Minimal volume and simplicial norm of visibility n-manifolds and compact 3-manifolds

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Abstract

In this survey paper, we shall derive the following result.

**Theorem A.** Let $M^n$ denote a closed Riemannian manifold with nonpositive sectional curvature and let $\tilde{M}^n$ be the universal cover of $M^n$ with the lifted metric. Suppose that the universal cover $\tilde{M}^n$ contains no totally geodesic embedded Euclidean plane $\mathbb{R}^2$ (i.e., $M^n$ is a visibility manifold). Then Gromov’s simplicial volume $\|M^n\|$ is non-zero. Consequently, $M^n$ is non-collapsible while keeping Ricci curvature bounded from below. More precisely, if $\text{Ric}_{\text{g}} \geq -(n-1)$, then $\text{vol}(M^n, \text{g}) \geq \frac{1}{(n-1)!n!} \|M^n\| > 0$.

Among other things, we also outline a proof for the following direct consequence of Perelman’s recent work on 3-manifolds.

**Theorem B.** (Perelman) Let $M^3$ be a closed a-spherical 3-manifold ($K(\pi, 1)$-space) with the fundamental group $\Gamma$. Suppose that $\Gamma$ contains no subgroups isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. Then $M^3$ is diffeomorphic to a compact quotient of real hyperbolic space $\mathbb{H}^3$, i.e., $M^3 \equiv \mathbb{H}^3/\Gamma$. Consequently, $\text{MinVol}(M^3) \geq \frac{1}{24} \|M^3\| > 0$.

Minimal volume and simplicial norm of all other compact 3-manifolds without boundary and singular spaces will also be discussed.

**Key words:** Non-positive curvature, visibility spaces, minimal volume, simplicial norm, Gromov-hyperbolic groups, hyperbolization of 3-manifolds.

1. Introduction.

In 1993, Professor S. T. Yau proposed studying the Martin boundary, the Gromov-norm and other semi-hyperbolic properties of manifolds with Bammann rank one (cf. Problem #47 and #97 of [Y93]). A nonpositively curved
A manifold $M^n$ is called to be of Ballmann rank one, if there is a geodesic $\sigma : \mathbb{R} \rightarrow M^n$ which does not admit any non-zero orthogonal parallel Jacobi field (cf. [Ba82]). Various progress has been made in the study of rank-one manifolds, for example, see [BaL94], [BCG96], [Ca00], [CCR01], [CCR04], [CFL07] and [LS06]. Manifolds of Ballmann rank-one can be divided into two sub-classes: collapsible manifolds and non-collapsing manifolds. For collapsing manifolds of nonpositive sectional curvature, the first author, Jeff Cheeger and Xiaochun Rong indeed showed that if a closed Riemannian manifold with nonpositive sectional curvature $-1 \leq \sec_{M^n} \leq 0$ and if the volume is sufficiently small $\text{vol}(M^n) \leq \varepsilon_n$, then $M^n$ must be a generalized graph-manifold and hence minimal volume $\text{MinVol}(M^n)$ and Gromov-norm $\| [M^n] \|$ of $M^n$ both vanish”, (cf. [CCR01], [CCR04]). In this paper, we consider non-collapsible manifolds of Ballmann rank-one. In particular, we consider visibility manifolds in the sense of Eberlein and O’Neill. Our results of this paper will be complementing to the results of [CCR01-04] on collapsing manifolds with nonpositive curvature.

In 1973 Eberlein and O’Neill introduced the so called ”visibility manifolds”.

**Definition 1.1.** ([EO73]) Let $M^n$ denote a closed Riemannian manifold with non-positive sectional curvature and let $\tilde{M}^n$ be the universal cover of $M^n$ with the lifted metric. If the universal cover $\tilde{M}^n$ contains no totally geodesic embedded Euclidean plane $\mathbb{R}^2$, then $M^n$ is called a visibility manifold.

In this paper, we always assume that all spaces have dimensions $n \geq 2$. It has been conjectured by various authors that the following assertion might be true.

**Conjecture 1.2.** Any closed visibility manifold $M^n$ must admit a metric $g$ of negative sectional curvature $\sec_g \leq -1$.

There is partial evidence to support this conjecture. For example, the Preissman theorem states that if a closed manifold $M^n$ admits a metric of negative sectional curvature, then its fundamental group $\pi_1(M^n)$ has no subgroup isomorphic to $\mathbb{Z} \times \mathbb{Z}$, (cf. [Y71]). In fact, it follows from Yau’s thesis [Y71] that there are no $\mathbb{Z} \times \mathbb{Z}$ subgroups contained in the fundamental group of a visibility manifold.

In this paper, we will provide another piece of evidence for the conjecture. It is known that if a closed manifold $M^n$ admits a metric of negative sectional curvature, then $M^n$ is non-collapsible, (cf. [Gr82] and [IY82]).
shall also show that any closed visibility manifold is non-collapsible. In fact, Gromov introduced the notion of minimal volume by setting:

\[ \text{MinVol}(M^n) = \inf \{ \text{vol}_n(M^n, g) \mid -1 \leq \text{sec}_g \leq 1 \} \quad (1.1) \]

More precisely, we shall show that any closed visibility manifold has non-zero Gromov-norm, non-zero minimal volume and hence is non-collapsible.

