An $SL_2(\mathbb{R})$-Casson invariant and Reidemeister torsions

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Abstract

We define an $SL_2(\mathbb{R})$-Casson invariant of closed 3-manifolds. We also observe procedures of computing the invariants in terms of Reidemeister torsions. We discuss some approach of giving the Casson invariant some gradings.

Keywords

Casson invariant, Reidemeister torsion, 3-manifolds, Chern-Simons class

1 Introduction

In a series of lectures [Cas], Casson defined a $\mathbb{Z}$-valued topological invariant of an integral homology 3-sphere $M$. Choose a Heegaard splitting $M = W_1 \cup_\Sigma W_2$, where $\Sigma$ is a connected closed surface. Roughly speaking, the Casson invariant counts equivalent classes of irreducible representations $\pi_1(M) \to SU(2)$, in contrast to $\pi_1(\Sigma) \to SU(2)$. Several topologists (see, e.g., [Ati, BN]) have generalized the invariant to count representations in a number of other Lie groups $G$; see [Cur1, Cur2, BH] for the cases $G = SO(3), U(2), SO(4), SL_2(\mathbb{C}), SU(3)$. The Casson invariant is a landmark topic in low-dimensional topology, and has been studied from many viewpoints, including the Chern-Simons theory; see, e.g., [AM, Sav].

This paper is inspired by the note of Johnson [John]. A difficult point of the Casson invariants is to explicitly determine appropriate weights appearing in the counts of representations. To solve this, under a condition, he suggested a procedure of computing the weights from Reidemeister torsions; see Theorem B.2. Since the note is unpublished, we give a proof of the theorem; see Appendix B.

In this paper, we mainly address the case $G = SL_2(\mathbb{R})$. Of particular interest of us is a relation to Reidemeister torsion and the Chern-Simons invariant. Since $SL_2(\mathbb{R}) = SU(1,1)$ is over $\mathbb{R}$ and non-compact, we need sensitive discussions as in [Lab, SW, Wit], e.g., we focus on Zariski-density instead of irreducibility of representations. Then, in an analogous way to the previous Casson invariants, we define an $SL_2(\mathbb{R})$-Casson invariant for closed 3-manifolds (Definition 2.1). In addition, similar to Theorem B.2 we also give an approach to the weight from Reidemeister torsions of $M$ (Theorem 3.2); as an application, we compute the $SL_2(\mathbb{R})$-Casson invariants of some Brieskorn manifolds; see 3.3. In 4.1 we further discuss a grading of weights appearing in the counts of representations $\pi_1(M) \to SL_2(\mathbb{R})$, and define a graded $SL_2(\mathbb{R})$-Casson invariant, which lies in the group ring $\mathbb{Z}[\mathbb{R}/\mathbb{Z}]$; see Section 4.1. Here the grading is obtained by Reidemeister torsions or the Chern-Simons 3-class of the Pontryagin class $p_1$; see Section 4.3 for some examples.

This paper is organized as follows. We introduce the $SL_2(\mathbb{R})$-Casson invariant in 2 and discuss some computation of the invariants in 3 In 4.1 we discuss some approaches to giving the Casson invariant some gradings. Finally, 5 gives the proofs of the theorems. In Appendix A we algebraically describe the symplectic structure on the flat moduli space in [G1].

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Conventional notation. We means by $M$ a connected closed 3-manifold with an orientation, and by $\Sigma$ a oriented closed surface. Let $g \in \mathbb{N}$ denote the genus of $\Sigma$.

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2 Definition; $SL_2(\mathbb{R})$-Casson invariant

Following the definition of the $SU(2)$-Casson invariant (see Appendix B for the definition), we will define an $SL_2(\mathbb{R})$-Casson invariants.

As a preliminary, let us explain the diagram below (1). Let $(W_1, W_2, \Sigma)$ be a Heegaard splitting of $M$, where $W_i$ is a handlebody with $\partial W_i = \Sigma$ and $M = W_1 \cup_{\Sigma} W_2$. For a Lie group $G$ and a connected CW complex $Z$ of finite type, we mean by $\text{Hom}(\pi_1(Z), G)$ the set of homomorphisms $\pi_1(Z) \to G$ with compact-open topology, and by $\text{Hom}(\pi_1(Z), G)/G$ the quotient space of $\text{Hom}(\pi_1(Z), G)$ by the conjugate action. Then, the pushout diagram

$$
\begin{array}{ccc}
\pi_1(\Sigma) & \xrightarrow{i_1} & \pi_1(W_1) \\
\downarrow{i_2} & & \downarrow{j_1} \\
\pi_1(W_2) & \xrightarrow{j_2} & \pi_1(M)
\end{array}
$$

of surjections of fundamental groups induces a commutative diagram

$$
\begin{array}{ccc}
\text{Hom}(\pi_1(\Sigma), G)/G & \xrightarrow{i_1^*} & \text{Hom}(\pi_1(W_1), G)/G \\
\downarrow{i_2^*} & & \downarrow{j_1^*} \\
\text{Hom}(\pi_1(W_2), G)/G & \xrightarrow{j_2^*} & \text{Hom}(\pi_1(M), G)/G
\end{array}
$$

of inclusions. Here, we should notice $\text{Hom}(\pi_1(M), G)/G = \cap_{i=1}^2 \text{Hom}(\pi_1(W_i), G)/G$.

In what follows, $G$ is meant by $SL_2(\mathbb{R})$, and $\mathfrak{g}$ is the Lie algebra of $G$.

Next, let us discuss an open subset of $\text{Hom}(\pi_1(Z), G)/G$ in terms of Zariski-density. Regarding $SL_2(\mathbb{R})$ as a real affine algebraic variety in $\mathbb{R}^4$, we canonically equip $SL_2(\mathbb{R})$ with Zariski topology. Let $\Lambda$ be an infinite group of $SL_2(\mathbb{R})$ generated by $\{a_1, \ldots, a_n\}$. Then, as is known (see, e.g., [Lab, Proposition 5.3.4]), $\Lambda$ is Zariski-dense in $SL_2(\mathbb{R})$ if and only if

$$
\bigcap_{i:j \leq n} \{W \in \text{Gr}_k(\mathfrak{g}) \mid a_i.W = W\} = \emptyset
$$

for any $k < 3$, where $\text{Gr}_k(\mathfrak{g})$ denotes the Grassmannian manifold of $k$-planes in $\mathfrak{g}$. Thus, the subset

$$
\text{Hom}(\pi_1(Z), G)^{zd} := \{\rho \in \text{Hom}(\pi_1(Z), G) \mid \text{Im}(\rho) \subset G \text{ is Zariski-dense.}\}
$$

is Zariski-open in $\text{Hom}(\pi_1(Z), G)$. It is known (see, e.g., [Lab, Theorem 5.2.6]) that if $Z$ is $\Sigma$ with $\text{Genus}(\Sigma) \geq 2$, then the conjugacy action of $PSL_2(\mathbb{R})$ on $\text{Hom}(\pi_1(\Sigma), G)^{zd}$ is proper and
free, and the quotient $\text{Hom}(\pi_1(\Sigma), G)^zd/G$ is an open manifold of dimension $6g - 6$, and the tangent space at $\rho \in \text{Hom}(\pi_1(\Sigma), G)^zd$ is identified with the cohomology $H^1_\rho(\Sigma; g)$ with local coefficients by $\rho$. Here, we should notice

$$H^0_\rho(\Sigma; g) = H^2_\rho(\Sigma; g) = 0, \quad H^1_\rho(\Sigma; g) \cong \mathbb{R}^{6g - 6}, \quad \text{for any } \rho \in \text{Hom}(\pi_1(\Sigma), G)^zd$$

by considering the Euler characteristic. Further, we recall from [G1] the symplectic structure on $\text{Hom}(\pi_1(\Sigma), G)^zd/G$; precisely, the cohomology $H^1_\rho(\Sigma; g)$ admits the alternating non-degenerate bilinear form defined by the composite

$$H^1_\rho(\Sigma; g)^2 \cong H^2_\rho(\Sigma; g \otimes g) \xrightarrow{\cdot \cap [\Sigma]} g \otimes g \xrightarrow{\text{Killing form}} \mathbb{R},$$

where $\sim$ is the cup product, and $\bullet \cap [\Sigma]$ is the pairing with the orientation 2-class $[\Sigma] \in H_2(\Sigma; \mathbb{Z})$. In particular, $\text{Hom}(\pi_1(\Sigma), G)^zd/G$ is oriented.

Next, we will observe the case $Z = W_i$. Since $\pi_1(W_i)$ is the free group of rank $g$, $\text{Hom}(\pi_1(W_i), G)$ is identified with $G^g$, and the conjugacy action of $PSL_2(\mathbb{R})$ on $G^g$ is also proper and free. Furthermore, the action preserves the Haar measure of $G^g$; thus it does the orientation. Therefore, the restricted action of the open set $\text{Hom}(\pi_1(W_i), G)^zd$ is proper and free, and preserves the orientation. In particular, the quotient $\text{Hom}(\pi_1(W_i), G)^zd/G$ is an oriented open manifold of dimension $3g - 3$.

Let us denote $\text{Hom}(\pi_1(Z), G)^zd/G$ by $R^zd(Z)$. Then, the restriction of (I) is written in

$$\begin{array}{ccc}
R^zd(\Sigma) & \xrightarrow{i^*_1} & R^zd(W_1) \\
\downarrow & & \downarrow
\end{array}$$

$$\begin{array}{ccc}
R^zd(W_2) & \xrightarrow{j^*_2} & R^zd(W_1) \cap R^zd(W_2) \subset R^zd(M).
\end{array}$$

Let us consider the union of 0-dimensional components in the intersection $\text{Im}(i^*_1) \cap \text{Im}(i^*_2)$, and denote the union by $\mathcal{I}_{0\text{-dim}}$, which is not always of finite order. Notice that the inclusion $SL_2(\mathbb{R}) \hookrightarrow SL_2(\mathbb{C})$ canonically gives rise to $\iota : R^zd(\Sigma) \hookrightarrow \text{Hom}(\pi_1(\Sigma), SL_2(\mathbb{C}))/SL_2(\mathbb{C})$. Define

$$\mathcal{I}_{\text{comp}} := \{ P \in \mathcal{I}_{0\text{-dim}} \mid \iota(P) \text{ is a 0-dimensional component in } \text{Im}(i^*_1 \otimes \mathbb{C}) \cap \text{Im}(i^*_2 \otimes \mathbb{C}) \}.$$

We claim that $\mathcal{I}_{\text{comp}}$ is of finite order, and there is its open tubular neighborhood of $\mathcal{I}_{\text{comp}}$ which does not meet other higher dimensional components of $\text{Im}(i^*_1) \cap \text{Im}(i^*_2)$. Indeed, as is shown in [Cur1 §2], the complexification of $\mathcal{I}_{0\text{-dim}}$ is of finite order, and admits its open tubular neighborhood which does not meet other higher dimensional components of $\text{Im}(i^*_1 \otimes \mathbb{C}) \cap \text{Im}(i^*_2 \otimes \mathbb{C})$ over $\mathbb{C}.$

Similarly to the $SU(2)$-case, the intersection points in $\mathcal{I}_{\text{comp}}$ are not always transversal. If not so, Transversality theorem ensures an isotopy $h : R^zd(\Sigma) \rightarrow R^zd(\Sigma)$ such that $h$ is supported in a compact neighborhood of $\mathcal{I}_{\text{comp}}$ which does not meet any higher dimensional component of the intersection, and $h(R^zd(W_1))$ meets $R^zd(W_2)$ transversally in $\text{supp}(h)$.

