GAUGE INVARIANT SURFACE HOLOMONY AND MONOPOLES

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Abstract. There are few known computable examples of non-abelian surface holonomy. In this paper, we give several examples whose structure 2-groups are covering 2-groups and show that the surface holonomies can be computed via a simple formula in terms of paths of 1-dimensional holonomies inspired by earlier work of Chan Hong-Mo and Tsou Sheung Tsun on magnetic monopoles [10]. As a consequence of our work and that of Schreiber and Waldorf [18], this formula gives a rigorous meaning to non-abelian magnetic flux for magnetic monopoles. In the process, we discuss gauge covariance of surface holonomies for spheres for any 2-group, therefore generalizing the notion of the reduced group introduced by Schreiber and Waldorf in [20]. Using these ideas, we also prove that magnetic monopoles have an abelian group structure.

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1. Introduction

1.1. Background, motivation, and overview. Ordinary holonomy along paths for principal group bundles has been studied for over 40 years in the context of gauge theories in physics and in the context of fiber bundles in mathematics. Recently, with ideas from higher category theory, it has been possible to extend these ideas to holonomy along surfaces. Although higher holonomy, and more generally higher gauge theory, has been studied in the context of abelian gauge theory for higher-dimensional manifolds, it was thought for some time that non-abelian generalizations were not possible [21]. Today, we understand this as being due to the fact that a group object in the category of groups is an abelian group. However, by “categorifying” well-known concepts, and considering group objects in the category of categories, one can avoid this restriction. Furthermore, for quite some time in the physics literature, it was not known how to define higher holonomy because different ways of taking the integral resulted in different answers. The language of higher categories and functors allows us to give a resolution to this problem.

A more rigorous approach to surface holonomy for abelian structure groups has been known for quite some time under the name abelian gerbes with connection with a formal presentation offered by Gawedzki [12] in 1988 in the context of the WZW model, with further work in 2002 with Reis [13]. Further development under the name of non-abelian gerbes, higher bundles, and so on were carried out in the following years. In [6], Baez and Schreiber give a definition of these generalizations in terms of parallel transport. The most up-to-date theoretical framework in terms of category theory, which provides a language easily adaptable for non-abelian generalizations, was established by Schreiber and Waldorf in [20]. In this categorical setting, higher principal bundles with connections are described by transport functors.

The motivation for transport functors comes from the following observation. In [17], Schreiber and Waldorf prove that a principal group bundle with connection over a smooth manifold determines, and is determined by, a transport functor defined on the thin path groupoid of that manifold with values in a fattened version of the structure group viewed as a one-object category. The upshot of this equivalence is that it is conceptually simple to go from categories and functors to 2-categories and 2-functors. In [18], [19], and [20], Schreiber and Waldorf take advantage of this
equivalence and abstract the definition so that it can be used to define principal 2-group 2-bundles allowing a conceptually simple formulation of surface holonomy.

In the present article, we sketch the notion of transport functors and transport 2-functors given by Schreiber and Waldorf. It is our hope that this less-detailed version will make it slightly easier for readers to follow the full-fledged theory of \cite{17}, \cite{18}, \cite{19}, and \cite{20}. We also give an explicit and pictorial representation for surface holonomy more suitable for computations, in hopes that it will be useful for lattice gauge theory.

The notion of gauge covariance for holonomy along spheres is discussed in detail. We illustrate these concepts by focusing on principal 2-bundles with a structure 2-group given by a Lie group $G$ and a covering space $H$ of $G$. We discuss a very simple formula, motivated by constructions in \cite{10}, for holonomy along surfaces in a local trivialization and show that this formula agrees with the surface-ordered integral in \cite{18}. This gives an interesting relationship between (i) well-known formulas in the physics literature for computing the magnetic flux in terms of a loop of holonomies and (ii) the surface-ordered holonomy in terms of 1- and 2-forms of \cite{18}. In terms of physics, we argue that the latter is the correct analogue to computing the magnetic flux as a surface integral and our formula tells us that this agrees with the usual definition given in the physics literature. This is all done without the introduction of a Higgs field, completing the ideas of \cite{14}.

Then we consider an entire collection of examples of transport 2-functors coming from an ordinary principal $G$-bundle with connection along with a choice of a subgroup $N$ of $\pi_1(G)$, the fundamental group of $G$ (such a choice of subgroup determines a covering of $G$). We show that when the subgroup $N$ is chosen to be $\pi_1(G)$ itself, our example reduces to the curvature 2-functor defined by Schreiber and Waldorf in \cite{20}. On the other extreme, when the subgroup $N$ is chosen to be the trivial group $\{1\}$, the structure Lie 2-groupoid associated to this 2-functor has associated Lie crossed module $\tau : \tilde{G} \to G$, the universal cover of $G$. Such 2-groups are called covering 2-groups.

It is this case we consider for our computations of magnetic charges giving several explicit calculations of surface holonomy in terms of group elements. But just as ordinary holonomy is not exactly group-valued (due to conjugation issues), surface holonomy isn’t in general either. However, we prove that for spheres, surface holonomies are completely gauge invariant once we pass to a certain quotient analogous to conjugacy classes.

1.2. Outline of paper along with main results. The outline of the paper is as follows.

In Section \cite{2} we review the main definitions of transport functors along with an equivalence between local descent data and global transport functors. Although some earlier work such as \cite{1} has been done in describing principal bundles with connection via their holonomies, we follow the recent work of Schreiber and Waldorf \cite{17} who describe it precisely and categorically in a framework that is suitable for higher generalizations. We briefly discuss the relationship to principal $G$-bundles with connection, where $G$ is a Lie group, in their usual formulation by introducing the category of $G$-torsors (manifolds with free and transitive right $G$-actions). The equivalence between the two descriptions was proven in \cite{17}. We also review the relationship between local descent data and differential cocycle data for principal
group bundles, recalling the well-known formula for parallel transport in terms of a path-ordered integral. Our presentation on obtaining group-valued holonomies in Section 2.8 differs only slightly from that of [17], but we feel that it is a very natural description given the language that Schreiber and Waldorf present.

In Section 3, we review how to ‘categorify’ the definitions and statements of Section 2 in order to discuss transport 2-functors. The main references for this section include [18], [19], and [20]. We only briefly review the technical points but spend more time on a computational understanding of higher holonomy and also supplying an iterated integral expression for higher holonomy including a picture (Figure 14) that we think will be useful for lattice gauge theory. Our treatment of obtaining group-valued higher holonomy again slightly differs from [20] in presentation and also in the sense that we restrict ourselves to holonomy along spheres (as opposed to any genus g surface). This lets us discuss gauge covariance and gauge invariance simply and in full detail without referring to the ‘reduced group’ of [20]. We show, in Theorem 3.29, that our higher holonomy lands in a set that surjects into the reduced group and give a simple example, in Lemma 3.33, where this surjection has nontrivial kernel.

In Section 4, we consider transport 2-functors with structure 2-group given by a covering 2-group. We give a new and simple formula for surface holonomy in a local trivialization in terms of homotopy classes of paths of ordinary holonomies motivated by constructions in [10] for computing magnetic charge as a topological number. In Definition 4.10, we give our main construction of a transport 2-functor associated to every principal G-bundle with connection and to any subgroup of π₁(G). We prove that this assignment is functorial. Furthermore, this example of a transport 2-functor is shown to reduce to the example of Schreiber and Waldorf known as the curvature 2-functor in [20] when the subgroup of π₁(G) is chosen to be π₁(G) itself. We describe this construction on four levels: (i) global transport functors (ii) functors with smooth trivialization data chosen (iii) descent data (iv) differential cocycle data. This allows one to work with either construction at whatever level he or she pleases. We then summarize our result as a list of commutative diagrams of functors. Finally, we discuss the relationship between our formula for surface holonomy and the surface-ordered integral formula from [18] and prove that the two formulas in fact coincide in Corollary 4.15.

In Section 5, we consider special cases of covering 2-groups and give several examples all of which are known as magnetic monopoles [10]. The first example is to start with any principal U(1)-bundle with connection over a smooth manifold M. It is shown that the surface holonomy along a sphere coming from the path-curvature 2-functor defined in Section 4 is precisely the integral of the curvature form of the principal U(1)-bundle along this sphere, which in this case is the integral of the first Chern class over the sphere. This example is precisely the Dirac monopole [8] and the surface holonomy gives the magnetic charge as the integral of a magnetic flux. We then discuss non-abelian examples starting with a principal SO(3)-bundle with connection over the sphere and compute the surface holonomy explicitly using both our simple formula and the formula in terms of path-ordered integrals using differential forms. In the case of a non-trivial bundle, the surface holonomy along the sphere is given by the element −1_{2×2} in SU(2), the universal cover of SO(3), which is the nontrivial element in the kernel of the covering map τ : SU(2) → SO(3). We do this same computation in other examples including SU(n) → SU(n)/Z(n),
where $Z(n)$ is the center of $SU(n)$, and also for the case $SU(n) \times \mathbb{R} \rightarrow U(n)$. This gives a rigorous meaning to the notion of non-abelian magnetic flux as a surface holonomy along a sphere (see Definition 5.2). Furthermore, it is shown that this magnetic flux is a gauge-invariant quantity in Corollary 5.3.

Finally, the appendix includes an overview of smooth/diffeological/Chen spaces which are used to describe several of the constructions involving infinite-dimensional manifolds and smooth maps between them. The appendix also includes some conventions on 2-category theory, 2-groups, and higher-dimensional algebra [5]. We explicitly use higher-dimensional algebra for ‘multiplying’ not only horizontally but also vertically. Our conventions for this multiplication are reviewed in the appendix.

In bullet format, this article contains the following results.

- **Theorem 2.25** allows one to define gauge-invariant holonomy in the language of transport functors by Definition 2.27. The image lands in conjugacy classes instead of the abelianization.
- **Theorem 3.29** accomplishes the analogous result for surface holonomy along spheres in Definition 3.30. The image lands in “α-conjugacy classes” (Definition 3.28) instead of the reduced 2-group of [21]. The set of α-conjugacy classes surjects to the reduced 2-group but is not in general injective as shown in Lemma 3.33. We also prove that the fixed points of this α action form a central subgroup of the group of surface holonomies in Lemma 3.35.
- The rest of the paper discusses particular cases of transport 2-functors with structure 2-groups being covering 2-groups (Definition 4.4) that are called path-curvature 2-functors (Definition 4.7). These transport 2-functors are defined without using surface integrals, and we show, in Theorem 4.14 and Corollary 4.15 that locally, any transport 2-functor (defined as in [18] using surface integrals) with structure 2-group a covering 2-group, coincides with ours, thus enabling a simple formula for calculating surface holonomy.
- **Section 5** includes several examples and explicit computations of surface holonomy. Due to the previously mentioned theorem, these examples can rightfully be called magnetic fluxes of magnetic monopoles from physics. We include several examples of non-abelian surface holonomy. We conclude with Corollary 5.3 that shows that the magnetic flux is a fixed point under the α action and therefore lands in the central subgroup mentioned earlier. In particular, this implies that the magnetic charge is an abelian group-valued quantity known as a topological number.

1.3. Disclaimer about category theory. Category theory has had an enormous influence on mathematics and has been around since about 1945. We understand that the typical physicist, and possibly even mathematician, may not be comfortable using this language. However, we believe that it is fundamental enough to devote some time to learning just a few of the basics. Fortunately, we review the basic ideas in the form of examples in Section 2.1.

There are books even aimed at high school students that teach ordinary category theory effectively such as [11] but also more advanced books such as [15] written by one of the founders of the subject. We do not require the sophistication of the latter reference (or even that of the former). The latter reference includes a short section on higher categories, but is not sufficient for our purposes. 2-categories on
the other hand are even less known, especially weak 2-categories. For a review of 2-categories, we have included a terse review in the appendix but also refer the reader to the appendix of [19]. A good pedestrian’s first exposure to 2-categories along with a friendly introduction to 2-bundles and higher gauge theory is in [4].

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1.5. Lack of acknowledgments. As with any body of work, it is not really feasible to reference everyone who has made a contribution in the field. The references we have chosen to use merely represent the ones that have influenced us greatly. The body of work done on magnetic monopoles is overwhelming and we have limited our sources to only a few. Furthermore, there has also been a lot of work on gerbes, higher bundles, and even their relationship to magnetic monopoles which we have not even mentioned. Our sincerest apologies for the lack of such acknowledgements.

2. Principal bundles with connection are transport functors

In this section, we review the notion of transport functor following [17]. We split up the discussion into several parts. We first discuss a Čech description of principal $G$-bundles (without connection), where $G$ is a Lie group, in terms of smooth functors. Then we attempt a guess for describing principal $G$-bundles with connections in terms of smooth functors. This attempt fails as it only gives trivialized bundles, motivating the need to use transport functors. We then proceed to describing local trivialization data, descent data, and finally transport functors. The key feature of descent data is that it enables us to encode smoothness while still allowing the ‘bundle’ to have nontrivial topology. We then discuss a reconstruction functor that takes us from the category of descent data to the category of transport functors with chosen trivializations. It is here that we introduce a version of the Čech groupoid
incorporating paths and ‘jumps’ that are necessary for transition functions. Then
we move in the other direction and go from smooth descent data to locally defined
differential forms, or more generally differential cocycle data. We also describe how
to go from differential cocycle data to smooth descent data. We then summarize
the four different levels describing transport functors and their relationship to one
another. Finally, we use these relationships to make sense of assigning group ele-
ments to holonomy and discuss its gauge covariance and invariance stressing the
use of conjugacy classes.

2.1. A Čech description of principal G-bundles. Let G be a Lie group. Principal
G-bundles over a manifold M can be described simply in terms of functors.
Furthermore, an equivalence of such bundles corresponds to a natural transfor-
mation of the corresponding functors. This is done as follows (this is an expansion
of [22] [Remark II.13.]).

Definition 2.1. Given an open cover \( \{ U_i \}_{i \in I} \) of M, define the Čech groupoid \( \mathcal{U} \) to
have as a smooth space of objects

\[ \mathcal{U}_0 := \coprod_{i \in I} U_i \]

and a smooth space of morphisms, called ‘jumps,’

\[ \mathcal{U}_1 := \coprod_{i,j \in I} U_{ij}, \]

where \( U_{ij} := U_i \cap U_j \) and the order of the index is kept track of in the disjoint
union. In other words, elements of \( \mathcal{U}_0 \) are written as \( (x, i) \) and elements of \( \mathcal{U}_1 \)
are written as \( (x, i, j) \). Let \( (x, i, j) \in \mathcal{U}_1 \). The source and target maps are given by
\( s((x, i, j)) := (x, i) \) and \( t((x, i, j)) := (x, j) \). The identity-assigning map is given by
\( i((x, i)) := (x, i, i) \). Now consider two morphisms \( (x, i, j) \) and \( (x', i', j') \) and suppose
we wish to define \( (x', i', j') \circ (x, i, j) \). This composition can only be defined when
\( t((x, i, j)) = s((x', i', j')) \), i.e. when \( (x, j) = (x', i') \). There is no condition on \( j' \) so
we rename that index \( k \). In this case the composition can be defined and is given by

\[ (x, j, k) \circ (x, i, j) := (x, i, k). \]

Now, another useful construction assigns to every group or Lie group G a one-
object groupoid \( \mathcal{B} \) as follows. Denote the one object by \( \bullet \). Let the set of morphisms
from \( \bullet \) to itself be given by the set \( G \). If G is a Lie group, we can also give this set
a smooth structure making \( \mathcal{B} \) a 2-space.

Definition 2.2. A 2-space/smooth category is a (small) category whose objects,
morphisms, and sets of composable morphisms all form smooth spaces (see Appendix
6.1). Furthermore, the source, target, identity-assigning, and composition maps are
all smooth. A Lie groupoid is a 2-space together with a smooth inverse map from
morphisms to morphisms that acts as an inverse under composition.

Example 2.3. The Čech groupoid mentioned above is also a 2-space.

\footnote{Our apologies for this double usage of the letter i to mean both the identity-inclusion map
and the index letter. We hope that it is not too confusing. Later, we will also use the letter i for
several other purposes.}
**Definition 2.4.** Let smooth functor from one 2-space to another is an ordinary functor that is smooth on objects and morphisms. Likewise, a smooth natural transformation is a natural transformation whose function from objects to morphisms is smooth.

Any smooth functor \( U \rightarrow BG \) gives the Čech cocycle data of a principal \( G \)-bundle over \( M \) subordinate to the cover \( \{ U_i \}_{i \in I} \). To see this, simply recall what a functor does. To each object \( (x, i) \) in \( U \) it assigns the single object • in \( BG \). To each jump \( (x, i, j) \), it assigns an element denoted by \( g_{ij}(x) \in G \) in such a way that we get a smooth 1-cochain \( g_{ij} : U_{ij} \rightarrow G \)

\[
\text{(4) } \begin{array}{c}
\bullet & \text{j} & \bullet & \text{i} \\
\end{array} \quad \begin{array}{c}
\text{g}_{ij} \\
\end{array}
\]

This picture should be interpreted as follows. We consider \( x \in U_{ij} \) and draw \( (x, i, j) \) as the figure on the left. Its image under \( U \rightarrow BG \) is \( g_{ij}(x) \) drawn on the right (without explicitly writing \( x \)). To each triple intersection \( U_{ijk} \), which corresponds to the composition of \( (x, i, j) \) in \( U_{ij} \) with \( (x, j, k) \) in \( U_{jk} \) as in (3), functoriality gives a cocycle condition

\[
\text{(5) } \begin{array}{c}
\bullet & \text{j} & \bullet & \text{k} & \bullet & \text{i} \\
\end{array} \quad \begin{array}{c}
\text{g}_{jk} & \text{g}_{ij} & \text{g}_{ik} \\
\end{array}
\]

which says

\[
\text{(6) } g_{jk}g_{ij} = g_{ik}.
\]

Although awkward at first, this convention was chosen to match that of [17] and [20] so that the reader who is interested in further details can consult without too much trouble.

We now discuss refinements and morphisms between two such functors. Let \( \{ U'_v \}_{v \in I'} \) be another cover of \( M \) with associated Čech groupoid \( \mathcal{U}' \). Let \( P : \mathcal{U} \rightarrow BG \) and \( P' : \mathcal{U}' \rightarrow BG \) be two smooth functors. A morphism from \( P \) to \( P' \) consists of a common refinement \( \{ V_\alpha \}_{\alpha \in A} \), with associated Čech groupoid \( \mathcal{U} \), of both \( \{ U_i \}_{i \in I} \) and \( \{ U'_v \}_{v \in I'} \).
and \( \{ U'_{\alpha} \}_{\alpha \in I'} \) along with a smooth natural transformation

\[
\begin{array}{c}
\alpha \\
\downarrow \downarrow \\
\Downarrow h \\
\Updownarrow P \\
\Omega \\
\downarrow \downarrow \\
\downarrow \downarrow \\
\End{array}
\]

\[BG.\]

The refinement condition means that there are associated functions \( \alpha : A \rightarrow I \) and \( \alpha' : A \rightarrow I' \) so that \( V_a \subset U_{\alpha(a)} \) and \( V_a \subset U'_{\alpha'(a)} \) for all \( a \in A \). These functions determine the functors \( \alpha : \Omega \rightarrow \Omega \) and \( \alpha' : \Omega \rightarrow \Omega' \) drawn above. We denote the restrictions of \( g_{\alpha(a)\alpha'(b)} \) and \( g'_{\alpha'(a)\alpha'(b)} \) to \( V_{ab} \) by \( g_{ab} \) and \( g'_{ab} \) respectively. Any such smooth natural transformation gives an equivalence of Čech cocycle data of principle \( G \)-bundles. To see this, simply recall what a natural transformation does.

To each object \( (x, a) \) in \( \Omega \) it assigns a group element \( h_a(x) \in G \) in a smooth way. In other words, it gives a smooth function \( h_a : V_a \rightarrow G \). To each jump \( (x, a, b) \) in \( \Omega \) the naturality condition

\[
\begin{array}{c}
\gamma_a \\
\downarrow \downarrow \\
\downarrow \downarrow \\
\End{array}
\]

say that

\[
h_{ab} g_{ab} = g'_{ab} h_a
\]

on \( V_{ab} \). This is precisely the condition that says the principal \( G \)-bundles \( P \) and \( P' \) are equivalent \[16\].

2.2. A naive guess for transport functors. Our goal in this section is to guess what a connection on a principal \( G \)-bundle over \( M \) should be in terms of functors. We will fail at this attempt, but will learn an important lesson which will motivate the modern definition in terms of transport functors.

A connection gives an assignment from paths in \( M \) to isomorphisms of fibers between the endpoints. In a principal \( G \)-bundle \( P \rightarrow M \), every fiber is a right \( G \)-torsor.

**Definition 2.5.** Let \( G \) be a Lie group. Let \( G \)-Tor be the category whose objects are right \( G \)-torsors, i.e. manifolds \( X \) equipped with a free and transitive right \( G \)-action, and whose morphisms are \( G \)-equivariant maps.

Furthermore, this assignment of an isomorphism of fibers over paths should only depend on the thin homotopy class of a path. Since this concept is important, we recall the definition of the thin path-groupoid of a manifold \( X \).

**Definition 2.6.** [The thin path-groupoid] Let \( X \) be a smooth manifold. Define the smooth category \( P_1(X) \) as follows. Define the set of objects of \( P_1(X) \) to be the points of the smooth manifold \( X \). To define the set of morphisms of \( P_1(X) \), we first consider the set of smooth maps \( \gamma : [0, 1] \rightarrow X \) with the property that there exists
an $\epsilon$ with $\frac{1}{2} > \epsilon > 0$ such that $\gamma(t)$ is constant for all $t \in [0, \epsilon] \cup [1 - \epsilon, 1]$. Such paths are called paths with sitting instances and for such paths $\gamma$ with $\gamma(0) = x$ and $\gamma(1) = y$, we write

$$\gamma : x \to y.$$

We then define an equivalence relation on such paths as follows. Two paths $\gamma$ and $\gamma'$ with the same endpoints, i.e. $\gamma(0) = \gamma'(0) = x$ and $\gamma(1) = \gamma'(1) = y$, are said to be thinly homotopic if there exists a smooth map $\Gamma : [0, 1] \times [0, 1] \to X$ with the following two properties.

(a) First, there exists an $\epsilon$ with $\frac{1}{2} \geq \epsilon > 0$ such that

$$\Gamma(t, s) = \begin{cases} 
  x & \text{for all } (t, s) \in [0, \epsilon] \times [0, 1] \\
  y & \text{for all } (t, s) \in [1 - \epsilon, 1] \times [0, 1] \\
  \gamma(t) & \text{for all } (t, s) \in [0, 1] \times [0, \epsilon] \\
  \gamma'(t) & \text{for all } (t, s) \in [0, 1] \times [1 - \epsilon, 1]
\end{cases}$$

Such a map is called a bigon in $X$ and is typically denoted by

(b) Second, the rank of $\Gamma$ is strictly less than 2, i.e. the differential $D(\Gamma(t,s)) : T_{(t,s)}([0,1] \times [0,1]) \to T_{y}X$, where $T_{y}X$ denotes the tangent space to $X$ at the point $y \in Y$, has kernel of dimension at least one for all $(t, s) \in [0, 1] \times [0, 1]$.

The set of morphisms of $P_{1}(X)$, denoted by $P_{1}X$, is defined to be the set of thin homotopy classes of paths with sitting instances. We sometimes call these thin paths. Such a set has a natural smooth structure in the category of Chen spaces by using the mapping space smooth structure, the subspace smooth structure, and the quotient smooth structure (a very short review of Chen spaces is given in the appendix while a simple review article for Chen spaces and related concepts using sheaves on sites is [3] and for more details regarding the path groupoid see [17]).

The composition in $P_{1}(X)$ is defined to be concatenation of paths in the usual sense via representatives and then taking the thin homotopy class associated to that composition. Namely, given two thin homotopy classes of paths of sitting instances

$$\gamma : x \to y \to z,$$

the composition is given by the class associated to

$$\gamma' \circ \gamma : \gamma (2t) \quad \text{for } 0 \leq t \leq \frac{1}{2}, \quad \gamma'(2t - 1) \quad \text{for } \frac{1}{2} \leq t \leq 1.$$

Under the sitting instance assumption and the thin homotopy equivalence relation, the composition is well-defined, smooth, associative, and has left and right units given by constant paths. $P_{1}(X)$ is a Lie groupoid since thin homotopy classes of paths are invertible and the function that assigns every class to its inverse is smooth.
With this definition of the thin path-groupoid of $M$, one might guess that a transport functor should be a functor $\mathcal{P}_1(M) \to G$-Tor. However, since $G$-Tor is not a smooth category, there is no obvious way of demanding such a functor to be smooth. One might therefore try to use $BG$ instead of $G$-Tor. Indeed, notice that there is a natural functor $i : BG \to G$-Tor given by

$$i(\bullet) := G$$

as a right $G$-torsor and on morphisms

$$i(g) := L_g,$$

left-multiplication on $G$ by $g$. It is easy to check that this is a right $G$-equivariant map. Furthermore, this functor $i$ is an equivalence of categories. Therefore, one can think of $G$-Tor as a ‘thickening’ of $BG$. We can then try to use $BG$ for our target instead of $G$-Tor so that we can ask for smoothness. Then one might guess that a transport functor should be a smooth functor $\mathcal{P}_1(M) \to BG$. Unfortunately, now that we have smoothness, we’ve lost local triviality because such smooth functors describe parallel transport on trivialized principal $G$-bundles.

In order to encode local instead of global triviality, we have to combine these ideas with the ideas of the previous section in terms of the Čech groupoid (we will also return to a more suitable combination of the path groupoid and the Čech groupoid in Section 2.5). To avoid a huge collection of indices again, we denote our open cover $\{U_i\}_{i \in I}$ of $M$ simply by $Y := \bigsqcup_{i \in I} U_i$ and we let $\pi : Y \to M$ be the inclusion of these open sets into $M$. Note that this map is a surjective submersion. Then, the next guess might be that we need to have a smooth functor $\mathcal{P}_1(Y) \to BG$, but we still need an assignment of fibers $\mathcal{P}_1(M) \to G$-Tor. These assignments should be compatible in terms of the functor $i : BG \to G$-Tor and the submersion $\pi$. This is exactly what is done in [17] and we therefore now proceed to discussing local triviality of functors.

2.3. Local triviality of functors. Our first goal is to discuss local triviality of functors without making any assumptions on smoothness, which is left to the next section. The fibers of principal $G$-bundles were right $G$-torsors, which led us to consider the category $G$-Tor of $G$-torsors. One of the great features of Schreiber’s and Waldorf’s work [17] is their generality on the different flavors of bundles. If one wants to work with vector bundles one simply replaces $G$-Tor with $\textbf{Vect}$, the category of vector spaces (over some appropriate field such as $\mathbb{R}$ or $\mathbb{C}$), and if this vector bundle is an associated bundle for some representation of $G$, then this representation is precisely encoded by a functor $i : BG \to \textbf{Vect}$. Fiber bundles can be defined similarly. Therefore, we’ve made two important observations. The first is the desire to have a category $T$ specifying the fibers of our bundle and the second is to encode the group structure by a functor $i : BG \to T$. Schreiber and Waldorf generalize this even further by considering any Lie groupoid $\mathcal{G}$ instead of the special one $BG$. They define a $\pi$-local trivialization as follows [17] [Definition 2.5.]

**Definition 2.7.** Let $\mathcal{G}$ be a Lie groupoid, $T$ a category, $i : \mathcal{G} \to T$ a functor, and $M$ a smooth manifold. Fix a surjective submersion $\pi : Y \to M$. A $\pi$-local
i-trivialization of a functor $F: \mathcal{P}_1(M) \to T$ is a pair $(\text{triv}, t)$ of a functor $\text{triv}: \mathcal{P}_1(Y) \to \text{Gr}$ and a natural isomorphism $t: \pi^* F \Rightarrow \text{triv}$, as in the diagram

\[
\begin{array}{c}
\begin{array}{c}
\mathcal{P}_1(M) \\
F
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\mathcal{P}_1(Y) \\
\text{triv}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
T \\
\pi
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{Gr}
\end{array}
\end{array}
\]

(17)

Here $\pi^* F := F \pi_*$ is the pullback of $F$ along $\pi$ and $\text{triv}_i := \text{itriv}$. The groupoid $\text{Gr}$ is called the structure groupoid for $F$.

Functors $F: \mathcal{P}_1(M) \to T$ equipped with $\pi$-local i-trivializations $(\text{triv}, t)$ form the objects, written as triples $(F, \text{triv}, t)$, of a category denoted by $\text{Triv}_\pi^1(i)$.

**Definition 2.8.** A morphism $\alpha: (F, \text{triv}, t) \to (F', \text{triv}', t')$ in $\text{Triv}_\pi^1(i)$ is a natural transformation $\alpha: F \Rightarrow F'$. Composition is given by vertical composition of natural transformations.

**Remark 2.9.** One might expect to also demand a natural transformation $h: \text{triv} \Rightarrow \text{triv}'$ in the definition of a morphism $(F, \text{triv}, t) \to (F', \text{triv}', t')$ satisfying some compatibility condition with $\alpha$, $t$, and $t'$. This natural compatibility condition completely determines $h$ which is why it is excluded in the definition.

Although it is possible to discuss smoothness for the functor $\text{triv}: \mathcal{P}_1(Y) \to \text{Gr}$, there is no way to discuss smoothness for transition functions. In fact, it is not even obvious what transition functions are in this description. This is the motivation for introducing descent objects [17][Definition 2.8.]

**Definition 2.10.** Let $\text{Gr}$ be a Lie groupoid, $T$ a category, and $i: \text{Gr} \to T$ a functor. Fix a surjective submersion $\pi: Y \to M$. A descent object is a pair $(\text{triv}, g)$ consisting of a functor $\text{triv}: \mathcal{P}_1(Y) \to \text{Gr}$, a natural isomorphism

\[
\begin{array}{c}
\begin{array}{c}
\mathcal{P}_1(Y) \\
\text{triv}_i
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\mathcal{P}_1(Y^2) \\
\pi_2\pi_1
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
P_1(Y) \\
\text{triv}_i
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
T
\end{array}
\end{array}
\]

(18)

where $Y^2$ along with $\pi_1$ and $\pi_2$ come from the pullback

\[
\begin{array}{c}
\begin{array}{c}
Y^2 \\
\pi_1
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
Y \\
\pi_2
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{Gr} \\
\pi
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
M
\end{array}
\end{array}
\]

(19)

The pair $(\text{triv}, g)$ must satisfy the following two conditions

\[
\begin{array}{c}
\begin{array}{c}
\pi_1^* g \pi_2^* g = \pi_{13}^* g,
\end{array}
\end{array}
\]

(20)
where again the left-hand-side is vertical composition of natural transformations (read from top to bottom), and

\[(21) \quad \text{id}_{\text{triv}} = \Delta^* \eta,\]

where \(\Delta\) is the diagonal \(\Delta : Y \to Y^{[2]}\) (which exists by the universal property of the pullback) and \(\pi_{12}, \pi_{23}, \text{and } \pi_{13}\) all come from the three projections

\[(22) \quad Y^{[2]} \xrightarrow{\pi_{23}} Y^{[3]},\]

where \(Y^{[3]} = Y \times_M Y \times_M Y\) is the limit of the diagram

\[(23) \quad \;

\text{Descent objects form the objects of a category denoted by } \mathcal{D}_{\text{triv}}^1(i).\]

**Definition 2.11.** A descent morphism \(h : (\text{triv}, g) \to (\text{triv}', g')\) is a natural transformation \(h : \text{triv} \Rightarrow \text{triv'}\) satisfying

\[(24) \quad \pi_1^* h = g \pi_2^* h.\]

Composition is defined by vertical composition of natural transformations.

There is a functor \(\text{Ex}^1_{\pi} : \text{Triv}^1_{\pi}(i) \to \mathcal{D}_{\text{triv}}^1(i)\) that extracts descent data from trivialization data. At the level of objects, this functor is defined as follows. Let \((F, \text{triv}, t)\) be an object in \(\text{Triv}^1_{\pi}(i)\). For the pair \((\text{triv}, g)\), take \(t\) to be exactly the same. For \(g\) take the composition \(g := \pi_1^* \pi^* t\) coming from the composition in the diagram

\[(25) \quad \;

where \(\tilde{t}\) is the (vertical) inverse of \(t\). In [17], it is proved that this defines a descent object. On a morphism \(\alpha : (F, \text{triv}, t) \to (F', \text{triv}', t')\), the functor \(\text{Ex}^1_{\pi}\) is defined by setting

\[(26) \quad h := \pi_1^* \pi^* \alpha.\]
coming from the composition in the diagram

\[
\begin{array}{c}
\text{triv.} \\
\text{triv.} \\
\end{array}
\begin{array}{c}
T_	ext{triv} \\
\downarrow \scriptstyle \alpha \\
\downarrow \scriptstyle \beta \\
\end{array}
\begin{array}{c}
P_1(M) \\
\downarrow \scriptstyle \pi \\
\downarrow \scriptstyle \pi \\
\end{array}
\begin{array}{c}
P_1(Y) \\
\phantom{\alpha}
\end{array}
\end{array}
\]

(27)

The functor \( \text{Ex}^\pi_1 \) is proven to be part of an equivalence of categories between \( \text{Triv}^1_\pi(i) \) and \( \text{Des}^1_\pi(i) \) in [17] [Theorem 2.9]. This is done by constructing an inverse functor \( \text{Rec}^1_\pi : \text{Des}^1_\pi(i) \to \text{Triv}^1_\pi(i) \), which we will describe in Section 2.5.

**Definition 2.12.** Let \( (F, \text{triv}, t) \) be a \( \pi \)-local i-trivialization of a functor \( F : P_1(M) \to T \), i.e. an object of \( \text{Triv}^1_\pi(i) \). The descent object associated to the \( \pi \)-local i-trivialization of \( F \) is \( \text{Ex}^\pi_1(F, \text{triv}, t) \). Let \( \alpha : (F, \text{triv}, t) \to (F', \text{triv}', t') \) be a morphism in \( \text{Triv}^1_\pi(i) \). The descent morphism associated to the \( \pi \)-local i-trivialization of \( \alpha \) is \( \text{Ex}^\pi_1(\alpha) \).

### 2.4. Transport functors

We now discuss smoothness of descent data and finally give a definition of transport functors.

**Definition 2.13.** A descent object \( (\text{triv}, g) \) as above is said to be smooth if \( \text{triv} : P_1(Y) \to \text{Gr} \) is a smooth functor and if there exists a smooth natural isomorphism \( \tilde{g} : \pi^* \text{triv} \Rightarrow \pi^* \text{triv} \) with \( g = 1, \tilde{g} \), where \( 1 \) is the identity natural transformation for the functor \( i \) and horizontal concatenation of natural transformations depicts the horizontal composition. A descent morphism \( h : (\text{triv}, g) \to (\text{triv}', g') \) as above is said to be smooth if there exists a smooth natural isomorphism \( h : \text{triv} \Rightarrow \text{triv}' \) with \( h = 1, \tilde{h} \).

Smooth descent objects and morphisms form the objects and morphisms of a category denoted by \( \text{Des}^1_\pi(i)^{\mathcal{Z}} \) and form a sub-category of \( \text{Des}^1_\pi(i) \).

**Definition 2.14.** A \( \pi \)-local i-trivialization \( (F, \text{triv}, t) \) is said to be smooth if the associated descent object \( \text{Ex}^\pi_1(F, \text{triv}, t) \) is smooth. A morphism \( \alpha : (F, \text{triv}, t) \to (F', \text{triv}', t') \) is said to be smooth if the associated descent morphism \( \text{Ex}^\pi_1(\alpha) \) is smooth.

Smooth local trivializations and their morphisms form the objects and morphisms of a category denoted by \( \text{Triv}^1_\pi(i)^{\mathcal{Z}} \) and form a sub-category of \( \text{Triv}^1_\pi(i) \). Schreiber and Waldorf show in [17] [Proposition 3.5] that \( \text{Ex}^\pi_1 \) restricts to an equivalence of categories \( \text{Triv}^1_\pi(i)^{\mathcal{Z}} \to \text{Des}^1_\pi(i)^{\mathcal{Z}} \) of smooth data. Again, we will discuss an inverse functor in Section 2.5 since it will be necessary in discussing gauge invariant holonomy in Section 2.8. We now come to the definition of a transport functor [17] [Definition 3.6].

**Definition 2.15.** Let \( \text{Gr} \) be a Lie groupoid, \( T \) a category, \( i : \text{Gr} \to T \) a functor, and \( M \) a smooth manifold. A transport functor on \( M \) with values in a category \( T \) and with Gr-structure is a functor \( \text{tra} : P_1(M) \to T \) that has the property that there exists a smooth surjective submersion \( \pi : Y \to M \) and a smooth \( \pi \)-local i-trivialization \( (\text{triv}, t) \).
Transport functors with values in $T$ with Gr-structure form the objects of a category $\text{Trans}^1_{\text{Gr}}(M, T)$. We also define the morphisms of transport functors.

**Definition 2.16.** A morphism $\eta$ of transport functors on $M$ from $F$ to $F'$ is a natural transformation $\eta : F \Rightarrow F'$ such that there exists a surjective submersion $\pi : Y \rightarrow M$ and smooth $\pi$-local i-trivializations $(\text{triv}, t)$, $(\text{triv}', t')$, and $h : (\text{triv}, t) \rightarrow (\text{triv}', t')$ of $F$, $F'$, and $\eta$ respectively.

By using pullbacks, one can define the composition of such morphisms. We will not explicitly describe this now since we will come back to it later when discussing limit categories over surjective submersions in Section 2.7.

### 2.5. The reconstruction functor: local to global.

The functor $\text{Ex}_{\pi}^1 : \text{Triv}_{\pi}^1(i) \rightarrow \text{Des}_{\pi}^1(i)$ is actually an equivalence of categories. To construct the (weak) inverse $\text{Rec}_{\pi}^1 : \text{Des}_{\pi}^1(i) \rightarrow \text{Triv}_{\pi}^1(i)$, we will need to introduce a new category that combines the Čech groupoid with the path groupoid, utilizing the surjective submersion $\pi : Y \rightarrow M$. This is described in full and complete detail in [17] [Section 2.3]. We will be content with pictures and an explanation suitable for our purposes. It will be useful to visualize the surjective submersion $\pi : Y \rightarrow M$ as an open cover, since this will be the case of interest for us later. The reason for even considering the inverses $\text{Rec}_{\pi}^1$ is that we will later build transport 2-functors from local differential cocycle data describing principal bundles with connections. In the examples that we give, we hope that both physicists and mathematicians can get something from these two formulations. The local computations in terms of differential forms and transition functions can be patched together to construct globally defined holonomies in terms of group elements. This is the motivation behind the present section.

We start by defining the Čech path groupoid.

**Definition 2.17.** Let $\mathcal{P}^1(M)$ be the category whose set of objects are the elements of $Y$. The set of morphisms are freely generated by two basic types of morphisms (the generators) which are given as follows

1) thin homotopy classes of paths $\gamma$ in $Y$ with sitting instances and
2) points $\alpha$ in $Y^{[2]}$ (thought of as morphisms $\pi_1(\alpha) \xrightarrow{\Theta} \pi_2(\alpha)$ and called jumps).

There are several relations imposed on the set of morphisms.

(a) For any thin homotopy class of paths with sitting instances $\Theta : \alpha \rightarrow \beta$ in $Y^{[2]}$ the diagram

$$
\begin{array}{ccc}
\pi_1(\beta) & \xleftarrow{\pi_1(\Theta)} & \pi_1(\alpha) \\
\pi_2(\beta) & \xleftarrow{\pi_2(\Theta)} & \pi_2(\alpha) \\
\beta & \downarrow & \alpha \\
\end{array}
$$

(28)

commutes (see Figure 4 for a visualization of this).

(b) For any point $\Xi \in Y^{[3]}$ the diagram

$$
\begin{array}{ccc}
\pi_2(\Xi) & \xleftarrow{\pi_2(\Theta)} & \pi_1(\Xi) \\
\pi_3(\Xi) & \xleftarrow{\pi_3(\Theta)} & \pi_1(\Xi) \\
\pi_3(\Xi) & \xleftarrow{\pi_3(\Theta)} & \pi_1(\Xi) \\
\end{array}
$$

(29)
Thinking in terms of an open cover as a submersion, condition i) above says that if a path $\Theta : \alpha \to \beta$ is in a double intersection, it doesn’t matter whether or not the jump is performed first and then the path is traversed or vice versa.

