Truncation of functional relations in the XXZ model.

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Abstract

The integrable XXZ model with a special open boundary condition is considered. We study Sklyanin transfer matrices after quantum group reduction in roots of unity. In this case Sklyanin transfer matrices satisfy a closed system of truncated functional equations. The algebraic reason for the truncation is found. The important role in proving of the result is performed by Zamolodchikov algebra introduced in the paper.

1 Introduction.

We consider the integrable XXZ model with a special open boundary condition. Its Hamiltonian

\[ H_{XXZ} = \sum_{n=1}^{N-1} \left[ \sigma_n^+ \sigma_{n+1}^- + \sigma_n^- \sigma_{n+1}^+ + \frac{cosh(\eta)}{2} \sigma_n^z \sigma_{n+1}^z + \frac{sinh(\eta)}{2} (\sigma_n^z - \sigma_{n+1}^z) \right]. \]
This Hamiltonian is invariant under quantum algebra $U_q(sl(2))$, whose generators $X$, $Y$ and $H$ satisfy the following commutation relations:

$$[H, X] = X, [H, Y] = -Y, [X, Y] = [2H]_q,$$

where the function $[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}$ and $q = e^\eta$. We have following representation for $X$, $Y$, and $H$ in the terms of Pauli matrices:

$$X = \sum_{n=1}^{N} q^{1/2} (\sigma_1^z \cdots + \sigma_{n-1}^z) \sigma_n^+ q^{-1/2} (\sigma_{n+1}^z \cdots + \sigma_N^z),$$

$$Y = \sum_{n=1}^{N} q^{-1/2} (\sigma_1^z \cdots + \sigma_{n-1}^z) \sigma_n^- q^{1/2} (\sigma_{n+1}^z \cdots + \sigma_N^z),$$

$$H = \sum_{n=1}^{N} \frac{\sigma_n^z}{2}.$$

We concentrate on the case where $q^{p+1} = -1$. It is known [1], [2] that if $q^{p+1} = -1$, then $X^{p+1} = 0$ and $Y^{p+1} = 0$; we can therefore consider the subfactor space $V_p = \text{Ker} X / \text{Im} X^p$.

The result of restricting the XXZ model on $V_p$ is the Minimal Model of Integrable Lattice Theory $LM(p, p + 1)$ [4], [5]. The thermodynamic limit is the ordinary Minimal Model of CFT $M(p, p + 1)$ [4] with the Virasoro central charge $c = 1 - \frac{6}{p(p+1)}$ [4].

The XXZ model is integrable [3], and as shown by Sklyanin [3], there exists a family of transfer matrices which commute with the Hamiltonian and between themselves. The Sklyanin transfer matrices are also $U_q(sl(2))$ invariant [6]. Therefore, they can also be restricted on $V_p$. As shown in [3], a result of this restriction is the truncation of the fusion functional relations [8] for the transfer matrices. This statement was proved in [3] using T-Q Baxter equation. In this work we prove this fact more directly and algebraically. Namely, we prove that if $q^{p+1} = -1$, then the Sklyanin transfer matrix $t_{p/2}(u)$ with the spin $j = p/2$ in the auxiliary space vanishes after the restriction on $V_p$ because

$$t_{p/2}(u) = X^p M + NX,$$  (1)
where $M$ and $N$ are some operators in the quantum space. In another words,

$$t_{p/2}(u) = 0 \pmod{\text{Ker} X / \text{Im} X^p}.$$ 

Then as it follows from Eq. (1) that the fusion functional relations are truncated and transformed to a system of functional equations that can be used to obtain the eigenvalues of the transfer matrices.

This paper is organized as follows.

In Sec. 2, we recall some basic formulas that are necessary for introducing the Sklyanin transfer matrices in Sec. 3.

In Sec. 4, we show how to obtain the operator $X$ from a Sklyanin monodromy matrix.

It will allow us then to obtain commutation relations between the generators of the quantum algebra $U_q(\hat{sl}(2))$ and the Sklyanin monodromy matrices in Sec. 5.

In Sec. 6, 7, 8, we express the $L$ operators in terms of the generators of the Zamolodchikov algebra and obtain commutation relations between these new operators and the generators of the quantum algebra $U_q(\hat{sl}(2))$.

In Sec. 9, we rewrite the Sklyanin transfer matrix in the new variables.

In Sec. 10, we introduce the operator $\text{ad} X$.

In Sec. 11, we write down some useful properties of $\text{ad} X$.

In Secs. 12 and 13 we prove (2) which is our main statement (the $\text{ad} X$ theorem).

