Higher order nonclassicalities of finite dimensional coherent states: A comparative study

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July 12, 2021

Abstract

Conventional coherent states (CSs) are defined in various ways. For example, CS is defined as an infinite Poissonian expansion in Fock states, as displaced vacuum state, or as an eigenket of annihilation operator. In the infinite dimensional Hilbert space, these definitions are equivalent. However, these definitions are not equivalent for the finite dimensional systems. In this work, we present a comparative description of the lower- and higher-order nonclassical properties of the finite dimensional CSs which are also referred to as qudit CSs (QCSs). For the comparison, nonclassical properties of two types of QCSs are used: (i) nonlinear QCS produced by applying a truncated displacement operator on the vacuum and (ii) linear QCS produced by the Poissonian expansion in Fock states of the CS truncated at \((d-1)\)-photon Fock state. The comparison is performed using a set of nonclassicality witnesses (e.g., higher order antibunching, higher order sub-Poissonian statistics, higher order squeezing, Agarwal-Tara parameter, Klyshko’s criterion) and a set of quantitative measures of nonclassicality (e.g., negativity potential, concurrence potential and anticlassicality). The higher order nonclassicality witness have found to reveal the existence of higher order nonclassical properties of QCS for the first time.

1 INTRODUCTION

Coherent states drew considerable attention of the quantum optics and atom optics community for various reasons. For example, a CS is known to be a quasi-classical state or the most classical state among the quantum states [1], and it has applications in almost all fields of physics [2, 3]. In quantum optics, CS has been traditionally defined in various ways, such as displacement of vacuum state, eigenket of annihilation operator, or infinite Poissonian superposition of Fock states [1, 4]. In the infinite dimensional Hilbert space, these different definitions of CS are equivalent. However, in the finite dimensional Hilbert space, different definitions lead to different finite dimensional coherent states which are referred to as qudit coherent states [5]. In general, a qudit may be viewed as a \(d\)-dimensional quantum state that can be expanded in Fock-state \(|n\rangle\) basis as

\[
|\psi\rangle_d = \sum_{n=0}^{d-1} c_n |n\rangle.
\]

With the recent developments in quantum state engineering [6, 7, 8, 9] and quantum computing and communication [see Ref. 10 and references therein], production and manipulation of these types of quantum states have become very important. Further, in the recent past, several applications of nonclassicality [10, 11, 12] and a few experimental demonstrations of higher order nonclassicality [13, 14, 15, 16] have been reported. Specifically, in the Laser Interferometer Gravitational-Wave Observatory (LIGO), squeezed vacuum state has been successfully used for the detection of the gravitational waves [17, 18] by reducing the noise [19, 20]. Squeezed state is also used in continuous variable quantum cryptography [11], teleportation of coherent state [21], etc. Anti-bunching is used for characterizing single photon sources [22] which are essential for the realization of various schemes for secure quantum communication. Further, entangled states have been established to be useful for various quantum information processing tasks ([10] and references therein). For example, entangled states are essential for quantum teleportation [23], densecoding [24], quantum cryptography [25], etc. In addition to the nonclassical states having the above mentioned applications, QCSs (being a finite superposition of Fock states, which are always...
with the Hermite polynomial references therein, possible ways of generating this QCS and a set of its nonclassical properties have been discussed. However, no attention has yet been provided to the higher order nonclassical properties of this state. 

\[ \beta \text{ of CS.} \] This type of QCSs for a complex amplitude periodic nature of applying a truncated displacement operator on the vacuum state as follows 

\[ \hat{D}_d(\alpha, \alpha^*) |0\rangle = \exp(\alpha \hat{a}_d^\dagger - \alpha^* \hat{a}_d) |0\rangle, \]

where the truncated displacement operator \( \hat{D}_d(\alpha, \alpha^*) \) operates on vacuum to generate QCS, and the qudit annihilation operator \( \hat{a}_d = \sum_{n=1}^{d-1} \sqrt{n} |n-1\rangle |n\rangle \) and the corresponding commutation relation is \( [\hat{a}_d, \hat{a}_d^\dagger] = d(d-1) \) which fundamentally differs from the standard creation and annihilation operators. The Fock-state expansion of the QCS in the form of Eq. (1) is given by 

