CONVERGENCE RATE OF SOLUTIONS TOWARDS THE
STATIONARY SOLUTIONS TO SYMMETRIC
HYPERBOLIC-PARABOLIC SYSTEMS IN HALF SPACE

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Abstract. In the present paper, we study a system of viscous conservation
laws, which is rewritten to a symmetric hyperbolic-parabolic system, in one-
dimensional half space. For this system, we derive a convergence rate of the
solutions towards the corresponding stationary solution with/without the sta-
bility condition. The essential ingredient in the proof is to obtain the a priori
estimate in the weighted Sobolev space. In the case that all characteristic
speeds are negative, we show the solution converges to the stationary solution
exponentially if an initial perturbation belongs to the exponential weighted
Sobolev space. The algebraic convergence is also obtained in the similar way.
In the case that one characteristic speed is zero and the other characteristic
speeds are negative, we show the algebraic convergence of solution provided
that the initial perturbation belongs to the algebraic weighted Sobolev space.
The Hardy type inequality with the best possible constant plays an essential
role in deriving the optimal upper bound of the convergence rate. Since these
results hold without the stability condition, they immediately mean the asymp-
totic stability of the stationary solution even though the stability condition does
not hold.

1. Introduction. We study the system of viscous conservation laws over the one-
dimensional half space $\mathbb{R}_+ \coloneqq \{ x \in \mathbb{R} \mid x > 0 \}$:

$$U_t + f(U)_x = (G(U)U_x)_x, \quad x \in \mathbb{R}_+, \ t > 0,$$

where $U = U(t,x)$ is an unknown $m$-vector valued function taking values in an open
convex set $\mathcal{O}_U \subset \mathbb{R}^m$; $f(U)$ is a smooth $m$-vector valued function defined on $\mathcal{O}_U$;
$G(U)$ is a smooth $m \times m$ matrix valued function defined on $\mathcal{O}_U$. The previous results
in [12] show an existence and an asymptotic stability of a stationary solution to the
system (1) under the suitable conditions, of which details are discussed later. The
main purpose of the present paper is to derive a convergence rate of a time-global
solution towards the stationary solution to the system (1). Following [3, 8, 12], we

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rewrite the system (1) to the normal form of the symmetric hyperbolic-parabolic systems. Here we state the several assumptions on the system (1). In order to rewrite (1) into the symmetric form, we assume

\[ [\text{A1}] \] the system (1) has an entropy function \( \eta = \eta(U) \) defined on \( \mathcal{O}_U \), which satisfies three conditions:

(i) \( \eta(U) \) is a smooth strictly convex scalar function, that is, the Hessian matrix \( D^2_U \eta(U) \) is positive definite for \( U \in \mathcal{O}_U \);
(ii) there exists a smooth scalar function \( q(U) \) defined on \( \mathcal{O}_U \), which is called an entropy flux, such that \( D_U q(U) = D_U \eta(U) D_U f(U) \) for \( U \in \mathcal{O}_U \);
(iii) the matrix \( G(U)(D^2_U \eta(U))^{-1} \) is real symmetric and non-negative definite for \( U \in \mathcal{O}_U \).

Owing to this assumption, we see from [3, 8, 12] that there exists a diffeomorphism \( U \mapsto \hat{U} \) from \( \mathcal{O}_U \) onto an open set \( \mathcal{O}_{\hat{U}} \) such that the system (1) is deduced to the symmetric form by setting \( U = \hat{U} \). Namely, the system (1) is rewritten to

\[
\hat{A}^0(\hat{U}) \hat{U}_t + \hat{A}(\hat{U}) \hat{U}_x = (\hat{B}(\hat{U}) \hat{U}_x)_x, \quad x \in \mathbb{R}_+, \ t > 0,
\]

where

\[
\hat{A}^0(\hat{U}) : \text{real symmetric and positive definite for } \hat{U} \in \mathcal{O}_{\hat{U}},
\]

\[
\hat{A}(\hat{U}) : \text{real symmetric for } \hat{U} \in \mathcal{O}_{\hat{U}},
\]

\[
\hat{B}(\hat{U}) : \text{real symmetric and non-negative definite for } \hat{U} \in \mathcal{O}_{\hat{U}}.
\]

We further assume the system (2) is deduced to the normal form of the symmetric hyperbolic-parabolic systems. Precisely, we assume there exists a diffeomorphism \( \hat{U} \mapsto u \) from \( \mathcal{O}_{\hat{U}} \) onto an open set \( \mathcal{O}_u \) such that letting \( U = \hat{U}(u) \), the system (2) is rewritten to a system of equations for \( u = \gamma(v, w), \ v = v(t, x) \in \mathbb{R}^{m_1}, \ w = w(t, x) \in \mathbb{R}^{m_2} \):

\[
A^0(u)u_t + A(u)u_x = B(u)u_{xx} + g(u, u_x), \quad x \in \mathbb{R}_+, \ t > 0,
\]

where

\[
A^0(u) = \begin{pmatrix} A^0_1(u) & 0 \\ 0 & A^0_2(u) \end{pmatrix}, \quad A(u) = \begin{pmatrix} A_{11}(u) & A_{12}(u) \\ A_{21}(u) & A_{22}(u) \end{pmatrix},
\]

\[
B(u) = \begin{pmatrix} 0 & 0 \\ 0 & B_2(u) \end{pmatrix}, \quad g(u, u_x) = \begin{pmatrix} g_1(u, w_x) \\ g_2(u, u_x) \end{pmatrix}.
\]

In (3), \( A^0(u) \) is an \( m \times m \) real symmetric and positive definite matrix for \( u \in \mathcal{O}_u \); \( A^0_1(u) \) and \( A^0_2(u) \) are an \( m_1 \times m_1 \) and an \( m_2 \times m_2 \), respectively, real symmetric and positive definite matrices for \( u \in \mathcal{O}_u \); \( A(u) \) is an \( m \times m \) real symmetric matrix for \( u \in \mathcal{O}_u \); \( A_{11}(u) \) and \( A_{22}(u) \) are an \( m_1 \times m_1 \) and an \( m_2 \times m_2 \) real symmetric matrices, respectively, and \( \gamma A_{12}(u) = A_{21}(u) \) for \( u \in \mathcal{O}_u \); \( B_2(u) \) is an \( m_2 \times m_2 \) real symmetric and positive definite matrix for \( u \in \mathcal{O}_u \); \( g(u, u_x) \) is a smooth remainder function and \( g_1 \) depends on \( u \) and only \( w \). Summarizing the above, we assume

\[ [\text{A2}] \] the system (2) is rewritten to the normal form of the symmetric hyperbolic-parabolic systems (3).

It is shown in [8] that if the null condition \([\text{N}]\), that is

\[ [\text{N}] \] the null space \( \mathcal{N} := \ker \hat{B}(\hat{U}) \) is independent of \( \hat{U} \in \mathcal{O}_{\hat{U}} \),
holds, then (2) is deduced to (3) with \( m_1 := \dim \mathcal{N}, m_2 := m - m_1, g_1(u, w_x) \equiv 0 \)
and \( g_2(u, w_x) = B_2(u)xw_x \). We discuss, in Section 4.4, the system for compressible and viscous gases, which satisfies the null condition [N] except for viscous and non-heat conductive gas. By a diffeomorphism \( u \mapsto U \) from \( \mathcal{O}_u \) onto \( \mathcal{O}_U \), we have expression of \( A^0(u), A(u) \) and \( B(u) \) as
\[
A^0(u) = \begin{pmatrix} U_u(u) & D_u^2 \eta(U(u))U_u(u) \end{pmatrix}, \quad (4a)
\]
\[
A(u) = \begin{pmatrix} U_u(u) & D_u^2 \eta(U(u))f_U(U(u))U_u(u) \end{pmatrix}, \quad (4b)
\]
\[
B(u) = \begin{pmatrix} U_u(u) & D_u^2 \eta(U(u))G(U(u))U_u(u) \end{pmatrix}. \quad (4c)
\]
Here and hereafter, we often abbreviate a Fréchet derivative \( D_U f(U) \) of a function \( f(U) \) to \( f_U(U) \).

We decompose the system (3) into the symmetric hyperbolic system for \( v \) and the symmetric strongly parabolic system for \( w \):
\[
A^0_1(u)v_t + A_{11}(u)v_x + A_{12}(u)w_x = g_1(u, w_x), \quad (5a)
\]
\[
A^0_2(u)w_t + A_{21}(u)v_x + A_{22}(u)w_x = B_2(u)w_{xx} + g_2(u, w_x). \quad (5b)
\]
The initial and the boundary conditions for the system (5) are prescribed as
\[
u(0, x) = u_0(x) = \begin{pmatrix} v_0, w_0 \end{pmatrix}(x), \quad \text{that is,} \quad (v, w)(0, x) = (v_0, w_0)(x), \quad (6)
\]
\[
w(t, 0) = w_b, \quad (7)
\]
where \( w_b \in \mathbb{R}^{m_2} \) is a constant. We assume a spatial asymptotic state of the initial data is a constant:
\[
\lim_{x \to \infty} u_0(x) = u_+ = \begin{pmatrix} v_+, w_+ \end{pmatrix}, \quad \text{that is,} \quad \lim_{x \to \infty} (v_0, w_0)(x) = (v_+, w_+), \quad (8)
\]
where \( u_+ \in \mathcal{O}_u \) is a certain constant. Moreover we impose the condition on \( A_{11}(u) \).

[A3] The matrix \( A_{11}(u_+) \) is negative definite.

This condition corresponds to the outflow problem for the model system of compressible viscous gases. This system is studied in [6, 7, 13]. Since we construct the solution to the problem (5)–(7) in a small neighborhood of \( u_+ \) and then the all characteristics for the hyperbolic equations are negative, the boundary condition (7) for the parabolic equations is necessary and sufficient. Hence the problem (5)–(7) is well-posed with the boundary condition only on \( w \).

1.1. Previous results. We mention several related results for (1) obtained in Nakamura-Nishibata [12], which discusses an existence of a stationary solution and its asymptotic stability. Letting \( U_+ := U(u_+) \), we define \( \bar{U} \) as a stationary solution for (1) which converges to \( U_+ \) as \( x \to \infty \). The solution \( \bar{U} \), therefore, satisfies a system of equations
\[
f(\bar{U})_x = (G(\bar{U})\bar{U}_x)_x, \quad x \in \mathbb{R}_+. \quad (9)
\]
Let \( \tilde{u} = \begin{pmatrix} \tilde{v}, \tilde{w} \end{pmatrix} \) be a stationary solution for (5) and then we have the expressions \( \tilde{u} = u(\bar{U}) \) and \( \tilde{U} = U(\tilde{u}) \) by using a diffeomorphism \( u \mapsto U \). We assume \( \tilde{u} \) satisfies the same conditions (7) and (8):
\[
\tilde{w}(0) = w_b, \quad (10a)
\]
\[
\lim_{x \to \infty} \tilde{u}(x) = u_+, \quad \text{that is,} \quad \lim_{x \to \infty} (\tilde{v}, \tilde{w})(x) = (v_+, w_+). \quad (10b)
\]
The paper [12] studies the boundary value problem (9) and (10) for the non-degenerate and the degenerate flows. We call the non-degenerate if the matrix
$D_U f(U_+)$ has no zero-eigenvalue. On the other hand, if $D_U f(U_+)$ has zero-eigenvalues, we call the degenerate. For the degenerate flow, the paper [12] supposes that $D_U f(U_+)$ has only one zero-eigenvalue to show an existence of a stationary solution. Under this assumption, let $\mu(U)$ be an eigenvalue of $D_U f(U)$ satisfying $\mu(U_+) = 0$ and let $R(U)$ be a real eigenvector of $D_U f(U)$ corresponding to $\mu(U)$.

For a matrix $A$, the notation $\#(A)$ designate the number of negative eigenvalues of the matrix $A$. An existence of a stationary solution for the boundary value problem (9) and (10) is summarized in Lemmas 1.1 and 1.2, which are proved in [12]. Lemmas 1.1 and 1.2 correspond to results for the non-degenerate and the degenerate flows, respectively.

**Lemma 1.1** ([12]). Assume $[A1]$–$[A3]$ hold and set $\delta := |w_+ - w_b|$. Moreover we assume

$$\#(D_U f(U_+)) > m_1$$

holds. Then there exists a local stable manifold $\mathcal{M} \subset \mathbb{R}^{m_2}$ around the equilibrium $w_+$ such that if $w_b \in \mathcal{M}$ and $\delta$ is sufficiently small, then there exists a unique smooth solution $\hat{u}$ to the problem (9) and (10) satisfying

$$|\partial_x^k(\hat{u}(x) - u_+)| \leq C\delta e^{-cx} \text{ for } k = 0, 1, \ldots.$$  \hfill (12)

**Lemma 1.2** ([12]). Assume $[A1]$–$[A3]$ hold and set $\delta := |w_+ - w_b|$. Moreover we assume $D_U f(U_+)$ has only one zero-eigenvalue and the characteristic field corresponding to $\mu(U_+) = 0$ is genuinely nonlinear, that is,

$$D_U \mu(U_+) R(U_+) \neq 0.$$  \hfill (13)

Then there exists a certain region $\mathcal{M} \subset \mathbb{R}^{m_2}$ such that if $w_b \in \mathcal{M}$ and $\delta$ is sufficiently small, then there exists a unique smooth solution $\hat{u}$ to the problem (9) and (10) satisfying

$$|\partial_x^k(\hat{u}(x) - u_+)| \leq C\frac{\delta^{k+1}}{(1 + \delta x)^{k+1}} \text{ for } k = 0, 1, \ldots.$$  \hfill (14)

The paper [12] also studies the asymptotic stability of the stationary solution assuming the condition $[K]$ or equivalently $[SK]$. The equivalence of the conditions $[K]$ and $[SK]$ is proved in [15].

$[K]$ There exists an $m \times m$ matrix $K$ such that $KA^0(u_+)$ is skew-symmetric and $L := [KA(u_+)] + B(u_+)$ is real symmetric and positive definite, where $[A] := (A + \, ^tA)/2$ is a symmetric part of a matrix $A$.

$[SK]$ Let $\lambda A^0(u_+) \phi = A(u_+) \phi$ and $B(u_+) \phi = 0$ for $\lambda \in \mathbb{R}$ and $\phi \in \mathbb{R}^m$. Then $\phi = 0$.

The condition $[K]$ is first proposed by [3, 17] to study the asymptotic stability of the constant state. We often call the condition $[SK]$ the stability condition. Moreover, to estimate the remainder term $g(u, u_x)$ in (3), the papers [3, 12] impose

$[A4]$ $g(u_+, 0) = Dg(u_+, 0) = 0$.

Here $Dg(u, u_x)$ stands for the Fréchet derivative of $g$ with respect to $u$ and $u_x$.

For the non-degenerate flow, the asymptotic stability of the stationary solution is summarized in the following lemma.
Lemma 1.3 ([12]). Assume the same assumptions as in Lemma 1.1, the stability condition [SK] (or [K]) and [A4] hold. Then there exists a positive constant $\varepsilon_0$ such that if
\begin{equation}
\|u_0 - \tilde{u}\|_{H^2} + \delta \leq \varepsilon_0,
\end{equation}
then the problem (5), (6) and (7) has a unique solution $u$ globally in time satisfying
\begin{equation}
u - \tilde{u} \in C([0, \infty); H^2(\mathbb{R}^+)).\end{equation}
Moreover the solution $u$ converges to the stationary solution $\tilde{u}$:
\begin{equation}
\lim_{t \to \infty} \|u(t) - \tilde{u}\|_{L^\infty} = 0.
\end{equation}

The paper [12] also shows asymptotic stability of the the degenerate flow under the assumption that all of the non-zero characteristics are negative, that is, the matrix $A(u_+)$ has only one zero-eigenvalue and the other eigenvalues are negative. Note that this assumption is same as [A6] to be stated later.

Lemma 1.4 ([12]). Assume the same assumptions as in Lemma 1.2, the stability condition [SK] (or [K]) and [A4] hold. Moreover, we assume that the matrix $A(u_+)$ is non-positive definite. Then the same conclusion as in Lemma 1.3 holds true.

In the remainder of this section, we formulate an initial boundary value problem for the perturbation from the stationary solution. The perturbation is defined by
\begin{equation}
(\varphi, \psi)(t, x) := (v, w)(t, x) - (\tilde{v}, \tilde{w})(x).
\end{equation}
We use a notation $\xi(t, x) := u(t, x) - \tilde{u}(x) = (\varphi, \psi)(t, x)$. From (3), $\xi$ satisfies a system of equations
\begin{equation}
A^0(\tilde{u})\xi_t + A(\tilde{u})\xi_x = B(\tilde{u})\xi_{xx} + h,
\end{equation}
where
\begin{equation}
h := g - \tilde{g} - (A - \tilde{A})\tilde{u}_x + (B - \tilde{B})\tilde{u}_{xx},
\end{equation}
\begin{equation}
\tilde{g} := g(\tilde{u}, \tilde{u}_x), \quad \tilde{A} := A(\tilde{u}), \quad \tilde{B} := B(\tilde{u}).
\end{equation}
The system (15) is rewritten to a system for $\varphi$ and $\psi$ as
\begin{align}
A^0_1(\tilde{u})\varphi_t + A_{11}(\tilde{u})\varphi_x + A_{12}(\tilde{u})\psi_x &= h_1, \\
A^0_2(\tilde{u})\psi_t + A_{21}(\tilde{u})\varphi_x + A_{22}(\tilde{u})\psi_x &= B_2(\tilde{u})\psi_{xx} + h_2,
\end{align}
where
\begin{equation}
\begin{aligned}
h_1 &:= g_1 - \tilde{g}_1 - (A_{11} - \tilde{A}_{11})\tilde{v}_x - (A_{12} - \tilde{A}_{12})\tilde{w}_x, \\
h_2 &:= g_2 - \tilde{g}_2 - (A_{21} - \tilde{A}_{21})\tilde{v}_x - (A_{22} - \tilde{A}_{22})\tilde{w}_x + (B_2 - \tilde{B}_2)\tilde{w}_{xx}, \\
\tilde{g}_1 &:= g_1(\tilde{u}, \tilde{w}_x), \quad \tilde{g}_2 := g_2(\tilde{u}, \tilde{w}_x), \quad \tilde{A}_{ij} := A_{ij}(\tilde{u}) (i, j = 1, 2), \quad \tilde{B}_2 := B_2(\tilde{u}).
\end{aligned}
\end{equation}
We solve the system (16) with the initial and the boundary conditions, derived from (6), (7) and (10a), as
\begin{align}
(\varphi, \psi)(0, x) &= (\varphi_0, \psi_0)(x) := (v_0, w_0)(x) - (\tilde{v}, \tilde{w})(x), \\
\psi(t, 0) &= 0.
\end{align}
Since the existence of solutions to (16)-(18) is shown in $H^2$ Sobolev space, the initial data is assumed to be compatible with the boundary condition:
\begin{equation}
\psi_0(0) = 0.
\end{equation}
1.2. Main results. We study a convergence rate of a time-global solution towards the stationary solution, of which existence is proved in Lemmas 1.1 and 1.2, to the system (1). Our analyses is divided into four cases: the non-degenerate flows with/without the stability condition and the degenerate flows with/without the stability condition. Notice that the time-global existence of solutions and its asymptotic stability have been shown, under the stability condition, in [12]. These results are summarized in Lemmas 1.3 and 1.4. On the other hand, stabilities are shown, for the first time, by the present paper without the stability condition. To obtain the convergence rate, we impose conditions [A5] or [A6] below for the matrix $D_U f(U_+)$.

