Approximating $L_p$ unit balls via random sampling

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Abstract

Let $X$ be an isotropic random vector in $\mathbb{R}^d$ that satisfies that for every $v \in S^{d-1}$, $\| \langle X, v \rangle \|_{L_q} \leq L \| \langle X, v \rangle \|_{L_p}$ for some $q \geq 2p$. We show that for $0 < \varepsilon < 1$, a set of $N = c(p, q, \varepsilon) d$ random points, selected independently according to $X$, can be used to construct a $1 \pm \varepsilon$ approximation of the $L_p$ unit ball endowed on $\mathbb{R}^d$ by $X$. Moreover, $c(p, q, \varepsilon) \leq c^p \varepsilon^{-2} \log(2/\varepsilon)$; when $q = 2p$ the approximation is achieved with probability at least $1 - 2 \exp(-cN \varepsilon^2 / \log(2/\varepsilon))$ and if $q$ is much larger than $p$—say, $q = 4p$, the approximation is achieved with probability at least $1 - 2 \exp(-cN \varepsilon^2)$.

In particular, when $X$ is a log-concave random vector, this estimate improves the previous state-of-the-art—that $N = c'(p, \varepsilon) d^{p/2} \log d$ random points are enough, and that the approximation is valid with constant probability.

1 Introduction

Let $\mu$ be a centred measure on $\mathbb{R}^d$ and for $p \geq 1$ set

$$B(L_p(\mu)) = \left\{ v \in \mathbb{R}^d : \int_{\mathbb{R}^d} |\langle v, x \rangle|^p d\mu(x) \leq 1 \right\}$$

to be the unit ball corresponding to the $L_p$ norm endowed on $\mathbb{R}^d$ by $\mu$.

Even at an intuitive level, the sets $B(L_p(\mu))$ seem significant because they “code” some information on the measure $\mu$. But the fact of the matter is that they are far more important than one might first suspect. Their dual bodies, the so-called $Z_p(\mu)$ bodies, were introduced by E. Lutwak and G. Zhang (under different normalization) in [12]. G. Paouris discovered in his seminar work [19] that when $\mu$ is log-concave\(^1\), the geometry of this family of bodies captures vital information on properties of the generating measure. An alternative, equivalent approach was developed independently by B. Klartag in [6], using the logarithmic Laplace Transform (for a presentation of a unified version of the two approaches, see [7]).

More information on the geometry of $B(L_p(\mu))$ and $Z_p$ bodies and their central role in modern Asymptotic Geometric Analysis can be found in the books [1] and [3].

The identity of the sets $B(L_p(\mu))$ is very useful when it comes to the analysis of statistical algorithms involving the measure $\mu$. However, in many statistical applications $\mu$ is not known,

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\(^1\)A log-concave measure has a density that is a log-concave function on $\mathbb{R}^d$. 

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and rather than knowing the measure, one is given a sample \(X_1, \ldots, X_N\), selected independently according to \(\mu\). Thus, it is natural to ask whether the sets \(B(L_p(\mu))\) can be recovered, or at least approximated using a random sample—hopefully, of a small cardinality, and that the recovery procedure is successful with high probability.

The question of estimating \(B(L_2(\mu))\) is called covariance estimation in statistical literature, and has been studied extensively in recent years (see, e.g., \([10, 8, 17, 18, 16, 13]\)). Classical results focus on situations where \(\mu\) is “well-behaved”, in the sense that linear functionals \(\langle v, \cdot \rangle\) are very light-tailed. Recently, sharp estimates were obtained in heavy-tailed situations. Roughly put, and without going into technical details, the covariance of a centred random vector \(X\) can be recovered under very mild assumptions: given a sample \(X_1, \ldots, X_N\) for \(N \geq c(\varepsilon)d\), one can find \(K = K(X_1, \ldots, X_N) \subset \mathbb{R}^d\), such that

\[(1 - \varepsilon)K \subset B(L_2(\mu)) \subset (1 + \varepsilon)K.\]

The linear dependence on \(d\) is clearly optimal, while the best estimate that is currently known on \(c(\varepsilon)\) is \(\sim \varepsilon^{-2} \log(2/\varepsilon)\) (see \([13]\)). At the same time, all the methods used for covariance estimation are valid only when \(p = 2\) and do not extend to any other value of \(p > 2\), even if \(X\) is a gaussian random vector, let alone in more general scenarios. As a result, the question of estimating \(B(L_p(\mu))\) using random data remained completely open, and in the few cases where partial results were known (e.g., Theorem \([13]\) below), the estimates were far from satisfactory.

Since the covariance can be effectively estimated from a sample whose cardinality is proportional to the dimension of the underlying space, let us fix one such structure:

**Definition 1.1.** A measure \(\mu\) on \(\mathbb{R}^d\) is isotropic if it is centred and has the identity as its covariance. In particular, \(B(L_2(\mu)) = B_d^2\), the Euclidean unit ball.

It is standard to verify that every measure \(\mu\) on \(\mathbb{R}^d\) has an affine image that is isotropic.

Given that normalization, the question we wish to address is:

**Question 1.2.** Let \(\mu\) be an isotropic measure on \(\mathbb{R}^d\) and set \(X\) to be the random vector distributed according to \(\mu\). For \(p > 2\) and \(0 < \varepsilon, \delta < 1\) find \(N = N(\varepsilon, \delta, p)\) and a mapping \(\Phi_p : (\mathbb{R}^d)^N \times \mathbb{R}^d \to \mathbb{R}_+\) for which, with \(\mu^N\)-probability at least \(1 - \delta\), for any \(v \in \mathbb{R}^d\),

\[(1 - \varepsilon)E |\langle X, v \rangle|^p \leq \Phi_p \left( (X_i)_{i=1}^N, v \right) \leq (1 - \varepsilon)E |\langle X, v \rangle|^p.\]

An example that would be of particular interest is when \(\mu\) is an isotropic, log-concave measure, and the state of the art estimate for such measures is due to Guédon and Rudelson:

**Theorem 1.3.** \([5]\) There exists an absolute constant \(c\) for which the following holds. Let \(\varepsilon \in (0, 1), p \geq 2\) and \(d \geq d_0(\varepsilon, p)\). Let \(X\) be an isotropic, log-concave random vector in \(\mathbb{R}^d\) and set

\[N \geq \frac{(cp)^p}{\varepsilon^2}q^{p/2} \log d.\]

Then for \(t > \varepsilon\), with probability at least \(1 - 2 \exp(-c(t/(\varepsilon c'_p))^{1/p})\), for any \(v \in \mathbb{R}^d\),

\[(1 - t)E |\langle X, v \rangle|^p \leq \frac{1}{N} \sum_{j=1}^N |\langle X_j, v \rangle|^p \leq (1 + t)E |\langle X, v \rangle|^p.\]
Theorem 1.3 implies that for log-concave random vectors, \( N = c(p, \varepsilon) d^{p/2} \log d \) random points suffice to construct a \( 1 \pm \varepsilon \) approximation that is valid with a constant confidence level—say \( \delta = 1/2 \). Moreover, the approximation procedure is the most natural choice—the \( p \)-empirical mean,

\[
\frac{1}{N} \sum_{i=1}^{N} |\langle X_i, v \rangle|^p.
\]

