FINDING CENTRAL DECOMPOSITIONS OF $p$-GROUPS

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Abstract. Polynomial-time algorithms are given to find a central decomposition of maximum size for a finite $p$-group of class 2 and for a nilpotent Lie ring of class 2. The algorithms use Las Vegas probabilistic routines to compute the structure of finite $*$-rings and also the Las Vegas C-MeatAxe. When $p$ is small, the probabilistic methods can be replaced by deterministic polynomial-time algorithms. The methods introduce new group isomorphism invariants including new characteristic subgroups.

1. Introduction

The main goal of this paper is to prove:

Theorem 1.1. There are deterministic and Las Vegas polynomial-time algorithms which, given a finite $p$-group $P$ of class 2, return a set $H$ of subgroups of $P$ where distinct members pairwise commute, and of maximum size such that $H$ generates $P$ and no proper subset does.

We call $H$ a central decomposition of $P$ since $P$ is a central product of the groups in $H$ (with centers permitted to overlap haphazardly) [1, (11.1)]. $P$ is input as a permutation, matrix, or (black-box) polycyclic group.

Theorem 1.1 applies a new group isomorphism invariant for $p$-groups: an associative ring with involution, i.e.: a $*$-ring. Central decompositions are a natural application of these $*$-ring methods and appear to be undetectable by conventional $p$-groups methods such as using factors of a characteristic central series. The $*$-rings convert the commutation structure of a $p$-group into classical questions about ring structure which can be computed using linear algebra. The "atoms" of a central decomposition (centrally indecomposable subgroups) have specific associated $*$-rings making them detectable and restricting their structure.

Theorem 1.1 applies broadly, but special groups are our main focus, specifically, $p$-groups $P$ with elementary abelian Frattini subgroup $\Phi(P) = P' = Z(P)$. These groups have few discernible characteristic subgroups, so group isomorphism invariants of any kind are helpful. Despite their name, special groups are diverse, comprising at least $p^{2n^3/27 - 4n^2/9}$ of the at most $p^{2n^3/27 + O(n^{8/3})}$ groups of order $p^n$ [12, Theorem 2.3], [29, p. 153]. While there are $p^{2n^3/27 + O(n^2)}$ centrally decomposable special groups of order $p^n$ (e.g.: $P \times Z_p$), using $*$-rings shows there are $p^{2n^3/27 + O(n^2)}$ centrally indecomposable special groups of order $p^n$ as well [35].
Using the group isomorphism invariants developed for Theorem 1.1 we introduce various other applications such as defining new characteristic and fully invariant subgroups of $p$-groups as well as algorithms to find generators for these subgroups. We also consider the problem of central products of general groups and explain the importance of P. Hall’s isoclinism to the study of central decompositions. The details of these applications as well as useful examples are provided in the closing Sections 7 and 8, and Appendices A and B.

Whereas it is customary to use nilpotent Lie rings in order to exploit the commutation of a $p$-group, this does not seem to be helpful for central decompositions. Indeed, our $*$-rings are nonnilpotent and can be simple, semisimple, or have large radicals. We use the radical and semisimple structure of $*$-rings for Theorem 1.1, and also for the obvious analogue, Theorem 7.1, for nilpotent Lie rings of class 2 (including characteristic 0).

Central products have various irregularities which set them apart from the more familiar but special case of direct products. If $H$ is a central decomposition of $P$ and $H \in \mathcal{H}$, then considering $P/H$ can omit the intricate intersections of the members of $\mathcal{H} - \{H\}$. Therefore, inductive proofs and greedy algorithms seem impossible with central products. There can be no Theorem of Krull-Remak-Schmidt type for central products, for example, $D_8 \circ D_8 \cong Q_8 \circ Q_8$ and similar examples for odd extraspecial $p$-groups [9, Theorem 5.5.2]. More strikingly, C. Y. Tang [30, Section 6] gives a group of order $2^{12}$ which is the central product of two centrally indecomposable subgroups, but also the central product of three centrally indecomposable subgroups (Example B.17). Theorem 1.1 finds a central decomposition of maximum length in one pass, rather than through the gradual refinement of an evolving central decomposition, and so avoids the latter problem.

**Remark 1.2.** There can be any number of Aut $P$-orbits of central decompositions as in Theorem 1.1, but if $P$ has class 2 and exponent $p$, these orbits can be classified using Jordan algebras [33, Theorem 1.1].

The algorithms for Theorem 1.1 perform with roughly the same asymptotic efficiency as algorithms for modules of a comparable size. Essential tools for our algorithms include the MeatAxe [15, 17, 26] and algorithms for rings introduced by Ronyai, Friedl, and Ivanyos [27, 16].

**Remark 1.3.** The author and P. A. Brooksbank recently revisited the essential algorithms for $*$-rings introduced in Sections 3 and 5 [6, 7]. The resulting algorithms make greater use of fast module theory methods, improve the complexity of those sections, and are implemented for use in MAGMA [4]. Early tests have handled randomized examples for $p$-groups of size $p^{15}$ with rank 36 and $p = 3, 5, 7, 11$, and used roughly five seconds of real-time on a conventional laptop, and examples of size $p^{190}$ with intentionally complex central decompositions took one hour on a laboratory computer with extensive memory; details are included in [6, 7].

**1.1. Survey of the paper.** Section 2 consists of background.

In Section 3 our algorithm passes from $P$ to the bilinear map $b : P/\mathbb{Z}(P) \times P/\mathbb{Z}(P) \rightarrow P'$ of commutation in $P$. It is shown that central decompositions of $P$ correspond to orthogonal decompositions of $b$ (Proposition 3.8 and Theorem 3.9). To find a fully refined orthogonal decomposition of $b$, the ring of adjoints of $b$ is computed. This is a natural $*$-ring. Continuing the translation of the problem,
orthogonal decompositions are related to self-adjoint idempotents of \(\text{Adj}(b)\) (Corollary 4.5). These translations occupy Section 4.

**Remark 1.4.** As suggested above, when \(b\) is a bilinear map (rather than a form) the ring of adjoints can be far from simple and can have a rich structure of radicals and semisimple factors. Examples can be constructed to demonstrate this structure occurs within our application to \(p\)-groups, even for \(p\)-groups of small order [33, Section 7].

In Section 5 we begin the process of constructing self-adjoint idempotents by using the semisimple and radical structure of \(\text{Adj}(b)\). This structure can be computed efficiently by reducing to rings of characteristic \(p\) and applying the algorithms of Ronyai, Friedl, and Ivanyos for finite \(\mathbb{Z}_p\)-algebras [27, 16, 18]. This stage uses Las Vegas polynomial-time algorithms for factoring polynomials over finite fields of characteristic \(p\), such as the methods of Berlekamp or Cantor-Zassenhaus [31, Chapter 14]. However, for a deterministic algorithm (for small \(p\)), Las Vegas algorithms can be avoided.

Section 6 includes the proof of Theorem 1.1 by first finding an orthogonal decomposition of \(b\) of maximum possible size and converting this to a central decomposition of \(P\) of maximum possible possible size.

Section 7 creates the analogue of Theorem 1.1 for nilpotent Lie rings of class 2, introduces the four families of centrally indecomposable \(p\)-groups, and presents new characteristic subgroups which are easily identified using \(\text{Adj}(b)\).

Section 8 shows how the nonabelian members of a central decompositions are preserved by group isoclinisms of any group, not only finite \(p\)-groups. There a conjecture is given concerning the uniqueness of central decompositions of maximum possible size. Then the rôle of adjoints is then expanded to central products of general groups is explained.

The appendices give examples which demonstrate that the cases considered in Section 5 do occur in the context of finite \(p\)-groups. We also provide an alternative proof of the example of C.Y. Tang [30, Section 6] using the methods of Theorem 1.1. Our proof extends the example to an infinite expanding family of examples.

## 2. Background

Throughout this work we assume \(p\) is a prime. Unless otherwise obvious, all our groups, rings, modules, and algebras are finite. All our associative rings are unital. We express abelian groups additively.

We use \(A \sqcup B\) for the disjoint union of sets \(A\) and \(B\), and \(A - B\) for the complement of \(A \cap B\) in \(A\). For details on computational complexity and rigorous treatments of polynomial-time and Las Vegas algorithms see [28, Chapter 1].

For a \(p\)-group \(P\), we let \(P' = [P, P]\) denote the derived subgroup of \(P\), \(Z(P)\) the center of \(P\), and \(\Phi(P)\) the Frattini subgroup of \(P\).

We have need in various places to apply homomorphisms and isomorphisms between finite abelian \(p\)-groups, rings, and algebras. We say a homomorphism is effective when it can be evaluated efficiently – for instance with the same cost as matrix multiplication – and a coset representative for the preimage of an element in the codomain can also be found efficiently. This means that effective isomorphisms are easily evaluated and inverted on any desired element.
2.1. Central products and central decompositions. The term central product was invented by P. Hall to describe a specific type of amalgamated product especially common when constructing $p$-groups [10, Section 3.2]. Specifically, a central product over a set $\mathcal{H}$ of groups is an epimorphism $\varphi : \prod_{H \in \mathcal{H}} H \to G$ such that $H \cap \ker \varphi = 1$ for all $H \in \mathcal{H}$ [11 (11.1)]. The problem with that definition is that it allows any epimorphism, for instance, $\mathbb{Z}_p^n \to \mathbb{Z}_p$ so that $\mathbb{Z}_p$ is a central product of an arbitrary number of groups. To avoid this obvious degeneracy, we consider only central products which have the added constraint: $\langle \mathcal{J} \rangle \varphi = G$ for $\mathcal{J} \subseteq \mathcal{H}$ implies $\mathcal{J} = \mathcal{H}$. All other central products will be known as degenerate so that by default central products are nondegenerate.

A central decomposition is a set $\mathcal{H}$ of subgroups of $P$ which generates $P$, no proper subset does, and distinct members commute. Note that 1 is never in a central decomposition. When $\{P\}$ is the only central decomposition of $P$, then $P$ is centrally indecomposable. A central decomposition is fully refined when its members are centrally indecomposable. If $\mathcal{H}$ is a central decomposition of $P$, then the direct product $\prod_{H \in \mathcal{H}} H$ maps homomorphically onto $P$ via $(x_H)_{H \in \mathcal{H}} \mapsto \prod_{H \in \mathcal{H}} x_H$, and the kernel of the map intersect each $H \in \mathcal{H}$ trivially. Thus, central decompositions give rise to central products, and vice-versa; compare [1, (11.1)].

Remark 2.1. These definitions are not sufficient to guarantee that a central decomposition of an abelian group is a direct product (e.g.: $\{(1,0),(1,1)\}$ is a central decomposition of $\mathbb{Z}_p^2 \times \mathbb{Z}_p$ but not a direct decomposition). Yet, all fully refined central decompositions of an abelian group have size equal to the rank of the group and our algorithms for Theorem 1.1 make an effort to return direct factors when possible.

2.2. Representing groups for computation. We assume throughout that $P$ is a finite $p$-group of class 2 (i.e.: $P' \leq Z(P)$) for a known prime $p$. Groups and subgroups will be specified with generators; so, $P = \langle S \rangle$. We will not consider the specific representation of $P$, but assume only that it can be input with $O(|S|n)$ bits of data (ex: $n = |\Omega|$ if $G$ acts faithfully on $\Omega$ and $n = d^2 \log q$ if $P \leq \text{GL}(d,q)$) and that there are polynomial-time, in $n$, algorithms which: multiply, invert, and test equality of elements in $P$; and also test membership, i.e.: given $g \in P$ and $T \subseteq P$, determine if $g \in \langle T \rangle$. The first three problems have standard $O(n^2)$-time algorithms (or better). However, the membership-test algorithms are considerably more involved, see [14, Section 3.1], [28, Chapters 3-4], and [23, Theorem 3.2].

Remark 2.2. Polycyclic groups can also be used as input; however, there are no known polynomial-time algorithms to multiply with such groups [21, p. 670]. Hence, Theorem 1.1 treats polycyclic group inputs as “black-box” groups so that polynomial-time refers to a the total number of group multiplications and membership tests.

The assumptions on $P$ given thus far lead to deterministic polynomial-time algorithms which: find $|\langle T \rangle|$ for any $T \subseteq P$, find generators for the normal closure $\langle T^G \rangle$ of $T \subseteq P$; find generators for $P'$, and find generators for $Z(P)$ [14, Section 3.3]. These are the additional algorithms we assume for our $p$-groups.

We will use the following in the timing of our algorithms:

(i) $\text{mem}(P)$ – the time to perform membership test in $P$,
(ii) $\text{rank } P$ – the rank of $P$, i.e. $\log_p |P : \Phi(P)|$,
(iii) $\exp(P)$ – the exponent of $P$, i.e. the smallest $p^e$ such that $P^{p^e} = 1$. 

Both $P'$ and $Z(P)$ can be computed once at the start of our algorithms, and will not contribute to the overall complexity. We store any relevant elements of our groups as words (straight-line-programs) in the original generating set of $P$. We define homomorphisms by the images of the generators and therefore pulling back elements of the images can be done by pulling back words in the appropriate generating sets.

2.3. Central products and discrete logs. Suppose that $P \leq \text{GL}(d,q)$ with $p > d$ and $(p,q) = 1$. This is enough to require that $P$ embeds in $A := \text{GF}(q^{e_1})^\times \times \cdots \times \text{GF}(q^{e_s})^\times$, and so $P$ is abelian. The centrally indecomposable abelian groups are cyclic of prime power order. However, to determine that a subgroup of $A$ is cyclic appears to be a very difficult number theory problem in general solved (in non-polynomial-time) by discrete logs [14, Section 7.1].

