On the distributed evaluation of recursive queries over graphs

Stéphane Grumbach∗  Fang Wang†  Zhilin Wu§

Abstract
Logical formalisms such as first-order logic (FO) and fixpoint logic (FP) are well suited to express in a declarative manner fundamental graph functionalities required in distributed systems. We show that these logics constitute good abstractions for programming distributed systems as a whole, since they can be evaluated in a fully distributed manner with reasonable complexity upper-bounds. We first prove that FO and FP can be evaluated with a polynomial number of messages of logarithmic size. We then show that the (global) logical formulas can be translated into rule programs describing the local behavior of the nodes of the distributed system, which compute equivalent results. Finally, we introduce local fragments of these logics, which preserve as much as possible the locality of their distributed computation, while offering a rich expressive power for networking functionalities. We prove that they admit tighter upper-bounds with bounded number of messages of bounded size. Finally, we show that the semantics and the complexity of the local fragments are preserved over locally consistent networks as well as anonymous networks, thus showing the robustness of the proposed local logical formalisms.

1 Introduction
Logical formalisms have been widely used in different fields of computer science to provide high-level programming abstractions. The relational calculus used by Codd to describe data-centric applications in an abstract way, is at the origin of the technological and commercial success of relational database management systems [16]. Datalog, an extension of Horn clause logic with fixpoints, has been widely used to specify functionalities involving recursion [17].

The development of distributed applications over networks of devices is generally a very tedious task, involving handling low level system details. The lack of high-level programming abstraction has been identified as one of the roadblocks for the deployment of networks of cooperating objects [15].

Recently, the use of queries to define network applications has been considered. Initially, the idea emerged in the field of sensor networks. It was suggested to see the network as a database, and interact with it through declarative queries. Several systems have been developed, among which Cougar [9] and TinyDB [14], supporting SQL dialects. Queries are processed in a centralized manner, leading to distributed execution plans.

∗INRIA-LIAMA, CASIA, PO Box 2728, Beijing 100190, PR China. Stephane.Grumbach@inria.fr
†Lab of Computer Science, Institute of Software, Chinese Academy of Sciences, Beijing 100190. wangf@ios.ac.cn
‡China Graduate School, Chinese Academy of Sciences, Beijing 100049, China
§LIAMA, CASIA, PO Box 2728, Beijing 100190, PR China. zlwu@liama.ia.ac.cn
More recently, query languages were proposed as a mean to express communication network problems such as routing protocols [13] and declarative overlays [12]. This approach, known as declarative networking is extremely promising for it offers a high-level abstraction to program networks. It was also shown how to use recursive queries to perform diagnosis of asynchronous systems [1], network monitoring [18], as well as self-organization protocols [10]. Distributed query languages provide new means to express complex network problems such as node discovery [3], route finding, path maintenance with quality of service [6], topology discovery, including physical topology [5], etc.

However, there is a lack of systematic theoretical investigations of query languages in the distributed setting, in particular on their semantics, as well as the complexity of their distributed computation. In the present paper, we consider a distributed evaluation of classical query languages, namely, first-order logic and fixpoint logic, which preserves their classical semantics.

First-order logic and fixpoint logic have been extensively investigated in the context of database theory [2] as well as finite model theory [7]. Since the seminal paper of Fagin [8], showing that the class NP corresponds exactly to problems which can be expressed in existential second-order logic, many results have linked Turing complexity classes with logical formalisms. Parallel complexity has also been considered for first-order queries which can be evaluated in constant time over circuits with arbitrary fan-in gates [11].

This raised our curiosity on the distributed potential of these classical query languages to express the functionalities of communication networks, which have to be computed in a distributed manner over the network itself. If their computation can be distributed efficiently, they can form the basis of a high level abstraction for programming distributed systems as a whole.

We rely on the classical message passing model [4]. Nodes exchange messages with their neighbors in the network. We consider four measures of complexity: (i) the in-node computational complexity, rarely addressed in distributed computing; (ii) the distributed time complexity; (iii) the message size; and (iv) the per-node message complexity. The behavior of the nodes is governed by an algorithm, the distributed query engine, which is installed on each node, and evaluates the queries by alternating local computation and exchange of queries and results with the other nodes.

We first consider the distributed complexity of first-order logic and fixpoint logic with inflationary semantics, which accumulates all the results of the different stages of the computation. Note that our result carry over for other formalisms such as least fixpoint. We prove that the distributed complexity of first-order queries is in $O(\log n)$ in-node time, $O(\Delta)$ distributed time ($\Delta$ is the diameter of the network), messages of size $O(\log n)$, and a polynomial number of messages per node. For fixpoint, a similar bound can be shown but with a polynomial distributed time.

We then consider the translation of logical formulae that express properties of graphs at a global level, into rule programs that express the behavior of nodes at a local level, and compute the same result. We introduce a rule language, Netlog, which extends Datalog, with communication primitives, and is well suited to express distributed applications, ranging from networking protocols to distributed data management. Netlog is supported by the Netquest system, on which the examples of this paper have been implemented. We prove that graph programs in Datalog$^-$ [2] can be translated to Netlog programs. Since it is well known that first-order and fixpoint logics can be translated in Datalog$^-$ [7], it follows that global logical formulae can be translated in behavioral programs in Netlog producing the same result.

Finally, we define local fragments of first-order and fixpoint logic, respectively $FO_{loc}$ and $FP_{loc}$. These fragments provide a good compromise in the trade-off between expressive power and efficiency.
of the distributed evaluation. Important network functionalities (e.g. spanning tree, on-demand routes etc.) can be defined easily in $F P_{loc}$. Meanwhile, its complexity is constant for all our measures, but the distributed time which is linear in the diameter for $F O_{loc}$ and in the size of the network for $F P_{loc}$.

Our results shed light on the complexity of the distributed evaluation of queries. Note that if the communication network is a clique (unbounded degree), our machinery resembles Boolean circuits, and we get constant distributed time, a result which resembles the classical $A C^0$ bound [11].

We have restricted our attention to bounded degree graphs and synchronous systems. Most of our algorithms carry over, or can be extended to unrestricted graphs, and asynchronous computation, but not necessarily the complexity bounds. Interestingly, the results for the local fragments carry over for other classes of networks, such as locally consistent networks or anonymous networks, thus showing the robustness of the languages $F O_{loc}$ and $F P_{loc}$.

The paper is organized as follows. In the next section, we recall the basics of first-order and fixpoint logics. In Section 3, the computation model is presented. Section 4 is devoted to distributed first-order query execution, and Section 5 to fixpoint query execution. In Section 6, we introduce a behavioral language, Netlog, and show that FP formulae can be translated into equivalent Netlog programs. In Section 7, we consider the restriction to the local fragments, and show that they can be evaluated over different types of networks.

2 Graph logics

We are interested in functions on graphs that represent the topology of communication networks. We thus restrict our attention to finite connected bounded-degree undirected graphs. Let $D$ be the bound on the degree.

We assume the existence of an infinite ordered set of constants, $U$, the universe of node Id’s. A graph, $G = (V,G)$, is defined by a finite set of nodes $V \subset U$, and a set of edges $G \subseteq V \times V$.

We express the functions on graphs as queries. A query of arity $\ell$ is a computable mapping from finite graphs to finite relations of arity $\ell$ over the domain of the input graph closed under graph isomorphisms. A Boolean query is a query with Boolean output.

Logical languages have been widely used to define queries. A formula $\varphi$ over signature $G$ with $\ell$ free variables defines a query mapping instances of finite graphs $G$ to relations of arity $\ell$ defined by: $A = \{(x_1, \ldots, x_\ell) | G \models \varphi(x_1, \ldots, x_\ell)\}$. We equivalently write $G, A \models \varphi$.

We denote by $F O$ the set of queries definable using first-order formulae. First-order queries can be used in particular to check locally forbidden configurations for instance. Their expressive power is rather limited though.

Fixpoint logics on the other hand allow to express fundamental network functionalities, such as those involving paths. If $\varphi(T; x_1, \ldots, x_\ell)$ is a first-order formula with $\ell$ free variables over signature $\{G, T\}$, where $T$ is a new relation symbol of arity $\ell$, called the fixpoint relation, then $\mu(\varphi(T))$ denotes a fixpoint formula whose semantics is defined inductively as the inflationary fixpoint $I$, of the sequence:

$$I_0 = \emptyset;$$
$$I_{i+1} = \varphi(I_i) \cup I_i, i \geq 0$$

3
where $\varphi(I_i)$ denotes the result of the evaluation of $\varphi(T)$ with $T$ interpreted by $I_i$. The $I_i$’s constitute the stages of the computation of the fixpoint. We write $G, I \models \mu(\varphi(T))$, whenever $I$ is the fixpoint of the formula $\varphi(T)$ as defined by the above induction.

It is well known [2] that on ordered domains, the class of graph queries defined by inflationary fixpoint, denoted FP, captures exactly all Ptime mappings, that is mappings that can be computed on a Turing machine in time polynomial in the size of the graph.

The following examples illustrate the expressive power of FP for distributed applications.

The formula $\mu(\varphi(T)(x, h, d))$ for instance where the formula $\varphi(T)(x, h, d)$ is defined by:

$$(G(x, h) \land h = d) \lor (G(x, h) \land \exists z(T(h, z, d) \land x \neq z) \land \neg \exists uT(x, u, d))$$

defines a table-based routing protocol (OLSR like) on the graph $G$, where $h$ is the next hop from $x$ to destination $d$.

A spanning tree from a node $x$ satisfying $\text{ReqNode}(x)$ can be defined by a fixpoint formula $\mu(\psi(\text{ST})(x, y))$, where the formula $\psi(\text{ST})(x, y)$ is defined by:

$$(G(x, y) \land \text{ReqNode}(x)) \lor
\neg \exists x' \text{ST}(x', y) \land \exists w(\text{ST}(w, x) \land w \neq y) \land G(x, y) \land \forall w' \forall x'(\text{ST}(w', x') \land G(x', y) \Rightarrow x' \geq x))$$

Similarly, an On-Demand Routing protocol (AODV like), can be defined by the fixpoint queries $\mu(\varphi(\text{RouteReq})(x, y, d))$ and $\mu(\psi(\text{NextHop})(x, y, d))$, where $d$ is a constant and $\varphi(\text{RouteReq})(x, y, d)$ is defined by:

$$(G(x, y) \land \text{ReqNode}(x) \land \text{dest}(d)) \lor
\exists w(\text{RouteReq}(w, x, d) \land w \neq y) \land G(x, y) \land x \neq d \land \neg \exists w' \text{RouteReq}(w', y, d))$$

and $\psi(\text{NextHop})(x, y, d)$ is defined by:

$$(\text{RouteReq}(x, d, d) \land y = d) \lor (\exists z \text{NextHop}(y, z, d) \land \text{RouteReq}(x, y, d))$$

where a route request is first emitted by a node $x$ satisfying $\text{ReqNode}(x)$, then a path defined by next hops from that node to destination $d$ is established by backward computation on the route request.

### 3 Distributed evaluation

We are interested in this paper in the distributed evaluation of queries. We assume that each query to the network is posed by a requesting node (the node satisfying the predicate $\text{ReqNode}(x)$ in the examples of the previous section).

The result of a query shall be distributed over all the nodes of the network. In a query $Q(x_1, x_2, \cdots, x_\ell)$, one of the attributes $x_i$ denotes the holding node, written explicitly as $@x_i$, that is the node which holds the results relative to $x_i$. More precisely, the tuple $(a_1, \cdots, a_{i-1}, a, a_{i+1}, \cdots, a_\ell)$ is held by node $a$, such that $Q(a_1, \cdots, a_{i-1}, a, a_{i+1}, \cdots, a_\ell)$ holds. For simplicity, we will choose the first variable as holding attribute.

