On The Slope of Hyperelliptic Lefschetz Fibrations and The Number of Separating Vanishing Cycles

Yusuf Z Gurtas

Abstract

In this article we find an upper bound for the slope of genus $g$ hyperelliptic Lefschetz fibrations, which is sharp when $g = 2$, and demonstrate the strong connection, in general, between the slope of hyperelliptic genus $g$ Lefschetz fibrations and the number of separating vanishing cycles. Specifically, we show that the slope is greater than $4 - \frac{4}{g}$ if and only if the fibration contains separating vanishing cycles. We also improve the existing bound on $\frac{s}{n}$, the ratio of number of separating vanishing cycles to the number of non-separating vanishing cycles, for hyperelliptic Lefschetz fibrations of genus $g \geq 2$. In particular we show that $s \leq n$ for such fibrations when $g \geq 6$.

1 Introduction

Let $X \rightarrow S^2$ be a genus $g$ Lefschetz fibration. (The reader is referred to [2] for a thorough review of Lefschetz fibrations.)

It’s known that the 4− manifold $X$ carries an almost complex structure; therefore it makes sense to define its holomorphic Euler characteristic and first Chern class. Let

$$\chi_h = \frac{1}{4} (\sigma + \chi) \quad \text{and} \quad c_1^2 = 2\chi + 3\sigma,$$

where $\chi$ is the Euler characteristic and $\sigma$ is the signature of the 4− manifold $X$. The slope $\lambda_f$ of $X$ is defined as $\lambda_f := K_f^2/\chi_f$ where $K_f^2 := c_1^2 + 8(g - 1)$ and $\chi_f := \chi_h + g - 1$. It’s known that

$$\lambda_f \geq 4 - \frac{4}{g}$$

for a genus $g$ Lefschetz fibration and this bound is sharp. For example, all of the known hyperelliptic Lefschetz fibrations over $S^2$ with no separating
vanishing cycles satisfy $\lambda_f = 4 - \frac{4}{g}$. We will write $\lambda$ for simplicity from now on and all the Lefschetz fibrations discussed in this article will be hyperelliptic.

The connection between $\lambda$ and the number of separating vanishing cycles of a Lefschetz fibration seems to be unaccounted for in the literature. Let $s$ be the number of separating vanishing cycles and $n$ be the number of those that are non-separating. In this article we will prove:

**Theorem 1.** A genus $g$ hyperelliptic Lefschetz fibration $X \to S^2$ satisfies $\lambda > 4 - \frac{4}{g}$ if and only if $s \neq 0$, i.e., it contains separating vanishing cycles.

Recall that a Lefschetz fibration can not contain only separating vanishing cycles. Therefore the theorem should be interpreted as a fibration containing a mixture of separating and non-separating vanishing cycles.

An interesting question that arises at this point is the proportion of the number of separating cycles within a fibration, in particular its ratio to the number of non-separating vanishing cycles, $\frac{s}{n}$. We do not find any estimates in the literature on this ratio except for

$$\frac{s}{n} \leq 5$$

(1)

due to A.Stipsicz, [5]. Since we have $n > 0$ in a given Lefschetz fibration, this ratio is always defined.

**Definition 2.**

$$\rho(g) = \max \{ r = \frac{s}{n} \mid \exists \ a \Sigma_g - \text{hyperelliptic Lefschetz fibration } X \to S^2 \text{ with } s \text{ separating and } n \text{ non-separating vanishing cycles} \}$$

There isn’t enough evidence to justify that the bound (1) could actually be sharp. On the contrary, all of the known examples suggest that $\rho(g)$ may not be too high.

In this article we will improve the bound on $\rho$ for hyperelliptic Lefschetz fibrations and show that:

**Theorem 3.** For an hyperelliptic Lefschetz fibration of genus $g \geq 2$ we have

$$\rho(g) \leq \frac{3g + 2}{4(g - 1)}.$$
The last result is about signature of hyperelliptic Lefschetz fibrations. Even though there is an explicit formula that gives the signature in terms of separating and non-separating vanishing cycles for genus $g$ hyperelliptic Lefschetz fibrations, it is desirable to have a formula that relates the signature to the total number of vanishing cycles, perhaps by a scalar multiplication.

**Theorem 4.** For a genus $g$ hyperelliptic Lefschetz fibration we have

$$\sigma = k (n + s),$$

where $k = -\frac{\lambda - 8}{\lambda - 12}$.

In the next section we will prove Theorem 1 and Theorem 4 and show some of their applications for genus 2. The following section will summarize similar results for genus 3. The case of low genus is handled separately because there is only one type of separating vanishing cycle when $g < 4$ and due to that reason general formulas don’t always give rise to results that are as sharp as could be when restricted to low genus. It is also intended to give the reader an easy preparation for the general case which will be addressed in the last section along with the proof of Theorem 3.

We prove all the results for hyperelliptic Lefschetz fibrations but some of them generalize to non-hyperelliptic case as well. Please see Remark 13 for results that generalize to non-hyperelliptic Lefschetz fibrations. Even though we found out that there are shorter proofs for some of the results, we chose to leave them in the original format they were written in. We pointed out to those shorter proofs in Remark 17. We don’t claim originality on most of the results but Theorem 3 has not appeared anywhere else to the best of our knowledge.

