1. Introduction

In this paper, we will examine the concept of an embedding of a quadratic space and analyze its connection to Spin Groups and Composition algebras. By a quadratic space $(V, q)$, we mean a free module $V$, over a commutative ring, equipped with a quadratic form $q$.

**Definition 1.1.** Let $(V, q)$ be a quadratic space and $A$ be an associative algebra. We will say that $(V, q)$ is embedded in $A$ if $V \subseteq A$ and $q(v) = v\alpha(v) = \alpha(v)v$ where $\alpha$ is an isometry of $(V, q)$.

Familiar examples of embeddings are given by the normed division algebras over $\mathbb{R}$ - the Real numbers, Complex numbers, Quaternions and Octonions. In the above cases, we have $V = A$ and $\alpha$ is the involution such that $x\alpha(x)$ is the norm of $x \in A$. These algebras are examples of Composition algebras.

For any embedding of $(V, q)$ (in an algebra $A$) to be useful, one should be able to capture some geometric properties of $(V, q)$ through the algebraic structure of $A$. The Clifford algebra is an important example in this respect - for example reflections in $(V, q)$ correspond to conjugation of $V$ by elements in $Cl$ [see V.1.3, [2]].

Given an embedding of a quadratic space, we will analyze it by linking it to the Clifford Algebra. By trying to understand when and what embeddings are possible, we learn further about the structure of the quadratic spaces and the algebras in which they are embedded. We make progress in two directions:

- **Limitations on Embeddings**: Every free module has a ring of endomorphisms associated with it. So given a non-degenerate quadratic space $(V, q)$, it is natural to ask if it can be *embedded* in $\text{End}(V)$?

  We will see that such embeddings are possible only in dimensions 1, 2, 4 or 8. It turns out that Composition algebras can be embedded in their respective endomorphism rings and therefore we have a different perspective of Hurwitz’s celebrated theorem that Composition algebras (whose quadratic form is non-degenerate) exist only in dimensions 1, 2, 4 and 8.

  This might be a natural setting for Hurwitz’s theorem as one can immediately ask when it is possible to embed $(V, q)$ in $\text{End}(\text{End}V)$. By a similar argument one can show that this may happen only in
dimensions 1, 2, 4, 8 and 16, immediately raising the question as to whether the 16- dimensional Cayley Dickson algebra can be embedded in $\text{End}(\text{End}V)$. Similarly one may conjecture that embeddings in $\text{End}^k(V) = \text{End}^{k-1}(\text{End}V)$ characterize higher dimensional Cayley-Dickson algebras.

- Connecting two different embeddings of the same quadratic space: We derive general properties of embeddings in Sections 4-5 and study the connection to Clifford Algebras. Given an embedding of $(V, q)$ in an algebra $A$, we describe the Spin representation in $A$. It turns out that when there is an involution of $A$ that acts trivially on the underlying quadratic space $V$, the Spin group acts faithfully.

The special case of the Hyperbolic space has been worked out in detail in [CV] using what are called Suslin Matrices. For the hyperbolic quadratic space, $H(V) = V \oplus V^*$, where $q(v, w) = v_1 w_1 + \cdots + v_n w_n$, the behavior of the Spin groups depends on the parity of $\dim V$. This is because there are two types of involution depending on whether $\dim V$ is odd or even. We will return to the hyperbolic case in the final section of the paper. The present paper generalizes the results of [CV] to general quadratic spaces.

Though Clifford Algebras have been studied in detail, they may not always be easy to work with. Sometimes it might be useful to switch to a more concrete embedding (as in the case of Suslin Matrices) to work with low dimensional Spin and Epin (or Elementary Spin) groups. For instance, one can easily compute using Suslin Matrices, that $\text{Spin}(H(R^3)) \cong \text{SL}_4(R)$ (Theorem 7.1, [CV]). We conclude this paper with a few questions that might be of general interest.

1.2. Notation. Given any quadratic form $(V, q)$, we can associate a bilinear form $\langle v, w \rangle = q(v + w) - q(v) - q(w)$, for $v, w \in V$.

Definition 1.3. The quadratic space $(V, q)$ is said to be non-degenerate when $\langle v, x \rangle = 0$ holds for all $x \in V$, if and only if $v = 0$.

Remark 1.4. All quadratic spaces in the paper are assumed to be non-degenerate. All modules considered are free-modules over a commutative ring $R$.

