APPROXIMATION PROPERTIES OF COMPLEX GENUINE 
\(\alpha\)-BERNSTEIN-DURRMEYER OPERATORS

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Abstract. Herein we propose a complex form of a genuine Bernstein-Durrmeeyer type operators depending on a non-negative real parameter \(\alpha\). We present the quantitative upper bound, Voronovskaja type result and exact order of approximation for these operators and for their derivatives attached to analytic functions on compact disks. These results validate the extension of approximation properties of complex genuine \(\alpha\)-Bernstein-Durrmeyer type operators from real intervals to compact disks in the complex plane.

Keywords: Complex Bernstein-Durrmeyer-type polynomials, quantitative estimates, Voronovskaja-type result, simultaneous approximation.

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1. Introduction

In [5], Chen et al. introduced a new generalization of Bernstein operators depending on a non-negative real parameter \(\alpha\), which are having some remarkable properties, given by

\[ B^\alpha_n(f; x) = \sum_{k=0}^{n} q^\alpha_{n,k}(x) f \left( \frac{k}{n} \right), \]  

(1.1)

for any function \(f(x)\) defined on \([0, 1]\), \(n \in \mathbb{N}\).

The \(\alpha\)-Bernstein polynomials \(q^\alpha_{n,k}(x)\) of degree \(n\) is defined by \(q^\alpha_{1,0}(x) = 1 - x, q^\alpha_{1,1}(x) = x\) and

\[ q^\alpha_{n,k}(x) = \left[ \binom{n-2}{k}(1-\alpha)x + \binom{n-2}{k-2}(1-\alpha)(1-x) + \binom{n}{k}\alpha x(1-x) \right]x^{k-1}(1-x)^{n-k-1} \]

for \(n \geq 2, x \in [0, 1]\). \(B^\alpha_n(f; x)\) are positive operators for \(0 \leq \alpha \leq 1\).

These operators have many useful properties such as preservation of linear (or constant) functions, monotonicity and convexity. To approximate the Lebesgue integrable functions by \(\alpha\)-Bernstein operators, Acar et. al [1] introduced a genuine \(\alpha\)-Bernstein-Durrmeyer operators as follows:

\[ G^\alpha_n(f; x) = q^\alpha_{n,0}(x)f(0) + q^\alpha_{n,n}(x)f(1) + \sum_{k=1}^{n-1} q^\alpha_{n,k}(x) \int_0^1 p_{n-2,k-1}(t)f(t)dt, \]  

(1.2)

where \(p_{n,k}(t) = \binom{n}{k}t^k(1-t)^{n-k}\). Authors studied local approximation, error estimation in terms of Ditzian-Totik modulus of smoothness and the convergence of these operators to certain functions by illustrative graphics.

Very recently, Çetin [1] introduced the complex form of the operators defined in [11]. These operators exhibit the approximation properties i.e. upper estimate, asymptotic formula, exact order of convergence and some shape preserving properties in the complex domain. These properties for complex Bernstein polynomials in compact disks were initially investigated by Lorentz [14]. Also, in [7], a very useful book by Gal and references therein, the overconvergence properties of the well known complex operators i.e. Bernstein-Stancu operators, \(q\)-Bernstein polynomials, Kantorovich operators etc. have been discussed.
He obtained quantitative estimates for these operators. He put in evidence the overconvergence phenomenon for several operators, namely the extensions of approximation properties with exact quantitative estimates, from the real interval to compact disks in the complex plane. Several type of complex operators were studied by a number of authors (see [2]–[4], [8]–[13]).

Motivated from the real case and overconvergence properties of above work on complex operators, we consider the complex form of the operators \((1.2)\), for \(n \in \mathbb{N}, \alpha \in [0,1], z \in \mathbb{C} \) and \(f\) is complex valued analytic function in an open disk \(D_R = \{z \in \mathbb{C}, |z| < R\},\ R > 1\), as follows:

\[
G_n^\alpha(f; z) = q^n_{0,0}(z)f(0) + q^n_{0,n}(z)f(1) + (n - 1)\sum_{k=1}^{n-1} q^n_{k,k}(z) \int_0^1 p_{n-2,k-1}(t)f(t)dt. \tag{1.3}
\]

The aim of this paper is to present the approximation properties of complex genuine \(\alpha\)-Bernstein-Durrmeyer operators. Here an upper estimate, Voronovskaja type result, exact order and simultaneous approximation are obtained for the order of approximation of these operators attached to analytic functions on a certain compact disk which give significant contribution in the theory of uniform convergence without depending on the parameter \(\alpha\).

