Eigenvalue estimates for the higher order buckling problem

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Abstract. In this paper, we consider lower order eigenvalues of Laplacian operator with any order in Euclidean domains. By choosing special rectangular coordinates, we obtain two estimates for lower order eigenvalues.

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1 Introduction

Let \( \Omega \) be a bounded domain in an \( n(\geq 2) \)-dimensional Euclidean space \( \mathbb{R}^n \) with smooth boundary \( \partial \Omega \). In the present article, we consider the eigenvalue estimate for the following problem:

\[
\begin{aligned}
(\Delta)^p u &= \Lambda(\Delta)u, & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = \cdots = \frac{\partial^{p-1} u}{\partial \nu^{p-1}} &= 0 & \text{on } \partial \Omega,
\end{aligned}
\]

(1.1)

where \( \nu \) denotes the outward unit normal vector field of \( \partial \Omega \) and \( p \) is a positive integer. Let \( 0 < \Lambda_1 < \Lambda_2 \leq \Lambda_3 \leq \cdots \rightarrow +\infty \) denote the successive eigenvalues for (1.1), where each eigenvalue is repeated according to its multiplicity.

When \( p = 2 \), the eigenvalue problem (1.1) is called the buckling problem. For the buckling problem, Payne-Pólya-Weinberger [11] proved, in the case of \( n = 2 \), that

\[
\Lambda_2 \leq 3\Lambda_1.
\]

(1.2)

Following the method of Payne-Pólya-Weinberger in [11], the inequality (1.2) can be generalized to \( \Omega \subset \mathbb{R}^n \) as (see [2]):

\[
\Lambda_2 \leq \left(1 + \frac{8}{n+2}\right)\Lambda_1.
\]

In 1984, Hile and Yeh [10] improved the above results as follows:

\[
\Lambda_2 \leq \frac{n^2 + 8n + 20}{(n+2)^2}\Lambda_1.
\]
On the other hand, Ashbaugh [1] proved another inequality as the following form:

$$\sum_{i=1}^{n} (\Lambda_{i+1} - \Lambda_1) \leq 4\Lambda_1. \quad (1.3)$$

To answer a question of Ashbaugh given in [1], Cheng-Yang [4] obtained in 2006 a universal inequality for higher eigenvalues of (1.1) with $p = 2$. In fact, they proved that

$$\sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 \leq \frac{4(n+2)}{n^2} \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)\Lambda_i. \quad (1.4)$$

As a generalization of (1.4), Huang and Li [7] proved the following inequality of eigenvalue estimate for the problem (1.1) with $p \geq 2$:

$$\sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 \leq \frac{4(p-1)(n+2p-2)}{n^2} \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)\Lambda_i. \quad (1.5)$$

Estimates for higher order eigenvalues of (1.1) has been recently studied by many mathematicians. For the other related development in this direction, we refer to [3, 5–8, 12, 13] and the references therein.

In particular, Cheng-Ichikawa-Mametsuka considered in [9] the eigenvalue estimate for the problem

$$\begin{cases} (-\Delta)^p u = \lambda u, & \text{in } \Omega, \\ u = \frac{\partial u}{\partial \nu} = \cdots = \frac{\partial^{p-1} u}{\partial \nu^{p-1}} = 0 & \text{on } \partial \Omega, \end{cases} \quad (1.6)$$

and proved the following inequalities:

$$\sum_{i=1}^{n} (\Lambda_{i+1} - \Lambda_1) \leq 4p(2p-1)\lambda_1 \quad \text{for } p \geq 2, \quad (1.7)$$

$$\sum_{i=1}^{n} (\Lambda_{i+1}^p - \Lambda_1^p)^{p-1} \leq (2p-1)^p \lambda_1^{\frac{p-1}{p}} \quad \text{for } p \geq 2. \quad (1.8)$$

Inspired by [9], we consider the eigenvalue problem (1.1) with $p \geq 2$ and wish to obtain the similar results as (1.7) and (1.8). Our main results of this paper are stated as follows:

