On the potential functions for the hyperbolic structures of a knot complement

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Abstract We explain how to construct certain potential functions for the hyperbolic structures of a knot complement, which are closely related to the analytic functions on the deformation space of hyperbolic structures.

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Dedicated to Professor Mitsuyoshi Kato for his 60th birthday

1 Introduction

Let $M$ be the complement of a hyperbolic knot $K$ in $S^3$. Through the study of Kashaev’s conjecture, we have found a complex function which gives the volume and the Chern-Simons invariant of the complete hyperbolic structure of $M$ at the critical point corresponding to the promised solution to the hyperbolicity equations for $M$, see [2, 4] for details.

The purpose of this article is to explain how to construct such complex functions for the non-complete hyperbolic structures of $M$. Such functions are closely related to the analytic functions on the deformation space of the hyperbolic structures of $M$, parametrized by the eigenvalue of the holonomy representation of the meridian of $K$, which reveal a complex-analytic relation between the volumes and the Chern-Simons invariants of the hyperbolic structures of $M$, see [3, 5] for details.

In this note, we suppose $K$ is $5_2$ for simplicity which is represented by the diagram $D$ depicted in Figure 1.
2 Geometry of a knot complement

2.1 Ideal triangulations

We first review an ideal triangulation of $M$ due to D. Thurston. Let $\hat{M}$ denote $M$ with two poles $\pm \infty$ of $S^3$ removed. Then, $\hat{M}$ decomposes into 5 ideal octahedra corresponding to the 5 crossings of $D$, each of which further decomposes into 4 ideal tetrahedra around an axis, as shown in Figure 2.

In fact, we can recover $\hat{M}$ by gluing adjacent tetrahedra as shown in Figure 3.
As usual, we put a hyperbolic structure on each tetrahedron by assigning a complex number, called \textit{modulus}, to the edge corresponding to the axis as shown in Figure 4. In what follows, we denote the tetrahedron with modulus $z$ by $T(z)$.

Let $B$ be the intersection between $T(a_1) \cup T(b_1)$ and $T(b_3) \cup T(c_3)$. Then, each of

$$T(a_1), T(b_1), T(b_3), T(c_3)$$

intersects $\partial N(B \cup K)$ in two triangles, and they are essentially one-dimensional objects in $S^3 \setminus N(B \cup K)$. On the other hand, each of

$$T(c_1), T(d_1), T(a_2), T(b_2), T(d_2), T(a_3), T(d_3), T(a_4), T(b_4), T(c_4), T(c_5)$$

intersects $\partial N(B \cup K)$ in two triangles and one quadrangle, and they are essentially two-dimensional objects in $S^3 \setminus N(B \cup K)$. Thus, by contracting these
15 tetrahedra, we obtain an ideal triangulation $S$ of $M$ with

$$T(c_2), T(d_4), T(a_5), T(b_5), T(d_5).$$

Figure 5 exhibits the triangulation of $\partial N(B \cup K)$ induced by $S$, where each couple of edges labeled with the same number are identified.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure5.png}
\caption{Figure 5}
\end{figure}

### 2.2 Hyperbolicity equations

If $c_2, d_4, a_5, b_5, d_5$ above give a hyperbolic structure of $M$, the product of the moduli around each edge in $S$ should be 1, which is called the hyperbolicity equations and can be read from Figure 5 as follows.

\[
\begin{align*}
   d_4 b_5 &= a_5 b_5 d_5 = 1, \\
   c_2 a_5 (1 - 1/d_4) &= (1 - 1/d_5)(1 - 1/c_2)(1 - 1/b_5), \\
   (1 - d_4) & \quad (1 - a_5)(1 - b_5) = 1, \\
   c_2 (1 - 1/a_5) &= (1 - 1/d_5)(1 - 1/b_5), \\
   (1 - d_5)(1 - c_2) \quad (1 - d_5)(1 - a_5)(1 - d_4) = 1, \\
   d_4 (1 - 1/a_5)(1 - 1/d_4) &= d_5 (1 - 1/c_2). \\
   1 - b_5 \quad 1 - c_2 & = 1.
\end{align*}
\]

