THE HOPF ALGEBRA OF (q)MULTIPLE POLYLOGARITHMS
WITH NON-POSITIVE ARGUMENTS

KURUSCH EBRAMI-FARD, DOMINIQUE MANCHON, AND JOHANNES SINGER

Abstract. We consider multiple polylogarithms in a single variable at non-positive integers. Defining a connected graded Hopf algebra, we apply Connes’ and Kreimer’s algebraic Birkhoff decomposition to renormalize multiple polylogarithms at non-positive integer arguments, which satisfy the shuffle relation. The q-analogue of this result is as well presented, and compared to the classical case.

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1. Introduction

Let \( n, k_1, \ldots, k_n \) be positive integers. Multiple polylogarithms (MPLs) in a single variable are defined by

\[
\text{Li}_{k_1,\ldots,k_n}(z) := \sum_{m_1 > \ldots > m_n \geq 0} \frac{z^{m_1}}{m_1^{k_1} \cdots m_n^{k_n}}
\]

when \( z \) is a complex number. The function \( \text{Li}_{k_1,\ldots,k_n}(z) \) is of depth \( \text{dpt}(k) := n \geq 2 \) and weight \( \text{wt}(k) := k_1 + \cdots + k_n \), for \( k := (k_1, \ldots, k_n) \). It is analytic in the open unit disk and, in the case \( k_1 > 1 \), continuous on the closed unit disk. In this case we observe the connection

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of MPLs and multiple zeta values (MZVs)

$$\zeta(k_1, \ldots, k_n) := \sum_{m_1 > \cdots > m_n > 0} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}} = \text{Li}_{k_1, \ldots, k_n}(z).$$

Equivalently we can define MPLs by induction on the weight \(w(t)\) as follows:

$$z \frac{d}{dz} \text{Li}_{k_1, \ldots, k_n}(z) = \text{Li}_{k_1 - 1, k_2, \ldots, k_n}(z) \quad \text{if} \quad k_1 > 1,$$

$$\left(1 - z\right) \frac{d}{dz} \text{Li}_{k_1, k_2, \ldots, k_n}(z) = \text{Li}_{k_2, \ldots, k_n}(z) \quad \text{if} \quad n > 1,$$

$$\text{Li}_{k_1, \ldots, k_n}(0) = 0.$$

Therefore one can observe an integral formula for MPLs using iterated Chen integrals. Indeed, let \(\varphi_1, \ldots, \varphi_p\) be complex-valued differential 1-forms defined on a compact interval. Then we define inductively for real numbers \(x\) and \(y\)

$$\int_x^y \varphi_1 \cdots \varphi_p := \int_x^y \varphi_1(t) \int_x^t \varphi_2 \cdots \varphi_p.$$

Now we set

$$\omega_{k_1, \ldots, k_n} := \omega_0^{k_1 - 1} \omega_1 \cdots \omega_0^{k_n - 1} \omega_1,$$

where

$$\omega_0(t) := \frac{dt}{t} \quad \text{and} \quad \omega_1(t) := \frac{dt}{1 - t}.$$

Using the differential equations (2), (3) and the initial conditions (4) we obtain

$$\text{Li}_{k_1, \ldots, k_n}(z) = \int_0^z \omega_{k_1, \ldots, k_n}$$

using the convention \(\text{Li}_0(z) = 1\). This representation gives rise to the well known shuffle products of MPLs and MZVs (see e.g. [Wal02, Wal11]).

Recall that the \(\mathbb{Q}\)-vector space spanned by MZVs forms an algebra equipped with two products. The quasi-shuffle product is obtained when one multiplies series (1) directly, which yields a linear combination of MZVs due to the product rule for sums. The aforementioned shuffle product between MZVs derives from integration by parts for iterated integrals. The resulting so-called double shuffle relations among MZVs arise from the interplay between these two products. An alternative characterization of MPLs can be given by the following formula

$$\text{Li}_{k_1, \ldots, k_n}(z) = J^{k_1}[y, J^{k_2}[y \cdots J^{k_n}[y] \cdots ]](z),$$

where \(y(z) := \frac{z}{1 - z}\) and \(J[f](z) := \int_0^z \frac{f(t)}{t} dt\) (see Lemma 3.2). Since \(J\) is a Rota–Baxter operator of weight zero, iterations of the operator \(J\) induce a product that coincides with the usual shuffle product for MPLs (see Lemma 3.7). For \(|z| < 1\) Equation (6) is valid for all \(k_1, \ldots, k_n \in \mathbb{Z}\). The inverse of \(J\) is given by \(J^{-1}[f](t) = \delta[f](t) := t^\frac{1}{2}\delta(t)\) (Proposition 3.1). We study the \(\mathbb{Q}\)-vector space

$$\mathcal{MP} := \langle z \mapsto \text{Li}_{-k_1, \ldots, -k_n}(z) : k_1, \ldots, k_n \in \mathbb{N}_0, n \in \mathbb{N}\rangle_\mathbb{Q},$$

which is indeed an algebra, where the product is induced by Equation (6) (see Lemma 3.5).
The algebra $\mathcal{MP}$ admits also an interpretation for MZVs at non-positive integers. Indeed, let $k_1,\ldots,k_n \in \mathbb{N}_0$. It is easily seen that $Li_{-k_1,\ldots,-k_n}(z)$ is convergent for $|z| < 1$ and divergent for $z = 1$. Nevertheless we can perceive the product induced for $|z| < 1$ as an analogue for the shuffle product of MZVs at non-positive integers. In order to make this connection more precise we have to establish a renormalization procedure. This permits us to extract explicit numbers for MZVs with non-positive arguments in a consistent way, such that they satisfy the shuffle product relations induced by the algebra structure of $\mathcal{MP}$.

We should keep in mind that a characterization of the shuffle product at non-positive integers – in contrast to the quasi-shuffle product – is a crucial point. Since the quasi-shuffle product is induced by the series representation of MZVs, the combinatorics is essentially the same as for positive arguments. On the other hand the shuffle product for positive indices is induced by the integral representation (5). The combinatorics behind this product comes from shuffling of integration variables. It could be illustrated by the shuffling of two decks of cards, say a deck of red and blue cards, each consecutively numbered such that the internal numbering of red and blue cards is preserved. In this approach, however, it is not clear how to handle non-positive arguments, which corresponds to a non-positive number of cards.

Extracting finite numbers for MZVs at non-positive integers is accomplished by the process of renormalization, which involves two steps:

(I) introduction of a regularization scheme,
(II) applying a subtraction method.

In step (I) we consider divergent MZVs $\zeta(-k_1,\ldots,-k_n)$ with $k_1,\ldots,k_n \in \mathbb{N}_0$, and introduce a so-called regularization parameter $z$, which systematically deforms the divergent MZV in order to obtain a meromorphic function in $z$ with the only singularity in $z = 0$. Step (II) involves a systematic procedure to eliminate singularities in terms of recursively defined subtractions. A rather natural way to achieve such eliminations is widely known as minimal subtraction scheme. The renormalization process is an integral part of perturbative quantum field theory (QFT). See e.g. [Col84]. The “right” choice of the regularization scheme in QFT is essential in the light of constraints coming from physics. In our context those constraints are of mathematical nature: The deformation of divergent MZVs has to be established in such a way that the regularized MZVs coincide with the meromorphic continuation of (M)ZVs. The recursively defined subtractions in step (II) involve combinatorial structures, which are concisely captured by the Connes–Kreimer Hopf algebraic approach to renormalization [CK00] [CK01] [Man04].

One of the key points in our approach is based on providing an adequate Hopf algebra together with an algebra morphism from that Hopf algebra into the space $\mathcal{MP}$ (Theorem 3.18), which permits to define a consistent renormalization process. Regarding regularization schemes, we will use the fact, that MPLs may be considered as regularized classical MZVs. However, we also consider a specific $q$-analogue of MZVs [OOZ12], where the variable $q$ takes the role of a natural regulator.

Remark 1.1. Renormalization of MZVs at non-positive integers appeared already in [GZ08] and [MP10]. The authors applied regularization schemes together with well-chosen subtraction methods suitable for preserving the quasi-shuffle product for renormalized MZVs – at non-positive arguments. Using Ecalle’s Mould calculus, Bouillot proposes in his work [Bon14]
a unifying picture of MZVs at non-positive arguments respecting the quasi-shuffle product. The common point of our approach with those presented in the aforementioned references is the use of the Connes–Kreimer Hopf algebraic approach to renormalization, and the corresponding algebraic Birkhoff decomposition, which encodes the subtraction procedure for singularities. However, we should emphasize that in our work it is the shuffle product, in a naturally extended sense, which is satisfied by renormalized MZVs at non-positive arguments.

In [CEM14b] the authors indicated that the $q$-parameter appearing in a specific $q$-analogue of MZVs [OOZ12] (see Equation (8) below) may be considered as a regularization parameter for MZVs at non-positive arguments. Indeed, we would like to demonstrate that under the $q$-parameter regularization [CEM14a, CEM14b, OOZ12] a proper renormalization of MZVs can be defined. The $q$-multiple polylogarithm ($q$MPL) in one variable is defined as

$$\text{Li}_{k_1,\ldots,k_n}^q(z) := \sum_{m_1>\ldots>m_n>0} \frac{z^{m_1}}{[m_1]_q^{k_1}\cdots[m_n]_q^{k_n}}$$

with $[m]_q := \frac{1-q^m}{1-q}$. It turns out that for $|z| < 1$ the series in (7) is convergent for $k_1,\ldots,k_n \in \mathbb{Z}$, and especially for $|q| < 1$ we obtain the formal power series

$$\mathfrak{Z}_q(k_1,\ldots,k_n) := \text{Li}_{k_1,\ldots,k_n}^q(q) = \sum_{m_1>\ldots>m_n>0} \frac{q^{m_1}}{[m_1]_q^{k_1}\cdots[m_n]_q^{k_n}} \in \mathbb{Z}[[q]].$$

These $q$MZWVs were introduced by Ohno, Okuda and Zudilin in [OOZ12], and further studied in [CEM14a, CEM14b], see also [Sin15, Sin14, Zha14]. For $k_1 > 1$ and $k_2,\ldots,k_n \geq 1$ we see that

$$\lim_{q \searrow 1} \mathfrak{Z}_q(k_1,\ldots,k_n) = \zeta(k_1,\ldots,k_n),$$

where $q \not\to 1$ means $q \to 1$ inside an angular sector $-\frac{\pi}{2} + \varepsilon \leq \text{Arg}(1-q) \leq \frac{\pi}{2} - \varepsilon$ with $\varepsilon > 0$ sufficiently small. It will be convenient for technical reasons to consider the modified $q$MZWVs introduced in [OOZ12]

$$\mathfrak{T}_q(k_1,\ldots,k_n) := (1-q)^{-(k_1+\cdots+k_n)} \mathfrak{Z}_q(k_1,\ldots,k_n).$$

They are used to establish a Hopf algebra structure on the space of modified $q$MPLs. The modification has to be reversed after renormalization, in order to relate the renormalized $q$MZWVs via (9) to renormalized MZWVs.

The paper is organized as follows. In Section 2 we recall the basic results on the meromorphic continuation of MZWVs. Section 3 contains the main result, i.e., the detailed construction of a Hopf algebra for MPLs at non-positive integers. A generalization of this finding to the $q$-analogue of MZWVs defined by Ohno, Okuda and Zudilin is presented as well. In Section 4 we recall the Hopf algebra approach to perturbative renormalization by Connes and Kreimer, and apply one of its main theorems to the renormalization of MPLs at non-positive integer arguments. The $q$-analogue of this result is as well presented, and compared to the classical case.

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2. Meromorphic Continuation of MZVs

In this section we review some well-known facts about the meromorphic continuation of MZVs. For \( n \in \mathbb{N} \) we consider the function

\[
\zeta_n : \mathbb{C}^n \to \mathbb{C}, \quad \zeta_n(s_1, \ldots, s_n) := \sum_{m_1 > \cdots > m_n > 0} \frac{1}{m_1^{s_1} \cdots m_n^{s_n}}.
\]

Proposition 2.1 ([Zha00]). The infinite sum (11) converges absolutely for \( \text{Re}(s_k) > 1 \) and \( \sum_{j=1}^k \text{Re}(s_j) > k, \ k = 1, \ldots, n \). In this domain \( \zeta_n \) defines an analytic function in \( n \) variables.

Theorem 2.2 ([AET01, AT01, Zha00]). The function \( \zeta_n(s_1, \ldots, s_n) \) admits a meromorphic extension to \( \mathbb{C}^n \). The subvariety \( S_n \) of singularities is given by

\[
S_n = \left\{ (s_1, \ldots, s_n) \in \mathbb{C}^n : s_1 = 1; s_1 + s_2 = 2, 1, 0, -2, -4, \ldots; \sum_{i=1}^j s_i \in \mathbb{Z}_{\leq j} \ (j = 3, 4, \ldots, n) \right\}.
\]

In the subsequent sections \( \zeta_n \) always denotes the meromorphic continuation of MZVs.

Remark 2.3. In this paper we discuss \( \zeta_n \) restricted to the set \( (\mathbb{Z}_{\leq 0})^n \). The Bernoulli numbers are defined by the following generating series

\[
\frac{te^t}{e^t - 1} = \sum_{m \geq 0} \frac{B_m}{m!} t^m.
\]

The first few values are \( B_0 = 1, B_1 = \frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}, B_5 = 0, \) etc., especially \( B_{2l+1} = 0 \) for \( l \in \mathbb{N} \). Therefore we have the following cases for \( \zeta_n \) restricted to non-positive arguments:

- Case \( n = 1 \): For \( l \in \mathbb{N}_0 \) we have the well known formula

  \[
  \zeta_1(-l) = \frac{B_{l+1}}{l+1}.
  \]

- Case \( n = 2 \): In the light of Theorem 2.2 we assume the sum \( k_1 + k_2 \) to be odd. Therefore we obtain from [AET01] that

  \[
  \zeta_2(-k_1, -k_2) = \frac{1}{2} (1 + \delta_0(k_2)) \frac{B_{k_1+k_2+1}}{k_1+k_2+1}.
  \]

- Case \( n \geq 3 \): From Theorem 2.2 we deduce that

  \[
  (\mathbb{Z}_{\leq 0})^n \subseteq S_n.
  \]

Therefore we obtain no information from the meromorphic continuation.

