Asymptotic symmetry and group invariance for randomization

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Abstract

Symmetry is a cornerstone of much of mathematics, and many probability distributions possess symmetries characterized by their invariance to a collection of group actions. Thus, many mathematical and statistical methods rely on such symmetry holding and ostensibly fail if symmetry is broken. This work considers under what conditions a sequence of probability measures asymptotically gains such symmetry or invariance to a collection of group actions. Considering the many symmetries of the Gaussian distribution, this work effectively proposes a non-parametric type of central limit theorem. That is, a Lipschitz function of a high dimensional random vector will be asymptotically invariant to the actions of certain compact topological groups. Applications of this include a partial law of the iterated logarithm for uniformly random points in an $\ell^n_\infty$-ball and an asymptotic equivalence between classical parametric statistical tests and their randomization counterparts even when invariance assumptions are violated.

1 Introduction

The central limit theorem stands as one of the most fundamental results in probability theory. In its essence, sequences of measures properly normalized asymptotically approach the Gaussian, and the standard Gaussian measure on $\mathbb{R}^n$ satisfies many symmetries: reflective, permutative, rotational, etc. This elicits a question: under what conditions will a sequence of measures asymptotically achieve certain symmetries?

In mathematics, the symmetry of an object is encoded by said object’s invariance to a collection of group actions. For example, the standard Gaussian measure on $\mathbb{R}^n$ is invariant to any element of the orthogonal group $O(n)$ comprised of all $n \times n$ matrices $M : \mathbb{R}^n \to \mathbb{R}^n$ such that $M^T M = M M^T = I$. Such symmetry is often leveraged in mathematical proofs, statistical estimation and testing procedures, and elsewhere. As a consequence, the lack of such group invariance can quickly invalidate a result. Thus, this work seeks to determine conditions under which group invariance is achieved in an asymptotic sense. That is, for a sequence of random vectors $X^{(n)} \in \mathbb{R}^n$ and a sequence of functions $T_n : \mathbb{R}^n \to \mathbb{R}$ and mappings $\pi_n : \mathbb{R}^n \to \mathbb{R}^n$, how close are the random variables $T_n(X^{(n)})$ and $T_n(\pi_n X^{(n)})$ as $n$ grows to infinity?

Example 1.1. Let $X^{(n)} = (X_1, \ldots, X_n)$ have independent and identically distributed mean zero unit variance entries and $T_n(X^{(n)}) = n^{-1/2} \sum_{i=1}^n X_i$. If $\pi_n$ is a permutation on $n$ elements,
then obviously
\[ T_n(\pi_n X^{(n)}) = n^{-1/2} \sum_{i=1}^{n} X_{\pi(i)} = n^{-1/2} \sum_{i=1}^{n} X_i = T_n(X^{(n)}). \]

If instead \( \pi_n \) is a reflection, \( \pi_n(X_1, \ldots, X_n) = (\varepsilon_1 X_1, \ldots, \varepsilon_n X_n) \) for some \( \varepsilon_i \in \{-1, 1\} \), then the distributions of \( T_n(\pi_n X^{(n)}) \) and \( T_n(X^{(n)}) \) only coincide if \( X_1 = -X_1 \) in distribution. Nevertheless, if such reflective symmetry fails to hold, \( T_n(\pi_n X^{(n)}) \) and \( T_n(X^{(n)}) \) still asymptotically converge in distribution to the same \( N(0, 1) \) distribution.

The main results of this work are contained in Section 2. Theorem 2.1 and Corollary 2.2 demonstrate the almost sure equality between a probability measure on some Hilbert space and Haar measure on some subset of a compact topological group \( G \). These results are highly reminiscent of similar results from Lehmann and Romano (2006) and Hemerik and Goeman (2018a) and others. The novel theorem in this work is Theorem 2.2 with its paired Corollary 2.6. These extend the previous results by considering conditions on the random vector \( X^{(n)} \), the function \( T_n \), and the group of symmetries \( G \) under which almost sure convergence and convergence in mean between the two probabilities occur.

The first application comes in Section 3, which considers the group of rotations \( SO(n) \) applied to uniform measures on \( \ell_p^n \)-balls. The result is a partial law of the iterated logarithm (LIL), which is an almost sure upper bound of the form
\[ \limsup_{n \to \infty} \frac{|\sum_{i=1}^{n} X_i|}{K n^{1/2 - 1/p} \sqrt{\log \log n}} \leq 1, \quad \text{a.s.} \]
for some constant \( K > 0 \) and \( (X_1, \ldots, X_n) \) a uniformly random point within \( \ell_p^n \). Of note, these \( X_i \) are not independent unlike the usual LIL setting. Showing the above equals 1 almost surely is left to future work as proving the lower bound typically relies on the second Borel-Cantelli lemma. This, in turn, relies on independence of events, which is not the case in this problem. Furthermore, the value of \( K \) resulting from certain concentration inequalities on \( SO(n) \) is suboptimal. For independent random variables, there is a long history of classical LIL results that are detailed in Chapter 5 of Stout (1974) and Chapter 8 of Ledoux and Talagrand (1991).

The second application comes in Section 4, which considers classical one-sample and two-sample statistical hypothesis testing. Nonparametric randomization tests can be performed using the group of reflections and permutations, respectively, but only when the data distribution is invariant to such group actions. Section 4 first applies Theorem 2.2 to these testing problem to show when, for large \( n \), the distribution of the test statistic is nearly invariant to the corresponding group actions. It secondly proves some quantitative versions of Theorem 2.2 in these specific settings; see Theorems 4.1 and 4.2. These are in the style of Berry-Esseen bounds for the central limit theorem (Feller, 2008).

Randomization testing in statistics arguably dates back to Sir Ronald Fisher himself with the formulation of the permutation test, but it has gained more popularity with the advent of modern computers (Pesaran and Salmaso, 2010; Good, 2013). There has been much modern work on the usage of random permutations for hypothesis testing (Hemerik and Goeman, 2018a,b; Hemerik et al., 2019; Kashlak and Yuan, 2022; Kashlak et al., 2022; Kim et al., 2022; Vesely et al., 2023). There has also been some very recent work on subgroup and subset selection for more efficient permutation testing (Barber et al., 2022; Koning, 2023; Koning and Hemerik, 2023). Beyond permutations, a few works have considered hypothesis testing with random rotations (Solari et al., 2014), the wild bootstrap effectively uses random reflections to construct confidence regions (Burak and Kashlak, 2023), and other recent works consider general group symmetry (Chiu and Bloem-Reddy, 2023).
To correctly apply a two-sample t-test for testing for equality of population means, the populations should be Gaussian distributed with homogeneous variances. This ensures that the classic student’s t-test is exact. That is, the desired type-I error rate is achieved for all finite sample sizes. However in practice, the t-test is applied to a vast number of scenarios without producing erroneous results. This is in part thanks to the central limit theorem and asymptotic normality. If the sample size is large enough, the sample means as they appear in the t-statistic will be close enough to Gaussian to ensure some trust in the final result. The notion of ‘close’ to Gaussian can be made more precise through results like the aforementioned Berry-Esseen bounds.

In contrast, to correctly apply a two-sample permutation test for testing for equality of population means, the joint distribution of the sample should be exchangeable, i.e. invariant to permutations, under the null hypothesis. This is a slightly weaker condition than assuming an iid sample, and thus reasonable in many practical settings. However, as an example, heterogeneous population variances will violate the exchangeability assumption. A motivating goal of this work is to determine if the permutation test is robust to slight deviations from exchangeability much like the t-test’s robustness to slight deviations from Gaussianity.

2 Asymptotic Invariance

Let \((\Omega, \mathcal{F}, P)\) be a probability space, \((H, \mathcal{H})\) be an Hilbert space space equipped with \(\mathcal{H}\), the Borel \(\sigma\)-field generated by the norm topology, and let \((\mathbb{R}, \mathcal{B})\) denote the real line with the standard Borel \(\sigma\)-field. Let \(X : \Omega \to H\) be an \(H\)-valued random variable, and let \(T : H \to \mathbb{R}\) be a measurable function. Let \(G\) be a compact topological group, which implies that the multiplication, \((g, h) \to gh\), and inversion, \(g \to g^{-1}\), operations are continuous. The group \(G\) comes equipped with normalized Haar measure \(\rho\) being the unique left-invariant measure on the Borel sets derived from the topology on \(G\); that is, \(\rho(E) = \rho(gE)\) for any \(g \in G\) and any measurable subset \(E\) of \(G\). Existence and uniqueness of \(\rho\) is discussed in chapter 2 of Hofmann and Morris (2020) among other sources. Elements of \(G\) are said to act on \(H\), i.e. for \(L(H)\), the space of \(H\)-endomorphisms with the strong operator topology, there exists a representation \(\pi : G \to L(H)\) with \(\pi(g) = \pi_g\) such that \(\pi_g : H \to H\) and \(\pi(g)\pi(h) = \pi(g)\pi(h)\) for any \(g, h \in G\). The Hilbert space \(H\) is called an Hilbert \(G\)-Module; see “Weyl’s Trick” in Theorem 2.10 of Hofmann and Morris (2020) for the existence of an inner product that makes all \(\pi_g\) unitary. A set \(S \in \mathcal{H}\) is said to be \(G\)-invariant if \(S = \pi_gS\) for all \(g \in G\). Then, it can be readily checked that the collection \(\mathcal{S}\) of \(G\)-invariant sets forms a \(\sigma\)-field and \(\mathcal{S} \subseteq \mathcal{H}\).

Two conditions on the above will be considered in this work:

- **C1** The measure on \(H\) induced by \(X(\omega)\) is \(G\)-invariant, i.e., for all \(g \in G\) and \(A \in \mathcal{H}\), \(P(X \in A) = P(\pi_gX \in A)\) where \(P(X \in A) := P(\{\omega \in \Omega : X(\omega) \in A\})\).

- **C2** The measure on \(\mathbb{R}\) induced by the mapping \(T(X(\omega))\) is \(G\)-invariant, i.e., for all \(g \in G\) and \(B \in \mathcal{B}\), \(P(T(X) \in B) = P(T(\pi_gX) \in B)\) where \(P(T(X) \in B) := P(\{\omega \in \Omega : T(X(\omega)) \in B\})\).

Condition C2 implies that \(P(X \in T^{-1}(B)) = P(X \in \pi_gT^{-1}(B))\) for all \(g \in G\) or, that is, that condition C1 holds restricted to \(T = \sigma(T)\) rather than on all of \(\mathcal{H}\) where \(\sigma(T)\) is the smallest \(\sigma\)-field on \(H\) such that \(T\) is measurable.

**Example 2.1.** As a simple example, let \(T : \mathbb{R}^2 \to \mathbb{R}\) be \(T((x, y)) = x - y\) and \(G = \mathbb{S}_2\), the symmetric group on two elements. Then, \(B \in \mathcal{S}\) if whenever \((x_0, y_0) \in B\), then \((y_0, x_0) \in B\); i.e. reflections across the diagonal. And \(A \in \mathcal{T}\) if whenever \((x_0, y_0) \in A\), then \((x_0 + r, y_0 + r) \in A\) for any \(r \in \mathbb{R}\); i.e. diagonal strips. This is displayed in Figure 1.
It is clear that condition C1 implies condition C2. Furthermore, C1 has been studied in Lehmann and Romano (2006) and others referred to as the Radomization Hypothesis or Total Radomization Hypothesis. However, this is typically unnecessarily strong for hypothesis testing. An example of this is the two sample t-test discussed below in Section 4.2 where C2 is achieved either when the two populations have homogeneous variances or when the sample sizes coincide. C2 is also achieved if \( T \) is a \( G \)-invariant-mapping, i.e. \( T(x) = T(\pi_g x) \) for all \( g \in G \) and \( x \in H \). However, in this case, the utility of \( T \) is lost as the invariance or lack thereof with respect to specific probability measures is of primary interest. For a simple example that is explored more in Section 4, the sample mean is invariant to permutations, but the difference between two sample means may not be invariant; i.e. addition is commutative, and subtraction is not.

