Some bounds on the capacity of communicating the sum of sources

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Abstract—We consider directed acyclic networks with multiple sources and multiple terminals where each source generates one i.i.d. random process over an abelian group and all the terminals want to recover the sum of these random processes. The different source processes are assumed to be independent. The solvability of such networks has been considered in some previous works. In this paper we investigate on the capacity of such networks, referred as sum-networks, and present some bounds in terms of min-cut, and the numbers of sources and terminals.

I. INTRODUCTION

The seminal work by Ahlswede et al. [1] started a new regime of communication in a network where intermediate nodes are allowed to combine incoming information to construct outgoing symbols/packets. This has been popularly known as network coding. It was shown that the capacity of a multicast network under network coding is the minimum of the min-cuts of the individual terminals from the source. The multicast capacity under routing may be strictly less than that with coding. The area has subsequently seen rapid developments. Linear coding was proved to be sufficient to achieve capacity of a multicast network in [2]. Koetter and Médard [3] proposed a different framework of random and deterministic linear network coding, and Jaggi et. al [4] proposed a polynomial time algorithm for designing a linear network code for a multicast network. The capacity of networks with routing and network coding was investigated in [5], [6], [7].

In this paper, we consider a directed acyclic network with multiple sources and terminals where the sources generate one random process each and the terminals require the sum of those processes. We call such a network as a sum-network. The alphabet of the source processes is assumed to be a finite abelian group \( G \), and the sum is defined as the operation in \( G \). We allow fractional vector network coding where the number \( k \) of sums communicated to the terminals may be different from the vector dimension \( l \). The capacity is then defined naturally as the supremum of all rates \( k/l \) which are achievable. When the alphabet is a field or more generally a module over a commutative ring with identity, the capacity achieved by using only linear codes over that ring is referred as the linear coding capacity.

The problem of distributed function computation has been considered previously in the literature in different flavors (see [8], [9], [10], [11], [12], [13] for example). In the context of network coding, and along the same line as our present work, communicating the sum of the sources has been considered in several past works. Ramamoorthy ([14]) showed that if the number of sources or the number of terminals is not more than two, then the sum of the sources can be communicated if and only if each source-terminal pair is connected. On the other hand, there are networks ([15], [16]) with more than two sources and terminals where the sum can not be communicated at rate one even though every source-terminal pair is connected. In [15], [17], [18], the authors showed the richness of this problem as a class by showing existence of networks which are linearly solvable only over finite fields of characteristics belonging to a given finite or Co-finite set of primes, existence of network which is solvably equivalent to any (non-function) general network coding, and thus equivalent to any given system of polynomial equations [19]. It was also shown that by using a code construction originally given in [20], any fractional coding solution of a sum-network also naturally provides a fractional coding solution of the same rate for the reverse network. The case of one terminal and more general functions have been considered in [21], [22].

In this paper, we consider the problem of communicating the sum of the sources over a network to a set of terminals and investigate the capacity of such networks. The exact characterization of the capacity seems to be difficult and we present some bounds, and find the capacity exactly for some interesting networks with three sources and three terminals.

The paper is organized as follows. In Section II we formally introduce the system model and some preliminary definition. The results of the paper, that is, the bounds on the capacity of sum-networks are presented in Section III. We end with a discussion in Section IV.