Figure 1.

(c) The free composition of a class of paths with another class of paths is the usual composition of paths and for every point $y \in Y$, the class representing the constant path at $y$ is equal to $\Delta(y) \in Y^{[2]}$ which is the formal identity for the composition (this together with the previous condition demands that the points $\alpha \in Y^{[2]}$ are isomorphisms).

The notation for the free composition will be $\ast$.

A typical morphism in $\mathcal{P}_I^\pi(M)$ is depicted in Figure 2.

Figure 2. A generic representative of a morphism in $\mathcal{P}_I^\pi(M)$ is shown above for $Y = \bigsqcup_{i \in I} U_i$, the disjoint union over an open cover. The larger ellipses indicate open sets and the smaller ones in the middle indicate intersections. The curves in the open sets indicate the paths and the dotted vertical lines indicate the “jumps.”

Associated to every descent object $(\text{triv}, g)$ in $\mathcal{D}_\mathcal{S}_i^1(i)$ is a functor $R_{(\text{triv}, g)} : \mathcal{P}_I^\pi(M) \to T$ defined in a rather obvious way by sending $y \in Y$ to $\text{triv}_i(y)$, thin homotopy classes of paths $\gamma$ in $Y$ to $\text{triv}_i(\gamma)$, and finally jumps $\alpha \in Y^{[2]}$ to $g(\alpha) : \text{triv}_i(\pi_1(\alpha)) \to \text{triv}_i(\pi_2(\alpha))$. As described in [17] [Lemma 2.15.], this assignment extends to a functor $R : \mathcal{D}_\mathcal{S}_i^1(i) \to \text{Funct}(\mathcal{P}_I^\pi(M), T)$. On a descent morphism $h : (\text{triv}, g) \to (\text{triv}', g')$ it gives a natural transformation $R_h : R_{(\text{triv}, g)} \Rightarrow R_{(\text{triv}', g')}$ defined by sending $y \in Y$ to $h(y)$ for all $y \in Y$. 


The functor $\text{Rec}_x^1 : \text{Des}_x^1(i) \to \text{Triv}_x^1(i)$ will be defined so that it factors through this functor $\mathcal{R}$. What will then remain is to define a functor $\text{Funct}(\mathcal{P}_1^\gamma(M), T) \to \text{Funct}(\mathcal{P}_1(M), T)$ and we will do this by choosing a “path-lifting section” $s^\gamma : \mathcal{P}_1(M) \to \mathcal{P}_1^\gamma(M)$ of the canonical projection functor $p^\gamma : \mathcal{P}_1^\gamma(M) \to \mathcal{P}_1(M)$ which sends objects $y \in Y$ to $\pi(y)$, paths $\gamma$ to $\pi(\gamma)$, and points $\alpha \in Y^{[2]}$ to the identity.

Since $\pi : Y \to M$ is surjective, for every $x \in M$, there exists a $y \in Y$ such that $\pi(y) = x$. Therefore, define $s^\gamma : \mathcal{P}_1(M) \to \mathcal{P}_1^\gamma(M)$ on objects to be this assignment. Later we will state a theorem of [17] that explains how the reconstruction functor exists an open cover local sections, we can define $s$ sends objects $P$ to $\text{Funct}$. Therefore, define $s^\gamma$ by $t$.

By choosing a decomposition of every path to land in open sets one can lift using the locally defined sections. At the beginning and end of the path, one must apply a jump since the map $s$ defined on objects might not coincide with the lift of the endpoint of the path.

Figure 3. By choosing a decomposition of every path to land in open sets one can lift using the locally defined sections. At the beginning and end of the path, one must apply a jump since the map $s$ defined on objects might not coincide with the lift of the endpoint of the path.
that is part of an adjoint equivalence given by the quadruple \((s^\pi, p^\pi, \zeta, \text{id}_{p^\pi s^\pi})\) since \(p^\pi s^\pi = \text{id}_{P_1(M)}\). This natural isomorphism \(\zeta\) is the rather obvious one which sends \(y \in Y\) to the unique jump, an isomorphism, from \(y\) to \(s^\pi(\pi(y))\). It is natural by the first relation i) in Definition 2.17.

**Remark 2.18.** Note that we’re not considering smoothness here—we have not put a smooth structure on \(P_1\pi\) nor will we (although it is done in [17]). Indeed, the choice of lifts for the points could be sporadic. Again, all the smoothness for transport functors is contained in the descent object.

The functor \(s^\pi : P_1(M) \to P_1^\pi(M)\) induces a pullback functor \(s^\pi* : \text{Funct}(P_1^\pi(M), T) \to \text{Funct}(P_1(M), T)\) defined by \(s^\pi*(F) := Fs^\pi\) on functors \(F : P_1^\pi(M) \to T\). For a natural transformation \(\rho : F \to G\) it is defined by \(\rho_1s^\pi\). Finally, \(\text{Rec}_\pi\) is defined as the composition in the diagram

\[
\begin{array}{c}
\text{Funct}(P_1(M), T) \\
s^\pi* \\
\downarrow \quad \downarrow \\
\text{Funct}(P_1^\pi(M), T)
\end{array}
\xrightarrow{\text{Rec}_\pi} \xrightarrow{\text{Des}_1^{\pi\pi\pi}(i)}
\]

The claim is that the image of \(\text{Des}_1^{\pi\pi\pi}(i)\) under \(\text{Rec}_\pi^1\) is actually in \(\text{Triv}_1^{\pi\pi\pi}(i)\). This means at the level of objects that associated to \(R_{(\text{triv}, g)}\) there exists a \(\pi\)-local \(i\)-trivialization. We obviously take \(\text{triv}\) itself for the first part of this datum. To define \(t : \pi^* s^\pi \pi^* R_{(\text{triv}, g)} = \text{triv}\), we take the composition defined by the diagram

\[
\begin{array}{c}
P_1(M) \\
\downarrow p^\pi \\
\cdots \\
\downarrow \text{id} \\
P_1^\pi(M) \\
\downarrow R_{(\text{triv}, g)} \\
T \\
\downarrow i \\
\text{Gr}
\end{array}
\xrightarrow{\text{Rec}_\pi^1} \xrightarrow{\text{triv}}
\]

where the functor \(P_1(Y) \hookrightarrow P_1^\pi(M)\) is the inclusion and the unwritten 2-morphisms in the blank spaces are identities. The rest of the proof, namely the fact that the image of a morphism lands in \(\text{Triv}_1^{\pi\pi\pi}(i)\) under \(\text{Rec}_\pi^1\), is explained in [17] [Appendix B.1.].

### 2.6. Differential cocycle data

We now switch gears a bit and go in the other (infinitesimal) direction. We describe this in several parts. We focus on a local description first, i.e. we consider ‘trivialized’ transport functors. We extract the differential cocycle data from functors and then we construct functors from differential cocycle data. This is a very brief account of what’s in [17] [Section 4] where we have simplified the presentation since we do not prove any results.

#### 2.6.1. From functors to 1-forms

Given a smooth functor \(F : P_1(X) \to BG\), we will define a \(g\)-valued 1-form \(A\) defined pointwise for every \(x \in X\) and \(v \in T_xX\) as follows. Let \(\gamma : \mathbb{R} \to X\) be a curve in \(X\) with \(\gamma(0) = x\) and \(\frac{d}{dt}\gamma(0) = v\). Every smooth map
such as $\gamma : \mathbb{R} \to X$ induces a smooth pushforward functor $\gamma_* : \mathcal{P}_1(\mathbb{R}) \to \mathcal{P}_1(X)$. At the level of morphisms, it turns out that the space of thin homotopy classes of paths in $\mathbb{R}$, denoted by $P^1\mathbb{R}$, is actually smoothly equivalent to $\mathbb{R} \times \mathbb{R}$ since every thin homotopy class of paths in $\mathbb{R}$ is determined by its starting and ending point. We write this diffeomorphism as $\gamma_\mathbb{R} : \mathbb{R} \times \mathbb{R} \to P^1\mathbb{R}$ as is done in [18]. This function is defined by sending $(s,t)$ to the thin homotopy class of a path in $\mathbb{R}$ determined by its source point $s$ and target $t$ as shown schematically in Figure 4.

![Figure 4](image)

**Figure 4.** A point $(s,t)$ in $\mathbb{R}^2$ is drawn as two points on $\mathbb{R}$ and this gets mapped to the path in $\mathbb{R}$ from the point $s$ to the point $t$ shown on the right under the map $\gamma_\mathbb{R}$.

Therefore, we obtain a function $F_1\gamma_*\gamma_\mathbb{R}$ from the composition

$$G \overset{F_1}{\to} P^1X \overset{\gamma_*}{\to} P^1\mathbb{R} \overset{\gamma_\mathbb{R}}{\to} \mathbb{R} \times \mathbb{R}.$$ (34)

Here $F_1$ is $F$ restricted to the set of morphisms $P^1X$. Using this, we define

$$A_x(v) := -\frac{d}{dt} |_{t=0} F_1(\gamma_*(\gamma_\mathbb{R}(0,t)))$$. (35)

In [17] [Section 4], Schreiber and Waldorf prove that this is independent of $\gamma$ and only depends on $x$ and $v$. They also show that this defines a 1-form $A \in \Omega^1(X; g)$.

2.6.2. *From 1-forms to functors.* Starting with a $g$-valued 1-form $A \in \Omega^1(X; g)$ on $X$ we want to define a smooth functor $\mathcal{P}_1(X) \to \mathcal{B}G$. To do this, we first define a function $k_A : PX \to G$ on paths in $X$ with sitting instances (we do not mod out by thin homotopy). Given a path $\gamma : [0,1] \to X$ with sitting instances, we can pull back the 1-form $A$ to $\mathbb{R}$, namely $\gamma^*(A) \in \Omega^1([0,1]; g)$. We then define $k_A(\gamma)$ using the path-ordered-exponential

$$k_A(\gamma) := \mathcal{P} \exp \left\{ \int_0^1 A_t \left( \frac{\partial}{\partial t} \right) dt \right\}.$$ (36)

Recall how this path-ordered exponential is defined

$$\mathcal{P} \exp \left\{ \int_0^1 A_t \left( \frac{\partial}{\partial t} \right) dt \right\} := \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^1 dt_1 \cdots \int_0^1 dt_n \mathcal{T} \left[ A_{t_n} \left( \frac{\partial}{\partial t_n} \right) \cdots A_{t_1} \left( \frac{\partial}{\partial t_1} \right) \right],$$ (37)

where the “time-ordering operator” $\mathcal{T}$ is defined by

$$\mathcal{T}[A_s A_t] := \begin{cases} A_tA_s & \text{if } t \geq s \\ A_s A_t & \text{if } s \geq t \end{cases}.$$ (38)

Also, by definition, at $n = 0$, the right-hand side of equation (37) is the identity. We can picture the path-ordered exponential schematically as a power series of graphs with marked points as in Figure 5.

![Figure 5](image)

**Figure 5.** The path-ordered exponential.

It turns out this map only depends on the thin homotopy class of $\gamma$ and therefore factors through a smooth map $F_A : P^1X \to G$ on thin homotopy classes of paths. This map defines a smooth functor $F_A : \mathcal{P}_1(X) \to \mathcal{B}G$. This is proved in [17] [Proposition 4.3. and Lemma 4.5.].
2.6.3. Local differential cocycles for transport functors. The above constructions can be generalized to smooth natural transformations between smooth functors. Given a smooth natural transformation \( h : F \Rightarrow F' \) of smooth functors \( F,F' : \mathcal{P}(X) \to \mathcal{B}G \) we obtain a function, written somewhat abusively also as \( h : X \to G \) satisfying

\[
h(y)F(\gamma) = F'(\gamma)h(x)
\]

for all paths \( \gamma : x \to y \) in \( X \). If we differentiate this condition, we obtain

\[
A' = \text{Ad}_h(A) - h^*\theta,
\]

where \( \theta \) is right Maurer-Cartan form, sometimes written as \( dgg^{-1} \) for matrix groups, \( A \) is the 1-form corresponding to \( F \), \( A' \) is the 1-form corresponding to \( F' \), and \( \text{Ad} \) is the adjoint action on the Lie algebra \( \mathfrak{g} \) defined by

\[
\text{Ad}_h(T) := \frac{d}{dt} \Big|_{t=0} (h \exp(tT)h^{-1})
\]

for all \( T \in \mathfrak{g} \).

This motivates the following definition.

**Definition 2.19.** Let \( Z^1_X(G) \) be the category defined as follows. An object of \( Z^1_X(G) \) is a 1-form \( A \in \Omega^1(X;\mathfrak{g}) \). A morphism from \( A \) to \( A' \) is a function \( h : X \to G \) satisfying

\[
A' = \text{Ad}_h(A) - h^*\theta.
\]

The composition is defined by

\[
A'' \overset{h'}{\leftarrow} A' \overset{h}{\leftarrow} A := A'' \overset{h' \cdot h}{\leftarrow} A,
\]

where \( h' \cdot h \) is (pointwise) multiplication of \( G \)-valued functions.

The above analysis defines two functors

\[
\begin{array}{ccc}
Z^1_X(G) & \overset{\mathcal{P}}{\longrightarrow} & \text{Funct}^\Xi(X,\mathcal{B}G) \\
\end{array}
\]

In [17] [Proposition 4.7.], Schreiber and Waldorf prove that these two functors are inverses of each other (not just an equivalence pair).
All of this was for globally defined smooth functors. Transport functors on \( M \) are not necessarily smooth globally. However, there must exist a surjective submersion \( \pi : Y \to M \) with a \( \pi \)-local \( i \)-trivialization. This means we have a smooth functor 
\[
\text{triv} : \mathcal{P}_1(Y) \to BG \quad \text{which corresponds to 1-form } A \in \Omega^1(Y;g), \text{which is an object in } Z^1_Y(G).
\]
But we also have a natural transformation \( g : \pi_1^\ast \text{triv} \Rightarrow \pi_2^\ast \text{triv} \) that factors through a smooth natural transformation \( \tilde{g} : \pi_1^\ast \text{triv} \Rightarrow \pi_2^\ast \text{triv} \) which is a morphism as explained above, but in the category \( Z^1_Y(S) \) from \( \pi_1^\ast A \) to \( \pi_2^\ast A \). This means
\[
\pi_2^\ast A = \text{Ad}_g(\pi_1^\ast A) - \tilde{g}^\ast \tilde{g}.
\]
The condition
\[
\pi_{12}^\ast g = \pi_{13}^\ast
\]
translates to
\[
\pi_{23}^\ast \tilde{g} \cdot \pi_{12}^\ast \tilde{g} = \pi_{13}^\ast g,
\]
where the \( \cdot \) indicates group multiplication in \( G \). A morphism of transport functors subordinate to the same surjective submersion is a natural transformation \( h : \text{triv} \Rightarrow \text{triv}' \) that factors through a smooth natural transformation \( \tilde{h} : \text{triv} \Rightarrow \text{triv}' \) and therefore defines a morphism from \( A \) to \( A' \) in \( Z^1_Y(G) \). This analysis motivates the following definition of local differential cocycles.

**Definition 2.20.** Let \( \pi : Y \to M \) be a surjective submersion. Define the category \( Z^1_Y(G) \) of differential cocycles subordinate to \( \pi \) as follows. An object of \( Z^1_Y(G) \) is a pair \((A, g)\), where \( A \) is an object in \( Z^1_Y(G) \), \( g \) is a morphism from \( \pi_1^\ast A \) to \( \pi_2^\ast A \) in \( Z^1_Y(S) \). A morphism from \((A, g)\) to \((A', g')\) is a morphism \( h \) from \( A \) to \( A' \) in \( Z^1_Y(G) \). The composition of morphisms in \( Z^1_Y(G) \) is defined by
\[
(A'', g'') \leftarrow h'' \quad (A', g') \leftarrow h' \quad (A, g) := (A'', g'') \leftarrow h \quad (A, g).
\]
The above generalizations imply that the functors \( \mathcal{P} \) and \( \mathcal{D} \) extend to equivalences
\[
Z^1_Y(G) \cong \begin{array}{c}
\mathcal{P} \quad \text{deg}_{\mathcal{P}}(i) \quad \mathcal{D} \end{array}
\]
whenever \( i : BG \to T \) is an equivalence. This is [Corollary 4.9.] in [17].

2.7. **Limit over surjective submersions.** Here we just give a brief summary of the four levels of construction introduced and the notation of the functors relating these categories. To do this, we get rid of the dependence on the surjective submersion in the categories introduced in the prequel. Several of our categories depended on the choice of a surjective submersion. These categories were \( \text{Triv}^i(i)^\ast, \text{deg}_{\text{Triv}}(i)^\ast, \) and \( Z^1_Y(G)^\ast \). On the contrast, the category of transport functors \( \text{Trans}_{BG}^\ast(M, T) \) does not depend on \( \pi \). To relate these categories better, we will take limits over \( \pi \). Changing the surjective submersion gives a collection of categories depending on such surjective submersions. One can take a limit over the collection of surjective submersions in this case.

The general construction is done as follows. Let \( S_\pi \) be a family of categories parametrized by smooth surjective submersions \( \pi : Y \to M \) and let \( F(\zeta) : S_\pi \to S_{\pi\zeta} \) be a family of functors for every refinement \( \zeta : Y' \to Y \) of \( \pi \) satisfying the condition that for any iterated refinement \( \zeta' : Y'' \to Y' \) and \( \zeta : Y' \to Y \) of \( \pi : Y \to M \) then...
In this case, an object of \( S_M := \lim_{\pi} S_\pi \) is given by a pair \((\pi, X)\) of a surjective submersion \( \pi : Y \to M \) and an object \( X \) of \( S_\pi \). A morphism from \((\pi_1, X_1)\) to \((\pi_2, X_2)\) consists of an equivalence class of a common refinement together with a morphism \( f : (F(y_1))(X_1) \to (F(y_2))(X_2) \) in \( S_\pi \). It is written as a pair \((\zeta, f)\). Two such \((\zeta, f)\) and \((\zeta', f')\) are equivalent if they agree (on the nose) on their common pullback. The composition

\[
\begin{array}{c}
\text{(51)} \\
(\pi_3, X_3) \leftrightarrow (\pi_2, X_2) \leftrightarrow (\pi_1, X_1)
\end{array}
\]

is defined by choosing representatives and taking the pullback refinement

\[
\begin{array}{c}
\text{(52)} \\
\text{written as } \zeta_{13} : Z_{13} \to M \text{ along with the composition } (F(pr_{23}))(g)(F(pr_{12}))(f).
\end{array}
\]

One can check this definition does not depend on the equivalence class chosen.

After getting rid of the specific choices of the surjective submersions, we can take the limits of all the categories we have introduced. We make the following notation, slightly differing from that of [20]:

\[
\begin{align*}
\text{(53)} & \quad \text{Triv}^1_M(i) := \lim_{\pi} \text{Triv}^1_\pi(i) \\
\text{(54)} & \quad \text{Des}^1_M(i) := \lim_{\pi} \text{Des}^1_\pi(i) \\
\text{(55)} & \quad Z^1(M; G) := \lim_{\pi} Z^1_\pi(G).
\end{align*}
\]

Then from our previous discussions, we just collect the functors we have introduced relating all these categories to \( \text{Trans}^1_{BG}(M, T) \) after taking such limits over surjective submersions:

\[
\begin{align*}
\text{(56)} & \quad Z^1(M; G) \xrightarrow{\text{Pr}^1} \text{Des}^1_M(i) \xrightarrow{\text{Rec}^1} \text{Triv}^1_M(i) \xrightarrow{\text{c}} \text{Trans}^1_{BG}(M, T),
\end{align*}
\]

where \( c \) chooses a local trivialization for a transport functor while \( v \) forgets the choice of local trivialization. Under the condition that \( i : BG \to T \) is an equivalence of categories, all of the above functors are equivalences. This is actually proved for the 2-categorical generalization in [19] [Proposition 4.2.1. and Theorem 4.2.2.] and
in the smooth setting this is proven in \cite{[20]} [Theorem 3.2.2., Lemma 3.2.3., and Lemma 3.2.4.]. We’ll discuss the 2-categorical version in more detail later.

2.8. **Parallel transport, holonomy, and gauge invariance.** Holonomy for ordinary bundles with connection is typically defined in several different ways. In all cases, it is a special case of parallel transport where one restricts attention to based loops. We will first define what we mean by holonomy with respect to a given transport functor and choice of local trivialization and then we will discuss its gauge invariance or covariance. The hope is to define a notion of holonomy that doesn’t depend on the choice of trivialization or on the isomorphism class of the transport functor. We will do this in three steps.

First, we will define a notion of holonomy for a transport functor $\text{tra} : \mathcal{P}_1(M) \rightarrow \mathcal{T}$ by choosing a local trivialization and using it to define a $G$-valued holonomy along based loops. In the second step, we consider how the value of this assignment changes upon a change of basepoint for the based loop and then consider how it changes when the local trivialization is chosen differently. Finally, we will discover that holonomy changes in both situations under conjugation and therefore one can define a holonomy map $\text{hol} : LM \rightarrow G/\text{Inn}(G)$ from the space of thin homotopy classes of free loops to the conjugacy classes of $G$.

The following presentation of this material slightly differs from that of \cite{[17]} but is close in spirit with the categories introduced earlier and has the same content. We first discuss ordinary holonomy for ordinary transport functors with $B\mathcal{G}$-structure and with value in $\mathcal{T}$. This can be done if $i : B\mathcal{G} \rightarrow \mathcal{T}$ is full and faithful meaning that it is an isomorphism on Hom-sets. To get group-valued holonomy functions on paths, we consider the following composition of functors (starting at the left and moving clockwise),

\[
\begin{array}{c}
\text{Trans}^1_{B\mathcal{G}}(M, T) \\
\text{Ex}^1 \\
\text{Rec}^1
\end{array}
\]

(57)

\[
\begin{array}{c}
c \\
\text{Triv}^1(i)\cong \\
\text{Triv}^1(i)\cong
\end{array}
\]

each of which is an equivalence of categories under the assumptions we’ve made in the preceding sections. We write this composition as $\mathcal{E} : \text{Trans}^1_{B\mathcal{G}}(M, T) \rightarrow \text{Trans}^1_{B\mathcal{G}}(M, T)$. Because all of these functors are equivalences, different choices of $c$ and $\text{Rec}^1$ (by choosing different local trivializations and different sections $s^\pi : \mathcal{P}_1(M) \rightarrow \mathcal{P}^\pi_1(M)$) give naturally equivalent functors (a 2-categorical generalization of this fact is proven in Appendix \[6.2\] in Lemma \[6.32\]). $\mathcal{E}$ will assign group elements to thin homotopy classes of paths (more on this below) for every transport functor $F$. $\mathcal{E}$ will also assign group-valued gauge transformations for every morphism $\eta : F \rightarrow F'$ of transport functors. Furthermore, we know that a natural isomorphism $\mathcal{E} : \text{id} \Rightarrow \mathcal{E}$ exists. Choosing such a natural isomorphism is what allows us to choose isomorphisms from our fibers to $G$ and allows us to relate our original parallel transports to the group elements defined from $\mathcal{E}$.

To see this, first recall what $\mathcal{E}$ does. It helps to quickly review the composition of functors that define $\mathcal{E}$ since earlier we only defined each piece with respect to a particular surjective submersion $\pi : Y \rightarrow M$. For a transport functor $F$, we
first choose a local trivialization $c(F) := (\pi, F, \text{triv}, t)$. Then we extract the local descent object $\text{Ex}(\pi, F, \text{triv}, t) := (\pi, \text{triv}, g)$. Then, we reconstruct a transport functor $\text{Rec}(\pi, \text{triv}, g)$ and then forget the trivialization data keeping just the functor $\text{rec}(\text{Rec}(\pi, \text{triv}, g))$. The resulting transport functor, written as $\mathcal{E}_F$ (as opposed to $\mathcal{E}(F)$ for instance), is defined by

\begin{equation}
M \ni x \mapsto G := \text{triv}_t(s^\pi(x)) =: \mathcal{E}_F(x)
\end{equation}

\begin{equation}
P^1M \ni \gamma \mapsto R_{\text{Ex}(c(F))}(s^\pi(\gamma)) =: \mathcal{E}_F(\gamma),
\end{equation}

where $s$ is a choice of section $s : \mathcal{P}_1(M) \to \mathcal{P}^*_1(M)$ and $\mathcal{E}_F(\gamma) := R_{\text{Ex}(c(F))}(s^\pi(\gamma))$ is defined by the trivialization $c(F)$, its associated descent object $\text{Ex}(c(F))$, and the section $s$ by choosing a lift of the path $\gamma$ and applying trivialized transport on the pieces and transition functions on the jumps (see Section 2.5).

To a morphism $\eta : F \to F'$ of transport functors, the resulting morphism of transport functors, written as $\mathcal{E}_F$, is defined as follows. First, we choose surjective submersions $\pi : Y \to M$ and $\pi' : Y' \to M$ for $F$ and $F'$ respectively along with local trivializations $(\text{triv}, t)$ and $(\text{triv}', t')$. This means that under $c$ the morphism $c(\eta)$ is defined on a common refinement $\zeta : Z \to M$ of both $\pi$ and $\pi'$. The same thing applies to the extracted descent morphism $\text{Ex}(c(\eta)) = (\zeta, h)$. Since our domain is changed under the refinement, $h$ is defined by the composition

\begin{equation}
\begin{tikzcd}
\text{triv}_t \ar[rr dd] & & \mathcal{P}_1(Y) \ar[ll] \ar[rr] \ar[dd] & & \mathcal{P}_1(Z) \ar[ll] \ar[dd] \\
& \pi_* & & \pi'_* & \\
\text{triv}_t' \ar[rr uu] & & \mathcal{P}_1(Y') \ar[ll] \ar[rr] \ar[uu] & & \mathcal{P}_1(Y) \ar[ll] \ar[uu]
\end{tikzcd}
\end{equation}

and it satisfies

\begin{equation}
y^{[2]*}g \zeta^*h = \zeta'^*h y^{[2]*}g'.
\end{equation}

Now, the reconstruction functor $\text{Rec} : \text{Desc}^1(i) \to \text{Triv}^1(i)$ assigns this morphism to $\text{Rec}(\zeta, h) := s^{\pi*}R_{\text{Ex}(c(\eta))}$ (using notation from Section 2.5) which is a morphism of transport functors from $\text{Rec}(y^\pi(\pi, \text{triv}, g))$ to $\text{Rec}(y^{\pi'}(\pi', \text{triv}', g'))$ with respect to this common refinement and where $s^{\pi} : \mathcal{P}_1(M) \to \mathcal{P}^*_1(M)$ is a section for this common refinement. $\text{Rec}(\zeta, h)$ is defined by sending $x \in M$ to $h(s^\pi(x))$ which is a morphism from $\text{triv}_t(y^\pi(s^\pi(x)))$ to $\text{triv}_t'(y^{\pi'}(s^{\pi'}(x)))$.

Now, the natural isomorphism $\varphi : \text{id} \Rightarrow \mathcal{E}$ assigns to every transport functor $F$ a morphism of transport functors $\varphi_F : F \to \mathcal{E}_F$ and remember (see Definition 2.16) that a morphism of transport functors is itself a natural transformation with certain smoothness conditions. This means that to every $x \in M$, we get an isomorphism $\varphi_F(x) : F(x) \to G$ satisfying naturality, which means that to every thin path
\[ \gamma \in P^1 M \text{ from } x \text{ to } y, \text{ the diagram} \]

\[
\begin{array}{c}
G \xleftarrow{r_F(x)} F(x) \\
\downarrow \\
\Phi(\gamma) \\
G \xleftarrow{r_F(y)} F(\gamma) \\
\end{array}
\]

(62)

commutes.

Since \( s \) itself is a natural transformation, to every morphism \( \eta : F \to F' \) of transport functors, the diagram

\[
\begin{array}{c}
\Phi \xleftarrow{s_F} F \\
\downarrow \eta \\
\Phi' \xleftarrow{s_{F'}} F' \\
\end{array}
\]

(63)

commutes.

We’d like to restrict parallel transport to thin homotopy classes of based loops, and eventually free loops. First recall the definition of the thin based loop space.

**Definition 2.21.** The thin based loop space of \( M \) is the equalizer of the diagram

\[
\begin{array}{c}
P^1 M \xrightarrow{s} M \\
\downarrow t \\
\end{array}
\]

and is written as \( \Omega^1 M \).

**Definition 2.22.** The \( s \)-holonomy of \( F \), written as \( \text{hol}^s_F \), is defined as the restriction of parallel transport of a transport functor \( F \) to the thin based loop space of \( M \):

\[
\text{hol}_{s}^F := \Phi \big|_{\Omega^1 M} : \Omega^1 M \to G.
\]

We now pose three questions that will motivate the rest of our discussion on 1-holonomy.

i) How does \( \text{hol}_{s}^F \) depend on the choice of basepoint? Namely, suppose that two thin based loops \( \gamma \) and \( \gamma' \), with possibly different basepoints, are thinly homotopic *without preserving the basepoint* (see Definition 2.23). Then, how is \( \text{hol}_{s}^F(\gamma) \) related to \( \text{hol}_{s}^F(\gamma') \)?

ii) How does \( \text{hol}_{s}^F \) depend on \( F \)? Namely, suppose that \( \eta : F \to F' \) is a morphism of transport functors. How is \( \text{hol}_{s}^F \) related to \( \text{hol}_{s}^{F'} \) in terms of \( \eta \)?

iii) How does \( \text{hol}_{s}^F \) depend on \( s \), the choice of trivialization? Namely, suppose that \( s' \) is another trivialization. Then how is \( \text{hol}_{s}^F \) related to \( \text{hol}_{s'}^F \)?

We will discover that the answers to all of these questions involve conjugation in \( G \). With this, we will be able to associate with every equivalence class of transport functors, a holonomy function with values in *conjugacy classes in \( G \)* on a certain thin free loop space. We therefore first define what we mean by thin free loop space and then we proceed to answer the above questions.

Denote the smooth space of loops in \( M \) by \( LM := \text{Man}(S^1, M) \).
Definition 2.23. Two smooth loops \( \gamma \) and \( \gamma' \) in \( M \) are thinly homotopic if there exists a smooth map \( h : S^1 \times I \to M \) such that

i) there exists an \( \epsilon > 0 \) with \( h(t, s) = \gamma(t) \) for \( s \leq \epsilon \) and \( h(t, s) = \gamma'(t) \) for \( s \geq 1 - \epsilon \) and for all \( t \in S^1 \) and

ii) the smooth map \( h \) has rank \( \leq 1 \).

The first condition is technical so that thin homotopy defines an equivalence relation. The second condition is where the thin structure is buried. We denote the loop space of \( M \) described by a function \( \beta, \beta' \) respectively, of an element \( [\gamma] \in L^1 M \) need not have the same image. In particular, their basepoints might not even be common points. For instance, one can examine Figure 6 which gives an example.

However, for representatives \( \gamma : x \to x' \) of \( i_{\beta([\gamma])} \) and \( i_{\beta'([\gamma])} \) respectively, where \( x' \) and \( x \) are not necessarily the same point, there is an unbased thin homotopy \( h : S^1 \times I \to M \) with \( h(t, s) = \gamma(t) \) for \( s \leq \epsilon \) and \( h(t, s) = \gamma'(t) \) for \( s \geq 1 - \epsilon \) for some \( \epsilon > 0 \). Due to such a homotopy, one can choose an unbased loop \( \bar{\gamma} \) and two paths (not necessarily loops) with sitting instances \( \bar{\gamma}_{xx}, \bar{\gamma}_{x'x} \) respectively, where \( x', x \) are thinly homotopic to \( \gamma \) preserving the basepoint \( x \). Third, the composition of \( \gamma_{xx} \circ \bar{\gamma}_{xx} \) is thinly homotopic to \( \gamma' \) preserving the basepoint \( x' \). This is depicted in Figure 6.

This says that given two based loops, with possibly different basepoints, that are thinly homotopic without preserving the basepoint, one can always choose a representative of such a thin homotopy class of a loop in \( M \) with two marked points so that the associated two based loops (coming from starting at either basepoint) are thinly homotopic to the original two with a thin homotopy that preserves the basepoint. More precisely, we proved the following fact.

Lemma 2.24. Let \( [\gamma] \in L^1 M \) be a thin homotopy class of loops in \( M \). Let \( \beta, \beta' : L^1 M \to M \) be two basepoint-choosing maps. Let \( \gamma \) and \( \gamma' \) be representatives of \( i_{\beta([\gamma])} \) and \( i_{\beta'([\gamma])} \) respectively. Denote the basepoint \( \beta([\gamma]) \) of \( \gamma \) by \( x \) and the basepoint \( \beta'([\gamma]) \) of \( \gamma' \) by \( x' \). Then, there exists an unbased loop \( \bar{\gamma} \) (that contains both \( x \) and \( x' \)) and two paths \( \bar{\gamma}_{xx} : x \to x' \) and \( \bar{\gamma}_{x'x} : x' \to x \) with sitting instances such that the following three properties hold.

i) \( \bar{\gamma} \) is the composition of \( \bar{\gamma}_{xx} \) and \( \bar{\gamma}_{x'x} \).

ii) \( \bar{\gamma}_{xx} \circ \bar{\gamma}_{x'x} \) is thinly homotopic to \( \gamma \) preserving the basepoint \( x \).

iii) \( \bar{\gamma}_{x'x} \circ \bar{\gamma}_{xx} \) is thinly homotopic to \( \gamma' \) preserving the basepoint \( x' \).
Therefore, without loss of generality, we can choose a single representative \( \gamma \) (called \( \tilde{\gamma} \) in the above Lemma) of a thin homotopy class of loops \([\gamma]\) with a decomposition as in the Lemma. Also to not have awkward notation, we write \( y \) instead of \( x' \) for the other basepoint. Denote the loop \( \gamma \) by \( \gamma_x \) once the basepoint \( x \) has been chosen and similarly \( \gamma_y \) if \( y \) is chosen. Denote the path from \( x \) to \( y \) along \( \gamma \) by \( \gamma_{yx} \) and the path from \( y \) to \( x \) along \( \gamma \) by \( \gamma_{xy} \). Note that when we mod out by thin homotopy, \( \gamma_y = \gamma_{xy} \circ \gamma_x \circ \gamma_{xy} \). For convenience, we will abuse notation a bit and not distinguish between the actual paths versus the thin homotopy classes. See Figure 8 for a picture of these facts.

By functoriality of the transport functor \( \mathcal{F} \), we have

\[
\begin{align*}
\text{hol}_F(\gamma_y) &= \mathcal{F}(\gamma_y) \\
(66) \quad &= \mathcal{F}(\gamma_{xy} \circ \gamma_x \circ \gamma_{xy}) \\
(67) \quad &= \mathcal{F}(\gamma_{ixy}) \mathcal{F}(\gamma_x) \mathcal{F}(\gamma_{xy}) \\
(68) \quad &= (\mathcal{F}(\gamma_{xy}))^{-1} \text{hol}_F(\gamma_x) \mathcal{F}(\gamma_{xy}) \\
(69) \quad &= \mathcal{F}(\gamma_{xy})^{-1} \text{hol}_F(\gamma_x) \mathcal{F}(\gamma_{xy})
\end{align*}
\]

so that \( \text{hol}_F \) depends on basepoint by conjugation in \( G \).
Figure 7. The domain of the homotopy \( h : S^1 \times I \to M \) is drawn as an annulus depicting the domain of \( \gamma \) as the inner circle and that of \( \gamma' \) as the outer circle. The homotopy allows us to choose a loop \( \tilde{\gamma} \), drawn somewhat in the middle, that contains both basepoints \( x \) and \( x' \) and is thinly homotopic to both \( \gamma \) and \( \gamma' \). This loop \( \tilde{\gamma} \) is decomposed into two paths \( \tilde{\gamma}_{xx} : x \to x' \) (drawn on the right) and \( \tilde{\gamma}_{xx'} : x' \to x \) (drawn on the left). The dashed lines indicate the \( \epsilon \) region where the paths do not change. All paths are oriented counter-clockwise. Note that the original homotopy \( h \) need not separate the two different basepoints into the northern and southern hemispheres as drawn, but by a reparametrization, we can always do this.

Figure 8. A based loop \( \gamma_x \) at \( x \) is shown on the left. By choosing another basepoint \( y \), one obtains a based loop \( \gamma_y \). The two are related by considering the paths \( \gamma_{yx} \) from \( x \) to \( y \) along \( \gamma_x \) and \( \gamma_{xy} \) from \( y \) to \( x \) along \( \gamma_x \). We have \( \gamma_y = \gamma_{yx} \circ \gamma_x \circ \gamma_{xy} \) but we also have \( \gamma_y = \gamma_{xy} \circ \gamma_x \circ \gamma_{yx} \).
ii) Suppose that $\eta : F \to F'$ is a morphism of transport functors. Then, for every thin path $\gamma : x \to y$ we have a commutative diagram

$\begin{array}{ccc}
\ell_{F'}(x) & \xrightarrow{\ell_{\eta}(x)} & \ell_{F}(x) \\
\downarrow & & \downarrow \\
\ell_{F'}(y) & \xrightarrow{\ell_{\eta}(y)} & \ell_{F}(y)
\end{array}
$

which says

(71) $\ell_{\eta}(y) \ell_{\gamma}(\eta) = \ell_{\gamma}(\eta') \ell_{\eta}(x)$.

If we restrict this to a loop $\gamma$ based at $x$, then

(72) $\text{hol}_{F'}(\gamma) = (\ell_{\eta}(x))^{-1} \text{hol}_{F}(\gamma) \ell_{\eta}(x)$

so that again, $\text{hol}_{F'}$ changes under conjugation when the functor $F$ is changed to a gauge equivalent one.

iii) Suppose that another trivialization $\ell'$ was chosen. Any two trivializations are naturally isomorphic. To see this, recall that one can choose natural isomorphisms $\mathfrak{s} : \text{id} \Rightarrow \ell$ and $\mathfrak{s}' : \text{id} \Rightarrow \ell'$ and with these choices we have a natural isomorphism $\mathfrak{s} : \ell \Rightarrow \ell'$. Therefore, for every transport functor $F$ we have a morphism of transport functors $\mathfrak{s} : \ell \Rightarrow \ell'$ satisfying naturality. This means to every $x \in M$ we have a morphism $\mathfrak{s}_{F}(x) : \ell'_{F}(x) \to \ell_{F}(x)$ satisfying naturality which means that to every path $\gamma : x \to y$ the diagram

$\begin{array}{ccc}
\ell_{F}(x) & \xrightarrow{\ell_{\mathfrak{s}_{F}(x)}} & \ell'_{F}(x) \\
\downarrow & & \downarrow \\
\ell_{F}(y) & \xrightarrow{\ell_{\mathfrak{s}_{F}(y)}} & \ell'_{F}(y)
\end{array}
$

commutes so that for a loop $\gamma : x \to y$, we have again

(74) $\text{hol}_{F'}(\gamma) = (\mathfrak{s}_{F}(x))^{-1} \text{hol}_{F}(\gamma) \mathfrak{s}_{F}(x)$.

In conclusion, the answer to every one of the three questions is conjugation. This is what is called gauge covariance. To get something gauge invariant, we first denote the quotient map from $G$ to its conjugacy classes by $q : G \to G/\text{Inn}(G)$, where $\text{Inn}(G)$ stands for the inner automorphisms of $G$ and the quotient $G/\text{Inn}(G)$ is given by the conjugation action of this group on $G$. All of the above considerations show that the following theorem holds.

**Theorem 2.25.** Let $F$ be a transport functor with $BG$ structure with values in $T$ over $M$ and $\ell$ a local trivialization functor. Let $L^{1}M, \Omega^{1}M, i_{\beta}$, and $q$ be defined as above. Then the composition

(75) $G/\text{Inn}(G) \xleftarrow{q} G \xrightarrow{\text{hol}_{F}} \Omega^{1}M \xleftarrow{i_{\beta}} L^{1}M$

is

i) smooth,

ii) independent of $\beta$,

iii) independent of the equivalence class of $F$,
iv) and independent of the equivalence class of $\mathcal{F}$.

**Remark 2.26.** Note that $i_3$ is certainly not smooth, and yet the above composition $\text{qhol}_t^{i_3}$ is smooth.

Notice that this theorem lets us make the following definition.

