EXTREMAL HOLOMORPHIC MAPS IN SPECIAL CLASSES OF DOMAINS

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ABSTRACT. In the paper we discuss three different notions of extremal holomorphic mappings: weak $m$-extremals, $m$-extremals and $m$-complex geodesics. We discuss relations between them in general case and in the special cases of unit ball, classical Cartan domains, symmetrised bidisc and tetrablock. In particular we show that weak 3-extremal maps in the symmetrised bidisc are rational thus giving the (partial) answer to a problem posed in a recent paper by J. Agler, Z. Lykova and N. J. Young.

1. Introduction

Throughout the paper $\mathbb{D}$ will always denote the unit disc in the complex plane.

Let $D$ be a domain in $\mathbb{C}^n$. Let $m \geq 2$ and let $\lambda_1, \ldots, \lambda_m \in \mathbb{D}$ be distinct points and $z_1, \ldots, z_m \in D$. Following [4] we say that the interpolation data

\[ \lambda_j \mapsto z_j, \mathbb{D} \to D, \quad j = 1, \ldots, m, \]

are extremally solvable if there is a map $h \in \mathcal{O}(\mathbb{D}, D)$ such that $h(\lambda_j) = z_j, j = 1, \ldots, m$, and there is no $f \in \mathcal{O}(\mathbb{D}, D)$ (i.e. $f$ is holomorphic on some neighborhood of $\mathbb{D}$ and its image lies in $D$) such that $f(\lambda_j) = z_j, j = 1, \ldots, m$.

We say that $h \in \mathcal{O}(\mathbb{D}, D)$ is $m$-extremal if for all choices of $m$ distinct points $\lambda_1, \ldots, \lambda_m \in \mathbb{D}$ the interpolation data

\[ \lambda_j \mapsto h(\lambda_j), \mathbb{D} \to D, \quad j = 1, \ldots, m, \]

are extremally solvable. Note that if $h$ is $m$-extremal then it is $(m+1)$-extremal.
Generally, the fact that for a fixed \( m \) the interpolation data are extremally solvable for some \( \lambda_1, \ldots, \lambda_m \) does not imply that the interpolation data are extremally solvable for all other \( m \) points \( \mu_1, \ldots, \mu_m \). This is already not the case generally for \( m = 2 \). In other words extremals with respect to the Lempert function for some pair of points need not be extremal for the Lempert function for any pair of points. In particular, there are domains, e.g. the annulus in the complex plane, possessing no 2-extremals. For basic properties of the Lempert function (and other holomorphically invariant functions) that we shall use we refer the Reader to [12].

Therefore, it is natural to introduce a weaker notion of \( m \)-extremal map which is equivalent with the notion of extremal in the sense of Lempert when \( m = 2 \). This may be done as follows: if an analytic disc \( f : \mathbb{D} \to D \) and fixed points \( \lambda_1, \ldots, \lambda_m \in \mathbb{D} \) are such that the problem \( \lambda_j \mapsto f(\lambda_j) \) is extremally solvable, then we shall say that \( f \) is a \textit{weak} \( m \)-extremal with respect to \( \lambda_1, \ldots, \lambda_m \). Naturally, \( f \) will be said to be a \textit{weak} \( m \)-extremal if it is a weak extremal with respect to some \( m \) distinct points in the unit disc.

Of course \( m \)-extremals are weak \( m \)-extremals for any system of \( m \) distinct points \( \lambda_1, \ldots, \lambda_m \in \mathbb{D} \). In general the class of weak \( m \)-extremals is strictly bigger than the class of \( m \)-extremals (as already mentioned even if \( m = 2 \) with \( D \) being for instance the annulus). Similar problems concerning some kind of \( m \) extremality in several variable context were considered for instance in [7], [9].

Certainly these two notions coincide in the case of domains for which the assertion of the Lempert theorem holds. Recall this theorem in the form it would be convenient to us (see [14] and [15]).

\textbf{Theorem 1.} Let \( D \) be a bounded convex or smooth strongly linearly convex domain in \( \mathbb{C}^n \). Then for any \( w, z \in D \), \( w \neq z \) there are a holomorphic mapping \( f : \mathbb{D} \to D \) such that \( w \) and \( z \) lie in the image of \( f \) and a holomorphic function \( F : D \to \mathbb{D} \) such that \( F \circ f \) is the identity \( \text{id}_\mathbb{D} \).

In fact the function \( f \) in the above result is extremal for the Lempert function of \( w \) and \( z \) whereas \( F \) is extremal for the Carathéodory distance of \( w \) and \( z \). Let us call the function \( F \) the \textit{left inverse of} \( f \). Recall also that the Lempert theorem holds for the symmetrized bidisc and the tetrablock (see [6], [8] and [10]).

Note also that (weak) \( m \)-extremals (for \( \lambda_1, \ldots, \lambda_m \)) are (weak) \((m + 1)\)-extremals (for \( \lambda_1, \ldots, \lambda_m, \lambda_{m+1} \) with arbitrarily chosen \( \lambda_{m+1} \in \mathbb{D} \) distinct from \( \lambda_1, \ldots, \lambda_m \)).
The existence of left inverses in the Lempert Theorem suggests another notion of extremal mappings. Namely, we generalize the notion of a complex geodesic (see e.g. [12]) as follows.

Let $f: \mathbb{D} \to D$ be a holomorphic mapping, $m \geq 2$. We say that $f$ is an $m$-complex geodesic if there is a function $F \in \mathcal{O}(D, \mathbb{D})$ such that $F \circ f$ is a non-constant Blaschke product of degree at most $m$. Note that 2-complex geodesics are simply complex geodesics and any $m$-complex geodesic is an $(m+1)$-complex geodesic.

The aim of the paper is to try to understand the relations between the notions of $m$-extremals, weak $m$-extremals and $m$-complex geodesics in special classes of domains (convex ones, classical Cartan domains, the unit ball, symmetrised bidisc and tetrablock).

We also see that in some class of domains (containing for examples classical Cartan domains) the notions of weak $m$-extremals and $m$-extremals are equivalent. Clearly, in the polydisc all $m$-extremals are $m$-complex geodesics. This is not the case for the the Euclidean unit ball. We show that any there are 4-extremals in the unit ball which are not 4-complex geodesics.

Finally we present a new method for describing (weak) $m$-extremals in the symmetrized bidisc. In our approach the crucial role is played by the geometry of the tetrablock - the domain that, similarly to the symmetrized bidisc, arises naturally in the control-engineering problems. Then some arguments allow us to reduce the problem to already investigated classical domains.

In particular, we show that all weak 3-extremals in the symmetrized bidisc are rational and map $\mathbb{T}$ into the Shilov boundary. As a corollary we get that all 3-extremals in the symmetrized bidisc are rational of degree at most 4 which gives answer to a problem posed in [4] in case $m = 3$ (this case was also studied in [5]).

Here is some notation: for $\alpha \in \mathbb{D}$ let $m_\alpha(\lambda) = \frac{\alpha - \lambda}{1 - \bar{\alpha}\lambda}$, $\lambda \in \mathbb{D}$, be a Blaschke factor. Moreover, $\mathbb{T}$ is the unit circle in the complex plane and $\mathbb{C}^{k \times l}$ stands for the space of $k \times l$ complex matrices. We shall denote by $\text{Aut}(D)$ the group of holomorphic automorphisms of a domain $D$ of $\mathbb{C}^n$. Moreover, $\partial_S D$ denotes the Shilov boundary (with respect to the algebra $\mathcal{O}(D) \cap \mathcal{C}(\overline{D})$) of a bounded domain $D$ of $\mathbb{C}^n$.

2. Results on (weak) $m$-extremals and $m$-complex geodesics. General case and classical Cartan domains

We start with some basic properties and relations between different notions of extremal mappings.
Proposition 2. Let $f: \mathbb{D} \to D$ be a holomorphic mapping. Assume that $F: D \to D$ is such that $F \circ f$ is a Blaschke product of degree $m$. Then $f$ is $(m + 1)$-extremal.

