A note on bosonic open strings in constant $B$ field

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Abstract

We sketch the main steps of old covariant quantization of bosonic open strings in a constant $B$ field background. We comment on its space-time symmetries and the induced effective metric. The low-energy spectrum is evaluated and the appearance of a new non-commutative gauge symmetry is addressed.

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1 Introduction

Starting with evidence has been gathering that ideas of noncommutative geometry may play an important role in string theory (for a review and more references see [2]). Especially interesting is the model of open strings propagating in a constant two-form ($B$-field) background. This model attracted attention since it can be studied from a perturbative string point of view. Previous studies show that this model is related to noncommutativity of D-branes [3], [4] and in the zero slope limit to noncommutative Yang-Mills theory [2]. In this note we would like to study this system from the old perturbative string perspective concentrating on its symmetries.

In the conformal gauge the model has the following action

$$S = -\frac{1}{4\pi\alpha'} \int_{\Sigma} d^2\sigma \left[ \eta_{\mu\nu} \partial_\alpha X^\mu \partial^\alpha X^\nu - \epsilon^{\alpha\beta} B_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu \right] + \int_{\partial \Sigma} d\tau A_\mu \dot{X}^\mu \quad (1)$$

where $\Sigma$ is an oriented world-sheet with boundaries and signature $(-1,1)$. For the open string case the $B$-field should appear in a gauge invariant combination with the $U(1)$ gauge field $B_{\mu\nu} = B + 2\pi\alpha' \partial_\mu A_\nu$. Thus the action has the form

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \left[ \eta_{\mu\nu} \partial_\alpha X^\mu \partial^\alpha X^\nu - \epsilon^{\alpha\beta} B_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu \right] \quad (2)$$

where $B$ is assumed to be constant. The equation of motion and the boundary conditions are given by the following expressions

$$\partial_\alpha \partial^\alpha X^\mu = 0, \quad \eta_{\mu\nu} X^\nu + B_{\mu\nu} \dot{X}^\nu |_{\sigma=0,\pi} = 0. \quad (3)$$

In what follows we will not impose any pure Dirichlet boundary conditions. All results can be straightforwardly generalized to the case with pure Dirichlet conditions. Thus using modern language one can say that we are looking at a D25-brane with constant $B$-field.

The paper is organized as follows: In section 2 we sketch some steps of the covariant quantization of the model. In sections 3 and 4 the symmetries and the mass spectrum are discussed. In the last two sections we consider noncommutative gauge transformations of massless states and explain how it might be related to deformation quantization.

2 The old covariant quantization

In this section we sketch some steps of the old covariant approach to quantization. Essential parts of this calculations have already appeared in [3]. However, for the sake of completeness
we prefer to go through this calculations once more. We will write the equations in such a way that one can analyze the possible modifications of them in different situations.

The general form of the bulk equation of motion \( \partial_\alpha \partial^\alpha X^\mu = 0 \) is given by

\[
X^\mu(\tau, \sigma) = q^\mu + a_0^\mu \tau + b_0^\mu \sigma + \sum_{n \neq 0} \frac{e^{-in\tau}}{n} (i a_n^\mu \cos n\sigma + b_n^\mu \sin n\sigma) \quad (4)
\]

By imposing boundary conditions (3) on (4) we get the following solution

\[
X^\mu(\tau, \sigma) = q^\mu + (a_0^\mu \tau - B_\nu^\mu a_0^\nu \sigma) + \sum_{n \neq 0} \frac{e^{-in\tau}}{n} (i a_n^\mu \cos n\sigma - B_\nu^\mu a_n^\nu \sin n\sigma) \quad (5)
\]

