Low and high-energy localization landscapes for tight-binding Hamiltonians in 2D lattices

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Localization of electronic wave functions in modern two-dimensional (2D) materials such as graphene can impact drastically their transport and magnetic properties. The recent localization landscape (LL) theory has brought many tools and theoretical results to understand such localization phenomena in the continuous setting, but with very few extensions so far to the discrete realm or to tight-binding Hamiltonians. In this paper, we show how this approach can be extended to almost all known 2D lattices, and propose a systematic way of designing LL even for higher dimension. We demonstrate in detail how this LL theory works and predicts accurately not only the location, but also the energies of localized eigenfunctions in the low and high energy regimes for the honeycomb and hexagonal lattices, making it a highly promising tool for investigating the role of disorder in these materials.

The promises of the expected electronic properties of new 2D materials often face the reality of genuine materials where disorder can be difficult to avoid [1]. Its influence might be large enough to switch the behavior of a material from metal to insulator [2], a transition which can be related to Anderson localization. The concept of Anderson localization, initially introduced in tight-binding models in the context of condensed matter physics [3], has been applied since in the continuous setting to all types of waves, being quantum [4], classical [5–9] or even gravitational [10]. In this setting, the recent theory of the localization landscape (LL) [11] has brought new insights and methods to address the wave localization properties in systems such as gases of ultracold atoms [12], disordered semiconductor alloys [13], or enzymes [14], and have been successfully extended to Dirac fermions [15] and non scalar field theory [16]. In this letter, we show that the whole machinery of the LL can be generalized to tight-binding systems for most known 1D and 2D lattices, allowing us to broaden the range of predictivity of this fruitful approach.

Tight-binding models are commonly used to study perfect [17] as well as disordered lattices [18–20]. The tight-binding Hamiltonian \( \hat{H} \) with on-site disorder and nearest-neighbor coupling is defined as

\[
(\hat{H}\psi)_n = -t \sum_{m \in \langle n \rangle} (\psi_m - \psi_n) + (V_n - b_n t) \psi_n
\]

(1)

where \( \psi \equiv (\psi_n)_{n \in \{1, N\}} \) is the wave function defined on the sites of the lattice (numbered from 1 to \( N \) here), \( V_n \) is the on-site potential at site \( n \), \( -t \) is the coupling constant between neighboring sites (assumed to be constant here), \( \langle n \rangle \) indicates the ensemble of nearest neighbors of site \( n \), and \( b_n \) is its cardinal. In the following, one will assume that \( t \) has value 1, thus setting the energy unit, and that \( V_n = W \nu_n \) where \( \nu_n \) is an i.i.d. random variable with uniform law in \([-0.5, 0.5]\), \( W \) being therefore the disorder strength for a given lattice.

Responsible for the remarkable properties of graphene, the honeycomb lattice will be the first paradigmatic structure that we study. Figures 1(a) and 1(e) show this lattice and its celebrated dispersion relation in the tight-binding approximation, respectively [21]. We solve the Schrödinger equation for the Hamiltonian defined in Eq. (1) on the honeycomb lattice, with the on-site potential depicted in Fig. 1(b). In Fig. 1(c) are displayed the first four eigenstates which, as expected, exhibit a finite spatial extension typical of Anderson-localized modes.

In the following, we show how to build the two discrete localization landscapes displayed in Fig. 1(d) and 1(i). These landscapes allow us to accurately predict the location of the localized modes near the two band edges (low and high energy), as well as their energies, without solving Eq. (1). We then generalize this method to the most common lattices encountered in 2D materials.

Let us first summarize the salient features of the LL theory in the continuous setting. For any positive definite Hamiltonian \( \hat{H} \) in such setting, the localization landscape \( u \) is defined as the solution to

\[
\hat{H}u = 1.
\]

(2)

One of the main results of the LL theory is that the
quantity $1/u$—which has the dimension of an energy—acts as an effective potential confining in its wells the localized states at low energy [23]. Moreover, the energy of the local fundamental state inside each well was found to be almost proportional to the value of the potential $1/u$ at its minimum inside the well [24],

$$E \approx \left( 1 + \frac{d}{4} \right) \min(1/u),$$

where $d$ is the embedding dimension of the system.

