DYNAMICS IN A DIFFUSIVE PREDATOR-PREY SYSTEM WITH STAGE STRUCTURE AND STRONG ALLEE EFFECT

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Abstract. In this paper, we consider the dynamics of a diffusive predator-prey system with stage structure and strong Allee effect. The upper-lower solution method and the comparison principle are used in proving the nonnegativity of the solutions. Then the stability and the attractivity basin of the boundary equilibria are obtained, by which we investigated the bistable phenomena. The existence and local stability of the positive constant steady-state are investigated, and the existence of Hopf bifurcation is studied by analyzing the distribution of eigenvalues. On the center manifold, we studied the criticality of the Hopf bifurcation by the normal form theory. Some numerical simulations are carried out for illustrating the theoretical results.

1. Introduction. Stage structure in a predator-prey model may improve the assumption that each individual predator has the same ability to capture prey. In fact, most immature predators do not have the ability to attack prey. Thus, immature individuals and mature individuals of the predator should be divided by a fixed age since the ability of attacking prey are obviously different. There has been a fair amount of work on modeling stage structured model consisting of immature and mature individuals for species (see \([5, 6, 22, 18]\) and the references therein). Wang and Chen \([19]\) found that an orbitally asymptotically stable periodic orbit exists in that model. They also established the condition for the permanence of the populations.

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Saito and Takeuchi [14] proposed a stage structured predator-prey model as follows:

\[
\begin{align*}
\dot{x}(t) &= x(t)[r_1 - a_{11}x(t) - a_{13}Y(t)], \\
\dot{y}(t) &= -r_2y(t) + a_{31}x(t)Y(t) - a_{31}e^{-r_2\tau}x(t-\tau)Y(t-\tau), \\
Y(t) &= -r_3Y^2(t) + a_{31}e^{-r_2\tau}x(t-\tau)Y(t-\tau).
\end{align*}
\]

(1)

Here, \(x(t), y(t), Y(t)\) denote the density of the prey, the immature and mature predator at time \(t\), respectively. The immature individuals and mature individuals are separated by age \(\tau\). The term \(a_{31}e^{-r_2\tau}x(t-\tau)Y(t-\tau)\) in the second equation represents the immature individuals (with the mature birth rate \(a_{31}\)) born at time \(t - \tau\) and alive time \(t\) (with the immature death rate \(-r_2\)). They obtained the conditions for the local asymptotic stability and global attractivity of the interior equilibrium of the model. Apart from this, the conditions under which the Hopf bifurcation may occur in system (1) are derived in [12].

Since the prey and predator always distribute inhomogeneously in different locations, the diffusion has been taken into considered in many ecological models (see [3, 27, 8, 24]). Xu and Wei [23] considered a diffusive budworm model with a structured population model subjected to the Neumann boundary conditions as follows:

\[
\frac{\partial u(x,t)}{\partial t} = d_1 \Delta u(x,t) - D u(x,t) - \frac{\beta h^2(x,t)}{\gamma^2 + u^2(x,t)} + q_1 e^{-d_\tau} u(x,t-\tau) e^{-\alpha_1 u(x,t-\tau)},
\]

(2)

where \(u(x,t)\) is the mature budworm density at location \(x\) and time \(t\), \(\tau\) is the maturation delay. They investigated the stability of the steady state and the existence of Hopf bifurcation. The global existence of the periodic solutions is also established.

In recent years, the Allee effect growth of a population has been studied extensively [17, 16]. A strong Allee effect refers to the phenomenon that one population has a negative growth when the size of the population is below the threshold value [15]. This is a universal phenomenon in nature since many preys are social animals, they can only live in groups, like ants, birds, rabbits and so on [1].

Therefore, in order to describe the predator-prey model more accurately, we combine the stage structure in predator and the strong Allee effect growth in prey, and impose a no-flux boundary condition, so it is a closed ecosystem. Such a diffusive system with strong Allee effect and stage structure takes the following form:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= d_1 \Delta u + \alpha u(1 - \frac{u}{N})(u - M) - \frac{ku^2v}{b + u^2}, \\
\frac{\partial v}{\partial t} &= d_2 \Delta v - rv + \theta e^{-\tau_0\tau} \frac{u^2(t-\tau)v(t-\tau)}{b + u^2(t-\tau)}, \\
\partial_n u &= 0, \quad \partial_n v = 0, \\
u(x,t) &= u_0(x,t) \geq 0, \quad v(x,t) = v_0(x,t) \geq 0,
\end{align*}
\]

(3)

where \(u(t,x)\) and \(v(t,x)\) are the population densities of the prey and the mature predator, respectively. \(d_i (i = 1, 2)\) is the diffusion coefficient. \(k\) represents the predation rate. \(r\) is the average mortality rate of the mature predator. \(\theta\) is the conversion rate. \(\sqrt{b}\) denotes the half saturation coefficient. \(\tau\) is the maturation time delay of the immature predator. \(r_0\) is the average mortality rate of the immature predator. The Laplacian operator \(\Delta\) shows the diffusion effect. The habitat \(\Omega \subset \mathbb{R}^n\) is a bounded domain with smooth boundary for \(n \geq 1\), and \(\nu\) is the outer normal
direction for \( x \in \partial \Omega \). \( d_1, d_2, \alpha, k, r, \theta, b \) and \( r_0 \) are all positive constants, \( M \) represents the Allee threshold of prey with \( 0 < M < N \), and \( N \) denotes the carrying capacity of the prey. The term
\[
\theta e^{-r_0 \tau} \frac{u^2(t-\tau)v(t-\tau)}{b+u^2(t-\tau)}
\]
represents the number of immature predators that were born at time \( t-\tau \) which still survive at time \( t \) and are transferred from the immature stage to the mature stage at time \( t \).

With a nondimensionalized change of variables:
\[
s = \alpha Mt, \quad \tilde{u} = \frac{u}{M}, \quad \tilde{v} = \frac{kv}{\tilde{\alpha}M},
\]
let \( d_1' = \frac{d_1}{\alpha M}, d_2' = \frac{d_2}{\alpha M}, \tilde{\tau} = \alpha \tau \), \( b' = \frac{b}{\alpha M} \), \( d = \frac{r_0}{\alpha M} \), we obtain
\[
\begin{align*}
\frac{\partial \tilde{u}}{\partial s} &= d_1' \Delta \tilde{u} + \tilde{u}(1 - \frac{\tilde{u}}{N/M})(u-1) - \frac{\theta}{\alpha M} \frac{\tilde{u}^2}{b' + \tilde{u}^2} \tilde{v}(s-\tau), \quad x \in \Omega, s > 0, \\
\frac{\partial \tilde{v}}{\partial s} &= d_2' \Delta \tilde{v} - r \frac{\tilde{v}}{\alpha M} \tilde{v}(s-\tau) - \frac{\theta}{\alpha M} e^{-d\tilde{\tau}} \tilde{u}^2(s-\tau) \tilde{v}(s-\tau) \tau, \quad x \in \Omega, s > 0, \\
\partial_s \tilde{u} &= 0, \partial_s \tilde{v} = 0, \quad x \in \partial \Omega, s > 0, \\
\tilde{u}(x,s) = u_0(x,s) \geq 0, \tilde{v}(x,s) = \tilde{v}_0(x,s) \geq 0, \quad x \in \Omega, s \in [-\tilde{\tau}, 0].
\end{align*}
\]
Let \( K = \frac{\alpha}{\alpha M}, m = \frac{\theta}{\alpha M}, \nu' = \frac{r}{\alpha M} \), and still denote \( s, d_1', d_2', b', r' \) and \( \tilde{\tau} \) as \( t, d_1, d_2, b, r \) and \( \tau \). Dropping the tildes of \( u \) and \( v \), we obtain the simplified dimensionless system of equations:
\[
\begin{align*}
\frac{\partial u}{\partial t} &= d_1 \Delta u + u(u-1)(1 - \frac{u}{K}) - \frac{mu^2v}{b+u^2}, \quad x \in \Omega, t > 0, \\
\frac{\partial v}{\partial t} &= d_2 \Delta v - rv + me^{-\nu't} \frac{u^2(t-\tau)v(t-\tau)}{b+u^2(t-\tau)}, \quad x \in \Omega, t > 0, \\
\partial_t u &= 0, \partial_t v = 0, \quad x \in \partial \Omega, t > 0, \\
u(x,t) = u_0(x,t) \geq 0, v(x,t) = v_0(x,t) \geq 0, \quad x \in \Omega, t \in [-\tau, 0].
\end{align*}
\]
In the remaining part of this article, we focus on system (4). For the new parameters, \( K \) is a rescaled carrying capacity of the prey. \( r \) is the mortality rate of the predator, and \( m \) is the strength of the interaction. To maintain the meaningful of the system, we assume that \( m > r \).

In this paper, we have proved the global existence and non-negativity of the solutions to (4). Several groups of conditions are given under which global asymptotical behaviour of the solutions can be determined. The attractivity basin of the extinction equilibrium and the local stability of boundary equilibria is investigated. Particularly, we show that large amount of predator will always drives both predator and prey into extinction, which is called over-predation. This is a character of predator-prey system with strong Allee effect. Over-predation exists widely in nature. For example, people have driven Emberiza aureola, Manis pentadactyla and Chinese sturgeon to the edge of extinction by capturing and eating them. We also proved that the bistable phenomenon may occur which means the system dynamics has strong selectivity with respect to the initial conditions. Besides, we investigated Hopf bifurcations at the positive steady state: when the delay \( \tau \) decreases passing through a critical value, the positive steady state loses its stability and the stable or unstable time-periodic oscillations appear. This explains why the number of the predator and prey vary periodically, as reported in [19, 12]. Finally,
by using a group of parameters, we illustrate the rich dynamical behaviours of our model.

Methods for analyzing diffusive predator-prey systems have been developed since past decades. In this paper we apply some classical techniques like upper-lower solution method, comparison methods, geometry criterion and bifurcation theory to proceed the analysis. We emphasize several points in our derivations: for one thing, the proof of over-predation requires subtle analysis, since stage structure makes coefficients dependant with delay. Besides, when seeking for the conditions under which the positive equilibrium $E_3$ is stable, we searched the stable region and then the critical bifurcation point with the decrease of $\tau$ and draw the conclusion.

The rest of the paper are structured in the following way. In section 2, we prove the nonnegativity of the solutions in system (4). In section 3, by analyzing the distribution of the roots of the associated characteristic equation, we study the stability of the boundary equilibria and the positive equilibrium. Taking $\tau$ as a parameter, using the approach of Beretta and Kuang [2], we show that the positive steady-state can be destabilized through a Hopf bifurcation. In section 4, we illustrate our results by numerical simulations and find that the system may exhibit stable spatially homogeneous periodic solutions or transient periodic solutions near different Hopf bifurcation points. In section 5, a conclusion is given to complete this paper.

2. **Nonnegativity of solutions.** In this section, the nonnegativity of the solutions is proved. Let $\Omega \in \mathbb{R}^N$ be a bounded domain with smooth boundary. Define

$$X := \{(u, v) \in W^{2,2}(\Omega) \mid \partial_x u = \partial_v v = 0 \text{ at } x \in \partial \Omega\},$$

with the standard inner product $(\cdot, \cdot)$, and

$$C := \{(\varphi_1, \varphi_2) \mid (\varphi_1, \varphi_2) \in C([\tau, 0], X)\}.$$

Now we give the conclusion as follows.

**Theorem 2.1.** The system (4) has a unique solution $(u(x, t), v(x, t))$ which satisfies $u(x, t) \geq 0, v(x, t) \geq 0$ for $x \in \Omega$ and $t > 0$. Particularly, $u(x, t) > 0, v(x, t) > 0$ for $x \in \Omega$ and $t > 0$ when $u_0(x, 0) \neq 0, v_0(x, 0) \neq 0$ for $x \in \Omega$.

