Representation theory of $C^*$-algebras for a higher-order class of spheres and tori

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Abstract
We construct $C^*$-algebras for a class of surfaces that are inverse images of certain polynomials of arbitrary degree. By using the directed graph associated to a matrix, the representation theory can be understood in terms of "loop" and "string" representations, which are closely related to the dynamics of an iterated map in the plane. As a particular class of algebras we introduce the "Hénon algebras", for which the dynamical map is a generalized Hénon map, and give an example where irreducible representations of all dimensions exist.

Introduction

In [1] fuzzy analogues of spheres and tori were constructed as $C^*$-algebras whose irreducible representations were then classified by using a graph method. In this paper, we extend those results to a larger class of surfaces defined as inverses images of certain polynomials of arbitrary degree. It turns out that the representation theory of these $C^*$-algebras can again be understood in terms of loop and string representations. Moreover, we will show that classifying irreducible representations amounts to finding periodic orbits and $N$-strings of a dynamical map $s: \mathbb{R}^2 \to \mathbb{R}^2$. As an important subclass of $C^*$-algebras, we introduce the Hénon algebras, for which the dynamical map $s$ will be a generalized Hénon map. In the cases we consider, every surface has a Hénon algebra as its fuzzy analogue. We will also give an example of a second order Hénon algebra for which irreducible representations of all dimensions exist for a fixed value of the parameter $\hbar$. 
1 The $C$-algebras

For $c, \alpha_0, \alpha_1, \ldots, \alpha_n \in \mathbb{R}$, we will consider subsets of $\mathbb{R}^3$ being inverse images of the polynomial

$$C(\vec{x}) = c + \frac{1}{2}z^2 + \sum_{k=1}^{n+1} \frac{\alpha_{k-1}}{2k}(x^2 + y^2)^k,$$

and we shall write $\Sigma = C^{-1}(0)$. When $\Sigma$ is a compact surface, it will have the topology of a sphere or a torus. Following [1], we introduce the Poisson bracket

$$\{f, g\} = \nabla C \cdot (\nabla f \times \nabla g),$$

and calculate

$$\{x, y\} = z$$
$$\{y, z\} = \alpha_0 x + x \sum_{k=1}^{n} \alpha_k (x^2 + y^2)^k$$
$$\{z, x\} = \alpha_0 y + y \sum_{k=1}^{n} \alpha_k (x^2 + y^2)^k.$$  

To define the corresponding $C$-algebra, we replace $\{\cdot, \cdot\}$ with $[\cdot, \cdot]/i\hbar$ and choose a particular ordering of the r.h.s. in (1.2). Setting $W = X + iY$ and $V = X - iY$, we will choose this ordering to be

$$[X, Y] = i\hbar Z$$
$$[Y, Z] = i\hbar \alpha_0 X + \frac{i\hbar}{2} \sum_{k=1}^{n} \left[ \tilde{\beta}_k V (VW)^k + (VW)^k W \right]$$
$$[Z, X] = i\hbar \alpha_0 Y + \frac{i\hbar}{2i} \sum_{k=1}^{n} \left[ \tilde{\gamma}_k \left( (VW)^k W - V (VW)^k \right) + \tilde{\gamma}_k \left( (WV)^k W - V (WV)^k \right) \right]$$

with $\tilde{\beta}_k + \tilde{\gamma}_k = \alpha_k$ for $k = 1, 2, \ldots, n$. From these equations, $Z$ can be eliminated and the two remaining equations can be rewritten entirely in terms of $W$ and $V$. The result appears in the following definition:

**Definition 1.** Let $\vec{\beta} = (\beta_1, \ldots, \beta_n)$ and $\vec{\gamma} = (\gamma_1, \ldots, \gamma_n)$ be vectors in $\mathbb{R}^n$ such that at least one of $\beta_n$ and $\gamma_n$ is non-zero, and let $\alpha \in \mathbb{R}$. Define $C_n(\vec{\beta}, \vec{\gamma})$ to be the quotient of the free algebra $C(V, W)$ with the two-sided ideal generated by the relations

$$W^2 V = \alpha W + \sum_{k=1}^{n} \beta_k (VW)^k W + \sum_{k=1}^{n} \gamma_k (WV)^k W$$
$$WV^2 = \alpha V + \sum_{k=1}^{n} \beta_k V (VW)^k + \sum_{k=1}^{n} \gamma_k V (WV)^k.$$  

We say that the algebra $C_n(\vec{\beta}, \vec{\gamma})$ has order $n$. 

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To go from (1.3)–(1.5) to (1.6)–(1.7) we set $\alpha = -2\hbar^2\alpha_0$, $\beta_1 = -2\hbar^2\tilde{\beta}_1 - 1$, $\gamma_1 = -2\hbar^2\tilde{\gamma}_1 + 2$ and $\beta_k = -2\hbar^2\tilde{\beta}_k$ and $\gamma_k = -2\hbar^2\tilde{\gamma}_k$ for $k \geq 2$.

As an important subclass of algebras, we introduce the Hénon algebras; a name that will later be justified by its relation to the generalized Hénon map.

**Definition 2.** Let $\vec{\beta}, \vec{\gamma} \in \mathbb{R}^n$ such that $\vec{\beta} = (b, 0, \ldots, 0)$ and $\gamma_n \neq 0$. Then we call $H_n(\vec{\gamma}) = C_n(\vec{\beta}, \vec{\gamma})$ a Hénon algebra of order $n$.

