ON BIFURCATION DELAY: AN ALTERNATIVE APPROACH USING GEOMETRIC SINGULAR PERTURBATION THEORY

TING-HAO HSU

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Abstract. To explain the phenomenon of bifurcation delay, which occurs in planar systems of the form
\[ \dot{x} = \epsilon f(x, z, \epsilon), \quad \dot{z} = g(x, z, \epsilon)z, \]
where \( f(x, 0, 0) > 0 \) and \( g(x, 0, 0) \) changes sign at least once on the \( x \)-axis, we use the Exchange Lemma in Geometric Singular Perturbation Theory to track the limiting behavior of the solutions. Using the trick of extending dimension to overcome the degeneracy at the turning point, we show that the limiting attracting and repulsion points are given by the well-known entry-exit function, and the maximum of \( z \) on the trajectory is of order \( \exp(-1/\epsilon) \). Also we prove smoothness the return map up to arbitrary finite order in \( \epsilon \).

1. Introduction

Consider the planar system
\[ \dot{x} = \epsilon f(x, z, \epsilon) \]
\[ \dot{z} = g(x, z, \epsilon)z \]
with \( x \in \mathbb{R}, z \in \mathbb{R}, f \) and \( g \) are \( C^1 \) functions satisfying
\[ f(x, 0, 0) > 0; \quad g(x, 0, 0) < 0 \text{ for } x < 0 \text{ and } g(x, 0, 0) > 0 \text{ for } x > 0. \]

Note that (1.1) is a slow-fast system [JK94, Kue15] with fast variable \( z \) and slow variable \( x \). Fix any \( x_0 < 0 \) and choose \( z_0 > 0 \) small enough so that \( g(x_0, z_0) < 0 \) for all \( z \in [0, z_0] \). When \( \epsilon = 0 \), it is clear that the trajectory starting at \((x_0, z_0)\) goes straight to \((x_0, 0)\). The \( x \)-axis is attracting when \( x < 0 \) and repelling when \( x > 0 \) since \( g(x, 0, 0) \) changes sign at \( x = 0 \). For \( \epsilon > 0 \), besides being attracted by the \( x \)-axis, the trajectory also moves right at speed of order \( \epsilon \). After the trajectory passes \( x = 0 \), the \( x \)-axis becomes repelling, so the trajectory tends to move away from the \( x \)-axis. See Fig 1a. However, it is well known that, for small \( \epsilon > 0 \), the trajectory does not immediately leave the vicinity of the \( x \)-axis after crossing the origin. Instead, the trajectory stays at the \( x \)-axis until it approaches the point \((x_1, 0)\) which satisfies
\[ \int_{x_0}^{x_1} \frac{g(x, 0, 0)}{f(x, 0, 0)} \, dx = 0. \]

See Fig 1b. This phenomenon has been called “bifurcation delay” [Ben91], “Pontryagin delay” [MKKR94], or “delay of instability” [Lin00], and the function \( x_0 \mapsto x_1 \) implicitly defined by (1.3) is called the entry-exit [Ben81] or way in-way out [Die84] function.

Bifurcation delay has been studied by various methods in the literature, including asymptotic expansion [Hab79, MKKR94], comparison to solutions constructed by separation of variables [Sch85], gradient estimates using the variational equation [DM08], and the blow-up method of geometric singular perturbation theory [DMS16].

Our results stated below are included in the literature, but the proof in this note provides a new approach using geometric singular perturbation theory.

Main Theorem. Consider (1.1), where \( f \) and \( g \) are \( C^{r+1} \), \( r \geq 1 \), and satisfy (1.2). Choose \( x_0 < 0 \) such that there exists \( x_1 > 0 \) satisfying (1.3). If \( z_0 > 0 \) is small enough, then the following holds. Let \( \gamma_\epsilon \) be the
Fig 1. (a) When \( \epsilon = 0 \), the x-axis is a line of equilibria for (1.1e). The trajectory starting at \((x_0, z_0)\) goes straight down to the x-axis. Separated by the turning point \((x, z) = (0, 0)\), the x-axis changes from attracting to repelling. (b) When \( \epsilon > 0 \) and small, the trajectory of (1.1e) starting at \((x_0, z_0)\) first tends to the x-axis, and then it turns at \((x_0, 0)\). The trajectory turns again when it approaches the point \((x_1, 0)\) as \(\epsilon\) approaches zero.

trajectory of (1.1e) that starts at the point \((x_0, z_0)\) and ends at the cross section \(\{x > 0, z = z_0\}\). Then

\[
\tilde{\gamma}_\epsilon \to \tilde{\gamma}_1 \cup \tilde{\gamma}_0 \cup \tilde{\gamma}_2 \equiv \left( \{x_0\} \times [0, z_0] \right) \cup \left( [x_0, x_1] \times \{0\} \right) \cup \left( \{x_1\} \times [0, z_0] \right) \quad \text{as} \quad \epsilon \to 0
\]

in the sense of point-sets, and

\[
\min_{(x, z) \in \gamma_\epsilon} \int_0^{x_0} \frac{g(x, 0, 0)}{f(x, 0, 0)} \, dx = 0.
\]