Given a fixed \( x \in H \) and some \( \alpha \in (0, 1) \), the randomization threshold \( t_\alpha(x) \in \mathbb{R} \) can be defined as

\[
t_\alpha(x) = \inf \{ t \in \mathbb{R} : \rho(\{ g \in G : T(\pi_g x) > t \}) \leq \alpha \}
\]

Therefore, \( \rho(\{ g \in G : T(\pi_g X) > t_\alpha(X) \}) \leq \alpha \) \( P \)-almost surely. This somewhat simple fact is critical for the following Corollaries 2.2 and 2.6.

The following is a slightly modified version of Theorem 15.2.2 from Lehmann and Romano (2006) with some similar results appearing in Hemerik and Goeman (2018a).

**Theorem 2.1.** Under condition C2,

\[
P(T(X) \in B \mid S \cap T) = \rho(\{ g \in G : T(\pi_g X) \in B \})
\]

\( P \)-almost surely for any \( B \in \mathcal{B} \).

**Proof.** For any \( G \)-invariant set \( S \in \mathcal{S} \cap \mathcal{T} \), there exists an \( A \in \mathcal{B} \) such that \( S = T^{-1}(A) \). Thus,
denoting $E_X[\cdot]$ the expected value with respect to $X$,

$$E_X \left[ \rho \left( \{ g \in G : T(\pi_g X) \in B \} \right) 1_S \right]$$

$$= \int_S \int_G 1_{\{T(\pi_g x) \in B\}} \, d\rho(g) \, dP(x)$$

$$= \int_G \int_S 1_{\{T(\pi_g x) \in B\}} \, dP(x) \, d\rho(g)$$

$$= \int \mathbb{P} (T(\pi_g X) \in B, X \in S) \, d\rho(g)$$

$$= \int \mathbb{P} (T(\pi_g X) \in B, \pi_g X \in S) \, d\rho(g)$$

$$= \int \mathbb{P} (\pi_g X \in T^{-1}(B) \cap T^{-1}(A)) \, d\rho(g)$$

$$= \int \mathbb{P} (X \in T^{-1}(B \cap A)) \, d\rho(g) = \mathbb{P} (T(X) \in B, X \in S).$$

Thus, by uniqueness, the conditional probability $\mathbb{P} (T(X) \in B \mid S \cap T)$ coincides with the random measure $B \to \rho \left( \{ g \in G : T(\pi_g X) \in B \} \right)$.

\[ \square \]

**Corollary 2.2.** Under condition C2, $\mathbb{P} (T(X) > t_{\alpha}(X)) \leq \alpha$.

**Proof.** Let $R = \{ x \in H : T(x) > t_{\alpha}(x) \}$. Then, $R \in \mathcal{H}$ and from Theorem 2.1

$$\mathbb{P} (T(X) > t_{\alpha}(X)) = \mathbb{E} \left[ \mathbb{P} (T(X) > t_{\alpha}(X) \mid S \cap T) \right]$$

$$= \mathbb{E} \left[ \rho \left( \{ g \in G : T(\pi_g X) > t_{\alpha}(X) \} \right) \right] \leq \alpha$$

as almost sure equality implies equality in mean. \[ \square \]

The validity of the above result hinges on condition C2. Upon removal of that condition, almost sure equality is lost, but can still be achieved asymptotically. In what follows, let $H = \mathbb{R}^\infty$. For all $n \in \mathbb{N}$, let $S_n$ be the $\sigma$-field of $G_n$-invariant sets in $\mathbb{R}^n$. Furthermore, let $T_n : \mathbb{R}^n \to \mathbb{R}$ be $c_n$-Lipschitz and $\mathcal{T}_n$ be the smallest $\sigma$-field on $\mathbb{R}^n$ such that $T_n$ is measurable. The choice of the sequence Lipschitz constants has intriguing implications that are discussed post-theorem in Section 2.2.

Functions $T_n$, groups $G_n$, and sets $S_n$ can be extended to $\mathbb{R}^\infty$. A set $S_n \in S_n$ can be written as $\{ S_n \otimes \mathbb{R} \otimes \ldots \} \subset \mathbb{R}^\infty$ which is invariant to elements of $G_n$ acting on the first $n$ coordinates and fixing the rest. Let $G := \bigcup_{n \geq 1} G_n$ which consists of all group actions from $G_n$ that only modify the first $n$ entries of $x \in \mathbb{R}^\infty$ for all $n \in \mathbb{N}$; e.g. $G$ may consist of all permutations that only permute a finite number of elements as arises in the Hewitt-Savage zero-one law (Rao and Rao, 1974; Dudley, 2002). Tychonoff’s theorem ensures compactness is maintained in the limit with respect to the product topology; of note, arbitrary products of compact groups are compact as are subgroups of such (Hofmann and Morris, 2020, Proposition 1.14). A tail set $E \subset \mathbb{R}^\infty$, as defined by Oxtoby (2013), is such that if $x \in E$ and if $y$ differs from $x$ in only a finite number of coordinates, then $y \in E$. Since any $g \in G$ necessarily modifies only a finite number of coordinates, $x \in E$ implies $\pi_g x \in E$ and thus tail sets are $G$-invariant. Furthermore, $\mathcal{S}$ is the $\sigma$-field on $\mathbb{R}^\infty$ of $G_n$-invariant sets for all $n \in \mathbb{N}$, and $T : \mathbb{R}^\infty \to \mathbb{R}$ is defined as $T := \lim_{n \to \infty} T_n$ where $T_n$ can be defined on $\mathbb{R}^\infty$ by projecting $x \in \mathbb{R}^\infty$ onto the first $n$ coordinates. A simple example is the sample mean $T_n(x) = n^{-1} \sum_{i=1}^{n} x_i$, which will be discussed in Section 4.
In what follows, the notion of a Lévy family is required (Gromov and Milman, 1983; Ledoux, 2001). Let \((M^{(n)}, d^{(n)}, \mu^{(n)})\) be a family of metric measure spaces for \(n \geq 1\). The open neighbourhood of a set \(A \subset M^{(n)}\) for some \(t > 0\) is \(A_t = \{ x \in M^{(n)} : d^{(n)}(x, A) < t \}\). This collection of metric measure spaces is said to be a normal Lévy family if

\[
\sup_{A \subset M^{(n)}} \left\{ 1 - \mu^{(n)}(A_t) : \mu^{(n)}(A) \geq 1/2 \right\} \leq Ke^{-kn^2}
\]

for some constants \(K, k > 0\). From the previous paragraph, \(M^{(n)} = G_n\) treated as a subgroup of \(G\) that acts as the identity on all coordinates \(i > n\). The measure \(\mu^{(n)} = \rho_n\) will be Haar measure for \(G_n\). The main results below require the family \((G_n, d_n, \rho_n)\) be a normal Lévy family. This, of course, covers a wide variety of groups. Most notably, the classical compact groups \(SO(n), SU(n),\) and \(Sp(2n)\) with the Hilbert-Schmidt metric satisfy this requirement (Meckes, 2019, Chapter 5). Furthermore, any sequence of topological groups corresponding to compact connected smooth Riemannian manifolds with geodesic distance and strictly positive Ricci curvature embedded in \(\mathbb{R}^n\) (Gromov et al., 1999; Ledoux, 2001; Milman and Schechtman, 2009). For discrete groups, the reflection group (Section 4.1) with the Hamming metric is normal Lévy following from Hoeffding’s inequality. The symmetric group (Section 4.2) and many other compact groups are also normal Lévy; see Corollary 4.3 and Theorem 4.4 in Ledoux (2001).

The following theorem shows that Theorem 2.1 can hold in an asymptotic sense. It is proven via the three subsequent lemmas below. Lastly, an asymptotic analogue of Corollary 2.2 is stated and proved below in Corollary 2.6.

**Theorem 2.2.** Let \(X \in \mathbb{R}^\infty\) and \(X^{(n)} \in \mathbb{R}^n\) be \(X\) projected onto its first \(n\) coordinates. Let \(T_n\) be \(c_n\)-Lipschitz such that for some \(p \geq 1\), \(E\|X^{(n)}\|^p < \infty\) for all \(n\) and \(\sum_{n=1}^\infty c_n^p < \infty\). Furthermore, let \(n^{-1/2}c_n\|X^{(n)}\| \xrightarrow{a.s.} 0\). Lastly, let the collection of \(G_n\) be a normal Lévy family. Then,

\[
P\text{-almost surely and in } L^1 \text{ as } n \to \infty.
\]

**Proof.** Decomposing the difference gives

\[
\left| P \left( \{ g \in G_n : T_n(\pi g X^{(n)}) > t \} \right) - P \left( T_n(X^{(n)}) > t | S_n \cap T_n \right) \right| \to 0
\]

The three pieces will be dealt with by the subsequent lemmas. Part (I) is handled by concentration of measure for compact topological groups in Lemma 2.5. Part (II) follows from Lemma 2.3, which implies condition C2 in the limiting case. Thus, by Theorem 2.1 \(\rho(\{ g \in G : T(\pi g X) > t \}) = P( T(X) > t | S \cap T \}) P\)-almost surely and thus equal in mean as well. Lastly, part (III) is handled by martingale convergence in Lemma 2.4.

**Lemma 2.3.** For \(X \in \mathbb{R}^\infty\), let \(X^{(n)} \) be \(X\) projected onto \(\mathbb{R}^n\) by taking the first \(n\) coordinates. Let \(T_n\) be \(c_n\)-Lipschitz with \(c_n \to 0\). Assuming \(E\|X^{(n)}\|^p < \infty\) for each \(n \in \mathbb{N}\), the function \(T_{n+m}(X)\) is asymptotically \(G_n\)-invariant in mean for any fixed \(n \in \mathbb{N}\), i.e. \(E[T_{n+m}(\pi g X) - T_{n+m}(X)] \to 0\) for any fixed \(g \in G_n\) and \(n \in \mathbb{N}\) as \(m \to \infty\). Furthermore, if for some \(p \geq 1\), \(E\|X^{(n)}\|^p < \infty\) and \(\sum_{n=1}^\infty c_n^p < \infty\) then \(T_{n+m}(X)\) is asymptotically \(G_n\)-invariant \(P\)-almost surely.
Proof. For a fixed \( n \) and any \( g \in G_n \) and any \( m \in \mathbb{N} \), there exists a unitary representation \( \pi_g : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m} \) that fixes the final \( m \) coordinates. That is,

\[
\pi_g = \begin{pmatrix} \pi_g^{(n)} & \mathbf{0} \\ \mathbf{0} & I^{(m)} \end{pmatrix}.
\]

Writing \( X^{(n+m)} = (X^{(n)}), X^{(m)}) \) and \( \pi_g = (\pi_g^{(n)}, I^{(m)}) \) where \( I^{(m)} \) is the identity mapping on \( \mathbb{R}^m \) results in \( \pi_g X^{(n+m)} = (\pi_g^{(n)} X^{(n)}, X^{(m)}) \). Thus, for any fixed \( n \)

\[ |T_{n+m}(\pi_g X) - T_{n+m}(X)| \leq c_{n+m} \| \pi_g X - X \| \leq 2c_{n+m} \| X^{(n)} \|, \]

Taking the expectation and \( m \to \infty \) proves asymptotic invariance in mean.

Secondly, by Markov’s inequality, for any \( t > 0 \)

\[ P \left( |T_{n+m}(\pi_g X) - T_{n+m}(X)| > t \right) \leq P \left( 2c_{n+m} \| X^{(n)} \| > t \right) \]

\[ \leq 2^{p} c_{n+m} E \| X^{(n)} \|^p t^{-p}. \]

Hence, almost sure convergence follows from the Borel-Cantelli lemma and the assumptions that \( \sum_{n=1}^{\infty} 2^p c_n < \infty \). \( \square \)

Lemma 2.4. The sequence of conditional probabilities \( P \left( T(X^{(n)}) > t \mid S_n \cap T_n \right) \) converges to \( P \left( T(X) > t \mid S \cap T \right) \) almost surely and in \( L^1 \) as \( n \to \infty \).

