On the local antimagic vertex coloring of sub-devided some special graph

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Abstract. A graph $G$ is an ordered pair of sets $G(V, E)$, where $V$ is a set of vertices and the elements of set $E$ are usually called edges. The concept of graph labeling has recently gained a lot of popularity in the area of graph theory. His popularity is due not only to the mathematical challenges of graph labelings but also to the wide range of applications that graph labeling offer to other branches of science, for instance, x-ray, cryptography, coding theory, circuit design and communication network design. A graph labeling we mean an assignment of integers to elements of a graph such as vertex, edge, and both. Local antimagic of a graph was motivated by Arumugam et al. Thus, we initiate to developed the concept of local antimagic vertex coloring for subdevided graphs. A graph $G$ called sub-devided if the graph $G$ be the graph obtained by inserting a vertex to each edge of the graph $G$. Definition the concept local antimagic vertex coloring is $f : E(G) \rightarrow \{1, 2, 3, ..., |E(G)|\}$ if for any two adjacent vertices $a_1$ and $a_2$, $w(a_1) \neq w(a_2)$, where for $v \in G$, $w(a) = \sum_{e \in E(a)} f(e)$, where $E(a)$ and $V(a)$ are respectively the set of edges incident to $a$. The local antimagic vertex labeling induces a proper vertex coloring of graph $G$ if each vertex $a$ is assigned the color $w(a)$. The minimum colors needed to coloring the vertices in graph $G$ called local antimagic vertex chromatic number, denoted by $\chi_{la}(G)$. In this paper we study the local antimagic vertex coloring of sub-devided some special graph as follows $SF_{n, m}$, $SS_{n, m}$, $SW_{n, m}$, and $SF_{n, m}$.

1. Introduction
A graph $G$ is an ordered pair of sets $G(V, E)$, where $V$ is a set of vertices and the elements of set $E$ are usually called edges. While it is entirely possible to deal with a graph as a mathematical structure $G(V, E)$, graphs also have natural pictorial representation, $V$ is drawn as an arbitrary scattering of vertices and $E$ is drawn as a set of lines between the vertices, a line can be drawn straight or curved but never crossing itself. The basic definition of graph can be seen in [5, 6]. The concept of graph labeling has recently
gained a lot of popularity in the area of graph theory. This popularity is due not only to the mathematical challenges of graph labelings but also to the wide range of applications that graph labeling offer to other brances of science, for instance, x-ray, cryptography, coding theory, circuit design and communication network design.

A graph labeling we mean an assignment of integers to elements of a graph such as vertex, edge, and both. By a labeling we mean a one-to-one mapping that carries a set of graph elements into a set of integer number, called labels. We call these labelings a vertex labeling, edge labeling and total labeling. These conditions are usually expressed on the basis of the values (called weights). In our case, we give the label on the edges of a graph and determine the vertex weights. If all the vertex weights are different then the graph called antimagic. The study of antimagic labeling of graphs was motivated by Hartsfield and Ringel [1]. Some result about antimagic labeling can be found in Dafik et al [2, 3].

Graph coloring is assigned colors the component of graph such as vertices, edges and face of a graph, so there are no vertices, edges and face adjacent have the same colors. Definition the concept local antimagic vertex coloring is $f : E(G) \rightarrow \{1, 2, 3, ..., |E(G)|\}$ if for any two adjacent vertices $a_1$ and $a_2$, $w(a_1) \neq w(a_2)$, where for $v \in G$, $w(a) = \sum_{e \in E(a)} f(e)$, where $E(a)$ and $V(a)$ are respectively the set of edges incident to $a$. The local antimagic vertex labeling induces a proper vertex coloring of graph $G$ if each vertex $a$ is assigned the color $w(a)$. The minimum colors needed to coloring the vertices in graph $G$ called local antimagic vertex chromatic number, denoted by $\chi_{la}(G)$.

Local antimagic of a graph was motivated by Arumugam et al. [4]. They introduce the concept of local antimagic and gave an exact value of local antimagic coloring of some graph. The result of local antimagic labeling of graph developed by Putri et al. [12]. They developed the concept local antimagic coloring namely local antimagic total vertex and determine the exact value of some families tree and amalgamation of path and star graph. The other results about local antimagic of graphs can be seen in [7, 8, 9, 10, 11].

