Revisiting novel symmetries in coupled $\mathcal{N} = 2$ supersymmetric quantum systems: examples and supervariable approach

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Abstract

We revisit the novel symmetries in $\mathcal{N} = 2$ supersymmetric quantum mechanical models by considering specific examples of coupled systems. Further, we extend our analysis to a general case and list out all the novel symmetries. In each case, we show the existence of two sets of discrete symmetries that correspond to the Hodge duality operator of differential geometry. Thus, we are able to provide a proof of the conjecture which points out the existence of more than one set of discrete symmetry transformations corresponding to the Hodge duality operator. Moreover, we derive on-shell nilpotent symmetries for a generalized superpotential within the framework of supervariable approach.

Keywords: $\mathcal{N} = 2$ SUSY quantum mechanical system, continuous and discrete symmetries, de Rham cohomological operators, (anti-)chiral supervariable approach, SUSY invariant restrictions

1. Introduction

Particle interactions in nature have always been a question of immense scientific interest. The quest to unify the four fundamental interactions, namely; gravitational, electromagnetic, strong and weak has drawn significant attention over the past century. However, the consistent unification of gravity with strong and electroweak interactions is still a profound task of interest [1–3]. This led to the birth of a popular principle known as supersymmetry [4]. It is one of the most beautiful principles which has a rich background support from physics as well as mathematics. It unifies two different classes of particles—fermions and bosons [5–7]—with mathematical substantiality encompassed within a graded Lie algebra which uses anti-commutation relations instead of commutation relations, thereby enabling the unification of fundamental interactions [8, 9].

The concept of supersymmetry has various applications beyond grand unification. In non-relativistic quantum mechanics, supersymmetry has paved the way for an elegant factorization technique of the Hamiltonian. This allows the calculation of energy spectrum of a given Hamiltonian without resorting to more complicated methods of solving the corresponding Schrödinger equation (see [4] for details).

The supersymmetric (SUSY) harmonic oscillator has been regarded as one of the simplest examples as far as $\mathcal{N} = 2$ SUSY quantum mechanical (QM) models are concerned (see e.g. [10–13]). In the recent past, the correspondence among the continuous symmetries of SUSY harmonic oscillator with the de Rham cohomological operators of differential geometry has been established at the algebraic level [14, 15] and it has been shown that the discrete symmetries correspond to the Hodge duality operator of differential geometry [16–18]. Further a detailed note about the existence of discrete symmetries in any general $\mathcal{N} = 2$ SUSY QM model has also been provided and it has been shown that they provide an example for Hodge theory [19]. It is worthwhile to mention that a similar kind of one-to-one correspondence among the symmetry transformations and de Rham cohomological operators has already been shown to exist in some of the field theoretic models within the framework of Becchi–Rouet–Stora–Tyutin (BRST) formalism [20–23].
In the BRST formalism, the Bonora–Tonin (BT) superfield approach [24, 25] enables us to derive off-shell nilpotent and absolutely anticommuting (anti-)BRST symmetries for a given p-form gauge theory (see, e.g., [26–28]). The BT formalism has been extended in the regime of SUSY theories (known as supervariable formalism) to derive the nilpotent symmetries [29–31]. In the supervariable approach, the off-shell nilpotent symmetries have been procured by making use of (anti-)chiral supervariables and SUSY invariant restrictions (SUSYIRs).

In the present endeavor, our prime motive is to extend the study of [14] to a general SUSY system. For this, first we superimpose the harmonic oscillator superpotential with other types of superpotentials (such as free particle and ‘Coulomb-type’) and explore the continuous and discrete symmetries of these systems. We then extend our analysis to a generalized form of superpotential from where any superposition of ‘polynomial-type’ superpotentials can be retrieved. Another key motive of the present work is to provide a proof for the conjecture (see [19] for details) which states that there might exist more than one set of discrete symmetry transformations which provide an analog for the Hodge duality operator of differential geometry. Furthermore, the idea of supervariable formalism has been mostly exploited to derive the off-shell nilpotent symmetries existing in the SUSY theory (see [30, 31]). Thus, the final motive of our present investigation is to make use of (anti-)chiral supervariable approach to derive the on-shell nilpotent symmetries for the general superpotential.

The contents of the paper is organized as follows. Section 2 gives the preliminaries for the study of on-shell symmetries existing for a generalized form of superpotential. Section 3 provides an account of the continuous and discrete symmetries for the superposition of harmonic oscillator superpotential with free particle as well as with the ‘Coulomb-type’ superpotential. Section 4 deals with the study of continuous and discrete symmetries for the ‘generalized’ superpotential, whereas the cohomological aspects of these discrete symmetries are discussed in section 5. Our section 6 contains the derivation of the above mentioned on-shell nilpotent fermionic continuous symmetries, for ‘generalized’ superpotential, within the framework of (anti-)chiral supervariable approach. Furthermore, we express the Lagrangian corresponding to ‘generalized’ superpotential in terms of supervariables and establish an equivalence among the translational generators and fermionic symmetries. Finally, in section 7, we summarize our work.

2. Preliminaries

We start with the generalized $N = 2$ SUSY Lagrangian [11] of the following form

$$L = \frac{\bar{q}^2}{2} - \frac{1}{2} W'^2 + i \bar{\psi} \dot{\psi} - W' \bar{\psi} \dot{\psi},$$

(1)

where $q$ is bosonic coordinate, $\psi(\bar{\psi})$ are fermionic variables and $W$ represents superpotential (prime indicates derivative with respect to the bosonic coordinate and over dot represents the time derivative). The above Lagrangian has following sets of fermionic symmetries ($s_1, s_2$)

$$s_1 q = -i \dot{\psi}, \quad s_1 \bar{\psi} = \bar{q} + i W',$$

(2)

$$s_2 q = i \dot{\psi}, \quad s_2 \bar{\psi} = \bar{q} - i W'.$$

(3)

We can easily verify that under the above symmetry transformations ($s_1, s_2$), the Lagrangian (1) transforms as

$$s_1 L = \frac{d}{dt}(-W' \bar{\psi}), \quad s_2 L = \frac{d}{dt}(i \dot{\psi} \bar{\psi}).$$

(4)

The above mentioned fermionic symmetries are on-shell nilpotent of order two (i.e. $s_1^2 = s_2^2 = 0$), which can be proven with the help of following Euler–Lagrange equations of motion arising from Lagrangian (1), as

$$\dot{q} = -W' W'' - W' \bar{\psi}, \quad \dot{\psi} = i W'' \bar{\psi}, \quad \bar{\psi} = -i W'' \psi.$$  

(5)

It is interesting to note that the Lagrangian (1) is also endowed with a bosonic symmetry ($s_0$) which can be constructed from the above mentioned fermionic symmetries as (see (2), (3))

$$s_0 \Phi = \{s_1, s_2\} \Phi = 2i \bar{\Phi},$$

(6)

where $\Phi$ being any of the $q, \psi, \bar{\psi}$. The explicit form of the symmetry transformations under $s_0$ is as follows

$$s_0 q = 2i \dot{q}, \quad s_0 \psi = i \dot{\psi} + W' \bar{\psi}, \quad s_0 \bar{\psi} = i \dot{\psi} - W'' \psi.$$  

(7)

It is worthwhile to mention that the study of symmetries of the SUSY Lagrangian leads to the existence of different possible combinations of superpotentials [11]. Thus, in our present work, we consider the generalized form of the superpotential as given below [11]

$$W' = \omega q + \mu + (\lambda/q),$$

(8)

where $\omega, \mu$ and $\lambda$ are arbitrary constants. It is clear from the above expression that various combinations of the constants $\omega, \mu$ and $\lambda$ yield different kinds of superpotentials. For example, $\lambda = 0$ returns the superposition of harmonic oscillator superpotential with free particle, whereas $\mu = 0$ gives the superposition of superpotentials such as harmonic oscillator with the ‘Coulomb-like’ [11]. We shall consider these two cases in detail in the next section.