[A5] The matrix $D_U f(U_+)$ is negative definite.

[A6] The matrix $D_U f(U_+)$ has only one zero-eigenvalue and the other eigenvalues of the matrix $D_U f(U_+)$ are negative.

The condition [A5] corresponds to the non-degenerate flow; [A6] is for the degenerate flow. We have the relation between $D_U f(U)$ and $A(u)$:

$$D_U f(U(u)) \sim A(u), \quad u \in \mathcal{O}_u.$$  \hfill (19)

Here, for matrices $A$ and $B$ of which all eigenvalues are real number, a notation $A \sim B$ means that the numbers of positive eigenvalues, zero eigenvalues and negative values of $A$ coincide with those of $B$. For the proof of (19), see [12]. Owing to (19), the assumption [A5] is equivalent to that $A(u_+)$ is negative definite; [A6] is equivalent to that $A(u_+)$ has only one zero-eigenvalue and the other eigenvalues are negative.

The results concerning the non-degenerate flows are summarized in Theorems 1.5 and 1.6.

**Theorem 1.5.** Assume the same assumptions as in Lemma 1.1, the stability condition [SK] (or [K]), [A4] and [A5] hold.

(i) (Exponential decay) We assume $u_0 - \tilde{u} \in H^2(\mathbb{R}^+) \text{ and } e^{\alpha x/2}(u_0 - \tilde{u}) \in L^2(\mathbb{R}^+)$ hold for a certain positive constant $\alpha$. Then, for a certain constant $\beta \in (0, \alpha]$, there exists a positive constant $\varepsilon_0$ such that if

$$\|u_0 - \tilde{u}\|_{L^2} + \delta \leq \varepsilon_0,$$

then the problem (5), (6) and (7) has a unique solution $u$ globally in time satisfying

$$u - \tilde{u} \in C([0, \infty); H^2(\mathbb{R}^+)) \text{ and } e^{\beta x/2}(u - \tilde{u}) \in C([0, \infty); L^2(\mathbb{R}^+)).$$

Moreover there exists a certain constant $\nu \in (0, \beta)$ such that the solution $u$ verifies the decay estimate

$$\|u(t) - \tilde{u}\|_{H^2} + \|e^{\beta x/2}(u(t) - \tilde{u})\|_{L^2} \leq C(\|u_0 - \tilde{u}\|_{H^2} + \|e^{\beta x/2}(u_0 - \tilde{u})\|_{L^2})e^{-\nu t/2}$$

for $t > 0$.

(ii) (Algebraic decay) We assume $u_0 - \tilde{u} \in H^2(\mathbb{R}^+) \text{ and } (1 + x)^{\alpha/2}(u_0 - \tilde{u}) \in L^2(\mathbb{R}^+)$ hold for a certain positive constant $\alpha$. Then there exists a positive constant $\varepsilon_0$ such that if

$$\|u_0 - \tilde{u}\|_{H^2} + \delta \leq \varepsilon_0,$$
Theorem 1.6. Assume the same assumptions as in Lemma 1.1, [A4] and [A5] hold.

(i) (Exponential decay) We assume \( e^{\alpha x/2}(u_0 - \tilde{u}) \in H^2(\mathbb{R}_+) \) holds for a certain positive constant \( \alpha \). Then, for a certain constant \( \beta \in (0, \alpha] \), there exists a positive constant \( \varepsilon_0 \) such that if
\[
(\| e^{\beta x/2}(u_0 - \tilde{u}) \|_{H^2} + \delta)\beta^{-1} \leq \varepsilon_0,
\]
then the problem \( (5), (6) \) and \( (7) \) has a unique solution \( u \) globally in time satisfying
\[
e^{\beta x/2}(u - \tilde{u}) \in C([0, \infty); H^2(\mathbb{R}_+)).
\]
Moreover there exists a certain constant \( \nu \in (0, \beta) \) such that the solution \( u \) verifies the decay estimate
\[
\| e^{\beta x/2}(u(t) - \tilde{u}) \|_{H^2} \leq C \| e^{\beta x/2}(u_0 - \tilde{u}) \|_{H^2} e^{-\nu t/2}
\]
for \( t > 0 \).

(ii) (Algebraic decay) We assume \((1+\gamma x)^{\alpha/2}(u_0 - \tilde{u}) \in H^2(\mathbb{R}_+)\) holds for a certain positive constant \( \gamma \) and a certain constant \( \alpha \geq 2 \). Then, for an arbitrary constant \( \theta \in (0, \alpha] \), there exists a positive constant \( \varepsilon_0 \) such that if
\[
(\| (1+\gamma x)^{\alpha/2}(u_0 - \tilde{u}) \|_{H^2} + \delta)\gamma^{-1} + \gamma \leq \varepsilon_0,
\]
then the problem \( (5), (6) \) and \( (7) \) has a unique solution \( u \) globally in time satisfying
\[
(1+\gamma x)^{\alpha/2}(u - \tilde{u}) \in C([0, \infty); H^2(\mathbb{R}_+)).
\]
Moreover the solution \( u \) verifies the decay estimate
\[
\| u(t) - \tilde{u} \|_{H^2} \leq C \| (1+\gamma x)^{\alpha/2}(u_0 - \tilde{u}) \|_{H^2} (1+t)^{-(\alpha-\theta)/2}
\]
for \( t > 0 \).

For the degenerate flows, we show the stability in Theorems 1.7 and 1.8. Here we assume [A6] in place of [A5].

Theorem 1.7. Assume the same assumptions as in Lemma 1.2, the stability condition [SK] (or [K]), [A4] and [A6] hold. We also assume \( |\sigma|^{-\alpha/2}(u_0 - \tilde{u}) \in H^2(\mathbb{R}_+) \) holds for a certain constant \( \alpha \in [1,5] \), where \( \sigma \) is the solution to the equation (35).

Then, for an arbitrary constant \( \theta \in (0, \alpha] \), there exists a positive constant \( \varepsilon_0 \) such that if
\[
\| |\sigma|^{-\alpha/2}(u_0 - \tilde{u}) \|_{H^2} + \delta^{1/2} \leq \varepsilon_0,
\]
then the problem \( (5), (6) \) and \( (7) \) has a unique solution \( u \) globally in time satisfying
\[
|\sigma|^{-\alpha/2}(u - \tilde{u}) \in C([0, \infty); H^2(\mathbb{R}_+)).
\]
Moreover the solution $u$ verifies the decay estimate
\[
\|u(t) - \tilde{u}\|_{H^2} \leq C \|\sigma|^{-\alpha/2}(u_0 - \tilde{u})\|_{H^2} (1 + t)^{-(\alpha-\theta)/4}
\]
for $t > 0$.

**Theorem 1.8.** Assume the same assumptions as in Lemma 1.2, [A4] and [A6] hold. We also assume $|\sigma|^{-\alpha/2}(u_0 - \tilde{u}) \in H^2(\mathbb{R}_+)$ holds for a certain constant $\alpha \in [2, 5)$, where $\sigma$ is the solution to the equation (35). Then the same conclusion as in Theorem 1.7 holds true.

**Related results.** The method of symmetrization for the hyperbolic equations by the entropy function are first proposed by well-known researches such as Godunov [2] and Friedrichs-Lax [1]. Then Kawashima and Shizuta in [8] generalize this method to the hyperbolic-parabolic equations. For the system (1) in the full space $\mathbb{R}^n$, Umeda, Kawashima and Shizuta in [17] show the asymptotical stability of the constant state under the condition [K]. The equivalence of conditions [K] and [SK] is proved by Shizuta and Kawashima in [15]. For the system (1) in the half space $\mathbb{R}_+$, Nakamura and Nishibata in [12] construct a general theory for the existence and the asymptotic stability of the stationary solution under the stability condition. Their results are summarized in Lemmas 1.1–1.4. Our main purpose is to show the asymptotic stability of the stationary solution and obtain the convergence rate. We utilize the weighted energy method and derive these results with/without the stability condition [SK] (or equivalency [K]).

This kind of problems are first studied by Kawashima, Nishibata and Zhu in [7] for the isothermal/isentropic model of compressible viscous gases in the half space $\mathbb{R}_+$. They prove the existence and the asymptotic stability of the stationary solution for the outflow problem. Then the convergence rate towards its stationary solution is studied by Nakamura, Nishibata and Yuge in [13].

For the scalar viscous conservation law in the half space $\mathbb{R}_+$, Kawashima and Kurata in [4] study the asymptotic stability of the degenerate stationary solution and derive the algebraic convergence rate under the assumption that the initial perturbation is in the weighted space $L^2_\omega(\mathbb{R}_+)$ with $\omega(x) = (1 + x)^\alpha$ for $\alpha < 3 + 2/q$, where $q$ is the degenerate exponent. The upper bound of $\alpha$ is best possible in the sense that the spectrum of the corresponding linearized operator is positive for $\alpha > 3 + 2/q$. We note that the upper bound of $\alpha$ in Theorems 1.7 and 1.8 is best possible too. In the paper [10], the system of viscous conservation laws with Burgers-type flux term is studied. This system is simplified system for the case of $q = 1$. Then the algebraic convergence rate toward the degenerate stationary solution is considered under the assumption $\alpha < 5$.

**Outline of the paper.** The remainder of the present paper is organized as follows. To discuss the convergence rate towards the stationary solution, we state the summary of results in [12] for the existence of the stationary solution in Section 2. The essential ingredient for deriving convergence rate is to obtain the a priori estimate in the suitable weighted Sobolev space. To this end, in Section 3, we obtain several estimates with the general weight function: basic estimates and higher order estimates up to second order derivatives. To derive the basic estimates, we employ an energy form, which is defined from the entropy function. Especially, for the degenerate flow, we also utilize the Hardy type inequality with the best possible constant in [4]. The higher order estimates are obtained by applying the energy method on the symmetric system. In the section 4, we get the estimates in the
suitable weighted Sobolev space by using the results in Section 3 to derive the convergence rate towards the stationary solution. In Section 4.4, we discuss the heat-conductive model for compressible and viscous gases as the application of the main results in the present paper. For viscous and non-heat-conductive gas, the stability condition does not hold. The stability of the stationary solution for this case has been open problem in [12] as it assumes the stability condition. Note that the results in the present paper solves this open problem.

Notation. The norm $|\cdot|$ denotes the Euclidean norm on vectors and the operator norm on matrices, and the notation $\langle \cdot, \cdot \rangle$ denotes the standard inner product on pairs of vectors. For $1 \leq p \leq \infty$, $L^p = L^p(\mathbb{R}_+)$ represents the Lebesgue space over $\mathbb{R}_+$ equipped with the norm $\| \cdot \|_{L^p}$. For a non-negative integer $s$, $H^s = H^s(\mathbb{R}_+)$ represents the $s$-th Sobolev space over $\mathbb{R}_+$ in the $L^2$ sense with the norm $\| \cdot \|_{H^s}$. We notice the relations $H^0(\mathbb{R}_+) = L^2(\mathbb{R}_+)$ and $\| \cdot \|_{H^0} = \| \cdot \|_{L^2}$. If $f(u)$ is a smooth $n$-vector function of the $n$-vector $u$, the notation $D_u f(u)$ designates a Fréchet derivative of $f$ with respect to $u$, that is, $D_u f(u)$ is represented as an $m \times n$ matrix $D_u f(u) = (\partial_{u_i} f(u))_{ij}$, where $f(u) = \begin{bmatrix} f_1(u), \ldots, f_m(u) \end{bmatrix}$ and $u = \begin{bmatrix} u_1, \ldots, u_n \end{bmatrix}$. We often abbreviate a Fréchet derivative $D_u f(u)$ to $f_u(u)$. If $f(u)$ is a smooth $m$-vector function of the $m$-vector $u$, we define the notations $f_{uu}(u)v$ and $f_{uu}(u)w$ by an $m \times m$ matrix $f_{uu}(u)v := (\sum_{k=1}^m \partial_{u_i u_k} f_i(u) v_k)_{ij}$ and an $m$-vector $f_{uu}(u)w := \sum_{i=1}^m \partial_{u_i} f_i(u) w_i$, respectively, where $f(u) = \begin{bmatrix} f_1(u), \ldots, f_m(u) \end{bmatrix}$, $u = \begin{bmatrix} u_1, \ldots, u_m \end{bmatrix}$ and $v = \begin{bmatrix} v_1, \ldots, v_m \end{bmatrix}$. We also use a notation $f_{uu}(u)v := f_{uu}(u)w$. $C(I; B)$ denotes the space of continuous functions on the interval $I$ taking values in the Banach space $B$. $L^2(I; B)$ denotes the space of $L^2$-functions on the interval $I$ with values in the Banach space $B$.

Let $\omega = \omega(x) > 0$ be a smooth scalar function defined on $\mathbb{R}_+$. For $1 \leq p < \infty$, we define the weighted $L^p$ space over $\mathbb{R}_+$ by

$$L^p_{\omega}(\mathbb{R}_+) := \left\{ \xi \in L^p(\mathbb{R}_+) \mid \| \xi \|_{L^p_{\omega}} := \left( \int_{\mathbb{R}_+} \omega(x) |\xi(x)|^p \, dx \right)^{1/p} < \infty \right\}.$$ 

For a non-negative integer $s$, we define the weighted $H^s$ space over $\mathbb{R}_+$ by

$$H^s_{\omega}(\mathbb{R}_+) := \left\{ \xi \in H^s(\mathbb{R}_+) \mid \| \xi \|_{H^s_{\omega}} := \left( \sum_{i=0}^s \| \partial_x^i \xi \|_{L^2_{\omega}}^2 \right)^{1/2} < \infty \right\}.$$ 

Note that the relations $H^0_{\omega}(\mathbb{R}_+) = L^2(\mathbb{R}_+)$ and $\| \cdot \|_{H^0_{\omega}} = \| \cdot \|_{L^2_{\omega}}$ hold; if $\omega$ satisfies $|\partial_x^k \omega(x)| \leq C |\omega(x)|$ for $k = 1, 2, \ldots$, the norm $\| \xi \|_{H^s_{\omega}}$ is equivalent to the norm defined by $\| \omega^{\alpha/2} \xi \|_{H^s}$, where $\alpha$ is a positive constant. $H^s_{\omega}(\mathbb{R}_+)$ represents the completion of $C_0^\infty(\mathbb{R}_+)$ with respect to the norm

$$\| \xi \|_{H^s_{\omega}} := \left( \int_{\mathbb{R}_+} \omega(x) |\partial_x^s \xi(x)|^2 \, dx \right)^{1/2}.$$ 

2. Summary for existence of stationary solution. The existence of the stationary solution to the system (1) is proved in [12]. This result is summarized in Lemmas 1.1 and 1.2. In this section, we state the outline for the proofs of Lemmas 1.1 and 1.2.

2.1. Non-degenerate flow. Integrating (9) over $[x, \infty)$ and multiplying the resultant equality by $U_u(\bar{u}) D^2_{\bar{u}} \eta(\bar{U})$ on the left, we have the equality

$$B(\bar{u}) \bar{u}_x = \begin{bmatrix} U_u(\bar{u}) D^2_{\bar{u}} \eta(\bar{U}) \end{bmatrix} (f(\bar{U}) - f(U_+)).$$ (20)
Here we have utilized (4c), $\dot{U}_x = U_u(\bar{u})\bar{u}_x$ and the property $\dot{U}_x(x) \to 0$ as $x \to \infty$. The system (20) is rewritten for $\bar{u}(x) := \bar{u}(x) - u_+, \bar{v}(x) := \bar{v}(x) - v_+$, $\bar{w}(x) := \bar{w}(x) - w_+$ as
\begin{align}
0 &= A_{11}(u_+\bar{v}) + A_{12}(u_+\bar{w}) + O(|\bar{u}|^2), \\
B_2(\bar{w})\bar{w}_x &= A_{21}(u_+\bar{v}) + A_{22}(u_+\bar{w}) + O(|\bar{u}|^2)
\end{align}
(21a)
(21b)
owing to Taylor’s Theorem and (4b). Due to the assumption [A3], we solve (21a) with respect to $\bar{v}$ by the implicit function theorem. Thus $\bar{v}$ is represented as the function of $\bar{w}$
\begin{equation}
\bar{v} = \Gamma\bar{w} + O(|\bar{w}|^2), \quad \Gamma := -A_{11}(u_+)^{-1}A_{12}(u_+).
\end{equation}
(22)
Substituting (22) in (21b), we have an $m_2 \times m_2$ autonomous system of first order ordinary differential equations for $\bar{w}$
\begin{equation}
\bar{w}_x = \bar{A}\bar{w} + O(|\bar{w}|^2), \\
\bar{A} := B_2(u_+)^{-1}(-A_{21}(u_+)A_{11}(u_+)^{-1}A_{12}(u_+) + A_{22}(u_+)).
\end{equation}
(23)
Since the condition [A5] implies that the matrix $\bar{A}$ is negative definite, the existence and the decay property of the solution to the boundary value problem (23) and (10) follow from the stable manifold theorem.

Lemma 2.1 ([12]). Assume [A5] holds. Then there exists a local stable manifold $\mathcal{M}^* \subset \mathbb{R}^{m_2}$ such that if $w_b \in \mathcal{M}^*$ and $\delta := |w_+ - w_b|$ is sufficiently small, then the problem (23) and (10) has a unique smooth solution $\bar{w}$ satisfying
\begin{equation}
|\partial_k^p(\bar{w}(x) - w_+)| \leq C\delta e^{-\chi x} \quad \text{for} \quad k = 0, 1, \ldots.
\end{equation}

2.2. Degenerate flow. To study the degenerate flow, we have to calculate the right-hand side of (21) up to quadratic terms in $\bar{u} = \bar{v}, \bar{w}$. Precisely, by using Taylor’s Theorem and (4b), we rewrite the equation (20) as
\begin{align}
0 &= A_{11}(u_+)\bar{v} + A_{12}(u_+\bar{w}) + F_1[\bar{u}] + R_1[\bar{u}] + O(|\bar{u}|^3), \\
B_2(\bar{u})\bar{w}_x &= A_{21}(u_+)\bar{v} + A_{22}(u_+)\bar{w} + F_2[\bar{u}] + R_2[\bar{u}] + O(|\bar{u}|^3),
\end{align}
(24a)
(24b)
where
\begin{align}
F[\bar{u}] := & \frac{1}{2}\bar{v}U_u(u_+)D_\xi^2\eta(U_+)(U_{uu}(u_+)\bar{u})^2, \\
R[\bar{u}] := & \frac{1}{2}\bar{v}U_u(u_+)D_\xi^2\eta(U_+)(U_{uu}(u_+)\bar{u})^2 \\
& + \left(\bar{v}U_u(u_+ + u_+)D_\xi^2\eta(U_+ + u_+) - \bar{v}U_u(u_+)D_\xi^2\eta(U_+)\right)U_{uu}(u_+),
\end{align}
\begin{equation}
F[\bar{u}] = \bar{v}(F_1[\bar{u}], F_2[\bar{u}]) = O(|\bar{u}|^3), \quad R[\bar{u}] = \bar{v}(R_1[\bar{u}], R_2[\bar{u}]) = O(|\bar{u}|^2).
\end{equation}
Substituting (22) in (24a), we have
\begin{equation}
\bar{v} = \Gamma\bar{w} - A_{11}(u_+)^{-1}(\hat{F}_1[\bar{w}] + \hat{R}_1[\bar{w}]) + O(|\bar{w}|^3),
\end{equation}
(25)
\begin{equation}
\hat{F}_i[\bar{w}] := F_i[\Gamma\bar{w}, \bar{w}], \quad \hat{R}_i[\bar{w}] := R_i[\Gamma\bar{w}, \bar{w}] \quad (i = 1, 2).
\end{equation}
Substituting (25) in (24b), we have an $m_2 \times m_2$ autonomous system of first order ordinary differential equations for $\bar{w}$
\begin{equation}
\bar{w}_x = \bar{A}\bar{w} + B_2(u_+)^{-1}(\bar{v}\hat{F}_1[\bar{w}] + \hat{F}_2[\bar{w}]) + \bar{Q}[\bar{w}] + O(|\bar{w}|^3),
\end{equation}
\begin{equation}
\bar{Q}[\bar{w}] := B_2(u_+)^{-1}(\bar{v}\hat{R}_1[\bar{w}] + \hat{R}_2[\bar{w}]) + (B_2(\bar{u})^{-1} - B_2(u_+)^{-1})B_2(u_+)\bar{A}\bar{w}.
\end{equation}
(26)
The existence and the decay property of the solution to the boundary value problem (26) and (10) follow from the center manifold theorem. Here note that the assumptions [A3] and [A6] imply that the non-positive matrix $A$ has only one zero-eigenvalue.

