Quantum Error Correction and Reversible Operations

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Abstract

I give a pedagogical account of Shor’s nine-bit code for correcting arbitrary errors on single qubits, and I review work that determines when it is possible to maintain quantum coherence by reversing the deleterious effects of open-system quantum dynamics. The review provides an opportunity to introduce an efficient formalism for handling superoperators. I present and prove some bounds on entanglement fidelity, which might prove useful in analyses of approximate error correction.

1 Introduction

A bit is the fundamental unit of information, represented by a choice between two alternatives, conventionally labeled 0 and 1. In the real world the abstract notion of a bit must be realized as a physical system. A classical bit, the unit of classical information processing, can be thought of as a two-state classical system. The two states, perhaps 0 or 1 printed on a page or the two positions of a particle in a double-well system, are distinguishable, and because they are distinguishable, they can be copied. A quantum bit or qubit, the unit of quantum information processing, is a two-state quantum system. The two basis states, labeled $|0\rangle$ and $|1\rangle$, are distinguishable and can be copied, just like the states of a classical bit. The difference arises from the superposition principle: the general state of a qubit is an arbitrary linear combination of $|0\rangle$ and $|1\rangle$. The many possible superposition states available to a qubit give it more information-processing power than a classical bit, even though the general superposition states cannot be distinguished reliably and cannot be copied—the no-cloning theorem forbids it.

The enhanced information-processing power of qubits can be harnessed to a variety of information-processing tasks including computation (for a review and extensive list of references, see [3]).

The price for the power of quantum information is eternal vigilance in maintaining quantum coherence, for the enhanced information-processing power comes from the ability to manipulate superposition states. Coupling to the environment, with the accompanying noise and decoherence, tends to destroy superpositions. Indeed, what we call a classical bit is just a qubit whose coupling

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to the environment keeps it from occupying superpositions of two orthogonal states. Decoherence strips the kets off $|0\rangle$ and $|1\rangle$, leaving the 0 and 1 of a classical bit, which cannot be superposed.

Quantum information processing got a tremendous boost with Shor’s surprising discovery [4] that quantum information stored in superpositions can be protected against decoherence. Shor’s announcement of a nine-bit quantum code, followed shortly by the discovery of five-bit [5] and seven-bit codes [6], ignited an explosion of activity on quantum error correction [7, 8, 9, 10, 11, 12] (see [3] for further references), which led to the demonstration that a quantum computer can perform arbitrarily complicated computations provided that the error per operation can be reduced below an error threshold [13, 14, 15].

In this article I give in Sec. 2 a pedagogical account of the simplest error-correction scheme, Shor’s nine-bit code [4], with the aim of illustrating the essential ideas of quantum error correction. In Sec. 3 I review work that determines when it is possible to maintain quantum coherence by reversing the deleterious effects of open-system quantum dynamics, using the review as an opportunity to introduce an efficient formalism for manipulating superoperators. In Sec. 4 I first review the information-theoretic description of error correction and then present and prove some bounds on entanglement fidelity, which might turn out to be useful in considering approximate error correction.

2 Quantum Error Correction: Shor’s Nine-Bit Code

Quantum error correction is closely related to classical error correction. Much of the theory of quantum error correction [7] [8] [9] comes from the theory of classical linear codes, in which a classical code word of length $k$, i.e., a string of $k$ 0’s and 1’s, becomes a vector in a $k$-dimensional vector space over the field consisting of 0 and 1. I do not discuss this theory here, but rather consider a particular quantum code, Shor’s nine-bit code, to illustrate the essential ideas of quantum error correction.

To get started, though, let’s first consider correcting errors on a classical bit. The only type of error is a bit flip, in which 0 and 1 are interchanged. If the errors are rare, they can be corrected using redundancy, because one only needs to correct single-bit errors. Suppose, for example, that a 0 is encoded as the three-bit string 000, and a 1 as the three-bit string 111. Then an error on a single bit can always be detected and corrected. The situation is summarized here:

| Error Type  | Bits  |
|-------------|-------|
| no error    | 000   | 111  |
| flip 1st bit| 010   | 101  |
| flip 2nd bit| 001   | 110  |

An error can be detected by polling the three bits. The minority bit suffered the flip, which can be corrected by flipping the minority bit again. If the probability for a bit flip on a single qubit is $p \ll 1$, then this scheme reduces the probability of error from $p$ to $3p^2$, a winning proposition if $p \leq \frac{1}{3}$.