The results of fixpoint queries are thus distributed on holding nodes. In the OLSR like example of the previous section, each node shall hold its routing table as a result of the evaluation of the query.
The nodes of the network are equipped with a distributed query engine to evaluate queries. It is a universal algorithm that performs the distributed evaluation of any network functionality expressed using queries. The computation relies on the message passing model for distributed computing [4].

The configuration of a node is given by a state, an in-buffer for incoming messages, an out-buffer for outgoing messages, and some local data and metadata used for the computation. We assume that the metadata on each node contain a unique identifier, the upper bound on the size of the network, \( n \), and the diameter of the network, \( \Delta \). We also assume that the local data of each node includes all its neighbors with their identifiers.

We distinguish between computation events, performed in a node, and delivery events, performed between nodes which broadcast their messages to their neighbors. A sequence of computation events followed by delivery events is called a round of the distributed computation.

A local execution is a sequence of alternating configurations and events occurring on one node. We assume that the network is static, nodes are not moving, and that the communication has no failure.

We assume that at the beginning of the computation of a query, all the nodes are idle, in initial state, with their in-buffers, and out-buffers empty. Note that, it is easy to extend the present computational framework to a multithreaded computation with several concurrent queries running in the network. The requesting node broadcasts its query to its neighbors. The incoming messages in the subsequent nodes trigger the start of their query engine computation.

The evaluation of a query terminates when the out-buffers of all nodes are empty. The result is distributed over the network in the memories of all nodes. Note that alternative termination modes are also possible.

We consider four measures of the complexity of the distributed computation:

- The per-round in-node computational complexity, IN-TIME/ROUND, is the maximal computational time of the in-node computation in one round;
- The distributed time complexity, DIST-TIME, is the maximum number of rounds of any local execution of any node till the termination;
- The message size, MSG-SIZE, is the maximum number of bits in messages;
- The per-node message complexity, #MSG/NODE, is the maximum number of messages sent by any node till the termination of the evaluation.

There is a trade-off between the in-node computation and the communication. Our objective is to distribute the workload in the network as evenly as possible, with a balanced amount of computation and communication on each node. Clearly, centralized computation can be carried on by loading the topology of the network on the requesting node, and performing the evaluation by in-node computation. The centralized evaluation of FO and FP admits the following complexity bounds.

**Proposition 1.** Let \( G \) be a network of diameter \( \Delta \), with \( n \) nodes. Let \( \varphi \) be a FO formula with \( v \) variables. The complexity of the centralized evaluation of the query \( \varphi \) on \( G \) is given by:

\[
\begin{align*}
\text{IN-TIME/ROUND} & \quad O(n^v \log n) \\
\text{DIST-TIME} & \quad O(\Delta) \\
\text{MSG-SIZE} & \quad O(\log n) \\
\text{#MSG/NODE} & \quad O(n)
\end{align*}
\]

Suppose \( \mu(\varphi(T)(x_1, \ldots, x_\ell)) \) is a FP formula such that \( T \) is a relational symbol of arity \( \ell \), and it contains \( v = \ell + k \) variables (\( \ell \) free and \( k \) bounded). Then the complexity upper-bound of the
centralized evaluation of the query \( \mu(\varphi(T)(x_1, \ldots, x_\ell)) \) on \( G \) is the same as the above complexity for FO formulae except for the IN-TIME/ROUND which is in \( O(n^{\ell+\epsilon} \log n) \).

Note that all nodes, but the requesting node, have \( O(\log n) \) per-round in-node complexity. The proof of this result follows from classical results on data complexity of query languages \([2]\). In the sequel, we focus exclusively on distributed query evaluation.

## 4 Distributed complexity of FO

In this section we show that the distributed evaluation of FO can be done with a polynomial number of messages but logarithmic in-node computation per round. The result relies on a naive distributed query engine for FO, \( \mathcal{QE}_{FO} \), which works as follows.

The requesting node starts the computation by submitting a query. The nodes broadcast Boolean answers to queries when they have them, and otherwise queries they cannot answer, to their neighbors. Each node reduces queries by instantiating variables. In \( \mathcal{QE}_{FO} \), nodes start instantiating from the leftmost quantified variable, and from the rightmost free variable. The last instantiated free variable therefore denotes the holding node of the query, on which the corresponding tuples will be stored. The nodes simplify the queries by removing all facts, or subformulae they can fully evaluate.

Let \( \varphi \) be a first-order formula with \( \ell \) free variables. The query engine handles the following message types: message \( \{? B \varphi\} \) for Boolean queries, message \( \{? x_1 \ldots ? x_i a_{i+1} \ldots a_\ell \varphi\} \) for non-Boolean queries, and message \( \{B \varphi\} \) for answers of Boolean queries.

Each node stores pairs \((\text{query}, \text{parent query})\), in a query table, associating the query being evaluated to the query from which it derives. Nodes also store the Boolean answers \( !B \varphi \) and non-Boolean answers \( \langle a_1 \ldots a_\ell \rangle \) to queries in an answer table.

We will see that the diameter \( \Delta \) of the graph induces an upper-bound on the response time of queries. The algorithm uses clocks that are defined according to this upper-bound. **Clocks** are associated to the evaluation of queries as well as subqueries. After the time of a clock associated to a query on a node has elapsed, the value of the query can be determined by the node. From now on, we assume that we are given a clock compliant with the communication graph. The value of the clocks will be defined in Definition 1 below.

The main steps of the query engine work as follows. Note that we assume for simplicity in the sequel that the system is synchronous. This assumption can be relaxed easily in asynchronous systems without impact on the complexity by using spanning trees rather than the clocks.

**Initial Boolean query emission** For a Boolean query, the requesting node, say \( a \), broadcasts the query, \( ?B \varphi \), adds \( (?B \varphi, nil) \) into the query table, and sets a **clock** for the answer. Meanwhile it instantiates the leftmost bounded variable and produces a subquery. For an existentially quantified formula \( \exists x \psi \), if \( \psi(a) \) is true then it is a witness that \( \exists x \psi \) is true. For a universally quantified formula \( \forall x \psi \), if \( \psi(a) \) is false then it is a counterevidence and \( \forall x \psi \) is false. If the node doesn’t have the answer to \( \psi(a) \), it inserts \( \psi(a) \) along with its parent query into the query table, e.g. \( (?B \psi(a), ?B \exists x \psi) \), broadcasts \( \psi(a) \) and also sets a **clock** for \( \psi(a) \). If no witness / counterevidence is received before the clock elapses, then \( \exists x \psi \) is false / \( \forall x \psi \) is true. It then recursively handles \( \psi(a) \) in the same way.

**Boolean query reception** Every node upon reception of a Boolean query, \( ?B \varphi \), checks at first its query table. If there is a record for this query, it does nothing. Otherwise its behavior is similar to the Boolean query emission of the requesting node, with the difference that it also broadcasts the answer.
**Boolean answer reception** Every node receiving an answer to a Boolean query, !Bϕ, checks its answer table. If there is a record, it does nothing. Otherwise, it stores the answer, checks the query table. If it is waiting for the answer, it then tries to evaluate the parent query (if it has one), stores and broadcasts its answer if it has; if it is not waiting for the answer, it broadcasts !Bϕ.

**Initial non-Boolean query emission** The requesting node submits and broadcasts the query ?x₁...?xℓϕ(x₁,...,xℓ). It sets the clock, inserts (?x₁...?xℓϕ(x₁,...,xℓ), nil) into the query table, instantiates the rightmost free variable to get the subquery, which is ?x₁...?xℓ−1!aϕ(x₁,...,xℓ−1,a), and broadcasts it. Meanwhile the subquery is inserted into the query table and handled further by the requesting node. When all the free variables are instantiated, the Boolean query !Bϕ(a₁...aℓ) is emitted and a record (?Bϕ(a₁...aℓ), !a₁...!aℓϕ(a₁...aℓ)) is inserted in the query table of node a₁.

**Non-Boolean query reception** Every node checks its query table when it receives a query ?x₁...?xℓ−1!a!aℓϕ(x₁,...,xℓ−1,a), nil) in the query table, its behavior is then similar to the initial non-Boolean query emission with i − 1 free variables.

**Distributed tuple answer collection** If the Boolean query !Bϕ(a₁,...,aℓ) receives a positive answer to it, and there is a record (?Bϕ(a₁...aℓ), !a₁...!aℓϕ(a₁...aℓ)) in the query table, ⟨a₁,...,aℓ⟩ is stored in the answer table of the current node which corresponds to the instantiation of the leftmost free variable, that is the holding node for the answer.

We now turn to the clocks which parameterize the first-order query engines. The following theorem provides an upper-bound on the distributed time complexity of the evaluation of a formula.

**Theorem 1.** For networks of diameter Δ, the distributed time complexity of the evaluation of a formula with w variables or constants by QFÖ is bounded by 2Δw.

**Proof.** The proof is done by induction on the number of variables and constants in the query ψ.

**Basis:** Assume w = 2. There are three possibilities: two constants, or two variables, or one constant and one variable in the query ψ.

- If there are two constants, say a and b, the query ψ is propagated to a and gets the value of the atom G(a,b) which takes at most Δ rounds. Then the answer of ψ is sent back to the requesting node which takes at most Δ rounds. The total time is at most 2Δ rounds.

- If there are one variable x and one constant a in ψ, the query is propagated to every node at which the variable is instantiated and we get the answers of G(x,a), which takes Δ rounds.

  - When the variable is free, the answer is stored in the local table of x.
  - When the variable is bounded, the witness/counter evidence of ψ is sent back to the requesting node which takes at most Δ rounds. Or if after Δ rounds, the requesting node does not receive any sub-answer, it is sound to consider that there are no witnesses or counterevidences.

So the total time is 2Δ rounds in both cases.

- If there are two variables then it takes Δ rounds to instantiate one variable at every node (suppose the formula obtained is η) and then Δ rounds for the other variable (suppose the formula obtained is ξ). Therefore 2Δ in all.
- If both of the variables are free variables, if \( \xi \) is true, then the tuple is stored in the local table.

- If the first variable is free and the second one is bounded, then it takes \( \Delta \) rounds for the witness/counter evidence (if there is one) to get to the first instantiating node from the second one, if \( \eta \) is true, suppose \( a \) is the instantiation of the free variable, then the answer is stored in the local table.

- If both variables are bounded, then it takes \( \Delta \) rounds for the answer to get to the first instantiating node and then \( \Delta \) to the requesting node.

So the total time is \( 4\Delta \) rounds.

Therefore, for \( w = 2 \), the time is bounded by \( 2\Delta w \) rounds.

**Induction:** Suppose that when the sum of variables and constants is \( w \), e.g. there are \( l \) free variables, \( k \) bounded variables, \( c \) constants and \( w = l + k + c \), the time is bounded by \( 2\Delta w \) rounds. We prove the result for \( w + 1 \):

- when there are \( c + 1 \) constants: there are \( \Delta \) rounds (at most) for the sub-query to get to the additional constant node and \( \Delta \) rounds for the answer to the sub-query getting back. Therefore the total time is at most \( 2\Delta(w + 1) \) rounds.

- when there are \( k + 1 \) bounded variables: w.l.o.g. we assume that the additional bounded variable is the leftmost bounded variable, then \( \Delta \) rounds are sufficient before instantiating the second variable to instantiate the first variable, and \( \Delta \) rounds for the answer getting to the first instantiating node from the second one. Therefore the total time is at most \( 2\Delta(w + 1) \) rounds.

- when there are \( l + 1 \) free variables: it takes \( \Delta \) rounds for instantiating the additional free variable. So the total time is \( 2\Delta w + \Delta \).

Therefore, the distributed time time is bounded by \( 2\Delta(w + 1) \).

We can now settle the values of the clocks in the query engine.

**Definition 1.** The value of the clock in a network of diameter \( \Delta \), for an FO query with \( w \) variables or constants is \( 2\Delta w \).

The next result shows the robustness of the algorithm: its independence from the order in which messages are handled by the query engine.

**Proposition 2.** The distributed first-order query engine is insensitive to the order of the incoming messages in a round.

