PARTIAL CROSSED PRODUCT DESCRIPTION OF THE C*-ALGEBRAS ASSOCIATED WITH INTEGRAL DOMAINS

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ABSTRACT. Recently, Cuntz and Li introduced the C*-algebra \( \mathfrak{A}[R] \) associated to an integral domain \( R \) with finite quotients. In this paper, we show that \( \mathfrak{A}[R] \) is a partial group algebra of the group \( K \rtimes K^\times \) with suitable relations, where \( K \) is the field of fractions of \( R \). We identify the spectrum of this relations and we show that it is homeomorphic to the profinite completion of \( R \). By using partial crossed product theory, we reconstruct some results proved by Cuntz and Li. Among them, we prove that \( \mathfrak{A}[R] \) is simple by showing that the action is topologically free and minimal.

1. Introduction

Fifteen years ago, motivated by the work of Julia [14], Bost and Connes constructed a C*-dynamical system having the Riemann \( \zeta \)-function as partition function [2]. The C*-algebra of the Bost-Connes system, denoted by \( C_Q \), is a Hecke C*-algebra obtained from the inclusion of the integers into the rational numbers. In [19], Laca and Raeburn showed that \( C_Q \) can be realized as a semigroup crossed product and, in [20], they characterized the primitive ideal space of \( C_Q \).

In [1], [4] and [15], by observing that the construction of \( C_Q \) is based on the inclusion of the integers into the rational numbers, Arledge, Cohen, Laca and Raeburn generalized the construction of Bost and Connes. They replaced the field \( \mathbb{Q} \) by an algebraic number field \( K \) and \( \mathbb{Z} \) by the ring of integers of \( K \). Many of the results obtained for \( C_Q \) were generalized to arbitrary algebraic number fields (at least when the ideal class group of the field is \( \text{h} = 1 \)) [16], [17].

Recently, a new construction appeared. In [5], Cuntz defined two new C*-algebras: \( Q_N \) and \( Q_Z \). Both algebras are simple and purely infinite and \( Q_N \) can be seen as a C*-subalgebra of \( Q_Z \). These algebras encode the additive and multiplicative structure of the semiring \( \mathbb{N} \) and of the ring \( \mathbb{Z} \). Cuntz showed that the algebra \( Q_N \) is, essentially, the algebra generated by \( C_Q \) and one unitary operator. In [25], Yamashita realized \( Q_N \) as the C*-algebra of a topological higher-rank graph.

The next step was given by Cuntz and Li. In [6], they generalized the construction of \( Q_Z \) by replacing \( \mathbb{Z} \) by a unital commutative ring \( R \) (which is an integral domain with finite quotients by principal ideals). This algebra was called \( \mathfrak{A}[R] \). Cuntz and Li showed that \( \mathfrak{A}[R] \) is simple and purely infinite (when \( R \) is not a field) and they related a C*-subalgebra of its with the generalized Bost-Connes systems (when \( R \) is the ring of integers in an algebraic number field having \( \text{h} = 1 \) and, at most, one real place). In [23], Li extended the construction of \( \mathfrak{A}[R] \) to an arbitrary unital ring.
The aim of this text is to show that the algebra \( \mathfrak{A}[R] \) can be seen as a partial crossed product (when \( R \) is an integral domain with finite quotients). We show that \( \mathfrak{A}[R] \) is isomorphic to a partial group algebra of the group \( K \rtimes K^\times \) with suitable relations, where \( K \) is the field of fractions of \( R \). By using the relationship between partial group algebras and partial crossed products, we see that \( \mathfrak{A}[R] \) is a partial crossed product by the group \( K \rtimes K^\times \). We characterize the spectrum of the commutative algebra arising in the crossed product and show that this spectrum is homeomorphic to \( \hat{R} \) (the profinite completion of \( R \)). Furthermore, we show that the partial action is topologically free and minimal. By using that the group \( K \rtimes K^\times \) is amenable, we conclude that \( \mathfrak{A}[R] \) is simple.

Recently, some similar results appeared. In [21] and [3], Brownlowe, an Huef, Laca and Raeburn showed that \( \mathbb{Q}N \) is a partial crossed product by using a boundary quotient of the Toeplitz (or Wiener-Hopf) algebra of the quasi-lattice ordered group \((\mathbb{Q} \rtimes \mathbb{Q}^\times, \mathbb{N} \rtimes \mathbb{N}^\times)\) (see [24] and [18] for Toeplitz algebras of quasi-lattice ordered groups). We observe that our techniques are different from theirs. We don’t use Nica’s construction [24] (indeed, our group \( K \rtimes K^\times \) is not a quasi-lattice, in general). From our results, in the case \( R = \mathbb{Z} \), we see that \( \mathbb{Q}Z \) is a partial crossed product by \( \mathbb{Q} \rtimes \mathbb{Q}^\times \). From this, it is immediate that \( \mathbb{Q}N \) is a partial crossed product by \( \mathbb{Q} \rtimes \mathbb{Q}^\times \) (as in [3]).

Before we go to the main result we give, in the section 2, a brief review about the algebra \( \mathfrak{A}[R] \) and the theories of partial crossed products and partial group algebras. In the section 3, we state our main theorem: the algebra \( \mathfrak{A}[R] \) is isomorphic to a partial group algebra. In the section 4, we study \( \mathfrak{A}[R] \) by using the techniques of partial crossed products. We recover the faithful conditional expectation constructed by Cuntz and Li in [6, Proposition 1] in a very natural way. Furthermore, we use the concepts of topological freeness and minimality of a partial action to show that \( \mathfrak{A}[R] \) is simple.

2. Preliminaries

2.1. The \( \mathbb{C}^* \)-algebra \( \mathfrak{A}[R] \) of an Integral Domain. Throughout this text, \( R \) will be an integral domain (unital commutative ring without zero divisors) with the property that the quotient \( R/(m) \) is finite, for all \( m \neq 0 \) in \( R \). We denote by \( R^\times \) the set \( R \setminus \{0\} \) and by \( R^\times \) the set of units in \( R \).