**Proof.** Clearly, $u(x, t) \equiv 0$ and $v(x, t) \equiv 0$ when $u_0(x, t) \equiv 0$ and $v_0(x, t) \equiv 0$ for $x \in \Omega, t \in [-\tau, 0]$. So without loss of generality, we assume that $u_0(x, t) \neq 0, v_0(x, t) \neq 0$ for $x \in \Omega, t \in [-\tau, 0]$. Define

$$M(u, v, u^t, v^t) = u(u - 1)(1 - \frac{u}{K}) - \frac{muv^2}{b + uv},$$

$$N(u, v, u^t, v^t) = -rv + me^{-du} \frac{(u^t)^2v^t}{b + (u^t)^2},$$

where

$$u^t = u(t - \tau), \quad v^t = v(t - \tau).$$

Obviously, $M_u = -\frac{muv^2}{b + uv} \leq 0, M_{u^t} = 0, M_{ut} = 0, N_u = 0, N_{u^t} = 0, N_{utt} = me^{-du} \frac{2uv^2}{b + (u^t)^2} \geq 0$, and $N_{u^t} = me^{-du} \frac{(u^t)^2}{b + (u^t)^2} \geq 0$ in $\mathbb{R}_+^2 \times \mathbb{R}_+^2$. Thus system (4) is a mixed quasi-monotone system[11].
Let $(\tilde{u}(x,t), \tilde{v}(x,t)) = (0,0)$ and $(\hat{u}(x,t), \hat{v}(x,t)) = (u^*(t), v^*(t))$, where $(u^*(t), v^*(t))$ is the unique solution to

$$
\begin{align*}
\frac{du}{dt} &= u(u-1)(1-\frac{u}{K}), \\
\frac{dv}{dt} &= -rv + me^{-d_T} \frac{(u^t)^2v^t}{b + (u^t)^2},
\end{align*}
$$

(5)

where $u_0^*(t) = \sup_{x \in \Omega} u_0(x,t)$, $v_0^*(t) = \sup_{x \in \Omega} v_0(x,t)$.

Since $u(u-1)(1-\frac{u}{K}) < 0$ for $u > K$, then for the first equation of (5), $u(t)$ exists globally. Moreover, for any $\epsilon > 0$, there exists $T > 0$ such that $u(t) < K + \epsilon$ for $t > T$. For the second equation of (5),

$$
\frac{dv}{dt} = -rv + me^{-d_T} \frac{(u^t)^2v^t}{b + (u^t)^2}.
$$

Let $V(t)$ be the unique solution of

$$
\begin{align*}
\dot{V} &= -rV + me^{-d_T} \frac{(u^t)^2V^t}{b + (u^t)^2}, \\
V(t) &= v_0^*(t), \quad t \in [-\tau, 0],
\end{align*}
$$

(6)

Note that for $t \in [0, \tau]$, $V^t = V(t-\tau) = v_0^*(t-\tau) \geq 0$, $u(t-\tau) = u_0^*(t-\tau) \geq 0$. Then the solution of (6) exists and $V(t) \geq 0$ for $t \in [0, \tau]$. One can obtain $\dot{V}(t)$ exists and $V(t) \geq 0$ for $t \in [0, 2\tau]$ in the same way. Hence by using the mathematical induction, we have $V(t)$ exists globally for $t \geq 0$. Hence $(\tilde{u}(x,t), \tilde{v}(x,t)) = (0,0)$ and $(\hat{u}(x,t), \hat{v}(x,t)) = (u^*(t), v^*(t))$ are a pair of lower-solution and upper-solution to (4), respectively, since

$$
\begin{align*}
\frac{\partial \tilde{u}}{\partial t} - d_1 \Delta \tilde{u} - \tilde{u}(\tilde{u}-1)(1-\frac{\tilde{u}}{K}) + \frac{m\tilde{u}^2\hat{v}}{b + \hat{u}^2} &\geq 0, \\
\frac{\partial \hat{u}}{\partial t} - d_1 \Delta \hat{u} - \hat{u}(\hat{u}-1)(1-\frac{\hat{u}}{K}) + \frac{m\hat{u}^2\hat{v}}{b + \hat{u}^2} &\leq 0,
\end{align*}
$$

and

$$
\begin{align*}
\frac{\partial \tilde{v}}{\partial t} - d_2 \Delta \tilde{v} + r\tilde{v} - me^{-d_T} \frac{(\tilde{u}^t)^2\hat{v}^t}{b + (\hat{u}^t)^2} &\geq 0, \\
\frac{\partial \hat{v}}{\partial t} - d_2 \Delta \hat{v} + r\hat{v} - me^{-d_T} \frac{(\hat{u}^t)^2\hat{v}^t}{b + (\hat{u}^t)^2} &\leq 0,
\end{align*}
$$

and they obviously satisfied boundary conditions. Using the definition of lower-upper solution in Definition 8.1.2 in [10], we can get that

$$
0 \leq u_0(x,t) \leq u_0^*(t), \quad 0 \leq v_0(x,t) \leq v_0^*(t) \quad \text{for} \quad x \in \Omega, \quad t \in [-\tau, 0].
$$

Thus, from Theorem 8.3.3 in [10] we get that system (4) has a unique and globally defined solution $(u(x,t), v(x,t))$ which satisfies

$$
0 \leq u(x,t) \leq u^*(t), \quad 0 \leq v(x,t) \leq v^*(t), \quad x \in \Omega, \quad t > 0.
$$

This proves the nonnegativity of solutions of system (4).

In the following, we will prove that if $u_0(x,0) \not\equiv 0 \text{ and } v_0(x,0) \not\equiv 0$ for $x \in \Omega$, then the unique globally defined solution $(u(t,x), v(t,x))$ of system (4) would satisfy

$$
u(x,t) > 0, \quad v(x,t) > 0 \quad \text{for} \quad x \in \Omega, \quad t > 0.$$
In fact, we can rewrite system (4) into the following form:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= d_1 \Delta u + u(u-1)(1 - \frac{u}{K}) - \frac{m u^2 v}{b + u^2}, \quad x \in \Omega, \ t > 0, \\
\frac{\partial v}{\partial t} &= d_2 \Delta v - rv + \frac{m(u')^2 u'}{b + (u')^2}, \quad x \in \Omega, \ t > 0, \\
u(x, 0) &= u_0(x, 0), \quad x \in \Omega, \\
v(x, 0) &= v_0(x, 0), \quad x \in \Omega.
\end{align*}
\tag{7}
\]

Denote \( \tilde{u} = u(x, t) \) and choose \( \hat{u} = 0 \). Obviously \( \tilde{u} \) is the solution of system (7) with \( u_0(x, 0) \neq 0 \), and \( \hat{u} = 0 \) is the solution of system (7) with \( u_0(x, 0) = 0 \). Obviously we have

\[
\begin{align*}
\frac{\partial \tilde{u}}{\partial t} - d_1 \Delta \tilde{u} - H(\tilde{u}, v) &\geq \frac{\partial \tilde{u}}{\partial t} - d_1 \Delta \hat{u} - H(\hat{u}, v), \quad x \in \Omega, \ t > 0, \\
\frac{\partial v}{\partial t} - d_2 \Delta v + rv &\geq \frac{\partial v}{\partial t} - d_2 \Delta \hat{v} + r\hat{v}, \quad x \in \Omega, \ t > 0, \\
u_0(x, 0) &\geq 0, \quad u_0(x, 0) \neq 0, \quad x \in \Omega,
\end{align*}
\]

where \( H(u, v) = u(u-1)(1 - \frac{u}{K}) - \frac{m u^2 v}{b + u^2} \). The comparison principle in Lemma 2.2.1 from [10] implies that \( u(x, t) \) in system (7) should satisfy \( u(x, t) > 0 \) for \( x \in \Omega \) and \( t > 0 \). This proves \( u(x, t) > 0 \) for \( x \in \Omega \) and \( t > 0 \).

On the other hand, \( v(x, t) \) in system (4) can be rewritten as:

\[
\begin{align*}
\frac{\partial v}{\partial t} &= d_2 \Delta v - rv + \frac{m(u')^2 u'}{b + (u')^2}, \quad x \in \Omega, \ t > 0, \\
\frac{\partial v}{\partial t} &= 0, \quad x \in \partial \Omega, \ t > 0, \\
v(x, t) &= v_0(x, t), \quad x \in \Omega, \ t \in [-\tau, 0].
\end{align*}
\tag{8}
\]

Let \( W(t) \) be the unique solution of

\[
\begin{align*}
\frac{\partial W}{\partial t} &= d_2 \Delta W - rW, \quad x \in \Omega, \ t \geq 0, \\
\frac{\partial W}{\partial t} &= 0, \quad x \in \partial \Omega, \ t \geq 0, \\
W(x, 0) &= v_0(x, 0), \quad x \in \Omega.
\end{align*}
\tag{9}
\]

Since \( v_0(x, t) \geq 0 \) and \( v_0(x, 0) \neq 0 \), using the comparison principle discussed above, we can get that the solution \( W(x, t) \) of Eq. (9) satisfies \( W(x, t) > 0 \) for \( x \in \Omega, \ t > 0 \). Since the solution \( v(x, t) \) of Eq. (8) satisfies \( v(x, t) \geq W(x, t) \), we get that

\[
u(x, t) > 0 \quad \text{for} \ x \in \Omega, \ t > 0.
\]

\[\square\]

From the proof above, we may get the following remark.

**Remark 1.** Since \( u(x, t) \leq u^*(t) \) and \( u^*(t) \) satisfies Eq. (5), we get that

\[
\begin{align*}
u^*(t) &\rightarrow 0, \quad \text{if} \ \sup_{\Omega} u_0(x, 0) < 1, \\
u^*(t) &\rightarrow 1, \quad \text{if} \ \sup_{\Omega} u_0(x, 0) = 1, \\
u^*(t) &\rightarrow K, \quad \text{if} \ \sup_{\Omega} u_0(x, 0) > 1.
\end{align*}
\tag{10}
\]

Therefore, if \( \sup_{\Omega} u_0(x, 0) < 1 \), \( u(x, t) \rightarrow 0 \) for all \( x \in \Omega \). And for any \( \epsilon > 0 \), there exists \( T_\epsilon > 0 \) such that \( u(x, t) < K + \epsilon \) in \( \Omega \times [T_\epsilon, +\infty) \).
3. Stability and bifurcation analysis of the constant steady states. In the following of this paper, we consider system (4) on $\Omega = (0, l\pi)$. Then

$$X = \{(u, v) \in W^{2,2}(0, l\pi) | \partial_u u = \partial_v v = 0 \text{ at } x = 0, l\pi\}.$$ 

Clearly, the constant equilibria of (4) are determined by the following equations

$$\begin{align*}
-\frac{u(u-1)(1-\frac{u}{K}) - \frac{mu^2v}{b+u^2}}{b+u^2} = 0, \\
-rv + me^{-dr} \frac{u^2v}{b+u^2} = 0.
\end{align*} \tag{11}$$

Solving Eq.(11), system (4) has three non-negative boundary constant steady states $E_0, E_1$ and $E_2$ and one possible positive constant steady state $E_3$. The steady states are given by $E_0 = (0, 0), E_1 = (1, 0), E_2 = (K, 0)$ and $E_3 = (u_*, v_*)$, where

$$u_* = \sqrt{\frac{b}{\frac{m}{r}e^{-dr} - 1}}, \quad v_* = \frac{e^{-dr}}{r}u_*(1 - \frac{u_*}{K}).$$

Notice that $K = \frac{N}{M} > 1$ and $m > r$, we have the following result.

**Lemma 3.1.** The positive equilibrium $E_3$ of system (4) exists if and only if

$$(H1) \quad \frac{m}{b+1} < re^{dr} < \frac{mK^2}{b+K^2}$$

is satisfied.

3.1. Characteristic equation. In this section, we aim at the stability analysis of the equilibria $E_0, E_1, E_2$ and $E_3$.