## Genus 2

The signature of a genus $g$ hyperelliptic Lefschetz fibration $X \to S^2$ is given by

$$-\frac{g + 1}{2g + 1}n + \sum_{h=1}^{[g/2]} \frac{4h (g - h) s_h}{2g + 1} - s.$$
Let
\[ x = \sum_{h=1}^{\lfloor g/2 \rfloor} h (g - h) s_h, \]
where \( s = \sum_{h=1}^{\lfloor g/2 \rfloor} s_h \). The other invariants of \( X \) that will be used throughout the article are:

- Euler characteristic
  \[ \chi = n + s - 4 (g - 1), \]

- holomorphic Euler characteristic
  \[ \chi_h = \frac{1}{4} (\chi + \sigma) = \frac{1}{4} \left( n + s - 4 (g - 1) - \frac{g + 1}{2g + 1} n + \frac{4x}{2g + 1} - s \right) = \frac{ng + 4x}{4(2g + 1)} - (g - 1), \]

- square of the first Chern class \( c_1^2 \)
  \[ c_1^2 = 2 \chi + 3 \sigma = 2 (n + s - 4 (g - 1)) + 3 \left( -\frac{g + 1}{2g + 1} n + \frac{4x}{2g + 1} - s \right) = 2n - s - 8 (g - 1) - \frac{3}{2g + 1} n + \frac{12x}{2g + 1} \]

where \( s \) is the number of separating vanishing cycles and \( n \) is the number of non-separating vanishing cycles.

**Lemma 5.** \( sg \leq 2x \) for \( g \geq 2 \).

**Proof.** It’s not difficult to see that \( s(g - 1) \leq x \) by definition of \( x \) and \( s \). Therefore
\[ s \leq \frac{x}{g - 1} \quad \text{and} \quad sg \leq \frac{gx}{g - 1}. \]

The proof follows from the fact that \( \frac{g}{g-1} \leq 2 \) for \( g \geq 2 \). \( \square \)

**Proof of Theorem** \( \square \) The slope \( \lambda \) of the fibration is given as
\[ \lambda = \frac{c_1^2 + 8 (g - 1)}{\chi_h + (g - 1)} = \frac{2n - s - \frac{3}{2g + 1} n + \frac{12x}{2g + 1}}{\frac{ng + 4x}{4(2g + 1)}} = \frac{4 \frac{n (g - 1) - s (2g + 1) + 12x}{ng + 4x}}{4}. \]
Assume $s \neq 0$. Then $x \neq 0$ and we have

$$\lambda - (4 - 4/g) = 4 \frac{n(g - 1) - s(2g + 1) + 12x}{ng + 4x} - 4 + 4/g$$

$$= 4 \frac{-2sg^2 - sg + 8gx + 4x}{(ng + 4x)g} = 4 \frac{(2g + 1)(4x - sg)}{(ng + 4x)g} > 0,$$

because $4x > sg$ by Lemma 5. and all other factors are positive. Therefore $\lambda - (4 - 4/g) = 0$ if and only if $4x = sg$; i.e., if and only if $s = 0$.  \[\square\]

**Corollary 6.** For a genus 2 Lefschetz fibration we have

$$\lambda = 2 \frac{n + 7s}{n + 2s} = 2 \frac{1 + 7r}{1 + 2r}. \quad (4)$$

**Proof.** From (3) we have

$$\lambda = 4 \frac{n(g - 1) - s(2g + 1) + 12x}{ng + 4x}.$$

Setting $g = 2$ and realizing that for a genus 2 Lefschetz fibration $x = s$ we obtain

$$4 \frac{n(2 - 1) - s(2 \cdot 2 + 1) + 12s}{2n + 4s} = 2 \frac{n + 7s}{n + 2s}.$$

Dividing through by $n$ gives

$$2 \frac{1 + 7r}{1 + 2r}. \quad \square$$

**Proposition 7** ((Corollary 10, [4])). For a genus 2 Lefschetz fibration we have

$$c_1^2 \leq 6\chi_h - 3.$$

**Proof.** We will use the bound

$$\sigma \leq n - s - 4 \quad (5)$$
for hyperelliptic Lefschetz fibrations given by Corollary 9, [4]. First, we write \( \chi \) in terms of \( \chi_h \):

\[
\chi_h = \frac{1}{4} (\sigma + \chi) = \frac{1}{4} \left( -\frac{3}{5} n - \frac{1}{5} s + n + s - 4 \right) = \frac{1}{10} n + \frac{1}{5} s - 1,
\]

therefore

\[
\chi = n + s - 4 = 10 \left( \frac{1}{10} n + \frac{1}{5} s - 1 \right) + 6 - s = 10 \chi_h + 6 - s.
\]

Then, since \( \sigma \leq n - s - 4 = n + s - 4 - 2s = \chi - 2s \), we have

\[
\chi_h = \frac{1}{4} (\sigma + \chi) \leq \frac{1}{4} (\chi - 2s + \chi) = \frac{1}{2} (\chi - s)
\]

\[
\chi_h \leq \frac{1}{2} (10 \chi_h + 6 - s - s) = 5 \chi_h + 3 - s,
\]

which can be written as

\[
s \leq 4 \chi_h + 3. \tag{6}
\]

Finally, we have

\[
c_1^2 = 12 \chi_h - \chi = 12 \chi_h - (10 \chi_h + 6 - s) = 2 \chi_h - 6 + s \leq 2 \chi_h - 6 + 4 \chi_h + 3 = 6 \chi_h - 3. \tag{7}
\]

\[\square\]

**Remark 1.** Solving the inequality (6) for \( \chi_h \) we get \( \frac{1}{4} (s - 3) \leq \chi_h \). This means that for genus 2 Lefschetz fibrations we have \( \chi_h \geq 0 \).

**Remark 2.** For genus 2 Lefschetz fibrations we have \( c_1^2 = 2 \chi_h + s - 6 \) by (7). Therefore all genus 2 fibrations with no separating vanishing cycles are necessarily on the Noether line. The manifold lands above Noether line if and only if it contains separating vanishing cycles.

