Formal Context Generation using Dirichlet Distributions

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Abstract. We suggest an improved way to randomly generate formal contexts based on Dirichlet distributions. For this purpose we investigate the predominant way to generate formal contexts, a coin-tossing model, recapitulate some of its shortcomings and examine its stochastic model. Building up on this we propose our Dirichlet model and develop an algorithm employing this idea. By comparing our generation model to a coin-tossing model we show that our approach is a significant improvement with respect to the variety of contexts generated. Finally, we outline a possible application in null model generation for formal contexts.

Keywords: Formal Concept Analysis, Dirichlet Distribution, Random Context

1 Introduction

Formal concept analysis (FCA) is often used as a tool to represent and extract knowledge from data sets expressed as cross-tables between a set of objects and a set of attributes, called formal contexts. Many real-world data sets can be transformed, i.e., scaled, with little effort to be subjected to methods from FCA. There has been a great effort to develop methods to efficiently compute both formal concepts and related properties. This has led to a multitude of algorithms at our disposal.

An important task for the investigation of data sets through FCA is to decide whether observed patterns are meaningful or not. Related fields, e.g., graph theory and ecology, employ null model analysis, cf. [13]. This method randomizes data sets with the constraint to preserve certain properties. Hence, this method may be applied to FCA through randomly generating formal contexts in a similar manner. One step in this direction is to develop a novel way to randomly generate formal contexts, which is the topic of this paper. Randomly generating formal contexts is also relevant for other applications, e.g., comparing the performance of FCA algorithms, as done in [1, 9]. However, methods for generating adequate random contexts are insufficiently investigated [3].

Authors are given in alphabetical order. No priority in authorship is implied.
A naïve approach for a random generation process is to uniformly draw from the set of all formal contexts for a given set of attributes $M$. This approach is infeasible as the number of formal contexts with pairwise distinct objects is $2^{2|M|}$. A related problem is the random generation of Moore families as investigated in [7]. There, the author suggested an approach to uniformly draw from the set of closure systems for a given set of attributes. However, this approach is not feasible on sets of more than seven elements.

The predominant method to randomly generate formal contexts is a coin-tossing process, mainly due to the ease of use and the lack of proper alternatives. Yet, this approach is biased to generate a certain class of contexts. Here we step in with a novel approach. Based upon a thorough examination of the coin-tosssing approach we suggest an improved stochastic model to randomly generate formal contexts using Dirichlet distributions. For this we analyze the influence of the distribution parameters on the resulting contexts. Afterwards we empirically evaluate our model on randomly generated formal contexts with six to ten attributes. We show that our approach is a significant improvement upon the coin-tossing process in terms of the variety of contexts generated.

As for the structure of this paper in Section 2 we first give a short problem description and recall some basic notions from FCA followed by a brief overview of related work in Section 3. We proceed by stochastically modeling and examining the coin-toss and suggest the Dirichlet model in Section 4. In Section 5 we evaluate our model empirically and discuss our findings followed by an outline on the application for null models in Section 6. Lastly in Section 7 we give our conclusions and an outlook.