\section{Yang–Baxter equation.}

The integrable structure of XXZ is confined in the well-known equation

$$R_{bm}^{an}(u-v)L_c^b(u)L_k^m(v) = L_m^a(v)L_b^u(u)R_{ck}^{bm}(u-v), \quad (2)$$

where $L$ is a monodromy matrix whose entries $L_n^m$ ($n, m = 1, \ldots, 2j + 1$) are operators acting in the so-called quantum space, which is a tensor product of $N$ two-dimensional linear spaces in our case. The dimension $2j + 1$ of the so-called auxiliary space can be different, strictly speaking we should write something like $(L_j)^m_n$ instead of $L_n^m$ and $(R_{j_1j_2})_{bm}^{an}$ instead of $R_{bm}^{an}$, but sometimes for simplicity we do not do that and assume that everything is
understood from the context. The $R_{j_1j_2}$-matrix acts in a tensor product of two auxiliary spaces and satisfies to the Yang–Baxter equation
\[ R_{an}^{bm}(u-v)R_{cp}^{mr}(u)R_{ks}^{mr}(v) = R_{mr}^{ap}(v)R_{bs}^{ar}(u)R_{ck}^{bm}(u-v) \]
and the unitarity relation
\[ (R_{j_1j_2})_{ck}^{an}(u)(R_{j_1j_2})_{bn}^{ck}(-u) = \phi_{j_1j_2}(u)\phi_{j_1j_2}(-u)\delta^a_b\delta^m_n. \]
The monodromy matrix $L$ can be realized through an N-uple product of $R$-matrixes.
\[ L(u) = R_N(u) \ldots R_1(u), \]
here a comultiplier $R_n$ stands for the $R$-matrix acting in a tenzor product of the auxiliary space and of the $n$-th comultiplier of the quantum space. We need the related matrix $\bar{L}^t(u)$,
\[ \bar{L}^t(u) = R_1(u) \ldots R_N(u) \sim L^{-1}(-u). \]
It is easy to show that
\[ \bar{L}^t_j(u) = \xi_{j}^N(-u)L_{-j}^{-1}(-u) \]
where $\xi_j(u) \equiv \phi_{1/2j}(u)\phi_{1/2j}(-u)$. Using this equality, it is possible to obtain the formulas
\[ R_{an}^{bm}(u-v)L_{ck}^{m}(v)L_{ck}^{k}(u) = L_{m}^{n}(u)L_{m}^{n}(v)R_{ck}^{bm}(u-v) \]
\[ L_{ck}^{a}(u)L_{ck}^{m}(v)R_{ck}^{cm}(u+v) = R_{cm}^{ak}(u+v)L_{ck}^{k}(v)L_{ck}^{k}(u) \]
In our case, the $R$-matrix
\[ R(u) = \begin{pmatrix} \sinh(u + ( \frac{1}{2} + \hat{H})\eta) & \sinh(\eta)\hat{F} \\ \sinh(\eta)\hat{E} & \sinh(u + ( \frac{1}{2} - \hat{H})\eta) \end{pmatrix}. \]
The $\hat{E}$, $\hat{F}$, $\hat{H}$ are generators of the quantum algebra $U_q(sl(2))$: $[\hat{H}, \hat{E}] = \hat{E}$, $[\hat{H}, \hat{F}] = -\hat{F}$, $[\hat{E}, \hat{F}] = [2\hat{H}]_q$.
In the representation with the spin $j$ we have
\[ \pi_j(\hat{H})_{mn} = (j + 1 - n)\delta_{m,n}, \quad m, n = 1, 2, \ldots, 2j + 1. \]
\[ \pi_j(\hat{E})_{mn} = \omega_m\delta_{m,n-1}, \quad \pi_j(\hat{F})_{mn} = \omega_n\delta_{m-1,n}. \]
where
\[ \omega_n \equiv \sqrt{[n]_q[2j + 1 - n]_q}. \]
The spin $j$ takes values from the set $(0, \frac{1}{2}, 1, \ldots)$. 
3 Sklyanin transfer matrix.

