SUSPENSION HOMOTOPY OF 6-MANIFOLDS

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Abstract. For a simply connected closed orientable manifold of dimension 6, we show its homotopy decomposition after double suspension. This allows us to determine its $K$- and $KO$-groups easily. Moreover, for a special case we refine the decomposition to show the rigidity property of the manifold after double suspension.

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1. Introduction

Let $M$ be a closed orientable smooth manifold of dimension $n$. There are tremendous investigations in geometric topology to classify the diffeomorphism or homeomorphism type of $M$ in various cases. For instance, in the general case, Wall [Wal1, Wal3] studied the $(s - 1)$-connected $2s$-manifolds and the $(s - 1)$-connected $(2s + 1)$-manifolds. For the concrete case with specified dimension $n$, Bardon [Bar] classified the simply connected 5-manifolds, and Wall [Wal2], Jupp [Jup] and Zhubr [Zhu1, Zhu2] classified the simply connected 6-manifolds. More recently, Kreck and Su [KS] classified certain non-simply connected 5-manifolds, while Crowley and Nordström [CN] and Kreck [Kre] studied the classification problem of various kinds of 7-manifolds.

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In the mentioned literature, the homotopy classification of $M$ was usually carried out as a byproduct in terms of a system of invariants. However, it is almost impossible to extract nontrivial homotopy information of $M$ directly from the classification. On the other way around, the unstable homotopy theory is a powerful tool to study the homotopy properties of manifolds. There are several interesting attempts in recent years along this way. For instance, Beben and Theriault [BT1] studied the loop decompositions of $(s - 1)$-connected $2s$-manifolds, while Beben and Wu [BW] and Huang and Theriault [HT] studied the loop decompositions of the $(s - 1)$-connected $(2s + 1)$-manifolds. The homotopy groups of these manifolds were also investigated by Sa. Basu and So. Basu [BB, Bas] from different point of view. Moreover, a theoretical method of loop decomposition was developed by Beben and Theriault [BT2], which is quite useful for studying the homotopy of manifolds. Additionally, the homotopy type of the suspension of a connected 4-manifold was determined by So and Theriault [ST].

In this paper, we study the homotopy aspect of simply connected 6-manifolds. Let $M$ be a simply connected closed orientable 6-manifold. By Poincaré duality and the universal coefficient theorem we have

$$H_*(M; \mathbb{Z}) = \begin{cases} 
\mathbb{Z}^d \oplus T & * = 2, \\
\mathbb{Z}^{2m} \oplus T & * = 3, \\
\mathbb{Z}^d & * = 4, \\
\mathbb{Z} & * = 0,6 \\
0 & \text{otherwise},
\end{cases}$$

(1)

where $m, d \geq 0$, and $T$ is a finitely generated abelian torsion group. Our first main theorem concerns the double suspension splitting of $M$. Denote $\Sigma X$ be the suspension of any CW-complex $X$. Denote $P^n(T)$ be the Moore space such that the reduced cohomology $\tilde{H}^*(P^n(T); \mathbb{Z}) \cong T$ if $* = n$ and 0 otherwise [N2].

**Theorem 1.1.** Let $M$ be a simply connected closed orientable 6-manifold with homology of the form (1). Suppose that $T$ has no 2 or 3-torsion. Then there is an integer $c$ with $0 \leq c \leq d$ determined by the cohomology ring of $M$ such that

- if $c = 0$,

$$\Sigma^2 M \cong \Sigma W_0 \vee \bigvee_{j=1}^{d-1} (S^4 \vee S^6) \vee \bigvee_{i=1}^{2m} S^5 \vee P^6(T) \vee P^5(T),$$

where $W_0 \cong (S^3 \vee S^5) \cup e^7$;

- if $c = d$,

$$\Sigma^2 M \cong \Sigma W_d \vee \bigvee_{i=1}^{d-1} \Sigma^2 \mathbb{C}P^2 \vee \bigvee_{i=1}^{2m} S^5 \vee P^6(T) \vee P^5(T),$$

where $W_d \cong \Sigma \mathbb{C}P^2 \cup e^7$. 
• if $1 \leq c \leq d - 1$,

$$\Sigma^2 M \simeq \Sigma W_c \vee \bigvee_{i=1}^{c-1} \Sigma^2 \mathbb{C}P^2 \vee \bigvee_{j=1}^{d-c-1} (S^4 \vee S^6) \vee \bigvee_{i=1}^{2m} S^5 \vee P^6(T) \vee P^5(T),$$

where $W_c \simeq (\Sigma \mathbb{C}P^2 \vee S^3 \vee S^5) \cup e^7$.

Notice the number $c$ is indeed determined by the Steenrod square $Sq^2 : H^2(M; \mathbb{Z}/2\mathbb{Z}) \to H^4(M; \mathbb{Z}/2\mathbb{Z})$, while there is an ambiguous term $W_c$ ($0 \leq c \leq d$). Since we only need the suspension of $W_c$ and the Hopf element $\eta_i \in \pi_{i+1}(S^i)$ is detected by $Sq^2$, the ambiguity reduces to the components of the attaching map of the top cell of $W_c$ to the wedge summand $\Sigma \mathbb{C}P^2$ and $S^3$.

Nevertheless, Theorem 1.1 is still useful, for instance, to calculate the $K$-group or the $KO$-group of $M$ in Corollary 1.2. In particular, when $M$ is a Calabi-Yau threefold it partially reproduces the result of Doran and Morgan [DM, Corollary 1.10] on its $K$-group by different method, and provides new computation on its $KO$-group. Moreover, there are many examples of simply connected Calabi-Yau threefolds. For instance, based on Kreuzer and Skarke [KSk], Batyrev and Kreuzer [BK] showed that there are exactly 473 800 760 families of simply connected Calabi-Yau 3-folds corresponding to 4-dimensional reflexive polytopes.

**Corollary 1.2.** Let $M$ be the manifold in Theorem 1.1. Then for the reduced $K$-group and $KO$-groups of $M$

$$\tilde{K}(M) \cong \mathbb{Z}^{\oplus 2d+1} \oplus T, \quad \tilde{KO}(M) \cong \bigoplus_d (\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}).$$

If we specify to the case when $d = 1$, we can obtain the complete description of $M$ after double suspension, based on the work of Yamaguchi [Yam] (also summarised and corrected by Baues [Bau]). Denote $\eta_i^3 = \eta_{i+2} \circ \eta_{i+1} \circ \eta_i \in \pi_{i+3}(S^i)$ [Tod], where $\eta_i \in \pi_{i+1}(S^i)$ is the Hopf element. Let $V_3$ be the manifold as the total space of the sphere bundle of the oriented $\mathbb{R}^3$-bundle over $S^4$ determined by its first Pontryagin class $p_1 = 12s_4$, where $s_4 \in H^4(S^4; \mathbb{Z})$ is a generator.