One can check that the form of Virasoro conditions are not modified

\[
\frac{1}{4\alpha'}(\dot{X} \pm X')^2 = 0 \quad (6)
\]

and using the expansion (5) the constraints (3) can be expanded as follows

\[
L_k = \frac{1}{4\alpha'} \sum_{n=-\infty}^{\infty} G_{\nu\rho} a_n^\nu a_k^{-n}, \quad G_{\nu\rho} = \eta_{\nu\rho} - B_{\nu\mu} \eta^{\mu\sigma} B_{\sigma\rho} \quad (7)
\]

where \( G_{\mu\nu} \) coincides with the notation of [2]. From the action (2) one can define the canonical momentum density

\[
P_\mu = \frac{1}{2\pi\alpha'}(\eta_{\mu\nu} \dot{X}^\nu + B^\mu_{\nu\nu} X^\nu) \quad (8)
\]

which has the following expansion

\[
P_\mu = \frac{1}{2\pi\alpha'} \sum_{n=-\infty}^{\infty} G_{\mu\nu} a_n^\nu e^{-in\tau} \cos n\sigma \quad (9)
\]

The canonical commutation relations have the standard form

\[
[P_\mu(\tau, \sigma), X^\nu(\tau, \sigma')] = -i\delta_\mu^\nu \delta(\sigma - \sigma'),
\]

\[
[X^\mu(\tau, \sigma), X^\nu(\tau, \sigma')] = [P_\mu(\tau, \sigma), P_\nu(\tau, \sigma')] = 0. \quad (10)
\]

They imply the commutation relations for the modes. From this point we would like to be more careful. One can start from the relations \([P_\mu(\tau, \sigma), P_\nu(\tau, \sigma')] = 0 \) and \([P_\mu(\tau, \sigma), X^\nu(\tau, \sigma')] = -i\delta_\mu^\nu \delta(\sigma - \sigma')\). These relations fix uniquely the following commutators

\[
G_{\mu\nu}[a_n^\rho, a_m^\nu] = 2\alpha' n \delta_\mu^\nu \delta_{m+n}, \quad G_{\mu\nu}[a_n^\rho, q^\nu] = -2i\alpha' \delta_\mu^\nu \delta_n \quad (11)
\]

\(^1\)In Euclidean signature the matrix \( G_{\mu\nu} \) has the same form.
So far we have not assumed anything definite about the matrix $G_{\mu \rho}$. To proceed one should assume that $G_{\mu \rho}$ is invertible. Now one can look at the commutator $[X^\mu(\tau, \sigma), X^\nu(\tau, \sigma')] = 0$. Using the mode expansion and commutation relations (11) we get

$$[X^\mu(\tau, \sigma), X^\nu(\tau, \sigma')] = [q^\mu, q^\nu] - 2i\alpha' \theta^{\mu\nu}(\sigma + \sigma') - 2i\alpha' \theta^{\mu\nu} \sum_{n \neq 0} \frac{1}{n} \sin n(\sigma + \sigma') = 0$$

(12)

where $\theta^{\mu\nu} = -B^{\mu}_{\sigma}(G^{-1})^{\sigma\nu}$ and it coincides with the notation of [2]. Using the following Fourier expansion on the interval $(0, 2\pi)$ for $(\sigma + \sigma')$

$$\sum_{n \neq 0} \frac{1}{n} \sin n(\sigma + \sigma') = \pi - (\sigma + \sigma')$$

(13)

we can fix commutators for $q^\mu$. Thus by requiring the commutation relations to be valid at all nonboundary points we get the commutators

$$[a^\rho_n, a^\nu_m] = 2\alpha' n \delta_{n+m}(G^{-1})^{\rho\nu}, \quad [a^\rho_n, q^\nu] = -2i\alpha' \delta_n(G^{-1})^{\rho\nu}, \quad [q^\mu, q^\nu] = 2\pi i\alpha' \theta^{\mu\nu}$$

(14)

Using (7) and (14) one can show that $L_k$ satisfy the Virasoro algebra

$$[L_k, L_n] = (k-n)L_{k+n} + A(n)\delta_{k+n}$$

(15)

where using standard arguments one can show that the anomaly has the same value as in the usual $(B = 0)$ case.