## 2 FCA Basics and Problem Description

We begin by recalling basic notions from formal concept analysis. For a thorough introduction we refer the reader to [8]. A formal context is a triple $\mathbb{K} := (G,M,I)$ of sets. The elements of $G$ are called objects and the elements of $M$ attributes of the context. The set $I \subseteq G \times M$ is called incidence relation, meaning $(g,m) \in I \iff$ the object $g$ has the attribute $m$. We introduce the common operators, namely the object derivation $\cdot : \mathcal{P}(G) \rightarrow \mathcal{P}(M)$ by $A \subseteq G \mapsto A' := \{m \in M \mid \forall g \in A : (g,m) \in I\}$, and the attribute derivation $\cdot : \mathcal{P}(M) \rightarrow \mathcal{P}(G)$ by $B \subseteq M \mapsto B' := \{g \in G \mid \forall m \in B : (g,m) \in I\}$. A formal concept of a formal context is a pair $(A,B)$ with $A \subseteq G$, $B \subseteq M$ such that $A' = B$ and $B' = A$. We then call $A$ the extent and $B$ the intent of the concept. With $\mathcal{B}(\mathbb{K})$ we denote the set of all concepts of some context $\mathbb{K}$. A pseudo-intent of $\mathbb{K}$ is a subset $P \subseteq M$ where $P \neq P'$ and $Q'' \subseteq P$ holds for every pseudo-intent $Q \subseteq P$. In the following we may omit formal when referring to formal contexts and formal concepts. Of particular interest in the following is the class of formal contexts called contranominal scales. These contexts are constituted by $([n],[n],\neq)$ where $[n] := \{1,\ldots,n\}$. The number of concepts for a contranominal scale with $n$ attributes is $2^n$, thus having $2^n$ intents and therefore zero pseudo-intents. If a context $(G,M,I)$ fulfills the property that for every $m \in M$ there exists an object $g \in G$ such that $g' = M \setminus \{m\}$, then $(G,M,I)$ contains a subcontext isomorphic to a contranominal scale of size $|M|$, i.e, $\exists G \subseteq G$ such that $(G,M,I \cap (G \times M)) \cong ([n],[n],\neq)$ with $n = |M|$. 
In this paper we deal with the problem of randomly generating a formal context given a set of attributes. Our motivation originates in Figure 1 where we show 25000 randomly generated contexts with ten attributes. The model used to generate these contexts is a *coin-toss*, as recalled more formally in Section 4. This method is the predominant approach to randomly generate formal contexts. In Figure 1 we plotted the number of intents versus the number of pseudo-intents for each generated context. We may call this particular plotting method *I-PI plot*, where every point resembles a particular combination of intent number and pseudo-intent number, called *I-PI coordinate*. Note that having the same I-PI coordinate does not imply that the corresponding contexts are isomorphic. However, different I-PI coordinates imply non-isomorphic formal contexts. The reason for employing intents and pseudo-intents is that they correspond to two fundamental features of formal contexts, namely *concept lattice* and *canonical implication base*, which we will not introduce in the realm of this work. We observe in Figure 1 that there appears to be a relation between the number of intents and the number of pseudo-intents. This was first mentioned in a paper by Borchmann [2]. Naturally, the question emerges whether this empirically observed correlation is based on a structural connection between intents and pseudo-intents rather than chance. As it turned out in a later study this apparent correlation is most likely the result of a systematic bias in the underlying random generation process [3].

We therefore strive after a novel approach that does not exhibit this or any other bias. Consistently with the above the I-PI coordinates and their distribution are used as an indicator for how diverse created contexts are. The coin-toss approach will serve as a baseline for this. We start by analyzing the coin-toss model which leads to a formalization fitted to the requirements of FCA in Section 4. This enables us to discover Dirichlet distributions as a natural generalization.

### 3 Related Work

The problem depicted in Section 2 gained not much attention in the literature so far. The first observation of the correlation between the number of intents and pseudo-intents in randomly generated contexts was by Borchmann as a side note in [2]. The phenomenon was further investigated in [3] with the conclusion
that it is most likely a result of the random generation process. Their findings suggest that the coin tossing approach as basis for benchmarking algorithms is not a viable option and other ways need to be explored. Related to this is a work by Ganter on random contexts in [7]. There the author looked at a method to generate closure systems in a uniform fashion, using an elegant and conceptually simple acceptance-rejection model. However, this method is infeasible for practical use. Furthermore, the authors in [12] developed a synthetic formal context generator that employs a density measure. This generator is composed of multiple algorithms for different stages of the generation process, i.e., initialization to reach minimum density, regular filling and filling close to the maximum density. However, the survey in [3] found that the generated contexts exhibited a different type of systematic bias.

4 Stochastic Modelling

In the following we analyze and formalize a stochastic model for the coin-toss approach. By this we unravel crucial points for improving the random generation process. To enhance the readability we write $Z \sim \text{Distribution}$ to denote that $Z$ is both a random variable following a certain Distribution and a realization of said random variable.

4.1 Coin Toss - Direct Model

Given a set of attributes $M$ we construct $I \subseteq G \times M$ utilizing a direct coin-toss model as follows. We let $G$ be a set of objects with $|G| \sim \text{DiscreteUniform}(|M|, 2^{|M|})$ and draw a probability $p \sim \text{Uniform}(0, 1)$. For every $(g, m) \in G \times M$ we flip a binary coin denoted by $\omega(g, m) \sim \text{Bernoulli}(p)$, i.e., $\omega \in \Omega = \{0, 1\}$ where $P(\omega = 1) = p$ and $P(\omega = 0) = 1 - p$, and let

$$X(g, m)(\omega) := \begin{cases} \{(g, m)\} & \text{if } \omega = 1 \\ \emptyset & \text{if } \omega = 0. \end{cases}$$

Then we obtain the incidence relation by $I := \{X(g, m)(\omega(g, m)) \mid (g, m) \in G \times M\}$. Hence, $I$ contains all those $(g, m)$ where the coin flip was a success, i.e., $\omega = 1$. If we partition the set of coin tosses though grouping, i.e., $\{\omega(g, m) \mid m \in M\} \mid g \in G\}$, and look for some object $g$ at the number of successful tosses, we see that they follow a $\text{Binomial}(|M|, p)$ distribution. In detail, a binomial distribution with $|M|$ trials and a success probability of $p$ in each trial. This means that no matter how $G, M$ and the probability $p$ are chosen, we always end up with a context where the number of attributes per object is the realization of a $\text{Binomial}(|M|, p)$ distributed random variable for every object $g \in G$.

