An Inverse Problem of the Calculus of Variations on Arbitrary Time Scales

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Abstract

We consider an inverse extremal problem for variational functionals on arbi-
trary time scales. Using the Euler–Lagrange equation and the strengthened Legen-
dre condition, we derive a general form for a variational functional that attains a
local minimum at a given point of the vector space.

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1 Introduction

We study an inverse problem associated with the following fundamental problem of the calculus of variations: to minimize

\[ L(y) = \int_a^b L(t, y(\sigma(t)), y(\Delta(t))) \Delta t \]  

subject to the boundary conditions \( y(a) = y_0(a), y(b) = y_0(b) \), on a given time scale \( \mathbb{T} \). More precisely, we describe a general form of a variational functional \( (1.1) \) having an extremum at a given function \( y_0 \) under the Euler–Lagrange and strengthened Legendre conditions on time scales \([1]\). Throughout the paper we assume the reader to be familiar with the basic definitions and results from the time scale theory \([3, 4, 9]\). For a review on general approaches to the calculus of variations on time scales see \([1, 5–8, 12, 13, 17]\). For analogous results in \( \mathbb{T} = \mathbb{R} \) see \([15, 16]\). The results here obtained are new even for simple (but important) time scales like \( \mathbb{T} = \mathbb{Z} \) or \( \mathbb{T} = q^\mathbb{N}_0, q > 1 \).

The paper is organized as follows. In Section 2 we collect some necessary definitions and results of the delta calculus on time scales, which are used throughout the text. The main results are presented in Section 3. We find a general form of the variational functional \( (1.1) \) that solves the inverse extremal problem (Theorem 3.2). In order to illustrate our results, we present the form of the Lagrangian \( L \) on an isolated time scale (Corollary 3.4). We end by presenting the form of the Lagrangian \( L \) in the periodic time scale \( \mathbb{T} = h\mathbb{Z}, h > 0 \) (Example 3.6) and in the \( q \)-scale \( \mathbb{T} = q^\mathbb{N}_0, q > 1 \) (Example 3.7).

2 Preliminaries

In this section we introduce basic definitions and theorems that will be useful in the sequel. A time scale \( \mathbb{T} \) is an arbitrary nonempty closed subset of \( \mathbb{R} \). Let \( a, b \in \mathbb{T} \) with \( a < b \). We define the interval \([a, b]_\mathbb{T} := [a, b] \cap \mathbb{T} = \{t \in \mathbb{T} : a \leq t \leq b\}\).

Definition 2.1 (See \([3]\)). The forward jump operator \( \sigma : \mathbb{T} \to \mathbb{T} \) is defined by \( \sigma(t) := \inf\{s \in \mathbb{T} : s > t\} \) for \( t \neq \sup \mathbb{T} \) and \( \sigma(\sup \mathbb{T}) := \sup \mathbb{T} \) if \( \sup \mathbb{T} < +\infty \). The backward jump operator \( \rho : \mathbb{T} \to \mathbb{T} \) is given by \( \rho(t) := \sup\{s \in \mathbb{T} : s < t\} \) for \( t \neq \inf \mathbb{T} \) and \( \rho(\inf \mathbb{T}) = \inf \mathbb{T} \) if \( \inf \mathbb{T} > -\infty \). The graininess function \( \mu : \mathbb{T} \to [0, \infty) \) is defined by \( \mu(t) := \sigma(t) - t \).

A point \( t \in \mathbb{T} \) is called right-dense, right-scattered, left-dense or left-scattered if \( \sigma(t) = t, \sigma(t) > t, \rho(t) = t, \rho(t) < t \), respectively. We say that \( t \) is isolated if \( \rho(t) < t < \sigma(t) \), that \( t \) is dense if \( \rho(t) = t = \sigma(t) \).

Example 2.2. The two classical time scales are \( \mathbb{R} \) and \( \mathbb{Z} \), representing the continuous and the purely discrete time, respectively. The other standard examples are \( h\mathbb{Z}, h > 0, \)
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and \( q^{\mathbb{N}_0}, q > 1 \). It follows from Definition 2.1 that if \( T = \mathbb{R} \), then \( \sigma(t) = t \) and \( \mu(t) = 0 \) for all \( t \in T \); if \( T = h\mathbb{Z} \), then \( \sigma(t) = t + h \) and \( \mu(t) = h \) for all \( t \in T \); if \( T = q^{\mathbb{N}_0} \), then \( \sigma(t) = qt \) and \( \mu(t) = t(q - 1) \) for all \( t \in T \).