**Lemma 2.2 ([12]).** Assume [A3] and [A6] hold. Let $\hat{r}$ be a right eigenvector of $A$ corresponding to the zero-eigenvalue. We also assume

\[
\langle r, F[r] \rangle \neq 0
\]

holds. Here we define the m-vector $r$ as

\[
r := \begin{pmatrix} \Gamma \hat{r} \\ \hat{r} \end{pmatrix}.
\]

Then there exists a certain region $\mathcal{M} \subset \mathbb{R}^{m_2}$ such that if $w_b \in \mathcal{M}$ and $\delta := |w_+ - w_b|$ is sufficiently small, then the problem (26) and (10) has a unique smooth solution $\bar{w}$ satisfying

\[
|\partial^k \bar{w}(x)| \leq C \frac{\delta^{k+1}}{(1 + \delta x)^{k+1}} \text{ for } k = 0, 1, \ldots
\]

To derive the optimal convergence rate towards the stationary solution, we have to utilize its properties in detail. Since $\Gamma(B_2(u_+)^{\dagger})\hat{r} = (B_2(u_+)^{\dagger}\hat{r}, \hat{r}) > 0$, we may normalize $\hat{r}$ as

\[
\bar{r} = 1, \quad \bar{\ell} := \Gamma(B_2(u_+)^{\dagger})\hat{r}.
\]

We notice that $\bar{\ell}$ is a left eigenvector of $A$ corresponding to the zero-eigenvalue. From (22), we have

\[
A(u_+)r = 0, \quad \Gamma r A(u_+) = 0.
\]

Hence $r$ and $\Gamma r$ are right and left eigenvectors of $A(u_+)$ corresponding to the zero-eigenvalue, respectively. We also have

\[
f_U(U_+)U_u(u_+)r = 0, \quad \Gamma \Gamma^T U_u(u_+)D_U^2 \eta(U_+)f_U(U_+) = 0
\]

from (4b) and (30). Therefore $U_u(u_+)r$ and $\Gamma \Gamma^T U_u(u_+)D_U^2 \eta(U_+)$ are right and left eigenvectors of $f_U(U_+)$ corresponding to the zero-eigenvalue, respectively.

As $A$ is transformed to the Jordan normal form, we see that there exists an $m_2 \times m_2$ invertible matrix $Q$ such that the $m_2 \times m_2$ matrix $Q^{-1}AQ$ satisfies

\[
Q^{-1}A Q = \begin{pmatrix} 0 & 0 \\ 0 & A' \end{pmatrix} := A,
\]

where the $(m_2 - 1) \times (m_2 - 1)$ matrix $A'$ is an upper triangular matrix

\[
A' = \begin{pmatrix} \tilde{\lambda}_2 & * \\ & \ddots \\ & & \tilde{\lambda}_{m_2} \end{pmatrix}.
\]

Here $\tilde{\lambda}_j$, $j = 2, \ldots, m_2$, are non-zero-eigenvalues of $A$. Setting $z(x) := Q^{-1}\bar{w}(x)$, we deduce (26) to the system for $z$ as

\[
\dot{z}_x = Q^{-1}A Qz + Q^{-1}B_2(u_+)^{-1}(\Gamma \hat{F}_1[Qz] + \hat{F}_2[Qz]) + Q^{-1}Q[Qz] + O(|z|^3).
\]

Letting $z = (z_1, z_2, \ldots, z_{m_2}) = (z_1, z')$, we decompose the system (33) into that for $z_1$ and $z'$:

\[
\begin{align}
\dot{z}_{1x} &= \langle r, F[r] \rangle z_1^2 + O(|z||z'|) + O(|z|^3) =: H(z_1, z'), \\
\dot{z}'_x &= A' z' + O(|z|^2).
\end{align}
\]
The system (34) has a local center manifold \( z' = \Phi^c(z_1) \ (= O(|z_1|^2)) \) corresponding to the zero-eigenvalue. Let \( \sigma \) be a solution to the system (34) restricted on the local center manifold satisfying
\[
\sigma_x = H(\sigma, \Phi^c(\sigma)) = \langle r, F[r] \rangle \sigma^2 + O(|\sigma|^3), \quad \sigma(x) \to 0 \text{ as } x \to \infty. \tag{35}
\]
Owing to (27), we see the problem (35) has a solution. Precisely, if \( \langle r, F[r] \rangle > 0 \) holds and \( |\sigma(0)| \) is sufficiently small, we have the monotone increasing solution \( \sigma \) satisfying \( \sigma(x) < 0 \) for \( x \in \mathbb{R}^+ \) and \( \sigma(x) \to 0 \) as \( x \to \infty \); on the other hand, we have the monotone decreasing solution \( \sigma \) satisfying \( \sigma(x) > 0 \) for \( x \in \mathbb{R}^+ \) and \( \sigma(x) \to 0 \) as \( x \to \infty \) if \( \langle r, F[r] \rangle < 0 \) holds and \( |\sigma(0)| \) is sufficiently small. Hence we also have
\[
\langle r, F[r] \rangle \sigma < 0. \tag{36}
\]
By virtue of the center manifold theorem again, the solution \( z = (z_1, z') \) to the system (34) is given by
\[
z_1(x) = \sigma(x) + O(\delta \varepsilon^{-x}),
\]
\[
z'(x) = \Phi^c(\sigma(x)) + O(\delta \varepsilon^{-x}). \tag{37}
\]
The solution \( \sigma \) to the problem (35) exists if the boundary data \( z(0) = z_b := Q^{-1}(w_b - w_+) \) corresponding to the system (34) belongs to a certain region \( \mathcal{M}' \subset \mathbb{R}^{m_2} \) related to the local stable manifold and the local center manifold. If \( |z_b| \) is sufficiently small, the solution \( \sigma \) also satisfies
\[
|\sigma(x)| \sim \frac{\delta}{1 + \delta^2 x}, \quad |\partial^k_x \sigma(x)| \leq C|\sigma(x)|^{k+1} \text{ for } k = 1, 2, \ldots. \tag{38}
\]
From (14) and (38), we have
\[
|\partial^k_x (\hat{U}(x) - u_+)| \leq C|\sigma(x)|^{k+1} \text{ for } k = 0, 1, \ldots. \tag{39}
\]
From (31), we see \( U_u(u_+)r \) is a right eigenvector of \( f_u(U_+) \) corresponding to the zero-eigenvalue. Hence we may take \( R(U_+) \) in (13) by
\[
R(U_+) := U_u(u_+)r. \tag{40}
\]
Moreover, due to (31) again, we see
\[
L(U_+) := \tau^T U_u(u_+) D^2_{U} \eta(U_+) \tag{41}
\]
is a left eigenvector of \( f_u(U_+) \) corresponding to the zero-eigenvalue. As \( D^2_{U} \eta \) is positive definite, the inequality \( L(U_+) R(U_+) > 0 \) holds. Then we have the equalities
\[
\langle r, F[r] \rangle = \frac{\kappa}{2} D_{U} \mu(U_+) R(U_+), \tag{42}
\]
\[
L(U_+) f_{U}(U_+) R(U_+)^2 = \kappa D_{U} \mu(U_+) R(U_+), \tag{43}
\]
where \( \kappa := L(U_+) R(U_+) > 0 \). The properties (35), (37) and (42) yield that
\[
\sigma_x = \frac{\kappa}{2} D_{U} \mu(U_+) R(U_+) \sigma^2 + O(|\sigma|^3), \tag{44}
\]
\[
\sigma_{xx} = \frac{\kappa^2}{2} |D_{U} \mu(U_+) R(U_+)|^2 \sigma^2 + O(|\sigma|^4), \tag{45}
\]
\[
\hat{U} - U_+ = R(U_+) \sigma + O(|\sigma|^2), \tag{46}
\]
\[
\hat{U}_x = \frac{\kappa}{2} D_{U} \mu(U_+) R(U_+) R(U_+) \sigma^2 + O(|\sigma|^3), \tag{48}
\]
\[
\hat{U}_x = \frac{\kappa}{2} (D_{U} \mu(U_+) R(U_+) R(U_+) \sigma^2 + O(|\sigma|^3), \tag{49}
\]
\[
\hat{U}_{xx} = \frac{\kappa^2}{2} |D_{U} \mu(U_+) R(U_+)|^2 \sigma^2 + O(|\sigma|^4), \tag{45}
\]
\[
\hat{U}_{xx} = \frac{\kappa^2}{2} |D_{U} \mu(U_+) R(U_+)|^2 \sigma^2 + O(|\sigma|^4), \tag{45}
\]
\[
\hat{U}_{xx} = \frac{\kappa^2}{2} |D_{U} \mu(U_+) R(U_+)|^2 \sigma^2 + O(|\sigma|^4). \tag{45}
\]
for $|\sigma| \ll 1$. See [12] for the details.

3. Weighted energy estimates. In the remainder of the present paper, we show the asymptotic stability and obtain the convergence rate towards the stationary solution, which is summarized in Theorems 1.5–1.8. We first introduce several function spaces. Letting $\omega_c(x) = e^{\alpha x}$ or $(1 + x)^\alpha$, we define function spaces $X(0, T)$ and $X_{\omega_c}(0, T)$ for $T > 0$ by

$$X(0, T) := \{(\varphi, \psi) \mid (\varphi, \psi) \in C([0, T]; H^2), (\varphi_t, \psi_t) \in C([0, T]; L^2),$$
$$\varphi_x \in L^2(0, T; H^1), \psi_x \in L^2(0, T; H^2),$$
$$\varphi_t \in L^2(0, T; L^2), \psi_t \in L^2(0, T; H^1)\};$$

$$X_{\omega_c}(0, T) := \{(\varphi, \psi) \mid \omega^{1/2}_c(\varphi, \psi) \in C([0, T]; H^2), (\omega^{1/2}_c, \varphi_t, \psi_t) \in C([0, T]; L^2),$$
$$\omega_c^{1/2} \varphi_x \in L^2(0, T; H^1), \omega_c^{1/2} \psi_x \in L^2(0, T; H^2),$$
$$\omega_c^{1/2} \varphi_t \in L^2(0, T; L^2), \omega_c^{1/2} \psi_t \in L^2(0, T; H^1)\}.\]
for $t \in [0, T]$, where $\mathcal{G}$, $\mathcal{R}_1$ and $\mathcal{R}_2$ are defined in (52).

**Proof.** Straightforward computations yield the equation for $\mathcal{E}$:

$$
\mathcal{E}_t + \mathcal{F}_x + \langle B_2(u)\psi_x, \psi_x \rangle + \mathcal{G} = B_x + \mathcal{R}_1 + \mathcal{R}_2,
$$

(52)

where

$$
\mathcal{F} := q(U) - q(\bar{U}) - D_U\eta(\bar{U})(f(U) - f(\bar{U})),
$$

$$
\mathcal{B} := \langle D_U\eta(U) - D_U\eta(\bar{U}), (G(U)U_x - G(\bar{U})\bar{U}_x) \rangle,
$$

$$
\mathcal{G} := \langle D_U\eta(\bar{U})_x f(U) - f(\bar{U}), (D_U\eta(U) - D_U\eta(\bar{U}))f(\bar{U})_x \rangle,
$$

$$
\mathcal{R}_1 := -\langle D_U^2\eta(U)U_u(u)\xi_x, (G(U)U_u(u) - G(\bar{U})\bar{U}_u(\bar{u}))\bar{u}_x \rangle
$$

$$
-\langle G(U)U_u(u)\xi_x, (D_U^2\eta(U)U_u(\bar{u}) - D_U^2\eta(\bar{U})\bar{U}_u(\bar{u}))\bar{u}_x \rangle,
$$

$$
\mathcal{R}_2 := -\langle (D_U^2\eta(U)U_u(\bar{u}) - D_U^2\eta(\bar{U})\bar{U}_u(\bar{u}))\bar{u}_x, (G(U)U_u(u) - G(\bar{U})\bar{U}_u(\bar{u}))\bar{u}_x \rangle.
$$

Multiplying (52) by the weight function $W(t, x) = \chi(t)\omega(x)$, we have

$$
(W\mathcal{E})_t + (W\mathcal{F})_x - W_x\mathcal{F} + W\langle B_2(u)\psi_x, \psi_x \rangle + W\mathcal{G} + W_x\mathcal{B} = W_t\mathcal{E} + (WB)_x + W(\mathcal{R}_1 + \mathcal{R}_2),
$$

(53)

$$
\mathcal{F} = \frac{1}{2}\langle A(\bar{u})\xi, \xi \rangle + O(\|\xi\|^3),
$$

(54)

$$
\mathcal{G} = \frac{1}{2}\langle D_U^2\eta(\bar{U})\bar{U}_x, f_{UU}(\bar{U})\Xi^2 \rangle + O(\|ar{u}_x\|\|\xi\|^3) + O(\|ar{u}_x\|^2\|\xi\|^2) + O(\|ar{u}_{xx}\|\|\xi\|^2),
$$

(55)

$$
|\mathcal{R}_1| \leq C\|ar{u}_x\|\|\xi\|\|\xi_x\|, \quad |\mathcal{R}_2| \leq C\|ar{u}_x\|^2\|\xi\|^2,
$$

(56)

where $\Xi := U - \bar{U}$. By using (18), (54), [A5] and taking $N(T)$ and $\delta$ sufficiently small, the integration of the second and third terms in the left-hand side of (53) are estimated as

$$
\int_{\mathbb{R}^+} (WF)_x dx = -(WF)|_{x=0} \geq c\chi(t)\omega(0)|\varphi(t, 0)|^2,
$$

(58)

$$
\int_{\mathbb{R}^+} -W_x\mathcal{F} dx \geq c\chi(t)|\varphi(t, \psi(t))|^2\mathcal{L}_{x, \omega_x}.
$$

(59)

Owing to the positivity of $B_2(u)$, we have the estimate for the fourth term

$$
\int_{\mathbb{R}^+} W\langle B_2(u)\psi_x, \psi_x \rangle dx \geq c\chi(t)|\psi_x(t)|^2\mathcal{L}_{x, \omega}.\tag{60}
$$

Since $\mathcal{E}$ is equivalent to $|(\varphi, \psi)|^2$, the first term in the right-hand side is estimated as

$$
\left|\int_{\mathbb{R}^+} W_t\mathcal{E} dx\right| \leq C\chi(t)|\varphi(t, \psi(t))|^2\mathcal{L}_{x, \omega}.
$$

(61)

Owing to the boundary condition (18) and (55), we have

$$
\left|\int_{\mathbb{R}^+} (WB)_x dx\right| = |(WB)|_{x=0} \leq C(\|\xi_x(t, 0)\| + |\bar{u}_x(0)|)\chi(t)\omega(0)|\varphi(t, 0)|^2
$$

$$
\leq C(N(T) + \delta)\chi(t)\omega(0)|\varphi(t, 0)|^2
$$

(62)

for the second term. Here we have also used the fact that $|\bar{u}_x| \leq C\delta$ and $|\xi_x| \leq N(T)$, which follows from (12). Estimating (55) gives

$$
|\mathcal{B}| \leq C\epsilon|\psi_x|^2 + C(\epsilon + N(T) + \delta)|\varphi, \psi|^2,
$$
where $\epsilon$ is an arbitrary positive constant. Then the integration of $W_2B$ over $\mathbb{R}_+$ is handled as
\[
\int_{\mathbb{R}_+} W_2B \, dx \leq C_\epsilon \chi(t) |\psi_x(t)|^2_{L^2,\omega_x} + C(\epsilon + N(T) + \delta) \chi(t) |(\varphi, \psi)(t)|^2_{L^2,\omega_x}.
\] (63)

Consequently, integrating (53) with respect to $x$ and $t$ over $\mathbb{R}_+ \times (0, t)$, substituting the above estimates (58)–(63) in the resultant equality and letting $N(T)$, $\delta$ and $\epsilon$ suitably small, we obtain the desired estimate (51).

3.2. Basic $L^2$ estimate for degenerate flow. To derive the basic $L^2$ estimate for the degenerate flow, we introduce the constant matrices $P'$ and $P$ as follows. Let $r$ be the $m$-vector defined in (28), and $Q$ and $A$ be the $m_2 \times m_2$ matrices in (32). We represent $A$ by $(0, A')$, where $A'_0 = \Gamma (0, A')$ is the $m_2 \times (m_2 - 1)$ constant matrix. Then let an $m \times (m - 1)$ constant matrix $P'$, and also an $m_1 \times (m - 1)$ and an $m_2 \times (m - 1)$ constant matrices $P'_1$ and $P'_2$, be
\[
P' = \begin{pmatrix} P'_1 \\ P'_2 \end{pmatrix} := \begin{pmatrix} I_{m_1} & O_{m_1, m_2-1} \\ O_{m_2, m_1} & QA'_0 \end{pmatrix}.
\] (64)

Moreover, let an $m \times m$ constant matrix $P$ be given by
\[
P := (P', r) = \begin{pmatrix} P'_1 & \Gamma \hat{r} \\ P'_2 & \hat{r} \end{pmatrix} = \begin{pmatrix} I_{m_1} & O_{m_1, m_2-1} & \Gamma \hat{r} \\ O_{m_2, m_1} & QA'_0 & \hat{r} \end{pmatrix}.
\] (65)

Here $I_m$ and $O_{m,n}$ denote the identity matrix of size $n$ and the $m \times n$ zero matrix, respectively. From straightforward calculations with (29)–(30) and (32), we see that

[C] The matrix $P$ defined in (65) satisfies the conditions
(i) $P$ is invertible;
(ii) the $m \times m$ constant matrix $\Gamma P A(u_+) P$ satisfies the equality
\[
\Gamma P A(u_+) P = \begin{pmatrix} \hat{A}' & 0 \\ 0 & 0 \end{pmatrix},
\]
where the $(m - 1) \times (m - 1)$ constant matrix $\hat{A}'$ is real symmetric and negative definite;
(iii) $\hat{P}'_2 B_2(u_+) \hat{r} = 0$.