3. Algebraic Framework

We briefly introduce Rota–Baxter algebras, since they conveniently relate to shuffle-type products on word algebras. Two such shuffle products are presented, which encode products of MPLs and \( q \)-MPLs at integer arguments. The main result of this section is the construction of a graded connected commutative and cocommutative shuffle Hopf algebra for \((q)\)MPLs at non-positive integer arguments.
3.1. Rota–Baxter Algebra and multiple zeta values. Let $k$ be a ring, $\lambda \in k$ and $A$ a $k$-algebra. A Rota–Baxter operator (RBO) of weight $\lambda$ on $A$ over $k$ is a $k$-module endomorphism $L$ of $A$ such that

$$L(x)L(y) = L(xL(y)) + L(L(x)y) + \lambda L(xy)$$

for any $x, y \in A$. A Rota–Baxter $k$-algebra (RBA) of weight $\lambda$ is a pair $(A, L)$ with a $k$-algebra $A$ and a Rota–Baxter operator $L$ of weight $\lambda$ on $A$ over $k$. On the algebra of continuous functions $C(\mathbb{R})$ the integration operator

$$R: C(\mathbb{R}) \to C(\mathbb{R}), \quad R[f](z) := \int_0^z f(x) \, dx$$

is a RBO of weight zero, which is an immediate consequence of the integration by parts formula. We consider the $C$-algebra of power series

$$\mathcal{P}_{\geq 1} := \left\{ f(t) := \sum_{k \geq 1} a_k t^k : R_f \geq 1 \right\} \subseteq t \mathbb{C}[t]$$

without a term of degree zero in $t$, and radius of convergence, $R_f$, of at least 1. We define the operator

$$J: \mathcal{P}_{\geq 1} \to \mathcal{P}_{\geq 1}, \quad J[f](t) := \int_0^t f(z) \frac{dz}{z}. $$

Further the Euler derivation $\delta$ is given by

$$\delta: \mathcal{P}_{\geq 1} \to \mathcal{P}_{\geq 1}, \quad \delta[f](t) := \frac{\partial f}{\partial t}(t). $$

Proposition 3.1.

(i) The pair $(\mathcal{P}_{\geq 1}, J)$ is a RBA of weight $\lambda = 0$.

(ii) The operator $\delta$ is a derivation, i.e., $\delta[fg] = \delta[f]g + f\delta[g]$, for any $f, g \in \mathcal{P}_{\geq 1}$.

(iii) The operators $J$ and $\delta$ are mutually inverse, i.e., $J \circ \delta = \delta \circ J = \text{Id}.$

Proof. Statement (i) follows from integration by part. The second claim is straightforward to show. Finally, item (iii) is an immediate consequence of the fundamental theorem of calculus together with the fact that $f(0) = 0$ for any $f \in \mathcal{P}_{\geq 1}$. \hfill \square

Lemma 3.2. Let $k_1, \ldots, k_n$ be integers. Then $\text{Li}_{k_1,\ldots,k_n}(t) \in \mathcal{P}_{\geq 1}$, explicitly

$$\text{Li}_{k_1,\ldots,k_n}(t) = J^{k_1}[y, J^{k_2}[y \cdots J^{k_n}[y] \cdots]](t),$$

where $y(t) := \frac{t}{1-t} \in \mathcal{P}_{\geq 1}$.

Proof. Using the fact that $J^{-1} = \delta$ we prove the claim for $k := (k_1, \ldots, k_n) \in \mathbb{Z}^n$ by induction on its depth, $\text{dpt}(k) = n$. For $\text{dpt}(k) = 1$ we easily compute

$$J^k[y](t) = \left\{ \begin{array}{ll}
\sum_{m \geq 1} \frac{t^m}{m^k}, & \text{for } k \geq 0 \\
\sum_{m \geq 1} \frac{t^m}{m^k}, & \text{for } k < 0
\end{array} \right\} = \text{Li}_k(t)$$

for any $k \in \mathbb{Z}$. In the inductive step we get

$$J^{k_1}[y, J^{k_2}[y \cdots J^{k_n}[y] \cdots]](t) = J^{k_1} \left[ \sum_{m > 0} \sum_{m_2 > \cdots > m_n > 0} \frac{t^{m_2} \cdots t^{m_n}}{m_2 \cdots m_n} \right]$$
\[
= j^{k_1} \left[ \sum_{m_1 > m_2 > \cdots > m_n > 0} \frac{t^{m_1}}{m_1 m_2^2 \cdots m_n^2} \right] \\
= \sum_{m_1 > m_2 > \cdots > m_n > 0} \frac{t^{m_1}}{m_1 m_2^2 \cdots m_n^2} \\
= \text{Li}_{k_1, \ldots, k_n}(t)
\]

using the induction hypothesis. \(\square\)

This lemma gives rise to the following algebraic formalism. Let \(X_0 := \{j, d, y\}\), and \(W_0\) denotes the set of words on the alphabet \(X_0\), subject to the rule \(j d = d j = 1\), where \(1\) denotes the empty word. Therefore any word \(w \in W_0\) can be uniquely written in the canonical form

\[w = j^{k_1} y j^{k_2} y \cdots j^{k_n-1} y j^{k_n}\]

for \(k_1, \ldots, k_n \in \mathbb{Z}\) using the notation \(j^{-1} = d\) and \(j^0 = 1\). The length of the word \(w\) above is \(|w| = k_1 + \cdots + k_n + n - 1\). Further, \(A_0\) denotes the vector space \(A_0 := \langle W_0 \rangle_{\mathbb{Q}}\) spanned by the words in \(W_0\). Next we define the product \(\cup_0 : A_0 \times A_0 \to A_0\) by \(1 \cup_0 w := w \cup_0 1 := w\) for any word \(w \in W_0\), and recursively with respect to the sum of the length of two words in \(W_0\):

1. \(y u \cup_0 v := u \cup_0 y v := y(u \cup_0 v)\),
2. \(j u \cup_0 j v := j(u \cup_0 j v) + j(j u \cup_0 v)\),
3. \(d u \cup_0 d v := d(u \cup_0 d v) - u \cup_0 d^2 v\),
4. \(d u \cup_0 j v := d(u \cup_0 j v) - u \cup_0 j d v\),
5. \(j u \cup_0 d v := d(j u \cup_0 v) - u \cup_0 j v\).

**Remark 3.3.** • Note that (iv) can be deduced from (iii) by replacing \(v\) by \(j^2 v\).
• (iii) does not really define \(d u \cup_0 d v\) by induction on the sum of lengths of two words, because \(|d u| + |d v| = |u| + |d^2 v|\). Using (i) and writing \(u' = du = d^k y w\) for some \(k \geq 1\), we can however get a recursive definition by iterating (iii) as follows:

\[d^k y w \cup_0 d v = d(d^{k-1} y w \cup_0 d v - d^{k-2} y w \cup_0 d^2 v + \cdots + (-1)^{k-1} y w \cup_0 d^k v) + (-1)^k y (w \cup_0 d^{k+1} v).\]

**Lemma 3.4. The \(\mathbb{Q}\)-vector space**

\[\mathcal{T} := \langle j^{k_1} y j^{k_2} y \cdots j^{k_n-1} y j^{k_n} \in W_0 : k_n \neq 0, n \in \mathbb{N} \rangle_{\mathbb{Q}}\]

is a two sided ideal of \((A_0, \cup_0)\).

**Proof.** Let \(a \in \{j, d\}\) and \(u := u' a \in W_0\) and \(v \in W_0\). We prove \(u \cup_0 v \in T\) by induction on \(r := |u| + |v|\). The base cases are true because we observe for \(r = 1\) that \(d \cup_0 1 = d, j \cup_0 1 = j\) and for \(r = 2\) that

\[d \cup_0 d = 0, \quad j \cup_0 j = 2 j^2, \quad j \cup_0 d = d \cup_0 j = 0.\]

For the inductive step we have several cases:
• 1st case: \(a = y u d\) or \(a = y v\). This is an immediate consequence of (i) and the induction hypothesis.
• 2nd case: \( u = juu d \) and \( v = jv \). We observe using (ii) and the induction hypothesis that 
\[ juu d \cup_0 jv = j(\uu u d \cup_0 jv + juu d \cup_0 v) \in \mathcal{T}. \]

• 3rd case: \( u = duu d \) and \( v = jv \). We observe using (iv) and the induction hypothesis that 
\[ duu d \cup_0 jv = d(\uu u d \cup_0 jv) - uu d \cup_0 v \in \mathcal{T}. \]

• 4th case: \( u = juu d \) and \( v = dv \). We observe using (v) and the induction hypothesis that 
\[ juu d \cup_0 dv = d(juu d \cup_0 v) - uu d \cup_0 v \in \mathcal{T}. \]

• 5th case: \( u = duu d \) and \( v = dv \). We observe using (iii) that 
\[ duu d \cup_0 dv = d(\uu u d \cup_0 dv) - uu d \cup_0 dv. \]

By induction hypothesis the first term is an element of \( \mathcal{T} \). If \( \uu u d \) starts with a \( j \) or \( y \) we are in one of the above cases. Therefore we only consider the case, where \( \uu u d \) is a word consisting purely of \( d \), i.e., \( \uu u d = d^n \) for \( n \in \mathbb{N} \). Now we prove \( d^n \cup_0 d^m w = 0 \) for any \( w \in W_0 \) and \( m \in \mathbb{N} \). For \( n = 1 \) we have \( d \cup_0 d^m \uu u d = d(1 \cup_0 d^m \uu u d) - 1 \cup_0 d^{m+1} \uu u d = 0 \). Therefore we obtain by induction hypothesis that 
\[ d^{n+1} \cup_0 d^m \uu u d = d(d^n \cup_0 d^m \uu u d) - d^n \cup_0 d^{m+1} \uu u d = 0. \]

All in all we have shown that \( \cup_0 (\mathcal{T} \otimes \mathcal{A}_0) \subseteq \mathcal{T} \).

Since \( \cup_0 \) is not commutative we also have to prove \( v \cup_0 u \in \mathcal{T} \) by induction on \( r := |v| + |u| \).

The base cases are true. The first four cases are completely analogous to the first four cases above. We only discuss the following case: \( v = dv \) and \( u = duu d \).

We observe using (iii) that 
\[ dv \cup_0 duu d = dv(\cup_0 duu d) - \uu u d \cup_0 \uu u d. \]

By induction hypothesis the first term is an element of \( \mathcal{T} \). If \( \uu u d \) starts with a \( j \) or \( y \) we are in one of the above cases. Again only the case \( \uu u d = d^n \) for \( n \in \mathbb{N} \) has to be considered. By the same induction as in the previous case we obtain that the last term is zero. This proves \( \cup_0 (\mathcal{A}_0 \otimes \mathcal{T}) \subseteq \mathcal{T} \). The proof is now complete. \( \square \)

Let \( Y_0 := \{1\} \cup W_0y \) be the set of admissible words, i.e. words which do not end up with a \( j \) or a \( d \). It is easily seen that \( \mathcal{A}_0' := Y_0/\mathbb{Q} \) is a subalgebra of \((\mathcal{A}_0, \cup_0)\) isomorphic to \( \mathcal{A}_0/\mathcal{T} \). A priori, the product \( \cup_0 \) on \( \mathcal{A}_0 \) is neither commutative nor associative. Now let \( \mathcal{L} \) (resp. \( \mathcal{L}' \)) be the ideal of \( \mathcal{A}_0 \) (resp. \( \mathcal{A}_0' \)) generated by 
\[ \{j^k(d(u \cup_0 v) - duu d \cup_0 v - u \cup_0 dv), u, v \in W_0y, k \in \mathbb{Z}\}. \]

Let \( \mathcal{B}_0 \) (resp. \( \mathcal{B}_0' \)) be the quotient algebra \( \mathcal{A}_0/\mathcal{L} \) (resp. \( \mathcal{A}_0'/\mathcal{L}' \)). We obviously have the isomorphism:
\[ \mathcal{B}_0' \sim \mathcal{A}_0/(\mathcal{T} + \mathcal{L}). \]

**Proposition 3.5.** The pair \((\mathcal{B}_0, \cup_0)\) is a commutative, associative and unital algebra.

**Proof.** We first prove commutativity \( u' \cup_0 v' = v' \cup_0 u' \in \mathcal{L} \) by induction on \( r = |u'| + |v'| \). The cases \( r = 0 \) and \( r = 1 \) are immediate. Several cases must be considered:

• 1st case: \( u' = yu \) or \( v' = yv \). The induction hypothesis immediately applies, using (i).

• 2nd case: \( u' = ju \) and \( v' = jv \). Then we have by induction hypothesis:
\[ ju \cup_0 jv - jv \cup_0 ju = j(ju \cup_0 v + u \cup_0 jv - jv \cup_0 u - v \cup_0 ju) \in \mathcal{L}. \]
• 3rd case: $u' = du$ and $v' = jv$ or vice-versa. We have then:

$$du \uplus_0 jv - jv \uplus_0 du = d(u \uplus_0 jv) - u \uplus_0 v - d(jv \uplus_0 u) + v \uplus_0 u$$

$$= d(u \uplus_0 jv - jv \uplus_0 u) - (u \uplus_0 v - v \uplus_0 u),$$

which belongs to $\mathcal{L}$ by induction hypothesis.