The key elements in this scheme, redundancy and polling, cannot be translated to quantum error correction: making redundant copies of a qubit is ruled out by the no-cloning theorem, which forbids making copies of arbitrary superposition states; polling requires ascertaining the state of each qubit, which destroys the quantum coherence one is aiming to protect. On closer examination, though, the situation is more promising. The eight strings that result from no error and from the three bit flips are all distinct. This distinguishability opens up the possibility of a quantum code in which $|0\rangle$ is encoded as a state of three qubits, the “logical zero” state $|0\rangle_L = |000\rangle$, and $|1\rangle$
is encoded as the “logical one” state $|1\rangle_L = |111\rangle$. The logical zero and logical one span a two-dimensional subspace, the “code subspace” to be used for quantum information processing. Notice now what happens to an arbitrary superposition of the logical one and logical zero states under single-qubit bit flips:

$$\begin{align*}
\text{no error} & : 1 \otimes 1 \otimes 1 \quad \alpha|000\rangle + \beta|111\rangle, \\
\text{flip 1st qubit} & : \sigma_1 \otimes 1 \otimes 1 \quad \alpha|100\rangle + \beta|011\rangle, \\
\text{flip 2nd qubit} & : 1 \otimes \sigma_1 \otimes 1 \quad \alpha|010\rangle + \beta|101\rangle, \\
\text{flip 3rd qubit} & : 1 \otimes 1 \otimes \sigma_1 \quad \alpha|001\rangle + \beta|110\rangle.
\end{align*}$$

(2)

Here the middle column writes the error in terms of the unit operator 1 and the bit-flip Pauli operator $\sigma_1 = |0\rangle\langle 1| + |1\rangle\langle 0|$ for the appropriate qubit.

The no-error operator and the three bit-flip errors map the code subspace unitarily to orthogonal two-dimensional subspaces within the eight-dimensional Hilbert space of a qubit triplet. The single-qubit errors can be detected and distinguished by determining in which two-dimensional subspace the system lies, without in any way disturbing the superposition, and the error can be corrected by mapping the error subspaces unitarily back to the code subspace. This is the essence of quantum error correction: find a code subspace such that the high probability errors map unitarily to orthogonal subspaces; then the errors can be detected and corrected without destroying quantum coherence. The superposition states in the code subspace are entangled states of the three qubits. The redundancy used by a classical code, which cannot be translated to quantum coding, is replaced by entanglement in a quantum code.

Having gotten this far, however, we now realize that the task is tougher than classical error correction, because there are quantum errors that have no classical counterpart. Specifically, there are “phase flips,” described by the Pauli operator $\sigma_3 = |0\rangle\langle 0| - |1\rangle\langle 1|$, and errors described by the Pauli operator $-i\sigma_2 = \sigma_1\sigma_3 = -|0\rangle\langle 1| + |1\rangle\langle 0|$, which is a phase flip followed by a bit flip. If we can correct these errors, we can correct all single-qubit errors, because all error operators can be written as a linear combination of the unit operator (no error) and the three Pauli operators.

Let’s concentrate first on the phase-flip errors, hoping that the combined phase-bit flips take care of themselves. The first thing to notice is that $\sigma_1$ and $\sigma_3$ switch roles in the transformed basis defined by

$$|\pm\rangle \equiv \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle) :$$

(3)

$\sigma_3$ becomes a bit flip, and $\sigma_1$ becomes a phase flip,

$$\sigma_3|\pm\rangle = |\mp\rangle, \quad \sigma_1|\pm\rangle = \pm|\pm\rangle.$$  

(4)

Thus we can correct single-qubit phase flips by using a quantum code whose logical basis states are $|++\rangle$ and $|--\rangle$, but this comes at the expense of being unable to correct the original bit-flip errors.

Here entanglement comes to the rescue again. The entire code subspace spanned by $|000\rangle$ and $|111\rangle$ is protected against single-qubit bit flips, so we can use any orthogonal basis in the code subspace as the logical zero and one. In particular, we can use “up” and “down” states

$$|\uparrow\rangle \equiv \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle),$$

$$|\downarrow\rangle \equiv \frac{1}{\sqrt{2}}(|000\rangle - |111\rangle),$$

(5)

which are defined in analogy to the way the states $|\pm\rangle$ are related to $|0\rangle$ and $|1\rangle$ for a single qubit.
Thus the nine-bit code is a degenerate code. The discussion above, the three phase-flip errors on each qubit triplet have exactly the same effect. In the code subspace is affected by all 27 single-qubit errors. Notice first that in accordance with this logical zero and one constitute Shor’s nine-bit code. Using the following nine-qubit states as the logical zero and one states:

This suggests correcting both bit- and phase-flip errors by again tripling the number of qubits and with the third triplet cycling through the eight up- and down-type states that span the triplet subspace unitarily to a two-dimensional subspace. Moreover, by examining the table, one sees that the no-error case plus 21 independent errors. Each error maps the code subspace unitarily back to the code subspace. The eight types of up and down states make up an orthogonal basis for the three-qubit Hilbert space.

