POSITIVE RATIONAL NODAL LEAVES ON SURFACES

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Abstract. We consider singular holomorphic foliations on compact complex surfaces with invariant rational nodal curve of positive self-intersection. Then, under some assumptions, we list all possible foliations.

1. Introduction

Let $X$ be a compact complex surface and $\mathcal{F}$ a codimension one singular holomorphic foliation on it. This work aims at generalizing the following result of Brunella (see [2] and [3]):

**Theorem 1.1.** Let $\mathcal{F}$ be a foliation on a compact complex surface $X$ and let $C \subset X$ be a rational curve with a node $p$, invariant by $\mathcal{F}$, and with $C^2 = 3$. Suppose that $p$ is a reduced nondegenerate singularity of $\mathcal{F}$, and that it is the unique singularity of $\mathcal{F}$ on $C$. Then $\mathcal{F}$ is unique up to birational transformations.

The unique foliation given by Theorem 1.1 will be called Brunella’s very special foliation (see subsection 3.1 for the definition).

But, what occurs if $C^2$ is an arbitrary positive integer? More specifically, we want to study/classify foliations on compact complex surfaces satisfying assumptions similar to the ones of Theorem 1.1 with the hypothesis $C^2 = 3$ replaced by $C^2 = n$, where $n$ is an arbitrary positive integer.

**Definition 1.2.** Let $\mathcal{F}$ be a foliation on a compact complex surface $X$. A link for $\mathcal{F}$ is a rational nodal curve $C \subset X$ with only one node $p \in C$ such that:

1. $C$ is positive, that is, $C^2 = n > 0$;
2. $C$ is $\mathcal{F}$-invariant;
3. $p$ is a reduced nondegenerate singularity of $\mathcal{F}$, and it is the unique singularity of $\mathcal{F}$ on $C$.

The existence of $C \subset X$, $C^2 = n > 0$, implies that $X$ is a projective surface (see [1], Theorem 6.2, page 160).

Our main purpose in this paper is to prove the following theorem:

**Theorem 1.3.** Let $\mathcal{F}$ be a foliation on a compact complex surface $X$ and let $C \subset X$ be a link for $\mathcal{F}$. Then we have only three possibilities, each one unique up to birational transformations:

1. $C^2 = 1$ and $\mathcal{F}$ is birational to a foliation $\mathcal{F}_1$ on $Bl_3(\mathbb{P}^2)/\alpha$, where $\alpha \in \text{Aut}(Bl_3(\mathbb{P}^2))$ and $Bl_3(\mathbb{P}^2)$ is a blow-up of $\mathbb{P}^2$ in three non-collinear points;
2. $C^2 = 2$ and $\mathcal{F}$ is birational to a foliation $\mathcal{F}_2$ on $\mathbb{P}^1 \times \mathbb{P}^1/\beta$, $\beta \in \text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$;

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(3) $C^2 = 3$ and $\mathcal{F}$ is birational to a foliation $\mathcal{F}_3$ on $\mathbb{P}^2/\gamma$ (Brunella’s very special foliation), $\gamma \in \text{Aut}(\mathbb{P}^2)$.

2. SOME RESULTS IN ALGEBRAIC AND COMPLEX GEOMETRY

For the reader’s convenience, we summarize here some classical fundamentals results which will be used along this paper.

2.1. Bimeromorphic geometry.

Definition 2.1 (Exceptional Curves). A compact, reduced, connected curve $C$ on a nonsingular surface $X$ is called exceptional, if there is a bimeromorphic map $\pi : X \to Y$ such that $C$ is exceptional for $\pi$, i.e., if there is an open neighbourhood $U$ of $C$ in $X$, a point $y \in Y$, and a neighbourhood $V$ of $y$ in $Y$, such that $\pi$ maps $U \setminus C$ biholomorphically onto $V \setminus \{y\}$, whereas $\pi(C) = y$. We shall express this situation also by saying that $C$ is contracted to $y$.

Theorem 2.2 (Grauert’s criterion, [1], page 91). A reduced, compact connected curve $C$ with irreducible components $C_i$ on a smooth surface is exceptional if and only if the intersection matrix $(C_i \cdot C_j)$ is negative definite.

Definition 2.3 (Exceptional curves of the first kind). These are nonsingular rational curves with self-intersection $-1$. Frequently we call such curves $(-1)$-curves. A very useful characterisation of $(-1)$-curves is given by

Theorem 2.4 ([1], page 97). Let $X$ be a nonsingular surface, $E \subset X$ a $(-1)$-curve and $\pi : X \to Y$ the map contracting $E$. Then $y = \pi(E)$ is nonsingular on $Y$.

Theorem 2.5 (Uniqueness of the $\sigma$-process, [1], page 98). Let $X$ and $Y$ be smooth surfaces and $\pi : X \to Y$ a bimeromorphic map. If $E = \pi^{-1}(y)$ is an irreducible curve, then near $E$, the map $\pi$ is equivalent to the $\sigma$-process with centre $y$.

Lemma 2.6 (Factorization lemma, [1], page 98). Let $\pi : X \to Y$ be a bimeromorphic map with $X$, $Y$ nonsingular surfaces. Unless it is an isomorphism, there is a factorization $\pi = \pi' \circ \sigma$, where $\sigma : X \to X$ is a $\sigma$-process.

Corollary 2.7 (Decomposition of bimeromorphic maps, [1], page 98). Let $X$, $Y$ be non-singular and $\pi : X \to Y$ a bimeromorphic map. Then $\pi$ is equivalent to a succession of $\sigma$-transforms, which locally (with respect to $Y$) are finite in number.

