On the Security of Index Coding with Side Information

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Abstract—Security aspects of the Index Coding with Side Information (ICSI) problem are investigated. Building on the results of Bar-Yossef et al. (2006), properties of linear index codes are further explored. The notion of weak security, considered by Bhattacharjee and Narayan (2005) in the context of network coding, is generalized to block security. It is shown that the linear index code based on a matrix \( L \), whose column space code \( C(L) \) has length \( n \), minimum distance \( d \) and dual distance \( d^\perp \), is \((d - 1 - t)\)-block secure (and hence also weakly secure) if the adversary knows in advance \( t \leq d - 2 \) messages, and is completely insecure if the adversary knows in advance more than \( n - d^\perp \) messages. Strong security is examined under the conditions that the adversary: (i) possesses \( t \) messages in advance; (ii) eavesdrops at most \( \mu \) transmissions; (iii) corrupts at most \( \delta \) transmissions. We prove that for sufficiently large \( q \), an optimal linear index code which is strongly secure against such an adversary has length \( \kappa_q + \mu + 2\delta \). Here \( \kappa_q \) is a generalization of the min-rank over \( \mathbb{F}_q \) of the side information graph for the ICSI problem in its original formulation in the work of Bar-Yossef et al.

I. INTRODUCTION

The problem of Index Coding with Side Information (ICSI) was introduced by Birk and Kol [1, 2]. It was motivated by applications such as audio and video-on-demand, and daily newspaper delivery. In these applications a server (sender) has to deliver some sets of data, audio or video files to a set of clients (receivers), different sets are requested by different receivers. Assume that before the transmission starts, the receivers have already (from previous transmissions) some files or movies in their possession. Via a slow backward channel, the receivers can let the sender know which messages they already have in their possession, and which messages they request. By exploiting this information, the amount of the overall transmissions can be reduced. As it was observed in [1], this can be achieved by coding the messages at the server before broadcasting them out.

Another possible application of the ICSI problem is in opportunistic wireless networks. These are networks in which a wireless node can opportunistically listen to the wireless channel. As a result, the node may obtain packets that were not designated to it (see [3–5]). This way, a node obtains some side information about the transmitted data. Exploiting this additional knowledge may help to increase the throughput of the system.

Consider the toy example in Figure 1. It presents a scenario with one sender and four receivers. Each receiver requires a different information packet (or message). The naive approach requires four separate transmissions, one transmission per an information packet. However, by exploiting the knowledge of the subsets of messages that clients already have, and by using coding of the transmitted data, the server can satisfy all the demands by broadcasting just one coded packet.

The ICSI problem has been a subject of several recent studies [3, 6–12]. This problem can be regarded as a special case of the well-known network coding (NC) problem [13, 14]. In particular, it was shown that every instance of the NC problem can be reduced to an instance of the ICSI problem [3, 10].

Fig. 1: An example of the ICSI problem

Several previous works focused on the design of an efficient index code for the ICSI problem. Given an instance of the ICSI problem, Bar-Yossef et al. [6] proved that finding the
best scalar linear binary index code is equivalent to finding the so-called min-rank of a graph, which is known to be an NP-hard problem (see [6], [15]). Here scalar linear index codes refer to linear index codes in which each message is a symbol in the field $\mathbb{F}_q$. By contrast, in vector linear index codes each message is a vector over $\mathbb{F}_q$. Lubetzky and Stav [7] showed that there exist instances in which scalar linear index codes outperform nonbinary fields and linear index codes over mixed fields outperform the scalar linear binary index codes. El Rouayheb et al. [3], [10] showed that for certain instances of the ICSI problem, vector linear index codes achieve strictly higher transmission rate than scalar linear index codes do. They also pointed out that there exist instances in which vector nonlinear index codes outperform vector linear index codes. Vector nonlinear index codes were also shown to outperform scalar nonlinear index codes for certain instances by Alon et al. [12]. Several heuristic solutions for the ICSI problem were proposed in [9], [11].

In this paper, we study the security aspects of linear index codes. We restrict ourselves to scalar linear index codes. It is known that vector linear index codes can achieve better transmission rate than their scalar counterparts, for certain instances of the ICSI problem [3], [10]. However, if the block length is fixed, one can model a vector index code as a scalar index code applied to another instance of the ICSI problem. If the block length is $\ell$, the number of messages is $n$, and the number of receivers is $m$ in the original (vector) instance, then the equivalent (scalar) instance can be viewed as having $\ell n$ messages and $\ell m$ receivers.

Let $\mathbb{F}_q$ be a finite field with $q$ elements. In its most general formulation, a linear index code maps $x \in \mathbb{F}_q^n$ onto $(x|g) L$, where $g \in \mathbb{F}_q^q$ is a random vector, $L$ is an $(n+\eta) \times N$ matrix over $\mathbb{F}_q$, and $n, \eta, N \in \mathbb{N}$. In this work, we show that each deterministic linear index code (i.e. $\eta = 0$) provides a certain level of information security. More specifically, let the code $\mathcal{C}(L)$ be spanned by the columns of $L$, and let $d$ and $d^\perp$ be its minimum distance and dual distance, respectively. We say that a particular adversary is of strength $t$ if it has $t$ messages in its possession. Then, we show that the index code based on $L$ is $(d−1+t)$-block secure against all adversaries of strength $t \leq d−2$ and is completely insecure against any adversary of strength at least $n−d^\perp + 1$. If $\mathcal{C}(L)$ is an MDS code, then the two bounds coincide. The technique used in the proof for this result is reminiscent of that used in the constructions of (multiple) secret sharing schemes from linear error-correcting codes [16], [17]. The results on the security of linear index codes can be further employed to analyze the existence of solutions for a natural generalization of the ICSI problem, so-called the Index Coding with Side and Restricted Information (ICSRIR) problem. In that problem, it is required that some receivers have no information about some messages.

In the sequel, we also consider a non-deterministic linear index code, based on the use of random symbols (i.e. $\eta \geq 1$). We show that the coset coding technique (which has been successfully employed in Secure Network Coding literature, see, for instance [18–22]) gives an optimal strongly secure linear index code of length $\kappa_q + \mu + 2\delta$. This index code is strongly secure against an adversary which:

(i) has $t$ arbitrary messages in advance;
(ii) eavesdrops at most $\mu$ transmissions;
(iii) corrupts at most $\delta$ transmissions.

Previous works on the security aspects (and on the error-correction aspect, as a special case) of network coding dealt with the multicast scenario. One of the main reasons for this limitation is that the optimal simultaneous transmission rates for non-multicast networks have not been fully characterized yet. The ICSI problem can be modeled as a special case of the non-multicast Network Coding problem ( [3], [12]). Moreover, being modeled in that way, it requires that there are directed edges from particular sources to each sink, which provide the side information. The symbols transmitted on these special edges are not allowed to be corrupted, where usually for network coding any edge can be corrupted. These two differences restrict the ability to derive the results on the security of the index coding schemes from the existing results on security of network coding schemes.

The paper is organized as follows. Notations and definitions, which are used in the rest of the paper, are introduced in Section II. The model and some basic results for the ICSI problem are presented in Section III. The block security of linear index codes is analyzed in Section IV. In particular, the Index Coding with Side and Restricted Information problem is presented and analyzed in Section V. Section VI is devoted to the analysis of strong security for index coding. The paper is concluded in Section VII.

II. PRELIMINARIES

Recall that we use the notation $\mathbb{F}_q$ for the finite field with $q$ elements, where $q$ is a power of prime. We also use $\mathbb{F}_q^n$ for the set of all nonzero elements of $\mathbb{F}_q$. Let $[n]$ denote the set of integers $\{1, 2, \ldots, n\}$. For the vectors $u = (u_1, u_2, \ldots, u_n) \in \mathbb{F}_q^n$ and $v = (v_1, v_2, \ldots, v_n) \in \mathbb{F}_q^n$, the (Hamming) distance between $u$ and $v$ is defined to be the number of coordinates where $u$ and $v$ differ, namely,

$$d(u, v) = |\{i \in [n]: u_i \neq v_i\}|.$$

The support of a vector $u \in \mathbb{F}_q^n$ is defined to be the set $\text{supp}(u) = \{i \in [n]: u_i \neq 0\}$. The (Hamming) weight of a vector $u$, denoted $\text{wt}(u)$, is defined to be $|\text{supp}(u)|$, the number of nonzero coordinates of $u$.

A $k$-dimensional subspace $C$ of $\mathbb{F}_q^n$ is called a linear $[n, k, d]_q$ $(q$-ary) code if the minimum distance of $C$, $d(C) \triangleq \min_{u \in C, v \in C, u \neq v} d(u, v)$, is equal to $d$. Sometimes we may use the notation $[n, k]_q$ for the sake of simplicity. The vectors in $C$ are called codewords. It is easy to see that the minimum weight of a nonzero codeword in a linear code $C$ is equal to its minimum distance $d(C)$. A generator matrix $G$ of an $[n, k]_q$-code $C$ is a $k \times n$ matrix whose rows are linearly independent codewords of $C$. Then $C = \{yG : y \in \mathbb{F}_q^n\}$. 
The dual code or dual space of $C$ is defined as $C^\perp = \{ u \in \mathbb{F}_q^n : u^T c = 0 \text{ for all } c \in C \}$. The minimum distance of $C^\perp$, $d(C^\perp)$, is called the dual distance of $C$.

The following upper bound on the minimum distance of a $q$-ary linear code is well-known (see [23] Chapter 1).

**Theorem 2.1 (Singleton bound):** For an $[n, k, d]_q$-code, we have $d \leq n - k + 1$.

Codes attaining this bound are called maximum distance separable (MDS) codes. For a subset of vectors

$$\{ c^{(1)}, c^{(2)}, \ldots, c^{(k)} \} \subseteq \mathbb{F}_q^n,$$

define its linear span over $\mathbb{F}_q$:

$$\text{span}_q \left( \{ c^{(1)}, c^{(2)}, \ldots, c^{(k)} \} \right) = \left\{ \sum_{i=1}^{k} \alpha_i c^{(i)} : \alpha_i \in \mathbb{F}_q, i \in [k] \right\}.$$ 

We use $e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{F}_q^n$ to denote the unit vector, which has a one at the $i$th position, and zeros elsewhere. We also use $I_n, n \in \mathbb{N}$, to denote the $n \times n$ identity matrix.

