A SEMICLASSICAL PERSPECTIVE ON MULTIVARIATE
ORTHOGONAL POLYNOMIALS

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Abstract. Differential properties for orthogonal polynomials in several
variables are studied. We consider multivariate orthogonal polynomials whose
gradients satisfy some quasi–orthogonality conditions. We obtain several char-
acterizations for these polynomials including the analogous of the semiclassi-
sical Pearson differential equation, the structure relation and a differential–
difference equation.

1. Introduction

In the univariate case, a semiclassical moment functional \( u \) is a quasi–definite
functional satisfying a distributional Pearson equation

\[
D(\phi u) = \psi u
\]

where \( \phi \) and \( \psi \) are polynomials with \( \deg(\psi) \geq 1 \). They constitute a natural ex-
stension of classical functionals (Hermite, Laguerre, Jacobi, and Bessel) and they
have been extensively analyzed during the last two decades (see \([6], [15]\)). Obvi-
ously, the sequence \( \{p_n\}_n \) of orthogonal polynomials associated with \( u \) is also called
semiclassical.

The distributional Pearson equation (1) plays a key role in the study of dif-
ferential properties of semiclassical orthogonal polynomials. In fact, this equation
allows us to characterize semiclassical polynomials as the only sequences of orthog-
onal polynomials satisfying one of the following equivalent properties:

(a) the so–called structure relation,
(b) the quasi–orthogonality of the derivatives,
(c) the second order differential–difference relation.

For further properties and characterizations see, for instance, \([15]\).

In the multivariate case, we will call semiclassical a quasi– definite moment func-
tional \( u \) satisfying the matrix Pearson–type equation

\[
\text{div} (\Phi u) = \Psi^t u,
\]

where \( \Phi \) is a \( d \times d \) symmetric polynomial matrix and \( \Psi \) is a \( d \times 1 \) polynomial matrix
with \( \deg(\Psi) \geq 1 \), and such that \( \det(\langle u, \Phi \rangle) \neq 0 \). Of course, this definition includes all
the “classical” moment functionals in the usual literature (\([2], [8], [9], [10], [13], [14], [16]\)).

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The aim of our contribution is to analyze the extension of characterizations (a), (b), and (c) to the multivariate case by means of a matrix formalism. In the bivariate case, the first of these three characterizations was the main result of a previous publication (see [1]). The two other characterizations constitute the main objective of this work.

The structure of the paper is the following. In Section 2, a basic background about moment functionals and multivariate orthogonal polynomials is given in order to allow the reader to be familiar with such concepts. In Section 3 we present two new characterizations for sequences of multivariate semiclassical orthogonal polynomials. Theorems 3.3 and 3.4 are, respectively, the extensions of the quasi–orthogonality of the derivatives and the differential–difference relation satisfied by semiclassical orthogonal polynomials in one variable. Finally, in Section 4 we explore two non–trivial examples of multivariate semiclassical orthogonal polynomials.

2. Background

We denote by \( P \) the linear space of polynomials in \( d \) variables with real coefficients, and by \( P' \) its topological dual (see [15]). Let \( P_n \) be the set of real polynomials of total degree not greater than \( n \), and \( \Pi_n \) the space of homogeneous polynomials of degree \( n \) in \( d \) variables. It can be observed that \( \dim P_n = \binom{n+d}{n} \) and \( \dim \Pi_n = \binom{n+d-1}{n} = r_n \).

Let \( \mathcal{M}_{h \times k}(\mathbb{R}) \) be the linear space of \( (h \times k) \) real matrices, and \( \mathcal{M}_{h \times k}(P) \) the linear space of \( (h \times k) \) polynomial matrices. In addition, \( I_h \) represents the identity matrix of order \( h \).

If \( M = (m_{i,j}(x,y))_{i,j=1}^{h,k} \) represents a \( (h \times k) \) polynomial matrix, we define the degree of \( M \) by

\[
\deg M = \max\{\deg m_{i,j}(x,y), 1 \leq i \leq h, 1 \leq j \leq k\} \geq 0.
\]

Now, we review some basic definitions and tools about multivariate orthogonal polynomials that we will need in the rest of the paper. For a complete description of this and other related subjects see, for instance, [1, 3, 5, 11, 12, 16, 17].