Theorem 2.8 ([1], page 192). Let $X$ be a compact surface and $C$ a smooth rational curve on $X$. If $C^2 = 0$, then there exists a modification $\pi : X \to Y$, where $Y$ is ruled, such that $C$ meets no exceptional curve of $\pi$, and $\pi(C)$ is a fibre of $\pi$.

2.2. Complex geometry.

Lemma 2.9 ([16], Lemma 5). Let $X$ be a compact complex manifold of dimension $n > 1$, $K$ a compact subset of $X$ and $E$ a holomorphic vector bundle over $X$. If $X$ is strongly pseudoconvex, then every section $s$ of $E$ over $X - K$ can be extended to a meromorphic section $\tilde{s}$ over all of $X$.

Lemma 2.10 ([11], page 32). Let $X$ be a compact complex surface and $C \subset X$ a compact irreducible curve. If $C^2 > 0$ then $X - C$ is strongly pseudoconvex.
3. Existence

For us a cycle of smooth rational curves (or simple a cycle) always means the union of a finite number of smooth rational curves in general position $C_i$, $i = 1, \ldots, m$, $m > 1$, such that: if $m = 2$, then $\#C_1 \cap C_2 = 2$; if $m > 2$, then $\#C_i \cap C_{i+1} = \#C_1 \cap C_m = 1$, $i = 1, \ldots, m - 1$, otherwise $\#C_i \cap C_j = 0$.

3.1. Existence for $C^2 = 3$ (Brunella’s very special foliation). Let $L$ be the linear foliation on $\mathbb{P}^2$ given in affine coordinates by the linear 1-form

$$\omega = \frac{1 \pm \sqrt{-3}}{2} ydx - xdy. $$

This foliation has an invariant cycle of three lines $C_1 \cup C_2 \cup C_3$. Moreover, the foliation $L$ is $\gamma$-invariant, where $\gamma: (s: t: u) \mapsto (u: s: t)$ is in $\text{Aut}(\mathbb{P}^2)$.

The quotient foliation $F_3 = L/\gamma$ obtained by taking the quotient of $(\mathbb{P}^2, L)$ by the group generated by $\gamma$ is, by definition, Brunella’s very special foliation.

Note that the choice of $\lambda$ don’t affect the birational class of $F_3$, since the involution $(x, y) \mapsto (y, x)$ conjugates the two possible constructions.

3.2. Existence for $C^2 = 2$. We take the foliation $M$ on $\mathbb{P}^1 \times \mathbb{P}^1$ given in affine coordinates $(x, y)$ by the linear 1-form

$$\omega = \lambda ydx - xdy = \pm \sqrt{-1} ydx - xdy.$$ 

where $\lambda = \pm \sqrt{-1}$. Then it leaves invariant the cycle of four lines

$$(\mathbb{P}^1 \times \{0\}) \cup (\mathbb{P}^1 \times \{\infty\}) \cup (\{0\} \times \mathbb{P}^1) \cup (\{\infty\} \times \mathbb{P}^1),$$

in which the only singularities are the crossing points, each one reduced nondegenerate. The automorphism of order 4

$$\beta: (u: v, z: w) \mapsto (z: w, u: v).$$

is such that, in affine coordinates $(x, y)$, $\beta(x, y) = (y, \frac{1}{x})$ and

$$\beta^* \omega = \beta^*(\lambda ydx - xdy) = \frac{1}{x} dy - y(-\frac{1}{x^2})dx,$$

hence, since $\lambda = \pm \sqrt{-1}$,

$$\omega \wedge \beta^* \omega = (\lambda ydx - xdy) \wedge (\lambda \frac{1}{x} dy + \frac{y}{x^2} dx) = (\lambda^2 + 1) \frac{y}{x} dx \wedge dy = 0.$$

Note that $\beta$ permutes cyclically the cycle of four lines

$$(\mathbb{P}^1 \times \{0\}) \cup (\mathbb{P}^1 \times \{\infty\}) \cup (\{0\} \times \mathbb{P}^1) \cup (\{\infty\} \times \mathbb{P}^1).$$

Then the quotient foliation $F_2$ obtained by taking the quotient of $(\mathbb{P}^1 \times \mathbb{P}^1, M)$ by the group generated by $\beta$ is the desired foliation, that is, $F_2$ has a link of self-intersection 2.

Again the choose of $\lambda$ don’t affect the birational class of $F_2$, since the involution $(u: v, z: w) \mapsto (z: w, u: v)$ conjugates the two possible constructions.
3.3. Existence for $C^2 = 1$. Let $\mathcal{L}$ and $\gamma$ as in subsection 3.1. Recall that $\mathcal{L}$ has a cycle of three invariant lines $C_1 \cup C_2 \cup C_3$, where $C_i = \{(z_1 : z_2 : z_3) \in \mathbb{P}^2 | z_i = 0\}$, $i = 1, 2, 3$. Consider the standard Cremona transformation $f : \mathbb{P}^2 \to \mathbb{P}^2$, $f([z_1 : z_2 : z_3]) = [z_2 z_3 : z_1 z_3 : z_1 z_2]$. Note that $\mathcal{L}$ is $f$-invariant.

