Finite-size effects in one-dimensional Bose-Einstein condensation of photons

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Abstract

The Bose-Einstein condensation (BEC) of photons has been realized in one- and two-dimensional systems. When considering the influence of finite-size effect, the condensation in the one-dimensional fibre is of special interest since such a condensation cannot occur in the thermodynamic limit due to the linear dispersion relation of photons. The finite-size effect must play a key role in this system and needs a detailed description. However, the previous theoretical analysis of finite-size effect is often not accurate enough and only gives the leading-order contribution due to a divergence difficulty. In this paper, by using an analytical continuation method to overcome the divergence difficulty, we give an analytical treatment for the finite-size effect in BEC. The analytical expressions of the critical temperature or critical particle number with higher order correction and the chemical potential below the transition point are presented. Our result shows that in a recent experiment, the deviation between experiment and theory is overestimated, most of which is caused by the inaccurate theoretical treatment of the finite-size effect. By taking into account the next-to-leading correction, we find that the actual deviation is much smaller.

1 Introduction

The Bose-Einstein condensation (BEC) of photons was generally believed to be impossible since the number of photons is not conserved and the extremely weak interaction between photons cannot thermalize the gas. However, the situation changed in recent years. By trapping photons in a dye-filled microcavity, the BEC has been realized in two-dimensional systems [1, 2, 3, 4]. In these experiments, the photons are trapped between two curved mirrors. The fixed longitudinal momentum gives an effective mass to the photon and a non-vanishing chemical potential to the photon gas. The repeated absorption and emission cycle of the dye molecules thermalizes the photon gas. Recently, a one-dimensional photon condensation is also reported [5]. In the experiment,

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the photons are confined in a closed erbium–ytterbium co-doped fiber with a cutoff wavelength. The existence of the cutoff wavelength gives the photons a nonvanishing chemical potential.

In the experiments of the photon condensation, the finite particle number makes the behavior of the phase transition different from the thermodynamic limit case. In particular, the finite-size effect in the one-dimensional condensation has some different feature since it cannot occur in thermodynamic limit due to the linear dispersion relation of photons. This implies that the finite-size effect has a significant influence in this case and needs to be carefully analyzed.

Many studies have been devoted to the finite-size effect in BEC. The corrections to the critical temperature and the condensate fraction have been found, but often only the leading order term can be obtained [6, 7, 8]. The main obstacle for accurately studying the finite-size effect is the divergence problem: When taking account into the contribution from the discrete energy levels of the trapped particles accurately, most terms in the expressions of thermodynamic quantities become divergent at the transition point.

To overcome the divergence difficulty, we will use an analytical continuation method to give an analytical treatment to the problem of photon condensation [9, 10]. By this way, we will obtain more accurate expressions of critical temperature and condensate fraction with next-to-leading corrections. We will also give the analytical expression of the chemical potential, which is hard to obtain before. At low temperature, the chemical potential is linear in temperature, which is quite different from the thermodynamic limit case. The comparison with the numerical solution confirms our result.

In the experiment of one-dimensional photon condensation [5], the deviation of the critical particle number between experiment and theory is about 5.6%. However, according to our result, this deviation is mainly caused by the inaccurate estimate of the finite-size effect in the previous studies. Our result shows that the contribution of the finite-size effect should be 4.2% higher than the previous prediction. Therefore the agreement of experiment and theory is actually very well.

This paper is organized as follows. In section 2 we give an analytical treatment to the finite-size effect of the photon condensation in one dimension. In section 3 we compare our result with experiment. Conclusions and some discussion are presented in section 4.

