A New Diffeomorphism Symmetry Group of Non-Barotropic Magnetohydrodynamics

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Abstract. The theorem of Noether dictates that for every continuous symmetry group of an Action the system must possess a conservation law. In this paper we discuss some subgroups of Arnold’s labelling symmetry diffeomorphism related to non-barotropic magnetohydrodynamics (MHD) and the conservations laws associated with them. Those include but are not limited to the metage translation group and the associated topological conservations law of non-barotropic cross helicity.

1. Introduction
The theorem of Noether dictates that for every continuous symmetry group of an Action the system must possess a conservation law. For example time translation symmetry results in the conservation of energy, while spatial translation symmetry results in the conservation of linear momentum and rotation symmetry in the conservation of angular momentum to list some well known examples. But sometimes the conservation law is discovered without reference to the Noether theorem by using the equations of the system. In that case one is tempted to inquire what is the hidden symmetry associated with this conservation law and what is the simplest way to represent it.

The concept of metage as a label for fluid elements along a vortex line in ideal fluids was first introduced by Lynden-Bell & Katz [1]. A translation group of this label was found to be connected to the conservation of Moffat’s [2] helicity by Yahalom [3]. The concept of metage was later generalized by Yahalom & Lynden-Bell [4] for barotropic MHD, but now as a label for fluid elements along magnetic field lines which are comoving with the flow in the case of ideal MHD. Yahalom & Lynden-Bell [4] has also shown that the translation group of the magnetic metage is connected to Woltjer [5, 6] conservation of cross helicity for barotropic MHD. Recently the concept of metage was generalized also for non barotropic MHD in which magnetic field lines lie on entropy surfaces [7]. This was later generalized by dropping the entropy condition on magnetic field lines [8].

Cross Helicity was first described by Woltjer [5, 6] and is give by:

\[ H_C \equiv \int \vec{B} \cdot \vec{v} d^3x, \]

in which \( \vec{B} \) is the magnetic field, \( \vec{v} \) is the velocity field and the integral is taken over the entire flow domain. \( H_C \) is conserved for barotropic or incompressible MHD and is given a topological
interpretation in terms of the knottiness of magnetic and flow field lines. A generalization of barotropic fluid dynamics conserved quantities including helicity to non-barotropic flows including topological constants of motion is given by Mobbs [9]. However, Mobbs did not discuss the MHD case.

Both conservation laws for the helicity in the fluid dynamics case and the barotropic MHD case were shown to originate from a relabelling symmetry through the Noether theorem [3, 10, 11, 4]. Webb et al. [13] have generalized the idea of relabelling symmetry to non-barotropic MHD and derived their generalized cross helicity conservation law by using Noether’s theorem but without using the simple representation which is connected with the metage variable. The conservation law deduction involves a divergence symmetry of the action. These conservation laws were written as Eulerian conservation laws of the form \( D_t + \nabla \cdot F = 0 \) where \( D \) is the conserved density and \( F \) is the conserved flux. Webb et al. [15] discuss the cross helicity conservation law for non-barotropic MHD in a multi-symplectic formulation of MHD. Webb et al. [12, 13] emphasize that the generalized cross helicity conservation law, in MHD and the generalized helicity conservation law in non-barotropic fluids are non-local in the sense that they depend on the auxiliary nonlocal variable \( \sigma \), which depends on the Lagrangian time integral of the temperature \( T(x, t) \). Notice that a potential vorticity conservation equation for non-barotropic MHD is derived by Webb, G. M. and Mace, R.L. [16] by using Noether’s second theorem.

It should be mentioned that Mobbs [9] derived a helicity conservation law for ideal, non-barotropic fluid dynamics, which is of the same form as the cross helicity conservation law for non-barotropic MHD, except that the magnetic field induction is replaced by the generalized fluid helicity \( \Omega = \nabla \times (\vec{v} - \sigma \nabla s) \). Webb et al. [12, 13] also derive the Eulerian, differential form of Mobbs [9] conservation law (although they did not reference Mobbs [9]). Webb and Anco [14] show how Mobbs conservation law arises in multi-symplectic, Lagrangian fluid mechanics.

