On Characterizations of Some General \((\alpha, \beta)\) Norms in a Minkowski Space.

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Abstract

General \((\alpha, \beta)\) norms are an important class of Minkowski norms which contains the original \((\alpha, \beta)\) norms. In this note, by studying the behavior of the Darboux curves (see Definition 2.1 below) of the indicatrix, we give a characterization of 3-dimensional general \((\alpha, \beta)\) norms. By studying the isoperimetric properties of the indicatrix, as well as the isoperimetric inequalities in a Minkowski space, we give some global geometric quantities which characterizes Randers norms of arbitrary dimensions.

Key words and phrases: General \((\alpha, \beta)\) norm; Randers norm; Minkowski norm; Blaschke structure.

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1 Introduction

Recent years, the study of Finsler geometry has attracted a lot of attention, for basic and advanced topics, see [4], [7], [18], etc. Methods from other fields of mathematics also have some interesting applications in Finsler geometry. For example, the Hamiltonian systems (cf. [2], [25]), the ergodic theory (cf. [1], [10]), etc. Roughly speaking, a Finsler metric \(F\) on a manifold \(M^n\) is a collection of Minkowski norms on each tangent space \(T_xM^n\), \(x \in M^n\), which varies smoothly on \(M^n\). The model of the tangent space of a Finsler manifold is a vector space equipped with a Minkowski norm, namely, a Minkowski space.

Euclidean norms are of course Minkowski norms. Geometers first characterized Euclidean norms among the Minkowski ones. In 1953, Deicke [8]
proved his famous theorem which states that a Minkowski norm on a vector space $V$ is Euclidean, if and only if its Cartan form vanishes. Later in 1964, Su gave another characterization by studying curves on the indicatrix:

**Theorem 1.1.** (23) Let $V$ be a vector space and $O$ its origin. Let $F$ be a Minkowski norm on it and $M$ the indicatrix of $F$. Then $F$ is a Euclidean norm if and only if: every geodesic of $M$ with respect to the metric induced by $F$ lies on a plane passing through $O$.

Among the non-Euclidean Minkowski norms, Randers norm seems to be the simplest one, it can be represented as $F = \alpha + \beta$. Where $\alpha$ is a Euclidean norm and $\beta$ is an 1-form (cf. [4], [17]). Matsumoto-Hojo gave a characterization of $n(\geq 3)$-dimensional Randers norms in [12] and [13]:

**Theorem 1.2.** ([13]) Let $F$ be a Minkowski norm on an $n(\geq 3)$-dimensional vector space, $F$ is a Randers norm if and only if its Matsumoto torsion (see (2.13) below) vanishes.

On the other hand, a Randers norm can also be viewed as a shift metric, i.e. a solution of the Zermelo's navigation problem with navigation data $(h, W)$ (see (2.14) below). Where $h$ is a Euclidean norm and $W$ a vector satisfying $h(W) < 1$ (cf. Chern-Shen [7], pp.24). From this point of view, one can see that the indicatrix of a Randers norm can be derived by shifting the indicatrix of a Euclidean norm. Hence an $n$-dimensional Minkowski norm is a Randers norm if and only if its indicatrix is an $(n-1)$-dimensional ellipsoid. This is true for all $n \in \mathbb{N}$. Based on those observations, Mo-Huang proved in [15] that

**Theorem 1.3.** ([15]) Let $F$ be a Minkowski norm on an $n$-dimensional vector space, $F$ is a Randers norm if and only if $L_1$ (see (2.25) below) is a constant along the indicatrix.

Mo-Huang further pointed out that the Matsumoto torsion (possibly multiplied by a positive factor) is just the cubic form of the indicatrix with its Blaschke structure (see (2.23) below). Hence the Matsumoto-Hojo Theorem (Theorem 1.2) is a corollary of the Pick-Berwald Theorem (cf. [16], pp.53).

A more general class of non-Euclidean norm is the $(\alpha, \beta)$ norm. A $(\alpha, \beta)$ norm is defined by $F = \alpha \phi \left( \frac{\beta}{\alpha} \right)$. Where $\alpha$ is a Euclidean norm, $\beta$ is a 1-form and $\phi$ a smooth function satisfying certain conditions(cf. [7], pp.5-7). In virtue of the navigation problem, Yu-Zhu in [26] investigated the so-called general $(\alpha, \beta)$ norm, which can be viewed as a solution of (2.14), where $F$ is an $(\alpha, \beta)$ norm and $U$ is a vector satisfies $F(U) < 1$. They mentioned that
in an $\alpha$-orthogonal coordinate, the indicatrix of an $n$-dimensional general $(\alpha, \beta)$ norm is an $(n-1)$-dimensional Euclidean hypersurface of revolution.

In this work, we will build up characterizations of some general $(\alpha, \beta)$ norms. We mainly use methods arising from affine differential geometry. We shall first review some basic affine properties of the indicatrix. In section 3, we will derive a characterization of 3-dimensional general $(\alpha, \beta)$ norms (Theorem 3.6). We will show that the indicatrix of a 3-dimensional general $(\alpha, \beta)$ norm is an affine surface of revolution (Proposition 3.3) and vice versa. Based on Su’s Theorem 3.4, we will then find the characterization by studying the Darboux curves of the indicatrix. In section 4, we will give new characterizations of Randers norms by proving a maximum property of Randers norms (see Theorem 4.2), and also some integral inequalities on the indicatrix (see Theorem 4.4-4.5). All the characterizations given in section 4 are global quantities of the indicatrix.

For convenience, we will mainly deal with in this paper an $(n + 1)$-dimensional Minkowski space $(V^{n+1}, F)$ equipped with its Busemann-Hausdorff volume form, since the results we obtained can be generalized to $(V^{n+1}, F)$ equipped with other kinds of volume forms with no essential differences.

2 Preliminaries

2.1 The Minkowski norm

Let $V^{n+1}$ be an $(n + 1)$-dimensional vector space and $F$ a Minkowski norm on it. Denote by $M^n$ the indicatrix of $F$, i.e.

$$M^n := \{ y \in V^{n+1} \mid F(y) = 1 \} .$$  \hspace{1cm} (2.1)

