The string little algebra

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Abstract

We consider the string, like point particles and branes, to be an irreducible representation of the semi-direct product of the Cartan involution invariant subalgebra of E11 and its vector representation. We show that the subalgebra that preserves the string charges, the string little algebra, is essentially the Borel subalgebra of E9. We also show that the known string physical states carry a representation of parts of this algebra.

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1. Introduction

Quite some time ago it was conjectured that the theory of strings and branes had a Kac-Moody symmetry that was very extended $E_8$, usually called $E_{11}$. In particular it was proposed that the non-linear realisation of the semi-direct product of $E_{11}$ and its vector representation, denoted $E_{11} \otimes_s l_1$, was the low energy effective action for strings and branes $[1,2]$. The resulting equations of motion are unique and, at low levels they are just the maximally supergravity theories. One finds the different theories by taking different decomposition of $E_{11}$ $[3,4]$. Indeed one can find all maximally supersymmetric supergravity theories and also all the gauged maximal supergravities, for a review see reference $[5]$. Put simply the low energy behaviour of strings and branes is encoded in the symmetry $E_{11} \otimes_s l_1$.

The $E_{11}$ theory treats strings and branes on an equal footing and also encodes an infinite number of duality symmetries. In particular the vector representation of $E_{11}$ contains all the brane charges $[2,6,7,8]$. At low levels these charges are those that we are familiar with, such as the M2 and M5 branes, but at higher levels one finds an infinite number of new brane charges which were not encountered in other approaches. Thus $E$ theory contains an infinite number of previously unknown branes and so new degrees of freedom. It also provides a framework in which to investigate these new branes.

There is a natural correspondence between the fields in the non-linear realisation of $E_{11} \otimes_s l_1$ and the brane charges which allows one to identify which field is the source for the brane charge in the vector representation $[6,7,8]$. In the non-linear realisation a given field arises from a positive root of $E_{11}$ and this corresponds to a weight in the vector representation that in turn corresponds to the brane charge. Indeed one can go further, from this root one can construct a group element of $E_{11}$ and which encodes a putative solution to the field equation with the corresponding brane charge $[10]$. This method gives the known solutions for the half BPS branes such as the M2 and M5 branes, but at higher levels it provides putative solutions for the infinite number of higher level branes encoded in the vector representation.

A general formula for the tension of these branes was found in $[8]$. In the case of the IIA and IIB theories this formula included the way the tensions depended on the string coupling constants of these two theories. While this dependence for the familiar branes, such as the D branes, was reproduced, the higher branes often had quite different dependencies on the string coupling. The construction was generalised to find solutions corresponding to two positive roots in eleven $[11]$ and ten dimensions $[12]$. These solutions for low level roots reproduced all the quarter BPS branes in these dimensions. A further generalisation to include three and more numbers of $E_{11}$ roots was given in reference $[11]$.

While the brane charges for the familiar branes carry Lorentz indices which are contained in one anti-symmetrised block, for example $Z^{a_1 a_2}$ and $Z^{a_1 \ldots a_5}$ for the M2 and M5 branes, the brane charges at higher levels contain many more indices and these are usually not contained in one anti-symmetrised block. Such branes have become known as exotic branes. The existence such charges was first indicated when examining the action of T duality on brane charges in lower dimensions, such as in three dimensions $[9]$. Although there is some literature on exotic branes it usually only utilises a part of the underlying $E_{11}$ structure. Much remains to be understood about the higher level branes and indeed
An important step in the development of quantum field theory was the construction by Wigner of all of the irreducible representations of the Poincare algebra, which is just the semi-direct product of the Lorentz algebra and the translations [13]. Indeed one can think of a particle as being defined as an irreducible representation of the Poincare algebra. From this approach one can deduce the on-shell states of a given particle and even the covariant field equations of motion for the free particle.

The symmetries of E theory are similar in structure to the Poincare algebra which can be written as $SO(1,D-1) \otimes T^D$ where $T^D$ are the translations. The role of the Lorentz algebra is played by the Cartan involution subalgebra of $E_{11}$, denoted by $I_c(E_{11})$, and the role of the translations is played by the elements of the vector representation of $E_{11}$, denoted by $l_1$. Thus the algebra $I_c(E_{11}) \otimes l_1$ has the same semi-direct structure as the Poincare algebra. While the Poincare algebra only contains the momentum generators for the point particles, the vector representation of $E_{11}$ contains all brane charges.

As such it is natural to consider irreducible representations of the $I_c(E_{11}) \otimes l_1$ and one may hope that they define what is meant by the point particles, strings and branes. Indeed mimicking the procedure for Poincare algebra we can choose a brane charge that we are interested in and then compute the little algebra that leaves the charge inert. We can then choose an irreducible representation of this little algebra and then hope that these contain the degrees of freedom corresponding to the brane of interest. Finally one can then boost the representation to find the irreducible representation of $I_c(E_{11}) \otimes l_1$ and so the on-shell conditions that define the brane and even find the covariant equations of motion of the free brane [14].

The programme just outlined was carried out for the massless particle, that is, the only non-zero charge in the vector representation was chosen to be the momenta and this was massless. The particle little algebra was found to be $I_c(E_9)$ and this was shown to have an irreducible representation that has dimension 128. It contains precisely the bosonic degrees of freedom of eleven dimensional supergravity [14,15]. The free equations of motion must be those of the above non-linear realisation of $E_{11} \otimes l_1$. This implies that this non-linear realisation only has degrees of freedom of eleven dimensional supergravity.

It would be interesting to further carry out the programme outlined above and in this paper we take the next step. We will consider the the IIA theory and in particular the IIA string from this view point. We will take the only non-zero charges in the vector representation to be those corresponding to the IIA string, that is, we will take only the momenta $P_a$ and the string charge $Z^a$ to be non-zero. We will do this in such a way as to preserve the SO(1,1) symmetry of the string. We find the subalgebra of $I_c(E_{11})$ that preserves this choice and so the string little algebra. It has an algebra that has essentially the same algebra as the Borel subalgebra of $E_9$. We will then argue that the known string physical states carry a representation of at least part of this little algebra.

In section two we will give the $E_{11} \otimes l_1$ algebra at low levels in the decomposition which leads to the IIA string. In section three we will derive the string little algebra at low levels and in section five we find it at all levels. In section six we explain how the string little algebra sits in $E_{11}$ and discuss how the physical states of the superstring carry a representation of parts of the string little algebra. Finally in section four we consider
the same steps for a toy string model based on the algebra \( SO(D, D) \otimes_s T^{2D} \) where \( D \) is the dimension of spacetime.

2. The IIA algebra

In this section we summarize the \( E_{11} \) algebra in the form corresponding to the IIA theory. The IIA theory arises from the non-linear realisation of \( E_{11} \otimes_s l_1 \) when we delete node 10 and decompose the \( E_{11} \) algebra in terms of the residual \( SO(10, 10) \) algebra. We will refer to node 10 as the IIA node. The resulting listing of the generators in this decomposition, which belong to representations of \( SO(10, 10) \), are more easily understood if also delete node 11 and further decompose with respect to the remaining \( SL(10) \) algebra. The resulting Dynkin diagram where the deleted nodes are indicated with a \( \oplus \) is given by

\[
\begin{array}{cccccccccc}
\oplus & 11 & \oplus & 10 \\
\bullet & - & \bullet & - & \bullet & - & \bullet & - & \bullet & - & \bullet
\end{array}
\]

The list of generators resulting from this decomposition can be found using the program SimpLie \[16\]. The generators are organized in terms of two levels \( (l_{(1)}, l_{(2)}) \) associated to nodes ten and eleven respectively. We will refer to the first level as the IIA level, or sometimes just the level. The number of up minus down indices on a given generator is equal to \( l_{(1)} + 2l_{(2)} \), and this can be used to determine the eleven level \( l_{(2)} \) in a generator in the list below once the IIA level is known.

One can also find the generators in the above IIA decomposition directly from the \( E_{11} \) algebra in its eleven dimension, M theory, formulation which appears by deleting just the node eleven. We will denote the latter generators with a hat \( \hat{\cdot} \) below. At level zero \( (l_{(1)} = 0) \) the \( E_{11} \) algebra contains the generators \[17\]

\[
K^{a}_{\underline{b}} , \quad \tilde{R} , \quad R^{ab} , \quad R_{ab} ,
\]

where \( a, b, \ldots = 0, 1, \ldots, 9 \). The relation between the two sets of generators is given by

\[
K^{a}_{\underline{b}} = \hat{K}^{a}_{\underline{b}} + \frac{1}{6} \delta^{a}_{\underline{b}} \tilde{R} , \quad \tilde{R} = - \sum_{a=0}^{9} \hat{K}^{a}_{a} + 2\hat{K}^{11}_{11} , \quad R^{ab} = \hat{R}^{ab}_{11} , \quad R_{ab} = \hat{R}_{ab11} .
\]

These generators belong to the adjoint representation of \( SO(10, 10) \), as indeed they must. As illustrated in this equation we will often use the number 11, rather than 10, to denote the eleventh dimension.

At IIA level one we have the generators

\[
R^{a}_{11} = \hat{K}^{a}_{11} , \quad R^{a_1 a_2 a_3}_{a_4 a_5} = \hat{R}^{a_1 a_2 a_3}_{a_4 a_5} , \quad R^{a_1 \cdots a_5}_{a_6} = \hat{R}^{a_1 \cdots a_5}_{a_6}^{11} ,
\]

\[
R^{a_1 \cdots a_7} = \hat{R}^{a_1 \cdots a_7}_{11} , \quad R^{a_1 \cdots a_9}_{a_{10} a_{11}} = \hat{R}^{a_1 \cdots a_9}_{a_{10} a_{11}}^{11,11} ,
\]

At IIA level level minus one the generators are

\[
R_{a} = \hat{K}^{11}_{a} , \quad R_{a_1 a_2 a_3} = \hat{R}_{a_1 a_2 a_3} , \quad R_{a_1 \cdots a_5} = \hat{R}_{a_1 \cdots a_5}^{11} ,
\]
These 512 generators at level one belong to the spinor representation of SO(10, 10) as do the generators at level minus one.

While at level two the IIA generators are given by

\[ R_{\alpha_1 \cdots \alpha_9} = \hat{R}_{\alpha_1 \cdots \alpha_9}, \quad R_{\alpha_1 \cdots \alpha_9} = R_{\alpha_1 \cdots \alpha_9}^{11}, \quad R_{\alpha_1 \cdots \alpha_7, \beta} = \hat{R}_{\alpha_1 \cdots \alpha_7, \beta}, \]

\[ R_{\alpha_1 \cdots \alpha_7}^{(1)} = \hat{R}_{\alpha_1 \cdots \alpha_7}^{(1)}(111) \quad R_{\alpha_1 \cdots \alpha_7}^{(2)} = \hat{R}_{\alpha_1 \cdots \alpha_7}^{(2)}, \]

\[ R_{\alpha_1 \cdots \alpha_7, \beta} = \hat{R}_{\alpha_1 \cdots \alpha_7, \beta}^{11}, \quad R_{\alpha_1 \cdots \alpha_7, \beta, \gamma} = \hat{R}_{\alpha_1 \cdots \alpha_7, \beta, \gamma}^{11}, \]

\[ R_{\alpha_1 \cdots \alpha_7, \beta, \gamma} = \hat{R}_{\alpha_1 \cdots \alpha_7, \beta, \gamma}^{11}, \quad R_{\alpha_1 \cdots \alpha_7, \beta, \gamma, \delta} = \hat{R}_{\alpha_1 \cdots \alpha_7, \beta, \gamma, \delta}^{11}, \]

and at level minus two the IIA generators are

\[ R_{\alpha_1 \cdots \alpha_9} = \hat{R}_{\alpha_1 \cdots \alpha_9}, \quad R_{\alpha_1 \cdots \alpha_9} = \hat{R}_{\alpha_1 \cdots \alpha_9}^{11}, \quad R_{\alpha_1 \cdots \alpha_7, \beta} = \hat{R}_{\alpha_1 \cdots \alpha_7, \beta}^{11}, \]

\[ R_{\alpha_1 \cdots \alpha_7}^{(1)} = \hat{R}_{\alpha_1 \cdots \alpha_7}^{(1)}(111) \quad R_{\alpha_1 \cdots \alpha_7}^{(2)} = \hat{R}_{\alpha_1 \cdots \alpha_7}^{(2)}, \]

\[ R_{\alpha_1 \cdots \alpha_7, \beta} = \hat{R}_{\alpha_1 \cdots \alpha_7, \beta}^{11}, \quad R_{\alpha_1 \cdots \alpha_7, \beta, \gamma} = \hat{R}_{\alpha_1 \cdots \alpha_7, \beta, \gamma}^{11}, \]

\[ R_{\alpha_1 \cdots \alpha_7, \beta, \gamma} = \hat{R}_{\alpha_1 \cdots \alpha_7, \beta, \gamma}^{11}, \quad R_{\alpha_1 \cdots \alpha_7, \beta, \gamma, \delta} = \hat{R}_{\alpha_1 \cdots \alpha_7, \beta, \gamma, \delta}^{11}, \]

Here the indices of the IIA generators in a given block are anti-symmetric. The generators at level two belong to the four rank tensor and a singlet representations of SO(10, 10) which have dimensions 4845 and 1 respectively. We note that there are two generators with ten anti-symmetric indices which are distinguished by the subscripts 1 and 2. The algebra of the IIA generators up to level one is given in Appendix A. We note that at IIA level three we have a spinor and a two tensor with a spinor index representations of SO(10, 10) which have dimensions 512 and 87040 respectively.

We now consider the action of the Cartan involution in the IIA theory. We recall that the Cartan involution \( I_c \) is an automorphism of the algebra which acts as \( I_c(AB) = I_c(A)I_c(B) \), for any two elements of the algebra \( A \) and \( B \), and satisfies \( I_c^2(A) = A \) for all elements \( A \). Its action on the IIA generators is inherited from its action in the eleven dimensional theory where \( I_c(R^\alpha) = -(-1)^{l+1} R^\alpha \) where \( l \) is the level of \( R^\alpha \) except at level zero where \( I_c(K_{\alpha \beta}) = -\eta^{-\alpha \beta}_{\gamma \delta} K_{\gamma \delta} \) and then using the relationship between the eleven dimensional and IIA generators of equations (2.2)-(2.6). Acting on the IIA level zero algebra generators we find that

\[ I_c(K_{\alpha \beta}) = -K_{\alpha \beta}^\gamma, \quad I_c(\hat{R}) = -\hat{R}, \quad I_c(R_{\alpha_1 \alpha_2}) = -\eta^{a_1 \beta_1} \eta^{a_2 \beta_2} R_{\alpha_1 \alpha_2 \beta_1 \beta_2}, \]

on the level one generators as

\[ I_c(R^\alpha) = -\eta^{\alpha \beta} R_\beta, \quad I_c(R_{\alpha_1 \alpha_2 \alpha_3}) = -\eta^{a_1 \beta_1} \eta^{a_2 \beta_2} \eta^{a_3 \beta_3} R_{\alpha_1 \alpha_2 \alpha_3 \beta_1 \beta_2 \beta_3}, \]
\[ I_c(R^{a_1 \cdots a_r}) = +\eta^{c_1 \cdots c_{r+1}} b_1 \cdots b_r R_{b_1 \cdots b_r}, \quad I_c(R^{a_1 \cdots a_r}) = -\eta^{c_1 \cdots c_{r+1}} b_1 \cdots b_r R_{b_1 \cdots b_r}, \]

\[ I_c(R^{a_1 \cdots a_r}) = +\eta^{c_1 \cdots c_{r+1}} b_1 \cdots b_r R_{b_1 \cdots b_r}, \quad I_c(R^{a_1 \cdots a_r}) = -\eta^{c_1 \cdots c_{r+1}} b_1 \cdots b_r R_{b_1 \cdots b_r}, \]

and on the level two generators as

\[ I_c(R^{a_1 \cdots a_6}) = +\eta^{a_1 \cdots a_6} b_1 \cdots b_6 R_{b_1 \cdots b_6}, \quad I_c(R^{a_1 \cdots a_6}) = -\eta^{a_1 \cdots a_6} b_1 \cdots b_6 R_{b_1 \cdots b_6}, \]

\[ I_c(R^{a_1 \cdots a_6}) = -\eta^{a_1 \cdots a_6} b_1 \cdots b_6 R_{b_1 \cdots b_6}, \quad I_c(R^{a_1 \cdots a_6}) = +\eta^{a_1 \cdots a_6} b_1 \cdots b_6 R_{b_1 \cdots b_6}, \]

\[ I_c(R^{a_1 \cdots a_6}) = +\eta^{a_1 \cdots a_6} b_1 \cdots b_6 R_{b_1 \cdots b_6}, \quad I_c(R^{a_1 \cdots a_6}) = -\eta^{a_1 \cdots a_6} b_1 \cdots b_6 R_{b_1 \cdots b_6}, \]

\[ I_c(R^{a_1 \cdots a_6} b_1 \cdots b_6) = -\eta^{a_1 \cdots a_6} b_1 \cdots b_6 R_{b_1 \cdots b_6}, \quad I_c(R^{a_1 \cdots a_6} b_1 \cdots b_6) = +\eta^{a_1 \cdots a_6} b_1 \cdots b_6 R_{b_1 \cdots b_6}, \]

\[ I_c(R^{a_1 \cdots a_6} b_1 \cdots b_6) = +\eta^{a_1 \cdots a_6} b_1 \cdots b_6 R_{b_1 \cdots b_6}, \quad I_c(R^{a_1 \cdots a_6} b_1 \cdots b_6) = -\eta^{a_1 \cdots a_6} b_1 \cdots b_6 R_{b_1 \cdots b_6}, \]

\[ I_c(R^{a_1 \cdots a_6} b_1 \cdots b_6) = -\eta^{a_1 \cdots a_6} b_1 \cdots b_6 R_{b_1 \cdots b_6}, \quad I_c(R^{a_1 \cdots a_6} b_1 \cdots b_6) = +\eta^{a_1 \cdots a_6} b_1 \cdots b_6 R_{b_1 \cdots b_6}, \]

In these equations \( \eta^{a_1 \cdots a_6} b_1 \cdots b_6 \cdots \equiv \eta^{a_1 \cdots a_6} b_1 \cdots b_6 \cdots \).

We now construct a subalgebra using Cartan involution invariant combinations of the generators. At level zero the combinations

\[ J_{a_1 a_2} = \eta_{a_1 a_2} K^{a_1}_{a_2} - \eta_{a_2 a_1} K^{a_2}_{a_1}, \quad S_{a_1 a_2} = R^{a_1 c_2} \eta_{c_1 a_1} \eta_{c_2 a_2} - R_{a_1 a_2}, \]

are invariant under \( I_c \). Similarly at level one the combinations

\[ S_{a_1} = R^{a c_3} \eta_{c a_1} - R_{a_1}, \quad S_{a_1 a_2 a_3} = R^{a_1 a_2 a_3} \eta_{c_1 c_2 c_3} a_1 a_2 a_3 - R_{a_1 a_2 a_3}, \]

\[ S_{a_1 \cdots a_7} = R^{a_1 \cdots a_7} \eta_{c_1 \cdots c_7} a_1 \cdots a_7 + R_{a_1 \cdots a_7}, \quad S_{a_1 a_2 a_3} = R^{a_1 a_2 a_3} \eta_{c_1 c_2 c_3} a_1 a_2 a_3 - R_{a_1 a_2 a_3}, \]

are also invariant. At level two the combinations

\[ S_{a_1 \cdots a_6} = R^{a_1 \cdots a_6} \eta_{c_1 \cdots c_6} a_1 \cdots a_6 + R_{a_1 \cdots a_6}, \quad S_{a_1 \cdots a_8} = R^{a_1 \cdots a_8} \eta_{c_1 \cdots c_8} a_1 \cdots a_8 - R_{a_1 \cdots a_8}, \]

\[ S_{a_1 a_2 a_3} = R^{a_1 a_2 a_3} \eta_{c_1 c_2 c_3} a_1 a_2 a_3 - R_{a_1 a_2 a_3}, \]

\[ S_{a_1 \cdots a_6} = R^{a_1 \cdots a_6} \eta_{c_1 \cdots c_6} a_1 \cdots a_6 + R_{a_1 \cdots a_6}, \quad S_{a_1 \cdots a_8} = R^{a_1 \cdots a_8} \eta_{c_1 \cdots c_8} a_1 \cdots a_8 - R_{a_1 \cdots a_8}, \]

\[ S_{a_1 a_2 a_3} = R^{a_1 a_2 a_3} \eta_{c_1 c_2 c_3} a_1 a_2 a_3 - R_{a_1 a_2 a_3}, \]

\[ S_{a_1 \cdots a_6} = R^{a_1 \cdots a_6} \eta_{c_1 \cdots c_6} a_1 \cdots a_6 + R_{a_1 \cdots a_6}, \quad S_{a_1 \cdots a_8} = R^{a_1 \cdots a_8} \eta_{c_1 \cdots c_8} a_1 \cdots a_8 - R_{a_1 \cdots a_8}, \]

\[ S_{a_1 a_2 a_3} = R^{a_1 a_2 a_3} \eta_{c_1 c_2 c_3} a_1 a_2 a_3 - R_{a_1 a_2 a_3}, \]
The commutators of the above generators, which are in the IIA formulation, can be found by the generators in terms of their eleven dimensional formulation using equations (2.2) to (2.6) and the known commutation relations for these latter generators. At level zero we find that

\[ [[J_{a_1 a_2}, J_{b_1 b_2}^*}] = -4 \delta^{[a_1}_{[b_1} J_{a_2] b_2}^* ] , \quad [[J_{a_1 a_2}, S_{b_1 b_2}^*}] = -4 \delta^{[a_1}_{[b_1} S_{a_2] b_2}^* ] , \]

\[ [S_{a_1 a_2}^*, S_{b_1 b_2}^*}] = -4 \delta^{[a_1}_{[b_1} J_{a_2] b_2}^* ] , \quad (2.13) \]

The commutators of the \( I_c(E_{11}) \) generators formed from IIA level plus and minus one \( E_{11} \) generators are given by

\[ [S^a, S_b^*] = -J_{a b}^* , \quad [S^a, S_{b_1 b_2}^*] = -\frac{3}{2} \delta^{a}_{[b_1} S_{b_2 b_3}^* ] , \quad [S^a, S_{b_1 b_2 ... b_7}^*] = -5 \delta^{a}_{[b_1} S_{b_2 ... b_7}^* ] , \]

\[ [S^a, S_{b_1 b_2 ... b_7}] = -2 S^a_{b_1 ... b_7} + 7 S^a_{b_1 b_2 b_3} , \quad [S^a_{b_1 b_2 b_3}, S_{b_4 b_5 b_6}] = -18 J^{a_1}_{[b_1} \delta^{a_2 a_3}_{[b_2 b_3]} + 2 S^a_{b_1 b_2 b_3} , \]

\[ [S^a_{b_1 b_2 b_3}, S_{b_4 b_5 b_6}] = -30 S_{[b_1 b_2} \delta^{a_1}_{b_3] b_4} S^a_{b_5 b_6} + S^a_{b_1 b_2 b_3} + 5 S^a_{b_1 b_2 b_3} (5 S_{b_1 b_2 b_3} - 5 ) , \]

\[ [S^a_{b_1 ... b_7}, S_{b_1 ... b_7}] = 0 , \quad [S^a_{b_1 ... b_7}, S_{b_1 ... b_7}] = -5 \cdot 30 J^{a_1}_{[b_1} \delta^{a_2 ... a_7}_{b_2 ... b_7}] , \]

\[ [S^a_{b_1 ... b_7}, S_{b_1 ... b_7}] = 9 \cdot 70 S_{[b_1 b_2} \delta^{a_1}_{b_3] b_4} S^a_{b_5 b_6 b_7} , \quad [S^a_{b_1 ... b_7}, S_{b_1 ... b_7}] = -7 \cdot 7 \cdot 180 J^{a_1}_{[b_1} \delta^{a_2 ... a_7}_{b_2 ... b_7}] , \]

(2.14)

and at level zero \( I_c(E_{11}) \) with the \( I_c(E_{11}) \) generators formed from IIA level plus and minus one \( E_{11} \) generators as

\[ [[J_{a_1 a_2}, S_b^*}] = -2 \eta^{a}_{b [a_1} S_{b a_2]}^* ] , \quad [[J_{a_1 a_2}, S_{b_1 b_2}^*}] = -2 \cdot 3 \delta^{a_{1}}_{[b_1} S_{a_2] b_2}^* ] , \]

\[ [[J_{a_1 a_2}, S_{b_1 b_2 ... b_7}^*}] = -2 \cdot 5 \delta^{a_{1}}_{[b_1} S_{b_2 ... b_7}^* ] , \quad [[J_{a_1 a_2}, S_{b_1 b_2}] = 16 S_{b_1 ... b_7}^*] , \]

\[ [S_{a_1 a_2}, S_b^*] = S_{a_1 a_2 b}^* , \quad [S^a_{b_1 b_2 b_3}, S_{b_4 b_5 b_6}] = +6 \delta^{a_{1}}_{b_1} S_{b_2 b_3}^* - 2 S^a_{b_1 b_2 b_3} , \]

\[ [S^a_{b_1 b_2 b_3}, S_{b_1 b_2 b_3}] = +10 \delta^{a_{1}}_{b_1} S_{b_2 b_3}^* - S^a_{b_1 b_2 b_3} b_3] , \]

\[ [S^a_{b_1 ... b_7}, S_{b_1 ... b_7}] = 42 S_{b_1 ... b_7}^* \delta^{a_{1}}_{b_1} b_7] , \quad (2.15) \]

We now consider how the vector \( (l_1) \) representation of \( E_{11} \) appears in the IIA theory. These generators can be related to the way the \( l_1 \) generators appear in the eleven dimensional theory which we will again denote by a hat.i.e. \( \hat{ } \). The generators in the \( l_1 \) representation in its IIA presentation at IIA level zero are [17]

\[ P_a = \hat{P}_a , \quad Q^a = -\hat{Z}^a_{11} . \quad (2.16) \]
We note that the coefficient in \( Q^a = -Z^a_{11} \) is different to that in the conventions in reference [17], see below (A.15) of that paper.

