Exact Critical Exponents for Pseudo-Particles in the Kondo Problem

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Abstract

Exact critical exponents of the Green functions for pseudo-fermions and slave bosons in the SU(N) Anderson model with \( U \to \infty \) are obtained by using the Bethe ansatz solution and boundary conformal field theory. They are evaluated exactly for mixed valence systems and Kondo systems with crystalline fields. The results agree with the prediction of Menge and Müller-Hartmann, which coincide with those of the X-ray problem. Some implication of our results in one-dimensional chiral systems is also discussed.
Some ten years ago, the slave boson method was proposed as a tool to deal with the strong correlation problem for interacting electron systems \cite{1,2}. The essence of the method is that in the presence of the strong Coulomb interaction, the original Fock space of electrons can be mapped to that of pseudo-particles, so-called the slave boson, $b$, and the pseudo-fermion, $f_m$ ($m = 1, 2, \cdots, N$), under the constraint for the conservation of particles, $b^\dagger b + \sum_m f_m^\dagger f_m = 1$. More recently, Menge and Müller-Hartmann studied low-energy dynamical properties of these pseudo-particles for the impurity Anderson model with infinite Coulomb interaction by using the bosonization method \cite{3}. In the simplest case with degeneracy $N = 1$, they obtained the critical exponents of the Green functions which coincide with those found in the X-ray problem \cite{4}. They also predicted the formula of the exponents for general SU($N$) cases, and the result for the $N = 2$ case was recently confirmed numerically by the renormalization group method \cite{5}. It is interesting to study the critical behavior of pseudo-particles exactly, because we can directly see the infrared catastrophe inherent in the Kondo problem through their asymptotic behavior.

In this paper, we obtain the exact critical exponents for pseudo-particles of the $U \to \infty$ SU($N$) Anderson model by combining the Bethe ansatz solution \cite{6} with the finite-size scaling in boundary conformal field theory \cite{7}. The form of the exponents we shall derive is in agreement with that obtained by Menge and Müller-Hartmann \cite{3}. We evaluate them explicitly for mixed-valence and Kondo regimes. In the previous paper, we pointed out that one can read anomalous exponents related to the X-ray edge singularity from the finite-size spectrum of the Anderson model, by using boundary conformal field theory \cite{8} (see also ref. \cite{9}). We apply this idea to the present problem, obtaining the exact critical exponents.

The Hamiltonian we consider is the $U \to \infty$ SU($N$) Anderson model,

\begin{equation}
H = \sum_{m=1}^{N} \int dx c_m^\dagger(x) \left( -i \frac{\partial}{\partial x} \right) c_m(x) + \epsilon_f \sum_{m=1}^{N} f_m^\dagger f_m \\
+ V \sum_{m=1}^{N} \int dx \delta(x) \left( f_m^\dagger b c_m(x) + c_m^\dagger(x)b^\dagger f_m \right),
\end{equation}

with the constraint $b^\dagger b + \sum_m f_m^\dagger f_m = 1$, where the impurity electron states have $N$-fold degeneracy. The Hamiltonian \cite{1} has been reduced to the one-dimensional one by using
partial-wave representation, and the spectrum for conduction electrons has been linearized around the Fermi energy. The pseudo-particle operators \( b \) and \( f_m \) \((m = 1, 2, \ldots, N)\) represent the unoccupied and single occupied states at the impurity site, and the double occupancy is strictly forbidden due to the infinite Coulomb repulsion.

The exact finite-size spectrum of the model \([8]\) was obtained from the Bethe-ansatz solution \([9]\). We briefly summarize the results necessary for the present discussions. The effective two-body \( S \)-matrix for host electrons which satisfies the Yang-Baxter equation is

\[
S(k_i - k_j) = \frac{k_i - k_j - iV^2P_{\alpha\beta}}{k_i - k_j - iV^2}.
\]

(2)

where \( P_{\alpha\beta} \) is a permutation operator of two coordinates \( x_\alpha \) and \( x_\beta \), and the effect of the impurity is incorporated through the phase shift due to the impurity, \( \exp(2i\phi(k_j)) \) with \( \phi(k) = \tan^{-1}(2(k - \epsilon_f)/V^2) \). Under periodic boundary conditions, one can then diagonalize the many-body \( S \)-matrix, and obtain the Bethe ansatz equations \([10]\). The finite-size excitation spectrum computed by the standard technique \([10, 13]\) is written in terms of the \( N \times N \) matrix \([8]\),

\[
\frac{1}{L}E_1 = \frac{2\pi v}{L} \frac{1}{2} \Delta M^T C_f \Delta M - \frac{\pi v}{L} N \frac{\delta F}{\pi} (\Delta M)^2,
\]

