Galaxy rotation curves from external influence on Schwarzschild geometry

A. Bhattacharyay

Indian Institute of Science Education and Research, Pune, India

Consequences of a postulate that the intergalactic space of spiral galaxies, showing rotation curves in nonconformity with Newtonian gravity, does not reach the Minkowski limit is introduced in this paper. It is shown here that a wide range of galaxy rotation curves can be obtained as a perturbation of the weak field limit of the Schwarzschild geometry. The theory explains why galaxy rotation curves are non Keplarian where the planetary motions are Keplarian. If the radius of a spiral galaxy is \( r \) and its Schwarzschild radius based on luminous mass is \( \lambda \), the present theory predicts that \( r \approx \sqrt{\lambda r_0} \) where \( r_0 \) is the radius of the universe. The present theory also predicts that the ordinary matter of the universe has a mass \( c^2 r_0 / 2G \), where \( G \) is the gravitational constant. Considering the radius of the universe to be \( 10^7 \) kpc, the total mass comes out to be \( 10^{53} \) kg which is consistent with present estimation.

PACS numbers:

Introduction: Galaxy rotation curves of spiral (disk) galaxies show that the magnitude of velocity \( v \) (which we will refer to as velocity in what follows) of distant stars from the center of galaxy is, to a good extent, independent of the distance \( r \) from the center of the galaxy. This is a regime supposed to be of Newtonian gravity where these rotation curves exhibit \( r \) independence of \( v \). According to Newton’s laws, the velocity of approximate mean circular motion of these stars should follow the relation

\[
\frac{v^2}{r} = \frac{GM}{r^2},
\]

where \( G \) is Newton’s constant of gravity and \( M \) is the luminous mass at the galaxy center. This gives \( v = \sqrt{GM/r} \) and it should fall with distance, however, it remains relatively constant or even increasing with distance in some cases as observations show [1, 2].

Existence of dark matter is invoked in this scenario (and in many other cases, e.g., gravitational lensing) where one considers the mass of the galaxy comprises of an otherwise invisible component which is spread over being proportional to the distance i.e. \( M(r) \propto r \) [3]. This can produce velocity profiles as seen at large \( r \). This idea of existence of dark matter is probably the most dominant paradigm followed at present.

People have tried to explain these rotation curves by invoking different ideas within the realm of Einstein’s gravity. An interesting idea in this respect is the one put forward by Carmeli [4] couple of decades ago. Carmeli tried to take into account the effects of the expanding universe on the rotation curves at large distances. More work has been done along this line following Carmeli’s 1998 work [5]. There are other recent approaches to explain galaxy rotation curves using conformal gravity [6, 7], Palatini formalism [8], using Grumiller’s modified gravity [9] etc.

Another competing idea to explain galaxy rotation curves is due to Milgrom who considered altering Newton’s law of motion or equivalently gravity when the acceleration is extremely small of the order of \( 10^{-8} \) cm/sec\(^2 \) [10–15]. This is an interesting idea applicable to galaxy rotation curves and is followed by many to explain such observations [16, 17]. This paradigm of MOND does not really justify (within the scope of known facts without taking into account unobserved fields) why Newton’s law of motion should alter, however, accepts the observed facts that it alters in the case of such galaxy rotation curves.

In this paper we introduce a radical idea that in the regime of weak fields where gravity is not Newtonian (in spiral galaxies), one cannot get this regime by perturbing the Minkowski space. The non-Newtonian weak field limit of gravity at the edge of the spiral galaxies indicates the fact that the Schwarzschild geometry of these galaxies does not attain the Minkowski limit in the intergalactic space. As a result, the weak field regime of the Schwarzschild metric, when used to model such systems, has to be directly perturbed. Based on this new assumption, we show in this paper that the galaxy rotation curves over a good range of variety can be captured by the metric we propose. Our theory explains why the Newtonian gravity limit gets modified at the scales of the galaxies whereas for planetary systems it works. Our theory correctly estimates the radius and the mass of the ordinary matter of the universe based primarily on the parameters of our own galaxy the Milky Way.

