Tikhonov and Landweber convergence rates: characterization by interpolation spaces

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Abstract

Algebraic convergences rates of (iterated) Tikhonov regularization for linear inverse problems in Hilbert spaces are characterized by the membership of the exact solution to intermediate spaces produced by the K-method of real interpolation. Similar results are obtained for the Landweber iteration.

Keywords: Tikhonov, Landweber, regularization, convergence rates, sharp converse results, interpolation spaces, \( K \)-functional

1. Introduction

It has long been known that the Hölder-type source conditions originally used to obtain algebraic convergence rates for Tikhonov regularization are only sufficient, not necessary. Consequently, other concepts have been introduced to completely characterize those rates such as the spectral decay condition [1 theorem 2.1], distance functions [2 p 3], and variational source conditions [3]; pointers to the origins of those concepts can be found therein. We propose here a more concise condition through the notion of interpolation spaces and establish links to the concepts just listed. Specifically, we argue that the intermediate spaces \((E_0, E_1)_{0,q}\) produced by the \( K \)-method of real interpolation [4, 5] with fine index \( q = \infty \) naturally capture the essential behavior of (iterated) Tikhonov regularization, that is, convergence rates, converse results, and saturation, in the noise-free and the noisy case. This is not unexpected, given the resemblance of the \( K \)-functional (1) and the Tikhonov functional (41), but we systematically quantify this connection by careful estimates with particular attention to the limiting cases \( \theta = 0, 1 \). In a similar vein, the relationship between near-minimizers for \( L \)-functionals and Tikhonov regularization was highlighted in [6 chapter 6]. We prove analogous convergence and converse results for the Landweber iteration, and comment on the applicability of the discrepancy principle as a stopping rule.
This note consists of two main parts. In the first part (section 2) we develop the required preliminaries of the $K$-method of real interpolation, introduce the different intermediate subspaces, and describe them and their interrelations using spectral theory in Hilbert spaces. In passing, we relate the concepts of distance functions and variational source conditions. In the second part (section 3) we elaborate on how the convergence rates of Tikhonov regularization and Landweber iteration are characterized in terms of the intermediate subspaces.

We write $A \lesssim B$ to mean that there is a constant $C \geq 0$ independent of the parameters specified by quantifiers such that $A \leq CB$. If, in addition, $B \lesssim A$, we write $A \sim B$.

2. Preliminaries

2.1. Interpolation spaces

Let $X$ be a Banach space (here and henceforth: over the reals). Let $X_0 \subset X$ be another Banach space, continuously embedded in $X$. We write $\| \cdot \|_0 := \| \cdot \|$ and $\| \cdot \|_1$ for the norms of $X$ and $X_0$, respectively. The $K$-functional is defined as

$$K_t(x) := \inf_{x_1 \in X} \left( \| x - x_1 \|_0^2 + t^2 \| x_1 \|_1^2 \right)^{1/2}, \quad x \in X, \quad t > 0. \quad (1)$$

For real $0 < \theta < 1$ and $1 \leq q \leq \infty$ the $K$-method of real interpolation defines intermediate subspaces $X_q \subset (X, X_0)_{\theta,q} \subset X$ based on the integrability of $t \mapsto K_t(x)$. Here we are only interested in the case $q = \infty$ with one of the spaces embedded into the other, and therefore refer to standard sources such as $[4, 5]$ for general definitions. For any $0 \leq \theta \leq 1$ and any $x \in X$ set

$$\| x \|_{\theta,1} := \sup_{t>0} t^{-\theta} K_t(x), \quad (2)$$

possibly infinite. We point out that we include the limiting cases $\theta = 0$ and $\theta = 1$ in this definition. The reason for the unusual notation will become apparent in section 2.6 where instead of $X_1$ we will consider a family of subspaces $X_\gamma \subset X$ parametrized by $\gamma$. Define the spaces

$$X_{\theta,1} := (X, X_0)^{\theta,1} := \left\{ x \in X : \| x \|_{\theta,1} < \infty \right\} \quad (3)$$

with the norm $\| \cdot \|_{\theta,1}$. For $0 < \theta < 1$, these are Banach spaces. Moreover, the following embeddings $X_0 \subset X_{\theta,1} \subset X$ are continuous. The space $X_{\theta,1}$ need not coincide with $X_\theta$ for $\theta = 0, 1$, see the remarks on the Gagliardo completion in $[5$ chapter 5 section 1], but it will be the case in the more specific setting of section 2.6.

2.2. The constant $N_\theta$

For $0 \leq \theta \leq 1$, the constant $1 \leq N_\theta \leq \sqrt{2}$ given by

$$N_\theta^{-2} := \theta^\theta (1 - \theta)^{1-\theta} = \sup_{\lambda \in [0,1]} \lambda^\theta (1 - \lambda)^{1-\theta} = \sup_{s>0} \frac{s^{2(1-\theta)}}{s^2 + 1} \quad (4)$$

will play a recurrent role. By convention, $N_0 := N_1 := 1$, making $\theta \mapsto N_\theta$ continuous on $[0, 1]$. For example, if $a, b > 0$ then Young’s inequality with exponents $p = 1/(1-\theta)$ and $q = 1/\theta$ gives

$$N_\theta^2 a^{1-\theta} b^\theta = (a/(1-\theta))^{1-\theta} (b/\theta)^\theta \leq a + b. \quad (5)$$
As an example, for any real \( \lambda \geq 0 \) and \( t > 0 \), any real \( k \geq 1/2 \), and any \( 0 \leq \theta \leq 1 \),
\[
N_{\theta}^{2(1-2k)} \frac{2^{2(1-\theta)}}{t^2 + \lambda^{2k}} \leq \left( \frac{a^{1-\theta}}{\alpha + \lambda} \right)^{2k} \quad \text{for} \quad \alpha = t^{1/k} \left( \frac{1 - \theta}{\theta} \right)^{1/(2k)}. \tag{6}
\]
Indeed, after algebraic simplification, the inequality in (6) is equivalent to
\[
(\alpha + \lambda)^{p} \leq (1 - \theta)^{1-p} \alpha^{p} + \theta^{1-p} \lambda^{p},
\]
itself a consequence of Hölder’s inequality with \( p = 2k \geq 1 \) as one of the exponents.

2.3. Distance function

Distance functions were introduced as a means to characterize the regularization error of
linear regularization operators in [2], previously also in [7 theorem 2.12]. We briefly comment
on the relation to the K-functional. Let \( 0 < \theta < 1 \). Fix \( x \in X \).
Define the distance function
\[
d(r) := \inf \left\{ \| x - x_1 \|_0 : x_1 \in X_1, \| x_1 \|_1 \leq r \right\}, \quad r \geq 0.
\tag{7}
\]
This function is nonnegative, nonincreasing, bounded by \( d(0) = \| x \|_0 \), and convex. To simplify the notation, we shall write \( \| x_1 \|_1 \leq r \), or similar, without mentioning that \( x_1 \in X_1 \). It is clear that \( d \) has compact support if and only if \( x \in X_1 \).

Inspection of the definitions of the K-functional (1) and the distance function (7) reveals that
\[
K_{\theta}(x) = \inf_{r > 0} d(r)^2 + r^2 t^2)^{1/2} \sim \inf_{r > 0} d(r) + tr = -d^*(t),
\tag{8}
\]
where \( d^* \) is the Legendre–Fenchel conjugate of \( d \). Thus the distance function (7) characterizes
the subspace \( X_{\theta,1} \subset X \). The first characterization, boundedness of \( \| t^{-\theta} d^*(-t) \|_1 \)
occurs from the definition (2) and the equivalence (8). The second characterization is the behavior of
\( d \) at infinity, more precisely the identity
\[
E = N_{\theta}^{-1} \| x \|_{\theta,1} = D,
\tag{9}
\]
where
\[
E := \sup_{r > 0} \inf_{\| x_1 \|_1 \leq r} \| x - x_1 \|_0 - \theta \| x_1 \|_1^\theta \quad \text{and} \quad D := \sup_{r > 0} d(r^1 - \theta) r^\theta.
\tag{10}
\]
From the identity (9) one infers the more qualitative observation that \( x \in X_{\theta,1} \) if and only if
the distance function (7) exhibits the asymptotic decay rate \( d(r) = O(r^{-\theta/(1-\theta)}) \) for \( r \to \infty \).

