Several results of black holes thermodynamics can be considered as firmly founded and formulated in a very general manner. From this starting point we analyse in which way these results may give us the opportunity to gain a better understanding in the thermodynamics of ordinary systems for which a pre-relativistic description is sufficient. First, we investigated the possibility to introduce an alternative definition of the entropy basically related to a local definition of the order in a spacetime model rather than a counting of microstates. We show that such an alternative approach exists and leads to the traditional results provided an equilibrium condition is assumed. This condition introduces a relation between a time interval and the reverse of the temperature. We show that such a relation extensively used in the black hole theory, mainly as a mathematical trick, has a very general and physical meaning here; in particular its derivation is not related to the existence of a canonical density matrix. Our dynamical approach of thermodynamic equilibrium allows us to establish a relation between action and entropy and we show that an identical relation exists in the case of black holes. The derivation of such a relation seems impossible in the Gibbs ensemble approach of statistical thermodynamics. From these results we suggest that the definition
of entropy in terms of order in spacetime should be more general that the Boltzmann one based on a counting of microstates. Finally we point out that these results are obtained by reversing the traditional route going from the Schrödinger equation to statistical thermodynamics.

PACS number:03.65Ca, 05.30-d, 05.70-a, 47.53+n.

I. INTRODUCTION

After the discovery of the Hawking radiation (1) it became clear that the similarity observed by Bekenstein (2) between the four laws of black hole mechanics and the ones of standard thermodynamics is much more than a simple analogy but represents a result having a deep significance. For a given black hole, the same values for temperature, \( T_{BH} \), and entropy, \( S_{BH} \) have been obtained from different approaches. Consequently, we may consider the black hole thermodynamics as firmly established; one exception, may be, concerns the derivation of the second law of thermodynamics that is replaced by the so-called general second law (3). The main question emerging from these results is relative to the physical origin of the black hole entropy. It is implicitly assumed that the physical content of \( S_{BH} \) can be understood, at least to some extent, from the usual ingredients of classical physics. This leads to search a relation between \( S_{BH} \) and a counting of microstates. After thirty years of intensive works in that direction we have still to deal with questions relative to the nature and the location in space of these microstates (4,5).

In this paper we reverse this problematic: Instead of trying to understand something concerning the black hole physics starting from the results obtained in ordinary physics, we try to see if it is possible to learn something about ordinary systems starting from results obtained in the domain of black holes. By ordinary systems we mean those for which a pre-relativistic description is sufficient. Obviously, we cannot expect a strict mapping between the thermodynamic properties of ordinary systems and those of systems including
black holes for which there are fundamental singularities, presence of horizons and existence of a holographic principle. Nevertheless, as we shall see, it is possible to reformulate some results obtained in the black hole domain in a so general manner that it appears natural to ask whether it exists similar results in the case of ordinary systems.

It is well known that the Hawking temperature, $T_H$, of a Schwarzschild black hole of mass $M$ measured at a large distance from the hole is given by $T_H = \frac{\hbar}{8\pi M}$. In the original derivation of this result Hawking (1) used techniques of quantum field theory on a given classical curved background spacetime. For this purpose it was useful to work in the "real Euclidean section" of the Schwarzschild geometry, in which the time is rotated to its imaginary value and the thermal Green function is periodic with a period $\beta = \frac{1}{k_B T_H}$ (see for instance (6) for a mathematical analysis of this result). Later, Gibbons and Hawking (7) were able to deduce thermodynamics of black hole from statistical mechanics. The main steps in their derivation are the following. A canonical partition function, $Z_{GH}$, is introduced using a path integral in which the action is the one of the gravitational field and the time on which the action is defined is $\beta \hbar$. Then a zero loop approximation is performed leading to a free energy defined by $F_{GH} = -k_B T_H \ln Z_{GH}$ directly proportional to the action and from the usual thermodynamic relations it is possible to calculate the entropy. For four different metrics, it has been shown that the results derived in this way agree with previous derivations although the methods are totally different. Thus the entropy derived from the spacetime properties has a geometrical origin, it is not based explicitly on the counting of microstates although it coincides with the entropy derived from the thermodynamics.