In a seminal paper ([Gr82]), Gromov introduced the notion of simplicial volume of a closed oriented manifold. In the same paper, the question was raised which manifolds have non-zero simplicial volume. For a closed manifold of negative curvature, Inoue and Yano ([IY82]) showed that the simplicial volume must be nonzero. It is a natural question whether the simplicial volume of visibility manifolds is nonzero. In this paper, we show that it turns out to be true. More precisely, we prove the following theorem:

**Main Theorem.** Let \( M^n \) be a closed manifold of nonpositive curvature. Suppose that \( M^n \) is a visibility manifold. Then Gromov’s simplicial volume \( \|M^n\| \) is non-zero. Consequently, \( M^n \) is non-collapsible while keeping Ricci curvature bounded from below. More precisely, if \( \text{Ric}_g \geq -(n-1) \), then

\[ \text{vol}(M^n, g) \geq \frac{1}{(n-1)^n n!} \|M^n\| > 0. \quad (1.2) \]

We are very grateful to Igor Belegradek for bringing the work of Bridson [Bri95] and Yamaguchi [Yama97] on singular spaces to our attention. The definition of curvatures for possibly singular spaces can be found in many graduate textbooks, (e.g. [BuB101]).

**Main Corollary.** Let \( X \) be a simply-connected and complete metric space of nonpositive curvature. Suppose that there does not exist an isometric embedding of the Euclidean plane into \( X \); \( X \) has no boundary and suppose that \( X \) has a co-compact lattice \( \Gamma \) such that \( X/\Gamma \) has co-homological dimension \( n \) and \( H_n(X/\Gamma, \mathbb{Z}) \neq 0 \). Then the simplicial norm of \( X/\Gamma \) is positive. If, in addition, \( X \) has curvature \( \geq -1 \), then

\[ \text{vol}(X/\Gamma) \geq \frac{1}{(n-1)^n n!} \|X/\Gamma\| > 0. \quad (1.3) \]

For spaces of dimension 3, the work of Perelman provides a more refined result to support Conjecture 1.2 above as well.

**Theorem 1.3.** (due to Perelman [Per02-03], [CZ06], [KL06], [GT07-08]) Let \( M^3 \) be a closed aspherical 3-manifold (\( K(\pi, 1) \)-space) with the fundamental group
Suppose that $\Gamma$ contains no subgroups isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. Then $M^3$ is diffeomorphic to a compact quotient of real hyperbolic space $\mathbb{H}^3$, i.e., $M^3 \cong \mathbb{H}^3/\Gamma$. Consequently, any compact 3-dimensional visibility manifold $M^3$ admits a metric of constant negative sectional curvature.

Perelman posted two important preprints on Ricci flows on compact 3-manifolds with surgery online ([Per02], [Per03]), in order to solve Thurston’s conjecture on Geometrization of 3-dimensional manifolds. Thurston’s Geometrization Conjectures states that “for any closed, oriented and connected 3-manifold $M^3$, there is a decomposition $[M^3 - \bigcup \Sigma_j^2] = N_1^3 \cup N_2^3 \ldots \cup N_m^3$, such that each $N_j^3$ admits a locally homogeneous metric with possible incompressible boundaries $\Sigma_j^2$, where $\Sigma_j^2$ is homeomorphic to a quotient of a 2-sphere or a 2-torus”. There are exactly 8 homogeneous spaces in dimension 3. The list of 3-dimensional homogeneous spaces includes 8 geometries: $\mathbb{R}^3$, $\mathbb{H}^3$, $S^3$, $\mathbb{H}^2 \times \mathbb{R}$, $S^2 \times \mathbb{R}$, $SL(2, \mathbb{R})$, Nil and Sol. Several teams of outstanding mathematicians generously made their efforts to fill in the detailed proof of Perelman’s densely written arguments in [Per02] and [Per03]. Among them, we should mention important contributions in the clarification of Perelman’s work by Cao-Zhou [CZ06], Kleiner-Lott [KL06] and Morgan-Tian [MT07]-[MT08]. The Ricci flow with surgery part of Perelman’s work has been well-understood. For example, see Ben Chow and his co-authors’ four different books on Ricci flows (e.g. [CLN06]). The metric geometric part of Perelman’s work (Perelman’s collapsing theorem for 3-manifolds) has been studied by four teams, including Besson’s team [BBBMP07], Cao-Ge [CG09], Morgan-Tian [MT08] and Shioya-Yamaguchi [ShiY05]. The papers [BBBMP07] and [MT08] did not use Perelman’s stability theorem (cf. [Ka07]), while the paper of the first author and Ge [CG09] used the critical point theory of distance functions. Shioya and Yamaguchi used Perelman’s stability theorem in their papers [ShiY00] and [ShiY05] before the publication of Vitali Kapovitch’s proof [Ka07]. Over all, no major errors have been found in Perelman’s papers [Per02]-[Per03]. Perelman’s work is very solid and has become increasingly appreciated by the geometry-topology community. We will explain how Perelman’s recent work ([Per02] and [Per03]) on Geometrization of 3-manifolds implies the above Theorem 1.3 in Section 4 below. Gromov’s norm of other compact 3-manifolds will also be discussed in Section 4.

There are several ways to obtain the positivity of simplicial volume. Following a suggestion of Thurston, it suffices to get a uniform upper bound of the volumes of suitably defined straighted simplices. Gromov [Gr82] outlined, and Inoue-Yano [InY82] showed in detail that the simplicial volume
of closed negative curved manifolds is nonzero. The recent work of Jean-François Lafont and Benjamin Schmidt verified the positivity of simplicial volume of a closed locally symmetric space of non-compact type ([LS06]) by using a similar ideas. F. Ledrappier and S. Lim recently studied volume entropy of hyperbolic buildings, cf. [LeL08].

