Universal amplitudes ratios for critical aging via functional renormalization group

Michele Vodret\textsuperscript{1,2}, Alessio Chiocchetta\textsuperscript{3} and Andrea Gambassi\textsuperscript{4,5,}\textsuperscript{*}

\textsuperscript{1} Chair of Econophysics and Complex Systems, École Polytechnique, 91128 Palaiseau Cedex, France
\textsuperscript{2} Ladhyx, Ecole Polytechnique, UMR CNRS 7646, 91128 Palaiseau Cedex, France
\textsuperscript{3} Institut für Theoretische Physik, Universität zu Köln, D-50937 Cologne, Germany
\textsuperscript{4} SISSA—International School for Advanced Studies, Via Bonomea 265, 34136 Trieste, Italy
\textsuperscript{5} INFN, Sezione di Trieste, Via Bonomea 265, 34136 Trieste, Italy

E-mail: gambassi@sissa.it

Received 6 December 2021, revised 6 May 2022
Accepted for publication 19 May 2022
Published 9 June 2022

Abstract
We discuss how to calculate non-equilibrium universal amplitude ratios in the functional renormalization group approach, extending its applicability. In particular, we focus on the critical relaxation of the Ising model with non-conserved dynamics (model A) and calculate the universal amplitude ratio associated with the fluctuation–dissipation ratio of the order parameter, considering a critical quench from a high-temperature initial condition. Our predictions turn out to be in good agreement with previous perturbative renormalization-group calculations and Monte Carlo simulations.

Keywords: nonequilibrium dynamics, functional renormalisation group, universality, critical dynamics, aging dynamics

(Some figures may appear in colour only in the online journal)

1. Introduction

Statistical mechanics is one of the most successful theoretical frameworks in physics, connecting the macroscopic behavior of equilibrium many-body systems to their microscopic dynamics. Working out this relation is, in general, challenging and it can be done either approximately or numerically at the cost of significant computational resources. This task is highly simplified when a system displays universality. In this case, the macroscopic, large-distance
behavior does not depend on microscopic details but rather only on some gross features of the system such as symmetries, spatial dimensionality, and range of interactions. Accordingly, different physical systems may correspond to the same universality classes, characterized by a set of critical exponents and dimensionless ratios of non-universal amplitudes [1], or, equivalently, by scaling relations of physical quantities. Universal behaviors have been also observed in systems out of equilibrium, much less understood than their equilibrium counterparts. Nonequilibrium universality emerges in classical [2–7] and quantum [8–20] systems, and even in socio-economic [21] and ecologic models [22–24].

A paradigmatic case of non-equilibrium universality is represented by the critical relaxational dynamics of statistical systems in contact with a thermal reservoir [25–27]. Perhaps, the simplest instance of critical dynamics is provided by the Glauber dynamics of the Ising model, characterized by the absence of conservation laws and corresponding to the so-called model A in the classification of reference [28].

A non-equilibrium universal behavior is displayed by this model when it is brought out of equilibrium via a critical quench, i.e., when it is prepared in an initial equilibrium state and then suddenly put in contact, at time $t_0$, with a bath at the critical temperature. As a consequence of critical scale invariance, the equilibration time diverges, and the system undergoes slow dynamics, referred to as aging [27]. Aging is revealed, in the simplest instance, by two-time functions, such as the response and the correlation functions, which only depend on the ratio $t/t'$ for $t/t' \gg 1$, with $t' > t_0$ the waiting time and $t$ the observation time. As a consequence, the larger the waiting time $t'$ is, the slower the system responds at time $t$ to an external perturbation applied at time $t'$. Moreover, the fluctuation–dissipation theorem, which links dynamic response and correlation functions of the system, does not apply [7], because of the breaking of time-translational symmetry (TTS) and time-reversal symmetry (TRS). In order to quantify the departure from equilibrium of the relaxing system, the fluctuation–dissipation ratio (FDR) $X$ is introduced, which differs from unity whenever the fluctuation–dissipation theorem does not apply. The long-time limit of the FDR, denoted by $X^\infty$, has been shown to be a universal amplitude ratio, whose value does not depend on the details of the system, but only on its universality class [27] and possibly on some gross features of the quench [29, 30].

Since exact solutions of the dynamics are available only for a few models, the predictions for $X^\infty$ were obtained predominantly either via a perturbative renormalization-group (pRG) analysis or Monte Carlo (MC) simulations [27]. Moreover, contrary to critical exponents, the determination of amplitude ratios requires the calculation of the full form of two-time functions and, thus, it is computationally more demanding [31].

In this work, we present an approach to calculate $X^\infty$ for the critical dynamics of model A, by using a functional renormalization-group (fRG) approach. The application of the fRG for studying the critical short-time dynamics of model A was introduced in reference [32], where it was used to calculate the critical initial-slip exponent. Here, instead, we address the calculation of the universal amplitude ratio $X^\infty$ by analyzing the long-time limit of the dynamics. The main result of our analysis is that, within the local potential approximation (LPA) [33], the universal amplitude ratio $X^\infty$ depends only on the critical initial-slip exponent, leading to predictions in good agreement with the available pRG and MC estimates. While the FDR has already been studied within the fRG approach for the stationary state of the KPZ equation [34], we present here the first computation of $X^\infty$ within the fRG technique for a regime where both TTS and TRS are absent.

The paper is organized as follows: the model A, the aging dynamics, and the universality of $X^\infty$ are reviewed in section 2. The fRG approach to the study of a system undergoing a quench, as well as its LPA approximation, is discussed in section 3. The calculation of the two-time functions and the connection to the aging features are discussed in section 4, while our
predictions for $X^\infty$ are presented in section 5. Finally, in section 6 we summarize the approach introduced here and we discuss future perspectives. All the relevant details of the calculations are provided in a number of appendices.

2. Aging and fluctuation-dissipation ratio for model A

2.1. Model A of critical dynamics

Model A of critical dynamics [7, 28] captures the universal features of relaxational dynamics in the absence of conserved quantities of a system belonging to the Ising universality class and coupled to a thermal bath. The effective dynamics of the coarse-grained order parameter (i.e., the local magnetization), described by the classical field $\varphi = \varphi(r, t)$, is given by the Langevin equation

$$\dot{\varphi} = -\frac{\delta H}{\delta \varphi} + \zeta;$$

(1)

here $\dot{\varphi}$ is the time derivative of $\varphi$, $\Omega$ is a kinetic coefficient, $\zeta$ is a zero-mean Markovian and Gaussian noise with correlation $\langle \zeta(r, t), \zeta(r', t') \rangle = D \delta(r-r') \delta(t-t')$ and $D$ a constant quantifying the thermal fluctuations induced by the bath at temperature $T$ (measured in units of Boltzmann constant). As long as one is interested in studying the system in the vicinity of its critical point $H$ is assumed to be of the Landau–Ginzburg form:

$$H = \int_r \left[ \frac{1}{2} (\nabla \varphi)^2 + \tau_0 \varphi^2 + \frac{g}{4!} \varphi^4 \right],$$

(2)

where $\int_r \equiv \int d^d r$ with $d$ the spatial dimensionality, $\tau$ parametrizes the distance from the critical point and $g \geq 0$ controls the strength of the interaction. The noise $\zeta$ is assumed to satisfy the detailed balance condition $D = 2\Omega T$ [27]: accordingly, the equilibrium state is characterized by a probability distribution $\approx e^{-H[\varphi]/T}$.

We assume that the system is prepared at $t = t_0$ in an equilibrium high-temperature state with temperature $T_0$. Accordingly, the initial value $\varphi_0 = \varphi(r, t_0)$ of the order parameter can be described by a Gaussian distribution $\approx e^{-\mathcal{H}_0[\varphi_0]/T_0}$ with

$$\mathcal{H}_0 = \frac{\tau_0}{2} \int_r \varphi_0^2,$$

(3)

In order to study the relaxation of the system, it is convenient to consider two-time functions [35], which are the simplest quantities retaining non-trivial information about the dynamics. In this work, we focus in particular on response and correlation functions of the order parameter $\varphi$. The response function is defined as the response of the order parameter to an external magnetic field $h$ applied at $r = 0$ after a waiting time $t' > t_0$, i.e.,

$$R_r(t, t') = \left. \frac{\delta \langle \varphi_r(t) \rangle_h}{\delta h(\tau')} \right|_{h=0},$$

(4)

where $\langle \cdot \rangle_h$ stands for the mean over the stochastic dynamics induced by $\mathcal{H}[\varphi; h] = \mathcal{H}[\varphi] - \beta \int_r h \varphi$. The response function vanishes for $t < t'$ because of causality. In order to simplify the notation, in what follows, we absorb the factor $\beta$ into the definition of the response function. The correlation function, instead, is defined as

$$C_r(t, t') = \langle \varphi_r(t) \varphi_0(t') \rangle,$$

(5)

3
The symbol $\langle \cdot \rangle$ in equations (4) and (5) indicates the average over both the initial condition $\varphi_0(r)$ and the realizations of the noise $\zeta$. Furthermore, in equations (4) and (5) we have taken advantage of spatial translational invariance, by setting one of the spatial coordinates to zero.

In order to characterize the distance from equilibrium of a system that is evolving in a bath at fixed temperature $T$, the FDR is usually introduced [27, 36]:

$$ X_r(t, t') \equiv \frac{TR_r(t, t')}{\partial_t C_r(t, t')}. $$

(6)

When the waiting time $t'$ is larger than the equilibration time $t_{eq}(T)$ the dynamics is TTS and TRS. Thus, the fluctuation–dissipation theorem holds and it implies $X_r(t, t') = 1$. The asymptotic value of the FDR,

$$ X^\infty \equiv \lim_{t' \to \infty} \lim_{t \to \infty} X_r(t, t'), $$

(7)

is a very useful quantity in the description of systems with slow dynamics, since $X^\infty = 1$ whenever TTS and TRS are recovered. Conversely, $X^\infty \neq 1$ is a signal of an asymptotic non-equilibrium dynamics. Accordingly, we distinguish such cases as a function of the temperature $T$ of the bath: for $T > T_c$, then $X^\infty = 1$ since $t_{eq}$ is finite; for $T < T_c$, on the basis of general scaling arguments, it has been argued that $X^\infty$ vanishes [37]. For the special case of $T = T_c$, there are no general arguments constraining the value of $X^\infty$ and therefore this quantity has to be determined for each specific model. However, $X^\infty$ is a universal quantity associated with critical dynamics and, in addition, the following identity holds for systems quenched to their critical point [27]:

$$ X^\infty = X^\infty_{q=0}. $$

(8)

where the quantity $X^\infty_{q=0}$ is defined as in equation (6) but replacing $R_r$ and $C_r$ with their Fourier transforms, i.e., $R_q$ and $C_q$, respectively. Its long-time limit $X^\infty_{q=0}$ is obtained as in equation (7).

If the temperature $T$ of the bath is equal to the critical temperature $T_c$, the equilibration time $t_{eq}$ diverges and the relaxational dynamics prescribed by model A exhibits self-similar properties, signaled by the emergence of algebraic singularities and scaling behaviors. The scaling behavior is displayed by the response and correlation functions in two regimes: first, the short-time one in which $t' \to t_m$ with fixed $t$, where $t_m$ is a microscopic time which depends on the specific details of the underlying microscopic model. Second, the long-time regime, in which $t \to \infty$ with fixed $t'$. The response and correlation functions for model A in the aging regime are thus given, respectively, by [27]

$$ R_{q=0}(t, t') = A_R(t - t')^{\alpha} \left( \frac{t'}{t} \right)^{\theta} F_R(t'/t), $$

(9a)

$$ C_{q=0}(t, t') = A_C(t - t')^{\alpha} \left( \frac{t'}{t} \right)^{\theta} C(t'/t), $$

(9b)

where $A_R$ and $A_C$ are non-universal amplitudes. The exponent $a = (2 - \eta - 2z)/\nu$ is associated with the time-translational invariant part of the scaling functions with $\eta$ the anomalous dimension of the order parameter $\varphi$, $z$ the dynamical critical exponent while $F_R(t'/t)$ are universal scaling functions which satisfy $F_R(t'/t) = 1$. The breaking of TTS and TRS in the scaling form (9) is characterized by the so-called initial-slip exponent $\theta$, which is generically independent of the static critical exponent $\eta, \nu$ and of the dynamical critical exponent $z$. Using the known
scaling forms (9) for the two-point functions and the relation given by equation (8), one finds the following expression for the asymptotic value of the FDR (8) [27]:

\[ X^\infty = \frac{A_R}{A_C (1 - \theta)} \]  

(10)

Accordingly, \( X^\infty \) is a universal amplitude ratio.