In the case of a tight-binding Hamiltonian, Lyra et al. [25] have studied a 1D chain with nearest-neighbor coupling and have shown that the positions of the localized modes are given by two different localization landscapes. The low energy LL is obtained by solving the analog of Eq. (2) in the discrete setting, i.e., $\hat{H}u = 1$ ($u \equiv (u_n)_{n \in [1,N]}$, $1$ is a vector filled with 1) with the same boundary conditions than the eigenvalue problem. Another LL, called the dual localization landscape (DLL), gives the positions of the envelope of the highly oscillating, high-energy, wave functions. More recently, Wang and Zhang [26] have proved mathematically that the reciprocals of these discrete LL and DLL act indeed as effective confining potentials in a tight-binding system at both low- and high-energy regimes, respectively.

Figure 1(d) shows the reciprocal of the LL, $1/u \equiv (1/u_n)_{n \in [1,N]}$ computed on the honeycomb lattice with the on-site disorder depicted in Fig. 1(b). Note that a shift $\hat{H} \rightarrow \hat{H} + V_{\text{shift}}$ has been performed in (1) to ensure a positive definite Hamiltonian, see Supplemental Material A. As already observed for continuous systems, the role of effective confining potential played by $1/u$ is revealed through its basins, labelled following their depth in Fig. 1(d). Indeed, one can observe the correspondence between the deepest wells of $1/u$ and the positions of the first eigenmodes plotted in Fig. 1(c). As analyzed by Arnold et al. [24] in the continuous setting, two almost-equal eigenvalues can lead to a different ordering in the values of the minima of $1/u$, thus inducing a mismatch in the correspondence. This effect, which does not affect the ability of the LL to predict the position of localized modes, is visible in Fig. 1(c) and (d) with the first and fourth eigenstates and minima, and has been analyzed in detail in the Supplemental Material B. Finally, we have quantitatively tested that, regardless of the lattice, the tight-binding LL efficiently pinpoint the localized modes (see Supplemental Material C).

The symmetry of the honeycomb lattice allows us a straightforward derivation of the landscape governing the high-energy localized states, namely the DLL. Indeed, the tight binding Hamiltonian (1) can be decomposed into $\hat{H} = \hat{H}_0 + \hat{V}$, where $\hat{H}_0$ stands for the uniform honeycomb lattice with zero on-site energy, and $\hat{V}$ accounts for the disordered on-site potential. The unperturbed part of the Hamiltonian displays the usual chiral symmetry for a bipartite lattice, $\Sigma_z \hat{H}_0 \Sigma_z = -\hat{H}_0$, where the Pauli-like matrix $\Sigma_z$ acts on the sublattice degree of freedom: it keeps the amplitudes on the $A$ sites fixed but inverts those on the $B$ sites ($\Sigma_z P_A = P_B$, the difference between the respective projectors on the two sub-lattices). Due the diagonal nature of the disordered potential, the complete Hamiltonian obeys the symmetry $\Sigma_z \left( \hat{H}_0 + \hat{V} \right) \Sigma_z = -\left( \hat{H}_0 - \hat{V} \right)$. The latter property is exemplified in Fig. 1(e) and (i): unlike the DOS of
the uniform lattice, the DOS of a given realization of the disordered system is not symmetric with respect to the origin, but the DOS obtained by inverting the sign of all on-site energies is the exact symmetric of the original situation.

Let us call \( \phi \equiv (\phi_n)_{n \in [1, N]} \) the eigenstates of the inverted Hamiltonian ordered by increasing eigenvalues. The low-energy states of the inverted Hamiltonian now correspond to the high-energy states of the original Hamiltonian through \( \phi = \Sigma_j \psi_j \). Since the high-energy eigenstates oscillate with a period equal to the nearest-neighbor distance, the new low-energy states appear as “demodulated” versions of their high-energy counterparts, see Fig. 1(g) and (h). We can now therefore use the localization landscape for the inverted system, but with \( u^* \) being now the solution to \( \hat{H}^* u^* = 1 \) with

\[
(\hat{H}^*\phi)_n = t \sum_{m \in \langle n \rangle} (\phi_m - \phi_n) + (V_{\text{shift}} - V_n)\phi_n.
\]

In the example of Fig. 1(i), one can clearly see how the deepest wells pinpoint the locations of the localized states. Beyond the honeycomb lattice, this spectrum inversion strategy can be deployed for other lattices with symmetric band structure, as the 1D dimer chain or the 2D Lieb lattice [see Supplemental Material Table S2].