Definition 2.1. [6, Definition 1] The regular \( \mathbb{C}^* \)-algebra of \( R \), denoted by \( \mathfrak{A}[R] \), is the universal \( \mathbb{C}^* \)-algebra generated by isometries \( \{s_m \mid m \in R^\times\} \) and unitaries \( \{u^n \mid n \in R\} \) subject to the relations

\[
\begin{align*}
\text{(CL1)} \quad & s_ms_{m'} = s_{mm'}, \\
\text{(CL2)} \quad & u^n u^{n'} = u^{n+n'}, \\
\text{(CL3)} \quad & s_m u^n = u^{mn} s_m, \\
\text{(CL4)} \quad & \sum_{l+(m) \in R/(m)} u^l s_m s_m^* u^{-l} = 1;
\end{align*}
\]

for all \( m, m' \in R^\times \) and \( n, n' \in R \).

We denote by \( e_m \) the range projection of \( s_m \), namely \( e_m = s_ms_m^* \). It is easily seen that, under (CL2) and (CL3), \( u^l e_m u^{-l} = u^{l'} e_m u^{-l'} \) if \( l + (m) = l' + (m) \). From this, we see that the sum in (CL4) is independent of the choice of \( l \).
Let \( \{ \xi_r \mid r \in R \} \) be the canonical basis of the Hilbert space \( \ell^2(R) \) and consider the operators \( S_m \) and \( U^n \) on \( \ell^2(R) \) given by \( S_m(\xi_r) = \xi_{mr} \) and \( U^n(\xi_r) = \xi^{n+r} \).

**Definition 2.2.** [6, Section 2] The **reduced regular** \( C^* \)-algebra of \( R \), denoted by \( \mathfrak{A}[R] \), is the \( C^* \)-subalgebra of \( B(\ell^2(R)) \) generated by the operators \( \{ S_m \mid m \in R^{\times} \} \) and \( \{ U^n \mid n \in R \} \).

One can check that \( S_m \) is an isometry, \( U^n \) is a unitary and satisfy (CL1)-(CL4). Hence, there exists a surjective \(*\)-homomorphism \( \mathfrak{A}[R] \rightarrow \mathfrak{A}_n[R] \).

In [6], Cuntz and Li showed that, when \( R \) is not a field, \( \mathfrak{A}[R] \) is simple; therefore the above \(*\)-homomorphism is a \(*\)-isomorphism. In the section 4, we will show that \( \mathfrak{A}[R] \) is simple (when \( R \) is not a field) by using the partial crossed product description of \( \mathfrak{A}[R] \).

For future references, we need the following lemma, proved by Cuntz and Li:

**Lemma 2.3.** [6, Lemma 1] For all \( n, n' \in R \) and \( m, m' \in R^{\times} \), the projections (in \( \mathfrak{A}[R] \)) \( u^n e_m u^{-n} \) and \( u^{n'} e_{m'} u^{-n'} \) commute.

More details about these algebras can be found in [5], [6], [7], [8], [22], [23] and [25].

### 2.2. Partial Crossed Products

Here, we review some basic facts about partial actions and partial crossed products.

**Definition 2.4.** [9, Definition 1.1] A **partial action** \( \alpha \) of a (discrete) group \( G \) on a \( C^* \)-algebra \( \mathcal{A} \) is a collection \( \{ D_g \}_{g \in G} \) of ideals of \( \mathcal{A} \) and \(*\)-isomorphisms \( \alpha_g : D_{g^{-1}} \rightarrow D_g \) such that

- (PA1) \( D_e = \mathcal{A} \), where \( e \) represents the identity element of \( G \);
- (PA2) \( \alpha_g^{-1}(D_h \cap D_{g^{-1}}) \subseteq D_{gh^{-1}} \);
- (PA3) \( \alpha_{g^{-1}} \circ \alpha_h(x) = \alpha_{gh}(x) \), \( \forall x \in \alpha_h^{-1}(D_h \cap D_{g^{-1}}) \).

In the above definition, if we replace the \( C^* \)-algebra \( \mathcal{A} \) by a locally compact space \( X \), the ideals \( D_g \) by open sets \( X_g \) and the \(*\)-isomorphisms \( \alpha_g \) by homeomorphisms \( \theta_g : X_{g^{-1}} \rightarrow X_g \), we obtain a **partial action** \( \theta \) of the group \( G \) on the space \( X \). A partial action \( \theta \) on a space \( X \) induces naturally a partial action \( \alpha \) on the \( C^* \)-algebra \( C_o(X) \). The ideals \( D_g \) are \( C_0(X_g) \) and \( \alpha_g(f) = f \circ \theta_{g^{-1}} \).

We say that a partial action \( \theta \) on a space \( X \) is **topologically free** if, for all \( g \in G \setminus \{ e \} \), the set \( F_g = \{ x \in X_{g^{-1}} \mid \theta_g(x) = x \} \) has empty interior. A subset \( V \) of \( X \) is **invariant** under the partial action \( \theta \) if \( \theta_g(V \cap X_{g^{-1}}) \subseteq V \), for every \( g \in G \). The partial action \( \theta \) is **minimal** if there are no invariant open subsets of \( X \) other than \( \emptyset \) and \( X \). It is easy to see that \( \theta \) is minimal if, and only if, every \( x \in X \) has dense orbit, namely \( O_x = \{ \theta_g(x) \mid g \in G \text{ for which } x \in X_g \} \) is dense in \( X \).

**Definition 2.5.** [9, Definition 6.1] A **partial representation** \( \pi \) of a (discrete) group \( G \) into a unital \( C^* \)-algebra \( \mathcal{B} \) is a map \( \pi : G \rightarrow \mathcal{B} \) such that, for all \( g, h \in G \),

- (PR1) \( \pi(e) = 1 \);
- (PR2) \( \pi(g^{-1}) = \pi(g)^* \);
- (PR3) \( \pi(g)\pi(h)\pi(h^{-1}) = \pi(gh)\pi(h^{-1}) \).