3 Symmetries of the model

Now let us look at the symmetries of the action (2). Naively one might expect the full ten dimensional Poincaré group to be a symmetry of the action. The action (2) is certainly invariant under ten dimensional translations $X^\mu \to X^\mu + b^\mu$. Using the Noether theorem we can find the corresponding conserved current. The zero component of this current coincides with the momentum density (8). Thus one can define the momentum operator for the string as follows

$$p_\mu = \int_0^\pi d\sigma P_\mu(\tau, \sigma) = \frac{1}{2\alpha'} G_{\mu\nu} a^\nu_0.$$  

(16)

One can check the conservation of momentum explicitly

$$p_\mu = \frac{1}{2\pi \alpha'} \int_0^\pi d\sigma (\eta_{\mu\nu} \dot{X}^\nu + B_{\mu\nu} \dot{X}^{\nu'}) = \frac{1}{2\pi \alpha'} (\eta_{\mu\nu} X^\nu + B_{\mu\nu} X^{\nu'})|_{\sigma=0,\pi} = 0.$$  

(17)

Invertibility is guaranteed as long as $B_{0i}$ components are not too large. See below.
We can rewrite the commutations relations for the zero modes as follows
\[ [q^\mu, p_\nu] = i\delta^\mu_\nu, \quad [q^\mu, q^\nu] = 2\pi i\alpha'\theta^{\mu\nu}. \]  
(18)

Now let us take a look at the Lorentz symmetry of the model. If the \( B \)-field is regarded as a constant background then the Lorentz symmetry is broken down to such transformations which preserve the form of \( B \). To analyze the question we can perform \( SO(d-1,1) \) transformation which brings \( B \) to the canonical block diagonal form
\[
(B_{\mu\nu}) = \begin{pmatrix}
0 & \lambda_1 & 0 & 0 & \ldots & 0 & 0 \\
-\lambda_1 & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & \lambda_2 & \ldots & 0 & 0 \\
0 & 0 & -\lambda_2 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 0 & \lambda_{d/2} \\
0 & 0 & 0 & 0 & \ldots & -\lambda_{d/2} & 0
\end{pmatrix}
\]  
(19)

and thus \( G_{\mu\nu} \) will have the following form
\[
(G_{\mu\nu}) = \begin{pmatrix}
\lambda_1^2 - 1 & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 1 - \lambda_1^2 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 1 + \lambda_2^2 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & 1 + \lambda_2^2 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 1 + \lambda_{d/2}^2 & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & 1 + \lambda_{d/2}^2
\end{pmatrix}
\]  
(20)

where \( \eta_{\mu\nu} = (-1, 1, \ldots, 1) \). In other words we rewrite the action (2) in coordinates such that \( B \) assumes the above form. In the Euclidian version of the theory one should replace \( G_{00} \) and \( G_{11} \) by \( 1 + \lambda_1^2 \). In the Minkowski case \( G_{\mu\nu} \) can acquire a 0 eigenvalue when \( \lambda_1 = 1 \). In this case we will have problems with imposing the canonical commutation relations (10) on the model. Thus one can conclude that the model is not well-defined at all values of \( \lambda_1 \).

By looking at (19) one can find the Lorentz group of the model. If \( \lambda_1 = \lambda_2 = \ldots = \lambda_r = 0 \) and the rest are different from zero then the Lorentz group will be \( \Lambda = SO(2r-1,1) \otimes (SO(2))^{(d/2-r)} \). By definition the new Lorentz group \( \Lambda \) will preserve the form of \( G_{\mu\nu} \) and \( \theta^{\mu\nu} \). Thus any constructions with these objects are Lorentz covariant in the present model. The natural scalar product of the Lorentz group is given by \( k_\mu(G^{-1})^{\mu\nu}p_\nu \) where \( k_\mu \) and \( p_\nu \) are vectors which transform under the new Lorentz group \( \Lambda \). As a result we will see that in mass spectrum all particles are classified under the irreducible representations of the new Poincaré group which is a semidirect product of the new Lorentz group \( \Lambda \) and the ten dimensional group of translations.
To understand better the commutation relations (18) it is useful to write down the explicit form of $\theta^{\mu\nu}$ in the basis where $B_{\mu\nu}$ has the form (19)

