REGULARITY OF THE EXTREMAL SOLUTION FOR SINGULAR P-LAPLACE EQUATIONS

DANIELE CASTORINA

Abstract. We study the regularity of the extremal solution $u^*$ to the singular reaction-diffusion problem $-\Delta_p u = \lambda f(u)$ in $\Omega$, $u = 0$ on $\partial \Omega$, where $1 < p < 2$, $0 < \lambda < \lambda^*$, $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain and $f$ is any positive, superlinear, increasing and (asymptotically) convex $C^1$ nonlinearity. We provide a simple proof of known $L^r$ and $W^{1,r}$ a priori estimates for $u^*$, i.e. $u^* \in L^\infty(\Omega)$ if $n \leq p+2$, $u^* \in L^{\frac{n}{n-p-2}}(\Omega)$ if $n > p+2$ and $|\nabla u^*|^{p-1} \in L^{\frac{n}{n-(p')-1}}(\Omega)$ if $n > pp'$.

1. Introduction and main result

The aim of this paper is the study of the following quasilinear reaction-diffusion problem:

\[
\begin{cases}
-\Delta_p u &= \lambda f(u) \quad \text{in } \Omega, \\
u &= 0 \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{cases}
\]

where the diffusion is driven by the singular $p$-Laplace operator $\Delta_p u := \text{div}(|\nabla u|^{p-2}\nabla u)$, $1 < p < 2$, $\Omega$ is a smooth bounded domain of $\mathbb{R}^n$, $n \geq 2$, $\lambda$ is a positive parameter and the nonlinearity $f$ is any $C^1$ positive increasing function satisfying

\[
\lim_{t \to +\infty} \frac{f(t)}{t^{p-1}} = +\infty.
\]

Typical reaction terms $f$ satisfying the above assumptions are given by the exponential $e^u$ and the power $(1+u)^m$ with $m > p-1$.

These reaction-diffusion problems appear in numerous models in physics, chemistry and biology. In particular, when $p = 2$ and $f(u)$ is the exponential, (1.1) is usually referred to as the Gelfand problem: it arises as a simplified model in a number of interesting physical contexts. For example, up to dimension $n = 3$, equation (1.1) can be derived from the thermal self-ignition model which describes the reaction process in a combustible material during the ignition period. We refer the interested reader to [2, 16] for the detailed derivation of the model, as well as other physical motivations for this problem. In the case of singular nonlinearities such as $f(u) = (1-u)^{-2}$, problem (1.1) is also relevant as a model equation to describe Micro Electro Magnetic System (MEMS) devices theory (see [22] for a complete account on this subject). Regarding the MEMS equation, the compactness of the
minimal branch of solutions and some spectral issues connected with it were investigated in [8] for general \( p > 1 \) and nonlinearities \( f(u) \), singular at \( u = 1 \), with growth comparable to \((1 - u)^{-m}\), \( m > 0 \).

In order to set up the problem, we will say that a (nonnegative) function \( u \in W_{0}^{1,p}(\Omega) \) is a weak solution of (1.1) if \( f(u) \in L^{1}(\Omega) \) and \( u \) satisfies

\[
\int_{\Omega} |\nabla u|^{p-2}\nabla u \cdot \nabla \varphi \, dx = \lambda \int_{\Omega} f(u) \varphi \, dx \quad \text{for all } \varphi \in C_{0}^{1}(\Omega).
\]

Moreover, if \( f(u) \in L^{\infty}(\Omega) \) we say that \( u \) is a regular solution of (1.1). By standard regularity results for non-uniformly elliptic equations, one has that every regular solution \( u \) belongs to \( C^{1,\alpha}(\Omega) \) for some \( 0 < \alpha < 1 \) (see [13, 19, 24]).