**Theorem 1.3.** Let $M$ be a simply connected closed orientable 6-manifold with homology of the form (1) such that $d = 1$. Let $x, y \in H^*(M; \mathbb{Z})$ be two generators such that $\deg(x) = 2$ and $\deg(y) = 4$. Denote $x^2 = ky$ for some $k \in \mathbb{Z}$. Suppose $T$ has no 2 or 3-torsion. Then

• if $k$ is odd, then $M$ is Spin, moreover

$$\Sigma^2 M \simeq \Sigma^2 \mathbb{C}P^3 \vee \bigvee_{i=1}^{2m} S^5 \vee P^6(T) \vee P^5(T),$$

when $k \equiv 1 \pmod{6}$, while

$$\Sigma^2 M \simeq \Sigma^2 V_3 \vee \bigvee_{i=1}^{2m} S^5 \vee P^6(T) \vee P^5(T),$$

when $k \equiv 3 \pmod{6}$;
• if $k$ is even and $V$ is non-Spin
\[
\Sigma^2 M \simeq S^4 \vee \Sigma^4 CP^2 \vee \bigvee_{i=1}^{2m} S^5 \vee P^6(T) \vee P^5(T);
\]

• if $k$ is even and $V$ is Spin
\[
\Sigma^2 M \simeq (S^4 \cup \lambda \eta_3^3 e^8) \vee S^6 \vee \bigvee_{i=1}^{2m} S^5 \vee P^6(T) \vee P^5(T),
\]

where $\lambda \in \mathbb{Z}/2$ is determined by $M$.

It should be remarked that there is no ambiguity in the term $(S^4 \cup \lambda \eta_3^3 e^8)$ in the last decomposition of $\Sigma^2 M$. Indeed, the stable cube element $\eta_3^3 \in \pi_{n+3}(S^n)$ ($n \geq 2$) is detected by the secondary operation $T$ [Har, Exercise 4.2.5], and in our case the homotopy decomposition has to preserve the module structure induced by the cohomology operations. Moreover, it is clear that the number $k \mod 2$ and the spin condition of $M$ are determined by the Steenrod square $Sq^2$. And we will also see that $\Sigma CP^3$ and $\Sigma V_3$ can be distinguished by the Steenrod power $P^1 : H^3(\Sigma M; \mathbb{Z}/3\mathbb{Z}) \to H^7(\Sigma M; \mathbb{Z}/3\mathbb{Z})$. Hence, we obtain the following rigidity result for manifolds in Theorem 1.3 after double suspension.

**Corollary 1.4.** Let $M$ and $M'$ be two manifolds in Theorem 1.3. Then $\Sigma^2 M \simeq \Sigma^2 M'$ if and only if $H^*(\Sigma^2 M; \mathbb{Z}) \cong H^*(\Sigma^2 M'; \mathbb{Z})$ as abelian groups, and $H^*(\Sigma^2 M; \mathbb{Z}/p\mathbb{Z}) \cong H^*(\Sigma^2 M'; \mathbb{Z}/p\mathbb{Z})$ as $\mathbb{Z}/2\mathbb{Z}(Sq^2, T)$-modules when $p = 2$, and as $\mathbb{Z}/3\mathbb{Z}(P^1)$-modules when $p = 3$.

The paper is organized as follows. In Section 2 we reduce the decomposition problem of 6-manifolds to that of ones whose third Betti numbers are zero. In Section 3, we give a detailed procedure to decompose 6-manifolds after double suspension by homology decomposition method. Section 4 and Section 5 are devoted to prove Theorem 1.1 and Theorem 1.3 respectively. In Section 6, we compute some homotopy groups of odd primary Moore spaces used in Section 3.

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2. Reducing to the case when $b_3(M) = 0$

The following well known splitting theorem of 6-manifolds was proved by Wall [Wal2] in smooth category, while Jupp [Jup] pointed out that the theorem holds in topological category by the same argument.
Theorem 2.1. [Wal2, Theorem 1] Let $M$ be a simply connected closed orientable 6-manifold with third Betti number $b_3(M) = 2m$. Then there exists a 6-manifold $M_1$ such that

$$M \cong M_1 \# m S^3 \times S^3.$$ 

Corollary 2.2. Let $M$ and $M_1$ be the manifolds in Theorem 2.1. Then

$$\Sigma M \cong \Sigma M_1 \vee \bigvee_{i=1}^m (S^4 \vee S^4).$$

Proof. Let $M'_1$ and $M'$ be the 5-skeletons of $M_1$ and $M$ respectively. It is known that $S^3 \vee S^3$ is the 5-skeleton of $S^3 \times S^3$ and $\Sigma(S^3 \times S^3) \cong \Sigma(S^3 \vee S^3) \vee S^7$. In particular, there is a homotopy retraction $r : \Sigma(S^3 \times S^3) \to \Sigma(S^3 \vee S^3)$. For the connected sum $M_1 \# (S^3 \times S^3)$, there are the obvious pinch maps $q_1 : M_1 \# (S^3 \times S^3) \to M_1$ and $q_2 : M_1 \# (S^3 \times S^3) \to S^3 \times S^3$. Consider the composition

$$\phi : \Sigma(M_1 \# (S^3 \times S^3)) \xrightarrow{\mu'} \Sigma(M_1 \# (S^3 \times S^3)) \vee \Sigma(M_1 \# (S^3 \times S^3)) \xrightarrow{EE_1 \vee (r)E_2} \Sigma M_1 \vee \Sigma(S^3 \vee S^3),$$

where $\mu'$ is the standard co-multiplication of suspension complex, and $E$ denotes the suspension of a map. It is easy to see that $\phi$ induces an isomorphism on homology and then is a homotopy equivalence by the Whitehead Theorem. Since

$$M' \cong M'_1 \vee \bigvee_{i=1}^m (S^3 \vee S^3),$$

by repeating the above argument, we can show the decomposition in the corollary.

In Theorem 2.1, the connected summand $M_1$ satisfies $b_3(M_1) = 0$. Hence by Corollary 2.2 it suffices to consider such 6-manifolds in the sequel.

3. Homology Decomposition of $M$ after Suitable Suspensions

Let $M$ be a simply connected closed orientable 6-manifold with $b_3(M) = 0$. By Poincaré duality and the universal coefficient theorem we have

$$H_*(M; \mathbb{Z}) = \begin{cases} \mathbb{Z}^\oplus d \oplus T & * = 2, \\ T & * = 3, \\ \mathbb{Z}^\oplus d & * = 4, \\ \mathbb{Z} & * = 0, 6 \\ 0 & \text{otherwise}, \end{cases}$$

(2)

where $d \geq 0$, and $T$ is a finitely generated abelian torsion group. We may denote

$$T = \bigoplus_{k=1}^r \mathbb{Z}/p_k^{r_k} \mathbb{Z},$$

(3)

where each $p_k$ is a prime and $r_k \geq 1$.