As mentioned in the introduction, the localization landscape also provides accurate estimates of the localized eigenvalues in the continuous setting [24]. However, the generalization of the simple Eq. (3) to tight-binding Hamiltonians has never been studied systematically, nor its extension to the higher part of the spectrum. We plot on Fig. 2(a) (resp. 2(b)) the lowest (resp. highest) eigenvalues of Eq. (1) versus the local minimum values of the effective potential \( 1/u \) (resp. \( 1/u^* \)) at the position of localized eigenstates for a honeycomb lattice with \( N = 2135 \) sites and for a given disorder \( W = 3 \). Each scatter plots corresponds to 100 realizations of the disordered potential. In both cases, a direct proportionality is clearly observed for the lowest part of the plots, with Pearson coefficients of the linear regression close to 0.99. A general study of the quality of the proportionality is presented in Supplemental Material B. Note also that in order to obtain this proportionality (which is more than a simple linear dependency), one has to choose \( V_{\text{shift}} \) so that the shifted potential has a minimum value close to zero (see Supplemental Material A).

These observations indicate that the discrete low-energy localization landscape performs as well as its continuous analog in predicting energy and spatial distribution of localized modes without resolving an eigenvalue equation. Moreover, the high-energy DLL also exhibits the same properties. For both range of energy, LL and DLL provide a good estimate of the integrated density of states, as shown in Supplemental Material D. All these results are not restricted to the honeycomb lattice.

**FIG. 2.** (a) Proportionality between \( \text{min}(1/u) \) and \( E \) for the lowest energies. The blue dots correspond to the 3% states of lowest energy for 100 different configurations, the orange dots to the 3-5% tier, and the pink dots to the 5-7% tier, respectively. The black line corresponds to a linear fit of the pink dots, the slope \( s \) and the Pearson coefficient \( \rho \) being given in the frame. (b) Proportionality between \( \text{min}(1/u^*) \) and \( V_{\text{shift}} - E \) for the highest energies. Similar plot to (a), but for the states of highest energy. (c) Slope \( s \) for the low-energy states, for different lattices in 1D (squares) and 2D (circles). Each symbol corresponds to a disorder strength \( W \), but instead of reporting \( W \) on the horizontal axis, we chose to use the average value of the IPR which is a better comparison parameter across different lattices. The dashed horizontal lines show the limits expected in the continuous case from Eq. (3). The black circle corresponds to the case displayed in (a). (d) Similar plot to (c) for the highest-energy states.
sus the mean IPR corresponding to the same 3% of the eigenstates.

With one noticeable exception for the Lieb lattice, the values of the slope \( s \) appear to evolve continuously between \( s = 1 + d/4 \) and \( s = 1 \), both for the lowest and the highest eigenvalues [see Fig. 2(c) and 2(d)]. Moreover, all the curves bunch into two smooth master curves, one for each space dimension. In the weak disorder limit, \( \langle \text{IPR} \rangle \to 0 \), that is to say when the influence of the disordered potential is small compared to the Laplacian term, one can reasonably expect that the continuous result of Eq. (3) still holds for both the lowest and the highest part of the spectrum. This is indeed observed: the slopes fall on the \( (1 + d/4) \) limit for \( \langle \text{IPR} \rangle \to 0 \). In the other limit case, when the disorder is so strong that an eigenstate is localized on a single site (\( \langle \text{IPR} \rangle \to 1 \)), the LL is essentially supported locally on the same site. This means that the eigenstate and the LL are locally proportional, a feature already observed in the continuous setting. Eq. (2) at the only site \( n \) supporting the wave function therefore becomes \( H u_n = 1 \approx E u_n \), hence \( E \approx 1/u_n \) and a slope \( s \approx 1 \) is expected.