Proof. Recall that it is well-known that the Blaschke product $F \circ f: \mathbb{D} \to \mathbb{D}$ is $(m + 1)$-extremal (see e.g. [16]). Suppose that $f$ is not an $(m + 1)$-extremal. Then there is a holomorphic mapping $g: \mathbb{D} \to D$ with $g(\mathbb{D}) \subset \subset D$ such that for some $m + 1$ pairwise different points $\lambda_1, \ldots, \lambda_{m+1}$ we have $g(\lambda_j) = f(\lambda_j)$, $j = 1, \ldots, m + 1$. Then $(F \circ g)(\lambda_j) = (F \circ f)(\lambda_j)$, $j = 1, \ldots, m + 1$ and $(F \circ g)(\mathbb{D}) \subset \subset \mathbb{D}$ which contradicts the $(m + 1)$-extremality of $F \circ f$. □

Corollary 3. Let $f \in \mathcal{O}(\mathbb{D}, D)$, $F \in \mathcal{O}(D, \mathbb{D})$. Assume that the function $B := F \circ f \in \mathcal{O}(\mathbb{D}, D)$ is a Blaschke product of degree $m$ and $B_1$ is a Blaschke product of degree $k$. Then the function $D \ni \lambda \mapsto f(B_1(\lambda)) \in D$ is an $(mk + 1)$-extremal.

The three notions introduced in the preliminary section have clear relations: an $m$-complex geodesic is an $m$-extremal and an $m$-extremal is a weak $m$-extremal (for any system of $m$ pairwise points). Recall that a weak $m$-extremal need not be an $m$-extremal. Already in the case $m = 2$ the example of the holomorphic covering of the annulus is a weak 2-extremal for (some) pairs of points and yet it is not 2-extremal for all pairs of points - it follows from the fact that the Lempert function of two different points from the annulus is equal to the Poincaré distance of some two points from the unit disc which belong to the preimages of the given points of a holomorphic covering of the annulus - consult [12] for details. Recall also that it follows from the celebrated Lempert theorem that all weak 2-extremals are 2-complex geodesics. In our paper we get some results on the lacking implications in special classes of domains.

We start with the analysis of properties of extremals in classical domains. We shall focus on classical Cartan domains of the first, second and third type denoted by $\mathcal{R}_I^{n,m}$ and $\mathcal{R}_II^n$ and $\mathcal{R}_{III}^n$, respectively. In particular, $\mathbb{B}_n = \mathcal{R}_I^{1,n}$ is the unit Euclidean ball. Note that results obtained here work for other classes of domains, not necessarily symmetric, like the Lie ball. In this section we shall use the letter $\mathcal{R}$ to denote any of these domains or their Cartesian products. We just demand $\mathcal{R}$ to be a bounded, convex and balanced domain in $\mathbb{C}^n$ whose group of holomorphic automorphisms acts transitively.

Since classical domains of $\mathbb{C}^{2 \times 2}$ play a crucial role in the paper, we put $\mathcal{R}_I := \mathcal{R}_I^{2,2}$ and $\mathcal{R}_{II} := \mathcal{R}_{II}^2$. For definitions and basic properties of the Cartan domains that we shall use in the paper we refer the Reader to [11].
Remark 4. Assume that $D$ is a balanced pseudoconvex domain and assume that $f : D \to D$ is a weak $m$-extremal for $\lambda_1, \ldots, \lambda_m$ such that $f(0) = 0$. We may write $f(\lambda) = \lambda \psi(\lambda)$, $\lambda \in \mathbb{D}$. Then either $\psi$ is an analytic disc in $D$ or $\psi$ is an analytic disc lying entirely in $\partial D$.

If the second possibility holds then $\mu_D(f(\lambda)) = |\lambda|$, where $\mu_D$ denotes the Minkowski functional of $D$, which easily implies that $f$ is a weak 2-extremal for 0 and arbitrary $\lambda \in \mathbb{D} \setminus \{0\}$.

If the first possibility holds, then in the case $\lambda_j \neq 0$ for any $j$ the mapping $\psi$ is a weak $m$-extremal for $\lambda_1, \ldots, \lambda_m$. If, on the other hand, $\lambda_m = 0$ and $m \geq 3$, then $\psi$ is a weak $(m - 1)$-extremal for $\lambda_1, \ldots, \lambda_{m-1}$.

On the other hand if $f : D \to D$, where $D$ is a balanced pseudoconvex domain, is a weak $m$-extremal for distinct points $\lambda_1, \ldots, \lambda_m \in \mathbb{D}$, then the function $g$ given by the formula $g(\lambda) := m_{\lambda_{m+1}}(\lambda)f(\lambda)$, $\lambda \in \mathbb{D}$, where $\lambda_{m+1} \neq \lambda_j$, $j = 1, \ldots, m$, is a weak $(m + 1)$-extremal for $\lambda_1, \ldots, \lambda_{m+1}$.

Proposition 5. Let $f : \mathbb{D} \to \mathcal{R}$ be a weak $m$-extremal for some $m$ pairwise different points in a classical Cartan domain $\mathcal{R}$, $m \geq 2$. Then $f$ is proper.

Proof. We proceed inductively with respect to $m$. In case $m = 2$ it follows from the transitivity of the group of automorphisms and the fact that any weak 2-extremal $f : \mathbb{D} \to \mathcal{R}$ with $f(0) = 0$ is such that $\mu_\mathcal{R}(f(\lambda)) = |\lambda|$, $\lambda \in \mathbb{D}$ where $\mu_D$ is the Minkowski functional of $\mathcal{R}$.

Let now $f$ be a weak $m$-extremal for $\lambda_1, \ldots, \lambda_m$ in $\mathcal{R}$, $m \geq 3$. Due to transitivity of $\mathcal{R}$ we may assume that $\lambda_m = 0$ and $f(0) = 0$. To finish the proof it is sufficient to make use of Remark 4 and inductive assumption for $\psi$ (in the case $\psi$ does not lie in the boundary of $\mathcal{R}$).

Remark 6. Recall (see e.g. [11]) that the classical domains are homogeneous (i.e. their group of holomorphic automorphisms act transitively, balanced and convex).

Proposition 7. Any weak $m$-extremal in any of the the classical Cartan domains $\mathcal{R}$ is an $m$-extremal.

Proof. We will proceed inductively. For $m = 2$ the assertion is a simple consequence of the Lempert Theorem.

So assume that $m \geq 3$, any $(m - 1)$-weak extremal is an $(m - 1)$-extremal and let $f$ be a weak $m$-extremal with respect to $\lambda_1, \ldots, \lambda_m$. Composing $f$ with a Möbius map we may assume that $\lambda_m = 0$. Thanks to transitivity of $\text{Aut}(\mathcal{R})$ one may moreover assume that $f(0) = 0$. Then $f(\lambda) = \lambda \varphi(\lambda)$, where $\varphi$ is an analytic disc in $\mathcal{R}$. If $\varphi(0) \in \partial \mathcal{R}$, then $f$ is 2-extremal for 0, $\lambda$ (see Remark 4) for any $\lambda \in \mathbb{D} \setminus \{0\}$ and
thus, in view of (for instance) the Lempert theorem 2-extremal. In
the other case \( \varphi \) is an analytic disc in \( \mathcal{R} \) and it is \((m - 1)\)-extremal
for some system of points and thus, due to the inductive assumption
\((m - 1)\)-extremal in \( \mathcal{R} \). Now applying Remark 4 we easily finish the proof.

Take any points \( \sigma_1, \ldots, \sigma_m \in \mathbb{D} \). We claim that \( f \) is a weak extremal
with respect to them. If \( \sigma_j = 0 \) for some \( j \) (without loss of generality
assume that \( j = m \)) then the fact that \( f \) would not be extremal for
\( \sigma_1, \ldots, \sigma_m \) would deliver an analytic disc lying relatively compactly in
\( \mathcal{R} \) coinciding with \( \varphi \) at points \( \sigma_1, \ldots, \sigma_{m-1} \) contradicting the \((m - 1)\)-extremality of \( \varphi \). So suppose that \( \sigma_j \) does not vanish. Seeking a
contradiction assume that one may find a holomorphic mapping \( g : \overline{\mathbb{D}} \to \mathcal{R} \) such that \( g(\sigma_j) = f(\sigma_j), \ j = 1, \ldots, m \). Let \( \Psi_a \) denote the
automorphism of \( \mathcal{R} \) such that \( \Psi_a(a) = 0, \Psi_a(0) = a \).