**Definition 2.1.** Let $(W_1, W_2, \Sigma)$ be a Heegaard decomposition of $M$ with $g > 1$, and $h$ be the isotopy as above. Then, we define the $SL_2(\mathbb{R})$-Casson invariant by the formula

$$\lambda_{SL_2(\mathbb{R})}(M) := \sum (-1)^g \varepsilon_f \in \mathbb{Z},$$
where the sum runs over $f$ of $h(R^d(W_1)) \cap R^d(W_2) \cap I_{\text{comp}}$. In addition, $\varepsilon_f$ equals $\pm 1$ depending on whether the orientations of the spaces $T_f h(R^d(W_1)) \oplus T_f (R^d(W_2))$ and $T_f(R^d(\Sigma))$ agree. If $g \leq 1$, we define $\lambda_{SL_2(\mathbb{R})}(M)$ to be zero.

In §5.1 we later show a topological invariance of $\lambda_{SL_2(\mathbb{R})}(M)$. To be precise,

**Theorem 2.2.** The invariant $\lambda_{SL_2(\mathbb{R})}(M) \in \mathbb{Z}$ depends only on the homeomorphism class of the 3-manifold $M$.

## 3 Computation of $SL_2(\mathbb{R})$-Casson invariants

The purpose of this section is to give a procedure of computing the $SL_2(\mathbb{R})$-Casson invariant, by means of Reidemeister torsions. As seen in Appendix B, the idea basically arises from the work of John in the case $G = SU(2)$. We begin by reviewing the torsions in §3.1

### 3.1 Review; Reidemeister torsions

Let us review algebraic torsions for cochain complexes. Let $\mathbb{F}$ be a commutative field of characteristic zero. Consider a cochain complex of length $m$,

$$C^*: 0 \to C^0 \xrightarrow{\partial^0} C^1 \xrightarrow{\partial^1} \cdots \xrightarrow{\partial^{m-2}} C^{m-1} \xrightarrow{\partial^{m-1}} C^m \to 0,$$

where $C^i$ is a vector $\mathbb{F}$-space of finite dimension. Let us select a basis $c_i$ for $C^i$, a basis $b_i$ for the boundaries $B^i$, and a basis $h_i$ for the cohomology $H^i$, where we sometimes regard $h_i$ as elements, $\tilde{h}_i$, of $C^i$ by lifts. In addition, we choose a lift, $\tilde{b}_{i+1} \in C^i$, of $b_{i+1}$, with respect to $\partial_i: C^i \to B^{i+1}$. By $b_i \tilde{h}_i \tilde{b}_{i+1}$ we mean the collection of elements given by $b_i, \tilde{h}_i$ and $\tilde{b}_{i+1}$. This set $b_i \tilde{h}_i \tilde{b}_{i+1}$ is indeed a basis for $C^i$. For bases $d, e$ of a finite dimensional $\mathbb{F}$-space, denote by $[d/e]$ the invertible matrix of a basis change, i.e. $[d/e] = (a_{ij})$ where $d_i = \sum_j a_{ij} e_j$. Then, the **algebraic torsion** (of the based complex $(C^*, c_i, h_i)$) is defined to be the alternating product

$$\mathcal{T}(C^*, c, h) := \prod_i \det [b_i \tilde{h}_i \tilde{b}_{i+1}/c_{2i}] \in \mathbb{F}^\times.$$ 

It is well-known that $\mathcal{T}(C^*, c, h)$ is independent of the choices of $b_i$ and $\tilde{b}_{i+1}$, but it does depend on the choices of $c_i$ and $h_i$. More precisely, if we select such other bases $c_i'$ and $h_i'$, we can verify

$$\mathcal{T}'(C^*, c', h') = \mathcal{T}(C^*, c, h) \prod_{j \geq 0} (\det [c_j/c'_j] \det [\tilde{h}_j/\tilde{h}_j'] (-1)^{j+1}) \in \mathbb{F}^\times. \quad (5)$$

If $C^*$ is acyclic, we often write $\mathcal{T}(C^*, c)$ instead of $\mathcal{T}(C^*, c, h)$.

Next, we review Reidemeister torsions. Let $X$ be a connected finite CW-complex. Take an $SL_n$-representation $\rho: \pi_1(X) \to SL_n(\mathbb{F})$, and regard $\mathbb{F}^n$ as a left $\mathbb{Z}[\pi_1(X)]$-module. Let $\tilde{X}$ be the universal covering space of $X$ as a CW complex, and $C_*(\tilde{X}; \mathbb{Z})$ be the cellular complex associated with the CW complex structure. This $C_*(\tilde{X}; \mathbb{Z})$ can be considered to be a left free $\mathbb{Z}[\pi_1(X)]$-module by Deck transformations. Then, the cochain complex with local coefficients is defined on

$$C_\rho^*(X; \mathbb{F}^n) := \text{Hom}_{\mathbb{Z}[\pi_1(X)]\text{-mod}}(C_*(\tilde{X}; \mathbb{Z}), \mathbb{F}^n).$$
Let us choose orientations, $c_X$, of the cells of $X$, and take a canonical basis of $\mathbb{F}^n$. If we regard a lift of $c_X$ as a basis of $C_*(\tilde{X};\mathbb{Z})$, then $C_*(X;\mathbb{F}^n)$ is a based chain complex over $\mathbb{F}$. Furthermore, with a choice of a basis $h$, of the cohomology $H^i_\rho(X;\mathbb{F}^n)$, the Reidemeister torsion of $(X, \rho)$ is defined to be

$$T(C_*(X;\mathbb{F}^n), c_X, h) \in \mathbb{F}^\times.$$

By \cite{Tur}, if two representations $\rho$ and $\rho'$ are conjugate, the resulting torsions are equal. However, discussions of signs are subtle, and this torsion does depend on the CW-complex.

To obtain topological invariants, we review the refined torsions by Turaev \cite[Chapter 3]{Tur}. Let $H^*(X;\mathbb{R})$ be the ordinary cohomology over $\mathbb{R}$. Suppose an orientation of $\bigoplus_{i\geq 0} H^i(X;\mathbb{R})$. Choose a basis $h^R \subset H^i(X;\mathbb{R})$ such that the sequence $(h^0, h^1, \ldots)$ is a positive basis in the oriented vector space $H^*(X;\mathbb{R})$. Let us define

$$\bar{\tau}(C^*(X;\mathbb{R}), c_X, h^R) := (-1)^{N(X)} T(C^*(X;\mathbb{R}), c_X, h^R) \in \mathbb{R}^\times,$$

where

$$N(X) = \sum_{i=0}^{\dim(X)} \left( \sum_{j=0}^{\dim(X)} \dim H^{i-j}(X;\mathbb{R}) \sum_{j=0}^{\dim(X)-j} \dim C^{i-j}(X;\mathbb{R}) \right) \in \mathbb{Z}/2\mathbb{Z}. \quad (6)$$

Then, the refined torsion is defined to be

$$\tau^0_\rho(X, h) := \text{sign}(\bar{\tau}(C^*(X;\mathbb{R}), c_X, h^R)) \cdot T(C^*(X;\mathbb{F}^n), c_X, h) \in \mathbb{F}^\times.$$

**Theorem 3.1** (cf. \cite[Lemma 18.1 and Theorem 18.3]{Tur}). If $n$ is even (resp. odd), then the torsion $T(C^*(X;\mathbb{F}^n), c_X, h)$ (resp. refined torsion $\tau^0_\rho(X, h)$) is independent of the order of the cells of $X$, their orientation, and the choice of $h^R$ (however, it does depends on the choice of $h$). Moreover, the torsion is invariant under simple homotopy equivalences preserving the homology orientation.

If $h = \emptyset$, the statement is exactly \cite[Lemma 18.1 and Theorem 18.3]{Tur}. Since their proofs are the same as the proof of Theorem \ref{thm:3.1} we omit the details. Recall that, any two triangulations of an oriented $C^\infty$-manifold $N$ are simple homotopy equivalent (see, e.g., \cite[§II.8]{Tur}); in conclusion, if $X$ is a triangulation of $N$, the refined torsion gives a topological invariant of $N$ associated with $\rho : \pi_1(N) \to SL_n(\mathbb{F})$ and $h$.

### 3.2 Statement

In this subsection, we give a procedure of computing the $SL_2(\mathbb{R})$-Casson invariants. For this, we shall develop methods of analyzing computation of $\varepsilon_f$, as an analog to Theorem \cite{Bor2}

**Theorem 3.2.** We assume $H_*(M;\mathbb{Q}) \cong H_*(S^3;\mathbb{Q})$, i.e., $M$ is a rational homology 3-sphere, and that, for any $f \in \mathcal{I}_{\text{comp}}$, the intersection of $\text{Im}(i^*_f)$ and $\text{Im}(i^*_f)$ at $f$ is transversal. Then, the equality $\varepsilon_f = (-1)^{g} \cdot \text{sign}(\tau^0_f(M))$ holds for any $f \in \mathcal{I}_{\text{comp}}$. In particular,

$$\lambda_{SL_2(\mathbb{R})}(M) = \sum_{f \in \mathcal{I}_{\text{comp}}} \text{sign}(\tau^0_f(M)). \quad (7)$$
Since the proof is technical, the proof will appear in [5.3] The assumption is characterized by the following lemma.

**Lemma 3.3.** Take \( f \in \mathcal{I}_{\text{comp}} \). Then, the intersection of \( \text{Im}(i_1^*) \) and \( \text{Im}(i_2^*) \) at \( f \) is transversal, if and only if \( H^1_f(M; g) = H^2_f(M; g) = 0 \).