Example 4.1. We generated 5000 contexts with the coin-tossing approach. A plot of their I-PI coordinates and a histogram showing the distribution of pseudo-intents are shown in Figure 2. In the histogram we omitted the high value for zero pseudo-intents. This value emerges from the large amount of generated contranominal scales
by the coin-toss model. In particular, 1714 of the contexts contain a contranominal scale and have therefore zero pseudo-intents.

We observe that most of the contexts have less than 100 pseudo-intents with varying numbers of intents between 1 and 1024. The majority of contexts has an I-PI coordinate close to an imaginary curve and the rest has, in most cases, less pseudo-intents, i.e., their I-PI coordinates lie below this curve. Looking at the histogram we observe a varying number of pseudo-intents. We have a peak at zero and a high number of 126 contexts with one pseudo-intent. Additionally there is a peak of 62 contexts with 36 pseudo-intents and a peak of 55 contexts with 73 pseudo-intents. In between we have a low between 18 to 23 pseudo-intents and one around 50.

4.2 Coin Toss - Indirect Model

In order to exhibit a critical point in the direct coin-tossing model we introduce an equivalent model using an indirect approach, called indirect coin-toss. Furthermore, this model serves as an intermediate stage to our proposed generation scheme.

As we just established, the number of successful coin tosses, i.e., number of attributes per object, follows a binomial distribution. An indirect model that generates the same kind of formal contexts as direct coin-tossing can therefore be obtained by using a binomial distribution. In contrast to the direct model we first determine the total number of successful coin-tosses per object and pick the specific attributes afterwards. We formalize this model as follows.

Given a set of attributes $M$, as before, we let $G$ be a set of objects with $|G| \sim \text{DiscreteUniform}([|M|, 2^{|M|}])$ and draw a probability $p \sim \text{Uniform}(0,1)$. For every $g \in G$ we let $\theta_g \sim \text{Binomial}(|M|, p)$ be the number of attributes associated to that object $g$. Hence, $\theta_g \in \{0, \ldots, |M|\}$. We let $\Theta_g := \{B \subseteq M \mid |B| = \theta_g\}$ be the set of all possible attribute combinations for $g$ and denote by $\text{DiscreteUniform}(\Theta_g)$ the discrete uniform distribution on $\Theta_g$. Now for every $g \in G$ we let $B_g \sim \text{DiscreteUniform}(\Theta_g)$ to obtain the set of attributes belonging to the object $g$ and define the incidence relation by $I := \bigcup \{(g, m) \mid m \in B_g\} \mid g \in G\}$. We present pseudocode for the indirect
Algorithm 1: Indirect Coin-Tossing

Input: a set of attributes $M$
Output: a formal context $(G,M,I)$

1. $N \sim \text{DiscreteUniform}[|M|, 2^{|M|}]$
2. $G := \{1, \ldots, N\}$
3. $p \sim \text{Uniform}(0,1)$
4. For all $g \in G$ do
5. \hspace{1em} $\theta_g \sim \text{Binomial}(|M|, p)$
6. \hspace{1em} $\Theta_g := \{B \subseteq M \mid |B| = \theta_g\}$
7. \hspace{1em} $B_g \sim \text{DiscreteUniform}(\Theta_g)$
8. \hspace{1em} $I_g := \{(g,m) \mid m \in B_g\}$
9. \hspace{1em} $I := \bigcup_{g \in G} I_g$
10. return $(G,M,I)$

Coin-tossing in Algorithm 1. This serves as a foundation for our proposed generation algorithm in Section 4.3. The indirect formulation reveals that the coin tossing approach is restricting the class of possible distributions for $\theta_g$, i.e., the number of attributes for the object $g$, to only the set of binomial ones. Thereby it introduces a systematic bias as to which contexts are being generated. An example for a context that is unlikely to be created by the coin-tossing model is a context with ten attributes where every object has either two or seven attributes.

4.3 Dirichlet Model

One way to improve the generating process is to use a broader class of discrete distributions to determine $\theta_g$ (cf. Algorithm 1, Line 5). In the indirect coin-tossing model we were drawing from the class of binomial distributions with a fixed number of trials. In contrast to that we now draw from the class of all discrete probability distributions on the same sample space, i.e., distributions that have the same support of $\{0, \ldots, |M|\}$, which in our case represents the possible numbers of attributes for an object. For finite sample spaces every probability distribution can be considered as a categorical distribution. Therefore, a common method to draw from the class of all discrete probability distributions is to employ a Dirichlet distribution. In Bayesian statistics this distribution is often utilized as prior distribution of parameters of a categorical or multinomial distribution [6].