Before wrapping up this section, we would like to point out that for a system having superpotential of the form (8) the Euler–Lagrange equations of motion turn out to be

$$\dot{q} = -\omega^2 q + \left(\frac{\lambda}{q^2} - \omega\right) \mu + \frac{\lambda^2}{q^4} - \frac{2 \lambda}{q^3} \bar{\psi} \psi,$$

$$\dot{\psi} = i \left(\omega - \frac{\lambda}{q^2}\right) \bar{\psi}, \quad \bar{\psi} = -i \left(\omega - \frac{\lambda}{q^2}\right) \psi.$$  

(9)

3. Examples: different cases

We consider linear combinations of the ‘generalized’ superpotential (see (8) above) as they return some familiar systems
Therefore, we have

\[ W' = \omega q + \mu. \]  (10)

The resultant Lagrangian with the superpotential (10) looks like

\[ L^{(1)} = \frac{\dot{q}^2}{2} - \frac{1}{2}\omega^2 q^2 + i\dot{\psi}\overline{\psi} - \omega\overline{\psi}\psi - \frac{\mu^2}{2} - \omega mq. \]  (11)

The above Lagrangian may be decomposed either into the harmonic oscillator or the free particle Lagrangian depending on the values of \( \omega \) and \( \mu \). The corresponding continuous symmetries \( s_1^{(1)}, s_2^{(1)} \) are of the following form

\[ s_1^{(1)} q = -i\dot{q}, \quad s_1^{(1)} \psi = \dot{\psi} + i\omega q + i\mu, \quad s_1^{(1)} \psi = 0, \]  (12)

\[ s_2^{(1)} q = i\dot{q}, \quad s_2^{(1)} \psi = -\dot{\psi} + i\omega q + i\mu, \quad s_2^{(1)} \psi = 0. \]  (13)

These symmetries are on-shell nilpotent and the action of the above stated symmetries on the Lagrangian (11) gives

\[ s_1^{(1)} L^{(1)} = -\frac{d}{dt}(\omega q + \mu)\psi, \quad s_2^{(1)} L^{(1)} = \frac{d}{dt}(iq\psi), \]  (14)

and the corresponding Noether conserved charges for the above symmetries are, respectively

\[ Q^{(1)} = (-i\dot{q} + \omega q + \mu)\psi, \quad \dot{Q}^{(1)} = \dot{\psi}(i\dot{q} + \omega q + \mu). \]  (15)

Furthermore, the following bosonic symmetry \( s_w^{(1)} = \{s_1^{(1)}, s_2^{(1)}\}; \)

\[ s_w^{(1)} q = 2i\dot{q}, \quad s_w^{(1)} \psi = i\dot{\psi} + \omega q, \quad s_w^{(1)} \psi = i\dot{\psi} - \omega q, \]  (16)

transforms the Lagrangian (11) as:

\[ s_w^{(1)} L^{(1)} = 2i\frac{d}{dt}\left(\frac{\dot{q}^2}{2} - \frac{\omega^2 q^2}{2} + i\dot{\psi}\overline{\psi} - \omega\overline{\psi}\psi - \frac{\mu^2}{2} - \omega mq\right). \]  (17)

It is worthwhile to mention couple of points here; first, the bosonic symmetry \( s_w^{(1)} \) listed in (16) reduces to the form (6) on the on-shell, i.e., by using equations of motion (see (19) below). Second, it can be easily verified that the Lagrangian (11) transforms into a total time derivative of itself, modulo a constant factor, under \( s_w^{(1)} \). The corresponding Noether conserved charge for \( s_w^{(1)} \) can be given as

\[ Q_w^{(1)} = 2i\frac{d}{dt}\left(\frac{\dot{q}^2}{2} + \frac{\omega^2 q^2}{2} + i\dot{\psi}\overline{\psi} + \omega\overline{\psi}\psi + \omega mq\right). \]  (18)

The equations of motion corresponding to \( L^{(1)} \) have the following form

\[ \ddot{q} = -\omega^2 q - \omega\mu, \quad \dot{\psi} = i\omega\overline{\psi}, \quad \dot{\overline{\psi}} = -i\omega\psi. \]  (19)

In addition to the above mentioned continuous symmetries, we also have following discrete symmetries in the system which leave the Lagrangian (11) quasi-invariant

\[ q \rightarrow -q, \quad t \rightarrow t + \lambda, \quad \psi \rightarrow \pm i\psi, \quad \overline{\psi} \rightarrow \mp i\overline{\psi}, \quad \omega \rightarrow -\omega, \quad \mu \rightarrow \mu + \lambda, \]  (20)

where the above set has only parity symmetry (as \( q \rightarrow -q \)). Furthermore, we have another set of discrete symmetry transformations that contains both parity and time-reversal which also leaves the Lagrangian (11) quasi-invariant

\[ q \rightarrow -q, \quad t \rightarrow -t, \quad \psi \rightarrow \pm i\psi, \quad \overline{\psi} \rightarrow \pm i\overline{\psi}, \quad \omega \rightarrow \omega + \mu, \quad \mu \rightarrow -\mu. \]  (21)

Finally, there exist two more sets of discrete symmetries containing time reversal symmetry, one with parity and other without parity symmetry, as follows:

\[ q \rightarrow \pm q, \quad t \rightarrow t, \quad \psi \rightarrow \pm \psi, \quad \overline{\psi} \rightarrow \mp \psi, \quad \omega \rightarrow \omega, \quad \mu \rightarrow \pm \mu. \]  (22)

At this juncture, we would like to mention that among the discrete symmetries listed in (20)–(22), only two sets have physical realization. We would comment on this, in detail, in section 5 where we discuss the case for a more general superpotential. Furthermore, it can be shown that the Hamiltonian \( (H^{(1)}) \) corresponding to \( L^{(1)} \) can be written as \( H^{(1)} = \frac{1}{2}(Q^{(1)}, \dot{Q}^{(1)}) \).

It is worthwhile to mention that we can directly obtain the symmetries and conserved charges for the two sets of individual superpotentials from these explicit expressions derived above in the following manner.

(a): when substituting the value of \( \mu = 0 \) in the above set of equations, we retrieve the expressions for the SUSY harmonic oscillator Lagrangian. This can be verified in a straightforward manner.

(b): whereas when \( \omega = 0 \), this leads to the non-relativistic particle case. In a more general sense, the expressions represent a particle under a constant potential of magnitude \( \mu \). This can also be individually verified.