II. SYSTEM MODEL AND DEFINITIONS

We consider a directed acyclic multigraph \( \mathcal{G} = (V, E) \), where \( V \) is a finite set of nodes and \( E \subseteq V \times V \) is the set of edges in the network. For any edge \( e = (i, j) \in E \), the node \( j \) is called the head of the edge and the node \( i \) is called the tail of the edge; and are denoted as head\( (e) \) and tail\( (e) \) respectively. For each node \( v \), In\( (v) = \{ e \in E : \text{head}(e) = v \} \) is the set of incoming edges at the node \( v \). Similarly, Out\( (v) = \{ e \in E : \text{tail}(e) = v \} \) is the set of outgoing edges from the node \( v \).
Each edge in the network is capable of carrying a symbol from the alphabet in each use. Each edge is used once per unit time and is assumed to be zero-error and zero-delay communication channel. A network code is an assignment of an edge function to each edge and a decoding function to each terminal. In a \((k, l)\) fractional network code, \(k\) symbols generated at each source are blocked and encoded into \(l\)-length vectors on the outgoing edges. All the internal edges also carry \(l\)-length vectors. Thus for a \((k, l)\) fractional network code over \(G\), an edge function for an edge \(e\), with \(\text{tail}(e) = v\), is defined as
\[
f_e : G^k \rightarrow G^l, \text{ if } v \in S
\]
and
\[
f_e : G^{l|\text{In}(v)|} \rightarrow G^l, \text{ if } v \notin S.
\]

A decoding function for a terminal \(v\) is defined as
\[
g_v : G^{l|\text{Out}(v)|} \rightarrow G^k.
\]

The goal in a sum-network is that the terminals should be able to recover the sum of the \(k\)-length vectors generated at the sources. A \((k, l)\) fractional network code over \(G\) is called a \(k\)-length or a \(k\)-dimension vector network code over \(G\) if \(k = l\) and called a scalar network code over \(G\) if \(k = l = 1\). A network code is called a linear network code when the alphabet is a field, or more generally a module over a commutative ring with identity, and all the edge functions and the decoding functions are linear functions over the alphabet field or the ring. Note that even if the alphabet is an abelian group, one can talk about a linear solution by considering the abelian group as a module over the integer ring and then a code can be linear over the integer ring.

A network has a solution over \(G\) using a \((k, l)\) fractional network code over \(G\) if the demand of each terminal node is fulfilled using some \((k, l)\) fractional network code over \(G\). The ratio \(k/l\) is the rate of the \((k, l)\) fractional network code. A rate \(k/l\) is said to be achievable if there is a \((k, l)\) fractional solution for the supremum of all achievable rates is defined to be the capacity. The linear network coding capacity of a network is the supremum of all rates that are achievable using fractional linear network codes. Clearly, the network coding capacity of a network is greater than or equal to the linear network coding capacity. A sum-network is said to be solvable (resp. linearly solvable) if it has a \((1, 1)\) coding (resp. linear coding) solution.

III. CAPACITY OF SUM-NETWORKS

First, we mention the following simple lower bound on the capacity of any sum-network.

**Theorem 1.** The capacity of a sum-network is bounded by the minimum of the min-cuts of all source-terminal pairs. That is,
\[
\text{Capacity} \leq \min_{i,j} \{ \text{min-cut} (s_i - t_j) \}.
\]

**Proof:** For any source \(s_i\) of the network, let us fix the source processes of the other sources to the all-zero ("zero" being the identity element of the alphabet group) sequence. Then the problem reduces to the multicast problem from the source \(s_i\) to all the terminals, and the capacity of this problem is the minimum of the min-cuts from \(S_i\) to all the terminals. The overall capacity of the sum-network must be less than or equal to each of these multicast capacities for different \(i\). □

In [17], [18], the reverse of a sum-network was considered where the direction of the edges are reversed and the role of sources and terminals is interchanged. It was shown, using a code-construction originally described in a basic form in [20] as the dual code in the language of codes on graphs, that if a sum-network has a \((k, n)\) fractional linear solution, then from such a network code, one can also construct a \((k, n)\) fractional linear solution of the reverse sum-network. This means that the linear coding capacity of the reverse sum-network is the same as the linear coding capacity of the original sum-network.

Our lower bounds on the capacity of sum-networks have varying degree of tightness depending on the number of sources and the number of terminals of the network. So we present these bounds in different subsections dealing with various numbers of sources and terminals. For the rest of the paper, \(m\) and \(n\) will denote the number of sources and the number of terminals respectively.