Under the above hypotheses, problem (1.1) has been extensively studied for \( p = 2 \). Crandall and Rabinowitz [11] prove the existence of an extremal parameter \( \lambda^{*} \in (0, +\infty) \) such that: if \( \lambda < \lambda^{*} \) then problem (1.1) admits a regular solution \( u_{\lambda} \) which is minimal among all other possible solutions, and if \( \lambda > \lambda^{*} \) then problem (1.1) admits no regular solution. Moreover, they prove that every minimal solution \( u_{\lambda} \) is semi-stable in the sense that the second variation of the energy functional associated to (1.1) is nonnegative definite. Subsequently, Brezis and Vázquez [4] prove that the pointwise increasing limit of minimal solutions given by

\[
(1.3) \quad u^{*} := \lim_{\lambda \uparrow \lambda^{*}} u_{\lambda},
\]

is a weak solution of (1.1), usually known as extremal solution. Apart from a detailed study of the model cases (exponential and power nonlinearities), they also raise some interesting open problems for general nonlinearities \( f \) satisfying the above assumptions. One of the most challenging questions proposed, which has not been answered completely yet, is to show show that the extremal solution \( u^{*} \) is bounded or in the energy class, depending on the range of dimensions. In this direction, Nedev [20] proved, in the case of convex nonlinearities, that \( u^{*} \in L^{\infty}(\Omega) \) if \( n \leq 3 \) and \( u^{*} \in L^{r}(\Omega) \) for all \( 1 \leq r < n/(n - 4) \) if \( n \geq 4 \). Subsequently, Cabré [5], Cabré and Sanchón [7], and Nedev [21] proved, in the case of convex domains and general nonlinearities, that \( u^{*} \in L^{\infty}(\Omega) \) if \( n \leq 4 \) and \( u^{*} \in L^{\frac{2n}{n-4}}(\Omega) \cap H_{0}^{1}(\Omega) \) if \( n \geq 5 \). More recently, Villegas [25] extends Nedev result to \( n = 4 \) thanks to a clever use of the a priori estimates of Cabré [5], without convexity assumptions on the domain.

While for the standard Laplacian the regularity of the extremal solution has been subject of a rich literature, in the case of the p-Laplacian, i.e. for \( p \neq 2 \), the available results are very few. García Azorero, Peral and Puel [17, 18] study (1.1) when \( f(u) = e^{u} \), obtaining the existence of the family of minimal regular solutions \( (\lambda, u_{\lambda}) \) for \( \lambda \in (0, \lambda^{*}) \) and that \( u^{*} \) is a weak energy solution independently of \( n \). In addition \( n < p + 4p/(p - 1) \) then \( u^{*} \in L^{\infty}(\Omega) \). Moreover, they prove that \( \lambda^{*} = p^{p-1}(n - p) \) and \( u^{*}(x) = \log(1/|x|^{p}) \) when \( \Omega = B_{1} \) and \( n \geq p + 4p/(p - 1) \). Cabré and Sanchón [9] proved the existence of an extremal parameter \( \lambda^{*} \in (0, \infty) \) such that problem (1.1) admits a minimal regular solution \( u_{\lambda} \in C^{1}_{0}(\Omega) \) for
λ ∈ (0, λ*) and admits no regular solution for λ > λ*. Moreover, every minimal solution uλ is semi-stable for λ ∈ (0, λ*).

One of the main difficulties is the fact that for arbitrary p > 1 it is unknown if the limit of minimal solutions u* is a (weak or entropy) solution of (1.1)_λ. In the affirmative case, it is called the extremal solution of (1.1)_λ. However, in [23] Sanchón has proved that the limit of minimal solutions u* is a weak solution (in the distributional sense) of (1.1)_λ whenever p ≥ 2 and (f(t) − f(0))^{1/(p−1)} is convex for t sufficiently large. Essentially under the same hypotheses, Bidaut-Veron and Hamid in [3] are able to show that u* is a locally renormalized solution of (1.1)_λ in the singular case p < 2.

In this paper, in the spirit of the clever proof of [25] for the case p = 2, we extend some of the results of [10] for the degenerate p-Laplacian to the singular case 1 < p < 2. We obtain the boundedness of the extremal solution up to a critical dimension n_p := p + 2 while we prove that it belongs to L^{n_p/(p−2)}(Ω) if n > p + 2, for any smooth bounded domain Ω, under a standard (asymptotic) convexity assumption on the nonlinearity. Unfortunately our dimensional n_p is not optimal in the singular case (n_p < pp' for 1 < p < 2), hence we are not able to match the one obtained in [3] for this case. However, the regularity results in [3] for the singular case are rather involved while our alternative proof is very simple and direct. In higher dimensions, we will establish in a different way the same Sobolev regularity obtained in [23] for the degenerate case. Our main result is the following:

**Theorem 1.1.** Suppose that 1 < p < 2 and let f be a positive, increasing and superlinear C^1 nonlinearity such that \( f^{1/p-1}(t) \) is convex for any t ≥ T. Let u* be the extremal solution of (1.1) and set n_p := p + 2. The following assertions hold:

(a) If n ≤ n_p then u* ∈ L^∞(Ω). In particular, u* is a regular solution to (1.1)_λ.

