We study the influence of a dissipation process on diffusion dynamics triggered by slow fluctuations. We study both strong- and weak-friction regime. When the latter regime applies, the system is attracted by the basin of either Gaussian or Lévy statistics according to whether the fluctuation correlation function is integrable or not. We analyze with a numerical calculation the border between the two basins of attraction.

I. INTRODUCTION

There is a growing interest in literature on the physical manifestation of Lévy diffusion. This interesting subject can be dealt with in two distinct ways. The first rests on the assumption that there exist in nature Lévy fluctuations, namely stochastic processes of the Lévy stable form. We refer to this approach as stochastic. There are many examples of this attitude, and we limit ourselves to quoting a sample of the early work based on this view. One of the most recent examples of this kind of approach is given by the work of Ref. 3.

The second way of dealing with Lévy fluctuations is based on the assumption that these processes have a Hamiltonian foundation. We refer to this kind of approach as dynamic. An outstanding champion of this view is Zaslavsky and the interested reader is referred to his recent book for a transparent illustration of this perspective. According to Zaslavsky’s analysis, the Hamiltonian dynamics yielding Lévy diffusion is characterized by specific weak-chaos properties. For calculation purposes these properties can be mimicked by either maps of the same kind as those used to study intermittency or a suitable non-linear transformation of a random noise generator. Therefore we shall refer ourselves to all these papers as examples of a dynamic approach to Lévy diffusion not to speak of those papers explicitly using Hamiltonian systems as diffusion generators.

As far as the free diffusion is concerned, the dynamic and stochastic approach yield almost indistinguishable results. The main purpose of this letter is to show that when the interesting case of a perturbation is considered, the dynamic approach and the stochastic approach can lead to totally different predictions. A reconciliation of the two approaches can only be obtained in the limiting case of external perturbations of extremely weak intensity.

II. STOCHASTIC APPROACH

We consider a free motion of a point particle subject to a Markovian time interval $dt$ there is a momentum change per unit of mass given by:

$$dv(t) = d\eta(t),$$

where $v(t)$ is the velocity of the particle. In the treatment of this letter the forcing term will be either a Wiener or a Lévy process. This means that the probability density for the fluctuation $\eta(t)$ to make a jump by the intensity $\eta$ in the time interval $t$, $P(\eta, t)$, defined through the inverse Fourier transform of its characteristic function, reads:

$$P(\eta, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ik\eta} e^{-\frac{1}{2}\sigma^2 k^2 t} dk$$

in the case of a Wiener process, and:

$$P(\eta, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ik\eta} e^{-b|k|^\alpha t} dk$$

in the case of a symmetric Lévy process. The Lévy process is characterized by a positive parameter $b$, determining the width of the distribution and by the coefficient $\alpha$ determining the specific Lévy statistics of the system.

As discussed in Ref. 4, the dynamic approach to Lévy processes has to be restricted to the range: $1 < \alpha < 2$, while, as well known, the full range of Lévy processes is: $0 < \alpha < 2$. Thus, to make a comparison between the stochastic and the dynamic approach we are forced to restrain our analysis to the restricted interval of $\alpha$ values: $1 < \alpha < 2$. In this interval we do not have available any known analytical expression for the density distribution $P(\eta, t)$. However, we know that the distribution is bell-shaped and that in the asymptotic limit of large $\eta$ is characterized by inverse power law tails, proportional to $t/|\eta|^{(1+\alpha)}$. As far as the width...
of the distribution is concerned, it cannot be defined by
the variance, which is infinite. In a loose sense the width
of the distribution is determined by the parameter $b$. It is
evident that Eq. (3) makes the distribution $P(v, t)$ iden-
tical to $P(\eta, t)$.

Let us consider the perturbation of the free diffusion
generated by Eq. (3) by means of:

$$dv(t) = K(v) dt + d\eta(t) ,$$  \hspace{1cm} (4)

where $K(v)$ is a generic perturbation. For simplicity
we restrict our analysis to the case of linear damping,
thereby replacing $K(v)$ in Eq. (4) with $-\lambda v(t)$:

$$dv(t) = -\lambda v(t) dt + d\eta(t) .$$  \hspace{1cm} (5)

When the fluctuation $d\eta(t)$ is the Wiener process of Eq.
(3), Eq. (4) becomes the ordinary Ornstein-Uhlenbeck
process, which yields the Fokker-Planck equation and with it the equilibrium Gaussian density, whose Fourier
transform is $e^{-\frac{\alpha^2}{2}k^2}$, with variance:

$$\sigma^2_{\lambda} = \frac{\sigma^2}{2\lambda} .$$  \hspace{1cm} (6)

