 1.4.5. Choose a point $R_1$ on $E_{d+1}$ different from $E_d \cap E_{d+1}$ and $E_{d+1} \cap E_{d+2}$ and blow up $R_1$ to obtain the exceptional curve $G_{d+2}$. Choose a point $R_2$ which is different from the point $G_{d+2} \cap E_{d+1}$ and blow it up to obtain the exceptional curve $G_{d+3}$. Repeat this kind of blowing-ups $(s - 1)$ times to obtain a linear chain $G_{d+2}, G_{d+3}, \ldots, G_{d+s-1}, G = G_{d+s}$. Call the resulting surface $W$. Let $Y = W - (l'_\infty + S' + l'_0 + E_1 + \cdots + E_{d+r-1} + G_{d+2} + \cdots + G_{d+s-1})$. Then $\text{Aut}(Y)$ has no torus group as a subgroup.

**Proof.** Crucial in the above construction is that the adjacent two components $E_d$ and $E_{d+1}$ are branching components. By the same arguments in the proof of Theorem 1.4.10 we can show that the $\mathbb{A}^1$-fibration $\rho_Y : Y \to A$ inherited from the $\mathbb{A}^1$-fibration $\rho : X \to A$ is a unique $\mathbb{A}^1$-fibration on $Y$. Hence if there is a torus action $\sigma : G_m \times Y \to Y$, we can extend it to a torus action on the surface $W$ which stabilizes each component of the boundary divisor $W - Y$. It is then clear that every branching component is pointwise fixed. Now consider the point $E_d \cap E_{d+1}$. Since two curves $E_d$ and $E_{d+1}$ with distinct tangential directions are pointwise fixed, the action is trivial in an open neighborhood of the point $E_d \cap E_{d+1}$. Hence $\sigma$ is trivial. This implies that, with the notations of Theorem 1.4.12 $\pi : \text{Aut}(Y) \to \text{Aut}(A - \{0\})$ is
trivial, i.e., $\text{Aut}(Y) = G$. Meanwhile, $G$ does not contain any torus group by Lemma 1.4.13. Q.E.D.

We can strengthen Theorem 1.4.10 to the following effect.

**Lemma 1.4.16** Let $X$ be a smooth affine surface with an $\mathbb{A}^1$-fibration $\rho : X \to A$, where $A$ is isomorphic to $\mathbb{A}^1$. Let \( \{m_1 F_1, \ldots, m_n F_n\} \) exhaust all multiple fibers of $\rho$, where the multiplicity $m$ is $\gcd(\mu_1, \ldots, \mu_r)$ for a fiber $F = \sum_{i=1}^{r} \mu_i C_i$ which is a disjoint sum of the irreducible components $C_i \cong \mathbb{A}^1$. Suppose that $n \geq 2$. Then there are no non-constant morphisms from $\mathbb{A}^1$ to $X$ whose image is transverse to the given $\mathbb{A}^1$-fibration $\rho$. In particular, there are no $\mathbb{A}^1$-fibrations whose general fibers are transverse to the given $\mathbb{A}^1$-fibration $\rho$.

**Proof.** Let $\varphi : B \to X$ be a non-constant morphism whose image is transverse to the given $\mathbb{A}^1$-fibration $\rho$, where $B \cong \mathbb{A}^1$. Let $V$ be a smooth compactification of $X$ such that $D := V - X$ is a divisor with simple normal crossings and $\rho$ extends to a $\mathbb{P}^1$-fibration $p : V \to \overline{A}$, where $\overline{A} \cong \mathbb{P}^1$. Then $D$ contains a cross-section $S$ of $p$ and the other components of $D$ are contained in the fibers of $p$. We may assume that the fiber $F_{\infty} := p^{-1}(P_{\infty})$ is a smooth fiber, where \( \{P_{\infty}\} = \overline{A} - A \). Let $\overline{B}$ be the closure of the image $\varphi(B)$ in $V$. Then $\overline{B}$ meets the fiber $F_{\infty}$, for otherwise $\varphi(B)$ is contained in a fiber of $\rho$. Since $B \cong \mathbb{A}^1$, the composite of the morphisms $\rho_B := \rho \circ \varphi : B \to A$ is surjective. Choose coordinates $x, y$ of $B, A$, respectively. Then $y = f(x)$, where $y$ is identified with $(\rho_B)^*(y)$ and $f(x)$ is a polynomial in $x$. Let $\rho(F_i)$ be defined by $y = a_i$ for $i = 1, 2$. Then $y - a_i = (f_i(x))^{m_i}$ for some non-constant polynomial $f_i(x)$. Then we have a polynomial relation $f_1(x)^{m_1} - f_2(x)^{m_2} = a_2 - a_1$. 

Let $m = \gcd(m_1, m_2)$. We consider the cases $m > 1$ and $m = 1$ separately.

Suppose that $m > 1$. Set $m_i = mn_i$ for $i = 1, 2$. Then $g_1(x)^m - g_2(x)^m = a_2 - a_1 \neq 0$, where $g_i(x) = f_i(x)^{m_i}$. Then we have

$$g_1(x)^m - g_2(x)^m = (g_1(x) - g_2(x)) (g_1(x) - \zeta g_2(x)) \cdots (g_1(x) - \zeta^{m-1} g_2(x)) = a_2 - a_1,$$

where $\zeta$ is a primitive $m$-th root of unity. Since this is a polynomial identity, it follows that $g_1(x) - \zeta^i g_2(x) = c_i \neq 0$ for $1 \leq i \leq m - 1$. Since $m \geq 2$, $g_1(x)$ and $g_2(x)$ are constants. This is a contradiction.

Suppose that $m = 1$. We may assume that $m_1 < m_2$. Let $C$ be an affine curve in $\mathbb{A}^2$ defined by $X^{m_1} - Y^{m_2} = 1$ and let $\overline{C}$ be a projective plane curve defined by $Z^{m_2}X^{m_1} - Y^{m_2} = Z^{m_2}$. Then the relation $f_1(x)^{m_1} - f_2(x)^{m_2} = a_2 - a_1$ implies that there is a non-constant morphism $\varphi : \mathbb{A}^1 \to C$. We shall show that the curve $C$ is irrational. The affine curve $C$ is smooth, and the curve $\overline{C}$ has a singular point $Q$ defined by $(X, Y, Z) = (1, 0, 0)$. The singularity type at $Q$ is a cuspidal singularity of type $z^{\ell_1} - y^n = 0$, where $\ell_1 = m_2 - m_1$ and $n = m_2$. By the Euclidean algorithm, define positive integers $\ell_2, \ldots, \ell_s$ and $a_1, \ldots, a_s$ by $n = a_1 \ell_1 + \ell_2$ ($0 < \ell_2 < \ell_1$), $\ell_1 = a_2 \ell_2 + \ell_3$ ($0 < \ell_3 < \ell_2$), $\ldots$, $\ell_{s-2} = a_{s-1} \ell_{s-1} + \ell_s$ ($1 = \ell_s < \ell_{s-1}$), $\ell_{s-1} = a_s \ell_s$.

Then the multiplicity sequence at $Q$ is $\{\ell_1^{a_1}, \ell_2^{a_2}, \ldots, \ell_s^{a_s-1}\}$, where $\ell^a$ signifies that the singular points of multiplicity $\ell$ appear $a$-times consecutively along the successive blowing-ups. Computing the genus drop by those singularities, we find that the geometric genus $g$ of $C$ is equal to

$$\frac{1}{2}(n-1)(n-\ell_1-1).$$

Since $n > 1$ and since $n = \ell_1 + 1$ implies $m_1 = 1$, we know that $g > 0$ and $C$
is irrational. Then the existence of the non-constant morphism \( \varphi : \mathbb{A}^1 \to C \) is a contradiction. Hence there are no \( \mathbb{A}^1 \)-fibrations whose general fibers are transverse to \( \rho \).

Q.E.D.

The following result is noteworthy in view of Theorem 1.4.10 and Lemma 1.4.16.

**Lemma 1.4.17** Let \( X \) be an affine pseudo-plane with an \( \mathbb{A}^1 \)-fibration \( \rho : X \to \mathbb{A} \). Then there exists a non-constant morphism \( \varphi : \mathbb{A}^1 \to X \) whose image is transversal to the \( \mathbb{A}^1 \)-fibration \( \rho \).

**Proof.** We consider a smooth compactification \( V \) as specified in Definition 1.4.5. Let \( p : V \to \overline{A} \) be the \( \mathbb{P}^1 \)-fibration which extends the \( \mathbb{A}^1 \)-fibration \( \rho \), where \( \overline{A} \cong \mathbb{P}^1 \). Let \( P_\infty = \overline{A} - A \) and let \( P_0 = \rho(dF) \), where \( dF \) is the unique multiple fiber of \( \rho \). Let \( \tau : \tilde{A} \to A \) be a \( d \)-ple cyclic covering ramifying over the points \( P_0 \) and \( P_\infty \). Let \( \tilde{P}_0 \) be the unique point of \( \tilde{A} \) lying over \( P_0 \). Let \( \tilde{X} \) be the normalization of the fiber product \( X \times_A \tilde{A} \). Then \( \tilde{X} \) has an \( \mathbb{A}^1 \)-fibration \( \tilde{\rho} : \tilde{X} \to \tilde{A} \) which has a unique reduced reducible fiber with all other fibers reduced and irreducible. Namely, the fiber \( \tilde{\rho}^{-1}(\tilde{P}_0) \) consists of \( d \) copies of the affine line, each of which is counted with multiplicity one. If one deletes all components but one from \( \tilde{\rho}^{-1}(\tilde{P}_0) \), the surface \( \tilde{X} \) with \( (d - 1) \) copies of the affine line removed is isomorphic to the affine plane. In particular, \( \tilde{X} \) is simply connected. Let \( Y \) be the affine plane obtained in this fashion. The remaining one component \( \ell \) of \( \tilde{\rho}^{-1}(\tilde{P}_0) \) is considered to be a coordinate line in \( Y \) (cf. Theorem 1.3.11). Let \( B \) be another coordinate line in \( Y \) which meets \( \ell \) transversally in one point. Let \( \varphi : B \to X \) be the restriction onto \( B \) of the covering morphism \( \tilde{X} \to X \). Then \( \varphi \) is a non-constant morphism. Q.E.D.
Some partial cases of affine pseudo-planes were observed in tom Dieck [10] as examples of affine surfaces without cancellation property. In order to state tom Dieck’s result, we shall recall and generalize a little bit his construction. Write \( \Sigma_n = \text{Proj} (\mathcal{O}_{\mathbb{P}^1}(-n) \oplus \mathcal{O}_{\mathbb{P}^1}) \) as the quotient of \((\mathbb{A}^2 \setminus 0) \times \mathbb{P}^1\) under the relation

\[
(z_0, z_1), [w_0, w_1] \sim (\nu z_0, \nu z_1), [\nu^nw_0, w_1]
\]

for \( \nu \in G_m = \mathbb{k}^* \). The projection \( \{(z_0, z_1), [w_0, w_1]\} \mapsto [z_0, z_1] \) induces a \( \mathbb{P}^1 \) fibration \( p_n : \Sigma_n \to \mathbb{P}^1 \). In the above definition by quotient and in what follows, the integer \( n \) could be negative. If \( n \geq 0 \), the curve \( w_0 = 0 \) (resp. \( w_1 = 0 \)) is a section \( M_1 \) of \( p_n \) with \((M_1^2) = n \) (resp. the minimal section \( M_0 \) with \((M_0^2) = -n \)). Meanwhile, if \( n < 0 \), then the curve \( w_0 = 0 \) (resp. \( w_1 = 0 \)) is the minimal section \( M_0 \) (resp. a section \( M_1 \) with \((M_1^2) = |n| \)) of \( \Sigma_{|n|} \). Let \( d \geq 2 \) and \( r = d + n \geq 1 \). With the notations of Lemma 1.4.6, we assume that the fiber \( \ell_0 \) is defined by \( z_0 = 0 \). Let \( w = w_0/w_1 \). Then \( \{z_0/z_1, w/z_1^n\} \) is a system of local coordinates at the point \( M_1 \cap \ell_0 \) (resp. \( M_0 \cap \ell_0 \)) if \( n \geq 0 \) (resp. \( n < 0 \)). Let \( \Lambda \) be a linear subsystem of \( |M_1 + d\ell_0| \) if \( n \geq 0 \) (resp. \( |M_0 + d\ell_0| \) if \( n < 0 \)) consisting of members which meet the curve \( M_1 \) (resp. \( M_0 \)) at the point \( M_1 \cap \ell_0 \) (resp. \( M_0 \cap \ell_0 \)) with multiplicity \( r \) if \( n \geq 0 \) (resp. \( n < 0 \)). Then any member of \( \Lambda \) is defined by an equation

\[
\frac{w}{z_1^n} \left\{ a_0 + a_1 \left( \frac{z_0}{z_1} \right) + \cdots + a_{d-1} \left( \frac{z_0}{z_1} \right)^{d-1} + a_d \left( \frac{z_0}{z_1} \right)^d \right\} + a_{d+1} \left( \frac{z_0}{z_1} \right)^r = 0
\]

or equivalently by

\[
w_0 \left( a_0 z_1^d + a_1 z_0 z_1^{d-1} + \cdots + a_{d-1} z_0^{d-1} z_1 + a_d z_0^d \right) + a_{d+1} z_0^r w_1 = 0 \quad (1)
\]

for \((a_0, a_1, \ldots, a_{d+1}) \in \mathbb{P}^{d+1}\). In fact, it is readily computed that \( \dim \Lambda = \)
$d + 1$. So, the curve $C_d$ in Lemma 1.4.6 is defined by such an equation. We shall verify the following result.

**Lemma 1.4.18** Let $X = X(d, r)$ be an affine pseudo-plane with $n = r - d > 0$. Let $\sigma : G_m \times X \to X$ be a non-trivial action of the algebraic torus $G_m$. Write $X = \Sigma \setminus M_1 \cup C_d$ as in Lemma 1.4.6. Then the following assertions hold true.

1. The action $\sigma$ induces an action $\sigma : G_m \times \Sigma_n \to \Sigma_n$ such that $\sigma(\lambda)M_i \subseteq M_i$ for $i = 0, 1$, $\sigma(\lambda)C_d \subseteq C_d$ and $\sigma(\lambda)\ell \sim \ell$ for $\lambda \in k^*$, where $\ell$ is a fiber of $\Sigma_n$.

2. The curve $C_d$ is defined by an equation

$$z_1^d w_0 + az_0^r w_1 = 0.$$

**Proof.** (1) Let $\rho : X \to A \cong A^1$ be the unique $A^1$-fibration (cf. Theorem 1.4.10). Then the fibers of $\rho$ are permuted by the action $\sigma$. Hence $\sigma$ extends to the cross-section $S'$ and sends $S'$ into itself. Let $W$ be a $G_m$-equivariant smooth normal compactification of $X$ whose existence is guaranteed by [49]. We may assume that $W \setminus X$ contains the cross-section $S'$. Let $F_0$ and $F_\infty$ be two fibers of the $\mathbb{P}^1$-fibration $p : W \to \mathbb{P}^1$ whose supports partly or totally lie outside of $X$, where $F_0$ contains the multiple fiber of $\rho$. We may assume that all $(-1)$ components of $F_0$ and $F_\infty$ are fixed componentwise under the action $\sigma$. Then we may assume that $F_\infty$ is irreducible and $F_0$ minus the component $F$ contains no $(-1)$ components, where $F \cap X$ gives rise to the multiple fiber of $\rho$. Then we may assume that $W \setminus X$ has the dual graph as in Definition 1.4.5. So, the action $\sigma$ induces a $G_m$-action on $\Sigma_n$ such
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that $\sigma(\lambda)(M_1) \subseteq M_1$, $\sigma(\lambda)(C_d) \subseteq C_d$ and $\sigma(\lambda)(\ell_0) \subseteq \ell_0$ because $M_1, C_d, \ell_0$ are the images on $\Sigma_n$ of the components $E_1, \ell', \overline{F}$, respectively. The minimal section $M_0$ is stable under the $\sigma$ action because the minimal section is unique on $\Sigma_n$.

(2) The $G_m$-action $\sigma$ on $\Sigma_n$ is given as follows in terms of the coordinates.

$$\mu \cdot ((z_0, z_1), [w_0, w_1]) = (\mu^{\alpha}z_0, \mu^{\beta}z_1), [\mu^{\gamma}w_0, \mu^{\delta}w_1])$$

for $\mu \in k^*$. Since $C_d$ is stable under the $\sigma$-action, the defining equation (1) must be semi-invariant. Note that $a_0a_{d+1} \neq 0$ because $C_d$ is irreducible. Hence we obtain $\alpha r + \delta = \beta d + \gamma$. Suppose that $(a_1, \ldots, a_d) \neq (0, \ldots, 0)$. Then we have an additional relation $\alpha i + \beta (d - i) + \gamma = \beta d + \gamma$ for some $1 \leq i \leq d$. The last relation implies $\alpha = \beta$. So, the first relation gives $\gamma = \alpha n + \delta$. Then we have

$$\mu \cdot ((z_0, z_1), [w_0, w_1]) = (\mu^{\alpha}z_0, \mu^{\beta}z_1), [\mu^{\gamma}w_0, \mu^{\delta}w_1])$$

$$= (\mu^{\alpha}z_0, \mu^{\beta}z_1), [\mu^{\alpha n + \delta}w_0, \mu^{\delta}w_1])$$

$$\sim ((z_0, z_1), [\mu^{\delta}w_0, \mu^{\delta}w_1])$$

$$= ((z_0, z_1), [w_0, w_1]).$$

Hence the $\sigma$-action is trivial. This proves the second assertion. Q.E.D.

After tom Dieck [10], we denote by $V(d, r)$ the affine pseudo-plane $X(d, r)$ obtained by using a curve defined by an equation

$$z_1^d w_0 + z_0^r w_1 = 0$$

where $r - d$ could be negative. Then $V(d, r)$ has a $G_m$-action defined by

$$\mu \cdot ((z_0, z_1), [w_0, w_1]) = (\mu z_0, z_1), [w_0, \mu^{-r} w_1])$$
for $\mu \in k^*$. One can show that any $G_m$-action on $V(d, r)$ is reduced to the one defined by (3) for

$$
\mu \cdot ((z_0, z_1), [w_0, w_1]) = ((\mu^\alpha z_0, \mu^\beta z_1), [\mu^\gamma w_0, \mu^\delta w_1])
$$

$$
= ((\mu^\alpha z_0, \mu^\beta z_1), [\mu^{\alpha r - \beta d + \delta} w_0, \mu^\delta w_1])
$$

$$
= ((\mu^\alpha z_0, \mu^\beta z_1), [\mu^{\beta n + (\alpha - \beta)r + \delta} w_0, \mu^\delta w_1])
$$

$$
\sim ((\mu^{\alpha - \beta} z_0, z_1), [\mu^{r(\alpha - \beta)} w_0, w_1])
$$

$$
= ((\mu^{\alpha - \beta} z_0, z_1), [w_0, \mu^{-r(\alpha - \beta)} w_1]).
$$

In tom Dieck [10], the following result is shown, where $A^1(c)$ stands for the $A^1$ with a $G_m$-action of weight $c$ and $\tilde{V}(d, r)$ for the universal covering of $V(d, r)$.

**Theorem 1.4.19** With the above notations, there are $G_m$-equivariant isomorphisms

$$
V(d, r) \times A^1(-s) \cong V(d, s) \times A^1(-r)
$$

and

$$
\tilde{V}(d, r) \times A^1(-s) \cong \tilde{V}(d, s) \times A^1(-r)
$$

for arbitrary positive integers $d, r$ and $s$.

This result inspires us a very interesting question.

**Question.** Let $X(d, r)$ be an affine pseudo-plane with $d \geq 2$ and $r \geq 1$. Does this surface have cancellation property?

Towards answering this question, we first consider the universal covering $\tilde{X}(d, r)$ of an affine pseudo-plane $X(d, r)$. 

Lemma 1.4.20 The following assertions hold true.

1. The universal covering $\tilde{X}(d, r)$ is isomorphic to an affine hypersurface in $\mathbb{A}^3 = \text{Spec } k[x, y, z]$ defined by an equation
   \[ x^r z + (y^d + a_1 xy^{d-1} + \cdots + a_{d-1} x^{d-1} y + a_d x^d) = 1. \]

2. The projection $(x, y, z) \mapsto x$ induces an $\mathbb{A}^1$-fibration $\tilde{\rho} : \tilde{X}(d, r) \to \mathbb{A}^1$ such that every fiber except for $\tilde{\rho}^{-1}(0)$ is smooth and the fiber $\tilde{\rho}^{-1}(0)$ consists of $d$ copies of $\mathbb{A}^1$ which are reduced.

3. There is a $G_a$-action on $\tilde{X}(d, r)$ defined by
   \[ c \cdot (x, y, z) = (x, y + cx^r, z - x^{-r} ((y + cx^r)^d + a_1 x (y + cx^r)^{d-1} + \cdots + a_d x^d)) \]
   where $c \in G_a = k$.

4. Let $\omega$ be a $d$-th root of unity. Then there exist uniquely determined polynomials $p_\omega(x), q_\omega(x) \in k[x]$ satisfying the following conditions.
   (i) $\deg p_\omega(x) \leq r - 1$.
   (ii) $p_\omega(0) = \omega$.
   (iii) $x^r q_\omega(x) + p_\omega(x)^d + a_1 x p_\omega(x)^{d-1} + \cdots + a_{d-1} x^{d-1} p_\omega(x) + a_d x^d = 1$.
   (iv) $p_{\lambda \omega}(\lambda x) = \lambda p_\omega(x), q_{\lambda \omega}(\lambda x) = \lambda^{-r} q_\omega(x)$ for any $d$-th root $\lambda$ of unity.

By making use of these polynomials, we define the morphism
\[ \varphi_\omega : \mathbb{A}^2 \cong \mathbb{A}^1 \times G_a \to \tilde{X}(d, r), \quad (x, c) \mapsto c \cdot (x, p_\omega(x), q_\omega(x)) \]
which is an open immersion onto an open set $U_\omega$ which is the complement of $\prod_{\gamma \neq \omega} G_{a} \cdot (0, \gamma, 0)$. The inverse morphism is defined by

\[
(x, y, z) \mapsto \begin{cases} (x, y - p_\omega(x)/x^r) & \text{if } x \neq 0 \\ (0, -z + q_\omega(0)/d_\omega d^{d-1}) & \text{if } x = 0 \end{cases}
\]

(5) $\tilde{X}(d, r)$ is obtained by gluing together the $d$-copies of the affine plane $\mathbb{A}^2$ by the transition functions

\[
g_{\lambda \omega} := \varphi_\lambda^{-1} \circ \varphi_{\omega} : \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1 \times \mathbb{A}^1, \quad (x, c) \mapsto (x, c + \frac{p_\omega(x) - p_\lambda(x)}{x^r}).
\]

(6) The Galois group is a cyclic group $H(d) := \mathbb{Z}/d\mathbb{Z}$ of order $d$ and acts as

\[
\lambda \cdot \varphi_{\omega}(x, c) = \varphi_{\lambda \omega}(\lambda x, \lambda^{1-r} c).
\]

Proof. (1) The surface $X(d, r)$ is the complement in $\Sigma_n$ of the curves $C_d$ defined by the equation (1) above and the curve $w_0 = 0$ which is $M_1$ if $r - d \geq 0$ (resp. $M_0$ if $r - d < 0$), where $n = |r - d|$. Since $w_0 \neq 0$, we can normalize to $w_0 = 1$. We can then normalize

\[
w_0 \left( a_0 z_1^d + a_1 z_0 z_1^{d-1} + \cdots + a_{d-1} z_0^{d-1} z_1 + a_d z_0^d \right) + a_{d+1} z_0^r w_1 \neq 0
\]

to the relation

\[
z_0^r w_1 + (a_0 z_1^d + a_1 z_0 z_1^{d-1} + \cdots + a_{d-1} z_0^{d-1} z_1 + a_d z_0^d) = 1,
\]

where $a_0 \neq 0$. This normalization comes from the defining equivalence relation

\[
(z_0, z_1), [w_0, w_1] \sim (\nu z_0, \nu z_1), [\nu^n w_0, w_1].
\]
We may assume that $a_0 = 1$. Now let $H(d) = \mathbb{Z}/d\mathbb{Z}$ be the cyclic group of order $d$. Then $H(d)$ is the Galois group of the covering $\widetilde{X}(d, r) \to X(d, r)$ acting as
\[ \lambda \cdot (x, y, z) \mapsto (\lambda x, \lambda y, \lambda^{-n} z) \]
for $\lambda \in H(d)$. In terms of $\varphi_\omega$'s, it is written as in the assertion (6).

(3) Let $\delta$ be a derivation on the coordinate ring defined by $\delta(x) = 0, \delta(y) = x^r$ and $\delta(z) = -(dy^{d-1} + (d - 1)a_1xy^{d-2} + \cdots + a_{d-1}x^{d-1})$. Then $\delta$ is locally nilpotent. Hence it defines a $G_a$-action on $\widetilde{X}(d, r)$ by Lemma 1.2.1 which is as specified as in the assertion.