2 Critical temperature and chemical potential

Consider photons in a one-dimensional closed fiber with length $L$ and index of refraction $n$. The possible frequencies of the photons are restricted by periodic boundary conditions as

$$\omega = m' \frac{2\pi c}{nL} \equiv m' \Delta, \quad (1)$$
where \( m' \) is an integer, and we have introduced \( \Delta = \frac{2\pi c}{nL} \) with \( c \) the speed of light in vacuum. If there is a cutoff frequency \( \omega_0 = m_0\Delta \), namely, only the photons with a frequency higher than \( \omega_0 \) can exist in the fiber and the quantum number \( m' \) in equation (1) must be larger than \( m_0 \). For convenience, we shift the energy spectrum to make the ground-state energy vanish. Then the spectrum of the photons in the fiber becomes

\[
\varepsilon_m = m\hbar\Delta, \quad (m = 0, 1, 2, 3, \cdots)
\]

Clearly, the photon in such a system has the same energy spectrum as that of nonrelativistic particles in a one-dimensional harmonic trap, so these two kinds of systems should show the same transition behavior.

As we know, the BEC occurs when the number of excited particles \( N_e \) equals the total number of particles \( N \) at the chemical potential \( \mu = 0 \). The excited photon number is

\[
N_e = \sum_{m=1}^{\infty} \frac{1}{e^{\beta(\varepsilon_m - \mu)} - 1},
\]

where \( \beta = 1/k_B T \) with \( k_B \) is the Boltzmann constant. In the thermodynamic limit, the energy spectrum becomes continuous and the density of states is \( \rho(\varepsilon) = 1/\hbar\Delta \), the summation in equation (3) is converted to an integral as

\[
N_e = \frac{1}{\hbar\Delta} \int_0^\infty \frac{1}{e^{\beta(\varepsilon - \mu)} - 1} d\varepsilon = \frac{1}{\beta\hbar\Delta} g_1(e^{\beta\mu}),
\]

where \( g_\sigma (z) = \sum_{\ell=1}^{\infty} z^\ell / \ell^\sigma \) is the Bose-Einstein integral, which has the following asymptotic behavior

\[
g_\sigma (e^{\beta\mu}) \approx \begin{cases} 
\zeta(\sigma), & (\sigma > 1) \\
-\ln (-\beta\mu), & (\sigma = 1) \\
\Gamma(-\sigma + 1) \frac{1}{(-\beta\mu)^{-\sigma+1}}, & (\sigma < 1)
\end{cases} \quad (\mu \to 0)
\]

Then in the thermodynamic limit, \( N_e \) in equation (4) is divergent at \( \mu = 0 \), so there is no phase transition.

On the other hand, in finite systems, the energy spectrum is discrete and the first excited energy is not 0, the summation in equation (3) should be convergent and a finite critical temperature can be obtained. In ref. [5], the summation is approximately converted to an integral similar to equation (4) but the lower limit of the integral is replaced by the first excited energy \( \hbar\Delta \). Then the critical particle number can be calculated as [5]

\[
N_e^{(0)} = \frac{k_B T}{\hbar\Delta} \ln \frac{k_B T}{\hbar\Delta}.
\]

In this treatment, the interval between the ground state and the first excited state is taken into account, but the higher levels are still regarded as continuous. In fact, many previous studies of BEC in finite systems along the similar line. The finite-size effect of the BEC in one-dimensional harmonic trap is also
discussed in refs. \[6, 7, 8\]. Though the treatments have some difference, they all depended on similar approximations and can only give the leading-order correction similar to equation (6) (may differ by a factor).

Obviously, a more rigorous treatment of equation (3) is to perform the summation directly. To do this, one can expand every term in the sum as

\[ N_e = \sum_{m=1}^{\infty} \frac{1}{e^{\beta (\varepsilon_m - \mu)} - 1} = \sum_{m=1}^{\infty} \sum_{\ell=1}^{\infty} \frac{e^{-\beta (\varepsilon_m - \mu)}}{\ell^2} = \sum_{\ell=1}^{\infty} e^{\ell \beta \mu} K (\ell \beta \Delta), \]  

(7)

where

\[ K (t) = \sum_{m=1}^{\infty} e^{-mt} = \frac{1}{e^t - 1} \]  

(8)

is the global heat kernel \[11, 12, 13\]. For small \( t \), the heat kernel (8) can be expanded as a series of \( t \),

\[ K (t) = \sum_{k=0}^{\infty} C_k t^{k-1}, \quad (t \to 0^+) \]  