Notice now that the bare up and down states are flipped by all three single-qubit phase flips:

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The eight types of up and down states make up an orthogonal basis for the three-qubit Hilbert space. Single-qubit errors can be corrected by determining in which of the orthogonal subspaces the nine qubits lie and then mapping that subspace unitarily back to the code subspace. The degeneracy leaves a situation where we must consider the no-error case plus 21 independent errors. Each error maps the code subspace unitarily back to the code subspace. Moreover, by examining the table, one sees that the code subspace and the 21 error subspaces are mutually orthogonal. The 22 states that arise from $|0\rangle_L$ are orthogonal to the 22 states that arise from $|1\rangle_L$: the former all have up states for two of the triplets, whereas the latter all have down states for two of triplets, so they disagree on up versus down in at least one position. The states that arise from $|0\rangle_L$ ($|1\rangle_L$) are mutually orthogonal because they are the 22 states that come from putting two of the triplets in the up (down) state, with the third triplet cycling through the eight up- and down-type states that span the triplet Hilbert space. Single-qubit errors can be corrected by determining in which of the orthogonal subspaces the nine qubits lie and then mapping that subspace unitarily back to the code subspace.

As noted above, error correction works for single-qubit errors that are not described by Pauli operators, because any error can be written as a linear combination of the unit operator and the Pauli operators. Consider, for example, decay from $|0\rangle$ to $|1\rangle$, with essentially instantaneous phase decoherence between $|0\rangle$ and $|1\rangle$, a situation described by three error operators: $A_1 = \sqrt{1-\gamma} |0\rangle \langle 0| = \sqrt{1-\gamma} (|0\rangle + |1\rangle)/2$, $A_2 = \sqrt{\gamma} |1\rangle \langle 0| = \sqrt{\gamma} (\sigma_1 - i\sigma_2)/2$, and $A_3 = \gamma |1\rangle \langle 1| = (1-\sigma_3)/2$, $\gamma$ being the probability of decay. If one of the nine qubits suffers a decay, a measurement of the error subspace reveals either no error or one of the Pauli errors on that qubit. All of these being correctable, it doesn’t matter which is the result of the measurement.

The effect of single-qubit bit flips on these states is summarized here:

$$
\begin{align*}
\sigma_1 \otimes 1 \otimes 1 |\uparrow\rangle &= \frac{1}{\sqrt{2}} (|100\rangle + |011\rangle) \equiv |\uparrow_0\rangle, \\
1 \otimes \sigma_1 \otimes 1 |\uparrow\rangle &= \frac{1}{\sqrt{2}} (|010\rangle + |101\rangle) \equiv |\uparrow_1\rangle, \\
1 \otimes 1 \otimes \sigma_1 |\uparrow\rangle &= \frac{1}{\sqrt{2}} (|001\rangle + |110\rangle) \equiv |\uparrow_3\rangle, \\
\sigma_1 \otimes 1 \otimes 1 |\downarrow\rangle &= \frac{1}{\sqrt{2}} (|100\rangle - |011\rangle) \equiv |\downarrow_0\rangle, \\
1 \otimes \sigma_1 \otimes 1 |\downarrow\rangle &= \frac{1}{\sqrt{2}} (|010\rangle - |101\rangle) \equiv |\downarrow_2\rangle, \\
1 \otimes 1 \otimes \sigma_1 |\downarrow\rangle &= \frac{1}{\sqrt{2}} (|001\rangle - |110\rangle) \equiv |\downarrow_3\rangle. 
\end{align*}
$$

(6)

The eight types of up and down states make up an orthogonal basis for the three-qubit Hilbert space.

Notice now that the bare up and down states are flipped by all three single-qubit phase flips:

$$
\left( \begin{array}{c}
\sigma_3 \otimes 1 \otimes 1 \\
1 \otimes \sigma_3 \otimes 1 \\
1 \otimes 1 \otimes \sigma_3
\end{array} \right) \times \left\{ \begin{array}{c}
|\uparrow\rangle \\
|\downarrow\rangle
\end{array} \right\} = \left\{ \begin{array}{c}
|\uparrow\rangle \\
|\downarrow\rangle
\end{array} \right\}.
$$

(7)

This suggests correcting both bit- and phase-flip errors by again tripling the number of qubits and using the following nine-qubit states as the logical zero and one states:

$$
|0\rangle_L = |\uparrow\uparrow\uparrow\rangle, \quad |1\rangle_L = |\downarrow\downarrow\downarrow\rangle.
$$

(8)

This logical zero and one constitute Shor’s nine-bit code.

To complete the discussion of the nine-bit code, I display in Table 1 how an arbitrary state in the code subspace is affected by all 27 single-qubit errors. Notice first that in accordance with the discussion above, the three phase-flip errors on each qubit triplet have exactly the same effect. Thus the nine-bit code is a degenerate code, one in which errors that are independent on the entire Hilbert space become the same on the code subspace. The degeneracy leaves a situation where we must consider the no-error case plus 21 independent errors. Each error maps the code subspace unitarily back to the code subspace. Moreover, by examining the table, one sees that the code subspace and the 21 error subspaces are mutually orthogonal. The 22 states that arise from $|0\rangle_L$ are orthogonal to the 22 states that arise from $|1\rangle_L$: the former all have up states for two of the triplets, whereas the latter all have down states for two of triplets, so they disagree on up versus down in at least one position. The states that arise from $|0\rangle_L$ ($|1\rangle_L$) are mutually orthogonal because they are the 22 states that come from putting two of the triplets in the up (down) state, with the third triplet cycling through the eight up- and down-type states that span the triplet Hilbert space. Single-qubit errors can be corrected by determining in which of the orthogonal subspaces the nine qubits lie and then mapping that subspace unitarily back to the code subspace.