\[ |\alpha\rangle_d = \frac{1}{\alpha^{d-1} \sqrt{\pi}} \exp(\sum_{n=0}^{d-1} \frac{i \phi_n}{\sqrt{n}} |\alpha^*\rangle |\alpha\rangle), \]

where the superposition coefficients are 

\[ c_n^{(d)}(\alpha) = f_n^{(d)} \frac{1}{\sqrt{n! \pi^{d-1}}} \exp(i |\alpha|), \]

with 

\[ f_n^{(d)} = \frac{(d-1)!}{n!^{1/2}} \exp[(in^{1/2})], \]

and the modified Hermite polynomial \( H_n(x) \) is related to the Hermite polynomial \( H_n(x) = \sum_{n=0}^{d-1} \frac{(-1)^n}{\sqrt{n! \pi}} \exp(i |\alpha|), \) which is parametrically dependent on the photon number for the QCS \( |\alpha\rangle \) is a periodic state. The complex parameter \( \alpha \) (with \( \phi_0 = 0 \)) is perfectly periodic in nature for \( d = 2, 3 \) and almost periodic for \( d > 3 \). The periods of \( \alpha \) for \( d = 2 \) and 3 are \( T_2 = \pi \) and \( T_3 = 2\pi/\sqrt{3} \), respectively, whereas the periods for \( d > 3 \) are \( \sqrt{4d+2} \). Due the periodic nature of \( \alpha \) the photon number for the QCS \( |\alpha\rangle \) is also periodic in nature with maximum value \( |\alpha|^2 = d-1 \) which corresponds to the photon number of the highest energy Fock state. In Ref. [5] and references therein, possible ways of generating this QCS and a set of its nonclassical properties have been discussed. However, no attention has yet been provided to the higher order nonclassical properties of this state.

The second type of QCSs studied here can be generated by truncating the Fock space superposition of CS. This type of QCSs for a complex amplitude \( \beta \) are defined as 

\[ |\beta\rangle_d = \mathcal{N} \exp(\beta \hat{a}_d^\dagger) |0\rangle = \mathcal{N} \sum_{n=0}^{d-1} \frac{\beta^n}{\sqrt{n!}} |n\rangle, \]

where \( \mathcal{N} = \frac{1}{(\sum_{n=0}^{d-1} \frac{\beta^n}{\sqrt{n!}})^{1/2}} \) is the normalization constant. This type of QCS is referred to as the linear QCS. The QCS \( |\beta\rangle_d \) can be written in the form of Eq. (1) with 

\[ c_n^{(d)}(\beta) = \mathcal{N} \frac{\beta^n}{\sqrt{n!}}. \]

This QCS is referred to as linear QCS and was studied earlier in Refs. [6] and references therein. In Ref. [5], it is explicitly shown that nonclassical properties (e.g., Wigner function and nonclassical volume) of linear QCS and nonlinear QCS are different. Here, we aim to extend the observation further by comparing the nonclassical properties of these QCSs using various witnesses and measures of nonclassicality with a specific focus on the witnesses of higher order nonclassicality. The remaining part of the paper is organized.
as follows. In Section 2 we compare nonclassical characters of linear and nonlinear QCSs using a set of witnesses of nonclassicality which generally reflects the presence of higher order nonclassicality (except Klyshko’s criterion), but does not provide any quantitative measure of nonclassicality. Specifically, in this section, we perform comparison of nonclassicality in QCSs using the criteria of higher order antibunching (HOA), higher order sub-Poissonian photon statistics (HOSPS), higher order squeezing (HOS) and Agarwal-Tara criterion and Klyshko’s criterion. In Section 3 we compare the amount of nonclassicality present in linear and nonlinear QCSs by using a set of quantitative measures of nonclassicality (e.g., concurrence potential, negativity potential, and anticlassicality). Finally, the paper is concluded in Section 4.