Moreover, for any compact interval \(K \subset (-\infty, 0)\) such that \(x_1\) is well-defined by (1.3) for each \(x_0 \in K\), there exist \(\epsilon_0 > 0\) such that if we set \(x_{1, \epsilon}\) by \(x_{1, 0} = x_1\) and

\[
(x_{1, \epsilon}, z_0) = \tilde{\gamma}_\epsilon \cap \{x > 0, z = z_0\}, \quad \epsilon \in (0, \epsilon_0],
\]

then \(x_{1, \epsilon}\) is a \(C^r\) function of \((x_0, \epsilon)\) on \(K \times [0, \epsilon_0]\).

In Section 2 we analyze the structure of (1.1e) as a slow-fast system. In Section 3 we state a simple case of the Exchange Lemma which can be applied in our context. In Section 4 we complete the proof of the Main Theorem.

2. SINGULAR CONFIGURATION

Fix \(z_0 > 0\) small enough so that \(g\) is negative on \(\tilde{\gamma}_1\) and is positive on \(\tilde{\gamma}_2\), where \(\tilde{\gamma}_i\) are defined in (1.4).

From the equation of \(\dot{x}\) in (1.1e), to prove (1.4) we expect the travel time of \(\gamma_0\) to be of order \(1/\epsilon\), so we set \(\tau = \epsilon t\), where \(t\) is the time variable in (1.1e), and expect the change in \(\tau\) along the \(\gamma_\epsilon\) to be of order 1. Fix any \(\Delta > 0\) with \(\Delta < \frac{1}{2} \min\{|x_0|, |x_1|\}\). Set

\[
I_\epsilon = \left\{ \begin{bmatrix} x \\ z \\ \tau \end{bmatrix} = \begin{bmatrix} x_0 \\ z_0 \\ \sigma \end{bmatrix} : |\sigma| < \Delta \right\}, \quad J_\epsilon = \left\{ \begin{bmatrix} x \\ z \\ \tau \end{bmatrix} = \begin{bmatrix} x_1 + \sigma \\ z_0 \\ \sigma \end{bmatrix} : |\sigma| < \Delta \right\},
\]

where \(\tau_1 > 0\) is defined later in (2.15). Our strategy is to show that the manifolds evolved from \(I_\epsilon\) and \(J_\epsilon\) along the flow (1.1e) have nonempty intersection.
I_0 (x_0; z_0; 0) = J_0 (x_1; z_0; = 1)

Fig 2. The manifolds evolved from $I_\epsilon$ and $J_\epsilon$ are two surfaces nearly parallel to $\{z = 0\}$ in $(x, z, \tau)$-space.

To track the slow time variable $\tau$, we append the equation $\dot{\tau} = \epsilon$ to the system (1.1\epsilon). That is, we write (1.1\epsilon) as

\[
\begin{align*}
\dot{x} &= \epsilon f(x, z, \epsilon) \\
\dot{z} &= g(x, z, \epsilon)z \\
\dot{\tau} &= \epsilon.
\end{align*}
\]

Intuitively, when $\epsilon > 0$ is small, $I_\epsilon$ and $J_\epsilon$ first go straight to the plane $\{z = 0\}$, and then go along the flow in $(x, \tau)$-plane. Although the two surfaces evolved from $I_\epsilon$ and $J_\epsilon$ may go along close to a common trajectory on the plane, they are both nearly parallel to $\{z = 0\}$. Thus it is not clear whether they intersect; see Fig 2. To overcome this difficulty, we introduce the new variable

\[
\zeta = \epsilon \log(1/z).
\]

Using $\dot{\zeta} = -\epsilon \dot{z}/z$, the system (2.7\epsilon) can be expressed as

\[
\begin{align*}
\dot{x} &= \epsilon f(x, z, \epsilon) \\
\dot{z} &= g(x, z, \epsilon)z \\
\dot{\zeta} &= -\epsilon g(x, z, \epsilon) \\
\dot{\tau} &= \epsilon.
\end{align*}
\]

Note that (2.9\epsilon) a slow-fast system, which has two distinguished limiting systems. The limiting fast system (or layer problem), obtained by setting $\epsilon = 0$ in (2.9\epsilon), is

\[
\begin{align*}
\dot{z} &= g(x, z, 0)z \\
\dot{x} &= 0, \quad \dot{\zeta} = 0, \quad \dot{\tau} = 0,
\end{align*}
\]

and the limiting slow system (or reduced problem) is

\[
\begin{align*}
z &= 0 \\
x' &= f(x, 0, 0) \\
\zeta' &= -g(x, 0, 0) \\
\tau' &= 1,
\end{align*}
\]

where $t$ is $\frac{d}{dt}$. The spirit of Geometric Singular Perturbation Theory is to first study the limiting systems, which have no parameter $\epsilon$ and have lower dimension, and then make conclusion about the full system for $\epsilon > 0$. 