Note that since we have the freedom of choosing $\tilde{\beta}_k$ and $\tilde{\gamma}_k$, as long as $\tilde{\beta}_k + \tilde{\gamma}_k = \alpha_k$, every surface of the form (1.1) has a Hénon algebra as its fuzzy counterpart.

Let us continue by noting a crucial fact about the algebra $C_n(\vec{\beta}, \vec{\gamma})$:

**Proposition 1.** In $C_n(\vec{\beta}, \vec{\gamma})$ it holds that $[WV, VW] = 0$.

**Proof.** Multiplying (1.6) from the left with $V$, and (1.7) to the right with $W$, one immediately obtains $WV^2W = VW^2V$, which is equivalent to $[WV, VW] = 0$.

## 2 Hermitian representations

We are interested in finding hermitian representations of the algebra generated by the relations (1.3)–(1.5). This is equivalent to finding representations $\phi$, of $C_n(\vec{\beta}, \vec{\gamma})$, such that $\phi(W)^\dagger = \phi(V)$. Let us therefore, by a slight abuse of terminology, call such representations of $C_n(\vec{\beta}, \vec{\gamma})$ hermitian. In the following, we will often write $W$ instead of $\phi(W)$, when there is no risk of confusion. Let us first show that any hermitian representation of $C_n(\vec{\beta}, \vec{\gamma})$ can be decomposed into irreducible representations.

**Proposition 2.** Any hermitian representation of $C_n(\vec{\beta}, \vec{\gamma})$ is completely reducible.

**Proof.** Let $\phi$ be a hermitian representation of $C_n(\vec{\beta}, \vec{\gamma})$. Moreover, let $A$ be the subalgebra, of the full matrix-algebra, generated by $\phi(W)$ and $\phi(V)$. First we note that since $\phi(V) = \phi(W)^\dagger$, the algebra $A$ is invariant under hermitian conjugation, thus given $M \in A$ we know that $M^\dagger \in A$.

We prove that $\text{Rad}(A)$ (the radical of $A$), i.e. the largest nilpotent ideal of $A$, vanishes, which implies, by the Wedderburn-Artin theorem (see, e.g. [2]), that $\phi$ is completely reducible. Let $M \in \text{Rad}(A)$. Since $\text{Rad}(A)$ is an ideal it follows that $M^\dagger M \in \text{Rad}(A)$. For a finite-dimensional algebra, $\text{Rad}(A)$ is nilpotent, which in particular implies that there exists a positive integer $m$ such that $(M^\dagger M)^m = 0$. It follows that $M = 0$, hence $\text{Rad}(A) = 0$.

In any hermitian representation, $WW^\dagger$ and $W^\dagger W$ will be two commuting hermitian matrices, by Proposition [1]. Therefore, we can always, by a unitary change of coordinates, choose a basis such that they are diagonal. We write $WW^\dagger = D$ and $W^\dagger W = \tilde{D}$, where

$$D = \text{diag}(d_1, \ldots, d_N) \quad \tilde{D} = \text{diag}(\tilde{d}_1, \ldots, \tilde{d}_N) \quad d_i, \tilde{d}_i \geq 0 \text{ for } i = 1, \ldots, N.$$
For hermitian representations, equation (1.7) is the hermitian transpose of equation (1.6). Hence, finding hermitian representations of $C_n(\vec{\beta},\vec{\gamma})$ is equivalent to solving the equations

$$WD = \alpha W + \sum_{k=1}^{n} \left[ \beta_k \tilde{D}^k W + \gamma_k D^k W \right],$$

$$D = WW^\dagger \quad \text{and} \quad \tilde{D} = W^\dagger W. \quad (2.2)$$

Together with the obvious relation $DW = W\tilde{D}$, we write out (2.1) in components:

$$W_{ij} \left[ \alpha + \sum_{k=1}^{n} \left( \beta_k d_i^k + \gamma_k d_i^k \right) - d_j \right] = 0$$

$$W_{ij} \left( d_i - \tilde{d}_j \right) = 0.$$ 

If $W_{ij} \neq 0$, we find that

$$d_j = \alpha + \sum_{k=1}^{n} \left( \beta_k d_i^k + \gamma_k d_i^k \right)$$

$$\tilde{d}_j = d_i.$$ 

If we define the map $s$ by

$$s : \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} \alpha + \sum_{k=1}^{n} \left( \beta_k y^k + \gamma_k x^k \right) \\ x \end{pmatrix} \equiv \begin{pmatrix} \alpha + q(y) + p(x) \\ x \end{pmatrix},$$

and $\tilde{x}_i = (d_i, \tilde{d}_i)$, we can write $\tilde{x}_j = s(\tilde{x}_i)$ when $W_{ij} \neq 0$. We call $s$ the dynamical map of $C_n(\vec{\beta},\vec{\gamma})$. For a Hénon algebra the dynamical map becomes

$$s : \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} \alpha + p(x) + by \\ x \end{pmatrix},$$

which is usually referred to as the generalized Hénon map.

From these considerations we get a necessary condition relating the eigenvalues of $D$ and $\tilde{D}$ through the dynamical map $s$ and the structure of $W$. This observation suggests that one should find a way to keep track of the non-zero matrix elements of $W$; for this reason, we introduce the directed graph of a matrix.

### 2.1 Graph representations

Let $G = (V, E)$ denote a directed graph with vertex set $V = \{1, 2, \ldots, N\}$ and edge set $E \subseteq V \times V$. We say that $G = (V, E)$ is the directed graph (digraph) of the $N \times N$ matrix $W$ if it holds that

$$(i, j) \in E \Leftrightarrow W_{ij} \neq 0$$

for all $i$ and $j$ in $V$. When $W$ is the matrix of a hermitian representation of $C_n(\vec{\beta},\vec{\gamma})$, we simply say that $G$ is a representation of $C_n(\vec{\beta},\vec{\gamma})$. If $G$ is connected, we say that the
representation is connected. In the following, we will call a directed path from a transmitter to a receiver a \emph{string}, and a directed cycle a \emph{loop}. It is a trivial fact that any finite digraph has at least one string or one loop.