2. Auxiliary results

Let \(e_p(z) = z^p, p \in \mathbb{N} \cup \{0\}, z \in \mathbb{C}\). From the definition of our operators, it can be easily shown that \(G_n^\alpha(e_0; z) = 1, \ G_n^\alpha(e_1; z) = z\).

We first mention the recurrence relation for moments:

**Proposition 1.** For all \(p \in \mathbb{N} \cup \{0\}\), \(n \in \mathbb{N}, z \in \mathbb{C}\) and \(\alpha \in [0,1]\), we have

\[
G_n^\alpha(e_{p+1}; z) = (1 - z) + nz + \frac{z}{n + p} (1 - z) + \frac{z}{n + p} S_1(e_p; z) + \frac{z}{n + p} S_2(e_p; z),
\]

where \(S_1(e_p; z) = \sum_{k=0}^{n-1} p_{n-2,k}(z)(1 - \alpha)((1 - z) + \frac{z}{n + p} S_1(e_p; z) + \frac{z}{n + p} S_2(e_p; z),
\]

and \(S_2(e_p; z) = \sum_{k=2}^{n} p_{n-2,k}(z)(1 - \alpha)z \int_0^1 p_{n-2,k-1}(t)t^p dt.
\]

**Proof.** If \(p = 0\), then the recurrence relation is immediate from \(G_n^\alpha(e_0; z) = 1, \ G_n^\alpha(e_1; z) = e_1(z), \ S_1(e_0; z) = (1 - z)(1 - \alpha)\) and \(S_2(e_0; z) = z(1 - \alpha)\). So, let us suppose that \(p \geq 1\).

Denote \(I = \int_0^1 p_{n-2,k-1}(t)t^p dt = \binom{n - 2}{k - 1} B(k + p, n - k)\), from the formula of definition, we can write

\[
G_n^\alpha(e_p; z) = \sum_{k=1}^{n-1} q^n_{k,k}(z) \ast I
\]

\[
+ (n - 1) \sum_{k=1}^{n} \alpha p_{n,k}(z) \ast I := S_1(e_p; z) + S_2(e_p; z) + S_3(e_p; z) \text{(say)}
\]

\[
\frac{d}{dz}(S_1(e_p; z)) = (n - 1) \sum_{k=1}^{n-1} (1 - \alpha) p_{n-2,k}(z)(1 - z) \ast I
\]

\[
= \frac{(n - 1)(1 - \alpha)}{z(1 - z)} \sum_{k=1}^{n-1} p_{n-2,k}(z)(1 - z)(k - (n - 1)z) \ast I
\]

\[
= \frac{(n - 1)}{1 - z} S_1(e_p; z) + \frac{(1 - \alpha)(n - 1)}{z(1 - z)} \sum_{k=1}^{n-1} p_{n-2,k}(z)(1 - z) \ast I.
\]
Now, from the formula for beta function \( pB(p, q) = (p + q)B(p + 1, q) \forall p, q > 0 \), we have \((k + p)B(k + p, n - k) = (n + p)B(k + p + 1, n - k)\). Therefore, we obtain

\[
(S_1(e_p; z))' = -\frac{n-1}{1-z}S_1(e_p; z) + \frac{(1-\alpha)(n-1)}{z(1-z)} \sum_{k=1}^{n-1} p_{n-2,k}(z)(1-z) \binom{n-2}{k} \times ((n+p)B(k+p+1, n-k) - pB(k+p, n-k))
\]

\[z(1-z)(S_1(e_p; z))' = (n+p)S_1(e_{p+1}; z) - ((n-1)z+p)S_1(e_p; z).
\]

(2.1)

Similarly,

\[z(1-z)(S_2(e_p; z))' = (n+p)S_2(e_{p+1}; z) - ((n-1)z+p+1)S_2(e_p; z)
\]