**Proof.** There are two fundamental steps in the algorithm of the query engine: query propagation and result construction. During query propagation, queries and subqueries arriving on one node have no interaction. They generate entries in the query table. During result construction, results of independent queries do not interfere, and results of the same query are handled with a set semantics.

We can now define the distributed inference.
Definition 2. Let \( G \) be a graph, \( \psi \) a formula with \( \ell \) free variables, and \( A \) a finite relation of arity \( \ell \). We write \( G, A \models_{\text{FO}} \psi \) if and only if \( A \) is the union of all the answers produced by the query engine \( \mathcal{QE}_{\text{FO}} \) on all nodes, upon request of \( \psi \) from any node.

We next prove the soundness and completeness of the query engine.

Theorem 2. For any network \( G \) of diameter at most \( \Delta \), and any first-order formula \( \psi \), \( G, A \models \psi \) if and only if \( G, A \models_{\text{FO}} \psi \).

Proof. First observe that it is sufficient to prove the result for Boolean formulae. Indeed, if there are \( \ell \) free variables in the query, they get instantiated by all possible \( n^\ell \) instantiation when the query travels around the system of \( n \) nodes, resulting in \( n^\ell \) Boolean first-order queries. The result of each query (tuple of \( \ell \) constants) is then stored at the key node if it satisfies the Boolean query.

The result is also rather obvious for variable-free formulae. Suppose that a query has \( c \) (\( c \geq 2 \)) constants and no variables. The query is broadcasted to every node and once it successively reaches nodes, it gets the Boolean value for the atoms containing the corresponding constants, replaces the corresponding atoms by their value and produces a new query which is broadcasted again. The result is obtained when the query has reached (at most) \( c - 1 \) of the constants. Then the answer is sent back to the requesting node. The total time required is at most \( 2\Delta(c - 1) \). The clock time being fixed at \( 2\Delta c \) rounds, it is suffices to get the result.

The rest of the proof is done by induction on the number of bounded variables for Boolean formulae.

Basis: Assume the query has one bounded variable. Then it must has at least one constants, so \( c \geq 1 \). First it is broadcasted by the requesting node and the variable is instantiated by every node, thus producing \( n \) sub-queries with at most \( c + 1 \) constants. After the sub-queries reach at most \( c - 1 \) of the constants (note that one of the constants stems from instantiating the variable and the sub-queries gets it immediately at the instantiating node) and get their answers, the witness for \( \exists \) or the counterevidence for \( \forall \) is sent back to the requesting node which then produces the final answer. If no witnesses/counterevidences are received before the clock time elapses, a negative/positive answer is produced by the requesting node.

Induction: Assume that if the query has \( k \) (\( k \geq 2 \)) bounded variables and \( c \) constants, i.e. the query is in the form:
\[
\psi_k = A_1 x_1 \ldots A_k x_k \varphi(x_1 \ldots x_k) \quad \text{(denoting } \exists \text{ or } \forall \text{ by } A) ,
\]
then \( G \models \psi_k \) if and only if \( G \models_{\text{FO}} \psi_k \).

We prove the result for the case where there are \( k + 1 \) bounded variables in the query
\[
\psi_{k+1} = A_1 x_1 \ldots A_{k+1} x_{k+1} \varphi(x_1 \ldots x_{k+1})
\]
After the first variable has been instantiated at each node, the \( n \) sub-queries of the form
\[
\psi'_k = A_2 x_2 \ldots A_{k+1} x_{k+1} \varphi(x_2 \ldots x_{k+1})
\]
are queries with \( k \) bounded variables and \( c + 1 \) constants. They are then further propagated by the instantiating node. By induction assumption, \( G \models \psi'_k \) if and only if \( G \models_{\text{FO}} \psi'_k \), so every node gets a sound answer to \( \psi'_k \). After one instantiating node gets the answer to \( \psi'_k \), it sends the answer to the requesting node. If it is true and \( A_1 \) is \( \exists \) then the requesting node takes it as a witness and \( \psi_{k+1} \) is true; if it is false and \( A_1 \) is \( \forall \) then the requesting node takes it as a counterevidence and \( \psi_{k+1} \) is false. If the requesting node does not receive any witnesses/counterevidences until the clock time has elapsed, it gives a negative/positive answer to \( \psi_{k+1} \). Therefore \( G \models \psi_{k+1} \) if and only if \( G \models_{\text{FO}} \psi_{k+1} \).
We next consider the complexity of the distributed evaluation. Theorem 3 is the fundamental result of this section. It shows the potential for distributed evaluation of first-order queries with logarithmic in-node time complexity, distributed time linear in the diameter of the graph, and polynomial amount of communication.

**Theorem 3.** Let $G$ be a graph of diameter $\Delta$, with $n$ nodes, and let $\varphi$ be a first-order formula with $v$ variables. The complexity of the distributed evaluation of the query $\varphi$ on $G$ by $\mathcal{QE}_{FO}$ is given by:

| IN-TIME/ROUND | DIST-TIME | MSG-SIZE | #MSG/NODE |
|----------------|-----------|-----------|------------|
| $O(\log n)$   | $O(\Delta)$ | $O(\log n)$ | $O(n^{v+1})$ |

**Proof.** (sketch)

We assume that $\varphi$ has $\ell$ free variables, $k$ bounded variables and $c$ constants. So $v = \ell + k$. Let $w = v + c$.

**IN-TIME/ROUND**

We consider the complexity in the size of the graph. The query is partially evaluated on the local data (identifiers of neighbors) of $O(\log n)$ size. It is rewritten in a systematic fashion into sub-queries by instantiating variables. Both operations can be performed in $O(\log n)$ time. The searching on the query table and answer table (both of size $O(n^v)$) can be done in $O(\log n)$ time as well by binary searching.

**DTIME**

As shown in Theorem 1, the distributed time for a query is $2\Delta w$, so the time complexity is in $O(\Delta)$.

**MSG-SIZE**

It is evident that MSG-SIZE is $O(\log n)$.

**#MSG/NODE**

During the distributed evaluation of queries, new queries can be generated by instantiating free and bounded variables. The total number of queries generated during the distributed evaluation is $O(\sum_{i=1}^{\ell} n^i)$, which is $O(n^{v+1})$. So the number of queries and answers received by each node is $O(n^{v+1})$. Therefore, the number of messages sent by each node is $O(n^{v+1})$.

Note that the first-order query engine relies on a naive evaluation of queries. It can be optimized by taking advantage of the patterns in the query to limit the propagation of subqueries, but this does not affect the global complexity upper bounds.

5 **Distributed complexity of FP**

We next consider the complexity upper bounds for FP. It relies on a query engine which is defined as follows. Note that we first assume that the system is synchronous and we discuss asynchronous systems at the end of the present section.

**Query engine for FP, $\mathcal{QE}_{FP}$**. At first, the requesting node broadcasts $\mu(\varphi(T)(x_1, \ldots, x_\ell))$ (where $T$ is a relational symbol of arity $\ell$). It takes $\Delta$ rounds for all nodes to receive the query. In order to coordinate the computation of the stages of the fixpoint on different nodes, a hop counter $c$ is broadcasted together with the query $\mu(\varphi(T)(x_1, \ldots, x_\ell))$, and a clock $\sigma$ is set for each node. Initially, the requesting node sets $\sigma = \Delta$, and broadcasts $(\mu(\varphi(T)(x_1, \ldots, x_\ell)), \Delta - 1)$ to its neighbors. Each node receiving messages of the form $(\mu(\varphi(T)(x_1, \ldots, x_\ell)), c)$ sets $\sigma = c$ and propagates the formula $(\mu(\varphi(T)(x_1, \ldots, x_\ell)), c - 1)$ to its neighbors, unless $c = 0$ or $\sigma$ has been set before.
When the clock $\sigma$ expires, each node $a$ sets a local table for $T$ and performs the recursion on $
abla(T)$ by iterating the use of the first-order query engine $\mathcal{Q}_F$ on the query $\varphi(T)$ as follows:

- $a$ sets a clock $\tau = 2\Delta w$ (where $w$ is the number of variables or constants in $\varphi(T)$), evaluates the query $\exists x_1 \ldots \exists x_{\ell-1} \!\! 
abla \varphi(T)(x_1, \ldots, x_{\ell-1}, a)$ using $\mathcal{Q}_F$, which takes time $2\Delta w$.

- If $a$ receives a query $\exists x_1 \!\! a_2 \ldots \exists a_{\ell} \varphi(T)(x_1, a_2, \ldots, a_{\ell})$ before $\tau$ expires, $x_1$ is instantiated by $a$ to get the subquery $\exists x_2 \!\! a_2 \ldots \exists a_{\ell} \varphi(T)(a, a_2, \ldots, a_{\ell})$, and the evaluation of the Boolean query $\psi(T)(a, a_2, \ldots, a_{\ell})$ starts. If $a$ gets a positive answer to that Boolean query, it stores $(a, a_2, \ldots, a_{\ell})$ in a temporary buffer.

- When the clock $\tau$ expires, node $a$ updates the local table for $T$ and sets another clock $\eta = \Delta$. If some new tuples $(a, a_2, \ldots, a_{\ell})$ have been produced, $a$ broadcasts an informing message to its neighbors, which will be propagated further to all the nodes in the network to inform them that the computation has not reached a fixpoint yet.

- If some new tuples have been produced in $a$ or $a$ has received some informing messages when the clock $\eta$ expires, it resets $\tau = 2\Delta w$ and starts the next iteration, otherwise the evaluation terminates.

**Definition 3.** Let $\mu(\varphi(T))$ be a fixpoint formula, $G, I \vdash_F \mu(\varphi(T))$ if and only if upon request of $\mu(\varphi(T))$ from any node $a$, the query engine $\mathcal{Q}_F$ produces answer $I$ distributed in the network.

As for FO, we show that the query engine is sound and complete.

**Theorem 4.** For a network $G$ and $\mu(\varphi(T))$ a fixpoint formula, $G, I \models \mu(\varphi(T))$ if and only if $G, I \vdash_F \mu(\varphi(T))$.

The proof of Theorem 4 follows easily from Theorem 2.

**Theorem 5.** Let $G$ be a graph of diameter $\Delta$, with $n$ nodes, $T$ a relation symbol of arity $\ell$, and $\mu(\varphi(T)(x_1, \ldots, x_{\ell}))$ be a FP formula with $v = \ell + k$ (first-order) variables ($\ell$ free and $k$ bounded). The complexity of the distributed evaluation of the query $\mu(\varphi(T))$ by $\mathcal{Q}_F$ on $G$ is given by:

| IN-TIME/ROUND | DIST-TIME | MSG-SIZE | #MSG/NODE |
|----------------|-----------|-----------|-----------|
| $O(\log n)$    | $O(n^\ell \Delta)$ | $O(\log n)$ | $O(n^\ell + 1)$ |

**Proof.** Let $w$ be the total number of variables and constants in $\varphi(T)(x_1, \ldots, x_{\ell})$.

Messages $(\mu(\varphi(T)(x_1, \ldots, x_{\ell})))$, hop are transferred in the network, before the clock $\sigma$ expires, which takes $O(\Delta)$ round and $O(1)$ messages for each node.

Queries $\exists x_1 \ldots \exists x_{\ell-1} \!\! \mu \varphi(T)(x_1, \ldots, x_{\ell-1}, a)$ are evaluated after the clock $\sigma$ expires, before $\tau$ expires. $O(n^v)$ messages are sent by each node for each such query (there are at most $v-1$ variables in $\varphi(T)(x_1, \ldots, x_{\ell-1}, a)$) by Theorem 2. Since there are $n$ such $\exists x_1 \ldots \exists x_{\ell-1} \!\! \mu \varphi(T)(x_1, \ldots, x_{\ell-1}, a)$ queries, the total number of messages sent by each node is $O(n^{v+1})$.