Denote $U(t) = (u(t), v(t))^T, U^i(\cdot) = U(t+\cdot)$. Then in the space $\mathcal{C} = C([-\tau, 0], X)$, system (4) can be written as an abstract differential equation as follows

$$\frac{dU(t)}{dt} = D\Delta U(t) + F(U_t), \tag{12}$$

where $D = \text{diag}(d_1, d_2)$, $\text{dom}(D\Delta) = \{(u, v) \in W^{2,2} | \partial_u u = \partial_v v = 0 \text{ at } x = 0, l\pi\}$, and $F : \mathcal{C} \rightarrow X$ is given by

$$F(U_t) = \begin{pmatrix}
-\frac{u(u-1)(1-\frac{u}{K}) - \frac{mu^2v}{b+u^2}}{b+u^2} \\
-rv + me^{-dr} \frac{(u^i)^2v^i}{b+(u_i)^2}
\end{pmatrix}.$$ 

Linearizing system (12) at one of the constant steady states, for simplicity, denote the constant steady state as $(u_0, v_0)$, thus we obtain

$$\frac{dU(t)}{dt} = D\Delta U(t) + L(U_t), \tag{13}$$

where $L : \mathcal{C} \rightarrow X$ is given by

$$L(U_t) = L_1U^i(0) + L_2U^i(-\tau),$$

with

$$L_1 = \begin{pmatrix}
-3u_0^2 + (2K + 2)u_0 - K \\
0
\end{pmatrix} \frac{2mbu_0v_0}{(b + u_0^2)^2} - \frac{mu_0^2}{b + u_0^2}$$

and

$$L_2 = \begin{pmatrix}
-3u_0^2 + (2K + 2)u_0 - K \\
0
\end{pmatrix} \frac{2mbu_0v_0}{(b + u_0^2)^2} - \frac{mu_0^2}{b + u_0^2}.$$
and

$$L_2 = \begin{pmatrix} 0 & 0 \\ e^{-dr} \frac{2mbu_0v_0}{(b + u_0^2)^2} & e^{-dr} \frac{mu_0^2}{b + u_0^2} \end{pmatrix}.$$ 

From Wu [21], we obtain that the characteristic equation for the linear system (13) satisfying

$$\lambda y - D\Delta y - L(e^{\lambda y}) = 0, y \in \text{dom}(D\Delta), y \neq 0. \quad (14)$$

It is well known that the operator $u \mapsto \Delta u$ with $\partial_x u = 0$ at $x = 0$ and $l \pi$ has eigenvalues $-\frac{n^2}{l^2}$ ($n \in \mathbb{N}_0$) with the corresponding eigenfunctions $\frac{n}{l} x, n \in \mathbb{N}_0$. Let

$$y = \sum_{n=0}^{\infty} \left( a_n \right) \cos \frac{n}{l} x$$

be an eigenfunction of the operator $D\Delta + L$ with eigenvalue $\lambda$, see [25, 26], and substituting this eigenfunction into (14), we get

$$\begin{align*}
\left| \begin{array}{cc}
\lambda + d_1 \frac{n^2}{l^2} + 3u_0^2 - (2K + 2)u_0 + K & \frac{2mbu_0v_0}{(b + u_0^2)^2} + \frac{mu_0^2}{b + u_0^2} \\
-e^{-dr} \frac{2mbu_0v_0}{(b + u_0^2)^2} e^{-\lambda r} & \lambda + d_2 \frac{n^2}{l^2} + r - e^{-dr} \frac{mu_0^2}{b + u_0^2} e^{-\lambda r} 
\end{array} \right| = 0.
\end{align*}$$

Therefore, the characteristic equation (14) at the equilibrium $(u_0, v_0)$ is equivalent to

$$\lambda^2 + T_n \lambda + D_n + M_n e^{-\lambda r} = 0, \quad n \in \mathbb{N}_0, \quad (15)$$

where

$$T_n = \frac{(d_1 + d_2)n^2}{l^2} + 3u_0^2 - (2K + 2)u_0 + K + \frac{2mbu_0v_0}{(b + u_0^2)^2} + r,$$

$$D_n = \frac{n^2}{l^2} + \frac{3u_0^2 - (2K + 2)u_0 + K}{K} + \frac{2mbu_0v_0}{(b + u_0^2)^2} d_2 \frac{n^2}{l^2} + r,$$

$$M_n = -e^{-dr} \frac{mu_0^2}{b + u_0^2} \left[ \lambda + d_1 \frac{n^2}{l^2} + 3u_0^2 - (2K + 2)u_0 + K \right].$$

The stability of the steady state $(u_0, v_0)$ can be determined by the distribution of the roots of Eq. (15): if all the roots of Eq. (15) have negative real parts, then $(u_0, v_0)$ is locally asymptotically stable; if at least one root of Eq. (15) has positive real parts, then $(u_0, v_0)$ is unstable; if any root has zero real part and other roots all have negative real parts, then the stability of $(u_0, v_0)$ cannot be determined by the linearized system directly.

For simplicity, denote

$$\tilde{a} = d_1 \frac{n^2}{l^2} + \frac{3u_0^2 - (2K + 2)u_0 + K}{K} + \frac{2mbu_0v_0}{(b + u_0^2)^2}, \quad \tilde{b} = \frac{mu_0^2}{b + u_0^2},$$

$$\tilde{c} = -e^{-dr} \frac{2mbu_0v_0}{(b + u_0^2)^2}, \quad \tilde{d} = d_2 \frac{n^2}{l^2} + r,$$

$$\tilde{f} = e^{-dr} \frac{mu_0^2}{b + u_0^2}.$$

Then, the characteristic Eq. (15) at $(u_0, v_0)$ turns out to be

$$[\lambda^2 + (\tilde{a} + \tilde{d})\lambda + \tilde{a}\tilde{d}] + (-\tilde{f}\lambda - \tilde{a}\tilde{f} - \tilde{b}\tilde{c})e^{-\lambda r} = 0.$$
3.2. Local stability of $E_0$, $E_1$ and $E_2$. First we investigate the local stability of $E_0(0,0)$ and $E_1(1,0)$. By direct calculation on the roots of Eq.(15) at $E_0, E_1$, we can get the following theorem.

**Theorem 3.2.** $E_0(0,0)$ is locally asymptotically stable; $E_1(1,0)$ is unstable.

As to the stability of $E_2$, first we introduce a lemma.

**Lemma 3.3.** ([7], Theorem A.5) All roots of the equation $(z + a)e^z + b = 0$, where $a$, $b$ are real, have negative real parts if and only if

$$\begin{align*}
    a > -1, \\
    a + b > 0, \\
    b < \xi \sin \xi - a \cos \xi,
\end{align*}$$

(16)

where $\xi$ is the root of $\xi = -a \tan \xi$, $0 < \xi < \pi$, if $a \neq 0$ and $\xi = \frac{\pi}{2}$ if $a = 0$.

Applying Lemma 3.3, we have the conclusion as follows.

**Theorem 3.4.** $E_2(K,0)$ is locally asymptotically stable when $re^{\sigma r} > \frac{mK^2}{b + K^2}$ and unstable when $re^{\sigma r} < \frac{mK^2}{b + K^2}$.

**Proof.** For $E_2(K,0)$, the characteristic equation becomes

$$(\lambda + \overline{a})(\lambda + \overline{d} - \overline{f}e^{-\lambda r}) = 0,$$

(17)

where

$$\overline{a} = d_1\frac{n^2}{l^2} + K - 1, \quad \overline{d} = d_2\frac{n^2}{l^2} + r, \quad \overline{f} = \frac{mre^{-d_2rK^2}}{b + K^2}.$$

Since $\lambda_{1(n)} = -\overline{a} < 0$, to get the stability of $E_2(K,0)$, we only need to check the sign of real part of the roots in $\lambda + \overline{d} - \overline{f}e^{-\lambda r} = 0$.

When $\tau = 0$, $\lambda + \overline{d} - \overline{f}e^{-\lambda r} = 0$ is equivalent to $\lambda + \overline{d} - \overline{f} = 0$. The roots of $\lambda + \overline{d} - \overline{f} = 0$ are $\lambda_{2(n)} = \overline{f} - \overline{d} = \frac{mK^2}{b + K^2} - d_2\frac{n^2}{l^2} - r$. The roots $\lambda_{2(n)}$, as a function of $n^2$, reaches the maximum when $n = 0$ and $\lambda_{2(0)} = \frac{mK^2}{b + K^2} - r$. Thus, for $\tau = 0$, if $r > \frac{mK^2}{b + K^2}$, then $E_2$ is stable; if $r < \frac{mK^2}{b + K^2}$, $E_2$ is unstable.

When $\tau > 0$, to apply Lemma 3.3, multiply $\lambda + \overline{d} - \overline{f}e^{-\lambda r} = 0$ by $\tau$ and get $\lambda \tau + \overline{d} \tau - \overline{f} e^{-\lambda r} = 0$. For simplicity, denote $\lambda \tau = z$, $\overline{d} \tau = a$, $-\overline{f} \tau = b$, then $\lambda \tau + \overline{d} \tau - \overline{f} e^{-\lambda r} = 0$ turns to be $(z + a)e^z + b = 0$. Clearly, $a$, $b$ are real numbers. From Lemma 3.3, we get that all the roots of $(z + a)e^z + b = 0$ have negative real parts if and only if Eq.(16) holds, that means,

$$\begin{align*}
    \overline{d} \tau > -1, \\
    (\overline{d} - \overline{f}) \tau > 0, \\
    -\overline{f} \tau < \xi \sin \xi - \overline{d} \tau \cos \xi,
\end{align*}$$

(18)

where $\xi = -\overline{d} \tau \tan \xi$, $0 < \xi < \pi$.

Obviously, $\overline{d} > 0$. Thus, $\overline{d} \tau > 0 > -1$, and the first inequality in (18) always hold. As to the second inequality in (18), $(\overline{d} - \overline{f}) \tau > 0$ requires $\overline{d} > \overline{f}$, that is,
r > e^{−dτ} \frac{m K^2}{b + K^2}$ should hold. As to the third inequality in (18), we know $0 < \xi < \pi$ and $\xi = -\tilde{d} τ \tan \xi$, thus $\frac{\pi}{2} < \xi < \pi$. It is easy to see that

$$\begin{cases} -\tilde{f} τ + d \tilde{τ} \cos \xi < 0, \\ \xi \sin \xi > 0, \end{cases}$$

thus $-\tilde{f} τ < \xi \sin \xi - \tilde{d} τ \cos \xi$ always holds true.

Summarizing the above discussions, we get that if $re^{dτ} > \frac{mK^2}{b + K^2}$, all the roots of $\lambda + d - \tilde{f}e^{-\lambda τ} = 0$ have negative real parts and $E_2$ is stable. If $re^{dτ} < \frac{mK^2}{b + K^2}$, at least one root of $\lambda + d - \tilde{f}e^{-\lambda τ} = 0$ has positive real part. Thus, $E_2(K, 0)$ is unstable. \Box

3.3. The attractivity of $E_0$. In this section we investigate the attractivity of $E_0(0, 0)$. For the extinction solution $E_0 = (0, 0)$, from Remark 1, we get that if $\sup_{\Omega} u_0(x, 0) < 1$, $u(x, t) \rightarrow 0$ for $x \in \Omega$.

To prove $v(x, t) \rightarrow 0$ when $\sup_{\Omega} u_0(x, 0) < 1$, we first introduce a lemma.

**Lemma 3.5.** ([20], Theorem 3.1) Assume that $d, \delta, \tau > 0$, $f \in C^1([0, \infty), [0, \infty))$ and $f(0) = 0$. If $\sup_{y \in (0, \infty)} f'(y) < \delta$, then every solution $u(x, t)$ of

$$\frac{∂u(x, t)}{∂t} = dΔu(x, t) - δu(x, t) + f(u(x, t - τ))$$

with homogeneous Neumann boundary conditions and non-negative initial conditions converges to zero (uniformly in $x$) as $t \rightarrow \infty$ if and only if $f(y) < \delta y$ for all $y > 0$.

As for the predator term $v(x, t)$ in system (4), we can rewrite it as follows:

$$\begin{cases} \frac{∂v}{∂t} = d_2 Δv - rv + me^{-dτ} \frac{(u')^2 v^t}{b + (u')^2}, & x \in \Omega, \ t > 0, \\ \frac{∂v}{∂t} = 0, & x \in \partial \Omega, \ t > 0, \\ v(x, t) = v_0(x, t) \geq 0, & x \in \Omega, \ t \in [-\tau, 0]. \end{cases} \tag{19}$$

Since $\lim_{t \rightarrow +\infty} u(x, t) = 0$ when $\sup_{\Omega} u_0(x, 0) < 1$, there exists $T_0 > 0$ such that $\frac{m(u')^2}{b + (u')^2} < \frac{r}{2}$ for $t > T_0$, and then

$$\frac{me^{-dτ}(u')^2}{b + (u')^2} < \frac{r}{2}.$$

Let $Q(x, t)$ be the unique solution of

$$\begin{cases} \frac{∂Q}{∂t} = d_2 ΔQ - rQ + \frac{r}{2} Q', & x \in \Omega, \ t > T_0, \\ \frac{∂Q}{∂t} = 0, & x \in \partial \Omega, \ t > T_0, \\ Q(x, t) = v(x, t), & x \in \Omega, \ t \in [-\tau, T_0]. \end{cases} \tag{20}$$

Then the comparison principle implies that $v(x, t) \leq Q(x, t)$ for $x \in \Omega, \ t > 0$. In order to apply Lemma 3.5, we denote by $g(y) = \frac{r}{2} y$. Clearly, $g(y) \in C^1([0, \infty), [0, \infty))$ and $g(0) = 0$, $g'(y) = \frac{r}{2} < r$ and $g(y) = \frac{r}{2} y < ry$ as $t \rightarrow \infty$ for all $y > 0$.

