Quantum logic as a sum over classical logic gates

Bruno Nachtergaele‡ and Vipul Periwal†

Department of Physics
Princeton University
Princeton, New Jersey 08544-0708

It is shown that certain natural quantum logic gates, i.e. unitary time evolution matrices for spin-$\frac{1}{2}$ quantum spins, can be represented as sums, with appropriate phases, over classical logic gates, in a direct analogy with the Feynman path integral representation of quantum mechanics. On the other hand, it is shown that a natural quantum gate obtained by analytically continuing the transfer matrix of the anisotropic nearest-neighbour Ising model to imaginary time, does not admit such a representation.

‡ bxn@math.princeton.edu
† vipul@puhepl.princeton.edu
Shor’s discovery[1] of polynomial time quantum algorithms for prime factorization and discrete logarithm has resulted in an upsurge of interest in the properties of quantum computation[2]. Significant results have been obtained concerning the physical realizability of quantum gates, and the realizability of classical universal 3-bit gates such as the Fredkin and Toffoli gates in terms of quantum 2-bit logic[3]. Furthermore, Barenco et al.[4] have shown that all quantum gates can be expressed as compositions of all one-bit quantum gates and the two-bit exclusive-or gate. The problems of error correction[5] and decoherence[6] in quantum computation have also been addressed.

Our aim here is to consider the properties of quantum logic in a different light. Instead of constructing classical logic in terms of quantum gates, we want to represent some rather general quantum gate arrays in terms of coherent sums over classical gate arrays, much as Feynman represented quantum mechanical amplitudes in terms of classical paths[7]. There are obvious reasons for wanting such a representation. An intuition for the efficiency of quantum computation is that quantum computers sum over many classical computations, and it is important to understand quantitatively how this works, and how it can be exploited. Further, it is likely that the true power of quantum computation will come from massively parallel computation, a point that has been considered from the very beginnings of the subject, and recently re-emphasized[8]. An intuition for the behaviour of such quantum logic, in terms of classical logic, is of great interest in this context, just as in the case of the Feynman path integral. Consider, for example, the quantum phenomenon of tunneling—one would like to know what the classical computational analogue of this might be, and how it should be used in the design of quantum logic and quantum programming. Indeed, programming quantum logic on the basis of what classical logic it replaces, is likely to be inefficient. It may be more efficient to program according to the properties of quantum logic, much as with writing code for parallel processors, and for this purpose it is again important to gain some classical intuition for the properties of quantum gates. Such classical logic representations also allow for a new type of simulation of quantum logic by
classical parallel processors, rather obviously.

We present two independent insights into classical representations of quantum logic. First, we show that for a natural set of Hamiltonians governing quantum spin-$\frac{1}{2}$ degrees of freedom, there is a simple representation of the unitary time evolution operator, in other words, the quantum logic gate, in terms of appropriately weighted sums over classical logic gates[9]. We describe properties of these ‘logic integrals’ (adapting the term ‘path integrals’ to the present context) which can be deduced from the physical properties of the spin systems, and we suggest some uses for such quantum logic. Certain general properties of such quantum logic for a one-dimensional chain of spins could be inferred by finite size scaling calculations around conformal field theories in two dimensions[10].

 Secondly, we consider an anisotropic Ising model on a two-dimensional square lattice. We show that the transfer matrix of this model, analytically continued, is unitary at a unique value of the ‘time’ coupling, and we show that this unitary quantum gate cannot be represented as a sum over classical logic gates in general. Thus, ‘logic integrals’ do not necessarily exist as representations of quantum logic. This will not come as a surprise to physicists[11].