We recall the following well-known result in coding theory.

**Theorem 2.2 ([22], p. 66):** Let $C$ be an $[n, k, d]_q$-code with dual distance $d^\perp$ and $M$ denote the $q^k \times n$ matrix whose $q^k$ rows are codewords of $C$. If $r \leq d^\perp - 1$ then each $r$-tuple from $\mathbb{F}_q^n$ appears in an arbitrary set of $r$ columns of $M$ exactly $q^{r-r}$ times.

For a random vector $Y = (Y_1, Y_2, \ldots, Y_n)$ and a subset $B = \{i_1, i_2, \ldots, i_k\}$ of $[n]$, where $i_1 < i_2 < \cdots < i_k$, let $Y_B$ denote the vector $(Y_{i_1}, Y_{i_2}, \ldots, Y_{i_k})$. For an $n \times k$ matrix $M$, let $M_{ij}$ denote the $i$th row of $M$, and $M[j]$ its $j$th column. For a set $E \subseteq [n]$, let $M_E$ denote the $|E| \times k$ submatrix of $M$ formed by rows of $M$ which are indexed by the elements of $E$. For a set $F \subseteq [k]$, let $M[F]$ denote the $n \times |F|$ submatrix of $M$ formed by columns of $M$ which are indexed by the elements of $F$.

Let $X$ and $Y$ be discrete random variables taking values in the sets $\Sigma_X$ and $\Sigma_Y$, respectively. Let $\Pr(X = x)$ denote the probability that $X$ takes a particular value $x \in \Sigma_X$. Let $H(X), H(X | Y), I(X; Y), \text{ and } I(X; Y | Z)$ denote the (binary) entropy, conditional entropy, mutual information, and conditional mutual information (see [25] for the background).

### III. INDEX CODING AND SOME BASIC RESULTS

The Index Coding with Side Information (ICSI) problem considers the following communications scenario. There is a unique sender (or source) $S$, who has a vector of messages $x = (x_1, x_2, \ldots, x_n) \in \mathbb{F}_q^n$ in his possession, which is a realized value of a random vector $X = (X_1, X_2, \ldots, X_n)$. $X_1, X_2, \ldots, X_n$ hereafter are assumed to be independent uniformly distributed random variables over $\mathbb{F}_q$. There are also $m$ receivers $R_1, R_2, \ldots, R_m$. For each $i \in [m]$, $R_i$ has some side information, i.e. $R_i$ owns a subset of messages $\{x_j\}_{j \in X_i}, X_i \subseteq [n]$. In addition, each $R_i, i \in [m]$, is interested in receiving the message $x_{f(i)}$, for some demand function $f : [m] \rightarrow [n]$. Here we assume that $f(i) \notin X_i$ for all $i \in [m]$. Let $X = (X_1, X_2, \ldots, X_m)$. An instance of the ICSI problem is given by a quadruple $(m, n, X, f)$. Here we assume that every receiver requests exactly one message. This assumption is not a limitation of the model, as we can consider an equivalent problem by splitting each receiver who requests multiple messages into multiple receivers, each of whom requests exactly one message and have the same set of side information (see [11], [6]).

**Definition 3.1:** An index code over $\mathbb{F}_q$ for an instance $(m, n, X, f)$ of the ICSI problem, referred to as an $(m, n, X, f)$-IC over $\mathbb{F}_q$, is an encoding function

$$\mathcal{E} : \mathbb{F}_q^n \rightarrow \mathbb{F}_q^N,$$

such that for each receiver $R_i, i \in [m]$, there exists a decoding function

$$\mathcal{D}_i : \mathbb{F}_q^N \times \mathbb{F}_q^{\lvert X_i \rvert} \rightarrow \mathbb{F}_q,$$

satisfying

$$\forall x \in \mathbb{F}_q^n : \mathcal{D}_i(\mathcal{E}(x), x_{X_i}) = x_{f(i)}.$$ 

The parameter $N$ is called the length of the index code. In the scheme corresponding to this code, $S$ broadcasts a vector $\mathcal{E}(x)$ of length $N$ over $\mathbb{F}_q$.

**Definition 3.2:** An index code of the shortest possible length is called optimal.

**Definition 3.3:** A linear index code is an index code, for which the encoding function $\mathcal{E}$ is a linear transformation over $\mathbb{F}_q$. Such a code can be described as

$$\forall x \in \mathbb{F}_q^n : \mathcal{E}(x) = xL,$$

where $L$ is an $n \times N$ matrix over $\mathbb{F}_q$. The matrix $L$ is called the matrix corresponding to the index code $\mathcal{E}$. We also refer to $\mathcal{E}$ as the index code based on $L$. Notice that the length of $\mathcal{E}$ is the number of columns of $L$.

Let $E \subseteq [n]$ and $u \in \mathbb{F}_q^n$. In the sequel, we write $u \prec E$ if $\text{supp}(u) \subseteq E$. Intuitively, this means that if some receiver knows $x_j$ for all $j \in E$ (and also knows $u$), then this receiver is also able to compute the value of $xu^T$.

Hereafter, we assume that the sets $X_i, \text{ for all } i \in [m]$, are known to $S$. Moreover, we also assume that the index code $\mathcal{E}$ is known to each receiver $R_i, i \in [m]$. In practice this can be achieved by a preliminary communication session, when the knowledge of the sets $X_i, \text{ for all } i \in [m]$, and of the code $\mathcal{E}$ are disseminated between the participants of the scheme.

Let $\mathcal{C}(L) = \text{span}_q(\{L[j]\}^T_{j \in [N]})$, the subspace spanned by the (transposed) columns of $L$. The following lemma was implicitly formulated in [6] for the case where $m = n, f(i) = i$ for all $i \in [m]$, and $q = 2$. This lemma specifies a sufficient condition on $\mathcal{C}(L)$ so that a receiver can reconstruct
a particular message. We reproduce this lemma with its proof in its general form for the sake of completeness of the presentation.

**Lemma 3.1:** Let \( L \) be an \( n \times N \) matrix over \( \mathbb{F}_q \). Assume that \( S \) broadcasts \( xL \). Then, for each \( i \in [m] \), the receiver \( R_i \) can reconstruct \( x_{f(i)} \) if there exists a vector \( u^{(i)} \in \mathbb{F}_q^n \) satisfying
1) \( u^{(i)} \prec X_i \);
2) \( u^{(i)} + e_{f(i)} \in C(L) \).

**Proof:** Assume that \( u^{(i)} \prec X_i \) and \( u^{(i)} + e_{f(i)} \in C(L) \). Since \( u^{(i)} + e_{f(i)} \in C(L) \), there exists \( \beta \in \mathbb{F}_q^n \) such that
\[
\beta L^T = x \beta L^T - x u^{(i)}.
\]
By taking the transpose and pre-multiplying by \( x \), we obtain
\[
x(u^{(i)} + e_{f(i)})^T = (xL)\beta^T.
\]
Therefore,
\[
x_{f(i)} = xe_{f(i)}^T = (xL)\beta^T - xu^{(i)}.
\]
Observe that \( R_i \) is able to find \( u^{(i)} \) and \( \beta \) from the knowledge of \( L \). Moreover, \( R_i \) is also able to compute \( xu^{(i)} \) since \( u^{(i)} \prec X_i \). Additionally, \( R_i \) knows \( xL \), which is transmitted by \( S \). Therefore, \( R_i \) is able to compute \( x_{f(i)} \).

**Remark 3.2:** It follows from Lemma 3.1 that \( L \) corresponds to a linear \((m,n,X,f)\)-IC over \( \mathbb{F}_q \) if \( C(L) \supseteq \operatorname{span}_q \{ (u^{(i)} + e_{f(i)}) \}_{i \in [m]} \) for some \( u^{(i)} \prec X_i \), \( i \in [m] \). We show later in Corollary 4.5 that this condition is also necessary. Finding such an \( L \) with minimal number of columns by careful selection of \( u^{(i)} \)'s is a difficult task (in fact it is NP-hard to do so, see [6], [15]), which, however, yields a linear coding scheme with the minimal number of transmissions.

IV. BLOCK SECURE LINEAR INDEX CODES

A. Block Security and Weak Security

In this section, we assume the presence of an adversary \( A \) who can listen to all transmissions. Assume that \( S \) employs a linear index code based on \( L \). The adversary is assumed to possess side information \( \{x_j\}_{j \in X_A} \), where \( X_A \subseteq [n] \). For short, we say that \( A \) knows (or possesses, owns) \( x_{X_A} \). The **strength** of \( A \) is defined to be \( |X_A| \). Denote \( \hat{X}_A \equiv ([n] \setminus X_A) \). Note that by listening to \( S \), the adversary also knows \( s \equiv \hat{C}(x) = xL \). We define below several levels of security for linear index codes.

**Definition 4.1:** Suppose that the sender \( S \) possesses a vector of messages \( x \in \mathbb{F}_q^n \), which is a realized value of a random vector \( X = (X_1, X_2, \ldots, X_n) \), whose coordinates \( X_i \), \( i \in [n] \), are all independent and uniformly distributed over \( \mathbb{F}_q \). An adversary \( A \) possesses \( x_{X_A} \). Consider a linear \((m,n,X,f)\)-IC over \( \mathbb{F}_q \) based on \( L \).

1) For \( B \subseteq \hat{X}_A \), the adversary is said to have no information about \( x_B \) if
\[
H(X_B \mid XL, X_{X_A}) = H(X_B).
\]

In other words, despite the partial knowledge on \( x \) that the adversary has (his side information and the transmissions he eavesdrops), the symbols \( x_B \) still looks completely random to him.

2) The index code is said to be \( b \)-block secure against \( X_A \) if for every \( b \)-subset \( B \subseteq \hat{X}_A \), the adversary has no information about \( x_B \).