**Definition 2.27.** Let $[F]$ be an equivalence class of transport functors and $[\mathcal{I}]$ an equivalence class of local trivialization functors. The gauge invariant holonomy of $[F]$ is defined to be the smooth map in the previous theorem, namely

$$\text{hol}^{[F]}_{\mathcal{I}} := \text{qhol}^{F}_{i_{\mathcal{I}}} : L^1 M \to \text{G/Inn}(G)$$

where $F$ is a representative of $[F]$, $\mathcal{I}$ is a choice of local trivialization, and $\beta : L^1 M \to M$ is a choice of basepoint for thin loops in $M$.

### 3. A review of transport 2-functors

In the present section, we will review the basics of transport 2-functors. As before, we split up the discussion into several parts and follow a similar pattern to the transport functor case. However, since we are now aware of what local triviality should mean, we skip the guess-work and head straight to the correct theory. We start with a Čech description of ordinary principal 2-group 2-bundles (without connection) in terms of smooth 2-functors. We then discuss how to add connection data by introducing transport functors, local triviality, and descent data. The discussion of the reconstruction functor is more involved, and because it is important for the calculation, we spend some time on it. Nevertheless, we skip some technical details (such as compositors and unifiers). Then we consider the differential cocycle data and discuss a formula for higher holonomy in terms of an iterated surface integral. We summarize the results as before. For a full treatment of this, please refer to [19] and [20]. Finally, we discuss surface holonomy in more detail and its gauge covariance. We assume the reader is familiar with the basics of 2-categories but we review some concepts in Appendix 6.2. We use the notation (with minor changes) from [19], where a review of basic notions from 2-category theory is given sufficient for our purposes.

#### 3.1. A Čech description of principal $\mathfrak{G}$-2-bundles

Let $\mathfrak{G}$ be a Lie 2-group and denote the associated crossed module by $H \xrightarrow{\tau} G \xrightarrow{\alpha} \text{Aut}(H)$ (crossed modules are reviewed in Appendix 6.3). Principal $\mathfrak{G}$-2-bundles over a manifold $M$ can be described in terms of 2-functors in terms of the Čech groupoid as well (this also comes from [22] [Remark II.13.]). However, since we are dealing with 2-categories we need to slightly modify the Čech groupoid. The way we do this is just by throwing on identity 2-morphisms. In other words, given an open cover $\{U_i\}_{i \in I}$ of $M$, a 2-morphism from $(x, i, j)$ to $(x', i', j')$ exists only if $x' = x$, $i' = i$, and $j' = j$ and in this case there is only the identity 2-morphism. Composition is uniquely defined by this. This defines the Čech 2-groupoid, also written as $\mathfrak{U}$.

Every 2-group or Lie 2-group $\mathfrak{G}$ can also be viewed as a one object 2-groupoid or Lie 2-groupoid $\mathfrak{B}\mathfrak{G}$ as follows. Denote the one object by $\bullet$. Let the set of morphisms from $\bullet$ to itself be given by the points of $G$. The entire set of 2-morphisms is given by $H \rtimes G$. Horizontal multiplication is the monoidal product and vertical multiplication is the composition. This different multiplications and our conventions are reviewed in Appendix 6.3. In this article, a Lie 2-groupoid means the following.
Definition 3.1. A Lie 2-groupoid is a strict 2-category $\mathcal{C}$ whose objects, 1-morphisms, and 2-morphisms are all smooth spaces and all structure maps are smooth. Furthermore, all 1- and 2-morphisms are all invertible.

Example 3.2. The Čech 2-groupoid mentioned above and $\mathcal{B}G$ for a Lie 2-group $G$ are Lie 2-groupoids.

Definition 3.3. 2-functors between Lie 2-groupoids are smooth if they assignment data smoothly. Similarly, pseudonatural transformations are smooth when the assignments defining them are smooth.

Any smooth 2-functor $\mathcal{U} \to \mathcal{B}G$ gives the Čech cocycle data of a principal $G$-2-bundle over $M$ subordinate to the cover $\{U_i\}_{i \in I}$. To see this, simply recall what a 2-functor does (see Definition 6.11 of Appendix 6.2). To each object $(x, i)$ in $\mathcal{U}$ it assigns the single object $\bullet$ in $BG$. To each jump $(x, i, j)$, it assigns an element denoted by $g_{ij}(x) \in G$ in such a way that we get a smooth 1-cochain $g_{ij} : U_{ij} \to G$ as in Section 2.1 but to each triple intersection $U_{ijk}$, which corresponds to the composition of $U_{ij}$ with $U_{jk}$, it assigns an element $h_{ijk}(x) \in H$ in such a way that we get a smooth 2-cochain $f_{ijk} : U_{ijk} \to H$

\[(77)\]

which says (compare with Section 4.1.3)

\[(78)\] $\tau(f_{ijk})g_{jk}g_{ij} = g_{ik}$.

The 2-functor satisfies an associativity condition which is translated into a condition on quadruple intersections where we get a “cocycle condition”

\[(79)\]
where \( f_{ijk} \) is short for \((f_{ijk}, g_{jk}g_{ij})\), etc. This condition says

\[
(f_{jkl}, g_{kl}g_{ik})(e, g_{ij}) = (e, g_{kl})(f_{ijk}, g_{jk}g_{ij})
\]

which gives a condition in the first factor after composing both sides (see the appendix for our conventions on 2-group multiplication)

\[
f_{ijk}f_{jkl} = f_{ikl}(f_{ijk})
\]

The 2-functor also assigns 0-cochains \( \psi_i : U_i \rightarrow H \)

\[
\begin{array}{c}
\bullet \\
\downarrow \\
\downarrow \\
\bullet
\end{array} 
\quad \mapsto 
\begin{array}{c}
\bullet \\
\downarrow \\
\uparrow \\
\bullet
\end{array}
\]

which immediately says (by definition)

\[
\tau(\psi_i) = g_{ii},
\]

satisfying two “degenerate” cocycle conditions on each double intersection \( U_{ij} \) of \( M \) for the two ways one edge can be collapsed on the triangle. One is

\[
\begin{array}{c}
\bullet \\
\downarrow \\
\uparrow \\
\bullet
\end{array} 
\quad \mapsto 
\begin{array}{c}
\bullet \\
\downarrow \\
\uparrow \\
\bullet
\end{array}
\]

which algebraically is written as

\[
h_{ijj}g_{ij}(\psi_i) = e.
\]

The other cocycle condition is

\[
\begin{array}{c}
\bullet \\
\downarrow \\
\uparrow \\
\bullet
\end{array} 
\quad \mapsto 
\begin{array}{c}
\bullet \\
\downarrow \\
\uparrow \\
\bullet
\end{array}
\]

which algebraically says

\[
h_{iij}\psi_i = e.
\]

Refinements and 1-morphisms between two such 2-functors is similar to the ordinary functor case from Section 2.1 but a bit more subtle due to modifications.
(which we won’t discuss now anyway). Let \( \{U'_i\}_{i \in I'} \) be another cover of \( M \). Let \( P : U \to BG \) and \( P' : U' \to BG \) be two smooth functors. A 1-morphism from \( P \) to \( P' \) consists of a common refinement \( \{V_a\}_{a \in A} \) of both \( \{U_i\}_{i \in I} \) and \( \{U'_i\}_{i \in I'} \) along with a smooth pseudo-natural transformation

\[
\begin{array}{c}
\alpha
\end{array}
\]

By definition (see Definition 6.13 of Appendix 6.2), to each object \((x, a)\) in \( \mathcal{V} \) such a pseudo-natural transformation gives a smooth function \( h_a : V_a \to G \) as before, but also to each jump \((x, a, b)\) in \( \mathcal{V} \), it gives another smooth function \( \epsilon_{ab} : V_{ab} \to H \)

which says that

\[
\tau(\epsilon_{ab})h_bg_{ab} = g'_{ab}h_a.
\]

The higher naturality conditions of a pseudo-natural transformation are given as follows. In general, to every 2-morphism, there is an associated naturality condition, but because our 2-morphisms in \( U \) are all identities, this condition is vacuously true. Now, to every pair of composable 1-morphisms \((x, i, j)\) and \((x, j, k)\) we get

\[
\begin{array}{c}
\alpha
\end{array}
\]

Commutativity of this diagram says

\[
(\epsilon, h_c)(f_{abc}, g_{bc}g_{ab}) = (\epsilon_{bc}, h_cg_{bc})(\epsilon, g_{ab})
\]

which gives a condition in the second factor after composing both sides

\[
\epsilon_{ac}h_b(\epsilon_{abc}) = f'_{abc}\alpha_{g'_{ac}}(\epsilon_{ab})\epsilon_{bc}.
\]
for all \( a, b, c \in A \).

Finally, to every object \((x, i)\) we get on each open set \(U_i\)

\[
\begin{array}{c}
\includegraphics{diagram}
\end{array}
\]

where the back face of the cylinder is just the identity 2-morphism, which reads

\[
\frac{(e, h_a) (\psi_a, e)}{(\epsilon_{aa}, h_q g_{aa})} = \frac{(e, h_a)}{(\psi'_a, e) (e, h_a)},
\]

which gives a condition in the first factor after composing both sides

\[
\epsilon_{aa} \alpha_{h_a} (\psi_a) = \psi'_a.
\]

Therefore, a 1-morphism of such principal 2-bundles as described above defines an equivalence of principal 2-bundles as described in [22].

We won’t discuss 2-morphisms now because we will see that the above construction is a special case of the concept of limits. We come back to it when we discuss limits of 2-categories in Section 3.6.

### 3.2. Local triviality of 2-functors.

Just as transport functors describe parallel transport along paths, transport 2-functors describe parallel transport along paths and surfaces. They exhibit a formulation of a generalization of bundles with connection that describe such transport. We start by generalizing the thin path groupoid \(\mathcal{P}_1(X)\) to the thin path 2-groupoid \(\mathcal{P}_2(X)\).

**Definition 3.4.** [The thin path-2-groupoid] Let \(X\) be a smooth manifold. Define the smooth 2-category \(\mathcal{P}_2(X)\) as follows. Define the set of objects and 1-morphisms of \(\mathcal{P}_2(X)\) to be the set of objects and 1-morphisms of \(\mathcal{P}_1(X)\) respectively. The set of 2-morphisms of \(\mathcal{P}_2(X)\), denoted by \(P^2 X\), is given by the set of all bigons in \(X\) (note that there is no assumption on the rank—see Definition 2.6) modulo an equivalence relation, which will again be thin homotopy, but for bigons. Two bigons \(\Gamma\) and \(\Gamma'\) are said to be thinly homotopic if there exists a map \(A : [0, 1] \times [0, 1] \times [0, 1] \to X\) with the following two properties.
i) First, there exists an $\epsilon$ with $\frac{1}{2} > \epsilon > 0$ such that

$$A(t, s, r) = \begin{cases} 
  x & \text{for all } (t, s, r) \in [0, \epsilon] \times [0, 1] \times [0, 1] \\
  y & \text{for all } (t, s, r) \in [1 - \epsilon, 1] \times [0, 1] \times [0, 1] \\
  \gamma(t) & \text{for all } (t, s, r) \in [0, 1] \times [0, \epsilon] \times [0, 1] \\
  \gamma'(t) & \text{for all } (t, s, r) \in [0, 1] \times [1 - \epsilon, 1] \times [0, 1] \\
  \Gamma(t, s) & \text{for all } (t, s, r) \in [0, 1] \times [0, 1] \times [0, \epsilon] \\
  \Gamma'(t, s) & \text{for all } (t, s, r) \in [0, 1] \times [0, 1] \times [1 - \epsilon, 1] 
\end{cases}$$

(97)

ii) Second, the rank of $A$ is strictly less than 3 for all $(t, s, r) \in [0, 1] \times [0, 1] \times [0, 1]$ and is strictly less than 2 for all $(t, s, r) \in [0, 1] \times ([0, \epsilon] \cup [1 - \epsilon, 1]) \times [0, 1]$.

The set of 2-morphisms of $\mathcal{P}_2(X)$ is defined to be the set of thin homotopy classes of bigons with sitting instances. We sometimes call these thin bigons. Such a set also has a natural smooth structure in a similar way to paths. The various compositions needed to define the 2-category $\mathcal{P}_2(X)$ are the usual ways of composing paths and homotopies by either stacking squares vertically or horizontally and parametrizing via double speed vertically or horizontally respectively. However, the definition is a bit subtle for vertical composition since thin homotopy classes were taken. Given two vertically composable thin homotopy classes of bigons of sitting instances (for the illustrative purposes of this example only, we write square brackets to denote the thin homotopy equivalence classes - otherwise we assume it implicitly)

$$\begin{align*}
\gamma &\Rightarrow [\gamma] \\
\delta &\Rightarrow [\delta] \\
\delta' &\Rightarrow [\delta'] \\
\Delta &\Rightarrow [\Delta] \\
x &\Rightarrow [x] \\
y &\Rightarrow [y] 
\end{align*}$$

(98)

the vertical composition is given by first choosing representatives $\delta$ for the target of $\Gamma$ and $\delta'$ for the source of $\Delta$. Then, there exists a thin (rank strictly less than 1) bigon $\Sigma : \delta \Rightarrow \delta'$. Using this thin bigon we can compose in the usual way by the class associated to

$$\begin{align*}
\Sigma(t, 3s) &\quad \text{for } 0 \leq s \leq \frac{1}{3} \\
[\Sigma(t, 3s - 1)] &\quad \text{for } \frac{1}{3} \leq s \leq \frac{2}{3}, \ t \in [0, 1]. \\
\Delta(t, 3s - 2) &\quad \text{for } \frac{2}{3} \leq s \leq 1.
\end{align*}$$

(99)

To avoid this subtlety, and because we can always choose representatives of $\Gamma$ and $\Delta$ so that $\delta = \delta'$, we can ignore $\Sigma$ for all practical purposes of this paper. Therefore, we will write the vertical composition as

$$\begin{align*}
\Gamma(2s) &\quad \text{for } 0 \leq s \leq \frac{1}{2}, \ t \in [0, 1]. \\
\Delta(2s - 1) &\quad \text{for } \frac{1}{2} \leq s \leq 1.
\end{align*}$$

(100)
Given two horizontally composable thin homotopy classes of bigons with sitting instances

\[ z \xrightarrow{\gamma} y \xrightarrow{\delta} x \]

the horizontal composition is given by the class associated to

\[ (\Gamma \Gamma)(t, s) := \begin{cases} \Gamma(2t, s) & \text{for } 0 \leq t \leq \frac{1}{2}, \\ \Gamma'(2t - 1, s) & \text{for } \frac{1}{2} \leq t \leq 1, \end{cases} s \in [0, 1]. \]

All such compositions are well-defined, smooth, associative, have left and right units given by constant bigons for horizontal composition and paths viewed as bigons for vertical composition respectively, and satisfy the interchange law. \( P_{2}(X) \) defines a 2-category internal to the category of smooth spaces. Such a thing is called a smooth 3-space. In fact, \( P_{2}(X) \) is a Lie 2-groupoid since thin homotopy classes of bigons are invertible in both ways and the functions that assign every class to its vertical and horizontal inverses are both smooth.

**Definition 3.5.** Let \( Gr \) be a Lie 2-groupoid, \( T \) be a 2-category, \( i : Gr \to T \) a 2-functor, and \( M \) a smooth manifold. Fix a surjective submersion \( \pi : Y \to M \). A \( \pi \)-local \( i \)-trivialization of a 2-functor \( F : P_{2}(M) \to T \) is a pair \((\text{triv}, t)\) of a strict 2-functor \( \text{triv} : P_{2}(Y) \to Gr \) and a pseudonatural equivalence

\[ \begin{array}{ccc} P_{2}(M) & \xleftarrow{\pi*} & P_{2}(Y) \\ F \downarrow & & \downarrow \alpha \text{triv} \\ T \xrightarrow{i} Gr \end{array} \]

meaning that there exist a weak inverse \( \tilde{t} \) along with modifications \( i_{\alpha} : \tilde{t} \Rightarrow \text{id}_{\pi*F} \) and \( j_{t} : \text{id}_{\text{triv}} \Rightarrow \tilde{t} \) satisfying the zig-zag identities (see [5] [Definition 7.] and particularly their discussion on string diagrams). The 2-groupoid \( Gr \) is called the structure 2-groupoid for \( F \).

2-functors \( F : P_{2}(M) \to T \) equipped with \( \pi \)-local \( i \)-trivializations \((\text{triv}, t)\) form the objects, written as triples \((F, \text{triv}, t)\), of a 2-category denoted by \( \text{Triv}^{2}_{\pi}(i) \).

**Definition 3.6.** A 1-morphism \( \alpha : (F, \text{triv}, t) \to (F', \text{triv}', t') \) in \( \text{Triv}^{2}_{\pi}(i) \) is a pseudo-natural transformation \( \alpha : F \Rightarrow F' \). A 2-morphism \( \alpha \Rightarrow \alpha' \) is a modification (see Definition 6.18 in Appendix 6.2).

**Definition 3.7.** Let \( Gr \) be a Lie 2-groupoid, \( T \) a 2-category, and \( i : Gr \to T \) a 2-functor. Fix a surjective submersion \( \pi : Y \to M \). A descent object is a quadruple \((\text{triv}, g, \psi, f)\) consisting of a strict 2-functor \( \text{triv} : P_{2}(Y) \to Gr \), a pseudonatural
equivalence

\[
\begin{array}{c}
\overset{\pi_{1*}}{\mathcal{P}_2(Y) \xrightarrow{\text{triv}} \mathcal{P}_2(Y[2])} \\
\downarrow g \quad \downarrow \pi_{2*} \\
T \xleftarrow{\text{triv}} \mathcal{P}_2(Y)
\end{array}
\]

and invertible modifications

\[
f : \pi^*_{12}g \Rightarrow \pi^*_{13}g
\]

and invertible modifications

\[
\psi : \text{id}_{\text{triv}} \Rightarrow \Delta^* g.
\]

These modifications must satisfy the coherence conditions which are explicitly given in \[19\] [Definition 2.2.1.] (in the examples of this current paper, the above modifications will actually be trivial and the coherence conditions will automatically be satisfied, which is why we leave them out).

Descent objects form the objects of a 2-category denoted by \(\mathfrak{D}^2\). Morphisms and 2-morphisms are defined as follows.

**Definition 3.8.** A descent 1-morphism from \((\text{triv}, g, \psi, f)\) to \((\text{triv}', g', \psi', f')\) is a pair \((h, \epsilon)\) with \(h\) a pseudo-natural transformation \(h : \text{triv} \Rightarrow \text{triv}'\) and \(\epsilon\) is an invertible modification

\[
\epsilon : \pi^*_{2h}g \Rightarrow \pi^*_{1h}g'
\]

These must satisfy certain identities \[19\] [Definition 2.2.2.].

**Definition 3.9.** Let \((h, \epsilon)\) and \((h', \epsilon')\) be two descent 1-morphisms from \((\text{triv}, g, \psi, f)\) to \((\text{triv}', g', \psi', f')\). A descent 2-morphism from \((h, \epsilon)\) to \((h', \epsilon')\) is a modification \(E : h \Rightarrow h'\) satisfying a certain identity \[19\] [Definition 2.2.3.].

There is a 2-functor \(\text{Ex}^2_\mathbb{P} : \text{Triv}^2(i) \to \mathfrak{D}^2(i)\) which is an equivalence of 2-categories \[19\] [Proposition 4.1.1.]. The inverse functor will be discussed in Section 3.4. \(\text{Ex}^2_\mathbb{P}\) extracts descent data from trivialization data which we now describe in more detail. At the level of objects, this functor is defined as follows. Let \((F, \text{triv}, t)\) be an object in \(\text{Triv}^2(i)\). For the quadruple \((\text{triv}, g, \psi, f)\), take \(t\) to be exactly the same. For \(g\) take the composition \(g := \pi^*_{1t}g\) coming from the composition in the diagram just as before.
but this time $\tilde{\tau}$ is the weak (vertical) inverse to $t$. By definition $f$ should be a modification $f : \pi_{13}^* g \Rightarrow \pi_{13}^* \hat{g}$ but using our definition of $g$, this means that we can break it down as follows

\[
(109) \quad f : \pi_{13}^* g = \pi_{12}^* (\pi_{23}^* \hat{g}) = \left( \begin{array}{c} \pi_{13}^* \\ \pi_{23}^* \\ \pi_{23}^* \hat{g} \end{array} \right) \Rightarrow \pi_{23}^* \hat{g} = \pi_{13}^* \hat{g}
\]

where all equalities hold by commutativity of certain diagrams and the leftover $\Rightarrow$ is what is to be specified. Therefore, we define this modification by the following sequence of modifications (see \[19\] Section 2.3 before [Lemma 2.3.1])

\[
(110) \quad \left( \begin{array}{c} \pi_{13}^* \xi \\ \pi_{23}^* \xi \\ \pi_{23}^* \eta \end{array} \right) \Rightarrow \left( \begin{array}{c} \pi_{13}^* \xi \\ \pi_{23}^* \xi \\ \pi_{23}^* \psi \end{array} \right) \Rightarrow \left( \begin{array}{c} \pi_{13}^* \xi \\ \pi_{23}^* \xi \\ \pi_{23}^* \eta \end{array} \right) \Rightarrow \left( \begin{array}{c} \pi_{13}^* \xi \\ \pi_{23}^* \xi \\ \pi_{23}^* \eta \end{array} \right) \Rightarrow \pi_{23}^* \psi \Rightarrow \pi_{23}^* \eta,
\]

where the associators are the ones from Lemma 6.27, $i_t$ is part of the pseudo-natural equivalence from $t$ and $\tilde{\tau}$, and $l$ is the left unifier in Lemma 6.29.

Finally, by definition $\psi$ should be a modification $\psi : \text{id}_{\text{triv}} \Rightarrow \Delta^* g$ but using our definition of $g$, we can break it down as follows

\[
(111) \quad \psi : \text{id}_{\text{triv}} = \Delta^* \pi_1^* \text{id}_{\text{triv}} \Rightarrow \Delta^* \left( \begin{array}{c} \pi_1^* \\ \pi_1^* \xi \\ \pi_1^* \eta \end{array} \right) = \Delta^* g
\]

and such a modification can be achieved by

\[
(112) \quad \Delta^* \pi_1^* \text{id}_{\text{triv}} \Rightarrow \Delta^* \pi_1^* \left( \begin{array}{c} \tilde{\tau} \\ \xi \\ \eta \end{array} \right) \Rightarrow \Delta^* \left( \begin{array}{c} \tilde{\tau} \\ \xi \\ \eta \end{array} \right)
\]

where $j_t$ is the other part of the pseudo-natural equivalence from $t$ and $\tilde{\tau}$.

It is proved in \[19\] [Lemma 2.3.1.] that this indeed defines a descent object and that this assignment of descent data to trivialization data extends to a 2-functor $\text{Ex}^2_\pi : \text{Triv}^2_\pi \rightarrow \mathcal{D} \text{Ex}^2_\pi(i)$ to include 1-morphisms and 2-morphisms. This is done in \[19\] [Lemma 2.3.2. and Lemma 2.3.3.].

**Definition 3.10.** Let $(F, \text{triv}, t)$ be a $\pi$-local i-trivialization of a 2-functor $F : \mathcal{P}_2(M) \rightarrow T$, i.e. an object of $\text{Triv}^2_\pi(i)$. The descent object associated to the $\pi$-local i-trivialization is $\text{Ex}^2_\pi(F, \text{triv}, t)$. A similar definition is made for 1- and 2-morphisms.

### 3.3. Transport 2-functors.

We now wish to discuss smoothness for descent data. However, to do this is not so simple as it was for ordinary functors. We will have to make a detour to describe how to think of natural transformations as functors and modifications as natural transformations by modifying the source and target categories. For the purposes of this document, we will make stricter assumptions than is done in \[20\] which are sufficient for our purposes and simplify several of the arguments and constructions.

Let $\mathcal{C}$ and $\mathcal{D}$ be two strict 2-categories. Let $\mathcal{C}_{0,1}$ denote the category whose objects and morphisms are the objects and 1-morphisms of $\mathcal{C}$ respectively. Because $\mathcal{C}$ is strict, this defines a category. Let $\mathcal{D}$ be the category whose objects are morphisms $X_f \xrightarrow{\sim} Y_f$ of $\mathcal{D}$. The set of morphisms in $\mathcal{D}$ from $X_f \xrightarrow{\sim} Y_f$ to $X_g \xrightarrow{\sim} Y_g$ are pairs
of morphisms \((x : X_f \to X_g, y : Y_f \to Y_g)\) along with a 2-morphism \(\varphi : gx \Rightarrow yf\) as in the diagram

\[
\begin{array}{c}
X_g \xleftarrow{x} X_f \\
g \downarrow \quad \varphi \downarrow \quad f \\
Y_g \xleftarrow{y} Y_f
\end{array}
\]

The composition is given by stacking

\[
\begin{array}{c}
X_h \xleftarrow{x'} X_g \xleftarrow{x} X_f \\
h \downarrow \quad \psi \downarrow \quad f \\
Y_h \xleftarrow{y'} Y_g \xleftarrow{y} Y_f
\end{array} = \begin{array}{c}
X_h \xleftarrow{x'x} X_f \\
h \downarrow \quad \psi \downarrow \quad f \\
Y_h \xleftarrow{y'y} Y_f
\end{array}.
\]

One can check that under our assumptions, this forms a category.

Notice that \(\Lambda D\) has a bit more structure. It also has a partially defined operation on objects and 1-morphisms given by “stacking vertically.” Suppose that \(X_f \xleftarrow{f} Y_f\) and \(Y_f \xleftarrow{f'} Z_f\) are two 1-morphisms in \(D\) then one can compose them and this gives a partially defined associative and unital operation on objects of \(\Lambda D\). Similarly, given morphisms in \(\Lambda D\) which can be vertically stacked as in the diagram

\[
\begin{array}{c}
X_g \xleftarrow{x} X_f \\
g \downarrow \quad \varphi \downarrow \quad f \\
Y_g \xleftarrow{y} Y_f \\
g' \downarrow \quad \varphi' \downarrow \quad f' \\
Z_g \xleftarrow{z} Z_f
\end{array} = \begin{array}{c}
X_g \xleftarrow{f'g} X_f \\
g' \downarrow \quad \varphi g' \downarrow \quad f'f \\
Y_g \xleftarrow{z} Y_f
\end{array}.
\]

This additional partially defined composition is written as \(\otimes\) in [20], which is confusing at first, but we’ll stick with this to avoid even more confusion when the reader compares notation.

Now consider two 2-functors \(F, G : C \to D\) and \(\rho : F \Rightarrow G\), a pseudo-natural transformation as in

\[
\begin{array}{c}
D \\
\rho \downarrow \\
G
\end{array} \xleftarrow{F} \begin{array}{c}
C
\end{array}.
\]
One can now define a functor $\mathcal{F}(\rho) : \mathcal{C}_{0,1} \to \Lambda \mathcal{D}$ associated to this pseudonatural transformation by the following assignment

\[
\begin{array}{c}
X \\
\mathcal{F}(\rho)
\end{array}
\begin{array}{c}
FX \\
\rho(X)
\end{array}
\begin{array}{c}
GX \\
\gamma
\end{array}
\]

on objects $X$ in $\mathcal{C}_{0,1}$, i.e. objects in $\mathcal{C}$, and

\[
\begin{array}{c}
\begin{array}{c}
Y \\
f
\end{array}
\mathcal{F}(\rho)
\begin{array}{c}
\begin{array}{c}
FX \\
f
\end{array}
\rho(Y)
\end{array}
\begin{array}{c}
GY \\
\gamma
\end{array}
\begin{array}{c}
GX \\
\gamma
\end{array}
\end{array}
\]

on morphisms in $\mathcal{C}_{0,1}$, i.e. 1-morphisms in $\mathcal{C}$. One can check this defines a functor.

Now consider a modification $A : \rho \Rightarrow \sigma$ between two pseudo-natural transformations $\rho, \sigma : F \Rightarrow G$ between two 2-functors $F, G : \mathcal{C} \to \mathcal{D}$ between two strict 2-categories $\mathcal{C}$ and $\mathcal{D}$ as in

\[
\begin{array}{c}
\begin{array}{c}
D \\
\sigma
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
F
\end{array}
\begin{array}{c}
\rho
\end{array}
\begin{array}{c}
\sigma
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\mathcal{C}
\end{array}
\begin{array}{c}
\mathcal{D}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\]
on objects and
\[
\begin{array}{c}
X_g \xrightarrow{x} X_f \\
\downarrow \simeq \downarrow \sigma \\
Y_g \xrightarrow{y} Y_f
\end{array}
\quad \quad \quad
\begin{array}{c}
FX_g \xrightarrow{F_x} FX_f \\
\downarrow \simeq \downarrow \sigma \\
FY_g \xrightarrow{F_y} FY_f
\end{array}
\]
(122)
on morphisms.

**Definition 3.11.** A descent object \((\text{triv}, g, \psi, f)\) as above is said to be smooth if

i) the 2-functor \(\text{triv} : \mathcal{P}_2(Y) \to \text{Gr}\) is smooth,

ii) the functor \(\mathcal{F}(g) : \mathcal{P}_2(Y[2]) \to \Lambda T\) is a transport functor with \(\Lambda\text{Gr}\)-structure, and

iii) the natural transformations \(\mathcal{F}(\psi) : \mathcal{F}(/\text{id}_{\text{triv}}) \Rightarrow \Delta^* \mathcal{F}(g)\) and \(\mathcal{F}(f) : \pi_2^* \mathcal{F}(g) \otimes \pi_1^* \mathcal{F}(g) \Rightarrow \pi_1^* \mathcal{F}(g)\) are morphisms between transport functors.

Smooth descent objects form the objects of a 2-category denoted by \(\mathcal{D}_{\text{es}}^\pi(i)^{\pi}\) and form a sub-2-category of \(\mathcal{D}_{\text{es}}^\pi(i)\). Smoothness of descent 1-morphisms and descent 2-morphisms is discussed in [20] following [Definition 3.1.2].

**Definition 3.12.** A \(\pi\)-local \(i\)-trivialization \((F, \text{triv}, t)\) is said to be smooth if the associated descent object \(\mathcal{E}_{\pi}(F, \text{triv}, t)\) is smooth. The same can be said of 1-morphisms and 2-morphisms.

Smooth local trivializations form the objects of a 2-category denoted by \(\mathcal{T}_{\text{riv}}^{\pi}(i)^{\pi}\) and with smooth 1-morphisms and 2-morphisms form a sub-2-category of \(\mathcal{T}_{\text{riv}}^{\pi}(i)\). Furthermore, \(\mathcal{E}_{\pi}^{\pi}\) restricts to an equivalence of 2-categories of smooth data [20] [Lemma 3.2.3].

After all these definitions, it should be more or less clear now what the definition of a transport 2-functor is by just abstracting what we did for the one-dimensional case. Schreiber and Waldorf [20] [Definition 3.2.1.] define a transport 2-functor as follows.

**Definition 3.13.** Let \(\text{Gr}\) be a Lie 2-groupoid, \(T\) a 2-category, \(i : \text{Gr} \to T\) a 2-functor, and \(M\) a smooth manifold. A transport 2-functor on \(M\) with values in a category \(T\) and with \(\text{Gr}\)-structure is a 2-functor \(\text{tra} : \mathcal{P}_2(M) \to T\) that has the property that there exists a smooth surjective submersion \(\pi : Y \to M\) and a smooth \(\pi\)-local \(i\)-trivialization \((\text{triv}, t)\).

Transport 2-functors over \(M\) with values in \(T\) with \(\text{Gr}\)-structure form the objects of a 2-category \(\mathcal{A}_{\text{Gr}}^{\pi}(M,T)\). A 1-morphism of transport functors is a pseudo-natural transformation of 2-functors for which there exists a common surjective submersion \(\pi\) and smooth \(\pi\)-local \(i\)-trivializations of both 2-functors so that the associated descent 1-morphism is smooth (see comments after [Definition 3.2.1.] in [20]). A similar definition exists for 2-morphisms.

As a short summary, in the past two sections we introduced three categories in discussing transport 2-functors. These were \(\mathcal{D}_{\text{es}}^\pi(i)^{\pi}\), \(\mathcal{T}_{\text{riv}}^{\pi}(i)^{\pi}\), and \(\mathcal{A}_{\text{Gr}}^{\pi}(M,T)\). The category \(\mathcal{T}_{\text{riv}}^{\pi}(i)^{\pi}\) was used to describe local triviality of transport 2-functors and their morphisms in \(\mathcal{A}_{\text{Gr}}^{\pi}(M,T)\). We then constructed a 2-functor \(\mathcal{E}_{\pi}^{\pi} : \mathcal{T}_{\text{riv}}^{\pi}(i)^{\pi} \to \mathcal{D}_{\text{es}}^\pi(i)^{\pi}\) that allowed us to describe smoothness of transport conditions by looking at subcategories \(\mathcal{D}_{\text{es}}^\pi(i)^{\pi} \subset \mathcal{D}_{\text{es}}^\pi(i)\) and \(\mathcal{T}_{\text{riv}}^{\pi}(i)^{\pi} \subset \mathcal{T}_{\text{riv}}^{\pi}(i)\).
3.4. The reconstruction 2-functor: from local to global. As stated before, the 2-functor $\text{Ex}_2^\pi : \text{Triv}_2^\pi(i) \to \text{Des}_2^\pi(i)$ is an equivalence of 2-categories. To construct a (weak) inverse $\text{Rec}_2^\pi : \text{Des}_2^\pi(i) \to \text{Triv}_2^\pi(i)$, we need to enhance the Čech path groupoid so that it includes more data.

We do not require the full general definition of $\mathcal{P}_2^\pi(M)$ in [20] [Section 3.1] for our purposes, but briefly the general definition is obtained by keeping the same objects and morphisms but replacing the relations that we imposed by 2-morphisms and setting relations on those. There are also additional 2-morphisms given by bigons, paths on intersections, and other formal 2-morphisms such as associators, unitors, and 2-morphisms relating the formal product to the usual composition of paths. We therefore warn the reader that although the following definition is not the same as that in [19] [Section 3.1], we use their general results and theorems which in fact rely on their more general definition.

**Definition 3.14.** Let $\mathcal{P}_2^\pi(M)$ be the category whose set of objects and 1-morphisms are the objects and morphisms of $\mathcal{P}_1^\pi(M)$ respectively. The set of 2-morphisms are freely generated by

i) thin homotopy classes of bigons $\Gamma$ in $Y$ with sitting instances,

ii) thin homotopy classes of paths $\Theta : \alpha \to \beta$ in $Y^{[2]}$ with sitting instances thought of as 2-isomorphisms

\[(123)\]

\[
\begin{array}{c}
\pi_1(\beta) \\
\pi_2(\beta)
\end{array}
\xleftarrow{\pi_2(\Theta)}
\begin{array}{c}
\pi_1(\alpha) \\
\pi_2(\alpha)
\end{array}
\xrightarrow{\Theta}
\begin{array}{c}
\pi_1(\alpha) \\
\pi_2(\alpha)
\end{array}
\xleftarrow{\pi_1(\Theta)}
\begin{array}{c}
\beta \\
\alpha
\end{array}
\]

(one should think of this as weakening the first relation in the definition of $\mathcal{P}_1^\pi(M)$—see Figure 9 for a visualization of this),

**Figure 9.** Thinking in terms of an open cover as a submersion, condition ii) above says that if a path $\Theta : \alpha \to \beta$ is in a double intersection, there is a relationship between going along the path first and then jumping versus jumping first and then going along the path. In particular, the two need not be equal.
iii) points $\Xi$ in $Y^{[3]}$ thought of as 2-isomorphisms

\[
\begin{array}{c}
\pi_2(\Xi) \\
\pi_3(\Xi)
\end{array}
\begin{array}{c}
\pi_1(\Xi) \\
\pi_2(\Xi)
\end{array}
\begin{array}{c}
\Xi \\
\Xi
\end{array}
\begin{array}{c}
\pi_1(\Xi) \\
\pi_3(\Xi)
\end{array}
\begin{array}{c}
\Xi \\
\Xi
\end{array}
\]

(one should think of this as weakening the second relation in the definition of $P^+_1(M)$).

iv) points $a$ in $Y$ thought of as 2-isomorphisms

\[
\begin{array}{c}
a \downarrow \Delta_a \\
a \\
\Delta(a)
\end{array}
\]

(one should think of this as weakening part of the third relation in the definition of $P^+_2(M)$).

v) and several other more technical generators discussed in more detail in [19].

There are several relations imposed on the set of 2-morphisms and 2-morphisms that are slightly different from that of $P^+_1(M)$. We will not discuss any of them, and the reader is referred to [20] [Section 3.1] for the details. As before, the compositions will be written with $\ast$ and will be drawn vertically or horizontally when dealing with 2-morphisms.

As before, we will now associate to every object $(\text{triv}, g, \psi, f)$ in $\mathcal{D}es^2_\ast(i)$ a functor $R_{(\text{triv}, g, \psi, f)}: P^+_{2}(M) \rightarrow T$. This functor is defined as follows. It sends $y \in Y$ to $\text{triv}_i(y)$, thin homotopy classes of paths $\gamma$ in $Y$ to $\text{triv}_i(\gamma)$, and jumps $\alpha \in Y^{[2]}$ to $g(\alpha) : \text{triv}_i(\pi_1(\alpha)) \rightarrow \text{triv}_i(\pi_2(\alpha))$. For the basic 2-morphisms, it makes the following assignments

\[
\begin{array}{c}
y \downarrow \Gamma \\
x
\end{array}
\begin{array}{c}
\text{triv}_i(y) \\
\text{triv}_i(\Gamma)
\end{array}
\begin{array}{c}
\text{triv}_i(x) \\
\text{triv}_i(\delta)
\end{array}
\]

for bigons $\Gamma : \gamma \Rightarrow \delta$ in $Y$,
for paths $\Theta : \alpha \to \beta$ in $Y^{[2]}$,

\[
\begin{array}{ccc}
\pi_2(\Xi) & \xrightarrow{\pi_{23}(\Xi)} & \pi_1(\Xi) \\
\pi_3(\Xi) & \xrightarrow{\pi_{13}(\Xi)} & \pi_1(\Xi) \\
\end{array}
\xrightarrow{R_{(\pi_{triv},p,\psi,f)}}
\begin{array}{ccc}
\text{triv}_{1}(\pi_2(\Xi)) & \xrightarrow{g(\pi_{23}(\Xi))} & \text{triv}_{1}(\pi_1(\Xi)) \\
\text{triv}_{1}(\pi_3(\Xi)) & \xrightarrow{g(\pi_{13}(\Xi))} & \text{triv}_{1}(\pi_1(\Xi)) \\
\end{array}
\]

for points $\Xi$ in $Y^{[3]}$, and

\[
\begin{array}{ccc}
\alpha & \xrightarrow{id_\alpha} & \Delta(a) \\
\Delta(a) & \xrightarrow{R_{(\pi_{triv},p,\psi,f)}} & \text{triv}_{1}(a) \\
\end{array}
\xrightarrow{1_{\text{triv}_{1}(a)}}
\begin{array}{ccc}
\text{triv}_{1}(a) & \xrightarrow{g(\Delta(a))} & \text{triv}_{1}(a) \\
\end{array}
\]

for points $a$ in $Y$. This defines a 2-functor $R : \mathcal{D}_{\mathcal{S}_{\mathbb{Z}}(i)} \to \text{Funct}(\mathcal{P}_{\mathbb{Z}}^2(M), T)$ at the level of objects. The rest of this 2-functor is defined in [20] [Proposition 3.3.2.]

We now move on to defining, as before, a section $s : \mathcal{P}_2(M) \to \mathcal{P}_{\mathbb{Z}}^2(M)$ (we write $s$ now instead of $s^\pi$) of the canonical projection functor $p^\pi : \mathcal{P}_{\mathbb{Z}}^2(M) \to \mathcal{P}_2(M)$ defined in the same way as $p^\pi : \mathcal{P}_1(M) \to \mathcal{P}_2(M)$ on the level of objects and morphisms. On the level of 2-morphisms, $p^\pi$ sends a bigon $\Gamma$ in $Y$ to a the bigon $\pi(\Gamma)$ in $M$. It sends a path $\Theta$ in $Y^{[2]}$ to the identity bigon $1_{\pi(\Theta)}$ (the vertical identity) in $M$ and it sends a point $\Xi$ in $Y^{[3]}$ to the constant bigon at the point $\pi(\Xi)$ in $M$. Finally, it sends a point $a$ in $Y$ to the constant bigon at the point $\pi(a)$ in $M$. To define the section, we will constantly use, but do not prove (its proof is quite involved), the following important lemma from [20] [Lemma 3.2.2.].