The generators in the \( l_1 \) representation at IIA level one are given by

\[
Z = \hat{P}_{11}, \quad Z^a_{1, a_2} = \hat{Z}^a_{1, a_2}, \quad Z^{a_1 \cdots a_4} = \hat{Z}^{a_1 \cdots a_4}, \quad Z^{a_1 \cdots a_5} = \hat{Z}^{a_1 \cdots a_5}_{11,11},
\]

\[
Z^{a_1 \cdots a_3} = \hat{Z}^{a_1 \cdots a_3}, \quad Z^{a_1 \cdots a_0} = \hat{Z}^{a_1 \cdots a_0}_{11}, \quad (2.17)
\]

and at IIA level two as

\[
Z^{a_1 \cdots a_5} = \hat{Z}^{a_1 \cdots a_5}, \quad Z^{a_1 \cdots a_7}_{(1)} = \hat{Z}^{a_1 \cdots a_7}_{11}, \quad Z^{a_1 \cdots a_7}_{(2)} = \hat{Z}^{a_1 \cdots a_7}_{11}, \quad Z^{a_1 \cdots a_0}_{(2)} = \hat{Z}^{a_1 \cdots a_0}_{11,11,11}, \quad Z^{a_1 \cdots a_0}_{(3)} = \hat{Z}^{a_1 \cdots a_0}_{11,11,11}, \quad (2.18)
\]

\[
Z^{a_1 \cdots a_7}_{(1)} = \hat{Z}^{a_1 \cdots a_7}_{11}, \quad Z^{a_1 \cdots a_7}_{(2)} = \hat{Z}^{a_1 \cdots a_7}_{11}, \quad Z^{a_1 \cdots a_0}_{(3)} = \hat{Z}^{a_1 \cdots a_0}_{11,11,11}, \quad Z^{a_1 \cdots a_0}_{(4)} = \hat{Z}^{a_1 \cdots a_0}_{11,11,11},
\]

\[
Z^{a_1 \cdots a_7}_{(2)} = \hat{Z}^{a_1 \cdots a_7}_{11}, \quad Z^{a_1 \cdots a_7}_{(3)} = \hat{Z}^{a_1 \cdots a_7}_{11}, \quad Z^{a_1 \cdots a_0}_{(4)} = \hat{Z}^{a_1 \cdots a_0}_{11,11,11},
\]

\[
Z^{a_1 \cdots a_7}_{(3)} = \hat{Z}^{a_1 \cdots a_7}_{11}, \quad Z^{a_1 \cdots a_7}_{(4)} = \hat{Z}^{a_1 \cdots a_7}_{11}, \quad Z^{a_1 \cdots a_0}_{(5)} = \hat{Z}^{a_1 \cdots a_0}_{11,11,11},
\]

All of the indices in a given block are anti-symmetric. The lowered subscripts 1, 2 and 3 indicate generators that occur with multiplicity more than one. In some cases this is because there are two ways to find a generator with the same indices from the eleven dimensional generators and in the other cases the generator in eleven dimensions has multiplicity two.

The commutators of the \( l_1 \) generators with the \( I_c(E_{11}) \) are most easily computed from their eleven dimensional origin given in the above equations. The commutators of the IIA level zero \( E_{11} \) with IIA level zero \( l_1 \) generators are given as

\[
[J_{a_1 a_2}, P^c] = -2 \delta^{[a_1}_{[a_2} P^{c]}_{a_2]}, \quad [J_{a_1 a_2}, Q^c] = -2 \delta^{[a_1}_{[a_2} Q^{c]}_{a_2]},
\]

\[
[S^{a_1 a_2}, P^c] = -2 \delta^{[a_1}_{[a_2} Q^{c]}_{a_2]}, \quad [S^{a_1 a_2}, Q^c] = -2 \delta^{[a_1}_{[a_2} P^{c]}_{a_2}], \quad (2.19)
\]

The commutators of the level zero \( I_c(E_{11}) \) generators with level one \( l_1 \) generators are

\[
[J_{a_1 a_2}, Z] = 0, \quad [J_{a_1 a_2}, Z^{b_1 b_2}_{a_3}] = -2 \cdot 2 \delta^{[b_1}_{[a_1} Z^{b_2]}_{a_2]},
\]

\[
[J_{a_1 a_2}, Z^{b_1 \cdots b_4}_{a_3}] = -4 \cdot 2 \delta^{[b_1}_{[a_1} Z^{b_2 b_3 b_4]}_{a_3}, \quad [J_{a_1 a_2}, Z^{b_1 \cdots b_6}_{a_3}] = -6 \cdot 2 \delta^{[b_1}_{[a_1} Z^{b_2 b_3 b_4 b_5}]_{a_3},
\]

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The level plus and minus one \( E_{11} \) generators with the level zero \( l_1 \) generators have the commutators

\[
[S_a, P_a] = -\eta_{a\alpha} Z, \quad [S_a, Z_{b\alpha}] = \mp 2\delta_a^b Z, \quad [S_a, Z_{b\alpha}] = \mp 2\delta_a^b Z, \quad [S_a, Z_{b\alpha}] = \mp 2\delta_a^b Z.
\]

The commutators of the \( I_c(E_{11}) \) generators formed from \( I_a \) level plus and minus one \( E_{11} \) generators with the level one \( l_1 \) generators are given by

\[
[S_a^2, Z] = P_a, \quad [S_a^2, Z_{b\alpha}] = -2\delta_a^b Q_{b\alpha}, \quad [S_a^2, Z_{b\alpha}] = -2\delta_a^b Q_{b\alpha}, \quad [S_a^2, Z_{b\alpha}] = -2\delta_a^b Q_{b\alpha},
\]

\[
[S_a^2, Z_{b\alpha}] = -2\delta_a^b Q_{b\alpha}, \quad [S_a^2, Z_{b\alpha}] = -2\delta_a^b Q_{b\alpha}, \quad [S_a^2, Z_{b\alpha}] = -2\delta_a^b Q_{b\alpha},
\]

\[
[S_a^2, Z_{b\alpha}] = -2\delta_a^b Q_{b\alpha}, \quad [S_a^2, Z_{b\alpha}] = -2\delta_a^b Q_{b\alpha}, \quad [S_a^2, Z_{b\alpha}] = -2\delta_a^b Q_{b\alpha},
\]

The general element of \( I_c(E_{11}) \), can be written as

\[
\Lambda^a S_a = \Lambda^a_{a\alpha} J_{a\alpha} + \Lambda^a_{a\alpha} S_{a\alpha} + \Lambda^a_{a\alpha} S_{a\alpha} + \Lambda^a_{a\alpha} S_{a\alpha} + \Lambda^a_{a\alpha} S_{a\alpha} + \Lambda^a_{a\alpha} S_{a\alpha} + \Lambda^a_{a\alpha} S_{a\alpha} + \Lambda^a_{a\alpha} S_{a\alpha} + \Lambda^a_{a\alpha} S_{a\alpha} + \cdots
\]
The transformation of the vector representation under $I_c(E_{11})$ transformations is given by

$$\delta l_A = [\Lambda^\alpha S_{\alpha}, l_A].$$  \hfill (2.24)

Evaluating equation (2.24) on the $l_1$ generators we find, at IIA level zero:

$$\delta P_\perp = -2\Lambda_{b^b} P_\perp - 2\tilde{\Lambda}_{b^b} Q_{b^b} - \Lambda_2 Z + 3\Lambda_{b^c b^c} Z_{b^1 b^2} - \frac{5}{2} \Lambda_{b^1 c_1 b_1 c_2} Z_{c_1 c_2} + \ldots$$

$$\delta Q^b = -2\Lambda_{b^b} Q^b - 2\tilde{\Lambda}_{b^b} P_\perp + \Lambda_2 Z^{b_1 b_2} - \Lambda_{b^c c_1 b^c c_2} Z^{b_1 b_2 c_1 c_2} + \frac{1}{6} \Lambda_{b_1 c_1 b_1 c_2} Z_{b_1 b_2 c_1 c_2} + \ldots; \text{ check (2.25)}$$

and at IIA level one

$$\delta Z = \Lambda^\alpha P_\perp + \tilde{\Lambda}_{b^b} Z^{b_1 b_2 c_1 c_2} + \frac{1}{2} \Lambda_{b_1 c_1 b_1 c_2} Z_{b_1 b_2 c_1 c_2} + \ldots$$

$$\delta Z^{b_1 b_2} = -6\Lambda_{b^b} b_1 b_2 P_\perp - 2\Lambda_{b^b} b_1 b_2 Z + 4\Lambda_{b^b} b_1 b_2 Z^{b_2} + \tilde{\Lambda}_{b^b} b_1 b_2 Z^{b_1 b_2} + \ldots$$

$$\delta Z^{b_1 b_2} = 60\Lambda_{b^b} b_1 b_2 P_\perp + 24\Lambda_{b^b} b_1 b_2 Q_{b_1 b_2} - 8\Lambda_{b_1 b_2} b_1 b_2 Z + 12\Lambda_{b_1 b_2} b_1 b_2 Z^{b_1 b_2} + \ldots$$

$$\delta Z^{b_1 b_2} = 8 \cdot 135 Q_{b_1 b_2} + 135 \Lambda_{b_1 b_2} b_1 b_2 P_\perp - 6 \cdot 15 Z_{b_1 b_2} b_1 b_2 + \ldots; \text{ (2.26)}$$

where we have only given results up the generators $Z^{b_1 b_2}$. At level two we find that

$$\delta Z^{b_1 b_2} = \frac{61}{2} \Lambda_{b^b} b_1 b_2 P_\perp - 60 \Lambda_{b_1 b_2} b_1 b_2 Z - 60 Z_{b_1 b_2} \Lambda_{b_1 b_2 b_1 b_2} - 5 Z_{b_1 b_2} b_1 b_2$$

$$+ 10 \Lambda_{b_1 b_2} b_1 b_2 P_\perp + \frac{1}{3} \tilde{\Lambda}_{b^b} b_1 b_2 Z^{b_1 b_2} + \frac{1}{3} \Lambda_{b^b b_1 b_2} b_1 b_2 c_1 b_1 c_2$$

$$\delta Z^{b_1 b_2} = 6 \cdot 7 \cdot 135 Q_{b_1 b_2} + 135 \Lambda_{b_1 b_2} b_1 b_2 P_\perp - 7 \cdot 135 \Lambda_{b_1 b_2} b_1 b_2 P_\perp + 7 \cdot 135 Z_{b_1 b_2} b_1 b_2 + \ldots$$

$$\delta Z^{b_1 b_2} = -2 \cdot 3 \cdot 7 \cdot 15 Q_{b_1 b_2} + 15 \Lambda_{b_1 b_2} b_1 b_2 P_\perp + 7 \cdot 15 \Lambda_{b_1 b_2} b_1 b_2 P_\perp + 7 \cdot 15 Z_{b_1 b_2} b_1 b_2 + \ldots$$

$$\delta Z^{b_1 b_2} = 135 \cdot 12 Q_{b_1 b_2} + 135 \cdot 12 \Lambda_{b_1 b_2} b_1 b_2 P_\perp + 135 \cdot 12 \Lambda_{b_1 b_2} b_1 b_2 P_\perp + 135 \cdot 12 \Lambda_{b_1 b_2} b_1 b_2 P_\perp + \ldots$$

$$(2.27)$$
where we have only computed up to and including transformations of the generator $Z^2_{\tilde{a}_1\ldots\tilde{a}_5}$.

The tangent space algebra is $I_c(E_{11})$. In eleven dimensions a bilinear $I_c(E_{11})$ invariant was found to be [18]

$$L^2_{E_{11}} = L_AL_BK^{AB} = P_aP_{\tilde{a}} + \frac{1}{2}Z_{\tilde{a}_1\tilde{a}_2}Z^{\tilde{a}_1\tilde{a}_2} + \frac{1}{5!}Z_{\tilde{a}_1\ldots\tilde{a}_5}Z^{\tilde{a}_1\ldots\tilde{a}_5} + \ldots \quad (2.28)$$

where $\tilde{a}, \tilde{b}, \ldots = 0, \ldots, 10$ and $K^{AB}$ is the tangent group metric. The corresponding IIA invariant can be found by using equations (2.22) which relates the IIA vector generators to those in the eleven dimensional theory. The result is given by

$$L^2_{IIA} = P_aP_{\tilde{a}} + Q_aQ_{\tilde{a}} + ZZ + \frac{1}{2}Z_{\tilde{a}_1\tilde{a}_2}Z^{\tilde{a}_1\tilde{a}_2} + \frac{1}{4!}Z_{\tilde{a}_1\ldots\tilde{a}_4}Z^{\tilde{a}_1\ldots\tilde{a}_4} + \frac{1}{5!}Z_{\tilde{a}_1\ldots\tilde{a}_5}Z^{\tilde{a}_1\ldots\tilde{a}_5} + \ldots \quad (2.29)$$

3. The string little algebra at low levels

In 1939 Wigner found the irreducible representations of the Poincare group $SO(1,3) \otimes_s \{P_\mu\}$, $\mu = 0, \ldots, 3$. In this paper we will study the irreducible representations of the semi-direct product of $I_c(E_{11})$ with its $l_1$ vector representation, denote by $I_c(E_{11}) \otimes_s l_1$. Given the similarities of Poincare algebra and the algebra $I_c(E_{11}) \otimes_s l_1$, the generators of the vector representation commute, we can follow a similar path to that taken by Wigner. In this section we will carry out the first step and find the string little algebra at low levels. We will first review the method for constructing irreducible representations in E theory [14] adapted to the IIA string.

In the case of the Poincare algebra, the Wigner method, at least as practiced by physicists, begins by choosing specific values for the momenta $P_\mu$, which are the conserved charges of the translation. We then find the subalgebra of $SO(1,3)$ which preserves this choice, that is, we find the little algebra, For a massive particle we can choose the momenta to take values $P_\mu = (m, 0, 0, 0)$ in which case the little algebra is $SO(3)$. We then choose an irreducible representation of $SO(1,3)$ which encodes the physical degrees of freedom of the particle. Finally one carries out a boost out of the rest frame to find the full representation of the Poincare group.

In E theory, the Poincare algebra is replaced by semi-direct products such as $I_c(E_{11}) \otimes_s l_1$ and the conserved charges $P_a$ by the charges in the vector representation. The lowest level elements of $I_c(E_{11})$ and the vector representation are the Lorentz algebra and the translations respectively and so the Poincare algebra is the lowest level part of $I_c(E_{11}) \otimes_s l_1$. However, now the situation is much more complicated, $I_c(E_{11})$ is an infinite dimensional algebra, whose properties are largely only understood at low levels, while the vector representation contains an infinite number of brane charges. Increasing in level the next elements in eleven dimensions, are the M2, M5 brane charges and the Taub-Nut charge which are followed by an infinite number of new charges.

In E theory we begin by choosing values for the brane charges in the vector representation that are those of interest, to be non-zero. Next we find the subalgebra of $I_c(E_{11})$, denoted by $\mathcal{H}$, which preserves this choice of brane charges. In keeping with the Poincare algebra case we refer to the subalgebra $\mathcal{H}$ as the little algebra. The next step is to choose
an irreducible representation of $\mathcal{H}$ and this should encode the degrees of freedom of the brane we are studying.

The irreducible representation of $I_c(E_{11}) \otimes s l_1$ corresponding to the point particle was studied in reference [14]. The non-zero brane charges were chosen to be that of the usual momenta which was taken to be massless. In particular this paper took $P_a = (m, 0, \ldots, 0, m)$ with all the other brane charges being zero. The resulting little algebra was found to be $\mathcal{H} = I_c(E_9)$ which is just affine $E_8$. Despite the infinite nature of this algebra it has a representation that only has a finite number of states and these are precisely the bosonic degrees of freedom of eleven dimensional supergravity [14].

In this paper we will apply the same strategy to find the irreducible representation of $I_c(E_{11}) \otimes s l_1$ that corresponds to the string. In particular we will study the IIA string. As explained in section two the IIA theory emerges when we consider the decomposition that results from deleting node ten in the $E_{11}$ Dynkin diagram to find the algebra $D_{10}$. The generators are then arranged according to their level with respect to this node. We often referred to this level as the IIA level.

The lowest members of the vector representation, that is the ones that have IIA level zero, are the usual translations $P_a$ and the string charge $Q^a$, and they belong to the vector representation of $D_8$. It might be tempting to take these charges to have the values $P_a = (m, 0, \ldots, 0)$ and $Q^a = (0, m, 0, \ldots, 0)$ where $a, b, \ldots = 0, 1, \ldots, 9$. However, unlike for the point particle, a string has a world volume which has a corresponding symmetry, namely $SO(1,1)$. We will take this symmetry to be preserved by our choice and so we take the brane charges in $l_A$ to obey

$$P_a = -\varepsilon_{ab}Q^b \quad \text{or equivalently} \quad Q^a = -\varepsilon^{ab}P_b \quad l_A = 0 \quad \text{otherwise} \quad (3.1)$$

In this equations $a, b, \ldots = 0, 1$. The choice first mentioned does satisfy these conditions but it is obviously not $SO(1,1)$ covariant.

The adoption of a relation among the brane charges, rather than particular values, is a new and important difference required when we find irreducible representations of $I_c(E_{11}) \otimes s l_1$ that correspond to branes rather than those of the Poincare algebra which concern the point particle. The choice of brane charges of equation (3.1) obeys $L^2_{IIA} = 0$ for the $I_c(E_{11})$ invariant of equation (2.29). For the known brane charges this condition is just the same as the well known half BPS condition that results from the supersymmetry algebra. It also obeys the condition $P_a Q^a = 0$ which means that it obeys the $I_c(E_{11})$ constraints discussed in reference [18] where the relation of these two conditions to the Casimirs is explained.

The next step is to find the little algebra $\mathcal{H}$ that preserves the choice of equation (3.1). To do this we examine the $I_c(E_{11})$ variation of the brane charges given in equations (2.26) and (2.27) and look for which of them are non-zero. Keeping only such terms, we find at IIA level zero that

$$\delta P_a = -2(\Lambda_a^e - \tilde{\Lambda}_a^e c^{ce})P_e , \quad \delta Q^b = 2(\Lambda^b_c c^{ce} - \tilde{\Lambda}^b_c c^{ce})P_e ; \quad (3.2)$$

at level one:

$$\delta Z = \Lambda^e P_e , \quad \delta Z^{b, c} = -6(\Lambda^b_c \delta^{c e} - \frac{1}{3} \Lambda_a^{[b} c^{e] c^{ce}})P_e ,$$

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\[ \delta Z^{b_1 \cdots b_4} = 12(5\Lambda^{eb_1 \cdots b_4} - 2\Lambda^{[eb_1 \cdots b_4]}\varepsilon^{ce})P_e, \]
\[ \delta Z^{b_1 \cdots b_6} = -8 \cdot 135(\delta^{[b_1} \Lambda^{b_2 \cdots b_6]}\varepsilon^{ce} + 7\Lambda^{eb_1 \cdots b_6})P_e \] (3.3)

and at level two:
\[ \delta Z^{b_1 \cdots b_5} = \frac{6!}{2}\Lambda^{eb_1 \cdots b_5}P_e, \delta Z^{b_1 \cdots b_7} = 7 \cdot 15(6\delta^{[b_1} \Lambda^{b_2 \cdots b_7]}\varepsilon^{pe} - 8\Lambda^{eb_1 \cdots b_7} + 7\Lambda^{[eb_1 \cdots b_7]}P_e, \]
\[ \delta Z^{b_1 \cdots b_5} = 135(-12\delta^{[b_1} \Lambda^{b_2 \cdots b_5]}\varepsilon^{pe} + 6\delta^{e} \Lambda^{b_1 \cdots b_5}\varepsilon^{pe} - 8 \cdot 7\Lambda^{eb_1 \cdots b_5} - 7\Lambda^{[eb_1 \cdots b_5]}P_e, \]
\[ \delta Z^{b_1 \cdots b_7} = 7 \cdot 135(-6\delta^{[b_1} \Lambda^{b_2 \cdots b_7]}\varepsilon^{pe} - 8 \cdot 7\Lambda^{eb_1 \cdots b_7} - 7\Lambda^{[eb_1 \cdots b_7]}P_e) \] (3.4)

In these equations we have omitted the variation of some of the more complicated charges and we have used equation (3.1) to eliminate \( Q^a \) in terms of \( P_a \).

Requiring that the variations of the charges to vanish results in restrictions on the \( \Lambda^{ab} \)'s and so the transformation. We first consider the variations of the level IIA zero charges \( P_b \) and \( Q^b \). These do not separately have to vanish but they must preserve the relation of equation (3.1), namely
\[ \delta(P_a + \varepsilon_{ab}Q^b) = 0 \] (3.5)

Inserting (3.2) for the case of \( b = b \) into (3.5) we see \( \Lambda^{ab} \) and \( \tilde{\Lambda}^{ab} \) must satisfy
\[ 2(-\Lambda^{ab} + \varepsilon_{ac}\Lambda^{cd}\varepsilon^{db})P_b + 2(\tilde{\Lambda}_{ac}\varepsilon^{cb} - \varepsilon_{ac}\tilde{\Lambda}^{cb})P_b = 0. \] (3.6)

However (3.6) vanishes identically since we may write \( \Lambda^{ab} = \varepsilon_{ab}\Lambda \) and \( \tilde{\Lambda}^{ab} = \varepsilon_{ab}\tilde{\Lambda} \). Hence there are no restrictions on \( \Lambda^{ab} \) and \( \tilde{\Lambda}^{ab} \). Part of this symmetry is the obvious SO(1,1) Lorentz world sheet symmetry that we already insisted the charges conditions should preserve but they also preserve the transformations corresponding to \( S_{ab} \).