3) The index code is said to be \( b \)-block secure against all adversaries of strength \( t \) \((0 \leq t \leq n-1)\) if it is \( b \)-block secure against \( X_A \) for every \( X_A \subseteq [n] \), \( |X_A| = t \).

4) The index code is said to be weakly secure against \( X_A \) if it is \( 1 \)-block secure against \( X_A \). In other words, after listening to all transmissions, the adversary has no information about each particular message that he does not possess in the first place.

5) The index code is said to be **weakly secure against all adversaries of strength \( t \) \((0 \leq t \leq n-1)\)** if it is weakly secure against \( X_A \) for every \( t \)-subset \( X_A \) of \([n] \).

6) The index code is said to be completely insecure against \( X_A \) if an adversary, who possesses \( \{x_i\}_{i \in X_A} \), by listening to all transmissions, is able to determine \( x_i \) for all \( i \in \hat{X}_A \).

7) The index code is said to be completely insecure against any adversary of strength \( t \) \((0 \leq t \leq n-1)\) if an adversary, who possesses an arbitrary set of \( t \) messages, is always able to reconstruct all of the other \( n-t \) messages after listening to all transmissions.

**Remark 4.1:** Even when the index code is \( b \)-block secure \((b \geq 1)\) as defined above, the adversary is still able to obtain information about dependencies between various \( x_i \)'s in \( \hat{X}_A \) (but he gains no information about any group of \( b \) particular messages). This definition of \( b \)-block security is a generalization of that of weak security (see [26], [27]). Obviously, if an index code is \( b \)-block secure against \( X_A \) \((b \geq 1)\) then it is also weakly secure against \( X_A \), but the converse is not always true.

B. Necessary and Sufficient Conditions for Block Security

In the sequel, we consider the sets \( B \subseteq [n], B \neq \emptyset, \) and \( E \subseteq [n], E \neq \emptyset \). Moreover, we assume that the sets \( X_A, B, \) and \( E \) are disjoint, and that they form a partition of \([n] \), namely \( X_A \cup B \cup E = [n] \). In particular, \( \hat{X}_A = B \cup E \).

**Lemma 4.2:** Assume that for all \( u \prec X_B \) and for all \( \alpha_i \in \mathbb{F}_q, i \in B \) (not all \( \alpha_i \)'s are zeros),
\[
u + \sum_{i \in B} \alpha_i e_i \notin C(L).
\]
Then,
1) for all \( i \in B \):
\[
L_i \in \operatorname{span}_q \{ (L_j)_{j \in E} \};
\]
2) the system
\[
yL_E = wL_B
\]
has at least one solution \( y \in \mathbb{F}_q^{[B]} \) for every choice of \( w \in \mathbb{F}_q^{[B]} \).
Proof:

1) If \( \mathrm{rank}_q(L_E) = N \) then the first claim follows immediately. Otherwise, assume that \( \mathrm{rank}_q(L_E) < N \). As the \( N \) columns of \( L_E \) are linearly dependent, there exists \( y \in \mathbb{F}_q^N \setminus \{0\} \) such that \( yL_E^T = 0 \).

- If for all such \( y \) and for all \( i \in B \) we have \( yL^T_i = 0 \), then \( L_i \in \langle \{C \rangle_\} \rangle_\rangle = \\mathrm{span}_q(\{L_j\}_{j \in B}) \) for all \( i \in B \).

- Otherwise, there exist \( y \in \mathbb{F}_q^N \) and \( i \in B \) such that \( yL^T_i = 0 \) and \( yL^T_i \neq 0 \). Without loss of generality, assume that

\[
L = \begin{bmatrix} L_{x_A} \\ L_B \\ L_E \end{bmatrix}.
\]

Let \( c = yL^T \in C(L) \). Then

\[
c = (c_{x_A} | c_B | c_E) = (yL^T_{x_A}, yL^T_B, yL^T_E).
\]

Hence \( c_B = yL^T_B \neq 0 \) and \( c_E = yL^T_E = 0 \). Let

\[
u = (c_{x_A} | \{0\} | \alpha) \in X_A \text{ and } \alpha = c_i \text{ for all } i \in B.
\]

Then \( \alpha_i \text{'s are not all zero and } u + \sum_{i \in B} \alpha_i e_i = c \in C(L) \), which contradicts (2).

2) By (3), each row of \( L_B \) is a linear combination of rows of \( L_E \). Hence \( wL_B \) is also a linear combination of rows of \( L_E \). Therefore, (4) has at least one solution.

While Lemma 3.1 does not discuss security, it provides sufficient conditions for successful reconstruction of the information by a legitimate receiver \( R_i \). Obviously, the legitimate receiver \( R_i \) can be replaced by an adversary \( A \) in the formulation of that lemma. Thus, the conditions in Lemma 3.1 can be viewed as sufficient conditions for absence of weak security with respect to this \( A \).

In Lemma 4.3, which appears below, we show that the same conditions are also necessary (for the absence of information security with respect to the adversary \( A \)). However, similarly, \( A \) can be replaced by a legitimate receiver \( R_i \) in the formulation of Lemma 4.3. Thus, this lemma also provides both necessary and sufficient conditions for successful reconstruction of the information by the legitimate receiver \( R_i \).

Additionally, in Lemma 4.3, the weak security is further generalized to block security.

Lemma 4.3: Let \( L \) be an \( n \times N \) matrix over \( \mathbb{F}_q \). Assume that \( S \) broadcasts \( xL \). For a subset \( B \subseteq \mathbb{X}_A \), an adversary \( A \) who owns \( x_{x_A} \), after listening to all transmissions, has no information about \( x_B \) if and only if

\[
\forall u \lhd X_A, \forall \alpha_i \in \mathbb{F}_q \text{ with } \alpha_i, i \in B, \text{ not all zero: } u + \sum_{i \in B} \alpha_i e_i \notin C(L).
\]

In particular, for each \( i \in \mathbb{X}_A, A \) has no information about \( x_i \) if and only if

\[
\forall u \lhd X_A : u + e_i \notin C(L).
\]

Proof: Assume that (5) holds. We need to show that

\[
\Pr(X_B = g | XL = s, X_{x_A} = x_{x_A}) = \frac{1}{q^{N-B}}.
\]

Consider the following linear system with the unknown \( z \in \mathbb{F}_q^n \)

\[
\begin{cases}
z_B = g \\
z_{x_A} = x_{x_A} \\
z_L = s
\end{cases}.
\]

which is equivalent to

\[
\begin{cases}
z_B = g \\
z_{x_A} = x_{x_A} \\
z_E L_E = s - g L_B - x_{x_A} L_{x_A}
\end{cases}.
\]

In order to prove that (5) holds, it suffices to show that for all choices of \( g \in \mathbb{F}_q^{B} \), (7) always has the same number of solutions \( z \). Notice that the number of solutions \( z \) of (7) is equal to the number of solutions \( z_E \) of

\[
z_E L_B = s - g L_B - x_{x_A} L_{x_A},
\]

where \( s, g, \) and \( x_{x_A} \) are known. For any \( g \in \mathbb{F}_q^{B} \), if (8) has a solution, then it has exactly \( q^{E - \mathrm{rank}_q(L_E)} \) different solutions. Therefore, it suffices to prove that (8) has at least one solution for every \( g \in \mathbb{F}_q^{B} \).

Since \( x \) is an obvious solution of (7), we have

\[
x E L_E = s - x B L_B - x_{x_A} L_{x_A}.
\]

Subtract (9) from (8) we obtain

\[
(z_E - x E)L_E = (x_B - g)L_B,
\]

which can be rewritten as

\[
y L_E = w L_B,
\]

where \( y \triangleq z_E - x E, w \triangleq x_B - g \). Due to Lemma 4.2 (10) always has a solution \( y \), for every choice of \( w \). Therefore (8) has at least one solution for every \( g \in \mathbb{F}_q^{B} \).

Now we prove the converse. Assume that (5) does not hold. Then there exist \( u \lhd X_A \) and \( \alpha_i \in \mathbb{F}_q, i \in B, \) where \( \alpha_i \text{'s are not all zero, such that } \sum_{i \in B} \alpha_i e_i = c - u, \) for some \( c \in C(L) \). Hence, similar to the proof of Lemma 3.1, the adversary obtains

\[
\sum_{i \in B} \alpha_i x_i = x (\sum_{i \in B} \alpha_i e_i)^T = x (c - u)^T = x c^T - x u^T.
\]
Note that the adversary can calculate $xe^T$ from $s$, and can also find $xe^T$ based on his own side information. Therefore, $A$ is able to compute a nontrivial linear combination of $x_i$'s, $i \in B$. Hence the entropy $H(X_B|X_L, X_{X_A}) < H(X_B)$. Thus, the adversary gains some information about the $x_B$.

Corollary 4.4 generalizes Lemma 3.1 by providing both necessary and sufficient conditions for a receiver’s ability to recover the desired message. (Note that this corollary can be equally applied to the legitimate receiver $R_i$ as well as to the adversary $A$.)

Corollary 4.4: Let $L$ be an $n \times N$ matrix over $\mathbb{F}_q$ and let $S$ broadcast $xL$. Then for each $i \in [m]$, the receiver $R_i$ can reconstruct $x_f(i)$ if and only if there exists a vector $u(i) \in \mathbb{F}_q^n$ such that

1) $u(i) \in C$;
2) $u(i) + e_{f(i)} \in C(L)$.

Corollary 4.5: The matrix $L$ corresponds to a linear $(m, n, X, f)$-IC over $\mathbb{F}_q$ if and only if for all $i \in [m]$, there exists a vector $u(i) \in \mathbb{F}_q^n$ satisfying

1) $u(i) \in \mathcal{X}_i$;
2) $u(i) + e_{f(i)} \in C(L)$.