Instead of using skeleton decomposition, we may apply homology decomposition to study the cell structure of $M$. For any finitely generated abelian group $A$, let $P^n(A)$ be the Moore space such that
the reduced cohomology $\tilde{H}^*(P^n(A); \mathbb{Z}) \cong A$ if $* = n$ and 0 otherwise [N2]. The information on the homotopy groups of $P^n(T)$ used in this section will be proved in Section 6.

**Theorem 3.1.** [Hat, Theorem 4H.3] Let $X$ be a simply connected CW-complex. Denote $H_i = H_i(X; \mathbb{Z})$. Then there is a sequence of complex $X(i)$ ($i \geq 2$) such that

- $H_j(X(i); \mathbb{Z}) \cong H_j(X; \mathbb{Z})$ for $j \leq i$ and $H_j(X(i); \mathbb{Z}) = 0$ for $j > i$;
- $X(2) = P^1(H_2)$, and $X(i)$ is defined by a homotopy cofibration $P^i(H_i) \xrightarrow{f_{i-1}} X(i-1) \xrightarrow{\iota_{i-1}} X(i)$,

where $f_{i-1}$ induces a trivial homomorphism $f_{i-1*} : H_{i-1}(P^i(H_i); \mathbb{Z}) \rightarrow H_{i-1}(X(i-1); \mathbb{Z})$;
- $X \cong \text{hocolim} \{X(2) \xrightarrow{\iota_2} \cdots \xrightarrow{\iota_{i-2}} X(i-1) \xrightarrow{\iota_{i-1}} X(i) \xrightarrow{\iota_i} \cdots \}$. □

From this theorem, it is clear that the homology decomposition is compatible with the suspension functor. That is, for $X$ in Theorem 3.1 the sequence of the triples $(\Sigma X(i), Ef_i, E\iota_i)$ contributes to a homology decomposition of $\Sigma X$.

### 3.1. Structure of $M(3)$.

By Theorem 3.1 and (2),

$$M(2) \cong \bigvee_{i=1}^d S^2 \vee P^3(T),$$

and

$$P^3(T) \xrightarrow{f_2} M(2) \xrightarrow{i_2} M(3).$$

Notice that from (3), we have $P^n(T) \cong \bigvee_{k=1}^\ell P^n(p^k)$ by [N3] or [N2].

**Lemma 3.2.** The map $f_2$ in (4) is null-homotopic, and hence

$$M(3) \cong \bigvee_{i=1}^d S^2 \vee P^3(T) \vee P^4(T).$$

**Proof.** Since $P^3(T) \cong \bigvee_{k=1}^\ell P^3(p^k)$, there is the embedding $j : \bigvee_{i=1}^\ell S^2 \rightarrow P^3(T)$ of the bottom cells. Consider the following commutative diagram:

$$
\begin{array}{ccc}
\pi_2(\bigvee_{i=1}^\ell S^2) & \cong & \pi_2(P^3(T)) \\
\xrightarrow{j_*} & \xrightarrow{f_{2*}} & \xrightarrow{\iota_*} \\
H_2(\bigvee_{i=1}^\ell S^2; \mathbb{Z}) & \cong & H_2(P^3(T); \mathbb{Z}) \\
\xrightarrow{j_*} & \xrightarrow{f_{2*}=0} & H_2(M(2); \mathbb{Z}),
\end{array}
$$

where the Hurewicz homomorphisms $\text{hur}$ are isomorphisms by the Hurewicz theorem, $f_{2*} = 0$ on homology by Theorem 3.1, and both $j_*$ are epimorphisms. In particular, $f_{2*} \circ j_*$ is trivial on
homotopy groups, and hence \( f_2 \circ j \) is null homotopic. Then with (4) we have the diagram of homotopy cofibrations

\[
\begin{array}{cccccc}
\vee_{i=1}^{\ell} S^2 & \rightarrow & * & \rightarrow & \vee_{i=1}^{\ell} S^3 \\
\bigvee_{i=1}^{\ell} p^r_k & \downarrow & \downarrow & \downarrow & \bigvee_{i=1}^{\ell} p^r_k \\
\vee_{i=1}^{\ell} S^2 & \rightarrow & M(2) & \rightarrow & M(2) \vee \bigvee_{i=1}^{\ell} S^3 \\
\downarrow & & \downarrow i_2 \circ \bigvee_{i=1}^{\ell} p^r_k & & \\
P^3(T) & \rightarrow & M(2) & \rightarrow & M(3),
\end{array}
\]

where \( p^r_k : S^n \rightarrow S^n \) is a map of degree \( p^r_k \), and \( i_2 : \bigvee_{i=1}^{\ell} S^3 \rightarrow M(2) \vee \bigvee_{i=1}^{\ell} S^3 \) is the injection onto the sphere summands. It follows that

\[
M(3) \simeq M(2) \vee P^4(T) \simeq \bigvee_{i=1}^{d} S^2 \vee P^3(T) \vee P^4(T),
\]

and the proof of the lemma is completed. \( \square \)

The following corollary follows from Lemma 3.2 and will be used in Lemma 3.6.

**Corollary 3.3.** The homotopy cofiber of the obvious inclusion \( j : P^3(T) \vee P^4(T) \rightarrow M(3) \rightarrow M \) is a Poincaré duality complex \( V \) with cell structure

\[
V = \bigvee_{i=1}^{d} S^2 \cup e^4_{(1)} \cup e^4_{(2)} \ldots \cup e^4_{(d)} \cup e^6.
\]

\( \square \)

Moreover, by [Wal2, Theorem 8] \( V \) is homotopy equivalent to a closed smooth manifold.

### 3.2. **Structure of** \( M(5) \).

By Theorem 3.1 and Lemma 3.2,

\[
M(3) \simeq \bigvee_{i=1}^{d} S^2 \vee P^3(T) \vee P^4(T),
\]

\( (5) \)

\[
\bigvee_{i=1}^{d} S^3 \xrightarrow{f_3} M(3) \xrightarrow{i_3} M(4) = M(5).
\]

We need to study the map

\[
f_3 : \bigvee_{i=1}^{d} S^3 \rightarrow \bigvee_{i=1}^{d} S^2 \vee P^3(T) \vee P^4(T).
\]