Proof of the identity (9) is in three steps:

(a) \( E \leq N_{\theta}^{-1} \| x \|_{\theta,1} \). Estimating the \( \inf \) of \( E \) as in (5) for
\( a := \| x - x_1 \|_0^2 \) and \( b := r^2 \| x_1 \|_1^2 \),
\[
\inf_{\| x_1 \|_1 \leq r} \left( \cdots \right) \leq N_{\theta}^{-1} r^{1-\theta} \inf_{\| x_1 \|_1 \leq r} \left( \| x - x_1 \|_0^2 + r^2 \| x_1 \|_1^2 \right)^{1/2} \leq N_{\theta}^{-1} r^{1-\theta} \sup_{r > 0} r^\theta K_{\theta}(x).
\]
Here, the condition \( \| x_1 \|_1 \leq r \) is redundant if \( t \) is large enough, so the infimum becomes
\( K_{\theta}(x) \), explaining the second inequality. Taking the supremum over \( r > 0 \) proves the claim.

(b) \( \| x \|_{\theta,1} \leq N_{\theta} D \). The definition (10) of \( D \) implies \( d(r) \leq (r^\theta / D)^{1/(\theta - 1)} \). Using this in the
equivalence (8), then computing the infimum yields \( K_{\theta}(x) \leq t^\theta N_{\theta} D \). Now multiply by \( t^{-\theta} \)
and take \( \sup_{r > 0} \).

(c) \( D \leq E \). If \( D \) is finite then \( d(\infty) = 0 \). Given that \( d \) is convex, \( d \) is strictly decreasing
(unless where it vanishes). Thus, for each \( r > 0 \), the infimum in \( d(r) \) is achieved at
\( \| x_1 \|_1 = r \). Concerning \( E \), we may suppose that the \( \supinf \) is assumed at \( \| x_1 \|_1 = r \),
adjusting $r$ if necessary. Hence, both $D$ and $E$ equal $\inf_{\|x\|_0 = r} \|x - x_1\|_0^{1-\theta} \|x\|_0^\theta$. This establishes the identity (9).

2.4. Interpolation inequality of operators

Let $Y$ be a Banach space. Let $S : X \to Y$ be a bounded linear operator with norm $C_0 \geq 0$. Assume that $S : X_1 \to Y$ is also a bounded linear operator, with norm $C_1 \geq 0$. Then, for any $0 < \theta < 1$,

$$\|Sx\|_Y \leq N_0 C_0^{1-\theta} C_1^\theta \|x\|_{0,1} \quad \forall x \in X_{\theta:1}.$$  \hspace{1cm} (11)

Indeed, let $x \in X_{\theta:1}$. Given any $x_1 \in X_1$, write $x = (x - x_1) + x_1$, apply the triangle inequality with boundedness of $S$, and estimate by Cauchy–Schwarz:

$$\|Sx\|_Y \leq C_0 \|x - x_1\|_0 + C_1 \|x_1\| \leq t^\theta (C_0^2 + t^{-2} C_1^2)^{1/2} x \|x - x_1\|_0^{1/2} + t^2 \|x_1\|_0^{1/2}.\]$$

Inserting an infimum over $x_1 \in X_1$ then a supremum over $t > 0$ in the second factor, in view of the definition (2) we obtain $\|Sx\|_Y \leq t^\theta (C_0^2 + t^{-2} C_1^2)^{1/2} \|x\|_{0,1}$. Minimization over $t > 0$ gives the estimate (11).

2.5. A lemma for measures

We will call a nonnegative finite measure $\mu$ on Borel subsets of $[0, \infty)$ a ‘Borel measure on $[0, \infty)$’. For any such $\mu$ and any real $\nu \geq 0$ we define $\|\mu\|_{\nu}$ by

$$\|\mu\|_{\nu}^2 := \sup_{\nu > 0} r^{-\nu} \mu([0, t)).$$  \hspace{1cm} (12)

The following lemma records two useful properties of $\|\cdot\|_{\nu}$.

**Lemma 2.1.** Let $\mu$ be a Borel measure on $[0, \infty)$. Let $\nu > 0$. Then

$$\mu([0, \Lambda)) + \Lambda^{2\gamma} \int_{[\Lambda, \infty)} \lambda^{-2\gamma} d\mu(\lambda) \leq \frac{\gamma}{\gamma - \nu} \Lambda^{2\nu} \|\mu\|_{\nu}^2 \quad \forall \Lambda > 0 \quad \forall \gamma > 0,$$  \hspace{1cm} (13)

and

$$\int_{[0, \Lambda)} \lambda^{-2r} d\mu(\lambda) \leq \frac{r + \nu}{\nu} \Lambda^{2r} \|\mu\|_{r+\nu}^2 \quad \forall \Lambda > 0 \quad \forall r > 0,$$  \hspace{1cm} (14)

whenever the right-hand-side is finite.

**Proof.** Define the left-continuous function $I(\Lambda) = \mu([0, \Lambda))$ for $\Lambda \geq 0$. Fix $\Lambda > 0$. Writing the integral as a Riemann–Stieltjes integral, and integrating by parts we have

$$\int_{[\Lambda, \infty)} \lambda^{-2\gamma} d\mu(\lambda) = \int_{[\Lambda, \infty)} \lambda^{-2\gamma} dI(\lambda) = -\Lambda^{-2\gamma} I(\Lambda) + 2\gamma \int_{[\Lambda, \infty)} \lambda^{-2\gamma+1} I(\lambda) d\lambda.$$  \hspace{1cm} (15)

Estimating $I(\lambda) \leq \lambda^{2\nu} \|\mu\|_{\nu}^2$ under the integral, evaluating, and rearranging leads to the estimate (13). Similarly,

$$\int_{[0, \Lambda)} \lambda^{-2r} d\mu(\lambda) \leq \int_{[0, \Lambda)} \lambda^{-2r} dI(\lambda) = \lambda^{2r} I(\lambda) \bigg|_{\lambda=0}^{\lambda=\Lambda} + 2r \int_{[0, \Lambda)} \lambda^{-2r-1} I(\lambda) d\lambda.$$  \hspace{1cm} (16)

Estimating $I(\lambda) \leq \lambda^{2(\nu+1)} \|\mu\|_{r+\nu}^2$ and evaluating the integral yields the estimate (14). \hspace{1cm} $\square$

Alternative proof. Fix $\Lambda > 0$. Ad (13): consider $A_s := [\Lambda, \infty) \cap \{\lambda \geq 0 : \lambda^{-2s} > s\}$. If $s \geq \Lambda^{-2r}$ then $A_s$ is empty, otherwise $A_s = [\Lambda, s^{-1/(2s)})$. Therefore
\[
\int_{(\Lambda, \infty)} \lambda^{-2\gamma} d\mu(\lambda) = \int_0^{\Lambda^{-2}} \mu(A_r) dr = -\Lambda^{-2} \mu([0, \Lambda]) + \int_0^{\Lambda^{-2}} \mu(0, s^{-1/2\gamma})) ds
\]

Estimating \( \mu([0, t]) \leq t^{2\nu} \|\mu\|_\nu^2 \) under the integral and evaluating yields (13).

Ad (14): the statement is trivial for \( r = 0 \), so suppose \( r > 0 \).

Consider \( B_r := [0, \Lambda) \cap \{ \lambda \geq 0; \lambda^{-2\nu} > s \} \). If \( s \leq \Lambda^{-2\nu} \) then \( B_r = [0, \Lambda) \) and \( \mu(B_r) \leq \Lambda^{2(1-\nu)} \|\mu\|_{\nu, r}^2 \). Otherwise \( B_r = [0, s^{-1/(2\nu)}) \), so that \( \mu(B_r) \leq s^{-1/(2\nu)} \|\mu\|_{\nu, r}^2 \). Using this in \( \int_{(0, \Lambda)} \lambda^{-2\nu} d\mu(\lambda) = \int_0^\infty \mu(B_r) dr \) yields (14).