In \((x, z, \zeta, \tau)\)-space, \(\mathcal{I}_e\) and \(\mathcal{J}_e\) are parametrised as

\[
\mathcal{I}_e = \left\{ \begin{pmatrix} x \\ z \\ \zeta \\ \tau \end{pmatrix} = \begin{pmatrix} x_0 \\ z_0 \\ 0 \\ 0 \end{pmatrix} + \sigma \begin{pmatrix} 0 \\ 0 \\ 0 \\ \log(1/z_0) \end{pmatrix} + \epsilon \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} : |\sigma| < \Delta \right\}
\]

and

\[
\mathcal{J}_e = \left\{ \begin{pmatrix} x \\ z \\ \zeta \\ \tau \end{pmatrix} = \begin{pmatrix} x_0 \\ z_0 \\ 0 \\ \tau_1 \end{pmatrix} + \sigma \begin{pmatrix} 0 \\ 0 \\ 0 \\ \log(1/z_0) \end{pmatrix} + \epsilon \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} : |\sigma| < \Delta \right\}.
\]

From now on we identify the \((x, \zeta, \tau)\)-space with the set \(\{z = 0\}\) in \((x, z, \zeta, \tau)\)-space (and temporarily ignore the relation (2.8) between \(z\) and \(\zeta\)). Let \(\Lambda_L\) and \(\Lambda_R\) be the \(\omega\)- and \(\alpha\)-limit sets of \(\mathcal{I}_0\) and \(\mathcal{J}_0\) along the flow of (2.10). Then

\[
\Lambda_L = \left\{ \begin{pmatrix} x \\ \zeta \\ \tau \end{pmatrix} = \begin{pmatrix} x_0 \\ 0 \\ \sigma \end{pmatrix} : |\sigma| < \Delta \right\}, \quad \Lambda_R = \left\{ \begin{pmatrix} x \\ \zeta \\ \tau \end{pmatrix} = \begin{pmatrix} x_1 + \sigma \\ 0 \\ \tau_1 \end{pmatrix} : |\sigma| < \Delta \right\}.
\]

The following proposition describes the manifolds evolved from \(\Lambda_L\) and \(\Lambda_R\) along the slow flow (2.11).

**Proposition 1.** Assume \(x_0\) and \(x_1\) satisfy (1.3). Set

\[
\tau_1 = \int_{x_0}^{x_1} \frac{1}{f(x, 0, 0)} \, dx.
\]

Let \(M_L\) and \(M_R\) be the manifolds evolved from \(\Lambda_L\) and \(\Lambda_R\) along the slow flow (2.11). Then \(M_L\) and \(M_R\) intersect transversally along a curve \(\gamma_0\) in \((x, \zeta, \tau)\)-space. Moreover, the projection of \(\gamma_0\) in \((x, z)\)-space is \(\hat{\gamma}_0\) defined in (1.4), and the minimum of \(\zeta\) on \(\gamma_0\) equals \(\zeta_0\) defined in (1.5).

See Fig 3 for an illustration of Proposition 1.

**Proof.** By integrating (2.11), a portion of \(M_L\) can be parametrized as

\[
M_L = \left\{ \begin{pmatrix} x \\ \zeta \\ \tau \end{pmatrix} = \begin{pmatrix} x \\ \zeta_{-}(x; \hat{\tau}_0) \\ \tau_{-}(x; \hat{\tau}_0) \end{pmatrix} : x \in [x_0, x_1], \hat{\tau}_0 \in [-\Delta, \Delta] \right\}
\]

where

\[
\zeta_{-}(x; \hat{\tau}_0) = \int_{x_0}^{x} \frac{g(r, 0, 0)}{f(r, 0, 0)} \, dr, \quad \tau_{-}(x; \hat{\tau}_0) = \hat{\tau}_0 + \int_{x_0}^{x} \frac{1}{f(r, 0, 0)} \, dr,
\]

and a portion of \(M_R\) can be parametrized as

\[
M_R = \left\{ \begin{pmatrix} x \\ \zeta \\ \tau \end{pmatrix} = \begin{pmatrix} x \\ \zeta_{+}(x; \hat{\tau}_1) \\ \tau_{+}(x; \hat{\tau}_1) \end{pmatrix} : x_1 \in [\hat{x}_1 - \Delta, \hat{x}_1 + \Delta] \right\}
\]

where

\[
\zeta_{+}(x; \hat{\tau}_1) = \int_{x}^{\hat{x}_1} \frac{g(r, 0, 0)}{f(r, 0, 0)} \, dr, \quad \tau_{+}(x; \hat{\tau}_1) = \tau_1 - \int_{x}^{\hat{x}_1} \frac{1}{f(r, 0, 0)} \, dr.
\]