Remark 4.6: It follows from Corollary 4.5 that $L$ corresponds to a linear $(m, n, X, f)$-IC over $\mathbb{F}_q$ if and only if $C(L) \supset \text{span}_q\{\{u(i) + e_{f(i)}\}_{i \in [m]}\}$, for some $u(i) \in \mathcal{X}_i$, $i \in [m]$. If we define

$$\kappa_q = \kappa_q(m, n, X, f) \triangleq \min\{\text{rank}_q(\{u(i) + e_{f(i)}\}_{i \in [m]}), u(i) \in \mathbb{F}_q^n, u(i) \in \mathcal{X}_i\},$$

then $\kappa_q$ is the shortest possible length of a linear $(m, n, X, f)$-IC over $\mathbb{F}_q$.

Corollary 4.7: The length of an optimal linear $(m, n, X, f)$-IC over $\mathbb{F}_q$ is $\kappa_q = \kappa_q(m, n, X, f)$.

Remark 4.8: The quantity $\kappa_q$ defined in (11) is precisely the min-rank over $\mathbb{F}_q$ of the side information graph of an ICSI instance in the case $m = n$ and $f(i) = i$ for all $i \in [n]$.

Idea of the proof: Recall that in [6], the side information graph $\mathcal{G}$ of an instance of the ICSI problem is defined by $\mathcal{G} = (\mathcal{V}_G, E_G)$, where $\mathcal{V}_G = [n]$ and $E_G = \{e = (i, j) : i, j \in [n], j \in \mathcal{X}_i\}$.

A matrix $A$ over $\mathbb{F}_q$ is said to fit $\mathcal{G}$ if

$$a_{i,j} \neq 0 \text{, if } i = j,$$
$$a_{i,j} = 0 \text{, if } i \neq j, (i,j) \notin E_G.$$

Then the min-rank of the side information graph $\mathcal{G}$ is defined by

$$\text{min}\{\text{rank}_q(A) : \text{ A fits } \mathcal{G}\}.$$

Suppose that $u(i) \in \mathcal{X}_i$ for all $i \in [n]$. Let $A = (a_{i,j})$ be the $n \times n$ matrix whose $i$th row is precisely $u(i) + e_i$, for each $i \in [n]$. Then $A$ fits $\mathcal{G}$. Conversely, if $A'$ fits $\mathcal{G}$ then by multiplying each row of $A'$ with a suitable nonzero constant (which does not change the rank of $A'$), one obtains a matrix $A$ which is of the aforementioned form. In other words, for each $i \in [n]$, the $i$th row of the resulting matrix $A$ equals $u(i) + e_i$ for some $u(i) \in \mathcal{X}_i$. Therefore, $\kappa_q$ defined above is indeed the minimum rank over $\mathbb{F}_q$ of a matrix which fits the side information graph $\mathcal{G}$. Thus, $\kappa_q$ is precisely the min-rank over $\mathbb{F}_q$ of $\mathcal{G}$.}

Theorem 4.9: Consider a linear $(m, n, X, f)$-IC over $\mathbb{F}_q$ based on $L$. Let $d$ be the minimum distance of $C(L)$.

1) This index code is $(d - 1 - t)$-block secure against all adversaries of strength $t \leq d - 2$. In particular, it is weakly secure against all adversaries of strength $t = d - 2$.

2) This index code is not weakly secure against at least one adversary of strength $t = d - 1$. Generally, if there exists a codeword of $C(L)$ of weight $w$, then this index code is not weakly secure against at least one adversary of strength $t = w - 1$.

3) Every adversary of strength $t \leq d - 1$ is able to determine a list of $q^{n-1-t-N}$ vectors in $\mathbb{F}_q^n$ which includes the vector of messages $x$.

Proof:

1) Assume that $t \leq d - 2$. By Lemma 4.3, it suffices to show that for every $t$-subset $\mathcal{X}_A$ of $[n]$ and for every $(d - 1 - t)$-subset $B$ of $X_A$,

$$\forall u, u_0 \in \mathcal{F}_q \text{ with } \alpha_i, i \in B, \text{ not all zero :}$$

$$u + \sum_{i \in B} \alpha_i e_i \notin C(L).$$

For such $u$ and $\alpha_i$'s, we have $\text{wt}(u + \sum_{i \in B} \alpha_i e_i) = \text{wt}(u) + \text{wt}(\sum_{i \in B} \alpha_i e_i) \leq t + (d - 1 - t) = d - 1 < d$. Moreover, as $\text{supp}(u) \cap B = \emptyset$ and $\alpha_i$'s, $i \in B$, are not all zero, we deduce that $u + \sum_{i \in B} \alpha_i e_i \notin \mathcal{C}(L)$. We conclude that $u + \sum_{i \in B} \alpha_i e_i \notin \mathcal{C}(L)$.

2) We now show that the index code is not weakly secure against at least one adversary of strength $t = d - 1$. The more general statement can be proved in an analogous way. Pick a codeword $c = (c_1, c_2, \ldots, c_n) \in \mathcal{C}(L)$ such that $\text{wt}(c) = d$ and let $\text{supp}(c) = \{i_1, i_2, \ldots, i_d\}$. Take $\mathcal{X}_A = \{i_1, i_2, \ldots, i_{d-1}\}$, $|\mathcal{X}_A| = d - 1$. Let

$$u = (c/c_{i_d} - e_{i_d}).$$

Then, $u \in \mathcal{X}_A$ and $u + e_{i_d} = c/c_{i_d} \in \mathcal{C}(L)$. By Lemma 3.1, $A$ is able to determine $x_{i_d}$. Hence the index code is not weakly secure against the adversary $A$, who knows $d - 1$ messages $x_i$’s in advance.

3) Let $s = xL$. Consider the following linear system of equations with unknown $z \in \mathbb{F}_q^n$

$$\begin{cases}
z_{\mathcal{X}_A} = x_{\mathcal{X}_A} \\
zL = s,
\end{cases}$$
which is equivalent to

\[
\begin{align*}
\mathbf{z}_{\mathcal{X}_A} & = \mathbf{x}_{\mathcal{X}_A} \\
\mathbf{z}_{\mathcal{X}_A} - \mathbf{L}_{\mathcal{X}_A} \mathbf{L}_{\mathcal{X}_A} & = \mathbf{s} - \mathbf{x}_{\mathcal{X}_A} \mathbf{L}_{\mathcal{X}_A}.
\end{align*}
\]  (12)

The adversary \( A \) attempts to solve this system. Given that \( s \) and \( \mathbf{x}_{\mathcal{X}_A} \) are known, the system (12) has \( n - t \) unknowns and \( N \) equations. Note that \( t \leq d - 1 \), and thus by applying Theorem 4.7 to \( C(L) \) we have \( n - t \geq n - d + 1 \geq N \). If \( \text{rank}_q(L_{\mathcal{X}_A}) = N \) then (12) has exactly \( q^{n-t} \) solutions, as required.

Next, we show that if \( \text{rank}_q(L_{\mathcal{X}_A}) = N \), then the index code is weakly secure against some adversary of strength \( d \), where \( d \) is the dual distance of \( C(L) \), and therefore by Theorem 4.9, if \( \text{rank}_q(L_{\mathcal{X}_A}) = N \), then the index code is weakly secure against any adversary of strength \( d \).

**Theorem 4.10:** The linear index code based on \( L \) is completely insecure against any adversary of strength \( t \geq n - d + 1 \), where \( d \) denotes the dual distance of \( C(L) \).

**Proof:** Suppose the adversary knows a subset \( \{x_j\}_{j \in \mathcal{X}_A \setminus \mathcal{X}_A} \), \( |\mathcal{X}_A \setminus \mathcal{X}_A| \leq n - d + 1 \). By Corollary 4.4, it suffices to show that for all \( j \in \mathcal{X}_A \setminus \mathcal{X}_A \), there exists \( u \in \mathbb{F}_q^n \) satisfying simultaneously \( u \not\in \mathcal{X}_A \) and \( u + e_j \in C(L) \).

Indeed, take any \( j \in \mathcal{X}_A \), and let \( \rho = n - t \leq d - 1 \). Consider the \( \rho \) indices which are not in \( \mathcal{X}_A \). By Theorem 2.3, there exists a codeword \( c \in C(L) \) with

\[
c_{t} = \begin{cases} 
1 & \text{if } t = j, \\
0 & \text{if } t \not\in \mathcal{X}_A \cup \{j\}.
\end{cases}
\]

Then \( \text{supp}(c) \subseteq \mathcal{X}_A \setminus \{j\} \). We define \( u \in \mathbb{F}_q^n \) such that \( u \not\in \mathcal{X}_A \), as follows. For \( t \in \mathcal{X}_A \), set \( u_t = c_t \), and for \( t \not\in \mathcal{X}_A \), we set \( u_t = 0 \). It is immediately clear that \( c = u + e_j \). Therefore, by Corollary 4.4, the adversary can reconstruct \( x_j \).

We have shown that the index code is completely insecure against an arbitrary set \( \mathcal{X}_A \) satisfying \( |\mathcal{X}_A| \geq n - d + 1 \), hence completing the proof.

When \( C(L) \) is an MDS code, we have \( n - d + 1 = n - d + 1 \), and hence the two bounds established in Theorems 4.9 and 4.10 are actually tight. The following example further illustrates the results stated in these theorems.

**Example 4.2:** Let \( n = 7, m = 7, q = 2 \), and \( f(i) = i \) for all \( i \in [m] \). Suppose that the receivers have in their possession sets of messages as appear in the third column of the table below. Suppose also, that the demands of all receivers are as in the second column of the table.

| Receiver | Demand | \( \{x_i\}_{i \in \mathcal{X}_i} \) |
|-----------|---------|----------------------------------|
| \( R_1 \)  | \( x_1 \) | \( \{x_6, x_7\} \)            |
| \( R_2 \)  | \( x_2 \) | \( \{x_5, x_7\} \)            |
| \( R_3 \)  | \( x_3 \) | \( \{x_5, x_6\} \)            |
| \( R_4 \)  | \( x_4 \) | \( \{x_5, x_6, x_7\} \)       |
| \( R_5 \)  | \( x_5 \) | \( \{x_1, x_2, x_6\} \)       |
| \( R_6 \)  | \( x_6 \) | \( \{x_1, x_3, x_4\} \)       |
| \( R_7 \)  | \( x_7 \) | \( \{x_2, x_3, x_6\} \)       |

For \( i \in [7] \), let \( u^{(i)} \in \mathbb{F}_2^n \) such that \( \text{supp}(u^{(i)}) = \mathcal{X}_i \). Assume that an index code based on \( L \) with \( C(L) = \text{span}_q\{u^{(i)} + e_i\}_{i \in [7]} \) is used. For instance, we can take \( L \) to be the matrix whose set of columns is \( \{L[i] = u^{(i)} + e_i\}_{i \in [7]} \).
It is easy to see that $C(L)$ is a $[7, 4, 3]_2$ Hamming code with $d = 3$ and $d^\perp = 4$.

