1. Introduction

Let $\Sigma$ be a connected closed oriented surface of genus $g$. In 1986 Goldman [Go] attached to $\Sigma$ a Lie algebra $L = L(\Sigma)$, later shown by Turaev ([Tu]) to have a natural structure of a Lie bialgebra. It is defined as follows. As a vector space, $L$ has a basis $e_\gamma$ labeled by conjugacy classes $\gamma$ in the fundamental group $\pi_1(\Sigma)$, geometrically represented by closed oriented curves on $\Sigma$ without a base point. To define the commutator $[e_\gamma_1, e_\gamma_2]$, one needs to bring the two curves $\gamma_1, \gamma_2$ into general position by isotopy, and then for each intersection point $p_i$ of the two curves, define $\gamma_3$ to be the curve obtained by tracing $\gamma_1$ and then $\gamma_2$ starting and ending at $p_i$. Then one defines $[e_\gamma_1, e_\gamma_2]$ to be $\sum_i \varepsilon_i e_{\gamma_3}$, where $\varepsilon_i = 1$ if $\gamma_1$ approaches $\gamma_2$ from the right at $p_i$ (with respect to the orientation of $\Sigma$), and $-1$ otherwise.

The combinatorial structure of $L$ has been much studied; see e.g. [C, Tu]. However, many problems about the structure of $L$ remained open. In particular, in 2001, M. Chas and D. Sullivan communicated to me the following conjecture.

Conjecture 1.1. The center of $L$ is spanned by the element $e_1$, where $1 \in \pi_1(\Sigma)$ is the trivial loop.

In this paper, we will prove this conjecture. In fact, we prove a more general result.

Theorem 1.2. The Poisson center of the Poisson algebra $S^*L$ is $Z = \mathbb{C}[e_1]$.

The proof of the theorem occupies the rest of the paper.

Remark. A quiver theoretical analog of Theorem 1.2 is given in [CEG]. It claims that if $\Pi$ is the preprojective algebra of a quiver $Q$ which is not Dynkin or affine Dynkin, then the Poisson center of $S^*\Pi$ (where $L = \Pi/[[\Pi, \Pi]$ is the necklace Lie algebra attached to $\Pi$) consists of polynomials in the vertex idempotents.

2. Proof of the theorem

2.1. Moduli spaces of flat bundles. We will assume that $g > 1$, since in the case $g \leq 1$ the theorem is easy.
Recall that the fundamental group $\Gamma = \pi_1(\Sigma)$ is generated by $X_1, \ldots, X_g$, $Y_1, \ldots, Y_g$ with defining relation

\[(1) \prod_{i=1}^{g} X_i Y_i X_i^{-1} Y_i^{-1} = 1.\]

Thus we can define the scheme of homomorphisms $\tilde{M}_g(N) = \text{Hom}(\Gamma, GL_N(\mathbb{C}))$ to be the closed subscheme in $GL_N(\mathbb{C})^{2g}$ defined by equation (1). One can also define the moduli scheme of representations (or equivalently, of flat connections on $\Sigma$) to be the categorical quotient $M_g(N) = \tilde{M}_g(N)/PGL_N(\mathbb{C})$.

The schemes $\tilde{M}_g(N)$ and $M_g(N)$ carry the Atiyah-Bott Poisson structure; its algebraic presentation may be found in [FR] (using r-matrices) and [AMM] (using quasi-Hamiltonian reduction); see also [Go].

Let us recall the following known results about these schemes, which we will use in the sequel.

**Theorem 2.1.** (i) $\tilde{M}_g(N)$ and $M_g(N)$ are reduced.

(ii) $\tilde{M}_g(N)$ is a complete intersection in $GL_N(\mathbb{C})^{2g}$.

(iii) $\tilde{M}_g(N)$ and $M_g(N)$ are irreducible algebraic varieties. Their generic points correspond to irreducible representations of $\Gamma$.

(iv) The Poisson structure on $M_g(N)$ is generically symplectic.

**Proof.** Let $\tilde{M}'_g(N)$ be the algebraic variety corresponding to the scheme $\tilde{M}_g(N)$. It is shown in [Li] that this variety is irreducible. Moreover, it is clear that the generic point of this variety corresponds to an irreducible representation of $\Gamma$ (we can choose $X_i, Y_i$ generically for $i < g$ and then solve for $X_g, Y_g$). It is easy to show that near such a point the map $\mu : GL(N)^{2g} \to SL(N)$ given by the left hand side of (1) is a submersion. This implies (ii). We also see that $\tilde{M}_g(N)$ is generically reduced. Since it is a complete intersection, it is Cohen-Macaulay and hence reduced everywhere. Thus we get (i) and (iii). Property (iv) is well known and is readily seen from [FR] or [AMM]. The theorem is proved. \(\Box\)

**2.2. Injectivity of the Goldman homomorphism.** Now let us return to the study of the Lie algebra $L$. To put ourselves in an algebraic framework, we note that $L$ is naturally identified with $A/[A,A]$, where $A = \mathbb{C}[\Gamma]$ is the group algebra of $\Gamma$. Thus, elements of $L$ can be represented by linear combinations of cyclic words in $X_i^{\pm 1}, Y_i^{\pm 1}$.