1.5. General References. We begin with a few preliminaries on Clifford algebras. All the relevant algebras are described quickly, whenever they appear, so that the article remains accessible to a wide readership. For general literature on Clifford algebras and Spin groups over a commutative ring, the reader is referred to [E1], [E2] by H. Bass. Another good source is the book [K] by M.-A. Knus. To learn more about Suslin Matrices, see [L, Chapter III.7]. A recent survey of Suslin Matrices can be found at [RJ] by Ravi Rao and Selby Jose. To learn more about Octonions, the reader is recommended to check out the delightful article [Ba] by John Baez.
2. Preliminaries on Clifford Algebras

Perhaps the most important example of an embedding of a quadratic space is given by the Clifford algebra. Given any quadratic space \((V, q)\), its Clifford algebra \(Cl(V, q)\) (or simply, \(Cl\)) is the “freest” algebra generated by \(V\) subject to the condition \(x^2 = q(x)\) for all \(x \in V\).

More precisely, the Clifford algebra \(Cl(V, q)\) is the quotient of the tensor algebra

\[ T(V) = R \oplus V \oplus V^2 \oplus \cdots \oplus V^n \oplus \cdots \]

by the two sided ideal \(I(V, q)\) generated by all \(x \otimes x - q(x)\) with \(x \in V\).

2.1. Basic Properties of the Clifford Algebra \(Cl(V, q)\):

- **Z\(_2\) Grading of \(Cl\)**: The Clifford algebra \(Cl(V, q)\) is an associative algebra (with unity) over \(R\) with a linear map \(i : V \to Cl(V, q)\) such that \(i(x)^2 = q(x)\). The terms \(x \otimes x\) and \(q(x)\) appearing in the generators of \(I(V, q)\) have degrees 0 and 2 in the grading of \(T(V)\). By grading \(T(V)\) modulo 2 by even and odd degrees, it follows that the Clifford algebra has a \(Z\(_2\)\)-grading \(Cl = Cl_0 \oplus Cl_1\) such that \(V \subseteq Cl_1\) and \(Cl_i Cl_j \subseteq Cl_{i+j} (i, j \text{ mod } 2)\).

- **Universal Property**: The Clifford algebras has the following universal property. Given any associative algebra \(A\) over \(R\) and any linear map \(j : V \to A\) such that \(j(x)^2 = q(x)\) for all \(x \in V\),

there is a unique algebra homomorphism \(f : Cl(V, q) \to A\) such that \(f \circ i = j\).

- **Basis of \(Cl\)**: The elements of \(V\) generate the Clifford algebra. Furthermore, the following result implies that if \(\text{rank}(V) = n\), then \(\text{rank}(Cl) = 2^n\).

**Theorem 2.2.** (Poincare-Birkhoff-Witt) Let \(\{v_1, \cdots, v_n\}\) be a basis of \((V, q)\). Then \(\{v_1^{e_1} \cdots v_n^{e_n} : e_i = 0, 1\}\) is a basis of \(Cl(V, q)\).

For a simple proof, see Theorem IV. 1.5.1, [K].

2.3. Structure of Clifford algebras. The Clifford algebra \(Cl(V, q)\) is an example of a graded Azumaya algebra. Since we will use this fact once in the paper (in Section 3), we will briefly introduce graded Azumaya algebras before moving on. To learn more about the structure of Clifford algebras over a commutative ring, go to [B2] or [K].

An algebra \(A = A_0 \oplus A_1\) is said to be a \((Z\(_2\))\) graded algebra if \(A_i\) is a subgroup of \((A, +)\) and \(A_i A_j \subseteq A_{i+j} (i, j \text{ mod } 2)\).
For a graded-algebra $A$ as above, the elements in $h(A) = A_0 \cup A_1$ will be called the homogeneous elements of $A$. If $a \in h(a)$, we write $d(A) = i$ if $a \in A_i$, ($i = 0, 1$). An ideal $I \subseteq A$ is said to be graded if it is a direct sum of the intersections $I_i = I \cap A_i$.

The graded center of $A$ is the graded-subspace $\hat{C}(A)$ such that $c \in h(\hat{C}) \iff ca = (-1)^{d(a)c}ac$ for all $a \in h(A)$.

An algebra $A$ (over $R$) is said to be a graded central simple algebra if $A$ has no nontrivial graded ideals and $\hat{C}(A) = R$.

**Definition 2.4.** A is a Graded Azumaya algebra over a commutative ring $R$ if

1. $A$ is finitely generated as an $R$-module and
2. $A/(mA)$ is a graded central simple algebra $\forall$ maximal ideals $m \subset R$.

The graded tensor product of two algebras $A \hat{\otimes} B$ is defined as :

$$(a \hat{\otimes} b)(a' \hat{\otimes} b') = (-1)^{d(b)d(a')}aa' \hat{\otimes} bb'$$

for all homogeneous elements $a, a' \in A$ and $b, b' \in B$.

One can use the universal property of Clifford Algebras to compute the Clifford algebra of an orthogonal sum of quadratic spaces.