Variational principles for magnetohydrodynamics were introduced by previous authors both in Lagrangian and Eulerian form. Sturrock [17] has discussed in his book a Lagrangian variational formalism for magnetohydrodynamics. Vladimirov and Moffatt [22] in a series of papers have discussed an Eulerian variational principle for incompressible magnetohydrodynamics. However, their variational principle contained three more functions in addition to the seven variables which appear in the standard equations of incompressible magnetohydrodynamics which are the magnetic field \( \vec{B} \) the velocity field \( \vec{v} \) and the pressure \( P \). Kats [23] has generalized Moffatt’s work for compressible non-barotropic fluids but without reducing the number of functions and the computational load. Sakurai [21] has introduced a two function Eulerian variational principle for force-free magnetohydrodynamics and used it as a basis of a numerical scheme, his method is discussed in a book by Sturrock [17]. Yahalom & Lynden-Bell [4] combined the Lagrangian of Sturrock [17] with the Lagrangian of Sakurai [21] to obtain an Eulerian Lagrangian principle for barotropic magnetohydrodynamics which will depend on only six functions. The variational derivative of this Lagrangian produced all the equations needed to describe barotropic magnetohydrodynamics without any additional constraints. The equations obtained resembled the equations of Frenkel, Levich & Stilman [24] (see also [25]). Yahalom [29] have shown that for the barotropic case four functions will suffice. Moreover, it was shown that the cuts of some of those functions [30] are topological local conserved quantities.

Previous work was concerned only with barotropic magnetohydrodynamics. Variational principles of non-barotropic magnetohydrodynamics can be found in the work of Bekenstein & Oron [26] in terms of 15 functions and V.A. Kats [23] in terms of 20 functions. Morrison [27] has suggested a Hamiltonian approach but this also depends on 8 canonical variables (see table 2 [27]). The variational principle introduced in [18, 19] show that only five functions will suffice to describe non-barotropic MHD in the case that we enforce a Sakurai [21] representation for the magnetic field.
The plan of this paper is as follows: First we introduce the basic quantities and equations of non-barotropic MHD. Then we describe the concept of magnetic metage for non-barotropic MHD. This is followed by a description of a Lagrangian variational principle for non-barotropic MHD. Finally we discuss some subgroups of Arnold’s labelling symmetry diffeomorphism related to non-barotropic MHD and the conservations laws associated with them.

2. Basic Equations
Consider the equations of non-barotropic MHD [17, 18]:

\[ \frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}), \]  
(2)

\[ \nabla \cdot \vec{B} = 0, \]  
(3)

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \]  
(4)

\[ \rho \frac{d\vec{v}}{dt} = \rho \left( \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} \right) = -\nabla p(\rho, s) + \frac{(\nabla \times \vec{B}) \times \vec{B}}{4\pi}, \]  
(5)

\[ \frac{ds}{dt} = \frac{\partial s}{\partial t} + (\vec{v} \cdot \nabla)s = 0. \]  
(6)

In the above the following notations are utilized: \( \frac{\partial}{\partial t} \) is the temporal derivative, \( \frac{d}{dt} \) is the temporal material derivative and \( \nabla \) has its standard meaning in vector calculus. \( \rho \) is the fluid density and \( s \) is the specific entropy. Finally \( p(\rho, s) \) is the pressure which depends on the density and entropy (the non-barotropic case). Equation (2) describes the fact that the magnetic field lines are moving with the fluid elements ("frozen" magnetic field lines), equation (3) describes the fact that the magnetic field is solenoidal, equation (4) describes the conservation of mass and equation (5) is the Euler equation for a fluid in which both pressure and Lorentz magnetic forces apply. Equation (6) describes the fact that heat is not created (zero viscosity, zero resistivity) in ideal non-barotropic MHD and is not conducted, thus only convection occurs. The number of independent variables for which one needs to solve is eight (\( \vec{v}, \vec{B}, \rho, s \)) and the number of equations (2,4,5,6) is also eight. Notice that equation (3) is a condition on the initial \( \vec{B} \) field and is satisfied automatically for any other time due to equation (2).

In non-barotropic MHD one can calculate the temporal derivative of the cross helicity (1) using the above equations and obtain:

\[ \frac{dH_C}{dt} = \int T \nabla s \cdot \vec{B} d^3x, \]  
(7)

in which \( T \) is the temperature. Hence, generally speaking cross helicity is not conserved.