What can we say about graphs being representations of \( C_n(\vec{\beta}, \vec{\gamma}) \)? In fact, it turns out that one can classify all representations by classifying their digraphs, and the fundamental building-blocks will be strings and loops. Let us now proceed and try to understand the structure of these graphs.

Let \( G \) be a representation of \( C_n(\vec{\beta}, \vec{\gamma}) \). To each vertex \( i \in V \), we assign the vector \( \vec{x}_i = (d_i, \tilde{d}_i) \). If there is an edge from \( i \) to \( j \), then \( W_{ij} \neq 0 \) and we must have \( \vec{x}_j = s(\vec{x}_i) \) by the argument in the previous section. Now, assume that the representation \( G \) has a loop on \( n \) vertices. Then there exists a sequence \((i_1, i_2, \ldots, i_n, i_{n+1} = i_1)\) such that \((i_k, i_{k+1}) \in E\) for \( k = 1, 2, \ldots, n \), which implies that \( s^n(\vec{x}_1) = \vec{x}_1 \). Thus, the existence of a loop implies the existence of a period point of the dynamical map.

Next, we shall prove that loops and strings are in fact exclusive subgraphs of any representation, i.e. the existence of a loop prohibits the existence of a string. We prove this by showing that a representation with a loop is strongly connected. For this, we need the following lemma.

**Lemma 1.** Let \( G = (V, E) \) be a representation of \( C_n(\vec{\beta}, \vec{\gamma}) \). Then \( i \in V \) is a transmitter iff \( \tilde{d}_i = 0 \), and \( i \in V \) is a receiver iff \( d_i = 0 \).

**Proof.** Let \( W \) be the matrix of a representation of \( C_n(\vec{\beta}, \vec{\gamma}) \), whose digraph is \( G \). Since \( D = WW^\dagger \) and \( \tilde{D} = W^\dagger W \), we have

\[
d_i = \sum_k W_{ik} W_{ik} = \sum_k |W_{ik}|^2
\]

and it follows that \( d_i = 0 \) if and only if \( W_{ik} = 0 \) for all \( k \), i.e. \( i \) is a receiver. In the same way \( \tilde{d}_i = 0 \) if and only if \( W_{ki} = 0 \) for all \( k \), i.e. \( i \) is a transmitter. \( \square \)

**Proposition 3.** Let \( G \) be a connected representation of \( C_n(\vec{\beta}, \vec{\gamma}) \) containing a loop. Then \( G \) is strongly connected, i.e. for every pair of vertices \( i, j \) there exists a directed path from \( i \) to \( j \).

**Proof.** Let \( i \) be a vertex in a loop, and define \( V_R(i) = \{ j \in V : \exists \text{ a dipath from } i \text{ to } j \} \). We first want to prove that \( V_R(i) = V \) and that no transmitters or receivers exist. Assume that there exists at least one vertex \( j \) such that \( j \notin V_R(i) \). Let us denote the vertices in \( V_R(i) \) by \( 1, \ldots, m \) and the vertices in \( V_c = V - V_R(i) \) by \( m+1, \ldots, N \). Since, by assumption there is no edge \emph{from} a vertex in \( V_R(i) \) \emph{to} a vertex in \( V_c \), the matrix \( W \) takes the form

\[
W = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}
\]

with \( B \neq 0 \) since \( G \) is assumed to be connected. We calculate \( D \) as

\[
D = WW^\dagger = \begin{pmatrix} AA^\dagger & AB^\dagger \\ BA^\dagger & BB^\dagger + CC^\dagger \end{pmatrix}
\]
Since $D$ is diagonal, we must have $AB^\dagger = 0$ and $AA^\dagger = \text{diag}(d_1, \ldots, d_m)$. If we can argue that $A$ is always invertible, then we get that $B = 0$, which contradicts that $G$ is connected, and hence, $V_c = \emptyset$. Therefore, let us now show that $A$ is invertible by showing that $d_i > 0$ for $i = 1, \ldots, m$.

We have assumed that there is a loop in $A$ and that $i$ is a vertex in a loop, i.e. we have that $s^n(\vec{x}_i) = \vec{x}_i$, where $n$ is the number of vertices in the loop. Assume that there is a transmitter (receiver) $j$ in $R(i)$. By definition of $R(i)$ there is a non-negative integer $k$ such that $s^k(\vec{x}_i) = \vec{x}_j$, and we also know that there is a vertex $l$ in the loop such that $\vec{x}_l = s^k(\vec{x}_i) = \vec{x}_j$. By Lemma 1 we conclude that $l$ is also a transmitter (receiver), which contradicts that $l$ is part of a loop. Hence, there are no transmitters or receivers in $A$, which, by Lemma 1 implies that $d_i > 0$ for $i = 1, \ldots, m$. This proves that $A$ is invertible, which implies that $B = 0$, which contradicts the fact that $G$ is connected. We conclude that $V_R(i) = V$ for all vertices $i$ in any loop.

Finally, let us argue that $V_R(i) = V$ for any $i \in V$. If we follow an outgoing dopath from $i$, we must, in a finite number of steps, reach a vertex contained in a loop, since there are no transmitters or receivers (and the graph is finite). From this vertex, by the argument above, we can reach any other vertex through a dopath. Hence, $V_R(i) = V$.