### 2.6. Spectral theory in Hilbert spaces

Suppose that \( X \) and \( Y \) are real Hilbert spaces. Let \( T : X \to Y \) be a nonzero bounded linear operator. Replacing \( X \) by \( X/\ker T \) if necessary (with the usual quotient norm), we assume that \( T \) is injective. (7)

Let \( E \) denote the projection valued spectral measure of \( T^*T \). Then \( E \) is compactly supported in \( [0, \infty) \) since \( T \) is bounded, and injectivity of \( T \) is equivalent to \( \|E(0)\| = 0 \), since \( \|E(t)\| = \ker(T^*T - \lambda I) \). For \( x \in X \) we define the Borel measure \( \mu_x \) on \( [0, \infty) \) by

\[
\|x\|_\nu := \|\mu_x\|_\nu = \sup_{t > 0} t^{-\nu} \|E(0,t)x\|, \quad \nu \geq 0.
\]

The subset

\[
\mathcal{X}_\nu = \{ x \in X : \|x\|_\nu < \infty \}
\]

of \( X \) is indeed a Banach space equipped with the norm \( \| \cdot \|_\nu \). A description of this subspace as an interpolation space is subsequently given in proposition 2.2. The definition of \( \mathcal{X}_\nu \) is inspired by the work [1].

For all real \( \gamma \geq 0 \), we define the Banach space \( X_\gamma \subset X \) as

\[
X_\gamma := \text{range}\left((T^*T)^\gamma\right) \text{ with the norm } \|x\|_\gamma := \|T^*T)^\gamma \| \text{, } x \in X_\gamma.
\]

In terms of the spectral measure \( E \), we can write (note \( E(0) = 0 \))

\[
\|x\|_\gamma^2 = \int_{(0, \infty)} \lambda^{-2\gamma} d\mu_x(\lambda) \quad \text{and} \quad \|x\|_\gamma^2 = \int_{(0, \infty)} \lambda^{-2\gamma} d\mu_x(\lambda) \quad \forall x \in X_\gamma,
\]

(21)

For any real \( 0 \leq \nu \leq \gamma \), we define the interpolation space

\[
X_{\nu;\gamma} := (X, X_\gamma)_{\nu/\gamma;\infty} \text{ with norm } \| \cdot \|_{\nu;\gamma},
\]

(22)

and will denote the corresponding \( K \)-functional by \( K_\gamma^{\nu} \). One can check (most easily in the case that \( T \) is compact) that for any \( x \in X \) and \( t > 0 \),

\[
|K_\gamma^{\nu}(x)|^2 = \int_{(0, \infty)} \inf_{t > 0} \{ (1 - \epsilon)^2 + \epsilon^2 t^2 \lambda^{-2\gamma} \} d\mu_x(\lambda) = \int_{(0, \infty)} \frac{t^2}{t^2 + \lambda^{2\gamma}} d\mu_x(\lambda).
\]

(23)

For the limiting cases \( \nu = 0 \) and \( \nu = \gamma \) we recover from equations (21) and (23) that

\[
X_{0;\gamma} = X_0 \quad \text{and} \quad X_{\gamma;\gamma} = X_\gamma \text{ with equality of norms.}
\]

(24)

The different spaces are related by the following proposition.
Proposition 2.2. Let $0 < \nu < \gamma$. Let $x \in X$. Then
\[
\sqrt{1 - \nu/\gamma} \| x \|_{\nu, \gamma} \leq \| x \|_{\nu} \leq N_{\nu/\gamma} \| x \|_{\nu, \gamma} \leq \| x \|_{\nu} \leq \| T^* T \|^{1 - \nu} \| x \|_{\gamma}.
\] (25)
Hence, the following embeddings are continuous:
\[
X_{\gamma} \subset X_{\nu} \subset X_{\nu, \gamma} = X_{\nu},
\] (26)
where $X_{\nu, \gamma} = X_{\nu}$ with equivalence of norms possibly not uniform in $\nu < \gamma$.

Proof. The inequality (25d) follows from the representation (21), restricting the domain of integration to $[0, \| T^* T \|]$. The inequality (25c) is obtained by identifying $s = t\lambda^{-\gamma}$ and $\theta = \nu/\gamma$ in (4), and using it in equation (23), viz.
\[
\| x \|_{\nu, \gamma}^2 = \sup_{t > 0} t^{-2\nu/\gamma} |K_\gamma (x)|^2 \leq N_{\nu/\gamma}^{-2} \int_{[0,\infty)} \lambda^{-2\nu} d\mu_\lambda (\lambda).
\] (27)
For the inequality (25b) we use the identity
\[
s^{-2\nu} = N_{\nu/\gamma}^2 t^{-2\nu/\gamma} \frac{t^2}{t^2 + s^{2\gamma}} \text{ for } t = s^{\frac{\gamma - \nu}{\nu}},
\] (28)
combined with $s \leq \nu$ in the first step of
\[
s^{-2\nu} \| E_{(0,\infty)} x \|^2 \leq N_{\nu/\gamma}^2 t^{-2\nu/\gamma} \int_{[0,\infty)} \frac{t^2}{t^2 + s^{2\gamma}} d\mu_\lambda (\lambda) \leq N_{\nu/\gamma}^2 \int_{[0,\infty)} \lambda^{-2\nu} d\mu_\lambda (\lambda).
\] (24)
Taking $\sup_{s > 0}$ on the right, then $\sup_{s > 0}$ on the left gives (25b). Finally, from equation (23), followed by the estimate (13), we have
\[
\| x \|_{\nu, \gamma}^2 \leq \sup_{t > 0} t^{-2\nu/\gamma} \left[ \| E_{(0,\infty)} x \|^2 + s^{2\gamma} \int_{(s,\infty)} \lambda^{-2\nu} d\mu_\lambda (\lambda) \right] \leq \frac{\gamma}{\gamma - \nu} \| x \|_{\nu}^2,
\]
with the choice $s = t^{1/\gamma}$, and this shows the inequality (25a). The last claim is a combination of (25a) and (25b). \qed

The choice $\gamma = 2\nu$ in the estimate (26) leads to the chain of inequalities
\[
\frac{1}{\sqrt{2}} \| \cdot \|_{\nu} \leq \inf_{\gamma > \nu} \| \cdot \|_{\nu, \gamma} \leq \| \cdot \|_{\nu, 2\nu} \leq \sqrt{2} \| \cdot \|_{\nu},
\] (29)
and therefore, $X_{\nu, 2\nu} = X_{\nu}$ with equivalence of norms uniformly in $\nu \geq 0$. The equality between $\inf_{\gamma > \nu} \| \cdot \|_{\nu, \gamma}$ and $\| \cdot \|_{\nu, 2\nu}$ does not hold in general. However, if $\mu_\lambda$ is a Dirac measure supported at some $\lambda_0 > 0$, and $\nu > 0$, then the infimum of
\[
\| x \|_{\nu, \gamma}^2 = \sup_{t > 0} t^{-2\nu/\gamma} \frac{t^2}{t^2 + \lambda_0^{2\gamma}} \leq N_{\nu/\gamma}^{-2} \lambda_0^{-2\nu},
\] (4)
over $\gamma > \nu$ is indeed achieved at $\gamma = 2\nu$.

The constants in (25a) and (25b) are sharp. For example, for $\mu_\lambda = \delta_{t_0}$ being the Dirac measure at $\lambda_0 = 1$,
\[
\| x \|_{\nu} = 1 \quad \text{and} \quad N_{\nu/\gamma} \| x \|_{\nu, \gamma} = 1.
\] (31)
On the other hand, for $d\mu_\lambda (\lambda) = 2\lambda^{2\nu-1} d\lambda$ (that this measure is not compactly supported is not essential) we find $\| x \|_{\nu} = 1$, while
\[ \|x\|_{\nu,\gamma}^2 = \sup_{t>0} t^{-2\nu} |K_\gamma(x)|^2 = \frac{\pi \nu / \gamma}{\sin(\pi \nu / \gamma)} = (1 + o(1)) \frac{\gamma}{\gamma - \nu} \quad \text{as} \quad \nu \to \gamma, \tag{32} \]

so that the norm equivalence in equation (25) does deteriorate as \( \nu \to \gamma \). The relation to the finite qualification of the (iterated) Tikhonov regularization is discussed in section 3.3.