**Corollary 8.** For a genus 2 Lefschetz fibration we have

\[
\lambda \leq 6 - \frac{1}{\chi_h + 1}.
\]

**Proof.** Using Proposition 7 we can write

\[
c_1^2 + 8 \leq 6 \chi_h - 3 + 8 = 6 (\chi_h + 1) - 1.
\]
Dividing through by $\chi_h + 1$ we obtain
\[
\lambda = \frac{\sigma^2 + 8}{\chi_h + 1} \leq 6 - \frac{1}{\chi_h + 1}.
\]

Note that $\chi_h + 1 > 0$ by Remark [ Remark 1].

**Corollary 9.** $\rho(2) \leq 2$.

*Proof.* Using Corollary [ Corollary 6] and Corollary [ Corollary 8] we can write
\[
\lambda = 2 \frac{1 + 7r}{1 + 2r} \leq 6 - \frac{1}{\chi_h + 1} \leq 6.
\]
for any genus 2 Lefschetz fibration. Solving it for $r$ gives $r \leq 2$. \qed

**Corollary 10.** The number of separating and non-separating vanishing cycles $s$ and $n$, respectively, in a genus 2 Lefschetz fibration satisfy
\[
2s + n = 10k,
\]
\[
2n - s \geq 5
\]
for some $k \in \mathbb{Z}^+$.

*Proof.*
\[
\chi_h = \frac{1}{4} (\sigma + \chi) = \frac{1}{4} \left( -\frac{3}{5}n - \frac{1}{5}s + n + s - 4 \right) = \frac{1}{10} (n + 2s) - 1
\]
Therefore $n + 2s = 10(\chi_h + 1)$ and $\chi_h + 1 > 0$ by Remark [ Remark 1]. This proves the equality. For the inequality we will use Corollary [ Corollary 6] and Corollary [ Corollary 8]:
\[
2\frac{n + 7s}{n + 2s} \leq 6 - \frac{1}{\chi_h + 1} = 6 - \frac{1}{10} (n + 2s) - 1 + 1 = 6 - \frac{10}{n + 2s}.
\]
Solving
\[
2\frac{n + 7s}{n + 2s} \leq 6 - \frac{10}{n + 2s}
\]
for $s$ we obtain $s \leq 2n - 5$ as claimed. \qed

It would be an interesting question to ask if this inequality is sharp.
Proposition 11. If the equations

\[
\begin{align*}
2s + n &= 10k \\
2n - s &= 5
\end{align*}
\]

are satisfied for a genus 2 Lefschetz fibration then

\[
\frac{s}{n} = \frac{4m + 3}{2m + 4}
\]  \hspace{1cm} (10)

for \(m \geq 0\).

Proof. Solving the given system of equations we obtain

\[n = 2 + 2k, \quad s = -1 + 4k\]

\(k \in \mathbb{Z}^+\). Therefore

\[
\frac{s}{n} = \frac{4k - 1}{2k + 2}.
\]

Now, let \(m = k - 1 \geq 0\). \(\square\)

First few values this sequence can take on are

\[
\frac{s}{n} = \frac{3}{4}, \frac{7}{6}, \frac{11}{8}, \frac{3}{2}, \frac{19}{12}, \frac{23}{14}.
\]

Xiao constructed examples realizing the values \(\frac{3}{4}, \frac{7}{6}\) and \(\frac{19}{12}\). \[6.\]

Remark 3. With \(\frac{s}{n} = \frac{4m + 3}{2m + 4}\) the slope becomes:

\[
\lambda = 2 + \frac{7(4m + 3)}{1 + 2(4m + 3)} = \frac{6m + 5}{m + 1} = 6 - \frac{1}{m + 1}.
\]

Invoking Corollary \(8\) we get

\[6 - \frac{1}{m + 1} \leq 6 - \frac{1}{\chi_0 + 1}, \quad \text{i.e.,} \quad 0 \leq m \leq \chi_0.\]

It's interesting to note that this bound is sharp for the examples that we know satisfy the equation \(2n - s = 5\), i.e., \(m = \chi_0\). Therefore we might conjecture that this is a characterizing feature for genus 2 fibrations satisfying \(2n - s = 5\). Indeed that is the case:
Proposition 12. For a genus 2 Lefschetz fibration we have

\[ 2n - s = 5 \quad \text{if and only if} \quad \lambda = 6 - \frac{1}{\chi_h + 1}. \]

Proof. Assume \( 2n - s = 5 \). Substitute \( \chi_h = \frac{1}{10} (n + 2s) - 1 \) into

\[ \frac{4\chi_h + 3}{2\chi_h + 4} \]  \hspace{1cm} (11)

and use \( 2n - s = 5 \) for both the numerator and denominator to see that it’s equal to \( \frac{s}{n} \). Then substitute (11) in place of \( \frac{s}{n} \) in (11) to obtain the desired equality. Conversely, assume that the bound on \( \lambda \) is sharp. Substitute \( \chi_h = \frac{1}{10} (n + 2s) - 1 \) into the bound and set it equal to (11). Solving that equality for \( s \) will result in \( s = 2n - 5 \). \( \square \)

Remark 4. We calculate the invariants of a genus 2 Lefschetz fibration with \( 2n - s = 5 \) as:

\begin{align*}
\sigma &= -n + 1 = -\frac{1}{2} (s + 3) \\
\chi &= 3n - 9 = \frac{3}{2} (s - 1) \\
\chi_h &= \frac{1}{2} n - 2 = \frac{1}{4} (s - 3) \\
c_1^2 &= 3n - 15 = \frac{3}{2} (s - 5)
\end{align*}

Remark 5. The bound (5) on signature is sharp and realized by genus 2 Lefschetz fibrations satisfying \( 2n - s = 5 \). Simply write \( 2n - s = 5 \) as \( n - s - 4 = -n + 1 = \sigma \).