(b) If n > n_p then u* ∈ L^{2n/(n−p)}(Ω).

(c) If n > pp' then |∇ u*|^{p−1} ∈ L^{n/(n−pp')} (Ω).

We will recall several auxiliary results as well as giving the proof of Theorem 1.1 in the next section. The main details of the proof of the technical lemmas can be found in the Appendix, section 3.

### 2. Proof of Theorem 1.1

Let us discuss a few preliminary results which will be used. First of all, the proof of Theorem 1.1 relies on the semistability of the minimal solution uλ for 0 < λ < λ*.

Recall that the linearization \( L_{u_\lambda} \) associated to (1.1)_λ at a given solution uλ is defined as

\[
L_{u_\lambda}(v, \varphi) := \int_\Omega |\nabla u_\lambda|^{p−2}(\nabla v, \nabla \varphi) + (p−2) \int_\Omega |\nabla u_\lambda|^{p−4}(\nabla u_\lambda, \nabla v)(\nabla u_\lambda, \nabla \varphi) − \int_\Omega \lambda f'(u_\lambda)v\varphi.
\]
for test functions $v, \varphi \in C_c^1(\Omega)$. Observe that the above linearization, in the degenerate case $1 < p < 2$, makes sense if $|\nabla u_\lambda|^{p-2} \in L^1(\Omega)$, which has been proved by Damascelli and Sciunzi in [12]. We then say that a solution of (1.1) is semistable if the linearized operator at $u_\lambda$ is nonnegative definite, i.e. $L_{u_\lambda}(\varphi, \varphi) \geq 0$ for any $\varphi \in C_c^1(\Omega)$. Equivalently, $u_\lambda$ is semistable if the first eigenvalue of $L_{u_\lambda}$ in $\Omega$, $\mu_1(L_{u_\lambda}, \Omega)$, is nonnegative. However, let us observe that for $p \neq 2$ the latter definition of semistability requires the spectral theory for $L_u$ that has been established by Esposito, Sciunzi and the author in [9].

Next, we will need two a priori estimates for the family of minimal solutions $u_\lambda, 0 < \lambda < \lambda^*$. For the reader’s convenience a sketch of the proofs of both auxiliary results can be found in the Appendix (section 3).

The first estimate gives uniform $L^\infty$ and $L^r$ bounds for $u_\lambda$ in terms of the $W_1, p+2$ norm in a neighborhood of the boundary, depending on the dimension. It can be directly derived from the a priori estimates for semistable solutions contained in Theorem 1.4 of [10], where Sanchón and the author extend to the case $p > 2$ the regularity results of [5, 7] for $u^*$ in convex domains.

**Lemma 2.1.** Let $u_\lambda$ be the minimal solution of (1.1). Then the following alternatives hold:

(a) If $n \leq p + 2$ then there exists a constant $C$ depending only on $n$ and $p$ such that

\[
\|u_\lambda\|_{L^\infty(\Omega)} \leq s + \frac{C}{s^{2/p}|\Omega|^{p-2}} \left( \int_{\{u_\lambda \leq s\}} |\nabla u_\lambda|^{p+2} \, dx \right)^{1/p} \quad \text{for all } s > 0.
\]

(b) If $n > p + 2$ then there exists a constant $C$ depending only on $n$ and $p$ such that

\[
\left( \int_{\{u_\lambda > s\}} \left( u_\lambda - s \right)^{\frac{-np}{n-p+2}} \, dx \right)^{\frac{n-p+2}{np}} \leq \frac{C}{s^{2/p}} \left( \int_{\{u_\lambda \leq s\}} |\nabla u_\lambda|^{p+2} \, dx \right)^{1/p}
\]

for all $s > 0$.

The second preliminary result is a uniform $L^1$ estimate for $(f(u_\lambda))' / u_\lambda$ when the power $\frac{1}{p-1}$ of the nonlinearity is asymptotically convex. This bound is a direct consequence of the estimates which have been proved in Proposition 5.28 of [3].