Let \( N_0 \) be the set of nonnegative integers. For \( \alpha = (\alpha_1, \ldots, \alpha_d) \in N_0^d \) and \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \) we write \( x^{\alpha} = x_1^{\alpha_1} \cdots x_d^{\alpha_d} \). The number \( |\alpha| = \alpha_1 + \cdots + \alpha_d \) is called the total degree of \( x^{\alpha} \).

Let \( \{\mu_\alpha\}_{\alpha \in N_0^d} \) be a multi–sequence of real numbers and let us denote by \( u \), the only linear functional in \( P' \) satisfying

\[
\langle u, x^{\alpha} \rangle = \mu_\alpha.
\]

where, as usual, \( \langle , \rangle \) stands for the duality bracket. Then, \( u \) is called the moment functional determined by \( \{\mu_\alpha\}_{\alpha \in N_0^d} \), and the number \( \mu_\alpha \) is called the moment of order \( \alpha \).

Let \( (P_n^\alpha)_{|\alpha|=n} \) be a sequence of polynomials of total degree \( n \). Using the matrix notation introduced in [11, 12] and [17], we can denote by \( P_n \) the vector polynomial

\[
P_n = (P_n^\alpha)_{|\alpha|=n} = (P_n^{\alpha(1)}, P_n^{\alpha(2)}, \cdots, P_n^{\alpha(r_n)})^t \in \mathcal{M}_{r_n \times 1}(P_n),
\]

where \( \alpha(1), \alpha(2), \cdots, \alpha(r_n) \) are the elements in \( \{\alpha \in N_0^d : |\alpha| = n\} \) arranged according to the lexicographical order.
When \( \{P_m\}_{m=0}^\infty \) is a basis of \( P_n \) for each \( n \geq 0 \), then \( \{P_n\}_{n=0}^\infty \) is called a polynomial system (PS).

**Definition 2.1.** We will say that a PS \( \{P_n\}_{n=0}^\infty \) is a weak orthogonal polynomial system (WOPS) with respect to \( u \) if

\[
\begin{align}
\langle u, P_n P_m \rangle &= 0, \quad m \neq n, \\
\langle u, P_n P_m^t \rangle &= H_n,
\end{align}
\]

where \( H_n \in M_{r_n}(\mathbb{R}) \) is a non singular matrix.

This definition means that every polynomial component in \( P_n \), is orthogonal to all the polynomials of lower degree, but two polynomial components of the same degree don’t have to be orthogonal.

A moment functional \( u \) is said quasi–definite if there exists a WOPS with respect to \( u \) (see [3]).

Now, we are going to recover the three term recurrence relation for orthogonal polynomials in several variables ([3, 11, 12, 17]). This relation takes a vector-matrix form and it plays an essential role in understanding the structure of orthogonal polynomials, as in the univariate case.

For \( n \geq 0 \), there exist unique matrices \( A_{n,i} \in M_{r_n \times r_{n+1}}(\mathbb{R}) \), \( B_{n,i} \in M_{r_n \times r_n}(\mathbb{R}) \) and \( C_{n,i} \in M_{r_n \times r_{n-1}}(\mathbb{R}) \), \( i = 1, 2, \ldots, d \), such that

\[
\begin{align}
x_i P_n &= A_{n,i} P_{n+1} + B_{n,i} P_n + C_{n,i} P_{n-1}, \quad 1 \leq i \leq d,
\end{align}
\]

where \( P_{-1} = 0 \) and \( C_{-1,i} = 0 \).

Moreover, \( \text{rank } A_{n,i} = \text{rank } C_{n+1,i} = r_n \) and \( \text{rank } A_n = \text{rank } C_{n+1} = r_{n+1} \),

where \( A_n = \begin{pmatrix} A_{n,1} & \vdots & A_{n,d} \end{pmatrix} \) and \( C_{n+1} = \begin{pmatrix} C_{n+1,1} & \vdots & C_{n+1,d} \end{pmatrix} \).

As a consequence of the three term recurrence relation, it is possible to obtain the polynomial \( P_{n+1} \) in terms of the vector polynomials \( P_n \) and \( P_{n-1} \) (see [3]).