**Proof.** Consider the Mayer-Vietoris sequence from \((\Sigma, W_1, W_2)\):

\[
H^0_f(\Sigma; g) \rightarrow H^1_f(M; g) \rightarrow H^1_f(W_1; g) \oplus H^1_f(W_2; g) \xrightarrow{i_1^* \oplus i_2^*} H^1_f(\Sigma; g) \rightarrow H^2_f(M; g) \rightarrow 0.
\]

Notice \( H^0_f(\Sigma; g) = 0 \) because of \( f \in R^{\text{ad}}(\Sigma) \). Since the intersection is transversal if and only if \( i_1^* \oplus i_2^* \) is an isomorphism, we get the conclusion. \( \square \)

### 3.3 Examples; some Brieskorn manifolds

Using Theorem 3.2, we will compute the \( SL_2(\mathbb{R}) \)-Casson invariants of some Brieskorn 3-manifolds.

Let us review the Brieskorn 3-manifolds. Fix integers \( m, p, q, d \in \mathbb{N} \) such that \( m, p, q \) are relatively prime and \( m, p \geq 3, q = dp + 1 \). Then, the Brieskorn 3-manifold

\[
\Sigma(m, p, q) := \{(x, y, z) \in \mathbb{C}^3 \mid x^m + y^p + z^q = 0, \ |x|^2 + |y|^2 + |z|^2 = 1 \}
\]

is a homology 3-sphere, and, \( \Sigma(m, p, q) \) is an Eilenberg-MacLane space if \( 1/m + 1/p + 1/q < 1 \). Furthermore, consider the group presentation \( \langle x_1, x_2, \ldots, x_m \mid r_1, \ldots, r_m \rangle \) with

\[
r_i := x_i^{x_{i+q}x_{i+2q}} \cdots x_i^{x_{i+(q-1)dq}} \big(x_i^{x_{i+q}x_{i+q+1}} \cdots x_i^{x_{i+(q-1)dq-2q+1x_{i+(q-1)dq-q+1}}^j} \big)^{-1};
\]

where the subscripts are taken by mod \( m \). According to [CHK], this group is isomorphic to \( \pi_1(\Sigma(m, p, q)) \), and this presentation is derived from a genus \( m \) Heegaard decomposition of \( \Sigma(m, p, q) \).

Let \( \widetilde{X} \) be the universal covering of \( \Sigma(m, p, q) \) as a contractible space, and let \( \pi_1 \) be \( \pi_1(\Sigma(m, p, q)) \) for short. We now address the cellular complex of \( \widetilde{X} \). By the Heegaard decomposition, the cellular complex is described as

\[
C_*(\widetilde{X}; \mathbb{Z}) : 0 \rightarrow \mathbb{Z}[\pi_1] \xrightarrow{\partial_3} \mathbb{Z}[\pi_1]^m \xrightarrow{\partial_2} \mathbb{Z}[\pi_1]^m \xrightarrow{\partial_1} \mathbb{Z}[\pi_1] \rightarrow 0 \quad \text{exacts}.
\]

Then, by a similar discussion to [Ko], we can verify that the boundary maps \( \partial_* \) have matrix presentations of the forms

\[
\partial_3 = \begin{pmatrix}
1 - x_1^{x_{1+(d_q-d-1)q}x_{1+(d_q-d-1)q}x_1 \cdots x_{1+(d_q-d-1)q}x_{1+(d_q-d-1)q}x_2 \cdots x_2, \\
1 - x_m^{x_{m+(d_q-d-1)q}x_{m+(d_q-d-1)q}x_m \cdots x_m}
\end{pmatrix} \in \text{Mat}(m \times 1; \mathbb{Z}[\pi_1]),
\]

\[
\partial_2 = \frac{\partial r_j}{\partial x_i} \big|_{1 \leq i, j \leq m} \in \text{Mat}(m \times m; \mathbb{Z}[\pi_1]), \tag{8}
\]

\[
\partial_1 = (1 - x_1, 1 - x_2, 1 - x_3, \ldots, 1 - x_m)^{\text{transpose}}. \tag{9}
\]

Here, \( \frac{\partial r_j}{\partial x_i} \) is the Fox derivative of \( r_j \) with respect to \( x_i \). As is known [Cur1, KY, Sav], if \( G = SL_2(\mathbb{R}) \), then \( R^{\text{ad}}(W_1) \cap R^{\text{ad}}(W_2) = R(\Sigma(m, p, q)) \) is true as a finite set, and satisfies
the assumption in Theorem 3.2. Given concrete \( m, p, q \in \mathbb{N} \) and a non-trivial Zariski-dense representation \( f : \pi_1(\Sigma(m, p, q)) \to SL_2(\mathbb{R}) \), by definition of torsion, we can compute the torsion \( \tau^0(\Sigma(m, p, q)) \) (Here, Theorem 2.2 in [Tu] makes the computation easier). When \( m, p, q \leq 9 \) or \( (m, p, q) = (m, 2, 3) \) with \( m < 25 \), we can verify \( \tau^0(\Sigma(m, p, q)) < 0 \) by the help of the computer program using Mathematica. Therefore, we suggest a conjecture.

**Conjecture 3.4.** Let \( m, p, q \in \mathbb{Z} \) be as above. Then, \( (-1)^g \text{sign}(\tau^0) = \varepsilon_f \in \{\pm 1\} \) would be negative for any \( f \in R^d(\mathcal{W}_1) \cap R^d(\mathcal{W}_2) = R^d(\Sigma(m, p, q)) \). In particular, Theorem 3.2 implies that the invariant \( \lambda_{SL_2(\mathbb{R})}(\Sigma(m, p, q)) \in \mathbb{Z} \) would be \(-|R^d(\Sigma(m, p, q))|\).

**Remark 3.5.** If we replace \( SL_2(\mathbb{R}) \) by \( SU(2) \), then \( \varepsilon_f = -1 \) is known; see [Sav]. Furthermore, as is shown [KY], Corollary 1.4, the order of \( R^d(\Sigma(m, p, q)) \) is equal to

\[
\frac{(m - 1)(p - 1)(q - 1)}{4} - 2\# \{(s, t, u) \in \mathbb{N}^3_0 \mid s < m, t < p, u < q, \frac{s}{m} + \frac{t}{p} + \frac{u}{q} < 1\}.
\]

4 Invariants graded by the Chern-Simons invariant.

We will discuss graded \( SL_2(\mathbb{R}) \)-Casson invariants.

4.1 Discussion; grading the invariant

In order to give a grading of the \( SL_2(\mathbb{R}) \)-Casson invariant, we first reconsider the isotopy \( h \) in [2]. Since \( 3g - 3 \geq 3 \), we can apply a Whitney trick in constructing \( h \). Hence, for any \( f \in \mathcal{I}_{\text{comp}} \), we can choose \( h \) such that \( h(f) = f \) if the local intersection number at \( f \) is \( \pm 1 \), and \( h(f) \) is not contained in \( \mathcal{I}_{\text{comp}} \) if the intersection number is 0. Therefore, if we have a map \( F : \text{Hom}(\pi_1(M), SL_2(\mathbb{R}))/SL_2(\mathbb{R}) \to K \) for some group \( K \), we can verify that the sum

\[
\lambda^{F}_{SL_2(\mathbb{R})}(M) := (-1)^g \sum_{f \in h(R^d(\mathcal{W}_1) \cap R^d(\mathcal{W}_2) \cap \mathcal{I}_{\text{comp}})} \varepsilon_f F(f) \in \mathbb{Z}[K]
\]

in the group ring is a topological invariant, where the proof is similar to that of Theorem 2.2. As examples of \( F \), the Reidemeister torsion and the Chern-Simons invariant are invariant with respect to the conjugacy action.

We now explain the definition of the Chern-Simons invariant in details. For a group \( G \), let \( BG \) be the Eilenberg-MacLane space. The classifying map \( c_M : M \to B\pi_1(M) \) gives rise to \( (c_M)_* : H_3(M; \mathbb{R}) \to H_3(B\pi_1(M); \mathbb{Z}) \). As is shown [Dup], the \( (p_1, \pi_1) \)-Chern-Simons class, \( P_1 \), is a representative 3-cocycle in the third cohomology \( H^3(\text{BSL}_2(\mathbb{R}); \mathbb{R}/\mathbb{Z}) \); see Theorem 4.1 below. Let \( [M] \in H_3(M; \mathbb{Z}) \) be the orientation 3-class of \( M \). Then, given a representation \( f : \pi_1(M) \to SL_2(\mathbb{R}) \), the Chern-Simons invariant is defined to be the pairing

\[
(P_1, f_* \circ (c_M)_*[M]) \in \mathbb{R}/\mathbb{Z}.
\]

Moreover, as is well-known, the Chern-Simons invariant is invariant with respect to the conjugacy action, and locally constant on \( \text{Hom}(\pi_1(M), SL_2(\mathbb{R}))/SL_2(\mathbb{R}) \).

In addition, when \( M \) is an integral homology 3-sphere, we can give an \( \mathbb{R} \)-valued lift of the invariant as follows. Let \( \tilde{G} \to PSL_2(\mathbb{R}) \) be the universal covering of \( SL_2(\mathbb{R}) \) associated with \( \pi_1(SL_2(\mathbb{R})) \cong \mathbb{Z} \), which is a central extension of fiber \( \mathbb{Z} \). Notice that every homomorphism
Theorem 4.1 (Dup, Theorem 1.11)

Let \( f \in \pi_1(M) \to SL_2(\mathbb{R}) \) uniquely admits a lift \( \tilde{f} : \pi_1(M) \to \tilde{G} \), since \( H_1(M; \mathbb{Z}) = H_2(M; \mathbb{Z}) = 0 \). Moreover, as seen [Dup, §1 and §4] (but this is noted by others), as a lift of \( P_1 \), there is a 3-cocycle \( \tilde{P}_1 \in H^3(BG; \mathbb{R}) \). To summarize, the sum

\[
\sum \epsilon_f \{ (\tilde{P}_1, \tilde{f} \circ (c_M)_*[M]) \} \in \mathbb{Z}[\mathbb{R}].
\]

gives a topological invariant of integral homology 3-spheres, as a graded \( SL_2(\mathbb{R}) \)-Casson invariant.

4.2 Computation of the graded invariant

We will give a procedure of computing the \( \mathbb{R}/\mathbb{Z} \)-valued invariant (11), if \( M \) is an Eilenberg-MacLane space.