Setting the variations of all the other brane charges to zero gives further restrictions on the \( I_c(E_{11}) \) transformation. Examining equations (3.2) we find the following restrictions on the \( \Lambda^{ab} \)'s. At level zero we find the conditions
\[ \Lambda^{ab} \neq 0, \; \Lambda^{ij} \neq 0, \; \tilde{\Lambda}^{ab} \neq 0, \; \tilde{\Lambda}^{ai} = \varepsilon^{ac}\Lambda^i_c, \; \tilde{\Lambda}^{ij} \neq 0. \] (3.7)

While varying only the charges \( Z, Z^{b_1 \cdots b_4}, Z^{b_1 \cdots b_5}, Z^{b_1 \cdots b_6}, Z^{b_1 \cdots b_7} \), at level one leads to the conditions
\[ \Lambda^a = 0, \; \Lambda^{ai_1i_2} = 0, \; \Lambda^{abi} = \frac{1}{6}\varepsilon^{ab}\Lambda^i, \; \Lambda^{ai_1 \cdots i_4} = 0, \]
\[ \Lambda^{abi_1i_2i_3} = \frac{2}{5}\varepsilon^{ab}\Lambda^{i_1i_2i_3}, \; \Lambda^{i_1 \cdots i_5} \neq 0, \; \Lambda^{e_1 \cdots i_6} = 0, \]
\[ \Lambda^{abi_1 \cdots i_5} = \frac{1}{7 \cdot 6}\varepsilon^{ab}\Lambda^{i_1 \cdots i_5}, \; \Lambda^{i_1 \cdots i_7} \neq 0, \] (3.8)

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At level two the computations become more difficult but it is straightforward to find the following restrictions

\[
\Lambda^{ai_1...i_5} = 0, \quad \Lambda^{ab_i...i_4} = 0, \quad \Lambda^{i_1...i_6} \neq 0, \\
\Lambda^{ai_1...i_7} = 0, \quad \Lambda^{ai_1...i_6,j} = 0, \quad \Lambda^{i_1...i_7,j} \neq 0, \quad \Lambda^{i_1...i_8} \neq 0
\] (3.9)

The generators of the little algebra are found by inserting these relations for the parameters into the generator of equation (2.23). Equations (3.7) to (3.9) imply that

\[
S_a, S_{a_1i_2}, S_{a_1...i_4}, S_{a_1...i_6}, S_{a_1a_2i_1...i_4}, S_{a_1i_7} \notin \mathcal{H}
\] (3.10)

but that \(\mathcal{H}\) contains the following generators

\[
J_{ij}, S_{ij}, J_{ab}, S_{ab}, \quad L_{ai}^{(0)} = J_{ai} + \varepsilon_a e S_{ei}
\] (3.11)

at level zero and

\[
L_{i_1...i_5}^{(1)} = S_{i_1...i_5} + \frac{1}{2} \varepsilon_{e_1e_2} S_{e_1e_2i_1...i_3}, \quad L_{i_1i_2i_3}^{(1)} = S_{i_1i_2i_3} + \varepsilon_{e_1e_2} S_{e_1e_2i_1i_2i_3}, \\
L_{i_1...i_5}^{(1)} = S_{i_1...i_5} + \frac{1}{2} \varepsilon_{e_1e_2} S_{e_1e_2i_1...i_5}
\] (3.12)

at level one. The superscript refers to the IIA level to which the generators belong.

Finding the rest of the relations between the \(\Lambda^{ai}\)'s resulting from the variations of the higher level charges becomes more and more difficult as to find the variations requires the commutation relations for higher and higher level generators which are generally not known. Instead one can exploit the fact that the little group algebra \(\mathcal{H}\) must close and as the generators of equation (3.11) and (3.12) belong to \(\mathcal{H}\) we can find in this way new elements by calculating their commutators. This algebra will be computed below but for completeness we now list some of the elements of the little algebra that we find in this way:

\[
L_{i_1...i_7}^{(1)} = S_{i_1...i_7} - \frac{1}{2} \varepsilon^{a_1a_2} S_{a_1a_2i_1...i_7},
\] (3.13)

which is also a IIA level one generator and also the level two generators

\[
L_{i_1...i_6}^{(2)} = S_{i_1...i_6} + \frac{1}{2} \varepsilon^{a_1a_2} S_{i_1...i_6a_1a_2} - \varepsilon^{a_1a_2} S_{i_1...i_6a_1a_2} - \frac{1}{8} \varepsilon^{a_1a_2} \varepsilon^{b_1b_2} S_{i_1...i_6a_1a_2b_1b_2}, \\
L_{i_1...i_8}^{(2)} = S_{i_1...i_8} + \varepsilon^{a_1a_2} S_{i_1...i_8a_1a_2} - \frac{1}{6} \varepsilon^{a_1a_2} S_{i_1...i_8a_1a_2} - \frac{1}{2} \varepsilon^{a_1a_2} S_{1(1)a_1a_2i_1...i_8} - \frac{4}{9} \varepsilon^{a_1a_2} S_{1(2)a_1a_2i_1...i_8} - \frac{1}{12} \varepsilon^{a_1a_2} S_{i_1...i_7,j,a_1a_2}
\]
\[-\frac{1}{18} \varepsilon_{a_1 a_2} S_{(2) a_1 a_2 i_1 \ldots i_7 j} \]  

(3.14)

The above results imply that the string little algebra is given by

\[ \mathcal{H} = \{ J_{ab}, L_{ai}^{(0)}, J_{ij}, S_{ab}, S_{ij}, L_i^{(1)}, L_{i1 i2 i3}^{(1)}, L_{i1 \ldots i_5}^{(1)}, L_{i1 i2 i3}^{(2)}, L_{i1 \ldots i_5}^{(2)}, L_{i1 \ldots i_7 j}^{(2)}, \ldots \} \]

(3.15)

Here the levels are separated by a semi-colon and the \ldots denotes higher level generators that we have not computed.

We will now compute the algebra of the generators of \( \mathcal{H} \). At IIA level zero we find that

\[
[J^{ab}, J_{cd}] = -4\delta^{[a}_{c} J^{b]} d], \quad [J^{ab}, L_{cl}^{(0)}] = -2\delta^{[a}_{c} L_{b]}^{(0)}, \quad [J^{ab}, J_{ij}] = 0,
\]

\[
[J^{ab}, S_{cd}] = -4\delta^{[a}_{c} S_{b]} d], \quad [J^{ab}, S_{ij}] = 0, \quad [L^{ai(i)}, L_{bj]}^{(0)}] = 0, \quad [L^{ai(i)}, J_{jk}] = 2\delta^{i}_{j} L_{a(0)}^{(0)},
\]

\[
[L^{ai(0)}, S_{bc}] = 2\varepsilon^{[a}_{b} L_{c]}^{(0)}, \quad [L^{ai(0)}, S_{jk}] = 2\varepsilon^{a}_{ij} L^{(0)}_{k}], \quad [J^{ij}, S_{kl}] = -4\delta^[i]_{[k} J^{j]} l],
\]

\[
[J^{ij}, S_{cd}] = 0, \quad [J^{ij}, S_{kt}] = 0, \quad [S^{ab}, S_{cd}] = -4\delta^{[a}_{c} S_{b]} d],
\]

\[
[S^{ab}, S_{ij}] = 0, \quad [S^{ij}, S_{kt}] = -4\delta^[i]_{[k} J^{j]} l] . \tag{3.16}
\]

The commutators of the generators in \( \mathcal{H} \) arising from IIA levels \( \pm 1 \) are given by

\[
[L_{i}^{(1)}, L_{j}^{(1)}] = 0, \quad [L_{i}^{(1)}, L_{j1 j2 j3}^{(1)}] = 0, \quad [L_{i}^{(1)}, L_{j1 \ldots j5}^{(1)}] = -L_{i1 j1 j5}^{(2)},
\]

\[
[L_{i}^{(1)}, L_{j1 \ldots j7}^{(1)}] = -L_{i1 j1 \ldots j7}^{(2)} + L_{j1 \ldots j7, i}^{(2)}, \quad [L_{i1 i2 i3}^{(1)}, L_{j1 j2 j3}^{(1)}] = 2L_{i1 i2 i3 j1 j2 j3}^{(2)},
\]

\[
[L_{i1 i2 i3}^{(1)}, L_{j1 \ldots j5}^{(1)}] = L_{i1 i2 i3 j1 \ldots j5}^{(2)} - 5L_{i1 i2 i3 j1 j4 j5}^{(2)} \tag{3.17},
\]

as well as higher level relations.

The algebra of the level zero generators with the level \( \pm 1 \) generators of \( \mathcal{H} \) is given by

\[
[J_{ab}, L_{i}^{(1)}] = 0, \quad [J_{ab}, L_{i1 i2 i3}^{(1)}] = 0, \quad [J_{ab}, L_{i1 \ldots i_5}^{(1)}] = 0, \quad [J_{ab}, L_{i1 \ldots i_7}^{(1)}] = 0,
\]

\[
[L_{ai}^{(0)}, L_{j}^{(1)}] = 0, \quad [L_{ai}^{(0)}, L_{j1 j2 j3}^{(1)}] = 0, \quad [L_{ai}^{(0)}, L_{j1 \ldots j5}^{(1)}] = 0, \quad [L_{ai}^{(0)}, L_{j1 \ldots j7}^{(1)}] = 0,
\]

\[
[J^{ij}, L_{k}^{(1)}] = -2\delta^{i}_{k} L_{j}^{(1)}, \quad [J^{ij}, L_{k1 k2 k3}^{(1)}] = -6\delta^{i}_{[k1} L_{j]}^{(1)} k2 k3],
\]

\[
[J^{ij}, L_{k1 \ldots k_7}^{(1)}] = -10\delta^{i}_{[k1} L_{j]}^{(1)} k2 \ldots k_7], \quad [J^{ij}, L_{k1 \ldots k_7}^{(1)}] = -14\delta^{i}_{[k1} L_{j]}^{(1)} k2 \ldots k_7],
\]

\[
[S^{ab}, L_{i}^{(1)}] = \varepsilon^{ab} L_{i}^{(1)}, \quad [S^{ab}, L_{i1 i2 i3}^{(1)}] = \varepsilon^{ab} L_{i1 i2 i3}^{(1)}, \quad [S^{ab}, L_{i1 \ldots i_5}^{(1)}] = \varepsilon^{ab} L_{i1 \ldots i_5}^{(1)},
\]

\[
[S^{ab}, L_{i1 \ldots i_7}^{(1)}] = \varepsilon^{ab} L_{i1 \ldots i_7}, \quad [S_{ij}, L_{k}^{(1)}] = -L_{ij k}^{(1)},
\]

\[
[S_{ij}, L_{k1 k2 k3}^{(1)}] = -2L_{ij}^{(1)} k1 k2 k3 + 6L_{ij}^{(1)} \delta^{ij}_{k1} k2 k3],
\]

\[
[S_{ij}, L_{k1 \ldots k_5}^{(1)}] = 10L_{[k1 k2 k3} \delta^{ij}_{k4 k5]} - L_{ij}^{(1)} k1 \ldots k_5 .
\]
\[ [S^{ij}, L^{(1)}_{k_1...k_7}] = 6 \cdot 7L^{(1)}_{[k_1...k_5} \delta^{ij}_{k_6k_7}] . \] (3.18)

4. A toy string as an irreducible representation

Rather than study the irreducible representations of \( I_c(E_{11}) \otimes_s l_1 \) we will, in this section, consider the simpler case of \( I_c(D_{10}^+) \otimes_s l_1 \). In this context \( l_1 \) is the vector representation which contains the generators \( P_{a,n}, Q^a_n \) with \( n \geq 0 \) and we take the real form of \( D_D \) to be \( SO(D,D) \). The algebra, \( I_c(D_{10}^+) \) occurs as subalgebra of \( I_c(E_{11}) \) in its IIA formulation and so our results will shed light on this latter case. In appendix B we formulated the algebra \( SO(D,D) \otimes_s (P_a, Q^a) \) in detail and using these results it is straightforward to formulate the algebra \( I_c(D_{10}^+) \otimes_s l_1 \).

4.1 The \( D_D^+ \otimes_s l_1 \) algebra

In this section we will give the explicit form of the affine Lie algebra \( SO(D,D)^+ \) as well as its Cartan invariant subalgebra. We will also discuss its vector representation. Given any finite dimensional semi-simple Lie algebra \( G \) we can divide the generators into those that are even and those that are odd under the Cartan involution operator \( I_c \) that the algebra possess. Let us denote these generators by \( S^a \) and \( T^i \) respectively in which case \( I_c(S^a) = S^a \) and \( I_c(T^i) = -T^i \). In terms of these generators the algebra takes the form

\[
[S^a, S^b] = f^{ab} c S^c, \quad [S^a, T^i] = f^{ai} c T^j, \quad [T^i, T^j] = f^{ij} c S^c
\] (4.1.1)

The affine algebra \( G^+ \) has the generators \( S^a_n \) and \( T^i_n \) which obey the algebra

\[
[S^a_n, S^b_m] = f^{ab} c S^c_{n+m}, \quad [S^a_n, T^i_m] = f^{ai} c T^j_{n+m}, \quad [T^i_n, T^j_m] = f^{ij} c S^c_{n+m}
\] (4.1.2)

where we omit to write the central term.

As explained in reference [20] the Cartan involution acts on \( G^+ \) as \( I_c(S_n) = S_{-n} \) and \( I_c(T_n) = -T_{-n} \) and so the Cartan involution invariant generators in \( I_c(G^+) \) are given by

\[
S^a_n = S^a_{-n}, \quad \text{and} \quad T^i_n = T^i_{-n}
\] (4.1.3)

and obey the commutators

\[
[S^a_n, S^b_m] = f^{ab} c S^c_{n+m} + f^{ab} c S^c_{-n-m}, \quad [S^a_n, T^i_m] = f^{ai} c T^j_{n+m} - f^{ai} c T^j_{-n-m},
\]

\[
[T^i_n, T^j_m] = f^{ij} c S^c_{n+m} - f^{ij} c S^c_{-n-m}
\] (4.1.4)

This is not an affine algebra.

We note that the generators \( S^a_n \) formed a closed subalgebra. The generators \( S^a \) form the algebra \( I_c(G) \) and so the generators \( S^a_n \) have the algebra \( (I_c(G))^+ \). However, the generators \( S^a_0 \) have the algebra \( I(I_c(G))^+ \) which is the subalgebra of \( (I_c(G))^+ \) that is invariant under the involution \( I \) which acts as \( I(S^a_0) = S^a_{-0} \). Obviously this is a subalgebra of \( I_c(G^+) \). For example, if \( G = A_D = SL(D+1) \) with generators \( K^i_j \), then \( I_c(SL(D+1)) = SO(D+1) \) contains the generators \( K^i_j - K^j_i \). In this case \( I(I_c(G^+)) = I(SO(D+1)^+) \) with the generators \( K_n^i_j - K_n^j_i + K_{-n}^i_j - K_{-n}^j_i \). The algebra \( I_c(SO(D+1)^+) \) contains in addition the generators \( K_n^i_j + K_n^j_i - K_{-n}^i_j - K_{-n}^j_i \).
As is well known one can formulate an affine algebra by taking $S^a_n = S^a e^{i n \sigma}$ and $T^i_n = T^i e^{i n \sigma}$. One then recovers the commutations of the affine algebra of equation (4.1.2). This allows us to interpret the affine algebra as a loop algebra, that is, as arising from a map of the closed loop, or closed string, into the finite dimensional semi-simple Lie group $G$. For the Cartan involution invariant algebra ($I_c(G^+)$) we have instead,

$$S^a_n = 2S^a \cos n \sigma, \quad T^i_n = 2i \sin n \sigma T^i$$

One readily finds that these generators do obey the commutators of equation (4.1.4). We can interpret $I_c(G^+)$ as a mapping of an open string into the finite dimensional semi-simple Lie group $G$. The open string associated with the $S^a_n$ is the usual string which obeys Neuman boundary conditions at both ends while the open string associated with $T^i_n$ obeys Dirichlet at each end.

We will now apply the above discussion to formulate the Cartan involution algebra of $D^+_D = SO(D, D)^+$. In appendix B we listed the Cartan even generators of $SO(D, D)$ as $J_{\underline{a}, \underline{b}}$ and $S_{\underline{a}, \underline{b}}$ and the odd generators as $T_{\underline{a}, \underline{b}}$ and $U_{\underline{a}, \underline{b}}$. We recall that $T_{\underline{a}, \underline{b}}$ and $U_{\underline{a}, \underline{b}}$ are symmetric and anti-symmetric in their indices respectively. Adding the level indices $n, m, \ldots$ to the generators of $SO(D, D)$ we find the generators of the affine algebra $SO(D, D)^+$. Their commutators can easily be read off from equations (B.1), (B.2) and (B.3) in appendix B to be given by

$$[J_n^{\underline{a}, \underline{b}}, J_{m, \underline{b}, \underline{b}_2}] = -4 \delta_{\underline{b}, \underline{b}_2}^{\underline{a}, \underline{a}_2} J_{n+m, \underline{a}_2, \underline{b}_2}, \quad [S_n^{\underline{a}, \underline{b}}, S_{m, \underline{b}, \underline{b}_2}] = -4 \delta_{\underline{b}, \underline{b}_2}^{\underline{a}, \underline{a}_2} S_{n+m, \underline{a}_2, \underline{b}_2}$$

and

$$[J_n^{\underline{a}_1, \underline{a}_2}, T_{m, \underline{b}, \underline{b}_2}] = -4 \delta_{\underline{b}, \underline{b}_2}^{\underline{a}_1, \underline{a}_2} T_{n+m, \underline{a}_2, \underline{b}_2}, \quad [J_n^{\underline{a}_1, \underline{a}_2}, U_{m, \underline{b}, \underline{b}_2}] = -4 \delta_{\underline{b}, \underline{b}_2}^{\underline{a}_1, \underline{a}_2} U_{n+m, \underline{a}_2, \underline{b}_2},$$

$$[S_n^{\underline{a}_1, \underline{a}_2}, T_{m, \underline{b}, \underline{b}_2}] = 4 \delta_{\underline{b}, \underline{b}_2}^{\underline{a}_1, \underline{a}_2} U_{n+m, \underline{a}_2, \underline{b}_2}, \quad [S_n^{\underline{a}_1, \underline{a}_2}, U_{m, \underline{b}, \underline{b}_2}] = 4 \delta_{\underline{b}, \underline{b}_2}^{\underline{a}_1, \underline{a}_2} T_{n+m, \underline{a}_2, \underline{b}_2}$$

as well as

$$[T_n^{\underline{a}_1, \underline{a}_2}, T_{m, \underline{b}, \underline{b}_2}] = 4 \delta_{\underline{b}, \underline{b}_2}^{\underline{a}_1, \underline{a}_2} T_{n+m, \underline{a}_2, \underline{b}_2}, \quad [U_n^{\underline{a}_1, \underline{a}_2}, U_{m, \underline{b}, \underline{b}_2}] = 4 \delta_{\underline{b}, \underline{b}_2}^{\underline{a}_1, \underline{a}_2} U_{n+m, \underline{a}_2, \underline{b}_2},$$

$$[T_n^{\underline{a}_1, \underline{a}_2}, U_{m, \underline{b}, \underline{b}_2}] = 4 \delta_{\underline{b}, \underline{b}_2}^{\underline{a}_1, \underline{a}_2} T_{n+m, \underline{a}_2, \underline{b}_2}$$

We have omitted the usual central terms.

The algebra can also be given in terms of the generators

$$M_n^{\underline{a}_1, \underline{a}_2} = \frac{1}{2} (J_n^{\underline{a}_1, \underline{a}_2} + S_n^{\underline{a}_1, \underline{a}_2}), \quad \bar{M}_n^{\underline{a}_1, \underline{a}_2} = \frac{1}{2} (J_n^{\underline{a}_1, \underline{a}_2} - S_n^{\underline{a}_1, \underline{a}_2}),$$

$$V_n^{\underline{a}_1, \underline{a}_2} = T_n^{\underline{a}_1, \underline{a}_2} - U_n^{\underline{a}_1, \underline{a}_2}, \quad \bar{V}_n^{\underline{a}_1, \underline{a}_2} = T_n^{\underline{a}_1, \underline{a}_2} + U_n^{\underline{a}_1, \underline{a}_2}$$

(4.1.9)
Their commutators are given by
\[
[M_n^{a_1b_1}, M_{m_1}^{b_1}] = -4\delta_{b_1}^{[a_1} M_{n+m}^{a_2]b_2}, \quad [\bar{M}_n^{a_1b_1}, \bar{M}_{m_1}^{b_1}] = -4\delta_{b_1}^{[a_1} \bar{M}_{n+m}^{a_2]b_2},
\]
\[
[M_n^{a_1b_1}, \bar{M}_{m_1}^{b_1}] = 0 \tag{4.1.10}
\]
and
\[
[M_n^{a_1b_1}, V_{m_1}^{b_1}] = -4\delta_{b_1}^{[a_1} V_{n+m}^{a_2]b_2}, \quad [M_n^{a_1b_1}, \bar{V}_{m_1}^{b_1}] = -4\delta_{b_2}^{[a_1} V_{n+m}^{a_2]b_1},
\]
\[
[\bar{M}_n^{a_1b_1}, \bar{V}_{m_1}^{b_1}] = -4\delta_{b_1}^{[a_1} \bar{V}_{n+m}^{a_2]b_2}, \quad [\bar{M}_n^{a_1b_1}, V_{m_1}^{b_1}] = -4\delta_{b_2}^{[a_1} \bar{V}_{n+m}^{a_2]b_1} \tag{4.1.11}
\]

as well as
\[
[V_n^{a_1b_1}, V_{m_1}^{b_1}] = 4\delta_{b_2}^{a_1} \bar{M}_{n+m}^{a_2} b_2 + 4\delta_{b_1}^{a_2} M_{n+m}^{a_1} b_1,
\]
\[
[\bar{V}_n^{a_1b_1}, \bar{V}_{m_1}^{b_1}] = 4\delta_{b_2}^{a_1} \bar{M}_{n+m}^{a_2} b_2 + 4\delta_{b_1}^{a_2} \bar{M}_{n+m}^{a_1} b_1,
\]
\[
[V_n^{a_1b_1}, \bar{V}_{m_1}^{b_1}] = 4\delta_{b_2}^{a_1} M_{n+m}^{a_1} b_2 + 4\delta_{b_1}^{a_2} \bar{M}_{n+m}^{a_2} b_1 \tag{4.1.12}
\]

We note that \(M_n^{a_1b_1}\) and \(\bar{M}_n^{a_1b_1}\) generate the algebra \(SO(D)^+ \otimes SO(D)^+\) which equals \((I_c SO(D,D))^+\).

The generators of \(I_c(D_D^+)\) are
\[
J_n^{a_1b_1} + J_{-n}^{a_1b_1}, \quad S_n^{a_1b_1} + S_{-n}^{a_1b_1}, \quad T_n^{a_1b_1} - T_{-n}^{a_1b_1}, \quad U_n^{a_1b_1} - U_{-n}^{a_1b_1} \tag{4.1.13}
\]
or equivalently
\[
M_n^{a_1b_1} + M_{-n}^{a_1b_1}, \quad \bar{M}_n^{a_1b_1} + \bar{M}_{-n}^{a_1b_1}, \quad V_n^{a_1b_1} - V_{-n}^{a_1b_1}, \quad \bar{V}_n^{a_1b_1} - \bar{V}_{-n}^{a_1b_1} \tag{4.1.14}
\]

Their commutators can be readily deduced from those above. The first two generators in the above equation belong to the algebra \(I(SO(D)^+ \otimes SO(D)^+)^+\) in terms of the above notation. The last two generators belong to a representation of this algebra.