From a partial action \( \alpha \), we can construct two **partial crossed products**: \( \mathcal{A} \rtimes_{\alpha} G \) (full) and \( \mathcal{A} \rtimes_{\alpha,r} G \) (reduced). We can define both as follows: let \( \mathcal{L} \) be the normed \(*\)-algebra of the finite formal sums \( \sum_{g \in G} a_g \delta_g \), where \( a_g \in D_g \). The operations and the
norm in $L$ are given by $(a_g \delta_g)(a_h \delta_h) = \alpha_g(a_{g^{-1}}a_g) a_h \delta_h$, $(a_g \delta_g)^* = \alpha_{g^{-1}}(a_g^*) \delta_{g^{-1}}$ and $\| \sum_{g \in G} a_g \delta_g \| = \sum_{g \in G} \|a_g\|$. If we denote by $B_g$ the vector subspace $D_g \delta_g$ of $L$, then the family $(B_g)_{g \in G}$ generates a Fell bundle. The full and the reduced crossed products are, respectively, the full and the reduced cross sectional algebra of $(B_g)_{g \in G}$. It is well known that $A \rtimes_\alpha G$ is universal with respect to a covariant pair $(\varphi, \pi)$, where $\varphi : A \to B$ is a *-homomorphism ($B$ is a unital $C^*$-algebra), $\pi : G \to B$ is a partial representation of $G$ and the covariant equations are $\varphi(\alpha_g(x)) = \pi(g) \varphi(x) \pi(g^{-1})$ for $x \in D_{g^{-1}}$ and $\varphi(x) \pi(g) \pi(g^{-1}) = \pi(g) \pi(g^{-1}) \varphi(x)$ for $x \in A$.

There exists a faithful conditional expectation $E : A \rtimes_\alpha G \to A$ given by $E(a \delta_g) = a$ if $g = e$, and $E(a \delta_g) = 0$ if $g \neq e$. When the Fell bundle $(B_g)_{g \in G}$ is amenable ($G$ amenable implies its), the full and reduced constructions are isomorphic and, in this case, there exists a faithful conditional expectation of $A \rtimes_\alpha G$ onto $A$.

There is a close relation between topological freeness and minimality of the partial action and ideals of the reduced crossed product. If $\theta$ is a topologically free partial action on a space $X$ then $\theta$ is minimal if, and only if, $C_0(X) \rtimes_\alpha G$ is simple, where $\alpha$ is the action induced by $\theta$. Under the amenability hypothesis, this result is valid for the full crossed product too.

For more details about partial crossed products, see [9], [10], [11], [12] and [13].

2.3. Partial Group Algebras. Let $G$ be a discrete group, let $G$ be the set $G$ without the group operations and denote the elements in $G$ by $[g]$ (namely, $G = \{[g] \mid g \in G\}$). The **partial group algebra** of $G$, denoted by $C_p^*(G)$, is defined to be the universal $C^*$-algebra generated by the set $G$ with the relations

$$R_p = \{[e] = 1\} \cup \{[g^{-1}] = [g]^*\}_{g \in G} \cup \{[g][h][h^{-1}] = [gh][h^{-1}]\}_{g,h \in G}.$$  

The algebra $C_p^*(G)$ is universal with respect to a partial representation. Observe that the relations in $R_p$ correspond to the partial representation axioms (PR1), (PR2) and (PR3). Sometimes, we will refer to a relation in $R_p$ by indicating the corresponding axiom.

Consider the natural bijection between $P(G)$ and $\{0, 1\}^G$, where $P(G)$ is the power set of $G$. With the product topology, $\{0, 1\}^G$ is a compact Hausdorff space. Give to $P(G)$ the topology of $\{0, 1\}^G$. Denote by $X_G$ the subset of $P(G)$ of the subsets $\xi$ of $G$ such that $e \in \xi$. Clearly, with the induced topology of $P(G)$, $X_G$ is a compact space. For each $g \in G$, let $X_g = \{\xi \in X_G \mid g \in \xi\}$. It is easy to see that $\theta_g : X_{g^{-1}} \to X_g$ given by $\theta_g(\xi) = g\xi$ is a homeomorphism. The collection of open sets $(X_g)_{g \in G}$ of $X_G$ with the homeomorphisms $\theta_g$ define a partial action $\theta$ of $G$ on $X_G$. The partial crossed product $C(X_G) \rtimes_\alpha G$ is isomorphic to $C_p^*(G)$ (where $\alpha$ is the partial action induced by $\theta$).

For each $g \in G$, we abbreviate $[g][g^{-1}]$ by $e_g$. Let $R$ be a set of relations on $G$ such that every relation is of the form

$$\sum_i \lambda_i \prod_j e_{g_{ij}} = 0.$$  

The **partial group algebra** of $G$ with relations $R$, denoted by $C_p^*(G, R)$, is defined to be the universal $C^*$-algebra generated by the set $G$ with the relations $R_p \cup R$. Given a partial representation $\pi$ of $G$, we can extend $\pi$ naturally to sums of products of elements in $G$. If this extension satisfies the relations $R$, we say that $\pi$ is a **partial**
representation that satisfies $\mathcal{R}$. The algebra $C_p^*(G, \mathcal{R})$ is universal with respect to a partial representation that satisfies the relations $\mathcal{R}$.

Denote by $1_g$ the function in $C(X_G)$ given by $1_g(\xi) = 1$ if $g \in \xi$ and $1_g(\xi) = 0$ otherwise. By an abuse of notation, we also denote by $\mathcal{R}$ the subset of $C(X_G)$ given by the functions $\sum_{i} \lambda_i \prod_j 1_{g_{ij}}$, where $\sum \lambda_i \prod_j e_{g_{ij}} = 0$ is a relation in (the original) $\mathcal{R}$. The spectrum of the relations $\mathcal{R}$ is defined to be the compact Hausdorff space

$$\Omega_{\mathcal{R}} = \{ \xi \in X_G \mid f(g^{-1}\xi) = 0, \forall f \in \mathcal{R}, \forall g \in \xi \}.$$ 

Let $\Omega_g = \{ \xi \in \Omega_{\mathcal{R}} \mid g \in \xi \}$. By restricting the above $\theta_g$ to $\Omega_g^{-1}$, we obtain a partial action (again denoted by $\theta$) of $G$ on $\Omega_{\mathcal{R}}$ (the open sets are the $\Omega_g$’s and the homeomorphisms are the restrictions of the $\theta_g$’s). The main result concerning $C_p^*(G, \mathcal{R})$ says that this algebra is isomorphic to the partial crossed product $C(\Omega_{\mathcal{R}}) \rtimes_{\alpha} G$ (again, $\alpha$ is the partial action induced by $\theta$).