**Definition 2.3** (See [14]). A time scale \( T \) is said to be an isolated time scale provided given any \( t \in T \), there is a \( \delta > 0 \) such that \((t - \delta, t + \delta) \cap T = \{t\} \).

**Remark 2.4.** If the graininess function is bounded from below by a strictly positive number, then the time scale is isolated [2]. Therefore, \( h\mathbb{Z}, h > 0 \), and \( q^{\mathbb{N}_0}, q > 1 \), are examples of isolated time scales. Note that the converse is not true. For example, \( T = \log(\mathbb{N}) \) is an isolated time scale but its graininess function is not bounded from below by a strictly positive number.

To simplify the notation, one usually uses \( f^{\sigma}(t) := f(\sigma(t)) \). The delta derivative is defined for points from the set

\[ T^\kappa := \begin{cases} T \setminus \{\sup T\} & \text{if } \rho(\sup T) < \sup T < \infty, \\ T & \text{otherwise.} \end{cases} \]

**Definition 2.5** (See [3]). A function \( f : T \to \mathbb{R} \) is \( \Delta \)-differentiable at \( t \in T^\kappa \) if there is a number \( f^{\Delta}(t) \) such that for all \( \varepsilon > 0 \) there exists a neighborhood \( O \) of \( t \) such that

\[ |f^{\sigma}(t) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s| \quad \text{for all } s \in O. \]

We call \( f^{\Delta}(t) \) the \( \Delta \)-derivative of \( f \) at \( t \).

**Example 2.6.** If \( T = h\mathbb{Z} \), then \( f : T \to \mathbb{R} \) is delta differentiable at \( t \in T \) if, and only if,

\[ f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} = \frac{f(t + h) - f(t)}{h} =: \Delta_h f(t). \]

In the particular case \( h = 1 \), \( f^{\Delta}(t) = \Delta f(t) \), where \( \Delta \) is the usual forward difference operator. If \( T = q^{\mathbb{N}_0} = \{q^k : q > 1, k \in \mathbb{N}_0\} \), then \( f^{\Delta}(t) = \frac{f(qt) - f(t)}{(q - 1)t} =: \Delta_q f(t) \), i.e., we get the usual Jackson derivative of quantum calculus [11].

**Theorem 2.7** (See [3]). Let \( f : T \to \mathbb{R} \) and \( t \in T^\kappa \). If \( f \) is delta differentiable at \( t \), then

\[ f^{\sigma}(t) = f(t) + \mu(t)f^{\Delta}(t). \]

**Definition 2.8** (See [3]). A function \( f : T \to \mathbb{R} \) is called rd-continuous provided it is continuous at right-dense points in \( T \) and its left-sided limits exists (finite) at all left-dense points in \( T \).
The set of all rd-continuous functions \( f : T \rightarrow \mathbb{R} \) is denoted by \( C_{rd} = C_{rd}(T) = C_{rd}(T, \mathbb{R}) \). The set of functions \( f : T \rightarrow \mathbb{R} \) that are \( \Delta \)-differentiable and whose derivative is rd-continuous is denoted by \( C_{1rd} = C_{1rd}(T) = C_{1rd}(T, \mathbb{R}) \).

A function \( F : T \rightarrow \mathbb{R} \) is called an antiderivative of \( f : T \rightarrow \mathbb{R} \) provided that \( F(\Delta t(t)) = f(t) \) for all \( t \in T \). Let \( T \) be a time scale and \( a, b \in T \). If \( F \) is an antiderivative of \( f \), then the Cauchy \( \Delta \)-integral is defined by

\[
\int_a^b f(t) \Delta t := F(b) - F(a).
\]

**Theorem 2.9** (See [3]). Every rd-continuous function has an antiderivative. In particular, if \( t_0 \in T \), then \( F \) defined by

\[
F(t) := \int_{t_0}^t f(\tau) \Delta \tau,
\]

\( t \in T \), is an antiderivative of \( f \).

**Example 2.10.** If \( T = h\mathbb{Z}, h > 0 \), and \( a, b \in T \) with \( a < b \), then

\[
\int_a^b f(t) \Delta t = \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} f(kh)h.
\]

If \( T = q\mathbb{N}_0, q > 1 \), then

\[
\int_a^b f(t) \Delta t = (q-1) \sum_{t \in [a,b) \cap T} tf(t).
\]

Let \( T \) be a given time scale with at least three points. Consider the following variational problem on the time scale \( T \):

\[
\mathcal{L}(y) = \int_a^b L \left( t, y^\sigma(t), y^\Delta(t) \right) \Delta t \longrightarrow \text{min}, \quad y(a) = \alpha, \quad y(b) = \beta, \quad (2.1)
\]

where \( a, b \in T \) with \( a < b \); \( \alpha, \beta \in \mathbb{R}^n \) with \( n \in \mathbb{N} \), and \( L : T \times \mathbb{R}^{2n} \rightarrow \mathbb{R} \).