Remark 6. Thanks to the computations in Remark 4 we can express the slope \( \lambda \) in terms of \( n \) and \( s \) only as

\[ \lambda = \frac{3n - 15 + 8}{2n - 2 + 1} = 2 \frac{3n - 7}{n - 2} \quad \text{and} \quad \lambda = \frac{\frac{3}{2} (s - 5) + 8}{\frac{3}{4} (s - 3) + 1} = 2 \frac{1 + 3s}{1 + s}, \]

respectively, for fibrations satisfying \( 2n - s = 5 \).

Combining the results on the slope of genus 2 Lefschetz fibrations so far with Proposition 4 and Proposition 20 we can prove:
**Corollary 13.** For a genus 2 Lefschetz fibration with \( n \) non-separating and \( s \) separating vanishing cycles we have

\[
\lambda = \frac{2n + 7s}{n + 2s} \leq \frac{2}{1 + s} \leq \frac{2}{2s + n} \leq \frac{3n - 7}{2} \leq \frac{10s + n - 2}{2s + n} \leq \frac{25n - s - 12}{n - 2}.
\]

**Proof.** All but the fourth inequality are equivalent to \( 2n - s \geq 5 \), which is true by Corollary 10. The fourth inequality turns out to be \( 0 \leq 2(n - 4)(2n - s - 5) \) but this is also true thanks to Corollary 10 and Remark 8. All five inequalities become equality when \( 2n - s = 5 \). \( \square \)

Now, we will prove Theorem 4.

**Proof of Theorem 4.** From (3) we have

\[
\lambda = 4n (g - 1) - s (2g + 1) + 12x.
\]

Cross multiplication gives

\[
4n (g - 1) - 4s (2g + 1) + 48x = \lambda ng + 4x \lambda.
\]

Solving this for \( x \) results in

\[
x = \frac{\lambda ng + 4s (2g + 1) - 4n (g - 1)}{4(12 - \lambda)}.
\]

We will substitute this into the signature formula to obtain the result:

\[
\sigma = -\frac{g + 1}{2g + 1}n + \sum_{h=1}^{[g/2]} \frac{4h (g - h) s_h}{2g + 1} - s
\]

\[
= -\frac{g + 1}{2g + 1}n + \frac{4x}{2g + 1} - s
\]

\[
= -\frac{g + 1}{2g + 1}n + \frac{\lambda ng + 4s (2g + 1) - 4n (g - 1)}{(2g + 1)(12 - \lambda)} - s
\]

\[
= -\frac{(g + 1)n (12 - \lambda) + \lambda ng + 4s (2g + 1) - 4n (g - 1) - s (2g + 1)(12 - \lambda)}{(2g + 1)(12 - \lambda)}
\]

\[
= \frac{(2g + 1)(\lambda - 8)(n + s)}{(2g + 1)(12 - \lambda)}
\]

\[
= -\frac{8 - \lambda}{12 - \lambda} (n + s).
\]

\( \square \)
Remark 7. We have $c^2 < 8\chi_h$ for genus 2 Lefschetz fibrations by Proposition [7]. Therefore $\lambda < 8$ and the signature is always negative for those fibrations by Theorem [4].

Corollary 14. For a genus 2 Lefschetz fibration we have

$$\sigma \leq -2\chi_h - 3 \quad \text{and} \quad \sigma \leq -\frac{1}{3}\chi - 2. \quad (12)$$

Proof. We have $\lambda \leq 6 - \frac{1}{\chi_h + 1}$ by Corollary [8] and $-\frac{8 - \lambda}{12 - \lambda}$ is an increasing function of $\lambda$. Therefore, substituting $6 - \frac{1}{\chi_h + 1}$ in place of $\lambda$ in Theorem [4] gives

$$\sigma \leq -\frac{8 - \left(6 - \frac{1}{\chi_h + 1}\right)}{12 - \left(6 - \frac{1}{\chi_h + 1}\right)} (n + s) \leq -\frac{2\chi_h + 3}{6\chi_h + 7} (n + s) = -\frac{2\chi_h + 3}{6\chi_h + 7} (\chi + 4).$$

Now, substitute $\chi = 4\chi_h - \sigma$ and cross multiply to get

$$\sigma \left(6\chi_h + 7\right) \leq -(2\chi_h + 3) \left(4\chi_h - \sigma + 4\right)$$

using $\chi_h \geq 0$. Solving this for $\sigma$ gives the first inequality. In order to obtain the second inequality simply substitute $\chi_h = \frac{1}{4} (\chi + \sigma)$ into the first one and solve for $\sigma$. Note that both inequalities are sharp for genus 2 fibrations with $2n - s = 5$ and they can also be obtained using Remark [4] in that case. □

Remark 8. We proved in Corollary [10] that $2n - s \geq 5$ for genus 2 Lefschetz fibrations. In fact $2n - s$ is divisible by 5:

$$2n - s = 2(n + 2s) - 5s = 20(\chi_h + 1) - 5s = 5 \left(4\chi_h + 4 - s\right) = 5 \left(\sigma + \chi + 4 - s\right) = 5 \left(n + \sigma\right).$$

(One can also use the local signature formula $\sigma = -\frac{2}{5}n - \frac{1}{5}s$ in order to see that, [3]) Let $t = n + \sigma$. It’s clear that $t \in \mathbb{Z}^+$. Solving the equations

$$2s + n = 10k$$
$$2n - s = 5t$$
for $n$ and $s$ we get $n = 2t + 2k, s = 4k - t$. In particular $n \geq 4$ because $t, k \in \mathbb{Z}^+$. Substituting these values of $n$ and $s$ in (4) we obtain

$$
\lambda = 2 \left(1 + \frac{7 \frac{k-t}{2t+2k}}{1 + \frac{2k}{2t+2k}}\right) = 6 - \frac{t}{k} = 6 - \frac{t}{\chi_h + 1} \leq 6 - \frac{1}{\chi_h + 1}
$$

as we proved in Corollary 8.