We first recall the (normalized) definition of group (co-)homology. For a group \( G \), the group homology, \( H_n(G; \mathbb{Z}) \), is defined to be \( \text{Tor}_{n}^{\mathbb{Z}(G)}(\mathbb{Z}; \mathbb{Z}) \). For example, if we let \( C^\text{Nor}_n(G; \mathbb{Z}) \) be the quotient \( \mathbb{Z} \)-free module of \( \mathbb{Z}(G^{n+1}) \) subject to the relation \((g_0, \ldots, g_n) \sim 0 \) if \( g_i = g_{i+1} \) for some \( i \), the complex \( C^\text{Nor}_n(G; \mathbb{Z}) \) with boundary map

\[
\partial_n(g_0, \ldots, g_n) = \sum_{0 \leq i \leq n} (-1)^i(g_0, \ldots, g_{i-1}, g_{i+1}, \ldots, g_n),
\]

is acyclic and the homology of \( C^\text{Nor}_n(G; \mathbb{Z}) \otimes_{\mathbb{Z}[G]} \mathbb{Z} \) is isomorphic to \( H_n(G; \mathbb{Z}) \). Dually, for an abelian group \( M \), we can define a coboundary map on \( \text{Map}(G^{n+1}, M) \), and define the cohomology \( H^*(G; M) \). Any cohomology class of \( H^*(G; M) \) is represented by a map \( G^{n+1} \to M \). As is well-known, \( H_*(G; \mathbb{Z}) \cong H_*(BG; \mathbb{Z}) \) and \( H^*(BG; \mathbb{Z}) \cong H^*(G; \mathbb{Z}) \).

Let us recall from [Dup] the 3-cocycle, which represents the \( P_1 \) in details. Given 4-tuples of distinct points \((a_0, a_1, a_2, a_3)\) in \( P^{\mathbb{R}1} \), the cross ratio is defined by

\[
(a_0, a_1, a_2, a_3) := \frac{a_0 - a_2}{a_0 - a_3} \cdot \frac{a_1 - a_3}{a_1 - a_2} \in \mathbb{R} \setminus \{0, 1\}.
\]

For \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}) \), we define \( g \infty \) by \( b/d \) if \( d \neq 0 \), and by \( a/c \) if \( d = 0 \). In addition, consider the real Rogers’ \( L \)-function

\[
L(x) := -\frac{\pi^2}{6} - \frac{1}{2} \int_{0}^{x} \left( \frac{\log(1-t)}{t} + \frac{\log t}{1-t} \right) dt
\]

for \( 0 \leq x \leq 1 \), which is extended to \( \mathbb{R} \) by

\[
L(x) := \begin{cases} -L(1/x) & \text{for } x > 1, \\ L(1-1/x) & \text{for } x < 0. \\ \end{cases}
\]

**Theorem 4.1** ([Dup, Theorem 1.11]). Take the map \( l : SL_2(\mathbb{R})^4 \to \mathbb{R}/\mathbb{Z} \) defined by

\[
l(g_0, g_1, g_2, g_3) := -\frac{1}{4\pi^2} L(\{0, g_0^{-1}g_1\infty, g_0^{-1}g_2\infty, g_0^{-1}g_3\infty\}).
\]

Here, we put \( l(\{a_0, a_1, a_2, a_3\}) = 0 \) whenever there are two equal among \( a_0, a_1, a_2, a_3 \in P^{\mathbb{R}1} \).

Then, \( l \) is a 3-cocycle, and coincides with the Chern-Simons 3-class associated with the first Pontryagin class modulo 1/24. That is, \( 24l \) and \( 24P_1 \) are equal in \( H^3(SL_2(\mathbb{R}); \mathbb{R}/\mathbb{Z}) \).
Next, we will address an algorithm to describe the fundamental 3-class in the group complex \( C_3(\pi_1(M); \mathbb{Z}) \). Take a genus \( g \) Heegaard decomposition of \( M \). Since the 1-skeleton consists of \( g \) one-handles, we have a presentation \( \langle x_1, \ldots, x_g | r_1, \ldots, r_g \rangle \) of \( \pi_1(M) \). Then, since \( M \) is an Eilenberg-MacLane space, the cellular complex of the universal cover \( \tilde{M} \) is described as

\[
C_*(\tilde{M}) : 0 \to \mathbb{Z}[\pi_1(M)]^g \xrightarrow{\partial_2} \mathbb{Z}[\pi_1(M)] \xrightarrow{\partial_1} \mathbb{Z}[\pi_1(M)] \to \mathbb{Z} \quad \text{(exact)}.
\]

Here, according to \cite{Lyn}, the boundary maps \( \partial_2 \) and \( \partial_1 \) are given by (8) and (9), respectively. Denote the basis of \( C_3(\tilde{M}) \) by \( O_M \). Then, if we can construct a chain map \( c : C_*(\tilde{M}) \to C_3^{\text{Nor}}(\pi_1(M); \mathbb{Z}) \) as a \( \mathbb{Z}[\pi_1(M)] \)-homomorphism which is unique up to homotopy, then \( [c(O_M)] \in C_3^{\text{Nor}}(\pi_1(M); \mathbb{Z}) \otimes_{\mathbb{Z}[\pi_1(M)]} \mathbb{Z} \) means the fundamental 3-class.

The chain map \( c_* \) can be constructed as follows. Let \( c_0 \) be the identity. Let \( A \in \mathbb{Z}[G] \) be any element. Define \( c_1(Ax_i) := (A, Ax_i) \). If \( r_i \) is expanded as \( x_i^{\epsilon_1}x_i^{\epsilon_2} \cdots x_i^{\epsilon_n} \) for some \( \epsilon_k \in \{\pm 1\} \), we define

\[
c_2(Ar_i) = \sum_{m:1 \leq m \leq n} \epsilon_m(A, Ax_i^{\epsilon_1}x_i^{\epsilon_2} \cdots x_i^{\epsilon_{m-1}}x_i^{\epsilon_m/2}, Ax_i^{\epsilon_1}x_i^{\epsilon_2} \cdots x_i^{\epsilon_{m-1}}x_i^{\epsilon_m+1/2}) \in C_2^{\text{Nor}}(\pi_1(M); \mathbb{Z}).
\]

Then, we can easily verify \( \partial_1 \circ c_1 = c_0 \circ \partial_1 \) and \( \partial_2 \circ c_2 = c_1 \circ \partial_2 \). Notice that \( \partial_2 \circ c_2 \circ \partial_3(O_M) = c_1 \circ \partial_2 \circ \partial_3(O_M) = 0 \), that is, \( c_2 \circ \partial_3(O_M) \) is a 2-cycle. If we expand \( c_2 \circ \partial_3(O_M) \) as \( \sum n_i(g_0^i, g_1^i, g_2^i) \) for some \( n_i \in \mathbb{Z}, g_j^i \in G \), then \( O'_M := -\sum n_i(1, g_0^i, g_1^i, g_2^i) \) satisfies \( \partial_3(O'_M) = c_2 \circ \partial_3(O_M) \). Therefore, the correspondence \( O_M \to O'_M \) gives rise to a chain map \( c_3 : C_*^{\tilde{M}} \to C_3^{\text{Nor}}(\pi_1(M)); \mathbb{Z}) \) as desired. In conclusion, the above discussion is summarized as follows:

**Proposition 4.2.** For \( f : \pi_1(M) \to SL_2(\mathbb{R}) \), the composite \( l(f_*(O'_M)) \in \mathbb{R}/\mathbb{Z} \) is equal to the pairing \( \langle P_1, f_* (c_M)_*[M] \rangle \) modulo 1/24. In particular, the graded \( SL_2(\mathbb{R}) \)-Casson invariant is computed as

\[
\lambda_{SL_2(\mathbb{R})}^{24P_1}(M) = \sum \varepsilon_f \{24l(f_*(O'_M)) \} \in \mathbb{Z}[\mathbb{R}/\mathbb{Z}].
\]

### 4.3 Examples; some Seifert manifolds

For odd numbers \( m, n \in \mathbb{Z} \), let us consider the Seifert manifolds \( M_{m,n} := \Sigma((m, 1), (n, 1), (2, -1)) \) over \( S^2 \), where the three singular fibers are characterised by the integral surgery coefficients \((m, 1), (n, 1)\) and \((2, -1)\). Then, if \( 1/m + 1/n < 1/2 \), the manifold is an Eilenberg-MacLane space and admits a genus two Heegaard diagram; see, e.g., \cite{Sav} §6. The fundamental group is presented as

\[
\langle x, y \mid r_1 := y^n(xy)^{-2}, \ r_2 := x^m(yx)^{-2} \rangle.
\]

Furthermore, we can verify that \( \partial_3(O_M) \) is given by \( (1 - y)r_1 + (1 - x)r_2 \in C_2(\tilde{M}) \). Therefore, by the above construction of \( O'_M \), we can easily verify that

\[
O'_M = -(1, x, 1, y) - (1, x, y, xy) - (1, x, xy, xyx) - (1, y, 1, x) - (1, y, x, xy) - (1, y, xy, xyx) + \sum_{0 \leq j \leq m-2} (1, x, x^j, x^{j+1}) + \sum_{0 \leq j \leq n-2} (1, y, y^j, y^{j+1}).
\]

Furthermore, it is not difficult to classify all the \( SL_2 \)-representations of \( M_{m,n} \). Precisely,
Lemma 4.3. For $k, \ell \in \mathbb{N}$ with $k \leq n/2$ and $\ell \leq m/2$, take
\[
\beta_k := \exp(2\pi k \sqrt{-1}/n) + \exp(-2\pi k \sqrt{-1}/n), \quad \gamma_\ell := \exp(2\pi \ell \sqrt{-1}/m) + \exp(-2\pi \ell \sqrt{-1}/m).
\]
When $\beta_k^2 + \gamma_\ell^2 > 4$, let us consider the correspondence
\[
f_{k,\ell}(y) = \left(\frac{\beta_k/2}{(\gamma_\ell + \sqrt{\beta_k^2 + \gamma_\ell^2 - 4})/2}, \quad \frac{-\gamma_\ell + \sqrt{\beta_k^2 + \gamma_\ell^2 - 4}}{\beta_k/2}\right), \quad f_{k,\ell}(xy) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\]
Then, this gives rise to a homomorphism $f_{k,\ell} : \pi_1(M_{m,n}) \to SL_2(\mathbb{R})$. Furthermore, the map $(k, \ell) \mapsto f_{k,\ell}$ yields a bijection
\[
\{ (k, \ell) \in \mathbb{Z}_{\geq 0}^2 \mid k \leq \frac{n}{2}, \ell \leq \frac{m}{2}, \beta_k^2 + \gamma_\ell^2 > 4 \} \leftrightarrow \text{Hom}(\pi_1(M_{m,n}), SL_2(\mathbb{R}))_{\text{adic}}/SL_2(\mathbb{R}).
\]
In summary, since $\mathcal{O}_M'$ and $f_{k,\ell}$ are explicitly described, for small $k, \ell$ we can numerically compute the pairings $24l(f_*(\mathcal{O}_M'))$ by the help of the computer program. Here, in a similar fashion to [3.3] we can verify $\varepsilon_{f_{k,\ell}} < 0$ for any $k, \ell$. We give some examples:

Example 4.4. (I) The case of $m = 3$ and $n \leq 15$. Then the set consists of $\{f_{1, (n-1)/2}\}$. By the help of the computer program, the resulting computations of the pairing are listed as

| $n$ | 7   | 9   | 11  | 13  | 15  |
|-----|-----|-----|-----|-----|-----|
| Pairing $\in \mathbb{R}/\mathbb{Z}$ | 0.100637... | 0.826310... | 0.660662... | 0.549320... | 0.950164... |

(II) The case of $m = 5$ and $n \leq 11$. The resulting computations of the pairing $24l(f_*(\mathcal{O}_M'))$ are listed as

| $(n, k, \ell)$ | (7,2,1) | (7,2,3) | (9,2,1) | (9,2,4) |
|----------------|---------|---------|---------|---------|
| Pairing $\in \mathbb{R}/\mathbb{Z}$ | 0.562345... | 0.275253... | 0.906666... | 0.979077... |

| $(n, k, \ell)$ | (11,1,5) | (11,2,1) | (11,2,4) | (11,2,5) |
|----------------|---------|---------|---------|---------|
| Pairing $\in \mathbb{R}/\mathbb{Z}$ | 0.658563... | 0.456043... | 0.111275... | 0.942540... |

As is seen in the examples, it is reasonable to hope that if $\pi_1(M)$ has a non-trivial $SL_2(\mathbb{R})$-representation, the graded invariant $\lambda^P_{SL_2(\mathbb{R})}(M)$ is a strong invariant.