If we blow-up the crossing points of the cycle of three $\mathcal{L}$-invariant projective lines $C_1 \cup C_2 \cup C_3$, we obtain a birational morphism $\pi_3 : Bl_3(\mathbb{P}^2) \to \mathbb{P}^2$ and a foliation $\mathcal{N} = \pi_3^* \mathcal{L}$ with an invariant cycle of six smooth rational (-1)-curves, say $\tilde{C}_1 \cup \tilde{C}_2 \cup \tilde{C}_3 \cup \tilde{C}_4 \cup \tilde{C}_5 \cup \tilde{C}_6$, in which the singularities of $\mathcal{N}$ are only the crossing points (and they are reduced nondegenerate). Note that $\alpha = \pi_3^{-1} \circ f \circ \pi_3 : Bl_3(\mathbb{P}^2) \to Bl_3(\mathbb{P}^2)$ becomes an automorphism of order six that preserves the foliation and permutes cyclically the cycle of six invariant rational curves.

The quotient foliation $\mathcal{F}_1 = \mathcal{N}/\alpha$ has a link of self-intersection 1, hence $\mathcal{F}_1$ is the desired foliation.

4. Riccati Foliations

We develop here the first tools to prove our main result.

Let $\mathcal{F}$ be a foliation on $X$ which is Riccati with respect to a fibration $\pi : X \to B$, where $B$ is a nonsingular curve. If $R$ is a regular fibre of $\pi$ which is $\mathcal{F}$-invariant, then ([2] Chapter 4): there are at most two singularities on $R$ and there exists coordinates $(x, y) \in D \times \mathbb{P}^1$ around $R$, where $D$ is a disc, such that the foliation is given by the 1-form

$$
\omega = (a(x)y^2 + b(x)y + c(x))dx + h(x)dy.
$$

Let $q$ be a singularity for $\omega$. After a change in the $y$ coordinate, we can suppose $q = (0, 0)$. Writing $h(x) = h_k x^k + \ldots$, where $k > 0$ and $h_k \neq 0$, we define the multiplicity of the fibre $R$ as $l(\mathcal{F}, R) = k$. We want to prove the following property of $\mathcal{F}$:

Lemma 4.1. The exceptional divisor of the reduction of singularities of $\mathcal{F}$ at $q = (0, 0)$ is a chain of rational curves $L_1, \ldots, L_n$ such that there is at most one non-invariant component, and if $L_i$ is such component then

$$
L_i \cap L_j \neq \emptyset \Rightarrow Sing(\tilde{\mathcal{F}}) \cap L_j = 1 - \delta_{ij}
$$

where $\tilde{\mathcal{F}}$ is the reduced foliation and $\delta_{ij}$ is the Kronecker’s delta, that is, $\delta_{ii} = 1$ and $\delta_{ij} = 0$ if $i \neq j$.

Proof. If the linear part of $\omega$ at $q$ is non trivial, the result can be checked directly. We then suppose that the linear part at $q$ is trivial. Then $b(0) = c(0) = c'(0) = 0$ and $l(\mathcal{F}, R) = k > 1$. Since $Sing(\omega) \subset Sing(\mathcal{F})$ has codimension two, we have $a(0) \neq 0$. Therefore $\omega$ has algebraic multiplicity two at $q$. Since $b(0)^2 - 4a(0)c(0) = 0$, $q$ is the unique singularity of $\mathcal{F}$ in $R$. The blow-up at $q$ has on $R' \cap E' \ (E' \text{ is the exceptional divisor and } R' \text{ is the strict transform of } R)$ a singularity of the type $d(xy) = 0$ and no more singularities on $R'$. If we collapse $R'$, then $E'$ becomes a new fibre $R_1$ of a new Riccati foliation $\mathcal{F}_1$. In this way, there may be at most two singularities on $R_1$, but now $l(\mathcal{F}_1, R_1) < l(\mathcal{F}, R) = k$.

Applying this procedure (flipping of fibre) a finite number of times, we obtain a foliation $\mathcal{F}_m$ and an invariant fibre $R_m$ such that a generating 1-form for the foliation has algebraic multiplicity one. That is, if $\omega$ is that 1-form, then

$$
\omega_m = (a_m(x)y^2 + b_m(x)y + c_m(x))dx + h_m(x)dy.
$$
with \( c_m(0) = h_m(0) = 0 \), but \( b_m(0) \neq 0 \) or \( c'_m(0) \neq 0 \) or \( h'_m(0) \neq 0 \). Now, if the singularity \( (0,0) \) is dicritical, then the generating vector field for the foliation has two non zero linearly independent eigenvectors, and the exceptional divisor of the reduction of singularities \( \mathcal{F}_m \) at \((0,0)\) is a chain of rational curves \( L_1, \ldots, L_n \), such that if \( L_i \) is the (unique) non-invariant component and \( L_i \cap L_j \neq \emptyset \) then \( \text{Sing}(\mathcal{F}_m) \cap L_j = 1 - \delta_{ij} \). Since we can come back by blow-ups at points not equal to the \((0,0)\) point of \( \mathcal{F}_m \) to the blow-up of the original foliation at the original singular point \( q = (0,0) \), the property is also true for the reduction at \( q \) and then we conclude the proof.

\[ \square \]

**Proposition 4.2.** Let \( \mathcal{F} \) be a foliation on a compact complex surface \( X \). Let \( C = C_1 \cup \ldots \cup C_n \) be a cycle of \( n \) invariant smooth rational curves, where \( n > 1 \). Suppose that \( C \cap \text{Sing}(\mathcal{F}) = \bigcup_{i \neq j} C_i \cap C_j \) are reduced non-degenerate singularities of \( \mathcal{F} \). If \( \mathcal{F} \) is Riccati with respect to a rational fibration \( \pi : X \to B \), then every fibre of \( \pi \) through a point of \( C \cap \text{Sing}(\mathcal{F}) \) is completely supported on \( C \).