**Lemma 3.15.** Let $\gamma : x \to x'$ be a path in $M$ and let $\tilde{\gamma}$ and $\tilde{\gamma}'$ be two lifts of $\gamma$ as 1-morphisms in $\mathcal{P}_1^2(M)$ (the existence follows from our choices above when we defined a section functor $s : \mathcal{P}_1(M) \to \mathcal{P}_1^2(M)$). Then there exists a unique 2-isomorphism $A : \tilde{\gamma} \Rightarrow \tilde{\gamma}'$ in $\mathcal{P}_2^2(M)$ such that $p^\pi(A) = 1_\gamma$.

We will use this to define $s : \mathcal{P}_2(M) \to \mathcal{P}_2^2(M)$ on bigons (we have already defined it above on objects and 1-morphisms). Let $\Gamma : \gamma \Rightarrow \delta$ be any bigon in $M$ as in Figure 10.

As in the case of a path, because the domain is compact, there exists a decomposition of the bigon $\Gamma$ into smaller bigons $\{\Gamma_j\}_j$, as in Figure 11 each of which fits into an open set $U_j$.

Therefore, it suffices to define $s(\Gamma_j)$ for a single one of these bigons provided that we match up all sources and targets for the individual ones. Denote the bigon by

\[
\begin{array}{ccc}
x_j' & \xrightarrow{\gamma_j} & x_j \\
\delta_j & \xrightarrow{\Gamma_j} & \end{array}
\]
Then the image of this under $s$ will be defined as

$$s(x_j') \xrightarrow{s_j(x_j')} s_j(x_j) \xrightarrow{s_j(\delta_j)} s_j(\gamma_j) \xrightarrow{s(\gamma_j)} s(x_j).$$

In other words, we have lifted $\Gamma_j$ using the section $s_j : U_j \rightarrow Y$, but to make sure that this image matches up with how $s$ was already defined on objects and
1-morphisms, we use the unique jumps and 2-isomorphisms to match everything (these are the unlabeled 1-morphisms and 2-morphisms). We are using the previous lemma here. The image of the entire bigon $\Gamma$ is then defined by vertical and horizontal compositions of all the $s(\Gamma_j)$ so that $s$ respects compositions.

In [19] [Proposition 3.2.1.], it is shown that this functor $s$ is a weak inverse to $p^\pi$ as in the case for the path groupoid. However, a weak inverse in 2-category theory in this case means that there exists one pseudo-natural equivalence $\zeta : sp^\pi \Rightarrow id_{\mathcal{P}_2^\pi(M)}$ since $p^\pi s = id_{\mathcal{P}_2(M)}$. This means there exists a weak inverse to $\zeta$ which is written as $\xi : id_{\mathcal{P}_2^\pi(M)} \Rightarrow sp^\pi$. The weakness condition means that there are invertible modifications $i_\zeta : \zeta \zeta \Rightarrow id_{sp^\pi}$ and $j_\zeta : id_{id_{\mathcal{P}_2^\pi(M)}} \Rightarrow \zeta \xi$ that satisfy the zig-zag identities. The details are irrelevant for our purposes but can be found in [19] [Section 3.2] as well as the appendix of this paper. We make note of one important consequence of this equivalence. This follows from Lemma 3.15 referred to in the appendix. We reproduce it here for convenience.

**Lemma 3.16.** Let $S$ and $T$ be two 2-categories. Two weak inverses to a 2-functor $F : S \to T$ are pseudonaturally equivalent.

For us, this means that any two section 2-functors $s, s' : \mathcal{P}_2(M) \to \mathcal{P}_2^\pi(M)$ that are weak inverses to $p^\pi$ are pseudo-naturally equivalent [19] [Corollary 3.2.5.]. This simple fact is crucial to understanding the gauge covariance of surface holonomy so we state it as a separate fact.

**Corollary 3.17.** Any two section 2-functors $s, s' : \mathcal{P}_2(M) \to \mathcal{P}_2^\pi(M)$ that are weak inverses to $p^\pi$ are pseudo-naturally equivalent.

Although the proof of Lemma 3.16 is purely a categorical one, we can briefly outline what it means for the two sections $s$ and $s'$ in Corollary 3.17. We can define such a pseudo-natural equivalence $\eta : s \Rightarrow s'$ by the following assignment $M \ni x \mapsto$ the jump from $s(x)$ to $s'(x)$ and $PM \ni \gamma \mapsto$ the unique 2-isomorphism $s(\gamma) \Rightarrow s'(\gamma)$ specified by Lemma 3.15. We will exploit this fact when discussing examples of higher holonomy in Section 4.

As before, the 2-functor $s : \mathcal{P}_2(M) \to \mathcal{P}_2^\pi(M)$ induces a 2-functor $s^* : \text{Funct}(\mathcal{P}_2^\pi(M), T) \to \text{Funct}(\mathcal{P}_2(M), T)$, the pullback along $s$. Similarly, $\text{Rec}^2_s$ is defined as the composition in the diagram

$$
\begin{array}{ccc}
\text{Funct}(\mathcal{P}_2(M), T) & \xrightarrow{\text{Rec}^2_s} & \mathcal{D}\text{es}_s^2(i) \\
\downarrow s^* & & \downarrow R \\
\text{Funct}(\mathcal{P}_2^\pi(M), T) & \xrightarrow{R} & 
\end{array}
$$

(132)

As before, the image of $\mathcal{D}\text{es}_s^2(i)$ under $\text{Rec}^2_s$ lands in $\text{Triv}_s^2(i)$ and the definition is the same as it was before, only this time $\zeta$ is a pseudo-natural equivalence between 2-functors between 2-categories.

As a short summary, in this section we introduced a weak inverse functor $\text{Rec}^2_s : \mathcal{D}\text{es}_s^2(i) \to \text{Triv}_s^2(i)$ for $\text{Ex}^2_s : \text{Triv}_s^2(i) \to \mathcal{D}\text{es}_s^2(i)$ by using a Čech groupoid category $\mathcal{P}_2^\pi(M)$ associated to the surjective submersion $\pi : Y \to M$ to lift points, paths, and bigons in $M$ to points, paths and/or jumps, and bigons and/or jumps in $\mathcal{P}_2^\pi(M)$ respectively.
3.5. Differential cocycle data. In this section, we will give a brief review of an equivalence between differential forms and smooth 2-functors following [18] [Section 2]. This will allow us to describe parallel transport locally in terms of differential cocycle data. We will leave out several proofs but will provide pictures that we find illustrate the necessary ideas behind the statements.

3.5.1. From 2-functors to 2-forms. As usual, let $\mathfrak{G}$ denote the Lie 2-group

\[ H \rtimes G \xrightarrow{\beta} G \]

corresponding to the Lie crossed module $(H, G, \tau, \alpha)$. Given a strict smooth 2-functor $F : P_2(X) \to B\mathfrak{G}$, we will obtain differential forms $A \in \Omega^1(X; \mathfrak{g})$ and $B \in \Omega^2(X; \mathfrak{h})$. These will form the objects of a 2-category $Z^2_{P}(\mathfrak{G})$. By our previous discussion and since our 2-categories $P_2(X)$ and $B\mathfrak{G}$ are strict and the 2-functor $F$ is strict, the smooth functions $F_0$ and $F_1$ furnish a smooth functor of the underlying smooth 1-categories. Therefore, we obtain a differential form $A \in \Omega^1(X; \mathfrak{g})$. To obtain the differential form $B \in \Omega^2(X; \mathfrak{h})$ we focus on the composition

\[ P^2X \xrightarrow{F_2} H \times G \xrightarrow{p_H} H, \]

where $p_H$ is the projection onto the $H$ factor.

Let $x \in X$ and $v_1, v_2 \in T_xX$ and let $\Gamma : \mathbb{R}^2 \to X$ be a smooth map such that

\[ \Gamma((0,0)) = x, \quad \frac{\partial}{\partial s} \bigg|_{s=0} \Gamma(s, t = 0) = v_1, \quad \text{and} \quad \frac{\partial}{\partial t} \bigg|_{t=0} \Gamma(s = 0, t) = v_2. \]

We will want to obtain a bigon out of this and since we’ll be taking derivatives, we’ll want the bigon to get smaller and smaller. We do this by defining a map $\Sigma : \mathbb{R}^2 \to P^2\mathbb{R}^2$ by sending $(s, t)$ to the thin homotopy class of the bigon in Figure 12. Note that this is unambiguously defined after modding out by thin homotopy because a bigon in $\mathbb{R}^2$ is determined by its source and target paths up to thin homotopy.

![Figure 12](image-url)

**Figure 12.** A point $(s, t)$ in $\mathbb{R}^2$ gets mapped to the bigon in $\mathbb{R}^2$ shown on the right under the map $\Sigma_{\mathbb{R}}$.

Then we use this to define a smooth map $F_{\Gamma}$ by the composition of smooth maps

\[ \mathbb{R}^2 \xrightarrow{\Sigma_{\mathbb{R}}} P^2\mathbb{R}^2 \xrightarrow{\Gamma_{\mathbb{R}}} P^2X \xrightarrow{F_2} H \times G \xrightarrow{p_H} H. \]

This gives an element of the Lie algebra $\mathfrak{h}$ by taking derivatives

\[ B_x(v_1, v_2) := -\frac{\partial^2 F_{\Gamma}}{\partial s \partial t} \bigg|_{(0,0)} \in \mathfrak{h}. \]
Furthermore, this element is independent of the choice of $\Gamma$ provided that the bigon still represents the element $(x,v_1,v_2) \in TX \times_X TX$ after evaluation and derivatives. In fact, we get a smooth differential form $B \in \Omega^2(X;\mathfrak{h})$ (this is not obvious). The source-target matching condition that says $\tau(p_H(F(\Gamma)))F(\gamma) = F(\delta)$ for a bigon $\Gamma : \gamma \Rightarrow \delta$ and it implies

\[(138) \quad dA + \frac{1}{2}[A,A] = \tau(B),\]

where $\tau : \mathfrak{h} \to \mathfrak{g}$ is the differential of the map $\tau : H \to G$. All of these claims are proved in [18] [Section 2.2.1].

3.5.2. From 2-forms to 2-functors. Starting with a $\mathfrak{g}$-valued 1-form $A \in \Omega^1(X;\mathfrak{g})$ on $X$ and a $\mathfrak{h}$-valued 2-form $B \in \Omega^2(X;\mathfrak{h})$ on $X$ we want to define a smooth functor $P^2_X : H \to B\mathfrak{g}$. From the previous work, namely Section 2.6.2, we have already defined the functor at the level of objects and paths. What remains is to define $F^2 : P^2_X \to H \times G$. To do this, we first define a function $k_{A,B} : BX \to G$ on bigons in $X$ with sitting instances (we do not mod out by thin homotopy). Given a bigon $\Sigma : [0,1] \times [0,1] \to X$ with sitting instances, we can pull back the 1-form $A$ and the 2-form $B$ to $[0,1] \times [0,1]$, obtaining $\Sigma^*(A) \in \Omega^1([0,1] \times [0,1];\mathfrak{g})$ and $\Sigma^*(B) \in \Omega^2([0,1] \times [0,1];\mathfrak{h})$.

Schreiber and Waldorf [19] define $k_{A,B}$ in several steps by first introducing an $\mathfrak{h}$-valued 1-form $A_\Sigma \in \Omega^1([0,1];\mathfrak{h})$ after integrating over one of the directions for the bigon. It is defined by

\[(139) \quad (A_\Sigma)_s \left( \frac{d}{ds} \right) := -\int_0^1 dt \alpha_{F_1(\Sigma_s(\gamma_{s,t}))}^{-1} \left( (\Sigma^*B)_{(s,t)} \left( \frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right) \right),\]

where $\gamma_{s,t}$ is defined to be the straight vertical path from $(s,0)$ to $(s,t)$ in $[0,1] \times [0,1]$ as in Figure 13.

![Figure 13](image)

**Figure 13.** The path $\gamma_{s,t}$ is the straight vertical path from the point $(s,0)$ to $(s,t)$ in $[0,1] \times [0,1]$.

Note that in the expression for $A_\Sigma$, there is a path-ordered integral expression in the term $F_1(\Sigma_s(\gamma_{s,t}))$. Otherwise, it is an ordinary integral. Also note that $A_\Sigma$ depends on $\Sigma$. In particular, it is not invariant under thin homotopy.

**Remark 3.18.** Incidentally, although Schreiber and Waldorf in [18] made their own arguments for how to obtain such a formula for $A_\Sigma$, this formula appears in a special case as early as 1977 in the work of Goddard, Nuyts, and Olive on...
magnetic monopoles \[14\] on the right-hand side of equation (2.9) and it may have been known earlier \[7\]. The special case considered is the case of the crossed module \( G \rightarrow G \) with \( \alpha \) being the ordinary conjugation action. We will return to this paper again when discussing magnetic monopoles because our analysis in terms of surface holonomy completely makes rigorous their claims about the topological quantum numbers associated to magnetic charges independent of a Higgs field.

Finally, we define

\[
k_{A,B}(\Sigma) := \alpha_{F_1(\gamma)} \left( \mathcal{P} \exp \left\{ - \int_0^1 A_{\Sigma} \right\} \right)
\]

for a bigon \( \Sigma : \gamma \Rightarrow \delta \).

We can picture this integral schematically as a power series of graphs with marked points and paths as in Figure 14 which is analogous to Figure 13.

![Figure 14](image)

**Figure 14.** The path-ordered integral \( \mathcal{P} \exp \left\{ - \int_0^1 A_{\Sigma} \right\} \) is depicted schematically as an infinite sum of terms expressed by placing \( B \) at the endpoints of the paths, along which we've computed parallel transport using \( A \) making sure to keep the later \( s \)-valued terms on the right. The picture is to be interpreted similarly to the one-dimensional case once we've integrated along the \( t \) direction (vertical) to obtain \( A_{\Sigma} \).

However, each of the paths drawn has a path-ordered integral expression attached to it, and therefore each expression has an additional power series of the form expressed when we discussed the ordinary path-ordered integral.

It turns out this map \( k_{A,B} \) only depends on the thin homotopy class of \( \Sigma \) and therefore factors through a smooth map \( F_2 : P^2 \chi \rightarrow H \) on thin homotopy classes of paths. This map together with \( F_1 \) define a strict smooth 2-functor \( F : \mathcal{P}_2(\chi) \rightarrow \mathcal{B}\Phi \). This is proven in \[18\] [Proposition 2.17.]

**Definition 3.19.** The group element \( k_{A,B}(\Sigma) \) is called the surface transport associated to the bigon \( \Sigma \) and the differential forms \( A \) and \( B \). Abusively, we will sometimes call this surface holonomy or 2-holonomy. Sometimes, it is also called higher holonomy. However, a more precise definition will be given in Section 3.7 when we discuss gauge covariance.

**Remark 3.20.** Historically, understanding the appropriate generalization of the path-ordered integral to surface-ordered integrals was a difficult task. It was not obvious which formulas were correct or even what the criteria for correctness should be. The language of functors allows one to make this precise. The criteria for
correctness is that 2-holonomy should be expressed in terms of a transport 2-functor. By the analysis of this section, any formula that satisfies these functorial properties, has the local constraint given by equation (138), and changes appropriately under gauge transformations (which we have so far only discussed globally but will discuss differentially soon), can be rightfully called 2-holonomy. The specific formula in equation (140) is only one such formula that works. However, there could be many other, potentially simpler formulas, that also describe 2-holonomy. In Section 4 for instance, we prove that for certain structure 2-groups, the formula (140) agrees with one that is easily computable in terms of homotopy classes of paths.

3.5.3. Local differential cocycles for transport 2-functors. By similar considerations to the previous sections, we can differentiate transport functors and use their properties to obtain relations among all the differential data. This section uses differential Lie crossed modules which are briefly reviewed in Appendix 6.3. All the information in this section is discussed in more detail in [20]. In particular, the functions, differential forms, and their relations are all derived in [20]. We merely reproduce the results here.

Definition 3.21. Let $Z^2_X(\mathfrak{G})$ be the category defined as follows. An object of $Z^2_X(\mathfrak{G})$ is a pair $(A, B)$ of a 1-form $A \in \Omega^1(X; g)$ and a 2-form $B \in \Omega^2(X; h)$ satisfying

$$\tau(B) = dA + \frac{1}{2}[A, A].$$

A 1-morphism from $(A, B)$ to $(A', B')$ is a pair $(h, \varphi)$ of a smooth map $h : X \to G$ and a 1-form $\varphi \in \Omega^1(X; h)$ satisfying

$$A' + \tau(\varphi) = \text{Ad}_h(A) - h^*\theta$$

(here $\tau$ is the differential of $\tau : H \to G$—see the appendix for more information) and

$$B' + \alpha_{A'}(\varphi) + d\varphi + \frac{1}{2}[\varphi, \varphi] = \alpha_h(B).$$

The composition is defined by

$$A'' + \alpha_{A'}(\varphi) + d\varphi + \frac{1}{2}[\varphi, \varphi] = \alpha_h(B).$$

A 2-morphism from $(h, \varphi)$ to $(h', \varphi')$, which are both 1-morphisms from $(A, B)$ to $(A', B')$, is a smooth map $f : X \to H$ satisfying

$$h' = \tau(f)h$$

and

$$\varphi' + (R^{-1}_f \circ \alpha_f)(A') = \text{Ad}_f(\varphi) - f^*\theta.$$

The vertical composition is defined by

$$\alpha_{h''}(\varphi'') + d\varphi' + \frac{1}{2}[\varphi', \varphi'] = \alpha_h(B).$$
The horizontal composition is defined by
\[
(148) \quad (A'', B'') \xrightarrow{f_2} (A', B') \xrightarrow{f_1} (A, B) = (A'', B'') \xrightarrow{f_1 \circ_{12} (f_2)} (A, B).
\]

As in Section 2.6.3 we use these arguments to define 2-functors
\[
(149) \quad Z^2_\pi(\mathfrak{G})^x \xrightarrow{P} \text{Funct}^x(X, B\mathfrak{G})
\]

which turn out to be inverses of each other (not just equivalences) as proven in [18, Theorem 2.21.]

As before, this was for globally defined differential data corresponding to globally trivial transport 2-functors. Transport 2-functors on \(M\) are not necessarily of this type, but they are locally trivializable via some surjective submersion \(\pi : Y \to M\) and a \(\pi\)-local \(i\)-trivialization. By similar arguments to the previous discussion in Section 2.6.3 we are led to the following, rather long and complicated, definition.

**Definition 3.22.** Let \(\pi : Y \to M\) be a surjective submersion. Define the category \(Z^2_\pi(\mathfrak{G})\) of differential cocycles subordinate to \(\pi\) as follows. An object of \(Z^2_\pi(\mathfrak{G})\) is a tuple \(((A, B), (g, \varphi), \psi, f)\), where \((A, B)\) is an object in \(Z^2_\pi(G)\), \((g, \varphi)\) is a 1-morphism from \(\pi^*_1(A, B)\) to \(\pi^*_2(A, B)\) in \(Z^2_\pi(\mathfrak{G})\), \(\psi\) is a 2-morphism from \(\text{id}_{(A, B)}\) to \(\Delta^*(g, \varphi)\) in \(Z^2_\pi(\mathfrak{G})\), and \(f\) is a 2-morphism from \(\pi^*_2(\mathfrak{G})\) to \(\pi^*_1(\mathfrak{G})\) to \(\pi^*_1(\mathfrak{G})\). A 1-morphism from \(((A, B), (g, \varphi), \psi, f)\) to \(((A', B'), (g', \varphi'), \psi, f')\) is a tuple \(((h, \phi), \epsilon)\), where \((h, \phi)\) is a 1-morphism from \((A, B)\) to \((A', B')\) in \(Z^2_\pi(\mathfrak{G})\) and \(\epsilon\) is a 2-morphism from \(\pi^*_2(h, \phi)\) to \((g', \varphi')\) in \(Z^2_\pi(\mathfrak{G})\). A 2-morphism from \(((h, \phi), \epsilon)\) to \(((h', \phi'), \epsilon')\) is a 2-morphism \(E(h, \phi)\) to \((h', \phi')\) in \(Z^2_\pi(\mathfrak{G})\).

The above generalizations show that the 2-functors \(P\) and \(D\) extend to and equivalence pair
\[
(150) \quad Z^2_\pi(\mathfrak{G})^x \xrightarrow{P} \text{Det}^x_{\pi}(i)^x
\]
whenever \(i : B\mathfrak{G} \to T\) is an equivalence.

### 3.6. Direct limits

In this section, we get rid of the dependence on the surjective submersion in the categories introduced in the prequel. Several of our 2-categories depended on the choice of a surjective submersion. These 2-categories were \(\text{Triv}^x(i)^x, \text{Det}^x_{\pi}(i)^x,\) and \(Z^2_\pi(\mathfrak{G})^x\). Changing the surjective submersion gives a collection of 2-categories dependent on this surjective submersion. One can take a limit over the collection of surjective submersions in this case. This will be a slight generalization of what was done in Section 2.7. However, there are subtle issues in terms of defining the many compositions.

The general construction is done as follows. Let \(S_\pi\) be a family of 2-categories parametrized by smooth surjective submersions \(\pi : Y \to M\) and let \(F(\zeta) : S_\pi \to S_{\pi\zeta}\) be a family of 2-functors for every refinement \(\zeta : Y' \to Y\) of \(\pi\) satisfying the condition that for any iterated refinement \(\zeta : Y'' \to Y'\) and \(\zeta : Y' \to Y\) of
\( \pi : Y \to M \) then \( F(\zeta') = F(\zeta)F(\zeta) \). In this case, an object of \( S_M := \varprojlim_{\pi} S_\pi \) is given by a pair \((\pi, X)\) of a surjective submersion \( \pi : Y \to M \) and an object \( X \) of \( S_\pi \). A 1-morphism from \((\pi_1, X_1)\) to \((\pi_2, X_2)\) consists of a common refinement

\[
\begin{array}{ccc}
Z & \overset{y_1}{\rightarrow} & Y_1 \\
\downarrow & & \downarrow \\
Y_2 & \overset{\pi_1}{\rightarrow} & M
\end{array}
\]

(151)

together with a 1-morphism \( f : (F(y_1))(X_1) \to (F(y_2))(X_2) \) in \( S_\zeta \). It is written as a pair \((\zeta, f)\). The composition

\[
(\pi_3, X_3) 
\xrightarrow{((\zeta_{23}, g), (\zeta_{12}, f))}
(\pi_2, X_2) 
\xrightarrow{((\zeta_{12}, f))}
(\pi_1, X_1)
\]

(152)

consists of the pullback refinement

\[
\begin{array}{ccc}
Z_{13} & \overset{pr_{12}}{\rightarrow} & Z_{12} \\
\downarrow & & \downarrow \\
Y_1 & \overset{\zeta_{12}}{\rightarrow} & M
\end{array}
\]

(153)

written as \( \zeta_{13} : Z_{13} \to M \) along with the composition \( (F(pr_{23}))(g)(F(pr_{12}))(f) \). A 2-morphism from \((\zeta, f)\) to \((\zeta', f')\) consists of an equivalence class of pairs \((\omega, \alpha)\) where \( \omega \) is a common refinement of \( \zeta \) and \( \zeta' \) as in the following diagram

\[
\begin{array}{ccc}
W & \overset{z'}{\rightarrow} & Z' \\
\downarrow & & \downarrow \\
\overset{y_1'}{Y_1} & \underset{\pi_1}{\rightarrow} & M
\end{array}
\]

(154)

and \( \alpha \) is a 2-morphism \( \alpha : F(z)(f) \Rightarrow F(z')(f') \). Two such pairs \((\omega_1, \alpha_1)\) and \((\omega_2, \alpha_2)\) are equivalent if they agree on the pullback.

After getting rid of the specific choices of the surjective submersions, we can take the limits of all the categories we have introduced. We make the following notation,
slightly differing from that of \[20\]:

\[
\begin{align*}
\text{Triv}_M^2(i)^\infty & := \lim_{\pi} \text{Triv}_M^2(i)^\infty \\
\text{Des}_M^2(i)^\infty & := \lim_{\pi} \text{Des}_M^2(i)^\infty \\
Z^2(M; G) & := \lim_{\pi} Z^2(M; G)\end{align*}
\]

Then from our previous discussions, we just collect the functors we have introduced relating all these categories to \(\text{Trans}^2(M, T)\) after taking such limits over surjective submersions:

\[
\begin{align*}
Z^2(M; G)^\infty \xrightarrow{p} \text{Des}_M^2(i)^\infty \xrightarrow{\text{Rec}_2^\infty} \text{Triv}_M^2(i)^\infty \xrightarrow{\nu} \text{Trans}^2_{BG}(M, T)
\end{align*}
\]

Under the conditions that \(i : BG \to T\) is an equivalence of categories, all of the above 2-functors are equivalence pairs. In the ordinary setting this is proven in \[19\] [Proposition 4.2.1. and Theorem 4.2.2.] and in the smooth setting this is proven in \[20\] [Theorem 3.2.2., Lemma 3.2.3., and Lemma 3.2.4.].

3.7. Higher parallel transport, 2-holonomy, and gauge invariance. For ordinary holonomy, we consider loops with a basepoint and computed parallel transport. We discussed in Section 2.8 how to obtain group-valued holonomies, how the value changes under a change of basepoint, how the value changes under a different choice of trivialization, and how it changes under a different choice of transport functor. We will analyze 2-holonomy in an analogous manner, but it is also important to notice that we have to choose not just basepoints but also based loops. This is all done very precisely in \[20\] [Section 5] and we will not pursue these questions in full generality here. For the examples we give later in this paper, we only care about the special case when our surfaces are spheres with a chosen basepoint. Such a surface is depicted as a bigon by choosing a basepoint \(x\) and considering the sphere as a loop from the constant loop to itself. One such visualization is given in Figure 15. However, since the case of more general base loops for spheres is not very difficult to analyze, we do so as well. This also lets us discuss gauge invariance more naturally. This analysis is completely independent of what types of Lie 2-groups \(G\) we use. In the ordinary holonomy case, we required that \(i : BG \to T\) was a fully
faithful functor. Here we require that $i : B\mathcal{S} \to T$ is a full and faithful 2-functor (this means that it is an equivalence on Hom-categories as described in Definition 6.33 in Appendix 6.2).

To get $G$-valued holonomy functions along paths and $H$-valued holonomy functions along surfaces, we consider the following composition of functors (starting at the left and moving clockwise),

$$
\begin{array}{ccc}
\text{Trans}^2_{\mathcal{B}\mathcal{S}}(M,T) & \xrightarrow{\text{Ex}^2} & \text{Des}^2(i)^{\mathcal{C}} \\
\text{Rec}^2 & \xleftarrow{\text{Triv}^2(i)^{\mathcal{C}}} & \end{array}
$$

just as in Section 2.8. Similar comments that applied there hold here as well. For instance, each of these 2-functors is an equivalence of 2-categories under the assumptions we’ve made in the preceding sections. We write this composition as $\mathcal{F} : \text{Trans}^2_{\mathcal{B}\mathcal{S}}(M,T) \to \text{Trans}^2_{\mathcal{B}\mathcal{S}}(M,T)$. This time, $\mathcal{F}$ will assign $G$-valued elements to thin paths for every transport functor $F$ as well as $H$-valued elements to thin bigons. $\mathcal{F}$ will also assign $G$-valued and $H$-valued gauge transformations for every 1-morphism $\eta : F \to F'$ of transport functors. In addition, $\mathcal{F}$ will assign $H$-valued 2-gauge transformations for every 2-morphism $A : \eta \Rightarrow \eta'$. A pseudo-natural equivalence $\mathcal{F} : \text{id} \Rightarrow \mathcal{F}$ describes how to relate the transport functor to the trivialized one. Although modifications of pseudo-natural transformations are allowed, we will not analyze them in this paper. Briefly, they are to be interpreted as relating the two different ways of choosing the pseudo-natural transformations that relate the transport functor to the trivialized one.

Just as before, we quickly review what the composition of 2-functors defining $\mathcal{F}$ are. For a transport 2-functor $F$, we choose a local trivialization $c(F) = (\pi, F, \text{triv}, t)$. Then we extract the local descent object $\text{Ex}(\pi, F, \text{triv}, t) = (\pi, \text{triv}, g, \psi, f)$. Then, we reconstruct a transport 2-functor $\text{Rec}(\pi, \text{triv}, g, \psi, f)$ and then forget the trivialization data keeping just the 2-functor $v(\text{Rec}(\pi, \text{triv}, g, \psi, f))$. The resulting transport 2-functor, written as $\mathcal{F}$, is defined by

$$
\begin{align*}
M \ni x &\mapsto G =: \text{triv}(s(x)) =: \mathcal{F}(x), \\
P^1 M \ni \gamma &\mapsto R_{\text{Ex}(c(F))}(s(\gamma)) =: \mathcal{F}(\gamma), \quad \text{and} \\
P^2 M \ni \Sigma &\mapsto R_{\text{Ex}(c(F))}(s(\Sigma)) =: \mathcal{F}(\Sigma),
\end{align*}
$$

where $s$ is a choice of section $s : \mathcal{P}_2(M) \to \mathcal{P}_2^0(M)$ and $\mathcal{F}(\gamma) := R_{\text{Ex}(c(F))}(s(\gamma))$ is defined by the trivialization $c(F)$, its associated descent object $\text{Ex}(c(F))$, and the section $s$ by choosing a lift of the path $\gamma$ and applying trivialized transport on the pieces and transition functions on the jumps as well as choosing a lift of the bigon $\Sigma$ and applying the necessary maps depending on the decomposition of the bigon (see Section 3.4). The interested reader can explicitly define the compositor and the unitor for the 2-functor $\mathcal{F}$. We won’t need the precise definitions for our analysis when studying 2-holonomy. All we need to know is that the 2-functors defining $\mathcal{F}$ are invertible.

We’d like to restrict higher parallel transport to thin homotopy classes of marked spheres for the purpose of this paper (in general, one would like to restrict to the more general space of thin homotopy classes of marked closed surfaces), and
eventually free thin spheres. First we make a definition of the thin marked sphere space, which should be thought of as analogous to the thin based loop space.

**Definition 3.23.** The thin marked sphere space of $M$ is the limit of the diagram

\[ P^2 M \xrightarrow{\tau} P^1 M \xrightarrow{\tau} M \]

The thin marked sphere space of $M$ is written as $\Omega S^2 M$.

Explicitly,

\[ \Omega S^2 M = \{ \Sigma \in P^2 M \mid s(\Sigma) = t(\Sigma) \text{ and } s(s(\Sigma)) = t(t(\Sigma)) \} \]

so that a thin marked sphere is nothing but a thin bigon from a based loop to itself.

**Remark 3.24.** Note that elements of $\Omega S^2 M$ need not look like embedded spheres in $M$. Indeed, they might look like pinched croissants as Figure 16 indicates. Nevertheless, since we will look at spheres in the end, this little subtlety will not matter.

![Figure 16. A pinched croissant is an example of a thin marked sphere.](image)

**Definition 3.25.** The $\ell$-2-holonomy of $F$, written as $\text{hol}_F^\ell$, is defined as the projection to $H$ of the restriction of parallel transport of a transport 2-functor $F$ to the thin marked sphere space of $M$:

\[ \text{hol}_F^\ell := p_H \left|_{\Omega S^2 M} \right.: \Omega S^2 M \to H. \]

We now pose three questions analogous to the 1-holonomy case but now for 2-holonomy.

i) How does $\text{hol}_F^\ell$ depend on the choice of a based loop? Namely, suppose that two thin based spheres $\Sigma$ and $\Sigma'$, with possibly different based loops, are thinly homotopic without preserving the based loops (see Definition 3.26). Then, how is $\text{hol}_F^\ell(\Sigma)$ related to $\text{hol}_F^\ell(\Sigma')$?

ii) How does $\text{hol}_F^\ell$ depend on $F$? Namely, suppose that $\eta: F \to F'$ is a morphism of transport functors. How is $\text{hol}_F^\ell$ related to $\text{hol}_{F'}^\ell$ in terms of $\eta$?

iii) How does $\text{hol}_F^\ell$ depend on $\ell$, the choice of trivialization? Namely, suppose that $\ell'$ is another trivialization. Then how is $\text{hol}_F^\ell$ related to $\text{hol}_{F}^{\ell'}$?

Due to the fact that we are restricting ourselves to marked spheres instead of arbitrary surfaces, the answer will be closely related to the 1-holonomy case and will be given by a generalized version of conjugation. As before, we need to define what we mean by thin free sphere space and then we’ll proceed to answer the above questions.

Denote the smooth space of spheres in $M$ by $SM := \text{Man}(S^2, M)$. 
Definition 3.26. Two smooth spheres $\Sigma$ and $\Sigma'$ in $M$ are thinly homotopic if there exists a smooth map $h : S^2 \times I \to M$ such that

i) there exists an $\epsilon > 0$ with $h(t, s) = \Sigma(t)$ for $s \leq \epsilon$ and $h(t, s) = \Sigma'(t)$ for $s \geq \epsilon$ and for all $t \in S^2$ and

ii) the smooth map $h$ has rank $\leq 2$.

We denote the space of such thin homotopy classes of spheres as $S^2M$ and call it the free thin sphere space of $M$. Notice that by choosing a marking, i.e. a based loop, for every element in $S^2M$, described by a function $\beta : S^2M \to \Omega^1M$, one obtains a function $i_\beta : S^2M \to \Omega_{S^2}M$. By similar arguments to the loop case, one can argue using thin homotopy that the image indeed lands in $P^2M$.

We now proceed to answering the above questions in order.

i) Just as in the case of loops, we consider two different thin based loops for thin spheres via two functions $\beta, \beta' : S^2M \to \Omega^1M$. Representatives $\Sigma$ and $\Sigma'$ of $i_\beta([\Sigma])$ and $i_{\beta'}([\Sigma])$, respectively, need not have associated based loops that lie on some common image. Figure 17 depicts such a possible situation.

![Figure 17](image.png)

**Figure 17.** Two different representatives of an unbased thin homotopy class of a sphere can have two based loops that do not lie on a common sphere. $\Sigma$ is drawn in green (the "inner" sphere) and has its based loop as $\gamma : x \to x'$ to the left while $\Sigma'$ is drawn in purple (the "outer" sphere) and has its based loop as $\gamma' : x' \to x'$ to the right. Both spheres are oriented the same way.

As in the case of loops, we can use thin homotopy to draw both based loops on the same sphere. To see this, first notice that if we have a representative $\Sigma : \gamma \Rightarrow \gamma$, where $\gamma : x \to x$, of $i_\beta([\Sigma])$ and a representative $\Sigma' : \gamma' \Rightarrow \gamma'$, where $\gamma' : x' \to x'$, of $i_{\beta'}([\Sigma])$, there is an unbased thin homotopy $h : S^2 \times I \to M$ with $h(\cdot, s) = \Sigma$ for $s \leq \epsilon$ and $h(\cdot, s) = \Sigma'$ for $s \geq 1 - \epsilon$ for some $\epsilon > 0$. Such a homotopy allows us to choose an unbased sphere $\tilde{\Sigma}$, a path $\gamma'_{x'} : x \to x'$, and three bigons (not necessarily spheres) with sitting instances $\Sigma_{x} : c_x \Rightarrow \gamma$ (as usual $c_x$ denotes the constant path at $x$), $\Sigma'_{x'} : \gamma' \Rightarrow c_{x'}$, and $\Delta : \gamma'_{x'} \circ \gamma \circ \overline{c_{x'}}$ with the following properties. First, as an unbased sphere, $\Sigma$ can be written
as either of the compositions

\[
\Sigma \gamma' x' \circ \Sigma \gamma x \circ 1_{\gamma' x'} \quad \text{or} \quad \frac{1_{\gamma' x'} \circ \Delta \circ 1_{\gamma' x'}}{\Sigma \gamma x}
\]

(in either order vertically because we are ignoring the based loops). Second, the composition of \textit{bigons}

\[
1_{\gamma' x'} \circ \Delta \circ 1_{\gamma' x'}
\]

\[
\frac{1_{\gamma' x'} \circ \Sigma \gamma x' \circ 1_{\gamma' x'}}{\Sigma \gamma x}
\]

is thinly homotopic to \(\Sigma\) preserving the based loop \(\gamma\). Third, the composition of \textit{bigons}

\[
\Sigma \gamma' x' \circ \Sigma \gamma x \circ 1_{\gamma' x'}
\]

\[
\frac{1_{\gamma' x'} \circ \Sigma \gamma x' \circ 1_{\gamma' x'}}{\Delta}
\]

is thinly homotopic to \(\Sigma'\) preserving the based loop \(\gamma'\). This is depicted in Figures 18 and 19.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure18.png}
\caption{The domain of the homotopy \(h : S^2 \times I \to M\) is drawn as a solid ball with a smaller solid ball removed from the center (like a peach without its seed). It depicts \(\Sigma\) as the inner sphere (not labeled to avoid clutter) and \(\Sigma'\) as the outer sphere. The based loop \(\gamma : x \to x\) of \(\Sigma\) is drawn on the northern hemisphere while the based loop \(\gamma' : x' \to x'\) of \(\Sigma'\) is drawn on the southern hemisphere. By a reparametrization, one can always place the loops on either hemisphere. The homotopy allows us to choose a sphere \(\tilde{\Sigma}\), drawn somewhat in the middle (in orange), that contains both based loops \(\gamma\) and \(\gamma'\) and is thinly homotopic to both \(\Sigma\) and \(\Sigma'\). As a result, there exists a path \(\gamma_{x' x} : x \to x'\) on \(\tilde{\Sigma}\). We continue this analysis in Figure 19.}
\end{figure}
Lemma 3.27. Let $[\Sigma] \in S^2M$ be a thin homotopy class of spheres in $M$. Let $\beta, \beta' : S^2M \to \Omega^3M$ be two based loop-choosing maps. Let $\Sigma$ and $\Sigma'$ be representatives of $i\beta([\Sigma])$ and $i\beta'([\Sigma])$ respectively. Denote the based loop $\beta([\Sigma])$ of $\gamma$ by $x$ and the basepoint $\beta'([\gamma])$ of $\gamma'$ by $x'$. Then, there exists an unbased sphere $\hat{\Sigma}$ (that contains both $\gamma$ and $\gamma'$), a path $\hat{\gamma}_{x'x} : x \to x'$, and three bigons $\Sigma_{x} : c_{x} \Rightarrow \gamma_{x}, \Sigma_{x'} : \gamma' \Rightarrow c'_{x'}, \Delta : \gamma_{x'x} \circ \gamma \circ \gamma_{x'x} \Rightarrow \gamma', \Gamma$ all with sitting instances, such that the following three properties hold.

i) $\hat{\Sigma}$ is the vertical composition of $\Sigma_{x'}, 1_{\gamma_{x'}} \circ \Sigma_{x} \circ 1_{\gamma_{x'}}$ and $\Delta$ in the order given or a cyclic permutation of this order.

ii) $\left(1_{\gamma_{x'}} \circ \Sigma_{x} \circ 1_{\gamma_{x'}}\right)$ is thinly homotopic to $\Sigma'$ preserving the based loop $\gamma'$.

iii) $\left(1_{\gamma_{x'}} \circ \Sigma_{x} \circ 1_{\gamma_{x'}}\right)$ is thinly homotopic to $\Sigma'$ preserving the based loop $\gamma'$.

Therefore, without loss of generality, we can choose a single representative $\Sigma$ (called $\hat{\Sigma}$ in the above Lemma) of a thin homotopy class of spheres $[\Sigma]$ with a decomposition as in the Lemma. Changing notation a bit, let $\Sigma$ be a thin sphere and let $(\Sigma_{x}, \gamma_{x}, x)$ denote one marking and $(\Sigma_{y}, \gamma_{y}, y)$ another marking. This means that $\gamma_{x}$ is a thin loop at $x$ and $\Sigma_{x} : \gamma_{x} \Rightarrow \gamma_{x}$ is the sphere $\Sigma$ written as a thin bigon from the loop $\gamma_{x}$ to itself (and similarly for $\Sigma_{y}$). Choose a path $\gamma_{yx}$ from $x$ to $y$. Then choose a bigon (see Figure 20) $\Delta : \gamma_{yx} \Rightarrow \gamma_{x} \circ \gamma_{x} \circ \gamma_{yx}$. 