It is well known that $M^n$ is an $n$-dimensional strictly convex hypersurface enclosing the origin $O$ (cf. [6], [9]). In a chosen coordinate of $V^{n+1}$, $M^n$ is uniquely determined by $F$ and vice versa (cf. [4], [7]). Since $M^n$ is diffeomorphic to the standard unit sphere $S^n(1)$, we can parameterize $M^n$ by

$$r : S^n(1) \to M^n \subset V^{n+1}$$

$$r = r(\theta^1, ..., \theta^n)$$

$$= (r^1, ..., r^{n+1}) ,$$  \hspace{1cm} (2.2)

where $\theta^1, ..., \theta^n$ are the $n$ angles of spherical coordinate system in $\mathbb{R}^{n+1}$. It follows that

$$r^A = r^A(\theta^1, ..., \theta^n) .$$  \hspace{1cm} (2.3)
Also, the unit Finsler ball with respect to $F$ is

$$B_F(1) := \{ y \in V^{n+1} \mid F(y) < 1 \},$$  \hspace{1cm} (2.4)

which is a strictly convex domain in $\mathbb{R}^{n+1}$ with $\partial B_F(1) = M^n$. Throughout the paper, for $V^{n+1}$, let the capitalized Latin indices $A, B, C...$ run from 1 to $n+1$, and the uncapitalized Latin indices $i, j, k...$ run from 1 to $n$. For an arbitrary fixed affine coordinate $\{y^A\}_{A=1}^{n+1}$ in $V^{n+1}$, the fundamental tensor of $F$ is given by

$$g_{AB}(y) := \frac{1}{2} \frac{\partial^2 F^2}{\partial y^A \partial y^B}(y).$$ \hspace{1cm} (2.5)

Differentiating $F^2(y)$ three times gives the Cartan torsion

$$A_{ABC}(y) := \frac{1}{4} F \frac{\partial^3 F^2}{\partial y^A \partial y^B \partial y^C}(y).$$ \hspace{1cm} (2.6)

The angular form of $F$ is

$$h_{AB}(y) := g_{AB}(y) - \frac{\partial F}{\partial y^A}(y) \frac{\partial F}{\partial y^B}(y).$$ \hspace{1cm} (2.7)

One can check that the restriction of $g(y)$ on $M^n$ is precisely $h(y)$. The Busemann-Hausdorff volume form, is given by

$$dV_{B-H} := \sigma_F dy^1 \wedge \ldots \wedge dy^{n+1},$$ \hspace{1cm} (2.8)

where

$$\sigma_F := \frac{\omega_{n+1}}{Vol_{\mathbb{R}^{n+1}}(B_F(1))}.$$ \hspace{1cm} (2.9)

Here in (2.9), we denote

$$Vol_{\mathbb{R}^{n+1}}(\Omega) := \int_\Omega 1 dy^1 \wedge \ldots \wedge y^{n+1}$$ \hspace{1cm} (2.10)

as the volume of a measurable set $\Omega \subset \mathbb{R}^{n+1}$, and $\omega_{n+1} = \frac{\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+3}{2})}$ is the volume of the standard $(n+1)$-dimensional Euclidean unit ball. The distorsion $\tau$ of $F$ with respect to $dV_{B-H}$ is defined by

$$\tau(y) := \log \frac{\sqrt{\det(g_{AB}(y))}}{\sigma_F}.$$ \hspace{1cm} (2.11)

It is easy to check that the Cartan form

$$I(y) := I_A(y) dy^A := g^{BC}(y) A_{ABC}(y) dy^A$$ \hspace{1cm} (2.12)
equals to $d\tau$, where $(g^{AB}) = (g_{AB})^{-1}$. The Matsumoto torsion defined in [12] and [13] is

$$M_{ABC}(y) := A_{ABC}(y) - \frac{1}{n + 3} (I_A(y)h_{BC}(y) + I_B(y)h_{CA}(y) + I_C(y)h_{AB}(y)).$$

(2.13)

Throughout this paper, by saying $\tilde{F}(y)$ is a Minkowski norm with navigation data $(F, U)$, we mean that $\tilde{F}(y)$ is the solution of the following navigation problem (cf. [14]):

$$F\left(\frac{y}{\tilde{F}(y)} + U\right) = 1,$$

(2.14)

where $U$ is a vector in $V^{n+1}$ satisfies $F(U) < 1$. It is proved that $\tilde{F}$ is uniquely determined by (2.14) and is a Minkowski norm on $V^{n+1}$ whenever $F(U) < 1$ is held (cf. [7], [14]). And $\tilde{M}^n$, the indicatrix of $\tilde{F}$, satisfies

$$\tilde{M}^n = M^n - U$$

(2.15)

as a point set.

2.2 Centro-affine structure of the indicatrix

Throughout the paper, vectors are considered as column vectors. For $n + 1$ vectors $\{v_A\}_{A=1}^{n+1}$ with $v_A = v_A^B \frac{\partial}{\partial y^B}$, we write simply $(v_1, ..., v_{n+1})$ in short of the $(n + 1) \times (n + 1)$ matrix

$$ (v_A^B) = \begin{pmatrix} v_1^1 & ... & v_1^{n+1} \\ ... & ... & ... \\ v_{n+1}^1 & ... & v_{n+1}^{n+1} \end{pmatrix}. $$

(2.16)

Choose $L = -y$ as the transversal field on $M^n$ and take the parametrization (2.2) of $M^n$, one gets the centro-affine immersion of $M^n$ into $V^{n+1}$ ([6], [15], [16]), the centro-affine fundamental form coincides precisely with the angular form:

$$h = h_{ij} d\theta^i \otimes d\theta^j, \quad h_{ij} = h_{AB} \frac{\partial r^A}{\partial \theta^i} \frac{\partial r^B}{\partial \theta^j}. $$

(2.17)

And by definition (cf. [16] §II.1) we have

$$h_{ij} = -\frac{\det \left( \frac{\partial r}{\partial \theta^1}, ..., \frac{\partial r}{\partial \theta^n}, D \frac{\partial r}{\partial \theta^1}, ..., D \frac{\partial r}{\partial \theta^n} \right)}{\det \left( \frac{\partial r}{\partial \theta^1}, ..., \frac{\partial r}{\partial \theta^n}, r \right)}. $$

(2.18)
The coefficients of the centro-affine connection $\nabla^{(c)}$ on $TM^n$ is given by

$$\Gamma^{(c)k}_{ij} = \frac{1}{2} h^{kl} \left( \frac{\partial h_{ij}}{\partial \theta^l} + \frac{\partial h_{il}}{\partial \theta^j} - \frac{\partial h_{ij}}{\partial \theta^l} \right) - \frac{1}{2} h^{kl} A_{ijkl},$$  \hspace{1cm} (2.19)$$

where $A_{ijkl} = A_{ABC} \frac{\partial \theta^A}{\partial r} \frac{\partial \theta^B}{\partial y^1} \frac{\partial \theta^C}{\partial y^i}$ and $(h^{ij}) = (h_{ij})^{-1}$. And the corresponding cubic form $\nabla^{(c)} h$ is just the Cartan torsion $A(\theta) = A_{ijk}(r(\theta)) \frac{d\theta^i \otimes d\theta^j \otimes d\theta^k}{d\theta^k}$, which is fully symmetric in $i, j, k$. For the centro-affine immersion of $M^n$, the shape operator is always $s^{(c)} = \text{Id}$ on $TM^n$. By (2.17) and the fact that $g_{AB} y^A y^B = F^2(y)$, we have for $\forall y (= r(\theta)) \in M^n$

$$\begin{pmatrix} (h_{ij})_{n \times n} & 0 \\ 0 & 1 \end{pmatrix} = \left( \begin{array}{c} \frac{\partial r}{\partial \theta^1}, \ldots, \frac{\partial r}{\partial \theta^n}, r \end{array} \right)^T (g_{AB}) \left( \begin{array}{c} \frac{\partial r}{\partial \theta^1}, \ldots, \frac{\partial r}{\partial \theta^n}, r \end{array} \right).$$  \hspace{1cm} (2.20)$$

### 2.3 The equiaffine structure of the indicatrix

Fix a volume form on the ambient affine space $V^{n+1}$, one can define the corresponding equiaffine structure (or Blaschke structure in other words) of a immersed hypersurface. For basic properties of the equiaffine structure of hypersurfaces, one can refer to Blaschke [3], Su [23] §I.5, Nomizu-Sasaki [16], etc. In the following, we just list some basic equiaffine properties of the indicatrix (cf. [15]).