It was shown by Sklyanin in [3] that if transfer matrices are defined as

\[ t_j(u) = \text{tr}_{\pi_j} \left( e^{-2(u+\eta)\hat{H}} L_j(u) e^{2\hat{H}u} \hat{L}_j(u) \right) \]

\[ \equiv \sum_{n,k=1}^{2j+1} e^{2(n-k)u} q^{-2(j+1-n)} (L_j)^n_k(u) \hat{L}_j^n_k(u), \]

where, as above

\[ (\hat{L}_n^m)_{iN\ldots i_1}^{jN\ldots j_1} = (L_n^m)_{jN\ldots j_1}^{iN\ldots i_1}, \]

then the transfer matrices commute between themselves,

\[ [t_j_1(u_1), t_j_2(u_2)] = 0 \]

and with the Hamiltonian,

\[ [t_j(u), H_{XXZ}] = 0 \]

The basic transfer matrix \( t_{1/2}(u) \) with the spin of the auxiliary space \( j = 1/2 \) is related to the \( XXZ \) Hamiltonian by

\[ H_{XXZ} = \frac{\sinh(\eta)}{2} \frac{d \log t_{1/2}(u)}{du} \bigg|_{u=0} - \frac{\sinh^2(\eta)}{2 \cosh(\eta)} - \frac{N}{2} \cosh(\eta). \]

4 Behavior of \( L \) as \( u \to \pm \infty \).

We consider the monodromy matrix in the representation with the spin 1/2. It is a \( 2 \times 2 \) matrix whose entries are some operators. Using the expression of the monodromy matrix as an \( N \)-uple product of \( R \)-matrices with same auxiliary and different quantum spaces it is easy to obtain its behavior as \( u \to \pm \infty \):

\[ L(u \to +\infty) = 2^{-N} e^{(u+\eta/2)(N-1)} \begin{pmatrix} e^{u+\eta/2}q^H & (q - q^{-1})X_0 \\ (q - q^{-1})X & e^{u+\eta/2}q^{-H} \end{pmatrix} \]

\[ L(u \to -\infty) = (-1)^N 2^{-N} e^{-(u+\eta/2)(N-1)} \begin{pmatrix} e^{-u+\eta/2}q^{-H} & -(q - q^{-1})Y \\ -(q - q^{-1})Y_0 & e^{-(u+\eta/2)}q^H \end{pmatrix}. \]
The explicit expressions for $X_0$ and $Y_0$ in terms of Pauli matrices can be derived from those for $X$ and $Y$ by changing $q \to q^{-1}$ and $\sigma^\pm \to \sigma^\mp$. The $X, Y, X_0, Y_0$ and $H$ are generators of the quantum affine algebra $U_q(\hat{sl}(2))$. Namely, if we denote: $x_1 \equiv X, y_0 \equiv Y_0, x_0 \equiv X_0, y_1 \equiv Y$, then the relations $[x_i, y_i] = [2H]_q, [x_i, y_j] = 0, [H, x_0] = -x_0, [H, y_1] = -y_1, [H, x_1] = x_1$ and $[H, y_0] = y_0$ are fulfilled as well as the Serre relations

\[(ad_q x_i)^3 x_j = 0, \quad (ad_q y_i)^3 y_j = 0,\]

where

\[(ad_q x_i)^3 x_j \equiv (x_i)^3 x_j - [3]_q (x_i)^2 x_j x_i + [3]_q x_i x_j (x_i)^2 - x_j (x_i)^3.\]

5 Transformation of $L^m_n$ and $\bar{L}^m_n$ under the quantum group action.

We consider the Eq-s. (2) and (3) in the limit as $u \to \pm \infty$. We take the monodromy matrix $L(u)$ in the representation with the spin $j = 1/2$. We take the second monodromy matrix $L(v)$ in the representation with an arbitrary spin $j$. The $L(u)$ matrix in the limit yields the generators of the algebra $U_q(\hat{sl}(2))$ with respect to the formulas in the previous section. We thus obtain the following commutation relations between the $L^m_n, \bar{L}^m_n$, and the generators of $U_q(\hat{sl}(2))$. We here write only those containing $X$ and $q^{\pm H}$:

\[q^H L^m_n = q^{n-k} L^m_{k-q^H}, \quad q^H \bar{L}^m_n = q^{k-n} \bar{L}^m_{k-q^H}, \quad (4)\]

\[X L^m_n - q^{2(j+1)-k-n} L^m_{k-H} X = \omega_{k-1} \lambda q^{j+1-n} L^m_{k-1} q^{-H}, \quad (5)\]

\[X \bar{L}^m_n - q^{k+n-2(j+1)} \bar{L}^m_{k-H} X = \omega_{n-1} \lambda^{-1} q^{k-j-1} \bar{L}^m_{n-1} q^H, \quad (6)\]

where $\lambda \equiv e^v$.  

5
These commutation relations can be used to prove (4). However, this way is rather difficult, and we chose another one. Equations (2) and (3) show that elements of the monodromy matrix \( L \) are transformed under the quantum group like a composition of two representations of it. We therefore introduce a new algebra (which is just the Zamolodchikov algebra) such that each element \( L_n^m \) or \( \bar{L}_n^m \) is a product of two generators and Eq-s. (2),(4) and (3) follow from the defining relations of this new algebra. Rewriting the Sklyanin transfer matrix \( t_{p/2}(u) \) in terms of the generators of the new algebra we can easily prove our main statement (4).