As noted above, error correction works for single-qubit errors that are not described by Pauli operators, because any error can be written as a linear combination of the unit operator and the Pauli operators. Consider, for example, decay from $|0\rangle$ to $|1\rangle$, with essentially instantaneous phase decoherence between $|0\rangle$ and $|1\rangle$, a situation described by three error operators: $A_1 = \sqrt{1-\gamma} |0\rangle \langle 0| = \sqrt{1-\gamma} (|0\rangle + |1\rangle)/2$, $A_2 = \sqrt{\gamma} |1\rangle \langle 0| = \sqrt{\gamma} (\sigma_1 - i\sigma_2)/2$, and $A_3 = |1\rangle \langle 1| = (1-\sigma_3)/2$, $\gamma$ being the probability of decay. If one of the nine qubits suffers a decay, a measurement of the error subspace reveals either no error or one of the Pauli errors on that qubit. All of these being correctable, it doesn’t matter which is the result of the measurement.
The nine-bit code wastes Hilbert space, for it uses only 44 of the $2^9 = 512$ dimensions in the nine-qubit Hilbert space. If one wants to correct $r$ errors per qubit, using a code with $N$ qubits, then to accommodate the code subspace and the $rN$ errors, one needs $2(1 + rN)$ dimensions. This leads to the quantum Hamming bound [4]:

$$2(1 + rN) \leq 2^N \iff r \leq \frac{2^{N-1} - 1}{N}.$$  

(9)

Three qubits permit correction of one error, the situation we started with in this section. Five qubits have the potential for correcting the three Pauli errors, a potential realized in an astonishing five-qubit quantum code [5].

3 Reversal of Open-System Dynamics

In this section I review the description of open-system dynamics in terms of quantum operations and the question of when quantum coherence can be maintained by reversing open-system dynamics. The review serves to introduce a formalism for handling quantum operations, which provides insight into their structure.

Throughout this section and the next I use a set of conventions introduced by Schumacher [16]. The primary quantum system is denoted by $Q$; it is assumed to be finite-dimensional, with a Hilbert space $\mathcal{H}_Q$ of dimension $D$. The primary system interacts with an environment $E$, and to deal with purifications of $Q$ states, there can be an additional, passive reference system $R$. Where it is necessary to avoid confusion, superscripts $R, Q, E$ are used to distinguish states and operators of these systems. Initial states are unprimed, and states after the dynamics are denoted by a prime.

3.1 Open-system dynamics and quantum operations

Consider a primary system $Q$, initially in state $\rho$, which is brought into contact with an environment $E$, initially in state $\rho_E = \sum_l \lambda_l |\phi_l\rangle \langle \phi_l |$, where the states $|\phi_l\rangle$ are the eigenstates of $\rho_E$. The two systems interact for a time, the interaction described by a unitary operator $U$, and then the environment is observed to be in a subspace spanned by orthogonal states $|g_k\rangle$, corresponding to a projection operator $P^E = \sum_k |g_k\rangle \langle g_k |$. The unnormalized state of the system after the observation is given by a partial trace over the environment,

$$\text{tr}_E(P^E U (\rho \otimes \rho_E) U^\dagger) \equiv \mathcal{A}(\rho) ,$$  

(10)

where $\mathcal{A}$ is a linear map on system density operators. Inserting the forms of the projector $P^E$ and the environment state $\rho_E$ leads to

$$\mathcal{A}(\rho) = \sum_{k,l} \sqrt{\lambda_l} \langle g_k | U | \phi_l \rangle \rho \langle \phi_l | U^\dagger | g_k \rangle \sqrt{\lambda_l} = \sum_\alpha A_\alpha \rho A_\alpha^\dagger .$$  

(11)

The system operators

$$A_\alpha = A_{kl} \equiv \sqrt{\lambda_l} \langle g_k | U | \phi_l \rangle ,$$  

(12)

where the Greek index $\alpha$ is as an abbreviation for $k$ and $l$, provide an operator decomposition of the map $\mathcal{A}$ and thus are called decomposition operators. The normalized post-dynamics system state is

$$\rho' = \frac{\mathcal{A}(\rho)}{\text{tr}(\mathcal{A}(\rho))} ,$$  

(13)
where one should notice that
\[ \text{tr}(P^E U (\rho \otimes \rho^E) U^\dagger) = \text{tr}(A(\rho)) = \text{tr}\left(\rho \sum_{\alpha} A^\dagger_{\alpha} A_{\alpha}\right) \]  

is the probability for the environment to be found in the specified subspace.

A primary quantum system that is exposed to an initially uncorrelated environment always has dynamics described by a map like \( A \). This includes both nonselective dynamics, where no observation is made on the environment (\( P^E = 1^E \)), and selective dynamics, where the system state is conditioned on the result of a measurement on the environment (\( P^E \neq 1^E \)). It is important to identify the mathematical conditions that characterize a suitable map \( A \). We can immediately identify three conditions that \( A \) must satisfy.