It is easy to observe that these equations are generated by

\[
\begin{align*}
   d_4 b_5 &= a_5 b_5 d_5 = 1, \\
   c_2 a_5 &= 1 - 1/a_5, \\
   d_4 &= (1 - 1/c_2)(1 - d_5) = (1 - 1/a_5)(1 - d_4) = c_2/d_5.
\end{align*}
\]

which suggests to put

\[
\begin{align*}
   c_2 &= y\xi, \\
   d_4 &= x/\xi, \\
   a_5 &= x/y, \\
   b_5 &= \xi/x, \\
   d_5 &= y/\xi.
\end{align*}
\]

and to rewrite the hyperbolicity equations as follows.

\[
\begin{align*}
   (1 - y/x)(1 - x/\xi) &= 1 - y/x, \\
   1 - \xi/x \quad (1 - y/\xi)(1 - 1/y\xi) & = \xi^2.
\end{align*}
\]
Note that the variables $x, y$ correspond to the interior edges of a graph depicted in Figure 6, which is $D$ with some edges deleted.

A solution to the equations above determines a hyperbolic structure of $M$, where $\xi$ is nothing but the eigenvalue of the holonomy representation of the meridian of $K$. The set $D$ of such solutions is called the deformation space of the hyperbolic structures of $M$ and can be parametrized by $\xi$ or the eigenvalue $\eta$ of the holonomy representation of the longitude of $K$. In our example, $\eta$ is given by

$$\eta = \frac{y\xi^6}{x} \cdot (1 - 1/y\xi) = \frac{y\xi^6}{x} \cdot \frac{1 - \xi/x}{(1 - x/\xi)(1 - y/\xi)}.$$ 

Note that the factors $1 - x/\xi, 1 - y/\xi, 1 - \xi/x$ and $1 - 1/y\xi$ correspond to the corners of $D$ which touch the unbounded regions.

### 3 Potential functions

Curious to say, we can always construct a potential function for the hyperbolicity equations and $\eta$ combinatorially by using Euler’s dilogarithm function

$$\text{Li}_2(z) = -\int_0^z \frac{\log(1 - w)}{w} dw,$$

where we remark that the volume of a tetrahedron with modulus $z$ is given by

$$D(z) = \text{Im} \text{Li}_2(z) + \log |z| \arg(1 - z).$$
3.1 Neumann-Zagier’s functions

In fact, we define \( V(x, y, \xi) \) by

\[-\text{Li}_2(1/y\xi) + \text{Li}_2(y/\xi) - \text{Li}_2(y/x) + \text{Li}_2(\xi/x) + \log \log \frac{x^2}{y^2\xi^6} - \frac{\pi^2}{6},\]

the principal part of which is nothing but the sum of dilogarithm functions associated to the corners of the graph as shown in Figure 7.

![Figure 7](image)

Then, we have

\[
\begin{align*}
\frac{\partial V}{\partial x} &= \log \frac{\xi^2(1 - \xi/x)}{(1 - y/x)(1 - x/\xi)}, \\
\frac{\partial V}{\partial y} &= \log \frac{1 - y/x}{\xi^2(1 - y/\xi)(1 - 1/y\xi)},
\end{align*}
\]

both of which vanish on \( \mathcal{D} \), and

\[
\xi \frac{\partial V}{\partial \xi} = \log \frac{x^2(1 - x/\xi)(1 - y/\xi)}{y^2\xi^{12}(1 - \xi/x)(1 - 1/y\xi)}
= \log \left\{ \frac{x}{y\xi^6} \cdot \frac{1}{1 - 1/y\xi} \right\}^2 - x \frac{\partial V}{\partial x} - y \frac{\partial V}{\partial y}
= \log \left\{ \frac{x}{y\xi^6} \cdot \frac{(1 - x/\xi)(1 - y/\xi)}{1 - \xi/x} \right\}^2 + x \frac{\partial V}{\partial x} + y \frac{\partial V}{\partial y},
\]

that is,

\[
\xi \frac{\partial V}{\partial \xi} = -\log \eta^2
\]

on \( \mathcal{D} \), which shows \( V(x, y, e^u) \) coincides with \( \Phi(u) \) given in [3, Theorem 3].