From Lemma 3.5, every solution $Q(x, t)$ of Eq. (20) converges to $Q = 0$ (uniformly in $x$) as $t \rightarrow \infty$, so does the solution $v(x, t)$ of equation (19). Combining the local stability of $E_0$, we have the following theorem.
Theorem 3.6. If \( \sup_{x \in \Omega} u_0(x, 0) < 1 \), then the corresponding solution \((u(x, t), v(x, t))\) tends to \((0, 0)\) as \( t \to \infty \) for all \( x \in \Omega \). That is, \( E_0(0, 0) \) has a basin of attraction containing \{(u, v) \in X| 0 \leq u(x, t) < 1, v(x, t) \geq 0, \text{ for any } x \in \Omega \}.

The following theorem states that for a given initial value \( u_0(x, t) \) on \( \Omega \times [-\tau, 0] \), if \( v_0(x, t) \) is big enough, then the corresponding solution \((u(x, t), v(x, t))\) of system (4) originating from \((u_0(x, t), v_0(x, t))\) will also be attracted by \((0, 0)\). In fact, this describes the over-predation phenomenon. Over-predation devotes the extinction of the prey.

**Theorem 3.7.** Suppose that \( m, b, r, d, \tau \) are fixed. For a given initial value of the prey population \( u_0(x, t) \geq 0 \), there exists a constant \( v_0^* \) which depends on the parameters and \( u_0(x, t) \), such that when the initial predator population \( v_0(x, t) \geq v_0^* \) for \((x, t) \in \Omega \times [-\tau, 0] \), the corresponding solution \((u(x, t), v(x, t))\) of system (4) tends to \((0, 0)\) uniformly for \( x \in \Omega \) as \( t \to \infty \).

**Proof.** For a fixed \( \epsilon > 0 \), there exists \( T_0 > 0 \) such that \( u(x, t) < K + \epsilon \) for \( t > T_0 \) from Remark 1. Therefore \( u(x, t) \) satisfies

\[
\begin{align*}
\frac{\partial u}{\partial t} &= d_1 \Delta u + \frac{u}{K} - \frac{mu^2v}{b+u^2}, & x \in \Omega, t > T_0, \\
\partial_n u &= 0, & x \in \partial \Omega, t > T_0, \\
u(x, T_0) &\leq K + \epsilon, & x \in \Omega.
\end{align*}
\]

If \( \sup_{x \in \Omega} u(x, T_0) < 1 \), then \((u(x, t), v(x, t)) \to (0, 0)\) apparently from Theorem 3.6. In the following, we will prove when \( \sup_{x \in \Omega} u(x, T_0) \geq 1 \), \((u(x, t), v(x, t)) \to (0, 0)\).

Since \( \sup_{x \in \Omega} u(x, T_0) \geq 1 \), it is easy to get \( u(x, T_0) \neq 0 \), then from Theorem 2.1, we have \( u(x, t) > 0 \) for \( t > T_0 \) and \( x \in \Omega \). For convenience, take \( T_2 > T_1 > T_0 \).

Let \( v_1(x, t) \) be the solution to

\[
\begin{align*}
\frac{\partial v}{\partial t} &= d_2 \Delta v - rv, & x \in \Omega, t > 0, \\
\partial_n v &= 0, & x \in \partial \Omega, t > 0, \\
v(x, 0) &= v_0(x, 0), & x \in \Omega.
\end{align*}
\]

Then \( v(x, t) \geq v_1(x, t) \) from the comparison principle of parabolic equation for any \( t > 0 \). That is,

\[
v(x, t) \geq v_1(x, t) = \sum_{n=0}^{\infty} c_n \cos \frac{n\pi x}{l} e^{-\frac{(2m^2x^2}{d_2}+r)t}
\]

\[
= e^{-rt} \sum_{n=0}^{\infty} c_n \cos \frac{n\pi x}{l} e^{-\frac{(2m^2x^2}{d_2})t},
\]

where \( c_n \) satisfies

\[
c_n = \begin{cases} 
\frac{1}{l^2} \int_0^l v_0(x, 0)dx, & n = 0, \\
\frac{1}{l} \int_0^l v_0(x, 0) \cos \frac{n\pi x}{l}dx, & n = 1, 2, \ldots
\end{cases}
\]
Since \( \sum_{n=0}^{\infty} c_n \cos \frac{n \pi x}{l} e^{-\frac{d_2 n^2}{l^2} t} \) is uniformly convergent for \( t \geq 0 \), denoting by \( g(x, t) = \sum_{n=0}^{\infty} c_n \cos \frac{n \pi x}{l} e^{-\frac{d_2 n^2}{l^2} t} \), then we have

\[
v(x, t) \geq v_1(x, t) = e^{-rt} g(x, t).
\]

Moreover, notice that if \( v_0(x, 0) \) has a lower bound as \( v_0^\ast \), then \( v(x, t) \geq v_1(x, t) \geq v_0^\ast e^{-rt} \) when \( t \geq 0 \), especially when \( t \in [0, T_1 + T_2] \),

\[
v(x, t) \geq v_0^\ast e^{-r(T_1 + T_2)}.
\]

Since \( (u - 1)(1 - \frac{u}{K}) \leq \frac{(K-1)^2}{4K} := M_1 \) for all \( u \geq 0 \) and \( \frac{m}{b + u^2(x,t)} \geq \frac{m}{b + (K+\epsilon)^2} \) for \( t > T_1 \), then \( u(x, t) \) satisfies that

\[
\begin{cases}
\frac{\partial u}{\partial t} \leq d_1 \Delta u + M_1 u - \frac{m z_{\min}}{b + (K + \epsilon)^2} v_0^\ast e^{-r(T_1 + T_2)} u^2, & x \in \Omega, T_1 < t < T_1 + T_2, \\
\frac{\partial u}{\partial n} = 0, & x \in \partial \Omega, T_1 < t < T_1 + T_2, \\
1 \leq u(x, T_1) \leq K + \epsilon, & x \in \Omega.
\end{cases}
\]

In the following, we will prove if \( v_0^\ast \) and \( T_2 \) are large enough, then \( u(x, T_1 + T_2) < 1 \). Take \( z_{\min} = \inf_{x \in \Omega, t \in [T_1, T_1 + T_2]} u(x, t) \), then clearly \( u(x, t) \geq z_{\min} > 0 \) for all \( (x, t) \in \bar{\Omega} \times [T_1, T_1 + T_2] \).

For a contradiction, suppose \( u(x, t) \geq 1 \) for all \( t > T_1 \) and some \( x \in \bar{\Omega} \). Clearly \( u(x, t) \) satisfies \( u(x, t) \leq \bar{u}(x, t) \), where \( \bar{u}(x, t) \) is the solution to

\[
\begin{cases}
\frac{\partial u}{\partial t} = d_1 \Delta u + [M_1 - \frac{m z_{\min}}{b + (K + \epsilon)^2} v_0^\ast e^{-r(T_1 + T_2)}] u, & x \in \Omega, T_1 < t < T_1 + T_2, \\
\frac{\partial u}{\partial n} = 0, & x \in \partial \Omega, T_1 < t < T_1 + T_2, \\
z_{\min} \leq u(x, T_1) \leq K + \epsilon, & x \in \Omega,
\end{cases}
\]

where

\[
\bar{u}(x, t) = \sum_{n=0}^{\infty} \tilde{c}_n \cos \frac{n \pi x}{l} e^{-\frac{d_2 n^2}{l^2} (t-T_1)} = e^{[M_1 - \frac{m z_{\min}}{b + (K + \epsilon)^2} v_0^\ast e^{-r(T_1 + T_2)}] (t-T_1)} \sum_{n=0}^{\infty} \tilde{c}_n \cos \frac{n \pi x}{l} e^{-\frac{d_2 n^2}{l^2} (t-T_1)},
\]

and

\[
\tilde{c}_n = \begin{cases} 
\frac{1}{l} \int_0^l u(x, T_1) dx, & n = 0, \\
\frac{1}{l} \int_0^l u(x, T_1) \cos \frac{n \pi x}{l} dx, & n = 1, 2, \ldots
\end{cases}
\]

Denote \( \bar{u}(x, t) = e^{[M_1 - \frac{m z_{\min}}{b + (K + \epsilon)^2} v_0^\ast e^{-r(T_1 + T_2)}] (t-T_1)} \), \( \bar{g}(x, t) \) is a continuous function on \( (x, t) \in \bar{\Omega} \times [T_1, T_1 + T_2] \), let \( \max_{x \in \Omega, t \in [T_1, T_1 + T_2]} \bar{g}(x, t) = H \), thus we have

\[
u(x, t) \leq \bar{u}(x, t) \leq H e^{[M_1 - \frac{m z_{\min}}{b + (K + \epsilon)^2} v_0^\ast e^{-r(T_1 + T_2)}] (t-T_1)}.
\]

Direct calculation implies that if we choose

\[
v_0^\ast > e^{T_1} \frac{H M_1}{2M_1 b + (K + \epsilon)^2},
\]

then
and
\[ T_2 > \frac{\ln H}{M_1}, \]
then
\[ M_1 - \frac{m\hat{z}_{\text{min}}}{b + (K + \epsilon)^2} v_0 e^{-r(T_1 + T_2)} \leq -M_1, \]
and for any \( x \in \Omega \),
\[ u(x, T_1 + T_2) \leq H e^{(M_1 - \frac{m\hat{z}_{\text{min}}}{b + (K + \epsilon)^2} v_0 e^{-r(T_1 + T_2)})(t - T_1)} < 1. \]

This contradicts with the assumption that \( u(x, t) \geq 1 \) for all \( t > T_1 \), which means for a given \( u_0(x, t) \), if \( v_0^* \) is large enough and \( v_0(x, t) \geq v_0^* \), the solution \( u(x, t) \) will be smaller than 1 eventually. Therefore, \( (u(x, t), v(x, t)) \) tends to \((0, 0)\) as \( t \to \infty \) from Theorem 3.6. Since \( \epsilon \) is chosen arbitrarily, it is clear that \( v_0^* \) depends only on the fixed parameters and \( T_0 \) which depends on \( u_0(x, t) \).

The above theorem implies that \((0, 0)\) is always a locally stable steady state with basin of attraction including all large \( v_0(x, t) \) for a given \( u_0(x, t) \). Thus system (4) is bistable if there is another locally stable steady state or periodic solutions. We have the following conclusion.

**Theorem 3.8.** System (4) is bistable when \( re^{\tau} > \frac{mK^2}{b + K^2} \) in the following sense: if \( re^{\tau} > \frac{mK^2}{b + K^2} \), the die-out equilibrium \( E_0(0, 0) \) of system (4) is locally asymptotically stable; meanwhile, the boundary equilibrium \( E_2(K, 0) \) is locally asymptotically stable.

### 3.4. Local stability and Hopf bifurcation of \( E_2(u_*, v_*) \)

For convenience, we rewrite the hypotheses in the following:

\[ (H1) \quad \frac{m}{b + 1} < re^{\tau} < \frac{mK^2}{b + K^2}. \]

From Lemma 3.1 we know that \( (H1) \) guarantees the existence of the positive equilibrium \( E_3 \).

**Remark 2.** \( (H1) \) holds provided that either of the following conditions is satisfied:

(i) When \( \frac{m}{b + 1} < r < \frac{mK^2}{b + K^2} \), \( \tau \in [0, \tau_{\text{max}}] \), where \( \tau_{\text{max}} = \frac{1}{r} \ln \left[ \frac{mK^2}{b + K^2} \right] \).

(ii) When \( r \leq \frac{m}{b + 1} \), \( \tau \in [\tau_{\text{min}}, \tau_{\text{max}}] \), where \( \tau_{\text{min}} = \frac{1}{r} \ln \left[ \frac{m}{b + K^2} \right] \),
\[ \tau_{\text{max}} = \frac{1}{r} \ln \left[ \frac{mK^2}{r(b + K^2)} \right]. \]

Since case (ii) can be dealt in the same way as case (i), in the following of this paper, we focus on the case (i).

For convenience, we further make the following hypotheses:

\[ (H2) \quad \frac{m}{b + 1} < r < \frac{mK^2}{b + K^2}. \]

First, we analyze the local stability of \( E_3 \) when \( \tau \) is close enough to \( \tau_{\text{max}} \).