• 4th case: $u' = du$ and $v' = dv$. Then $dv \uplus_0 du - d(dv \uplus_0 u) + d^2v \uplus_0 u \in \mathcal{L}$, hence:

$$du \uplus_0 dv - dv \uplus_0 du = d(u \uplus_0 dv) - u \uplus_0 d^2v - d(dv \uplus_0 u) + d^2v \uplus_0 u \mod \mathcal{L}$$

$$= d(dv \uplus_0 du - dv \uplus_0 u) - (u \uplus_0 d^2v - d^2v \uplus_0 u) \mod \mathcal{L}.$$ 

The first term belongs to $\mathcal{L}$ by induction hypothesis. We further suppose that $u'$ is written $d^k yw$ for some $k \geq 1$ and $w \in Y_0$. Iterating the process using (iii) we finally get $dv \uplus_0 du = (-1)^k(yw \uplus_0 d^{k+1}v - d^{k+1}v \uplus_0 yw) \mod \mathcal{L}$. We are then back to the first case.

Associativity follows by showing $u' \uplus_0 (v' \uplus_0 w') = (u' \uplus_0 v') \uplus_0 w'$ via induction on the sum $|u'| + |v'| + |w'|$. If one of the words is the empty one nothing is to show. Now let $u' = au$, $v' = bv$ and $w' = cw$ with $a, b, c \in \{d, j, y\}$.

• 1st case: one of the letters is $y$, for example $u' = yu$. Using the induction hypothesis, we obtain

$$(yu \uplus_0 v') \uplus_0 w' = (y(u \uplus_0 v')) \uplus_0 w' = y((u \uplus_0 v') \uplus_0 w')$$

$$= y(u \uplus_0 (v' \uplus_0 w')) \mod \mathcal{L}$$

$$= yu \uplus_0 (v' \uplus_0 w') \mod \mathcal{L}.$$ 

Note that the other cases $v' = yv$ or $w' = yw$ are similar, and the arguments are completely analogous.

• 2nd case: $a = b = c = j$. On the one hand we have

$$(ju \uplus_0 jv) \uplus_0 jw = j((u \uplus_0 jv) \uplus_0 jw) + j((ju \uplus_0 v) \uplus_0 jw)$$

$$+ j((ju \uplus_0 jv) \uplus_0 w) + j((ju \uplus_0 v) \uplus_0 w)$$

$$= j((u \uplus_0 jv) \uplus_0 jw) + j((ju \uplus_0 v) \uplus_0 jw) + j((ju \uplus_0 jv) \uplus_0 w),$$

on the other hand

$$ju \uplus_0 (jv \uplus_0 jw) = j(u \uplus_0 jv \uplus_0 jw) + j(u \uplus_0 jv \uplus_0 w)$$

$$+ j(ju \uplus_0 (v \uplus_0 jw)) + j(ju \uplus_0 (jv \uplus_0 w))$$

$$= j(u \uplus_0 (jv \uplus_0 jw)) + j(ju \uplus_0 (v \uplus_0 jw)) + j(ju \uplus_0 (jv \uplus_0 w)).$$

Hence $(ju \uplus_0 jv) \uplus_0 jw = ju \uplus_0 (jv \uplus_0 jw) \mod \mathcal{L}.$

• 3rd case: two $j$’s and one $d$. On the one hand we have

$$(ju \uplus_0 jv) \uplus_0 dw = d(j(u \uplus_0 jv) \uplus_0 w) - (u \uplus_0 jv) \uplus_0 w$$

$$+ d(jju \uplus_0 v) \uplus_0 w) - (ju \uplus_0 v) \uplus_0 w$$

$$= d((ju \uplus_0 jv) \uplus_0 w) - (u \uplus_0 jv) \uplus_0 w - (ju \uplus_0 v) \uplus_0 w,$$
on the other hand
\[ jw \equiv_0 (jv \equiv_0 dw) = ju \equiv_0 d(jv \equiv_0 w) - ju \equiv_0 (v \equiv_0 w) \]
\[ = d(ju \equiv_0 (jv \equiv_0 w)) - u \equiv_0 (jv \equiv_0 w) - ju \equiv_0 (v \equiv_0 w). \]
then \((ju \equiv_0 jv) \equiv_0 dw = ju \equiv_0 (jv \equiv_0 dw) \mod L.

- 4th case: two \(d\)'s and one \(j\). We have to prove
\[ (du \equiv_0 dv) \equiv_0 dw = du \equiv_0 (dv \equiv_0 jw) \mod L. \]
It suffices to show that
\[ (d^k yu \equiv_0 dv) \equiv_0 dw = d^k yu \equiv_0 (dv \equiv_0 jw) \mod L \]
with \(u \in W_0\) and \(k \in \mathbb{N}\). Using (iii) we observe
\[ (d^k yu \equiv_0 dv) \equiv_0 jw = d((d^{k-1} yu \equiv_0 dv) \equiv_0 jw) \equiv_0 jw - (d^{k-1} yu \equiv_0 d^2 v) \equiv_0 jw \]
\[ = d((d^{k-1} yu \equiv_0 dv) \equiv_0 jw) \equiv_0 jw - (d^{k-1} yu \equiv_0 d^2 v) \equiv_0 jw \mod L \]
\[ = d^{k-1} yu \equiv_0 (dv \equiv_0 jw) + d^{k-1} yu \equiv_0 (d^2 v \equiv_0 jw) - (d^{k-1} yu \equiv_0 d^2 v) \equiv_0 jw \mod L. \]
For the difference of the two terms in the previous line to belong to \(L\), it suffices to prove
\[ d^{k-1} yu \equiv_0 (d^2 v \equiv_0 jw) = (d^{k-1} yu \equiv_0 d^2 v) \equiv_0 jw \mod L. \]
Applying the above procedure iteratively this could be reduced to
\[ yu \equiv_0 (d^k v \equiv_0 jw) = (yu \equiv_0 d^k v) \equiv_0 jw \mod L, \]
which is true by using (i) and the induction hypothesis.

- 5th case: \(a = b = c = d\). We have to prove
\[ (du \equiv_0 dv) \equiv_0 dw = du \equiv_0 (dv \equiv_0 dw) \mod L. \]
It suffices to show that
\[ (d^k yu \equiv_0 dv) \equiv_0 dw = d^k yu \equiv_0 (dv \equiv_0 dw) \mod L, \]
with \(u \in W_0\) and \(k \in \mathbb{N}\). Using (iii) we observe
\[ (d^k yu \equiv_0 dv) \equiv_0 dw = d((d^{k-1} yu \equiv_0 dv) \equiv_0 dw) \equiv_0 dw - (d^{k-1} yu \equiv_0 d^2 v) \equiv_0 dw \]
\[ = d((d^{k-1} yu \equiv_0 dv) \equiv_0 dw) \equiv_0 dw - (d^{k-1} yu \equiv_0 d^2 v) \equiv_0 dw \mod L \]
\[ = d^k yu \equiv_0 (dv \equiv_0 dw) + d^{k-1} yu \equiv_0 (d^2 v \equiv_0 dw) + d^{k-1} yu \equiv_0 (dv \equiv_0 d^2 w) \]
\[ - (d^{k-1} yu \equiv_0 d^2 v) \equiv_0 dw - (d^{k-1} yu \equiv_0 dv) \equiv_0 d^2 w \mod L. \]
Iteratively applying this procedure leads – as in the 4th case – to the claim using (i) and the induction hypothesis. Proposition 3.5 is thus proven.

Now we define the map \(\zeta^\omega_t: \mathcal{B}'_0 \to \mathbb{Q}[\![t]\!]\) by \(\zeta^\omega_t(1) := 1\), and for any \(k_1, \ldots, k_n \in \mathbb{Z}\),
\[ j^{k_1} y \cdots j^{k_n} y \mapsto \zeta^\omega_t(j^{k_1} y \cdots j^{k_n} y) := L_{k_1, \ldots, k_n}(t). \]

Lemma 3.6. The map \(\zeta^\omega_t\) is multiplicative, i.e. is an algebra morphism.
Proof. From Proposition \(3.1\) (ii) and (iii) we obtain for any \(f, g \in \mathcal{P}_{\geq 1}\)
\[\delta[Jfg] = J[f]g + J[g]f.\]
Therefore the definition of \(\sqcup_0\) and Proposition \(3.1\) (i), (ii), (iii) and (12) imply that
\[\zeta'_t : \mathcal{B}'_0 \rightarrow \mathbb{Q}[t], \quad j^{k_1} y \cdots j^{k_n} y \mapsto J^{k_1} [y] \cdots J^{k_n} [y] \cdots (t)\]
with \(k_1, \ldots, k_n \in \mathbb{Z}\) is an algebra morphism. \(\square\)

Next we show that if we restrict the shuffle product \(\sqcup_0\) to admissible words corresponding to positive arguments we obtain the ordinary shuffle product. Let \(\mathcal{C} := \mathbb{Q}1 \oplus j\mathbb{Q}(j, y)y\) and \(\mathcal{D} := \mathbb{Q}1 \oplus x_0 \mathbb{Q}(x_0, x_1) x_1\).

Lemma 3.7. The algebras \((\mathcal{C}, \sqcup_0)\) and \((\mathcal{D}, \sqcup)\) are isomorphic, where \(\sqcup\) denotes the ordinary shuffle product.

Proof. It is easily seen that \(\Phi: (\mathcal{D}, \sqcup) \rightarrow (\mathcal{C}, \sqcup_0)\) given by \(1 \mapsto 1\) and
\[x_0^{k_1-1} x_1 x_0^{k_2-1} x_1 \cdots x_0^{k_n-1} x_1 \mapsto j^{k_1} y j^{k_2} y \cdots j^{k_n} y\]
is an algebra morphism, for \(k_1, \ldots, k_n \in \mathbb{N}\), with \(k_1 > 1, n \in \mathbb{N}\). Since \(\Phi\) is bijective the proof is complete. \(\square\)

3.2. \(q\)-multiple zeta values. For a formal power series \(f \in \mathbb{Q}[\lbrack t \rbrack]\) we define the \(q\)-dilation operator as \(E_q[f](t) := f(qt)\). Let \(\mathcal{A} := t\mathbb{Q}[\lbrack t, q \rbrack]\) be the space of formal power series in the variables \(t\) and \(q\), without a term of degree zero in \(t\). We can interpret \(\mathcal{A}\) as the \(\mathbb{Q}[\lbrack q \rbrack]\)-algebra \(\mathbb{Q}[\lbrack \lbrack t \rbrack \rbrack]\). Then the \(\mathbb{Q}[\lbrack q \rbrack]\)-linear map \(P_q: \mathcal{A} \rightarrow \mathcal{A}\) is defined by
\[P_q[f](t) := \sum_{n \geq 0} E_q^n [f](t),\]
Furthermore, the \(q\)-difference operator \(D_q: \mathcal{A} \rightarrow \mathcal{A}\) is defined as \(D_q := \text{Id} - E_q\). We have the following known result:

Proposition 3.8 (\cite{CEM14b}).

(i) The pair \((\mathcal{A}, P_q)\) is a RBA of weight \(\lambda = -1\).
(ii) For any \(f, g \in \mathcal{P}_{\geq 1}\) the operator \(D_q\) satisfies the generalized Leibniz rule, i.e.
\[D_q[fg] = D_q[f]g + f D_q[g] - D_q[f] D_q[g].\]
(iii) The operators \(P_q\) and \(D_q\) are mutually inverse, i.e. \(D_q \circ P_q = P_q \circ D_q = \text{Id}\).

Remark 3.9. Recall that the Jackson integral
\[\mathcal{J}[f](x) := \int_0^x f(y) dy = (1 - q) \sum_{n \geq 0} f(q^n x) q^n x\]
is the \(q\)-analogue of the classical indefinite Riemann integral \(R\). For functions \(\frac{f(x)}{x}\) – where the Jackson integral is well defined – it reduces to
\[(1 - q) \sum_{n \geq 0} f(q^n x) = \int_0^x \frac{f(y)}{y} dy = (1 - q) P_q[f](x),\]
which is the \(q\)-analogue of the integral operator \(J\). Correspondingly, the \(q\)-analogue of the Euler derivation \(\delta\) reduces to \((\text{Id} - E_q)\).
Lemma 3.10 (CEM14b). Let $k_1, \ldots, k_n \in \mathbb{Z}$. Then we have
$$\bar{\zeta}_q(k_1, \ldots, k_n) = P_q^{k_1}[yP_q^{k_2}[y \cdots P_q^{k_n}[y] \cdots]](q).$$

Surprisingly enough, the algebraic formalism for $q$MZVs is simpler than in the classical case. Let $X_{-1} := \{p, d, y\}$. By $W_{-1}$ we denote the set of words on the alphabet $X_{-1}$, subject to the rule $pd = dp = 1$, where $1$ denotes the empty word. Again, $A_{-1}$ denotes the algebra spanned by the words in $W_{-1}$, i.e., $A_{-1} := \langle W_{-1} \rangle_{\mathbb{Q}}$. Then we define the product $\sqcup_1 : A_{-1} \otimes A_{-1} \to A_{-1}$ by $\sqcup_1 w := w \sqcup_1 1 := w$ for any $w \in W_{-1}$, and for any words $u, v \in W_{-1}$

(i) $yu \sqcup_1 v := u \sqcup_1 yv := y(u \sqcup_1 v)$,
(ii) $pu \sqcup_1 pv := p(u \sqcup_1 pv) + p(pu \sqcup_1 v) - p(u \sqcup_1 v)$,
(iii) $du \sqcup_1 dv := u \sqcup_1 dv + du \sqcup_1 v - d(u \sqcup_1 v)$,
(iv) $du \sqcup_1 pv = pv \sqcup_1 du := d(u \sqcup_1 pv) + du \sqcup_1 v - u \sqcup_1 v$.