**Corollary 15.** For genus 2 Lefschetz fibrations we have

$$
\frac{s}{n} \leq \frac{4\chi_h + 3}{2\chi_h + 4}.
$$

**Proof.** Using (9) and $s \leq 2n - 5$ we have

$$
\chi_h = \frac{1}{10} (2s + n) - 1 \leq \frac{1}{10} (2(2n - 5) + n) - 1 = \frac{1}{2} n - 2.
$$

Thus $2\chi_h + 4 \leq n$. Taking the reciprocal of this and combining it with (6) yields the result.

**Remark 9.** The least number of vanishing cycles for a genus 2 Lefschetz fibration has been narrowed down to a number that is equal to 7 or 8, [4]. Remark 8 gives a minimum value for $n$, which is 4, as well as Corollary 15. With that value of $n$ the smallest $s$ can be is 3 by Corollary 10. Therefore the fibration with $n + s = 4 + 3 = 7$ vanishing cycles constructed by Xiao in [6] realizes that minimum number.

From geographical perspective there are three important regions for genus 2 Lefschetz fibrations that are distinct in some ways from one another:

1. $2 \leq \lambda \leq 4$,
2. $4 < \lambda < 5$,
3. $5 \leq \lambda < 6$.

In the first region we see most of the known genus 2 Lefschetz fibrations that come from topological constructions and mapping class group considerations. These are the fibrations satisfying $0 \leq \frac{s}{n} \leq \frac{1}{3}$. In particular $\lambda = 2$ corresponds to the classical examples that do not contain any separating
vanishing cycles. \( \lambda = 4 \) corresponds to the fibrations satisfying \( 3s = n \). The well known construction by Matsumoto has been the only known example satisfying this ratio. The author of this article has recently given many more examples satisfying \( 3s = n \).

The second region is the loci of fibrations satisfying \( \frac{1}{3} < \frac{s}{n} < \frac{3}{4} \). To the best of our knowledge there are no known examples of genus 2 Lefschetz fibrations in this region coming from topological constructions or mapping class group considerations. The author of this article has constructed an example with \( \frac{s}{n} = \frac{17}{36} \). All genus 2 Lefschetz fibrations in the first two regions satisfy \( 2n - s > 5 \) because Proposition 12 requires \( \lambda = 6 - \frac{1}{\chi h + 1} \) for fibrations satisfying \( 2n - s = 5 \) and \( 6 - \frac{1}{\chi h + 1} \geq 5 \).

The third region is the region of fibrations satisfying \( \frac{3}{4} \leq \frac{s}{n} < 2 \). The fibrations satisfying the relation \( 2n - s = 5 \) are in this region. The only known, to the author, examples of this sort come from algebro-geometric constructions and are due to Xiao, [6]. They correspond to ratios \( \frac{s}{n} = \frac{3}{4}, \frac{7}{6}, \frac{19}{12} \). It’s an open question how high this ratio can be. It would also be interesting to find a fibration in this region with \( 2n - s > 5 \) that is not a fiber sum of fibrations satisfying \( 2n - s = 5 \).

### 3 Summary of genus 3 case

Almost all of the calculations in the previous section can be carried out for genus 3 in much the same manner. We will just list the results in the sequence they appeared for genus 2 instead of redoing all of them.

Formula (3) gives

\[
\lambda = 4 - \frac{2n + 17s}{3n + 8s} = 4 - \frac{2 + 17r}{3 + 8r}
\]

when we substitute \( g = 3, x = 2s \).

Proposition 7 (Corollary 10, [4]) becomes

\[
c_1^2 \leq \frac{29}{4} \chi h - \frac{11}{4} = 7.25 \chi h - 2.75.
\]

Remark 1 becomes

\[-1 \leq \chi h.\]
Corollary 8 gives
\[ \lambda \leq \frac{29}{4} - \frac{5}{4} \chi h + 2. \]  
(14)

Corollary 9 turns out to be \[ \rho(3) \leq \frac{11}{8} = 1.375. \]

Corollary 10 takes the form
\[
\begin{align*}
3n + 8s &= 28k \\
11n - 8s &\geq 28
\end{align*}
\]

and solving the system with equalities gives
\[ s = -\frac{3}{4} + \frac{11}{4}k, n = 2 + 2k, k = \chi h + 2 \in \mathbb{Z}^+. \]

After letting \( k = 4m + 1, m \geq 0 \), we obtain
\[
\frac{s}{n} = \frac{11m + 2}{8m + 4},
\]  
(15)

which is the genus 3 version of (10). Combining (13) and (14) and using
\[
\chi h = \frac{3}{28}n + \frac{2}{7}s - 2 \quad \text{and} \quad 11n - 8s = 28
\]
together we see that the bound (14) on \( \lambda \) would be sharp if there were fibrations satisfying the equation \( 11n - 8s = 28 \) but we do not know any example of that. For such fibrations the signature bound (5) would also be sharp and realized by genus 3 hyperelliptic Lefschetz fibrations satisfying \( 11n - 8s = 28 \):

\[
\begin{align*}
11n - 8s &= 28 \\
11n - 8s &= 28 - 10n + 7s \\
&= 24 - 10n + 7 \left( \frac{11}{8}n - \frac{7}{2} \right) \\
&= \frac{3}{8}n - \frac{1}{2} \\
&= \frac{-4}{7}n + \frac{1}{7} \left( \frac{11}{8}n - \frac{7}{2} \right) \\
&= \frac{4}{7}n + \frac{1}{7}s \\
&= \sigma.
\end{align*}
\]
In fact, $11n - 8s$ is divisible by 28:

$$11n - 8s = 28t,$$  \hspace{1cm} (16)

where $t = \frac{1}{4}(n - s - \sigma) \in \mathbb{Z}^+$ and the calculation above is just $t = 1$ case (See Remark 16). Solving a similar system as in Remark 8 gives

$$\lambda = \frac{29}{4} - \frac{5}{4 \chi_h + 2} \leq \frac{29}{4} - \frac{5}{4 \chi_h + 2}.$$