3.2. Case II: \( \mu = 0 \)

In this case, the superpotential (8), which is analogous to the superposition of harmonic oscillator with ‘Coulomb-like’ takes the following form

\[ W' = \omega q + \lambda/q, \]  (23)

consequently, the Lagrangian can be given as

\[ L^{(2)} = \frac{\dot{q}^2}{2} - \frac{\lambda^2}{2\omega^2 q^2} + i\dot{\psi}\overline{\psi} - \omega\overline{\psi}\psi - \omega\lambda. \]  (24)

It is straightforward to check that the above Lagrangian (24) has the following sets of fermionic continuous symmetries \( (s_1^{(2)} \) and \( s_2^{(2)} \))

\[ s_1^{(2)} q = -i\dot{q}, \quad s_1^{(2)} \psi = 0, \quad s_2^{(2)} \psi = i\dot{q} + i\omega q + \frac{i\lambda}{q}. \]  (25)
\[ s_1^{(2)} q = i \psi, \quad s_2^{(2)} \tilde{\psi} = 0, \quad s_2^{(2)} q = -i + \omega q + \frac{i \lambda}{q}, \quad (26) \]

where on-shell nilpotency can be proven by virtue of Euler–Lagrange equations of motion (see (32) below). Here, the Lagrangian (24) transforms into a total derivative under these fermionic continuous symmetries and takes the following forms, respectively

\[ s_1^{(2)} L^{(2)} = -\frac{d}{dt} \left( (\omega q + \frac{\lambda}{q}) \psi \right), \quad s_2^{(2)} L^{(2)} = \frac{d}{dt} (i q \tilde{\psi}). \quad (27) \]

In a similar fashion, as we have done in the last subsection, we can construct the bosonic symmetry \( s_w^{(2)} = \{ s_1^{(2)}, s_2^{(2)} \} \) as follows

\[ s_w^{(2)} q = 2 i \dot{q}, \quad s_w^{(2)} \dot{\psi} = i \dot{\psi} + \left( \omega - \frac{\lambda}{q^2} \right) \psi, \quad (28) \]

It is easy to check that under the above bosonic symmetry, the Lagrangian (24) transforms in the following manner

\[ s_w^{(2)} L^{(2)} = i \frac{d}{dt} \left( \dot{q}^2 - \omega^2 q^2 - \frac{\lambda^2}{q^2} + i \dot{\psi} \psi - \omega \dot{\psi} \psi + \frac{i \lambda \dot{\psi} \dot{\psi}}{q^2} \right). \quad (29) \]

It is interesting to note that the symmetries listed in (28) can be recast into the form (6) with the help of equations of motion (see (32) below). In this scenario, the Lagrangian (24) transforms into a total time derivative of itself, modulo a constant factor, under the bosonic symmetry \( s_w^{(2)} \). The Noether conserved charges corresponding to the fermionic symmetries (25), (26) are, respectively, found to be

\[ Q^{(2)} = \left( -i \dot{q} + \omega q + \frac{\lambda}{q} \right) \psi, \quad \bar{Q}^{(2)} = \tilde{\psi} \left( i \dot{q} + \omega q + \frac{\lambda}{q} \right), \quad (30) \]

whereas for the bosonic symmetry (see (28)), the conserved charge \( Q_w^{(2)} \) is given by

\[ Q_w^{(2)} = 2 i \left( \frac{\dot{q}^2}{2} + \frac{\omega^2 q^2}{2} + \frac{\lambda^2}{2 q^2} + \omega \dot{\psi} \psi - \frac{\lambda \dot{\psi} \dot{\psi}}{q^2} \right). \quad (31) \]

The Lagrangian (24) has the following Euler–Lagrange equations of motion

\[ \ddot{q} = -\omega^2 q + \frac{\lambda^2}{q^3} - \frac{2 \lambda \dot{\psi} \dot{\psi}}{q^3}, \quad \ddot{\psi} = i (\omega - \frac{\lambda}{q^2}) \tilde{\psi}, \quad \ddot{\tilde{\psi}} = -i (\omega - \frac{\lambda}{q^2}) \psi. \quad (32) \]

It is again straightforward to check that, \( H^{(2)} = \frac{1}{2} \left( \dot{Q}^{(2)}, \dot{\bar{Q}}^{(2)} \right) \), where \( H^{(2)} \) represents the Hamiltonian corresponding to \( L^{(2)} \). Besides these continuous symmetries there also exists a set of discrete symmetry transformations as

\[ q \longrightarrow - q, \quad t \longrightarrow t, \quad \psi \longrightarrow \pm i \tilde{\psi}, \quad \tilde{\psi} \longrightarrow \mp i \psi, \quad \omega \longrightarrow - \omega, \quad \lambda \longrightarrow - \lambda. \quad (33) \]

This set of discrete symmetry transformations contains only parity symmetry, but no time-reversal symmetry. Moreover, we also have discrete symmetry transformations which consist of both parity and time-reversal symmetry as follows

\[ q \longrightarrow - q, \quad t \longrightarrow - t, \quad \psi \longrightarrow \pm i \tilde{\psi}, \quad \tilde{\psi} \longrightarrow \mp i \psi, \quad \omega \longrightarrow \omega, \quad \lambda \longrightarrow \lambda. \quad (34) \]

Furthermore, there exist two more sets of discrete symmetry transformations one with parity symmetry and another without it along with time-reversal symmetry, as

\[ q \longrightarrow \pm q, \quad t \longrightarrow - t, \quad \psi \longrightarrow \pm \tilde{\psi}, \quad \tilde{\psi} \longrightarrow \mp \psi, \quad \omega \longrightarrow \omega, \quad \lambda \longrightarrow \lambda. \quad (35) \]

Here all the above listed discrete symmetry transformations (33)–(35) leave the Lagrangian (24) quasi-invariant. It is worthwhile to mention that among the above mentioned sets of discrete symmetries, only two sets correspond to be physically significant (similar to the previous case). We shall discuss this aspect in section 5.

Before ending this subsection we would like to comment that in this combination of superpotentials, we retrieve two individual cases.

(a): when substituting \( \lambda = 0 \) in the above set of equations, we retrieve the corresponding expressions for the SUSY harmonic oscillator. This can be verified by individually evaluating fermionic and bosonic symmetries (and their corresponding charges) for a harmonic oscillator superpotential.

(b): when substituting \( \omega = 0 \) in the above set of equations, we retrieve the expressions for a classical potential equivalent to \( V(q) = \frac{k}{2} q^2 \) (see [11] for details). This can be again verified by separately evaluating fermionic and bosonic symmetries (and their corresponding charges) for a Coulomb-like superpotential.

4. Superpotentials of type \( W' = \sum_{j=a}^{b} \beta_j q^j \)

In this section, we take a more general case with ‘generalized’ superpotential of the form

\[ W' = \sum_{j=a}^{b} \beta_j q^j, \quad (36) \]

where \( a, b \) could be positive or negative integers in an ‘appropriate order’ and \( \beta_j \)'s are constant coefficients. If \( a, b \) are taken in ‘appropriate order’ (i.e. \( a, a + 1, a + 2, \ldots, b = a + n \)) with \( n \) being a positive integer, we obtain a series of superpotentials with the powers of \( q \) changing in geometric
progression. It is straightforward to obtain the superpotential \( A \) from this ‘generalized’ superpotential with the following choices (as a special case).