On the other hands, the Casson invariant with grading by the Reidemeister torsions is computed as follows:

Proposition 4.5. Let $M = \mathbb{R}^2$ be the standard $SL_2(\mathbb{R})$-representation. Then, for the homomorphism $f_{k,\ell} : \pi_1(M_{m,n}) \to SL_2(\mathbb{R})$, the complex $C^*_{f_{k,\ell}}(M_{m,n}; \mathbb{R}^2)$ is acyclic, and the torsion is given by
\[
\mathcal{T}(C^*_{f_{k,\ell}}(M_{m,n}; \mathbb{C}^2)) = \frac{4}{(2 - \beta_k)(2 - \gamma_\ell)} \in \mathbb{R}^\times.
\]

Proof. Consider the complexification $f_{k,\ell} \otimes \mathbb{C} : \pi_1(M_{m,n}) \to SL_2(\mathbb{R}) \hookrightarrow SL_2(\mathbb{C})$. According to the main theorem in [K1], the complex $C^*_{f_{k,\ell} \otimes \mathbb{C}}(M_{m,n}; \mathbb{C}^2)$ is acyclic and the torsion $\mathcal{T}(C^*_{f_{k,\ell} \otimes \mathbb{C}}(X; \mathbb{C}^2), \mathbb{C}^X)$ is $4/((2 - \beta_k)(2 - \gamma_\ell))$. Since, from the definition of $\mathcal{T}$, the acyclicity and the torsion is invariant under the complexification, we conclude the desired.

5 Proofs of theorems

We will give the proofs of Theorems 2.2 and 3.2. Throughout this section, we let $G = SL_2(\mathbb{R})$. 


5.1 Proofs of Theorem 2.2

Proof of Theorem 2.2: The proof is almost same as discussions of [AM, Chapter IV] or [Sav, §16.3]. First, consider the case where $M$ is one of the lens spaces, $S^3$ and $S^1 \times S^2$. Then, $R^{zd}(M)$ is empty for any Heegaard decomposition of $M$. Hence, $\lambda_{SL_2(\mathbb{R})}(M) = 0$ by definition. However, we may assume $M \neq S^3$, and $g > 1$, in what follows.

Let $(W_1', W_2', \Sigma')$ be another Heegaard decomposition of $M$. If $(W_1', W_2', \Sigma)$ and $(W_1', W_2', \Sigma')$ are isotopic, we can easily verify the invariance of $\lambda_{SL_2(\mathbb{R})}(M)$. Thanks to the famous theorem [Rei], it is enough to show the invariance of $\lambda_{SL_2(\mathbb{R})}$ if $(W_1', W_2', \Sigma')$ is a Heegaard decomposition obtained from $(W_1, W_2, \Sigma)$ by attaching an unknotted handle; see Figure 1. Then, we have the following identifications: $\pi_1(W_1') = \mathbb{Z} \ast \pi_1(W_1)$ and $\pi_1(W_2') = \mathbb{Z} \ast \pi_1(W_2)$, where the $\mathbb{Z}$ are generated by the loop $a_0$ and $b_0$ in Figure 1 respectively.

Let $\Sigma_0 = \Sigma \setminus D^2$ and $\Sigma_0' := \Sigma' \setminus D^2$, where $D^2$ is the 2-disc which is removed in the handle-attaching; see Figure 1. Then, $\pi_1(\Sigma_0) = \mathbb{Z} \ast \mathbb{Z} \ast \pi_1(\Sigma_0')$, where the factor $\mathbb{Z} \ast \mathbb{Z}$ is freely generated by $a_0, b_0$. Then, we get the identifications

$$R(W_k') = G \times R(W_k), \quad R(\Sigma_0') = G \times G \times R(W_k).$$

Consider the inclusions

$$G \times \text{Hom}(\pi_1(W_1), G) \hookrightarrow G \times G \times \text{Hom}(\pi_1(\Sigma_0), G); \quad (a, \alpha) \mapsto (a, 1, \alpha),$$

$$G \times \text{Hom}(\pi_1(W_2), G) \hookrightarrow G \times G \times \text{Hom}(\pi_1(\Sigma_0), G); \quad (b, \alpha) \mapsto (1, b, \alpha),$$

which factor through $\text{Hom}(\pi_1(\Sigma'), G)$, Then, we have the following identifications:

$$\text{Hom}(\pi_1(W_1'), G) \cap \text{Hom}(\pi_1(W_2'), G) = 1 \times 1 \times \text{Hom}(\pi_1(W_1), G) \cap \text{Hom}(\pi_1(W_2), G)$$

$$= 1 \times 1 \times \text{Hom}(\pi_1(M), G).$$

We see that

$$\text{Hom}(\pi_1(W_1'), G)^{zd} \cap \text{Hom}(\pi_1(W_2'), G)^{zd} = 1 \times 1 \times (\text{Hom}(\pi_1(W_1), G)^{zd} \cap \text{Hom}(\pi_1(W_2), G)^{zd}).$$

Since these identifications are equivariant with respect to the conjugacy $\text{PSL}_2(\mathbb{R})$-action, we have

$$R^{zd}(W_1') \cap R^{zd}(W_2') = 1 \times 1 \times (R^{zd}(W_1) \cap R^{zd}(W_2)).$$

We discuss the isotopy $h$. By a similar discussion to [AM, pages 70–78], we can verify that there is an isotopy $\tilde{h} : R^{zd}(\Sigma') \to R^{zd}(\Sigma')$ such that

$$\tilde{h}(R^{zd}(W_1')) \cap R^{zd}(W_2') = 1 \times 1 \times (h(R^{zd}(W_1)) \cap R^{zd}(W_2)).$$

Therefore, we have

$$\lambda_{SL_2(\mathbb{R})}(M)' = (-1)^{g+1} \sum_f \varepsilon_f, \quad \lambda_{SL_2(\mathbb{R})}(M) = (-1)^g \sum_f \varepsilon_f. \quad (13)$$

Hence, it is enough to show $\varepsilon_f = -\varepsilon_f$ for any $f$. However, the proof is the same as that in the case $G = SU(2)$; see, e.g., [Sav, pages 155–156]. Thus, we omit the details.
5.2 Proof of Theorem 3.2

Next, to prove Theorem 3.2, let us review a theorem of [Mil1]. Consider a short exact sequence
\[ 0 \to C^* \xrightarrow{\beta} \overline{C}^* \xrightarrow{k} C^* \to 0, \]
in the category of bounded chain complexes and chain mappings over \( \mathbb{F} \). Then, the long exact homology sequence
\[ H^*: H^0 \xrightarrow{j} P^0 \xrightarrow{k_0} H^0 \to \cdots \to H^m \xrightarrow{j^m} P^n \xrightarrow{k^n} H^n \] (14)
can be thought of as an acyclic chain complex of length \( 3m + 3 \). Hence, if we fix bases \( h, h, \overline{h} \) of \( H^*, H^*, P^* \), respectively, we can define the torsion \( T(H^*, h, h, \overline{h}) \).

Theorem 5.1 ([Mil1 Theorem 3.2]). We now assume that, \( C^i, \overline{C}^i, C^i \) have distinguished bases \( c_i, \overline{c}_i, c_i \) such that \( \det[c_i/\overline{c}_i/c_i] = 1 \) for all \( i \). Then, there is \( \eta \in \mathbb{Z}/2 \) such that
\[ (-1)^\eta T(\overline{C}^*, c, \overline{h}) = T(C^*, h)T(\overline{C}^*, c, h)T(H^*, h, h, \overline{h}) \in \mathbb{F}^x. \]

Remark 5.2. In the original paper, \( \eta \) is not clarified. However, thoughtfully following the proofs of [Thur Theorem 1.5] and [Mil1 Theorem 3.2], we can verify that \( \eta \) is formulated as
\[ \eta = \sum_{i=0}^m \dim(\text{Im}(j_i)) \dim B^i + \dim(\text{Im}(k_i)) \dim B^{i+1} + \dim B^{i+1} \dim B^i \in \mathbb{Z}/2. \]

Moreover, we discuss the refined torsions on closed surfaces. Recall from (3) that, for any representation \( \rho \in R^{\text{ad}}(\Sigma) \), the cohomology \( H^1(\Sigma; \mathfrak{g}) \cong \mathbb{R}^{6g-6} \) admits a symplectic structure; we can choose a symplectic basis \( h_{\text{sym}} \subset H^1(\Sigma; \mathfrak{g}) \). Moreover, concerning ordinary cohomology, choose a symplectic basis \( h_1 \subset H^1(\Sigma; \mathbb{R}) \cong \mathbb{R}^{2g} \), which is compatible with the orientation of \( H^*(\Sigma; \mathbb{R}) \). Then, we can define the refined torsion,
\[ \tau^0_\rho(\Sigma, h_{\text{sym}}) \in \mathbb{R}^x. \]

By (3) and symplecticity of \( h_{\text{sym}} \), this torsion does not depend on the choice of \( h_{\text{sym}} \).

Remark 5.3. In [SW Section 3.4.4] (see also [Lab Proposition 4.3.6] or [Witt §4.5]), the function \( R^{\text{ad}}(\Sigma) \to \mathbb{R}^x \) which takes \( \rho \) to \( \tau^0_\rho(\Sigma, h_{\text{sym}}) \) is mathematically shown to be constant on each connected component of \( R^{\text{ad}}(\Sigma) \). For more details,
Proposition 5.4 (cf. [SW, Wit]). For any $\rho \in R^{sd}(\Sigma)$, the torsion $\tau^0_\rho(\Sigma, h_{sym})$ equals $1/2^g-1$.