(4) Write $p_\omega(x) = \omega + c_1(\omega)x + \cdots + c_{r-1}(\omega)x^{r-1}$, where the coefficients are to be determined by the relation
\[ x^r q_\omega(x) + p_\omega(x)^d + a_1xp_\omega(x)^{d-1} + \cdots + a_{d-1}x^{d-1}p_\omega(x) + a_dx^d = 1 \] (5)
which is obtained from the equation (4) above by substituting $p_\omega(x), q_\omega(x)$ for $y, z$. By the condition (i), it is easy to see that $p_\omega(x)$ is uniquely determined. Namely the coefficients $c_1(\omega), \ldots, c_{r-1}(\omega)$ are uniquely determined by putting the coefficients of the terms $x^i$ ($1 \leq i \leq r - 1$) to be zero in the left-hand side of the equation (5) above. Then $q_\omega(x)$ is uniquely determined as well. By multiplying $\lambda^d = 1$ to the relation (5), we obtain
\[ (\lambda x)^r \lambda^{-r} q_\omega(\lambda^{-1}(\lambda x)) \]
\[ + (\lambda p_\omega(\lambda^{-1}(\lambda x)))^d + a_1(\lambda x)(\lambda p_\omega(\lambda^{-1}(\lambda x)))^{d-1} + \cdots + a_d(\lambda x)^d = 1. \]
Replace $\lambda x$ by $x$ in the above relation. Then the uniqueness of the polynomials $p_{\omega}(x), q_{\omega}(x)$ imply that $p_{\lambda\omega}(x) = \lambda p_\omega(\lambda^{-1}x)$ and $q_{\lambda\omega}(x) = \lambda^{-r} q_\omega(\lambda^{-1}x)$. Now replace $x$ by $\lambda x$. Then we obtain the relation (iv). Note that $\varphi_\omega : \mathbb{A}^2 \to \cdots$
$U_\omega$ is injective and $U_\omega \cong \mathbb{A}^2$. Hence $\varphi_\omega$ is an isomorphism by [5]. The other assertions are verified in a straightforward manner. Q.E.D.

**Example 1.4.21** The following are easy cases to compute.

1. If $r = 3$, $d = 2$, then $p_\omega(x) = \omega - \frac{a_1}{2}x + \frac{(a_1^2 - 4a_2)\omega}{8}x^2$ and $q_\omega(x) = -\{2c_1(\omega)c_2(\omega) + a_1c_2(\omega) + c_2(\omega)^2\}$.

2. If $r = 4$, $d = 2$, then $p_\omega(x) = \omega - \frac{a_1}{2}x + \frac{(a_1^2 - 4a_2)\omega}{8}x^2, c_3(\omega) = 0$ and $q_\omega(x) = -c_2(\omega)^2x$.

Concerning the isomorphism classes of the hypersurfaces $\tilde{X}(d, r)$, we have the following result.

**Lemma 1.4.22** Let $\tilde{X}_1(d, r)$ and $\tilde{X}_2(d, r)$ be the hypersurfaces defined by the equations $x^r z + y^d + a_2 x^2 y^{d-2} + \cdots + a_d x^d = 1$ and $x^r z + y^d + b_2 x^2 y^{d-2} + \cdots + b_d x^d = 1$, respectively. Let $f : \tilde{X}_1(d, r) \to \tilde{X}_2(d, r)$ be an isomorphism. Suppose $r > d$. Then the induced homomorphism of the coordinate rings $\varphi := f^*: \mathbb{k}[x, y, z] \to \mathbb{k}[x, y, z]$ is given as $\varphi(x) = cx, \varphi(y) = uy + x^r G(x)$ and

$$
\varphi(z) = c^{-r}z - (cx)^{-r} \left\{ \left( (uy + x^r G(x))^d + b_2 c^2 x^2 (uy + x^r G(x))^{d-2} + \cdots + b_d c^d x^d \right) - \left( y^d + a_2 x^2 y^{d-2} + \cdots + a_d x^d \right) \right\},
$$

where $c, u \in \mathbb{k}^*$ with $u^d = 1$ and $G(x) \in \mathbb{k}[x]$ which satisfy the condition $a_i = c^i u^{d-i} b_i$.

**Proof.** Note that $f$ preserves the unique $\mathbb{A}^1$-fibrations of $\tilde{X}_1(d, r)$ and $\tilde{X}_2(d, r)$ as well as the reduced fibers. Hence $\varphi(x) = cx$ with $c \in \mathbb{k}^*$. Then
\[ \varphi \text{ induces an automorphism of the polynomial ring } k[x, x^{-1}][y]. \]

So, one can write \( \varphi(y) = u x^e y + F(x, x^{-1}) \) with \( u \in k^*, e \in \mathbb{Z} \) and \( F(x, x^{-1}) \in k[x, x^{-1}] \).

Since \( \varphi \) is a homomorphism of the coordinate rings, it follows that \( e \geq 0 \) and \( F(x, x^{-1}) = F(x) \in k[x] \). Then we have

\[
c^r x^r \varphi(z) + (u x^e y + F(x))^d + b_2 c^2 x^2 (u x^e y + F(x))^{d-2} + \cdots + b_d c^d x^d = 1.
\]

This relation should coincide with the relation

\[
x^r z + y^d + b_2 x^2 y^{d-2} + \cdots + b_d x^d = 1.
\]

The comparison of the coefficients of the terms \( y^d \) and \( xy^{d-1} \) implies that \( u^d = 1, e = 0 \) and \( F(x) = 0 \) is divisible by \( x^r \). Furthermore, with the hypothesis \( r > d \), we readily see that \( a_i = c^i b_i (2 \leq i \leq d) \) and \( \varphi(z) \) is given as in the statement. Q.E.D.

Let \( \tilde{X}(d, r) \) be the affine hypersurface in \( \mathbb{A}^3 \) defined by the equation (4) in Lemma 1.4.20 which has the transition functions given in the assertion (5) of the same lemma. Let \( \tilde{X}'(d, s) \) be a similar affine hypersurface in \( \mathbb{A}^3 \) with the equation

\[
x^s z + (y^d + a'_1 x y^{d-1} + \cdots + a'_d x^{d-1} y + a'_d x^d) = 1
\]

and the transition functions

\[
g_{\lambda \omega}' := \varphi_{\lambda}^{-1} \circ \varphi_{\omega}' : \mathbb{A}_s^1 \times \mathbb{A}_s^1 \to \mathbb{A}_s^1 \times \mathbb{A}_s^1, \quad (x, c) \mapsto (x, c + \frac{p_{\omega}'(x) - p_{\lambda}'(x)}{x^s}).
\]

As in [10], we define a 3-dimensional affine variety \( W(d, r, s) \) by gluing together \( d \)-copies of the affine 3-space \( \{\omega\} \times \mathbb{A}^3 (\omega \in H(d)) \) by the following identification

\[
(\omega, x, c_1, c_2) \sim (\lambda, x, c_1 + \frac{p_{\omega}(x) - p_{\lambda}(x)}{x^r}, c_2 + \frac{p_{\omega}'(x) - p_{\lambda}'(x)}{x^s}), \quad x \neq 0.
\]
The projection \((\omega, x, c_1, c_2) \mapsto (\omega, x, c_1)\) yields a morphism \(\pi_1 : W(d, r, s) \to \tilde{X}(d, r)\) which is a principal \(G_a\)-bundle over \(\tilde{X}(d, r)\) with \(G_a\)-action coming.

Similarly, the projection \((\omega, x, c_1, c_2) \mapsto (\omega, x, c_2)\) gives rise to a principal \(G_a\)-bundle \(\pi_2 : W(d, r, s) \to \tilde{X}'(d, s)\). The Galois group action of \(H(d)\) onto \(\tilde{X}(d, r)\) and \(\tilde{X}'(d, s)\) as specified in the assertion (6) of Lemma 1.4.20 is lifted to \(W(d, r, s)\) as follows so that \(\pi_1\) and \(\pi_2\) are \(H(d)\)-equivariant:

\[\lambda \cdot (\omega, x, c_1, c_2) = (\lambda \omega, \lambda x, \lambda^{1-r} c_1, \lambda^{1-s} c_2) \quad \text{for} \quad \lambda \in H(d).\]

Since every principal \(G_a\)-bundle over an affine variety is trivial (cf. [48]), we have splittings of \(W(d, r, s)\) in two ways.

**Theorem 1.4.23** Let \(\tilde{X}(d, r)\) and \(\tilde{X}'(d, s)\) be as above. Then we have isomorphisms

\[\tilde{X}(d, r) \times \mathbb{A}^1 \cong W(d, r, s) \cong \tilde{X}'(d, s) \times \mathbb{A}^1.\]

We would like to descend these isomorphisms down to the level of \(H(d)\)-quotient varieties. For this purpose, one looks for \(H(d)\)-equivariant sections of \(\pi_1 : W(d, r, s) \to \tilde{X}(d, r)\) and \(\pi_2 : W(d, r, s) \to \tilde{X}'(d, s)\).

**Lemma 1.4.24** For the principal \(G_a\)-bundle \(\pi_1 : W(d, r, s) \to \tilde{X}(d, r)\), an \(H(d)\)-equivariant section is given by a family of polynomials \(\sigma_\omega \in k[x, c], \omega \in H(d)\) satisfying the following conditions:

1. For all \(\omega, \lambda \in H(d)\) and \((x, c) \in \mathbb{A}^1 \times \mathbb{A}^1\),
   \[\sigma_\omega(x, c) + \frac{p'_\omega(x) - p'_\lambda(x)}{x^s} = \sigma_\lambda(x, c) + \frac{p_\omega(x) - p_\lambda(x)}{x^r}.\]

2. For \(\omega, \lambda \in H(d)\),
   \[\lambda^{1-s} \sigma_\omega(x, c) = \sigma_{\lambda \omega}(\lambda x, \lambda^{1-r} c).\]
We can use the relation (2) in the above lemma to compute $\sigma_\lambda$ from $\sigma_1$

$$\sigma_\lambda(x, c) = \lambda^{1-s} \sigma_1(\lambda^{-1}x, \lambda^{-1}c).$$  

(6)

In terms of the function $\sigma_1$, the conditions (1) and (2) are reformulated as in the following result. The proof is essentially the same as in [10] if one takes into account the relation (4) (iv) of Lemma 1.4.20.

**Lemma 1.4.25** Given a polynomial $\sigma = \sigma_1 \in k[x, c]$, define polynomials

$$\{\sigma_\lambda \mid \lambda \in H(d)\}$$

by the equation (6) above. Then the conditions (1) and (2) in Lemma 1.4.24 are satisfied if and only if $\sigma$ satisfies

$$\lambda^{1-s} x^s \sigma(\lambda^{-1}x, \lambda^{-1}c + \frac{p_1(x) - p_\lambda(x)}{x^r}) = x^s \sigma(x, c) + p_1'(x) - p_\lambda'(x)$$

(7)

for all $\lambda \in H(d), (x, c) \in \mathbb{A}^1_x \times \mathbb{A}^1$.

If the polynomial $\sigma$ in Lemma 1.4.25 exists, then the principal $G_a$-bundle $\pi_1: W(d, r, s) \rightarrow \tilde{X}(d, r)$ splits $H(d)$-equivariantly. Namely, define isomorphisms on the $\omega$-charts $\theta_\omega: (\omega, \mathbb{A}^2) \times \mathbb{A}^1 \rightarrow (\omega, \mathbb{A}^3)$ by $((\omega, x, c_1), c_2) \mapsto (\omega, x, c_1, \sigma_\omega(x, c_1) + c_2)$. Then the $\theta_\omega$ glue together to give an $H(d)$-equivariant isomorphism $\theta: \tilde{X}(d, r) \times \mathbb{A}^1(1-s) \rightarrow W(d, r, s)$, where $\mathbb{A}^1$ signifies $\mathbb{A}^1$ considered as an $H(d)$-module (i.e., $H(d)$-vector space of dimension 1) with weight $1-s$.

In order to find solutions of $\sigma(x, c)$ in $k[x, c]$ is a kind of Diophantine Problem, and there are no answers to the existence problem of solution in general cases but there is an elaborate treating by tom Dieck [10] in the case where $\tilde{X}(d, r) = \tilde{V}(d, r)$ and $\tilde{X}'(d, s) = \tilde{V}(d, s)$. In general, we write $\sigma(x, c)$ as a polynomial in $c$ with coefficients in $k[x]$

$$\sigma(x, c) = f_0(x) + f_1(x)c + \cdots + f_t(x)c^t.$$  

(8)
We shall show the existence of solutions in the cases where $r = s$ and either $\tilde{X}(d, r) = \tilde{V}(d, r)$ or $\tilde{X}'(d, r) = \tilde{V}(d, r)$.

**Lemma 1.4.26** The following assertions hold true.

1. Suppose that $\tilde{X}(d, r) = \tilde{V}(d, r)$, i.e., $p_\lambda(x) = \lambda$ for all $\lambda \in H(d)$ and that the $p'_{\lambda}(x)$ satisfy the relation

   $$p'_1(\lambda x) - p'_\lambda(\lambda x) = p'_1(x) - p'_\lambda(x) \quad \text{for any } \lambda \in H(d).$$

   Put $\sigma(x, c) = f_1(x)c$, where

   $$f_1(x) = \frac{1}{1 - \lambda}\{p'_1(\lambda x) - p'_\lambda(\lambda x)\}.$$

   Then $\sigma(x, c)$ satisfies the relation (7) in Lemma 1.4.25. If $d = 2$ and $r = 3$ then the $p_\lambda(x)$ satisfy the relation above for the $p'_\lambda(x)$ (see Example 1.4.21).

2. Suppose that $\tilde{X}'(d, r) = \tilde{V}(d, r)$ and that $d = 2, r = 3$. With the notations of Example 1.4.21 let

   $$\sigma(x, c) = \left(\frac{a^2 - 4a_2}{8}\right)^3 x^3 + \left\{1 - \left(\frac{a^2 - 4a_2}{8}\right) x^2 + \left(\frac{a^2 - 4a_2}{8}\right)^2 x^4\right\}c.$$

   Then $\sigma(x, c)$ satisfies the relation (7) in Lemma 1.4.25.

**Proof.** (1) Let

$$\sigma(x, c) = f_0(x) + f_1(x)c$$

and put it in the relation (7) of Lemma 1.4.25. The relation (7) reads as

$$\lambda^{1-s}x^r f_0(\lambda^{-1}x) + x^r f_1(\lambda^{-1}x)c + (1 - \lambda)f_1(\lambda^{-1}x)$$

$$= x^r(f_0(x) + f_1(x)c) + p'_1(x) - p'_\lambda(x).$$
Then we may put

\[ f_0(\lambda^{-1}x) = \lambda r^{-1}f_0(x), \quad f_1(\lambda^{-1}x) = f_1(x) \]

\[ (1 - \lambda)f_1(\lambda^{-1}x) = p'_1(x) - p'_\lambda(x). \]

Now put \( f_0(x) = 0 \) and

\[ f_1(x) = \frac{1}{1 - \lambda} \{ p'_1(\lambda x) - p_\lambda(\lambda x) \}, \]

where \( p'_1(\lambda x) - p'_\lambda(\lambda x) \) has a factor \( 1 - \lambda \). Then, under the hypothesis in the assertion (1), \( \sigma(x, c) \) satisfies the relation (7).

(2) The computation is straightforward. Q.E.D.

A similar argument as in Lemma 1.4.26 gives the following result.

**Proposition 1.4.27** Suppose that \( d < r < 2d \) and that \( \tilde{X}(d, r) \) is defined by

\[ x^r z + y^d + ax^d = 1 \]

with \( a \in k \). Then the following assertions hold true.

1. \( p_{\lambda}(x) = \lambda - \frac{a}{d} \lambda x^d \) for \( \lambda \in \text{H}(d) \).

2. Define \( \sigma(x, c) \) and \( \tau(x, c) \) as follows:

\[ \sigma(x, c) = \frac{1}{1 - \lambda} \{ p_1(\lambda x) - p_\lambda(\lambda x) \} c \]

\[ \tau(x, c) = \frac{-a^2}{d^2} x^{2d-r} + \left( 1 + \frac{a}{d} x^d \right) c. \]

Then we have:

\[ \lambda^{1-r}x^r \sigma(\lambda^{-1}x, \lambda^{-1}(c + \frac{1 - \lambda}{x^r}x^d)) = x^r \sigma(x, c) + p_1(x) - p_\lambda(x) \]

\[ \lambda^{1-r}x^r \tau(\lambda^{-1}x, \lambda^{-1}(c + \frac{p_1(x) - p_\lambda(x)}{x^r})) = x^r \sigma(x, c) + 1 - \lambda \]
for all $\lambda \in H(d)$. Hence we have an isomorphism

$$\widetilde{X}(d, r) \times A^1 \cong \widetilde{\mathcal{V}}(d, r) \times A^1.$$  

**Proof.** The relation for $\sigma(x, c)$ follows from the assertion (1) of Lemma 1.4.26. The relation for $\tau(x, c)$ can be verified by a straightforward computation. Q.E.D.
Chapter 2

Jacobian Conjecture

§1. Generalities

§2. Generalized Jacobian Conjecture

§3. Affirmative results

§4. Normalization by étale endomorphisms

§5. Affine pseudo-coverings
In this chapter, the ground field is an algebraically closed field $k$ of characteristic zero. We often confuse $k$ with the complex field $\mathbb{C}$ if the use of $\mathbb{C}$ is more suitable. We shall gather together some of the well-known results on the Jacobian conjecture in 2.1 and then shift to the generalized Jacobian conjecture in 2.2, 2.3 and 2.4. In the last section 2.5, we deal with affine pseudo-coverings, and the treatment contains new results.

2.1 Generalities

The following is a renown

**Jacobian Conjecture.** Let $f_1, \ldots, f_n$ be elements of a polynomial ring in $n$ variables $\mathbb{C}[x_1, \ldots, x_n]$ defined over the complex field $\mathbb{C}$. Suppose that the Jacobian determinant

$$J(F; X) = \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{vmatrix} \in \mathbb{C}^*.$$  

Then $\mathbb{C}[x_1, \ldots, x_n] = \mathbb{C}[f_1, \ldots, f_n]$.

Here $F$ and $X$ signifies a set of polynomials $(f_1, \ldots, f_n)$ and a set of variables $(x_1, \ldots, x_n)$ and $\mathbb{C}^*$ stands for the set of nonzero complex numbers $\mathbb{C} \setminus \{0\}$. We also denote the Jacobian determinant by

$$J \left( \frac{f_1, \ldots, f_n}{x_1, \ldots, x_n} \right).$$

In geometric terms, the conjecture can be expressed as follows. Define a
polynomial morphism (or endomorphism) $F : \mathbb{A}^n \to \mathbb{A}^n$ by

$$(x_1, \ldots, x_n) \mapsto (f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n)).$$

We also denote this morphism by $F = (f_1, \ldots, f_n)$. If $F$ is an automorphism of the affine space $\mathbb{A}^n$, there exists then another polynomial morphism $G = (g_1, \ldots, g_n) : \mathbb{A}^n \to \mathbb{A}^n$ such that $G \circ F = \text{id}$ and $F \circ G = \text{id}$. By a theorem of Ax [5, 7] which asserts that an injective algebraic endomorphism $\varphi : X \to X$ of an algebraic variety $X$ is an automorphism, a polynomial mapping $F = (f_1, \ldots, f_n) : \mathbb{A}^n \to \mathbb{A}^n$ is an automorphism if there exists a polynomial morphism $G = (g_1, \ldots, g_n) : \mathbb{A}^n \to \mathbb{A}^n$ such that $G \circ F = \text{id}$.

Denote by $J(G \circ F; F)$ the Jacobian determinant $J(G; X)$ with the set of polynomials $F = (f_1, \ldots, f_n)$ substituted for the variable $X = (x_1, \ldots, x_n)$. Partial differentiation of a composite function then yields the following relation

$$J(G \circ F; F) \cdot J(F; X) = 1.$$

Hence the Jacobian determinant $J(F; X)$ is an invertible element of $\mathbb{C}[x_1, \ldots, x_n]$. Since $\mathbb{C}[x_1, \ldots, x_n]^* = \mathbb{C}^*$, it follows that if a polynomial endomorphism $F : \mathbb{A}^n \to \mathbb{A}^n$ is an automorphism, then the Jacobian determinant $J(F; X)$ is a nonzero constant.

The Jacobian Conjecture asserts that the converse of this result holds true. Namely, it asserts that

**Jacobian Conjecture (the second form).** A polynomial endomorphism $F = (f_1, \ldots, f_n) : \mathbb{A}^n \to \mathbb{A}^n$ is an automorphism if the Jacobian determinant $J(F; X)$ is a nonzero constant.
In the case \( n = 1 \), the conjecture holds because the condition \( \partial f / \partial x \in \mathbb{C}^* \) for a polynomial in one variable \( x \)

\[
f = a_0 x^m + a_1 x^{m-1} + \cdots + a_m, \quad a_0 \neq 0
\]

implies \( \deg f = 1 \). In the case \( n \geq 2 \), the conjecture remains unsolved.

Set \( y_i = f_i(x_1, \ldots, x_n) \) (\( i = 1, \ldots, n \)), where we assume that \( f_i(0, \ldots, 0) = 0 \). With the condition \( J(F; X) \in \mathbb{C}^* \), one can express \( x_1, \ldots, x_n \) as formal power series in \( y_1, \ldots, y_n \). In fact, we can determine the coefficients of the terms of a formal power series \( \varphi_i \) by the method of undetermined coefficients

\[
x_i = \varphi_i(y_1, \ldots, y_n) = \sum_{\alpha=0}^{\infty} c_{\alpha} y^\alpha
\]

after substituting \( f_i(x_1, \ldots, x_n) \) for \( y_i \), where \( y^\alpha = y_1^{\alpha_1} \cdots y_n^{\alpha_n} \) for \( \alpha = (\alpha_1, \ldots, \alpha_n) \).

By the inverse mapping theorem, the formal power series \( \varphi_i(y_1, \ldots, y_n) \) is a holomorphic function in variables \( y_1, \ldots, y_n \) in a small open neighborhood of the origin. Namely, the polynomial endomorphism \( F \) induces a local isomorphism between the small open neighborhoods of the point \( P \) and its image \( Q \). This follows from the observation that if \( J(F; X) \in \mathbb{C}^* \), the polynomial endomorphism \( F : \mathbb{A}^n \to \mathbb{A}^n \) induces an isomorphism \( T_F : T_{\mathbb{A}^n, P} \to T_{\mathbb{A}^n, Q} \) of the tangent spaces or an isomorphism \( dF : \Omega^1_{\mathbb{A}^n, P} \to \Omega^1_{\mathbb{A}^n, Q} \) of the cotangent spaces. Hence
2.1. GENERALITIES

the condition $J(F; X) \in \mathbb{C}^*$ is equivalent to the condition that $F : \mathbb{A}^n \to \mathbb{A}^n$ is unramified (or étale since $F$ is the morphism between two smooth varieties).

Although $F$ is locally an isomorphism, one cannot conclude that the endomorphism $F$ gives, therefore, an isomorphism between $\mathbb{A}^n$ and its image $F(\mathbb{A}^n)$. In fact, there might be two or more points $P_1, \ldots, P_d$ of $\mathbb{A}^n$ mapped to the same image $Q$. Note that there is no subvariety mapped to a point by $F$, for otherwise a tangential direction along the subvariety is in the kernel of the tangential mapping $T_F$ above for a point $P$ of the subvariety. So, only finitely many points are mapped to $Q$. If one takes a point $Q$ to be sufficiently general then the number $d$ is equal to the degree of the field extension

$$[\mathbb{C}(x_1, \ldots, x_n) : \mathbb{C}(f_1, \ldots, f_n)].$$

A polynomial endomorphism $F : \mathbb{A}^n \to \mathbb{A}^n$ is called a local isomorphism. We also say that $F$ is unramified or étale at every point of $\mathbb{A}^n$. With this terminology, the Jacobian conjecture asserts equivalently the following:

**Jacobian Conjecture (the third form).** *A polynomial endomorphism $F$ of $\mathbb{A}^n$ is an isomorphism if it is unramified everywhere on $\mathbb{A}^n$.*

We can define an unramified morphism $\varphi : X \to Y$ between algebraic varieties. A morphism $\varphi$ is said to be unramified if, for any point $P \in X$ and $Q := \varphi(P) \in Y$, the morphism $\varphi$ induces a local isomorphism between small open neighborhoods of $P$ and $Q$ (the case $k = \mathbb{C}$), or if, for any point $P \in X$ and $Q := \varphi(P) \in Y$, $\varphi$ induces an isomorphism $\hat{\varphi}^* : \hat{\mathcal{O}}_{Y,Q} \to \hat{\mathcal{O}}_{X,P}$ of the local rings (if $k$ is arbitrary), where $\hat{\mathcal{O}}_{Y,Q}$ and $\hat{\mathcal{O}}_{X,P}$ are respectively the completions of the local rings $\mathcal{O}_{Y,Q}$ and $\mathcal{O}_{X,P}$. An unramified morphism
\( \varphi : X \to Y \) is a \textit{finite} morphism if, for any point \( Q \in \varphi(X) \), the inverse image \( \varphi^{-1}(Q) \) consists of \( d \) points for certain fixed number \( d \). Then it follows that the image \( \varphi(X) \) coincides with \( Y \) itself because a finite morphism is a closed mapping. We then say that the morphism \( \varphi : X \to Y \) is a \textit{finite covering} of degree \( d \). Such coverings are familiar in the study of the fundamental group. If one notes that \( \mathbb{C}^n \) is simply connected, it suffices to show that \( F \) is a finite morphism. Hence we can generalize the above form of the Jacobian conjecture and state the following more general conjecture.

**Generalized Jacobian Conjecture.** Let \( \varphi : X \to X \) be an unramified endomorphism. Then \( \varphi \) is a finite morphism.

The Jacobian conjecture looks, at the first glance, much of analytic nature, but it is more algebro-geometric or topological. The author feels that the conjecture is deeply related to a (yet non-existent) theory of branched coverings (ramified along the hidden part, i.e., the boundary at infinity) of algebraic varieties. In other words, the author believes that the study of unramified (or étale) endomorphisms of algebraic varieties is significant in order to solve the Jacobian conjecture. We shall list up below some of known results on the Jacobian conjecture which have been proved by far.