**Theorem 2.5.** The map $f : (V_1, q_1) \perp (V_2, q_2) \to Cl(V_1, q_1) \hat{\otimes} Cl(V_2, q_2)$ defined by $f(x_1 + x_2) = x_1 \hat{\otimes} 1 + 1 \hat{\otimes} x_2$ induces an isomorphism

$Cl(V_1 \perp V_2, q_1 \perp q_2) \cong Cl(V_1, q_1) \hat{\otimes} Cl(V_2, q_2)$.

**Theorem 2.6.** Let $(V, q)$ be a non-degenerate quadratic form. Then $Cl(V, q)$ is a graded Azumaya algebra.

A proof of the above result can be found in Section 2.3, [B2].

3. **Hurwitz’s theorem as an embedding theorem**

Every quadratic space $(V, q)$ can be embedded in its Clifford algebra. Another algebra associated with any $R$-module $V$ is its ring of endomorphisms $End(V)$. If we fix a basis for the free module $V$ with $\text{rank}(V) = n$, then $End(V)$ can be seen as the ring of $n \times n$ matrices with entries in $R$.

Given the extra structure of the quadratic space $(V, q)$, one may wonder if it is possible to relate the quadratic form to the multiplicative structure of $End(V)$.

**Theorem 3.1.** Suppose there is an embedding of a (non-degenerate) quadratic space $(V, q)$ in its ring of endomorphisms $End(V)$. Then $\text{rank}(V) \in \{1, 2, 4, 8\}$.

**Proof.** Suppose $End(V)$ contains a copy of $V$ and $\alpha$ is an isometry of $V$ such that

$q(v) = v\alpha(v) = \alpha(v)v$.

Let $A = End(V)$. Let $T : V \to M_2(A)$ defined by $T(x) = \left( \begin{array}{cc} \alpha(v) & x \\ 0 & 0 \end{array} \right)$. 

Clearly $T^2(v) = q(v)$ for all $v \in V$. By the universal property of Clifford algebras, there exists an $R$-linear homomorphism $\phi : Cl \rightarrow M_2(A)$ such that $\phi(v) = T(v)$. In fact, $\phi$ is a graded-homomorphism, where the even and odd elements of $M_2(A)$ are respectively of the form $(\alpha \beta)$ and $(\alpha \gamma)$.

Since $\phi$ is a $R$-linear homomorphism, by tensoring with $R/m$ for a maximal ideal $m$, we get a corresponding $R/m$-linear homomorphism $\phi \otimes R/m : Cl \otimes R/m \rightarrow M_2(A \otimes R/m)$. Since $Cl$ and $M_2(A)$ are free modules over $R$, we have

$$\dim(Cl \otimes R/m) = \text{rank}(Cl) = 2^n$$

and

$$\dim M_2(A \otimes R/m) = \text{rank}(M_2(A)) = 4n^2.$$

Since both $Cl \otimes R/m$ and $M_2(A) \otimes R/m$ are graded central simple algebras, it follows from the graded-version of the Double Centralizer Theorem [see Page 60, Theorem 2.35 of [TW]] that $\dim(Cl \otimes R/m)$ divides $\dim(M_2(A \otimes R/m))$ i.e.,

$$2^n \mid 4n^2$$

which is possible only for $n = 1, 2, 4, 8$. Summarizing, if a quadratic space $(V, q)$ can be embedded in $End(V)$ then $\text{rank}(V) = 1, 2, 4$ or 8.

3.2. Composition algebras. By an algebra over $R$, we mean a (not necessarily associative) algebra over $R$ with multiplicative identity.

Definition 3.3. A Composition algebra $A$ over a $R$ is an algebra equipped with a non-degenerate quadratic form $q$ such that, for all $x$ and $y$ in $A$,

$$q(xy) = q(x)q(y) \quad \text{(Composition law)}$$

There is an involution on $A$ given by

$$\overline{x} = -x + \langle x, 1 \rangle.$$

An easy check shows that $x\overline{x} = q(x)$ for $x \in A$. The following are a few consequences of the Composition law:

1. $\langle xy, xz \rangle = q(x)\langle y, z \rangle$. (Scaling law)

Replacing $y$ by $y + z$ in equation (1) gives

$$q(xy) + q(xz) + \langle xy, xz \rangle = q(x)[q(y) + \langle y, z \rangle + q(z)]$$

from which we cancel some terms.

2. $\langle xy, z \rangle = \langle y, xz \rangle$.

Put $x + 1$ instead of $x$ in the Scaling law. Then we have

$$\langle xy, z \rangle + \langle y, xz \rangle + \langle xy, xz \rangle + \langle y, z \rangle = [q(x) + \langle x, 1 \rangle + 1][y, z].$$

Cancelling terms, we get

$$\langle xy, z \rangle + \langle y, xz \rangle = \langle x, 1 \rangle \langle y, z \rangle$$
which implies that $\langle xy, z \rangle = \langle y, xz \rangle$.