Disk galaxies showing spiral structures are relatively young galaxies and are considered to be somewhere in the process of evolution to stable elliptical galaxy structures. However, the spiral arms of these galaxies are quite stable over very large times and this stability also requires the stars in the arms of these galaxies to move faster than what the Newtonian gravity would allow those to do. Although, the structure of these galaxies are rather flat, trying to
capture the essential features of the rotation curves by making use of the Schwarzschild geometry is not a bad idea. The reason being that it is an exact solution of Einstein’s equation with spherical symmetry, hence, building the intended reality upon its weak fields is a reasonable approach particularly when one does not want to do that as a perturbation on Minkowski space.

In the present paper, the perturbation of the Schwarzschild metric is considered due to external sources. In such a situation, the boundary (large \( r \)) region of the Schwarzschild spacetime gets altered/perturbed due to external sources. The vacuum Einstein equations are not satisfied by the perturbed metric, because it effectively takes into account the existence of an average mass density at large scales. We are giving up here the asymptotic limit of the Schwarzschild metric at large distances from the source. In general this linear equation of \( \Phi \) comes from the same equation obeyed otherwise constant potential \(-\Phi_0\). We are interested in finding the gravitational potential at a distance \( R_1 \) from the system \((M_1)\) and \( R_1 \) is large enough for the rules of Newtonian gravity and superposition to hold good. Therefore, in the Newtonian limit where the superposition works

\[
\Phi(R_1) = -\frac{GM_1}{R_1} - \Phi_0 \simeq -\frac{GM_1}{R_1} \left[ 1 + \frac{R_1}{R_0} \right],
\]

where \( R_0 = \Phi_0/GM_1 \) is a length scale that has emerged from this background field.

The superposition works in the weak field Newtonian limit of gravity because the potential obeys linear equation \( \nabla^2 \Phi \simeq 4\pi G \rho \), where \( \rho \) is the source density. The Newtonian potential \( \Phi(R_1) \) obtained by perturbing the Minkowski metric at large distances from the source. In general this linear equation of \( \Phi \) comes from the same equation obeyed by \( g_{00} \) in the weak field small velocity limit of the Schwarzschild metric as

\[
\nabla^2 g_{00} = \frac{8\pi G}{c^4} T_{00},
\]

where \( g_{00} \) and \( T_{00} \) the time element (diagonal) of the metric and the energy-momentum tensor respectively. Consideration of \( T_{00} = \rho c^2 \) and \( g_{00} = 1 + \frac{2\Phi}{c^2} \) results in the equation of \( \Phi \) where \( \rho \) is the density of the source, \( c \) is velocity of light and \( \Phi \) turns out to be Newton’s gravitational potential. \( \Phi \) obeys superposition because so does \( g_{00} \) in the presence of the same source apart from a factor of \( c^2 \). Consider in analogy, therefore, a perturbed Schwarzschild metric in the weak field Newtonian limit (small velocity) as

\[
ds^2 = (1 - \frac{\lambda}{r})(1 + \beta r)c^2 dt^2 - dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2),
\]

where \( \beta \) is an inverse length scale to be found out from the scales of a typical disk galaxy. The structure of the perturbation is guessed from the simple example of Newtonian gravity as given above. Here, the isotropy of the space is in place, the background field is due to everything else in the universe and the length scale \( \beta^{-1} \) will capture this background. Note that, in the strong gravity regime where superposition does not work, this method of perturbation should break down. However, \( \beta \) coming out to be extremely small it practically does not matter even if one keeps this perturbation at small distances.

Let us have a look at the elements of the nonzero elements (diagonal) of the Ricci tensor corresponding to this metric. These elements are

\[
R_{00} = \frac{c^2 \left( 3r^4 \beta^2 - 6r^2 \beta \lambda - \lambda^2 - 4r^3 \beta (\beta \lambda - 1) \right)}{4r^3(1 + \beta r)(r - \lambda)},
\]

\[
R_{11} = \frac{r^4 \beta^2 + 6r^2 \beta \lambda - 3\lambda^2 - 4r\lambda(\beta \lambda - 1)}{4r^2(1 + \beta r)^2(r - \lambda)^2},
\]

\[
R_{22} = \frac{r^2 \beta + \lambda}{2(1 + \beta r)(r - \lambda)},
\]

\[
R_{33} = \frac{(r^2 \beta + \lambda) \sin^2 \theta}{2(1 + \beta r)(r - \lambda)}.
\]
In the following, we will see that, for the galaxy rotation curves to result at the distance \( r \), \( \beta r^2 \sim \lambda \). This indicates \( R_{22}, R_{33} \sim 1/r \) (and \( R_{00}, R_{11} \) are even smaller) which is as bad a situation as that in the case of deviation form exact solution in the Newtonian limit of the Schwarzschild metric. It can also be checked that all of the elements of the Riemann tensor are devoid of any divergence and are regular. The only singularity is at \( r = 0 \) which is standard. The \( \beta r \) perturbation will appear, in what follows, to be order unity at the edge of the universe, hence, there is no large \( r \) issue.