**Theorem 2.1.1** The following assertions hold for a polynomial mapping \( F : \mathbb{A}^n \to \mathbb{A}^n \) with \( J(F; X) \in \mathbb{C}^* \).

1. The image of the polynomial mapping \( F : \mathbb{A}^n \to \mathbb{A}^n \) is a Zariski open set and contains all codimension one points.

2. If \( \mathbb{C}(x_1, \ldots, x_n) = \mathbb{C}(f_1, \ldots, f_n) \) holds additionally, so does the conjec-
ture, where \( \mathbb{C}(x_1, \ldots, x_n) \) (resp. \( \mathbb{C}(f_1, \ldots, f_n) \)) is the quotient field of the polynomial ring \( \mathbb{C}[x_1, \ldots, x_n] \) (resp. \( \mathbb{C}[f_1, \ldots, f_n] \)).

(3) More generally, if \( \mathbb{C}(x_1, \ldots, x_n) \) is a Galois extension of \( \mathbb{C}(f_1, \ldots, f_n) \), then the conjecture holds.

**Proof.** A related result is taken up again in Lemma 2.2.1. We shall prove only the assertions (1) and (3). The assertion (2) follows from (3).

(1) If \( \varphi : X \to X \) is an endomorphism of an algebraic variety, we write it as \( \varphi : X_1 \to X_2 \) to distinguish the source from the target, where \( X_1 = X_2 = X \). Similarly, the induced endomorphism of the coordinate rings is written as \( \varphi^* : R_2 \to R_1 \), where \( R_1 = R_2 = \Gamma(X, \mathcal{O}_X) \). Let \( R = \mathbb{C}[x_1, \ldots, x_n] \).

Then \( \varphi^* \) is injective and \( \varphi^*(R_2) = \mathbb{C}[f_1, \ldots, f_n] \), which we denote by \( S \). Let \( p \) be a height 1 prime ideal of \( S \). Since \( S \) is factorial, \( p = (h) \) for \( h \in S \).

Since \( R^* = S^* = \mathbb{C}^* \), \( hR \) is a proper ideal of \( R \). Let \( \mathfrak{P}_1, \ldots, \mathfrak{P}_d \) be all the prime divisors of \( hR \). Then \( \mathfrak{P}_1, \ldots, \mathfrak{P}_d \) have height one. Let \( \mathfrak{P} = \mathfrak{P}_1 \) and let \( q = \mathfrak{P} \cap S \). Then \( q \supseteq p \). We have only to show that \( q = p \). Suppose the contrary. Consider \( S/q \to R/\mathfrak{P} \). The associated morphism \( \text{Spec } R/\mathfrak{P} \to \text{Spec } S/q \) is the restriction \( F_{\mathfrak{P}} \) of \( F \) onto the subvariety \( V(\mathfrak{P}) \).

If \( \text{ht } (q) \geq 2 \), then the general fiber of \( F_{\mathfrak{P}} \) has positive dimension. This contradicts the hypothesis that \( F \) is unramified. The openness of \( F(\mathbb{A}^n) \) follows from the fact that \( F \) is étale (= unramified + flat) and that a flat morphism is an open mapping.

(3) Set \( X = \mathbb{A}^n \) and consider a polynomial endomorphism \( F : X_1 \to X_2 \).

Let \( G \) be the Galois group of the field extension \( \mathbb{C}(x_1, \ldots, x_n)/\mathbb{C}(f_1, \ldots, f_n) \).

Let \( \tilde{X}_2 \) be the normalization of \( X_2 \) in the function field of \( X_1 \). Then \( \tilde{X}_2 \) is a normal algebraic variety containing \( X_1 \) as a Zariski open set by Zariski’s
Main Theorem and the normalization morphism $\tilde{F} : \tilde{X}_2 \to X_2$ is a finite morphism. Then the Galois group $G$ acts on $\tilde{X}_2$ in such a way that

$$\tilde{F} \circ \sigma = \tilde{F}, \quad \forall \sigma \in G.$$  

Furthermore, the quotient variety of $\tilde{X}_2$ under the $G$-action is isomorphic to $X_2$. By the assertion (1), the image $F(X_1)$ is a Zariski open set of $X_2$, which contains all codimension one points. Consider a codimension one point $Q$ of $X_2$ and its closure $W$ in $X_2$ which is a hypersurface of $\mathbb{A}^n$. Since $Q \in F(X_1)$, there exists a codimension one point $P$ of $X_1$ such that $Q = F(P)$. If we set $\tilde{F}^{-1}(Q) = \{P_1, \ldots, P_d\}$ with $P = P_1$, it is known that $G$ acts transitively on the set $\tilde{F}^{-1}(Q)$. Namely, for any point $P_i \in \tilde{F}^{-1}(Q)$, there exists $\sigma_i \in G$ such that $P_i = \sigma_i(P_1)$. Since $F$ induces a local isomorphism between the points $P_1$ and $Q$, $F$ induces a local isomorphism between the automorphic image $P_i$ of $P_1$ and $Q$. Hence the finite morphism $\tilde{F} : \tilde{X}_2 \to X_2$ is unramified at every codimension one point of $\tilde{X}_2$. Since $X_2$ is smooth, the finite morphism $\tilde{F} : \tilde{X}_2 \to X_2$ is then unramified by the purity of branch loci. Now since $\pi_1(X) = \{1\}$, it follows that $\tilde{F}$ is an isomorphism. Hence $F : X_1 \to X_2$ is also an isomorphism. Q.E.D.

When $n = 2$ we have more specified results. For more comprehensive treatments, see van den Essen [14].

**Theorem 2.1.2** Suppose $n = 2$. Let $\deg f_1 = m, \deg f_2 = n$, where $\deg$ stands for the total degree with respect to variables $x_1, x_2$. Then the following assertions hold.

(4) In one of the following cases, $\mathbb{C}[f_1, f_2] = \mathbb{C}[x_1, x_2]$ holds (Nakai-Baba [43]):
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(i) Either \( m \) or \( n \) is a prime number.

(ii) Either \( m = 4 \) or \( n = 4 \).

(iii) \( m = 2p > n \) and \( p \) is an odd prime.

(5) If either \( m \) or \( n \) is a product of at most two prime numbers, then \( \mathbb{C}[x_1, x_2] = \mathbb{C}[f_1, f_2] \) holds (Applegate-Onishi [1]).

(6) If \( \max(m, n) \leq 100 \), then \( \mathbb{C}[f_1, f_2] = \mathbb{C}[x_1, x_2] \) holds (Moh [37]).

(7) If \( \mathbb{C}[x_1, x_2] \nsubseteq \mathbb{C}[f_1, f_2] \), \( \gcd(\deg f_1, \deg f_2) \geq 16 \) (Heitmann [23]).

(8) With the affine plane \( \mathbb{A}^2 \) embedded naturally into the projective plane \( \mathbb{P}^2 \), we denote the line at infinity by \( \ell_\infty \). Denote by \( \overline{V(f_1)} \) the closure in \( \mathbb{P}^2 \) of the affine plane curve \( V(f_1) \) defined by \( f_1 = 0 \). Similarly, we define \( \overline{V(f_2)} \). Suppose that \( \overline{V(f_1)} \cap \ell_\infty \) as well as \( \overline{V(f_2)} \cap \ell_\infty \) consists of a single point for every pair \( (f_1, f_2) \) satisfying \( J(F; X) \in \mathbb{C}^* \). Then the Jacobian conjecture in the case \( n = 2 \) holds (Abhyankar [1]).

We shall explain below analytic, algebraic and geometric approaches. For a set of polynomials \( F = (f_1, \ldots, f_n) \), we define a derivation \( \delta_i \) of the polynomial ring \( \mathbb{C}[x_1, \ldots, x_n] \) by \( h \mapsto \delta_{f_i}(h) \), where we set

\[
\delta_{f_i}(h) = J\left( \frac{f_1, \ldots, h, \ldots, f_n}{x_1, \ldots, x_n} \right)
\]

by substituting \( h \) for \( f_i \). If a derivation \( \delta \) of \( \mathbb{C}[x_1, \ldots, x_n] \) satisfies the condition that \( \delta^N(h) = 0 \) (\( N \gg 0 \)) for any element \( h \in \mathbb{C}[x_1, \ldots, x_n] \), we say that \( \delta \) is locally nilpotent. If \( n = 2 \), the following result is known (cf. Bass-Connell-Wright [6]).
Lemma 2.1.3 With the above notations, the following assertion holds.

(9) Suppose that \( J(F; X) \in \mathbb{C}^* \) for \( F = (f_1, f_2) \). Then \( \mathbb{C}[x_1, x_2] = \mathbb{C}[f_1, f_2] \) if and only if \( \delta_1 \) or \( \delta_2 \) is a locally nilpotent derivation.

We shall consider a formulation of the Jacobian conjecture in terms of partial differentiations. We consider the following operators \( x_i \) and \( \partial_i \) which act from the left on the polynomial ring \( A_n := \mathbb{C}[x_1, \ldots, x_n] \) in \( n \) variables. They are defined by

\[
x_i : f \mapsto x_i f, \quad \partial_i : f \mapsto \frac{\partial f}{\partial x_i}
\]

The operator algebras over \( \mathbb{C} \) generated by the operators \( x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \)

\[
D_n := \mathbb{C}[x_1, \ldots, x_n, \partial_1, \ldots, \partial_n]
\]

is called the Weyl algebra. It is easy to verify the following commuting relations among those operators: D

\[
[\partial_i, x_j] = \delta_{ij}, [\partial_i, \partial_j] = [x_i, x_j] = 0 \ (\forall i, j),
\]

where the bracket product of the operators \( \rho \) and \( \sigma \) is defined by \([\rho, \sigma] = \rho \sigma - \sigma \rho\). An element of the Weyl algebra \( D_n \) is written as

\[
P = \sum_{\alpha} a_{\alpha} \partial^{\alpha}, \quad a_{\alpha} \in \mathbb{C}[x_1, \ldots, x_n],
\]

where we put \( \partial^{\alpha} = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} \) for a multi-index \( \alpha = (\alpha_1, \ldots, \alpha_n) \). Let \( |\alpha| = \alpha_1 + \cdots + \alpha_n \) and let

\[
D_n(v) = \{ \sum_{\alpha} a_{\alpha} \partial^{\alpha} ; |\alpha| \leq v \}.
\]
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Then \( \{ D_n(v) \}_{v \geq 0} \) is an increasing sequence of left \( A_n \)-modules of the Weyl algebra \( D_n \), which we call the \( \Gamma \)-filter. Let \( \varphi : D_n \rightarrow D_n \) be a ring endomorphism of the Weyl algebra \( D_n \), which is uniquely determined if the elements \( \varphi(x_i) \) and \( \varphi(\partial_j) \) are assigned so that the following relations are satisfied:

\[
[\varphi(\partial_i), \varphi(x_j)] = \delta_{ij}, [\varphi(\partial_i), \varphi(\partial_j)] = [\varphi(x_i), \varphi(x_j)] = 0.
\]

It is known that any ring endomorphism of \( D_n \) is injective. We then have the following conjecture due to Diximier.

**Diximier Conjecture.** Any ring endomorphism of the Weyl algebra \( D_n \) is surjective.

A weakened version of the Diximier conjecture is stated as follows:

**Weak Diximier Conjecture.** Any ring endomorphism \( \varphi \) of the Weyl algebra \( D_n \) is surjective provided it preserves the \( \Gamma \)-filter, i.e.,

\[
\varphi(D_n(v)) \subseteq D_n(v) \quad (\forall v \geq 0).
\]

There are mutual implications between the Diximier conjecture and the Jacobian conjecture.

**Lemma 2.1.4** The following assertions hold.

1. If the Diximier conjecture holds, so does the Jacobian conjecture (Vaserstein-Kac [6]).

2. The weak Diximier conjecture holds if and only if the Jacobian conjecture holds (van den Essen [13]).
We have observed the Jacobian conjecture over an algebraically closed field. But we can state the same conjecture over a non-closed field, e.g., over the real field $\mathbb{R}$. In the real case, there is a counterexample by Pinchuk [44].

**Theorem 2.1.5** We have the following result.

(12) There exists a pair of polynomials $(f_1, f_2)$ in two variables $x_1, x_2$ with real coefficients such that the polynomial mapping $F = (f_1, f_2) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a local isomorphism but not a homeomorphism. The Jacobian determinant is not invertible but does not have solutions in $\mathbb{R}^2$.

Given a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$, let $f = f_0 + f_{(1)} + \cdots + f_{(d)}$ be the homogeneous decomposition with respect to the total degree, where $f_{(i)}$ is the $i$-th homogeneous part. For a system of polynomials $F = (f_1, \ldots, f_n)$, set $F_i = (f_{1,(i)}, \ldots, f_{n,(i)})$, where $f_{j,(i)}$ is the $i$-th homogeneous part of $f_j$. Then we have a decomposition into the homogeneous parts $F = F_{(0)} + \cdots + F_{(d)}$. Set also $\deg F = \max(\deg f_1, \ldots, \deg f_n)$. We shall recall first a result of Wang [52].

**Theorem 2.1.6** Let $F = (f_1, \ldots, f_n) : \mathbb{A}^n \rightarrow \mathbb{A}^n$ be a polynomial endomorphism such that $J(F; X) \in \mathbb{C}^*$ and $\deg F \leq 2$. Then $F$ is an automorphism.

**Proof.** By a theorem of Ax [5], it suffices to show that $F : \mathbb{A}^n \rightarrow \mathbb{A}^n$ is injective. Suppose the contrary. Then there exist two points $P, Q$ of $\mathbb{C}^n$ such that $F(P) = F(Q)$. Define $G = (g_1, \ldots, g_n)$ by $G(X) = F(X + P) - F(P)$ and let $R := Q - P$, where $X + P$ or $Q - P$ is the point obtained by the coordinate-wise addition or subtraction. Then $G(O) = G(R) = 0$. Since $\deg F \leq 2$, we have $\deg G \leq 2$. Hence we can write $G = G_{(1)} + G_{(2)}$. Set
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\[ t_0 = 1/2. \text{ Then we can compute as follows.} \]

\[
0 = G(R) = G(1)(R) + G(2)(R)
\]
\[
= G(1)(R) + 2t_0G(2)(R)
\]
\[
= \frac{d}{dt} \left( tG(1)(R) + t^2G(2)(R) \right)_{t=t_0}
\]
\[
= \frac{d}{dt} \left( G(1)(tR) + G(2)(tR) \right)_{t=t_0}
\]
\[
= \frac{d}{dt} G(tR) \bigg|_{t=t_0} = J(G; X)(t_0 R) \cdot R.
\]

Since \( J(G; X) = J(F; X) \), we have \( J(G; X)(t_0 R) \neq 0 \) and hence \( R = 0 \). But this is absurd. Q.E.D.

For a given polynomial endomorphism \( F = (f_1, \ldots, f_n) : \mathbb{A}^n \to \mathbb{A}^n \), we can define a new set of polynomials by adding new variables \( x_{n+1}, \ldots, x_{n+m} \) and the associated polynomial mapping \( F^{[m]} = (f_1, \ldots, f_n, x_{n+1}, \ldots, x_{n+m}) : \mathbb{A}^{n+m} \to \mathbb{A}^{n+m} \). Then we have the following Reduction Theorem [6].

**Theorem 2.1.7** Given a polynomial endomorphism \( F : \mathbb{A}^n \to \mathbb{A}^n \), there exist an integer \( m > 0 \) and polynomial automorphisms \( G, H : \mathbb{A}^{n+m} \to \mathbb{A}^{n+m} \) such that the composite \( F' := G \circ F^{[m]} \circ H \) satisfies the condition \( \deg F' \leq 3 \).

If \( J(F; X) \in \mathbb{C}^* \) holds, so does the condition \( J(F'; X^{[m]}) \in \mathbb{C}^* \). Hence the Jacobian conjecture holds in general if it holds for polynomial endomorphisms satisfying \( \deg F \leq 3 \).

This result can be elaborated in the following fashion.

**Theorem 2.1.8** The Jacobian conjecture holds in general if it holds for a polynomial endomorphism \( F : \mathbb{A}^n \to \mathbb{A}^n \) of the special form \( F = X + K, K = (k_1, \ldots, k_n) \), where \( k_i \) is either 0 or a homogeneous polynomial of degree 3.
Drużkowski [11] made the following improvement of Theorem 2.1.8.

**Theorem 2.1.9** If the Jacobian conjecture holds for the $k_i$ in the following form
\[
k_i = \left(\sum_{j=1}^{n} a_{ji}x_j\right)^3, \quad a_{ji} \in \mathbb{C}
\]
then it holds in general.

We say that a polynomial mapping $F = X + K$ is of Drużkowski type if the homogeneous part $K$ of degree 3 is the third power of a linear form as in the above theorem. Drużkowski [12] proved the following result.

**Lemma 2.1.10** A polynomial endomorphism $F = X + K$ of Drużkowski type is an automorphism if the coefficient matrix $A = (a_{ji})$ of the linear form in $K$ satisfies either $\text{rank } A \leq 2$ or $n - \text{rank } A \leq 2$.

We shall restrict ourselves to the case of two variables in the following discussions. We employ the notations $f, g$ preferably to $f_1, f_2$ and $x, y$ to $x_1, x_2$. We set $\deg f = m, \deg g = n$ and let $f_m, g_n$ be the top homogeneous parts of $f, g$. If $m + n > 2$, the hypothesis $J(F; X) \in \mathbb{C}^*$ implies
\[
J\left(\frac{f_m}{x, y}, \frac{g_n}{x, y}\right) = 0,
\]
where $F = (f, g)$ and $X = (x, y)$. If $d = \gcd(m, n)$, this implies that there exists a homogeneous polynomial $h$ of degree $d$ satisfying $f_m \sim h^{m/d}$ and $g_n \sim h^{n/d}$, where $p \sim q$ signifies $p/q \in \mathbb{C}^*$. Hence if the hypothesis $J(F; X) \in \mathbb{C}^*$ implies either $m \mid n$ or $n \mid m$, it will follow that $g_n \sim f_m^{n/m}$ or $f_m \sim g_n^{m/n}$. Suppose, for example, $g_n \sim f_m^{n/m}$. Then we can choose $c \in \mathbb{C}$ in such a way
that with $f_1 = f, g_1 = g - cf^{n/m}$, we have

$$J \left( \frac{f_1, g_1}{x, y} \right) = J \left( \frac{f, g}{x, y} \right) \in \mathbb{C}^*$$

and $\max(\deg f_1, \deg g_1) < \max(\deg f, \deg g)$. Namely the proof of the Jacobian conjecture will proceed by induction on $\max(\deg f, \deg g)$.

Note that the homogeneous polynomials $f_m$ and $g_n$ are decomposed into the products of linear forms

$$f_m \sim \prod_{i=1}^{m} (a_i x + b_i y), \quad g_n \sim \prod_{j=1}^{n} (c_j x + d_j y).$$

Now embed the affine plane $\mathbb{A}^2$ into the projective plane $\mathbb{P}^2$ by $(x, y) \mapsto (x, y, 1)$. Then the set of points $I(f) := \{ (b_i, -a_i, 0) ; 1 \leq i \leq m \}$ is the set of points where the closure of the affine curve $V(f)$ meets the line at infinity $\ell_\infty := \{ z = 0 \}$ and the set of points $I(g) := \{ (d_j, -c_j, 0) ; 1 \leq j \leq n \}$ is the set $V(g) \cap \ell_\infty$, where $V(g)$ is the closure of the affine curve $V(g)$. So, the above result on the existence of a homogeneous polynomial $h$ of degree $d$ implies that the hypothesis $J(F; X) \in \mathbb{C}^*$ yields the coincidence of two sets of intersection $I(f)$ and $I(g)$.

In the above arguments, we used the weights $\deg x = \deg y = 1$. But we may use different weights

$$\deg_\omega x = \omega_1, \quad \deg_\omega y = \omega_2,$$

where $\omega = (\omega_1, \omega_2)$ is a pair of integers, and define the degree of a monomial $cx^\alpha y^\beta$ ($c \neq 0$) by

$$\deg_\omega cx^\alpha y^\beta = \alpha \omega_1 + \beta \omega_2.$$

With this degree, we can consider the $\omega$-degree of polynomials, $\omega$-homogeneous polynomials and the decompositions of polynomials into the $\omega$-homogeneous
CHAPTER 2. JACOBIAN CONJECTURE

parts. We denote the top $\omega$-homogeneous parts of $f$ and $g$ by $f_\omega^+$ and $g_\omega^+$, respectively. If $f$ and $g$ are not $\omega$-homogeneous but satisfy the condition $J(F; X) \in \mathbb{C}^*$, then we know

$$J\left(\frac{f_\omega^+}{x}, \frac{g_\omega^+}{y}\right) = 0.$$ 

If we set $m = \deg_\omega f, n = \deg_\omega g, d = \gcd(m, n)$ then there exists, as in the previous case, a $\omega$-homogeneous polynomial $h$ such that

$$f_\omega^+ \sim h^{m/d}, \quad g_\omega^+ \sim h^{n/d}$$

(see Abhyankar [1]). If we change the value of $\omega$, the combinations of monomials appearing in $f_\omega^+$ and $g_\omega^+$ will change accordingly. Hence we can get more informations on the polynomials $f$ and $g$ satisfying the hypothesis $J(F; X) \in \mathbb{C}^*$. The following results of Magnus [27] is shown by using the idea of $\omega$-degree (see Nakai-Baba [43]).

**Lemma 2.1.11** The following assertion holds.

(13) If $J(F; X) \in \mathbb{C}^*$ and $\min(m, n) > 1$, then $\gcd(m, n) > 1$. Hence if either $m$ or $n$ is a prime number, then the Jacobian conjecture holds.

Given a polynomial $f = \sum_{i,j} c_{ij}x^iy^j$, the set $S(f) = \{(i, j) : c_{ij} \neq 0\}$ is called the support of $f$. In the first quadrant of the coordinate plane, consider the smallest convex polygon containing the origin $(0, 0)$ and the points of $S(f)$. We call it the Newton polygon of $f$ and denote it by $N(f)$. The following result is due to Abhyankar [1].

**Lemma 2.1.12** The Jacobian conjecture is equivalent to the following condition: For any pair of polynomials $F = (f, g)$ satisfying the condition
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\[ J(F; X) \in \mathbb{C}^*, \] the newton polygon \( N(f) \) (or \( N(g) \)) is a triangle with three points \((0, q), (0, 0), (p, 0)\) as summits, where \( p, q \) are non-negative integers.

### 2.2 Generalized Jacobian Conjecture

We shall consider the generalized Jacobian conjecture in a bit elaborated form:

**Generalized Jacobian Conjecture.** Let \( X \) be a smooth algebraic variety defined over an algebraically closed field \( k \) of characteristic zero and let \( \varphi : X \to X \) be an étale endomorphism. Then \( \varphi \) is an étale finite morphism.

We shall begin with the following result, of which the first assertion is already given in Theorem 2.1.1.

**Lemma 2.2.1** Let \( X \) and \( \varphi \) be the same as in the above conjecture. Then the following assertions hold:

1. If \( \varphi \) is injective or if \( \varphi \) is birational, then \( \varphi \) is an automorphism.

2. If the logarithmic Kodaira dimension \( \kappa(X) = \dim X \), i.e., \( X \) is of general type, then \( \varphi \) is an automorphism. We refer to [34] for the definition of logarithmic Kodaira dimension and relevant results.

3. If \( X \) is complete and has nonzero Euler number \( e(X) \) then \( \varphi \) is an automorphism.

**Proof.** (1) If \( \varphi \) is injective, the assertion follows from Ax’s theorem [5, 7]. If \( \varphi \) is birational, Zariski’s Main Theorem implies that \( \varphi \) is injective because \( X \) is smooth and \( \varphi \) is quasi-finite. Hence follows the assertion.
(2) The assertion follows from [24].

(3) Let \( d := \deg \varphi \). Since \( \varphi \) is a finite morphism when \( X \) is complete, we have \( e(X) = de(X) \). If \( e(X) \neq 0 \) then \( d = 1 \). Namely \( \varphi \) is birational. Hence \( \varphi \) is an automorphism by (1). 

\[ \text{Q.E.D.} \]

**Remark.** (1) If \( X \) has logarithmic Kodaira dimension \( \kappa(X) \geq 0 \) and if \( \varphi : X \to X \) is a dominant morphism, then \( \varphi \) is an étale endomorphism (cf. [24 Th. 2]).

(2) If \( X \) is a commutative group variety, then the *multiplication by m* endomorphism for a positive integer \( m > 1 \) is a finite étale morphism, and it is an automorphism if and only if \( X \) is a vector group. Furthermore, \( \kappa(X) = 0 \) if and only if \( X \) has no unipotent subgroups. It is well-known that \( X \) has no unipotent groups if and only if \( X \) is an extension of an abelian variety by an algebraic torus.

If \((X, \varphi)\) is a pair of a smooth algebraic variety \( X \) and a non-finite étale endomorphism \( \varphi \) of \( X \), then we say that \((X, \varphi)\) is a counterexample to the generalized Jacobian Conjecture (GJC in short). It is clear that if \((X, \varphi)\) is a counterexample to GJC then so is \((X \times Y, \varphi \times \text{id}_Y)\) for any smooth variety \( Y \).

The appearance of a finite étale endomorphism for an abelian variety or an algebraic torus is related to the group structure on \( X \) which comes from a lattice structure on the universal covering space. Thus we are tempted to look for a counterexample to GJC which is not related to a group structure. The first of such examples was constructed in [33].