We will now consider the vector representation of \(D_D^+\) which contains the generators \(P_{n\alpha}\) and \(Q_{n\alpha}\). Their commutators with the generators of \(D_D^+\) follow from those of equation (B.8) and (B.9) and are as follows

\[
[J_{n\alpha\beta}, P_{m\gamma}] = -2\delta_{\gamma\gamma} P_{n+m\beta}, \quad [J_{n\alpha\beta}, Q_{m\gamma}] = -2\delta_{\gamma\gamma} Q_{n+m\beta},
\]
\[
[S_{n\alpha\beta}, P_{m\gamma}] = -2\delta_{\gamma\gamma} Q_{n+m\beta}, \quad [S_{n\alpha\beta}, Q_{m\gamma}] = -2\delta_{\gamma\gamma} P_{n+m\beta} \tag{4.1.15}
\]
and
\[
[T_{n\alpha\beta}, P_{m\gamma}] = -2\delta_{\gamma\gamma} P_{n+m\beta}, \quad [T_{n\alpha\beta}, Q_{m\gamma}] = +2\delta_{\gamma\gamma} Q_{n+m\beta},
\]
\[
[U_{n\alpha\beta}, P_{m\gamma}] = -2\delta_{\gamma\gamma} Q_{n+m\beta}, \quad [U_{n\alpha\beta}, Q_{m\gamma}] = 2\delta_{\gamma\gamma} P_{n+m\beta} \tag{4.1.16}
\]

It will be advantageous to formulate the commutators in terms of the generators
\[
\varphi_{n\alpha} = P_{n\alpha} + Q_{n\alpha}, \quad \bar{\varphi}_{n\alpha} = P_{n\alpha} - Q_{n\alpha}, \tag{4.1.17}
\]

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Then the commutators with the generators of $D_D^+$ follow from those of equation (B.11) and (B.12) and are
\[
[M_{n_a^1 a^2}, \varphi_{m_b}] = -2\eta_{b[a^3}, \varphi_{n+m a^2]}, \quad [M_{n_a^1 a^2}, \varphi_{m_b}] = 0,
\]
\[
[M_{n_a^1 a^2}, \bar{\varphi}_{m_b}] = -2\eta_{b[a^3}, \bar{\varphi}_{n+m a^2]}, \quad [M_{n_a^1 a^2}, \bar{\varphi}_{m_b}] = 0,
\]
and
\[
[V_{n_a^1 a^2}, \varphi_{m_b}] = -2\eta_{b[a^3}, \varphi_{n+m a^2]}, \quad [V_{n_a^1 a^2}, \bar{\varphi}_{m_b}] = -2\eta_{b[a^3}, \bar{\varphi}_{n+m a^2]},
\]
\[
[V_{n_a^1 a^2}, \bar{\varphi}_{m_b}] = -2\eta_{b[a^3}, \bar{\varphi}_{n+m a^2]}, \quad [V_{n_a^1 a^2}, \varphi_{m_b}] = -2\eta_{b[a^3}, \varphi_{n+m a^2}].
\]
(4.1.18)

The $D_D^+$ variations of the vector representation are given by $\delta P_{n\bar{a}} = [\Lambda, P_{n\bar{a}}]$ and $\delta Q_{n\bar{a}} = [\Lambda, Q_{n\bar{a}}]$ where
\[
\Lambda = \Lambda_{n}^{ab} J_{n ab} + \bar{\Lambda}_{n}^{ab} \bar{J}_{n ab} + \Omega_{n}^{ab} T_{n ab} + \hat{\Omega}_{n}^{ab} V_{n ab}
\]
\[
= \lambda_{n}^{ab} M_{n ab} + \bar{\lambda}_{n}^{ab} \bar{M}_{n ab} + \mu_{n}^{ab} V_{n ab} + \bar{\mu}_{n}^{ab} \bar{V}_{n ab}
\]
(4.1.20)

We find, generalising equations (B.15) and (B.16), that the transformations on the vector representation are given by
\[
\delta \varphi_{n\bar{a}} = -2(\Lambda_m + \bar{\Lambda}_m)_{a}^{b} \varphi_{n+m b} - 2(\Omega_m + \bar{\Omega}_m)_{a}^{b} \bar{\varphi}_{n+m b}
\]
\[
= -2\lambda_{m}^{a} \varphi_{n+m b} - 2(\eta_{m a}^{a} + \bar{\mu}_{m a}^{b}) \bar{\varphi}_{n+m b},
\]
\[
\delta \bar{\varphi}_{n\bar{a}} = -2(\Lambda_m - \bar{\Lambda}_m)_{a}^{b} \bar{\varphi}_{n+m b} - 2(\Omega_m + \bar{\Omega}_m)_{a}^{b} \bar{\varphi}_{n+m b}
\]
\[
= -2\lambda_{m}^{a} \bar{\varphi}_{n+m b} - 2(\eta_{m a}^{a} + \bar{\mu}_{m a}^{b}) \varphi_{n+m b},
\]
(4.1.21)

Their transformations under $I_c(D_D^+)$ are then given by
\[
\delta \varphi_{n\bar{a}} = -2(\Lambda_m + \bar{\Lambda}_m)_{a}^{b} (\varphi_{n+m b} + \varphi_{n-m b}) - 2(\Omega_m - \bar{\Omega}_m)_{a}^{b} (\bar{\varphi}_{n+m b} - \bar{\varphi}_{n-m b})
\]
\[
= -2\lambda_{m}^{a} (\varphi_{n+m b} + \varphi_{n-m b}) - 2(\eta_{m a}^{a} + \bar{\mu}_{m a}^{b}) (\bar{\varphi}_{n+m b} - \bar{\varphi}_{n-m b}),
\]
\[
\delta \bar{\varphi}_{n\bar{a}} = -2(\Lambda_m - \bar{\Lambda}_m)_{a}^{b} (\varphi_{n+m b} + \varphi_{n-m b}) - 2(\Omega_m + \bar{\Omega}_m)_{a}^{b} (\bar{\varphi}_{n+m b} - \bar{\varphi}_{n-m b})
\]
\[
= -2\lambda_{m}^{a} (\bar{\varphi}_{n+m b} + \bar{\varphi}_{n-m b}) - 2(\eta_{m a}^{a} + \bar{\mu}_{m a}^{b}) (\varphi_{n+m b} - \varphi_{n-m b}),
\]
(4.1.22)

The quantity
\[
\sum_{n} (P_{n a}^2 + Q_{n a}^2) = \sum_{n} (P_{n a}^2 - Q_{n a}^2), \quad \text{or equivalently} \quad \sum_{n} \varphi_{n a}^2 + \sum_{n} \bar{\varphi}_{n a}^2
\]
is invariant under $I_c(D_D^+)$ transformations.

**4.2 The toy string little algebra**
Since we are seeking to mimic the behaviour of a string we expect that a component of its momentum $P_a$ and its string charge $Q_a$ are non-zero while all the other charges vanish. It may be tempting to take $P_0 = m$ and $Q_1 = m$ non-zero however, this does not take into account the fact that the string has a SO(1,1) symmetry in its world sheet. As such we take the condition

$$P_a + \epsilon_a b Q_b = 0$$

(4.2.1)

with all other charges zero, that is,

$$P_{na} = 0 = Q_{na}, \quad \text{and} \quad P_{ni} = 0 = Q_{ni}, \quad n = 1, 2, \ldots$$

(4.2.2)

We have split the original indices $a, b, \ldots$ into $a, b, \ldots = 0, 1$ and $i, j, \ldots = 2, \ldots, D - 1$.

For this choice of charges the invariant of equation (4.1.23) vanishes. We note that $P_a$ and $Q_a$ are non-zero and are not specified other than that they are related by equation (4.2.1). Since the original Wigner method of irreducible representations applied to point particle this subtlety would not have been encountered before.

The little algebra is the subalgebra of $D_{D-1}^+$ that preserves the choice of equations (4.2.1) and (4.2.2). To find it we consider the variations of the combinations $(P \pm Q)_{in} = 0, n \geq 1$ and take $(P \pm Q)_a = (1 \mp \epsilon) a^b P_b \neq 0$ as required by equation (4.2.1). One finds the conditions

$$\Lambda_{nia} = \tilde{\Lambda}_{nib} e^b_a, \quad \Omega_{nia} = \hat{\Omega}_{nib} e^b_a$$

(4.2.3)

Taking the same equation for $(P \pm Q)_{an} = 0, n \geq 1$ we find the conditions

$$\Omega_{na}^b - \hat{\Omega}_{na}^c e^c_b = 0, \quad \Omega_{nab} = \hat{\Omega}_{nac} e^c_b$$

(4.2.4)

From these results, and using the expression of the algebra element in equation (4.1.20), we find the transformations which preserve equations (4.2.1) and (4.2.2) are given by

$$\mathcal{H} = \{ E_{n_ia}^{(1)} \equiv J_{nia} + \epsilon_a b S_{nib} + J_{nia} + \epsilon_a b S_{nib}, \quad E_{n_ia}^{(2)} \equiv T_{nia} + \epsilon_a b U_{nib} - U_{nia} - \epsilon_a b U_{nib}, \quad E_{nab} \equiv T_{nab} - \eta_{ab} U_n - T_{nab} + \eta_{ab} U_n, \quad J_{ab}, \quad S_{ab}, \quad I_c(SO(D-2, D-2)^+) \}$$

(4.2.5)

where $U_{nab} = \epsilon_{ab} U_n$ and $I_c(SO(D-2, D-2)^+)$ consists of the generators

$$I_c(D_{D-2}) = \{ M_{nij} + M_{nij}, \quad \bar{M}_{nij} + \bar{M}_{nij}, \quad T_{nij} - T_{nij}, \quad U_{nij} - U\bar{n}_{nij}, \quad n \geq 0 \}$$

(4.2.6)

The generators $E_{n_ia}^{(1)}$, $E_{n_ia}^{(2)}$ and $E$ obey the commutators

$$[E_{n_ia}^{(1)}, E_{m_ia}^{(1)}] = 0, \quad [E_{n_ia}^{(2)}, E_{m_ia}^{(2)}] = 0, \quad [E_{n_ia}^{(1)}, E_{mb_1 b_2}] = 0, \quad [E_{n_ia}^{(2)}, E_{mb_1 b_2}] = 0$$

(4.2.7)
Hence these three generators form a closed sub algebra.

4.3 The toy string representation of the little algebra

We now wish to find an irreducible representation of \( I_c(D_D) \otimes_s l_1 \) following a similar path to that taken by Wigner to find the irreducible representations of the Poincare group. The reader can look at section four of reference [14] to see how this works for the case of \( I_c(E_{11}) \otimes_s l_1 \) and the massless representation. In the last section we found the subalgebra \( \mathcal{H} \) of \( I_c(D_D^+) \) that preserves equation (2.1) which corresponds to a static string. The next step is to choose an irreducible representation of \( \mathcal{H} \otimes_s l_1 \). Given this we can then boost this irreducible representation to find an irreducible representation of \( I_c(D_D^+) \otimes_s l_1 \) As the generators \( E^{(1)}_{n_{ia}}, E^{(2)}_{n_{jib}} \) and \( E_{nab} \) belong to a closed subalgebra we can take them to be trivially realised. This step is similar to the case of the massless representations of Poincare algebra where one finds subalgebras which we can take to be trivially realised so as to ensure unitarity. We can also take the generators \( J_{ab} \) and \( S_{ab} \) to be trivially realised.

This leaves us to find an irreducible representation of \( I_c(SO(D-2, D-2)^+) \). The algebra \( SO(D-2, D-2)^+ \) contains the subalgebra \( SO(D-2)^+ \otimes SO(D-2)^+ \) and so, as discussed above, the algebra \( I_c(SO(D-2, D-2)^+) \) contains the algebra \( I(SO(D-2)^+) \otimes I(SO(D-2)^+) \).

In section five we will find the little algebra for the string and in section six we will examine if the string states carry a representation of this algebra. Here we will see if string like states can carry a representation of the little algebra of the toy string. Our discussion will prove useful for the case of the real superstring. We will first consider the open superstring and in particular the NS sector which has the oscillators \( b_{ir} \). There is a well known relation between the oscillators that appear in string theory and Lie algebras. These works involve strings whose momenta belong to the corresponding root, or weight, lattices. Indeed the suitably integrated string vertex operators have commutators that are those of the Lie algebra corresponding to the root lattice. For a review see reference [21].

In our situation in this paper the momenta do not belong to a root lattice, however, it is also known that given the \( b_{ir} \) oscillators we can form the objects

\[
\mathcal{M}_n^{ij} = \sum_r : b_{n-r}^i b_r^j : \quad i,j, \ldots = 2, \ldots, D - 1 \tag{4.3.1}
\]

which generate the affine algebra \( SO(D-2)^+ \) [22,23] This follows simply from the fact that these oscillators obey their usual anti-commutation relations

\[
\{b_{r}^i, b_{s}^j\} = \delta_{r+s,0} \delta^{ij} \tag{4.3.2}
\]

As such if we consider the \( b_{r}^i \) with \( r < 0 \) acting on the vacuum \( |0> \), which obeys \( b_{r}^i |0> = 0 \) for \( r > 0 \), then these states will form a representation of \( SO(D-2)^+ \). Clearly they will also carry a representation of the subalgebra \( I(SO(D-2)^+) \). These states do not, however, satisfy the physical state conditions of string theory even if we assigned the vacuum to carry a momentum. Acting on such a state with a generator, say with \( \mathcal{M}_n^{12} \), will change the level of the oscillators which act on the vacuum state, and so the level of the state, without changing the momentum of the state. However, the momentum of a
physical state conditions. This problem can be rectified if we consider instead of the usual oscillators, the corresponding DDF operators which in the N-S sector are denoted by $B^i_r$ [24]. These obey the commutator of equation (4.3.2) with $b^i_r$ replaced by $B^i_r$ and so we can construct the generator $\mathcal{M}_{n}^{ij} = \sum_r : B^{[i}_{n-r} B^{j]}_r :$ and these will obey the algebra of $SO(D - 2)^+$. Thus the string states

$$B^{i_1}_{-r_1} \ldots B^{i_n}_{-r_n} |p^{(0)}, 0 > , \text{ with } B^i_r |p^{(0)} >= 0, \ r > 0$$

will belong to a representation of $SO(D - 2)^+$. In this equation $(p^{(0)})^2 = \frac{1}{2\alpha'}$ and $B^i_r$ injects a momentum $-rk^i$ where $p^{(0)} \cdot k = \frac{1}{2\alpha'}$ and $k^2 = 0$. These states clearly also carry a representation of $I(SO(D - 2)^+)$ which is part of the little algebra $I_c(SO(D - 2, D - 2)^+)$ for the toy string found in the previous section.

Let us now consider the closed superstring, and particular the NS-NS sector, which has the oscillators $b^I_r \equiv (b^I_r, \bar{b}^I_r)$ where the indices $I, J, \ldots$ each take $D - 2$ values. From these we can construct the generators

$$\mathcal{M}_c^{I J} = \sum_r : b^{[I}_{n-r} b^{J]}_r :$$

which obey the algebra $SO(D - 2, D - 2)^+$. Hence the states formed by $b^I_r$ with $r < 0$ acting on the vacuum $|0 >$, which obeys $b^I_r |0 >= 0$ for $r > 0$, carry a representation of $SO(D - 2, D - 2)^+$. Clearly they also carry a representation of $I_c(SO(D - 2, D - 2)^+)$ which is our little algebra.

However, as for the open string these states do not obey the physical state conditions. As in this case we can use the DDF operators $B^I_r$ instead of the $b^I_r$. They have the same anti-commutation relations and so the objects $\hat{\mathcal{M}}_c^{I J} = \sum_r : B^{[I}_{n-r} B^{J]}_r :$ generate the algebra $SO(D - 2, D - 2)^+$. We can act with the closed string oscillators $B^I_r$, for $r > 0$, on a vacuum $|0, p^{(0)} >$ which carries a suitable momentum $p^{(0)}$ and these states will carry a representation of $SO(D - 2, D - 2)^+$. These states will obey all the physical states conditions with the exception for the GSO projection and the requirement of level matching. That this condition must be imposed by hand is a feature of the DDF construction of the physical states of all closed strings. One approach is to consider the enveloping algebra of $SO(D - 2, D - 2)^+$ and then only consider the action of those elements which preserved level matching and the GSO condition. This would be a symmetry which preserve the physical state conditions.

We can think of the open toy string as a restriction of the closed toy string and in particular by imposing the restriction $\sigma \rightarrow -\sigma$. This requires the two oscillators, say $b^I_r$ and $\bar{b}^I_r$, to transform in the same way. This means that the algebra should be is restricted to $I(SO(D - 2)^+)$, as indeed we found.

We will now present an alternative way of realising the symmetry on the physical states of the open superstring. Let is choose $p^{(0)} = (0, 1, 0, \ldots, 0)\frac{1}{\sqrt{2\alpha'}}$ and $k = (-1, 1, 0, \ldots, 0)\frac{1}{\sqrt{2\alpha'}}$ for the momenta of our vacuum physical state, that is, the tachyon.
Then the generators \((J_{01} - S_{01})\) acts on the momenta \(p^{(0)}\) to give \(p^{(0)} - k\). Instead of the oscillators \(b^i_r\) we consider the oscillators

\[
\hat{b}^i_r = e^{r(J_{01} - S_{01})} b^i_r
\]

(4.3.5)
to act on the vacuum. These will carry a momentum injection that is the one required by the physical state conditions. These oscillators also obey the same anti-commutators as \(b^i_r\) and the bilinear of equation (4.3.1) with \(b^i_r\) replaced by \(\hat{b}^i_r\) also obey the algebra of \(SO(D-2)^+\). As such these states do indeed carry a representation of \(I(SO(D-2)^+)\), that is, the part of the little algebra appropriate for the open superstring. In the above we have chosen the string charge of the vacuum state to be given by \(q^{(0)} = (1,0,0,\ldots,0)\sqrt{2\alpha'}\). Then under the above boost it changes by \(q^{(0)} \rightarrow q^{(0)} + k\) and so preserves the condition of equation (4.2.1).

Thus, neglecting level matching and the GSO projection, we have seen that the fermionic oscillators NS-NS states of the closed superstring do indeed form a representation of the little algebra of the toy string and the NS states of the open string the corresponding subalgebra. As we will see in section 6, the little algebra for the actual superstring has some very significant differences but the discussions in this section will prove useful.

We will close this section with some remarks concerning the oscillators \(\alpha^i_n, \bar{\alpha}^i_n, i = 2,\ldots,D-1\) of the closed string which obey the usual commutation relations

\[
[\alpha^i_n, \alpha^j_m] = n\delta^{ij}\delta_{n+m,0}, \ [\bar{\alpha}^i_n, \bar{\alpha}^j_m] = n\delta^{ij}\delta_{n+m,0}, \ [\alpha^i_n, \bar{\alpha}^j_m] = 0
\]

(4.3.6)

We could take them to have the same commutators with the generators of \(SO(D-2, D-2)^+\) as do \(\varphi^i_n\) and \(\bar{\varphi}^i_n\), that is, take them to belong to the same representation. Making such a replacement in equations (4.1.18) and (4.1.19) we find that

\[
[M_{\alpha n i_1 i_2}, \alpha_{m j}] = -2\eta_{j[i_1} \alpha_{n m i_2]} , \ [\bar{M}_{\alpha n i_1 i_2}, \alpha_{m j}] = 0,
\]

\[
[\bar{M}_{\alpha n i_1 i_2}, \bar{\alpha}_{m j}] = -2\eta_{j[i_1} \bar{\alpha}_{n m i_2]} , \ [M_{\alpha n i_1 i_2}, \bar{\alpha}_{m j}] = 0,
\]

(4.3.7)

and

\[
[V_{\alpha n i_1 i_2}, \alpha_{m j}] = -2\eta_{j[i_1} \alpha_{n m i_2} , \ [V_{\alpha n i_1 i_2}, \bar{\alpha}_{m j}] = -2\eta_{j[i_1} \alpha_{n m i_1} , \]

\[
[\bar{V}_{\alpha n i_1 i_2}, \bar{\alpha}_{m j}] = -2\eta_{j[i_1} \bar{\alpha}_{n m i_2} , \ [\bar{V}_{\alpha n i_1 i_2}, \alpha_{m j}] = -2\eta_{j[i_1} \bar{\alpha}_{n m i_1}
\]

(4.3.8)

At first sight we can apply these transformations to the physical states involving the bosonic oscillators and then conclude that these states which have the above bosonic oscillators carry a representation of the string little algebra \(SO(D-2, D-2)^+\). However, the transformations of equation (4.3.7) and (4.3.8) do not respect the commutators of equation (4.3.6) and so it is inconsistent to apply the above transformations of the oscillators. Nonetheless this discussion will be useful in the case for the actual superstring in section six which has a significantly different little algebra.
5. The string little algebra at all levels

In this section we will find the generators in the string little algebra and calculate their algebra.

5.1 The generators in the string little algebra

In section three we found all of the generators of the IIA string little algebra $H$ at level zero and level one, and three of the generators at level two. These are listed in equations (3.11-3.14). While these generators appear to be rather different they can, with two exceptions mentioned below, all be written in the form

$$L^\alpha = e^{-R^\alpha e^R} + I_c(e^{-R^\alpha e^R}) , \quad R = \frac{1}{2} \varepsilon_{ab} R^{ab} .$$

(5.1.1)

Here $\alpha$ is a positive root of $E_9$ and so the generator $R^\alpha$ has SL(10) indices which only take the values $i, j, ... = 2, ..., 9$. Equation (5.1.1) can be re-written with the action of $I_c$ in the second term carried out to find

$$L^\alpha = e^{-R^\alpha e^R} + e^{+\overline{R} R^\alpha e^{-R}} ,$$

(5.1.2)

where $\overline{R} = -I_c(R) = \frac{1}{2} \varepsilon^{ab} R_{ab}$ and $\overline{R}^\alpha = I_c(R^\alpha) = \pm R^\alpha$. Which sign it is in this last equation can be read off from the action of $I_c$ given in equations (2.7) to (2.9).

The reader can, for example, check that the generators $L^{(1)}_i$ and $L^{(1)}_{i_1 i_2 i_3}$ in equations (3.12) can be written in the form of equation (5.1.1);

$$L^{(1)}_i = e^{-R^i e^R} + I_c(e^{-R^i e^R}) = S_i + \frac{1}{2} \varepsilon^{a_1 a_2} S_{a_1 a_2 i} ,$$

(5.1.3)

$$L^{(1)}_{i_1 i_2 i_3} = e^{-R^{i_1 i_2 i_3} e^R} + I_c(e^{-R^{i_1 i_2 i_3} e^R}) = S_{i_1 i_2 i_3} + \varepsilon^{a_1 a_2} S_{a_1 a_2 i_1 i_2 i_3} .$$

(5.1.4)

The remaining generators in (3.12-3.15) at levels one and two can also be shown to be of the form of equation (5.1.1).

The IIA level zero generators in $H$ which were given in equation (3.15) and so they are well understood but we now comment on how they fit into the framework of equation (5.1.1). The generators $J_{ij}$ and $S_{ij}$ commute with $R$ and $\overline{R}$ and, by taking $R^\alpha$ to be the positive root $E_9$ generators $K^{ij}$, $j > i$ and $R^{ij11}$ respectively, we see that they are of the form of equation (5.1.1). The $J_{ij}$ and $S_{ij}$ generate $SO(8) \otimes SO(8)$. The generators $L^\alpha$ at a given IIA level belong to a representation of this algebra. The generators $J_{ij}$ transform the indices $i, j, ... = 2, ..., 9$ indices on the generators in the expected way while the generators $S_{ij}$ transform the different generators at the same IIA level into each other. Thus the generators of equation (5.1.1) at a given IIA level belong to a representation of $SO(8) \otimes SO(8)$.

The generators $J_{ab}$ and $S_{ab}$ belong to $H$ and have the algebra $SO(2) \otimes SO(2)$. The generator $J_{ab}$ commutes with $R$ and $\overline{R}$ and so they can also be written in the form of equation (5.1.1). It just transforms any $a, b, ...$ indices on the generators but such generators only occur at IIA level zero and their commutators are given in section three. The
generator $S_{ab}$ is contained in $S = \frac{1}{2} \epsilon_{ab} S^{ab} = R - \bar{R}$. Using the equations later in this section it can be shown that $[S, L_\alpha] = -l L_\alpha$ if $R^\alpha$ has level $l$. As such it is consistent to take the generators $J_{ab}$ and $S_{ab}$ to be trivially realised.

The IIA level zero generators $L_{ci}^{(0)}$ given in equation (3.11) are in $H$, are of the form of equation (5.1.1) as they can be written as

$$L_{ci}^{(0)} = J_{ci} + \epsilon_c e S_{ci} = e^{-R}(-\hat{K}_c^i)e^R + I_c(e^{-R}(-\hat{K}_c^i)e^R) \quad (5.1.5)$$

However, the generator $-\hat{K}_c^i$ does not belong to $E_9$ and so it is an exception to the formulation below equation (5.1.1). We will show later in this paper that these generators commute with themselves and also all higher level generators of the string little algebra. As such we can take them to be trivially realised and so they will also play no role in our further discussions on the string little algebra. Hence all the generators in the string little algebra of interest are indeed of the form of equation (5.1.1).

The generator $L_\alpha$ of equation (5.1.1) consists of two parts, namely the first and second terms. If the generator $R^\alpha$ of $E_9$ has IIA level $l \geq 1$ then all the terms in the first part will have level $l$. This follows from the fact that $R$ has level zero and so its commutator with $R^\alpha$ also has level $l$. The second part in equation (5.1.1) will then have level $-l$ by a similar argument. Thus a generator $L^\alpha$, as given in equation (5.1.1), has one term of level $l$ and another term of level $-l$. We will refer to such a generator as being of levels $\pm l$. Since the generator $R$ is inert under commutators with the SL(8) generators of $E_9$ all the contributions that appear in the first part of $L^\alpha$ have the same SL(8) character and similarly for the second piece.

At first sight the expression of equation (5.1.1) involves an infinite number of terms as it involves more and more commutators with $R$, or $\bar{R}$, coming form the exponentials. However, these series always terminate. We previously found the IIA levels $\pm l$ generators in $H$ that we found in equations (4.1.3) and (4.1.4) had only two terms and so these series terminate after only one commutator. While the IIA level $\pm 2$ generators in $H$ that we previously found had three terms. We will prove below in section (5.4) that the IIA levels $\pm l$ generators in $H$ have only $l + 1$ terms.

We now show that generators of the form of equation (5.1.1), with the $R^\alpha$ as specified, belong to the string little algebra $H$. For this to be true they must correspond to transformations of the brane charges which vanish when only the brane charges for the string are non-zero. The non-zero string brane charges are the level one brane charges $P_a$ and $Q^a$ which obey the relation of equation (3.1). To be more precise a generator $L^\alpha$ belongs to $H$ if the commutator $[L_\alpha, Z^\beta]$, where $Z^\beta$ is any any brane charge, vanishes when we take the brane charges to which it equals to be those for the string.