\[
(\theta^{\mu\nu}) = \\
\begin{pmatrix}
0 & \frac{\lambda_1}{1-\lambda_1} & 0 & 0 & \ldots & 0 & 0 \\
-\frac{\lambda_1}{1-\lambda_2} & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & -\frac{\lambda_2}{1+\lambda_2} & \ldots & 0 & 0 \\
0 & 0 & \frac{\lambda_2}{1+\lambda_2} & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 0 & -\frac{\lambda_{d/2}}{1+\lambda_{d/2}} \\
0 & 0 & 0 & 0 & \ldots & \frac{\lambda_{d/2}}{1+\lambda_{d/2}} & 0 \\
\end{pmatrix}
\] (21)

Ignoring the components of $B$ field along the time direction we can see that the components of $|\theta^{\mu\nu}|$ are bounded by $1/2$ and $|\theta^{\mu\nu}|$ takes the maximal value when B-field is 1 ($|\lambda| = 1$). Thus from (18) we get the following uncertainty relation

\[
\Delta q^\mu \Delta q^{\nu} \geq \frac{1}{2} \pi \alpha'
\] (22)

at $\lambda = 1$. We see that the maximal noncommutative effect appears when $B$ is of order one and the maximal cells have a size of order $\sqrt{\alpha'}$. Therefore in this situation only one degree of freedom can be per Plank cell. The general situation can be summarized as follows: as the $B$ field ($\lambda$) varies from 0 to $\infty$ we are going from pure Neumann boundary conditions to pure Dirichlet ones. As we are approaching these limiting situations the noncommutativity disappears.

## 4 Mass spectrum

We can now analyze the mass spectrum. The Hamiltonian $L_0$ of the system is:

\[
L_0 = \alpha' p_\mu (G^{-1})^{\mu\nu} p_\nu + \frac{1}{2} \sum_{n \neq 0} n G_{\mu\nu} \alpha_n^\mu \alpha_n^{\nu} - n
\] (23)

where $\alpha_n^\mu = \sqrt{2\alpha'} n |\alpha_n^\mu|$ (for $n \neq 0$) and $p_\mu$ is defined by (16). Arguments from the previous section indicate that the correct Casimir for the new Poincaré group is $M^2 = -p_\mu (G^{-1})^{\mu\nu} p_\nu$. We assume that $\lambda_1^2 < 1$ (so the metric $G_{\mu\nu}$ does not become degenerate) the quantization can be done in the standard way by imposing Virasoro conditions on the physical states: $(L_0 - 1) |\text{Phys} \rangle = 0$ and $L_n |\text{Phys} \rangle = 0$ for $n > 0$. Therefore the expression for the masses of string states is the following

\[
\alpha' M^2 = \sum_{n=1}^{\infty} n (\alpha_n^+)^{\mu} G_{\mu\nu} (\alpha_n)^{\nu} - 1
\] (24)
where we used \((\alpha^+_n)^\mu = \alpha^\mu_n\). Thus one can see that we get the usual spectrum except that the states will transform under a non-standard Poincaré group.  

There is a massless state in the model: \(\epsilon_\mu(k)\alpha^\mu_{-1}|0,k\rangle\). Requiring that \(L_0 - 1\) and \(L_1\) annihilate the state implies:

\[
\begin{align*}
k_\mu(G^{-1})^{\mu\nu}k_\nu &= 0, & \epsilon_\mu(k)(G^{-1})^{\mu\nu}k_\nu &= 0.
\end{align*}
\]

The second condition in (25) should be interpreted in same way as it is done in QED when by imposing condition \(\partial_\mu A_\mu = 0\) on the Fock space one kills unwanted states. It is important to stress that \(\epsilon_\mu\) is not a massless vector in the normal sense. It transforms as a massless vector under the new Lorentz group \(\Lambda\).