A. The case of \(\min\{m, n\} = 1\)

If the sum-network has only one source, then the network is a multicast network. The capacity of a multicast network is known to be equal to the minimum of the min-cuts of the source-terminal pairs, and thus the capacity achieves the min-cut upper bound. Moreover, this capacity is achieved by linear codes if alphabet is a finite field. Now, over a finite field, for the case of \(n = 1\), let us consider the reverse network of a sum-network obtained by reversing the direction of the edges and interchanging the role of the sources and the terminals. The reverse network is a multicast network and thus has linear coding capacity equal to the minimum of the min-cuts of the source-terminal pairs. So the linear coding capacity of the original one-terminal sum-network is the minimum of the min-cuts of the source-terminal pairs. Since the coding capacity is also upper bounded by the min-cut, the coding capacity of a one-terminal sum-network is the minimum of the min-cuts of the source-terminal pairs.

So we have

**Theorem 2:** The capacity (and the linear coding capacity) of a one-source or one-terminal sum-network is the minimum of the min-cuts of all source-terminal pairs.

B. The case of \(\min\{m, n\} = 2\)

It was proved in [14] that for a network with \(\min\{m, n\} = 2\) where every source-terminal pair is connected, it is possible to communicate the sum of the sources to the terminals (at rate 1). Which means that for \(\min\{m, n\} = 2\), \(\text{Capacity} \geq 1\) if the minimum of the min-cuts of the source-terminal pairs is at least 1. So, the min-cut bound is tight in this case if the min-cut is 1. However, if the min-cut is greater than 1, then it is not known if this upper bound is achievable.
However, we can always achieve the half of the min-cut upper bound by time-sharing. For example, for \( m = 2 \), each source can communicate its symbols in one slot at the rate of min-cut and then after two time-slots, the terminals can add the symbols received from the two sources. So, we have

**Theorem 3:** For \( \min \{m, n\} = 2 \), the capacity of a sum-network is bounded as

\[
\text{Capacity} \geq \max \left\{ \min \{1, \min_{i,j} (\text{min-cut} \ (s_i - t_j))\},
0.5 \times \min_{i,j} (\text{min-cut} \ (s_i - t_j)) \right\}.
\]

**C. The case of** \( m = n = 3 \)

The case of \( m = n = 3 \) is intriguing. On one hand, these are the smallest values of \( m, n \) for which there is a network (called \( S_3 \) in [15] and shown in Fig. 1) where every source-terminal pair is connected, i.e., which has min-cut \( \geq 1 \), but still does not have a linear [15] or non-linear [16] solution of rate 1. So, these are the smallest parameters for which the min-cut upper bound is known to be not achievable. (Though it is still not clear at this point if the min-cut upper bound may still be achievable in the limit as the supremum of achievable rates.) On the other hand, from elaborate investigation of possible networks with these parameters, there seems to be very limited types of networks. The \( S_3 \) and its extensions (essentially the network shown in Fig. 2) seem to be the only "non-solvable" sum-networks for \( m = n = 3 \). The network (let us call it \( X_3 \)) shown in Fig. 3 was presented in [17] and was shown to be solvable by scalar linear code over all fields except the binary field \( F_2 \).