We provide next another illustration of the fact that \( X_\nu \) is in general strictly larger than \( X_\gamma \), here for \( \nu = 1 \). We will revisit this example in section 3.5, example 3.5.

**Example 2.3.** Consider the diagonal operator \( T := \text{diag}((n^{-1/2})_{n \geq 1}) \) on the sequence space \( X := l_2(\mathbb{N}) \). Then \( T^*T \) has eigenvalues \( \lambda_n = n^{-1}, \ n \geq 1 \). Let \( x^i := (n^{-3/2})_{n \geq 1} \). Then \( \mu_{x^i} = \sum_{n \geq 1} n^{-2} \delta_{x^i} \), so that

\[ \|x^i\|_1^2 = \sum_{n \geq 1} \lambda_n^2 \mu_{x^i}(\{0, \lambda_n\}) = \sum_{n \geq 1} n^{-2} < \infty, \tag{33} \]

yet, \( \|x^i\|_2^2 = \sum_{n \geq 1} n^{-3} \) is not finite. Therefore \( x^i \in X_0 \setminus X_i \). The sup is assumed at \( N = 1 \), so that \( \|x^i\|_2^2 = \zeta(3) \approx (1.096)^2 \).

In [8 proposition 11] the variational inequality

\[ \exists \ \beta \geq 0: \ |\langle x, \omega \rangle| \leq \beta \| (T^*T)^\gamma \omega \|^{\nu/\gamma} \| \omega \|^{1-\nu/\gamma} \quad \forall \ \omega \in X, \tag{34} \]

was shown for \( 0 < \nu < \gamma \) to hold if and only if \( x \in X_\nu \). We prove a more precise statement, in particular including the limiting cases \( \nu = 0 \) and \( \nu = \gamma \). The first part of the proof (the inequality \( \leq \)) simplifies and sharpens the corresponding part of [8 proof of proposition 11]. The second part (the inequality \( \geq \)) draws from [3].

**Proposition 2.4.** Let \( x \in X \) and \( 0 \leq \nu \leq \gamma \). Then

\[ \sup_{\|\omega\|=1} \| (T^*T)^\gamma \omega \|^{\nu/\gamma} |\langle x, \omega \rangle| = N_{\nu,\gamma} \| x \|_{\nu,\gamma}, \tag{35} \]

**Proof.** For \( \nu = 0 \) the statement is trivial due to \( \| x \|_{0,\gamma} = \| x \| \). For \( \nu = \gamma \) the statement follows from [9 lemma 8.21] and \( \| x \|_{\nu,\gamma} = \| x \| \). In both cases, recall \( N_0 = N_1 = 1 \). For the remainder of the proof we assume \( 0 < \nu < \gamma \).

Let \( \omega \in X \) and consider the linear mapping \( S: x \mapsto \langle x, \omega \rangle \). Then \( |Sx| \leq \| \omega \| \| x \| \), and \( |Sx| \leq \| (T^*T)^\gamma \omega \| \| x \| \) for all \( x \in X_\nu \). By the operator interpolation inequality (11) we have \( |\langle x, \omega \rangle| \leq N_{\nu,\gamma} \| x \|_{\nu,\gamma} \| (T^*T)^\gamma \omega \|^{\nu/\gamma} \| \omega \|^{1-\nu/\gamma} \). This implies \( \langle \cdot, \cdot \rangle \) in equation (35).

To verify \( \geq \) in equation (35), it suffices to establish the case \( \gamma = 1 \), then apply it with \( (T^*T)^\gamma \) replacing \( T^*T \) (also in the definitions of the norms in section 2.6). Thus we assume that the variational inequality (34) holds with \( 0 < \nu < \gamma = 1 \). To verify \( N_{\nu,\gamma} \| x \|_{\nu,\gamma} \leq \beta \) we check \( D \leq N_{\nu,\nu}^{-2} \beta \) for the quantity \( D \) from the identity (9). In other words we show the bound \( d(r)^{1-\nu} r^{\nu} \leq N_{\nu,\nu}^{-2} \beta \) for the distance function \( d(\cdot, \cdot) \) from (7). To that end we combine [3 proposition 2.10], [3 theorem 4.1] and [3 theorem 4.5]: The inequality

\[ |\langle x, \omega \rangle| \leq \beta \| (T^*T)^\gamma \omega \|^{\nu/\gamma} \quad \forall \ \omega \in X \quad \text{with} \quad \| \omega \| = 1, \tag{36} \]
with
\[ \kappa := \frac{2\nu}{1+\nu} \quad \text{and} \quad \beta := \frac{2-\kappa}{2} \left(\frac{3-\kappa}{1-\kappa}\right)^{1/2} a^{\frac{1-\kappa}{2}} \quad \text{for some} \ 0 < \beta \leq 1 \quad \text{and} \quad a > 0, \] (37)

implies \( d^2(r) \leq \beta (-\varphi)^\varphi(-2r) \) for all \( r > 0 \), where \( (-\varphi)^\varphi \) is the Legendre–Fenchel conjugate of \( t \mapsto -\varphi(t) = -at^\varphi \) defined for \( t > 0 \). Straightforward but tedious algebra yields the desired estimate \( d(r)^{1-\nu-r\nu} \leq N^\nu \beta \).

3. Application to linear inverse problems in Hilbert spaces

3.1. Linear inverse problem

Let \( X \) and \( Y \) be Hilbert spaces. Let \( T : X \to Y \) be a bounded linear operator, with possibly nonclosed range. We write \( \| \cdot \| \) for the norm of \( X \) and for that of \( Y \). As in section 2.6, we assume that \( T \) is injective. Fix \( y^\dagger \in TX \) and let \( x^\dagger \) denote the solution to
\[ Tx^\dagger = y^\dagger. \] (38)

Let \( x_0 \in X \), called a prior, be given. The task is to find an approximation of \( x^\dagger \), given \( y^\dagger \) (noise-free case) or \( y^\dagger \approx y^\dagger \) (noisy case) with
\[ \| y^\dagger - y^\dagger \| \leq \delta. \] (39)

We use the notation from section 2.6, including the spectral measure \( E \), the Borel measure \( \mu_\alpha \), the spaces \( X_\gamma, X_{\nu,\gamma} \), etc.

3.2. Spectral cut-off regularization

The spectral cut-off regularization of \( x^\dagger \) is defined as \( x_\alpha = x_\alpha + E_{(0,\infty)}(x^\dagger - x_0) \) for a parameter \( \alpha > 0 \). From the definition (18) of \( \| \cdot \|_\nu \) it is immediate that the error of this regularization is
\[ \| (I - E_{(0,\infty)})(x^\dagger - x_0) \| = \| E_{(0,\infty)}(x^\dagger - x_0) \| \leq \alpha \| x^\dagger - x_0 \|_\nu \quad \forall \alpha > 0 \quad \forall \nu \geq 0. \] (40)

Since there are no restrictions on the possible convergence rate \( \nu \geq 0 \) (referred to as in infinite qualification), and no further constants are involved, we may view the performance of this regularization as a reference.