Let \( i_3 : P^4(T) \rightarrow \bigvee_{i=1}^{d} S^2 \vee P^3(T) \vee P^4(T) \) be the inclusion. Define the complex \( Y \) by the homotopy cofibration

\[
P^4(T) = M(5) \rightarrow Y.
\]
Lemma 3.4. The map $f_3$ in (5) factors as

$$f_3 : \bigvee_{i=1}^d S^3 \xrightarrow{f'_3} \bigvee_{i=1}^d S^2 \vee P^3(T) \xrightarrow{i_1 \lor i_2} \bigvee_{i=1}^d S^2 \vee P^3(T) \vee P^4(T),$$

for some $f'_3$, where $i_1$ and $i_2$ are inclusions. Moreover, there is the homotopy cofibration

$$\bigvee_{i=1}^d S^3 \xrightarrow{f_3} \bigvee_{i=1}^d S^2 \vee P^3(T) \xrightarrow{i'_3} Y,$$

and

$$M(5) \simeq Y \vee P^4(T).$$

Proof. First there is the diagram of homotopy cofibrations

$$\begin{array}{ccc}
\ast & \rightarrow & P^4(T) \\
\downarrow & & \downarrow i_3 \quad i_3 \\
\bigvee_{i=1}^d S^3 & \xrightarrow{f_3} & M(3) \\
\downarrow q_{1,2} & & \downarrow \\
\bigvee_{i=1}^d S^3 & \xrightarrow{f'_3} & \bigvee_{i=1}^d S^2 \vee P^3(T) \xrightarrow{i'_3} Y,
\end{array}$$

where $q_{1,2}$ is the obvious projection, $i'_3$ is induced from $i_3$, and $f'_3 := q_{1,2} \circ f_3$. The diagram immediately implies that (6) is a homotopy cofibration.

Denote $q_3 : \bigvee_{i=1}^d S^2 \vee P^3(T) \vee P^4(T) \rightarrow P^4(T)$ be the canonical projection. By Theorem 3.1, $f_{3*} : H_3(\bigvee_{i=1}^d S^3; \mathbb{Z}) \rightarrow H_3(\bigvee_{i=1}^d S^2 \vee P^3(T) \vee P^4(T); \mathbb{Z})$ is trivial. In particular, $q_3 \circ f_{3*} = 0$. Then by the Hurewicz Theorem $q_3 \circ f_3$ is null homotopic. Further, by the Hilton-Milnor Theorem (see §XI.6 of [Whi] for instance), $\pi_3(S^2 \vee P^3(T) \vee P^4(T)) \cong \pi_3(S^2 \vee P^3(T)) \oplus \pi_3(P^4(T))$, and hence

$$[\bigvee_{i=1}^d S^3, S^2 \vee P^3(T) \vee P^4(T)] \cong [\bigvee_{i=1}^d S^3, S^2 \vee P^3(T)] \oplus [\bigvee_{i=1}^d S^3, P^4(T)].$$

Under this isomorphism, the homotopy class of $f_3$ corresponds to $[f'_3] + [q_3 \circ f_3]$. However since we already show that $[q_3 \circ f_3] = 0$, we have $f_3 \simeq (i_1 \lor i_2) \circ f'_3$, and $M(5) \simeq Y \vee P^4(T)$ as required. \qed

3.3. Structure of $\Sigma M(5)$. From this point, we may need extra conditions on the torsion group $T$. First recall that we already showed that by Lemma 3.4

$$M(5) \simeq Y \vee P^4(T),$$

(7)

$$\bigvee_{i=1}^d S^3 \xrightarrow{f'_3} \bigvee_{i=1}^d S^2 \vee P^3(T) \xrightarrow{i'_3} Y.$$
Let \( q_1: \bigvee_{i=1}^d S^2 \vee P^3(T) \to \bigvee_{i=1}^d S^2 \) be the canonical projection. Denote \( f''_3 := q_1 \circ f'_3 = q_1 \circ q_{1,2} \circ f_3 \).

Also recall \( P^n(T) \simeq \bigvee_{k=1}^t P^n(p_{rk}^T) \). Let us suppose each \( p \geq 3 \) from now on.

**Lemma 3.5.** Suppose \( T \) has no \( 2 \)-torsion.

\[
\Sigma M_{(5)} \simeq \Sigma X \vee P^5(T) \vee P^4(T),
\]

where \( X \) is the homotopy cofiber of the map \( f''_3: \bigvee_{i=1}^d S^3 \to \bigvee_{i=1}^d S^2 \).

**Proof.** There is the diagram of homotopy cofibrations

\[
\begin{array}{ccc}
* & \rightarrow & P^3(T) \\
\downarrow & & \downarrow \iota_2 \\
\bigvee_{i=1}^d S^3 & \xrightarrow{f'_3} & \bigvee_{i=1}^d S^2 \vee P^3(T) \\
\downarrow & & \downarrow \iota'_3 \circ \iota_2 \\
\bigvee_{i=1}^d S^3 & \xrightarrow{f''_3} & \bigvee_{i=1}^d S^2 \\
\downarrow & & \downarrow q_1 \\
\bigvee_{i=1}^d S^3 & \xrightarrow{f'_3} & \bigvee_{i=1}^d S^2 \\
\downarrow & & \downarrow X,
\end{array}
\]

where \( \iota_2 \) is the canonical inclusion. Since \( \pi_4(P^4(p^T)) = 0 \) for odd \( p \) by Lemma 6.3, we have \( \Sigma Y \simeq \Sigma X \vee P^4(T) \). The lemma then follows from (7).

3.4. Structure of \( \Sigma^2 M \). Recall we have when \( T \) has no 2 torsion by Lemma 3.5

\[
\Sigma M_{(5)} \simeq \Sigma X \vee P^5(T) \vee P^4(T),
\]

(8)

\[
S^5 \xrightarrow{f_5} M_{(5)} \xrightarrow{\iota_5} M.
\]

Further by Corollary 3.3 we have the homotopy cofibration

\[
S^5 \rightarrow X \rightarrow V,
\]

where \( X \) is defined in Lemma 3.5 without restriction on \( T \), and \( V \) is a closed smooth manifold. We may further suppose \( T \) has no 3-torsion.

**Lemma 3.6.** Suppose \( T \) has no 2 or 3-torsion. Then

\[
\Sigma^2 M \simeq \Sigma^2 V \vee P^6(T) \vee P^5(T).
\]

**Proof.** By the Hilton-Milnor theorem, we may write the suspension of \( f_5 \) as

\[
Ef_5 = g^{(1)}_5 + g^{(2)}_5 + g^{(3)}_5 + \theta: S^6 \rightarrow \Sigma M_{(5)} \simeq \Sigma X \vee P^5(T) \vee P^4(T),
\]
for some \( \theta \), where \( E\theta = 0, \ g_5^{(i)} = q_i \circ Ef_5 \), and \( q_i \) is the canonical projection of \( \Sigma X \vee P^5(T) \vee P^4(T) \) onto its \( i \)-th summand. Then \( g_5^{(2)} = 0 \) by Lemma 6.4, and \( Eg_5^{(3)} = 0 \) by Lemma 6.6. It follows that \( f_5 = Eg_5^{(1)} \). Furthermore, there is the diagram of homotopy cofibrations

\[
\begin{array}{cccccc}
\ast & \longrightarrow & P^4(T) \vee P^3(T) & \longrightarrow & P^4(T) \vee P^3(T) \\
& & f_5 & & j_5 \\
& & S^5 & & M(5) \\
& & & \pi_5 & & \pi \\
& & S^5 & & X & & V,
\end{array}
\]

where the homotopy cofibration in the last column is defined in Corollary 3.3 by using Lemma 3.2, and similarly the homotopy cofibration in the middle column can be also defined by using Lemma 3.2. Then it is clear \( g_5^{(1)} \simeq E(\pi_5 \circ f_5) \) and the lemma follows. \( \square \)