Our proof takes a different approach. We will use Gromov’s bounded cohomology method, Gromov-hyperbolic group theory [Gr87] and a theorem of Miniyev [M01] to verify our Theorem. According to Gromov, to verify that the simplicial volume of $M^n$ is positive, it suffices to show that $M^n$ has nonzero $n$-dimensional bounded cohomology. A theorem of Igor Miniyev asserts that, if the fundamental group $\pi_1(M^n)$ of $M^n$ is Gromov-hyperbolic and if $M^n$ is a closed oriented aspherical manifold (a $K(\pi, 1)$-space), then $M^n$ has nonzero $n$-dimensional bounded cohomology. Finally, we will appeal to an earlier result of the first author, which states that, for a closed manifold $M^n$ with nonpositive sectional curvature, $M^n$ is a visibility manifold if and only if its fundamental group $\pi_1(M^n)$ is Gromov-hyperbolic, cf [Ca95] and [Ca00]. This completes the outline of proof of our main theorem.

The organization of this short article goes as follows. In Section 2, we review basic properties of simplicial volume and bounded cohomology needed for our proof. In Section 3, we discuss the relations among visibility manifolds and Gromov-hyperbolic groups. The proof of the Main Theorem will be completed in Section 4.

2. Simplicial volume and bounded cohomology.

We begin with the definition of simplicial volume.

**Definition 2.1.** Let $M^n$ be a $n$-dimensional closed manifold, $C^0(\Delta^k, M^n)$ be the set of singular $k$-simplices and $C_k(M^n, \mathbb{R})$ be the set of singular $k$-chains with real coefficient. For any $c = \sum_{i=1}^{j} r_i f_i$ with each $r_i \in \mathbb{R}$ and $f_i \in C^0(\Delta^k, M^n)$ be a singular real chain, the $l^1$-norm of $c$ is defined by $\|c\|_1 = \sum_{i=1}^{j} |r_i|$. The $l^1$-norm of a real singular homology class $[\alpha]$ is defined by

$$\|[\alpha]\|_1 = \inf \{\|c\|_1 : \partial(c) = 0, [c] = [\alpha]\}. \quad (2.1)$$
To study the volume collapsing property of a given manifold, we concentrate on the norm of the top-dimensional cohomology class.

**Definition 2.2.** (Gromov’s simplicial norm) Let \( M^n \) be an oriented closed connected \( n \)-dimensional manifold. The simplicial volume of \( M^n \) is defined as 
\[
\| M^n \| = \| i([M^n]) \|_1,
\]
where \( i: H_n(M, \mathbb{Z}) \rightarrow H_n(M, \mathbb{R}) \) is the change of coefficient homomorphism, and \([M^n]\) is the fundamental class arising from the orientation of \( M^n \).

The following fundamental theorem of Gromov indicates that there are deep relations between minimal volume (defined by equation (1.1) above) and Gromov’s simplicial norm.

**Theorem 2.3.** (Gromov’s estimate [Gr82, page 220]) Let \((M^n, g)\) be a complete \( n \)-dimensional Riemannian manifold with Ricci curvature bounded from below by 
\[
\text{Ric}_g \geq -(n-1).
\]
Then the volume of \((M^n, g)\) is bounded below by 
\[
\frac{1}{(n-1)^n n!} \| M^n \|.
\]
Consequently, one has 
\[
\text{MinVol}(M^n) \geq \frac{1}{(n-1)^n n!} \| M^n \|. \tag{2.2}
\]

Erwann Aubry further shown a similar estimate holds if the Ricci curvature of \( M^n \) is \( L^p \) bounded from below by \(-(n-1)\). We state his estimate as follows:

**Theorem 2.4.** (Erwann Aubry’s estimate [Au08]) Let \( M^n \) be a complete \( n \)-dimensional Riemannian manifold. If for any \( \epsilon > 0 \), \( \exists \) a constant \( C(p, n, D, \epsilon) > 0 \) such that \((M^n, g)\) satisfies 
\[
D^2 \| \rho \|_p \leq C(p, n, D, \epsilon),
\]
then
\[
\text{vol}(M^n, g) \geq \frac{1}{(n-1)^n n!} \| M^n \|. \tag{2.3}
\]

Here \( D \) denotes the diameter of \( M^n \) and \( p > \frac{n}{2}, \| \rho \|_p = \left( \frac{1}{\text{Vol}(M^n)} \int_M \rho^p \right)^\frac{1}{p}, \rho = (\hat{\text{Ric}} + (n-1))_-, \hat{\text{Ric}}(x) \) is the least eigenvalue of the Ricci tensor \( \text{Ric} \) at \( x \) and \( f_- = \max(-f, 0) \).

When \( M^n \) is diffeomorphic to a locally symmetric space of negative sectional curvature, Besson-Courtois-Gallot improved (2.2) by considering metrics of scalar curvature bounded from below. We thank Professor D. Kotschick [Ko04] for bringing [BCG96] to our attention again.

**Theorem 2.5.** (Besson-Courtois-Gallot [BCG96]) Let \( M^n \) be diffeomorphic to a compact locally symmetric space of negative sectional curvature and \( g_{can} \) be the
canonical metric of scalar curvature \(-n(n-1)\). Suppose that \(g\) is a smooth metric of scalar curvature \(\geq -n(n-1)\) on \(M^n\). Then

\[
Vol(M^n, g) \geq Vol(M^n, g_{can}) = c_n \|M^n\| > 0.
\]

(2.4)

We need to borrow another brilliant idea of Gromov to calculate the simplicial norm via the dual-space method. It is clear that the \(\ell^\infty\)-space is dual to the \(\ell^1\)-space. Gromov introduced the so-called bounded cohomology as follows.