(9)

with the coefficients

\[ C_0 = 1, C_1 = -\frac{1}{2}, C_2 = \frac{1}{12}, C_3 = 0, C_4 = -\frac{1}{720}, \cdots \]  

(10)

Substituting the heat kernel expansion (9) into equation (7), we have

\[ N_e = \sum_{k=0}^{\infty} C_k (\beta h \Delta)^{k-1} g_{1-k} (e^{\beta \mu}). \]  

(11)

A similar treatment can also apply to the grand potential and other thermodynamical quantities, and these quantities are also expressed as series of the Bose-Einstein integrals. The higher order correction terms can describe the effects of boundary, potential, or topology, depending on the specific systems. This heat kernel expansion approach has been applied to various problems in statistical physics \[11, 14\]. However, a serious difficulty arises when considering the problem of BEC phase transition. Due to the asymptotic form of the Bose-Einstein integral equation (5), every term in equation (11) is divergent at \( \mu \to 0 \), and the divergence becomes more severe in the higher orders. This divergence problem is the main obstacle for treating the problem of phase transition in finite systems. As mentioned above, in ref. \[5\], the divergence is avoided by replacing the summation of excited states by an integral approximately, and only gives the leading-order correction to the critical temperature. If we want to obtain a more accurate result, the divergence problem in equation (11) must be solved. In the following, we will use an analytical continuation method \[9, 10\] based on the heat kernel expansion to overcome the divergence problem.

First, substituting the leading term of the asymptotic expansion of each Bose-Einstein integral (14) into equation (11) gives

\[ N_e = -C_0 - \frac{1}{\beta h \Delta} \ln (-\beta \mu) + \sum_{k=1}^{\infty} C_k (\beta h \Delta)^{k-1} \Gamma (k) \frac{1}{(-\beta \mu)^k}. \]  

(12)
The summation in the second term can be represented by the heat kernel if the gamma function is replaced by the integral form
\[ \Gamma (\xi) = \int_0^{\infty} x^{\xi-1+s} e^{-x} dx, \quad (s \to 0) \tag{13} \]
where we have introduced a small parameter \( s \) which will be taken as 0 at the end of the calculation. Then equation (12) becomes
\[
N_e = -C_0 \frac{1}{\beta \hbar \Delta} \ln (-\beta \mu) + \int_0^{\infty} dx e^{-x} x^s \left[ K \left( \frac{x \hbar \Delta}{-\mu} \right) - C_0 \frac{-\mu}{x \hbar \Delta} \right] \\
= -C_0 \frac{1}{\beta \hbar \Delta} \ln (-\beta \mu) + \frac{1}{-\beta \mu} \Gamma (1 + s) \sum_{m=1}^{\infty} \left( 1 + \frac{m \hbar \Delta}{-\mu} \right)^{-1-s} - C_0 \frac{1}{\beta \hbar \Delta} \Gamma (s). \tag{14} \]

In the last step, the definition of heat kernel (8) has been employed to perform the integral. For \( \mu \to 0 \), the summation in equation (14) becomes
\[
\sum_{m=1}^{\infty} \left( 1 + \frac{m \hbar \Delta}{-\mu} \right)^{-1-s} \approx \sum_{m=1}^{\infty} \left( \frac{m \hbar \Delta}{-\mu} \right)^{-1-s} = \frac{(-\mu)^{1+s}}{(\hbar \Delta)^{1+s}} \zeta (1 + s), \tag{15} \]
where \( \zeta (s) = \sum_{n=1}^{\infty} n^{-s} \) is the Riemann \( \zeta \)-function.

Now taking the limit \( s \to 0 \) in equation (14), we have
\[
N_e \approx -\frac{1}{\beta \hbar \Delta} \ln (-\beta \mu) + \frac{1}{\beta \hbar \Delta} \left( \ln \frac{-\mu}{\hbar \Delta} + \gamma_E \right) \\
= \frac{1}{\beta \hbar \Delta} \left( \ln \frac{1}{\beta \hbar \Delta} + \gamma_E \right), \tag{16} \]
where the Euler constant \( \gamma_E = 0.577216 \). In this result, all the divergent terms of \( s \) and \( \mu \) are canceled, and the expression of the excited-state particle number is completely analytical. Therefore, with the help of the idea of analytical continuous, the heat kernel expansion is successfully applied to the phase transition point and the divergence is eliminated.