Figure 19. From the sphere $\hat{\Sigma}$ in Figure 18 the top cap defines a bigon $\Sigma_{\gamma_{x}} : c_{x} \Rightarrow \gamma_{x}$, drawn on the left in this figure. The path $\gamma_{x'x} : x \to x'$ in Figure 18 defines a bigon $\Delta : \gamma_{x'x} \circ \gamma \circ \gamma_{x'x} \Rightarrow \gamma'$ drawn in the middle of this figure. The bottom cap defines a bigon $\Sigma_{\gamma'x'} : \gamma' \Rightarrow c_{x'}$ drawn on the right.

These last two equations let us write the bigon $\Sigma$ in terms of $\Sigma'$ and vice versa. In fact, we have

$$\Sigma' = 1_{\gamma_{x'}} \circ \Sigma \circ 1_{\gamma_{x'}}$$

up to thin homotopy preserving the based loop $\gamma'$. 

The above argument says that given two based spheres, with possibly different based loops, that are thinly homotopic without preserving the based loops, one can always choose a representative of such a thin homotopy class of a sphere in $M$ with two marked loops so that the associated two based spheres (coming from starting at either based loop) are thinly homotopic to the original two with a thin homotopy that preserves the based loop. More precisely, we proved the following fact.
Figure 20. A based sphere $\Sigma_x$ at $\gamma_x$ and $x$ is shown on the left. The shaded region depicts the surface swept out between $s = 0$ and some small $s$. By choosing another based loop $\gamma_y$ with basepoint $y$ one has another based sphere $\Sigma_y$ with the same orientation. One can choose a path $\gamma_{yx}$ from $x$ to $y$. One can also choose a bigon $\Delta : \gamma_{yx} \circ \gamma_x \circ \gamma_{yx} \Rightarrow \gamma_y$ relating the two choices of markings. This relates the two based spheres via the relation \[(171)\].

With these choices, we have

\[
\begin{align*}
\gamma_y & \quad \Downarrow \quad \Sigma_y \quad y \\
\gamma_x & \quad \Downarrow \quad \Sigma_x \quad x
\end{align*}
\]

\[=\]

\[
\begin{align*}
\gamma_y & \quad \Downarrow \quad \Sigma_y \quad y \\
\gamma_x & \quad \Downarrow \quad \Sigma_x \quad x
\end{align*}
\]

\[\Rightarrow \Delta \]

i.e.

\[
\Sigma_y = 1_{\gamma_y} \Sigma_x 1_{\gamma_{yx}}
\]

By functoriality of the transport 2-functor $F$, we have

\[
\text{hol}_p^F(\Sigma_y) = pH \left( F(\Sigma_y) \right)
\]

\[
= pH \left( 1_{F(\gamma_{yx})} F(\Sigma_x) 1_{F(\gamma_{yx})} \right)
\]

where $C : F(\gamma_{yx}) F(\gamma_x) F(\gamma_{yx}) \Rightarrow F(\gamma_{yx} \circ \gamma_x \circ \gamma_{yx})$ is a combination of the compositor and associators. Writing out this composition in the 2-group $BG$ by using the projection map $p_H$, this gives

\[
\left( (p_H(\xi(\Delta)))^{-1}, \xi(\gamma_y) \right),
\left( p_H(C)^{-1}, F(\gamma_{yx}) F(\gamma_x) F(\gamma_{yx}) \right),
\left( 1, F(\gamma_{yx}) \right),
\left( p_H(C), F(\gamma_{yx}) F(\gamma_x) F(\gamma_{yx}) \right),
\left( p_H(\xi(\Delta)), F(\gamma_{yx} \circ \gamma_x \circ \gamma_{yx}) \right)
\]

\[
(174)
\]
which just has $H$ component given by
\begin{equation}
 p_H(\mathcal{F}(\Delta)) p_H(\mathcal{C}) \alpha_{\mathcal{F}(\gamma_x)} \left( \text{hol}_F(\Sigma_x) \right) p_H(\mathcal{C})^{-1} \left( p_H(\mathcal{F}(\Delta)) \right)^{-1},
\end{equation}
which can be written as
\begin{equation}
 \alpha_{\tau(p_H(\mathcal{F}(\Delta))) p_H(\mathcal{C}) \alpha_{\mathcal{F}(\gamma_x)}} \left( \text{hol}_F(\Sigma_x) \right).
\end{equation}
This result says that the 2-holonomy changes by $\alpha$-conjugation under a change of based loop.

ii) Now suppose that $\eta : F \to F'$ is a 1-morphism of transport $2$-functors. Then, for every thin path $\gamma : x \to y$ we have a $2$-isomorphism
\begin{equation}
\mathcal{F'}(x) = B\mathcal{B} \xrightarrow{\eta(x)} B\mathcal{B} = \mathcal{F}(x)
\end{equation}
\begin{equation}
\mathcal{F'}(y) = B\mathcal{B} \xrightarrow{\eta(y)} B\mathcal{B} = \mathcal{F}(y)
\end{equation}
satisfying the condition that for any bigon $\Sigma : \gamma \Rightarrow \delta$, with $\delta : x \to y$ another path, the following diagram
\begin{equation}
\begin{array}{ccc}
B\mathcal{B} & \xrightarrow{\eta(x)} & B\mathcal{B} \\
\downarrow \mathcal{F'}(\gamma) & & \downarrow \mathcal{F'}(\gamma) \\
B\mathcal{B} & \xrightarrow{\eta(y)} & B\mathcal{B}
\end{array}
\end{equation}
\begin{equation}
\mathcal{F'}(\delta) \xrightarrow{\eta(\Sigma)} \mathcal{F'}(\gamma)
\end{equation}
commutes. In this diagram, the $\eta(\delta)$ in the back is not shown. This diagram commuting means that
\begin{equation}
\frac{\eta(\gamma)}{\mathcal{F'}(\Sigma)} \frac{1}{\mathcal{F'}(x)} \frac{\eta(x)}{\mathcal{F'}(\gamma)} = \frac{1}{\eta(y)} \frac{\eta(\delta)}{\mathcal{F'}(\Sigma)} \frac{\eta(\delta)}{\mathcal{F'}(\gamma)}
\end{equation}
and writing this out using group elements gives
\begin{equation}
(p_H(\mathcal{F'}(\gamma)), \eta(y) \mathcal{F'}(\gamma)) = (1, \eta(y))(p_H(\mathcal{F'}(\Sigma)), \mathcal{F'}(\gamma))
\end{equation}
\begin{equation}
(p_H(\mathcal{F'}(\Sigma)), \mathcal{F'}(\gamma))(1, \eta(x)) = (p_H(\mathcal{F'}(\delta)), \eta(y) \mathcal{F'}(\delta)),
\end{equation}
which says after evaluating both sides and projecting to $H$ yields
\begin{equation}
p_H(\mathcal{F'}(\Sigma)) p_H(\mathcal{F'}(\gamma)) = p_H(\mathcal{F'}(\delta)) \alpha_{\mathcal{F'}(\Sigma)} (p_H(\mathcal{F'}(\Sigma))).
\end{equation}
Solving for $p_H(\mathcal{F'}(\Sigma))$ gives
\begin{equation}
p_H(\mathcal{F'}(\Sigma)) = p_H(\mathcal{F'}(\delta)) \alpha_{\mathcal{F'}(\Sigma)} (p_H(\mathcal{F'}(\Sigma)))^{-1}.
\end{equation}
Now, after specializing to the case where we have based loops at the same basepoint, i.e. $y = x$ and $\delta = \gamma$, so that we are comparing this transport along based spheres, this reduces to
\begin{equation}
\text{hol}_F^x(\Sigma) = p_H(\mathcal{F'}(\gamma)) \alpha_{\mathcal{F'}(\Sigma)} \left( \text{hol}_F^x(\Sigma) \right) p_H(\mathcal{F'}(\gamma))^{-1}
\end{equation}
\begin{equation}
= \alpha_{\tau(p_H(\mathcal{F'}(\gamma))) \mathcal{F'}(\Sigma)} \left( \text{hol}_F^x(\Sigma) \right).
\end{equation}
This says that $\text{hol}_F^E$ when restricted to based spheres changes under $\alpha$-conjugation when the functor $F$ is changed to a gauge equivalent one $F'$.

iii) Suppose that another trivialization $\mathcal{E}'$ was chosen. Any two trivializations are pseudonaturally equivalent, i.e. if $\mathcal{E}'$ was another trivialization then there exists a pseudonatural transformation $\mathcal{S}: \mathcal{E}' \Rightarrow \mathcal{E}$. This follows from the fact that each 2-functor in the composition of 2-functors that define $\mathcal{E}$ is an equivalence of 2-categories and so Lemma 6.32 applies. Therefore, for every transport 2-functor $F$ we have a 1-morphism of transport functors $\mathcal{S}_F: \mathcal{E}_F'' \rightarrow \mathcal{E}_F'$. Of course, we also have a map assigning to every 1-morphism of transport functors $\eta: \mathcal{E}_F \rightarrow \mathcal{E}_F'$ a 2-morphism $\mathcal{S}_\eta: \mathcal{E}_F \Rightarrow \mathcal{E}_F'$ satisfying naturality, but we will not need this fact for the following observation because we are dealing with strict Lie 2-groups. The 1-morphism of transport functors $\mathcal{S}_F$ assigns to every point $\pi \in M$ a morphism $\mathcal{S}_F(\pi): \mathcal{E}_F''(\pi) \rightarrow \mathcal{E}_F'$ and to every path $\gamma: \pi \rightarrow \pi'$ a 2-isomorphism

$\mathcal{S}_F(\pi):
\begin{array}{ccc}
\mathcal{E}_F(\pi) & \xrightarrow{\mathcal{S}_F(\pi)} & \mathcal{E}_F'(\pi) \\
\downarrow & & \downarrow \\
\mathcal{E}_G(\pi) & \xrightarrow{\mathcal{S}_F'(\pi)} & \mathcal{E}_G'(\pi)
\end{array}
$

satisfying the condition that for a bigon $\Sigma: \gamma \rightarrow \delta$ between two paths $\gamma, \delta: \pi \rightarrow \pi'$ the diagram

$\begin{array}{ccc}
\mathcal{E}_F(\pi) & \xrightarrow{\mathcal{S}_F(\pi)} & \mathcal{E}_F'(\pi) \\
\downarrow & & \downarrow \\
\mathcal{E}_G(\pi) & \xrightarrow{\mathcal{S}_F'(\pi)} & \mathcal{E}_G'(\pi)
\end{array}$

commutes. This result is very similar to the previous one and is therefore given by

$\text{hol}_F^E(\Sigma) = \alpha_{\tau(p_H(\mathcal{S}_F(\gamma)))}\mathcal{S}_F(\pi) \left( \text{hol}_F^E(\Sigma) \right)$,

which is again just $\alpha$-conjugation.

In conclusion, when restricted to a sphere, every such situation resulted in $\alpha$-conjugation. This should therefore also be called gauge covariance. This motivates the following definition.

**Definition 3.28.** Let $H \xrightarrow{\tau} G \xrightarrow{\alpha} \text{Aut}(H)$ be a crossed module. The $\alpha$-conjugacy classes in $H$, denoted by $H/\alpha$, is defined to be the quotient of $H$ under the equivalence relation

$h \sim h' \iff \text{there exists a } g \in G \text{ such that } h = \alpha_g(h')$.

Denote the quotient map by $q: H \rightarrow H/\alpha$.

As before, we have a similar theorem for gauge-invariance of 2-holonomy.
Theorem 3.29. Let $F$ be a transport 2-functor with $B\mathcal{G}$ structure with values in $T$ over $M$ and $\mathcal{L}$ a local trivialization 2-functor. Let $S^2M, \Omega S^2M,i_{\beta},$ and $q$ be defined as above. Then the composition

\begin{equation}
H/\alpha \xrightarrow{\beta} H \xleftarrow{\text{hol}^F_F} \Omega S^2M \xleftarrow{i_{\beta}} S^2M
\end{equation}

is

i) smooth,

ii) independent of $\beta$,

iii) independent of the equivalence class of $F$,

iv) and independent of the equivalence class of $\mathcal{L}$.

Notice that this theorem let’s us make the following definition.

Definition 3.30. Let $[F]$ be an equivalence class of transport 2-functors and $[\mathcal{L}]$ an equivalence class of local trivialization 2-functors. The gauge invariant 2-holonomy of $[F]$ is defined to be the smooth map in the previous theorem, namely

\begin{equation}
\text{hol}^F_{[\mathcal{L}]} := q\text{hol}^F_F i_{\beta} : S^2M \rightarrow H/\alpha
\end{equation}

where $F$ is a representative of $[F]$, $\mathcal{L}$ is a choice of local trivialization, and $\beta : S^2M \rightarrow \Omega^1 M$ is a choice of thin based loop for thin spheres in $M$.

We now compare this result to that in [20], where Schreiber and Waldorf introduce the reduced group associated to a 2-group in order to obtain a well-defined 2-holonomy independent of the marking as well as the representative of the transport functor used.

Definition 3.31. If a 2-group is of the form $B\mathcal{G}$ as above where the associated crossed module is $H \xrightarrow{\alpha} G \xrightarrow{\gamma} \text{Aut}(H)$, then the reduced group is defined to be $\mathcal{G}_{\text{red}} := H/[G,H]$, where $[G,H] = \{h^{-1}\alpha g(h) \mid g \in G, h \in H\}$.

The analogous object of the reduced 2-group in the case of ordinary holonomy for principal $G$ bundles with connection is actually $G/[G,G]$, the abelianization of $G$. Recall, $[G,G] = \{gg'g^{-1}g'^{-1} \mid g,g' \in G\}$ is a normal subgroup, called the commutator subgroup, of $G$ so the quotient is an abelian group, in fact in a universal sense.

Lemma 3.32. Let $G$ be a group, $[G,G]$ its commutator subgroup, and $G/\text{Inn}(G)$ conjugacy classes in $G$. The map $G/\text{Inn}(G) \rightarrow G/[G,G]$ given by taking a conjugacy class $[g]$, choosing a representative, and mapping it down to the quotient $G/[G,G]$, is

i) well-defined,

ii) surjective,

iii) and need not be injective in general.

Proof.

i) The map $G/\text{Inn}(G) \rightarrow G/[G,G]$ is well-defined because if $g'$ was another representative of $[g]$, then there would be a $\tilde{g} \in G$ such that $\tilde{g}g\tilde{g}^{-1} = g'$, and under the quotient map, the difference between $g$ and $g'$ is $g'g^{-1} = \tilde{g}g\tilde{g}^{-1}g^{-1} \in [G,G]$.

ii) Since $G \rightarrow G/[G,G]$ is surjective, and the map $G/\text{Inn}(G) \rightarrow G/[G,G]$ defined by choosing a representative is well-defined, the map $G/\text{Inn}(G) \rightarrow G/[G,G]$ is surjective.
iii) To see why the map $G/\text{Inn}(G) \to G/[G,G]$ is, in general, not injective, consider the following example [9]. Let $S_n$ be the symmetric group on $n$ letters, i.e. it is the permutation group of $n$ elements. Let $A_n$ be the alternating group on $n$ letters. This group is defined as the kernel of the homomorphism $S_n \to \{\pm 1\}$ given by taking the sign of the permutation. It turns out this kernel is also the commutator subgroup of $S_n$. Furthermore, it’s index is $[S_n/[S_n,S_n]] = [S_n/A_n] = [S_n : A_n] = 2$. On the other hand, let’s compute the conjugacy classes of $S_n$ for some small $n$. The simplest case actually suffices, although we’ll quote some results for higher $n$ to indicate that the difference between conjugacy classes and abelianization gets bigger. For $n = 3$, the set of conjugacy classes in $S_3$ is given by the following elements. The identity element, written as $( )$ is in its own class. The elements $(1,2), (1,3),$ and $(2,3)$ are in their own class. Finally, the elements $(1,2,3)$ and $(1,3,2)$ are in their own class. Therefore, the set of conjugacy classes for $S_3$ is given by a $3$-element set whereas the abelianization is a $2$-element group. For $S_4$, the set of conjugacy classes is a set of $5$ elements. For $S_5$, the set of conjugacy classes is a set of $7$ elements. The abelianization, however, is always of order $2$.

Therefore, conjugacy classes contain at least as much information about ordinary holonomy as do elements of the abelianization, and they are exactly the elements needed to define holonomy in a gauge invariant way due to Theorem 2.25.

In a similar way, the reduced group $\mathfrak{g}_{\text{red}}$ of a $2$-group $\mathfrak{g}$ is analogous to the abelianization and does not contain the full information of $2$-holonomy in general. One needs an analogue of conjugacy classes for $2$-holonomy. The candidate, for spheres at least, is $\alpha$-conjugacy classes, $H^{\alpha}$. In fact, we have a similar fact concerning $\alpha$-conjugacy classes and the reduced group.

**Lemma 3.33.** Let $H \xrightarrow{\alpha} G \xrightarrow{\phi} \text{Aut}(H)$ be a crossed module, $\mathfrak{g}$ the associated $2$-group, $\mathfrak{g}_{\text{red}} := H/[G,H]$ the reduced group of $\mathfrak{g}$, and $H/\alpha$ the $\alpha$-conjugacy classes in $H$. The map $H/\alpha \to \mathfrak{g}_{\text{red}}$ given by taking a conjugacy class $[h]$, choosing a representative, and mapping it down to the quotient $H/[G,H]$, is

i) well-defined,

ii) surjective,

iii) and need not be injective in general.

**Proof.**

i) Let $h$ and $h'$ be two representatives. Then there exists a $g \in G$ such that $\alpha_g(h) = h'$ and so the difference between $h$ and $h'$ is $h' h^{-1} = \alpha_g(h) h^{-1} \in [G,H]$.

ii) Since $H \to H/[G,H]$ is surjective, and the map $H/\alpha \to H/[G,H]$ defined by choosing a representative is well-defined, the map $H/\alpha \to H/[G,H]$ is surjective.

iii) To see why the map $H/\alpha \to \mathfrak{g}_{\text{red}}$ is, in general, not injective, consider the special case where $H = G$ and $\alpha$ is the ordinary conjugation. Then this case reduces to the previous case of Lemma 3.32.

\[ \square \]
In this case, one can make sense of gauge-invariant quantities coming from 2-holonomy without passing to the reduced group as is done in [20]. In the case of the examples considered in Section 5 one even gets a fixed point under the $\alpha$ action, in which case one does not need to pass to the $\alpha$-conjugacy classes.

**Definition 3.34.** Let $H \xrightarrow{\tau} G \xrightarrow{\alpha} \text{Aut}(H)$ be a crossed module. Denote the fixed points of $H$ under the $\alpha$ action by

$$\text{Inv}(\alpha) := \{ h \in H \mid \alpha_g(h) = h \text{ for all } g \in G \}.$$  

**Lemma 3.35.** In the notation from the previous definition, $\text{Inv}(\alpha)$ is a central subgroup of $H$.

**Proof.** Let $h, h' \in \text{Inv}(\alpha)$. Then

$$\alpha_g(hh') = \alpha_g(h)\alpha_g(h') = hh'$$

for all $g \in G$. Thus, $\text{Inv}(\alpha)$ is closed. $\alpha_g(e) = e$ for all $g \in G$ says $e \in \text{Inv}(\alpha)$. Let $h \in \text{Inv}(\alpha)$, then $\alpha_g(h^{-1}) = (\alpha_g(h))^{-1} = h^{-1}$ showing that $h^{-1} \in \text{Inv}(\alpha)$. Finally, $\text{Inv}(\alpha)$ is central because

$$hkh^{-1} = \alpha_{\tau(h)}(k) = k$$

for all $h \in H$ and $k \in \text{Inv}(\alpha)$. \qed

This will have physical relevance when discussing monopoles—it says that magnetic monopoles have gauge invariant charges that take values in a central subgroup of the pure gauge group.

4. The path-curvature 2-functor associated to a transport functor

In this section we define a particular principal 2-bundle with connection with structure 2-group given by a covering 2-group $\tau : H \to G$. This particular example comes from an ordinary principal $G$-bundle with connection and a choice of a subgroup of $\pi_1(\text{G})$ (defining the covering space $H$). This assignment is functorial. It is described on all levels introduced in the review, namely as a globally defined transport functor, as descent data, and as differential cocycle data. These constructions respect all of the functors relating these different levels.

4.1. The path-curvature 2-functor. The transport 2-functor defined later in this section is motivated by the study of magnetic monopoles in gauge theories as described in [10]. Some of the earlier accounts of similar descriptions can be found in the work of Wu and Yang in [23] under the name ‘total circuit’ and also in the work of Goddard, Nuyts, and Olive in [14]. Of course, several others worked on understanding the “topological quantum number” due to a magnetic charge in terms of just the magnetic charge alone, and the three references mentioned are the ones that have influenced us. We argue in Section 5 that in the case where the base space is a 3-manifold, this transport 2-functor has 2-holonomy along a sphere which is given by the magnetic flux through that sphere. Therefore, we give a mathematically rigorous description for magnetic flux for non-abelian monopoles. A more detailed description of the physics will be given in that section, but first we explain the mathematical structure.

The starting data is a principal $G$-bundle, where $G$ is a connected Lie group, with connection over $M$, which corresponds to a transport functor $\text{tra} : \mathcal{P}_1(M) \to G\text{-Tor}$ with $BG$ structure, but we also include the choice of a subgroup $N$ of $\pi_1(G)$. The
transport 2-functor that we will construct will be called the ‘path-curvature 2-functor.’ We will discuss two interesting cases for the choice of \( N \) although other choices are important for applications in physics so we keep this generality. When \( N = \pi_1(G) \), the path-curvature 2-functor coincides with the curvature 2-functor of Schreiber and Waldorf [20]. The choice \( N = \{1\} \), the trivial group, will be more appropriate in the context of gauge theory and computing invariants. This is the case we focus on for all our computations in Section 5 although we keep the generality since it is relevant in other situations.

To set up this example, we introduce the following Lie 2-group associated to any connected Lie group \( G \). Let \( \tilde{G} \) be the universal cover of \( G \) (we will describe what happens for arbitrary covers later) and denote the covering map by \( \tau : \tilde{G} \to G \). An explicit construction of \( \tilde{G} \) in terms of homotopy classes of paths will be useful for our purposes

\[
\tilde{G} := \text{Top}([0,1], G)/\sim = \{ h : [0,1] \to G \mid h(0) = e \}/\sim
\]

where \( h \sim h' \) if \( h(1) = h'(1) \) and there exists a homotopy \( h \Rightarrow h' \) relative the endpoints. Denote the equivalence class representing a path with square brackets as in \([h]\) or \([t \mapsto h(t)]\), where it is understood that \( t \) takes values in \([0,1]\). The multiplication in \( \tilde{G} \) is defined by choosing representatives and multiplying them pointwise (later we will show that this multiplication can be described in another way that is sometimes more convenient). Let \( \alpha : G \to \text{Aut}(\tilde{G}) \) be the conjugation map \( \alpha_g([h]) := [ghg^{-1}] \), where technically we mean

\[
\alpha_g([h]) := [t \mapsto gh(t)g^{-1}],
\]

where the concatenation means multiplication in \( G \). Define \( \tau : \tilde{G} \to G \) to be evaluation at the endpoint, \( \tau([h]) := h(1) \). It is easy to see that this defines a Lie crossed module.

**Definition 4.1.** The crossed module \( \tau : \tilde{G} \to G \) defined above is called the universal cover crossed module associated to a Lie group \( G \). Denote the associated Lie 2-group by \( \mathcal{G}_{(1)} \) and call it the universal cover 2-group associated to the Lie group \( G \).

In fact, the only way to give a covering map a crossed module structure is the way we have done so above. This follows from the following Lemma.

**Lemma 4.2.** Let \( H \xrightarrow{\tau} G \to \text{Aut}(H) \) be a crossed module with \( \tau \) a surjective map. Then \( \alpha \) is conjugation in \( H \) by a choice of lift, namely

\[
\alpha_g(h') = hh'h^{-1}, \text{ for all } g \in G, h' \in H
\]

for some \( h \) with \( \tau(h) = g \).

**Proof.** This action is well-defined since for another choice \( \tilde{h} \in H \) with \( \tau(\tilde{h}) = g \), we have

\[
\begin{align*}
hh'h^{-1} & (\tilde{h}h'\tilde{h}^{-1})^{-1} \\
& = hh'h^{-1}\tilde{h}h'^{-1}\tilde{h}^{-1} \\
& = \alpha_{\tau(h)}(h')\alpha_{\tau(h)}(h'^{-1}) \\
& = \alpha_g(h')\alpha_g(h'^{-1}) \\
& = \alpha_g(h'h'^{-1}) \\
& = \alpha_g(e) \\
& = e.
\end{align*}
\]
Given any subgroup \( N \subset \pi_1(G) \), we can construct another Lie 2-group in a similar way but by using a different equivalence relation. Define

\[
\tilde{G}_N := \text{Top}([0,1], G)/\sim_N = \{ h : [0,1] \to G \mid h(0) = e \}/\sim_N,
\]

where \( h \sim_N h' \) if \( h(1) = h'(1) \) and \( \left[ \frac{h}{h'} \right] \in N \) where \( h' \) denotes the reverse path and we use a vertical representation for the concatenation of paths in this context.

\[
h(t) := \begin{cases} 
h(2t) & \text{for } 0 \leq t \leq \frac{1}{2} \\ h'(2-2t) & \text{for } \frac{1}{2} \leq t \leq 1 \end{cases}
\]

**Caution:** The notation for path composition might seem a bit confusing at first. However, we will later see that this way of writing it is more in line with a vertical composition defined in a 2-category that will be relevant later. Note that we will never write inverses as fractions when discussing paths.

**Definition 4.3.** An equivalence class representing a path under the \( \sim_N \) equivalence relation will be denoted by \([h]_N\) or \([t \mapsto h(t)]_N\) and will be called an \( N \)-class.

Because \( \pi_1(G) \) is abelian, \( N \) is normal and this defines a Lie crossed module using the same definitions as above except replacing \( \tilde{G} \) with \( \tilde{G}_N \). The corresponding Lie 2-group is denoted by \( \tilde{G}_N \) and is called the \( N \)-cover 2-group associated to the Lie group \( G \) and subgroup \( N \subset \pi_1(G) \).

We use the same notation, namely \( \tau \) and \( \alpha \), to denote the covering map and conjugation action respectively. Since \( N \) will typically be fixed in any context, this should not cause confusion. It will be useful for us to have the following general definition summarizing these results.

**Definition 4.4.** Let \( G \) be a Lie group and \( N \) a subgroup of \( \pi_1(G) \). Then \( \tau : \tilde{G}_N \to G \) as discussed above, with \( \alpha \) the conjugation action, defines a crossed module called the \( N \)-cover crossed module of \( G \). Its associated 2-group is called the \( N \)-covering 2-group. We sometimes abusively say covering crossed module or covering 2-group without referring to \( N \) explicitly.

Let \( N < \pi_1(G) \) be a subgroup of the fundamental group of a Lie group \( G \). We will now define a 2-category \( G\text{-Tor}_N \) whose underlying 1-category \( (G\text{-Tor}_N)_{0,1} \) (see the beginning of Section 3.3) is \( G\text{-Tor} \). Although the category \( G\text{-Tor} \) as a whole is not a 2-space, notice that the set of morphisms between any two \( G \)-torsors is isomorphic to \( G \) and therefore has a unique smooth structure. Furthermore, the composition is a smooth map and is modeled by the group multiplication map \( G \times G \to G \). By this we mean that by choosing basepoints \( a, b, \) and \( c \) in \( G \)-Torsors \( A, B, \) and \( C \) respectively, the composition

\[
G\text{-Tor}(B, C) \times G\text{-Tor}(A, B) \to G\text{-Tor}(A, C)
\]

agrees with the multiplication \( G \times G \to G \) under the isomorphisms specified by the choice of basepoints. Therefore, the composition is smooth. Thus, \( G\text{-Tor} \) is enriched in smooth manifolds. Using this fact, we can extend \( G\text{-Tor} \) to an interesting 2-category \( G\text{-Tor}_N \) in a non-trivial way. Let \( A \) and \( B \) be two \( G \)-torsors and let
\( \varphi, \psi : A \to B \) be two morphisms of \( G \)-torsors as in

\[
\begin{array}{c}
\varphi \\
\downarrow \\
\psi
\end{array}
\quad
\begin{array}{c}
\varphi \\
\downarrow \\
\psi
\end{array}
\quad
\begin{array}{c}
\varphi \\
\downarrow \\
\psi
\end{array}
\]

\( \varphi, \psi \):

\[
\begin{array}{c}
B \\
\downarrow \\
A
\end{array}
\]

We define the set of 2-morphisms, drawn as

\[
\begin{array}{c}
\varphi \\
\downarrow \\
\psi
\end{array}
\quad
\begin{array}{c}
\varphi \\
\downarrow \\
\psi
\end{array}
\quad
\begin{array}{c}
\varphi \\
\downarrow \\
\psi
\end{array}
\quad
\begin{array}{c}
\varphi \\
\downarrow \\
\psi
\end{array}
\]

(207)

\[
\begin{array}{c}
B \\
\downarrow \\
A
\end{array}
\]

(208)

to be the set of \( N \)-classes of paths from \( \varphi \) to \( \psi \) in \( G\)-Tor(\( A, B \)). This means the following. Two paths \( \Sigma : \varphi \to \psi \) and \( \Sigma' : \varphi \to \psi \) in \( G\)-Tor(\( A, B \)), drawn as

are said to be \( N \)-equivalent if under the diffeomorphism defined by

\[
\begin{array}{c}
\varphi \\
\downarrow \\
\psi
\end{array}
\quad
\begin{array}{c}
\varphi \\
\downarrow \\
\psi
\end{array}
\quad
\begin{array}{c}
\varphi \\
\downarrow \\
\psi
\end{array}
\]

(209)

the homotopy class of the loop \( \Sigma : \varphi \to \varphi \), which gets sent to an element of \( \pi_1(G) \) under this isomorphism, is an element of \( N \). This isomorphism where \( \varphi \mapsto e \) is merely for convenience. In particular, the element defined this way is independent of the diffeomorphism defined by \( G\)-Tor(\( A, B \)) \( \varphi \mapsto e \in G \). To see this, if any other isomorphism was chosen, say sending some other morphism \( \varphi' : A \to B \) to \( e \in G \), then there exists a \( g \in G \) so that \( \varphi' \cdot g = \varphi' \) so that \( \varphi \mapsto g^{-1} \). In this case, one gets a loop based at \( g^{-1} \). To get one at \( e \), we merely multiply by \( g \) obtaining a loop based at \( e \in G \). This loop is exactly the same as \( \Sigma \) under the diffeomorphism defined by \( \varphi \mapsto e \). Therefore, the homotopy class is independent of the isomorphism chosen.

Vertical composition is defined on representatives as concatenation of paths.

Horizontal composition can be defined using the \( G \times G \to G \) multiplication. More explicitly, for two composable 2-morphisms as in

choose representatives of such paths so that \( \Sigma : I \to G\)-Tor(\( A, B \)) and \( \Sigma' : I \to G\)-Tor(\( B, C \)) with \( \Sigma(0) = \varphi, \Sigma(1) = \psi, \Sigma'(0) = \varphi', \) and \( \Sigma'(1) = \psi' \). Define the horizontal composition to be the \( N \)-class of the path \( \Sigma' \Sigma \) defined by

\[
\begin{array}{c}
\varphi \\
\downarrow \\
\psi
\end{array}
\quad
\begin{array}{c}
\varphi \\
\downarrow \\
\psi
\end{array}
\quad
\begin{array}{c}
\varphi \\
\downarrow \\
\psi
\end{array}
\quad
\begin{array}{c}
\varphi \\
\downarrow \\
\psi
\end{array}
\]

(211)

(212) \( s \mapsto (\Sigma' \Sigma)(s) := \Sigma'(s) \Sigma(s) \) for \( s \in [0,1] \).
where the composition on the right-hand-side is the usual composition of morphisms in \(G\)-Tor. We check that horizontal composition is well-defined. Suppose that \(\Sigma \sim_N \Omega\) and \(\Sigma' \sim_N \Omega'\). We must show that \(\Sigma'\Sigma \sim_N \Omega\Omega', \) i.e. 

\[
(213) \quad \left[ \frac{\Sigma'\Sigma}{\Omega\Omega'} \right] \in N
\]

but this representative is given by 

\[
(214) \quad \frac{\Sigma'\Sigma}{\Omega\Omega'}(s) = \begin{cases} 
\Sigma'(2s)\Sigma(2s) & \text{for } 0 \leq s \leq \frac{1}{2} \\
\Omega'(2-2s)\Omega(2-2s) & \text{for } \frac{1}{2} \leq s \leq 1
\end{cases}
\]

\[
(215) \quad \left( \frac{\Sigma'}{\Omega} \right)(s) \left( \frac{\Sigma}{\Omega} \right)(s)
\]

which gives two elements of \(N\) (after taking the homotopy class) and since \(N\) is a subgroup the result is also an element of \(N\). A similar argument is used to show that the interchange law holds. Therefore, \(G\)-Tor\(_N\) defines a strict 2-category. We summarize this as a definition for easy reference.

**Definition 4.5.** Let \(G\) be a Lie group and \(N < \pi_1(G)\) a subgroup of the fundamental group. The 2-category \(G\)-Tor\(_N\) has objects and 1-morphisms that of \(G\)-Tor. The composition of 1-morphisms is the same as that in \(G\)-Tor. The set of 2-morphisms between \(G\)-torsor morphisms \(\varphi\) and \(\psi\) in \(G\)-Tor\((A,B)\) are \(N\)-classes of paths from \(\varphi\) to \(\psi\). The vertical composition of 2-morphisms is ordinary composition of homotopy classes of paths. The horizontal composition of 2-morphisms is the pointwise composition of \(G\)-torsor morphisms after choosing representatives.

**Remark 4.6.** One can easily see that when \(N = \pi_1(G)\) the 2-categories \(G\)-Tor\(_N\) and \(G\)-Tor of [20] are equivalent because there is a unique \(\pi_1(G)\)-class of paths between any two morphisms of \(G\)-torsors (since every loop is \(\pi_1(G)\)-equivalent to every other loop).

We will now start by slowly describing the path curvature 2-functor, the structure 2-groupoid, and prove that it is indeed a transport 2-functor in the sense of [20].

**Definition 4.7.** The path-curvature 2-functor \(K_N(\text{tra})\) associated to an ordinary transport functor \(\text{tra}\) in \(\text{Trans}^1_{BG}(M, G\text{-Tor})\) and a subgroup \(N < \pi_1(G)\) is a 2-functor \(K_N(\text{tra}) : \mathcal{P}_2(M) \to G\text{-Tor}_N\) defined as follows. At the level of objects and 1-morphisms \(K_N(\text{tra})\) agrees with \(\text{tra} : \mathcal{P}_1(M) \to G\text{-Tor}\). For every thin bigon \(\Gamma : \gamma \Rightarrow \delta\) in \(\mathcal{P}_2(M)\), choose a representative \(\Gamma \in (P^1M)^I\) (smooth paths in \(P^1M\), denoted by the same letter, and define 

\[
(216) \quad K_N(\text{tra})(\Gamma) := [s \mapsto \text{tra}(\Gamma(\cdot, s))]_N,
\]

i.e. the \(N\)-class of the path from \(\text{tra}(\gamma)\) to \(\text{tra}(\delta)\) going along \(\text{tra}(\Gamma(\cdot, s))\) as a function of \(s \in [0,1]\). The notation means that \(\Gamma(\cdot, s)\) is a path with respect to the first coordinate for each fixed \(s\), and is depicted as a one-parameter family of \(G\)-torsor morphisms

\[
(217) \quad B \xrightarrow{\Gamma(\cdot, s)} A.
\]
This assignment is well-defined since ordinary homotopy is a special case of thin homotopy. It is also easy to check that vertical and horizontal compositions are respected under this assignment. Therefore, this defines a strict 2-functor.

Define a 2-functor \( i_N : \mathcal{B}G_N \to G^\text{Tor}_N \) as follows. At the level of objects, define it to agree with the functor \( i : BG \to G_{\text{Tor}} \), namely the single object gets sent to \( G \) viewed as a right \( G \)-torsor and any morphism, i.e. an element \( g \) of \( G \), gets sent to the \( G \)-torsor morphism from \( G \) to itself given by left multiplication by \( g \) defined by

\[
L_g(g') := gg' \quad \text{for all } g' \in G.
\]

By definition, a 2-morphism in \( \mathcal{B}G_N \) is of the form

\[
\begin{array}{c}
\bullet \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
(h)_{(N)}
\end{array}
\]

\[
\begin{array}{c}
\bullet \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\text{b(1)g}
\end{array}
\]

where \([h]_N\) is viewed as an \( N \)-class of a path \( h \) in \( G \) starting at the identity \( e \) in \( G \) and ending at a point written as \( \tau([h]_N) = b(1) \). The image of this under \( i_N \) is defined to be

\[
\begin{array}{c}
\bullet \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
G
\end{array}
\]

\[
\begin{array}{c}
\bullet \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
L_{g}
\end{array}
\]

\[
\begin{array}{c}
\bullet \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
G
\end{array}
\]

where \( s \mapsto L_{b(s)g} \) is the path in \( G^\text{Tor}(G,G) \cong G \) corresponding to the path \( s \mapsto b(s)g \) in \( G \) under this isomorphism. At this point it is unclear why the vertical composition is respected under \( i_N \). It is here that we need a Lemma that will make this fact obvious.

**Lemma 4.8.** Let \( \tau : H \to G \) be a covering 2-group with elements of \( H \) thought of as homotopy classes of paths in \( G \) starting at the identity \( e \in G \). Let \( h \) and \( h' \) be two representatives of elements \([h], [h'] \in H\). Denote the targets of \( h \) and \( h' \) by \( g \) and \( g' \), respectively. Then (the first equality is a definition)

\[
[h']h = [h' \cdot h] = \left[ \begin{array}{c} h' \\ h_g \end{array} \right]
\]

where \((h'g)(t) = h'(t)g \) for all \( t \in [0,1] \), and the vertical composition is the composition of paths starting with the one on top.

**Proof.** The homotopy is given by

\[
(t,s) \mapsto \begin{cases} 
  h'(st)h((2-s)t) & \text{for } 0 \leq t \leq \frac{1}{2} \\
  h'((2-s)t - 1 + s)h(st + 1 - s) & \text{for } \frac{1}{2} \leq t \leq 1 
\end{cases}
\]

and all \( s \in [0,1] \). \( \square \)

**Theorem 4.9.** The path-curvature 2-functor \( K_N(\text{tra}) \) defined above is a transport 2-functor with \( \mathcal{B}G_N \)-structure.
Proof. To prove this, we must provide a $\pi_N$-local $i_N$-trivialization of $K_N(\text{tra})$ and show that the associated descent object is smooth. This will be done in several steps, outlined as follows.

i) Define $\text{triv}_N : \mathcal{P}_2(Y) \to \mathcal{B}_N G$ and show it is a smooth strict 2-functor.

ii) Define a natural equivalence $\iota_N : \pi^* K_N(\text{tra}) \Rightarrow i_N \text{triv}_N$.

iii) Explicitly construct the associated descent object $(\text{triv}_N, g_N, \psi_N, f_N)$.

iv) Show that the descent object is smooth.

i) To start, $\text{tra} : \mathcal{P}_1(M) \to G$-Tor is assumed to be a transport functor, so there exists a $\pi$-local $i$-trivialization $(\text{triv} : \mathcal{P}_1(Y) \to BG, t : \pi_1^* \text{triv}_i \Rightarrow \pi_1^* \text{triv}_i)$, where $\pi : Y \to M$ is a surjective submersion, and whose associated descent object $\text{Ex}_N^1(\text{tra}, \text{triv}, t)$ is smooth. We first define $\pi_N : Y \to M$ to be $\pi$. Then we define $\text{triv}_N : \mathcal{P}_2(Y) \to \mathcal{B}_N G$ by making it agree with $\text{triv}$ on the 1-category $\mathcal{P}_1(Y)$ inside $\mathcal{P}_2(Y)$. For a bigon $\Gamma : \gamma \Rightarrow \delta$ in $Y$ we define

$$\text{triv}_N(\Gamma) := \left( [s \mapsto \text{triv}(\Gamma(\cdot, s))\text{triv}(\gamma)^{-1}]_N, \text{triv}(\gamma) \right) \in \tilde{G}_N \times G$$

Note that $[s \mapsto \text{triv}(\Gamma(\cdot, s))\text{triv}(\gamma)^{-1}]_N$ makes sense as an element of $\tilde{G}_N$ because $\tilde{G}_N$ is precisely defined to be the set of $N$-classes of paths in $G$ starting at the identity of $G$. Again, this is well-defined because thin homotopy factors through ordinary homotopy.