**Theorem 1.1.** Let $\Omega$ be a bounded domain in an $n(\geq 2)$-dimensional Euclidean space $\mathbb{R}^n$. Assume that $\Lambda_i$ is the $i$-th eigenvalue of the problem (1.1) with $p \geq 2$. Then,

$$\sum_{i=1}^{n} (\Lambda_{i+1} - \Lambda_1) \leq 4[p(2p+n-2)-n]\Lambda_1. \quad (1.9)$$

**Theorem 1.2.** Let $\Omega$ be a bounded domain in an $n(\geq 2)$-dimensional Euclidean space $\mathbb{R}^n$. Assume that $\Lambda_i$ is the $i$-th eigenvalue of the problem (1.1) with $p \geq 3$. Then,

$$\sum_{i=1}^{n} (\frac{\Lambda_{i+1}^p}{\Lambda_1^p} - \frac{\Lambda_{i+1}^{p-1}}{\Lambda_1^{p-1}})^{p-2} \leq (2p-2)\Lambda_1^{\frac{p-2}{p-1}}. \quad (1.10)$$
2 Proof of Theorem 1.1

Let \( u_i \) be the orthonormal eigenvalue function of the problem (1.1) with respect to \( L^2 \) inner product corresponding to \( \Lambda_i \), that is,

\[
\left\{ \begin{array}{ll}
(\Delta)^p u_i = \Lambda_i (\Delta) u_i & \text{in } \Omega, \\
u_i = \frac{\partial u_i}{\partial \nu} = \cdots = \frac{\partial^{p-1} u_i}{\partial \nu^{p-1}} = 0 & \text{on } \partial \Omega, \\
\int_{\Omega} \langle \nabla u_i, \nabla u_j \rangle = \delta_{ij}.
\end{array} \right.
\]

We first choose rectangular coordinates for \( \mathbb{R}^n \) by taking as origin the center of gravity of \( \Omega \) with mass-distribution \( |\nabla u_1|^2 \) such that

\[
\int_{\Omega} \langle \nabla (x_i u_1), \nabla u_1 \rangle = 0 \quad \text{for } i = 1, 2, \cdots, n.
\]

Then, by a rotation of the coordinate system if necessary, we may also assume

\[
\int_{\Omega} \langle \nabla (x_i u_1), \nabla u_j \rangle = 0 \quad \text{for } 2 \leq j \leq i \leq n,
\]

and hence we arrive at

\[
\int_{\Omega} \langle \nabla (x_i u_1), \nabla u_j \rangle = 0 \quad \text{for } 1 \leq j \leq i \leq n.
\]

Let \( \varphi_i = x_i u_1 \). Then \( \varphi_i = \frac{\partial \varphi_i}{\partial \nu} = \cdots = \frac{\partial^{p-1} \varphi_i}{\partial \nu^{p-1}} = 0 \) on \( \partial \Omega \) and

\[
\int_{\Omega} \langle \nabla \varphi_i, \nabla u_j \rangle = 0 \quad \text{for } 1 \leq j \leq i \leq n.
\]

From the Rayleigh-Ritz inequality, one gets

\[
\Lambda_{i+1} \leq \frac{\int_{\Omega} \varphi_i (\Delta)^p \varphi_i}{\int_{\Omega} |\nabla \varphi_i|^2}.
\]

(2.1)

Note that

\[
(\Delta)^p \varphi_i = (\Delta)^p (x_i u_1) = \Lambda_1 x_i (\Delta) u_1 - 2p (\Delta)^{p-1} u_{1,x_i},
\]