(3) $\overline{x} \cdot xy = q(x)y$. (Inverse law)

Using the first two properties, notice that

$$\langle \overline{x} \cdot xy, t \rangle = \langle xy, xt \rangle = q(x) \langle y, t \rangle = \langle q(x)y, t \rangle$$

for all $t \in A$. Since the bilinear form is non-degenerate, we have $\overline{x} \cdot xy = q(x)y$.

**Theorem 3.4.** Let $(A, q)$ be a Composition algebra over $R$. If $q$ is non-degenerate, then the rank of $A$ is $1, 2, 4$ or $8$.

**Proof.** We will show that composition algebras can be embedded in their respective endomorphism rings, then follows by Theorem 3.1 that they exist only in dimensions $1, 2, 4$ or $8$.

Consider the map $\phi : A \rightarrow \text{End}(A)$ given by $x \rightarrow L_x$, where $L_x(y) = xy$ is the left multiplicative operator. For $\phi$ to be an embedding, we require that $L_xL_y = L_{xy}$, which is nothing but the Inverse law $\overline{x} \cdot xy = [x]y$. □

**Corollary 3.5.** If $A$ is associative, then $\text{rank}(A) = 1, 2$ or $4$. Octonions are non-associative.

**Proof.** If $A$ is an associative algebra, then $A$ (as a quadratic space) can be embedded in itself instead of $\text{End}A$. By comparing ranks as in Theorem 3.1 we infer that $2^n|4n$. Therefore $\text{rank}(A) = 1, 2, 4$. □

**Remark 3.6.** The $R$-module $\text{End}(\text{End}(V))$ has rank $n^4$, where $n = \text{rank}(V)$. Suppose there is an embedding of $(V, q)$ in $\text{End}(\text{End}(V))$. Then, by exactly the same proof as in Theorem 3.1 it follows that

$$\text{rank}(V) \in \{1, 2, 4, 8, 16\}.$$

This raises the question as to whether the 16-dimensional Cayley-Dickson algebra $S$, also known as Sedenions, can be embedded in $\text{End}(\text{End}(S))$.

4. **General properties of embeddings of Quadratic Spaces**

Let $R$ be a commutative ring and $V$ a free $R$-module, equipped with a quadratic form $q$. Let $\text{Cl}$ denote the Clifford algebra of $(V, q)$. To avoid any confusion, we will denote the copy of $V$ in its Clifford algebra by $V_{\text{Cl}}$.

Let $(V, q) \subseteq A$ be an embedding with $q(v) = \overline{v}v = \overline{v}v$, for some isometry $v \rightarrow \overline{v}$.

Let $\phi : V_{\text{Cl}} \rightarrow M_2(A)$ defined by $v_{\text{cl}} \rightarrow \left( \begin{array}{cc} 0 & v \\ \overline{v} & 0 \end{array} \right)$. As $\phi^2(v_{\text{cl}}) = q(v)$ for all $v \in V$, the map $\phi$ extends to an $R$-algebra homomorphism $\phi : \text{Cl} \rightarrow M_2(A)$.

**Theorem 4.1.** Let $(V, q)$ be a quadratic space embedded in an algebra $A$. Let $v, w \in V$. Then $vwv \in V$ and

$$\overline{vwv} = \overline{v} \cdot \overline{w} \cdot \overline{v}.$$
Remark 4.2. The Standard involution. The map \( \phi : C_l \rightarrow M_2(A) \) given by \( \nu_{cl} \rightarrow \begin{pmatrix} 0 & v \\ w & 0 \end{pmatrix} \).

Take \( z_1 \rightarrow \begin{pmatrix} 0 & v \\ w & 0 \end{pmatrix} \) and \( z_2 \rightarrow \begin{pmatrix} 0 & w \\ v & 0 \end{pmatrix} \). Then \( z_1 z_2 z_1 \rightarrow \begin{pmatrix} 0 & wvw \\ vww & 0 \end{pmatrix} \). \( \square \)

Rings such that \( 1 \in V, aba \in V \) are called Special Jordan algebras. The reader is referred to [J], [M2] to learn more about them.

4.3. The Standard involution. The map \( \nu_{cl} \rightarrow -\nu_{cl} \) can be viewed as an inclusion of \( V_{cl} \) in the opposite algebra of \( C_l \). By the universal property of the Clifford algebra, this map extends to an involution of \( C_l \). This is called the standard involution on \( C_l \). We will soon describe how the standard involution acts on the algebra \( M_4(A) \).