**Lemma 2.2.** Let $u_\lambda$ be the minimal solution of (1.1) and suppose that $f^{\frac{1}{p-1}}(t)$ is convex for any sufficiently large $t$. Then there exists a constant $M$ independent of $\lambda \in (0, \lambda^*)$ such that

\[
\int_{\{u_\lambda > 1\}} \frac{(f(u_\lambda))'}{u_\lambda} \, dx \leq M.
\]

Finally, we will need a uniform gradient estimate which we can derive from the regularity results for the linear problem. Let us consider

\[\text{Lemma 2.3.}\]
where $g \in L^m(\Omega)$ for some $m > 1$. The following result gives the regularity of the gradient of a weak energy solution of (2.4) (see for instance [1]).

**Lemma 2.3.** Suppose that $n > q$ and let $q^* := \frac{np}{n - q}$ be the usual critical Sobolev exponent. Let $u$ be a weak energy solution of (2.4). Then there exists a constant $C$ depending on $n$, $p$, $q$ and $|\Omega|$ such that:

$$
\|\nabla u\|_{L^{q^*}(\Omega)} \leq C \|g\|_{L^q(\Omega)}.
$$

**Remark 2.4.** Notice that for $p < 2$ it might happen that $(p - 1)q^* < 1$. Hence the Sobolev norm $W_0^{1,(p-1)q^*}$ of the solution $u$ might not make sense: however, for the sake of simplicity, we will still keep this notation intending throughout the discussion that:

$$
\|u\|_{W_0^{1,(p-1)q^*}} := \|\nabla u\|_{L^{q^*}}.
$$

We are now ready to prove the regularity statements for $u^*$ given in Theorem 1.1.

**Proof of Theorem 1.1** part (a).

Let us begin by noticing that a trivial consequence of (2.1) is

$$
(2.5) \quad \|u_\lambda\|_{L^\infty(\Omega)} \leq s + \frac{C}{s^{2/p}} |\Omega|^{\frac{p+2-n}{np}} \left( \int_\Omega |\nabla u_\lambda|^{p+2} \, dx \right)^{1/p}
$$

where the constant $C$ is given by Lemma 2.1. Setting $A := \|u\|_{W_0^{1,p+2}(\Omega)}$, for any $s > 0$ consider the RHS of (2.5) as given by the function

$$
\Phi(s) := s + CA^{\frac{p+2}{p}} s^{-\frac{2}{p}}.
$$

By explicit computation we see that

$$
\Phi'(s) = 1 - \frac{2C}{p} A^{\frac{p+2}{p}} s^{-\frac{p+2}{p}}.
$$

Notice that $\Phi$ is a strictly convex function with a unique global minimum at

$$
s = \left( \frac{2C}{p} \right)^{\frac{p}{p+2}} A.
$$

By direct substitution we see that

$$
\Phi \left( \left( \frac{2C}{p} \right)^{\frac{p}{p+2}} A \right) = \left[ \left( \frac{2C}{p} \right)^{\frac{p}{p+2}} + \frac{C^{\frac{4-n}{2-n}} p^{\frac{2}{p-2}}}{2^{\frac{2}{p-2}}} \right] A.
$$
In particular, by the estimate (2.5) we have thus deduced that for any $0 < \lambda < \lambda^*$ there exists a positive constant $D$ independent of $\lambda$ such that we have

\[(2.6) \quad \|u_\lambda\|_{L^\infty(\Omega)} \leq D \|u_\lambda\|_{W_0^{1,p+2}(\Omega)}\]

Now let us observe that since $u_\lambda$ is weak energy solution of (1.1), from Lemma 2.3 we know that there exists a positive constant $E$ such that:

\[(2.7) \quad \|u_\lambda\|_{W_0^{1,p+2}(\Omega)} \leq E \|\lambda f(u_\lambda)\|_{L^q(\Omega)}\]

for $q = \frac{n(p+2)}{(p-1)n+p+2}$. Thus, taking into account (2.3), (2.6) and (2.7) and recalling that $f$ is increasing, we obtain that for any $0 < \lambda < \lambda^*$ the following chain of inequalities holds:

\[\|u_\lambda\|_{L^\infty(\Omega)} \leq D \|u_\lambda\|_{W_0^{1,p+2}(\Omega)} \leq (DE)^{q(p-1)} \|\lambda f(u_\lambda)\|_{L^q(\Omega)} \leq C \left( |\Omega| + \int_{\{u_\lambda > 1\}} \frac{f(u_\lambda)^{q'}}{u_\lambda^{q'}} \, dx \right)\]

with $C := (DE)^{q(p-1)}(\lambda^*)^q$. Let us point out that in the third line of the above calculations we have applied Holder inequality under the condition $q < p'$, which is true for $n < q_p := \frac{p(p+2)}{2(p-1)}$ and in fact $n \leq n_p < q_p$ for $1 < p < 2$.

Therefore there exist two positive constants $A$ and $B$ independent of $\lambda$ such that:

\[\|u_\lambda\|_{L^\infty(\Omega)} \leq A + B \|u_\lambda\|_{L^\infty(\Omega)}\]

Observe that thanks to (2.2) for any $s > 0$ we have:

**Proof of Theorem 1.1 part (b).**
\[
\int_{\Omega} |u_{\lambda}|^{-\frac{np}{n-(p+2)}} dx = \int_{\{u_{\lambda} \leq s\}} |u_{\lambda}|^{-\frac{np}{n-(p+2)}} dx + \int_{\{u_{\lambda} > s\}} |u_{\lambda}|^{-\frac{np}{n-(p+2)}} dx
\]

\[
= \int_{\{u_{\lambda} \leq s\}} |u_{\lambda}|^{-\frac{np}{n-(p+2)}} dx + \int_{\{u_{\lambda} > s\}} (u_{\lambda} - s)^{-\frac{np}{n-(p+2)}} dx
\]

\[
\leq s^{-\frac{np}{n-(p+2)}} |\{u_{\lambda} \leq s\}| + s^{-\frac{np}{n-(p+2)}} |\{u_{\lambda} > s\}| + \int_{\{u_{\lambda} > s\}} (u_{\lambda} - s)^{-\frac{np}{n-(p+2)}} dx
\]

\[
\leq s^{-\frac{np}{n-(p+2)}} |\Omega| + \frac{C}{s^{-\frac{2n}{n-(p+2)}}} \left( \int_{\Omega} |\nabla u_{\lambda}|^{p+2} dx \right)^{-\frac{n}{n-(p+2)}}
\]

From the above chain of inequalities we easily deduce that

\[(2.8) \quad \|u_{\lambda}\|_{L^{-\frac{n}{n-(p+2)}}(\Omega)} \leq s^{-\frac{np}{n-(p+2)}} |\Omega| + \frac{C}{s^{-\frac{2n}{n-(p+2)}}} \left( \int_{\Omega} |\nabla u_{\lambda}|^{p+2} dx \right)^{-\frac{n}{n-(p+2)}} \]

At this point, we note that the RHS of the above inequality (2.8), up to multiplicative constant depending only on \(|\Omega|\) and a change of variables \(t = s^{-\frac{np}{n-(p+2)}}\), is given by the same function \(\Phi(t)\) which has been optimized at the beginning of the previous proof. Then, in order to prove part (b) of the theorem, we can proceed essentially as in the proof of part (a) using (2.8) in place of (2.5).

**Proof of Theorem 1.1 part (c).**

Let us now suppose that \(n > np^{'}\). Applying again Lemma 2.3 but this time with exponent \(q = \frac{n(p-1)}{n(p-1)-p}\) (which spells \(q^* = \frac{n}{n-(p'+1)}\)), we see that:

\[
\|u_{\lambda}\|_{W_{0}^{1,(p-1)q^*}(\Omega)}^{q(p-1)} \leq E\|\lambda f(u_{\lambda})\|_{L^{q}(\Omega)}^{\frac{1}{p-1}}
\]