Let \( D_n^t = (D_{n,1}^t \ldots D_{n,d}^t) \in M_{r_{n+1} \times d r_n}(\mathbb{R}) \) be a left generalized inverse of \( A_n \). There exist matrices \( E_{n+1}^n \) and \( E_{n-1}^{n+1} \) such that

\[
\begin{align}
P_{n+1} = \sum_{i=1}^d x_i D_{n,i}^t P_n + E_{n+1}^n P_n + E_{n-1}^{n+1} P_{n-1}.
\end{align}
\]

Moreover, since the matrices \( C_{n,i} \), for \( i = 1, 2, \ldots, d \), are full rank, we can obtain \( P_{n-1} \) in terms of the vector polynomials \( P_{n+1} \) and \( P_n \). In fact, using the left generalized inverse of \( C_{n,i} \), that we will denote by \( G_{n,i} \), we deduce the following relations:

\[
\begin{align}
P_{n-1} = -G_{n,i} A_{n,i} P_{n+1} + (x_i G_{n,i} - G_{n,i} B_{n,i}) P_n, \quad 1 \leq i \leq d.
\end{align}
\]

3. **Semiclassical multivariate orthogonal polynomials**

First, we introduce the concept of semiclassical moment functional. For \( d = 2 \), this definition was given in [4].
Definition 3.1. A quasi–definite moment functional \( u \) is said to be **semiclassical** if it satisfies the matrix *Pearson–type* equation

\[
\text{div} (\Phi \ u) = \Psi^t \ u,
\]

where

\[
\Phi = (\phi_{ij})_{i,j=1}^d \in \mathcal{M}_d(P), \quad \Psi = (\psi_i)_{i=1}^d \in \mathcal{M}_{d \times 1}(P),
\]

are polynomial matrices such that \( \Phi \) is symmetric, \( \deg \Phi = p \geq 0 \), \( \deg \Psi = q \geq 1 \), and

\[
\det\langle u, \Phi \rangle \neq 0.
\]

We denote by \( s = \max\{p - 2, q - 1\} \geq 0 \).

Expression (6) means

\[
\langle \text{div} (\Phi \ u), f \rangle = \langle \Psi^t \ u, f \rangle,
\]

that is,

\[
\langle u, \Phi \nabla f + \Psi \ f \rangle = 0, \quad \forall f \in \mathcal{P}.
\]

The natural extension of the above property for matrices involves the Kronecker product (see, for instance [7], p. 242),

\[
\text{div} ((\Phi \otimes I_h) \ u) = (\Psi^t \otimes I_h) \ u, \quad h \geq 1.
\]

Relation (8) is equivalent to

\[
(\Phi \otimes I_h) \nabla u = (\tilde{\Psi} \otimes I_h) \ u, \quad h \geq 1,
\]

where \( \tilde{\Psi} = \Psi - (\text{div} \Phi)^t \).

**Remark** If \( s = 0 \), that is, \( \deg \Phi = p \leq 2 \) and \( \deg \Psi = 1 \), we recover the definition of classical WOPS given in [5], which includes the classical bivariate orthogonal polynomials studied by H. L. Krall and I. M. Sheffer ([13]), and other authors ([3, 8, 9, 14, 16]).

A WOPS with respect to a semiclassical moment functional \( u \) is called **semiclassical**.

Now, we are going to prove three characterizations for multivariate semiclassical orthogonal polynomials: structure relation, quasi–orthogonality relation for the gradients, and a differential–difference relation. From now on, we will denote by \( \{\mathcal{P}_n\}_{n \geq 0} \) a given WOPS associated with a quasi–definite moment functional \( u \).

**Theorem 3.1** (Structure relation). The moment functional \( u \) is semiclassical if and only if \( \{\mathcal{P}_n\}_{n \geq 0} \) satisfy

\[
\Phi \nabla \mathcal{P}^t_n = \sum_{j=n-s-1}^{n+p-1} (I_d \otimes \mathcal{P}^j) F^n_j, \quad \text{for} \quad n \geq s + 1,
\]

where \( F^n_j \in \mathcal{M}_{d r_j \times r_n}(\mathbb{R}) \).

**Proof.** See Theorem 5 in [1]. \( \square \)

Using relations (14) and (15), a shorter structure relation whose coefficients are polynomial matrices can be deduced.

**Corollary 3.2.** If \( u \) is semiclassical, then \( \{\mathcal{P}_n\}_{n \geq 0} \) satisfy

\[
\Phi \nabla \mathcal{P}^t_n = (I_d \otimes \mathcal{P}^t_{n+1}) M^n_1 + (I_d \otimes \mathcal{P}^t_n) M^n_2 \quad \text{for} \quad n \geq s + 1,
\]

where \( M^n_i \) are polynomial matrices with \( \deg(M^n_1) \leq s \) and \( \deg(M^n_2) \leq s + 1 \).
From Theorem 3.1, we can obtain the second characterization for semiclassical orthogonal polynomials.