3. Load and Metage
The following section follows closely a similar section in [4]. Consider a thin tube surrounding a magnetic field line, the magnetic flux contained within the tube is:

\[ \Delta \Phi = \int \vec{B} \cdot d\vec{S} \]  
(8)

and the mass contained with the tube is:

\[ \Delta M = \int \rho d\vec{v} \cdot d\vec{S}, \]  
(9)
in which $dl$ is a length element along the tube. Since the magnetic field lines move with the flow by virtue of equation (2) and equation (4) both the quantities $\Delta \Phi$ and $\Delta M$ are conserved and since the tube is thin we may define the conserved magnetic load:

$$\lambda = \frac{\Delta M}{\Delta \Phi} = \oint \rho B \, dl,$$

(10)

in which the above integral is performed along the field line. Obviously the parts of the line which go out of the flow to regions in which $\rho = 0$ have a null contribution to the integral. Notice that $\lambda$ is a **single valued** function that can be measured in principle. Since $\lambda$ is conserved it satisfies the equation:

$$\frac{d\lambda}{dt} = 0.$$

(11)

By construction surfaces of constant magnetic load move with the flow and contain magnetic field lines. Hence the gradient to such surfaces must be orthogonal to the field line:

$$\vec{\nabla} \lambda \cdot \vec{B} = 0.$$

(12)

Now consider an arbitrary comoving point on the magnetic field line and denote it by $i$, and consider an additional comoving point on the magnetic field line and denote it by $r$. The integral:

$$\mu(r) = \int_{i}^{r} \rho B \, dl + \mu(i),$$

(13)

is also a conserved quantity which we may denote following Lynden-Bell & Katz [1] as the magnetic metage. $\mu(i)$ is an arbitrary number which can be chosen differently for each magnetic line. By construction:

$$\frac{d\mu}{dt} = 0.$$

(14)

Also it is easy to see that by differentiating along the magnetic field line we obtain:

$$\vec{\nabla} \mu \cdot \vec{B} = \rho.$$

(15)

Notice that $\mu$ will be generally a **non single valued** function, we will show later in this paper that symmetry to translations in $\mu$; will generate through the Noether theorem the conservation of the magnetic cross helicity.

At this point we have two comoving coordinates of flow, namely $\lambda, \mu$ obviously in a three dimensional flow we also have a third coordinate. However, before defining the third coordinate we will find it useful to work not directly with $\lambda$ but with a function of $\lambda$. Now consider the magnetic flux within a surface of constant load $\Phi(\lambda)$. The magnetic flux is a conserved quantity and depends only on the load $\lambda$ of the surrounding surface. Now we define the quantity:

$$\chi = \frac{\Phi(\lambda)}{2\pi}.$$

(16)

Obviously $\chi$ satisfies the equations:

$$\frac{d\chi}{dt} = 0, \quad \vec{B} \cdot \vec{\nabla} \chi = 0.$$

(17)

Let us now define an additional comoving coordinate $\eta^*$ since $\vec{\nabla} \mu$ is not orthogonal to the $\vec{B}$ lines we can choose $\vec{\nabla} \eta^*$ to be orthogonal to the $\vec{B}$ lines and not be in the direction of the $\vec{\nabla} \chi$
lines, that is we choose $\eta^*$ not to depend only on $\chi$. Since both $\vec{\nabla} \eta^*$ and $\vec{\nabla} \chi$ are orthogonal to $\vec{B}$, $\vec{B}$ must take the form:

$$\vec{B} = A \vec{\nabla} \chi \times \vec{\nabla} \eta^*. \quad (18)$$

However, using equation (3) we have:

$$\vec{\nabla} \cdot \vec{B} = \vec{\nabla} A \cdot (\vec{\nabla} \chi \times \vec{\nabla} \eta^*) = 0. \quad (19)$$

Which implies that $A$ is a function of $\chi, \eta^*$. Now we can define a new comoving function $\eta$ such that:

$$\eta = \int_0^{\eta^*} A(\chi, \eta^*)d\eta^*, \quad \frac{d\eta}{dt} = 0. \quad (20)$$

In terms of this function we obtain the Sakurai (Euler potentials) presentation:

$$\vec{B} = \vec{\nabla} \chi \times \vec{\nabla} \eta. \quad (21)$$

And the density is now given by the Jacobian:

$$\rho = \vec{\nabla} \mu \cdot (\vec{\nabla} \chi \times \vec{\nabla} \eta) = \frac{\partial (x, y, z)}{\partial (\chi, \eta, \mu)}. \quad (22)$$

It can easily be shown using the fact that the labels are comoving that the above forms of $\vec{B}$ and $\rho$ satisfy equation (2), equation (3) and equation (4) automatically.