**Condition 1.** \( A \) is a linear map on operators; i.e., it is a superoperator.

**Condition 2′.** \( A \) maps positive operators to positive operators. (A positive operator \( G \) is one such that \( \langle \psi | G | \psi \rangle \geq 0 \) for all vectors \( | \psi \rangle \); density operators are positive operators.)

**Condition 3.** \( A \) is trace-decreasing, i.e., \( \text{tr}(A(\rho)) \leq 1 \) for all density operators \( \rho \). This condition, which follows immediately from Eq. (14), can be expressed as an operator inequality, 
\[ \sum_{\alpha} A^\dagger_{\alpha} A_{\alpha} \leq 1 \]  

The trace is preserved for nonselective dynamics, but generally decreases for selective dynamics.

It might be thought that the above three conditions are sufficient to characterize \( A \), but it turns out that Condition 2′ must be strengthened. The more restrictive condition can be motivated physically. Suppose that \( R \) is a reference system that, though it does not take part in the dynamics, cannot be neglected because the initial state \( \rho \) of \( Q \) is the partial trace over \( R \) of a joint state \( \rho_{RQ} \). We certainly want the map \( I^R \otimes A \), where \( I^R \) is the unit superoperator on \( R \), to take \( \rho_{RQ} \) to a positive operator, which can be normalized to be a density operator. This condition holds trivially for a map of the form (11), 
\[ (I^R \otimes A)(\rho_{RQ}) = \sum_{\alpha} (1^R \otimes A_{\alpha})\rho_{RQ}(1^R \otimes A^\dagger_{\alpha}) \geq 0 \]  

so we replace Condition 2′ with a stronger condition.

**Condition 2.** \( A \) is completely positive; i.e., \( (I^R \otimes A)(\rho_{RQ}) \geq 0 \) for all joint density operators \( \rho_{RQ} \) of \( Q \) and arbitrary reference systems \( R \).

A map on operators that satisfies Conditions 1–3 is called a quantum operation. The description of open-system dynamics in terms of quantum operations was pioneered by Hellwig and Kraus \[17, 18, 19\]. Thus far our discussion has established that any open-system dynamics of the sort introduced above is described by a quantum operation. We need two further properties: first, that any quantum operation has an operator decomposition, as in Eq. (11), and second, that any quantum operation can be realized by a unitary coupling to an environment. Once the former result is established, the latter is easy. Any linear, trace-decreasing map that has an operator decomposition can be realized by a unitary coupling to an initially pure-state environment. One partially defines a joint operator \( U \) on \( RQ \) as in Eq. (12) and uses the trace-decreasing condition (15) to show that the definition can be extended so that \( U \) is unitary. We turn now to the connection between complete positivity and the existence of an operator decomposition, which requires us to step back and consider the properties of superoperators.
3.2 Superoperators and complete positivity

The space of linear operators acting on $H_Q$ is a $D^2$-dimensional complex vector space $\mathcal{L}(H_Q)$. Let us introduce operator “kets” $|A\rangle = A$ and “bras” $\langle A| = A^\dagger$, distinguished from vector kets and bras by the use of smooth brackets. Then the natural inner product on $\mathcal{L}(H_Q)$, the trace-norm inner product, can be written as $\langle A|B\rangle = \text{tr}(A^\dagger B)$. An orthonormal basis $|e_j\rangle$ induces an orthonormal operator basis $|e_j\rangle\langle e_k|$ $= \tau_{jk} = \tau_{\alpha\beta}$, where the Greek index is again an abbreviation for two Roman indices. Not all orthonormal operator bases are of this outer-product form.

The space of superoperators on $Q$, i.e., linear maps on operators, is a $D^4$-dimensional complex vector space $\mathcal{L}(\mathcal{L}(H_Q))$. Any superoperator $S$ is specified by its “matrix elements” $S_{\alpha\beta \gamma\delta} \equiv \langle \tau_{\alpha\beta}| S| \tau_{\gamma\delta}\rangle = \langle e_i| S| e_j\rangle \langle e_k| e_m\rangle$, (17)

for the superoperator can be written in terms of its matrix elements as

$$ S = \sum_{i,j,m,k} S_{ij,mk} |e_i\rangle\langle e_j| \otimes |e_k\rangle|e_m\rangle = \sum_{\alpha,\beta} S_{\alpha\beta} \tau_{\alpha}\otimes \tau_{\beta}^\dagger = \sum_{\alpha,\beta} S_{\alpha\beta} |\tau_{\alpha}\rangle\langle \tau_{\beta}| .$$

(18)

The ordinary action of $S$ on an operator $A$, used above to generate the matrix elements, is obtained by dropping an operator $A$ into the center of the representation of $S$, in place of the tensor-product sign,

$$ S(A) = \sum_{\alpha,\beta} S_{\alpha\beta} |\tau_{\alpha}\rangle A |\tau_{\beta}\rangle^\dagger .$$

(19)

There is clearly another way that $S$ can act on $A$, the left-right action,

$$ S|A\rangle \equiv S|\tau_{\alpha}\rangle |A\rangle = \sum_{\alpha,\beta} S_{\alpha\beta} |\tau_{\alpha}\rangle |\tau_{\beta}| A\rangle ,$$

(20)

in terms of which the matrix elements are

$$ S_{\alpha\beta} = \langle \tau_{\alpha}| S| \tau_{\beta}\rangle = \langle e_i| S| e_j\rangle \langle e_k| e_m\rangle .$$

(21)

This expression provides the fundamental connection between the two actions of a superoperator.