When $\tau$ expires, each node sets a clock $\eta = \Delta$, and broadcasts informing messages to its neighbors if some new tuples are produced. Each node receives the informing message will broadcast it to its neighbors unless it has done that before. Each node sends $O(1)$ informing messages before $\eta$ expires.

When $\eta$ expires, if a node has produced some new tuples or received some informing messages during the previous iteration, it starts the next iteration.
So before the evaluation terminates, in each iterating period $2\Delta w + \Delta$ after the expiration of $\sigma$, at least one new tuple in $T$ is produced in some node, thus there are at most $n^\ell$ such periods before the termination of the evaluation since there are at most $n^\ell$ tuples in $T$.

Consequently the total time of the evaluation is in $\Delta + n^\ell(2\Delta w + \Delta) = O(n^\ell \Delta)$.

Because in each such period, $O(n^{v+1})$ messages are sent by each node, so the total number of messages sent by each node before the termination of the evaluation is $O(n^{\ell+v+1})$. □

Although the complexity upper-bound for DIST-TIME and #MSG/NODE is polynomial, the exponent relates to the number of variables. For most networking functionalities, this number is small, and the dependencies between the variables, might even lower it.

The algorithm $Q\mathcal{E}_{FP}$ above can be adapted to an asynchronous system by using a breadth-first-search (BFS) spanning tree (with the requesting node as the root), without impact on the complexity bounds. If an arbitrary spanning tree, not necessarily a BFS tree, is used, then the complexity bounds does not change, except the distributed time, which becomes $O(n^{\ell+1})$.

Note that with $Q\mathcal{E}_{FP}$, nodes are coordinated to compute every stage of the fixpoint simultaneously by using the clock $2\Delta w$, which is critical for preserving the centralized semantics of $FP$ formulae. However if $\varphi$ is monotone on $T$, the centralized semantics of the fixpoint is preserved no matter whether the stages are computed simultaneously or not. Similar results can be shown for alternative definitions of the fixpoint logic, such as Least Fixpoint.

6 In-node behavioral compilation

In this section, we see how to transform FO and FP formulae, which express queries at the global level of abstraction of the graph, to equivalent rule programs that model the behavior of nodes. We first introduce the Netlog language.

A Netlog program is a finite set of rules of the form:

$$(\uparrow) \gamma_0 : \neg \gamma_1 ; \ldots ; \gamma_l,$$

where $l \geq 0$. The head of the rule $\gamma_0$ is an atomic first-order formula. The body, $\gamma_1 ; \ldots ; \gamma_l$ is constituted of literals, i.e., atomic ($R(\overline{x})$) or negated atomic ($\neg R(\overline{x})$) formulae. Each atomic formula $\gamma_i$ has a holding variable, which is written explicitly as $@x$ and specifies the node on which the evaluation is performed. The communication construct, $\uparrow$, is added before the head if the result is to be pushed to neighbors.

In the sequel we denote the head of a rule $r$ as $\text{head}_r$ and the body as $\text{body}_r$ and denote the holding variable of a formula $\gamma_i$ as $hv_{\gamma_i}$. The relations occurring in the head of the rules are called intentional relations.

Some localization restrictions are imposed on the rules to ensure the effectiveness of the distributed evaluation.

(i) All literals in the body have the same holding variable;

(ii) the head is not pushed (by $\uparrow$) if the holding variable of the head is the holding variable of the body;

(iii) if the head is pushed (by $\uparrow$), assuming the holding variable of the head is $x$ and the holding variable of the body is $y$, then $G(\@y, x)$ is in the body.
A Netlog program is running on each node of the network concurrently. All the rules are applied simultaneously on a node. The holding variable of literals in the body is instantiated by the node ID itself. Facts deduced are stored on the node if the rule is not modified by ↑. Otherwise, they are sent to nodes interpreting the holding variable of the head.

On each node, (i) phases of executions of the rules on the node and (ii) phases of communication with other nodes are alternating till no new facts are deduced on each node. The global semantics is defined as the union of the facts obtained on each node.

For a graph \( G = (V, E) \), an instance \( I \) such that \( I = \bigcup_{v \in V} I_v \) where \( I_v \) is the fragment of \( I \) stored on node \( v \), a rule:
\[
\begin{align*}
 r : & \quad Q(\overline{x}) : -R_1(\overline{y}_1); \ldots; R_m(\overline{y}_m); -R_{m+1}(\overline{y}_{m+1}); \ldots; -R_l(\overline{y}_l).
\end{align*}
\]

and an instantiation \( \sigma \) of the variables occurring in \( r \),
\[
(I, \sigma) \models_G R_1(\overline{y}_1); \ldots; R_m(\overline{y}_m); -R_{m+1}(\overline{y}_{m+1}); \ldots; -R_l(\overline{y}_l)
\]
if and only if
\[
R_i(\sigma(\overline{y}_i)) \begin{cases} 
\in I_{\sigma(y)} \cup E, & \text{for } i \in [1, m] \\
\notin I_{\sigma(y)} \cup E, & \text{for } i \in [m+1, l]
\end{cases}
\]
where \( y \) is the holding variable of \( \text{body}_r \).

We define the immediate consequence operator of a Netlog program \( P \) as a mapping from an instance \( I \) to an instance:
\[
\Psi_{P,G}(I) = \bigcup_{v \in V} \left\{ Q(\overline{u}) \mid \exists r \in P : Q(\overline{x}) : -\text{body}_r, \exists \sigma \text{s.t. } (I, \sigma) \models_G \text{body}_r; \overline{u} = \sigma(\overline{x}); \sigma(hvQ(\overline{x})) = v. \right\}
\]

The computation of a Netlog program \( P \) on a graph \( G \) is given by the following sequence:
\[
I_0 = \emptyset; \\
I_{i+1} = \Psi_{P,G}(I_i), i \geq 0
\]

The computation of \( P \) on \( G \) terminates if the sequence \( (I_i)_{i \geq 0} \) converges to a fixpoint. If the computation of \( P \) on \( G \) terminates, we define \( P(G) \) to be the least fixpoint obtained by the computation sequence \( (I_i)_{i \geq 0} \).

Before we see how FO or FP formulae can be rewritten into Netlog programs, let us first illustrate the technique on the examples of Section 2.

**Example 1.** The following program computes the OLSR like table-based routing protocol as defined in Section 2:

\[
\begin{align*}
T(\overline{a}, d, d) & : = G(\overline{a}, d). \\
T(\overline{a}, h, d) & : = -\text{exist}T(\overline{a}, d); G(\overline{a}, h); \text{ask}T(\overline{a}, h, d). \\
\text{exist}T(\overline{a}, d) & : = T(\overline{a}, u, d). \\
\uparrow \text{ask}T(\overline{a}, h, d) & : = T(\overline{a}, z, d); G(\overline{a}, h); x \neq z. \\
T(\overline{a}, d, d) & : = T(\overline{a}, d, d).
\end{align*}
\]
New predicates (\textit{askT}) are introduced to store partial results that are computed on some nodes, and used by other nodes to which they have been forwarded. The last rule ensures the inflationary behavior (accumulation of results).

**Example 2.** The following program computes spanning trees as defined in Section 2. Several new predicates are introduced to reduce the complexity of the formula (\textit{delay, rej}) and to ensure the transfer of data between the nodes involved in the computation (\textit{askST}).

\begin{align*}
\uparrow ST(x, @y) & : = G(@x, y); \text{ReqNode}(@x). \\
ST(x, @y) & : = \neg \text{existST}(@y); \text{delay}(x, @y); \neg \text{rej}(x, @y). \\
\uparrow \text{askST}(x, @y) & : = ST(w, @x); G(@x, y); w \neq y. \\
\text{existST}(@y) & : = ST(x, @y). \\
\text{rej}(x', @y) & : = \text{askST}(x, @y); \text{askST}(x', @y); x' \geq x. \\
\text{delay}(x, @y) & : = \text{askST}(x, @y). \\
ST(x, @y) & : = ST(x, @y).
\end{align*}

**Example 3.** The following program computes the AODV like on-demand routing protocol as defined in Section 2.

\begin{align*}
\uparrow \text{RouteReq}(x, @y, d) & : = G(@x, y); \text{ReqNode}(@x); \text{dest}(d). \\
\text{RouteReq}(x, @y, d) & : = \text{askRouteReq}(x, @y, d); \neg \text{existRR}(@y, d). \\
\uparrow \text{askRouteReq}(x, @y, d) & : = \text{RouteReq}(w, @x, d); G(@x, y); x \neq d; w \neq y. \\
\text{existRR}(@y, d) & : = \text{RouteReq}(w', @y, d). \\
\uparrow \text{Nexthop}(@x, d, d) & : = \text{RouteReq}(x, @d, d); G(@d, x). \\
\uparrow \text{Nexthop}(@x, @y, d) & : = \text{RouteReq}(x, @y, d); \text{Nexthop}(@y, z, d); G(@y, x). \\
\text{RouteReq}(x, @y, d) & : = \text{RouteReq}(x, @y, d). \\
\text{Nexthop}(@x, d, d) & : = \text{Nexthop}(@x, d, d).
\end{align*}

We now consider the general translation of FO and FP formulae to \textit{Netlog} programs. It has been shown in [7] that FP is equivalent to \textit{Datalog}\textsuperscript{\neg} both with inflationary semantics. Moreover, both FO and FP formulae can be translated effectively to \textit{Datalog}\textsuperscript{\neg} programs. We therefore consider the translation of \textit{Datalog}\textsuperscript{\neg} programs into equivalent \textit{Netlog} programs. The main difficulty relies in the distribution of the computation.

The syntax and semantics of \textit{Datalog}\textsuperscript{\neg} is similar to the one of \textit{Netlog}, but without the communication primitives. Indeed, unlike \textit{Netlog}, a program in \textit{Datalog}\textsuperscript{\neg}, is processed in a centralized manner. The computation of a \textit{Datalog}\textsuperscript{\neg} program $P$ on a graph $G$ is given by the following sequence:

\begin{align*}
I_0 & = \emptyset; \\
I_{i+1} & = \Psi_{P,G}(I_i) \cup I_i, i \geq 0
\end{align*}

where $\Psi_{P,G}(I_i)$ is defined in a similar way as for \textit{Netlog}. 

14
The following algorithm rewrites a Datalog program $P_{DL}$ into a Netlog program $P_{NL}$. To synchronize stages of the recursion, there is a fact “start(a)” stored on each node $a$ at the beginning of the computation which triggers a clock used to coordinate stages.

In the sequel we do not distinguish between $G(@x,y)$ and $G(@y,x)$.

**Rewriting Algorithm:**

The algorithm rewrites the input program step by step.

**Step 1: Distributing Data**

Input: $P_{DL}$. Output: $P_1$.

Algorithm $Localize(P_{DL})$ chooses one variable as the holding variable for each relation in $P_{DL}$. $P_1$ is obtained by marking the holding variable of each literal in $P_{DL}$.

The Rewriting Algorithm supports different assignment of holding variables. For simplicity, we assume the left most variable of each relation is chosen as holding variable. For lack of space, we do not address the associated optimization problem.

**Step 2: Distributing Computation**

Input: $P_1$. Output: <$P_2, \kappa>$

Let $\Delta$ be the diameter of $G$.

For each rule $r\in P_1$, assume

- $hv_{head_r}$ is the holding variable of $head_r$,
- $h_r := hv_{head_r}$, and
- $CN_r := \{h_r\}$.

$Rewrite(r, h_r, CN_r)$ recursively rewrites the rule $r$ into several rules until the output rules satisfy the localization restriction (i). $body_r$ is divided into several parts: the local part that can be evaluated locally and the non-local part that cannot be evaluated locally. $h_r$ is the holding variable of the literals in the local part. The non-local part is partitioned into several disconnected parts which share no variables except the variables in $CN_r$ and are evaluated by additional rules $r_i$ on different nodes in parallel. The deduced facts of $r_i$ are pushed to the node where the rule $r$ is evaluated. Meanwhile, it calculates the number of rounds $\kappa_r$ for evaluating $r$.

