Strings and Two-dimensional QCD for Finite $N$

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Abstract

The string theory description of $SU(N)$ Yang-Mills on an arbitrary two-dimensional manifold, previously developed for the large $N$ asymptotic expansion, is extended to include finite values of $N$. The theory is considered from two points of view, first using a canonical Hamiltonian formulation, second using a global description of the partition function. In both formalisms, the effect on the string theory of taking a finite value of $N$ is described by a local projection operator which has a simple description in terms of the symmetric group $S_n$.

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1 Introduction

Recently, there has been a renewed interest in Yang-Mills theory in two dimensions. The explicit description of the asymptotic $1/N$ expansion of the partition function in terms of an equivalent string theory gives hope that a similar connection might be found for four-dimensional gauge theories, in particular for four-dimensional QCD. A review of the recent work in this area is given in [1]. In this paper we extend the string description of the $SU(N)$ gauge theory to obtain an exact string-theoretic description for arbitrary finite $N$.

It was first shown by Migdal [2] almost 20 years ago that in two dimensions, the pure Yang-Mills gauge theory with any gauge group is exactly soluble, and can be described by a triangulation-invariant lattice theory. This work was later expanded upon [3, 4, 5, 6] to describe the pure gauge theory on an arbitrary Riemann surface. The general formula for the partition function of the two-dimensional Yang-Mills theory on a Riemann surface $M$ of genus $G$ and area $A$ is given by

$$Z(G, \lambda A, N) = \sum_R \frac{(\dim R)^{2-2G} e^{-\frac{2\pi}{\lambda A} C_2(R)}}{N^C_2(R)},$$

where the sum is taken over all irreducible representations of the gauge group, with $\dim R$ and $C_2(R)$ being the dimension and quadratic Casimir of the representation $R$. The coupling constant $\lambda$ is related to the gauge coupling $\tilde{g}$ by $\lambda = \tilde{g}^2 N$.

It was recently shown [7, 8, 9, 10] that by expanding the sum over representations in (1.1) in powers of the gauge group parameter $N$, the asymptotic expansion for the partition function as $N \to \infty$ can be written to all orders in $1/N$ in terms of a sum over maps from a string world sheet onto the manifold $M$. The partition function given by this asymptotic expansion can be regarded as a theory in its own right, the “large $N$ theory.” The string maps summed over are maps that are singular only at a finite number of points. The partition function (1.1) can then be rewritten in the form

$$Z(G, \lambda A, N) = \int_{\Sigma(M)} d\nu W(\nu),$$

where $\Sigma(M)$ is a set of branched covering maps

$$\nu : \mathcal{N} \to \mathcal{M},$$

(1.3)
from Riemann surfaces \( \mathcal{N} \) of arbitrary genus \( g \) onto \( \mathcal{M} \), and the weight of each map in the partition function is given by

\[
W(\nu) = \pm N^{2-2g} \left| S_\nu \right| e^{-\frac{n\lambda A}{2}}. \tag{1.4}
\]

In the weight, \( n \) is the degree of the map \( \nu \) and \( \left| S_\nu \right| \) is the symmetry factor of the map (the number of diffeomorphisms \( \pi \) of \( \mathcal{N} \) which satisfy \( \nu \pi = \nu \)). The sign of the weight depends upon the set of singular points in the map \( \nu \). The types of singularities allowed for maps \( \nu \) in the set \( \Sigma(\mathcal{M}) \) depend upon the gauge group and the genus \( G \). In [10], it was shown that in the large \( N \) expansion, the vacuum expectation values of Wilson loops can also be expressed in terms of a sum over open string maps, where the boundary of the string world sheet is mapped to the Wilson loop.

In the initial work [9, 10], the large \( N \) expansion was studied for the gauge groups \( SU(N) \) and \( U(N) \). The \( SU(N) \) theory describes two-dimensional QCD (QCD\(_{2}\)). Recently, this work has been expanded to the other gauge groups \( SO(N) \) and \( Sp(N) \) [11, 12]. For these gauge groups, the string world sheet may not be orientable; the exponent of \( N \) in (1.4) is then simply the Euler characteristic of the string world sheet. In other recent work, connections have been made between QCD\(_{2}\) and several other theories of current interest [13, 14, 15]. The theory has also been studied on the lattice, and shown to give rise to an equivalent theory of triangulated strings [16]. The behavior of the partition function of the \( U(N) \) theory on the sphere has been studied by Douglas and Kazakov [17]. They showed that for small areas \( \lambda A < \sqrt{\pi} \), the partition function in the large \( N \) limit undergoes a phase transition, and that in the small area phase the partition function becomes trivial. Such a phase transition does not occur on higher genus Riemann surfaces, and does not occur for finite values of \( N \). Recently, using similar methods the vacuum expectation value for a simple Wilson loop on the sphere has been studied and shown to behave similarly to the partition function [18, 19, 20, 21].

In this paper, we extend the earlier results to describe a string theory for finite \( N \) Yang-Mills on an arbitrary two-dimensional manifold \( \mathcal{M} \). We restrict attention in this paper to \( SU(N) \), although the results are easily extended to other gauge groups. The primary result is that this theory can also be described in terms of a weighted sum over string maps; however, in the finite \( N \) theory, it is necessary to introduce an additional projection operator into the string theory picture. Although this projection operator complicates the string theory somewhat, one of the other complicating features of the large \( N \) theory disappears for finite \( N \). Specifically, in the string theory description of large \( N \) QCD\(_{2}\), it is necessary to consider
two “chiral” sectors, corresponding to strings with two distinct orientations. When the string partition function is written in terms of a sum over maps from a string world sheet to the target space $\mathcal{M}$, the existence of these two sectors makes it necessary to give an orientation to the string world sheet and to distinguish between orientation-preserving and orientation-reversing maps. In the finite $N$ theory, only one chiral sector need be counted, so that the resulting string theory is essentially a theory of unoriented strings.