It is not uncommon to have \( 1_A \in V, \overline{1_A} = 1_A \). This is indeed the case for many examples like Suslin Matrices and Composition algebras. Suppose \( 1_A \in V \) and \( \overline{1_A} = 1_A \). Then we have the following nice implications:

- For \( v \in V \), we have \( q(1 + v) = (1 + v)(1 + \overline{v}) \) to be a scalar. Therefore \( v + \overline{v} \) is a scalar for all \( v \in V \).
- You will see that in all the examples of embeddings in this paper, \( \alpha \), the isometry that gives us the embedding, has degree 2. This is not an accident. We have

\[
\alpha^2(x) \alpha(x) = \alpha(x^2), \quad \alpha^2(x + 1) \alpha(x + 1) = \alpha(x + 1)(x + 1).
\]

Using the fact that \( \alpha \) is linear and cancelling terms, it follows that \( \alpha^2(x) = x \) for all \( x \in V \), i.e. \( \alpha \) has degree 2. The interested reader can see the paper [M], which clarifies the connection between degree 2 algebras and the existence of a scalar involution.

- Now suppose \( A \subseteq C_l \), i.e., \( C_l \) contains all elements \( (\overline{a}, 0) \) with \( a \in A \); Then we will show that the standard involution restricts to \( A \).

First notice that the even part of the Clifford algebra \( C_l \) is closed under the standard involution, and its image in \( M_2(A) \) consists of matrices of the form \( (\overline{x}, 0) \). We will simply write \( (x, y) \) instead of \( (\overline{x}, 0) \).

Let \( (a, a)^* = (x, y) \). For \( A \) to be closed under the involution, we need \( x = y \).
Let $e = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We have $e^* = -e$ as $e \in V_{M_2(A)}$. Since $e(a, a)e = (a, a)$, we have $e^*(a, a)e^* = (a, a)^*$. Therefore

$$(y, x) = e \cdot (a, a)^* \cdot e = (x, y)$$

and so the standard involution on $M_2(A)$ restricts to $A$.

Conversely, given an involution of $A$, one might ask if it can be extended to the standard involution on $M_2(A)$. To see this, let us write $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as a $2 \times 2$ matrix and analyze its conjugate in terms of its blocks. The tables below illustrate, for a few examples, how the action of an involution $\ast$ on $A$ can be extended to the standard involution in $M_2(A)$.

Note that in order to show that an involution of $M_2(A)$ corresponds to the standard involution of the Clifford algebra, it is enough to check that its action on the elements of $V_{Cl}$ is multiplication by $-1$. In other words, if $z = \begin{pmatrix} a & v \\ v & 0 \end{pmatrix}$, then we need $z^* = -z$.

**Standard Involution of form 1: $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$**

| $v^*$ | $v^* = u \cdot v$ | $u^2 = 1, u \in R$ | $M^* = \begin{pmatrix} d^* & -b^* \\ -c^* & a^* \end{pmatrix}$ | $M^* = \begin{pmatrix} d^* & -uc^* \\ -ub^* & a^* \end{pmatrix}$ |
|-------|------------------|---------------------|-----------------------------|-----------------------------|

**Standard Involution of form 2: $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$**

| $v^*$ | $v^* = \overline{v}$ | $u^2 = 1, u \in R$ | $M^* = \begin{pmatrix} a^* & -c^* \\ -b^* & d^* \end{pmatrix}$ | $M^* = \begin{pmatrix} a^* & -uc^* \\ -ub^* & d^* \end{pmatrix}$ |
|-------|------------------|---------------------|-----------------------------|-----------------------------|

Since the involution acts trivially on scalar matrices, note that if $1_A \in V$, and $v^* = uv$ or $v^* = u\overline{v}$, then $u = 1$.

### 5. The Spin Representation: When $v^* = v$ for all $v \in V$

Motivated by the discussion in the previous section, we will now analyze embeddings $(V, q) \subseteq A$ with the following conditions:

1. $1_A \in V_A$ and $\overline{1_A} = 1_A$.
2. There is an involution of $A$ that restricts to the identity map on $V$, i.e., $v^* = v$ for all $v \in V$.

We will continue to identify $Cl$ with its image in $M_2(A)$. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(A)$. Then $M^* = \begin{pmatrix} d^* & -b^* \\ -c^* & a^* \end{pmatrix}$ gives us the standard involution on $Cl$.

In particular, $\begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}$ for convenience, we will sometimes write $(g_1, g_2)$ instead of $\begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}$. 


5.1. **Spin group.** The following groups are relevant to our discussion:

\[ U^0(V) := \{ x \in Cl_0 \mid xx^* = 1 \} \]
\[ Spin(V) := \{ x \in U^0(R) \mid xVx^{-1} = V \}. \]

Notice that the action of the Spin group on \( V \) is an isometry of \( V \).

5.2. **The Spin representation:** We will define a group \( SG(A) \subset A \) and show that it is isomorphic to the Spin group, when \( v^* = v \) for all \( v \in V \).

Let \( (g_1, g_2) \in Spin(V) \). Then \( (g_1, g_2)^* = (g_2^*, g_1^*) \). Since any element in \( U^0(R) \) has unit norm, we have \( g_2 = g_1^{-1} \).