Moreover we utilize the following Hardy type inequality with the best possible constant, which is proved in [4].

Lemma 3.3 ([4]). Let $\zeta \in C^1[0, \infty)$ and assume $\zeta > 0$, $\zeta_x > 0$ and $\zeta(x) \to \infty$ for $x \to \infty$. Then we have
\[
\int_{\mathbb{R}_+} \psi^2 \zeta_x \, dx \leq 4 \int_{\mathbb{R}_+} \psi_x^2 \zeta_x \, dx
\] (66)
for $\psi \in H^1_{0,\omega}(\mathbb{R}_+)$ with $\omega = \zeta^2 / \zeta_x$. Here the coefficient 4 is the best possible constant and there is no function $\psi \in H^1_{0,\omega}(\mathbb{R}_+)$ with $\omega = \zeta^2 / \zeta_x$, $\psi \neq 0$, which attains the equality in (66).

To show the following lemma, we utilize the weight function
\[W_2(t, x) = \chi_\sigma(t) \omega_\sigma(x) := (1 + t)^{\nu} |\sigma(x)|^{-\beta}
\] instead of the general weight function $W(t, x) = \chi(t) \omega(x)$. 
Lemma 3.4. Assume [A3] and [A6] hold. Let \((\varphi, \psi)\) be a solution to the problem (16), (17) and (18) such that \(|\sigma|^{-\alpha/2}(\varphi, \psi) \in C([0, T]; H^2(\mathbb{R}_+))\) for a certain positive constant \(T\) and a constant \(\alpha\) with \(\alpha \in [1, 5]\). Here \(\sigma\) is the solution to the equation (35). Then, for an arbitrary constant \(\theta \in (0, \alpha]\), there exists a positive constant \(\varepsilon_1\) such that if \(N(T) + \delta^{1/2} \leq \varepsilon_1\), then the solution \((\varphi, \psi)\) satisfies
\[
(1 + t)\nu \|\sigma|^{-\beta/2}(\varphi, \psi)(t)\|^2_{L_2^2} + \int_0^t (1 + \tau)\nu |\sigma(\tau)|^{-\beta}|\varphi(\tau, 0)|^2 \, d\tau \\
+ \int_0^t (1 + \tau)\nu \left(\|\sigma|^{-\beta-2/2}(\varphi, \psi)(\tau)\|^2_{L_2^2} + \|\sigma|^{-\beta/2}\psi_x(\tau)\|^2_{L_2^2}\right) \, d\tau \\
\leq C\|\sigma|^{-\beta/2}(\varphi_0, \psi_0)\|^2_{L_2^2} + C\nu \int_0^t (1 + \tau)\nu^{-1} \|\sigma|^{-\beta/2}(\varphi, \psi)(\tau)\|^2_{L_2^2} \, d\tau \\
+ C \int_0^t \int_{\mathbb{R}_+} (1 + \tau)\nu \left\{\|\sigma|^{-\beta}(|\mathcal{R}_1| + |\mathcal{R}_2|) + |\mathcal{R}_3|\right\} \, dx \, d\tau
\]
for \(\nu \geq 0\), \(\beta \in [\theta, \alpha]\) and \(t \in [0, T]\), where \(\mathcal{R}_1\), \(\mathcal{R}_2\) and \(\mathcal{R}_3\) are defined in (52) and (78).

Remark 1. The upper bound “5” for the index \(\alpha\) is optimal. This fact is proved in [4] for the scalar viscous conservation laws, which is the special case of (1).

Proof. For simplicity, we use the notations \(|\cdot|_{\sigma, \beta}\) and \(|\cdot|_{\sigma, \beta, s}\) which stand for \(|\cdot|_{L_2^2, \omega}\) and \(|\cdot|_{H^s, \omega}\) with \(\omega(x) = |\sigma(x)|^{-\beta}\), respectively. To derive the estimate (67), we employ the weight function \(W_\sigma(t, x) = \chi_\sigma(t)|\omega_\sigma(x)| = (1 + t)\nu|\sigma(x)|^{-\beta}\), where \(\nu \geq 0\) and \(\beta \in [\theta, \alpha]\). From (44) and (45), we have
\[
\partial_x((|\sigma(x)|^{-\beta})) = \beta \frac{\kappa}{2} D_U\mu(U_+)R(U_+) |\sigma(x)|^{-(\beta-1)} + O(\beta|\sigma|^{-(\beta-2)}),
\]
\[
\partial_x^2(|\sigma(x)|^{-\beta}) = \beta(\beta-1) \frac{\kappa^2}{4} D_U\mu(U_+)R(U_+)^2 |\sigma(x)|^{-(\beta-2)} + O(\beta(\beta+1)|\sigma|^{-(\beta-3)}).
\]
Multiplying (52) by the weight function \(W_\sigma\) yields the equation (53) with \(W_\sigma\) in place of \(W\). The second term \((W_\sigma F)_x\) is estimated similarly as in the derivation of (58). The other terms in the left-hand side are estimated by using the matrix \(P\) in (65) as follows. Let
\[
\tilde{\xi} = \left(\begin{array}{c}
\xi' \\
\hat{\psi}
\end{array}\right) := P^{-1} \xi,
\]
where \(\tilde{\xi}' \in \mathbb{R}^{n-1}\) and \(\hat{\psi} \in \mathbb{R}\). Note that \(c|\tilde{\xi}| \leq |\xi| \leq C|\tilde{\xi}|\) and \(\xi = P\tilde{\xi} = P^{\dagger}\tilde{\xi}' + r\hat{\psi}\). We rewrite \(A(\tilde{u})\) to estimate the third term \(W_{\sigma x} F\).

Owing to (4b), (47) and (39), \(A(\tilde{u})\) is rewritten as
\[
A(\tilde{u}) = \begin{array}{l}
\begin{aligned}
\end{array}
= \{A(u(t)) + (\begin{array}{l}
U(u(t)) D_U^2 \eta(U(t)) f_U(U(t)U_u(t))
\end{array}) \}
\end{array}
+ O(|\sigma|^2).
\]
Substituting this equality in (54), we have

\[
\mathcal{F} = \frac{1}{2} \langle A(u_+) \xi, \xi \rangle + \frac{1}{2} \left\langle \nabla^2 u(u_+ + \eta(U_+) (f_{UU}(U_+) R(U_+)) U_u(u_+) \xi, \xi \right\rangle \\
=: \mathcal{F}_1
\]

and

\[
\frac{1}{2} \left\langle \nabla^2 u(U_+ R(U_+)) U_u(u_+) \xi, \xi \right\rangle \\
=: \mathcal{F}_2
\]

Thus we have

\[
\mathcal{F} = \frac{1}{2} \left\langle \nabla^2 u(U_+ R(U_+)) U_u(u_+) \xi, \xi \right\rangle \\
+ \frac{1}{2} \left\langle \nabla^2 u(U_+ R(U_+)) U_u(u_+) \xi, \xi \right\rangle
\]

The second term \( \mathcal{F}_2 \) is computed as

\[
\mathcal{F}_2 = \frac{1}{2} \left\langle \nabla^2 u(U_+ R(U_+)) U_u(u_+) \xi, \xi \right\rangle
\]

The first term \( \mathcal{F}_1 \) is rewritten, with \( |C| \) and \( \xi = P \xi \), as

\[
\mathcal{F}_1 = \frac{1}{2} \langle A(Pu) \xi, \xi \rangle = \frac{1}{2} \langle A\xi', \xi' \rangle.
\]

The second term \( \mathcal{F}_2 \) is estimated as

\[
\mathcal{F}_2 = \frac{1}{2} \left\langle \nabla^2 u(U_+ R(U_+)) U_u(u_+) \xi, \xi \right\rangle
\]

The remainder terms \( \mathcal{F}_3 \) and \( \mathcal{F}_4 \) are estimated as

\[
\mathcal{F}_3 = O(|\sigma|^{1/2} |\xi|^2) + O(|\sigma|^{3/2} |\psi|^2), \quad \mathcal{F}_4 = O(|\sigma|^{1/2} |\xi|^2) + O(|\sigma|^{3/2} |\psi|^2)
\]

due to (31) and (39). Thus we have

\[
\mathcal{F} = \frac{1}{2} \langle A\xi', \xi' \rangle - \frac{\kappa}{2} \left| D U \mu(U_+) R(U_+) \right| |\sigma||\psi|^2
\]

Owing to (68), (71) and \( |C| \), the integral of \(-W_{\sigma x}\mathcal{F}\) over \( \mathbb{R}_+ \) is estimated below as

\[
\int_{\mathbb{R}_+} -W_{\sigma x} \mathcal{F} dx
\]

where the remainder terms satisfy

\[
|\mathcal{R}_{31}| \leq C|\sigma|^{-\beta(-1)}|\xi|^3, \quad |\mathcal{R}_{41}| \leq C|\sigma|^{-\beta-3}|\xi|^2,
\]

\[
|\mathcal{R}_{51}| \leq C(|\sigma|^{-\beta-3} |\xi|^2 + |\sigma|^{-\beta-5/2} |\psi|^2).
\]
We derive the estimate of the fourth term of (53). Since $B_2(u_+)$ is real symmetric and positive definite, we can define the square root $B_2(u_+)^{1/2}$, where $B_2(u_+)^{1/2}$ is real symmetric and positive definite. Thus we have
\[
\langle B_2(u)\psi_x, \psi_x \rangle = \langle B_2(u_+)^{1/2} \psi_x, \psi_x \rangle + \langle (B_2(u) - B_2(u_+)) \psi_x, \psi_x \rangle = |B_2(u_+)^{1/2} \psi_x|^2 + O(|\xi| |\psi_x|^2) + O(|\sigma| |\psi_x|^2).
\]
(73)
The equality (73) yields
\[
\int_{\mathbb{R}_+} W_\sigma \langle B_2(u_+)^{1/2} \psi_x, \psi_x \rangle dx = (1 + t)^\nu |B_2(u_+)^{1/2} \psi_x(t)|^2 \sigma_{\sigma, \beta}^2 + (1 + t)^\nu \int_{\mathbb{R}_+} R_{42} dx,
\]
\[
|R_{42}| \leq C(|\sigma|^{-\beta} |\xi|^2 + |\sigma|^{-\beta-1} |\psi_x|^2).
\]
(74)
The fifth term $G$ in (53) is computed as
\[
G = \frac{\kappa}{4} \left[ D_{U\mu}^2 \mu(U_+)|R(U_+)|^2 + O(|\sigma|^2 |\psi|^2) + O(|\sigma|^3 |\xi|^2) \right]
\]
\[
+ \frac{\kappa}{4} \left[ D_{U\mu} \mu(U_+)|R(U_+)|^2 + O(|\sigma|^2 |\xi|^2) + O(|\sigma|^3 |\xi|^2) \right]
\]
\[
+ \frac{\kappa^2}{4} \left[ D_{U\mu} \mu(U_+)|R(U_+)|^2 + O(|\sigma|^2 |\xi|^2) + O(|\sigma|^3 |\xi|^2) \right]
\]
\[
+ O((|\sigma|^2 |\xi|^2) + O(|\sigma|^3 |\xi|^2)).
\]
(75)
Here we have used (40), (41), (43), (49), (56), (39), $\Xi = U_+ \tilde{U} \xi + O(|\xi|^2)$ and $\xi = \tilde{U} \tilde{U} + \tilde{U} \tilde{U}$. Then the integration of $W_\sigma G$ over $\mathbb{R}_+$ yields the equality
\[
\int_{\mathbb{R}_+} W_\sigma G dx = \frac{\kappa^2}{4} |D_{U\mu} \mu(U_+)|R(U_+)|^2 (1 + t)^\nu |\tilde{U} \xi|^2 \sigma_{\sigma, \beta-2}
\]
\[
+ (1 + t)^\nu \int_{\mathbb{R}_+} (R_{32} + R_{43} + R_{52}) dx,
\]
\[
|R_{32}| \leq C|\sigma|^{-(\beta-2)} |\xi|^3, \quad |R_{43}| \leq C|\sigma|^{-(\beta-1)} |\xi|^2,
\]
\[
|R_{52}| \leq C(|\sigma|^{-\beta-1/2} |\xi|^2 + |\sigma|^{-\beta-1/2} |\psi|^2).
\]
(76)
We derive the estimate of the sixth term. From (55), (39), (68) and (69), $W_\sigma B$ satisfies
\[
W_\sigma B = \left( \frac{1}{2} W_{\sigma x} \langle B_2(u_+)^{1/2} \psi, B_2(u_+)^{1/2} \psi \rangle \right)_x
\]
\[
- \frac{\kappa}{8} |D_{U\mu} \mu(U_+)|R(U_+)|^2 \beta(\beta - 1)(1 + t)^\nu |\sigma(x)|^{-(\beta-2)} |B_2(u_+)^{1/2} \psi|^2
\]
\[
= \tilde{B}_1 + (1 + t)^\nu O(|\sigma|^{-(\beta-1)} |\xi|^2) + O(|\sigma|^{-(\beta-3)} |\xi|^2) + O(|\sigma|^{-(\beta-2)} |\psi|^2).
\]
By using (29) and $\psi = \tilde{U} \tilde{U} + \tilde{U} \tilde{U}$, we have
\[
|B_2(u_+)^{1/2} \psi|^2 = \langle B_2(u_+)^{1/2} \tilde{U} \tilde{U}, B_2(u_+)^{1/2} \tilde{U} \tilde{U} \rangle |\tilde{U} \tilde{U}|^2 + O(|\tilde{U} \tilde{U}|^2) + O(|\tilde{U} \tilde{U}|^2)
\]
\[
= |\tilde{U} \tilde{U}|^2 + O(|\tilde{U} \tilde{U}|^2).
\]
Thus we have
\[
B_1 = -\frac{\kappa^2}{8} \left| D_U \mu(U_+) R(U_+) \right|^2 \beta \gamma - (1 + t) \gamma \left| \sigma(x) \right|^{(\beta - 2)} \psi^0 \left| \psi^0 \right|^2 \\
+ (1 + t) \gamma \left( O(\left| \sigma \right|^{(\beta - 3/2)} \tilde{\xi}(t)^2) + O(\left| \sigma \right|^{(\beta - 5/2)} \tilde{\psi}(t)^2) \right).
\]

Hence, integrating \( W_\sigma B \) over \( \mathbb{R}_+ \) with the boundary condition (18), we have
\[
\int_{\mathbb{R}_+} W_\sigma B dx = -\frac{\kappa^2}{8} \left| D_U \mu(U_+) R(U_+) \right|^2 \beta \gamma - (1 + t) \gamma \left| \sigma(x) \right|^{(\beta - 2)} \psi^0(t)^2_{\sigma, \beta - 2} \\
+ (1 + t) \gamma \int_{\mathbb{R}_+} (R_{33} + R_{44} + R_{53}) dx,
\]

where
\[
|R_{33}| \leq C \left| \sigma \right|^{-(\beta - 1)} \left| \xi \right| \left| \xi \right|^2, \\
|R_{44}| \leq C \left( |\sigma|^{-(\beta - 3)} \left| \xi \right|^2 + |\sigma|^{-(\beta - 2)} \left| \psi \right| \left| \psi \right| \right), \\
|R_{53}| \leq C \left( |\sigma|^{-(\beta - 3/2)} \left| \xi \right|^2 + |\sigma|^{-(\beta - 5/2)} \left| \psi \right|^2 \right).
\]

Next we estimate the right hand side of (53). The first and the second terms \( W_\sigma E \) and \( (W_\sigma B)_x \) are estimated in the similar way to obtain (61) and (62). Since \( \left| \tilde{u}_x \right| \leq C \delta^2 \) holds for the degenerate flow due to (14), we have (61) and (62). Integrating (53) with \( W_\sigma \) in place of \( W \) over \( \mathbb{R}_+ \), substituting the above estimates (72), (74), (76) and (77) in the resultant equality and letting \( N(T) \) and \( \delta \) sufficiently small yield
\[
\frac{d}{dt} \left( 1 + t \right)^\gamma \int_{\mathbb{R}_+} \left| \sigma(x) \right|^{\gamma - \beta} E dx \\
+ C(1 + t)^\gamma \left| \sigma(0) \right|^{\gamma - \beta} \left| \varphi(t, 0) \right|^{2} + C(1 + t)^\gamma \left| \tilde{\xi}(t) \right|_{\sigma, \beta - 1}^{2} + I \\
\leq C \nu(1 + t)^{\gamma - 1} \left| \left( \varphi, \psi \right)(t) \right|_{\sigma, \beta}^{2} \\
+ \int_{\mathbb{R}_+} W_\sigma \left( |R_1| + |R_2| \right) dx + (1 + t)^\gamma \int_{\mathbb{R}_+} \left( |R_3| + |R_4| + |R_5| \right) dx,
\]

where
\[
I := \frac{\kappa^2}{4} \left| D_U \mu(U_+) R(U_+) \right|^2 \left( \beta + 1 - \frac{1}{2} \beta \gamma \right) \left( 1 + t \right)^\gamma \left| \psi^0(t) \right|_{\sigma, \beta - 2}^{2} \\
+ (1 + t)^\gamma \left| B_2(u_+) \right|^{1/2} \left| \psi_x(t) \right|_{\sigma, \beta}^{2},
\]

\[
R_3 = \sum_{i=1}^{3} R_{3i}, \quad R_4 = \sum_{i=1}^{4} R_{4i}, \quad R_5 = \sum_{i=1}^{3} R_{5i}.
\]

The term \( I \) is estimated below as
\[
I \geq c(1 + t)^\gamma \left| \psi^0(t) \right|_{\sigma, \beta - 2}^{2} + c(1 + t)^\gamma \left| \psi_x(t) \right|_{\sigma, \beta}^{2} \quad \text{for} \quad \beta \in [\theta, \alpha].
\]

We admit this estimate for the moment and proceed to show the estimate (67). By virtue of the estimate (79), we have
\[
\frac{d}{dt} \left( 1 + t \right)^\gamma \int_{\mathbb{R}_+} \left| \sigma(x) \right|^{\gamma - \beta} E dx \\
+ C(1 + t)^\gamma \left| \sigma(0) \right|^{\gamma - \beta} \left| \varphi(t, 0) \right|^{2} + \left| \tilde{\xi}(t) \right|_{\sigma, \beta - 1}^{2} + \left| \psi^0(t) \right|_{\sigma, \beta - 2}^{2} + \left| \psi_x(t) \right|_{\sigma, \beta}^{2} \\
\leq C \nu(1 + t)^{\gamma - 1} \left| \left( \varphi, \psi \right)(t) \right|_{\sigma, \beta}^{2} \\
+ \int_{\mathbb{R}_+} W_\sigma \left( |R_1| + |R_2| \right) dx + (1 + t)^\gamma \int_{\mathbb{R}_+} \left( |R_3| + |R_4| + |R_5| \right) dx.
\]
Due to $|\xi| \leq N(T)$, $|\sigma| \leq C\delta$ and Schwarz’s inequality, we have
\[
(1 + t)^\nu \int_{\mathbb{R}_+^n} |R_4| \, dx \leq C(N(T) + \delta)(1 + t)^\nu \left( |(\varphi, \psi)(t)|^2_{\sigma, \beta - 2} + |\psi_x(t)|^2_{\sigma, \beta} \right). \tag{81}
\]
The remainder term $R_5$ is estimated, with $|\sigma| \leq C\delta$, as
\[
(1 + t)^\nu \int_{\mathbb{R}_+^n} |R_5| \, dx \leq C\delta^{1/2}(1 + t)^\nu \left( |\xi'(t)|^2_{\sigma, \beta - 1} + |\psi_0(t)|^2_{\sigma, \beta} \right). \tag{82}
\]
Consequently, substituting (81), (82) and $|\xi(t)|^2_{\sigma, \beta - 2} \leq C(\xi'(t)|^2_{\sigma, \beta - 1} + |\psi_0(t)|^2_{\sigma, \beta - 2})$ in (80), letting $N(T)$ and $\delta^{1/2}$ sufficiently small and finally integrating the resultant inequality over $(0, t)$, we obtain the desired estimate (67).