**Proof:** Without loss of generality, let us assume that the number of terminals is 3 (otherwise consider the reverse network). Let us consider two symbols at each source: \( X_{1i}, X_{i2} \) at \( s_i \) for \( i = 1, 2, \ldots, m \). Let the two sums be denoted as \( \text{Sum}_1 = \sum_{i=1}^{m} X_{1i} \) and \( \text{Sum}_2 = \sum_{i=1}^{m} X_{i2} \). If we take two terminals at a time, the resulting network has a capacity \( \geq 1 \) as discussed in Section III-B using scalar linear network coding as proposed in [14]. Now, the two sums \( \text{Sum}_1 \) and \( \text{Sum}_2 \) can be communicated to all the terminals in three time slots. In the first time slot, \( \text{Sum}_1 \) is communicated to \( t_1 \) and \( t_2 \). In the second time slot, \( \text{Sum}_2 \) is communicated to \( t_2 \) and \( t_3 \). In the third time slot, \( \text{Sum}_1 + \text{Sum}_2 = \sum_{i=1}^{m} (X_{1i} + X_{i2}) \) is communicated to \( t_1 \) and \( t_3 \). Having received \( \text{Sum}_1 \) (respectively \( \text{Sum}_2 \)) and \( \text{Sum}_1 + \text{Sum}_2 \), the terminal \( t_1 \) (respectively \( t_3 \)) can recover \( \text{Sum}_2 \) (respectively \( \text{Sum}_1 \)) as well. So all the terminals recover the two sums in three time slots, thus achieving a rate \( 2/3 \) using linear coding.

It was proved in [16] that if a sum-network with \( m = n = 3 \) has two edge-disjoint paths between any source-terminal pairs, then the network is linearly solvable, that is, rate 1 is
achievable by scalar linear coding. This gives the following bound.

**Proposition 5:** The linear coding capacity of any sum-network with \( m = n = 3 \) with min-cut \( \geq 2 \) is at least 1.

Now we show that the network \( S_3 \) and its extension \( S'_3 \) shown in Fig. 2 both have capacity exactly 2/3 whereas their min-cut upper bound is 1. So, there is a gap between the capacity and the min-cut upper bound.

**Theorem 6:** The capacity and linear coding capacity of \( S_3 \) and \( S'_3 \) is 2/3.

**Proof:** Clearly, the network \( S'_3 \) is obtained from \( S_3 \) by adding one direct edge from \( s_1 \) to \( t_1 \), and subdividing the edge \((s_2,t_1)\) and adding one edge into it from \( s_1 \). So, the capacity and the linear coding capacity of the network \( S'_3 \) is at least that of \( S_3 \). By the previous theorem, the rate 2/3 is achievable by linear network coding in \( S_3 \). Now we will show that the capacity of \( S'_3 \) is bounded from above by 2/3. This will prove that both the networks have the same capacity and linear coding capacity and that these are both 2/3.

Consider any \((k,l)\) fractional network coding solution of \( S'_3 \), Let \( X_1, X_2, X_3 \in G^k \) be the message blocks generated at the three sources. Let the edges \((u_1, v_1)\) and \((u_2, v_2)\) carry the functions \(f(X_1, X_3)\) and \(g(X_2, X_3)\) respectively. For any fixed values of \( X_1 \) and \( X_2 \), the set of messages received by the terminal \( t_1 \) should be a one-one function of \( X_3 \) since the terminal can recover the sum \( X_1 + X_2 + X_3 \) which is a one-one function of \( X_3 \). Since the messages on \((s_1, t_1)\) and \((u_2, t_1)\) are fixed by the values of \( X_1 \) and \( X_2 \), the message on \((v_1, t_1)\) and thus \(f(X_1, X_3)\) must be a one-one function of \( X_3 \) for a fixed value of \( X_1 \).

Now clearly, for \( t_2 \) to be able to recover the sum, the function \( g \) should be such that one can recover \( X_2 + X_3 \) from \( g(X_2, X_3) \). Since \( t_2 \) recovers the sum \( X_1 + X_2 + X_3 \), and it can recover \( X_2 + X_3 \) from the message in \((v_2, t_3)\), it can also recover \( X_1 \) by subtracting. Now, \( t_3 \) receives \( f(X_1, X_3) \) on \((v_1, t_3)\) (WLOG) and this is a one-one function of \( X_3 \) for any given \( X_1 \). So, having recovered \( X_1, t_3 \) can recover \( X_3 \) from \( f(X_1, X_3) \). Then by using \( g(X_2, X_3) \) received on \((v_2, t_3)\) and the value of \( X_3, t_3 \) can also recover \( X_2 \). So, \( t_3 \) can recover all the original messages \( X_1, X_2, X_3 \). Now \((X_1, X_2, X_3)\) takes a total of \( |G|^{3k} \) possible values as a triple. On the other hand \(\{(u_1, v_1), (u_2, v_2)\}\) is a cut between the sources and \( t_3 \), and this cut can carry at most \( |G|^{2l} \) possible different message-pairs. So, we have \( |G|^{2l} \geq |G|^{3k} \Rightarrow k/l \leq 2/3 \).