Let \( (g, g^{-1}) \in Spin(V) \) and \( v \in V \). By definition, there exists an element \( w \in V \) such that \( (g, g^{-1})(\begin{pmatrix} o & v \\ v^* & o \end{pmatrix})(g^{-1}, g^*) = (\begin{pmatrix} o & w \\ v^* & o \end{pmatrix}) \), i.e.,

\[
\begin{pmatrix}
0 & gv^* \\
sv^{-1}gv^{-1} & 0
\end{pmatrix}
\begin{pmatrix}
o \\
0
\end{pmatrix}
=
\begin{pmatrix}
o \\
w
\end{pmatrix}.
\]

Let \( g \circ v = gv^g \).

If \( (g, g^{-1}) \in Spin(V) \), then \( g \circ v \in V \). Let \( A^x \) denote the invertible elements of \( A \). Consider the set

\[ G(A) = \{ g \in A^x \mid g \circ v \in V \forall v \in V \}. \]

**Theorem 5.3.** \( G(A) \) is a group and is closed under the involution \( * \).

**Proof.** Since \( 1 \in V \), we have \( gg^* \in V \). Therefore it follows from Theorem 4.1 that

\[ g \circ (g^* \circ v) = gg^*vg^* \in V. \]

Since the action \( \circ \) is a bijection of \( V \) when \( g \) is invertible, we have \( g^* \in G(A) \).

For the same reason \( g^{-1} \) is also in \( G(A) \) and so \( g^{-1} \in G(A) \). The set \( G \) is a group. \( \square \)

One has the homomorphism

\[ \chi : Spin(V) \to G(A) \]

given by \( (g, g^{-1}) \to g \).

Next, we will use the quadratic form \( q \) on \( V \) to define a ‘norm’ on \( G(A) \). The Spin group will turn out to be isomorphic to the subgroup of \( G(A) \) whose elements have unit norm.

Let us begin with three simple lemmas which help us show that

\[ q(g \circ v) = q(gg^*)q(v). \]

**Lemma 5.4.** Let \( v \in V \) such that \( \{v, 1\} \) are linearly independent. Suppose there exists an element \( v' \in V \) such that \( v + v' \) and \( vv' \) are scalars. Then \( v' = t \).
Proof. Since \( v + \overline{v} \in \mathbb{R} \), it follows that \( \overline{v} = v' + r \) for some \( r \in \mathbb{R} \). Since \( v\overline{v} = q(v) \), it follows that \( v(v' + r) \in \mathbb{R} \), implying \( rv \in \mathbb{R} \). Therefore \( r = 0 \) and \( v' = \overline{v} \). \( \square \)

Lemma 5.5. Suppose \( v_1, v_2 \in V \) and \( q(v_2) = v_2 \overline{v_2} = 1 \). Then \( \overline{v_1} + v_1 v_1 v_2 \in v_2 \mathbb{R} \).

Proof. Since \((\overline{v_1} + v_2)(v_1 + \overline{v_2}) \in \mathbb{R} \), it follows that
\[
\langle \overline{v_1}, v_2 \rangle = \overline{v_1} \cdot v_2 + v_1 v_1 \in \mathbb{R}.
\]
Multiplying by \( v_2 \) on the right, this implies that \( \overline{v_1} + v_1 v_1 v_2 \in v_2 \mathbb{R} \). \( \square \)

Lemma 5.6. Suppose \( q(g^* g) = 1 \) for some \( g \in G(A) \). Then
\[
q(g^* g) = q(g^{-1} g^{-1}) = q(g^{*^{-1}} g^{-1}) = 1.
\]

Proof. Let \( X = g^* g \).

If \( X \in \mathbb{R} \), then \( g^* g = g^* g \) and the result follows easily. Suppose \( X \notin \mathbb{R} \). We will show that \( X + X^{-1} \in \mathbb{R} \) and infer from Lemma 5.4 that \( X^{-1} = \overline{X} \).

Now,
\[
X^{-1} = g^{-1} g^{*^{-1}} = g^* \bullet (g^{*^{-1}} g^{-1})^2.
\]
Since \( q(g^* g) = 1 \), we also have
\[
q(g^{*^{-1}} g^{-1}) = q(g^* g) = 1.
\]
Therefore (using Lemma 5.5),
\[
X + X^{-1} = g^* \bullet (1 + (g^{*^{-1}} g^{-1})^2)
= g^* \bullet (r g^{*^{-1}} g^{-1}), \quad \text{for some } r \in \mathbb{R}.
= r \in \mathbb{R}
\]
Clearly we also have \( q(g^{-1} g^{-1}) = q(X^{-1}) = q(X) = 1 \). \( \square \)

Theorem 5.7. Let \( g \in G(A) \). For all \( v \in V \), we have
\[
q(g \bullet v) = q(g^* g) q(v).
\]

Proof. Case 1: \( q(g^* g) = 1 \).