Given this proposition, we can now finish the proof of Theorem 3.2.

**Proof of Theorem 3.2.** Let $G = SL_2(\mathbb{R})$. We will apply two situations to Theorem 5.1. Here, the first one is

$$C^* := C^*_f(\Sigma; g), \quad \overline{C}^* := C^*_f(W_1; g) \oplus C^*_f(W_2; g), \quad C^* := C^*_f(M; g).$$

Here, let $c, \overline{c}, g$ be the basis obtained from the orientations of the cellular structure of $M, W_1 \sqcup W_2, \Sigma$, respectively. Then, by the proof of Lemma 5.3, the acyclic complex $\mathcal{H}_*$ in (14) is equivalent to the isomorphism $i_1^* \oplus i_2^* : H^1_j(W_1; g) \oplus H^1_j(W_2; g) \to H^1_j(\Sigma; g)$. Let $h$ be $\emptyset, \overline{h}$ be $h_{sym}$, $h_i \in H^1_j(W_i; g)$ be bases which gives the orientation of $R^{sd}(W_i)$, and let $h$ be $h_1 \cup h_2$. Then, by definition of $\varepsilon_f$, we have

$$\varepsilon_f = \text{sign}(\det(i_1^* \oplus i_2^*)) = \text{sign}(\mathcal{T}(\mathcal{H}_*, h \cup h \cup \overline{h})) \in \{\pm 1\}.$$ 

On the other hand, another situation is given by the ordinary cellular complexes of the forms

$$C^* := C^*(\Sigma; \mathbb{R}), \quad \overline{C}^* := C^*(W_1; \mathbb{R}) \oplus C^*(W_2; \mathbb{R}), \quad C^* := C^*(M; \mathbb{R}).$$

Here, let 1-dimensional parts of $h^\mathbb{R}, \overline{h}^\mathbb{R}$ be the dual bases represented by the curves $a_1, b_1, \ldots, a_g, b_g$ in Figure 1. Then, since $H^*(M; \mathbb{R}) \cong H^*(S^3; \mathbb{R})$, we can easily check $\mathcal{T}(\mathcal{H}_*, h^\mathbb{R} \cup \overline{h}^\mathbb{R} \cup \overline{h}^\mathbb{R})$ is equal to 1. Furthermore, we give some examples of the number $N(X)$ in (3):

$$N(D) = N(S^1) = 1, \quad N(\Sigma) = N(\Sigma_0) = 0, \quad N(W_i) = N(M) = g \in \mathbb{Z}/2,$$

where the cellular complexes of $\Sigma, \Sigma_0, W_1, M$ are canonically obtained from the Heegaard decomposition of $M$.

Then, considering the ratio of the applications from the two situations to Theorem 5.1 we have

$$(-1)^g \cdot \tau^0_\rho(W_1, h_1)\tau^0_\rho(W_2, h_2) = \tau^0_\rho(\Sigma, h_{sym})\tau^0_\rho(M)\det(i_1^* \oplus i_2^*) \in \mathbb{R}^\times. \quad (15)$$

Note $\tau^0_\rho(\Sigma, h_{sym}) = 1/2^g-1 > 0$ from Proposition 5.4. Therefore, if $\text{sign}(\tau^0_\rho(W_1, h_1)) = \text{sign}(\tau^0_\rho(W_2, h_2))$ holds, the signs of (15) lead to the conclusion (7).

Finally, it suffices to show $\text{sign}(\tau^0_\rho(W_1, h_1)) = \text{sign}(\tau^0_\rho(W_2, h_2))$. Note that the function $\tau^0_\rho(W_i, h_i)$ is a continuous function on the connected space $R^{sd}(W_i)$ by Lemma 5.5 below. By duality of handle-attaching of $M$, there are $f_1 \in R^{sd}(W_1)$ and $f_2 \in R^{sd}(W_2)$ such that $\tau^0_\rho(W_1, h_1) = \tau^0_\rho(W_2, h_2)$, which implies the desired $\text{sign}(\tau^0_\rho(W_1, h_1)) = \text{sign}(\tau^0_\rho(W_2, h_2))$ by connectivity.

**Lemma 5.5.** $\text{Hom}(\pi_1(W_i), G)_{ad}$ is connected.

**Proof.** Let $C \subset \text{Hom}(\pi_1(W_i), G) = G^g$ be the complement of $\text{Hom}(\pi_1(W_i), G)^{ad}$. For the proof, it is enough to show that $C$ is of codimension $> 1$ over $\mathbb{R}$.

For this, recall the classification theorem of algebraic subgroups $K$ of $SL_2(\mathbb{R})$ with $\text{dim}(K) < 3$. More precisely,

- If $\text{dim}(K) = 2$, $K$ is isomorphic to either $\mathbb{R} \times \mathbb{R}^\times$ or $\mathbb{R} \times \mathbb{R}^\times_0$. 


• If \( \dim(K) = 1 \), \( K \) is either abelian or isomorphic to \( \mathbb{R} \times \{ \pm 1 \} \).

• If \( \dim(K) = 0 \), \( K \) is a cyclic group.

The conjugacy action of \( G \) on \( K \) has the stabilizer subgroup whose dimension is more than zero. Therefore, if \( f \in C \), the orbits of \( f \) in \( \text{Hom}(\pi_1(W_i), G) \) are of dimension less than 3. Notice that the quotient \( C/G \) by conjugacy action is a union of real varieties dimension less than \( 3g - 3 \). Hence, the dimension of \( C \) is \( 3g - 1 \) at most, as required.

5.3 Proof of Proposition 5.4

For the proof, we often use the theorem below. To describe this, for a \( PSL_2(\mathbb{R}) \)-representation \( \phi : \pi_1(\Sigma) \to PSL_2(\mathbb{R}) \), consider the associated \( P^1\mathbb{R} \)-bundle over \( \Sigma \), and let \( e(\phi) \in H^2(\Sigma; \mathbb{Z}) \cong \mathbb{Z} \) be the Euler class. Furthermore, let \( p : SL_2(\mathbb{R}) \to PSL_2(\mathbb{R}) \) be the projection. Then, for an \( SL_2 \)-representation \( f : \pi_1(\Sigma) \to SL_2(\mathbb{R}) \), the Euler class \( e(p \circ f) \) is known to be even (see (16) below).

**Theorem 5.6 (\([G2] \) Theorems A, B, and D).** The connected components of \( \text{Hom}(\pi_1(\Sigma), PSL_2(\mathbb{R}))/PSL_2(\mathbb{R}) \) is in 1:1-correspondence with \( \{ m \in \mathbb{Z} | 2g - 2 \geq |m| \} \) by the map \( \phi \mapsto e(\phi) \).

Moreover, \( |e(\phi)| = 2g - 2 \) if and only if \( \phi : \pi_1(\Sigma) \to PSL_2(\mathbb{R}) \) is a discrete and faithful representation.

Furthermore, we will explain a method of computing the Euler classes \( e(\phi) \), for \( \phi : \pi_1(\Sigma) \to PSL_2(\mathbb{R}) \). Let \( \tilde{G} \to PSL_2(\mathbb{R}) \) be the universal covering associated with \( \pi_1(PSL_2(\mathbb{R})) \cong \mathbb{Z} \), which is a central extension of fiber \( \mathbb{Z} = \{ z^m \}_{m \in \mathbb{Z}} \). Choose a set-theoretical lift \( \tilde{\phi}(a_i) \in \tilde{G} \) of \( \phi(a_i) \). Then, Milnor [Mi2] p. 218–220] showed the equality

\[
[\tilde{\phi}(a_1), \tilde{\phi}(b_1)] [\tilde{\phi}(a_2), \tilde{\phi}(b_2)] \cdots [\tilde{\phi}(a_g), \tilde{\phi}(b_g)] = z^{e(\phi)}. \tag{16}
\]

In particular, the left hand side is independent of the choice of the lifts.

**Proof of Proposition 5.4** The proof will be divided into four steps.

**Step 1** We first consider the cases \( g = 2, 3 \). Then, given even \( N \) with \( |N| \leq 2g - 2 \), we can construct concretely \( \phi_N : \pi_1(\Sigma) \to SL_2(\mathbb{R}) \) with \( e(p \circ \phi_N) = N \). For such \( \phi_N \), using Proposition A.2, we can verify that \( \tau^0_{\phi_N}(\Sigma, h_{\text{sym}}) = 1/2^{g-1} \) by the help of the computer, although the program is a bit intricate. Thanks to Remark 5.3 for any \( \rho : \pi_1(\Sigma) \to SL_2(\mathbb{R}) \), we directly have \( \tau^0_\rho(\Sigma, h_{\text{sym}}) = 1/2^{g-1} \) as required.

**Step 2** Recall the notation \( D \subset \Sigma, \Sigma' \), and \( \Sigma_0, \Sigma'_0 \) in 5.1. Let \( T \) be the torus with two circle boundaries such that \( \Sigma'_0 = T \cup S^1 \Sigma_0 \); see Figure 1. Let \( h_{S^1} \cup h_{S^1} \) be a basis of \( H^*(\partial T; \mathbb{R}) \otimes g \cong H^*(S^1; \mathbb{R}) \otimes g^2 \) as dual bases of a dual of the orientation class \([S^1]\). When \( \rho_T : \pi_1(T) \to SL_2(\mathbb{R}) \) is trivial, we will show \( \tau^0_{\rho_T}(T, h_{\text{sym}}|_{T} \cup h_{S^1} \cup h_{S^1}) = 1/2 \).

Consider \( \Sigma = \Sigma_0 \cup S^1 D \). By the Mayer-Vietoris argument and Theorem 5.1 we notice

\[
\begin{align*}
\tau^0_f(D, h_D)\tau^0_f(\Sigma, h_{\text{sym}}) & = \tau^0_f(\Sigma_0, h_{\text{sym}} \cup h_{S^1})\tau^0_f(S^1, h_{S^1})\mathcal{T}(\mathcal{H}_*, h \cup h \cup \overline{h}), \\
\tau^0_f(D', h_D)\tau^0_f(\Sigma', h_{\text{sym}}') & = \tau^0_f(\Sigma_0', h_{\text{sym}}' \cup h_{S^1})\tau^0_f(S^1, h_{S^1})\mathcal{T}(\mathcal{H}_*, h \cup h \cup \overline{h}).
\end{align*}
\]
We can easily see $\tau_0^0(D,h_D) = \tau_0^0(S^1,h_{S^1}) = -1$ by definition. Furthermore, we can verify that $\mathcal{T}(\mathcal{H}_s)$ and $\mathcal{T}(\mathcal{H}_s')$ are equal to 1 by the choice of the bases $h_{sym}, h_{S^1}, h_D$. Therefore, the two equalities are rewritten in

$$\tau_0^0(\Sigma, h_{sym}) = \tau_0^0(\Sigma_0, h_{sym} \cup h_{S^1}), \quad \tau_0^0(\Sigma', h_{sym}') = \tau_0^0(\Sigma_0', h_{sym}' \cup h_{S^1}).$$

By Step 1, these terms are $1/2$ and $1/4$, respectively, if $g = 2$. Let $\rho_T$ be the restriction $f_T$. Therefore, the Mayer-Vietoris sequence with Theorem 5.1 gives rise to

$$\tau_0^0(T, h_T)/2 = \tau_0^0(\Sigma_0, h_{sym} \cup h_{S^1})\tau_0^0(\Sigma_0', h_{sym}' \cup h_{S^1})$$

$$= -\tau_0^0(\Sigma_0', h_{sym}' \cup h_{S^1})\tau_0^0(S^1, h_{S^1}) = \tau_0^0(\Sigma_0', h_{sym}' \cup h_{S^1}) = 1/4.$$

Hence, $\tau_{\rho_T}^0(T, h_{sym}|T \cup h_{S^1} \cup h_{S^1}) = 1/2$ as required.