where \( u_{1,x_i} = \partial u_1 / \partial x_i \). It follows that

\[
\int_{\Omega} \varphi_i (\Delta)^p \varphi_i = \int_{\Omega} \varphi_i [\Lambda_1 x_i (\Delta) u_1 - 2p (\Delta)^{p-1} u_{1,x_i}]
= \int_{\Omega} \varphi_i [\Lambda_1 (\Delta) (x_i u_1) + 2\Lambda_1 u_{1,x_i} - 2p (\Delta)^{p-1} u_{1,x_i}]
= \Lambda_1 \int_{\Omega} |\nabla \varphi_i|^2 + 2\Lambda_1 \int_{\Omega} x_i u_{1,u_{1,x_i}} - 2p \int_{\Omega} \varphi_i (\Delta)^{p-1} u_{1,x_i}
= \Lambda_1 \int_{\Omega} |\nabla \varphi_i|^2 - \Lambda_1 \int_{\Omega} u_1^2 - 2p \int_{\Omega} \varphi_i (\Delta)^{p-1} u_{1,x_i}.
\]

(2.2)
Combining with (2.1) and (2.2) yields
\[
(\Lambda_{i+1} - \Lambda_1) \int_\Omega |\nabla \varphi_i|^2 \leq -\Lambda_1 \int_\Omega u_1^2 - 2p \int_\Omega \varphi_i (-\Delta)^{p-1} u_{1,x}. \tag{2.3}
\]
Using integration by parts, we have
\[
\int_\Omega \langle \nabla \varphi_i, \nabla u_{1,x} \rangle = - \int_\Omega \langle \nabla (x_i u_1), \nabla u_1 \rangle = -1 - \int_\Omega u_{1,x}^2 - \int_\Omega \langle [\nabla (x_i u_1) - u_1 \nabla x_i], \nabla u_{1,x} \rangle
\]
\[
= -1 - 2 \int_\Omega u_{1,x}^2 - \int_\Omega \langle \nabla \varphi_i, \nabla u_{1,x} \rangle.
\]
Hence,
\[
1 \leq 1 + 2 \int_\Omega u_{1,x}^2 = -2 \int_\Omega \langle \nabla \varphi_i, \nabla u_{1,x} \rangle. \tag{2.4}
\]
By Cauchy inequality one knows from (2.4) that
\[
1 \leq 4 \left( \int_\Omega \langle \nabla \varphi_i, \nabla u_{1,x} \rangle \right)^2 \leq 4 \int_\Omega |\nabla \varphi_i|^2 \int_\Omega |\nabla u_{1,x}|^2. \tag{2.5}
\]
Then from (2.3) and (2.5), it is easily seen that
\[
\sum_{i=1}^n (\Lambda_{i+1} - \Lambda_1) \leq 4 \sum_{i=1}^n \left\{ (\Lambda_{i+1} - \Lambda_1) \int_\Omega |\nabla \varphi_i|^2 \int_\Omega |\nabla u_{1,x}|^2 \right\}
\]
\[
\leq 4 \sum_{i=1}^n \left\{ -\Lambda_1 \int_\Omega u_1^2 - 2p \int_\Omega \varphi_i (-\Delta)^{p-1} u_{1,x} \right\} \int_\Omega |\nabla u_{1,x}|^2 \right\}
\]
\[
\leq 4 \left\{ -\Lambda_1 \int_\Omega u_1^2 - 2p \int_\Omega \varphi_i (-\Delta)^{p-1} u_{1,x} \right\} \left\{ \sum_{i=1}^n \int_\Omega |\nabla u_{1,x}|^2 \right\}. \tag{2.6}
\]
Denote
\[
\nabla^r = \begin{cases} 
\Delta^{r/2} & \text{when } r \text{ is even,} \\
\nabla (\Delta^{(r-1)/2}) & \text{when } r \text{ is odd.}
\end{cases}
\]
Then we have the following lemma:

**Lemma 2.1.** [7] Let \( u_1 \) be the eigenfunction of the problem (1.3) corresponding to the eigenvalue \( \Lambda_1 \). Then we have
\[
\int_\Omega |\nabla^r u_1|^2 \leq \Lambda_1^{(r-1)/(p-1)} \quad \text{for } r = 2, 3, \ldots, p. \tag{2.7}
\]