2 COMPARISON OF NONCLASSICALITY IN QCSs USING THE WITNESSES OF NONCLASSICALITY

A quantum state is referred to as nonclassical if its Glauber Sudarshan $P$-function cannot be written like a classical probability distribution. In other words, negative values of $P$-function implies that the state does not have classical analogue, and can be referred to as a nonclassical state. As there does not exist any general procedure for the measurement of $P$-function, several operational criteria are designed to identify the signatures of nonclassicality. Most of these criteria are only sufficient in the sense that if one of the criteria is satisfied then the state is definitely nonclassical, but if it’s not satisfied then we cannot conclude anything about the nonclassicality of the state. For example, in this section, we would discuss nonclassical properties of linear and nonlinear QCSs using the criteria of HOA, HOS, HOSPS, Agarwal-Tara and Klyshko’s criteria. All these criteria are only sufficient. Further, they are only witnesses of nonclassicality (only provides the signature of nonclassicality) as none of them provide any quantitative measures of the amount of nonclassicality present in a state. Of course, there exist a handful of measures of nonclassicality, but each of them has some limitations (see [41] for a discussion). We will get back to the issues of nonclassicality measures in the next section, where we will compare linear and nonlinear QCSs using some of those measures. This section is focused on witnesses of nonclassicality and in what follows we would compare nonclassical features of linear and nonlinear QCSs using the criteria of HOA, HOS, HOSPS and Agarwal-Tara and Klyshko’s criteria. All these criteria are based on moments of photon number and/or quadrature and are thus experimentally measurable with the general measurement scheme proposed in Ref. [42]. It may be further noted that an infinite set (or hierarchy) of such moment based criteria is equivalent to the $P$-function, i.e., is both necessary and sufficient in nature [13]. Recently, Miranowicz et al., have summarized all the existing nonclassicality criteria for both single- and multi-mode bosonic fields and proposed a unified approach to generate new operational inequalities to characterize nonclassicality in the radiation [14]. In what follows, we would use a small set of these nonclassicality witnesses to perform the proposed comparison. To begin with, we would like to discuss the possibility of observing HOA in linear and nonlinear QCSs.

2.1 Higher order antibunching

Different well-known criteria of higher order nonclassicality can be expressed in compact forms for the finite dimensional states given by Eq. (1). In this section, we would focus on HOA. The concept of HOA was introduced by Lee in 1990 [15]. In fact, he provided a criterion for HOA using the theory of majorization. Subsequently, in 2002, Lee’s criterion was modified by Ba An [46], who introduced a simplified criterion for HOA and used that to show the existence of HOA in the trio coherent state. Later, in 2006, Pathak and Garcia [47] provided a clear physical meaning to HOA and further simplified the criterion of HOA. This criterion is now known as Pathak-Garcia criterion, and in what follows, we use Pathak-Garcia criterion [47] of HOA as a witness of nonclassicality. According to this criterion, $l$th order antibunching is observed if following inequality is satisfied by a quantum state

$$D(l) = \langle N^{l+1} \rangle - \langle N \rangle^{l+1} = \langle a^{\dagger(l+1)} a^{l+1} \rangle - \langle a^{\dagger} a \rangle^{l+1} < 0. \tag{7}$$

For $l = 1$, one can obtain a usual antibunching criterion. Before proceeding further, we would like to note that until the recent past, HOA was supposed to be a very rare phenomenon, but in 2006, some of the present authors have established that HOA is not so rare, and subsequently HOA has been reported in various infinite dimensional quantum states (e.g., nonlinear squeezed state [48], photon-added coherent state [49], photon added and subtracted squeezed coherent state [50]) and a set of finite dimensional states (e.g., binomial state [48], generalized binomial state [49], reciprocal binomial state [49], negative binomial state [49], hypergeometric state [49]). It has also been reported in a set of physical systems (e.g., in asymmetric nonlinear optical coupler [51, 52], BEC systems [53, 54], coupled anharmonic...
Hillery’s criterion for higher order squeezing using Hong-Mandel’s criterion which can be described by the following inequality

\[ \langle (\Delta X)^n \rangle = \sum_{r=0}^{n} \sum_{l=0}^{\frac{n}{2}} \sum_{k=0}^{r-2i} (-1)^r \binom{2i}{r-1}!! \binom{r-2i}{k} \binom{n}{r} \binom{r}{l} \binom{r}{k} \langle a^d + a^{d-1} \rangle < \frac{1}{2^n} (n-1)!!, \]

