Measurements of semi-local and non-maximally entangled states

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Abstract

Consistency with relativistic causality narrows down dramatically the class of measurable observables. We argue that by weakening the preparation role of ideal measurements, many of these observables become measurable. Particularly, we show by applying entanglement assisted remote operations, that all Hermitian observables of a $2 \times 2$-dimensional bipartite system are measurable.

1 Introduction

Most of the states in the Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ of two systems (a bipartite system) are entangled states. Similarly, the eigenstates of most observables that act on $\mathcal{H}_{AB}$, are entangled states. This poses no problem as long as the systems $A$ and $B$ are both localized in the same region of space – any observable can in principle be measured by some joint interaction between the systems $A$ and $B$ and a localized measuring device. However, once we move $A$ and $B$ apart to separate locations in space, we must introduce a new constraint: the measurements should be consistent with relativistic causality. Hence, only the causal observables, which do not give rise to superluminal effects, are physical. This restriction to causal observables, turns out to narrows down dramatically the class of measurable observables. For a pair of two-level systems (qubits) this seems to leave us only the observables whose eigenstates are the trivial non-entangled basis ($|0_A0_B\rangle, |1_A1_B\rangle, |1_A0_B\rangle, |0_A1_B\rangle$), or the non-local maximally entangled Bell-states\cite{1, 2, 3}. Other operators with non-maximally entangled eigenstates like

$$\alpha |0_A0_B\rangle + \beta |1_A1_B\rangle$$

(1)
\[ |\alpha| \neq |\beta|, \text{ and even the “local” operators with non-entangled one-side “twisted” eigenstates basis} \]
\[ |0_A\rangle|1_B\rangle, \ |0_A\rangle|0_B\rangle, \ |1_A\rangle(|0_B\rangle + |1_B\rangle), \ |1_A\rangle(|0_B\rangle - |1_B\rangle) \]

(2)
cannot be measured without violating causality. The situation becomes even more disturbing for higher dimensional bipartite systems. For a \(4 \times 4\) it has been argued that certain causal operators are in fact not measurable\[^3\].

The purpose of this article is to show that the limitations described above can be avoided by relaxing the requirements that measurements needs to satisfy. The key point is that ideal quantum measurements play a dual role. They serve us to observe an unknown quantity, and to prepare an initial state of the system. Indeed the above mentioned causal restrictions on measurability, have been derived under the assumption that after a measurement is completed, the system collapses to an eigenstates of the observable\[^4\]. This preparation assumption makes it simpler to verify the result of a measurement by repeating again the measurement. However the roles of observation and preparation, are logically independent. In the present article we hence assume that:

A measurement of an observable does not necessarily prepare eigenstates of that same observable.

Instead our measurement procedure will prepare either one of the Bell-states or, with equal unbiased probabilities, one of the untwisted direct product states. Hence, in accordance with causality\[^3\], in both cases the local information is erased, and the local reduced mixed state is a unit matrix. This turns out not only to removes the conflict with causality, but also allows us to construct measurements for non-trivial cases. In fact, we show that at least for \(2 \times 2\) dimensional bipartite systems all the observable can be measured.

Other requirements from the measurement remain intact: 1) When the initial state is an eigenstate of the measured observable, the outcome is with certainty given by the corresponding eigenvalue. 2) For a general state, the corresponding eigenvalues are observed with the usual quantum probability. 3) The measurement is instantaneous in the sense that the local interactions with measuring devices is completed in time \(\Delta t\), that can be taken to be arbitrarily small. 4) The measurement is performed locally at two space-time regions that are causally disconnected. Particularly \(c\Delta t \ll L\), where
$L$ is the distance between the systems A and B. 5) The outcome of the measurement is recorded locally by the observers at A and B, and becomes known only after the local recordings are compared. Finally, we assume that the resources used to perform the measurement include an unlimited supply of entangled pairs which have been earlier distributed between the parties. We will see that different operators require different entanglement resources.