This paper discusses the chromatic number of local antimagic vertex coloring of sub-devided some special graph. A graph $G$ called sub-devided if the graph $G$ be the graph obtained by inserting a vertex to each edge of the graph $G$. In this paper we study the local antimagic vertex coloring of sub-devided some special graph and determined the chromatic number of $\chi_{la}(SF_{n,m})$, $\chi_{la}(SS_{n,m})$, $\chi_{la}(SW_{n,m})$ and $\chi_{la}(SF_{n,m})$.

2. Previous Result

The following definition and theorem of previous result, we take to show our results.

**Definition 2.1.** [4] Let $G = (V, E)$ be a connected graph with $|V| = p$ and $|E| = q$. A bijection $f : E \rightarrow \{1, 2, ..., q\}$ is called a local antimagic labeling if for any two adjacent vertices $a$ and $b$, $w(a) \neq w(b)$, where $w(a) = \sum_{e \in E(a)} f(e)$, and $E(a)$ is the set of edges incident to $a$.

**Theorem 2.1.** [4] For any tree $T$ with $l$ leaves, $\chi_{la}(T) \geq l + 1$.

**Theorem 2.2.** [4] For the friendship graph $F_n$, $\chi_{la}(F_n) = 3$. 

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3. Main Result
In this study we have 4 theorems about the local antimagic vertex coloring of sub-devided of some graph as follows $SF_{n,m}$, $SS_{n,m}$, $SW_{n,m}$ and $SF_{n,m}$. We show our result by the following.

Remark 3.1. Let $G$ be a special graph of order $n \geq 3$, the local antimagic vertex chromatic number of $G$ is $\chi_{la}(G)$ and the local antimagic vertex chromatic number of subdevided graph $G$ is $\chi_{la}Sub(G)$, we have $\chi_{la}Sub(G) \geq \chi_{la}(G)$.

Theorem 3.1. Let $SF_{n,m}$ be sub-devided friendship graph. For any integer number $n \geq 2$ and $m \geq 3$, the local antimagic vertex coloring of $SF_{n,m}$ is $\chi(SF_{n,m}) = 3$.

Proof. Let the vertex set and edge set of $SF_{n,m}$ be a $\{a, a^j_1; 1 \leq i \leq n, 1 \leq j \leq m\}$ and $\{aa^1_i; 1 \leq i \leq n\} \cup \{a^j_i a^j_{i+1}; 1 \leq i \leq n, 1 \leq j \leq m - 1\} \cup \{aa^1_i; 1 \leq i \leq n\} = e^j_i; 1 \leq i \leq n, 1 \leq j \leq m$. The cardinality of vertices and edges of $SF_{n,m}$ are $|V(SF_{n,m})| = nm + 1$ and $|E(SF_{n,m})| = nm + n$. The local antimagic vertex coloring of $SF_{n,m}$ is $\chi_{la}(SF_{n,m}) = 3$. To proof $\chi_{la}(SF_{n,m}) = 3$, we will show $\chi_{la}(SF_{n,m}) \geq 3$ and $\chi_{la}(SF_{n,m}) \leq 3$. To show the $\chi_{la}(SF_{n,m}) \geq 3$ we identify the vertex color of a graph $SF_{n,m}$ and the minimum colors needed to coloring the vertices of a graph $SF_{n,m}$ is 3 colors. It concludes that $\chi_{la}(SF_{n,m}) \geq 3$. Furthermore, we will show $\chi_{la}(SF_{n,m}) \leq 3$.