The Blaschke metric of $M^n$ is given by

$$G = G_{ij} d\theta^i \otimes d\theta^j, \quad G_{ij} = \left[ \frac{\sigma_F^2}{\det(g_{AB})} \right] \frac{1}{n+2} h_{ij}. \hspace{1cm} (2.21)$$

While the affine norm field is given by

$$\xi(y) := \frac{1}{n} \triangle_G y = - \left[ \frac{\sigma_F^2}{\det(g_{AB})} \right] \frac{1}{n+2} \left( y^A + h^{AB} y^B \right) \frac{\partial}{\partial y^A} \hspace{1cm} (2.22)$$

for $\forall y \in M^n$, where $\triangle_G$ is the Beltrami-Laplacian of the Blaschke metric $G$. Also, the cubic form at $y \in M^n$ induced by the affine norm field $\xi$ is

$$C(y) = 2 \left[ \frac{\sigma_F^2}{\det(g_{AB})} \right] \frac{1}{n+2} M(y), \hspace{1cm} (2.23)$$

where $M(y)$ is the Matsumoto torsion defined in (2.13). In the case of $n = 2$, we recall the definition of the Darbous curve (cf. [23], §1.5):
Definition 2.1. A curve \( \gamma \subset M^2 \) is called a Darboux curve if \( \gamma \) is the integral curve of the null direction of \( C \).

The shape operator with respect to the equiaffine structure is

\[
s^j_i = \left( \delta^j_i + \frac{2}{n+2} h^{jk} I_{i;k} - \frac{2n}{(n+2)^2} h^{jk} I_i I_k \right),
\]

(2.24)

where \( I_i = I_A \frac{\partial A}{\partial \theta^i} \) and ";" is the covariant derivative with respect to the Levi-Civita connection of the angular form \( h \). Finally, let \( \{\lambda_1, ..., \lambda_n\} \) be \( n \) eigenvalues of \( s^j_i \), they are the affine principle curvatures of \( M^n \). For \( 1 \leq k \leq n \), we define here

\[
L_k := \frac{k!(n-k)!}{n!} \sum_{i_1 < ... < i_k} \lambda_{i_1}...\lambda_{i_k}
\]

(2.25)

the k-th affine mean curvature of \( M^n \).

3 Characterization of 3-dimensional general \((\alpha, \beta)\) norms

In this section, we will give a characterization of 3-dimensional general \((\alpha, \beta)\) norms by studying the induced norm on a certain 2-dimensional subspace.

In Yu-Zhu\cite{26}, the general \((\alpha, \beta)\) norm is investigated. Roughly speaking, it can be derived from an \((\alpha, \beta)\) norm by solving the navigation problem (2.14). Hence its indicatrix can be derived by shifting the indicatrix of an \((\alpha, \beta)\) norm. In\cite{26}, Yu-Zhu proved that in an \(\alpha\)-orthogonal coordinate, the indicatrix of an \((\alpha, \beta)\) norm is a hypersurface of revolution whose axis passes through the origin. Further, the indicatrix of a general \((\alpha, \beta)\) norm is a hypersurface with its axis possibly does not pass the origin.

Theorem 3.1. (Theorem 2.2 in\cite{26}) Let \( F \) be a Minkowski norm on a vector space \( V \) of dimension \( n \geq 2 \). Then \( F \) is an \((\alpha, \beta)\) norm if and only if \( F \) is \( G \)-invariant, where

\[
G = \{ g \in GL(n, \mathbb{R}) \mid g = diag(A, 1), \quad A \in O(n-1) \}.
\]

Theorem 3.1 is confusing, because of the uncertainty of coordinates of \( V \), as well as the uncertainty of the group action. Yu-Zhu’s proof of the above
theorem further showed that if $F$ is $G$-invariant, then $\alpha$ is expressed by

$$\alpha(y) = \sqrt{\sum_{A=1}^{n} (y^A)^2}$$  \hspace{1cm} (3.26)$$

and $\beta$ by $\beta(y) = by^n$ for some constant $b \in \mathbb{R}$. So their theorem actually only works for $\alpha$-orthogonal coordinates which can not be chosen a priori in the proof of sufficiency. At least, the proof in [26] leads to

**Theorem 3.2.** Let $F$ be a Minkowski norm on a vector space $V$ of dimension $n \geq 2$ with $\{y^i\}_{i=1}^{n}$ a fixed coordinate. Then $F$ is an $(\alpha, \beta)$ norm expressed by

$$F(y) = \alpha \phi \left( \frac{\beta}{\alpha} \right),$$

$$\alpha(y) = \sqrt{\sum_{A=1}^{n} (y^A)^2},$$

$$\beta(y) = by^n, \quad b \in \mathbb{R}$$  \hspace{1cm} (3.27)$$

if and only if $F$ is $G$-invariant, where

$$G = \{ g \in GL(n, \mathbb{R}) \mid g = \text{diag}(A, 1), \quad A \in O(n - 1) \}. $$

In order to give a characterization of the general $(\alpha, \beta)$ norms independent of the choice of coordinates, let’s first check the 3-dimensional case. In this case the indicatrix of the Minkowski is a surface. The following affine description of the indicatrix is crucial in this section:

**Proposition 3.3.** Let $V^3$ be a 3-dimensional Minkowski space and $\bar{F}$ a Minkowski norm on it. Let $\bar{M}^2$ be the indicatrix of $\bar{F}$ in $V^3$, then $\bar{F}$ is a general $(\alpha, \beta)$ norm if and only if $\bar{M}^2$ is an affine surface of revolution.

**Proof.** ($\implies$) By definition, for some suitable vector $v \in V^3$, $\bar{M}^2 + v$ is the indicatrix of some $(\alpha, \beta)$ norm. Now it is obviously seen by Theorem 3.2 that $\bar{M}^2 + v$ is an affine surface of revolution, hence for $\bar{M}^2$.