Consider the Geodesic equations of this system

\[
\frac{d^2 r}{d\tau^2} - r \left( \frac{d\theta}{d\tau} \right)^2 - r \sin^2 \theta \left( \frac{d\phi}{d\tau} \right)^2 + \frac{c^2(\beta r^2 + \lambda)}{2r^2} = 0, \tag{5}
\]

\[
\frac{d^2 \theta}{d\tau^2} + \frac{2}{r} \frac{dr}{d\tau} \frac{d\theta}{d\tau} - \cos \theta \sin \theta \left( \frac{d\phi}{d\tau} \right)^2 = 0, \tag{6}
\]

\[
\frac{d^2 \phi}{d\tau^2} + \frac{2}{r} \frac{dr}{d\tau} \frac{d\phi}{d\tau} + 2 \cot \theta \frac{d\theta}{d\tau} \frac{d\phi}{d\tau} = 0 \tag{7}
\]

and

\[
\frac{dt}{d\tau^2} + \frac{2(\beta r^2 + \lambda)}{2r(1 + \beta r)(r - \lambda)} \frac{dr}{d\tau} \frac{dt}{d\tau} = 0. \tag{8}
\]

Consider the case of a circular orbit (for the sake of simplicity) on the equatorial plane \( \theta = \pi/2 \). This makes \( t \propto \tau \) and from eqn.(5)

\[
v = \frac{c}{\sqrt{2}} \sqrt{\left( \frac{\beta r + \lambda}{r} \right)} \tag{9}
\]

where on this circular orbit \( v = r \frac{d\phi}{d\tau}, \frac{d^2 r}{d\tau^2} = \frac{dr}{d\tau} \frac{d\phi}{d\tau} = \frac{d^2 \theta}{d\tau^2} = \frac{d\theta}{d\tau} = \frac{d^2 \phi}{d\tau^2} = 0 \). The eqn.(9) immediately tells us that \( \beta r \) has to be comparable and within an order of magnitude of \( \lambda/r \) to effectively raise the rotation curves above what is given by the Newtonian gravity.

The expression of the velocity, at a closer query, immediately reveals that if \( \beta \) is too small then for smaller \( r \) and corresponding \( \lambda \) there is no reason to not get Keplerian orbits. For example, this is the case of our solar system. The Newtonian gravity, as obtained from perturbation of Minkowski space would remain intact for the planetary systems for a very small \( \beta \) despite the fact that orbits of disk galaxies can get altered from what Newtonian gravity predicts at the scale of these spiral galaxies. \( \beta \) must be small for this phenomenological model to work and let us have an estimate of how small this \( \beta \) is using some known facts.

Take the example of our galaxy the Milky Way. It has a central star Sagittarius A* (considered to be a super massive black hole) of radius of the order of \( 10^7 \) km and a mass of about 4 million solar masses. The luminous mass of the same galaxy is estimated to be of that due to about 200 billion stars and therefore the \( \lambda \) for such a system can be roughly estimated to be \( 10^{10} \) km considering the mass of the core of the galaxy to be a billion solar masses. Now, 1 kpc being \( \times 10^{16} \) km, the distance \( r \) in the scale of \( \lambda \) is of the order of \( 10^9 \times n \) where \( n \) is a number that is safely considered to be within order 10. This immediately tells us that \( r_0 = \beta^{-1} \) is of the order of \( 10^{13} \) in the units of \( \lambda \). 1 kpc being \( 10^6 \) in \( \lambda \), \( r_0 = 10^7 \) kpc. The radius of the observable universe being about \( 14.3 \times 10^9 \) parsec is of the order of \( r_0 \).