**Proof.** First we prove the inequality
\[
\left( \int_\Omega u_1 (-\Delta)^r u_1 \right)^{1/(r-1)} \leq \left( \int_\Omega u_1 (-\Delta)^{r+1} u_1 \right)^{1/r}. \tag{2.8}
\]
For $r = 2$, we have
\[
\int_{\Omega} u_1 (-\Delta)^2 u_1 = \int_{\Omega} \langle \nabla u_1, \nabla (-\Delta) u_1 \rangle \\
\leq \left( \int_{\Omega} |\nabla u_1|^2 \right)^{1/2} \cdot \left( \int_{\Omega} |\nabla (-\Delta) u_1|^2 \right)^{1/2} \\
= \left( \int_{\Omega} u_1 (-\Delta)^3 u_1 \right)^{1/2}.
\]

Suppose that inequality (2.7) holds for $r-1$, that is,
\[
\left( \int_{\Omega} u_1 (-\Delta)^{r-1} u_1 \right)^{1/(r-2)} \leq \left( \int_{\Omega} u_1 (-\Delta)^r u_1 \right)^{1/(r-1)}.
\]

Then, for integer $r$,
\[
\int_{\Omega} u_1 (-\Delta)^r u_1 = - \int_{\Omega} \langle \nabla^{r-1} u_1, \nabla^{r+1} u_1 \rangle \\
\leq \left( \int_{\Omega} |\nabla^{r-1} u_1|^2 \right)^{1/2} \cdot \left( \int_{\Omega} |\nabla^{r+1} u_1|^2 \right)^{1/2} \\
= \left( \int_{\Omega} u_1 (-\Delta)^{r-1} u_1 \right)^{1/2} \cdot \left( \int_{\Omega} u_1 (-\Delta)^{r+1} u_1 \right)^{1/2} \\
\leq \left( \int_{\Omega} u_1 (-\Delta)^r u_1 \right)^{(r-2)/2(r-1)} \cdot \left( \int_{\Omega} u_1 (-\Delta)^{r+1} u_1 \right)^{1/2},
\]
which gives
\[
\left( \int_{\Omega} u_1 (-\Delta)^r u_1 \right)^{1/(r-1)} \leq \left( \int_{\Omega} u_1 (-\Delta)^{r+1} u_1 \right)^{1/r}.
\]

This means that inequality (2.8) holds. Repeatedly using inequality (2.8), we deduce
\[
\left( \int_{\Omega} u_1 (-\Delta)^r u_1 \right) \leq \left( \int_{\Omega} u_1 (-\Delta)^{p+1} u_1 \right)^{1/(p-1)} = \Lambda_1^{1/(p-1)}.
\]