Remark 3.11. We can deduce (iv) from (iii).

Lemma 3.12. The pair $(A_{-1}, \sqcup_1)$ is a commutative, associative and unital algebra.

Proof. The proof is similar to that of [CEM14b, Theorem 7], and left to the reader. □

Next we introduce the set of words ending in the letter $y$ and containing the empty word
$$Y_{-1} := W_{-1}y \cup \{1\} \subseteq W_{-1},$$
subject to the rule $pd = dp = 1$. Note that $(\langle Y_{-1} \rangle_{\mathbb{Q}}, \sqcup_1)$ is a subalgebra of $(A_{-1}, \sqcup_1)$. Moreover we introduce the map $\bar{\zeta}_q^\omega : \langle Y_{-1} \rangle_{\mathbb{Q}} \to \mathbb{Q}[q]$ by
$$p^{k_1}y \cdots p^{k_n}y \mapsto \bar{\zeta}_q^\omega(p^{k_1}y \cdots p^{k_n}y) := \bar{\zeta}_q(k_1, \ldots, k_n)$$
for any integers $k_1, \ldots, k_n$.

Lemma 3.13 (CEM14b). The map $\bar{\zeta}_q^\omega$ is an algebra morphism.

3.3. General Word Algebraic Part. In this section we explore the algebraic structure that is related to non-positive arguments for MZVs and $q$MZVs. For this reason we introduce a parameter $\lambda \in \mathbb{Q}$. The case $\lambda = 0$ corresponds to MZVs and the case $\lambda = -1$ to (modified) $q$MZVs.

Let $L := \{d, y\}$ be an alphabet of two letters. The free monoid of $L$ with empty word $1$ is denoted by $L^\ast$. We denote the free algebra of $L$ by $\mathbb{Q}\langle L \rangle$ and define the subspace of words ending in $d$ by
$$\mathcal{T}_- := \mathcal{T} \cap \mathbb{Q}\langle L \rangle = \langle \{wd : w \in L^\ast\} \rangle_{\mathbb{Q}} \subseteq \mathbb{Q}\langle L \rangle,$$
with $\mathcal{T}$ defined in Lemma 3.4. The set of admissible words is defined as
$$Y := L^\ast y \cup \{1\},$$
and the $\mathbb{Q}$-vector space spanned by $Y$ is notated as $\mathcal{H} := \langle Y \rangle_{\mathbb{Q}}$. It is isomorphic to the quotient $\mathbb{Q}\langle L \rangle / \mathcal{T}_-$. The weight $\text{wt}(w)$ of a word $w \in Y$ is given by the number of letters of $w$, and we use the convention $\text{wt}(1) := 0$. Furthermore, the depth $\text{dpt}(w)$ of a word $w \in Y$ is given by the number of $y$ in $w$. The $\mathbb{Q}$-vector space $\mathcal{H}$ is graded by depth, i.e.
$$\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_{(n)}$$
with $\mathcal{H}_{(n)} := \langle w \in Y : \text{dpt}(w) = n \rangle_{\mathbb{Q}}$. 
3.3.1. The algebra $\mathcal{H}_{\lambda}$, $\lambda \neq 0$. Let $\lambda \in \mathbb{Q}\setminus \{0\}$. We define the product

$$\shuffle_{\lambda} : \mathbb{Q}\langle L \rangle \otimes \mathbb{Q}\langle L \rangle \to \mathbb{Q}\langle L \rangle$$

iteratively by

(P1) $\shuffle_{\lambda} w := w \shuffle_{\lambda} \mathbf{1} := w$ for any $w \in L^{*}$;
(P2) $\gamma u \shuffle_{\lambda} v := u \shuffle_{\lambda} \gamma v := y(u \shuffle_{\lambda} v)$ for any $u, v \in L^{*}$;
(P3) $d_{u} \shuffle_{\lambda} d_{v} := \frac{1}{\lambda}[d(u \shuffle_{\lambda} v) - d_{u} \shuffle_{\lambda} v - u \shuffle_{\lambda} d_{v}]$ for any $u, v \in L^{*}$.

Furthermore, we define the unit map $\eta : \mathbb{Q} \to \mathbb{Q}\langle L \rangle$, $\mathbf{1} \mapsto \mathbf{1}$.

Proposition 3.14. For $\lambda \in \mathbb{Q}$, the triple $(\mathbb{Q}\langle L \rangle, \shuffle_{\lambda}, \eta)$ is a commutative, associative, and unital $\mathbb{Q}$-algebra. The subspace $\mathcal{T}_{-}$ is a two-sided ideal of $\mathbb{Q}\langle L \rangle$.

Proof. In the case $\lambda = -1$ the proof is a consequence of Lemma 3.12. We give a proof for any $\lambda \neq 0$ for completeness, although it could be derived from the case $\lambda = -1$ by appropriate rescaling. Commutativity is clear from the definition. We only have to verify associativity if all words begin with a letter $d$. We apply induction on the sum of the lengths of the words. The base case is trivial. For the inductive step we observe for $a, b, c \in L^{*}$ that

$$(da \shuffle_{\lambda} db) \shuffle_{\lambda} dc = \frac{1}{\lambda} [d(a \shuffle_{\lambda} b) - da \shuffle_{\lambda} b - a \shuffle_{\lambda} db] \shuffle_{\lambda} dc$$

and

$$da \shuffle_{\lambda} (db \shuffle_{\lambda} dc) = \frac{1}{\lambda} [da \shuffle_{\lambda} [d(b \shuffle_{\lambda} c) - db \shuffle_{\lambda} c - b \shuffle_{\lambda} dc]]$$

which shows associativity. From definition we obtain that $\shuffle_{\lambda}(\mathcal{T}_{-} \otimes \mathbb{Q}\langle L \rangle) = \shuffle_{\lambda}(\mathbb{Q}\langle L \rangle \otimes \mathcal{T}_{-}) \subseteq \mathcal{T}_{-}$ and therefore $\mathcal{T}_{-}$ is a two-sided ideal of $\mathbb{Q}\langle L \rangle$. $\square$

3.3.2. The coproduct $\Delta_{\lambda}$, $\lambda \in \mathbb{Q}$. Now we define the coproduct

$$\Delta_{\lambda} : \mathbb{Q}\langle L \rangle \to \mathbb{Q}\langle L \rangle \otimes \mathbb{Q}\langle L \rangle$$

by

(C1) $\Delta_{\lambda}(y) := \mathbf{1} \otimes y + y \otimes \mathbf{1}$,
(C2) \( \Delta_\lambda(d) := 1 \otimes d + d \otimes 1 + \lambda d \otimes d \),

which extends uniquely to an algebra morphism (with respect to concatenation) on the free algebra \( Q\langle L \rangle \). The counit map \( \varepsilon : Q\langle L \rangle \to Q \) is given by \( \varepsilon(1) = 1 \) and \( \varepsilon(w) = 0 \) for any word \( w \in L^* \setminus \{1\} \).

**Example 3.15.** We have

\[
\Delta_\lambda(dy) = 1 \otimes dy + dy \otimes 1 + d \otimes y + y \otimes d + \lambda dy \otimes d + \lambda d \otimes dy.
\]

**Proposition 3.16.** For \( \lambda \in Q \) the triple \((Q\langle L \rangle, \Delta_\lambda, \varepsilon)\) is a cocommutative and counital coalgebra, and \( T_- \) is a coideal of \( Q\langle L \rangle \).

**Proof.** Cocommutativity is clear by definitions (C1) and (C2). The counit axiom is not hard to verify. Finally we have to check coassociativity. We have

\[
(\text{Id} \otimes \Delta_\lambda)\Delta_\lambda(d) = (\text{Id} \otimes \Delta_\lambda)(1 \otimes d + d \otimes 1 + \lambda d \otimes d)
\]

\[
= 1 \otimes 1 \otimes d + 1 \otimes d \otimes 1 + \lambda 1 \otimes d \otimes d + d \otimes 1 \otimes 1 + \lambda d \otimes 1 \otimes d + \lambda d \otimes d \otimes 1 + \lambda^2 d \otimes d \otimes d
\]

and

\[
(\Delta_\lambda \otimes \text{Id})\Delta_\lambda(d) = (\Delta_\lambda \otimes \text{Id})(1 \otimes d + d \otimes 1 + \lambda d \otimes d)
\]

\[
= 1 \otimes 1 \otimes d + 1 \otimes d \otimes 1 + d \otimes 1 \otimes 1 + \lambda d \otimes d \otimes 1 + \lambda 1 \otimes d \otimes d + \lambda d \otimes 1 \otimes d + \lambda^2 d \otimes d \otimes d.
\]

The case \((\text{Id} \otimes \Delta_\lambda)\Delta_\lambda(y) = (\Delta_\lambda \otimes \text{Id})\Delta_\lambda(y)\) is easy to see. We immediately obtain

\[
\Delta_\lambda(T_-) \subseteq T_- \otimes Q\langle L \rangle + Q\langle L \rangle \otimes T_-,
\]

which concludes the proof. \( \square \)

3.3.3. **Compatibility properties of the coproduct** \((\lambda \neq 0 \text{ case})\).

**Lemma 3.17.** For words \( u, v \in L^* \) we have

\[
\Delta_\lambda(y)[\Delta_\lambda(u) \triangledown \Delta_\lambda(v)] = \Delta_\lambda(yu) \triangledown \Delta_\lambda(v) = \Delta_\lambda(u) \triangledown \Delta_\lambda(yv)
\]

and

\[
\Delta_\lambda(d)[\Delta_\lambda(u) \triangledown \Delta_\lambda(v)] = \Delta_\lambda(du) \triangledown \Delta_\lambda(v) + \Delta_\lambda(u) \triangledown \Delta_\lambda(dv) + \lambda[\Delta_\lambda(du) \triangledown \Delta_\lambda(dv)].
\]

**Proof.** Using (P2) and Sweedler’s notation, \( \Delta_\lambda(u) = \sum_{(u)} u_1 \otimes u_2 \), we obtain for the first equality of (14)

\[
\Delta_\lambda(y)[\Delta_\lambda(u) \triangledown \Delta_\lambda(v)] = \Delta_\lambda(y) \left[ \sum_{(u),(v)} (u_1 \triangledown u_2 \triangledown v_1) \otimes (u_2 \triangledown u_1 \triangledown v_2) \right]
\]

\[
= \sum_{(u),(v)} [(u_1 \triangledown v_1) \otimes y(u_2 \triangledown v_2) + y(u_1 \triangledown v_1) \otimes (u_2 \triangledown v_2)]
\]

\[
= \sum_{(u),(v)} [(u_1 \triangledown v_1) \otimes (yu_2 \triangledown v_2) + (yu_1 \triangledown v_1) \otimes (u_2 \triangledown v_2)]
\]

\[
= \Delta_\lambda(yu) \triangledown \Delta_\lambda(v).
\]

The second equality follows completely analogously. For (15) we observe

\[
\Delta_\lambda(d)[\Delta_\lambda(u) \triangledown \Delta_\lambda(v)]
\]
On the one hand Theorem 3.18. which concludes the proof. 

\[
\Delta_\lambda(d) \left[ \sum_{(u),(v)} (u_1 \shuffle_{\lambda} v_1) \otimes (u_2 \shuffle_{\lambda} v_2) \right] \\
= \sum_{(u),(v)} \left[ d(u_1 \shuffle_{\lambda} v_1) \otimes (u_2 \shuffle_{\lambda} v_2) + (u_1 \shuffle_{\lambda} v_1) \otimes d(u_2 \shuffle_{\lambda} v_2) + \lambda d(u_1 \shuffle_{\lambda} v_1) \otimes d(u_2 \shuffle_{\lambda} v_2) \right] \\
= \sum_{(u),(v)} \left[ (1 \otimes d + d \otimes 1 + \lambda d \otimes d)(u_1 \otimes u_2) \shuffle_{\lambda} (v_1 \otimes v_2) \right] \\
+ \sum_{(u),(v)} \left[ (u_1 \otimes u_2) \shuffle_{\lambda} (1 \otimes d + d \otimes 1 + \lambda d \otimes d)(v_1 \otimes v_2) \right] \\
+ \lambda \sum_{(u),(v)} \left[ (1 \otimes d + d \otimes 1 + \lambda d \otimes d)(u_1 \otimes u_2) \shuffle_{\lambda} (1 \otimes d + d \otimes 1 + \lambda d \otimes d)(v_1 \otimes v_2) \right] \\
= \Delta_\lambda(du) \shuffle_{\lambda} \Delta_\lambda(v) + \Delta_\lambda(u) \shuffle_{\lambda} \Delta_\lambda(dv) + \lambda \left[ \Delta_\lambda(du) \shuffle_{\lambda} \Delta_\lambda(dv) \right], \\
\]

which yields the claim. \( \square \)

3.3.4. The Hopf algebra \( H_\lambda, \lambda \neq 0 \).

**Theorem 3.18.** The quintuple \( H_\lambda = (H, \shuffle_{\lambda}, \eta, \Delta_\lambda, \varepsilon) \) is a Hopf algebra with

\[
\Delta_\lambda(w) := \Delta_\lambda(w) \mod (T_+ \otimes \mathbb{Q}\langle L \rangle + \mathbb{Q}\langle L \rangle \otimes T_-)
\]

for any word \( w \in Y \), where \( H = \mathbb{Q}\langle L \rangle T_- \) is always identified with \( \langle Y \rangle \mathbb{Q} \).