**Definition 2.6.** Let \(C^k(M^n, \mathbb{R})\) be the set of singular \(k\)-cochains with real coefficient. For any \(c \in C^k(M^n, \mathbb{R})\), set the \(\ell^\infty\) norm of \(c\) by \(|c|_\infty = \sup \{|c(\sigma)| : \sigma \in C_k(M^n, \mathbb{R})\}\). The set of bounded \(k\)-cochains with real coefficient \(C^k_b(M^n, \mathbb{R})\) is defined by

\[
C^k_b(M^n, \mathbb{R}) = \{c : c \in C^k(M^n, \mathbb{R}), |c|_\infty < \infty\}. \tag{2.5}
\]

Assume that \(\delta\) is the co-chain operator from \(C^k(M, \mathbb{R})\) to \(C^{k+1}(M, \mathbb{R})\). It can be easily checked that \(\delta(C^k_b(M, \mathbb{R})) \subseteq C^{k+1}_b(M, \mathbb{R})\) and we define the \(k\)-th bounded cohomology group of \(M^n\) as

\[
H^k_b(M, \mathbb{R}) = \text{Ker}(\delta(C^k_b(M, \mathbb{R}))) / \text{Im}(\delta(C^{k-1}_b(M, \mathbb{R}))). \tag{2.6}
\]

**Remark 2.7.** If \(G\) is an arbitrary group, the bounded cohomology group of \(G\) can be defined as the corresponding bounded cohomology group of the Eilenberg-Maclane space \(K(G, 1)\). For a further discussion of the bounded cohomology theory of groups, see [Iv87] and [Bro94].

As shown by Gromov, the theory of bounded cohomology sheds new light on the notion of simplicial volume. In fact, it gives the cohomological definition of simplicial volume. More precisely, we have the following proposition.

**Proposition 2.8.** (Gromov [Gr82]) Let \(M^n\) be a closed oriented \(n\)-dimensional manifold. Then \(\|M\|^{-1} = \inf \{\|\beta\|_\infty : \beta \in H^k_b(M, \mathbb{R}), [\beta, [M]] = 1\}\). Consequently, \(\|M\|\) is nonzero if and only if there exists a bounded \(\beta \in H^k_b(M, \mathbb{R})\) which does not vanish on \([M]\).

For the proof of the above proposition, see [Gr82].

In the next section, we set the stage to show that “if \(M^n\) is a closed visibility manifold then \(\pi_1(M^n)\) must be Gromov-hyperbolic and hence \(H^n_b(M^n, \mathbb{R}) \neq 0\).”
3. Visibility manifolds and Gromov-hyperbolic spaces.

There are at least three equivalent definitions of visibility manifolds. We will also use the following one of equivalent definitions for visibility manifolds.

**Definition 3.1.** ([EO73])

1. A simply-connected manifold $\tilde{M}^n$ of nonpositive curvature is said to be a visibility manifold if for each point $p \in \tilde{M}^n$ and $\varepsilon > 0$, there exists a constant $R(p, \varepsilon) > 0$ such that if $\sigma : [a, b] \to \tilde{M}^n$ is a geodesic segment satisfying the condition $d(p, \sigma) \geq R(p, \varepsilon)$, then $\angle_p(\sigma(a), \sigma(b)) \leq \varepsilon$, where $\angle_p(\sigma(a), \sigma(b))$ denotes the angle based at $p$ between $\sigma(a)$ and $\sigma(b)$. $\tilde{M}^n$ is called a “uniform visibility manifold” if the constant $R(p, \varepsilon)$ may be chosen to be independent of $p$.

2. A closed manifold $M^n$ of nonpositive curvature is said to be a visibility (uniform visibility) manifold if its universal covering does.

Examples of uniform visibility manifolds include all complete Riemannian manifolds with sectional curvature $\leq -a^2 < 0$.

**Example 3.2. (Negative curvature implies visibility).** A simply-connected and complete Riemannian manifold $\tilde{M}^n$ with strictly negative sectional curvature $\sec \tilde{M}^n \leq -1$ is a uniform visibility manifold. Since $\tilde{M}^n$ does not contain any totally geodesic flat $\mathbb{R}^2$, by Definition 1.1 in Section 1, we see that $\tilde{M}^n$ is a visibility manifold. By an explicit calculation below, we can further show that $\tilde{M}^n$ is a uniform visibility manifold described in Definition 3.1. In fact, when $\sec \tilde{M}^n \leq -1$, we can choose that $R(p, \varepsilon) = R(\varepsilon) = |\cosh^{-1}(\frac{\pi}{\varepsilon} + 1)|$. (3.1)

To see this, we use two elementary facts. For $p \in \tilde{M}^n$ and a geodesic segment $\sigma : [a, b] \to M^n$, we let $\Delta_{p, \sigma}$ be the cone with the apex $p$ over $\sigma([a, b])$. It is easy to check that $\Delta_{p, \sigma}$ is a ruled sub-manifold of $M^n$. Let $\hat{g}_\Delta$ be the induced metric on the solid triangle $\Delta_{p, \sigma}$. There is a theorem of Gauss on the relation between intrinsic curvature and extrinsic curvature. Let $II_\Delta$ be the second fundamental form of $\Delta$ in $\tilde{M}^n$. Suppose that $X$ is the unit ruled direction of $\Delta$ and $Y$ is its orthonormal complement of $X$ in $T_x(\Delta)$. Then a calculation shows that

$$\sec(\Delta, \hat{g}) = \sec_{\tilde{M}^n} + \langle II_\Delta(X, X), II_\Delta(Y, Y) \rangle - \|\langle II_\Delta(X, Y)\| \leq -1, \quad (3.2)$$

for all $x \in \Delta$. 