4. Proof of Theorem 1.1 and Corollary 1.2

In Lemma 3.6 we have established the double suspension splitting of \( M \) when \( b_3(M) = 0 \), and are left to consider the homotopy of \( V \) after suspension. Recall that \( V \) is a Poincaré Duality complex of dimension 6, and its 5-skeleton \( V_5 = X \) is the homotopy cofiber of the map \( f'' : d \bigvee_{i=1}^d S^3 \rightarrow d \bigvee_{i=1}^d S^2 \) by Lemma 3.5. The following lemma, as a special case of [H, Lemma 6.1], determines the suspension homotopy type of \( X \).

**Lemma 4.1.** [H, Lemma 6.1]

\[\Sigma X \simeq \bigvee_{i=1}^c \Sigma CP^2 \bigvee_{j=1}^{d-c} (S^3 \vee S^5),\]

for some \( 0 \leq c \leq d \). \( \square \)

We may apply the method in [H, Section 3] to decompose \( \Sigma^2 V \), in the same way that we used it to prove [H, Lemma 6.4 and Lemma 6.6].

**Lemma 4.2.** Suppose \( \Sigma X \) decomposes as in Lemma 4.1.

- If \( c = 0 \),

\[\Sigma^2 V \simeq \Sigma W_0 \bigvee_{j=1}^{d-1} (S^4 \vee S^6),\]

where \( W_0 \simeq (S^3 \vee S^5) \cup e^7 \);

- if \( c = d \),

\[\Sigma^2 V \simeq \Sigma W_d \bigvee_{i=1}^{d-1} \Sigma^2 CP^2,\]

where \( W_d \simeq \Sigma CP^2 \cup e^7 \);
• if $1 \leq c \leq d - 1$,
  \[ \Sigma^2 V \simeq \Sigma W_c \vee \bigvee_{i=1}^{c-1} \Sigma^2 \mathbb{C}P^2 \vee \bigvee_{j=1}^{d-c-1} (S^4 \vee S^6), \]
  where $W_c \simeq (\Sigma \mathbb{C}P^2 \vee S^3 \vee S^5) \cup e^7$.

Proof. As we pointed out that the proof is similar to that of [H, Lemma 6.4 and Lemma 6.6], we may only sketch it. The interested reader can find the details of the method in [H, Section 3]. With Lemma 4.1 let $g : S^6 \to \Sigma X \simeq \bigvee_{i=1}^c \Sigma \mathbb{C}P^2 \vee \bigvee_{j=1}^{d-c} (S^3 \vee S^5)$ be the attaching map of the top cell of $\Sigma V$. To apply the method in [H, Section 3], we only need the information of homotopy groups $\pi_6(\Sigma \mathbb{C}P^2) \simeq \mathbb{Z}/6\mathbb{Z}$ by [Muk, Proposition 8.2(i)], $\pi_6(S^3) \simeq \mathbb{Z}/12\mathbb{Z}$ and $\pi_6(S^5) \simeq \mathbb{Z}/2$ which are all finite cyclic groups. Then we can represent the attaching map $Eg$ of the top cell of $\Sigma^2 V$ by a matrix $B$, and apply [H, Lemma 3.1] to transform $B$ to a simpler one $C$. The new matrix representation $C$ of the attaching map, corresponding to a base change of $\Sigma X$ through a self homotopy equivalence, will give the desired decomposition.

Now we can prove Theorem 1.1.

Proof of Theorem 1.1. First by Theorem 2.1 and Corollary 2.2, we have
  \[ \Sigma M \simeq \Sigma M_1 \vee \bigvee_{i=1}^m (S^4 \vee S^4), \]
  where $M_1$ is a closed 6-manifold with homology of the form (2). In particular $b_3(M_1) = 0$. Hence by Lemma 3.6 and Lemma 4.2, we have that if $1 \leq c \leq d - 1$
  \[ \Sigma^2 M \simeq \Sigma W_c \vee \bigvee_{i=1}^{c-1} \Sigma^2 \mathbb{C}P^2 \vee \bigvee_{j=1}^{d-c-1} (S^4 \vee S^6) \vee P^6(T) \vee P^5(T) \vee \bigvee_{i=1}^{2m} S^5, \]
  where $W_c \simeq (\Sigma \mathbb{C}P^2 \vee S^3 \vee S^5) \cup e^7$. The decompositions for the other two cases when $c = 0$ or $d$ can be obtained similarly. Finally, notice that $c$ records the number of the non-trivial Steenrod square $Sq^2 : H^2(\Sigma^2 M; \mathbb{Z}/2\mathbb{Z}) \to H^4(\Sigma^2 M; \mathbb{Z}/2\mathbb{Z})$, which is preserved by the decomposition and the suspension operator. Since $Sq^2$ is the cup square on the elements of $H^2(M; \mathbb{Z}/2\mathbb{Z})$, this completes the proof of Theorem 1.1.

To prove Corollary 1.2, we need the Bott periodicity showed in the following table:

**Table 1.** $K^{-i}(S^0)$ and $\widetilde{KO}^{-j}(S^0)$

| $i$ mod 2 | 0 | 1 |
|------------|---|---|
| $K^{-i}(S^0)$ | $\mathbb{Z}$ | 0 |

| $j$ mod 8 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|------------|---|---|---|---|---|---|---|---|
| $\widetilde{KO}^{-j}(S^0)$ | $\mathbb{Z}$ | $\mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z}/2\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | 0 | 0 |

From Table (1) we can easily calculate the following, where only $\widetilde{KO}^{-2}(P^5(T)) = 0$ requires that $T$ has no 2-torsion.
Lemma 4.3. Let $W_c$ be the complex defined in Lemma 4.2 for $0 \leq c \leq d$.