**Proof.** Let \( p \in C \cap \text{Sing}(\mathcal{F}) \). If \( R = \pi^{-1}(\pi(p)) \) is the fibre through \( p \), we can write

\[ R = C_{i_1} \cup \ldots \cup C_{i_k} \cup E_1 \cup \ldots \cup E_l \]

where \( i_1, \ldots, i_k \in \{1, \ldots, n\} \) and \( E_1, \ldots, E_l \) are smooth rational curves not in \( \{C_1, \ldots, C_n\} \), and, by Theorem \[2, \text{N} \] (see \[1 \], page 192), there is a birational transformation

\[ \sigma = \sigma_m \circ \ldots \circ \sigma_1 : X \to Y \]

where each \( \sigma_i, \ i = 1, \ldots, m \), is a blow-up at a point \( p_i \), such that \( S = \sigma(R) \) is a regular fibre for the fibration \( \rho = \pi \circ \sigma^{-1}(\sigma \text{ is contraction of components of } R) \).

Note that if we blow-up a regular point of a foliation, the exceptional divisor is invariant, with only one singularity on it, of type \( xdy + ydx \). Therefore if \( p_i \) is a regular point for the induced foliation \((\sigma_m \circ \ldots \circ \sigma_1)_* \mathcal{F} \), then \((\sigma_m \circ \ldots \circ \sigma_1)^{-1}(p_i) = D_1 \cup \ldots \cup D_r \) is \( \mathcal{F} \)-invariant.

If \( \#D_1 \cap (D_1 \cup \ldots \cup D_l \cup \ldots \cup D_r) = \#D_1 \cap \text{Sing}(\mathcal{F}) = 1 \). Now, if \( C \cap (\sigma_m \circ \ldots \circ \sigma_1)^{-1}(p_i) \neq \emptyset \), then, since \((\sigma_m \circ \ldots \circ \sigma_1)^{-1}(p_i) \) is connected and \( \mathcal{F} \)-invariant, we conclude that \((\sigma_m \circ \ldots \circ \sigma_1)^{-1}(p_i) \subset C \), hence \( D_l = C_{i_l} \), which result the contradiction \( 1 = \#D_k \cap \text{Sing}(\mathcal{F}) = \#C_i \cap \text{Sing}(\mathcal{F}) = 2 \). Then, if we contract \((\sigma_m \circ \ldots \circ \sigma_1)^{-1}(p_i) \), we don’t affect the cycle \( C \).

So we can look at \( \sigma \) as a reduction of singularities of \( \sigma_*(\mathcal{F}) \) in \( S \) and use Lemma \[4, \text{L} \] to conclude: if \( p \in C_i \cap C_j \) then \( C_i \) or \( C_j \) is a component of \( R \), otherwise we will have a non-invariant component of \( R \) with singularity.

If the set \( \{E_1, \ldots, E_l\} \) is not empty, since \( R \) is connected, there exist \( C_i \) and \( E_j \) components of \( R \) such that \( C_i \cap E_j \neq \emptyset \). Then \( E_j \) is not \( \mathcal{F} \)-invariant. But \( C_i \) has two singularities, then by Lemma \[4, \text{L} \] \( C_i \) cannot intersect \( E_j \). Then we have \( \{E_1, \ldots, E_l\} = \emptyset \).

\[ \square \]

**Definition 4.3.** Let \( \mathcal{F} \) be a foliation on a compact complex surface \( X \). A \((k,l)\)-cycle for \( \mathcal{F} \) is a cycle of \( k > 1 \) smooth rational curves \( C = C_1 \cup \ldots \cup C_k \subset X \) such that:

1. \( C^2 = n > 0 \);
2. \( C_i^2 = l, i = 1, \ldots, n \);
3. \( C \) is \( \mathcal{F} \)-invariant;
(4) \( C \cap \text{Sing}(\mathcal{F}) = \bigcup_{j \neq j} C_i \cap C_j \) are reduced nondegenerate singularities of \( \mathcal{F} \).

**Corollary 4.4.** Let \( \mathcal{F} \) be a foliation on a compact complex surface \( X \) and let \( C = C_1 \cup \ldots \cup C_k \subset X \) be a \((k,l)\)-cycle for \( \mathcal{F} \). Then \((k,l) \in \{(2,-1),(3,-1),(3,1),(6,-1)\} \cup \{(2m,0) \mid m \in \mathbb{N}\}\).

**Proof.** The proof is just an easy application of Proposition 4.2 using suitable blow-ups at the crossing points of the cycle or blow-downs of exceptional curves.

Let \( C = C_1 \cup \ldots \cup C_k \subset X \) be a \((k,l)\)-cycle for a foliation \( \mathcal{F} \) on \( X \). We can suppose that \( C_i \cap C_{i+1} = \{p_i\}, i = 1, \ldots, k-1 \), and \( C_k \cap C_1 = \{p_k\} \), where the \( k \) points \( p_1, \ldots, p_k \) are distinct.

If \( l > 0 \), choose \( z \in C \) a crossing point. After a suitable sequence of \( l \) blow-ups beginning at \( z \), we obtain a new cycle of rational curves

\[
\tilde{C} = E_l \cup \ldots \cup E_1 \cup D_1 \cup D_2 \cup \ldots \cup D_k
\]

where \( D_1^2 = 0, E_l^2 = -1, E_{l-1}^2 = -2, \ldots, E_2^2 = -2, D_2^2 = l, D_3^2 = l, \ldots, D_k^2 = l-1 \), and \( D_1 \cap E_1 = \{p\} \). Then, the foliation \( \mathcal{F} \) is Riccati with respect to a rational fibration that has \( D_1 \) as a regular fibre.

