A note on Abelian varieties embedded in quadrics

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Abstract: We show that if $A$ is a $d$-dimensional abelian variety in a smooth quadric of dimension $2d$ then $d = 1$ and $A$ is an elliptic curve of bidegree $(2, 2)$ on a quadric. This extends a result of Van de Ven which says that $A$ only can be embedded in $\mathbb{P}^{2d}$ when $d = 1$ or 2.

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1 Introduction.

Let $A$ be a $d$-dimensional abelian variety embedded in $\mathbb{P}^{N}$. It is well known that $2d \leq N$. Moreover, in [7] Van de Ven proved that the equality holds only when $d = 1$ or 2.

It is a natural question to study the possibilities for $d$ when the abelian variety $A$ is embedded in any other smooth $2d$-dimensional variety $V$. In particular, here we study the embedding in smooth quadrics. We obtain the following result:

Theorem 1.1 If $A$ is a $d$-dimensional abelian variety in a smooth quadric of dimension $2d$ then $d = 1$ and $A$ is an elliptic curve of bidegree $(2, 2)$ on a quadric.

We will use similar methods to Van de Ven’s proof. The calculation of the self intersection of $A$ in the quadric and the Riemann-Roch theorem for abelian varieties allow only the cases $d = 1, 2, 3$.

The case $d = 1$ is the classical elliptic curve of type $(2, 2)$ contained in the smooth quadric of $\mathbb{P}^{3}$.

When $d = 2$, $A$ is an abelian surface in $\mathbb{P}^{5}$. We see that is the projection of an abelian surface $A' \subset \mathbb{P}^{6}$ given by a $(1, 7)$ polarization. By a result [6] due

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to R. Lazarsfeld, this is projectively normal and it is not contained in quadrics. Therefore, $A$ is not contained in quadrics either.

Finally, two results [4], [5] of J.N.Iyer allow us to discard the case $d = 3$.

## 2 Proof of the Theorem.

Let $j : A \hookrightarrow Q$ be an embedding of a $d$-dimensional abelian variety into a $2d$-dimensional smooth quadric, with $d > 1$. The Chow ring of the smooth quadric in codimension $d$ is generated by cocycles $\alpha$ and $\beta$ with the relations $\alpha^2 = \beta^2 = 1$, $\alpha\beta = 0$. Thus, $A$ will be equivalent to $a\alpha + b\beta$ and

$$A.A = a^2 + b^2.$$  \hspace{1cm} (1)

On the other hand, by the self-intersection formula ([3], pag 431) we have $A.A = j^* c_d(N_{A,Q})$. To obtain $c_d(N_{A,Q})$, let us consider the normal bundle sequence:

$$0 \to T_A \to j^* T_Q \to N_{A,Q} \to 0$$

Since the tangent bundle of an abelian variety is trivial, we see that $c(T_Q) = (1 + H)^{n+2}(1 + 2H)^{-1}$, where $\overline{H} = i^* H$ and $H$ is a hyperplane in $\mathbb{P}^{n+1}$. We compute the class of the tangent bundle of a quadric in the following lemma:

**Lemma 2.1** Let $i : Q \hookrightarrow \mathbb{P}^{n+1}$ be an $n$-dimensional smooth quadric in $\mathbb{P}^{n+1}$. Then

$$c(T_Q) = (1 + \overline{H})^{n+2}(1 + 2\overline{H})^{-1}$$

where $\overline{H} = i^* H$ and $H$ is a hyperplane in $\mathbb{P}^{n+1}$.

**Proof:** We have an exact sequence:

$$0 \to T_Q \to i^* T_{\mathbb{P}^{n+1}} \to N_{Q,\mathbb{P}^{n+1}} \to 0$$

Since $Q$ is a hypersurface $N_{Q,\mathbb{P}^{n+1}} \cong \mathcal{O}_Q(Q) \cong \mathcal{O}_Q(2\overline{H})$ and the total class of the normal bundle is $c(N_{Q,\mathbb{P}^{n+1}}) = 1 + 2\overline{H}$. On the other hand, it is well known that $c(T_{\mathbb{P}^{n+1}}) = (1 + H)^{n+2}$. Now, from the splitting principle the claim follows.