From the assumption (1.3) and the definition (2.15), we have

\[
\zeta_{-}(0; 0) = \int_{x_0}^{0} \frac{|g(r, 0, 0)|}{f(r, 0, 0)} \, dr = \zeta_{+}(0; x_1), \quad \tau_{-}(0; 0) = \int_{x_0}^{0} \frac{1}{f(r, 0, 0)} \, dr = \tau_{+}(0; x_1).
\]

From uniqueness of solution of (2.11), it follows that \(M_L\) and \(M_R\) intersect along the curve

\[
\gamma_0 = \left\{ \begin{pmatrix} x \\ \zeta \\ \tau \end{pmatrix} = \begin{pmatrix} x \\ \zeta_{-}(x; 0) \\ \tau_{-}(x; 0) \end{pmatrix} = \begin{pmatrix} x \\ \zeta_{+}(x; x_1) \\ \tau_{+}(x; x_1) \end{pmatrix} : x \in [x_0, x_1] \right\}.
\]
Similarly, taking derivatives in (2.23) γ to show that the intersection is transversal. Clearly the projection of γ₀ in (x, z)-space is ̂γ₀ and the minimum of ζ on γ₀ is ζ₀ defined in (1.5). It remains to show that the intersection is transversal.

Fix any ̂x ∈ [x₀, x₁]. Let ̂q = γ₀ ∩ {x = ̂x}. Taking derivatives in x and τ₀ in (2.16), we obtain

\[ T_0 M_L = \text{Span} \left\{ \begin{bmatrix} f(̂x, 0, 0) \\ -g(̂x, 0, 0) \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}. \]

Similarly, taking derivatives in x and x₁ in (2.18) we obtain

\[ T_0 M_R = \text{Span} \left\{ \begin{bmatrix} f(̂x, 0, 0) \\ -g(̂x, 0, 0) \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}. \]

Since f(̂x, 0, 0) > 0 and g(x₁, 0, 0) > 0, the union of T₀M_L and T₀M_R spans the (x, ζ, τ)-space. This means the intersection is transversal.

For the limiting fast system (2.10), the forward trajectory of (x₀, z₀, 0, 0) and the backward trajectory of (x₁, z₀, 0, 0) are

\[
\gamma_1 = \left\{ \begin{bmatrix} x \\ z \\ ζ \\ τ \end{bmatrix} = \begin{bmatrix} x₀ \\ z₀ \\ 0 \\ 0 \end{bmatrix} : z ∈ [0, z₀] \right\} \quad \text{and} \quad \gamma_2 = \left\{ \begin{bmatrix} x \\ z \\ ζ \\ τ \end{bmatrix} = \begin{bmatrix} x₁ \\ z₀ \\ 0 \\ 0 \end{bmatrix} : z ∈ [0, z₀] \right\},
\]

respectively. Combined with γ₀ given in Proposition 1, now we have the configuration

\[(x₀, z₀, 0, 0) \xrightarrow{γₙ} (x₀, 0, 0, 0) \xrightarrow{γ₀} (x₁, 0, 0, 0) \xrightarrow{γ₂} (x₁, z₀, 0, 0),\]

where γ₁ and γ₂ are trajectories of the limiting fast system (2.10), and γ₀ is a trajectory of the limiting slow system (2.11).

Note that for each γᵢ (i = 0, 1, 2), the projection in (x, z)-space is ̂γᵢ defined in (1.4), and the projection in (x, z, τ)-space is shown in Fig 2. In (x, ζ, τ)-space, γ₁ and γ₂ collapse to points, and γ₀ is as shown in Fig 3.

To track the manifolds evolved from Λ and Λ, we make the following simple observation.

**Proposition 2.** Consider the system (2.9e) in (x, z, ζ, τ)-space. Suppose ε > 0. Let Λ* and Λ* be the manifolds evolved from Λ and Λ. Then the following statements are equivalent.

(i) Λ* and Λ* have nonempty intersection in (x, z, ζ, τ)-space.
(ii) The projections of Λ* and Λ* in (x, z, τ)-space have nonempty intersection.
(iii) The projections of Λ* and Λ* in (x, ζ, τ)-space have nonempty intersection.
Proof. On $I^*$ and $J^*$ the relation $z = \epsilon \log(1/|z|)$ holds, so these statements are equivalent. \hfill \Box

From this proposition, to prove (1.4) we only need to check that the projections of $I^*$ and $J^*$ in $(x, \zeta, \tau)$-space have nonempty intersection. This will be confirmed in Section 4 using the Exchange Lemma stated in Section 3.

