Quasinormal modes of the electric potential in the 4 dimensional anti de Sitter Reissner-Norstöm black hole spacetime with scalar hair

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Abstract. In this study, analytical calculations are performed to assess the quasinormal modes (qnm) of the perturbed electromagnetic potential within the a four-dimensional spacetime of the Reissner-Nordström (RN) anti-de Sitter (AdS) black holes which are dressed with scalar hair. The black holes serve as the solutions for the Einstein-Maxwell theory under a negative cosmological constant and conformally-coupled real self-interacting scalar field. When the strength of the scalar field is suitable, phase transition of the black holes occurs to create a new form of MTZ black holes. The electromagnetic field can then be perturbed and the qnm and their frequencies calculated in order to meet the boundary conditions both at the horizon and at the far distant region from the new MTZ black hole. In order to address a simplified version of this problem, this study investigates the case whereby the mass of the MTZ black hole is significantly less than the AdS radius.

1. Introduction
There has been growing interest recently in black hole thermodynamics and phase transition [1–4]. It was demonstrated in [5] that the laws of thermodynamics govern the behavior of black holes. These laws of thermodynamics have an important implication, which is the existence of black hole radiation, or Hawking radiation. It has long been questioned whether or not black holes release radiation, and Hawking radiation has been an important component of this debate. The work of Maldacena [6] proposed a correspondence between the theory of general relativity in anti-de Sitter (AdS) black hole spacetimes and a conformal field theory (CFT). Later studies which investigated these links in greater depth [7–9] proposed procedures which might allow the study of quantum systems via solutions to Einstein field equations within the corresponding AdS black hole spacetimes. Earlier studies [10] have provided clearly defined wormhole solutions for the AdS spacetimes, while Hawking [11] understood the potential for information within AdS black holes using quantum tunneling to leak out, thereby lowering the entropy levels. It is held by the no-hair theorem that it is only possible to fully describe a black hole solution through the three classical external parameters of charge, mass, and angular momentum. When Hawking radiation occurs, however, it allows information to be lost from within the black holes, thus changing the properties of the black holes under the Hair theorem. Numerous studies have examined this phenomenon of black hole perturbation causing changes in the properties [12]. When black holes are perturbed by a particular physical influence, such as a scalar field, phase transition will take place, and under the right circumstances this can result in a change to a
new kind, while there can also be a disruption in the smoothness of entropy changing when the transition occurs. It has been shown that upon perturbation of a Reissner-Nordström anti-de Sitter black hole by a scalar hair, there is a new exact solution which results [13]. This is called an MTZ black hole, and the evidence of entropy discontinuity or/and non smoothness, when varying the temperature, indicates the existence of a phase transition from a Reissner-Nordström black hole to a new form of black hole.

In this paper, part 2 provides a reconfirmation of the Einstein-Maxwell theory using a conformally-coupled self-interacting scalar field as described in an earlier work [4]. Part 3 the thermodynamics of the both black holes and the phase transition between them in [13] are reviewed. Part 4 describes the process of perturbing the electromagnetic potential. The perturbed potential differential equation for quasinormal modes (qmn) is solved in part 5. This research differs from an earlier study [14] which did not include the black hole phase transition. The study has similarities one other piece of research [15], although a different approach in employed for calculations before the results are compared. The conclusion is given on part 6.

2. Einstein-Maxwell theory with a conformally coupled self-interacting scalar field

An earlier study [4] describes the four-dimensional Einstein-Maxwell system with a cosmological constant $\Lambda$ and a real conformally-coupled self-interacting scalar field as shown below:

$$I[g_{\mu\nu}\phi, A_{\mu}] = \int d^4x\sqrt{-g}\left[R - 2\Lambda - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{12}R\phi^2 - \frac{1}{16\pi}F^{\mu\nu}F_{\mu\nu}\right].$$ (1)

Newton’s constant is given by $G$ and $\alpha$ serves as an arbitrary coupling constant. A total of three field equations are associated with the action:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G\left(T^{\phi}_{\mu\nu} + T^{em}_{\mu\nu}\right),$$ (2)