where \( n \) is an even number, quadrature variable is defined as \( X = \frac{1}{\sqrt{2}} (a + a^\dagger) \) and \( (x) \) is conventional Pochhammer symbol. Now, for the finite dimensional states of the form (1), we can have a compact

\[ D(l) = \langle N^{(l+1)} \rangle - \langle N \rangle^{l+1} = \sum_{j=0}^{d-1} \frac{j!}{(j-l-1)!} |c_j|^2 - \left( \sum_{j=0}^{d-1} |c_j|^2 \right)^{l+1} < 0. \]

Now, we can obtain \( D(l) \) for nonlinear and linear QCSs with the help of Eqs. (7), (6) and (8), respectively and the corresponding expressions of \( D(l) \)’s are plotted in Fig. (1) which clearly shows the variation of higher order antibunching in QCS with parameter \( \alpha/\beta \) for nonlinear (linear) QCSs. All the plots in Fig. (1) show that the states are antibunched (of different orders) for the particular values of the parameters used here. Specifically, Figs. (1a) and (1c) show that the depth of nonclassicality increases with the order of antibunching in linear and nonlinear QCSs, respectively. Similarly, Figs. (1b) and (1d) show the effect of dimensions on antibunching in linear and nonlinear QCSs, respectively.

2.2 Higher order squeezing

Further, higher order squeezing can be studied using two independent criteria: Hillery’s criterion for amplitude powered squeezing [60] and Hong-Mandel’s criterion [61]. Here, we investigate higher order squeezing using Hong-Mandel’s criterion which can be described by the following inequality

\[ \langle (\Delta X)^n \rangle = \sum_{r=0}^{n} \sum_{l=0}^{\frac{n}{2}} \sum_{k=0}^{r-2i} (-1)^r \binom{2i}{r-1}!! \binom{r-2i}{k} \binom{n}{r} \binom{r}{l} \binom{r}{k} \langle a^d + a^{d-1} \rangle^{n-r} \langle a^{d-k} \rangle^{2i-k} < \frac{1}{2^n} \left( \frac{1}{2} \right)^{n-1} (n-1)!!, \]
2.3 Higher Order sub-Poissonian Photon Statistics

Higher order sub-Poissonian photon statistics [62, 63] is an important aspect of the study of the existence of higher order nonclassicality and quantum statistical properties of the radiation field. In the recent past, HOSPS has been reported in a set of infinite dimensional states (e.g., nonlinear squeezed state [48], of higher order nonclassicality and quantum statistical properties of the radiation field. In the recent

expression for HOS using Eqs. (11), (12) and \( \frac{1}{2} \) in the form \[ 15 \]

\[
\langle (\Delta X)^n \rangle = \sum_{r=0}^{n} \sum_{j=0}^{r} \sum_{k=0}^{r-2i} \frac{(-1)^r}{2^r} \left( \frac{r-2i}{2!} \right) \alpha \beta \gamma \delta \epsilon \eta \theta \nu \rho \sigma \tau \upsilon \xi \zeta \chi \psi \Omega \Xi \Phi \Theta \Psi \Upsilon \Omega
\]

Using Eq. (10) and the expressions of \( c_j \) from Eqs. (8) and (9), we investigate the existence of HOS in nonlinear and linear coherent states. The same is performed through the corresponding plots shown in Fig. 2 which illustrates the existence of Hong-Mandel type HOS in both linear and nonlinear QCSs. Figs. 2(a) and 2(d) show the higher order squeezing in linear and nonlinear QCSs, respectively, for \( n = 4 \). Similarly, Figs. 2(b) and 2(e) show higher order squeezing for \( n = 4 \). Comparing Fig. 2(c) with Fig. 2(f) and Fig. 2(b) with Fig. 2(d) we can easily conclude that the higher order nonclassical properties of linear and nonlinear coherent states are different.
For the nonlinear and linear QCSs, the analytic expressions for the coefficients \( \alpha \) and \( \beta \) are used from Eqs. (4) and (5), respectively. After performing few step algebra, we find out analytic expression for \( D_n(\ell - 1) \). We plot \( D_n(\ell - 1) \) with respect to \( \alpha \) and \( \beta \) for the different values of \( \ell \), the result is depicted in Figs. (3)(a) and (3)(c). The results shown in Figs. (3)(a) and (3)(c) illustrate the existence of HOSPS through the presence of negative regions. Specifically, of the figure ensures the existence of higher order sub-Poissonian photon statistics for \( \ell > 1 \) and sub-Poissonian photon statistics for \( \ell = 1 \). Further, we can see that for a linear coherent state the nonclassicality witness for HOSPS is found to vary monotonically with \( \beta \) and finally to approach a saturation. However, in case of nonlinear QCS, the nonclassicality witness is found to show a kind of oscillation with frequency same as \( \alpha \).

