Quark Droplets in the NJL Mean Field

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We study the conditions for the existence of stable quark matter in the Nambu–Jona-Lasinio mean field at zero temperature and discuss its interpretation.

1. INTRODUCTION

Because of its relatively simple structure the Nambu–Jona-Lasinio (NJL) model \cite{1} is one of the most popular models for studying the spontaneous breaking of chiral symmetry and its restoration at finite temperatures or densities \cite{2} - \cite{6}. However, most of its predictions suffer from the fact that the model is a pure quark model without confinement, which in general leads to rather unrealistic scenarios. In mean field approximation for $T = 0$ we can distinguish three different cases \cite{7}:

Case I: The phase transition is of second order. Then for all densities there is a uniform phase of quarks. The quarks are massive at low densities and (almost) massless at high densities. The pressure is always positive and if we do not apply an external force to the system it will expand to arbitrarily large volumes, i.e. arbitrarily small densities.

Case II: The phase transition is of first order. In this case there will be a regime in which two phases coexist. In general, this mixed phase consists of a low-density phase ($\rho = \rho_l$) with massive and a high density phase ($\rho = \rho_h$) with massless quarks. However, for $\rho < \rho_l$ we have again a uniform phase of massive quarks. As in case I, low densities are energetically favored and quark matter is unstable against expansion.

Case III: The phase transition is of first order, but with $\rho_l = 0$. Then the low-density component of the mixed phase is the vacuum, i.e. we find droplets of massless quarks surrounded by the non-trivial vacuum. At least in a schematic way this is very reminiscent of the MIT bag model. The droplets are the energetically most favored configuration and therefore stable against expansion or collapse. In contrast, any uniform distribution of quarks at lower densities is unstable against break-up into the droplet-vacuum phase. This scenario was also studied in ref. \cite{8} within a similar model.

For NJL-like models it depends on the model parameters which of the three cases is realized. However, the non-existence of a uniform dilute gas of quarks in nature excludes the cases I and II, which both predict such a phase, from being realistic. In case III there is no stable uniform quark gas at low densities but instead we have the droplet phase. At least if we adopt the interpretation of quark droplets as schematic bag model baryons this seems to be the most realistic scenario of the model. In this article we want to discuss whether it is also compatible with “realistic” model parameters. To large extent this will
be done following ref. [7].

2. FORMALISM

We consider a Lagrangian for quarks with \( n_f = 2 \) flavors and \( n_c = 3 \) colors interacting by NJL-like four-fermion vertices:

\[
\mathcal{L} = \bar{\psi}(i\slashed{\partial} - m_0)\psi + G_S[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\vec{\tau}\psi)^2] - G_V(\bar{\psi}\gamma^\mu\psi)^2.
\]  

(1)

The first two interaction terms (scalar-isoscalar and pseudoscalar-isovector) were taken from the original NJL-Lagrangian [1]. Since vector interactions are known to be important at finite densities, e.g., like in the Walecka model [9], we have also included a vector-isoscalar term. Linearising \( \bar{\psi}\psi \) and \( \psi^\dagger\psi \) about their thermal expectation values we can calculate the mean field thermodynamic potential (per volume) at temperature \( T \) and chemical potential \( \mu \) [5]. We restrict ourselves to the Hartree approximation. The result for \( T = 0 \) and \( \mu \geq 0 \) reads:

\[
\omega_{MF}(\mu; m, \mu_R) = -12 \int \frac{d^3 p}{(2\pi)^3} \left[ E_p + (\mu_R - E_p) \theta(\mu_R - E_p) \right] + \frac{(m - m_0)^2}{4G_S} - \frac{(\mu - \mu_R)^2}{4G_V},
\]  

(2)

with \( E_p = \sqrt{m^2 + \vec{p}^2} \). The integral is strongly divergent and has to be regularized. For simplicity we use a 3-dimensional sharp cutoff. The auxiliary variables \( m \) (“constituent quark mass”) and \( \mu_R \) are defined as \( m = m_0 - 2G_S(\bar{\psi}\psi) \) and \( \mu_R = \mu - 2G_V(\bar{\psi}\gamma^\mu\psi) \). Since on the other hand the condensates \( \langle \bar{\psi}\psi \rangle \) and \( \langle \bar{\psi}\gamma^\mu\psi \rangle \) have to be calculated from the thermodynamic potential by taking the appropriate derivatives we encounter a selfconsistency problem. It can be shown that it is solved by the stationary points of \( \omega_{MF} \) with respect to \( m \) and \( \mu_R \). This can be used to eliminate \( \mu_R \) for given values of \( \mu \) and \( m \), \( \mu_R = \mu_R(\mu, m) \). Furthermore, we subtract a constant in order to shift the value of the minimum of the vacuum thermodynamic potential to zero, i.e. we define:

\[
\tilde{\omega}(\mu, m) := \omega_{MF}(\mu; m, \mu_R(\mu, m)) - \omega_{MF}(0; m_{vac}, 0).
\]  