The next important question is the following, what is the real gauge transformation for this massless gauge boson. Recall that in ordinary \((B = 0)\) open string theory gauge transformations amount to a shift of the state by a null state (a physical state which is orthogonal to all physical states and therefore of zero norm):

\[
\begin{align*}
\epsilon_\mu(k)\alpha^\mu_{-1}|0,k\rangle \rightarrow \epsilon_\mu(k)\alpha^\mu_{-1}|0,k\rangle + ik_\mu\gamma(k)\alpha^\mu_{-1}|0,k\rangle
\end{align*}
\]

In this case one can show that the shifted state has zero norm on shell, \(k^2 = 0\). In the next section we will explicitly show that a noncommutative gauge transformation in the \(B \neq 0\) case results in a shift by a null state as well.

## 5 Noncommutative gauge transformation of massless state

In these two last sections we would like to take a naive (but hopefully transparent) look at the noncommutative transformations and its relation to the deformation quantization.

Let us sketch some important properties of commutative and noncommutative gauge transformations which we are going to use. For the case of usual abelian gauge transformations we have the following invariance

\[
A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu \lambda(x)
\]

or in momentum representation

\[
\epsilon_\mu(k) \rightarrow \epsilon_\mu(k) + ik_\mu\gamma(k)
\]

\(^3\)If one tries to define the mass operator as \(M^2 = -p_\mu p^\mu\), the spectrum of the string turns out to be continuous which is not one would expect for a sensible string theory.
where we are using the following notation
\[ A_\mu(x) = \frac{1}{(2\pi)^{d/2}} \int d^d k \ e^{ikx} \epsilon_\mu(k), \quad \epsilon_\mu(k) = \frac{1}{(2\pi)^{d/2}} \int d^d x \ e^{-ikx} A_\mu(x) \] (29)
and \( \gamma \) is the Fourier transform of \( \lambda \). The reality condition for \( A_\mu(x) \) and \( \lambda(x) \) corresponds to the following condition \( \epsilon^*_\mu(k) = \epsilon_\mu(-k) \), \( \gamma^*(k) = \gamma(-k) \). The condition \( k_\mu \epsilon^\mu(k) = 0 \) corresponds to the Lorentz condition for \( A^\mu \).

Noncommutative \( U(1) \) gauge transformations act as follows:
\[ A_\mu \rightarrow A_\mu + \partial_\mu \lambda + i \lambda^* A_\mu - i A^\mu \lambda \] (30)
where the star product has the following standard definition
\[ f(x) \ast g(x) = e^{i\pi\alpha'^{\theta\mu\nu} \frac{\partial}{\partial p^\mu} \frac{\partial}{\partial \bar{p}^\nu} f(p) \bar{g}(k - p)} \] (31)
In the momentum representation we have
\[ \frac{1}{(2\pi)^{d/2}} \int d^d x \ e^{-ikx} (f(x) \ast g(x)) = \frac{1}{(2\pi)^{d/2}} \int d^d p \ e^{-i\pi\alpha'^{\theta\mu\nu} p^\mu k^\nu} f(p) \bar{g}(k - p) \] (32)
where
\[ f(x) = \frac{1}{(2\pi)^{d/2}} \int d^d p \ e^{ipx} \bar{f}(p), \quad g(x) = \frac{1}{(2\pi)^{d/2}} \int d^d l \ e^{ilx} \bar{g}(l) \] (33)
Using (30) and (32) one can get the noncommutative gauge transformation in momentum representation
\[ \epsilon_\mu(k) \rightarrow \epsilon_\mu(k) + i k_\mu \gamma(k) + \frac{2}{(2\pi)^{d/2}} \int d^d p \gamma(p) \epsilon_\mu(k - p) \sin(\pi\alpha'^{\theta\nu\rho} p^\nu k^\rho) \] (34)
for the case when \( d \) is even (in odd case there are some problems). It is important to note that in noncommutative gauge transformations (30) the gauge parameter is not completely arbitrary (unlike the commutative case). In noncommutative case the field strength is not a gauge invariant object and therefore to impose the gauge invariance on the action one should require that the gauge potential and gauge parameter fall off at infinity fast enough. In the momentum space it means that they have appropriate behavior at zero momentum. Also all functions in the noncommutative case should be infinitely differentiable.