For Theorem we assume $P$ has class 2; hence, $p < d$ or $p|q$ and thus the algorithms of Theorem 3.2 can be applied instead of discrete logs. Thus, there are no discrete log type problems to consider for matrix $p$-groups of class 2.

2.4. Abelian $p$-groups, bases, and solving systems of equations. We outline the obvious generalizations of linear algebra we require to work with abelian $p$-groups. A careful exposition is given in [24] Chapter I,Section I.G.

Let $V$ be a finite abelian $p$-group. A set $X \subseteq V$ is linearly independent if $0 = \sum_{x \in X} s_x x$, $s_x \in \mathbb{Z}$, implies $s_x \equiv 0 \mod |x|$, for all $x \in X$. A basis $X$ for $V$ is a linearly independent generating set of $V$; hence, $V = \bigoplus_{x \in X} (x)$. Every basis of $V$ determines an isomorphism to an additive representation $\mathbb{Z}_{p^{e_1}} \oplus \cdots \oplus \mathbb{Z}_{p^{e_s}}$ for $e_1 \leq \cdots \leq e_s \in \mathbb{Z}^+$. Operating in the latter representation is preferable to $V$’s original representation and we assume that all abelian groups (including subgroups) are specified with a basis.

Each endomorphism $f$ of $V$ can be represented by an integer matrix $F = [F_{ij}]$ such that $p^{e_j-e_i}F_{ij}$, $1 \leq i \leq j \leq s$, and furthermore, every such matrix induces an endomorphism of $V$ (with respect to $X$) [13] Theorem 3.3.

To row-reduce an $m \times n$ matrix $A$ with entries in $\mathbb{Z}_{p^e}$ is a modification of Gaussian elimination: first sort the rows so that the least residue classes satisfy $A_{11}|A_{1i}$ as integers, for all $i \geq 1$, then continue with standard row reduction noting that it may be impossible to clear entries above a pivot entry. That process uses $O(m^2n)$ operations in $\mathbb{Z}_{p^e}$ and leads to algorithms which convert generators of $V$ into a basis, extend linearly independent subsets of $V$ to bases, and compute the intersection of subgroups. Improvements on these methods can be had, consider and [25] Theorem 8.3.

Recently, P. A. Brooksbank and E. M. Luks created a polynomial-time algorithm which, given a module $M$ and nontrivial submodule $N$, returns a direct decomposition $M = X \oplus Y$ with $N \leq X$ and $X$ minimal with that property [3] Theorem 3.6. We use that result in the specific context of $\mathbb{Z}_{p^e}$-modules.

2.5. Bilinear maps, $\bot$-decompositions, and isometry. A $\mathbb{Z}_{p^e}$-bilinear map $b : V \times V \to W$ is a function of $\mathbb{Z}_{p^e}$-modules $V$ and $W$ where

$$b(su + u', tv + v') = sb(u,v) + sb(u',v') + tb(u',v) + b(u',v'),$$

for each $u, u', v, v' \in V$ and $s, t \in \mathbb{Z}_{p^e}$. A $\bot$-decomposition of $b$ is a decomposition $V$ of $V$ into a direct sum of submodules which are pairwise orthogonal, i.e. $b(X,Y) = 0$ for distinct $X, Y \in V$. 


Let $X$ and $Z$ be ordered bases of the $V$ and $W$ respectively. Set $B_{xy}^{(z)} \in \mathbb{Z}_p^r$ so

$$b \left( \sum_{x \in X} s_x x, \sum_{y \in X} t_y y \right) = \sum_{x, y \in X} \sum_{z \in Z} s_x t_y B_{xy}^{(z)} z, \quad \forall s_x, s_y \in \mathbb{Z}_p^r, x, y \in X.$$  

Set

$$B_{xy} = \sum_{z \in Z} B_{xy}^{(z)} z, \quad \forall x, y \in X;$$  

so that $B = [B_{xy}]_{x, y \in X}$ is an $n \times n$-matrix with entries in $W$, where $n = |X|$. Writing the elements of $V$ as row vectors with entries in $\mathbb{Z}_p^r$ with respect to the basis $X$ we can then write:

$$b(u, v) = uBv^t, \quad \forall u, v \in V.  

Take $F, G \in \text{End} V$ represented as matrices. Define $FB$ and $BG$ by the usual matrix multiplication, but notice the result is a matrix with entries in $W$. Evidently, $(F + G)B = FB + GB$, $F(GB) = (FG)B$, and similarly for the action on the right. The significance of these operations is seen by their relation to $b$:

$$b(uf, v) = uFBv^t \text{ and } b(u, vg) = uBG^t v^t;$$  

for all $u, v \in V$.

An isometry between two bilinear maps $b : V \times V \to W$ and $b' : V' \times V' \to W$ is an isomorphism $\alpha : V \to V'$ such that $b'(u\alpha, v\alpha) = b(u, v)$ for all $u, v \in V$. Evidently, isometries map $\perp$-decompositions of $b$ to $\perp$-decompositions of $b'$.

Finally, we call a bilinear map $\theta$-symmetric if there is $\theta \in \text{GL}(W)$ of order at most 2 such that

$$b(u, v) = b(v, u) \theta, \quad \forall u, v \in V.  

This meaning of $\theta$-symmetric includes the usual symmetric, $b(u, v) = b(v, u)$; and skew symmetric, $b(u, v) = -b(v, u)$ flavors of bilinear maps. If $W = \langle b(u, v) : u, v \in V \rangle$ then $\theta$ is uniquely determined by $b$ and so we make no effort to specify $\theta$ explicitly.

2.6. Rings. All our rings are subrings of $\text{End} V$, for a given abelian $p$-group $V$ (as in Section 2.4). These rings will be specified by a set of matrices which generate the ring under addition and multiplication. Multiplication and addition are handled in the usual matrix manner.

3. Reducing central decompositions to orthogonal decompositions

In this section we reduce the problem of finding a central decomposition of a $p$-group of class 2 to the related problem of finding a $\perp$-decomposition of an associated bilinear map. Throughout we assume that $P$ is a $p$-group of class 2.

3.1. The bilinear maps $\text{Bi}(P)$. R. Baer [2] associated to $P$ various bilinear maps including: $b := \text{Bi}(P)$ defined by $b : P/Z(P) \times P/Z(P) \to P'$ where

$$b(Z(P)x, Z(P)y) := [x, y], \quad \forall x, y \in P.  

It is evident that $b$ is $\mathbb{Z}_{p^r}$-bilinear where $p^r = \exp(P)$. Notice that $b$ is alternating: $b(Z(P)x, Z(P)x) = 0$, for all $x \in P$.  


3.2. Central decompositions from orthogonal decompositions. Let $\mathcal{H}$ be a central decomposition of $P$. The following related sets are useful:

\begin{align}
(3.3) & \quad \mathcal{H}Z(P) := \{HZ(P) : H \in \mathcal{H}\} - \{Z(P)\}, \\
(3.4) & \quad \mathcal{H}Z(P)/Z(P) := \{HZ(P)/Z(P) : H \in \mathcal{H}\} - \{Z(P)/Z(P)\}, \text{ and} \\
(3.5) & \quad Z(\mathcal{H}) := \{H \in \mathcal{H} : H \leq Z(P)\}.
\end{align}

Note that $\mathcal{H} - Z(\mathcal{H})$ is in bijection with $\mathcal{H}Z(P)/Z(P)$ so that

\begin{equation}
|\mathcal{H}| = |\mathcal{H}Z(P)/Z(P)| + |Z(\mathcal{H})|.
\end{equation}

Since $P = \langle H \rangle$ and $|H, \mathcal{H} - \{H\}| = 1$, it follows that $H \cap \mathcal{H} - \{H\} \leq Z(P)$. Thus, $\mathcal{H}Z(P)/Z(P)$ is a direct decomposition of $P/Z(P)$.

Suppose that $\mathcal{V}$ is a direct decomposition of $P/Z(P)$. Define

\begin{equation}
\mathcal{H}(\mathcal{V}) := \{H \leq P : Z(P) \leq H, H/Z(P) \in \mathcal{V}\}.
\end{equation}

Note that $\mathcal{V}$ and $\mathcal{H}(\mathcal{V})$ are in a natural bijection.

**Proposition 3.8.** Let $P$ be a $p$-group of class $2$ and $b := \text{Bi}(P)$.

(i) If $\mathcal{H}$ is a central decomposition of $P$ then $\mathcal{H}Z(P)/Z(P)$ is a $\perp$-decomposition of $b$.

(ii) If $\mathcal{V}$ is a $\perp$-decomposition of $b$ then $\mathcal{H}(\mathcal{V})$ is a central decomposition of $P$ where $\mathcal{H}(\mathcal{V})Z(P) = \mathcal{H}(\mathcal{V})$ and $\mathcal{H}(\mathcal{V})/Z(P) = \mathcal{V}$.

**Proof.** (i). If $\mathcal{H}$ is a central decomposition of $P$ then $\mathcal{H}Z(P)/Z(P)$ is a direct decomposition of $\mathcal{V} := P/Z(P)$. Furthermore, if $H$ and $K$ are distinct members of $\mathcal{H}$ then $[H, K] = 1$, so that $b(\mathcal{H}Z(P)/Z(P), KZ(P)/Z(P)) = 0$. Thus, $\mathcal{H}Z(P)/Z(P)$ is a $\perp$-decomposition of $b$.

(ii). Let $\mathcal{V}$ be a $\perp$-decomposition of $b$ and set $\mathcal{K} := \mathcal{H}(\mathcal{V})$. By definition, $\mathcal{K} = KZ(P)$ and $K/Z(P) = \mathcal{V}$, so that $K \cap (K - \{K\}) = Z(P)$ for all $K \in \mathcal{K}$. It remains to show that $\mathcal{K}$ is a central decomposition of $P$. As $\mathcal{V} \neq \emptyset$ it follows that $\mathcal{K} \neq \emptyset$. Furthermore, $\mathcal{V} = \langle \mathcal{V} \rangle$ so $P = \langle K, Z(P) \rangle = \langle \mathcal{K} \rangle$, as $Z(P) \leq K$ for any $K \in \mathcal{K}$. Since $\mathcal{K}$ is in bijection with $\mathcal{V}$, if $J$ is a proper subset of $\mathcal{K}$ then $J/Z(P)$ is a proper subset of $\mathcal{V}$ and as $J/Z(P)$ does not generate $V$ it follows that $J$ does not generate $P$. Finally, if $H$ and $K$ are distinct members of $\mathcal{K}$ then $0 = b(H/M, K/M) = [H, K]$. Thus, $\mathcal{K}$ is a central decomposition of $P$. \qed

**Theorem 3.9.** If $P$ is a $p$-group of class $2$, then $P$ is centrally indecomposable if, and only if, $\text{Bi}(P)$ is $\perp$-indecomposable and $Z(P) \leq \Phi(P)$.

**Proof.** Assume that $P$ is centrally indecomposable.

Let $\mathcal{V}$ be a $\perp$-decomposition of $\text{Bi}(P)$. By Proposition 3.8 (ii), $\mathcal{H}(\mathcal{V})$ is a central decomposition of $P$ and therefore $\mathcal{H}(\mathcal{V}) = \{P\}$. Hence, $\mathcal{V} = \mathcal{H}(\mathcal{V})/Z(P) = \{P/Z(P)\}$. As $\mathcal{V}$ was an arbitrary $\perp$-decomposition of $\text{Bi}(P)$, it follows that $\text{Bi}(P)$ is $\perp$-indecomposable.

Next let $\Phi(P) \leq Q \leq P$ be such that $P/\Phi(P) = Q/\Phi(P) \oplus Z(P)\Phi(P)/\Phi(P)$ as $\mathbb{Z}_p$-vector spaces. Set $\mathcal{H} = \{Q, Z(P)\}$. Clearly $[Q, Z(P)] = 1$ and $P$ is generated by $\mathcal{H}$. Therefore, $\mathcal{H}$ contains a subset which is a central decomposition of $P$. As
Let $H$ be a central decomposition of $P$. By Proposition $3.8(i)$ we know $HZ(P)Z(P)$ is a $\perp$-decomposition of $Bi(P)$. Thus, $HZ(P)Z(P) = \{P/Z(P)\}$ so that $HZ(P) = \{P\}$. Hence, for all $H \in H$, either $H \leq Z(P)$ or $HZ(P) = P$. As $Z(P) \leq \Phi(P) < P$, it follows that at least one $H \in H$ is not contained in $Z(P)$ and furthermore, $P = HZ(P) = H$ as $Z(P)$ consists of non-generators. Since no proper subset of $H$ generates $P$ and $P \in H$, it follows that $\mathcal{H} = \{P\}$. Since $H$ was an arbitrary central decomposition of $P$ it follows that $P$ is centrally indecomposable. \hfill $\square$

**Lemma 3.10.** For a $p$-group $P$ of class 2 where $Bi(P)$ is $\perp$-indecomposable, every central decomposition of $P$ has exactly one nonabelian member.