(3)

where \( \Delta M^{(l)} \equiv \Delta M^{(l)}_h = \frac{-\delta F}{\pi} (N - l) \) for \( 1 \leq l \leq N - 1 \), and \( \Delta M^{(0)} = \Delta N_h - N\delta F/\pi \) with \( \delta F \) being the phase shift at the Fermi level. Here \( \Delta N_h \) is the number of charge excitations, \( \Delta M^{(l)}_h \)'s are quantum numbers related to spin degrees of freedom, and the \( N \times N \) matrix \( C_f \) is given by

\[
C_f = \begin{pmatrix}
1 & -1 & 0 \\
-1 & 2 & \ddots \\
& \ddots & \ddots & -1 \\
0 & -1 & 2
\end{pmatrix}.
\]

(4)

One can check that the last term in eq.(3), which has been evaluated from the excited states, is equal to the shift of the ground-state energy due to the presence of the impurity \([14]\). Therefore the increment of the ground state energy cancels the last term of eq.(3),
which is thus irrelevant for the discussions of critical exponents and will be dropped in the following.

We wish to generalize the above results to more general cases with magnetic fields and crystalline fields. In these cases, the phase shifts \( \delta_l \) depend on the total angular momentum, \( l \). The first-order energy corrections in \( 1/L \) due to external fields read

\[
E^{(1)} = \sum_{m=0}^{N-1} \frac{\delta_l \Delta N_l}{\pi L},
\]

where \( \Delta N_l \) is the number of added conduction electrons with total angular momentum \( l \). This term together with the \( 1/L \)-corrections to the finite-size spectrum of host electrons gives the total finite-size spectrum, which is given by eq.\((3)\) with the quantum numbers replaced by, \( \Delta M^{(0)} \rightarrow \Delta N_h - \sum_{l=0}^{N-1} \delta_l / \pi \), and \( \Delta M^{(l)} \rightarrow \Delta M_h^{(l)} - \sum_{i=l}^{N-1} \delta_i / \pi \) for \( 1 \leq l \leq N - 1 \). These formulae are most general and are useful to discuss the case with a crystalline field.

We are now ready to study the long-time behavior of the Green functions for pseudo-particles; \( \langle f^\dagger_m(t) f_m(0) \rangle \sim t^{-\alpha_f} \), and \( \langle b^\dagger(t) b(0) \rangle \sim t^{-\alpha_b} \). Applying the finite-size scaling we can determine the critical exponents of correlation functions. As discussed previously \[8\], we can neglect the phase shifts, if we are concerned with canonical exponents characterizing the local Fermi liquid, because the phase-shift effect is equivalent to imposing twisted boundary conditions, and such effect can be incorporated into the redefinition of the charge quantum number \[8\]. In order to derive critical exponents for pseudo-particles, however, we must regard the fractional number of \( f \)-electrons, \( n_l = \delta_l / \pi \), as quantum numbers, and thus the phase shifts play an essential role to determine the critical exponents. For instance, to obtain the Green function of pseudo-fermions, we set the quantum numbers as \( \Delta N_h = 1 \) and \( \Delta M_h^{(l)} = 0 \). We then read off the corresponding critical exponent,

\[
\alpha_f = 1 - \frac{2\delta_F}{\pi} + N \left( \frac{\delta_F}{\pi} \right)^2,
\]

in the absence of crystalline fields and magnetic fields. The exponent for slave-boson Green function, \( \alpha_b \), is obtained in a similar manner. Since the slave boson expresses a vacancy, it carries neither charge nor spin. Thus by taking \( \Delta N_h = \Delta M_h^{(l)} = 0 \), one has
\[ \alpha_b = N \left( \frac{\delta_F}{\pi} \right)^2. \]  

(7)

The formulae for \( \alpha_f \) and \( \alpha_b \) agree with those predicted by Menge and Müller-Hartmann by means of bosonization [3], and take the same form as those in the X-ray problem: the exponent of pseudo-fermion is equal to the X-ray absorption exponent, and that of slave boson to the X-ray photoemission exponent. We have evaluated the critical exponents \( \alpha_{f,b} \) exactly for the SU\((N)\) model with total angular momentum \( J = 1/2, 3/2, 5/2, 7/2 \) \((N = 2J + 1)\), which have been shown as a function of the renormalized energy level of \( f \)-electrons, \( \epsilon_f^* \), in figs.1 and 2. Note that \( J = 5/2 \) case \((J = 7/2 \) case\) corresponds to Ce (Yb) impurities in a metal.