Since a strongly connected graph can not have any transmitters or receivers, we get the following result.

**Corollary 1.** Let $G = (V, E)$ be a connected representation of $C_n(\vec{\beta}, \vec{\gamma})$ containing a loop on $n$ vertices. Then $G$ does not contain a string.

Moreover, since either a string or a loop must exist in a finite directed graph, the following definition is natural.

**Definition 3.** A String representation of $C_n(\vec{\beta}, \vec{\gamma})$ is a representation whose graph does not contain a loop. A Loop representation is a representation whose graph does not contain a string.

From Corollary 1 we conclude that every connected hermitian representation of $C_n(\vec{\beta}, \vec{\gamma})$ is either a string representation or a loop representation. In general, any hermitian representation is a direct sum of string and loop representations.

Let us show that the structure of loop and string representations is preserved among equivalent representations.

**Proposition 4.** Let $\phi$ and $\phi'$ be two equivalent hermitian representations of $C_n(\vec{\beta}, \vec{\gamma})$. If $\phi$ is a loop representation then $\phi'$ is a loop representation.

**Proof.** Since the representations are equivalent, there exists an invertible matrix $P$ such that

$$\phi'(W) = P\phi(W)P^{-1} \quad \text{and} \quad \phi'(W^\dagger) = P\phi(W^\dagger)P^{-1},$$

from which it follows that

$$\phi'(D) = P\phi(D)P^{-1} \quad \text{and} \quad \phi'(\tilde{D}) = P\phi(\tilde{D})P^{-1}.$$

Hence, $\phi'(D)$ and $\phi(D)$ have the same eigenvalues, and the same is true for $\phi'(\tilde{D})$ and $\phi(\tilde{D})$. By assumption, $\phi$ is a loop representation, which implies that no eigenvalues of $\phi(D)$ or $\phi(\tilde{D})$ are zero, by Lemma 1. Hence, no eigenvalues of $\phi'(D)$ or $\phi'(\tilde{D})$ are zero, which implies that $\phi'$ is a loop representation. \qed
3 The structure of locally injective representations

Let us introduce the spectrum of a representation.

**Definition 4.** Let \( \phi \) be a \( N \)-dimensional hermitian representation of \( C_n(\beta, \gamma) \). We set

\[
\text{Spec}(\phi) = \{ \vec{x}_i = (d_i, \tilde{d}_i) : i = 1, 2, \ldots, N \}
\]

and we call this set the spectrum of \( \phi \).

We will now show that the spectrum is preserved among equivalent representations. This follows from the next lemma.

**Lemma 2.** Let \( D \) and \( \tilde{D} \) be diagonal matrices and assume that there exists an invertible matrix \( P \) such that \( PDP^{-1} \) and \( \tilde{D}P^{-1} \) are diagonal. Then there exists a permutation \( \sigma \) such that \( PDP^{-1} = \sigma^\dagger D \sigma \) and \( \tilde{D}P^{-1} = \sigma^\dagger \tilde{D} \sigma \).

**Proof.** By an overall permutation, we can write \( D \) and \( \tilde{D} \) in the following block-diagonal form:

\[
D = \begin{pmatrix}
d_1 \mathbb{1}_{n_1} & \cdots & \\
& \ddots & \\
& & d_k \mathbb{1}_{n_k}
\end{pmatrix}
\quad \text{and} \quad
\tilde{D} = \begin{pmatrix}
\tilde{D}_1 & \cdots & \\
& \ddots & \\
& & \tilde{D}_k
\end{pmatrix}
\]

with \( d_i \neq d_j \) whenever \( i \neq j \). Since \( PDP^{-1} \) is diagonal, it has the same eigenvalues as \( D \), including multiplicities. Therefore, there exists a permutation \( \sigma_0 \) such that \( PDP^{-1} = \sigma_0^\dagger D \sigma_0 \). From this it follows that \( [\sigma_0 P, D] = 0 \) which imposes the following form of \( \sigma_0 P \):

\[
\sigma_0 P = \begin{pmatrix}
P_1 & \cdots & \\
& \ddots & \\
& & P_k
\end{pmatrix}
\]

Since \( P\tilde{D}P^{-1} \) is diagonal, \( \sigma_0 P\tilde{D}P^{-1}\sigma_0^\dagger \) will also be diagonal. On the other hand

\[
\sigma_0 P\tilde{D}P^{-1}\sigma_0^\dagger = \begin{pmatrix}
P_1 \tilde{D}_1 P_1^{-1} & \cdots & \\
& \ddots & \\
& & P_k \tilde{D}_k P_k^{-1}
\end{pmatrix},
\]

which implies that \( P_i \tilde{D}_i P_i^{-1} \) is diagonal for \( i = 1, \ldots, k \). Hence, there exists permutations \( \gamma_1, \ldots, \gamma_k \) such that \( P_i \tilde{D}_i P_i^{-1} = \gamma_i \tilde{D}_i \gamma_i \). Now, let us set

\[
\gamma = \begin{pmatrix}
\gamma_1 & \cdots & \\
& \ddots & \\
& & \gamma_k
\end{pmatrix}
\]

and define \( \sigma = \gamma \sigma_0 \). We then get

\[
\sigma^\dagger D \sigma = \sigma_0^\dagger \gamma^\dagger D \gamma \sigma_0 = \sigma_0^\dagger D \sigma_0 = PDP^{-1},
\]

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there exists a non-negative integer $k$.