2.4 Agarwal-Tara Criterion

In Ref. [63], Agarwal and Tara introduced a moment based criterion to investigate the witness of the nonclassical characteristics of a given quantum state. They introduced a parameter \( A_3 \). The parameter \( A_3 \) varies between zero to one with respect to \( \alpha \) and \( \beta \), and thus depict the presence of nonclassicality. The periodic nature of \( A_3 \) with respect to \( \alpha \) and \( \beta \) is observed to be similar with that for the witness of HOSPS. The zero value of \( A_3 \) for \( \alpha \) (\( \beta \)) in Fig. (3)(b) (3)(d)) is consistent with the fact that vacuum state is a classical state having non-negative \( P \)-function.

2.5 Klyshko’s criterion

Klyshko [64] introduced a nonclassicality witness involving probability, \( p_n = \langle n | \rho | n \rangle \) of obtaining Fock state \( |n\rangle \), as follows

\[
B(n) \equiv (n + 2) p_n p_{n+2} - (n + 1) |p_{n+1}|^2 < 0. \tag{15}
\]

This inequality has an advantage over other existing criteria, as in this criteria we need only the photon number distribution \( p(n) \) for the three successive values of \( n \), whereas other criteria a complete description is required. Further, this criterion has been recently employed to reveal nonclassicality present in various quantum states [60]. Here, we observe \( B(n) \) to be negative for both linear and nonlinear QCS which indicate the existence of a nonclassical photon statistics. Such signatures of nonclassical photon statistics have already been found through the investigation on HOA and HOSPS. However, the satisfaction of any of them (HOA and HOSPS) does not warrant the satisfaction of Klyshko’s criterion. Further, for a nonlinear QCS, detectability of the nonclassical character via Klyshko’s criterion is found to depend on the dimension of the state and choice of \( \alpha \). We calculate the inequality \( B(n) \) for the both types of the QCSs with the help of Eqs. (4), (5), (6) and (15), respectively. In Figs. (4)(a) and (4)(b), we clearly visualize the Klyshko’s criteria for QCSs \( |\alpha \rangle \) and \( |\beta \rangle \) for different values of \( \alpha \) and \( \beta \) where the negative values of the \( B(n) \), indicates the existence of the nonclassicality. It is noticeable that \( B(n) \) is negative for particular Fock dimension for the given values of \( \alpha \) and \( \beta \).
Figure 3: The variation of $D_h(l)$ parameter illustrating HOSPS and $A_3$ parameter for nonclassicality criterion are shown with the parameter $\alpha$ or $\beta$. (a) and (c) show HOSPS in state $|\alpha\rangle_d$ and $|\beta\rangle_d$ for $d = 3$ with $l = 2$ (smooth blue line), $l = 3$ (dashed red line) and $l = 4$ (dotted dashed magenta line). (b) and (d) show $A_3$ criterion in state $|\alpha\rangle_d$ and $|\beta\rangle_d$ for $d = 3$ (smooth blue line), $d = 4$ (dashed red line) and $d = 5$ (dotted dashed magenta line).