As a concentration result for \( p \)-empirical means, the estimate in Theorem 1.3 is close to the best that one can hope for. However, the choice of the \( p \)-empirical mean as an approximation procedure happens to be far from optimal. The problem with that choice can be seen even for a single function: if \( f : \Omega \to \mathbb{R} \) is relatively heavy-tailed, a typical sample \( (|f(X_i)|^p)_{i=1}^{N} \) contains enough atypically large values, making the \( p \)-empirical average too high. As an example, consider \( p = 2 \) and let \( f \) be a function for which the Chebychev bound

\[
P\left( \left| \frac{1}{N} \sum_{i=1}^{N} f^2(X_i) - E f^2 \right| \geq \sqrt{\frac{Ef^4(X)}{\delta N}} \right) \leq \delta
\]

is sharp. Thus, for the 2-empirical mean to be a \( 1 \pm \varepsilon \) approximation of \( \|f\|_{L_2}^2 \) with confidence \( 1 - \delta \), the sample size scales as \( 1/\delta \). In contrast, as we explain in what follows, the optimal (subgaussian) dependence on \( \delta \) can be achieved: the right estimate scales like \( \sqrt{\log(2/\delta)} \) (at least in the range we are interested in). Naturally, the way a recovery procedure performs on a single function says very little on the dependence of \( N \) on the dimension in Question 1.2 rather, it indicates that the \( p \)-empirical mean is likely to be a suboptimal way of approximating the \( L_p \) norm. And indeed, our main result is that a different procedure requires a sample size that scales linearly in the dimension rather than like \( d^{p/2} \log d \), and performs with very high (subgaussian) probability.

To formulate the result we require a standard definition.

**Definition 1.4.** Let \( q > p \geq 1 \). The random vector \( X \) satisfies an \( L_q - L_p \) norm equivalence with constant \( L \) if for every \( v \in \mathbb{R}^d \), \( \| (X, v) \|_{L_q} \leq L \| (X, v) \|_{L_p} \).

**Remark 1.5.** One should keep in mind that log-concave random vectors are absolutely continuous and satisfy \( L_q - L_p \) norm equivalence with constant \( cq/p \) for a suitable absolute constant \( c \) (see, for example, [3]).

The procedure we use is as follows:

Given a tuning parameter \( \frac{1}{N} \leq \theta < 1 \) and \( p \geq 2 \), set

\[
\Psi_{p,\theta}(v) = \frac{1}{N} \sum_{j \geq \theta N} \langle \langle X_i, v \rangle \rangle_j^p
\]

where \( (z_j^*)_{j=1}^{N} \) is the non-increasing rearrangement of \( (|z_j|)_{j=1}^{N} \).

**Remark 1.6.** Note that \( \Psi_{p,\theta} \) is \( p \)-positive homogeneous; therefore, it suffices to show that

\[
(1 - \varepsilon)E|\langle v, X \rangle|^p \leq \Psi_{p,\theta}(v) \leq (1 + \varepsilon)E|\langle v, X \rangle|^p
\]

for every \( v \) in some Euclidean sphere to ensure that (1.2) holds for every \( v \in \mathbb{R}^d \).
Theorem 1.7. There is an absolute constant $c_0$ for which the following holds. Let $X$ be an isotropic random vector in $\mathbb{R}^d$ that is absolutely continuous. For any $\varepsilon \in (0, 1)$ and $\theta = c_0 \varepsilon^2 N$,

\begin{equation}
(1 - \varepsilon) \mathbb{E} | \langle X, v \rangle |^p \leq \Psi_{p, \theta}(v) \leq (1 + \varepsilon) \mathbb{E} | \langle X, v \rangle |^p \quad \text{for every } v \in \mathbb{R}^d,
\end{equation}

provided that

\[ N \geq c_1^p d \frac{\log(2/\varepsilon)}{\varepsilon^2} \]

in the following cases:

(a) If $X$ satisfies an $L_{2p}$-$L_p$ norm equivalence with constant $L$, then $c_1$ depends only on $L$ and \[(1.3)\) holds with probability at least $1 - 2 \exp(-c_2(L, \varepsilon) N)$ for $c_2(L, \varepsilon) = c(L) \frac{\varepsilon^2}{\log(2/\varepsilon)}$.

(b) The logarithmic factor in $c_2(L, \varepsilon)$ from (a) is not needed if $X$ satisfies an $L_q$-$L_p$ norm equivalence with constant $L$ for some $q > 2p$. The claim holds with probability at least $1 - 2 \exp(-c_2^2 \varepsilon^2 N)$ and $c_2^2$ depends on $q - 2p$ and $L$.

(c) In particular, if $X$ is log-concave then $c_1$ is an absolute constant and \[(1.3)\) holds with probability at least $1 - 2 \exp(-c_3^2 \varepsilon^2 N)$ for a suitable absolute constant $c_3$.

Part (c) of Theorem 1.7 follows immediately from Part (b) because a log-concave vector is absolutely continuous and satisfies an $L_{4p}$-$L_p$ norm equivalence with an absolute constant. Thus, the number of vectors that suffice for the construction of a $1 \pm \varepsilon$ approximation of $B(L_p(\mu))$ for a log-concave measure $\mu$ scales linearly in the dimension $d$.

Observe that for every $\theta$ and $p$,

\[ \Psi_{p, \theta}(v) \leq \frac{1}{N} \sum_{i=1}^{N} | \langle X_i, v \rangle |^p; \]

thus, Theorem 1.7 leads to a one-sided (lower) bound on the empirical mean:

Corollary 1.8. In the situations described in Theorem 1.7 we have that for every $v \in \mathbb{R}^d$,

\begin{equation}
(1 - \varepsilon) \mathbb{E} | \langle X, v \rangle |^p \leq \frac{1}{N} \sum_{i=1}^{N} | \langle X_i, v \rangle |^p.
\end{equation}

Corollary 1.8 is one in a long line of results which show that one-sided inequalities for the empirical mean are almost universally true—under only minimal assumptions on $X$ (see, for example, [15, 9, 13] for results of a similar flavour). And while the lower bound is universal, the upper one is highly restrictive, and is false in general. The “truncation” functional $\Psi$ addresses the problem of atypically large values that are likely to appear in each vector $| \langle X_i, v \rangle |_{i=1}^{N}$ when $X$ is heavy-tailed, and that leads to the two-sided estimate of Theorem 1.7.

The crucial point is that $\Psi$ endows an adaptive truncation level—based on the nonincreasing rearrangement of the vector $| \langle X_i, v \rangle |_{i=1}^{N}$—rather than at a fixed value.