From equation (16), the critical particle number for a given temperature \( T \) is obviously
\[
N_c = \frac{k_B T}{\hbar \Delta} \left( \ln \frac{k_B T}{\hbar \Delta} + \gamma_E \right), \tag{17} \]
and the critical temperature for a fixed particle number \( N \) is
\[
T_c = \frac{\hbar \Delta}{k_B W (N e^{\gamma_E})}, \tag{18} \]
where \( W (z) \) is the Lambert \( W \) function, satisfying \( z = W (ze^z) \). This critical temperature is a little lower than the previous result corresponding to the critical particle number (6)
\[
T_0 = \frac{\hbar \Delta}{k_B W (N)}. \tag{19} \]
According to the asymptotic expansion of the Lambert function $W(x) \approx \ln x - \ln \ln x$ for $x \to \infty$, the critical temperature can be approximated as

$$T_c \approx \frac{\hbar \Delta}{k_B} \frac{N}{\ln N + \gamma E - \ln (\ln N + \gamma E)}.$$  \hspace{1cm} (20)

We retain the second term in the denominator since for a relative small particle number, e.g. $N \sim 10^4$, $\ln \ln N$ is not much smaller than $\ln N$.

The condensate fraction is straightforward from equation (16),

$$\frac{N_0}{N} = 1 - \frac{1}{N} \frac{k_B T}{\hbar \Delta} \left( \ln \frac{k_B T}{\hbar \Delta} + \gamma E \right).$$  \hspace{1cm} (21)

This result is not very accurate, especially near the transition point. The reason is that the chemical potential has been assumed to be vanished below the transition point in the above calculation. As the temperature tends to the transition point, the deviation of the chemical potential from 0 becomes larger and larger. To describe the phase transition more accurately, we need to find the form of the chemical potential.

The expression of chemical potential $\mu$ can be addressed by the help of the analytical result of the excited-state number (16). For a small but nonzero chemical potential $\mu$, the ground-state particle number which is about $1/(-\beta \mu)$ is not zero at the phase transition point. Then the total particle number $N$ should contain the contributions from both the ground state and the excited states:

$$N = \frac{1}{-\beta \mu} + N_e.$$  \hspace{1cm} (22)

Here the excited-state particle number $N_e$ takes the same form as equation (14), but in the summation (15), an extra term which is proportional to $\mu$ should be added. Similar to the above procedure, we can obtain

$$N_0 = \frac{1}{-\beta \mu} - \frac{\beta \mu}{(\beta h \Delta)^2} - \frac{1}{2} + \frac{1}{2} \frac{1}{\beta h \Delta},$$  \hspace{1cm} (23)

where $N_0$ has been given in equation (21). In the right hand side of equation (23), the last two terms are small. After neglecting these two terms, the chemical potential can be solved as

$$\mu = -\frac{\sqrt{6} \hbar \Delta}{\pi} \left[ 1 + \left( \frac{\sqrt{3} T_0 N_0}{T N/\ln N} \right)^2 - \frac{\sqrt{3} T_0}{\sqrt{2} \pi} \frac{N_0}{T N/\ln N} \right].$$  \hspace{1cm} (24)

An interesting feature of this result is that at low temperature $T \ll T_c$, the chemical potential is

$$\mu \approx -\hbar \Delta \frac{1}{\ln N} \frac{T}{T_0}, \quad (T \ll T_c)$$  \hspace{1cm} (25)
Figure 1: The chemical potential below the critical temperature for total particle number \( N = 10^4 \). Our result of critical temperature \( T_c \) is lower than the previous one \( T_0 \). At low temperature, the chemical potential is approximately linear related to the temperature.

which is linearly related to the temperature. This is different from the thermodynamic limit result

\[
\mu = -k_B T e^{-\frac{\Delta E}{k_B T}} N, \quad (T \ll T_c)
\]  

which is exponential small. In figure 1 we compare the chemical potential in equation (24) with the thermodynamic limit one and the exact numerical solution, and it confirms the above low-temperature result.