The definition and the properties of the low-energy LL are valid for any lattice, in any dimension, and are not restricted to nearest-neighbor coupling. We simulated thoroughly many different “canonical” lattices, for which details are provided in Supplemental Material C. The construction of the high-energy DLL, however, used explicitly in our case the chiral symmetry of the honeycomb lattice, hence the central symmetry of its DOS. We will show in the last part of the letter that when this property cannot be used, there remains a general procedure which consists in “demodulating” the Schrödinger equation around a local maximum of the dispersion relation of the Laplacian. This procedure that can be applied to any lattice, leading to a DLL that relies on the specific band structure of the given lattice.

To illustrate this, we now focus on the hexagonal lattice [Fig. 3(a)] whose DOS is not symmetric [Fig. 3(c)]. In Fig. 3(d), we display the states corresponding to the 4 highest eigenvalues. Similarly to what was observed for the honeycomb lattice [Fig. 1(g)], we first note that the high-energy eigenstates are spatially oscillating with a period equal to the nearest-neighbor distance. Without on-site energy disorder, the highest eigenvalues are located at the vertices \( K \) of the first Brillouin zone (BZ) [Fig. 3(b) and 3(c)]. The corresponding wave vectors are \( \vec{k}_{\text{max}} = \pm \frac{4\pi}{3a} \hat{x}, \pm \left( \frac{2\pi}{3a} \hat{y} + \frac{2\pi}{\sqrt{3}a} \hat{y} \right) \). To remove the rapidly oscillating part of the high-energy eigenstates, we define the envelope \( \phi \) of an eigenmode \( \psi \) whose wave vector is close to \( \vec{k}_{\text{max}} \) by \( \psi_n = e^{i \vec{k}_{\text{max}} \cdot \vec{r}_n} \phi_n \), where \( \vec{r}_n \) is the position of site \( n \) (see Supplemental Material E for the derivation of the landscape around a local maximum of the dispersion relation). By injecting \( \phi \) in Eq. (1), we obtain a “demodulated” equation that reads

\[
\begin{align*}
\mathcal{H}_{\vec{k}_{\text{max}}}^* \phi &= (V_{\text{shift}} - E) \phi. \\
\end{align*}
\]  

We then compute the landscape \( u^* \) associated to this Hamiltonian, i.e. \( \mathcal{H}_{\vec{k}_{\text{max}}}^* u^* = 1 \), and obtain a complex confining potential \( 1/|u^*| \) whose absolute value is plotted in Fig. 3(e). The comparison between the deepest wells of \( 1/|u^*| \) \( \equiv \left( 1/|u^*_n| \right)_{n \in \{1, N\}} \) of the landscapes around the specific local maxima of the dispersion relation. FIG. 3. (a) Hexagonal lattice and (b) reciprocal lattice with the first Brillouin zone and the high symmetry points. (c) Dispersion relation (d) Four states with the highest eigenvalues of a hexagonal lattice with on-site disorder, \( N = 1068 \) and \( W = 3 \); (e) Effective potential where the 4 deepest minima are marked.

\[
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More challenging is the construction of the systems. It bears the same efficiency than its continuous effective confining potential defined for continuous of the spectrum is described by a discrete extension of Hamiltonian, with a focus on 2D lattices. The low-energy part discrete systems described by a tight-binding Hamiltonian [27]. In this letter, we have extended its scope to accessible form the information contains in a Hamiltonian has proven its remarkable efficiency to bring in a more cases.

Born a decade ago, the localization landscape theory has proven its remarkable efficiency to bring in a more accessible form the information contains in a Hamiltonian [27]. In this letter, we have extended its scope to accessible form the information contains in a Hamiltonian has proven its remarkable efficiency to bring in a more cases.

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Appendix A: Energy shift

For the simulations presented in the paper, we considered the Hamiltonian given by Eq. (1) in the main body of the paper, but written in a more concise way:

$$\langle \hat{H}\psi \rangle_n = -t \sum_{m \in \{nn\}} \psi_m + (V_n + V_{\text{shift}}) \psi_n ,$$

(S1)

where $V_n$ is the on-site potential at site $n$, and $t$ is the coupling constant between neighboring sites. Additionally, $V_n = W \nu_n$ where $\nu_n$ is an i.i.d. random variable with uniform law in $[-0.5, 0.5]$, $W$ being therefore the disorder strength for a given lattice, and $V_{\text{shift}}$ the energy shift that avoids negative eigenvalues. Finally $V_{\text{shift}} = -\min(\mathcal{E}) + W/2$ where $\mathcal{E}$ is the energy of the system without disorder at zero on-site energy. The considered values are shown in Table S1.