Let’s choose \( a = -1 \) and \( b = 1 \), this implies
\[
\sum_{j=1}^{1} \beta_j q^j = \beta_0 q^1 + \beta_0 q^0 + \beta_1 q^1,
\]
with identifying \( \beta_0 = \lambda \), \( \beta_0 = \mu \) and \( \beta_1 = \omega \), we obtain the superpotential \( A \) as
\[
\sum_{j=1}^{1} \beta_j q^j = \frac{\lambda}{q} + \mu + \omega q.
\]

Thus, we start with ‘generalized’ superpotential \( A \), so the Lagrangian can be given as
\[
L^{(\ell)} = \frac{\tilde{q}^2}{2} - \frac{1}{2} \left( \sum_{j=1}^{b} \beta_j q^j \right)^2 + \tilde{\psi} \tilde{\psi} - \left( \sum_{j=1}^{b} j \beta_j q^{j-1} \right) \tilde{\psi} \tilde{\psi}.
\]
The fermionic symmetries \( s_1^{(\ell)}, s_2^{(\ell)} \) associated with the above mentioned Lagrangian are
\[
s_1^{(\ell)} q = -i \psi, \quad s_1^{(\ell)} \psi = 0, \quad s_1^{(\ell)} \tilde{\psi} = \tilde{\psi} + i \sum_{j=1}^{b} \tilde{\beta}_j q^{j-1} \tilde{\psi},
\]
\[
s_2^{(\ell)} q = i \psi, \quad s_2^{(\ell)} \psi = -i \psi, \quad s_2^{(\ell)} \tilde{\psi} = -i \sum_{j=1}^{b} \tilde{\beta}_j q^{j-1} \tilde{\psi}.
\]

Moreover, apart from the above fermionic symmetries as listed in \( (40) \), we have the following bosonic symmetry \( s_w^{(\ell)} \) which can be explicitly given as
\[
s_w^{(\ell)} q = 2i \tilde{q}, \quad s_w^{(\ell)} q = i \tilde{\psi} + \sum_{j=1}^{b} \beta_j q^{j-1} \tilde{\psi},
\]
\[
s_w^{(\ell)} \tilde{\psi} = i \tilde{\psi} - \sum_{j=1}^{b} \beta_j q^{j-1} \tilde{\psi}.
\]

It is now straightforward to check that under the above set of symmetry transformations, \( s_1^{(\ell)}, s_2^{(\ell)}, s_w^{(\ell)} \) the Lagrangian \( (39) \) transforms as follows
\[
s_1^{(\ell)} L^{(\ell)} = -\frac{d}{dt} \left( \sum_{j=1}^{b} \left( \beta_j q^j \right) \tilde{\psi} \right), \quad s_2^{(\ell)} L^{(\ell)} = \frac{d}{dt} \left( iq \tilde{\psi} \right),
\]
\[
s_w^{(\ell)} L^{(\ell)} = i \frac{d}{dt} \left( \tilde{q}^2 - \sum_{j=1}^{b} \beta_j q^j \right) + i \tilde{\psi} \tilde{\psi} - \left( \sum_{j=1}^{b} j \beta_j q^{j-1} \tilde{\psi} \tilde{\psi} \right).
\]

As it can be seen that the Lagrangian \( (39) \) transforms to a total derivative, hence \( s_1^{(\ell)}, s_2^{(\ell)} \) and \( s_w^{(\ell)} \) are symmetries of the theory. It is important to point out that the Lagrangian \( (39) \) transforms into the total time derivative of itself, modulo a constant factor, under the bosonic symmetry \( s_w^{(\ell)} \) on the on-shell, where we make use of the equations of motion (see \( (47) \)). Therefore, the conserved charges associated with these symmetries \( s_1^{(\ell)}, s_2^{(\ell)}, s_w^{(\ell)} \) can be given, respectively
\[
Q^{(g)} = -iq + \sum_{j=1}^{b} \beta_j q^j \tilde{\psi}, \quad \bar{Q}^{(g)} = i \tilde{q} + \sum_{j=1}^{b} \tilde{\beta}_j q^j \tilde{\psi}.
\]

\[
Q_w^{(g)} = 2i \left( \frac{\tilde{q}^2}{2} + \frac{1}{2} \sum_{j=a}^{b} \beta_j q^j \right)^2 + \left( \sum_{j=a}^{b} j \beta_j q^{j-1} \tilde{\psi} \tilde{\psi} \right).
\]

The Euler–Lagrange equations of motion can be deduced from \( (39) \), as follows
\[
\tilde{q} = -\sum_{j=a}^{b} \left( \beta_j q^j \left( k \beta_j q^{j-1} \right) - \sum_{j=a}^{b} j (j - 1) \beta_j q^{j-2} \right) \tilde{\psi},
\]
\[
\psi = i \sum_{j=a}^{b} \left( j \beta_j q^{j-1} \right) \tilde{\psi}, \quad \bar{\psi} = i \sum_{j=a}^{b} \left( j \beta_j q^{j-1} \right) \tilde{\psi}.
\]

Before wrapping up this section, we would like to comment that the symmetries \( s_1^{(\ell)} \) and \( s_2^{(\ell)} \) are on-shell nilpotent (i.e. \( s_1^{(\ell)} s_2^{(\ell)} = 0 \) and \( s_2^{(\ell)} s_1^{(\ell)} = 0 \) on using above equations of motion).

Apart from the above mentioned continuous symmetries \( (40)–(42) \), the Lagrangian \( (39) \) also possesses following discrete symmetries
\[
q \rightarrow -q, \quad t \rightarrow -t, \quad \psi \rightarrow \pm i \tilde{\psi}, \quad \beta_{2n+1} \rightarrow -\beta_{2n+1}, \quad \beta_{2n} \rightarrow -\beta_{2n},
\]
where \( n \) is any integer.

The above listed discrete symmetry transformations endow only parity symmetry without any time-reversal symmetry. Let us now consider another set of discrete symmetry transformations which possess both parity and time-reversal symmetries, as
\[
q \rightarrow -q, \quad t \rightarrow -t, \quad \psi \rightarrow \pm i \tilde{\psi}, \quad \beta_{2n+1} \rightarrow \beta_{2n+1}, \quad \beta_{2n} \rightarrow -\beta_{2n},
\]
where \( n \) is any integer.

Moreover, there exist another two sets of discrete symmetry transformations, as
\[
q \rightarrow \pm q, \quad t \rightarrow -t, \quad \psi \rightarrow \pm i \tilde{\psi}, \quad \beta_{2n+1} \rightarrow \beta_{2n+1}, \quad \beta_{2n} \rightarrow \pm \beta_{2n},
\]
where \( n \) is any integer.

In the above, one set is endowed with parity symmetry and another without it. However time-reversal symmetry is present in both the sets. It is easy to verify that under these sets of discrete symmetry transformations \( (48)–(50) \), the Lagrangian \( (39) \) remains quasi-invariant.

5. Cohomological aspects

The continuous symmetries defined in \( (40)–(42) \) satisfy the following algebra
\[
s_1^{(\ell)} s_2^{(\ell)} = 0, \quad s_1^{(\ell)} s_1^{(\ell)} = 0, \quad s_2^{(\ell)} s_2^{(\ell)} = 0, \quad s_1^{(\ell)} \left[ s_1^{(\ell)}, s_2^{(\ell)} \right] = 0, \quad s_2^{(\ell)} \left[ s_1^{(\ell)}, s_2^{(\ell)} \right] = 0.
\]
At this juncture, we would like to mention that the two sets of continuous symmetries listed in the previous section (see (12), (13), (16) and (25), (26), (28)) also satisfy the same algebra as listed in (51). Moreover, the above mentioned algebraic relations are satisfied only on the on-shell where we make use of equations of motion.

On the other hand, the de Rham cohomological operators \((d, \delta, \Delta)\) of differential geometry satisfy the relations given below [16, 17]

\[
d^2 = 0, \quad \delta^2 = 0, \quad \Delta = [d, \delta] = d\delta + \delta d, \quad \Delta = (d + \delta)^2, \quad [\Delta, d] = 0, \quad [\Delta, \delta] = 0, \quad (52)
\]

where, \((\delta, d)\) are (co-)exterior derivatives and \(\Delta\) is the Laplacian operator. In addition to the expressions listed above, the co-exterior derivative is related with exterior derivative through the Hodge duality operator (*) and it is expressed as

\[
\delta = \pm \ast d \ast, \quad (53)
\]

where the \(\pm\) signs in the above expression are determined by the degree of the form and the dimensionality of the manifold [16, 17]. Thus, from (51) and (52), we can see that the algebra obeyed by the symmetry transformations is same as the algebra obeyed by the de Rham cohomological operators. In order to have perfect analogy between symmetries and cohomological operators the equation (53) should also be satisfied [19].