- $\widetilde{K}(P^6(T)) \cong T$, $\widetilde{K}(P^6(T)) = 0$,
- $\widetilde{KO}^2(P^5(T)) = \widetilde{KO}^2(P^6(T)) = 0$,
- $\widetilde{KO}^1(\Sigma CP^2) \cong \widetilde{KO}^1(S^3 \vee S^5) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, $\widetilde{KO}^1(\Sigma^2 CP^2) \cong \widetilde{KO}^1(S^4 \vee S^6) = 0$,
- $\widetilde{KO}^1(W_0) \cong \widetilde{KO}^1(W_d) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, $\widetilde{KO}^1(W_c) \cong \bigoplus \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Proof of Theorem 1.2. Let us only compute the $\widetilde{KO}$-group of $M$ when $1 \leq c \leq d - 1$, while the other cases can be computed similarly. By Theorem 1.1 and Lemma 4.3, we have

$$
\widetilde{KO}(M) \cong \widetilde{KO}^2(\Sigma^2 M) \cong \widetilde{KO}^2(\Sigma W_c) \oplus \bigoplus_{i=1}^{c-1} \widetilde{KO}^2(\Sigma^2 CP^2) \oplus \bigoplus_{j=1}^{d-c-1} \widetilde{KO}^2(S^4 \vee S^6)
$$

$$
\oplus \bigoplus_{j=1}^{2m} \widetilde{KO}^2(S^5) \oplus \widetilde{KO}^2(P^6(T)) \oplus \widetilde{KO}^2(P^5(T))
$$

$$
\cong \bigoplus_2 (\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}) \oplus \bigoplus_c (\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}) \oplus \bigoplus_{d-c-1} (\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})
$$

$$
= \bigoplus_d (\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}).
$$


5. The case when $H^2(M; \mathbb{Z}) \cong \mathbb{Z}$

By Lemma 3.6, we may consider the torsion free case first. In [Yam], Yamaguchi classified the homotopy types of CW-complexes of the form $V \simeq S^2 \cup e^4 \cup e^6$. Specifying to the case when $V$ is a manifold, we can summarized the necessary result in the following theorem (c.f. [Bau, Section 1]).

Theorem 5.1. [Yam, Corollary 4.6, Lemma 2.6, Lemma 4.3] Let $V \simeq S^2 \cup_{k\eta_2} e^4 \cup_b e^6$ be a closed smooth manifold, where $k\eta_2$ with $k \in \mathbb{Z}$ and $b$ are the attaching maps of the cells $e^4$ and $e^6$ respectively. Then the top attaching map $b$ is determined by a generator $bW \in \pi_5(S^2 \cup_{k\eta_2} e^4)$ of infinite order, the second Stiefel-Whitney class $\omega_2(V) \in H^2(V; \mathbb{Z}/2)$ of $V$, and an indeterminacy term $b' \in \mathbb{Z}/2$ which depends on the following three cases.

- If $k$ is odd, then $V$ is Spin and the homotopy type of $V$ is unique determined by $k$, and $b = bW$;
- If $k$ is even and $V$ is non-Spin, then the homotopy type of $V$ is unique determined by $k$, and $b = bW + \eta_4$ with $\eta_4$ representing the generator of a $\mathbb{Z}/2$ summand determined by $\omega_2(V)$;
- If $k$ is even and $V$ is Spin, then $V$ has precisely two homotopy types depending on the value of $b' \in \mathbb{Z}/2$, and $b = bW + b'$.

Remark 5.2. In Theorem 5.1, $bW$, as a generator of the $\mathbb{Z}$-summand of $\pi_5(S^2 \cup_{k\eta_2} e^4)$, is indeed a relative Whitehead product when $k \neq 0$ by [Yam, Lemma 2.6]. It is possible that the suspension
map $Eb_W$ is not null-homotopic. $\tilde{\eta}_4$ is derived from the homotopy class of
\[ S^5 \xrightarrow{b} S^2 \cup_{k\eta_2} e^4 \xrightarrow{q} S^4, \]
where $q$ is the quotient map onto the 4-cell of $S^2 \cup_{k\eta_2} e^4$ (c.f. [Bau, Section 1]). $b'$ is from a class of $\pi_5(S^2) \cong \mathbb{Z}/2\{\eta_2^3\}$ by [Yam, Lemma 2.6] or [Bau, Section 1]. Also, as pointed out in Mathematical Reviews [MR], the original theorem of [Yam] was misstated which is corrected here and in [Bau, Section 1] as well.

Thanks to Theorem 5.1 and Remark 5.2, we can describe the suspension homotopy type of $V$. Recall that $\pi_6(\Sigma CP^2) \cong \mathbb{Z}/6\{E\pi_2\}$ [Muk, Theorem 8.2(i)], where $\pi_2 : S^5 \to CP^2$ is the Hopf map, and $E$ is the suspension of a map.

**Proposition 5.3.** Let $V \simeq S^2 \cup_{k\eta_2} e^4 \cup_b e^6$ be a closed smooth manifold.

- If $k$ is odd, then $V$ is Spin and
  \[ \Sigma V \simeq \Sigma CP^2 \cup k'E\pi_2 e^7, \]
  where $k' = 1$ or $3$ such that $k' \equiv \pm k \mod 6$;
- If $k$ is even and $V$ is non-Spin
  \[ \Sigma V \simeq S^3 \vee \Sigma^3 CP^2; \]
- If $k$ is even and $V$ is Spin
  \[ \Sigma V \simeq (S^3 \cup b'\eta_3^3 e^7) \vee S^5, \]
  where $b' \in \mathbb{Z}/2$ is from Theorem 5.1.

**Proof.** It is clear that the decompositions for the two cases when $k$ is even follows immediately from Theorem 5.1 and Remark 5.2. When $k$ is odd, $V$ is spin and $\Sigma V \simeq \Sigma CP^2 \cup Eb_W e^7$ by Theorem 5.1. Also notice that $\Sigma CP^2 \cup Eb_W e^7 \simeq \Sigma CP^2 \cup_{-Eb_W} e^7$. Hence, to prove the statement in the proposition it suffices to show that the suspension map
\[ E : \pi_5(S^2 \cup_{k\eta_2} e^4) \to \pi_6(\Sigma CP^2) \]
sends the generator $b_W$ to $kE\pi_2$ up to sign.

For this purpose, start with the diagram of homotopy cofibrations
\[
\begin{array}{cccccccc}
S^3 & \xrightarrow{k\eta_2} & S^2 & \xrightarrow{k} & S^2 & \xrightarrow{q} & S^2 \cup_{k\eta_2} e^4 & \xrightarrow{r} & S^4 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
S^3 & \xrightarrow{\eta_2} & S^2 & \xrightarrow{} & \Sigma CP^2 & \xrightarrow{} & \Sigma CP^2 & \xrightarrow{} & \Sigma CP^2 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
P^4(k) & \xrightarrow{} & \ast & \xrightarrow{} & P^5(k), & & & & \\
\end{array}
\]

(9)
which defines the map $r$. Then there is the diagram of homotopy fibrations

\[
\begin{array}{ccccccccc}
S^1 & \longrightarrow & Z & \longrightarrow & S^2 \cup_{kn_2} e^4 & \longrightarrow & K(\mathbb{Z}, 2) \\
\| & & \| & & \| & & \| \\
S^1 & \longrightarrow & S^5 & \longrightarrow & CP^2 & \longrightarrow & K(\mathbb{Z}, 2),
\end{array}
\]