(2.2)

and

\[z(1-z)(S_3(e_p; z))' = (n+p)S_3(e_{p+1}; z) - (nz+p)S_3(e_p; z).
\]

(2.3)

By combining equations (2.1) – (2.3), we get the required result. \( \square \)

**Remark 1.** By using proposition 1 we get

\[G_n^\alpha(e_2; z) = z^2 + \frac{2z(1-z)}{n+1}\left(1 + \frac{1-\alpha}{n}\right).
\]

**Lemma 1.** (i) For \( n \in \mathbb{N}, p \in \mathbb{N} \cup \{0\} \) and \( \alpha \in [0,1] \), we have \( G_n^\alpha(e_p; 1) = 1 \).

(ii) For \( n, p \in \mathbb{N}, z \in \mathbb{C} \) and \( \alpha \in [0,1] \), we have

\[G_n^\alpha(e_p; z) = \frac{(n-1)!}{(n-1+p)!}\left((1-\alpha) \sum_{s=0}^{n-1} \binom{n-1}{s} z^s \triangle_1^z F_p(0) + \alpha \sum_{s=0}^{n} \binom{n}{s} z^s \triangle_1^z E_p(0)\right)
\]

\[= \frac{(n-1)!}{(n-1+p)!}\left((1-\alpha) \sum_{s=0}^{\min(n-1,p)} \binom{n-1}{s} z^s \triangle_1^z F_p(0) + \alpha \sum_{s=0}^{\min(n,p)} \binom{n}{s} z^s \triangle_1^z E_p(0)\right),
\]

where \( F_p(k) = \left(1 - \frac{k}{n-1}\right)E_p(k) + \frac{k}{n-1}E_p(k + 1), E_p(k) = \prod_{j=0}^{p-1} (k + j) \forall k \geq 0 \) and \( \triangle_1^z F_p(0) = \sum_{k=0}^{n} (-1)^k \binom{k}{s} F_p(s-k), \triangle_1^z E_p(0) = \sum_{k=0}^{n} (-1)^k \binom{k}{s} E_p(s-k) \) and \( \triangle_1^z F_p(0), \triangle_1^z E_p(0) \geq 0 \) for all \( s \) and \( p \).

**Proof.** (i) From the definition of the operators 1.3

\[G_n^\alpha(e_p; z) = \nu_{n,\alpha}(z) + (n-1) \sum_{k=1}^{n-1} q_{n,k}^{(\alpha)}(z) \ast I
\]

\[G_n^\alpha(e_p; 1) = \nu_{n,\alpha}(1) + (n-1) \sum_{k=1}^{n-1} q_{n,k}^{(\alpha)}(1) \ast I = 1.
\]

(ii) Let us denote \( \prod_{j=0}^{p-1} (\mu + j) := E_p(\mu) \). So, \( I \) can be written as

\[I = \binom{n-2}{k-1} B(k+p, n-k) = \frac{(n-2)!}{(k-1)!(n-k-1)!} \frac{(k+p-1)!(n-k-1)!}{(n-1+p)!}
\]

\[= \frac{(n-2)!}{(n-1+p)!} \frac{(k+p-1)!(n-k-1)!}{(k+1)(k+2) \cdots (k+p-1)} := \frac{(n-2)!}{(n-1+p)!} E_p(k),
\]

It is clear that \( E_p(\mu) \) and its derivative of any order are \( \geq 0 \forall \mu \geq 0 \), i.e. \( \triangle_1^z E_p(0) \geq 0 \forall k, p.\)
where

\[ F_j \]

This completes the proof.