It is tempting to retain from these results some general aspects that we may try to extend to ordinary systems. First, in parallel to the usual definition of entropy it might exist an alternative definition connected with the geometry of space-time and that we might establish without explicit reference with a counting of microstates. In what follows we shall see that such an alternative definition of entropy exists also for ordinary systems provided a spacetime model is introduced as suggested in (8). This spacetime is formed by a discrete manifold on which exists a structural relation that mimics the uncertainty relations and
a given kinematics. An equilibrium condition introduces a time $\beta \hbar$, which is proportional to the reverse of the temperature. For this time our entropy is identical to the traditional one expressed in terms of path integral. Second, black hole theory suggests the existence of a relation between action and spacetime free energy leading to a relation between action and entropy. Hereafter we deduce a similar relation that is verified for black holes and for ordinary systems. For the simple case considered here, the result is quite general i.e. it exists beyond the zero loop approximation. It is noteworthy that the existence of a relation between action and entropy has been suggested by Eddington a long time ago and reinvestigated later by de Broglie searching a relation between two quantities that are considered as relativistic invariants in restricted relativity (for a review in this domain see (9)).

This paper is organized as follows. In Section 2 we summarize the properties of the spacetime model. In Section 3 we introduce the path-entropy and other quantity giving a global information about the spacetime structure. In Section 4 we analyse the relation time / temperature from which we establish a link with ordinary thermodynamics. We point out the analogies but also the differences with the founding of a similar relation existing in the case a thermal Green function defined in imaginary time that are commonly used in the black hole theory. In Section 4 we consider the relation between action and entropy. In the last Section we give some concluding remarks.

II. SPACETIME MODEL

In his book with Hibbs, Feynman (10) developed a given number of fundamental remarks concerning the derivation of the partition function in terms of path-integral. He suggested a possible new foundation of statistical mechanics directly in terms of path integral as he did for quantum mechanics. In this context ”directly” means without using the Schrödinger equation. Note that this Feynman’s conjecture implies to give a physical meaning to the paths and also to justify why one path integral is sufficient to describe statistical physics while only a product of path integral via the square of the amplitude of the wave function
has a physical meaning in quantum physics.

One the most economical way to give a physical meaning to the paths consists in assuming the existence of a primarily discrete spacetime \(^{(8)}\). Although this is probably not strictly needed we assume that the spacetime points \((t_i, x_i)\) are located on the sites of a regular lattice as in the chessboard model investigated in \(^{(10)}\). The spacetime structure is characterized by the existence of a relation between the elementary length \(\Delta x\) and time interval \(\Delta t\) corresponding to the lattice spacing. When a mass is introduced in this lattice we assume that \((\Delta x)^2/\Delta t = \hbar/m\), a relation mimicking the Heisenberg uncertainty relations.

We assume that the free motion in this spacetime model is as simple as possible. By definition, a path corresponds to a set of sites \((t_i, x_i)\); the values of \(t_i\) are such as \(t_{i+1} > t_i\) whatever \(i\) and the coordinate positions, \(x_{i+1}\) is necessarily one of the nearest neighbours of \(x_i\), thus a path corresponds to a random walk.

In this spacetime model we have a discrete manifold, the quantification appears via the relation between \(\Delta x\) and \(\Delta t\) and the kinematics is defined in terms of paths on which exist a causal relation. There is no metric as in the usual mathematical model of the general relativity. However, from the results obtained by Sorkin et al (see for instance \(^{(11, 12)}\)) we know that a causal structure may determine almost all the information needed to specify the metric up to a multiplicative function called the conformal factor; the lack of conformal factor means that we have no quantitative measure for lengths and volume in spacetime.

Our approach is reminiscent of the of causal sets theory on several points: we have a discrete spacetime, the paths defined above obey to causal relation and even the relation between \(\Delta x\) and \(\Delta t\) is not sufficient to fix a measure for lengths. As in the causal sets theory to define a length we have to say, for instance, how many spacetime points exists in a given volume. This can be done if, from a physical argument, we may fix the value of \(\Delta x\), for instance.