For \( E_3(u_*(\tau), v_*(\tau)) \), the characteristic equation becomes
\[ [\lambda^2 + (\bar{a} + \bar{d})\lambda + \bar{a}\bar{d}] + (\bar{f} \lambda - \bar{a}\bar{f} - \bar{b}\epsilon)e^{-\lambda \tau} = 0. \]
\[ \text{(22)} \]
where
\[
\bar{e} = d_1 \frac{n^2}{I^2} + \frac{3u^2 - (2K + 2)u_* + K}{K} + \frac{2b(u_* - 1)(1 - \frac{u_*}{K})}{b + u_*^2}, \\
\bar{b} = re^{d\tau}, \\
\bar{d} = d_2 \frac{n^2}{I^2} + r,
\]
\[
\bar{f} = r.
\]

**Theorem 3.9.** $E_3(u_*, v_*)$ is locally asymptotically stable, provided that
\[1 < u_* < K\) and $3u_*^2 - (2K + 2)u_* + K > 0.
\]

**Proof.** For simplicity, denote $s_1 = \frac{3u_*^2 - (2K + 2)u_* + K}{K}$, $s_2 = \frac{2b(u_* - 1)(1 - \frac{u_*}{K})}{b + u_*^2}$. Then the characteristic equation (22) becomes
\[
(\lambda + d_1 \frac{n^2}{I^2} + s_1 + s_2)(\lambda + d_2 \frac{n^2}{I^2} + r) = r(\lambda + d_1 \frac{n^2}{I^2} + s_1)e^{-\lambda \tau}, \quad n = 0, 1, \cdots \tag{23}
\]
For a contradiction, suppose $\lambda_p = a + ib$ ($a, b \in R, a \geq 0$) is one root of Eq.(23) with nonnegative real part, then there must exists an integer $k$ such that $\lambda_p = a + ib$ is the root of the $(k+1)$th equation in (23). Substituting $\lambda_p = a + ib$ into the $(k+1)$th equation of (23), we have
\[
\left(1 + \frac{s_2}{\lambda_p + d_1 \frac{n^2}{I^2} + s_1}\right)(\lambda_p + d_2 \frac{n^2}{I^2} + r) = re^{-\lambda_p \tau}. \tag{24}
\]
the right side of the Eq.(24) is $re^{-(a+ib)\tau}$, thus we have
\[|re^{-\lambda_p \tau}| \leq r.
\]
The left side of Eq.(24) becomes
\[
(a + r + d_2 \frac{k^2}{I^2}) \left[1 + \frac{s_2(a + d_1 \frac{k^2}{I^2} + s_1) + b^2s_2}{(a + d_1 \frac{k^2}{I^2} + s_1)^2 + b^2} \right] + ib \left[1 + \frac{s_2(d_1 \frac{k^2}{I^2} + s_1) - s_2(r + d_2 \frac{k^2}{I^2})}{(a + d_1 \frac{k^2}{I^2} + s_1)^2 + b^2}\right],
\]
since $s_1, s_2 > 0$ and $a \geq 0$, we have
\[
\left|\left(1 + \frac{s_2}{\lambda_p + d_1 \frac{n^2}{I^2} + s_1}\right)(\lambda_p + d_2 \frac{n^2}{I^2} + r)\right| > a + r + d_2 \frac{k^2}{I^2} > r.
\]
This contradicts with the assumption. Thus, all roots of Eq.(22) have negative real part, therefore, $E_3(u_*, v_*)$ is locally asymptotically stable. \(\square\)

**Theorem 3.10.** Let $\bar{\tau}$ be such that
\[
\frac{K + 1}{3} < u_*(\bar{\tau}) < K \) and $3u_*^2(\bar{\tau}) - (2K + 2)u_*(\bar{\tau}) + K = 0.
\]
Then for all $\tau$ satisfying $\max\{\bar{\tau}, 0\} < \tau < \tau_{max}$, $3u_*^2(\tau) - (2K + 2)u_*(\tau) + K > 0$ holds. That is, $E_3(u_*(\tau), v_*(\tau))$ is stable.

This can be proved easily by the fact that $u_*(\tau)$ is an increasing function of $\tau$. Here we omit the proof.

**Remark 3.** If $\bar{\tau} \leq 0$, then $E_3(u_*, v_*)$ is stable for all $\tau \in (0, \tau_{max})$. For $\bar{\tau} > 0$ and taking $\tau$ as the bifurcation parameter, we may investigate the existence of Hopf bifurcation between 0 and $\bar{\tau}$. 
Without loss of generality, here we only consider the case when $\bar{\tau} > 0$.

Since $\frac{2b(u_\ast - 1)(1 - \frac{u_\ast}{K})}{b + u_\ast^2} > 0$, $3u_\ast^2(\tau) - (2K + 2)u_\ast(\tau) + K = 0$ and $u_\ast(\tau)$ is an increasing function of $\tau$, therefore, when $\tau$ reduces from $\bar{\tau}$, there must exists a critical value $\tau_c$ such that when $\tau \in (\tau_c, \bar{\tau})$,

$$\frac{3u_\ast^2(\tau) - (2K + 2)u_\ast(\tau) + K}{K} + \frac{2b(u_\ast(\tau) - 1)(1 - \frac{u_\ast(\tau)}{K})}{b + u_\ast^2(\tau)} > 0$$

and when $\tau = \tau_c$,

$$\frac{3u_\ast^2(\tau) - (2K + 2)u_\ast(\tau) + K}{K} + \frac{2b(u_\ast(\tau) - 1)(1 - \frac{u_\ast(\tau)}{K})}{b + u_\ast^2(\tau)} = 0.$$

Let $\tau = \max\{\tau_c, 0\}$, now we can investigate the Hopf bifurcation when $\tau \in (\tau_c, \bar{\tau})$.

Obviously $\frac{3u_\ast^2 - (2K + 2)u_\ast + K}{K} + \frac{2b(u_\ast - 1)(1 - \frac{u_\ast}{K})}{b + u_\ast^2} > 0$ for all $\tau \in (\tau_c, \bar{\tau})$.

Denote by

$$P(\lambda, \tau) = \lambda^2 + (\bar{a} + \bar{d})\lambda + \bar{ad}, \quad Q(\lambda, \tau) = -\bar{f}\lambda - \bar{af} - \bar{bc}.$$

When $\tau \in (\tau_c, \bar{\tau})$, we can apply the geometry criterion due to Beretta and Kuang [2] to study the stability of $E_3$. Here we need to verify the following properties for $n \in \mathbb{N}_0$ and $\tau \in (\tau_c, \bar{\tau})$:

(i) $P(0, \tau) + Q(0, \tau) \neq 0$;
(ii) $P(\lambda, \tau) + Q(\lambda, \tau) \neq 0$;
(iii) $\lim_{|\lambda| \to \infty} \sup \{ \frac{|Q(\lambda, \tau)|}{|P(\lambda, \tau)|} : Re\lambda \geq 0 \} < 1$;
(iv) $F(\omega, \tau) := |P(\lambda, \tau)|^2 - |Q(\lambda, \tau)|^2$ has a finite number of zeros;
(v) Each positive root $\omega(\tau)$ of $F(\omega, \tau) = 0$ is continuous and differentiable in $\tau$ whenever it exists.

Proof. Given the fact that

$$P(0, \tau) + Q(0, \tau) = \bar{ad} - \bar{af} - \bar{bc}$$

$$= [d_1 \frac{n^2}{l^2} + \frac{3u_\ast^2 - (2K + 2)u_\ast + K}{K} \frac{2b(u_\ast - 1)(1 - \frac{u_\ast}{K})}{b + u_\ast^2} + \frac{2b(u_\ast - 1)(1 - \frac{u_\ast}{K})}{b + u_\ast^2}] d_2 \frac{n^2}{l^2}$$

$$> 0,$$

we have

$$P(\lambda, \tau) + Q(\lambda, \tau) = (\bar{a} + \bar{d})\lambda + \bar{ad} - \bar{af} - \bar{bc}$$

$$= -\omega^2 + \bar{a}(\bar{d} - \bar{f}) - \bar{bc} + \lambda(\bar{a} + \bar{d} - \bar{f})$$

$$= -\omega^2 + \bar{a}(\bar{d} - \bar{f}) - \bar{bc} + \lambda\omega(\bar{a} + \bar{d} - \bar{f})$$

$$= -\omega^2 + P(0, \tau) + Q(0, \tau) + \lambda\omega(\bar{a} + \bar{d} - \bar{f}).$$

Since

$$\bar{a} + \bar{d} - \bar{f} = (d_1 + d_2) \frac{n^2}{l^2} + \frac{3u_\ast^2 - (2K + 2)u_\ast + K}{K} \frac{2b(u_\ast - 1)(1 - \frac{u_\ast}{K})}{b + u_\ast^2} > 0,$$

(i) and (ii) are satisfied.

Moreover, we have

$$\lim_{|\lambda| \to \infty} \frac{|Q(\lambda, \tau)|}{|P(\lambda, \tau)|} = \lim_{|\lambda| \to \infty} \frac{|-f\lambda - af - bc|}{|\lambda^2 + (a + d)\lambda + ad|} = 0.$$
which means (iii) follows.

Let \( F_n(\omega, \tau) \) be defined as in (iv) with the following expression
\[
F_n(\omega, \tau) = |P(i\omega, \tau)|^2 - |Q(i\omega, \tau)|^2
= (\bar{a}d - \omega^2)(\bar{a}f - \omega\bar{c})
= \omega^4 + (\bar{a}d - \omega^2)\omega^2 - (\bar{a}f + \bar{b}c)^2
\]

It is obvious that property (iv) is satisfied, and by the Implicit Function Theorem, (v) is also satisfied.

In the rest part of this section, taking \( \tau \) as the bifurcation parameter, we analyze the Hopf bifurcation at the positive equilibrium \( E_3 \) when \( \tau \in (\bar{\tau}, \bar{\tau}] \). Denote
\[
F_n(\omega, \tau) = \omega^4 + a_1(\tau)\omega^2 + a_2(\tau),
\]
where \( a_1(\tau) = \bar{a}d - \bar{f}^2 - \bar{d}^2, a_2(\tau) = \bar{a}d\bar{d} - (\bar{a}f + \bar{b}c)^2 \).

Now, let \( \lambda = i\omega(\omega > 0) \) be a root of Eq. (22). Substituting it into Eq. (22) and separating the real and imaginary parts, we have
\[
\sin \omega \tau = \frac{(\bar{a}d - \omega^2)\bar{f}\omega - (\bar{a} + \bar{d})(\bar{a}f + \bar{b}c)\omega}{(\bar{a}f + \bar{b}c)^2 + (\omega)^2},
\]
\[
\cos \omega \tau = \frac{(\bar{a}d - \omega^2)(\bar{a}f + \bar{b}c) + (\bar{a} + \bar{d})f\omega^2}{(\bar{a}f + \bar{b}c)^2 + (\omega)^2}.
\]

Since \( \lambda = i\omega(\omega > 0) \) is the root of Eq. (22), then
\[
|P(i\omega, \tau)|^2 = |Q(i\omega, \tau)|^2,
\]
that is, \( \omega \) is a positive root of \( F_n(\omega, \tau) = |P(i\omega, \tau)|^2 - |Q(i\omega, \tau)|^2 = \omega^4 + a_1(\tau)\omega^2 + a_2(\tau) \).

Since \( a_1(\tau) > 0 \), we get that \( F_n(\omega, \tau) = 0 \) has positive root only and if only \( a_2(\tau) < 0 \). In fact we have \( a_2(\tau) = (\bar{a}d - \bar{a}f - \bar{b}c)(\bar{a}d + \bar{a}f + \bar{b}c) \) and \( \bar{a}d - \bar{a}f - \bar{b}c > 0 \). Thus, \( F_n(\omega, \tau) = 0 \) has positive root only and if only \( \bar{a}d + \bar{a}f + \bar{b}c < 0 \). Denote the only positive root as \( \omega_n(\tau) \), then \( \omega_n(\tau) \) satisfies
\[
\omega_n^2(\tau) = \frac{-a_1(\tau) + \sqrt{a_1^2(\tau) - 4a_2(\tau)}}{2}.
\]

This means that \( \omega_n(\tau) \) makes sense if and only if \( \sqrt{a_1^2(\tau) - 4a_2(\tau)} > a_1(\tau) \), which requires \( \bar{a}d + \bar{a}f + \bar{b}c < 0 \). For simplicity, denote \( H_n(\tau) = \bar{a}d + \bar{a}f + \bar{b}c \).