So, we can write

\[
G_n^\alpha (e_p; z) = q_{n,n}^\alpha (z) + \frac{(n-1)!}{(n+p-1)!} \sum_{k=1}^{n-1} q_{n,k}^\alpha (z) * E_p(k)
\]

\[
= q_{n,n}^\alpha (z) + \frac{(n-1)!}{(n+p-1)!} \left( \sum_{k=0}^{n} q_{n,k}^\alpha (z) * E_p(k) - q_n^{(0)}(z) E_p(0) - q_n^{(0)}(z) E_p(n) \right)
\]

\[
= q_{n,n}^\alpha (z) - \frac{(n-1)!}{(n+p-1)!} q_{n,n}^\alpha (z) E_p(n) + \frac{(n-1)!}{(n+p-1)!} \sum_{k=0}^{n} q_{n,k}^\alpha (z) * E_p(k)
\]

\[
= \frac{(n-1)!}{(n+p-1)!} \sum_{k=0}^{n} q_{n,k}^\alpha (z) * E_p(k) = \frac{(n-1)!}{(n+p-1)!} \left( 1 - \alpha \right) \sum_{k=0}^{n-1} p_{n-2,k}(z)(1-z) * E_p(k)
\]

\[
+ (1 - \alpha) \sum_{k=0}^{n-1} p_{n-2,k-1}(z) z * E_p(k+1) + \alpha \sum_{k=0}^{n} p_{n,k}(z) * E_p(k)
\]

\[
= \frac{(n-1)!}{(n+p-1)!} \left[ (1 - \alpha) \sum_{k=0}^{n-1} \binom{n-1}{k} \left( 1 - \frac{k}{n-1} \right) E_p(k) + \frac{k}{n-1} E_p(k+1) \right] z^k (1-z)^{n-k-1}
\]

\[
+ \alpha \sum_{k=0}^{n} p_{n,k}(z) * E_p(k)
\]

\[
= \frac{(n-1)!}{(n+p-1)!} \left[ (1 - \alpha) \sum_{k=0}^{n-1} p_{n-1,k}(z) * F_p(k) + \alpha \sum_{k=0}^{n} p_{n,k}(z) * E_p(k) \right],
\]

where \( F_p(k) = \left( 1 - \frac{k}{n-1} \right) E_p(k) + \frac{k}{n-1} E_p(k+1). \)

\[
G_n^\alpha (e_p; z) = \frac{(n-1)!}{(n+p-1)!} \left( 1 - \alpha \right) \sum_{k=0}^{n-1} \binom{n-1}{k} z^k E_p(k) \sum_{j=0}^{n-1-k} (-1)^j \binom{n-1-k}{j} z^j
\]

\[
+ \alpha \sum_{k=0}^{n} \binom{n}{k} z^k * E_p(k) \sum_{j=0}^{n-k} (-1)^j \binom{n-k}{j} z^j
\]

Assume \( j = s - k \) in both the summations of above equation, then we get

\[
= \frac{(n-1)!}{(n+p-1)!} \left( 1 - \alpha \right) \sum_{s=0}^{n-1} \binom{n-1}{s} z^s \sum_{k=0}^{s} (-1)^{s-k} \binom{s}{k} F_p(k)
\]

\[
+ \alpha \sum_{s=0}^{n} \binom{n}{s} z^s \sum_{k=0}^{s} (-1)^{s-k} \binom{s}{k} E_p(k)
\]

\[
= \frac{(n-1)!}{(n+p-1)!} \left( 1 - \alpha \right) \sum_{s=0}^{n-1} \binom{n-1}{s} z^s \sum_{k=0}^{s} (-1)^{s-k} \binom{s}{k} F_p(s-k)
\]

\[
+ \alpha \sum_{s=0}^{n} \binom{n}{s} z^s \sum_{k=0}^{s} (-1)^{s-k} \binom{s}{k} E_p(s-k)
\]

\[
= \frac{(n-1)!}{(n+p-1)!} \left( 1 - \alpha \right) \min_{\{n-1,p\}} \sum_{s=0}^{\min\{n-1,p\}} \binom{n-1}{s} z^s \Delta_s^p F_p(0) + \alpha \sum_{s=0}^{\min\{n,p\}} \binom{n}{s} \sum_{q=0}^{s} \Delta_s^q E_p(0)
\].

This completes the proof. \( \square \)
3. Main results

The first main result of this section is the following upper estimate:

3.1. Upper estimate.

Theorem 1. (i) For all $p, n \in \mathbb{N} \cup \{0\}$, $|z| \leq r$ and $\alpha \in [0,1]$, we have $|G_n^\alpha(e_p; z)| \leq r^p$.