3.3. Tikhonov regularization

For \( \alpha > 0 \), the regularized solution \( x_\alpha \in X \) in the noise-free case is defined as the unique minimizer of the Tikhonov functional
\[ J_\alpha(x; x_0) := \| y^\dagger - Tx \|^2 + \alpha \| x - x_0 \|^2, \quad x \in X. \] (41)

Replacing \( y^\dagger \) by \( y^\delta \) defines the regularized solution \( x_\alpha \in X \) in the noisy case. They are equivalently characterized by the first-order optimality conditions
\[ x_\alpha = (T^*T + \alpha I)^{-1}(T^*y^\dagger + \alpha x_0) \quad \text{and} \quad x_\alpha = (T^*T + \alpha I)^{-1}(T^*y^\delta + \alpha x_0). \] (42)

Writing \( e^\delta := x_\alpha - x_\alpha \), for the moment, we have \( \| Te^\delta \|^2 + \alpha \| e^\delta \|^2 = \langle (T^*T + \alpha I)e^\delta, e^\delta \rangle = \langle T^*(y^\dagger - y^\delta), e^\delta \rangle \leq \| y^\dagger - y^\delta \| \| Te^\delta \| \leq \frac{1}{2} \delta^2 + \| Te^\delta \|^2, \quad \text{and} \]
cancelation of $\| T e^\delta \|^2$ on both ends gives the error splitting

$$
\| x^\delta - x^\alpha \| \leq \| x^\delta - x_0 \| + \| e^\delta \| \leq \| x^\delta - x_\alpha \| + \frac{1}{2} \frac{\delta}{\sqrt{n}}.
$$

(43)

The parameter $\alpha > 0$ is determined by a parameter choice strategy

$$
\tilde{\alpha}: (\delta, y^\delta, \ldots) \mapsto \tilde{\alpha}(\delta, y^\delta, \ldots).
$$

(44)

The one that minimizes $\alpha \mapsto \| x^\delta - x^\alpha \|$ whenever $y^\delta$ and $y^\dagger$ are fixed may be considered the optimal strategy. We shall suppose that the parameter choice strategy satisfies

$$
\delta^{-2\nu} \sup_{(40)} \| x^\dagger - x^\alpha(\delta, y^\delta, \ldots) \|_{2^{\nu+1}} \sim \sup_{\alpha>0} \alpha^{-\nu} \| x^\dagger - x_\alpha \| \quad \forall \delta > 0,
$$

(45)

where the hidden constants do not depend on $\delta$, the exact solution $x^\dagger$, or the prior $x_0$, but may depend on $\nu \geq 0$. Here, $\sup_{(40)}$ means the supremum over all $y^\delta \in Y$ which satisfy the estimate (39). As an example, the a priori parameter choice strategy defined by

$$
\sqrt{\tilde{\alpha}(\delta, y^\dagger)} \| x^\dagger - x_{\tilde{\alpha}(\delta, y^\dagger)} \| = \frac{1}{2} \delta \quad \forall \delta > 0
$$

(46)

satisfies (45). Specifically, the error splitting (43) quickly yields $\text{LHS} \leq \text{RHS}$ in the equivalence (45) and an inspection of [1 proof of theorem 2.6] yields $\frac{1}{2}(5/\sqrt{2})^{2\nu+1} \text{LHS} \geq \text{RHS}$. That proof also shows that the optimal strategy, see above, satisfies the equivalence (45). Of course, of practical interest are parameter choice strategies that do not access the exact data $y^\dagger$ or the exact solution $x^\dagger$; in that regard the notion of quasioptimality by Raus and Hämarik [10] is useful, see comments following proposition 3.1 below. In any case, equation (45) formalizes the equivalence

$$
\| x^\dagger - x^\alpha(\delta, y^\delta, \ldots) \| \lesssim \delta^{-2\nu/(2^\nu+1)} \quad \forall \delta > 0 \quad \Leftrightarrow \quad \| x^\dagger - x_\alpha \| \lesssim \alpha^{-\nu} \quad \forall \alpha > 0
$$

(47)

of the error estimates in the noisy and in the noise-free cases, and in the following we assume the equivalence (45), and focus on its RHS.

From the abstract theory of interpolation, convergence rates of the Tikhonov regularization error $\| x^\dagger - x_\alpha \|$ for $x^\dagger \in X_{\nu-1}$ when $0 < \nu < 1$ quickly follow. Indeed, for $\alpha > 0$ consider the linear mapping $S_{\alpha}: X \to X$, $(x^\dagger - x_0) \mapsto (x^\dagger - x_\alpha)$. From the first-order conditions (42),

$$
S_{\alpha} = \alpha (T^*T + \alpha I)^{-1},
$$

(48)

so that $\| S_{\alpha} \| \leq 1$. Under the classical source condition

$$
x^\dagger \in x_0 + X_1 \quad \text{where} \quad X_1 = \text{range}(T^*T),
$$

(49)

it is known (and shown below in the characterization (52) for $\nu = 1$) that

$$
\| S_{\alpha}(x^\dagger - x_0)\| \leq \alpha \| x^\dagger - x_0 \|,
$$

(50)

The operator interpolation inequality (11) implies

$$
\| x^\dagger - x_\alpha \| = \| S_{\alpha}(x^\dagger - x_0)\| \leq \alpha^\nu N_{\nu} \| x^\dagger - x_0 \|_{\nu-1} \quad \forall x^\dagger \in x_0 + X_{\nu-1} \quad \forall \alpha > 0.
$$

(51)

The following proposition shows that $x_0 + X_{\nu-1} \subset X$ is precisely the (affine) subspace that allows those convergence rates, which is the main observation of this note.
Proposition 3.1. Let $0 \leq \nu \leq 1$. Then
\[ N_{\nu}^{-1} \| x^* - x_0 \|_{\nu,1} \leq \sup_{\alpha > 0} \alpha^{-\nu} \| x^* - x_\alpha \| \leq \| x^* - x_0 \|_{\nu,1}. \tag{52} \]

Equalities hold for $\nu = 0$ and $\nu = 1$. If $\nu = 1$ then $\sup_{\alpha > 0}$ can be replaced by $\lim_{\alpha \downarrow 0}$.

The result is a special case of proposition 3.2 below, and the proof is therefore omitted. Several remarks are in order.

Combining the inequalities (25a)-(25b) and (52), for $0 < \nu < 1$, we have that 
\[ \| x^* - x_\alpha \| \leq C \alpha^{-\nu} \] for all $\alpha > 0$ if and only if $x^* \in x_0 + X_{\alpha}$. In essence, this was already shown in [1 theorem 2.1]. We emphasize, however, that the inequalities (25a) and (52) yield the more precise upper bound
\[ \| x^* - x_\alpha \| \leq \frac{1}{2(1-\nu)} \| x^* - x_0 \| \quad \forall \alpha > 0. \tag{53} \]

In particular, although the rate of convergence of the Tikhonov regularization is at least $\nu$ whenever $x^* \in x_0 + X_{\nu}$, the constant in (53) may deteriorate as $\nu \downarrow 1$ compared to the error (40) of the spectral cut-off regularization. This may be interpreted as a quantitative description of the finite qualification of Tikhonov regularization, that is, its inability to provide convergence larger than $\nu = 1$. Of course, by equation (52), the rate of $\nu = 1$ does hold if (and only if) $x^* \in x_0 + X_{\alpha}$, but this condition is more restrictive than $x^* \in x_0 + X_1$, cf proposition 3.1 and example 2.3.

From equation (52) we see that Tikhonov regularization with a parameter choice rule satisfying the equivalence (45) is order optimal on $M_{\nu,\rho} := \{ x \in X : \| x \|_{\nu,1} \leq \rho \}$ if $T$ has nonclosed range. Indeed, equations (45) and (52) imply the error estimate 
\[ \| x^* - x_\alpha \| \leq C \delta^{\nu/(\nu+1)} \| x^* - x_0 \| \quad \forall \alpha > 0 \] whenever $x^* \in M_{\nu,\rho}$; on the other hand, [11 proposition 3.15] and comments therein show that this estimate is optimal because $M_{\nu,\rho}$ contains the classical source set $\{ x \in X : \| x \|_{\nu} \leq \rho \}$ by inequality (25c). Now we can invoke [10 theorem 2.3] to assert that any parameter choice rule that is strongly quasioptimal in the sense of [10 definition 2.2] will again rise to an order optimal method on $M_{\nu,\rho}$.

The final statement of proposition 3.1 implies the saturation result: if $\nu \| x^* - x_\alpha \| = o(\alpha)$ as $\alpha \downarrow 0$ then $x^* = x_0$. An analogous statement holds in the noisy case for any parameter choice strategy satisfying the equivalence (45).