**Example 2.2.2** Let \( C \) be a nonsingular cubic curve in \( \mathbb{P}^2 \) and let \( X = \)
\[ \mathbb{P}^2 - C. \] Then the following assertions hold:

1. \( \pi(X) = 0 \) and \( \text{Pic}(X) \cong \mathbb{Z}/3\mathbb{Z}. \)

2. There exists a surjective, non-finite, étale endomorphism \( \varphi : X \to X \) of degree 3.

3. Let \( \tilde{X} \) be the normalization of the lower \( X \) in the function field of the upper \( X \). Then \( \tilde{X} \) is a smooth affine surface containing \( X \) as a Zariski open set, and \( \tilde{X} - X \) is a disjoint union of six affine lines.

**Proof.** We shall prove the assertion (2). Let \( \pi : W \to \mathbb{P}^2 \) be a triple covering which ramifies totally over the curve \( C \). Then \( W \) is a smooth projective surface with

\[ K_W \sim \pi^*(K_{\mathbb{P}^2} + 2H), \]

where \( H \) is a hyperplane in \( \mathbb{P}^2 \). Then \( K_W \sim -\pi^*(H) \), hence \( -K_W \) is ample and \( (K_W^2) = 3 \). So, \( W \) is a del Pezzo surface of degree 3. It is well-known that a del Pezzo surface of degree 3 is obtained from \( \mathbb{P}^2 \) by blowing up six points in general position and that there are 27 lines contained in the surface. Conversely, if one contracts mutually disjoint six lines on \( W \), then one obtains a birational morphism from \( W \) to \( \mathbb{P}^2 \). The 27 lines on \( W \) are obtained as follows. The curve \( C \) has 9 flexes \( P_1, \ldots, P_9 \). Let \( \ell_i \) (1 \( \leq \) i \( \leq \) 9) be the tangent line of \( C \) at the point \( P_i \). Let \( \tilde{C} \) be the inverse image of \( C \) on \( W \) and let \( \tilde{P}_i \) be a point on \( \tilde{C} \) lying over \( P_i \). Then \( \pi^{-1}(\ell_i) \) consists of three lines \( L_i^{(j)} \) (j = 1, 2, 3) such that \( L_i^{(1)}, L_i^{(2)}, L_i^{(3)} \) and \( \tilde{C} \) meet each other transversally only in the point \( \tilde{P}_i \). Then \( \{L_i^{(j)} \mid 1 \leq i \leq 9, 1 \leq j \leq 3\} \) are the 27 lines on \( W \). Choose six disjoint lines \( L_1, \ldots, L_6 \) among them and contract them to
obtain a birational morphism $\rho : W \to \mathbb{P}^2$. Let $\tilde{C} = \rho(C)$. Then $\mathbb{P}^2 - \tilde{C}$ is isomorphic to $X$ and $\rho$ induces an isomorphism

$$\rho' : W - (\tilde{C} + L_1 + \cdots + L_6) \xrightarrow{\sim} \mathbb{P}^2 - \tilde{C}.$$ 

Let $\varphi$ be the composite of $\rho'^{-1}$ and $\pi|_{W-(\tilde{C}+L_1+\cdots+L_6)}$. Then $\varphi : X \to X$ is a required étale endomorphism. Let $\tilde{X} = W - \tilde{C}$. Then $\rho|_{\tilde{X}} : \tilde{X} \to X$ is the normalization morphism of $X$ in the function field of $W$. Hence $\tilde{X} - X$ consists of six affine lines.

This example with due specialization gives rise to a new counterexample with a $\mathbb{Q}$-homology plane $^1$.

**Example 2.2.3** Let $C := \{X^3 + Y^3 + Z^3 = 0\}$ be a cubic curve in $\mathbb{P}^2$ and let $S := \mathbb{P}^2 - C$. Let $T := \{X^3 + Y^3 + Z^3 + W^3 = 0\}$ be a cubic hypersurface in $\mathbb{P}^3$ and let $\pi : \mathbb{P}^3 \to \mathbb{P}^2$ be the projection $\pi([x, y, z, w]) = [x, y, z]$. Denote by $T_0$ the curve $\{W = 0\}$ on $T$. Then the following assertions hold:

1. $\pi : T - T_0 \to S$ is the universal covering morphism.

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1 A closed immersion of an elliptic curve $C$ into $\mathbb{P}^2$ is given by $\Phi_{[3P]} : C \hookrightarrow \mathbb{P}^2$ with a closed point $P$. If $f : C \to C'$ is an isomorphism of elliptic curves with $P' := f(P)$, then there exists an automorphism $g : \mathbb{P}^2 \to \mathbb{P}^2$ such that $g \cdot \Phi_{[3P]} = \Phi_{[3P']} \cdot f$. Suppose that $C$ (resp. $C'$) is written as $Y^2Z = 4X^3 - g_2XZ^2 - g_3Z^3$ (resp. $Y'^2Z' = 4X'^3 - g'_2X'Z'^2 - g'_3Z'^3$) with respect to a system of homogeneous coordinates $(X, Y, Z)$ (resp. $(X', Y', Z')$) such that the point $P$ (resp. $P'$) is given as $(0, 1, 0)$. Then the $j$-invariant of $C$ (resp. $C'$) is given as $j = g_2^3(g_2^3 - 27g_3^3)^{-1}$ (resp. $j' = g'_2^3(g'_2^3 - 27g'_3^3)^{-1}$). Since $j = j'$, we have $(g/g_2)^3 = (g_3/g'_3)^2$. Namely, $g_2 = cg'_2$ and $g_3 = dg'_3$ with $c, d \in k^*$. By a change of coordinates $(X, Y, Z) \mapsto (cX, \sqrt[3]{c^3/d}Y, dZ)$, we may assume $g_2 = g'_2$ and $g_3 = g'_3$. Then the complements $\mathbb{P}^2 - C$ and $\mathbb{P}^2 - C'$ are isomorphic to each other.
(2) $T$ is a del Pezzo surface with $K_T = O(-T_0)$. There exist six points in general position on $\mathbb{P}^2$, say $P_1, \ldots, P_6$, such that $T$ is obtained from $\mathbb{P}^2$ by blowing up these points. Let $E_i$ be the exceptional curve lying over $P_i$. Then each $E_i$ meets $T_0$ transversally in exactly one point and hence each $E_i$ is a straight line in $\mathbb{P}^3$. Let $\tau : T \to \mathbb{P}^3$ be the blowing-up morphism and let $C' = \tau(T_0)$. We have an isomorphism

$$T - (T_0 \cup E_1 \cup \cdots \cup E_6) \to \mathbb{P}^2 - C'.$$

Since $T_0 \cong C'$, there is an isomorphism $C \cong C'$. Hence there exists an automorphism of $\mathbb{P}^2$ which maps $C$ onto $C'$. Clearly, the morphism $\pi : T - (T_0 \cup E_1 \cup \cdots \cup E_6) \to S$ is an étale but non-finite morphism. Thus we obtain a non-finite étale endomorphism $\tilde{f} := \pi \circ \tau^{-1} : S \to S$.

(3) Consider the following action of $\mathbb{Z}/3\mathbb{Z} = \langle \sigma \rangle$ on $T$ induced by an action on $\mathbb{P}^3$,

$$\sigma([x,y,z,w]) = [\theta x, \theta^2 y, \theta z, w].$$

Here $\theta$ is a primitive cube root of unity. This action has no fixed points on $T - T_0$ and it commutes with the projection $\pi : \mathbb{P}^3 \to \mathbb{P}^2$ and the covering transformation $h : T - T_0 \to T - T_0$, where

$$\sigma([x,y,z]) = [\theta x, \theta^2 y, \theta z] \quad \text{and} \quad h([x,y,z,w]) = [x,y,z,\theta w].$$

(4) There exist six disjoint lines $F_1, \ldots, F_6$ on $T$ such that $\sigma$ keeps this set of lines stable. We can use these six skew lines on $T$ to get a morphism $g : T \to \mathbb{P}^2$ such that these six lines are the exceptional curves for the blowing up morphism $g$. Then $\sigma$ acts fixed-point freely on $T - (T_0 \cup E_1 \cup \cdots \cup E_6)$ and commutes with the projection $\pi : T - T_0 \to S$. 
Thus we get an induced non-finite étale endomorphism from $S/(\sigma)$ to itself.

(5) $V := S/(\sigma)$ is an NC-minimal $\mathbb{Q}$-homology plane with $\kappa(V) = 0$.

**Proof.** We shall prove the assertions (4) and (5). The rest is rather straightforward to verify.

(4) Let $F_1 := \{X + Y = 0 = Z + W\}$. Then $F_2 := \sigma(F_1) = \{\theta X + Y = 0 = Z + \theta W\}$ and $F_3 := \sigma^2(F_1) = \{X + \theta Y = 0 = Z + \theta^2 W\}$. It is easy to see that these three lines are mutually disjoint. Let $F_4 := \{X + \theta W = 0 = Y + Z\}$. Then $F_5 := \sigma(F_4) = \{\theta X + W = 0 = Y + \theta Z\}$ and $F_6 := \sigma^2(F_4) = \{X + W = 0 = \theta Y + Z\}$. Then $F_4, F_5, F_6$ are mutually disjoint and $F_1$ is disjoint from each of them. It follows that the six lines $F_i$ are mutually disjoint and $\sigma$ keeps the set of these lines stable.

(5) Recall from [17, §6] that a pair $(Y, D)$ of a smooth projective surface $Y$ and a connected simple normal crossing divisor $D$ on $Y$ with $\kappa(Y - D) \geq 0$ is said to be **NC-minimal** if in the Zariski decomposition $K_Y + D = P + N$, the negative part $N = Bk^*(D)$. Fujita has proved that if $(Y, D)$ is not NC-minimal then $Y$ contains a $(-1)$-curve $L$ which meets $D$ transversally in only one point which is a point on a maximal rational twig of $D$. Further, $\kappa(Y - D - L) = \kappa(Y - D)$ (cf. [17] Lemma 6.20). Clearly, since $\kappa(S) = 0$ and $\sigma$ acts fixed-point freely on $S$ we see that $\kappa(V) = 0$ by Iitaka’s result. Since $e(S) = 3$, we get $e(V) = 1$. Since $\pi_1(S) \cong \mathbb{Z}/(3)$, we see that $b_1(V) = 0$. Hence $b_2(V) = 0$, which shows that $V$ is a $\mathbb{Q}$-homology plane. If $V$ is not NC-minimal, then by Fujita’s lemma $V$ contains a curve $L \cong \mathbb{A}^1$. Then for some integer $\ell > 0$ the divisor $\ell L$ is the divisor of a regular function $\varphi$ on $V$. The
morphism \( \varphi : V \to \mathbb{A}^1 \) is a \( \mathbb{A}^1 \)-fibration because \( \pi(V - L) = 0 \) by Fujita’s lemma and using Kawamata’s inequality \([34]\), \( \pi(V - L) \geq \pi(F) + \pi(\mathbb{A}^1_*) \) to the morphism \( V - L \to \mathbb{A}^1_* \), where \( F \) is a general fiber. The pull-back of this morphism to \( S \) gives an \( \mathbb{A}^1_* \)-fibration on \( S \). We will show that \( S \) does not admit any \( \mathbb{A}^1_* \)-fibration. Suppose that \( \tau : S \to B \) is an \( \mathbb{A}^1_* \)-fibration, where \( B \) is a smooth curve. Since there is no non-constant morphism from \( \mathbb{P}^2 \) to a curve, there are one or two points of indeterminacy in \( \mathbb{P}^2 \). After resolving the indeterminacies we get a \( \mathbb{P}^1 \)-fibration on the blown-up surface such that the proper transform of the curve \( C \) is contained in a singular fiber of the \( \mathbb{P}^1 \)-fibration. But it is well-known that every irreducible component of a \( \mathbb{P}^1 \)-fibration on a smooth projective surface is a (smooth) rational curve. This contradiction shows that \( V \) is NC-minimal. Q.E.D.

In the following two examples the endomorphisms \( \varphi \) are finite étale endomorphisms, but its appearance is less related to the group structure of a surface which has logarithmic Kodaira dimension zero. We shall recall some definitions on \( \mathbb{A}^1_* \)-fibrations which are necessary in the subsequent arguments. See \([34]\) for more explanations and relevant results.

Let \( X \) be a smooth affine surface and let \( \pi : X \to B \) be an \( \mathbb{A}^1_* \)-fibration. Here we note that \( X \) has always an \( \mathbb{A}^1_* \)-fibration provided \( \pi(X) = 1 \) and that there are no other \( \mathbb{A}^1_* \)-fibrations. There is a smooth normal compactification \( V \) of \( X \) such that \( \pi \) extends to a \( \mathbb{P}^1 \)-fibration \( p : V \to \overline{B} \). Let \( D = V - X \), which is a reduced effective divisor with simple normal crossings. Since the general fibers of \( \pi \) are isomorphic to \( \mathbb{A}^1_* \), \( D \) contains components which are transverse to the fibration \( p \). There are two cases to consider. The first case is that there are two cross-sections \( H_1, H_2 \) of \( p \), and the second case is that \( D \)
has a unique component $H$ which is transverse to $p$ and is a 2-section. The $\mathbb{A}^1$-fibration $\pi$ is called *untwisted* (resp. *twisted*) in the first (resp. second) case.

**Example 2.2.4** Let $\pi : X \to B$ be an untwisted $\mathbb{A}^1$-fibration such that every fiber is isomorphic to $\mathbb{A}^1_*$ if taken with the reduced structure. Assume that $X$ has a compactification $V$ such that $V$ is normal and $V - X$ consists of one point and a connected divisor. Assume furthermore that $B$ is an elliptic curve and that $\pi$ has no multiple fibers. Then there exists a nontrivial étale endomorphism $\psi : X \to X$ if and only if there exist an endomorphism $\beta$ of $B$, an invertible sheaf $L$ on $B$ of positive degree and a positive integer $d > 1$ satisfying the condition: $L^\otimes d \cong \beta^*(L)$. We have necessarily $d = \deg \beta$ and $\deg \varphi = d^2$. Furthermore, $\varphi$ is a finite morphism.

**Proof.** There exists an invertible sheaf $L$ such that $X = \mathbb{P}(\mathcal{O}_B \oplus L) - (S_0 \cup S_1)$, where $S_0, S_1$ are the cross-sections of the $\mathbb{P}^1$-bundle. The invertible sheaf $L$ is determined uniquely up to isomorphisms by the $\mathbb{A}^1$-bundle $X$. We may assume that $(S_0^2) < 0$. Then $S_0$ (resp. $S_1$) corresponds to the projection from $\mathcal{O}_B \oplus L$ to $\mathcal{O}_B$ (resp. $L$). Suppose there exists an étale endomorphism $\varphi : X \to X$. Then $\pi \circ \varphi = \beta \circ \pi$ for an endomorphism $\beta$ of $B$. Let now $Y$ be a fiber product $X \times_B (B, \beta)$ and $\pi_Y : Y \to B$ be the induced $\mathbb{A}^1$-fibration. Then $Y = \mathbb{P}(\mathcal{O}_B \oplus \beta^*(L)) - (S_{0,Y} \cup S_{1,Y})$, where $S_{0,Y} = S_0 \times_B (B, \beta)$ and $S_{1,Y} = S_1 \times_B (B, \beta)$. Let $\beta_Y : Y \to X$ be the base change of $\beta$. Then $\varphi$ splits as a composite $\varphi = \beta_Y \circ \psi$, where $\psi : X \to Y$ is an étale $B$-morphism. Then $\psi$ extends to a $B$-morphism of $\mathbb{P}^1$-bundles $\overline{\psi} : \mathbb{P}(\mathcal{O}_B \oplus L) \to \mathbb{P}(\mathcal{O}_B \oplus \beta^*(L))$ such that $\overline{\psi}^{-1}(S_{0,Y}) = S_0$ and $\overline{\psi}^{-1}(S_{1,Y}) = S_1$. Take an affine
open covering \( U = \{ U_i \}_{i \in I} \) of \( B \) such that \( \mathcal{L} \big|_{U_i} = \mathcal{O}_{U_i} e_i \) for every \( i \in I \).

Write \( \mathcal{O}_B = \mathcal{O}_{B e} \). Then \( \pi^{-1}(U_i) = \text{Spec} \ A_i[z_i, z_i^{-1}] \), where \( z_i = e_i/e \) and \( A_i = \Gamma(U_i, \mathcal{O}_B) \). Let \( \{ a_{ij} \} \) be the transition functions of \( \mathcal{L} \) with respect to \( U \). Then \( \psi \big|_{U_i} : \pi^{-1}(U_i) \to \pi^{-1}(U_i) \) is given by \( \beta^*(z_i) = \lambda_i z_i^d \) with \( \lambda_i \in A_i^* \) and \( d := \deg \psi \). Then it is easy to deduce a relation \( \beta^*(a_{ji}) = a_{ji}^d \lambda_j \lambda_i^{-1} \) for \( i, j \in I \). This implies that \( \mathcal{L} \otimes d \cong \beta^*(\mathcal{L}) \). Since \( \deg \mathcal{L} \otimes d = d \deg \mathcal{L} \) and \( \deg \beta^*(\mathcal{L}) = \deg \beta \deg \mathcal{L} \), we have necessarily \( d = \deg \beta \).

Conversely, if we are given \( \beta, \mathcal{L}, d \) as above, we can construct an endomorphism \( \varphi : X \to X \) as a composite of \( \beta_Y \) and \( \psi \), where \( \psi \) is a \( d \)-th power morphism on the generic fibers. This \( d \)-th power morphism is extended to a \( B \)-morphism \( \psi : X \to Y \) because of the condition \( \mathcal{L} \otimes d \cong \beta^*(\mathcal{L}) \). Since \( \varphi = \beta_Y \circ \psi \), \( \deg \beta_Y = \deg \beta \) and \( \deg \psi = d \), we have \( \deg \varphi = d^2 \). Q.E.D.

Taking the quotients of \( X \) and \( B \) in Example 2.2.4 with respect to suitable involutions, we can construct a new example. More precisely, we have the following example (cf. [28]).

**Example 2.2.5** Let \( n \) be a positive integer \( > 1 \) such that \( d := (n^2 - n)/2 \) is odd. Let \( C \) be an elliptic curve which we view as an abelian variety by fixing one point as a point of origin. Let \( \mathcal{L} \) be an invertible sheaf on \( C \) such that \( \deg \mathcal{L} > 0 \) and \( n_C^*(\mathcal{L}) \cong \mathcal{L} \otimes n^2 \). Let \( \mathcal{M} := \mathcal{L} \otimes (n^2 - n)/2 \). Let \( (-1)_C : C \to C \) be the involution \( p \mapsto -p \). Then the following assertions hold:

(1) \( n_C^*(\mathcal{M}) \cong \mathcal{M} \otimes n^2 \) and \( (-1)_C^*(\mathcal{M}) \cong \mathcal{M} \).

(2) Let \( Y = \mathbb{P}(\mathcal{O}_C \oplus \mathcal{M}) - (S_0 \cup S_1) \), where \( S_0 \) (resp. \( S_1 \)) is the cross-section corresponding to the projection \( \mathcal{O}_C \oplus \mathcal{M} \to \mathcal{O}_C \) (resp. \( \mathcal{O}_C \oplus \mathcal{M} \to \mathcal{M} \)).
Let \( q : Y \to C \) be the canonical projection of \( \mathbb{A}_1^* \)-bundle. Then there exists an étale endomorphism \( \psi : Y \to Y \) such that \( n_C \circ q = q \circ \psi \).

(3) Let \( \{U_i\}_{i \in I} \) be an open covering of \( C \) such that \( \mathcal{M} |_{U_i} = \mathcal{O}_{U_i}[e_i, e_i^{-1}] \) for every \( i \in I \). Then \( q^{-1}(U_i) = \text{Spec} \mathcal{O}_{U_i}[e_i, e_i^{-1}] \). Define an involution \( \iota : Y \to Y \) locally by \( \iota^*(e_i) = -e_i \) for \( i \in I \). Then the involution \( \iota \) is well-defined and satisfies the conditions \( (-1)_C \circ q = q \circ \iota \) and \( \psi \circ \iota = \iota \circ \psi \).

(4) Let \( B := C/((-1)_C) \) and let \( X := Y/\langle \iota \rangle \). Then the projection \( q : Y \to C \) induces an \( \mathbb{A}_1^* \)-fibration \( \pi : X \to B \), where \( B \cong \mathbb{P}^1 \) and \( \pi \) has four multiple fibers of multiplicity two. Furthermore, the multiplication by \( n \) endomorphism \( n_C \) of \( C \) induces an endomorphism \( \beta : B \to B \).

(5) The étale endomorphism \( \psi : Y \to Y \) induces an étale endomorphism \( \varphi : X \to X \) such that \( \pi \circ \varphi = \beta \circ \pi \). The endomorphism \( \varphi \) is not an automorphism.

Proof. (1) By [39, Corollary 3, page 59], we have for any invertible sheaf \( \mathcal{L} \)
\[
n^*_C(\mathcal{L}) \cong \mathcal{L}^{(\frac{n^2+n}{2})} \otimes (-1)^*_C \mathcal{L}^{(\frac{n^2-n}{2})}.
\]
Since \( n^*_C(\mathcal{L}) \cong \mathcal{L}^{\otimes n^2} \) we have
\[
(-1)^*_C \mathcal{L}^{(\frac{n^2-n}{2})} \cong \mathcal{L}^{(\frac{n^2-n}{2})}.
\]
Hence we have \( n^*_C(\mathcal{M}) \cong \mathcal{M}^{\otimes n^2} \) and \( (-1)^*_C \mathcal{M} \cong \mathcal{M} \).

(2) See Lemmas 5.3 and 5.4 in [28].

(3) Clear by the definition.

(4) The fixed points in \( C \) by \( (-1)_C \) are the 2-torsion points, and there are four of them. The quotient morphism \( C \to B \) ramifies at these four points,
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and $B$ is therefore isomorphic to $\mathbb{P}^1$. Since $(-1)_C \circ n_C = n_C \circ (-1)_C$, $n_C$ induces an endomorphism $\beta : B \to B$.

(5) The involution $\iota : Y \to Y$ has no fixed points by the definition. Hence the quotient morphism $Y \to X$ is a finite étale morphism. This implies that the fibers of $q : Y \to C$ lying over the four 2-torsion points of $C$ produce the four multiple fibers of $\pi : X \to B$ with multiplicity 2. The rest of the assertion is now readily ascertained. Q.E.D.

The following two examples show that there do exist counterexamples to GJC in the cases where the logarithmic Kodaira dimension is not necessarily zero.

Example 2.2.6 Let $C$ be a smooth complete curve of genus $g$ and let $T = \text{Spec } \mathbb{C}[\xi, \xi^{-1}]$ be a one-dimensional algebraic torus. Let $Q_1$ and $Q_2$ be the points of $T$ defined by $\xi = 1$ and $\xi = -1$, respectively. Let $P_1$ and $P_2$ be two distinct points of $C$. Let $Y = C \times T$, let $C_i = C \times \{Q_i\}$ and let $T_i = \{P_i\} \times T$ for $i = 1, 2$. Let $\sigma : Z \to Y$ be the blowing-up of the points $(P_1, Q_1)$ and $(P_2, Q_2)$ and let $E_i = \sigma^{-1}((P_i, Q_i))$ ($i = 1, 2$). Let $X = Z - \sigma^*T_1 - \sigma^*T_2$, where $\sigma^*T_i$ is the proper transform of $T_i$ by $\sigma$. Let $q : X \to T$ be a morphism induced by the projection $C \times T \to T$.

Let $g : T \to T$ be the endomorphism defined by $g^*(\xi) = \xi^n$ for odd $n > 2$, let $\tilde{X} = X \times_T (T, g)$ and let $\tilde{q} : \tilde{X} \to T$ be the canonical projection. Then $\tilde{q}$ has $2n$ singular fibers $L_{1j} = \tilde{q}^*(Q_{1j})$ and $L_{2j} = \tilde{q}^*(Q_{2j})$ (1 $\leq$ $j$ $\leq$ $n$), where $Q_{1j}$ and $Q_{2j}$ are defined respectively by $\xi = \omega^{j-1}$ and $\xi = -\omega^{j-1}$ with $\omega$ being a primitive $n$-th root of the unity. The fibers $L_{1j}$ and $L_{2j}$ have the same forms as the fibers $L_1 := q^*(Q_1)$ and $L_2 := q^*(Q_2)$, respectively. Write $L_{1j} = M_{1j} + \Delta_{1j}$ and $L_{2j} = M_{2j} + \Delta_{2j}$, where $\Delta_{1j} \cong \Delta_{2j} \cong \mathbb{A}^1$ and $M_{1j}$ and
$M_{2j}$ are considered as open sets of $C$. Let $X_1 := \tilde{X} - \sum_{i=1}^{2} \sum_{j=2}^{n} \Delta_{ij}$. Then the following assertions hold:

1. $X_1$ is isomorphic to $X$ and the composite $\varphi$ of the open immersion $X_1 \hookrightarrow \tilde{X}$ and the canonical projection $\tilde{X} \to X$ is a surjective non-finite étale endomorphism of degree $n$.

2. We have $\kappa(X_1) = 1$ if $g > 0$ and $\kappa(X_1) = 0$ if $g = 0$. If $g = 0$ then $X \cong F_0 - (D_1 + D_2)$, where $F_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $D_1, D_2$ are the curves of type $(1,1)$.