(ii) Let $f(z) = \sum_{p=0}^\infty c_p z^p \forall |z| < R$, $1 \leq r < R$. For all $|z| \leq r$, $n \in \mathbb{N}$ and $\alpha \in [0,1]$, we have

\[
|G_n^\alpha(f; z) - f(z)| \leq \frac{C_r(f)}{n}, \text{ where } C_r(f) = 2 \sum_{p=2}^\infty |c_p|p(p-1)r^p < \infty.
\]

Proof. From Lemma 1 (i) and (ii), we have

\[
G_n^\alpha(e_p; 1) = \frac{(n-1)!}{(n-1 + p)!} \left(1 - \alpha \right) \sum_{s=0}^{\min\{n-1,p\}} \binom{n-1}{s} \Delta_s^1 F_p(0) + \alpha \sum_{s=0}^{\min\{n,p\}} \binom{n}{s} \Delta_s^1 E_p(0) = 1,
\]

then

\[
|G_n^\alpha(e_p; z)| \leq \frac{(n-1)!}{(n-1 + p)!} \left(1 - \alpha \right) \sum_{s=0}^{\min\{n-1,p\}} \binom{n-1}{s} \Delta_s^1 F_p(0) + \alpha \sum_{s=0}^{\min\{n,p\}} \binom{n}{s} \Delta_s^1 E_p(0) r^s
\]

\[
\leq \frac{(n-1)!}{(n-1 + p)!} \left(1 - \alpha \right) \sum_{s=0}^{\min\{n-1,p\}} \binom{n-1}{s} \Delta_s^1 F_p(0) + \alpha \sum_{s=0}^{\min\{n,p\}} \binom{n}{s} \Delta_s^1 E_p(0) r^\alpha = r^p
\]

proves (i).

(ii) To find the upper estimate for the operators $G_n^\alpha(f; z)$, we first prove that $G_n^\alpha(f; z) = \sum_{p=0}^\infty c_p G_n^\alpha(e_p; z)$.

For this let $f_m(z) = \sum_{p=0}^m c_p e_p(z)$, $|z| \leq r$, $m \in \mathbb{N}$.

By linearity property of $G_n^\alpha(f; z)$, we can write $G_n^\alpha(f_m; z) = \sum_{p=0}^m c_p G_n^\alpha(e_p; z) \forall |z| \leq r$.

So, $\forall |z| \leq r$, $n \in \mathbb{N}$, it is sufficient to prove that $\lim_{m \to \infty} G_n^\alpha(f_m; z) = G_n^\alpha(f; z)$.

Now, for all $|z| \leq r$, $r \geq 1$, we have

\[
|G_n^\alpha(f_m; z) - G_n^\alpha(f; z)|
\]

\[
\leq |f_m(0) - f(0)| |(1-\alpha)(1-z)^{n-1} + \alpha(1-z)^n| + |f_m(1) - f(1)| |(1-\alpha)z^{n-1} + \alpha z^n|
\]

\[
+ (n-1) \sum_{k=1}^{n-1} q_{n,k}^\alpha \int_0^1 p_{n-2,k-1}(t) \cdot |f_m(t) - f(t)| dt
\]

\[
\leq |f_m(0) - f(0)| ((1-\alpha)(1+r)^{n-1} + \alpha(1+r)^n) + |f_m(1) - f(1)| ((1-\alpha)r^{n-1} + \alpha r^n)
\]

\[
+ (n-1) \sum_{k=1}^{n-1} \left\{ \binom{n-2}{k} (1-\alpha)r + \binom{n-2}{k-2} (1-\alpha)(1+r) + \binom{n}{k} \alpha r(1+r) \right\} r^{k-1}(1+r)^{n-k-1}
\]

\[
\times \int_0^1 p_{n-2,k-1}(t) \cdot |f_m(t) - f(t)| dt \leq C_{r,n} \|f_m - f\|_r,
\]
where

\[ C_{r,n}^\alpha = \alpha(r^n + (1 + r)^n) + (1 - \alpha)(r^{n-1} + (1 + r)^{n-1}) \\
+ (n - 1) \sum_{k=1}^{n-1} \left\{ (1 - \alpha) \binom{n-2}{k} r + (1 - \alpha) \binom{n-2}{k-2} (1 + r) + \alpha \binom{n}{k} r(1 + r) \right\} \int_0^1 p_{n-2,k-1}(t) dt. \]

Therefore, we get

\[ |G_n^\alpha(f; z) - f(z)| \leq \sum_{p=0}^{\infty} |e_p||G_n(e_p; z) - e_p(z)| = \sum_{p=2}^{\infty} |e_p||G_n(e_p; z) - e_p(z)| \]

as \(G_n^\alpha(e_0; z) = e_0(z) = 1\), \(G_n(e_1; z) = e_1(z) = z\).