Proof. Let \( Z_{n,m} := E[1_{T(X^{(n)}) > t} | S_m \cap T_m], \) \( Z_{n,\infty} := E[1_{T(X^{(n)}) > t} | S_{\infty} \cap T], \) \( Z_{\infty,m} := E[1_{T(X) > t} | S_{m} \cap T_m], \) and \( Z := E[1_{T(X) > t} | S \cap T]. \) For \( S \in S_m, S = \{ S^{(m)} \times \mathbb{R} \times \ldots \} \) and similarly for sets in \( T_m \). Hence, \( S_m \cap T_m \subset S_{m+1} \cap T_{m+1} \), and thus the sequence \( Z_{n,m} \to Z_n \) almost surely and in \( L^1 \) for any fixed \( n \) as a consequence of Levy’s Upward Lemma; see Rogers and Williams (2000) section II.50. The same holds for \( Z_{\infty,m} \to Z \). But furthermore, \( T(X^{(n)}) \) is, of course, \( T_n \)-measurable. Hence, for any fixed \( n \) and all \( m_1, m_2 \geq n \), \( Z_{n,m_1} = Z_{n,m_2} \) almost surely. Hence,

\[ E[1_{T(X^{(n)}) > t} | S_n \cap T_n] = E[1_{T(X^{(n)}) > t} | S \cap T] \quad (2.1) \]

almost surely, and by the conditional dominated convergence theorem (Rogers and Williams (2000) section II.41),

\[ E[1_{T(X^{(n)}) > t} | S_m \cap T_m] \xrightarrow{a.s.} E[1_{T(X) > t} | S_m \cap T_m], \text{ and} \]

\[ E[1_{T(X^{(n)}) > t} | S \cap T] \xrightarrow{a.s.} E[1_{T(X) > t} | S \cap T] \]

as \( n \to \infty \).

As a consequence of Equation 2.1, the sequence \( \{ Z_{n,n} \}_{n=1}^{\infty} \) is almost surely equal to \( \{ Z_{n,n+k} \}_{n=1}^{\infty} \) for any \( k \in \mathbb{N} \). As equality holds for all \( k \), \( \{ Z_{n,n} \}_{n=1}^{\infty} \) is almost surely equal to \( \{ Z_{n,\infty} \}_{n=1}^{\infty} \). As noted above, \( Z_{n,\infty} \xrightarrow{a.s.} Z \) via dominated convergence. Hence, \( Z_{n,n} \) converges almost surely as well.

Lastly, a classic theorem of Doob (Rogers and Williams, 2000, section II.44) implies that the \( Z_{n,m} \) are uniformly integrable. Hence, uniform integrability and convergence almost surely (in probability) implies convergence in \( L^1 \) by Theorem 21.2 in chapter II of Rogers and Williams (2000). \( \square \)
Lemma 2.5. For each $n$, let $G_n$ be a normal Lévy family with respect to normalized Haar measure. Let $X \in \mathbb{R}^\infty$ be a random variable with projection $X^{(n)} \in \mathbb{R}^n$ onto the first $n$ coordinates, and let $T_n$ be $c_n$-Lipschitz. If $n^{-1/2}c_n\|X^{(n)}\| \xrightarrow{a.s.} 0$ then

$$\rho(\{g \in G_n : T_n(\pi_g X^{(n)}) > t\}) - 1_{T(X) > t} \to 0$$

$P$-almost surely and in $L^1$ as $n \to \infty$.

Proof. Let $f_{x,n} : G_n \to \mathbb{R}$ be defined as $f_{x,n}(g) = T_n(\pi_g x)$. Then, for any $g, h \in G_n$ with unitary representations $\pi_g, \pi_h \in L(\mathbb{R}^n)$,

$$|f_{x,n}(g) - f_{x,n}(h)| = |T(\pi_g x) - T(\pi_h x)|$$

$$\leq c_n\|\pi_g x - \pi_h x\|$$

$$\leq c_n\|x\|\|\pi_g - \pi_h\|_{L(\mathbb{R}^n)}.$$ 

Thus, $f_{x,n}$ is $c_n\|x\|$-Lipschitz on $L(\mathbb{R}^n)$ with respect to the operator norm. As a consequence of the $G_n$ forming a normal Lévy family, there exists fixed constants $K, k > 0$ such that for all $t \geq 0$ and $n \geq 1$

$$\rho \left( |f_{x,n}(g) - \int f_{x,n}(g)d\rho(g)| > t \right) \leq K \exp \left( -\frac{mnt^2}{2c_n^2\|x\|^2} \right).$$

Consequently, 

$$\rho(\{g \in G_n : T_n(\pi_g X^{(n)}) > t\}) \leq K \exp \left( -\frac{nk}{2c_n^2\|X^{(n)}\|^2} \left( t - \int f_{X^{(n)}}(g)d\rho(g) \right)^2 \right).$$

(2.2)

By Jensen’s inequality, the right hand side of inequality 2.2 is bounded above by

$$K \int \exp \left( -\frac{nk}{2c_n^2\|X^{(n)}\|^2} \left( t - f_{X^{(n)}}(g) \right)^2 \right) d\rho(g).$$

For any fixed $\omega \in \Omega$ and any $\varepsilon > 0$, let $t_{\omega,\varepsilon} = T(X(\omega)) + \varepsilon$. Then, by dominated convergence, the assumption that $n^{-1/2}c_n\|X^{(n)}\| \xrightarrow{a.s.} 0$, and Lemma 2.3,

$$\rho(\{g \in G_n : T_n(\pi_g X^{(n)}) > t_{\omega,\varepsilon}\}) \to 0$$

Hence, for all $\omega \in \Omega$ such that $T(X(\omega)) \leq t$, $\rho(\{g \in G_n : T_n(\pi_g X^{(n)}(\omega)) > t\}) \to 0$. As the concentration inequality is agnostic to direction, the above argument can be redone for $1 - \rho(\{g \in G_n : T_n(\pi_g X^{(n)}) > t\}) = \rho(\{g \in G_n : T_n(\pi_g X^{(n)}) \leq t\})$ to conclude that $\rho(\{g \in G_n : T_n(\pi_g X^{(n)}) > t\}) \xrightarrow{a.s.} 1_{T(X) > t}$.

For convergence in $L^1$, it is trivial to note that $\sup_n |\rho(\{g \in G_n : T_n(\pi_g X^{(n)}) > t\})| \leq 1$. Thus, the sequence $\rho(\{g \in G_n : T_n(\pi_g X^{(n)}) > t\})$ is uniformly integrable and converges almost surely, and hence in probability, from the first part of this lemma. Hence,

$$E\left| \rho(\{g \in G_n : T_n(\pi_g X^{(n)}) > t\}) - 1_{T(X) > t} \right| \to 0$$

by Theorem 10.3.6 of Dudley (2002).

\[
\text{Corollary 2.6. Under the setting of Theorem 2.2,}
\]

\[
\lim_{n \to \infty} P \left( T_n(X^{(n)}) > t_n(X^{(n)}) \right) \leq \alpha.
\]
Proof. From Theorem 2.2, for any \( \varepsilon > 0 \), there exists an \( N \in \mathbb{N} \) such that for all \( n > N \),
\[
P \left( T_n(X^{(n)}) > t_\alpha(X^{(n)}) \right) = E \left[ P \left( T_n(X^{(n)}) > t_\alpha(X^{(n)}) \mid S_n \cap T_n \right) \right]
\leq E \left[ \rho \left( \{ g \in G_n : T_n(\pi_g X^{(n)}) > t_\alpha(X^{(n)}) \} \right) \right] + \varepsilon \leq \alpha + \varepsilon.
\]
Taking \( \varepsilon \to 0 \) finishes the proof. \( \square \)

2.1 Remark on Group Selection

The above theorems and corollaries can hold for a multitude of groups. In particular, if they hold for a group \( G \), then they hold for any subgroup of \( G \). The choice of \( G \) directly results in a choice of \( S \), the \( \sigma \)-field of invariant sets. Indeed, a “larger” group \( G \) will make \( S \) “smaller”, and thus, the randomized \( 1[T_n(\pi_g X^{(n)}) > t] \) can be used to extract more information about \( 1[T_n(X^{(n)}) > t] \). When conditioning on \( S \), the smaller \( S \) is, the more restricted the conditional probability will be.

For illustrative purposes, let \( G \) be the trivial group. In such a scenario, the random measures \( \rho(\{ g \in G_n : T_n(\pi_g X^{(n)}) > t \}) \) and \( P(T_n(X^{(n)}) > t | S_n \cap T_n) \) coincide with \( 1[T_n(X^{(n)}) > t] \) and no meaningful inference is achievable. In particular, the randomization threshold for a fixed \( x \) is \( t_\alpha(x) = T(x) \), and the conclusion of Corollary 2.6 is the immensely unhelpful fact that \( \lim_{n \to \infty} 0 \leq \alpha \).

For a richer discrete group \( G \) with cardinality \( |G| \), the random measure \( \rho(\{ g \in G : T(\pi_g X) > t \}) \) can take on at most the values \( i/|G| \) for \( i = 0, 1, \ldots, |G| \). Hence, the finer granularity of, say, the symmetric group over the alternating group or the cyclic group may be preferable.

In a statistical context, group selection for randomization tests is intimately connected to the null hypothesis under examination. The tail probability \( P(T(X) > t_\alpha(X)) \) from the above corollaries corresponds to a p-value concerned with whether or not condition C2 holds. That is, the p-value is for the following hypotheses:
\[
H_0 : T(X) \overset{d}{=} T(\pi_g X) \quad \forall g \in G,
H_1 : \exists g \in G \text{ s.t. } T(X) \overset{d}{\neq} T(\pi_g X).
\]

One approach may be to select the maximal invariant group of transformations that preserve the distribution of \( T(X) \) under the null hypothesis. However, the recent work of Koning and Hemerik (2023) proposes a clever approach to subgroup selection for improving the performance of the classical permutation test, which opens up more research questions into the best choice of \( G \).

2.2 Remark on Lipschitz constants

As far as Theorem 2.2 is concerned, the sequence of Lipschitz constants \( c_n \) can be arbitrary as long as the conditions continue to hold. Thus, the choice of \( c_n \) is dependent on the problem under consideration.

In Section 4.1, a simple example of a one-sample location test is considered for independent and identically distributed \( X_1, \ldots, X_n \). The function \( T \) is chosen to be \( T(X^{(n)}) = n^{-1/2} \sum_{i=1}^n X_i \) in order to contrast the result with the Berry-Esseen Theorem. However, the standard sample mean \( \bar{X} = n^{-1} \sum_{i=1}^n X_i \) is another valid choice for \( T \), which would, in contrast to the \( n^{-1/2} \)-normalization, require weaker moment assumptions on that random variables \( X_i \).

Venturing deeper into this Lipschitzian rabbit hole, one could easily consider the function \( T \) defined as \( T(X^{(n)}) = n^{-2} \sum_{i=1}^n X_i \). As far as Theorem 2.2 is concerned, this function is perfectly valid. Of course, it is well known that \( T(X^{(n)}) \overset{d}{\to} 0 \) as long as \( E|X_1|^{1/2} < \infty \) and, of note,
irregardless of the mean of \( X_1 \) (Stout, 1974, Theorem 3.2.3). Thus, this choice of \( T \) is perfectly useless for statistical inference purposes. The conclusion is that \( T \) must be chosen preemptively for the problem at hand and not to post-hoc make Theorem 2.2 applicable. Such selection of \( c_n \) occurs in the subsequent Section 3 where \( c_n \) is chosen to achieve the right convergence properties.

In summary, the requirement of Theorem 2.2 that \( c_n \| X^{(n)} \| / \sqrt{n} \overset{a.s.}{\to} 0 \) dictates the relation between \( c_n \) and \( X^{(n)} \). The faster \( c_n \) tends to zero, the fewer conditions are required on the moments of \( X^{(n)} \). However, if \( c_n \) decreases too fast, the utility of \( T \) is lost. And furthermore, if \( X^{(n)} \) is a well-behaved random vector, then there are many more valid choices of \( c_n \).

3 Uniform points in an \( \ell_p^n \)-ball

As a toy application of Theorem 2.2, convergence properties can be derived for sums of coordinates for random vectors within an \( \ell_p^n \)-ball. The goal of this section is to show a law of the iterated logarithm for uniformly random points in an \( \ell_p^n \)-ball, i.e. uncorrelated and bounded but not independent. Theorems 2.1 and 2.2 allow for a quick proof of an almost sure upper bound with correct asymptotic rate, but sub-optimal constant dependent on the concentration behaviour of random group elements.