If $\eta(t)$ is the Lévy process of Eq. (3), the Fourier transform of the equilibrium distribution is $e^{-b_k|k|^\alpha}$, with the same
exponent $\alpha$ as that of free diffusion. The parameter $b_\lambda$
(see [85]) is defined by:

$$b_\lambda = \frac{b}{\alpha \lambda} .$$  \hspace{1cm} (7)

We note that $b_\lambda$ and $\sigma^2_{\lambda}$ share the same structure. In both
cases the effect of the friction term is that of quenching
the free diffusion so as to generate a time inde-
pendent, or equilibrium, distribution. This means that
$\lim_{t \to \infty} P(v, t) = P(v)$.

**III. DYNAMIC APPROACH**

Both the Wiener and Lévy noise illustrated in Section
II are mathematical abstractions with a limit of validity.
The discrepancy between this mathematical abstraction
and physical reality can become significant as a result of
an external perturbation forcing the dynamics of the
system to produce physical effects stemming from the
time scale where the mathematical idealization departs
from physical reality. Here we focus our attention on the
case where the physical reality seems to be properly
described by the dichotomous fluctuation $\xi$ used in earlier
work for a dynamic derivation of Lévy processes [4].

We assign to this dichotomous variable the values $W$ and $-W$. According to Ref. [14] the statistics of the dif-
fusion process generated by this variable are determined
by both its dichotomous nature and its correlation func-
tion $\Phi_\xi(t)$. We assign to this correlation function the form:

$$\Phi_\xi(t) = \frac{(\beta T)^\beta}{(\beta T + t)^{\beta/2}} , \beta > 0 .$$  \hspace{1cm} (8)

According to the theoretical analysis of Ref. [14], the
dichotomous nature of this fluctuation makes especially
relevant the physical meaning of the function $\psi_\xi(t)$
defined by:

$$\psi_\xi(t) = T \frac{d^2}{dt^2} \Phi_\xi(t) = \frac{(\beta + 1)(\beta T)^{\beta+1}}{(\beta T + t)^{\beta+2}} .$$  \hspace{1cm} (9)

This function is the distribution of sojourn times in one of
the two equiprobable states of the dichotomous variable
$\xi$. The parameter $T$ denotes the mean waiting sojourn
time.

It is convenient to illustrate some aspects of the free
diffusion process generated by this fluctuation, namely,
the case described by:

$$dv(t) = \xi(t) dt .$$  \hspace{1cm} (10)

A realization of this process of free diffusion is:

$$v(t) = \xi_n (t - t_{n-1}^{(0)}) + \sum_{k=0}^{n-1} \xi_k \tau_k ,$$  \hspace{1cm} (11)

where $t_{n-1}^{(0)} < t < t_{n-1}^{(0)} + \tau_n$ and \( t_{n-1}^{(0)} = \sum_{k=0}^{n-1} \tau_k \).

Here the $\tau_k$'s denote the time durations of the sojourns in
the accelerating states occurring prior to time $t$, and the $\xi_k$'s denote either the value $W$ or the value $-W$ of the
dichotomous fluctuation. The distribution of the sojourn
times is given by Eq. (8). Thus we easily derive the
distribution of the velocity jumps $\Delta v_k = \xi_k \tau_k$, which
turns out to be:

$$\psi(\Delta v) = \frac{(\beta + 1)(\beta T)^{\beta+1}}{2 W (\beta T + |\Delta v|/W)^{\beta+2}} .$$  \hspace{1cm} (12)

This distribution is of central importance to predict the
statistics of the resulting diffusion process. If $\beta > 1$
the second moment of this distribution is finite. Thus we are
allowed to apply the conventional central limit theorem,
thereby recovering a diffusion process, indistinguishable
in the long-time limit from those triggered by the Wiener
fluctuations, with variance per unit of time given by:

$$\sigma^2 = \frac{2 \beta W^2 T}{\beta - 1} .$$  \hspace{1cm} (13)

We say [17] that the diffusion process is attracted by the
basin of Gauss statistics. If $0 < \beta < 1$ the sec-
ond moment of the distribution of Eq. (12) is not finite,
and we are therefore forced to use the Lévy-Khintchine-
Gnedenko theorem [17] (that is, the extension of the cen-
tral limit theorem for random variables with no finite
variance). According to Ref. [17] and to the result of
the Appendix, this condition yields a Lévy process with parameters $b$ and $\alpha$ given by:

$$b = W (\beta TW)^\beta \sin \left(\frac{\pi \beta}{2} \right) \Gamma(1 - \beta)$$

and

$$\alpha = \beta + 1.$$  \hspace{1cm} (15)

In this case we say that the diffusion process is attracted by the basin of Lévy statistics.