Notice however, that $\eta$ is defined in a non unique way since one can redefine $\eta$ for example by performing the following transformation: $\eta \rightarrow \eta + f(\chi)$ in which $f(\chi)$ is an arbitrary function. The comoving coordinates $\chi, \eta$ serve as labels of the magnetic field lines. Moreover the magnetic flux can be calculated as:

$$\Phi = \int \vec{B} \cdot d\vec{S} = \int d\chi d\eta. \quad (23)$$

In the case that the surface integral is performed inside a load contour we obtain:

$$\Phi(\lambda) = \int_{\lambda} d\chi d\eta = \chi \int_{\lambda} d\eta = \left\{ \begin{array}{l} \chi[\eta] \quad \chi(\eta_{\text{max}} - \eta_{\text{min}}) \end{array} \right. \quad (24)$$

There are two cases involved; in one case the load surfaces are topological cylinders; in this case $\eta$ is not single valued and hence we obtain the upper value for $\Phi(\lambda)$. In a second case the load surfaces are topological spheres; in this case $\eta$ is single valued and has minimal $\eta_{\text{min}}$ and maximal $\eta_{\text{max}}$ values. Hence the lower value of $\Phi(\lambda)$ is obtained. For example in some cases $\eta$ is identical to twice the latitude angle $\theta$. In those cases $\eta_{\text{min}} = 0$ (value at the "north pole") and $\eta_{\text{max}} = 2\pi$ (value at the "south pole").

Comparing the above equation with equation (16) we derive that $\eta$ can be either single valued or not single valued and that its discontinuity across its cut in the non single valued case is $[\eta] = 2\pi$.

The triplet $\chi, \eta, \mu$ will suffice to label any fluid element in three dimensions. But for a non-barotropic flow there is also another label $s$ which is comoving according to equation (6). The question then arises of the relation of this label to the previous three. As one needs to make a choice regarding the preferred set of labels it seems that the physical ones are $\chi, \eta, s$ in which we use the surfaces on which the magnetic fields lie and the entropy, each label has an obvious physical interpretation. In this case we must look at $\mu$ as a function of $\chi, \eta, s$. If the magnetic field lines lie on entropy surface then $\mu$ regains its status as an independent label. The density can now be written as:

$$\rho = \frac{\partial \mu}{\partial s} \frac{\partial (\chi, \eta, s)}{\partial (x, y, z)}. \quad (25)$$
Now as $\mu$ can be defined for each magnetic field line separately according to equation (13) it is obvious that such a choice exist in which $\mu$ is a function of $s$ only. One may also think of the entropy $s$ as a functions $\chi, \eta, \mu$. However, if one change $\mu$ in this case this generally entails a change in $s$ and the symmetry described in equation (13) is lost.

4. Lagrangian variational principle of MHD

Consider the action [8]:

$$A \equiv \int L d^3 x dt,$$

$$L \equiv \rho \left(\frac{1}{2} \vec{v}^2 - \epsilon(\rho, s)\right) - \frac{\vec{B}^2}{8\pi} - \rho \sigma \frac{ds}{dt},$$  \hspace{1cm} (26)

A variation with respect to the Lagrange multiplier $\sigma$ will obviously result in equation (6). A variation with respect to $s$ will result in:

$$\delta_s A = \int d^3 x dt \delta s \left[ \frac{\partial(\rho \sigma)}{\partial t} + \vec{\nabla} \cdot (\rho \sigma \vec{v}) - \rho T \right] + \int dt \oint d\vec{S} \cdot \rho \sigma \vec{v} \delta s - \int d^3 x \rho \sigma \delta s|_{t_0}^{t_1},$$  \hspace{1cm} (27)

Taking into account the continuity equation (4) we obtain for locations in which the density $\rho$ is not null the result:

$$\frac{d\sigma}{dt} = T,$$  \hspace{1cm} (28)

provided that $\delta_s A$ vanished for an arbitrary $\delta s$. Now let us turn our attention to the variation with respect to the fluid element displacement which takes the form:

$$\delta_{\vec{\xi}} A = \int \delta L_{\vec{\xi}} d^3 x dt,$$

$$\delta L_{\vec{\xi}} = \delta \rho \left(\frac{1}{2} \vec{v}^2 - w(\rho, s)\right) - \rho \sigma \delta \vec{v} \cdot \vec{\nabla} s + \rho \vec{v} \cdot \delta \vec{v} - \frac{\vec{B} \cdot \delta \vec{B}}{4\pi},$$  \hspace{1cm} (29)