(3)

Here \( m_{vac} \) is the constituent quark mass which minimizes the thermodynamic potential in vacuum. For a given value of \( \mu \) the extrema of \( \tilde{\omega} \) correspond to the selfconsistent solutions and the stable solution is given by the absolute minimum. Other thermodynamic quantities, like the pressure \( p \), baryon number density \( \rho_B = \frac{1}{n_c} \langle \bar{\psi}\gamma^\mu\psi \rangle \) and energy density \( \varepsilon \), can be calculated from the thermodynamic potential in the standard way.

3. STABILITY OF QUARK MATTER AT T=0

With the appropriate choice of parameters chiral symmetry is spontaneously broken in the NJL-vacuum, i.e. the quarks acquire a constituent mass \( m_{vac} \) much larger than the current mass \( m_0 \). Unless stated otherwise we will work in the chiral limit, \( m_0 = 0 \). Then the thermodynamic potential \( \tilde{\omega} \) is symmetric about \( m = 0 \). In vacuum \( m = 0 \) corresponds to a local maximum and \( \tilde{\omega} \) is minimal at \( m = \pm m_{vac} \). For large chemical potentials chiral symmetry becomes restored. Thus, with increasing \( \mu \), the maximum at \( m = 0 \) eventually has to turn into an absolute minimum. If this happens at a critical chemical potential less than the vacuum mass, \( \mu_{crit} < m_{vac} \), we have a first-order phase transition of the type “case III”.
An example for this case is shown in fig. 1 where we find a critical chemical potential $\mu_{\text{crit}} = 376$ MeV, well below the vacuum mass $m_{\text{vac}} = 400$ MeV. Since for $T = 0$ and $G_V = 0$ the chemical potential is equal to the Fermi energy of the system, for a given mass $m$, a chemical potential less than $m$ corresponds to zero density and vice versa. Hence the degenerate minima of $\tilde{\omega}$ at $\mu = \mu_{\text{crit}}$ correspond to two phases of equal pressure in chemical equilibrium, one being the nontrivial vacuum ($m = m_{\text{vac}}$ and $\rho_B = 0$) and the second consisting of massless quarks at finite density. In our example we find $\rho_B = 0.47 \, fm^{-3} = 2.75 \rho_0$ for the dense phase. This means, droplets of massless quarks and a baryon number density of $2.75 \rho_0$ are stable in vacuum, as far as our mean field approach is justified.

![Figure 1](image1.png)  
Figure 1. Thermodynamic potential $\tilde{\omega}$ as a function of the constituent quark mass $m$ for $m_0 = 0$, $\Lambda = 587.9$ MeV, $G_S \Lambda^2 = 2.48$ and $G_V = 0$. The different curves correspond to three different chemical potentials: $\mu = 0$ (dashed line), $\mu = 376$ MeV (solid line) and $\mu = 450$ MeV (dotted line).

![Figure 2](image2.png)  
Figure 2. Energy per baryon number as a function of the baryon number density $\rho_B$ for the same parameters as in fig. 1. The solid line corresponds to massive, the dotted line to massless quarks.