**Proof.** Let $\mathcal{H}$ be central decomposition of $P$. Since $P \neq Z(P)$ and $Bi(P)$ is $\perp$-indecomposable, there is a nonabelian $H \in \mathcal{H}$ and $HZ(P) = \{P\}$ so that $P = HZ(P)$. If $K \in \mathcal{H} - \{H\}$ then $[K, P] = [K, HZ(P)] = [K, H] = 1$, since distinct members of $H$ commute. Thus $K \leq Z(P)$, which proves that $H$ is the only nonabelian group in $\mathcal{H}$. \hfill $\square$

**Proposition 3.11.** There is a deterministic polynomial-time algorithm which, given a $p$-group $P$ of class 2 such that $Bi(P)$ is $\perp$-indecomposable, returns a nonabelian centrally indecomposable group $Q$ such that $P = Q$ or $\{Q, Z(P)\}$ is a central decomposition of $P$.

**Proof.** Algorithm. If $Z(P) \leq \Phi(P)$ then return $P$; otherwise, compute generators for a vector space complement $Q/Z(P)$ to $Z(P)\Phi(P)/\Phi(P)$ in $P/\Phi(P)$, $\Phi(P) \leq Q < P$. Recurse with $Q$ in the role of $P$ and return the result of this recursive call.

**Correctness.** If $Z(P) \leq \Phi(P)$ then Theorem $3.9$ proves that $P$ is centrally indecomposable. Otherwise, $Z(P)\Phi(P)/\Phi(P)$ is a proper subspace of the vector space $P/\Phi(P)$. The group $Q$ satisfies $P = QZ(P)$. Hence, $P' = [QZ(P), QZ(P)] = Q'$ (so $Q$ is nonabelian) and $[Z(Q), P] = [Z(Q), QZ(P)] = 1$, so that $Z(Q) = Q \cap Z(P) \geq P'$. In particular, the isomorphism of $P/Z(P) = QZ(P)/Z(P) \cong Q/Z(Q)$ gives an isometry between $Bi(P)$ and $Bi(Q)$ which implies that $Bi(Q)$ is $\perp$-indecomposable. Thus we may recurse with $Q$. By induction, the return of a recursive call is a centrally indecomposable subgroup $P' \leq R \leq P$ such that $Q = RZ(Q)$ and so $P = RZ(P)$, which proves that $\{R, Z(P)\}$ is a central decomposition of $P$.

**Timing.** The number of recursive calls is bounded by the log of the exponent $p^c$ of $P/P'$. To find a vector space complement amounts to finding a basis of $Z(P)\Phi(P)/\Phi(P)$ and extending the basis to one for $P/\Phi(P)$, and so it uses $O(\log^3|P : \Phi(P)|)$ operations in $\mathbb{Z}_p$. Hence, the total number of operations in $\mathbb{Z}_p$ is in $O(\epsilon \log^3|P : \Phi(P)|) \leq O(\log^4|P : P'|)$.

**Corollary 3.12.** There are deterministic polynomial-time algorithms which, given a $p$-group $P$ of class 2 and $V$ a fully refined $\perp$-decomposition of $Bi(P)$, return a fully refined central decomposition $\mathcal{J}$ of $P$ such that:

(i) $JZ(P)/Z(P) = V$ and

(ii) $Z(\mathcal{J})$ is a direct decomposition of $Z(P)$. 

$\square$
In particular, if \( V \) has maximum size amongst the set of \( \perp \)-decompositions of \( \text{Bi}(P) \), then \( \mathcal{H} \) has maximum size amongst the set of central decompositions of \( P \).

**Proof. Algorithm.** Compute the pullback \( \mathcal{H} := \mathcal{H}(V) \). Set \( \mathcal{K} = \emptyset \). For each \( H \in \mathcal{H} \), use the algorithm of Proposition 3.11 to find a nonabelian centrally indecomposable subgroup \( K \leq H \) such that \( H = KZ(P) \) and add \( K \) to \( \mathcal{K} \). Next, find bases for \( Z(P) \) and for \( Z(\langle \mathcal{K} \rangle) \) and apply the algorithm for \( [5] \) Theorem 3.6] to find a direct factor \( X \) of \( Z(P) \) which is minimal with respect to containing \( Z(\langle \mathcal{K} \rangle) \). Find a basis \( \mathcal{X} \) for \( X \) and \( \mathcal{Y} \) of a complement \( Y \) to \( X \) in \( Z(P) \), and return

\[
\mathcal{J} := \mathcal{K} \cup \{ \langle x \rangle : x \in \mathcal{X}, x \notin Z(\langle \mathcal{K} \rangle) \} \cup \{ \langle y \rangle : y \in \mathcal{Y} \}.
\]

**Correctness.** By Proposition 3.8 we know that \( \mathcal{H} \) is a central decomposition of \( P \) in which every member \( H \) has \( Z(H) = Z(P) \) and \( \text{Bi}(H) \) is \( \perp \)-indecomposable. Thus the algorithm of Proposition 3.11 can be applied to \( H \) and the set \( \mathcal{K} \) consists of nonabelian centrally indecomposable subgroups where distinct members pairwise commute; thus, \( \mathcal{K} \) is a fully refined central decomposition of \( \langle \mathcal{K} \rangle \) of maximum possible size. Notice \( \mathcal{K} = \mathcal{J} - Z(\mathcal{J}) \) and the members of \( Z(\mathcal{J}) \) are cyclic and a direct decomposition of \( \langle \mathcal{J} \rangle \). Hence, \( \mathcal{J} \) is a fully refined central decomposition of \( P \). Furthermore, \( KZ(P) = \mathcal{H} \). By Proposition 3.8(ii) we have:

\[
\begin{align*}
\mathcal{J}Z(P)/Z(P) &= KZ(P)/Z(P) = \mathcal{H}/Z(P) = \mathcal{V} \quad \text{and} \\
Z(\mathcal{J}) &= \{ \langle x \rangle : x \in \mathcal{X}, x \notin \langle \mathcal{K} \rangle \} \cup \{ \langle y \rangle : y \in \mathcal{Y} \}.
\end{align*}
\]

Thus, (i) and (ii) is proved. It remains to prove that \( \mathcal{J} \) has maximum size amongst central decompositions of \( P \).

First \( |\mathcal{J}| = |\mathcal{K}| + |Z(\mathcal{J})| \). Also, \( X \) is a minimal direct factor of \( Z(P) \) which contains \( P' \) and so \( Z(P)^p P' = Z(P)^p X \). As, \( Z(P) = X \oplus Y \) and \( \langle \mathcal{K} \rangle = P' \), it follows that \( |Z(\mathcal{J})| = \text{rank } Z(P) - \text{rank } Z(P)^p P'/Z(P)^p \). If \( \mathcal{L} \) is any other central decomposition of \( P \), then \( |\mathcal{L}| = |\mathcal{J}| = |\mathcal{L}| \). By the maximality of \( \mathcal{V} \), \( |\mathcal{J}Z(P)/Z(P)| \leq |\mathcal{V}| = |\mathcal{K}| \). As \( Z(P) \) is abelian and \( \langle \mathcal{L} \rangle \leq Z(P) \), it follows that \( |Z(\mathcal{L})| \leq \text{rank } Z(P) - \text{rank } Z(P)^p P'/Z(P)^p \). Thus, \( \mathcal{J} \) has maximum possible size.

**Timing.** There are \( |V| \) calls made to the algorithm of Proposition 3.11 which uses \( O(|\log \exp(H)| \log^3 [H : \Phi(H)]) \) operations in \( \mathbb{Z}_p \) for each \( H \in \mathcal{H} \). The algorithm of \( [5] \) Theorem 3.6 runs in \( O(\text{rank}^5 Z(P))-\text{time} \). Thus, the number of field operations lies in \( O(|V| \log \exp(P) \log^3 [P : \Phi(P)] + |S| \text{memb}(P) + \text{rank}^5 Z(P)) \leq O(\log^{5}|P : P'| + \log^6 Z(P)/Z(P)^p) \). \( \square \)

4. The *-ring of adjoints of a bilinear map

The translations of Section 3 lead us to consider how to find a fully refined \( \perp \)-decomposition of a bilinear map. For this we introduce the ring of adjoints.

Throughout this section we assume that \( b : V \times V \rightarrow W \) is a \( \theta \)-symmetric \( \mathbb{Z}_p^\theta \)-bilinear map.

4.1. **Adjoints:** \( \text{Adj}(b) \), \( \text{Sym}(B) \), and \( \mathfrak{A}(R, \ast) \). The ring of adjoints of \( b \) is:

\[
(4.1) \quad \text{Adj}(b) := \{(f, g) \in \text{End} V \oplus (\text{End} V)^{op} : b(u, v) = b(u, vg), \forall u, v \in V\}.
\]

There is a natural subset of \( \text{Adj}(b) \) of self-adjoint elements:

\[
(4.2) \quad \text{Sym}(b) := \{(f, f) \in \text{End} V \oplus (\text{End} V)^{op} : b(u, v) = b(u, vf), \forall u, v \in V\}.
\]
Remark 4.3. Notice that Sym(b) is not an associative subring but rather a Jordan ring, quadratic in the case of characteristic 2, cf. [33 Section 4.5]. This is a vital observation for answering questions surrounding \( \perp \)-decompositions; however, for algorithmic purposes this nonassociative perspective is not necessary.

If \( b \) is \( \theta \)-symmetric then \((f, g) \in \text{Adj}(b)\) if, and only if, \((g, f) \in \text{Adj}(b)\). Hence, \((f, g) \mapsto (g, f)\) is an anti-isomorphism \(\ast\). Indeed, \(\ast\) has order 1 or 2 so that \(\text{Adj}(b)\) is a \(\ast\)-ring.

In general, for a \(\ast\)-ring \((R, \ast)\) and additive subgroup \(S \subseteq R\), we define \(\mathfrak{S}(S, \ast) = \{s \in S : s^\ast = s\}\) which is again a subgroup of \(S\) as \(\ast\) is additive. \(\mathfrak{S}\) is for Hermite and is a notation encouraged by Jacobson [20 Section 1.4]. Evidently, \(\text{Sym}(b) = \mathfrak{S}(\text{Adj}(b))\).

4.2. Self-adjoint idempotents. An endomorphism \(e \in \text{End} V\) is an idempotent if \(e^2 = e\). This makes \(V = Ve \oplus V(1-e)\). Indeed, every direct decomposition \(V\) of \(V\) is parameterized by the set \(E := \mathcal{E}(V)\) of projection idempotents; that is, for each \(U \in \mathcal{V}, e_U \in E\) with kernel \(\langle V - \{U\} \rangle\) and where the restriction of \(e_U\) to \(U\) is the identity. It follows that distinct members \(e\) and \(f\) of \(E\) are orthogonal (i.e. \(ef = 0 = fe\)) and \(1 = \sum e \in \mathcal{E} e\).

Evidently \(1 \in \text{Sym}(b)\), so \(\text{Sym}(b)\) contains idempotents. All idempotents in \(\text{Sym}(b)\) are self-adjoint. The significance of \(\text{Sym}(b)\) is the following:

Theorem 4.4. A direct decomposition \(V\) of \(V\) is a \(\perp\)-decomposition of \(b : V \times V \rightarrow W\) if, and only if, \(\mathcal{E}(V) \subseteq \text{Sym}(b)\).

Proof. Suppose that \(V\) is a \(\perp\)-decomposition of \(b\). Take \(e \in \mathcal{E}(V)\). Then \(b(ue, v) = b(ue, ve + v(1-e)) = b(ue, ve) + b(ue, v(1-e))\), for all \(u, v \in V\). As \(1 - e = \sum_{f \in \mathcal{E}(V) - \{e\}} f\), and \(Ve\) is perpendicular to \(Vf\) for each \(f \in \mathcal{E}(V)\), it follows that \(Ve\) is perpendicular to \(V(1-e)\); hence, \(b(ue, v) = b(ue, ve)\). Similarly, \(b(ue, ve) = b(u, ve)\), so that \(e \in \text{Sym}(b)\).

Now suppose that \(V\) is a direct decomposition of \(V\) with \(\mathcal{E}(V) \subseteq \text{Sym}(b)\). If \(e \in \mathcal{E}(V)\) then \(b(ue, v(1-e)) = b(u, v(1-e)e) = 0\), for all \(u, v \in V\). So \(Ve\) is perpendicular to \(V(1-e)\). Thus every subspace of \(V(1-e)\) is perpendicular to \(Ve\), which includes \(Vf\) for every \(f \in \mathcal{E}(V) - \{e\}\). So \(V\) is a \(\perp\)-decomposition of \(b\).

A self-adjoint idempotent \(e \in \text{Sym}(b)\) is self-adjoint-primitive if it is not the sum of proper (i.e.: not 0 or 1) pairwise orthogonal self-adjoint idempotents in \(\text{Sym}(b)\). (Such idempotents need not be primitive in \(\text{Adj}(b)\) under the usual meaning of primitive idempotents.) A set of pairwise orthogonal self-adjoint-primitive idempotents of \(\text{Sym}(b)\) which sum to 1 is called a frame of \(\text{Sym}(b)\). More generally, in a \(\ast\)-ring \((R, \ast)\), a self-adjoint frame is a set of self-adjoint-primitive pairwise orthogonal idempotents which sum to 1.

Corollary 4.5. There is a natural bijection between the set of fully refined \(\perp\)-decompositions of \(b\) and the set of all frames of \(\text{Sym}(b)\).

Proof. This follows directly from Theorem 4.4.