To show that the transformation (34) is a gauge transformation we should show that the following state is null:
\[ \int d^dp \gamma(p) \epsilon_\mu(k - p) \sin(\pi\alpha'^{\theta\nu\rho} p^\nu k^\rho) \alpha^\mu_{-1} |o, k\rangle = f_\mu(k) \alpha^\mu_{-1} |o, k\rangle \] (35)
or in other words one has to prove that the following two properties hold
\[ f_\mu(k) k^\mu = 0, \quad f_\mu(k) f^\mu(k) = 0, \quad \epsilon^*_\mu(k) f^\mu(k) = 0 \] (36)
when the properties (25) hold. The properties (36) insure that the state (35) is a physical and it is orthogonal to all physical states. Let us show that equations (36) are true. Introducing the notation

\[ A(k) = f_\mu(k)k^\mu = \int d^4p \gamma(p)\epsilon_\mu(k-p)k^{\mu} \sin(\pi\alpha'\theta^\rho p_\rho p_\mu) \] (37)

one can show that \( A^*(k) = A(-k) \) using the reality conditions for \( A_\mu \) and \( \lambda \). One can equivalently write (37) as:

\[ A(k) = f_\mu(k)k^\mu = \int d^4p \gamma(p)\epsilon_\mu(k-p)p^{\mu} \sin(\pi\alpha'\theta^\rho p_\rho p_\mu) \] (38)

where we used \( \epsilon_\mu(k-p)(k-p)^\mu = 0 \). From (38) again using the reality conditions one can show that \( A^*(k) = -A(-k) \) and therefore \( A(k) = 0 \). Since \( k_\mu f^\mu(k) = 0 \) is just Lorentz gauge condition we require this condition to be true off-shell (i.e. even when \( k^2 \neq 0 \)). In the present proof any subtleties are excluded since all functions have nice behavior in general including at zero momentum.

Now we can prove that the second property in (36) is true. The combination \( f_\mu(k)f^{*\mu}(k) = G(k^2) \) is Lorentz invariant and therefore it can depend only on \( k^2 \). We are only interested in the value of \( G(k^2) \) when \( k^2 = 0 \). Thus it is just a constant which one can calculate by picking some concrete value of \( k_\mu \) which satisfies \( k^2 = 0 \). We can do it for \( k_\mu = (0, 0, ..., 0) \) and find that it is zero. The last condition conditions in (36) can be proven in the same way. Thus we have proven all conditions in (36).

To avoid confusion one should understand that properties (36) are not true as equalities on functions. Since we are dealing with a Poincaré invariant quantum theory one should look at the correlation functions of operators. For instance, the last expression in (36) should be understood as the following correlator in coordinate space

\[ \langle A^\mu(x)\delta A_\mu(y) \rangle = G((x-y)^2) \] (39)

or in momentum space

\[ \langle \epsilon^\mu(-k)\delta\epsilon_\mu(k) \rangle = G(k^2) \] (40)

where by \( \delta \) we mean a gauge transformation. It is the fact that we are dealing with correlation functions in a Poincaré invariant vacuum give us such a simple dependence on momentum.

6 General gauge transformation and deformation quantization

One can try to address the following question: what is the general form of a null state which corresponds to gauge transformations. In this section we would like to argue that in general a
gauge transformation is related to deformation quantization with respect to $\alpha'$. The relation between open string model in $B$ field background and the deformation quantization was originally pointed out in [3].

There is only one local solution for conditions (36) which correspond to $k_\mu \gamma(k)$. All other solutions will be nonlocal in momentum space. Since we are interested in infinitesimal gauge transformations, a null state should be linear in the gauge field $\epsilon_\mu$ and gauge parameter $\gamma$.