Set
\[
I_n = \{ \tau | \tau \in (\bar{\tau}, \bar{\tau}], \text{satisfies } H_n(\tau) < 0 \}.
\]

Assume that \( I_n \) is nonempty. Then for \( \tau \in I_n \), there exists a unique \( \omega_n = \omega_n(\tau) > 0 \), which satisfies Eq. (26), such that \( F_n(\omega_n, \tau) = 0 \). Let \( \theta_n(\tau) \in [0, 2\pi) \) be defined for \( \tau \in I_n \) by
\[
\sin \theta_n(\tau) = \frac{(\bar{a}d - \omega_n^2)\bar{f}\omega - (\bar{a} + \bar{d})(\bar{a}f + \bar{b}c)\omega}{(\bar{a}f + \bar{b}c)^2 + (\omega_n^2)^2},
\]
\[
\cos \theta_n(\tau) = \frac{(\bar{a}d - \omega_n^2)(\bar{a}f + \bar{b}c) + (\bar{a} + \bar{d})f\omega_n^2}{(\bar{a}f + \bar{b}c)^2 + (\omega_n^2)^2}.
\]

We have \( \theta_n(\tau) \in [\frac{3\pi}{2}, 2\pi) \) by the fact that \( \tau \in (\bar{\tau}, \bar{\tau}] \) implies \( \sin \theta_n(\tau) < 0 \) and \( \cos \theta_n(\tau) > 0 \). From the above definitions, it follows that \( \theta_n \) is well and uniquely defined for all \( \tau \in I_n \).
One can check that \( i\omega_n(\tau^*)/\omega_n(\tau^*) > 0 \) is a purely imaginary root of Eq. (22) if and only if \( \tau^* \) is a root of the function \( S_n^m(\tau) \), defined by

\[
S_n^m(\tau) = \tau - \frac{\theta_n(\tau) + 2m\pi}{\omega_n(\tau)}, \quad \tau \in I_n, \text{ with } n, m \in \mathbb{N}_0.
\]

The following is the result introduced by Beretta and Kuang [2].

**Lemma 3.11.** For a fixed \( n_0 \in \mathbb{N}_0 \), assume that the function \( S_{n_0}^m(\tau) \) has a simple positive root \( \tau^* \in I_{n_0} \) for some \( m \in \mathbb{N}_0 \), then a pair of simple purely imaginary roots \( \pm i\omega_{n_0}(\tau^*) \) of equation (22) exists at \( \tau = \tau^* \) and

\[
\text{Sign}\left\{ \frac{d \Re(\lambda)}{d\tau} \right\}_{\lambda = i\omega_{n_0}(\tau^*)} = \text{Sign}\left\{ \frac{dF_{n_0}}{d\omega_{n_0}}(\omega_{n_0}(\tau^*), \tau^*) \right\} \times \text{Sign}\left\{ \frac{dS_{n_0}^m(\tau)}{d\tau} \right\}_{\tau = \tau^*}. \tag{28}
\]

Since

\[
\frac{dF_{n_0}}{d\omega_{n_0}}(\omega_{n_0}(\tau), \tau) = 4\omega_{n_0}^4 + 2(\alpha^2 + d^2 - f_{n_0}^2)\omega_{n_0} > 0,
\]

condition (28) is equivalent to

\[
\delta(\tau^*) = \text{Sign}\left\{ \frac{d \Re(\lambda)}{d\tau} \right\}_{\lambda = i\omega_{n_0}(\tau^*)} = \text{Sign}\left\{ \frac{dS_{n_0}^m(\tau)}{d\tau} \right\}_{\tau = \tau^*}.
\]

Therefore, this pair of simple conjugate purely imaginary roots crosses the imaginary axis from left to right if \( \delta(\tau^*) = 1 \) and from right to left if \( \delta(\tau^*) = -1 \).

**Remark 4.** It can be easily observed that \( I_{n+1} \subset I_n \), \( S_n^m(\tau) > S_{n+1}^m(\tau) \) for all \( \tau \in I_n \), \( S_n^m(\tau) > S_{n+1}^m(\tau) \) for all \( \tau \in I_{n+1} \) and \( S_n^m(\tau) \) approaches to negative infinity at \( \tau \). Thus, if \( S_{n_0}^m \) has no zero in \( I_{n_0} \), then for any \( n \geq n_0 \), \( m \geq m_0 \), \( \tau \in I_n \), we have \( S_n^m(\tau) \leq S_{n_0}^m(\tau) \) and \( S_n^m \) have no zeros in \( I_n \).

**Remark 5.** For \( n, m \in \mathbb{N}_0 \), denote the set of the zeros of \( S_n^m \) by

\[
J_n^m = \{ \tau_n^m | \tau_n^m \in I_n, S_n^m(\tau_n^m) = 0 \}.
\]

In what follows, we always assume \( \frac{dS_n^m}{d\tau}(\tau_n^m) \neq 0 \) and \( J_{n_1}^{m_1} \cap J_{n_2}^{m_2} = \emptyset \) for \( n_1 > n_2 \) and \( m_1 < m_2 \). Rearrange these roots in the set

\[
J = \bigcup_{n, m \in \mathbb{N}_0} J_n^m = \{ \tau_0, \tau_1, \cdots, \tau_k \}, \text{ with } \tau_i < \tau_{i+1}, \ 0 \leq i \leq k + 1.
\]

Applying Corollary in [13], we can draw the conclusion: If \( \frac{m_{n+1}}{n_{n+1}} < \alpha < \frac{m_K}{n_K} \) and \( \tau \in (\tau_k, \tau_{max}) \), then all roots of Eq. (22) have negative real parts when \( \tau \in (\tau_k, \tau_{max}) \) and at least a pair of roots have positive real parts when \( \tau \in (\tau_k, \tau_{max}) \). Furthermore, all other roots of Eq. (22), except a pair of purely imaginary roots, have negative real parts when \( \tau = \tau_k \).

Now we can state the following theorem on the existence of a Hopf bifurcation at the positive steady state.

**Theorem 3.12.** Assume that (H2) holds.

(i) If either \( I_0 \) is empty or the function \( S_0^0 \) has no positive zero in \( I_0(\emptyset) \), then for all \( \tau \in (\tau_k, \tau_{max}) \), the steady state \( E_3 \) of Eq. (4) is locally asymptotically stable;

(ii) If \( J \neq \emptyset \), then the steady state \( E_3 \) of Eq. (4) is locally asymptotically stable for \( \tau \in (\tau_k, \tau_{max}) \) and unstable for \( \tau \in (\tau_k, \tau_{max}) \) with a Hopf bifurcation occurring at \( E_3 \) when \( \tau = \tau_i \in J \).
Theorem 3.12 gives some sufficient conditions to ensure that Eq. (4) undergoes a Hopf bifurcation at $E_3$. Next, under the conditions of Theorem 3.12(ii), we shall use the center manifold and normal form theories presented by Wu [21] and Faria [4] to study the direction of Hopf bifurcation and the stability of the bifurcating periodic solutions from $E_3$. As the details are given in the Appendix, we summarize the results in the following theorem.

**Theorem 3.13.** Assume that the conditions ensuring that Hopf bifurcation at $E_3$ occurs in Theorem 3.12(ii) are fulfilled. Then the periodic solutions bifurcated from $E_3$ are asymptotically stable/unstable on the center manifold if $\text{Re}(c_1(0)) < 0(> 0)$, where $c_1(0)$ is derived in the Appendix.

4. Numerical simulations. We choose a set of parameter values of Eq. (4):

$$K = 11, \; r = 1, \; m = 2, \; b = 38, \; d = 0.01, \; d_1 = d_2 = 0.1, \; l = 1.$$  \hfill (29)

Under this parameter set, one can easily see that

$$\frac{m}{b+1} \approx 0.0513, \quad \frac{mK^2}{b+K} \approx 1.5220$$

and thus

$$\frac{m}{b+1} < r < \frac{mK^2}{b+K}.$$  

By calculation, we obtain that

$$\tau_{\text{max}} \approx 42.0034, \quad \bar{\tau} \approx 17.6558.$$

**Figure 1.** The graphs of $s_1(u) + s_2(u)$ and $H_n(\tau)$ on $(\tau, \bar{\tau})$.

**Figure 2.** The graphs of $S^m_{\sigma_i}$ on $I_n$.

In order to gain the set $I_n$, we picture the graphs of $(s_1 + s_2)(u)$ and $H_n(\tau)$ in Figure 1. From the graph of $(s_1 + s_2)(u)$ in Figure 1, we get that $u(\tau_c) \approx 5.9108$, by
calculation, we can get that \( \tau \approx 4.2894 \). Note that \( \tau \in I_n \) is equivalent to \( H_n(\tau) < 0 \). Hence we have

\[
I_0 = (4.29, 11.51), \quad I_1 = (4.29, 10.13) \quad I_2 = (4.29, 5.37), \quad I_n = \emptyset, \quad n \geq 3.
\]

Accordingly, the pictures of \( S^m_n \) on \( I_n \) can be drawn clearly (see Figure 2). From Figure 2, we obtain the set of zeros of \( S^m_n \)

\[
J_0^0 = \{10.8427\}, \quad J_1^0 = \{8.9733\},
\]

and \( \frac{dS_n^m(\tau)}{d\tau} \neq 0 \). Hence we have \( k = 1 \) and

\[
\tau_0 \approx 8.9733, \quad \tau_1 \approx 10.8427.
\]

Therefore, the positive steady state \( E_3 \) is asymptotically stable when \( \tau \in (\tau_1, \tau_{\text{max}}) \), and unstable when \( \tau \in (\tau_0, \tau_1) \), as well as Hopf bifurcation takes place when \( \tau = \tau_i \in J \). The stability of \( E_3 \) is illustrated by Figure 3, the bifurcating periodic solutions are illustrated by Figure 4 and Figure 5.

**Figure 3.** For system (4), the positive steady state \( E_3 \) is locally asymptotically stable, where \( \tau = 20 \in (\tau_k, \tau_{\text{max}}) \), \( u_0(x, t) = 7.7 + 0.5 \cos 2x \), \( v_0(x, t) = 12 - 0.5 \cos 2x \).

**Figure 4.** For system (4), the bifurcating periodic solutions are asymptotically stable, where \( \tau = 10.8426 < \tau_1 \approx 10.8427 \) and is close to \( \tau_1 \). The initial values are \( u_0(x, t) = 6.92 + 0.5 \cos(2x) \), \( v_0(x, t) = 13.63 - 0.5 \cos(2x) \).

By using the algorithm given in Appendix, we can obtain \( \text{Rec}_1(0) \) corresponding to \( \tau = \tau_0 \) and \( \tau = \tau_1 \), as

\[
\text{Rec}_1^0(0) \approx 0.1456 \quad \text{and} \quad \text{Rec}_1^1(0) \approx -0.1389,
\]
For system (4), the transient spatially homogeneous periodic solutions occur, where \( \tau = 8.9734 > \tau_0 \approx 8.9733 \) and is close to \( \tau_0 \). The initial values are \( u_0(x,t) = 6.77 + 0.8 \cos 2x \), \( v_0(x,t) = 13.73 - 0.8 \cos 2x \).

\[ E_2(K,0) \] is locally asymptotically stable, where \( \tau = 43 > \tau_{\text{max}} \approx 42.0034 \), \( u_0(x,t) = 9.5 + 0.3 \cos 2x \), \( v_0(x,t) = 8 + \cos 2x \).

\[ E_0(0,0) \] attracts the solutions which have big initial value \( v_0(x,t) \), where \( \tau = 43 > \tau_{\text{max}} \approx 42.0034 \), \( u_0(x,t) = 9.5 + 0.3 \cos 2x \), \( v_0(x,t) = 16 + \cos 2x \).

respectively. It follows that the direction of Hopf bifurcation is forward at \( \tau_0 \) and backward at \( \tau_1 \). The bifurcating periodic solutions exist for \( \tau < \tau_1 \) and close to \( \tau_1 \), these bifurcating periodic solutions are orbitally asymptotically stable. The bifurcating periodic solutions exist for \( \tau > \tau_0 \) and close to \( \tau_0 \) and these bifurcating
periodic solutions are unstable. These are illustrated in Figure 4 and Figure 5. When $\tau > \tau_{\text{max}}$, the positive steady state $E_3$ disappears, and by Theorem 3.4, the steady state $E_2$ is local stable (see Figure 6). By theorem 3.7, when the initial predator value is large enough, the corresponding solution of system (4) will tend to $(0,0)$ uniformly for $x \in \Omega$ as $t \to \infty$ (see Figure 7). Figure 7 illustrates the excessive predation phenomenon.

5. Conclusion. In this paper, a diffusive predator-prey system with strong Allee effect and stage structure is investigated. Theoretical analysis indicates that both strong Allee effect and stage structure would make the dynamics of the system more complicated. These contains over-predation phenomenon, bistable phenomenon caused by the strong Allee effect and time periodic orbits caused by the stage structure. The attractivity basin of the die-out equilibrium and the local stability of boundary equilibria are investigated, suggesting that under certain condition, the initial value of the prey is important as well as the Allee threshold. If the initial value of the prey is smaller than the Allee threshold, the prey species will die out, and so does the predator species, see Theorem 3.6; if the initial value of the prey is larger than the Allee threshold, and when the initial value of the predator is large enough, both the prey and the predator will die out, see Theorem 3.7. Moreover, the existence and stability of the positive steady state are studied, the Hopf bifurcation at the positive steady state is investigated. We show that when the delay $\tau$ decreases to a critical value, the positive steady state will lose its stability and Hopf bifurcation will occur. Since the coefficients of the corresponding characteristic equation depend on the delay $\tau$, we prove that all roots of the characteristic equation have negative real parts when $\tau$ is close enough to $\tau_{\text{max}}$, see Theorem 3.9 and 3.10. By the theory of normal form and center manifold method, the conditions for determining the direction of the bifurcation and stability of bifurcating periodic solutions are derived in the appendix. We conclude that the time delay $\tau$ can affect the coexistent of the prey and the predator. Under certain condition, when $\tau_k < \tau < \tau_{\text{max}}$, these two species will coexist; when $\tau \in (\tau_{k-1}, \tau_k)$, the prey and the predator will exhibit oscillatory behaviour.