Since \( g(\sigma_1) = \sigma_1 \varphi(\sigma_1) \) we see that an analytic disc \( \psi \) given by the
formula \( \psi(\lambda) = \frac{1}{m_{\sigma_1}(\lambda)} \Psi_{\sigma_1, \varphi(\sigma_1)}(g(\lambda)) \) maps \( \mathbb{D} \) into \( \mathcal{R} \) (note that \( \psi \) cannot land in \( \partial\mathcal{R} \)). We shall show that \( \psi \) is a weak \((m - 1)\)-extremal in
\( \mathcal{R} \). This would give a contradiction as \( \psi(\overline{\mathbb{D}}) \subset \mathcal{R} \). It suffices to show
that \( \chi : \lambda \mapsto \frac{1}{m_{\sigma_1}(\lambda)} \Psi_{\sigma_1, \varphi(\sigma_1)}(\lambda \varphi(\lambda)) \) is a weak \((m - 1)\)-extremal, as it
goes with \( \psi \) at points \( \sigma_j, j = 2, \ldots, m \). Note that \( \chi(0) = \varphi(\sigma_1) \).

If \( \chi \) were not an \((m - 1)\)-extremal then we would be able to find
an analytic disc \( \tilde{\chi} : \overline{\mathbb{D}} \to \mathcal{R} \) such that \( \chi(0) = \tilde{\chi}(0) \) and \( \chi(\sigma_j) = \tilde{\chi}(\sigma_j) \) for \( j = 3, \ldots, m \). Then \( \tilde{\varphi}(\lambda) := \frac{1}{\chi} \Psi_{\sigma_1, \varphi(\sigma_1)}(m_{\sigma_1}(\lambda) \tilde{\chi}(\lambda)) \) would be
well defined (we remove singularity at 0) and would agree with \( \varphi \) at
\( \sigma_3, \ldots, \sigma_m \). Moreover, it follows immediately from the definition that
\( \tilde{\varphi}(\sigma_1) = \varphi(\sigma_1) \). This gives a desired contradiction.

Problem: Does a similar result hold for the symmetrized bidisc \( \mathbb{G}_2 \)
or the tetrablock \( \mathbb{E} \) if \( m \geq 3 \)? Does a similar result hold for any convex
domain?

Remark 8. In the classical domains the fact that all weak extremals
are extremals allows us to produce new extremals from the existing
ones. For instance, let \( f : \mathbb{D} \to \mathcal{R} \) be an \( m \)-extremal. Then the
function \( m_{\alpha}(\lambda) f(\lambda) \) is a weak \((m + 1)\)-extremal for \( \lambda_1, \ldots, \lambda_m, \alpha \) for \( m \)
pairwise different points \( \lambda_1, \ldots, \lambda_m, \alpha \neq \lambda_j \) and thus \((m + 1)\)-extremal.
Consequently, the function \( B \cdot f \) where \( B \) is a Blaschke product
of degree \( k \) is an \((m + k)\)-extremal. In particular, the function \( g \) given by
the formula \( g(\lambda) = \lambda^k f(\lambda), \lambda \in \mathbb{D} \), is an \((m + k)\)-extremal.
3. \textit{m}-complex geodesics and \textit{m}-extremals in the unit Euclidean ball

Note that any 3-extremal in the unit ball is, up to a composition with an automorphism of \(B_n\), of the form \(\lambda \mapsto (\lambda a_1, \sqrt{1 - a_1^2\lambda m(\lambda)}, 0, \ldots, 0)\), where \(m\) is a Möbius map and \(a_1 \geq 0\). We do not know such 3-extremals are complex geodesics but we are able to show at least in the case when \(m\) is a rotation.

\textbf{Proposition 9.} Let \(f \in \mathcal{O}(\mathbb{D}, B_n)\), \(n \geq 2\) be a 3-extremal of the form \(\lambda \mapsto (\lambda a_1, \sqrt{1 - a_1^2\lambda^2}, 0, \ldots, 0)\), where \(a_1 \geq 0\). Then \(f\) is a 3-complex geodesic.

\textbf{Proof.} It suffices to observe that the left inverse to the 3-extremal \(f\) (given by the formula \(f(\lambda) = (a_1\lambda, \sqrt{1 - a_1^2\lambda^2}, 0, \ldots, 0)\)) may be chosen as follows:

\begin{equation}
F(z) := \frac{z_1^2}{2 - a_1^2} + \frac{2\sqrt{1 - a_1^2}}{2 - a_1^2}z_2, \quad z \in B_n.
\end{equation}

It is simple to see that \(F \in \mathcal{O}(B_n, \mathbb{D})\) and \(F(f(\lambda)) = \lambda^2, \lambda \in \mathbb{D}\). \(\square\)

It is not clear from the first view why the left inverse in the above result is of the form as given above. In fact, the idea of the form of the function that resulted in that form will be more clear after the study of the proof of the next result where we shall prove that there are 4-extremals in the unit ball which are not 4-complex geodesics. Note that one may relatively easily produce necessary form for \(m\)-extremals. Another way of the necessary form of \(m\)-extremals (in more general domains called complex ellipsoids) was presented in [9].

\textbf{Proposition 10.} Let \(k \geq 2\). The function

\begin{equation}
f : \mathbb{D} \ni \lambda \mapsto (a_1\lambda^k, \sqrt{1 - a_1^2\lambda^{k+1}})
\end{equation}

where \(a_1 \in (0, 1)\) is a \((k + 2)\)-extremal in the unit ball \(B_2\) which is not a \((k + 2)\)-geodesic.

\textbf{Proof.} We already know that the analytic disc \(f\) is a \((k + 2)\)-extremal (use Remark 8). Suppose that there is an \(F \in \mathcal{O}(B_2, \mathbb{D})\) such that \(F(0) = 0\) and \(F \circ f\) is a non-constant Blaschke product of degree at most \(k + 1\). First we consider the case \(k \geq 3\). Expanding around 0 and comparing the Taylor coefficients on both sides of the equality:

\begin{equation}
F \left( \lambda^k a_1, \lambda^{k+1} \sqrt{1 - a_1^2} \right) = B(\lambda)
\end{equation}
we easily get (multiplying by a unimodular constant) that \( B(\lambda) = \lambda^k \) or \( B(\lambda) = \lambda^km_\gamma(\lambda) \) for some \( \gamma \in \mathbb{D} \). Write the Taylor expansion of \( F \) in the form \( \alpha z_1 + \beta z_2 + \ldots \). In the first case \( |\alpha a_1| = 1 \) which gives \( |\alpha| > 1 \) - contradiction. Consider now the second case. The coefficient at the left side at \( \lambda^k+2 \) is 0 which means that \( \gamma = 0 \). Write \( F(z_1, z_2) = \alpha z_1 + \beta z_2 + o(z_1, z_2) \). Since \( \limsup_{|\lambda| \to 1} |(F(\lambda(z_1, z_2)))/\lambda| \leq 1 \) we easily get that \( \alpha z_1 + \beta z_2 \in \mathbb{D} \) for any \( z \in \mathbb{B}_2 \) which shows that \( |\alpha|^2 + |\beta|^2 \leq 1 \). Then the coefficient at \( \lambda^{k+1} \) on the left side is \( \beta \sqrt{1 - a_1^2} \) which cannot have absolute value one as it does on the right side.

Now we assume that \( k = 2 \).

\[
F \left( \lambda^2 a_1, \lambda^3 \sqrt{1 - a_1^2} \right) = B(\lambda).
\]

where \( B(\lambda) = \lambda^2m_\gamma(\lambda) \) (the case when the degree of \( B \) is one or two is simple). Write the Taylor expansion of \( F \) as \( \alpha z_1 + \beta z_2 + \ldots \). Comparing the coefficients at \( \lambda^2 \) and at \( \lambda^3 \) leads us to equalities \( |\alpha a_1| = |\gamma|, |\beta| \sqrt{1 - a_1^2} = 1 - |\gamma|^2 \). We also know that \( \alpha z_1 + \beta z_2 \in \mathbb{D} \) for any \( z \in \mathbb{B}_2 \) which is equivalent to the inequality \( |\alpha|^2 + |\beta|^2 \leq 1 \). Consequently,

\[
\frac{|\gamma|^2}{a_1^2} + \frac{(1 - |\gamma|^2)^2}{1 - a_1^2} \leq 1.
\]

The last inequality is equivalent to

\[
|\gamma|^2(1 - a_1^2) + (1 - |\gamma|^2)^2a_1^2 - a_1^2(1 - a_1^2) \leq 1
\]

or

\[
(a_1^2 - |\gamma|^2)^2 + |\gamma|^2(1 - |\gamma|^2)(1 - a_1^2) \leq 0.
\]

The last inequality does not hold for any \( \gamma \in \mathbb{D}, a_1 \in (0,1) \) - contradiction.

\[\square\]

Remark 11. Note that for any \( f \) as in the previous proposition there is an \( F \) (of the form \( A\alpha_1^{k+1} + Bz_2^k \)) such that \( F \circ f \) is the Blaschke product of degree \( k(k + 1) \).