Let \( w = g \bullet v \) and \( w' = g^{*^{-1}} \bullet \overline{v} \). Since \( w \cdot w' = v \overline{v} \), it is enough to prove that \( \overline{w} = w' \).

Let us assume for now that \( \{w, 1\} \) are linearly independent. We will first show that \( w + w' \in \mathbb{R} \) and use it to prove that \( w' = \overline{w} \).
Let $X = g^*g$. Since $q(gg^*) = 1$, it follows from Lemma 5.6 that $q(X) = 1$. We have
\[
w = gvg^* \\
= g^{-1}(XvX)g^{-1} \\
= g^{-1} \cdot (XvX).
\]

Since $q(X) = 1$, we know (from Lemma 5.5) that $v + XvX = rX$ for some $r \in R$. Therefore
\[
w' + w = g^{-1} \cdot (v + XvX) \\
= g^{-1} \cdot rg^* \\
= r \in R
\]
Since $w' + w$ and $ww'$ are scalars, it follows from Lemma 5.4 that $w' = w$.

Now suppose \{w, 1\} are linearly dependent. Then we can write $w = (w_0 + w) - w_0$, where \{w_0, 1\} are linearly independent. As $\cdot$ is a linear action it follows that $w' = \overline{w}$.

**Case 2 :** $q(gg^*) = a$.

Clearly $a$ is invertible since $g \in G(A)$.

Suppose there is an $x \in R$ such that $x^2 = a^{-1}$. Take $h = xg$. Then $q(h \cdot v) = x^2 \cdot q(g \cdot v)$ and $q(hh^*) = 1$. The result follows immediately from Case 1.

Now suppose $x^2 = a^{-1}$ has no solutions in $R$. Then one has the identity $q(g \cdot v) = q(gg^*)q(v)$ over the ring $\frac{R[x]}{(x^2 - a^{-1})}$. Since each term of the equation lies in $R$, the result follows in this case too. □

**Remark 5.8.** Let $R^\times$ denote the group of units in $R$. Define $d : G_{n-1}(R) \to R^\times$ as $d(g) = q(gg^*)$.

As a consequence of Theorem 5.7 we have, for $g, h \in G_{n-1}(R)$,
\[
d(gh) = q(ghh^*g^*) = q(gg^*)q(hh^*) = d(g)d(h).
\]
Thus $d$ is a group homomorphism and 
\[
ker(d) = SG(A) \equiv Spin(V)
\]
in the case when the identity map on $V$ can be lifted to an involution of $A$.

## 6. The Suslin embedding

Let $R$ be any commutative ring and $H(R^n) := R^n \oplus R^{n*}$. By fixing a basis of $R^n$, one can then write the quadratic form on $H(R^n)$ as
\[
q(v, w) = v \cdot w^\top = a_1b_1 + \cdots + a_nb_n.
\]
for $v = (a_1, \cdots, a_n), \ w = (b_1, \cdots, b_n)$. This quadratic space $(H(R^n), q)$ is referred to as the hyperbolic space. We will now define Suslin matrices
which give an embedding of the hyperbolic space into the ring of matrices $M_{2^{n-1}}(\mathbb{R})$.

The Suslin matrix $S_n(v, w)$ of size $2^n \times 2^n$ is constructed from two vectors $v, w$ in $\mathbb{R}^{n+1}$ as follows:

Let $v = (a_0, v_1), w = (b_0, w_1)$ where $v_1, w_1$ are vectors in $\mathbb{R}^n$. Define

$$
S_1(v, w) = \begin{pmatrix} a_0 & v_1 \\ -w_1 & b_0 \end{pmatrix},
$$

$$
S_n(v, w) = \begin{pmatrix} a_0 I_{2^{n-1}} & S_{n-1}(v_1, w_1) \\ -S_{n-1}(v_1, w_1) & b_0 I_{2^{n-1}} \end{pmatrix}
$$

and

$$
\overline{S}_n(v, w) = \begin{pmatrix} b_0 I_{2^{n-1}} & -S_{n-1}(v_1, w_1) \\ S_{n-1}(v_1, w_1) & a_0 I_{2^{n-1}} \end{pmatrix}
$$

It easily follows that $S_n = S_n(v, w)$ satisfies the following properties:

1. $S_n \overline{S}_n = \overline{S}_n S_n = (v \cdot w^T) I_{2^n}$, for $n \geq 1$.
2. $\det S_n = (v \cdot w^T) 2^{n-1}$, for $n \geq 1$.

In his paper [S], A. Suslin then describes a sequence of matrices $J_n \in M_{2^n}(\mathbb{R})$ such that $J J^T = I$.