In order to implement a measurement we use the following method. We perform a remote instantaneous transformation of the unmeasurable set of eigenstates to a measurable set. The map between the sets is not deterministic. Different maps can be generated. However in all cases, we can use the local outcomes of the measurements, to infer the relation between the unmeasurable states and the new measurable states. On the latter states we can then perform ordinary measurements. The final result is then obtained by combining the information on the map, with the local measurement outcomes.

We will demonstrate our method on a number of examples. In section 2. and 3. we consider $2 \times 2$ twisted product basis. In section 4. we apply our method to the case of non-maximal eigenstates, and in section 5. to the twisted set[3] in a $4 \times 4$ dimensional Hilbert space.

2 2×2 twisted product basis

Consider a $2 \times 2$ Hilbert space of a bipartite system, held by Alice and Bob, and spanned by the local basis vectors $|0_A\rangle$, $|1_A\rangle$ and $|0_B\rangle$, $|1_B\rangle$, respectively. In the forthcoming, we shall handle the basis $|0\rangle$, $|1\rangle$ as spin $\frac{1}{2}$-like states, $|0\rangle \equiv | \uparrow_z\rangle$, $|1\rangle \equiv | \downarrow_z\rangle$, and define accordingly the Pauli operators.

The first example which we discuss is a non-degenerate operator, defined by the four mutually orthogonal direct-product eigenstates:

$$
\begin{align*}
|\Psi^1_{AB}\rangle &= |0_A\rangle |0_B\rangle, \\
|\Psi^2_{AB}\rangle &= |0_A\rangle |1_B\rangle, \\
|\Psi^3_{AB}\rangle &= |1_A\rangle \frac{1}{\sqrt{2}} (|0_B\rangle + |1_B\rangle), \\
|\Psi^4_{AB}\rangle &= |1_A\rangle \frac{1}{\sqrt{2}} (|0_B\rangle - |1_B\rangle).
\end{align*}
$$

This operator was shown to be semilocalizable[3, 5], i.e. it can be measured with the assistance of one-directional classical communication from Alice to Bob. We now show that this operator may be measured \textit{instantaneously},
without any exchange of information, by utilizing one ebit of shared entanglement.

The main idea here, is to perform a conditional $\pi/2$ rotation of Bob’s state when Alice’s state is $|1_A\rangle$. This maps the twisted basis on Bob’s side

$$\{ |0_B\rangle + |1_B\rangle, |0_B\rangle - |1_B\rangle \} \rightarrow \{ |0_B\rangle, |1_A\rangle \}$$

(4)

On the other hand, no such mapping will be performed if Alice’s state turns out to be $|0_A\rangle$.

The “switch” which controls the remote rotation is in Alice’s hands. Bob on the other hand will perform a fixed set of manipulations independently of Alice’s choice, and finally measures $\sigma_z$. Having done so, Alice and Bob obtain local records that when combined allow us to distinguish with certainty between the elements of the basis (3). The final state of Alice’s and Bob’s spin is always given by a unit density matrix. Hence, although the whole process takes place instantaneously, no violations of causality occur.

Let now consider the process in some detail. Several methods for generating remote operations have been suggested [7, 8, 9, 10, 11]. Here we follow the state-operator (stator) method of Ref. [10], which allows simple and intuitive construction of a large class of remote operations. For the present case, Alice and Bob need one shared ebit in order to perform a remote rotation. The initial state at the hands of Alice and Bob is then

$$\frac{1}{\sqrt{2}} \left( |0_a\rangle \otimes |0_b\rangle + |1_a\rangle \otimes |1_b\rangle \right) \otimes |\Psi_{AB}\rangle$$

(5)

where by the small letters, $a$ and $b$, we denote the ancillary ebit, shared by Alice and Bob, respectively. Bob starts by performing a local CNOT interaction (with respect to $\sigma_y$) between the entangled qubit $b$ and his state $B$, described by the unitary transformation

$$U_{bB} = |0_b\rangle \langle 0_b| \otimes I_B + |1_b\rangle \langle 1_b| \otimes \sigma_y$$