($\impliedby$) Let $\{\bar{y}^A\}$ be the original coordinate of $V^3$. It was proved in §IV.2 of Su [23] that there are only three types of affine surfaces of revolution, namely parabolic (Type (I)), elliptic (Type (II)) and hyperbolic (Type (III)). As the indicatrix $\bar{M}^2$ is a compact surface, it must be of Type (II), i.e. one branch of the affine curvature curves are parallel curves and they are ellipses. These ellipses also coincide with one branch of Darboux curves. Moreover,
in [23](pp.120-122), it is proved that for any affine surface of revolution of elliptic type, one can choose an affine coordinate \{y^A\} (which probably changes the origin of \(V^3\)) such that the surface \(\bar{M}^2\) is given by

\[
\begin{align*}
{y^1}(\theta^1, \theta^2) &= \theta^2 \\
{y^2}(\theta^1, \theta^2) &= \frac{a_2}{a_1} \exp \left( - \int \left( \psi(\theta^2) - \theta^2 \right)^{-1} d\theta^2 \right) \cos(\kappa \theta^1) \\
{y^3}(\theta^1, \theta^2) &= \frac{a_3}{a_1} \exp \left( - \int \left( \psi(\theta^2) - \theta^2 \right)^{-1} d\theta^2 \right) \sin(\kappa \theta^1)
\end{align*}
\]

(3.28)

where \(a_1, a_2, a_3\) and \(\kappa > 0\) are constants and \(\psi : \mathbb{R} \rightarrow \mathbb{R}\) is a function. By a translating along the \(y^A_1\)-direction, one can assume that \(O^*\), the origin of the coordinate system \(\{y^A\}_{A=1}^3\), is enclosed by \(\bar{M}^n\).

Let

\[
\tilde{y}^1 = y^1, \quad \tilde{y}^2 = \frac{y^2}{a_2}, \quad \tilde{y}^3 = \frac{y^3}{a_3},
\]

(3.29)

now Theorem 3.2 applies in the affine coordinate \(\{\tilde{y}^A\}\). Choose \(\{\tilde{y}^A\}\) as the coordinate of \(V^3\), and let \(\tilde{F}(\tilde{y})\) be the Minkowski norm on \(V^3\) whose indicatrix is \(\bar{M}^2\), then Theorem 3.2 implies \(\tilde{F}(\tilde{y})\) is an \((\alpha, \beta)\) norm. Let the origin of the coordinate \(\{\tilde{y}^A\}\) be \(\tilde{O}\) and that of \(\{y^A\}\) be \(O\), then \(F\) is a Minkowski norm with navigation data \((\tilde{F}, \overrightarrow{OO})\). Hence \(F\) is a general \((\alpha, \beta)\) norm.

Su had showed in his pioneering works that

**Theorem 3.4.** (Theorem 22 of [20], Theorem 36 of [21]) One branch of the Darboux curves of a surface lies on parallel planes if and only if the surface is an affine surface of revolution or an affine sphere of Type (I), (II) or (III) defined in [21](also, see [22], pp.120-122).

Now we are going to character 3-dimensional general \((\alpha, \beta)\) norms by using Theorem 3.4 as mentioned in Proposition 3.3. Type (I) and (III) are excluded in our case. First, let’s find out that how can a plane curve on \(\bar{M}^2\) be its Darboux curve. Suppose \(W^2\) is a 2-dimensional subspace of \(V^3\), then \(\tilde{F}\) induces a Minkowski norm \(F\) on \(W^2\). From now on, denote objects with respect to \(\tilde{F}\) by adding a bar, and corresponding objects of \(F\) without it. Without loss of generality, one can assume that

\[
W^2 = \text{Span} \left\{ \frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2} \right\}
\]

\[
V^3 = \text{Span} \left\{ \frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}, \frac{\partial}{\partial y^3} \right\}
\]

(3.30)
By definition, the indicatrix of $F$, denoted by $M$, is
\[ M^1 = M^2 \cap W^2. \]  
(3.31)
which is a strongly convex closed curve lies in the 2-plane $V$. The fundamental tensor of $F$ is
\[ g_{ij} = \frac{1}{2} \partial_y^2 F = \frac{1}{2} \partial_y^2 \tilde{F} = \tilde{g}_{ij}, \]  
(3.32)
the angular form of $F$ is
\[ h_{ij} = g_{ij} - \frac{\partial F}{\partial y^i} \frac{\partial F}{\partial y^j} = \tilde{h}_{ij}, \]  
(3.33)
and the Cartan tensor of $F$ is
\[ A_{ijk} = F \frac{1}{2} \partial_y g_{ij} = \tilde{F} \frac{1}{2} \partial_y \tilde{g}_{ij} = \tilde{A}_{ijk}, \]  
(3.34)
where $1 \leq i, j, k \leq 2$.

Nearby $M^1$, one can choose a local coordinate $(\theta^1, \theta^2)$ of $\tilde{M}^2$ such that $M^1$ is represented by $\theta^2 = 0$. This can be done since (i) $M^1 \subset W^2$ is diffeomorphic to the standard circle $S^1$, so it can be parameterized as
\[ \gamma : S^1 (\simeq [0,1]/\{0,1\}) \to W^2 (\subset V^3) \]  
(3.35)
\[ \begin{cases}
  \gamma^1(\theta^1) = \rho(\theta^1) \cos \theta^1 \\
  \gamma^2(\theta^1) = \rho(\theta^1) \sin \theta^1 \\
  \gamma^3(\theta^1) = 0
\end{cases} \]  
(3.36)
where $\rho(\theta^1)$ is a $C^\infty$ function on $S^1$; and (ii) $\tilde{M}^2$ is transverse to $W^2$, so one can take $\theta^2 = y^3$ nearby $M^1$.