Proceeding exactly as in the proof of part (a) and applying Holder inequality (notice that for \(n > np^{'}\) and \(p < 2\) we have that \(q < p^{'}\)) and Sobolev inequality we obtain:

\[
\|u_{\lambda}\|_{W_{0}^{1,(p-1)q^*}(\Omega)}^{q(p-1)} \leq E^{q(p-1)}(\lambda^*)^{q} \left( (f(1))^{q} |\Omega| + \int_{\{u_{\lambda} > 1\}} \frac{f(u_{\lambda})^{q}}{u_{\lambda}^{\frac{q}{2}}} dx \right)
\]

\[
\leq E^{q(p-1)}(\lambda^*)^{q} \left( (f(1))^{q} |\Omega| + M^{\frac{q}{p-1}} \left( \int_{\Omega} u_{\lambda}^{\frac{q}{2}} dx \right)^{\frac{p-1}{q}} \right)
\]

\[
\leq E^{q(p-1)}(\lambda^*)^{q} \left( (f(1))^{q} |\Omega| + M^{\frac{q}{p-1}} S\|u_{\lambda}\|_{W_{0}^{1,(p-1)q^*}(\Omega)}^{\frac{q}{2}} \right)
\]

Once again there exist two positive constants \(A\) and \(B\) independent of \(\lambda\) such that:

\[
\|u_{\lambda}\|_{W_{0}^{1,(p-1)q^*}(\Omega)}^{q(p-1)} \leq A + B\|u_{\lambda}\|_{W_{0}^{1,(p-1)q^*}(\Omega)}^{\frac{q}{2}}
\]
We thus see that $u_λ$ is uniformly bounded in $W^{1,(p-1)q^*}_0(Ω)$ and, taking the limit as $λ \uparrow λ^*$, we get that $u^* \in W^{1,(p-1)q^*}_0(Ω)$. This is exactly statement (c) of Theorem 1.1 so the proof is done.

3. Appendix

Sketch of the proof of Lemma 2.1. Let us recall that the semistability of $u_λ$ reads as

\[(3.9) \int_Ω |∇u_λ|^p|∇ϕ|^2 + (p-2)\int_Ω |∇u_λ|^{p-4}(∇u_λ, ∇ϕ)^2 - \int_Ω λf'(u_λ)ϕ^2 ≥ 0\]

for any $ϕ ∈ C^1_c(Ω)$. Considering $ϕ = |∇u_λ|η$ as a test function in (3.9), we obtain

\[(3.10) \int_Ω [(p-1)|∇u_λ|^{p-2}|∇T,u_λ|^2 + B_{u_λ}^2 \eta^2] dx ≤ (p-1) \int_Ω |∇u_λ|^p|∇η|^2 dx\]

for any Lipschitz continuous function $η$ with compact support. Here $∇T,u_λ$ is the tangential gradient along a level set of $|v|$ while $B_{u_λ}^2$ denotes the $L^2$-norm of the second fundamental form of the level set of $|v|$ through $x$. The fact that $ϕ = |∇u_λ|η$ is an admissible test function as well as the computations behind (3.10) can be found in [14] (see also Theorem 1 in [15]).

On the other hand, noting that $(n-1)H_v^2 ≤ B_{v}^2$ (with $H_v(x)$ denoting the mean curvature at $x$ of the hypersurface $\{y ∈ Ω : |v(y)| = |v(x)|\}$), and

$$|∇u_λ|^{p-2}|∇T,u_λ|^2 = \frac{4}{p^2}|∇T,u_λ|^{\frac{2}{p}};$$

we obtain the key inequality

\[(3.11) \int_Ω \left(\frac{4}{p^2}|∇T,u_λ|^p/2| + \frac{n-1}{p-1} \frac{H_v^2}{|∇u_λ|^p}\right) η^2 dx ≤ \int_Ω |∇u_λ|^p|∇η|^2 dx\]

for any Lipschitz continuous function $η$ with compact support. By taking $η = T,s,u_λ = \min\{s, u_λ\}$ in the semistability condition (3.11) we obtain

$$\int_{\{u_λ > s\}} \left(\frac{4}{p^2}|∇T,u_λ|^p/2| + \frac{n-1}{p-1} \frac{H_v^2}{|∇u_λ|^p}\right) dx ≤ \frac{1}{s^2} \int_{\{u_λ < s\}} |∇u_λ|^p+2 dx$$

for a.e. $s > 0$. In particular,

$$\min\left(\frac{4}{(n-1)p}, 1\right) I_p(u_λ - s; \{x ∈ Ω : u_λ > s\})^p ≤ \frac{p-1}{(n-1)s^2} \int_{\{u_λ < s\}} |∇u_λ|^p+2 dx$$

for a.e. $s > 0$, where $I_p$ is the functional defined as follows

$$I_p(v; Ω) := \left(\int_Ω \left(\frac{1}{p^2}|∇T,v|^p/2| + |H_v|^2|∇v|^p\right) dx\right)^{1/p}, \quad p ≥ 1$$