Now, we have two cases: (A) \(2 \leq p \leq n\),  (B) \(p > n\).

Case (A): From Lemma 1 we obtain

\[ G_n^\alpha(e_p; z) - e_p(z) = \frac{(n-1)!}{(n-1+p)!} \left( (1 - \alpha) \sum_{s=0}^{p-1} \binom{n-1}{s} z^s \Delta^s_1 F_p(0) + \alpha \sum_{s=0}^{p-1} \binom{n}{s} z^s \Delta^s_1 E_p(0) \right) \]

\[ + \frac{(n-1)!}{(n-1+p)!} \left( (1 - \alpha) \binom{n-1}{p} \Delta^p_1 F_p(0) + \alpha \binom{n}{p} \Delta^p_1 E_p(0) \right) \]

\[ |G_n^\alpha(e_p; z) - e_p(z)| \leq r^p \left( 1 - \frac{(n-1)!}{(n-1+p)!} \right) \left\{ (1 - \alpha) \binom{n-1}{p} \Delta^p_1 F_p(0) + \alpha \binom{n}{p} \Delta^p_1 E_p(0) \right\} \]

\[ + \frac{(n-1)!}{(n-1+p)!} \left( (1 - \alpha) \binom{n-1}{p} \Delta^p_1 F_p(0) + \alpha \binom{n}{p} \Delta^p_1 E_p(0) \right) \]

\[ - (1 - \alpha) \binom{n-1}{p} z^p \Delta^p_1 F_p(0) - \alpha \binom{n}{p} z^p \Delta^p_1 E_p(0) \]

\[ \leq r^p \left( 1 - \frac{(n-1)!}{(n-1+p)!} \right) \left\{ (1 - \alpha) \binom{n-1}{p} \Delta^p_1 F_p(0) + \alpha \binom{n}{p} \Delta^p_1 E_p(0) \right\} \]

\[ + \frac{(n-1)!}{(n-1+p)!} r^p \left( (1 - \alpha) \binom{n-1}{p} \Delta^p_1 F_p(0) + \alpha \binom{n}{p} \Delta^p_1 E_p(0) \right) \]

\[ - \left( (1 - \alpha) \binom{n-1}{s} \Delta^s_1 F_p(0) + \alpha \sum_{s=0}^{p} \binom{n}{s} \Delta^s_1 E_p(0) \right) \]

\[ \leq 2r^p \left( 1 - \frac{(n-1)!}{(n-1+p)!} \right) \left\{ (1 - \alpha) \binom{n-1}{p} \Delta^p_1 F_p(0) + \alpha \binom{n}{p} \Delta^p_1 E_p(0) \right\}. \]
Now, by using the relationship between derivatives and differences of a function (see [5], pg-251), we get
\[
\frac{(n-1)!}{(n-1+p)!} \left\{ (1-\alpha) \binom{n-1}{p} \Delta^R_1 F_p(0) + \alpha \binom{n}{p} \Delta^R_1 E_p(0) \right\}
\]
\[
= \frac{(n-1)!}{(n-1+p)!} \left\{ (1-\alpha) \binom{n-1}{p} \left( 1 + \frac{p}{n-1} \right) + \alpha \binom{n}{p} \right\}
\]
\[
= \frac{(n-1)!}{(n-1+p)!} \left\{ (1-\alpha) \binom{n-2}{n-1+p} \binom{n}{n-1+p} + \alpha \binom{n}{n-1+p} \right\}
\]
\[
= \prod_{j=1}^{p-1} \left( (1-\alpha) \frac{n+(j-1)-p}{n+j-1} + \alpha \frac{n+j-p}{n+j} \right).
\]