The above results are proved in [12] and [13].

3. Partial Group Algebra Description of $\mathfrak{A}[R]$

Let $R$ be an integral domain satisfying the conditions stated in the previous section. Denote by $K$ the field of fractions of $R$ and consider the semidirect product $K \rtimes K^\times$. The elements of $K \rtimes K^\times$ will be denoted by a pair $(u, w)$, where $u \in K$ and $w \in K^\times$. Recall that $(u, w)(u', w') = (u + u'w, uw')$ and $(u, w)^{-1} = (-u/w, 1/w)$. We denote by $[u, w]$ an element of set $K \rtimes K^\times$ without the group operations (as the set $\mathcal{G}$ associated to $G$ in the previous section).

Again, denote $[g][g^{-1}]$ by $e_g$. Consider the sets of relations

$$\mathcal{R}_1 = \{ e_{(n,1)} = 1 \mid n \in R \}, \quad \mathcal{R}_2 = \{ e_{(0,1/m)} = 1 \mid m \in R^\times \},$$

$$\mathcal{R}_3 = \left\{ \sum_{n+(m) \in R/(m)} e_{(n,m)} = 1 \mid m \in R^\times \right\}$$

and $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$. We observe that, under the relations $\mathcal{R}_1$ and $\mathcal{R}_p$ (relations stated in the previous section), the sum in $\mathcal{R}_3$ does not depend on the choice of $n$. Indeed, for $k \in R$,

$$e_{(n+km,m)} = [n + km, m][(n + km, m)^{-1}] \overset{\mathcal{R}_1}{=} [(n, m)(k, 1)]e_{(-k,1)}[(k, 1)^{-1}(n, m)^{-1}] = [(n, m)(k, 1)][(k, 1)^{-1}][k, 1][(k, 1)^{-1}(n, m)^{-1}] \overset{\text{PR3}}{=} [n, m][k, 1][(k, 1)^{-1}][k, 1][(k, 1)^{-1}][(n, m)^{-1}] = [n, m]e_{(k,1)}e_{(k,1)}[(n, m)^{-1}] = e_{(n,m)}.$$

**Remark 3.1.** The relations in $\mathcal{R}_1$ are unnecessary. They can be obtained from $\mathcal{R}_3$ with $m = 1$.

Consider the partial group algebra $C_p^*(K \rtimes K^\times, \mathcal{R})$. We will show that this algebra is isomorphic to $\mathfrak{A}[R]$.

**Proposition 3.2.** There exists a $\ast$-homomorphism $\Psi : \mathfrak{A}[R] \to C_p^*(K \rtimes K^\times, \mathcal{R})$ such that $\Psi(u^n) = [n, 1]$ and $\Psi(s_m) = [0, m]$.

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1Sometimes, we work with the element $(u, w)^{-1}$ or the element $(u_1, w_1)(u_2, w_2)$. For these elements, our corresponding notations will be $[(u, w)^{-1}]$ and $[(u_1, w_1)(u_2, w_2)]$. 

Let Claim 3.3. We need to show that \([n, 1] \in R\), that \([0, m] = \text{is an isometry (for } m \in R^x)\) and that the relations (CL1)-(CL4) are satisfied. From \(\mathcal{R}_1\) and (PR2), we have \([n, 1][n, 1]^* = e_{(n, 1)} = 1\) and \([n, 1]^*[n, 1] = e_{(-n, 1)} = 1\), ie, \([n, 1] \in R\). Similarly, from \(\mathcal{R}_2\) and (PR2) we see that \([0, m] = \text{is an isometry. By using this fact,}\)

\[
\Psi(s_1 s_2) = [0, m][0, m'] = [0, m][0, m'][0, m]^* [0, m'] = [0, m'][0, m'] = [0, mn] = \Psi(s_{mn})
\]

hence (CL1) is satisfied. We can prove (CL2) in the same way. To show (CL3), note that

\[
\Psi(s_m s_m) = [0, m][n, 1] = [0, m][n, 1][n, 1]^*[n, 1] = [mn, m][n, 1][n, 1]^*[n, 1] = [mn, m],
\]

because \([n, 1] \in R\). On the other hand,

\[
\Psi(u_{mn} s_m) = [mn, 1][0, m] = [mn, 1][mn, 1]^*[mn, 1][0, m] = [mn, m].
\]

Finally, (CL4) follows from \(\mathcal{R}_3\) and\(^2\)

\[
\Psi(u^n e_m u^{-n}) = [n, 1][0, m][n, 1]^*[-n, 1] = [n, m][0, 1/m][-n, 1][-n, 1]^*[-n, 1] = [n, m][(n, m)^{-1}][-n, 1][n, 1]^*[-n, 1] = [n, m][(n, m)^{-1}] = e_{(n, m)}.
\]

\[\square\]

Now, we will construct an inverse for \(\Psi\). In the next claim, note that every element in \(K \times K^\times\) can be written under the form \((\frac{n}{m}, \frac{m}{m'})\), where \(n \in R\) and \(m, m' \in R^x\).

Claim 3.3. The map \(\pi : K \times K^\times \rightarrow \mathfrak{A}[R]\) given by \(\pi \left( \left( \frac{n}{m'}, \frac{m}{m'} \right) \right) = s_m u^n s_m\) is independent of the representation of \((\frac{n}{m}, \frac{m}{m'})\).

Proof. Let \(\left( \frac{n}{m'}, \frac{m}{m'} \right) = \left( \frac{q}{p'}, \frac{p'}{p'} \right)\), ie, \(pm' = p'm\) and \(m'q = p'n\). Hence,

\[
s_{p'} u^n s_p = s_{p'} s_{m'} s_m u^n s_p \overset{\text{(CL3)}}{=} s_{p'} s_{m'} u^{m'q} s_{m'} s_p \overset{\text{(CL1)}}{=} s_{p'} u^{m'q} s_{m'} s_p \overset{\text{(CL1)}}{=} s_{m'} s_{p'} u^{m'q} s_{m'} s_p \overset{\text{(CL3)}}{=} s_{m'} s_{p'} s_{m'} u^n s_{m'} = s_{m'} u^n s_{m'}.
\]

\[\square\]

Proposition 3.4. The map \(\pi\) defined above is a partial representation of \(K \times K^\times\) that satisfies \(\mathcal{R}\).