In section 3 we briefly review the approach of reference [32], for studying the critical dynamics based on the fRG technique, leaving to section 4 the calculation of the two-time functions (4) and (5) in the aging regime, in which they are given by equation (9), that finally allows us to calculate \( X^\infty \) via equation (10).

2.2. Gaussian approximation

In the absence of interaction (\( g = 0 \)), equation (1) is linear and therefore, by using equations (3)–(5), it is possible to calculate exactly the response and the correlation functions. Here and in what follows the coefficient \( \Omega \) has been absorbed into the definition of the effective Hamiltonian \( H \). After a Fourier transform in space with momentum \( q \), defining \( \omega_q = q^2 + r \),

\[ R_q(t, t') = R_q(t - t') = \delta(t - t')e^{-\omega_q |t - t'|}, \]
\[ C_q(t, t') = C^D_q(t, t') + \tau_0^{-1} e^{-\omega_q (t + t' - 2\tau_0)}, \]  

(11)

where \( \delta(t) \) is the Heaviside step function and \( C^D \) is the Dirichlet correlation function

\[ C^D_q(t, t') = \frac{1}{\omega_q} \left[ e^{-\omega_q |t - t'|} - e^{-\omega_q (t + t' - 2\tau_0)} \right], \]

(12)

which corresponds to a high-temperature (\( \tau_0 = +\infty \)) system that instantaneously loses the correlation with the initial state, i.e.,

\[ C^D_q(t, t_0) = 0. \]  

(13)

Within the Gaussian approximation, TTS is broken by the correlation function but not by the response function. In addition, the dynamics (1) becomes critical for \( r = 0 \): in this case, comparing equation (11) with the scaling functions (9), one finds \( A_R = 1, A_C = 2 \) and \( \theta = 0 \), which imply \( X^\infty = 1/2 \) from equation (10).

2.3. Response functional

As a result of a finite interaction strength (\( g \neq 0 \)), the Gaussian value of the universal quantities like \( \theta \) and \( X^\infty \) may acquire sizable corrections [27] and a dependence on the spatial dimensionality \( d \) of the model. An analytical approach to treat interactions (\( g \neq 0 \)) in equation (1) is provided by field-theoretical methods. From the Langevin equation (1) with initial condition (3) it is possible to construct the response action \( S = S[\hat{\phi}, \varphi] \) given by [7]

\[ S = \mathcal{H}_0[\hat{\varphi}_0, \varphi_0] + \int_r \int_t^{+\infty} \hat{\varphi} \left( \frac{\delta \mathcal{H}}{\delta \varphi} - D\hat{\varphi} \right), \]

(14)

The scalar field \( \hat{\varphi} = \hat{\varphi}(r, t) \) is the so-called response field and it appears in the definition of the response function (4) as:
\[ R_r(t, t') = \langle \varphi_r(t) \tilde{\varphi}_0(t') \rangle, \quad (15) \]

where the symbol \( \langle \cdot \rangle \) denotes a functional integral, which, for a generic observable \( O[\varphi, \tilde{\varphi}] \), can be evaluated as

\[ \langle O \rangle = \int \mathcal{D}\varphi \mathcal{D}\tilde{\varphi} O[\varphi, \tilde{\varphi}] e^{-S[\varphi, \tilde{\varphi}]} . \quad (16) \]

This kind of average cannot, in general, be exactly computed in the presence of interactions, and one has to resort to approximations. The action \( S \) is the most suitable quantity to carry out pRG calculations of the response and the correlation function. However, for the purpose of using fRG it is convenient to introduce a generating function associated with the response functional \( S \), i.e., the effective action \( \Gamma \). For future convenience we denote by \( \Psi \) the vector having \( \varphi \) and \( \tilde{\varphi} \) as components, with \( \Psi^t = (\varphi, \tilde{\varphi}) \) and then indicate \( S[\varphi, \tilde{\varphi}] \) as \( S[\Psi] \).

3. Functional renormalization group for a quench

In this section we introduce the fRG approach and the LPA of it, emphasizing its physical interpretation and how it can be applied to a temperature quench. The calculation of the two-time functions, instead, is presented in section 4.

3.1. fRG equation and LPA

The fRG implements Wilson’s idea of momentum shells integration at the level of the effective action \( \Gamma \), for which it provides an exact flow equation. In order to implement the fRG scheme [38, 39], it is necessary to supplement the response functional \( S[\Psi] \) with a cutoff function \( R_k(q^2) \) over momenta \( q \), which is introduced as a quadratic term in the modified action

\[ S_k[\Psi] \equiv S[\Psi] + \Delta S_k[\Psi], \]

where \( \Delta S_k[\Psi] = \int_x \Psi^t \sigma \Psi R_k / 2 \) with the matrix \( \sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) acting in the two dimensional space of the variable \( \Psi \). The cutoff function \( R_k \) as a function of \( k \) is characterized by the following limiting behaviors [38, 39]:

\[ R_k(q^2) \simeq \begin{cases} \Lambda^2 & \text{for } k \to \Lambda, \\ 0 & \text{for } k \to 0, \end{cases} \]

where \( \Lambda \) is the ultraviolet cutoff, which can be identified, for instance, with the inverse of the lattice spacing of an underlying microscopic lattice model. The effect of \( R_k \) is to supplement long-distance modes with an effective \( k \)-dependent quadratic term, and thus allowing a smooth approach to the critical point when the response action \( S \) is recovered for \( k \to 0 \). In fact, this finite \( k \)-dependent quadratic term regularizes the infrared divergences which would arise from loop corrections evaluated at criticality [38, 39].

The running effective action \( \Gamma_k \), defined from \( S_k \) as explained in appendix A, can be interpreted as a modified effective action which interpolates between the microscopic one \( S \), given here by equation (14), and the long-distance effective one \( \Gamma \); accordingly, \( \Gamma_k \) has the following limiting behavior:

\[ \Gamma_k \simeq \begin{cases} S & \text{for } k \to \Lambda, \\ \Gamma & \text{for } k \to 0. \end{cases} \]
The running effective action $\Gamma_k$ obeys the following equation [38, 39]

$$\frac{d\Gamma_k}{dk} = \frac{1}{2} \int_x \vartheta(t-t_0) \text{tr} \left( G_k \sigma \frac{dR_k}{dk} \right),$$

(19)

where $\int_x = \int_r \int_{-\infty}^{+\infty} dt, x = (r, t)$, and

$$G_k^{-1} = \Gamma^{(2)} + \sigma R_k.$$

(20)

In equation (20) and in what follows a superscript $(n)$ to $\Gamma$ denotes the $n$-derivative with respect to the two-dimensional variable $\Phi = (\tilde{\phi}, \phi)$. Notice that Heaviside theta in equation (19) enforces the quench protocol described in section 2 [32]. Equation (19) is a (functional) first-order differential equation in $k$. Accordingly, in order to be solved for $\Gamma_k$, one has to specify an initial condition for the RG flow, i.e., one has to set the coupling constants which appear in equation (2) at scale $k = \Lambda$ in the corresponding effective action $\Gamma_\Lambda = S$ (see equation (18)).

While equation (19) is exact, it is generally not possible to solve it and thus one has to resort to approximation schemes that render equation (19) amenable to analytic and numerical calculations. The first step in this direction is to use an ansatz for the form of the effective action $\Gamma_k$ which, once inserted into equation (19), results in a set of coupled non-linear differential equations for the couplings which parametrize it. We consider the following LPA ansatz [33] for the modified effective action $\Gamma_k$ of model A:

$$\Gamma_k = \Gamma_{0,k}[^{\tilde{\phi}, \phi}] + \int_x \vartheta(t-t_0) \tilde{\phi} \left( Z_k \tilde{\phi} + K_k \nabla^2 \phi + \frac{\partial U_k}{\partial \phi} - D_k \tilde{\phi} \right).$$

(21)

The initial-time action $\Gamma_{0,k} = \Gamma_{0,k}[^{\tilde{\phi}, \phi}]$ accounts for the initial conditions (3) and will be discussed later on. The field and time-independent factors $Z_k, K_k$ and $D_k$ in equation (21) account for a possible renormalization of the derivatives and of the noise term, while the generic potential $U_k = U_k(\phi)$ encompasses the interactions of the model. Higher-order terms in the response field or in the derivative terms can be included in the ansatz for the effective action, leading to a renormalization of parameters and representing, for example, the emergence of non-Gaussian contributions to the noise, which in turn affects the two-point functions. However, motivated by arguments of relevance, we decided to stick to the lowest order approximation, for simplicity.

The potential $U_k = U_k(\phi)$ is assumed to be a $\mathbb{Z}_2$-symmetric local truncated polynomial, i.e.,

$$U_k(\phi) = \sum_{n=1}^{n_U} \frac{g_{2n,k}}{(2n)!} (\phi^2 - \bar{\phi}_k^2)^n,$$

(22)

where $\bar{\phi}_k$ is a background field chosen to be the minimum of $U_k$ and $n_U \geq 2$ is the truncation order. Every coupling $g_{2n,k}$ can be obtained from the expansion of the effective action as

$$g_{2n,k} = \left. \frac{\delta^{2n} \Gamma_k}{\delta \phi^{2n} \delta \tilde{\phi}} \right|_{\tilde{\phi} = 0 = \phi = \bar{\phi}_k},$$

(23)

where the derivatives of $\Gamma$ are evaluated at the homogeneous field configuration $\tilde{\phi} = 0$ and $\phi = \bar{\phi}_k$. Finally, in order to derive the RG equations for the couplings appearing in the effective action (21), one has to take the derivative with respect to $k$ of both sides of equation (23) and, by using equation (19), one finds
\[
\frac{dg_{2n,k}}{dk} = \frac{\delta^{2n}}{\delta \phi^{2n-1} \delta \phi} \frac{d\Gamma_k}{dk}_{\phi=0} + \frac{\delta^{2n+1}}{\delta \phi^{2n} \delta \phi} \frac{d\phi_k}{dk},
\]

from which one can evaluate the flow equation for the couplings \(g_{2n,k}\).

While in reference [32] the exact form of \(\Gamma_k\) was required to determine the initial critical exponent \(\theta\) via a short-time analysis of the dynamics, here it will not play a major role since we are interested in the long-time limit of the aging regime. In the following we introduce a different approach which does not make use of the explicit form of \(\Gamma_k\) in order to calculate \(\theta\): we implement the initial condition via the Dirichlet condition on the correlation function at any fixed scale \(k\), as detailed in appendix B1. This fact is justified by the fact that, due to the canonical dimension of \(\tau_0\), its fixed-point value is known to be \(\tau_0 = +\infty\) [25, 32]. Correspondingly, the average and the fluctuations of the initial field \(\phi_0\) vanish together with all correlation function involving it.