4.3. Computing \(\text{Adj}(b)\) and \(\text{Sym}(b)\). Let \(V\) and \(W\) be finite abelian \(p\)-groups specified with bases \(X\) and \(Z\) respectively. Take \(b : V \times V \rightarrow W\) to be a \(\mathbb{Z}_{p^r}\)-bilinear map. Assume that \(b\) is input with structure constant matrix \(B\) with respect to the bases \(X\) and \(Z\) (cf. [2,5]).
If \( \text{End} V \) is expressed as matrices (see Section 2.3) with respect to \( \mathcal{X} \) then
\[
(4.6) \quad \text{Adj}(B) = \{(X, Y) \in \text{End} V \oplus \text{End} V : XB = BY^t\}.
\]
To find a basis for \( \text{Adj}(B) \) we solve for \( X \) and \( Y \) such that:
\[
(4.7) \quad 0 = \sum_{x \in \mathcal{X}} X_{xx}^t B_{x'y}^{(z)} - \sum_{y \in \mathcal{X}} Y_{yy}^t B_{xy}^{(z)}, \quad \forall x, y \in \mathcal{X}, z \in \mathbb{Z}.
\]
This amounts to solving \(|\mathcal{X}|^2 |\mathbb{Z}| \) linear equations over \( \mathbb{Z}_{p^e} \), each in \( 2|\mathcal{X}| \) variables and can be done using \( O(|\mathcal{X}|^4 |\mathbb{Z}|) \) operations in \( \mathbb{Z}_{p^e} \) (cf. Section 2.4). Computing a basis of \( \text{Sym}(b) \) can be done in similar fashion.

**Remark 4.8.** If \( b \) is \( \theta \)-symmetric then the number of equations determining \( \text{Adj}(b) \)
be decreased by 2 by considering the ordering of the basis \( \mathcal{X} \) and using only the equations (4.4) for \( x \leq y, x, y \in \mathcal{X} \) and \( z \in \mathbb{Z} \).

5. **Algorithms for \( \ast \)-rings**

In Section 4 the self-adjoint idempotents of the \( \ast \)-ring \( \text{Adj}(b) \) where linked with \( \perp \)-decompositions of \( b \), and through the theorems of Section 5 also to central decompositions of \( P \). In this section we show how to find self-adjoint idempotents by appealing to the semisimple and radical structure of \( \ast \)-rings. Most of the algorithms reduce to known algorithms for the semisimple and radical structure theorems of finite algebras over \( \mathbb{Z}_p \).

5.1. **A fast Skolem-Noether algorithm.** Let \( K \) be a field of characteristic \( p \).
The Skolem-Noether theorem states that every ring automorphism \( \varphi \) of \( M_n(K) \)
satisfies \( X \varphi = D^{-1} X \sigma D \) for some \( (D, \sigma) \in \text{GL}_n(K) \times \text{Gal}(K/\mathbb{Z}_p) \), and for all \( X \in M_n(K) \), [3] (3.62)]. Given an effective automorphism \( \varphi \), there is a straightforward method to find \( (D, \sigma) \) which involves solving a system of \( n^2 \) linear equations over \( K \) and thus uses \( O(n^6) \) field operations. We offer the following improvement by analyzing the proof the the Skolem-Noether theorem in [14] Chapter VIII.

**Proposition 5.1.** Given an effective ring automorphism \( \varphi \) of \( M_n(K) \), \( K \) a finite field of characteristic \( p \), there is a deterministic algorithm using \( O(n^4 + \dim_{\mathbb{Z}_p} K) \)
operations in \( \mathbb{Z}_p \) which returns \( (D, \sigma) \in \text{GL}_n(K) \times \text{Gal}(K/\mathbb{Z}_p) \) such that \( X \varphi = D^{-1} X \sigma D \), for all \( X \in M_n(K) \).

**Proof. Algorithm.** Define \( g : K^n \to M_n(K) \) by \( x \mapsto \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \) and \( \tau : K^n \to M_n(K) \) by \( x \tau = x g \varphi \). Fix a basis \( \{x_1, \ldots, x_n\} \) of \( K^n \) and find the first \( 1 \leq i \leq n \) such that \( x_i(x_j \tau) \neq 0 \) for all \( 1 \leq j \leq n \). Set \( D := \begin{bmatrix} x_i(x_1 \tau) \\ \vdots \\ x_i(x_n \tau) \end{bmatrix} \in M_n(K) \). Induce \( \sigma : K \to K \) by \( \alpha \mapsto [(\alpha I_n) \varphi]_{11} \), then return \( (D, \sigma) \).

**Correctness.** We summarize how the steps in this algorithm perform the various stages of the proof of Skolem-Noether, given in [14] Chapter VIII.

Let \( I \) be the image of \( g \). As \( I \) is a minimal right ideal, the image \( J := I \varphi \) is also a minimal right ideal. Thus, there is an \( 1 \leq i \leq n \) such that \( x_i J \neq 0 \). Since \( x_i J \) is a simple right \( M_n(K) \)-module, it follows that \( x_i J \cong K^n \). As \( \{x_1 g, \ldots, x_n g\} \) is a basis of \( I \), \( \{x_1 \tau, \ldots, x_n \tau\} \) is a basis of \( J \) and so \( \{x_i(x_1 \tau), \ldots, x_i(x_n \tau)\} \) is a basis
of \( x, J \). Thus \( D \) is an invertible matrix in \( M_n(K) \). Finally, \((\alpha I_n)\varphi = (\alpha\sigma)I_n\), for \( \alpha \in K \), defines a field automorphisms of \( K \). It follows that \( X\varphi = D^{-1}X\circ D \) for each \( X \in M_n(K) \).

**Timing.** The algorithm searches over the set of all \( 1 \leq i, j \leq n \) and tests whether \( x_i(x, \tau) \neq 0 \), a test which uses \( O(n^2) \) field operations in \( K \). The additional task of inducing \( \sigma \) uses \( O(\dim_{Z_p} K) \) operations in \( Z_p \).

\[ \square \]

### 5.2. Constructive recognition of finite simple \( \ast \)-rings.

The classification (up to \( \ast \)-isomorphism) of simple \( \ast \)-rings appears to have developed from multiple disciplines simultaneously (most involving rings over infinite or arbitrary fields). Key players included A.A. Albert, N. Jacobson, and A. Weil; see [22]. We attempt to give an ersatz proof which condenses the various ideas distributed amongst the sources. In particular, we include the elements that will be used in our algorithms.

**Lemma 5.2.** The Jacobson radical of a \( \ast \)-ring is a \( \ast \)-ideal.

**Proof.** The Jacobson radical is the intersection over the set of maximal left ideals as well as the set of maximal right ideals; \( \ast \) interchanges these sets. \( \square \)

**Theorem 5.3.** A finite \( \ast \)-simple ring \( (R, \ast) \) is either simple as a ring or the direct product of two isomorphic simple rings. Thus, there is a field \( K \), a vector space \( V \) over \( K \), and an involution \( \circ \) on \( \text{End}_K V \) such that \( (R, \ast) \) is \( \ast \)-ring isomorphic to one of the following:

(i) Classical: \( (\text{End}_K V, \circ) \).

(ii) Exchange: \( (\text{End}_K V \oplus \text{End}_K V, \bullet := \circ \otimes 2) \) where \( (f, g)^\ast := (g^\circ, f^\circ) \), for all \( f, g \in \text{End}_K V \). Furthermore, any two exchange type \( \ast \)-simple \( \ast \)-rings which are isomorphic as rings are isomorphic as \( \ast \)-rings.

**Proof.** (The proof is implicit in [20, p. 178].) By Lemma 5.2, \( J(R) \) is a \( \ast \)-ideal. As \( (R, \ast) \) is \( \ast \)-simple, \( J(R) = 0 \). By the Wedderburn theorems, \( R \) is a direct product of its minimal ideals. Fix a minimal ideal \( M \) of \( R \) and \( I \) a minimal left ideal of \( M \). Thus, \( M^\ast \) is also a minimal ideal of \( R \) with minimal right ideal \( I^\ast \). As \( M \) is a simple ring its center \( K := Z(M) \) is a field. Evidently \( I \) is a left \( K \)-vector space and by Wedderburn’s theorems, the left action of \( M \) on \( I \) produces a ring isomorphism \( \varphi : M \rightarrow \text{End}_K I \). Define, \( \varphi : M^\ast \rightarrow \text{End}_K I \) by \( v(xg) := x^s v \) for all \( x \in M^\ast \) and \( v \in I \). Evidently \( \varphi \) is also a ring isomorphism. Thus, \( f \mapsto f^c := (f\varphi^{-1})^\ast \varphi \), for all \( f \in \text{End}_K I \), is an involution on \( \text{End}_K I \).

Finally, \( M + M^\ast \) is a nontrivial \( \ast \)-ideal and \( (R, \ast) \) is \( \ast \)-simple; therefore, \( R = M + M^\ast \). If \( M = M^\ast \) then \( (R, \ast) \) is of classical type and \( \varphi \) is a \( \ast \)-ring isomorphism to \( (\text{End}_K I, \circ) \). Otherwise, \( R = M \oplus M^\ast \) and \( (R, \ast) \) is of exchange type and \( \varphi \oplus \varphi \) is a \( \ast \)-ring isomorphism to \( (\text{End}_K I \oplus \text{End}_K I, \bullet) \).

If \( \ast \) is another involution on \( \text{End}_K I \) and \( \circ := \ast \otimes 2 \). Define \( \mu : \text{End}_K I \otimes \text{End}_K I \rightarrow \text{End}_K I \otimes \text{End}_K I \) by

\[(f, g) \mapsto (f, g^\circ \ast), \quad \forall f, g \in \text{End}_K I.\]

Evidently \( \mu \) is a ring isomorphism. Furthermore,

\[(f, g)^\ast \mu = (g^\circ, f^\circ \ast) = (f, g^\circ \ast \circ = (f, g)\mu^\circ, \quad \forall f, g \in \text{End}_K I.\]

Thus \( \mu \) is a \( \ast \)-ring isomorphism. \( \square \)
Remark 5.6. The map $\mu$ defined in (5.4) need not be $K$-linear, but rather only $K$-semilinear. Our algorithms do not require $K$-linear isomorphisms, but they can be modified to detect these distinctions when necessary.

From the coordinatization of $*$-simple algebras given in Theorem 5.3, it is now an application of the Skolem-Noether theorem and classical forms to produce the following Proposition 5.14; compare [10, IX.10-11].

Proposition 5.7. There is a deterministic polynomial-time algorithm which, given a finite classical $*$-simple $*$-ring $(M_n(K), \sigma)$, returns a $*$-ring isomorphism $\varphi : (M_n(K), \sigma) \to \text{Adj}(d)$, where $d : K^n \times K^n \to K$ is a nondegenerate symmetric or alternating bilinear, or Hermitian sesquilinear $K$-form.

Proof. Apply the algorithm of Proposition 5.1 to find $(D, \sigma) \in \text{GL}_n(K) \rtimes \text{Gal}(K/Z_p)$ such that $(X^o)^t = D^{-1}X^oD$, for all $X \in M_n(K)$. If $\sigma \neq 1$ and $D^{-1}D^\sigma = -I$ then find $\gamma \in K$ such that $\gamma^\sigma \neq \gamma$, and reset $D := (\gamma - \gamma^\sigma)D$. Define $d : K^n \times K^n \to K$ by $d(u, v) := uD(v^\sigma)^t$, for all $u, v \in K^n$. Return $\mu : (M_n(K), \sigma) \to \text{Adj}(d) \subseteq \text{End}_K K^n \otimes \text{End}_K K^n$ defined by $\mu := (a, a^\circ)$.

Correctness. As $D$ is invertible, $d$ is biadditive, linear in the first variable, and nondegenerate. For all $\alpha \in K$, $\alpha \sigma^2 I = D^{-2}(\alpha \sigma^2 I)D^2 = ((\alpha I)^\circ)^o = \alpha I$. Hence, $\sigma^2 = 1$. Also,

$$ (5.8) \quad X = (X^o)^o = D^t(D^t X^o D^{-1})^o D^{-t} = D^t D^{-\sigma} X D^\sigma D^{-t}, \quad \forall X \in M_n(K). $$

Thus, $D^{-\sigma}D^t = \alpha I$, for some $\alpha \in K$. As $D = (D^t)^t = D^2$, it follows that $\alpha = \pm 1$. Therefore $D^t = \pm D^\sigma$. If $\sigma \neq 1$ then $d$ is $\pm 1$-symmetric. If $\sigma \neq 1$ and $\alpha = 1$ then $d$ is Hermitian. Otherwise, $\alpha = -1 \neq 1$, char $K \neq 2$, and $K$ is a quadratic field extension over the subfield fixed by $\sigma$. So there is a $\beta := \gamma - \gamma^\sigma \in K$ such that $\beta^\sigma = -\beta$. Evidently, $(\beta D)^t(\beta D)^{-\sigma} = -\beta \beta^{-\sigma} I = I$. Thus, reseting $D$ to $\beta D$ makes $D^t = D^\sigma$ and $d$ is Hermitian.

Finally, $XD = D(X^o)^o$ so that $d(uX, v) = d(uX^o, v)$ for each $X \in M_n(K)$ and $u, v \in K^n$. Thus $(M_n(K), \sigma)$ is $*$-isomorphic to $\text{Adj}(d)$ via $X \mapsto (X, X^o)$.