In absence of such cutoff the limits \(\Delta x, \Delta t \to 0\) taken with the constraint \((\Delta x)^2/\Delta t = \hbar/m\) leads to a continuous diffusion process \(^{(13)}\) for which the diffusion coefficient is \(D = \hbar/2m\). In presence of an external potential, \(u(t, x)\), this one is simply added to the diffusion equation.
Using the Feynman-Kac formula, the fundamental solution, \( q(t_0, x_0; t, x) \), of this new equation appears as a weighted sum of all the paths connecting the space-time points \((x_0, t_0)\) to \((x, t)\); we have

\[
q(t_0, x_0; t, x) = \int \mathcal{D}x(t) \exp \left( -\frac{1}{\hbar} A^E[x(t); t, t_0] \right)
\]

(1)

where \( \mathcal{D}x(t) \) means the measure for the functional integral and

\[
A^E[x(t); t, t_0] = \int_{t_0}^{t} \left[ \frac{1}{2} m \left( \frac{dx(t')}{dt'} \right)^2 + u(t', x(t')) \right] dt'.
\]

(2)

At this level it is important to underline several points. The paths are associated with processes that occur in real time. These processes are generated from the euclidean action \( A^E[x(t); t, t_0] \) and they are such that, in average, there is no derivative i.e. no velocity in the usual sense on the paths \(^{(14)}\). The continuous limits \( \Delta x, \Delta t \to 0 \) are useful to give a physical meaning to the processes but, at least in principle, the explicit calculation of the path integral can be performed with finite values of \( \Delta x \) and \( \Delta t \) that appear as natural cutoffs in the discretization procedure needed to calculate (1). The function \( q(t_0, x_0; t, x) \) is not a density of probability although it verifies a Chapman-Kolmogorov law of composition. In what follows we will show that the function \( q(t_0, x_0; t, x) \) is sufficient to describe a lot of spacetime properties. In particular we will introduce it in a functional over paths used to describe the order in spacetime.

### III. PATH-ENTROPY

To define the order - or disorder - in spacetime we first adopt a local definition. Around a point \( x_0 \) we count the number of closed paths that we can form during a given time interval \( \tau \). If there is only one possible path we can say that we have a perfect order, no fluctuation around this path is accepted. However, after introducing a given measure, some fluctuations can take place and we have to deal with a given number of acceptable paths. For this measure, we may associate the order in spacetime with this number of paths. The
total order in our system will be obtained by summing the result of this procedure on all the points \( x_i \) existing in the spacetime. This definition seems quite natural anytime we have to deal with processes occurring in a given spacetime. Of course such a definition is not unique but it is probably the simplest one.

By analogy with the thermodynamic entropy, which is defined for given values of internal energy and volume we consider that our spacetime system is prepared with a given energy \( U \) and filled a volume \( V \). We define a path-entropy by counting the number of paths for which the euclidean action that we note hereafter as \( A^E[x(t); \tau] \), does not deviate too much from the action \( \tau U \). In reference with the standard thermodynamics we define a path-entropy, \( S_{\text{path}} \), according to

\[
S_{\text{path}} = k_B \ln \int dx_0 \int Dx(t) \exp - \frac{1}{\hbar} [A^E[x(t); \tau] - \tau U].
\]

(3)

\( S_{\text{path}} \) can be also rewritten as

\[
S_{\text{path}} = \frac{k_B \tau}{\hbar} U + k_B \ln Z_{\text{path}}
\]

(4)

with

\[
Z_{\text{path}} = \int dx_0 \int Dx(t) \exp - \frac{1}{\hbar} A[x(t); \tau] = \int dx_0 q(0, x_0; \tau, x_0)
\]