Alternatively this can be seen if we calculate the energy per baryon number, $\varepsilon/\rho_B$ as a function of density $\rho_B$. The result is shown in fig. 2 for the same set of parameters as in fig. 1. At not too high densities there are solutions with massive and with massless quarks, and the massive ones are always energetically favored. In the limit $\rho_B \to 0$ the energy per baryon number is just 3 times the constituent quark mass. However, in agreement with our considerations above we find the absolute minimum of $\varepsilon/\rho_B$ at $2.75 \rho_0$ which is in the regime where only massless solutions exist. At this point the pressure vanishes and the energy per baryon number is 3 times $\mu_{\text{crit}}$. This shows again that $\mu_{\text{crit}} < m_{\text{vac}}$ is necessary for this minimum to be stable.
The results can become qualitatively different if we change the model parameters. For instance, with the parameters of fig. 1 but with a vector coupling $G_V = 0.5 G_S$ we find a first-order phase transition at $\mu_{\text{crit}} = 410.3$ MeV $> m_{\text{vac}}$. This corresponds to what we called “case II” and there is no dense matter solution which can coexist with the vacuum. The energy per baryon number as a function of density looks similar to fig. 2 but with the minimum of the massless solutions being only metastable: The corresponding energy, $3 \mu_{\text{crit}}$, is now larger than the energy per baryon number for $\rho_B \to 0$, which is $3 m_{\text{vac}}$. Finally, if we further increase the vector coupling, e.g. $G_V = G_S$, we find a second-order phase transition and the energy per baryon number is a strictly rising function of density.

A more systematic overview is given in fig. 3 where three lines of constant binding energy per quark, $E_{\text{bind}} = m_{\text{vac}} - \mu_{\text{crit}}$, are plotted. For each value of $m_{\text{vac}}$ the cutoff $\Lambda$ was fixed by fitting the pion decay constant to its empirical value $f_\pi = 92.4$ MeV. We are left with two model parameters, which can be chosen to be $m_{\text{vac}}$ and the ratio of vector and scalar coupling constant, $G_V/G_S$. We see that the vector interaction reduces the binding, which is not surprising since vector mean fields are known to be repulsive. We also find that the binding becomes stronger with increasing constituent quark masses. For $m_{\text{vac}} < 343$ MeV there is no bound quark matter, even without vector interaction. This agrees quite well with an approximate relation derived in ref. 7 which states that stable quark matter is only possible in the NJL-model if $m_{\text{vac}} \gtrsim 4 f_\pi$. 

![Figure 3](image1.png)  
Figure 3. Lines of constant binding energy per quark for fixed $f_\pi = 92.4$ MeV and varying constituent quark masses $m_{\text{vac}}$ and coupling constants $G_V/G_S$: 0 MeV (solid line), 50 MeV (dashed) and 100 MeV (dotted).

![Figure 4](image2.png)  
Figure 4. Energy per baryon number as a function of baryon number density for non-strange quark matter in the 3-flavor model. The parameters are: $m_0^u = m_0^d = 5.5$ MeV, $m_0^s = 140.7$ MeV, $\Lambda = 602.3$ MeV and $G_S \Lambda^2 = 1.84$, $K \Lambda^5 = 12.4$ (solid line), $G_S \Lambda^2 = 2.31$, $K = 0$ (dashed line).
The region shown in fig. 3 corresponds more or less to the regime of “realistic parameters”. One could try to further constrain \( m_{vac} \) by fitting the quark condensate. However, \( \langle \bar{\psi}\psi \rangle \) is not known very precisely and its dependence on \( m_{vac} \) is only weak, once \( f_\pi \) is fixed. The vector coupling constant could be determined by fitting vector meson masses. In the literature this usually leads to \( G_V/G_S \) of the order of 0.5 to 1 [2,6] although much higher values are also found [10]. On the other hand a realistic vector coupling constant in dense matter could be rather different from the vacuum one.

Thus the scenario of stable quark droplets in vacuum (“case III”) is possible but not a necessary or very probable consequence of restricting the model parameters to realistic values. This is in some contradiction to ref. [8] where stable droplets are found to be most likely. The model of ref. [8] is almost identical to our model with \( G_V = 0 \), with the only difference that instead of using a sharp cutoff the vertices are multiplied with form factors of the form \( (\frac{\Lambda^2}{p^2+\Lambda^2} )^\nu \). As an example we looked at \( \nu = 1 \) and \( \Lambda = 800 \) MeV. It turned out that we are able to produce very similar results with our model if we use a sharp cutoff of about 350 MeV. However, for this cutoff parameter we find a pion decay constant of less than 60 MeV. This suggests that the parameters chosen in ref. [8] might also correspond to a too small value of \( f_\pi \).