There are several possible ways to view the projection operator for the finite $N$ QCD$_2$ string theory. We explore two such views in this paper. First, we analyze the partition function of the finite $N$ theory in the Hamiltonian formulation. In the Hamiltonian picture, the Hilbert space of QCD$_2$ on a slice $S^1$ of $\mathcal{M}$ is simply the set of class functions on the gauge group $SU(N)$. Two natural sets of states spanning this Hilbert space are the set of characters of irreducible representations $\chi_R(U)$, and the set of products of traces

$$\Upsilon_\sigma(U) = \prod_{j=1}^s (\text{Tr} U^{k_j}), \quad (1.5)$$

where the matrices $U$ are in the fundamental representation, and $\sigma = \{k_1, \ldots, k_s\}$ is a set of positive integers with $0 < k_1 \leq \cdots \leq k_s$. The class functions given by products of traces can be interpreted in terms of a set of $s$ strings winding around the circle $k_j$ times; we will refer to these functions as “string” states. As $N \to \infty$, any finite set of distinct string states become linearly independent. For fixed finite $N$, however, the string states have linear dependencies, given by the Mandelstam identities [22]. To develop a string-theoretic interpretation of the finite $N$ gauge theory it is desirable to use the string states. The redundancy of these states can be eliminated by considering an “extended” state space consisting of finite linear combinations of formal string states corresponding to all the $\Upsilon_\sigma$ functions. As described in [23], the “physical” Hilbert space is then the quotient of this extended state space by the Mandelstam identities, suitably completed to form a Hilbert space. In this paper we follow an alternative method. First we complete the extended state space to obtain a Hilbert space which has the string states as an orthogonal basis, and then we identify the physical Hilbert space with a subspace. We explicitly construct the operator on the extended Hilbert space that projects down to this subspace; this projection operator has a simple form in terms of the string basis.

There is a natural Hamiltonian on the extended Hilbert space, which is most simply expressed in terms of the character basis. In fact, the Hamiltonian operator is diagonal in this basis and acts on the representation $R$ by multiplication by $C_2(R)$ times a constant.
In terms of the string basis, the Hamiltonian is no longer diagonal but contains interaction terms which describe the change in string windings caused by singular points such as branch points and infinitesimal tubes \[24, 14\]. In the finite \( N \) theory, we can use precisely the same Hamiltonian, restricted to the subspace of physical states.

The second view of the projection operator in the finite \( N \) theory arises when we analyze the partition function in a purely group-theoretic fashion, using techniques analogous to those presented in \[9, 10\]. These previous papers dealt with the large \( N \) expansion. If one replaces the sum over irreducible representations in (1.1) by a sum over all Young diagrams with finitely many boxes one obtains the partition function for the large \( N \) theory (or, more precisely, a single “chiral sector” thereof). Using the theory of Young diagrams this sum can be rewritten as a sum over elements of the symmetric groups \( S_n \). The group-theoretic terms in this expression can all be grouped into a single \( \delta \) function of a product of elements of \( S_n \). This \( \delta \) function can then be interpreted as a topological condition on permutations of sheets of a covering space, where a permutation \( \sigma \in S_n \) at a point \( x \in \mathcal{M} \) describes a multiple branch point singularity in the covering map; the positions of the singularities are integrated over, giving factors of the area of \( \mathcal{M} \) in the partition function. This gives a string-theoretic interpretation of the partition function for large \( N \) QCD in two dimensions.

Irreducible representations of \( SU(N) \) correspond to Young diagrams with \(< N \) rows. In the finite \( N \) case, we find that the restriction to irreducible representations of \( SU(N) \) in the sum (1.1) changes the \( \delta \) function of the group theoretic terms in the partition function in a way that corresponds to the insertion of a projection operator at a single fixed but arbitrary point on the manifold \( \mathcal{M} \). We find that this projection operator is mathematically equivalent to the projection operator constructed in the Hilbert space picture, when both are expressed in terms of the symmetric group. The “projection point” plays a role similar to that of the \( \Omega \)-points which arise in the large \( N \) theory on higher genus Riemann surfaces \[6, 14\]. (\( \Omega \)-points also appear in the finite \( N \) theory when \( \mathcal{M} \) is not a torus.) The projection points, like the \( \Omega \)-points, are points at which the string maps are allowed to have extra singularities. Just as the \( \Omega \)-points probably indicate some global structure in the string theory (as opposed to local interactions, which correspond to integrals over \( \mathcal{M} \)), the projection point will hopefully give some insight into what features must be added to a string action formalism for finite \( N \) QCD\(_2\).

In Section 2, we review the Hamiltonian description of QCD\(_2\) and the string interpretation of this theory in terms of bosonic creation and annihilation operators. Within this framework,
we explicitly construct the projection operator from the extended state space to the physical Hilbert space and describe the resulting string picture of the partition function for finite N QCD$_2$ on the torus. We observe that for SU(2), the projection operator has a particularly simple form. In Section 3, we rederive the projection operator of Section 2 by considering the partition function of the finite $N$ gauge theory from a purely group-theoretic point of view. We expand (1.1) as a formal power series and show that the restriction to irreducible representations of SU($N$) can be described algebraically by the insertion of a projection operator which is mathematically equivalent to the one derived in Section 2. Finally, in Section 4 we review our results and discuss the possible application of this work to Euclidean quantum gravity in 3 dimensions.