**Definition 2.11.** We say that \( y \in C_{1rd}(T) \) is admissible for problem \( (2.1) \) if it satisfies the boundary conditions \( y(a) = \alpha \) and \( y(b) = \beta \).

**Definition 2.12.** An admissible function \( \hat{y} \) is called a local minimizer of problem \( (2.1) \) provided there exists \( \delta > 0 \) such that \( \mathcal{L}(\hat{y}) \leq \mathcal{L}(y) \) for all admissible \( y \) with \( ||y - \hat{y}||_{C_{rd}} < \delta \), where

\[
||f||_{C_{rd}} = \sup_{t \in [a,b]} ||f^\sigma(t)|| + \sup_{t \in [a,b]} ||f^\Delta(t)||
\]

with \( || \cdot || \) a norm in \( \mathbb{R}^n \).
In what follows the Lagrangian \( L \) is understood as a function \( (t, x, v) \to L(t, x, v) \) and by \( L_x \) and \( L_v \) we denote the partial derivatives of \( L \) with respect to \( x \) and \( v \), respectively. Similar notation is used for second order partial derivatives.

**Theorem 2.13 (The Euler–Lagrange equation [10])**. Assume that \( L(t, \cdot, \cdot) \) is differentiable in \((x, v)\) and \( L(t, \cdot, \cdot), L_x(t, \cdot, \cdot), L_v(t, \cdot, \cdot) \) are continuous at \((y^\sigma, y^\Delta)\), uniformly in \( t \) and rd-continuous in \( t \) for any admissible \( y \). If \( \hat{y}(t) \) is a local minimizer of the variational problem (2.1), then there exists a vector \( c \in \mathbb{R}^n \) such that the Euler–Lagrange equation  
\[
L_v(t, \hat{y}^\sigma(t), \hat{y}^\Delta(t)) = \int_a^t L_x(\tau, \hat{y}^\sigma(\tau), \hat{y}^\Delta(\tau)) \Delta \tau + c^T
\]  
holds for \( t \in [a, b]^\kappa_T \).

**Theorem 2.14 (The Legendre condition [1])**. If \( \hat{y} \) is a local minimizer of the variational problem (2.1), then  
\[
A(t) + \mu(t) \left\{ C(t) + C^T(t) + \mu(t) B(t) + (\mu(\sigma(t)))^\dagger A(\sigma(t)) \right\} \geq 0,
\]  
\( t \in [a, b]^\kappa_T \), where \( A(t) = L_{vv} \left( t, \hat{y}^\sigma(t), \hat{y}^\Delta(t) \right), B(t) = L_{xx} \left( t, \hat{y}^\sigma(t), \hat{y}^\Delta(t) \right), C(t) = L_{xv} \left( t, \hat{y}^\sigma(t), \hat{y}^\Delta(t) \right) \), and where \( \alpha^\dagger = \frac{1}{\alpha} \) if \( \alpha \in \mathbb{R} \setminus \{0\} \) and \( 0^\dagger = 0 \).

**Remark 2.15.** If (2.3) holds with the strict inequality \( > \), then it is called the strengthened Legendre condition.

**Definition 2.16 (See [3])**. We say that a function \( p : \mathbb{T} \to \mathbb{R} \) is regressive provided  
\[
1 + \mu(t)p(t) \neq 0
\]  
holds for all \( t \in \mathbb{T}^\kappa \). The set of all regressive and rd-continuous functions \( f : \mathbb{T} \to \mathbb{R} \) is denoted by \( \mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R}) \).

**Theorem 2.17 (See [4])**. Let \( p \in \mathcal{R}, f \in C_{rd}, t_0 \in \mathbb{T} \) and \( y_0 \in \mathbb{R} \). Then, the unique solution of the initial value problem  
\[
y^\Delta = p(t)y + f(t), \quad y(t_0) = y_0,
\]  
is given by  
\[
y(t) = e_p(t, t_0)y_0 + \int_{t_0}^t e_p(t, \sigma(\tau))f(\tau) \Delta \tau,
\]  
where \( e_p(\cdot, \cdot) \) denotes the exponential function on time scales.
Remark 2.18 (See [3]). An alternative form of the solution of the initial value problem
\( (2.4) \) is given by
\[
y(t) = e_p(t, t_0) \left[ y_0 + \int_{t_0}^{t} e_p(t_0, \sigma(\tau)) f(\tau) \Delta \tau \right].
\]

For more properties of the delta exponential function we refer the reader to [3, 4].