For our first problem, we consider quantum spin-$\frac{1}{2}$ degrees of freedom defined on a finite set of sites $\Gamma$. The Hilbert space at each site is $\mathcal{H}_x \cong \mathbb{C}^2$, and observables are elements of the bounded operators on this Hilbert space, just the set of $2 \times 2$ complex matrices $M(2, \mathbb{C})$. The Hamiltonian $H$ for such a system can be written in general as

$$H = - \sum_{b \in \mathcal{B}} J_b h_b$$

where $\mathcal{B}$, the set of ‘bonds’, is a collection of subsets of $\Gamma$, and $h_b$ is an arbitrary element in $\bigotimes_{x \in b} M(2, \mathbb{C})$. For much of our discussion, it will suffice to take $\Gamma$ as a subset of the integers, say $\{0, \ldots, L\}$, and $\mathcal{B} = \{\{0, 1\}, \ldots, \{L - 1, L\}\}$, which is the case easiest to visualize, but it is important to observe that our approach holds in all generality. Physically important observables are usually expressed in terms of the spin matrices $S^1, S^2, S^3$ which are the
generators of the fundamental representation of SU(2). They satisfy the commutation relations

$$[S^\alpha, S^\beta] = \sqrt{-1} \sum_\gamma \epsilon_{\alpha\beta\gamma} S^\gamma$$

where $\alpha, \beta, \gamma \in \{1, 2, 3\}$ and $\epsilon_{\alpha\beta\gamma}$ is the completely antisymmetric tensor with $\epsilon_{123} = 1$.

Aizenman and Nachtergaele[9] have given a ‘quasi-state’ decomposition for the quantum statistical mechanics of this system, starting from a Poisson integral formula. Using this decomposition for ‘imaginary temperatures’, we obtain the following expression for the unitary evolution operator

$$\exp(\sqrt{-1} \beta H) = \int D\beta\omega K(\omega),$$

where $D\beta\omega$, up to normalization and analytic continuation, is the probability measure of a product of independent Poisson processes for each bond in $B$, running over the time interval $[0, \beta]$ with rates $J_b$. More explicitly, the integration measure is given by

$$D\beta\omega = \prod_b \sum_{n_b=0}^{\infty} (-\sqrt{-1}J_b)^{n_b} \int_{0<t_1\leq t_2\leq \beta} \prod_{j=1}^{n_b} dt_j$$

$K(\omega)$ is a time ordered product of operators, one for each bond in $\omega$:

$$K(\omega) = \prod^* h_{b_n} h_{b_{n-1}} \ldots h_{b_1},$$

if $\omega$ is the set of bonds $\{(b_1, t_1), \ldots, (b_n, t_n)\}$ with $t_1 < t_2 < \ldots t_n$. $\prod^*$ indicates a time ordered product. Fig. 1 shows a configuration $\omega$ in the case that $\Gamma$ is a one-dimensional lattice and $B$ are the nearest neighbour bonds on this lattice.

Now consider, for example, $h$ to be the operator that interchanges the states of the two sites,

$$h\phi \otimes \psi = \psi \otimes \phi$$

for any two vectors $\phi, \psi \in \mathbb{C}^2$. This is the exchange gate on 2 bits,

$$E \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

4
but it is equivalent to the Heisenberg spin $\frac{1}{2}$ ferromagnet! Fig. 2 shows a configuration $\omega$ in this model.

From the expressions (1-3) it is obvious that the quantum evolution operator can be decomposed as a linear superposition of classical logic gates of the form $K(\omega)$. For concreteness, consider a three spin system (equivalently, a three bit gate). By performing the integrals in (2), we obtain series expansions for the coefficients of the various classical logic gates appearing in the decomposition:

$$
\exp(\sqrt{-1}\beta H) = (1 - \beta^2 J^2 + \ldots)1 + (-\sqrt{-1}\beta J + \ldots)E_{12} + (-\sqrt{-1}\beta J + \ldots)E_{23} + (-\frac{1}{2}\beta^2 J^2 + \ldots)E_{12} + (-\frac{1}{2}\beta^2 J^2 + \ldots)E_{12}^2 + + (\frac{1}{3}\sqrt{-1}\beta^3 J^3 + \ldots)E_{13}.
$$

Here $E_{ij}$ is the exchange gate on the $i$ and $j$ bits, and

$$
E_{123} = E_{23}E_{12} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

is the matrix that permutes the three bits cyclically. This example illustrates the utility of classical logic representations of quantum gates—by varying $\beta$, one can single out contributions of different classical logic gates from the quantum gate. $\beta$ is just the time of evolution of the quantum system, so no external classical ‘switches’ are needed, which helps in minimizing the effects of decoherence[6].