Since \( 1 - \sum_{j=1}^{k} a_j \leq \sum_{j=1}^{k} (1-a_j) \), \( 0 \leq a_j \leq 1 \), \( j = 1, 2, \cdots, k \),
then
\[
1 - \frac{(n-1)!}{(n-1+p)!} \left\{ (1-\alpha) \binom{n-1}{p} \Delta^R_1 F_p(0) + \alpha \binom{n}{p} \Delta^R_1 E_p(0) \right\}
\]
\[
= 1 - \prod_{j=1}^{p-1} \left( (1-\alpha) \frac{n+(j-1)-p}{n+j-1} + \alpha \frac{n+j-p}{n+j} \right)
\]
\[
\leq \sum_{j=1}^{p-1} \left( 1 - \left( (1-\alpha) \frac{n+(j-1)-p}{n+j-1} + \alpha \frac{n+j-p}{n+j} \right) \right)
\]
\[
= p \sum_{j=1}^{p-1} \left( (1-\alpha) \frac{n+j-1}{n+j} + \frac{\alpha}{n+j} \right) \leq p(p-1) \left( \frac{1-\alpha}{n} + \frac{\alpha}{n+1} \right)
\]
\[
\Rightarrow |G^\alpha_n(e_p; z) - e_p(z)| \leq 2p(p-1) r^p \left( \frac{1-\alpha}{n} + \frac{\alpha}{n+1} \right) \leq \frac{2p(p-1)}{n} r^p.
\]

Case(B): By (i) and \( p > n \), we get
\[
|G^\alpha_n(e_p; z) - e_p(z)| \leq |G^\alpha_n(e_p; z)| + |e_p(z)| \leq 2r^p \leq \frac{2p(p-1)}{n} r^p.
\]

Finally, by combining both the cases (A) and (B), \( \forall p, n \in \mathbb{N} \), we obtain
\[
|G^\alpha_n(e_p; z) - e_p(z)| \leq \sum_{p=2}^{\infty} |e_p||G^\alpha_n(e_p; z) - e_p(z)| \leq \frac{2}{n} \sum_{p=2}^{\infty} |e_p|p(p-1) r^p,
\]

which gives the desired result. \( \square \)

Next, we have the following qualitative asymptotic formula for complex genuine \( \alpha \)-Bernstein-Durrmeyer operators \( G^\alpha_n(f; z) \).

3.2. Voronovskaja type result.

**Theorem 2.** Let \( 1 \leq r < R \) and suppose that \( f(z) = \sum_{p=0}^{\infty} e_p z^p \), \( \forall |z| < R \). Then \( \forall n \in \mathbb{N}, \alpha \in [0,1] \) and \( |z| \leq r \), we have
\[
\lim_{n \to \infty} n(G^\alpha_n(f; z) - f(z)) = z(1-z)f''(z), \text{ uniformly in } \overline{D_r},
\]
Proof. From Acar et al. \[1\], it is known that

\[
\lim_{n \to \infty} \left( n(G_n^\alpha(f; z) - f(z)) - \frac{(n+1-\alpha)}{n+1} z(1-z)f''(z) \right) = 0, \forall x \in [0, 1].
\]

According to Vitali’s theorem stated in \[7\], i.e. if a sequence \(\{f_n\}_{n \in \mathbb{N}}\) of analytic functions in \(D_R\) is bounded in each \(\overline{D}_r\), then it is uniformly convergent in \(\overline{D}_r\).

So, it is sufficient to show that the sequence \(\left\{ n(G_n^\alpha(f; z) - f(z)) - \frac{(n+1-\alpha)}{n+1} z(1-z)f''(z) \right\}_{n \in \mathbb{N}}\) is bounded in each \(\overline{D}_r\).

By using Theorem 1, we get

\[
\left| n(G_n^\alpha(f; z) - f(z)) - \frac{(n+1-\alpha)}{n+1} z(1-z)f''(z) \right| \leq C_r(f) + \frac{(n+1-\alpha)}{n+1} r(1+r) \|f''\|_r \leq C_r(f) + r(1+r) \|f''\|_r,
\]

for all \(z \in \overline{D}_r\), \(1 \leq r < R\) and \(\alpha \in [0, 1]\), where \(C_r(f)\) is the same constant as defined in Theorem \[1\] which proves the theorem.