(10)

where $f_c$ and $f_x$ represent the generators $c \in H^2(CP^2; \mathbb{Z})$, and $x \in H^2(S^2 \cup_{kn_2} e^4; \mathbb{Z})$ respectively, and $Z$ is the homotopy fibre of $f_x$ mapping to $S^5$ by the induced map $\tilde{r}$. By analyzing the Serre spectral sequences of the homotopy fibrations in Diagram (10), it can be showed that $Z \simeq P^4(k) \cup e^5$ and $\tilde{r}^* : H^5(S^5; \mathbb{Z}) \to H^5(Z; \mathbb{Z})$ is of degree $k$. Since by Lemma 6.3 $\pi_4(P^4(k)) = 0$ when $k$ is odd, we see that $Z \simeq P^4(k) \lor S^5$, and then $\tilde{r}_*$ is of degree $k$ on homology. Moreover, by the naturality of the Hurewicz homomorphism and Lemma 6.4, it is easy to see that $\tilde{r}_* : \pi_5(Z) \simeq \pi_5(S^5)$ is of degree $k$. It follows that $r_* : \pi_5(S^2 \cup_{kn_2} e^4) \simeq \pi_5(CP^2) \simeq \mathbb{Z}$ is of degree $k$ by Diagram (10).

Now the naturality of suspension map induces the following commutative diagram

\[
\begin{array}{ccc}
\pi_5(S^2 \cup_{kn_2} e^4) & \longrightarrow & \pi_5(CP^2) \\
E & \downarrow & E \\
\pi_6(\Sigma CP^2) & \longrightarrow & \pi_6(\Sigma CP^2),
\end{array}
\]

(11)

where $E : \pi_5(CP^2) \simeq \mathbb{Z} \to \pi_6(\Sigma CP^2) \simeq \mathbb{Z}/6\mathbb{Z}$ is surjective by [Muk, Theorem 8.2(i)]. We have showed that $r_*$ in Diagram (11) is of degree $k$. On the other hand, from the last column of Diagram (9) we have the homotopy cofibratior

\[
\Sigma CP^2 \xrightarrow{Er} CP^2 \longrightarrow P^6(k).
\]

Applying the Blakers-Massey Theorem [BM], we obtain the exact sequence

\[
\pi_6(\Sigma CP^2) \xrightarrow{Er} \pi_6(\Sigma CP^2) \longrightarrow \pi_6(P^6(k)).
\]

(12)

Since $\pi_6(P^6(k)) = 0$ by Lemma 6.3 and $\pi_6(\Sigma CP^2) \simeq \mathbb{Z}/6\mathbb{Z}\{E \pi_2\}$, we see that $Er_*$ is an isomorphism from (12). Then by Diagram (11), it follows that $E : \pi_5(S^2 \cup_{kn_2} e^4) \to \pi_6(\Sigma CP^2)$ sends the generator $b_W$ to $kE \pi_2$ up to sign. This proves the statement in the case when $k$ is odd, and we have completed the proof of the proposition.

Now we can prove Theorem 1.3 and Corollary 1.4.

\textbf{Proof of Theorem 1.3.} First by Theorem 2.1, Corollary 2.2 and Lemma 3.6, we have

\[
\Sigma^2 M \simeq \Sigma^2 M_1 \lor \bigvee_{i=1}^m (S^5 \lor S^5) \simeq \Sigma^2 V \lor P^6(T) \lor P^5(T) \lor \bigvee_{i=1}^m (S^8 \lor S^5),
\]

\textbf{Proof of Theorem 1.4.} First by Theorem 2.1, Corollary 2.2 and Lemma 3.6, we have
where $M_1$ is a closed 6-manifold with homology of the form (2) such that $b_3(M_1) = 0$ and $d = 1$. Moreover, By Corollary 3.3 and the assumption on the ring structure of $H^*(M; \mathbb{Z})$, $V \simeq S^2 \cup_k e_6$ for some attaching map $b$. Denote $\lambda = b'$ in Theorem 5.1. The theorem for the two cases when $k$ is even then follows immediately from Proposition 5.3. For the case when $k$ is odd, recall that there is the fibre bundle [HBJ, Section 1.1]

$$S^2 \to \mathbb{C}P^3 \xrightarrow{\sigma} S^4,$$

with its first Pontryagin class $p_1 = 4s_4$ where $s_4 \in H^4(S^4; \mathbb{Z})$ is a generator. Then pullback this bundle along the self-map of $S^4$ of degree 3, we obtain the 6-manifold $V_3$ with bundle projection $\sigma_3$ onto $S^4$ in the following diagram of $S^2$-bundles

$$\begin{array}{ccc}
S^2 & \to & V_3 \\
\downarrow & & \downarrow \\
S^2 & \to & \mathbb{C}P^3 \\
\downarrow & & \downarrow \\
& & \sigma_3 \\
& & S^4
\end{array}$$

From this diagram, it is easy to see that the first Pontryagin class of $\sigma_3$ is $12s_4$ as required and $x^2 = 3y$, where by abuse of notation $x, y \in H^*(V_3; \mathbb{Z})$ are two generators such that $\deg(x) = 2$ and $\deg(y) = 4$. Hence by Proposition 5.3, $\Sigma V \simeq \Sigma \mathbb{C}P^3$ when $k \equiv 1 \mod 6$ and $\Sigma V \simeq \Sigma V_3$ when $k \equiv 3 \mod 6$, and then the two decompositions when $k$ is odd in the theorem follows. This completes the proof of the theorem.

**Proof of Corollary 1.4.** As the discussions before Corollary 1.4, the number $k$ mod 2 and the spin condition of $M$ are determined by the Steenrod square $Sq^2$. Since the attaching maps of the top cells of $\Sigma \mathbb{C}P^3$ and $\Sigma V_3$ are $E\pi_2$ of order 6 and $3E\eta_2$ of order 2 respectively by Proposition 5.3, after localization at 3 we can consider the Steenrod power $P^1 : H^3(\Sigma M; \mathbb{Z}/3\mathbb{Z}) \to H^7(\Sigma M; \mathbb{Z}/3\mathbb{Z})$. Then since $\Sigma V_3 \simeq_{(3)} S^3 \cup e^7 \vee S^7$, $P^1$ acts trivially on its cohomology. In contrast, $\Sigma \mathbb{C}P^3 \simeq_{(3)} S^3 \cup_{\alpha_1} e^7 \vee S^5$ where $\alpha_1$ an element detected by $P^1$ [Har, Section 1.5.5]. Hence, $\Sigma \mathbb{C}P^3$ and $\Sigma V_3$ can be distinguished by $P^1$. Moreover, the stable cube element $\eta_n^3 \in \pi_{n+3}(S^n)$ ($n \geq 2$) is detected by the secondary operation $T$ [Har, Exercise 4.2.5]. And there is no indeterminacy since in the either case $S^4 \cup_{\alpha_3} e^7$ splits off as a wedge summand of the double suspension of the manifold. From the above discussions on cohomology operations, we can prove the corollary easily by the decompositions in Theorem 1.3.