If $\Gamma = \Gamma_A \cup \Gamma_B$ is a bipartite lattice, and $B$ is a set with elements of the form $\{a, b\}, a \in \Gamma_A$ and $b \in \Gamma_B$, then we consider

$$
h = \sum_{m,m' = \pm 1/2} (-1)^{m-m'}|m,-m\rangle\langle m',-m'|.
$$

This is the Heisenberg anti-ferromagnet. In terms of classical logic, this operator $h$ corresponds to $1 - E$, and is shown in Fig. 3. In this case, a new phenomenon that contributes
to the quantum logic gate, but would not appear in classical logic, becomes apparent in
the quasi-state representation. Notice that $1 - E$ is proportional to a projection of rank
1: $(1 - E)^2 = 2(1 - E)$. The factor of 2 corresponds to the fact that there are closed loops
in a typical $\omega$, as shown in Fig. 3. The sum over classical configurations that gives the
quantum amplitude therefore includes sums over ‘virtual’ states of the classical logic.

Such logic integral decompositions of quantum logic can be extended to a much
wider class of Hamiltonians with ease\[9\], providing simple classical logic representations
with component classical gates that are $n$-bit gates. In the one-dimensional case, with
$\Gamma = \{0, \ldots, L\}$, this amounts to taking $B = \{\{0, \ldots, n\}, \{1, \ldots, n + 1\}, \ldots\}$. Quasi-state
decompositions for such cases have been reported in detail elsewhere\[9\].

Simple properties of such massively parallel quantum logic can be extracted from
physical properties of these systems. Some quantum spin systems exhibit phase transitions
at (imaginary) values of $\beta$ in the infinite volume limit. There are two aspects of this that
will be useful for quantum computation:

1. The existence of phase transitions implies that the quantum gate will exhibit different
characteristics depending on the sign of $\beta - \beta_c$. In other words, letting the quantum
evolution of the initial states run for a short time, or a long time, effectively leads to
two different quantum gates. When $\beta > \beta_c$, one expects long range order, implying
algebraically decaying correlations between the input and the output, and for $\beta < \beta_c$
one expects exponential decay of correlations.

2. At finite lattice sizes, one can still get a good handle on properties of the quantum
gate for $\beta$ close to $\beta_c$ by calculating finite size corrections to the correlation functions
at criticality\[10\].

For the converse of our first problem, we turn now to the anisotropic Ising model
in two dimensions, to exhibit another aspect to representations of quantum logic as ‘logic
integrals’ of classical logic. Recall that this model is a classical statistical mechanics model,
with spins taking values $\pm 1$ living on the sites of a square two dimensional lattice. For
our purposes, we take the system to be of finite extent in the space direction. The time direction’s extent will not be relevant for us, but for the nonce we assume periodic boundary conditions in the time direction. The partition function of this model is

\[ Z \equiv \sum_{\{\sigma\}} \exp \left( -\beta_1 \sum_{i=0}^{N} \sum_{t} \sigma_{i,t}\sigma_{i,t+1} - \beta \sum_{t} \sum_{i=0}^{N-1} \sigma_{i,t}\sigma_{i+1,t} \right), \]

where the sum over \( t \) is a sum over the time slices of the lattice. Introduce a transfer matrix \( T \), defined by

\[ \langle \tilde{\sigma}_0, \ldots \tilde{\sigma}_N | T | \sigma_0, \ldots \sigma_N \rangle \equiv 2^{-N/2} \exp \left( -\beta_1 \sum_{i=0}^{N} \tilde{\sigma}_i\sigma_i - \beta \sum_{i=0}^{N-1} \sigma_i\sigma_{i+1} \right). \]