Let us have another look at the estimate of the parameter \( \beta \). If we put the expression of modified \( g_{00} = 1 - \lambda/r + \beta r + \beta \lambda \) in eqn.\( (3) \) and estimate the effective mass density which has to be present on top of the central mass \( \lambda \) due to the \( \beta r \) term, we get an effective density \( \rho_{\text{eff}} = \frac{c^2 \beta r_0}{3} \) where we have taken \( T_{00} = \rho c^2 \). Note that, this density is visible at large \( r \) (where superposition holds) and the singularity at \( r = 0 \) is of no consequence. This will result in the total mass of the universe \( M_{\text{uni}} = \frac{c^2 \beta r_0^2}{3} \simeq \frac{c^2 r_0}{3} \) since \( \beta \sim r_0^{-1} \). Putting the typical values, \( c^2 = 9 \times 10^{16} \text{ m}^2/\text{sec}^2 \), \( r_0 = 10^7 \times 10^9 \text{ m} \) and \( G = 6.6 \times 10^{-11} \text{ m}^3/\text{kg} \cdot \text{sec}^2 \), the mass of the universe comes out to be \( M_{\text{uni}} \simeq 10^{63} \text{ kg} \). This is exactly the estimated order of magnitude of the mass of the ordinary matter in the universe. So, we get an expression of the ordinary matter of the universe in terms of the universal constants \( c \), \( G \) and the radius of the universe \( r_0 \). Moreover, an observer at the center of the galaxy does not see the isotropic universe to be homogeneous because the
effective density falls as $1/r$. If true, this result would have implication for the existing standard cosmological model. It is important to note that in the above estimation no dark matter contribution is taken into account which actually makes the mass of our galaxy estimated to be 1.5 trillion solar masses. The important lesson this order of magnitude calculations gives us is that if the size of the galaxy scaled by its mass has some universality then that probably is due to the rest of the universe it is in causal connection with which fixes the $r_0$. Let’s get the velocity $v$ in km/sec using these numbers.

$$v = \frac{c}{\sqrt{2}} \sqrt{\left( \frac{r}{r_0} + \frac{1}{r} \right)} = \frac{10^5}{\sqrt{2} \times 10^3} \sqrt{\left( \frac{n}{10} + \frac{1}{n} \right)} = \frac{100}{\sqrt{2}} \sqrt{\left( \frac{n}{10} + \frac{1}{n} \right)},$$

where $r_0 = 1/\beta \lambda$ when all the distances are scaled by $\lambda$. It can immediately be recognized that we are getting the right orders of magnitude for the velocity. For a visual representation of the effect refer to the Fig.1 where the velocity

$$v = \frac{100}{\sqrt{2}} \sqrt{\left( \frac{n}{m} + \frac{1}{n} \right)}$$

has been plotted for two different choices of $m$ and those curves are compared with the unperturbed one corresponding to $\beta = 0$.

![FIG. 1: Velocity vs distance plot.](image)

Now that we have got an idea about the rotation curves of the massive objects moving with velocity smaller than velocity of light, let us have a look at the same traveling with velocity comparable to $c$. In this case, the metric is

$$ds^2 = (1 - \frac{\lambda}{r})(1 + \beta r)c^2 dt^2 - \frac{dr^2}{1 - \frac{\lambda}{r}} - r^2(d\theta^2 + \sin^2 \theta d\phi^2),$$

and the corresponding equation of dynamics of $r$ (equivalent one to eqn.(5)) is

$$\frac{d^2 r}{d\tau^2} - \frac{\lambda}{2r^2 - 2r\lambda} \left( \frac{dr}{d\tau} \right)^2 - (r - \lambda) \left( \frac{d\theta}{d\tau} \right)^2 - (r - \lambda) \sin^2 \theta \left( \frac{d\phi}{d\tau} \right)^2 + \frac{c^2(r - \lambda)(\beta r^2 + \lambda)}{2r^3} = 0.$$  

Under the similar conditions as mentioned for eqn.(9), the velocity in the present case is determined by the same relation as the eqn.(9). Thus, this velocity relation apparently works for all the range of velocities of massive objects.

Discussion: Galaxy rotation curves from a perturbed Schwarzschild metric are shown in this paper. The entire effect occurs as a result of existence of a phenomenological length scale $r_0$ of the order of the size of the universe which influences the weak field limit of a Schwarzschild geometry. The method relies on a radical idea of not trying to capture this weak field limit of the Schwarzschild geometry as a perturbation of the Minkowski space which is done for the Newtonian gravity. The underlying assumption is that, in the intergalactic space the metric does not reach the Minkowski limit of the Schwarzschild metric due to proximity of the galaxies in scales of their typical size.
The present method is based on a perturbation which would not affect smaller length scales (for example solar system) due to the very large value of $r_0$. For the systems which have at large $r$, $\lambda/r^2$ considerably bigger than the $1/r_0$ where $r$ is distance from the center, the Newtonian limit of gravity will work. This is the regime of the planetary systems which show Keplerian orbits.