We first prove that $\text{triv}_N$ as defined is a strict 2-functor. It is already a strict 2-functor at the level of objects and 1-morphisms. We first check that vertical composition of bigons goes to vertical composition of bigons. Consider two bigons $\Gamma : \gamma \Rightarrow \delta$ and $\Delta : \delta \Rightarrow \epsilon$. Their respected images under the assignment above gives

$$\text{triv}(\gamma) \quad \overset{\text{triv}(\Delta)}{\overset{\text{triv}(\gamma)}{\overset{\text{triv}(\gamma)}{\text{triv}(\gamma)}}} \quad \text{triv}(\epsilon)$$

which, after taking the $\tilde{G}_N$ component, gives

$$\left( s \mapsto \text{triv}(\Delta(\cdot, s))\text{triv}(\gamma)^{-1}\text{triv}(\Gamma(\cdot, s))\text{triv}(\gamma)^{-1} \right)_N$$

while first composing in $\mathcal{P}_2(Y)$ and then applying $\text{triv}_N$ gives$^2$

$$\text{triv}_N \left( \begin{array}{l} \Gamma \\ \Delta \end{array} \right) = \left( \left[ s \mapsto \text{triv} \left( \begin{array}{l} \Gamma \\ \Delta \end{array} \right)(\cdot, s) \right] \text{triv}(\gamma)^{-1} \right)_N, \text{triv}(\gamma)$$

A homotopy between these two representatives is achieved by the following map $H(s, r) :=$

$$\begin{cases} \text{triv}(\Gamma(\cdot, (r + 1)s))\text{triv}(\gamma)^{-1} & \text{for } 0 \leq s \leq \frac{r}{2} \\ \text{triv}(\Delta(\cdot, (r + 1)s - r))\text{triv}(\gamma)^{-1}\text{triv}(\Gamma(\cdot, (r + 1)s))\text{triv}(\gamma)^{-1} & \text{for } \frac{r}{2} \leq s \leq 1 - \frac{r}{2} \\ \text{triv}(\Delta(\cdot, (r + 1)s - r))\text{triv}(\gamma)^{-1} & \text{for } 1 - \frac{r}{2} \leq s \leq 1 \end{cases}$$

$^2$Technically, $\Delta : \delta' \Rightarrow \epsilon$ and there is a thin homotopy $\Sigma : \delta \Rightarrow \delta'$ but this means $\text{triv}(\delta) = \text{triv}(\delta')$ so the above statement still holds.

$^3$Again, this is technically not correct. One has to use a thin homotopy $\Sigma : \delta \Rightarrow \delta'$ but the reader can check that the proof follows through with a slightly modified homotopy.
which indeed satisfies

\[(228) \quad H(s, 0) = \text{triv}(\Delta(\cdot, s))\text{triv}(\delta^{-1})\text{triv}(\Gamma(\cdot, s))\text{triv}(\gamma^{-1})\]

and

\[(229) \quad H(s, 1) = \begin{cases} 
\text{triv}(\Gamma(\cdot, 2s))\text{triv}(\gamma^{-1}) & \text{for } 0 \leq s \leq \frac{1}{2} \\
\text{triv}(\Delta(\cdot, 2s - 1))\text{triv}(\gamma^{-1}) & \text{for } 1 - \frac{1}{2} \leq s \leq 1
\end{cases}
\]

This proves more than what we needed since all we had to show was that the two elements are in the same \(N\)-class. Showing that the two representatives are homotopic is stronger and implies they’re in the same \(N\)-class.

Now consider the horizontal composition of \(\Gamma : \gamma \Rightarrow \delta\) and \(\Pi : \alpha \Rightarrow \beta\) written as \(\Pi\Gamma : \alpha\gamma \Rightarrow \beta\delta\). First composing the bigons and then applying the map \(\text{triv}_N\) gives

\[(230) \quad \text{triv}_N(\Pi\Gamma) = \left[ s \mapsto \text{triv}(\Pi(\cdot, s))\text{triv}(\alpha\gamma)^{-1}\right]_N, \alpha\gamma\]

while first applying the map \(\text{triv}\) to each bigon and then multiplying in \(BG_N\) gives

\[(231) \quad p\tilde{G}_N(\text{triv}_N(\Pi)\text{triv}_N(\Gamma))
\]

\[(232) \quad =\left[ s \mapsto \text{triv}(\Pi(\cdot, s))\text{triv}(\alpha)^{-1}\text{triv}(\alpha)\text{triv}(\Gamma(\cdot, s))\text{triv}(\gamma)^{-1}\text{triv}(\alpha)^{-1}\right]_N
\]

\[(233) \quad =\left[ s \mapsto \text{triv}(\Pi(\cdot, s))\text{triv}(\Gamma(\cdot, s))\text{triv}(\alpha\gamma)^{-1}\right]_N
\]

\[(234) \quad =\left[ s \mapsto \text{triv}(\Pi(\cdot, s))\text{triv}(\alpha\gamma)^{-1}\right]_N
\]

because for every fixed \(s\) parallel transport of paths is a homomorphism. This shows that the two representatives are actually equal, which again is more than what we need. Therefore, \(\text{triv}_N\) defines a strict 2-functor.

We now show that this 2-functor is smooth. We already know this 2-functor is smooth at the level of objects and 1-morphisms. To see that it is smooth at the level of 2-morphisms, let \(U \subset \mathbb{R}^k\) be an open set and let \(\varphi : U \to P^2Y\) be a plot (at this point, the reader should review Appendix 6.1). Our goal will be to show that the composition \(U \xrightarrow{\varphi} P^2Y \xrightarrow{\text{triv}_N} \tilde{G}_N \times G\) is smooth in the usual sense (a plot for a finite-dimensional manifold). First, we simplify this by just showing that the projection to \(\tilde{G}_N\) is smooth. For the purposes of this proof, we write this composition \(p\tilde{G}_N\text{triv}_N =: f\). Then, we recall that in defining \(f\), we chose representatives of bigons. Therefore, we think of \(P^2Y\) as the quotient space of smooth paths \((P^1Y)^I\) by homotopy. Therefore, we have \(f\) defined on \((P^1Y)^I\) and we write this map as \(\tilde{f} : (P^1Y)^I \to \tilde{G}_N\).

A map \(\varphi : U \to P^2Y\) as above is a plot if and only if there exists a cover \(\{U_j\}_{j \in J}\) and plots \(\varphi_j : U_j \to (P^1Y)^I\) such that

\[(235) \quad (P^1Y)^I \xrightarrow{\varphi_j} U_j \xleftarrow{\varphi} P^2Y \xrightarrow{\varphi} U\]

commutes for all \(j \in J\). By definition, \(\tilde{f}\) is constant on homotopy classes of paths in \(P^1Y\). Therefore, it suffices to show that \(\tilde{f}\) is smooth which means it
suffices to show that \( \tilde{f} \varphi_j \) is smooth. But this composition is given by

\[
(236) \quad u \mapsto \tilde{f}(\varphi_j(u)) = \left[ s \mapsto \text{triv}(\varphi_j(u)(s))\text{triv}(\varphi_j(u)(0))^{-1} \right]_N,
\]

which can be rewritten using the mapping-cartesian adjunction as

\[
(237) \quad (u, s) \mapsto \text{triv}(\varphi(u)(\cdot, s))\text{triv}(\varphi(u)(0))^{-1}
\]

which is smooth by the definition of what a plot is for the mapping space \( P^1 Y \) and the fact that \( \text{triv} \) is smooth on this space.

ii) Our goal now is to define a natural equivalence \( t_N: \pi^* K_N(\text{tra}) \Rightarrow i_N \text{triv}_N \).

Note that since \( \text{tra} \) is a transport functor, we have a natural isomorphism \( t: \pi^* \Rightarrow i \text{triv} \). Therefore, we define \( t_N \) by \( y \in Y \), an object of \( \mathcal{P}_2(Y) \), gets sent to \( t_N(y) := t(y) \). For \( \gamma \in P^1 Y \), since \( t \) was a natural transformation for ordinary functors, the required diagram already commutes so we set \( t_N(\gamma) := \text{id} \).

iii) Because of our definition of \( \text{triv}_N \) and \( t \) and since our target category is a strict 2-category, the associated descent data will not be too different from the ordinary transport functor case. Namely, the modifications \( \psi_N \) and \( f_N \) are both trivial, i.e. they are the identity 2-morphisms on objects. \( g_N \) is also completely specified by \( g \) since \( t_N \) is specified by \( t \).

iv) As mentioned above, \( \text{triv}_N \) is smooth. What’s left to show is that \( F(g_N) : \mathcal{P}_1(Y[2]) \to \Lambda G^{-\text{Tor}}_N \) is a transport functor with \( \Lambda B\mathcal{G}_N \)-structure. First let’s explicitly describe \( \Lambda B\mathcal{G}_N \) and \( \Lambda G^{-\text{Tor}}_N \). The objects of \( \Lambda B\mathcal{G}_N \) are 1-morphisms of \( B\mathcal{G} \) which are precisely elements of \( G \).

A morphism from \( g_1 \) to \( g_2 \) in \( \Lambda B\mathcal{G}_N \) is a pair of elements \( g_3 \) and \( g_4 \) of \( G \) along with an element \( h \in H \) fitting into the diagram

\[
(238)
\]

Similarly an object of \( \Lambda G^{-\text{Tor}}_N \) is a pair of objects \( P \) and \( P' \) in \( G^{-\text{Tor}}_N \) and a \( G \)-equivariant map \( P \to P' \). A morphism from \( P \to Q \) to \( P' \to Q' \) in \( \Lambda G^{-\text{Tor}}_N \) is a pair of \( G \)-equivariant maps \( p: P \to P' \) and \( q: Q \to Q' \) along with an \( N \)-class of a path \( \alpha : gp = qf \) as in the diagram

\[
(239)
\]

By applying the general definition of \( F(g_N) \), we know that this functor is defined by

\[
(240) \quad Y[2] \ni y \xrightarrow{F(g_N)} i(\text{triv}(\pi_1(y))) = G \quad i(\text{triv}(\pi_2(y))) = G
\]
The above construction is functorial. Namely, for any morphism of parallel transport functors $h : \text{tra} \Rightarrow \text{tra}'$ with $BG$-structure and values in $G\text{-Tor}$, there exists a smooth natural isomorphism $\tilde{g} : \pi_1^*_\text{tra} \Rightarrow \pi_1^*\text{tra}'$ such that $g = \tilde{g}$. Using this fact, one can define $\tilde{g}_N : \pi_1^*\text{tra}_N \Rightarrow \pi_2^*\text{tra}_N$ in an analogous way to how $g_N$ was defined from $g$ but this time using $\tilde{g}$. Furthermore, $F(g_N)$ factors through $\Lambda(i_{G,N})$ via $F(g_N) = \Lambda(i_{G,N})F(\tilde{g}_N)$ since $g = \tilde{g}$.

Therefore this defines a global trivialization with the identity surjective submersion $\text{id} : Y^{[2]} \to Y^{[2]}$ with the trivialization functor being $F(\tilde{g}_N) : \mathcal{P}_1(Y^{[2]}) \to \Lambda BG_N$. This functor is smooth since $\tilde{g}$ is smooth. Furthermore, the descent object associated to this transport functor is trivial because of the global trivialization. Thus $F(g_N)$ defines a transport functor.

This concludes the proof that $K_N(\text{tra})$ defines a transport 2-functor with $BG_N$ structure.

We summarize this by formally writing the definition of the path-curvature 2-functor.

**Definition 4.10.** Let $\text{tra} : \mathcal{P}_1(M) \to G\text{-Tor}$ be a transport functor over $M$ with $BG$ structure and values in $G\text{-Tor}$ and let $N < \pi_1(G)$ be a subgroup. Then the strict 2-functor $K_N(\text{tra}) : \mathcal{P}_2(M) \to G\text{-Tor}_N$ defined by

$$\gamma \quad \Rightarrow \quad \begin{array}{c}
\begin{array}{c}
\mathcal{P}_1(M)
\end{array}
\begin{array}{c}
\mathcal{P}_2(M)
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{tra}(\gamma)
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
x \mapsto \text{tra}(y)
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{tra}(x)
\end{array}
\end{array}
\end{array}$$

is a transport 2-functor and is called the path-curvature transport 2-functor associated to $\text{tra}$ and $N$.

More can be said, although we will not prove the details since the proof is simple. The above construction is functorial. Namely, for any morphism of parallel transport functors $h : \text{tra} \Rightarrow \text{tra}'$ with $BG$-structure with values in $G\text{-Tor}$, there is a corresponding 1-morphism of parallel transport 2-functors $h_N : K_N(\text{tra}) \Rightarrow K_N(\text{tra}')$ with $BG_N$-structure with values in $G\text{-Tor}_N$. By viewing $\text{Trans}_{BG}(M, G\text{-Tor})$ as a 2-category whose 2-morphisms are all identities, this defines a 2-functor

$$\mathcal{L}_N^{\text{Trans}} = K_N : \text{Trans}_{BG}^1(M, G\text{-Tor}) \to \text{Trans}_{BG_N}^2(M, G\text{-Tor}_N).$$

In fact, in the above proof, in steps i) and ii), we have also outlined the definition of a 2-functor (see equation (223) and surrounding text)

$$\mathcal{L}_N^{\text{Triv}} : \text{Triv}^1_{\pi}(i)^{\mathbb{Z}} \to \text{Triv}^2_{\pi}(i_N)^{\mathbb{Z}}$$

given by the assignment

$$(\text{tra}, \text{triv}, t) \mapsto (K_N(\text{tra}), \text{triv}_N, t_N := t)$$

and

$$y' \xrightarrow{F(Y^{[2]}) \circ \gamma} y \xrightarrow{F(g_N)} \begin{array}{c}
\gamma \quad \Rightarrow \quad \begin{array}{c}
\begin{array}{c}
\mathcal{P}_1(M)
\end{array}
\begin{array}{c}
\mathcal{P}_2(M)
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{tra}(\gamma)
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
x \mapsto \text{tra}(y)
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{tra}(x)
\end{array}
\end{array}
\end{array}$$
on objects (see Definitions 2.7 and 3.5) and
\[(246)\]
\[\alpha \mapsto \alpha_N := \alpha\]
on morphisms (see Definitions 2.8 and 3.6). In these two assignments, we are viewing a natural transformation as a pseudonatural transformation by assigning the identity 2-morphism to every 1-morphism.

In steps iii) and iv) we have also outlined the definition of a 2-functor
\[(247)\]
\[\mathcal{L}_N^{Des} : \mathfrak{Des}_1^\infty(i)^\infty \to \mathfrak{Des}_2^\infty(i_N)^\infty\]
given by the assignment
\[(248)\]
\[(\text{triv}, g) \mapsto (\text{triv}_N, g_N := g, \psi_N := 1, f_N := 1)\]
on objects (see Definitions 2.10 and 3.7) and
\[(249)\]
\[h \mapsto (h_N := h, \epsilon_N := 1)\]
on morphisms (see Definitions 2.11 and 3.8).

By definition, both squares in the diagram
\[(250)\]
\[
\begin{array}{ccc}
\mathfrak{Des}_1^\infty(i)^\infty & \xleftarrow{\text{Ex}} & \text{Triv}_1^\infty(i)^\infty & \xleftarrow{c} & \text{Trans}^1_{BG}(M, G\text{-Tor}) \\
\mathcal{L}_N^{Des} & \downarrow & \mathcal{L}_N^{Triv} & \downarrow & K_N \\
\mathfrak{Des}_2^\infty(i_N)^\infty & \xleftarrow{\text{Ex}} & \text{Triv}_2^\infty(i_N)^\infty & \xleftarrow{c} & \text{Trans}^2_{BG_N}(M, G\text{-Tor}_N)
\end{array}
\]
commute, on the nose.

The path-curvature 2-functor associated to a transport functor is flat. We define this in a moment, but first we define a modified version of the thin path 2-groupoid following [20].

**Definition 4.11.** Let $X$ be a manifold. If one drops condition ii) from Definition 3.4, then one obtains a 2-groupoid $\Pi_2(X)$ that has points of $X$ as objects, thin homotopy classes of paths for 1-morphisms, and (ordinary) homotopy classes of bigons for 2-morphisms.

[20] call this 2-groupoid the *fundamental 2-groupoid*. Although we prefer to use that terminology for the usual fundamental 2-groupoid (whose 1-morphisms are also ordinary homotopy classes of paths), we use this terminology for the purposes of this paper to avoid confusion.

**Definition 4.12.** A transport 2-functor $F : \mathcal{P}_2(M) \to T$ with Gr-structure is said to be flat if it factors through the fundamental 2-groupoid $\Pi_2(M)$.

The curvature 2-functor $K(\text{tra}) = K_{z(G)}(\text{tra})$ introduced in [20] is completely determined on bigons by the boundary of the bigon. It is therefore obviously flat, but it is even more restrictive than just that. Not only does it not depend on the homotopy class of the bigon, it doesn’t depend on the bigon at all. On the other hand, the path-curvature 2-functor $K_N(\text{tra})$ introduced here depends on the homotopy class of the bigon, but it is still flat because it factors through $\Pi_2(M)$ at the level of 2-morphisms.

**Corollary 4.13.** The path-curvature 2-functor $K_N(\text{tra})$ is flat.

We will explore this further when we discuss 2-holonomy for the path-curvature 2-functor in Section 5.
4.2. A description in terms of differential form data. In this section, we prove several important and useful facts. The first theorem says that locally transport functors whose structure 2-group is a covering 2-group can be described in terms of the path-curvature 2-functor. The second part of this section contains a discussion about the relationship between the path-curvature 2-functor specifically and its differential cocycle data. As before, let $\pi : Y \to M$ denote a surjective submersion, $G$ a connected Lie group, $N < \pi_1(G)$ a subgroup, and $\tau : \tilde{G}_N \to G$ the cover of $G$ defined by $N$. Denote the 2-category associated to the crossed module from this covering 2-group by $\tilde{G}_N$.

First, we define a 2-functor
\[
\mathcal{L}_N^Z : Z^1_\pi(G) \to Z^2_\pi(\tilde{G}_N)
\]
by
\[
(A, g) \mapsto ((A, B := \tau^{-1}(F)), (g, \varphi := 1), (\psi := 1, f := 1))
\]
on objects and
\[
h \mapsto (h, \varphi := 0)
\]
on morphisms. Here $\tau : \tilde{G}_N \to G$ is the map of Lie algebras induced by $\tau$. Please note that we sometimes denote Lie algebras with underlines as described in Appendix 6.3.

Second, we notice that specifically for the path-curvature 2-functor $K_N(\text{tra})$, and particularly its associated descent object $\mathcal{D}(\mathcal{L}_N^{\text{Des}}(K_N(\text{tra})))$, the analysis in Section 3.5 gives the following differential cocycle data associated to $K_N(\text{tra})$. The assignment on thin paths induces a 1-form $A$ with values in $G$ since the functor $K_N(\text{tra})$ agrees precisely with tra on thin paths. On thin bigons, the assignment induces a 2-form $B$ with values in $\tilde{G}_N$ satisfying $dA + \frac{1}{2}[A, A] = \tau(B)$. Because $\tau$ is a local diffeomorphism, $\tau$ is an isomorphism. Therefore, $B$ is determined by this condition and is given by $B = \tau^{-1}\left(dA + \frac{1}{2}[A, A]\right)$. Therefore, the associated differential cocycle data to the path-curvature 2-functor $K_N(\text{tra})$ is
\[
\mathcal{D}(\mathcal{L}_N^{\text{Des}}(K_N(\text{tra}))) = \left(A, B := \tau^{-1}\left(dA + \frac{1}{2}[A, A]\right), g, \varphi = 0, f = 1, \psi = 1\right).
\]
Therefore, the two descriptions agree showing that the diagram
\[
Z^1_\pi(G) \xrightarrow{\mathcal{D}} \mathcal{D}(\mathcal{L}_N^{\text{Des}}(i_G)])[\pi]
\]
\[
\downarrow
\]
\[
Z^2_\pi(\tilde{G}_N) \xrightarrow{\mathcal{D}} \mathcal{D}(\mathcal{L}_N^{\text{Des}}(i_{G,N})\pi)
\]
commutes.

This analysis is actually a bit more general as the following theorem shows.

**Theorem 4.14.** Let $X$ be a manifold and $F_N : \mathcal{P}_2(X) \to B\tilde{G}_N$ be any smooth strict 2-functor. Then there exists a smooth functor $F : \mathcal{P}_1(X) \to BG$ such that $F_N = K_N(F)$.

**Proof.** The functor $\mathcal{D} : \text{Func}^\pi(\mathcal{P}_2(X), B\tilde{G}_N) \to Z^2_\pi(\tilde{G}_N)\pi$ (defined in Section 3.5) produces $(A \in \Omega^1(X; g), B \in \Omega^2(X; h))$ satisfying $dA + \frac{1}{2}[A, A] = \tau(B)$. Since
ξ : h → g is an isomorphism, B = ξ⁻¹ \left( dA + \frac{1}{2} [A, A] \right). Restricting F_N to \( \mathcal{P}_1 X \) gives \( F : \mathcal{P}_1(X) → BG \) that satisfies \( D(F) = A \). By the same token, we have \( D(K_N(F)) = (A, ξ^{-1} (dA + \frac{1}{2} [A, A])) \) which coincides with \( D(F_N) \). Since \( \mathcal{P} : Z^2_\pi(G_N)^\infty → \text{Funct}(\mathcal{P}_1(X), B\mathcal{G}_N) \) is a strict inverse to \( D \) by [Theorem 2.21] of [18], we conclude that \( F_N = K_N(F) \).

This theorem implies the following interesting result about the local formula for 2-holonomy for transport 2-functors with covering 2-groups as their structure 2-groupoids.

**Corollary 4.15.** The formula for local parallel transport for any bigon under any smooth transport functor \( F : \mathcal{P}_2(X) → B\mathcal{G}_N \) is given by the formula

\[
F \left( \begin{array}{c}
\gamma \\
\delta
\end{array} \right) = \bullet \Rightarrow \left( \begin{array}{c}
E(\gamma) \\
E(\delta)
\end{array} \right).
\]

Finally, by [18], the functors \( \mathcal{P} \) in each row of

\[
\begin{array}{c}
Z^2_\pi(G) \xrightarrow{\mathcal{P}} \text{Des}_{\pi}^2(i_G)^\infty \\
Z^2_\pi(G_N) \xrightarrow{\mathcal{P}} \text{Des}_{\pi}^2(i_{G,N})^\infty
\end{array}
\]

are (weak) inverses to \( D \). Therefore, the diagram (257) involving \( \mathcal{P} \) commutes weakly.

5. **Examples and magnetic monopoles**

As briefly mentioned above, the path-curvature transport 2-functor is motivated by constructions in physics. In [23], [10], and [14] the authors define a magnetic charge of a magnetic monopole in terms of a magnetic flux through a sphere by considering a loop of parallel transports enclosing the sphere. More precisely, the general idea was to consider a loop of loops as in Figure 15 and compute the holonomy along each loop and consider the path of such holonomies. This defines a loop at the identity. Taking the homotopy class of this loop was the definition of the magnetic charge in the physics literature. [14] was closer to defining it as a double-path-ordered integral, but stopped short and used other means to analyze it. We want to point out here that it is not obvious that these methods make sense. For instance, is it necessary to begin with the constant loop? What should this loop have anything to do with a magnetic flux, which was defined in the abelian case to be just \( \int_{S^2} F \), the integral of the curvature? Is the resulting quantity gauge invariant? And how does one know that these concepts are even correct?

As we show in this section, the path-curvature transport 2-functor introduced in the previous section gives a precise meaning to magnetic flux in terms of surface holonomy. Furthermore, since this magnetic flux is defined using surface holonomy, for which we have proven gauge covariance in Section 3.7 specifically Theorem 3.29, we can meaningfully ask if the magnetic flux is a gauge invariant quantity.
This would be the case if it is invariant under $\alpha$-conjugation. We review the interesting cases considered in the physics literature, those of $U(1)$ monopoles, $SO(3)$ monopoles, and $SU(n)/Z(n)$ for all $n$. We also consider the cases $U(n)$ for all $n$.

We do this in two ways. We first start with a transport functor, described in terms of its differential cocycle data, and use the methods of Section 3.4 and Section 3.7 to reconstruct a transport functor with group-valued holonomies. We then apply construct the path-curvature 2-functor and compute surface holonomy.

The other method we use, which is equivalent by Theorem 4.14 and Corollary 4.15, is to use the surface-ordered integral of equation (140) from [18] and the definition of the differential cocycle data of the path-curvature 2-functor discussed in Section 4.2. This is unnecessary due to Corollary 4.15 but we do it anyway for the reader’s convenience. In the process, we must discuss section 2-functors $s : \mathcal{P}_2(M) \to \mathcal{P}_2^N(M)$ associated to some surjective submersion $\pi : Y \to M$. We will define the section 2-functor for the paths and bigons of interest to us (rather than defining it for all paths and bigons) in the case of the first example of $U(1)$ monopoles. We then use the same section 2-functor for all other examples.

For the following discussions, we will be using the following conventions depicted in Figure 21 for describing coordinates on the sphere.

![Figure 21](image)

**Figure 21.** The azimuthal angle $\phi$ is drawn in red and extends from the $x$ axis (pointing to the left) and goes counterclockwise in the $xy$-plane. The zenith angle $\theta$ is drawn in blue and extends from the $z$ axis (pointing vertically) towards the $xy$-plane.

### 5.1. Abelian $U(1)$ monopoles

First, we will give an explicit example coming from abelian magnetic monopoles. Let $P[n] \to S^2$ be the principal $U(1)$-bundle described by the following local trivialization. Denote the northern and southern hemispheres by $U_N$ and $U_S$ respectively. Define the transition function $g_{NS} : U_{NS} \simeq S^1 \to U(1)$ along the equator to be

$$g_{NS}(\phi) = e^{in\phi},$$

where $\phi$ is the azimuthal angle and $n$ is an integer. Equip this bundle with a connection such as

$$A_N = \frac{n}{2i}(1 - \cos \theta) d\phi \quad \& \quad A_S = \frac{n}{2i}(1 + \cos \theta) d\phi.$$
This satisfies the correct property, namely
\[(260) \quad A_N = g_{NS} A s g_{NS}^{-1} - d g_{NS} g_{NS}^{-1}\]
on $U_N \cap U_S$. This describes the local differential cocycle data of a principal $U(1)$-bundle with connection. Since $i : BU(1) \to U(1)$-Tor is an equivalence of categories, this differential cocycle data corresponds to a global transport functor by \[17\] [Theorem 5.4].

We now consider the path-curvature 2-functor where $N = \pi_1(S^1) \cong \mathbb{Z}$ so that we consider the universal cover $\mathbb{R} \to U(1)$ defined by $\phi \mapsto e^{2\pi i \phi}$. The surjective submersion is given by the inclusion of open sets $\pi : U_N \sqcup U_S \to S^2$. The functor $P : Z_1^1(G) \to \text{Des}_1^1(i)$ sends the differential cocycle object $(g, A)$ to $\text{triv}_A : P_1(U_N \sqcup U_S) \to BG$ defined by the path-ordered exponential and the natural transformation $g : \pi^1_1(\text{triv}_i) \to \pi^1_2(\text{triv}_i)$ defined on components $\phi \in S^1$ by $i(g_{NS}(\phi))$. We now choose a path-lifting section $s : P_1(S^2) \to \mathcal{P}_1^\bullet(S^2)$ which we will explicitly define on a certain collection of paths. But first we define it on objects. We make the choice
\[(261) s(x) := \begin{cases} s_N(x) = (x, N) & \text{if } x \in U_N \\ s_S(x) = (x, S) & \text{if } x \in S^2 \setminus U_N \end{cases}\]
for objects. For the paths, we only consider paths of the form depicted in Figure 22. The reason for this is because we will consider a sequence of such loops starting at

![Figure 22. A loop on the sphere is made to always start at the equator at the point \(\bullet\). In this figure, the loop is drawn for some \(\theta\) in the range \(\frac{\pi}{2} < \theta < \pi\).](image)

the constant loop at the point \(\bullet\) on the equator enclosing the sphere going from $U_N$ to $U_S$ and finally ending on the constant loop at the point \(\bullet\) as depicted in Figure 23. Therefore, we will assign to a loop in $U_N$ to be precisely that loop in $U_N$. To a loop in $S^2 \setminus U_N$, we use the jump taking us from $U_S$ to $U_N$. Thus,
\[(262) s(\gamma_\theta) := \alpha_{NS}(\bullet) \ast s_S(\gamma_\theta) \ast \alpha_{SN}(\bullet)\]
for $\theta > \frac{\pi}{2}$. In Section 5.5, we will see that all of these choices do not matter for the final result of the 2-holonomy.

We can also extend this section $s : \mathcal{P}_1(S^2) \to \mathcal{P}_1^\bullet(S^2)$ to include lifts for bigons $s : \mathcal{P}_2(S^2) \to \mathcal{P}_2^\bullet(S^2)$, but we only care about two bigons in our calculation. The
Figure 23. Loops along the $\phi$ direction on the sphere of constant $\theta$ are drawn for $\theta = \frac{\pi}{2}$ and two intermediate values in the range $0 < \theta < \frac{\pi}{2}$. However, each loop is made to start at the point $\bullet$ so that the sphere is thought of as a bigon $S^2 : c_\bullet \Rightarrow c_\bullet$.

first is the bigon given by

\begin{equation}
(\phi, \theta) \mapsto \gamma_\theta(\phi), \quad \text{where} \quad (\phi, \theta) \in [0, 2\pi] \times [0, \pi/2]
\end{equation}

and is a bigon $c_\bullet \Rightarrow \gamma_{\pi/2}$ which lands in $U_N$. It covers the northern hemisphere. We send this bigon to the same bigon in $U_N$. In other words, it is given by (see (131))

\begin{equation}
s(\Sigma_N) := s_N(\Sigma_N)
\end{equation}

in $\mathcal{P}_2^\pi(S^2)$. We do a similar thing for

\begin{equation}
(\phi, \theta) \mapsto \gamma_\theta(\phi), \quad \text{where} \quad (\phi, \theta) \in [0, 2\pi] \times [\pi/2, \pi]
\end{equation}

for the bigon $\gamma_{\pi/2} \Rightarrow c_\bullet$ which lands in $U_S$. This is the bigon covering the southern hemisphere. However, our boundary data need to match up so that we’ll be able to compose in $\mathcal{P}_2^\pi(S^2)$. How to do this is outlined in our prescription above (see
and is explicitly given here

\[(\ref{131})\] and is explicitly given here

\[
\begin{align*}
\text{(266)} & \quad s(\bullet) \xrightarrow{!} s_S(\bullet) \xrightarrow{!} s_S(\bullet) \xrightarrow{!} s(\bullet) \\
& \quad s(\tau_{\gamma/2}) \xrightarrow{!} s_d(\tau_{\gamma/2}) \xrightarrow{!} s_d(\tau_{\gamma/2}) \xrightarrow{!} s(\tau_{\gamma/2})
\end{align*}
\]

where the ! signifies the unique 2-isomorphisms from Lemma 3.15. For the full bigon \(\Sigma : c_\bullet \Rightarrow c_\bullet\) depicting the full sphere, we simply break it up into the two pieces defined above and compose vertically. The result of this is

\[
\begin{align*}
\text{(267)} & \quad s(\bullet) := s_N(\bullet) \xrightarrow{!} s_S(\bullet) \xrightarrow{!} s_S(\bullet) \xrightarrow{!} s(\bullet) := s_N(\bullet) \\
& \quad s(\tau_{\gamma/2}) := s_N(\tau_{\gamma/2}) \xrightarrow{!} s_d(\tau_{\gamma/2}) \xrightarrow{!} s_d(\tau_{\gamma/2}) \xrightarrow{!} s(\tau_{\gamma/2})
\end{align*}
\]

where the unspecified arrows are the unique ones. Applying our prescription from above to define the global transport functor applied to the sphere, we obtain the following diagram in \(G-\text{Tor}_{(1)}\)

\[
\begin{align*}
\text{(268)} & \quad G \xrightarrow{\text{id}_G} G \xrightarrow{[s \mapsto L_{\text{cwve}}(\Sigma_N(\cdot, \cdot))]} G \xrightarrow{\text{id}_G} G \\
& \quad \xrightarrow{[s \mapsto L_{\text{cwve}}(\Sigma_S(\cdot, \cdot))]} \xrightarrow{\text{id}_G} \xrightarrow{\text{id}_G}
\end{align*}
\]
since $i_{\{1\}} \text{triv}(y) = G$ for all $y$ and so on for paths and bigons (see the definition of $R_{(\text{triv},g,\alpha,f)}$ in Section [3.4] and $g_{NS}(\phi = 0) \equiv g_{NS}(\bullet) = 1$. Here $c_f$ for a $G$-Torsor morphism $f$ means the constant path at $f$. Furthermore, $g_{NS}$ on paths is the identity since $g_{NS}$ came from a natural transformation of ordinary functors between between categories. But now this is simple to compute because of all the simplifications. The result is just the path

\[
\begin{align*}
(269) \quad s \mapsto \begin{cases} 
L_{\text{triv}}(\Sigma_N(\cdot,2s)) & \text{for } 0 \leq s \leq \frac{1}{2} \\
L_{\text{triv}}(\Sigma_N(\cdot,2s-1)) & \text{for } \frac{1}{2} \leq s \leq 1
\end{cases}
\end{align*}
\]

which reduces to a computation on the group level. Therefore, all we have to do is compute the homotopy class of the path

\[
(270) \quad s \mapsto \begin{cases} 
\text{triv}(\Sigma_N(\cdot,2s)) & \text{for } 0 \leq s \leq \frac{1}{2} \\
\text{triv}(\Sigma_N(\cdot,2s-1)) & \text{for } \frac{1}{2} \leq s \leq 1
\end{cases}
\]

in the group $G$ thanks to Lemma [4.8]. But remember that $s = \frac{\theta}{\pi}$ and by using the differential forms we’ve obtained, this is easily calculable

\[
(271) \quad \text{triv}(\Sigma_N(\cdot,2s)) = \text{triv} \left( \Sigma_N \left( \cdot, \frac{2\theta}{\pi} \right) \right)
\]

\[
(272) \quad = e^{\frac{\theta}{\pi} \int_{s}^{1} (1-\cos \theta) \, \text{d}\phi}
\]

\[
(273) \quad = e^{-\text{in}(1-\cos \theta)}
\]

since the paths going along $\theta$ do not contribute to the parallel transport since the connection form only has a $\text{d}\phi$ contribution. Similarly,

\[
(274) \quad \text{triv}(\Sigma_N(\cdot,2s-1)) = \text{triv} \left( \Sigma_N \left( \cdot, \frac{2\theta}{\pi} - 1 \right) \right) = e^{\text{in}(1+\cos \theta)}.
\]

As a sanity check, notice that

\[
(275) \quad e^{-\text{in}(1-\cos \frac{\pi}{2})} = e^{-\text{in}} = e^{\text{in}} = e^{\text{in}(1+\cos \frac{\pi}{2})}
\]

showing that the matching condition (so that our path is continuous) is satisfied. This matching condition was the one used, for instance, in [23] (see equation (47)).

**Remark 5.1.** It is important to note that one could make sense of this even if the path was not continuous. For instance, this could happen if the transition function were nontrivial so that the conjugation action would result in a discontinuous path. This would not change the resulting conclusion.

Notice that $1-\cos \theta$ is a monotonically increasing function of $\theta$ for $0 \leq \theta \leq \frac{\pi}{2}$ starting at 0 when $\theta = 0$ and ending at 1 when $\theta = \frac{\pi}{2}$. Therefore, $e^{-\text{in}(1-\cos \theta)}$ winds around the circle starting at 1 and ending at $e^{-\text{in}} = (-1)^n$ winding around monotonically $\frac{n}{2}$ times clockwise if $n$ is positive and counterclockwise otherwise. Now, the function $1+\cos \theta$ is a monotonically decreasing function of $\theta$ for $\frac{\pi}{2} \leq \theta \leq \pi$ starting at $(-1)^n$ when $\theta = 0$ and ending at 1 when $\theta = \pi$. Therefore, $e^{\text{in}(1+\cos \theta)}$ winds around the circle starting at $e^{\text{in}} = (-1)^n$ and ending at 1 winding around monotonically $\frac{n}{2}$ times clockwise if $n$ is positive and counterclockwise otherwise. In other words, the loop goes a total of $n$ times around clockwise if $n$ is positive and $n$ times counterclockwise if $n$ is negative:

\[
(276) \quad \text{hol}^{[n]}(S^2) = -n.
\]
If we wanted to, we could have also computed this using differential forms and the formula for 2-transport \((140)\) of Schreiber and Waldorf \([18]\) locally and pasted the group elements together vertically as above. Of course, by the equivalence between local smooth functors and differential forms, our formula in terms of ordinary holonomy bypasses the rather (a-priori) complicated surface holonomy formula \((140)\) due to Corollary 4.15. It’ll actually turn out that the surface holonomy formula \((140)\) is not so complicated in this particular case due to our choice of bigon representing the sphere and the differential forms representing the connection. We will subsequently do this analysis strictly in terms of the differential forms associated to the path-curvature 2-functor discussed in Section 4.2.

The curvature is given by

\[ F_N = \frac{n}{2\pi} \sin \theta d\theta \wedge d\phi \]

and similarly for \(F_S\). Therefore, the connection 2-form is given by

\[ B_N = \frac{1}{4\pi} F_N = -\frac{n}{4\pi} \sin \theta d\theta \wedge d\phi \]

and similarly for \(B_S\). The 1-form \(A_{\Sigma_N}\) is given by

\[ (A_{\Sigma_N})_\theta \left(\frac{d}{d\theta}\right) = -\int_0^{2\pi} d\phi B_{(\theta,\phi)} \left(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi}\right) = \frac{n}{2} \sin \theta \]

and the 2-transport along \(\Sigma_N\) is given by

\[ k_{A,B}(\Sigma_N) = \mathcal{P} \exp \left\{-\int_0^{\pi/2} d\theta (A_{\Sigma_N})_\theta \left(\frac{d}{d\theta}\right)\right\} \]

\[ = -\int_0^{\pi/2} d\theta \frac{n}{2} \sin \theta \]

because the exponential map \(\mathbb{R} \to \mathbb{R}\) is the identity. The 2-transport along \(\Sigma_S\) is done similarly and is given by

\[ k_{A,B}(\Sigma_N) = -\int_{\pi/2}^{\pi} d\theta \frac{n}{2} \sin \theta. \]

Vertically composing these results as was done earlier yields

\[ k_{A,B}(\Sigma_S) + k_{A,B}(\Sigma_N) = -\int_{\pi/2}^{\pi} d\theta \frac{n}{2} \sin \theta - \int_0^{\pi/2} d\theta \frac{n}{2} \sin \theta \]

\[ = -\int_0^{\pi/2} d\theta \frac{n}{2} \sin \theta \]

\[ = -n \]

because the group operation in \(\mathbb{R}\) is addition. Therefore, the result obtained in terms of the path-curvature 2-functor in terms of homotopy classes of paths in \(G\) agrees with the double path-ordered exponential formula \((140)\) of Schreiber and Waldorf \([18]\) from the differential cocycle data, which is what we expect due to Corollary 4.15.