6 \( L_n^m \) and \( \bar{L}_n^m \) as a composition of elements of the Zamolodchikov algebra.

We introduce the Zamolodchikov algebra generated by the operators \( \theta^a_j(u) \), \( \bar{\theta}^a_j(u) \), \( \bar{\theta}^n_j(u) \) \( \), where \( j \) is any positive half integer and \( n = 1, \ldots, 2j+1 \), with the following defining relations:

\[
(R_{j_1j_2})^{an}_{bm}(u-v)\theta^b_{j_1}(u)\theta^m_{j_2}(v) = \phi_{j_1j_2}(u-v)\theta^n_{j_1}(v)\theta^a_{j_2}(u),
\]

\[
(R_{j_1j_2})^{bn}_{an}(u-v)\theta^n_{j_1}(v)\theta^m_{j_2}(v) = \phi_{j_1j_2}(u-v)\theta^b_{j_1}(u)\theta^a_{j_2}(v),
\] (7)

\[
\theta^a_{j_1}(u)\theta^m_{j_2}(v) = \tau_{j_1j_2}(u-v)\theta^m_{j_2}(v)\theta^a_{j_1}(u),
\]

\[
(R_{j_1j_2})^{am}_{bn}(u-v)\bar{\theta}^m_{j_1}(v)\bar{\theta}^b_{j_2}(u) = \phi_{j_1j_2}(u-v)\theta^b_{j_1}(u)\bar{\theta}^m_{j_2}(v),
\]

\[
(R_{j_1j_2})^{bn}_{am}(u-v)\bar{\theta}^n_{j_1}(u)\bar{\theta}^m_{j_2}(v) = \phi_{j_1j_2}(u-v)\theta^b_{j_1}(u)\bar{\theta}^n_{j_2}(v),
\] (8)

\[
\bar{\theta}^m_{j_1}(v)\bar{\theta}^n_{j_2}(u) = \tau_{j_1j_2}(u-v)\bar{\theta}^n_{j_2}(v)\bar{\theta}^m_{j_1}(u),
\]

\[
(R_{j_1j_2})^{am}_{bn}(u+v)\bar{\theta}^m_{j_2}(v)\bar{\theta}^b_{j_1}(u) = \chi_{j_1j_2}(u+v)\theta^a_{j_1}(u)\bar{\theta}^m_{j_2}(v),
\]

\[
(R_{j_1j_2})^{bn}_{am}(u+v)\bar{\theta}^n_{j_2}(v)\bar{\theta}^a_{j_1}(u) = \bar{\chi}_{j_1j_2}(u+v)\theta^a_{j_1}(u)\bar{\theta}^n_{j_2}(v),
\]

\[
\theta^a_{j_1}(u)\bar{\theta}^m_{j_2}(v) = \rho_{j_1j_2}(u+v)\bar{\theta}^m_{j_2}(v)\theta^a_{j_1}(u),
\]

\[
\theta^a_{j_1}(u)\bar{\theta}^n_{j_2}(v) = \bar{\rho}_{j_1j_2}(u+v)\bar{\theta}^n_{j_2}(v)\theta^a_{j_1}(u),
\] (9)

where \( \phi_{j_1j_2}, \tau_{j_1j_2}, \rho_{j_1j_2} \) and \( \chi_{j_1j_2} \) are some functions. These functions depend on the specific form of \( R_{j_1j_2} \). For example \( \phi_{j_1j_2} \) is defined from the unitarity
equation and is equal to the multiplier in the r.h.s. of this equation. We don’t know how to obtain the form of the three other functions in general case, but we can find the needed combination of them from the self-consistency requirement. The Zamolodchikov algebra is associative for arbitrary $\phi_{j_1j_2}$, $\tau_{j_1j_2}$, $\rho_{j_1j_2}$ and $\chi_{j_1j_2}$ provided only that the $R$-matrix satisfies the triangle equation. A similar algebra was considered in [9].

It is easy to verify using (7)–(9) that if

$$
(L_j)^a_b(u) = \theta^a_j(u)\theta^b_j(u),
$$

$$
(\bar{L}_j)^a_b(u) = \bar{\theta}^a_j(u)\bar{\theta}^b_j(u),
$$

then $L$ and $\bar{L}$ satisfy the Yang–Baxter equations (2) and (3).