As most of the terms were calculated previously we will only calculate the term $-\rho \sigma \delta \vec{v} \cdot \vec{\nabla} s$ which is equal to:

$$-\rho \sigma \delta \vec{v} \cdot \vec{\nabla} s = \vec{\xi} \cdot \rho T \vec{\nabla} s - \vec{\nabla} \left(\frac{\partial (\rho \sigma \vec{\nabla} s \cdot \vec{\xi})}{\partial t}\right) - \vec{\nabla} \cdot \left(\rho \sigma (\vec{\nabla} s \cdot \vec{\xi}) \vec{v}\right).$$  \hspace{1cm} (30)

The above result was obtained using equation (6) and equation (28). Hence the variation of the action with respect to a displacement of the fluid elements is:

$$\delta_{\vec{\xi}} A = \int d^3 \rho (\vec{v} - \sigma \vec{\nabla} s) \cdot \vec{\xi}|_{t_0}^{t_1}$$

$$+ \int dt \left\{ \int d\vec{S} \cdot \left[ -\rho \vec{\xi} \left(\frac{1}{2} \vec{v}^2 - w(\rho, s)\right) + \rho \vec{v} \left(\vec{v} - \sigma \vec{\nabla} s \cdot \vec{\xi}\right) + \frac{1}{4\pi} \vec{B} \times (\vec{\xi} \times \vec{B}) \right]\right\}$$

$$+ \int d^3 x \vec{\xi} \cdot \left[ -\rho \vec{\nabla} w + \rho T \vec{\nabla} s - \frac{\partial(\rho \vec{v})}{\partial t} - \frac{\partial(\rho \vec{v} v_k)}{\partial x_k} - \frac{1}{4\pi} \vec{B} \times (\vec{\nabla} \times \vec{B}) \right],$$  \hspace{1cm} (31)

in which a summation convention is assumed. Taking into account the continuity equation (4) and thermodynamic identities we obtain:

$$\delta_{\vec{\xi}} A = \int d^3 \rho (\vec{v} - \sigma \vec{\nabla} s) \cdot \vec{\xi}|_{t_0}^{t_1}$$
\[
+ \int dt \left\{ \int d\vec{S} \cdot [-\rho \vec{\xi} \left( \frac{1}{2} \vec{v}^2 - w(\rho, s) \right) + \rho \vec{v} \vec{\xi} + \frac{1}{4\pi} \vec{B} \times (\vec{\xi} \times \vec{B})] \right\} + \int d^3\vec{x} \left\{ \left[ -\vec{\nabla} P - \frac{\partial \vec{\xi}}{\partial t} - \rho \vec{v} \vec{\xi} \right] \cdot \vec{B} \right\},
\]

(32)

Hence we obtain the correct dynamical equations for an arbitrary \( \vec{\xi} \). Now suppose that the equations and boundary conditions hold. Then:

\[
\delta \vec{\xi} \cdot A = \int d^3x \rho (\vec{v} - \sigma \vec{\nabla} s) \cdot \vec{B}
\]

(33)

If in addition \( \vec{\xi} \) is a small symmetry displacement such that \( \delta \vec{\xi} \cdot A = 0 \) we obtained a conserved Noether current:

\[
\delta J = \int d^3x \rho (\vec{v} - \sigma \vec{\nabla} s) \cdot \vec{B}
\]

(34)

5. The labelling symmetry group and its subgroups

It is obvious that the choice of fluid labels is quite arbitrary. However, when enforcing the \( \chi, \eta, \mu \) coordinate system such that:

\[
\rho = \frac{\partial (\chi, \eta, \mu)}{\partial (x, y, z)}.
\]

(35)

The choice is restricted to \( \tilde{\chi}, \tilde{\eta}, \tilde{\mu} \):

\[
\frac{\partial (\tilde{\chi}, \tilde{\eta}, \tilde{\mu})}{\partial (\chi, \eta, \mu)} = 1.
\]

(36)