The commutator $[L_\alpha, Z^\beta]$ is of the form

$$[L_\alpha, Z^\beta] = [e^{-R} R^\alpha e^R, Z^\beta] + [e^{\bar{R}} \bar{R}_\alpha e^{-\bar{R}}, Z^\beta] \quad (5.1.6)$$

If $Z^\beta$ is an element in the $l_1$ representation with level $\bar{l} = 1, 2, \ldots$and $L_\alpha$ is a generator with levels $\pm l$ then the first term in equation (5.1.6) has level $\bar{l} + l$. This has a level which greater than one and as such it can not be equal to the string brane charges, which are the
only non-zero charges, Hence it must vanish. We are therefore left with the second term of equation (5.1.5). This has level $\tilde{l} - l$ and it will only be non-zero if $\tilde{l} - l = 0$. in which case it must be of the form

$$[e^{\overline{R}_\alpha}e^{-\overline{R}}, Z^\beta] = \lambda^a P_a + \mu_b Q^b = (\lambda^b - \mu_a \varepsilon^{ab}) P_b$$  \hspace{1cm} (5.1.7)$$

where $\lambda^a$ and $\mu_b$ are numbers and we have used equation (3.1) in the last step.

The brane charge $Z^\beta$ has its $SL(8) \otimes SL(2)$ indices which are up and consist of the $SL(8)$ indices $i, j, \ldots = 2, \ldots 9$ and $SL(2)$ indices $a, b, \ldots = 0, 1$. However the $E_9$ negative level generator $\overline{R}_\alpha$ has only SL(8) indices $i, j, \ldots = 2, \ldots 9$. which are down. The $E_{11}$ commutators respect the up and down positions of the $SL(8) \otimes SL(2)$ indices carried by the generators. Indeed the result of the commutator $[\overline{R}_\alpha, Z^\beta]$ is equal to a delta symbol relating the up and down indices times a brane charge which has its indices up. It follows that this commutator can only result in the brane charge $Q^a$ and not the brane charge $P_a$ as the latter world index is a down index and this is not present on the left-hand side of this commutator. In this case the $\beta$ indices on $Z^\beta$ must consist of $i, j, \ldots = 2, \ldots, 9$ indices, denoted $\beta'$, and one index $a = 0, 1$. The $i, j, \ldots = 2, \ldots, 9$ indices on $Z^\beta$ form a delta symbol with the $i, j, \ldots = 2, \ldots, 9$ indices on $\overline{R}_\alpha$, denoted $\delta^{\beta'}_\alpha$, and the $a$ index is inherited onto $Q^a$. Using these arguments we find that the commutator between $\overline{R}_\alpha$ and $Z^\beta$ takes the form

$$[\overline{R}_\alpha, Z^\beta] = f \delta^{\beta'}_\alpha Q^a$$  \hspace{1cm} (5.1.8)$$

where $f$ is a number.

The generator $R = -R^{0111}$ where $R^{0111}$ is the generator in the eleven dimensional theory, while $\overline{R} = -R_{0111}$. It follows, using the previous type of arguments, that $[\overline{R}, Z^\beta] = 0$ for the $Z^\beta$ discussed above which gave a non-zero commutator with $\overline{R}_\alpha$ when only the string brane charges were non-zero. Using this result we can re-write (5.1.7) as

$$e^{\overline{R}[\overline{R}_\alpha, Z^\beta]}e^{-\overline{R}} = e^{\overline{R}} f \delta^{\beta'}_\alpha Q^a e^{-\overline{R}} = f \delta^{\beta'}_\alpha (Q^a + \varepsilon^{ab} P_b) = 0$$  \hspace{1cm} (5.1.9)$$

where we used the commutators in equation (A.7) in the appendix and, in the last step, we used the relation between the string charges $P_a$ and $Q^a$ given in equation (3.1).

Thus we find that a generator $L_\alpha$ of equation (5.1.1) obeys the relation

$$[L_\alpha, Z^\beta] = 0.$$  \hspace{1cm} (5.1.10)$$

for all brane charges $Z^\beta$ when only the string charges are non-zero after the commutator has been evaluated. It follows that $L_\alpha$ corresponds to transformations of the brane charges that preserve the string charges and and so $L_\alpha$ belongs to the string little algebra $\mathcal{H}$. We observe that this is an all orders result in the context of $E_{11}$. This does not show that all generators in $\mathcal{H}$ are of the form of (5.1.1). However, we believe this is the case as it is true at IIA levels one and two.

5.2 The commutators of the string little algebra

In this section we will derive the commutators of generators of the string little algebra, that is, generators of the form of equation (5.1.1), or equivalently equation (5.1.2), for the
$R^\alpha$ with $\alpha$ a positive root of $E_9$. Using equation (5.1.2) the commutator of $L_\alpha$ with $L_\beta$ can be written as

$$[L_\alpha, L_\beta] = e^{-R}[R^\alpha, R^\beta]e^R + e^{R}[R_\alpha, R_\beta]e^{-R}$$

$$+ [e^{-R}R^\alpha e^R, e^{R}R_\beta e^{-R}] + [e^{-R}R_\alpha e^{-R}, e^{-R}R^\beta e^R].$$  (5.2.1)

If $R^\alpha$ and $R^\beta$ have IIA levels $l$ and $l'$ then the first, second, third and fourth terms in equation (5.2.1) have IIA levels $l + l'$, $-l - l'$, $l - l'$ and $-l + l'$ respectively. We will show below that the sum of the third and fourth terms of levels $l - l'$ and $-l + l'$ respectively vanish. We will now refer to these terms as **cross terms**. This will be proved in the next section but we will assume it to be the case for the rest of this section. As such the commutator of equation (5.2.1) is equal to

$$[L_\alpha, L_\beta] = e^{-R}[R^\alpha, R^\beta]e^R + e^{R}[R_\alpha, R_\beta]e^{-R} = e^{-R}[R^\alpha, R^\beta]e^R + I_c(e^{-R}[R^\alpha, R^\beta]e^R)$$  (5.2.2)

The commutator of two positive level generators $R^\alpha$ and $R^\beta$ of $E_9$ leads to another such positive level generator and so we may write it as

$$[R^\alpha, R^\beta] = f^{\alpha\beta\gamma}R^\gamma$$  (5.2.3)

It follows, using the properties of the $I_c$ involution, that the commutator of $L^\alpha$ and $L^\beta$ is given by

$$[L^\alpha, L^\beta] = f^{\alpha\beta\gamma}L^\gamma$$  (5.2.4)

Thus the commutator of two elements of the string algebra is of the form given in equation (5.1.1) and so it also belongs to $I_c(E_{11})$ and to the string little algebra $H$. This shows that the string little algebra of equation (5.1.1) does indeed form a closed algebra.

Since the generators are of the form of equation (5.1.1), that is, they are convoluted with $R$, or $R_i$, it is easy to compute their commutators. For example, for the generators of equations (5.1.3) and (5.1.4) we readily find that

$$[L_i^{(1)}, L_j^{(1)}] = e^{-R}[R^i, R^j]e^R + I_c(e^{-R}[R^i, R^j]e^R) = 0,$$  (5.2.5)

$$[L_i^{(1)}, L_{j_{ijj_3}}^{(1)}] = e^{-R}[R^i, R^{j_{1j2j_3}}]e^R + I_c(e^{-R}[R^i, R^{j_{1j2j_3}}]e^R) = 0$$  (5.2.6)

and

$$[L_{i_1i_2i_3}^{(1)}, L_{j_{1j2j_3}}^{(1)}] = e^{-R}[R^{i_1i_2i_3}, R^{j_{1j2j_3}}]e^R + I_c(e^{-R}[R^{i_1i_2i_3}, R^{j_{1j2j_3}}]e^R)$$

$$= 2e^{-R}R^{i_1i_2i_3j_{1j2j_3}}e^R + 2I_c(e^{-R}R^{i_1i_2i_3j_{1j2j_3}}e^R) = 2L^{(2)}_{i_1i_2i_3j_{1j2j_3}}$$  (5.2.7)

A more complicated example is given by

$$[L^{(1)}_{i_{1i_2i_3}}, L^{(1)}_{j_{1...j_5}}] = e^{-R}[R^{i_1i_2i_3}, R^{j_{1...j_5}}]e^R + I_c(e^{-R}[R^{i_1i_2i_3}, R^{j_{1...j_5}}]e^R)$$

$$= e^{-R}R^{i_1i_2i_3j_{1...j_5}}e^R + I_c(e^{-R}R^{i_1i_2i_3j_{1...j_5}}e^R)$$

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5.3 The vanishing of the cross terms

In this section we will show that the cross terms, that is, the level \( l - l' \) and level \(-l + l'\) terms in the commutator of equation (5.2.1) vanish. In section three we computed the algebra of the level \( \pm 1 \) generators of the string little algebra and we did indeed find that the cross terms do vanish, see equations (3.16). We recall that in these sections we are no longer considering the level zero generators in the string little algebra unless stated to the contrary. To show the cross terms vanish in general we will need a streamlined notation. We use the notation

\[ 1 \wedge B = B, \quad A \wedge B = [A, B], \quad A^{(2)} \wedge B = [A, [A, B]], \quad \text{etc...} \quad (5.3.1) \]

so that

\[ e^{-R} R^\alpha e^R = e^{-R} \wedge R^\alpha = \sum_p R^p \wedge R^\alpha \quad (5.3.2) \]

In fact this sum terminates when \( p = l \) as we will show in section 5.4 that

\[ R^{l+1} \wedge R^\alpha = 0, \quad \overline{R}^{l'+1} \wedge \overline{R}_\beta = 0. \quad (5.3.3) \]

if \( R^\alpha \) and \( \overline{R}_\beta \) have levels \( l \) and \( l' \) respectively. The second equation is just the Cartan involution of the first equation. As such the sum in equation (5.3.2) terminates when \( p = l \).

Using the above notation and the above result the first cross term in equation (5.2.1) can be written as

\[ [e^{-R} R^\alpha e^R, e^{-\overline{R}} \overline{R}_\beta e^{-\overline{R}}] = \sum_{p=0}^{\ell} \sum_{q=0}^{\ell'} \frac{(-1)^p}{p!} \frac{1}{q!} [X^{(p)}, Y^{(q)}] \quad (5.3.4) \]

where

\[ X^{(p)} \equiv R^p \wedge R^\alpha, \quad Y^{(p)} \equiv \overline{R}^p \wedge R^\beta. \quad (5.3.5) \]

To evaluate the expression of equation (5.3.4) we will need some identities. We define

\[ \tilde{D} = K^0_0 + K^1_1 = \hat{K}^0_0 + \hat{K}^1_1 + \frac{1}{3} \hat{R} = \frac{2}{3} (\hat{K}^0_0 + \hat{K}^1_1 + \hat{K}^{11}_{11}) - \frac{1}{3} \sum_{a=2}^{9} \hat{K}^a_a \quad (5.3.6) \]

where we have used the correspondence with the generators in the eleven dimensional theory given in equations (2.2). We recall that a hat refers to generators as they appear in the eleven dimensional theory. This generator obeys the following commutators

\[ [R, \overline{R}] = -\tilde{D}, \quad [\tilde{D}, R] = +2R, \quad [\tilde{D}, \overline{R}] = -2\overline{R}. \quad (5.3.7) \]
It can be shown that

\[ [\tilde{R}, R^\alpha] = -3l R^\alpha \ , \quad [\tilde{R}, \overline{R}_\beta] = +3l' \overline{R}_\beta \ . \quad (5.3.8) \]

where \( R^\alpha \) is an \( E_{11} \) generator with IIA positive level \( l \geq 0 \) and \( \overline{R}_\beta \) is a an \( E_{11} \) generator with negative level \( -l' \leq 0 \). It follows from equation (5.3.6) that

\[ [\tilde{D}, R^\alpha] = -l R^\alpha \ , \quad [\tilde{D}, \overline{R}_\beta] = +l' \overline{R}_\beta \ . \quad (5.3.9) \]

as there generators have no 0 or 1 indices as they belong to \( E_9 \). Using the Jacobi identity and equation (5.3.9) we find that

\[ [\tilde{D}, Y^{(p)}] = [\tilde{R}, [\tilde{D}, Y^{(p-1)}]] - 2[\tilde{R}, Y^{(p-1)}] = (l' - 2p)Y^{(p)} \quad (5.3.10) \]

Using the Cartan involution, in particular \( I_c(\tilde{D}) = -\tilde{D} \) and \( I_c(Y^{(p)}) = (-1)^p X^{(p)} \), on this last equation we find that

\[ [\tilde{D}, X^{(p)}] = (l - 2p)X^{(p)} \quad (5.3.11) \]

One can show that

\[ [R, Y^{(p)}] = -p(l' - (p - 1))Y^{(p-1)} \quad \text{and} \quad [\overline{R}, X^{(p)}] = -p(l - (p - 1))X^{(p-1)} \quad (5.3.12) \]

To derive the first equation we commute \( R \) through the \( \overline{R} \)’s in \( Y^{(p)} \) using equation (5.3.11) and then use equation \([R, \overline{R}_\beta] = 0\). Using similar arguments one can show that

\[ R^{(q)} \wedge Y^{(p)} = (-1)^q \frac{p!}{(p-q)!} \frac{(l' - (p - q))!}{(l' - p)!} Y^{(p-q)} \quad (5.3.13) \]

and

\[ \overline{R}^{(q)} \wedge X^{(p)} = (-1)^q \frac{p!}{(p-q)!} \frac{(l - (p - q))!}{(l - p)!} X^{(p-q)} \quad (5.3.14) \]

Finally, using the Jacobi identities and equation (5.3.12) one can show two recursion relations for \([X^{(p)}, Y^{(q)}] \):

\[ [X^{(p)}, Y^{(q)}] = [R, [X^{(p-1)}, Y^{(q)}]] + q(l' - (q - 1))[X^{(p-1)}, Y^{(q-1)}] \quad (5.3.15) \]

and

\[ [X^{(p)}, Y^{(q)}] = [\overline{R}, [X^{(p)}, Y^{(q-1)}]] + p(l - (p - 1))[X^{(p-1)}, Y^{(q-1)}] \quad (5.3.16) \]

Finally we can evaluate the object of interest, the commutator of equation (5.3.4). We will do this for a level one generator with a level \( l' \) generator. Using equation (5.3.3) this commutator becomes

\[ [e^{-R} R^\alpha e^{R}, e^{\overline{R}} \overline{R}_\beta e^{-\overline{R}}] = \sum_{p=0}^{l'} \sum_{q=0}^{l} \frac{(-1)^p}{p!} \frac{1}{q!} [X^{(p)}, Y^{(q)}] \quad (5.3.17) \]
Using the recursion relation (5.3.15) we find that the terms involving $X^{(1)}$ can be evaluated as
\[
- \frac{1}{(q+1)!} [X^{(1)}, Y^{(q+1)}] = - \frac{1}{(q+1)!} \{ [R, [R^\alpha, Y^{(q+1)}]] + (q+1)(l'-q)[R^\alpha, Y^{(q)}] \}
\]
\[
= - \frac{1}{(q+1)!} \{ R \wedge R^{(q+1)} \wedge [R^\alpha, R_\beta] + (q+1)(l'-q)[R^\alpha, Y^{(q)}] \}
\]
\[
= - \frac{1}{(q+1)!} \{ -(q+1)(l'-1-q') + (q+1)(l'-q)][R^\alpha, Y^{(q)}] \} = - \frac{1}{q!} [R^\alpha, Y^{(q)}]. \tag{5.3.18}
\]

This term cancels the $\frac{1}{(q)!} [X^{(0)}, Y^{(q)}]$ term in equation (5.3.17). Carrying out all such cancelations we find that only the $p = 0$, $q = l'$ term remains, and so
\[
[e^{-R^\alpha e^R}, e^{R_\beta}e^{-R}] = \frac{1}{l'!} [R^\alpha, Y^{(l')} = \frac{1}{l'!} [R^\alpha, R_\beta'] = \frac{1}{l'!} R_\beta' \wedge [R^\alpha, R_\beta] = 0. \tag{5.3.19}
\]

In the above we have used the Jacobi identity, as well as the relation $[R^\alpha, R] = 0$. The second to last step in the above equation contains the generators $R^\alpha$ of level 1 and $R_\beta$ of level $-l'$ and so the commutator $[R^\alpha, R_\beta]$ is at level $-(l'-1)$. Using equation (5.3.3) we find that the final expression vanishes. The same result holds for the other cross term in equation (5.2.2) as it is the Cartan involution of the term we just showed vanished.

Thus we have shown that the cross terms vanish for $[L_\alpha, L_\beta]$ when $L_\alpha$ is at levels $\pm 1$ and $L_\beta$ is at levels $\pm l'$. However, from the commutators of these two elements of the string little algebra we can find all elements of this algebra and as a result show that their commutators have no cross terms. The commutator of any two generators $L_\alpha$ and $L_\beta$ can be written as
\[
[L_\alpha, L_\beta] = [L_\gamma, L_\alpha'], [L_\beta] = [L_\gamma, [L_\alpha', L_\beta]] + [[L_\gamma, L_\beta], L_\alpha'] \tag{5.3.20}
\]

where $L_\alpha$ has been given by $L_\alpha = [L_\gamma, L_\alpha']$ where $L_\gamma$ has level $\pm 1$. The generator $L_\alpha'$ has a level which is one less than $L_\alpha$. Proceeding in this way we can write the commutator $[L_\alpha, L_\beta]$ in terms of commutators of multiple level one generators and one other element of the string little algebra. From the result just above it follows that the commutator has no cross terms.

The reader may like to show directly that the commutator of two generators in the string little algebra that have any levels have no cross terms. A useful identity is given by
\[
[X^{(p)}, Y^{(q)}] = \begin{cases} 
\frac{1}{l'!} [R^\alpha, R_\beta], & \text{if } l' > 1 \\
\frac{1}{l'!} [\hat{K}^{ij}, [R^\alpha, R_\beta]], & \text{if } l > l' \end{cases} \tag{5.3.21}
\]

While evaluating the cross terms of the commutator $[X^{(p)}, Y^{(q)}]$ for general $p$ and $q$ directly, rather than using the above use of level $\pm 1$ generators, one has to be careful when the commutator $[R^\alpha, R_\beta]$ not only has IIA level zero but also $E_{11}$ level zero, that is, when it equals a combination of the generators $\hat{K}^{ij}$ and $\hat{K}^{11}$. The problem is that these
latter generators do not commute with either $R$ or $\bar{R}$. However keeping such terms in the calculation one finds as already shown that the cross terms vanish.

There is one important exception to the fact that the cross terms in the commutators of the generators of the string little algebra vanish, namely for those generators that are at level zero. These are the generators

\[ J_{ij} = \eta_{ik} K^k_j - \eta_{jk} K^k_i, \quad S^{ij} = \hat{R}^{ij11} - \eta^{il} \eta^{jm} \hat{R}_{kl11}, \quad L^{(0)}_{ai} = -e^{-R} \hat{K}_a^i e^R \eta_{ki} + e^{\bar{R}} \hat{K}_d^i e^{-\bar{R}} \eta_{id} \]

(5.3.22)

The cross terms in the commutators of these generators are non-zero. The generators $J_{ij}$ and $S^{ij}$ obey the algebra of $SO(8) \otimes SO(8)$ while the commutators of the generators $L^{(0)}_{ai}$ with $J_{ij}$ and $S^{ij}$ give again $L^{(0)}_{ai}$, see equation (3.16).

As we will now show the level zero generator $L^{(0)}_{ai}$ commutes with all the level greater than zero generators in the string algebra. The general commutator is of the form

\[ [L^{(0)}_{ai}, L_\beta] = [X_{ia}, e^{-R} R^\beta e^R] + [X_{ia}, e^R \bar{R}_\beta e^{-\bar{R}}] + I_c([X_{ia}, e^{-R} \eta R^\beta e^R] + [X_{ia}, e^R \bar{R}_\beta e^{-\bar{R}}]) \]

(5.3.23)

where $X_{ia} \equiv -e^{-R} \hat{K}_a^i e^R \eta_{ki}$. The first term has the simple form

\[ -e^{-R} [\hat{K}_a^i, R^\beta] e^R = 0 \]

(5.3.24)

It is zero as $R^\beta$ belongs to $E_9$, with $\beta$ being a positive root, which means that it possesses no $a, b$ indices and as a result it commutes with $\hat{K}_a^i$. The second term can be treated much like the level one and level $l'$ generators in equation (5.3.17) and evaluated as in equation (5.3.18) using the equation $[\hat{D}, \hat{K}_a^i] = -\hat{K}_a^i$. As a result we find very similar cancellations leaving the expression $\frac{1}{16} \bar{R}^{l'} \wedge [\hat{K}_a^i, \bar{R}_\beta] \eta_{lk} = 0$. One can show that this vanishes at low levels. We believe it vanishes at all levels and we take this to be the case. As such the generators $L^{(0)}_{ai}$ commute with all the generators in the string little group except for $J_{ij}$ and $S^{ij}$ under which it is a representation. As such it is consistent to take the generators $L^{(0)}_{ai}$ to be trivially realised.

Thus the non-trivial part of the string little algebra $\mathcal{H}$ contains at level zero $SO(8) \otimes SO(8)$ and at other levels the generators are in one to one correspondence with the positive root generators of $E_9$. They also obey the same algebra as these generators. Put another way the generators have the same algebra as the Borel subalgebra of $E_9$ except for those at IIA level zero which have the algebra of $SO(8) \otimes SO(8)$. This little algebra is quite different to the little algebra found for the massless point particle which was found to be $I_c(E_9)$ [14]. Although the generators $L_\alpha$ given in equation (5.1.1) contain the expression $R^\alpha + I_c(R^\alpha)$, which is in $I_c(E_9)$, they also contains commutators with $R$ and $\bar{R}$ and as a result $L_\alpha$ belong to $I_c(E_{11})$ rather than $I_c(E_9)$. Indeed it is this latter point that changes the commutators from those of $I_c(E_9)$ to those of the Borel subalgebra of $E_9$. Thus although the commutators are the same as those of the Borel subalgebra of $E_9$ the generators are not the Borel generators of the $E_9$ that is evident from the Dynkin diagram of $E_{11}$.

5.4 Proof of $R^{i+1} \wedge R^\beta = 0$
In this subsection we will prove the relation

\[ R^{l+1} \wedge R^\beta = 0 \]  \hspace{1cm} (5.4.1)

In this equation \( \beta \) is a positive root of \( E_9 \) with IIA level \( l \) and \( R = \frac{1}{2} \epsilon_{ab} \hat{R}^{ab} \). This equation was stated in equation (5.3.3) of subsection 4.3 and used to prove some of the crucial results in that section. Here we will show that the root associated to the generator of equation (5.4.1) is not present in the underlying \( E_{11} \) algebra and so the commutators in equation (5.4.1) must vanish.

The generator \( R = -\hat{R}_{12}^{11} \) can be found by taking suitable commutators of \( \hat{K}_{ab} \) with the generators \( R_{910}^{11} \) and in particular

\[ R_{12}^{11} = \hat{K}^2 3 \wedge \hat{K}^4 4 \wedge \ldots \hat{K}^9 10 \wedge \hat{K}^1 2 \wedge \hat{K}^2 3 \wedge \ldots \wedge \hat{K}^8 9 \wedge R_{910}^{11} \]  \hspace{1cm} (5.4.2)

In agreement with much of the previous literature we will be somewhat contrary-wise and take the indices to run from 1, 2, \ldots, 11 rather than 0, 1, \ldots, 10. The root \( \alpha_{11} \) is associated to \( \hat{R}^{91011} \), while the roots \( \alpha_1 \) and \( \alpha_2 \) to \( \hat{K}_1^2 \) and \( \hat{K}_2^3 \) respectively etc. It follows that the root associated to \( R_{12}^{11} \) is

\[ \delta = (\alpha_1 + 2\alpha_2 + 2\alpha_3 + \ldots + 2\alpha_8 + \alpha_9) + \alpha_{11} \]  \hspace{1cm} (5.4.3)

It is straightforward to show that \( \delta^2 = 2 \) which must be the case as it belongs to the subalgebra \( D_{10} \) of \( E_{11} \). We note that \( \delta \) possess no root \( \alpha_{10} \) and has level 2 with respect to node 2.