Finally, we find the exact order of approximation by using the above asymptotic result.

3.3. Exact order of approximation.

**Theorem 3.** Let \(\alpha \in [0, 1], f : D_R \to \mathbb{C}\) is analytic in \(D_R, R > 1\) with \(f(z) = \sum_{p=0}^{\infty} c_p z^p\). If \(f\) is not a polynomial of degree \(\leq 1\), then for all \(1 \leq r < R\), we have

\[
\|G_n^\alpha(f) - f\|_r \sim \frac{1}{\alpha}, n \in \mathbb{N}
\]

in \(\overline{D}_r\), where the constants in the equivalence depend on \(f\) and \(r\).

**Proof.** By using Theorem \[2\] there exist constants \(0 < C_1, C_2 < \infty\) independent of \(n\) such that

\[
C_1 \leq \|G_n^\alpha(f) - f\|_r \leq C_2, \Rightarrow \frac{C_1}{n} \leq \|G_n^\alpha(f) - f\|_r \leq \frac{C_2}{n}
\]

holds in \(\overline{D}_r\).

Thus, we get the required result. \(\square\)

In the following result, we find the exact order of approximation for the derivatives of our operators \(G_n^\alpha(f; z)\):

3.4. Simultaneous approximation.

**Theorem 4.** Let \(\alpha \in [0, 1], f : D_R \to \mathbb{C}\) is analytic in \(D_R, R > 1\) with \(f(z) = \sum_{p=0}^{\infty} c_p z^p\). If \(f\) is not a polynomial of degree \(\leq \max\{1, 1-l\}\), then for all \(1 \leq r < r_1 < R\) and \(l \in \mathbb{N}\), we have

\[
\left\|G_n^{\alpha(l)}(f) - f^{(l)}\right\|_r \sim \frac{1}{\alpha}, n \in \mathbb{N}
\]

in \(\overline{D}_r\), where the constants in the equivalence depend on \(f, r, r_1\) and \(l\).

**Proof.** Denoting by \(\Gamma\) the circle of radius \(r_1 > r\) and centre 0, by Cauchy’s formula, it follows that \(\forall z \leq r, \mu \in \Gamma\) with \(|\mu - z| \geq r_1 - r\), and \(n \in \mathbb{N}\), we have

\[
|G_n^{\alpha(l)}(f; z) - f^{(l)}(z)| \leq \frac{l!}{2\pi i} \int_{\Gamma} \frac{|G_n^\alpha(f; \mu) - f(\mu)|}{|\mu - z|^{l+1}} |d\mu| \leq \frac{l!}{2\pi} \frac{2\pi r_1}{(r_1 - r)^{l+1}} + \|G_n^\alpha(f) - f\|_r \leq C_{r_1}(f) \frac{l!}{n} \frac{r_1}{(r_1 - r)^{l+1}}.
\]

From Theorem \[2\] we obtain

\[
\lim_{n \to \infty} n(G_n^{\alpha(l)}(f; z) - f^{(l)}(z)) = (z(1-z)f''(z))^{(l)}, \text{ uniformly in } \overline{D}_r.
\]
So, \( \exists \) constants \( 0 < M_1, M_2 < \infty \) independent of \( n \) such that
\[
M_1 \leq n\|G_n^\alpha(t) - f(t)\|_r \leq M_2, \quad \Rightarrow \quad \frac{M_1}{n} \leq \|G_n^\alpha(t) - f(t)\|_r \leq \frac{M_2}{n},
\]
holds in \( D_r \).

This completes the proof. \( \square \)

4. Further Application

As applications of our complex operators (1.3), one can study some shape preserving properties. Thus, reasoning exactly as it was done in the case of complex \( \alpha \)-Bernstein polynomials in [6], one can prove that beginning with an index, the summation-integral operators \( G_n^\alpha(f; z) \) in this paper approximate the analytic functions, preserving in addition, the geometric properties of star likeness, convexity, univalence and spirallikeness in a certain disk.

5. Data Availability Statement

Data sharing not applicable to this manuscript as no datasets were generated or analysed during the current study.

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