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\phi = \frac{1}{6}R\phi + 4\alpha\phi^3,$$ (3)

$$0 = \partial_\nu\left(\sqrt{-g}F^{\mu\nu}\right),$$ (4)

the scalar field energy-momentum tensor $\phi$ and for the electromagnetic field can be expressed as:

$$T^{\phi}_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + \frac{1}{6}\left[g_{\mu\nu}g^{\alpha\beta}\nabla_\alpha\nabla_\beta - \nabla_\mu\nabla_\nu + G_{\mu\nu}\right]\phi^2 - g_{\mu\nu}\alpha\phi^4,$$ (5)

$$T^{em}_{\mu\nu} = \frac{1}{4\pi}\left[F_{\mu\alpha}F_{\nu\beta} - \frac{1}{4}g_{\mu\nu}g^{\gamma\delta}F_{\gamma\alpha}F_{\delta\beta}\right]g^{\alpha\beta}. $$ (6)

The system has constant curvature, i.e., $R = 4\Lambda$. An exact static solution exists for the field equation (2) to equation (4), for which the metric is:

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2d\sigma^2, $$ (7)

$$f(r) = \frac{r^2}{l^2} + \gamma \left(1 + \frac{G\mu}{r}\right)^2, $$ (8)

$l$ is the AdS radius and the $d\sigma^2$ represents the line element for a two-dimensional space, for which the curvature is assumed to be constant and may be normalized as $\gamma = 1, 0, -1$. It can be inferred that the $d\sigma^2$-space surface is locally isometric to the sphere given by $S^2$, the flat space given by $\mathbb{R}^2$, and the hyperbolic $H^2$ for $\gamma = 1, 0, -1$. 


An exact solution to $\phi$ is obtained through equations (2) to equation (6).

$$\phi = \sqrt{\frac{1}{2\alpha l^2} \left( \frac{G\mu}{r + G\mu} \right)},$$

(9)

in which the particular point of interest will be the conditions of AdS spacetime, i.e. $\Lambda = -3l^{-2}$ where the negative curvature $\gamma = -1$ is selected. An earlier study [4] gave the potential $A^\mu$ only as electric given by $A^\mu = A_0 dx^0 = -\frac{q}{r} dt$. To achieve a static solution for equation (7) and equation (8), the integration constants $q$ and $\mu$ cannot be independent:

$$q^2 = \gamma G\mu^2 \left( 1 + \frac{2\pi \Lambda}{9\alpha} \right).$$

(10)

$M$ indicates the mass while $Q$ is the electric charge; both are related to these parameters

$$M = -\gamma \frac{\sigma}{4\pi} \mu; \quad Q = \frac{\sigma}{4\pi} q,$$

(11)

in which $\sigma$ represents the total solid angle or unit area of the $d\sigma^2$-space; for example, the sphere $S^2 \sigma = 4\pi$.

3. Phase transition

In this section, the free energy and entropy of the hyperbolic Reissner-Nordström black hole and the hairy black hole in the previous section are reviewed from [13]. Both free energies are compared and discussed at the transition phase. The free energy of a black hole can be written as

$$F = F(T, \Phi) = M - TS - \Phi Q$$

(12)

where $M$, $S$, $Q$ and $T$ are the black hole mass, entropy, charge and temperature respectively. $\Phi$ is the potential difference $\Phi = A_0(\infty) - A_0(\text{horizon})$. The hairy black hole gives

$$T = \frac{1}{2\pi l} \left( \frac{2r_h}{l} - 1 \right), \quad \Phi = \frac{q}{r_h}$$

(13)

where $r_h$ is the horizon of the hairy black hole (see equation (29)),

$$M = \frac{\sigma}{4\pi} \mu, \quad Q = \frac{\sigma}{4\pi} q, \quad S = \frac{\sigma r_h^2}{4G} \left( 1 - \frac{4\pi G}{3} \phi^2(r_h) \right)$$

(14)

The hRN black hole gives

$$T = \frac{1}{2\pi l} \left( \frac{\rho_h}{l} + \frac{G\mu_0}{\rho_h^2} - \frac{G q_0^2 l}{\rho_h^3} \right), \quad \Phi = \frac{q_0}{\rho_h}$$

(15)

$$M = \frac{\sigma}{4\pi} \mu_0, \quad Q = \frac{\sigma}{4\pi} q_0, \quad S = \frac{\sigma \rho_h^2}{4G}$$

(16)

where $\rho_h$ is its horizon.