This concludes the proof of Lemma 2.1. \qed

From (2.7) and Schwarz inequality it follows that
\[
1 = \left( \int_{\Omega} |\nabla u_1|^2 \right)^2 = \left( \int_{\Omega} u_1 (-\Delta) u_1 \right)^2 \\
\leq \int_{\Omega} u_1^2 \int_{\Omega} u_1 (-\Delta)^2 u_1 \leq \Lambda_1^{p-1} \int_{\Omega} u_1^2.
\] (2.9)
A direct computation yields
\[
\int_{\Omega} \varphi_i(-\Delta)^{p-1} u_{1,x_i} = \int_{\Omega} u_{1,x_i} (-\Delta)^{p-1}(x_i u_1)
\]
\[
= \int_{\Omega} u_{1,x_i} \left( x_i (-\Delta)^{p-1} u_1 - 2(p-1)(-\Delta)^{p-2} u_{1,x_i} \right)
\]
\[
= - \int_{\Omega} u_1 \left( (-\Delta)^{p-1} u_1 + x_i (-\Delta)^{p-1} u_{1,x_i} \right)
\]
\[
+ 2(p-1) \int_{\Omega} u_1 (-\Delta)^{p-2} u_{1,x_i}
\]
\[
= - \int_{\Omega} \varphi_i(-\Delta)^{p-1} u_{1,x_i} - \int_{\Omega} u_1 (-\Delta)^{p-1} u_1
\]
\[
+ 2(p-1) \int_{\Omega} u_1 (-\Delta)^{p-2} u_{1,x_i},
\]
(2.10)
which shows that
\[
\sum_{i=1}^{n} \int_{\Omega} \varphi_i(-\Delta)^{p-1} u_{1,x_i}
\]
\[
= \sum_{i=1}^{n} \left( -\frac{1}{2} \int_{\Omega} u_1 (-\Delta)^{p-1} u_1 + (p-1) \int_{\Omega} u_1 (-\Delta)^{p-2} u_{1,x_i} \right)
\]
\[
= -\frac{2p+n-2}{2} \int_{\Omega} u_1 (-\Delta)^{p-1} u_1
\]
\[
\geq -\frac{2p+n-2}{2} \Lambda_{1}^{\frac{p-2}{p-1}}.
\]
(2.11)
On the other hand, it easy to see that
\[
\sum_{i=1}^{n} \int_{\Omega} |\nabla u_{1,x_i}|^2 = -\sum_{i=1}^{n} \int_{\Omega} u_1 (-\Delta) u_{1,x_i} = \int_{\Omega} u_1 (-\Delta)^2 u_1 \leq \Lambda_{1}^{\frac{p}{p-1}}. \tag{2.12}
\]
Finally, applying (2.9), (2.11) and (2.12) to (2.5), one finds
\[
\sum_{i=1}^{n} (\Lambda_{i+1} - \Lambda_1) \leq 4[p(2p + n - 2) - n] \Lambda_1,
\]
completing the proof of Theorem 1.1. \qed

3 Proof of Theorem 1.2

By virtue of (2.9), it holds that
\[
\Lambda_1 \int_{\Omega} u_1^2 \geq \Lambda_{1}^{\frac{p-2}{p-1}}. \tag{3.1}
\]
And from (2.10) one finds that
\[
\int_{\Omega} \varphi_i(-\Delta)^{p-1} u_{1,x_i} = -\frac{1}{2} \int_{\Omega} u_1 (-\Delta)^{p-1} u_1 + (p-1) \int_{\Omega} u_1 (-\Delta)^{p-2} u_{1,x_i}
\]
\[
\geq -\frac{1}{2} \Lambda_{1}^{\frac{p-2}{p-1}} - (p-1) \int_{\Omega} |\nabla^{p-2} u_{1,x_i}|^2,
\]
(3.2)
Lemma 3.1. Let $\Lambda_i$ be the $i$-th eigenvalue of problem (1.1) with $p \geq 2$, and $u_i$ be the orthonormal eigenfunction corresponding to $\Lambda_i$. Then for $1 \leq i \leq n$, either
\[
\sum_{k=1}^{p-2} \frac{1}{\Lambda_{i+1}^{p-1}} \left( \Lambda_{i+1}^{p-1} - \Lambda_{i+1}^{p-k} \right) \leq 2p(p-1) \int_{\Omega} |\nabla^{p-2} u_{1,x_i}|^2, \tag{3.4}
\]
or
\[
\Lambda_{i+1}^{\frac{1}{p-1}} - \Lambda_{i}^{\frac{1}{p-1}} \leq 4 \int_{\Omega} |u_{1,x_i}|^2. \tag{3.5}
\]

Proof. Suppose that there exists an $i$ such that neither (3.4) nor (3.5) holds. Then by (3.3)
\[
(L_{i+1} - L_{1}) \int_{\Omega} |\nabla \varphi_i|^2 \leq (p-1)L_{1}^{\frac{p-2}{p-1}} + 2p(p-1) \int_{\Omega} |\nabla^{p-2} u_{1,x_i}|^2
\]
\[
< (p-1)L_{1}^{\frac{p-2}{p-1}} + \sum_{k=1}^{p-2} \Lambda_{k-1}^{1} \left( \Lambda_{i+1}^{\frac{p-1-k}{p-1}} - \Lambda_{i}^{\frac{p-1-k}{p-1}} \right)
\]
\[
= \sum_{k=1}^{p-1} \Lambda_{k-1}^{\frac{p-1-k}{p-1}} \Lambda_{i+1}^{\frac{k-1}{p-1}} = \frac{\Lambda_{i+1} - \Lambda_{i}}{\Lambda_{i+1}^{\frac{1}{p-1}} - \Lambda_{i}^{\frac{1}{p-1}}},
\]
which shows that
\[
(L_{i+1}^{\frac{1}{p-1}} - L_{i}^{\frac{1}{p-1}}) \int_{\Omega} |\nabla \varphi_i|^2 < 1. \tag{3.6}
\]