It remains to show (79) to complete the proof, which depends on the Hardy type inequality with the best possible constant in Lemma 3.3. It is easy to see (79) holds if $\alpha \in [1, 3]$ since $\beta + 1 - \beta(\beta - 1)/2 \geq 1$ for $\beta \in [0, 3]$ and $B_2(u_\pm)$ is positive definite. It remains to show (79) for $\alpha \in (3, 5)$. If $\beta \in [3, 5]$, (79) holds as $\beta + 1 - \beta(\beta - 1)/2 \geq 1$ also holds. Hence, we consider the case with $\beta \in (3, \alpha]$. From (29), (C), the positivity of $B_2(u_\pm)$ and $\psi_x = P_2 \tilde{\xi}_x + \tilde{\psi}_x$, the following estimate holds:
\[
|B_2(u_\pm)^{1/2}\psi_x|^2 \geq (\tilde{\gamma}_2 B_2(u_\pm) \tilde{\psi}_x^2 + 2 \tilde{\psi}_x P_2 \tilde{\xi}_x^2 + 2 \tilde{\xi}_x \tilde{\xi}_x^\top \tilde{\psi}_x) \geq \tilde{\psi}_x^2 + c|P_2 \tilde{\xi}_x^2|^2.
\]

Thus the second term of $I$ satisfies
\[
(1 + t)^\nu |B_2(u_\pm)^{1/2}\psi_x(t)|^2_{\sigma, \beta} \geq (1 + t)^\nu |\tilde{\psi}_x(t)|^2_{\sigma, \beta} + c(1 + t)^\nu |P_2 \tilde{\xi}_x(t)|^2_{\sigma, \beta}. \tag{83}
\]
To estimate the first term in the right-hand side of (83), we utilize Lemma 3.3. Letting
\[
\zeta_\sigma(x) := |\sigma(x)|^{-(\beta - 1)}, \tag{84}
\]
we see the function $\zeta_\sigma$ satisfies the conditions for $\zeta$ in Lemma 3.3 if $\delta$ is sufficiently small. From the boundary condition (18) and (65), we have $\psi_0(0, 0) = 0$. This property yields that $\psi_0 \in H_{1/2, \omega}^0(\mathbb{R}_+)$ with $\omega = |\sigma|^{-(\beta - 2)} / |\partial_\sigma|$. Hence we can apply Lemma 3.3 and have
\[
\frac{(\beta - 1)^2}{4} \int_{\mathbb{R}_+} |\psi_0|^2 |\sigma|^{-2} |\partial_\sigma| \, dx \leq \int_{\mathbb{R}_+} |\psi_0|^2 |\sigma|^{-2} |\partial_\sigma| \, dx \leq \int_{\mathbb{R}_+} |\psi_0|^2 |\sigma|^{-2} |\partial_\sigma| \, dx. \tag{85}
\]
By (85), $|\tilde{\psi}_x^2|_{\sigma, \beta}$ is estimated as
\[
|\tilde{\psi}_x(t)|^2_{\sigma, \beta} \geq \frac{\kappa^2}{16} |Du_\mu(U_\pm) R(U_\pm)|^2 (\beta - 1)^2 |\psi_0(t)|^2_{\sigma, \beta - 2} - C\delta (|\psi_0(t)|^2_{\sigma, \beta - 2} + |\tilde{\psi}_x(t)|^2_{\sigma, \beta}). \tag{86}
\]
Here we have also used the fact that $|\partial_\sigma| = \kappa |Du_\mu(U_\pm) R(U_\pm)||\sigma|^2 / 2 + O(|\sigma|^3)$, which follows from (44). We define the functions $g_1(\beta)$ and $g_2(\beta)$ by $g_1(\beta) := \beta + 1 - \beta(\beta - 1)/2$ and $g_2(\beta) := (\beta - 1)^2/4$. It is easy to see the condition $g_1(\beta) + g_2(\beta) > 0$ is equivalent that $-1 < \beta < 5$. Letting a constant $\gamma_*$ by
\[
\gamma_* := \min \left\{ \min_{3 \leq \beta \leq \alpha} \frac{g_1(\beta) + g_2(\beta)}{2 + g_2(\beta)}, 1 - \kappa \right\}
\]
for a sufficiently small positive constant $\kappa_*$, an inequality $0 < \gamma_* < 1$ holds. Then $\gamma_*$ satisfies

$$g_1(\beta) + (1 - \gamma_*)g_2(\beta) \geq 2\gamma_*$$  \hspace{1cm} (87)

for $\beta \in (3, \alpha)$. By using (83), (86), (87) and $|\psi_x(t)|^2_{\sigma, \beta} \leq C(|P_2^0 \tilde{\psi}_0(t)|^2_{\sigma, \beta} + |\tilde{\psi}_0(t)|^2_{\sigma, \beta})$, $I$ is estimated, for sufficiently small $\delta$, as

$$I \geq \frac{\kappa^2}{4} |D_\mu(U_+)R(U_+)|^2 (g_1(\beta) + (1 - \gamma_*)g_2(\beta))(1 + t)^{\nu}|\tilde{\psi}_0(t)|^2_{\sigma, \beta} - 2 + c(1 + t)^{\nu}|\tilde{\psi}_0(t)|^2_{\sigma, \beta},$$

which immediately yields (79).

3.3. Estimate for the first order derivatives. We next derive the estimate for the first order derivatives. Note that the computations in Subsections 3.3 and 3.4 hold for both the non-degenerate and the degenerate flows. We first show the estimate for $\varphi_x$ and $\psi_x$.

**Lemma 3.5.** Assume [A3] holds. Let $(\varphi, \psi)$ be a solution to the problem (16), (17) and (18) in a time interval $[0, T]$. If $(\varphi, \psi)$ satisfies $\omega^1/2(\varphi, \psi) \in C([0, T]; H^2(\mathbb{R}))$, then we have

$$\chi(t)\langle \varphi_x, \psi_x \rangle(t)_{L^2, \omega} + \int_0^t \chi(\tau)\langle \omega(0)\varphi_x(\tau, 0) \rangle^2 + |\varphi_x(\tau)|^2_{L^2, \omega} + |\psi_x(\tau)|^2_{L^2, \omega} d\tau$$

$$\leq C\chi(0)\langle \varphi_0x, \psi_0x \rangle_{L^2, \omega}^2 + \int_0^t \chi(\tau)(C|\varphi_x(\tau)|^2_{L^2, \omega} + C_\epsilon|\psi_x(\tau)|^2_{L^2, \omega} + |\psi_x(\tau)|^2_{L^2, \omega}) d\tau$$

$$+ C \int_0^t \chi(\tau)\langle \varphi_x, \psi_x \rangle(\tau)_{L^2, \omega}^2 d\tau + C \int_0^t \int_{\mathbb{R}^+} W(\|H_1\| + |H_2|) dx d\tau$$  \hspace{1cm} (88)

for $t \in [0, T]$, where $\epsilon$ is an arbitrary positive constant; $H_1$ and $H_2$ are defined in (89) and (90).

**Proof.** Differentiating (16a) with respect to $x$ and then taking an inner product of the resultant equality with $\varphi_x$ yield

$$\left(\frac{1}{2}(A_0^0 \varphi_x, \varphi_x)\right)_t + \left(\frac{1}{2}(A_{11} \varphi_x, \varphi_x)\right)_x + (A_{12} \psi_x, \varphi_x) = H_1,$$

$$H_1 := h_1^1 + h_2^1,$$

$$h_1^1 := h_{1x} - A_{12}^0 \psi_t - A_{11x}^0 \varphi_x - A_{12x}^0 \psi_x.$$  \hspace{1cm} (89)

Taking an inner product of (16b) with $-\psi_{xx}$, we have

$$\left(\frac{1}{2}(A_0^0 \psi, \psi)\right)_t = \left(\langle A_0^0 \psi_t, \psi_x \rangle\right)_x + B_2 \psi_{xx},$$

$$= \langle A_{12} \psi_{xx}, \varphi_x \rangle + \langle A_{22} \psi_x, \psi_{xx} \rangle + H_2,$$$$

H_2 := -\langle A_{12}^0 \psi_t, \psi_x \rangle + \left(\frac{1}{2}(A_{22} \psi_x, \psi_x)\right) - (h_2, \psi_{xx}).$$  \hspace{1cm} (90)

Here we have used $T A_{21} = A_{12}$. Adding (89) to (90), we obtain

$$\left(\frac{1}{2}(A_1^0 \varphi_x, \varphi_x) + \frac{1}{2}(A_2^0 \psi, \psi)\right)_t + \left(\frac{1}{2}(A_{11} \varphi_x, \varphi_x) + A_{12} \psi_x, \varphi_x \right)_x + B_2 \psi_{xx},$$

$$= \langle A_{22} \psi_x, \psi_{xx} \rangle + H_1 + H_2.$$  \hspace{1cm} (91)
Multiplying (91) by the weight function \( W(t, x) = \chi(t)\omega(x) \), we have
\[
\left(W\left(\frac{1}{2}(A_0^0 \varphi_x, \varphi_x) + \frac{1}{2}A_2^0 \psi_x, \psi_x\right)\right)_t + \left(W\left(\frac{1}{2}(A_{11} \varphi_x, \varphi_x) - (A_2^0 \psi_x, \psi_x)\right)\right)_x
- W_x \frac{1}{2}(A_1^0 \varphi_x, \varphi_x) + W_x(A_2^0 \psi_x, \psi_x) + W(B_2 \psi_{xx}, \psi_{xx})
= W_t\left(\frac{1}{2}(A_1^0 \varphi_x, \varphi_x) + \frac{1}{2}(A_2^0 \psi_x, \psi_x)\right) + W(A_{22} \psi_x, \psi_{xx}) + W(H_1 + H_2). 
\]
Since \( \psi(t, 0) = 0 \) due to the boundary condition (18), the integration of the second term in the left-hand side satisfies
\[
\int_{\mathbb{R}_+} W\left(\frac{1}{2}(A_{11} \varphi_x, \varphi_x) - (A_2^0 \psi_t, \psi_x)\right) dx \geq c\chi(t)\omega(0)\|\varphi_x(t, 0)\|^2. 
\]
Here we have also used \([A_3]\). The third term is also estimated by \([A_3]\) as
\[
\int_{\mathbb{R}_+} -W_x \frac{1}{2}(A_{11} \varphi_x, \varphi_x) dx \geq c\chi(t)\|\varphi_x(t)\|^2. 
\]
The fourth term in the left-hand side and the second term in the right-hand side of (92) are handled as
\[
\int_{\mathbb{R}_+} W_t\left(\frac{1}{2}(A_1^0 \varphi_x, \varphi_x) + \frac{1}{2}(A_2^0 \psi_x, \psi_x)\right) dx \leq C\varepsilon \chi(t)\|\varphi_x(t)\|^2 + \varepsilon \chi(t)\|\psi_t(t)\|^2, 
\]
\[
\int_{\mathbb{R}_+} W(A_{22} \psi_x, \psi_{xx}) dx \leq C\varepsilon \chi(t)\|\psi_{xx}(t)\|^2 + \varepsilon \chi(t)\|\psi_{xx}(t)\|^2, 
\]
where \( \varepsilon \) and \( \epsilon \) are arbitrary positive constants. The integral of the first term in the right-hand side is estimated as
\[
\int_{\mathbb{R}_+} W_t\left(\frac{1}{2}(A_1^0 \varphi_x, \varphi_x) + \frac{1}{2}(A_2^0 \psi_x, \psi_x)\right) dx \leq C\chi_t(t)(\varphi_x(t), \psi_x(t))^2. 
\]
Consequently, integrating (92) with respect to \( x \) and \( t \) over \( \mathbb{R}_+ \times (0, t) \) and then using the above estimates (93)–(97) with sufficiently small constant \( \epsilon \), we obtain the desired estimate (88).

In Lemma 3.5, we have the estimate for the dissipation \( \varphi_x \) with the weight function \( \omega_x \). However, this estimate disappears if \( \omega \equiv 1 \). The stability condition \([SK]\) allows us to get the estimate \( \| \varphi_x \|_{L^2, \omega} \), which equals \( \| \varphi_x \|_{L^2, \omega} \) if \( \omega \equiv 1 \). This estimate is necessary in the proof of Theorem 1.5.

**Lemma 3.6.** Assume the stability condition \([SK]\) (or \([K]\)) holds. Let \((\varphi, \psi)\) be a solution to the problem (16), (17) and (18) in a time interval \([0, T]\). If \((\varphi, \psi)\) satisfies \( \omega^{1/2}(\varphi, \psi) \in C([0, T]; H^2(\mathbb{R}_+))\), then we have
\[
\int_0^t \chi(\tau)(\varphi_x(\tau))^2 d\tau 
\leq C\chi(0)(\varphi_0, \psi_0)^2 + C\chi(t)(\varphi, \psi(t))^2_{H^1, \omega} + C\int_0^t \chi(\tau)(\varphi, \psi(\tau))^2_{H^1, \omega} d\tau 
+ C\int_0^t \chi(\tau)(\omega(0)(\varphi, \varphi_x(\tau, 0))^2 + |\varphi_x(\tau)|^2_{H^1, \omega}) d\tau 
+ C\int_0^t \int_{\mathbb{R}_+} \chi(\tau)\omega_x(\xi(t))|\xi(t)|^2 dxd\tau + C\int_0^t \int_{\mathbb{R}_+} W(|Kh', \xi_x)| dxd\tau
\]
for \( t \in [0, T] \), where \( K \) and \( h' \) are defined in \([K]\) and (99).
Proof. Letting $\tilde{A}^0 := A^0(u_+)$, $\tilde{A} := A(u_+)$ and $\tilde{B} := B(u_+)$, we rewrite the system (15) as

$$
\tilde{A}^0 \xi_t + \tilde{A} \xi_x = \tilde{B} \xi_{xx} + h',
$$

$$
h' := h - (A^0 - \tilde{A}^0) \xi_t - (A - \tilde{A}) \xi_x + (B - \tilde{B}) \xi_{xx}.
$$

(99)

Multiplying (99) by the compensation matrix $K$ and taking an inner product of the resultant equality with $\xi_x$ yield

$$
\left( \frac{1}{2} \langle K \tilde{A}^0 \xi, \xi_x \rangle \right)_t - \left( \frac{1}{2} \langle K \tilde{A}^0 \xi, \xi_t \rangle \right)_x + \langle L \xi_x, \xi_x \rangle
$$

$$
= \langle B \xi_x, \xi_x \rangle + \langle K \dot{B} \xi_{xx}, \xi_x \rangle + \langle K h', \xi_x \rangle
$$

(100)

since $K \tilde{A}^0$ is skew-symmetric. Multiplying (100) by the weight function $W(t, x) = \chi(t) \omega(x)$, we have

$$
\left( W \frac{1}{2} \langle K \tilde{A}^0 \xi, \xi_x \rangle \right)_t - \left( W \frac{1}{2} \langle K \tilde{A}^0 \xi, \xi_t \rangle \right)_x + W \langle L \xi_x, \xi_x \rangle
$$

$$
= W_t \frac{1}{2} \langle K \tilde{A}^0 \xi, \xi_x \rangle + W \langle \dot{B} \xi_x, \xi_x \rangle + W \langle K \dot{B} \xi_{xx}, \xi_x \rangle + W \langle K h', \xi_x \rangle.
$$

(101)

From a direct computation with (16a), the estimate

$$
|\varphi_t(t, 0)|^2 \leq C |(\varphi, \varphi_x, \psi_x)(t, 0)|^2
$$

(102)

holds. Owing to the boundary condition $\psi(t, 0) = \psi,l(t, 0) = 0$, (102) and Sobolev’s lemma, the second term in the left-hand side is estimated as

$$
\left| \int_{\mathbb{R}_+} \left( W \frac{1}{2} \langle K \tilde{A}^0 \xi, \xi_t \rangle \right)_x \right| \leq C \chi(t) \omega(0) |(\varphi, \varphi_t)(t, 0)|^2
$$

$$
\leq C \chi(t) \omega(0) |(\varphi, \varphi_x)(t, 0)|^2 + C \chi(t) |\psi_x(t)|^2_{L^2, \omega}.
$$

(103)

The third term is estimated as

$$
\left| \int_{\mathbb{R}_+} W \frac{1}{2} \langle K \tilde{A}^0 \xi, \xi_t \rangle \right| \leq C \chi(t) \int_{\mathbb{R}_+} \omega(x) |x| |\xi| |\xi_t| dx.
$$

(104)

Due to the positivity of the matrix $L$ in the condition $[\mathbf{K}]$, we have

$$
\int_{\mathbb{R}_+} W \langle L \xi_x, \xi_x \rangle dx \geq c \chi(t) |(\varphi_x, \psi)(t)|_{L^2, \omega}^2.
$$

(105)

The first term in the right-hand side of (101) is estimated as

$$
\left| \int_{\mathbb{R}_+} W \frac{1}{2} \langle K \tilde{A}^0 \xi, \xi_x \rangle \right| \leq C \chi(t) |(\varphi, \psi)(t)|_{H^1, \omega}^2.
$$

(106)

The second and third terms are estimated as

$$
\left| \int_{\mathbb{R}_+} W \langle \dot{B} \xi_x, \xi_x \rangle \right| \leq C \chi(t) |\psi_x(t)|_{L^2, \omega}^2,
$$

(107)

$$
\left| \int_{\mathbb{R}_+} W \langle K \dot{B} \xi_{xx}, \xi_x \rangle \right| \leq c \chi(t) |(\varphi_x, \psi_x)(t)|_{L^2, \omega}^2 + C \epsilon \chi(t) |\psi_{xx}(t)|_{L^2, \omega}^2,
$$

(108)

where $\epsilon$ is an arbitrary positive constant. Integrate (101) with respect to $x$ and $t$ over $\mathbb{R}_+ \times (0, t)$, substitute the above estimate (103)–(108) and then take $\epsilon$ sufficiently small. These procedures yield the desired estimate (98). \qed
3.4. Estimate for the second order derivatives. The main purpose of this subsection is to derive the estimate for the second order derivatives. Precisely, we obtain the estimates for \( \varphi_t \), \( \psi_t \) and \( \varphi_{xx} \) and the dissipative estimate for \( \varphi_{xx} \). In deriving these estimates, the regularity of the time local solution is insufficient. Hence we need to employ a difference approximation with respect to \( x \) as in [7] and a mollifier with respect to \( t \) (see the paper [12] for the details). We, however, assume the perturbation \( (\varphi, \psi) \) is sufficiently smooth and omit these arguments since these procedures are standard. We first show the estimate for the time derivatives \( \varphi_t \) and \( \psi_t \) in Lemma 3.7. In fact, these computations are executed with the mollifier with respect to the time variable.