3. A Simple Case of the Exchange Lemma

To track manifolds for the system (2.9e), we will apply the following simple case of the Exchange Lemma.

**Theorem 1.** Consider a system for $(b, c) \in \mathbb{R} \times \mathbb{R}^l$, $l \geq 1$, of the form
\[
\begin{align*}
\dot{b} &= \rho(b, c, \epsilon)b \\
\dot{c} &= \epsilon h(c, \epsilon)
\end{align*}
\]
defined on a set of the form $[0, \Delta] \times U$, where $\Delta > 0$ and $U$ is a bounded open set in $\mathbb{R}^l$. Assume the coefficients are $C^r$ functions, $r \geq 1$, satisfying
\[
(3.25) \quad \rho(b, c, \epsilon) < -\nu \quad \text{and} \quad |h(c, \epsilon)| > \nu
\]
for some $\nu > 0$. Let $\Lambda$ be a $(\sigma - 1)$-dimensional compact manifold in $\mathbb{R}^l$, $1 \leq \sigma \leq l$. Suppose for each $c \in \Lambda$ we have $h(c, \epsilon) \notin T_c \Lambda$. That is, the slow flow
\[
(3.26) \quad c' = h(c, 0)
\]
is not tangent to $\Lambda$. Fix any $\tau_+ > \tau_- > 0$ satisfying
\[
\Lambda \circ [0, \tau_+ ] \subset U,
\]
where $\circ$ is the solution operator for (3.26), and set $M = \Lambda \circ [\tau_-, \tau_+]$. Then there exists a positive number $\Delta_1 < \Delta$ such that for any $b_0 \in (0, \Delta_1)$ and $C^r$ function $\varphi(c, \epsilon)$, if we set
\[
(3.27) \quad I^*_{\text{in}} = \{(b, c) \in (b_0, \epsilon \varphi(\hat{c}, \epsilon)) : \hat{c} \in \Lambda\},
\]
then there is on neighborhood $V$ of $M$ such that
\[
(3.28) \quad I^* \cap V \text{ is } C^r \text{ O}(\epsilon)-\text{close to } M,
\]
where $I^*$ is the manifold evolved from $I^*_{\text{in}}$ along (3.24e).

**Proof.** The assertion (3.28) with $r = 1$ follows from the $(k + \sigma)$-Exchange Lemma [JT09, Theorem 6.7] with $(k, m) = (0, 1)$. The $C^r$ smoothness follows from the General Exchange Lemma [Sch08]. \hfill \Box

See Fig 4 for an illustration of the theorem. The assertion (3.28) means that there is a $C^r$ function $\tilde{b}(c, \epsilon)$ defined on $M \times [0, \epsilon_0]$, where $\epsilon_0 > 0$, satisfying
\[
\tilde{b}(c, \epsilon), \epsilon) \in I^* \quad \forall c \in M, \epsilon \in (0, \epsilon_0]
\]
and $\|	ilde{b}(\cdot, \epsilon)\|_{C^r(M)} = O(\epsilon)$.

**Remark 1.** The theorem can also be interpreted as a special case of the General Exchange Lemma in Schecter [Sch08], the Strong $\lambda$-Lemma in Deng [Den89], or the $C^r$-Inclination Theorem in Brunovsky [Bru99].

**Remark 2.** Let $\{\Lambda_\mu\}_{\mu \in A}$, where $A$ is a compact interval, be a $C^r$ family of $(\sigma - 1)$-dimensional manifolds. If we replace $\Lambda$ by $\Lambda_\mu$ and denote the corresponding $I_\mu$ by $I_{\epsilon, \mu}$. Then, as a result of the General Exchange Lemma [Sch08], $I_{\epsilon, \mu} \cap V$ is uniformly $C^r$ for $(\mu, \epsilon) \in A \times (0, \epsilon_0]$.

**Corollary 3.** Assume all the assumptions in Theorem 1 except we replace (3.24e) by
\[
(3.29e) \quad \begin{align*}
\dot{b} &= \rho(b, c, \epsilon)b \\
\dot{c} &= \epsilon h(c) + \epsilon bE(b, c, \epsilon),
\end{align*}
\]
where the coefficients are $C^{r+1}$ functions, $r \geq 1$. Then conclusion in the theorem still holds. Moreover, the mapping
\[
\Pi_\epsilon : I^*_\epsilon \cap V \to I_\epsilon \times [\tau_-, \tau_+], \quad q_\epsilon \mapsto (q^*_\epsilon, \epsilon),
\]
defined by
\[
q_\epsilon = q^*_\epsilon \cdot (\tau_\epsilon/\epsilon),
\]
where $\cdot$ is the solution operator for (3.29e), is uniformly $C^r$ in $\epsilon$.