By Proposition 4.2, a fibre \( R \) through a point not in \( D_1 \) must be supported on \( \tilde{C} \), and such a fibre must be also disjoint from \( D_1 \), since \( D_1 \) is a fibre. That is, we must have \( R \subset \tilde{C} - D_1 \subset E_l \cup \ldots \cup E_1 \cup D_2 \cup \ldots \cup D_k \). Since \( D_1 \cap E_1 \neq \emptyset \) and \( D_1 \cap D_2 \neq \emptyset \), \( R \subset \tilde{C} - (D_1 \cup E_1 \cup D_2) \subset E_l \cup \ldots \cup E_2 \cup D_3 \cup \ldots \cup D_k \). If \( k = 2 \) and \( l = 1 \), then, in fact, \( R \subset \tilde{C} - (D_1 \cup E_1 \cup D_2) = \emptyset \), and we obtain a contradiction, since \( R \) cannot be empty. For \( k > 2 \) or \( l > 1 \), every connected curve supported on \( E_l \cup \ldots \cup E_2 \cup D_3 \cup \ldots \cup D_k \) cannot be contracted to a rational curve of zero self-intersection, hence cannot be a fibre of a rational fibration. Therefore, there is no \((k,l)\)-cycle if \( l > 0 \).

Now, suppose \( l = 0 \). Then, since \( C_i^2 = 0 \), \( i = 1, \ldots, k \), we don’t need take blow-ups to produce rational fibrations. Just choose, for example, \( C_1 \) as the fibre \( R_1 \) of a rational fibration and \( \mathcal{F} \) Riccati with respect to this fibration. Suppose that \( k = 2m+1 \) is odd. Take the fibre \( R_2 \) through the crossing point \( p_3 \). Since \( R_2 \) must be supported on \( C \), we obtain \( R_2 = C_3 \). By the same reason, the fibre \( R_3 \) through the crossing point \( p_5 \) is \( R_3 = C_5 \). Inductively, we obtain that the fibre \( R_i \) through \( p_{2i-1} \) is \( R_i = C_{2i-1} \). Then \( R_{m+1} = C_{2m+1} = C_k \) is the fibre through \( p_{2m+1} = p_k \), which is impossible since the fibre through \( p_k = p_{2m+1} \) is just \( C_1 \neq C_k \). Hence, if \( l = 0 \), then \( k \) must be even.

Finally, using contractions instead of blow-ups, we can conclude that there is no \((k,-1)\)-cycle if \((k,-1)\) is not in \( \{(2,-1),(3,-1),(6,-1)\}\).

We can now give here a different proof of [22 Chapter 3, Proposition 4].

**Proposition 4.5.** Let \( \mathcal{F} \) be one of the foliations \( \mathcal{F}_1, \mathcal{F}_2 \) or \( \mathcal{F}_3 \). Then \( \mathcal{F} \) is not birational to a Riccati foliation.

**Proof.** Just like before, after one blow-up at the nodal point in the link of \( \mathcal{F} \), we conclude, by Proposition 4.2 that \( \mathcal{F} \) cannot be Riccati.

\[ \square \]

5. **Proof of the Theorem** [13]

5.1. **Preliminary computations.** Let \( p \) be the node of \( C \) and \( C^2 = n \) a positive integer. If the hypotheses for the foliation are as in the Introduction (that is, \( C \)
is a link for $F$), we can use the Camacho-Sad formula to calculate the quotient of eigenvalues of $F$ at $p$ (see [2] Chapter 3):

$$n = C^2 = CS(F, Y, p) = \lambda + \frac{1}{\lambda},$$

Then we have the equation

$$\lambda^2 + (2 - n)\lambda + 1 = 0$$

whose solution is

$$\lambda = \frac{n - 2 \pm \sqrt{n(n - 4)}}{2}.$$ 

Therefore:

1. if $C^2 = 1$ then $-\lambda$ is a 6th primitive root of unit;
2. if $C^2 = 2$ then $-\lambda$ is a 4th primitive root of unit;
3. if $C^2 = 3$ then $-\lambda$ is a 3rd primitive root of unit;
4. if $C^2 = 4$ then $\lambda = 1$;
5. if $C^2 > 4$ then $\lambda$ is a positive irrational number.

### 5.2. Basic lemmas and propositions.

Here we will develop some more "technology" for the proof of our main result.

The next lemma is the generalization of [2, Chapter 3, Lemma 1]. The proof is essentially the same.

**Lemma 5.1.** Let $F$ be a foliation on a compact complex surface $X$ and let $C \subset X$ be a link for $F$ with node $p \in C$. Let $L = N_F^* \otimes \mathcal{O}_X(C)$ and $\lambda$ be the quotient of eigenvalues at $p$. Suppose that $-\lambda$ is a $k$th primitive root of unit, $k \geq 2$. Then there exists a neighbourhood $U$ of $C$ such that $L^{\otimes k}|_U = \mathcal{O}_U$.

**Proof.** Since $\lambda$ is non-real, given a point $q \in C - \{p\}$ and a transversal $T$ to $F$ at $q$, the corresponding holonomy group of $F$, $\text{Hol}_F \subset \text{Diff}(T, q)$, is infinite cyclic, generated by an hyperbolic diffeomorphism with linear part $\exp(2\pi i \lambda)$ ([4] or [10]). Hence, there exists on $T$ a $\text{Hol}_F$-linearising coordinate $z$, $z(q) = 0$. We extend this coordinate to a full neighbourhood of $q$ in $X$, constantly on the local leaves of $F$. The logarithmic 1-form $\eta_q = \frac{dz}{z}$ defines $F$, is closed, and $\eta_q|_T$ is $\text{Hol}_F$-invariant.