In the following, we show how the LPA ansatz simplifies the fRG equation (19) for the modified effective action \(\Gamma_k\). By taking advantage of the locality in space and time of the LPA ansatz (21), i.e., of the fact that it is written as an integral over \(x\), one can rewrite the second of equation (20) as

\[
G_k^{-1}(x, x') = G_{0,k}^{-1}(x, x') - \delta(x - x') \Sigma_k(x),
\]

where we separated \(\Gamma_k^{(2)} + R_k\sigma\) in equation (20) in a field-independent and a field-dependent part \(G_{0,k} = G_{0,k}(x, x')\) and \(\Sigma_k = \Sigma_k(x)\), respectively. In order to derive \(G_{0,k}\) one should invert the field-independent part in equation (20), while imposing equation (12) on the correlation function (the response function does not depend on the initial condition within the Gaussian approximation, see section 2.2). The analytical expression of \(G_{0,k}\) and \(\Sigma_k\) is reported in appendix B1. Inverting equation (25) leads to a Dyson equation for \(G_k\):

\[
G_k(x, x') = G_{0,k}(x, x') + \int_y G_{0,k}(x, y) \Sigma_k(y) G_k(y, x').
\]

One can then cast the fRG equation (19) in a simpler form by using equation (26). In fact, by formally solving the Dyson equation (26), one can express \(G_k\) as a series which, together with equation (19), renders

\[
\frac{d\Gamma_k}{dk} = \sum_{n=1}^{+\infty} \Delta \Gamma_{n,k},
\]

where the functions \(\Delta \Gamma_{n,k}\) are given by

\[
\Delta \Gamma_{n,k} = \frac{1}{2} \int_{x_1, \ldots, x_n} \text{tr} \left[ G_{0,k}(x, y_1) \Sigma_k(y_1) G_{0,k}(y_1, y_2) \times \cdots \times \Sigma_k(y_n) G_{0,k}(y_n, x) \frac{dR_k}{dk} \sigma \right].
\]

As a simple but instructive example, we consider a truncation of the effective potential \(U_k\) up to the fourth power of the field \(\phi\) (i.e., \(n_{tr} = 2\)) around a configuration with vanishing minimum \(\phi_k = 0\), i.e.,
\[ U_k = \frac{r_k}{2} \phi^2 + \frac{g_k}{4!} \phi^4. \] (29)

The field-independent part of \( \Gamma^{(2)} \), i.e., \(-\Sigma_k\), can be explicitly evaluated as

\[ \Sigma_k = -g_k \bar{\vartheta}(t-t_0) \begin{pmatrix} \bar{\phi}\phi_i/\sqrt{2} & \phi_i^2/2 \\ 0 & 0 \end{pmatrix}. \] (30)

Since \( \Sigma_k \) appears \( n \) times in the convolution (28) which defines \( \Delta \Gamma_{n,k} \) on the rhs of equation (27), it follows that \( \Delta \Gamma_{n,k} \) contains a product of \( 2n \) fields (grouped in \( n \) pairs which depend on different times \( t_i \) for \( i = 1, \ldots, n \)). Accordingly, the only terms on the rhs of equation (27) which contribute to renormalize \( U_k \) are \( \Delta \Gamma_{1,k} \) and \( \Delta \Gamma_{2,k} \). Also the lhs of equation (27) is a polynomial of the fields, because of the LPA ansatz (21), and therefore each term of the expansion on the lhs is uniquely matched by a term of the expansion on the rhs. We note that \( \Delta \Gamma_{2,k} \), at variance with \( \Delta \Gamma_{1,k} \), is a quantity that depends on the fields \( \phi \) and \( \bar{\phi} \) evaluated at different times. Since the lhs of equation (27), with the LPA ansatz (21), is local in time and space, one has to retain only local contributions in the rhs. In appendix B2 we detail how the \( \Delta \Gamma_{n,k} \)'s terms with \( i = 1, 2 \) can be calculated. In the following sections, we investigate how different truncations lead to time-dependent effective potential as an effect of the quench.

### 3.2. Time-dependent effective action

We derive here the RG equations from the ansatz (21) with the quartic potential \( U \) introduced in equation (29). Since this ansatz corresponds to a LPA [38, 39], the anomalous dimensions of the derivative terms \( (K_k, Z_k) \) and of the noise strength \( D_k \) vanish, and therefore for simplicity, we set \( K_k = Z_k = D_k = 1 \) in what follows. Accordingly, the anomalous dimension \( \eta \) and the dynamical critical exponent \( z \) are equal to their Gaussian values \( \eta = 0 \) and \( z = 2 \), respectively.

The only non-relevant terms which are renormalized within this scheme are those proportional to quadratic and quartic powers of the fields \( \phi \) and \( \bar{\phi} \), i.e., those associated with the post-quench parameter \( r_k \) and the coupling \( g_k \). The renormalization of the quadratic terms is determined by the contribution \( \Delta \Gamma_{1,k} \) appearing on the rhs of equation (27), while the renormalization of the quartic one by the contribution \( \Delta \Gamma_{2,k} \). The calculations of these two contributions are detailed in appendix B, and in particular appendix B3, using an optimized cutoff function \( R_k \), and leads to the following IRG flow equations:

\[ \frac{d r_k(t)}{d k} = -k^{d+1} \frac{d}{d \omega_k} \left[ 1 - e^{-2z \omega f_1(\omega_k t)} \right], \] (31a)

\[ \frac{d g_k(t)}{d k} = 6k^{d+1} \frac{d}{d \omega_k} \left[ 1 - e^{-2z, f_2(\omega_k t)} \right], \] (31b)

where \( \omega_k = r_k + k^2 \), \( f_1(x) = 1 + 2x \), \( f_2(x) = 1 + 2x^2 + 2x \), \( a_d = 2/[(d/2)(4\pi)^{d/2}] \), \( d \) is the spatial dimensionality of the system and \( \Gamma(x) \) is the gamma function. Within the standard LPA approximation to non-equilibrium systems, the time dependence of \( r_k \) and \( g_k \) on the lhs is exclusively given by the non-equilibrium initial condition (3) via \( \Delta \Gamma_{1,k} \) and \( \Delta \Gamma_{2,k} \) in equation (27) and ultimately by \( G_{0,k} \), via the Dirichlet correlation function (12). Remarkably, the time dependence of the couplings \( r_k \) and \( g_k \) vanishes exponentially in time: for \( t \to \infty \) one thus obtains the RG flow equations for the bulk part of the ansatz (21), which determine the equilibrium...
fixed points. This structure is preserved for higher-order truncation in LPA approximation, as we detail in appendix B5.

In the long-time equilibrium regime, the fRG flow equation (31) can be further simplified by rewriting them in terms of the dimensionless couplings \( \tilde{r}_k = r_k k^{-2} \) and \( \tilde{g}_k = (a_d/d) g_k k^{d-4} \). The fixed points for the various values of the spatial dimensionality \( d \) are found by solving the system of equations corresponding to requiring vanishing derivatives of the dimensionless couplings, i.e.,

\[
\begin{align*}
0 &= 2\tilde{r}^* - \frac{\tilde{g}^*}{(1 + \tilde{r}^*)^2}, \\
0 &= (d - 4)\tilde{g}^* + 6 \frac{\tilde{g}^*}{(1 + \tilde{r}^*)^2},
\end{align*}
\]

where the superscript * indicates fixed-point quantities. Equations (32a) and (32b) admit two solutions: the Gaussian fixed point \((\tilde{r}^*_G, \tilde{g}^*_G) = (0, 0)\) and the Wilson–Fisher (WF) one, which, at leading order in \( \epsilon = 4 - d \), reads \((\tilde{r}^*_WF, \tilde{g}^*_WF) = (-\epsilon/12, \epsilon/6) + O(\epsilon^2)\). By linearizing equations (32a) and (32b) around these solutions, one finds that the Gaussian fixed point is stable only for \( d > 4 \), while the WF fixed point is stable only for \( d < 4 \). The latter has an unstable direction, and from the inverse of the negative eigenvalue of the associated stability matrix, one derives the critical exponent \( \nu \), which reads \( \nu = 1/2 + \epsilon/12 + O(\epsilon^2) \) and is the same as in equilibrium [7].

### 3.3. The case with \( \bar{\phi}_k \neq 0 \)

In this section, we consider the approximation for the effective potential (22) with a non-vanishing background. This case differs from the one considered in equation (29), since it corresponds to an expansion around a finite homogeneous value \( \bar{\phi}_k \); this choice allows one to capture the leading divergences of two-loop corrections in a calculation which is technically done at one-loop, as typical of background field methods (see, e.g., references [38–40]); accordingly, one calculate, for instance, the renormalization of the factors \( Z_k, K_k, \) and \( D_k \). In fact, the presence of the background field \( \bar{\phi}_k \) reduces two-loop diagrams to one-loop ones in which an internal classical line (corresponding to a correlation function, \( C_{0,k} \)) has been replaced by the insertion of two expectation values \( \bar{\phi}_k \). This is illustrated in the two diagrams below in which the external straight lines stand for the field \( \phi \), while the wiggly lines indicate the response field \( \tilde{\phi} \); see also, e.g., reference [7]. For instance, in the case \( n_r = 2 \), where equation (22) can be conveniently rewritten as

\[
U_k(\phi) = \frac{g_k}{4!} \left( \phi^2 - \bar{\phi}_k^2 \right),
\]

the renormalization of \( Z_k \) and \( K_k \) for come from the diagram

\[
\includegraphics[width=\textwidth]{diagram.png}
\]
while the renormalization of the noise strength $D_k$ comes from the diagram:

In these diagrams, those on the right indicate how the corresponding perturbative diagrams encountered in pRG, reported on the left, are reproduced within fRG (see, e.g., reference [32]). The flow equations for $Z_k$ and $K_k$ can be conveniently expressed in terms of the corresponding anomalous dimensions $\eta_D$, $\eta_Z$, and $\eta_K$, defined as $\eta_D \equiv -d \log D_k/d \log k$ and similarly for the others. We note that $\eta_D = \eta_Z$: this a consequence of detailed balance [7, 40], which characterizes the equilibrium dynamics of model A. In fact, while the short-time dynamics after the quench violates detailed balance inasmuch as time-translational invariance is broken, in the long-time limit (in which the flow equations are valid) detailed balance is restored. From general scaling arguments [7], the static anomalous dimension $\eta$ and the dynamical critical exponent $z$ are given by

$$\eta = \eta_k^*, \quad z = 2 - \eta_k^* + \eta_Z.$$  

The calculation of the anomalous dimension $\eta$ and the dynamic critical exponent $z$, discussed in reference [32] (to which we refer the reader for details), gives these anomalous dimensions in terms of the fixed-point values of the couplings which define the LPA ansatz.

Let us discuss how the RG equations can be derived in the presence of a non-vanishing background field $\bar{\phi}_k$. First, equation (24) shows that the $k$-derivative of $\bar{\phi}_k$ enters the fRG flow equations for the couplings (23). In order to define unambiguously $\bar{\phi}_k$, it is possible to change the variable $\phi$, which the effective potential $U_k$ depends on, to the $Z_2$-invariant $\rho = \phi^2/2$. One then defines the minimum as the configuration which correspondsto a vanishing $\rho$-derivative $U_k(\rho) = U_k(\phi^2/2)$. The computations that follows in order to obtain the $k$-derivative of $\bar{\phi}_k$ are detailed in appendix B4. There, it is shown how the fRG flow equation for $\bar{\phi}_k$ is given in terms of $\Gamma_1, k$, while the one for $g_k$ is given in terms of $\Gamma_2, k$. Finally, using the computation of these terms, detailed in appendix B2, one determines the fRG equation of the parameters which define the ansatz (33). The time dependence of these fRG flow equations enters via decreasing exponentials, as in equation (31). Accordingly, this allows a straightforward analysis of the equilibrium fixed point of the Ising universality class, as done for the case of vanishing background field approximation. This consideration still holds true for the case of higher-order truncation, as explained in appendix B5.

The fixed point can be determined numerically, similarly to what is done in equation (32), once the fRG equation for the coupling $g_k$ and the background field $\phi_k$ are accompanied by that for the derivative parameter $K_k$, to which they are coupled [32] and by taking advantage of the relation $\eta_D = \eta_Z$.

### 4. Aging regime

In this section we discuss how to obtain the two-point correlation and response functions of the order parameter within the LPA approximation, detailing how the aging regime (9) is recovered.
4.1. Two-time functions in LPA

Let us denote by $G(t, t')$ the $2 \times 2$ matrix of the physical two-point correlations (cf appendix B), whose entries are given by the response $R$ and correlation $C$ functions, defined in equations (4) and (5), respectively. $G(t, t')$ can be calculated as the inverse of the physical effective action $\Gamma_{k=0} \equiv \Gamma$, evaluated in the configuration of $\Phi$ minimizing $\Gamma$ (see equation (20) and reference [7]). This minimum configuration is given by $\Phi = 0$ for the critical quench, since the background field $\bar{\phi}_k$ vanishes at $k = 0$, as a consequence of criticality. Moreover, within the LPA, $\Gamma_{(2)}$ is assumed to be a local quantity in space and time, i.e., $\Gamma_{(2)}(x, x') = \delta(x - x')\Gamma_{(2)}(x)$. With these two simplifications, the equation of motion for the physical two-point function $G(t, t')$ becomes

$$\Gamma_{(2)}(t)|_{\Phi = 0} G_q(t, t') = \delta(t - t'),$$

(35)

where we exploited the space-translational invariance of model A, and Fourier-transformed with respect to this spatial dependence.