**Lemma 3.7.** Assume \([A3]\) holds. Let \((\varphi, \psi)\) be a solution to the problem (16), (17) and (18) in a time interval \([0, T]\). If \((\varphi, \psi)\) satisfies \( \omega^{1/2}(\varphi, \psi) \in C([0, T]; H^2(\mathbb{R}_+)) \), then we have

\[
\chi(t)|\varphi_t(t)|^2_{L^2, \omega} + \int_0^t \chi(t)\left(\omega(0)|\varphi_t(\tau, 0)|^2 + |\psi_{xt}(\tau)|^2_{L^2, \omega}\right) d\tau \\
\leq C\chi(0)|\varphi(0, 0)|^2_{H^2, \omega} + C \int_0^t \chi(t)|\varphi_t(\tau, \tau)|^2_{L^2, \omega} d\tau \\
+ C \int_0^t \chi(t)|\varphi_t(\tau)|^2_{L^2, \omega} d\tau + C \int_0^t \int_{\mathbb{R}_+} W|\mathcal{H}_3| dxd\tau, \tag{109}
\]

for \( t \in [0, T] \), where \( \mathcal{H}_3 \) is defined in (110).

**Proof.** Differentiating (15) with respect to \( t \) and taking an inner product of the resultant equality with \( \xi_t \), we have

\[
\left(\frac{1}{2}(A^0_t \xi_t, \xi_t)\right)_t + \left(\frac{1}{2}(A\xi_t, \xi_t) - \langle B\xi_{xt}, \xi_t \rangle\right)_x + \langle B\xi_{xt}, \xi_t \rangle = \mathcal{H}_3, \]

\( \mathcal{H}_3 := \langle h^1, \xi_t \rangle + \left(\frac{1}{2}(A^0_t + A_x)\xi_t - B_x\xi_{xt}, \xi_t \right) \), \( h^1 := h_t - A^0_t \xi_t - A_t \xi_x + B_t \xi_{xx} \). \hspace{1cm} (110)

Multiplying (110) by the weight function \( W(t, x) = \chi(t)\omega(x) \) yields

\[
\left(W\frac{1}{2}(A^0_t \xi_t, \xi_t)\right)_t + \left(W\left(\frac{1}{2}(A\xi_t, \xi_t) - \langle B\xi_{xt}, \xi_t \rangle\right)_x + \langle B\xi_{xt}, \xi_t \rangle\right) \\
- W\frac{1}{2}(A\xi_t, \xi_t) - \langle B\xi_{xt}, \xi_t \rangle = W\frac{1}{2}(A^0_t \xi_t, \xi_t) + W\mathcal{H}_3. \tag{111}
\]

Owing to \([A3]\) and the boundary condition \( \psi_t(t, 0) = 0 \), the second term in the left-hand side is estimated as

\[
\int_{\mathbb{R}_+} \left(W\left(\frac{1}{2}(A\xi_t, \xi_t) - \langle B\xi_{xt}, \xi_t \rangle\right)_x \right) dx \geq c\chi(t)\omega(0)|\varphi_t(t, 0)|^2. \tag{112}
\]

The fourth term is estimated as

\[
\left|\int_{\mathbb{R}_+} W\left(\frac{1}{2}(A\xi_t, \xi_t) - \langle B\xi_{xt}, \xi_t \rangle\right) dx\right| \leq C\chi(t)|\varphi_t(t, \psi_t(t)|^2_{L^2, \omega_x} + C\chi(t)|\psi_{xt}(t)|^2_{L^2, \omega_x}, \tag{113}
\]

where \( \epsilon \) is an arbitrary positive constant. Integrating (111) with (112), (113) and taking \( \epsilon \) sufficiently small, we derive the desired estimate (109). \( \square \)

The estimate for \( \varphi_{xx} \) is summarized in Lemma 3.8. To prove Lemmas 3.8 and 3.9, the regularity of the time-local solution is not enough. To cover this problem, we employ the difference quotient with respect to the spatial variable. However, this
Assume omit this discussion and proceed to estimate as if the solution has enough regularity. Hence, we

The second and the third terms in the right-hand side are estimated as

\[
\frac{1}{2} W(t, x) = \chi(t)\omega(x),
\]

The following equality holds:

\[
\frac{1}{2} W(A_1^0 \varphi_{xx}, \varphi_{xx})_t + \frac{1}{2} W(A_1^1 \varphi_{xx}, \varphi_{xx})_x - \frac{1}{2} W_x(A_1^1 \varphi_{xx}, \varphi_{xx}) + W(B_2 \varphi_{xx}, \varphi_{xx}) + W(A_{22} \varphi_{xx}, \varphi_{xx}) + W(A_{22} \varphi_{xx}, \varphi_{xx}) + W(H_4 - \langle h_2, \varphi_{xxx} \rangle).
\]

Multiplying (117) by the weight function \(W(t, x) = \chi(t)\omega(x)\), the following inequality holds:

\[
\int_{\mathbb{R}_+} W(A_2^0 \varphi_{xt}, \varphi_{xxx}) dx \leq C \epsilon(t) |\varphi_{xt}(t)|_{L^2, \omega}^2 + \epsilon(t) |\varphi_{xxx}(t)|_{L^2, \omega}^2,
\]

for \(t \in [0, T]\), where \(H_4\) and \(h_2^2\) are defined in (115) and (116).

Proof. Applying \(\partial_t^2\) and \(\partial_x\) to (16a) and (16b) and then taking an inner product of the resultant equalities with \(\varphi_{xx}\) and \(\varphi_{xxx}\), respectively, we have

\[
\frac{1}{2} \langle A_1^0 \varphi_{xx}, \varphi_{xx} \rangle_t + \frac{1}{2} \langle A_1^1 \varphi_{xx}, \varphi_{xx} \rangle_x + \langle A_2 \varphi_{xxx}, \varphi_{xxx} \rangle = H_4,
\]

\[
H_4 := \langle h_2^2, \varphi_{xx} \rangle + \frac{1}{2} \langle (A_1^0 + A_{1x}) \varphi_{xx}, \varphi_{xx} \rangle,
\]

\[
h_2^2 := h_1 - (A_1^0 \varphi_x + A_{1x} \varphi_x) - (A_{11} \varphi_x + 2A_{1x} \varphi_x) - (A_{12} \varphi_x + 2A_{12} \varphi_x),
\]

and

\[
\langle B_2 \varphi_{xxx}, \varphi_{xxx} \rangle
\]

\[
= \langle A_{12} \varphi_{xxx}, \varphi_{xxx} \rangle + \langle A_1^0 \varphi_{xt}, \varphi_{xxx} \rangle + \langle A_{22} \varphi_{xxx}, \varphi_{xxx} \rangle - \langle h_2^2, \varphi_{xxx} \rangle,
\]

Adding (115) to (116) yields

\[
\frac{1}{2} \langle A_1^0 \varphi_{xx}, \varphi_{xx} \rangle_t + \frac{1}{2} \langle A_1^1 \varphi_{xx}, \varphi_{xx} \rangle_x + \langle B_2 \varphi_{xxx}, \varphi_{xxx} \rangle
\]

\[
= \langle A_2^0 \varphi_{xt}, \varphi_{xxx} \rangle + \langle A_{22} \varphi_{xxx}, \varphi_{xxx} \rangle + H_4 - \langle h_2^1, \varphi_{xxx} \rangle.
\]

The second and the third terms in the right-hand side are estimated as
Proof. Differentiating (99) with respect to $K$ by the compensation matrix $K$ for

$$\parallel \cdot \parallel_{L^2,\omega}$$

is equal to the norm $\parallel \cdot \parallel_{L^2}$ if $\omega \equiv 1$, under the stability condition.

We derive the dissipative estimate for $\varphi_{xx}$ with respect to the norm $\parallel \cdot \parallel_{L^2,\omega}$, which is equal to the norm $\parallel \cdot \parallel_{L^2}$ if $\omega \equiv 1$, under the stability condition.

\textbf{Lemma 3.9.} Assume the stability condition \textbf{[SK]} (or \textbf{[K]}) holds. Let $(\varphi, \psi)$ be a solution to the problem (16), (17) and (18) in a time interval $[0, T]$. If $(\varphi, \psi)$ satisfies $\omega^{3/2}(\varphi, \psi) \in C([0, T]; H^2(\mathbb{R}_+))$, then we have

$$\int_0^t \chi(\tau)|\varphi_{xx}(\tau)|^2_{L^2,\omega} d\tau$$

$$\leq C\chi(0)|\varphi_0|^2_{H^1,\omega} + C\chi(t)|\varphi_{xx}(t)|^2_{L^2,\omega} + C\int_0^t \chi(\tau)|\varphi_{xx}(\tau)|^2_{L^2,\omega} d\tau$$

$$+ \int_0^t \chi(\tau)(c'\varphi_{xt}(\tau))^2_{L^2,\omega} + C\epsilon'\varphi_{xx}(\tau)|^2_{L^2,\omega} + C\epsilon'|\psi_{xx}(\tau)|^2_{L^2,\omega} d\tau$$

$$+ C\int_0^t \chi(\tau)(\omega(0)||(\varphi, \varphi_{xx}, \varphi_{xt}, \varphi_{xx})(\tau, 0)||^2 + |\psi_{xx}(\tau)|^2_{L^2,\omega} + |\psi_{xt}(\tau)|^2_{L^2,\omega}) d\tau$$

$$+ C\int_0^t \int_{\mathbb{R}_+} W(|Kh'_x, \xi_{xx}|) dxd\tau$$

(121)

for $t \in [0, T]$, where $c'$ is an arbitrary positive constant; $K$ and $h'$ are defined in \textbf{[K]} and (99).

\textbf{Proof.} Differentiating (99) with respect to $x$ and multiplying the resultant equality by the compensation matrix $K$ on the left yield

$$K\bar{A}^0\xi_{xt} + K\bar{A}\xi_{xx} = KB\psi_{xxx} + Kh'_x.$$  

(122)

The skew-symmetric matrix $K\bar{A}^0$ is represented as

$$K\bar{A}^0 = \begin{pmatrix} K_1 & K_2 \\ -K_2 & K_3 \end{pmatrix},$$

where $K_1$ and $K_3$ are $m_1 \times m_1$ and $m_2 \times m_2$ skew-symmetric matrices, respectively; $K_2$ is an $m_1 \times m_2$ matrix. Then the following equalities hold:

$$\langle K\bar{A}^0\xi_{xt}, \xi_{xx} \rangle = \langle K_1\varphi_{xt}, \varphi_{xx} \rangle + \langle K_3\psi_{xx}, \psi_{xxx} \rangle - \langle \bar{K}_2\varphi_{xt}, \psi_{xx} \rangle + \langle K_3\psi_{xx}, \psi_{xxx} \rangle,$n

$$\langle K_1\varphi_{xt}, \varphi_{xx} \rangle = \left( \frac{1}{2} \langle K_1\varphi_{x}, \varphi_{xx} \rangle \right)_t - \left( \frac{1}{2} \langle K_1\varphi_{x}, \varphi_{xt} \rangle \right)_x.$$n

Taking an inner product of (122) with $\xi_{xx}$ and utilizing the above equalities, we have

$$\left( \frac{1}{2} \langle K_1\varphi_{x}, \varphi_{xx} \rangle \right)_t - \left( \frac{1}{2} \langle K_1\varphi_{x}, \varphi_{xt} \rangle \right)_x + \langle L\xi_{xx}, \xi_{xx} \rangle$$

$$= -\langle K_2\psi_{xt}, \varphi_{xx} \rangle + \langle \bar{K}_2\varphi_{xt}, \psi_{xx} \rangle - \langle K_3\psi_{xt}, \psi_{xx} \rangle + \langle B\psi_{xxx}, \psi_{xxx} \rangle$$

$$+ \langle KB\psi_{xxx}, \psi_{xxx} \rangle + \langle Kh'_x, \xi_{xx} \rangle.$$  

(123)
Multiplying (123) by the weight function $W(t, x) = \chi(t)\omega(x)$ yields
\[
\left(\frac{1}{2}(K_1\varphi_x, \varphi_{xx})\right)_t - \left(\frac{1}{2}(K_1\varphi_x, \varphi_{xt})\right)_x + W_x \left(\frac{1}{2}K_1\varphi_x, \varphi_{xt}\right) + W(L\xi_{xx}, \xi_{xx})
= W\left(\frac{1}{2}K_1\varphi_x, \varphi_{xx}\right) - W\left(K_2\psi_{xt}, \varphi_{xx}\right) + W\left(\nabla K_2\varphi_{xt}, \psi_{xx}\right) - W\left(K_3\psi_{xt}, \psi_{xx}\right)
+ W\left(\mathcal{B}\xi_{xx}, \xi_{xx}\right) + W\left(\mathcal{B}^2\xi_{xx}, \xi_{xx}\right) + W\left(K\xi_{xx}, \xi_{xx}\right).
\tag{124}
\]
From straightforward calculations with (16a), the estimate
\[
|\varphi_{xt}(t, 0)|^2 \leq C(|\varphi, \varphi_x, \psi_t, \varphi_{xx}, \psi_{xx})(t, 0)|^2
\tag{125}
\]
holds. Then the second term in the left-hand side is estimated as
\[
\left|\int_{\mathbb{R}_+} W_x \left(\frac{1}{2}K_1\varphi_x, \varphi_{xt}\right) dx\right| \leq C\epsilon\chi(t)|\varphi_{xt}(t)|_2^2 + \epsilon'\chi(t)|\varphi_{xt}(t)|_2^2,
\tag{126}
\]
by Sobolev’s lemma and (125). The third terms in both sides are estimated as
\[
\left|\int_{\mathbb{R}_+} W \left(K_2\varphi_{xt}, \psi_{xx}\right) dx\right| \leq \epsilon'\chi(t)|\varphi_{xt}(t)|_2^2 + C\epsilon'\chi(t)|\psi_{xx}(t)|_2^2,
\tag{127}
\]
where $\epsilon'$ is an arbitrary positive constant. The terms in the right-hand side except the first, the third and the last one are estimated as
\[
\left|\int_{\mathbb{R}_+} W \left(K_3\psi_{xt}, \psi_{xx}\right) + W\left(\mathcal{B}\xi_{xx}, \xi_{xx}\right) + W\left(\mathcal{B}^2\xi_{xx}, \xi_{xx}\right) dx\right|
\leq C\epsilon\chi(t)|\varphi_{xx}(t)|_2^2 + C\epsilon\chi(t)|\psi_{xx}(t)|_2^2 + |\psi_{xt}(t)|_2^2,
\tag{129}
\]
where $\epsilon$ is an arbitrary positive constant. Consequently, integrating (124), utilizing the above estimates (126)–(129) and letting $\epsilon$ sufficiently small, we obtain the desired estimate (121).

4. Convergence rate. In this section, we derive the decay estimates, which are main results in the present paper. The essential ingredient in the proofs is to obtain the estimates for the perturbation $(\varphi, \psi)$ in the weighted Sobolev space. In deriving them, we utilize the estimates of the remainder terms $g_1$ and $g_2$ in (5). If $N(T)$ is bounded, the terms $g_1$ and $g_2$ are estimated from [A4] as
\[
|g(u, u_x)| \leq C(|\hat{u} - u_+| + |\hat{u}_x|) + |\xi| + |\xi_x|(|\xi| + |\xi_x|),
\]
\[
|Dg(u, u_x)| \leq C(|\hat{u} - u_+| + |\hat{u}_x|) + |\xi| + |\xi_x|.
\tag{130}
\]

4.1. Non-degenerate flow with stability condition. The estimate for Theorem 1.5 is summarized in Proposition 1. In proving this proposition, we utilize the following Poincaré type inequalities
\[
\int_{\mathbb{R}_+} e^{-\alpha t}|\xi(x)|^2 dx \leq C(|\xi(0)|^2 + ||\xi_x||_2^2),
\tag{131a}
\]
\[
\int_{\mathbb{R}_+} (1 + x)^\beta e^{-\alpha t}|\xi(x)|^2 dx \leq C(|\xi(0)|^2 + ||\xi_x||_2^2)
\tag{131b}
\]
for $\xi \in H^1(\mathbb{R}_+)$, where $\beta$ is a positive constant. These inequalities are proved in the paper [7].
Proposition 1. Assume [A4], [A5] and the stability condition [SK] (or [K]) hold.

(i) (Exponential decay) Let \((\varphi, \psi)\) be a solution to the problem (16), (17) and (18) such that \((\varphi, \psi) \in C([0, T]; H^2(\mathbb{R}_+))\) and \(e^{\alpha x/2}(\varphi, \psi) \in C([0, T]; L^2(\mathbb{R}_+))\) for certain positive constants \(\alpha\) and \(T\). Then, for a certain constant \(\beta \in (0, \alpha]\), there exist positive constants \(\varepsilon_1\) and \(C\) such that if 
\[
\sup_{0 \leq \tau \leq T} \|(\varphi, \psi)(\tau)\|_{H^2} + \delta \leq \varepsilon_1,
\]
then the solution \((\varphi, \psi)\) satisfies
\[
e^{\nu t} \left( \|(\varphi, \psi)(t)\|_{H^2}^2 + \|(1 + x)^{\alpha/2}(\varphi, \psi)(t)\|_{L^2}^2 \right) + \int_0^t e^{\nu \tau} \left( \|(1 + x)^{\alpha/2}(\varphi, \psi)(\tau)\|_{L^2}^2 \right) d\tau
\leq C \left( \|(\varphi_0, \psi_0)\|_{H^2}^2 + \|(1 + x)^{\alpha/2}(\varphi_0, \psi_0)\|_{L^2}^2 \right) + \int_0^t \|(1 + x)^{\alpha/2}(\varphi, \psi)(\tau)\|_{L^2}^2 d\tau
\] (132)
for \(t \in [0, T]\), where \(\nu\) is a positive constant satisfying \(\nu \ll \beta\).