**Theorem 3.3** (Quasi–orthogonality relation for gradients). *The functional \( u \) is semiclassical if and only if, for \( n \geq s + 1 \),

\[
\langle u, (\nabla P^t_m)^t \Phi \nabla P^t_n \rangle = 0, \quad 0 \leq m < n - s.
\]

**Proof.** Suppose that \( u \) is semiclassical. From (10), we can write

\[
\langle u, (\nabla P^t_m)^t (I_d \otimes P^t_j) \rangle = \sum_{j=n-s-1}^{n+p-1} \langle u, (\nabla P^t_m)^t (I_d \otimes P^t_j) \rangle F_j^n.
\]

Taking into account that

\[
(\nabla P^t_m)^t (I_d \otimes P^t_j) = (\partial_1 P^t_m) P^t_j \cdots (\partial_d P^t_m) P^t_j,
\]

we obtain

\[
\langle u, (\nabla P^t_m)^t (I_d \otimes P^t_j) \rangle = 0, \quad m - 1 < j,
\]

that is,

\[
\langle u, (\nabla P^t_m)^t \Phi \nabla P^t_n \rangle = 0, \quad m < n - s.
\]

Reciprocally, assume that the quasi–orthogonality relations hold, we define

\[
\Psi = -\sum_{i=0}^{s+1} \langle u, (\nabla P^t_1)^t \Phi \nabla P^t_i \rangle H_{i}^{-1} P_i.
\]

Since \( \nabla P^t_1 = I_2 \), we obtain

\[
\langle \text{div} (\Phi u), P^t_n \rangle = -\langle u, \Phi \nabla P^t_n \rangle = -\langle u, (\nabla P^t_1)^t \Phi \nabla P^t_n \rangle.
\]

On the other hand,

\[
\langle \Psi^t u, P^t_n \rangle = \langle u, \Psi P^t_n \rangle = -\sum_{i=0}^{s+1} \langle u, (\nabla P^t_1)^t \Phi \nabla P^t_i \rangle H_{i}^{-1} \langle u, P_i P^t_n \rangle.
\]

If \( n \geq s + 2 \), using the above relations, we get:

\[
\langle \text{div} (\Phi u), P^t_n \rangle = 0 = \langle \Psi^t u, P^t_n \rangle.
\]

Furthermore, for \( 0 \leq n \leq s + 1 \),

\[
\langle \text{div} (\Phi u), P^t_n \rangle = -\langle u, (\nabla P^t_1)^t \Phi \nabla P^t_n \rangle = \langle \Psi^t u, P^t_n \rangle,
\]

and then, the result follows. \( \square \)

Now, we deduce a matrix differential–difference relation for semiclassical orthogonal polynomials in several variables.

Let us define the differential operator

\[
L[f] = \text{div} (\Phi \nabla f) + \Psi^t \nabla f, \quad \forall f \in \mathcal{P}.
\]

Therefore, the Lagrange adjoint of \( L \) is given by

\[
L^*[u] = \text{div} (\Phi \nabla u) - \text{div} (\Psi u),
\]

since it satisfies

\[
\langle L^*[u], f \rangle = \langle u, L[f] \rangle, \quad \forall f \in \mathcal{P}.
\]

Using the explicit expression of the polynomial matrices \( \Phi \) and \( \Psi \), the operator \( L \) can be written as follows
\[ L[f] = \sum_{i,j=1}^{d} \phi_{ij} \partial_{ij}^2 f + \sum_{i=1}^{d} \psi_{i} \partial_{i} f, \]

and, from the above expression, we deduce \( \deg L[f] \leq s + \deg f \).

**Theorem 3.4** (Matrix differential–difference relation). A functional \( u \) is semiclassical if and only if there exist matrices \( \Lambda_n^m \in M_{r \times r} (\mathbb{R}) \), such that

\[
L[P^d_n] = \sum_{i=n-s}^{n+s} P^d_i \Lambda_i^n, \quad n \geq s + 1.
\]

When \( n \leq s \), relation (12) reads

\[
L[P^d_n] = \sum_{i=1}^{n+s} P^d_i \Lambda_i^n,
\]

that is, \( \Lambda_n^0 = 0, \forall n \geq 0 \).