(6)

This yields the state

$$\frac{1}{\sqrt{2}} \left( |0_a\rangle \otimes |0_b\rangle \otimes I_B + |1_a\rangle \otimes |1_b\rangle \otimes \sigma_y \right) |\Psi_{AB}\rangle$$

(7)

Next he performs a measurement of $\sigma_x$ of the entangled qubit $b$ and keeps the result $v(\sigma_x)$. The resulting state is now

$$\left( |0\rangle_a \otimes I_B \pm |1\rangle_a \otimes \sigma_y \right) |\Psi_{AB}\rangle$$

$$\equiv \mathcal{S} |\Psi_{AB}\rangle$$

(8)
where the ± above corresponds to \( v(\sigma_{y_b}) \). The object \( S \) is the state-operator (stator) defined in [10]. The stator satisfies the eigenoperator equation:

\[
\sigma_{x_a} S = v(\sigma_{x_b}) \sigma_{y_B} S
\]

This equation captures the correlations between a unitary transformation which is acted by Alice on \( a \) and the equivalent rotation on an arbitrary state of Bob. Particularly, we also have that if Alice acts with the unitary transformation \( \exp(i\alpha \sigma_{x_a}) \), that is equivalent to the unitary transformation \( \exp(i\alpha \sigma_{y_B}) \) on Bob’s qubit (up to a trivial \( \pi \) rotation around the \( y \)-axis).

Having “prepared” the above stator, Alice proceeds and measures \( \sigma_{z_A} \). Consider the two possible outcomes. If it turns out that \( |\Psi_A\rangle = |0_A\rangle \), she next measures \( \sigma_{z_a} \) and keeps the result \( v(\sigma_{z_a}) \). This will induce on Bob’s qubit the transformation

\[
\frac{1 + v(\sigma_{z_a})}{2} \sigma_{y_B} + v(\sigma_{x_b}) \frac{1 - v(\sigma_{z_a})}{2} \sigma_{y_B}
\]

If, on the other hand, she gets \( |\Psi_A\rangle = |1_A\rangle \), she acts by \( e^{i\pi/4} \sigma_{x_a} \) on \( a \) in order to untwist Bob’s qubit, and then measures \( \sigma_{z_a} \) of \( a \). The induced remote transformation is in this case

\[
\left( \frac{1 + v(\sigma_{z_a})}{2} \sigma_{y_B} + v(\sigma_{x_b}) \frac{1 - v(\sigma_{z_a})}{2} \sigma_{y_B} \right) e^{i\pi/4} v(\sigma_{y_b}) \sigma_{y_B}
\]

The four possible outcomes of this map are summarized in the following table:

| \( \sigma_{z_A} \) \( \sigma_{z_B} \) | \( v(\sigma_{x_B}) = +1 \) | \( v(\sigma_{x_B}) = -1 \) |
|-----------------|-----------------|-----------------|
| \( v(\sigma_{z_A}) = +1 \) | \( |\Psi_{AB}^1\rangle \rightarrow |0_A\rangle |0_B\rangle \) | \( |\Psi_{AB}^4\rangle \rightarrow |0_A\rangle |0_B\rangle \) |
| \( |\Psi_{AB}^2\rangle \rightarrow |0_A\rangle |1_B\rangle \) | \( |\Psi_{AB}^2\rangle \rightarrow |0_A\rangle |1_B\rangle \) |
| \( |\Psi_{AB}^3\rangle \rightarrow |1_A\rangle |0_B\rangle \) | \( |\Psi_{AB}^3\rangle \rightarrow |1_A\rangle |0_B\rangle \) |
| \( |\Psi_{AB}^4\rangle \rightarrow |1_A\rangle |1_B\rangle \) | \( |\Psi_{AB}^4\rangle \rightarrow |1_A\rangle |1_B\rangle \) |