$Rewrite(r, h_r, CN_r) : output <T_r, \kappa_r>$

Begin
Assume

$$r : \gamma : -\gamma_1; \ldots; \gamma_l.$$ where $l \geq 1$.

Let $S = \{\gamma_1, \ldots, \gamma_l\}$, $S' = \{\gamma_i | \gamma_i \in S \text{ and } hv_{\gamma_i} = h_r\}$, so that $S'$ contains all the literals in $body_r$ whose holding variable is the same as the one of the head, $h_r$.

- If $S' = S$, then $T_r := \{r\}$, and $\kappa_r := 1$.
- If $S' \neq S$,
  Begin
  Let $S'' := S - S'$, so that $S''$ contains all the literals in $body_r$ whose holding variables are not $h_r$. 


15
For $\gamma_j, \gamma_k \in S''$, let $\gamma_j \approx \gamma_k$ if $\gamma_j$ and $\gamma_k$ have some common variables besides the variables in $CN_r$. Assume $\{S'_1, \ldots, S''_n\}$ ($n \geq 1$) is a partition of $S''$ in minimal subsets closed under $\approx$, so that the literals in $S''$ are divided into disconnected "subgraph" components.

For each $S''_i, i \in [1, n]$, let

$$T_i := \{ hv_{\gamma_iw} | \gamma_iw \in S''_i \ \text{and} \ G(@h_r, hv_{\gamma_iw}) \in S' \}.$$ 

so that $T_i$ contains the variables which are the holding variable of one literal in $S''_i$ and are also a neighbor of $h_r$.

- If $T_i \neq \emptyset$, which means the non-local part $S''_i$ is connected with the local part $S'$. Choose one variable $hv_{\gamma_iw}$ from $T_i$. Let $S''_i := S''_i \cup \{ G(@h_{v_{\gamma_iw}}, h_r) \}$. Let $h_{r_i} := hv_{\gamma_iw}$. Let $CN_{r_i} := CN_r \cup \{ h_{r_i} \}$. Let $d_{r_i} := 1$. Assume $S''_i = \{ \gamma_{i,1}, \ldots, \gamma_{i,m_i} \}$. Let

$$r_i : Q_i(\overline{y_i}) : - \gamma_{i,1; \ldots; \gamma_{i,m_i}}.$$ 

where $Q_i$ is a new relation name and $\gamma_{i,m_i}$ contains all the variables occurring both in $S''_i$ and in either $S'$ or $head_r$, that is in $var(S''_i) \cap (var(S') \cup var(head_r))$, with $h_r$ as holding variable.

- If $T_i = \emptyset$, then the non-local part $S''_i$ is disconnected from the local part $S'$. Choose one literal $\gamma_iw \in S''_i$. Assume $y$ is a variable not occurring in $r$, let $S''_i := S''_i \cup \{ y = hv_{\gamma_iw} \}$. Let $h_{r_i} := hv_{\gamma_iw}$. Let $CN_{r_i} := CN_r \cup \{ h_{r_i} \}$. Let $d_{r_i} := 1 + \Delta$. Assume $S''_i = \{ \gamma_{i,1}, \ldots, \gamma_{i,m_i} \}$. Let

$$r_i : Q_i(\overline{y_i}) : - \gamma_{i,1; \ldots; \gamma_{i,m_i}}.$$ 

where $Q_i$ is a new relation name and $\gamma_{i,m_i}$ contains all the variables occurring both in $S''_i$ and in either $S'$ or $head_r$, that is in $var(S''_i) \cap (var(S') \cup var(head_r))$, with $y$ as holding variable. Moreover, let

$$r'_i : Q_i(\overline{y_i}) : - Q_i(\overline{y_i}); G(\overline{y_i}).$$

Assume $S' = \{ \gamma_{1,1}, \ldots, \gamma_{k,1} \}$ ($k \geq 0$), let

$$r'_i : \gamma : - \gamma_{i,1; \ldots; \gamma_{k,1}} : Q_i(\overline{y_i}); \ldots : Q_n(\overline{y_n}).$$

$Q_i(\overline{y_i}), i \in [1, n]$, is called sub-query.

Assume $< T_{r_i}, \kappa_{r_i} > = \text{Rewrite}(r_i, h_{r_i}, CN_{r_i})$, let

- $T_r := \{ r' \} \cup \bigcup_{i \in [1, n]} \{ r'_i \} \cup T_{r_i}$, and
- $\kappa_r := \max \{ \kappa_{r_i} + d_{r_i} | i \in [1, n] \}$,

End

Finally, let

- $P_2 := \bigcup_{r \in P_1} T_r$, and

16
\[ \kappa := \max\{\Delta, \max\{\kappa_r \mid r \in \mathcal{P}_1\}\}. \]

**Step 3: Communication**

Input: \( < \mathcal{P}_2, \kappa > \). Output: \( < \mathcal{P}_3, \kappa > \).

\( \mathcal{P}_3 \) is obtained by adding \( \uparrow \) in the head of each rule \( r \) where \( r \in \mathcal{P}_2 \) with the holding variable of the head different from the holding variable of the body. So that rules in \( \mathcal{P}_3 \) satisfy the localization restriction (ii) and (iii).

**Step 4: Stage coordination with clocks**

Input: \( < \mathcal{P}_3, \kappa > \). Output: \( \mathcal{P}_4 \).

The rules in \( \mathcal{P}_3 \) are modified as follows:

- Add the literals "clock(@x,q)" and "q \neq 0" to the body of each rule, where \( x \) is the holding variable of the body.

- For each rule with an intensional relation \( R \) of \( \mathcal{P}_{DL} \) in its head, replace \( R \) in the head with \( tempR \) and add

  \[ R(\overline{x}) : - \ tempR(\overline{x}); clock(@x,0). \]

  \[ continue(@x) : - \ tempR(\overline{x}); -R(\overline{x}); clock(@x,0). \]

  \[ \uparrow inf(@y,x) : - \ tempR(\overline{x}); -R(\overline{x}); clock(@x,0); G(@x,y). \]

  in \( \mathcal{P}_4 \) where \( x \) is the holding variable of both \( R \) and \( tempR \).

- Add

  \[ continue(@x) : - \ start(@x). \]

  \[ \uparrow inf(@y,x) : - \ start(@x); G(@x,y). \]

  \[ clock(@x,\kappa) : - \ start(@x). \]

  \[ clock(@x,p) : - \ clock(@x,q); q \geq 1; p = q - 1; \neg stop(@x). \]

  \[ clock(@x,\kappa) : - \ clock(@x,0); \neg stop(@x). \]

  \[ \uparrow inf(@z,x) : - \ inf(@y,x); G(@y,z); x \neq z; clock(@x,q); q \geq \Delta. \]

  \[ continue(@x) : - \ inf(@x,y); clock(@x,q); q \neq 0. \]

  \[ continue(@x) : - \ continue(@x); clock(@x,q); q \neq 0. \]

  \[ stop(@x) : - \ \neg continue(@x); clock(@x,0). \]

in \( \mathcal{P}_4 \).

**Step 5: Inflationary result**

Input: \( \mathcal{P}_4 \). Output: \( \mathcal{P}_{NL} \).

\( \mathcal{P}_{NL} \) contains rules in \( \mathcal{P}_4 \) and the following rules:

- For each relation \( R \) in \( \mathcal{P}_4 \) except \( start, clock, continue, inf \) and \( stop \) but not in \( \mathcal{P}_{DL} \), add

  \[ R(\ldots @x \ldots) : - R(\ldots @x \ldots); clock(@x,q); q \neq 0. \]

in \( \mathcal{P}_{NL} \).
- For each intensional relation $R$ of $\mathcal{P}_{DL}$, add

$$R(\overline{x}) : \neg R(\overline{x}).$$

in $\mathcal{P}_{NL}$.

It is obvious that each rule in a program $\mathcal{P}_{NL}$ produced by the Rewriting Algorithm satisfies the localization restrictions, and can thus be computed effectively on one node. We can now state the main result of this section which shows that the global semantics of $\mathcal{P}_{DL}$ coincides with the distributed semantics of $\mathcal{P}_{NL}$.

**Theorem 6.** For a graph $G = \{V,G\}$, a Datalog program $\mathcal{P}_{DL}$ and its rewritten Netlog program $\mathcal{P}_{NL}$ produced by the Rewriting Algorithm, the computation of $\mathcal{P}_{NL}$ on $G$ terminates iff the computation of $\mathcal{P}_{DL}$ on $G$ terminates, and $\mathcal{P}_{NL}(G) = \mathcal{P}_{DL}(G)$.

$\mathcal{P}_{NL}$ slows down the computation of $\mathcal{P}_{DL}$. During one stage ($\kappa$ rounds) of the computation of $\mathcal{P}_{NL}$, the clock turns from $\kappa$ to 0, the sub-queries are evaluated and the sub-results are transmitted. At the end of each stage, the deduced facts for the intensional relations of $\mathcal{P}_{DL}$ are cumulated and all the sub-results are cleared. Hence, one such stage of $\mathcal{P}_{NL}$ is equivalent to one stage of $\mathcal{P}_{DL}$. For an intensional relation $R$ of $\mathcal{P}_{DL}$, $R(\overline{c}) \in I_{DLi}$ if and only if $R(\overline{c}) \in I_{NLi(\kappa+i)+1}$, $i \geq 0$, where $I_{DLi}$ and $I_{NLi}$ are the stages of respectively the fixpoints of $\mathcal{P}_{DL}$ and $\mathcal{P}_{NL}$.

The termination of the computation of $\mathcal{P}_{NL}$ is ensured by the predicate $\text{stop}$ as follows: the computation starts with a fact $\text{start}(a)$ on each node $a$, which triggers $\text{clock}(a, \kappa)$, $\text{continue}(a)$ and $\text{inf}(b, a)$ where $b$ is a neighbor of $a$. When the clock decreases from $\kappa$ to 0, the evaluation of the sub-queries is done. The facts of an intensional relation $R$ of $\mathcal{P}_{DL}$ are stored in $\text{tempR}$. Meanwhile, $\text{inf}(v, a)$ is pushed to all the other nodes $v$ to inform that the computation on $a$ continues, so that $\text{continue}(v)$ is deduced. $\text{continue}(a)$ for one stage is maintained to the end of the stage. When the clock turns to zero, (i) the program checks if $\text{continue}(a)$ is true. If false, $\text{stop}(a)$ is deduced. Since $\neg \text{stop}(a)$ is a precondition for decreasing the clock and the clock is a precondition for deducing facts of all the other relations except $R$, so only the facts of $R$ are preserved along the stages. Thus the fixpoint is obtained and the computation terminates. Otherwise ($\text{stop}(a)$ is not deduced), the computation continues. (ii) The program compares facts of $\text{tempR}$ and $R$. If there are newly deduced facts, these facts are added into $R$. Meanwhile $\text{continue}(a)$ and $\text{inf}(b, a)$ are deduced for the next stage.

The proof of Theorem 6 relies on the following Lemma and the fact that the Rewriting Algorithm produces only rules satisfying the localization restrictions.

**Lemma 7.** For a graph $G = \{V,G\}$, a Datalog program $\mathcal{P}_{DL}$ and its rewritten Netlog program $\mathcal{P}_{NL}$ produced by the Rewriting Algorithm, the computation sequence $(I_{NLj})_{j \geq 0}$ for $\mathcal{P}_{NL}$ satisfies:

1. For each relation $R$ in $\mathcal{P}_{NL}$, $R(\overline{c}) \in I_{NLP}$ if $R(\overline{c}) \in I_{NLi,c_1}$ and $R(\overline{c}) \notin I_{NLi,c'}$, where $c_1$ is the holding node of $R(\overline{c})$ and $c' \neq c_1$.

2. $I_{NLO} = \{\text{start}(v)|v \in V\}$.