Following the coding scheme, $S$ broadcasts the following four bits:

\[
\begin{align*}
s_1 &= x(u^{(1)} + e_1)^T, \\
s_2 &= x(u^{(2)} + e_2)^T, \\
s_3 &= x(u^{(3)} + e_3)^T, \\
s_4 &= x(u^{(4)} + e_4)^T.
\end{align*}
\]

Each $R_i$, $i \in [7]$, can compute $x(u^{(i)} + e_i)^T$ by using a linear combination of $s_1, s_2, s_3, s_4$. Then, each $R_i$ can subtract $x(u^{(i)})^T$ (his side information) from $x(u^{(i)} + e_i)^T$ to retrieve $x_i = xe_i^T$.

For example, consider $R_5$. Since

\[
x \left(u^{(5)} + e_5\right)^T = x \left((u^{(1)} + e_1) + (u^{(2)} + e_2)\right)^T = s_1 + s_2,
\]

$R_5$ subtracts $x_1 + x_2 + x_6$ from $s_1 + s_2$ to obtain

\[
\begin{align*}
(s_1 + s_2) - (x_1 + x_2 + x_6) &= (x_1 + x_2 + x_5 + x_6) - (x_1 + x_2 + x_6) \\
&= x_5.
\end{align*}
\]

If an adversary $A$ has a knowledge of a single message $x_i$, then by Theorem 4.9 $A$ is not able to determine any other message $x_{\ell}$, for $\ell \neq i$. Indeed, $d(C(L)) = 3$, while $t = 1$, the code is weakly secure against all adversaries of strength $t = 1$. If none of the messages are leaked, then the adversary has no information about any group of 2 messages. On the other hand, the code is completely insecure against any adversary of strength $t \geq 4$; in that case $A$ is able to determine the remaining $7 - t$ messages.

Remark 4.11: So far we only discuss the case when the adversary can listen to all $N$ transmissions. If we consider an adversary, which can eavesdrop at most $\mu$ ($\mu \leq N$) messages, then analogous results can also be obtained. Consider a linear index code based on $L$. Let

\[
d_{\mu} \triangleq \min \{d(C(L[W])) : W \subseteq [N], |W| = \mu\},
\]

and

\[
d_{\mu}^\perp \triangleq \min \{d((C(L[W]))^\perp) : W \subseteq [N], |W| = \mu\}.
\]

Then it is straightforward to see that the results in Theorems 4.9 and 4.10 still hold, with $d$ and $d^\perp$ being replaced by $d_{\mu}$ and $d_{\mu}^\perp$, respectively.

D. Role of the Field Size

The following example demonstrates that the use of index codes over larger fields might have a positive impact on the security level. More specifically, in that example, index codes over large fields significantly enhance the security, compared with index codes over small fields.

Example 4.3: Suppose that the source $S$ has $n$ messages $x_1, x_2, \ldots, x_n$. Assume that there are $m < n$ receivers $R_1, R_2, \ldots, R_m$, and each receiver $R_i$ has the same set of side information, $X_i = \{m + 1, m + 2, \ldots, n\}$. Assume also that each $R_i$ requires $x_i$, for $i \in [m]$. Any index code for this instance must have length at least $m$, since all the vectors $u^{(i)} + e_i$, for some $u^{(i)} \in X_i$, $i \in [m]$, are linearly independent over any field.

If we employ an index code over $\mathbb{F}_2$, by the fact that there are no nontrivial binary MDS codes, we deduce that the minimum distance $d$ of $C(L)$ is at most $n - m$. Hence index codes over $\mathbb{F}_2$ is not secure against some adversaries of strength $t = n - m - 1$. However, if we consider index codes over $\mathbb{F}_q$ for sufficiently large $q (q \geq n - 1)$, there exists a $q$-ary MDS code $C$ with minimum distance exactly $n - m - 1$. By choosing $L$ so that $C(L) = C$, the index code based on $L$ is secure against all adversaries of strength at most $t = n - m - 1$, which is strictly more secure than the those over $\mathbb{F}_2$. To find such an $L$, let $M = (I_m | P)$ be a generator matrix in standard form of an $[n, m]_q$-MDS code, and then take $L = M^T$. Then $L[i] = u^{(i)} + e^{(i)}$, for some $u^{(i)} \in X_i = \{m+1, m+2, \ldots, n\}$, $i \in [m]$. Therefore, by Corollary 4.5, $L$ corresponds to a linear index code for this instance.

Note that if we employ an index code over $\mathbb{F}_2$, then for large values of $n$ the minimum distance $d$ of $C(L)$ is bounded from above by the sphere-packing bound

\[
d \leq 2n \cdot (H^{-1}(1 - m/n) - \varepsilon),
\]

where $\varepsilon \rightarrow 0$ as $n \rightarrow \infty$. There is a variety of stronger upper bounds on the minimum distance of binary codes, such as the Johnson bound, the Elias bound, and the McEliece-Rodemich-Rumsey-Welch bound (see [29] Chapter 4.5 for more details). These bounds provide even stronger bounds on the security of the binary scheme for this instance of the ICSI problem. By contrast, as shown above, by using a $q$-ary MDS code, the distance $d$ of $C(L)$ can achieve the Singleton bound. It is well known that there is a significant gap between the Singleton bound and the sphere-packing bound (see [29], p. 111) for details). Therefore, for this instance of the ICSI problem, index codes over large fields provide significantly higher levels of security than those over binary field.

E. Application: Index Coding with Side and Restricted Information

In this section, we consider an extension of the ICSI problem, which we call the Index Coding with Side and Restricted Information (ICSRI) problem. This problem arises in applications such as audio and video-on-demand. Consider a client who has subscribed for certain media content (audio or video programs, movies, newspapers, etc.) At the same time, this client has not subscribed to some other content. The content provider wants to restrict this client from obtaining a content which he is not eligible for, even though he might be able to obtain it “for free” from the transmissions provided by the server. As we show in sequel, the solution for the ICSRI problem is a straight-forward application of the results in Corollary 4.4.

More formally, the arguments of an instance $(m, n, \mathcal{X}, \mathcal{Z}, f)$ of the ICSRI problem are similar to their counterparts for
the ICSI problem. The new additional parameter, \( Z = (Z_1, Z_2, \ldots, Z_m) \), represents the sets \( Z_i \subseteq [n] \) of message indices that the respective receivers \( R_i, i \in [m] \), are not allowed to obtain. The goal is that at the end of the communication round, the receiver \( R_i \) has the message \( x_{f(i)} \) in its possession, for all \( i \in [m] \), and it has no information about \( x_j \) for all \( j \in Z_i \). The notion of a linear \((m, n, \mathcal{X}, f)\)-IC over \( \mathbb{F}_q \) is naturally extended to that of a linear \((m, n, \mathcal{X}, Z, f)\)-IC over \( \mathbb{F}_q \).

Let
\[
\mathcal{F}(m,n,\mathcal{X},Z,f) \overset{\Delta}{=} \bigcup_{i=1}^{m} \{ u + e_j : u \prec \mathcal{X}_i, j \in Z_i \}.
\]

The following proposition provides a necessary and sufficient condition for a linear index code to be also a solution to an instance of the ICSI problem.

**Proposition 4.12:** The linear \((m, n, \mathcal{X}, f)\)-IC over \( \mathbb{F}_q \) based on \( L \) is also a linear \((m, n, \mathcal{X}, Z, f)\)-IC if and only if \( C(L) \cap \mathcal{F}(m,n,\mathcal{X},Z,f) = \emptyset \).

**Proof:** Let \( S \) employ the \((m, n, \mathcal{X}, f)\)-IC over \( \mathbb{F}_q \) based on \( L \). Then clearly \( R_i \) can recover \( x_{f(i)} \) for all \( i \in [m] \). Due to Lemma 4.3 for each \( i \in [m] \) and \( j \in Z_i \), \( R_i \) has no information about \( x_j \) if and only if
\[
\forall u \prec \mathcal{X}_i : u + e_j \notin C(L).
\]

Hence we complete the proof.

**Example 4.4:** Consider an instance \((m, n, \mathcal{X}, f)\) of the ICSI problem where \( m, n, \mathcal{X}, \) and \( f \) are defined as in Example 4.2 Moreover, let \( Z = (Z_1, Z_2, \ldots, Z_7) \), where \( Z_1 = \{2, 3, 4, 5\}, Z_2 = \{1, 3, 4, 6\}, Z_3 = \{1, 2, 4, 7\}, \) and \( Z_4 = Z_5 = Z_6 = Z_7 = \emptyset \). Consider the index code based on \( L \) constructed in Example 4.2. It is straightforward to verify that \( C(L) \cap \mathcal{F}(m,n,\mathcal{X},Z,f) = \emptyset \). Therefore, by Proposition 4.12 this index code also provides a solution to this instance of the ICSI problem.

Let
\[
\kappa_q^* = \kappa_q^*(m, n, \mathcal{X}, f) \overset{\Delta}{=} \min \{ \text{rank}_q((u^{(i)} + e_{f(i)}))_{i \in [m]} \},
\]

where the minimum is taken over all choices of \( u^{(i)} \prec \mathcal{X}_i \), \( i \in [m] \), which satisfy
\[
\text{span}_q \left( (u^{(i)} + e_{f(i)})_{i \in [m]} \right) \cap \mathcal{F}(m,n,\mathcal{X},Z,f) = \emptyset.
\]

Let \( \kappa_q^* = +\infty \) if there are no choices of \( u^{(i)} \)'s, \( i \in [m] \), which satisfy (13). The following proposition follows immediately.

