Deciding separability with a fixed error

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Abstract

We give a short proof of the cross norm characterization of separability due to O. Rudolph and show how its computation, for a fixed chosen error, can be reduced to a linear programming problem whose dimension grows polynomially with the inverse of the error.

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1 Introduction

Entanglement plays a key role in many of the most interesting applications of quantum computation and quantum information [1]. However, there is still no procedure to efficiently distinguish separable and entangled states. There are two main analytical characterizations of separability: the first one [2] uses positive maps and the other [3] uses tensor norms. The problem with these characterizations is that they are not easy to compute. This is the reason that they are associated (by relaxing some conditions) to a number of related computable necessary criteria of separability, such as the PPT criterion [4] or the CCN criterion [5], [6], [7]. For a review see, e.g., [8]. However, up to date there is no computable characterization of separability. In this letter we work in this direction, by dealing with the problem of making computable the separability characterization of [3]. For two recent algorithmical approaches to the separability characterization based on positive maps (only for bipartite systems) we refer to [9] and [10].

To begin, let us recall some basic facts. A multipartite state can be seen as a positive operator on a tensor product of finite dimensional Hilbert spaces $H_1 \otimes \cdots \otimes H_k$ with trace one. We are going to call $n_j = \dim(H_j)$ and

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\[ n = \max\{n_j\}. \] Once we have fixed an orthonormal basis in each \( H_j \), we can see it as \( \ell^2_n(C^n) \) with the euclidean norm \( \| \cdot \|_2 \). A multipartite state \( \rho \) is said to be separable if it can be prepared in a “classical” way, that is, if it can be written as a convex combination \( \rho = \sum_{i=1}^n \omega_i \rho_i^1 \otimes \cdots \otimes \rho_i^n \), with \( \rho_i^j \) being a positive operator on \( H_j \) with trace one.

In the sequel we are going to exploit some basic facts of tensor norms that we briefly recall here (for more information we refer to \([11]\)):

If \( X_1, \ldots, X_k \) are finite dimensional (real or complex) normed spaces, by \( \bigotimes_{j=1,\pi} X_j \) we denote the algebraic tensor product \( \bigotimes_{j=1}^k X_j \) endowed with the projective norm

\[
\pi(u) := \inf \left\{ \sum_{i=1}^m \| u_i^1 \| \cdots \| u_i^k \| : u = \sum_{i=1}^m u_i^1 \otimes \cdots \otimes u_i^k \right\}.
\]

This tensor norm is both commutative and associative, in the sense that \( \bigotimes_{j=1,\pi} X_j = \bigotimes_{j=1,\pi} X_{\sigma(j)} \) for any permutation of the indices \( \sigma \) and that \( \bigotimes_{j=1,\pi} \left( \bigotimes_{j=1,\pi} X_{ij} \right) = \bigotimes_{j=1,\pi} X_{ij} \). The projective norm \( \pi \) is in duality with the injective norm \( \epsilon \), defined on \( \bigotimes_{j=1}^k X_j \) as

\[
\epsilon(u) := \sup \left\{ \left| \sum_{i=1}^m \phi^1(u^1_i) \cdots \phi^k(u^k_i) \right| : \phi^j \in X_j^*, \| \phi_j \| \leq 1 \right\},
\]

where \( X_j^* \) denotes the topological dual of \( X_j \) and \( u = \sum_{i=1}^m u_i^1 \otimes \cdots \otimes u_i^k \).

Moreover, this norm is injective, in the sense that, if \( Y_j \) is a subspace of \( X_j \) and \( u \in \bigotimes_{j=1}^k Y_j \), we have that \( \epsilon(u) \) is the same if we consider \( u \) in \( \bigotimes_{j=1,\epsilon} Y_j \) or in \( \bigotimes_{j=1,\epsilon} X_j \). Finally, \( \epsilon(u) \) is just the norm of \( u \) if we see it as a \((k-1)\)-linear operator \( u : X_1 \times \cdots \times X_{k-1} \to X_k^* \).