3.2 Dehn fillings

Furthermore, for a slope \( \alpha \in \mathbb{Q} \), we put

\[
V_\alpha(x, y, \xi) = V(x, y, \xi) + \frac{\log \xi(2\pi\sqrt{-1} - p \log \xi)}{q},
\]

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where $p, q \in \mathbb{Z}$ denote the numerator and the denominator of $\alpha$. Then, we have

$$\xi \frac{\partial V_\alpha}{\partial \xi} = \xi \frac{\partial V}{\partial \xi} + \frac{2 \pi \sqrt{-1} - p \log \xi^2}{q} = \frac{2 \pi \sqrt{-1} - p \log \xi^2 - q \log \eta^2}{q},$$

and so a solution $(x_\alpha, y_\alpha, \xi_\alpha)$ to the equations

$$d V_\alpha(x, y, \xi) = 0$$

determines the complete hyperbolic structure of the closed 3-manifold $M_\alpha$ obtained from $M$ by $\alpha$ Dehn filling. Note that, by choosing $r, s \in \mathbb{Z}$ such that $ps - qr = 1$, we can compute the logarithm of the eigenvalue of the holonomy representation of the core geodesic $\gamma_\alpha$ of $M_\alpha$ which is related to the length and the torsion of $\gamma_\alpha$ as follows, see [3, Lemma 4.2].

$$\log \frac{x_r y_s}{\eta} = \frac{s \pi \sqrt{-1} - \log \xi}{q} = \frac{\text{length}(\gamma_\alpha) + \sqrt{-1} \cdot \text{torsion}(\gamma_\alpha)}{2}.$$  

**Volumes and Chern-Simons invariants**

**3.3 Yoshida’s functions**

As in [4], we can observe

$$\text{Im } V_\alpha(x, y, \xi) = -D(1/y\xi) + D(y/\xi) - D(y/x) + D(\xi/x) + D(x/\xi)$$

$$+ \log |x| \cdot \text{Im } x \frac{\partial V_\alpha}{\partial x} + \log |y| \cdot \text{Im } y \frac{\partial V_\alpha}{\partial y} + \log |\xi| \cdot \text{Im } \xi \frac{\partial V_\alpha}{\partial \xi},$$

and so

$$\text{Im } V_\alpha(x_\alpha, y_\alpha, \xi_\alpha) = \text{vol}(M_\alpha).$$

To detect $\text{Re } V_\alpha(x_\alpha, y_\alpha, \xi_\alpha)$, we shall consider

$$R(x, y, \xi) = -R(1/y\xi) + R(y/\xi) - R(y/x) + R(\xi/x) + R(x/\xi) - \frac{\pi^2}{6},$$

where $R(z)$ denotes Roger’s dilogarithm function defined by

$$R(z) = \text{Li}_2(z) + \log z \log (1 - z)/2.$$  

Then, $R(x, y, \xi)$ can be expressed as

$$- \text{Li}_2(1/y\xi) + \text{Li}_2(y/\xi) - \text{Li}_2(y/x) + \text{Li}_2(\xi/x) + \text{Li}_2(x/\xi)$$

$$- \frac{\log x}{2} \left( x \frac{\partial V}{\partial x} - \log \xi^2 \right) - \frac{\log y}{2} \left( y \frac{\partial V}{\partial y} + \log \xi^2 \right) - \frac{\log \xi}{2} \left( \xi \frac{\partial V}{\partial \xi} - \log \frac{x^2}{y^2 \xi^2} \right),$$