On the other hand, it follows from (2.5) that
\[
1 \leq 4 \int_{\Omega} |\nabla \varphi_i|^2 \int_{\Omega} |u_{1,x_i}|^2 < (L_{i+1}^{\frac{1}{p-1}} - L_{i}^{\frac{1}{p-1}}) \int_{\Omega} |\nabla \varphi_i|^2,
\]
which contradicts with (3.6). This concludes the proof of Lemma 3.1. □

Lemma 3.2. Let $\Lambda_i$ be the $i$-th eigenvalue of problem (1.1) with $p \geq 2$, and $u_i$ be the orthonormal eigenfunction corresponding to $\Lambda_i$. Then
\[
\left( \int_{\Omega} |\nabla^r u_{1,x_i}|^2 \right)^{\frac{1}{r+1}} \leq \left( \int_{\Omega} |\nabla^{r+1} u_{1,x_i}|^2 \right)^{\frac{1}{r+2}} \quad \text{for } r = 1, 2, \ldots, p-2. \tag{3.7}
\]

Proof. For $r = 1$, we have
\[
\int_{\Omega} |\nabla u_{1,x_i}|^2 = - \int_{\Omega} \langle \nabla^2 u_{1,x_i}, u_{1,x_i} \rangle \leq \left( \int_{\Omega} |\nabla^2 u_{1,x_i}|^2 \right)^{\frac{1}{2}} \left( \int_{\Omega} |u_{1,x_i}|^2 \right)^{\frac{1}{2}} \leq \left( \int_{\Omega} |\nabla^2 u_{1,x_i}|^2 \right)^{\frac{1}{2}} = \left( \int_{\Omega} |\nabla u_{1,x_i}|^2 \right)^{\frac{1}{2}}.
\]
Assume that (3.7) is true for $r - 1$, that is,
\[
\left( \int_\Omega |\nabla^{r-1} u_{1,x_i}|^2 \right)^{\frac{1}{r-1}} \leq \left( \int_\Omega |\nabla^r u_{1,x_i}|^2 \right)^{\frac{1}{r}}.
\]
Then for $r$
\[
\int_\Omega |\nabla^r u_{1,x_i}|^2 = - \int_\Omega \langle \nabla^{r-1} u_{1,x_i}, \nabla^{r+1} u_{1,x_i} \rangle \\
\leq \left( \int_\Omega |\nabla^{r-1} u_{1,x_i}|^2 \right)^{\frac{1}{r-1}} \left( \int_\Omega |\nabla^{r+1} u_{1,x_i}|^2 \right)^{\frac{1}{r+1}},
\]
which gives that
\[
\left( \int_\Omega |\nabla^r u_{1,x_i}|^2 \right)^{\frac{1}{r}} \leq \left( \int_\Omega |\nabla^{r+1} u_{1,x_i}|^2 \right)^{\frac{1}{r+1}},
\]
and Lemma 3.2 is obtained. □

Making use of Lemma 3.1 and Lemma 3.2, we can prove the following lemma:

**Lemma 3.3.** If $p \geq 3$, then
\[
\Lambda_{i+1}^{\frac{1}{p-1}} - \Lambda_1^{\frac{1}{p-1}} \leq 2p \left( \int_\Omega |\nabla^{p-2} u_{1,x_i}|^2 \right)^{\frac{1}{p-2}} \tag{3.8}
\]
holds for $1 \leq i \leq n$.