Corollary 13 would take the form

$$\lambda = 4 \frac{2n + 17s}{3n + 8s} \leq \frac{29s + 8}{4s + 3} \leq \frac{187n + 232s - 140}{4n + 8s} \leq \frac{129n - 68}{4n - 2} \leq \frac{2}{3} \frac{15n + 26s - 28}{3n + 8s} \leq \frac{25n - s - 12}{n - 2}. $$

All but the fourth inequality above are equivalent to $0 \leq 11n - 8s - 28$. The fourth one comes down to $0 \leq (3n - 16) (11n - 8s - 28)$ but $n \geq 8$ for genus 3 hyperelliptic Lefschetz fibrations. Remark 7 would still be valid for genus 3 hyperelliptic Lefschetz fibrations.

Genus 3 equivalent of the bounds in Corollary 14 are

$$\sigma \leq -\frac{3}{4} \chi_h - \frac{11}{4} \text{ and } \sigma \leq -\frac{3}{19} \chi - \frac{44}{19}. $$

Finally, genus 3 version of Corollary 15 is

$$\frac{s}{n} \leq \frac{11 \chi_h + 19}{8 (\chi_h + 3)}.$$ 

using $s \leq \frac{1}{4} (11 \chi_h + 19)$, which is equivalent to $11n - 8s \geq 28$, and $2 \chi_h + 6 \leq n$, (23).

4 General Case

**Proposition 16.** For a genus $g$ hyperelliptic Lefschetz fibration the slope is given by

$$\lambda = 12 - \frac{n + s}{\chi_h + g - 1},$$  \hspace{1cm} (17)
Proof. By definition
\[
\lambda = \frac{c_1^2 + 8 (g - 1)}{\chi_h + g - 1} = \frac{12 \chi_h - \chi + 8 (g - 1)}{\chi_h + g - 1}
= \frac{12 \chi_h + 12g - 12 - \chi - 4 (g - 1)}{\chi_h + g - 1}
= 12 + \frac{-(n + s - 4 (g - 1)) - 4 (g - 1)}{\chi_h + g - 1}
= 12 - \frac{n + s}{\chi_h + g - 1}
\]
\[
\square
\]

Remark 10. To see that (17) agrees with (3) and (13) for genus 2 and 3 simply substitute \(\frac{10}{19} (n + 2s) - 1\) and \(\frac{3}{28} n + \frac{3}{7} s - 2\) for \(\chi_h\), respectively. The proof when \(s = 0\) is straightforward:
\[
\chi_h + g - 1 = \frac{1}{4} \left( -\frac{g + 1}{2g + 1} n + n - 4 (g - 1) \right) + g - 1 = \frac{1}{4} \frac{ng}{2g + 1}
\]
and
\[
12 - \frac{n}{\chi_h + g - 1} = 12 - \frac{n}{4 \frac{2}{2g + 1}} = 4 \frac{g - 1}{g}.
\]

Remark 11. (17) can also be written as
\[
\lambda = 12 - \frac{4}{1 + \frac{\sigma}{n+s}} = 12 - 4 \frac{n + s}{\sigma + n + s} = 8 + 4 \frac{\sigma}{\sigma + n + s},
\]
either by solving the formula given by Theorem 4 for \(\lambda\) or using the relation
\[
\sigma + n + s = 4 (\chi_h + g - 1).
\]

Remark 12. The first formula in Remark 11 shows how the slope depends on the (unweighted) "average \(\frac{\sigma}{n+s}\) of signature per vanishing cycle". When \(\lambda = 10\), this average must be 1. This can never happen because the "signature contribution" of each vanishing cycle is either \(-1\), or 0, or +1 and according to the handlebody decomposition of Lefschetz fibrations the first handle attached along the first vanishing cycle, which can be arranged to be a non-separating one by cyclically permuting, will always result in a 4–manifold with 0 signature, [4]. This is proved in the following proposition.
Proposition 17. For a genus $g$ hyperelliptic Lefschetz fibration we have

$$\lambda \leq 10 - \frac{2 + s}{\chi_h + g - 1}. \quad (19)$$

Proof. First we estimate $\chi_h$ as

$$\chi_h = \frac{1}{4} (\sigma + \chi) = \frac{1}{4} \left( -\frac{g + 1}{2g + 1} n + \sum_{h=1}^{[g/2]} \frac{4h (g - h) s_h}{2g + 1} - s + n + s - 4 (g - 1) \right)$$

$$\leq \frac{1}{4} \left( \frac{ng}{2g + 1} + \frac{4} {2g + 1} \left( g - \frac{g}{2} \right) s - 4 (g - 1) \right)$$

$$= \frac{1}{4} \frac{ng}{2g + 1} + \frac{1}{4} \frac{sg^2}{2g + 1} - (g - 1) := M, \quad (20)$$

using the fact that $h(g - h) \leq \frac{g}{2} \left( g - \frac{g}{2} \right)$ and $\sum_{h=1}^{[g/2]} s_h = s$. Now, use this to write $\chi$ as

$$\chi = n + s - 4 (g - 1)$$

$$= \frac{4 (2g + 1)}{g} \left( \frac{1}{4} \frac{ng}{2g + 1} + \frac{1}{4} \frac{sg^2}{2g + 1} - (g - 1) \right) + (1 - g) s + 4g - \frac{4}{g}$$

$$= \frac{4 (2g + 1)}{g} M + (1 - g) s + 4g - \frac{4}{g}. \quad (21)$$

The estimate

$$\sigma \leq n - s - 4 = n + s - 4 (g - 1) - 2s + 4 (g - 2) = \chi - 2s + 4 (g - 2),$$