**Proof.** On the one hand \( H \) is the quotient by a two-sided ideal \( T_- \) and therefore \( (H, \shuffle_{\lambda}, \eta) \) is an algebra. On the other hand \( T_- \) is a coideal. Hence, \( (H, \Delta_\lambda, \varepsilon) \) is a coalgebra. Since \( H \) is connected it suffices to prove that \( (H, \shuffle_{\lambda}, \eta, \Delta_\lambda, \varepsilon) \) is a bialgebra. We show that

\[
\Delta_\lambda(u' \shuffle_{\lambda} v') = \Delta_\lambda(u') \shuffle_{\lambda} \Delta_\lambda(v') \mod (T_+ \otimes \mathbb{Q}\langle L \rangle + \mathbb{Q}\langle L \rangle \otimes T_-)
\]

by induction on the sum of weights \( \text{wt}(u) + \text{wt}(v) \), of the words \( u, v \in Y \). The base cases are straightforward.

- 1st case: \( u' = yu \) or \( v' = yv \). We have with Lemma [3.17]

  \[
  \Delta_\lambda(yu \shuffle_{\lambda} v') = \Delta_\lambda(y(u \shuffle_{\lambda} v')) = \Delta_\lambda(y) \Delta_\lambda(u \shuffle_{\lambda} v') \\
  = \Delta_\lambda(y) \Delta_\lambda(u) \shuffle_{\lambda} \Delta_\lambda(v') \mod (T_+ \otimes \mathbb{Q}\langle L \rangle + \mathbb{Q}\langle L \rangle \otimes T_-) \\
  = \Delta_\lambda(yu) \shuffle_{\lambda} \Delta_\lambda(v') \mod (T_+ \otimes \mathbb{Q}\langle L \rangle + \mathbb{Q}\langle L \rangle \otimes T_-).
  \]

- 2nd case: \( u' = du \) and \( v' = dv \). We have with Lemma [3.17] and the induction hypothesis

  \[
  \Delta_\lambda(du \shuffle_{\lambda} dv) = \frac{1}{\lambda} \left[ \Delta_\lambda(d(u \shuffle_{\lambda} v) - du \shuffle_{\lambda} v - u \shuffle_{\lambda} dv) \right] \\
  = \frac{1}{\lambda} \left[ \Delta_\lambda(d) \Delta_\lambda(u) \shuffle_{\lambda} \Delta_\lambda(v) - \Delta_\lambda(du) \shuffle_{\lambda} \Delta_\lambda(v) \\
  - \Delta_\lambda(u) \shuffle_{\lambda} \Delta_\lambda(dv) \right] \mod (T_+ \otimes \mathbb{Q}\langle L \rangle + \mathbb{Q}\langle L \rangle \otimes T_-) \\
  = \frac{1}{\lambda} \left[ \Delta_\lambda(du) \shuffle_{\lambda} \Delta_\lambda(v) + \Delta_\lambda(u) \shuffle_{\lambda} \Delta_\lambda(dv) + \lambda \Delta_\lambda(du) \shuffle_{\lambda} \Delta_\lambda(dv) \\
  - \Delta_\lambda(du) \shuffle_{\lambda} \Delta_\lambda(v) - \Delta_\lambda(u) \shuffle_{\lambda} \Delta_\lambda(dv) \right] \mod (T_+ \otimes \mathbb{Q}\langle L \rangle + \mathbb{Q}\langle L \rangle \otimes T_-) \\
  = \Delta_\lambda(du) \shuffle_{\lambda} \Delta_\lambda(dv) \mod (T_+ \otimes \mathbb{Q}\langle L \rangle + \mathbb{Q}\langle L \rangle \otimes T_-),
  \]

which concludes the proof. \( \square \)
Example 3.19. For $n\in\mathbb{N}$ we have $\Delta_{\lambda}(y^n) = \sum_{i=0}^{n} \binom{n}{i} y^i \otimes y^{n-i}$, and
\[
\Delta_{\lambda}(d^m y) = 1 \otimes d^m y + d^n y \otimes 1;
\]
\[
\Delta_{\lambda}(yd^n y) = 1 \otimes yd^m y + y \otimes d^n y + d^n y \otimes y + yd^n y \otimes 1;
\]
\[
\Delta_{\lambda}(dy^m y) = 1 \otimes dy^m y + y \otimes d^n y + d^n y \otimes dy + dy \otimes d^n y + d^n y \otimes dy + \lambda dy \otimes d^n y + \lambda d^n y \otimes dy.
\]

3.3.5. Compatibility between the product and the coproduct ($\lambda = 0$ case). Let us now focus on the case $\lambda = 0$. Recall from Paragraph 3.11 that $L$ is the two-sided ideal of the (noncommutative and nonassociative) algebra $(A_0, \mathfrak{m}_0)$ generated by the elements
\[
j^k(d(u \mathord{\shuffle} v) - du \mathord{\shuffle} v - u \mathord{\shuffle} dv), \quad k \in \mathbb{Z}, \ u, v \in W_0y.
\]
Now let $L_-$ be the two-sided ideal of the (noncommutative and nonassociative) subalgebra $(\mathbb{Q}\langle L \rangle, \mathfrak{m}_0)$ generated by the elements
\[
d^k(d(u \mathord{\shuffle} v) - du \mathord{\shuffle} v - u \mathord{\shuffle} dv), \quad k \in \mathbb{N}_0, \ u, v \in L^*.
\]
Further let $L_0^2 := L_- \otimes \mathbb{Q}\langle L \rangle + \mathbb{Q}\langle L \rangle \otimes L_-.$

Proposition 3.20. For any $u', v' \in L^*$ we have:
\[
\overline{\Delta}_0(u' \mathord{\shuffle} v') = \overline{\Delta}_0(u') \mathord{\shuffle} \overline{\Delta}_0(v') \mod L_0^2.
\]

Proof. We use induction on $r := |u'| + |v'|$. The cases $r = 0$ and $r = 1$ being immediate. The case $u' = yu$ or $v' = yv$ is easy and left to the reader. In the case $u' = du$ and $v' = dv$ we compute:
\[
\overline{\Delta}_0(du \mathord{\shuffle} dv) = \overline{\Delta}_0(du \mathord{\shuffle} dv) - u \mathord{\shuffle} d^2 v
\]
\[
= \overline{\Delta}_0(d \overline{\Delta}_0(u \mathord{\shuffle} dv) - u \mathord{\shuffle} d^2 v)
\]
\[
= \overline{\Delta}_0(d \overline{\Delta}_0(u) \mathord{\shuffle} \overline{\Delta}_0(dv)) - \overline{\Delta}_0(u \mathord{\shuffle} d^2 v) \mod L_0^2.
\]

hence we get:
\[
\overline{\Delta}_0(du \mathord{\shuffle} dv) - \overline{\Delta}_0(du \mathord{\shuffle} dv) = -\left(\overline{\Delta}_0(u \mathord{\shuffle} d^2 v) - \overline{\Delta}_0(u \mathord{\shuffle} d^2 v) \right) \mod L_0^2.
\]
Iterating this process we return to the case when one of the arguments starts with a $y$. 

3.3.6. The Hopf algebra $H_0$. Let $H_0 := \mathbb{Q}\langle L \rangle/(L_- + T_-)$.

Proposition 3.21. The ideal $L_-$ is a coideal of $(\mathbb{Q}\langle L \rangle, \overline{\Delta}_0)$, where $\overline{\Delta}_0$ is defined by $(C1)$ and $(C2)$ with $\lambda = 0$.

Proof. Using Proposition 3.20 we compute
\[
\overline{\Delta}_0(du \mathord{\shuffle} dv) = \overline{\Delta}_0(d \overline{\Delta}_0(u \mathord{\shuffle} dv)) - \overline{\Delta}_0(du \mathord{\shuffle} dv) \mod L_0^2.
\]
\[
= \overline{\Delta}_0(d \overline{\Delta}_0(u) \mathord{\shuffle} \overline{\Delta}_0(v)) - \overline{\Delta}_0(du \mathord{\shuffle} dv) \mod L_0^2.
\]
Hence, $\overline{\Delta}_0(du \mathord{\shuffle} dv) \in L_0^2$. 

Corollary 3.22. \( \mathcal{H}_0 \) is a commutative Hopf algebra.

Proof. From Proposition \(3.16\) and Proposition \(3.21\) we get that the ideal \( \mathcal{T}_- + \mathcal{L}_- \) is also a coideal of \( \mathbb{Q}\langle \mathcal{L} \rangle \). Hence the quotient \( \mathcal{H}_0 \) is a bialgebra. It is graded by the depth (and weight), hence connected, thus \( \mathcal{H}_0 \) is a Hopf algebra.

We will denote by \( \Delta_0 \) the coproduct on \( \mathcal{H}_0 \). By a slight abuse of notation the coproduct on \( \mathbb{Q}\langle \mathcal{L} \rangle / \mathcal{T}_- \), which was deduced from \( \overline{\Delta}_0 \), will also be denoted \( \Delta_0 \). Note that the ideal \( \mathcal{T}_- + \mathcal{L}_- \) is also graded by depth. One then gets a grading on the quotient \( \mathcal{H}_0 \), which we still denote by \( \text{dpt} \).

3.3.7. Shuffle factorization. Let \( \lambda \in \mathbb{Q} \), including the case \( \lambda = 0 \). From connectedness we can always write

\[
\Delta_\lambda([w]) = 1 \otimes [w] + [w] \otimes 1 + \Delta_\lambda([w]) \quad \text{with} \quad \Delta_\lambda([w]) \in \bigoplus_{p+q = n, p \neq 0, q \neq 0} \mathcal{H}(p) \otimes \mathcal{H}(q).
\]

Therefore in the following we use two variants of Sweedler’s notation

\[
\Delta_\lambda([w]) = \sum_{(\lambda)} [w]_1 \otimes [w]_2 \quad \text{and} \quad \tilde{\Delta}_\lambda([w]) = \sum_{(\lambda)} [w]' \otimes [w]''.
\]

The following theorem, valid for any \( \lambda \), including \( \lambda = 0 \), provides a nice example of the theory outlined in [Pat93].

Theorem 3.23. Let \( \lambda \in \mathbb{Q} \). Then for all \( w \in \mathbb{L}^* \) we have

\[
\varpi_\lambda \circ \Delta_\lambda([w]) = 2^{\text{dpt}([w])}[w],
\]

where \( [w] \) stands for the class of \( w \) modulo \( \mathcal{T}_- \) in the case \( \lambda \neq 0 \) (resp. modulo \( \mathcal{T}_- + \mathcal{L}_- \) in the case \( \lambda = 0 \)).

Proof. We prove this by induction on the weight of \( [w] \). For \( \text{wt}([w]) = 0 \) we have \( [w] = 1 \) and obtain \( \varpi_\lambda \circ \Delta_\lambda(1) = \varpi_\lambda(1 \otimes 1) = 1 \). For the inductive step we consider two cases:

1. 1st case: \( w = yv \) with \( v \in \mathbb{L}^* \)
We have \( \text{dpt}([w]) = \text{dpt}([v]) + 1 \) and obtain

\[
\varpi_\lambda \circ \Delta_\lambda([yv]) = \varpi_\lambda \circ (\Delta_\lambda([y])\Delta_\lambda([v])) = \varpi_\lambda \left( \sum_{(\lambda)} [v]_1 \otimes [yv]_2 + \sum_{(\lambda)} [yv]_1 \otimes [v]_2 \right)
= \sum_{(\lambda)} \varpi_\lambda ([v]_1 \otimes [yv]_2) + \sum_{(\lambda)} \varpi_\lambda ([yv]_1 \otimes [v]_2)
= 2[y](\varpi_\lambda \circ \Delta_\lambda([v]))
= 2^{\text{dpt}([v]) + 1}[yv] = 2^{\text{dpt}([w])}[w].
\]

2. 2nd case: \( w = dv \) with \( v \in \mathbb{L}^* \)
Since \( \text{dpt}([w]) = \text{dpt}([v]) \) we observe

\[
\varpi_\lambda \circ \Delta_\lambda([dv]) = \varpi_\lambda \circ (\Delta_\lambda([d])\Delta_\lambda([v]))
= \varpi_\lambda \left( \sum_{(\lambda)} [dv]_1 \otimes [v]_2 + [v]_1 \otimes [dv]_2 + \lambda([dv]_1 \otimes [dv]_2) \right)
= \varpi_\lambda \left( \sum_{(\lambda)} [dv]_1 \otimes [v]_2 + [v]_1 \otimes [dv]_2 + \lambda([dv]_1 \otimes [dv]_2) \right)
\]
Let \( S \) (resp. in \( Q \)) both.

The linear map \( K : \mathbb{1} \to \mathcal{H} \) is defined by

\[
K = [d] \left( \sum_{\{v\}} \varpi_{\lambda} ([v]_1 \otimes [v]_2) \right) = [d] \left( \varpi_{\lambda} \circ \Delta_{\lambda}([v]) \right)
\]

\[= 2^{dpt([v])} [dv] = 2^{dpt([w])} [w].\]

\[\square\]

**Corollary 3.24.** Let \( \lambda \in \mathbb{Q} \). Then for all \( w \in L^* \), we have

\[
(2^{dpt([w])} - 2) [w] = \varpi_{\lambda} \circ \tilde{\Delta}_{\lambda}([w])
\]

\[
= \sum_{\{v\}} [w]' \varpi_{\lambda} [w]'' = K \cdot K([w]).
\]

The linear map \( K := \text{Id} - \eta \circ \varepsilon \in \text{End}_{\mathbb{Q}}(\mathcal{H}_{\lambda}) \) is a projector to the augmentation ideal \( \mathcal{H}' := \bigoplus_{n>0} \mathcal{H}_n \), and \( f \ast g := \varpi_{\lambda} \circ (f \otimes g) \circ \Delta_{\lambda}, f, g \in \text{End}_{\mathbb{Q}}(\mathcal{H}_{\lambda}). \)

3.3.8. A combinatorial description of the coproduct \( \Delta_{\lambda} \). In the following we give a combinatorial description of the coproduct \( \Delta_{\lambda} \). However, note that we consider the construction only on an admissible representative \( w \in Y \) of a given equivalence class in \( \mathcal{Q}(L)/\mathcal{T}_{-} \) for \( \lambda \neq 0 \) (resp. in \( \mathcal{Q}(L)/((\mathcal{L}_{-} + \mathcal{T}_{-}) \) for \( \lambda = 0 \).