3 Main Results

The problem under our consideration is to find a general form of the variational functional
\[
L(y) = \int_{a}^{b} L \left( t, y^\sigma(t), y^\Delta(t) \right) \Delta t,
\]
\( L : [a, b]_T \times \mathbb{R}^2 \to \mathbb{R} \), subject to the boundary conditions \( y(a) = y(b) = 0 \), possessing a local minimum at zero, under the Euler–Lagrange and the strengthened Legendre conditions. We assume that \( L(t, \cdot, \cdot) \) is a \( C^2 \)-function with respect to \( (x, v) \) uniformly in \( t \), and \( L, L_x, L_v, L_{vv} \in C_{rd} \) for any admissible path \( y(\cdot) \). Observe that under our assumptions, by Taylor’s theorem, we may write \( L \), with the big \( O \) notation, in the form
\[
L(t, x, v) = P(t, x) + Q(t, x)v + \frac{1}{2}R(t, x, 0)v^2 + O(v^3),
\]
where
\[
\begin{align*}
P(t, x) &= L(t, x, 0), \\
Q(t, x) &= L_v(t, x, 0), \\
R(t, x, 0) &= L_{vv}(t, x, 0).
\end{align*}
\]
Let \( R(t, x, v) = R(t, x, 0) + O(v) \). Then, one can write \( (3.2) \) as
\[
L(t, x, v) = P(t, x) + Q(t, x)v + \frac{1}{2}R(t, x, v)v^2.
\]
Now the idea is to find general forms of \( P(t, y^\sigma(t)), Q(t, y^\sigma(t)) \) and \( R(t, y^\sigma(t), y^\Delta(t)) \) using the Euler–Lagrange and the strengthened Legendre conditions. Note that the Euler–Lagrange equation \( (2.2) \) at the null extremal, with notation \( (3.3) \), is
\[
Q(t, 0) = \int_{a}^{t} P_x(\tau, 0) \Delta \tau + C,
\]
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$t \in [a, b]^\mathbb{T}_r$. Therefore, choosing an arbitrary function $P(t, y^\sigma(t))$ such that $P(t, \cdot) \in C^2$ with respect to the second variable, uniformly in $t$, $P$ and $P_x$ are rd-continuous in $t$ for all admissible $y$, and by (3.5) we can write a general form of $Q$:

$$Q(t, y^\sigma(t)) = C + \int_a^t P_x(\tau, 0) \Delta \tau + q(t, y^\sigma(t)) - q(t, 0), \quad (3.6)$$

where $C \in \mathbb{R}$ and $q$ is an arbitrary function such that $q(t, \cdot) \in C^2$ with respect to the second variable, uniformly in $t$, and $q$ and $q_x$ are rd-continuous in $t$ for all admissible $y$. With notation (3.3), the strengthened Legendre condition (2.3) at the null extremal has the form

$$R(t, 0, 0) + \mu(t) \left\{ 2Q_x(t, 0) + \mu(t)P_{xx}(t, 0) + (\mu^\sigma(t))^\dagger R(\sigma(t), 0, 0) \right\} > 0, \quad (3.7)$$

$t \in [a, b]^\mathbb{T}_r$, where $\alpha^\dagger = \frac{1}{\alpha}$ if $\alpha \in \mathbb{R} \setminus \{0\}$ and $0^\dagger = 0$. Hence, we set

$$R(t, 0, 0) + \mu(t) \left\{ 2Q_x(t, 0) + \mu(t)P_{xx}(t, 0) + (\mu^\sigma(t))^\dagger R(\sigma(t), 0, 0) \right\} = p(t) \quad (3.8)$$