Remark 1.9. The definition of $\Psi_{p, \theta}$ can be extended beyond the class of linear functionals by setting for $f : \Omega \to \mathbb{R}$,

\[ \Psi_{p, \theta}(f) = \frac{1}{N} \sum_{j \geq \theta N} (|f(X_i)|^p)_j^*. \]
Moreover, Theorem 1.7 can be extended to far more general function classes than the set of linear functionals on $\mathbb{R}^d$. However, that requires the development of a rather involved technical machinery that is not needed for addressing Question 1.2. We defer the study of the general scenario to [11], and devote this work to the “shortest path” leading to the proof of Theorem 1.7.

In what follows we denote the expectation $\mathbb{E} f(X)$ by $\mathbb{P}(f)$ and $\mathbb{P}_N(f) = \frac{1}{N} \sum_{i=1}^{N} f(X_i)$ is the empirical mean of $f$. We use the same notation—$\mathbb{P}(A)$ and $\mathbb{P}_N(A)$ to denote the actual and empirical measures of a set $A$. Absolute constants are denoted by $c, C$, etc.; their value may change from line to line. $a \lesssim b$ implies that there is an absolute constant $c$ such that $a \leq cb$.

Contrary to what one may expect, the proof of Theorem 1.7 is rather simple. It is based on two facts: first, that for any function $f$, $\Psi_{p,\theta}(f)$ is a sharp estimate of $\mathbb{E}|f|^p$ if $X_1,...,X_N$ satisfies certain ratio estimates of the form

$$\sup_{\{t: \mathbb{P}(|f| > t) \geq \Delta\}} \left| \frac{\mathbb{P}_N(|f| > t)}{\mathbb{P}(|f| > t)} - 1 \right| < \varepsilon.$$  

Second, that there is an event of high $\mu^N$-probability for which the required ratio estimates are satisfied uniformly by all the linear functionals $F = \{\langle v, \cdot \rangle : v \in \mathbb{R}^d\}$. This relies heavily on the fact that $F$ is small in an appropriate sense.

The two components of the proof are presented in the next two sections.

2 Empirical tail integration

The goal of this section is to show that, for an arbitrary function $f$, $\Psi_{p,\theta}(f)$ is a good estimator of $\mathbb{E}|f|^p$ for a well-chosen $\theta$, as long as there is enough information on the ratios $\mathbb{P}_N(|f| \in I)/\mathbb{P}(|f| \in I)$ for generalized intervals $I$; here an in what follows generalized intervals are open/closed, half-open/closed intervals in $\mathbb{R}$—including rays.

| Let $0 \leq \lambda, \Delta < 1$ and $C \geq 1$. For a function $f$ let $\mathcal{A}_{\lambda,C,\Delta}$ be the event on which the following holds: |
|---|
| (1) For any $t > 0$ such that $\mathbb{P}(|f| > t) \geq \Delta$, we have that |
| \[ \left| \frac{\mathbb{P}_N(|f| > t)}{\mathbb{P}(|f| > t)} - 1 \right| \leq \lambda; \]
| (2) If $j \in \mathbb{N}$ and $t > 0$ satisfy that $2^{-j} \mathbb{P}(|f| > t) \geq \Delta$, then |
| \[ \left| \frac{\mathbb{P}_N(|f| > t)}{\mathbb{P}(|f| > t)} - 1 \right| \leq 2^{-j/2}; \]  
| (3) For any generalized interval $I \subset \mathbb{R}$, |
| \[ \mathbb{P}_N(|f| \in I) \leq \frac{3}{2} \mathbb{P}(|f| \in I) + C\Delta. \] |
Ratio estimates are natural in the context of Question 1.2 because
\[ \mathbb{E}|f|^p = \int_0^\infty pt^{p-1} \mathbb{P}(|f| > t) dt. \]

If sufficiently sharp ratio estimates are available, it is possible to approximate this integral by an empirical functional of the form
\[ \int_0^T pt^{p-1} \mathbb{P}(|f| > t) dt. \]

Note that Property (2) is a collection of ‘isomorphic’ estimates that become closer to an isometry for larger sets. For example, if \( \mathbb{P}(|f| > t) \) is of the order of constant, then the allowed distortion in (2.1) can be as small as
\[ \left| \frac{\mathbb{P}_N(|f| > t)}{\mathbb{P}(|f| > t)} - 1 \right| \lesssim \sqrt{\Delta}, \]
whereas when \( \mathbb{P}(|f| > t) \sim \Delta \), the allowed distortion in (2.1) is 1/2. That fits the idea of approximating the integral by an empirical counterpart: for a well-chosen \( T \), \( \int_0^T pt^{p-1} \mathbb{P}_N(|f| > t) dt \) can be very close to \( \int_0^T pt^{p-1} \mathbb{P}(|f| > t) dt \) even when the distortion is relatively large for sets \( \{|f| > t\} \) whose measure is small; at the same time, minimal distortion is essential for sets of relatively large measure, as those have a much higher impact on the two integrals.

**Definition 2.1.** For a function \( f, p \geq 1 \) and \( T > 0 \) set
\[ \mathcal{E}_{T,p}(f) = 2\sqrt{\Delta} \int_0^T pt^{p-1} \mathbb{P}(|f| > t) dt. \]

Also, for \( 0 < \eta < 1 \), let
\[ Q_{1-\eta}(f) = \inf \{ t : \mathbb{P}(f > t) < \eta \}, \]
i.e., \( Q_{1-\eta}(f) \) is the \( \eta \) quantile of \( f \).

**Theorem 2.2.** There are absolute constants \( c_1, \ldots, c_5 \) for which the following holds. Let \( p \geq 1 \) and assume that \( f(X) \in L_p \) is nonnegative and absolutely continuous. Set \( \frac{\sqrt{\Delta}}{N} < \Delta \leq 1/2 \) and assume that \( f \) satisfies properties (1)-(3) on the event \( A = A_{\lambda, C, \Delta} \) for \( \lambda = 1/2 \) and \( C = 2 \). Setting \( \theta = c_2 \Delta \) and \( \Lambda = Q_{1-c_3 \Delta} \) we have that on the event \( A \),
\[ \frac{1}{N} \sum_{j \geq \theta N} (f^p(X_i))^*_j \leq \mathbb{E}f^p + c_4 \mathcal{E}_{\Lambda,p}(f), \tag{2.2} \]
and
\[ \frac{1}{N} \sum_{j \geq \theta N} (f^p(X_i))^*_j \geq \mathbb{E}f^p - c_4 \left( \mathcal{E}_{\Lambda,p}(f) + \mathbb{E}f^p 1_{\{f > Q_{1-c_3 \Delta}\}} \right). \tag{2.3} \]