Let \( X^{(n)} \) be a uniformly random point inside the \( \ell_p^n \)-ball, i.e. \( \sum_{i=1}^n |X_i|^p \leq 1 \), and function \( T_n(X^{(n)}) = c_n \langle 1_n, X^{(n)} \rangle \) for some normalizing constant depending on \( n \) such as \( c_n = n^{-1/2} \) and the \( n \)-long vector \( 1_n^T = (1, \ldots, 1) \). Thus, \( T_n : \mathbb{R}^n \to \mathbb{R} \) is \( c_n \)-Lipschitz. Two simple examples follow.

**Example 3.1** (\( \ell_\infty \) and \( \ell_p \)-balls for \( p \leq 1 \)). Let \( X^{(n)} \) be a uniformly random point within the \( \ell_\infty^n \)-ball, i.e. \( \max_{i=1,\ldots,n} |X_i| \leq 1 \). Then, it is well known via the central limit theorem, Chebyshev’s inequality, and the strong law of large numbers, respectively, that

\[
\sum_{i=1}^n \frac{X_i}{n^{1/2}} \xrightarrow{d} Z \sim \mathcal{N}(0,1), \quad \sum_{i=1}^n \frac{X_i}{n^q} \xrightarrow{p} 0, \quad \text{and} \quad \sum_{i=1}^n \frac{X_i}{n^{q+1/2}} \xrightarrow{a.s.} 0
\]

for any choice of \( q > 1/2 \). Furthermore, as the collection of \( X_i \) are independent and identically distributed \( \text{Uniform}[-1,1] \) random variables, the Law of the Iterated Logarithm (see, for example, de Acosta (1983) or Ledoux and Talagrand (1991) Chapter 8) implies that

\[
\limsup_{n \to \infty} \frac{1}{\sqrt{2n \log \log n}} \sum_{i=1}^n \frac{X_i}{\sqrt{2n \log \log n}} = 1 \quad \text{and} \quad \liminf_{n \to \infty} \frac{1}{\sqrt{2n \log \log n}} \sum_{i=1}^n \frac{X_i}{\sqrt{2n \log \log n}} = -1
\]

and thus \( n^{-q} \sum_{i=1}^n X_i \xrightarrow{a.s.} 0 \) for any choice of \( q > 1/2 \).

Let \( X^{(n)} \) be a uniformly random point within the \( \ell_p^n \)-ball for \( p \leq 1 \). Then, \( \sum_{i=1}^n |X_i| \leq 1 \). Hence,

\[
\sum_{i=1}^n \frac{X_i}{n^{p/2}} \xrightarrow{p} 0
\]

for any choice of \( q > 0 \). Furthermore, a quick calculation\(^1\) for the \( \ell_2^n \)-ball shows that \( \mathrm{E}X_1^2 \leq C/n^2 \) for some constant \( C > 0 \). Thus, \( n^{-q} \sum_{i=1}^n X_i \xrightarrow{a.s.} 0 \) for any \( q > 0 \) via Chebyshev’s inequality and the first Borel-Cantelli lemma. The goal of what follows is to extend this idea to other \( \ell_p^n \)-balls and achieve more precise rates of convergence.

\(^1\)See Appendix A for tedious details.
The group of interest is the special orthogonal group, \( SO(n) = \{ M \in \mathbb{R}^{n \times n} : M^T M = I, \det(M) = 1 \} \). Integrating over \( SO(n) \) with respect to its normalized Haar measure \( \rho \) yields
\[
E_{SO(n)} T_n(MX^{(n)}) = 0 \quad \text{and} \quad E_{SO(n)} T_n(MX^{(n)})^2 = c_n^2 \|X\|^2_2.
\]
Indeed, for the second moment calculation,
\[
E_{SO(n)} T_n(MX^{(n)})^2 = c_n^2 E_{SO(n)} \left[ (MX^{(n)})^T 1_n 1_n^T (MX^{(n)}) \right],
\]
and the calculation becomes a consequence of the following lemma.

**Proposition 3.2.** Let \( A \in \mathbb{R}^{n \times n} \) be a symmetric matrix with spectrum \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \), and let \( b_A : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^+ \) be a bilinear form defined by
\[
b_{A,SO(n)}(x, y) = \int_{SO(n)} (Mx)^T A(My) d\rho(M)
\]
where integration is taken over \( SO(n) \) with respect to Haar measure \( \rho \). Then, \( b_{A,SO(n)}(x, y) \) is rotationally invariant and furthermore
\[
b_{A,SO(n)}(x, y) = \bar{\lambda} \langle x, y \rangle
\]
where \( \langle \cdot, \cdot \rangle \) is the standard Euclidean inner product and \( \bar{\lambda} = n^{-1} \sum_{i=1}^n \lambda_i \).

**Proof.** Any bilinear form on a real Hilbert Space is of the form \( \langle Mx, y \rangle \) for some bounded operator \( M \). Hence, \( b(x, y) = \sum_{i=1}^n c x_i y_i \) for some \( c > 0 \) by rotational invariance. Without loss of generality, let \( A \) be diagonal with entries \( \lambda_1, \ldots, \lambda_n \). Then, choosing \( x = y \) to be any unit vector results in
\[
c = \int_{\|v\|=1} v^T A v d\mu = \int_{\|v\|=1} \sum_{i=1}^n \lambda_i v_i^2 d\mu
\]
where \( \mu \) is the uniform surface measure of the \((n-1)\)-sphere. By symmetry, the integral can be restricted to fraction of the sphere where \( v_i \geq 0 \). Furthermore, \( \{ v_i = v_j : i \neq j \} \) is a measure zero event. Thus,
\[
c = 2^n \int_{v_1 \geq v_2 \geq \ldots \geq v_n \geq 0} \sum_{\pi \in S_n} \sum_{i=1}^n \lambda_i v_{\pi(i)}^2 d\mu
\]
\[
= 2^n \int_{v_1 \geq v_2 \geq \ldots \geq v_n \geq 0} \sum_{i=1}^n \lambda_i \sum_{\pi \in S_n} v_{\pi(i)}^2 d\mu
\]
\[
= 2^n \int_{v_1 \geq v_2 \geq \ldots \geq v_n \geq 0} \sum_{i=1}^n \lambda_i (n-1)! d\mu
\]
\[
= \frac{2^n}{n!2^n} \sum_{i=1}^n \lambda_i (n-1)! = \frac{1}{n!} \sum_{i=1}^n \lambda_i
\]
as the sum is over \( n! \) permutations in \( S_n \), which is grouped into \((n-1)!\) sets of \( n \) \( v_i^2 \)'s that sum to 1. \( \square \)

Let \( f_{n,x} : SO(n) \to \mathbb{R} \) be defined as \( f_{n,x}(M) := T_n (Mx) \). Then, as a consequence of the Cauchy-Schwarz inequality,
\[
|f_{n,x}(M_1) - f_{n,x}(M_2)| \leq c_n \sqrt{n} \|x\|_2 \|M_1 - M_2\|_{\text{HS}}
\]
making \( f_{n,x} \) a \( c_n \sqrt{n} \|x\|_2 \)-Lipschitz function on \( SO(n) \) with respect to the Hilbert-Schmidt metric. Concentration of measure for the classical compact groups (Meckes, 2019, Theorem 5.17) implies that

\[
\rho (f_{n,x}(M) \geq t) \leq \exp \left[ - \left( \frac{n-2}{n} \right) \frac{t^2}{24 c_n^2 \|x\|_2^2} \right] \leq \exp \left[ - \frac{\kappa_0 t^2}{c_n^2 \|x\|_2^2} \right]
\]

for \( n \geq 3 \) and some dimension independent constant \( \kappa_0 > 0 \), e.g. \( \kappa_0 = 1/72 \).

In what follows, it is shown that

\[
\limsup_{n \to \infty} \frac{|\sum_{i=1}^n X_i|}{K n^{1/2-1/p} \sqrt{\log \log n}} \leq 1, \text{ a.s.} \tag{3.1}
\]

for some constant \( K > 0 \). Our conjecture is \( K = 2^{1/2+1/p} \). Figure 2 displays the value of \( |\sum_{i=1}^n X_i|/2^{1/2+1/p} n^{1/2-1/p} \sqrt{\log \log n} \) computed for 1000 simulated uniform vectors within the \( \ell_p^n \)-ball for \( p = 1, 2, \infty \) and \( n = 10^1, 10^2, \ldots, 10^6 \). An algorithm for simulating such random vectors is detailed in Appendix B.

### 3.1 Case \( p = 2 \)

If \( X^{(n)} \) is a uniformly random point in the \( \ell_2^n \)-ball, then the measure induced by \( X^{(n)} \) is invariant to any rotation \( M \in SO(n) \). Thus, Theorem 2.1 and the above concentration of measure result imply that, for any fixed \( t > 0 \),

\[
P \left( n^{-q} \sum_{i=1}^n X_i > t \right) \leq e^{-\kappa_0 n^{2q} \epsilon^2}
\]

where \( c_n = n^{-q} \). Thus, \( n^{-q} \sum_{i=1}^n X_i \) converges to zero in probability for any \( q > 0 \). Furthermore, from a standard application of the first Borel-Cantelli Lemma, \( n^{-q} \sum_{i=1}^n X_i \to 0 \) almost surely for any choice of \( q > 1/2 \) similar to the direct computation approach performed in the appendix.

Going further, it can be quickly shown that for some \( K > 0 \),

\[
\limsup_{n \to \infty} \frac{|\sum_{i=1}^n X_i|}{K \sqrt{\log \log n}} \leq 1
\]

using similar arguments as is Stout (1974), de Acosta (1983), and others. Indeed, fix \( \epsilon > 0 \), and note that the sequence \( (X_i)_{i=1}^n \) is a symmetric sequence, and hence Lévy’s maximal inequalities are applicable (Ledoux and Talagrand, 1991, Proposition 2.3). Let \( n_k = \lfloor c_0 k \rfloor \) for some \( c_0 > 1 \). Then,

\[
P \left( \frac{\max_{n \leq n_k} |\sum_{i=1}^n X_i|}{K \sqrt{\log \log n_k}} \geq 1 + \epsilon \right) \leq 2P \left( \frac{|\sum_{i=1}^{n_k} X_i|}{K \sqrt{\log \log n_k}} \geq 1 + \epsilon \right) \leq 2 \exp \left( - \kappa_0 (1 + \epsilon)^2 K^2 \log \log n_k \right) \leq 2 (\log n_k)^{-\kappa_0 (1+\epsilon)^2 K^2}.
\]

Hence, the series

\[
\sum_{k=1}^\infty P \left( \frac{\max_{n \leq n_k} |\sum_{i=1}^n X_i|}{K \sqrt{\log \log n_k}} \geq 1 + \epsilon \right) < \infty
\]

given that \( K \) is chosen such that \( \kappa_0 K^2 \geq 1 \). Thus, by invoking the first Borel-Cantelli lemma and taking \( \epsilon \to 0 \) results in

\[
\limsup_{n \to \infty} \frac{|\sum_{i=1}^n X_i|}{K \sqrt{\log \log n}} \leq 1, \text{ a.s.}
\]

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Figure 2: For dimensions $n = 10^1, \ldots, 10^6$, 1000 vectors are generated from a uniform distribution on the $\ell_p^n$-ball, and for each vector, Equation 3.1 with conjectured optimal constant is computed (black circles). The red lines are for $\pm 1$; the blue are for the 1st and 3rd quartile; the dashed black are for the min and max values.
3.2 Case \( p > 2 \)

If \( X^{(n)} \) is a uniformly random point in the \( \ell_p^n \)-ball for \( 2 < p < \infty \), then the induced measure is not rotationally invariant. However, Theorem 2.2 can still be applied. Indeed, let \( c_n = n^{1/p-1/2}/\sqrt{\log \log n} \). As \( X^{(n)} \) is restricted to a compact set, \( E\|X^{(n)}\|^p' < \infty \) for any choice of \( p' \geq 1 \). Hence, \( \sum_{i=1}^{\infty} n^{p'(1/p-1/2)}/\sqrt{\log \log n} < \infty \) for any choice of \( p' > (p-2)/2p \). Furthermore, \( \|X^{(n)}\| \leq n^{1/2-1/p} \), so \( n^{-1/2}c_n\|X^{(n)}\| \leq 1/\sqrt{n}\log \log n \), which converges to zero almost surely. Therefore, 

\[
P \left( \frac{n^{1/p-1/2}}{\sqrt{\log \log n}} \sum_{i=1}^{n} X_i > t \right) - \rho [f_{n,X}(M) \geq t] \to 0
\]

almost surely and in \( L^1 \) by Theorem 2.2. Sub-Gaussian concentration on \( SO(n) \) as a consequence of Theorem 5.17 from Meckes (2019), results in

\[
\rho [f_{n,X}(M) \geq t] \leq \exp \left( -\kappa_0 t^2 \frac{1}{\|X^{(n)}\|^2} n^{1-2/p} \log \log n \right) \leq e^{-\kappa t^2 \log \log n} \quad \text{and}
\]

\[
\rho \left[ c_n \langle 1_n, (I-M)X^{(n)} \rangle \geq t \right] \leq e^{-\kappa t^2 \log \log n}.
\]