In order to retrieve the two-point function $G$ from equation (35), one should first compute $\Gamma_{(2)}|_{\Phi = 0}$ given the LPA ansatz (21) for the effective action. By implementing the Dirichlet boundary condition (12), the equation of motion (35) can be cast in the form of a set of equations for the response and correlation functions. Thus, one obtains

$$[Z\partial_t + r(t)]R_{q=0}(t, t') = \delta(t - t'),$$

(36a)

$$C_{q=0}(t, t') = \frac{2}{D} \int_{t_0}^{\infty} ds R_{q=0}(t, s) R_{q=0}(t', s),$$

(36b)

where $r(t)$ is defined as the $k \to 0$ limit of the renormalized parameter $r_k$ defined by

$$r_k \equiv \frac{\partial^2 U_k}{\partial \phi^2}(\phi = 0) \simeq \begin{cases} r_\Lambda & \text{for } k \to \Lambda, \\ r(t) & \text{for } k \to 0. \end{cases}$$

(37)

The time dependence of $r(t) = r_{k=0}(t)$ in equation (36) is given by the fRG flow equation, as shown, for instance, in equation (31a). In order to access the aging regime, one requires the critical point to be reached at long times, i.e., $r(t) \to 0$ for $t \to +\infty$. This condition can be achieved by properly fine-tuning the microscopic value $r_{k=0}$, as discussed in more detail in section 4.2.

4.2. Long-time aging dynamics

We first consider the case $\bar{\phi}_k = 0$ and $n_g = 2$ discussed in section 3.2. We consider the case of a critical quench and we denote by $r^*(t)$ the fine-tuned value of $r(t)$ which vanishes in the long-time limit. By integrating equation (31a) in $k$ from the microscopic scale $\Lambda$ to $k = 0$ and evaluating the couplings $r_k$ and $g_k$ at their fixed point values, one obtains

$$r^*(t) = -\frac{\theta}{\bar{\gamma}} \left[ 1 - e^{-2\Lambda^2(1 + \bar{\gamma})^2} \bar{F}_1(\Lambda^2 t) \right],$$

(38)

with $\bar{F}_1(x) = 1 + x(1 + \bar{\gamma})$. The value of $\theta$, the meaning of which will be discussed further below, is given by

$$\theta = \frac{\bar{g}^*}{z(1 + \bar{\gamma})^2},$$

(39)
with \( z = 2 \) and the fixed-point values \( \tilde{r}^* \) and \( \tilde{g}^* \) given by the solution of equation (32). In order to obtain equation (38), the microscopic value \( r_{\Lambda} \) has been fine-tuned to \( r_{\Lambda}^* = -\Lambda \tilde{g}^*(1 + \tilde{r}^*)^{-2}/z \), implying that the critical temperature is shifted towards a smaller value compared to the Gaussian one, \( r_{\Lambda}^* = 0 \), as expected because of the presence of fluctuations [7].

Let us now solve the equations of motion for the two-time functions given in equation (36). Since we are solely interested in the aging regime, we shall assume \( t \gg t' \gg \Lambda^{-2} \). In this case, one can use the asymptotic value of the parameter \( r^*(t) \) given by

\[
r^*(t) = -\frac{\theta}{t},
\]

as detailed in appendix C1, and the solutions reads:

\[
R_{q=0}(t, t') = \left( \frac{t}{t'} \right)^{\theta} \quad \text{and} \quad C_{q=0}(t, t') = \frac{2t'}{1-2\theta} \left( \frac{t}{t'} \right)^{\theta}
\]  

(41)

By comparing equation (41) with their scaling form (9), one finds that the non-universal amplitudes of the response and the correlation functions are given by \( A_R = 1 \), \( A_C = 2(1-\theta)^{-1} \), respectively, and that \( \theta \) introduced in equation (39) is nothing but the critical initial-slip exponent. This method to derive \( \theta \) is alternative to that employed in reference [32], which was based on the renormalization of the initial-time action \( \Gamma_0 \). Our result for \( \theta \) exactly matches the one obtained in reference [32] within the same ansatz for \( U_k \), as explained in appendix C3.

The expression of the FDR \( X^\infty \), given in equation (10) in terms of \( A_R, A_C, \) and \( \theta \), finally reads:

\[
X^\infty = \frac{1/2 - \theta}{1 - \theta}.
\]  

(42)

Accordingly, \( X^\infty \) is given only in terms of \( \theta \), which, in turn is fully determined by the fixed points of equation (32). We note that, from equations (39) and (42), one obtains \( \theta = \epsilon/6 + O(\epsilon^2) \) and \( X^\infty = 1 - \epsilon/12 + O(\epsilon^2) \) to leading order in \( \epsilon = 4-d \), thus retrieving the known one-loop result [27]. Indeed, the functional renormalization group equation has a simple one-loop like structure. At the lowest LPA approximation, the diagrams accounted for are essentially the same as those of a perturbative calculation, yielding to same results.

The validity of equation (42) goes beyond the simple ansatz \( \bar{\phi}_k = 0 \) discussed in this section. In fact, it is a general consequence of the LPA approximation and it does not depend on the specific truncation of the potential \( U_k \), as we prove in the next section. Accordingly, different LPA truncations will affect the specific value assumed by \( X^\infty \) only through the value of \( \theta \), as we discuss in what follows for a different truncation.

### 4.3. Finite background field \( \bar{\phi}_k \neq 0 \)

In the following, we discuss the case of the LPA ansatz with the generic effective potential \( U_k \) given by equation (33). The parameter \( r_k \) defined in equation (37), is now given by

\[
r_k = -\frac{1}{3!} g_k \bar{\phi}_k^3.
\]  

(43)

The form of \( dr_k/dk \) is now determined by the flow equation of the parameters \( g_k \) and \( \bar{\phi}_k \), discussed in section 3.3. The equation for the two-point function (36) can be cast in the following useful form
Figure 1. Numerical estimates of the equilibrium critical exponent $\nu$, of the non-equilibrium critical initial-slip exponent $\theta$ and of the universal amplitude ratio $X^\infty$ in spatial dimension $d = 3$, as functions of the truncation order $n_{tr}$. Red dots, instead, correspond to the non-vanishing background field approximation for $U_k$ discussed in section 3.3. The shaded cyan areas indicate the numerical estimates derived from the MC simulations available in the literature.

\[
\begin{align*}
\left[ \partial_t + \frac{r(t)}{Z} \right] \tilde{R}_q = & \delta(t - t'), \\
\tilde{C}_q = & \frac{2D}{Z} \int_0^\infty ds \tilde{R}_q = \phi(t, s) \tilde{R}_q = \phi(t', s),
\end{align*}
\] (44a)

with $Z$ assumed to be time-independent, $\tilde{R}_q(t, t') = ZR_q(t, t')$, and $\tilde{C}_q(t, t') = ZC_q(t, t')$. The same derivation as in section 4.2 applies, upon replacing $r_k$ with $r_k/Z_k$ in equation (44). A canonical power counting shows that $r/Z$ has the dimension of an inverse time, and therefore equations similar to equations (38) and (40) can be obtained, as detailed in appendix C2.

The final prediction for $\theta$ is different from equation (39), but it agrees order by order with the one found previously by means of the short-time analysis of the dynamics in reference [32], as we detail in appendix C3. By solving equation (44) for the reduced response and correlation function, $\tilde{R}$ and $\tilde{C}$, respectively, one obtains a form similar to the one given in equation (41) in the aging regime, using the fact that $D_k/Z_k = \text{const.}$, which is a consequence of the fact that the flow equation of $D_k$ and $Z_k$ are identical [32] (the constant can then be reabsorbed in a suitable renormalization of the fields). We find consequently that equation (42) encompasses also the case in which the potential $U_k$ is expanded around its non-vanishing minimum.

We emphasize that, because of the definition (37), the value of $r_k$ depends on the order $n_{tr}$ of the truncation in the ansatz (22). For instance, for $n_{tr} = 3$, it is given by

\[
r_k = -\frac{1}{3!} g_k \phi_k^2 + \frac{1}{5!} \lambda_k \phi_k^4,
\] (45)
Table 1. Summary of available estimates for the critical initial-slip exponent and the asymptotic value $X^\infty$ of the FDR, for $d = 3$.

| Method      | $\theta$  | $X^\infty$ |
|-------------|-----------|------------|
| fRG (this work) | 0.144     | 0.415      |
| MC          | 0.14(1) [41] | 0.40(1) [37] |
| MC          | 0.135(1) [42] | 0.380(13) [43] |
| MC          | 0.135(1) [44] | 0.429(6) [31] |

where $\lambda_k = g_6$.$^k$. In reference [32] the term proportional to $\lambda_k$ in equation (45) was neglected, resulting in an inconsequential discrepancy in the value of $\theta$ of 2% for $d = 3$ compared to the one obtained here (see appendix D for a detailed comparison between the results of reference [32] and those presented here).

5. Discussion of the results

Here we discuss our predictions for the values of the critical initial-slip exponent $\theta$ and for the universal amplitude ratio $X^\infty$.

We first consider the convergence of the predictions in equations (39) and (42) upon increasing the order $n_t$ of the truncation in equation (22). In figure 1 we report the estimates of $\nu$, $\theta$, and $X^\infty$ for $d = 3$, obtained with a non-vanishing background field, for various truncation orders $n_t$. The values (red dots) display a rather small variation upon increasing $n_t$, suggesting that their values have effectively converged to those corresponding to a full LPA solution. This fast convergence is due to the use of the optimized cutoff given by equation (B8), as pointed out in reference [45]. While the values of $\nu$ and $\theta$ appear to be in very good agreement with the available MC data (shaded areas), the values of $X^\infty$ lie outside (though close to) the MC estimates.

In table 1 we report the values of our best approximation $n_t = 6$ for $d = 3$ of $\theta$ and $X^\infty$, comparing them with the MC and pRG estimates available in literature. For what concerns the values given by pRG, we report for $\theta$ the result of the three-loop Borel summation procedure [43], while for $X^\infty$ we report the low Padé approximation of the two-loop calculation [31].

In figure 2, we report the predictions for $\theta$ and $X^\infty$ obtained with $n_t = 6$ (red solid curve) as functions of the spatial dimensionality $d$, and we compare them with MC (symbols) estimates and pRG predictions (black curves). The agreement between fRG and pRG predictions for $\theta$ is remarkable for $d \gtrsim 3.2$, while increasing discrepancies emerge at smaller values of $d$. The fRG predictions cannot be extended down to $d = 2$ with the ansatz considered. In fact, for $d \leq 3$ additional stable fixed points appear beyond the WF one, while for $d \leq 2.5$ the latter disappears. This is not surprising, since for $d < 3$ there exist more relevant terms than those considered in the truncation, which therefore is no longer sufficient and justified.

Concerning the values of $X^\infty$, the predictions of fRG correspond to that of pRG close to $d = 4$, but it departs quite soon from it. The fRG prediction is much closer to the MC estimate in $d = 3$ than the pRG one, although still not within its error bars.

The significant improvement provided by the fRG compared to the pRG can be traced back to two reasons. First, the LPA approximation provides a resummation of one-loop diagrams, resulting in a more precise determination of the amplitudes $A_p$ and $A_C$, compared to the pRG predictions at order $\epsilon$; a comparison of the analytical expressions between pRG and fRG at
Figure 2. Estimates of the critical initial-slip exponent \( \theta \) (left panel) and of the asymptotic value \( X^\infty \) of the FDR (right panel) as a function of the spatial dimensionality \( d \) of the system. Red lines are obtained with the non-vanishing background field LPA approximation of the fRG, with \( n_{tr} = 6 \). The black lines indicate Borel resummations of, respectively, three-loop (left panel, from reference \([44]\)) and two-loop (right panel, from reference \([27]\)) pRG. For \( d = 3 \) the MC results for \( \theta \) are from references \([41]\) (black) and \([42]\) (magenta), while for \( X^\infty \) are from references \([37]\) (black) and \([43]\) (magenta). For \( d = 2 \), all the MC data are derived from reference \([27]\). Insets: magnification of the main plots for \( d \approx 3 \).

Table 2. Comparison of predictions of the pRG treatment with those of fRG within the LPA approximation, for the various relevant quantities.

| \[ fRG \] in LPA | \[ pRG \] at order \( \epsilon \) |
|-------------------|------------------|
| \( A_R \)         | \( 1 + O(\epsilon^2) \) |
| \( A_C \)         | \( 2(1 + 2\theta) + O(\epsilon^2) \) |
| \( X^\infty \)    | \( (1 - \theta)/2 + O(\epsilon^2) \) |

The non-perturbative nature of the fRG provides a more accurate estimate of \( \theta \) (see figure 2, left panel) and therefore of \( X^\infty \).