**Proposition 4.13:** The length of an optimal linear \((m, n, \mathcal{X}, f)\)-IC over \( \mathbb{F}_q \) is \( \kappa_q^* \). If \( \kappa_q^* = +\infty \) then there exist no linear \((m, n, \mathcal{X}, f)\)-ICs over \( \mathbb{F}_q \).

V. STRONGLY SECURE INDEX CODES WITH SIDE INFORMATION

In this section, we consider a different model of adversary. Similarly to its counterpart in Section IV, the adversary \( A \) in this section owns some prior side information. Additionally, \( A \) can listen to \( \mu \leq N \) transmissions of \( S \). It can also corrupt some transmissions of \( S \), received by any of \( R_i, i \in [m] \).

We start the analysis with some basic definitions of error-correcting index codes. This type of index codes was studied very recently by the authors of this paper in [30]. We repeat some basic results for the sake of completeness.

A. Error-Correcting Index Codes

Assume that some of the symbols received by \( R_i, i \in [m] \), are in error. Consider an ICSI instance \((m, n, \mathcal{X}, f)\), and assume that \( S \) broadcasts a vector \( \mathbf{e}(x) \in \mathbb{F}_q^N \). Let \( \mathbf{e}(i) \in \mathbb{F}_q^N \) be the error affecting the information received by \( R_i, i \in [m] \). Then \( R_i \) actually receives the vector
\[
\mathbf{y}(i) = \mathbf{e}(x) + \mathbf{e}(i) \in \mathbb{F}_q^N,
\]

instead of \( \mathbf{e}(x) \). The following definition is a generalization of Definition 3.1.

**Definition 5.1:** A \( \delta \)-error-correcting index code over \( \mathbb{F}_q \) for an instance \((m, n, \mathcal{X}, f)\) of the ICSI problem, referred to as a \( \delta \)-error-correcting \((m, n, \mathcal{X}, f)\)-IC over \( \mathbb{F}_q \), is an encoding function
\[
\eta : \mathbb{F}_q^n \to \mathbb{F}_q^N,
\]
such that for each receiver \( R_i, i \in [m] \), there exists a decoding function
\[
\mathcal{D}_i : \mathbb{F}_q^N \times \mathbb{F}_q^{\lvert \mathcal{X}_i \rvert} \to \mathbb{F}_q,
\]
satisfying
\[
\forall x, \mathbf{e}(i) \in \mathbb{F}_q^n, \quad \text{wt}(\mathbf{e}(i)) \leq \delta : \mathcal{D}_i(\mathbf{e}(x) + \mathbf{e}(i), x_{\mathcal{X}_i}) = x_{f(i)}.
\]

The definitions of the length, of a linear index code, and of the matrix corresponding to an index code are naturally extended to \( \delta \)-error-correcting index codes.

We define the following sets
\[
\mathcal{I}(q, m, n, \mathcal{X}, f) \overset{\Delta}{=} \{ z \in \mathbb{F}_q^n : \exists i \in [m] \text{ such that } z_{\mathcal{X}_i} = 0 \text{ and } z_{f(i)} \neq 0 \}.
\]

For each \( i \in [m] \) we also define
\[
\mathcal{Y}_i \overset{\Delta}{=} [n] \setminus \{ f(i) \} \cup \mathcal{X}_i.
\]

Then the collection of supports of all vectors in \( \mathcal{I}(q, m, n, \mathcal{X}, f) \) is precisely
\[
\mathcal{I}(m, n, \mathcal{X}, f) \overset{\Delta}{=} \bigcup_{i \in [m]} \{ f(i) \} \cap \mathcal{Y}_i.
\]

The necessary and sufficient condition for a matrix \( L \) to correspond to a linear \( \delta \)-error-correcting index code is given in the following lemma.
Lemma 5.1: The matrix $L$ corresponds to a linear $\delta$-error-correcting $(m, n, X, f)$-IC over $F_q$ if and only if
\[ \text{wt}(zL) \geq 2\delta + 1 \text{ for all } z \in I(q, m, n, X, f). \]  
(15)
Equivalently, $L$ corresponds to a linear $\delta$-error-correcting $(m, n, X, f)$-IC over $F_q$ if and only if
\[ \text{wt} \left( \sum_{i \in K} z_i L_i \right) \geq 2\delta + 1, \]
for all $K \in \mathcal{J}(m, n, X, f)$ and for all choices of nonzero $z_i \in F_q$, $i \in K$.

Proof: For each $x \in F^n_q$, we define
\[ B(x, \delta) = \{ \hat{c} \in F^n_q : \hat{c} = xL + \xi, \text{wt}(\xi) \leq \delta, \xi \in F^n_q \}, \]
the set of all vectors resulting from at most $\delta$ errors in the transmitted vector associated with the information vector $x$. Then the receiver $R_i$ can recover $x_{f(i)}$ correctly if and only if
\[ B(x, \delta) \cap B(x', \delta) = \emptyset, \]
for every pair $x, x' \in F^n_q$ satisfying:
\[ x_i = x'_i, \text{ and } x_{f(i)} \neq x'_{f(i)}. \]
Therefore, $L$ correspond to a linear $\delta$-error-correcting $(m, n, X, f)$-IC over $F_q$ if and only if the following condition is satisfied: for all $i \in [m]$ and for all $x, x' \in F^n_q$ such that $x_i = x'_i$ and $x_{f(i)} \neq x'_{f(i)}$, it holds
\[ \forall \xi, \xi' \in F^N_q, \text{wt}(\xi) \leq \delta, \text{wt}(\xi') \leq \delta : \]
\[ xL + \xi \neq x'L + \xi'. \]
(16)
Denote $z = x' - x$. Then, the condition in (16) can be reformulated as follows: for all $i \in [n]$ and for all $z \in F^n_q$ such that $z_i = 0$ and $z_{f(i)} \neq 0$, it holds
\[ \forall \xi, \xi' \in F^N_q, \text{wt}(\xi) \leq \delta, \text{wt}(\xi') \leq \delta : zL \neq \xi - \xi'. \]
(17)
The equivalent condition is that for all $z \in I(q, m, n, X, f)$,
\[ \text{wt}(zL) \geq 2\delta + 1. \]
Since for $z \in I(q, m, n, X, f)$ we have
\[ zL = \sum_{i \in \text{supp}(z)} z_i L_i, \]
the condition (15) can be restated as
\[ \text{wt} \left( \sum_{i \in K} z_i L_i \right) \geq 2\delta + 1, \]
for all $K \in \mathcal{J}(m, n, X, f)$ and for all choices of nonzero $z_i \in F_q$, $i \in K$.

The next corollary follows directly from Lemma 5.1 by considering an error-free setup, i.e. $\delta = 0$. It is easy to verify that the conditions stated in this corollary and in Corollary 4.3 are equivalent, as expected.

Corollary 5.2: The matrix $L$ corresponds to an $(m, n, X, f)$-IC over $F_q$ if and only if
\[ \text{wt} \left( \sum_{i \in K} z_i L_i \right) \geq 1, \]
for all $K \in \mathcal{J}(m, n, X, f)$ and for all choices of nonzero $z_i \in F_q$, $i \in K$.

B. A Lower Bound on the Length

We start this section with a generalization of the definition of index codes to randomized index codes. Consider $\eta \in \mathbb{N}$ random variables $G_1, G_2, \ldots, G_\eta$, which are distributed independently and uniformly over $F_q$. Let $G = (G_1, G_2, \ldots, G_\eta)$ and let $g = (g_1, g_2, \ldots, g_\eta)$ be a realization of $G$.

Definition 5.2: An $\eta$-randomized $(m, n, X, f)$-IC over $F_q$ for an instance $(m, n, X, f)$ is an encoding function
\[ \mathcal{E} : F^m_q \times F^n_q \to F^N_q, \]
such that for each receiver $R_i$, $i \in [m]$, there exists a decoding function
\[ \mathcal{D}_i : F^N_q \times \{ \mathcal{X}_i \} \to F_q, \]
satisfying
\[ \forall x \in F^n_q : \mathcal{D}_i(\mathcal{E}(x, g), x_{\mathcal{X}_i}) = x_{f(i)}, \]
for any $g \in F^n_q$, which is a realization of the random vector $G$.

The definition of a $\delta$-error-correcting index code can be naturally extended to that of a $\delta$-error-correcting randomized index code. We simply replace $\mathcal{E} : F^m_q \to F^N_q$ by $\mathcal{E} : F^m_q \times F^n_q \to F^N_q$, and $\mathcal{E}(x)$ by $\mathcal{E}(x, g)$ in Definition 5.1.

An $\eta$-randomized index code is linear over $F_q$ if it has a linear encoding function $\mathcal{E}$,
\[ \mathcal{E}(x, g) = (x | g)L, \]
where $L$ is an $(n + \eta) \times N$ matrix over $F_q$. In the sequel we assume that any message $x_i$, $i \in [n]$ is requested by at least one receiver. Observe that by simply treating $x_1, x_2, \ldots, x_n, g_1, g_2, \ldots, g_\eta$ as messages, the results from previous sections still apply to linear randomized index codes.