One way to define the Dirichlet distribution is to use gamma distributions [6]. A Gamma$(\rho, \tau)$ distribution with shape parameter $\rho \geq 0$ and scale parameter $\tau > 0$ can be characterized on the real line with respect to the Lebesgue measure by a density function

$$f(z | \rho, \tau) = \frac{1}{\Gamma(\rho) \tau^{\rho}} \exp^{-z/\tau} z^{\rho-1} 1_{(0, \infty)}(z)$$

if $\rho > 0$, where $1_S$ denotes the indicator function on some set $S$ and $\Gamma$ denotes the gamma function. In the case of $\rho = 0$ the gamma distribution degenerates at zero.
The Dirichlet($\beta\alpha$) distribution with parameters $\beta\alpha = (\beta\alpha_1, \ldots, \beta\alpha_K)$, where $\beta > 0$, $\alpha_i \geq 0$ for all $i$ and $\alpha_i > 0$ for some $i \in \{1, \ldots, K\}$ and $\sum_{i=1}^{K}\alpha_i = 1$, is a probability distribution on the set of $K$-dimensional discrete distributions. Given independent random variables $Z_1, \ldots, Z_K$ with $Z_i \sim \text{Gamma}(\beta\alpha_i, 1)$ it is defined as the distribution of a random vector $(Y_1, \ldots, Y_K)$ where $Y_i = Z_i / \sum_{j=1}^{K} Z_j$ for $i \in \{1, \ldots, K\}$. Note that this allows for some of the variables to be degenerate at zero which will be useful in the application for null models, as we will describe in Section 6. If $\alpha_i > 0$ for all $i$ the random vector $(Y_1, \ldots, Y_K)$ has a density

$$f(y_1, \ldots, y_K \mid \beta\alpha_1, \ldots, \beta\alpha_K) = \frac{\Gamma(\beta)}{\prod_{i=1}^{K}\Gamma(\beta\alpha_i)} \prod_{i=1}^{K} y_i^{\beta\alpha_i - 1} 1_S(y_1, \ldots, y_K)$$

(1)

on the simplex $S = \{ (y_1, \ldots, y_K) \in \mathbb{R}^K \mid y_i \geq 0, \sum_{i=1}^{K} y_i = 1 \}$. Note that $f$ in Equation (1) is a density with respect to the $(K-1)$-dimensional Lebesgue measure and we can rewrite $f$ as a $(K-1)$-dimensional function by letting $y_K = 1 - \sum_{i=1}^{K-1} y_i$ and using an appropriate simplex representation. Also note that the elements of $(Y_1, \ldots, Y_K)$ have the expected value $E(Y_i) = \alpha_i$, the variance $\text{Var}(Y_i) = \frac{\alpha_i(1-\alpha_i)}{\beta+1}$ and the co-variance $\text{Cov}(Y_i, Y_j) = \frac{\alpha_i\alpha_j}{\beta+1}$ for $i \neq j$. Hence, the parameter $\alpha$ is called base measure or mean as it describes the expected value of the probability distribution and $\beta$ is called precision parameter and describes the variance of probability distributions with regard to the base measure. A large value for $\beta$ will cause the drawn distributions to be close to the base measure, a small value will cause them to be distant. A realization of a Dirichlet distributed random variable is an element of $S$ and can therefore be seen as probability vector of a $K$-dimensional categorical distribution.

In Figure 3 we illustrate the effects of varying $\beta$. We show different distributions of probabilities for three categories drawn from 3-dimensional Dirichlet distributions. The base measure $\alpha$ in each case is the uniform distribution, i.e., $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, the precision parameter $\beta \in \{30, 3, \frac{3}{10}\}$ varies. The choice of $\beta = 3$ then results in a uniform distribution on the probability simplex. For comparison we also chose $\beta = 30$ and $\beta = \frac{3}{10}$. A possible interpretation of the introduced simplex is the following. Each corner of the simplex can be thought of as one category. The closer a point in the simplex is to a corner the more likely this category is to be drawn.
Algorithm 2: Dirichlet Approach

```plaintext
Input : a set of attributes $M$
Output: a formal context $(G,M,I)$

1. $N \sim \text{DiscreteUniform}[|M|, 2^{|M|}]$
2. $G := \{1, \ldots, N\}$
3. $\nu := (1, \ldots, 1)$  // $(|M|+1 \text{ ones})$
4. $\alpha := \nu/\|\nu\|_1$
5. $\beta := |M| + 1$
6. $p \sim \text{Dirichlet}(\beta \alpha)$
7. for all $g \in G$ do
8. \hspace{1em} $\theta_g \sim \text{Categorical}(p)$
9. \hspace{1em} $\Theta_g := \{B \subseteq M | |B| = \theta_g\}$
10. \hspace{1em} $B_g \sim \text{DiscreteUniform}(\Theta_g)$
11. \hspace{1em} $I_g := \{(g,m) | m \in B_g\}$
12. \hspace{1em} $I := \bigcup_{g \in G} I_g$
13. return $(G,M,I)$
```