Then, the $L^∞$ and $L^*$ estimates of Lemma 2.1 follow directly from the Morrey and Sobolev type inequalities involving $I_p$ proved in [10], namely taking $v := u_λ - s$ in

$$\|v\|_{L^∞(Ω)} ≤ C_1 |Ω|^{-\frac{p+2-n}{np}} I_p(v; Ω)$$
if $n < p + 2$, for some constant $C_1 = C_1(n, p)$, or
\[ \|v\|_{L^r(\Omega)} \leq C_2|\Omega|^{\frac{1}{r}} \frac{n-(p+2)}{np} I_p(v; \Omega) \quad \text{for every } 1 \leq r \leq \frac{np}{n-(p+2)}, \]
if $n > p + 2$, where $C_2 = C_2(n, p, r)$. The borderline case $n = p + 2$ is slightly more involved, but we are still able to prove a Morrey type inequality as for the case $n < p + 2$ (see page 22 in [10] for the details). Lemma 2.1 is thus proved.

Sketch of the proof of Lemma 2.2.

Define $\psi(t) := (f(t) - f(0))^{\frac{1}{p-1}}$. By our assumptions we have that $\psi$ is increasing, convex for $t$ sufficiently large and superlinear at infinity. Choosing $\varphi = \psi(u_\lambda)$ in the semistability condition (3.9) and observing that $(p-1)\psi(u_\lambda)^p\psi'(u_\lambda) = f'(u_\lambda)\psi(u_\lambda)^2$, we have
\[
\lambda \int_\Omega \psi(u_\lambda)^p \psi'(u_\lambda) \leq \int_\Omega |\nabla u_\lambda|^p \psi'(u_\lambda)^2 \tag{3.12}
\]
On the other hand, multiplying (1.1) by $g(u_\lambda) := \int_0^u \psi(s)^2 \, ds$ and integrating by parts in $\Omega$ we get:
\[
\int_\Omega |\nabla u_\lambda|^p \psi'(u_\lambda)^2 = \lambda \int_\Omega (f(u_\lambda) - f(0))g(u_\lambda) + \lambda f(0) \int_\Omega g(u_\lambda) \tag{3.13}
\]
Comparing (3.12) and (3.13) it is then easy to see that:
\[
\int_\Omega \psi(u_\lambda)^p \psi'(u_\lambda)^p \leq f(0) \int_\Omega g(u_\lambda) \tag{3.14}
\]
where $h(t) := \int_0^t (\psi'(t) - \psi'(s)) \psi'(s) \, ds$. Now, thanks to the fact that $h(t) \gg \psi'(t)$ for $t$ large (see page 765 in [3]), we deduce from (3.14) that there exists a constant $C$ independent of $\lambda$ such that
\[
\int_\Omega \psi(u_\lambda)^p \psi'(u_\lambda)^p \leq C \tag{3.15}
\]
In particular, since the asymptotic convexity implies $2t\psi'(t) \geq \psi(t)$ for any $t$ sufficiently large, from (3.15) we arrive at
\[
\int_\Omega \frac{\psi(u_\lambda)^p}{u_\lambda} \leq 2C \tag{3.16}
\]
The desired estimate (2.3) is then just a direct consequence of (3.16) and $p' = p/p - 1 > 1$ since:
\[
\int_{\{u_\lambda > 1\}} \frac{(f(u_\lambda))'}{u_\lambda} \, dx = \int_{\{u_\lambda > 1\}} \frac{(f(u_\lambda) - f(0) + f(0))'}{u_\lambda} \, dx \leq \int_{\{u_\lambda > 1\}} \frac{(f(u_\lambda) - f(0))'}{u_\lambda} \, dx + \int_{\{u_\lambda > 1\}} \frac{f(0)'}{u_\lambda} \, dx \leq \int_\Omega \psi(u_\lambda) \, dx + f(0) \int_{\{u_\lambda > 1\}} \frac{1}{u_\lambda} \, dx \leq 2C + f(0)'|\Omega| := M
\]
which proves Lemma 2.2.