We present now the characterization of separability given in \([3]\) with a simplified proof.

**Theorem 1.1.** A multipartite state \( \rho \) is separable if and only if \( \rho \) is in the closed unit ball of \( \bigotimes_{j=1,\pi} T(H_j) \), where \( T(H_j) \) is the Banach space of all trace class operators on the Hilbert space \( H_j \).

**Proof.** We only write the non-trivial part. By definition it is clear that the closed unit ball \( B \) of \( \bigotimes_{j=1,\pi} T(H_j) \) is the closed convex hull of \( A := \{ \rho^1 \otimes \cdots \otimes \rho^k : \| \rho^j \| \leq 1 \} \). Since \( A \) is clearly compact, its convex hull is closed and hence coincides with \( B \). Then \( \rho \), being in \( B \), can be written as a convex combination

\[
\rho = \sum_{i=1}^n \omega_i \rho_i^1 \otimes \cdots \otimes \rho_i^k,
\]

with \( \| \rho_i^j \| \leq 1 \). Now we reason as in \([3]\):

\[
1 = Tr(\rho) = \sum_{i=1}^n \omega_i \prod_{j=1}^k Tr(\rho_i^j) \leq \sum_{i=1}^n \omega_i \prod_{j=1}^k \| \rho_i^j \| \leq 1.
\]
Therefore $\text{Tr}(\rho_i^j) = \|\rho_i^j\|$ for every $i, j$, which means that $\rho_i^j$ are positive and with trace one.

With this in hand, fixing orthonormal systems in the Hilbert spaces and using the fact that $\mathcal{T}(H_j)$ is isometric to $H_j \otimes \pi H_j$, deciding the separability of a density operator is equivalent to computing the norm of the corresponding element of $\bigotimes_{j=1}^k (\rho_{n,j}^{C} \otimes \pi \ell_2^{n,j,c}) = \bigotimes_{j=1}^{2k} \rho_{n,j}^{C}$.

The main aim of this letter then is to show how the problem of computing a norm in $\bigotimes_{j=1}^k \ell_n^{\infty} \otimes \pi \ell_2^{N,j}$ can be reduced (once we have fixed the error we want to obtain) to a linear programming problem (LPP), which can be efficiently solved.

We recall some terminology. By $\ell_n^{\infty,\mathbb{R}}$ we will denote $\mathbb{R}^n$ with the sup-norm $\| \cdot \|_{\infty}$, $(e_i)_{i=1}^n$ will denote the canonical basis of $\mathbb{C}^n$ or $\mathbb{R}^n$ and $\langle x, \phi \rangle$ or $\langle \phi, x \rangle$ will denote the duality relation $\phi(x)$, whenever $x \in X$ and $\phi \in X^*$. Finally, we will write $\hookrightarrow$ instead of simply $\rightarrow$ to point out that an operator is injective (and therefore admits an inverse).

\section{Reduction to a LPP}

As a first step we treat the real case:

\textbf{Lemma 2.1.} For any $m \in \mathbb{N}$ and $n \in \mathbb{N}$, one can find (constructively) an $N \in \mathbb{N}$ and a linear operator $I : \ell_n^{\infty,\mathbb{R}} \hookrightarrow \ell_2^{N,\mathbb{R}}$ such that $\|I\| \leq 1$ and $\|I^{-1}\| \leq \frac{m}{m-1}$. Moreover, we can take $N \leq (2nm + 1)^n$.

\textbf{Proof.} We take the set

$$A := \left\{ (a_1, \ldots, a_n) : a_i = \frac{h}{nm}, h = 0, \pm 1, \pm 2, \ldots, \pm nm \right\}.$$ 

Clearly, the cardinality of $A$ is $(2nm + 1)^n$. Now, we define

$$B := \left\{ \frac{a}{\|a\|_2} : a \in A \setminus \{0\} \right\}.$$ 

We will see that $B$ is a $\frac{1}{m}$-covering of the unit sphere of $\ell_2^n$, that is, for every $x \in S_{\ell_2^n}$, there exists $b \in B$ with $\|x - b\|_2 \leq \frac{1}{m}$.