3. In the case $g = 0$, $X$ is isomorphic to $\text{Spec} \, k[x, y, z, z^{-1}]/(xy = z^2 - 1)$. A surjective étale endomorphism $\varphi : X := \text{Spec} \, k[x', y', z', z'^{-1}]/(x'y' = z'^2 - 1) \to X := \text{Spec} \, k[x, y, z, z^{-1}]/(xy = z^2 - 1)$ is given by

$$x = x', \quad y = y'(z'^{2(n-1)} + z'^{2(n-2)} + \cdots + z'^2 + 1), \quad z = z'^n,$$

where $n$ is a positive integer.

4. $X$ has an untwisted $\mathbb{A}_1^*$-fibration $\phi : X \to C$ induced by the projection $Y \to C$.

Remark. The non-finite étale endomorphism given in the assertion (3) above can be generalized to the universal coverings of some classes of affine pseudo-planes. Let $V(d, r)$ be tom Dieck's affine pseudo-plane of type $(d, r)$ (cf. Lemma 1.4.18) and let $\tilde{V}(d, r)$ be its universal covering which is defined by $x^r z + y^d = 1$ in $\mathbb{A}^3$. Let $X = \text{Spec} \, k[x, y, z, y^{-1}]/(x^r z + y^d = 1)$. Define a surjective étale endomorphism $\varphi : X = \text{Spec} \, k[x', y', z', y'^{-1}]/(x'^r z' + y'^d = 1) \to X = \text{Spec} \, k[x, y, z, y^{-1}]/(x^r z + y^d = 1)$ by $x = x', y = y'^n, z = \quad \cdots$
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\[ z' \left( y^{d(n-1)} + y^{d(n-2)} + \cdots + y^d + 1 \right) \]. Then \( \varphi \) has degree \( n \), and there are \( n^2 - n \) affine lines missing in the above \( X \) for \( \varphi \) to be finite.

In Example 2.2.6, the surface \( X \) has an untwisted \( \mathbb{A}^1_* \)-fibration. By taking a quotient of the surface \( X \) by an involution, we obtain a surface \( \hat{X} \) with a twisted \( \mathbb{A}^1_* \)-fibration.

**Example 2.2.7** Take \( C = \mathbb{P}^1 \) in Example 2.2.6. Let \( \eta \) be an inhomogeneous coordinate on \( C \) such that \( \eta = 0 \) (resp. \( \infty \)) at \( P_1 \) (resp. \( P_{\infty} \)). Let \( \iota : Y \to Y \) be an involution defined by \( \iota^* (\xi) = \xi^{-1} \) and \( \iota^* (\eta) = -\eta \). Since \( \iota((P_i, Q_i)) = (P_i, Q_i) \) for \( i = 1, 2 \), the involution \( \iota \) lifts to an involution \( \iota \) of \( X \) such that \( \iota \cdot \varphi = \varphi \cdot \iota \). Let \( \hat{X} = X / \langle \iota \rangle \). Then the following assertions hold:

1. The surface \( \hat{X} \) is a smooth affine surface with a twisted \( \mathbb{A}^1_* \)-fibration \( \hat{\phi} : \hat{X} \to \mathbb{P}^1 \) which is induced by the untwisted \( \mathbb{A}^1_* \)-fibration \( \phi : X \to C \).

2. The \( \acute{e}tale \) endomorphism \( \varphi : X \to X \) descends to an \( \acute{e}tale \) endomorphism \( \hat{\varphi} : \hat{X} \to \hat{X} \) of degree \( n \). Furthermore, \( \hat{\varphi} \) is not finite.

3. The surface \( \hat{X} \) is isomorphic to \( F_0 - D \) with an irreducible curve \( D \), where \( F_0 \cong \mathbb{P}^1 \times \mathbb{P}^1 \) and \( D \sim 2M + \ell \) with \( M \) and \( \ell \) being the fibers of two distinct projections from \( F_0 \) to \( \mathbb{P}^1 \). The \( \mathbb{A}^1_* \)-fibration \( \hat{\phi} \) is given by restricting the \( \mathbb{P}^1 \)-fibration \( p := p|_{\ell} \) onto \( \hat{X} \).

4. If \( \theta : F_0 \to F_0 \) is a double covering ramifying over \( \ell_1, \ell_2 \) which are the fibers of \( p \) passing through the points of ramifications of \( p|_D : D \to \mathbb{P}^1 \). Then \( \theta^* (D) = D_1 + D_2 \), where \( D_1 \sim D_2 \sim M + \ell \) and \( X = F_0 - (D_1 + D_2) \).
(5) $\pi(\hat{X}) = -\infty$, $\text{Pic}(\hat{X}) = \mathbb{Z}$ and $\Gamma(\hat{X}, \mathcal{O}_{\hat{X}})^* = k^*$. 

(6) An explicit construction of $X, \iota, \hat{X}$ and $\hat{\varphi}$ are given as follows:

$$X = \text{Spec } k[x, y, z, z^{-1}]/(xy = z^2 - 1), \quad \iota \left( \begin{array}{c} x \\ y \\ z \end{array} \right) = \left( \begin{array}{c} x \\ -yz^{-2} \\ z^{-1} \end{array} \right)$$

$$\hat{X} = \text{Spec } k[x, t, u]/(x^2u = t^2 - 4), \quad \hat{\varphi} \left( \begin{array}{c} x \\ t \\ u \end{array} \right) = \left( \begin{array}{c} x \\ g(t) \\ uh(t) \end{array} \right),$$

where $g(t), h(t)$ are polynomials in $t$ such that

$$g \left( z + \frac{1}{z} \right) = \frac{z^{2n} + 1}{z^n}, \quad h \left( z + \frac{1}{z} \right) = \left( \frac{z^{2n-2} + \cdots + z^2 + 1}{z^{n-1}} \right)^2.$$ 

In the assertion (4), it is straightforward to ascertain the commutativity $\varphi \cdot \iota = \iota \cdot \varphi$. Set $A = k[x, y, z, z^{-1}]/(xy = z^2 - 1)$. Let

$$t = z + \frac{1}{z} \quad \text{and} \quad u = \frac{y^2}{z^2}.$$ 

Then the $\iota$-invariant subring $B$ is generated by three elements $x, t, u$ which satisfy a relation $x^2u = t^2 - 4$. In fact, taking the squares of both sides of the relation $xy = z^2 - 1$, we obtain the said relation in $x, t, u$.

As indicated in the above examples, given a pair $(X, \varphi)$ of a smooth surface $X$ and an étale endomorphism $\varphi$ together with a finite group action $G$ on $X$ which commutes with the endomorphism $\varphi$ and whose fixed point locus is either the empty set or a disjoint union of smooth one-dimensional components, we can produce another pair $(\hat{X}, \hat{\varphi})$ of a smooth surface $\hat{X}$ and an étale endomorphism $\hat{\varphi}$, where $\hat{X} = X/G$. Reversing this construction, we have the following result:
Theorem 2.2.8 Let $X$ be a smooth algebraic variety and let $\varphi : X \to X$ be a non-finite étale endomorphism. Suppose that $\pi_1(X)$ is a finite group. Let $q : \tilde{X} \to X$ be the universal covering morphism. Then the following assertions hold:

1. There exists a non-finite étale endomorphism $\tilde{\varphi} : \tilde{X} \to \tilde{X}$ such that $q \circ \tilde{\varphi} = \varphi \circ q$.

2. There exists a group endomorphism $\chi : \pi_1(X) \to \pi_1(X)$ such that $\tilde{\varphi}(gu) = \chi(g)\tilde{\varphi}(u)$ for any $g \in \pi_1(X)$ and $u \in \tilde{X}$.

3. Conversely, if there exists a non-finite étale endomorphism $\tilde{\varphi} : \tilde{X} \to \tilde{X}$ satisfying the condition that $\tilde{\varphi}(gu) = \chi(g)\tilde{\varphi}(u)$ with a group endomorphism $\chi : \pi_1(X) \to \pi_1(X)$ for $g \in \pi_1(X)$ and $u \in \tilde{X}$, then there exists a non-finite étale endomorphism $\varphi : X \to X$ such that $q \circ \tilde{\varphi} = \varphi \circ q$.

Proof. (1) Let $Z = (X, \varphi) \times_X (\tilde{X}, q)$ and let $Z = Z_1 \coprod \cdots \coprod Z_r$ be the decomposition into connected components. Let $\rho_i : Z_i \to X$ and $\sigma_i : Z_i \to \tilde{X}$ be respectively the restriction onto $Z_i$ of the projections from $Z$ to the first factor $X$ and the second factor $\tilde{X}$ of the fiber product. Then $\rho_i$ is a finite étale morphism but $\sigma_i$ is a non-finite étale morphism. In fact, if $\sigma_i$ is finite then $q \circ \sigma_i = \varphi \circ \rho_i$ is a finite morphism. Hence $\varphi$ is a finite morphism as well, a contradiction. Since $\rho_i : Z_i \to X$ is a finite étale covering and since $\tilde{X}$ is the universal covering space of $X$ there exists a finite étale morphism $\tau_i : \tilde{X} \to Z_i$ such that $q = \rho_i \circ \tau_i$. Let $\tilde{\varphi} = \sigma_i \circ \tau_i$. Then $q \circ \tilde{\varphi} = \varphi \circ q$ and $\tilde{\varphi}$ is a non-finite étale endomorphism.

(2) Since $q \circ \tilde{\varphi}(gu) = \varphi \circ q(gu) = \varphi \circ q(u) = q \circ \tilde{\varphi}(u)$, we have $\tilde{\varphi}(gu) = \chi(g, u)\tilde{\varphi}(u)$ for any $g \in \pi_1(X)$ and $u \in \tilde{X}$. Fix an element $g \in \pi_1(X)$ and
move $u \in \tilde{X}$. Then $u \mapsto \chi(g, u)$ is a morphism from $\tilde{X}$ to a finite group $\pi_1(X)$. Hence $\chi(g, u)$ is independent of the choice of $u \in \tilde{X}$. So, we have $\tilde{\varphi}(gu) = \chi(g)\tilde{\varphi}(u)$. Then it is easy to show that $\chi$ is a group endomorphism. Q.E.D.

This result inspires us to think of the following:

**Conjecture.** Let $X$ be a smooth algebraic variety and let $q : U \to X$ be the universal covering, where $U$ is not necessarily an algebraic variety. Then $X$ has a non-finite étale endomorphism if and only if $U$ has a non-finite unramified endomorphism.

Theorem 2.1.8 implies that some of the counterexamples already treated above have étale, non-finite endomorphisms.

**Theorem 2.2.9** Let $\tilde{X}$ be either the universal covering of $\mathbb{P}^2 - C$ in Example 2.2.2 or $T - T_0$ in Example 2.2.3. Then $\tilde{X}$ has an étale, non-finite endomorphism.

In Theorem 2.2.9 the universal covering $\tilde{X}$ is not rational as $\tilde{X}$ is the complement of a smooth K3-surface. Nevertheless, the following example shows that there exists a simply connected, smooth, affine surface $\tilde{X}$ which has an étale non-finite endomorphism.

**Example 2.2.10** Let $V_0 = \mathbb{P}^1 \times \mathbb{P}^1$. Let $M_0$ be a cross-section and let $\ell_0, \ell_1, \ell_\infty$ be distinct three fibers with respect to the second projection $\pi_2 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$. Let $\varphi : V \to V_0$ be a sequence of blowing-ups with centers at $\ell_0 \cap M_0, \ell_1 \cap M_0$ and their infinitely near points such that $\varphi^*(\ell_0) = \ell'_0 + E_1 + 2E_2 + 2E_3$ and $\varphi^*(\ell_1) = \ell'_1 + F_1 + 2F_2 + 2F_3$, where $(\ell'_0)^2 = (\ell'_1)^2 =$
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\((E_i^2) = (F_i^2) = -2\) for \(i = 1, 2\) and \((E_3^2) = (F_3^2) = -1\). Let

\[ X := V - (\ell_\infty + M_0' + \ell_1' + E_1' + F_1' + E_2' + F_2'). \]

Hence \(X\) has an \(\mathbb{A}^1\)-fibration \(\rho : X \to B\) with two multiple fibers \(2E_3 \cap X, 2F_3 \cap X\) of multiplicity 2. Then \(X\) has a degree two, non-finite étale endomorphism.

**Proof.** Let \(\sigma : B' \to B\) be a degree two covering ramifying over the point at infinity \(p_\infty\) and \(p_0\), where \(p_0 = \rho(E_3 \cap X)\). Let \(\bar{X}\) be the normalization of \(X \times_B B'\), let \(\tau : \bar{X} \to X\) be the composite of the normalization morphism and the first projection \(X \times_B B' \to X\) and let \(\bar{\rho} : \bar{X} \to B'\) be the \(\mathbb{A}^1\)-fibration induced naturally by \(\rho\). Then \(\bar{\rho}^*(q_0)\) is a disjoint sum \(G_1 + G_2\) of two affine lines and \(\tau : \bar{X} \to X\) is a finite étale morphism, where \(q_0\) is a point of \(B'\) lying over \(p_0\). Then \(\bar{X} - G_1 \cong \bar{X} - G_2 \cong X\), and \(\tau |_{\bar{X} - G_1}\) and \(\tau |_{\bar{X} - G_2}\) induce a non-finite étale endomorphism of \(X\). Q.E.D.

2.3 Affirmative results

Most results ever known in the affirmative are in the surface case. So we restrict ourselves to the surface case.

**Lemma 2.3.1** Let \(X\) be a smooth affine surface with \(\kappa(X) = -\infty\). Suppose that one of the following conditions is satisfied:

1. \(X\) is irrational but not elliptic ruled.
2. \(\Gamma(X, \mathcal{O}_X)^* \neq k^*,\) and \(\text{rank } (\Gamma(X, \mathcal{O}_X)^*/k^*) \geq 2\) if \(X\) is rational.
Then any étale endomorphism \( \varphi : X \to X \) is an automorphism.

**Proof.** **Case** (1) Since \( \pi(X) = -\infty \), there exists an \( \mathbb{A}^1 \)-fibration \( \rho : X \to C \), where \( C \) is a smooth curve of positive genus and \( C = \rho(X) \). Let \( \varphi : X \to X \) be an étale endomorphism. Then there exists an endomorphism \( \beta : C \to C \) such that \( \rho \cdot \varphi = \beta \cdot \rho \). Since the genus of \( C \) is greater than 1 by the hypothesis, it follows from the Riemann-Hurwitz formula that \( \beta \) is an automorphism. Since \( \beta \) is of finite order, we may assume after replacing \( \varphi \) by its suitable iteration that \( \beta \) is the identity. Hence \( \rho = \rho \circ \varphi \). Now consider the generic fiber \( X_K \) of \( \rho \) with the function field \( K \) of \( C \). The endomorphism \( \varphi \) induces an étale endomorphism \( \varphi_K : X_K \to X_K \). Since \( X_K \) is isomorphic to \( \mathbb{A}^1_K \), \( \varphi_K \) is an automorphism. This implies that \( \varphi \) is a birational morphism. Hence it is injective by Zariski’s main theorem. So, \( \varphi \) is an automorphism by Ax’s theorem \[5, 7\].

**Case** (2) Let \( A = \Gamma(X, \mathcal{O}_X) \). Since \( \pi(X) = -\infty \), \( X \) contains a cylinder-like open set \( U_0 \times \mathbb{A}^1 = \text{Spec } B[x] \), where \( U_0 = \text{Spec } B \) is an affine normal curve. We have \( A \subset B[x] \) and \( A^* \subset B^* \). Let \( R_0 \) be the \( k \)-subalgebra of \( A \) generated by all elements of \( A^* \). Since rank \( A^*/k^* < \infty \), the algebra \( R_0 \) is finitely generated over \( k \). Let \( R \) be the normalization of \( R_0 \) in \( A \). We have \( R \subset B \). Let \( \overline{C} = \text{Spec } R \) and let \( \rho : X \to C \subset \overline{C} \) be the morphism induced by the injection \( R \subset A \), where \( C = \rho(X) \). Since \( R^* \supset A^* \not\subset k^* \), it follows that \( \pi(C) \geq 0 \). Let \( F \) be a general fiber of \( \rho \). By Kawamata’s addition

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\footnote{Consider a compactification \( V \) of \( X \) such that \( V - X \) has pure codimension one. Write \( V - X = \sum_{i=1}^{r} D_i \). If \( f \in A^* \), then the divisor \((f)\) is supported by the components \( D_i \) and \((f) = (g)\) if and only if \( g = cf \) for some \( c \in k^* \). Hence \( A^*/k^* \) is embedded into a free abelian group generated by the \( D_i \). Hence \( A^*/k^* \) is also a finitely generated free abelian group.}
theorem, we know that $\pi(F) = -\infty$. Namely, $F \cong \mathbb{A}^1$. Hence $\rho$ is an $\mathbb{A}^1$-fibration. Let $\varphi : X \to X$ be an étale endomorphism. Since $C$ is irrational or $C$ is a rational curve with at least three places at infinity, $\varphi$ induces an endomorphism $\beta$ such that $\rho \cdot \varphi = \beta \cdot \rho$. In fact, $\beta$ is an automorphism by the Riemann-Hurwitz theorem. The rest of the proof is the same as in the case (1).

Q.E.D.

Let $X$ be a smooth affine surface with an $\mathbb{A}^1_*$-fibration $\rho : X \to C$. A singular fiber $S$ is, by definition, a fiber which is not isomorphic to $\mathbb{A}^1_*$ as a scheme. Let $S$ be a singular fiber. Then, by [33, Lemma 4], $S$ is written as $S = \Gamma + \Delta$, where

(1) $\Gamma = 0, \Gamma = \alpha \Gamma_1$ with $\alpha \geq 1$ and $\Gamma_1 \cong \mathbb{A}^1_*$, or $\Gamma = \alpha_1 \Gamma_1 + \alpha_2 \Gamma_2$, where $\alpha_1 \geq 1, \alpha_2 \geq 1, \Gamma_1 \cong \Gamma_2 \cong \mathbb{A}^1$ and $\Gamma_1$ and $\Gamma_2$ meet each other in one point transversally.

(2) $\Delta \geq 0$ and $\text{Supp} \Delta$ is a disjoint union of curves isomorphic to $\mathbb{A}^1$.

When an étale endomorphism $\varphi : X \to X$ is given, we write it $\alpha : X_1 \to X_2$ to distinguish the source $X$ from the target $X$.

**Lemma 2.3.2** Let $\rho : X \to C$ be an untwisted $\mathbb{A}^1_*$-fibration and let $\varphi : X \to X$ be an étale endomorphism such that $\rho \cdot \varphi = \rho$ and $\text{codim}_X(X - \varphi(X)) \geq 2$. Let $\nu : \tilde{X} \to X$ be the normalization of $X_2$ in the function field of $X_1$. Then we have:

(1) There exists an open immersion $\iota : X \hookrightarrow \tilde{X}$ such that $\varphi = \nu \cdot \iota$.

(2) $\nu : \tilde{X} \to X$ is an étale Galois covering of $X$ with a cyclic group $G$ of order $n$ as the Galois group, where $n = \text{deg} \varphi$. 


(3) Let $S = \Gamma + \Delta$ be a singular fiber of $\rho$. Then $\varphi$ is finite over $\Gamma$, i.e., $\varphi^*(\Gamma)$ is $G$-invariant, and $\varphi$ is totally decomposable over $\Delta$, i.e., the stabilizer group of each connected component of $\Delta$ is trivial.

Proof. Our proof consists of several steps.

(1) Let $K$ be the function field of $C$ and let $X_K$ be the generic fiber of $\rho$. Then $X_K = \text{Spec } K[x, x^{-1}]$, and $\varphi$ induces an étale $K$-endomorphism $\varphi_K : X_{1,K} \to X_{2,K}$. Clearly, $\varphi_K$ is given by a $K$-algebra endomorphism $\theta_K : x \mapsto ax^{\pm n}$ of $K[x, x^{-1}]$, where $a \in K$ and $n = \text{deg } \varphi$. Let $G$ be the group of $n$-th roots of the unity in $\mathbb{C}$, which is a cyclic group of order $n$. The group $G$ acts on $X_{1,K}$ by $(x, \zeta) \mapsto x\zeta$, where $\zeta \in G$, and $X_{2,K}$ is clearly the quotient variety $X_{1,K}/G$. The $G$-action on $X_{1,K}$ is extended to an action on the normalization $\tilde{X}$ and $X_1$ is embedded into $\tilde{X}$ as an open set. Note that $\tilde{X}$ is $G$-equivariant.

(2) Let $P$ be a closed point of $C$ such that the fiber $F := \rho^*(P)$ is a smooth fiber. Let $\mathcal{O} := \mathcal{O}_{C, P}$ and let $X_\mathcal{O} := X \times_C \text{Spec } \mathcal{O}$. Then we can choose the element $x$ above so that $X_\mathcal{O} = \text{Spec } \mathcal{O}[x, x^{-1}]$ and the induced endomorphism $\varphi_\mathcal{O} : X_\mathcal{O} \to X_\mathcal{O}$ is given by an $\mathcal{O}$-endomorphism $x \mapsto ax^{\pm n}$, where $a \in \mathcal{O}^*$. So the $G$-action extends over $X_\mathcal{O}$ and $X_{1,\mathcal{O}}/G = X_{2,\mathcal{O}}$. In this case, we have $\tilde{X}_\mathcal{O} = X_\mathcal{O}$, where $\tilde{X}_\mathcal{O} = \tilde{X} \times_C \text{Spec } \mathcal{O}$.

(3) Now let $S := \rho^*(P)$ be a singular fiber and write $S = \Gamma + \Delta$ as above. Since there are no non-constant morphisms from $\mathbb{A}^1$ to $\mathbb{A}^1_*$ and since $\text{codim}_X (X - \rho(X)) \geq 2$ by the hypothesis, it follows that $\varphi_*(\Gamma) = \Gamma$ and $\varphi(\Delta) = \Delta$ as cycles. In particular, $\varphi$ is surjective. We claim that if $\Gamma \neq 0$ then $\Gamma$ is $G$-invariant. In fact, take a nonsingular completion $p : V \to B$ as follows. Let $\mu : \tilde{X} \to \tilde{X}$ be a $G$-equivariant resolution of singularities.
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of $\tilde{X}$ such that $X_1$ is an open set of $\tilde{X}$ and that $\tilde{X} - X_1$ is a divisor with simple normal crossings. Then $\tilde{X}$ has an $\mathbb{A}^1$-fibration $\rho \cdot \mu : \tilde{X} \to C$. We can find a $G$-equivariant smooth completion $p : V \to B$ of this $\mathbb{A}^1$-fibration such that $V$ is $G$-equivalent, $V - \tilde{X}$ is a divisor with simple normal crossings, $B$ is a smooth complete curve containing $C$ as an open set and $p|_{\tilde{X}} = \rho \cdot \mu$. The $G$-equivariant completion is possible by Sumihiro’s theorem [49]. Let $\Sigma = p^*(P)$, where $P = \rho(S)$. Then $\Sigma \cap X_1 = S$ and $\Sigma$ is $G$-invariant. If $\Gamma$ were not $G$-invariant then the translation $g^*\Gamma$ of $\Gamma$ by some element $g$ of $G$ would be a divisor disjoint from $\Gamma$ and $\Sigma$ would therefore contain a loop. This is impossible. So, $\Gamma$ is $G$-invariant. Now suppose $\Delta \neq 0$, and let $\Delta_1$ be an irreducible component of $\Delta$. Since $\Delta_1$ and $\varphi(\Delta_1)$ are isomorphic to $\mathbb{A}^1$ and since $\varphi_{\Delta_1} : \Delta_1 \to \varphi(\Delta_1)$ is an étale morphism, it is an isomorphism. Note that $G$ acts transitively on the components of $\nu^{-1}(\varphi(\Delta_1))$ and that the isotropy group of $\Delta_1$ is trivial by the previous remark. Hence $g(\Delta_1) \neq \Delta_1$ for any non-unit element $g$ of $G$ and $\nu : \tilde{X} \to X$ is étale above the singular fiber $S$. So, $\nu : \tilde{X} \to X$ is a finite étale morphism. Q.E.D.

We can obtain some noteworthy consequences from Lemma 2.3.2. We need one preparatory result.

Lemma 2.3.3 Let $\pi : Y \to X$ be an étale finite Galois covering of a smooth algebraic variety $X$ with a cyclic group $G$ of order $n$ as the Galois group. Then there exists an invertible $O_X$-module $L$ such that $L^\otimes n \cong O_X$ and $X \cong \text{Spec} \left( \bigoplus_{i=0}^{n-1} L^\otimes i \right)$. Furthermore, we have $\pi^*(L) \cong O_Y$.

Proof. Since the assertion is of local nature we may assume that $X$ is affine. So, let $X = \text{Spec} A$ and let $Y = \text{Spec} B$, where we view $A$ as a subalgebra
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of $B$. It is well-known that $G$ is written as a group scheme in the following form:

$$G = \text{Spec } \mathbb{C}[t] \quad \text{with} \quad t^n = 1, \mu(t) = t \otimes t,$$

$$\varepsilon(t) = 1 \quad \text{and} \quad \eta(t) = t^{-1},$$

where $\mu, \varepsilon$ and $\eta$ are respectively the comultiplication, the augmentation and the coinverse. The action of $G$ on $X$ is given in terms of the following coaction:

$$\Delta : B \rightarrow B[t], \ b \mapsto \Delta(b) = \sum_{i=0}^{n-1} \Delta_i(b) t^i.$$

The property that $\Delta$ is a coaction is equivalent to the following properties:

(1) The mapping $\Delta_i$ defined by $b \mapsto \Delta_i(b)$ is a $\mathbb{C}$-endomorphism of $B$.

(2) $\Delta_i \Delta_j = \delta_{ij} \Delta_j$, where $\delta_{ij} = 1$ if $i = j$ and 0 if $i \neq j$, and $\sum_{i=0}^{n-1} \Delta_i = 1$.