**Proof.** If \( u \) is a semiclassical functional, Lemma 4.1 in [5] provides

\[
\langle u, P_m \, \text{div} \, (\Phi \nabla P^d_n) \rangle = \langle u, \text{div} \, ((I_d \otimes P_m) \Phi \nabla P^d_n) \rangle - \langle u, (\nabla P^d_m)^t \Phi \nabla P^d_n \rangle,
\]

for \( n, m \geq 0 \). Besides, from (13), we deduce

\[
\langle u, \text{div} \, ((I_d \otimes P_m) \Phi \nabla P^d_n) \rangle = -\langle (\nabla u, (\Phi \otimes I_{r_m}) (I_d \otimes P_m) \nabla P^d_n) \rangle
\]

\[
= -\langle (\Phi \otimes I_{r_m}) \nabla u, (I_d \otimes P_m) \nabla P^d_n \rangle
\]

\[
= -\langle (\Phi \otimes I_{r_m}) u, (I_d \otimes P_m) \nabla P^d_n \rangle
\]

\[
= -\langle u, (\Phi \otimes I_{r_m}) (I_d \otimes P_m) \nabla P^d_n \rangle
\]

\[
= -\langle u, P_m \Phi \nabla P^d_n \rangle.
\]

Therefore, we have

\[
\langle u, P_m L[P^d_n] \rangle = -\langle u, (\nabla P^d_m)^t \Phi \nabla P^d_n \rangle.
\]

Now, since \( L[P^d_n] \) is a polynomial matrix of degree at most \( n + s \), we obtain the expansion

\[
L[P^d_n] = \sum_{i=0}^{n+s} P^d_i \Lambda_i^n,
\]

where \( \Lambda_i^n \) is given by

\[
\langle u, P_m L[P^d_n] \rangle = \langle u, \sum_{i=0}^{n+s} P^d_i \Lambda_i^n \rangle = H_m \Lambda_m^n,
\]

and using (13), we get

\[
H_m \Lambda_m^n = -\langle u, (\nabla P^d_m)^t \Phi \nabla P^d_n \rangle.
\]

In the case \( n \geq s + 1 \), from Theorem 3.3

\[
H_m \Lambda_m^n = 0, \quad 0 \leq m < n - s,
\]

so, relation (12) follows.

When \( n \leq s \),

\[
H_0 \Lambda_0^n = -\langle u, (\nabla P^d_0)^t \Phi \nabla P^d_n \rangle = 0.
\]
Corollary 3.5. If a quasi–definite moment functional \( N \) exist polynomial matrices \( \Phi \) and \( \Psi \) are given by
\[
\Phi = \begin{pmatrix}
\alpha_1 x_1 - 1 & x_1 x_2 & \cdots & x_1 x_d \\
x_2 x_1 & \alpha_2 x_2 - 1 & \cdots & x_2 x_d \\
\vdots & \vdots & \ddots & \vdots \\
x_d x_1 & x_d x_2 & \cdots & \alpha_d x_d - 1
\end{pmatrix},
\]
\[
\Psi = \begin{pmatrix}
(|\alpha| + d)x_1 - (\alpha_1 + 1) \\
(|\alpha| + d)x_2 - (\alpha_2 + 1) \\
\vdots \\
(|\alpha| + d)x_d - (\alpha_d + 1)
\end{pmatrix}.
\]

Finally, Theorem 3.3 provides the desired result.

\( \square \)

4. EXAMPLES

Example 1: Appell–type orthogonal polynomials

The so–called Appell polynomials (12) and (13) allow us to express \( L[\mathbb{P}_n^t] \) in terms of the vector polynomials \( \mathbb{P}_n^t \) and \( \mathbb{P}_n^t + \mathbb{P}_n^t \).

Corollary 3.5. If a quasi–definite moment functional \( u \) is semiclassical, then there exist polynomial matrices \( \mathbb{N}_1^n \) and \( \mathbb{N}_2^n \), satisfying
\[
(14)
L[\mathbb{P}_n^t] = \mathbb{P}_n^{t+1} \mathbb{N}_1^n + \mathbb{P}_n^t \mathbb{N}_2^n
\]
with \( \deg(\mathbb{N}_i^n) \leq s - 1 \) and \( \deg(\mathbb{N}_2^n) \leq s \).