(17), can be used to write

$$\chi_h = \frac{1}{4} (\sigma + \chi) \leq \frac{1}{4} (\chi - 2s + 4 (g - 2) + \chi) = \frac{1}{2} \chi - \frac{1}{2} s + g - 2 \quad (22)$$

and using (21) we obtain

$$\chi_h \leq \frac{1}{2} \left( \frac{4 (2g + 1)}{g} M + (1 - g) s + 4g - \frac{4}{g} \right) - \frac{1}{2} s + g - 2$$

$$= \frac{2g + 1}{g} M - \frac{1}{2} sg + 3g - 2 - \frac{2}{g}.$$
We will solve this for $sg$
\[ sg \leq 4 \frac{2g + 1}{g} M - 2\chi_h + 6g - 4 - \frac{4}{g} \]
and use it in estimating
\[
\begin{align*}
c_1^2 & = 12\chi_h - \chi = 12\chi_h - \left( 4 \frac{2g + 1}{g} M + (1 - g) s + 4g - \frac{4}{g} \right) \\
& = 12\chi_h - 4 \frac{2g + 1}{g} M + (g - 1) s - 4g + \frac{4}{g} \\
& \leq 12\chi_h - 4 \frac{2g + 1}{g} M + 4 \frac{2g + 1}{g} M - 2\chi_h + 6g - 4 - \frac{4}{g} - s - 4g + \frac{4}{g} \\
& = 10\chi_h + 2g - 4 - s.
\end{align*}
\]
Now,
\[
\lambda = \frac{c_1^2 + 8(g - 1)}{\chi_h + g - 1} \leq \frac{10\chi_h + 2g - 4 - s + 8(g - 1)}{\chi_h + g - 1} = \frac{10\chi_h + 10g - 10 - 2 - s}{\chi_h + g - 1}
\]
and we have
\[
\lambda \leq 10 - \frac{2 + s}{\chi_h + g - 1}.
\]

**Corollary 18.** The slope $\lambda$ of an hyperelliptic genus $g$ Lefschetz fibration satisfies $\lambda \leq 10$.

**Remark 13.** Proposition 16 is true in general, i.e., the assumption that the Lefschetz fibration is hyperelliptic is not necessary. Therefore the formulas in Remark 11 are also true in general and using Remark 12 we can say that Corollary 18 extends to non-hyperelliptic Lefschetz fibrations as well. Because of Remark 11 we also conclude that Theorem 4 extends to non-hyperelliptic fibrations.

**Remark 14.** One can show that
\[
\chi_h \leq \frac{n}{2} - g, \quad \text{i.e.,} \quad 2\chi_h + 2g \leq n,
\]
for hyperelliptic genus $g$ Lefschetz fibrations using (22):
\[
\chi_h \leq \frac{1}{2} \chi - \frac{1}{2} s + g - 2 = \frac{1}{2} (n + s - 4 (g - 1)) - \frac{1}{2} s + g - 2 = \frac{1}{2} n - g.
\]
Corollary 19. Let $X \to S^2$ be a simply connected genus $g \geq 2$ hyperelliptic Lefschetz fibration with $b_2^+ \geq 1$. Then the minimum number of non-separating vanishing cycles is $2g + 2$. If furthermore $b_2^+ > 1$ then this minimum becomes $2g + 4$.

Proof. By definition of $\chi_h$ we have

$$\chi_h = \frac{1}{4} (\sigma + \chi) = \frac{1}{4} \left( b_2^+ - b_2^- + 2 - 2b_1 + b_2^+ + b_2^- \right) = \frac{1}{2} \left( b_2^+ + 1 - b_1 \right).$$

Using (23) and the assumption $b_1 = 0$ we get

$$\frac{1}{2} \left( b_2^+ + 1 \right) \leq \frac{1}{2} n - g.$$

Solving this inequality for $n$ after using $b_2^+ \geq 1$ yields $2g + 2 \leq n$. Clearly $2g + 4 \leq n$ when $b_2^+ > 1$ because $b_2^+$ must be odd. \qed

Proposition 20. The slope of an hyperelliptic genus $g$ Lefschetz fibration satisfies

$$\frac{4g - 1}{g} + \frac{4s}{g} \cdot \frac{(2g + 1)(3g - 4)}{ng + 4s(g - 1)} \leq \lambda \leq 10 - 2 + s \quad (24)$$

Proof. The signature satisfies the bound

$$\sigma = -\frac{g + 1}{2g + 1} n + \frac{4x}{2g + 1} - s \geq -\frac{g + 1}{2g + 1} n + \frac{4s(g - 1)}{2g + 1} - s = -\frac{g + 1}{2g + 1} n + \frac{2g - 5}{2g + 1}$$

because $s(g - 1) \leq x$ by definition of $x$ and $s$. Now, using Theorem 4 we can write

$$-\frac{g + 1}{2g + 1} n + \frac{2g - 5}{2g + 1}s \leq -\frac{8 - \lambda}{12 - \lambda} (n + s)$$

and solving this for $\lambda$ gives the first inequality. To prove the second inequality we begin with the fact that $\chi_h + g - 1 > 0$, as we mentioned in the proof of Corollary 14. Using this and (23) we can write

$$\frac{1}{\chi_h + g - 1} \leq \frac{-2}{n - 2}.$$

Now, adding 10 to both sides after multiplying by $2 + s$ proves the second inequality thanks to Proposition 17. \qed
Remark 15. We wrote (24) in that particular form instead of simplifying it in order to emphasize the fact that it is another proof for Theorem 1 and that \(4 - \frac{4}{g} \leq \lambda \leq 10\) for hyperelliptic Lefschetz fibrations. The lower bound in (24) gives (4) when we set \(g = 2\) and it gives the genus 3 version of (4) when \(g\) is set equal to 3. The reason this estimate is sharp for low genus is the fact that there is only one type of separating vanishing cycle for low genus and due to that reason the estimate \(s(g - 1) \leq x\) becomes equality for genus \(g = 2, 3\).