By the Poincaré linearisation theorem, in a neighbourhood of $p$ the foliation is defined by a closed logarithmic 1-form $\eta_p = \frac{dz}{z} - \lambda \frac{dw}{w}$ ([4] or [10]). If $q$ is close to $p$, then $\eta_p|_T$ is $\text{Hol}_F$-invariant.

We obtain a neighbourhood $U$ of $C$ by the union of the open sets $U_j$, such that in each $U_j$ the foliation is defined by a logarithmic 1-form $\eta_j$, with poles on $C$, which is closed and $\text{Hol}_F$-invariant at the transversals. On $U_i \cap U_j$ we have $\eta_i = f_{ij} \eta_j$, $f_{ij} \in \mathcal{O}^*$. The closedness of $\eta_i$ and $\eta_j$ implies that $df_{ij} \wedge \eta_j = 0$, then $f_{ij}$ is constant along the local leaves of $F$. Moreover, $f_{ij}|_T$ is $\text{Hol}_F$-invariant and hence constant because the holonomy is hyperbolic.

Thinking $\eta_j$ as local sections of $L = N^*_F \otimes \mathcal{O}_X(C)$, then the previous property shows that $L|_U$ is defined by a locally constant cocycle. Hence, to show that $L^{\otimes k}|_U = \mathcal{O}_U$ it is sufficient to show that $L^{\otimes k}|_C = \mathcal{O}_C$. We can now use the residue of $\eta_j$ along $C$ to calculate the cocycle. For $\eta_q$ with $q \in C - \{p\}$ we can choose the 1-form to produce any non-zero residue. But we have a restriction around $p:$
the residue of $\eta_p$ on one separatrix is $-\lambda$ times the residue on the other separatrix. Since $(-\lambda)^k = 1$, its is clear that $L^{\otimes k}|_C = O_C$.

Also the next proposition is an easy adaptation of Brunella’s argument in [2] Chapter 3, page 61-62.

**Proposition 5.2.** Let $\mathcal{F}$ be a foliation on a compact complex surface $X$ and let $C \subset X$ be a link for $\mathcal{F}$ with node $p \in C$. Let $\lambda$ be the quotient of eigenvalues at $p$. Suppose that $-\lambda$ is a $k^{th}$ primitive root of unit, $k > 2$. Then there exists a compact surface $Z$, a transformation $f : Z \to X$, a neighbourhood $U$ of $C$ and an open set $V \subset Z$ such that $f|_V : V \to U$ is a regular $k$-covering over $U$. Moreover, $f|_V^{-1}(C)$ is a cycle of $k$ smooth rational curves, each one with self-intersection $|_C$, a transformation $f$ is conjugated to $(k, C^2 - 2)$-cycle, and the deck transformations of $f|_V$ permutes cyclically the curves in the cycle.

**Proof.** By the above lemma, the line bundle $L^{\otimes k}$ has a nontrivial section over $U$ without zeroes. Since $C^2 > 0$, the open set $X - C$ is strictly pseudoconvex by Lemma [2.11]. Then, by Lemma [2.9], that section can be extended to the full $X$ as a global meromorphic section $s$ of $L^{\otimes k}$. Consider $E(L^{\otimes k})$ the compactification of the total space of $L^{\otimes k}$. Let $\tilde{s}$ the compactification of the graph of $s$ in $E(L^{\otimes k})$. Let $\tau : E(L) \to E(L^{\otimes k})$ be the map defined by the $k^{th}$ tensor power.

Let $Z$ be the desingularisation of $\tau^{-1}(\tilde{s})$ and elimination of indeterminacies of the projection $\tau^{-1}(\tilde{s}) \dashrightarrow X$. Take $f : Z \to X$ the induced projection.

**Lemma 5.3.** Let $p_1 = (1 : 0 : 0), p_2 = (0 : 1 : 0), p_3 = (0 : 0 : 1)$ be three non collinear points in $\mathbb{P}^2$. Let $\gamma \in \text{Aut}(\mathbb{P}^2)$ given by $\gamma(z_1 : z_2 : z_3) = (z_3 : z_1 : z_2)$. If $J \in \text{Aut}(\mathbb{P}^2)$ is another automorphism such that $J(p_1) = p_2, J(p_2) = p_3$ and $J(p_3) = p_1$, then $J$ is conjugated to $\gamma$, that is, there is $g \in \text{Aut}(\mathbb{P}^2)$ such that $\gamma = g \circ J \circ g^{-1}$.

**Proof.** In homogeneous coordinates, $J(z_1 : z_2 : z_3) = (xz_3 : yz_1 : zz_2)$, where $xyz \neq 0$. Note that we can suppose $xyz = 1$. Since $\text{Aut}(\mathbb{P}^2) = \text{PGL}(3, \mathbb{C})$, writing $J$ and $\gamma$ as matrices, $J = \begin{pmatrix} 0 & 0 & x \\ y & 0 & 0 \\ 0 & z & 0 \end{pmatrix}$ and $\gamma = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, we need to show that there is a matrix $A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \in \text{GL}(3, \mathbb{C})$, such that $AJ = \gamma A$ in $\text{PGL}(3, \mathbb{C})$.

If $a = (a_1, a_2, a_3), b = (b_1, b_2, b_3), c = (c_1, c_2, c_3)$, it’s easy to see that the equality $AJ = \gamma A$ is equivalent to $x\gamma(c) = a, y\gamma(a) = b, z\gamma(b) = c$. Take $a \neq 0$ and define $b = y\gamma(a)$ and $c = z\gamma(b) = yz\gamma^2(a)$. Then the matrix $A = (a, b, c) \in \text{GL}(3, \mathbb{C})$ is a solution.