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Using (3.2) and applying Gauss-Bonnet formula to the geodesic triangle $(\Delta, \hat{g})$, we have the area estimate of $(\Delta, \hat{g})$:

$$\text{Area}(\Delta, \hat{g}) \leq \pi. \quad (3.3)$$

Let $B(p, r)$ be the metric ball in $\tilde{M}^n$ centered at $p$ with radius $r$, and let $\varepsilon = \angle_p(\sigma(a), \sigma(b))$ be the visibility angle of $\sigma([a, b])$ viewed from $p$. Let $R$ be the largest radius of inscribed fan in $\Delta$ given by

$$R = \inf_{a \leq t \leq b} \{\|\text{Exp}_p^{-1}(\sigma(t))\|\}, \quad (3.4)$$

where $\text{Exp}_p$ is the exponential map of $\tilde{M}^n$ at $p$.

Using area comparison theorem for $\sec \leq -1$, we have

$$[\cosh R - 1] \varepsilon \leq \text{Area}[B(p, R) \cap \Delta_{p, \sigma}] \leq \pi. \quad (3.5)$$

The formula (3.1) now follows from (3.4)-(3.5). Hence, when $\tilde{M}^n$ has strictly negative sectional curvature $\sec_{\tilde{M}^n} \leq -1$, such a manifold $\tilde{M}^n$ must be a uniform visibility manifold with the visibility function

$$R(p, \varepsilon) = R(\varepsilon) = |\cosh^{-1}(\frac{\pi}{\varepsilon} + 1)|.$$

This completes the proof of the assertion that “strictly negative curvature implies uniform visibility”.

It is a very interesting fact that a uniform visibility manifold turns out to be a Gromov hyperbolic space, which is very important in the study of large scale geometry. Roughly speaking, Gromov hyperbolic space characterizes coarse features of hyperbolic plane, and can be extended to the category of metric space. We need to recall some basic properties of Gromov-hyperbolic theory, which are needed in our paper. For a detailed exposition of Gromov-hyperbolic groups and Gromov-hyperbolic spaces, see [Sho91].

**Definition 3.3.** Let $(X, d)$ be a metric space with underlying set $X$ and metric $d$, and let $x, y \in X, \gamma : [0, 1] \to X$ be a curve in $X$ connecting $x$ and $y$, define the length of $\gamma$ by

$$\ell(\gamma) = \sup\left\{\sum_{i=0}^{n} d(x_i, x_{i+1}) : x = x_0 < x_1 < \cdots < x_n = y\right\}.$$

The length distance between $x$ and $y$ is defined by

$$d_\ell(x, y) = \inf\left\{\ell(\gamma) : \gamma(0) = x, \gamma(1) = y\right\}.$$
We say \((X, d)\) is a geodesic metric space if for any two points \(x, y \in X\), there exists a curve \(\gamma\) connecting \(x\) and \(y\) such that \(\ell(\gamma) = d(x, y)\). In this case, we say \(\gamma\) is a geodesic connecting \(x\) and \(y\).

It is clear that a \(n\)-dimensional complete Riemannian manifold \(M^n\) is always a geodesic metric space with respect to the metric induced by the Riemann metric on \(M^n\).

Let \((X, d)\) be a geodesic metric space. By a geodesic triangle \(\sigma\) in \(X\), we mean a collection of three points \(x, y, z\) connected by three geodesics \(\sigma_1, \sigma_2, \sigma_3\) and we call \(\sigma_i (\ i = 1, 2, 3)\) are the sides of geodesic triangle \(\sigma\).

There are several equivalent definitions of Gromov-hyperbolic spaces and Gromov-hyperbolic groups. We will use the following definition in terms of thin-triangles.

**Definition 3.4.** (Gromov-hyperbolic spaces and groups [Sho91])

1. Let \((X, d)\) be a geodesic metric space, \((X, d)\) is said to be a Gromov-hyperbolic space if all geodesic triangles in \((X, d)\) are \(\delta\)-thin. More precisely, there is \(\delta > 0\) such that each side of any given geodesic triangle in \((X, d)\) is contained in the \(\delta\)-neighborhood of the union of the two other sides.

2. A finitely generated group \(\Gamma\) is said to be Gromov-hyperbolic if a Cayley graph \(G\Gamma\) of \(\Gamma\) is Gromov-hyperbolic as a geodesic metric space.

The relations between a Gromov-hyperbolic space \(\tilde{M}^n\) and its co-compact lattice group \(\Gamma\) can be seen by the following proposition.

**Proposition 3.5.** ([Sho91]) Gromov-hyperbolicity is preserved under quasi-isometries. More precisely, suppose that there exists map \(f : (X, d) \to (Y, d_1)\) such that \(\frac{1}{\lambda}d(x, y) - C \leq d_1(f(x), f(y)) \leq \lambda d(x, y) + C\) for some constant \(C, \lambda > 0\) and \(f(X)\) is \(C\)-dense in \(Y\). Then \((X, d)\) is Gromov-hyperbolic if and only if \((Y, d_1)\) is Gromov-hyperbolic.

Suppose that \(M^n\) is a compact Riemannian manifold with its fundamental group \(\Gamma\). Then the universal cover \(\tilde{M}^n\) of \(M^n\) must be quasi-isometric to any Cayley-graph \(G\Gamma\) of the fundamental group \(\Gamma\).

**Proposition 3.6.** Let \(M^n\) be a compact Riemannian manifold without boundary. Suppose that \(\tilde{M}^n\) is the universal cover of \(M^n\) with the lifted metric and \(\Gamma = \pi_1(M^n)\) is the fundamental group of \(M^n\). Then \(\Gamma\) is Gromov-hyperbolic if and only if \(\tilde{M}^n\) is a Gromov-hyperbolic metric space.
Under the assumption of nonpositive sectional curvature, our following result shows that Gromov-hyperbolicity condition is equivalent to the visibility condition.