In the rest of this section we describe the model for our proposed random formal context generator. Given a set of attributes $M$, we let $G$ be a set of objects with $|G| \sim \text{DiscreteUniform}[|M|, 2^{|M|}]$. We then use a probability vector $p \sim \text{Dirichlet}(\beta \alpha)$ to determine the probabilities for an object to have 0 to $|M|$ attributes, where $\alpha := \nu/\|\nu\|_1$ with $\nu := (1, \ldots, 1) \in \mathbb{R}^{|M|+1}$. By using $\alpha$ as base measure and $\beta := |M| + 1$, which implies $\beta \alpha = (1, \ldots, 1)$, we draw uniformly from the set of discrete probability distributions. As a different way to determine $\theta_g$ we can therefore use $p = (p_0, \ldots, p_{|M|})$ as probabilities of a $|M| + 1$ dimensional categorical distribution $\theta_g \sim \text{Categorical}(p)$. These categories are the numbers of attributes for an object, i.e., $P(\theta_g = c) = p_c$ for $c \in \{0, \ldots, |M|\}$. Looking back at Algorithm 1, Lines 3 and 5 we replace the binomial distribution based on a uniformly distributed random variable by a categorical distribution based on a Dirichlet distributed one. Afterwards we proceed as in Algorithm 1. We present pseudocode for the Dirichlet approach in Algorithm 2 to emphasize the changes we made compared to Algorithm 1 and as a further reference for the experiments in Section 5.

## 5 Experiments

In this section we present a first experimental investigation of Algorithm 2. We evaluated the results by examining the numbers of intents and pseudo-intents of generated contexts. The contexts were generated using Python 3 and all further computations, i.e., the I-PI coordinates, were done using conexp-clj. The generator code as well as the generated contexts can be found on GitHub.

3 https://github.com/exot/conexp-clj

4 https://github.com/maximilian-felde/formal-context-generator
5.1 Observations

For each experiment we generated 5000 formal contexts with an attribute set $M$ of ten attributes using Algorithm 2. We also employed slightly altered versions of this algorithm. Those alterations are concerned with the choice of $\beta$, as we will see in the following. We plotted the resulting I-PI coordinates and a histogram of the pseudo-intents for each experiment. In the histogram we omitted the value for zero pseudo-intents, i.e., the peak for contexts containing a contranominal scale of size $|M|$. A comparable experiment on ten attributes is described in Example 4.1, where a (direct) coin-toss model was utilized. The results of Example 4.1 are shown in Figure 2. This will serve as a baseline in terms of variety and distribution of I-PI coordinates.

First we used Algorithm 2 without alterations. The results are depicted in Figure 4. We can see that most of the generated contexts have less than 75 pseudo-intents and the number of intents varies between 1 and 1024. There is a tendency towards contexts with fewer pseudo-intents and we cannot observe any context with more than 101 pseudo-intents. The number of generated contexts containing contranominal scales of size $|M|$ was 2438. The histogram shows that the number of contexts that have a certain quantity of pseudo-intents decreases as the number of pseudo-intents increases with no significant dips or peaks. In this form the Dirichlet approach does not appear to be an improvement over the coin-tossing method. In contrary, we observe the spread of the number of pseudo-intents to be smaller than in Example 4.1.

Our next experiment was randomizing the precision parameter $\beta$ between 0 and $|M|+1$, i.e., let $\beta \sim \text{Uniform}(0,|M|+1)$ in Algorithm 2, Line 5. We will refer to this alteration as variation A. The results are shown in Figure 5. We can see that again many contexts have less than 100 pseudo-intents and the number of intents once again varies over the full possible range. There are 1909 contexts that contain a contranominal scale of size $|M|$. However, we notice that there is a not negligible number of contexts with over 100 and up to almost 252 pseudo-intents, which constitutes theoretical maximum [3]. Most of these gather around nearly vertical lines close to 75, 200, 380, 600 and 820 intents. Even though most of the contexts have an I-PI coordinate along one of those lines there are a few contexts in-between

Figure 4: Dirichlet, $\alpha = (\frac{1}{|M|+1}, \ldots, \frac{1}{|M|+1})$, $\beta = |M|+1$.  

\[\text{Figure 4: Dirichlet, } \alpha = (\frac{1}{|M|+1}, \ldots, \frac{1}{|M|+1}), \beta = |M|+1.\]
100 and 175 pseudo-intents that do not fit this description. Looking at the histogram we can observe again that while the number of pseudo-intents increases the number of generated contexts to that pseudo-intent number decreases. This is in contrast to Example 4.1. This time, however, we can clearly see a peak at seven to ten pseudo-intents with 190 contexts having ten pseudo-intents. Apart from this we observed no other significant dips or peaks. We also tried randomizing the base measure $\alpha$ using Dirichlet distributions. However, this did not improve the results.