To compute the dual landscape, we use $V_{\text{shift}} = \max(\mathcal{E}) + W/2$. Note that for the 1D chain with 2nd neighbors coupling the energy is

$$\mathcal{E} = -2 [t_1 \cos (k_x a) + t_2 \cos (2k_x a)] ,$$

(S2)

where the corresponding $k_{\text{max}}$ is the solutions of

$$t_1 \sin (k_x a) = -2t_2 \sin (2k_x a).$$

(S3)

Then, values shown in Table S1 correspond to the particular case $t_2 = t_1/\sqrt{8}$.

| Lattice              | $-\min(\mathcal{E})$ | $\max(\mathcal{E})$ |
|----------------------|-----------------------|----------------------|
| 1D Chain             | 2                     | 2                    |
| 1D dimer chain       | $t_1 + t_2$           | $t_1 + t_2$          |
| 1D chain with 2nd neighbors coupling 2 $(t_1 + t_2) \sqrt{2t_1}$ |
| Square               | 4                     | 4                    |
| Lieb                 | $\sqrt{8}$            | $\sqrt{8}$           |
| tts                  | 5                     | 3                    |
| Hexagonal            | 6                     | 3                    |
| Honeycomb            | 3                     | 3                    |
| Kagome               | 4                     | 2                    |

TABLE S1. Smallest and largest eigenvalues of the tight binding Hamiltonian without on-site potential and with $V_{\text{shift}} = 0$. The couplings are all $t = 1$, except for 1D dimer chain and the 1D chain with 2nd neighbor coupling cases where they are explicitly written.

Appendix B: Accuracy of the predictions

In this section, we quantify the quality of the localization prediction of the localized states in a situation of small (Fig. S1) and large disorder (Fig. S2). This is done by calculating the distances between the position of the maximum of an eigenstate, and the positions of all the wells of the effective potential. The wells are ordered by their depths, and we then find the rank of the well corresponding to the minimum distance, and report it in the 2D histograms of Fig. S1 and S2. In the high-energy case, the computations are done using the symmetry of the bandstructure for the honeycomb lattices [Figs. S1(c) and S2(c)], and using the explicit demodulation for the hexagonal lattices [Figs. S1(d) and S2(d)].

Similarly, we quantify energy predictions for all the methods presented in the Main Text. In Fig. S3, we plot the Pearson coefficient of the linear regression as a function of the number of minima taken to compute the slope for the honeycomb lattice, and in Fig. S4 the same quantity for the hexagonal lattice. We see that whatever the disorder, the maximum correlation is obtained when we chose the 3% states with the lowest (or highest) eigenvalues.

In Fig. S5, we plot a 2D histogram showing the distribution of the eigenstates in the eigenvalues-IPR plane. We clearly see the more localized states for higher disorder, and the asymmetry of the density of states for the hexagonal lattices.

Figure S6 shows that in every case considered in the paper, the eigenvalues and the minima of the effective potential.
potential are highly correlated (Pearson coefficient very close to 1).

Finally, Fig. S7 is similar to Main Text Fig. 2(d), except that the calculations are done using the explicit demodulation. This explains why there are more cases in Fig. S7 than in Fig. 2 as the constraint on the symmetry of the DOS is lifted.

Appendix C: Exhaustive study of various 1D and 2D lattices and simulations details

In Table. S2, we list the different lattices studied in this study with their properties. The values of $k_{\text{max}}$ shown here for the 1D chain with 2nd neighbor coupling correspond to the case $t_2 = t_1/\sqrt{8}$. For each lattice, we computed 100 different configurations, see Table S3 for details. In the case of symmetric DOS, we calculated the landscape prediction both using the symmetry of the DOS and using the explicit demodulation. In Fig. 2(c) and (d) of the Main text, 3% of the eigenstates correspond to 64 states, and because we consider 100 configurations, the slopes are calculated using 6400 points.