Using our definitions (14, 15), \( u \) is a classical moment functional (i.e., it is semiclassical with \( s = 0 \)), since it satisfies the matrix Pearson–type equation (6), where the matrices \( \Phi \) and \( \Psi \) are given by
Now, we introduce the *Appell-type polynomials* as the orthogonal polynomials in $d$ variables with respect to the moment functional

$$v = u + \lambda \delta(x),$$

where $\lambda \geq 0$ is a positive real number, and $\delta(x)$ is the usual Dirac distribution at $0 \in \mathbb{R}^d$. The action of $v$ over polynomials is defined as follows,

$$\langle v, f \rangle = \int_{\mathbb{R}^d} f(x) \omega_{\alpha}(x) \, dx + \lambda f(0).$$

The moment functional $v$ is semiclassical with $s = 1$, since $v$ satisfies the matrix Pearson–type equation (15)

$$\text{div}(\hat{\Phi} v) = \hat{\Psi}^t v,$$

where

$$\hat{\Phi} = x_1 \Phi, \quad \hat{\Psi} = (x_1(x_1 - 1), x_1x_2, \ldots, x_1x_d)^t + x_1 \Psi.$$

In fact, using that $x_1 \delta(x) = 0$, we get

$$\text{div}(\hat{\Phi} v) = \text{div}(x_1 \Phi (u + \lambda \delta(x))) = \text{div}(x_1 \Phi u) + \text{div}(x_1 \Phi \lambda \delta(x)) = (1, 0, \cdots, 0) \Phi u + x_1 \text{div}(\Phi u) = (x_1(x_1 - 1), x_1x_2, \cdots, x_1x_d) v + x_1 \Psi^t v.$$

Observe that the matrix Pearson–type equation for $v$ is not unique. The moment functional $v$ also satisfies (15) with

$$\hat{\Phi} = x_i \Phi, \quad \hat{\Psi} = (x_i x_1, \cdots, x_i(x_i - 1), \cdots, x_i x_d)^t + x_i \Psi,$$

for $1 \leq i \leq d$, since $x_i \delta(x) = 0$.

**Example 2: A multivariate analogue of the classical orthogonal polynomials**

Examples of two–variables analogues of the Jacobi polynomials are studied in [10] by T. Koornwinder. Using similar tools, we present an example of a semiclassical weight function with unbounded support.

Let $\alpha_i$ be real numbers with $\alpha_i > -1$, for $1 \leq i \leq d$. Then, for $k_i \geq 0$, $1 \leq i \leq d$, and $k_1 \geq k_d$, we define the polynomials

$$P_{k_1, \ldots, k_d}^{(\alpha_1, \ldots, \alpha_d)}(x) = L_{k_1-1}^{(\alpha_1+2k_d+1)}(x_1) L_{k_2}^{(\alpha_2)}(x_2) \cdots L_{k_{d-1}}^{(\alpha_{d-1})}(x_{d-1}) x_d^{k_d} P_{d}^{(\alpha_d, 0)}(x_1^{-1} x_d),$$

where $L_{k_i}^{(\alpha_i)}$ is a Laguerre polynomial in one variable, and $P_{d}^{(\alpha_d, 0)}$ is a Jacobi polynomial.

The polynomials $P_{k_1, \ldots, k_d}^{(\alpha_1, \ldots, \alpha_d)}(x)$ are orthogonal with respect to the weight function

$$w(x) = x_1^{\alpha_1} \cdots x_d^{\alpha_d-1} e^{-(x_1 + \cdots + x_{d-1})} (1 - x_1^{-1} x_d)^{\alpha_d},$$

on the region $\{ x = (x_1, \ldots, x_d) / -x_1 < x_d < x_1, \ x_i > 0, \ i = 1, \ldots, d - 1 \}$.