(ii) (Algebraic decay) Let \((\varphi, \psi)\) be a solution to the problem (16), (17) and (18) such that \((\varphi, \psi) \in C([0, T]; H^2(\mathbb{R}_+))\) and \((1 + x)^{\alpha/2}(\varphi, \psi) \in C([0, T]; L^2(\mathbb{R}_+))\) for certain positive constants \(\alpha\) and \(T\). Then there exist positive constants \(\varepsilon_1\) and \(C\) such that if 
\[
\sup_{0 \leq \tau \leq T} \|(\varphi, \psi)(\tau)\|_{H^2} + \delta \leq \varepsilon_1,
\]
then the solution \((\varphi, \psi)\) satisfies
\[
(1 + t)^{\alpha} \left( \|(\varphi, \psi)(t)\|_{H^2}^2 + \|(1 + x)^{\alpha/2}(\varphi, \psi)(t)\|_{L^2}^2 \right)
+ \int_0^t \left( (1 + x)^{\alpha/2}(\varphi, \psi)(\tau)\right)_{L^2}^2 d\tau
+ \int_0^t \left( \|\varphi_\tau(\tau)\|_{H^2}^2 + \|\psi_\tau(\tau)\|_{H^2}^2 + \|(1 + x)^{\alpha/2}\psi_\tau(\tau)\|_{L^2}^2 \right) d\tau
\leq C \left( \|(\varphi_0, \psi_0)\|_{H^2}^2 + \|(1 + x)^{\alpha/2}(\varphi_0, \psi_0)\|_{L^2}^2 \right)
\] (133)
for \(t \in [0, T]\).

Proof. To show (132) and (133), it suffices to derive the estimates for \(\psi_{xx}\) and the dissipation \(\varphi_t\), \(\psi_t\) and \(\varphi_{xt}\) in \(\|\cdot\|_{L^2}\) since the other terms have already been handled in Lemmas 3.2, 3.5, 3.6, 3.7, 3.8 and 3.9. The straightforward computations with (15), (16a) and (16b) yield that
\[
\chi(t)\|\psi_{xx}(t)\|_{L^2}^2 \leq C \chi(t)\left( \|(\varphi, \psi)(t)\|_{H^2}^2 + \|\psi_t(t)\|_{L^2}^2 \right),
\] (134)
\[
\int_0^t \chi(\tau)\|\varphi_t(\tau)\|_{L^2}^2 d\tau
\leq C \int_0^t \chi(\tau)\|\varphi_{xx}(\tau)\|_{L^2}^2 d\tau + C \int_0^t \chi(\tau)|h|^2 dx d\tau,
\] (135)
\[
\int_0^t \chi(\tau)\|\varphi_{xt}(\tau)\|_{L^2}^2 d\tau
\leq C \int_0^t \chi(\tau)\|\varphi_{xx}(\tau)\|_{L^2}^2 d\tau + C \int_0^t \chi(\tau)|h|^2 dx d\tau
\] (136)
for $t \in [0, T]$, where $h$ and $h_1$ are defined in (15) and (89). Substitute $\omega \equiv 1$ in (88), (98), (109), (114) and (121), multiply these results, (51) and (134)–(136) by suitable constants, respectively, sum up the resultant inequalities and then take $\varepsilon'$ from (121) sufficiently small, to obtain the $H^2$-energy inequality

$$
\chi(t)(\|\varphi_x, \psi_x(t)\|_{H^1}^2 + \|\varphi, \psi(t)\|_{L^2, \omega}^2 + \|\varphi_t, \psi_t(t)\|_{L^2}^2) \\
+ \int_0^t \chi(\tau) (\|\varphi(\tau, 0)\|^2 + \|\varphi(\tau)\|_{L^2, \omega}^2) \, d\tau \\
+ \int_0^t \chi(\tau) (\|\varphi_x, \varphi_t, \psi_t, \psi_{xx}(\tau)\|_{H^1}^2 + \|\psi_x(\tau)\|_{L^2, \omega}^2) \, d\tau \\
\leq C(\|\varphi_0, \psi_0\|_{H^2}^2 + \|\varphi_0, \psi_0\|_{L^2, \omega}^2) + C \int_0^t \chi(\tau) \|\varphi_x(\tau)\|_{L^2, \omega}^2 \, d\tau \\
+ C \int_0^t \chi(\tau) (\|\varphi(\tau)\|_{H^1}^2 + \|\psi_x, \varphi_t, \psi_t(\tau)\|_{L^2}^2) \, d\tau \\
+ C \int_0^t \int_{\mathbb{R}^+} (W(|\mathcal{G}| + |\mathcal{R}_1| + |\mathcal{R}_2|) + \chi(\tau)(|\mathcal{R}_6| + |\mathcal{R}_7|)) \, dx \, d\tau \tag{137}
$$

for $t \in [0, T]$, where

$$
\mathcal{R}_6 := \|\mathcal{H}_1| + |\mathcal{H}_2| + |\mathcal{H}_3| + |\mathcal{H}_4| + |\mathcal{H}_5| + |\mathcal{H}_6| + |\mathcal{H}_7| + |\mathcal{H}_8| + |\mathcal{H}_9| + |h_1^2| + |h_t^2| + |h_1^2|^2,
$$

$$
\mathcal{R}_7 := \|K h_1^\prime, \xi_z\| + \|K h_2^\prime, \xi_{xx}\|.
$$

The remainder terms $\mathcal{R}_6$ and $\mathcal{R}_7$ are estimated as

$$
|\mathcal{R}_6| \leq C(|\psi_{xx}|^2 + |\varphi_{xx}|^2 + |\varphi_{xx}| + |\varphi_{xx}| + |\varphi_{xx}| + |\varphi_{xx}|) \xi_z^2 \\
+ (|\xi| + |\xi_z|) (|\xi_z|^2 + |\xi_t|^2 + |\xi_{xx}|^2 + |\psi_{xx}|^2) \\
+ (|\xi| + |\xi_z|) (|\xi_z|^2 + |\xi_t|^2 + |\xi_{xx}|^2 + |\psi_{xx}|^2) \tag{138}
$$

$$
|\mathcal{R}_7| \leq C(|\varphi_{xx}| + |\psi_{xx}| + |\psi_{xx}|) \xi_z^2 \\
+ (|\xi| + |\xi_z|) (|\xi_z|^2 + |\xi_t|^2 + |\xi_{xx}|^2 + |\psi_{xx}|^2) \\
+ (|\xi| + |\xi_z|) (|\xi_z|^2 + |\xi_t|^2 + |\xi_{xx}|^2 + |\psi_{xx}|^2) \tag{139}
$$

due to (15), (89), (90), (99), (110), (115), (116) and (130).

To prove Proposition 1-(i), let the weight function $W(t, x) = \chi(t) \omega(x) = e^{\nu t} e^{\beta x}$, where $\nu > 0$ and $\beta \in (0, \alpha]$. Substituting (56), (57), (138) and (139) in (137) and applying (12), (131a), Sobolev’s lemma and the inequalities $|\xi|, |\xi_z|, \|\xi_{xx}\|_{L^2} \leq N(T)$ to the resultant inequality with $\nu \ll \beta \ll 1$, we get the desired estimate (132) if $N(T)$ and $\delta$ are sufficiently small.

To prove Proposition 1-(ii), let the weight function $W(t, x) = \chi(t) \omega(x) = (1 + t)^{\nu} (1 + x)^{\nu}$, where $\nu \geq 0$ and $\beta \in [0, \alpha]$. The last term in the right-hand side of (137) is handled similarly as above except to use (131b) instead of (131a). The result is

$$(1 + t)^{\nu} (\|\varphi_x, \psi_x(t)\|_{H^1}^2 + \|(1 + x)^{\beta/2}(\varphi, \psi(t))\|_{L^2}^2 + \|\varphi_t, \psi_t(t)\|_{L^2}^2) \\
+ \int_0^t (1 + t)^{\nu} \beta \|(1 + x)^{(\beta - 1)/2}(\varphi, \psi(t))\|_{L^2}^2 + \|\varphi_x(\tau)\|_{H^1}^2) \, d\tau \\
+ \int_0^t (1 + t)^{\nu} \|\psi_{xx}(\tau)\|_{L^2}^2 + \|\psi_{xx}(\tau)\|_{H^1}^2 + \|\psi_t, \psi_t(\tau)\|_{L^2}^2) \, d\tau$$

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\[
\leq C\left(\|(\varphi_0, \psi_0)\|_{H^2}^2 + \|(1 + x)^{\beta/2}(\varphi_0, \psi_0)\|_{L^2}^2\right) \\
+ C\beta \int_0^t (1 + \tau)^{\nu}(1 + x)^{(\beta-1)/2}\psi_x(\tau)\|_{L^2}^2 d\tau \\
+ C\nu \int_0^t (1 + \tau)^{\nu-1}(1 + x)^{\beta/2}(\varphi, \psi)(\tau)\|_{L^2}^2 d\tau \\
+ C\nu \int_0^t (1 + \tau)^{\nu-1}(\|\varphi_x(\tau)\|_{H^1}^2 + \|\psi_x, \varphi_t, \psi_t)(\tau)\|_{L^2}^2) d\tau
\]

for \( \nu \geq 0, \beta \in [0, \alpha] \) and \( t \in [0, T] \). Thus we obtain the desired estimate (133) by applying the induction with respect to \( \nu \) and \( \beta \) to (140). As these procedures are similar to [5, 9, 14], we omit the details. \( \square \)

4.2. Non-degenerate flow without stability condition. We next derive the following estimate to show Theorem 1.6.

**Proposition 2.** Assume [A4] and [A5] hold.

(i) (Exponential decay) Let \((\varphi, \psi)\) be a solution to the problem (16), (17) and (18) such that \(e^{\alpha x/2}(\varphi, \psi) \in C([0, T]; H^2(\mathbb{R}^+))\) for certain positive constants \(\alpha\) and \(T\). Then, for a certain constant \(\beta \in (0, \alpha]\), there exist positive constants \(\varepsilon_1\) and \(C\) such that if

\[
\left(\sup_{0 \leq t \leq T} \|e^{\beta x/2}(\varphi, \psi)(t)\|_{H^2} + \delta\right) \beta^{-1} \leq \varepsilon_1,
\]

then the solution \((\varphi, \psi)\) satisfies

\[
e^{\nu t}\|e^{\beta x/2}(\varphi, \psi)(t)\|_{H^2}^2 + \int_0^t e^{\nu \tau}(\|e^{\beta x/2}(\varphi, \psi)(\tau)\|_{L^2}^2 + \|e^{\beta x/2}\varphi_x(\tau)\|_{H^1}^2 + \|e^{\beta x/2}\psi_x(\tau)\|_{H^2}^2) d\tau \\
\leq C\|e^{\beta x/2}(\varphi_0, \psi_0)\|_{H^2}^2
\]

for \( t \in [0, T] \), where \( \nu \) is a positive constant satisfying \( \nu \ll \beta \).

(ii) (Algebraic decay) Let \((\varphi, \psi)\) be a solution to the problem (16), (17) and (18) such that \((1 + \gamma x)^{\alpha/2}(\varphi, \psi) \in C([0, T]; H^2(\mathbb{R}^+))\) for certain positive constants \(T\) and \(\gamma\) and a certain constant \(\alpha \geq 2\). Then, for an arbitrary constant \(\theta \in (0, \alpha]\), there exist positive constants \(\varepsilon_1\) and \(C\) such that if

\[
\left(\sup_{0 \leq t \leq T} \|(1 + \gamma x)^{\alpha/2}(\varphi, \psi)(t)\|_{H^2} + \delta\right) \gamma^{-1} + \gamma \leq \varepsilon_1,
\]

then the solution \((\varphi, \psi)\) satisfies

\[
\|(1 + \gamma x)^{\alpha/2}(\varphi, \psi)(t)\|_{H^2}^2 + (1 + t)^{\alpha-\theta}\|(1 + \gamma x)^{\theta/2}(\varphi, \psi)(t)\|_{H^2}^2 \\
+ \int_0^t \|(1 + \gamma x)^{(\alpha-1)/2}(\varphi, \psi)(\tau)\|_{L^2}^2 d\tau \\
+ \int_0^t \|(1 + \gamma x)^{(\alpha-1)/2}\varphi_x(\tau)\|_{H^1}^2 + \|(1 + \gamma x)^{\alpha/2}\psi_x(\tau)\|_{H^2}^2 d\tau \\
\leq C\|(1 + \gamma x)^{\alpha/2}(\varphi_0, \psi_0)\|_{H^2}^2
\]

for \( t \in [0, T] \).
Proof. Since we have already obtained the estimates as in Lemmas 3.2, 3.5, 3.7 and 3.8, it suffices to derive the estimates for $\psi_{xx}$ and the dissipation $\varphi_t$, $\psi_t$ and $\varphi_{xt}$ in the suitable norms, respectively, to prove Proposition 2. Owing to the similar calculations to (134)–(136), we have

$$\chi(t)|\psi|_{L^2,\omega} \leq C \chi(t)|((\varphi, \psi)(t)|^2_{H^1,\omega} + |\psi(t)|^2_{L^2,\omega}),$$

(143)

$$\int_0^t \chi(\tau)|((\varphi, \psi(\tau)|^2_{H^1,\omega} + |\psi(\tau)|^2_{L^2,\omega}) d\tau \leq C \int_0^t \chi(\tau)|((\varphi, \psi(\tau)|^2_{H^1,\omega} + |\psi(\tau)|^2_{L^2,\omega}) d\tau + C \int_0^t \chi(\tau)\omega(x)|h|^2 d\tau,$$

(144)

$$\int_0^t \chi(\tau)|\varphi_{xt}(\tau)|^2_{L^2,\omega} d\tau \leq C \int_0^t \chi(\tau)|((\varphi_{xx}, \psi_{xx})(\tau)|^2_{L^2,\omega} d\tau + C \int_0^t \chi(\tau)\omega(x)|h|^2 d\tau$$

(145)

for $t \in [0, T]$. Multiplying (51), (88), (109), (114) and (134)–(135) by suitable constants, respectively, summing up the resultant inequalities and then letting $\varepsilon$ from (88) sufficiently small, we get the $H^2$-energy inequality

$$\chi(t)|((\varphi, \psi)(t)|^2_{H^2,\omega} + |(\varphi_t, \psi_t)(t)|^2_{L^2,\omega})$$

$$+ \int_0^t \chi(\tau)|((\varphi, \psi)(\tau)|^2_{H^2,\omega} + |(\psi, \psi_t)(\tau)|^2_{L^2,\omega}) d\tau$$

$$+ \int_0^t \chi(\tau)|((\varphi_t, \psi_t)(\tau)|^2_{H^1,\omega} + |\psi(\tau)|^2_{L^2,\omega} + |\psi_{xt}(\tau)|^2_{L^2,\omega}) d\tau$$

$$\leq C |(\varphi, \psi)(t)|^2_{H^2,\omega} + C \int_0^t \chi(\tau)|\psi(\tau)|^2_{L^2,\omega} d\tau$$

$$+ C \int_0^t \chi(\tau)|((\varphi, \psi)(\tau)|^2_{H^2,\omega} + |\psi(\tau)|^2_{H^1,\omega} + |(\varphi_t, \psi_t)(\tau)|^2_{L^2,\omega}) d\tau$$

$$+ C \int_0^t \int \mathcal{W}(|G| + |\mathcal{R}_1| + |\mathcal{R}_2| + |\mathcal{R}_3|) d\tau$$

(146)

for $t \in [0, T]$.

To prove Proposition 2-(i), let the weight function $W(t, x) = \chi(t)\omega(x) = e^{\nu t}e^{\beta x}$, where $\nu > 0$ and $\beta \in (0, \alpha]$, and $N_{\nu}(T)$ be

$$N_{\nu}(T) := \sup_{0 \leq t \leq T} \|e^{\beta x/2}(\varphi, \psi)(t)\|_{H^2}.$$  

(147)

Substituting (56), (57) and (138) in (146) yields the desired estimate (141) if $\nu, N_{\nu}(T) + \delta \ll \beta \ll 1$. Here we have used (12) and the inequalities $|\xi| + |\xi_x| + \|e^{\beta x/2}\xi_{xx}\|_{L^2} \leq N_{\nu}(T)$.

To prove Proposition 2-(ii), let the weight function $W(t, x) = \chi(t)\omega(x) = (1 + t)\nu(1 + \gamma x)^\beta$, where $\nu > 0$ and $\beta \in [\theta, \alpha]$, and $N_{\alpha}(T)$ be

$$N_{\alpha}(T) := \sup_{0 \leq t \leq T} \|(1 + \gamma x)^{\alpha/2}(\varphi, \psi)(t)\|_{H^2}.$$  

(148)

We deal with the second term in the right-hand side of (146) by taking $\gamma$ sufficiently small. Owing to the inequalities $|(1 + \gamma x)\xi|, |(1 + \gamma x)\xi_x|$, $\|e^{\beta x/2}\xi_{xx}\|_{L^2} \leq N_{\nu}(T)$
for \( \alpha \geq 2 \), the last terms are also dealt with if \( N_\alpha(T) + \delta \ll \gamma \ll 1 \). Thus we get the following estimate
\[
(1 + t)^\nu \left( \|(1 + \gamma x)\beta/2(\varphi, \psi)(t)\|_{H^2}^2 + \|(1 + \gamma x)\beta/2(\varphi_t, \psi_t)(t)\|_{L^2}^2 \right) \\
+ \int_0^t (1 + \tau)^\nu \gamma \|(1 + \gamma x)\beta/2(-\varphi(\tau))\|_{H^2}^2 \, d\tau \\
+ \int_0^t (1 + \tau)^\nu \gamma \|(1 + \gamma x)\beta/2(\psi, \varphi_t, \psi_t)(\tau)\|_{L^2}^2 + \|(1 + \gamma x)\beta/2(\psi_x(\tau))\|_{H^2}^2 \, d\tau \\
\leq C \|(1 + \gamma x)\beta/2(\varphi_0, \psi_0)\|_{H^2}^2 \\
+ C \nu \int_0^t (1 + \tau)^\nu^{-1} \|(1 + \gamma x)\beta/2(\varphi(\tau))\|_{H^2}^2 + \|(1 + \gamma x)\beta/2(\psi(\tau))\|_{H^2}^2 \, d\tau \\
+ C \nu \int_0^t (1 + \tau)^\nu^{-1} \|(1 + \gamma x)\beta/2(\varphi_t, \psi_t)(\tau)\|_{L^2}^2 \, d\tau.
\]
(149)

By using the induction as in the proof of Proposition 1-(ii), we derive the desired estimate (142).

\[\square\]

4.3. Degenerate flow with stability condition. To prove Theorem 1.7, we derive the following estimate.

**Proposition 3.** Assume \([A3], [A4], [A6]\) and the stability condition \([SK]\) (or \([K]\)) hold. Let \((\varphi, \psi)\) be a solution to the problem (16), (17) and (18) such that \(|\sigma|^{-\alpha/2}(\varphi, \psi) \in C([0, T], H^2(\mathbb{R}^n))\) for a certain positive constant \(T\) and a certain constant \(\alpha \in [1, 5]\). Here \(\sigma\) is the solution to the equation (35). Then, for an arbitrary constant \(\theta \in (0, \alpha]\), there exist positive constants \(\varepsilon_1\) and \(C\) such that if
\[
\sup_{0 \leq t \leq T} \|\sigma|^{-\alpha/2}(\varphi, \psi)(t)\|_{H^2}^2 + \delta^{1/2} \leq \varepsilon_1,
\]
then the solution \((\varphi, \psi)\) satisfies
\[
\|\sigma|^{-\alpha/2}(\varphi, \psi)(t)\|_{H^2}^2 + (1 + t)^{(\alpha - \theta)/2} \|\sigma|^{-\theta/2}(\varphi, \psi)(t)\|_{H^2}^2 \\
+ \int_0^t \left( \|\sigma|^{-\alpha/2}(\varphi, \psi)(\tau)\|_{L^2}^2 + \|\sigma|^{-\alpha/2}(\varphi_x(\tau))\|_{H^2}^2 + \|\sigma|^{-\alpha/2}(\psi_x(\tau))\|_{H^2}^2 \right) \, d\tau \\
\leq C \|\sigma|^{-\alpha/2}(\varphi_0, \psi_0)\|_{H^2}^2
\]
(150)

for \(t \in [0, T]\).