(3) $\Delta_i(b_1) \Delta_j(b_2) \in \Delta_{i+j}(B)$ for $b_1, b_2 \in B$, where we replace $i + j$ by an integer $\ell$ with $0 \leq \ell < n$ and $\ell \equiv i + j \pmod{n}$ if $i + j \geq n$.

Set $B_i = \Delta_i(B)$ for $0 \leq i < n$. Then $B_i$ is an $A$-module and $B_0 = A$, which is the $G$-invariant subalgebra of $B$. In view of the above properties, we have $B = \sum_{i=0}^{n-1} B_i$ and $B_i \cdot B_j \subseteq B_{i+j}$. Now the property that $\pi$ is étale implies that $B_1$ is a projective $A$-module of rank 1, $B_i \cong B_1^{\otimes i}$ ($1 \leq i < n$) and $B_1^{\otimes n} \cong A$. Conversely, if $B_1$ is a projective $A$-module of rank 1 such that $B_1^{\otimes n} \cong A$ then $B := \sum_{i=0}^{n-1} B_1^{\otimes i}$ is given an étale $A$-algebra structure if an isomorphism $\theta : B_1^{\otimes n} \rightarrow A$ is assigned. The group $G$ acts on $B$ as follows:

$$(\sum_{i=0}^{n-1} b_i)\zeta^i = \sum_{i=0}^{n-1} b_i \zeta^i \text{ if } b_i \in B_1^{\otimes i} \text{ and } \zeta \text{ is a primitive } n\text{-th root of unity.}$$

Clearly, $\pi^* \mathcal{L} \cong \mathcal{O}_Y$ because $B_1 B \cong B$. 
Theorem 2.3.4 Let $X$ be a smooth affine surface with an $\mathbb{A}^1_*$-fibration $\rho : X \to C$. Assume that $\text{Pic} \, X = (0)$ and $\Gamma(X, \mathcal{O}_X)^* = \mathbb{C}^*$. Then any étale endomorphism $\varphi : X \to X$ is an automorphism provided $\rho \circ \varphi = \rho$.

Proof. Since $\text{Pic} \, X = (0)$ and $\Gamma(X, \mathcal{O}_X)^* = \mathbb{C}^*$, it follows that the $\mathbb{A}^1_*$-fibration $\rho$ is untwisted and codim $X(X - \varphi(X)) \geq 2$. Let $\tilde{X}$ be the normalization of $X_2$ in the function field of $X_1$. By Lemma 2.3.2, the normalization morphism $\nu : \tilde{X} \to X$ is an étale finite Galois covering with a cyclic group of order $n$ as the Galois group. By Lemma 2.3.3 there exists an invertible sheaf $L$ such that $L \otimes n \cong \mathcal{O}_X$ and $\tilde{X} \cong \text{Spec} \left( \bigoplus_{i=0}^{n-1} L \otimes i \right)$. Since $\text{Pic} \, X = (0)$ the invertible sheaf $L$ is trivial. Then $\tilde{X}$ is a disjoint union of $n$-copies of $X$. Since $X$ is an dense open set of $\tilde{X}$, we know that $n = 1$. Hence $\varphi$ is an automorphism.\[Q.E.D.\]

In order to look more closely into the case of a smooth affine surface with an $\mathbb{A}^1_*$-fibration, we say that a singular fiber $S = \Gamma + \Delta$ is of the simple type (resp. non-simple type) if $\Gamma = \alpha \Gamma_1$ with $\Gamma_1 \cong \mathbb{A}^1_*$ and $\Delta = 0$ (resp. otherwise). Let $\rho : X \to C$ be an untwisted $\mathbb{A}^1_*$-fibration and let $\varphi : X \to X$ be an étale endomorphism. We need the following result.

Lemma 2.3.5 Let $X$ and $\varphi$ be the same as above. Suppose that $\pi(X) = 1$.

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\(^3\)Let $X$ be a smooth affine surface with a twisted $\mathbb{A}^1_*$-fibration $\rho : X \to C$. Since there is only a 2-section $H$ in the boundary at infinity of $X$ which is transverse to the $\mathbb{P}^1$-fibration extending $\rho$, $\text{Pic} \, X$ modulo the subgroup generated by all fiber components of $\rho$ has a nonzero 2-torsion element, and $\text{Pic} \, X$ is not zero.

\(^4\)Suppose that there is a curve $A$ on $X$ such that $A \cap \varphi(X) = \emptyset$. Since $\text{Pic} \, X = (0)$, there is an element $a$ of $\Gamma(X, \mathcal{O}_X)$ such that $A = V(a)$. Then the element $\varphi^*(a)$ is invertible on the upper $X$, which is impossible because $\Gamma(X, \mathcal{O}_X)^* = \mathbb{C}^*$. 
Then the following assertions hold:

(1) The endomorphism $\varphi$ preserves the $\mathbb{A}^1_\ast$-fibration $\rho$, i.e., there exists an endomorphism $\beta : C \to C$ such that $\beta \cdot \rho = \rho \cdot \varphi$ holds.

(2) $\varphi$ sends a singular fiber of non-simple type to a singular fiber of non-simple type.

(3) After replacing $\varphi$ by its iteration, we may view the endomorphism $\beta$ as the identity morphism in one of the following cases:

(i) $\kappa(C) = 1$.

(ii) $\kappa(C) = 0$ and there exists a singular fiber in the fibration $\rho$.

Proof. The assertion follows from [20]. In order to prove the assertion (2), note that there is no non-constant morphism from $\mathbb{A}^1$ to $\mathbb{A}^1_\ast$. Hence it follows that $\varphi$ sends any singular fiber of non-simple type to a fiber of non-simple type. Suppose that there exists a singular fiber of non-simple type. Let $T$ be the set of points $P$ of $C$ such that $\rho^*(P)$ is a fiber of non-simple type. Then it is easy to see that there exists a nonempty subset $T_1$ of $T$ such that $\beta$ induces an automorphism on $T_1$. Hence replacing $\varphi$ by its iteration, we may assume that $\beta$ induces the identity automorphism on the subset $T_1$. Suppose that there are only singular fibers of simple type. Then the same argument works in this case as well because $\varphi$ sends a multiple fiber to a multiple fiber.

Consider the assertion (3). If $\kappa(C) = 1$ then $C$ is a log curve of general type. Hence $\beta$ is an automorphism. Furthermore, $\beta$ is of finite order. Suppose $\kappa(C) = 0$ and there exists a singular fiber of non-simple type. By the above observation, there exists a point $P \in C$ such that $\rho^*(P)$ is a singular fiber.
of non-simple type and $\beta(P) = P$. Then $\varphi$ induces an endomorphism of $C - \{P\}$. Since $\pi(C - \{P\}) = 1$ we are done by the case $\pi(C) = 1$. If there are only singular fibers of simple type, we are done by the above remark.

Q.E.D.

With the setting before Lemma 2.3.5, we consider the case where $C$ is a rational curve. Suppose further that $\text{rank} \text{Pic} (X) = 0$ and $\Gamma(X, \mathcal{O}_X)^* = \mathbb{C}^*$. Then we can specify the structure of the $\mathbb{A}^1_*$-fibration $\rho : X \to C$.

**Lemma 2.3.6** Under the above assumptions the following assertions hold:

1. $C$ is isomorphic to $\mathbb{P}^1$ or $\mathbb{A}^1$.

2. If $C \cong \mathbb{P}^1$ then every fiber of $\rho$ is irreducible. Hence every fiber is isomorphic to $\mathbb{A}^1_*$ or $\mathbb{A}^1$ if taken with the reduced structure.

3. If $C \cong \mathbb{A}^1$ then every fiber of $\rho$ except for only one fiber $S$ is irreducible and isomorphic to either $\mathbb{A}^1_*$ or $\mathbb{A}^1$ if taken with the reduced structure. The unique reducible fiber $S$ consists of two irreducible components $F_0, F_1$ such that either $\Gamma = F_0 + F_1$ or $\Gamma = F_0$ and $\Delta = F_1$.

**Proof.**

1. Since $C$ is rational by the hypothesis and since $\Gamma(X, \mathcal{O}_X)^* = \mathbb{C}^*$, we have $C \cong \mathbb{P}^1$ or $C \cong \mathbb{A}^1$.

2. The dimension counting of $\text{Pic} (X)_\mathbb{Q}$ by making use of the $\mathbb{A}^1_*$-fibration proves the assertions (2) and (3). See [36]. Q.E.D.

**Lemma 2.3.7** With the same assumptions as in Lemma 2.3.6, suppose further that $C$ is isomorphic to $\mathbb{P}^1$. Let $m_1 \Gamma_1, \ldots, m_s \Gamma_s, m_{s+1} \Delta_{s+1}, \ldots, m_{s+t} \Delta_{s+t}$ exhaust all singular fibers of $\rho$, where $\Gamma_i \cong \mathbb{A}^1_*$ $(1 \leq i \leq s)$ and
\[ \Delta_j \cong \mathbb{A}^1 \ (s + 1 \leq j \leq s + t). \] We set \( r = s + t. \) Let \( \varphi \) be an \( \acute{e}tale \) endomorphism of \( X \) such that \( \rho \cdot \varphi = \beta \cdot \rho \) for an endomorphism \( \beta \) of \( C. \) Suppose \( r \geq 3. \) Then \( \beta \) is an automorphism except when the multiplicity sequence is one of the following:

\[ \{m_1, \ldots, m_r\} = \{2, 2, 2, 2\}, \{2, 3, 6\}, \{2, 4, 4\}, \{3, 3, 3\} \]

**Proof.** The proof is the same as in [28]. Q.E.D.

Let \( P_i = \rho(\Gamma_i) \) and \( Q_j = \rho(\Delta_j) \) with the above notations. If \( \beta \) is an automorphism, then \( \beta \) induces a permutation on the finite set \( \{P_1, \ldots, P_s, Q_{s+1}, \ldots, Q_r\}. \) Hence, by replacing \( \varphi \) by its suitable iteration, we may assume that \( \beta \) is the identity morphism. Namely, we have \( \rho \cdot \varphi = \rho. \) Then Lemma 2.3.2 yields the following result.

**Theorem 2.3.8** Let \( X \) be a smooth affine surface with an \( \mathbb{A}^1_+ \)-fibration \( \rho : X \rightarrow \mathbb{P}^1. \) Suppose that \( \kappa(X) = 1, \) Pic \( (X) = 0 \) and \( \Gamma(X, \mathcal{O}_X)^* = \mathbb{C}^*. \) Suppose further that there are at least three singular fibers and that the multiplicity sequence of the singular fibers of \( \rho \) is none of the following:

\[ \{m_1, \ldots, m_r\} = \{2, 2, 2, 2\}, \{2, 3, 6\}, \{2, 4, 4\}, \{3, 3, 3\} \]

Then any \( \acute{e}tale \) endomorphism of \( X \) is an automorphism.

**Proof.** Since \( \kappa(X) = 1, \) an \( \acute{e}tale \) endomorphism \( \varphi \) of \( X \) preserves the \( \mathbb{A}^1_+ \)-fibration \( \rho. \) Namely there exists an endomorphism \( \beta \) of the base curve \( \mathbb{P}^1 \) such that \( \rho \cdot \varphi = \beta \cdot \rho. \) Then \( \beta \) is an automorphism by Lemma 2.3.7. After replacing \( \varphi \) by its iteration, we may assume that \( \beta \) is the identity morphism. The rest of the proof is the same as in Theorem 2.3.4 Q.E.D.
We consider the case of affine surfaces with $\mathbb{A}^1$-fibrations. In [19], the generalized Jacobian conjecture for $\mathbb{Q}$-homology planes is considered. It is shown that any étale endomorphism of a $\mathbb{Q}$-homology plane $X$ is an automorphism if one of the following conditions is satisfied:

1. $\pi(X) = 2$ or $1$.
2. $\pi(X) = -\infty$ and $X$ has an $\mathbb{A}^1$-fibration $\rho : X \to B$ with at least two multiple fibers.

The second case above has one case slipping off. Namely, the surface $X$ in Example 2.2.10 is a $\mathbb{Q}$-homology plane with an $\mathbb{A}^1$-fibration which has two multiple fibers of multiplicity 2. It is a counterexample to the generalized Jacobian conjecture. We shall here consider the case (2) above. We recall the following two lemmas (cf. [19, Lemma 6.1] and [19, 28, Lemma 3.1]).

**Lemma 2.3.9** Let $\rho : X \to B$ be an $\mathbb{A}^1$-fibration on a $\mathbb{Q}$-homology plane. Suppose that $\rho$ has at least two singular fibers. Let $g : \mathbb{A}^1 \to X$ be a non-constant morphism. Then the image of $g$ is a fiber of $\rho$.

**Proof.** In fact, note that the base curve $B$ is isomorphic to $\mathbb{A}^1$ (cf. [36, Lemma 2.5]). Then the assertion follows from Lemma [3.16]. Q.E.D.

**Lemma 2.3.10** For $i = 1, 2$, let $\rho_i : X_i \to B_i$ be $\mathbb{A}^1$-fibrations on $\mathbb{Q}$-homology planes. Let $\varphi : X_1 \to X_2$ and $\beta : B_1 \to B_2$ be dominant morphisms such that $\rho_2 \circ \varphi = \beta \circ \rho_1$. Let $m\Gamma$ be an irreducible fiber of $\rho_2$ lying over a point $P \in B_2$ with $m \geq 1$ and $\Gamma$ reduced, and let $Q \in B_1$ be a point such that $\beta(Q) = P$. Suppose $\rho_1^*(Q) = \ell \Delta$, where $\Delta$ is reduced and irreducible and $\ell$
is its multiplicity. Suppose furthermore that $\varphi$ is an étale morphism. If the ramification index of $\beta$ at $Q$ is $e$ then $\ell e = m$. In particular, if $m = 1$ then $\ell = e = 1$.

Applying these lemmas, we shall show the following result.

**Lemma 2.3.11** Let $X$ be a $\mathbb{Q}$-homology plane with an $\mathbb{A}^1$-fibration $\rho : X \to B$. Let $m_1A_1, \ldots, m_nA_n$ exhaust all multiple fibers of $\rho$. Let $\varphi : X \to X$ be an étale endomorphism. Then the following assertions hold:

1. If $n \geq 2$, then there exists an endomorphism $\beta$ of $B$ such that $\rho \cdot \varphi = \beta \cdot \rho$.

2. The above endomorphism $\beta$ is an automorphism provided $n \geq 3$ or $n = 2$ and $\{m_1, m_2\} \neq \{2, 2\}$.

**Proof.** The first assertion is an immediate consequence of Lemma 2.3.9. So, we consider the second assertion. We employ the arguments in [28, Lemmas 3.1 and 3.3]. Note that $\beta : B \to B$ is a finite morphism because $B$ is the affine line. By Lemma 2.3.10, the set $\{P_1, \ldots, P_n\}$ is mapped to itself by $\beta$, where $P_i = \rho(A_i)$. Suppose, furthermore, that the points $Q_1, \ldots, Q_s$, none of which belongs to $\{P_1, \ldots, P_n\}$, are mapped to $\{Pp_1, \ldots, P_n\}$. Then, by Lemma 2.3.10, the ramification index of $\beta$ at $Q_j$, say $e_j$, is larger than 1. In fact, if $\beta(Q_j) = P_i$ then $e_j = m_i$.

Since $\beta$ induces an étale finite morphism

$$\beta : B - \{P_1, \ldots, P_n, Q_1, \ldots, Q_s\} \longrightarrow B - \{P_1, \ldots, P_n\},$$

the comparison of the Euler numbers gives rise to an equality

$$1 - (n + s) = d(1 - n),$$

(1)
where $d = \deg \beta$. On the other hand, by summing up the ramification indices, we have an inequality

$$2s + n \leq dn.$$  \hspace{1cm} (2)

So, by combining (1) and (2) together, we have an inequality

$$2(d - 1)(n - 1) = 2s \leq (d - 1)n.$$ \hspace{1cm} (3)

Suppose $d > 1$. Then $n \leq 2$. Hence, if $n \geq 3$ then $d = 1$ and $\beta$ is an automorphism. Suppose that $d > 1$ and $n = 2$. Then the equality occurs in (3), and hence the equality occurs in (2). Namely, the ramification index $e_j$ at $Q_j$ is two for all $j$, and $s = d - 1$. Since $d > 1$ implies $s > 0$, we may assume that $Q_1$ is mapped to $P_1$. Then $m_1 = 2$. Suppose $d \geq 3$. Then $2s = 2(d - 1) > d$. Hence one of the $Q_j$ is mapped to $P_2, \ldots, P_n$, say $P_2$. Hence $m_2 = 2$. In this case, after a suitable change of indices, one of the following two cases is possible:

1. $s = s_1 + s_2 = d - 1$, and $Q_1, \ldots, Q_{s_1}, P_1$ (or $P_2$) (resp. $Q_{s_1+1}, \ldots, Q_s, P_2$ (or $P_1$)) are mapped to $P_1$ (resp. $P_2$).

2. $s = s_1 + s_2, d = 2s_1 = 2s_2 + 2$, and $Q_1, \ldots, Q_{s_1}$ (resp. $Q_{s_1+1}, \ldots, Q_s, P_1, P_2$) are mapped to $P_1$ (resp. $P_2$).

Finally, suppose that $d = n = 2$ and $s = 1$. Then we may assume that $\beta(Q_1) = P_1$ and $\beta(P_1) = \beta(P_2) = P_2$. Then $m_2 = 2$ as well by Lemma 2.3.10. So, if $\{m_1, m_2\} \neq \{2, 2\}$, then $d = 1$ and $\beta$ is an automorphism.

Q.E.D.

As a consequence of Lemma 2.3.11, we can prove the following result, which rectifies Theorem 6.1 in [19].
Theorem 2.3.12 Let $X$ be a $\mathbb{Q}$-homology plane with an $\mathbb{A}^1$-fibration $\rho : X \to B$. Let $m_1A_1, \ldots, m_nA_n$ exhaust all multiple fibers of $\rho$. Suppose that either $n \geq 3$ or $n = 2$ and $\{m_1, m_2\} \neq \{2, 2\}$. Then any étale endomorphism $\varphi : X \to X$ is an automorphism.

Proof. By Lemma 2.3.11 there exists an automorphism $\beta$ of $B$ such that $\rho \cdot \varphi = \beta \cdot \rho$. Since $\beta$ is an automorphism, Lemma 2.3.10 implies that $\beta$ induces a permutation of the finite set $\{P_1, \ldots, P_n\}$. By replacing $\beta$ by its suitable iteration $\beta^r$, we may assume that $\beta$ induces the identity on $\{P_1, \ldots, P_n\}$. Since $n \geq 2$ and $\beta$ (or rather an induced automorphism of the smooth compactification $\overline{B}$ of $B$) fixes the point at infinity $P_\infty$. Hence $\beta$ is then the identity automorphism.

Let $K = k(B)$ be the function field of $B$ and let $X_K$ be the generic fiber of $\rho$. Then $X_K$ is isomorphic to the affine line over $K$, and $\varphi$ induces an étale endomorphism $\varphi_K$ of $X_K$. Since $\varphi_K$ is then finite, $\varphi_K$ is an automorphism. Hence $\varphi$ is birational. Then Zariski’s Main Theorem implies that $\varphi$ is an open immersion. Note that $\text{Pic} \ (X)_\mathbb{Q} = 0$ and $\Gamma(\mathcal{O}_X)^* = \mathbb{C}^*$. Suppose that $X \neq \varphi(X)$. Then $X - \varphi(X)$ has pure codimension one. Since $\text{Pic} \ (X)_\mathbb{Q} = 0$, there exists a regular function $h$ on $X$ such that the zero locus $(h)_0$ of $h$ is supported by $X - \varphi(X)$. Then $\varphi^*(h)$ is a non-constant invertible function on $X$, which contradicts the property $\Gamma(\mathcal{O}_X)^* = \mathbb{C}^*$. So, $\varphi$ is an automorphism.

Q.E.D.
2.4 Normalization by étale endomorphisms

Given an étale endomorphism $\varphi : X_1 \to X_2$ with $X_1 \cong X_2 \cong X$, consider the normalization $\nu : \tilde{X}_2 \to X_2$ of $X_2$ in the function field of $X_1$. If $X$ is normal, $X_1$ is a Zariski open set of $\tilde{X}_2$ with an open immersion $\iota : X_1 \hookrightarrow \tilde{X}_2$ such that $\varphi = \nu \circ \iota$. We are interested in the complement $\tilde{X}_2 - X_1$. First of all, we shall generalize Lemma 2.3.2.

Lemma 2.4.1 Let $X$ and $Y$ be smooth affine surfaces with $\mathbb{A}_1$-fibrations $\rho_X : X \to C$ and $\rho_Y : Y \to C$, where $C$ is a smooth algebraic curve. Let $\varphi : Y \to X$ be an étale morphism such that $\rho_Y = \rho_X \circ \varphi$ and $\mathrm{codim}_X(X - \varphi(Y)) \geq 2$. Let $\nu : Z \to X$ be the normalization of $X$ in the function field of $Y$. Then the following assertions hold:

1. $Z$ is a smooth affine surface containing $Y$ as a Zariski open set.

2. The normalization morphism $\nu : Z \to X$ is an étale Galois covering with a cyclic group $G$ of order $n$ as the Galois group, where $n := \deg \varphi$.

3. Let $S = \Gamma + \Delta$ be a singular fiber of $\rho_X$, where $\Gamma$ is $\emptyset$ or isomorphic to $\mathbb{A}_1^1$ or $\mathbb{A}_1 \cup \mathbb{A}_1$ with two $\mathbb{A}_1$’s meeting in one point transversally and $\Delta$ is a disjoint union of the affine lines. If $\Gamma \neq \emptyset$ then $\varphi$ is finite over $\Gamma$, i.e., $\varphi^*(\Gamma)$ is $G$-invariant, and $\varphi$ is totally decomposable over $\Delta$, i.e., the stabilizer group of each connected component of $\Delta$ is trivial.

Proof. If the $\mathbb{A}_1$-fibrations $\rho_X$ and $\rho_Y$ are both untwisted, the proof is exactly the same as for Lemma 2.3.2. Let $K$ be the function field of $C$ and let $X_K, Y_K$ be the generic fibers of $\rho_X, \rho_Y$, respectively. Let $\overline{X}_K, \overline{Y}_K$ be the
smooth completions of $X_K, Y_K$, respectively. Then $\overline{X}_K$ and $\overline{Y}_K$ are isomorphic to the projective line defined over $K$, and $\rho_X$ (resp. $\rho_Y$) is untwisted or twisted according as $\overline{X}_K$ (resp. $\overline{Y}_K$) consists of two $K$-rational points or one non $K$-rational point. Since $\varphi$ induces a finite $K$-morphism $\overline{\varphi}_K : \overline{Y}_K \to \overline{X}_K$, $\rho_X$ is untwisted if so is $\rho_Y$.

Consider the case where $\rho_Y$ is twisted. Then there exists a double covering $C' \to C$ such that $\rho_Y' : Y' \to C'$ is an untwisted $\mathbb{A}_1^1$-fibration, where $Y'$ is the normalization of the fiber product $Y \times_C C'$ and $\rho_Y'$ is a composite of the normalization morphism $Y' \to Y \times_C C'$ and the projection $Y \times_C C' \to C'$. Considering all possible singular fibers of $\rho_Y$, we can show that $Y'$ is a smooth affine surface. Similarly, we consider the normalization $X'$ of the fiber product $X \times_C C'$ and the induced $\mathbb{A}_1^1$-fibration $\rho_X' : X' \to C'$. The étale morphism $\varphi : Y \to X$ induces an étale morphism $\varphi' : Y' \to X'$ such that $\rho_Y' = \rho_X' \circ \varphi'$. Furthermore, $\deg \varphi' = \deg \varphi$ and $\codim X'(X' - \varphi'(Y')) \geq 2$.

Let $Z'$ be the normalization of $X'$ in the function field of $Y'$. It is readily verified that $Z'$ is the normalization of the fiber product of $Z \times_C C'$. More precisely, $Z'$ has an involution $i : Z' \to Z'$ induced by the involution of the double covering $C' \to C$, and $Z$ is the quotient variety with respect to $i$. As shown in the untwisted case, $Z'$ is smooth and the normalization morphism $\nu' : Z' \to X'$ is an étale Galois covering with a cyclic group $G$ of order $n$ as the Galois group, where $n = \deg \varphi'$. Since the involution $i$ commutes with the Galois group action, the assertions for $\varphi : Y \to X$ follow from the corresponding assertions for $\varphi' : Y' \to X'$.

Q.E.D.

Let $\rho : X \to C$ be an $\mathbb{A}_1^1$-fibration on a smooth affine surface $X$. Let $S = \Gamma + \Delta$ be a singular fiber of $\rho$. We call $S$ a singular fiber of the first kind (of
the second kind, of the third kind, respectively) if \( \Gamma_{\text{red}} \cong \mathbb{A}_1 (\Gamma_{\text{red}} \cong \mathbb{A}_1 \cup \mathbb{A}_1, \Gamma_{\text{red}} = \emptyset, \) respectively).

**Corollary 2.4.2** With the same notations and assumptions as in Lemma 2.4.1, let \( \varphi : Y \to X \) be an étale endomorphism satisfying \( \rho_Y = \rho_X \circ \varphi. \) Then \( \varphi \) is an isomorphism if \( \rho_X \) has a singular fiber of the second kind.