**Remark 2.3.** There is no hope of obtaining an empirical-based estimator of \( \mathbb{E}f^p \) if the contribution of the tail \( \mathbb{E}f^p 1_{\{f > Q_{1-\kappa}(f)\}} \) is too big. The reason is that the set \( \{ f > Q_{1-\kappa}(f) \} \) may be under-represented in the sample: if one is interested in an estimate that holds with \( \mu^N \)-probability of at least \( 1 - 2 \exp(-c \Delta N) \), then
\[ \mathbb{P}_N(f > Q_{1-\kappa}(f)) = \frac{1}{N} | \{ i : f(X_i) > Q_{1-\kappa}(f) \} | \]
may be much smaller than \( \kappa = \mathbb{P}(f > Q_{1-\kappa}(f)) \) unless \( \kappa \) is of the order of \( \Delta \). To see that, let \( \kappa \lesssim \Delta \). Then with probability at least \((1 - \kappa)^N \geq \exp(-c\kappa N) \geq \exp(-c\Delta N)\) we have that \( f(X_i) \leq Q_{1-\kappa}(f) \) for every \( 1 \leq i \leq N \). On such samples one cannot distinguish between \( f \) and of \( f^1_{\{f \leq Q_{1-\kappa}(f)\}} \); however, there can be a significant difference between \( \mathbb{E}f^p \) and \( \mathbb{E}f^p_{\{f < Q_{1-\kappa}(f)\}} \). As a result, the term \( \mathbb{E}f^p_{\{f \geq Q_{1-c5\Delta}(f)\}} \) in (2.3) is essential.

Before we turn to the proof of Theorem [2.2] let us examine the two parameters that are featured in it—namely, \( \mathcal{E}_{T,p}(f) \) for \( T = Q_{1-\kappa}(f) \) and \( \mathbb{E}f^p_{\{f \geq Q_{1-\kappa}(f)\}} \) for some \( 0 < \kappa < 1 \).

To ease notation we remove the dependence of the two parameters on \( f \), and write \( \mathcal{E}_{T,p} \) and \( Q_{1-\kappa} \) instead.

**Lemma 2.4.** Let \( f(X) \in L_2 \) be nonnegative and absolutely continuous. Then

\[
\mathbb{E}f^p_{\{f > Q_{1-\kappa}\}} \leq \|f\|_{L_2}^p \sqrt{\kappa}
\]

and

\[
\mathcal{E}_{Q_{1-\kappa},p} \leq c\sqrt{\Delta} \left( \log \left( \frac{1}{\kappa} \right) \right) \|f\|_{L_2}^p
\]

for an absolute constant \( c \).

Moreover, if \( f \in L_q \) for \( q > 2p \) then

\[
\mathcal{E}_{Q_{1-\kappa},p} \leq c_{q,p} \sqrt{\Delta} \|f\|_{L_q}^p
\]

for \( c_{q,p} = 2p/(q - 2p) \).

**Proof.** The proofs of the claims are straightforward. For the first claim observe that

\[
\mathbb{E}f^p_{\{f > Q_{1-\kappa}\}} \leq (\mathbb{E}f^{2p})^{1/2} \mathbb{E}^{1/2}(f > Q_{1-\kappa}) = \|f\|_{L_2}^p \sqrt{\kappa}.
\]

Turning to the two estimates on \( \mathcal{E}_{Q_{1-\kappa},p} \), let \( T = Q_{1-\kappa} \) and consider the following two cases. If \( T \geq \|f\|_{L_p} \) then by the Cauchy-Schwarz inequality,

\[
\mathcal{E}_{T,p} = 2\sqrt{\Delta} \left( \int_0^{\|f\|_{L_p}} pt^{p-1} \sqrt{\mathbb{P}(f > t)} \, dt + \int_0^T \frac{1}{\sqrt{t}} \cdot pt^{p-1/2} \sqrt{\mathbb{P}(\|f\| > t)} \, dt \right)
\]

\[
\leq 2\sqrt{\Delta} \left( \int_0^{\|f\|_{L_p}} pt^{p-1} \, dt + \left( \log \frac{T}{\|f\|_{L_p}} \right)^{1/2} \sqrt{\frac{p}{2}} \left( \int_0^T 2pt^{2p-1} \mathbb{P}(\|f\| > t) \, dt \right)^{1/2} \right)
\]

\[
\leq 2\sqrt{\Delta} \left( \|f\|_{L_p}^p + \left( \frac{p}{2} \log \left( \frac{T}{\|f\|_{L_p}} \right) \right)^{1/2} \|f\|_{L_2}^p \right) = (*).
\]

Recalling that \( f \) is absolutely continuous, \( \kappa = \mathbb{P}(f \geq Q_{1-\kappa}) \leq \left( \frac{\|f\|_{L_p}}{Q_{1-\kappa}} \right)^p \).

In particular, \( T/\|f\|_{L_p} = Q_{1-\kappa}/\|f\|_{L_p} \leq 1/\kappa^{1/p} \) and

\[
\frac{p}{2} \log \left( \frac{T}{\|f\|_{L_p}} \right) \leq \frac{p}{2} \log \left( \frac{1}{\kappa^{1/p}} \right) \leq \frac{1}{2} \log \left( \frac{1}{\kappa} \right).
\]
Therefore,
\[ (\ast) \lesssim \sqrt{\Delta} \left( \|f\|_{L^p}^p + \|f\|_{L^2}^p \sqrt{\log \left( \frac{1}{\kappa} \right)} \right) \lesssim \sqrt{\Delta \log \left( \frac{1}{\kappa} \right)} \|f\|_{L^2}^p, \]
as claimed. The proof in the case \( Q_{1-\kappa} \leq \|f\|_{L^p} \) requires only the trivial estimate on the integral in \([0, \|f\|_{L^p}]\) used above.

Finally, if \( f \in L_q \) for \( q > 2p \) then
\[ \int_0^{\|f\|_{L^q}} pt^{p-1} \sqrt{\mathbb{P}(f > t)} dt \leq \|f\|_{L_q}^p. \]
and since \( \mathbb{P}(f > t) \leq \left( \|f\|_{L_q}/t \right)^q \), it is evident that
\[ \int_{\|f\|_{L_q}}^T pt^{p-1} \sqrt{\mathbb{P}(f > t)} dt \leq \|f\|_{L_q}^{q/2} \int_{\|f\|_{L_q}}^T pt^{p-1-q/2} dt \leq \|f\|_{L_q}^{q/2} \cdot \frac{p}{(q/2 - p)} \|f\|_{L_q}^{p-q/2} = \frac{2p}{q - 2p} \|f\|_{L_q}^p. \]

\( \square \)

Combining Theorem 2.2 with Lemma 2.4 leads to the following, more user-friendly corollary:

**Corollary 2.5.** There are absolute constants \( c_1, \ldots, c_6 \) for which the following holds. Set \( p \geq 1 \) and let \( f(X) \in L_{2p} \) be nonnegative and absolutely continuous. Set \( \frac{4}{N} \leq \Delta \leq 1/2 \), let \( \theta = c_2 \Delta \), and put
\[ \Psi_{p,\theta}(X_1, \ldots, X_N) = \frac{1}{N} \sum_{j \geq \theta N} (|f(X_i)|^p)_{j}. \]
Let \( A \) be the event on which \( f \) satisfies properties (1)-(3) with constants \( \lambda = 1/2 \) and \( C = 2 \). Then on the event \( A \),
\[ |\Psi_{p,\theta}(X_1, \ldots, X_N) - E f^p| \leq c_4 \sqrt{\Delta \log \left( \frac{1}{\Delta} \right)} \|f\|_{L_{2p}}^p. \tag{2.4} \]
Moreover, if \( f \in L_q \) for \( q > 2p \) then with the same probability,
\[ |\Psi_{p,\theta}(X_1, \ldots, X_N) - E f^p| \leq c_{q,p} \sqrt{\Delta \log \left( \frac{1}{\Delta} \right)} \|f\|_{L_q}^p, \]
where \( c_{q,p} \sim p/(q - 2p) \).