Timing. Applying the algorithm for Proposition 5.1 uses $O(n^4 + \dim_{Z_p} K)$ operations in $Z_p$. Determining if $\sigma \neq 1$ discovers some $\gamma \in K$ such that $\gamma^\sigma \neq \gamma$, and can be carried out within the algorithm of Proposition 5.1. Therefore, the remaining computations involve only matrix multiplication. So the overall time lies in $O(n^4 + \dim_{Z_p} K)$. \hfill $\square$

Remark 5.9. The $*$-simple $*$-rings of exchange type can also be treated as adjoints of a form. Specifically, let $C := (K \oplus K, [\alpha, \beta] := (\beta, \alpha))$. Then define $d : C^n \times C^n \to C$ by $d(u, v) = uv^\sigma$. Evidently, $\text{Adj}(d)$ is $*$-ring isomorphic to $(M_n(K) \oplus M_n(K), (X, Y)^* := (Y^t, X^t))$. See [33, Section 4.2] for uniform treatment of these forms using associative composition algebras.

5.3. Computing the $*$-semisimple and $*$-radical structure of $\text{Adj}(b)$. We require the following generalization of the algorithm of [10] using effective homomorphism (Section 2.4).

Theorem 5.10. There is a Las Vegas polynomial-time algorithm which, given $R \subseteq \text{End}_V$, for a finite abelian $p$-group $V$, returns a set $\Omega$ of effective ring epimorphisms such that:

(i) for each $\pi \in \Omega$, $\ker \pi$ is a maximal ideal of $R$ and the image of $\pi$ is $M_n(K)$ for some field $K$ and integer $n$ (dependent on $\pi$),
(ii) for each maximal ideal \( M \) of \( R \) there is a unique \( \pi \in \Omega \) such that \( M = \ker \pi \), and

(iii) if \( x, y \in R \) such that \( x \pi = y \pi \) then the representatives \( x', y' \in R \) of the pullbacks to \( R \) of \( x \pi \) and \( y \pi \) given by the effective \( \pi \in \Omega \), satisfy \( x' \equiv y' \) (mod \( pR \)). Each evaluation or computation of preimages of \( \pi \) uses \( O(\text{rank}^3 R) \) operations.

Proof. Algorithm. Pass to \( \tilde{R} := R/pR \subset \text{End} \tilde{V}, \tilde{V} = V/pV, \) and using \cite{16} Corollary 1.5] compute a Wedderburn complement decomposition \( \tilde{R} = \tilde{S} \oplus J(\tilde{R}) \), where \( \tilde{S} \) is a subring of \( \tilde{R} \) and \( \tilde{S} \cong \tilde{R}/J(\tilde{R}) \) as rings (note that the direct decomposition is as vector spaces not necessarily as rings).

Apply the C-MeatAxe \cite{26}, to \( S \) to find a the set \( \mathcal{X} \) of irreducible \( S \)-submodules of \( V := V/pV \). As \( \tilde{S} \) is semisimple, \( \mathcal{X} \) is a direct decomposition of \( \tilde{V} \). Conjugate \( \tilde{R} \) by the change of basis matrix resulting from the basis exhibiting the submodules in \( \mathcal{X} \) so that \( \tilde{R} \) is block lower triangular. Use a greedy algorithm to find a minimal subset \( \mathcal{W} \) of \( \mathcal{X} \) such that \( \tilde{S} \) acts faithfully on \( \langle \mathcal{W} \rangle \). Let \( \tau : \tilde{R} \to \tilde{S} \) be the projection of \( \tilde{x} \in \tilde{R} \) to \( \tilde{S} \) given by the vector space decomposition \( \tilde{R} = \tilde{S} \oplus J(\tilde{R}) \). Pull-backs of \( \tau \) are defined by means of the image of basis elements and the linearity of \( \tau \).

For each \( \tilde{W} \in \mathcal{W} \), define \( \pi_{\tilde{W}} : \tilde{R} \to \text{End} \tilde{W} \) by \( \pi_{\tilde{W}}(x) := (x + pR)\tau|_{\tilde{W}}, \) for \( x \in R \). The coset representative of the inverse image of \( \tilde{t} \in \text{End} \tilde{W} \) is created by extending \( \tilde{t} \) to \( V \) as \( s \) acting as \( 0 \) on each \( \tilde{V}_i \neq \tilde{W} \), \( 1 \leq i \leq l \) (i.e., \( s \) has \( \tilde{t} \) in the \( \tilde{W} \) diagonal block of the matrix and 0’s elsewhere), and then returning a coset representative of \( s\tau^{-1} \). Thus \( \pi \) is an effective homomorphism. The algorithm returns the set \( \{ \pi_{\tilde{W}} : \tilde{W} \in \mathcal{W} \} \).

Correctness. If \( M \) is a maximal ideal of \( R \) then \( R/M \cong \text{End}_K W \) for some field extension \( K/\mathbb{Z}_p \) and \( K \)-vector space \( W \). Hence, \( R/M \) is a \( \mathbb{Z}_p \)-vector space and so \( R/J(R) \) is a \( \mathbb{Z}_p \)-vector space, which proves that \( pR \leq J(R) \) and \( J(R) = J(R)/pR \). Therefore, it suffices to find the projections of \( \tilde{R} \) onto its simple factors.

Since \( R/pR \subset \text{End} V \) we can apply \cite{16} Corollary 1.5]. Hence, we obtain a Wedderburn complement decomposition \( R = S \oplus J(R) \). As \( S \) is semisimple its action on \( V \) is completely reducible and the C-MeatAxe \cite{26} finds a decomposition \( V = V_1 \oplus \cdots \oplus V_l \) as above. For each \( \tilde{W} \in \mathcal{W} \), the map \( \pi_{\tilde{W}} \) is a ring homomorphism as \( \tau \) is a ring homomorphism and \( \tilde{W} \) is an \( S \)-module. Since \( \tilde{W} \) is also irreducible it follows that \( \tilde{T} := R\pi_{\tilde{W}} \leq \tilde{S} \) is a simple subring of \( \text{End}_{\mathbb{Z}_p} \tilde{W} \). The appropriate field of scalars is the center \( K \) of \( \tilde{T} \). Thus \( \tilde{W} \) is a \( K \)-vector space and \( \pi_{\tilde{W}} \) is a ring epimorphism onto \( \text{End}_K \tilde{W} \) with kernel a maximal ideal of \( R \), proving (i). Since \( \mathcal{W} \) is minimal with respect to having \( \tilde{S} \) represented faithfully on \( \langle \mathcal{W} \rangle \), the returned set of epimorphism has one epimorphism for each maximal ideal of \( R \), thus proving (ii).

Finally, for (iii) we note that the representative matrix for the inverse image under \( \pi \in \Omega \) of a point in \( \text{End}_K \tilde{W} \) is trivial in every block except the block on which \( \pi \) is projected. Furthermore, to evaluate \( \pi \) requires we compute \( (x + pR)\tau \) which is done by writing \( x + pR \) in the bases of the block decomposition given by \( \{V_1, \ldots, V_l\} \) and uses \( O(\text{dim}^3 V) \) operations. To compute a preimage of \( \tilde{t} \) under \( \pi \) requires we write \( \tilde{t} \) in the basis \( \mathcal{X} \tau \) where \( \mathcal{X} \) is the fixed basis of \( R \). Therefore the algorithm returns correctly.

Timing. The significant tasks are computing the Wedderburn decomposition and the use of the C-MeatAxe, both which use \( O(\text{dim}^3 V) \) operations in \( \mathbb{Z}_p \); see \cite{16} Corollary 1.4]. \( \square \)
Lemma 5.11. There is a deterministic polynomial-time algorithm which, given a $\ast$-ring epimorphism $\gamma : (R, \ast) \to (T, \ast)$ and $t \in T$ such that $t^\ast = t$, returns an $s \in R$ such that $s\gamma = t$ and $s^\ast = s$.

Proof. Algorithm. Use a basis $X$ for $\mathfrak{S}(R, \ast)$ and write $t = \sum_{x \in X} s_x x\gamma$, with $s_x \in \mathbb{Z}_{p^\ast}$. Return $\sum_{x \in X} s_x s$.

Correctness. Since $\gamma$ is an $\ast$-ring homomorphism, $x\gamma^\ast = x^\ast \gamma = x\gamma$. As $\gamma$ is an epimorphism, $X\gamma$ spans the submodule of self-adjoint elements of $(T, \ast)$. Therefore, $t = \sum_{x \in X} s_x x\gamma = (\sum_{x \in X} s_x x) \gamma$. So the return is correct.

Timing. Assuming a basis for $\mathfrak{S}(R, \ast)$ is provided, the task required $O(|X|)$ evaluations of $\gamma$, and Gaussian elimination to write $t$ as a linear combination of $X\gamma$, which uses $O(|X|^3)$ operations in $T$.

Corollary 5.12. There is a Las Vegas polynomial-time algorithm which, given a $\ast$-ring $(R, \ast)$ where $R \subseteq \text{End} V$ for an abelian $p$-group $V$, returns a set $\Gamma = \{ \gamma : (R, \ast) \to (T, \ast) \}$ of $\ast$-ring epimorphisms where:

(i) there is exactly one $\gamma \in \Gamma$ for each maximal $\ast$-ideal $M$ of $(R, \ast)$, and $\ker \gamma = M$, 

(ii) for each $\gamma : (R, \ast) \to (T, \ast) \in \Gamma$ either: 

(a) $T = (M_n(K) \oplus M_n(K), (X, Y) \mapsto (Y^\ast, X^\ast))$, or 
(b) $T = \text{Adj}(d)$ for a nondegenerate symmetric, alternating, or Hermitian form $d : K^m \times K^m \to K$.

(iii) If $x, y \in (R, \ast)$ such that $x\gamma = y\gamma$ then the representatives $x', y' \in (R, \ast)$ of the pullbacks to $(R, \ast)$ of $x\gamma$ and $y\gamma$ given by the effective homomorphism $\gamma \in \Gamma$, satisfy $x' \equiv y' \pmod{pR}$. Furthermore, if $x \in \mathfrak{S}(R, \ast)$ then $x' \in \mathfrak{S}(R, \ast)$.

Proof. Algorithm. Let $\Gamma = \emptyset$. Using the algorithm of Theorem 5.10, compute a representative set of ring epimorphisms $\Omega = \{ \pi : R \to M_n(K) \}$ corresponding to the maximal ideals of $R$. Take $\pi \in \Omega$ and set $M := \ker \pi$. Test if $M^\ast = M$. If so then apply the algorithm of Proposition 5.7 to construct an effective isomorphism $\varphi : (M_n(K), \ast) \to \text{Adj}(d)$. Add $\varphi$ to $\Gamma$ and continue. Otherwise, find $\pi' \in \Omega$ where $\ker \pi' = M^\ast$. Then remove $\pi'$ from $\Omega$ and define $\gamma : R \to (M_n(K) \oplus M_n(K), \ast)$ by $r\gamma := (r\pi, (r^\ast \pi')^\ast)$. Add $\gamma$ to $\Gamma$ and continue.

Correctness. By Theorem 5.11, Proposition 5.7, and Theorem 5.10 the algorithm returns correctly.

Timing. The number of operations is dominated by the algorithm for Theorem 5.10.

5.4. Finding self-adjoint frames. Let $(R, \ast)$ be a finite $\ast$-ring. We outline how to find a self-adjoint frame of $\mathfrak{S}(R, \ast) = \{ r \in R : r^\ast = r \}$. To do this we require the following lemma:

Lemma 5.13 (Lifting idempotents). Suppose that $e \in R$ such that $e^2 - e \in \text{rad} R$ and $(e^2 - e)^n = 0$ for some $n \in \mathbb{Z}$. Setting

$$e := e^n \sum_{j=0}^{n-1} \binom{2n - 1}{j} e^{n-1-j} (1 - e)^j$$

it follows that:

(i) $e^2 = e$,

(ii) $e \equiv e \pmod{\text{rad} R}$.
(iii) $1 - e = 1 - \hat{e}$, and
(iv) If $*$ is an involution on $R$ and $e^* = e$ then $\hat{e}^* = \hat{e}$.

Proof. (i) through (iii) can be verified directly, compare [2, (6.7)]. For (iv) notice that $\hat{e}$ is a polynomial in $\mathbb{Z}[e]$. As $1^* = 1$ and $e^* = e$ it follows that $\hat{e}^* = \hat{e}$. □

Proposition 5.15. (i) There is a deterministic polynomial-time algorithm which, given $\text{Adj}(d)$ for a nondegenerate symmetric, alternating, or Hermitian form $d : K^n \times K^n \to K$, returns a self-adjoint frame of $\text{Adj}(d)$ of maximum possible size.

(ii) If $(M_n(K) \oplus M_n(K), \bullet)$ a simple $*$-ring with a standard exchange involution, then $\mathcal{E} = \{(E_{ii}, E_{ii}) : 1 \leq i \leq n\}$ is a self-adjoint frame of $(M_n(K) \oplus M_n(K), \bullet)$ of maximum possible size.

Proof. (i). Algorithm. If $d$ is alternating, compute a hyperbolic basis $\mathcal{X}$ for $d$. If $d$ is symmetric (and non-alternating if $K$ has characteristic 2) or Hermitian, then compute an orthogonal basis $\mathcal{X}$ for $d$. Return $\mathcal{E} = \{(x : x \in \mathcal{X})\}$.