Problem. It would be interesting to know whether 3-extremals in the unit ball are 3-complex geodesic. Even if the answer to this question is positive we think that a counterpart of the Lempert theorem for 3-extremals in the convex domains does not hold, i.e. there is a convex domain \( D \) and a 3-extremal \( f \) in \( D \) for which there is no left inverse \( F \) such that the composition of \( F \circ f \) is a non-constant Blaschke product of degree at most two.
4. (Weak) $m$-extremals in the symmetrized bidisc

4.1. Geometry of the tetrablock. Recall that the tetrablock is a domain in $\mathbb{C}^3$ denoted by $E$ and given by the formula

$$E = \{(x_1, x_2, x_3) \in \mathbb{C}^3 : |x_1 - \bar{x}_2x_3| + |x_2 - \bar{x}_1x_3| + |x_3|^2 < 1 \}.$$ 

It is well known that $E$ may be given as the image of the Cartan domain of the first type $\mathcal{R}_I$ under the mapping $\pi : \mathbb{C}^{2 \times 2} \to \mathbb{C}^3$

$$z = (z_{i,j}) \mapsto (z_{11}, z_{22}, \det z).$$

Basic properties of the tetrablock that we use may be found in [2], [3], [10], [13], [17] and [19].

What is more, $\pi$ restricted to $\mathcal{R}_{II}$ maps properly $\mathcal{R}_{II}$ onto $E$ and the locus set of this proper mapping $\{(x_1, x_2, x_3) \in \mathbb{D}^3 : x_1x_2 = x_3 \}$ coincides with the royal variety of $E$ which we denote by $T$.

In the sequel we shall make use of the structure of the group of automorphisms of $E$ and its connection with the group of automorphisms of $\mathcal{R}_{II}$ (see [13]). Recall that the group of automorphisms of $\mathcal{R}_{II}$ is generated by the linear isomorphisms $L_U : x \mapsto UxU^t$, where $U$ is a unitary matrix, and the mappings

$$\Phi_a : x \mapsto (1 - aa^*)^{-\frac{1}{2}}(x - a)(1 - a^*x)^{-1}(1 - a^*a)^{\frac{1}{2}},$$

where $a \in \mathcal{R}_{II}$ (see [1]). Note that $\Phi_a(0) = -a$ and $\Phi_a(a) = 0$. Moreover, $\Phi^{-1}_a = \Phi_{-a}$.

It follows from [13] that any automorphism $\psi$ of the tetrablock is of the form

$$\psi \circ \pi = \pi \circ \Phi$$

for some automorphism $\Phi$ of $\mathcal{R}_{II}$. Moreover, such an automorphism $\Phi$ is generated either by $\Phi_A$, where $A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$, $a_1, a_2 \in \mathbb{D}$, or by $L_U$, where $U = \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix}$ or $U = \begin{pmatrix} 0 & \omega_1 \\ \omega_2 & 0 \end{pmatrix}$, where $\omega_1, \omega_2 \in \mathbb{T}$.

Some simple computations lead to the description of the group of automorphism of $E$ (see also [17]). More precisely, the automorphism $\Phi_A$ of $\mathcal{R}_{II}$, where $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, $a, b \in \mathbb{D}$, induces the automorphism
\psi of \mathbb{E} given by
\begin{equation}
\psi(x_1, x_2, x_3) = \left( \frac{x_1 - a - bx_3 + abx_2}{1 - \bar{a}x_1 - bx_2 + \bar{a}bx_3}, \frac{x_2 - b - \bar{a}x_3 + \bar{ab}x_1}{1 - \bar{a}x_1 - bx_2 + \bar{a}bx_3}, \frac{x_3 - ax_2 - bx_1 + ab}{1 - \bar{a}x_1 - bx_2 + \bar{a}bx_3} \right).
\end{equation}

Moreover, \( L_U \), where \( U \) is a unitary diagonal or anti-diagonal matrix, induces automorphisms of \( \mathbb{E} \) generated by
\[(x_1, x_2, x_3) \mapsto (\omega x_1, \eta x_2, \omega \eta x_3), \]
where \( \omega, \eta \in \mathbb{T} \), and by
\[(x_1, x_2, x_3) \mapsto (x_2, x_1, x_3). \]

In the sequel we shall make use of the following

Remark 12. Note that \( \pi(x) \) lies in the topological boundary of \( \mathbb{E} \), where \( x \in \partial \mathcal{R}_I \), if an only if \( |x_{12}| = |x_{21}| \) (this is an immediate consequence of Lemma 9 in \cite{10} and the properness of \( \pi|_{\mathcal{R}_{II}} : \mathcal{R}_{II} \to \mathbb{E} \)).

In particular, for \( x \in \overline{\mathcal{R}_I} \) the following statement is an immediate consequence of results from \cite{13}:
\[ x \in \partial_S \mathcal{R}_I \quad \text{iff} \quad \pi(x) \in \partial_S \mathbb{E}. \]

4.2. (Weak) \( m \)-extremals intersecting \( \Sigma \). At this point we may outline the idea of study of weak \( m \)-extremals in the symmetrised bidisc. First we concentrate on a more difficult problem of description of \( m \)-extremals intersecting the royal variety \( \Sigma \) (definition is given below).

Since \( \pi : \mathcal{R}_{II} \to \mathbb{E} \) is proper, \( \pi \) restricted to \( \mathcal{R}_{II} \setminus \pi^{-1}(\mathcal{T}) \) is a holomorphic covering of \( \mathbb{E} \setminus \mathcal{T} \). Therefore, an analytic disc in \( \mathbb{E} \) omitting its royal variety may be lifted to an analytic disc in the classical Cartan domain of the second type. Of course this does not have to be true if an analytic disc intersects the royal variety of the tetrablock. But then we may lift it to an analytic disc in \( \mathcal{R}_I \) or, up to a composition with an automorphism of the tetrablock, it is of the form \( \lambda \mapsto (0, 0, a(\lambda)) \) for some \( a \in \mathcal{O}(\mathbb{D}, \mathbb{D}) \) (see \cite{10}, Lemma 7).

At this point it would be reasonable to recall the close relationship between the tetrablock and the symmetrized bidisc \( \mathbb{G}_2 \). Recall that symmetrised bidisc (denoted by \( \mathbb{G}_2 \)) is a domain which may be defined as the image under the (proper and holomorphic) mapping \((z_1, z_2) \mapsto (z_1 + z_2)\) of the bidisc \( \mathbb{G}_2 \) or equivalently
\begin{equation}
\mathbb{G}_2 := \{ (s, p) \in \mathbb{C}^2 : |s - \overline{sp}| + |p|^2 < 1 \}.
\end{equation}
A similar role to that of $T$ (in $E$) is played by the royal variety of the symmetrised bidisc, i.e. the set $\Sigma := \{ (2\lambda, \lambda^2) : \lambda \in \mathbb{D} \}$. First recall that there is a natural embedding

\[ \iota : G_2 \ni (s, p) \mapsto (s/2, s/2, p) \in E. \]

On the other hand the mapping

\[ p : E \ni x \mapsto (x_1 + x_2, x_3) \in G_2 \]

satisfies the equality $p \circ \iota = \text{id}_{G_2}$. We shall use these facts extensively.

Thus if $f : \mathbb{D} \to G_2$ is an analytic disc, then $(f_1/2, f_1/2, f_2)$ is analytic disc in the tetrablock $E$. Therefore, we may lift to an analytic disc in $\mathbb{R}_{II}$. It is much more comfortable and natural to lift $f$ to an analytic disc in the classical Cartan domain of the second type.

To do it let us denote the action permuting columns of a given matrix $a \in \mathbb{C}^{2 \times 2}$ by $\tau a$. These observations lead us to the following lemma enabling us to transport the problem of the study of $(m$-extremals) analytic discs in the symmetrised bidisc to that in the Cartan domain of the second type.