We will now consider the \( E_{11} \) algebra when decomposed into \( A_1 \otimes D_8 \) as indicated by the deleted nodes in the \( E_{11} \) Dynkin diagram given by

\[
\begin{array}{ccccccccccc}
\bullet & 11 & \otimes & 10 \\
\bullet & - & \oplus & - & \bullet & - & \bullet & - & \bullet & - & \bullet & - & 8 & 9 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\end{array}
\]

The \( E_{11} \) root \( \alpha \) associated to the generator \( R^s \wedge R^\beta \) can be written as

\[ \alpha = \epsilon + 2s\alpha_2 + \gamma + l\alpha_{10} \]  \hspace{1cm} (5.4.4)

Here \( \epsilon \) is a root of the Lie algebra \( A_1 \) which is associated with node 1, while \( \gamma \) is a root of \( D_8 \) which is associated with nodes 3,4,5,6,7,8,9 and 11. The roots \( \alpha_2 \) and \( \alpha_{10} \) are the simple roots of \( E_{11} \) and their coefficients \( 2s \) and \( l \) correspond to the fact that the root has levels \( 2s \) and \( l \) corresponding to nodes 2 and 10 respectively. We recall that node ten is the node that we delete to find the ten dimensional IIA theory and so level \( l \) relates to the fact that \( R^\beta \) has IIA level \( l \).

By construction \( R^\beta \) belongs to \( E_9 \) which is associated with nodes 3-10 and node 11 and it has, by assumption, IIA level \( l \). However, it does not contain the root \( \alpha_2 \) and so it has level zero for this node. In contrast we see from equation (5.4.3) that \( R \) has level two with respect to node 2 but it has IIA level zero, Consequently \( R^s \) has level \( 2s \) and 0.
with respect to nodes 2 and 10. As such the root of equation (5.4.4), which corresponds to $R^s \wedge R^\beta$, has levels $2s$ and $l$ with respect to nodes 2 and 10.

Following, for example, the review of chapter 16 of [19] we can analyse the $A_1 \otimes D_8$ representations that occur in the roots of the form of equation (5.4.4). The first step is to write the roots $\alpha_2$ and $\alpha_{10}$ in the form

$$\alpha_2 = y - \nu_1 - \mu \quad \alpha_{10} = x - \nu_7$$

(5.4.5)

where $\mu$ is the fundamental weight of $A_1$ and $\nu_1$ and $\nu_7$ are the fundamental weights of $D_8$ corresponding to nodes 3 and 9 in the above Dynkin diagram respectively. The quantities $x$ and $y$ are orthogonal to all the roots, and so the weights, of $A_1$ and $D_8$. These roots must have length squared two and so

$$\alpha_2^2 = y^2 + \nu_1^2 + \mu^2 = y^2 + 1 + \frac{1}{2} = 2 \quad \text{and so} \quad y^2 = \frac{1}{2}$$

(5.4.6)

and

$$\alpha_{10}^2 = x^2 + \nu_7^2 = x^2 + 2 = 2 \quad \text{and so} \quad x^2 = 0$$

(5.4.7)

In evaluating these equations we used the scalar products of the fundamental weights which can be found, for example, in appendix D of [19]. It is straightforward to check that they have the correct scalar products with the other $E_{11}$ roots with one exception, namely $\alpha_2 \cdot \alpha_9$ which requires more care. We have that

$$\alpha_2 \cdot \alpha_9 = \nu_1 \cdot \nu_7 + x \cdot y = \frac{1}{2} + x \cdot y = 0 \quad \text{and so} \quad x \cdot y = -\frac{1}{2}$$

(5.4.8)

This novel feature arises as we have deleted two nodes.

Substituting these expressions for the roots $\alpha_2$ and $\alpha_{10}$ in the root $\alpha$ of equation (5.4.4) is given by

$$\alpha = \epsilon - 2s\mu + 2s\nu_7 + \gamma - 2s\nu_1 - l\nu_7 + lx$$

(5.4.9)

We can analyse the $A_1 \otimes D_8$ representations that occur in the roots of the form of equation (5.4.4). The part of the root in equation (5.4.4) in the space of the $A_1$ algebra is $\epsilon - s\mu$ and this must, for suitable $\epsilon$, be a highest weight of $A_1$. We therefore find the equation

$$q\mu = 2s\mu - \epsilon = 2s\mu - n\hat{\alpha}_1, \quad \text{and so} \quad \frac{q}{2} = s - n, \quad \text{for} \quad n \geq 0$$

(5.4.10)

where $q$ is the Dynkin index of the highest representation and $\hat{\alpha}_1$ is the simple root of $A_1$. Proceeding in a similar way for the $D_8$ part of the algebra we find that

$$\sum_{j \in \{3, \ldots, 9, 11\}} p_j \lambda_j = l\nu_7 + 2s\nu_1 - \gamma = l\nu_7 + 2s\nu_1 - \sum_{j \in \{3, \ldots, 9, 11\}} m_j \hat{\alpha}_j, \quad \text{for} \quad m_j \geq 0$$

(5.4.11)
We have written \( \gamma = \sum_{j \in \{3, \ldots, 9, 11\}} m_j \hat{\alpha}_j \) where \( \hat{\alpha}_j \), \( j = 3, \ldots, 9, 11 \) are the simple roots of \( D_8 \) and \( \lambda_j \) its fundamental weights. Taking the dot product of equation (5.4.11) with \( \lambda_i \) and using the definition of the inverse \( D_8 \) Cartan matrix \( (A_{ij}^{D_8})^{-1} = \lambda_i \cdot \lambda_j \), we find

\[
\sum_{j \in \{3, \ldots, 9, 11\}} p_j (A_{ji}^{D_8})^{-1} = l \nu_7 \cdot \lambda_i + 2 s \nu_1 \cdot \lambda_i - m_i \quad (5.4.12)
\]

By analysing equations (4.4.11) and (4.4.12) one can find at low levels the possible representations of \( A_1 \otimes D_8 \) that can occur.

In any Kac-Moody algebra the lengths of the roots are bounded by 2. Using equations (4.4.10) and (4.4.11) we find that

\[
\alpha^2 = \frac{q^2}{2} + \sum_{i,j} p_i (A_{ij}^{D_8})^{-1} p_j + 2 s^2 - 2 s l \leq 2. \quad (5.4.13)
\]

This will only be less than or equal to 2 if \( s \leq l \) holds. This indeed proves the identity of equation (5.4.1) since the root corresponding to \( R_{l+1} \wedge R^\beta \) does not exist in the \( E_{11} \) algebra.

6 String states and an irreducible representation of the string little algebra

We begin this section by clarifying how the string little algebra sits in \( E_{11} \) in relation to the well known subalgebras \( E_9, E_8 \) and its Cartan involution subalgebra \( \text{SO}(16) \).

6.1 Relationship of the string little algebra to \( E_9 \)

In section five we found the subalgebra \( \mathcal{H} \) of \( I_c(E_{11}) \otimes_s l_1 \) that preserves the string charges. We referred to this as the string little algebra. It contained the generators \( J_{ab} \) and \( S_{ab}, a, b = 0, 1 \), which generate \( \text{SO}(2) \otimes \text{SO}(2) \), and the generator \( L_{ai}^{(1)} \). It also contained the generators

\[
\mathcal{H}_r = \{ J_{ij}, S_{ij}; L_\alpha ; \alpha \in \text{positive roots of } E_9 \}. \quad (6.1.1)
\]

where \( L_\alpha \) is defined in equation (5.1.1). As explained in section 5.2 the generators in equation (6.1.1) obey an algebra that has the algebra \( \text{SO}(8) \otimes \text{SO}(8) \) at IIA level zero but at higher levels it has commutators that have the same algebra as the Borel subalgebra of \( E_9 \).

In this paper we wish to construct the string analogue of the irreducible particle representations first constructed by Wigner [13]. To do this we must choose a representation of the string little algebra and then carry out the corresponding boost. As discussed in the previous section we may choose the generators \( J_{ab}, S_{ab} \) and \( L_{ai}^{(1)} \) of \( \mathcal{H} \) to be trivially realised. As such we just have to choose a representation of the algebra obeyed by the algebra \( \mathcal{H}_r \) of equation (6.1.1).

To better understand the algebra \( \mathcal{H}_r \) it will be useful to understand in detail its relation to \( E_9 \). The Dynkin diagram of \( E_9 \) is found by deleting the first two nodes, labelled one and two and indicated by a \( \oplus \), from the Dynkin diagram of \( E_{11} \):

```
+11+10
1  2  3  4  5  6  7  8  9
```

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Since we are considering the ten dimensional IIA theory we delete node ten, called the IIA node, in the above Dynkin diagram. We will decompose the $E_9$ algebra with respect to the remaining $D_8 = \text{SO}(8, 8)$ subalgebra as shown by the diagram

\[
\begin{array}{cccccccccc}
\bullet & 11 & \oplus & 10 & | & | \\
\oplus & - & \oplus & - & \bullet & - & \bullet & - & \bullet & - & - & - & - & - & - & - & - \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9
\end{array}
\]

The low level generators in such a decomposition can be found by using Simple [16]. Deleting node 10 we find that $E_9$ decomposes into the 120 dimensional adjoint representation of $D_8$ at level zero and the 128 dimensional spinor representation of $D_8$ at level one. This pattern repeats itself, at all even levels and odd levels we find the adjoint and spinor representations of $D_8$ respectively. In order to recognise this result in terms of generators we are familiar with we will not only delete the IIA node ten but also node eleven to result in a decomposition to the algebra $\text{SL}(8)$ which corresponds in the Dynkin diagram consisting of nodes 3 to 9 inclusive. At IIA level zero, the 120 generators which belong to the adjoint of $D_8 = \text{SO}(8, 8)$ is given by

\[
K^{(0)}_{ij} = K^{i}_{j} \, , \, R^{ij}_{(0)} = R^{ij} \, , \, R_{(0)ij} = R_{ij} \, , \, i, j, \ldots = 3, 4, \ldots, 10 \quad (6.1.2)
\]

Here the $K^{i}_{j}$ generate $\text{GL}(8)$ and together with $R^{ij}$ and $R_{ij}$ they generate $\text{SO}(8,8)$. The subscripts here, and in the equations that follow, denote the IIA level of the generators. At level zero only one finds an additional generator with no $\text{SL}(8)$ indices which we have omitted to write. It is the analogue of the dilaton generator in the IIA theory.

At IIA level one we find the generators

\[
R^{i}_{(1)} = R^{i} \, , \, R^{i_1 i_2 i_3}_{(1)} = R^{i_1 i_2 i_3} \, , \, R^{i_1 \ldots i_5}_{(1)} = R^{i_1 \ldots i_5} \, , \, R^{i_1 \ldots i_7}_{(1)} = R^{i_1 \ldots i_7} \, , (6.1.3)
\]

These 128 generators belong to the Majorana-Weyl spinor representation of $D_8$. Similarly at level minus we find another 128 dimensional spinor representation that is given by

\[
R_{(-1)i} = R_{i} \, , \, R_{(-1)i_1 i_2 i_3} = R_{i_1 i_2 i_3} \, , \, R_{(-1)i_1 \ldots i_5} = R_{i_1 \ldots i_5} \, , \, R_{(-1)i_1 \ldots i_7} = R_{i_1 \ldots i_7} \, , (6.1.4)
\]

At level two we find the generators

\[
K^{(2)}_{ij} = \frac{1}{7!} \varepsilon_{ik_1 \ldots k_7} R^{k_1 \ldots k_7, j} \, , \, \tilde{R}^{(2)} = \frac{1}{8!} \varepsilon_{k_1 \ldots k_8} R^{k_1 \ldots k_8} \, , \\
R^{ij}_{(2)} = \frac{1}{6!} \varepsilon^{ij}_{k_1 \ldots k_6} R^{k_1 \ldots k_6} \, , \, R_{(2)ij} = \frac{1}{8!} \varepsilon_{k_1 \ldots k_8} R^{k_1 \ldots k_8, ij} \, , (6.1.5)
\]

which belong to another copy of the 120 dimensional adjoint representation of $D_8$. The first and second generator generate $\text{GL}(8)$. While at level minus we have the same representation

\[
K^{(-2)}_{ij} = \frac{1}{7!} \varepsilon^{ik_1 \ldots k_7} R_{k_1 \ldots k_7, j} \, , \, \tilde{R}^{(-2)} = \frac{1}{8!} \varepsilon_{k_1 \ldots k_8} R_{k_1 \ldots k_8} \, , 
\]
\[ R^{ij}_{(-2)} = \frac{1}{6!} \varepsilon^{ijk_1 \ldots k_6} R_{k_1 \ldots k_6}^{k_1 \ldots k_6}, \quad R^{ij}_{(-2)} = \frac{1}{8!} \varepsilon^{k_1 \ldots k_8} R_{k_1 \ldots k_8}^{k_1 \ldots k_8} R_{i j}, \quad (6.1.6) \]

At level three we find that the generators belong to the 128 dimensional spinor representation of \( D_8 \). They are given by

\[ R^{i_1 i_2 i_3}_{(3)} = \frac{1}{8!} \varepsilon^{k_1 \ldots k_8} R_{k_1 \ldots k_8}^{k_1 \ldots k_8} R_{i_1 i_2 i_3}, \]

\[ R^{i_1 \ldots i_5}_{(3)} = \frac{1}{8!} \varepsilon^{k_1 \ldots k_8} R_{k_1 \ldots k_8}^{k_1 \ldots k_8} i_1 i_2 i_3, \]

\[ R^{i_1 \ldots i_7}_{(3)} = \frac{1}{8!} \varepsilon^{k_1 \ldots k_8} R_{k_1 \ldots k_8}^{k_1 \ldots k_8} i_1 i_2 i_3, \quad (6.1.7) \]

Similarly at level minus three we find another spinor of \( D_8 \),

\[ R^{i}_{(-3)} = \frac{1}{8!} \varepsilon^{k_1 \ldots k_8} R_{k_1 \ldots k_8}^{k_1 \ldots k_8} i, \quad R^{i}_{(-3)} = \frac{1}{8!} \varepsilon^{k_1 \ldots k_8} R_{k_1 \ldots k_8}^{k_1 \ldots k_8} i, \]

\[ R^{i_1 \ldots i_5}_{(-3)} = \frac{1}{8!} \varepsilon^{k_1 \ldots k_8} R_{k_1 \ldots k_8}^{k_1 \ldots k_8} i_1 i_2 i_3, \quad R^{i_1 \ldots i_7}_{(-3)} = \frac{1}{8!} \varepsilon^{k_1 \ldots k_8} R_{k_1 \ldots k_8}^{k_1 \ldots k_8} i_1 i_2 i_3, \quad (6.1.8) \]

In section three we did not list the correspondence between the \( E_{11} \) generators that appear in the eleven dimensional theory at IIA level three and the generators as they appear in the IIA theory, that is, when the node ten is deleted. We now give this correspondence

\[ R^{a_1 \ldots a_7 \ b} = \tilde{R}^{a_1 \ldots a_7 \ b}, \quad R^{a_1 \ldots a_7 \ b} = \tilde{R}^{a_1 \ldots a_7 \ b}, \]

\[ R^{a_1 \ldots a_7 \ b_1 \ldots b_7} = \tilde{R}^{a_1 \ldots a_7 \ b_1 \ldots b_7}, \quad R^{a_1 \ldots a_7 \ b_1 \ldots b_7} = \tilde{R}^{a_1 \ldots a_7 \ b_1 \ldots b_7}, \quad (6.1.9) \]

The identifications at level minus three are given by

\[ R^{a_1 \ldots a_7 \ b} = \tilde{R}^{a_1 \ldots a_7 \ b}, \quad R^{a_1 \ldots a_7 \ b_1 \ b_2 \ b_3} = \tilde{R}^{a_1 \ldots a_7 \ b_1 \ b_2 \ b_3}, \]

\[ R^{a_1 \ldots a_7 \ b_1 \ldots b_7} = \tilde{R}^{a_1 \ldots a_7 \ b_1 \ldots b_7}, \quad R^{a_1 \ldots a_7 \ b_1 \ldots b_7} = \tilde{R}^{a_1 \ldots a_7 \ b_1 \ldots b_7}, \quad (6.1.10) \]

The pattern that emerges is obvious, at even \( E_9 \) levels \( 2n = 0, \pm 2, \pm 4, \ldots \) the \( E_9 \) algebra contains the adjoint representation of \( SO(8, 8) \), which we denote by \( SO(8, 8)_{2n} \), with generators

\[ K^{i}_{(2n) j}, \quad R^{ij}_{(2n)}, \quad (6.1.11) \]

While at each odd \( E_9 \) level, \( 2n + 1 = 1, 3, \ldots \), we find a Majorana-Weyl spinor representation of \( D_8 \), which we denote by \( 128_{2n+1} \), with generators

\[ R^{i}_{(2n+1)} , \quad R^{i_1 i_2 i_3}_{(2n+1)} , \quad R^{i_1 \ldots i_5}_{(2n+1)} , \quad R^{i_1 \ldots i_7}_{(2n+1)} , \quad n = 1, \pm 3, \pm 5, \ldots. \quad (6.1.12) \]

We can thus write \( E_9 \) as

\[ E_9 = \sum_{n \in \mathbb{Z}} SO(8, 8)_{2n} \oplus 128_{2n+1}. \quad (6.1.13) \]
The algebra of $E_9$ reflects its affine character and can be written as

$$[\mathbf{120}_{2n}, \mathbf{120}_{2m}] = \mathbf{120}_{2(n+m)}, \quad [\mathbf{120}_{2n}, \mathbf{128}_{2m+1}] = \mathbf{128}_{2(n+m)+1},$$

$$[\mathbf{128}_{2n+1}, \mathbf{128}_{2m+1}] = \mathbf{120}_{2(n+m+1)}$$

(6.14)

The Cartan involution invariant algebra of $E_9$ has generators that are given by $R^\alpha + I_c(R^\alpha)$ where $R^\alpha$ is a positive root generator of $E_9$. Using the action of the Cartan involution given in equation (2.7) we find at IIA level zero that the Cartan involution invariant generators are given by

$$K^i_j - K^j_i, \quad R^{i_1i_2} - R^{i_2i_1}$$

(6.15)

which generate $SO(8) \otimes SO(8)$. While examining equation (2.8) the Cartan involution invariant generators arising from IIA levels $\pm 1$ are given by

$$R^i - R_i, \quad R^{i_1i_2i_3} - R^{i_3i_2i_1}, \quad R^{i_1\ldots i_5} + R^{i_1\ldots i_5}, \quad R^{i_1\ldots i_7} - R^{i_1\ldots i_7}$$

(6.16)

The pattern at higher levels is obvious.

We can now discuss the string little algebra in more detail and in particular the generators given in equation (6.1.1). The generator $L_\alpha$ has the form

$$L_\alpha = R^\alpha + I_c(R^\alpha) + \text{terms involving commutators with } R \text{ or } \overline{R}$$

(6.17)

We recognise the two terms shown explicitly in $L_\alpha$ in equation (6.1.17) as an element of $I_c(E_9)$ but, as indicated, it also contains terms involving the commutators with $R$ or $\overline{R}$. Thus the generators in the string little algebra $\mathcal{H}_r$ are in one to one correspondence with those of $I_c(E_9)$. However, $L_\alpha$ belongs to $I_c(E_{11})$ rather than $I_c(E_9)$ as the generators $R$ and $\overline{R}$ are not in $E_9$.

At IIA level zero there are no $R$ and $\overline{R}$ commutators as these vanish and the generators in $\mathcal{H}_r$ are those of equation (6.1.15). Thus, at level zero, the algebra of $\mathcal{H}_r$ and $I_c(E_9)$ are the same, namely they just contain the generators of $SO(8) \otimes SO(8)$. However, for all higher levels the generators are very different. The commutators of the generators in $\mathcal{H}_r$ with IIA level greater than zero have the same algebra as $R^\alpha$, where $\alpha$ is a positive root of $E_9$, that is, their algebra is isomorphic to the part of the Borel subalgebra of $E_9$ that contains generators corresponding to positive roots. In particular the generators in $\mathcal{H}_r$ that contain the positive roots of $D_8^+$ obey the same algebra as the commutators of the Borel algebra of $D_8^+$.

As is well known $E_9$ is nothing but affine $E_8$, that is, $E_8^+$. However, so far we have hardly mentioned $E_8$, or it’s Cartan involution invariant subalgebra $I_c(E_8) = SO(16)$ which both play such an important role in the eleven dimensional $E_{11}$ theory. In this theory one deletes node 11 to leave a $GL(11)$ algebra. This contrasts with the IIA theory being studied in this paper in which one deletes the IIA node, that is, node ten which leaves the algebra $D_{10}$. To be relevant to our current context we will first delete the nodes 1 and 2 as shown in the Dynkin diagram in figure above to leave $E_9$ rather than consider
the full $E_{11}$ algebra. Further deleting node eleven we find the algebra $A_8 = SL(9)$ however, this is not the deletion that corresponds to the IIA theory which is of interest to us here.

The algebra $E_9$ is affine $E_8$ but this $E_8$ arises in the $E_{11}$ Dynkin diagram consisting of nodes 4-10 and node 11 and is found by deleting node three. It is useful to first analyse how the generators of $E_8$ appear from the eleven dimensional theory by deleting node eleven and the resulting $SL(8)$ algebra which corresponds to nodes 4-10. The resulting Dynkin diagram is given by

\[ \begin{array}{cccccccccccc}
\oplus & 11 & & \oplus & & 10 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & & \\
\end{array} \]

The generators of $E_8$ are given by

\[
\hat{K}_{\underline{ij}}^\underline{\ell}, \hat{R}_{\underline{i} \underline{j} \underline{k} \underline{l}},  \epsilon_{\underline{i} \underline{j} \underline{k} \underline{l} \underline{m} \underline{n} \underline{o}},  \hat{R}_{\underline{i} \underline{j} \underline{k} \underline{l} \underline{m} \underline{n} \underline{o}},  \hat{R}_{\underline{i} \underline{j} \underline{k} \underline{l} \underline{m} \underline{n} \underline{o}} \quad (6.1.18)
\]

where the indices take the values $\underline{i}, \underline{j}, \underline{k}, \underline{l}, \ldots = 4, \ldots, 11$. These generators have levels zero, one, two and three respectively with respect to the level corresponding to node eleven. Their negative level analogues are given by

\[
\hat{R}_{\underline{i} \underline{j} \underline{k} \underline{l} \underline{m} \underline{n} \underline{o}},  \epsilon_{\underline{i} \underline{j} \underline{k} \underline{l} \underline{m} \underline{n} \underline{o}},  \hat{R}_{\underline{i} \underline{j} \underline{k} \underline{l} \underline{m} \underline{n} \underline{o}},  \hat{R}_{\underline{i} \underline{j} \underline{k} \underline{l} \underline{m} \underline{n} \underline{o}} \quad (6.1.19)
\]

The Cartan involution acts on these generators as specified in equation (2.7) and the invariant algebra is generated by

\[
\hat{K}^\underline{ij} - \hat{K}^\underline{ji}, \hat{R}^\underline{ij} \underline{k} \underline{l},  \epsilon_{\underline{i} \underline{j} \underline{k} \underline{l} \underline{m} \underline{n} \underline{o}},  \hat{R}^\underline{l} \underline{k} \underline{i} \underline{j} \underline{m} \underline{n} \underline{o},  \hat{R}^\underline{k} \underline{l} \underline{i} \underline{j} \underline{m} \underline{n} \underline{o} \quad (6.1.20)
\]

The second and third generators obey the algebra of $SO(8) \otimes SO(8)$ while the second and fourth generators belong to the $(8_s, 8_c)$ representations of this algebra where $8_s$ and $8_c$ are the spinor representations of $SO \otimes SO(8)$. Taken together these 120 generators obey the algebra of $SO(16)$. A detailed discussion can be found in section four of reference [15]. The 128 Cartan involution odd generators in $E_8$ have the opposite signs to those in equation (6.1.18) and they belong to the Majorana-Weyl spinor representation of this $SO(16)$.

The $E_9$ is then given by the affinisation of the $E_8$ algebra. It generators are

\[
E_9 = \sum_n 248_n = \sum_n (SO(16)_n \oplus 128_n), \quad n = 0, \pm 1, \ldots \quad (6.1.21)
\]

where 248 is the adjoint representation of $E_8$ and the second line contains the decomposition of $E_9$ that we just discussed into its Cartan involution invariant subalgebra $SO(16)$ of $E_8$.