It is clear that, for every $x \in S_{\ell_2^n}$, there exists an element $a \in A \setminus \{0\}$ with $\|a - x\|_{\infty} \leq \frac{1}{2nm}$. Now

$$\left\| \frac{a}{\|a\|_2} - x \right\|_2 \leq \left\| x - a + a - \frac{a}{\|a\|_2} \right\|_2 \leq \|x - a\|_2 + \|\|a\|_2 - 1\|.$$ 

Using the fact that $\| \cdot \|_2 \leq \sqrt{n} \| \cdot \|_{\infty}$ we obtain that

$$\left\| \frac{a}{\|a\|_2} - x \right\|_2 \leq \frac{1}{2m} + \frac{1}{2m} = \frac{1}{m}.$$ 

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Now, it is known (see for instance [12, page 56]) that $I : \ell_2^m \leftrightarrow \ell_\text{card} B$, given by $I(x) = ((x, b))_{b \in B}$ verifies that $(1 - \frac{1}{m}) \|x\| \leq \|I(x)\| \leq \|x\|$, which means that $\|I\| \leq 1$ and $\|I^{-1}\| \leq \frac{m}{m-1}$.

It is important to say that having $n$ in the exponent is essential in Lemma 2.1 [12].

**Proposition 2.2.** If we call $c_{ij} = I_j(e_{ij})$, being $I_j : \ell_2^{\mathbb{N}_j,\mathbb{R}} \leftrightarrow \ell_\infty^{\mathbb{N}_j,\mathbb{R}}$ as in Lemma 2.1, we have that the norm of $\rho$ in $\bigotimes_{j=1}^k \ell_2^{\mathbb{N}_j,\mathbb{R}}$ is, with a relative error bounded by \((\frac{m}{m-1})^k - 1\), the solution to the following LPP:

Maximize $\sum_{i_1,\ldots,i_k=1}^{n_1,\ldots,n_k} \rho_{i_1,\ldots,i_k} \lambda_{i_1,\ldots,i_k}$

subject to the conditions:

\[-1 \leq \sum_{i_1,\ldots,i_k=1}^{n_1,\ldots,n_k} \lambda_{i_1,\ldots,i_k} c_{1i_1}^1(s_1) \cdots c_{ki_k}^k(s_k) \leq 1, \quad 1 \leq s_j \leq N_j, \quad 1 \leq j \leq k.\]

**Proof.** By duality, we see the element $\rho \in \bigotimes_{j=1}^k \ell_2^{\mathbb{N}_j,\mathbb{R}}$ as an operator $\rho : \bigotimes_{j=1}^k \ell_2^{\mathbb{N}_j,\mathbb{R}} \rightarrow \mathbb{R}$. Using Lemma 2.1 and the injectivity of the $\epsilon$ norm, we have that $I = \bigotimes_{j=1}^k I_j : \bigotimes_{j=1}^k \ell_2^{\mathbb{N}_j,\mathbb{R}} \leftrightarrow \bigotimes_{j=1}^k \ell_\infty^{\mathbb{N}_j,\mathbb{R}}$ verifies that $\|I\| \leq 1$ and $\|I^{-1}\| \leq \left(\frac{m}{m-1}\right)^k$. Now, as $\bigotimes_{j=1}^k \ell_\infty^{\mathbb{N}_j,\mathbb{R}}$ is canonically isometric to $\ell_\infty^{N_1 \cdots N_k,\mathbb{R}}$, we have that the solution to the LPP gives us exactly

$$\sup_{\|I(\lambda)\| \leq 1} \langle \rho, \lambda \rangle,$$

where $\lambda \in \bigotimes_{j=1}^k \ell_2^{\mathbb{N}_j,\mathbb{R}}$. Finally, the above comments tell us that

$$\|\rho\| = \sup_{\|\lambda\| \leq 1} \langle \rho, \lambda \rangle \leq \sup_{\|I(\lambda)\| \leq 1} \langle \rho, \lambda \rangle \leq \left(\frac{m}{m-1}\right)^k \sup_{\|\lambda\| \leq 1} \langle \rho, \lambda \rangle = \left(\frac{m}{m-1}\right)^k \|\rho\|.$$