Proof. Let $\sigma_0 = \gamma_1^\dagger \tilde{D}_1 \gamma_1 \cdots \gamma_k^\dagger \tilde{D}_k \gamma_k$.

We will now introduce the concept of locally injective representations. It is a technical condition that is needed as an assertion in Theorem 1.

Definition 5. Let $s : \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \alpha + \sum_{k=1}^n (\beta_k y^k + \gamma_k x^k)$ be the dynamical map of $C_n(\tilde{\beta}, \tilde{\gamma})$, and let $\phi$ be a hermitian representation of $C_n(\tilde{\beta}, \tilde{\gamma})$. If $s|_{\text{Spec}(\phi)} : \text{Spec}(\phi) \rightarrow \mathbb{R}^2$ is injective, we say that $\phi$ is a locally injective representation.

Note that if $s$ is invertible, then any representation is locally injective. In particular, this is true for the Hénon algebras. It also turns out to be true for all loop representations.

Proposition 5. Let $G$ be a connected loop representation of $C_n(\tilde{\beta}, \tilde{\gamma})$. Then $G$ is locally injective.

Proof. Assume that $s(\vec{x}_i) = s(\vec{x}_j)$ for some vertices $i$ and $j$. Since $G$ is a loop representation, there exists a positive integer $n$ such that $s^n(\vec{x}_i) = \vec{x}_i$. From Proposition [5] we know that there exists a non-negative integer $k$ such that $s^k(\vec{x}_i) = \vec{x}_j$. Since $s(\vec{x}_i) = s(\vec{x}_j)$ we get that

$\vec{x}_i = s^n(\vec{x}_i) = s^n(\vec{x}_j) = s^{n+k}(\vec{x}_i) = s^k(\vec{x}_i) = \vec{x}_j$. \hfill $\square$

Now, let us prove the main theorem, giving the structure of locally injective representations. It enables us to show that every representation is a direct sum of loops and strings.

Theorem 1. Let $\phi$ be an $N$-dimensional connected locally injective hermitian representation of $C_n(\tilde{\beta}, \tilde{\gamma})$. Then there exists a positive integer $k$ dividing $N$, a unitary $N \times N$ matrix $T$, unitary $N/k \times N/k$ matrices $U_0, \ldots, U_{k-1}$ and $x_0, y_0, \hat{e}_0, \ldots, \hat{e}_{k-1} \in \mathbb{R}$ such that

$$T \phi(W) T^\dagger = \begin{pmatrix} 0 & \sqrt{e_1} U_1 & 0 & \cdots & 0 \\ 0 & 0 & \sqrt{e_2} U_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \sqrt{e_{k-1}} U_{k-1} \\ \sqrt{e_0} U_0 & 0 & \cdots & 0 & 0 \end{pmatrix} \quad (3.1)$$

$$\hat{e}_i = s^i(x_0, y_0) \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (3.2)$$

with $\hat{e}_1, \ldots, \hat{e}_{k-1} > 0$. 

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Moreover, we take Lemma 3. Let \( U \) be a unitary matrix such that \( UDU^\dagger \) and \( U\tilde{D}U^\dagger \) are diagonal, set \( \tilde{W} = U\tilde{\phi}(W)U^\dagger \) and let \( G \) be the digraph of \( W \). Let \( \hat{x}_0, \ldots, \hat{x}_{k-1} \) be an enumeration of \( \text{Spec}(\phi) \) such that \( \hat{x}_{i+1} = s(\hat{x}_i) \) for \( i = 0, \ldots, k - 2 \). This can be done since \( G \) is connected and locally injective, which in particular means that if \( s(\hat{x}_i) = s(\hat{x}_j) \) then \( \hat{x}_i = \hat{x}_j \). Moreover, let us write \( \hat{x}_i = (e_i, \tilde{e}_i) \). We note that if \( G \) has a transmitter, it must necessarily correspond to the vector \( \hat{x}_0 \), in which case \( \tilde{e}_0 = 0 \). In particular this means that no vertex corresponding to \( \hat{x}_i \), for \( i > 0 \), can be a transmitter and hence, by Lemma 1, \( \tilde{e}_1, \ldots, \tilde{e}_{k-1} = 0 \). Now, define

\[
V_i = \{ j \in V : \hat{x}_j = \hat{x}_i \} \quad i = 0, \ldots, k - 1,
\]

and set \( l_i = |V_i| \). Since \( \hat{x}_{i+1} = s(\hat{x}_i) \) and \( \phi \) is locally injective, a necessary condition for \( (i, j) \in E \) is that \( j = i + 1 \). This implies that there exists a permutation \( \sigma \in S_N \) (permuting vertices to give the order \( V_0, \ldots, V_{k-1} \)) such that

\[
W' := \sigma \tilde{W} \sigma^\dagger = \begin{pmatrix} 0 & W_1 & 0 & \cdots & 0 \\ 0 & 0 & W_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & W_{k-1} \\ W_0 & 0 & \cdots & 0 & 0 \end{pmatrix}
\]