**Step 3** We suppose that Proposition 5.4 is true if $k = g$. First, consider the case of $|e(p \circ f')| \leq 2k - 4$, where $f' : \pi_1(\Sigma') \to SL_2(\mathbb{R})$. By Theorem 5.6, there is $f_0 : \pi_1(\Sigma') \to SL_2(\mathbb{R})$ such that $f_0$ and $f'$ lay in the same connected components of $\text{Hom}(\pi_1(\Sigma'), SL_2(\mathbb{R}))$ and the restriction $f_0|\pi_1(T)$ is constant. Since $\tau_0^0(T, h_{sym}|T \cup h_{S^1} \cup h_{S^1}) = 1/2$ by Step 2, a similar Mayer-Vietoris argument shows that

$$\tau_{f_0}^0(\Sigma', h_{sym}) = \tau_{f_0}^0(\Sigma, h_{sym})\tau_{f_0}^0(T, h_{sym}|T) = \frac{1}{2k} \frac{1}{2} = \frac{1}{2k+1}.$$

Hence, by induction on $g$, we complete the proof with $|e(p \circ f')| \leq 2g - 4$.

**Step 4** It reminds to consider the case $g = k + 1$ and $|e(p \circ f)| = 2g - 2$. By Theorem 5.6 again, $f : \pi_1(\Sigma') \to SL_2(\mathbb{R})$ is a faithful discrete representation. Let $h_{S^1}' \cup h_{S^1}'$ be a basis of $H^*(\partial T; g) \cong H^*(S^1; \mathbb{R})^2$ as 2-copies of a dual of the orientation class $[S^1]$. By a similar way to Step 2, we can show $\tau_{\rho_T}^0(T, h_{sym}|T \cup h_{S^1} \cup h_{S^1}') = 1/2$ as well. Hence, as the same discussion as Step 3, we can show Proposition 5.4 with $|e(p \circ f)| = 2g - 2$ by induction to $g$.

A Computation of the symplectic structures on flat moduli spaces

We give an algebraic description of the non-degenerate alternating 2-form in [H]. Although such a discussion can be seen in [GTI] §3.10, it contains minor errors; we reformulate such a description in a simplified way.

Take the standard presentation $\pi_1(\Sigma) = \langle a_1, b_1, \ldots, a_g, b_g | r \rangle$, where $r = [a_1, b_1] \cdots [a_g, b_g]$. Since $\Sigma$ is an Eilenberg-MacLane space, the cellular complex of the universal covering space is known to be denoted by a complex of group homology:

$$C_* : 0 \to \mathbb{Z}[\pi_1(\Sigma)] \xrightarrow{\partial_2} \mathbb{Z}[\pi_1(\Sigma)]^{|g|} \xrightarrow{\partial_1} \mathbb{Z}[\pi_1(\Sigma)] \xrightarrow{\epsilon} \mathbb{Z} \to 0 \quad (\text{exact}).$$

Let us fix the canonical basis of $C_2$ and $C_1$ by $R$ and $x_1, y_1, \ldots, x_g, y_g$, respectively. Then, the boundary maps are known to be

$$\partial_1(x_i) = 1 - a_i, \quad \partial_1(y_i) = 1 - b_i \quad \text{and} \quad \partial_1(aR) = a \sum_{i=1}^g \frac{\partial_r}{\partial a_i} x_i + \frac{\partial_r}{\partial b_i} y_i$$

for $a \in \mathbb{Z}[\pi_1(\Sigma)]$. Here, $\frac{\partial_r}{\partial a_i}$ is the Fox derivative. Then, given a left $\mathbb{Z}[\pi_1(\Sigma)]$-module $M$, any 1-cocycle in local coefficient $M$ can be regarded as a left $\mathbb{Z}[\pi_1(\Sigma)]$-homomorphism $f : \mathbb{Z}[\pi_1(\Sigma)]^{|g|} \to M$ satisfying $\sum_{i=1}^g \frac{\partial_r}{\partial a_i} f(x_i) + \frac{\partial_r}{\partial b_i} f(y_i) = 0$. 

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Next, we will give a description of the cup product $H^1 \otimes H^1 \to H^2$. Let $F$ be the free group $\langle a_1, b_1, \ldots, a_g, b_g \rangle$. Consider the function

$$\kappa : F \times F \to \mathbb{Z}[\pi_1(\Sigma)]^{2g} \otimes \mathbb{Z}[\pi_1(\Sigma)]^{2g}; \quad (u, v) \mapsto \alpha(u) \otimes u\alpha(v).$$

Here $\alpha(w)$ is define by $\sum_{i=1}^{g} \frac{\partial w}{\partial x_i} x_i + \frac{\partial w}{\partial y_i} y_i$. Then, according to [Tro] Lemma in §2.3, there is uniquely a map $\Upsilon : F \to \mathbb{Z}[\pi_1(\Sigma)]^{2g} \otimes \mathbb{Z}[\pi_1(\Sigma)]^{2g}$ satisfying

$$\Upsilon(uv) = \Upsilon(u) + u\Upsilon(v) + \kappa(u, v), \quad \Upsilon(1) = \Upsilon(a_i) = \Upsilon(b_i) = 0, \quad \text{for any } u, v \in F.$$

**Proposition A.1** (A special case of [Tro] §2.4). Let $\rho : \mathbb{Z}[\pi_1(\Sigma)] \to \text{End}(M)$ be a homomorphism, and regard $M$ as a left $\mathbb{Z}[\pi_1(\Sigma)]$-module. For any two 1-cocycles $f, f' : \mathbb{Z}[\pi_1(\Sigma)]^{2g} \to M$, the cup product $f \smile f'$ as a 2-cocycle is represented by a map $\mathbb{Z}[\pi_1(\Sigma)] \to M \otimes M$ give by

$$f \smile f'(a \cdot R) = (f \otimes f')( (a \otimes a) \cdot \Upsilon(r)), \quad \text{for } a \in \mathbb{Z}[\pi_1(\Sigma)].$$

As a special case, we consider a bilinear map $\psi : M \otimes M \to A$ which is diagonally invariant with respect to $\mathbb{Z}[\pi_1(\Sigma)]$. Then,

**Proposition A.2.** Let $M$ be as above and $f, f'$ be 1-cocycles. Suppose $\psi(a, b) = \psi(\rho(g)a, \rho(g)b)$ for any $a, b \in M$ and $g \in \pi_1(\Sigma)$. Then, the composite of $\psi$ and the cup product

$$H^1(\pi_1(\Sigma); M)^{\otimes 2} \longrightarrow H^2(\pi_1(\Sigma); M^{\otimes 2}) \cong M^{\otimes 2} \longrightarrow A$$

is represented by the map $\mathbb{Z}[\pi_1(\Sigma)]^{2g} \otimes \mathbb{Z}[\pi_1(\Sigma)]^{2g} \to A$ which sends $(\sum_{i=1}^{g} k_i x_i + \ell_i y_i) \otimes (\sum_{j=1}^{g} k'_j x_j + \ell'_j y_j)$ to

$$\sum_{i=1}^{g} \psi(f(k_i x_i), \rho(a_i + b_i^{-1} a_i^{-1} b_i^{-1} f'(\ell'_i y_i))) - \psi(f(\ell_i y_i), \rho(b_i a_i^{-1}) f'(k'_i x_i)))$$

$$+ \psi(f(k_i x_i), \rho(1 - a_i b_i^{-1} a_i^{-1} f'(k'_i x_i))) + \psi(f(\ell_i y_i), \rho(1 - b_i a_i^{-1} b_i^{-1} f'(\ell'_i y_i)))$$

$$+ \sum_{m:1 \leq m < i} \psi(\rho(I_m - I_m a_m b_m a_m^{-1}) f(k_m x_m) + \rho(I_m a_m - I_m + 1) f(\ell_i y_m),$$

$$\rho(I_i - I_i a_i b_i^{-1} f'(k'_i x_i) + \rho(I_i a_i - I_i + 1) f'(\ell'_i y_i)),$$

where $k_i, k'_i, \ell_i, \ell'_i \in \mathbb{Z}[\pi_1(\Sigma)]$ and $I_i = [a_i, b_i] \cdots [a_{i-1}, b_{i-1}].$

This proof can be done by a direct computation of $\Upsilon$, thanks to Proposition A.1.

**B The work of Johnson** [John]

In this appendix, we explain the work of Johnson [John] in details (see Theorem B.2), which gives a way of computing the $SU(2)$-Casson invariant under a certain assumption. Let $G$ be $SU(2)$ hereafter. We suppose knowledge in §3.2 and 3.1.

First, we briefly review the $SU(2)$-Casson invariant of an integral homology 3-sphere $M$. Let $\text{Hom}(\pi_1(Z), G)^{irr}$ be the open subset consisting of irreducible representations $\pi_1(Z) \to G$. Denote the conjugacy quotient $\text{Hom}(\pi_1(Z), G)^{irr}/G$ by $R^{irr}(Z)$. It is well-known (see, e.g., [AM Sav]) that if $Z$ is $\Sigma$ with $g \geq 2$, then the conjugacy action of $PSU(2)$ on $\text{Hom}(\pi_1(\Sigma), G)^{irr}$
is proper and free, and the quotient $R^{\text{irr}}$ is an oriented manifold of dimension $6g - 6$, and the tangent space at $\rho \in R^{\text{irr}}$ is identified with the first cohomology $H^1_\rho(\Sigma; \mathfrak{g})$ with local coefficients by $\rho$. Furthermore, $R^{\text{irr}}(W_k)$ is known to be an oriented manifold of dimension $3g - 3$. To summarize, the restriction of $\mathfrak{h}$ is written in

$$
\begin{array}{c}
R^{\text{irr}}(\Sigma) \\
\downarrow i_1 \\
R^{\text{irr}}(W_1) \\
\downarrow j_1 \\
R^{\text{irr}}(W_2) \\
\downarrow i_2 \\
R^{\text{irr}}(W_1) \cap R^{\text{irr}}(W_2) \subset R^{\text{irr}}(M),
\end{array}
$$

of $C^\infty$-embeddings. The intersection $R^{\text{irr}}(W_1) \cap R^{\text{irr}}(W_2)$ are compact, but not always transversal. If not so, by Transversality theorem, we can choose an isotopy $h : R^{\text{irr}}(\Sigma) \to R^{\text{irr}}(\Sigma)$ such that $h$ is supported in a compact neighborhood of $R^{\text{irr}}(W_1) \cap R^{\text{irr}}(W_2)$ and $h(R^{\text{irr}}(W_1))$ meets $R^{\text{irr}}(W_2)$ transversally in $\text{supp}(h)$.