To see the complex case, we are going to reduce it to the real case. The idea is the following. Again by duality, we consider the element $\rho \in \bigotimes_{j=1}^k \ell_2^{\mathbb{N}_j,\mathbb{C}}$ as an operator $\rho : \bigotimes_{j=1}^k \ell_2^{\mathbb{N}_j,\mathbb{C}} \rightarrow \mathbb{C}$. 

\[\text{4}\]
The norm of $\rho$ in $\bigotimes_{j=1}^{k} \ell_{2}^{n_{j},\mathbb{C}}$ is, with a relative error bounded by $\left(\left(\frac{m}{m-1}\right)^{k} - 1\right)$, the solution of the following LPP:

**Theorem 2.3.** The norm of $\rho$ in $\bigotimes_{j=1}^{k} \ell_{2}^{n_{j},\mathbb{C}}$ is, with a relative error bounded by $\left(\left(\frac{m}{m-1}\right)^{k} - 1\right)$, the solution of the following LPP:

Maximize:

$$\sum_{i_{1},\ldots,i_{k}=1}^{n_{1},\ldots,n_{k}} \Re(\rho_{i_{1},\ldots,i_{k}}) \tilde{\lambda}_{i_{1},\ldots,i_{k}} - \sum_{i_{1},\ldots,i_{k}=1}^{n_{1},\ldots,n_{k-1}} \sum_{i_{k}=n_{k}+1}^{2n_{k}} \Im(\rho_{i_{1},\ldots,i_{k}}) \tilde{\lambda}_{i_{1},\ldots,i_{k}}$$

subject to the conditions:

$$-1 \leq \sum_{i_{1},\ldots,i_{k}=1}^{2n_{1},\ldots,2n_{k}} \tilde{\lambda}_{i_{1},\ldots,i_{k}} c_{i_{1}}^{s_{1}} \cdots c_{i_{k}}^{s_{k}} \leq 1, \quad 1 \leq s_{j} \leq N_{j}, \quad 1 \leq j \leq k,$$

where the variables are $\tilde{\lambda}_{i_{1},\ldots,i_{k}}, 1 \leq i_{1} \leq n_{1}, \ldots, 1 \leq i_{k-1} \leq n_{k-1}, 1 \leq i_{k} \leq 2n_{k}$, and the other $\tilde{\lambda}_{i_{1},\ldots,i_{k}}$ are obtained from the variables using the conditions (2).
Conclusion

We have shown how the separability characterization given in [3] can be reduced to a linear programming problem. Though it is a first step towards a complete computational solution to the separability problem, it is still far from being efficient, in the sense that, as $n$ and $k$ appear as exponents in the dimension of the LPP, we need to assume that the number of spaces and the dimension of them is low. That the dependence in $n$ is exponential is not such an inconvenience in the possible applications to quantum computing, where the dimension $n$ is usually supposed to be 2. Moreover, as the separability problem has been shown to be NP-hard [13], this exponential dependence in $n$ is essential to the problem and not just to our approach. On the other hand, the exponential dependence in $k$ is difficult to avoid, just because the dimension of the space $\bigotimes_{j=1}^{k} \ell_2^n$ is $n^k$. Our approach has the advantage that, for fixed $n$ and $k$, the error $\left(\frac{m}{m-1}\right)^k - 1$ can be seen to be of order $\frac{1}{m}$, which makes the dimension of the LPP depend polynomially in the inverse of the error. Moreover, our approach works for arbitrary multipartite systems. Finally, we think that the new techniques used here are interesting in their own right and can lead to more efficient solutions to the separability problem.

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