Proof. By Fenichel’s Theorem [Fen79], there exists $\Delta_1 > 0$ and a $C^{r+1}$ function $\pi_\epsilon^+$ of the form  

$$\pi_\epsilon^+(b, c) = (0, c + \epsilon b \theta(b, c, \epsilon))$$

defined on a neighborhood of  

$$[0, \Delta_1] \times (\Lambda \circ [0, \tau_+])$$

such that  

$$\left. \pi_\epsilon^+ \left( (b, c) \cdot t \right) = \left( \pi_\epsilon^+(b, c) \right) \cdot t, \quad t \geq 0, \right.$$  

(3.30)

where $\cdot$ is the solution operator of (3.29e). This implies that the new coordinates  

$$(\hat{b}, \hat{c}) = (b, c - \epsilon b \theta(b, c, \epsilon))$$

brings (3.29e) into a system of the form  

$$(3.32e) \quad \begin{align*}
\dot{\hat{b}} &= \hat{\rho}(\hat{b}, \hat{c}, \epsilon) \hat{b} \\
\dot{\hat{c}} &= \hat{\epsilon} \hat{h}(\hat{c}, \epsilon),
\end{align*}$$

where $\hat{\rho}$ and $\hat{h}$ are $C^r$ functions, and $\hat{h}$ satisfies  

$$\hat{h}(\hat{c}, \epsilon)|_{b=0, \epsilon=0} = h(c, 0).$$

(3.33)

Decrease $\Delta_1$ if necessary so that the conclusion in Theorem 1 holds. Choose any $b_0 \in (0, \Delta_1]$ and $C^{r+1}$ function $\varphi(c, \epsilon)$, and set $\mathcal{I}^m_\epsilon$ by (3.27). Then we can write $\mathcal{I}^m_\epsilon$ as  

$$\mathcal{I}^m_\epsilon = \{(b, c) : (\hat{b}, \hat{c}) = (b_0, \hat{c} + \epsilon \hat{\varphi}(\hat{c}, \epsilon)) : \hat{c} \in \Lambda\}$$

for some $C^{r+1}$ function $\hat{\varphi}$. By Theorem 1 and (3.33) there exists a neighborhood $V$ of  

$$\tilde{M} = \{(b, c) : (\hat{b}, \hat{c}) \in \Lambda \circ [\tau_-, \tau_+]\},$$

where $\circ$ is the solution operator of (3.26), such that  

$$\mathcal{I}^m_\epsilon \cap V \text{ is } C^r O(\epsilon)\text{-close to } \tilde{M}.$$  

From (3.31) we know $\tilde{M}$ is $C^{r+1} O(\epsilon)$-close to $M$, so we obtain (3.28).

Given $q = (b_\epsilon, c_\epsilon) \in \mathcal{I}^m_\epsilon \cap V$, write $(q^m_\epsilon, \tau_\epsilon) = ((b_0, c^m_\epsilon), \tau_\epsilon) = \Pi_{s}(q)$. Let  

$$\pi_\epsilon^+(b_\epsilon, c_\epsilon) = (0, c^1_\epsilon) \quad \text{and} \quad \Lambda_\epsilon = \pi_\epsilon^+(\mathcal{I}^m_\epsilon)$$

FIG. 4. Consider the system $$(\dot{b}, \dot{c}_1, \dot{c}_2) = (-b, \epsilon, 0)$. Suppose $\mathcal{I}^m_\epsilon = \{b_0\} \times \Lambda$. Then the projection of $\mathcal{I}^m_\epsilon$ along the stable fibers is $\Lambda$. The Exchange Lemma assures that $\mathcal{I}^m_\epsilon$, the manifold evolved from $\mathcal{I}^m_\epsilon$, is $C^1 O(\epsilon)$-close to $\Lambda \circ [\tau_-, \tau_+]$, where $\circ$ is the solution operator for the slow flow $$(c'_1, c'_2) = (1, 0).$$. }
Since $\pi^+$ is uniformly $C^r$ and the flow $c' = h(c, \epsilon)$ is non-tangential to $\Lambda_\epsilon$, we can uniquely define a uniformly $C^r$ function $c^1_\epsilon \mapsto (c^0_\epsilon, \tau_\epsilon)$ by

$$c^1_\epsilon = c^0_\epsilon \bullet \tau_\epsilon, \quad c^0_\epsilon \in \Lambda_\epsilon,$$

where $\bullet$ is the solution operator for $c' = h(c, \epsilon)$. From (3.30) we have

$$c^i_{\epsilon} = ((\pi^+_{\epsilon})^{-1}(0, c^0_{\epsilon})) \cap \{b = b_0\}.$$  

Note that the restriction of $\pi^+_{\epsilon}$ on $\{b = b_0, c \in V\}$ is a local diffeomorphism into $\{b = 0, c \in V\}$. Hence the function $c^i_{\epsilon} \mapsto c^i_{\epsilon}$ is well-defined and is uniformly $C^r$.