We see that in the long-time limit the diffusion generated by $\xi$ is essentially indistinguishable from that generated by the Lévy fluctuation $\eta$. What about the influence of an external perturbation on these two distinct fluctuations? One might be tempted to make the conjecture that both processes result in the same equilibrium distribution. However, for this conjecture to be correct it would be necessary to apply first the generalized central limit theorem and, afterwards, the external perturbation. In general, the result is different from what one would obtain reversing this order, namely, applying the perturbation first, and the time asymptotic analysis next. Thus the naive prediction that both fluctuations produce the same equilibrium distribution, in general is wrong.

To substantiate our view, let us consider the dynamic counterpart of Eq.(3):

$$\frac{d}{dt} v(t) + \lambda v(t) = \xi(t).$$

It is easy to solve Eq.(16). In a single motion event we have:

$$v(t) = \frac{\pm W}{\lambda} (1 - e^{-\lambda(t - t^{(0)}_{n-1})}) + v(t^{(0)}_{n-1}) e^{-\lambda(t - t^{(0)}_{n-1})},$$

$$\hspace{1cm} (17)$$

with $t^{(0)}_{n-1} < t < t^{(0)}_{n-1} + \tau_n$ and $t^{(0)}_{n-1} = \sum_{k=0}^{n-1} \tau_k$.

Here $v(t)$ is forced to remain in the interval $[-W/\lambda, +W/\lambda]$. The computer simulation of this letter is based, as done in Ref. 19, on generating, by means of a suitable non-linear deformation of the random numbers with a uniform distribution in the interval $[0,1]$, a sequence of sojourn times $\tau_k$ with the density of Eq.(3), and with another random number generator, equivalent to tossing a coin, a random sequence of velocity signs. For each trajectory realization we set the initial condition $v(0) = 0$ and the trajectory is observed till to a fixed stop time, and subsequently recorded in a bin. Each trial was repeated $10^4$ times. In all the numerical calculations behind the results illustrated in the enclosed figures, we assume that equilibrium is reached after a time $> 20/\lambda$. The following subsections are devoted to illustrating the results obtained in three distinct physical conditions. Note that to illustrate the equilibrium distribution shape corresponding to different values of $\lambda$ we use a variable $z$, which is the rescaled velocity obtained by multiplying the original velocity $v$ by the factor $W/\lambda$. Thus the rescaled velocity ranges from $z = -1$ to $z = 1$.

A. Large perturbation

In the strong damping case, say when $\lambda T \beta > 1$, for any of the two accelerating states equilibrium is reached before the state comes to its end. This means that the preferred velocity values will be either $W/\lambda$ or $-W/\lambda$. This also means that the equilibrium distribution is expected to be $\cup$-shaped, as confirmed by the curve of Fig. 3 corresponding to $\lambda = 0.05$. This damping-induced $\cup$-shaped distribution is generated by system’s dynamics regardless of whether the corresponding free diffusion is located in the Lévy or the Gauss basin of attraction.

B. Weak perturbation

We have seen that in the case of very large friction the distribution has a distinct $\cup$-shaped form: This means two peaks enclosing an almost empty central region. From Fig. 1 we see also that weakening the perturbation intensity has first the effect of populating this empty central region with a uniform distribution (see the curve corresponding to $\lambda = 5 \cdot 10^{-3}$). A further decrease of the perturbation intensity ($\lambda \leq 4 \cdot 10^{-4}$) makes a bell-shaped Lévy distribution of decreasing width and increasing amplitude appear in this central part of the equilibrium distribution.

Fig. 2 is devoted to illustrating the effect that changing $\beta$ has on the distribution shape, in a case of weak friction, namely, with $\lambda = 10^{-5}$. Note that in the Gauss basin of attraction, with $\beta > 1$, the distributions are Gaussian functions with no algebraic tails, whereas in the Lévy basin of attraction, with $\beta < 1$, the equilibrium distribution is markedly characterized by the emergence of slow tails. The transition from the one to the other condition takes place at $\beta = 1$.

C. Transition from the Lévy to the Gauss basin of attraction

As pointed out in subsection 3 B, in the weak-friction regime, the statistics of the equilibrium distribution are determined by the values of the parameter $\beta$. According to whether system’s dynamics belongs to the Gauss or the Lévy basin of attraction, we are led to adopt different criteria to determine the distribution width using the stochastic analysis, i.e. the variance $\sigma^2_\lambda$ or the parameter $b_\lambda$. In the former case, $\beta > 1$, the variance $\sigma^2_\lambda$ is obtained using Eq.(9) and Eq.(13). This yields:
FIG. 1. Equilibrium probability densities for different values of the friction $\lambda$ at $\beta = 0.6$. The Lévy shape emerges upon friction decrease.