An additional consequence of equation (52) is that, for $0 \leq \nu \leq \gamma$,
\[ N_{\nu/\gamma}^{-1} \| x^* - x_0 \|_{\nu,\gamma} \leq \sup_{\alpha > 0} \alpha^{-\nu} \| x^* - x_{\alpha,\gamma} \| \leq \| x^* - x_0 \|_{\nu,\gamma} \tag{54} \]
where $x_{\alpha,\gamma}$ is given by
\[ x_{\alpha,\gamma} := \arg\min_{x \in X} \left\{ \| T^* T \|^\nu/2 \| x^* - x \|^2 + \alpha \| x^* - x_0 \|^2 \right\}. \tag{55} \]

This is simply equation (52) for the operator $(T^* T)^{\nu/2}$ instead of $T$. Therefore, letting $x_0 = 0$, the mapping $x^* \mapsto \sup_{\alpha > 0} \alpha^{-\nu} \| x^* - x_{\alpha,\gamma} \|$ defines a norm on $X_{\nu,\gamma}$ which is equivalent to $\| \cdot \|_{\nu,\gamma}$ uniformly in $0 \leq \nu \leq \gamma$.

The variational inequality (34) was used as a starting point in [8 lemmas 2(ii) and 7] for an elementary proof of convergence rates for Tikhonov regularization without invoking spectral theory. This is certainly of interest, but leads to the natural question whether equation (34) itself can be established in particular cases without spectral theory; the characterization (35) in terms of interpolation spaces is a step in that direction. That elementary proof, however, seems to be limited to rates $\nu \leq 1/2$ in the error bound (51), as it uses the variational inequality (34) with $\gamma = 1/2$. 

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As an example consider the following integral operator on square integrable functions \( X := Y := L_2 \) of the unit interval \((-1, 1)\),
\[
T : X \to Y, \quad (Tx)(t) := \int_{-1}^{t} x(s)ds - \frac{1}{2} \int_{-1}^{1} \int_{-1}^{\tau} x(s)dsd\tau.
\]
One can check that \( u := T^*Tx \) solves the boundary value problem \(-uu'' = x, u \big|_{\pm 1} = 0\). Hence, \( X_1 \) from the definition (20) is the Sobolev space \( H^2 \cap H^1_0 \). Let now \( y^i(t) = |t| - 1 \), so that equation (38) holds with \( x^i(s) := \text{sign}(s) \). This means, solving equation (38) with noisy data \( y^i \) amounts to taking the derivative of \( y^i \) perturbed by \( L_2 \) noise. Set \( \nu := 1/4 \). The interpolation space \( X_{\nu, 1} = (X, X_1)_{\nu, 4} \) is the Besov space \( B^{1/2}_{2, \infty} \) [12 propositions 1.25 and 1.30]. In view of [12 definition 1.12 of \( B \)], to check that \( x^i \in X_{\nu, 1} \), it suffices to check that \( L_2 \) step functions are in \( B^{1/2}_{2, \infty}(\mathbb{R}) \), but this is immediate from the intrinsic definition [12 definition 1.1]. On the other hand, [13 chapter 1 theorem 11.7] implies that \( t \mapsto |x(t)|^2/(t^2 - 1) \) is integrable for any \( x \in X_{1/4} \) (denoted by \( H^{1/2}_1 \) in [13]), which is clearly not the case for \( x^i \). In summary, \( x^i \in X_{1/4} \setminus X_{1/4} \), and proposition 3.1 implies the convergence rate \( \nu = 1/4 \), familiar from [2 example 5].

On a more general note, let \( M \) be a connected smooth finite-dimensional Riemannian manifold with bounded geometry (complete, injectivity radius bounded away from zero, bounded covariant derivatives of the curvature tensor). For instance, the flat space \( \mathbb{R}^d \) or any compact manifold without boundary has bounded geometry. On \( M \), Sobolev \( H^s \) and Besov \( B^s_{r, \infty} \) spaces can be defined intrinsically through local means [14 § 1.11.4], and by [14 § 1.11.3] one has \( (H^s, H^{s+k})_{r, \infty} = B^{s+k}_{r, \infty} \). Thus, for \( T^*T \) with domain \( X = H^s \) and range \( X_0 = H^{s+2m} \), \( m > 0 \), we have \( X_{\nu, 1} = B^{1/2}_{2, \infty} \), so that the convergence rate \( \nu \) in the error bound (51) is characterized in terms of Besov smoothness of \( x^i \). This covers a wide class of operators \( T \), for instance bijective elliptic pseudo-differential operators \( H^s \to H^{s+m} \) of order \(-m\) such as \((I - \Delta)^{-m/2}\).

### 3.4. Stationary iterated Tikhonov regularization

Fixing \( \alpha > 0 \), for each integer \( k \geq 1 \), define \( x_{\alpha, k} \) as the minimizer of the Tikhonov functional \( x \mapsto J_\alpha(x; x_{\alpha,k-1}) \), where \( J_\alpha \) is defined in (41) and \( x_{\alpha, 0} := x_0 \). Thus, each iterate is used as a prior for the next one with the same (stationary) choice of the regularization parameter \( \alpha > 0 \).

Then
\[
x^i - x_{\alpha, k} = S_{\alpha} \left( x^i - x_0 \right) \quad \forall \ k \geq 1,
\]
with \( S_{\alpha} \) from equation (48). Since \( S_{\alpha} \) commutes with powers of \((T^*T)\), we obtain from the estimate (50) the shift estimate \( \| S_{\alpha} (x^i - x_0) \|_{k-\ell} \leq \alpha \| x^i - x_0 \|_{k-\ell} \), and repeating this argument for each iteration \( \ell = 1, 2, \ldots, k \), of the \( k \)-fold Tikhonov regularization then \( \| S_{\alpha} (x^i - x_{\alpha, k}) \| \leq \alpha^k \| x^i - x_0 \|_k \). The operator interpolation inequality readily yields the convergence rate \( \alpha^\nu \) for \( x^i \in X_0 + X_{\nu, k} \) and \( 0 \leq \nu \leq k \). The following extension of proposition 3.1 is more precise.

**Proposition 3.2.** Let \( k \geq 1 \) be an integer, let \( 0 \leq \nu \leq k \) be real. Then
\[
N_{\nu/k}^{1-2k} \| x^i - x_0 \|_{\nu, k} \leq \sup_{\alpha > 0} \alpha^{-\nu} \| x^i - x_{\alpha, k} \| \leq \| x^i - x_0 \|_{\nu, k}.
\]

If \( \nu = k \) then \( \sup_{\alpha > 0} \) can be replaced by \( \lim_{\alpha \to 0} \).
Proof. From equations (57) and (48),
\[ \alpha^{-2\nu} \| x^l - x_{n,k} \|^2 = \alpha^{-2\nu} \int_{(0,\infty)} \left( \frac{\alpha}{\alpha + \lambda} \right)^{2k} d\mu_{x^l - x_0}(\lambda). \] (59)

The first inequality in (58) is therefore a consequence of the inequality (6) with \( \theta = \nu/k \), and the representation (23) of the interpolation norm. The second inequality in (58) follows by identifying \( \alpha = t^{1/k} \), estimating \( 1/(\alpha + \lambda)^{2k} \leq 1/(t^{2} + \lambda^{2k}) \), and invoking the representation (23). Finally, if \( \nu = k \), then \( \sup_{\alpha > 0} (59) = \lim_{\alpha \searrow 0} (59) \) by the Lebesgue monotone convergence theorem.

Remarks analogous to those following proposition 3.1 apply. For instance, the \( k \)-fold Tikhonov regularization saturates: If \( \| x^l - x_{n,k} \| = o(\alpha^2) \) as \( \alpha \searrow 0 \) then \( x^l = x_0 \).