In this basis we get

\[
D = \text{diag}(e_0, \ldots, e_0, \ldots, e_{k-1}, \ldots, e_{k-1}) = W'W'^\dagger = \text{diag}(W_1W_1^\dagger, \ldots, W_{k-1}W_{k-1}^\dagger, W_0W_0^\dagger)
\]

\[
\tilde{D} = \text{diag}(\tilde{e}_0, \ldots, \tilde{e}_0, \ldots, \tilde{e}_{k-1}, \ldots, \tilde{e}_{k-1}) = W'^\dagger W' = \text{diag}(W_0^\dagger W_0, W_1^\dagger W_1, \ldots, W_{k-1}^\dagger W_{k-1}),
\]

which gives \( W_i'W_i'^\dagger = e_{i-1}I_{l_{i-1}} \) and \( W_i'^\dagger W_i' = \tilde{e}_iI_{l_i} \). Since \( \hat{x}_{i+1} = s(\hat{x}_i) \) we know that \( \tilde{e}_{i+1} = e_i \), which implies that \( W_i'W_i'^\dagger = \tilde{e}_iI_{l_i-1} \) for \( i = 1, \ldots, k - 1 \). Any matrix satisfying such conditions must be a square matrix, i.e., \( l_i = l_{i-1} \) for \( i = 1, \ldots, k - 1 \). Hence, \( W_i' \) is a square matrix of dimension \( N/k \), and there exists a unitary matrix \( U_i \) such that \( W_i' = \sqrt{\tilde{e}_i}U_i \). Moreover, we take \( T \) to be the unitary matrix \( \sigma U \).

Having obtained this result, we can proceed to the task of classifying all representations up to equivalence. We start by proving the following simple lemma:

**Lemma 3.** Let \( W_1 \) and \( W_2 \) be matrices such that

\[
W_1 = \begin{pmatrix} 0 & w_1U_1 & 0 & \cdots & 0 \\ 0 & 0 & w_2U_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & w_{n-1}U_{n-1} \\ w_0U_0 & 0 & \cdots & 0 & 0 \end{pmatrix} ; \quad W_2 = \begin{pmatrix} 0 & w_1I & 0 & \cdots & 0 \\ 0 & 0 & w_2I & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & w_{n-1}I \\ w_0V & 0 & \cdots & 0 & 0 \end{pmatrix}
\]

where \( U_0, \ldots, U_{n-1} \) are unitary matrices, \( w_0, \ldots, w_{n-1} \in \mathbb{C} \) and \( V \) a diagonal matrix such that

\[
SVS^\dagger = U_1U_2\cdots U_{n-1}U_0
\]


for some unitary matrix $S$. Then there exists a unitary matrix $P$ such that
\[ W_1 = PW_2 P^\dagger \quad \text{and} \quad W_1^\dagger = PW_2^\dagger P. \]

Proof. Let us define $P$ as $P = \text{diag}(S, P_1, \ldots, P_{n-1})$ with
\[ P_l = (U_1 U_2 \ldots U_l)^\dagger S \]
for $l = 1, \ldots, n - 1$. Then one easily checks that $W_1 = PW_2 P^\dagger$ and $W_1^\dagger = PW_2^\dagger P^\dagger$. \hfill \Box

Digraphs of the matrix $W_2$, appearing in Lemma 3, correspond to digraphs with several disconnected components, each being either a loop or a string. Therefore, using this lemma together with Theorem 1 we obtain the following classification:

**Theorem 2.** Let $\phi$ be a locally injective hermitian representation of $C_n(\tilde{\beta}, \tilde{\gamma})$. Then $\phi$ is unitarily equivalent to a representation whose graph is such that every connected component is either a string or a loop.

Moreover, we can prove that a representation can be reduced no further.

**Proposition 6.** Let $\phi$ be a hermitian representation of $C_n(\tilde{\beta}, \tilde{\gamma})$ written in the form of Theorem 4 and assume that the corresponding graph is a loop or a string. Then $\phi$ is irreducible.

Proof. Let us start by proving the statement when $\phi$ is a loop representation. For a representation in the form given by Theorem 4 and for which the corresponding graph is a loop, the dimension $N$ is the smallest positive integer such that $s^N(\vec{x}_i) = \vec{x}_i$ for all vertices $i$ in the loop. If a smaller such $N$ existed, then the corresponding graph would not be a loop, when brought to the form of Theorem 4. Now, assume that $\phi$ is reducible. Then there exists a representation $\phi'$ of dimension $N' < N$, and by Lemma 2 we know that $\{\vec{x}_1', \ldots, \vec{x}_{N'}\} \subseteq \{\vec{x}_1, \ldots, \vec{x}_N\}$. This implies that $\phi'$ is also a loop representation; hence, there must exist an integer $k \leq N'$ such that $s^k(\vec{x}_1') = \vec{x}_1'$, which is impossible by the above argument. This proves that $\phi$ is irreducible.

Next, we assume that $\phi$ is a reducible string representation of dimension $N$. Then there exists a string representation $\phi'$ of dimension $N' < N$, which implies that there exists a string of length $k < N$. Since $\text{Spec}(\phi') \subseteq \text{Spec}(\phi)$, this contradicts the existence of the original string of length $N$. Hence, $\phi$ is irreducible. \hfill \Box

The next proposition tells us when two irreducible representations of the same dimension are equivalent.

**Proposition 7.** Let $\phi$ and $\phi'$ be $N$-dimensional irreducible hermitian representations of $C_n(\tilde{\beta}, \tilde{\gamma})$. Then $\phi$ and $\phi'$ are equivalent if and only if $\text{Spec}(\phi) = \text{Spec}(\phi')$ and $\det \phi(W) = \det \phi'(W)$.

Proof. First, assume that $\phi$ is equivalent to $\phi'$. It follows directly that $\det \phi(W) = \det \phi'(W)$. Furthermore, it follows from Lemma 2 that $\text{Spec}(\phi) = \text{Spec}(\phi')$.