From the IIA perspective studied in this paper we have the same $E_9$ but we deleted the IIA node, node 10, to find $D_8 = SO(8, 8)$ and we decomposed the $E_9$ algebra in terms of this algebra. To aid our analysis we also deleted node eleven and decomposed to the remaining $A_8$ algebra associated with nodes 3-9. While the algebra $D_8$ appears in both
the eleven dimensional and IIA viewpoints they are not the same algebra. This is easily seen once we realise that in the IIA viewpoint node ten is deleted but from the eleven dimensional viewpoint this node it is part of $E_8$ Dynkin diagram. Thus in the IIA and eleven dimensional viewpoints different nodes are deleted. Hence the $D_8 = SO(8,8)$ we discussed in the IIA theory is not subalgebra of $E_8$ as it involves node three which is not a node of the $E_8$ that appears from the elven dimensional viewpoint. In contrast the $E_8$ of the eleven dimensional theory requires node ten to be active but this is deleted in the IIA theory. It is intriguing that despite this $E_9$ has a very similar decomposition into the two different $D_8$’s. In the IIA case we can write $E_9$ as in equation (6.1.13) while from the eleven dimensional viewpoint we write it as in equation (6.1.19). One can think of it as a kind of duality.

### 6.2 The relation of the string little algebra to the string states.

As we explained in the previous section the string little algebra involves an algebra that is the same as the Borel subalgebra of $E_9$ at higher IIA levels and $SO(8) \otimes SO(8)$ at IIA level zero. Such a little algebra is unexpected and is quite different to the little algebras for the point particles. At first sight it is not clear why it has such a simple form.

In principle we should now choose an irreducible representation of the string little algebra and then carry out the boost to find the full irreducible representation of $I_c(E_{11}) \otimes_s l_1$. We will not be able to show that the well known string states do carry a representation of the full string little algebra, rather we will make progress towards this goal. In particular the algebra $SO(8,8)^+$ is a subalgebra of $E_9$ and we will focus on considering a representation of the Borel subalgebra of $SO(8,8)^+$ part of the string little algebra.

We first consider the open superstring and in particular the NS sector. As explained in section 4.3, from the $b^I_r$ oscillators one can construct a bilinear quantity whose generators are those of $SO(8)^+$, see equation (4.3.1). However, these generators do not act on physical states to give physical states as they do not boost the momentum of the states to that required by the physical state conditions. As explained at the end of section four, one can replace them by their DDF analogues, see equation (4.3.3), or use the boost of equation (4.3.5), at first sight it is not clear why it has such a simple form.

Since the generators are bilinear in the oscillators they act on the physical states so as to take states with an even (odd) number of oscillators to states which also have an even (odd) number of oscillators. As such the physical states do not carry an irreducible representation of $SO(8)^+$ but contain two irreducible representations which have an odd and even in the number of the oscillators. If we choose to have the irreducible representation that is odd in oscillators then we make the GSO projection in this sector. Thus the GSO projection can be understood as a restriction to have an irreducible representation of $SO(8)^+$.

Let us now consider the NS-NS sector of the closed superstring as discussed in section 4.3. Now we have the oscillators $b^I_r \equiv (b^I_r, \bar{b}^I_r)$ where the indices $I,J,\ldots$ take $8 + 8 = 16$ values and the generators $\mathcal{M}_{IJ} = \sum_r b^I_{r-j} b^J_{r+j}$ which obey the algebra $SO(8,8)^+$. As for the open string these do not take physical states to physical states, however, we can use their DDF analogues or do a boost similar to that in equation (4.3.5). At first sight
the string states in the NS-NS sector do carry a representation of $SO(8,8)^+$ and so the Borel subalgebra of $SO(8,8)^+$ at IIA levels greater than one and $SO(8) \otimes SO(8)$ at level zero which a large part of the string little algebra.

However, this statement does not take account of the GSO projection which ensures that one has odd number of $b^i$ oscillators and also odd numbers of $\bar{b}^i$. This choice is preserved by $SO(8)^+ \otimes SO(8)^+$ but not the generators $\mathcal{M}^I_{0J}$ for $I = 1, \ldots, 8$ and $J = 9, \ldots, 16$. However, it is preserved by the action of bilinears of the latter generators. There is also the level matching condition to take account of. This would only be preserved if we took suitable combinations of generators. However, one could also allow windings, or achieve matching by also acting with the bosonic oscillators. The string states come so close to carrying a representation of $SO(8,8)^+$ that it is tempting to think that in a more complete treatment would work better.

One could also consider generators that transform the NS-NS into the R-R sectors. Such generators could take the generic form

$$-2 D^i_k |R> < N| b^{(i} b^{k)}_{\frac{1}{2}} + D^i_{j_1 j_2} |R> < N| b^{i_1} b^{j_2}_{\frac{1}{2}} + \ldots$$

$$9 D^{i_1} |R> < N| b^{i_2} b^{i_3}_{\frac{1}{2}} - 6 D^{k i_1 i_2} |R> < N| b^{i_3} b^{k}_{\frac{1}{2}} + \ldots$$

where $|R>$ and $|N>$ are the vacuum states in the R-R and NS-NS sectors, $D^k = d^k_0 + i d^k_0$ and $\ldots$ are higher level terms. In fact these generators are the Borel part of the transformations of the massless states that appear when one considers the irreducible representations of $I_c(E_{11}) \otimes l_1$ suitable to the IIA theory. This can be deduced from the eleven dimensional case given in references [14,15] by carrying out a dimensional reduction.

In the R and R-R sectors of the open and closed sectors of the superstring we have the $d^i_n$ and the $d^I_n = (d^i_n, d^I_n)$ oscillators respectively. From these one can form bilinear expressions that generate the algebras $SO(8)^+$ and $SO(8,8)^+$ respectively. It would be interesting to find their replacement by the analogue DDF operators, or include momentum boosts, which take physical states to physical states.

We now study the situation from the view point of the Green-Schwarz formulation of the IIA superstring. In the light-cone gauge one has a residual $SO(8)$ symmetry of the Lorentz algebra. In addition to the bosonic oscillators the theory contains two fermionic spinor oscillators $S^\alpha_n$ and $\bar{S}^\alpha_n$ with $\alpha, \beta, \ldots = 1, \ldots, 8$. These are Majorana-Weyl spinors of $Spin(8)$ of opposite chirality. They obey the anti-commutators

$$\{ S^\alpha_n, S^\beta_m \} = \delta^{\alpha, \beta} \delta_{n+m,0}, \quad \{ \bar{S}^\alpha_n, \bar{S}^\beta_m \} = \delta^{\alpha, \beta} \delta_{n+m,0}, \quad \{ S^\alpha_n, \bar{S}^\beta_m \} = 0 \quad (6.2.1)$$

We will need the properties of such spinors. Looking at section 5.6 of the book [19] on finds for this Euclidean case that we may choose $B = C = I$ and $\rho = 1$. Then $\epsilon = 1$ and $(\gamma^i)^* = \gamma^i$ and $(\gamma^i)^T = \gamma^i$. It follows that the matrices

$$\gamma^i, \gamma^{i_1 \ldots i_4}, \gamma^{i_1 \ldots i_5}, \gamma^{i_1 \ldots i_8}, \quad (6.2.2)$$

are symmetric and the others are antisymmetric. We also find that if $\gamma_9 \chi = \chi$, where $\gamma_9 = \gamma_1 \ldots \gamma_8$, then $\chi \gamma_9 = \bar{\chi}$ for a Majorana-Weyl spinor and as such only the expressions
\( \chi \gamma^{i_1 i_2} \) and \( \bar{\chi} \gamma^{i_1 \ldots i_6} \chi \) are non zero. However, these two expressions are related by a duality transformation.

Taking into account the above properties we can form the operators

\[
M_n^{i_1 i_2} = \sum_p S_{n-p} \gamma^{i_1 i_2} S_p, \quad \bar{M}_n^{i_1 i_2} = \sum_p \bar{S}_{n-p} \gamma^{i_1 i_2} \bar{S}_p \quad (6.2.3)
\]

In this equation \( S_{n-p} \gamma^{i_1 i_2} S_p = S_n^{\alpha} (\gamma^{i_1 i_2})_{\alpha \beta} S_p^{\beta} \). The first expression is the only bilinear object in \( S_n^{\alpha} \) alone that we can construct. The same applies for \( \bar{S}_n^{\alpha} \). Using the relations of equation (6.2.1) we find that the generators of equation (6.2.3) have the algebra \( SO(8)^+ \otimes SO(8)^+ \) which we can think of acting on the left and right moving parts of the string.

For the mixed bilinear expressions we have spinors of opposite Weyl character and so we can only have

\[
L_n^I = \sum_p S_{n-p} \gamma^I \bar{S}_p \quad (6.2.4)
\]

where \( \gamma^I = (\gamma^i, \gamma^{i_1 i_2 i_3}) \). We have taken account of the fact that the gamma matrices between the spinors obey duality relations.

The commutators of the \( L_n^I \) with \( M_n^{i_1 i_2} + \bar{M}_n^{i_1 i_2} \) is given by

\[
[M_n^{i_1 i_2} + \bar{M}_n^{i_1 i_2}, L_n^I] = 2 \sum_p S_{n+m-p} [\gamma^{i_1 i_2}, \gamma^I] \bar{S}_p \quad (6.2.5)
\]

which is just the usual affine rotation. With \( M_n^{i_1 i_2} - \bar{M}_n^{i_1 i_2} \) we find that

\[
[M_n^{i_1 i_2} - \bar{M}_n^{i_1 i_2}, L_n^I] = 2 \sum_p S_{n+m-p} \gamma^{i_1 i_2} [\gamma^I, \gamma^I] \bar{S}_p \quad (6.2.6)
\]

Thus the \( L_n^I \) belong to a representation of \( SO(8)^+ \otimes SO(8)^+ \) and indeed they belong to the \((8_s, 8_c)_n\) representation.

The commutators of the \( L_n^I \) are given by

\[
[L_n^I, L_m^J] = \sum_p S_{n+m-p} \gamma^J \gamma^I S_p \eta_I - \sum_p \bar{S}_{n+m-p} \gamma^I \gamma^J \bar{S}_p \eta_I \quad (6.2.7)
\]

where \( \eta_I = -1 \) if \( \gamma^I \) equals \( \gamma^I \) and \( \eta_I = 1 \) if \( \gamma^{i_1 \ldots i_5} \). In view of the above comments these commutators can only belong to \( SO(8)^+ \otimes SO(8)^+ \). As such the generators of equations (6.2.3) and (6.2.4) close to give the algebra \( SO(8, 8)^+ \).

In the open superstring, the states are formed by the oscillators \( S_n^{\alpha} \) and the bosonic oscillators acting on the 16 dimensional \( 8_v \oplus 8_c \) states which belong to a representation of \( S_0^{\alpha} \) which itself belongs to the \( 8_s \) representation of \( \text{spin}(8) \). This is part of the spacetime supersymmetry and it takes the 8 spin one states into the 8 spin "one" "half" states and vice-versa. The \( 8_v \oplus 8_c \) are the massless states of the open string. For a review of states in the superstring in the Green-Schwarz formulation see section 5.3 of reference [26].

The \( SO(8)^+ \) generators \( M_n^{i_1 i_2} \) clearly act on the states of the open superstring to give other states which will be physical states provided we also inject a corresponding momenta.
The latter could either be achieved by the analogue of a DDF construction or the boosts we considered at the end of section four. Assuming this to be possible the open string states belong to a representation of $SO(8)^+$ in agreement with our result in the Neveu-Schwarz formulation. We note that this is a representation of the bosons and also the fermion as the generator $M^{ij}_{n}$ does not mix these two.

The closed string physical states result from the left moving oscillators $S^\alpha_n$ and the right moving oscillators $\bar{S}^\alpha_n$ and also the bosonic oscillators acting on the $256 = 16 \otimes 16$ dimensional formed from the tensor product of the left and right moving states $(8_v + 8_c) \otimes (8_v + 8_s)$. The states $8_v \otimes 8_v$ belong to the $1 + 28 + 35_v$ dimensional representations of spin(8) which are the massless states in the NS-NS sector. While the $8_c \otimes 8_s$ belong to the $8_v + 56_v$ representation of spin(8) which are the massless states of the R-R sector. The generators of equation (6.2.3) and (6.3.4) act on the physical states. The left moving generators $M^{ij}_{n}$ and $\bar{M}^{ij}_{n}$ give affine Lorentz rotations of the left moving states and the right moving sectors. The generators $L^I$ act on the left and right movers so as to transform the NS-NS states into the R-R states into each other. Assuming we can provide the required boost of the momentum when these generators act on the physical states they will carry a representation of the algebra $SO(8,8)^+$ and also take account of the level matching. This latter constraint can perhaps be achieved by taking certain combinations of generators, or involving the bosonic operators, or allowing winding.

So far we did not discuss the role of the bosonic oscillators $\alpha^i_n$, $\bar{\alpha}^i_n$, $i = 2, \ldots, 9$. They should also play an important and one would expect the above symmetries to also act on these oscillators. However, as we explained at the end of section four, the natural transformations of $SO(8,8)^+$ of the oscillators, see equations (4.3.7) and (4.3.8), do not respect their commutation relations of equation (4.3.6). However, the string little algebra for the superstring is not $I_c(E_9)$ but is essentially the Borel subalgebra of $E_9$. A subalgebra of this algebra is the Borel subalgebra of $SO(8,8)^+$ which we can take to have the generators $M^{ij}_{n}$, $\bar{M}^{ij}_{n}$, $V^{ij}_{n}$ and $\bar{V}^{ij}_{n}$ for $n \leq 0$. These act on the negatively moded oscillators as in equations (4.3.7) and (4.3.8) so as to take the negatively moded into themselves. The physical states contain negatively moded bosonic oscillators acting on the vacuum and so, with a suitable momentum boost, these generators take physical states to physical states. While somewhat unconventional the action of these generators on the positively moded oscillators can be chosen to preserve the commutation relations. Thus if would appear that one can have the action of a significant part of the string little algebra. One could also apply this strategy to the fermionic oscillators.

In this section we have made a fragmented and partial attempt to understand if a representation of the string little algebra is carried by the well known physical states of the superstring. While there are many encouraging features there are also some unresolved issues. We have only investigated the Borel part of $SO(8,8)^+$ and not the 128 dimensional spinor representation that occurs at even levels. We have also not taken a unified approach in that we did not consider the bosonic and fermionic oscillators in a unified way. It would be good to pursue these matters further.

7 Discussion

E theory provides, for the first time, a unified approach to branes within which one can hope to treat point particles, string and branes in the same way. Indeed one can take
the attitude that they are all irreducible representations of $I_c(E_{11}) \otimes_s l_1$ in much the same way as particles, in our familiar quantum field theories, are irreducible representations of the Poincare algebra [14]. In this paper we have applied this strategy to strings and in particular the IIA string. Taking the corresponding charges in the vector representation to be non zero we found the little algebra for the string. It has an unusual form in that it is essentially the Borel subalgebra of $E_9$. However, this $E_9$ is not the $E_9$ that is obviously apparent form the Dynkin diagram of $E_{11}$ by deleting node two, but only emerged after a rather involved calculation involving generators in $E_{11}$ which are outside this obvious $E_9$. The Borel character of the little algebra has its origins in the fact that the string charges involve the momenta and the string charge which satisfy a relation rather than each have specific values. This is a novel feature of finding the irreducible representations for extended objects. This contrasts with the massless case where it is $I_c(E_9)$.

The next step in the procedure would be to take a representation of the little algebra. However in section six we investigated if the known states of the superstring did indeed carry a representation of essentially the Borel subalgebra of $E_9$. While we were not able to show that this was the case we did find quite a bit of evidence that there are additional symmetries which are in this algebra. Starting from the fact that bilinears in the known oscillators can generate the algebra $SO(8,8)^+$, whose Borel subalgebra, is a substantial subalgebra of the string little algebra we investigated if the physical states carried a representation of this algebra. If one allowed winding or some other way of relaxing the level matching requirements, and one could carry out the required momentum boosts, then it would appear that this was true. Our treatment was also provisional in that it did not consider the symmetry to act in a unified way on all the oscillators. We note that it is only required to show that the physical states carry a representation of this algebra and this may not have a simple expression in terms of the known oscillators.

Assuming that we had a representation of the string little algebra the next step would be to boost it up to a representation of $I_c(E_{11}) \otimes_s l_1$. Such a step was discussed in reference [14] and it would be very interesting to do this for the string. The net result should be a wavefunction that depends on the string coordinates and obeys the physical state conditions. These conditions also follow from the string dynamics which can also be constructed from and $E_{11}$ viewpoint as a non-linear realisation of $I_c(E_{11}) \otimes_s l_1$ with a suitable local subalgebra [7,27]. This local subalgebra has been found at low levels and it essentially agrees with the string little algebra found in this paper even though the two derivations are very different. It would be interesting to join up these two different approaches and this may help illuminate the unknown parts of each approach.

It would be very interesting to extend the approach of this paper to the branes and in particular the M2 and M5 branes. The first step would be to find their little algebras. We would expect the calculation to have many points in common with the derivation of the string little algebra found in this paper.

String theory on a torus has additional symmetries which can, for a suitable torus, be any of the Lie algebras in the Killing-Cartan list. It is believed that these are a quirk of taking the string on the torus and not a sign of some underlying symmetry of string theory. A similar attitude was taken to the Cremmer-Julia symmetries in the maximal supergravity theories; they were thought just to be a consequence of the dimensional reduction on a
torus. However, we now know that they are symmetries of the underlying theory, namely E theory, although this theory was not discovered by thinking in this way [1]. Perhaps we should think that the torus probes aspects of string theory not present in our usual spacetime and this allows us to see some hidden symmetries.

In E theory space time has an infinite number of coordinates [2]. Taking string theory on a torus introduces the winding coordinates that are just a few of these coordinates. It is useful to think of the infinite coordinates as a kind of effective spacetime reflecting some of the properties of the underlying structure. The idea that taking particular situations may display hidden symmetries has a precedent in particle physics. For example symmetries that are hidden at low energy in particle physics, as they are spontaneously broken, will show up at high energy. Another example is the discovery of asymptotic charges in general relativity when we consider spacetime that are not flat but asymptotically flat.

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Appendix A: The IIA Commutators

In this appendix we will construct at low levels the commutators of the $E_{11}$ algebra in its decomposition that leads to the IIA theory. As described in section two of this paper this is in the decomposition to $SO(10,10)$ which is for ease of recognition further decomposed into representation of $GL(10)$. Given the generators at a low IIA level this algebra can be determined directly by considering all possible expressions consistent with the $SL(10)$ character of the generators involved in a given commutator, the fact that the commutators preserve the level and then requiring that these expressions satisfy the Jacobi identities. Equivalently the algebra can be determined from the algebra of the $E_{11}$ algebra in its eleven dimensional decomposition as given, for example, in [19]. We give the IIA algebra to level one, but only include those generators which involve the underlying $E_{11}$ generators to $E_{11}$ level three for simplicity. This means for example that we will ignore $\tilde{R}^{a_1 ... a_9} = R^{a_1 ... a_9, 11, 1111}$ in the algebra in this approximation as it has level four with respect to the deletion of node eleven. The algebra of the generators at IIA level zero is

$$\left[ K^{a_d}_{b_d}, K^{c_d}_{d} \right] = \delta^{c_d}_{b_d} K^{a_d}_{d} - \delta^{a_d}_{b_d} K^{c_d}_{d} , \quad \left[ K^{a_d}_{b_d}, \tilde{R} \right] = 0 , \quad \left[ K^{a_d}_{b_d}, R^{cd} \right] = 2 \delta^{a_d}_{b_d} \tilde{R}^{[a|d]} ,$$

$$\left[ K^{a_d}_{b_d}, R_{cd} \right] = -2 \delta^{a_d}_{b_d} \tilde{R}_{[d|c]} , \quad \left[ \tilde{R}, \tilde{R} \right] = 0 , \quad \left[ \tilde{R}, R^{ab} \right] = 0 , \quad \left[ \tilde{R}, R_{ab} \right] = 0 ,$$

$$\left[ R^{ab}, R^{cd} \right] = 0 , \quad \left[ R^{ab}, R_{cd} \right] = 4 \delta^{a_d}_{b_d} K^{c_d}_{d} , \quad \left[ R_{ab}, R_{cd} \right] = 0 . \quad (A.1)$$

These generators obey the algebra of $D_{10} \otimes GL(1) = SO(10,10) \otimes GL(1)$ as they must.

The commutators between the IIA level one generators are given by

$$\left[ R^{a_a}, R^{b_b} \right] = 0 , \quad \left[ R^{a_a}, R^{b_b}_{b_2 b_3} \right] = 0 , \quad \left[ R^{a_a}, R^{b_b}_{b_2 ... b_5} \right] = - R^{b_b}_{b_2 ... b_5} ,$$

$$\left[ R^{a_a}, R^{b_b}_{b_3 ... b_7} \right] = -2 R^{a_a}_{b_3 ... b_7} - 7 R^{a_a}_{b_1 ... b_3} , \quad \left[ R^{a_a}_{a_2 a_3}, R^{b_b}_{b_2 b_3} \right] = 2 R^{a_a}_{a_2 a_3} R^{b_b}_{b_2 b_3} ,$$

$$\left[ R^{a_a}_{a_2 a_3}, R^{b_b}_{b_2 b_3} \right] = 2 R^{a_a}_{a_2 a_3} R^{b_b}_{b_2 b_3} ,$$
\[ [R^a_1 a_1 a_2 b_2 \cdots b_5] = -3R^b_1 \cdots b_5 \{ a_1 a_2 a_3 \}, \quad [R^a_1 a_2 a_3, R^b_1 \cdots b_5] = 0, \]
\[ [R^a_1 \cdots a_5, R^b_1 \cdots b_5] = 0, \quad [R^a_1 \cdots a_5, R^b_1 \cdots b_7] = 0, \quad [R^a_1 \cdots a_7, R^b_1 \cdots b_7] = 0, \quad (A.2) \]

and for those with IIA level minus one by
\[ [R^a_1, R^b_1] = 0, \quad [R^a_1, R^b_1 b_2 b_3] = 0, \quad [R^a_1, R^b_1 \cdots b_5] = +R^a_1 b_1 \cdots b_5, \]
\[ [R^a_1, R^b_1 \cdots b_7] = +2R^a_1 b_1 \cdots b_7 + 7R^{a_1} b_1 \cdots b_7 \cdots b_5, \quad [R^a_1 a_2 a_3, R^b_1 b_2 b_3] = 2R^a_1 a_2 a_3 b_2 b_3, \]
\[ [R^a_1 a_2 a_3, R^b_1 b_2 b_3] = -3R^a_1 \cdots a_3 b_2 b_3, \quad [R^a_1 a_2 a_3, R^b_1 \cdots b_7] = 0, \quad [R^a_1 \cdots a_5, R^b_1 \cdots b_5] = 0, \quad [R^a_1 \cdots a_7, R^b_1 \cdots b_7] = 0. \quad (A.3) \]

The algebra of IIA level zero generators with IIA level one generators are given by
\[ [K^a_2, R^a_2] = R^a_2 \delta^a_2 = -\frac{1}{2} \delta^a_2 R^a_2, \quad [K^a_2, R^a_1 \cdots a_5] = 3\delta^a_2 R^a_1 \{ a_1 \cdots a_5 \} - \frac{1}{2} \delta^a_2 R^a_1 \cdots a_3, \]
\[ [K^a_2, R^a_1 \cdots a_7] = 5\delta^a_2 \{ a_1 \cdots a_7 \} - \frac{1}{2} \delta^a_2 R^a_1 \cdots a_5, \]
\[ [K^a_2, R^a_1 \cdots a_7] = 7\delta^a_2 \{ a_1 \cdots a_7 \} - \frac{1}{2} \delta^a_2 R^a_1 \cdots a_5. \]
\[ \tilde{R}, R^a_1 = -3R^a_1, \quad \tilde{R}, R^a_1 \cdots a_5 = -3R^a_1 \{ a_5 \}, \quad \tilde{R}, R^a_1 \cdots a_7 = -3R^a_1 \cdots a_7, \]
\[ \tilde{R}, R^a_1 \cdots a_7 = -3R^a_1 \cdots a_7, \quad [R^a_1 a_2, b_1 \cdots b_7] = -R^a_1 a_2, \quad [R^a_1 a_2, b_1 \cdots b_7] = -R^a_1 a_2 b_1 \cdots b_7, \]
\[ [R^a_1 \cdots a_2, b_1 \cdots b_7] = -2R^a_1 \cdots a_2 b_1 \cdots b_7, \quad [R^a_1 \cdots a_2, b_1 \cdots b_7] = -42R^a_1 \cdots a_2 b_1 \cdots b_7. \quad (A.4) \]