Noting that \( \langle 1_n, X^{(n)} \rangle = \langle 1_n, MX^{(n)} \rangle + \langle 1_n, (I-M)X^{(n)} \rangle \), the same argument as in the previous sub-section where \( p = 2 \) can be applied to show the existence of some \( K > 0 \) such that

\[
\limsup_{n \to \infty} \frac{\|\sum_{i=1}^{n} X_i\|}{K n^{1/2-1/p} \sqrt{\log \log n}} \leq 1, \quad \text{a.s.}
\]

3.3 Case \( p < 2 \)

The case of a uniform point inside an \( \ell_p^n \)-ball with \( 1 \leq p < 2 \) poses some additional problems to surmount. Of note, the condition in Theorem 2.2 that the normalizing sequence \( c_n \to 0 \), and furthermore that there exists a \( p' \geq 1 \) such that \( \sum_{i=1}^{\infty} c_n < \infty \), fails to hold when the desired \( c_n = n^{1/p-1/2}/\sqrt{\log \log n} \). This suggests an alternative to Lemma 2.3 when \( E\|X^{(n)}\| \) tends to zero as \( n \to \infty \). That is, noting that

\[
|T_n(\pi_g X^{(n)}) - T_n(X^{(n)})| \leq c_n\|\pi_g X^{(n)} - X^{(n)}\| \leq 2c_n\|X^{(n)}\|,
\]

allows for the condition \( c_n \to 0 \) to be replaced by \( c_n E\|X^{(n)}\| \to 0 \) to conclude that

\[
E|T_n(\pi_g X^{(n)}) - T_n(X^{(n)})| \to 0.
\]

It similarly follows that \( c_n\|X^{(n)}\| \overset{a.s.}{\to} 0 \) implies that \( |T_n(\pi_g X^{(n)}) - T_n(X^{(n)})| \overset{a.s.}{\to} 0 \). This can be applied to uniformly random points within the \( \ell_p^n \)-ball for \( 1 \leq p < 2 \).

In this setting, a theorem of Schechtman and Zinn proposes concentration of the Euclidean norm on such an \( \ell_p^n \)-ball (Schechtman and Zinn, 1990, 2000). That is, for \( Y^{(n)} = (Y_1, \ldots, Y_n) \) a uniformly random point on the surface of an \( \ell_p^n \)-ball with \( 1 \leq p < 2 \), then there exist constants \( T, c > 0 \) such that for all \( t > T n^{1/2-1/p} \), \( P (\|Y\| > t) \leq e^{-ctn^{p'}} \). This implies that

\[
E\|X^{(n)}\| \leq E\|Y^{(n)}\| = \int_0^{T n^{1/2-1/p}} P (\|Y\| > t) dt + \int_{T n^{1/2-1/p}}^{1} P (\|Y\| > t) dt \leq T n^{1/2-1/p} + \int_{T n^{1/2-1/p}}^{1} e^{-ctn^{p'}} dt
\]

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and the last integral, with $\delta_n = Tn^{1/2-1/p}$, is bounded as follows:

$$
\int_{\delta_n}^{1} e^{-cn^p} dt \leq \int_{\delta_n}^{1} \frac{1}{\delta_n^{p-1}} e^{-cn^p} dt \leq \frac{n^{1/2-p/2-1/p}}{T^{p-1}cn^{p/2}}.
$$

The result is

$$
E\|X^{(n)}\| \leq n^{1/2-1/p} \left( T + \frac{1}{T^{p-1}cn^{p/2}} \right) = O(n^{1/2-1/p}).
$$

Thus, in the context of the problem at hand, $c_nE\|X^{(n)}\| = O((\log \log n)^{-1/2})$ and via Markov’s inequality, $c_n\|X^{(n)}\| \xrightarrow{p} 0$, and via the Lévy-Itô-Nisio theorem (Ledoux and Talagrand, 1991, Theorem 2.4), $c_n\|X^{(n)}\| \xrightarrow{a.s.} 0$. Thus, $T_n(X^{(n)})$ is once again asymptotically invariant to rotations, and invoking the same argument as before shows the existence of some $K > 0$ such that

$$
\limsup_{n \to \infty} \frac{\sum_{i=1}^{n} X_i}{K^{1/2-1/p} \sqrt{\log \log n}} \leq 1, \ a.s.
$$

### 4 One and two sample testing

A different application of Theorems 2.1 and 2.2 is statistical hypothesis testing. In the following sections, classical examples of one and two sample testing are considered. For comparison with each case, quantitative versions of Theorem 2.2 are proved assuming third moment conditions, which rely on results like the Berry-Esseen theorem. The constants in the following theorems can likely be improved with more careful arguments, but this is not investigated in this work. These results are included for contrast with the previously mentioned theorems and not assumed to be state-of-the-art.

#### 4.1 One Sample Location Test

A simple example of asymptotic invariance arises in the one sample location test (see Lehmann and Romano (2006) examples 15.2.1, 15.2.4, and 15.2.5). Given $X = (X_1, \ldots, X_n)$ independent and identically distributed real valued random variables with mean $\mu$, the hypotheses under consideration are

$$
H_0 : \mu = 0 \quad \text{and} \quad H_1 : \mu \neq 0.
$$

Let $G = \{-1, +1\}^n$ be the group corresponding to the vertices of the $n$-dimensional hypercube. For $\{\theta_i\}_{i=1}^{n}$ such that $\sum_{i=1}^{n} \theta_i^2 = 1$, let $T : \mathbb{R}^n \to \mathbb{R}$ be $T(x) = \sum_{i=1}^{n} \theta_i x_i$. And lastly, let $\pi_g x = (\pm x_1, \ldots, \pm x_n)$. To apply a randomization test based on the the group $G$, the additional assumption that the univariate distribution of the $X_i$ is symmetric about the origin is required, i.e. $P(X_i \in B) = P(X_i \in -B)$ where $-B = \{x \in \mathbb{R} : -x \in B\}$. In which case, condition C2 from above is satisfied, i.e. $T(\pi_g X) = T(X)$ in distribution, and thus the conclusions of Theorem 2.1 and Corollary 2.2 are valid.

Even in the absence of symmetry, Theorem 2.2 outlines conditions under which significance thresholds based on random sign flips asymptotically achieve the desired test size. In the above setting, the groups $G$ when paired with the Hamming metric form a normal Lévy family (Ledoux, 2001, Theorem 2.11). The function $T$ is Lipschitz with constant $c_n = \max_{i=1,\ldots,n} |\theta_i|$. Thus, the $\theta_i$ must be chosen such that for some $p \geq 1, \sum_{i=1}^{\infty} \max_{i=1,\ldots,n} |\theta_i| < \infty$. Moreover, the condition that $c_n \|X^{(n)}\|/\sqrt{n} \xrightarrow{a.s.} 0$ has further implications on the moments of $\|X^{(n)}\|$ and the choice of $\theta_i$. By invoking Theorem 2.1.3. of Stout (1974), almost sure convergence to 0 is achieved if for
some $p > 0$,

$$
\sum_{i=1}^{\infty} c_n^p E[\|X_{(n)}\|^p] < \infty.
$$

In particular, as the $X_i$’s are iid, choosing $p = 2q$ results in a simplified convergence condition:

$$
\sum_{n=1}^{\infty} \frac{c_n^{2q}}{n^{q/2}} E\left(\sum_{i=1}^{n} X_i^2\right)^q \leq E[X_1^{2q}] \sum_{n=1}^{\infty} \frac{c_n^{2q}}{n^{q/2}} < \infty.
$$

Hence, if $X_1$ has a finite $p$th moment, then the $\theta_i$ must be chosen so that $\max_{i=1,\ldots,n} |\theta_i| = o(n^{1/p})$. For the common choice of $\theta_1 = \ldots = \theta_n = 1/\sqrt{n}$, Theorem 2.2 requires the relatively mild condition that $EX_1^{3+\varepsilon}$ for some $\varepsilon > 0$ in order to achieve asymptotic equivalence.

In contrast, assuming a finite third moment allows for a quantitative version of Theorem 2.2 to be proved directly for this specific setting. Indeed, a simple application of the Berry-Esseen theorem (Feller, 2008, Section XVI.5) under the appropriate assumptions demonstrates the asymptotic validity of the randomization test. More recent work on Berry-Esseen bounds can be used to generalize beyond the iid setting and make use of other “natural characteristics” beyond merely the third absolute moment (Bobkov et al., 2014, 2018), but this is not explored in this work.

Theorem 4.1. Let $X = (X_1, \ldots, X_n)$ be iid mean zero random variables with variance $\sigma^2$ and $E|X_1|^3 = \omega < \infty$. Then, for $T$ as above and for all $t \in \mathbb{R}$ and some universal constant $C > 0$,

$$
|P(T(X) > t) - EX\rho\{g \in G : T(\pi_g X) > t}\} \leq \frac{2C\omega}{\sigma^3} \sum_{i=1}^{n} |\theta_i|^3.
$$

Furthermore, if $\sum_{i=1}^{n} |\theta_i|^3 \rightarrow 0$ as $n \rightarrow \infty$, then the probabilities coincide asymptotically. In particular, if $\theta_1 = \ldots = \theta_n = n^{-1/2}$, then the right hand side is $O(n^{-1/2})$.

Proof. Let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{-1, +1\}^n$. Let $\Phi(t)$ be the cumulative distribution function for a univariate standard normal random variable. In this setting,

$$
EX\rho\{g \in G : T(\pi_g X) > t\} = 2^{-n} EX\left(\sum_{\varepsilon \in \{-1, +1\}^n} 1 \left[\sum_{i=1}^{n} \varepsilon_i \theta_i X_i > t\right]\right)
$$

$$
= 2^{-n} \sum_{\varepsilon \in \{-1, +1\}^n} P\left(\sum_{i=1}^{n} \varepsilon_i \theta_i X_i > t\right)
$$

Irregardless of $\varepsilon_i$, $EX(\varepsilon_i \theta_i X_i) = 0$, $E(\varepsilon_i \theta_i X_i)^2 = \theta_i^2 \sigma^2$, and $E|\varepsilon_i \theta_i X_i|^3 = |\theta_i|^3 \omega$. Consequently, the Berry-Esseen theorem (Feller, 2008, Theorem 2, Section XVI.5) implies that there exists a universal constant $C > 0$ such that for any fixed choice of $\varepsilon$

$$
\left|P\left(\frac{1}{\sigma} \sum_{i=1}^{n} \varepsilon_i \theta_i X_i \leq t\right) - \Phi(t)\right| \leq \frac{C\omega}{\sigma^3} \sum_{i=1}^{n} |\theta_i|^3.
$$

And thus, for $\Phi^c = 1 - \Phi$,

$$
|EX\rho\{g \in G : T(\pi_g X) > t\} - \Phi^c(t/\sigma)| \leq \frac{C\omega}{\sigma^3} \sum_{i=1}^{n} |\theta_i|^3.
$$
Finally,
\[
|P(T(X) > t) - \text{Exp}\{\{g \in G : T(\pi_g X) > t\}\}| \\
\leq |P(T(X) > t) - \Phi^c(t/\sigma)| + |\text{Exp}\{\{g \in G : T(\pi_g X) > t\}\} - \Phi^c(t/\sigma)| \leq \frac{2C\omega}{\sigma^3} \sum_{i=1}^{n} |\theta_i|^3.
\]

4.2 Two Sample t-Test

The two sample t-test stands as a prototypical hypothesis test (Lehmann and Romano, 2006, Section 11.3). The goal is to determine if two populations have the same mean. Let \(X = (X_1, \ldots, X_n, X_{n+1}, \ldots, X_{n+m})\) be independent Gaussian real valued random variables such that
\[
\mathbb{E}X_i = \begin{cases} 
\mu_1, & i \leq n \\
\mu_2, & i > n
\end{cases}
\text{ and } \text{Var}(X_i) = \begin{cases} 
\sigma_1^2, & i \leq n \\
\sigma_2^2, & i > n
\end{cases}.
\]

The sample means are defined as \(\bar{X}_1 = \frac{1}{n} \sum_{i=1}^{n} X_i\) and \(\bar{X}_2 = \frac{1}{m} \sum_{i=n+1}^{n+m} X_i\). The standard two sample t-test statistic under the assumption of homogeneous variances is to compute the test statistic
\[
T_{\text{hom}}(X) = \frac{\bar{X}_1 - \bar{X}_2}{s_p \sqrt{n^{-1} + m^{-1}}}
\]
with \(s_p^2 = \frac{(n-1)s_1^2 + (m-1)s_2^2}{n+m-2}\) where \(s_p^2\) is the pooled estimator for the population variance based on the sample variances \(s_1^2\) and \(s_2^2\) calculated for each population. Under the null hypothesis that \(\mu_1 = \mu_2\), the test statistic, \(T_{\text{hom}}(X)\), has a t-distribution with \(n + m - 2\) degrees of freedom.