In the context of Theorem 1.7 we show in what follows that there is a high probability event on which, for every \( v \in \mathbb{R}^d \), \( f_v(X) = | \langle X, v \rangle | \) satisfies properties (1)-(3), as long as \( \Delta \geq c \frac{N}{d} \log \left( \frac{N}{d} \right) \). Once that fact is established (see Section 3), Theorem 1.7 follows immediately from Corollary 2.5.

Let us prove the following version of Theorem 2.2 which gives some freedom in the choice of parameters \( \lambda \), \( C \) and \( \Delta \).
Theorem 2.6. Set $p \geq 1$ and let $f(X) \in L_p$ be nonnegative and absolutely continuous. Assume that $X_1, ..., X_N$ is a sample for which $f$ satisfies Properties (1) – (3) with constants $\lambda$, $C$ and $\Delta$. Set $\theta \geq 4\Delta \max\{(1 + \lambda), C + 3/2\}$ and let

$$
\theta_1 = \frac{\theta + 2C\Delta}{1 - \lambda} \quad \text{and} \quad \theta_2 = \frac{\theta - 2C\Delta}{1 + \lambda}.
$$

Then

$$
\frac{1}{N} \sum_{j \geq \theta N} (f^p(X_i))^* \leq \mathbb{E} f^p + 2\sqrt{\Delta} \int_0^{Q_{1-\theta_2}(f)} pt^{p-1} \sqrt{\mathbb{P}(f > t)} dt,
$$

and

$$
\frac{1}{N} \sum_{j \geq \theta N} (f^p(X_i))^* \geq \mathbb{E} f^p - \left(1 + \frac{1}{1 - \lambda}\right) \mathbb{E} f^p \mathbb{1}_{\{f \geq Q_{1-\theta_1}(f)\}} - 2\sqrt{\Delta} \int_0^{Q_{1-\theta_1}(f)} 2t \sqrt{\mathbb{P}(f > t)} dt.
$$

Theorem 2.2 follows directly from Theorem 2.6 with the choice of $\lambda = 1/2$, $C = 2$ and for $A_{\lambda,C,\Delta}$ that is the set of samples for which $f(X)$ satisfies Properties (1)-(3) with those parameters.

The proof of Theorem 2.6 requires several preliminary steps, starting with a straightforward observation: clearly, $\theta_2 \geq 2\Delta$, and therefore,

$$
\mathbb{P}(f > t) \geq 2\Delta \quad \text{for any} \quad 0 < t \leq Q_{1-\theta_2}(f).
$$

(2.5)

As a result, all the level sets $\{f > t\}$ for $0 < t \leq Q_{1-\theta_2}(f)$ satisfy Property (1).

Lemma 2.7. Using the notation of Theorem 2.6, let $\hat{Q} = (f(X_i))^*_{\theta N}$. Then

$$
Q_{1-\theta_1}(f) < \hat{Q} < Q_{1-\theta_2}(f).
$$

Proof. There are at least $\theta N$ indices $i$ such that $f(X_i) \geq \hat{Q}$; thus

$$
\mathbb{P}_N(f \geq \hat{Q}) = \frac{1}{N} \left| \{i : f(X_i) \geq \hat{Q}\} \right| \geq \theta.
$$

(2.6)

Therefore, by property (3) for $I = [\hat{Q}, \infty)$,

$$
\theta \leq \mathbb{P}_N(f \geq \hat{Q}) \leq \frac{3}{2} \mathbb{P}(f \geq \hat{Q}) + C\Delta,
$$

and as $\theta \geq (C + 3/2)\Delta$ it follows that

$$
\mathbb{P}(f > \hat{Q}) = \mathbb{P}(f \geq \hat{Q}) \geq \frac{2}{3} (\theta - C\Delta) \geq \Delta.
$$

Hence, by Property (1) for $t = \hat{Q}$,

$$
\frac{\mathbb{P}_N(f > \hat{Q})}{\mathbb{P}(f > \hat{Q})} - 1 \leq \lambda.
$$

(2.7)
Next, using Property (3) once again, we have that for any $t > 0$ and any $\gamma \leq |t|/2$,
\[
P_N(f \in [t - \gamma, t]) \leq \frac{3}{2} P(f \in [t - \gamma, t]) + C\Delta.
\]
Taking $\gamma \to 0$ and by the absolute continuity of $f(X)$, $P_N(f = t) \leq C\Delta$ for any $t > 0$. In particular, for $t = \hat{Q}$
\[
\left| \{i : f(X_i) = \hat{Q} \} \right| \leq C\Delta N.
\]
Hence,
\[
\theta - 2C\Delta < P_N(f > \hat{Q}) < \theta + 2C\Delta.
\]
Using (2.7),
\[
P(f > \hat{Q}) \leq \frac{P_N(f > \hat{Q})}{1 - \lambda} < \frac{\theta + 2C\Delta}{1 - \lambda} = \theta_1,
\]
and
\[
P(f > \hat{Q}) \geq \frac{P_N(f > \hat{Q})}{1 + \lambda} > \frac{\theta - 2C\Delta}{1 + \lambda} = \theta_2,
\]
implicating that
\[
Q_{1-\theta_1} < \hat{Q} < Q_{1-\theta_2}, \quad (2.8)
\]
as claimed.