Defining the matrices

$$\Phi = \begin{pmatrix}
 x_1(x_1 - x_d) & 0 & 0 & \cdots & 0 \\
 x_2 & x_1(x_1 - x_d) & 0 & \cdots & 0 \\
 \vdots & \ddots & \ddots & \ddots & \vdots \\
 0 & \cdots & 0 & x_{d-1} & x_1(x_1 - x_d) \\
 0 & \cdots & 0 & 0 & x_1(x_1 - x_d)
 \end{pmatrix},$$

and

$$\Psi = \begin{pmatrix}
 1 & 0 & 0 & \cdots & 0 \\
 0 & 1 & 0 & \cdots & 0 \\
 \vdots & \ddots & \ddots & \ddots & \vdots \\
 0 & \cdots & 0 & 1 & 0 \\
 0 & \cdots & 0 & 0 & 1
 \end{pmatrix}.$$
\[ \Psi = \begin{pmatrix} -x_1^2 + x_1 x_d + (\alpha_1 + 2)x_1 + (\alpha_d - \alpha_1 - 1)x_d \\ \alpha_2 - x_2 \\ \vdots \\ \alpha_{d-1} - x_{d-1} \\ -(\alpha_d + 1)x_1^2 \end{pmatrix}, \]

we can prove that the weight function \( w(x) \) satisfies (6) and so, the polynomials \( P_{k_1, \ldots, k_d}(x) \) are semiclassical.

Another examples for semiclassical orthogonal polynomials in two variables appear in [1].

**References**

[1] M. Álvarez de Morales, L. Fernández, T. E. Pérez and M. A. Piñar, Semiclassical orthogonal polynomials in two variables, J. Comput. Appl. Math. (to appear).

[2] P. Appell and J. Kampé de Fériet, *Fonctions hypergémétriques et hypersphériques. Polynomes d’Hermite*, Gauthier-Villars, Paris, 1926.

[3] C. F. Dunkl and Y. Xu, *Orthogonal Polynomials of several variables*, Encyclopedia of Mathematics and its Applications 81. Cambridge University Press, 2001.

[4] L. Fernández, T. E. Pérez and M. A. Piñar, Classical Orthogonal Polynomials in two variables: a matrix approach, Numerical Algorithms 39 (2005), 131–142.

[5] L. Fernández, T. E. Pérez and M. A. Piñar, On multivariate classical orthogonal polynomials, Rendiconti del Circolo Matematico di Palermo Serie II, Suppl. 76 (2005), 315–329.

[6] E. Hendriksen and H. van Rossum, Semi–classical orthogonal polynomials, *Polynômes Orthogonaux et Applications*, Bar-le-Duc 1984, C. Brezinski et al. Eds. Lecture Notes in Math. n. 1171, Springer-Verlag, Berlin, 1985, pp. 354–361.

[7] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, 1991.

[8] Y. J. Kim, K. H. Kwon and J. K. Lee, Orthogonal polynomials in two variables and second-order partial differential equations, J. Comput. Appl. Math. 82 (1997), 239–260.

[9] Y. J. Kim, K. H. Kwon and J. K. Lee, Partial differential equations having orthogonal polynomial solutions, J. Comput. Appl. Math. 99 (1998), 239–253.

[10] T. Koornwinder, Two variable analogues of the classical orthogonal polynomials. Theory and application of special functions (Proc. Advanced Sem., Math. Res. Center, Univ. Wisconsin, Madison, Wis., 1975), pp. 435–495. Math. Res. Center, Univ. Wisconsin, Publ. No. 35, Academic Press, New York, 1975.

[11] M. A. Kowalski, The recursion formulas for orthogonal polynomials in n variables, SIAM J. Math. Anal. 13 (1982), pp. 309–315.

[12] M. A. Kowalski, Orthogonality and recursion formulas for polynomials in n variables, SIAM J. Math. Anal. 13 (1982), pp. 316–323.

[13] H. L. Krall and I. M. Sheffer, Orthogonal polynomials in two variables, Ann. Mat. Pura Appl. Serie 4 76 (1967), 325–376.

[14] L. L. Littlejohn, Orthogonal polynomial solutions to ordinary and partial differential equations, in *Orthogonal Polynomials and their Applications. Proceedings Segovia (Spain), 1986*, Lecture Notes in Mathematics, vol. 1329, Springer, Berlin, (1988), 98–124.

[15] P. Maroni, Problèmes à l’étude des polynômes orthogonaux semi–classiques, *Ann. Mat. Pura Appl. Ser. 4* 149 (1987) 165–184.

[16] P. K. Suetin, *Orthogonal Polynomials in Two Variables*, Gordon and Breach, Amsterdam (1999).

[17] Y. Xu, On multivariate orthogonal polynomials, Siam J. Math. Anal. 24 (1993), 783–794.
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