Proof. First, we will show that \(\pi\) is a partial representation. Since \(\pi((0, 1)) = s_1^* u^0 s_1 = 1\), we have (PR1). Observe that

\[
\pi \left( \frac{n}{m'}, \frac{m}{m'} \right)^{-1} = \pi \left( \frac{-n}{m}, \frac{m'}{m} \right) = s_m u^{-n} s_m = \pi \left( \frac{n}{m'}, \frac{m}{m'} \right)^*.
\]

\(^2\)Be careful with the \(e\)'s! The notation \(e_m\) represents \(s_m s_m^*\) in \(\mathfrak{A}[R]\) and \(e_{(n, m)}\) represents \([n, m][n, m]^*\) in \(C_b(K \times K^\times, \mathcal{R})\).
which shows (PR2). To see (PR3), let \( g = \left( \frac{q}{p}, \frac{p}{p'} \right) \) and \( h = \left( \frac{n}{m'}, \frac{m}{m} \right) \). We have \( gh = \left( \frac{m'q + pm}{p'm'}, \frac{pm'}{p'm} \right) \) and, therefore,

\[
\pi(gh)\pi(h^{-1}) = \pi(gh)\pi(h)^* = (s^*_m u^m q + pm s_{pm}) s_m u^{-n} s_{m'} \quad \text{(CL1),(CL2),(CL3)}
\]

\[
s^*_m u^m s_{pm} u^n s_m s^*_m u^{-n} s_{m'} = s^*_p u^q s^*_m s_p u^n s_{pm} s^*_m u^{-n} s_{m'} \quad \text{Lemma 2.3}
\]

This shows that \( \pi \) is a partial representation. It remains to show that the extension of \( \pi \) satisfies the relations in \( \mathcal{R} \). By remark 3.1, it suffices to show that the relations in \( \mathcal{R}_2 \) and \( \mathcal{R}_3 \) are satisfied. It follows from

\[
\pi(e_{(0,1/m)}) = \pi([0, 1/m][0, m]) = s^*_m u^0 s_1^* u^0 s_m = 1
\]

and

\[
\pi \left( \sum_{n+(m) \in \mathbb{R}/(m)} e_{(n,m)} \right) = \sum_{n+(m) \in \mathbb{R}/(m)} s^*_m u^n s_m s^*_m u^{-n} s_1 = 1.
\]

\[\qed\]

**Remark 3.5.** We can define \( \pi \) for a general representation of a element in \( K \times K^\times \) by

\[
\pi \left( \left( \frac{n}{m'}, \frac{m}{m} \right) \right) = s^*_m u^n s^*_m s_{m'} s_m.
\]

**Theorem 3.6.** The \(*\)-homomorphism \( \Psi \) defined above is a \(*\)-isomorphism. Its inverse \( \Phi : C^*_p(K \times K^\times, \mathcal{R}) \to \mathfrak{A}[R] \) is given by \( \Phi \left( \left[ \frac{n}{m'}, \frac{m}{m} \right] \right) = s^*_m u^n s_m \).

**Proof.** The existence of \( \Phi \) follows from \( \pi \) and the universal property of \( C^*_p(K \times K^\times, \mathcal{R}) \). It remains to show that \( \Psi \) and \( \Phi \) are inverses each other. Indeed, \( \Phi(\Psi(u^n)) = \Phi([n, 1]) = s^*_1 u^n s_1 = u^n, \Phi(\Psi(s_m)) = \Phi([0, m]) = s^*_m u^0 s_m = s_m \) and

\[
\Psi \left( \Phi \left( \left[ \frac{n}{m'}, \frac{m}{m'} \right] \right) \right) = \Psi(s^*_m u^n s_m) = [0, 1/m'][0, 1] [n, 1] [0, m] = [0, 1/m'][0, 1/m'][0, 1/m'][0, 1] [n, 1] [n, 1] [n, 1] [0, m] = \left[ \frac{n}{m'}, \frac{m}{m'} \right] .
\]

\[\qed\]

# 4. Partial Crossed Product Description of \( \mathfrak{A}[R] \)

Before characterizing \( \mathfrak{A}[R] \) as a partial crossed product, note that the group \( K \times K^\times \) is solvable and, hence, amenable. Therefore, there exists a faithful conditional expectation (imported from the partial crossed product realization) \( E : C^*_p(K \times K^\times, \mathcal{R}) \to C^*(\{e_g\}_{g \in K \times K^\times}) \) given by

\[
E([g_1][g_2] \cdots [g_k]) = \delta_{g_1g_2 \cdots g_k} e_{[g_1]} [g_2] \cdots [g_k].
\]

In [6, Proposition 1], Cuntz and Li constructed a faithful conditional expectation \( \Theta \) on \( \mathfrak{A}[R] \) given by \( \Theta(s^*_m u^n s_m s^*_m u^{-n} s_{m'}) = \delta_{m,m'} \delta_{n,n} s^*_m u^n s_m s^*_m u^{-n} s_{m'} \). The next proposition shows that, under the \(*\)-isomorphism \( \Psi \), \( E \) and \( \Theta \) are the same conditional expectation.
Proposition 4.1. $E \circ \Psi = \Psi \circ \Theta$.

Proof. First of all, observe that $(\frac{n}{m''}, \frac{m}{m''}) (\frac{-n'}{m'}, \frac{m'}{m''}) = (0, 1)$ if, and only if, $m' = m''$ and $n = n'$. Hence,

$$E \circ \Psi(s_{m''}^{n} s_{m}^{-n'} s_{m''}) = E \left( \frac{n}{m''}, \frac{m}{m''} \right) \left( \frac{-n'}{m'}, \frac{m'}{m''} \right) =$$

$$\delta_{m', m''} \delta_{n, n'} \left[ \frac{n}{m'}, \frac{m'}{m''} \right] \left[ \frac{-n'}{m'}, \frac{m'}{m''} \right].$$

On the other hand

$$\Psi \circ \Theta(s_{m''}^{n} s_{m}^{-n'} s_{m''}) = \Psi(\delta_{m', m''} \delta_{n, n'} s_{m''}^{n} s_{m}^{-n'} s_{m''}) =$$

$$\delta_{m', m''} \delta_{n, n'} \left[ \frac{n}{m'}, \frac{m'}{m''} \right] \left[ \frac{-n'}{m'}, \frac{m'}{m''} \right].$$

We already know that $\mathfrak{A}[R]$ is a partial crossed product. Indeed, every partial group algebra is a partial crossed product (see section 2.3). From now on, our goal is to study $\mathfrak{A}[R]$ by this way.