Proposition 21. Let \(X \to S^2\) be a genus \(g\) hyperelliptic Lefschetz fibration with \(n\) non-separating vanishing cycles. Then

- \(n\) is divisible by 4, if \(g\) is odd;
- \(n\) is even, if \(g \equiv 2 \pmod{4}\).

Proof. \(\sigma + s + n\) is divisible by 4 by (18). Write the signature

\[
\sigma = \frac{g + 1}{2g + 1} n + \frac{4x}{2g + 1} - s,
\]

where \(x = \sum_{h=1}^{[g/2]} h (g - h) s_h\), as

\[
(2g + 1) (\sigma + s) + (g + 1) n = 4x. \tag{25}
\]

Equivalently,

\[
(2g + 1) (\sigma + s + n) - gn = 4x,
\]

which shows that \(gn\) is divisible by 4 and the proof follows from that.

Divisibility of \(n\) by 4 when \(g\) is odd also follows from Proposition 4.10 of [1].

Remark 16. If \(g\) is not divisible by 4 then \(n\) is even by Proposition [21]. In that case we conclude from (25) that \(s + \sigma\) is also even. We use this and the fact that \(\sigma + s + n\) is divisible by 4 to prove that \(n - s - \sigma\) is divisible by 4 as well when \(g\) is not divisible by 4:

\[
n - s - \sigma = \sigma + s + n - 2 (s + \sigma).
\]

Then

\[
\frac{1}{4} (n - s - \sigma) = \frac{1}{4} \left( n - s - \left( \frac{g + 1}{2g + 1} n + \frac{4x}{2g + 1} - s \right) \right) = \frac{1}{4} \left( \frac{3g + 2}{2g + 1} n - 4x \right) \in \mathbb{Z}^+. \tag{26}
\]
When $g = 2$, (26) becomes $2n - s = 5(n + \sigma)$ as we found in Remark 8. When $g = 3$ then (26) is the same as (16). The integer (26) is positive because of (5).

Proof of Theorem 3. Using the bound (5) we get $\frac{1}{4}(n - s - \sigma) \geq 1$. Then (26) gives

$$1 \leq \frac{1}{4}(3g + 2)n - 4x,$$

and hence

$$x \leq \frac{1}{4}n(3g + 2) - (2g + 1).$$

Using the estimate $(g - 1)s \leq x$ one more time, we have

$$(g - 1)s \leq \frac{1}{4}n(3g + 2) - (2g + 1).$$

Dividing through by $n(g - 1)$ gives

$$r = \frac{s}{n} \leq \frac{3g + 2}{4(g - 1)} - \frac{2g + 1}{n(g - 1)}.$$ 

Since $s$ and $n$ are arbitrary, we conclude

$$\rho(g) \leq \frac{3g + 2}{4(g - 1)}.$$ 

\[ \square \]

Corollary 22. For an hyperelliptic Lefschetz fibration of genus $g \geq 6$ we have $s \leq n$.

Remark 17. One can prove Theorem 3 by solving

$$4\frac{g - 1}{g} + \frac{4s}{g} \cdot \frac{(2g + 1)(3g - 4)}{ng + 4s(g - 1)} \leq 10 - 2\frac{2 + s}{n - 2}$$

for $\frac{s}{n}$ as well, (24). Also, solving

$$\lambda = 12 - 4\frac{n + s}{n + s + \sigma} \leq 10 - 2\frac{2 + s}{n - 2}$$

for \[ \square \]
for $\sigma$ results in \(5\), which is another proof for Proposition \(17\). Finally, solving

$$4 g - 1 \leq \lambda = 12 - \frac{4}{1 + \frac{\sigma}{n+s}}$$

for $\frac{\sigma}{n+s}$ gives

$$\frac{\sigma}{n+s} \geq -\frac{g + 1}{2g + 1},$$

which shows that ”the average signature per vanishing cycle” is at least $-\frac{g+1}{2g+1}$ for Lefschetz fibrations satisfying $\lambda \geq 4 - 4/g$ and it is greater than that whenever $s > 0$ by virtue of Theorem \(1\). Based on this observation we conclude the following bound on $\rho (g)$ in general, without assuming hyperellipticity:

**Corollary 23.** For a Lefschetz fibration of genus $g \geq 2$ we have

$$\rho (g) \leq 3 + 2.\frac{g}{g}.$$

**Proof.** By Corollary 7 in \(4\) we have $\sigma \leq n - s$. Combining that with Remark \(17\) we conclude

$$-\frac{g + 1}{2g + 1} < \frac{n - s}{n + s} = \frac{1 - r}{1 + r}.$$

The result follows once we solve this inequality for $r$. 

**Acknowledgements.** Many thanks to Hurşit Önsiper for helpful and encouraging conversations and for referring the author to the article written by Xiao. The author is also grateful to Hisaaki Endo for his insightful comments.

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Yusuf Z. Gürtas
Mathematics Department
DePauw University
602 S. College Avenue
Greencastle, IN 46135
U.S.A.
yusufgurtas@depauw.edu