Definition 5.3: The linear $\eta$-randomized $(m, n, X, f)$-IC over $F_q$ based on $L$ is said to be $(\mu, t, \delta)$-strongly secure if it has the following two properties:
1) This code is $\delta$-error-correcting. In other words, upon receiving $(x | g)L$ with at most $\delta$ coordinates in error, the receiver $R_i$ can still recover $x_{f(i)}$, for all $i \in [m]$.
2) This code is $(\mu, t)$-strongly secure. In other words, an adversary $A$ who possesses $x_{\mathcal{X}_A}$, for $\mathcal{X}_A \subseteq [n], |\mathcal{X}_A| = t$, and listens to at most $\mu$ transmissions, $\mu \leq N$, gains no information about other messages. Equivalently,
\[ H(X_{\mathcal{X}_A} | (X | G)L[W], X_{\mathcal{X}_A}) = H(X_{\mathcal{X}_A}), \]
for any $W \subseteq [N], |W| \leq \mu$. \


Remark 5.3:
1) If $\mu = t = \eta = 0$, then a $(\mu, t, \delta)$-strongly secure $\eta$-randomized $(m, n, \mathcal{X}, f)$-IC over $\mathbb{F}_q$ is simply a $\delta$-error-correcting $(m, n, \mathcal{X}, f)$-IC over $\mathbb{F}_q$.
2) If $\delta = 0$, the index code is strongly secure, but has no error-correcting capability. In that case, we simply say that the code is "$(\mu, t)$-secure" instead of "$(\mu, t, 0)$-secure".
3) A simple concatenation of an error-correcting index coding scheme and a secure index coding scheme may not necessarily yield a $(\mu, t, \delta)$-secure $\eta$-randomized $(m, n, \mathcal{X}, f)$-IC over $\mathbb{F}_q$.

In the lemma below, we assume that each message is requested by at least one receiver. Otherwise, that "useless" message can be discarded without affecting the model.

Lemma 5.4: If $L$ corresponds to a $(\mu, t)$-secure linear $\eta$-randomized $(m, n, \mathcal{X}, f)$-IC over $\mathbb{F}_q$, then $\eta \geq \mu$.

Proof: We prove this lemma by contradiction. Suppose that $L$ corresponds to a $(\mu, t, \delta)$-secure $\eta$-randomized $(m, n, \mathcal{X}, f)$-IC over $\mathbb{F}_q$, and that $\eta < \mu$. Let $E = \{n + 1, n + 2, \ldots, n + \eta\}$.

For $W \subseteq [N]$ let $C(L[W])$ be the space spanned by columns of $L$ indexed by elements of $W$. Then, for all $W \subseteq [N]$ with $|W| \leq \mu$, it holds that

$$H(X_{\hat{X}_A} | (X|G)L[W], X_{\hat{X}_A}) = H(X_{\hat{X}_A}),$$

i.e. an adversary who owns $x_{\hat{X}_A}$ gains no information about $x_{\hat{X}_A}$ after eavesdropping the transmissions corresponding to the set of indices $W$. From Lemma 4.3 with $C(L)$ being replaced by $C(L[W])$, we conclude that $C(L[W])$ does not contain a vector $e$ which satisfies $e_{\hat{X}_A} \neq 0$ and $e_{\hat{X}_A} = 0$. In the sequel, we refer to this property of $C(L[W])$ as Property A.

Let $L' = (L_{\hat{X}_{\hat{X}_A}})^T$ be the matrix obtained from $L$ by first deleting rows of $L$ indexed by $X_A$, and then taking its transpose. We show that $\text{rank}_q(L') \leq \mu - 1$. Indeed, take any $\mu$ rows of $L'$, denote them $L'_{j_1}, \ldots, L'_{j_\mu}$. Let $L''$ be the submatrix of $L'$ formed by the last $\eta$ columns. Since $\eta < \mu$, the $\mu$ rows $L''_{j_1}, \ldots, L''_{j_\mu}$ are linearly dependent. Hence, there exist $\alpha_1, \alpha_2, \ldots, \alpha_\mu$, not all zeros, such that

$$\sum_{\ell=1}^{\mu} \alpha_{\ell} L''_{ji} = 0.$$

This implies

$$\sum_{\ell=1}^{\mu} \alpha_{\ell} L'_{ji} = 0,$$

due to Property A. Thus, $\text{rank}_q(L') \leq \mu - 1$.

Now let $r = \text{rank}_q(L') < \mu$, and let

$$\{L'_{j_1}, L'_{j_2}, \ldots, L'_{j_r}\}$$

be a basis of the space spanned by the rows of $L'$. Suppose that the receiver $R_i$ requests $x_{j(i)}$ where $j(i) \in \hat{X}_A$. On the one hand, by Corollary 4.5, $C(L)$ contains a vector $c = u^{(i)} + e_{j(i)}$ where $u^{(i)} \in X_i$. Therefore, $c_{\hat{X}_A} = 0$ and $c_{\hat{X}_A} \neq 0$.

On the other hand, there exist $\beta_1, \beta_2, \ldots, \beta_r$ such that

$$c_{\hat{X}_A} | e_{\hat{X}_A} = \sum_{\ell=1}^{r} \beta_{\ell} L'_{j_{\ell}}.$$

Since $r < \mu$ and $c_{\hat{X}_A} = 0$, by Property A we have $c_{\hat{X}_A} = 0$.

We obtain a contradiction. 

Remark 5.5: From Lemma 5.4, a $(\mu, t, \delta)$-secure linear randomized index code requires at least $\mu$ random symbols. We show in Section V.C that there exists such a code that uses precisely $\mu$ random symbols.

Lemma 5.6: Suppose that $L$ corresponds to a linear randomized $(m, n, \mathcal{X}, f)$-IC over $\mathbb{F}_q$. If this randomized index code is $(\mu, t)$-secure, then for all $i \in [\mu]$, there exists a vector $v^{(i)} \in \mathbb{F}_q^{n+\mu}$ satisfying

1) $v^{(i)} \notin [n]$;
2) $v^{(i)} + e_{n+i} \notin C(L)$.

Proof: Assume, by contradiction, that for some $i \in [\mu]$, we have $v^{(i)} + e_{n+i} \notin C(L)$ for all $v^{(i)} \in [n]$. Consider a virtual receiver, which has a side information set \{\{x_j\}_{j \in [n]}\}, and requests the symbol $g_i$. By Corollary 4.3 this virtual receiver has no information about $g_i$ after listening to all transmissions. In other words, we have

$$H(G_i | (X|G)L, X) = H(G_i),$$

and, in particular, for a smaller set of side information,

$$H(G_i | (X|G)L, X_{\hat{X}_A}) = H(G_i).$$

We recall Definition 5.3 for every $\mu$-subset $W \subseteq [N]$ and every $t$-subset $X_A \subseteq [n]$, we have

$$H(X_{\hat{X}_A} | (X|G)L[W], X_{\hat{X}_A}) = H(X_{\hat{X}_A}).$$

In the sequel we show that if the value of $G_i$ is known to the adversary, this randomized index code is still $(\mu, t)$-secure. In other words, we aim to show that

$$H(X_{\hat{X}_A} | (X|G)L[W], X_{\hat{X}_A}, G_i) = H(X_{\hat{X}_A}),$$

for every $\mu$-subset $W \subseteq [N]$ and every $t$-subset $X_A \subseteq [n]$. Indeed, the left-hand side of (21) is equal to

$$H(X_{\hat{X}_A} | (X|G)L[W], X_{\hat{X}_A}) = I(X_{\hat{X}_A}; G_i | (X|G)L[W], X_{\hat{X}_A}),$$

which is

$$H(X_{\hat{X}_A}) - I(X_{\hat{X}_A}; G_i | (X|G)L[W], X_{\hat{X}_A}),$$

due to (20). Hence, it suffices to show that

$$I(X_{\hat{X}_A}; G_i | (X|G)L[W], X_{\hat{X}_A}) = 0.$$
We have
\[ I(X_{\hat{X}_a}; G_i | (X|G)L[W], X_{\hat{X}_a}) = H(G_i | (X|G)L[W], X_{\hat{X}_a}) - H(G_i | X_{\hat{X}_a}) = H(G_i | X_{\hat{X}_a}L[W], X_{\hat{X}_a}) = H(G_i) - H(G_i) = 0, \]
where the third transition is due to (18) and (19).

To this end, we have shown that the randomized index code is still \((\mu, t)\)-strongly secure if the adversary knows the realized value of \(G_i\). Equivalently, discarding the random variable \(G_i\) from the scheme does not affect its strong security. However, this contradicts Lemma 5.4 since the resulting code has less than \(\mu\) random symbols.

The following theorem proves a lower bound on the length of a \((\mu, t)\)-strongly secure linear randomized index code.

**Theorem 5.7:** The length of a \((\mu, t)\)-strongly secure linear \(\eta\)-randomized \((m, n, \mathcal{X}, f)\)-IC over \(\mathbb{F}_q\) is at least \(\kappa_q + \mu\).

**Proof:** Suppose the linear randomized index code is based on \(L\). We divide the proof into several cases.

Case 1: \(\eta = \mu\). Then, by Corollary 4.5 and Lemma 5.6 the subspace \(C(L)\) must contain:
- the vectors \(u^{(i)} + e_f(i)\) for some \(u^{(i)} \in \mathcal{X}_i\), for all \(i \in [m]\);
- the vectors \(v^{(i)} + e_{n+i}\) for some \(v^{(i)} \in [n]\), for all \(i \in [\mu]\).

Due to linear independence of these vectors and to the definition of \(\kappa_q\), the length of the code is at least
\[ \dim(C(L)) \geq \text{rank}_q(\{u^{(i)} + e_f(i)\}_{i \in [m]}) + \text{rank}_q(\{v^{(i)} + e_{n+i}\}_{i \in [\mu]}) \geq \kappa_q + \mu. \]

Case 2: \(\eta > \mu\), and for all \(i \in [\eta]\) there exists some vector \(v^{(i)} \in [n]\) such that \(u^{(i)} + e_{n+i} \in C(L)\) for all \(i \in [\mu]\). By following exactly the same argument as in the proof of Lemma 5.6 we deduce that discarding \(G_i\) does not affect the strong security of the randomized index code. By doing so, we obtain a new randomized \((\mu, t)\)-strongly secure index code, which has \(\eta - 1\) random variables. This code is based on \(L'\), which is obtained from \(L\) by deleting its \((n+i)\)-th row.

The above argument can be applied until either the number of random variables decreases to \(\mu\), or the code in consideration satisfies the condition of Case 2. In both cases, the resulting randomized index code has length at least \(\kappa_q + \mu\). As the length of the code do not change during the process, we conclude that the length of the original code is at least \(\kappa_q + \mu\).