Define the labeling $f : E(SF_{n,m}) \rightarrow \{1, 2, 3, ..., |E(SF_{n,m})|\}$, the function of labeling edges as follows

$$f(e^j_i) = \begin{cases} 1 + \frac{n(i+1)}{2} - n, & \text{for } j = 1, 3, 5, \ldots \text{ and } i = 1, 2, 3, \ldots, n \\ nm - i + 1 + \frac{n}{2} - n, & \text{for } j = 2, 4, 6, \ldots \text{ and } i = 1, 2, 3, \ldots, n \end{cases}$$

From the function of edge labeling above, the vertex weights are as follows:

$$w(v) = \begin{cases} n + \frac{n(n-1)}{2}, & \text{for } v = a \\ 1 + nm, & \text{for } v = a^j_i, \text{ for } j = 1, 3, 5, \ldots \text{ and } i = 1, 2, 3, \ldots, n \\ n + 1 + nm, & \text{for } v = a^j_i, \text{ for } j = 2, 4, 6, \ldots \text{ and } i = 1, 2, 3, \ldots, n \end{cases}$$

Hence, the function of total vertex weights above, we induces a proper vertex coloring of $SF_{n,m}$ and gives $\chi_{la}(SF_{n,m}) \leq 3$. Since $\chi_{la}(SF_{n,m}) \geq 3$ and $\chi_{la}(SF_{n,m}) \leq 3$, thus $\chi_{la}(SF_{n,m}) = 3$. \hfill \Box

Theorem 3.2. Let $SS_{n,m}$ be sub-devided star graph. For any integer number $n \geq 2$ and $m \geq 3$, the local antimagic vertex coloring of $SS_{n,m}$ is $n + 1 \leq \chi(SS_{n,m}) \leq n + 2$.

Proof. Let the vertex set and edge set of $SS_{n,m}$ be a $\{a, a^j_1; 1 \leq i \leq n, 1 \leq j \leq m\}$ and $\{aa^1_i; 1 \leq i \leq n\} \cup \{a^j_i a^j_{i+1}; 1 \leq i \leq n, 1 \leq j \leq m - 1\}$. The cardinality of vertices and edges of $SS_{n,m}$ are $|V(SS_{n,m})| = nm + 1$ and $|E(SS_{n,m})| = nm$. The local antimagic vertex coloring of $SS_{n,m}$ is $\chi_{la}(SS_{n,m}) \leq l + 2$. To proof $\chi_{la}(SF_{n,m}) \leq l + 2$, we will show based on Arumugam’s theorem [3], this theorem explain that for any tree with $l$
leaves, have the chromatic number of $T$ is $\chi_{la}(T) \geq \ell + 1$, hence all the leaves receive distinct colors. Further, for this condition, sub-devided star graph has $n$ leaves and one central vertex. We identify the vertex color of a graph $SS_{n,m}$ and the minimum colors needed to coloring the vertices of a graph $SS_{n,m}$ is $n+1$ colors. It concludes that $\chi_{la}(SS_{n,m}) \geq n+1$. Furthermore, we will show $\chi_{la}(SS_{n,m}) \leq 3$.

Define the labeling $f : E(SS_{n,m}) \rightarrow \{1, 2, 3, ..., |E(SS_{n,m})|\}$, the function of labeling edges as follows

$$f(e^j) = \begin{cases} 
    i + \frac{n(j+1)}{2} - n, & \text{for } j = 1, 3, 5, \ldots \text{ and } i = 1, 2, 3, \ldots, n \\
    nm - i + 1 + \frac{nj}{2} - n, & \text{for } j = 2, 4, 6, \ldots \text{ and } i = 1, 2, 3, \ldots, n 
\end{cases}$$

From the function of edge labeling above, the vertex weights are as follows:

$$w(v) = \begin{cases} 
    n + \frac{n(n-1)}{2}, & \text{for } v = a \\
    1 + nm, & \text{for } v = a^j_i, \text{ for } j = 1, 3, 5, \ldots, m-1 \text{ and } i = 1, 2, 3, \ldots, n \\
    n + 1 + nm, & \text{for } v = a^j_i, \text{ for } j = 2, 4, 6, \ldots, m-1 \text{ and } i = 1, 2, 3, \ldots, n \\
    i + \frac{n(m+1)}{2} - n, & \text{for } v = a^m_i, \text{ for } m \text{ is odd and } i = 1, 2, 3, \ldots, n \\
    nm - i + 1 + \frac{nm}{2} - n, & \text{for } v = a^m_i, \text{ for } m \text{ is even and } i = 1, 2, 3, \ldots, n 
\end{cases}$$

Hence, the function of total vertex weights above, we induces a proper vertex coloring of $SS_{n,m}$ and gives $\chi_{la}(SS_{n,m}) \leq n+2$. Since $\chi_{la}(SS_{n,m}) \geq n+1$ and $\chi_{la}(SS_{n,m}) \leq n+2$, thus $n+1 \leq \chi_{la}(SS_{n,m}) \leq n+2$. \hfill $\square$

**Theorem 3.3.** Let $SW_{n,m}$ be sub-devided wheel graph. For any integer number $n \geq 2$, $n$ even and $m \geq 3$, the local antimagic vertex coloring of $SW_{n,m}$ is $\chi_{la}(SW_{n,m}) \leq 6$.