Let \( w := d^{n_1-1}y \cdots d^{n_k-1}yd^{n_k-1}y \in Y \) be a word, with \( n := \sum_{i=1}^{k} n_i \), and define for \( 1 \leq m \leq k \), \( N^m_w := \{n_1, n_1 + n_2, \ldots, n_1 + \cdots + n_m\} \). The coproduct \( \Delta_{\lambda}(w) \) can be calculated as follows. Let \( S := \{s_1 < \cdots < s_l \} \subseteq \{n\} := \{1, \ldots, n\} \) and \( S := [n] \setminus S = \{s_1 < \cdots < s_{n-l}\} \).

Define the words \( w_S := w_{s_1} \cdots w_{s_l} \) and \( w_{\bar{S}} := w_{s_1} \cdots w_{s_{n-l}} \). The set \( S \) is called admissible if both \( w_S \) and \( w_{\bar{S}} \) are in \( Y \), i.e., if \( s_1, \ldots, s_{n-l} \in N^k_w \) with \( w \in Y \). The coproduct is then given by

\[\Delta_{\lambda}(w) = \sum_{\mathcal{S} \subseteq \{n\}, \mathcal{S} \text{ adm}} w_S \otimes w_{\bar{S}} + \sum_{\mathcal{S} \subseteq \{n\}, \mathcal{S} \text{ adm}} \sum_{\mathcal{J} \subseteq \{1, \ldots, n\}, \mathcal{J} \neq \emptyset, \mathcal{J} \cap \mathcal{N}_w^k = \emptyset} \lambda^{[\mathcal{J}]} w_S \otimes w_{[n]\setminus(S\cup(J)).}
\]

For \( \lambda = 0 \) this reduces to the coproduct corresponding to MPLs at non-positive integer arguments

\[\Delta_0(w) := \sum_{\mathcal{S} \subseteq \{n\}, \mathcal{S} \text{ adm}} w_S \otimes w_{\bar{S}}.
\]

We introduce now a graphical notation, which should make the above more transparent. The set of vertices \( V := \{\bullet, \circ\} \) is used to define a polygon. The black vertex \( \bullet \sim d \) and the white one \( \circ \sim y \). To each admissible word \( w = d^{n_1-1}y \cdots d^{n_k-1}yd^{n_k-1}y \in Y \) corresponds an polygon with clockwise oriented edges, and the vertices colored clockwise according to the word \( w \). For instance, the word \( w = dydy \) corresponds to

![Polygon](https://example.com/polygon.png)

An admissible subset \( S \subseteq \{n\} \) corresponds to a sub polygon. The admissible subsets for the above example are as follows: \( \{1, 3\}, \{2, 3\}, \{4, 5\} \) correspond respectively to

![Sub Polygons](https://example.com/sub_polygons.png)
\{1, 2, 3\}, \{2, 4, 5\}, \{1, 4, 5\}\ correspond\ respectively\ to\ 

\{1, 2, 4, 5\} and \{3\} correspond\ respectively\ to\ 

In\ the\ last\ polygon\ we\ have\ marked\ the\ single\ vertex\ subpolygon\ by\ a\ circle\ around\ the\ white\ vertex.\ The\ coproduct\ 

\[
\Delta_0(dddy) = \sum_{S \subseteq [n]} w_S \otimes w_{\bar{S}}
\]

\[
= dddy \otimes 1 + 1 \otimes dddy + 3ddy \otimes ddy + 3ddy \otimes dy + ddy \otimes y + y \otimes ddy.
\]

The\ first\ two\ terms\ on\ the\ right-hand\ side\ correspond\ to\ \(S = [n]\)\ and\ \(S = \emptyset\),\ respectively.\ The\ pictorial\ description\ of\ the\ weight-\(\lambda\)\ coproduct\ is\ captured\ as\ follows.\ The\ second\ term\ on\ the\ right-hand\ side\ of\ the\ coproduct\ reflects\ the\ term\ \(\lambda d \otimes d\)\ in\ the\ coproduct\ 

\[
\overline{\Delta}_\lambda(d) := 1 \otimes d + d \otimes 1 + \lambda d \otimes d
\]

defined\ further\ above.\ It\ amounts\ to\ a\ certain\ doubling\ of\ those\ black\ vertices\ in\ an\ admissible\ word\ \(w = d^{n_1-1}y \cdots d^{n_k-1}y d^{n_k-1}y \in Y\),\ which\ appear\ before\ the\ \(y\)\ at\ position\ \(n_1 + \cdots + n_{k-1}\).\ Algebraically\ this\ means\ that\ extracting\ a\ subpolygon\ corresponding\ to\ the\ admissible\ subset\ \(S' \subseteq [n]\)\ leads\ to\ a\ splitting\ of\ the\ word\ \(w\)\ into\ \(w_S\)\ and\ \(w_{\bar{S}}\),\ where\ the\ augmented\ complement\ sets\ \(\bar{S}'\)\ contain\ \(\bar{S}\),\ i.e.,\ \(\bar{S} \subseteq \bar{S}'\).\ This\ is\ due\ to\ not\ eliminating\ several\ black\ vertices,\ i.e.,\ \(d's\)\ that\ appear\ before\ the\ \(y\)\ at\ position\ \(n_1 + \cdots + n_{k-1}\).\ Pictorially\ we\ denoted\ this\ by\ doubling\ black\ vertices.\ Returning\ to\ the\ example\ above.\ We\ find\ for\ the\ admissible\ word\ \(w = dddy\)

\[
\begin{align*}
\text{which\ correspond\ to\ the\ admissible\ subsets\ and\ related\ augmented\ complement\ sets:}\ \{1, 3\}, \ &\bar{S}' = \{1, 2, 4, 5\},\ \text{and}\ \{2, 3\}, \ &\bar{S}' = \{1, 2, 4, 5\},\ \text{respectively.\ Next\ we\ consider}\ \\
\text{where\ we\ have}\ \{2, 4, 5\}, \ &\bar{S}' = \{1, 2, 3\},\ \text{and}\ \{1, 4, 5\}, \ &\bar{S}' = \{1, 2, 3\},\ \text{respectively.\ For}\ \\
\text{we\ have}\ \{1, 2, 3\}, \ &\bar{S}' = \{1, 4, 5\},\ \text{and}\ \{1, 2, 3\}, \ &\bar{S}' = \{2, 4, 5\},\ \text{and}\ \{1, 2, 3\}, \ &\bar{S}' = \{1, 2, 4, 5\},\ \text{respectively.\ Finally}\ 
\end{align*}
\]
where we have \( \{1, 2, 4, 5\} \), \( \bar{S}' = \{1, 3\} \), and \( \{1, 2, 4, 5\} \), \( \bar{S}' = \{2, 3\} \), and \( \{1, 2, 4, 5\} \), \( \bar{S}' = \{1, 2, 3\} \), respectively.

The coproduct
\[
\Delta_{\lambda}(ddydy) = \sum_{S \subseteq [n]} w_S \otimes w_{\bar{S}} + \sum_{S \subseteq [n]} \sum_{J = \{j_1 < \cdots < j_p\} \subseteq S, J \neq \emptyset, J \cap N_{ddydy} = \emptyset} \lambda(J) w_S \otimes w_{[5]}(S \setminus J)
\]
\[
= ddydy \otimes 1 + 1 \otimes ddydy + 3dy \otimes ddy + 3ddy \otimes dy + dddy \otimes y + y \otimes dddy
\]
\[
+ 2\lambda dy \otimes ddy + 4\lambda ddy \otimes ddy + 2\lambda dddy \otimes dy + \lambda^2 dddy \otimes ddy + \lambda^2 dddy \otimes ddy.
\]

Again, the first two terms on the right-hand side correspond to \( S = [n] \) and \( S = \emptyset \), respectively.

4. Renormalization of regularized MZVs

Alain Connes and Dirk Kreimer discovered a Hopf algebraic approach to the BPHZ renormalization method in perturbative quantum field theory \cite{Connes-Kreimer-2000, Connes-Kreimer-2001}. See \cite{Manchon-2004} for a review. One of the fundamental results of these seminal works is the formulation of the process of perturbative renormalization in terms of a factorization theorem for regularized Hopf algebra characters. We briefly recall this theorem, and apply it in the context of the Hopf algebra introduced on \( t \)- and \( q \)-regularized MPLs, when considering them at non-positive arguments.

4.1. Connes–Kreimer Renormalization in a Nutshell. We regard the commutative algebra \( A := \mathbb{Q}[z^{-1}, z] \) with the renormalization scheme \( A = A_- \oplus A_+ \), where \( A_- := z^{-1} \mathbb{Q}[z^{-1}] \) and \( A_+ := \mathbb{Q}[[z]] \).

On \( A \) we define the corresponding projector \( \pi : A \to A_- \) by
\[
\pi \left( \sum_{n=-k}^{\infty} a_n z^n \right) := \sum_{n=-k}^{-1} a_n z^n
\]
with the common convention that the sum over the empty set is zero. Then \( \pi \) and \( \text{Id} - \pi : A \to A_+ \) are Rota–Baxter operators of weight \(-1\). See e.g. \cite{Connes-Kreimer-2000, Ebrahimi-Fard-2002, Ebrahimi-Fard-Gould-2007}.

Let \( (H, m_H, \Delta) \) be a bialgebra and \( (A, m_A) \) an algebra. Then we define the convolution product \( \ast : \text{Hom}(H, A) \otimes \text{Hom}(H, A) \to \text{Hom}(H, A) \) by the composition
\[
H \xrightarrow{\Delta} H \otimes H \xrightarrow{\varphi \otimes \psi} A \otimes A \xrightarrow{m_A} A
\]
for \( \varphi, \psi \in \text{Hom}(H, A) \), or in Sweedler’s notation
\[
(\varphi \ast \psi)(x) := m_A(\varphi \otimes \psi)(\Delta(x)) = \sum_{(x)} \varphi(x_1)\psi(x_2).
\]

The Connes–Kreimer Hopf algebra approach unveiled a beautiful encoding of one of the key concepts of the renormalization process, i.e., Bogoliubov’s counter term recursion, in terms of an algebraic Birkhoff decomposition:

**Theorem 4.1 (\cite{Connes-Kreimer-2000, Connes-Kreimer-2001, Manchon-2004, Ebrahimi-Fard-Gould-2007}).** Let \( (H, m_H, \Delta) \) be a connected filtered Hopf algebra and \( A \) a commutative unital algebra equipped with a renormalization scheme \( A = A_- \oplus A_+ \) and corresponding idempotent Rota–Baxter operator \( \pi \), where \( A_- = \pi(A) \) and \( A_+ = (\text{Id} - \pi)(A) \). Further let \( \phi : H \to A \) be a Hopf algebra character. Then:
a) The character $\phi$ admits a unique decomposition

$$\phi = \phi_-^{(-1)} \star \phi_+$$

called algebraic Birkhoff decomposition, in which $\phi_- : \mathcal{H} \to \mathbb{Q} \oplus \mathcal{A}_-$ and $\phi_+ : \mathcal{H} \to \mathcal{A}_+$ are characters.

b) The maps $\phi_-$ and $\phi_+$ are recursively given fixed point equations

$$\phi_- = e - \pi (\phi_+ \star (\phi - e)),$$

$$\phi_+ = e + (\text{Id} - \pi) (\phi_- \star (\phi - e)),$$

where the unit for the convolution algebra product is $e = \eta_{\mathcal{A}} \circ \varepsilon$, and $\eta_{\mathcal{A}} : \mathbb{Q} \to \mathcal{A}$ is the unit map of the algebra $\mathcal{A}$.

4.2. Renormalization of MZVs. An important remark is in order. To improve readability we skip brackets in the notation of classes of words, that is, in the following a word $w$ stands for the class $r_w$.

Let $k_1, \ldots, k_n \in \mathbb{N}_0$. Then we define a map $\phi : \mathcal{H}_0 \to \mathbb{Q}[z^{-1}, z]$ by

$$d^{k_1}y \cdots d^{k_n}y \mapsto \phi(d^{k_1}y \cdots d^{k_n}y)(z) := \partial_z^{k_1}x \cdots \partial_z^{k_n}x \cdots (z),$$

where $x(z) := \frac{e^z}{1 - e^z}$.