Correctness. By Corollary 5.12 we know that the set of frames of $\text{Sym}(d)$ is in bijection with the fully refined $\perp$-decompositions of $d$. As $d$ is a classical form the fully refined $\perp$-decomposition of $d$ are parameterized by standard bases; i.e., a bases $\mathcal{X}$ of $d$ such that for each $x \in \mathcal{X}$ there is a unique $y \in \mathcal{X}$ such that $d(x, y) \neq 0$. If $d$ is alternating, this makes $\mathcal{X}$ a hyperbolic basis, and any two hyperbolic bases of $d$ have the same size $2m$, where $m$ is the Witt index of $d$ (and also the size of a maximal $\perp$-decomposition of $d$). If $d$ is symmetric not in characteristic 2, or Hermitian in any characteristic, then $d$ has an orthogonal basis. The resulting $\perp$-decomposition of $d$ has maximum possible size. Finally, if $K$ has characteristic 2 and $d$ is symmetric but non-alternating, then $d$ has an orthogonal basis, and that produces a $\perp$-decomposition of maximum possible size.

Timing. Finding a standard basis of $d$ can be done by standard linear algebra at a cost of $O(n^3)$ operations in $K$.

(ii). Fix $1 \leq i \leq n$. Clearly $(E_{ii}, E_{ii})^* = (E_{ii}, E_{ii})^t = (E_{ii}, E_{ii}) = (E_{ii}, E_{ii})^2 = (E_{ii}, E_{ii})$ is a self-adjoint idempotent. The proper idempotents of $(E_{ii}, E_{ii})$ are parameterized by standard bases; i.e., each $\mathcal{X}$ of $d$ such that for each $x \in \mathcal{X}$ there is a unique $y \in \mathcal{X}$ such that $d(x, y) \neq 0$. If $d$ is alternating, this makes $\mathcal{X}$ a hyperbolic basis, and any two hyperbolic bases of $d$ have the same size $2m$, where $m$ is the Witt index of $d$ (and also the size of a maximal $\perp$-decomposition of $d$). If $d$ is symmetric not in characteristic 2, or Hermitian in any characteristic, then $d$ has an orthogonal basis. The resulting $\perp$-decomposition of $d$ has maximum possible size. Finally, if $K$ has characteristic 2 and $d$ is symmetric but non-alternating, then $d$ has an orthogonal basis, and that produces a $\perp$-decomposition of maximum possible size.

Theorem 5.16. There is a Las Vegas polynomial-time algorithm which, given a $*$-ring $(R, *)$ with $R \subseteq \text{End}V$, $V$ an abelian $p$-group, returns a self-adjoint frame of $(R, *)$ of maximum possible size.

Proof. Algorithm. Use the algorithm for Corollary 5.12 to compute a set $\Gamma$ of $*$-epimorphisms onto simple $*$-algebras, one for each maximal $*$-ideal of $(R, *)$. For each $\gamma : (R, *) \to (T, *) \in \Gamma$, use the algorithm for Proposition 5.15 to compute a self-adjoint frame $\mathcal{E}_\gamma$ of $(T, *)$ of maximum possible size. Use the algorithm for Corollary 5.12 (iii) to pullback $\mathcal{E}_\gamma$ to a set

$$\mathcal{F}_\gamma = \{f \in R : f^2 \equiv f \ (\text{mod } pR), f^* \equiv f \ (\text{mod } pR), \gamma \in \Gamma\},$$

with $\mathcal{F}_\gamma \equiv \mathcal{E}_\gamma$. Apply [5,14] to the members of $\mathcal{F}_\gamma$ to create $\mathcal{G} = \{\hat{f} : f \in \mathcal{F}_\gamma, \gamma \in \Gamma\}$. Return $\mathcal{G}$.

Correctness. Evidently, $\mathcal{E} = \bigsqcup_{\gamma \in \Gamma} \mathcal{E}_\gamma$ is a self-adjoint frame of $(R/J(R), *)$ of maximum possible size. The pullback $\mathcal{F} := \bigsqcup_{\gamma \in \Gamma} \mathcal{F}_\gamma$ consists of self-adjoint elements
of \((R, *)\) for which \(\mathcal{F} \gamma = \mathcal{E}\) and the two sets are in bijection. By Lemma 5.13 the return \(\mathcal{G}\) is a self-adjoint frame of \((R, *)\) of maximum possible size.

**Timing.** The algorithm for Corollary 5.12 uses \(O(\text{rank}^3 V)\) operations in \(\mathbb{Z}_{p^e}\). Fix \(\gamma : (R, *) \to (T_\gamma, *) \in \Gamma\) with \(T_\gamma = \text{End}_K W_\gamma\). Proposition 5.13 uses \(O(\text{rank}^3 W_\gamma)\) operations in \(K_\gamma\); thus, \(O(\text{log}^3 W_\gamma)\) operations in \(\mathbb{Z}_p\). Since \(\sum_{\gamma \in \Gamma} \text{rank} W_\gamma\) is at most \(\text{rank} V\), it follows that this stage takes at most \(O(\text{log}^3 |V|)\) operations in \(\mathbb{Z}_p\).

The algorithms for Lemma 5.11 uses \(O(\text{rank}^3 T_\gamma)\) operations in \(\mathbb{Z}_{p^e}\). Since the bases computed in Lemma 5.11 can be reused for each application with respect to a fixed \(\gamma\), it follows that the total number of operations in \(\mathbb{Z}_{p^e}\) uses

\[
O \left( \sum_{\gamma \in \Gamma} \text{rank}^3 T_\gamma \right) = O \left( \sum_{\gamma \in \Gamma} \text{rank}^6 W_\gamma \right) = O(\text{log}^6 |V|)
\]

operations in \(\mathbb{Z}_{p^e}\). \(\square\)

6. Proof of Theorem 6.1

**Proof of Theorem 6.1**

Algorithm. Given a finite \(p\)-group \(P\) of class 2, compute bases for \(P/Z(P)\) and \(P'\) and compute a structure constant representation of \(b := \text{Bi}(P)\) (which is straightforward from the definitions in Section 3.1 and (2.3)).

Next, compute a basis for \(\text{Adj}(b)\) (Section 4.3). Apply Theorem 5.10 to find a self-adjoint frame \(\mathcal{E}\) of \(\text{Adj}(b)\) of maximum possible size. Induce a fully refined \(\perp\)-decomposition \(\mathcal{V} = \{(P/Z(P))e : e \in \mathcal{E}\}\) of \(b\) (cf. Corollary 4.5).

Apply Corollary 3.12 to produce a fully refined central decomposition of \(P\).

Correctness. This follows from Corollary 3.12 Corollary 4.3 and Theorem 5.10.

Timing. Since rank \(\text{Adj}(b) \leq \log^2 p |P : Z(P)|^2 \leq \log^2 p |P : P'|\), the total number of operations in \(\mathbb{Z}_{p^e}\) lies in \(O(\text{log}^6 |P : P'|)\).

Deterministic version Suppose that \(p\) is small \((p \leq \text{log}^e |P|\) for some constant \(c\)). Here, the Las Vegas method of [16] can be replaced by the deterministic methods of [27] in the algorithm of Theorem 5.10. Consequently, every Las Vegas algorithm is replaced by a deterministic algorithm. \(\square\)

6.1. Bottlenecks. The main bottleneck in practice is computing generators for \(\text{Adj}(\text{Bi}(P))\) for a given \(p\)-group \(P\). Examples carried with in collaboration with P. A. Brooksbank [6, 7] show that with a group of size \(p^{40}\), for \(p \in \{5, 7, 11\}\), a conventional laptop used roughly 5 seconds of real-time to compute generators for \(\text{Adj}(\text{Bi}(P))\) and only milliseconds to determine the \(\ast\)-ring structure of \(\text{Adj}(\text{Bi}(P))\). Sometimes this occurs because the rank of \(\text{Adj}(\text{Bi}(P))\) can be small as compared to the rank of \(P\). However, examples of groups of order \(p^{196}\) with intentionally large adjoint \(\ast\)-rings with radicals and multiple \(\ast\)-simple factors still spend most of the time computing generators for \(\text{Adj}(\text{Bi}(P))\), roughly 1 hour as compared to the 1 minute spent in identify the ring structure. For details see [7].

7. Related results

We summarize some of the related applications of the algorithm and methods for Theorem 6.1.
7.1. Central decompositions of Lie rings. There is related problem of central
decomposition of nilpotent Lie ring \( L \) of class 2; see [3, p.608-609]. Though we
do not require \( L \) be an algebra over a field, we assume that multiplication in \( L \)
is \( K \)-bilinear for some commutative ring \( K \) (not necessarily finite or of positive
characteristic) for which computation is feasible either in polynomial-time or toler-
able in practice, so we call \( L \) a Lie \( K \)-algebra. \( L \) should be specified by reasonable
means, for instance, generated by matrices under the usual commutator bracket,
or given with a basis and structure constants.

**Theorem 7.1.** Suppose that \( K \) is a commutative local ring with an oracle to factor
polynomials in \( K[x] \). Then, there is a Las Vegas polynomial-time algorithm which,
given a finite rank nilpotent Lie \( K \)-algebra of class 2, returns a central decomposition
of \( L \) of maximum size.

**Proof.** Algorithm. Define \( Bi(L) : L/Z(L) \times L/Z(L) \to [L, L] \) by \( Bi(L)(Z(L) +
x, Z(L) + y) := [x, y] \), for all \( x, y \in L \). Compute \( Adj(Bi(L)) \) and use Theorem 5.16
to find a self-adjoint frame \( \mathcal{E} \) of \( Bi(L) \) (which requires the polynomial factorization
oracle [27, Section 4.5]). Pullback the decomposition to \( L \) and apply the algorithms
for Proposition 3.11 and Corollary 3.12 using \( \Phi(L) := J(K)L + [L, L] \) in the rôle of
\( \Phi(P) \).

Correctness. The proof is the same as Theorem 1.1

Timing. The overall number of operations spent in computing \( Adj(b) \) and in
Theorem 5.16 which both lie in \( O(\text{rank}^6 L) \). \( \square \)

**Remark 7.2.** The practicality of Theorem 7.1 depends on the practicality of the
oracle to factor polynomials and working in \( K \). Over \( \mathbb{Q} \), factoring polynomials is as
difficult as factoring integers and therefore not a polynomial-time process. However,
in practice that “glitch” is often of little distress.

7.2. Determining the types of centrally indecomposables.

**Theorem 7.3.** A \( p \)-group \( P \) of class 2 is centrally indecomposable if, and only
if, \( Z(P) \leq \Phi(P) \) and \( Adj(Bi(P))/J(Adj(Bi(P))) \) is \( * \)-isomorphic to one of the
following:

1. Orthogonal type: \( GF(p^n) \) with identity involution,
2. Unitary type: \( GF(p^n) \), with field involution of order 2,
3. Exchange type: \( GF(p^n) \oplus GF(p^n) \) with involution \( (x, y)^* := (y, x) \), for all
   \( x, y \in GF(p^n) \); or
4. Symplectic type: \( M_2(GF(p^n)) \) with involution

\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}^* :=
\begin{pmatrix}
d & -b \\
-c & a
\end{pmatrix}, \quad \forall a, b, c, d \in GF(p^n).
\]

**Proof.** By Theorem 5.9 it remains to show that \( (S, *) := Adj(Bi(P))/J(Adj(Bi(P))) \)
is one of the algebras listed. By Corollary 4.5 we know \( (S, *) \) has no proper self-
adjoint-primitive idempotents. This makes \( (S, *) \) a \( * \)-simple ring.

If \( S \) is classical, then by Proposition 5.7 it follows that \( S \) is \( * \)-isomorphic to \( Adj(d) \)
for a nondegenerate symmetric, alternating, or Hermitian form \( d : K^n \times K^n \to K \).
By Corollary 4.5 \( d \) must be \( 1 \)-indecomposable, so \( n = 1 \) if \( d \) is symmetric or
Hermitian, or \( n = 2 \) if \( d \) is alternating. This handles cases (1), (2), and (4).

If \( S \) has an exchange involution, then by Proposition 6.15(ii), \( S \) must be \( * \)-ring
isomorphic to the \( * \)-ring in (3). \( \square \)
Remark 7.5. Examples of the orthogonal, exchange, and symplectic type were first given in [33, Section 7]. Appendix A includes new examples including the first examples of unitary type.

Corollary 7.6. There is polynomial-time algorithm which, given a finite centrally indecomposable $p$-group of class 2, returns the type of the group as listed in Theorem 7.3.

Proof. This is immediate from Theorem 5.10, Proposition 5.7, and Theorem 7.3. □

7.3. Testing indecomposability. Suppose that we are only interested in testing if a $p$-group $P$ of class 2 is centrally indecomposable. By Theorem 7.3, the key step is to determine that $\text{Adj}(b)/J(\text{Adj}(b))$ is one of the four algebras in that list. That process is easier in the present framework as it requires that there be at most 2 isomorphism types of simple modules in the composition series of $V$ as an $\text{Adj}(b)$-module. Furthermore, the simple modules have dimension 1 or 2 when viewed over the correct field, i.e. $Z(\text{Adj}(b)/J(\text{Adj}(b)))$. This can be determined using the absolute irreducibility test of the MeatAxe [15], thus reducing the time in those stages to $O(\log^4 |V|)$-time. Unfortunately, the bottleneck remains in computing generators for $\text{Adj}(b)$, which still requires $O(\log^5 |V|)$-time.

7.4. Finding orbits of central decompositions. In [33], the action of the automorphism group of a $p$-group $P$ of class 2 and exponent $p$ was studied. Though not presented in detail, it is clear that the methods here can be used to find a representative fully refined central decomposition for each $C_{\text{Aut}P}(Z(P))$-orbit as described in [33, Corollary 5.23.(iii)]. The necessary step is to choose an orthogonal basis in Proposition 5.15 with the desired address in the sense of [33, Definition 5.1].