Along $M^1$, the Matsumoto torsion of $\tilde{M}^2$ is
\[ \mathbf{M} (\gamma(\theta^1)) \left( \frac{\partial \gamma}{\partial \theta^1}, \frac{\partial \gamma}{\partial \theta^1}, \frac{\partial \gamma}{\partial \theta^1} \right) \]
\[ = \mathbf{A} (\gamma(\theta^1)) \left( \frac{\partial \gamma}{\partial \theta^1}, \frac{\partial \gamma}{\partial \theta^1}, \frac{\partial \gamma}{\partial \theta^1} \right) - \frac{3}{4} \mathbf{I} (\gamma(\theta^1)) \left( \frac{\partial \gamma}{\partial \theta^1} \right) \tilde{h} (\gamma(\theta^1)) \left( \frac{\partial \gamma}{\partial \theta^1}, \frac{\partial \gamma}{\partial \theta^1} \right) \]
\[ = \mathbf{A} (\gamma(\theta^1)) \left( \frac{\partial \gamma}{\partial \theta^1}, \frac{\partial \gamma}{\partial \theta^1}, \frac{\partial \gamma}{\partial \theta^1} \right) - \frac{3}{4} \mathbf{I} (\gamma(\theta^1)) \left( \frac{\partial \gamma}{\partial \theta^1} \right) h (\gamma(\theta^1)) \left( \frac{\partial \gamma}{\partial \theta^1}, \frac{\partial \gamma}{\partial \theta^1} \right) \]
\[ = \mathbf{I} (\gamma(\theta^1)) \left( \frac{\partial \gamma}{\partial \theta^1} \right) h (\gamma(\theta^1)) \left( \frac{\partial \gamma}{\partial \theta^1}, \frac{\partial \gamma}{\partial \theta^1} \right) - \frac{3}{4} \mathbf{I} (\gamma(\theta^1)) \left( \frac{\partial \gamma}{\partial \theta^1} \right) h (\gamma(\theta^1)) \left( \frac{\partial \gamma}{\partial \theta^1}, \frac{\partial \gamma}{\partial \theta^1} \right) \]
\[ = \left[ \mathbf{I} (\gamma(\theta^1)) - \frac{3}{4} \mathbf{I} (\gamma(\theta^1)) \right] \left( \frac{\partial \gamma}{\partial \theta^1} \right) h (\gamma(\theta^1)) \left( \frac{\partial \gamma}{\partial \theta^1}, \frac{\partial \gamma}{\partial \theta^1} \right) \]  
(3.37)
where the second inequality holds by (3.32)-(3.34) and the third comes from the fact that the Matsumoto torsion of a 2-dimensional Minkowski norm always vanishes (cf. [18] §2.2.2).

Recall (2.23), we see that $M^1$ (or equivalently $\gamma(\theta^1)$) is a Darboux curve of $\bar{M}^2$ if and only if $\bar{M}(\gamma(\theta^1))\left(\frac{\partial\gamma}{\partial\theta^1}, \frac{\partial\gamma}{\partial\theta^1}, \frac{\partial\gamma}{\partial\theta^1}\right)$ always vanishes, by (3.37), we have

\[ \bar{M}(\gamma(\theta^1))\left(\frac{\partial\gamma}{\partial\theta^1}, \frac{\partial\gamma}{\partial\theta^1}, \frac{\partial\gamma}{\partial\theta^1}\right) \equiv 0 \]

\[ \iff I(\gamma(\theta^1)) - \frac{3}{4}\bar{I}(\gamma(\theta^1)) \equiv 0 \]

\[ \iff \frac{d}{d\theta^1} \log \left[ \frac{\det(g_{ij})^{\frac{1}{4}}/\sigma_F}{\det(\bar{g}_{AB})^{\frac{1}{4}}/\sigma_{\bar{F}}}(\gamma(\theta^1)) \right] \equiv 0 \]

\[ \iff \log \left[ \frac{\det(g_{ij})^{\frac{1}{4}}/\sigma_F}{\det(\bar{g}_{AB})^{\frac{1}{4}}/\sigma_{\bar{F}}}(\gamma(\theta^1)) \right] \equiv \text{constant}. \] (3.38)

Which can be rewritten in terms of distortions as

\[ \frac{1}{3}\tau(\gamma(\theta^1)) - \frac{1}{4}\bar{\tau}(\gamma(\theta^1)) \equiv \text{constant}. \] (3.39)

We summarize the above discussions in the following

**Proposition 3.5.** $M^1$ is the Darboux curve of $\bar{M}^2$ if and only if

\[ T := \frac{1}{3}\tau - \frac{1}{4}\bar{\tau} \] (3.40)

is a constant along $M^1$.

We are going to prove the main theorem of this section, before this, let’s agree with some notations. Suppose $U$ is a vector in $V^3$ and

\[ \bar{F}(U) < 1 \] (3.41)

and let $\bar{F}_U(y)$ be the Minkowski norm with navigation data $(\bar{F}, U)$ (cf. 2.14), then the indicatrix of $\bar{F}_U$, denoted by $\bar{M}^2_U$, equals to $\bar{M}^2 - U$ as point sets.

Denote $F_U$ the norm on $W^2$ induced by $\bar{F}_U$, with its indicatrix denoted by $M^1_U$, then

\[ M^1_U = \bar{M}^2_U \cap W^2 \] (3.42)

which is a strictly convex closed curve on $W^2$.

For each $p \in \bar{M}^2$ (except two points at which the tangent plane of $\bar{M}^2$ is parallel to $W^2$), $\bar{M}^2 \cap \{W^2 + \bar{O}p\}$ is a strictly convex closed curve parallel
to $W^2$. On the other hand, for any planer curve $\Gamma$ on $\bar{M}^2$ parallel to $W^2$, one can choose a vector $U$ such that

$$\Gamma - U = M_U^1,$$

(3.43)

for example, let $\Gamma = \bar{M}^2 \cap \{W^2 + U'\}$ for some vector $U' \in V^3$, take a point $q$ in the domain bounded by $\Gamma$ in $\{W^2 + U'\}$, then we can choose $U = \overrightarrow{Oq}$ since $F(\overrightarrow{Oq}) < 1$.

Let’s state and prove the following:

**Theorem 3.6.** Suppose $\bar{F}$ is a Minkowski norm on a 3-dimensional vector space $V^3$. Then $\bar{F}$ is a general $(\alpha, \beta)$ norm if and only if

(*) There exists a 2-dimensional subspace $W^2$ of $V^3$ such that for any $\bar{F}_U$ determined by (2.14) and (3.41), the norm $F_U$ induced by $\bar{F}_U$ on $W^2$ satisfies:

$$\frac{1}{3} \tau_U - \frac{1}{4} \bar{\tau}_U = \text{constant}$$

(3.44)

on $W^2 \setminus \{O\}$, where $\tau_U$ and $\bar{\tau}_U$ are the distortions with respect to $F_U$ and $\bar{F}_U$.

Furthermore, if $\bar{F}$ is not a Randers norm and the condition (*) is satisfied, then

(i) $W^2$ is defined by $\beta = 0$;

(ii) $F_U$’s are Randers norms.

**Proof.** $(\Longrightarrow)$ Suppose $\bar{F}$ is a general $(\alpha, \beta)$ norm which is not of Randers type, one can choose a coordinate $\{y^A\}_{i=1}^3$ on $V^3$ such that

$$\begin{cases}
\alpha(y) = \sqrt{\sum_{A=1}^3 (y^A)^2} \\
\beta(y) = by^3, b = \text{constant}.
\end{cases}$$

(3.45)

Take $W^2 = \text{Span} \left\{ \frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2} \right\}$, then (*) follows immediately. For (i) and (ii), Since Proposition 3.3 implies that $\bar{M}^2$ is an affine surface of revolution, it sufficient to show that when $\bar{F}$ is not a Randers norm, the axis of $\bar{M}^2$ is uniquely determined. One can check that (see [23], §IV.1 and §IV.2):

(a) The $y^3$-axis is parallel to the axis of $\bar{M}^2$ (denoted by $L$);

(b) All curves on $\bar{M}^2$ parallel to $W^2$ are planer affine lines of curvature;
(c) All the meridian curves of $M^2$ are affine lines of curvature, and each meridian line intersects with $L$ at exactly two points.