For a lattice of time extent \( \tau \), the partition function can now be written as \( Z \propto \text{tr} T^\tau \). This transfer matrix \( T \) essentially allows one to interpret the Ising model as a discrete-time one-dimensional quantum system, with \( T \equiv \exp(-H) \). We can now analytically continue this matrix to imaginary time, and ask if there are imaginary values of \( \beta \) and \( \beta_1 \) such that \( T \) is a unitary matrix.

To this end, we evaluate

\[ \langle \tilde{\sigma}|TT^\dagger|\sigma \rangle = 2^{-N} \sum_{\sigma'} \exp \left( -\sum_{i=0}^{N} [\beta_1 \tilde{\sigma}_i + \beta_1^* \sigma_i]\sigma_i' - (\beta + \beta^*) \sum_{i=0}^{N-1} \sigma_i'\sigma_{i+1}' \right). \]

It follows then that if \( \beta = \sqrt{-1}\gamma \), and \( \beta_1 = \pm\sqrt{-1}\pi/4 \), \( T \) is a unitary matrix for any value of \( \gamma \).

For \( N = 2 \), this matrix is

\[ T = \begin{pmatrix} \sqrt{-1} & 1 & 1 & \sqrt{-1} \\ 1 & \sqrt{-1} & 1 & -\sqrt{-1} \\ 1 & -\sqrt{-1} & \sqrt{-1} & 1 \\ -\sqrt{-1} & 1 & 1 & \sqrt{-1} \end{pmatrix} \times \text{diag}(\Delta, \Delta^*, \Delta^*, \Delta), \]

where \( \Delta \equiv \exp(-\sqrt{-1}\gamma) \). If \( \Delta = 1 \), it is clear that \( T \) can be written as

\[ \begin{bmatrix} \sqrt{-1} & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \sqrt{-1} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \]

\[ T(\gamma = 0) = \sqrt{-1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} - \sqrt{-1} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \]

\[ T(\gamma = 0), \]
which is readily recognizable as a sum over classical logic gates, with appropriate phase factors. Here, we have restricted ourselves to decompositions with coefficients of modulus 1. It is easy to see that there are two such decompositions.

However, when $\Delta \neq 1$, such a decomposition is not possible in general. Indeed, there is no reason to expect that it should be. Classical logic gates on $N$ bits are matrices in the $2^N \times 2^N$ permutation representation of the permutation group on $2^N$ letters. By Schur’s lemma, the complex linear span of the permutation representation on $n$ letters is a strict subalgebra of the algebra of complex $n \times n$ matrices, since the permutation representation is reducible, but the defining representation of $U(n)$ is certainly irreducible, so its complex linear span is all of the algebra of complex $n \times n$ matrices.

In conclusion, we have shown that there is a natural representation of parallel quantum gates in terms of sums over classical logic gates, analogous to the Feynman sum over paths representation of quantum mechanical amplitudes. We have shown that this viewpoint on quantum logic allows a whole host of tools from statistical mechanics and quantum spin chains to be used to obtain a better intuition for the characteristics of quantum logic. We have explicitly shown that such representations may not always be possible, indicating some of the limits of this approach.

References

1. P. W. Shor, “Algorithms for quantum computation: discrete logarithms and factoring”, in S. Goldwasser, ed., Proc. 35th Symp. on Foundations of Computer Science, Santa Fe, NM, 20–22 November 1994 (Los Alamitos, CA: IEEE Computer Society Press 1994) 124–134.