**Lemma 4.2.** The map $\phi : (\mathcal{H}_0, \sqcup_0) \to (\mathbb{Q}[z^{-1}, z], \cdot)$ is a Hopf algebra character. Furthermore, the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{H}_0, \sqcup_0 & \xrightarrow{\phi} & \mathbb{Q}[[t]], \cdot \\
\downarrow_{\zeta_t^{\mu}} & & \downarrow_{t \mapsto e^z} \\
\mathbb{Q}[z^{-1}, z], \cdot & \end{array}$$

**Proof.** From the chain and product rule of differentiation we easily obtain that $\phi = \zeta_t^{\mu}$. Furthermore, the evaluation map $t \mapsto e^z$ and $\zeta_t^{\mu}$ are both algebra morphisms (see Lemma 3.6). Therefore $\phi$ is – as a composition of multiplicative maps – itself a character. □

Next we apply Theorem 4.1 to the character $\phi$ (Lemma 4.2). Then we define renormalized MZVs $\zeta_+$ – using the character $\phi_+$ with image in $\mathbb{Q}[[z]]$ in the Birkhoff decomposition (19) of $\phi$ – by

$$\zeta_+(-k_1, \ldots, -k_n) := \lim_{z \to 0} \phi_+(d^{k_1}y \cdots d^{k_n}y)(z)$$

for $k_1, \ldots, k_n \in \mathbb{N}_0, n \in \mathbb{N}$. The first values of $\zeta_+$ in depth two are given in Table 1 (for an explicit calculation example see Example 4.5). Note that $\zeta_+$ respects the shuffle product $\sqcup_0$ as $\phi_+$ is a character with respect to the algebra $(\mathcal{H}_0, \sqcup_0)$. However, note that the quasi-shuffle relations are not verified because it would require $\zeta_+(0, 0) = \frac{3}{8}$.

Next we show that the renormalized MZVs coincide with the meromorphic continuation of MZVs discussed in Section 2.

**Theorem 4.3.** The renormalization procedure is compatible with the meromorphic continuation of MZVs, i.e. for $k \in \mathbb{N}_0$

$$\zeta_+(-k) = \zeta_1(-k)$$

(23)
Table 1. The renormalized MZVs $\zeta_+(k_1, k_2)$.

| $k_1$ | 0 | $-1$ | $-2$ | $-3$ |
|-------|---|------|------|------|
| 0     | $\frac{1}{4}$ | $\frac{1}{24}$ | 0    | $-\frac{1}{240}$ |
| $-1$  | $\frac{1}{12}$ | $\frac{1}{144}$ | $-\frac{1}{240}$ | $-\frac{1}{1440}$ |
| $-2$  | $\frac{1}{24}$ | $-\frac{1}{240}$ | $-\frac{1}{720}$ | $\frac{1}{504}$ |
| $-3$  | $-\frac{1}{120}$ | $-\frac{1}{360}$ | $\frac{1}{504}$ | $\frac{107}{100800}$ |

and for $a, b \geq 0$ with $a + b$ odd

(24) $\zeta_+(-a, -b) = \zeta_2(-a, -b)$.

Note that for $dpt(w) > 2$ there is no information from the meromorphic continuation (see Remark 2.3).

Proof. We begin with the proof of (23). From Equation (22) we obtain

\[
\phi(d^k y)(z) = \partial_z^k \left( \frac{e^z}{1 - e^z} \right) \\
= -\partial_z^k \left( \frac{1}{z} \frac{ze^z}{e^z - 1} \right) \\
= -\partial_z^k \left( \frac{B_0}{z} + \sum_{n \geq 0} \frac{B_{n+1}}{(n+1)!} z^n \right) \\
= -\left( \frac{(-1)^k k! B_0}{z^{k+1}} + \sum_{n \geq 0} \frac{B_{n+k+1}}{n + k + 1} \frac{1}{n!} z^n \right).
\]

Since $d^k y \in Y$ is a primitive element for the coproduct $\Delta_0$ we obtain with Remark 2.3 that

\[
\phi_+(d^k y)(z) = (\text{Id} - \pi) \phi(z) = -\frac{B_{k+1}}{k+1} + O(z) = \zeta_1(-k) + O(z).
\]

For Equation (24) we calculate for $a + b$ odd with $a, b \geq 0$ (see Remark 2.3)

\[
\phi(d^a y d^b y)(z) = \partial_z^a \left[ x(z) \partial_z^b \left[ x(z) \right] \right] \\
= \partial_z^a \left[ \left( \frac{B_0}{z} + \sum_{m \geq 0} \frac{B_{m+1}}{(m+1)!} z^m \right) \left( \frac{(-1)^b b! B_0}{z^{b+1}} + \sum_{n \geq 0} \frac{B_{n+b+1}}{n + b + 1} \frac{1}{n!} z^n \right) \right] \\
= \partial_z^a \left[ \text{pole part} + \sum_{n \geq 0} \frac{B_0 B_{n+b+2}}{n + b + 2} \frac{z^n}{(n+1)!} + \sum_{m \geq 0} \frac{(-1)^b b! B_0 B_{m+b+2}}{(m + b + 2)!} z^m \right].
\]
Hence, we obtain
\[ \phi \square \text{the weight graduation.} \]

\textbf{Corollary 4.4.} The renormalized MPL, which is presented in the above calculation that (25) side of (25) one can continue to apply Theorem 3.23 to the word \( s \), since

\[ \sum_{a,b \geq 0} B_{a+b} \frac{z^a b^b}{a! b!} \]

\[ = \text{pole part} + \frac{B_0 B_{a+b+2}}{(a+b+2)(a+1)} + (-1)^a a! B_0 B_{a+b+2} \]

\[ + \sum_{a,b \geq 0} B_{a+b} \frac{z^a b^b}{a! b! (a+b+2)!} \]

\[ \phi \square \text{the weight graduation.} \]

The second and third summand are zero since \( a + b + 2 \geq 3 \) is an odd number and therefore \( B_{a+b+2} = 0 \). We have three possibilities for the last sum to be different from zero:

- Case 1: \( m + 1 \) and \( n + b + 1 \) are even numbers. Then we have \( m + n + b + 2 = a + b + 2 \) even, which contradicts that \( a + b \) is odd.
- Case 2: \( m = 0 \). The last summand is equal to

\[ \frac{B_1 B_{a+b+1}}{a+b+1}. \]

- Case 3: \( n + b = 0 \). Then \( n = b = 0 \) and we have for the last summand

\[ \frac{B_1 B_{a+1}}{a+1}. \]

Therefore we obtain together with Remark 2.3 that

\[ \phi(d^a y^b y)(z) = \text{pole part} + \frac{1}{2} (1 + \delta_0(b)) \frac{B_{a+b+1}}{a+b+1} + O(z) \]

\[ = \text{pole part} + \zeta_2(-a,-b) + O(z). \]

Let \( s := a + b \) be an odd number and \( c,d \geq 0 \) with \( c + d = s \). Then we observe using the above calculation that

\[ \phi_-(p^c y)(z) \phi(p^d y)(z) = \frac{(-1)^c c! B_0}{z^c+1} \left( \frac{(-1)^d d! B_0}{z^d+1} + \sum_{n \geq 0} \frac{B_{a+d+1}}{n+d+1 n!} z^n \right) \]

\[ = \text{pole part} + (-1)^c \frac{B_0 B_{c+d+2}}{(c+1)(c+d+2)} + O(z) \]

Since \( c + d + 2 = a + b + 2 \geq 3 \) is an odd number \( B_{c+d+2} = 0 \) and the constant term is zero. Hence, we obtain \( \phi_+(d^a y^b y)(z) = \zeta_2(-a,-b) + O(z) \). Here we have used that \( \hat{\Delta}_0 \) respects the weight graduation.

In the light of Theorem 3.23 and Corollary 3.4 we deduce a simple way to calculate the renormalized MPL, which is presented in

\textbf{Corollary 4.4.} For \( w \in Y \), \( \text{dpt}(w) > 1 \)

\[ \phi_+(w) = \frac{1}{2^\text{dpt}(w) - 2} \sum_{(w')} \phi_+(w') \phi_+(w'). \]
decomposed into primitive elements. The renormalization of \( w \) respectively the corresponding MZV reduce to the simple renormalization of single MPLs at non-positive arguments corresponding to primitive words in \( \mathcal{H} \). For example

\[
\phi_+(dyd^n y) = \phi_+(y)\phi_+(d^{n+1} y) + \phi_+(dy)\phi_+(d^n y).
\]

**Proof of Corollary 4.4.** Statement (25) follows directly from Theorem 3.23, since \( \phi_+ \) is a character by construction. See (21) in Theorem 4.1. Observe that (25) it is compatible with (21), since \((\text{Id} - \pi)\phi_+ = \phi_+\).

**Example 4.5.** Let us calculate the renormalized MZVs \( \zeta_+ (0, -2) \) and \( \zeta_+ (-1, -1) \). From Example 3.19 we find

\[
\Delta_0 (yd^2 y) = y \otimes d^2 y + d^2 y \otimes y \quad \text{and} \quad \Delta_0 (dydy) = y \otimes d^2 y + d^2 y \otimes y + 2dy \otimes dy.
\]

Therefore we obtain from the iterative formulas (20) and (21) of Theorem 4.1

\[
\begin{align*}
\phi_+(yd^2 y) &= (\text{Id} - \pi) \left[ \phi(yd^2 y) - (\phi_- (y)\phi(d^2 y) + \phi_-(d^2 y)\phi(y)) \right], \\
\phi_+(dydy) &= (\text{Id} - \pi) \left[ \phi(dydy) - (\phi_- (y)\phi(d^2 y) + \phi_-(d^2 y)\phi(y) + 2\phi_- (dy)\phi(dy)) \right].
\end{align*}
\]

Using

\[
\begin{align*}
\phi(y)(z) &= -z^{-1} - \frac{1}{2} \frac{1}{12} z + \frac{1}{720} z^3 + O(z^4), \\
\phi(dy)(z) &= z^{-2} - \frac{1}{12} \frac{1}{240} z^2 + O(z^4), \\
\phi(d^2 y)(z) &= -2 z^{-3} + \frac{1}{120} z - \frac{1}{1512} z^3 + O(z^4)
\end{align*}
\]

and

\[
\begin{align*}
\phi(yd^2 y)(z) &= 2 z^{-4} + z^{-3} + \frac{1}{6} z^{-2} - \frac{1}{90} \frac{1}{240} z + \frac{1}{30240} z^2 + \frac{1}{3024} z^3 + O(z^4), \\
\phi(dydy)(z) &= 3 z^{-4} + z^{-3} + \frac{1}{240} \frac{1}{240} z - \frac{1}{1008} z^2 + \frac{1}{3024} z^3 + O(z^4),
\end{align*}
\]

we observe that

\[
\begin{align*}
(\text{Id} - \pi) \left[ \phi_- (y)\phi(d^2 y) + \phi_-(d^2 y)\phi(y) \right](z) &= -\frac{1}{90} + O(z) \\
(\text{Id} - \pi) \left[ \phi_- (y)\phi(d^2 y) + \phi_-(d^2 y)\phi(y) + 2\phi_- (dy)\phi(dy) \right](z) &= -\frac{1}{360} + O(z).
\end{align*}
\]

Hence,

\[
\phi_+(yd^2 y)(z) = O(z) \quad \text{and} \quad \phi_+(dydy)(z) = \frac{1}{240} + \frac{1}{360} + O(z) = \frac{1}{144} + O(z),
\]

which results in \( \zeta_+ (0, -2) = 0 \) and \( \zeta_+ (-1, -1) = \frac{1}{144} \).

Alternatively, we can use the shuffle product to calculate \( \zeta_+ (0, -2) \) and \( \zeta_+ (-1, -1) \). Note that since \( y \otimes_0 dy = ydy \) we have \( \zeta_+ (0, -2) = \zeta_+ (0)\zeta_+ (-2) = 0 \). Because of \( dy \otimes_0 dy = dydy - yd^2 y \) we see that \( \zeta_+ (-1, -1) = \zeta_+ (0, -2) + \zeta_+ (-1)^2 = \frac{1}{144} \).
A third way to calculate, say, $\zeta_+(-1, -1)$, is based on Corollary \ref{cor:renormalization} and described in Corollary \ref{cor:renormalization}:

$$\zeta_+(-1, -1) = \frac{1}{2} (\zeta_+(0)\zeta_+(-2) + \zeta_+(-2)\zeta_+(0) + 2\zeta_+(-1)\zeta_+(1)) = \frac{1}{144}.$$ 

4.3. Renormalization of $q$MZVs. The Hopf algebra $(\mathcal{H}, \shuffle, \Delta_{-1})$ is related to the modified $q$-analogue $\mathfrak{H}_q$, whereas the relation between MZVs and $q$MZVs (see Equation \ref{eq:1}) relies on a limit process involving the non-modified $q$MZVs $\mathfrak{H}_q$. Therefore the renormalization related to the $q$MZV deformation is more involved than the renormalization described in the previous section. First of all we apply Theorem \ref{thm:shuffle-renormalization} in the framework of modified $q$MZVs. We define the map $\psi: (\mathcal{H}, \shuffle_{-1}) \to \mathbb{Q}[z^{-1}, z]$ by

$$d^{k_1}y \cdots d^{k_n}y \mapsto \psi(d^{k_1}y \cdots d^{k_n}y)(z) := \sum_{m_1, \ldots, m_n \geq 0} \frac{B_{m_1}}{m_1!} \cdots \frac{B_{m_n}}{m_n!} \cdot C_{m_1, \ldots, m_n}^{k_1, \ldots, k_n} z_1^{m_1} \cdot \cdots \cdot z_n^{m_n}$$

for $k_1, \ldots, k_n \in \mathbb{N}_0$, where

$$C_{m_1, \ldots, m_n}^{k_1, \ldots, k_n} := \sum_{l_1, \ldots, l_n = 0}^{k_1} \left( \prod_{i=1}^{n} \frac{k_i}{l_i} \right) (-1)^{l_1 + 1} (l_1 + \cdots + l_i + 1)^{m_i - 1}.$$

**Lemma 4.6.** The map $\psi: (\mathcal{H}, \shuffle_{-1}) \to (\mathbb{Q}[z^{-1}, z], \cdot)$ is a character. Furthermore, the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{H}, \shuffle_{-1} & \xrightarrow{\mathfrak{H}_q} & \mathbb{Q}[[q]], \\
\psi \downarrow & & \downarrow q \mapsto e^z \\
(\mathcal{H}, \shuffle_{-1}) & \rightarrow & (\mathbb{Q}[z^{-1}, z], \cdot)
\end{array}$$