Clearly $M^* = J M^T J^T$ is an involution of $M_{2^n}(\mathbb{R})$ (as $J J^T = 1$). Thus there are two types of involution for the Suslin embedding, depending on the parity of $\text{rank}(V) = n$.

The map $\phi : H(\mathbb{R}^n) \to M_{2^n}(\mathbb{R})$ defined by $\phi(v, w) = \begin{pmatrix} 0 & S_{n-1}(v, w) \\ S_{n-1}(v, w) & 0 \end{pmatrix}$ induces an $\mathbb{R}$-algebra homomorphism $\phi : Cl \to M_{2^n}(\mathbb{R})$. In fact $\phi$ is an isomorphism (Section 3.1, [CV]); the elements $\phi(v, w)$ give a set of generators of the Clifford algebra.

**Remark 6.1.** There cannot be two involutions $*_1, *_2$ of $M_{2^n}(\mathbb{R})$ (for a fixed $n$) such that $S^*_1 = S$ and $S^*_2 = \overline{S}$. Otherwise both involutions can be lifted to the standard involution as in Table 4.3. This is not possible as the two involutions act differently on $\begin{pmatrix} S(v, w) & 0 \\ 0 & S(v, w) \end{pmatrix} \in Cl$, when $S(v, w) \neq S(v, w)$.

**Remark 6.2.** When $v \cdot w^T = 1$, the kernel of the map $\mathbb{R}^n \to \mathbb{R}$, defined by $w \to v \cdot w^T$, is a projective module. This projective module is not isomorphic to its dual when the row $v$ has odd size $> 3$. However, these projective modules are self-dual when $n$ is even. (See [NRS]).

Perhaps this difference in duality in the odd and even cases can give a deeper explanation for the corresponding behavior of the involution (Equation 2).
The above connection between the two embeddings of \( H(\mathbb{R}^n) \) into Suslin Matrices and Clifford Algebras has been studied in detail in the author’s paper \([CV]\). Suslin matrices were first introduced by A. Suslin in his paper \([S]\), in connection with unimodular rows and K-Theory. The recent work of A. Asok and J. Fasel (see \([AF]\)) uses Suslin Matrices in the context of \( A^1 \)-homotopy theory and Bott periodicity.

### 6.3. Applications to Quadratic Spaces

In a similar fashion, using Suslin Matrices, one can construct an explicit set of generators of the Clifford Algebra for other classes of quadratic spaces. Here are a few examples:

| Quadratic Space \((V, q)\) | Embedding | \(Cl\) |
|--------------------------|-----------|------|
| \((\mathbb{R}^{2n}, \sum_{i=1}^{n} v_i w_i)\) | \((v, w) \mapsto \left( \begin{array}{cc} o & S(v, w) \\ S(v, w) & o \end{array} \right)\) | \(M_{2^n}(\mathbb{R})\) |
| \((\mathbb{R}^{2n+1}, -x^2 + \sum_{i=1}^{n} v_i w_i)\) | \((x, v, w) \mapsto \left( \begin{array}{c} xi \\ S(v, w) \end{array} \right)\) | \(M_{2^n}(\mathbb{R}[i])\) |
| \((\mathbb{R}^{2n+2}, -x^2 - y^2 + \sum_{i=1}^{n} v_i w_i)\) | \((x, y, v, w) \mapsto \left( \begin{array}{c} xi+yj \\ S(v, w) \end{array} \right)\) | \(M_{2^n}(\mathbb{R}[i, j])\) |

(In the above table, we have \(i^2 = j^2 = -1\) and \(ij + ji = 0\).)

### 7. Conclusion

We have analyzed general embeddings of quadratic spaces by connecting it to their respective Clifford algebras. In particular we have focussed on the case where \(v^* = v\) lifts to an involution of \(A\). One can then define a norm function and describe the action of the Spin group on \(V_A\). Along the way, our analysis raised a few questions, which might be of general interest. Let’s conclude the paper by stating them here:

- Composition algebras can be embedded in their Endomorphism rings and so their rank should be \(1, 2, 4\) or \(8\). Are there any analogues of this result for higher dimensional Cayley-Dickson algebras? In particular, can the 16-dimensional Cayley-Dickson algebra \(S\) be embedded in \(\text{End}(\text{End}(S))\)?
- Given an embedding of \((V, q) \subseteq A\), we know that \(v w v \in V\) and so \(V\) becomes a Special Jordan algebra. It would be very interesting to learn the conditions under which the identity map on \(V \subseteq A\) (or more generally, maps of the type \(v^* = u v, u \in \mathbb{R}\)) can be lifted to an involution of \(A\). We know that this is not always the case for Suslin matrices (see Remark \[6.1\]).

The classification and structure of Special Jordan algebras has been worked out in the 20th century (see \([M2]\) for a survey). But their relationship with the overlying associative algebras, in different embeddings, remains to be explored further.
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