Let us apply this lemma to the previous situation. We obtain

$$c(N_{A,Q}) = (1 + h)^{2d+2}(1 + 2h)^{-1} = \sum_{k=0}^{2d+2} \binom{2d+2}{k} h^k \sum_{l=0}^{\infty} (-2h)^{-l}$$

where $h = j^* \overline{H}$. In particular, the top class is

$$c_d = F_d h^d, \text{ with } F_d = \sum_{k=0}^{d} \binom{2d+2}{k} (-2)^{(d-k)}.$$
Substituting this into the self-intersection formula, we have:
\[ A.A = F_d j_*(j^*H^d) = F_d H^d j_* A = F_d (a\alpha + b\beta).H^d = F_d (a + b). \]
Combining this expression with (1) we obtain the following relation
\[ a^2 + b^2 = F_d (a + b) \tag{2} \]
or equivalently,
\[ (a - \frac{F_d}{2})^2 + (b - \frac{F_d}{2})^2 = \frac{F_d^2}{2}. \]
We are interested in bounding the degree of \( A \), when \((a, b)\) satisfy this equation. Note that this is a circle of center \((\frac{F_d}{2}, \frac{F_d}{2})\) and radius \(\frac{F_d}{\sqrt{2}}\). Since \( \deg(A) = a + b \), it is clear that the maximal degree is reached when \((a, b) = (F_d, F_d)\), that is,
\[ \deg(A) \leq 2F_d. \]
On the other hand, the abelian variety is embedded in \( Q \subset \mathbb{P}^{2d+1} \). When \( d > 2 \), by Van de Ven’s Theorem, it spans \( \mathbb{P}^{2d+1} \). Furthermore, by the Riemann-Roch theorem for abelian varieties, we know that \( h^0(\mathcal{O}_A(h)) = \frac{\deg(A)}{d!} \). Thus, we have the following inequality:
\[ \deg(A) \geq 2(d + 1)! \]
Comparing the two bounds we see that a sufficient condition for the non-existence of the embedding \( j \) is \( F_d < (d + 1)! \). Now,
\[ F_d = \sum_{k=0}^{d} \binom{2d+2}{k} (-2)^{d-k} \leq \sum_{k=0}^{d} \binom{2d+2}{k} (2)^d \leq 2^d 2^{2d+1} = 2^{3d+1}. \]
We see that \( (d + 1)! > 2^{3d+1} \geq F_d \) when \( d = 17 \). A simple inductive argument shows that this holds if \( d \geq 17 \).

If \( d \leq 17 \), using the exact value of \( F_d \), we see that \( (d + 1)! > F_d \) for any \( d > 3 \).

We conclude that the unique possibilities are \( d = 2 \) or \( d = 3 \).

First, suppose that \( A \) is an abelian surface contained in a quadric. \( F_2 = 7 \) and we can check that the unique positive integer solution of the equation (2) is \( a = b = 7 \). Thus \( A \) must be an abelian surface of degree 14 given by the polarization \((1, 7)\). Note that \( A \subset Q \subset \mathbb{P}^5 \) is not linearly normal, that is, it is the projection of a linearly normal abelian surface \( A' \subset \mathbb{P}^6 \). The quadric \( Q \) can be lifted to a quadric containing the surface \( A' \).

Lazarsfeld proved in [6] that a very ample divisor of type \((1, d)\) with \( d \geq 13 \) or \( d = 7, 8, 9 \) is projectively normal. From this the following sequence is exact:
\[ 0 \rightarrow H^0(I_{A'}).\mathbb{P}^6(2) \rightarrow H^0(\mathcal{O}_{\mathbb{P}^6}(2)) \rightarrow H^0(\mathcal{O}_{A'}(2)) \rightarrow 0 \]
Since $h^0(O_{P^\infty}(2)) = h^0(O_{A'}(2)) = 28$, there are not quadrics containing the abelian surface $A'$ and we obtain a contradiction.

Finally, suppose that $d = 3$. Now, $F_3 = 24 = (3 + 1)!$, so the degree of the abelian variety is exactly $2F_3 = 48$. The line bundle $O_A(h)$ corresponds to a divisor of type $(1, 1, 8)$ or $(1, 2, 4)$. But J.N.Iyer prove in [4] that a line bundle of type $(1, \ldots, 1, 2d + 1)$ is never very ample. Moreover, in [5] she studies the map defined by a line bundle of type $(1, 2, 4)$ in a generic abelian threefold. She obtains that it is birational but not an isomorphism onto its image. Note that the very ampleness is an open condition for polarized abelian varieties (see [1]). It follows that a linear system of type $(1, 2, 4)$ cannot be very ample on any abelian threefold and this completes the proof.

**Remark 2.2** The sequence $F_d = \sum_{k=0}^{d} \binom{2d+2}{k} (-2)^{(d-k)}$ is related to the Fine numbers. For a reference see [2].

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