There exists a natural partial order on $R^\times$ given by the divisibility: we say that $m \leq m'$ if there exists $r \in R$ such that $m' = rm$. Whenever $m \leq m'$, we can consider the canonical projection $p_{m, m'} : R/(m') \rightarrow R/(m)$. Since $(R^\times, \leq)$ is a directed set, we can consider the inverse limit

$$\hat{R} = \lim_{\leftarrow} \{ R/(m), \ p_{m, m'} \},$$

which is the profinite completion of $R$. In this text, we shall use the following concrete description of $\hat{R}$:

$$\hat{R} = \left\{ (r_{m} + (m))_{m} \in \prod_{m \in R^\times} R/(m) \mid p_{m, m'}(r_{m} + (m')) = r_{m} + (m), \text{ if } m \leq m' \right\}.$$

Give to $R/(m)$ the discrete topology, to $\prod_{m \in R^\times} R/(m)$ the product topology and to $\hat{R}$ the induced topology of $\prod_{m \in R^\times} R/(m)$. With the operations defined componentwise, $\hat{R}$ is a compact topological ring. There exists a canonical inclusion of $R$ into $\hat{R}$ given by $r \mapsto (r + (m))_{m}$ (to see injectivity, take $r \neq 0$, $m$ non-invertible and note that $r \notin (rm)$).

The above partial order can be extended to $K^\times$. For $w, w' \in K^\times$, we say that $w \leq w'$ if there exists $r \in R$ such that $w' = wr$. Denote by $(w)$ the fractional ideal generated by $w$, namely $(w) = wR \subseteq K$. As before, if $w \leq w'$, we can consider the canonical projection $p_{w, w'} : (R + (w'))/(w') \rightarrow (R + (w))/(w)$. As before, we consider the inverse limit

$$\hat{R}_{K} = \lim_{\leftarrow} \{ (R + (w))/(w), \ p_{w, w'} \} \cong$$

$$\left\{ (u_{w} + (w))_{w} \in \prod_{w \in K^\times} (R + (w))/(w) \mid p_{w, w'}(u_{w'} + (w')) = u_{w} + (w), \text{ if } w \leq w' \right\}.$$

By the second isomorphism theorem, it could be $p_{w, w'} : R/(R \cap (w')) \rightarrow R/(R \cap (w))$. 
It is a compact crossed product ring too. In fact, $\hat{R}_K$ is naturally isomorphic to $\hat{R}$ as topological ring. In this text, we use $\hat{R}_K$ instead of $\hat{R}$ to simplify our proofs.

It is easy to see that, when $R$ is a field, then $\hat{R} \cong \hat{R}_K \cong \{0\}$.

Let $\Omega$ be the spectrum of the relations $\mathcal{R}$ (see section 2.3). We will show that $\Omega$ is homeomorphic to $\hat{R}_K$ (hence, homeomorphic to $\hat{R}$). Define

$$\rho : \hat{R}_K \longrightarrow \mathcal{P}(K \rtimes K^\times)$$

$$(u_w + (w))_w \longmapsto \{(u_w + rw, w) \mid w \in K^\times, r \in R\}.$$

Note that the definition is independent of the choice of $u_w$ in $u_w + (w)$.

**Claim 4.2.** $\rho(\hat{R}_K) \subseteq \Omega$.

**Proof.** Let $(u_w + (w))_w \in \hat{R}_K$. By the definition of $\hat{R}_K$, if $w \leq w'$, then $u_{w'} = u_w + kw$ for some $k \in R$. Denote $\rho((u_w + (w)))$ by $\xi$. Clearly, $(0, 1) \in \xi$. We need to show that $f(g^{-1}\xi) = 0$, for all $f \in \mathcal{R}$ and $g \in \xi$. Fix $g = (u_w + rw, w) \in \xi$. Let $f = 1_{(n,1)} - 1$ in $\mathcal{R}_1$ and note that $f(g^{-1}\xi) = 0$ is equivalent to $g(n,1) \in \xi$. Since $g(n,1) = (u_w + rw, w)(n,1) = (u_w + (r+n)w, w)$, we have $g(n,1) \in \xi$. Now, let $f = 1_{(0,1/m)} - 1$ in $\mathcal{R}_2$. Similarly, we must show that $g(0,1/m) \in \xi$. Observe that $g(0,1/m) = (u_w + rw, w)(0,1/m) = (u_w + rw, w/m)$. Since $w/m \leq w$, then $g(0,1/m) = (u_{w/m} + k(w/m) + rw, w/m) = (u_{w/m} + (k + rm)(w/m)) \in \xi$. To finish, fix $m \in R^\times$ and let $f = \sum_{n+(m) = 1_{(n,m)}} - 1$ in $\mathcal{R}_3$. We must show that there exists one, and only one class $n + (m)$ such that $g(n,m) \in \xi$. Indeed, $g(n,m) = (u_w + rw, w)(n,m) = (u_w + (n + r)w, w/m) = (u_{w/m} + (n + r - km)(w/m))$ and, for it belongs to $\xi$, we must have $(n + r - km) \in (w/m)$. Hence, $n \equiv k - r \mod m$, in other words, there exists only one class $n + (m)$ such that $g(n,m) \in \xi$. Since $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$, the proof is completed. \hfill \square

**Proposition 4.3.** $\rho : \hat{R}_K \longrightarrow \Omega$ is a homeomorphism.

**Proof.**

**Injectivity.** Let $(u_w + (w))_w, (v_w + (w))_w \in \hat{R}_K$ such that $\rho((u_w + (w))) = \rho(v_w + (w)))$. By the definition of $\rho$, the elements in $\rho((u_w + (w)))$ whose second component equals $w$ are of the form $(u_w + rw, w)$. Since $(v_w, w) \in \rho((v_w + (w)))$ and, therefore, $(v_w, w) \in \rho((u_w + (w)))$, we must have $v_w = u_w + rw$ for some $r \in R$. This show that $(u_w + (w))_w = (v_w + (w))_w$.