**Proof.** A graph $SW_{n,m}$ called sub-devided if the graph $W_n$ be the graph obtained by inserting $m$ vertex to each edge of spokes of a wheel graph. Let the vertex set and edge set of $SW_{n,m}$ be a $\{a, x_{i,j}; 1 \leq i \leq n, 1 \leq j \leq m\}$ and $\{x_{i,1}x_{i+1,1}; 1 \leq i \leq n-1\} \cup \{x_{1,1}x_{1,n}\} \cup \{x_{i,j}x_{i,j+1}; 1 \leq i \leq n, 1 \leq j \leq m-1\} \cup \{ax_{i,m}; 1 \leq i \leq n\}$. The cardinality of vertices and edges of $SW_{n,m}$ are $|V(SW_{n,m})| = nm + 1$ and $|E(SW_{n,m})| = nm + n$. We will show $\chi_{la}(SW_{n,m}) \leq 6$.

Define the labeling $f : E(SW_{n,m}) \rightarrow \{1, 2, 3, ..., |E(SW_{n,m})|\}$, the function of labeling edges as follows

$$f(x_{i,1}x_{i+1,1}) = \begin{cases} 
    \frac{i+1}{2}, & \text{for } i = 1, 3, 5, \ldots, n-1 \\
    \frac{n+1}{2}, & \text{for } i = 2, 4, 6, \ldots, n-2 
\end{cases}$$

$$f(x_{1,1}x_{n,1}) = n$$
Based on the vertex weights above, it can be seen that the cycle on the wheel graph has 3 vertex weights (indicated by $w(x_{i,1})$), for the vertices on the spoke of wheel graph

$$w(x_{i,j}) = \left\{ \begin{array}{ll}
  nm + 3n + 1, & \text{for } i = 1, 2, 3, \ldots, n \text{ and } j = 3, 5, 7, \ldots, m \\
  nm + 2n + 1, & \text{for and } i = 1, 2, 3, \ldots, n \text{ and } j = 2, 4, 6, \ldots, m \\
\end{array} \right.$$

$$w(a) = \left\{ \begin{array}{ll}
  \frac{n^2m}{2} + \frac{3n^2}{2} + \frac{n}{2}, & \text{for } m \text{ is odd} \\
  n^2 \left( \frac{m-1}{2} \right) + \frac{3n^2}{2} + \frac{n}{2}, & \text{for } m \text{ is even} \\
\end{array} \right.$$
has 2 vertex weights (indicated by \( w(x_{i,j}) \)) and 1 different point weights on central vertex (indicated by \( w(a) \)). We can see that a proper vertex coloring of \( SW_{n,m} \) and gives \( \chi_{la}(SW_{n,m}) \leq 6 \). It is concludes.

The following figure shows the labeling of local antimagic vertex coloring of sub-devided of wheel graph.

\[\text{Figure 1. The labeling of } SW_{6,4}\]

**Theorem 3.4.** Let \( SF_{n,m} \) be sub-devided fan graph. For any integer number \( n \geq 2 \), \( n \) even and \( m \geq 3 \), the local antimagic vertex coloring of \( SF_{n,m} \) is \( \chi(SF_{n,m}) \leq 6 \).