If, however, the population variances are heterogeneous, then the above test statistic will not have a t-distribution under the null. This is the so-called Behrens–Fisher problem. A standard solution to this problem is to use Welch’s t-test. The test statistic in this case is
\[
T_{\text{het}}(X) = \frac{\bar{X}_1 - \bar{X}_2}{\sqrt{s_1^2/n + s_2^2/m}}.
\]

The distribution under the null hypothesis of equal population means can be roughly approximated by a t-distribution with degrees of freedom equal to
\[
\frac{(s_1^2/n + s_2^2/m)^2}{s_1^2/[n^2(n-1)] + s_2^2/[m^2(m-1)]}.
\]
As, for example, when \(n \to \infty\) with \(m\) fixed, the degrees of freedom tend towards \(m - 1\).

The standard two sample permutation test arises from the unnormalized difference of means \(T(X) = X_1 - X_2\) and uniformly random permutations from \(G = S_{n+m}\), the symmetric group on \(n + m\) elements. Then, conditioned on the observed data \(X = x\), one computes
\[
p\text{-value} = \frac{|\{\pi_g : T(\pi_g x) \geq T(x)\}|}{(n + m)!}.
\]
Of course, this is computationally infeasible. Thus, the typical solution is to sample some permutations \(\{\pi_1, \ldots, \pi_r\}\) uniformly at random from \(S_{n+m}\) and compute
\[
p\text{-value} \approx \frac{1 + \sum_{i=1}^{r} 1[T(\pi_i x) \geq T(x)]}{1 + r}.
\]
The consequences of such sampling are discussed in Hemerik and Goeman (2018a). Otherwise, Kashlak et al. (2022) develops analytic methods for computing exact permutation test p-values for two-sample and k-sample tests for data in Banach spaces by making use of Khintchine-Kahane-type inequalities.

It can be seen that condition C2 holds in this setting assuming that either the variances are homogeneous, i.e. \( \text{Var}(X_1) = \ldots = \text{Var}(X_{n+m}) \), or the sample sizes are equal, i.e. \( n = m \). If both of these assumptions fail to hold, Theorem 2.2 may still be applicable. This conclusion is similar to the one-sample test setting of the previous section. Indeed, the symmetric group with the Hamming metric is a normal Lévy family (Ledoux, 2001, Corollary 4.3). The function \( T \) is Lipschitz with constant \( c_{n+m} = \min\{1/n, 1/m\} \), which, without loss of generality, taking \( n \geq m := m_n \) as a function of \( n \) gives \( c_{n+m} = 1/m_n \). Furthermore, for \( p = 2q \geq 1 \),

\[
\left( \frac{c_{n+m} \|X^{(n+m)}\|}{\sqrt{n+m}} \right)^{2q} = \frac{\left( \sum_{i=1}^{n+m} X_i^2 \right)^q}{m_n^{2q}(n+m)^q}.
\]

Hence, Theorem 2.1.3 of Stout (1974) again implies almost sure convergence to zero of the above sequence if

\[
E[X_1^{2q}] \sum_{n=1}^{\infty} \frac{1}{m_n^{2q}} < \infty.
\]

Thus, for proportional sample sizes \( m_n = \lfloor c_0 n \rfloor \) for \( 0 < c_0 < 1 \), convergence is achieved when \( p = 2q = 1 + \epsilon \) for some \( \epsilon > 0 \) and \( E[X_1^{2q+\epsilon}] < \infty \).

For comparison with Theorem 2.2, the following theorem quantitatively bounds how poorly a permutation test can perform when the assumption of exchangeability is violated for Gaussian data. The subsequent corollary passes through the Berry-Essen bounds to achieve a quantitative version of Theorem 2.2 for non-exchangeable random variables with finite third absolute moment.

Of note, if the sample sizes are proportional as discussed above, e.g. \( m_n = \lfloor c_0 n \rfloor \), then the next Theorem concludes that

\[
\left| P(T(X) > t) - E_{X_\rho}(\{g \in G : T(\pi g X) > t\}) \right| = O \left( \frac{\sqrt{\log n}}{n} \right).
\]

Though, it is worth future consideration as to whether or not the \( \log n \) term is necessary.

**Theorem 4.2.** Let \( X_1, \ldots, X_n \) be iid univariate random variables with mean \( \eta \), variance \( \sigma_1^2 \), and finite absolute third moment. Similarly, let \( X_{n+1}, \ldots, X_{n+m} \) be iid and independent of the first collection with mean \( \eta \), variance \( \sigma_2^2 \), and finite absolute third moment. Let \( T(X) = n^{-1} \sum_{i=1}^{n} X_i - m^{-1} \sum_{i=n+1}^{n+m} X_i \). Then, for any \( t \in \mathbb{R} \)

\[
\left| P(T(X) > t) - E_{X_\rho}(\{g \in G : T(\pi g X) > t\}) \right| \leq \sqrt{2 \left( \frac{1}{m} - \frac{1}{n} \right) (\sigma_1^2 - \sigma_2^2) \log \left( \frac{nm/\sqrt{n^2 - m^2}}{\sqrt{2\pi(\sigma_1^2 - \sigma_2^2)}} + 1 \right)} + O \left( \frac{\sqrt{1/n + 1/m}}{.}ight).
\]

**Lemma 4.1.** Let \( \mu \) and \( \nu \) be centred Gaussian measures on \( \mathbb{R} \) with variances \( \sigma_1^2 \) and \( \sigma_2^2 \), respectively, with \( \sigma_1^2 \geq \sigma_2^2 > 0 \). Then, denoting the Lévy-Prokhorov metric by \( d_{\text{LP}} \),

\[
d_{\text{LP}}(\mu, \nu) \leq \sqrt{2(\sigma_1^2 - \sigma_2^2) \log \left( \frac{1}{\sqrt{2\pi(\sigma_1^2 - \sigma_2^2)}} + 1 \right)}.
\]

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Proof. The Lévy-Prokhorov metric is defined as
\[
d_{LP}(\mu, \nu) = \inf \{ \varepsilon > 0 : \mu(A) \leq \nu(A) + \varepsilon \text{ and } \nu(A) \leq \mu(A) + \varepsilon \}
\]
where \(d_{KF}(X, Y) = \inf \{\varepsilon > 0 : P(|X - Y| > \varepsilon) < \varepsilon\}\) is the Ky Fan metric for random variables \(X\) and \(Y\).

Considering \((X, Y)\) bivariate Gaussian, the variance of \(X - Y\) is minimized when \(\text{cov}(X, Y) = \sigma^2\) and then \(\text{Var}(X - Y) = \sigma^2 - \sigma_i^2\). Thus,
\[
P(|X - Y| > \varepsilon) = 2 \int_{\varepsilon}^{\infty} \frac{1}{\sqrt{2\pi(\sigma^2_1 - \sigma^2_2)}} \exp \left( -\frac{t^2}{2(\sigma^2_1 - \sigma^2_2)} \right) dt
\]

and thus,
\[
d_{LP}(\mu, \nu) \leq \varepsilon \text{ such that } \exp \left( -\frac{\varepsilon^2}{2(\sigma^2_1 - \sigma^2_2)} \right) - \frac{\varepsilon^2}{2} \sqrt{\frac{2\pi}{\sigma^2_1 - \sigma^2_2}} = 0.
\]
The solution to this equation is
\[
\varepsilon = \sqrt{2(\sigma^2_1 - \sigma^2_2)W \left( 1 / \sqrt{2\pi(\sigma^2_1 - \sigma^2_2)} \right)} \leq \sqrt{2(\sigma^2_1 - \sigma^2_2) \log \left( \frac{1}{\sqrt{2\pi(\sigma^2_1 - \sigma^2_2)}} + 1 \right)}
\]
where \(W\) is Lambert's W function.

Lemma 4.2. Let \(X = (X_1, \ldots, X_n, X_{n+1}, \ldots, X_{n+m})\) be independent Gaussian real valued random variables such that
\[
E X_i = \eta_i, \forall i \text{ and } \text{Var} (X_i) = \begin{cases} \sigma^2_i, & i \leq n \\ \sigma^2_2, & i > n \end{cases}
\]
assuming without loss of generality that \(n > m\). Let \(T(X) = n^{-1} \sum_{i=1}^{n} X_i - m^{-1} \sum_{i=n+1}^{n+m} X_i\) and \(G = S_{n+m}\) be the symmetric group on \(n + m\) elements. Then,
\[
|P(T(X) > t) - E_X \rho \{ g \in G : T(\pi_g X) > t \})|
\]
\[
\leq \sqrt{2 \left( \frac{1}{m} - \frac{1}{n} \right) (\sigma^2_1 - \sigma^2_2) \log \left( \frac{nm/\sqrt{n^2 - m^2}}{\sqrt{2\pi(\sigma^2_1 - \sigma^2_2)}} + 1 \right)}.\]

Proof of Lemma 4.2. The Gaussian measure induced by \(T(X)\) has zero mean and variance \(\sigma^2_i/n + \sigma^2_j/m\) and will be denoted by \(\mu\). In turn, the Gaussian measure averaged over the group \(G\) is a weighted mixture of \(m\) Gaussian distributions and will be denoted \(\nu = \sum_{j=0}^{m} w_j \nu_j\). All component measures have zero mean and a variance of
\[
\sigma^2_1 \left( \frac{n - j}{n^2} + \frac{j}{m^2} \right) + \sigma^2_2 \left( \frac{m - j}{m^2} + \frac{j}{n^2} \right) = \frac{\sigma^2_1}{n} + \frac{\sigma^2_2}{m} + j \left( \frac{m^2 - n^2}{m^2 n^2} \right) (\sigma^2_1 - \sigma^2_2).
\]
with hypergeometric weights \( w_j = \binom{n}{m-j}/\binom{n+m}{m} \). From Lemma 4.1, the Lévy-Prokhorov metric between \( \mu \) and \( \nu_j \) for \( j \neq 0 \) is

\[
d_{LP}(\mu, \nu_j) \leq \sqrt{2j \left( \frac{n^2 - m^2}{n^2 m^2} \right) (\sigma_1^2 - \sigma_2^2) \log \left( \frac{nm/\sqrt{n^2 - m^2}}{\sqrt{2\pi(\sigma_1^2 - \sigma_2^2)}} + 1 \right)}
\]

Applying the triangle inequality and Jensen’s inequality finishes the proof:

\[
d_{LP}(\mu, \nu) \leq \sum_{j=0}^{m} \binom{n}{m-j} \binom{m-j}{n+m-j} \sqrt{2j \left( \frac{n^2 - m^2}{n^2 m^2} \right) (\sigma_1^2 - \sigma_2^2) \log \left( \frac{nm/\sqrt{n^2 - m^2}}{\sqrt{2\pi(\sigma_1^2 - \sigma_2^2)}} + 1 \right)}
\]