Let us explicitly check the validity of relation (53) in the context of symmetries (48)–(50), connecting continuous nilpotent symmetries \((s_1^{(q)}, s_2^{(q)})\) with discrete symmetry transformations \((\ast)\) in the following fashion

\[
s_1^{(q)} \phi = \pm \ast s_2^{(q)} \ast \phi, \quad \text{where} \quad \phi = q, \psi, \tilde{\psi}, \quad (54)
\]

and also the existence of inverse of the above relation, as

\[
s_2^{(q)} \phi = \mp \ast s_1^{(q)} \ast \phi. \quad (55)
\]

Before checking the validity of (53), we would like to comment that generally the two successive operations of discrete symmetry transformations on any general variable \(\phi\) of the theory yield

\[
\ast(\ast \phi) = \pm \phi. \quad (56)
\]

As far as the discrete symmetry transformations (48) are concerned, it is found to satisfy

\[
\ast(\ast \phi) = + \phi. \quad (57)
\]

Moreover, it can be checked explicitly that the transformations (48) do not satisfy the relation, \(s_1^{(q)} \phi = + \ast s_2^{(q)} \ast \phi\) (and also \(s_2^{(q)} \phi = - \ast s_1^{(q)} \ast \phi\)). So, this set of discrete symmetry transformations cannot be considered to be a perfect analog of the de Rham cohomological operators [19].

Now let us consider the discrete symmetry transformations (49), as they satisfy

\[
\ast[\ast \phi_1] = + \ast \phi_1, \quad \ast[\ast \phi_2] = - \ast \phi_2, \quad (58)
\]

where \(\phi_1\) and \(\phi_2\) are the bosonic and fermionic variables, respectively. However, the discrete symmetry transformations (49) do not satisfy the relation \(s_1^{(q)} \phi_1 = + \ast s_2^{(q)} \ast \phi_1\), (and the inverse relation \(s_2^{(q)} \phi_1 = - \ast s_1^{(q)} \ast \phi_1\), too). Additionally, for the transformations (49), the relation \(s_1^{(q)} \phi_2 = - \ast s_2^{(q)} \ast \phi_2\) (and its inverse relation \(s_2^{(q)} \phi_2 = + \ast s_1^{(q)} \ast \phi_2\)) is also not obeyed. Thus, we can ignore these discrete symmetry transformations while looking for a perfect analogy [19].

Finally, let us concentrate on the discrete symmetries listed in (50) which have special significance. These symmetries satisfy the relations listed in (58). Now considering a set of discrete symmetry transformations from (50), the one without parity symmetry, i.e.

\[
q \longrightarrow + q, \quad t \longrightarrow - t, \quad \psi \longrightarrow \pm \tilde{\psi}, \quad \tilde{\psi} \longrightarrow \mp \psi, \quad \beta_{2n+1} \longrightarrow \beta_{2n+1}, \quad \beta_{2n} \longrightarrow + \beta_{2n}, \quad \text{where} \ n \ \text{is any integer}. \quad (59)
\]

These transformations are found to obey relations analogous to (53), as listed below

\[
s_1^{(q)} \phi_1 = \pm \ast s_2^{(q)} \ast \phi_1, \quad s_1^{(q)} \phi_1 = \mp \ast s_2^{(q)} \ast \phi_1, \quad s_1^{(q)} \phi_2 = \pm \ast s_2^{(q)} \ast \phi_2, \quad s_1^{(q)} \phi_2 = \mp \ast s_2^{(q)} \ast \phi_2. \quad (60)
\]

Let us further concentrate on the upper signature of discrete symmetry transformations in (59), i.e.

\[
q \longrightarrow + q, \quad t \longrightarrow - t, \quad \psi \longrightarrow \pm \tilde{\psi}, \quad \tilde{\psi} \longrightarrow \mp \psi, \quad \beta_{2n+1} \longrightarrow \beta_{2n+1}, \quad \beta_{2n} \longrightarrow + \beta_{2n}, \quad \text{where} \ n \ \text{is any integer}. \quad (61)
\]

Here, we observe that the discrete symmetries given in (61) satisfy the relations as

\[
s_1^{(q)} \phi_1 = + \ast s_2^{(q)} \ast \phi_1, \quad s_1^{(q)} \phi_1 = - \ast s_2^{(q)} \ast \phi_1, \quad s_1^{(q)} \phi_2 = - \ast s_2^{(q)} \ast \phi_2, \quad s_1^{(q)} \phi_2 = + \ast s_2^{(q)} \ast \phi_2. \quad (62)
\]

These relations are the appropriate ones, in accordance with (53) (see [19] for details). Thus, these discrete symmetry transformations (61) along with continuous symmetries (40)–(42) provide the physical realization of de Rham cohomological algebra existing in the context of differential geometry. Whereas, the discrete symmetries with lower signature in (59) lead to \(s_1^{(q)} \phi_1 = - \ast s_2^{(q)} \ast \phi_1\) and \(s_1^{(q)} \phi_2 = + \ast s_2^{(q)} \ast \phi_2\) (and also the inverse relations) which are not in accordance with (53).

Now, considering the discrete symmetry transformations in (50) with parity inversion, i.e.: \(q \longrightarrow - q, \quad t \longrightarrow - t, \quad \psi \longrightarrow \pm \tilde{\psi}, \quad \tilde{\psi} \longrightarrow \mp \psi, \quad \beta_{2n+1} \longrightarrow \beta_{2n+1}, \quad \beta_{2n} \longrightarrow - \beta_{2n}, \quad \text{where} \ n \ \text{is any integer}. \quad (63)

The above mentioned discrete symmetry transformations also satisfy relations which are analogous to (53), as

\[
s_1^{(q)} \phi_1 = \mp \ast s_2^{(q)} \ast \phi_1, \quad s_1^{(q)} \phi_1 = \pm \ast s_2^{(q)} \ast \phi_1, \quad s_1^{(q)} \phi_2 = \pm \ast s_2^{(q)} \ast \phi_2, \quad s_1^{(q)} \phi_2 = \mp \ast s_2^{(q)} \ast \phi_2. \quad (64)
\]
Out of two sets of discrete symmetries listed in (63), let us focus on the lower signature, as explicitly given below

\[ q \rightarrow -q, \quad t \rightarrow -t, \quad \psi \rightarrow -\psi, \quad \bar{\psi} \rightarrow +\bar{\psi}, \]

\[ \beta_{2n+1} \rightarrow \beta_{2n+1}, \quad \beta_{2n} \rightarrow -\beta_{2n}, \text{ where } n \text{ is any integer.} \]  

(65)

This set of discrete symmetry transformations satisfy the same relations as in (62). Thus, we conclude that the discrete symmetry transformations (65) also provide a physical analog to the Hodge duality (\(\ast\)) operator [19]. Whereas, the discrete symmetries with upper signature in (63) guide to the relations such as \( s_{1}^{(g)} \phi_{1} = -s_{2}^{(g)} \ast \phi_{1} \) and \( s_{1}^{(g)} \phi_{2} = +s_{2}^{(g)} \ast \phi_{2} \) (and also the inverse relations), which are not in line with (53).