With respect to the left-right action, a superoperator works just like an operator. Multiplication of superoperators $T$ and $S$ is given by

$$ T \circ S = \sum_{\alpha,\beta,\gamma,\delta} T_{\alpha\gamma} S_{\gamma\delta} |\tau_{\alpha}\rangle |\tau_{\beta}| \otimes |\tau_{\gamma\delta}\rangle ,$$

(22)

and the adjoint is defined by

$$ (A|S^\dagger|B) = (B|S|A)^* \iff S^\dagger = \sum_{\alpha,\beta} S^*_{\beta\alpha} |\tau_{\alpha}\rangle |\tau_{\beta}| .$$

(23)

With respect to the ordinary action, superoperator multiplication, denoted as a composition $T \circ S$, is given by

$$ T \circ S = \sum_{\alpha,\beta,\gamma,\delta} T_{\gamma\delta} S_{\alpha\beta} |\tau_{\gamma\delta}\rangle \tau_{\alpha}\otimes |\tau_{\gamma\delta}\rangle^\dagger .$$

(24)

The adjoint with respect to the ordinary action, denoted by $S^\times$, is defined by

$$ \text{tr}([S^\times(B)]^\dagger A) = \text{tr}(B^\dagger S(A)) \iff S^\times = \sum_{\alpha,\beta} S^*_{\alpha\beta} |\tau_{\alpha}\rangle \otimes |\tau_{\beta}| .$$

(25)
To deal with complete positivity, we need to introduce a reference system $R$, which we choose now and henceforth to have the same dimension as $Q$, and we need to have a way of turning operators (superoperators) on $Q$ into vectors (operators) on $RQ$. To do so, introduce the unnormalized maximally entangled state

$$|\Psi\rangle \equiv \sum_j |f_j \rangle \otimes |e_j \rangle = \sum_j |f_j, e_j \rangle,$$

where the vectors $|f_j \rangle$ comprise an orthonormal basis for $R$. The VEC of an operator $A$ on $Q$ is the vector

$$|\Phi_A\rangle \equiv 1^R \otimes A|\Psi\rangle = \sum_j |f_j \rangle \otimes A|e_j \rangle.$$

The VEC map is a one-to-one, linear map from $L(H_Q)$ to $H_{RQ}$; we recover $A$ from $|\Phi_A\rangle$ via

$$\langle f_j, e_k | \Phi_A \rangle = \langle e_k | A | e_j \rangle.$$

It is easy to see that VEC preserves inner products,

$$\langle \Phi_A | \Phi_B \rangle = \text{tr}(A^\dagger B) = \langle A | B \rangle.$$

One further aspect of VEC deserves mention. Given a density operator $\rho$ for $Q$, applying VEC to $\sqrt{\rho}$,

$$|\Phi_{\sqrt{\rho}}\rangle = 1^R \otimes \sqrt{\rho} |\Psi\rangle = \sum_j |f_j \rangle \otimes \sqrt{\rho} |e_j \rangle,$$

generates a purification of $\rho$, i.e.,

$$\text{tr}_R(|\Phi_{\sqrt{\rho}}\langle \Phi_{\sqrt{\rho}}|) = \rho.$$

The analogous OP map is a one-to-one, linear map from $L(L(H_Q))$ to $L(H_{RQ})$. It takes a superoperator $S$ on $Q$ to the operator

$$(I^R \otimes S)(|\Psi\rangle\langle \Psi|) = \sum_{j,k} |f_j \rangle \langle f_k | \otimes S(|e_j \rangle \langle e_k|),$$

and we recover $S$ via

$$\langle f_j, e_l | (I^R \otimes S)(|\Psi\rangle\langle \Psi|) | f_k, e_m \rangle = \langle e_l | S(|e_j \rangle \langle e_k|) | e_m \rangle = S_{lj,mk}.$$

Looking at OP in a slightly different way,

$$(I^R \otimes S)(|\Psi\rangle) = \sum_{\alpha,\beta} S_{\alpha\beta}|\Phi_{\tau_\alpha}\rangle\langle \Phi_{\tau_\beta}|,$$

we find that

$$\langle \Phi_A | (I^R \otimes S)(|\Psi\rangle) | \Phi_B \rangle = \langle A | S | B \rangle.$$

Thus the OP of a superoperator $S$ operates in the same way as the left-right action of $S$.