7 Transformation of $\theta_n$, $\theta^n$, $\bar{\theta}_n$ and $\bar{\theta}^n$ under the quantum group action.

The following commutation relations between $L^b_a$, $\bar{L}^b_a$ and the generators of the Zamolodchikov algebra, which follow from (7)–(10), are sufficient for our purpose:

$$
R^{bn}_m(u-v)L^b_c(u)\theta^m_j(v) = \theta^n_j(v)L^a_c(u)\phi(u-v),
$$

$$
R^{bm}_m(u-v)\bar{\theta}^m_j(v)L^b_a(u) = \phi(u-v)L^a_c(u)\theta^b_j(v),
$$

$$
R^{am}_m(u-v)\bar{\theta}^m_j(v)\bar{L}^b_c(u) = \bar{L}^a_c(u)\theta^n_j(v)\phi(u-v),
$$

$$
R^{bm}_m(u-v)\bar{L}^c_a(u)\bar{\theta}^m_j(v) = \phi(u-v)\bar{\theta}^n_j(v)\bar{L}^c_a(u).
$$

Here we write $L^b_\pm$ as just $L$, $R^b_{\pm j}$ as $R$ and set $\phi(u) = \phi^b_\pm(u)\tau^b_\pm(u)$. We already know how to obtain the operator $X$ from $L(u)$. Having taken the limit $u \to \pm\infty$ we can obtain the commutation relation between $X$ and $\theta$. To
do this we must know what the function $\phi(u)$ is equal to. But the function $\phi(u)$ depends on the function $\tau_{x_j}(u)$, which is unknown. Actually, we only need to know the behavior of $\phi(u)$ in the limit $u \to +\infty$. Taking this limit we obtain

\[
q^H \theta^n_j(v) = \varepsilon(v) q^{n-j-1} \theta^n_j(v) q^H, \\
q^{-H} \theta^n_j(v) = \varepsilon(v) q^{j+1-n} \theta^n_j(v) q^{-H}, \\
q^H \theta^n_i(v) = \varepsilon(v) q^{j+1-n} \theta^n_i(v) q^H, \\
q^{-H} \theta^n_i(v) = \varepsilon(v) q^{n-j-1} \theta^n_i(v) q^{-H},
\]

where

\[
\varepsilon(v) \equiv \lim_{u \to +\infty} (2\phi(u - v) e^{-(u-v+n/2)}).
\]

It is obvious that these equations are self-consistent only if

\[
\varepsilon(v) = 1.
\]

Supposing that this is true we also obtain the commutation relations between $X$ and $\theta$, thus,

\[
q^H \theta^n_j = q^{n-j-1} \theta^n_j q^H, \\
q^H \theta^n_i = q^{j+1-n} \theta^n_i q^H, \\
q^H \bar{\theta}^n_j = q^{j+1-n} \bar{\theta}^n_j q^H, \\
q^H \bar{\theta}^n_i = q^{n-j-1} \bar{\theta}^n_i q^H,
\]

(15)

\[
X \theta^n_j - q^{j+1-n} \theta^n_j X = -\omega_n \lambda q \theta^n_j q^H, \\
X \bar{\theta}^n_j - q^{j+1-n} \bar{\theta}^n_j X = \omega_{n-1} \lambda \bar{\theta}^n_{j-1} q^{-H}, \\
X \theta^n_i - q^{n-j-1} \theta^n_i X = \omega_{n-1} \lambda \bar{\theta}^{n-1}_j q^H, \\
X \bar{\theta}^n_i - q^{n-j-1} \bar{\theta}^n_i X = -\omega_n \lambda^{-1} q^{-1} \bar{\theta}^{n+1}_j q^{-H}.
\]

(16)

It is easy to verify that these formulas are consistent with (4) and (5) if (10) is taken into account.
8 New variables $\psi$. 

We can make our formulas more convenient if we introduce the new variables $\psi$ by

\[
\begin{align*}
\theta^n_j &= \psi^n_j q^{-nH}, \\
\bar{\theta}^n_j &= \bar{\psi}^n_j q^{nH}, \\
\theta^n_n &= \psi^n_n q^{-nH}, \\
\bar{\theta}^n_n &= \bar{\psi}^n_n q^{nH},
\end{align*}
\]  

\begin{align*}
X \psi^n_j - q^{j+1-2n} \psi^n_j X &= -\omega_n \lambda q^{\psi^{n+1}_j}, \\
X \bar{\psi}^n_j - q^{2n-j-1} \bar{\psi}^n_j X &= \omega_{n-1} \lambda \bar{\psi}^{n-1}_j, \\
X \psi^n_n - q^{j+1-2n} \psi^n_n X &= \omega_{n-1} \lambda \psi^{j}_{n-1}, \\
X \bar{\psi}^n_n - q^{2n-j-1} \bar{\psi}^n_n X &= -\omega_n \lambda^{-1} q^{-1} \bar{\psi}^{n+1}_n.
\end{align*}