Consider the function $q_\epsilon \mapsto (c^i_{\epsilon}, \tau_\epsilon)$ as the composition of the following sequence

$$q_\epsilon \mapsto c^1_\epsilon \mapsto (c^0_\epsilon, \tau_\epsilon) \mapsto (c^i_{\epsilon}, \tau_\epsilon).$$

We have seen that each mapping in the sequence is uniformly $C^r$ in $\epsilon$. Hence the function $q_\epsilon \mapsto q^i_{\epsilon} = ((b_0, c^i_{\epsilon}), \tau_\epsilon)$ is uniformly $C^r$.

\section*{4. Completing the Proof of the Main Theorem}

Let $\mathcal{I}_\epsilon^*$ and $\mathcal{J}_\epsilon^*$ be the manifolds evolved from $\mathcal{I}_\epsilon$ and $\mathcal{J}_\epsilon$, respectively, defined in (2.12)-(2.13), along the flow (2.9ς).

\begin{proposition}
Fix any positive number $\delta < \min\{\frac{|x_0| - \Delta}{4}, \frac{x_1 - \Delta}{4}\}$. Let

$$\gamma^0_\delta = \gamma_0 \cap \{x \in [x_0 + \delta, x_1 - \delta]\} \quad \text{and} \quad M^\delta_{L,R} = M_{L,R} \cap \{x \in [x_0 + \delta, x_1 - \delta]\}.$$

Then there is a neighborhood $V$ of $\gamma^0_\delta$ such that

\begin{align}
\mathcal{I}^*_\epsilon \cap V &\quad \text{is } C^r \ O(\epsilon)\text{-close to } M^\delta_{L,R} \cap V \quad \text{(4.34)} \\
\mathcal{J}^*_\epsilon \cap V &\quad \text{is } C^r \ O(\epsilon)\text{-close to } M^\delta_{L,R} \cap V \quad \text{(4.35)}
\end{align}

Moreover, if we consider $\mathcal{I}_\epsilon = \mathcal{I}_{\epsilon, x_0}$ as a function of $x_0$ in the definition (2.12), then there exists $\epsilon_0 > 0$ and $\delta_1 > 0$ such that

$$\mathcal{I}^*_{\epsilon, x} \cap V \quad \text{is uniformly } C^r \text{ for } (x, \epsilon) \in [x_0 - \delta_1, x_0 + \delta_1] \times (0, \epsilon_0].$$

\end{proposition}

\begin{proof}
Choose $\Delta_1 > 0$ so that the conclusions of Corollary 3 holds for (2.9ς) restricted on

$$z \in [0, \Delta_1], \quad \text{dist}((x, \zeta, \tau), \Lambda_L) \leq \delta,$$

(by the choice of $\delta$, this region lies in $\{x < 0\}$) with

$$\Lambda = \Lambda_L, \quad \tau_- = \delta, \quad \tau_+ = 2\delta.$$

Set

$$\mathcal{I}^i_{\epsilon} = \mathcal{I}^*_{\epsilon} \cap \{x < 0, z = \Delta_1\}. \quad \text{(4.36)}$$

It is clear that

$$\mathcal{I}^i_{\epsilon} = \left\{ \begin{array}{c} x \\ z \\ \zeta \\ \tau \end{array} \right\} = \left( \begin{array}{c} x_0 \\ \Delta_1 \\ 0 \\ 0 \end{array} \right) + \sigma \left( \begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \end{array} \right) + \epsilon \left( \begin{array}{c} \varphi_1 \\ 0 \\ \log(1/\Delta_1) \\ \varphi_2 \end{array} \right) : |\sigma| < \Delta \right\}$$

for some $C^{r+1}$ functions $\varphi_i = \varphi_i(x, \tau, \epsilon)$. By Corollary 3, there exists a neighborhood $\tilde{V}_L$ of

$$M_L \cap \{x \in [x_0 + \frac{\delta}{2}, x_0 + 2\delta]\}$$

such that

$$\mathcal{I}^*_\epsilon \cap V_L \quad \text{is } C^1 \ O(\epsilon)\text{-close to } M_L \cap V_L \quad \text{(4.37)}$$

In particular, setting

$$\mathcal{I}^\delta_{\epsilon} \equiv \mathcal{I}^*_\epsilon \cap \{x = x_0 + \delta\} \quad \text{and} \quad M^\delta_{L} \equiv M_L \cap \{x = x_0 + \delta\}$$

we have

$$\mathcal{I}^\delta_{\epsilon} \quad \text{is } C^1 \ O(\epsilon)\text{-close to } M^\delta_{L} \quad \text{(4.38)}$$
Next we will track $I_e^*$ along $\gamma_0^\delta$ as the manifold evolved from $I_e^\delta$.