(5)

in which \( q(0, x_0; \tau, x_0) \) corresponds to closed paths for which \( t_0 = 0 \) and the time interval \( \tau \). \( Z_{\text{path}} \) is the total number of closed paths that we may count during \( \tau \) irrespective the value of \( U \). \( S_{\text{path}} \) contains two external parameters \( \tau \) and \( U \) while \( Z_{\text{path}} \) is only function of \( \tau \). We may consider the dependence of \( S_{\text{path}} \) versus these two parameters by considering the two derivatives : \( \frac{dS_{\text{path}}}{dU} \) defined as \( \frac{1}{T_{\text{path}}} \) and \( \frac{dS_{\text{path}}}{d\tau} \). From the results given in (8) we have :

\[
\frac{\hbar}{k_B} \frac{dS_{\text{path}}}{dU} = \frac{\hbar}{k_B T_{\text{path}}} = \tau + [U - (< u_K >_{\text{path}} + < u_P >_{\text{path}})] \frac{d\tau}{dU}
\]

(6)

and

\[
\frac{\hbar}{k_B} \frac{dS_{\text{path}}}{d\tau} = [U - (< u_K >_{\text{path}} + < u_P >_{\text{path}})] + \tau \frac{dU}{d\tau}
\]

(7)
in which a relation between $U$ and $\tau$ is assumed. The average over paths that appear in (6) and (7) are defined according to

$$< u_P >_{\text{path}} = \frac{1}{Z_{\text{path}}} \int dx_0 u(x_0) q(0, x_0; \tau, x_0)$$

and

$$\frac{m}{2} < \left( \frac{\delta x}{\delta t} \right)^2 >_{\text{path}} = \frac{\hbar}{2\delta t} - < u_K >_{\text{path}}$$

in which $< u_K >_{\text{path}}$ is a well behaved function in the limit $\delta t \to 0$ moreover we have checked on examples that $< u_K >_{\text{path}}$ is just the usual thermal kinetic energy (8).

The quantities, $S_{\text{path}}$, $Z_{\text{path}}$, $T_{\text{path}}$ and $\frac{dS_{\text{path}}}{d\tau}$ are well defined, they gives us a global characteristic of the spacetime structure but none of them corresponds to a thermodynamic quantity. In the next Section we show that such a correspondence can be established for a given value of $\tau$ and we discuss the existence of a general relation between time and temperature.

**IV. RELATION BETWEEN TIME AND TEMPERATURE**

In (6) the sum $< u_K >_{\text{path}} + < u_P >_{\text{path}}$ is only dependent on the parameter $\tau$ and we may choose $\tau$ in such a way that the previous sum coincide with $U$, thus $\tau$ corresponds to a condition of thermal equilibrium. Now from (6) we conclude that the relation between $\tau$ and the temperature $T_{\text{path}}$ is $\tau = \frac{\hbar}{k_B T_{\text{path}}}$ whatever the value of $\frac{d\tau}{dU}$. If we identify $T_{\text{path}}$ with the usual temperature we can see that $Z_{\text{path}}$ defined in (5) becomes identical to the traditional partition function expressed in terms of path integral (10) and thus we may recover all the results of thermodynamics. It is quite simple to verify that (7) is now reduced to $\frac{\hbar}{k_B} \frac{dS_{\text{path}}}{d\tau} = \tau \frac{dU}{d\tau}$ that we can rewrite as $dU = T dS$. Here we may interpret this relation as follows: if we increase the energy $U$ for the system preparation we increase the number of paths avaible and therefore the entropy in spacetime. It has also been shown that $\tau$ is the time interval that we have to wait in order to relax the quantum fluctuations and to reach a thermal regime (15). Finally we may also derive the relation $\tau = \beta h$ from the following
heuristic argument. From standard thermodynamics we know that if a change of energy \( \Delta U \) produces on a moving body a change of momentum \( \Delta P \), the corresponding change of entropy \( \Delta S \) is given by (16):

\[
\Delta S = \frac{1}{T} \Delta U - \left( \frac{V}{T} \right) \Delta P
\]