**Proof of Theorem 4.2.** The function \( T \) can be written equivalently as

\[
T(X) = \frac{1}{n} \sum_{i=1}^{n} (X_i - \eta) - \frac{1}{m} \sum_{i=n+1}^{n+m} (X_i - \eta).
\]

As a result, for \( i \leq n \) and \( i > n \), respectively,

\[
E \left[ \frac{(X_i - \eta)}{n} \right]^2 = \frac{\sigma_1^2}{n^2}, \quad E \left[ \frac{(X_i - \eta)}{m} \right]^2 = \frac{\sigma_2^2}{m^2}
\]

\[
E \left[ \frac{(X_i - \eta)}{n} \right]^3 = \frac{\nu_1}{n^3}, \quad E \left[ \frac{(X_i - \eta)}{m} \right]^3 = \frac{\nu_2}{m^3}
\]

Thus, the Berry-Esseen Theorem (Feller, 2008, Theorem 2, Section XVI.5) states that for some universal constant \( C > 0 \),

\[
\left| P \left( \left[ \frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n} \right]^{-1/2} T(X) \geq t \right) - \Phi(t) \right| \leq C \frac{\frac{\nu_1}{n^2} + \frac{\nu_2}{m^2}}{\left( \frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m} \right)^{3/2}} = C \frac{m^2 \nu_1 + n^2 \nu_2}{(m \sigma_1^2 + n \sigma_2^2)^2} \sqrt{\frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m}} = O \left( \sqrt{\frac{1}{n} + \frac{1}{m}} \right).
\]

Thus, let \( Z \in \mathbb{R}^{n+m} \) be multivariate Gaussian with mean zero and covariance matrix with zero
off-diagonal entries and main diagonal of $\sigma_i^2$ for the first $n$ entries and $\sigma_i^2$ for the final $m$ entries.

\[
|P(T(X) > t) - \mu_X\rho\{g \in G : T(\pi_g X) > t}\}| \\
\leq |P(T(X) > t) - P(Z > t)| \\
+ |P(T(Z) > t) - \mu_Z\rho\{g \in G : T(\pi_g Z) > t}\}| \\
+ |\mu_Z\rho\{g \in G : T(\pi_g Z) > t}\| - \mu_X\rho\{g \in G : T(\pi_g X) > t\}|
\]

The first line is $O(n^{-1/2} + m^{-1/2})$ by the Berry–Esseen theorem as discussed above. The second line is bounded by Lemma 4.2. The third line is also $O(n^{-1/2} + m^{-1/2})$ by the Berry–Esseen theorem. Indeed, the measure $\mu_Z\rho\{g \in G : T(\pi_g Z) > t\}$ is a weighted mixture of Gaussians as discussed in the proof of Lemma 4.2.

For comparison with Theorem 4.2, the following result provides a quantitative bound on the total variation distance between a Gaussian average and the randomly permuted average. Of note, the inclusion of the term $\sqrt{(n-m)/(n+m)}$ necessitates a stronger asymptotic agreement between $n$ and $m$; i.e. $m = cn$ is no longer sufficient. Such a result suggests that a total variation version of Theorem 2.2 may be of future interest.

**Theorem 4.3.** Let $X = (X_1, \ldots, X_n, X_{n+1}, \ldots, X_{n+m})$ be independent Gaussian real valued random variables such that

\[
\mu_X = \eta, \ \forall i \text{ and } \text{Var}(X_i) = \begin{cases} \sigma_i^2, & i \leq n \\ \sigma_i^2, & i > n \end{cases}
\]

assuming without loss of generality that $n \geq m$. Let $T(X) = n^{-1}\sum_{i=1}^n X_i - m^{-1}\sum_{i=n+1}^m X_i$ and $G = S_{n+m}$ be the symmetric group on $n + m$ elements. For measures $\mu$ and $\nu$ on $(\mathbb{R}, \mathcal{B})$ defined by $\mu(B) = P(T(X) \in B)$ and $\nu(B) = \mu_X\rho\{g \in G : T(\pi_g X) \in B\}$ for any $B \in \mathcal{B},$

\[
\|\mu - \nu\|_{TV} \leq \frac{1}{2} \left( \frac{n-m}{n+m} \right)^{1/2} |\sigma_2^2 - \sigma_1^2|^{1/2} \max \left\{ \sqrt{\frac{1}{\sigma_2^2} + \frac{m}{\sigma_1^2}}, \sqrt{\frac{1}{\sigma_2^2}} \right\}.
\]

**Proof of Theorem 4.3.** For two equivalent measures, $\mu$ and $\nu$, the Kullback–Leibler Divergence is $D_{KL}(\mu, \nu) = \int \log \frac{d\nu}{d\mu} d\mu$, and the symmetric KL-divergence is defined to be $H(\mu, \nu) = 0.5[D_{KL}(\mu, \nu) + D_{KL}(\nu, \mu)]$. From Pinkser’s Inequality,

\[
\|\mu - \nu\|_{TV} := \sup_{B \in \mathcal{B}} |\mu(B) - \nu(B)| \leq \sqrt{\frac{1}{2} \min\{D_{KL}(\mu, \nu), D_{KL}(\nu, \mu)\}} \leq \sqrt{\frac{1}{2} H(\mu, \nu)}.
\]

For two independent centred univariate Gaussian measures on $\mathbb{R}$, $\gamma_1$ and $\gamma_2$, with variances $\sigma_1^2$ and $\sigma_2^2$, the symmetric KL-divergence is

\[
H(\gamma_1, \gamma_2) = \frac{1}{2} D_{KL}(\gamma_1 | \gamma_2) + \frac{1}{2} D_{KL}(\gamma_2 | \gamma_1) = \frac{\sigma_1^2}{4\sigma_2^2} + \frac{\sigma_2^2}{4\sigma_1^2} - \frac{1}{2} = \frac{1}{4} \left( \frac{\sigma_1}{\sigma_2} - \frac{\sigma_2}{\sigma_1} \right)^2.
\]

In the context of the two-sample t-test for Gaussian data, the measure $\mu$ induced by $T(X)$ has zero mean and variance $\sigma_1^2/n + \sigma_2^2/m$. In turn, the measure $\nu$ averaged over the group $G$ is a weighted mixture of $m$ Gaussian distributions denoted as $\nu = \sum_{j=1}^m w_j \nu_j$. All component measures $\nu_j$ have zero mean and a variance of

\[
\sigma^2 = \frac{(n-j) m^2 + j n^2}{n^2 m^2} + \frac{(m-j) n^2 + j m^2}{n^2 m^2} = \frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m} + j \left( \frac{m^2 - n^2}{n^2 m^2} \right) (\sigma_1^2 - \sigma_2^2)
\]
with hypergeometric weights \( w_j = \binom{n}{j} \binom{m}{m-j} \binom{n+m}{m}^{-1} \). The symmetric KL-divergence between \( \mu \) and the \( j \)th mixture component \( \nu_j \) is

\[
H(\mu, \nu_j) = \frac{j^2 (m^{-2} - n^{-2})^2 (\sigma_1^2 - \sigma_2^2)^2}{4 \left[ \frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m} + j (m^{-2} - n^{-2}) (\sigma_1^2 - \sigma_2^2) \right]}
= \frac{1}{4} \left( j^2 (m^{-2} - n^{-2})^2 (\sigma_1^2 - \sigma_2^2)^2 \sigma_1^2 (\nu_j^{-1} + \sigma_2^2 m^{-1})^{-2} \right)
= \frac{1}{4} \left( \frac{C_{n,m,j}^2 (m^{-2} - n^{-2})^2}{1 + C_{n,m,j} (m^{-2} - n^{-2})} \right),
\]

where \( C_{n,m} = (\sigma_1^2 - \sigma_2^2) (\sigma_1^2 n^{-1} + \sigma_2^2 m^{-1})^{-1} \) for notational convenience.

Joint convexity of the KL-divergence implies that \( D_{KL}(\mu, \nu) \leq \sum_{j=1}^{M} w_j D_{KL}(\mu, \nu_j) \), which translates into the same for \( H(\mu, \nu) \). Thus, considering the extreme cases of \( n = m \) and \( n \to \infty \) for \( m \) fixed results in

\[
H(\mu, \nu) \leq \sum_{j=0}^{m} \frac{\binom{n}{j} \binom{m}{m-j} 1}{\binom{n+m}{m}} \frac{C_{n,m,j}^2 (m^{-2} - n^{-2})^2}{1 + C_{n,m,j} (m^{-2} - n^{-2})}
\leq \frac{1}{4} \sum_{j=0}^{m} \frac{\binom{n}{j} \binom{m}{m-j}}{\binom{n+m}{m}} \frac{C_{n,m,j} (m^{-2} - n^{-2})}{1 + C_{n,m,j} (m^{-2} - n^{-2})}
= \frac{1}{4} \frac{mn}{n+m} (m^{-2} - n^{-2}) \frac{\sigma_2^2 - \sigma_1^2}{\sigma_2^2 - \sigma_1^2} \frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m} \frac{1}{1 + \frac{m}{n} \sigma_1^2 + \frac{n}{m} \sigma_2^2}^{-1}
= \frac{1}{4} \frac{n-m}{n+m} \frac{\sigma_2^2 - \sigma_1^2}{\sigma_2^2 + \sigma_1^2} \frac{1}{1 + \frac{m}{n} \sigma_1^2 + \frac{n}{m} \sigma_2^2}^{-1}
= \frac{1}{4} \frac{n^2 - m^2}{(n+m)^2} \frac{\sigma_2^2 - \sigma_1^2}{\sigma_2^2 + \sigma_1^2} \frac{(nm + m^2) \sigma_1^2 + (nm + n^2) \sigma_2^2}{(\sigma_1^2 + \sigma_1^2)(n+m)^2}^{-1}
= \frac{1}{4} \frac{n-m}{n+m} \frac{\sigma_2^2 - \sigma_1^2}{\sigma_2^2 + \sigma_1^2} \frac{nm + m^2 \sigma_1^2 + (nm + n^2) \sigma_2^2}{(\sigma_2^2 + \sigma_1^2)(n+m)^2}^{-1}
\leq \frac{1}{4} \frac{n-m}{n+m} \frac{\sigma_2^2 - \sigma_1^2}{\sigma_2^2 + \sigma_1^2} \max \left\{ \frac{2}{4}, \frac{1}{n+m} \right\} \frac{1}{1 - \frac{\sigma_1^2}{\sigma_2^2}}
= \max \left\{ \frac{1}{2} \left( \frac{n-m}{n+m} \right) \frac{1}{\sigma_2^2 + \sigma_1^2} \left( \frac{1}{4} \frac{n-m}{n+m} \right) \left( 1 - \frac{\sigma_1^2}{\sigma_2^2} \right) \right\}
\]

Applying Pinsker’s inequality from above concludes the proof.

**4.3 Simulation Experiments**

The following subsections contain brief simulation experiments to illustrate the behaviour of the above hypothesis tests.
4.3.1 One Sample Location Test

To examine Theorem 4.1 in a simulation setting, the exponentially modified Gaussian distribution (EMGD) will be considered. A random variable $Z$ is said to be EMGD if it can be written as $Z = X + Y$ where $X$ is Gaussian, $Y$ is exponential, and $X$ and $Y$ are independent. This convolution of Gaussian and exponential distributions has some popularity in modelling problems within chemistry and cellular biology (Grushka, 1972; Golubev, 2010). In what follows, $X \sim \mathcal{N}(0, 1)$ and $Y \sim \text{Exponential}(\lambda)$, and $Z$ will be centred by $1/\lambda$ to have mean zero.

For sample sizes $n \in \{10, 100\}$, samples of $Z_1, \ldots, Z_n$ were generated 200 times for each exponential rate parameter $\lambda \in \{\infty, 10, 1, 0.1, 0.01, 0.001\}$ where $\lambda = \infty$ corresponds to $Y = 0$ almost surely. Thus, as $\lambda$ tends towards zero, the skewness of the centred EMGD increases. For each set of simulated $Z_1, \ldots, Z_n$, a standard one-sample $t$-test was performed via the `t.test()` function in the `stats` R package (R Core Team, 2019). Secondly, a randomization test was performed by generating 2000 random sign vectors $\varepsilon \in \{\pm 1\}^n$ to approximate the value of $E_X \rho(g \in G : T(\pi g X) > t)$ from Theorem 4.1.