9 Transfer matrix in the new variables. 

We rewrite the Sklyanin transfer matrices in the new variables $\psi$. The result suggests that the transfer matrices consist of two other objects. Replacing the old variables with the new ones, 

\[
(L_j)^n_k (\bar{L}_j)_k^n = \theta^n_j \bar{\theta}^n_k \bar{\psi}^n_j \psi^n_k,
\]

we obtain

\[
t_j(u) = \sum_{n=1}^{2j+1} \lambda^{2n} q^{-(n+1)(j+1-n)} \psi^n_j \left( \sum_{k=1}^{2j+1} \lambda^{-2k} q^{k(j+1-k)} \bar{\psi}^n_k \psi^n_k \right) \bar{\psi}^n_j
\]

(here and after $\lambda = e^u$). We see that the transfer matrix consists of two independent structures. If we set
\[ g_j^-(u) \equiv \sum_{k=1}^{2j+1} \lambda^{-2k} q^{k(j+1-k)} \psi_j^k \bar{\psi}_k, \] (20)

\[ g_j^+(u) \equiv \sum_{n=1}^{2j+1} \lambda^{2n} q^{-(n+1)(j+1-n)} \psi_j^n \bar{\psi}_n, \]

then the transfer matrix has the form

\[ t_j(u) = \sum_{n=1}^{2j+1} \lambda^{2n} q^{-(n+1)(j+1-n)} \psi_j^n \bar{\psi}_n. \]

The new objects \( g_j^-(u) \) and \( g_j^+(u) \) are remarkable because they are separately invariant with respect to the quantum algebra \( U_q(sl(2)) \),

\[ [g_j^+(u), U_q(sl(2))] = 0. \]

We prove this formula after introducing of the \textit{ad X} operator.

## 10 The \textit{ad X} operator.

It is convenient to introduce a linear operator (we call it the \textit{ad X} operator) such that the following properties hold

\[ \text{ad} X(\Psi + \Psi') = \text{ad} X(\Psi) + \text{ad} X(\Psi') \] (21)

and also if \( \Psi \) and \( \Psi' \) have definite degrees,

\[ \text{ad} X(\Psi \Psi') = \text{ad} X(\Psi) \Psi' + q^{\text{deg}(\Psi)} \Psi \text{ad} X(\Psi'), \]

\[ \text{ad} X(\Psi) = X \Psi - q^{\text{deg}(\Psi)} \Psi X, \]

\[ \text{deg}(\Psi \Psi') = \text{deg}(\Psi) + \text{deg}(\Psi'). \] (22)

By definition we set

\[ \text{ad} X(\psi_j^n) = X \psi_j^n - q^{j+1-2n} \psi_j^n X = -\omega_n \lambda q \psi_j^{n+1} \]

\[ \text{ad} X(\bar{\psi}_j^n) = X \bar{\psi}_j^n - q^{2n-j-1} \bar{\psi}_j^n X = \omega_{n-1} \lambda^{-1} \psi_{j-1}^{n-1} \]

\[ \text{ad} X(\bar{\psi}_n) = X \bar{\psi}_n - q^{j+1-2n} \bar{\psi}_n X = \omega_{n-1} \lambda \psi_{n-1} \]

\[ \text{ad} X(\psi_n) = X \psi_n - q^{2n-j-1} \psi_n X = -\omega_n \lambda^{-1} q^{-1} \bar{\psi}_{n+1} \]
(compare with (18)). We define the degrees of these operators by
\[
\begin{align*}
\deg(\psi^n_j) &= j + 1 - 2n, \\
\deg(\bar{\psi}^n_j) &= 2n - j - 1, \\
\deg(\psi^n_j) &= j + 1 - 2n, \\
\deg(\bar{\psi}^n_j) &= 2n - j - 1,
\end{align*}
\]
and also
\[
\begin{align*}
\deg(X^N \psi^n_j) &= \deg(\psi^n_j) - 2N, \\
\deg(X^N \bar{\psi}^n_j) &= \deg(\bar{\psi}^n_j) - 2N, \\
\deg(X^N \psi^n_j) &= \deg(\psi^n_j) + 2N, \\
\deg(X^N \bar{\psi}^n_j) &= \deg(\bar{\psi}^n_j) + 2N.
\end{align*}
\]
Pay your attention that applying the operator \(X\) to \(\psi\) and \(\bar{\psi}\) with upper index \(n\) decrease their degrees and conversely applying to the ones with lower index \(n\) increase their degrees. It is now easy to show that \(\text{ad } X(g^\pm_j(u)) = 0\).