At the transition between two black holes, the temperature is assumed to be continuous. Then from equation (13) and equation (15), the relation between $r_h$ and $\rho_h$ is

$$\frac{2r_h}{l} - 1 = \frac{\rho_h}{l} + \frac{G\mu_0}{\rho_h^2} - \frac{G q_0^2 l}{\rho_h^3}, \quad \frac{q}{r_h} = \frac{q_0}{\rho_h}$$

(17)
The free energies of both black holes, therefore, can be written as

\[ F_{\text{hRN}} = -\frac{\sigma l}{8\pi G} \left[ 2 \left( \frac{\rho_h}{l} - \pi l T \right) \frac{\rho_h^2}{l^2} \right] \]  
(18)

\[ F_{\text{hairy}} = -\frac{\sigma l}{8\pi G} \left[ \left( \pi l T + \frac{1}{2} \right)^2 + a \left( \pi l T - \frac{1}{2} \right)^2 \right] \]  
(19)

At the transition the free energy is also continuous, then the relation between \( \rho_h \) and temperature \( T \) can be written as

\[ \frac{\rho_h}{l} = \frac{2\pi l T}{3} \left[ 1 + \frac{1}{\sqrt{1 + \frac{3}{4(\pi l T)^2}}} \left[ 1 + (a - 1) \left( \pi l T - \frac{1}{2} \right)^2 \right] \right] \]  
(20)

The difference between the both black hole free energies is defined as

\[ \Delta F \equiv F_{\text{hRN}} - F_{\text{hairy}} \]  
(21)

At the transition \( \Delta F = 0 \) and the temperature \( T \) in which is satisfied equation (20) is \( T_c = \frac{1}{2\pi l} \), called the critical temperature, because \( \Delta F \) near the transition can be written as a function of \( T - T_c \)

\[ \Delta F = -\frac{a\sigma l^4}{64G} (T - T_c)^3 + O(T - T_c)^4. \]  
(22)

Equation (22) presents the second order phase transition, with the critical temperature \( T_c \).

4. Perturbation of electromagnetic potential

Of particular interest in this study is the process of perturbing the electrical field surrounding these novel black hole types. In this section we do the following, 1. perturb the electromagnetic potential as in equation (23) and derive its wave equation in equation (28), 2. find all singularities in equation (29) and their relations equation (33) and equation (34) and separate and extract them from the wave equation and solution equation (32), 3. in order to compare to other work we consider only for large black hole mass and reduce the wave equation to equation (37), where its solutions are the hypergeometric function equation (39).

It is observed that a simple-form electromagnetic-potential component, \( A = A_\mu dx^\mu \) in addition to \( A_0 \) might hold the static solution reported for equations (7) and equation (8):

\[ A = A_\mu dx^\mu = -\frac{q}{r} dt + A_\chi(t, r, \chi). \]  
(23)

The element \( A_\chi(t, r, \chi) \) acts as a function of a number of variables including \( t, r \) and \( \chi \), although not \( \chi \), given that \( \chi \) and \( \chi \) represent the variables for the axes in \( d\sigma^2 = d\chi^2 + d\chi^2 \).

Accordingly, it is possible to rewrite equation (4) using components from equation (8) and equation (23), in which \( F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \)

\[ \partial_\nu (\sqrt{-g} F^{\mu\nu}) = -\sqrt{-g} g^{\chi\chi} g^{tt} \partial_\chi A_\chi - \partial_\nu \sqrt{-g} g^{\chi\chi} g^{rr} \partial_r A_\chi - \sqrt{-g} g^{\chi\chi} g^{\gamma\gamma} \partial_\gamma A_\chi + 0. \]  
(24)

The separation variable technique may be employed to obtain a solution for equation (24):

\[ A_\chi(t, r, \gamma) = e^{-i\omega t} R(r)Q(\gamma), \]  
(25)
ω indicates the frequency of oscillation for the perturbed potential surrounding the black hole. Equation (24) then turns into:

\[ \frac{r^2}{R(r)} \partial_r [f(r) \partial_r R(r)] + \frac{r^2}{f(r)} \omega^2 + \frac{1}{Q(\gamma)} \partial^2 \gamma Q(\gamma) = 0. \] (26)