Figure 3 and Figure 4 display the 200 computed p-values for $n = 10$ and $n = 100$, respectively. The $t$-test p-value is plotted against the randomization test p-value. When $n = 10$ and the exponential rate parameter is small, i.e. the skewness is large, the p-values produced by the two tests begin to disagree. However, when $n = 100$, the two tests produce nearly identical p-values regardless of skewness.

4.3.2 Two Sample $t$-Test

To test the performance of the bound derived in Theorem 4.2, the permutation test and Welch’s two sample $t$-test are compared on simulated data. Figure 5 compares the cumulative distribution functions of $T(X)$ and $T(\pi_g X)$ with respect to the bounds via the Lévy-Prokhorov metric.
for \((n, m)\) values of \((200, 100)\), \((2000, 1000)\), and \((20000, 10000)\) always in a two-to-one ratio. For each of these three choices for \((n, m)\), 200 data vectors are simulated from a Gaussian distribution where the larger sample has a variance of 1 and the smaller sample has a variance of 16. As the total sample size \(N = n + m\) grows large, the difference between the two cdf functions becomes vanishingly small.

Secondly, for each of 200 replications, an iid standard Gaussian dataset is generated with sample size \(n = 200\), and iid Gaussian datasets with mean zero, variance 16, and sample sizes \(m = 25, 50, 100, 200\) are generated. The p-value for Welch’s two sample t-test, as described above, is computed in R via the \texttt{t.test()} function. The permutation test p-value is computed via 2000 permutations. Figure 6 displays the result of these simulations. When the smaller sample has a larger variance, the permutation test is anti-conservative. That is, it produces p-values that are smaller than desired and thus would lead to more frequent false rejections of the null hypothesis. But as \(m\) approaches \(n\), both the discrepancy between the two tests and the bounds from Theorem 4.3 vanish. If the sample with \(n = 200\) observations came from the population with the larger variance, then the permutation test would instead be too conservative and the plots would be reflected across the diagonal.

5 Discussion

It was demonstrated that nice functions of high dimensional random variables are often nearly invariant to the actions of a compact topological group. This leads to interesting applications such as random rotations of \(\ell_p^n\)-balls allowing for the application of concentration inequalities for \(SO(n)\) (Meckes, 2019). The motivating example of statistical hypothesis testing demonstrates that randomization tests can be widely applicable even if the invariance assumption does not
Figure 5: A comparison of the cumulative distribution functions for $T(X)$ (black) and $T(\pi_g X)$ (blue) with the upper and lower bounds (dashed lines) from Theorem 4.2.

Figure 6: A comparison of the two measures in Theorem 4.3 for $n = 200$ and $m = 25, 50, 100, 200$. The red lines correspond to the bounds on the total variation norm in Theorem 4.3.
A deeper question in need of future investigation is that of appropriate group selection. Many groups can satisfy the theorem conditions outlined in Section 2. However, each will induce a different collection of null hypothesis distributions. As noted, some very recent work has considered this question (Koning and Hemerik, 2023; Koning, 2023). Secondly, it is not always possible to parse the properties of a given statistical test using direct methods. The equivalence or asymptotic equivalence of randomization tests offers a novel approach to understanding the strengths and limitations of a given statistical test.

While Theorem 2.2 and Corollary 2.6 are certainly useful, the quantitative investigation in Section 4.2 hints that future results concerning convergence in total variation will also be of interest. In particular, a comparison of the conditions required to prove such theorem with those presented in this work would give insight into different types of convergence.

Lastly, a broader treatment of locally compact topological groups is of future interest. Of course, Haar measure would no longer be a finite measure leading to problems concerning how to think about a random element of a locally compact group.

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A Direct Computation of $\ell^p_n$-balls

As noted in Section 3, one can directly compute the second moment of a uniform point within an $\ell^p_n$-ball and thus derive convergence properties via a standard application of Chebyshev’s inequality and the first Borel-Cantelli lemma. Such tedious calculations for the $\ell^1_n$-ball and $\ell^2_n$-ball are included here for completeness. Note that the volume of an $\ell^p_n$-ball of radius $r > 0$ is

$$\text{vol}_{n,p}(r) = (2r)^n \frac{\Gamma(1 + 1/p)^n}{\Gamma(1 + n/p)}.$$ 

which appears in a variety of sources such as Wang (2005); Rabiei and Saleeby (2018). Thus, $\text{vol}_{n,1}(r) = (2r)^n / n!$ and $\text{vol}_{n,2}(r) = r^n \pi^{n/2} / \Gamma(1 + n/2)$. 

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A.1 The $\ell^1_n$-ball

Let $X \in \mathbb{R}^n$ be a uniformly random point within the $\ell^1_n$-ball; that is, $\sum_{i=1}^n |X_i| \leq 1$. By symmetry, $E X_i = 0$ for all $i$ and $E X_i X_j = 0$ for all $i \neq j$. For the second moment,

$$E X_i^2 = \frac{n!}{2^n} \int_{\ell^1_n} x_i^2 \, dx$$

$$= \frac{n!}{2^n} \int_{-1}^1 x_i^2 \int_{\ell^1_{n-1} |x_i|} d_{n-1} \, dx_1$$

$$= \frac{n!}{2^n} \int_{-1}^1 x_i^2 \frac{(2(1-|x_i|))^{n-1}}{(n-1)!} \, dx_1$$

$$= n \int_0^1 x_i^{n-1}(1-x_i)^{n-1} \, dx_1$$

$$= \frac{\Gamma(3)\Gamma(n)}{\Gamma(n+3)} = \frac{2}{(n+1)(n+2)}.$$

Thus, $\text{Var} \left( \sum_{i=1}^n X_i \right) = 2n/(n+1)(n+2) \leq 2/n$, and by Chebyshev’s inequality, for any $t > 0$,

$$P \left( n^{-q} \sum_{i=1}^n X_i \geq t \right) \leq \frac{2}{n^{1+2q}}.$$

As $\sum_{i=1}^\infty n^{-1-2q} < \infty$ for any choice of $q > 0$, we have that

$$n^{-q} \sum_{i=1}^n X_i \xrightarrow{a.s.} 0$$

as a consequence of the first Borel-Cantelli lemma.

A.2 The $\ell^2_n$-ball

A similar calculation can be performed for the $\ell^2$-ball. Let $X \in \mathbb{R}^n$ be a uniformly random point within the $\ell^2_n$-ball; that is, $\sum_{i=1}^n X_i^2 \leq 1$. By symmetry, $E X_i = 0$ for all $i$ and $E X_i X_j = 0$ for all $i \neq j$. For the second moment,

$$E X_i^2 = \frac{\Gamma(1+n/2)}{\pi^{n/2}} \int_{\ell^2_n} x_i^2 \, dx$$

$$= \frac{\Gamma(1+n/2)}{\pi^{n/2}} \int_{-1}^1 x_i^2 \int_{\ell^1_{n-1} \sqrt{1-x_i^2}} d_{n-1} \, dx_1$$

$$= \frac{\Gamma(1+n/2)}{\pi^{n/2}} \int_{-1}^1 x_i^2 \frac{(1-x_i^2)^{(n-1)/2} n(n-1)/2!}{\Gamma(1+(n-1)/2)!} \, dx_1$$

$$= 2 \frac{\Gamma(1+n/2)}{\sqrt{\pi} \Gamma((n+1)/2)} \int_0^1 x_i^2 (1-x_i^2)^{(n-1)/2} \, dx_1$$

$$= \frac{1}{\sqrt{\pi} \Gamma((n+1)/2)} \int_0^1 u^{1/2}(1-u)^{(n-1)/2} \, du$$

$$= \frac{1}{\sqrt{\pi} \Gamma((n+1)/2)} \frac{\Gamma(3/2)\Gamma((n+1)/2)}{\Gamma(n/2+2)} = \frac{1}{n+2}.$$
Thus, \( \text{Var} \left( \sum_{i=1}^{n} X_i \right) = n/(n+2) \leq 1 \), and by Chebyshev’s inequality, for any \( t > 0 \),

\[
P \left( n^{-q} \left| \sum_{i=1}^{n} X_i \right| \geq t \right) \leq \frac{1}{n^{2q}}.
\]

As \( \sum_{i=1}^{\infty} n^{-2q} < \infty \) for any choice of \( q > 1/2 \), we have that

\[
n^{-q} \left| \sum_{i=1}^{n} X_i \right| \overset{a.s.}{\rightarrow} 0
\]
as a consequence of the first Borel-Cantelli lemma.

### B Simulating uniform points in an \( \ell_p^n \) ball

For the sake of general interest and future simulations, it is shown in this appendix how to generate uniform random points within an \( \ell_p^n \) ball using the ratio of uniforms method (Fishman, 2013, Section 3.7 and references therein). This is of interest, in particular, because the naive acceptance-rejection approach of simulating \( U = (U_1, \ldots, U_n) \) with entries independent and identically distributed Uniform \([-1, 1]\) and accepting if \( \sum_{i=1}^{n} U_i^p \leq 1 \) will yield an acceptance with vanishingly small probability for large \( n \) as the volume of the \( \ell_p^n \)-ball is dwarfed by the volume of the hypercube. That is,

\[
\frac{\text{vol}_{n,p}(1)}{2^n} = \frac{\Gamma(1+1/p)^n}{\Gamma(1+n/p)} \ll 1
\]

for large \( n \).

Instead, let \( Y = (Y_1, \ldots, Y_n) \) be a vector of independent and identically distributed real valued random variables with probability density proportional to \( \exp(-|t|^p) \), and let \( Z \sim \text{Exponential}(1) \) be independent of the \( Y_i \)'s. Then, the vector

\[
X = \frac{Y}{\left( \sum_{i=1}^{n} |Y_i|^p + Z \right)^{1/p}}
\]
is a uniformly random point within the \( \ell_p^n \)-ball (Barthe et al., 2005). Hence, the problem of generating such \( X \)'s is reduced to generating such \( Y \)'s. For the special cases of \( p = 1, 2, \infty \) specific generation methods exist. Otherwise, Algorithm 1 can be used for arbitrary \( p \).

The ratio of uniforms method is an acceptance-rejection style algorithm where the ratio of two independent uniform random variables \( U \) and \( V \) is shown to have the desired distribution given an inequality is satisfied. In this case, let \( r(t) = \exp(-|t|^p) \), \( U \sim \text{Uniform}[0,1] \) and \( V \sim \text{Uniform}[-(2/ep)^{1/p}, (2/ep)^{1/p}] \). Then, \( Y = V/U \) has probability density proportional to \( \exp(-|t|^p) \) if

\[
U^2 \leq r(V/U) = \exp(-|V/U|^p)
\]

Some examples of this acceptance region are displayed in Figure 7. The probability of such a random pair \( (U, V) \) satisfying the above inequality can be quickly calculated to be

\[
P(\text{Accept for } p) = \frac{\Gamma(1+1/p)}{21^{1+1/p}} (ep)^{1/p},
\]

which varies between 0.5 and 0.75 for values of \( p \geq 1 \).
Figure 7: The acceptance region (blue) and probability for the ratio of uniforms method applied for $p = 1, 2, 4, 8$. 

$p = 1$, $P(\text{Accept}) = 0.6796$

$p = 2$, $P(\text{Accept}) = 0.7306$

$p = 4$, $P(\text{Accept}) = 0.692$

$p = 8$, $P(\text{Accept}) = 0.6345$
Algorithm 1 Ratio of uniforms method of uniform points within the $\ell_p^n$-ball

Initialize $\eta = 0$ (Count how many variates are accepted)

While $\eta < n$

Generate $U_1, \ldots, U_{n-\eta} \overset{iid}{\sim} \text{Uniform } [0, 1]$

Generate $V_1, \ldots, V_{n-\eta} \overset{iid}{\sim} \text{Uniform } \left[-(2/ep)^{1/p}, (2/ep)^{1/p}\right]$

for each $i \in \{1, \ldots, n - \eta\}$

if $\{ U_i^2 \leq \exp(-|V_i/U_i|^p) \}$

Keep the pair $(U_i, V_i)$

$\eta \leftarrow \eta + 1$

else

Reject the pair $(U_i, V_i)$

Return $Y = (V_1/U_1, \ldots, V_n/U_n)$

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