If there is another 2-dimensional subspace $W^2 \neq W^2$ satisfies (*), then one can get another branch of planer curvature lines parallel to $W^2$. So one will get at least three different branches affine lines of curvature on $\bar{M}^2$, this forces $\bar{M}^2$ to be an affine sphere, hence $\bar{F}$ is a Randers norm, a contradiction. $(\Leftarrow)$ Suppose that (*) is held, for each $p \in \bar{M}^2$, take $\Gamma_p = \bar{M}^2 \cap \{W^2 + \bar{O}p\}$. By the discussion before this theorem, we can represent $\Gamma_p$ as $M^1_U + U$ for some $U \in V^3$. While (*) and Proposition 3.5 implies that $\Gamma_p - U$ is a Darboux curve of $\bar{M}^2$, then $\Gamma_p$ is a Darboux curve of $\bar{M}^2$ passes through $p$. Now we’ve found one branch of Darboux curves on $\bar{M}^2$ parallel to $W^2$, by Theorem 3.4, $\bar{M}^2$ is an affine surface of revolution, hence $\bar{F}$ is a general $(\alpha, \beta)$ norm by Proposition 3.3. 

4 Some global characterizations of Randers norms

In this section, we will derive some global geometric quantities which characterizes Randers norms of arbitrary dimensions. The ideal is using the affine isoperimetric inequalities to characterize affine hyperspheres.

We define the affine volume of the indicatrix as the following:

**Definition 4.1.** Let $V^{n+1}$ be an $(n+1)$-dimensional vector space and $F$ a Minkowski norm on it. Let $dV_{B-H}$ the Busemann-Hausdorff volume form on $V^{n+1}$. Denote the indicatrix of $F$ by $M^n$. The affine volume of $M^n$ is

$$S(M^n) := \int_{M^n} 1\xi \cdot dV_{B-H}. \quad (4.46)$$

where $\xi$ is the affine norm field on $M^n$ defined in (2.22).

Let’s first take a look at Randers norms. For a Randers norm $F_R$, let $(G, W)$ be its navigation data, i.e. $G(y) = \sqrt{G_{AB}y^Ay^B}$ is a Euclidean norm and $W = w^A \frac{\partial}{\partial y^A}$ is a vector with $\|W\|^2_G = G_{AB}W^AW^B < 1$. Plugging $G$ and $W$ into (2.14) and by a direct computation, we have

$$F_R(y) = \alpha(y) + \beta(y), \quad (4.47)$$
with
\[
\alpha(y) = \sqrt{\left(\frac{G_{AB}}{\lambda} + \frac{G_{AC}G_{BD}W^C W^D}{\lambda^2}\right) y^A y^B},
\]
\[
\beta(y) = \frac{G_{AB} W^A y^B}{\lambda},
\]
\[
\lambda = 1 - \|W\|_G^2.
\]

Thus, we have by [7] §1.1
\[
\det (g_{AB}) = \left(\frac{F_R(y)}{\lambda \alpha(y)}\right)^{n+1} \det (G_{AB}),
\]
\[
I^A(y) = \frac{(n+1)\alpha(y)}{2F_R^2(y)} \left(\lambda W^A - \frac{y^A \beta(y)}{\alpha^2(y)} - \frac{(1-\lambda) y^A}{F_R(y)} + \frac{\beta^2(y) y^A}{\alpha^2(y) F_R(y)}\right),
\]
\[
\sigma_{F_R} = \sqrt{\det (G_{AB})}.
\]

By restricting the above (4.47)-(4.49) on the indicatrix of \(F_R(y)\) (denoted by \(M^n_R\)) and combining (2.5)-(2.12) and (2.22), we get the affine norm of \(M^n_R\)
\[
\xi(y) = - \left( y^A + W^A \right) \frac{\partial}{\partial y^A}.
\]

On the other hand, (2.15) and (3.41) imply that
\[
M^n_R = \{ y \in V^{n+1} \mid G_{AB} y^A y^B < 1 \} - W
\]
as point set. Hence \(M^n_R\) is an affine hypersphere centred at \(-W\) with constant radii 1 in \(V^{n+1}\). It is also easy to check that \(S(M^n_R) = (n+1)\omega_{n+1}\) and \(L_r = 1\) for all \(r = 1, \ldots, n\). The case of \(r = 1\) then implies that if \(L_1\) in Theorem [1.3] is a constant along the indicatrix, then it must be precisely 1.

For an arbitrary Minkowski norm, we have the following:

**Theorem 4.2.**
\[
S(M^n) \leq (n+1)\omega_{n+1}
\]
with the equality holds if and only if \(F\) is a Randers metric.

**Proof.** Choose an inner product \(\langle -, - \rangle\) on \(V^{n+1}\) such that \(\langle \frac{\partial}{\partial y^A}, \frac{\partial}{\partial y^B} \rangle = \delta_{AB}\). Choose \(\{\theta^i\}_{i=1}^n\) as the parameter of \(M^n\), therefore the Blaschke metric on \(M^n\) is
\[
G = G_{ij} d\theta^i \otimes d\theta^j, \quad G_{ij} = \left[\frac{\sigma_{F_R}^2}{\det (g_{AB})}\right]^{\frac{1}{n+2}} h_{ij}.
\]
We obtain the affine volume element of $M^n$ as
\[ \xi_{ij}dV_{B-H} = \sqrt{\det (G_{ij})} d\theta^1 \wedge \cdots \wedge d\theta^n \]
\[ = \frac{\sigma_{F+2}^{n+2}}{\det (g_{AB})} \sqrt{\det (h_{ij})} d\theta^1 \wedge \cdots \wedge d\theta^n. \] (4.54)

By (2.20),
\[ \det (h_{ij}) = \det (g_{AB}) \left[ \det \left( \frac{\partial r}{\partial \theta} \bigg| r \right) \right]^2, \] (4.55)
where
\[ \left( \frac{\partial r}{\partial \theta} \bigg| r \right) = \left( \frac{\partial r}{\partial \theta^1}, \ldots, \frac{\partial r}{\partial \theta^n}, r \right). \] (4.56)
so we have
\[ \sqrt{\det (G_{ij})} d\theta^1 \wedge \cdots \wedge d\theta^n \]
\[ = \frac{\sigma_{F+2}^{n+2}}{\det (h_{ij})} \left[ \det \left( \frac{\partial r}{\partial \theta} \bigg| r \right) \right]^{\frac{n}{n+2}} d\theta^1 \wedge \cdots \wedge d\theta^n. \] (4.57)