in which \( V \) is the velocity of the mobile. From (10) we may learn two different kinds of results. First, in terms of variations we may assume that \( \Delta U \) results from quantum fluctuations and its estimation is \( \Delta U = \frac{\hbar}{2 \tau} \). In (10) we can write \( V \Delta P \) as \( \Delta \left( \frac{1}{2} m V^2 \right) \) and if we assume that the quantum fluctuations lead to the thermal equilibrium we have \( \Delta \left( \frac{1}{2} m V^2 \right) = \frac{1}{2 \tau} \). Since the system is at equilibrium we must have no net change of entropy during these fluctuations i.e. \( \Delta S = 0 \), but we can see that \( \Delta S = 0 \) implies \( \tau = \beta \hbar \). Second, the relation (10) is adequate for an interpretation in the domain of restricted relativity. Since the entropy is considered as a relativistic invariant the right hand side of equation (10) must be relativistic invariant. It appears as the scalar product of the the energy momentum tensor by a quantity \((\frac{1}{T}, (\frac{V}{T}))\) that must be a four-vector showing that \((\frac{1}{T})\) must behave as a time in a Lorentz transformation. This result also supports the idea that it exists a fundamental relation between time and the reverse of temperature.

The relation \( \tau = \beta \hbar \) is extensively used in the derivation of the black hole properties but it appears more as a mathematical trick than a relation having a strong physical basis. In this domain it is usual to investigate the properties of analytic functions defined on a complex time plane. For a given spatial distance between two points it exists on the real time axis a given domain on which the interval is spacelike. On this domain the field commutator vanishes and this allows us to define a unique analytic continuation of the Green functions through the complex plane (6). Working at a non-zero temperature the Green functions defined on the euclidean sector exhibits a periodicity \( \tau = \beta \hbar \). To derive this result the crux is that the effect of temperature is determined via the thermal density matrix \( \exp - \beta H \) in which \( H \) is the hamiltonian operator.

In our derivation of the relation \( \tau = \beta \hbar \) we do not need the hamiltonian operator nor the
Schrödinger equation or the canonical form of the density matrix. What we need from quantum mechanics is the existence of a relation that mimics the Heisenberg uncertainty relation and defines the spacetime fine structure. This is reminiscent of a strong result obtained by Wald (3): some results of black hole theory can be derived without using the detailed expression of the Einstein equations.

From all arguments developed above it seems normal to conclude that the relation \( \tau = \beta h \) has a very general and deep physical meaning.

V. RELATION BETWEEN ACTION AND ENTROPY

In the Gibbs ensemble approach of statistical mechanics we cannot expect a relation between action and free energy because these two quantities have a very different nature: free energy is considered as an equilibrium quantity that we have to calculate by integration over the phase space while the action is a dynamical quantity and its definition requires the introduction of a time interval. In the previous Section we have developed a dynamical approach of thermodynamic equilibrium in which a time interval \( \tau \) is associated with the reverse of the temperature. We have shown that the free energy defined according to \( F = -k_B T \ln Z \) is a functional of an euclidean action \( A^E[x(t); t, t_0] \) as in the case of black hole.

Note that in our approach the zero loop approximation cannot be used without losing the main part of the underlying physics therefore we will stay with the functional form. Before setting up a relation between action and entropy we analyze the relation between the lagrangian and euclidean version of the action.

In (8) we have defined a potential energy \( < u_P >_{path} \) by an average over paths and the mean value of the kinetic energy over paths will be defined by \( \frac{m}{2} < (\frac{dx}{dt})^2 >_{path} \) that we have introduced in (9). As usually we may define a lagrangian by the difference between kinetic and potential energy. For an elementary time interval the corresponding action \( < A(T, \delta t) >_{path} \) will be
\[ <A(T, \delta t) >_{\text{path}} = \delta t \left[ \frac{m}{2} < \left( \frac{\delta x}{\delta t} \right)^2 >_{\text{path}} - <u_P >_{\text{path}} \right] = \frac{\hbar}{2} - \left[ <u_K >_{\text{path}} + <u_P >_{\text{path}} \right] = \frac{\hbar}{2} - U \delta t \]  

(11)