2. P. Benioff, J. Stat. Phys. 22 (1980) 563; 29 (1982) 515; R. P. Feynman, “Simulating physics with computers”, Int. J. Theor. Phys. 21 (1982) 467–488; “Quantum mechanical computers”, Found. Phys. 16 (1986) 507–531; D. Deutsch, “Quantum theory, the Church–Turing principle and the universal quantum computer”, Proc. Roy. Soc. Lond. A 400 (1985) 97–117; A. Peres, “Reversible logic and quantum computation”, Phys.
Rev. A32 (1985) 3266; D. Deutsch and R. Jozsa, “Rapid solution of problems by quantum computation”, Proc. Roy. Soc. Lond. A 439 (1992) 553–558; E. Bernstein and U. Vazirani, “Quantum complexity theory”, in Proc. 25th ACM Symp. on Theory of Computing, San Diego, CA, 16–18 May 1993 (New York: ACM Press 1993) 11–20 A. Berthiaume and G. Brassard, “The quantum challenge to structural complexity theory”, in Proc. 7th Structure in Complexity Theory Conference, Boston, MA, 22–25 June 1992 (Los Alamitos, CA: IEEE Computer Society Press 1992) 132–137; D. R. Simon, “On the power of quantum computation”, in S. Goldwasser, ed., Proc. 35th Symp. on Foundations of Computer Science, Santa Fe, NM, 20–22 November 1994 (Los Alamitos, CA: IEEE Computer Society Press 1994) 116–123

3 S. Lloyd, “A potentially realizable quantum supercomputer”, Science 261 (1993) 1569; R. Landauer, “Is quantum mechanics useful?”, Phil. Trans. Roy. Soc. Lond. A 353 (1995) 367–376; M. B. Plenio and P. L. Knight, “Realistic lower bounds for the factorization time of large numbers on a quantum computer”, preprint (1995) quant-ph/9512001; D. Beckman, A. N. Chari, S. Devabhaktuni and J. Preskill, “Efficient networks for quantum factoring”, preprint (1996) CALT-68-2021, quant-ph/9602016; N. Margolus, “Quantum computation”, Ann. NY Acad. Sci. 480 (1986) 487–497; D. A. Meyer, “From quantum cellular automata to quantum lattice gases”, UCSD preprint (1995), quant-ph/9604003; to appear in J. Stat. Phys.; I. L. Chuang and Y. Yamamoto, “A simple quantum computer”, Phys. Rev. A 52 (1995) 3489–3496; T. Sleator and H. Weinfurter, “Realizable Universal Quantum Logic Gates”, Phys. Rev. Lett. 74 (1995) 4087–4090; H. Körner and G. Mahler, “Optically driven quantum networks: applications in molecular electronics”, Phys. Rev. B 48 (1993) 2335–2346; A. Barenco, D. Deutsch, A. Ekert and R. Jozsa, “Conditional quantum dynamics and logic gates”, Phys. Rev. Lett. 74 (1995) 4083-4086; J. I. Cirac and P. Zoller, “Quantum computations with cold trapped ions”, Phys. Rev. Lett. 74 (1995) 4091–4094; C. Monroe, D.M. Meekhof, B.E. King, W.M. Itano and D.J. Wineland, “Demonstration
of a universal quantum logic gate”, *Phys. Rev. Lett.* **75** (1995) 4714; W.H. Zurek and R. Laflamme, “Quantum Logical Operations on Encoded Qubits”, quant-ph/9605013; D. P. DiVincenzo, “Two-bit gates are universal for quantum computation”, *Phys. Rev. A* **51** (1995) 1015–1022; H.F. Chau and F. Wilczek, “Realization of the Fredkin gate using a series of one- and two-body operators”, quant-ph/9503003

4 A. Barenco, C. H. Bennett, R. Cleve, D. P. DiVincenzo, N. Margolus, P. Shor, T. Sleator, J. Smolin and H. Weinfurter, “Elementary gates for quantum computation”, *Phys. Rev. A* **52** (1995) 3457–3467;