Following Blaschke [3] and Li-Zhao [11], let’s transform the affine volume form of $M^n$ into a "Euclidean" one. Denote $dV_E$ the volume form of $M^n$ induced by the inner product $\langle -,- \rangle$, it is easy to check that
\[ dV_E = \sqrt{\det \left( \frac{\partial r}{\partial \theta} \bigg| \nu \right) \det \left( \frac{\partial r}{\partial \theta} \bigg| \nu \right)} \frac{1}{n} d\theta^1 \wedge \cdots \wedge d\theta^n, \] (4.58)

where
\[ \left( \frac{\partial r}{\partial \theta} \bigg| \nu \right) = \left( \frac{\partial r}{\partial \theta^1}, \ldots, \frac{\partial r}{\partial \theta^n}, \nu \right) \] (4.59)
and $\nu$ is the unit normal field of $M^n$ with respect to $\langle -,- \rangle$. Set
\[ \Pi = \Pi_{ij} d\theta^i \otimes d\theta^j \]
the second fundamental form of $M^n$ with respect to $\langle -,- \rangle$ by
\[ D_{\frac{\partial r}{\partial \theta^i}} \frac{\partial r}{\partial \theta^j} = \Gamma^k_{ij} \frac{\partial r}{\partial \theta^k} - \Pi_{ij} \nu. \] (4.60)

We have by (2.18) and (4.60) that
\[ h_{ij} = \frac{-\det \left( \frac{\partial r}{\partial \theta^m}, \ldots, \frac{\partial r}{\partial \theta^m}, D_{\frac{\partial r}{\partial \theta^i}} \frac{\partial r}{\partial \theta^j} \right)}{\det \left( \frac{\partial r}{\partial \theta^m}, \ldots, \frac{\partial r}{\partial \theta^m}, r \right)} \]
\[ = \frac{\Pi_{ij} \det \left( \frac{\partial r}{\partial \theta^m}, \nu \right)}{\det \left( \frac{\partial r}{\partial \theta^m}, \ldots, \frac{\partial r}{\partial \theta^m}, r \right)} \] (4.61)
Plugging (4.61) into (4.57), we obtain
\[ \sqrt{\det (G_{ij})} d\theta^1 \wedge ... \wedge d\theta^n = \sigma_F^{-\frac{n}{n+2}} \left[ \det (\Pi_{ij}) \det \left( \frac{\partial r}{\partial \theta} \mid \nu \right) \right]^{\frac{1}{n+2}} d\theta^1 \wedge ... \wedge d\theta^n \]
\[ = \sigma_F^{-\frac{n}{n+2}} K^{\frac{1}{n+2}} \det \left( \frac{\partial r}{\partial \theta} \mid \nu \right) d\theta^1 \wedge ... \wedge d\theta^n \]
\[ = \sigma_F^{-\frac{n}{n+2}} K^{\frac{1}{n+2}} dV_E \]
where \( K = \frac{\det(\Pi_{ij})}{\det(\frac{\partial r}{\partial \theta} \mid \nu)^3(\frac{\partial r}{\partial \theta} \mid \nu)} \) is the Gauss-Kronecker curvature of \( M^n \). By (4.61), we have
\[ K = \det(h_{ij}) \left[ \det \left( \frac{\partial r}{\partial \theta}, ..., \frac{\partial r}{\partial \theta}, r \right) \right]^n \left[ \det \left( \frac{\partial r}{\partial \theta} \mid \nu \right) \right]^{n+2}. \] 
We can conclude that \( K > 0 \) everywhere on \( M^n \), since:

(i) The angular form \( (h_{ij}) \) is strictly positive definite;

(ii) The origin \( O \) locates strictly in the interior of the domain enclosed by \( M^n \) and \( \nu \) is the outer norm, hence
\[ \frac{\det \left( \frac{\partial r}{\partial \theta}, ..., \frac{\partial r}{\partial \theta}, r \right)}{\det \left( \frac{\partial r}{\partial \theta} \mid \nu \right)} = (r, \nu) > 0. \] 
Hence \( M^n \) is an ovaloid with respect to the chosen inner product, and the classical theory of ovaloids may then apply to \( M^n \). By the above computations, we obtain
\[ S(M^n) = \sigma_F^{-\frac{n}{n+2}} \int_{M^n} K^{\frac{1}{n+2}} dV_E \] 
To estimate RHS of (4.65), we use the Hölder inequality and the convexity of \( M^n \), precisely we have
\[ \left( \frac{\int_{M^n} K^{\frac{1}{n+2}} dV_E}{\int_{M^n} 1 dV_E} \right)^{n+2} \leq \frac{\int_{M^n} K dV_E}{\int_{M^n} 1 dV_E} = \frac{(n+1)\omega_{n+1}}{\int_{M^n} 1 dV_E} \]
Hence, we have
\[ S(M^n) \leq \sigma_F^{-\frac{n}{n+2}} \left[ (n+1)\omega_{n+1} \left( \int_{M^n} 1 dV_E \right)^{n+1} \right]^{\frac{1}{n+2}} \]
Set a standard Euclidean ball $B$ (with $\Sigma^n$ its boundary) of volume $Vol_{\mathbb{R}^{n+1}}(B) = Vol_{\mathbb{R}^{n+1}}(B_F(1))$, we have the $n$-dimensional volume of $\Sigma^n$ is

$$S_E(\Sigma^n) := \int_{\Sigma^n} 1 dV_E = (n+1)\omega_{n+1}^\frac{1}{n+1} Vol_{\mathbb{R}^{n+1}}^\frac{n}{n+1}(B). \quad (4.68)$$

Due to the convergency theorem of W.Gross (cf. Blaschke [3], §115), for $\forall \epsilon > 0$, by taking sufficiently many suitable Steiner symmetrization of $M^n$, one can construct a new ovaloid $\hat{M}^n$ such that

$$S_E(\hat{M}^n) := \int_{M^n} 1 dV_E < S_E(\Sigma^n) + \epsilon \quad (4.69)$$

and

$$S(M^n) \leq S(\hat{M}^n) \leq \sigma_F^\frac{n}{n+2} \left[(n+1)\omega_{n+1} S_E(\hat{M}^n)^{n+1} \right]^{\frac{1}{n+2}} \leq \sigma_F^\frac{n}{n+2} \left[(n+1)\omega_{n+1} (S_E(\Sigma^n) + \epsilon)^{n+1} \right]^{\frac{1}{n+2}}, \quad (4.70)$$

where we’ve used the analogous inequality of (4.67) for $\hat{M}^n$. Take $\epsilon \to 0$, we have

$$S(M^n) \leq (n+1)\sigma_F^\frac{n}{n+2} \left[\omega_{n+1}^2 Vol_{\mathbb{R}^{n+1}}^n(B_F(1)) \right]^{\frac{1}{n+2}}. \quad (4.71)$$

By definition of the Busemann-Hausdorff volume form, we have

$$\sigma_F Vol_{\mathbb{R}^{n+1}}(B_F(1)) = \omega_{n+1}. \quad (4.72)$$

Plugging the above equation into (4.71), (4.52) is proved.