**Lemma 13.** Let $f : \mathbb{D} \to G_2$ be an analytic disc. Then there is an analytic disc $\varphi : \mathbb{D} \to \mathbb{R}_{II}$ such that

\[ \left( \frac{f_1}{2}, \frac{f_1}{2}, f_2 \right) = \pi \circ \tau \varphi. \]

Of course, the result presented above is the most interesting in the case when $f(\mathbb{D})$ intersects $\Sigma$. Note that it is trivial if $f(\mathbb{D})$ is contained in $\Sigma$, because then $f = (f_1, f_2)$, where $f_1 \in \mathcal{O}(\mathbb{D}, \mathbb{D})$.

Lemma 13 is a crucial tool in our considerations. Its importance is a consequence of the following simple observations:

- $f : \mathbb{D} \to G_2$ is a (weak) $m$-extremal in $G_2$ if and only if $(\frac{f_1}{2}, \frac{f_1}{2}, f_2)$ is (weak) $m$-extremal in the tetrablock,
- if $\pi \circ \tau \varphi : \mathbb{D} \to E$ is a (weak) $m$-extremal, where $\varphi \in \mathcal{O}(\mathbb{D}, \mathbb{R}_{II})$, then $\varphi$ is $m$-extremal in $\mathbb{R}_{II}$.

Note that a lifting map $\varphi$ appearing in Lemma 13 may be chosen in a specific way.

**Lemma 14.** Let $f : \mathbb{D} \to G_2$ be an analytic disc. Then there is an analytic disc $\varphi : \mathbb{D} \to \mathbb{R}_{II}$ such that

\[ \left( \frac{f_1}{2}, \frac{f_1}{2}, f_2 \right) = \pi \circ \tau \varphi \]

and either
• \(\varphi\) lands in the boundary of \(\mathcal{R}_I\) and then \(f\) is, up to a composition with an automorphism of the symmetrized bidisc, of the form \((0, f_2)\), or

• \(\varphi\) is an analytic disc in \(\mathcal{R}_{II}\) such that \(|\varphi_{11}^*| = |\varphi_{22}^*|\) a.e. on \(T\) and \(\varphi_{11}\) vanishes nowhere on \(T\).

Moreover, if \(f\) is a weak \(m\)-extremal, then \(\varphi\) is an \(m\)-extremal in \(\mathcal{R}_{II}\).

**Remark 15.** Suppose that \(f : \mathbb{D} \to \mathbb{E}\) is a (weak) \(m\)-extremal in the tetrablock such that \(f(0) = 0\) and both \(f_1\) and \(f_2\) are not identically equal to 0. Suppose additionally that \(f = \pi \circ G\), where \(G : \mathbb{D} \to \mathcal{R}_I\) is of the form \(G = \begin{pmatrix} f_1 & B_1g_1 \\ B_2g_2 & f_2 \end{pmatrix}\), and \(B_1\) and \(B_2\) are Blaschke products.

Then, of course, the mapping \(G\) is an \(m\)-extremal in \(\mathcal{R}_I\). Comparing tangential limits we also see that the mapping \(H := \begin{pmatrix} f_1 & g_1 \\ B_1B_2g_2 & f_2 \end{pmatrix}\) maps \(\mathbb{D}\) into \(\overline{\mathcal{R}_I}\). Note also that \(g_1(0)\) lies in the unit disc. Actually, otherwise \(g_1\) would be a unimodular constant and therefore \(f_1 \equiv f_2 \equiv 0\) (as tangential limits of \(f_1\) and \(f_2\) would vanish on \(T\)). Thus \(H\) is an analytic disc in \(\mathcal{R}_I\). Since \(\pi \circ H = f\), we get that \(H = \begin{pmatrix} f_1 & g_1 \\ B_1B_2g_2 & f_2 \end{pmatrix}\) is an \(m\)-extremal in \(\mathcal{R}_I\).

**Proof of Lemma 14.** The existence of \(\varphi\) is guaranteed by Lemma 13. If \(f\) does not intersect \(\Sigma\), then one may assume that \(\tau^* \varphi\) is an analytic disc in \(\mathcal{R}_{II}\) and the assertion is clear. So suppose that \(f(\mathbb{D}) \cap \Sigma \neq \emptyset\). Composing \(f\) with an automorphism of \(G_2\) we may assume that \(f(0) = 0\).

If \(\varphi\) lands in \(\partial \mathcal{R}_I\), then the assertion follows from Lemma 7 in [10]. In the other case let us denote \(g = \tau^* \varphi\) and \(g = (g_{ij})\). Note that if \(g_{12}g_{21} \equiv 0\) then we may assume that \(g_{12} = g_{21} = 0\) and we are done.

If \(g_{12}g_{21} \neq 0\), then one may find a Blaschke product such that \(g_{12}g_{21} = bh\), where \(h : \mathbb{D} \to \overline{\mathbb{D}}\) does not vanish. Then \(\tilde{g}\) given by the formula

\[
\tilde{g} = \begin{pmatrix} g_{11} & \sqrt{b} \\ b\sqrt{h} & g_{22} \end{pmatrix}
\]

satisfies \(\pi \circ \tilde{g} = f\).

The second part of the lemma is a direct consequence of Remark 15. \(\square\)

The above lemma suggests that \(m\)-extremals \(\varphi\) such that \((\dagger)\) \(\varphi\) is, up to \(\text{Aut}(G_2)\), of the form \((0, \varphi_2)\) should be considered separately.
Using Lemma 13 and the description of complex geodesics in $R_{11}$ we get the following

**Theorem 16.** Let $\varphi : \mathbb{D} \to \mathbb{G}_2$ be a weak $m$-extremal. Assume that $\varphi$ is not of the form $(\dagger)$. Then there are $a_1, \ldots, a_m \in R_{11}$, a holomorphic function $Z : \mathbb{D} \to \mathbb{D}$ fixing the origin and unitary matrix $U$ such that
\[
\left(\frac{\varphi_1(\lambda)}{2}, \frac{\varphi_1(\lambda)}{2}, \varphi_2(\lambda)\right) = \pi(\tau \varphi_{a_1}(\lambda \varphi_{a_2}(\cdots \lambda \varphi_{a_n}(U \left( \begin{array}{cc} \lambda & 0 \\ 0 & Z(\lambda) \end{array} \right) U^t))), \quad \lambda \in \mathbb{D}.
\]

It seems that $Z$ appearing in the above theorem must be rational but we could not prove it. However we are able to show it for $m = 3$:

**Theorem 17.** Let $\varphi : \mathbb{D} \to \mathbb{G}_2$ be a weak 3-extremal with respect to $0, \sigma_1, \sigma_2 \in \mathbb{D}$, intersecting $\Sigma$. Then, either

1. $\varphi$ is up to a composition with an automorphism of $\mathbb{G}_2$ of the form $(0, \varphi_2)$, where $\varphi_2$ is a Blaschke product of degree 2, or
2. $\varphi$ lies entirely in $\Sigma$, i.e. $\varphi = \left(\varphi_1, \frac{\varphi_2}{2}\right)$, where $\varphi_1$ is a Blaschke product of degree at most 2, or
3. there are $a_1, a_2 \in R_{11}$, and a unitary symmetric matrix $U$ such that
\[
\left(\frac{\varphi_1(\lambda)}{2}, \frac{\varphi_1(\lambda)}{2}, \varphi_2(\lambda)\right) = \pi(\tau \Phi_{a_1}(\lambda \Phi_{a_2}(U \lambda))), \quad \lambda \in \mathbb{D}.
\]

In the case when weak 3-extremals do not touch the royal variety we get the following

**Theorem 18.** Let $f : \mathbb{D} \to \mathbb{G}_2$ is a weak 3-extremal in $\mathbb{G}_2$ such that $f(\mathbb{D}) \cap \Sigma = \emptyset$. Then there are Blaschke products $B_1, B_2$ of degree at most 2 such that
\[
f = (B_1 + B_2, B_1 B_2).
\]

As a consequence we obtain an affirmative answer to a question posed in [5].

**Theorem 19.** Let $\varphi : \mathbb{D} \to \mathbb{G}_2$ be a 3-extremal mapping. Then:

- $\varphi$ is a rational function of degree at most 4;
- $\varphi$ is $\mathbb{G}_2$-inner, i.e. $\varphi(\mathbb{T}) \subset \partial_s \mathbb{G}_2$, where $\partial_s \mathbb{G}_2 = \{ (z + w, zw) : |z| = |w| = 1 \}$ is the Shilov boundary of $\mathbb{G}_2$.