Figure 4: Illustration of Klyshko’s criterion (a) variation of $B(n)$ with respect to $n$ for QCS $|\alpha\rangle_d$ where thick (red) bar, medium (blue) bar and thin (green) bar correspond to $0.06 \times B(n)$, $B(n)$ and $0.5 \times B(n)$ with $\alpha = T_d/2, T_d/4$ and 2.5, respectively, and (b) variation of $B(n)$ with respect to $n$ for QCS $|\beta\rangle_d$ where thick (red) bar, medium (blue) bar and thin (green) bar correspond to $\beta = T_d/2, T_d/4$ and 2.5, respectively.
3 Measures of nonclassicality

So far we have discussed the possibilities of observing nonclassicality in QCS using a set of witnesses of nonclassicality. In the process, we have tried to compare the nonclassical nature of linear and nonlinear coherent states, but such a comparison was only qualitative in nature. To perform a quantitative comparison, we would require one or more quantitative measures of nonclassicality. In fact, many quantitative measures of nonclassicality (e.g., nonclassical distance, nonclassical depth, negative volume of Wigner function, negativity potential, concurrence potential) are in existence, but each of them have their own limitations (see [41] for a review). For example, Hillery introduced a measure of nonclassicality, nonclassical distance [63], as trace norm distance between the state under consideration and closest classical state. Computation of such a measure has drawback in requirement of optimization over infinite number of parameters. Subsequently, Lee [66] proposed another universal measure of nonclassicality known as nonclassical depth which is equal to the amount of Gaussian noise required to transform corresponding $P$-function into a classical probability distribution function. It’s established that this measure is not useful for non-Gaussian pure states [67]. The volume of the negative part of the Wigner function is also used as a quantitative measure of nonclassicality, known as nonclassical volume [68]. However, this measure reflects the drawbacks of the Wigner function, which fails to detect nonclassicality present in the Gaussian states (especially, squeezed states).

With the advancement of the quantum information and computation, various measures of entanglement (such as negativity, concurrence) have been introduced. In 2005, Asboth [69] introduced an excellent idea to use the measures entanglement for quantifying nonclassicality present in a quantum state. This whole idea relies on the conjecture that linear optics conserves nonclassicality [70, 71], therefore, it maps the amount of nonclassicality in the input ports of the beam splitter (BS) to the same amount of entanglement at its output ports. Thus, if we send the vacuum state through one of the input ports and a single mode nonclassical state through the other port of the BS, the amount of bipartite entanglement present in the output ports would quantify the amount of single mode nonclassicality in the quantum state under consideration. Note that if a classical state is inserted in the same manner, then the output will always be separable. This observation, guided Asboth to introduce a measure of the nonclassicality of the input single mode state, which was referred to as entanglement potential. In the original paper [69], logarithmic negativity and entropic entanglement potential are used as entanglement measures, but there exists a number of entanglement measures. In principle, one of these entanglement measures can be used to quantify nonclassicality. Here, we use entanglement potential as measure of single mode nonclassicality. For example, if logarithmic negativity (concurrence) is used as the measure of entanglement, then we would refer to it as negativity (concurrence) potential. In what follows, we have used negativity and concurrence potential to compare the amount of nonclassicality present in the linear and nonlinear QCS.

In brief, a BS transformation can be described by the Hamiltonian $H = \frac{1}{2}(a^\dagger b + ab^\dagger)$, where $a$ and $b$ are the annihilation operators of the two input modes. Following Asboth’s treatment, we combine an input state $\rho_n$ with the vacuum state $\langle 0 | 0 \rangle$ at a symmetric BS to obtain the output state as $\rho_{\text{out}} = U_{\text{BS}} (\rho_n \otimes \langle 0 | 0 \rangle) U_{\text{BS}}^\dagger$, where $U_{\text{BS}} = \exp[-iHt]$. Thus, the output state of the BS for the input state $|n\rangle \otimes |0\rangle$ can be obtained as [72]

$$
|n\rangle \otimes |0\rangle \mapsto_{\text{BS}} \frac{1}{2^n/2} \sum_{j=0}^{n} \sqrt{C_j} |j, n-j\rangle.
$$

Using Eq. (16) we can obtain the output of the BS for a more general scenario where a finite superposition of Fock states described by Eq. (1) is inserted from one port of the BS while a vacuum state is inserted from the other port. In this case, the two-mode output state is obtained as

$$
|\psi\rangle_d \otimes |0\rangle = |\psi, 0\rangle \mapsto_{\text{BS}} \sum_{n=0}^{d-1} \sum_{j=0}^{n} \sqrt{C_j} |j, n-j\rangle.
$$

It is already stated (see Section 2) that a finite superposition of Fock state is always nonclassical. Consequently we can expect the output state (17) to be entangled [69]. Therefore, in what follows, we quantify the amount of entanglement in the output state obtained in Eq. (17).