Since the last experiment revealed that small values for $\beta$ resulted in a larger variety of contexts we will now investigate those in more detail. For this we introduce a constant factor $c$ such that $\beta = c \cdot (|M| + 1)$. We find for the experiment called variation B the factor $c = 0.1$ suitable, as we will explain in Section 5.2. A plot of the results can be found in Figure 6. We can see that most of the contexts have less than 150 pseudo-intents and the number of intents is between 1 and 1024. Furthermore, the quantity of contexts containing a contranominal scale of size $|M|$ is 1169. This number is about 700 lower than in variation A, roughly 500 lower compared to the coin-tossing results in Example 4.1, and over 1200 lower than in the unaltered Dirichlet approach. We can again observe the same imaginary lines as mentioned for variation A, with even more contrast. Finally, we observe that the space between these lines contains significantly more I-PI coordinates. Choosing even smaller values for $c$ may result in less desirable sets of contexts. In particular, we found that lower values for $c$ appear to increase the bias towards the imaginary lines.

The histogram (Figure 6) of variation B differs distinguishably to the one in Figure 5. The distribution of pseudo-intent numbers is more volatile and more evenly distributed. There is a first peak of 366 contexts with ten pseudo-intents, followed by a dip to eleven contexts with seventeen pseudo-intents and more relative peaks of 50 to 60 contexts each at 28, 36 and 45 pseudo-intents. After 62 pseudo-intents the number of contexts having this amount of pseudo-intents or more declines with the exception of the peak at 120 pseudo-intents.

We established that both variations of Algorithm 2 with $\beta \sim \text{Uniform}(0,|M|+1)$ and $\beta = c \cdot (|M| + 1)$ are improvements upon the coin-tossing approach. In order to further increase the confidence in our Dirichlet approach we have generated 100,000
contexts with the coin-tossing approach as well as with both variations for six, seven and eight attributes. We compared the distinct I-PI coordinates after certain numbers of generated contexts. The results of this experiment is shown in Figure 7. Each subfigure shows the results for one attribute set size. We have plotted the number of generated contexts versus the number of distinct I-PI coordinates for the coin-toss (green solid line), variation A (orange dashed line) and variation B (blue dotted line). In all three plots we recognize that there is a steep increase of distinct I-PI coordinates at the beginning followed by a fast decline in new I-PI coordinates, i.e., a slow increase in the total number of distinct I-PI coordinates, for all three random generation methods. The graphs remind of sublinear growth. For all three attribute set sizes we can observe that the graphs of variation A and B lie above the graph of the coin-toss. Hence, variation A and B generated more distinct contexts compared to the coin-toss. Exemplary for \(|M| = 7\) the coin-tossing approach resulted in 1963 distinct I-PI coordinates and reached them after around 99,000 generated contexts. Variation A generated around 19,000 contexts until it hit 1963 distinct I-PI coordinates and reached a total of around 2450 after 100,000 contexts generated. Variation B reached the same number of distinct I-PI values already at around 13,000 generated contexts and resulted in 2550 distinct I-PI coordinates.

5.2 Discussion

We begin the discussion by relating the parameters of the Dirichlet approach to the variety of generated contexts. Afterwards we explore the discrepancy in the quantities of contexts that contain a contranominal scale and discuss the observed imaginary lines. Lastly, we discuss the ability of the different approaches for generating pairwise distinct I-PI coordinates efficiently.

The Dirichlet approach has two parameters, one being the \(\beta\) parameter related to the variance of the Dirichlet distribution, the other being the \(\alpha\) parameter which describes the expected value of the Dirichlet distribution, as explained more formally in Section 4.3. A large value for \(\beta\) results in categorical distributions that have probability vectors close to the base measure \(\alpha\), following from the definition.

\[
\alpha = \left(\frac{1}{|M|+1}, \ldots, \frac{1}{|M|+1}\right), \beta = 0.1(|M|+1).
\]
A small value for $\beta$ results in categorical distributions where the probability vectors are close to the corners or edges of the simplex, see Figure 3c. As already pointed out in Section 4.3, those corners of the simplex can be thought of as the categories, i.e., the possible numbers of attributes an object can have. This implies that for large $\beta$ the categories are expected to be about as likely as the corresponding probabilities in the base measure. Whereas, for small $\beta$ one or few particular categories are expected to be far more likely than others.