6. Conclusions and perspectives

In this work we presented an approach, based on the functional renormalization group (fRG), to predict the universal FDR \( X^\infty \) for the aging dynamics of model A. By calculating the two-time correlation functions from the renormalized effective action, we showed that, within the LPA, the universal ratio \( X^\infty \) depends only on the critical initial-slip exponent \( \theta \). As a side result, we proposed an alternative way to calculate \( \theta \) within the fRG by focusing on the long-time behavior of the two-time response and correlation functions. Moreover, we showed that the values of \( \theta \) and \( X^\infty \) converge quickly upon increasing the order of the truncation in the ansatz for the potential. Finally, we compared our prediction with the existing estimates obtained from MC simulations and from the perturbative renormalization group (pRG), finding that the fRG results are closer to the MC than those of pRG ones.

Since the value of \( X^\infty \) in \( d = 3 \) is still not compatible with the MC results (although closer to it than predicted by previous approaches), a natural future direction is the computation of the two-time correlation functions beyond the LPA. Non-local terms are in fact expected to provide sizeable corrections, as they contribute with a term proportional to \( \epsilon^2 \) in the pRG \([31]\). This could be achieved by using different approximations for closing the infinite hierarchy of
equations generated by the fRG equation, following, for instance, the approach of reference
[34] or considering mode-coupling approximations [46].

Other natural directions consist in applying our method to characterize the FDR of other
static universality classes, e.g., $O(N)$ and Potts models, or dynamics with conserved quantities
[47, 48]. Moreover, the extension of the FDR analysis to non-equilibrium quantum systems
[11–14, 16–18] represents an intriguing issue.

Acknowledgments

We are indebted to R Ben Ali Zinati and M Scherer for useful discussions. A C acknowledges
support by the funding from the European Research Council (ERC) under the Horizon 2020
research and innovation program, Grant Agreement No. 647434 (DOQS). M V warmly thanks
the University of Milan and SISSA for their hospitality during the first phases of this project.
This research was partly conducted within the Econophysics & Complex Systems Research
Chair, under the aegis of the Fondation du Risque, the Fondation de l’Ecole polytechnique, the
Ecole polytechnique and Capital Fund Management.

Data availability statement

The data that support the findings of this study are available upon reasonable request from the
authors.

Appendix A. Effective action

Starting from the $k$-dependent action $S_k$, we define the associated generating function $W_k[J]$ as

$$W_k[J] = \log \left[ \int D\Psi e^{-S_k[\Psi] + \int_{t,r} \Psi J} \right], \quad (A1)$$

where $D\Psi$ denotes functional integration over both the fields $\varphi$ and $\tilde{\varphi}$, while $J = (j, \tilde{j})$ is an
external field. Introducing the expectation value $\langle \cdot \rangle_k$, where the average is taken with respect
to the action $-S_k[\Psi] + \int_{t,r} \Psi J$, it is straightforward to check that the second derivative $W^{(2)}_k$
of $W_k$ with respect to $J$ is given by [38]

$$W^{(2)}_k = \langle \Psi \Psi^\dagger \rangle_k - \langle \Psi \rangle_k \langle \Psi^\dagger \rangle_k = G_k, \quad (A2)$$

where $G_k$ denotes the $k$-dependent two-point function, whose entries are given by the $k$ and $J$-
dependent version of the response (4) and the correlation (5) functions. Finally, the $k$-dependent
effective action $\Gamma_k[\Phi]$ is defined as the modified Legendre transform of $W_k[\Psi]$, given by

$$\Gamma_k[\Phi] = -W_k[J] + \int J \Phi - \Delta S_k[\Phi], \quad (A3)$$

where $J$ is fixed by the condition

$$\Phi = W^{(1)}_k = \langle \Psi \rangle_k. \quad (A4)$$

The definition of $\Gamma_k[\Phi]$ in equation (A3) is such that [38] $\Gamma_{k=\Lambda}[\Phi] \approx S[\Phi]$, i.e., when $k$ is equal
to the ultraviolet cutoff $\Lambda$ of the theory, the effective action $\Gamma_k$ reduces to the ‘microscopic’
action $S[\Phi]$ evaluated on the expectation value $\Phi$. The following relationship can then be derived [38]:

$$\Gamma_k^{(1)} = J - \sigma R_k \Phi.$$  \hspace{1cm} (A5)

### Appendix B. fRG flow equations

In this appendix we first present the detailed calculation of $\Delta \Gamma_n, k$ equation (28) with $n = 1$ and 2. Then, the fRG flow equations for the corresponding couplings $g_{2n,k}$ which appear in equation (22) are obtained explicitly for the case of vanishing background fields. Details concerning the non-vanishing background ansatz are then discussed. Finally, we focus on the case $n > 2$, providing analytical formula for the time-dependent part of the fRG flow equations for the various couplings.

#### B.1. Calculation of $G_{0,k}$ and $\Sigma_k$

Here we want to determine $G_{0,k}$ in equation (25), defined as $G_k$ in equation (20), but after replacing $\Gamma_k^{(2)}$ with its field-independent part. In order to do so, the matrix structure of the second variation $\Gamma_k^{(2)}$ of the modified effective action $\Gamma_k$ is set to

$$\Gamma_k^{(2)}(x, x') = \begin{pmatrix} \frac{\delta^2 \Gamma_k}{\delta \phi(x) \delta \phi(x')} & \frac{\delta^2 \Gamma_k}{\delta \phi(x) \delta \tilde{\phi}(x')} \\ \frac{\delta^2 \Gamma_k}{\delta \phi(x) \delta \tilde{\phi}(x')} & \frac{\delta^2 \Gamma_k}{\delta \tilde{\phi}(x) \delta \tilde{\phi}(x')} \end{pmatrix}.$$  \hspace{1cm} (B1)

In order to implement the LPA truncation, $\Gamma_k^{(2)}$ above has to be computed on the basis of the expression for $\Gamma_k$ given in equation (21). Moreover, in order to separate the field-independent contribution $G_{0,k}$ from the field-dependent one $\Sigma_k$, we proceed as explained in what follows. We first set

$$G_{0,k}^{-1}(q, t, t') = \begin{pmatrix} 0, & -Z_k \partial_t + \omega_{k,q} \\ Z_k \partial_t + \omega_{k,q}, & D_k \end{pmatrix},$$  \hspace{1cm} (B2)

where the dispersion relation $\omega_{k,q}$ is given by

$$\omega_{k,q} \equiv K_k q^2 + m_k + R_k(q^2),$$  \hspace{1cm} (B3)

with

$$m_k = \frac{\partial^2 U_k}{\partial \phi^2}(\phi = \tilde{\phi}_k).$$  \hspace{1cm} (B4)

Accordingly, we define the field-dependent part $\Sigma_k$ (we assume $t_0 = 0$ for simplicity), as

$$\Sigma_k = \partial(t) \begin{pmatrix} -\frac{\partial^2 U_k}{\partial \phi^2} + \frac{\partial^2 U_k}{\partial \phi^2} - m_k \\ \frac{\partial^2 U_k}{\partial \phi^2} - m_k, \quad 0 \end{pmatrix}.$$  \hspace{1cm} (B5)

The condition that $\phi = \tilde{\phi}_k$ in the definition of $m_k$ in equation (B4) is chosen in order to expand the remaining field-dependent part $\Sigma_k$ as a power series of $(\phi^2 - \tilde{\phi}_k^2)^2$. This allows one to use a
truncation procedure similar to that in the simple case of vanishing background field discussed at the end of section 3.1, as we discuss also below in appendix B4.

After a Fourier transform in space, \( G_{0,k} \) is obtained from equation (B2):
\[
G_{0,k}(q,t,t') = \begin{pmatrix} C_{0,k}(q,t,t') & R_{0,k}(q,t,t') \\ R_{0,k}(q,t',t) & 0 \end{pmatrix},
\]
where
\[
R_{0,k}(q,t,t') = \partial(t-t') \frac{1}{Z_k} e^{-\omega_k t' t}, \tag{B7a}
\]
\[
C_{0,k}(q,t,t') = 2D_k \int_0^{+\infty} R_{0,k}(q,t,s)R_{0,k}(q,t',s).
\]

We note that in equation (B6) the vanishing entry is due to the fact that the field-independent part of the second derivative of \( \Gamma \) with respect to \( \phi \) evaluated in the minimum configuration vanishes, as a consequence of causality [7]. Moreover, the correlation function in equation (B7) respects the Dirichlet condition, which is enforced in our ansatz, as described near equation (24).

B.2. Calculation of \( \Delta \Gamma_{1,k} \) and \( \Delta \Gamma_{2,k} \)

In order to calculate the \( \Delta \Gamma_{1,k} \)'s terms in equation (28) we use the so-called optimized regulator [49] \( R_k(q) \), given by
\[
R_k(q) = Z_k \vartheta(k^2 - q^2)(k^2 - q^2). \tag{B8}
\]
First, note that \( R_k(q) \) fulfills the conditions in equation (17), introduced in order to construct the \( \Phi^4 \) equation. By using the optimized cutoff function above it is possible to compute analytically the relevant integrals appearing in the \( \Delta \Gamma_{1,k} \)'s terms. This is due to two simplifications: first, the \( k \)-derivative of the cutoff function \( R_k \) in equation (28) is given by a theta function which vanishes for \( q^2 > k \), i.e., \( k \partial_k R_k \propto \vartheta(k^2 - q^2) \). This implements a modified ultraviolet cutoff in the \( q \) integrals in equation (28). The second simplification, which is a consequence of the first, is due to the fact that the modified dispersion relation \( \omega_{k,q} \) in equation (B3) simplifies to \( \omega_k = \omega_{k,q=q} = K k^2 + m_0 \) in the \( q \) interval allowed by \( \partial_k R_k(q) \). The regulator \( R_k(q^2) \) thus implements an infrared cutoff \( \sim k^2 \), rendering the dispersion \( \omega_{k,q} \) independent of \( q \) for \( q \leq k \). Accordingly, the two-time function \( G_{0,k}(q,t,t') \) in equation (B6) becomes independent of the momentum \( q \), and will be simply indicated by \( G_{0,k}(t,t') \).

The first term that we need to calculate is \( \Delta \Gamma_{1,k} \) given by equation (28) with \( n = 1 \). Taking advantage of these two simplifications, the \( q \) dependence, contained in the \( k \)-derivative of the cutoff function, can be factorized and the following expression for \( \Delta \Gamma_{1,k} \) is obtained:
\[
\Delta \Gamma_{1,k} = \frac{1}{2} \int_q \partial_k R_k(q) \int_{t_1 \to t} \text{tr} \left[ G_{0,k}(t_1) \Sigma_k(t_1) G_{0,k}(t_1) \right], \tag{B9a}
\]
\[
= K_k k^{d+1} \frac{a_d}{d} \left( 1 - \frac{\eta}{d+2} \right) \int_0^{+\infty} \text{d} t_1 \Sigma_{11,k}(t_1) \mathcal{F}_{1,k}(t_1), \tag{B9b}
\]
where \( \Sigma_{11,k} \) is the (1,1) entry of the matrix \( \Sigma_k \) defined in equation (B5), while the time-dependent function \( \mathcal{F}_{1,k}(t_1) \) is defined as
\[ \mathcal{F}_{1,k}(t_1) \equiv \int_0^{t_1} dt \, tr \left[ G_{0,k}(t,t_1) \sigma_2 G_{0,k}(t_1, t) \sigma \right], \]  
(B10)

with \( \sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \). The non-diagonal terms of \( \Sigma_k \) do not appear in the final result (B9b), as required by causality [7], since they would be multiplied by a factor \( \psi(t-t') \psi(t'-t) = 0 \).

Then, by computing the trace in equation (B10) one finds

\[ \mathcal{F}_{1,k}(t_1) = 2 \int_0^{t_1} dt' C_{0,k}(t_1, t) R_{0,k}(t_1, t) \]  
(B11a)

\[ = 4 D_k \int_0^\infty ds R_{0,k}^2(t_1, s)(t_1 - s) \]  
(B11b)

\[ = \frac{D_k}{Z_k \omega_k^2} \left[ 1 - e^{-\omega_k^2 f_1(\omega_k t_1)} \right], \]  
(B11c)

where, in order to obtain equation (B11b), we have used the expression of \( C_{0,k} \) in equation (B7b) together with the LPA property, which follows from equation (B7a), that \( R_{0,k}(t_1, t) R_{0,k}(t, s) = R_{0,k}(t, s) \psi(t_1 - t) \psi(t - s) \), and we have performed the integral over \( t \); in order to obtain equation (B11c), the time integral over \( s \) has been computed by using the analytical expression of \( R_{0,k} \) given in equation (B7a). Finally, the expression of \( \Delta \Gamma_{1,k} \), where the ansatz for \( \mathcal{U}_k \) is still unspecified, is given by equation (B9b), together with equation (B11c).