**Lemma 2.8.** Let $f(X)$ be nonnegative and absolutely continuous. Using the notation of Theorem 2.6 for $p \geq 1$,
\[
\int_0^{Q_{1-\theta_1}} pt^{p-1} P_N(f > t)dt - \theta \hat{Q}^p \leq \frac{1}{N} \sum_{j \geq \theta N} (f^p(X_i))^*_j \leq \int_0^{Q_{1-\theta_2}} pt^{p-1} P_N(f > t)dt \quad (2.9)
\]

**Proof.** Recall that $\hat{Q} = (f(X_i))^*_\theta$, and therefore,
\[
\frac{1}{N} \sum_{i=1}^N f^p 1_{\{f \leq \hat{Q} \}}(X_i) - \theta \hat{Q}^p \leq \frac{1}{N} \sum_{j \geq \theta N} (f^p(X_i))^*_j \leq \frac{1}{N} \sum_{i=1}^N f^p 1_{\{f \leq \hat{Q} \}}(X_i).
\]

By tail integration,
\[
\frac{1}{N} \sum_{i=1}^N f^p 1_{\{f \leq \hat{Q} \}}(X_i) = \int_0^\infty pt^{p-1} P_N \left( f 1_{\{f \leq \hat{Q} \}} > t \right) dt = \int_0^{\hat{Q}} pt^{p-1} P_N(f > t)dt;
\]

Lemma 2.7 shows that $Q_{1-\theta_1} < \hat{Q} < Q_{1-\theta_2}$ and the wanted estimate follows. ■

To control (2.9), let us obtain an estimate on $\int_0^T pt^{p-1} P_N(f > t)dt$ that holds as long as the probabilities $P(f > t)$, $t \in (0, T)$ are large enough and the nonnegative function $f$ satisfies Property (2). To formulate the claim, recall that
\[
\mathcal{E}_{T,p}(f) = 2\sqrt{\Delta} \int_0^T pt^{p-1} \sqrt{P(|f| > t)} dt.
\]
Lemma 2.9. Let $T$ be such that $\mathbb{P}(f > T) \geq \Delta$. Let $(X_1, ..., X_N)$ satisfy that, for any $0 < t < T$ and $j \in \mathbb{N}$,

\[
\text{if } 2^{-j} \mathbb{P}(f > t) \geq \Delta \text{ then } \left| \frac{\mathbb{P}_N(f > t)}{\mathbb{P}(f > t)} - 1 \right| \leq 2^{-j/2}.
\]  

(2.10)

Then

\[
\mathbb{E} f^p 1_{\{f \leq T\}} - \mathcal{E}_{T,p}(f) \leq \int_0^T pt^{p-1} \mathbb{P}_N(f > t) dt \leq \mathbb{E} f^p + \mathcal{E}_{T,p}(f).
\]

We present the proof for $p = 2$ and write $\mathcal{E}_T$ instead of $\mathcal{E}_{T,2}(f)$. The proof for $p \neq 2$ is identical and is omitted.

Proof. For every $T > 0$ let $j_T$ be the largest integer such that $\mathbb{P}(f > T) \geq 2^{j_T} \Delta$, and set $j_0$ to be the smallest integer such that $2^{j_0} \Delta \geq 1$. Therefore, $2^{j_0-1} \Delta \geq 1/2$ and

\[
\frac{1}{2} \mathbb{P}(f > T) \leq 2^{j_T} \Delta \leq \mathbb{P}(f > T).
\]

For $j_T \leq j \leq j_0 - 1$ let

\[
I_j = \{t > 0 : \Delta 2^j \leq \mathbb{P}(f > t) < \Delta 2^{j+1}\}
\]

and observe that

\[
\bigcup_{j=j_T}^{j_0-1} I_j \supset (0, T).
\]

Moreover, by (2.10), for $t \in I_j$

\[
(1 - 2^{-j/2}) \mathbb{P}(f > t) \leq \mathbb{P}_N(f > t) \leq (1 + 2^{-j/2}) \mathbb{P}(f > t),
\]

implying that

\[
\int_0^T 2t \mathbb{P}_N(f > t) dt = \sum_{j=j_T}^{j_0-1} \int_{I_j \cap (0, T)} 2t \mathbb{P}_N(f > t) dt \leq \sum_{j=j_T}^{j_0-1} \int_{I_j \cap (0, T)} 2t(1 + 2^{-j/2}) \mathbb{P}(f > t) dt
\]

\[
\leq \int_0^\infty 2t \mathbb{P}(f > t) dt + \sum_{j=j_T}^{j_0-1} \int_{I_j \cap (0, T)} 2t \cdot \Delta 2^{(j/2)+1} dt
\]

\[
\leq \mathbb{E} f^2 + 2\sqrt{\Delta} \sum_{j=j_T}^{j_0-1} \int_{I_j \cap (0, T)} 2t \sqrt{\mathbb{P}(f > t)} dt
\]

\[
\leq \mathbb{E} f^2 + 2\sqrt{\Delta} \int_0^T 2t \sqrt{\mathbb{P}(f > t)} dt = \mathbb{E} f^2 + \mathcal{E}_T.
\]

In the reverse direction and using the same argument,

\[
\int_0^T 2t \mathbb{P}_N(f > t) \geq \int_0^T 2t \mathbb{P}(f > t) dt - 2\sqrt{\Delta} \int_0^T 2t \sqrt{\mathbb{P}(f > t)} dt = \mathbb{E} f^2 - \mathcal{E}_T.
\]
Proof of Theorem 2.6. Apply Lemma 2.9 for $T = Q_{1-\theta_1}(f) \equiv Q_{1-\theta_1}$ and $T = Q_{1-\theta_2}(f) \equiv Q_{1-\theta_2}$, which are both valid choices, as
\[ P(f > Q_{1-\theta_1}) \geq P(f > Q_{1-\theta_2}) \geq \Delta. \]

Thus,
\[ \int_0^{Q_{1-\theta_2}} pt^{p-1} \mathbb{P}_N(f > t)dt \leq \mathbb{E} f^p + \mathcal{E}_{Q_{1-\theta_2}}(f), \quad (2.11) \]
and
\[ \int_0^{Q_{1-\theta_1}} pt^{p-1} \mathbb{P}_N(f > t)dt \geq \mathbb{E} f^p - \left( \mathbb{E} f^p \mathbb{P}(f > Q_{1-\theta_1}) + \mathcal{E}_{Q_{1-\theta_1}}(f) \right). \quad (2.12) \]

Next, let us show that
\[ \hat{Q}^p \theta \leq \frac{1}{1 - \lambda} \mathbb{E} f^p \mathbb{1}_{\{f \geq Q_{1-\theta_1}\}}, \quad (2.13) \]
which, by Lemma 2.8 completes the proof. To that end, recall that $P(f > Q_{1-\theta_2}) \geq \Delta$, and that by Lemma 2.7,
\[ Q_{1-\theta_1} < \hat{Q} < Q_{1-\theta_2}; \]
thus,
\[ \mathbb{E} f^p \mathbb{1}_{\{f \geq \hat{Q}\}} \leq \mathbb{E} f^p \mathbb{1}_{\{f \geq Q_{1-\theta_1}\}}. \quad (2.14) \]
Also, since $\mathbb{P}_N(f \geq \hat{Q}) \geq \theta$ and
\[ \mathbb{P}(f \geq \hat{Q}) \geq \mathbb{P}(f \geq Q_{1-\theta_2}) \geq \Delta, \]
it follows from Property (1) that
\[ \mathbb{E} f^p \mathbb{1}_{\{f \geq \hat{Q}\}} \geq \hat{Q}^p \mathbb{P}(f \geq \hat{Q}) \geq \hat{Q}^p (1 - \lambda) \mathbb{P}_N(f \geq \hat{Q}) \geq (1 - \lambda) \hat{Q}^p \theta, \]
proving (2.13).