**Proposition 3.7.** ([Ca95], [Ca00]) Let $M^n$ be a closed Riemann manifold with nonpositive curvature. Then $M^n$ is a uniform visibility manifold if and only if its fundamental group $\pi_1(M^n)$ is Gromov-hyperbolic.

**Proof.** For the convenience of readers, we include a short proof here.

**Step 1. To verify that Gromov-hyperbolicity implies visibility.**

Suppose that $\tilde{M}^n$ is Gromov-hyperbolic. By Definition 1.1, it is sufficient to verify that “the universal cover $\tilde{M}^n$ contains no totally geodesic embedded Euclidean plane $\mathbb{R}^2$”. Suppose contrary, $\tilde{M}^n$ had a totally geodesic embedded Euclidean plane $\Sigma^2 = \mathbb{R}^2$. We could consider equilateral geodesic triangles $\Delta_\ell$ of side length $\ell$ in $\Sigma^2$. As $\ell \to \infty$, the family of triangles $\{\Delta_\ell\}$ can not be $\delta$-thin for $\ell > 4\delta$. Thus, the geodesic triangles in $\tilde{M}^n$ can not be uniformly $\delta$-thin for any finite number $\delta$. Hence, $\tilde{M}^n$ can not be a Gromov-hyperbolic space, a contradiction.

**Step 2. To verify that uniform visibility implies Gromov-hyperbolicity.**

We now use an equivalent definition for our uniform visibility manifold $M^n$. Let us choose $\delta = R(p, \frac{\pi}{2}) = R(\frac{\pi}{2})$ given by Definition 3.1 for the uniform visibility manifold $\tilde{M}^n$. Suppose that $\Delta_{ABC}$ is a geodesic triangles in $\tilde{M}^n$ with three vertices $\{A, B, C\}$ and three sides $\{\sigma_a, \sigma_b, \sigma_c\}$. We are going to show that our geodesic triangle $\Delta_{ABC}$ is indeed $\delta$-thin with $\delta = R(\frac{\pi}{2})$. To see this, we may assume that the side $\sigma_a$ of $\Delta$ has two endpoints $B$ and $C$. For any interior point point $x \in \sigma_a$ of the side $\sigma_a$, we draw a new geodesic segment from $x$ to $A$, say $\sigma_{xA}$. It is clear that

$$\angle_x(B, A) + \angle_x(A, C) \geq \pi. \quad (3.6)$$

Thus, we have

$$\max\{\angle_x(B, A), \angle_x(A, C)\} \geq \frac{\pi}{2}. \quad (3.7)$$

Let $\sigma_{pq} : [0, d(p, q)] \to \tilde{M}^n$ be the unique geodesic segment from $p$ to $q$ of unit speed. By (3.7), we may assume that

$$\angle_x(B, A) \geq \frac{\pi}{2} \quad (3.8)$$

after re-indexing if needed. We now apply Definition 3.1 to the new geodesic triangle $\Delta_{xAB}$. By (3.8) and our assumption on the uniform visibility manifold $\tilde{M}^n$, we see that

$$d(x, \sigma_{AB}) \leq R(\frac{\pi}{2}). \quad (3.9)$$
Since $\sigma_c = \sigma_{AB}$, we have verified that “any point $x$ on one side $\sigma_a$ of the geodesic triangle $\Delta$ is contained in the $\delta$-neighborhood of the union of the two other sides $\{\sigma_c, \sigma_b\}$ for some $\delta = R\left(\frac{\pi}{2}\right) > 0$. Therefore, any geodesic triangle $\Delta$ in $\hat{M}^n$ is $R\left(\frac{\pi}{2}\right)$-thin. It follows from Definition 3.3 that $\hat{M}^n$ is Gromov-hyperbolic.

This completes the proof of Proposition 3.7. □

The definition of singular spaces with curvature bounded from above $curv \leq k$ can be found in many textbooks, e.g. [BuBI01] Chapter 9. If a complete metric space $X$ has curvature bounded from above $curv \leq k$, then $X$ is called a $CAT(k)$-space.

**Proposition 3.8.** ([Bri95]) Let $X$ be a simply-connected and complete metric space of non-positive curvature. Suppose that there does not exist an isometric embedding of the Euclidean plane into $X$; $X$ has no boundary and suppose that $X$ has a co-compact lattice $\Gamma$ such that $X/\Gamma$ has co-homological dimension $n$ and $H_n(X/\Gamma, \mathbb{Z}) \neq 0$. Then $\Gamma$ is Gromov-hyperbolic.

**Proof.** Proposition 3.7 was extended to possibly singular spaces by Martin Bridson [Bri95]. In fact, the proof of Proposition 3.7 is applicable to singular spaces as well. □

We will discuss the Gromov-norm and minimal volume of visibility manifolds in next section.

**4. Gromov’s simplicial norm of visibility $n$-manifolds and compact 3-manifolds.**

Our proof of main theorem and theorem 1.3 relies on a theorem of Igor Mineyev:

**Theorem 4.1.** (Mineyev [M01]) If $G$ is a hyperbolic group, then the map $H^n_b(G, \mathbb{R}) \to H^n(G, \mathbb{R})$, induced by inclusion, must be surjective for $n \geq 2$.

Now we are in position to prove our Main Theorem and Theorem 1.3.

**Proof of Main Theorem:**

Suppose that $M^n$ is a closed visibility manifold. It follows from Proposition 3.7 that the fundamental group $\pi_1(M^n)$ of $M^n$ is hyperbolic. It follows
from Cartan-Hadamard theorem ([BGS85]) that $M^n$ must be a closed aspherical manifold ($(K(\pi,1)$-space). For any compact aspherical manifold $M^n$ without boundary, its cohomology ring $H^*(M^n, \mathbb{Z})$ is uniquely determined by its fundamental group $\Gamma = \pi_1(M^n)$, see Brown’s book [Bro94]. In particular, we have

$$H^*(M^n, \mathbb{Z}) = H^*(\Gamma, \mathbb{Z}),$$

(4.1)

where $H^*(\Gamma, \mathbb{Z})$ is the co-homology of the group $\Gamma$.