3. If $\text{clock}(a,c) \in I_{NLP}$, then $\text{clock}(v,c) \in I_{NLP}$ for all $v \in V$. If $\text{stop}(a) \in I_{NLP}$, then $\text{stop}(v) \in I_{NLP}$ for all $v \in V$.
4. If \( \text{stop}(a) \in I_{NLs} \), then (i) \( \text{clock}(a, \kappa) \in I_{NLs} \), (ii) for \( q \in [1, s] \), \( \text{clock}(a, \kappa - p) \in I_{NLq} \), \( p \in [0, \kappa] \), if \( q = n(\kappa + 1) + p + 1 \) and (iii) if \( R(\overline{c}) \in I_{NLF} \) where \( f > s + 1 \), then \( R \) is an intensional relation of \( P_{DL} \). Continue \( (a) \notin I_{NLn(k+1)} \) for any \( a \in V \) and any \( p \geq s - (\kappa + 1) \) if \( \text{stop}(a) \in I_{NLs} \). (iii) continue \( a \notin I_{NLs} \).

5. For each relation \( R \) in \( P_{NL} \) but not in \( P_{DL} \), except the relations start, clock, continue, \( \inf \) and \( \text{stop} \), (i) if \( R(\overline{c}) \in I_{NLp} \), then \( \text{clock}(a, \kappa) \notin I_{NLp} \), and (ii) if \( p = n(\kappa + 1) + q \), \( q \in [2, \kappa + 1] \), then \( R(\overline{c}) \in I_{NLn(\kappa+1)+q} \), \( q' \in [q, \kappa + 1] \).

6. For each intensional relation \( R \) of \( P_{DL} \), if \( R(\overline{c}) \in I_{NLp} \) then \( R(\overline{c}) \in I_{NLp'} \) where \( p' \geq p \). Assume \( q = \min\{p|R(\overline{c}) \in I_{NLp}\} \), then \( \text{clock}(a, \kappa) \in I_{NLq} \).

Now we prove Theorem 6.

**Proof.** Assume the computation sequence for \( P_{DL} \) is \( (I_{DL1})_{i \geq 0} \) and for \( P_{NL} \) is \( (I_{NL1})_{j \geq 0} \). We prove for any intensional relation \( Q \) of \( P_{DL} \), \( Q(\overline{c}) \in I_{NLi(\kappa + 1) + 1} \) if \( Q(\overline{c}) \in I_{DLi} \).

Basis: \( i = 0 \), \( I_{DL0} = \emptyset \) and \( I_{NL1} = \{\text{continue}(a), \inf(b, a), \text{clock}(a, \kappa) | a \in V, G(a, b)\} \).

Induction: Suppose for \( n \geq 0 \), and each intensional relation \( Q \) of \( P_{DL} \),

\[
Q(a_1, \ldots, a_k) \in I_{DLn} \text{ if } Q(a_1, \ldots, a_k) \in I_{NL(n(\kappa + 1) + 1)}.
\]

First we prove that for \( n + 1 \), if \( Q(b_1, \ldots, b_k) \in I_{DLn+1} \), then \( Q(b_1, \ldots, b_k) \in I_{NL(n+1)(\kappa+1)+1} \).

If \( Q(b_1, \ldots, b_k) \in I_{DLn+1} \), then (i) \( Q(b_1, \ldots, b_k) \in I_{DLn} \) or (ii) \( Q(b_1, \ldots, b_k) \) is a newly deduced fact in \( I_{DLn+1} \).

If \( Q(b_1, \ldots, b_k) \in I_{DLn} \) then \( Q(b_1, \ldots, b_k) \in I_{NLn(\kappa + 1) + 1} \) by the induction hypothesis, and \( Q(b_1, \ldots, b_k) \in I_{NLp} \) where \( p \geq n(\kappa + 1) + 1 \) by Lemma 7.6, therefore \( Q(b_1, \ldots, b_k) \in I_{NL(n+1)(\kappa+1)+1} \).

Otherwise \( Q(b_1, \ldots, b_k) \notin I_{DLn} \), then there is one rule \( r \in P_{DL} \)

\[
r : Q(x_1, \ldots, x_k) : -R_1(y_1); \ldots; R_m(y_m); -R_{m+1}(y_{m+1}); \ldots; -R_l(y_l).
\]

and an instantiation \( \sigma \) of the variables in \( r \) such that \( \sigma(x_i) = b_i \) for \( i \in [1, k] \)

\[
R_i(\sigma(\overline{y}_i)) \left\{ \begin{array}{l} \in I_{DLn} \cup G, \text{ for } i \in [1, m] \\ \notin I_{DLn} \cup G, \text{ for } i \in [m + 1, l] \end{array} \right.
\]

and for some \( e \in [1, m] \), \( R_e(\sigma(\overline{y}_e)) \notin I_{DLn} \cup G \). By the induction hypothesis and Lemma 7.1,

\[
R_i(\sigma(\overline{y}_i)) \left\{ \begin{array}{l} \in I_{NLn(\kappa + 1) + 1$sigma(\overline{h}_v)$} \cup G, \text{ for } i \in [1, m] \\ \notin I_{NLn(\kappa + 1) + 1$sigma(\overline{h}_v)$} \cup G, \text{ for } i \in [m + 1, l] \end{array} \right.
\]

and \( R_e(\sigma(\overline{y}_e)) \notin I_{NL(n-1)(\kappa+1)+1$sigma(\overline{h}_v)$} \cup G \). According to Lemma 7.6

\[
R_i(\sigma(\overline{y}_i)) \left\{ \begin{array}{l} \in I_{NLp$sigma(\overline{h}_v)$} \cup G, \text{ for } i \in [1, m] \\ \notin I_{NLp$sigma(\overline{h}_v)$} \cup G, \text{ for } i \in [m + 1, l] \end{array} \right.
\]

where \( p \in [n(\kappa + 1) + 1, (n + 1)(\kappa + 1)] \) and \( R_e(\sigma(\overline{y}_e)) \) is newly deduced in \( I_{NLn(\kappa+1)+1} \). So \( \text{continue}(\sigma(\overline{h}_v)) \in I_{NLn(\kappa+1)+1} \). By Lemma 7.4, \( \text{stop}(a) \notin I_{NLn+1} \) and \( \text{clock}(a, \kappa - p) \in I_{NLn(\kappa+1)+1+p} \) for \( p \in [0, \kappa] \) and for any \( a \in V \).
Because
\[ Q(\overline{x_1},\ldots,x_k) : \neg tempQ(\overline{x_1},\ldots,x_k); clock(\overline{x_1},0). \]
is in \( P_{NL} \), so if \( tempQ(b_1,\ldots,b_k) \in n(\kappa + 1) + 1 + p, p \in [\kappa_r, \kappa] \), then \( Q(b_1,\ldots,b_k) \in I_{NL}(n(\kappa + 1) + 1 \kappa_r \leq \kappa. \)

According to Rewriting Algorithm, \( h_r = hv_{Q}, \) \( CN_r = \{ h_r \} \) and

- if all the holding variables of the literals in \( body_r \) are the same with \( h_r \) (\( S' = S \)), then
  \[ tempQ(\overline{x_1},\ldots,x_k) : -R_1(\overline{y_1}); \ldots; R_m(\overline{y_m}); -R_{m+1}(\overline{y_{m+1}}); \ldots; -R_l(\overline{y_l}); clock(\overline{x_1},q); q \neq 0. \]  
  and
  \[ tempQ(\overline{x_1},\ldots,x_k) : -tempQ(\overline{x_1},\ldots,x_k); clock(\overline{x_1},q); q \neq 0. \]  
  are in \( P_{NL} \). \( \kappa_r = 1. \) Therefore \( tempQ(b_1,\ldots,b_k) \in I_{NL}(n(\kappa + 1) + 1 + p \) for each \( p \in [1, \kappa] \), and \( Q(b_1,\ldots,b_k) \in I_{NL}(n(\kappa + 1) + 1) \) by Lemma 7.1 and 7.5.

- Otherwise, not all of the holding variables of the literals in \( body_r \) are the same with \( h_r \) (\( S' \neq S \)). Assume \( hv_{R_1} = \cdots = hv_{R_u} = hv_{R_{m+1}} = \cdots = hv_{R_{m+u}} = h_r. \) Then
  \[ \begin{align*}
  tempQ(\overline{x_1},\ldots,x_k) : -R_1(\overline{y_1}); \ldots; R_m(\overline{y_m}); -R_{m+1}(\overline{y_{m+1}}); \ldots; -R_m(\overline{y_{m+1}}); \\
  Q_1(\overline{z_1}); \ldots; Q_l(\overline{z_l}); clock(\overline{x_1},q); q \neq 0.
  \end{align*} \]
  and
  \[ tempQ(\overline{x_1},\ldots,x_k) : -tempQ(\overline{x_1},\ldots,x_k); clock(\overline{x_1},q); q \neq 0. \]  
  are in \( P_{NL} \) where \( Q_i(\overline{z_i}) \) is in \( head_{r_i} \) for \( r_i \in P_{NL} \). If for each \( i \in [1, t], Q_i(\overline{c_i}) \in I_{NL}(n(\kappa + 1) + 1 + p, p \in [\kappa_r, \kappa] \) then \( tempQ(b_1,\ldots,b_k) \in I_{NL}(n(\kappa + 1) + 1 + p, p \in [\kappa_r, \kappa] \) then \( tempQ(b_1,\ldots,b_k) \in I_{NL}(n(\kappa + 1) + 1 + p, p \in [\kappa_r, \kappa] \) is as follows:

Literals \( R_{w_{m+1}}(\overline{y_{m+1}}), \ldots, R_{m+1}(\overline{y_{m+1}}); \ldots, -R_t(\overline{y_t}) \) are grouped into subsets \( S'_l, \ldots, S''_l \) such that he literals in different subsets have no common variables except the variable in \( CN_r \) which is \( x_1 \).

For each \( S''_l \),

- if some of the holding variables of the literals in \( S''_l \) are the neighbors of \( h_r \), \( (T_i \neq \emptyset) \), then \( G_{(\overline{hv}_{\gamma_{r_i}}, h_r)} \) where \( hv_{\gamma_{r_i}} \) is one of such variables, is added into \( S''_l \). Then \( h_{r_i} = hv_{\gamma_{r_i}} \) and \( CN_{r_i} = CN_r \cup \{ h_{r_i} \}. \) Literals in \( S''_l \) along with "\( clock(\overline{hv}_{\gamma_{r_i}}, q)\)" "\( q \neq 0\)" constitute \( body_{r_i} \) "\( Q_i(\overline{z_i})\)" constitute \( head_{r_i} \) where \( \overline{z_i} \) contains all the variables both in \( body_{r_i} \) and in any of \( R_1(\overline{y_1}), \ldots, R_u(\overline{y_u}), -R_{m+1}(\overline{y_{m+1}}), \ldots, -R_{m+u}(\overline{y_{m+u}}) \) or \( head_r \) with \( h_r \) as the holding node. If the evaluation for \( r_i \) is finished, the result for the sub-query \( Q_i \) gets to \( \sigma(h_r) \) in the next round. \( d_{r_i} = 1. \)

- Otherwise (non of the holding variables of the literals in \( S''_l \) are the neighbors of \( h_r \), "\( y = hv_{\gamma_{r_i}}\)" is added into \( S''_l \) where \( \gamma_{r_i} \) is one literal in \( S''_l \) and \( y \) does not occur in \( r \). \( h_{r_i} = hv_{\gamma_{r_i}} \) \( CN_{r_i} = CN_r \cup \{ h_{r_i} \}. \) Literals in \( S''_l \) along with "\( clock(\overline{y}, q)\)" "\( y \neq 0\)" constitute \( body_{r_i} \) \( head_{r_i} \) is "\( Q_i(\overline{z_i})\)" where \( \overline{z_i} \) contains all the variables both in \( body_{r_i} \) and in any of \( R_1(\overline{y_1}), \ldots, R_u(\overline{y_u}), -R_{m+1}(\overline{y_{m+1}}), \ldots, -R_{m+u}(\overline{y_{m+u}}) \) or \( head_r \), with \( y \) as holding node.
Moreover, because the following rule is in $\mathcal{P}_{NL}$

$$\uparrow Q_i(@x \ldots) : -Q_i(@y \ldots); G(@y, x); clock(@y, q); q \neq 0.$$ 

therefore if the evaluation of $r_i$ is finished, then the result for the sub-query $Q_i$ is obtained locally in the next round and then is broadcast to every node in $\Delta$ rounds. $d_{r_i} = 1 + \Delta$.