Thus, we conclude that none of the two discrete symmetries (48), (49) can be considered as perfect symmetry in the realm of differential geometry, since they do not pave the way to the correct relations as in (54) and (55). Thus, among the eight sets of discrete symmetry transformations (48)–(50), two of them (as explicitly listed in (61) and (65)) have an important physical significance where we obtain a physical realization among the symmetries and the de Rham cohomological operators at the algebraic level. Taking the above relations (53)–(55) into consideration, we can easily identify continuous symmetries \( s_{2}^{(g)} \), \( s_{1}^{(g)} \) as analog of \( \delta, d \) of differential geometry and the relations (51) and (52) imply that \( s_{1}^{(g)} \) stands for \( \Delta \), the Laplacian operator. Moreover, the Hodge duality (\( \ast \)) operator can be realized in terms of two sets of discrete symmetries as listed in (61) and (65).

Before wrapping up this section, we would like to comment upon the physically relevant discrete symmetries in the case of superposition of harmonic oscillator superpotential with free particle and also with ‘Coulomb-like’ superpotential (i.e. Case I and Case II, respectively). As far as Case I is concerned, the discrete symmetries in (22) without parity and having upper signature provide the perfect analog of differential geometrical operators as they satisfy relation (62). Whereas the discrete symmetries with parity and having lower signature in (22) also obey the relations listed in (62) and hence also provide the physical analog. Likewise in Case II, the discrete symmetries in (35) without parity symmetry having upper signature and with parity symmetry having lower signature satisfy the relation (62) and hence provide physical analog for the Hodge duality operator.

6.1. On-shell nilpotent symmetries: anti-chiral supervariables

In order to obtain nilpotent fermionic continuous symmetries \( s_{1}^{(g)} \), we focus on the anti-chiral super-submanifold (see [30] for details). Therefore, we generalize ordinary variables to their corresponding anti-chiral supervariables in the following fashion

\[ Q(t, \bar{t}) = q(t) + \bar{t} \lambda(t), \]

\[ \psi(t, \bar{t}) = \psi(t) + i \bar{t} \lambda_{2}(t), \]

\[ \bar{\psi}(t, \bar{t}) = \bar{\psi}(t) + \bar{t} \lambda_{2}(t), \]

(66)

here, \( \lambda \) is a fermionic secondary variable, whereas \( \Omega_{1} \) and \( \Omega_{2} \) are bosonic secondary variables. To evaluate \( \Lambda(t), \Omega_{1}(t) \) and \( \Omega_{2}(t) \) we make use of the following SUSYIRs [30]. First of all, the invariance of \( \psi \) under transformation \( s_{1}^{(g)} \) (i.e. \( s_{1}^{(g)} \bar{\psi} = 0 \)) implies the following SUSYIR (see [19, 30] for details)

\[ \Psi(t, \bar{t}) = \psi(t). \]  

(67)

Thus, using (66) into (67), we obtain

\[ \Omega_{1}(t) = 0. \]  

(68)

Second, it is easy to verify that \( s_{1}^{(g)}(q\psi) = 0 \) and \( s_{1}^{(g)}(q\bar{\psi}) = 0 \). Thus, the invariant quantities \( q\psi \) and \( q\bar{\psi} \) lead to the following SUSYIRs, respectively

\[ Q(t, \bar{t})\psi(t, \bar{t}) = q(t)\psi(t), \quad Q(t, \bar{t})\bar{\psi}(t, \bar{t}) = q(t)\bar{\psi}(t). \]  

(69)

Making use of (68), we obtain \( \Lambda(t)\psi(t) = 0 \) and \( \Lambda(t)\bar{\psi}(t) = 0 \). As a result, the obvious choice we make is: \( \Lambda(t) \) must be proportional to \( \psi(t) \), i.e.

\[ \Lambda(t) = -i\psi(t). \]  

(70)

Finally, we focus our attention to the combination of following quantities: \( \frac{1}{2}q^{2}(t) + i\psi(t)\psi(t) - \frac{1}{2}q^{2}(t) + i\sum_{j=1}^{b}B_{j}q^{(j-1)}(t)\bar{\psi}(t)\bar{\psi}(t) + i\sum_{j=1}^{b}B_{j}q^{(j)}(t)\bar{\psi}(t) \equiv Y(t). \)

Here, \( Y(t) \) is itself an invariant quantity under \( s_{1}^{(g)} \) (i.e. \( s_{1}^{(g)}Y(t) = 0 \) without using equations of motion). Thus, we have the following SUSYIR

\[ \Gamma(t, \bar{t}) = Y(t), \]  

(71)

where, \( \Gamma(t, \bar{t}) \) represents the generalization of \( Y(t) \) on the anti-chiral super-submanifold and can be explicitly expressed as

\[ \Gamma(t, \bar{t}) = \frac{1}{2}Q^{2}(t, \bar{t}) + i\bar{\psi}(t, \bar{t})\psi(t, \bar{t}) - \frac{1}{2}i\sum_{j=1}^{b}B_{j}Q^{(j-1)}(t, \bar{t})\psi(t, \bar{t})\psi(t, \bar{t}) + i\sum_{j=1}^{b}B_{j}Q^{(j)}(t, \bar{t})\bar{\psi}(t, \bar{t}). \]  

(72)

Substituting for \( Q(t, \bar{t}), \psi(t, \bar{t}), \bar{\psi}(t, \bar{t}) \) from (66) and with the help of (68), equating both sides of (71), we obtain the

6. (Anti-)chiral supervariable approach: on-shell nilpotent symmetries

To derive the full set of on-shell nilpotent fermionic symmetries we start with \( \mathcal{N} = 2 \) SUSY QM model with the 'generalized’ superpotential (see section 4) on (1,2)-dimensional supermanifold. Here, the supermanifold is characterized by the bosonic variable \( t \) and the Grassmann variables \( \theta, \bar{\theta} \) (\( \theta^{2} = 0, \bar{\theta}^{2} = 0, \theta\bar{\theta} + \bar{\theta}\theta = 0 \)). We make use of SUSYIRs to derive on-shell nilpotent symmetries.
following expression
\[
\hat{q}\hat{\Lambda} = \left( \sum_{j,a} b_j q^j \right) \left( \sum_{k,a} k_b q^{k^{-1}} \right) \Lambda \\
- \left( \sum_{j,a} j(j-1) b_j q^{j-2} \right) \Lambda \hat{\psi} + i \left( \sum_{j,a} b_j q^j \right) \hat{\Lambda} \\
+ i \left( \sum_{j,a} j b_j q^{j^{-1}} \right) \hat{\Lambda} \hat{q} - \Omega_2 \left( \hat{\psi} + i \left( \sum_{j,a} b_j q^{j^{-1}} \right) \hat{\psi} \right) = 0.
\]

(73)

Substituting for \(\Lambda(t)\) from (70), we obtain
\[
-i\hat{q}\hat{\psi} + i \left( \sum_{j,a} b_j q^j \right) \left( \sum_{k,a} k_b q^{k^{-1}} \right) \psi + \left( \sum_{j,a} j b_j q^{j^{-1}} \right) \hat{q} \hat{\psi} \\
+ \left( \sum_{j,a} b_j q^j \right) \hat{\psi} = \Omega_2 \left( \hat{\psi} + i \left( \sum_{j,a} b_j q^{j^{-1}} \right) \hat{\psi} \right),
\]

which yields
\[
\Omega_2(t) = -\left( \hat{q} \psi + i \left( \sum_{j,a} b_j q^j \right) \psi \right).
\]

(75)

The equations (68), (70) and (75) lead to the following expressions on the anti-chiral super-submanifold
\[
Q(t, \theta) = q(t) + \theta \partial(-\hat{\psi}) \equiv q(t) + \theta \partial(s_1^{(q)} q),
\]
\[
\Psi(t, \theta) = \psi(t) + \theta \partial(s_1^{(\psi)} \psi),
\]
\[
\hat{\psi}(t, \theta) = \hat{\psi}(t) + \theta \left( i \left( \sum_{j,a} b_j q^j \right) \psi \right) \equiv \hat{\psi}(t) + \theta (s_1^{(\psi)} \hat{\psi}).
\]

(76)

While examining (76), it is clear that if we perform the translation along the Grassmannian direction \(\theta\), it gives the same result as \(s_1^{(q)}\) acting on the corresponding ordinary variable. Thus, the equivalence between the translational generator \(\partial\theta\) and \(s_1^{(q)}\) can be made as: \(\frac{\partial}{\partial \theta} \Phi(t, \theta) = s_1^{(q)} \Phi(t)\).