Thus the general solution has the form

$$ f_\mu(k) = \int d^d p \gamma(p) \epsilon_\mu(k-p) G(p,k-p) $$

where $G(p,k-p)$ is a kernel with the following property

$$ G^*(-p,k+p) = G(p,-k-p) = G(-p,k+p). $$

The property comes from the reality condition $f_\mu^*(k) = f_\mu(-k) = -f_\mu(k)$. We require two extra properties for the function $G(p,k-p)$

$$ G(p,-k-p) = -G(p,k-p), \quad G(-p,k+p) = -G(-p,k+p) $$

which ensures the condition $k^\mu f_\mu = 0$ off-shell (i.e. even when $k^2 \neq 0$). As a result of (43) we have $G(p,-p) = 0$ and it gives us the two properties in (36) on-shell. It is important to notice that the function $G(p,k-p)$ is defined up to gauge field and gauge parameter redefinitions

$$ f_\mu(k) = \int d^d p \gamma(p)s(p) \epsilon_\mu(k-p)s(k-p) \frac{G(p,k-p)}{s(p)s(k-p)}. $$

These redefinitions should not modify the on-shell physics. In other words it should not modify the on-shell state

$$ \epsilon_\mu(k) \alpha^-_{\mu-1}|o,k) \rightarrow s(k) \epsilon_\mu(k) \alpha^-_{\mu-1}|o,k), \quad s(0) = 1 $$

The function $s(k)$ has a natural expansion in powers of $\alpha'$. In the coordinate space these redefinitions correspond to the following transformation:

$$ A_\mu \rightarrow A_\mu + \alpha' D_1(A_\mu) + (\alpha')^2 D_2(A_\mu) + ... $$

where $D_i$ are differential operators. These redefinitions produce the group of automorphisms on the space of gauge fields. Since we would like to understand the general form of off-shell gauge transformation it is important to look at the possible field redefinitions which do not affect the on-shell physics.

$G(p,k-p)$ is a dimensionless scalar function and it can be expanded in powers of $\alpha'$. Lorentz invariance requires that it is a function of momenta with all indices contracted using either $\theta^{\mu\nu}$ or $(G^{-1})^{\mu\nu}$. In coordinate representation the expansion has the form

$$ \delta_\alpha A_\mu = \alpha' B_1(\alpha, A_\mu) + (\alpha')^2 B_2(\alpha, A_\mu) + ... $$
where $B_i$ are bidifferential operators, the general possible form can be easily written down. The null states (41) should correspond to some gauge symmetry. The finite gauge transformations belong to a group. Therefore at level of infinitesimal transformations one has to require:

$$\delta_\alpha \delta_\beta A_\mu - \delta_\beta \delta_\alpha A_\mu = \delta_{(\alpha,\beta)} A_\mu$$

where $\{\alpha, \beta\}$ is the bracket with respect to some associative algebra. It turns out that the requirement (48) is quite restrictive for the form of the kernel $G(p, k - p)$. From (48) in the momentum space one can read off the bracket as follows

$$\{\alpha, \beta\}(p) = \int d^d p \alpha(\tilde{p})\beta(p - \tilde{p}) \left[ \frac{G(p - \tilde{p})G(\tilde{p}, k - p) - G(\tilde{p}, k - p)G(p - \tilde{p}, k - p + \tilde{p})}{G(p, k - p)} \right]$$

(49)

The kernel $K(\tilde{p}, p - \tilde{p})$ for the bracket (expression in square brackets) does not depend on the vector $k$ and should be antisymmetric ($K(\tilde{p}, p - \tilde{p}) = -K(p - \tilde{p}, \tilde{p})$). Using these requirements one can arrive at the conclusion that $K(\tilde{p}, p - \tilde{p}) = G(\tilde{p}, p - \tilde{p})$. Thus in coordinate space the gauge transformation has the form of bracket: $\delta_\alpha A_\mu = \{\alpha, A_\mu\}$ and the requirement (48) is just the Jacobi identity for this bracket. Using antisymmetry of $G$ we can write down the leading term of the gauge transformation as follows

$$\delta_\alpha A_\mu = \alpha' G_1 \theta^{\mu\nu} \partial_\mu \alpha \partial_\nu A_\mu + O(\alpha'^2).$$