For each $i \in [1, t]$, if $Q_i(\sigma(h_{r_i}) \ldots) \in I_{NL}(n+1) + \kappa_{r_i}$, then $Q_i(c_i) \in I_{NL}(n+1) + \kappa_{r_i} + d_{r_i} - 1$, and because $\kappa_r = \max\{\kappa_{r_i} + d_{r_i}\}$, so $Q_i(c_{i}) \in I_{NL}(n+1) + (p' - 1)$ for $p' \in [\kappa_{r_i} + d_{r_i}, \kappa_r]$.

Each $r_i$ is then rewritten by $Rewrite$ function and the output rules are modified by the Rewriting Algorithm.

A set of rules $T_r \in \mathcal{P}_{NL}$ is obtained by applying the Rewriting Algorithm on $r$. For $r' \in T_r$ with some sub-queries

$$\kappa_{r'} = \max\{\kappa_{r_i'} + d_{r_i'}\}$$

and for $r'' \in T_r$, without sub-queries $\kappa_{r''} = 1$. The answers to $r \in T_r$ is in $I_{NL}(n+1) + \kappa_r$. Therefore $Q_i(\sigma(h_{r_i}) \ldots) \in I_{NL}(n+1) + \kappa_{r_i}$ for each $i \in [1, t]$, so finally $Q(b_1, \ldots, b_k) \in I_{DL}(n+1) + 1$.

Then we proof that if $Q(b_1, \ldots, b_k) \in I_{NL}(n+1) + 1$ then $Q(b_1, \ldots, b_k) \in I_{DL}(n+1)$ for $n + 1$.

If $Q(b_1, \ldots, b_k) \in I_{NL}(n+1)(n+1) + 1$, then

(i) $Q(b_1, \ldots, b_k) \in I_{NL}(n+1)(n+1)$

(ii) $tempQ(b_1, \ldots, b_k) \in I_{NL}(n+1)(n+1)$ and $clock(b_1, 0) \in I_{NL}(n+1)(n+1)$.

If $Q(b_1, \ldots, b_k) \in I_{NL}(n+1)(n+1)$, according to Lemma 7.6, $Q(b_1, \ldots, b_k) \in I_{NL}(n+1)+1$. By the induction hypothesis, $Q(b_1, \ldots, b_k) \in I_{DL}(n+1)$. So $Q(b_1, \ldots, b_k) \in I_{DL}(n+1)$.

Otherwise $(Q(b_1, \ldots, b_k) \notin I_{NL}(n+1)(n+1))$

is in $\mathcal{P}_{NL}$, $tempQ(b_1, \ldots, b_k) \in I_{NL}(n+1)(n+1)$, $clock(b_1, 0) \in I_{NL}(n+1)(n+1)$. Therefore $stop(a) \notin I_{NL}(n+1)(n+1)+1$, and by Lemma 7.4, $clock(a, \kappa - p) \in I_{NL}(p, \kappa)$ if $q = n(k+1) + 1+p$. A set of rules in $\mathcal{P}_{NL}$ of the following form, with $Q_1(\bar{x}_i)$, $Q_{i1}(\bar{z}_{i1})$, \ldots, $Q_{io}(\bar{z}_{io})$ as sub-queries, is used and only used for deducing $tempQ(b_1, \ldots, b_k)$

$$\uparrow Q_i(\bar{x}_i) : -R_{i1}(\bar{y}_i); \ldots; -R_{im}(\bar{y}_{im}); -R_{m+1}(\bar{y}_{m+1}); \ldots; -R_{it}(\bar{y}_t);$$

$$Q_{i1}(\bar{z}_{i1}); \ldots; Q_{io}(\bar{z}_{io}); clock(@y, q); q \neq 0.$$ 

and $tempQ(b_1, \ldots, b_k) \in I_{NL}(p, p) \in [p', (n+1)(\kappa + 1)]$ for some $p' \in [2, (n+1)(\kappa + 1)]$.

According to Rewriting Algorithm, all of these rules are rewritten from a rule in $\mathcal{P}_{DL}$ with $Q(x_1, \ldots, x_k)$ as the head and the literals $R_i(\bar{y}_i)$ and $-R_u(\bar{y}_u)$ occurring in these rules as the body. Assume the rule is

$$r : Q(x_1, \ldots, x_k) : -R_1(\bar{y}_1); \ldots; R_m(\bar{y}_m); -R_{m+1}(\bar{y}_{m+1}); \ldots; -R_t(\bar{y}_t).$$

By Lemma 7.6,

$$R_i(\sigma(\bar{y}_i)) \in I_{NL}(n+1) \cup G, \text{ for } i \in [1, m]$$

$$\notin I_{NL}(n+1) \cup G, \text{ for } i \in [m+1, t]$$

21
where \( \sigma(x_i) = b_i \) for \( i \in [1,k] \), and for some \( e \in [1,m], R_e(\sigma(y_e)) \notin I_{NL(n-1)(\kappa+1)+1} \cup G \). By the induction hypothesis

\[
\begin{align*}
R_i(\sigma(y_i)) \in I_{DLn} \cup G, & \quad \text{for } i \in [1,m] \\
& \notin I_{DLn} \cup G, \quad \text{for } i \in [m+1,l]
\end{align*}
\]

and \( R_e(\sigma(y_e)) \notin I_{DLn-1} \cup G \). So \( Q(b_1, \ldots, b_k) \in I_{DL+1} \).

Therefore for an intensional relation \( Q \) of \( P_{DL} \) on \( G \) if and only if \( Q(c) \in I_{NL}(\kappa+1) \).

We now prove that the computation of \( P_{NL} \) on \( G \) terminates if the computation of \( P_{DL} \) on \( G \) terminates.

The computation of \( P_{DL} \) on \( G \) terminates,

if \( (I_{DLj})_{j \geq 0} \) converges,

if no new facts in any intensional relation of \( P_{DL} \) are deduced in \( I_{DLi} \) for the minimal \( i \),

if no new facts in any intensional relation of \( P_{DL} \) are deduced in \( I_{NL(i+1)(\kappa+1)} \) for the minimal \( i \),

if \( continue(v) \notin I_{NL(i+1)(\kappa+1)} \) and \( continue(v) \in I_{NL(i+1)} \),

if \( stop(v) \in I_{NL(i+1)(\kappa+1)+1} \),

if \( clock(v,c) \notin I_{NL(i+1)(\kappa+1)+2} \),

if only the facts in the intensional relations of \( P_{DL} \) are in \( I_{NLp}, p > (i + 1)(\kappa + 1) + 2 \),

if \( (I_{NLj})_{j \geq 0} \) converges,

if the computation of \( P_{NL} \) on \( G \) terminates.

Therefore the computation of \( P_{NL} \) on \( G \) terminates if the computation of \( P_{DL} \) on \( G \) terminates and \( P_{NL}(G) = P_{DL}(G) \).

\[\square\]

7 Restriction to neighborhood

We next consider a restriction of FO and FP to bounded neighborhoods of nodes which ensures that the distributed computation can be performed with only a bounded number of messages per node.

Let \( \text{dist}(x, y) \leq k \) be the first-order formula stating that the distance between \( x \) and \( y \) in the graph is no more than \( k \). Let \( N^k(x) = \{ y | \text{dist}(x, y) \leq k \} \) denote the \( k \)-neighborhood of \( x \).

Let \( \varphi(x, \overline{y}) \) be an FO formula with free variables \( x, \overline{y} \), then \( \varphi^{(k)}(x, \overline{y}) \) denotes the formula with all the variables occurring in \( \varphi \) relativized to the \( k \)-neighborhood of \( x \), that is each quantifier \( \forall/\exists z \) is replaced by \( \forall/\exists z \in N^k(x) \), and \( y \in N^k(x) \) is added for each free variable \( y \).

The local fragments of FO and FP can be defined as follows.

**Definition 4.** \( \text{FO}_{loc} \) is the set of FO formulae of the form \( \varphi^{(k)}(x, \overline{y}) \).
The local fragment of FP can be defined as fixpoint of $FO_{loc}$ formulae.

**Definition 5.** $FP_{loc}$ is the set of $FP$ formulae of the form $\mu(\varphi(k)(T)(x, y))$, where $y = y_1 \ldots y_\ell$ and $T$ is of arity $\ell + 1$.

Consider again the examples of Section 2. It is easy to verify that the formula $\mu(\varphi(T)(x, h, d))$ defining the OLSR like table-based routing is not in $FP_{loc}$. On the other hand, the formula $\mu(\varphi(ST)(x, y))$ defining the spanning tree is in $FP_{loc}$, as well as the formulae $\mu(\varphi(RouteReq)(x, y, d))$ and $\mu(\varphi(NextHop)(x, y, d))$ defining the AODV like On-Demand Routing.

### 7.1 Distributed complexity

We now show that the distributed computation of the local fragments, $FO_{loc}$ and $FP_{loc}$, can be done very efficiently. We assume that the nodes are equipped with ports for each of their neighbors. The ports allow to bound the message size to a constant independent of the network size. The proof relies as previously on specific query engines for $FO_{loc}$ and $FP_{loc}$. The query engine for $FO_{loc}$ works both in synchronous and asynchronous systems.

**Query engine for $FO_{loc}$ ($QE_{FO_{loc}}$)** The requesting node broadcasts the $FO_{loc}$ formula $\varphi(k)(x, y)$.

For each node $a$, when it receives the query $\varphi(k)(x, y)$, it collects the topology information of its $k$-neighborhood by sending messages of $O(1)$ size, then evaluates $\varphi(k)(a, y)$ (where $x$ is instantiated by $a$) by in-node computation. Since all nodes collect their $k$-neighborhood topology information concurrently, these computations may interfere with each other. To avoid the interferences between concurrent local computations of different nodes, the traces of traversed ports are incorporated in all messages.

Each node collects the topology of its $k$-neighborhood as follows.

- For each node $a$, when it receives the query $\varphi(k)(x, y)$, it sends a message (“collect”, $k$, $j$) to its neighbor though port $j$, and waits for replies.

- Upon reception of a message (“collect”, $i$, $j_1 \ldots j_{2(k-i)+1}$) by port $j'$, $a$ adds $j_1 \ldots j_{2(k-i)+1} j'$ into a table $tracelist_a$, and
  - if $i > 0$, $a$ sends on each port $j''$ s.t. $j'' \neq j'$ the message (“collect”, $i-1$, $j_1 \ldots j_{2(k-i)+1} j''$), and waits for replies;
  - otherwise ($i = 0$), $a$ sends on port $j'$ the message (“reply”, $j_1j_2\ldots j_{2k+1}$, $j'$, $tracelist_a$).

- Upon reception of a message (“reply”, $j_1 \ldots j_{2r+1}, j_{2r+2} \ldots j_{2k+2}, tracelist_1 \ldots tracelist_{k-r+1}$) on port $j_{2r+1}$, and replies from all the other ports have been received
  - if $r = 0$, for $1 \leq s \leq k + 1$, $a$ stores in the local memory ($j_1 \ldots j_{2s}, tracelist_a$);
  - otherwise $a$ sends on port $j_{2r}$ a message (“reply”, $j_1 \ldots j_{2r-1}, j_{2r} \ldots j_{2k+2}, tracelist_a tracelist_1 \ldots tracelist_{k-r+1}$).