For Applying Theorem 4.1 to $\Gamma = \pi_1(M^n)$, we obtain $H^n_b(\Gamma, \mathbb{R}) \to H^n(\Gamma, \mathbb{R})$ is surjective. However, $H^n(\Gamma, \mathbb{R}) \simeq H^n(M^n, \mathbb{R}) \simeq \mathbb{R}$. It follows that $H^n_b(\Gamma, \mathbb{R})$ must be nonzero. Thus $H^n_b(M^n, \mathbb{R})$ is nonzero. By Gromov’s observation (cf. Proposition 2.8 above), we see that the simplicial volume $\|M^n\|$ must be non-zero. Main Theorem now follows from Gromov’s estimate described in Theorem 2.3 above.

**Proof of Main Corollary:** The assertion that the simplicial norm $\|X/\Gamma\| > 0$ follows from Proposition 3.8 and Theorem 4.1. For smooth Riemannian manifolds with Ricci curvature $\geq -(n-1)$, Gromov derived inequality (1.2). For singular spaces, Yamaguchi [Yama97] consider the case of curvature $\geq -1$. He further proved that the inequality (1.3) holds.

We now turn our attention to compact 3-manifolds. In the 3-dimensional case, we can compute the Gromov-simplicial norm for any compact 3-manifolds due to Perelman’s work on Thurston’s Geometrization Conjecture. Thurston’s proposed decomposition of a 3-manifold has the property that each summand $N_i^3$ is a locally homogeneous space with possible incompressible surface boundary. The definition of incompressible surfaces can be recalled as following.

**Definition 4.2.** (1) An embedded two-dimensional sphere $S^2$ in $M^3$ is said to be incompressible if $S^2$ does not bound a 3-dimensional ball $B^3$ in $M^3$.

(2) An embedded two-dimensional torus $T^2$ in $M^3$ is said to be incompressible if the inclusion map induces an injective homomorphism from the fundamental group of $T^2$ to the fundamental group of $M^3$, i.e., the homomorphism $i_* : \pi_1(T^2) \to \pi_1(M^n)$ is injective.

There are exactly 8 homogeneous spaces in dimension 3. The list of 3-dimensional homogeneous spaces includes 8 geometries: $\mathbb{R}^3, \mathbb{H}^3, S^3, \mathbb{H}^2 \times \mathbb{R}, S^2 \times \mathbb{R}, \bar{SL}(2, \mathbb{R}), Nil$ and $Sol$.  

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Theorem 4.3. (Perelman [Per02], [Per03]) Let $M^3$ be a closed, oriented and connected 3-manifold. There is an embedding of a disjoint union $\bigcup \Sigma_j^2$ of incompressible surfaces $\{S^2, \mathbb{RP}^2, T^2, T^2/\mathbb{Z}_2\}$ such that each component of the decomposition $[M^3 - \bigcup \Sigma_j^2]$ admits a locally homogeneous Riemannian metric of finite volume.

It is known that the compact quotient $H^3/\Gamma$ of 3-dimensional real hyperbolic space $H^3$ has non-zero Gromov simplicial norm $\|H^3/\Gamma\| > 0$, see [Gr82]. Any compact quotient of the remaining seven 3-dimensional homogeneous spaces $\{\mathbb{R}^3, S^3, \mathbb{H}^2 \times \mathbb{R}, S^2 \times \mathbb{R}, SL(2, \mathbb{R}), Nil, Sol\}$ is collapsible while keeping sectional curvature bounded from below. In fact, collapsible 3-manifolds are related to Seifert fiber space and graph-manifolds.

Definition 4.4. (Seifert fibered spaces and graph-manifolds) (1) A Seifert fiberation structure on a compact 3-manifold $M^3$ is a locally-free circle action on a finite normal covering $\hat{M}^3$ of $M^3$ such that, denote the covering transformation on $\hat{M}^3$ by $\tau$, we have $\tau(x, \zeta) = \bar{\zeta} \cdot x$ for all $x \in \hat{M}^3$ and $\zeta \in S^1$.

(2) A graph-manifold is a compact 3-manifold that is connected sum of manifolds each of which is either diffeomorphic to a solid torus or can be cut apart along a finite collection of incompressible tori into Seifert fibered 3-manifolds.

The celebrated Cheeger-Gromov collapsing theory implies that each graph-manifold $M^3$ admits a polarized $F$-structure, cf. [ChG86] and [ChG90]. Hence, for any graph-manifold $M^3$, both minimal volume and simplicial norm of $M^3$ vanish, i.e.,

$$\text{MinVol}(M^3) = \|M^3\| = 0. \quad (4.2)$$

Combining Perelman’s work (Theorem 4.3) and Cheeger-Gromov’s collapsing theory, we have

Corollary 4.5. (Perelman [Per02], [Per03]) Let $M^3$ be an oriented connected compact 3-manifold with a Perelman-Thurston the decomposition given by Theorem 4.3. The simplicial norm of $M^3$ is non-zero if and only if at least one of its components of $[M^3 - \bigcup \Sigma_j^2]$ has hyperbolic geometry.