**Proof.** Let \(N_\alpha(T)\) be
\[
N_\alpha(T) := \sup_{0 \leq t \leq T} \|\sigma|^{-\alpha/2}(\varphi, \psi)(t)\|_{H^2}^2.
\]
(151)

To prove Proposition 3, it suffices to derive the estimates for \(\psi_{xx}\) and the dissipation \(\varphi_t, \psi_t\) and \(\varphi_{xt}\) in \(\|\cdot\|_{L^2, \omega}\) in addition to the estimates in Lemmas 3.4, 3.5, 3.6, 3.7, 3.8 and 3.9. The same computations to obtain (143)–(145) yield that
\[
\chi(t) \|\psi_{xx}(t)\|_{L^2, \omega}^2 \leq C \chi(t) \|\psi(\varphi, \psi)(t)\|_{H^1, \omega}^2 + \|\psi_t(\varphi, \psi)(t)\|_{L^2, \omega}^2,
\]
(152)
the similar way to (138) and (139). To estimate $R$ for inequalities and letting $\varepsilon$ are defined in (67), in (88), (98), (109), (114), (121) and (152)–(154), multiplying where $\varepsilon$

\[
\leq C \int_0^t \chi(\tau)|\varphi(x, \psi_x, \psi_{xx})(\tau)|^2_{L^2} d\tau + C \int_0^t \int_{\mathbb{R}_+} \chi(\tau)|\omega(x)|^2 dx d\tau, 
\]

\[
\int_0^t \chi(\tau)|\varphi_{xx}(\tau)|^2_{L^2} d\tau 
\]

\[
\leq C \int_0^t \chi(\tau)|\varphi(x, \psi_x, \psi_{xx})(\tau)|^2_{L^2} d\tau + C \int_0^t \int_{\mathbb{R}_+} \chi(\tau)|\omega(x)|^2_{H^1} d\tau 
\]

for $t \in [0, T]$. Substituting $W(t, x) = \chi(t)|\omega(x)| = (1 + t)^{\nu}|\sigma(x)|^{-\beta}$, where $\nu$ and $\beta$ are defined in (67), in (88), (98), (109), (114), (121) and (152)–(154), multiplying these results and (67) by suitable constants, respectively, summing up the resultant inequalities and letting $\varepsilon$ from (88) and $\varepsilon'$ from (121) sufficiently small yield the $H^2$-energy estimate

\[
(1 + t)^{\nu}(|||\sigma|^{-\beta/2}(\varphi, \psi)(t)||^2_{H^2} + |||\sigma|^{-\beta/2}(\varphi_t, \psi_t)(t)||^2_{L^2}) 
\]

\[
+ \int_0^t (1 + t)^{\nu}(|||\sigma|^{-(\beta-2)/2}(\varphi, \psi)(\tau)||^2_{L^2} + |||\sigma|^{-\beta/2}\varphi_x(\tau)||^2_{H^1}) d\tau 
\]

\[
+ \int_0^t (1 + t)^{\nu}(|||\sigma|^{-(\beta-2)/2}\varphi_x(\tau)||^2_{H^1} + |||\sigma|^{-\beta/2}(\varphi_t, \psi_t)(\tau)||^2_{H^1}) d\tau 
\]

\[
\leq C(|||\sigma|^{-\beta/2}(\varphi_0, \psi_0)||^2_{H^2} 
\]

\[
+ C\nu \int_0^t (1 + t)^{\nu-1}(|||\sigma|^{-\beta/2}\varphi(\tau)||^2_{H^2} + |||\sigma|^{-\beta/2}\psi(\tau)||^2_{H^1}) d\tau 
\]

\[
+ C \int_0^t (1 + t)^{\nu}(|||\sigma||(|\mathcal{R}_1| + |\mathcal{R}_2| + |\mathcal{R}_6| + |\mathcal{R}_7|) + |\mathcal{R}_5|) d\tau 
\]

for $t \in [0, T]$. Here we have used the inequality

\[
\int_{\mathbb{R}_+} \partial_x (|||\sigma|^{-\beta}|\xi_t||\xi_t||d) 
\]

\[
\leq C_{\varepsilon''}(|||\sigma|^{-(\beta-2)/2}(\varphi, \psi)(t)||^2_{L^2} + \varepsilon''|||\sigma|^{-\beta/2}(\varphi_t, \psi_t)(t)||^2_{L^2}, 
\]

where $\varepsilon''$ is a sufficiently small positive constant. The remainder terms $\mathcal{R}_6$ and $\mathcal{R}_7$ are estimated as

\[
|\mathcal{R}_6| \leq C( |||\varphi_x||^2 + |||\varphi_{xx}||^2 + (||\tilde{u}_x||^3/2 + ||\tilde{u}_x|| + ||\tilde{u}_{xx}||)\xi_2 + ||\tilde{u}_x||^2||\psi_x||^2 
\]

\[
+ (||\tilde{u}_x|| + ||\tilde{u}_{xx}|| + ||\tilde{u}_{xxx}||)||\xi_2 + \xi_t||^2 + \xi_{xx}||^2 + \xi_{tt}||^2 + \psi_{xxx}||^2)) 
\]

\[
+ (||\xi + \xi_x||)(||\xi_2 + \xi_t||^2 + \xi_{xx}||^2 + \xi_{tt}||^2 + \psi_{xxx}||^2)) 
\]

\[
|\mathcal{R}_7| \leq C(|||\tilde{u}_x|^3/2 + ||\tilde{u}_x|| + ||\tilde{u}_{xx}||)\xi_2 
\]

\[
+ (||\tilde{u} - u_x||^3/2 + ||\tilde{u}_x||^3/2 + ||\tilde{u}_{xx}||)||\xi_2 + \xi_t||^2 + \xi_{xx}||^2 + \xi_{tt}||^2 + \psi_{xxx}||^2 
\]

\[
+ (||\xi + \xi_x||)(||\xi_2 + \xi_t||^2 + \xi_{xx}||^2 + \xi_{tt}||^2 + \psi_{xxx}||^2)) 
\]

in the similar way to (138) and (139). To estimate $\mathcal{R}_5$, we utilize the inequality as

\[
\int_{\mathbb{R}_+} |||\sigma|^{-\beta-1}|\xi_t||^3|d| 
\]

\[
\leq C(|||\sigma|^{-1/2}(\varphi, \psi)(t)||^2_{L^2} |||\sigma(0)|^{-\beta}\varphi(t, 0)||^2 + |||\sigma|^{-(\beta-2)/2}(\varphi, \psi)(t)||^2_{L^2} + |||\sigma|^{-\beta/2}(\varphi, \psi_x)(t)||^2_{L^2}), 
\]
which is proved by the same way as the paper [16]. Utilizing (39), (156)–(158), the inequality \( \| \sigma^{-1/2}(\varphi, \psi) \|_{L^2} \leq N_s(T) \) for \( \alpha \geq 1 \) and the argument for the induction in Proposition 1-(ii), we get the desired estimate (150) if \( N_s(T) \) and \( \delta^{1/2} \) are sufficiently small.

\[ \square \]

4.4. Degenerate flow without stability condition. The estimate for Theorem 1.8 is derived as follows.

**Proposition 4.** Assume [A3], [A4] and [A6] hold. Let \((\varphi, \psi)\) be a solution to the problem (16), (17) and (18) such that \( |\sigma|^{-\alpha/2}(\varphi, \psi) \in C([0, T]; H^2(\mathbb{R}_+)) \) for a certain positive constant \( T \) and a certain constant \( \alpha \in [2, 5] \). Here \( \sigma \) is the solution to the equation (35). Then, for an arbitrary constant \( \theta \in (0, \alpha] \), there exist positive constants \( \varepsilon_1 \) and \( C \) such that if

\[
\sup_{0 \leq t \leq T} \| \sigma|^{-\alpha/2}(\varphi, \psi)(t) \|_{H^2} + \delta^{1/2} \leq \varepsilon_1,
\]

then the solution \((\varphi, \psi)\) satisfies

\[
\begin{align*}
\| \sigma|^{-\alpha/2}(\varphi, \psi)(t) \|_{H^2}^2 &+ (1 + t)^{(\alpha-\theta)/2} \| \sigma|^{-\theta/2}(\varphi, \psi)(t) \|_{H^2}^2 \\
&+ \int_0^t \| \sigma|^{-(\alpha-2)/2}(\varphi, \psi)(\tau) \|_{L^2}^2 d\tau \\
&+ \int_0^t \| \sigma|^{-(\alpha-1)/2}\varphi_x(\tau) \|_{H^1}^2 + \| \sigma|^{-\alpha/2}\psi(\tau) \|_{H^2}^2 \| \psi(\tau) \|_{H^2} d\tau \\
&\leq C \| \sigma|^{-\alpha/2}(\varphi_0, \psi_0) \|_{H^2}^2
\end{align*}
\]

for \( t \in [0, T] \).

**Proof.** Substitute \( W(t, x) = \chi(t)\omega(x) = (1 + t)^{\nu}|\sigma(x)|^{-\beta} \), where \( \nu \) and \( \beta \) are defined in (67), in (88), (109), (114) and (143)–(145), multiply these results and (67) by suitable constants, respectively, sum up the resultant inequalities and then let \( \varepsilon \) from (88) sufficiently small, to obtain the \( H^2 \)-energy estimate

\[
\begin{align*}
(1 + t)^\nu \left( \| \sigma|^{-\beta/2}(\varphi, \psi) \|_{H^2}^2 &+ \| \sigma|^{-\beta/2}(\varphi_t, \psi_t) \|_{L^2}^2 \right) \\
&+ \int_0^t (1 + \tau)^\nu \left( \| \sigma|^{-(\beta-2)/2}(\varphi, \psi) \|_{L^2}^2 + \| \sigma|^{-(\beta-1)/2}\varphi_x \|_{H^1}^2 + \| \sigma|^{-\beta/2}\varphi \|_{H^3}^2 \right) d\tau \\
&+ \int_0^t (1 + \tau)^\nu \left( \| \sigma|^{-\beta-1/2}\varphi_t \|_{L^2}^2 + \| \sigma|^{-(\beta-1)/2}\psi \|_{L^2}^2 + \| \sigma|^{-\beta/2}\psi \|_{L^2}^2 \right) d\tau \\
&\leq C \| \sigma|^{-\beta/2}(\varphi_0, \psi_0) \|_{H^2}^2 \\
&+ C\nu \int_0^t (1 + \tau)^{\nu-1} \left( \| \sigma|^{-\beta/2}\varphi \|_{H^2}^2 + \| \sigma|^{-\beta/2}\psi \|_{H^1}^2 + \| \sigma|^{-\beta/2}(\varphi, \psi) \|_{L^2} \right) d\tau \\
&+ C \int_{\mathbb{R}_+} \int_0^t (1 + \tau)^\nu \left( |\varphi|(\|\varphi_1\| + |\varphi_2\| + |\varphi_3\|) + |\varphi_4| \right) d\tau d\tau
\end{align*}
\]

for \( t \in [0, T] \). Then we obtain the desired estimate (159) in the same way to derive (150) with the inequalities \( \| \sigma|^{-1}\xi \|_{L^2}, \| \sigma|^{-1}\xi_x \|_{L^2}, \| \sigma|^{-1/2}\xi_x \|_{L^2} \leq N_s(T) \) for \( \alpha \geq 2 \) if \( N_s(T) \) and \( \delta^{1/2} \) are sufficiently small.

\[ \square \]
Application. In this section, as the application of the main theorems, we treat the heat-conductive model for compressible and viscous gases in the one-dimensional half space $\mathbb{R}^+$. Precisely, we study the system:

$$
\rho_t + (\rho u)_x = 0, \quad (161a)
$$

$$
(\rho u)_t + (\rho u^2 + p)_x = (\mu u)_x, \quad (161b)
$$

$$
\left\{ \rho \left( e + \frac{u^2}{2} \right) \right\}_t + \left\{ \rho u \left( e + \frac{u^2}{2} \right) + pu \right\}_x = (\mu uu_x + \kappa \theta)_x, \quad (161c)
$$

where $\rho$, $u$ and $\theta$ are the mass density, the fluid velocity and the absolute temperature, respectively; the pressure $p = p(\rho, \theta)$ and the internal energy $e = e(\rho, \theta)$ are given smooth functions of $\rho > 0$ and $\theta > 0$; $\mu = \mu(\rho, \theta) > 0$ and $\kappa = \kappa(\rho, \theta) > 0$ are the viscosity coefficient and the heat conduction coefficient $\kappa$, respectively. We assume $\rho$, $\theta$, $p$, $e$ and the entropy of fluid $s = s(\rho, \theta)$ are governed by the first law of the thermodynamics

$$
de = \theta ds - pd \left( \frac{1}{\rho} \right). \quad (162)
$$

Moreover, we impose the following assumptions on the pressure $p$ and the internal energy $e$:

$$
p_{\rho}(\rho, \theta) > 0, \quad e_{\theta}(\rho, \theta) > 0. \quad (163)
$$

We consider the three cases concerning $\mu$ and $\kappa$:

(i) $\mu(\rho, \theta) > 0, \kappa(\rho, \theta) > 0$,

(ii) $\mu(\rho, \theta) \equiv 0, \kappa(\rho, \theta) > 0$,

(iii) $\mu(\rho, \theta) > 0, \kappa(\rho, \theta) \equiv 0$.

The first case (i) corresponds to viscous and heat conductive gas; (ii) corresponds to inviscid and heat conductive gas; (iii) corresponds to viscous and non-heat conductive gas. Following [8, 12], we rewrite the system (161) to the normal form and verify that this model satisfies the assumptions in theorems. We note that the assumption $[A5]$ is called the outflow and supersonic condition; the pair of $[A3]$ and $[A6]$ is called the outflow and transonic condition in gas dynamics.

Let $U = \gamma(\rho, p, \rho(e + \frac{u^2}{2}))$ and $u = \gamma(\rho, u, \theta)$. Let $u_+ = \gamma'(\rho_+, u_+, \theta_+)$ be a spatial asymptotic state of $u_+$. We define the entropy function $\eta$ for (161) by

$$
\eta(U) := -\rho s.
$$

Due to the thermodynamics law, $\eta(U)$ is regarded as a function of $U$. The system (161) is deduced to the normal form for $u$ as

$$
A^0(u) u_t + A(u) u_x = B(u) u_{xx} + g(u, u_x), \quad (164)
$$

where

$$
A^0(u) = \frac{1}{\rho \theta^2} \begin{pmatrix}
\theta p_{\rho} & 0 & 0 \\
0 & \rho^2 \theta & 0 \\
0 & 0 & \rho^2 e_{\theta}
\end{pmatrix}, \quad (165a)
$$

$$
A(u) = \frac{1}{\rho \theta^2} \begin{pmatrix}
u \theta p_{\rho} & \rho \theta p_{\rho} & 0 \\
\rho \theta p_{\rho} & \rho^2 u \theta & \rho \theta p_{\rho} \\
0 & \rho \theta p_{\rho} & \rho^2 u e_{\theta}
\end{pmatrix}, \quad (165b)
$$

$$
\[ B(u) = \frac{1}{\theta^2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mu \theta & 0 \\ 0 & 0 & \kappa \end{pmatrix}, \] (165c)

\[ g(u, u_x) = \frac{1}{\theta^2} \begin{pmatrix} 0 \\ \theta \mu u_x \\ \mu u_x^2 + \kappa u_x \end{pmatrix}. \] (165d)

Apparently, we see from (165d) that \([A4]\) holds. The characteristics of (161) are given by

\[ \lambda_1(u) = u + c_s(\rho, \theta), \quad \lambda_2(u) = u, \quad \lambda_3(u) = u - c_s(\rho, \theta), \] (166)

where \(c_s(\rho, \theta) := \sqrt{p_\theta + \theta p_\theta^2/(\rho^2 e_\theta)}\) is the sound speed.

In the case (i), the system is the compressible Navier–Stokes equations with the heat-conductivity which consists of one hyperbolic equation \((m_1 = 1)\) and two parabolic equations \((m_2 = 2)\). The paper [12] show the existence and the asymptotic stability of the stationary solution for supersonic, transonic and subsonic flows. Our theorems are applicable for the supersonic flow \(u_+ < -c_s(\rho_+, \theta_+)\) and the sonic flow \(u_+ = -c_s(\rho_+, \theta_+).\) Precisely, for the former flow, we have the exponential or the algebraic convergence rate subject to the spatial decay rate of the initial perturbation owing to Theorem 1.5. For the latter, we have algebraic convergence rate due to Theorem 1.7. The asymptotic stability is shown in [11] for the inflow problem.

In the case (ii), the system is the compressible Euler equations with the heat-conductivity which consists of two hyperbolic equations \((m_1 = 2)\) and one parabolic equation \((m_2 = 1).\) If \(\rho_0 \neq 0,\) the stability condition holds. Hence we see the stationary solution is stable for the supersonic flow \(u_+ < -c_s(\rho_+, \theta_+)\) and the sonic flow \(u_+ = -c_s(\rho_+, \theta_+).\) Precisely, for the former flow, we have the exponential or the algebraic convergence rate subject to the spatial decay rate of the initial perturbation owing to Theorem 1.5. Even though \(\rho_0 \neq 0,\) we have the stationary solution by assuming an additional condition, for example \(\rho_0 \theta \neq 0.\) For the supersonic flow, we can apply Theorem 1.6 and have the decay rates. For the transonic flow, Theorem 1.8 is applicable.

In the case (iii), we change the order of the component in \(u\) as \(u = \gamma(\rho, \theta, u).\) Then the coefficient matrices (165) are rewritten as

\[ A^0(u) = \frac{1}{\rho^2} \begin{pmatrix} \rho^2 \theta & 0 & 0 \\ 0 & \theta p_\rho & 0 \\ 0 & 0 & \rho^2 e_\theta \end{pmatrix}, \quad A(u) = \frac{1}{\rho^2} \begin{pmatrix} u \theta p_\rho & 0 & \rho \theta p_\theta \\ 0 & \rho^2 u e_\theta & \rho \theta p_\theta \\ \rho \theta p_\rho & \rho \theta p_\theta & \rho^2 u \theta \end{pmatrix}, \]

\[ B(u) = \frac{1}{\theta^2} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad g(u, u_x) = \frac{1}{\theta^2} \begin{pmatrix} 0 \\ \mu u_x^2 \\ \theta \mu u_x u_x \end{pmatrix}. \]

Hence the system consists of two hyperbolic equations \((m_1 = 2)\) and one parabolic equation \((m_2 = 1).\) We note that the condition \([N]\) does not hold in this case. Although the paper [12] shows the existence of the stationary solution for supersonic, transonic and subsonic flows, the asymptotic stability is not proved since the system does not admit the stability condition \([SK]\). However Theorems 1.6 and 1.8 are applicable to the supersonic and the transonic flows, respectively, and show the asymptotic stability with the convergence rates.

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