The term \( \Delta \Gamma_{2,k} \) is more complicated than the previous one since, unlike \( \Delta \Gamma_{1,k} \), it involves fields evaluated at different time coordinates. In fact, following the steps with which we have obtained equation (B9b), but for the case \( n = 2 \) in equation (28), one obtains

\[ \Delta \Gamma_{2,k} \propto K_k k^{d+1} d \frac{d}{d + 2} \left[ 1 - \frac{\eta}{d + 2} \right] \int_{t_1 t_2}^t 2 \Sigma_{11,k}(t_1) \Sigma_{12,k}(t_2) F_{2,k}(t_1, t_2), \]  
(B12)

where the two-time-dependent function \( F_{2,k}(t_1, t_2) \) is given by

\[ F_{2,k}(t_1, t_2) = \int_0^{t_1} dt' \left[ G_{0,k}(t, t_1) \sigma_2 G_{0,k}(t_1, t_2) \sigma G_{0,k}(t_2, t) \sigma \right]. \]  
(B13)

In equation (B12) we have retained only the contribution proportional to \( \bar{\psi} \), i.e., proportional to \( \Sigma_{11,k} \) (see equation (B5)), since they are the only ones which renormalize the couplings \( \Sigma_{2n,k} \), as one can see from their definition in equation (23). Other contributions, which renormalize the noise term \( D_k \), are discussed in reference [32]. As one can see from equation (B12), the kernel \( F_{2,k}(t_1, t_2) \) is a two-time function from which one has to extract its local part, accordingly to the LPA ansatz (21), in order to obtain the contribution of the \( \Delta \Gamma_{2,k} \)'s proportional to the terms which contain the coupling terms. This can be achieved by substituting \( \phi(t_2) \to \phi(t_1) \) in equation (B12) [32]. Other contributions, which renormalize the derivative term \( Z_k \), are discussed in reference [32]. If the previous substitution is made, equation (B12) becomes

\[ \Delta \Gamma_{2,k}^l \propto K_k k^{d+1} d \frac{d}{d + 2} \left[ 1 - \frac{\eta}{d + 2} \right] \int_0^t dt_1 2 \Sigma_{11,k}(t_1) \Sigma_{12,k}(t_1) F_{2,k}^l(t_1), \]  
(B14)

where the superscript \( l \) denotes the local part of \( F_{2,k} \) defined in equation (B13), i.e.,
The construction of the fRG equations for the couplings

In order to obtain equation (B15b) we have computed the trace in equation (B13) and then the intermediate time integrals over \( t \) and \( t_2 \), similarly to what has been done in the computation of equation (B10) leading to equation (B11c). In order to obtain equation (B15c), we computed the time integral over \( s \) in equation (B15b) by using the analytic expression of \( \mathcal{R}_{0,k} \) in equation (B7a). Finally, the local part \( \Delta \Gamma_{2,\hat{k}} \) of \( \Delta \Gamma_{2,k} \), proportional to \( \phi \), is given by equation (B14) together with \( F_{3,\hat{k}}(t_1) \) in equation (B15c). We have thus shown explicitly how the extraction of the local contribution in \( \Delta \Gamma_{2,k} \) is implemented in the simple case of \( \phi_k = 0 \).

**B.3. Case with \( \phi = 0 \) and \( n_y = 2 \)**

Here we want to derive the fRG flow equation for the parameter \( r_k \) and \( g_k \) introduced in the ansatz for the effective potential \( U_k \) given by equation (29), for which \( K_k = D_k = Z_k = 1 \), as discussed in section 3.2. The field-dependent part \( \Sigma_k \) of \( \Gamma^{(2)} \) is given by equation (30). Then, by taking advantage of equation (B9b), with equations (B14) and (30), one obtains

\[
\frac{d \Gamma_{1,k}}{d k} \propto \left( k^{d+1} \frac{d k}{d \phi} \right) \int_0^\infty \! dt \, \langle \phi^\dagger(t) \rangle \, \left[ g_k \phi(t) F_{1,k}(t) g_k^\dagger \phi^\dagger(t) F_{3,\hat{k}}(t_1) \right], \tag{B16}
\]

where we have considered only linear contributions in \( \tilde{\phi} \), in order to obtain the fRG flow equations of the coupling terms \( g_{2n,k} \) in equation (23), given by \( r_k \) and \( g_k \). The time-dependent functions \( F_{1,k} \) and \( F_{3,\hat{k}} \) are the same as those obtained previously, given by equations (B11c) and (B15c) respectively, with the dispersion relation \( \omega_k = k^2 + r_k \), with \( m_k = r_k \).

Finally, the fRG equation for the coupling \( r_k \) and \( g_k \) in equation (31), are obtained by means of equation (24) with \( \frac{d k}{d \phi} = 0 \) and the \( k \)-derivative of \( \Gamma_k \) given by equation (B16), setting \( n = 1 \) and \( n = 2 \), respectively.

**B.4. Case with \( \phi \neq 0 \) and \( n_y = 2 \)**

The construction of the fRG equations for the couplings \( g_{2n,k} \) which define the ansatz for the effective potential \( U_k \), given in equation (33), is discussed here for the case of non-vanishing background analyzed in section 3.3. According to the definition of \( \Sigma_k \) in equation (B5), one finds

\[
\Sigma_k(x) = -g_k \phi(t) \begin{pmatrix} \tilde{\phi}(x) & \rho(x) - \hat{\rho} \\ \rho(x) - \hat{\rho} & 0 \end{pmatrix}, \tag{B17}
\]

where we define

\[
\rho \equiv \phi^2, \quad \tilde{\rho} \equiv \phi \phi, \quad \text{and} \quad \hat{\rho} \equiv \frac{\phi^2}{2}, \tag{B18}
\]

while \( G_{0,k} \) is defined according to equation (B6) with \( m_k \) as in equation (B4) given by

\[
m_k = \frac{2}{3} \hat{\rho} g_k. \tag{B19}
\]
ensuing from definition (B4). The use of the \( \mathbb{Z}_2 \) invariants \( \rho \) and \( \bar{\rho} \) introduced in equation (B18) is customary in the context of fRG [39] and it helps in simplifying the notation in what follows. The form of \( \Sigma_k(x) \) in equation (B17) allows us to express the rhs of the fRG equation (27) as a power series of \( \rho - \bar{\rho}_k \); this is done in the spirit of the discussion below equation (30). Accordingly, together with the vertex expansion (23), this provides a way to unambiguously identify the renormalization of the terms appearing in the potential \( U_k \) in equation (33). In fact, \( \bar{\rho}_k \) and the coupling \( g_k \) are identified as [39]

\[
\frac{dU_k}{d\rho} \bigg|_{\rho = \bar{\rho}_k} = 0, \quad \frac{g_k}{3} = \frac{d^2U_k}{d\rho^2} \bigg|_{\rho = \bar{\rho}_k}, \quad \text{(B20)}
\]

where the first condition actually defines \( \bar{\rho}_k \) as the minimum of the potential. In terms of the effective action \( \Gamma_k \), equation (B20) become

\[
\frac{\delta \Gamma_k}{\delta \bar{\rho}} \bigg|_{\bar{\rho} = 0} = 0, \quad \text{and} \quad \frac{g_k}{3} = \frac{\delta^2 \Gamma_k}{\delta \bar{\rho} \delta \rho} \bigg|_{\bar{\rho} = 0}. \quad \text{(B21)}
\]

By taking a total derivative with respect to \( k \) of each equality in equation (B21), one finds

\[
\frac{\delta}{\delta \bar{\rho}} \frac{d\Gamma_k}{dk} \bigg|_{\bar{\rho} = 0} + \frac{\delta^2 \Gamma_k}{\delta \bar{\rho} \delta \rho} \bigg|_{\bar{\rho} = 0} \frac{d\bar{\rho}_k}{dk} = 0, \quad \text{(B22)}
\]

\[
\frac{1}{3} \frac{dg_k}{dk} = \frac{\delta^2}{\delta \bar{\rho} \delta \rho} \frac{d\Gamma_k}{dk} \bigg|_{\bar{\rho} = 0} + \frac{\delta^3 \Gamma_k}{\delta \bar{\rho} \delta \rho \delta \rho} \bigg|_{\bar{\rho} = 0} \frac{d\bar{\rho}_k}{dk}, \quad \text{(B23)}
\]

which, after replacing \( d\Gamma_k / dk \) with the fRG equation (19), render the flow equations for \( \bar{\rho}_k \) and \( g_k \). For the case of the potential \( U_k \) in equation (33), by using equation (B21), the set of flow equations (B22) and (B23) simplifies as

\[
\frac{d\bar{\rho}_k}{dk} = -3 \frac{\delta}{g_k} \frac{d\Gamma_k}{dk} \bigg|_{\bar{\rho} = 0} = -3 \frac{\delta \Delta \Gamma_{1,k}}{g_k} \bigg|_{\bar{\rho} = 0}, \quad \text{(B24)}
\]

\[
\frac{1}{3} \frac{dg_k}{dk} = \frac{\delta^2}{\delta \bar{\rho} \delta \rho} \frac{d\Gamma_k}{dk} \bigg|_{\bar{\rho} = 0} = \frac{\delta^2 \Delta \Gamma_{2,k}}{\delta \bar{\rho} \delta \rho} \bigg|_{\bar{\rho} = 0}, \quad \text{(B25)}
\]

where we used equation (27) with \( \Delta \Gamma_{1,k} \) and \( \Delta \Gamma_{2,k} \) defined in equation (28) in terms of the \( \Sigma_k(x) \) in equation (B17). The explicit form of the flow equations comes from a calculation analogous to the one discussed in section 3.2 and then detailed in appendix B2 (see equations (B9) and (B14)). In particular, the flow of \( m_k \), defined in equation (B19), takes contributions from the flow equations for both \( \bar{\rho}_k \) and \( g_k \).

Then, it is possible to construct the fRG flow equations for the dimensionless couplings

\[
\tilde{m}_k = \frac{2}{3} \bar{\rho}_k g_k, \quad \text{and} \quad \tilde{g}_k = \frac{a_d}{d} \frac{D}{Z_k K_k^2} \tilde{g}_k, \quad \text{(B26)}
\]

analogous to equation (32). Considering the solutions of the fRG equations with vanishing \( k \)-derivative of these dimensionless parameters one obtains equations similar to equation (32) that finally allows the determination of the fixed point values. In order to solve these fRG equations, one has to supplement them with those involving the anomalous dimension \( \eta \) (34), as one can see from the expression of the \( \Delta \Gamma_{n,k} \)’s terms computed in equations (B9b) and (B14).
B.5. Higher-order $\Delta \Gamma_{n,k}$’s terms

We discuss here what happens in the general case, i.e., with $n_T \geq 2$. Correspondingly, all the $\Delta \Gamma_{n,k}$’s terms with $n$ up to $n_T$ have to be computed. For these terms the associated time-dependent function $F_{n,k}$, that appear in equations (B9) and (B12) for $n = 1$ and $n = 2$ respectively, depend upon $i = 1, \ldots, n$ different times. The procedure for extracting the local contributions, with which we have obtained the last equation in equation (B14), has to be extended to all the times $t_i$ other than $t_1$. In order to retain linear contribution in $\tilde{\phi}$, as before, it suffices to replace $\phi(t_i) \to \phi(t_1)$. In fact, one can show that the structure of equations (B11c) and (B15c) is robust for higher-order terms ($n > 2$) and all of them leads to terms of the form

$$F_{n,k}^l(t_1) = \frac{D_k}{Z_k \omega_k^{n+1}} \left[ 1 - e^{-2\omega_k t_1/Z_k} f_n \left( \frac{\omega_k}{Z_k} \right) \right],$$

(B27)

with $f_n(x)$ being a polynomial in $x$. Accordingly, the exponential decay over time of the fRG flow equation (31) is a general feature within LPA approximation. Because of this, one can recover the long-time equilibrium fRG equations. This last equation is the proof of the statement done in the main text about the fact that any time-dependence in the fRG flow equation is of the exponentially decaying form as the one in equation (31); accordingly, the equilibrium flow equations are retrieved in the long-time limit, since $\eta_0 = \eta_Z$, as discussed around equation (34) in the main text.