with $p \in C^1_{rd}([a, b]^\mathbb{T})$, $p(t) > 0$ for all $t \in [a, b]^\mathbb{T}_r$, chosen arbitrary. Note that there exists a unique solution of (3.8) with respect to $R(t, 0, 0)$. If $t$ is a right-dense point, then $\mu(t) = 0$ and $R(t, 0, 0) = p(t)$. Otherwise, $\mu(t) \neq 0$, and using Theorem 2.7 with $f(t) = R(t, 0, 0)$ we modify equation (3.8) into a first order delta dynamic equation, which has a unique solution $R(t, 0, 0)$ in agreement with Theorem 2.17 (see details in the proof of Corollary 3.4). We derive a general form of $R$ from Legendre’s condition (3.7), as a sum of the solution $R(t, 0, 0)$ of equation (3.8) and function $w$, which is chosen arbitrarily in such a way that $w(t, \cdot, \cdot) \in C^2$ with respect to the second and the third variable, uniformly in $t$; $w_x, w_v$ and $w_{vv}$ are rd-continuous in $t$ for all admissible $y$. Concluding: a general form of the integrand $L$ for functional (3.1) follows from (3.4), (3.6) and (3.8), and is given by

$$L\left(t, y^\sigma(t), y^\Delta(t)\right) = P\left(t, y^\sigma(t)\right)$$

$$+ \left( C + \int_a^t P_x(\tau, 0) \Delta \tau + q(t, y^\sigma(t)) - q(t, 0) \right) y^\Delta(t)$$

$$+ \left( p(t) - \mu(t) \left\{ 2Q_x(t, 0) + \mu(t)P_{xx}(t, 0) + (\mu^\sigma(t))^\dagger R(\sigma(t), 0, 0) \right\} \right.$$  

$$+ w(t, y^\sigma(t), y^\Delta(t)) - w(t, 0, 0) \right) \frac{y^\Delta(t)^2}{2}. \quad (3.9)$$

We have just proved the following result.
Theorem 3.1. Let $\mathbb{T}$ be an arbitrary time scale. If functional (3.1) with boundary conditions $y(a) = y(b) = 0$ attains a local minimum at $\dot{y}(t) \equiv 0$ under the strengthened Legendre condition, then its Lagrangian $L$ takes the form (3.9), where $R(t, 0, 0)$ is a solution of equation (3.8), $C \in \mathbb{R}$, $\alpha^\dagger = \frac{1}{\alpha}$ if $\alpha \in \mathbb{R} \setminus \{0\}$ and $0^\dagger = 0$. Functions $P$, $p$, $q$ and $w$ are arbitrary functions satisfying:

(i) $P(t, \cdot), q(t, \cdot) \in C^2$ with respect to the second variable uniformly in $t$; $P$, $P_x$, $q$, $q_x$ are rd-continuous in $t$ for all admissible $y$; $P_{xx}(\cdot, 0)$ is rd-continuous in $t$; $p \in C^1_{rd}$ with $p(t) > 0$ for all $t \in [a, b]^2_{\mathbb{T}}$;

(ii) $w(t, \cdot, \cdot) \in C^2$ with respect to the second and the third variable, uniformly in $t$, $w_x, w_v, w_{vv}$ are rd-continuous in $t$ for all admissible $y$.

Now we consider the general situation when the variational problem consists in minimizing (3.1) subject to arbitrary boundary conditions $y(a) = y_0(a)$ and $y(b) = y_0(b)$, for a certain given function $y_0 \in C^2_{rd}([a, b]_\mathbb{T})$.

Theorem 3.2. Let $\mathbb{T}$ be an arbitrary time scale. If the variational functional (3.1) with boundary conditions $y(a) = y_0(a)$, $y(b) = y_0(b)$, attains a local minimum for a certain given function $y_0(\cdot) \in C^2_{rd}([a, b]_\mathbb{T})$ under the strengthened Legendre condition, then its Lagrangian $L$ has the form

$$L(t, y^\sigma(t), y^\Delta(t)) = P(t, y^\sigma(t) - y^\sigma_0(t)) + (y^\Delta(t) - y^\Delta_0(t))$$

$$\times \left( C + \int_a^t P_x(\tau, -y^\sigma_0(\tau)) \Delta \tau + q(t, y^\sigma(t) - y^\sigma_0(t)) - q(t, -y^\sigma_0(t)) \right)$$

$$+ \frac{1}{2} \left( p(t) - \mu(t) \left\{ 2Q_x(t, 0) + \mu(t) P_{xx}(t, 0) + (\mu^\sigma(t))^\dagger R(\sigma(t), 0, 0) \right\} ight.$$

$$+ w(t, y^\sigma(t) - y^\sigma_0(t), y^\Delta(t) - y^\Delta_0(t)) - w(t, -y^\sigma_0(t), -y^\Delta_0(t))$$

$$\left( y^\Delta(t) - y^\Delta_0(t) \right)^2,$$

where $R(t, 0, 0)$ is the solution of equation (3.8), $C \in \mathbb{R}$ and functions $P$, $p$, $q$, $w$ satisfy conditions (i) and (ii) of Theorem 3.1.