5 I.L. Chuang and R. Laflamme, “Quantum Error Correction by Coding”, quant-ph/9511003; R. Laflamme, C. Miquel, J.P. Paz and W.H. Zurek, “Perfect Quantum Error Correction Code”, quant-ph/9602019; E. Knill and R. Laflamme, “A Theory of Quantum Error-Correcting Codes”, quant-ph/9604034; A.R. Calderbank and P.W. Shor, “Good quantum error correcting codes exist”, quant-ph/9512032; P.W. Shor and J.A. Smolin, “Quantum Error Correcting Codes Need Not Completely Reveal the Error Syndrome”, quant-ph/9604000; A.M. Steane, “Multiple Particle Interference and Quantum Error Correction”, quant-ph/9601029; “Simple Quantum Error Correcting Codes”, quant-ph/9605021

6 W. G. Unruh, “Maintaining coherence in quantum computers”, *Phys. Rev. A* **51** (1995) 992–997; P.W. Shor, “Scheme for reducing decoherence in quantum memory”, *Phys. Rev. A* **52** (1995) 2493; G. M. Palma, K.-A. Souminen and A. Ekert, “Quantum computers and dissipation”, *Proc. Roy. Soc. Lond. A* **452** (1996) 567–584; I. L. Chuang, R. Laflamme, P. Shor and W. H. Zurek, “Quantum computers, factoring and decoherence”, *Science* **270** (1995) 1633-1635; C. Miquel, J. P. Paz and R. Perazzo, “Factoring in a dissipative quantum computer”, preprint (1996) quant-ph/9601021; R.J. Hughes, D.F.V. James, E.H. Knill, R. Laflamme, A.G. Petschek, “Decoherence Bounds on Quantum Computation with Trapped Ions”, quant-ph/9604020; I.L. Chuang, R. Laflamme and J.P. Paz, “Effects of Loss and Decoherence on a Simple
Quantum Computer”, quant-ph/9602018

7 R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (New York: McGraw-Hill 1965)

8 R.P. Feynman, Caltech Physics X lectures (1981); W.D. Hillis, “New computer architectures and their relationship to physics or why computer science is no good”, *Int. J. Theor. Phys.* 21 (1982) 255–262; N. Margolus, “Parallel quantum computation”, in W. H. Zurek, ed., *Complexity, Entropy, and the Physics of Information*, proceedings of the SFI Workshop, Santa Fe, NM, 29 May–10 June 1989, *SFI Studies in the Sciences of Complexity VIII* (Redwood City, CA: Addison-Wesley 1990) 273–287; N. Margolus, “Physics-like models of computation”, *Physica D* 10 (1984) 81–95; R. Mainieri, “Design constraints for nanometer scale quantum computers”, preprint (1993) LA-UR 93-4333, cond-mat/9410109

9 M. Aizenman and B. Nachtergaele, “Geometric Aspects of Quantum Spin States”, *Comm. Math. Phys.* 164 (1994) 17–63; B. Nachtergaele, “Quasi-state decompositions for quantum spin systems”, *Prob. Theory and Math. Stat.*, Proc. Sixth Vilnius Conference, (eds. B. Grigelionis et al.) (Publishing Services Group, Vilnius, 1994)

10 See, for example, J. Cardy, “Critical Percolation in Finite Geometries”, *J. Phys.* A25 (1992) L201–L206

11 A recent discussion of the sum over paths can be found in A. Anderson, “The use of exp(iS[x]) in the sum over histories”, *Phys. Rev.* D49 (1994) 4049, and references therein
Figure Captions

Fig. 1 A space-time configuration $\omega$ for a general Hamiltonian

Fig. 2 A configuration $\omega$ for the one-dimensional Heisenberg spin-$\frac{1}{2}$ ferromagnet

Fig. 3 A configuration $\omega$ for the one-dimensional Heisenberg antiferromagnet, showing virtual loops
Figure 1
Figure 2
Figure 3