In addition, by inspection of the time-dependent part of the $\Delta \Gamma_{n,k}$’s terms and based on previous considerations, an analytical formula for any localized kernel $F_{n,k}^l(t_1)$ can be obtained as

$$F_{n,k}^l(t_1) = \frac{2n+1}{n!} D_k \int_0^{t_1} ds \mathcal{R}_{0,k}^2(t_1, s)(t_1 - s)^n$$

(B28a)

$$= \frac{2n+1}{n!} D_k \frac{1}{d(-2\omega_k/Z_k)^n} F \left( \frac{\omega_k}{Z_k} \right),$$

(B28b)

where in the second equality $F$ is defined as

$$F \left( \frac{\omega_k}{Z_k} t_1 \right) = \int_0^{t_1} dt \mathcal{R}_{0,k}^2(t_1, t)$$

(B29a)

$$= \frac{1}{2Z_k \omega_k} \left( 1 - e^{-2\omega_k t_1/Z_k} \right),$$

(B29b)

and we used the analytical expression of $\mathcal{R}_{0,k}$ given by equation (B7a). We note that equation (B28a) reproduces equations (B11b) and (B15b) respectively for $n = 1$ and $n = 2$. Finally, equation (B29b) is simply obtained by computing the integral over $t$ in equation (B29a) with $\mathcal{R}_{0,k}$ given in equation (B7a).

The formulas in equations (B28b) and (B29b) are very useful in order to implement analytically the LPA approach discussed here for the case of higher-order terms ($n_T > 2$), thus
calculating the corresponding \( f_{nk}(x) \) from the comparison with equation (B27). Taking advantage of these analytical formulas a code in Mathematica has been used to compute the \( \Delta \Gamma_{k,n}'s \) terms for \( n \) up to \( n_{\text{max}}^{n} = 6 \), as explained in section 5.

### Appendix C. Long-time aging regime

In this appendix, we first detail how in the absence of the background field the aging regime for the response and the correlation functions, given by equation (41), is obtained. Then we consider the effect of the presence of a background field, detailing the way in which we obtain the expression for the reduced parameter \( r_{k}/Z_{k} \), needed for solving equation (44). Finally, we compare our results for \( \theta \) with those obtained in reference [32], proving the equivalence of these two approaches.

#### C.1. Case \( \tilde{\phi} = 0 \)

Here we detail the calculations that led to the predictions of the response and the correlation functions in the aging regime, given by equation (41), for the case of vanishing background field with \( n_{\text{tr}} = 2 \), starting from equation (36) and \( r \) given by equation (40). The response function in equation (41) is readily obtained in the long-time limit of the aging limit, solving equation (36a) with \( r(t) \) given by equation (40) with \( t > t' \gg \Lambda^{-2} \). For the correlation function it is less straightforward to extract the behavior in the aging regime: one can rewrite equation (36b) as

\[
\mathcal{C}(t,t') = 2\mathcal{R}(t,t')\int_{0}^{t'} ds\mathcal{R}^{2}(t',s), \tag{C1}
\]

where we used the identity \( \mathcal{R}(t,s) = \mathcal{R}(t,t')\mathcal{R}(t',s) \), which follows from the LPA approximation (36a). Accordingly, the integral in the equation above should be computed in the regime \( t' \gg \Lambda^{-2} \). This amount of studying

\[
\lim_{t' \to \infty} \int_{0}^{t'} ds\mathcal{R}^{2}(t',s), \tag{C2}
\]

where

\[
\mathcal{R}(t',s) = \exp\left(-\int_{s}^{t'} ds' r^s(s')\right). \tag{C3}
\]

The LPA prediction for the parameter \( r(t) \) in the case of a critical quench of the model is given by equation (38), that we report here

\[
r^s(t) = -\frac{\theta}{t}\left[1 - \tilde{F}_r(\Lambda^2 t)\right], \tag{C4}
\]

where \( \tilde{F}_r \) is a function which decays exponentially fast as \( t \) grows. Let us simplify the integral in equation (C2), taking advantage of equations (C3) and (C4):

\[
\int_{0}^{t'} ds\mathcal{R}^{2}(t',s) = t\int_{0}^{1} d\tau \tau^{-2\theta} \exp\left[-2\theta\int_{\tau}^{1} \frac{d\tau'}{\tau'} \tilde{F}_r(\Lambda^2 \tau')\right], \tag{C5}
\]

where, in the last equality, we made the following changes of variable: \( s = t' \tau \) and \( s' = t' \tau' \).
We recall that the aging regime is reached for $A = \Lambda \tau_0 \rightarrow \infty$. We break the integral in $\tau$ which appears in equation (C5) as $\int_0^1 d\tau = \int_0^{1/k} d\tau + \int_0^{A^{-1}} d\tau$. The integral $\int_0^{1/k} d\tau$ is given, in the aging limit, by $\int_0^1 d\tau \tau^{-2\theta} = (1 - 2\theta)^{-1}$, where we assumed that $\theta < 1/2$. In the following, we prove that the remaining integral over $\int_0^{A^{-1}} d\tau$ gives a vanishing contribution. To do so, we break the integral in $\tau'$ which appear in equation (C5), as $\int_0^1 d\tau' = \int_0^{1/k} d\tau' + \int_0^{A^{-1}} d\tau'$. Since the integral over $\int_0^{1/k} d\tau'$ converges, it gives an overall vanishing contribution when integrated over $\int_0^{A^{-1}} d\tau$ in the aging regime. Let us now focus on the remaining integral given by

$$\int_0^{A^{-1}} d\tau' \tau'^{-2\theta} \exp\left[-2 \theta \int_0^{A^{-1}} \frac{d\tau'}{\tau'} f(A\tau')\right]. \quad (C6)$$

With the change of variable $u = A\tau$ we obtain

$$A^{-1+2\theta} \int_0^1 du u^{-2\theta} \exp\left[-2 \theta \int_{u/A}^{1/k} \frac{d\tau'}{\tau'} f(A\tau')\right], \quad (C7)$$

and, as long as $-1 + 2\theta < 0$, this integral gives a vanishing contribution in the aging limit $A = \Lambda \tau_0 \rightarrow \infty$ to the limit expression given by equation (C2).

Summarizing, we have obtained

$$\lim_{t' \rightarrow \infty} \int_0^{t'} ds R^2(t', s) = \frac{t'}{1 - 2\theta}, \quad (C8)$$

which, when inserted in equation (C1), with equation (C4), gives equation (41), as anticipated in the main text.

C.2. Case $\bar{\phi} \neq 0$

Here we discuss how, in the presence of a background field, one can solve the equation (44) for the reduced two-time functions.

First, we prove that equations similar to equations (38) and (40) can be obtained for the reduced parameter $r/\bar{Z}$ which enters equation (44). In order to obtain these equations one can use a simplification that appears at the level of the flow equation for it, i.e.,

$$\frac{d}{dk} \frac{r_k}{\bar{Z}_k} = \frac{1}{\bar{Z}_k} \frac{dr_k}{dk} - \frac{\eta_k}{\bar{Z}_k} r_k. \quad (C9)$$

In fact, according to our previous analysis, we retain only the explicit time dependence of $G_{0,k}$ which appears in the $\Delta\Gamma_{n,k}$’s terms, as we have discussed in section 3.1 near equation (31). This amounts to the fact that the second term on the rhs in the previous equation is time-independent, thus it will simply renormalize $r_k$ and therefore the value it has to take in order to obtain a vanishing long-time limit of $r(t)$, as we did in order to obtain equation (38) from equation (31a) in section 4.2.

The fRG equation (C9) for the reduced parameter $r_k/\bar{Z}_k$ is obtained, e.g., for the case of the non-vanishing background field ansatz with $n_k = 2$, once the corresponding flow equation for $\phi_k$ and $g_k$ are derived, as explained in section 4.3. Accordingly, the flow equation of $r_k/\bar{Z}_k$ is proportional to field derivatives of $\Delta\Gamma_{1,k}$ and $\Delta\Gamma_{2,k}$, given by equations (B24) and (B25). Thus, it is proportional to $F_{1,k}$ and $F_{2,k}$, given explicitly by equations (B11c) and (B15c). These
two terms are equivalent for the analysis that follows, since their time dependence is always through $\omega_k t / Z_k$.

Considering only one of these terms, the flow equation for the reduced parameter $r_k / Z_k$ given by

$$\frac{1}{Z_k} \frac{dr_k(t)}{dk} = \tilde{A}_k k D_k \left[ 1 - F_k \left( \frac{\omega_k t}{Z_k} \right) \right], \quad (C10)$$

where $\tilde{A}_k$ is given by the dimensionless time-independent part of the corresponding fRG flow equation ($\tilde{A}_k = \tilde{g}_k / (1 + \tilde{r}_k)^2$ in equation (31a)), and $F_k$ is an exponentially vanishing function of its argument. In the vicinity of the infrared fixed point, i.e., for $k \to 0$, the factor $\omega_k i Z_k$ behaves as $\sim k^z$, while $k K_k Z_k D_k Z_k \sim k^{z-1}$, as a consequence of equation (34).

The critical initial-slip exponent $\theta$ is calculated from the integral over the cutoff $k$ of equation (C10) (see the discussion which leads to equation (39) and apply it to the case of non-vanishing background ansatz, i.e., replacing $r_k$ with its reduced version $r_k / Z_k$, as explained in section 4.3). In the vicinity of the infrared fixed point, one finds

$$-\frac{\theta}{t} \sim \lim_{t \to \infty} \frac{1}{Z_k} \frac{dr^*_k(t)}{dr}, \quad (C11a)$$

$$= \lim_{t \to \infty} \frac{1}{Z_k^{1/\tilde{A}_k}} \int_0^\Lambda dx \left[ -F_k^*(x) \right], \quad (C11b)$$

where $r^*_k$ means that the initial parameter $r_\Lambda$ is tuned in order to have $r^*(t) \to 0$ for $t \to \infty$, as explained in the main text around equation (37), and the substitution $x = k^z t$ is made in order to obtain equation (C11b). This equation is the proof of the statement done in the main text about the fact that relations similar to equations (38) and (40) are obtained also in the non-vanishing background field approximation for the reduced parameter $r / Z$.

### C.3. Comparison with the short-time analysis

Here we compare the predictions derived in section 4 for $\theta$ with those obtained in reference [32]. There, $\theta$ was calculated via an analysis of the short-time behavior, i.e., focusing on the limit in which the waiting time $t'$ is approximately the initial time $t_0$, i.e., $t' \sim t_0 = 0$. From general scaling arguments for the aging dynamics [35] the anomalous dimension $\eta_0$ of the boundary field $\tilde{\varphi}_0$ is related to $\theta$ by [32]

$$\theta = -\eta_0 / z. \quad (C12)$$

In the case with $\tilde{\varphi}_k = 0$ and $n_{\infty} = 2$ given by equation (29), $z = 2$, according to the fact that in the lowest order of LPA no anomalous dimension arises. Moreover, the anomalous dimension $\eta_0$ of the boundary field, given in reference [32] by equation (46) with $\tau_0 = +\infty$, matches the one which can be extracted from equation (39) by comparing it with equation (C12). Accordingly, the analysis of the short-time behaviour done in reference [32] and at long times presented here provide the same prediction for $\theta$, as it should be, given that the scaling functions (9) are attained in the regime $t / t' \gg 1$, which encompasses both cases. The equivalence between the two methods for calculating $\theta$ is also valid in the non-vanishing background field approximation, as described in what follows.
In the general case, it follows from the comparison of equation (C12) with equation (C11b) that

\[ \eta_0 = \tilde{A}^* \lim_{t \to \infty} \int_0^t \frac{dx}{x} \left[ -F^*_t(x) \right] . \]  

(C13)

From the previous identification of \( \tilde{A}_k \) (see below equation (C10)) and computing the integral over \( x \) from \( \infty \) to 0 of \( F^*(x) \) which appear in equation (C13), where \( F^*_t(x) = e^{-2x(1+\tilde{r}^2)} f_t(x) \) with \( f_t(x) = 1 + 2(1+\tilde{r})x \), it follows that \( \eta_0 = -\tilde{A}^*/(1+\tilde{r}) \); this expression is exactly equation (46) in reference [32] and equation (39) obtained here with \( z = 2 \).