**Proof.** Suppose that \( \rho_X \) has a singular fiber of the second kind \( S = \Gamma + \Delta. \) Write \( \Gamma = a_1 F_1 + a_2 F_2 \) with \( F_1 \cong F_2 \cong \mathbb{A}_1. \) Then \( F_1 \) and \( F_2 \) meet each other in a single point \( P \) transversally. Since \( \varphi^*(\Gamma) \) is \( G \)-invariant, the fiber \( T := \rho_Y^{-1}(\rho_X(S)) \) is a singular fiber of the second kind, and \( \Gamma' = b_1 G_1 + b_2 G_2 \) with \( G_1 \cong G_2 \cong \mathbb{A}_1 \) and \( G_1, G_2 \) meeting in a single point \( Q \) transversally if we write \( T = \Gamma' + \Delta'. \) Furthermore, the group \( G \) acts on \( G_1 \cup G_2 \) freely because the morphism \( \varphi \) induces an étale finite morphism from \( G_1 \cup G_2 \) onto \( F_1 \cup F_2. \) Nevertheless, it is clear that the point \( Q \) is fixed under the \( G \)-action. This implies that the order \( n \) of \( G \) equals one. Since \( n = \deg \varphi, \) it follows that \( \varphi \) is a birational morphism. Then \( Z \) is isomorphic to \( X. \) Hence \( \varphi : Y \to X \) is an open immersion. Meanwhile, since \( X \) and \( Y \) are affine, the complement \( Z - Y \) is of pure codimension one. Since \( \text{codim}_X(X - \varphi(Y)) \geq 2, \) it follows that \( \varphi \) is an isomorphism. Q.E.D.

**Lemma 2.4.3** Let \( X \) be a smooth affine surface with \( \pi(X) = 1 \) and let \( \varphi : X \to X \) be an étale endomorphism, which we write \( \varphi : X_1 \to X_2. \) Let \( \nu : \tilde{X}_2 \to X_2 \) be the normalization of \( X_2 \) in the function field of \( X_1. \) Then the following assertions hold:

1. \( \tilde{X}_2 \) is a smooth affine surface with \( \pi(\tilde{X}_2) = 1 \) and \( X_1 \) is an open set
of $\tilde{X}_2$. Furthermore, $\nu : \tilde{X}_2 \to X_2$ is an étale Galois covering with a cyclic group $G$ of order $n := \deg \varphi$ as the Galois group.

(2) The $\mathbb{A}^1$-fibration $\rho_1 : X_1 \to C_1$ extends to an $\mathbb{A}^1$-fibration $\tilde{\rho}_1 : \tilde{X}_2 \to C_1$ such that $\beta \circ \tilde{\rho}_1 = \rho_2 \circ \nu$, where $\beta : C_1 \to C_2$ is an automorphism.

(3) $\tilde{X}_2 - X_1$ is a disjoint union of irreducible curves isomorphic to the affine line. The number $N$ of irreducible components of $\tilde{X}_2 - X_1$ is zero or given by the following formula:

$$N = \sum_{i=1}^{r} (nd_i - d_i'),$$

where $r$ is the number of singular fibers of $\rho_2$ (and hence of $\rho_1$) and $d_i$ (resp. $d_i'$) is the number of irreducible components in $\rho_2^*(P_i)$ (resp. $\rho_1^*(\beta^{-1}(P_i))$) isomorphic to $\mathbb{A}^1$ with $\{P_1, \ldots, P_r\}$ exhausting all points of $C_2$ such that $\rho_2^*(P_i)$ is a singular fiber.

**Proof.** Consider smooth completions $X_1 \hookrightarrow V_1$ and $X_2 \hookrightarrow V_2$ such that $D_1 := V_1 - X_1$ and $D_2 := V_2 - X_2$ are divisors with simple normal crossings and that the endomorphism $\varphi$ extends to a morphism $\Psi : V_1 \to V_2$. By the logarithmic ramification formula, we have, for every $m > 0$,

$$|m(D_1 + K_{V_1})| = |m\Psi^*(D_2 + K_{V_2})| + mR_{\Phi},$$

with an effective divisor $R_{\Phi}$. Let $\Phi_1 := \Phi|_{m(D_1 + K_{V_1})}$ and $\Phi_2 := \Phi|_{m(D_2 + K_{V_2})}$ be the morphisms defined by the respective linear systems. Since $\kappa(X) = 1$, $\Phi_1$ and $\Phi_2$ define respectively the $\mathbb{A}^1$-fibrations $\rho_1 : X_1 \to C_1$ and $\rho_2 : X_2 \to C_2$ provided $m \gg 0$, where $C_1$ and $C_2$ are isomorphic to one and the same smooth curve $C$. Furthermore there exists an automorphism $\beta : C_1 \to C_2$
such that $\beta \circ \rho_1 = \rho_2 \circ \varphi$. Now take $Y$ and $X$ in Lemma 3.1 to be $X_1$ and $(C_1, \beta) \times_{C_2} X_2$. We denote by the same symbol $\varphi$ the induced étale morphism $X_1 \to (C_1, \beta) \times_{C_2} X_2$. Then $Z$ is isomorphic to $\tilde{X}_2$. The assertions (1) and (2) then follow from Lemma 2.4.1 and its proof.

Let $S = \Gamma + \Delta$ be a singular fiber of $\rho_2$ lying over a point $P \in C_2$. Then $\tilde{\rho}_1'(Q) = \nu^*(\Gamma) + \nu^*(\Delta)$ is a singular fiber of $\tilde{\rho}_1$ such that $\nu^*(\Gamma) = \varphi^*(\Gamma)$ is $G$-invariant and $\nu^*(\Delta)$ is totally decomposable, where $\beta(Q) = P$. If we write $\rho_1'(Q) = \Gamma' + \Delta'$ then $\Gamma' = \nu^*(\Gamma)$ and $\nu^*(\Delta)$ is the $G$-translate of $\Delta'$. Hence we conclude by a fiberwise argument that $\tilde{X}_2 - X_1$ consists of connected components each of which is isomorphic to $\mathbb{A}^1$. Now, by the above argument, we have $N = \sum_{i=1}^{r}(nd_i - d_i')$. Q.E.D.

Apart from the generalized Jacobian conjecture, we shall make the following remark (cf. [32]).

**Lemma 2.4.4** Let $X$ be a normal affine surface with an $\mathbb{A}^1$-fibration $\rho : X \to C$. Let $S$ be a singular fiber, i.e., a fiber which is scheme-theoretically not isomorphic to $\mathbb{A}^1$. Then $S$ is written as $S = \Gamma + \Delta$, where

1. $\Gamma = \emptyset$, $\Gamma$ is an irreducible curve with possibly one cyclic quotient singularity and the normalization of $\Gamma$ is isomorphic to $\mathbb{A}^1$, or $\Gamma = C_1 + C_2$ with the normalizations $\tilde{C}_1, \tilde{C}_2$ being isomorphic to $\mathbb{A}^1$ and $C_1, C_2$ meeting in one point which might be a cyclic quotient singular point of $X$;

2. $\Delta = \emptyset$ or $\Delta = \bigsqcup D_i$ with the normalization of $D_i$ being isomorphic to $\mathbb{A}^1$;

3. A connected component $D_i$ of $\Delta$ might have at most one singular point which is a cyclic quotient singularity.
(4) $X$ has at worst cyclic quotient singularities.

Proof. Let $\pi : \hat{X} \to X$ be a resolution of singularity. Then $\hat{\rho} := \rho \circ \pi : \hat{X} \to C$ is an $\mathbb{A}^1_*$-fibration. Furthermore, $\hat{X}$ has a smooth completion $\hat{X} \hookrightarrow W$ such that $W - \hat{X}$ is a divisor of simple normal crossings and the $\mathbb{A}^1_*$-fibration $\hat{\rho}$ extends to a $\mathbb{P}^1$-fibration $p : W \to \overline{C}$, where $\overline{C}$ is a smooth completion of $C$. Then $\overline{S} := p^{-1}(\rho(S))$ is a degenerate fiber of $p$. We may assume that $\overline{S} - S$ does not contain $(-1)$ components. Let $T$ be a maximal connected union of irreducible components of $\overline{S} - S$ such that $T \cap (W - (\hat{X} \cup \overline{S})) = \emptyset$. Since $X$ is affine, $T$ meets at least one irreducible component of $S$. If $T$ meets one irreducible component of the proper transform of $S$, then the dual graph of $T$ is a linear chain and $T$ contracts to a cyclic quotient singular point. In fact, if the dual graph of $T$ has a branching point then the successive contractions of $(-1)$ curves in $\overline{S}$ will produce either three components meeting in one point or two components meeting in the same point of a cross-section (or a double point of 2-section). This is a contradiction. A similar argument shows that if $T$ meets two irreducible components of $S$ then $T$ again contracts to a cyclic quotient singular point. Since $\rho$ is an $\mathbb{A}^1_*$-fibration, $T$ cannot meet three or more irreducible components of $S$. These considerations imply the stated assertions. Q.E.D.

2.5 Affine pseudo-coverings

Let $X$ be a smooth affine variety and let $f : Y \to X$ be a morphism of algebraic varieties. We say that $f$ is almost surjective if $\text{codim}_X (X - f(Y)) \geq 2$ and that $Y$ is an affine pseudo-covering of $X$ if $Y$ is affine, $f$ is étale and
almost surjective. Note that an affine pseudo-covering is not necessarily an ordinary finite covering. If the field extension $k(Y)/k(X)$ is a Galois extension, the affine pseudo-covering is called Galois. We are interested in determining all affine pseudo-coverings of the affine plane. We recall some known results [35].

**Lemma 2.5.1** Let $f : Y \to X$ be an étale morphism of smooth affine varieties. Suppose that $\text{Pic}(X) = 0$ and $\Gamma(Y, \mathcal{O}_Y)^* = \mathbb{C}^*$. Then $Y$ is an affine pseudo-covering of $X$.

**Proof.** It suffices to show that $f$ is almost surjective. Let $A$ (resp. $B$) be the coordinate ring of $X$ (resp. $Y$). Since $f$ is a dominant morphism, we may and shall identify $A$ with a subalgebra of $B$ via $f^*$. Then $A$ is factorial and $B^* = k^*$ by the hypothesis. Let $Z$ be an irreducible subvariety of $X$ of codimension one. Since $A$ is factorial, $Z$ is defined by an element $a \in A$. Let $\mathfrak{p} = aA$ be the defining ideal of $Z$. Since $B^* = k^*$, the ideal $aB$ is not equal to $B$. Let

$$aB = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$$

be the primary decomposition of the ideal $aB$. Since $B$ is normal, each primary ideal $\mathfrak{q}_i$ ($1 \leq i \leq r$) has height one. Let $V_i$ be the subvariety in $Y$ defined by the radical $\mathfrak{P}_i := \sqrt{\mathfrak{q}_i}$. Let $W_i$ be the subvariety defined by a prime ideal $\mathfrak{P}_i \cap A$. Note that $W_i \subseteq Z$ because $\mathfrak{p} \subseteq \mathfrak{P}_i \cap A$. It suffices to show that $W_i = Z$ for every $1 \leq i \leq r$. Suppose to the contrary that $W_i$ is a proper subvariety of $Z$. Then $f|_{V_i} : V_i \to W_i$ is a dominant morphism with $\dim V_i > \dim W_i$. Hence a general fiber of $f|_{V_i}$ has positive dimension. This is, however, impossible because $f$ is étale. Q.E.D.
This lemma implies that if $X$ is a smooth affine variety with $\text{Pic} \,(X) = 0$ and $\Gamma(X, \mathcal{O}_X)^* = \mathbb{C}^*$, then an étale endomorphism $\varphi : X \to X$ is almost surjective. This is, in particular, the case for $X = \mathbb{A}^n$. Let $f : Y \to X$ be a Galois affine pseudo-covering of smooth affine varieties with Galois group $G := \text{Gal}(\mathbb{C}(Y)/\mathbb{C}(X))$ and let $\tilde{Y}$ be the normalization of $X$ in $\mathbb{C}(Y)$. Then $Y$ is a Zariski open set of $\tilde{Y}$ and the group $G$ acts freely on $\tilde{Y}$ so that $X = \tilde{Y}/G$. The purity of branch loci implies that the normalization morphism $\tilde{f} : \tilde{Y} \to X$ is a finite étale morphism and that $\tilde{Y}$ is thereby smooth.

We shall consider various examples of affine pseudo-coverings mostly in the case where $X$ is a smooth affine surface. Let $X := X(d, r)$ be an affine pseudo-plane. Then the universal covering $\tilde{X} := \tilde{X}(d, r)$ of $X$ has a Galois group $H(d) \cong \mathbb{Z}/d\mathbb{Z}$. By Lemma 1.4.20, $\tilde{X}$ contains an open set $U_\omega$ which is isomorphic to $\mathbb{A}^2$ and mapped surjectively onto $X$ by the covering mapping, where $\omega \in H(d)$. Hence $\mathbb{A}^2$ is a Galois affine pseudo-covering of $X$. Slightly generalizing this result, we can prove the following result.

**Lemma 2.5.2** Let $X$ be a $\mathbb{Q}$-homology plane with $\kappa(X) = -\infty$. Hence there exists an $\mathbb{A}^1$-fibration $\rho : X \to C$, where $C \cong \mathbb{A}^1$. Suppose that $\rho$ has a unique multiple fiber $dF$, where $F \cong \mathbb{A}^1$ and $d \geq 2$. Then $\mathbb{A}^2$ is a Galois affine pseudo-covering of $X$ with Galois group $H(d)$.

**Proof.** The proof is just a repetition of the previous argument used in the construction of the universal covering of an affine pseudo-plane. Let $P = \rho(F)$ and let $\mu : \tilde{C} \to C$ be a $d$-ple cyclic covering which ramifies totally over $P$ and the point at infinity $P_\infty$ of $C$. Let $\tilde{Y}$ be the normalization of the fiber product $X \times_C \tilde{C}$ and $\tilde{f} : \tilde{Y} \to X$ be the composite of the normalization
morphism \( \tilde{Y} \to X \times_C \tilde{C} \) and the first projection \( X \times_C \tilde{C} \to X \). Then \( \tilde{f} \) is a Galois étale covering, and the multiple fiber \( dF \) splits into a disjoint union of \( d \) copies of the affine line on \( \tilde{Y} \). Let \( Y \) be an open set of \( \tilde{Y} \) obtained by removing \( d - 1 \) copies of \( \mathbb{A}^1 \) from \( \tilde{f}^{-1}(F) \). Then \( Y \) is isomorphic to \( \mathbb{A}^2 \) and mapped surjectively onto \( X \) by \( \tilde{f} \). Q.E.D.

We consider if the converse to the above remark holds. Let \( X \) be a smooth affine surface and let \( f : \mathbb{A}^2 \to X \) be a Galois affine pseudo-covering with Galois group \( G \). Let \( Y := \mathbb{A}^2 \) and let \( \tilde{f} : \tilde{Y} \to X \) be the normalization of \( X \) in the function field of \( \mathbb{A}^2 \). Then \( \tilde{f} \) is a finite étale Galois covering with Galois group \( G \) and \( \tilde{Y} \) contains \( Y \) as a Zariski open set. Furthermore, \( \tilde{Y} - Y \) is the disjoint union \( \bigsqcup_{i=1}^{r} C_i \) of the curves isomorphic to \( \mathbb{A}^1 \).

Lemma 2.5.3 Let the notations and assumptions be the same as above. We assume, furthermore, that the following hypothesis (H) holds:

\[
g(C_i) \cap h(C_j) = \emptyset \quad \text{whenever} \quad g(C_i) \neq h(C_j) \quad \text{for} \quad 1 \leq i, j \leq r \quad \text{and} \quad g, h \in G.
\]

Then there exists an \( \mathbb{A}^1 \)-fibration \( \tilde{\rho} : \tilde{Y} \to \tilde{C} \) such that the following conditions are satisfied.

1. The group \( G \) preserves the \( \mathbb{A}^1 \)-fibration \( \tilde{\rho} \). Namely, for any fiber component \( A \) of \( \tilde{\rho} \), the translate \( g(A) \) is a fiber component of \( \tilde{\rho} \) for \( g \in G \).

2. \( g(C_i) \) is a fiber component of \( \tilde{\rho} \) for \( 1 \leq i \leq r \) and \( g \in G \).

\footnote{In fact, consider any \( \mathbb{A}^1 \)-fibration \( \rho : Y := \mathbb{A}^2 \to C \). Then \( \rho \) extends to an \( \mathbb{A}^1 \)-fibration \( \tilde{\rho} : \tilde{Y} \to \tilde{C} \), where \( \rho = \tilde{\rho} \mid_Y \) and \( \tilde{C} \) contains \( C \) as an open set. Then it is readily shown that \( \tilde{Y} - Y \) consists of the fiber components of \( \tilde{\rho} \) which are necessarily isomorphic to \( \mathbb{A}^1 \).}
(3) The $\mathbb{A}^1$-fibration $\tilde{\rho}$ restricts to an $\mathbb{A}^1$-fibration $\rho : Y \to C$, where $C$ is a Zariski open set of $\tilde{C}$ and isomorphic to $\mathbb{A}^1$ and where $\rho$ is surjective.

(4) The curve $\tilde{C}$ is isomorphic to either $\mathbb{P}^1$ or $\mathbb{A}^1$.

(5) There exists a group homomorphism $\alpha : G \to \text{Aut} \tilde{C}$ which is injective.

Let $\tilde{K}$ be the function field of $\tilde{C}$. Then $G$ is the Galois group of the field extension $\tilde{K}/K$, where $K = \tilde{K}^G$.

**Proof.** Since $f : Y \to X$ is almost surjective, there exists a translate $g(C_i)$ for some $g \in G$ such that $g(C_i) \cap Y \neq \emptyset$. Then $g(C_i) \subset Y$. In fact, if not, $g(C_i) \cap C_j \neq \emptyset$ for some $j$. This contradicts the hypothesis. Since $g(C_i)$ is isomorphic to $\mathbb{A}^1$, Theorem of Abhyankar-Moh-Suzuki implies that there exists a trivial $\mathbb{A}^1$-bundle structure $\rho : Y \to C$ with $g(C_i)$ as a fiber, where $C \cong \mathbb{A}^1$. Note that if $h(C_j) \cap Y \neq \emptyset$ for $h \in G$, then $h(C_j)$ is a fiber of $\rho$.

Let $F$ be a fiber which is not of the form $h(C_j)$ with $1 \leq j \leq r$ and $h \in G$. Then, for any $g \in G$, the translate $g(F)$ meets none of the $h(C_j)$. In fact, if $g(F) \cap h(C_j) \neq \emptyset$, then $F \cap g^{-1}h(C_j) \neq \emptyset$, which implies that $F = g^{-1}h(C_j)$ because both are the fibers of $\rho$. This is a contradiction to the choice of $F$.

It is now clear that $\rho : Y \to C$ extends to an $\mathbb{A}^1$-fibration on $\tilde{\rho} : \tilde{Y} \to \tilde{C}$, where the parameter space $\tilde{C}$ for $\tilde{\rho}$ contains $C$ as a Zariski open set.

We shall show the assertion (5). Since $G$ preserves the $\mathbb{A}^1$-fibration $\tilde{\rho} : \tilde{Y} \to \tilde{C}$, there exists a natural group homomorphism $\alpha : G \to \text{Aut} \tilde{C}$. We

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6We may choose coordinates \{x, y\} on $Y$ so that the curve $g(C_i)$ is defined by $y = 0$. Suppose that $h(C_j)$ is defined by $f = 0$. Then $f$ as well as $f - \alpha$ is irreducible for all $\alpha \in k$ by the theorem of Abhyankar-Moh-Suzuki. Suppose $h(C_j) \cap g(C_i) = \emptyset$. Then we have $f(x, 0) = c \in k$. Namely, $f - c = yg(x, y)$ with $g(x, y) \in k[x, y]$. Since $f - c$ is irreducible, we must have $g(x, y) \in k^*$. 

shall show that $\alpha$ is injective. Since $\rho : Y \to C$ is a trivial $\mathbb{A}^1$-bundle, the generic fiber $Y_{\tilde{K}}$ of $\rho$ is the affine line $\mathbb{A}^1_{\tilde{K}} := \text{Spec} \, \tilde{K}[x]$, where $\tilde{K}$ is the function field of $\tilde{C}$. Then we have

$$g(x) = a(g)x + b(g) \quad \text{with} \quad a(g) \in \tilde{K}^*, \, b(g) \in \tilde{K} \quad \text{for} \quad g \in G,$$

where

$$a(gh) = a(h)^{g}a(g) \quad \text{and} \quad b(gh) = a(h)^{g}b(g) + b(h)^{g} \quad \text{for} \quad g, h \in G.$$

Suppose that $\ker \alpha \neq \{0\}$. Then it follows by induction that

$$b(g^{m}) = (a(g)^{m-1} + \cdots + a(g) + 1) b(g) \quad \text{for} \quad g \in \ker \alpha,$$

where $b(1) = 0$. If $a(g) \neq 1$, then the point given by $x = -b(g)/(a(g) - 1)$ is left fixed by the action of $g$, but this is not possible because $G$ acts on $\tilde{Y}$ freely. If $a(g) = 1$ and $m$ is the order of $g$, then $b(g) = 0$ because $a(g)^{m-1} + \cdots + a(g) + 1 = m \neq 0$ and $b(g^{m}) = 0$. Then $g$ acts on $\tilde{Y}$ trivially, and this is a contradiction. Hence $\ker \alpha = \{1\}$ and $\alpha$ is injective.

**Q.E.D.**

**Theorem 2.5.4** Let $X$ be a smooth affine surface and let $f : \mathbb{A}^2 \to X$ be a Galois affine pseudo-covering with Galois group $G$. Assume that the hypothesis (H) in Lemma 2.5.3 is satisfied. Then the following assertions hold.

1. There exist an $\mathbb{A}^1$-fibration $\rho : X \to C$ and a finite $G$-morphism of smooth curves $\pi : \tilde{C} \to C$ such that the given morphism $f : \mathbb{A}^2 \to X$ factors as $f = \tilde{f} \circ \iota$, where $\tilde{f} : \tilde{Y} \to X$ is a composite of the normalization morphism $\tilde{Y} \to X \times_C \tilde{C}$ with the normalization $\tilde{Y}$ of $X \times_C \tilde{C}$ in the function field of $\mathbb{A}^2$ and the canonical projection $X \times_C \tilde{C} \to X$ and where $\iota : \mathbb{A}^2 \to \tilde{Y}$ is an open immersion.
(2) $C$ is isomorphic to either $\mathbb{A}^1$ or $\mathbb{P}^1$.

(3) If $C$ is isomorphic to $\mathbb{A}^1$ and every fiber of $\rho$ is irreducible, then $X$ is a $\mathbb{Q}$-homology plane and the Galois group is a cyclic group. Furthermore, the $\mathbb{A}^1$-fibration $\rho : X \to C$ has a single multiple fiber.

(4) If $C$ is isomorphic to $\mathbb{P}^1$ and every fiber of $\rho$ is irreducible, then the $\mathbb{A}^1$-fibration $\rho : X \to C$ has at most three multiple fibers. If there are three multiple fibers $m_iF_i$ with $m_i > 1$ and $F_i \cong \mathbb{A}^1$, then $\{m_1, m_2, m_3\}$ is $\{2, 2, n\}, \{2, 3, 3\}, \{2, 3, 4\}$ or $\{2, 3, 5\}$ up to permutations.

Proof. Our proof consists of several steps. We may assume that the Galois group $G$ is non-trivial. If $G$ is trivial, the morphism $f$ is an isomorphism.

(I) Let $Y := \mathbb{A}^2$ and let $\tilde{Y}$ be the normalization of $X$ in the function field of $Y$. We follow the proof of Lemma 2.5.3. By the construction of the $\mathbb{A}^1$-fibration $\rho : Y \to C$, we find a $G$-stable open set $\tilde{W}$ of $Y$ such that $Y - \tilde{W}$ is a union of finitely many fibers of the $\mathbb{A}^1$-bundle and $\tilde{W} = \tilde{U} \times \mathbb{A}^1$. Note that $\tilde{U}$ is an open set of the affine line. Hence we may write $\tilde{U} = \text{Spec } k[t, u(t)^{-1}]$, where $u(t) \in k[t]$ and where we may assume that $t \mid u(t)$. Then $\tilde{W} = \text{Spec } A$ with $A = k[t, u(t)^{-1}, x]$ and $A^*$ is generated by $k^*$ and distinct prime factors of $u(t)$. Since $A$ is a $G$-algebra over $k$, the $G$-action preserves $A^*$. Hence $G$ acts on the function field $\tilde{K} := k(t)$. Let $K = \tilde{K}^G$ and $A_{\tilde{K}} = A \otimes_{k[t]} k(t) = \tilde{K}[x]$. Then $G$ is the Galois group of the field extension $\tilde{K}/K$ and $G$ acts on $A_{\tilde{K}}$. Hence, as in the proof of Lemma 2.5.3, we may write

$$g(x) = a(g)x + b(g) \quad \text{with} \quad a(g), b(g) \in K \quad \text{for} \quad g \in G,$$
where

\[ a(gh) = a(h)^g a(g) \quad \text{and} \quad b(gh) = a(h)^g b(g) + b(h)^g \quad \text{for} \quad g, h \in G. \]

By Theorem 90 of Hilbert, we find \( c \in \tilde{K}^* \) such that \( a(g) = c(c^g)^{-1} \) for \( g \in G \).

By replacing \( x \) by \( cx \), we may then assume that \( a(g) = 1 \). Then \( b(gh) = b(g) + b(h)^g \) for \( g, h \in G \). Hence \( b(g) = d - d^g \) with \( d = \left( \sum_{h \in G} b(h) \right) / |G| \).