7.5. Finding some new characteristic and fully invariant subgroups. We now show how the $*$-ring $\text{Adj}(\text{Bi}(P))$ can be used to uncover new characteristic and fully invariant subgroups of $P$.

Recall that $\text{Adj}(\text{Bi}(P))$ is a subring of $\text{End} V \times (\text{End} V)^{op}$ where $V := P/Z(P)$. Thus, $\text{Adj}(\text{Bi}(P))$ acts on $V$ by $v(f, g) := vf$, for all $v \in V$ and all $(f, g) \in \text{Adj}(\text{Bi}(P))$. If $I$ is a right ideal of $\text{Adj}(\text{Bi}(P))$ then $VI$ is a submodule of $V$. Recall that an ideal $I$ of a $*$-ring $R$ is $*$-characteristic ($*$-fully invariant) if $I\varphi = I$ for all $*$-ring automorphisms (endomorphisms) of $R$. We prove:

Theorem 7.7. For a $p$-group $P$ of class 2,

(7.8) $L := \{ Z(P) \leq L \leq P : L/Z(P) = (P/Z(P))I, I$ $*$-characteristic in $\text{Adj}(\text{Bi}(P)) \}$

is a lattice of characteristic subgroups of $P$.

Proof. $\text{Aut} P$ acts on $\text{Adj}(\text{Bi}(P))$ via

(7.9) $(f, g)^\varphi := (\varphi^{-1}_V)^\varphi f \varphi_v, \varphi_v g \varphi^{-1}_V), \forall (f, g) \in \text{Adj}(\text{Bi}(P)), \varphi \in \text{Aut} P.$

That action commutes with the $*$ involution on $\text{Adj}(\text{Bi}(P))$; so every $*$-characteristic $*$-ideal $I$ of $\text{Adj}(\text{Bi}(P))$ is acted on by $\text{Aut} P$. Thus, $0 \leq VI \leq V$ is an $\text{Aut} P$-submodule of $V = P/Z(P)$. As $Z(P)$ is characteristic in $P$, pulling back to $P$ proves our claim. □
Remark 7.10. There is a bilinear map $\text{Bi}(P, P')$ from $P/P' \times P/P' \to P'$ defined analogously to $\text{Bi}(P)$. This bilinear map may be degenerate; thus, $\text{Adj}(\text{Bi}(P, P'))$ is not necessarily a $*$-ring. However, because $P'$ is fully invariant, it follows that

$$L := \{P' \leq L \leq P : L/P' = (P/P')I, \text{I fully invariant in } \text{Adj}(\text{Bi}(P, P'))\}$$

is a lattice of fully invariant subgroups of $P$.

Using the radical and semisimple structure of $\text{Adj}(\text{Bi}(P))$ it is easy to identify various specific $*$-characteristic and $*$-fully invariant $*$-ideals of $\text{Adj}(\text{Bi}(P))$.

Example 7.12. Given a $*$-ring $(R, *)$:

(i) if $J$ is the Jacobson radical of $R$, then $\{J^i : i \in \mathbb{Z}^+\}$ is a flag of $*$-fully invariant $*$-ideals of $(R, *)$; and

(ii) the intersection of all maximal $*$-ideals with $*$-ring isomorphic quotients is a $*$-fully invariant $*$-ideal of $(R, *)$.

Corollary 7.13. There are polynomial-time algorithms which, given a $p$-group $P$ of class 2, return the characteristic and fully invariant subgroups of $P$ resulting from Example 7.12 and Theorem 7.7 or Remark 7.10.

Proof. This is an obvious application of the $*$-ring structure algorithms given in Section 8.

\[\Box\]

8. Central products of general groups

We deviate from our focus on $p$-groups of class 2 to address some of the situation for central decompositions of general finite groups.

8.1. Central products and isoclinism. We apply an equivalence relation on groups introduced by P. Hall [11] which is compatible with central products. This allows for a partial generalization of the concepts in Section 8.

The proof of Theorem 11 concentrates on the bilinear map $\text{Bi}(P) : P/Z(P) \times P/Z(P) \to P'$. Evidently nonisomorphic $p$-groups can have equivalent bilinear maps. Equivalence of bilinear maps $b : V \times V \to W$ and $b' : V' \times V' \to W'$ is defined by pairs of linear maps $(f : V \to V', \hat{f} : W \to W')$ such that:

$$b'(u, v) \hat{f} = b(u, v)\hat{f}, \quad \forall u, v \in V.$$

More generally, an isoclinism [11] of groups $G$ and $H$ is a pair $(\alpha : G/Z(G) \to H/Z(H), \hat{\alpha} : G' \to H')$ of group isomorphisms such that

$$[Z(G)\alpha x, Z(G)\alpha y] = [x, y]\hat{\alpha}, \quad \forall x, y \in G.$$  

Isomorphic groups are immediately isoclinic, but the converse is false (abelian groups are isoclinic to the trivial group). Clearly, $\text{Bi}(P)$ and $\text{Adj}(\text{Bi}(P))$ are group isoclinism invariants of $P$. Moreover, if $G$ and $H$ are general groups and $K$ is a central decomposition of $G$, then

$$K\alpha := \{Z(H) \leq J \leq H : \exists K \in K, J/Z(H) = KZ(G)/Z(G)\alpha\}$$

is a central decomposition of $H$. Call a central decomposition $\mathcal{H}$ of $G$ a $Z(G)$-central decomposition if $\mathcal{H} = \mathcal{H}Z(G)$. Thus we have proved:

Proposition 8.4. An isoclinism from a group $G$ to a group $H$ induces a bijection from the set of $Z(G)$-central decompositions of $G$ and the set of $Z(H)$-central decompositions of $H$. In particular, if $G$ is centrally indecomposable, then every central decomposition of $H$ has at most one nonabelian member.
Using group isoclinism we can generalize a conjecture made in [33].
Examples such as $D_8 \circ D_8 \cong Q_8 \circ Q_8$ and the similar problem for odd extraspecial groups of exponent $p^2$ (see [9, Theorem 5.5.2]) demonstrate that the group isomorphism classes of a central decomposition of maximum possible size need not be the same. However, we ask:

Is the multiset of group isoclinism types of a central decomposition of maximum possible size a group isoclinism invariant?

We conjecture that this is true for $p$-groups of class 2. If so, then it is probably true for all groups; in particular, the problem for nilpotent groups of larger class and groups with no center can benefit from the uniqueness afforded by the Krull-Remak-Schmidt theorem; compare [32, Section 4.3.4].

8.2. Idempotents in central products of general groups. Let $G$ be any group. We can define $c := c(G) : G/Z(G) \times G/Z(G) \to G'$ by

$$
c(Z(G)x, Z(G)y) := [x, y], \quad \forall x, y \in G.
$$

We also have:

$$
\text{Adj}(c) := \{(f, g) \in \text{End } G/Z(G) \times (\text{End } G/Z(G))^\text{op} : c((Z(G)xf, Z(G)y) = c(Z(G)x, Z(G)y)g), \forall x, y \in G\}.
$$

Obviously, $\text{Adj}(c)$ is closed to products and has an anti-automorphism $*: (f, g) \mapsto (g, f)$ of order 2. However, unlike $\text{Adj}(\text{Bi}(P))$, $\text{Adj}(G)$ need not be a ring since we cannot generally add endomorphisms of $G/Z(G)$. Nonetheless, it follows that:

**Proposition 8.7.** The set of $Z(G)$-central decompositions of $G$ is in bijection with the set of sets of self-adjoint idempotents of $\text{Adj}(c(G))$.

**Proof.** The proof is the same as that of Proposition 3.8 and Corollary 4.5. □

8.3. Finding central products of general groups. To find central decompositions of groups $G$ which are either non-nilpotent or nilpotent of class greater than 2, it is may be possible to begin by finding direct decompositions of $G/Z(G)$, and then reduce to central decomposition of $G$. The first polynomial-time algorithm to find a direct product decomposition of a finite group appeared in [32, Chapter IV] along with the algorithms of Theorem 1.1 [32, Chapter III]. A preliminary inspection supports the conjecture that a combination of these two results will produce a polynomial-time algorithm to find fully refined central decompositions of arbitrary finite groups.

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Appendix A. Examples of centrally indecomposable groups

We give examples which demonstrate some of the important aspects of the algorithm of Theorem A.1. As evidence that all cases considered here can occur, we give\( p\)-groups which are centrally indecomposable of each of the types listed in Theorem A.3. Furthermore, our proofs apply the techniques of Theorem 1.1 in a symbolic fashion illustrating how the methods can be used beyond a computer.

**Example A.1** (A centrally indecomposable group of orthogonal type).
\[
\text{Or}_p = \langle a, b, c \mid a^p, b^p, c^p, [a, b]^p, [a, c]^p, [b, c]^p, \text{ class 2} \rangle.
\]
is a special group of order \( p^6 \) and is centrally indecomposable of orthogonal type.

**Proof.** Let \( P := \text{Or}_p \). Clearly, \( P^p \leq P' = Z(P) \) so \( P \) is a special \( p \)-group of order \( p^6 \) and rank 3. Therefore \( P/Z(P) \cong Z^3_p \) and \( P' \cong Z^3_p \). So \( \text{Bi}(P) : Z^3_p \times Z^3_p \rightarrow Z^3_p \). Using \( \{Z(P)s, Z(P)t, Z(P)u\} \) and \( \{x := [a, b], y := [a, c], z := [b, c]\} \) as ordered bases for \( P/Z(P) \) and \( P' \) respectively, it is evident from (A.2) that \( \text{Bi}(P) \) is defined by \( \text{Bi}(P)(u, v) = uBu^t \) for all \( u, v \in Z^3_p \), where:

\[
B := \begin{bmatrix} 0 & x & y \\
-x & 0 & z \\
-y & -z & 0 \end{bmatrix}.
\]

Computing \( \text{Adj}(B) \) as in Section 4.3 (which is easily done with symbolic computation on an example of this size; compare [33, Lemma 7.1]) we find

\[
\text{Adj}(B) = \{(\alpha I_3, \alpha I_3) \in M_3(Z_p) \times M_3(Z_p) : \alpha \in Z_p\} = \text{Sym}(b),
\]
which is clearly \( * \)-isomorphic to \( Z_p \) with identity involution. By Theorem A.3, \( P \) is centrally indecomposable of orthogonal type. \( \square \)

**Example A.5** (A centrally indecomposable group of exchange type).
\[
E_p := \langle a, b, c, d \mid a^p, b^p, c^p, d^p, [a, c]^p, [a, d]^p, [b, c]^p, [b, d]^p, [a, b], [c, d], \text{ class 2} \rangle
\]
is a special group of order \( p^8 \) and is centrally indecomposable of exchange type.

**Proof.** \( \text{Bi}(E_p) \) is bilinear map \( Z^4_p \times Z^4_p \rightarrow Z^4_p \). With respect to the bases

\[
\{Z(E_p)a, Z(E_p)b, Z(E_p)c, Z(E_p)d\}
\]

and

\[
\{x := [a, c], y := [a, d], z := [b, c], w := [b, d]\},
\]

\( \text{Bi}(E_p) \) is defined by

\[
B := \begin{bmatrix} 0 & 0 & x & y \\
0 & 0 & z & w \\
-x & -z & 0 & 0 \\
-y & -w & 0 & 0 \end{bmatrix}.
\]

Evidently,

\[
\text{Adj}(B) = \left\{ \left( \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \otimes I_2, \begin{bmatrix} \beta & 0 \\ 0 & \alpha \end{bmatrix} \otimes I_2 \right) : \alpha, \beta \in Z_p \right\}.
\]

This is \( * \)-ring isomorphic to \( (Z_p \oplus Z_p, (\alpha, \beta)^* = (\beta, \alpha)) \). Thus, \( E_p \) is a centrally indecomposable group of exchange type. \( \square \)
Example A.9 (A centrally indecomposable group of unitary type). Let $p$ be odd and $\omega \in \mathbb{Z}$ be a non-square modulo $p$.

$$U_p := \langle a, b, c, d, e, f \mid a^p, b^p, c^p, d^p, e^p, f^p, [a, b]^p, [a, c]^p, [b, c]^p, [a, b]^\omega [d, e], [a, c]^\omega [f, e], [a, e], [b, d], [b, e], [b, f], [c, d], [c, e], [c, f], [c, e], \text{class } 2 \rangle \tag{A.10}$$

is a special group of order $p^{12}$ and is centrally indecomposable of unitary type.