2 Hilbert Space Picture

We will now proceed to describe the string theory of QCD$_2$ from the Hamiltonian point of view. We begin by considering the theory on the cylinder $S^1 \times [0, 1]$. For a fixed finite value of $N$, the Hilbert space $\mathcal{H}_N$ of QCD$_2$ associated with a spatial $S^1$ slice is simply the linear space of class functions on the gauge group SU($N$). As described in Section 1, there are two natural bases for this Hilbert space. One basis is the set of characters $\chi_R(U)$ of irreducible representations of SU($N$). This basis is orthonormal under the inner product

$$\int dU \chi_R(U) \chi_{R'}(U^\dagger) = \delta_{R,R'},$$

(2.1)

where $dU$ is the invariant Haar measure on SU($N$) normalized to $\int dU = 1$. Another basis for $\mathcal{H}_N$ is the set of string functions $\Upsilon_\sigma$ defined by (1.5). For each $n$, there is a set of string functions indexed by (conjugacy classes of) permutations $\sigma \in S_n$. For finite $N$, the set of string functions spans $\mathcal{H}_N$, but also contains linear dependencies. For $N > n$, the set of string functions $\Upsilon_\sigma$ with $\sigma \in S_n$ is linearly independent; these class functions are related to the irreducible characters of SU($N$) by the Frobenius relations

$$\chi_R(U) = \sum_{\sigma \in S_n} \frac{\chi_R(\sigma)}{n!} \Upsilon_\sigma(U),$$

(2.2)

$$\Upsilon_\sigma(U) = \sum_{R \in Y_n} \chi_R(\sigma) \chi_R(U),$$

(2.3)

where $\chi_R(U)$ is the character of $U$ in the representation $R$ of SU($N$) associated with a Young diagram containing $n$ boxes, and $\chi_R(\sigma)$ is the character of the permutation $\sigma$ in the
representation of $S_n$ associated with the same Young diagram. (We denote by $Y_n$ the set of all Young diagrams with $n$ boxes.)

Rather than using the actual Hilbert space associated with irreducible representations of $SU(N)$ for finite $N$, we will find it more convenient to consider an extended Hilbert space $\tilde{H}$, as in [23]. This Hilbert space has an orthonormal basis given by the set of all Young diagrams with a finite number of boxes; we write $|R\rangle$ for the state corresponding to the Young diagram $R$. Given a permutation $\sigma \in S_n$, we then define the “string state” $|\sigma\rangle$ in $\tilde{H}$ by the Frobenius relation

$$|\sigma\rangle = \sum_{R \in Y_n} \chi_R(\sigma)|R\rangle.$$  

If the permutation $\sigma$ has cycles of length $k_1, \ldots, k_s$, the state $|\sigma\rangle$ corresponds physically to a state in which there are $s$ strings with winding numbers $k_1, \ldots, k_s$. Note that the string states corresponding to conjugate elements in $S_n$ are equivalent, so that

$$|\sigma\rangle = |\tau \sigma \tau^{-1}\rangle \quad \forall \sigma, \tau \in S_n.$$  

One can show that the other Frobenius relation,

$$|R\rangle = \sum_{\sigma \in S_n} \frac{\chi_R(\sigma)}{n!}|\sigma\rangle,$$

follows from this definition. Moreover, one has the following formula for the inner product of string states

$$\langle \sigma | \tau \rangle = \frac{\delta_{T_{\sigma} T_{\tau}} n!}{|T_{\sigma}|},$$

where $T_{\sigma}$ is the conjugacy class of elements in $S_n$ that contains $\sigma$. We can therefore rewrite the identity operator in $\tilde{H}$ as

$$1 = \sum_{n} \frac{1}{n!} \sum_{\sigma \in S_n} |\sigma\rangle \langle \sigma|.$$  

For a fixed finite value of $N$, the irreducible representations of $SU(N)$ are in one-to-one correspondence with the Young diagrams having $< N$ rows. Thus the Hilbert space $H_N$ for the finite $N$ theory may be identified with the subspace of $\tilde{H}$ spanned by states $|R\rangle$ corresponding to Young diagrams with $< N$ rows. We call this the “physical” subspace of $\tilde{H}$. Note that the inner product on $H_N$ coincides with the inner product on the physical subspace inherited from that on $\tilde{H}$. In terms of the string basis, the projection operator onto the physical subspace is written

$$P_N = \sum_{n} \sum_{R \in Y_n^N} |R\rangle \langle R|.$$
\[ = \sum_n \sum_{R \in Y_n^N} \sum_{\sigma, \tau \in S_n} \frac{1}{n!^2} |\sigma\rangle \langle R| R\rangle \langle \tau| \langle \tau| \]  
\[ = \sum_n \sum_{R \in Y_n^N} \sum_{\sigma, \tau \in S_n} \frac{d_R}{n!^2} \chi_R(\sigma \tau^{-1}) |\sigma\rangle \langle \tau|, \]  

where \( Y_n^N \) is the set of Young diagrams in \( Y_n \) with fewer than \( N \) boxes in each column, and \( d_R = \chi_R(1) \) is the dimension of the representation \( R \) of \( S_n \).

We will now discuss the Hamiltonian on the physical and extended Hilbert spaces \( \mathcal{H}_N \) and \( \tilde{\mathcal{H}} \). We assume that the manifold \( \mathcal{M} \) on which we are describing the theory (in this case the cylinder) is equipped with a metric \( g \), such that \( L \) is the length of the slice \( S^1 \) associated with the Hilbert space, and such that \( g_{tt} = 1 \). In terms of the basis of characters \( \chi_R(U) \) for the physical Hilbert space, the Hamiltonian \( H \) is a diagonal operator which acts by multiplication by \( \lambda LC_2(R)/2N \), where \( C_2(R) \) is the quadratic Casimir of the representation \( R \). (See for example [13].)