Appendix. In this appendix, by applying the center manifold theorem and normal form theory of the partial differential equations with delay[4, 21], we study the direction of Hopf bifurcation and stability of bifurcating periodic solutions from the positive equilibrium $E_3$ near $\tau = \tau^*$ under the conditions of Theorem 3.12(ii).

Denote

\[
L_1(\tau) = \begin{pmatrix} \frac{-3u_*^2 + (2K + 2)u_* - K}{K} & -2mbu_*v_* \left(\frac{b + u_*^2}{b + u_*^2}\right)^2 & -r e^{d\tau} \\ 0 & 0 & -r \end{pmatrix} = \begin{pmatrix} l_{11} & l_{12} \\ 0 & l_{22} \end{pmatrix}
\]

(30)

\[
L_2(\tau) = \begin{pmatrix} 0 & 0 \\ me^{-d\tau} & \frac{2bu_*v_*}{(b + u_*^2)^2} \end{pmatrix} r = \begin{pmatrix} 0 & 0 \\ k_{21} & k_{22} \end{pmatrix}
\]

(31)
Rescaling the time \( t \to t/\tau \), and let \( \tilde{u}(x, t) = u(x, t) - u_*, \ \tilde{v}(x, t) = v(x, t) - v_* \), then we have

\[
\begin{align*}
\frac{\partial \tilde{u}}{\partial t} &= \tau [d_1 \Delta \tilde{u} + l_{11} \tilde{u} + l_{12} \tilde{v} + f_1(\tilde{u}, \tilde{v})], \quad x \in (0, l\pi), \ t > 0, \\
\frac{\partial \tilde{v}}{\partial t} &= \tau [d_2 \Delta \tilde{v} + l_{22} \tilde{v} + k_{21} \tilde{u}'(-1) + k_{22} \tilde{v}'(-1) + f_2(\tilde{u}, \tilde{v})], \quad x \in (0, l\pi), \ t > 0,
\end{align*}
\]

where

\[
\begin{align*}
\tilde{u}(x, t) &= \tilde{u}_0(x, t), \ \tilde{v}(x, t) = \tilde{v}_0(x, t), \\
\end{align*}
\]

and for \((\phi_1, \phi_2) \in \mathcal{C} := C([-1, 0], X)\),

\[
\begin{align*}
f_1(\phi_1, \phi_2) &= a_1 \phi_1^2(0) + a_2 \phi_1(0) \phi_2(0) + a_3 \phi_2^2(0) \phi_2(0) + a_4 \phi_1^3(0) + \mathcal{O}(4), \\
f_2(\phi_1, \phi_2) &= a_5 \phi_1(-1) \phi_2(-1) + a_6 \phi_1^2(-1) + a_7 \phi_2^2(-1) \phi_2(-1) + a_8 \phi_1^3(-1) + \mathcal{O}(4),
\end{align*}
\]

where

\[
\begin{align*}
a_1 &= \frac{1}{K} - \frac{3 \hat{u}_*}{K} + 1 - \frac{mb \hat{u}_*(b - 3 \hat{u}_*)}{(b + \hat{u}_*)^3}, & a_2 &= -\frac{2mb \hat{u}_*}{(b + \hat{u}_*)^2}, \\
a_3 &= \frac{mb(b - 3 \hat{u}_*)}{(b + \hat{u}_*)^3}, & a_4 &= \frac{1}{K} + \frac{4mb \hat{u}_* \hat{v}_*(b - \hat{u}_*)}{(b + \hat{u}_*)^4}, \\
a_5 &= \frac{2br}{u_* (b + u_*)^3}, & a_6 &= \frac{br \hat{v}_*(b - 3 \hat{u}_*)^2}{(b + \hat{u}_*)^4}, \\
a_7 &= \frac{u_*^2 (b + u_*)^2}{br (b - 3 \hat{u}_*)^3}, & a_8 &= \frac{u_*^2 (b + u_*)^2}{4br \hat{v}_* (\hat{u}_*^2 - b)},
\end{align*}
\]

and \( \hat{u}_* \) and \( \hat{v}_* \) are the values of \( u_* \) and \( v_* \) at \( \tau = \tau^* \).

Let \( \tau = \tau^* + \epsilon \). From the above discussion, we know that when \( \epsilon = 0 \), system (32) undergoes a Hopf bifurcation at the equilibrium \((0, 0)\). Then we can rewrite system (32) in a abstract form in the space \( \mathcal{C} = C([-1, 0], X) \) as

\[
\frac{dU(t)}{dt} = \hat{D} \Delta U(t) + L_\epsilon(U^\epsilon) + F(\epsilon, U^\epsilon), \tag{34}
\]

where

\[
\hat{D} = (\tau^* + \epsilon) D \text{ and } L_\epsilon : \mathcal{C} \to X, \ F : \mathcal{C} \to X \]

are defined, respectively, by

\[
L_\epsilon(\phi(\theta)) = (\tau^* + \epsilon)L_1(\tau^* + \epsilon)\phi(0) + (\tau^* + \epsilon)L_2(\tau^* + \epsilon)\phi(-1),
\]

\[
F(\epsilon, \phi(\theta)) = (F_1(\epsilon, \phi(\theta)), F_2(\epsilon, \phi(\theta)))^T,
\]

with

\[
(F_1(\epsilon, \phi(\theta)), F_2(\epsilon, \phi(\theta)))^T = (\tau^* + \epsilon)(f_1(\phi_1(\theta), \phi_2(\theta)), f_2(\phi_1(\theta), \phi_2(\theta)))^T,
\]

where \( f_1 \) and \( f_2 \) are defined by (33).

The linearized equation at the origin \((0, 0)\) has the form

\[
\frac{dU(t)}{dt} = \hat{D} \Delta U(t) + L_\epsilon(U^\epsilon). \tag{35}
\]
According to the theory of semigroup of linear operator [9], we know that the solution operator of (35) is a $C_0-$semigroup, and the infinitesimal generator $A_\epsilon$ is given by

$$
A_\epsilon \phi = \begin{cases} 
\phi(\theta), & \theta \in [-1,0), \\
\hat{D}\Delta \phi(0) + L_\epsilon(\phi), & \theta = 0,
\end{cases}
$$

(36)

with

$$
dom(A_\epsilon) := \{ \phi \in \mathcal{C} : \hat{\phi} \in \mathcal{C}, \phi(0) \in dom(\Delta), \hat{\phi}(0) = \hat{D}\phi(0) + L_\epsilon(\phi) \}.
$$

Define the extended Banach space:

$$
\mathcal{BC} := \left\{ \psi : [-1,0] \to X_C, \psi \text{ is continuous on } [-1,0), \exists \lim_{\theta \to 0^-} \psi(\theta) \in X_C \right\},
$$

rewrite (34) as the abstract ODE in $\mathcal{BC}$, then

$$
\frac{dU_t}{dt} = A_\epsilon U_t + X_0 F(\epsilon, U^t),
$$

(37)

where

$$
X_0(\theta) = \begin{cases} 
0, & \theta \in [-1,0), \\
I, & \theta = 0.
\end{cases}
$$

Denote $\beta_n^1(x) = (b_n(x), 0)^T, \beta_n^2(x) = (0, b_n(x))^T$, $\beta_n = (\beta_n^1(x), \beta_n^2(x))$, while

$$
b_n(x) = \frac{\cos \frac{\pi x}{2}}{\|\cos \frac{\pi x}{2}\|_{2,2}}.
$$

Define $A_{\epsilon,n}$ as

$$
A_{\epsilon,n}(\phi_n(\theta)b_n) = \begin{cases} 
\hat{\phi}_n(\theta)b_n, & \theta \in [-1,0), \\
\int_{-1}^\theta d\eta(\epsilon, \theta)\phi_n(\theta)b_n, & \theta = 0,
\end{cases}
$$

(38)

and

$$
L_{\epsilon,n}(\phi_n) = (\tau^* + \epsilon) L_1(\tau^* + \epsilon) \phi_n(0) + (\tau^* + \epsilon) L_2(\tau^* + \epsilon) \phi_n(-1).
$$

Using the Riesz representation theorem, we see that there exists a bounded variation $2 \times 2$ matrix function $\eta_n(\epsilon, \theta)(\theta \in [-1,0])$ such that

$$
\int_{-1}^\theta d\eta_n(\epsilon, \theta)\phi_n(\theta) = -\frac{n^2}{T} (\tau^* + \epsilon) D\phi_n(0) + L_{\epsilon,n}(\phi_n),
$$

for $\phi \in C([-1,0], \mathbb{R}^2)$. In fact, we can choose

$$
\eta_n(\epsilon, \theta) = \begin{cases} 
-(\tau^* + \epsilon) L_2(\tau^* + \epsilon), & \theta = -1, \\
0, & \theta \in (-1,0), \\
(\tau^* + \epsilon)(L_1(\tau^* + \epsilon) - \frac{n^2}{T^2} D), & \theta = 0.
\end{cases}
$$

Introduce the bilinear form $(\cdot, \cdot)$ on $\mathcal{C}^* \times \mathcal{C}$

$$
(\psi, \phi) = \sum_{k,j=0}^\infty (\psi_k, \phi_j) \int_\Omega b_k b_j dx,
$$

while

$$
\psi = \sum_{k=0}^\infty \psi_k b_k \in \mathcal{C}^*, \phi = \sum_{j=0}^\infty \phi_j b_j \in \mathcal{C},
$$
and
\[ \psi_n \in C^* := C([0, 1], \mathbb{R}^2), \phi_n \in C := C([-1, 0], \mathbb{R}^2). \]

Since for \( k \neq j \),
\[ \int_{\Omega} b_kb_j dx = 0, \]
we get that
\[ (\psi, \phi) = \sum_{n=0}^{\infty} (\psi_n, \phi_n)_c |b_n|^2, \]
where \((\cdot, \cdot)_c\) is the bilinear form defined on \( C^* \times C \)
\[ (\psi_n, \phi_n)_c = \overline{\psi_n^T(0)}\phi_n(0) - \int_{-1}^{0} \int_{\xi=0}^{\theta} \overline{\psi_n(\xi)}d\eta_n(0, \theta)\phi_n(\xi)d\xi. \]

Define the adjoint operator \( A^* \) of \( A_0 \) on \( C^* := C([0, 1], X) \),
\[ A^* \psi(s) = \begin{cases} -\psi(s), & s \in (0, 1], \\ \infty \int_{n=0}^{s} d\eta_n^T(0, \theta)\psi_n(-\theta)b_n, & s = 0. \end{cases} \]

Let
\[ q(\theta)b_{n_0} = q(0)e^{i\omega_{n_0}\tau^*}b_{n_0}, \quad q^*(s)b_{n_0} = q^*(0)e^{i\omega_{n_0}\tau^*}b_{n_0} \]
be eigenfunctions of \( A_0 \) and \( A^* \) corresponding to the eigenvalues \( i\omega_{n_0}\tau^* \) and \( -i\omega_{n_0}\tau^* \).

By calculation, we can choose
\[ q(0) = (1, q_1)^T, \quad q^*(0) = M(q_2, 1)^T, \]
such that \((q^*, q)_c = 1\), where
\[ q_1 = \frac{\omega_{n_0} + \frac{d}{2} - l_{11}}{l_{12}}, \quad q_2 = \frac{-\omega_{n_0} + \frac{d}{2} - l_{22} - k_{22\ell}e^{i\omega_{n_0}\tau^*}}{l_{12}}, \]
\[ M = \frac{1}{(q_1 + q_2) + \tau^*(k_{21} + k_{22\ell})e^{-i\omega_{n_0}\tau^*}}, \]
with \( l_{11}, l_{12}, l_{22}, k_{21}, k_{22\ell} \) defined by (30) and (31).