To complete the measurement of the semi-local set, Bob measures the operator \( \sigma_{z_B} \) for his state \( |\Psi_B\rangle \). Finally, in order to infer which eigenstate did they measure, Alice and Bob use \( v(\sigma_{z_a}) \) and \( v(\sigma_{y_b}) \) to identify which of the above four blocks (maps) has been realized, and then use the results for \( \sigma_{z_A} \) and \( \sigma_{z_B} \) to isolate one of the four possible states in that block.
3 General 2 × 2 product basis

We can implement our method to the more general case by applying a general rotation to Bob’s qubit spin in (3):

\[ \begin{align*}
|\Psi^1_{AB}\rangle &= |0_A\rangle |0_B\rangle, \\
|\Psi^2_{AB}\rangle &= |0_A\rangle |1_B\rangle \\
|\Psi^3_{AB}\rangle &= |1_A\rangle \left( \cos \frac{\alpha}{2} |0_B\rangle + \sin \frac{\alpha}{2} |1_B\rangle \right), \\
|\Psi^4_{AB}\rangle &= |1_A\rangle \left( \sin \frac{\alpha}{2} |0_B\rangle - \cos \frac{\alpha}{2} |1_B\rangle \right),
\end{align*} \]

with \(0 < \alpha < \pi\). For simplicity, and without loss of generality, we have ignored possible relative phases in \( |\Psi^3_{AB}\rangle \) and \( |\Psi^4_{AB}\rangle \).

Now our task is more complicated, because the rotation of \( |\Psi^3_B\rangle \) and \( |\Psi^4_B\rangle \) performed in the previous section succeeds here only with probability \(1/2\) to map Bob’s set to \( |0_B\rangle \) and \( |1_B\rangle \). With probability \(1/2\) the transformation takes us to \( (\cos \alpha |0_B\rangle + \sin \alpha |1_B\rangle) \) and \( (\sin \alpha |0_B\rangle - \cos \alpha |1_B\rangle) \). Hence we fail for the case \( \alpha \neq \frac{\pi}{2} \). Fortunately, the information about which map has been realized is available to Bob. If \( v(\sigma_{x_B}) = +1 \), Bob knows that they succeeded. If the result was \( v(\sigma_{x_B}) = -1 \), Alice and Bob can perform an additional rotation with an angle \(2\alpha\) in order correct the error. This second step again succeeds with probability \(1/2\). Thus the total probability of success is now \(3/4\), and the process can be repeated again until the probability of failure is sufficiently small. This method of corrections has been proposed in Ref. [12].

The correction steps require the supply of additional ebits. For the general case, \( n \) ebits allow us to perform a successful measurement with probability \(1 - \frac{1}{2^{n+1}}\). The ebits are necessary even if the process truncates after a finite small number of steps once Bob obtained \( v(\sigma_{x_B}) = +1 \). Since we do not allow any communication, Alice must apply a fixed set of rotations \( e^{i(n-1)\alpha \sigma_z} \) on the \( n \)’th ebit.

Nevertheless, not for all angles the infinite number of ebits is needed in order to get probability infinitesimally close to one. If \( \alpha = \frac{2k\pi}{2^{n+1}} \) then in the \( n+1 \) step they will always succeed. Thus, for certain angles the number of ebits \( n \) is relatively small: \( n = 2 \) for \( \alpha = \frac{\pi}{8}, \frac{3\pi}{8} \), \( n = 3 \) for \( \alpha = \frac{\pi}{16}, \frac{3\pi}{16}, \frac{5\pi}{16} \), etc.
4 Nonmaximal entangled eigenstates