For a representation \( R \) associated with a Young diagram with \( n \) boxes and with a set of symmetric group characters \( \chi_R(\sigma), \sigma \in S_n \), the \( SU(N) \) quadratic Casimir is given by

\[ C_2(R) = nN + \frac{n(n - 1)\chi_R(T_2)}{d_R} - \frac{n^2}{N}, \]  

where \( T_2 \) is the conjugacy class of elements in \( S_n \) containing one cycle of length 2 and \( n - 2 \) cycles of length 1. We can thus define an associated Hamiltonian \( \tilde{H} \) on the extended Hilbert space \( \tilde{\mathcal{H}} \) by

\[ \tilde{H}|R\rangle = \frac{\lambda L}{2} \left( n + \frac{n(n - 1)\chi_R(T_2)}{Nd_R} - \frac{n^2}{N^2} \right) |R\rangle, \]  

where \( n \) is the number of boxes in the Young diagram \( R \). From (2.10), it is clear that \( \tilde{H} \) agrees with \( H \) on the physical Hilbert space.

The Hamiltonian \( \tilde{H} \) has a simple description in terms of bosonic raising and lowering operators [24, 14]. In the string basis, the extended Hilbert space \( \tilde{\mathcal{H}} \) is equivalent to a bosonic Fock space, where the ground state \( |0\rangle \) corresponds to a state with no string excitations, and the bosonic operators \( a_k, a_k^\dagger \) satisfying \([a_k, a_l^\dagger] = k\delta_{k,l}\) correspond to the annihilation and creation respectively of a string winding \( k > 0 \) times around the circle. Note that we are only including strings winding a positive number of times around the circle; this corresponds to the fact that we are constructing a theory of unoriented strings. In terms of these operators, the Hamiltonian on the extended Hilbert space can be written

\[ \tilde{H} = \frac{\lambda L}{2} \left[ \sum_{n>0} a_n^\dagger a_n + \frac{2}{N} \sum_{n,m>0} (a_{n+m}^\dagger a_n a_m + a_n^\dagger a_m^\dagger a_{n+m}) - \frac{1}{N^2} (\sum_{n>0} a_n^\dagger a_n)^2 \right]. \]  

(2.12)
It is convenient for the string interpretation to write this Hamiltonian as a sum of terms

$$\tilde{H} = H_0 + \frac{1}{N} H_1 - \frac{1}{N^2} (H_h + H_t), \quad (2.13)$$

where

$$H_0 = H_h = \frac{\lambda L}{2} \sum_{n>0} a_n^\dagger a_n \quad (2.14)$$

$$H_1 = \lambda L \sum_{n,m>0} \left( a_{n+m}^\dagger a_n a_m + a_n^\dagger a_m^\dagger a_{n+m} \right) \quad (2.15)$$

$$H_t = \lambda L \left[ \left( \sum_{n>0} a_n^\dagger a_n \right)^2 - \sum_{n>0} a_n^\dagger a_n \right]. \quad (2.16)$$

To interpret these terms string-theoretically we first consider the partition function for a torus. On the torus of area $A$ with cycles of length $L$ and $\beta$, the partition function for the finite $N$ theory is given by

$$Z(1, \lambda A, N) = \text{Tr}_{\tilde{H}} P_N e^{-\beta \tilde{H}}. \quad (2.17)$$

Note that dropping the projection operator $P_N$ from this expression gives precisely the partition function for a single chiral sector of the asymptotic expansion of the full theory for large $N$, as described in [9, 10]. We can now describe this partition function in terms of a sum over string maps with the types of singularities familiar from the large $N$ theory; the projection operator $P_N$ gives rise to a single additional singular point in the string covering map. The leading term in the Hamiltonian, $H_0$, can be interpreted as the “free” string Hamiltonian. This term gives a contribution of $e^{-\frac{\lambda A}{2} n}$ for a string configuration of total winding number $n$ on a manifold of area $A$. By expanding the exponential of the remaining terms in the Hamiltonian in (2.17), we can interpret the interaction terms $H_1, H_t, H_h$ as describing singularities associated with interactions between strings. For instance, the interaction term $H_1$ precisely describes the string interaction due to a simple branch point in the map from the string world sheet onto the target space. At such a branch point, either two strings with winding numbers $k, l$ combine to form a single string with winding number $k + l$, or a single string with winding $k + l$ splits to form 2 strings with winding numbers $k$ and $l$. An example of such a branch point interaction is shown in Figure [1]. The factor of $1/N$ associated with this interaction term is precisely the factor which one expects for a singularity which decreases the Euler characteristic of the string world sheet by 1. Similarly, the remaining interaction terms $H_t$ and $H_h$ give the contributions from adding a string-string interaction between any pair of strings such as would arise from an infinitesimal tube connecting the sheets of
the string world sheet, and a self-interaction term coming from the addition of infinitesimal handles to the string world sheet. In this way, the partition function can be rewritten as a sum over possible covering maps associated with sequences of string interactions located at arbitrary points on the target space $\mathcal{M}$. Because the interaction terms carry factors of $\lambda A$, this gives a natural measure to the space $\bar{\Sigma}(\mathcal{M})$ of covering spaces, where each singularity is given an integration measure proportional to the area measure on $\mathcal{M}$.