Then, the $SU(2)$-Casson invariant, $\lambda_{SU(2)}(M)$, is defined to be $(-1)^g \sum \varepsilon_f$. Here, the sum runs over $h(R^{\text{irr}}(W_1)) \cap R^{\text{irr}}(W_2)$, and the number $\varepsilon_f$ equals $\pm 1$ depending on whether the orientations of the spaces $T_f h(R^{\text{irr}}(W_1)) \oplus T_f(R^{\text{irr}}(W_2))$ and $T_f(R^{\text{irr}}(\Sigma))$ agree. It is well known that $\lambda_{SU(2)}(M)$ is a topological invariant of $M$.

While it is not so easy to compute $\varepsilon_f$, Johnson [John] suggested a procedure of computing $\varepsilon_f$ from Reidemeister torsions. For this, we shall mention a proposition as in Lemma 3.3.

**Proposition B.1** ([Sav Theorem 16.4]). Take $f \in R^{\text{irr}}(W_1) \cap R^{\text{irr}}(W_2)$. Then, the intersection of $R^{\text{irr}}(W_1) \cap R^{\text{irr}}(W_2)$ at $f$ is transversal, if and only if $C^*_f(M; \mathfrak{g})$ is acyclic, i.e., $H^*_f(M; \mathfrak{g}) = 0$.

Thus, under the transversality, by the definition, in order to compute the invariants, it is enough to compute the sign $\varepsilon_f$ with respect to $f \in R^{\text{irr}}(M)$, since the isotopy $h$ may be the identity. In the note [John], Johnson gave the following theorem:

**Theorem B.2** ([John]). Suppose $H_*(M; \mathbb{Z}) \cong H_*(S^3; \mathbb{Z})$ and that, for any $f \in R^{\text{irr}}(W_1) \cap R^{\text{irr}}(W_2)$, the intersection of $\text{Im}(i_1^*)$ and $\text{Im}(i_2^*)$ at $f$ is transversal.

Then, the $SU(2)$-Casson invariant is formulated as

$$
\lambda_{SU(2)}(M) = \sum_{f \in R^{\text{irr}}(W_1) \cap R^{\text{irr}}(W_2)} \text{sign}(\tau^0_f(M)). \tag{17}
$$

This theorem might be a folklore; however, since the note [John] is unpublished, we now give a proof of this theorem:

**Proof.** The proof is almost same as the proof of Theorem 3.2. We suppose that the reader read [17.2]. Let $G$ be $SU(2)$, and $\mathfrak{g}$ be $\mathfrak{su}(2)$.

By (5), we have a symplectic structure on $H^1_\rho(\Sigma; \mathfrak{g}) \cong \mathbb{R}^{6g - 6}$ for any $\rho \in R^{\text{irr}}(\Sigma)$. With a choice of a symplectic basis $\mathfrak{h}_{\text{sym}} \subset H^1_\rho(\Sigma; \mathfrak{g})$, we can also define the refined torsion $\tau^0_\rho(\Sigma, \mathfrak{h}_{\text{sym}}) \in \mathbb{R}^\times$. By (5) and symplecticity of $\mathfrak{h}_{\text{sym}}$, this torsion does not depend on the choice of $\mathfrak{h}_{\text{sym}}$.

In a similar way to (15), we can obtain

$$
(-1)^g \cdot \tau^0_f(W_1, \mathfrak{h}_1) \tau^0_f(W_2, \mathfrak{h}_2) = \tau^0_f(\Sigma, \mathfrak{h}_{\text{sym}}) \tau^0_f(M) \text{det}(i_1^* \oplus i_2^*) \in \mathbb{R}^\times. \tag{18}
$$
We can show \( \text{sign}(\tau^0_0(W_1, h_1)) = \text{sign}(\tau^0_0(W_2, h_2)) \) as in Lemma 5.2. Since these equalities can be proven by the same way as in \([5.2]\) we omit the details.

By the construction, the function \( R^{irr}(\Sigma) \rightarrow \mathbb{R}^\times \) which takes \( \rho \) to \( \tau^0_0(\Sigma, h_{\text{sym}}) \) is continuous. Hence, for the proof, it is sufficient to show \( \tau^0_0(\Sigma, h_{\text{sym}}) > 0 \) in the case \( G = SU(2) \). For this, note a well known fact that the open set \( R^{irr}(\Sigma) \) is connected; see, e.g., \([GX]\). Therefore, we may show \( \tau^0_0(\Sigma, h_{\text{sym}}) > 0 \) for appropriate \( f_0 \in R^{irr}(\Sigma) \). Moreover, by a similar discussion in \([5.3]\) we may discuss only the case \( g = 2 \). For this, let us consider \( f_0 \) defined by

\[
 f_0(a_1) = \begin{pmatrix} 2\sqrt{-1+2-\sqrt{10}} & -2+\sqrt{-1(2+\sqrt{10})} \\ 2+\sqrt{-1(2+\sqrt{10})} & -2\sqrt{-1+2-\sqrt{10}} \end{pmatrix}, \quad f_0(b_1) = \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix},
\]

\[
 f_0(a_2) = \begin{pmatrix} 1-\sqrt{-1} & -1+\sqrt{-1} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \quad f_0(b_2) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}.
\]

By Proposition A.2 and the help of the computer, we can verify \( \tau^0_0(\Sigma, h_{\text{sym}}) > 0 \) as desired. \( \square \)

**Remark B.3.** In the \( SU(2) \)-case, we can show \( \tau^0_0(\Sigma, h_{\text{sym}}) = 1 \) for any \( g > 1 \) and any irreducible representation \( f : \pi_1(\Sigma) \rightarrow SU(2) \). The proof can be done in the same way as Proposition 5.4.

**References**

[Ati] M. F. Atiyah, *New invariants of 3- and 4-dimensional manifolds*, Proc. Sympos. Pure Math., vol. 48, Amer. Math. Soc., Providence, R.I., 1988, pp. 285–299.

[AM] S. Akbulut, J. D. McCarthy, *Casson’s invariant for oriented homology 3-spheres, an exposition*, Mathematical Notes 36, Princeton University Press, Princeton (1990).

[BH] H. Boden, C. Herald, *The SU(3) Casson invariant for integral homology 3-spheres*, J. Differential Geom. 50 (1998), 147–206.

[BN] S. Boyer, A. Nicas, *Varieties of group representations and Casson’s invariant for rational homology 3-spheres*, Trans. Amer. Math. Soc. 322 (1990), 507–522.

[Cas] A. Casson, Lectures at MSRI, 1985.

[CHK] A. Cavicchioli, F. Hegenbarth, A. Kim, *On cyclic branched coverings of torus knots*, Journal of Geometry, 64 (1999), 55–66.

[Cur1] C. L. Curtis, *An intersection theory count of the SL_2(\mathbb{C})-representations of the fundamental group of a 3-manifold*, Topology 40 (2001), 773–787.

[Cur2] C. L. Curtis, *Generalized Casson invariants for SO(3), U(2), Spin(4), and SO(4)*, Trans. Amer. Math. Soc. 343(1): 49–86, 1994.

[Dup] J. L. Dupont. *The dilogarithm as a characteristic class for flat bundles*. In Proceedings of the Northwestern conference on cohomology of groups (Evanston, Ill., 1985), volume 44, pages 137–164, 1987.

[G1] W. M. Goldman, *The symplectic nature of fundamental groups of surfaces*, Adv. Math. 54 (1984) 200–225.

[G2] W. M. Goldman, *Topological components of spaces of representations*, Invent. Math. 93 (1988), no. 3, 557–607.

[GX] W. M. Goldman, E. Z. Xia, *Ergodicity of mapping class group actions on SU(2)-character varieties*. In Geometry, rigidity, and group actions, Chicago Lectures in Math., pages 591–608.

[John] D. Johnson, *A geometric form of Casson’s invariant and its connection to Reidemeister torsion*, unpublished lecture notes.
[Ki] T. Kitano, *Reidemeister torsion of Seifert fibered spaces for SL(2; C)-representations*, Tokyo J. Math. 17 (1994), 59–75.

[KY] T. Kitano, Y. Yamaguchi, *SL(2; R)-representations of a Brieskorn homology 3-sphere*, ArXiv e-prints, February 2016.

[Ko] Y. Koda, *Spines, Heegaard splittings and the Reidemeister-Turaev torsion of Euler structure*. Tokyo J. Math. 30 (2007), 417–439.

[Lab] F. Labourie, *Lectures on representations of surface groups*, European Mathematical Society (EMS), Zürich, (2013), Zürich Lectures in Advanced Mathematics

[Lyn] R. Lyndon, *Cohomology theory of groups with a single defining relation*, Ann. of Math. 52 (1950), 650–665.

[Mil1] J. Milnor, *Whitehead torsion*, Bull. Amer. Math. Soc. 72 (1966), 358–426.

[Mil2] J. Milnor, *On the existence of a connection of curvature zero*, Comm. Math. Helv. 21, 215–223 (1958)

[Rei] K. Reidemeister, *Zur dreidimensionalen Topologie*, Abh. Math. Sem. Univ. Hamburg 9 (1933), 189–194.

[Sav] N. Saveliev, *Invariants for homology 3-spheres*, Encyclopaedia of Math. Sci. 140, SpringerVerlag, Berlin Heidelberg (2002).

[SW] D. Stanford, E. Witten, *JT Gravity and the Ensembles of Random Matrix Theory*, preprint

[Tro] H. F. Trotter, *Homology of group systems with applications to knot theory*, Ann. of Math. 76 (1962), 464–498.

[Tur] V. Turaev, *Introduction to combinatorial torsions*, Lectures in Mathematics, ETH Zürich, Birkhäuser, Basel, 2001. MR 2001m:57042 Zbl 0970.57001

[Wit] E. Witten, *On quantum gauge theories in two dimensions*, Commun. Math. Phys. 141 (1991) 153–209.