So for convenience we can allow:

\[ \frac{1}{Q(\gamma)} \partial^2 \gamma Q(\gamma) = -m^2 ; Q(\gamma) \sim e^{im\gamma}, \] (27)

in which Q is taken to be periodic, this leading to \( m^2 \geq 0 \). On the basis of equation (26), the ordinary differential equation becomes:

\[ r^2 f(r) \frac{d}{dr} \left[ f(r) \frac{dR(r)}{dr} \right] + \omega^2 r^2 R(r) - m^2 f(r) R(r) = 0. \] (28)

In order to simplify equation (28) to find a solution, the steps are carried out as follows: solve for \( f(x_i) = 0, i = 1, 2, 3, 4 \), i.e.,

\[ r_h \equiv r_1 = \frac{1}{2} + \frac{1}{2} \sqrt{1 + 4G_\mu/l}, \quad r_3 = \frac{1}{2} - \frac{1}{2} \sqrt{1 + 4G_\mu/l}, \]

\[ r_2 = -\frac{1}{2} - \frac{1}{2} \sqrt{1 - 4G_\mu/l}, \quad r_4 = -\frac{1}{2} + \frac{1}{2} \sqrt{1 - 4G_\mu/l}. \] (29)

The sole positive real number is given by \( r_h \) which represents the horizon radius. A new variable, \( x \equiv \frac{r}{r_h} \), can be defined, along with new parameters \( x_1 \equiv \frac{r_1}{r_h} = 1, x_2 \equiv \frac{r_2}{r_h}, x_3 \equiv \frac{r_3}{r_h}, x_4 \equiv \frac{r_4}{r_h} \) before rewriting \( f(x) \) as:

\[ f(x) = -\left( \frac{G_\mu}{r_h^2 x^2} \right) (x - x_1)(x - x_2)(x - x_3)(x - x_4). \] (30)

Equation (28) is thus changed to the \( x \) differential equation expressed as:

\[ (x - x_1)(x - x_2)(x - x_3)(x - x_4) \frac{d}{dx} \left[ (x - x_1)(x - x_2)(x - x_3)(x - x_4) \frac{dR}{dx} \right] \]

\[ + (\omega r_h)^2 \left( \frac{r_h}{G_\mu} \right)^4 R + \left( \frac{mr_h}{G_\mu} \right)^2 (x - x_1)(x - x_2)(x - x_3)(x - x_4) R = 0. \] (31)

Meanwhile, it is possible to simplify equation (31) by:

\[ R = (x - x_1)^{\alpha_1} (x - x_2)^{\alpha_2} (x - x_3)^{\alpha_3} (x - x_4)^{\alpha_4} F, \]

in which \( \alpha_i = -i \omega r_h \left( \frac{r_h}{G_\mu} \right)^2 a_i \) and \( a_1 = \frac{1}{(x_3 - x_1)(x_3 - x_2)(x_3 - x_4)}, a_2 = \frac{1}{(x_2 - x_1)(x_2 - x_3)(x_2 - x_4)}, a_3 = \frac{1}{(x_4 - x_1)(x_4 - x_2)(x_4 - x_3)} \) and \( a_4 = \frac{1}{(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)} \). Helpful relationships among \( a_i \) are

\[ \sum_{i=1}^{4} a_i = \sum_{i=1}^{4} a_i x_i = \sum_{i=1}^{4} a_i x_i^2 = 0 ; \quad \sum_{i=1}^{4} a_i x_i^3 = 1, \] (33)

also \( x_i \) are

\[ \sum_{i=1}^{4} x_i = \frac{2r_h}{G_\mu}, \quad x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 = \frac{r_h^2}{G_\mu}, \]

\[ x_1x_2x_3x_4 = -\left( \frac{r_h^2}{G_\mu} \right)^2, \quad x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 = 0, \] (34)
Having substituted $R$ into equation (31) from equation (32) and applying relationships obtained from equation (33) with basic algebraic principles, it is possible to extract the terms $(x - x_1)^{\alpha_1+1}(x - x_2)^{\alpha_2+1}(x - x_3)^{\alpha_3+1}(x - x_4)^{\alpha_4+1}$ and thereby reduce equation (31) to

$$(x - x_1)(x - x_2)(x - x_3)(x - x_4)\frac{d^2 F}{dx^2} + \left\{ (1 + 2\alpha_1)(x - x_2)(x - x_3)(x - x_4) + (1 + 2\alpha_2)(x - x_1)(x - x_3)(x - x_4) \right\} \frac{dF}{dx} + \left( \frac{m r_H}{G \mu} \right)^2 F = 0.$$  