Here \(\Phi(t, \theta)\) is the general supervariable defined on the (1,1)-dimensional super-submanifold parametrized by \((t, \theta)\) and \(\phi(t)\) is the corresponding ordinary variable [30].

6.2. On-shell nilpotent symmetries: chiral supervariables

To derive the fermionic continuous symmetries \(s_2^{(q)}\), we move to the chiral super-submanifold characterized by \((t, \theta)\). The ordinary variables are generalized to their corresponding chiral supervariables in terms of secondary variables as (see [30])
\[
Q(t, \theta) = q(t) + \theta \hat{\Lambda}(t),
\]
\[
\Psi(t, \theta) = \psi(t) + i \partial \hat{\Omega}_1(t),
\]
\[
\hat{\Psi}(t, \theta) = \hat{\psi}(t) + i \partial \hat{\Omega}_2(t),
\]

(77)

where \(\Lambda\) is a fermionic secondary variable and \(\Omega_1, \Omega_2\) are bosonic variables. We consider following SUSYIRs to evaluate \(\hat{\Lambda}(t), \hat{\Omega}_1(t)\) and \(\hat{\Omega}_2(t)\); first, the invariance of \(\hat{\psi}\) under \(s_2^{(q)}\) (i.e. \(s_2^{(q)} \hat{\psi} = 0\)) leads to the SUSYIR \(\Psi(t, \theta) = \hat{\psi}(t)\). Thus, with the help of (77), we have
\[
\hat{\Omega}_2(t) = 0.
\]

(78)

Second, we note that the quantities \((q^q)\) and \((q^\psi)\) remain invariant under \(s_2^{(q)}\) (i.e. \(s_2^{(q)} (q^q) = 0\) and \(s_2^{(q)} (q^\psi) = 0\)) which lead to the following SUSYIRs, respectively
\[
Q(t, \theta) \Psi(t, \theta) = q(t) \hat{\psi}(t), \quad \hat{Q}(t, \theta) \Psi(t, \theta) = \hat{q}(t) \hat{\psi}(t).
\]

(79)

Since \(\hat{\Omega}_2(t) = 0\) from (78), thus using it in (79) along with (77), we obtain \(\hat{\Lambda}(t) \hat{\psi}(t) = 0\) and \(\hat{\Lambda}(t) \hat{\psi}(t) = 0\). So, we conclude that \(\hat{\Lambda}(t)\) must be proportional to \(\hat{\psi}(t)\) and we appropriately choose
\[
\hat{\Lambda}(t) = i \hat{\psi}(t).
\]

(80)

Finally, we have the following combination \(\frac{1}{2} \hat{q}^2(t) - i \hat{\psi}(t) \Psi(t) - \frac{1}{2} \left( \sum_{j,a} b_j q^j(t) \right)^2 - \sum_{j,a} j b_j q^{j^{-1}}(t) \hat{\psi}(t) \Psi(t) - i \left( \sum_{j,a} b_j q^j(t) \right) \hat{\psi}(t) \Psi(t) \equiv \hat{T}(t)\). Here, \(\hat{T}(t)\) is itself an invariant quantity under \(s_2^{(q)}\) (i.e. \(s_2^{(q)} \hat{T}(t) = 0\) without using equations of motion). Thus, we have the following SUSYIR
\[
\hat{T}(t, \theta) = \hat{T}(t),
\]

(81)

where \(\hat{T}(t, \theta)\) represents the generalization of \(\hat{T}(t)\) on the chiral super-submanifold and can be explicitly given by
\[
\hat{T}(t, \theta) = \frac{1}{2} \hat{q}^2(t) - i \hat{\psi}(t) \Psi(t, \theta) - \frac{1}{2} \left( \sum_{j,a} b_j q^j(t) \right)^2 - \sum_{j,a} j b_j q^{j^{-1}}(t) \hat{\psi}(t) \Psi(t, \theta) - i \left( \sum_{j,a} b_j q^j(t) \right) \hat{\psi}(t) \Psi(t, \theta).
\]

(82)

Making substitutions for \(Q(t, \theta), \Psi(t, \theta), \hat{\psi}(t, \theta)\) from (77) and with the help of (78), equating both sides of (81), we obtain
\[
\hat{q} \hat{\Lambda} - \left( \sum_{j,a} b_j q^j \right) \left( \sum_{k,a} k_b q^{k^{-1}} \right) \Lambda \\
- \left( \sum_{j,a} j(j-1) b_j q^{j-2} \right) \hat{\psi} + i \left( \sum_{j,a} b_j q^j \right) \hat{\Lambda} \\
+ i \left( \sum_{j,a} j b_j q^{j^{-1}} \right) \hat{\Lambda} \hat{q} - \hat{\Omega}_1 \left( \hat{\psi} + i \left( \sum_{j,a} b_j q^{j^{-1}} \right) \hat{\psi} \right) = 0.
\]

(83)

After substituting for \(\hat{\Lambda}(t)\) from (80), we obtain
\[
i \hat{q} \hat{\psi} - i \left( \sum_{j,a} b_j q^j \right) \left( \sum_{k,a} k_b q^{k^{-1}} \right) \hat{\psi} + \left( \sum_{j,a} j b_j q^{j^{-1}} \right) \hat{q} \hat{\psi} \\
+ \left( \sum_{j,a} b_j q^j \right) \hat{\psi} = \hat{\Omega}_4 \left( \hat{\psi} - i \left( \sum_{j,a} j b_j q^{j^{-1}} \right) \hat{\psi} \right).
\]

(84)
which finally yields
\[ \Omega_i(t) = -i\left( -\dot{q}(t) + i \sum_{j=a}^{b} \beta_j q_j(t) \right). \]

Thus, the obtained expressions for \( \lambda_i(t) \), \( \Omega_1(t) \) and \( \Omega_2(t) \) enable us to write expansion of supervariables in chiral super-submanifold as follows
\[ Q(t, \theta) = q(t) + \theta(i\bar{\psi}) \equiv q(t) + \theta(q_0^s q), \]
\[ \Psi(t, \theta) = \psi(t) + \theta\left( -\dot{q} + i \sum_{j=a}^{b} \beta_j q_j \right) \equiv \psi(t) + \theta(q_0^s \psi), \]
\[ \bar{\Psi}(t, \theta) = \bar{\psi}(t) + \theta(0) \equiv \bar{\psi}(t) + \theta(s_0^{\bar{\psi}}). \]

Before wrapping up this subsection, we would like to comment that it is evident from (86) that \( \frac{d}{dt} \Phi(t, \theta) = s_i^\theta \phi(t) \). Here \( \Phi(t, \theta) \) is the general supervariable in the super-submanifold characterized by \( (t, \theta) \) and \( \phi(t) \) is the corresponding ordinary variable.