FIG. 2. Equilibrium probability densities for different values of $\beta$ at $\lambda = 10^{-5}$. The distribution change shape from Lévy to Gauss one when $\beta$ crosses the critical value $\beta = 1$. 
FIG. 3. The distribution widths as a function of \( \beta \). The curves are parametrized by the friction \( \lambda \). For \( \beta < 1 \) the ordinates refer to \( b_\lambda \). For \( \beta > 1 \) the ordinates refer to \( \sigma_\lambda^2 \). The lines illustrate the theoretical prediction according to the stochastic analysis. The points are the numerical results.

\[ \mathcal{P}(|z| > 0.8) \]

FIG. 4. The population of the region \( 1 > |z| > 0.8 \) as a function of \( \beta \), for different values of the friction \( \lambda \). In the small friction condition the population quickly drops to zero with \( \beta \) moving from \( \beta < 1 \) to \( \beta > 1 \).
\[ \sigma^2 = \frac{\beta W^2 T}{\lambda(\beta - 1)} , \]  

thereby resulting in a divergence for \( \beta \to 1^+ \). On the r.h.s. of Fig. 3 we establish a comparison between this theoretical prediction and the corresponding numerical result. For \( \beta >> 1 \) we find an excellent agreement between theory and numerical simulation. We note however that the numerical calculations do not produce any divergence at the transition from the Gauss to the Lévy basin of attraction. In the latter case, when \( \beta < 1 \), the expected Lévy equilibrium distribution (see [20]) should be characterized by the value of the parameter \( b_\lambda \) obtained using Eqs. (7), (14), (13). This value is:

\[ b_\lambda = \frac{W(\beta TW)^\beta}{\lambda(\beta + 1)} \sin \left( \frac{\pi}{2}\beta \right) \Gamma(1 - \beta) , \]  

which results again in a divergence at the transition from the Lévy to the Gauss basin of attraction. In the left side of Fig. 3 the comparison between theoretical prediction and numerical calculation shows a good agreement around \( \beta \approx 0.5 \).

Although the numerical calculation does not result in any divergence, we see that decreasing the friction intensity has the effect of improving the agreement between theoretical prediction and numerical finding. The measured values of the parameters \( b_\lambda \) and \( \sigma^2 \) becomes increasingly larger at the transition from one basin of attraction to the other, and the overall behavior becomes increasingly similar to that of a phase transition.

Fig. 3 is devoted to illustrating the peak contributions to the equilibrium distribution. We have evaluated numerically the amount of population for \( 0.8 \leq |z| \leq 1 \). We see that decreasing \( \lambda \) has the effect depleting this region if \( \beta > 1 \), while a significant amount of population is left in this region if \( \beta < 1 \). This numerical result shows that at extremely small friction values, moving from \( \beta < 1 \) to \( \beta > 1 \) has the significant effect of making the peak intensity drop to zero, even if the phase-transition character of the passage from \( \beta < 1 \) to \( \beta > 1 \) is characterized by rare intense fluctuations and, consequently, by a large numerical error.

**IV. CONCLUDING REMARKS**

As its main contribution, this paper sheds light into the difference between the conventional stochastic approach and the dynamic approach. Of some relevance is the numerical method adopted. Rather than using as stochastic generators intermittent maps we have founded our numerical treatment on the random generation of the waiting time distribution, a fact that allowed us to settle numerically problems that would have implied otherwise hard numerical difficulties. Setting apart the case of strong friction where the stochastic and the dynamic method yield strikingly different results, the numerical method used made it possible to settle two much more delicate problems:

(i) We have pointed out the residual differences between the two methods surviving in the extremely weak friction regime.

(ii) We have made accessible to numerical investigation the delicate issue of the transition occurring at \( \beta = 1 \) from the Gaussian \((\beta > 1)\) to the Lévy \((\beta < 1)\) statistics.

It has to be pointed out that this transition region is not yet well understood [20] and further research work along the lines of this letter might serve the interesting purpose of establishing whether in the limiting case of very small friction the equilibrium distribution yields the divergencies predicted by Eq. (18) and Eq. (19) or a finite values, as suggested by the recent theoretical analysis of Ref. [20]. We see from Fig. 3 that the numerical analysis yields finite rather than divergent widths at \( \beta = 1 \), but at the moment it is not yet clear if this is due to \( T > 0 \), as argued in Ref. [20], or to the adoption of not yet sufficiently small values of \( \lambda \).