(35)

Application of the relationships in equation (34) followed by multiplication of equation (35) be $-(G\mu)^2/r_h^2$, the result is:

$$\left[ \frac{r_h^2}{l^2} - x^2 + \frac{2G\mu}{r_h} x^3 - \frac{(G\mu)^2}{r_h^2} x^4 \right] \frac{d^2 F}{dx^2} + \left[ 2i\omega r_h - 2x - \frac{6G\mu}{r_h} x^2 - \frac{4(G\mu)^2}{r_h^2} x^3 \right] \frac{dF}{dx} - m^2 F = 0.$$  

(36)

This study focuses solely on the case where the black hole mass is significantly lower than the AdS radius, $\frac{G\mu}{r} << 1$. Accordingly, it is possible to approximate $\frac{r_h^2}{l^2} = \frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4G\mu}{r} + \frac{G\mu}{r}} \approx 1 + \frac{2G\mu}{r} + \frac{(G\mu)^2}{r_h^4} + O\left( \frac{(G\mu)^3}{r_h^4} \right)$. It is possible to approximate the initial term from equation (36) as:

$$\frac{r_h^2}{l^2} - x^2 + \frac{2G\mu}{r_h} x^3 - \frac{(G\mu)^2}{r_h^2} x^4 \approx 1 - x^2 + \frac{2G\mu}{r_h} (1 + x^3) + \frac{(G\mu)^2}{r_h^2} (1 - x^4) + O\left( \frac{(G\mu)^3}{r_h^4} \right),$$

and equation (36) maintains the lowest power of $G\mu/r_h$, as $(G\mu/r_h)^0$ (although $r_h$ also has $G\mu$)

$$(1 - x^2)\frac{d^2 F}{dx^2} + (2i\omega r_h - 2x) \frac{dF}{dx} - m^2 F = 0.$$  

(37)

By substituting $F = (1 + x)^\beta F$, in which $\beta = -i\omega r_h$, and altering the variable $v = x^2$ and extracting the term $(1 + x)^\beta$ from equation (37), then it is possible to obtain:

$$v(1 - v) \frac{d^2 F}{dv^2} + \left[ 1 - \left( \frac{3}{2} + \beta \right) v \right] \frac{dF}{dv} - \frac{\beta(\beta + 1) + m^2}{4} F = 0.$$  

(38)

When equation (38) is solved, the solutions are hypergeometric functions, so there are two independent solutions:

$$F_1 = _2 F_1(a, b; c; v), \quad F_2 = v^{1-c} _2 F_1(1 + a - c, 1 + b - c; 2 - c; v),$$

(39)

for which we have parameters $a, b = \frac{1}{4} + \frac{\beta}{2} \pm \frac{1}{4}\sqrt{1 - 4m^2}$ and $c = \frac{1}{2}$.

5. Quasinormal modes of the perturbed electromagnetic potential

In this section, the boundary conditions of the quasinormal modes are applied to equation (39) as the following 1. find the acceptable solution at far away zone in equation (40), i.e., a decaying wave in this region, 2. expand the acceptable far-away solution to the solutions at near the horizon by using the hypergeometric function property in equation (41), 3. apply the condition
at the horizon, i.e. only the ingoing wave or eliminate the outgoing at the horizon in equation (42). 4. this gives the frequencies as in equation (44), 5. we compare to another result for large $n$ in equation (45). In this black hole system, boundary conditions comprise the ingoing wave at the horizon and the outgoing or decaying waves at the distant region. Waves capable of meeting these criteria are known as quasinormal modes.

In the far distant region, $v = x^2 = r_0^2/r^2 \rightarrow 0$, from equation (32) and equation (39)

$$R(v \to 0) \sim A \, 2F_1(a; b; c; v) + B v^{1-c} 2F_1(1 + a - c, 1 + b - c; 2 - c; v) \rightarrow A,$$  \hspace{1cm} (40)

in which $A$ and $B$ act as arbitrary constants. In equation (40) the far distant region is given a constant potential, while in order to meet the conditions for the far distant zone boundary it is possible to set $A = 0$. The following solution is accepted at the far distant region: $v^{1-c} 2F_1(1 + a - c, 1 + b - c; 2 - c; v)$ and the properties of the hypergeometric function can be applied to assess the solution close to the horizon, $1 - v = (1 - x)(1 + x) \rightarrow 0$.