Appendix D: IDOS estimate

Figure S8 shows the counting function for the Hexagonal and Honeycomb lattices with a strong on-site disorder ($W=240$ and $W=480$ for the Honeycomb and Hexagonal lattice, respectively). Blue lines correspond to the solution of Eq. (1) while orange lines are the minima of both

FIG. S2. The number of states whose position match with the maxima of the landscape [(a) and (b)] and with the maxima of the dual landscape [(c) and (d)]. We consider the 3% of the states for each lattice, the strongest disorder shown in Table S3 ($W = 480$ and $W = 240$ for the Hexagonal and Honeycomb lattice, respectively) and 100 disorder configurations.

FIG. S3. Pearson correlation coefficient as a function of number of minima taken into account for the honeycomb lattice with different strengths of disorder $W$: (a) The landscape prediction; (b) The dual landscape prediction using the symmetry of the bandstructure; The linear regression quality do not evolve when the number of minima considered is larger than the actual number of minima of $1/\mathbf{u}$ ($1/\mathbf{u}^*$).

FIG. S4. Pearson correlation coefficient as a function of number of minima taken into account for the hexagonal lattice with different strengths of disorder $W$: (a) The landscape prediction; (b) The dual landscape prediction using the explicit demodulation; The linear regression quality do not evolve when the number of minima considered is larger than the actual number of minima of $1/\mathbf{u}$ ($1/\mathbf{u}^*$).
Honeycomb

(a) $W = 6$

(b) $W = 12$

(c) $W = 120$

(d) $W = 240$

FIG. S5. 2D histogram showing how the eigenvalues and the IPR are distributed.

confining and dual confining potentials.

Appendix E: The localization landscape around a local maximum of the dispersion relation

Let us assume that $\vec{k}_{\text{max}}$ is a wave vector at which the dispersion relation $E(\vec{k})$ of the Hamiltonian without potential (i.e., minus the Laplacian) exhibits a local maximum. Then one can write locally the dispersion relation as

$$E(\vec{k}) = E(\vec{k}_{\text{max}}) - \frac{1}{2}(\vec{k}_{\text{max}} - \vec{k})^\top \mathbf{A}(\vec{k}_{\text{max}} - \vec{k}) + o \left( \|\vec{k} - \vec{k}_{\text{max}}\|^2 \right),$$

(S1)

where $\mathbf{A}$ is a definite positive 2-by-2 matrix whose eigenvalues are the inverse of the effective masses in both directions.

One can then write any eigenfunction $\psi$ of the full Hamiltonian (i.e., with potential $V = W\nu$) as

$$\psi = \exp\left( j\vec{k}_{\text{max}} \cdot \vec{r} \right) \phi \quad (S2)$$

where $\phi$ is an envelope function satisfying the following equation

$$- \sum_{m \in \{nn\}} \left( e^{j\vec{k}_{\text{max}} \cdot \vec{r}_m} \phi_m - \phi_n \right) + W\nu_n \phi_n = E\phi_n, \quad (S3)$$

$E$ being the energy of $\psi$. The local maximum of the dispersion relation $E(\vec{k})$ at $\vec{k}_{\text{max}}$ implies that

$$\sum_{m \in \{nn\}} \vec{r}_m e^{j\vec{k}_{\text{max}} \cdot \vec{r}_m} = 0; \quad (S4)$$

FIG. S6. Pearson correlation coefficient as a function of the (IPR) corresponding to the data plotted in Main Text Fig. 2; (a) The landscape prediction; (b) The dual landscape prediction for the symmetric lattices using the symmetry of the DOS property; (c) The dual landscape prediction for all the lattices using the explicit demodulation.

FIG. S7. Proportionality factor between $V_{\text{shift}} - E$ and the minima of $1/|\mathbf{u}^\star|$ for the different 1D (squares) and 2D (circles) lattices studied. The linear fits are done on states with the 3% highest energy. The dashed horizontal lines show the limit expected using the approximate form Eq. (3). Each symbol correspond to a disorder strength. The results are plotted as a function of the mean IPR calculated over the 3% states found for a given disorder strength.
where the sum is taken over all interacting neighbors of one site of the lattice assumed to be located at $\mathbf{r}' = 0$.