Proof. A theorem of Gromov [Gr82] implies that $\|M^3_1 \# \bigcup \Sigma_j^2 M^3_2\| = \|M^3_1\| + \|M^3_2\|$, where $\Sigma_j^2$ are homeomorphic to quotients of $S^2$ or $T^2$. It follows that if $[M^3 - \bigcup \Sigma_j^2] = \tilde{N}_1^3 \cup \tilde{N}_2^3 \cup ... \cup \tilde{N}_m^3$, then

$$2\|M^3\| = \|\tilde{N}_1\| + ... + \|\tilde{N}_m\|, \quad (4.3)$$
where $\hat{N}_j^3$ is a closed manifold obtained by gluing two copies of $N_j^3$ along their boundaries.

If $N_j^3$ is a compact graph-manifold, then $\|\hat{N}_j^3\| = 0$ by a theorem of Cheeger-Gromov [ChG86-90]. If $\hat{N}_j^3$ is a compact quotient of the remaining seven geometries $\{\mathbb{R}^3, S^3, \mathbb{H}^2 \times \mathbb{R}, S^2 \times \mathbb{R}, SL(2, \mathbb{R}), Nil, Sol\}$, then $\hat{N}_j^3$ is a graph-manifold. Thus, $\|\hat{N}_j^3\| \neq 0$ if and only if $N_j^3 = \mathbb{H}^3/\Gamma_j$ is diffeomorphic a quotient of real hyperbolic space $\mathbb{H}^3$ with finite volume.

We enclose this paper by the proof of Theorem 1.3.

**Proof of Theorem 1.3 due to Perelman:**

Recall that our 3-manifold $M^3$ has a decomposition: $[M^3 - \bigcup \Sigma_j^2] = N_1^3 \cup N_2^3 \ldots \cup N_m^3$, where $\{\Sigma_j^2\}$ are either incompressible 2-spheres $S^2$ or incompressible 2-tori $T^2$. Since $M^3$ is aspherical, there is no incompressible 2-spheres in $M^3$. By our assumption, the fundamental group $\Gamma$ of $M^m$ contains no $\mathbb{Z} \oplus \mathbb{Z}$. Thus, there is no incompressible 2-tori in $M^3$ either. Thus, $M^3$ is a locally homogeneous space. Because $M^3$ is aspherical, $M^3$ can not be covered by $S^3$ or $S^2 \times S^1$. It is known that $Sol$ has no co-compact lattice. If $M^3$ is a compact quotient of $\mathbb{H}^2 \times \mathbb{R}$, then, by a theorem of Eberlein [Eb83], a finite normal cover of $M^3$ is diffeomorphic to $\Sigma^2 \times S^1$, where $\Sigma$ is a closed surface of genus $\geq 1$. In summary, we have the following fact.

**Fact 4.6.** If $M^3$ is a compact quotient of $\{\mathbb{R}^3, \mathbb{H}^2 \times \mathbb{R}, SL(2, \mathbb{R}), Nil\}$, then the fundamental group $\Gamma$ of $M^3$ has a subgroup $\hat{\Gamma}$ of finite index such that $\hat{\Gamma}$ has a nontrivial center $C$ containing $\mathbb{Z}$.

The verification of Fact 4.6 goes as follows. (i) For $\mathbb{R}^3$, the Biebarbach theorem (cf. [BaGS85]) states that any co-compact lattice $\hat{\Gamma}$ contains a subgroup $\hat{\Gamma}$ isomorphic to $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. (ii) We already discussed the compact quotient of $\mathbb{H}^2 \times \mathbb{R}$ by using Eberlein’s result. (iii) It is known that $SL(2, \mathbb{R})$ is isometric to the unit tangent bundle $S\mathbb{H}^2$ of the Poincare disk. There is a fibration $S^1 \rightarrow S\mathbb{H}^2 \rightarrow \mathbb{H}^2$. The fundamental subgroup corresponding to the $S^1$ is the center of $\pi_1(S\mathbb{H}^2/\Gamma)$. For the nilpotent group $Nil$, it is well-known that the co-compact group $\pi_1(Nil/\Gamma)$ has a subgroup $\hat{\Gamma}$ of finite index such that $\hat{\Gamma}$ has a nontrivial center isomorphic to $\mathbb{Z}$.

Since the cohomological dimension of $M^3$ is equal to 3 (cf. [Bro94]), its fundamental group $\Gamma$ can not be isomorphic to $\mathbb{Z}$. Thus, by Fact 4.6, we see that if $M^3$ is a compact quotient of $\{\mathbb{R}^3, \mathbb{H}^2 \times \mathbb{R}, SL(2, \mathbb{R}), Nil\}$, then its fundamental group $\hat{\Gamma}$ contains a subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, which contradicts to our assumption.
Therefore, by our assumptions on $M^3$, there is only one possibility: $M^3$ is diffeomorphic to a compact quotient of $\mathbb{H}^3$. Thus, $\text{MinVol}(M^3) \geq \frac{1}{24} \|M^3\| > 0$.

Recall that, by a theorem of Yau [Y71], the fundamental group of a visibility manifold contains no subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. The above argument shows that $M^3$ is diffeomorphic to a compact quotient $\mathbb{H}^3/\Gamma$ of $\mathbb{H}^3$.

This completes the proof of Theorem 1.3. □

**Corollary 4.7.** Let $M^3$ be a closed 3-dimensional aspherical manifold. Suppose that the fundamental group $\pi_1(M^3)$ contains no sub-group isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, suppose that $g$ is a smooth metric of scalar curvature $\geq -6$. Then $M^3$ is diffeomorphic to a compact quotient of $\mathbb{H}^3$ and the volume of $(M^3, g)$ has a lower bound:

$$\text{Vol}(M^3, g) \geq \text{Vol}(\mathbb{H}^3/\Gamma) = c_3 \|\mathbb{H}^3/\Gamma\| > 0.$$  

Corollary 4.7 is a direct consequence of Theorem 1.3 and Theorem 2.5.

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