By replacing \( x \) by \( x + d \), we may assume that \( g(x) = x \) for \( g \in G \). Namely, a suitable choice of the fiber coordinate \( x \) on the generic fiber (hence for all fibers over a sufficiently small open set of \( \tilde{U} \)), we may assume that the group \( G \) acts trivially on the fibers. We may assume that the curve \( \tilde{U} \) is chosen in such a way that \( G \) acts trivially on the factor \( \mathbb{A}^1 \) in the product \( \tilde{W} = \tilde{U} \times \mathbb{A}^1 \).

(II) Let \( U \) be the quotient \( \tilde{U} / G \). Then \( \tilde{W} := U \times \mathbb{A}^1 \) is an open set of \( X \), and \( \tilde{W} = W \times_U \tilde{U} \). The morphism \( f|_{\tilde{W}} : \tilde{W} \to W \) is induced by the quotient morphism \( q : \tilde{U} \to U \). Let \( \tilde{B} \) and \( B \) be the smooth completions of \( \tilde{U} \) and \( U \), respectively. Then \( \tilde{B} \) and \( B \) are isomorphic to \( \mathbb{P}^1 \). The morphism \( q \) extends to a morphism \( \pi : \tilde{B} \to B \). The \( G \)-action on \( \tilde{U} \) extends naturally to \( \tilde{B} \) and \( \pi \) is the quotient morphism \( \tilde{B} \to \tilde{B}/G \), where \( B = \tilde{B}/G \). On the other hand, the projection \( p_1 : W \to U \) extends to a morphism \( \rho : X \to C \), where \( C \) is an open set of the curve \( B \). Since \( \rho \circ f : \mathbb{A}^2 \to C \) is a dominant morphism, \( C \) is isomorphic to \( \mathbb{A}^1 \) or \( \mathbb{P}^1 \). Let \( \tilde{C} \) be the inverse image \( \pi^{-1}(C) \) in the curve \( \tilde{B} \). Then \( \pi : \tilde{C} \to C \) is a finite (possibly ramified) \( G \)-morphism with \( C = \tilde{C}/G \). By the above construction, it follows that \( f : \mathbb{A}^2 \to X \) factors as \( \mathbb{A}^2 \xrightarrow{\sigma} X \times_C \tilde{C} \xrightarrow{p_X} X \), where \( \sigma \) is a birational morphism and the canonical projection \( p_X \) is a finite morphism. Hence \( \tilde{Y} \) is the normalization of \( X \times_C \tilde{C} \).
in the function field of \( Y = \mathbb{A}^2 \), and \( f : Y \to X \) factors as indicated in the statement, where \( \iota \) is an open immersion by Zariski Main Theorem. Note by the above construction that \( \tilde{Y} \) has an \( \mathbb{A}^1 \)-fibration \( \tilde{\rho} : \tilde{Y} \to \tilde{C} \) such that the restriction \( \tilde{\rho} |_Y : Y \to \tilde{\rho}(Y) \) is an \( \mathbb{A}^1 \)-bundle over \( \mathbb{A}^1 \). Thus the first and second assertions are verified.

(III) Suppose that \( C \) is isomorphic to \( \mathbb{A}^1 \) and every fiber of \( \rho \) is irreducible. Then \( X \) is a \( \mathbb{Q} \)-homology plane because \( \text{Pic} (X) \) is a finite torsion group. Since \( G \) is a finite subgroup of \( \text{Aut} \mathbb{A}^1 \), \( G \) is a cyclic group. We shall show that the \( \mathbb{A}^1 \)-fibration \( \rho : X \to C \) has a single multiple fiber. If there are no multiple fibers, then \( \rho : X \to C \) is a trivial \( \mathbb{A}^1 \)-bundle over \( \mathbb{A}^1 \). Hence \( X \) is isomorphic to \( \mathbb{A}^2 \). Then \( \tilde{f} : \tilde{Y} \to X \) is an isomorphism. This case was excluded at the beginning. Let \( m_i F_i \) \((1 \leq i \leq s)\) exhaust all singular fibers of \( \rho \). Let \( P_i = \rho(F_i) \) and let \( P_\infty = B - C \), where \( B = \tilde{B}/G \cong \mathbb{P}^1 \). By the construction of \( \tilde{Y} \) in the assertion (2) (see also Lemma 2.5.5 below), we know that

(i) The Galois covering \( \pi : \tilde{B} \to B \) ramifies at the points \( \tilde{\pi}^{-1}(P_i) \) with ramification index \( N/m_i \) for \( 1 \leq i \leq s \) and totally ramifies over the point \( P_\infty \);

(ii) The curve \( \tilde{C} \) is isomorphic to \( \mathbb{A}^1 \) and \( \tilde{\rho} : \tilde{Y} \to \tilde{C} \) has \( m_i \) connected components over every point of \( \tilde{\pi}^{-1}(P_i) \) for \( 1 \leq i \leq s \), each of which is isomorphic to \( \mathbb{A}^1 \);

(iii) In order to obtain \( Y \), we remove \( m_i - 1 \) components from the fiber of \( \tilde{\rho} \) over every point of \( \tilde{\pi}^{-1}(P_i) \) for \( 1 \leq i \leq s \).
Let $N$ be the order of $G$. By the Riemann-Hurwitz theorem, we have
\[-2 = -2N + \sum_{i=1}^{s} N \frac{m_i - 1}{m_i} + (N - 1),\]
from which we obtain
\[\frac{1}{m_1} + \cdots + \frac{1}{m_s} = (s - 1) + \frac{1}{N}.
\]
This implies that $s = 1$ and $m_1 = N$.

(IV) We shall consider the assertion (4). Suppose first that there are no multiple fibers of $\rho$. Hence $\rho : X \to C$ is an $\mathbb{A}^1$-bundle over $C \cong \mathbb{P}^1$. Hence $\tilde{Y} \cong X \times_C \tilde{C}$ and $\tilde{Y} \to \tilde{C}$ is an $\mathbb{A}^1$-bundle over $\tilde{C} \cong \mathbb{P}^1$ as well. Since $\tilde{\rho} \mid_Y$ is an $\mathbb{A}^1$-bundle over $\mathbb{A}^1$, only one fiber of $\tilde{\rho}$ is removed to obtain $Y$. On the other hand, since $\pi : \tilde{C} \to C$ is a finite $G$-morphism with non-trivial $G$, the morphism $f : Y \to X$ is not an étale morphism. This is a contradiction. So, $\rho$ has at least one multiple fiber. Let $m_i F_i$ $(1 \leq i \leq s)$ exhaust all the multiple fibers of $\rho$, where $C \cong \mathbb{P}^1$ and let $P_i = \rho(F_i)$. Suppose $s > 2$. Let $g : D \to C$ be a Galois covering which ramifies over the points $P_i$ with ramification index $m_i$. Such a covering exists by [8] and [15]. Let $Z$ be the normalization of $X \times_C D$ and let $h : Z \to X$ be a composite of the normalization morphism $Z \to X \times_C D$ and the projection $X \times_C D \to X$. Then $h$ is an étale finite morphism. Since $\tilde{f} : \tilde{Y} \to X$ is the universal covering morphism, there exists a finite étale morphism $\tau : \tilde{Y} \to Z$ such that $\tilde{f} = h \circ \tau$. Hence the curve $D$ is a rational curve. This implies that
\[2N - 2 = \sum_{i=1}^{s} N \frac{m_i - 1}{m_i}.
\]
It then follows that $s = 3$ and \{m_1, m_2, m_3\} is \{2, 2, n\}, \{2, 3, 3\}, \{2, 3, 4\} or \{2, 3, 5\} up to permutations. Hence the assertion (4) holds. Q.E.D.
Let $X$ be a smooth affine variety. We define the *Makar-Limanov invariant* $ML(X)$ to be the intersection of the invariant subrings of the coordinate ring of $X$ under all possible $G_a$-actions on $X$. Let $X$ be a $\mathbb{Q}$-homology plane having the properties in the assertion (3) of Theorem 2.5.4. The Makar-Limanov invariant $ML(X)$ is either a polynomial ring $k[x]$ or the constant field $k$. By virtue of Theorem 1.4.10, the affine pseudo-plane $X := X(d, r)$ has the Makar-Limanov invariant $k[x]$ provided $r \geq 2$. On the other hand, we call a smooth affine surface $X$ a *Platonic $A_1$-fiber space* if $X$ has an $A_1$-fibration having the properties in the assertion (4) of Theorem 2.5.4. Let $X$ be a smooth affine surface with $A_2$ as a Galois affine pseudo-covering. Lemma 2.5.5 below implies that the Picard rank $\dim \text{Pic}(X)_\mathbb{Q}$ can be arbitrarily big.

**Lemma 2.5.5** Let $\rho : X \to C$ be an $A_1$-fibration with $C \cong A_1$ with one multiple irreducible fiber and $r$ reducible fibers with respective multiplicity one. Write $\rho^*(P_0) = mF_0$ with $m > 1$ and $\rho^*(P_i) = \sum_{j=1}^{n_i} F_{ij}$ with connected components $F_{ij}$ for $1 \leq i \leq r$ and $n_i > 1$, where $F_0$ and $F_{ij}$ are isomorphic to $A^1$. Suppose $m \geq n_i$ for all $i$. Let $\pi : \tilde{C} \to C$ be a cyclic Galois covering of degree $m$ ramifying totally over $P_0$ and the point at infinity $P_{\infty}$. Let $\tilde{Y}$ be the normalization of $X \times_C \tilde{C}$ and let $\tilde{\rho} : \tilde{Y} \to \tilde{C}$ be the composite of the normalization morphism and the canonical projection, where $\tilde{C} \cong A^1$. Let $Q_0 = \pi^{-1}(P_0)$ and let $\pi^{-1}(P_i) = \{Q_i^{(1)}, \ldots, Q_i^{(m)}\}$. Then $\tilde{\rho}^{-1}(Q_0)$ consists of $m$ affine lines with multiplicity one and $\tilde{\rho}^{-1}(Q_i^{(\ell)})$ has also $n_i$ disjoint affine lines $\sum_{j=1}^{n_i} \tilde{F}_{ij}^{(\ell)}$ for $1 \leq i \leq r$ and $1 \leq \ell \leq m$. Leave only one component among the $m$ affine lines in $\tilde{\rho}^{-1}(Q_0)$ and leave one component, say $G_i^{(\ell)}$, in $\sum_{j=1}^{n_i} \tilde{F}_{ij}^{(\ell)}$ which is chosen so that the images of $G_i^{(\ell)}$ ($1 \leq \ell \leq m$) by $\tilde{f} : \tilde{Y} \to X$ cover $\sum_{j=1}^{n_i} F_{ij}$ for each $1 \leq i \leq r$. Let $Y$ be the open set
of $\tilde{Y}$ with only one irreducible component left in $\tilde{\rho}^{-1}(Q_0)$ and $\tilde{\rho}^{-1}(Q_i^{(0)})$ for $1 \leq i \leq r$ and $1 \leq \ell \leq m$. Then $Y$ is isomorphic to $\mathbb{A}^2$ and the restriction $f = \tilde{f} \mid_Y : Y \to X$ is a Galois étale pseudo-covering. It is easy to compute $\dim \text{Pic} (X)_\mathbb{Q} = \sum_{i=1}^{r} (n_i - 1)$. Hence $\dim \text{Pic} (X)_\mathbb{Q}$ can be arbitrarily big.

\textbf{Proof.} The proof should be clear from the construction. Q.E.D.

It is almost clear that a Galois affine pseudo-covering of a topologically simply-connected smooth algebraic variety is trivial, that is to say, the covering morphism is an isomorphism. There are, nonetheless, non-trivial affine pseudo-coverings of the affine space. We just give few examples in the case of $\mathbb{A}^1$. It is obvious that we can produce various examples of affine pseudo-coverings of $\mathbb{A}^n$ by taking direct products.

\textbf{Example 2.5.1} Let $C$ be a smooth cubic curve in $\mathbb{P}^2$. Let $P$ be a flex of $C$ and let $\ell_P$ be the tangent line of $C$ at $P$. Choose a point $Q$ on $\ell$ such that $Q \neq P$ and that the other tangent lines of 8 other flexes do not meet $\ell_P$ on $Q$. Consider the projection of $\mathbb{P}^2$ to $\mathbb{P}^1$ with center $Q$. Since every line $\ell$ through $Q$ other than $\ell_P$ meets the curve $C$ either in three distinct points or in two distinct points with intersection multiplicities 1 and 2, we obtain an open set $Y$ of $C$ by omitting the point $P$ and the points where the line through $Q$ meets $C$ with intersection multiplicity 2 and a surjective étale morphism $f : Y \to \mathbb{A}^1$ by restricting the projection onto $Y$. This morphism

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7Let $f : Y \to X$ be a Galois affine pseudo-covering, where $\pi_1(X) = (1)$. Let $\tilde{f} : \tilde{Y} \to X$ be the induced Galois étale finite covering, where $Y$ is an open set of $\tilde{Y}$ and $f$ is the restriction of $\tilde{f}$ onto $Y$. Then $\tilde{f}$ is an isomorphism because $\pi_1(X) = (1)$. Hence $f : Y \to X$ is an open immersion such that $\text{codim}_X (X - f(Y)) \geq 2$. Since $Y$ is affine, it follows that $X = f(Y)$. 


has degree 3 and is non-Galois.

**Example 2.5.2** By the same construction as above, we can assume that $Y$ is rational. In fact, start with a cuspidal cubic $X_0^2X_2 = X_1^3$ and take the point $Q := (0, 0, 1)$ as the center of the projection. The lines through $Q$ are given by $\{a(X_0 - X_1) = X_2 \mid a \in k \cup \{\infty\}\}$. Let $Y$ be the curve $C$ with the cusp, the point $(0, 0, 1)$ and one more point removed off and let $f$ be the restriction of the projection from $Q$ onto $Y$. Then $f : Y \to \mathbb{A}^1$ is a surjective étale morphism.

The following result shows that the surfaces appearing in Theorem 2.5.4 cannot be affine pseudo-coverings of $\mathbb{A}^2$.

**Theorem 2.5.6** Let $X$ be either a $\mathbb{Q}$-homology plane with $\text{ML}(X) = k$ or a Platonic $\mathbb{A}^1$-fiber space possibly except for the case where the multiplicity sequence is $\{2, 2, m\}$ with $m > 5$. Then there are no étale morphisms from $X$ to $\mathbb{A}^2$.

**Proof.** We first treat the case where $X$ is a $\mathbb{Q}$-homology plane with $\text{ML}(X) = k$. Let $f : X \to \mathbb{A}^2$ be an étale morphism which we do not have to assume to be almost surjective. Let $p : S \to X$ be the universal covering, where $S$ is an affine hypersurface $xy = z^n - 1$ in $\mathbb{A}^3$. We refer to [29] for relevant results. Let $\varphi = f \cdot p$. Then $\varphi : S \to \mathbb{A}^2$ is an étale morphism. We show that such $\varphi$ does not exist. Note that the canonical divisor $K_S$ is trivial and $\Gamma(O_S)^* = k^*$. By a straightforward computation, one can show that a differential 2-form $\omega = (1/z^{n-1})dx \wedge dy$ is an everywhere nonzero regular form. On the other hand, since $\varphi$ is étale, there exist elements $f, g$ of $\Gamma(O_S)$ such that $\varphi^*(df \wedge dg)$ is an everywhere nonzero regular 2-form. Hence we
have \( \omega = c\varphi^*(df \wedge dg) \). Meanwhile, there exists a free \( \mathbb{Z}/n\mathbb{Z} \)-action given by \( (x, y, z) \mapsto (\zeta x, \zeta^{-1} y, \zeta^d z) \), where \( \zeta \) is a primitive \( n \)-th root of unity with \( n = |G| \) and \( 0 < d < n \). Then \( \zeta(\omega) = \zeta^{-d(n-1)}\omega \) and \( \varphi^*(df \wedge dg) \) is invariant under this \( \mathbb{Z}/n\mathbb{Z} \)-action because \( \varphi \) splits via \( X \). So, \( d(n-1) \) is divisible by \( n \), which is not the case.

Next consider the case where \( X \) is a Platonic \( \mathbb{A}^1 \)-fiber space. We consider, in general, the case where \( X \) has an \( \mathbb{A}^1 \)-fibration \( \rho : X \to \mathbb{P}^1 \) whose fibers are all irreducible. Let \( m_1 F_1, \ldots, m_r F_r \) exhaust all multiple fibers with \( m_i > 1 \) and \( F_i \cong \mathbb{A}^1 \). As in the first case, the assertion holds provided \( K_X \not\sim \mathcal{O}_X \).

We consider a smooth compactification \( V \) of \( X \) such that \( V \) is obtained by a sequence \( \sigma \) of blowing-ups from a minimal ruled surface \( \Sigma_n \) with \( n \geq 0 \) and the closures \( \overline{F}_i \) of \( F_i \) are unique \((-1)\) components of the singular fibers \( \Phi_i \). Let \( T \) be the image under \( \sigma \) of the unique boundary component which is transversal to the \( \mathbb{P}^1 \)-fibration on \( V \) which extends the given \( \mathbb{A}^1 \)-fibration \( \rho \). Let \( M \) be a minimal section of \( \Sigma_n \) and write \( T \sim M + a\ell \) with a fiber \( \ell \). Let \( k_i \) be the coefficient of \( \overline{F}_i \) in \( K_V \) which is written as

\[
K_V \sim -2M - (n + 2)\ell + \sum_{i=1}^{r} (k_i \overline{F}_i + \cdots).
\]

Since \( T \sim M + a\ell \), we have

\[
K_X \sim \left\{ (2a - n - 2) + \sum_{i=1}^{r} \frac{k_i}{m_i} \right\} \ell, \tag{1}
\]

where \( \ell \) signifies abusively the restriction of \( \ell \) on \( X \). The dual graph of \( \Phi_i \) together with the proper transform \( T' \) of \( T \) looks like:
Let $h_i$ be the number of blowing-ups to be needed to produce this dual graph from the single fiber $\ell_i$ of $\Sigma_n$. The center of the first blowing-up is called the starting point. We may assume that the starting point on the fiber $\ell$ does not lie on the section $T$. We omit the suffix $i$ to denote $m_i, k_i$ and $h_i$. If $m$ is small, we can express $k$ in terms of $h$. Suppose $m = 2$. Then $k = h$. Suppose $m = 3$. Then $k = h + 1$ if $(-3) \cap T' = \emptyset$ and $k = h$ if $(-3) \cap T' \neq \emptyset$, where $T'$ is the proper transform of $T$ on $V$ and $(-3) \cap T' = \emptyset$ (resp. $\neq \emptyset$) means that the adjacent component of $T'$ in the graph is not (resp. is) a $(-3)$-component. Suppose $m = 4$. Then $k = h + s - 1$ with $s \geq 3$ and $h \geq s + 2$ if the dual graph of $\Phi$ has more than one branching vertices, where the vertices are connected to more than two adjacent vertices, which is necessarily as in Figure 1 below with the proper transform $\ell'$ of $\ell$.

![Figure 1](image-url)
If the dual graph of Φ has only one branching vertex and \((-4) \cap T' = \emptyset\) (resp. \(\neq \emptyset\)), then \(k = h + 2\) (resp. \(k = h\)). Suppose \(m = 5\). Then \(k = h, k = h + 2\) or \(k = h + 3\), resp. if \((-5) \cap T' \neq \emptyset\), \((-3) \cap T' \neq \emptyset\), or \((-2) \cap T' \neq \emptyset\), resp. There are two cases when \((-2) \cap T' \neq \emptyset\). Namely, the proper transform \(\ell'\) meets either a \((-2)\)-component of \((-3)\)-component. On the other hand, it is rather easy to show by induction on the number of blowing-ups that \(k > m\).

We shall prove the assertion of Theorem 2.5.6 in the second case. Recall that \(T \sim M + a \ell\), where \(a \geq n\) or \(a = 0\). Suppose \(T \neq M\). Since \(X\) is a Platonic \(\mathbb{A}^1\)-fiber space, we have \(r = 3\). Hence \(I > 0\) in the relation (1), where

\[
I = \left\{ (2a - n - 2) + \sum_{i=1}^{3} \frac{k_i}{m_i} \right\}.
\]

In fact, \(k_i/m_i > 1\) and \(I > (2a - n - 2) + 3 \geq (2n - n - 2) + 3 = n + 1 \geq 1\).

So, we assume that \(T = M\). In this case, we show that the following three conditions lead to a contradiction.

1. For every \(1 \leq i \leq 3\), \(k_i\) is divisible by \(h_i\) and \(k_i > m_i\).

2. \(\sum_{i=1}^{3} \frac{k_i}{m_i} = n + 2\).

3. The intersection matrix of \(T' + \sum_{i=1}^{3} (\Phi_i - m_i F_i)\) has a positive eigenvalue.

The conditions (1) and (2) are equivalent to saying that \(K_X\) is linearly equivalent to 0. The condition (3) is equivalent to saying that \(X\) is an affine surface. Let \(\Phi\) represent one of the three singular fibers \(\Phi_i\) with \(1 \leq i \leq 3\). Then \(\Phi\) is supported by \(\ell'\) and the irreducible exceptional curves \(E_1, \ldots, E_h\), where \(E_h = F\) is a unique \((-1)\) component. Note that \(T = M\) and \(M\) is untouched
under the sequence $\sigma$ of blowing-ups. We shall determine the coefficient $\alpha$ in
the expression $A := M + \alpha \ell' + \sum_{i=1}^{h-1} \beta_i E_i$, which is subject to the relation
$(A \cdot \ell') = (A \cdot E_i) = 0$ for $1 \leq i \leq h - 1$. A straightforward computation gives
the following result.

**Lemma 2.5.7** We have the following table.

| $m$ | $k$       | $\alpha$               |
|-----|-----------|------------------------|
| 2   | $h$       | $\frac{1}{4}h$         |
| 3   | $h + 1$   | $\frac{h+3}{9}$        |
| 3   | $h$       | $\frac{h}{9}$          |
| 4   | $h + s - 1$ | $\frac{(8s-3)(h-s)-(12s-5)}{16(2(h-s)-3)}$ |
| 4   | $h + 2$   | $\frac{h+8}{16}$       |
| 4   | $h$       | $\frac{h}{16}$         |
| 5   | $h + 3$   | $\frac{h+15}{25}$      |
| 5   | $h + 3$   | $\frac{h+11}{25}$      |
| 5   | $h + 2$   | $\frac{h+6}{25}$       |
| 5   | $h$       | $\frac{h}{25}$         |

Let $\alpha_i$ be the coefficient of $\ell'_i$ as computed as above for the fiber $\Phi_i$ for
$1 \leq i \leq 3$. Then the condition (3) is equivalent to $-n + \alpha_1 + \alpha_2 + \alpha_3 > 0$.
We just indicate an outline of the proof in the case where $\{m_1, m_2, m_3\} = \{2, 3, 5\}$,
$k_1 = h_1, k_2 = h_2$ and $k_3 = h_3$. The conditions (1), (2) and (3)
impose

\[ h_1 \geq 4, h_2 \geq 6, h_3 \geq 10, \]
\[ \frac{h_1}{2} + \frac{h_2}{3} + \frac{h_3}{5} = n + 2, \]
\[ \frac{h_1}{4} + \frac{h_2}{9} + \frac{h_3}{25} > n. \]

Then we have

\[ 2 > \frac{h_1}{4} + \frac{2}{9} h_2 + \frac{4}{25} h_3 > 3. \]

This is a contradiction. Q.E.D.

We excluded the case \( \{m_1, m_2, m_3\} = \{2, 2, m\} \) with \( m > 5 \). This is because we are not able to prove an inequality \( \alpha m > k \) with the above notations. So, we make the following conjecture.

**Conjecture 2.5.8** Let \( mF \) be a multiple fiber of a smooth \( \mathbb{A}^1 \)-fiber space with irreducible fibers. Let \( k \) be the coefficient of \( F \) in the canonical divisor \( K_X \). Define a rational number \( \alpha \) by completing the fiber \( F \) in a singular fiber of a \( \mathbb{P}^1 \)-fibration. Then we have \( \alpha m > k \).

**Remark 2.5.9** Theorem 2.5.6 asserts, in fact, the following result.

Let \( X \) be either a \( \mathbb{Q} \)-homology plane with \( \text{ML} (X) = k \) or a Platonic \( \mathbb{A}^1 \)-fiber space possibly except for the case where the multiplicity sequence is \( \{2, 2, m\} \) with \( m > 5 \). Then there are no \( \text{\'{e}tale} \) morphisms from \( X \) to an affine surface with trivial canonical divisor.

We shall next consider the case of affine pseudo-planes.

**Theorem 2.5.10** Let \( X \) be an affine pseudo-plane of type \( (d, n, r) \). If \( r \neq 2 \) then there are no \( \text{\'{e}tale} \) morphism from \( X \) to \( \mathbb{A}^2 \).
Proof. Let $\rho : X \to A$ be the given $\mathbb{A}^1$-fibration with a multiple fiber $dF_0$, where $A \cong \mathbb{A}^1$. A simple computation using the smooth compactification $(V, D)$ in Definition 1.4.5 shows that Pic $(X)$ is isomorphic to $\mathbb{Z}/d\mathbb{Z}$ with a generator $F_0$ and the canonical divisor $K_X$ is linearly equivalent to $(r - 2)F_0$. Hence $K_X \not\sim 0$ unless $r = 2$. If there is an étale morphism $\varphi : X \to \mathbb{A}^2$, then $K_X$ is the inverse image of the canonical divisor of $\mathbb{A}^2$. Hence $K_X \sim 0$. So, we obtain the stated assertion if $r \neq 2$. Q.E.D.

We note that the differential module $\Omega^1_X$ is the direct sum of an invertible sheaf and $\mathcal{O}_X$ by virtue of a result due to Murthy-Swan [40]. So, $\Omega^1_X$ is trivial if and only if $K_X \sim 0$. 
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