- After receiving replies from all ports, $a$ computes the topology of the $k$-neighborhood of $a$ by utilizing the stored tuples ($j_1 \ldots j_{2r}, tracelist_a$) as follows:

  Let $$T^k(a) := \{j_1 \ldots j_{2r} | (j_1 \ldots j_{2r}, tracelist_a) \text{ is stored in local memory of } a\}. $$
Define an equivalence relation $\approx$ on $T^k(a)$ as follows: let $j_1\ldots j_{2r}, j_1^* \ldots j_{2s}^* \in T^k(a)$, then $j_1\ldots j_{2r} \approx j_1^* \ldots j_{2s}^*$ if and only if $\exists (j_1\ldots j_{2r}, \text{tracelist}), (j_1^* \ldots j_{2s}^*, \text{tracelist}')$ s.t. $j_1\ldots j_{2r} \in \text{tracelist}'$ or $j_1^* \ldots j_{2s}^* \in \text{tracelist}$. The requesting node sets a clock $\tau$.

The vertex set of the $k$-neighborhood of $a$ is

$$\left\{j_1\ldots j_{2r} | j_1\ldots j_{2r} \in T^k(a), r \leq k\right\} / \approx,$$

namely equivalence classes $[j_1 \ldots j_{2r}]$ of $\approx$ on elements $j_1\ldots j_{2r}$ ($r \leq k$) of $T^k(a)$.

Let $[j_1\ldots j_{2r}], [j_1^* \ldots j_{2s}^*]$ be two vertices of the $k$-neighborhood of $a$, then there is an edge between $[j_1\ldots j_{2r}]$ and $[j_1^* \ldots j_{2s}^*]$ if and only if $j_1^* j_2^* \ldots j_{2t+1}^* j_{2t+2}^* \in T^k(a)$ such that $j_1^* \ldots j_{2t}^* \approx j_1\ldots j_{2t}$ and $j_1^* \ldots j_{2s}^* \approx j_1^* \ldots j_{2s}^*$.

We can now state our main result for $\text{FO}_{\text{loc}}$.

**Theorem 8.** Let $G = (V, E)$ be a network with $n$ nodes and diameter $\Delta$. $\text{FO}_{\text{loc}}$ formulae $\phi^{(k)}(x, \overline{y})$ can be evaluated on $G$ with the following complexity upper bounds:

| TIME/ROUND | DIST-TIME | MSG-SIZE | #MSG/NODE |
|------------|-----------|----------|-----------|
| $O(1)$     | $O(\Delta)$ | $O(1)$   | $O(1)$    |

Note that the distributed time $O(\Delta)$ comes from the initial broadcasting of the formula. The computation itself is fully local, and can be done in $O(1)$ distributed time. In the case of an asynchronous system, DIST-TIME is bounded by $O(n)$.

We now consider $\text{FP}_{\text{loc}}$ which admits the same complexity bounds as $\text{FO}_{\text{loc}}$ except for the distributed time. We first assume that the system is synchronous, and discuss the asynchronous system later.

**Query engine for $\text{FP}_{\text{loc}}$ ($\text{QE}_{\text{FP}_{\text{loc}}}$)**

**Request flooding** The requesting node sets a clock $\sigma$ of value $\Delta$ and broadcasts the message $(\mu(\phi^{(k)}(T)(x, \overline{y})), \Delta - 1)$ to its neighbors. For each node $a$, if it receives message $(\mu(\phi^{(k)}(T)(x, \overline{y})), c)$ and it haven’t set the clock $\sigma$ before, then it sets a clock $\sigma$ of value $c$, and if $c > 0$, it broadcasts message $(\mu(\phi^{(k)}(T)(x, \overline{y})), c - 1)$ to all its neighbors.

**Topology collection** When the clock $\sigma$ expires, each node $a$ sets a clock $\sigma'$ of value $4k$ and starts collecting all the topology information in its $2k$-neighborhoods by sending messages and tracing the traversed ports (like for Theorem 5). Now each node $a$ gets a $2k$-local name for each $a'$ in its $k$-neighborhood, which is a set of traces from $a$ to $a'$ of length no more than $2k$, denote this $2k$-local name of $a'$ at $a$ by $\text{Name}_{a}^{2k}(a')$.

**Fixpoint Computation** In each node $a$, there is a local table to store the tuples $(a, \overline{b})$ in $T$, which uses the $k$-local names $\text{Name}_{a}^{k}(a')$ of $a'$.

When the clock $\sigma'$ expires, each node $a$ sets a clock $\tau = 3k$ and starts evaluating the FO formula $\phi^{(k)}(T)(a, \overline{y})$ (where $x$ is instantiated by $\text{Name}_{a}^{2k}(a)$, the $2k$-local name of $a$ at $a$). Node $a$ evaluates $\phi^{(k)}(T)(a, \overline{y})$ by instantiating all the (free or bounded) variables in $\phi^{(k)}(T)(a, \overline{y})$ by its $2k$-local names $\text{Name}_{a}^{2k}(a')$ for nodes in its $k$-neighborhood and considering all the possible instantiations one by one.

Suppose $a$ instantiates $(x, \overline{y})$ by $(a, \overline{b})$ and also instantiates all the bounded variables, then a variable-free formula $\psi$ is obtained. Since there may be atomic formulae $T(a', \overline{b'})$, $a$ should send the query $\text{?BT}(a', \overline{b'})$ to $a'$, then $a'$ should check whether $T(a', \overline{b'})$ holds or not and send the answer.
to $a$. It works since from $Name^2_k(b'_1)$, the 2k-local names of $b'_1$ at $a$, $a'$ can get $Name^k_a(b'_1)$, the k-local names of $b'_1$ at $a'$.

During the above evaluation of $\varphi^{(k)}(T)(a, \overline{y})$, if a new tuple $(a, \overline{b})$ satisfying $\varphi^{(k)}(T)(x, \overline{y})$ is obtained, $a$ stores it in a temporary buffer (the local table for $T$ will be updated later) by using the k-local names of $a$ and $\overline{b}$ at $a$, and sends messages to inform other nodes in its k-neighborhood that new facts are produced.

For each node $a$, when the clock $\tau$ expires, it sets the value of $\tau$ by $3k$ again; if some new tuples are produced, $a$ updates the local table for $T$, and empties the temporary buffer; if some new tuples are produced or some informing messages are received, it evaluates $\varphi^{(k)}(T)(a, \overline{y})$ again.

**Theorem 9.** Let $G = (V, G)$ be a network with $n$ nodes and diameter $\Delta$. $FP_{loc}$ formulae $\mu(\varphi^{(k)}(T)(x, \overline{y}))$ can be evaluated on $G$ with the following complexity upper bounds

| IN-TIME/ROUND | DIST-TIME | MSG-SIZE | #MSG/NODE |
|---------------|-----------|----------|------------|
| $O(1)$        | $O(n)$    | $O(1)$   | $O(1)$     |

**Proof.** It is easy to see that messages sent during the computation of $QE_{FP_{loc}}$ are of size $O(1)$.

Before the clock $\sigma$ expires, it is evident that each node sends only $O(1)$ messages of the format $(\mu(\varphi(T)(x, \overline{y})), c)$.

Then each node sets the clock $\sigma'$ and collects topology information of its 2k-neighborhood, since the degree of nodes is bounded and in the 2k-neighborhood of $a$ there are only $O(1)$ nodes, each node sends only $O(1)$ messages as well.

After the clock $\sigma'$ expires, each node $a$ sets the clock $\tau$ and starts evaluating $\varphi^{(k)}(T)(a, \overline{y})$. During each period $3k$ of $\tau$, node $a$ considers all the possible instantiations of the (free or bounded) variables in $\varphi^{(k)}(T)(a, \overline{y})$ one by one and evaluate the instantiated formula. During each such period, since the total number of different instantiations are $O(1)$ and only $O(1)$ messages are sent during the evaluation of each such instantiated formula $\varphi^{(k)}(T)(a, \overline{b})$, the total number of messages sent by $a$ is $O(1)$.

Moreover, after the clock $\sigma'$ expires and before the distributed computation terminates, each node $a$ only sends $O(1)$ messages: $a$ only be able to receive informing messages from nodes in its k-neighborhood, the total number of tuples $(a, \overline{b})$ produced on nodes in the k-neighborhood of $a$ is $O(1)$, so the total number of informing messages received by $a$ is $O(1)$, consequently $a$ evaluates $\varphi^{(k)}(x, \overline{y})$ at most $O(1)$ times, thus the total number of messages sent by $a$ is $O(1)$.

After the clock $\sigma'$ expires, during each period $3k$ of $\tau$, there should be at least one informing message sent by some node, which means at least one new tuple in $T$ is produced. Since there are at most $O(n)$ number of tuples in $T$, the total distributed time for the evaluation of $\mu(\varphi^{(k)}(T)(x, \overline{y}))$ is $O(n)$.

For asynchronous systems, a spanning tree rooted at the requesting node can be used to evaluate $FP_{loc}$, and the complexity bounds DIST-TIME and #MSG/NODE become respectively $O(n^2)$ and $O(n)$.

### 7.2 Networks with no global identifiers

The query engines $QE_{FO_{loc}}$ and $QE_{FP_{loc}}$ evaluate $FO_{loc}$ and $FP_{loc}$ queries by using only local names in the bounded neighborhoods of nodes, which suggests that for the evaluation of the local fragments of FO and FP, unique global identifiers for nodes are unnecessary. In this section, we
show that this is essentially the case, and consider their evaluation on networks with identifiers which are only locally consistent and on anonymous networks with ports.

**Definition 6.** A network $G = (V, E, L)$ with a labeling function $L : V \to C$ assigning identifiers to nodes, is k-locally consistent if for each node $a \in V$, for any $b_1, b_2 \in N^k(a)$, $L(b_1) \neq L(b_2)$.

Ports have been used to construct local names in the previous sub-section. They are not needed to evaluate $FO_{loc}$ and $FP_{loc}$ on locally-consistent networks since these networks have locally unique identifiers for nodes.

**Theorem 10.** A $FO_{loc}$ formula $\varphi^{(k)}(x, \bar{y})$ can be evaluated on k-locally consistent networks with the following complexity upper bounds:

| IN-TIME/ROUND | DIST-TIME | MSG-SIZE | #MSG/NODE |
|---------------|-----------|----------|-----------|
| $O(1)$        | $O(\Delta)$ | $O(1)$  | $O(1)$    |

**Theorem 11.** A $FP_{loc}$ formula $\mu(\varphi^{(k)}(T)(x, \bar{y}))$ can be evaluated on k-locally consistent networks with the following complexity upper bounds:

| IN-TIME/ROUND | DIST-TIME | MSG-SIZE | #MSG/NODE |
|---------------|-----------|----------|-----------|
| $O(1)$        | $O(n)$    | $O(1)$  | $O(1)$    |

Local fragments of FO and FP can also be evaluated with the same complexity bounds on anonymous networks with ports since local names can be obtained by tracing the traversed ports of messages.

Note that in general, FO and FP queries cannot be evaluated over locally consistent or anonymous networks.

## 8 Conclusion

Fixpoint logic expresses at a global level and in a declarative way the interesting functionalities of distributed systems. We have proved that fixpoint formulae over graphs admit reasonable distributed complexity upper-bounds.

Moreover, we showed how global formulae can be translated into rule programs describing the behavior of the nodes of the network and computing the same result. The examples given in the paper have been implemented on the Netquest system which supports the Netlog language. Finally, we proved the potential of restricted fragments of fixpoint logic to local neighborhood, that are still very expressive, but admit much tighter distributed complexity upper-bounds with bounded number of messages of bounded size, independent of the size of the network.

These results show how classical logical formalisms can help designing high level programming abstractions for distributed systems that allows to state the desired global result, without specifying its computation mode. We plan to pursue this investigation in the following directions. (i) Investigate the distributed complexity of other logical formalisms such as monadic Second Order Logic, which is very expressive on graphs. (ii) Study the optimization of the translation from fixpoint logic to Netlog, to obtain efficient programs. (iii) Extend these results to other distributed computing models.

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