Moreover the Euler potential magnetic field representation:

\[
\vec{B} = \vec{\nabla} \chi \times \vec{\nabla} \eta,
\]

(37)

reduces the choice further to:

\[
\frac{\partial (\tilde{\chi}, \tilde{\eta})}{\partial (\chi, \eta)} = 1.
\]

(38)

5.1. Metage translations

In what follows we consider the transformation (see also equation (13)):

\[
\tilde{\chi} = \chi, \tilde{\eta} = \eta, \tilde{\mu} = \mu + a(\chi, \eta)
\]

(39)

Hence \( a \) is a label displacement which may be different for each magnetic field line, as the field line is closed one need not worry about edge difficulties. This transformation satisfies trivially the conditions (36,38). If \( a = \delta \mu \) is small we can calculate the associated fluid element displacement with this relabelling.

\[
\vec{\xi} = -\frac{\partial \vec{r}}{\partial \mu} \delta \mu = -\delta \mu \frac{\vec{B}}{\rho}.
\]

(40)

Inserting this expression into the boundary term in equation (32) will result in:

\[
\delta A_B = \int dt \int d\vec{S} \cdot [\vec{B} \left( \frac{1}{2} \vec{v}^2 - w(\rho, s) \right) - \vec{v} \vec{\xi} + \vec{B} \times (\vec{\xi} \times \vec{B})] \delta \mu = 0,
\]

(41)

which is the condition for magnetic cross helicity conservation. Inserting equation (40) into equation (34) we obtain the conservation law:

\[
\delta J = \int d^3x \rho (\vec{v} - \sigma \vec{\nabla} s) \cdot \vec{B} = -\int d^3x \delta \mu (\vec{v} - \sigma \vec{\nabla} s) \cdot \vec{B}
\]

(42)
In the simplest case we may take $\delta \mu$ to be a small constant, hence:

$$\delta J = -\delta \mu \int d^3 x (\vec{v} - \sigma \vec{\nabla} s) \cdot \vec{B} = -\delta \mu H_{CNB}$$  \hspace{1cm} (43)$$

Where $H_{CNB}$ is the non barotropic global cross helicity [12, 31, 32] defined as:

$$H_{CNB} \equiv \int d^3 x (\vec{v} - \sigma \vec{\nabla} s) \cdot \vec{B} = \int d^3 x \vec{v}_t \cdot \vec{B}$$  \hspace{1cm} (44)$$

in which $\vec{v}_t \equiv \vec{v} - \sigma \vec{\nabla} s$ is the topological velocity field. We thus obtain the conservation of non-barotropic cross helicity using the Noether theorem and the symmetry group of metage translations. Of course one can perform a different translation on each magnetic field line, in this case one obtains:

$$\delta J = - \int d^3 x \delta \mu \vec{v}_t \cdot \vec{B} = - \int d\chi d\eta \delta \mu \oint_{\chi, \eta} d\mu \rho^{-1} \vec{v}_t \cdot \vec{B}$$  \hspace{1cm} (45)$$

Now since $\delta \mu$ is an arbitrary (small) function of $\chi, \eta$ it follows that:

$$I = \oint_{\chi, \eta} d\mu \rho^{-1} \vec{v}_t \cdot \vec{B}$$  \hspace{1cm} (46)$$

is a conserved quantity for each magnetic field line. Along a magnetic field line the following equations hold:

$$d\mu = \vec{\nabla} \mu \cdot d\vec{r} = \vec{\nabla} \mu \cdot \hat{B} dr = \frac{\rho}{B} dr$$  \hspace{1cm} (47)$$

in the above $\hat{B}$ is an unit vector in the magnetic field direction an equation (15) is used. Inserting equation (47) into equation (46) we obtain:

$$I = \oint_{\chi, \eta} d\vec{r} \vec{v}_t \cdot \hat{B} = \oint_{\chi, \eta} d\vec{r} \cdot \vec{v}_t.$$  \hspace{1cm} (48)$$

which is just the circulation of the topological velocity along the magnetic field lines. This quantity can be written in terms of the generalized Clebsch representation of the velocity [18]:

$$\vec{v} = \vec{\nabla} \nu + \alpha \vec{\nabla} \chi + \beta \vec{\nabla} \eta + \sigma \vec{\nabla} s, \hspace{1cm} \vec{v}_t = \vec{\nabla} \nu + \alpha \vec{\nabla} \chi + \beta \vec{\nabla} \eta$$  \hspace{1cm} (49)$$

as:

$$I = \oint_{\chi, \eta} d\vec{r} \cdot \vec{v}_t = \oint_{\chi, \eta} d\vec{r} \cdot \vec{\nabla} \nu = [\nu].$$  \hspace{1cm} (50)$$