**Proof.** By Lemma 3.1 either (3.4) or (3.5) holds.

(1) If (3.4) holds, then
\[
2p(p-1) \int_\Omega |\nabla^{p-2} u_{1,x_i}|^2 \geq \sum_{k=1}^{p-2} \Lambda_1^{\frac{k-1}{p-1}} \left( \Lambda_{i+1}^{\frac{p-1-k}{p-1}} - \Lambda_1^{\frac{p-1-k}{p-1}} \right) \\
\geq \sum_{k=1}^{p-2} \Lambda_1^{\frac{k-1}{p-1}} \left( \Lambda_{i+1}^{\frac{p-1-k}{p-1}} - \Lambda_1^{\frac{p-1-k}{p-1}} \right) \\
= \sum_{k=1}^{p-2} \Lambda_1^{\frac{p-3-k}{p-1}} \left( \Lambda_{i+1}^{\frac{1}{p-1}} - \Lambda_1^{\frac{1}{p-1}} \right) \\
= (p-2) \Lambda_1^{\frac{p-3}{p-1}} \left( \Lambda_{i+1}^{\frac{1}{p-1}} - \Lambda_1^{\frac{1}{p-1}} \right). \tag{3.9}
\]

Since
\[
\int_\Omega |\nabla^{p-2} u_{1,x_i}|^2 \leq \sum_{i=1}^n \int_\Omega |\nabla^{p-2} u_{1,x_i}|^2 = \int_\Omega |\nabla^{p-1} u_{1}|^2 \leq \Lambda_1^{\frac{p-2}{p-1}},
\]
we obtain from (3.9) that
\[
2p(p-1) \int_\Omega |\nabla^{p-2} u_{1,x_i}|^2 \geq (p-2) \Lambda_1^{\frac{p-3}{p-1}} \left( \Lambda_{i+1}^{\frac{1}{p-1}} - \Lambda_1^{\frac{1}{p-1}} \right) \\
\geq (p-2) \left( \int_\Omega |\nabla^{p-2} u_{1,x_i}|^2 \right)^{\frac{p-3}{p-2}} \left( \Lambda_{i+1}^{\frac{1}{p-1}} - \Lambda_1^{\frac{1}{p-1}} \right).
\]
Thus, for \( p \geq 3 \), one gets

\[
\Lambda_{i+1}^{\frac{1}{p}} - \Lambda_i^{\frac{1}{p}} \leq \frac{2p(p-1)}{p-2} \left( \int_{\Omega} |\nabla^{p-2}u_{1,x_i}|^2 \right)^{\frac{1}{p-2}} \leq 2p \left( \int_{\Omega} |\nabla^{p-2}u_{1,x_i}|^2 \right)^{\frac{1}{p-2}}. \tag{3.10}
\]

(2) If \((3.5)\) holds, then using \((3.7)\), it is easy to see

\[
\Lambda_{i+1}^{\frac{1}{p}} - \Lambda_i^{\frac{1}{p}} \leq 4 \int_{\Omega} |\nabla u_{1,x_i}|^2 \leq 2p \int_{\Omega} |\nabla u_{1,x_i}|^2 \leq \ldots \leq 2p \left( \int_{\Omega} |\nabla^{p-2}u_{1,x_i}|^2 \right)^{\frac{1}{p-2}}. \tag{3.11}
\]

Thus \((3.8)\) holds anyway. \(\square\)

Now summing up \((3.8)\) over \(i\) from 1 to \(n\) yields

\[
\sum_{i=1}^{n} (\Lambda_{i+1}^{\frac{1}{p}} - \Lambda_i^{\frac{1}{p}}) \leq (2p)^{p-2} \sum_{i=1}^{n} \int_{\Omega} |\nabla^{p-2}u_{1,x_i}|^2 = (2p)^{p-2} \sum_{i=1}^{n} \int_{\Omega} |\nabla^{p-1}u_{1,x_i}|^2 \leq (2p)^{p-2} \Lambda_{1}^{\frac{1}{p-1}},
\]

concluding the proof of Theorem 1.2. \(\square\)

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