**Proposition 5.4.** Let $\mathcal{F}$ be a foliation on a compact complex surface $Z$ and let $C_1 \cup C_2 \cup C_3 \subset Z$ be a $(3, 1)$-cycle for $\mathcal{F}$. Suppose that there exists a birational $\mathcal{F}$-automorphism $\phi : Z \dashrightarrow Z$ of order three permuting cyclically the rational curves. Then $\mathcal{F}$ is birational to the linear foliation $\mathcal{L}$ on $\mathbb{P}^2$ from subsection [3.1] and the quotient foliation $\mathcal{F}/\phi$ is birational to $\mathcal{F}_3 = \mathcal{L}/\gamma$. 
Proof. We can suppose $\phi(C_1) = C_2$, $\phi(C_2) = C_3$ and $\phi(C_3) = C_1$. Take, for each $i$, a section $s_i$ of $\mathcal{O}_Z(C_i)$ vanishing on $C_i$. Since $C_1$, $C_2$, $C_3$ are linearly equivalent, we can define a rational map

$$(s_1 : s_2 : s_3) : Z \dashrightarrow \mathbb{P}^2.$$ 

It’s easy to see that this map is birational and biregular in a neighbourhood of the cycle $C_1 \cup C_2 \cup C_3$, whose image is a cycle of three lines in $\mathbb{P}^2$. The induced foliation $\tilde{F}$ on $\mathbb{P}^2$ is linear because the degree of the foliation is 1. The birational automorphism $\phi$ is mapped to a birational automorphism $\tilde{\phi}$ of $\mathbb{P}^2$ which is birational in a neighbourhood of the three lines and hence everywhere; moreover these automorphism permutes cyclically the three lines. By Lemma 5.5 $\tilde{\phi}$ is conjugated to the automorphism $\gamma(z_1 : z_2 : z_3) = (z_3 : z_1 : z_2)$, that is, there is $g \in \text{Aut}(\mathbb{P}^2)$ such that $\gamma = g \circ \tilde{\phi} \circ g^{-1}$. Since $\gamma$ is an $g, \tilde{F}$-automorphism, an easy computation shows that $g_* \tilde{F} = \mathcal{L}$ in homogeneous coordinates $[z_1 : z_2 : z_3]$. In particular, $\tilde{F}/\phi$ is birational to $F_3 = \mathcal{L}/\gamma$. 

Analogously we can prove the following two results.

**Lemma 5.5.** Let $p_1 = (1 : 0 : 1 : 0)$, $p_2 = (0 : 1, 1 : 0)$, $p_3 = (0 : 1, 0 : 1)$, $p_4 = (1 : 0, 0 : 1)$ be four points in $\mathbb{P}^1 \times \mathbb{P}^1$. Let $\beta \in \text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ given by $\beta(z_1 : z_2, z_3 : z_4) = (z_4 : z_3, z_1 : z_2)$. If $J \in \text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ is another automorphism such that $J(p_1) = p_2$, $J(p_2) = p_3$, $J(p_3) = p_4$ and $J(p_4) = p_1$, then $J$ is conjugated to $\beta$, that is, there is $g \in \text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ such that $\beta = g \circ J \circ g^{-1}$. 

**Proposition 5.6.** Let $\mathcal{H}$ be a foliation on a compact complex surface $W$ and let $D_1 \cup D_2 \cup D_3 \cup D_4 \subset W$ be a $(4,0)$-cycle for $\mathcal{H}$. Suppose that there exists a birational $\mathcal{H}$-automorphism $\phi : W \dashrightarrow W$ of order four permuting cyclically the rational curves. Then $\mathcal{H}$ is birational to the linear foliation $\mathcal{M}$ on $\mathbb{P}^1 \times \mathbb{P}^1$ from subsection 2.2 and the quotient foliation $W/\phi$ is birational to $F_3 = \mathcal{M}/\beta$.

**Proof.** Take, for every $i$, a section $s_i$ of $\mathcal{O}_Z(D_i)$ vanishing on $D_i$. We define a rational map

$$(s_1 : s_2, s_3 : s_4) : W \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1.$$ 

It’s easy to see that this map is birational and biregular in a neighbourhood of the cycle $D_1 \cup D_2 \cup D_3 \cup D_4$, whose image is a cycle of four lines in $\mathbb{P}^1 \times \mathbb{P}^1$. Therefore, the induced foliation $\mathcal{H}$ on $\mathbb{P}^1 \times \mathbb{P}^1$ leaves invariant the cycle of four lines $(\mathbb{P}^1 \times \{0\}) \cup (\mathbb{P}^1 \times \{\infty\}) \cup (\{0\} \times \mathbb{P}^1) \cup (\{\infty\} \times \mathbb{P}^1)$ whose singularities on the cycle are only the crossing points, each one reduced nondegenerate. According to [2, Chapter 4, Proposition 1] (see also [8] and [9]) we have that this foliation on $\mathbb{P}^1 \times \mathbb{P}^1$ is given in affine coordinates $(x, y)$ by a linear 1-form

$$\omega = \lambda y dx - x dy.$$ 

The birational automorphism $\phi$ is mapped to a birational automorphism $\tilde{\phi}$ of $\mathbb{P}^1 \times \mathbb{P}^1$ which is birational in a neighbourhood of the four lines and hence everywhere; moreover these automorphism permutes cyclically the four lines. By Lemma 5.5 $\tilde{\phi}$ is conjugated to the automorphism $\beta(z_1 : z_2, z_3 : z_4) = (z_3 : z_4, z_2 : z_1)$, that is, there is $g \in \text{Aut}(\mathbb{P}^2)$ such that $\beta = g \circ \tilde{\phi} \circ g^{-1}$. Since $\beta$ is an $g, \tilde{F}$-automorphism, an easy
computation shows that $g_\ast \mathcal{H} = \mathcal{M}$ in homogeneous coordinates $[z_1 : z_2 : z_3 : z_4]$. In particular, $\mathcal{H}/\phi$ is birational to $\mathcal{F}_2 = \mathcal{M}/\beta$.