The only feature which remains to be interpreted in the string picture is the projection operator $P_N$. This operator can be simply interpreted in the same string language as being associated with an extra singularity point where a nontrivial string interaction occurs. Because this singularity point does not carry a factor of the target space area, we will fix a point $p \in \mathcal{M}$ where this singularity occurs. From (2.9), we see that a singularity associated with a permutation $\sigma \in S_n$ of sheets on the string covering space must give an extra factor of

$$P^{(N)}(\sigma) = \sum_{R \in Y_n^N} \frac{d_R}{n!} N^{n-K_\sigma} \chi_R(\sigma),$$

(2.18)

where $K_\sigma$ is the number of cycles in the permutation $\sigma$. The factor of $N^{n-K_\sigma}$ arises because of the change in the Euler characteristic of the covering space due to the permutation $\sigma$.

Thus, we have derived a string picture for the finite $N SU(N)$ gauge theory on the torus. In this theory, the set of maps $\bar{\Sigma}(\mathcal{M})$ is equal to the set of all topologically distinct covering maps $\nu$ from a surface $\mathcal{N}$ of arbitrary genus $g$ onto $\mathcal{M}$, with an arbitrary number of branch point, tube, and handle singularities, the positions of which are a set of modular parameters for $\nu$ each carrying a measure equal to $\lambda dA$. Because the Hilbert space only contains strings with positive winding, we need not consider $\mathcal{N}$ to have a fixed orientation; however, the manifold $\mathcal{N}$ must be orientable, since an orientation on $\mathcal{M}$ can always be pulled back to $\mathcal{N}$. The maps in $\bar{\Sigma}(\mathcal{M})$ are allowed to have an additional singularity at the fixed point

Figure 1: Branch point singularity associated with $H_1$
\( p \in \mathcal{M} \) associated with an arbitrary permutation \( \sigma \) in the sheets of the covering space. Such a singularity carries a weight factor of \( P^{(N)}(\sigma) \). By a careful analysis of the combinatorics, one can verify that the constant factor associated with a given cover \( \nu \) is precisely \( 1/|S_\nu| \). Thus, we find that the partition function for the finite \( N \) SU\((N)\) gauge theory is given by

\[
Z(1, \lambda A, N) = \int_{\Sigma(\mathcal{M})} d\nu W(\nu),
\]

with the weight of each map in the partition function being given by

\[
W(\nu) = (-1)^i P^{(N)}(\sigma) \frac{N^{2-2g}}{|S_\nu|} e^{-\frac{4\pi A}{g}},
\]

where \( i \) is the number of branch points in the map \( \nu \), and \( \sigma \) is the permutation associated with the singularity at the point \( p \).

With the exception of the singularity at \( p \), which we will refer to as a “projection point” singularity, this is precisely the formula for the asymptotic expansion of the partition function in a single chiral sector of the large \( N \) theory. The projection point, which exists at a fixed point in \( \mathcal{M} \) and does not carry a factor of the area, is analogous to the \( \Omega \)-points which arise in the large \( N \) theory on higher genus Riemann surfaces.

As a particularly simple example of the finite \( N \) theory, we consider the case \( N = 2 \). In this case, for any value of \( n \), there is only a single Young diagram in the set \( Y^2_n \). The corresponding representations of the symmetric group are the completely symmetric representations, which are trivial and have \( \chi_R(\sigma) = 1 \) for all \( \sigma \). Thus, in this case the weight of any permutation \( \sigma \) at a projection point simplifies to

\[
P^{(2)}(\sigma) = \frac{2^{n-K_\sigma}}{n!}.
\]

It is fairly straightforward to generalize the Hilbert space formalism used in this section to describe the finite \( N \) theory on higher genus Riemann surfaces. Along the lines of the axiomatic approach to conformal field theory \[25\], one can introduce a state

\[
D \in \hat{\mathcal{H}},
\]

corresponding to the state given on the boundary of a disk with zero area, and an operator

\[
T : \hat{\mathcal{H}} \otimes \hat{\mathcal{H}} \to \hat{\mathcal{H}},
\]

corresponding to the intertwining operator associated with a pair of pants (three-punctured sphere) of zero area. By combining these objects with the propagator described by the
cylinder of nonzero area, it is possible to build up an arbitrary Riemann surface. Although the forms of $D$ and $T$ are trivial in the basis of representations $|R\rangle$, they have a more complicated structure in terms of the string basis. For instance, the components of $D$ in the string basis are given by

$$D_\sigma = \langle \sigma | \exp(N \sum_{n \geq 1} a_n^\dagger) |0\rangle,$$

as was noted in [13]. In the string language, the state $D$ corresponds to the insertion of an $\Omega$-point singularity in the set of coverings, where the covering maps can have a singularity associated with an arbitrary permutation, and the operator $T$ can be associated with an $\Omega^{-1}$-point type singularity (we will review the description of these objects in the next section).

Again, to calculate the partition function in the finite $N$ theory, one must insert a projection operator at a single point on the manifold. Although this approach gives a fairly simple understanding of the finite $N$ result on a Riemann surface of general genus, we will not elaborate on the details of this construction here; rather, we will now derive the partition function for a Riemann surface of any genus using the global group-theoretic techniques originally used in [9, 10], and verify that the result is in agreement with that achieved by the Hilbert space approach in this section.