Indeed,
\[
\begin{align*}
\text{ad } X g^\pm_j(u) &= \sum_{k=1}^{2j+1} \lambda^{-2k} q^{k(j+1-k)} \left( \text{ad } X(\psi^i_k)\bar{\psi}^j_k + q^{j+1-2k} \psi^i_k \text{ad } X(\bar{\psi}^j_k) \right) \\
&= \sum_{k=1}^{2j+1} \lambda^{-2k} q^{k(j+1-k)} \left( \omega_{k-1} \lambda \psi^i_k \bar{\psi}^j_k - q^{j+1-2k} \omega_k \lambda^{-1} q^{-1} \psi^i_k \bar{\psi}^j_k \right) \\
&= 0.
\end{align*}
\]
Because \(\deg(g^\pm_j(u)) = 0\), we have \(X g^\pm_j(u) = g^\pm_j(u)X\). In fact, even more general statement
\[
[g^\pm_j(u), U_q(sl(2))] = 0
\]
is true.

\section{Properties of the \(\text{ad } X\) operator.}

Using properties (21) and (22) of the operator \(\text{ad } X\) it is easy to prove the identities
\[
(\text{ad } X)^N(\psi^A \bar{\psi}^B) = \sum_{n=0}^{N} q^{n(\deg(\psi^A) + n - N)} C_n^N \text{ad } X^{N-n}(\psi^A) \text{ad } X^n(\bar{\psi}^B),
\]

11
\[(\text{ad } X)^N(\psi_A \bar{\psi}_B) = \sum_{n=0}^{N} q^{n(\deg(\psi^A) - n + N)} C_n^N (\text{ad } X)^{N-n}(\psi_A)(\text{ad } X)^n(\bar{\psi}_B),\]

\[(\text{ad } X)^N(\bar{\psi}^A) = \sum_{n=0}^{N} (-1)^n q^{n(\deg(\bar{\psi}^A) + 1 - N)} C_n^N X^{N-n} \bar{\psi}^A X^n, \quad (25)\]

\[(\text{ad } X)^N(\psi_A) = \sum_{n=0}^{N} (-1)^n q^{n(\deg(\psi_A) - 1 + N)} C_n^N X^{N-n} \psi_A X^n,\]

where

\[C_n^N = \frac{[N]_q!}{[n]_q! [N-n]_q!}\]

is the \(q\)-binomial coefficient. We can observe a light difference in these formulas for \(\psi\) and \(\bar{\psi}\) with the upper and lower indexes.

12 Proof of the \(\text{ad } X\) theorem in terms of \(\theta\) operators.

Using the definition of the operator \(\text{ad } X\), we can obtain

\[\psi^n_j = a_n (\text{ad } X)^{n-1}(\psi^A_j),\]

\[\psi^j_n = b_n (\text{ad } X)^{2j+1-n}(\bar{\psi}^A_{2j+1}),\]

\[\bar{\psi}^n_j = c_n (\text{ad } X)^{2j+1-n}(\bar{\psi}^A_{2j+1}),\]

\[\bar{\psi}^j_n = d_n (\text{ad } X)^{n-1}(\bar{\psi}^A),\]

where

\[a_n = \prod_{k=1}^{n-1}(-\lambda^{-1} q^{-1} \omega^{-1}_n), \quad b_n = \prod_{k=1}^{2j+1-n}(\lambda^{-1} \omega^{-1}_{2j+1-n-k}),\]

\[c_n = \prod_{k=1}^{2j+1-n}(\lambda \omega^{-1}_{2j+1-k}), \quad d_n = \prod_{k=1}^{n-1}(\lambda q \omega^{-1}_n).\]

It is easy to verify that

\[a_n c_n = (-1)^{n-1} \omega^{-1} q^{1-n} \lambda^{2(j+1-n)},\]

\[b_n d_n = (-1)^{n-1} \omega^{-1} q^{n-1} \lambda^{2(n-j-1)},\]
where
\[ \omega = \prod_{k=1}^{2j} \omega_k. \]

Therefore,
\[ \psi_j \bar{\psi}_j^n = (-1)^{n-1} \omega^{-1} q^{n-1} \lambda 2^{(n-j-1)} (\text{ad } X)^{2j+1-n} (\psi_{2j+1}) (\text{ad } X)^{-1} (\bar{\psi}_j^n), \]
\[ \psi_j^{n+1} \bar{\psi}_j^n = (-1)^{n-1} \omega^{-1} q^{n-1} \lambda 2^{(n-j-1)} (\text{ad } X)^{2j+1-n} (\psi_{2j+1}) (\text{ad } X)^{-1} (\bar{\psi}_j^{n+1}). \]