We are ready now to return to complete positivity. Recall that we are trying to show that any completely positive superoperator $A$ has an operator decomposition. The key point is that complete positivity requires that the OP of $A$, i.e., $(I^R \otimes A)(|\Psi\rangle\langle \Psi|)$, be a positive operator, but Eq. (35) now shows this to be equivalent to the requirement that $A$ be positive relative to its left-right action, which I write as $A \geq 0$. Any such positive superoperator has many operator decompositions, including its orthogonal decomposition. Moreover, we get for free, by using the decomposition theorem for positive operators [20], the following result, originally due to Choi [21]:
two decompositions $A_\alpha$ and $B_\alpha$ give rise to the same completely positive superoperator if and only if they are related by a unitary matrix $V_{\beta\alpha}$, i.e.,

$$B_\beta = \sum_\alpha V_{\beta\alpha} A_\alpha$$  \hspace{1cm} (36)

(if one decomposition has a smaller number of operators, it is extended by appending zero operators).

What we have shown in this subsection is that a map is completely positive if and only if it is a positive superoperator relative to the left-right action. For a quantum operation we must add the trace-decreasing condition (15), which now can be put in the compact form

$$A \times (1) = \sum_\alpha A_\alpha^\dagger A_\alpha \leq 1,$$  \hspace{1cm} (37)

with equality if and only if the operation is trace preserving. Thus we can now characterize a quantum operation as a superoperator that is positive relative to the left-right action ($A \geq 0$, complete positivity) and that satisfies $A \times (1) \leq 1$ (trace-decreasing).

We should introduce one more ingredient before leaving our discussion of superoperators. Suppose the initial state $\rho$ of $Q$ is VEC’ed to an initial joint state $|\Phi\sqrt{\rho}\rangle$, as in Eq. (30). The joint state of $RQ$ after the dynamics described by $I_R \otimes A$ is given by

$$\rho_{RQ} = (I_R \otimes A)(|\Phi\sqrt{\rho}\rangle\langle\Phi\sqrt{\rho}|) = \left( I_R \otimes \frac{A \circ \sqrt{\rho} \otimes \sqrt{\rho}}{\text{tr}(A(\rho))} \right) (|\Psi\langle\Psi|).$$  \hspace{1cm} (38)

Referring to Eq. (35), we see that the superoperator

$$A_\rho = \frac{A \circ \sqrt{\rho} \otimes \sqrt{\rho}}{\text{tr}(A(\rho))}$$  \hspace{1cm} (39)

is equivalent to the joint density operator $\rho_{RQ}$; i.e., $\rho_{RQ}$ operates the same as the left-right action of $A_\rho$.

Straightforward consequences of the definition of $A_\rho$ are that

$$A_\rho(1) = \frac{A(\rho)}{\text{tr}(A(\rho))} = \rho', \hspace{1cm} \sigma = A_\rho^X(1) = \frac{\sqrt{\rho} A^X(1) \sqrt{\rho}}{\text{tr}(A(\rho))} \leq \frac{\rho}{\text{tr}(A(\rho))}. \hspace{1cm} (40)$$

If $A$ is trace preserving, then the density operator $\sigma = \rho$. The physical significance of $\sigma$ can be ferreted out with a bit more work. First we note that

$$\langle e_k | \sigma | e_j \rangle = \text{tr}(A_\rho^X(1) |e_j\rangle\langle e_k|) = \text{tr}(A_\rho(|e_j\rangle\langle e_k|)).$$  \hspace{1cm} (42)

Writing the joint state of $RQ$ after the dynamics as

$$\rho_{RQ} = (I_R \otimes A_\rho)(|\Psi\langle\Psi|) = \sum_{j,k} |f_j\rangle\langle f_k| \otimes A_\rho(|e_j\rangle\langle e_k|),$$  \hspace{1cm} (43)

we find that the state of $R$ after the dynamics,

$$\rho_R = \text{tr}_Q(\rho_{RQ}) = \sum_{j,k} |f_j\rangle\langle f_k| |e_k\rangle |\sigma e_j\rangle,$$  \hspace{1cm} (44)

is the “transpose” of $\sigma$ with respect to the bases $|e_j\rangle$ and $|f_j\rangle$.  


3.3 Reversal of quantum operations

We are now ready to formulate the problem of reversing open-system dynamics—i.e., correcting errors due to coupling to an environment. In this subsection I follow closely the formulation found in [23]. The mathematical statement of the problem is the reversibility of a quantum operation \( \mathcal{A} \) on a “code subspace” \( C \) of the system Hilbert space \( \mathcal{H}_Q \). The reversal must be accomplished by a physical process, so it, too, is described by an operation, \( \mathcal{R} \). We want the reversal definitely to occur, so we require that \( \mathcal{R} \) be a trace-preserving operation. Thus we say that \( [10, 22, 23] \) a quantum operation \( \mathcal{A} \) is reversible on the code subspace \( C \) if there exists a trace-preserving reversal operation \( \mathcal{R} \), acting on the total state space of \( \mathcal{Q} \), such that