3 Partition Function

In [9, 10], the expression (1.1) for the partition function was rewritten as an asymptotic series in $1/N$ by expressing the coefficients in this expansion in terms of characters of the symmetric group $S_n$, and then giving a geometric interpretation in terms of the permutations of sheets of a covering space realized by transport around nontrivial loops in the manifold $\mathcal{M}$. The expansion of the partition function in $1/N$ was accomplished by writing the dimension as well as the quadratic Casimir of a representation associated with a Young diagram $R$ of $SU(N)$ in terms of characters of the symmetric group representation associated with the same Young diagram. In this section, we show that using this group-theoretic approach we can write the partition function for the finite $N$ theory on a manifold of any genus, and that the projection operator defined in Section 2 appears naturally as an object in the algebra of the symmetric group.

It was shown in [10] that using elementary formulae from the theory of symmetric group representations, the contribution to the partition function (1.1) from all Young diagrams
with a finite number of boxes can be rewritten as

\[
Z_Y(G, \lambda A, N) = \sum_n \sum_{R \in Y_n} (\dim R)^{2-2G} e^{-\frac{\lambda A}{2G} C_2(R)}
\]

\[
= \sum_{n,i,t,h \geq 0} \frac{e^{-n \lambda A}}{i! t! h!} N^{n(2-2G)-i-2(t+h)} \frac{(-1)^i n^h (n^2 - n)^t}{2^{t+h}}
\]

\[
\cdot \sum_{p_1,\ldots,p_i \in T_2} \sum_{s_1,t_1,\ldots,s_G,t_G \in S_n} \left[ \sum_{R \in Y_n} \frac{d_R}{n!^2} \chi_R(p_1 \cdots p_i \Omega_n^{2-2G} \prod_{j=1}^G s_j t_j s_j^{-1} t_j^{-1}) \right],
\]

where

\[
\Omega_n = \sum_{\sigma \in S_n} \sigma N^{K_\sigma - n}
\]

is an element of the group algebra on \(S_n\) with the property that

\[
\dim R = \frac{N^n}{n!} \chi_R(\Omega_n).
\]

When discussing the large \(N\) \(SU(N)\) gauge theory, the contribution from (3.1) is only part of the complete partition function. In addition, one must consider representations corresponding to Young diagrams with a finite number of columns with order \(N\) boxes (furthermore, it was shown by Douglas and Kazakov [17] that for \(G = 0\) and \(\lambda A < \pi^2\) even this set of representations is insufficient due to the phase transition at the point \(\lambda A = \pi^2\)). In the large \(N\) case, (3.1) can be described as giving the contribution to the partition function from a single chiral sector of the theory, corresponding to orientation-preserving string maps from a world sheet onto \(\mathcal{M}\). The interpretation of this expression in terms of string maps arises from taking the sum over all representations \(R\) of the symmetric group, giving

\[
\delta(\rho) = \frac{1}{n!} \sum_R d_R \chi_R(\rho),
\]

where the \(\delta\) function on the symmetric group algebra picks out the coefficient of the identity permutation. Inserting this result into (3.1), the contribution from the group-theoretic terms reduces to

\[
\sum_{p_1,\ldots,p_i \in T_2} \sum_{s_1,t_1,\ldots,s_G,t_G \in S_n} N^{n(2-2G)-i} \left[ \frac{1}{n!} \delta(p_1 \cdots p_i \Omega_n^{2-2G} \prod_{j=1}^G s_j t_j s_j^{-1} t_j^{-1}) \right].
\]

This expression has a simple geometric interpretation as the sum of a factor \(N^{2-2g}/|S_\nu|\) over all \(n\)-fold covers \(\nu\) of \(\mathcal{M}\) by (possibly disconnected) surfaces of any genus \(g\), where the covering map \(\nu\) has elementary branch points at a set of points \(w_1, \ldots, w_i\), and singularities
associated with arbitrary permutations of the covering sheets at a set of $2 - 2G$ fixed points called “$\Omega$-points” on $\mathcal{M}$ (if $G > 1$, there are $2G - 2$ “$\Omega^{-1}$-points”; at these points there may be an arbitrary number $x$ of $\Omega$-point type singularities all coalesced to a single point, each carrying a factor of $-1$). The geometric interpretation of (3.5) can be derived by associating with $s_j, t_j$ the permutations on sheets of a cover associated with transport about a set of loops $a_j, b_j$ generating the first homotopy group $\pi_1(\mathcal{M})$, and similarly associating with $p_j$ the permutations associated with transport around the elementary branch points $w_j$. Using this description of (3.5) in terms of covering maps, it is possible to rewrite (3.1) in terms of a sum of string maps. We define $\Sigma(\mathcal{M})$ to be the set of all orientation-preserving maps from a string world sheet to $\mathcal{M}$ which are locally covering maps at all but a finite number of singular points, where the singular points may be elementary branch points, $\Omega$-points or $\Omega^{-1}$-points, or infinitesimal tubes or handles contracted to points on $\mathcal{M}$. We use again the natural measure $d\nu$ on $\Sigma(\mathcal{M})$ given by associating with each singular point (other than the $\Omega$-points, which are taken to be fixed) a measure factor proportional to the area measure on $\mathcal{M}$ with proportionality constant $\lambda$. In terms of this set of string maps, the partition function in a single chiral sector of the theory for large $N$ can be written as

$$Z_Y(G, \lambda A, N) = \int_{\Sigma(\mathcal{M})} d\nu \ e^{-\frac{n\lambda A}{2} \frac{(-1)^i N^{2-2g}}{|S_\nu|} \sum j x_j},$$

where $n$ is the winding number of the map $\nu$, $i$ is the number of branch points in $\nu$, $|S_\nu|$ is the symmetry factor, $g$ is the genus of the covering space, and $x_j$ are the numbers of nontrivial twists at the $\max(0, 2G - 2)$ $\Omega^{-1}$-points in the map $\nu$.