The operator with four non-degenerate Bell-state eigenstates is known to be measurable instantaneously [1, 2, 3]. However, for the case of ideal measurements, an observable with the nonmaximal eigenstates,

\[ |\Psi_{1AB}\rangle = \cos \frac{\alpha}{2} |0_A0_B\rangle + \sin \frac{\alpha}{2} |1_A1_B\rangle, \]

\[ |\Psi_{2AB}\rangle = \sin \frac{\alpha}{2} |0_A0_B\rangle - \cos \frac{\alpha}{2} |1_A1_B\rangle, \]

\[ |\Psi_{3AB}\rangle = \cos \frac{\alpha}{2} |0_A1_B\rangle + \sin \frac{\alpha}{2} |1_A0_B\rangle, \]

\[ |\Psi_{4AB}\rangle = \sin \frac{\alpha}{2} |0_A1_B\rangle - \cos \frac{\alpha}{2} |1_A0_B\rangle, \]

(13)

it was shown to be inconsistent with causality. Here we show that our method allows a precise measurement of a non-maximal operator with the above eigenstates. We later consider the somewhat more complicated case of non-maximal eigenstates with non-equal entanglement.

The measurement has two main steps. First, one ebit is used to perform a remote CNOT, which acts to transform the above set, to a set of mutually orthogonal non-entangled product states, reminiscent to the twisted set considered in section 3. Then, in the second step, the twisted set, is “untwisted” by a remote rotation. After this step Alice and Bob can apply ordinary measurements on their qubits.

Let us spell-out the details of the process. Alice and Bob begin as before by preparing the stator (8). Then Alice generates remote CNOT-like transformation on Bob’s qubit by applying the local unitary transformation \( \exp(-i\frac{\pi}{4}(1-\sigma_{zA})(1-\sigma_{xa})) \) on the half \( a \) of the ebit and her qubit \( A \), followed by a measurement of \( \sigma_{za} \). This activates on Bob’s qubit the unitary transformation

\[
\left( \frac{1 + v(\sigma_{za})}{2} \right)_B + v(\sigma_{xb}) \frac{1 - v(\sigma_{za})}{2} \sigma_{xB} \right) e^{-i\frac{\pi}{4}(1-\sigma_{zA})(1-v(\sigma_{xb})\sigma_{xB})}. \]

(14)

Notice that for \( v(\sigma_{za}) = v(\sigma_{xb}) = 1 \) that is the usual CNOT transformation. Otherwise we have to incorporate some trivial corrections. In the case \( v(\sigma_{za}) = +1 \), the state transforms to

\[ |\Psi_{1AB}\rangle = \left( \cos \frac{\alpha}{2} |0_A\rangle + v(\sigma_{xb}) \sin \frac{\alpha}{2} |1_A\rangle \right) |0_B\rangle, \]
\begin{align}
|\Psi_2^{AB}\rangle &= (\sin \frac{\alpha}{2}|0_A\rangle - v(\sigma_{zb}) \cos \frac{\alpha}{2}|1_A\rangle)|0_B\rangle, \\
|\Psi_3^{AB}\rangle &= (\cos \frac{\alpha}{2}|0_A\rangle + v(\sigma_{zb}) \sin \frac{\alpha}{2}|1_A\rangle)|1_B\rangle, \\
|\Psi_4^{AB}\rangle &= (\sin \frac{\alpha}{2}|0_A\rangle - v(\sigma_{zb}) \cos \frac{\alpha}{2}|1_A\rangle)|1_B\rangle, \end{align}
(15)

If Alice gets \(\sigma_{za} = -1\), the set is similar with Bob’s spin is flipped.