3.5. Landweber iteration

Fix \( \sigma > 0 \) with
\[ 0 < \sigma \| T^* T \| \leq 1. \] (60)

The Landweber iterates \( x_k \) are defined by
\[ x_k := x_{k-1} + \sigma T^* (y^l - T x_{k-1}), \quad k \in \mathbb{N}, \quad \text{with a given} \quad x_0 \in X, \] (61)
and \( x_k^\delta \) by the same iteration with \( y^l \) replaced by \( y^\delta \) in the noisy case. One motivation for this iteration is that \( x^\delta \) is a fixed point. By induction one finds
\[ x^l - x_k = (I - \sigma T^* T)(x^l - x_{k-1}) = (I - \sigma T^* T)^k(x^l - x_0) \quad \forall \ k \in \mathbb{N}, \] (62)
and similarly the residual representation
\[ y^l - T x_k = (I - T T^* T)^k(y^l - T x_0) \quad \text{and} \quad y^\delta - T x_k^\delta = (I - T T^* T)^k(y^\delta - T x_0). \] (63)

The condition (60) therefore guarantees nondivergence of the iterates.

For the noisy case, an error splitting analogous to (43) is true [11 lemma 6.2]:
\[ \| x^l - x_k^\delta \| \leq \| x^l - x_k \| + \delta \sqrt{k} \quad \forall \ k \geq 0, \] (64)
thus one often ‘morally’ identifies \( k \) with \( 1/\alpha \). We call a mapping
\[ k: (\delta, y^\delta, ...) \mapsto k(\delta, y^\delta, ...) \] (65)
a stopping rule, and in analogy to (44)–(45) we shall suppose that it satisfies
\[ \delta^{-2\nu} \sup_{(40)} \| x^l - x_k^\delta(\delta, y^\delta, ...) \|^{2\nu+1} \sim \sup_{k \geq 0} (1 + k/\nu) \| x^l - x_k \| \quad \forall \ \delta > 0, \] (66)
where the hidden constants do not depend on \( \delta \), \( x^l \), or \( x_0 \), but may depend on \( \nu \geq 0 \). The factor \( (1 + k/\nu) \) is motivated by proposition 3.4. The stopping rule \( k \) may be based on the knowledge of some of the iterates \( x_k \). For example it may be the smallest \( k \geq 0 \) for which the discrepancy principle
\[ \| y^\delta - T x_k^\delta \| \leq \delta \tau \] (67)
is satisfied with some fixed threshold \( \tau > 1 \), which in particular is not allowed to depend on \( x^l \). We will comment on this stopping rule at the end of this section.

We shall show that convergence rates of the Landweber iteration (61) in the noise-free case characterize the spaces \( X_\nu \). Before formalizing this, we need a lemma.
Lemma 3.3. The constant
\[ c_2 := \sup_{a > 0} \sup_{s > 0} \sqrt{s} I(s, a), \quad I(s, a) := \left( \frac{s + a}{a} \right)^{s-1} \int_0^1 s(1 - t)^{s-1} \, dt, \] (68)
is finite with \( c_2 \approx 1.135. \)

Proof. First, \( I(s, a) \) is well-defined for all real \( s > 0 \) and \( a > 0 \). The integral, known as the beta function [15 section 6.2], evaluates to \( \Gamma(s + 1) \Gamma(a + 1) / \Gamma(s + a + 1) \). One has \( \lim_{s \to \infty} I(s, a) = 1 \) for all \( a > 0 \). Moreover, the function \( s \mapsto I(s, a) \) is increasing for \( 0 \leq a \leq 1 \) and decreasing for \( 1 \leq a \). So we need to consider the pointwise limit \( I(s, a) \) as \( s \to \infty \) for \( 0 \leq a \leq 1 \). Stirling’s formula [15 section 6.1.38] for \( \Gamma(\cdot) \) yields
\[ \tilde{I}(a) := \lim_{s \to \infty} I(s, a) = a^{-a} \Gamma(a + 1) \quad \forall \, a > 0, \] (69)
which clearly is bounded in \( 0 \leq a \leq 1 \). We find numerically that \( \tilde{I} \) is maximized at \( a^\ast \approx 0.3164 \) with \( \tilde{I}(a^\ast) \approx 1.288 \). The constant \( c_2 \) is the square root of the latter. \( \square \)

We can now state the announced rate characterization for the Landweber iteration.

Proposition 3.4. Let \( \{x_k\}_{k \geq 0} \) be the Landweber iterates generated by equation (61) with step size \( \sigma > 0 \) as in (60). Let \( \nu > 0 \) and \( r \geq 0 \). Set
\[ \Delta'_\nu(x^t - x_0) := \sup_{k \geq 0} \| \varepsilon_k^{(r+\nu)} \| (T^*T)'(x^t - x_k) \| \quad \text{where} \quad \varepsilon_k := \frac{1}{\sigma} \frac{r + \nu}{k + r + \nu}. \] (70)

Then,
\[ c_1 \sqrt{\frac{r + \nu}{k + r + \nu}} \| x^t - x_0 \| \leq \Delta'_\nu(x^t - x_0) \leq c_2 \| x^t - x_0 \|, \] (71)
where \( c_2 \) is defined in (68) and
\[ c_1 := \inf_{k \geq 0} \left( (1 - \sigma \varepsilon_k)^k \varepsilon_k^{(1/k)} \right) \geq (\sigma \varepsilon_k^{1/k})^{r+\nu}. \] (72)

Proof. Fix \( x^t \in X \). We abbreviate \( \mu := \mu_{x^t - x_0} \) and \( \Delta := \Delta'_\nu(x^t - x_0) \). The quantity \( \varepsilon \) has been defined such as to satisfy
\[ \varepsilon_k = \arg \max_{\lambda > 0} f(\lambda) \quad \text{with} \quad f(\lambda) := \lambda^{2(r+\nu)}(1 - \sigma \lambda)^{2k} \quad \forall \, k \geq 0. \] (73)

To enhance readability we will assume for the remainder of the proof that \( x_0 = 0 \) and \( \sigma = 1 \). For the first inequality of the claim (71), define \( \mu^{2r}(\Lambda) := \int_{\Lambda} \lambda^{2r} \, d\mu(\lambda) \) on Borel subsets \( A \subset [0, \infty) \). Given \( 0 \leq \Lambda \leq \| T^*T \| \), determine the integer \( k \geq 0 \) by \( \varepsilon_{k+1} < \lambda \leq \varepsilon_k \). Then
\[ \mu^{2r}([0, \Lambda]) \leq \mu^{2r}([0, \varepsilon_k]) \leq (1 - \varepsilon_k)^{-2k} \int_{[0, \varepsilon_k]} \lambda^{2r}(1 - \lambda)^{2k} \, d\mu(\lambda) \] (74)
\[ \leq (1 - \varepsilon_k)^{-2k} \varepsilon_k^{2(r+\nu)} \Delta^2 \quad \text{(by definition of} \, \Delta) \] (75)
\[
\leq (1 - \varepsilon_k)^{-2k} \left[ \frac{\varepsilon_k}{\varepsilon_{k+1}} \right]^{2(r+\nu)} \Lambda^{2(r+\nu)} \Delta^2.
\]  
(76)

\[
\leq c_1^{-2} \Lambda^{2(r+\nu)} \Delta^2.
\]  
(77)

Since \( \Lambda > 0 \) was arbitrary, we obtain the bound

\[
\|\mu^{2r}\|_{r+\nu} \leq c_1^{-1} \Delta.
\]  
(78)

The first inequality of the claim (71) now follows from:

\[
\|\mu\|_\nu^2 = \sup_{r>0} r^{-2\nu} \int_{(0,r)} \lambda^{-2\nu} d\mu^2(\lambda) \leq \frac{(r+\nu)}{r} \|\mu^{2r}\|_\nu^2 \leq \frac{r+\nu}{r} c_1^{-2} \Delta^2.
\]

We now prove the second inequality in the claim (71). In view of equation (60), the measure \( \mu \) is supported on \([0, 1/\sigma]\), and recall that we have assumed \( \sigma = 1 \). So consider

\[
\varepsilon_k^{-2(r+\nu)} \left\| (T^*T)^k \left( x^* - x_k \right) \right\|^2 = \varepsilon_k^{-2(r+\nu)} \int_{[0,1]} \lambda^{2\nu} \left( 1 - \lambda \right)^{2k} d\mu(\lambda).
\]  
(79)

In the case \( k = 0 \), this is majorized by \( \mu([0, 1]) \leq \|\mu\|_\nu^2 \). Otherwise, defining \( I(\lambda) = \mu([0, \lambda)) \) and integrating by parts as in the proof of lemma 2.1, we estimate

\[
= \varepsilon_k^{-2(r+\nu)} \int_{[0,1]} \left( \lambda^{2\nu} \left( 1 - \lambda \right)^{2k} \right) I(\lambda) d\lambda
\]  
(80)

\[
\leq \varepsilon_k^{-2(r+\nu)} \int_{[0,1]} 2k \lambda^{2\nu} \left( 1 - \lambda \right)^{2k-1} I(\lambda) d\lambda.
\]  
(81)

Estimating \( I(\lambda) \leq \lambda^{2\nu} \|\alpha\|_\nu^2 \), and employing lemma 3.3 with \( s = 2k \) and \( a = 2(r+\nu) \), the second inequality in the claim (71) is obtained. This completes the proof of the error estimate (71).