**Proof.** If $\varphi$ omits $\Sigma$, the assertion is a direct consequence of Theorem 18. Similarly, the assertion is clear if $\varphi$ lies entirely in $\Sigma$.

Suppose that $\varphi(\mathbb{D})$ touches $\Sigma$ and $\varphi(\mathbb{D}) \not\subset \Sigma$. Let $\sigma_0$ be such that $\varphi(\sigma_0) \in \Sigma$. Since a composition with a Möbius map does not change a
degree of a rational mapping we may assume that \( \sigma_0 = 0 \). Then, there are \( a \in \mathcal{R}_{II} \), and \( \Phi \in \text{Aut}(\mathcal{R}_{II}) \) such that

\[
\left( \frac{\varphi_1(\lambda)}{2}, \frac{\varphi_1(\lambda)}{2}, \varphi_2(\lambda) \right) = \pi(\tau \Phi_a(\lambda \Phi(\lambda))), \ \lambda \in \mathbb{D}.
\]

Note that if \( \varphi(0) = 0 \), then all but one entries of the matrix \( a \) are equal to 0 (the element lying on the diagonal may not vanish). Then a straightforward calculation shows that \( \varphi_{ij} \) are of the form \( \frac{p_{ij}}{q} \), where \( p_{ij}, q \) are polynomials of degree at most 4.

Therefore, in a general case, that is when \( \varphi(0) \in \Sigma \), it suffices to compose \( \varphi \) with an automorphism of \( \mathbb{E} \).

To prove the second part recall that for any \( x \) lying in the Shilov boundary of \( \mathcal{R}_I \) its image \( \pi(x) \) lies in the Shilov boundary of \( \mathbb{E} \) (Remark 12).

To prove Theorem 17 we shall need some preparatory results. The first observation is

**Lemma 20.** Any \( m \)-extremal in the tetrablock or in the symmetrized bidisc is proper.

**Proof.** Since the embedding of \( G_2 \) into \( \mathbb{E} \) is proper it suffices to show the assertion for the tetrablock. First recall that any \( m \)-extremal in \( \mathcal{R}_{II} \) or in \( \mathcal{R}_I \) is proper (see Proposition 5).

Note also that \( \pi(x) \) lies in the topological boundary of \( \mathbb{E} \), where \( x \in \partial \mathcal{R}_I \), if and only if \( |x_{12}| = |x_{21}| \) (this is an immediate consequence of Lemma 9 in [11] and the properness of \( \pi|_{\mathcal{R}_{II}} : \mathcal{R}_{II} \to \mathbb{E} \)).

Now assume that \( f : \mathbb{D} \to \mathbb{E} \) is \( m \)-extremal. Then, using Lemma 14 we may lift it to an \( m \)-extremal \( g \) in \( \mathcal{R}_I \) such that \( |g_{12}| = |g_{21}| \) almost everywhere on \( \mathbb{T} \). Now it suffices to make use of the fact that \( g \) is proper. \( \square \)

**Proof of Theorem 17.** Composing \( \varphi \) with an automorphism of \( G_2 \) we may assume that \( \varphi(\sigma_0) = 0 \) for some \( \sigma_0 \in \mathbb{D} \). If \( \varphi(\mathbb{D}) \subset \Sigma \) or \( \varphi \) is of the form (†) the assertion is clear.

Since the automorphism \( \Phi_{\Lambda} \), where \( \Lambda = \begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix} \), maps \( \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \) to \( \begin{pmatrix} \lambda & 0 \\ 0 & m_{\alpha}(\lambda) \end{pmatrix} \), it suffices to show that there are an \( a \in \mathcal{R}_{II} \) and \( \Phi \in \text{Aut}(\mathcal{R}_{II}) \) and a Möbius map \( m \) such that

\[
(\varphi_1(\lambda)/2, \varphi_1(\lambda)/2, \varphi_2(\lambda)) = \pi(\tau \Phi_a(\lambda \Phi(\lambda))).
\]
Lifting $\varphi$ as in Theorem [16] we get a 3-extremal $F : \mathbb{D} \rightarrow \mathcal{R}_{11}$ such that $\pi \circ \sigma F = \varphi$, where

$$F(\lambda) = \Phi_a \left( \lambda \Phi \left( \begin{pmatrix} \lambda & 0 \\ 0 & Z(\lambda) \end{pmatrix} \right) \right), \quad \lambda \in \mathbb{D},$$

for some $Z \in \mathcal{O}(\mathbb{D}, \mathbb{D})$. The choice of $F$ implies, in particular, that functions $|F_{11}|$ and $|F_{22}|$ are different.

Note that our aim is to show that $Z$ is a Möbius map. Seeking a contradiction suppose that it is not the case. Then there is a holomorphic function $\tilde{Z}$ defined on a neighbourhood of $\bar{\mathbb{D}}$ such that $\tilde{Z}(\sigma_i) = Z(\sigma_i)$, $i = 1, 2$, which is not a Nash function (for a definition and basic properties of Nash functions we refer the Reader to [18]). Putting

$$\tilde{F}(\lambda) = \Phi_a \left( \lambda \Phi \left( \begin{pmatrix} \lambda & 0 \\ 0 & \tilde{Z}(\lambda) \end{pmatrix} \right) \right), \quad \lambda \in \mathbb{D},$$

we get a 3-extremal in $\mathcal{R}_{11}$ such that $\pi \circ \sigma \tilde{F}$ is a 3-extremal in $\mathbb{E}$. Note that $F_{11}F_{22} \neq 0$ on $\mathbb{T}$.

Using Lemma [20] we find that $|F_{11}| = |F_{22}|$ on $\mathbb{T}$. Thus there are finite Blaschke products or unimodular constants $B_1, B_2$ and a holomorphic function $g$ such that $\tilde{F}_{11} = B_1g$ and $\tilde{F}_{22} = B_2g$. Put $f := \tilde{F}_{12} = \tilde{F}_{21}$. Then

$$\Phi_a \left( \lambda \Phi \left( \begin{pmatrix} \lambda & 0 \\ 0 & \tilde{Z}(\lambda) \end{pmatrix} \right) \right) = \begin{pmatrix} B_1(\lambda)g(\lambda) & f(\lambda) \\ f(\lambda) & B_2(\lambda)g(\lambda) \end{pmatrix}.$$

We will modify the above relation so that $B_1 = 1$. Then some technical arguments will provide us with a contradiction.

First note that at least one of $B_1$ or $B_2$ is not a unimodular constant. Indeed, if it were not true, then one may use arguments involving the concept of Nash functions. More precisely, we proceed as follows. Put

$$\tilde{F}(\lambda, z) := \Phi_a(\lambda \Phi \left( \begin{pmatrix} \lambda & 0 \\ 0 & z \end{pmatrix} \right)), \quad \lambda, z \in \mathbb{D}.$$

Since $B_1, B_2 \in \mathbb{T}$, we find that $\tilde{F}_{11}(\lambda, \tilde{Z}(\lambda)) = \omega \tilde{F}_{22}(\lambda, \tilde{Z}(\lambda))$ on $\mathbb{D}$ where $\omega$ is a unimodular constant $\omega = B_1/B_2$. Since $\tilde{Z}$ is not Nash, the equality $\tilde{F}_{11}(\lambda, z) = \omega \tilde{F}_{22}(\lambda, z)$ holds for all $(\lambda, z) \in \mathbb{D}^2$. In particular, $\tilde{F}_{11}(\lambda, Z(\lambda)) = \omega \tilde{F}_{22}(\lambda, Z(\lambda)), \lambda \in \mathbb{D}$, which gives a contradiction (as already mentioned, $|F_{11}| \neq |F_{22}|$).

Note that

$$\lambda \mapsto \begin{pmatrix} g(\lambda) & f(\lambda) \\ f(\lambda) & B(\lambda)g(\lambda) \end{pmatrix}$$

is a 3-extremal mapping in $\mathcal{R}_{11}$, where $B = B_1B_2$. Actually, this is just Remark [15]. Now we are in a position which gives us a contradiction.
Note that here we do not work in the tetrablock - our settings is in $\mathcal{R}_{II}$. The advantage of this is that weak extremals and extremals coincide on $\mathcal{R}_{II}$.