Now we can decompose the space \( BC \) by \( \Lambda = \{ \pm i\omega_{n_0}\tau^* \} \) as follows
\[ BC = P + Q, \]
where
\[ P = \{ zb_{n_0} + \overline{z}b_{n_0} | z \in \mathbb{C} \}, \]
\[ Q = \{ \phi \in BC | (q^*b_{n_0}, \phi) = 0 \text{ and } (\overline{q^*}b_{n_0}, \phi) = 0 \}. \]

This means \( P \) is the 2-dimensional center subspace spanned by the basis vectors of the linear operator \( A_0 \) associated with purely imaginary eigenvalues \( \pm i\omega_{n_0}\tau^* \), and \( Q \) is the complement space of \( P \).

Thus, system (37) could be written as
\[ U^t = z(t)q^\ast b_{n_0} + \overline{z(t)}\overline{q}(\cdot)b_{n_0} + W(t, \cdot), \]
while
\[ z(t) = (q^*b_{n_0}, U^t), \quad W(t, \cdot) \in Q, \quad (39) \]
\[ W(t, \theta) = U^t - 2\text{Re}\{z(t)q(\theta)b_{n_0}\}, \quad (40) \]
thus we get that
\[ \dot{z}(t) = i\omega_{n_0}\tau^* z(t) + \overline{q}^T(0) \left\langle F(0, U^t), \beta_{n_0} \right\rangle, \quad (41) \]
Therefore the system restricted to the center manifold is given by
\[ W(t, \theta) = W(z(t), \bar{z}(t), \theta) = W_{20}(\theta) \frac{z^2}{2} + W_{11}(\theta) z \bar{z} + W_{02}(\theta) \frac{\bar{z}^2}{2} + \cdots, \quad (42) \]

For solution \( U_t \in \mathcal{C}_0 \), denote
\[ F(0, U^t)|_{\mathcal{C}_0} = \tilde{F}(0, z, \bar{z}), \]
\[ \tilde{F}(0, z, \bar{z}) = F_{20}(\frac{z^2}{2}) + F_{11}(z \bar{z}) + F_{02}(\frac{\bar{z}^2}{2}) + \cdots \]

Therefore the system restricted to the center manifold is given by
\[ \dot{z}(t) = i \bar{\omega}_n \tau^* z(t) + g(z, \bar{z}), \]

by calculation, we can get that
\[ g_{20} = \tau^* \mathcal{M} \int_0^{l \pi} b_{n_0}^3 dx [\bar{q}_2(2a_1 + 2a_2 q_1) + 2a_5 q_1 e^{-2i\bar{\omega}_n \tau^*} + 2a_6 e^{-2i\bar{\omega}_n \tau^*}], \]
\[ g_{11} = \tau^* \mathcal{M} \int_0^{l \pi} b_{n_0}^3 dx \{ [\bar{q}_2(2a_1 + a_2(q_1 + \bar{q}_1))] + a_5(\bar{q}_1 + q_1) + 2a_6 \}, \]
\[ g_{02} = \tau^* \mathcal{M} \int_0^{l \pi} b_{n_0}^3 dx [\bar{q}_2(2a_1 + 2a_2 q_1) + 2a_5 q_1 e^{2i\bar{\omega}_n \tau^*} + 2a_6 e^{2i\bar{\omega}_n \tau^*}], \]
\[ g_{21} = \tau^* \mathcal{M} \left( Q_1 \int_0^{l \pi} b_{n_0}^4 dx + Q_2 \int_0^{l \pi} b_{n_0}^2 dx \right), \]

where
\[ Q_1 = \bar{q}_2 [6a_4 + 2a_3(\bar{q}_1 + q_1)] + 6a_8 e^{-i\bar{\omega}_n \tau^*} + 2a_7(2q_1 e^{-i\bar{\omega}_n \tau^*} + \bar{q}_1 e^{-i\bar{\omega}_n \tau^*}], \]
\[ Q_2 = \bar{q}_2 \left[ 2a_1(W^{(1)}_{11}(0) + 2W^{(1)}_{11}(0)) + a_2(W^{(2)}_{20}(0) + 2W^{(2)}_{11}(0) + \bar{q}_1 W^{(1)}_{20}(0) \right. \]
\[ + 2q_1 W^{(1)}_{11}(0)) \right] + a_5 \left[ \bar{q}_1 W^{(1)}_{20}(-1) e^{i\bar{\omega}_n \tau^*} + 2q_1 W^{(1)}_{11}(-1) e^{-i\bar{\omega}_n \tau^*} \right. \]
\[ + 2W^{(2)}_{11}(-1) e^{-i\bar{\omega}_n \tau^*} + W^{(2)}_{20}(-1) e^{i\bar{\omega}_n \tau^*} \right] + 2a_6 \left[ W^{(1)}_{20}(-1) e^{i\bar{\omega}_n \tau^*} \right. \]
\[ + 2W^{(1)}_{11}(-1) e^{-i\bar{\omega}_n \tau^*} \left. \right]. \]

To get \( g_{21} \), we should first calculate \( W_{20} \) and \( W_{11} \). Take derivative of \( W(t) \) in (40), we get
\[ \dot{W} = U^{\tau} z q b_{n_0} - \bar{z} q b_{n_0} \]
\[ = \begin{cases} \mathcal{A} W - 2 \text{Re} \{ g(z, \bar{z}) q(\theta) b_{n_0} \}, & \theta \in [-r, 0), \\ \mathcal{A} W - 2 \text{Re} \{ g(z, \bar{z}) q(\theta) b_{n_0} + \tilde{F} \}, & \theta = 0, \end{cases} \]
\[ = \mathcal{A} W + H(z, \bar{z}, \theta), \]

where
\[ H(z, \bar{z}, \theta) = H_{20} \frac{z^2}{2} + H_{11} z \bar{z} + H_{02} \frac{\bar{z}^2}{2} + \cdots \]
Apparently,
\[
H_{20}(\theta) = \begin{cases} 
-g_{20}(\theta)\bar{b}_{n_0} - g_{02}(\theta)\bar{b}_{n_0}, & \theta \in [-r, 0), \\
-g_{20}(0)\bar{b}_{n_0} - g_{02}(0)\bar{b}_{n_0} + \hat{F}_{zz}, & \theta = 0,
\end{cases}
\]
\[
H_{11}(\theta) = \begin{cases} 
-g_{11}(\theta)\bar{b}_{n_0} - g_{11}(\theta)\bar{b}_{n_0}, & \theta \in [-r, 0), \\
-g_{11}(0)\bar{b}_{n_0} - g_{11}(0)\bar{b}_{n_0} + \hat{F}_{zz}, & \theta = 0.
\end{cases}
\]
Comparing the coefficients of (43) with the derived function of (40), we can obtain that
\[
(A_0 - 2i\omega_{n_0} I)W_{20}(\theta) = -H_{20}(\theta), \quad A_0 W_{11}(\theta) = -H_{11}(\theta), \quad \ldots \quad (44)
\]
From (36) and (44), for \(\theta \in [-1, 0)\), we have
\[
W_{20}(\theta) = -\frac{g_{20}}{i\omega_{n_0} \tau^*} \left( \frac{1}{q_1} \right) e^{i\omega_{n_0} \tau^* \theta} \bar{b}_{n_0} - \frac{g_{02}}{3i\omega_{n_0} \tau^*} \left( \frac{1}{q_1} \right) e^{-i\omega_{n_0} \tau^* \theta} \bar{b}_{n_0} + E_1 e^{2i\omega_{n_0} \tau^* \theta},
\]
\[
W_{11}(\theta) = \frac{g_{11}}{i\omega_{n_0} \tau^*} \left( \frac{1}{q_1} \right) e^{i\omega_{n_0} \tau^* \theta} \bar{b}_{n_0} - \frac{g_{11}}{i\omega_{n_0} \tau^*} \left( \frac{1}{q_1} \right) e^{-i\omega_{n_0} \tau^* \theta} \bar{b}_{n_0} + E_2,
\]
in which \(E_1\) and \(E_2\) are both 2-dimension vectors in \(X\), and can be determined by setting \(\theta = 0\) in \(H\). Set \(\theta = 0\) and by (44) and (45), we have
\[
(A_0 - 2i\omega_{n_0} \tau^* I)E_1 e^{2i\omega_{n_0} \tau^* \theta} |_{\theta = 0} + \hat{F}_{20} = 0, \quad A_0 E_2 |_{\theta = 0} + \hat{F}_{11} = 0. \quad (46)
\]
The terms \(\hat{F}_{20}\) and \(\hat{F}_{11}\) are elements in the space \(\mathcal{C}\), and they can be denoted by
\[
\hat{F}_{20} = \sum_{n=1}^{\infty} \langle \hat{F}_{20}, \beta_n \rangle b_n, \quad \hat{F}_{11} = \sum_{n=1}^{\infty} \langle \hat{F}_{11}, \beta_n \rangle b_n.
\]
Denote
\[
E_1 = \sum_{n=0}^{\infty} E_1^n b_n, \quad E_2 = \sum_{n=0}^{\infty} E_2^n b_n,
\]
substituting \(E_1, E_2\) into the equation (46), then for \(n = 0, 1, \ldots\), we have
\[
(A_0 - 2i\omega_{n_0} \tau^* I)E_1^n e^{2i\omega_{n_0} \tau^* \theta} |_{\theta = 0} = -\langle \hat{F}_{20}, \beta_n \rangle b_n,
\]
\[
A_0 E_2^n |_{\theta = 0} = -\langle \hat{F}_{11}, \beta_n \rangle b_n.
\]
Thus, \(E_1^n\) and \(E_2^n\) could be calculated by
\[
E_1^n = \left( 2i\omega_{n_0} \tau^* I - \int_{-1}^{0} e^{2i\omega_{n_0} \tau^* \theta} d\eta_n(0, \theta) \right)^{-1} \langle \hat{F}_{20}, \beta_n \rangle b_n,
\]
\[
E_2^n = -\left( \int_{-1}^{0} d\eta_n(0, \theta) \right)^{-1} \langle \hat{F}_{11}, \beta_n \rangle b_n,
\]
where
\[
\langle \hat{F}_{20}, \beta_n \rangle = \begin{cases} 
1, & n_0 \neq 0, \quad n = 0, \\
\frac{1}{\sqrt{2\pi}}, & n_0 \neq 0, \quad n = 2n_0, \\
0, & other,
\end{cases}
\]
\[
\langle \hat{F}_{11}, \beta_n \rangle = \begin{cases} 
\frac{1}{\sqrt{4\pi}}, & n_0 \neq 0, \quad n = 0, \\
\frac{1}{\sqrt{4\pi}}, & n_0 \neq 0, \quad n = 2n_0, \\
0, & other,
\end{cases}
\]
\[
\langle \hat{F}_{11}, \beta_n \rangle = \begin{cases} 
\frac{1}{\sqrt{4\pi}} \hat{F}_{11}, & n_0 \neq 0, \quad n = 0, \\
\frac{1}{\sqrt{2\pi}} \hat{F}_{11}, & n_0 \neq 0, \quad n = 2n_0, \\
\frac{1}{\sqrt{4\pi}} \hat{F}_{11}, & n_0 = 0, \quad n = 0, \\
0, & \text{other}.
\end{cases}
\]

\[
\hat{F}_{20} = \left(2a_5q_1e^{-2i\omega_n\tau^*} + 2a_6e^{-2i\omega_n\tau^*}\right),
\]

\[
\hat{F}_{11} = \left(2a_1 + a_2(q_1 + \bar{q}_1)\right).
\]

Hence, \(g_{21}\) could be represented explicitly.

Denote

\[
c_1(0) = \frac{1}{2\omega_{n_0} \tau^*} \left( g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3} |g_{02}|^2 \right) + \frac{1}{2} g_{21},
\]

\[
\mu_2 = -\frac{\text{Re}(\lambda(\tau^*))}{\text{Re}(c_1(0))},
\]

\[
\beta_2 = 2\text{Re}(c_1(0)),
\]

\[
T_2 = -\frac{1}{\omega_{n_0} \tau^*} \left( \text{Im}(c_1(0)) + \mu_2 \text{Im}(\lambda(\tau^*)) \right).
\]

Then by the general Hopf bifurcation theory (see [20]), we know that \(\mu_2\) determined the directions of the Hopf bifurcation: if \(\mu_2 > 0(<0)\), then the direction of the Hopf bifurcation is forward(backward), correspondingly, this means, when \(\tau > \tau^*(\tau < \tau^*)\) the bifurcating periodic solutions exist; \(\beta_2\) determines the stability of the bifurcating periodic solutions: when \(\beta_2 < 0(>0)\), the bifurcating periodic solutions are orbitally asymptotically stable(unstable); \(T_2\) determines the period of the bifurcating periodic solutions: when \(T_2 > 0(<0)\), the period of the bifurcating periodic solutions will increase (decrease) as \(\tau\) keeping away from \(\tau^*\).

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