The second equality results from (9) and the third is the consequence of the equilibrium condition discussed in the previous Section. Thus from (11) we can see that the elementary lagrangian action is related to \(-U \delta t\) and it seems natural of considering \(U \delta t\) as the elementary euclidean action. In the limit \(\delta t \to 0\) the lagrangian action has a finite value \(\frac{\hbar}{2}\) corresponding to the quantum of action, thus the action is not differentiable as expected. If we increase the temperature by \(\delta T\) we have

\[ <A(T + \delta T, \delta t) >_{\text{path}} - <A(T, \delta t) >_{\text{path}} = -\delta t [U(T + \delta T) - U(T)] = -\delta t T \delta S \]  

(12)

in which the last equality is a consequence of the usual thermodynamic relation. The net change of action \(\delta A_{\text{path}}\) on the time interval \(\tau\) will be obtained by summing up \([<A(T + \delta T, \delta t) >_{\text{path}} - <A(T, \delta t) >_{\text{path}}]\) on all the elementary time interval covering the total time interval \(\tau\). For the part of (12) containing \(U(T)\) this will be simply done by multiplying the previous result by the number of elementary steps \(i.e. \frac{\tau}{\delta t}\). It is easy to see that the final result can be written as

\[ \frac{\delta A_{\text{path}}}{\hbar} = -\frac{\delta S}{k_B} \]  

(13)

For a black hole having an area \(A\) the entropy is given by \(S = \frac{k_B c^3}{4G\hbar} A\) and the euclidean action is \(A^E = -\frac{c^3}{4G} A\) leading to the relation \(\frac{S}{k_B} = -\frac{A^E}{\hbar}\) from which we get immediatly (13) provided we identify the change of \(A_{\text{path}}\) with the change of \(A^E\). This identification is justified since the introduction of an imaginary time in the lagrangian defined in (11) leads to minus the hamiltonian and therefore, in comparison with the usual definition an extra minus sign is introduced. It is important to note that in the case of black hole the first loop approximation leads to a direct relation between action and entropy but the relation (13) is also verified. A relation like (13) that introduces a link between variations fits quite well with the spirit of thermodynamics and we may think that it represents a result more general than \(\frac{S}{k_B} = -\frac{A^E}{\hbar}\).
VI. CONCLUDING REMARKS

Entropy is a fundamental quantity in physics and the law of evolution of entropy may be considered as the actual law of evolution for a system. But recent results show that, possibly, the concept of entropy is far to be well understood (3, 17). This is the case in the black hole domain in which we are not able to associate the entropy with a counting of microstates. In addition, as a consequence of the Unruh effect (17), it appears that the number of microstates might be related to the motion of the observer. This suggests that a definition of entropy more general that the one introduced by Boltzmann might exist. This paper is an attempt to introduce a definition of entropy that we can use both for ordinary systems and systems including black hole. It is also an illustration showing that general results obtained in the domain of black hole can be extended to ordinary systems producing an improvement in our general understanding about thermodynamics.

We show that for ordinary systems the entropy can be calculated as in the case of black hole i.e. by inspecting the properties of a spacetime model. Of course, starting from this new approach based on a dynamic description of the thermodynamic equilibrium we recover the standard results. But we also establish two results used in the domain of black hole physics. First, it exists a relation between time and temperature; this is much more than a mathematical trick as it may appear in the derivation of the KMS condition (6). Here this relation is based on an equilibrium condition, it appears as very general since it is connected neither with the existence of a canonical density matrix nor with the existence of the Schrödinger equation. Second, we show that it exists a relation between action and entropy that is also verified in the case of black hole.

It is interesting to note that in our approach we do not follow the traditional route going from the Schrödinger equation to the calculation of eigen states for energy and then to the estimation of the partition function via the density matrix. In fact we follow the opposite route. We start from a spacetime model in which the quantification appears via a relation that determines the spacetime fine structure and consider directly the paths in spacetime,
the motion is then characterize by a positive semi-group but not by a unitary transformation. However, if we force the system to have a time-reversible behavior then, as shown in (8), we may recover the Schrödinger equation and an unitary law of evolution.

Finally, the results obtained in this paper suggest the following conjecture: it is possible that the general definition of entropy is connected with a definition of the order in spacetime rather than associated with a counting of microstates.
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