The present method is different in its construction from those based on the consideration of dark matter distribution. This treatment is more like that of MOND, the present method arrives at the modified Newtonian gravity at the length scales of size of galaxies on the basis of the consideration of giving away the Minkowski limit at these length scales. The estimation of $\beta$ based on the scales of our galaxy actually results in the correct order of magnitude of the ordinary matter of the universe which is an exceptional connection to show up. The whole emergent physics here is based on a very plausible physical consideration of abandoning Minkowski limit at galaxy scales and the verification/falsification of which can be done on the basis of observations.

As this method is shown to be able to qualitatively capture the whole range of rotation curves for the complete range of velocity of stars, it is also capable of making a prediction which could be tested by observation. The analysis indicates that there is no reason for the length scale $r_0$ not to be universal particularly when it comes out to be of the order of the scale of the universe. If $r_0$ is universal, then for all spiral galaxies the quantity $\lambda/r$ where $r$ is the radius of the galaxy should come out to be of the order of $1/r_0$ which can be verified from observations. It is being predicted here that, given its mass, the size of a spiral galaxy is determined by $\sqrt{\lambda r_0}$ where $\lambda$ is the Schwarzschild radius of the galaxy. Note that, $\lambda$ of a galaxy must be estimated on the basis of its luminous mass and no dark matter is involved. The present phenomenology also predicts that the universe as seen by an observer at the center of a galaxy is although isotropic, but, is not homogeneous as the effective density falls as $1/r$. The consequence of this result on cosmological models could be enormous.

Let us look at one more possibility of extrapolation of our present understanding. Given the nature of the perturbation $\beta r$ (being small) well inside a galaxy, since $\beta r$ takes into account the entire external mass of the universe, the real metric of a galaxy should be the one as given by eqn.(12). The Schwarzschild metric, although is an exact solution of Einstein’s equation, does not take into account the external mass of the universe. However, it turns out that, the Schwarzschild geometry is the backbone of the real one which is only a perturbation away from it. It would be interesting to explore the effects of this perturbation on the internal structures of a disk galaxy and even on a gravastar (Schwarzschild star) model of Mazur and Mottola [18] where one does not take the Schwarzschild metric to be that outside a gravastar, rather, take into account a $1/r$ fall of matter density outside the gravastar.

[1] S. M. Faber and J. S. Gallagher, Annu. Rev. Astron. Astrophys. 17, 135 (1979).
[2] V. C. Rubin, N. Thonnard and W. K. Jr. Ford, Astrophys. J. 238, 471 (1980).
[3] M. Roos, arXiv:1001.0316 (2010).
[4] M. Carmeli, Int. J. Theor. Phys., 37, 2621 (1998).
[5] J. G. Hartnett, Int. J. Theor. Phys., 45, 2118 (2006).
[6] P. D. Mannheim and J. G. O’Brien, J. Phys.: Conf. Ser. 437 012002 (2013).
[7] Q. Li and L. Modesto, arXiv:1906.05185v1 (2019).
[8] C. A. Sporea, A. Borowiec and A. Wojnar, Eur. Phys. J C Part. Fields 78, 308 (2018).
[9] H-Nan Lin, M-Hua Li, X Li and Z Chang, MNRAS 430, 450 (2013).
[10] M. Milgrom, Ap. J., 270, 365 (1983).
[11] M. Milgrom, Ap. J., 270, 371 (1983).
[12] M. Milgrom, Ap. J., 270, 384 (1983).
[13] J. Bekenstein and M. Milgrom, Ap. J., 286, 7 (1984).
[14] M. Milgrom Can. J. Phys. 93, 107 (2015).
[15] R. H. Sanders and S. S. McGaugh, Annu. Rev. Astron. Astrophys. 40, 263 (2003).
[16] L. Acido, Galaxies 5, 74 (2017).
[17] R. Scarpa, AIP Conf. Proc. 822, 253 (2006).
[18] P. O. Mazur and E. Mottola Class. Quantum Grav. 32 215024 (2015).