(50)

In general a bracket can be defined in terms of some associative product $\ast$ as follows:

$$\{f_1, f_2\} = f_1 \ast f_2 - f_2 \ast f_1,$$

(51)

and let $g$ be the kernel of the product in momentum space:

$$\frac{1}{(2\pi)^d/2} \int d^d x e^{-ikx} (f_1(x) \ast f_2(x)) = \frac{1}{(2\pi)^d/2} \int d^d p e^{-i\alpha' \theta^{\mu\nu} p_\mu k_\nu} f_1(p) f_2(k - p) g(p, k - p)$$

(52)

where

$$f_i(x) = \frac{1}{(2\pi)^d/2} \int d^d p e^{ipx} \tilde{f}_i(p).$$

(53)

The $G$ and $g$ kernels are related in the following simple way

$$G(p, k - p) = g(k, k - p) - g(k - p, p)$$

(54)

Thus one can see that by looking for the general gauge transformation we are arriving at the problem of classification of associative product which is defined as power series in $\alpha'$. We are interested in this product up to field redefinitions (46). This is exactly the star product construction considered by Kontsevich [7]. Due to Kontsevich in the present case the appropriate star product (up to field redefinitions) is given by Moyal product.
However we can analyze the concrete form of the star product in the straightforward way. Since $g(p, k - p)$ is a scalar it only depends on scalars constructed from $k$ and $p$ by contracting indices with either the metric $G_{\mu\nu}$ or the anti-symmetric $\theta^{\mu\nu}$. Thus $g$ is the function of four possible kinematic invariants which one can construct from vectors $p$ and $k - p$. Associativity of the $*$-product implies the following constraint on the kernel $g$:

\[ g(q, p - q)g(p, k - p) = g(p - q, k - p)g(q, k - q). \]  

(55)

To analyze (55) one should rewrite everything in terms of all possible kinematic invariants. It turns out that associativity written in this form is strong enough to restrict $g$ to be of the form:

\[ g(p, k - p) = e^{iC\alpha' p_{\mu} \theta^{\mu\nu} q_{\nu}} F(k^2). \]  

(56)

where we have used the field redefinitions in form (44).

Expressing $G$ as the kernel of the bracket with respect to the $*$ product then gives exactly the non-commutative gauge transformations of the previous section up to some trivial redefinitions. Thus we have shown in the other direction that null states corresponding to gauge transformations are always of the form (54).

Certainly the present analysis is naive and to make it precise one should show that the corresponding null states have the form $Q_{BRST}\Psi$ where $Q_{BRST}$ is the BRST generator. The standard BRST analysis for the open string theory is not the right one to address the question of noncomutativity. As we saw the noncomutativity involves all orders of $\alpha'$. Therefore one should modify somehow the standard BRST approach to address these questions. Previously from other reasoning it was argued that standard BRST complex should be modified [8]. We hope to come back to this question elsewhere.

7 Conclusions and discussion

In this paper we have analyzed open string theory in the presence of a constant $B$-field. We discussed the space-time symmetries of the problem and discussed the gauge invariance of the massless vector boson state using naive covariant quantization. Our main motivation for doing this was to re-derive the non-commutative gauge invariance of the system in a transparent way. Following the logic of two last sections one can expect as well some non-commutative properties for gauge transformations of gravitational and antisymmetric fields. This noncommutativity for $B$-field was indicated recently in [9].

It is straightforward to analyze closed strings in the presence of a $B$-field which has a flux through a non-trivial 2-cycle. From the logic presented in last two sections one might
expect as well some noncommutative properties for the closed strings. This theory has some interesting features and will be discussed in a subsequent article.

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