**Proof.** A graph \( SF_{n,m} \) called sub-devided if the graph \( F_n \) be the graph obtained by inserting \( m \) vertex to each edge of spokes of a fan graph. Let the vertex set and edge set of \( SF_{n,m} \) be \( \{a, x_{i,j}; 1 \leq i \leq n, 1 \leq j \leq m\} \) and \( \{x_{i,1}x_{i+1,1}; 1 \leq i \leq n-1\} \cup \{x_{i,j}x_{i,j+1}; 1 \leq i \leq n, 1 \leq j \leq m-1\} \cup \{ax_{i,m}; 1 \leq i \leq n\} \). The cardinality of vertices and edges of \( SF_{n,m} \) are \( |V(SF_{n,m})| = nm + 1 \) and \( |E(SF_{n,m})| = nm + n - 1 \). We will show \( \chi_{la}(SF_{n,m}) \leq 6 \).

Define the labeling \( f : E(SF_{n,m}) \rightarrow \{1, 2, 3, ..., |E(SF_{n,m})|\} \), the function of labeling edges as follows

\[ f(x_{i,1}x_{i+1,1}) = \begin{cases} 
\frac{i}{2}, & \text{for } i = 2, 4, 6, ..., n - 2 \\
\frac{n+i-1}{2}, & \text{for } i = 1, 3, 5, ..., n - 1 
\end{cases} \]
From the function of edge labeling above, the vertex weights are as follows:

\[
\begin{align*}
\text{for } m \text{ is odd and } i &= 1, 3, 5, ..., n - 1, \\
\frac{n(m+1)}{2} + n - i - 1 &= f(ax_{i,m}) \\
\text{for } m \text{ is odd and } i &= 2, 4, 6, ..., n, \\
\frac{n(m+1)}{2} + n - i + 1 &= f(ax_{i,m}) \\
\text{for } m \text{ is even and } i &= 1, 3, 5, ..., n - 1, \\
\frac{nm}{2} + n + i &= f(x_{i,j}) \\
\text{for } m \text{ is even and } i &= 2, 4, 6, ..., n, \\
\frac{nm}{2} + n + i - 2 &= f(x_{i,j}) \\
\end{align*}
\]

From the function of edge labeling above, the vertex weights are as follows:

\[
\begin{align*}
\text{for } m \text{ is odd, } i &= 1, 3, 5, ..., n - 1, \\
\frac{n(j+1)}{2} + n - j - 1 &= f(x_{i,j}) \\
\text{for } m \text{ is odd, } i &= 2, 4, 6, ..., n, \\
\frac{n(j+1)}{2} + n - j + 1 &= f(x_{i,j}) \\
\text{for } m \text{ is even, } i &= 1, 3, 5, ..., n - 1, \\
\frac{nm}{2} + n + i - \frac{nj}{2} &= f(x_{i,j}) \\
\text{for } m \text{ is even, } i &= 2, 4, 6, ..., n, \\
\frac{nm}{2} + n + i - 2 - \frac{nj}{2} &= f(x_{i,j}) \\
\end{align*}
\]

w(x_{i,1}) = \left\{ \begin{array}{ll}
2n, & \text{for } i = n \\
\frac{n}{2} + 2n, & \text{for } i = 2, 4, 6, ..., n - 2 \\
\frac{n}{2} + 2n - 2, & \text{for and } i = 1, 3, 5, ..., n - 1 \\
\end{array} \right.

w(x_{i,j}) = \left\{ \begin{array}{ll}
nm + 3n - 1, & \text{for } i = 1, 2, 3, ..., n \text{ and } j = 3, 5, 7, ..., m \\
nm + 2n - 1, & \text{for } i = 1, 2, 3, ..., n \text{ and } j = 2, 4, 6, ..., m \\
\end{array} \right.

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Based on the vertex weights above, it can be seen that the path on the fan graph has 3 vertex weights (indicated by \( w(x_{i,1}) \)), for the vertices on the spoke of fan graph has 2 vertex weights (indicated by \( w(x_{i,j}) \)) and 1 different point weights on central vertex (indicated by \( w(a) \)). We can see that a proper vertex coloring of \( SF_{n,m} \) and gives \( \chi_{la}(SF_{n,m}) \leq 6 \). It is concludes. □

4. Conclusion
In this paper we study the local antimagic vertex coloring of sub-devided some special graph and determined the chromatic number of graphs as follows \( SF_{n,m}, SS_{n,m}, SW_{n,m} \) and \( SF_{n,m} \).

Open Problem 4.1. find the chromatic number of local antimagic vertex coloring of some graphs operation.

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