We have seen that the Dirichlet approach without alterations generated around 2450 contexts containing a contranominal scale of size $|M|$. This number was 1900 for variation A and 1200 for variation B. One reason for the huge number of contranominal scales generated by the base version of the Dirichlet approach is that most of the realizations of the Dirichlet distribution (Algorithm 2, Line 6) are inner points of the probability simplex, i.e., they lie near the center of the simplex. These points or probability vectors result in almost balanced categorical distributions (Algorithm 2, Line 8), i.e., every category is drawn at least a few times for a fixed number of draws. This fact may explain the frequent occurrence of contranominal scales. The expected number of objects with $|M| - 1$ attributes that need to be generated for a context to contain a contranominal scale is low. In more detail, we only need to hit the $|M|$ equally likely distinct objects, having $|M| - 1$ attributes during the generation process. To be more precise, the mean $\mu_N$ and the standard deviation $\sigma_N$ of the number of required objects with $|M| - 1$ attributes can easily be computed via $\mu_N = N \sum_{k=1}^{N} \frac{1}{k}$ and $\sigma_N^2 = N \sum_{k=1}^{N} \frac{N-k}{k^2}$ with $N := (|M| - 1) = |M|$, cf. [5], as this is an instance of the so-called Coupon Collector Problem. For example for a context with ten attributes we get $\mu_{10} \approx 29.3$ and $\sigma_{10} \approx 11.2$, hence we need to generate on average around 30 objects with nine attributes to create a contranominal scale. Although, there is already a high probability of obtaining a contranominal scale after generating around 18 objects. This means if we generate a context with $|G| = 300$ objects and the probability for the category with nine attributes is around 10% we can expect the context to contain a contranominal scale.

If we use a lower value for $\beta$ we tend to get less balanced probability vectors from the Dirichlet distribution and therefore generate less contexts that contain a contranominal scale. The pathological case is a $\beta$ close to zero, which leads to contexts where all or nearly all objects have the same number of attributes. Even then we could expect at least $\frac{1}{|M|+1}$ of the contexts generated to contain a contranominal scale. In this case we basically draw from the set of categories, i.e., from the possible numbers of attributes. Those are related as corners of the simplex and the probability to land in the corner belonging to the category of $|M| - 1$ attributes is approximately $\frac{1}{|M|+1}$.

Contexts where every object has the same number of attributes are referred to as contexts with fixed row-density in [3]. They were used to show that the coin-tossing approach in practice does not generate a whole class of contexts. An explanation for the imaginary lines observed in Figures 5 and 6 is that they correspond to contexts with fixed row-density, cf. [3, Fig. 5]. As pointed out in the last paragraph very low values of $\beta$ the Dirichlet approach predominantly generates contexts where all objects belong to few or even only one category. This explanation is further supported by the increasing bias of the context’s I-PI coordinates to form those imaginary lines for
decreasing values of $\beta$. It also accounts for the peak at ten pseudo-intents in the histograms for variations A and B. This is due to the fact that a fixed row-density context with density $\frac{8}{10}$ that contains all possible objects has exactly ten pseudo-intents, cf. [3, Prop. 1]. This is again related to the Coupon Collector Problem. The solution to this problem yields the expected number of objects that we need in order to hit every possible combination. In particular for the case of the peak at ten pseudo-intents, $N = \binom{10}{8} = 45$, $\mu_{45} \approx 198$ and $\sigma_{45} \approx 56$, meaning if we generate a fixed row-density context with around 200 objects we can expect it to contain all possible combinations and therefore have ten pseudo-intents. This fits well with the observed 366 contexts with ten pseudo-intents in variation B. Consider the case that we only generate fixed row-density contexts containing all possible attribute combinations and all densities are equally likely. The expected number of contexts with eight attributes and therefore ten pseudo-intents for 5000 generated contexts then is $\frac{5000}{11} \approx 455$. Naturally, variation B does not predominantly generate fixed row-density contexts or even fixed row-density contexts with all possible attribute combinations. Hence, the aforementioned 366 observed contexts with ten pseudo-intents seem reasonable.

Lastly we discuss the observations from counting distinct I-PI coordinates. In Figure 7 we can see that the Dirichlet approach results in a broader variety of contexts in comparison to the coin-tossing for any fixed number of generated contexts. All three plots show that there is an initial phase where contexts with new I-PI coordinates are frequently generated followed by a far longer part where contexts with new I-PI coordinates become increasingly rare. This is not surprising, since the number of possible I-PI coordinates is finite and the probability to re-hit increases with every distinct generated I-PI coordinate. Note that this is an effect that would also be observed when using a perfectly uniform sampling mechanism. Nonetheless what we can observe is that it takes the Dirichlet approach significantly longer, compared to the coin-tossing model, to reach a point where only few new I-PI coordinates are generated.