**APPENDIX:**

In this Appendix we show how to determine the parameter \( b \) defining the width of the diffusion process \( \b_1 \) generated, in the long-time limit, by the dichotomous variable of Eq. (10). To realize this goal we use some fundamental theorems established by Lévy, Kintchine and Gnedenko, whose detailed demonstration can be found in Ref. [17].

The central issue is the assessment of the limit distribution of the normalized sum of the independent and identically distributed random variables \( \zeta_1, \zeta_2, \cdots, \zeta_n \):

\[ \omega_n = \frac{\zeta_1 + \zeta_2 + \cdots + \zeta_n}{B_n} - A_n , \]  

where \( A_n \) and \( B_n \) are suitable normalization constants. To establish this limit condition we rest on the following three properties.

First, we define what a stable distribution is all about. A distribution \( V(x) \) is stable if given the arbitrary real numbers \( a_1 > 0, b_1, a_2 > 0, b_2 \) there exist the numbers \( a_3 > 0 \) and \( b_3 \) such that the equality

\[ V(a_1 x + b_1) \ast V(a_2 x + b_2) = V(a_3 x + b_3) \]  

holds. Here \( \ast \) denotes the convolution product.

Second, we use a theorem by Kintchine and Lévy [17] which establishes that the stable distributions are those, and only those, corresponding to the sum of Eq. (A1) converging to a finite limit as \( n \to \infty \).

Third, using another theorem Kintchine and Lévy [17], we express the stable distribution by means of characteristic functions with the form \( \exp(i \gamma k - c|k|^\alpha) \). The range of \( \alpha \) is: \( 0 < \alpha < 2 \), namely wider than that compatible with the dynamic treatment of this letter that only
focuses on the interval: $1 < \alpha < 2$. Our treatment is restricted to the case of symmetric distribution, thereby setting $\gamma = 0$. The same theorem by Kintchine and Lévy establishes that:

$$c = -\alpha M(\alpha)(c_1 + c_2) \cos \left( \frac{\pi \alpha}{2} \right) \quad c > 0, \quad 1 < \alpha < 2,$$

(A3)

where

$$M(\alpha) = \int_0^\infty (e^{-y} - 1 + y) \frac{dy}{y^{1+\alpha}} = \frac{\Gamma(2 - \alpha)}{\alpha(\alpha - 1)}$$

(A4)

and $c_1$ and $c_2$ are two constants establishing the distribution asymptotic properties.

Let us consider the distribution function $F(x)$ of the random variable $\zeta$. According to a theorem established by Gnedenko [17], the distribution $F(x)$ converges to a stable function $\mathcal{V}(x)$ in the sense of Eq. (A2) if, and only if, the following relations hold. Note that the function $q_1(x)$ and $q_2(x)$ vanish for $x \to -\infty$ and $x \to \infty$, respectively. The constant $a$ depends on the choice of $B_n$ appearing in Eq. (A4). We make the choice $B_n = n^{1/\alpha}$ yielding $a = 1$. Thus in the the asymptotic limit $|x| \to \infty$ the distribution $F(x)$ only depends on $c_1$ and $c_2$. Furthermore, the earlier choice of a symmetric distribution yields $c_1 = c_2$.

Note that to realize our purposes we have to adopt the non-normalized form:

$$\omega' = \Delta v_1 + \Delta v_2 + \cdots + \Delta v_n.$$

(A7)

It is evident that this non-normalized form can be related to the normalized form of Eq. (A3) by setting $\Delta v_k = \zeta_k n^{-1/\alpha}$ and $A_n = 0$. This is equivalent to replacing the parameter $c_1$, responsible as earlier shown, for the asymptotic distribution properties, with $c_1 n$. Claiming no rigour, we set $n = t/T$, where $T$ is the mean value of the interval between two realizations of $\Delta v_k$, so $n$ is the mean number of realizations of the random variable $\Delta v$. Thus we obtain for the stochastic Lévy process $\exp(-bt|k|^\alpha)$:

$$b = c/T.$$

(A8)

By identifying $F(x)$ with the function distribution of $\zeta$, we obtain:

$$F(x) = 1 - \frac{1}{2} \frac{(\beta T)^{\beta+1}}{(\beta T + x/W)^{\beta+1}} \quad \text{if } x > 0.$$  

(A9)

Finally, by applying Eq. (A6) to Eq. (A9) and using Eqs. (A3) and (A8), we obtain the result of Eq. (14).

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1 Due to a misprint in Eq.(11) of chap.7 §34 of Ref. [17], the factor $\alpha$ of Eq. (A3) is missing.