References

[1] Alvino, A., Boccardo, L., Ferone, V., Orsina, L., and Trombetti, G.: Existence results for nonlinear elliptic equations with degenerate coercivity, Annali di Matematica 182, 53–79 (2003).

[2] Bebernes, J., Eberly, D.: Mathematical problems from combustion theory, Appl. Math. Sci. 83, Springer-Verlag, New York, (1989).

[3] Hamid, H., Bidaut-Veron, M.F.: On the connection between two quasilinear elliptic problems with source terms of order 0 or 1, Commun. Contemp. Math. 12, 727-88 (2010).

[4] Brezis, H., Vázquez, J.L.: Blow-up solutions of some nonlinear elliptic problems, Rev. Mat. Univ. Complut. Madrid 10 , 443–469 (1997).

[5] Cabré, X.: Regularity of minimizers of semilinear elliptic problems up to dimension four, Comm. Pure Appl. Math 63, 1362–1380 (2010).

[6] Cabré, X., Sanchón, M.: Semi-stable and extremal solutions of reaction equations involving the $p$-Laplacian, Comm. Pure Appl. Anal. 6, 43–67 (2007).

[7] Cabré, X., Sanchón, M.: Geometric-type Hardy-Sobolev inequalities and applications to the regularity of minimizers, J. Funct. Anal. 264, 303–325 (2013).

[8] Castorina, D., Esposito, P., Sciunzi, B.: Degenerate elliptic equations with singular nonlinearities, Calc. Var. PDE 34, 279–306 (2009).

[9] Castorina, D., Esposito, P., and Sciunzi, B.: Spectral theory for linearized $p$-Laplace equations, Nonlinear Anal. 74, 3606–3613 (2011).

[10] Castorina, D., Sanchón, M.: Regularity of stable solutions of $p$-Laplace equations through geometric Sobolev type inequalities, Jour. Europ. Math. Soc., to appear: arXiv:1201.3480v1

[11] Crandall, M.G., Rabinowitz, P.H.: Some continuation and variational methods for positive solutions of nonlinear elliptic eigenvalue problems, Arch. Rational Mech. Anal. 58, 207–218 (1975).

[12] Damascelli, L., Sciunzi, B.: Regularity, monotonicity and symmetry of positive solutions of $m$-Laplace equations. J. Differential Equations 206, 483–515 (2004)

[13] DiBenedetto, E.: $C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations, Nonlinear Anal. 7, 827–850 (1983).

[14] Farina, A., Sciunzi, B., Valdinoci, E.: Bernstein and De Giorgi type problems: new results via a geometric approach, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 7, 741–791 (2008)

[15] Farina, A., Sciunzi, B., Valdinoci, E.: On a Poincaré type formula for solutions of singular and degenerate elliptic equations, Manuscripta Math. 132, 335–342 (2010)

[16] Frank-Kamenetskii, D.A.: Diffusion and Heat Exchange in Chemical Kinetics, Princeton Univ. Press, Princeton, NJ, (1955).

[17] García, J., and Peral, I.: On an Emden-Fowler type equation, Nonlinear Anal. 18, 1085–1097 (1992).

[18] García, J., Peral, I., Puel, J.P.: Quasilinear problemas with exponential growth in the reaction term, Nonlinear Anal. 22 , 481–498 (1994).

[19] Lieberman, G.M.: Boundary regularity for solutions of degenerate elliptic equations, Nonlinear Anal. 12, 1203–1219 (1988).
[20] Nedev, G.: Regularity of the extremal solution of semilinear elliptic equations, C. R. Acad. Sci. Paris Sér. I Math., 330, 997–1002 (2000).

[21] Nedev, G.: Extremal solution of semilinear elliptic equations, Preprint 2001.

[22] Pelesko, J.A., Bernstein, D.H.: Modeling MEMS and NEMS, Chapman Hall and CRC Press (2002).

[23] Sanchón, M.: Existence and regularity of the extremal solution of some nonlinear elliptic problems related to the $p$-Laplacian, Potential Anal. 27, 217–224 (2007).

[24] Tolksdorf, P.: Regularity for a more general class of quasilinear elliptic equations, J. Differential Equations 51, 126–150 (1984).

[25] Villegas, S.: Boundedness of the extremal solution in dimension 4, Adv. Math. 235 (2013), 126-133.