Convex $p$-partitions of bipartite graphs

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Abstract

A set of vertices $X$ of a graph $G$ is convex if it contains all vertices on shortest paths between vertices of $X$. We prove that for fixed $p \geq 1$, all partitions of the vertex set of a bipartite graph into $p$ convex sets can be found in polynomial time.

Keywords: bipartite graph, convex partition, graph convexity, geodesic convexity

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1. Introduction

Given a graph $G = (V, E)$, a set $X$ of vertices is called convex if $G[X]$, the graph induced by $X$, contains all shortest paths between any two of its vertices. (All graphs here are undirected and simple.) The notion probably first appeared in [8], see also [10], and later became also known as geodesic convexity, or $d$-convexity, in order to distinguish it from different notions of convexity in graphs and other combinatorial structures (see [7] for an early overview). The book [11] gives an up-to-date survey of results on convexity in graphs.

One of the approaches to convexity in graphs comes from the viewpoint of computational complexity. Clearly, computing the distances between all

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pairs, one can decide in polynomial time if a given set of vertices is convex. To determine the size of a largest convex set not covering the whole graph, however, is an NP-complete problem, even for bipartite graphs, albeit linear for cographs \[5\]. The same phenomenon occurs (NP-completeness even for bipartite graphs, but linearity for cographs) if we wish to determine related invariants such as the hull number and the geodetic number of a graph \([1, 4, 6]\).

We focus here on the notion of a **convex** \(p\)-**partition** of a graph, that is, a partition of the vertex set into \(p\) convex sets. For instance, any graph on \(n\) vertices containing a matching of size \(m\) has a convex \((n - m)\)-partition, and trivially, any graph has a convex 1-partition. Deciding whether a graph has a convex \(p\)-partition, for fixed \(p \geq 2\), is NP-complete for arbitrary graphs, and linear time solvable for cographs \([2]\). Also, chordal graphs have convex \(p\)-partitions for all \(p \geq 1\) \([2]\).

In view of the above described panorama, it was conjectured in \([11]\) that also for bipartite graphs, it should be NP-complete to decide whether they have a convex \(p\)-partition. We show that, for any fixed \(p \geq 1\), this is not the case. More precisely, we prove that for \(p \geq 1\), all convex \(p\)-partitions of a bipartite graph can be enumerated in polynomial time. This extends a recent result of Glantz and Mayerhenke \([9]\), who prove the same for the case \(p = 2\). They also showed that all convex 2-partitions of a planar graph can be found in polynomial time.

### 2. Bipartite graphs with convex \(p\)-partitions

We start by reproving the result for bipartite graphs from \([3]\) in a slightly different way. At the same time, this will serve as a base for the general case. We denote the distance between two vertices \(u\) and \(v\) in a graph \(G\) by \(d_G(u,v)\).

**Lemma 1.** Given a convex set \(C\) in a connected bipartite graph \(G\), and an edge \(uv\) with \(u \in C\), \(v \notin C\) we have that \(d_G(u',u) < d_G(u',v)\), for each \(u' \in C\).

**Proof.** Suppose otherwise. Observe that since \(G\) is bipartite, \(d_G(u',u) \neq d_G(u',v)\), and thus we may assume \(d_G(u',u) > d_G(u',v)\). Then there is a shortest path \(P\) from \(u'\) to \(v\) not containing \(u\). Extending \(P\) to \(u\) through the edge \(vu\), gives a shortest path from \(u'\) to \(u\), a contradiction, as \(u\) and \(u'\) lie in the convex set \(C\), but \(v \notin C\). \(\square\)
Let $e = uv$ be an edge of $G$ and denote by $X_{uv}$ the set of vertices that are closer to $u$ than to $v$. If $G$ is a connected bipartite graph, then $V$ is the disjoint union $X_{uv} \cup X_{vu}$. From Lemma [1] we get the following corollaries.

**Corollary 2.** Let $uv$ be an edge of a connected bipartite graph $G$. If $C$ is a convex set containing $u$ and not containing $v$, then $C \subseteq X_{uv}$.

**Corollary 3.** Let $G = (V, E)$ be a connected bipartite graph, with a partition of $V$ into convex sets $X_1, X_2$. Let $uv \in E$, with $u \in X_1$ and $v \in X_2$. Then $X_1 = X_{uv}$ and $X_2 = X_{vu}$.

From the previous corollary it is direct that there are at most $|E|$ convex 2-partitions and that we can enumerate all convex 2-partitions in polynomial time.

**Proposition 4.** We can enumerate in polynomial time all convex 2-partitions of a connected bipartite graph.

We now prove that for fixed $p \geq 3$ we can enumerate in polynomial time all convex $p$-partitions of a connected bipartite graph. To this purpose we extend the idea present in Corollary [3].

We write $[p]$ for the set $\{1, \ldots, p\}$. For a set $F$ of edges, let $V(F)$ denote the set of all endvertices of edges of $F$.

Given a convex $p$-partition $\mathcal{X} = \{X_1, X_2, \ldots, X_p\}$ of a graph $G = (V, E)$, we call a pair $(F, \phi)$ an $\mathcal{X}$-skeleton, if $F \subseteq E$ and $\phi : V(F) \rightarrow [p]$ satisfy the following:

- all edges of $F$ go between distinct parts of $\mathcal{X}$;
- if there is at least one edge in $E$ between $X_i$ and $X_j$, then there is exactly one edge of $F$ between $X_i$ and $X_j$;
- $\phi(v) = i$ if $v \in X_i$.