Unfortunately, the gap between our ‘constants’ in the error estimate (71) is not robust in \( \nu \). For example, if \( r = 0 \), we have \( c_2/c_1 \sim e^\nu \) as \( \nu \to \infty \). Qualitatively though, the error estimate (71) means that it is necessary and sufficient for the asymptotic convergence rate \( \nu \) that the data be in \( X_\nu \), which explains the notoriously slow convergence of the Landweber iteration. In the limit \( \nu \to \infty \), the upper estimate in (71) yields the asymptotics

\[
\left\| \left( T^*T \right)^k \left( x^* - x_k \right) \right\| \leq c_2 e^{-k} \left( 1 + \frac{1}{2} k^2 \nu^{-1} + O(\nu^{-2}) \right) \| x^* - x_0 \|_\nu \quad \text{as} \quad \nu \to \infty,
\]  
(82)

for any fixed iteration \( k \geq 0 \). This bound indicates that, for a fixed iteration \( k \), only a limited amount of smoothness can be exploited (cf [16 section 3 example 3]).

From [2 theorem 1 and corollary 1], an estimate similar to the error estimate (71) with \( r = 0 \) can be obtained if we assume \( d(r) \sim r^{-\nu/(1-\nu)} \) for sufficiently large \( r \) > 0 for the distance function (see section 2.3).

We believe it is natural for \( \| \cdot \|_\nu \) to appear in the error estimate (71) rather than the interpolation norm \( \| \cdot \|_{\nu,\gamma} \). The following example illustrates this in the limiting case \( \nu = \gamma \). As in the \( k \)-fold Tikhonov regularization, however, it might be possible and meaningful to relate the error of the \( k \)th iterate to the \( \| \cdot \|_{\nu,k} \) norm of the data.

**Example 3.5.** Let \( T \) and \( x^* \) be as in example 2.3. Recall that \( \| x^* \|_1 = \infty \), while \( \| x^* \|_\nu \approx 1.096 \). We will estimate \( \Delta_0^\nu(x^*) \) for \( \nu = 1 \). Since \( \| T \| = 1 \), we set \( \sigma = 1 \) to satisfy (60). Then, for any integer \( k \geq 0 \),
\[ \varepsilon_k^2 \| x^k - x_k \|^2 = \varepsilon_k^2 \int_{(0,\infty)} (1 - \lambda)^{2k} d\mu_0(\lambda) = (k + 1)^2 \sum_{n \geq 1} n^{-3} (1 - n^{-1})^{2k}. \] (83)

Splitting the sum at \( n = k \), and exploiting in the finite subsum the fact that \( t \mapsto t^{-3} (1 - t^{-1})^{2k} \) peaks at \( t = 1 + \frac{2}{3} k \) we have

\[ \sum_{n \geq 1} (\cdots) \leq \sum_{n \leq k} \left( 1 + \frac{2}{3} k \right)^{-3} + \sum_{n > k} n^{-3} \leq C(k + 1)^{-2} \quad \forall \ k \geq 1 \] (84)

for some \( C \geq 0 \) independent of \( k \), meaning that the expression (83) is bounded uniformly in \( k \geq 0 \). We find numerically that \( |\Delta_0^n(x^t)|^2 = (83)_{k=0} \approx (1.096)^2 \), \( (83)_{k=1} \approx (0.5453)^2 \), \( (83)_{k=2} \approx (0.5475)^2 \), and (83) is decreasing for \( k \geq 2 \) with \( \lim_{k \to \infty} (83) = (1/2)^2 \). This limit can also be verified using equation (69). The infimum in \( c_1 \) is achieved at \( k = 2 \), so \( c_1 = 1/3 \).

Thus estimate (71) reads \( 1/3 \times 1.096 \leq 1.096 \leq 1.135 \times 1.096 \). If we use any of the iterates \( (n^{-3/2}(1 - n^{-1})^k)_{k \geq 1} \) as the initial guess \( x_0 \), then estimate (71) reads \( 1/3 \times 1/\sqrt{2} \leq 1/2 \leq 1.135 \times 1/\sqrt{2} \).

Our primary motivation for introducing \( r \geq 0 \) in the error measure (70) was to estimate the number of iterations needed for the stopping rule based on the discrepancy principle (67). We briefly comment on this. Fix \( \tau > 1 \). In view of the residual representation (63) and the assumption (60) we have

\[ \| y^k - Tx_k - y^k - Tx_k \| \leq \| y^k - Tx_k \| = \delta (85) \]

for each iteration \( k \geq 0 \). Hence, if \( \| y^k - Tx_k \| \leq \delta (\tau - 1) \) then the stopping criterion (67) is satisfied. Using estimate (71) with \( r = 1/2 \), this is fulfilled for any \( k \geq 0 \) such that \( \delta (\tau - 1) \leq \varepsilon_k^{e+1/2} c_2 \| x^k - x_0 \| \). This implies at most \( k^* \sim \delta^{-2/((2r+1))} \) iterations until the stopping criterion (67) is met, as in [11:theorem 6.5] but with the slightly weaker assumption that the data is in \( X_0 \) rather than in \( X_\alpha \). To reproduce the conclusion of [11:theorem 6.5], we still need to verify that the discrepancy principle (67) implies the error rate \( \| y^k - x^k \| \leq \delta^{2r/(2r+1)} \).

Here we cannot use the inequality [11 (4.66)],

\[ \| y^k - x_k \| \leq \| y^k - x_0 \|^{1/2r+1} \| y^k - Tx_k \|^{2r/(2r+1)} \] (86)

as in [11 proof of theorem 6.5] for the corresponding error rate with data in \( X_\alpha \). Instead, we estimate

\[ \| x^k - x_k \|^2 = \int_{(0,\infty)} (1 - \sigma \lambda)^{2k} d\mu(\lambda) \] (87)

\[ \leq \int_{(0,\infty)} \sigma^{-1} \int_{(r,\infty)} \lambda (1 - \sigma \lambda)^{2k} d\mu(\lambda) \] (88)

\[ \leq \varepsilon_k^2 \| x^k \|^2 + \varepsilon_k^{-1} \| y^k - Tx_k \|^2 \leq \delta^{2r/(2r+1)} \] (89)

and using this in the error splitting (64) yields \( \| y^k - x_k \| \leq \delta^{2r/(2r+1)} \).

4. Conclusions

We have introduced and investigated families of Banach spaces and their interrelations: the Hilbert scale \( X_\alpha \), the interpolation spaces \( X_{r/2} \), and the spaces \( X_\alpha = X_{r/2} \). We have shown that the interpolation spaces \( X_{r/3} \) are most adequate for the characterization of convergence rates in (iterated) Tikhonov regularization, while \( X_\alpha \) are better suited for the Landweber iteration. This insight should facilitate the verification of convergence rates. For instance, in
many situations, $X_{\gamma}$ can be understood as (a subspace of) the Besov space $B^1_{2,\infty}$, for which many characterizations are known.

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