3.1 Negativity Potential

The negativity potential ($E_N(\rho)$) is one of the useful quantitative measure of nonclassicality, which uses logarithmic negativity as the measure of entanglement. Specifically, the negativity of the two mode
entangled state with a density matrix $\rho$ is defined as
\[ N(\rho) = \max \{0, -2 \min\text{eig} (\rho_{\text{out}}^\Gamma)\}, \]
where $\rho_{\text{out}}^\Gamma$ is the partial transpose of the output state $\rho_{\text{out}}$ in Eq. (17). Further, logarithmic negativity, to quantify entanglement in the units of bits, is defined as
\[ E_N(\rho) = \log_2 ||\rho_{\text{out}}^\Gamma||_1, \] (18)
where $||\cdot||_1$ is the trace norm. It is related to negativity as $E_N(\rho) = \log_2 (2N + 1)$. Hence, using Eqs. (17) and (18) and doing a bit of algebra we can obtain an analytic expression for the logarithmic negativity as
\[ E_N(\rho) = 2 \log_2 \left( \frac{d - 1}{2} \sum_{n=0}^{d-1} |c_n|^2 \sum_{j=0}^{n} (n C_j)^2 \right). \] (19)

Using Eqs. (4), (6) and (19) and simplifying, we obtain the analytic expression for the negativity potential.

Further, we show the variation of logarithmic negativity with respect to $\alpha$ and $\beta$ in Figs. 5(a) and 5(c) where the positive values of the negativity potential in both cases ensure that the output states of the BS are entangled, and thus, in consistency with our expectation, both linear and nonlinear QCSs are found to be nonclassical. We observed that with an increase in the dimensions (i.e., for the larger value of Fock basis $|n\rangle$ superposition) negativity potential for the both the cases increases logarithmically and attain a maximum value. The present results are consistent with that reported by Asboth for the Fock state [69].

**3.2 Concurrence potential**

Concurrence is a universal quantitative measure of entanglement [74]. For the finite-dimensional cases, the bipartite pure state $|\psi\rangle \in H_A \otimes H_B$ with $\text{dim}[H_A \otimes H_B] < +\infty$, the concurrence $C(\rho)$ of $|\psi\rangle$ is defined as
\[ C(\rho) = \sqrt{2 [1 - \text{Tr} (\rho_A^2)]}, \] (20)
where $\rho_A = \text{Tr}_B (|\psi\rangle\langle\psi|)$ is a mixed state which is obtained by taking the partial trace of the output state of the BS. Therefore, in order to get the concurrence of the finite superposition of Fock states, we need to compute the value of $\rho_A^2$. The computation yields
\[ \text{Tr}_A \rho_A^2 = \sum_{n=0}^{d-1} \frac{|c_n|^4}{2^{2n}} \left\{ \sum_{j=0}^{n} (n C_j)^2 \right\}, \]
and using this equation along with the Eqs. (4), (6), (17), and (20), we can obtain concurrence potential for the input nonclassical QCSs $|\alpha\rangle_d$ and $|\beta\rangle_d$ in (a) and (b). The obtained results are illustrated in Figs. 5(b) and 5(d).

### 3.3 Anticlassicality

Anticlassicality is a distance based measure of nonclassicality introduced by Dodonov et al., in [76]. It quantifies the amount of nonclassicality by measuring the distance from the Fock states which are considered to be the most nonclassical states. Now for any arbitrary state $\rho$, the degree of anticlassicality is defined as [76]

$$A = \max_n \langle n | \rho | n \rangle,$$

where the integer $n$ runs over all non-negative integers. Excluding vacuum state $|0\rangle$, the degree of anticlassicality is denoted as $A_1$. In the present work, we have computed the degree of anticlassicality for QCSs. The results are shown in the Figs. 6(a) and 6(b), where the maximum values of the bar correspond to the degree of the anticlassicality ($A_1$) for linear and nonlinear QCSs. It is clear from the Figs. 6(a) and 6(b) that for a given value of the $\alpha$ and $\beta$, $A_1$s are different for QCSs. For the both QCSs, we observe that $A_1$s are always obtained at $n = 1$ for $\alpha = \beta = T_d/2$. The values of $A_1$s for different values of $\alpha$ and $\beta$ for QCSs are calculated, and the results are given in the Table 1.