We next discuss the crystal-field effects on the critical exponents. The effect of crystalline fields are particularly important for Kondo systems where \( \epsilon_f \) lies far below the Fermi level. As such an example, we consider the SU\((6)\) Coqblin-Schrieffer model \((\epsilon_f \to -\infty \) limit\) with cubic symmetry, which is considered as an appropriate model for Ce impurities in a metal [6,15]. In this model, the charge fluctuation is completely suppressed, and six-fold degenerate states split into a \( \Gamma_7 \) doublet and a \( \Gamma_8 \) quartet in a cubic crystalline field. For the case of the \( \Gamma_7 \)-ground state, we set the phase shifts as \( \delta_0 = \delta_1 = \delta_2 = \delta_4 = \delta_5 \equiv \delta_{\Gamma_7} \). Thus from the finite-size spectrum with a crystalline field, we obtain the exponent of pseudo-fermions of the \( \Gamma_7 \) doublet by taking \( \Delta N_h = 1, \Delta M^{(l)}_h = 0 \),

\[ \alpha_f^7 = \frac{5}{6} - \frac{4}{3} (n_{\Gamma_7} - n_{\Gamma_8}) + \frac{4}{3} (n_{\Gamma_7} - n_{\Gamma_8})^2, \]  

(8)

where \( n_{\Gamma_7} = \delta_{\Gamma_7}/\pi \), and \( n_{\Gamma_8} = \delta_{\Gamma_8}/\pi \). In the case of the \( \Gamma_8 \) ground state, we set \( \delta_0 = \delta_1 = \delta_2 = \delta_3 = \delta_4 = \delta_5 = \delta_{\Gamma_8} \). Then taking \( \Delta N_h = 1, \Delta M^{(l)}_h = 0 \), we obtain the exponent of pseudo-fermion of the \( \Gamma_8 \) quartet,

\[ \alpha_f^8 = \frac{5}{6} + \frac{2}{3} (n_{\Gamma_7} - n_{\Gamma_8}) + \frac{4}{3} (n_{\Gamma_7} - n_{\Gamma_8})^2. \]  

(9)

On the other hand, for the case of slave boson, by taking \( \Delta N_h = 0, \Delta M^{(l)}_h = 0 \), one gets

\[ \alpha_b = \frac{1}{6} + \frac{4}{3} (n_{\Gamma_7} - n_{\Gamma_8})^2. \]  

(10)
We computed the critical exponents $\alpha_{7,8}$ and $\alpha_b$ by the exact solution, and plotted them in fig.3 as a function of the crystal-field splitting, $\Delta \epsilon = \epsilon_{\Gamma_8} - \epsilon_{\Gamma_7}$.

Finally, we comment on some implication of the above results in one-dimensional (1D) chiral systems. The Fermi-edge singularity problem in 1D electron systems has attracted current interest [16–18]. If host electrons move only in one direction and the backward scattering due to the impurity is irrelevant, the results obtained above are applicable to 1D systems with a slight modification. Such a situation may be realized in the edge state of the quantum Hall effect [19]. Thus the following results may be the case for the Fermi-edge singularity in the fractional quantum Hall effect. In 1D Luttinger liquids, the Luttinger parameter (charge correlation exponent), $K_\rho$, appears in the finite-size spectrum of the charge sector. Thus the spectrum is given by eq.(3) with $C_f$ replaced by

$$
C_f = \begin{pmatrix} 
\frac{1}{NK_\rho} + \frac{N-1}{N} & -1 & 0 \\
-1 & 2 & \ddots \\
& \ddots & \ddots & -1 \\
0 & -1 & 2
\end{pmatrix}.
$$

(11)

Then the critical exponent for the X-ray absorption in this system is

$$
\alpha_f = \frac{1}{NK_\rho} \left(1 - \frac{N\delta}{\pi}\right)^2 + \frac{N-1}{N},
$$

(12)

and that for the photoemission is

$$
\alpha_b = \frac{N}{K_\rho} \left(\frac{\delta}{\pi}\right)^2.
$$

(13)

Note that for the edge state of the fractional quantum Hall effect with filling $\nu = N/(Nm+1)$ ($m$ even), $K_\rho$ is solely determined by the filling factor $\nu$ as $K_\rho = \nu/N$. We expect such anomalous exponents may be observed in the X-ray problem in the edge state of the fractional quantum Hall effect.

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FIGURES

FIG. 1. Critical exponent for pseudo-fermion $\alpha_f$ as a function of $\epsilon_f^*/2\Delta$ for the SU$(2J+1)$ Anderson model. The resonance width is $\Delta = V^2/2$.

FIG. 2. Critical exponent for slave boson $\alpha_b$ as a function of $\epsilon_f^*/2\Delta$.

FIG. 3. Critical exponents $\alpha_f$ for $\Gamma_7$ (broken line) and $\Gamma_8$ (solid line), and $\alpha_b$ as a function of the crystal-field splitting $\Delta \epsilon/T_0$ ($\Delta \epsilon = \epsilon_{\Gamma_7} - \epsilon_{\Gamma_8}$) for the SU$(6)$ Coqblin-Schrieffer model with cubic symmetry. Here $T_0$ is the Kondo temperature for $\Delta \epsilon = 0$. 