Suppose $S(M^n) = (n+1)\omega_{n+1}$ for some $M^n$, then the midpoints of parallel chords along any direction must lie on an $n$-dimensional hyperplane, hence $M^n$ must be an ellipsoid (cf. [11] §5.1).

Theorem 4.2 is actually an analog of the classical affine isoperimetric inequalities (see Su [23] §II.5, Blaschke [3] §65, §72, §73, Li-Zhao [11] §5.1, etc) in an arbitrary Minkowski space.

**Remark 4.3.** For an arbitrary volume form $d\bar{V} = \sigma_F dy^1 \wedge ... \wedge dy^{n+1}$ defined on $\mathbb{V}^{n+1}$, denote

$$Vol_{\sigma_F}(\Omega) := \int_{\Omega} 1 d\bar{V} \quad (4.73)$$

the F-volume for a measurable set $\Omega$ with respect to $d\bar{V}$. Then by a similar argument, Theorem 4.2 is still available in the sense that

$$\bar{S}(M^n) \leq (n+1) \left[\omega_{n+1}^2 Vol_{\sigma_F}^n(B_F(1)) \right]^{\frac{1}{n+2}} \quad (4.74)$$

with $S(M^n) := \int_{M^n} \xi_d d\bar{V}$ the corresponding affine area of $M^n$. The equality holds if and only if $F$ is a Randers metric.
The above (4.71) (or (4.74) equivalently) actually leads to the following integral inequalities of affine mean curvatures (Theorem 4.4 and Theorem 4.5 below), which can again give characterizations of Randers norms. Because the proofs are similar to those of Theorem 2.3 and Theorem 2.4 in Chapter 5 of [11], we will just give sketches.

**Theorem 4.4.** For any integers \( k \) and \( k^* \) satisfy \( 0 \leq k < k^* \leq n + 1, k < \frac{n+2}{2} \), we have

\[
\left( \int_{M^n} L_{k^*-1}\xi dV_{\mathcal{B}^{-H}} \right)^{n+2-2k} \left( \int_{M^n} L_{k-1}\xi dV_{\mathcal{B}^{-H}} \right)^{2k^*-n-2} \leq ((n + 1)\omega_{n+1})^{2(k^*-k)}.
\]

(4.75)

The equality holds if and only if \( F \) is a Randers norm. The \( L_i \)'s are defined in (2.25).

**Proof.** Step 1. We first construct a new hypersurface \( \Theta^n \) in \( V^{n+1} \) by

\[
\xi : M^n \to \Theta^n \\
y \to -\xi(y).
\]

(4.76)

As \( M^n \) is strictly convex, \( L_n > 0 \) everywhere on \( M^n \), hence \( \Theta^n \) is diffeomorphic to \( M^n \) and is an ovaloid. Denote \( \Xi \) the convex domain enclosed by \( \Theta^n \). Let \( M^n_t \) be a series of hypersurfaces defined by

\[
r_t : M^n \to M^n_t \\
y \to y - t\xi(y),
\]

(4.77)

and \( \Omega_t \) the domain enclosed by \( M^n_t \). Note that \( \Omega_0 = B_F(1) \). Denote the mixed volume (cf. [5], pp.136-138) by

\[
V_k := Vol_\sigma_F \left( \underbrace{B_F(1), ..., B_F(1)}_{(n+1-k)-times}, \underbrace{\Xi, ..., \Xi}_{k-times} \right).
\]

(4.78)

**Step 2.** It can be showed that \( M^n_t \)'s are all ovaloids, hence \( \Omega_t \)'s are convex bodies. By \( \partial \Omega_t = M^n_t \), we have

\[
Vol_\sigma_F(\Omega_t) = \frac{1}{(n+1)!} \int_{S^n(1)} \det \left( \frac{\partial r_t}{\partial \theta^1}, ..., \frac{\partial r_t}{\partial \theta^n}, r_t \right) d\theta^1 \wedge ... \wedge d\theta^n,
\]

(4.79)

where

\[
r_t(\theta) = r(\theta) - t\xi(r(\theta)).
\]

(4.80)
On the other hand, since $\Omega_t$’s are convex bodies, we have

$$\Omega_t = \Omega_0 + t\Xi,$$  \hspace{1cm} (4.81)

hence

$$Vol_{\sigma_{F}}(\Omega_t) = \sum_{k=0}^{n} \frac{(n+1)!}{k!(n+1-k)!} V_k.$$ \hspace{1cm} (4.82)

Comparing the coefficients of $t^k$ and combining (2.24) (2.25), we have

$$V_{k+1} = \frac{1}{n+1} \int_{M^n} L_k \xi \cdot dV_{B-H} \hspace{0.5cm} k \geq 1$$
$$V_1 = \frac{1}{n+1} S(M^n), \quad V_0 = Vol_{\sigma_{F}}(B_F(1)), \quad \text{(4.83)}$$

where we’ve used that

$$\frac{\partial \xi}{\partial \theta^i} = -s^{ij} \frac{\partial r}{\partial \theta^j}$$ \hspace{1cm} (4.84)

in computing the first equality of (4.83).

**Step 3.** The Alexanderov-Fenchel inequality (cf. [5], pp.143) yields that

$$V_k^2 \geq V_{k-1}V_{k+1}, \quad 1 \leq k \leq n$$ \hspace{1cm} (4.85)

and recall (4.71) that

$$V_1^{n+2} \leq \omega_{n+1}^2 V_0^n.$$ \hspace{1cm} (4.86)

Iterating (4.85) and (4.86) yields

$$V_k^{2k-n-2}V_{n+2-2k} \leq \omega_{n+1}^{2(k^*-k)}$$ \hspace{1cm} (4.87)

and combining (4.83), (4.75) is proved. While the equality holds in (4.75) if and only if the equalities hold in (4.85) and (4.86), hence in (4.71) as well, which implies that $M^n$ is an ellipsoid, and hence $F$ is a Randers norm.

The final characterization is given by integral of $\sqrt{L_n}$:

**Theorem 4.5.**

$$\int_{M^n} \sqrt{L_n} \xi \cdot dV_{B-H} \leq (n+1)\omega_{n+1}.$$ \hspace{1cm} (4.88)

where

$$L_n = \det \left((\delta^i_j + \frac{2}{n+2} h^{jki}- \frac{2n}{(n+2)^2} h^{jki} I_k, I_k) \right),$$ \hspace{1cm} (4.89)

as defined in (2.25). The equality holds if and only if $F$ is a Randers metric.
Proof. As $L_n > 0$ everywhere on $M^n$, (4.88) is a consequence of the Hölder inequality and (4.87). The proof is about the same as that of Theorem 2.4 of [11], Chapter 5, hence omitted.

Remark 4.6. Note that by Remark 4.3, Theorem 4.4 and Theorem 4.5 hold independently with the choice of $\sigma_F$, hence available for any volume form on $V^{n+1}$.

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