$$R(x \to 1) \Rightarrow (1 - x)^{\alpha_1} v^{1/2} 2F_1(1 + a - c, 1 + b - c; 2 - c; v)$$
$$= \frac{\Gamma[2 - c]\Gamma[c - a - b]}{\Gamma[1 - a]\Gamma[1 - b]} (1 - x)^{\alpha_1} 2F_1(b, a; a + b - c + 1, 1 - v)$$
$$+ \frac{\Gamma[2 - c]\Gamma[a + b - c]}{\Gamma[1 + a - c]\Gamma[1 + b - c]} (1 - x)^{\alpha_1} (1 - v)^{c - a - b}$$
$$\times 2F_1(c - b, c - a; c - a - b + 1, 1 - v).$$  \hspace{1cm} (41)

If $G\mu/l << 1$, it is possible to approximate $\alpha_1 \approx -\frac{i\omega r_h}{2}$ and $\alpha_1 + c - a - b \approx +\frac{i\omega r_h}{2}$, while the leading expansion close to the horizon in equation (41) can be expressed as:

$$R(x \to 1) \Rightarrow \frac{\Gamma[2 - c]\Gamma[c - a - b]}{\Gamma[1 - a]\Gamma[1 - b]} (1 - x)^{-i\frac{\omega r_h}{2}} + \frac{\Gamma[2 - c]\Gamma[a + b - c]}{\Gamma[1 + a - c]\Gamma[1 + b - c]} (1 - x)^{+i\frac{\omega r_h}{2}}. \hspace{1cm} (42)$$

The ingoing and outgoing activity at the horizon was given by the terms $(1 - x)^{\mp i\frac{\omega r_h}{2}}$ in equation (42). To satisfy the boundary condition at the horizon, the outgoing wave can be eliminated by setting $1/\Gamma[-n] = 0$ for which $n = 0, 1, 2, 3, \ldots$ i.e.,

$$1 + a - c = -n; \hspace{1cm} 1 + b - c = -n.$$  \hspace{1cm} (43)

At the end, the quasinormal frequencies are obtained as:

$$\omega r_h = -i(2n + \frac{3}{2}) \pm \frac{i}{2} \sqrt{1 - 4m^2}.$$  \hspace{1cm} (44)

If $m^2 > 1/4$, $\omega$ which appears in equation (44) can be rewritten as:

$$\omega r_h = \pm \sqrt{m^2 - \frac{1}{4}} - i(2n + \frac{3}{2}).$$  \hspace{1cm} (45)

Another study [15] calculates the quasinormal modes for the electromagnetic potential of the MTZ spacetime through the monodromy method, [16] for large values of $n$

$$\omega \approx -2ni \left(1 + \frac{2G\mu}{\pi i} \ln G\mu \right),$$

in which our own findings for large $n$, $\omega \rightarrow -2ni/l + O(G\mu/l)^1$. Another thing worth mentioned is our result in equation (45) also covering for any number $n$, while the analytic results in [15]
is approximated for the large number $n$. The leading orders resulting in both these cases were the same, but this did not apply to the higher order of $O(G\mu/l)$ which is as anticipated since the higher order terms were discarded in equation (36). The mathematical technique employed in the monodromy method in [15] is taking the advantage of letting the spacetime be complex and perform some contours around the black hole singularities and connect them through the approximated solutions near each singularity. This would give the quasinormal modes and their frequencies. However the method gives not much of physical interpretation. Contrary to our procedure, we more physically approach in solving the problem. For example we separate and extract the singularities from the wave equation (28) by using equation (29) and their relations in equation (33) and equation (34).

6. Conclusion
In this study, a solution is provided for the electromagnetic potential differential equation in the context of quasinormal modes, which concurs with the outcomes obtained from an alternative approach to calculation. The outcome is therefore not only a suitable result, but also a novel alternative process which can be applied to solve problems involving the quasinormal mode. Future research may focus on a number of interesting points. For instance, this result may be considered as the zero order perturbation, whereupon the techniques and procedures can be applied to perform the first order perturbation. There are very interesting aspects related to the perturbed electromagnetic potential, such as the particular ways the properties and behavior may influence the phase transition, which will be worthy of further investigation.

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