3 Comparison with the experiment

In equation (17), we present the critical particle number of BEC in the one-dimensional photon system. Compared with the previous result [5] given in equation (10), the leading-order term is the same, but our approach also gives a new next-to-leading correction term. This next-to-leading correction leads to a relative deviation as

\[
\frac{N_c - N_c^{(0)}}{N_c^{(0)}} = \frac{\gamma E}{\ln \frac{\theta_B}{\Delta E}} \sim \frac{\gamma E}{\ln N_c^{(0)}}.
\]  

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It indicates that the previous treatment in which the excited states are regarded as continuous gives a lower prediction of the order of $1/\ln N$, which is usually not very small in realistic systems.

The relevant experimental parameters are as follows \cite{5}. The length of the fiber $L = 27m$, the refraction coefficient $n = 1.444$, the critical temperature $T = 296K$, and the cutoff wavelength $\lambda_0 = 1568nm$. Then the critical particle numbers given by equations (6) and (17) are

$$N_c^{(0)} = 1.09 \times 10^7, \quad N_c = 1.14 \times 10^7.$$  \hspace{1cm} (28)

Our prediction of $N_c$ is about 4.2\% higher than $N_c^{(0)}$ given in Ref. \cite{5}.

In the experiment \cite{5}, the measured quantity is the pump power, which is proportional to the photon number, and the measurement result is $P_{\text{exp}}^c = 9.5\mu W$. Compared with the theoretical prediction $P_{c}^{(0)} = 9.0\mu W$ \cite{5}, the experimental result is about 5.6\% higher. This is not a large deviation, but according to the above analysis, most of the deviation is caused by the inaccurate theoretical prediction. Our result shows that the theoretical value of the critical particle number should increase by about 4.2\%. Consequently, including the next-to-leading contribution of the finite-size effect greatly improves the agreement between experiment and theory.

4 Conclusion and discussion

In this paper we give a more systematic and accurate discussion on the finite-size effect in one-dimensional Bose-Einstein condensation of photons. By using an analytical continuous method based on the heat kernel expansion, we overcome the divergence difficulty and obtain the next-to-leading order finite-size corrections on thermodynamic quantities. In the experiment of one-dimensional photon BEC \cite{5}, the measurement value of the critical particle number is about 5.6\% higher than the previous theoretical prediction. However, our result shows that most part of the deviation arise from the inaccurate analysis of the finite-size effect. When taking account of the next-to-leading correction, the deviation between experiment and theory becomes about 1.4\%.

The magnitude of the finite-size effect is closely related to the spatial dimension of the system. In fact, the most important factor determining the statistical properties is the density of states, and the density of states is strongly affected by the spatial dimension. For example, in a two-dimensional harmonic trap, the leading term of the finite-size correction to the critical temperature is of the order of $\ln N/\sqrt{N}$, and the next-to-leading correction has the order of magnitude $1/\sqrt{N}$ \cite{10}, which is often negligible. However, in the one-dimensional case, the leading correction is about $N/\ln N$, and our calculation gives the next-to-leading term of the order of $1/\ln N$, which is much larger than the two-dimensional case. In fact, in the thermodynamic limit, photon BEC cannot occur in one dimension, so the correction of the finite-size effect in one-dimensional system must be significant. The same behavior also appears in similar systems, e.g., non-
relativistic particles in one-dimensional harmonic traps or in two-dimensional boxes.

The method used in this paper is based on the heat kernel expansion. We know that the heat kernel expansion is a short-wavelength expansion (high-energy expansion). In principle, it is only applicable to the high temperature and low density case. When applying the heat kernel expansion to the problem of phase transition, the divergence problem arises indeed. In this paper, however, we show that by the help of the analytical continuation method, the application range of heat kernel expansion can be extended to below the transition point, and the thermodynamic quantities can also be obtained analytically.

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