At this point, Alice or Bob still cannot extract any information because the directions are still “twisted”. However Alice’s qubit is in either of the two orthogonal states of a spin that has been rotated by an angle \(v(\sigma_{zb})\alpha\) around the y-axis. To untwist the set, Bob now performs a remote rotation of Alice’s spin by the angle \(v(\sigma_{zb})\alpha\). As before, the number of ebits needed to utilized this step depends on the angle \(\alpha\), but with unlimited ebit resources we can always succeed with probability arbitrary close to unity. After completing this step (assuming that we succeeded) we obtain the following four maps summarized in the table:

| \(\sigma_{zb}\) | \(\sigma_{z_b}\) | \(v(\sigma_{z_b}) = +1\) | \(v(\sigma_{z_b}) = -1\) |
|---------------|---------------|-----------------|-----------------|
| \(v(\sigma_{za}) = +1\) | \(|\Psi_1^{AB}\rangle \rightarrow |0_A\rangle|0_B\rangle\) | \(|\Psi_1^{AB}\rangle \rightarrow |0_A\rangle|0_B\rangle\) | \(|\Psi_2^{AB}\rangle \rightarrow |0_A\rangle|0_B\rangle\) | \(|\Psi_2^{AB}\rangle \rightarrow |0_A\rangle|0_B\rangle\) |
| | \(|\Psi_2^{AB}\rangle \rightarrow |1_A\rangle|0_B\rangle\) | \(|\Psi_2^{AB}\rangle \rightarrow |1_A\rangle|0_B\rangle\) | \(|\Psi_3^{AB}\rangle \rightarrow |0_A\rangle|1_B\rangle\) | \(|\Psi_3^{AB}\rangle \rightarrow |0_A\rangle|1_B\rangle\) |
| | \(|\Psi_3^{AB}\rangle \rightarrow |0_A\rangle|1_B\rangle\) | \(|\Psi_3^{AB}\rangle \rightarrow |0_A\rangle|1_B\rangle\) | \(|\Psi_4^{AB}\rangle \rightarrow |1_A\rangle|1_B\rangle\) | \(|\Psi_4^{AB}\rangle \rightarrow |1_A\rangle|1_B\rangle\) |
| | \(|\Psi_4^{AB}\rangle \rightarrow |1_A\rangle|0_B\rangle\) | \(|\Psi_4^{AB}\rangle \rightarrow |1_A\rangle|0_B\rangle\) | | |

The information about columns is located only on the Bob’s side, while information about rows is located on Alice’s side. Finally Alice and Bob measure \(\sigma_z\) of their qubit \(A\) and \(B\) and by comparing results can identify the relevant non-maximal eigenstate.

The final state of the qubits \(A\) and \(B\) is in this case one of the equally probable direct product states. It is interesting to note that one can also perform, a measurement which achieves the same goal, but collapses the state to one of the Bell-states. To this end, we notice that the basis (13) can also be written as

\begin{align}
|\Psi_1^{AB}\rangle &= U_A|0_A(0 + 1)_B\rangle + U_A^\dagger|1_A(0 - 1)_B\rangle, \\
|\Psi_2^{AB}\rangle &= U_A|1_A(0 + 1)_B\rangle + U_A^\dagger|0_A(0 - 1)_B\rangle, \end{align}
(16)
\[ |\Psi_{AB}^{3}\rangle = U_A|1_A(0 + 1)B\rangle - U_A^\dagger|1_A(0 - 1)B\rangle, \]
\[ |\Psi_{AB}^{4}\rangle = U_A|0_A(0 + 1)B\rangle - U_A^\dagger|0_A(0 - 1)B\rangle, \]
where \( U_A = \exp(i\frac{\alpha}{2}\sigma_y) \). Hence Bob can perform a controlled rotation in order to undo the unitary transformation \( U_A \) and \( U_A^\dagger \). To complete the measurement, Alice and Bob need two more ebits in order to distinguish between the four Bell states. The resources needed for this method require one more ebit. However, the final state of the system is in this case a maximally entangled state, therefore both methods actually consume the same amount of entanglement. As in the previous case, the reduced density operator of a single qubit is given by a unit matrix.