Note that the first two conditions might be equivalently expressed by saying that after contracting the sets $X_i$ and deleting all remaining edges that are not in $F$, we are left with a (simple) graph $H_{(F, \phi)}$ whose edges represent the edges of $G$ that cross the partition. The last condition says $\phi$ assigns the same colour to all vertices of $V(F)$ that become identified in $H_{(F, \phi)}$.

Note that for a connected graph $G$ the second condition implies that $V(F) \cap X_j \neq \emptyset$, for each $j \in [p]$. Then, the third condition implies that $\phi$ is a surjective function.
**Theorem 5.** Let $G = (V, E)$ be a connected bipartite graph, let $F \subseteq E$ and let $\phi : V(F) \to [p]$. If $G$ has a convex $p$-partition with skeleton $(F, \phi)$, then this partition is unique. We can find such partition or show it does not exist in polynomial time.

**Proof.** Define lists $L(u)$ for each vertex $u \in V$ by setting

$$L(u) := [p] - \{\phi(w) : u \in X_{vw} \text{ for some } vw \in F\}.$$ 

For each pair of vertices $u$ and $w$ define $I[u, w]$ as the set of vertices in shortest paths between $u$ and $w$. For each vertex $u$, define

$$L'(u) := L(u) - \{\phi(w) : v \in [w] \text{ and } \phi(w) \notin L(v) \text{ for some } v \in I[u, w]\}.$$ 

We will prove that if $G$ has a convex $p$-partition $X = \{X_1, \ldots, X_p\}$ with skeleton $(F, \phi)$, then, for each $i \in [p]$,

$$L'(u) = \{i\} \text{ for every } u \in X_i.
\tag{1}$$

We first observe that

$$i \in L(u) \text{ for every } u \in X_i.
\tag{2}$$

Otherwise, there are $u \in X_i$ and $vw \in F$ such that $\phi(w) = i$ and $u \in X_{vw}$. Hence, $u, w \in X_i$ and $v \in I[u, w]$. Since $X_i$ is convex, $v \in X_i$, contradicting the fact that the edge $vw$ of $F$ must join distinct parts of $X$. This contradiction proves (2).

Moreover,

$$i \in L'(u) \text{ for every } u \in X_i.
\tag{3}$$

Otherwise, by (2), there are $u \in X_i$, $w \in V(F)$ and $v \in I[u, w]$ such that $\phi(w) = i$ and $i \notin L(v)$. Now, on the one hand, since $u, w \in X_i$ and $v \in I[u, w]$, the convexity of $X_i$ implies that $v \in X_i$. On the other hand, since $i \notin L(v)$, we know by (2) that $v \notin X_i$. This contradiction proves (3).

Next, we now show that, for each $j \in [p]$,

$$\text{if } v'w' \in E, \text{ with } w' \in X_j \text{ and } v' \notin X_j, \text{ then } j \notin L(v').
\tag{4}$$

This is immediate if $v'w' \in F$, by the definition of $L(v')$. Otherwise, there is $vw \in F$ such that $w \in X_j$, and $v, v' \in X_i$ for some $i \neq j$. Lemma 1 applied.
to the convex set $X_i$ and the edge $vw$ yields that $d(v', v) < d(v', w)$; i.e., $v' \in X_{vw}$. Thus, the definition of $L(v')$ gives that $j \notin L(v')$, proving (1).

We now prove (2). Consider $u \in X_i$ and $j \in [p] - \{i\}$. Let $w \in V(F) \cap X_j$ (as $G$ is connected, this set is non-empty) and let $P$ be a shortest path between $u$ and $w$. By construction, $P$ has some edge $vw'$ such that $v \notin X_j$ and $w' \in X_j$. By (1), we have that $j \notin L(v)$. As $v \in I[u, w]$, and as $\phi(w) = j$, the definition of $L'(u)$ implies that $j \notin L'(u)$. This completes the proof of (2).

Therefore, a convex $p$-partition with skeleton $(F, \phi)$ exists if and only if the following conditions hold: (i) $|L'(u)| = 1$ for each vertex $u$ of $G$; (ii) the parts of the corresponding partition are convex. The time needed to find a convex $p$-partition with skeleton $(F, \phi)$ is dominated by the time needed to compute the distance function of the graph. Indeed, once the distance function is known, the construction of each list $L(u)$ takes constant time and the construction of the each list $L'(u)$ takes linear time.

When given a connected bipartite graph $G$ and an integer $p$, we can decide whether $G$ has a convex $p$-partition as follows. We first guess a candidate skeleton $(F, \phi)$ and then, by using Theorem 5, we compute in polynomial time the unique (if any) partition $\{X_1, \ldots, X_p\}$ associated to $(F, \phi)$. The choices for $(F, \phi)$ are bounded from above by a function that depends only on $p$. In fact, if $(F, \phi)$ is a skeleton of some partition, then it must satisfy the following properties.

- The size of $F$ satisfies $|F| \in \{p - 1, \ldots, \binom{p}{2}\}$.
- The function $\phi$ is surjective.
- Identifying all vertices $v \in V(F)$ of the same colour under $\phi$ yields a connected simple graph.

From the first condition we know that there are roughly at most $p^2 \binom{n^2}{p^2}$ choices for $F$. From the second condition we know that there are roughly $\binom{p^2}{|F|} \leq p^{2p}$ functions $\phi$. Since the problem of determining the convex $p$-partitions of a graph can be reduced in polynomial time to computing the convex $p'$-partitions of its components for $p' \in \{1, \ldots, p\}$ [2, 3], we conclude the following.

**Corollary 6.** For each fixed $p \geq 1$, all convex $p$-partitions of a bipartite graph can be enumerated in polynomial time.
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