3 Ratio estimates for linear functionals

Let us turn to the second component needed for the proof of Theorem 1.7. Set $F = \{ \langle v, \cdot \rangle : v \in \mathbb{R}^d \}$ and denote
\[ U = \left\{ \mathbb{1}_{\{(v, \cdot) \in I\}} : v \in \mathbb{R}^d, \ I \subset \mathbb{R}_+ \ is \ a \ generalized \ interval \right\}. \quad (3.1) \]
For a binary-valued function $u$, let $\mathbb{P}_N(u) = \frac{1}{N} \sum_{i=1}^N u(X_i)$ be its empirical mean and set $\mathbb{P}(u) = \mathbb{E} u$.

To complete the proof of Theorem 1.7 it suffices to find, for $\Delta \geq \frac{d}{2} \log \left( \frac{eN}{d} \right)$, a high probability event $\mathcal{A}$ on which every $f \in F$ satisfies Properties (1)-(3) for $\lambda = 1/2$ and $C = 2$. In terms of the class of the indicator functions $U$, we show that:
For $\Delta \geq c_0 \frac{d}{N} \log \left( \frac{eN}{d} \right)$, with probability at least $1 - 2 \exp(-c_1 \Delta N)$, any $u \in U$ satisfies the following:

(a) If $P(u) \geq \Delta$ then
$$\left| \frac{P_N(u)}{P(u)} - 1 \right| \leq \frac{1}{2};$$

(b) If $j \in \mathbb{N}$ and $2^{-j}P(u) \geq \Delta$ then
$$\left| \frac{P_N(u)}{P(u)} - 1 \right| \leq 2^{-j/2};$$

(c) $P_N(u) \leq \frac{3}{2}P(u) + 2\Delta$.

The proof of this claim is based on standard tools in empirical processes theory.

**Definition 3.1.** Let $U$ be a class of $\{0, 1\}$-valued functions on $\Omega$. A set $\{x_1, \ldots, x_n\}$ is shattered by $U$ if for every $I \subset \{1, \ldots, n\}$ there is some $u_I \in U$ for which $u_I(x_i) = 1$ if $i \in I$ and $u_I(x_i) = 0$ otherwise.

The VC dimension of $U$ is the maximal cardinality of a subset of $\Omega$ that is shattered by $U$; it is denoted by $VC(U)$.

We refer the reader to [21] for basic facts on VC classes and on the VC-dimension.

The connection between the VC dimension and our problem is that the class of indicator $U$ defined in (3.1) satisfies that $VC(U) \leq cd$ for a suitable absolute constant $c$. The proof of this fact can be found, for example, in [21].

The validity of Properties (a)-(c) can be verified for any class of binary valued functions whose VC dimension is at most $d$.

**Theorem 3.2.** There are absolute constants $c_0$ and $c_1$ for which the following holds. Let $U$ be a class of binary valued functions, and assume that $VC(U) \leq d$. Then for $c_0 \frac{d}{N} \log \left( \frac{eN}{d} \right) \leq \Delta \leq \frac{1}{2}$, with probability at least $1 - 2 \exp(-c_1 \Delta N)$, the function class $U$ satisfied Properties (a)-(c).

**Remark 3.3.** It is very likely that Theorem 3.2 is known to experts: ratio estimates of that flavour have been used implicitly in [14], and more general ratio estimates, such as Proposition 2.8 in [4], can also be used to prove Theorem 3.2. However, we could not locate in literature a simple proof of a suitable version of Theorem 3.2 and the rest of this section is devoted to such a proof.

The proof of Theorem 3.2 is based on Talagrand’s concentration inequality for empirical processes indexed by a class of uniformly bounded functions.
Theorem 3.4. There exist an absolute constant $\kappa$ for which the following holds. Let $U$ be a class of functions, set

$$ \sigma_U^2 = \sup_{u \in U} \text{var}(u) \quad \text{and} \quad b = \sup_{u \in U} \|u\|_{L_\infty}, $$

and denote by $(\epsilon_i)_{i=1}^N$ independent, symmetric, $\{-1,1\}$-valued random variables that are also independent of $(X_i)_{i=1}^N$.

Then for every $x > 0$, with probability at least $1 - 2 \exp(-x)$,

$$ \sup_{u \in U} \left| \frac{1}{N} \sum_{i=1}^N u(X_i) - \mathbb{E}u \right| \leq \kappa \left( \mathbb{E} \sup_{u \in U} \left| \frac{1}{N} \sum_{i=1}^N \epsilon_i u(X_i) \right| + \sigma_U \sqrt{\frac{x}{N} + b \frac{x}{N^2}} \right). $$

Moreover, if $U$ is a class of binary valued functions and $\text{VC}(U) \leq d$ then

$$ \mathbb{E} \sup_{u \in U} \left| \frac{1}{N} \sum_{i=1}^N \epsilon_i u X_i \right| \leq \kappa \left( \sigma_U \sqrt{\frac{d}{N} \log \left( \frac{e}{\sigma_U} \right)} + \frac{d \log \left( \frac{e}{\sigma_U} \right)}{N} \right). \quad (3.2) $$

The proof of Theorem 3.4 can be found in [20] (see also [2] for a detailed exposition on related concentration inequalities).

Proof of Theorem 3.4. Let $c_0$ be a well-chosen absolute constant, set $c_0 \frac{d}{N} \log \left( \frac{eN}{d} \right) < \Delta < 1$ and let $j \geq 0$ such that $2^j \Delta \leq 1$. Set $\epsilon_j = 2^{-j/2}$ and let

$$ U_j = \{u \in U : 2^j \Delta < \mathbb{P}(u) \leq 2^{j+1} \Delta \}. $$

Consider the random variables $\sup_{u \in U_j} |\mathbb{P}_N(u) - \mathbb{P}(u)| = (*)_j$. For every $u \in U_j$ we have that $\mathbb{P}(u) \geq 2^j \Delta$ and therefore it suffices to show that, with high probability, $(*_j) \leq \epsilon_j 2^j \Delta = 2^{j/2} \Delta$.

To that end, observe that $\sigma_{U_j}^2 \leq 2^{j+1} \Delta$, set

$$ E_j = \mathbb{E} \sup_{u \in U_j} \left| \frac{1}{N} \sum_{i=1}^N \epsilon_i u(X_i) \right|, $$

and clearly $\text{VC}(U_j) \leq \text{VC}(U) \leq d$. It follows from the second part of Theorem 3.4 that

$$ E_j \leq \kappa \left( \sqrt{2^{j+1} \Delta} \frac{d}{N} \log \left( \frac{e}{2^{j+1} \Delta} \right) + \frac{d}{N} \log \left( \frac{e}{2^{j+1} \Delta} \right) \right) \leq \frac{1}{4} \epsilon_j 2^j \Delta = \frac{2^{j/2} \Delta}{4} $$

by the lower bound on $\Delta$ and the choice of $c_0$.