Our scheme works also for the most general operators with entangled eigenstates.

\[ |\Psi_{AB}^{1}\rangle = \cos \frac{\alpha}{2}|0_A0_B\rangle + e^{i\phi_1} \sin \frac{\alpha}{2}|1_A1_B\rangle, \]
\[ |\Psi_{AB}^{2}\rangle = \sin \frac{\alpha}{2}|0_A0_B\rangle - e^{i\phi_1} \cos \frac{\alpha}{2}|1_A1_B\rangle, \]
\[ |\Psi_{AB}^{3}\rangle = \cos \frac{\beta}{2}|0_A1_B\rangle + e^{i\phi_2} \sin \frac{\beta}{2}|1_A0_B\rangle, \]
\[ |\Psi_{AB}^{4}\rangle = \sin \frac{\beta}{2}|0_A1_B\rangle - e^{i\phi_2} \cos \frac{\beta}{2}|1_A0_B\rangle, \]

In the following we can set the phases \( \phi_1 = \phi_2 = 0 \) without loss of generality. The initial step remains the same. Alice and Bob apply a remote CNOT. However in the next step, Bob has to apply one of two possible remote rotations on Alice’s qubit. If his local qubit is \( |0_B\rangle \) he rotates Alice’s qubit, with respect to the \( y \) axis, by \( v(\sigma_{x_b})\alpha \). When his local qubit is \( |1_B\rangle \) he rotates Alice’s qubit by \( v(\sigma_{x_b})\beta \). (More generally, for non-vanishing \( \phi_1 \) and \( \phi_2 \) the axes of rotation must be chosen appropriately.) After this step the four possible sets become:

| \( \sigma_{x_a} \backslash \sigma_{x_b} \) | \( v(\sigma_{x_a}) = +1 \) | \( v(\sigma_{x_a}) = -1 \) |
|---|---|---|
| \( v(\sigma_{x_a}) = +1 \) | \( |0_A\rangle|0_B\rangle \) | \( |0_A\rangle|0_B\rangle \) |
|  | \( |1_A\rangle|0_B\rangle \) | \( |1_A\rangle|0_B\rangle \) |
|  | \( |0_A\rangle|1_B\rangle \) | \( |0_A\rangle|1_B\rangle \) |
|  | \( |1_A\rangle|1_B\rangle \) | \( |1_A\rangle|1_B\rangle \) |
| \( v(\sigma_{x_a}) = -1 \) | \( (\cos \gamma|0_A\rangle + \sin \gamma|1_A\rangle)|1_B\rangle \) | \( (\cos \gamma|0_A\rangle - \sin \gamma|1_A\rangle)|1_B\rangle \) |
|  | \( (\sin \gamma|0_A\rangle - \cos \gamma|1_A\rangle)|1_B\rangle \) | \( (\sin \gamma|0_A\rangle + \cos \gamma|1_A\rangle)|1_B\rangle \) |
|  | \( (\cos \gamma|0_A\rangle - \sin \gamma|1_A\rangle)|0_B\rangle \) | \( (\cos \gamma|0_A\rangle + \sin \gamma|1_A\rangle)|0_B\rangle \) |
|  | \( (\sin \gamma|0_A\rangle + \cos \gamma|1_A\rangle)|0_B\rangle \) | \( (\sin \gamma|0_A\rangle - \cos \gamma|1_A\rangle)|0_B\rangle \) |
where $\gamma = (\alpha - \beta)/2$. The two upper blocks are obtained for $v(\sigma_{az}) = +1$ while the lower row for $v(\sigma_{az}) = -1$. For the latter case, an additional rotation on the angle $\beta - \alpha$ is needed in order to rotate the four sets from second row to the $z$-direction. To obtain this, Alice and Bob use additional entanglement. Alice engages Bob’s particle into this transformation only if $v(\sigma_{az}) = -1$. Bob determines the rotation angle according to his previous outcomes as follows: if $v(\sigma_{xb})v(\sigma_{xB}) = 1$ he rotates Alice’s qubit by $\beta - \alpha$. If $v(\sigma_{xb})v(\sigma_{xB}) = -1$ he rotates by $\alpha - \beta$. Thus, finally we arrive to the sets:

| $\sigma_{z_a}\backslash\sigma_{x_b}$ | $v(\sigma_{x_b}) = +1$ | $v(\sigma_{x_b}) = -1$ |
|----------------------------------|----------------------|----------------------|
| $v(\sigma_{z_a}) = +1$          | $|0_A\rangle|0_B\rangle$ | $|0_A\rangle|0_B\rangle$ |
|                                  | $|1_A\rangle|0_B\rangle$ | $|1_A\rangle|0_B\rangle$ |
|                                  | $|0_A\rangle|1_B\rangle$ | $|0_A\rangle|1_B\rangle$ |
|                                  | $|1_A\rangle|1_B\rangle$ | $|1_A\rangle|1_B\rangle$ |
| $v(\sigma_{z_a}) = -1$          | $|0_A\rangle|1_B\rangle$ | $|0_A\rangle|1_B\rangle$ |
|                                  | $|1_A\rangle|1_B\rangle$ | $|1_A\rangle|1_B\rangle$ |
|                                  | $|0_A\rangle|0_B\rangle$ | $|0_A\rangle|0_B\rangle$ |
|                                  | $|1_A\rangle|0_B\rangle$ | $|1_A\rangle|0_B\rangle$ |

The full process is summarized in Figure 1.
Figure 1: Schematic plot of the steps in general case of non-maximal entangled eigenstates
5 4 × 4 Twist

Another interesting example of a measurable nonlocal operator is the twisted operator[3] living in 4 ⊗ 4 dimensional Hilbert space. This operator is defined by the following 16 entangled eigenstates:

|          | 0_B1_B | 2_B3_B          |
|----------|--------|-----------------|
| 0_A1_A   | 0_A0_B ± 1_A1_B | 0_A2_B ± 1_A3_B |
|          | 0_A1_B ± 1_A0_B | 0_A3_B ± 1_A2_B |
| 2_A3_A   | 2_A0_B ± 3_A1_B | U_B(2_A2_B ± 3_A3_B) |
|          | 2_A1_B ± 3_A0_B | U_B(2_A3_B ± 3_A2_B) |

where $U_B$ is some unitary rotation operator acting on the state of Bob’s particle. Alice’s and Bob’s Hilbert spaces are partitioned to two two-dimensional subspaces. Each quadrant contains four Bell states. If we can perform the transformation $U_B^\dagger$ on the $2_A3_A \times 2_B3_B$-quadrant, then the original set transforms to a measurable basis. To this end, Alice and Bob perform a measurement to project the local states of their particles into the subspaces $0_A, 1_A$ or $2_A, 3_A$ and $0_B, 1_B$ or $2_B, 3_B$. As a result the initial 16-set collapses to a 4-state set. Of course, the information about it is distributed between the parties: Alice knows the row, Bob knows the column. But we have encountered a very similar situation in the case of discussed in section 3. Hence Alice and Bob can untwist this transformation using one shared ebit. Bob performs a CNOT interaction (6) only if his state was projected to $2_B, 3_B$ subspace, and Alice acts on her qubit $a$ with an appropriate transformation $U_a$ (which generates acting on the stator $U_B^\dagger$) only if her state was projected into the subspace $2_A, 3_A$. To complete the measurement Alice and Bob perform a remote measurement of the Bell operator.

6 Summary

We have reconsidered the measurability problem for nonlocal operators which have been previously considered to be in conflict with relativistic causality and hence unmeasurable. We argued that, if the preparation role of an ideal measurement is relaxed, some observables do become measurable. In fact we showed that every bipartite operator in $2 \otimes 2$ Hilbert space can be measured instantaneously. We also demonstrated that the operator discussed in [3] for a bipartite $4 \otimes 4$ dimensional system is measurable in this sense.
The approach suggested here, seems to broaden considerably the class of measurable observables. However, the general question of measurability is still open. It remains to be seen, whether similar methods are applicable for general bipartite and multi-particle cases.

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