Now, assume that $\text{Spec}(\phi) = \text{Spec}(\phi')$ and that $\det \phi(W) = \det \phi'(W)$. Since the representations are irreducible, the spectrum consists of $N$ distinct vectors. Therefore, there exists a permutation $\sigma$ such that
\[ \phi(D) = \sigma^\dagger \phi'(D) \sigma \quad \text{and} \quad \phi(\tilde{D}) = \sigma^\dagger \phi'(\tilde{D}) \sigma. \]
Let us define a representation \( \psi \) by \( \psi(W) = \sigma^\dagger \phi'(W) \sigma \); clearly, \( \psi \) is equivalent to \( \phi' \). Since 
\[
|\phi(W)_{i,i+1}|^2 = \psi(D)_{ii} = |\psi(W)_{i,i+1}|^2,
\]
we have that \( |\phi(W)_{ij}| = |\psi(W)_{ij}| \) for \( i,j = 1,2,\ldots,N \). Moreover, since \( \text{det} \phi(W) = \text{det} \psi(W) \), there exists a diagonal unitary matrix \( P \) such that
\[
\phi(W) = P^\dagger \psi(W) P = P^\dagger \sigma^\dagger \phi'(W) \sigma P.
\]
Note that for a string representation \( \phi_S \), it is always true that \( \text{det} \phi_S(W) = 0 \).

Let us present the matrices of irreducible representations and their corresponding digraphs. The matrix of an irreducible loop representation has the form
\[
W = \begin{pmatrix}
0 & W_{12} & 0 & \cdots & 0 \\
0 & 0 & W_{23} & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & W_{N-1,N} \\
W_{N,1} & 0 & \cdots & \cdots & 0
\end{pmatrix}
\]
and the corresponding digraph is

For an irreducible string representation we get
\[
W = \begin{pmatrix}
0 & W_{12} & 0 & \cdots & 0 \\
0 & 0 & W_{23} & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & W_{N-1,N} \\
0 & 0 & \cdots & \cdots & 0
\end{pmatrix}
\]
with the following digraph

4 Constructing representations

Let \( \vec{x} \in \mathbb{R}^2 = \{(x,y) \in \mathbb{R}^2 : x > 0, y > 0\} \) be a periodic point of period \( N \), i.e. \( s^N(\vec{x}) = \vec{x} \) but \( s^k(\vec{x}) \neq \vec{x} \) for \( k = 1,\ldots,N-1 \), for the map
\[
s : \begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} \alpha + \sum_{k=1}^{n} \left( \beta_k y^k + \gamma_k x^k \right) \\ x \end{pmatrix},
\]
such that \( s^k(\vec{x}) \in \mathbb{R}_+^2 \) for \( k = 1, \ldots, N - 1 \). Writing \( s^k(\vec{x}) = (d_k, \tilde{d}_k) \), it is easy to check that \( W \) with

\[
W_{k,k+1} = \sqrt{d_k} \quad \text{for} \quad k = 1, \ldots, N - 1
\]
\[
W_{N,1} = e^{i\gamma} \sqrt{d_N}
\]
is an irreducible \( N \)-dimensional loop representation of \( C_n(\vec{\beta}, \vec{\gamma}) \) for all \( \gamma \in \mathbb{R} \). Hence, by Proposition 7, every periodic orbit contained in \( \mathbb{R}_+^2 \), gives rise to a one-parameter family of inequivalent loop representations, since \( \det W = e^{i\gamma} \sqrt{d_1 d_2 \cdots d_N} \). Moreover, disjoint orbits of the same period correspond to inequivalent irreducible representations of the same dimension.

In the same way, each \( \vec{x} = (a, 0) \) with \( a > 0 \) such that \( s^{N-1}(\vec{x}) = (0, b) \), with \( b > 0 \), and \( s^k(\vec{x}) \in \mathbb{R}_+^2 \) for \( k = 1, \ldots, N - 2 \), gives an irreducible \( N \)-dimensional string representation of \( C_n(\vec{\beta}, \vec{\gamma}) \). Namely, we define \( W \) through

\[
W_{k,k+1} = \sqrt{d_k}
\]

with \( (d_k, \tilde{d}_k) = s^{k-1}(\vec{x}) \) for \( k = 1, \ldots, N - 1 \). Since \( \det W = 0 \), there is no parameter giving inequivalent representations. If we define a \( N \)-string for \( s \) to be such a set of points, each \( N \)-string correspond to an irreducible string representation of \( C_n(\vec{\beta}, \vec{\gamma}) \).

From these considerations we conclude that finding all irreducible hermitian representations of \( C_n(\vec{\beta}, \vec{\gamma}) \) is equivalent to finding all periodic orbits in \( \mathbb{R}_+^2 \) and \( N \)-strings for the dynamical map.

5 Representations of \( C_1(\vec{\beta}, \vec{\gamma}) \)

In the case when the algebra is of order one, the dynamical map will be an affine map from the plane to itself. This allows us to work out the representations quite explicitly. In particular, \( s \) is invertible which implies that every representation is locally injective. Hence, we can classify representations according to Theorem 2.