**Surjectivity.** Let $\xi \in \Omega$. The relations in $\mathcal{R}_1$ and $\mathcal{R}_2$ together implies that if $g \in \xi$, then $g(q/p, 1/p) \in \xi$ for all $q \in R$ and $p \in R^\times$ (fix $g$ and apply $f(g^{-1}\xi) = 0$ for various $f$). For each $m \in R^\times$, let $f = \sum_{n+(m) = 1} 1_{(n,m)} - 1$ in $\mathcal{R}_3$ and apply $f(g^{-1}\xi) = 0$ with $g = (0, 1)$ to see that there exists only one class $n + (m)$ such that $(n, m) \in \xi$. Denote this class by $u_{m} + (m)$. Since $g(0,1/p) \in \xi$ if $g \in \xi$, then $\rho_{m, mp}(u_{mp} + (mp)) = (u_{m} + (m))$. From this, we can define unambiguously $u_w + (w) = u_m + (w)$ for $w = m/m' \in K^\times$. One can see that the classes $u_w + (w)$ are compatible with the projections $p_{w, w'}$ by using that $g(q/p, 1/p) \in \xi$ if $g \in \xi$. Hence, we have constructed $(u_w + (w))_w \in \hat{R}_K$.

We claim that $\rho((u_w + (w))) = \xi$. Since $(u_w, w) \in \xi$, $(u_w, w)(q, 1) = (u_w + qw, w)$ must belongs to $\xi$. This shows that $\rho((u_w + (w))) \subseteq \xi$. Suppose, by contradiction, $\rho((u_w + (w))) \neq \xi$. Hence, there exists $h \in \xi$ such that $h \not\in \rho((u_w + (w)))$. If we write $h = (n'/m', m/m')$, then $h \not\in \rho((u_w + (w)))$ is equivalent to $n' - m'u_m \not\in (m)$. Let
which is an open set of $\Omega$ (recall that the topology on $\Omega$ is induced by the product topology). Since $\rho$ is discrete, it suffices to show that $\rho^{-1}(\xi)$ is open. Let $g = (u_m, 1/m')$, $h' = (u_m, m/m')$ and note that both belong to $\rho((u_w + (w)))$ (hence, belong to $\xi$). Since $g^{-1} h = (-m'u_m, m')(n'/m', m/m') = (n' - m'u_m, m)$, $g^{-1} h' = (0, m)$ and $n' - m'u_m \notin (m)$, then $f(g^{-1} \xi) \neq 0$ if $f = \sum_{n+(m)} 1_{(n,m)} - 1$, which contradicts the fact that $\xi \in \Omega$. Hence, $\rho((u_w + (w))) = \xi$.

To finish the proof, observe that $\hat{R}_K$ and $\Omega$ are compact Hausdorff, therefore it suffices to show that $\rho$ (or $\rho^{-1}$) is continuous to conclude that $\rho$ is a homeomorphism. We will prove that $\rho^{-1}$ is continuous by showing that $\pi_w \circ \rho^{-1}$ is continuous for all $w \in K^\times$, where $\pi_w : \hat{R}_K \to (R + (w))/\langle w \rangle$ is the canonical projection. Since $(R + (w))/\langle w \rangle$ is discrete, it suffices to show that $\rho \circ \pi_w^{-1}(\{u_w + (w)\})$ is an open set of $\Omega$, for all $u_w + (w) \in (R + (w))/\langle w \rangle$. To see this, note that

$$\rho \circ \pi_w^{-1}(\{u_w + (w)\}) = \{\xi \in \Omega \mid (u_w, w) \in \xi\},$$

which is an open set of $\Omega$ (recall that the topology on $\Omega$ is induced by the product topology of $\{0,1\}^{K \times K^\times}$).

Following the section 2.3, there exists a partial action of $K \times K^\times$ on $\Omega$. By the above proposition, we can define this partial action on $\hat{R}_K$. Let $\hat{R}_g = \rho^{-1}(\Omega_g)$, where $\Omega_g = \{\xi \in \Omega \mid g \in \xi\}$, and $\theta_g$ be the homeomorphism between $\hat{R}_{g^{-1}}$ and $\hat{R}_g$. It is easy to see that

$$\hat{R}_{(u,w)} = \{(u_{w'}, (w'))_{w'} \in \hat{R}_K \mid u_w + (w) = u + (w)\}$$

and

$$\theta_{(u,w)}((u_{w'}, (w'))_{w'}) = (u + wu_{w'}, (w'))_{w'} = (u + wu_{w^{-1}w'} + (w'))_{w'},$$

ie, $\theta_{(u,w)}$ acts on $\hat{R}_{(u,w)^{-1}}$ by the affine transformation corresponding to $(u, w)$. The next proposition, whose proof is trivial, will be useful later.

**Proposition 4.4.** We have that

(i) $\hat{R}_{(u,w)} = \emptyset \iff u \notin R + (w)$;

(ii) $\hat{R}_{(u,w)} = \hat{R}_K \iff R \subseteq u + (w)$.

Now, we describe the topology on $\hat{R}_K$. Since $\hat{R}_K$ is a singleton set when $R$ is a field, we shall assume that $R$ is not a field in this paragraph. For $w \in K^\times$ and $C_w \subseteq (R + (w))/\langle w \rangle$, we define the open set

$$V^C_w = \{(u_{w'}, (w'))_{w'} \in \hat{R}_K \mid u_w + (w) \in C_w\}.$$ 

Clearly, if $w \leq w'$, then $V^C_w = V^{C_{w'}}_{w'}$, where $C_{w'} = \{u + (w') \in (R + (w'))/\langle w' \rangle \mid u + (w) \in C_w\}$. From the product topology, we know that the finite intersections of open sets $V^C_w$ form a basis for the topology on $\hat{R}_K$. By taking a common multiple of the $w$’s in the intersection, we see that every basic open set is of the form $V^C_w$ (since $V^C_{w_1} \cap V^C_{w_2} = V^{C_{w_1} \cap C_{w_2}}_{w_1}$). Furthermore, if $C_w \neq \emptyset$, $r$ is a non-invertible element in $R$ and $V^C_{w_r} = V^C_{wr}$, then $C_{wr}$ has, at least, two elements. Indeed, let $u + (w) \in C_w$ and $r_1, r_2 \in R$ such that $r_1 + (r) \neq r_2 + (r)$. It is easy to see that $u + wr_1 + (wr)$ and $u + wr_2 + (wr)$ are in $C_{wr}$ and that $u + wr_1 + (wr) \neq u + wr_2 + (wr)$. This shows that, if $V^C_w$ is non-empty, we can suppose that $C_w$ has more than one element.