Inserting this in the expression for \( g_j^-(u) \), we obtain
\[
g_j^-(u) = \sum_{k=1}^{2j+1} \lambda^{-2k} q^{k(j+1-n)} \psi_j^k \bar{\psi}_j^k
= \omega^{-1} \lambda^{-2(j+1)} q^j \sum_{n=0}^{2j} (-1)^n q^{n(j-n)} (\text{ad } X)^{2j-n} (\psi_{2j+1}) (\text{ad } X)^n (\bar{\psi}_j^n).
\]

It makes sense to compare this with the similar formula
\[
(\text{ad } X)^{2j} (\psi_{2j+1} \bar{\psi}_1) = \sum_{n=0}^{2j} q^{-(2j+1)n} q^n (\text{ad } X)^{2j-n} (\psi_{2j+1}) (\text{ad } X)^n (\bar{\psi}_j^n).
\]

Taking into account that if \( q^{p+1} = -1 \) then \( C_n^p = 1 \) and \( q^{+(p+1)n} = (-1)^n \), we obtain the conclusion
\[
g_{p/2}^-(u) = \omega^{-1} \lambda^{-2(p/2+1)} q^{p/2} (\text{ad } X)^p (\psi_{p+1} \bar{\psi}_1). \quad (28)
\]

Similarly, we have
\[
g_j^+(u) = \sum_{n=1}^{2j+1} \lambda^{2n} q^{-(n+1)(j+1-n)} \psi_j^n \bar{\psi}_j^n
= (-1)^{2j} \omega^{-1} \lambda^{2(j+1)} q^{j(2j+1)} \sum_{n=0}^{2j} (-1)^{-n} q^{n(j+n)} (\text{ad } X)^{2j-n} (\psi_j^n) (\text{ad } X)^n (\bar{\psi}_j^{2j+1}).
\]

for \( g_j^+(u) \); as above if \( q^{p+1} = -1 \), we obtain
\[
g_{p/2}^+(u) = (-1)^p \omega^{-1} \lambda^{2(p/2+1)} q^{p(p+1)/2} (\text{ad } X)^p (\psi_{p+1} \bar{\psi}_p^p). \]
If $q^{p+1} = -1$, we can use (29) and $\text{ad}X(g_j^-(u)) = 0$ to rewrite formula (19) for the transfer matrix $t_{p/2}(u)$ if as

$$t_{p/2}(u) = (\text{ad}X)^p(G_{p/2}(u)), \quad (30)$$

where we introduce the notation

$$G_{p/2}(u) \equiv (-1)^p \omega ^{-1} \lambda ^{2(p/2+1)} q^{p(p+1)/2} \left( \psi _{p/2}^1 g_{p/2}^- (u) \bar{\psi }_{p/2}^{p+1} \right) \quad (31)$$

Because of property (25) of $\text{ad}x$ and because $q^{p+1} = -1$ implies $C_n^p = 1$, we can also rewrite (30) as

$$t_{p/2}(u) = \sum _{n=0} ^p X^{p-n} G_{p/2}(u) X^n. \quad (32)$$

We have thus proved in fact our main result (the ad $X$ theorem). In the next section we return from the ”virtual” variables $\psi$ and $\bar{\psi}$ to the variables $L$ and $\bar{L}$.

13 Return to $L$ and $\bar{L}$.

By returning from the variables $\psi$ and $\bar{\psi}$ to the variables $L$ and $\bar{L}$ in (31), we can easily show that

$$G_{p/2}(u) = (-1)^p \omega ^{-1} q^{p^2/2} \sum _{k=1} ^{p+1} \left( \lambda ^{2(p/2+1-k)} (L_{p/2})_k ^1 (L_{p/2})_k ^{p+1} \right) q^{-pH}$$

Although we have proved the main theorem using intermediate calculations with the ”virtual” operators $\theta$, which somehow do not exist, this proof is nevertheless valid. Indeed, to prove Eq. (32), we could permute the operator $X$ with the entire $L$ and $\bar{L}$ to the right without splitting $L$ and $\bar{L}$ to the operators $\theta$. Thus acting, we would not meet these operators at all, but the result of this permutation must be the same (and Eq. (32) must be fulfilled) because of the associativity of the Zamolodchikov algebra.

This completes the proof of the ad $X$ theorem.
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