In [Go], Goldman defined a homomorphism of Poisson algebras

$$\phi_N : S^*L \to \mathbb{C}[M_g(N)]$$

defined by the formula $\phi_N(w)(\rho) = \text{Tr}(\rho(w))$, where $\rho$ is an $N$-dimensional representation of $\Gamma$ and $w$ is any cyclic word representing an element of $L$. It follows from Weyl’s fundamental theorem of invariant theory that the Goldman homomorphism is surjective.
Let \( L_+ \subset L \) be the linear span of the elements \( e_\gamma - e_1 \). Obviously, we have \( L = L_+ \oplus \mathbb{C}e_1 \).

**Proposition 2.2.** For any finite dimensional subspace \( Y \subset S^\bullet L_+ \), there exists an integer \( N(Y) \) such that for \( N \geq N(Y) \), the map \( \phi_N|_Y \) is injective.

**Proof.** Let \( K(N) \) be the kernel of \( \phi_N \) on \( S^\bullet L_+ \). It is clear that \( K(N + 1) \subset K(N) \) (as \( \phi_N(e_\gamma - e_1)(\rho \oplus \mathbb{C}) = \phi_N(e_\gamma - e_1)(\rho) \)). Thus it suffices to show that \( \cap_{N \geq 1} K(N) = 0 \).

Assume the contrary. Then there exists an element \( 0 \neq f \in S^\bullet L_+ \) such that \( \phi_N(f) = 0 \) for all \( N \).

Recall that according to [FI], the group \( \Gamma \) is *conjugacy separable*, i.e., if elements \( \gamma_0, ..., \gamma_m \) are pairwise not conjugate in \( \Gamma \) then there exists a finite quotient \( \Gamma' \) of \( \Gamma \) such that the images of \( \gamma_0, ..., \gamma_m \) are not conjugate in \( \Gamma' \).

Now let \( \gamma_0 = 1 \) and \( f = P(e_{\gamma_1} - e_1, ..., e_{\gamma_m} - e_1) \), where \( P \) is some polynomial. Let \( \Gamma' \) be the finite group as above, \( V_1, ..., V_s \) be the irreducible representations of \( \Gamma' \), and \( \chi_1, ..., \chi_s \) be their characters. Let \( V = \oplus_j N_j V_j \); we regard \( V \) as a representation of \( \Gamma \) and let \( N = \dim V \). Then \( \phi_N(f)(V) = P(w_1, ..., w_m) \), where \( w_i = \sum_j N_j \chi_j(\gamma_i) - \chi_j(1) \). By representation theory of finite groups, the matrix with entries \( a_{ij} = \chi_j(\gamma_i) - \chi_j(1) \) has rank \( m \); thus, there exist \( N_j \geq 0 \) such that \( P(w_1, ..., w_m) \neq 0 \). For such \( N_j \), \( \phi_N(f) \neq 0 \), which is a contradiction. \( \square \)

**2.3. Proof of Theorem 1.2.** Now we are ready to prove Theorem 1.2. Let \( z \) be a central element of the Poisson algebra \( S^\bullet L \). Consider the element \( \phi_N(z) \). This is a regular function on \( M_g(N) \) which Poisson commutes with all other functions (since \( \phi_N \) is surjective). Since by Theorem 2.1 the scheme \( M_g(N) \) is in fact a variety, which is irreducible and generically symplectic, any Casimir on this variety must be a scalar.

Since \( S^\bullet L = S^\bullet L_+ \otimes \mathbb{C}[e_1] \), we can write \( z \) as

\[
    z = \zeta(e_1) + \sum_{j=1}^m \zeta_j(e_1)f_j,
\]

were \( f_j \) are linearly independent elements which belong to the augmentation ideal of \( S^\bullet L_+ \), and \( \zeta, \zeta_j \in \mathbb{C}[t] \). Applying \( \phi_N \) to this equation, and using that \( \phi_N(e_1) = N \), we get that

\[
    \zeta(N) + \sum_{j=1}^m \zeta_j(N)\phi_N(f_j) = \gamma_N.
\]

Let \( Y \) be the linear span of 1 and \( f_j, j = 1, ..., m \) in \( S^\bullet L_+ \). By Proposition 2.2 for \( N \geq N(Y) \), we have

\[
    \zeta(N) + \sum_{j=1}^m \zeta_j(N)f_j = \gamma_N.
\]
Thus $\zeta_j(N) = 0$ for $N \geq N(Y)$. Hence $\zeta_j = 0$ for all $j$ and $z = \zeta(e_1)$. The theorem is proved.

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