Now we are ready to finish the proof of the theorem.

5.3. **Self-intersection 1.** Since $-\lambda$ is a $6^{th}$ primitive root of unit, by Proposition 5.2 we obtain a covering $F : Z \to X$, regular and of order six in a neighbourhood $U$ of $C$. The deck transformations over $U$ extend, by construction, to birational transformations of $Z$. Let $\alpha : Z \to Z$ be the extended deck transformation of order six.

Now, we lift $\mathcal{F}$ to $Z$ via $F$, obtaining a new foliation $\mathcal{G}$ which leaves invariant six smooth rational curves $C_i, i = 1, \ldots, 6$, forming a cycle over $C$. We have $C_i^2 = -1$, because $C^2 = 1$. The only singularities of $\mathcal{G}$ at the cycle are the six crossing points, all reduced nondegenerate as well as $p$.

We can contract three disjoint $(-1)$-curves of the cycle, say $C_1, C_3$ and $C_5$, obtaining a foliation $(\tilde{\mathcal{G}}, \tilde{Z})$ birational to $(\mathcal{G}, Z)$. Note that $\tilde{\mathcal{G}}$ has an invariant cycle of three smooth rational curves with self-intersection 1. Furthermore, $\alpha^2 = \alpha \circ \alpha$ induces a birational $\tilde{\mathcal{G}}$-automorphism that permutes cyclically this cycle. Therefore, by Proposition 5.4, $\tilde{\mathcal{G}}$ is birational to the linear foliation $\mathcal{L}$ on $\mathbb{P}^2$ given in subsection 3.1. In the same way, contracting the three disjoint $(-1)$-curves $C_2, C_4$ and $C_6$, we also obtain a foliation birational to $(\mathcal{L}, \mathbb{P}^2)$. Then $\alpha : Z \to Z$ induces a $\mathcal{L}$-automorphism $\tilde{\alpha} : \mathbb{P}^2 \to \mathbb{P}^2$. Since $\tilde{\alpha}$ is unique up to conjugation (Lemma 5.1), the same is true for $\alpha$. Therefore $\mathcal{F}$ is birational to the foliation $\mathcal{F}_1$ from subsection 3.3.

5.4. **Self-intersection 2.** In this case, $-\lambda$ is a $4^{th}$ primitive root of unit. By Lemma 5.2 we have a covering $G : W \to X$, which is regular and of order 4 on a neighbourhood of $C$. Lifting $\mathcal{F}$ to $W$, we obtain a foliation $\mathcal{H}$ which leaves invariant four smooth rational curves $D_i, i = 1, \ldots, 4$, forming a cycle over $C$. Analogously, $D_i^2 = 0$, because $C^2 = 1$. The only singularities of $\mathcal{H}$ at the cycle are the four crossing points, all reduced nondegenerate as well as $p$. Hence Proposition 5.6 implies that $\mathcal{F}$ is birational to $\mathcal{F}_2$.

5.5. **Self-intersection 3.** This case is covered by Theorem 1.1. Anyway, the proof is just Lemma 5.2 plus Proposition 5.4.

5.6. **Self-intersection 4.** In this case, $\lambda = 1$, therefore $p$ is a dicritical linerizable singularity (in particular, after a blow-up at $p$, the self-intersection of the strict transform of $C$ is $C^2 - 4 = 0$, so we obtain a rational fibration over $\mathbb{P}^1$) by [14] or [10]. But, since $\lambda$ is rational positive, the foliation is not reduced nondegenerate at $p$, hence this case is not possible in our assumptions.

5.7. **Self-intersection greater than 4.** Since $k > 4$ we have that $\lambda$ is a positive irrational number, hence the singularity is non-dicritical linerizable.

After $k$ suitable blow-ups the self-intersection of the strict transform of $C$ is $\tilde{C}^2 = C^2 - 4 - k + 1 = n - 3 - k$ (the first blow-up at $p$ and the following blow-ups at one of the two singular points of the foliation in the strict transform of $C$). Therefore, after $n - 3$ blow-ups we obtain $C^2 = 0$. Let $\sigma : \tilde{X} \to X$ be the transformation obtained by composing these blow-ups, $\tilde{C} = \sigma^*(C), E = \sigma^{-1}(p) = C_1 + \ldots + C_{(n-3)}$, where the $C_i$ are rational curves, with $C_i^2 = -1$ and $C_j^2 = -2$ if $j > 1$, and $\tilde{\mathcal{F}} = \sigma^*(\mathcal{F})$. 

□
Since $Z(\tilde{F}, \tilde{C}) = 2$, $\tilde{F}$ is a Riccati foliation with respect to a fibration $\pi : \tilde{X} \to B$, where $B$ is a smooth curve (by [2, Chapter 4, Proposition 1]). We can suppose that the fibration has connected fibres. Since the exceptional divisor $E$ is a union of smooth rational curves, the base $B$ is a smooth rational curve.

Let $q = C_1 \cap C_2$, which is a singularity of the foliation, and $R$ the fibre (possibly singular) through $q$. By Proposition 4.2, $R$ must by supported on $E$, which is impossible, since $E$ has negative definite matrix of intersection.

\[ \square \]

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