Using the relation $z = \exp(-\zeta/\epsilon)$, we write (2.9e) as
\[
\begin{align*}
\dot{x} &= \epsilon f(x, e^{-\zeta/\epsilon}, \epsilon) \\
\dot{\zeta} &= -\epsilon g(x, e^{-\zeta/\epsilon}, \epsilon) \\
\dot{\tau} &= \epsilon.
\end{align*}
\] (4.39e)

By a rescaling of time, the system is equivalent to
\[
\begin{align*}
x' &= f(x, e^{-\zeta/\epsilon}, \epsilon) \\
\zeta' &= -g(x, e^{-\zeta/\epsilon}, \epsilon) \\
\tau' &= 1.
\end{align*}
\] (4.40e)

Note that the system (4.40e) restricted in a neighborhood of $M_0^\delta$ is a regular perturbation of (2.11) since $\inf_{t \in I} \zeta > 0$. Also note that $M_0^\delta$ is evolved from $M_L^\delta$ along (2.11) and $I_e^*$ is evolved from $I_e^\delta$ along (4.39e), so it follows from (4.38) that there is neighborhood $V_L$ of $M_0^\delta$ such that the following is true:
\[
(4.41)
\]

The projection of $I_e^* \cap V_L$ in $(x, \zeta, \tau)$-space is $C^r O(\epsilon)$-close to $M_0^\delta$.

Using again the relation $z = \exp(-\zeta/\epsilon)$, we then obtain (4.34) with $V$ replaced by $V_L$. The $C^r$ smoothness of $I_e^* \cap V_L$ in $(x_0, \epsilon)$, as indicated in Remark 2, is a result of the General Exchange Lemma [Sch08]. Similarly, (4.41) holds with $V$ replaced by some neighborhood $V_R$ of $M_R^\delta$. Now take $V = V_L \cap V_R$. Then $V$ is a neighborhood of $\gamma_0^\delta$ and (4.34)-(4.35) hold.

In Proposition 1 we have seen that $M_L$ and $M_R$ intersect transversally along $\gamma_0$ in $(x, \zeta, \tau)$-space, since transversal intersections persist under $C^1$ perturbation (see e.g. [Lee13, Theorem 6.35]), it follows from (4.34)-(4.35) in Proposition 4 that the projections of $I_e^* \cap V$ and $J_e^* \cap V$ in $(x, \zeta, \tau)$-space intersect transversally along a curve $\gamma_e^\delta$ which is $C^1 O(\epsilon)$-close to $\gamma_0^\delta$ and is uniformly $C^r$ for $(x_0, \epsilon)$. Using the relation $z = \exp(-\zeta/\epsilon)$, as indicated in Proposition 2, we then recover an intersection curve $\gamma_e$ of $I_e^*$ and $J_e^*$ in $(x, z, \zeta, \tau)$-space.

From the uniqueness of solution of boundary value problems for (1.1e), the projection of $\gamma_e$ in $(x, z)$-space is the trajectory $\tilde{\gamma}_e$ defined in the statement of the Main Theorem. By construction $\gamma_e$ lies in a $O(\epsilon)$-neighborhood of the configuration (2.23), so (1.4) holds.

To prove the smoothness of the return map $(x_0, \epsilon) \mapsto x_{1, \epsilon}$, it suffices to show that
\[
q_e \equiv \gamma_e \cap \{ x > 0, z = 0 \}
\]
is uniformly $C^r$ for $(x_0, \epsilon) \in [x_0 - \Delta, x_0 + \Delta] \times (0, \epsilon_0]$. Let
\[
q_e^\delta \equiv \gamma_e \cap \{ x = x_1 - \delta \} \quad \text{and} \quad q_e^\text{out} \equiv \gamma_e \cap \{ x > 0, z = \Delta_1 \}
\]
where $\Delta_1$ is small enough so that the conclusion of Corollary 3 holds near $A_R$. Now consider the mapping $(x_0, \epsilon) \mapsto q_{1, \epsilon}$ as the composition of the following sequence:
\[
(x_0, \epsilon) \mapsto \psi_1 \mapsto \psi_2 \mapsto \psi_3 \mapsto q_{1, \epsilon}
\]
From Proposition 4 we know $\psi_1$ is uniformly $C^*$ from Corollary 3 we know $\psi_2$ is uniformly $C^r$. On the other hand, it is clear that $\psi_3$ is uniformly $C^r$ since it is the Poincaré map of a regularly perturbed flow along $\gamma_1$. Hence we conclude that the mapping $(x_0, \epsilon) \mapsto q_e$ is uniformly $C^r$ for $(x, \epsilon) \in [x_0 - \Delta, x_0 + \Delta] \times (0, \epsilon_0]$. This proves the Main Theorem.

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Department of Mathematics, The Ohio State University, Columbus, OH 43210
E-mail address: hsu.296@osu.edu