The next theorem establishes a lower bound on the length of a \((\mu, t, \delta)\)-strongly secure linear randomized index code.

**Theorem 5.8:** The length of a \((\mu, t, \delta)\)-strongly secure linear \(\eta\)-randomized \((m, n, \mathcal{X}, f)\)-IC over \(\mathbb{F}_q\) is at least \(\kappa_q + \mu + 2\delta\).

**Proof:** Let \(L\) correspond to a \((\mu, t, \delta)\)-strongly secure \(\eta\)-randomized \((m, n, \mathcal{X}, f)\)-IC over \(\mathbb{F}_q\). Let \(L'\) be the matrix obtained by deleting any \(2\delta\) columns of \(L\). Since \(L\) corresponds to a \(\delta\)-error-correcting index code, by Lemma 5.1 it satisfies
\[ \text{wt}\left(\sum_{i \in K} z_i L_i\right) \geq 2\delta + 1, \]
for all \(K \in \mathcal{J}(m, n, \mathcal{X}, f)\) and all choices of nonzero \(z_i \in \mathbb{F}_q\), \(i \in K\). We obtain that the rows of \(L'\) satisfy
\[ \text{wt}\left(\sum_{i \in K} z_i L'_i\right) \geq 1. \]

By Corollary 5.2 \(L'\) corresponds to an \(\eta\)-randomized \((m, n, \mathcal{X}, f)\)-IC over \(\mathbb{F}_q\). Since all entries of \(L'\) are contained in \(L\), we deduce that \(L'\) corresponds to a \((\mu, t, \delta)\)-strongly secure \(\eta\)-randomized \((m, n, \mathcal{X}, f)\)-IC over \(\mathbb{F}_q\). Therefore, by Theorem 5.7 \(L'\) has at least \(\kappa_q + \mu\) columns. Therefore, \(L\) has at least \(\kappa_q + \mu + 2\delta\) columns.

**C. A Construction of Optimal Strongly Secure Index Codes**

In this section, we present a construction of an optimal \((\mu, t, \delta)\)-strongly secure \(\mu\)-randomized linear \((m, n, \mathcal{X}, f)\)-IC over \(\mathbb{F}_q\), which has length attaining the lower bound established in Theorem 5.8. \(q\) it requires \(q\) to be at least \(\kappa_q + \mu + 2\delta + 1\). The proposed construction is based on the coset coding technique, originally introduced by Ozarow and Wyner [31]. This technique has been adopted in a variety of network coding applications, such as [18–22].

**Construction A:** Let \(L^{(0)}\) correspond to a linear \((m, n, \mathcal{X}, f)\)-IC over \(\mathbb{F}_q\) of optimal length \(\kappa_q\). Let \(M\) be a generator matrix of an \([N = \kappa_q + \mu + 2\delta, \kappa_q + \mu + 2\delta + 1]_q\) MDS code, so that the last \(\mu\) rows of \(M\) form a generator matrix of another MDS code. For instance, take
\[
M = \begin{pmatrix}
1 & 1 & \cdots & 1 \\
\alpha_1 & \alpha_2 & \cdots & \alpha_N \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{\kappa_q-1} & \alpha_{\kappa_q} & \cdots & \alpha_{\kappa_q-N} \\
\alpha_1 & \alpha_2 & \cdots & \alpha_N \\
\alpha_{\kappa_q+1} & \alpha_{\kappa_q+2} & \cdots & \alpha_{\kappa_q+N} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{\kappa_q+\mu-1} & \alpha_{\kappa_q+\mu} & \cdots & \alpha_{\kappa_q+\mu+N-1}
\end{pmatrix}.
\]
where \(\alpha_1, \alpha_2, \ldots, \alpha_N\) are pairwise distinct nonzero elements in \(\mathbb{F}_q\). Let \(P\) be the submatrix of \(M\) formed by the first \(\kappa_q\) rows, and \(Q\) the submatrix formed by the last \(\mu\) rows of \(M\). Take

\[
L = \left( \frac{L^{(0)} P}{Q} \right).
\]

**Lemma 5.9:** The matrix \(L\) in Construction A corresponds to a \(\delta\)-error-correcting \(\mu\)-randomized \((m, n, X, f)\)-IC over \(\mathbb{F}_q\).

**Proof:** Recall that \(g \in \mathbb{F}_q^m\) is a random vector. The encoding function \(E\) has a form

\[
E(x, g) = (x|g)L = xL^{(0)}P + gQ = (xL^{(0)})M.
\]

Since \(M\) is a generator matrix of a \(\delta\)-error-correcting code, each receiver \(R_i, i \in [m]\), is able to recover \((xL^{(0)})\) if the number of errors in \(E(x, g)\) is less than or equal to \(\delta\). Therefore, each receiver \(R_i\) can recover \(xL^{(0)}\), and hence, it can also recover \(x_{f(i)}, i \in [m]\), as \(L^{(0)}\) corresponds to a linear \((m, n, X, f)\)-IC over \(\mathbb{F}_q\).

**Lemma 5.10:** The matrix \(L\) in Construction A corresponds to a \((\mu, t)\)-strongly secure randomized \((m, n, X, f)\)-IC over \(\mathbb{F}_q\).

**Proof:** Suppose that the adversary \(A\) possess a message vector \(x_{X_{\lambda}A} : x_{X_{\lambda}A} = x\). Additionally, \(A\) can eavesdrop \(\mu\) transmissions, i.e., it has a knowledge of \(b \equiv (x|g)L[W]\), for some \(W \subseteq [N], |W| = \mu\). Below, we show that the entropy of \(X_{\lambda}A\) is not changed given the knowledge of \((X|G)L[W]\) and of \(X_{\lambda}A\). It suffices to show that for all \(a \in \mathbb{F}_q^{n-t}:

\[
\Pr(X_{\lambda}A = a \mid (X|G)L[W] = b, X_{\lambda}A = x_{X_{\lambda}A}) = \frac{1}{q^{n-t}}.
\]

The left-hand side of (22) can be re-written as

\[
\Pr(X_{\lambda}A = a, (X|G)L[W] = b \mid X_{\lambda}A = x_{X_{\lambda}A}).
\]

The numerator in (23) is given by

\[
\Pr(X_{\lambda}A = a, (X|G)L[W] = b \mid X_{\lambda}A = x_{X_{\lambda}A}) = \Pr(X_{\lambda}A = a \mid X_{\lambda}A = x_{X_{\lambda}A}) \times \Pr((X|G)L[W] = b \mid x_{\lambda}A = a, X_{\lambda}A = x_{X_{\lambda}A})
\]

\[
= \frac{1}{q^{n-t}} \times \frac{1}{q^{\mu}} = \frac{1}{q^{n-t+\mu}}.
\]

The penultimate transition can be explained as follows. We have

\[
b = (X|G)L[W] = XL^{(0)}P[W] + GQ[W].
\]

The matrix \(Q[W]\) is invertible due to the fact that \(Q\) is a generator matrix of an \([N, \mu]\)-MDS code. Since \(X\) is known, the system (25) has a unique solution given by

\[
G = (b - XL^{(0)}P[W])(Q[W])^{-1}.
\]

Since \(G\) is uniformly distributed over \(\mathbb{F}_q^\mu\),

\[
\Pr((X|G)L[W] = b \mid X_{\lambda}A = a, X_{\lambda}A = x_{X_{\lambda}A}) = \Pr(G = (b - XL^{(0)}P[W])(Q[W])^{-1}) = \frac{1}{q^\mu}.
\]

Similarly to (24), the denominator in (23) is

\[
\sum_{c \in \mathbb{F}_q^{n-t}} \Pr(X_{\lambda}A = c \mid X_{\lambda}A = x_{X_{\lambda}A}) \times \Pr((X|G)L[W] = b \mid X_{\lambda}A = c, X_{\lambda}A = x_{X_{\lambda}A})
\]

\[
= q^{n-t} \frac{1}{q^\mu} \frac{1}{q^\mu} = \frac{1}{q^\mu}.
\]

From (23), (24), and (26), we obtain (22), as claimed.

From Theorem 5.8, Lemma 5.9, and Lemma 5.10 we obtain the main result of this section.

**Theorem 5.11:** The length of an optimal \((\mu, t, \delta)\)-strongly secure linear \(\eta\)-randomized \((m, n, X, f)\)-IC over \(\mathbb{F}_q (q \geq \kappa_q + \mu + 2\delta + 1)\) is \(\kappa_q + \mu + 2\delta\). Moreover, the code in Construction A achieves this optimal length.

VI. CONCLUSIONS AND OPEN QUESTIONS

In this paper, we initiate a study of the security aspects of linear index coding schemes. We introduce a notion of block security and establish two bounds on the security level of a linear index code based on the matrix \(L\). These analysis makes use of the minimum distance and the dual distance of \(C(L)\), the code spanned by the columns of \(L\). While the dimension of this code corresponds to the number of transmissions in the scheme, the minimum distance characterizes its security strength.

Our second contribution is the analysis of the strong security of linear index codes. New bounds on the length of linear index codes, which are resistant to errors, eavesdropping, and information leaking, are established. Index codes that achieve these bounds are constructed. These new bounds cannot be deduced directly from the existing results in network coding literature.

One important problem, which remains open, deals with a design of an optimal secure index coding scheme. This problem can be formulated as follows: given an instance of the ICSI problem, how to design \(L\), such that \(C(L)\) has the largest possible minimum distance? More specifically, let us define the binary side information matrix \(A = (a_{i,j})_{i \in [n], j \in [n]}\) as in [6], namely

\[
a_{i,j} = \begin{cases} 1 & \text{if } j = i \text{ or } j \in X_i \\ 0 & \text{otherwise} \end{cases}
\]

The problem is equivalent to finding a way to turn certain off-diagonal 1’s in \(A\) into 0’s, such that the rows of the resulting matrix generate an error-correcting code of the largest possible
minimum distance. It is very likely that this task is a hard problem. For comparison, even finding the minimum distance of an error-correcting code given by its generating matrix is known to be NP-hard [32].

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