The commutators of the IIA level zero with the IIA level minus one are given by
\[ [K^a_2, R^a_2] = -R^a_2 \delta^a_2 + \frac{1}{2} \delta^a_2 R^a_2, \quad [K^a_2, R^a_1 \cdots a_5] = -3\delta^a_2 \{ a_1 \cdots a_5 \} + \frac{1}{2} \delta^a_2 R^a_1 \cdots a_3, \]
\[ [K^a_2, R^a_1 \cdots a_7] = -5\delta^a_2 \{ a_1 \cdots a_7 \} + \frac{1}{2} \delta^a_2 R^a_1 \cdots a_5, \]
\[ [K^a_2, R^a_1 \cdots a_7] = -7\delta^a_2 \{ a_1 \cdots a_7 \} + \frac{1}{2} \delta^a_2 R^a_1 \cdots a_5. \]
\[ \tilde{R}, R^a_1 = 3R^a_1, \quad \tilde{R}, R^a_1 \cdots a_5 = 3R^a_1 a_2 a_3, \quad \tilde{R}, R^a_1 \cdots a_7 = 3R^a_1 a_2 a_3, \]
\[ \tilde{R}, R^a_1 \cdots a_7 = 3R^a_1 \cdots a_7, \quad [R^a_1 a_2, b_1 \cdots b_7] = 6\delta^a_2 \{ a_1 \cdots a_2 \} b_1 \cdots b_7, \]
\[ [R^a_1 a_2, b_1 \cdots b_7] = -10R^a_1 b_1 \cdots b_7 \delta^a_2 \{ a_1 \cdots a_2 \} b_1 \cdots b_7, \quad [R^a_1 a_2, b_1 \cdots b_7] = -42R^a_1 b_1 \cdots b_7 \delta^a_2 \{ a_1 \cdots a_2 \} b_1 \cdots b_7, \]
\[ [R^a_1 a_2, b] = R^a_1 a_2 b, \quad [R^a_1 a_2, R^b_1 b_2 b_3] = -2R^a_1 a_2 b_1 \cdots b_5, \]
\[ [R^a_{[a_1 \ldots a_n]}, R_{b_1 \ldots b_m}] = -2R_{[a_1 \ldots a_n b_1 \ldots b_m]} \quad , \quad [R^a_{[a_1 \ldots a_n]}, R_{b_1 \ldots b_m}] = 0 \quad . \quad (A.5) \]

The IIA level one generators with the IIA level minus generators are given by

\[ [R^a_{\alpha \beta}, R_{\alpha \beta}] = K^a_{\alpha \beta} - \frac{1}{6} \delta^a_{\alpha \beta} (3D - R) \quad , \quad [R^a_{\alpha \beta \gamma \delta}, R_{\alpha \beta \gamma \delta}] = -3 \delta^a_{\alpha \beta} R_{\alpha \beta \gamma \delta} \quad , \]

\[ [R^a_{\alpha \beta \gamma \delta}, R_{\alpha \beta \gamma \delta}] = 0 \quad , \quad [R^a_{\alpha \beta \gamma \delta}, R_{\alpha \beta \gamma \delta}] = 0 \quad , \quad [R^a_{\alpha \beta \gamma \delta}, R^b_{\alpha \beta \gamma \delta}] = -3 \delta^a_{\alpha \beta} R^b_{\alpha \beta \gamma \delta} \quad , \]

\[ [R^a_{\alpha \beta \gamma \delta}, R^b_{\alpha \beta \gamma \delta}] = 18 \delta^a_{\alpha \beta} K^2_{\gamma \delta} - \delta^a_{\alpha \beta} (3D - R) \quad , \]

\[ [R^a_{\alpha \beta \gamma \delta}, R_{\alpha \beta \gamma \delta}] = 30 \delta^a_{\alpha \beta} R^b_{\alpha \beta \gamma \delta} - [R^a_{\alpha \beta \gamma \delta}, R_{\alpha \beta \gamma \delta}] = 0 \quad , \]

\[ [R^a_{\alpha \beta \gamma \delta}, R_{\alpha \beta \gamma \delta}] = 0 \quad , \quad [R^a_{\alpha \beta \gamma \delta}, R^b_{\alpha \beta \gamma \delta}] = -30 \delta^a_{\alpha \beta} R^b_{\alpha \beta \gamma \delta} \quad , \]

\[ [R^a_{\alpha \beta \gamma \delta}, R^b_{\alpha \beta \gamma \delta}] = -5 \cdot 5 \delta^a_{\alpha \beta} K^2_{\gamma \delta} + 5 \delta^a_{\alpha \beta} (3D - R) \quad , \]

\[ [R^a_{\alpha \beta \gamma \delta}, R^b_{\alpha \beta \gamma \delta}] = -9 \cdot 70 \delta^a_{\alpha \beta} R^b_{\alpha \beta \gamma \delta} - [R^a_{\alpha \beta \gamma \delta}, R_{\alpha \beta \gamma \delta}] = 0 \quad , \]

\[ [R^a_{\alpha \beta \gamma \delta}, R^b_{\alpha \beta \gamma \delta}] = 7 \cdot 7 \cdot 180 \delta^a_{\alpha \beta} R^b_{\alpha \beta \gamma \delta} - [R^a_{\alpha \beta \gamma \delta}, R^b_{\alpha \beta \gamma \delta}] = 7 \cdot 6 \cdot 5 \delta^a_{\alpha \beta} (3D - R) \quad , \quad (A.6) \]

where \( D = \sum_{\alpha=0}^{9} K^a_{\alpha \alpha} \).

We now give the commutators of the IIA generators with the \( l_1 \) generators. Again these results can be derived directly or equivalently from the commutators of the \( E_{11} \otimes s l_1 \) algebra in its eleven dimensional formulation [19]. Here we consider the algebra to level one in the IIA and \( l_1 \) representations and again only to \( E_{11} \) level three. The IIA level zero generators with the IIA level zero \( l_1 \) generators have the commutators

\[ [K^a_{\alpha \beta}, P_{\gamma \delta}] = -\delta^a_{\alpha \beta} P_{\gamma \delta} \quad , \quad \tilde{R}, P_{\gamma \delta} = -3P_{\gamma \delta} \quad , \quad [R^a_{\alpha \beta}, P_{\gamma \delta}] = -2 \delta^a_{\alpha \beta} Q^{\gamma \delta} \quad , \]

\[ [R^a_{\alpha \beta}, P_{\gamma \delta}] = 0 \quad , \quad [K^a_{\alpha \beta}, Q_{\gamma \delta}] = +\delta^a_{\alpha \beta} Q^{\gamma \delta} \quad , \quad \tilde{R}, Q_{\gamma \delta} = -3Q_{\gamma \delta} \quad , \]

\[ [R^a_{\alpha \beta}, Q_{\gamma \delta}] = 0 \quad , \quad [R^a_{\alpha \beta}, Q_{\gamma \delta}] = +2 \delta^a_{\alpha \beta} P_{\gamma \delta} \quad , \quad (A.7) \]

The IIA level zero IIA generators with the IIA level one \( l_1 \) generators have the commutators

\[ [K^a_{\alpha \beta}, Z] = -\frac{1}{2} \delta^a_{\alpha \beta} Z \quad , \quad [K^a_{\alpha \beta}, Z^{\gamma \delta \epsilon}] = 2 \delta^a_{\alpha \beta} Z^{\gamma \delta \epsilon} - \frac{1}{2} \delta^a_{\alpha \beta} Z^{\gamma \delta \epsilon} \quad , \]

\[ [K^a_{\alpha \beta}, Z^{\gamma \delta \epsilon}] = 4 \delta^a_{\alpha \beta} Z^{\gamma \delta \epsilon} - \frac{1}{2} \delta^a_{\alpha \beta} Z^{\gamma \delta \epsilon} \quad , \quad [K^a_{\alpha \beta}, Z^{\gamma \delta \epsilon}] = 6 \delta^a_{\alpha \beta} Z^{\gamma \delta \epsilon} - \frac{1}{2} \delta^a_{\alpha \beta} Z^{\gamma \delta \epsilon} \quad , \]

\[ [\tilde{R}, Z] = -6Z \quad , \quad [\tilde{R}, Z^{\alpha \beta}] = -6Z^{\alpha \beta} \quad , \quad [\tilde{R}, Z^{\alpha \beta \gamma \delta}] = -6Z^{\alpha \beta \gamma \delta} \quad , \]

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\[ [\hat{R}, Z_{\alpha_1 \cdots \alpha_n}] = -6 Z_{\alpha_1 \cdots \alpha_n}, \quad [R_{\alpha_1 \alpha_2}, Z] = Z_{\alpha_1 \alpha_2}, \quad [R_{\alpha_1 \alpha_2}, Z] = 0, \]
\[ [R_{\alpha_1 \alpha_2}, Z_{\beta_1 \beta_2}] = Z_{\alpha_1 \alpha_2 \beta_1 \beta_2}, \quad [R_{\alpha_1 \alpha_2}, Z_{\beta_1 \beta_2}] = 2 \delta_{\beta_1 \beta_2} Z, \]
\[ [R_{\alpha_1 \alpha_2}, Z_{\beta_1 \cdots \beta_n}] = \frac{1}{3} Z_{\alpha_1 \alpha_2 \beta_1 \cdots \beta_n}, \quad [R_{\alpha_1 \alpha_2}, Z_{\beta_1 \cdots \beta_n}] = 12 \delta_{\beta_1 \beta_2} Z_{\beta_1 \cdots \beta_n}, \]
\[ [R_{\alpha_1 \alpha_2}, Z_{\beta_1 \cdots \beta_n}] = 0, \quad [R_{\alpha_1 \alpha_2}, Z_{\beta_1 \cdots \beta_n}] = 2 \cdot 45 \delta_{\beta_1 \beta_2} Z_{\beta_1 \cdots \beta_n}, \quad (A.8) \]

The commutators of the IIA level one IIA generators with level zero \( l_1 \) generators are given by
\[ [R_{\alpha l}, P_{b}] = -\delta_{\alpha l} Z, \quad [R_{\alpha_1 \alpha_2 \alpha_3}, P_{b}] = 3 \delta_{\alpha_1 b} Z_{\alpha_2 \alpha_3}, \]
\[ [R_{\alpha_1 \alpha_2 \cdots \alpha_n}, P_{b}] = -\frac{5}{2} \delta_{\alpha_1 b} Z_{\alpha_2 \cdots \alpha_n}, \quad [R_{\alpha_1 \alpha_2 \cdots \alpha_n}, P_{b}] = \frac{7}{6} \delta_{\alpha_1 b} R_{\alpha_2 \cdots \alpha_n}, \]
\[ [R_{\alpha}, Q_{\beta}] = -Z_{\alpha \beta}, \quad [R_{\alpha_1 \alpha_2 \alpha_3}, Q_{\beta}] = -Z_{\alpha_1 \alpha_2 \alpha_3 \beta}, \]
\[ [R_{\alpha_1 \cdots \alpha_n}, Q_{\beta}] = \frac{1}{6} Z_{\alpha_1 \cdots \alpha_n \beta}, \quad [R_{\alpha_1 \cdots \alpha_n}, Q_{\beta}] = 0, \quad (A.9) \]
and the level minus one IIA with level zero \( l_1 \) commutators are given by
\[ [R_{\alpha l}, P_{b}] = 0, \quad [R_{\alpha_1 \alpha_2 \alpha_3}, P_{b}] = 0, \quad [R_{\alpha_1 \alpha_2 \cdots \alpha_n}, P_{b}] = 0, \quad [R_{\alpha_1 \cdots \alpha_n}, P_{b}] = 0, \]
\[ [R_{\alpha l}, Q_{\beta}] = 0, \quad [R_{\alpha_1 \alpha_2 \alpha_3}, Q_{\beta}] = 0, \quad [R_{\alpha_1 \cdots \alpha_n}, Q_{\beta}] = 0, \quad [R_{\alpha_1 \cdots \alpha_n}, Q_{\beta}] = 0. \quad (A.10) \]

The commutators of the level one IIA generators with the level one \( l_1 \) commutators are given by
\[ [R_{\alpha l}, Z] = 0, \quad [R_{\alpha}, Z_{\beta_1}] = 0, \quad [R_{\alpha_1 \alpha_2}, Z_{\beta_1 \beta_2}] = Z_{\alpha_1 \alpha_2 \beta_1 \beta_2}, \quad [R_{\alpha_1 \alpha_2 \alpha_3}, Z] = 0, \]
\[ [R_{\alpha_1 \alpha_2}, Z_{\beta_1 \cdots \beta_n}] = Z_{\alpha_1 \alpha_2 \beta_1 \cdots \beta_n}, \quad [R_{\alpha_1 \alpha_2 \cdots \alpha_n}, Z_{\beta_1 \cdots \beta_n}] = Z_{\alpha_1 \alpha_2 \cdots \alpha_n \beta_1 \cdots \beta_n}, \]
\[ [R_{\alpha_1 \alpha_2 \cdots \alpha_n}, Z_{\beta_1 \cdots \beta_n}] = -Z_{\alpha_1 \cdots \alpha_n \beta_1 \cdots \beta_n}, \quad [R_{\alpha_1 \alpha_2 \cdots \alpha_n}, Z_{\beta_1 \cdots \beta_n}] = 0, \]
\[ [R_{\alpha_1 \cdots \alpha_n}, Z] = \frac{1}{2} R_{\alpha_1 \cdots \alpha_n}, \quad [R_{\alpha_1 \cdots \alpha_n}, Z_{\beta_1 \cdots \beta_n}] = -Z_{\alpha_1 \cdots \alpha_n \beta_1 \cdots \beta_n}, \quad [R_{\alpha_1 \cdots \alpha_n}, Z_{\beta_1 \cdots \beta_n}] = 0, \]
\[ [R_{\alpha_1 \cdots \alpha_n}, Z_{\beta_1 \cdots \beta_n}] = 0, \quad [R_{\alpha_1 \cdots \alpha_n}, Z_{\beta_1 \cdots \beta_n}] = 0, \quad \text{and level minus one IIA with the level one } l_1 \text{ commutators are given by} \]
\[ [R_{\alpha l}, Z] = -P_{\alpha l}, \quad [R_{\alpha}, Z_{\beta_1 \beta_2}] = 2 \delta_{\beta_1 \beta_2} Q_{\beta_3 \beta_4}, \quad [R_{\alpha_1 \alpha_2}, Z_{\beta_1 \beta_2}] = 0, \quad [R_{\alpha_1 \alpha_2}, Z_{\beta_1 \cdots \beta_n}] = 0, \]
\[ [R_{\alpha_1 \cdots \alpha_n}, Z] = 0, \quad [R_{\alpha_1 \cdots \alpha_n}, Z_{\beta_1 \beta_2}] = 6 \delta_{\beta_1 \beta_2} P_{\alpha_3 \cdots \alpha_n}, \quad (A.12) \]
[\text{In this section we will give some of commutators of the algebra } I_c(E_{11}) \otimes s l_1 \text{ at level zero which will be useful in section four of this paper. In appendix A of this paper the } E_{11} \otimes s l_1 \text{ in its decomposition corresponding to the IIA theory was found by deleting node ten (the IIA node) from the } E_{11} \text{ Dynkin diagram, leaving the algebra } SO(10, 10) \otimes GL(1), \text{ followed by deleting node eleven to decompose the result into representations of } GL(10). \text{ The algebra of the generators at IIA level zero was given in equation (A.1). In what follows we will discard the } GL(1) \text{ generator } \hat{R}.\]

At IIA level zero the vector representation contains the generators \( P_a = \hat{P}_a \) and \( Q^a = -\hat{Q}^{a11} \) and their commutators with \( E_{11} \) was given in equation (A.13). The Cartan involution acts as in equation (2.7).

Defining \( J_{ab} = K^a_b - \eta^{ac} \eta_{bd} K^d_c \), \( S_{ab} = R^{ab} - \eta^{ac} \eta_{bd} R_{cd} \) we find the Cartan involution invariant algebra, \( I_c(SO(10, 10)) = I_c(D_{10}) \) is given by

\[
\begin{align*}
[J^{a1}{}_{a2}, J_{b1}{}_{b2}] &= -4\delta^{[a1}{}_{[b1} J_{a2]}{}_{b2]}, \quad [J^{a1}{}_{a2}, S_{b1}{}_{b2}] = -4\delta^{[a1}{}_{[b1} S_{a2]}{}_{b2]}, \\
[S^{a1}{}_{a2}, S_{b1}{}_{b2}] &= -4\delta^{[a1}{}_{[b1} J_{a2]}{}_{b2}],
\end{align*}
\]

(B.1)

Defining \( M^{a1}{}_{a2} = \frac{1}{2} (J^{a1}{}_{a2} + S^{a1}{}_{a2}) \), \( \bar{M}^{a1}{}_{a2} = \frac{1}{2} (J^{a1}{}_{a2} - S^{a1}{}_{a2}) \) these commutators become

\[
\begin{align*}
[M^{a1}{}_{a2}, M_{b1}{}_{b2}] &= -4\delta^{[a1}{}_{[b1} M_{a2]}{}_{b2]}, \quad [\bar{M}^{a1}{}_{a2}, \bar{M}_{b1}{}_{b2}] = -4\delta^{[a1}{}_{[b1} \bar{M}_{a2]}{}_{b2}], \quad [M^{a1}{}_{a2}, \bar{M}_{b1}{}_{b2}] = 0 \quad (B.2)
\end{align*}
\]

which we recognise as the algebra \( I_c(SO(10, 10)) = SO(10) \otimes SO(10) \). In this section all our equations apply equally well to \( SO(D) \otimes SO(D) \) with suitable adjustment of the range of the indices.

The Cartan involution odd generators are \( T_{ab} = \eta_{ac} K^c_b + \eta_{bc} K^c_a \), \( U_{ab} = R^{ab} + \eta_{ac} \eta_{bd} R_{cd} \) and they obey the algebra

\[
\begin{align*}
[J^{a1}{}_{a2}, T_{b1}{}_{b2}] &= -4\delta^{[a1}{}_{[b1} T_{a2]}{}_{b2]}, \quad [J^{a1}{}_{a2}, U_{b1}{}_{b2}] = -4\delta^{[a1}{}_{[b1} U_{a2]}{}_{b2]}, \\
[S^{a1}{}_{a2}, T_{b1}{}_{b2}] &= 4\delta^{[a1}{}_{[b1} U_{a2]}{}_{b2]}, \quad [S^{a1}{}_{a2}, U_{b1}{}_{b2}] = 4\delta^{[a1}{}_{[b1} T_{a2]}{}_{b2}]
\end{align*}
\]

(B.3)

and

\[
\begin{align*}
[T^{a1}{}_{a2}, T_{b1}{}_{b2}] &= 4\delta^{(a1}{}_{b1} J_{a2)}{}_{b2]}, \quad [U^{a1}{}_{a2}, U_{b1}{}_{b2}] = 4\delta^{(a1}{}_{b1} J_{a2)}{}_{b2]}, \quad [T^{a1}{}_{a2}, U_{b1}{}_{b2}] = 4\delta^{(a1}{}_{b1} S_{a2)}{}_{b2}]
\end{align*}
\]

(B.4)
Defining the generators
\[ V_{a_1a_2} = (T_{a_1a_2} - U_{a_1a_2}), \quad \bar{V}_{a_1a_2} = (T_{a_1a_2} + U_{a_1a_2}) \] (B.5)
the above commutators become
\[ \left[ M^{a_1a_2}, V_{b_1b_2} \right] = -4\delta^{[a_1}_{b_1} V^{a_2]}_{b_2}, \quad \left[ M^{a_1a_2}, \bar{V}_{b_1b_2} \right] = -4\delta^{[a_1}_{b_2} V^{a_2]}_{b_1}, \]
\[ \left[ \bar{M}^{a_1a_2}, V_{b_1b_2} \right] = -4\delta_{b_1}^{[a_1} \bar{V}^{a_2]}_{b_2}, \quad \left[ \bar{M}^{a_1a_2}, \bar{V}_{b_1b_2} \right] = -4\delta_{b_2}^{[a_1} \bar{V}^{a_2]}_{b_1} \] (B.6)
While the odd with odd generators obey
\[ [V^{a_1a_2}, V_{b_1b_2}] = 4\delta_{b_1}^{a_1} M^{a_2}_{b_2} + 4\delta_{b_2}^{a_2} M^{a_1}_{b_1}, \quad [\bar{V}^{a_1a_2}, \bar{V}_{b_1b_2}] = 4\delta_{b_1}^{a_1} M^{a_2}_{b_2} + 4\delta_{b_2}^{a_2} \bar{M}^{a_1}_{b_1}, \]
\[ [V^{a_1a_2}, \bar{V}_{b_1b_2}] = 4\delta_{b_1}^{a_2} \bar{M}^{a_1}_{b_2} + 4\delta_{b_2}^{a_1} \bar{M}^{a_2}_{b_1}, \] (B.7)
We will now consider the vector representation and it commutators with the odd and even parts of \( D_{10} \). The Cartan involution invariant algebra \( I_c(D_{10}) \) acts on the vector representation as
\[ [J_{ab}, P_c] = -2\eta_{c[a} P_{b]}, \quad [J_{ab}, Q_c] = -2\eta_{c[a} Q_{b]}, \quad [S_{ab}, P_c] = -2\eta_{c[a} Q_{b]}, \quad [S_{ab}, Q_c] = -2\eta_{c[a} P_{b}] \] (B.8)
The above equations correct a number of typographical errors in reference [17]. While with the Cartan involution odd generators we have
\[ [T_{ab}, P_c] = -2\eta_{c[a} P_{b]}, \quad [T_{ab}, Q_c] = +2\eta_{c[a} Q_{b]}, \quad [U_{ab}, P_c] = -2\eta_{c[a} Q_{b]}, \quad [U_{ab}, Q_c] = 2\eta_{c[a} P_{b}] \] (B.9)
The commutators of the vector representation simplify if we write the commutators in terms of
\[ \varphi_a = P_a + Q_a, \quad \tilde{\varphi}_a = P_a - Q_a, \] (B.10)
Then
\[ [M_{a_1a_2}, \varphi_b] = -2\eta_{c[a_1} \varphi_{a_2]} , \quad [\bar{M}^{a_1a_2}, \varphi_b] = 0, \quad [\bar{M}^{a_1a_2}, \tilde{\varphi}_b] = -2\eta_{c[a_1} \tilde{\varphi}_{a_2]} , \quad [M^{a_1a_2}, \tilde{\varphi}_b] = 0, \] (B.11)
and
\[ [V_{a_1a_2}, \varphi_b] = -2\eta_{ba_1} \varphi_{a_2}, \quad [V_{a_1a_2}, \tilde{\varphi}_b] = -2\eta_{ba_1} \tilde{\varphi}_{a_2}, \]
\[ [\bar{V}_{a_1a_2}, \varphi_b] = -2\eta_{ba_1} \varphi_{a_2}, \quad [\bar{V}_{a_1a_2}, \tilde{\varphi}_b] = -2\eta_{ba_1} \tilde{\varphi}_{a_2}, \] (B.12)
Thus the generators of \( I_c(D_{10}) \otimes \mathfrak{s}_1 \) split into two irreducible algebras which are
\[ M_{a_1a_2}, \varphi_a, \quad \text{and} \quad \bar{M}^{a_1a_2}, \tilde{\varphi}_b \] (B.13)
Finally we give the explicit transformations of the vector representation which were given by
\[ \delta P_a = [\Lambda, P_a] \] and \( \delta Q_a = [\Lambda, Q_a] \) where we take the Lie algebra element of \( D_{10} \) to be given by
\[ \Lambda = \lambda^{ab} J_{ab} + \tilde{\lambda}^{ab} S_{ab} + \Omega^{ab} T_{ab} + \hat{\Omega}^{ab} U_{ab} = \lambda^{ab} M_{ab} + \tilde{\lambda}^{ab} \bar{M}^{ab} + \mu^{ab} V_{ab} + \tilde{\mu}^{ab} \bar{V}_ab \] (B.14)

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We find that under $D_{10}$ transformations

$$\delta \varphi^a = -2(\Lambda + \tilde{\Lambda})^a_b \varphi_b - 2(\Omega - \tilde{\Omega})^a_b \bar{\varphi}_b = -2\lambda^a_b \varphi_b - 2(\mu^a_b + \tilde{\mu}^b_a)\bar{\varphi}_b,$$

$$\delta \bar{\varphi}^a = -2(\Lambda - \tilde{\Lambda})^a_b \bar{\varphi}_b - 2(\Omega + \tilde{\Omega})^a_b \varphi_b = -2\tilde{\lambda}^a_b \bar{\varphi}_b - 2(\tilde{\mu}^a_b + \mu^b_a)\varphi_b, \quad (B.15)$$

It is straightforward to verify that

$$P^2_a + Q^2_a, \quad \text{and} \quad P^a Q_a \text{ or equivalently } \varphi^2_a, \quad \text{and} \quad \bar{\varphi}^2_a \quad (B.16)$$

are invariant under $I_c(D_{10})$ transformations.

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