$[\nu]$ is the discontinuity of $\nu$. This was shown to be equal to the amount of non barotropic cross helicity per unit of magnetic flux [31, 32].

$$I = [\nu] = \frac{dH_{CNB}}{d\Phi}.$$  \hspace{1cm} (51)$$

5.2. Transformations of magnetic surfaces
Consider the following transformations:

$$\tilde{\eta} = \eta + \delta \eta(\chi, \eta), \hspace{1cm} \tilde{\chi} = \chi + \delta \chi(\chi, \eta), \hspace{1cm} \tilde{\mu} = \mu$$  \hspace{1cm} (52)$$
in which $\delta \eta, \delta \chi$ are considered small in some sense. Inserting the above quantities into equation (38) and keeping only first order terms we arrive at:

$$\partial_\eta \delta \eta + \partial_\chi \delta \chi = 0.$$  

This equation can be solved as follows:

$$\delta \eta = \partial_\chi \delta f, \quad \delta \chi = -\partial_\eta \delta f,$$

in which $\delta f = \delta f(\chi, \eta)$ is an arbitrary small function. In this case we obtain a particle displacements of the form:

$$\xi = -\frac{\partial^2 \eta}{\partial \chi \partial \eta} \delta \chi - \frac{\partial^2 \eta}{\partial \eta \partial \eta} = -\frac{1}{\rho} \left( \nabla \eta \times \nabla \mu \delta \chi + \nabla \mu \times \nabla \chi \delta \eta \right) = \frac{\nabla \mu}{\rho} \times \left( \nabla \eta \delta \chi - \nabla \chi \delta \eta \right)$$

A special case that satisfies equation (53) is the case of a constant $\delta \chi$ and $\delta \eta$, those two independent displacements lead to two new topological conservation laws:

$$\delta J_\chi = \delta \chi \int d^3x \bar{v}_t \cdot \nabla \mu \times \nabla \eta = \delta \chi H_{CNB\chi}, \quad \delta J_\eta = \delta \eta \int d^3x \bar{v}_t \cdot \nabla \chi \times \nabla \mu = \delta \eta H_{CNB\eta}.$$  

Where the new non barotropic global cross helicities are defined as:

$$H_{CNB\chi} \equiv \int d^3x \bar{v}_t \cdot \nabla \mu \times \nabla \eta, \quad H_{CNB\eta} \equiv \int d^3x \bar{v}_t \cdot \nabla \chi \times \nabla \mu$$

We remark that those topological constants will only be conserved under special boundary conditions satisfying:

$$\oint d\vec{S} \cdot \left[ -\rho \frac{1}{2} \bar{v}^2 - w(\rho, s) \right] + \rho \bar{v}(\bar{v}_t \cdot \tilde{\xi}) + \frac{1}{4\pi} \bar{B} \times (\tilde{\xi} \times \bar{B}) = 0$$

for:

$$\tilde{\xi}_\chi = \frac{1}{\rho} \left( \nabla \mu \times \nabla \eta \right) \delta \chi, \quad \tilde{\xi}_\eta = \frac{1}{\rho} \left( \nabla \chi \times \nabla \mu \right) \delta \eta$$

This is more plausible for magnetic field lines which lie on topological torii. In this case $\eta$ is non single valued [4] and thus the translation in this direction resembles moving fluid elements along closed loops. Finally we remark that for barotropic MHD $\bar{v}_t$ can be replaced with $\bar{v}$.

6. Conclusion

We have shown the connection of the translation symmetry groups of labels to both the global non barotropic cross helicity conservation law and the conservation law of circulations of topological velocity along magnetic field lines. The latter were shown to be equivalent to the amount of non barotropic cross helicity per unit of magnetic flux [31, 32]. Further more we have shown that two additional cross helicity conservation laws exist the $\chi$ and $\eta$ cross helicities. Possible applications for MHD constraints of the current constants of motion are described in [32]. The importance of constants of motion for stability analysis is also discussed in [33].

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