Let $x_j = c_2 e^2 2^j \Delta N = c_2 \Delta N$. Invoking the first part of Theorem 3.4 it is evident that with probability at least $1 - 2 \exp(-c_3 \Delta N)$

$$ \sup_{u \in U_j} |\mathbb{P}_N(u) - \mathbb{P}(u)| \leq 2^{-j/2} \Delta, $$

as required. Thus, Property (b) follows with the wanted probability thanks to union bound for $\{ j \geq 0 : 2^j \Delta \leq 1 \}$ and recalling that $\Delta \geq c_0 \frac{d}{N} \log \left( \frac{eN}{d} \right)$ for a well-chosen absolute constant $c_0$.  

\[14\]
Repeating the same argument for $\varepsilon_j = \lambda$ and $x_j = c_4 \lambda^2 2^j \Delta$ and using the union bound once again, it is evident that with probability at least $1 - 2\exp(-c_5 \lambda^2 \Delta N)$,

$$\sup_{\{u \in U : \mathbb{P}(u) \geq \Delta\}} \left| \frac{\mathbb{P}_N(u)}{\mathbb{P}(u)} - 1 \right| \leq \lambda.$$ 

Property (a) is verified with the wanted probability by setting $\lambda = 1/2$.

Finally, let us turn to Property (c). When $\mathbb{P}(u) \geq \Delta$, Property (c) follows from Property (a), and when $\mathbb{P}(u) \leq \Delta$ we use Talagrand’s concentration inequality again. Indeed, by the lower bound on $\Delta$,

$$E \sup_{\{u \in U : \mathbb{P}(u) \leq \Delta\}} \left| \frac{1}{N} \sum_{i=1}^{N} \varepsilon_i u(X_i) \right| \leq \kappa \left( \sqrt{\frac{\Delta d}{N} \log \left( \frac{e}{\Delta} \right)} + \frac{d}{N} \log \left( \frac{e}{\Delta} \right) \right) \leq \frac{\Delta}{2\kappa}$$

Therefore, with probability at least $1 - \exp(-x)$,

$$\sup_{\{u \in U : \mathbb{P}(u) \leq \Delta\}} |\mathbb{P}_N(u) - \mathbb{P}(u)| \leq \kappa \left( \frac{\Delta}{2\kappa} + \sqrt{\frac{\Delta x}{N}} + \frac{x}{N} \right) \leq 2\Delta$$

by setting $x = c_5 \Delta N$. Hence, Property (c) is verified with $C = 2$ and with the wanted probability. ■

**Proof of Theorem 1.7.** With probability at least $1 - 2\exp(-c\Delta N)$, every indicator function in $U$ satisfies Properties (a)-(c). As a result, on that event, for every $v \in \mathbb{R}^d$, $|\langle v, X \rangle|$ satisfies Properties (1)-(3). Now Theorem 1.7 follows immediately from Corollary 2.5. ■

### 4 Concluding Remarks

The proof of Theorem 1.7 is relatively straightforward, but that is due to good fortune—that the class of indicator functions

$$U = \left\{ \mathbb{1}_{\{\langle v, i \rangle \in I\}} : v \in \mathbb{R}^d, I \subset \mathbb{R}_+ \text{ is a generalized interval} \right\},$$

is very simple — it has VC-dimension that is proportional to the algebraic dimension of the underlying space. For more general classes of functions the situation is far more complex: the class of indicators generated by tails of functions in $F$ need not have a finite VC dimension, let alone a well behaved one. In [11] we develop a theory that allows one to overcome that obstacle. We show that under minimal condition on the class $F$ and with high probability, Properties (1)-(3) hold uniformly in the class. As a result, a more general version of Theorem 1.7 happens to be true.

### References

[1] Shiri Artstein-Avidan, Apostolos Giannopoulos, and Vitali D. Milman. *Asymptotic geometric analysis. Part I*, volume 202 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2015.
[2] Stéphane Boucheron, Gábor Lugosi, and Pascal Massart. *Concentration inequalities*. Oxford University Press, Oxford, 2013.

[3] Silouanos Brazitikos, Apostolos Giannopoulos, Petros Valettas, and Beatrice-Helen Vritsiou. *Geometry of isotropic convex bodies*, volume 196 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2014.

[4] Evarist Giné and Vladimir Koltchinskii. Concentration inequalities and asymptotic results for ratio type empirical processes. *Ann. Probab.*, 34(3):1143–1216, 2006.

[5] Olivier Guédon and Mark Rudelson. $L_p$-moments of random vectors via majorizing measures. *Adv. Math.*, 208(2):798–823, 2007.

[6] B. Klartag. Uniform almost sub-Gaussian estimates for linear functionals on convex sets. *Algebra i Analiz*, 19(1):109–148, 2007.

[7] Bo’az Klartag and Emanuel Milman. Centroid bodies and the logarithmic Laplace transform—a unified approach. *J. Funct. Anal.*, 262(1):10–34, 2012.

[8] Vladimir Koltchinskii and Karim Lounici. Concentration inequalities and moment bounds for sample covariance operators. *Bernoulli*, 23(1):110–133, 2017.

[9] Vladimir Koltchinskii and Shahar Mendelson. Bounding the smallest singular value of a random matrix without concentration. *Int. Math. Res. Not. IMRN*, (23):12991–13008, 2015.

[10] Karim Lounici. High-dimensional covariance matrix estimation with missing observations. *Bernoulli*, 20(3):1029–1058, 2014.

[11] Gabor Lugosi and Shahar Mendelson. work in progress.

[12] Erwin Lutwak and Gaoyong Zhang. Blaschke-Santaló inequalities. *J. Differential Geom.*, 47(1):1–16, 1997.

[13] Shahar Mendelson. Approximating the covariance ellipsoid. *Commun. Contemp. Math.*, to appear.

[14] Shahar Mendelson. Improving the sample complexity using global data. *IEEE Trans. Inform. Theory*, 48(7):1977–1991, 2002.

[15] Shahar Mendelson. Learning without concentration. *J. ACM*, 62(3):Art. 21, 25, 2015.

[16] Shahar Mendelson and Nikita Zhivotovskiy. Robust covariance estimation under $L_4-L_2$ norm equivalence. *Ann. Statist.*, 48(3):1648–1664, 2020.

[17] Stanislav Minsker. Sub-Gaussian estimators of the mean of a random matrix with heavy-tailed entries. *Ann. Statist.*, 46(6A):2871–2903, 2018.

[18] Stanislav Minsker and Xiaohan Wei. Robust modifications of U-statistics and applications to covariance estimation problems. *Bernoulli*, 26(1):694–727, 2020.

[19] G. Paouris. Concentration of mass on convex bodies. *Geom. Funct. Anal.*, 16(5):1021–1049, 2006.
[20] M. Talagrand. Sharper bounds for Gaussian and empirical processes. *Ann. Probab.*, 22(1):28–76, 1994.

[21] Aad W. van der Vaart and Jon A. Wellner. *Weak convergence and empirical processes*. Springer Series in Statistics. Springer-Verlag, New York, 1996.