**Proof.** $\text{Bi}(U_p)$ is bilinear map $\mathbb{Z}_p^6 \times \mathbb{Z}_p^6 \rightarrow \mathbb{Z}_p^4$. With respect to the bases

$$\{Z(U_p)a, Z(U_p)b, Z(U_p)c, Z(U_p)d, Z(U_p)e, Z(U_p)f\}$$

and $\{x := [a, b], y := [a, c], z := [b, c], u := [a, d]\}$, $\text{Bi}(U_p)$ is defined by

$$B := \begin{bmatrix}
0 & x & y & u & 0 & 0 \\
-x & 0 & z & 0 & 0 & 0 \\
-y & -z & 0 & 0 & 0 & 0 \\
-u & 0 & 0 & 0 & -\omega x & -\omega y \\
0 & 0 & 0 & \omega x & 0 & -\omega z \\
0 & 0 & 0 & \omega y & \omega z & 0
\end{bmatrix}. \tag{A.11}$$

By computing we find:

$$\text{Adj}(B) = \left\{ \left( \begin{array}{cc}
\alpha & \beta \\
\omega \beta & \alpha
\end{array} \right) \otimes I_3, \left( \begin{array}{cc}
\alpha & -\beta \\
-\omega \beta & \alpha
\end{array} \right) \otimes I_3 \right\} : \alpha, \beta \in \mathbb{Z}_p. \tag{A.12}$$

This is $*$-ring isomorphic to $GF(p^2) = \mathbb{Z}_p[x]/(x^2 - \omega)$ with field involution $\sqrt{\omega} \mapsto -\sqrt{\omega}$. Thus, $U_p$ is a centrally indecomposable group of unitary type. \hfill \Box

Example A.13 (A centrally indecomposable group of exchange type).

$$p_+^{1+2} := \langle a, b \mid a^p, b^p, [a, b]^p, \text{class } 2 \rangle \tag{A.14}$$

is an extraspecial group of order $p^3$ and is centrally indecomposable of symplectic type.

**Proof.** $\text{Bi}(p_+^{1+2})$ is bilinear map $\mathbb{Z}_p^2 \times \mathbb{Z}_p^4 \rightarrow \mathbb{Z}_p^4$. With respect to the bases

$$\{Z(E_p)a, Z(E_p)b\}$$

and $\{x := [a, b]\}$, $\text{Bi}(p_+^{1+2})$ is defined by

$$B := \begin{bmatrix}
0 & x \\
-x & 0
\end{bmatrix}. \tag{A.15}$$

Clearly,

$$\text{Adj}(B) = \left\{ \left( \begin{array}{cc}
\alpha & \beta \\
\gamma & \delta
\end{array} \right), \left( \begin{array}{cc}
\delta & -\beta \\
-\gamma & \alpha
\end{array} \right) \right\} : \alpha, \beta, \gamma, \delta \in \mathbb{Z}_p. \tag{A.16}$$

This is $*$-ring isomorphic to $M_2(\mathbb{Z}_p)$ with symplectic involution. Thus, $p_+^{1+2}$ is a centrally indecomposable group of symplectic type. \hfill \Box
APPENDIX B. EXAMPLES OF CENTRALLY DECOMPOSABLE GROUPS

We now demonstrate how central products can be used to characterize $p$-groups. We conclude by reproving an example of Taft and generalizing it to an infinite family.

The most common central product is one where a group is created as a central product of a single group $G$ with center identified in a natural fashion:

$$G \circ G := G \times G/\langle (x, y) \in Z(G \times G) : xy = 1 \rangle.$$  

A subtle generalization is to include exponents in the identification:

**Definition B.1.** Fix $(a_1, \ldots, a_n) \in \mathbb{Z}^n$. For a group $G$ define:

$$(B.2)\quad G^{\alpha(a_1,\ldots,a_n)} = G^n/\langle (x_1, \ldots, x_n) \in Z(G^n) : x_1^{a_1} \cdots x_n^{a_n} = 1 \rangle.$$  

Evidently, $G^{\alpha(a_1,\ldots,a_n)}$ is an $n$-fold central product of $G$ but where the centers are identified according to the given exponents.

**Example B.3.** Define

$$R_p := \langle a, b, c, d, e, f \mid a^p, b^p, c^p, d^p, e^p, f^p,$$

$$[a, b][a, e]^{-2}, [a, c][a, f]^{-2}, [a, d], [a, e]^p, [a, f]^p,$$

$$[b, c][b, f]^{-2}, [b, d][a, e], [b, e], [b, f]^p,$$

$$[c, d][a, f], [c, e][b, f], [c, f],$$

$$[d, e][a, e]^{-2}, [d, f][a, f]^{-2},$$

$$[e, f][b, f]^{-2}, \text{class } 2 \rangle.$$  

Then $|R_p| = p^9$, $R_p' \cong \mathbb{Z}_3^3$, $R_p/\mathcal{R}_p \cong \mathbb{Z}_6^3$. Furthermore,

(i) $R_2$ is a special group and centrally indecomposable of symplectic type,

(ii) $R_3 \cong \text{Or}_3 \times \mathbb{Z}_3^3$ and is almost special (i.e.: $R_3^3 = \Phi(R_3) \leq Z(R_3)$ and $Z(R_3)$ is elementary abelian), and

(iii) if $p > 3$ then $R_p \cong \text{Or}^\circ_p(1,3)$. Furthermore, $R_p \cong \text{Or}_p \circ \text{Or}_p$ if, and only if, $3$ is a square modulo $p$.

**Proof.** Set $x := [a, e]$, $y := [a, f]$, and $z := [b, f]$. Evidently $R_p' = \langle x, y, z \rangle \cong \mathbb{Z}_3^3$ and $R_p$ has order $p^9$.

When $p \neq 3$, $R_p' = Z(R_p)$. Furthermore,

$$(B.5)\quad R_p/Z(R_p) = \langle (Z(R_p)a, Z(R_p)b, Z(R_p)c, Z(R_p)d, Z(R_p)e, Z(R_p)f) \cong \mathbb{Z}_6^6.$$  

With respect to the given generators, $b := \text{Bi}(R_p) : \mathbb{Z}_6^6 \times \mathbb{Z}_6^6 \to \mathbb{Z}_6^6$ (Section 3.1)) is defined by $b(u, v) = uBu^t$, for all $u, v \in \mathbb{Z}_p$, where:

$$(B.6)\quad B := \begin{bmatrix} 0 & 2x & 2y & 0 & x & y \\ -2x & 0 & 2z & -x & 0 & z \\ -2y & -2z & 0 & -y & -z & 0 \\ 0 & x & y & 0 & 2x & 2y \\ -x & 0 & z & -2x & 0 & 2z \\ -y & -z & 0 & -2y & -2z & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \otimes \begin{bmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{bmatrix}.$$  

(The tensor notation is the usual Kronecker product and we use it here to compress the data; for more on adjoints and tensors see [33, Section 7.2].) As $p \neq 3$, $D :=
Thus, computing \( \text{Adj}(B) \) as in Section 4.3 shows that

\[
\text{Adj}(B) = \{(A \otimes I_3, DA^t D^{-1} \otimes I_3) : A \in M_2(\mathbb{Z}_p)\}
\]

which is *-isomorphic to \( \text{Adj}(D) \). The bilinear map \( d : \mathbb{Z}_p^2 \times \mathbb{Z}_p^2 \to \mathbb{Z}_p \) given by \( d(u, v) := uDV^t \), for all \( u, v \in \mathbb{Z}_p^2 \), is a symmetric nondegenerate bilinear form. If \( p = 2 \), then \( d \) is also a nondegenerate alternating bilinear form of dimension 2, and \( d \) is \( \perp \)-indecomposable. Thus \( \text{Adj}(d) \cong \text{Adj}(D) \cong \text{Adj}(B) \) has no proper self-adjoint-primitive idempotents; that is, \( R_2 \) is centrally indecomposable of symplectic type; see Theorem 7.3.

(iii). If \( p > 3 \) then \( d \) has an orthogonal basis, for instance \( \{(1, -1), (1, 1)\} \). This produces the following self-adjoint frame for \( \text{Adj}(d) \):

\[
\begin{bmatrix}
\frac{1}{2} & -\frac{1}{2} \\
-\frac{1}{2} & \frac{1}{2}
\end{bmatrix}, \begin{bmatrix}
\frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2}
\end{bmatrix}
\]

Thus, \( E := \{e \otimes I_3, f \otimes I_3\} \) is a self-adjoint frame of \( \text{Adj}(B) \) and decomposes \( \mathbb{Z}_p^6 \) into:

\[
E := \langle (1, 0, 0, -1, 0, 0), (0, 1, 0, 0, -1, 0), (0, 0, 1, 0, 0, -1) \rangle,
\]

and \( b(E, F) = 0 \); thus, \( E, F \) is a fully refined \( \perp \)-decomposition of \( b \) of maximum possible length. Pulling back to subgroups of \( R_p \) we have:

\[
H_{1, -1} = \langle ad^{-1}, be^{-1}, cf^{-1} \rangle \leq R_p \text{ and } H_{1, 1} = \langle ad, be, cf \rangle \leq R_p.
\]

So \( \{H_{1, -1}, H_{1, 1}\} \) is a fully refined central decomposition of \( R_p \). Indeed, if we change the basis of \( b \) so to the bases given for \( E \) and \( F \) we have:

\[
\begin{bmatrix}
2 & 0 \\
0 & 6
\end{bmatrix} \begin{bmatrix}
0 & x & y \\
-x & 0 & z \\
-y & z & 0
\end{bmatrix}.
\]

Let \( \tilde{x} := 2x = [a, b], \tilde{y} := 2y = [a, c] \), and \( \tilde{z} := 2z = [b, c] \). Thus,

\[
\begin{bmatrix}
1 & 0 \\
0 & 3
\end{bmatrix} \begin{bmatrix}
0 & \tilde{x} & \tilde{y} \\
-\tilde{x} & 0 & \tilde{z} \\
-\tilde{y} & -\tilde{z} & 0
\end{bmatrix}.
\]

Thus, it is clear that \( H_{1, -1} \) and \( H_{1, 1} \) are isomorphic to \( \text{Or}_p \) and furthermore, \( R_p = H_{1, -1} H_{1, 1} \cong \text{Or}_p^{\alpha(1, 3)} \). If \( 3 \equiv \alpha^{-2} (p) \), for some \( \alpha \in \mathbb{Z} \), then set:

\[
H_{(\alpha, \alpha)} := \langle a^\alpha, b^\alpha, e^\alpha, f^\alpha \rangle.
\]

Thus, \( R_p = H_{(1, -1)} H_{(\alpha, \alpha)} \cong \text{Or}_p^{\alpha(1, 1)} \).

(ii). If \( p = 3 \) we can compute \( Z(R_3) \) directly to verify the properties. However, an alternative approach is to use the related bilinear map \( Bi(R_3, R_3') : R_3/R_3' \times R_3/R_3' \to R_3 \). This produces a \( \mathbb{Z}_3 \)-bilinear map exactly as in (B.6). The only exception is that \( B \) is degenerate. Computing a basis for the radical of \( B \) can be done by computing a basis for the radical of \( D = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \) e.g. write \( D \) with respect
Proof. Let \( \text{given by (B.4) and (A.2), produces the following matrix defining} (B.19) \)
\[ B := \begin{bmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{bmatrix}. \]
Pulling back to subgroups of \( R_p \) we have the following central factors:
\[ (B.16) \quad H_{(1,0)} := \langle a, b, c \rangle \cong O_{r_3}, \quad \text{and} \quad H_{(1,1)} := \langle ad, be, cf \rangle R_3' \cong \mathbb{Z}_{3}^{6}. \]
So \( R_3 = H_{(1,0)}H_{(1,1)} \cong O_{r_3} \times \mathbb{Z}_{3}^{3}. \)

We now construct the example of C.Y. Tang \([30, \text{Section 6}]\) of a 2-group with fully refined central decompositions of different sizes. This demonstrates where the algorithm for Theorem 1.1 must properly select a central decomposition of maximum possible length.

Example B.17 (C.Y. Tang).
\[ \text{(B.18) } R_2 \circ O_{r_2} := \langle a, b, c, d, e, f \rangle \times \langle s, t, u \rangle / \langle [a, e][s, t]^{-1}, [a, f][s, u]^{-1}, [b, f][t, u]^{-1} \rangle \]
is isomorphic to \( O_{r_2} \circ O_{r_2} \circ O_{r_2} \). Yet \( R_2 \) and \( O_{r_2} \) are both centrally indecomposable.

We provide an alternative proof using the approach of Theorem 1.1.

**Proof.** Let \( P := R_2 \circ O_{r_2} \). Using the obvious bases of \( R_2 / Z(R_2) \times O_{r_2} / Z(O_{r_2}) \)
given by \([B.3] \text{ and } [A.2]\), produces the following matrix defining \( B(P): \)
\[ (B.19) \quad B := D \otimes \begin{bmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{bmatrix}, \quad D := \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \]
Thus, \( \text{Adj}(B) \cong \text{Adj}(D) \). The map \( d : \mathbb{Z}_{3}^{2} \times \mathbb{Z}_{2}^{2} \to \mathbb{Z}_{2} \) defined by \( d(u, v) = udv^4 \) is a nondegenerate symmetric bilinear form which has an orthonormal basis \( \{(0, 1, 1), (1, 1, 1), (1, 0, 1)\} \). Evidently this produces a \( \perp \)-decomposition of \( d \) (and \( b \)) of maximum possible length. The corresponding fully refined central decomposition of \( P \) has the following factors:
\[ (B.20) \quad H_{(0,1,1)} := \langle ds, et, fu \rangle \cong O_{r_2}, \]
\[ (B.21) \quad H_{(1,1,1)} := \langle ads, bet, cfu \rangle \cong O_{r_2}, \quad \text{and} \]
\[ (B.22) \quad H_{(1,0,1)} := \langle as, bt, cu \rangle \cong O_{r_2}. \]

**Remark B.23.** Our proof of Example B.17 can be applied to central products where \( O_{r_2} \) is replaced by any 2-group of orthogonal type. Asymptotically, there are \( 2^{2n^2/27+O(n^2)} \) such groups of order \( 2^n \); thus, there are infinite expanding families of examples of the type introduced by Tang.

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