**6. Some computations on homotopy groups of odd primary Moore spaces**

In this section, we work out the homotopy groups of Moore spaces used in Section 3. Consider the Moore space $P^{2n+1}(p^r)$ with $n \geq 1$, $p \geq 3$ and $r \geq 1$. We have the homotopy fibration

$$F^{2n+1}(p^r) \to P^{2n+1}(p^r) \xrightarrow{\sigma} S^{2n+1},$$

where
where $q$ is the pinch map of the bottom cell. Cohen-Moore-Neisendorfer proved the following the famous decomposition theorem.

**Theorem 6.1.** [CMN, N] Let $p$ be an odd prime. There is a $p$-local homotopy equivalence

$$
\Omega F^{2n+1}\{p^r\} \simeq_{(p)} S^{2n-1} \times \prod_{k=1}^{\infty} S^{2p^k n-1}\{p^r\} \times \Omega \Sigma \bigvee_{\alpha} P^{n_{\alpha}}(p^r),
$$

where $S^i\{p^r\}$ is the homotopy fibre of the degree map $p^r : S^i \rightarrow S^i$, and $\bigvee_{\alpha} P^{n_{\alpha}}(p^r)$ is an infinite bouquet of mod $p^r$ Moore spaces with only finitely many in each dimension and the least value of $n_{\alpha}$ is $4n - 1$.

We also need the following classical result.

**Lemma 6.2.** [N2, Proposition 6.2.2] Let $p$ be an odd prime.

$$
P^m(p^r) \wedge P^n(p^r) \simeq P^{m+n}(p^r) \vee P^{m+n-1}(p^r).
$$

**Lemma 6.3.** [ST, So] Let $p$ be an odd prime.

$$
\pi_3(P^3(p^r)) = \mathbb{Z}/p^r\mathbb{Z}, \quad \pi_n(P^n(p^r)) = 0,
$$

for $n \geq 4$.

**Proof.** The cases when $n = 3$ and 4 were already proved in [ST, Lemma 2.1] and [So, Lemma 3.3] respectively, while the remaining cases follow immediately from the Freudenthal Suspension Theorem.

**Lemma 6.4.** Let $p$ be an odd prime.

$$
\pi_{n+1}(P^n(p^r)) = 0,
$$

for $n \geq 3$.

**Proof.** $\pi_4(P^3(p^r)) = 0$ was showed in [So, Lemma 3.3]. Let us consider $\pi_5(P^4(p^r))$. By the classical EHP-sequence (Chapter XII, Theorem 2.2 of [Whi]), there is the exact sequence

$$
0 = \pi_4(P^3(p^r)) \rightarrow \pi_5(P^4(p^r)) \xrightarrow{H} \pi_5(P^4(p^r) \wedge P^3(p^r)) \xrightarrow{P} \pi_5(P^3(p^r)) \rightarrow \pi_4(P^4(p^r)) = 0.
$$

By Lemma 6.2,

$$
\pi_5(P^4(p^r) \wedge P^3(p^r)) \cong \pi_5(P^6(p^r) \vee P^7(p^r)) \cong \mathbb{Z}/p^r\mathbb{Z}.
$$

Hence, by Lemma 6.3 and the above exact sequence, $P$ is an isomorphism and then $\pi_5(P^4(p^r)) = 0$. The remaining cases follow immediately from the Freudenthal Suspension Theorem, and this completes the proof of the lemma.

In the remaining two lemmas, we exclude the case when $p = 3$. 

Lemma 6.5. Let \( p \geq 5 \),
\[
\pi_{n+2}(P^n(r)) = 0,
\]
for \( n \geq 6 \).

Proof. By the Freudenthal Suspension Theorem, it suffices to show \( \pi_9(P^7(r)) = 0 \). For that let us compute \( \pi_9(F^7\{p\}) \) first. By Theorem 6.1,
\[
\pi_9(F^7\{p\}) \cong \pi_8(\Omega F^7\{p\}) \cong \pi_8(S^5)_{(p)}.
\]
Since \( \pi_8(S^5) \cong \mathbb{Z}/24\mathbb{Z} \) and \( p \geq 5 \), \( \pi_9(F^7\{p\}) = 0 \). Now from the exact sequence of homotopy groups of the homotopy fibration (13) \((n = 3)\)
\[
0 = \pi_9(F^7\{p\}) \to \pi_9(P^7(r)) \to \pi_9(S^7)_{(p)} = 0,
\]
we see that \( \pi_9(P^7(r)) = 0 \).

Lemma 6.6. Let \( p \geq 5 \). The suspension morphism
\[
E : \pi_6(P^4(r)) \to \pi_7(P^5(r)) \cong \mathbb{Z}/p^r\mathbb{Z}
\]
is trivial.

Proof. On the one hand there is the EHP-sequence of \( P^4(r) \)
\[
\pi_6(P^4(r)) \xrightarrow{E} \pi_7(P^5(r)) \xrightarrow{H} \pi_7(P^5(P^r) \wedge P^4(r)) \to \pi_5(P^4(r)) = 0,
\]
where \( \pi_5(P^4(r)) = 0 \) by Lemma 6.4, and
\[
\pi_7(P^5(P^r) \wedge P^4(r)) \cong \pi_7(P^8(P^r) \wedge P^9(r)) \cong \mathbb{Z}/p^r\mathbb{Z}
\]
by Lemma 6.2. It follows that
\[
(14) \quad \pi_7(P^5(r))/\text{Im}(E) \cong \mathbb{Z}/p^r\mathbb{Z}.
\]
On the other hand there is the EHP-sequence of \( P^5(r) \)
\[
\pi_9(P^6(r) \wedge P^5(r)) \xrightarrow{P} \pi_7(P^5(r)) \to \pi_8(P^6(r)) = 0,
\]
where \( \pi_8(P^6(r)) = 0 \) by Lemma 6.5, and
\[
\pi_9(P^6(P^r) \wedge P^5(P^r)) \cong \pi_9(P^{10}(P^r) \wedge P^{11}(P^r)) \cong \mathbb{Z}/p^r\mathbb{Z}
\]
by Lemma 6.2. It follows that
\[
(15) \quad \pi_7(P^5(r)) \cong \mathbb{Z}/p^r\mathbb{Z}/\text{Ker}(P).
\]
Combining (14) and (15), we see that \( \pi_7(P^5(r)) \cong \mathbb{Z}/p^r\mathbb{Z} \), and \( \text{Im}(E) = \text{Ker}(P) = 0 \). The proof of the lemma is completed.
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