One can thus define a new Hamiltonian $\hat{H}^\star_{k_{\text{max}}}$ by:

$\langle \hat{H}^\star_{k_{\text{max}}} \phi \rangle_n = E(k_{\text{max}}) + \sum_{m \in \{\text{nn}\}} (e^{i k_{\text{max}} \cdot (\mathbf{r}_m - \mathbf{r}_n)} \phi_m - \phi_n) - W \nu_n \phi_n.$

(S5)

The energy of a plane wave $\phi = \exp(j \mathbf{k} \cdot \mathbf{r})$ for this Hamiltonian $\hat{H}^\star_{k_{\text{max}}}$ is therefore

$$E^\star(\mathbf{k}) = \langle \phi | \hat{H}^\star_{k_{\text{max}}} | \phi \rangle = \frac{1}{2} \mathbf{k}^\top \mathbf{A} \mathbf{k} - \frac{W}{2} + o\left(\|\mathbf{k}\|^2\right).$$

(S6)

One can therefore apply the localization landscape formalism to this Hamiltonian shifted by a quantity $W/2$. 

FIG. S8. The counting function computed by [(a) and (b)] the confining potential, and by [(c) and (d)] the dual confining potential. We consider the 3% of the states for each lattice. Here, the strongest disorder was chosen ($W=240$ and $W=480$ for the Honeycomb an Hexagonal lattice, respectively).
Lattice | Bravais lattice  | $\ell$ of bands | DOS Sym? | $k_{\text{max}}a$
--- | --- | --- | --- | ---
1D chain | 1D | 1 | $y$ | $\pm \pi$
1D dimer chain ($t_2 = t_1/2$) | 1D | 2 | $y$ | $\pm 2\pi$
1D chain with 2nd neighbour coupling | 1D | 1 | $n$ | $\pm \frac{3\pi}{2}$
Square | sq | 1 | $y$ | $M (\pm \pi, \pm \pi)$
Lieb | sq | 3 | $y$ | $(\pm 2\pi, \pm 2\pi)$ (flat band in the middle)
tts | sq | 4 | $n$ | $(\pm 2\pi, \pm 2\pi)$
hexagonal | hxl | 1 | $n$ | $K \left( \pm \frac{4\pi}{3}, 0 \right)$, $\pm \left( \frac{2\pi}{3}, \frac{2\pi}{\sqrt{3}} \right)$, $\pm \left( \frac{2\pi}{3}, -\frac{2\pi}{\sqrt{3}} \right)$
Honeycomb | hxl | 2 | $y$ | $\left( 0, \pm \frac{4\pi}{\sqrt{3}} \right)$, $\pm \left( 2\pi, \frac{2\pi}{\sqrt{3}} \right)$, $\pm \left( 2\pi, -\frac{2\pi}{\sqrt{3}} \right)$
Kagome | hxl | 3 | $n$ | flat band

TABLE S2. High energy wave vector for the different lattices studied.

| Lattice | Number of sites $N$ | Disorder calculated $W$ | Number of configurations calculated |
|---|---|---|---|
1D chain | 1001 | 2, 4, 10, 20, 40, 80, 160 | 100 |
1D dimer chain ($t_2 = t_1/2$) | 2001 | 2, 4, 10, 20, 40, 80, 160 | 100 |
1D chain with 2nd neighbour coupling | 1001 | 2, 4, 10, 20, 40, 80, 160 | 100 |
Square | 961 | 4, 8, 20, 40, 80, 160, 320 | 100 |
Lieb | 2821 | $\frac{8}{3}$, $\frac{16}{3}$, $\frac{40}{3}$, $\frac{80}{3}$, $\frac{160}{3}$, $\frac{320}{3}$, $\frac{640}{3}$ | 100 |
tts | 3661 | 5, 10, 25, 50, 100, 200, 400 | 100 |
Hexagonal | 1068 | 6, 12, 30, 60, 120, 240, 480 | 100 |
Honeycomb | 2135 | 3, 6, 15, 30, 60, 120, 240 | 100 |
Kagome | 3185 | 4, 8, 20, 40, 80, 160, 320 | 100 |

TABLE S3. Summary of the simulations