The defining relations for \( C_1(\vec{\beta}, \vec{\gamma}) \) are

\[
W^2 V = \alpha W + \beta_1 VW^2 + \gamma_1 VW W
\]
\[
W V^2 = \alpha V + \beta_1 V^2 W + \gamma_1 VW V
\]
or, in terms of \( X, Y, Z \)

\[
[X, Y] = Z
\]
\[
[Y, Z] = \frac{\alpha}{2} X + \frac{i}{2} (\beta_1 + \gamma_1 - 1) X^3 + \frac{1}{2} (\beta_1 - \gamma_1 + 3) Y XY
\]
\[
\quad + \frac{i}{2} (\beta_1 + 1) (X^2 Y - Y X^2) + \frac{1}{2} (\gamma_1 - 2) \left( X Y^2 + Y^2 X \right)
\]
\[
[Z, X] = \frac{\alpha}{2} Y + \frac{1}{2} (\beta_1 + \gamma_1 - 1) Y^3 + \frac{1}{2} (\beta_1 - \gamma_1 + 3) X Y X
\]
\[
\quad + \frac{i}{2} (\beta_1 + 1) \left( X Y^2 - Y^2 X \right) + \frac{1}{2} (\gamma_1 - 2) \left( X Y^2 + Y^2 X \right).
\]

\(^1\text{Note that similar equations are under consideration in [3].}\)
For convenience, let us write \( \beta_1 = q - p^2 \) and \( \gamma_1 = 2p \). Finding irreducible hermitian representations of this algebra corresponds to finding periodic orbits and \( N \)-strings for the affine map

\[
\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} 2p & q - p^2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \alpha \\ 0 \end{pmatrix}.
\]

In the following, let us assume that \( q < 0 \) for simplicity. The representations for these algebras were found and studied in [1], but let us recall some basic facts. We introduce the following notation

\[
s(\vec{x}) = A\vec{x} + \vec{c} = \begin{pmatrix} 2p & q - p^2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \alpha \\ 0 \end{pmatrix}.
\]

It follows from the fact that \( q < 0 \) that this map will always have a unique fix-point \( \vec{x}_f \), which implies that \( s \) amounts to a linear map around \( \vec{x}_f \). Let us study the periodic orbits of \( s \). A necessary condition for a periodic orbit in \( \mathbb{R}^2 \) to exist is that the eigenvalues \( \lambda, \mu \) of \( A \) satisfy \( \lambda^n = \mu^n = 1 \), which restricts the existence of loop representations to certain algebras. Let us find a way to parametrize them. The (complex) eigenvalues of \( A \) are

\[
\lambda = p + \sqrt{q}, \\
\mu = p - \sqrt{q},
\]

and demanding \( |\lambda| = |\mu| = 1 \) gives us a parametrization through

\[
p = \cos 2\theta \\
q = -\sin^2 2\theta
\]

with \( 0 < \theta < \pi/2 \). Furthermore, \( \lambda^n = \mu^n = 1 \) implies that \( e^{i2n\theta} = 1 \), which gives \( \theta = k\pi/n \) for some \( k \in \mathbb{Z} \). Requiring that \( n \) is the least period of the orbit gives \( \gcd(k, n) = 1 \) and if \( \alpha > 0 \) we can always find periodic orbits in \( \mathbb{R}^2 \). Hence, for these algebras there are only irreducible loop representations of dimension \( n \). As we will see in the next part, this fact is changed for higher order algebras, where irreducible representations of all dimension might exist.

## 6 Representations of a second order Hénon algebra

The first order Hénon algebras are just the algebras presented in the previous section. For these algebras, everything can be explicitly calculated since the map \( s \) is an affine map. When we turn to the next order Hénon algebras, things become much more involved. Nevertheless, the dynamical map is still invertible, which implies that all representations can be decomposed into irreducible loop and string representations. We will now continue to construct a second order Hénon algebra for which irreducible loop representations of all dimensions exist. Consider the Hénon map

\[
f : \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} a - by - x^2 \\ x \end{pmatrix}
\]
For certain values of $a$ and $b$, e.g. $a = 5$ and $b = 0.3$ there is a bounded subset $\Lambda \subset \mathbb{R}^2$ such that $f|_{\Lambda}$ is topologically conjugate to a two-sided shift on two symbols (see, e.g. [4]). By definition, this means that there exists a bijection $h : \Lambda \to \Sigma_2$, where $\Sigma_2 = \{(\ldots, a_{-1}, a_0, a_1, \ldots) : a_k \in \{0, 1\} \text{ for all } k \in \mathbb{Z}\}$, and a shift map $\sigma : \Sigma_2 \to \Sigma_2$ such that $\sigma((\ldots, a_{-1}, a_0, a_1, \ldots)) = (\ldots, a_0, a_1, a_2, \ldots)$ and $\sigma \circ h = h \circ f$. Thus, the periodic orbits of $f|_{\Lambda}$ is in one-to-one correspondence with the periodic orbits of $\sigma$ on $\Sigma_2$. Moreover, for the shift map on $\Sigma_2$ there exists periodic points of all periods. However, for our purposes, we need periodic orbits contained in $\mathbb{R}^2_+$, which is the case if $\Lambda \subset \mathbb{R}^2_+$. This is not true for the map $f$, but we can easily create a new map with this property. Since $\Lambda$ is a bounded set, there exists a positive number $r$ (for $a = 5$ and $b = 0.3$ we can take $r > 2.5$) such that $\Lambda \subset \{(x, y) \in \mathbb{R}^2 : |x| < r \text{ and } |y| < r\}$. If we define

$$s : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a + r - b(y - r) - (x - r)^2 \\ x \end{pmatrix}$$

one can easily check that $f^n(x, y) + (r, r) = s^n((x, y) + (r, r))$. This implies that every periodic orbit of $f$ in $\Lambda$ will appear as a periodic orbit for $s$, contained in $\mathbb{R}^2_+$. Hence, the second order Hénon algebra defined by

$$W^2V = (a + r + br - r^2)W - bVW + 2rWVW - (WV)^2W$$
$$WV^2 = (a + r + br - r^2)V - bV^2W + 2rVWV - V(WV)^2$$

has irreducible loop representations of all dimensions for certain values of $a$, $b$ and $r$.

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