and so $R(x, y, \xi)$ agrees with

$$V(x, y; \xi) + \log \xi \log \eta$$

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on $\mathcal{D}$ and with
\[ V_\alpha(x, y, \xi) = \frac{\log \xi (2\pi \sqrt{-1} - p \log \xi)}{q} + \log \xi \log \eta = V_\alpha(x, y, \xi) - \frac{\pi \sqrt{-1} \log \xi}{q} \]
at $(x_\alpha, y_\alpha, \xi_\alpha) \in \mathcal{D}$. Therefore, we have
\[ R(x_\alpha, y_\alpha, \xi_\alpha) = V_\alpha(x_\alpha, y_\alpha, \xi_\alpha) - \frac{s\pi^2 + \pi \sqrt{-1} \log \xi}{q} + \frac{s\pi^2}{q} \]
\[ = V_\alpha(x_\alpha, y_\alpha, \xi_\alpha) + \frac{\pi \sqrt{-1}}{2} \{ \text{length}(\gamma_\alpha) + \sqrt{-1} \cdot \text{torsion}(\gamma_\alpha) \} + \frac{s\pi^2}{q}. \]
In particular,
\[ \text{Im} \frac{2}{\pi} R(x, y, \xi_\alpha) = \text{Im} \frac{2}{\pi} V_\alpha(x, y, \xi_\alpha) + \frac{2 \log |\xi_\alpha|}{q} \]
\[ = \frac{2}{\pi} \cdot \text{vol}(M_\alpha) + \text{length}(\gamma_\alpha), \]
which shows that, up to a pure imaginary constant,
\[ \frac{2}{\pi \sqrt{-1}} R(x, y, e^u) \]
must coincide with $2\pi f(u)$ of [3, Theorem 2], and that
\[ \text{Re} \frac{2}{\pi} R(x, y, \xi_\alpha) = \text{Re} \frac{2}{\pi} \left\{ V_\alpha(x, y, \xi_\alpha) + \frac{s\pi^2}{q} \right\} - \text{torsion}(\gamma_\alpha) \]
must coincide with $-4\pi CS(M_\alpha) - \text{torsion}(\gamma_\alpha)$. Consequently, up to some constant which is independent of $\alpha$, we have
\[ \text{Re} \left\{ V_\alpha(x, y, \xi_\alpha) + \frac{s\pi^2}{q} \right\} = -2\pi^2 CS(M_\alpha). \]

4 Concluding remarks

We redefine $V_\alpha(x, y, \xi)$ as follows.
\[ V_\alpha(x, y, \xi) = V(x, y, \xi) + \frac{\log \xi (2\pi \sqrt{-1} - p \log \xi)}{q} + s\pi^2. \]
Then, $dV_\alpha(x, y, \xi) = 0$ gives the hyperbolicity equations for $M_\alpha$, and
\[ V_\alpha(x_\alpha, y_\alpha, \xi_\alpha) = -2\pi^2 CS(M_\alpha) + \text{vol}(M_\alpha) \sqrt{-1} \]
up to a real constant, where $(x_\alpha, y_\alpha, \xi_\alpha)$ is a solution to the equations above.

We finally remark that such a construction always works, even for a link, and the analytic functions in [3, 5] are now combinatorially constructed up to a constant. For the figure-eight knot and $\alpha \in \mathbb{Z}$, our potential function coincides with the function in [1] which appears in the “optimistic” limit of the quantum SU(2) invariants of $M_\alpha$. 

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References

[1] H Murakami, Optimistic calculations about the Witten-Reshetikhin-Turaev invariants of closed three-manifolds obtained from the figure-eight knot by integral Dehn surgeries, “Recent Progress Toward the Volume Conjecture”, RIMS Kokyuroku 1172 (2000) 70–79

[2] H Murakami, J Murakami, M Okamoto, T Takata, Y Yokota, Kashaev’s conjecture and the Chern-Simons invariants of knots and links, preprint

[3] W D Neumann, D Zagier, Volumes of hyperbolic 3-manifolds, Topology 24 (1985) 307–332

[4] Y Yokota, On the volume conjecture for hyperbolic knots, preprint available at http://www.comp.metro-u.ac.jp/~jojo/volume-conjecture.ps

[5] T Yoshida, The $\eta$-invariant of hyperbolic 3-manifolds, Invent. Math. 81 (1985) 473–514

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