Proof. The result follows as a corollary of Theorem 3.1. In order to reduce the problem to the case of null boundary conditions $y(a) = 0$ and $y(b) = 0$, we introduce the auxiliar variational functional

$$\tilde{L}(y) := L(y + y_0) = \int_a^b L(t, y^\sigma(t) + y^\sigma_0(t), y^\Delta(t) + y^\Delta_0(t)) \Delta t$$

$$= : \int_a^b \tilde{L}(t, y^\sigma(t), y^\Delta(t)) \Delta t.$$
subject to boundary conditions \( y(a) = 0 \) and \( y(b) = 0 \). The result follows by application of Theorem 3.1 to the auxiliary Lagrangian \( \tilde{L} \).

For the classical situation \( \mathbb{T} = \mathbb{R} \), Theorem 3.2 gives a recent result of [15].

**Corollary 3.3 (Theorem 4 of [15]).** If the variational functional

\[
\mathcal{L}(y) = \int_{a}^{b} L(t, y(t), y'(t)) dt
\]

attains a local minimum at \( y_0(\cdot) \in C^2[a,b] \) satisfying boundary conditions \( y(a) = y_0(a) \) and \( y(b) = y_0(b) \) and the classical Legendre condition \( R(t, y_0(t), y_0'(t)) > 0, \ t \in [a, b] \), then its Lagrangian \( L \) has the form

\[
L(t, y(t), y'(t)) = P(t, y(t) - y_0(t)) \\
+ (y'(t) - y_0'(t)) \left( C + \int_{a}^{t} P_x(\tau, -y_0(\tau)) d\tau + q(t, y(t) - y_0(t)) - q(t, -y_0(t)) \right) \\
+ \frac{1}{2} \left( p(t) + w(t, y(t) - y_0(t), y'(t) - y_0'(t)) - w(t, -y_0(t), -y_0'(t)) \right) (y'(t) - y_0'(t))^2,
\]

where \( C \in \mathbb{R} \).

**Proof.** Follows from Theorem 3.2 with \( \mathbb{T} = \mathbb{R} \). \( \square \)

Theorem 3.2 seems to be new for any time scale other than \( \mathbb{T} = \mathbb{R} \). In the particular case of an isolated time scale, where \( \mu(t) \neq 0 \) for all \( t \in \mathbb{T} \), we get the following corollary.

**Corollary 3.4.** Let \( \mathbb{T} \) be an isolated time scale. If functional (3.1) subject to the boundary conditions \( y(a) = y(b) = 0 \) attains a local minimum at \( \hat{y}(t) \equiv 0 \) under the strengthened Legendre condition, then the Lagrangian \( L \) has the form

\[
L \left( t, y^\sigma(t), y^\Delta(t) \right) = P \left( t, y^\sigma(t) \right) \\
+ \left( C + \int_{a}^{t} P_x(\tau, 0) \Delta \tau + q(t, y^\sigma(t)) - q(t, 0) \right) y^\Delta(t) \\
+ \left( e_\tau(t, a) R_0 + \int_{a}^{t} e_\tau(t, \sigma(\tau)) s(\tau) \Delta \tau + w(t, y^\sigma(t), y^\Delta(t)) - w(t, 0, 0) \right) \frac{y^\Delta(t)^2}{2},
\]

where \( C, R_0 \in \mathbb{R} \) and \( r(t) \) and \( s(t) \) are given by

\[
r(t) := -\frac{1 + \mu(t)(\mu^\sigma(t))^\dagger}{\mu^2(t)(\mu^\sigma(t))^\dagger}, \quad s(t) := \frac{p(t) - \mu(t)[2Q_x(0, 0) + \mu(t)P_{xx}(0, 0)]}{\mu^2(t)(\mu^\sigma(t))^\dagger},
\]

(3.10)
with \( \alpha \in \mathbb{R} \setminus \{0\} \) and \( 0^\dagger = 0 \), where functions \( P, p, q, w \) satisfy assumptions of Theorem 3.1.