4. QUARK DROPLETS AS SCHEMATIC MIT-BAGS

In case III dense matter of massless quarks can coexist with the non-trivial vacuum. As pointed out in the introduction it is very attractive to identify these quark droplets with baryons in a bag-model picture. In fact, the energy of a spherical MIT-bag [11] with radius \( R \),

\[
E_{MIT} = \frac{4\pi}{3} R^3 B + \frac{3x - z_0}{R} = \frac{B}{\rho_B} + \frac{3x - z_0}{R} \left( \frac{4\pi}{3} \right)^{1/3} \rho_B^{1/3},
\]

has a similar structure as the energy per baryon number for massless quarks in the NJL mean field

\[
\frac{\varepsilon}{\rho_B} = \frac{B}{\rho_B} + \frac{3n_c}{4} \left( \frac{3\pi^2}{n_f} \right)^{1/3} \rho_B^{1/3} + G_V n_c^2 \rho_B,
\]

if we switch off the vector interaction. Here the bag constant is given by \( B = \tilde{\omega}(0,0) \). Of course our thermodynamic approach is valid only for a large number of particles in a large volume whereas eq. (4) was derived for three quarks in a volume which can be small. This leads to different coefficients in front of \( \rho_B^{1/3} \). However, taking \( x = 2.04 \) and \( z_0 = 1.84 [11] \), the deviation turns out to be surprisingly small (\( \sim 20\% \)). One should also keep in mind that eq. (4) is only valid for the massless quark solutions. As we have seen, if the density is low enough the quarks can lower their energy by acquiring a finite constituent mass (cf. fig. 2). In particular, only a finite amount of energy is needed to lower the density to zero, corresponding to an infinite bag radius: the NJL mean field does not confine. As an important consequence the quark droplets become immediately unstable against evaporation of massive quarks at any finite temperature and one has to be very careful with the interpretation of the model for \( T > 0 \).
5. THREE-FLAVOR MODEL

In this section we extend the model to three quark flavors \[12\]:

\[
\mathcal{L} = \bar{\psi}(i\partial - \hat{m}_0)\psi + G_S \sum_{k=0}^{8} \left[ (\bar{\psi}\lambda_k\psi)^2 + (\bar{\psi}i\gamma_5\lambda_k\psi)^2 \right] - K \left[ \det_f(\bar{\psi}(1 + \gamma_5)\psi) + \det_f(\bar{\psi}(1 - \gamma_5)\psi) \right].
\]

Here \(\psi = (u, d, s)^T\) is a 3-dimensional vector and \(\hat{m}_0 = \text{diag}(m_u^0, m_d^0, m_s^0)\) a 3 × 3 matrix in flavor space. In addition to the 4-point interaction the Lagrangian contains a t’Hooft-type 6-point interaction which is a determinant in flavor space and breaks the \(U_A(1)\) symmetry. For simplicity we neglect vector interactions in this section. Since each flavor is separately conserved we have in principle three different chemical potentials and the thermodynamic potential contains the term \(\mu N \equiv \mu_u u^\dagger u + \mu_d d^\dagger d + \mu_s s^\dagger s\).

In the following we consider non-strange isospin-symmetric matter, \(m_u^0 = m_d^0, \mu_u = \mu_d\) and \(\mu_s = 0\). We adopt the model parameters of ref. \[13\] which have been fitted to the pseudoscalar mass spectrum and \(f_\pi\). For these parameters we find stable quark matter with \(\rho = 2.36 \rho_0\) and a binding energy per quark of 6.5 MeV. The energy per baryon number is plotted in fig. 4 (solid line). Because of the determinant interaction this result is influenced by the strange quark even though the net strangeness \(\langle s^\dagger s \rangle\) is zero. To show this we switch off the determinant interaction and increase \(G_S\) such that the vacuum properties of the non-strange sector, \(\langle \bar{u}u \rangle\), \(f_\pi\) and \(m_\pi\), remain unchanged. The result is also shown in fig. 4 (dashed line). Here the binding energy is 2.9 MeV at \(\rho = 2.43 \rho_0\). Thus the flavor mixing gives some extra binding but the effect is not very large. A more systematic discussion of the \(SU(3)\) model will be published elsewhere.

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\[1\]Here we use the notation of ref. \[6\]. Notice that \(\mu_s\) is different from the strangeness chemical potential \(\mu_{\text{strange}}\) which is often introduced as the deviation from a common baryon chemical potential \(\mu_B\): \(\mu_N = \mu_B(u^\dagger u + d^\dagger d + s^\dagger s) + \mu_{\text{strange}} s^\dagger s\). In particular in our notation \(\mu_s = 0\) corresponds to \(\langle s^\dagger s \rangle = 0\).