### 4 CONCLUSION

In conclusion, we would like to note that the present work reports the nonclassical features of linear and nonlinear QCSs through different witnesses of lower- and higher-order nonclassicalities with a focus on the higher order nonclassicality. Further, it quantifies the amount of nonclassicalities present in the linear and nonlinear QCSs by using negativity potential, concurrence potential, and anticlassicality. The uniqueness of this work lies in the fact that higher-order nonclassicalities of QCSs, had not been studied prior to the present work. Further, neither nonclassicalities present in QCS had been quantified earlier, nor anticlassicality had been used to quantify nonclassicality of similar systems in any of the existing works.

In the present work, various inequalities and measurement techniques are used which established the existence and quantification of the nonclassicality in QCSs. The obtained results are plotted and analyzed to reveal that in light of every nonclassical witness and measurement technique for linear and nonlinear QCSs have different characteristics. It is well known that any finite superposition of Fock states is nonclassical which can be explained with the idea of the ‘hole burning’ [77]. However, finite superposition of Fock states do not ensure the existence of higher order nonclassicality. Here, we have explicitly established the existence of higher-order nonclassicality in QCSs using various witnesses of higher- order nonclassicality, e.g., the criteria of HOA, HOS, HOSPS and Agarwal-Tara criterion. In case of lower-order nonclassicality, we have used Klyshko’s criterion and have compared the results (for both
Table 1: Degree of anticlassicality ($A_1$) for QCSs $|\alpha\rangle_d$ and $|\beta\rangle_d$ for different values of parameters $\alpha$ and $\beta$.

| Serial number | Values of $\alpha$ or $\beta$ | Anticlassicality ($A_1$) of QCS $|\alpha\rangle$ | Anticlassicality ($A_1$) of QCS $|\beta\rangle$ |
|---------------|--------------------------------|---------------------------------|---------------------------------|
| 1.            | $T_d/2$                        | 0.473                           | 0.217                           |
| 2.            | $T_d/4$                        | 0.233                           | 0.247                           |
| 3.            | 2.5                            | 0.171                           | 0.164                           |

lower- and higher-order nonclassicalities) for linear and nonlinear QCSs. Interestingly, the comparison led to the observation that the nonclassicality witnesses show oscillation respect to $\alpha$ for nonlinear QCS, but no such oscillation is observed for linear QCS. We have also quantized the quantumness (amount of nonclassicality) present in the QCSs by converting a nonclassical state to an entangled state at the output of the BS. We calculate negativity potential and concurrence potential of the nonclassical state and we have shown that both of the cases amount of nonclassicality increases with the dimension ($d$) and for a particular dimension it gradually approaches a maximum value (cf. Figs. 5(a)-5(d)). The anticlassicality for both types of QCSs are obtained and used for comparison (cf. Figs. 6(a) and 6(b)). The degree of anticlassicality are also calculated for different values of $\alpha$ and $\beta$. The comparison result of the degree of anticlassicality for QCSs are given in the Table 1. The table (and also the results illustrated in Figs. 6(a) and 6(b)) restricts us from making a statement like linear (nonlinear) QCS is more nonclassical than the nonlinear (linear) QCS. The origin of this restriction on the comparative statement can be attributed to the fact that nonclassicality witnesses and measures are observed to oscillate only for nonlinear QCS.

We conclude the paper with the hope that present work may be helpful for further research and the experimental findings specifically in the study of higher order nonclassicality for the QCSs having applications in the quantum information processing.

Acknowledgment

A.P. and N.A. thank the Department of Science and Technology (DST), India, for support provided through the DST project No. EMR/2015/000393. A.P. also thanks K. Thapliyal and A. Miranowicz for some useful technical discussions.

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