5.2. \(SO(3)\) monopoles. Now we will give examples for non-abelian magnetic monopoles. The first example will be similar to the abelian case since we will consider the following principal \(SO(3)\) bundle over \(S^2\) defined by the two open sets \(U_N\) and \(U_S\) with transition function \(g_{NS} : U_{NS} \approx S^1 \to SO(3)\) to be

\[ g_{NS}(\phi) := e^{-\phi J_3} \]
where

\[(287) \quad J_1 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_2 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad J_3 := \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\]

form a set of generators for the Lie algebra \(\mathfrak{so}(3)\). One can give explicit connection forms \(A_N\) and \(A_S\) on \(U_N\) and \(U_S\) respectively as follows

\[(288) \quad A_N := \frac{J_3}{2} (1 - \cos \theta) d\phi \quad \& \quad A_S := -\frac{J_3}{2} (1 + \cos \theta) d\phi.\]

These define local curvature 2-forms \(F_N\) and \(F_S\). Indeed, the gauge transformation defined above shows that

\[(289) \quad g_{NS}^{-1} A_S g_{NS}^{-1} = A_S + J_3 d\phi\]

\[(290) \quad = -\frac{J_3}{2} (1 + \cos \theta) d\phi + J_3 d\phi\]

\[(291) \quad = \frac{J_3}{2} (1 - \cos \theta) d\phi\]

\[(292) \quad = A_N\]

because all elements commute. The curvature 2-form is given by

\[(293) \quad F_N = dA_N + \frac{1}{2} [A_N, A_N] = \frac{J_3}{2} \sin \theta \, d\theta \wedge d\phi\]

again because the elements commute. Since \(F_N = F_S\), this defines a \(\mathfrak{so}(3)\)-valued closed 2-form on \(S^2\). Let \(\tau : SU(2) \to SO(3)\) be the double cover map so that \(N = \pi_1(SO(3)) \cong \mathbb{Z}_2\). Recall that the induced map on the level of Lie algebras \(\tau : \mathfrak{su}(2) \to \mathfrak{so}(3)\) is an isomorphism and is given by

\[(294) \quad \tau \left( \frac{1}{2i} \sigma_i \right) = J_i\]

where the \(\sigma_i\) are the Pauli matrices

\[(295) \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \& \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

As in the general case, define \(B_N := \tau^{-1}(F_N)\) and \(B_S := \tau^{-1}(F_S)\), or explicitly

\[(296) \quad B = \frac{\sigma_3}{4i} \sin \theta \, d\theta \wedge d\phi\]

since \(B_N = B_S\) on their common domain of definition. By our analysis in Section 4.2, this defines the differential cocycle data of the path-curvature 2-functor. We will compute the 2-holonomy in two different ways. We will follow the same procedure as in the \(U(1)\) case and compute 2-holonomy in terms of homotopy classes of paths and then we will use formula (140).

To help us with the first task, we first recall how \(SU(2)\) the way described above in terms of the Pauli spin matrices is isomorphic to the universal cover of \(SO(3)\) described in terms of homotopy classes of paths starting at the identity in \(SO(3)\). An isomorphism \(PSO(3)/\sim \cong SU(2)\) can be given by using the fact that \(SU(2)\) is the universal cover and therefore satisfies the universal lifting property. Given any path \(\gamma : [0, 1] \to SO(3)\) starting at \(\gamma(0) = I_{3 \times 3}\), there exists a unique lift
\( \tilde{\gamma} : [0, 1] \to SU(2) \) starting at \( \tilde{\gamma}(0) = I_{2 \times 2} \) and such that the diagram

\[
\begin{array}{ccc}
SU(2) & \xrightarrow{\tilde{\gamma}} & SO(3) \\
\downarrow & & \\
[0, 1] & \xrightarrow{\gamma} & [0, 1]
\end{array}
\]

commutes. In this way, we can define a map

\[
PSO(3)/\sim \to SU(2)
\]

\[
[\gamma] \mapsto \tilde{\gamma}(1).
\]

By using the universal property one more time, one can show that this map is well-defined. Finally, it is a smooth diffeomorphism of covering spaces.

We can now check what the value of the path-curvature transport 2-functor is on the sphere by doing the same computations as above but using the new \( \sigma(3) \)-valued differential forms. The result for the bigon describing the northern hemisphere is given by

\[
\text{triv}(\Sigma_N(\cdot, 2s)) = \text{triv} \left( \Sigma_N \left( \cdot, \frac{2 \theta}{\pi} \right) \right) = e^{\frac{2}{\pi} J_3^\theta (1 - \cos \theta)} d\phi = e^{\pi J_3 (1 - \cos \theta)}
\]

since the paths going along \( \theta \) do not contribute to the parallel transport since the connection form only has a \( d\phi \) contribution. The path-ordered exponential is reduced to an ordinary exponential of an integral because only \( J_3 \) is involved and \( J_3 \) commutes with itself. Similarly, the southern hemisphere gives

\[
\text{triv}(\Sigma_S(\cdot, 2s - 1)) = \text{triv} \left( \Sigma_S \left( \cdot, \frac{2 \theta}{\pi} - 1 \right) \right) = e^{-\pi J_3 (1 + \cos \theta)}.
\]

Again, as a sanity check we show that the boundary values match up between the two hemispheres along the equator:

\[
e^{\pi J_3 (1 - \cos \frac{\pi}{2})} = e^{\pi J_3} = -I_3 = e^{-\pi J_3} = e^{-\pi J_3 (1 + \cos \frac{\pi}{2})}.
\]

Now we can compute the homotopy class of the path as \( \theta \) ranges from 0 to \( \pi \). Using similar arguments, namely that \( 1 - \cos \theta \) is a monotonically increasing function of \( \theta \) for \( \theta \) between 0 and \( \frac{\pi}{2} \), we see that this defines a nontrivial loop in \( SO(3) \) at the identity which agrees with our previous calculation. Therefore,

\[
\text{hol}_{11}(S^2) = -I_{2 \times 2}.
\]

Now we will use the differential cocycle data and integrate using formula (140). First, we compute \( A_{\Sigma_N} \) for the northern hemisphere bigon. Because only \( \sigma_3 \) is involved in the computation, everything commutes and conjugation is trivial. Therefore,

\[
(A_{\Sigma_N})_\theta \left( \frac{d}{d\theta} \right) = -\int_0^{2\pi} d\phi B_{(\theta, \phi)} \left( \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi} \right) = -\frac{\pi \sigma_3}{2i} \sin \theta
\]
and the 2-transport along $\Sigma_N$ is given by

$$k_{A,B}(\Sigma_N) = \mathcal{P} \exp \left\{ - \int_{\theta=0}^{\theta=\pi/2} (A_{\Sigma_N})_\theta \left( \frac{d}{d\theta} \right) \right\}$$

$$= \exp \left\{ \int_{\theta=0}^{\theta=\pi/2} \frac{\pi \sigma_3}{2i} \sin \theta \right\}.$$  

The 2-transport along $\Sigma_S$ is done similarly and is given by

$$k_{A,B}(\Sigma_S) = \exp \left\{ \int_{\theta=0}^{\theta=\pi} \frac{\pi \sigma_3}{2i} \sin \theta \right\}$$

Vertically composing these results yields

$$k_{A,B}(\Sigma_S)k_{A,B}(\Sigma_N) = \exp \left\{ \int_{\theta=0}^{\theta=\pi} \frac{\pi \sigma_3}{2i} \sin \theta \right\} = e^{\pi i \sigma_3} = -I_{2 \times 2}$$

because again every term commutes. We will discuss what these group elements mean after we finish a few more examples.

5.3. $SU(n)/Z(n)$ monopoles. Another collection of non-abelian examples arise from the Lie group $SU(n)$ and noting that the center of $SU(n)$ is $Z(n)$ where, in the fundamental representation, elements in $Z(n)$ are of the form

$$\exp \left\{ \frac{2\pi ki}{n} \right\} I_n,$$

where $k \in \{0, 1, \ldots, n-1\}$ and $I_n$ is the $n \times n$ unit matrix. The previous example was the special case $n = 2$. The equivalence relation on $SU(n)/Z(n)$ says that two elements $A$ and $B$ of $SU(n)$ are equivalent if there exists a $k \in \{0, 1, \ldots, n-1\}$ such that

$$AB^{-1} = \exp \left\{ \frac{2\pi ki}{n} \right\} I_n.$$

We denote the elements of equivalence classes with square brackets such as $[A]$.

The possible $SU(n)/Z(n)$ principal bundles over the sphere are determined by the clutching function along the equator, which is a homotopy class of a loop $S^1 \to SU(n)/Z(n)$ which itself is precisely an element of $Z(n)$. The quotient map is written as $\tau : SU(n) \to SU(n)/Z(n)$ and is a covering map of Lie groups. Therefore, it defines a Lie 2-group.

Let’s first consider the case for $n = 3$, which is relevant in the theory of quarks and gluons (see Section 1.4 of [10]). Define $X$ to be the element in the Lie algebra of $SU(3)$ to be

$$X := \frac{i}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$  

The exponential of this matrix is unitary. We define transition functions by

$$g_{k;NS}(\phi) := \exp \{-k\Sigma(X)\} = [\exp\{-k\phi X\}] = \begin{pmatrix} e^{-\frac{k\phi}{3}} & 0 & 0 \\ 0 & e^{-\frac{k\phi}{3}} & 0 \\ 0 & 0 & e^{\frac{2k\phi}{3}} \end{pmatrix}.$$
for each $k \in \{0, 1, 2\}$. The element $X$ is a scalar multiple of the Gell-mann matrix $\lambda_k$. Note we have

$$g_{k,NS}(0) = g_{k,NS}(2\pi) = g_{k,NS}(4\pi) = [I_3] \in SU(3)/Z(3).$$

The transition function defines a map $\phi \mapsto g_{k,NS}(\phi)$ whose homotopy class determines a principal $SU(3)/Z(3)$ bundle characterized by the integer $k \in \{0, 1, 2\}$.

We define a connection on this bundle analogously to the $SO(3)$ case by setting

$$A_{k,N} := \frac{k\tau}{2}(1 - \cos \theta)d\phi \quad \& \quad A_{k,S} := -k\tau(1 + \cos \theta)d\phi.$$

A similar computation shows that this collection of 1-forms is consistent with the transition function. The connection 2-form is similarly given by

$$B_{k,N} = \frac{kX}{2} \sin \theta \, d\theta \wedge d\phi$$

and likewise for $B_{k,S}$. This defines a $SU(3)/Z(3)$-valued closed 2-form on $S^2$.

Again, we can do the computation for the 2-holonomy in the two ways described earlier. The first case is done by computing the homotopy class of the path of holonomies using the definition of the path-curvature 2-functor of Definition 4.7. The second way is via the differential forms associated to the path-curvature 2-functor described in Section 4.2 and equation (140). The computation is completely analogous to the previous two examples.

For the first case, we have

$$\text{hol}_{k,\{1\}}(S^2) = \left\{ \begin{array}{ll}
\theta & \mapsto \left\{ \begin{array}{ll}
e^{\frac{k}{n}X_{ij}(1-\cos \theta)d\phi} & \text{if } 0 \leq \theta \leq \frac{\pi}{2} \\
e^{-\frac{k}{n}X_{ij}(1+\cos \theta)d\phi} & \text{if } \frac{\pi}{2} \leq \theta \leq \pi 
\end{array} \right. \\
\theta & \mapsto \left\{ \begin{array}{ll}
e^{k\pi X(1-\cos \theta)} & \text{if } 0 \leq \theta \leq \frac{\pi}{2} \\
e^{-k\pi X(1+\cos \theta)} & \text{if } \frac{\pi}{2} \leq \theta \leq \pi 
\end{array} \right.
\end{array} \right. = e^{\frac{2\pi i k}{3} I_3}. $$

As for the computation in terms of differential forms, also by analogous computations to previous cases,

$$\left( A_{k,\Sigma_N} \right)_\theta \left( \frac{d}{d\theta} \right) = -\int_0^{2\pi} d\phi \frac{kX}{2} \sin \theta = -k\pi X \sin \theta$$

and likewise for $\left( A_{k,\Sigma_S} \right)_\theta \left( \frac{d}{d\theta} \right)$. Also

$$k_{A,B}(\Sigma_N) = \exp \left\{ \int_0^{\pi/2} k\pi X \sin \theta \, d\theta \right\}$$

and finally the 2-holonomy along the sphere is

$$\text{hol}_{k,\{1\}}(S^2) = k_{A,B}(\Sigma_S)k_{A,B}(\Sigma_N) = \exp(2\pi kX) = e^{\frac{2\pi i k}{3} I_3}.$$ 

For the general case of $SU(n)$, by using the matrix

$$X := \frac{i}{n} \begin{pmatrix}
1 & 1 \\
& \ddots \\
& & 1 \\
& & & 1 - n
\end{pmatrix}$$
the formulas for the transition function, connection 1-forms, and connection 2-forms are all the same with this new $X$ replacing the old one. Completely analogous computations lead to a 2-holonomy along the sphere

$$\text{hol}_{k}(S^{2}) = e^{\frac{2\pi i k}{n} \mathbb{1}_n},$$

which again is a rigorously defined non-abelian magnetic flux for $SU(n)/Z(n)$ gauge theories.

5.4. $U(n)$ monopoles. We now discuss yet another collection of examples generalizing the $U(1)$ case. Consider the group $U(n)$ of unitary $n \times n$ matrices. The Lie algebra, $\mathfrak{u}(n) = U(n)$ consists of Hermitian matrices. The universal cover of $U(n)$ is $SU(n) \times \mathbb{R}$. The covering map $\tau : SU(n) \times \mathbb{R} \to U(n)$ is defined by $\tau(A,t) := Ae^{2\pi it}$. The image of $\tau$ is clearly a $U(1)$ subgroup of $U(n)$. The fiber of this covering map is given by the kernel which is

$$\ker \tau = \left\{ (A,t) \mid A = e^{-2\pi i t} \text{ and } \det A = e^{-2\pi i nt} = 1 \iff t = \frac{k}{n}, \ k \in \mathbb{Z} \right\}$$

$$\tau(X) = 2\pi i \mathbb{1}_n.$$

Consider the Lie algebra element along this real line

$$X := (0_n, 1),$$

where $0_n$ is the $n \times n$ zero matrix. Then its image in $\mathfrak{u}(n)$ under $\tau$ is

$$\tau(X) = 2\pi i \mathbb{1}_n.$$

With this, for every integer $k$, we define the transition function, connection 1-forms, and connection 2-forms completely analogously to the previous examples (specifically the $\mathbb{R} \to U(1)$ example), namely

$$g_{NS}(\phi) = e^{ik\phi \mathbb{1}_n},$$

$$A_N = \frac{k}{2i} (1 - \cos \theta) \mathbb{1}_n d\phi \quad \& \quad A_S = -\frac{k}{2i} (1 + \cos \theta) \mathbb{1}_n d\phi,$$

and

$$B = \tau^{-1} \left( \frac{k}{2i} \sin \theta \mathbb{1}_n \ d\theta \wedge d\phi \right) = -\frac{k}{4\pi} \sin \theta (0_n, 1) \ d\theta \wedge d\phi.$$

In terms of the path of holonomies via the path-curvature 2-functor, the surface holonomy is

$$\text{hol}_{k}(S^{2}) = \begin{cases} \theta \mapsto & \left\{ \begin{array}{ll} \frac{k}{2i} \mathbb{1}_n \cos^\theta(1-\cos \theta) d\phi & \text{if } 0 \leq \theta \leq \frac{\pi}{2} \\ \frac{k}{2i} \mathbb{1}_n \cos^\theta(1+\cos \theta) d\phi & \text{if } \frac{\pi}{2} \leq \theta \leq \pi \end{array} \right. \\ \theta \mapsto & \left\{ \begin{array}{ll} e^{\frac{k}{2i} \mathbb{1}_n (1-\cos \theta) d\phi} & \text{if } 0 \leq \theta \leq \frac{\pi}{2} \\ e^{-\frac{k}{2i} \mathbb{1}_n (1+\cos \theta) d\phi} & \text{if } \frac{\pi}{2} \leq \theta \leq \pi \end{array} \right. \\ \theta \mapsto & -k \in \mathbb{Z}. \end{cases}$$

If we want to compute the surface holonomy in terms of formula (140), we first compute

$$\left( A_{\Sigma_{n}} \right)_{\theta} \left( \frac{d}{d\theta} \right) = \int^{2\pi}_{0} d\phi \frac{k}{4\pi} \sin \theta (0_n, 1) = \frac{k}{2} \sin \theta (0_n, 1)$$
so that we get
\[ k_{A,B}(\Sigma_N) = \mathcal{P} \exp \left\{ - \int_0^{\pi/2} d\theta \frac{k}{2} \sin \theta \ (0_n, 1) \right\} = \left( I_n, - \int_0^{\pi/2} d\theta \frac{k}{2} \sin \theta \right) \]
and the 2-holonomy along the sphere is
\[ \text{hol}_{k_{11}}(S^2) = k_{A,B}(\Sigma_S) k_{A,B}(\Sigma_N) \]
\[ = \left( I_n, - \int_0^\pi d\theta \frac{k}{2} \sin \theta \right) \left( I_n, - \int_0^{\pi/2} d\theta \frac{k}{2} \sin \theta \right) \]
\[ = \left( I_n, - \int_0^\pi d\theta \frac{k}{2} \sin \theta \right) \]
\[ = (I_n, -k). \]

5.5. Magnetic flux is a gauge-invariant quantity. In this section we state a theorem that is trivial to prove in the formalism presented above but gives an interesting physical interpretation. As a little background, consider the Dirac monopole. This is a magnetic monopole that defines a magnetic field \( \vec{B} \) in \( \mathbb{R}^3 \setminus \{0\} \). This magnetic field is divergence-free in \( \mathbb{R}^3 \setminus \{0\} \) but has a singularity at the origin. There does not exist a globally-defined vector potential \( \vec{A} \) on \( \mathbb{R}^3 \setminus \{0\} \) such that \( \nabla \times \vec{A} = \vec{B} \).

The magnetic flux through a sphere enclosing the origin is the integral of the curvature along that sphere. It is also written as \( \oint_{S^2} \vec{B} \cdot d\vec{n} \) where \( \vec{n} \) is the unit normal to the sphere. Its value is independent of the homotopy class of the sphere. This magnetic flux is the magnetic charge of the magnetic monopole.

As soon as one considers “non-abelian” Dirac monopoles, one would like an analogous concept of a magnetic flux measuring the magnetic charge of a magnetic monopole enclosed by a sphere. However, it no longer makes sense to compute the integral of the curvature along the sphere because the curvature is no longer a globally-defined quantity. The theory of transport 2-functors allows us to make sense of such an integral. Previous to this, the best way around this obstacle was by using a homotopy class of loops of holonomies, which produces an element in the fundamental group of the gauge group \([10]\) as explained earlier. One of the points of our work is that these concepts are made precise in terms of the path-curvature 2-functor. In fact, we take this as our primary definition of magnetic flux since this is the one that resembles the one in the physics literature.

Definition 5.2. Let \( P \to S^2 \) be a non-trivial principal \( G \)-bundle over \( S^2 \). The magnetic flux of the magnetic monopole associated to \( P \) is the 2-holonomy \( \text{hol}_{P_{11}}(S^2) \) associated to the path-curvature 2-functor along \( S^2 \).

All the previous examples relied on choices for the open cover, paths and bigons used to describe the sphere, and choices of lifts of paths and bigons. It is not clear that the surface holonomy computed is independent of these choices. Theorem [14] gives us two important results, the first of which tells us the magnetic flux is indeed independent of these choices.

Corollary 5.3. Let \( P \to S^2 \) be a non-trivial principal \( G \)-bundle over \( S^2 \). Let \( \hat{G} \to G \) denote the universal cover, \( BG \) the Lie 2-group associated to this, and \( C(P) \) the path-curvature transport 2-functor associated to \( P \). Then the magnetic flux is a gauge-invariant quantity, i.e., is a fixed point under \( \alpha \)-conjugation. In terms of the
notation of Section 3.7 Definition 3.34,

\[(343) \hol_{P;1}(S^2) \in \Inv(\alpha).\]

Proof. On the one hand, in general (for any structure 2-group), 2-holonomy along a sphere for any gauge 2-group is well-defined up to \(\alpha\)-conjugation as discussed in Section 3.7 and particularly Theorem 3.29. On the other hand, the 2-holonomy along a sphere for covering 2-groups lands in a central subgroup and is therefore invariant under the \(\alpha\) action, which coincides with ordinary conjugation since the map covering map \(\tau\) is surjective.

To see that the 2-holonomy lands in a central subgroup, first note that
\[
\hol_{P;1}(S^2) \in \ker \tau
\]
because the source and target of a sphere viewed as a bigon are equal. Now, for any crossed module, Lemma 6.41 says \(\ker \tau\) is a normal subgroup. Furthermore, for covering 2-groups specifically, \(\ker \tau\) is a discrete subgroup of \(\tilde{G}\) because \(\tau\) is a covering map. But discrete normal subgroups of connected Lie groups are central as the following argument shows. Let \(N\) denote such a subgroup of \(\tilde{G}\). Given any \(n \in N\), define the continuous map \(\varphi : \tilde{G} \to \tilde{G}\) by \(\varphi(g) := gng^{-1}n^{-1}\) for all \(g \in \tilde{G}\). Since \(N\) is normal, \(gng^{-1}n^{-1} \in N\) and so \(gng^{-1}n^{-1} \in N\). Thus \(\varphi(\tilde{G}) \subset N\). The fact that \(\varphi\) is continuous and that \(\varphi(e) = e\) implies that \(\varphi(\tilde{G}) = \{e\}\) since \(N\) is discrete and \(\tilde{G}\) is connected. Therefore, \(gng^{-1}n^{-1} = e\), i.e. \(gn = ng\) for all \(g \in \tilde{G}\) and all \(n \in N\). Thus \(N\) is central. \(\Box\)

A corollary of this and Theorem 4.14 is the following which relates the magnetic flux to a surface integral of the magnetic field. This is more of a physics corollary than a math corollary.

Corollary 5.4. The magnetic flux (Definition 5.2) can be computed as a surface integral by using (140) locally. This surface integral, which lands in the covering group, is the analogue of \(\int_{S^2} F_A\) where in electromagnetism \(F_A\) is the electromagnetic field strength due to the local potential \(A\).

The theorem can be extended rather trivially to spheres embedded in other spaces, but first we state a preliminary result regarding pullbacks.

Proposition 5.5. Let \(M\) be any 3-manifold and let \(P \to M\) be a principal \(G\)-bundle over \(M\). Let \(\sigma : S^2 \to M\) be a map of a sphere into \(M\). Denote the pullback bundle by \(\sigma^*(P) \to S^2\). Then the 2-holonomy with respect to the transport 2-functor \(L(P)\) along a bigon describing the sphere in \(M\) equals the 2-holonomy with respect to the transport 2-functor \(L(\sigma^*(P))\) along \(S^2\).

Definition 5.6. Let \(P \to M\) be a non-trivial principal \(G\)-bundle over a 3-manifold \(M\). Let \(\sigma : S^2 \to M\) be a smooth map of a sphere into \(M\). The magnetic flux along \(\sigma\) of a magnetic monopole defined by this principal bundle is its associated 2-holonomy from the path-curvature transport 2-functor.

Corollary 5.7. Under the same hypothesis in the definition, the magnetic flux is a gauge invariant quantity.

As we have shown, the surface holonomies of transport 2-functors give a mathematically rigorous explanation for the topological quantum number (the magnetic charge) associated to magnetic monopoles for gauge theories with any structure/gauge group in the language of magnetic flux. Furthermore, it expresses this
quantity as a group element in the universal cover of the gauge group. We emphasize that no Higgs field was introduced to do these computations. This therefore gives a rigorous mathematical result first mentioned by Goddard, Nuyts, and Olive at the end of Section 2 of their paper [14] by using the notion of transport 2-functors introduced by Schreiber and Waldorf in [20] to describe magnetic flux generalizing the notion from the theory of electromagnetism to non-abelian gauge theories.

6. Appendix

6.1. Smooth spaces. We will very briefly state important definitions and smooth structures needed in this paper. The category of finite-dimensional manifolds is not suitable for our purposes, nor is the category of certain infinite-dimensional manifolds. This section reviews diffeological spaces, which constitute one candidate for a notion of smooth spaces. For a review of smooth spaces that also compares several other candidates, please refer to [3].

Definition 6.1. A smooth space is a set $X$ together with a collection of plots $\{\varphi : U \to X\}$, called its smooth structure, where each $U$ is an open set in some $\mathbb{R}^n$ ($n$ can vary) satisfying the following conditions.

i) If $\varphi : U \to X$ is a plot and $\theta : V \to U$, where $V$ is an open set of some $\mathbb{R}^m$, is a smooth map, then $\varphi \theta : V \to X$ is a plot.

ii) Every map $\mathbb{R}^0 \to X$ is a plot.

iii) Let $\varphi : U \to X$ be a function and let $\{U_j\}_{j \in J}$ be a collection of open sets covering $U$ with $i_j : U_j \to U$ denoting the inclusion. Then if $\varphi i_j : U_j \to X$ is a plot for all $j \in J$, then $\varphi : U \to X$ is a plot.

Example 6.2. Let $M$ be a smooth manifold. Define the set of plots to be the collection of all smooth maps $\varphi : U \to M$ for various open sets $U$ in Euclidean space. Then $M$ with this collection of plots forms a smooth space.

Example 6.3. Let $A$ be a subset of a smooth space $X$ and denote the inclusion map by $i : A \to X$. Define the subspace smooth structure on $A$ to be the collection of functions $\varphi : U \to A$ such that $i \varphi : U \to X$ are plots of $X$.

Example 6.4. Let $X$ be a smooth space and $\sim$ an equivalence relation on $X$. The quotient smooth structure on $X/\sim$ is defined as follows. A function $\varphi : U \to X/\sim$ is a plot if there exists an open cover $\{U_j\}_{j \in J}$ along with plots $\varphi_j : U_j \to X$ for $X$ such that

$$\varphi$$

commutes for all $j \in J$. Here the unspecified maps are the obvious ones.

Example 6.5. Let $X$ and $Y$ be smooth spaces. Define the smooth structure on $X \times Y$ to be the collection of maps $\varphi : U \to X \times Y$ such that $\pi_X \varphi : U \to X$ and $\pi_Y \varphi : U \to Y$ are both plots of $X$ and $Y$, respectively. Here $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are the projection maps and are smooth (in the sense of the following definition) with respect to this smooth structure.
Definition 6.6. A function \( f : X \to Y \) between two smooth spaces is smooth if for every plot \( \varphi : U \to X \) of \( X \), \( f \varphi : U \to Y \) is a plot of \( Y \).

Example 6.7. Let \( X \) and \( Y \) be two smooth spaces. The set of maps \( X \to Y \) acquires a natural smooth structure. A function \( \varphi : U \to \text{Maps}(X, Y) \) is a plot if and only if the associated function \( \tilde{\varphi} : U \times X \to Y \) is smooth. Recall, \( \tilde{\varphi} \) is defined by \( \tilde{\varphi}(u, x) := \varphi(u)(x) \).

The above ideas are important in the proof of Theorem 4.9.

6.2. 2-categories. The Appendix of [19] explains all the categorical background we need in this paper. However, for completeness and convenience, we spell out several important definitions and key facts. These include 2-categories, 2-functors, their compositions, pseudonatural transformations, their two compositions, modifications, and their three compositions. We also spell out what an equivalence between 2-categories is and what a pseudonatural equivalence between 2-functors is. Simultaneously, we discuss what the strict versions are as special cases. We also describe what it means for a 2-functor to be full and faithful by describing 2-categories in terms of categories with “hom-categories” instead of “hom-sets.” This last description of 2-categories is more commonly known as “bi-categories” first introduced by Bénabou [2].

To set the notation, we write the pullback of two morphisms \( f : X \to Z \) and \( g : Y \to Z \) as follows

\[
\begin{array}{ccc}
X_f \times_g Y & \xrightarrow{\pi_X} & X \\
\downarrow{\pi_Y} & & \downarrow{f} \\
Y & \rightarrow & Z \\
g & \rightarrow & g
\end{array}
\]

Definition 6.8. A (small) 2-category \( \mathcal{C} \) consists of

i) a set \( C_0 \) of objects,

ii) a set \( C_1 \) of 1-morphisms,

iii) a set \( C_2 \) of 2-morphisms,

iv) functions

\[
C_2 \overset{s}{\underset{t}{\xrightarrow{i}}} C_1 \overset{s}{\underset{t}{\xrightarrow{i}}} C_0,
\]

where \( s, t, \) and \( i \) stand for source, target, and identity-assignment, respectively,

v) compositions

\[
C_1 \circ_i C_1 \to C_1, \quad C_2 \circ_i C_2 \to C_2, \quad C_2 \circ_{ss} \times_{tt} C_2 \to C_2
\]

called ordinary composition of 1-morphisms, vertical composition of 2-morphisms, and horizontal composition of 2-morphisms, and drawn as

\[
\begin{array}{ccc}
z & \xrightarrow{\alpha} & y \\
\downarrow{\beta} & & \downarrow{\gamma} \\
x & \xrightarrow{\alpha \beta} & x
\end{array}
\]

\[\text{4These drawings place restrictions on the above mentioned functions.}\]
vi) for every triple \((\alpha, \beta, \gamma)\) of composable 1-morphisms, a 2-morphism
\[
(\alpha \beta) \gamma
\]
called the associator,

vii) and finally, for every morphism \(y \xleftarrow{\alpha} x\), two 2-morphisms
\[
\alpha 1_x \quad \text{&} \quad 1_y \alpha
\]
called the left and right unifiers, respectively. Here we write \(1_x\) instead of \(i(x)\).

This data must satisfy the following conditions.

(a) The functions \(s, t,\) and \(i\) have to satisfy the following equalities
\[
(353) \quad si = \text{id}_{C_\alpha} = ti, \quad si = \text{id}_{C_\beta} = ti, \quad ss = st, \quad \text{&} \quad ts = tt.
\]

(b) Vertical composition is associative and the identity-assigning map gives units with respect to this composition. The latter is drawn as
\[
(354)
\]

(c) For every quadruple \((\alpha, \beta, \gamma, \delta)\) of composable 1-morphisms, the diagram
\[
(355)
\]
commutes. This is called the pentagon axiom.

(d) For every pair \((z \xrightarrow{\alpha} y, y \xleftarrow{\beta} x)\) of composable 1-morphisms, the diagram

\[
\begin{array}{c}
\alpha(1_y\beta) \\
\downarrow \quad \downarrow \quad \downarrow \\
\alpha \\
\end{array}
\begin{array}{c}
\alpha \\
(\alpha 1_y)\beta \\
\end{array}
\]

(356)

commutes. Furthermore,

\[(357) \quad 1_{\alpha 1} = 1_{\alpha \beta}.
\]

(e) For every triple

\[
\begin{array}{c}
\alpha \\
\downarrow \\
z \\
\end{array}
\begin{array}{c}
\Sigma \\
\end{array}
\begin{array}{c}
\downarrow \\
y \\
\end{array}
\begin{array}{c}
\beta \\
\end{array}
\begin{array}{c}
\downarrow \\
x \\
\end{array}
\begin{array}{c}
\Omega \\
\end{array}
\begin{array}{c}
\downarrow \\
w \\
\gamma \\
\end{array}
\]

of horizontally composable 2-morphisms, the diagram

\[
\begin{array}{c}
(\alpha'\beta')\gamma' \\
\downarrow \quad \downarrow \quad \downarrow \\
\alpha'\beta' \\
\end{array}
\begin{array}{c}
(\alpha\beta)\gamma \\
\downarrow \quad \downarrow \quad \downarrow \\
\alpha\beta \\
\end{array}
\]

(359)

commutes.

(f) For every quadruple

\[
\begin{array}{c}
\alpha \\
\downarrow \\
z \\
\end{array}
\begin{array}{c}
\Sigma \\
\end{array}
\begin{array}{c}
\downarrow \\
y \\
\end{array}
\begin{array}{c}
\beta \\
\end{array}
\begin{array}{c}
\downarrow \\
x \\
\Omega \\
\end{array}
\]

(360)

of 2-morphisms composable in the fashion indicated above, then

\[
(361) \quad (\Sigma\Omega) = \left(\frac{\Sigma}{\Sigma'}\right)\left(\Omega \right)
\]

This is called the interchange law.

(g) For every 2-morphism

\[
\begin{array}{c}
\alpha \\
\downarrow \\
y \\
\end{array}
\begin{array}{c}
\Sigma \\
\end{array}
\begin{array}{c}
\downarrow \\
x \\
\end{array}
\begin{array}{c}
\beta \\
\end{array}
\]

(362)

the 2-morphisms \(1_{1_x}\) and \(1_{1_y}\) act as right and left identities, respectively, i.e.

\[(363) \quad \Sigma 1_{1_x} = \Sigma \quad \& \quad 1_{1_y} \Sigma = \Sigma.
\]
(h) All associators and unifiers are vertically invertible 2-morphisms in the following sense. A 2-morphism

\[ \alpha \]
\[ \beta \]
\[ \sigma \]
\[ \gamma \]
\[ \delta \]

is said to be vertically invertible if there exists a 2-morphism

\[ \alpha' \]
\[ \beta' \]
\[ \gamma' \]
\[ \delta' \]

so that

\[ \sigma' = \alpha' \beta' \delta' = \sigma \]

In this appendix, we shall write \( \Sigma^{-1}_v \) for the vertical inverse of \( \Sigma \).

Sometimes we abuse notation in the body of the paper and simply write \( \Sigma^{-1}_v \) for the vertical inverse \( \Sigma^{-1}_v \) in hopes that it is clear from the context which inverse it means. However, in this appendix, we will use the more accurate \( \Sigma^{-1}_v \).

**Definition 6.9.** A strict 2-category is a 2-category whose associators and unifiers are all the identity 2-morphism.

**Definition 6.10.** A strict 2-groupoid is a strict 2-category where all 1-morphisms and 2-morphisms are invertible.

**Definition 6.11.** Let \( \mathcal{C} \) and \( \mathcal{D} \) be two 2-categories. A 2-functor \( F \) from \( \mathcal{C} \) to \( \mathcal{D} \), written as \( F : \mathcal{C} \rightarrow \mathcal{D} \), consists of

i) functions

\[ F_i : C_i \rightarrow D_i \]

for \( i = 0, 1, 2 \), that assign objects, 1-morphisms, and 2-morphisms in the following manner

\[ F_0(y) \]
\[ F_1(\alpha) \]
\[ F_2(\sigma) \]
\[ F_1(\beta) \]

ii) for every pair \( (\alpha, \beta) \) of 1-morphisms in \( \mathcal{C} \), a 2-morphism

\[ F_1(\alpha)F_1(\beta) \]

called the compositor,
iii) and for every object \( x \) in \( \mathcal{C} \), a 2-morphism

\[
F_1(1_x) \quad \downarrow^u \quad 1_{F_0(x)}
\]

called the unitor.

This data must satisfy the following conditions.

(a) For every triple \( p, \alpha, \beta, \gamma \) of composable 1-morphisms in \( \mathcal{C} \), the diagram

\[
F((\alpha \beta) \gamma) \xleftarrow{c_{\alpha,\beta,\gamma}} F(\alpha \beta) F(\gamma) \xleftarrow{c_{\alpha \beta} 1_{F(\gamma)}} (F(\alpha) F(\beta)) F(\gamma)
\]

commutes. Note that we write \( F \) instead of \( F_0, F_1, \) or \( F_2 \) from now on since it will be clear from the context.

(b) For every 1-morphism \( y \xrightarrow{\alpha} x \) in \( \mathcal{C} \), the diagrams

\[
1_{F(y)} F(\alpha) \xleftarrow{c_{\alpha, y, a}} F(1_y) F(\alpha) \quad \text{and} \quad F(\alpha) 1_{F(x)} \xleftarrow{c_{\alpha, 1_x}} F(\alpha) F(1_x)
\]

both commute.

(c) For every pair \( (\Sigma, \Omega) \) of vertically composable 2-morphisms in \( \mathcal{C} \)

\[
F(\Sigma) = F(\Omega)
\]

(d) For every 1-morphism \( \alpha \) in \( \mathcal{C} \),

\[
F(1_\alpha) = 1_{F(\alpha)}.
\]

(e) For every pair

of horizontally composable 2-morphisms in \( \mathcal{C} \), the diagram

\[
F(\gamma) F(\delta) \xleftarrow{c_{\gamma, \delta}} F(\alpha) F(\beta)
\]

commutes.
Definition 6.12. Let $\mathcal{C}, \mathcal{D},$ and $\mathcal{E}$ be 2-categories and let $F : \mathcal{D} \to \mathcal{E}$ and $G : \mathcal{C} \to \mathcal{D}$ be two 2-functors. The composition of $F$ and $G$, written as $FG : \mathcal{C} \to \mathcal{E}$, is the 2-functor defined as follows.

i) The functions $(FG)_i : C_i \to E_i$ are defined to be

\[(FG)_i := F_i G_i.\]

ii) For every pair $(\alpha, \beta)$ of 1-morphisms in $\mathcal{C}$, the compositor $e_{\alpha, \beta}^{FG}$ is defined to be the vertical composite of the 2-morphisms

\[
\begin{array}{ccc}
(FG)(\alpha) & (FG)(\beta) \\
\downarrow & \downarrow \\
F(G(\alpha)G(\beta)) & F(c_{\alpha, \beta}^G)
\end{array}
\]

where we have used superscripts to distinguish the compositors.

iii) For every object $x$ in $\mathcal{C}$, the unitor $u_x^{FG}$ is defined to be the vertical composite of the 2-morphisms

\[
\begin{array}{ccc}
(FG)(1_x) \\
\downarrow & \\
F(1_{G(x)}) & u_{G(x)}^G
\end{array}
\]

It is not immediately clear from this definition that the data defines a 2-functor $FG : \mathcal{C} \to \mathcal{E}$. We therefore check the necessary axioms.

Proof. We check the properties one at a time.
(a) Let \((\alpha, \beta, \gamma)\) be a triple of composable 1-morphisms in \(C\). We need the outer part of the following diagram to commute.

The right hexagon commutes by condition (a) for the 2-functor \(F\) applied to the three 1-morphisms \(G(\alpha), G(\beta),\) and \(G(\gamma)\). The left hexagon commutes by condition (c) for the 2-functor \(F\), associativity of vertical composition, and by condition (a) for the 2-functor \(G\) applied to the three 1-morphisms \(\alpha, \beta,\) and \(\gamma\). The top square commutes because \(1_{(FG)(\gamma)} = F(1_{G(\gamma)})\) by condition (d) for the 2-functor \(F\) and by condition (e) applied to the pair \((\epsilon^G_{\alpha, \beta}, 1_{G(\gamma)})\). The bottom square commutes by condition (d) again and condition (e) applied to the pair \((1_{G(\alpha)}, \epsilon^G_{\beta, \gamma})\). Therefore, the outer part of the diagram commutes.

(b) Let \(y \xRightarrow{a} x\) be a 1-morphism in \(C\). We need the outer part of the following diagram to commute.

The top right corner commutes by condition (e) for the 2-functor \(F\) applied to the pair \((u^G_y, 1_{G(\alpha)})\) of horizontally composable 2-morphisms. The left corner commutes by condition (b) for the 2-functor \(F\) applied to the 1-morphism \(G(y) \xRightarrow{G(\alpha)} G(x)\). The bottom corner commutes by condition (c) for the 2-functor \(F\), associativity of vertical composition, and by condition (b) for the 2-functor \(G\) applied to the 1-morphism \(y \xRightarrow{\alpha} x\). Therefore, the outer part of the diagram commutes.

A similar argument shows that the other required diagram also commutes.
(c) Let \((\Sigma, \Omega)\) be a pair of vertically composable 2-morphisms in \(\mathcal{C}\). Then

\[
(FG)(\Sigma) = F\left( G\left( \frac{\Sigma}{\Omega} \right) \right) = F(G(\Sigma)) = F(G(\Omega)) = (FG)(\Omega).
\]

(d) Let \(\alpha\) be a 1-morphism in \(\mathcal{C}\). Then

\[
(FG)(1_{\alpha}) = F(1_{G(\alpha)}) = 1_{F(G(\alpha))} = 1_{(FG)(\alpha)}.
\]

(e) Let

\[
\begin{array}{ccc}
\bullet & \xrightarrow{\gamma} & \bullet \\
\downarrow{\alpha} & & \downarrow{\beta} \\
\bullet & \xrightarrow{\delta} & \bullet
\end{array}
\]

be a pair of horizontally composable 2-morphisms. We need the outer part of the following diagram to commute.

\[
\begin{array}{rcl}
F(\gamma)F(\delta) & \xleftarrow{\Delta_{\Sigma}(\Sigma)} & (FG)(\alpha)(FG)(\beta) \\
F(G(\gamma)G(\delta)) & \xleftarrow{\Delta_{\Omega}(\Omega)} & F(G(\alpha)G(\beta)) \\
F(c_{\alpha, \beta}) & \xleftarrow{\Delta_{\Sigma}(\Sigma)} & F(c_{\alpha, \beta}) \\
(\Delta_{\Omega}(\Omega)) & \xleftarrow{\Delta_{\Sigma}(\Sigma)} & (FG)(\alpha\beta)
\end{array}
\]

The top square commutes by condition (e) for the 2-functor \(F\) applied to the pair \((G(\Sigma), G(\Omega))\). The bottom square commutes by condition (e) for the 2-functor \(F\) and by condition (e) for the 2-functor \(G\) applied to the pair \((\Sigma, \Omega)\). Therefore, the outer part of the diagram commutes.