6 Applications

We see for our Dirichlet approach at least two major applications. For one, an improved random generation process can be used to facilitate more reliable time performance comparisons of FCA algorithms. The obtained contexts exhibit less bias and therefore a greater variety concluding in more robust runtime results.

A second application is the generation of null models for formal contexts, on which we elaborate for the rest of this section. Null models are well employed in graph theory and are an adaption of a statistical concept. The idea there is to generate random graphs that match some structural properties of some reference graph. A null model is therefore not a single graph but denotes a whole family of graphs that share said properties or an algorithm that generates such a family. Given some reference graph, null models are used as a baseline to decide whether an observed property in the graph is meaningful. In a nutshell, the idea of null models is to generate random graphs that are in some way similar to some reference graph in order to investigate some other property in this very same reference graph. There is extensive literature on random graphs available, including a thorough review article by Newman [11].
Two basic structural features in graph theory are the number of nodes and the degree distribution. Hence, one way to define a null model for some graph is to generate random graphs with an equal number of nodes where the degree distribution as the structural property is preserved. This is a well known idea, e.g., suggested by Newman when he criticizes the shortcomings of Bernoulli random graphs. [10] An extension to this null model employs expected degree distributions [4]. This model is able to capture certain peculiarities arising from real-world graphs more robustly.

The idea of null models can be translated to the realm of formal contexts in principle as sketched in the following paragraphs. A possible connection is that for every formal context one may construct a bipartite graph. The vertex set is constituted by the disjoint union of the object set and the attribute set. The edge set is constructed from the incidence relation in a natural way. Applying now the idea of a graph theoretical null model that preserves the degree distribution to this bipartite graph can be understood as a family of contexts where the row- and column-sums are invariant. However, in bipartite graphs it is reasonable to only look at one of the elements of the partition. This relates to analyzing a context with focus on either the attributes
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or the objects. Then a null model can be defined by keeping structural properties, e.g., preserving the absolute degree distribution, on only one of the elements of the partition. In the following we focus on row-sums, i.e., the number of attributes per object. This corresponds very well to our model for randomly generating formal contexts, cf. Section 4.3. Of course this choice is arbitrary to some extend. Interchanging the roles of attributes and objects would allow to focus on column-sums.

An easy way to generate similar contexts, with respect to row-sum distribution, is to permute the attributes of every object in the context. This clearly leaves the row-sums intact and therefore also the row-sum distribution. A null model that allows a bit more variation is one where we use the object set node degree distribution of some reference context. One possibility is to use that distribution as prior for a categorical distribution considering the node degrees as categories. From this distribution we then draw for each object in the reference context a new node degree. Using this we draw accordingly many elements from the attribute set. Hence, every attribute gets an edge to this object. This procedure does not necessarily preserve the object degree distribution, but the expected distribution equates the distribution of the reference context.

One further step is to use Dirichlet distributions. We can use the normalized object degree distribution of the reference context as a base measure for the Dirichlet distribution. Here we might need the ability to deal with zero components in the base measure. In this case a large value for $\beta$ is necessary to control the deviation from the object degree distribution. Again the realizations of such a Dirichlet distribution are used as priors for a categorical distribution with the object node degrees as categories. For each object in the reference context we then generate new node degrees utilizing this categorical distribution. By definition does the expected object degree distribution resemble the distribution of the reference context. For this null model approach we can employ our results from Section 4.3. One may apply Algorithm 2 with the appropriate number of objects in Line 1, the reference contexts object degree distribution as $\alpha$ in Line 4 and a large value for $\beta$, e.g., $\beta = 1000 \cdot (|M|+1)$, in Line 5.

7 Conclusions and Outlook

Analyzing a stochastic model for the coin-tossing approach for randomly generating formal contexts lead in a natural way to a more sophisticated context generator. By addressing the carved out limitations of the underlying binomial model we comprehended the versatility of Dirichlet distributions for random contexts. Based on this we developed an algorithm which can easily be implemented. This algorithm draws random contexts from a significantly larger class of contexts compared to the common coin-toss approach. We empirically evaluated this new approach for different sizes of attribute sets. The conducted experiments showed that we generated a significantly broader variety of contexts. This increased variety may increase the reliability of random context based investigations, like algorithm performance comparisons.

The novel Dirichlet approach also allows us to generate null models. Basically in this generation model we can tailor the contexts generated with respect to a given
row-sum-distribution. This method can be employed for empirical investigations related to formal contexts, like social network analysis with FCA or ecology.

There are still various unsolved problems remaining. For example, how can one minimize the amount of generated contranominal scales? This could be done through considering different base measures. Investigating those is itself a fruitful next step in order to understand their relation to real-world context generation, like null models. Also this novel approach to random formal contexts raises many more questions about the relation of the data table context and Dirichlet distributions.

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