In order to prove the equivalence between the two methods we consider the analytical formula that, in reference [32], has been used in order to calculate the anomalous dimension \( \eta_0 \) of the boundary fields \( \tilde{\varphi}_0 \), which is given by

\[ \eta_0 = -\int_0^\infty \frac{dk}{Z_k} \frac{d\bar{r}^2(k)}{dk} , \]

(C14)

where only the time dependent part in equation (C10), i.e., \( F_k(x) \) is retained. Accordingly, equation (C14) matches exactly with equation (C13) if the substitution \( x = k^2 t \) is made. The equivalence of the two approaches then follows.

**Appendix D. Expression of \( \theta \) for \( \partial_k \neq 0 \) and \( n_{tr} = 3 \)**

Here we provide the correct expression of \( \theta \) for the case of non-vanishing background field with \( n_{tr} = 3 \), fixing a mistake in reference [32] (see equation (45) and discussion around it). For completeness, we further provide the details which allow the numerical computation of \( \theta \).

First, the equations for the fixed-point dimensionless couplings \( m^*, g^* \) and \( \lambda^* \) defined above equation (32) are given by (the superscript * which henceforth we omit denotes fixed point values),

\[ 0 = (-2 + \eta) \tilde{m} + \left( 1 - \frac{\eta}{d+2} \right) \frac{2g}{1 + \tilde{m}^2} \times \left[ 1 + \frac{3}{2} \left( \frac{\tilde{m}^*}{g^2} \right)^2 + \frac{3m}{1 + \tilde{m}} \left( 1 + \frac{\tilde{m}^*}{g^2} \right)^2 \right] , \]

(D1a)

\[ 0 = \tilde{g} \left[ d - 4 + 2\eta + \left( 1 - \frac{\eta}{d+2} \right) \frac{6\tilde{g}}{1 + \tilde{m}^3} \left( 1 + \frac{\tilde{m} \lambda}{g^2} \right)^2 \right] \]

\[ + \left( 1 - \frac{\eta}{d+2} \right) \frac{\tilde{\lambda}}{(1 + \tilde{m})^3} \left( -2 + 3 \frac{\tilde{m} \lambda}{g^2} \right) , \]

(D1b)

\[ 0 = \tilde{\lambda} \left[ 2d - 6 + 3\eta + 30 \left( 1 - \frac{\eta}{d+2} \right) \frac{\tilde{g}}{(1 + \tilde{m})^{3/2}} \left( \frac{1 + \tilde{m} \lambda}{g^2} \right) \right] - 18 \left( 1 - \frac{\eta}{d+2} \right) \frac{\tilde{g}^3}{(1 + \tilde{m})^{3/2}} \left( 1 + \frac{\tilde{m} \lambda}{g^2} \right)^3 , \]

(D1c)
where \( \eta = \eta_k \) and the dimensionless couplings \( \tilde{m}_k \) and \( \tilde{g}_k \) are defined in equation (B26), while \( \tilde{\lambda}_k \) has been defined as

\[
\tilde{\lambda}_k = a_d \frac{D_k^2}{dZ_k^2K_k^3} \lambda_k^{d-2d}. \tag{D2}
\]

Note that in reference [32] this definition of \( \tilde{\lambda}_k \) was used, although the text therein reported a definition without the factor 5 at the denominator. Since equation (D1) depend upon the anomalous dimension \( \eta_K \), one has to supplement them with the equation for it, given by

\[
\eta_K = \frac{3\tilde{m}\tilde{g}}{(1 + \tilde{m})^4} \left( 1 + \frac{\tilde{m}\tilde{\lambda}}{\tilde{g}} \right)^2, \tag{D3a}
\]

\[
\eta_Z = \left( 1 - \frac{\eta_K}{d+2} \right) \frac{9\tilde{m}\tilde{g}}{2(1 + \tilde{m})^2} \left( 1 + \frac{\tilde{m}\tilde{\lambda}}{\tilde{g}} \right)^2, \tag{D3b}
\]

where we have reported also the anomalous dimension related to the derivative parameter \( Z_k \) and the noise term \( D_k \). From equation (D1), using equation (D3a), one can calculate numerically the fixed point values \((\tilde{m}^*, \tilde{g}^*, \tilde{\lambda}^*)\) depending on the spatial dimensionality \( d \), as we have done using equation (32) in the main text. We find numerically (using Wolfram Mathematica 12.3.1) the following fixed point values of the rescaled couplings in \( d = 3 \) (up to the second significant digit):

\[
\tilde{m}^* \simeq 0.30, \quad \tilde{g}^* \simeq 0.26, \quad \tilde{\lambda}^* \simeq 0.04. \tag{D4}
\]

The values \( \eta_{K,Z} \) of the anomalous dimensions at this fixed point are found by replacing directly equation (D4) into the expressions (D3a) and (D3b). The dynamical critical exponent \( z \) is given by the second of equation (34). In order to compute \( \theta \), we use the general scaling relation for \( \theta \), given by equation (C12), and the value of \( z \) obtained via a long-time analysis of the dynamics, one can focus on the anomalous dimension \( \eta_0 \) of the boundary field. Following the calculations discussed in appendices C2 and C3 one obtains

\[
\eta_0 = - \left( 1 - \frac{\eta}{d+2} \right) \frac{\tilde{g}}{(\tilde{m} + 1)^3} \left[ \frac{27\tilde{m}^2 \left( \frac{\tilde{m}}{\tilde{g}} + 1 \right)}{2(\tilde{m} + 1)^2} \right]^{3/2}
+ \frac{9\tilde{m}}{2} \left( 1 - \frac{\tilde{m}}{2\tilde{g}} \right) \left( \frac{\tilde{m}}{\tilde{g}} + 1 \right) \left[ 1 - \frac{3\tilde{m}}{2\tilde{g}} \right]. \tag{D5}
\]

We note that the analysis presented in reference [32] led to a wrong expression for \( \eta_0 \), given by equation (G6) (not reported here). In fact, as discussed here in the main text, they missed to add to \( r \) the term proportional to \( \lambda \) (see equation (45) for the correct equation for \( \eta_0 \)). The computation of \( \theta \) follows via equation (C12), once the dimensionless fixed-point values of \( m_k, g_k \) and \( \lambda_k \) are calculated. For instance, using the values equation (D4), one obtains \( \theta \) for \( d = 3 \). Finally the universal amplitude ratio \( X^\infty \) is calculated accordingly to equation (42).
ORCID iDs

Alessio Chiocchetta © https://orcid.org/0000-0003-1782-966X

References

[1] Privman V, Hohenberg P and Aharony A 1991 Phase Transitions and Critical Phenomena vol 14 ed C Domb and J L Lebowitz (New York: Academic)
[2] Henkel M, Hinrichsen H and Lübbeck S 2008 Non-Equilibrium Phase Transitions vol 1 (Berlin: Springer)
[3] Henkel M, Hinrichsen H, Pleimling M and Lübbeck S 2011 Non-Equilibrium Phase Transitions vol 2 (Berlin: Springer)
[4] Schmittmann B and Zia R 1995 Statistical Mechanics of Driven Diffusive Systems (Phase Transitions and Critical Phenomena vol 17) ed C Domb and J L Lebowitz (New York: Academic)
[5] Bouchaud J-P, Cugliandolo L F, Kurchan J and Mézard M 1997 Out of equilibrium dynamics in spin-glasses and other glassy systems Spin Glasses and Random Fields (Singapore: World Scientific) pp 161–223
[6] Hinrichsen H 2000 Adv. Phys. 49 815
[7] Täuber U C 2014 Critical Dynamics (Cambridge: Cambridge University Press)
[8] Mitra A, Takei S, Kim Y B and Millis A J 2006 Phys. Rev. Lett. 97 236808
[9] Scheppach C, Berges J and Gasenzer T 2010 Phys. Rev. A 81 033611
[10] Scholl J, Nowak B and Gasenzer T 2012 Phys. Rev. A 86 013624
[11] Langen T, Gasenzer T and Schmiedmayer J 2016 J. Stat. Mech. 064009
[12] Sieberer L M, Huber S D, Altman E and Diehl S 2013 Phys. Rev. Lett. 110 195301
[13] Altman E, Sieberer L M, Chen L, Diehl S and Toner J 2015 Phys. Rev. X 5 011017
[14] Nicklas E, Karl M, Höfer M, Johnson A, Muessel W, Strobel H, Tomkovič J, Gasenzer T and Oberthaler M K 2015 Phys. Rev. Lett. 115 245301
[15] Chiocchetta A, Tavora M, Gambassi A and Mitra A 2015 Phys. Rev. B 91 220302
[16] Chiocchetta A, Gambassi A, Diehl S and Marino J 2017 Phys. Rev. Lett. 118 135701
[17] Prüfer M, Kunkel P, Strobel H, Lannig S, Linnemann D, Schmied C-M, Berges J, Gasenzer T and Oberthaler M K 2018 Nature 563 217
[18] Erne S, Bücker R, Gasenzer T, Berges J and Schmiedmayer J 2018 Nature 563 225
[19] Young J T, Gorshkov A V, Foss-Feig M and Maghrebi M F 2020 Phys. Rev. X 10 011039
[20] Diesell O K, Diehl S and Chiocchetta A 2021 arXiv:2103.01947 [cond-mat.quant-gas]
[21] Bouchaud J-P, Farmer J D and Lillo F 2009 Handbook of Financial Markets: Dynamics and Evolution (Handbooks in Finance) ed T Hens and K R Schenk-Hoppé (Amsterdam: North-Holland)
[22] Szabó G and Czárán T 2001 Phys. Rev. E 63 061904
[23] Mobilia M, Georgiev I T and Täuber U C 2007 J. Stat. Phys. 128 447
[24] Fruchart M, Hanai R, Littlewood P B and Vitelli V 2021 Nature 592 363
[25] Janssen H K, Schaub B and Schmittmann B 1989 Z. Phys. B 73 539
[26] Janssen H K 1992 From Phase Transitions to Chaos: Topics in Modern Statistical Physics ed G Györgyi, I Kondor, L Sasvári and T Tél (Singapore: World Scientific)
[27] Calabrese P and Gambassi A 2005 J. Phys. A: Math. Gen. 38 R133
[28] Hohenberg P C and Halperin B I 1977 Rev. Mod. Phys. 49 435
[29] Calabrese P, Gambassi A and Krzakala F 2006 J. Stat. Mech. P06016
[30] Calabrese P and Gambassi A 2007 J. Stat. Mech. P01001
[31] Calabrese P and Gambassi A 2002 Phys. Rev. E 66 066101
[32] Chiocchetta A, Gambassi A, Diehl S and Marino J 2016 Phys. Rev. B 94 174301
[33] Dupuis N, Canet L, Eichhorn A, Metzner W, Pawlowski J M, Tissier M and Wschebor N 2021 The nonperturbative functional renormalization group and its applications Phys. Rep. 910 1
[34] Kloss T, Canet L and Wschebor N 2012 Phys. Rev. E 86 051124
[35] Calabrese P and Gambassi A 2004 J. Stat. Mech. P07013
[36] Cugliandolo L F, Kurchan J and Parisi G 1994 J. Phys. I 4 1641
[37] Godrèche C and Luck J M 2000 J. Phys. A: Math. Gen. 33 9141
[38] Berges J, Tetradis N and Wetterich C 2002 Phys. Rep. 363 223
[39] Delamotte B 2012 Lect. Notes Phys. 852 49
[40] Canet L and Chaté H 2007 J. Phys. A: Math. Theor. 40 1937
[41] Grassberger P 1995 Physica A 214 547
[42] Jaster A, Mainville J, Schülke L and Zheng B 1999 J. Phys. A: Math. Gen. 32 1395
[43] Prudnikov V V, Prudnikov P V and Mamonova M V 2017 Phys. Usp. 60 762
[44] Prudnikov V V, Prudnikov P V, Kalashnikov I A and Tsirkin S S 2008 J. Exp. Theor. Phys. 106 1095
[45] Litim D F 2002 Acta Phys. Slov. 52 181
[46] Bouchaud J-P, Cugliandolo L, Kurchan J and Mézard M 1996 Physica A 226 243
[47] Oerding K and Janssen H K 1993 J. Phys. A: Math. Gen. 26 5295
[48] Oerding K and Janssen H K 1993 J. Phys. A: Math. Gen. 26 3369
[49] Litim D F 2000 Phys. Lett. B 486 92