We will now study how this analysis must be modified when we have a gauge group $SU(N)$ with finite $N$. Restricting the sum over representations to $Y^N_n$ for each $n$, the group theoretic terms in expression (3.1) take the form

$$\sum_{p_1, \ldots, p_i \in T_2} \sum_{s_1, t_1, \ldots, s_G, t_G \in S_n} N^{n(2-2G)-i} \sum_{R \in Y^N_n} \frac{d_R}{n!^2} \chi_R(p_1 \cdots p_i \Omega^{-2G} \prod_j s_j t_j s_j^{-1} t_j^{-1} \prod_{j} s_j t_j s_j^{-1} t_j^{-1}).$$

(3.7)

We now introduce a set of projection operators $P^{(N)}_n$. For each set of values $n, N$, $P^{(N)}_n$ is an element of the algebra over the symmetric group $S_n$, defined by

$$P^{(N)}_n = \sum_{\sigma \in S_n} \sum_{R \in Y^N_n} \frac{d_R}{n!} \chi_R(\sigma) \sigma = \sum_{\sigma \in S_n} N^{K_n-n} P^{(N)}(\sigma) \sigma.$$

(3.8)

These operators are projection operators in the sense that

$$(P^{(N)}_n)^2 = P^{(N)}_n,$$

(3.9)
which follows directly from the relation
\[ \frac{d_R}{n!} \sum_\sigma \chi_R(\sigma) \chi_S(\sigma^{-1} \tau) = \delta_{RS} \chi_R(\tau). \] (3.10)

Note that the projection operator \( P_n^{(N)} \) commutes with all elements of \( S_n \). By inserting the projection operator into a \( \delta \) function on the symmetric group, we have
\[ \delta(\sigma P_n^{(N)}) = \sum_{R \in Y_n} \frac{d_R}{n!} \chi_R(\sigma). \] (3.11)

Using this relation in the expression (3.7) for the group theoretic terms in the finite \( N \) partition function, we get
\[ \sum_{p_1, \ldots, p_i \in T_2} \sum_{s_1, t_1, \ldots, s_G, t_G \in S_n} N_n^{(2-2G)-i} \left[ \frac{1}{n!} \delta(p_1 \cdots p_i \Omega_n^{2-2G} p_n^{(N)} \prod_j s_j t_j s_j^{-1} t_j^{-1}) \right]. \] (3.12)

Just as the expression (3.5) can be described in terms of a sum over covering maps with \( s_j, t_j \) representing the permutations on the sheets of the cover given by transporting around the generators of \( \pi_1(\mathcal{M}) \), \( p_j \) representing permutations around elementary branch points, and the term \( \Omega_n^{2-2G} \) giving the contribution of \( 2-2G \) \( \Omega \)-points, we can interpret (3.12) in a similar fashion as a sum over covering maps. All the possible singularities for covering maps in the large \( N \) case still appear here; in addition, however, there is a single fixed “projection” point on \( \mathcal{M} \) where a singularity corresponding to an arbitrary permutation can occur. The weight associated with a singularity giving a permutation \( \sigma \in S_n \) is given by \( P^{(N)}(\sigma) \), just as we found in the Hamiltonian formulation. Here, this result arises by taking the expression from (3.11) and dividing by the factor \( N^{K_\sigma - n} \) which appears because this singularity increases the genus of the covering space. We can now proceed to define a string theory of maps for the finite \( N \) theory for any genus \( G \). We define \( \Sigma(\mathcal{M}) \) to be the set of all covering maps of \( \mathcal{M} \) with a finite number of the usual types of movable singularities (branch points, infinitesimal tubes, and infinitesimal handles), \( |2-2G| \) singularities of a general type at \( \Omega \)-points or \( \Omega^{-1} \)-points, and an additional singularity at the projection point associated with a permutation \( \tau \) on the \( n \) sheets of the cover. The partition function for finite \( N \) of the QCD$_2$ gauge theory is then given by
\[ Z(G, \lambda A, N) = \int_{\Sigma(\mathcal{M})} d\nu e^{-\frac{nA}{2} \frac{N^{2-2g}}{|S|}} (-1)^i \sum_j x_j P^{(N)}(\tau). \] (3.13)

We have thus given a description of the partition function for finite \( N \) QCD$_2$ on an arbitrary Riemann surface \( \mathcal{M} \) in terms of a weighted sum over a certain class of maps from
a string world sheet onto \( \mathcal{M} \). We have derived this result from two rather different points of view and arrived at the conclusion that the essential feature in the string picture for the partition function of the finite \( N \) theory is the insertion at a single point of a singularity weighted with the coefficients of the projection operator \( P_n^{(N)} \). Otherwise, the string picture is extremely similar to that in the large \( N \) theory, with the change that the string world sheet is not oriented in the finite \( N \) theory.