**Proposition 4.5.** The partial action $\theta$ on $\hat{R}_K$ is topologically free if, and only if, $R$ is not a field.
Proof. If \( R \) is a field, then \( \hat{R}_K = \{0\} \) and, hence, \( \theta \) is not topologically free. Conversely, suppose that \( R \) is not a field. We need to show that \( F_g = \{ x \in \hat{R}_{g^{-1}} \mid \theta_g(x) = x \} \) has empty interior, for all \( g \in K \rtimes K^\times \setminus \{(0,1)\} \). We shall consider two cases: \( g = (u,1) \) and \( g = (u,w), \ w \neq 1 \).

**Case 1.** If \( u \notin R \), then the proposition 4.4 says that \( \hat{R}_{g^{-1}} = \emptyset \). So, we can suppose \( u \in R \). If \( F_g \neq \emptyset \), then equation \( \theta_g(x) = x \) implies that \( u \in (m) \) for every \( m \in R^\times \).

Since \( R \) is not a field, then \( u = 0 \). This show that \( F_g = \emptyset \) if \( g = (u,1) \) and \( u \neq 0 \).

**Case 2.** Let \( g = (u,w) \) such that \( w \neq 1 \) and \( u \in R + (w) \) (if \( u \notin R + (w) \), then \( \hat{R}_{g^{-1}} = \emptyset \)). Let \( V \) be a non-empty open set contained in \( \hat{R}_{g^{-1}} \). We will show that there exists \( x \in V \) such that \( \theta_g(x) \neq x \). By shrinking \( V \) if necessary, we can suppose that \( V = V_w^{C_{w'}} \). Futhermore, we can assume that \( C_{w'} \) has more than one element. Let \( u_1 + (w') \) and \( u_2 + (w') \) be distinct elements of \( C_{w'} \), hence \( u_1 - u_2 \notin (w') \). Suppose, by contradiction, \( \theta_g(x) = x \) for all \( x \in V \). Since \( (u_1 + (w'''))_{w''} \in V, i = 1, 2, \) then

\[
\theta_{(u,w)}((u_1 + (w'''))_{w''}) = (u_1 + (w'''))_{w''} \implies (u + wu_i + (w'''))_{w''} = (u_i + (w'''))_{w''}.
\]

By choosing \( w'' = (w - 1)w' \) (note that \( w \neq 1 \)), we see that \( u + (w - 1)u \in ((w - 1)w') \), for \( i = 1, 2 \). By subtracting the equations (for different \( i \)'s), we have \( (w - 1)(u_1 - u_2) \in ((w - 1)w') \) and, therefore \( u_1 - u_2 \notin (w') \); which is a contradiction! This show that \( F_g \) has empty interior.

**Proposition 4.6.** The partial action \( \theta \) is minimal.

**Proof.** If \( R \) is a field, then the result is trivial. Now, suppose that \( R \) is not a field. We will prove that every \( x \in \hat{R}_K \) has dense orbit (see section 2.2) by showing that if \( V \) is a non-empty open set, then there exists \( g \in K \rtimes K^\times \) such that \( x \in \hat{R}_{g^{-1}} \) and \( \theta_g(x) \in V \). Let \( x = (u_w + (w))_{w} \in \hat{R}_K \) and \( V = V_w^{C_{w'}} \) non-empty. Take \( u' + (w') \in C_{w'} \) and observe that we can suppose, without loss of generality, \( u' \in R \) and \( u_w \in R \). Let \( g = (u' - u_w, 1) \). By the proposition 4.4, \( \hat{R}_{g^{-1}} = \hat{R}_K \) and, hence, \( x \in \hat{R}_{g^{-1}} \). To finish, note that \( \theta_g(x) = \theta_{(w'-u_w,1)}((u_w + (w))_w) = (u' - u_w + u_w + (w))_w \in V \).

Following, we summarize the results of this section.

**Theorem 4.7.** The algebra \( \mathfrak{A}_x[R] \) is *-isomorphic to the partial crossed product \( C(\hat{R}_K) \rtimes_{\alpha} K \rtimes K^\times \), where \( \alpha \) is the partial action induced by \( \theta \). The *-isomorphism is given by \( u^n \mapsto 1_{(n,1)} \) and \( s_m \mapsto 1_{(0,m)} \delta_{(0,m)}, \) where \( 1_{(0,m)} \) is the characteristic function of \( \hat{R}_g \).

**Theorem 4.8.** \( \mathfrak{A}_x[R] \) is simple.

**Proof.** By the propositions 4.5 and 4.6, the reduced crossed product \( C(\hat{R}_K) \rtimes_{\alpha} r K \rtimes K^\times \) is simple. Since \( K \rtimes K^\times \) is amenable, then \( C(\hat{R}_K) \rtimes_{\alpha} K \rtimes K^\times \cong C(\hat{R}_K) \rtimes_{\alpha} K \rtimes K^\times \) and, therefore, \( C(\hat{R}_K) \rtimes_{\alpha} K \rtimes K^\times \) is simple. The result follows from the previous theorem.

**Corollary 4.9.** \( \mathfrak{A}_x[R] \cong \mathfrak{A}_x[r][R] \).

When \( R = \mathbb{Z} \), we can restrict our partial action to the subgroup \( \mathbb{Q} \times \mathbb{Q}_+^\times \) of \( \mathbb{Q} \times \mathbb{Q}_+^\times \) and the corresponding partial crossed product is the algebra \( \mathcal{Q}_N \) introduced by Cuntz in [5] and realized as a partial crossed product in [3] by Brownlowe, an Huef, Laca and Raeburn.
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