Composing $[14]$ with a Möbius map we may assume that $B$ vanishes at the origin. Moreover, after a composition with $\Phi_{\alpha}$, where $\tilde{\alpha} = \begin{pmatrix} 0 & f(0) \\ f(0) & 0 \end{pmatrix}$, replacing $f$ and $g$ with other we may assume that $f(0) = 0$.

Therefore, thanks to the procedure described in Remark 4 there are $Z \in O(\mathbb{D}, \mathbb{D})$, and an automorphism $\Phi$ of $\mathcal{R}_{II}$ such that

$$
(15) \quad \begin{pmatrix} g(\lambda) & f(\lambda) \\ f(\lambda) & B(\lambda)g(\lambda) \end{pmatrix} = \Phi_C(\lambda \Phi \begin{pmatrix} \lambda & 0 \\ 0 & Z(\lambda) \end{pmatrix}), \quad \lambda \in \mathbb{D}
$$

where $C = \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix}$ and $c = g(0)$. Since the set of Nash functions on a domain of $\mathbb{C}^n$ forms a subring of the ring of holomorphic functions (see [18]) we easily find that $Z$ is not a Nash function. The goal is to derive a contradiction directly from (15). To do it we will find an explicit formula for $\Phi$ and then, to simplify the computations, we will pass to the tetrablock.

Put

$$
(16) \quad G(\lambda, z) := \Phi_C \left( \lambda \Phi \begin{pmatrix} \lambda & 0 \\ 0 & z \end{pmatrix} \right), \quad \lambda, z \in \mathbb{D}.
$$

Since $G_{22}(\lambda, Z(\lambda)) = B(\lambda)G_{11}(\lambda, Z(\lambda)), \quad \lambda \in \mathbb{D}$, making use once again of the fact that automorphisms of the classical domain are rational and $Z$ is not a Nash function we find that $G_{22}(\lambda, z) = B(\lambda)G_{11}(\lambda, z), \quad \lambda, z \in \mathbb{D}$.

By (13), $G_{11}(0, \cdot) \equiv -c$ and $G_{12}(0, \cdot) \equiv G_{21}(0, \cdot) \equiv 0$. Moreover, $G_{12} \equiv G_{21}$.

Write $G_{11}(\lambda, z) = -c + \lambda g_{11}(\lambda, z)$, $G_{12}(\lambda, z) = \lambda g_{12}(\lambda, z)$, $G_{21}(\lambda, z) = \lambda g_{21}(\lambda, z), \quad \lambda, z \in \mathbb{D}$. Additionally, put $g_{12} := g_{21}$.

The explicit formula for $\Phi_C^{-1}$ is

$$
\Phi_C^{-1}(x) = \begin{pmatrix} \frac{x_{11} + c}{1 + |c|^2} & \sqrt{1 - |c|^2} \frac{x_{21}}{1 + |c|^2} \\ \sqrt{1 - |c|^2} \frac{x_{21}}{1 + |c|^2} & \frac{1}{1 + |c|^2} \frac{x_{11} + c}{1 + |c|^2} \end{pmatrix}, \quad x = (x_{ij}) \in \mathcal{R}_{I}.
$$

Write $B(\lambda) = \lambda b(\lambda)$. Composing the relation (13) with $\Phi_C^{-1}$, dividing by $\lambda$ and putting $\lambda = 0$ we infer that

$$
\Phi \begin{pmatrix} 0 & 0 \\ 0 & z \end{pmatrix} = \begin{pmatrix} \frac{g_{11}(0)}{1 - |c|^2} & \sqrt{1 - |c|^2} \frac{g_{21}(0)}{1 + |c|^2} \\ \sqrt{1 - |c|^2} \frac{g_{21}(0)}{1 + |c|^2} & -b(0)c \end{pmatrix}, \quad z \in \mathbb{D}.
The first observation is that the component \((\Phi \begin{pmatrix} 0 & 0 \\ 0 & z \end{pmatrix})_{22}\) is a constant not depending on \(z\). Therefore the composition \(\Phi \) with \(\Phi_D\), where \(D = \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}\) and \(d = -b(0)c\) has the property \((\Phi_D \circ \Phi \begin{pmatrix} 0 & 0 \\ 0 & z \end{pmatrix})_{22} = 0\). This means that the function \(a := (\Phi \begin{pmatrix} 0 & 0 \\ 0 & \cdot \end{pmatrix})_{12}\) satisfies

\[
\det(\Phi_D \circ \Phi \begin{pmatrix} 0 & 0 \\ 0 & z \end{pmatrix}) = a^2(z), \quad z \in \mathbb{D}.
\]

Thanks to the form of automorphisms of \(\mathcal{R}_{II}\) we see that the function in the left side in the equation (17) is either constant or it is a rational function of degree 1. Hence, \(a\) is constant.

Therefore

\[
\Phi_D \circ \Phi \begin{pmatrix} 0 & 0 \\ 0 & \cdot \end{pmatrix} = \left( \begin{array}{cc} \alpha(\cdot) & a \\ a & 0 \end{array} \right)
\]

for some holomorphic function \(\alpha\). Composing the above relation with \(\Phi_A\), where \(A = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}\) we find that

\[
\Phi_A \circ \Phi_D \circ \Phi \begin{pmatrix} 0 & 0 \\ 0 & \cdot \end{pmatrix} = \left( \begin{array}{cc} \frac{\alpha(\cdot)}{1-|a|^2} & 0 \\ 0 & 0 \end{array} \right).
\]

Put \(\Psi = \Phi_A \circ \Phi_B \circ \Phi\). Using the description of \(\text{Aut}(\mathcal{R}_{II})\) we see that \(\Psi\) is of the form \(\Psi = U\Phi_{\Gamma}U^t\), where \(\Gamma = \begin{pmatrix} \gamma_1 & \gamma_0 \\ \gamma_0 & \gamma_2 \end{pmatrix}\) and \(U\) is unitary. Comparing the determinants in (18) we see that

\[
-\frac{\gamma_1(z - \gamma_2) - \gamma_0^2}{1 - \bar{\gamma}_1z} = 0 \quad \text{for any} \quad z \in \mathbb{D},
\]

whence \(\gamma_1 = \gamma_0 = 0\). Simple calculations show that \(U\) is anti-diagonal.

To simplify the notation let us assume that \(U = 1\) (the argument is the same in the general case).

Summing up we have obtained the relation

\[
\Phi = \Phi_{-D} \circ \Phi_{-A} \circ (U\Phi_{\Gamma}U^t),
\]

whence

\[
\Phi_C \left( \begin{array}{c} \lambda \Phi_{-D} \left( \Phi_{-A} \left( U\Phi_{\Gamma} \left( \begin{array}{cc} \lambda & 0 \\ 0 & z \end{array} \right) U^t \right) \right) \right) = \begin{pmatrix} G_{11}(\lambda, z) & G_{12}(\lambda, z) \\ G_{21}(\lambda, z) & \lambda b(\lambda) G_{11}(\lambda, z) \end{pmatrix}
\]

for $\lambda, z \in \mathbb{D}$. In particular, replacing $z$ with $m_{\gamma_z}(z)$, we get that the equality

$$\begin{align*}
(19) \quad \left( \Phi_C \left( \lambda \Phi_D \left( \Phi_{-A} \left( \begin{array}{cc} z & 0 \\ 0 & \lambda \end{array} \right) \right) \right) \right)_{22} &= \lambda b(\lambda) \Phi_C \left( \lambda \Phi_D \left( \Phi_{-A} \left( \begin{array}{cc} z & 0 \\ 0 & \lambda \end{array} \right) \right) \right)_{11}
\end{align*}$$

holds for any $\lambda$ and $z$ (whenever it is well defined).