At this point, a natural question to ask is whether the composition of 2-functors is associative. It is also not immediately obvious whether or not the composition with the identity 2-functor doesn’t change the 2-functor that it is composed with. However, before we discuss this, we first discuss pseudonatural transformations and pseudonatural equivalences.

**Definition 6.13.** Let \(\mathcal{C}\) and \(\mathcal{D}\) be two 2-categories and let \(F, G : \mathcal{C} \to \mathcal{D}\) be two 2-functors. A psuedonatural transformation \(\rho\) from \(F\) to \(G\), written as \(\rho : F \Rightarrow G\), consists of

i) a function \(\rho : C_0 \to D_1\) assigning a 1-morphism to an object \(x\) in the following manner

\[
\begin{array}{ccc}
x & \xrightarrow{\rho} & F(x) \\
\downarrow{\rho(x)} & & \downarrow{\rho(x)} \\
G(x)
\end{array}
\]
ii) and a function $\rho: C_1 \to D_2$ assigning a vertically invertible 2-morphism to every 1-morphism $y \xleftarrow{\alpha} x$ in the following manner

\[
\begin{array}{c}
y \xleftarrow{\alpha} x \\
\rho \downarrow \downarrow \\
F(y) \xleftarrow{\rho(y)} F(x) \\
G(y) \xleftarrow{\rho(y)} G(x)
\end{array}
\]

The data must satisfy the following conditions.

(a) For every pair $(z \xleftarrow{\alpha} y, y \xleftarrow{\beta} x)$ of composable 1-morphisms in $C$, the diagram

\[
\begin{array}{c}
\alpha \downarrow \\
\rho(z)(F(\alpha)F(\beta)) \xleftarrow{\rho(\alpha)^{-1}\rho(y)} (G(\alpha)\rho(y))F(\beta) \\
\downarrow \\
G(\alpha)(G(\beta)\rho(x)) \xleftarrow{\rho(\alpha)^{-1}} (G(\alpha)G(\beta))\rho(x)
\end{array}
\]

commutes.

(b) For every 2-morphism

\[
\begin{array}{c}
y \xleftarrow{\alpha} x \\
\rho \downarrow \\
G(\alpha)\rho(x) \xleftarrow{\rho(\gamma)} \rho(y)F(\alpha)
\end{array}
\]

the diagram

\[
\begin{array}{c}
y \xleftarrow{\alpha} x \\
\rho \downarrow \\
G(\alpha)\rho(x) \xleftarrow{\rho(y)} \rho(y)F(\alpha)
\end{array}
\]

commutes.

**Remark 6.14.** There was no condition on $\rho$ in the previous definition for the identity 1-morphism $x \xleftarrow{1} x$ for an object $x$ of $C$. This condition would require that
the diagram

\[
\begin{array}{c}
\rho(x)^1_F(x) \leftrightarrow 1_{\rho(x)} \rho(x) \quad \rho(x) \quad \rho(1_x) \\
\rho(x) \quad \rho(x) \quad 1_{\rho(x)} \rho(x) \leftrightarrow G(1_x) \rho(x)
\end{array}
\]  
(391)

commute. However, as is shown in [19] [Lemma A.7.], this condition follows from the conditions in the definition.

**Definition 6.15.** Let \( \mathcal{C} \) and \( \mathcal{D} \) be two 2-categories and let \( F, G, H : \mathcal{C} \to \mathcal{D} \) be three 2-functors and let \( \rho : F \Rightarrow G \) and \( \sigma : G \Rightarrow H \) be two pseudonatural transformations. The vertical composition of \( \rho \) with \( \sigma \), written as \( \rho \sigma : F \Rightarrow H \), is defined as follows.

i) To every object \( x \) in \( \mathcal{C} \), assign

\[
F(x) \quad \rho(x) \quad G(x) \quad \sigma(x) \quad H(x)
\]

the composition \( \sigma(x) \rho(x) \).

ii) To every 1-morphism \( y \leftarrow x \) in \( \mathcal{C} \), assign the 2-morphism

\[
(\sigma(y) \rho(y)) F(\alpha) \xRightarrow{\rho_{\alpha}(y,y)} H(\alpha) (\sigma(x) \rho(x))
\]

defined by the vertical composition of the following 2-morphisms in \( \mathcal{D} \)

\[
(\sigma(y) \rho(y)) F(\alpha) \xRightarrow{a_{\sigma(y),\rho(y),F(\alpha)}} \sigma(y) (\rho(y) F(\alpha)) \xRightarrow{1_{\sigma(y)} \rho(\alpha)} \sigma(y) (G(\alpha) \rho(x))
\]

\[
H(\alpha) (\sigma(x) \rho(x)) \xRightarrow{a_{\rho(\alpha),\sigma(x),\rho(x),\rho(\alpha)}} (H(\alpha) \sigma(x)) \rho(x) \xRightarrow{\sigma(1_{\rho(x)})} (\sigma(y) G(\alpha)) \rho(x)
\]

Again, it is not obvious that this definition of vertical composition of pseudonatural transformations results in another pseudonatural transformation. We leave it to the reader to check that conditions (a) and (b) hold. Instead, we move on to discussing the horizontal composition of pseudonatural transformations.

**Definition 6.16.** Consider a collection of 2-categories, 2-functors, and pseudonatural transformations fitting into a diagram of the form

\[
\begin{array}{c}
\mathcal{E} \quad F \quad \mathcal{D} \quad G \\
\mathcal{C}
\end{array}
\]

(395)

The horizontal composition of \( \rho \) with \( \sigma \), written as \( \rho \sigma : FG \Rightarrow HJ \), is defined as follows.
i) To every object \( x \) in \( C \), assign

\[
\begin{array}{c}
x \\
\xrightarrow{\rho \sigma} \\
F(J(x)) \xrightarrow{F(\sigma(x))} F(G(x))
\end{array}
\]

(396)

ii) To every 1-morphism \( y \xrightarrow{\alpha} x \) in \( C \), assign the 2-morphism

\[
\begin{array}{c}
\rho(J(y))(\sigma(y))(FG)(\alpha) \\
\xrightarrow{(\rho \sigma)(y)(FG)(\alpha)} \\
(HJ)(\alpha)(\rho \sigma)(x)
\end{array}
\]

(397)

defined by the vertical composition of the following 2-morphisms in \( D \)

\[
\begin{array}{c}
\rho(J(y))(\sigma(y))(FG)(\alpha) \\
\xrightarrow{(\rho \sigma)(y)(FG)(\alpha)} \\
\rho(J(y))(\alpha)(\rho \sigma)(x)
\end{array}
\]

(398)

\[
\begin{array}{c}
\rho(J(y))(\sigma(y))(FG)(\alpha) \\
\xrightarrow{(\rho \sigma)(y)(FG)(\alpha)} \\
\rho(J(y))(\sigma(x))
\end{array}
\]

\[
\begin{array}{c}
\rho(J(y))(\sigma(x)) \\
\xrightarrow{(\rho \sigma)(y)(FG)(\alpha)} \\
(HJ)(\alpha)(\rho \sigma)(x)
\end{array}
\]

Remark 6.17. There are actually two natural choices for the composition of pseudonatural transformations. The other one involves assigning to every object \( x \) in \( C \)

\[
\begin{array}{c}
x \\
\xrightarrow{\rho \sigma} \\
F(G(x))
\end{array}
\]

(399)

and a similar modification for the assignment on morphisms. By the existence of the vertical isomorphism \( \sigma(\rho(x)) : \rho(J(x))F(\sigma(x)) \Rightarrow H(\sigma(x))\rho(G(x)) \), these two results are isomorphic. We will not discuss in more detail the relationship between the two and we will stick with the first definition.

As before, one should check that the definition above indeed defines a pseudonatural transformation. Again, we skip the proof.

Definition 6.18. Let \( C \) and \( D \) be two 2-categories, \( F, G : C \to D \) be two 2-functors, and \( \rho, \sigma : F \Rightarrow G \) be two pseudonatural transformations. A modification \( A \) from \( \sigma \) to \( \rho \), written as \( A : \sigma \Rightarrow \rho \) and drawn as

\[
\begin{array}{c}
D \\
\xrightarrow{\rho \sigma} \\
\xrightarrow{A} \\
C
\end{array}
\]

(400)
consists of a function \( A : C_0 \to D_2 \) assigning a 2-morphism to an object \( x \) in the following manner

\[
\begin{array}{c}
\xymatrix{
F(x) \\
\rho(x) & A(x) & \sigma(x) \\
G(x)
}
\end{array}
\]

This assignment must satisfy the condition that for every 1-morphism \( y \xleftarrow{\alpha} x \), the diagram

\[
\begin{array}{c}
\xymatrix{
G(\alpha)\sigma(x) & \sigma(y)F(\alpha) \\
1_{C(\alpha)}A(x) & A(y)1_{F(\alpha)} \\
G(\alpha)\rho(x) & \rho(y)F(\alpha)
}
\end{array}
\]

commutes.

Modifications have three types of compositions, all of which are used in understanding an important Lemma later on in this section.

**Definition 6.19.** Consider 2-categories, 2-functors, pseudonatural transformations, and modifications as in the following diagram

\[
\begin{array}{c}
\xymatrix{
F \\
\rho & A & \sigma \\
G
}
\end{array}
\]

The internal composition of \( A \) and \( B \), written as \( \mathcal{B} \circ A : \sigma \Rightarrow \lambda \), is the modification defined by the assignment

\[
\begin{array}{c}
\xymatrix{
F(x) \\
\lambda(x) & B(x) & A(x) & \sigma(x) \\
G(x)
}
\end{array}
\]

i.e.

\[
(\mathcal{B} \circ A)(x) := \frac{A(x)}{B(x)}.
\]

**Remark 6.20.** In [19], this composition is called vertical, but we prefer to call it internal since the other two compositions (to be defined momentarily) are more naturally called vertical and horizontal since they include vertically and horizontally composing pseudonatural transformations.

As usual, it’s not obvious that this defines a modification. Since there is only one condition to check, we do this now.
Proof. Let \( y \xrightarrow{\alpha} x \) be a 1-morphism in \( C \). We need the outer part of the following diagram to commute.

\[
\begin{array}{c}
\sigma_p \downarrow \\
G(\alpha) \sigma(x) & \hspace{1cm} & \hspace{1cm} \sigma(y) F(\alpha) \\
\downarrow & \hspace{1cm} & \hspace{1cm} \downarrow \\
G(\alpha) \rho(x) & \hspace{1cm} & \hspace{1cm} \rho(y) F(\alpha) \\
\downarrow & \hspace{1cm} & \hspace{1cm} \downarrow \\
G(\alpha) \lambda(x) & \hspace{1cm} & \hspace{1cm} \lambda(y) F(\alpha) \\
\end{array}
\]

Each square commutes by the only condition for \( \mathcal{A} \) and \( \mathcal{B} \) being modifications and by the interchange law in \( D \). \( \square \)

**Definition 6.21.** Consider 2-categories, 2-functors, pseudonatural transformations, and modifications as in the following diagram

\[
\begin{array}{c}
\sigma_p \downarrow \\
F \downarrow \\
A \downarrow \\
\mathcal{D} \leftarrow \mathcal{C} \leftarrow \mathcal{C} \leftarrow \mathcal{B} \leftarrow \lambda \downarrow \rho \downarrow \\
\end{array}
\]

The vertical composition of \( \mathcal{A} \) and \( \mathcal{B} \), written as \( \mathcal{A} \mathcal{B} : \sigma \Rightarrow \rho \), is the modification defined by the assignment

\[
\begin{array}{c}
A(x) \xrightarrow{\sigma(x)} \mathcal{A}(x) \xrightarrow{\rho(x)} B(x) \\
\end{array}
\]

\( i.e. \)

\[
\begin{array}{c}
\mathcal{A} \mathcal{B}(x) := B(x) \mathcal{A}(x),
\end{array}
\]

the horizontal composition of 2-morphisms in \( D \).

**Remark 6.22.** Again, this notation is a bit confusing. The modification was defined using vertical compositions of natural transformations but the actual definition involved horizontal composition in the 2-category \( D \). The reader is encouraged to draw more pictures of diagrams to avoid further confusion.
Definition 6.23. Consider 2-categories, 2-functors, pseudonatural transformations, and modifications as in the following diagram

\[
\begin{align*}
F & \circlearrowleft_{\rho} E \to H \\
& \circlearrowright_{\sigma} \downarrow \circlearrowright_{\Delta} \downarrow \circlearrowright_{\sigma} \downarrow \circlearrowright_{\Delta} \\
& \circlearrowleft_{\rho} \downarrow \circlearrowleft_{\Delta} \downarrow \circlearrowleft_{\rho} \downarrow \circlearrowleft_{\Delta} \\
G & \circlearrowleft_{\lambda} D \to J \\
& \circlearrowright_{\epsilon} \downarrow \circlearrowright_{\Delta} \downarrow \circlearrowright_{\epsilon} \downarrow \circlearrowright_{\Delta} \\
& \circlearrowleft_{\lambda} \downarrow \circlearrowleft_{\epsilon} \downarrow \circlearrowleft_{\lambda} \downarrow \circlearrowleft_{\epsilon} \\
C & \circlearrowleft_{\lambda} \downarrow \circlearrowleft_{\epsilon} \downarrow \circlearrowleft_{\lambda} \downarrow \circlearrowleft_{\epsilon} .
\end{align*}
\]

The horizontal composition of \( \mathcal{A} \) and \( \mathcal{B} \), written as \( \mathcal{A} \mathcal{B} : \sigma \lambda \Rightarrow \rho \epsilon \), is the modification defined by the assignment

\[
\begin{align*}
(\mathcal{A} \mathcal{B})(x) := \mathcal{A}(J(x)) \mathcal{B}(x)
\end{align*}
\]

for all objects \( x \) in \( C \).

We now come back to answering the question posed about the associativity, or lack thereof, of composition of 2-functors and pseudonatural transformations.

Lemma 6.24. Consider the following sequence of 2-categories and 2-functors

\[
\begin{align*}
\mathcal{F} & \xleftarrow{F} \mathcal{E} \xleftarrow{G} \mathcal{D} \xleftarrow{H} \mathcal{C}.
\end{align*}
\]

Then \( (FG)H = F(GH) \), i.e. the composition of 2-functors is associative, or equivalently the associator (a-priori a nontrivial pseudonatural transformation) for 2-functor composition is the identity.

Proof. We prove this by checking that each of the pieces of data that specify the 2-functors \( (FG)H \) and \( F(GH) \) actually coincide (see Definition 6.12).

i) Because ordinary composition of functions is associative,

\[
((FG)H)_i = (FG)_i H_i = (F_i G_i) H_i = F_i (G_i H_i) = F_i (GH)_i = (F(GH))_i.
\]
ii) For every pair \((\alpha, \beta)\) of 1-morphisms in \(C\), we have the following list of equalities

\[
\begin{align*}
    c_{\alpha, \beta}^{(FG)H} & = c^{FG}_{H(\alpha), H(\beta)}(c^H_{\alpha, \beta}) \\
    & = \left( c^F_{G(H(\alpha)), G(H(\beta))} \right) F(c^G_{H(\alpha), H(\beta)}) (c^H_{\alpha, \beta}) \\
    & = \left( c^F_{G(H(\alpha)), G(H(\beta))} \right) (F G)(c^H_{\alpha, \beta}) \\
    & = \left( c^F_{G(H(\alpha)), G(H(\beta))} \right) F(c^G_{H(\alpha), H(\beta)}) (c^H_{\alpha, \beta}) \\
    & = \left( c^F_{G(H(\alpha)), G(H(\beta))} \right) (F G)(c^H_{\alpha, \beta}) \\
    & = \left( c^F_{G(H(\alpha)), G(H(\beta))} \right) F(c^G_{H(\alpha), H(\beta)}) (c^H_{\alpha, \beta}) \\
    & = c_{\alpha, \beta}^{(GH)H} \\
    \end{align*}
\]

\(\square\)

iii) For every object \(x\) in \(C\), we have the following list of equalities

\[
\begin{align*}
    u^{(FG)H}_x = (FG)(u^H_x) & = (FG)(u^H_x) (F G)(u^H_x) \\
    & = (F G)(u^H_x) (u^G_{H(x)}) \\
    & = (F G)(u^H_x) u^G_{H(x)} \\
    & = F \left( F G \right)(u^H_x) u^G_{H(x)} \\
    & = F \left( u^G_{H(x)} \right) u^G_{H(x)} \\
    & = F \left( u^G_{H(x)} \right) u^G_{H(x)} \\
    & = F \left( u^G_{H(x)} \right) u^G_{H(x)} \\
    & = F \left( u^G_{H(x)} \right) u^G_{H(x)} \\
    & = u^F_{GH}(x) \\
    & = u^F_{GH}(x) \\
    & = u^F_{GH}(x) \\
    & = u^F_{GH}(x) \\
\end{align*}
\]

because vertical composition is associative and because 2-functors respect vertical composition.

\[\square\]

**Definition 6.25.** Let \(C\) be a 2-category. Define a functor \(1_C : C \to C\) by the following.

i) Define the assignment on objects, 1-morphisms, and 2-morphisms to be the identity functions

\[
(1_C)_j := \text{id}_j : C_j \to C_j
\]
for all $j = 0, 1, 2$.

ii) For every pair $(\alpha, \beta)$ of composable 1-morphisms in $\mathcal{C}$, define the compositor to be

\[
\alpha_{\alpha, \beta}^{1_C} := 1_{\alpha\beta}
\]

the identity 2-morphism.

iii) For every object $x$ in $\mathcal{C}$, define the unitor to be

\[
u^1_C(x) := 1_x
\]

the identity 2-morphism.

$1_C$ is called the identity 2-functor for $\mathcal{C}$. It is sometimes written as $\text{id}_C$.

**Lemma 6.26.** Let $F : \mathcal{C} \to \mathcal{D}$ be a 2-functor between two 2-categories $\mathcal{C}$ and $\mathcal{D}$. Then $F1_C = F = 1_D F$, i.e. the left and right unifiers for the composition of 2-functors are both the identity.

**Proof.** We prove this in a similar manner to the previous Lemma.

i) Because ordinary composition of functions by an identity function results in that same function,

\[
(F1_C)_j = F_j (\text{id}_C)_j = F_j = (\text{id}_D)_j F_j = (1_D F)_j
\]

for all $j = 0, 1, 2$.

ii) For every pair $(\alpha, \beta)$ of 1-morphisms in $\mathcal{C}$, we have the following list of equalities

\[
c_{\alpha, \beta}^{F1_C} = c_{\alpha(\alpha), 1}\frac{\alpha(\beta)}{1_C}\frac{\beta}{\beta}
\]

\[
c_{\alpha, \beta}^{F1_C} = c_{\alpha, \beta}^{1_D} F(\alpha(\beta))
\]

\[
c_{\alpha, \beta}^{F1_C} = 1_{\alpha(\beta)}
\]

\[
c_{\alpha, \beta}^{F1_C} = c_{\alpha(\beta), \beta}
\]

\[
c_{\alpha, \beta}^{F1_C} = 1_F(\alpha(\beta))
\]

\[
c_{\alpha, \beta}^{F1_C} = c_{\alpha(\beta), \beta}
\]

\[
c_{\alpha, \beta}^{F1_C} = 1_{F(\alpha)} F(\beta)
\]

because the vertical identity 2-morphism is an identity for vertical composition and because 2-functors respect vertical identities.
iii) For every object \( x \) in \( C \), we have the following list of equalities

\[
\begin{align*}
\tag{438}
& u^F_{1C} = F(u^F_z) \\
& = F(1_{1z}) \\
& = u^F_z \\
\tag{439}
& = u^F \\
\tag{440}
& = u^F_z \\
& = 1_{1F(x)} \\
\tag{441}
& = 1D(u^F_z) \\
& = u^F_{1C} \\
\tag{442}
& = 1D(p^F_z) \\
& = u^F_{1F(x)} \\
\tag{443}
& = u^F_{1F(x)}
\end{align*}
\]

because the vertical identity 2-morphism is an identity for vertical composition and because 2-functors respect vertical identities.

\[\square\]

Although composition of 2-functors has no surprises, vertical composition of pseudonatural transformations is a bit more complicated. In particular, there are associators and unifiers.

**Lemma 6.27.** Let \( C \) and \( D \) be two 2-categories, \( F,G,H,J : C \to D \) be four 2-functors, and \( \rho : F \Rightarrow G \), \( \sigma : G \Rightarrow H \), and \( \lambda : H \Rightarrow J \) be three pseudonatural transformations. Then the assignment

\[
\begin{align*}
\tag{444}
x \mapsto a_{\lambda,\sigma,\rho}(x) &:= a_{\lambda(x),\sigma(x),\rho(x)}, \\
\end{align*}
\]

the associator in the category \( D \), for any object \( x \) in \( C \), defines a modification

\[
\begin{align*}
\tag{445}
a_{\lambda,\sigma,\rho} : \rho_1 \Rightarrow \rho.
\end{align*}
\]

Furthermore, \( a_{\rho,\sigma,\lambda} \) is invertible and satisfies the pentagon axiom of condition (c) in Definition 6.8.

**Definition 6.28.** Let \( C \) and \( D \) be two 2-categories and let \( F : C \to D \) be a 2-functor. Define a pseudonatural transformation \( 1_F : F \Rightarrow F \) as follows.

i) The assignment \( x \mapsto 1_{F(x)} : F(x) \to F(x) \) defines the map \( 1_F : C_0 \to D_1 \).

ii) The assignment sending \( y \sim x \) to \( r_{F(y)}^{F(x)} : 1_{F(y)}F(a) \Rightarrow F(a) \Rightarrow 1_{F(x)} \)

defines \( 1_F : C_1 \to D_2 \).

**Lemma 6.29.** Let \( C \) and \( D \) be two 2-categories, \( F,G : C \to D \) be two 2-functors, and \( \rho : F \Rightarrow G \) a pseudonatural transformation. Then the assignments

\[
\begin{align*}
\tag{446}
x \mapsto l_{\rho}(x) &:= l_{\rho(x)}, \\
\end{align*}
\]

the left unifier in \( D \), and

\[
\begin{align*}
\tag{447}
x \mapsto r_{\rho}(x) &:= r_{\rho(x)}, \\
\end{align*}
\]

the right unifier in \( D \), define modifications

\[
\begin{align*}
\tag{448}
l_{\rho} : F_1 \Rightarrow \rho
\end{align*}
\]
and
\[(449) \quad r_\rho : \rho_{1_F} \Rightarrow \rho\]
respectively. Furthermore, \(l_\rho\) and \(r_\rho\) are invertible and satisfy condition (d) in Definition 6.8.

Finally, after giving the numerous definitions that arise in basic 2-category theory, we come to the important notion of a pseudonatural equivalence of 2-functors between two 2-categories. This notion is a key ingredient in understanding how surface holonomy changes under a different choice of transition functions (see Corollary 3.17 and the surrounding discussion).

**Definition 6.30.** Let \(\mathcal{C}\) and \(\mathcal{D}\) be two 2-categories and \(F,G : \mathcal{C} \rightarrow \mathcal{D}\) two 2-functors. A pseudonatural equivalence from \(F\) to \(G\) is a quadruple \(\rho, \sigma, i, j\) consisting of pseudonatural transformations \(\rho : F \Rightarrow G\), \(\sigma : G \Rightarrow F\), and invertible modifications \(i : \rho \sigma \Rightarrow 1_F\) and \(j : 1_G \Rightarrow \sigma \rho\) such that the diagrams
\[(450) \quad \begin{array}{c}
\rho \quad \rho \\
\downarrow \\
\rho
\end{array} \quad \begin{array}{c}
\sigma \\
\downarrow \\
\sigma
\end{array} \quad \begin{array}{c}
i \quad i \\
\downarrow \\
i
\end{array} \quad \begin{array}{c}
\downarrow \\
\rightarrow
\end{array} \quad \begin{array}{c}
\rho \\
\downarrow \\
\rho
\end{array} \quad \begin{array}{c}
\downarrow \\
\rightarrow
\end{array} \quad \begin{array}{c}
\rho \\
\downarrow \\
\rho
\end{array}
\]
both commute. In these diagrams, \(l\) and \(r\) are the left and right unifiers of Lemma 6.29 respectively and \(a\) is the associator of Lemma 6.27. The above two diagrams are known as zig-zag identities due to their representation in terms of string diagrams in the strict case (see [5]). We abusively say that \(\rho : F \Rightarrow G\) is a pseudonatural equivalence without writing out the other pieces of data. \(F\) and \(G\) are said to be pseudonaturally equivalent if there exists a pseudonatural equivalence between them.

**Definition 6.31.** Let \(\mathcal{S}\) and \(\mathcal{T}\) be two 2-categories and \(F : \mathcal{S} \rightarrow \mathcal{T}\) a 2-functor. \(F\) is called an equivalence of categories if there exists a functor \(G : \mathcal{T} \rightarrow \mathcal{S}\) together with pseudonatural equivalences \(\rho_\mathcal{S} : GF \Rightarrow 1_\mathcal{S}\) and \(\rho_\mathcal{T} : FG \Rightarrow 1_\mathcal{T}\). The functor \(G : \mathcal{T} \rightarrow \mathcal{S}\) along with the pseudonatural equivalences is called a weak inverse of \(F\).

An important lemma that is used frequently in Section 3.4 and Section 3.7 is the following.

**Lemma 6.32.** Let \(\mathcal{S}\) and \(\mathcal{T}\) be two 2-categories. Two weak inverses to a 2-functor \(F : \mathcal{S} \rightarrow \mathcal{T}\) are pseudonaturally equivalent.

**Proof.** By assumption, there exist \(G, G' : \mathcal{T} \rightarrow \mathcal{S}\) with pseudonatural equivalences
\[(451) \quad \begin{cases}
\rho_\mathcal{S} : GF \Rightarrow 1_\mathcal{S}, & \sigma_\mathcal{S} : 1_\mathcal{S} \Rightarrow GF, \\
i_\mathcal{S} : \sigma_\mathcal{S} \Rightarrow \rho_\mathcal{S} \otimes \rho_\mathcal{S}, & j_\mathcal{S} : 1_\mathcal{S} \Rightarrow \sigma_\mathcal{S} \otimes \sigma_\mathcal{S},
\end{cases}
\]
\[(452) \quad \begin{cases}
\rho_\mathcal{T} : FG \Rightarrow 1_\mathcal{T}, & \sigma_\mathcal{T} : 1_\mathcal{T} \Rightarrow FG, \\
i_\mathcal{T} : \sigma_\mathcal{T} \Rightarrow \rho_\mathcal{T} \otimes \rho_\mathcal{T}, & j_\mathcal{T} : 1_\mathcal{T} \Rightarrow \sigma_\mathcal{T} \otimes \sigma_\mathcal{T},
\end{cases}
\]
\[(453) \quad \begin{cases}
\rho_\mathcal{S}' : G'F \Rightarrow 1_\mathcal{S}, & \sigma_\mathcal{S}' : 1_\mathcal{S} \Rightarrow G'F, \\
i_\mathcal{S}' : \sigma_\mathcal{S}' \Rightarrow \rho_\mathcal{S}', & j_\mathcal{S}' : 1_\mathcal{S} \Rightarrow \sigma_\mathcal{S}' \otimes \sigma_\mathcal{S}',
\end{cases}
\]
and
\[
(454) \quad \left( \rho_T' : FG' \rightarrow 1_T, \sigma_T' : 1_T \rightarrow FG', i_T' : \sigma_T' \rightarrow 1_{FG'}, j_T' : 1_{1_T} \rightarrow \sigma_T' \rho_T' \right).
\]
We define a pseudonatural transformation \( \rho : G \rightarrow G' \) by taking the vertical composition of pseudonatural transformations
\[
(455) \quad G = 1_S G \xrightarrow{\sigma'_T G} (G' F) G = G'(FG) \xrightarrow{1_{G' \rho_T}} G' 1_T = G'
\]
and a pseudonatural transformation \( \sigma : G' \rightarrow G \) by
\[
(456) \quad G' = G' 1_T \xrightarrow{1_{G' \sigma_T}} G'(FG) = (G' F) G \xrightarrow{\rho'_T G} 1_S G = G,
\]
both of which have been simplified by Lemma 6.24 and Lemma 6.26. We also define a modification \( i : \rho' \Rightarrow \text{id}_G \) by the internal composition
\[
(457) \quad \rho' = \left( \begin{array}{c}
\sigma'_T G \\
\rho'_T G \\
\end{array} \right)
\]
and a modification \( j : 1_G \Rightarrow \sigma' \rho \) by the composition
\[
(458) \quad \sigma' \rho = \left( \begin{array}{c}
1_{1_G} \\
\sigma' T G \\
\rho' T G \\
\end{array} \right)
\]
We leave it to the reader to check that all the required diagrams from Definition 6.30 commute.

We conclude this section with a definition that is important in extracting group-valued holonomies. Recall that an ordinary functor \( F : \mathcal{C} \rightarrow \mathcal{D} \) is full and faithful or fully faithful if for any two objects \( x \) and \( y \) in \( \mathcal{C} \), the induced map of sets
\[
(459) \quad \text{Hom}_\mathcal{C}(y, x) \rightarrow \text{Hom}_\mathcal{D}(F(y), F(x))
\]
\[
(460) \quad \alpha \mapsto F(\alpha)
\]
is a bijection. The analogous property for 2-categories and 2-functors is a bit more subtle.
First note that for any two objects \( x \) and \( y \) of a 2-category \( C \), one can define a category \( \text{Hom}_C(y, x) \) by setting

\[
\text{Hom}_C(y, x)_0 := \{ \alpha \in C_1 \mid s(\alpha) = x \text{ and } t(\alpha) = y \}
\]

and

\[
\text{Hom}_C(y, x)_1 := \{ \Sigma \in C_2 \mid ss(\Sigma) = x \text{ and } tt(\Sigma) = y \}.
\]

One can define the source, target, and identity-assigning maps by restricting the ones from \( C \). Composition in \( \text{Hom}_C(y, x) \) is the restriction of the vertical composition in \( C \). It is associative and unital by condition (b) of Definition 6.8.

**Definition 6.33.** Let \( F : C \rightarrow D \) be a 2-functor between two 2-categories. \( F \) is said to be fully faithful if the restriction of \( F \) to the induced functor

\[
\text{Hom}_C(y, x) \rightarrow \text{Hom}_D(F(y), F(x))
\]

is an equivalence of categories, or equivalently, if the above functor on \( \text{Hom} \)-categories is both essentially surjective and fully faithful, for all objects \( x \) and \( y \) in \( C \).

6.3. 2-group conventions. For completeness, we very briefly review the definitions of 2-groups assuming some familiarity with ordinary category theory as well as some 2-category theory from the previous section.

**Definition 6.34.** A strict monoidal category is a category \( C \) equipped with

i) a functor \( \mu : C \times C \rightarrow C \)

ii) and an object \( I \)

such that

(a) the diagram

\[
\begin{array}{ccc}
C \times C \times C & \xrightarrow{\mu \times \text{id}_C} & C \times C \\
\downarrow{\text{id}_C \times \mu} & & \downarrow{\mu} \\
C \times C & \xrightarrow{\mu} & C
\end{array}
\]

commutes (this axiom is the axiom of associativity)

(b) and

\[
\mu(x, I) = x = \mu(I, x)
\]

for all objects \( x \) in \( C \) (this is the unit axiom). Typically, the monoidal product is written as

\[
x \otimes y
\]

for objects and

\[
\alpha \otimes \beta
\]

for morphisms.

**Remark 6.35.** There is a weaker notion of a strict monoidal category and goes by the name of just monoidal category. Since we are working with 2-categories, we do not need to introduce monoidal categories.
**Definition 6.36.** A strict monoidal category \((\mathcal{C}, \mu, I)\) is said to be a strict groupal category if there exists a functor \(\text{inv} : \mathcal{C} \to \mathcal{C}\) such that the diagrams

\[
\begin{array}{c}
\xymatrix{
\mathcal{C} \ar[r]^-{\Delta} & \mathcal{C} \times \mathcal{C} \\
\mathcal{C} \ar[r]_-{\text{inv} \times \text{id}_C} & \mathcal{C} \times \mathcal{C} \\
\mathcal{C} \ar[r]^-{\mu} & \mathcal{C}
}
\end{array}
\]

and

\[
\begin{array}{c}
\xymatrix{
\mathcal{C} \ar[r]^-{\Delta} & \mathcal{C} \times \mathcal{C} \\
\mathcal{C} \ar[r]_-{\text{id}_C \times \text{inv}} & \mathcal{C} \times \mathcal{C} \\
\mathcal{C} \ar[r]^-{\mu} & \mathcal{C}
}
\end{array}
\]

both commute. Here \(\bullet\) represents the category with one object and 1 morphism, \(I : \bullet \to \mathcal{C}\) is the functor that sends \(\bullet\) to \(I\), and \(\Delta\) is the diagonal functor sending objects \(x\) to \((x, x)\) and similarly for morphisms. It is common to write \(\text{inv}(x) = x^{-1}\). These diagrams are sometimes then written as

\[
x^{-1} \otimes x = I
\]

and

\[
x \otimes x^{-1} = I,
\]

respectively for any object \(x\) in \(\mathcal{C}\) and similarly for morphisms.

**Definition 6.37.** A strict 2-group is a strict groupal groupoid (a groupoid is a category whose morphisms are invertible with respect to the composition). In this article, when we say “2-group” we will always mean a strict 2-group.

There is a 2-category of strict 2-groups denoted by \(2\text{-Grp}\) whose 1-morphisms and 2-morphisms are functors and natural transformations (that respect the monoidal structure) respectively. It is useful to relate this higher-categorical definition to one involving ordinary groups.

**Example 6.38.** Every strict groupal category \(\mathcal{G}\) gives rise to a strict 2-category \(\mathcal{B}\mathcal{G}\) in the following way. Let \(\mathcal{B}\mathcal{G}\) have a single object \(\bullet\). Define the 1-morphisms of \(\mathcal{B}\mathcal{G}\) to be the objects of \(\mathcal{G}\). Define the 2-morphisms of \(\mathcal{B}\mathcal{G}\) to be the morphisms of \(\mathcal{G}\). This is pictured as

\[
\begin{array}{c}
\xymatrix{
\text{y} & \text{x} \\
\text{y} \ar[u] & \text{x} \\
\text{y} \ar[u] & \text{x} \\
\text{y} \ar[u] & \text{x} \\
\text{y} \ar[u] & \text{x}
}
\end{array}
\]

The composition of 1-morphisms in \(\mathcal{B}\mathcal{G}\) is defined to be the monoidal product of objects in \(\mathcal{G}\). The vertical composition of 2-morphisms in \(\mathcal{B}\mathcal{G}\) is defined to be the ordinary composition of morphisms in \(\mathcal{G}\). The horizontal composition of 2-morphisms in \(\mathcal{B}\mathcal{G}\) is defined to be the monoidal product of morphisms in \(\mathcal{G}\). This is drawn as

\[
\begin{array}{c}
\xymatrix{
\text{z} & \text{w} \\
\text{z} \ar[u] & \text{w} \\
\text{z} \ar[u] & \text{w} \\
\text{z} \ar[u] & \text{w} \\
\text{z} \ar[u] & \text{w}
}
\end{array}
\]
for horizontal composition and

\[
\begin{array}{c}
\begin{array}{c}
\bullet \\
\downarrow^g \\
\downarrow\tau(h) \\
\downarrow\tau(h)g \\
\end{array}
\end{array}
\quad \mapsto \quad
\begin{array}{c}
\begin{array}{c}
\bullet \\
\downarrow^{\alpha g} \\
\downarrow\tau(h)g \\
\end{array}
\end{array}
\]

for vertical composition. The associators and unifiers are all the identities. The interchange law follows from functoriality of \(\mu\).

**Definition 6.39.** A crossed module is a quadruple \((H, G, \tau, \alpha)\) of two groups, \(G\) and \(H\), group homomorphisms \(\tau : H \to G\) and \(\alpha : G \to \text{Aut}(H)\), satisfying the two conditions

\[
\alpha_{\tau(h)}(h') = hh' h^{-1}
\]

and

\[
\tau(\alpha_g(h)) = g \tau(h) g^{-1}.
\]

Occasionally, a crossed module as above will also be written as \(H \xrightarrow{\tau} G \xrightarrow{\alpha} \text{Aut}(H)\).

The collection of crossed modules form the objects of a 2-category \(\text{CrsMod}\).

**Theorem 6.40.** The 2-categories \(\text{CrsMod}\) and \(\text{2-Grp}\) are equivalent.

This theorem has been known for quite some time in several different forms. A simple place to start for this is in the article [4] with more information in [5].

**Proof.** We only prove the equivalence at the level of objects and in only one direction. This will set up our conventions throughout the paper. Given a crossed module \((H, G, \tau, \alpha)\) the associated 2-group is defined to have a single object \(\bullet\), \(G\) as its set of 1-morphisms, and \(H \ltimes G\) as its set of 2-morphisms. Composition of 1-morphisms is given by multiplication in \(G\). The source and target maps of 2-morphisms are defined pictorially by

\[
\begin{array}{c}
\begin{array}{c}
\bullet \\
\downarrow^g \\
\downarrow\tau(h)g \\
\end{array}
\end{array}
\]

Vertical and horizontal compositions are defined pictorially by

\[
\begin{array}{c}
\begin{array}{c}
\bullet \\
\downarrow^g \\
\downarrow\tau(h)g \\
\downarrow\tau(h)\tau(h)g \\
\end{array}
\end{array}
\quad = \quad
\begin{array}{c}
\begin{array}{c}
\bullet \\
\downarrow^g \\
\downarrow\tau(hh)g \\
\end{array}
\end{array}
\]
and

\[(479) \begin{array}{ccc}
g' & \bullet & g \\
\tau(h')g' & \circlearrowright & \tau(h)g \\
\end{array} = \begin{array}{ccc}
g'g & \bullet & \bullet \\
\tau(h')g' \tau(h)g & \circlearrowright & \circlearrowright \\
\end{array} \]

respectively. \qed

For more information, see [5] and [4]. The above proof sets up our convention for 2-group multiplication. Namely equation (478) defines vertical composition and equation (479) defines horizontal composition. Please be aware that different authors have different conventions (since the 2-categories CrsMod and 2-Grp are equivalent in many ways).

A trivial but important fact that we need in studying gauge invariance, mainly Corollary 5.3, is the following.

**Lemma 6.41.** Let \( H \xrightarrow{\tau} G \xrightarrow{\alpha} \text{Aut}(H) \) be a crossed module. Then \( \ker \tau \) is a normal subgroup of \( H \).

**Proof.** Let \( k \in \ker \tau \) and \( h \in H \). Then

\[(480) \tau(hkh^{-1}) = \tau(h)\tau(k)\tau(h)^{-1} = \tau(h)\tau(h)^{-1} = e. \]

Given a Lie crossed module \( H \xrightarrow{\tau} G \xrightarrow{\alpha} \text{Aut}(H) \) there is an associated differential Lie crossed module \( \mathfrak{h} \xrightarrow{\alpha_\tau} \mathfrak{g} \xrightarrow{\alpha} \text{Der}(\mathfrak{h}) \). It satisfies the following two identities

\[(481) \alpha_{\tau(B')}(B) = [B', B] \]

and

\[(482) \tau(\alpha_A(B)) = [A, \tau(B)] \]

for all \( A \in \mathfrak{g} \) and \( B, B' \in \mathfrak{h} \).

Note that the action \( \alpha \) also induces another map on the Lie algebras once an element \( g \) is plugged in. More precisely, \( \alpha_g : H \rightarrow H \). By differentiating this map at the Lie algebra, we obtain a Lie algebra homomorphism \( \alpha_g : \mathfrak{h} \rightarrow \mathfrak{h} \). Both \( \alpha \) and \( \alpha_g \) are important for understanding the differential cocycle data of Section 3.5.

As a final remark on notation, occasionally in the article \( G \) is sometimes written to denote the Lie algebra of \( G \) instead of \( \mathfrak{g} \). This is why \( \tau \) is written in this way. This is done for two reasons. The first is because this is sometimes common notation in physics articles. The second, better, reason is because some groups are written in a rather complicated fashion such as \( SU(pN)q\{Z\}pNq \). It is much better to write \( SU(N)/Z(N) \) than it is to write \( su(n)/Z(N) \) or something of this sort.

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