**Proof.** In the case of an isolated time scale \( \mathbb{T} \), we may obtain the form of function \( Q \) in the same way as it was done in the proof of Theorem 3.1. We derive a general form for \( R \) from Legendre’s condition. By relation \( f^\sigma = f + \mu f^\Delta \) (Theorem 2.7), one may write equation (3.8) as

\[
R(t, 0, 0) + \mu(t)(\mu^\sigma(t))^\dagger \left( R(t, 0, 0) + \mu(t)R^\Delta(t, 0, 0) \right)
+ \mu(t) \left\{ 2Q_x(t, 0) + \mu(t)P_{xx}(t, 0) \right\} - p(t) = 0.
\]

Hence,

\[
\mu^2(t)(\mu^\sigma(t))^\dagger R^\Delta(t, 0, 0) + \left[ 1 + \mu(t)(\mu^\sigma(t))^\dagger \right] R(t, 0, 0)
+ \mu(t)[2Q_x(t, 0) + \mu(t)P_{xx}(t, 0)] - p(t) = 0. \tag{3.12}
\]

For an isolated time scale \( \mathbb{T} \), equation (3.12) is a first order delta dynamic equation of the following form:

\[
R^\Delta(t, 0, 0) + \frac{1 + \mu(t)(\mu^\sigma(t))^\dagger}{\mu^2(t)(\mu^\sigma(t))^\dagger} R(t, 0, 0)
+ \frac{\mu(t)[2Q_x(t, 0) + \mu(t)P_{xx}(t, 0)] - p(t)}{\mu^2(t)(\mu^\sigma(t))^\dagger} = 0.
\]

With notation (3.11) we have

\[
R^\Delta(t, 0, 0) = r(t)R(t, 0, 0) + s(t). \tag{3.13}
\]

Observe that \( r(t) \) is regressive. Indeed, if \( \mu(t) \neq 0 \), then

\[
1 + \mu(t)r(t) = 1 - \frac{1 + \mu(t)(\mu^\sigma(t))^\dagger}{\mu(t)(\mu^\sigma(t))^\dagger} = 1 - \frac{\mu^\sigma(t) + \mu(t)}{\mu(t)} = -\frac{\mu^\sigma(t)}{\mu(t)} \neq 0
\]

for all \( t \in [a, b]^\kappa \). Therefore, by Theorem 2.17, there is a unique solution to equation (3.13) with initial condition \( R(a, 0, 0) = R_0 \in \mathbb{R} \):

\[
R(t, 0, 0) = e_r(t, a)R_0 + \int_a^t e_r(t, \sigma(\tau))s(\tau)\Delta\tau. \tag{3.14}
\]

Thus, a general form of the integrand \( L \) for functional (3.1) is given by (3.10). \( \square \)

**Remark 3.5.** Instead of (3.14), we can use an alternative form for the solution of the initial value problem (3.13) subject to \( R(a, 0, 0) = R_0 \) (cf. Remark 2.18):

\[
R(t, 0, 0) = e_r(t, a) \left[ R_0 + \int_a^t e_r(a, \sigma(\tau))s(\tau)\Delta\tau \right].
\]
Then the Lagrangian \( L \) \( (3.10) \) can be written as

\[
L(t, y^\sigma(t), y^\Delta(t)) = P(t, y^\sigma(t)) \\
+ \left( C + \int_a^t P_x(\tau, 0) \Delta \tau + q(t, y^\sigma(t)) - q(t, 0) \right) y^\Delta(t) \\
+ \left( e_r(t, a) \left[ R_0 + \int_a^t e_r(a, \sigma(\tau)) s(\tau) \Delta \tau \right] + w(t, y^\sigma(t), y^\Delta(t)) - w(t, 0, 0) \right) \frac{y^\Delta(t)^2}{2}.
\]

Based on Corollary \(3.4\), we present the form of Lagrangian \( L \) in the periodic time scale \( T = h\mathbb{Z} \).

**Example 3.6.** Let \( T = h\mathbb{Z}, h > 0 \), and \( a, b \in h\mathbb{Z} \) with \( a < b \). Then \( \mu(t) \equiv h \). We consider the variational functional

\[
\mathcal{L}(y) = h \sum_{k=\frac{a}{h}}^{\frac{b}{h}} L(kh, y(kh + h), \Delta y(kh))
\]

subject to the boundary conditions \( y(a) = y(b) = 0 \), which attains a local minimum at \( \hat{y}(kh) \equiv 0 \) under the strengthened Legendre condition

\[
R(kh, 0, 0) + 2hQ_x(kh, 0) + h^2 P_{xx}(kh, 0) + R(kh + h, 0, 0) > 0,
\]

\( kh \in [a, b - 2h] \cap h\mathbb{Z} \). Functions \( r(t) \) and \( s(t) \) (see \( (3.11) \)) have the following form:

\[
r(t) = \frac{-2}{h} \in \mathcal{R}, \quad s(t) = \frac{p(t)}{h} - \left( 2Q_x(t, 0) + hP_{xx}(t, 0) \right).
\]