Here it is convenient to pass to $\mathbb{E}$ (as it lies in 3-dimensional space). The equation (10) gives the following formula for the automorphism induced by $\Phi_C$:

$$\psi_1(z) = \left( \frac{z_1 - c}{1 - \bar{c}z_1}, \frac{z_2 - \bar{c}z_3}{1 - \bar{c}z_1}, \frac{z_3 - c\bar{z}_2}{1 - \bar{c}z_1} \right)$$

and the one induced by $\Phi_{-D}$:

$$\psi_2(z) = \left( \frac{z_1 + \bar{d}z_3}{1 + dz_2}, \frac{z_2 + d}{1 + dz_2}, \frac{z_3 + dz_1}{1 + dz_2} \right).$$

Then (19) may be rewritten in the following way:

$$\begin{align*}
B(\lambda) \left( \psi_1(\lambda). \psi_2 \left( \frac{z(1 - |a|^2)}{1 - \bar{a}^2z}, \frac{\lambda(1 - |a|^2)}{1 - \bar{a}^2z}, \frac{(1 - |a|^2)^2\lambda z - (\bar{a}\lambda z - a)^2}{(1 - \bar{a}^2\lambda z)^2} \right) \right)_{2} &= B(\lambda) \left( \psi_1(\lambda). \psi_2 \left( \frac{z(1 - |a|^2)}{1 - \bar{a}^2z}, \frac{\lambda(1 - |a|^2)}{1 - \bar{a}^2z}, \frac{(1 - |a|^2)^2\lambda z - (\bar{a}\lambda z - a)^2}{(1 - \bar{a}^2\lambda z)^2} \right) \right)_{1},
\end{align*}$$

where $\lambda \mapsto \lambda x$ is an action on $\mathbb{C}^3$ given by $\lambda.x = (\lambda x_1, \lambda x_2, \lambda^2x_3)$, $x \in \mathbb{C}^3$, $\lambda \in \mathbb{C}$.

Thus we have the equality

$$\frac{y_2 - \bar{c}\lambda y_3}{\lambda y_1 - c} = b(\lambda),$$

where $y = \psi_2 \left( \frac{z(1 - |a|^2)}{1 - \bar{a}^2z}, \frac{\lambda(1 - |a|^2)}{1 - \bar{a}^2z}, \frac{(1 - |a|^2)^2\lambda z - (\bar{a}\lambda z - a)^2}{(1 - \bar{a}^2\lambda z)^2} \right)$. Consequently

$$\begin{align*}
(20) \quad \frac{x_2 + d - \bar{c}x_3 - \lambda \bar{c}dx_1}{-c - cdx_2 + \lambda x_1 + \lambda dx_3} &= b(\lambda),
\end{align*}$$

where $x = \left( \frac{z(1 - |a|^2)}{1 - \bar{a}^2z}, \frac{\lambda(1 - |a|^2)}{1 - \bar{a}^2z}, \frac{(1 - |a|^2)^2\lambda z - (\bar{a}\lambda z - a)^2}{(1 - \bar{a}^2\lambda z)^2} \right)$.

Recall that $-cb(0) = d$ (or put $\lambda = 0$ in (20)).

Assume first that $a \neq 0$. Then letting $z \to \infty$ we find that

$$\begin{align*}
(21) \quad \frac{x_2 + d - \bar{c}x_3 - \lambda \bar{c}dx_1}{-c - cdx_2 + \lambda x_1 + \lambda dx_3} &= b(\lambda)
\end{align*}$$
holds for \( x = \left( \frac{1-|a|^2}{-a^2}, 0, \frac{1}{a} \right) \). Putting this \( x \) into (21), we get

\[
\frac{d + \lambda \bar{c} \frac{1-|a|^2}{a^2} + \bar{c} d \frac{1-|a|^2}{a^2}}{-c + \frac{1-|a|^2}{a^2} - \lambda d \frac{1-|a|^2}{a^2}} = b(\lambda).
\]

Putting \( \lambda = 0 \) we find that

\[
1 + \frac{\bar{c} \frac{1-|a|^2}{a^2} + cd}{\frac{1-|a|^2}{a^2}} = \frac{1}{c},
\]

whence either \(|a| = 1\) or \(|c| = 1\).

If \( a = 0 \) the situation is simpler. Indeed, then \( x = (z, \lambda, \lambda z) \). Taking \( z = 0 \) and putting it into (20) we get that

\[
\frac{\lambda + d}{-c - c d \lambda} = b(\lambda).
\]

Since \( b \) is a Blaschke product, \(|c| = 1\); a contradiction. \( \Box \)

4.3. (Weak) 3-extremals omitting \( \Sigma \).

**Proof of Theorem 18.** If \( f : \mathbb{D} \to G_2 \) is a weak 3-extremal such that \( f(\mathbb{D}) \cap \Sigma = \emptyset \), then we may lift it to \( \mathbb{D}^2 \). Namely, there are \( \varphi_1, \varphi_2 \in \mathcal{O}(\mathbb{D}, \mathbb{D}) \) such that \( f = (\varphi_1 + \varphi_2, \varphi_1 \varphi_2) \).

Since \( f \) is weak \( m \)-extremal we get that \( \varphi = (\varphi_1, \varphi_2) \) is \( m \)-extremal in the bidisc. Therefore one of functions \( \varphi_1 \) or \( \varphi_2 \) is a Blaschke product of degree at most 2. Losing no generality assume that it is \( \varphi_1 \).

We claim that \( \varphi_2 \) is a Blaschke product of degree at most 2, too. Otherwise, one can find a non-rational holomorphic function \( \psi : \mathbb{D} \to \mathbb{D} \) which agrees with \( \varphi_2 \) at 3 given points and such that \( \psi(\mathbb{D}) \subset \subset \mathbb{D} \). Then \( (\varphi_1 + \psi, \varphi_1 \psi) \) is a weak 3-extremal in \( G_2 \). It follows from Rouché theorem that the equation \( \varphi_1 = \psi \) has at least one solution in \( \mathbb{D} \), whence \( (\varphi_1 + \psi, \varphi_1 \psi) \) is an irrational weak 3-extremal in \( G_2 \) intersecting the royal variety \( \Sigma \). This gives a contradiction. \( \Box \)

5. Identity is \( m \)-extremal

We conclude the paper with a simple observation on a more general notion of \( m \)-extremals introduced in [5] and we present a solution of a problem posed there.

Similarly to the case of mappings defined on the unit disc.

Let \( D \) be a bounded domain in \( \mathbb{C}^N \) and \( \Omega \) a domain in \( \mathbb{C}^M \). Let \( m \geq 2 \) and let \( \lambda_1, \ldots, \lambda_m \in D \) be pairwise different points and \( z_1, \ldots, z_m \in \Omega \). Following [4] we say that the interpolation data

\[
(22) \quad \lambda_j \mapsto z_j, \quad D \to \Omega, \quad j = 1, \ldots, m
\]
is extremally solvable if there is a map $h \in \mathcal{O}(D, \Omega)$ such that $h(\lambda_j) = z_j$, $j = 1, \ldots, m$ and for any open neighborhood $U$ of $\overline{D}$ there is no $f \in \mathcal{O}(U, \Omega)$ such that $f(\lambda_j) = z_j$, $j = 1, \ldots, m$.

We say that $h \in \mathcal{O}(D, \Omega)$ is $m$-extremal if for all choices of $m$ pairwise distinct points $\lambda_1, \ldots, \lambda_m \in D$ the interpolation data

$$\lambda_j \mapsto h(\lambda_j), \ D \rightarrow \Omega, \quad j = 1, \ldots, m$$

is extremally solvable. Note that if $h$ is $m$-extremal then it is $m + 1$ extremal.

The question posed (and partially solved) in [4] (Proposition 2.5 and remark just before it) is whether the identity is $m$-extremal. The answer is yes.

**Proposition 21.** Let $D$ be a bounded domain in $\mathbb{C}^N$. Then the identity mapping $\text{id}_D : D \rightarrow D$ is $m$-extremal for any $m \geq 2$.

**Proof.** It is sufficient to show that $\text{id}_D$ is 2-extremal. Suppose that it does not hold. Then there are points $w, z \in D$, $w, z \in D$, an open $U \supset \overline{D}$ and $h \in \mathcal{O}(U, D)$ such that $h(w) = w, h(z) = z$. It easily follows from the definition of the Lempert function and standard reasoning employing the Montel theorem that there is a mapping $f : \overline{D} \rightarrow \overline{D}$ such that $f(0) = w, f(\sigma) = z$ and $p(0, \sigma) = k_D(w, z)$. Now the function $g := h \circ f \in \mathcal{O}(\overline{D}, D)$, $g(0) = w, g(\sigma) = z$ and $g(\overline{D}) \subset h(\overline{D}) \subset D$ which easily implies that $k_D(w, z) < p(0, \sigma)$ - contradiction. □

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