6.3. Invariance of Lagrangian: supervariable approach

We can generalize the starting Lagrangian (39) onto the (1,1)-dimensional (anti-)chiral super-submanifold in the following manner
\[ \tilde{L}^{(ac)}_0 = -\frac{1}{2} \tilde{Q}_0^{(1)}(t, \tilde{\theta}) \tilde{Q}_0^{(1)}(t, \tilde{\theta}) + i\tilde{\Psi}_0^{(1)}(t, \tilde{\theta})\tilde{\Psi}_0^{(1)}(t, \tilde{\theta}) \]
\[ - \frac{1}{2} \tilde{W}^{\prime}(\tilde{Q}_0^{(1)}) \tilde{W}^{\prime}(\tilde{Q}_0^{(1)}) \]
\[ - \tilde{W}^{\prime\prime}(\tilde{Q}_0^{(1)}) \tilde{\Psi}_0^{(1)}(t, \tilde{\theta})\tilde{\Psi}_0^{(1)}(t, \tilde{\theta}), \]
and
\[ \tilde{L}^{(c)}_0 = -\frac{1}{2} \tilde{Q}_0^{(2)}(t, \theta) \tilde{Q}_0^{(2)}(t, \theta) + i\tilde{\Psi}_0^{(2)}(t, \theta)\tilde{\Psi}_0^{(2)}(t, \theta) \]
\[ - \frac{1}{2} \tilde{W}^{\prime}(\tilde{Q}_0^{(2)}) \tilde{W}^{\prime}(\tilde{Q}_0^{(2)}) \]
\[ - \tilde{W}^{\prime\prime}(\tilde{Q}_0^{(2)}) \tilde{\Psi}_0^{(2)}(t, \theta)\tilde{\Psi}_0^{(2)}(t, \theta). \]

Here, the superscripts (1,2) denote the expansion of the supervariables in the (anti-)chiral directions and the superscripts (ac,c) denote the generalization of Lagrangian along the (anti-)chiral directions. The Taylor expansion for the superpotential derivatives \( \tilde{W}^{\prime} \) and \( \tilde{W}^{\prime\prime} \) for the anti-chiral case are given as
\[ \tilde{W}^{\prime}(\tilde{Q}_0^{(1)}) = W^{\prime}(q) - i\partial W^{\prime\prime}(q)\psi(t), \]
\[ \tilde{W}^{\prime\prime}(\tilde{Q}_0^{(1)}) = W^{\prime\prime}(q) - i\partial W^{\prime\prime\prime}(q)\psi(t). \]

And for the chiral case, we have
\[ \tilde{W}^{\prime}(\tilde{Q}_0^{(2)}) = W^{\prime}(q) + i\partial W^{\prime\prime}(q)\bar{\psi}(t), \]
\[ \tilde{W}^{\prime\prime}(\tilde{Q}_0^{(2)}) = W^{\prime\prime}(q) + i\partial W^{\prime\prime\prime}(q)\bar{\psi}(t). \]

Now, substituting the expressions (76), (89) into \( \tilde{L}^{(ac)}_0 \) and (86), (90) into \( \tilde{L}^{(c)}_0 \), we obtain the following expressions
\[ \tilde{L}^{(ac)}_0 = \frac{\tilde{q}^2}{2} - \frac{1}{2} \sum_{j=a}^{b} \beta_j q_j^2 + \bar{\psi}\psi \]
\[ - \left( \sum_{j=a}^{b} j\beta_j q_j^{-1} \right) \psi + \frac{d}{dt} \left( \sum_{j=a}^{b} \beta_j q_j \psi \right), \]
\[ \tilde{L}^{(c)}_0 = \frac{\tilde{q}^2}{2} - \frac{1}{2} \sum_{j=a}^{b} \beta_j q_j^2 + i\bar{\psi}\psi \]
\[ - \left( \sum_{j=a}^{b} j\beta_j q_j^{-1} \right) \psi + i\partial \frac{d}{dt} (q\bar{\psi}). \]

It is evident from the above expressions that, taking the translational derivative of (91) and (92) along the \( \tilde{\theta}, \theta \) direction, respectively we obtain
\[ \frac{\partial}{\partial \tilde{\theta}} \tilde{L}^{(ac)}_0 = -\frac{d}{dt} \left( \sum_{j=a}^{b} \beta_j q_j \psi \right), \quad \frac{\partial}{\partial \theta} \tilde{L}^{(c)}_0 = \frac{d}{dt} (i\bar{q}\bar{\psi}). \]

Thus, the translation along the anti-chiral direction (\( \tilde{\theta} \)) on the superspace Lagrangian (87) gives the same result as \( s_i^\theta \) acting on the SUSY Lagrangian (39). Similarly, the translation of superspace Lagrangian (88) along the chiral direction (\( \theta \)) also provides the same result as \( s_i^{\tilde{\theta}} \) acting on the SUSY Lagrangian (39). From these observations, we can draw the equivalences \( \frac{\partial}{\partial \tilde{\theta}} \equiv s_i^\theta \) and \( \frac{\partial}{\partial \theta} \equiv s_i^{\tilde{\theta}} \). Furthermore, these translational generators (\( \partial_{\tilde{\theta}}, \partial_{\theta} \)) are nilpotent (see [30]). These translational generators along the (anti-)chiral directions in super-submanifold provide the proof for nilpotency of fermionic symmetries in a straightforward manner.

7. Conclusions

In our present investigation, we have extensively studied about the genre of symmetries existing in the coupled \( N=2 \) SUSY QM models. In particular, we have examined the superposition of harmonic oscillator superpotential with free particle and ‘Coulomb-type’ superpotentials. Moreover, we have considered a ‘generalized’ superpotential which, in turn, contains the previous two examples as a limiting case (see section 4 for details). We have inferred that all the above mentioned coupled systems are endowed with two sets of on-shell nilpotent, fermionic continuous symmetries as well as a set of bosonic symmetry transformations. The corresponding conserved charges have also been derived.

Further we have shown that, in each case, there exist eight sets of discrete symmetry transformations that leave the corresponding Lagrangian quasi-invariant. However, among these eight sets of discrete symmetries only two of them (in each case) provide a physical realization of the Hodge duality operator. It is clear from (51) and (52) that the algebra satisfied by the fermionic and bosonic symmetries are same as that of the algebra existing among the de Rham cohomological operators.
(d, δ, Δ). Furthermore, on a given manifold the exterior (d) and co-exterior (δ) derivatives are related with one another through the Hodge duality operator *(•) see, (53). We have shown that this Hodge duality operator finds its physical analog in terms of two sets of discrete symmetries of the theory, e.g. (61) and (65). Thus, we have been able to provide a proof of the conjecture (by considering three different examples) which endorses the existence of more than one set of discrete symmetry transformations as the analog of Hodge duality operation [19].

Moreover, we have applied the supervariable approach to \( \mathcal{N} = 2 \) SUSY QM model with a ‘generalized’ superpotential. For this, the ordinary variables of the theory have been generalized to the corresponding supervariables onto the (1,1)-dimensional (anti-)chiral super-submanifold. The on-shell nilpotent fermionic symmetries existing in the above mentioned model have been derived by making use of SUSYIRs. The form of SUSYIRs employed here gives a general norm for the invariant restrictions for any kind of superposition of potentials falling under the category discussed in the present investigation. Furthermore, we have generalized the starting Lagrangian (39) onto the (anti-)chiral super-submanifold and with the help of that we have provided an equivalence among fermionic symmetries and translational generators. Finally, the nilpotency of the fermionic symmetries has been captured in a straightforward manner with the aid of nilpotent translational generators.

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