Hence,

\[
\int_a^t P_x(\tau, 0) \Delta \tau = h \sum_{i=\frac{a}{h}}^{\frac{b}{h}} P_x(ih, 0),
\]

\[
\int_a^t e_r(t, \sigma(\tau)) s(\tau) \Delta \tau = \sum_{i=\frac{a}{h}}^{\frac{b}{h}} (-1)^{\frac{b}{h} - i - 1} \left( p(ih) - 2hQ_x(ih, 0) - h^2 P_{xx}(ih, 0) \right).
\]
Therefore, the Lagrangian $L$ of the variational functional (3.15) on $\mathbb{T} = h\mathbb{Z}$ has the form

$$L(kh, y(kh + h), \Delta_h y(kh)) = P(kh, y(kh + h))$$

$$+ \left( C + \sum_{i=\frac{1}{h}}^{k-1} hP_x(ih, 0) + q(kh, y(kh + h)) - q(kh, 0) \right) \Delta_h y(kh)$$

$$+ \frac{1}{2} \left( (-1)^{k-\frac{3}{h}} R_0 + \sum_{i=\frac{1}{h}}^{k-1} (-1)^{k-i-1} (p(ih) - 2hQ_x(ih, 0) - h^2 P_{xx}(ih, 0)) \right)$$

$$+ w(kh, y(kh + h), \Delta_h y(kh)) - w(kh, 0, 0) \right) (\Delta_h y(kh))^2,$$

where functions $P, p, q, w$ are arbitrary but satisfy assumptions of Theorem 3.1.

Now we consider the $q$-scale $\mathbb{T} = q^{N_0}, q > 1$. In order to present the form of Lagrangian $L$, we use Remark 3.5.

**Example 3.7.** Let $\mathbb{T} = q^{N_0} = \{q^k : q > 1, k \in \mathbb{N}_0\}$ and $a, b \in \mathbb{T}$ with $a < b$. We consider the variational functional

$$L(y) = (q - 1) \sum_{t \in [a, b)} tL(t, y(qt), \Delta_q y(t))$$

subject to the boundary conditions $y(a) = y(b) = 0$, which attains a local minimum at $\hat{y}(t) \equiv 0$ under the strengthened Legendre condition

$$R(t, 0, 0) + (q - 1)t \{2Q_x(t, 0) + (q - 1)tP_{xx}(t, 0)\} + \frac{1}{q} R(qt, 0, 0) > 0$$

at the null extremal, $t \in \left[ a, \frac{b}{q^2} \right] \cap q^{N_0}$. Functions given by (3.11) may be written as

$$r(t) = \frac{q + 1}{t(1 - q)}, \quad s(t) = \frac{qp(t)}{t(q - 1) - 2qQ_x(t, 0) - q(q - 1)tP_{xx}(t, 0)}.$$

Hence,

$$\int_a^t P_x(\tau, 0) \Delta \tau = (q - 1) \sum_{\tau \in [a, t]} \tau P_x(\tau, 0), \quad e_r(t, a) = \prod_{s \in [a, t]} (-q),$$

$$\int_a^t e_r(a, \sigma(\tau)) s(\tau) \Delta \tau$$

$$= \sum_{\tau \in [a, t]} \frac{(1 - q)^{\tau}}{q \prod_{s \in [a, \tau]} (-q)} \left[ \frac{qp(\tau)}{\tau(q - 1) - 2qQ_x(\tau, 0) - q(q - 1)\tau P_{xx}(\tau, 0)} \right].$$
Therefore, the Lagrangian $L$ of the variational functional (3.16) has the form

$$L(t, y(qt), \Delta_qy(t)) = P(t, y(qt))$$

$$+ \left( C + (q-1) \sum_{\tau \in [a,t)} \tau P_x(\tau, 0) + q(t, y(qt)) - q(t, 0) \right) \Delta_qy(t) + \left\{ \prod_{s \in [a,t)} (-q) \right\} \Delta_qy(t)$$

$$\times \left[ R_0 + \sum_{\tau \in [a,t)} \frac{(1-q)\tau}{q} \prod_{s \in [a,\tau)} (-q) \left( \frac{qp(\tau)}{\tau(q-1)} - 2qQ_x(\tau, 0) - q(q-1)\tau P_{xx}(\tau, 0) \right) \right]$$

$$+ w(t, y(qt), \Delta_qy(t)) - w(t, 0, 0) \right\} \frac{(\Delta_qy(t))^2}{2},$$

where functions $P, p, r, w$ are arbitrary but satisfy assumptions of Theorem 3.1.

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