A feature of this string description of the theory which complicates the picture somewhat is the explicit appearance of factors of \( N \), the inverse string coupling, in the weights \( P^{(N)}(\sigma) \) associated with singularities at the projection point. This dependence means that the terms in the expansion cannot simply be ordered by genus according to the power of \( N \) they carry. In the string picture, this would correspond to string interaction terms with coefficients proportional to the inverse string coupling; such terms make the calculation of explicit results more complicated. This also leads to an interesting and somewhat nontrivial question about the relationship between the finite \( N \) and large \( N \) theories. Clearly, we expect that the coefficients of the asymptotic \( 1/N \) expansion should really be calculating the behavior of the finite \( N \) theory as \( N \to \infty \). From this string picture, however, this correspondence is not obvious. For example, in the large \( N \) theory, when \( G > 1 \) we expect the leading nontrivial term in the asymptotic expansion of the partition function to be given by

\[
Z_\infty(G, \lambda A, N) = 1 + 2N^{2G-2}e^{-\frac{\lambda A}{2}} + O(N^{2G-2}).
\]  
(3.14)

(We use the notation \( Z_\infty \) to differentiate the asymptotic expansion for large \( N \) from the partition function at finite \( N \).) This implies that

\[
\lim_{N \to \infty} N^{2G-2} (Z(G, \lambda A, N) - 1) = 2e^{-\frac{\lambda A}{2}}.
\]  
(3.15)

Showing that (3.15) is true from the string expression (3.13) is a rather nontrivial proposition. In fact, once one has removed the \( 1/N \) ordering of the asymptotic expansion for the partition function, it is difficult to calculate anything concrete because in general an arbitrarily large number of terms contribute to the partition function. One might try ordering in powers of \( e^{-\lambda A} \), but this is impossible because arbitrarily large powers of \( \lambda A \) appear.

4 Conclusions

We have seen that a string-theoretic description of the partition function for finite \( N \) QCD\(_2\) can be attained at the expense of introducing an arbitrary “projection point” at which a
permutation of the sheets of the cover occurs. Like the \( \Omega \)-points which occur already in the large \( N \) theory, the fact that these singularities do not appear with area factors suggests that rather than arising from a local interaction, they represent a global phenomenon. Indeed, this is rather clear in the case of the projection point, since it is the expression of the fact that we may obtain the finite \( N \) theory by imposing the Mandelstam identities. In string-theoretic terms, the Mandelstam identities may be regarded as a constraint \([23]\). One complicating feature of the finite \( N \) string picture is that the weights associated with singularities at the projection point contain explicit factors of \( N \), the inverse string coupling. This implies that a Lagrangian description of the theory on the world sheet will include string interaction terms with coefficients proportional to the inverse string coupling.

Recently it has been shown by Cordes, Moore, and Ramgoolam that the counting of string maps with \( \Omega \)-point type singularities can be expressed in a much simpler language by interpreting the sum over \( \Omega \)-points as an orbifold Euler characteristic of a space of covering maps, which appears naturally in the context of a topological-type field theory on the world sheet \([27]\); a similar world sheet topological theory was described in \([28]\) using harmonic maps. It would be interesting to explore the possibility of finding a similar interpretation for the extra terms arising from the projection point singularities; the fact that these singularities carry factors of \( N \) rather then \( 1/N \), however, may complicate such an interpretation.

An interesting challenge will be to calculate the vacuum expectation values of Wilson loops for finite \( N \) QCD\(_2\) in the string-theoretic framework. From the results for Wilson loops in the large \( N \) theory \([10]\) we expect the result to be expressed in terms of a sum over maps from open string world sheets with boundaries which are taken to the Wilson loops by the string maps. For \( SU(2) \), it seems that such a description of Wilson loop VEV’s should be possible. In this case, the Wilson loops are not oriented, and a Wilson loop can thus bound a string world sheet on either side. Mathematically, this is simply a result of the fact that the fundamental and the conjugate representations are identical for \( SU(2) \). Even for \( SU(2) \), a rigorous description of Wilson loop VEV’s in terms of a sum over string maps is complicated by self-intersections of the Wilson loops. In principle, one should be able to construct such a description with self-intersections described by twists in the string world sheet, as was done for the large \( N \) theory in \([10]\). It would be interesting to work through the details of this calculation and compare to results on Wilson loops in the \( SU(2) \) theory achieved through other methods, such as in \([29]\). For finite values of \( N \) greater than 2, the problem is complicated further by the fact that the tensor product of \( N \) fundamental
representations contains the trivial representation. Thus, it is difficult to see how to bound a string world sheet by single Wilson loops. Physically, this obstruction corresponds to the existence of baryons, which are nonexistent in the large $N$ theory, and which are essentially equivalent to mesons for $SU(2)$. Finding a clean string-theoretic description of Wilson loop VEV’s for $N > 2$ may require an alternative to the “projection point” formalism we have described here.

Finding a string interpretation of Wilson loop VEV’s for finite values of $N$ would, however, shed interesting new light on theories other than QCD$_2$, as described in [23]. First, if one could simply handle the $\lambda \to 0$ limit of the theory one could obtain a string-theoretic interpretation of $BF$ theory, also known as topological Yang-Mills theory. (For a review of this theory, see for example [26].) When the gauge group is $SO(3)$, the Wilson loop vacuum expectation value $\langle W(\gamma_1) \cdots W(\gamma_n) W(\eta_1) \cdots W(\eta_m) \rangle$ in $BF$ theory may also be interpreted as the inner product of string states $\langle \gamma_1, \ldots, \gamma_n | \eta_1, \ldots, \eta_m \rangle$ in Euclidean quantum gravity on the manifold $\mathbf{R} \times \mathcal{M}$. In the quantum gravity context, one may think of the curves $\eta_i$ as living in $\{t_1\} \times \mathcal{M}$, the curves $\gamma_i$ as living in $\{t_2\} \times \mathcal{M}$, and the string world sheets as being mapped into $[t_1, t_2] \times \mathcal{M}$. (Since the Hamiltonian constraint in quantum gravity effectively makes the Hamiltonian zero, the results are independent of $t_1$ and $t_2$.) These expectation values can then be interpreted as describing an inner product on the physical Hilbert space of gravity in the loop variable formulation. Such a construction would give a natural interpretation of Euclidean quantum gravity in 3 dimensions as a string theory.

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