The following observations lead us to believe that the network \( S'_3 \) is essentially the only maximal extension of \( S_3 \) which has the same capacity.

1) Further subdividing \((u_3, t_1)\) and adding an edge from it to \( t_2 \) makes the network \( X'_3 \) a subgraph of the resulting network, and thus the capacity of the network increases to 1.

2) Also subdividing \((s_1, t_2)\) and adding an edge from it into \( t_1 \) does not change its capacity since there is already an edge \((s_1, t_1)\) and the new edge can not carry any extra information to \( t_1 \).

3) Also subdividing \((s_1, t_1)\) and adding an edge to it from \( s_2 \) gives a strictly richer network than \( X'_3 \), and thus the capacity of the network increases to 1.

4) Instead of the edge \((s_1, t_1)\), if an edge \((s_2, t_2)\) is added, then the resulting network (shown in Fig. 4) is strictly richer than \( X'_3 \) because the edges \((s_1, t_2)\) and \((s_2, t_2)\) can jointly carry more information than an edge from \( u_3 \) to \( t_2 \). So the resulting network has capacity 1 even though it does not have a binary scalar solution (like \( X'_3 \)).

These observations also lead us to believe that

**Conjecture 7:** The capacity of a sum-network with \( m = n = 3 \) is either 0, 2/3 or at least 1.

![Fig. 4. The network \( X'_3 \)](image)

**D. The case of \( m, n > 3 \)**

This is the most ill-understood class of sum-networks. We only have what we suspect to be a very loose lower bound on the capacity of this class of networks. This lower bound is obtained by similar coding by time-sharing scheme as in the proof of Theorem 4.

**Theorem 8:** The linear coding capacity of a sum-network with \( \min\{m,n\} \geq 2 \) and min-cut \( \geq 1 \) is at least \( 2/\min\{m,n\} \).

**Proof:** The case of \( \min\{m,n\} = 2 \) follows from Theorem 4. For \( \min\{m,n\} > 2 \), without loss of generality, let us assume that \( n \leq m \). For even \( n \), we can group the terminals into \( n/2 \) pairs and in each time slot communicate the sum of the source symbols to one pair of terminals. So, in \( n/2 \) time slots we can communicate one sum of the source symbols to all the terminals thus achieving a rate \( 2/n \). For odd \( n \), we can group the terminals into \( (n-3)/2 \) pairs and one triple. We can communicate one sum to each pair in one time slot. So, we can communicate two sums to all the pairs in \( (n-3) \) slots. Then using the same scheme as in the proof of Theorem 4 we can communicate two sum to the group of three terminals in three time slots. So, in overall \( n \) time
slots, we can communicate two sums to all the terminals in the network. This gives us a rate $2/n = 2/\min\{m,n\}$.

We believe that this bound is very loose and there is scope for improvement. Even though we failed to come up with any achievable scheme of higher rate, we also failed to construct a network satisfying the bound with equality.

### IV. DISCUSSION

Some upper and lower bounds on the capacity of communicating the sum of sources to a set of terminals are presented in this paper. A decreasing degree of tightness is observed or suspected in these bounds as the numbers of sources and terminals increase. We summarize the bounds in Table I to bring out this observation. The parenthetic comments in the table entries indicate the tightness of the bound as known or conjectured (indicated with an exclamation (!) mark.) The interrogation (?) mark as an entry indicates that nothing is known about the case.

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