**Proof.** Let $k_1, \ldots, k_n \in \mathbb{N}_0$. First we observe that

$$\mathfrak{H}_q(d^{k_1}y \cdots d^{k_n}y)$$

$$= \sum_{m_1, \ldots, m_n > 0} q^{m_1} (1 - q^{m_1})^{k_1} (1 - q^{m_2})^{k_2} \cdots (1 - q^{m_n})^{k_n}$$

$$= \sum_{m_1, \ldots, m_n > 0} q^{m_1 + \cdots + m_n} (1 - q^{m_1 + \cdots + m_n})^{k_1} (1 - q^{m_2 + \cdots + m_n})^{k_2} \cdots (1 - q^{m_n})^{k_n}$$

$$= \sum_{l_1=0}^{k_1} \sum_{l_n=0}^{k_n} (-1)^{l_1 + \cdots + l_n} \binom{k_1}{l_1} \cdots \binom{k_n}{l_n} \sum_{m_1, \ldots, m_n > 0} q^{m_1(l_1 + 1) + \cdots + m_n(l_1 + \cdots + l_n + 1)}$$

$$= \sum_{l_1=0}^{k_1} \sum_{l_n=0}^{k_n} (-1)^{l_1 + \cdots + l_n + n} \binom{k_1}{l_1} \cdots \binom{k_n}{l_n} q^{l_1 + 1} \cdots q^{l_1 + \cdots + l_n + 1} \frac{q^{l_1 + \cdots + l_n + 1} - 1}{q^{l_1 + \cdots + l_n + 1} - 1}.$$ 

This leads to

$$\mathfrak{H}_q(d^{k_1}y \cdots d^{k_n}y) \mapsto \sum_{l_1=0}^{k_1} \sum_{l_n=0}^{k_n} \prod_{j=1}^{n} (-1)^{l_j + 1} \binom{k_j}{l_j} \frac{e^{z(l_1 + \cdots + l_j + 1)} - 1}{e^{z(l_1 + \cdots + l_j + 1)} - 1}.$$
\[
= \sum_{i_l=0}^{k_i} \left( \prod_{j=1}^{n} (-1)^{l_j+1} \binom{k_j}{l_j} \sum_{m_j \geq 0} \frac{B_{m_j}}{m_j!} (z(l_1 + \cdots + l_j + 1))^{m_j-1} \right)
= \sum_{m_1, \ldots, m_n \geq 0} \frac{B_{m_1}}{m_1!} \cdots \frac{B_{m_n}}{m_n!} \cdot C_{m_1, \ldots, m_n} z^{m_1 + \cdots + m_n - n}.
\]

The map \(\psi\) is a character since it is a composition of algebra morphisms.

Next we reverse the modification process applied in Equation (20). Therefore we apply Theorem 4.1 to the character \(C\) with

\[
\text{Lemma 4.8. Let } k_1, \ldots, k_n \in \mathbb{N}_0. \text{ Then } 3_\psi(-k_1, \ldots, -k_n) \text{ is well defined, and we have}
\]

\[
3_\psi(-k_1, \ldots, -k_n) = \psi_+(d^{k_1}y \cdots d^{k_n}y)(z)
\]

for \(k_1, \ldots, k_n \in \mathbb{N}_0\).

**Theorem 4.7.** Let \(k_1, \ldots, k_n \in \mathbb{N}_0\). Then \(3_\psi(-k_1, \ldots, -k_n)\) is well defined, and we have

\[
3_\psi(-k_1, \ldots, -k_n) = \zeta_+(k_1, \ldots, -k_n).
\]

Especially, the renormalized MZVs \(3_\psi\) respect the shuffle product \(\shuffle_0\).

For the proof of this theorem we need an auxiliary result:

**Lemma 4.8.** We have \(\psi_+(d^{k}y)(z) = (-1)^k \zeta_+(k) z^k + O(z^{k+1})\) for all \(k \in \mathbb{N}_0\).

**Proof.** Since \(d^k y\) is primitive with respect to the coproduct \(\Delta_{-1}\), we obtain from Lemma 4.6 that

\[
\psi_+(d^{k}y)(z) = (\text{Id} - \pi) \psi(d^{k}y)(z) = (\text{Id} - \pi) \left( \sum_{m \geq 0} \frac{B_m}{m!} \cdot C_{m} z^{m-1} \right) = \sum_{m \geq 0} \frac{B_m}{m!} \cdot C_{m} z^{m-1}
\]

with \(C_{m} = \sum_{l=0}^{k} \binom{k}{l} (-1)^{l+1} (l+1)^{m-1} \). We have

\[
C_{m} = \frac{1}{k+1} \delta_z \left. \left( \sum_{l=0}^{k} \binom{k+1}{l} (-z)^l \right) \right|_{z=1} = \frac{1}{k+1} \delta_z (1-z)^{k+1}.
\]

This shows that \(C_{m} = 0\) for \(m = 1, \ldots, k\). Furthermore, we observe that \(C_{k+1} = (-1)^k \frac{(k+1)!}{k+1} \), which completes the proof.

**Proof of Theorem 4.7.** Let \(w := d^{k_1}y \cdots d^{k_n}y\) with \(k_1, \ldots, k_n \in \mathbb{N}_0\). In order to prove that \(3_\psi\) is well defined we show \(\psi_+(w) \in O(z^{\text{wt}(w) - n})\). We split up the coproduct \(\Delta_{-1}(w)\) into two parts

\[
\Delta_{-1}(w) = \Delta_0(w) + (\Delta_{-1}(w) - \Delta_0(w)).
\]

From Corollary 4.4 we deduce that \(\Delta_{-1}(w)\) induces a \(\mathbb{Q}\)-linear combination of products with \(n\) factors of \(\psi_+\) in primitive elements of \(\mathcal{H}\). The part of \(\psi_+\) corresponding to \(\Delta_0(w)\) in the coproduct factorization is homogeneous in weight \(\text{wt}(w)\) and the one related to \(\Delta_{-1}(w) - \Delta_0(w)\) has weight greater than \(\text{wt}(w)\). Therefore Lemma 4.8 implies that \(\psi_+(w) \in O(z^{\text{wt}(w) - n})\). Hence, the limit in (26) exists. On the one hand we can apply Corollary 4.4 to \(\phi_+(w)\), defined in the previous section, which corresponds to the factorization induced by \(\Delta_0(w)\). On
the other hand we can do the same with $\psi(w)$. However, this factorization is related to $\Delta_1(w)$. After dividing by $z^{wt(w)-n}$ and taking the limit $z \to 0$ only the first part in the decomposition \(\tilde{\Delta}_1(w)\) of $\Delta_1(w)$ makes a contribution in the factorization of $\psi(w)$. Using the fact that the leading factor of $\psi_+(d^k y)(z)$ equals $(-1)^k \zeta_+(\bar{k}) z^k$ concludes the proof. \(\square\)

**Example 4.9.** Let us calculate the renormalized $q$MZV $\mathfrak{b}_+(1, -1)$. From Example 3.19 we obtain

$$\tilde{\Delta}_0(d^y dy) = y \otimes d^2 y + d^2 y \otimes y + 2 d y \otimes d y - d y \otimes d^2 y - d^2 y \otimes d y,$$

which gives

$$\psi_+(d^y dy) = (\text{Id} - \pi) \left( \psi(d^y dy) - (\psi(-y) \psi(d^y) + \psi(-d^y) \psi(y) + 2 \psi(-d^y) \psi(dy)ightharpoonup \right.$$

$$\left. - \psi(dy) \psi(d^y) - \psi(dy) \psi(dy) \rightharpoonup \right)$$

Using

$$\psi(y)(z) = - z^{-1} - \frac{1}{2} z + \frac{1}{12} z^2 - \frac{1}{720} z^3 + \frac{1}{30240} z^4 + O(z^5),$$

$$\psi(dy)(z) = - \frac{1}{2} z^{-1} - \frac{1}{12} z - \frac{7}{720} z^2 + \frac{31}{30240} z^3 + O(z^4),$$

$$\psi(d^2 y)(z) = - \frac{1}{3} z^{-1} + \frac{1}{60} z^2 - \frac{1}{168} z^3 + O(z^4),$$

and

$$\psi(d^y dy)(z) = \frac{5}{12} z^{-2} + \frac{1}{6} z^{-1} - \frac{1}{36} z + \frac{1}{216} z^2 + \frac{1}{120} z^3 - \frac{19}{9072} z^4 + O(z^5),$$

we observe that

$$(\text{Id} - \pi) \left[ \psi_-(y) \psi(d^y) + \psi_-(d^y) \psi(y) + 2 \psi_-(d^y) \psi(dy) \rightharpoonup \right.$$

$$\left. - \psi(dy) \psi(d^y) - \psi(dy) \psi(dy) \rightharpoonup \right] (z) = - \frac{1}{18} - \frac{1}{135} z^2 + O(z^3)$$

and

$$\left[ - \psi_-(y) \psi(d^y) - \psi_-(d^y) \psi(y) \rightharpoonup \right.$$

$$\left. - \psi(dy) \psi(d^y) - \psi(dy) \psi(dy) \rightharpoonup \right] (z) = \frac{1}{36} + \frac{11}{2160} z^2 + O(z^3).$$

Therefore we have $\psi_+(d^y dy)(z) = \frac{1}{144} z^2 + O(z^3)$ and consequently $\mathfrak{b}_+(1, -1) = \frac{1}{144}$, which coincides with $\zeta_+(-1, -1)$.

**Remark 4.10.** The crucial point in the renormalization of $q$MZVs is the fact that $\psi_+(w)(z) \in O(z^{wt(w)-n})$. We used a corollary of Theorem 3.23 to prove this. However, in the light of Theorem 4.11 the previous example shows that this is obtained by non-trivial cancellations.

**References**

[AET01] S. Akiyama, S. Egami, and Y. Tanigawa. Analytic continuation of multiple zeta-functions and their values at non-positive integers. *Acta Arith.*, 98(2):107–116, 2001.

[AT01] S. Akiyama and Y. Tanigawa. Multiple zeta values at non-positive integers. *Ramanujan J.*, 5(4):327–351 (2001).

[Bou14] O. Bouillot. Multiple Bernoulli Polynomials and Numbers. *preprint*, 2014.

[CEM14a] J. Castillo Medina, K. Ebrahimi-Fard, and D. Manchon. On euler’s decomposition formula for qMZVs. To be published in The Ramanujan Journal, *arXiv:1309.2759*, 2014.

[CEM14b] J. Castillo Medina, K. Ebrahimi-Fard, and D. Manchon. Unfolding the double shuffle structure of q-multiple zeta values. To be published in Bulletin of the Australian Mathematical Society, arXiv:1310.1330, 2014.
[CK00] A. Connes and D. Kreimer. Renormalization in quantum field theory and the Riemann-Hilbert problem. I. The Hopf algebra structure of graphs and the main theorem. Comm. Math. Phys., 210(1):249–273, 2000.

[CK01] A. Connes and D. Kreimer. Renormalization in quantum field theory and the Riemann-Hilbert problem. II. The β-function, diffeomorphisms and the renormalization group. Comm. Math. Phys., 216(1):215–241, 2001.

[Col84] J. Collins. Renormalization: An introduction to renormalization, the renormalization group and the operator product expansion. Cambridge University Press, Cambridge, 1984.

[Ebr02] K. Ebrahimi-Fard. Loday-type Algebras and the Rota-Baxter Relation. Letters in Mathematical Physics, 61:139–147, 2002.

[EG07] K. Ebrahimi-Fard and L. Guo. Rota-Baxter algebras in renormalization of perturbative quantum field theory. Fields Institute Communications, 50:47–105, 2007.

[EGP07] K. Ebrahimi-Fard, J. M. Gracia-Bondía, and F. Patras. A lie theoretic approach to renormalization. Commun. Math. Phys., 276:821–845, 2007.

[GZ08] L. Guo and B. Zhang. Renormalization of multiple zeta values. J. Algebra, 319(9):3770–3809, 2008.

[Man04] D. Manchon. Hopf algebras, from basics to applications to renormalization. ArXiv Mathematics e-prints, 2004.

[MP10] D. Manchon and S. Paycha. Nested sums of symbols and renormalized multiple zeta values. Int. Math. Res. Not. IMRN, (24):4628–4697, 2010.

[OOZ12] Y. Ohno, J. Okuda, and W. Zudilin. Cyclic q-MZSV sum. J. Number Theory, 132(1):144 – 155, 2012.

[Pat93] F. Patras. La décomposition en poids des algèbres de Hopf. Annales de l’Institut Fourier, 43(2):1067–1087, 1993.

[Sin14] J. Singer. On Bradley’s q-MZVs and a Generalized Euler Decomposition Formula. submitted, 2014.

[Sin15] J. Singer. On q-analogues of Multiple Zeta Values. To appear in Funct. Approx. Comment. Math., 2015.

[Wal02] M. Waldschmidt. Multiple polylogarithms: an introduction. In Number theory and discrete mathematics (Chandigarh, 2000), Trends Math., pages 1–12. Birkhäuser, Basel, 2002.

[Wal11] M. Waldschmidt. Lectures on Multiple Zeta Values, 2011.

[Zha00] J. Zhao. Analytic continuation of multiple zeta functions. Proc. Amer. Math. Soc., 128(5):1275–1283, 2000.

[Zha14] J. Zhao. Uniform approach to double shuffle and duality relations of various q-analogs of multiple zeta values via rota-baxter algebras. ArXiv e-prints, 2014.