\[
\mathcal{R}(\rho') = \frac{\mathcal{R} \circ \mathcal{A}(\rho)}{\text{tr}(\mathcal{A}(\rho))} = \rho
\]

for all \( \rho \) whose support is confined to \( C \). An immediate consequence of the linearity of \( \mathcal{R} \circ \mathcal{A} \) is that

\[
\text{tr}(\mathcal{A}(\rho)) = \text{constant} \equiv \mu^2
\]

for all \( \rho \) whose support is confined to \( C \) [22], where \( \mu \) is a real constant satisfying \( 0 < \mu \leq 1 \). This allows us to rewrite the reversibility condition (45). To do so, we introduce the restriction of \( \mathcal{A} \) to \( C \),

\[
\mathcal{A}_C \equiv \mathcal{A} \circ P \otimes P
\]

(47)

where \( P \) is the projector onto \( C \). Then the reversibility condition becomes

\[
\mathcal{R} \circ \mathcal{A}_C = \mu^2 P \otimes P
\]

(48)

It is not hard to show (for proofs see [11, 12, 23]) that a quantum operation \( \mathcal{A} \), with decomposition operators \( A_\alpha \), is reversible on \( C \) if and only if there exists a positive matrix \( m_{\alpha\beta} \), having unit trace, such that

\[
P_C A_\alpha^\dagger A_\beta P_C = \mu^2 m_{\alpha\beta} P_C
\]

(49)

It is instructive to rewrite this condition as \( \langle e_j | A_\alpha^\dagger A_\beta | e_k \rangle = \mu^2 m_{\alpha\beta} \delta_{jk} \), where the vectors \( |e_j\rangle \) make up an orthonormal basis on \( C \). We can think of each decomposition operator \( A_\alpha \) as an “error operator.” Though each error operator acts like a multiple of a unitary operator within \( C \), different error operators are not required to map to orthogonal subspaces. How do we square this with the discussion in Sec. 2?

The puzzle is resolved by realizing that the decomposition of \( \mathcal{A} \) is not unique. We need to choose the decomposition operators to represent independent, indeed orthogonal errors within \( C \). To do so, take any density operator \( \rho \) whose support is the entirety of \( C \), and use a unitary matrix transformation of the decomposition operators \( A_\alpha \) to diagonalize the matrix \( \mu^2 m_{\alpha\beta} = \text{tr}(A_\alpha \rho A_\beta^\dagger) \).

In the new decomposition, called a canonical decomposition [23], the decomposition operators \( A_\alpha \) satisfy

\[
(A_\beta \sqrt{\rho} | A_\alpha \sqrt{\rho} \rangle = \text{tr}(A_\alpha \rho A_\beta^\dagger) = \mu^2 m_{\alpha\beta} = \mu^2 \lambda_\alpha \delta_{\alpha\beta}, \quad (50)
\]

where the eigenvalues \( \lambda_\alpha \) satisfy \( \sum \lambda_\alpha = 1 \). In terms of this canonical decomposition, the reversal condition (49) becomes

\[
P_C A_\beta^\dagger A_\alpha P_C = \mu^2 \lambda_\alpha \delta_{\alpha\beta} P_C
\]

(51)

This condition has a ready interpretation. When \( \alpha = \beta \), the operator polar-decomposition theorem [24] implies that there exists a unitary operator \( U_\alpha \) such that

\[
A_\alpha P_C = U_\alpha \sqrt{P_C A_\alpha^\dagger A_\alpha P_C} = \mu \sqrt{\lambda_\alpha} U_\alpha P_C = \mu \sqrt{\lambda_\alpha} P_\alpha U_\alpha
\]

(52)
In the last equality we introduce the projector \( P_\alpha \equiv U_\alpha P_C U_\alpha^\dagger \) onto the subspace that is the unitary image of \( C \) under \( U_\alpha \). In terms of the canonical decomposition the restriction of \( A \) to \( C \) becomes

\[
A_C = \sum_\alpha \tilde{A}_\alpha P_C \otimes P_C \tilde{A}_\alpha^\dagger = \mu^2 \sum_\alpha \lambda_\alpha U_\alpha P_C \otimes P_C U_\alpha^\dagger = \mu^2 \sum_\alpha \lambda_\alpha P_\alpha U_\alpha \otimes U_\alpha^\dagger P_\alpha . \tag{53}
\]

When one or more of the eigenvalues \( \lambda_\alpha \) is zero, we are dealing with a degenerate code \[9, 23\]: as far as the action of \( A \) within \( C \) is concerned, the \( \lambda_\alpha = 0 \) operators are irrelevant. We discard those operators henceforth, remembering that this is legitimate so long as we restrict attention to the code subspace. The irrelevant decomposition operators discarded, the content of the \( \alpha \neq \beta \) terms in Eq. \((49)\) is that the image subspaces \( P_\alpha \) are orthogonal, i.e.,

\[
P_\alpha P_\beta = \delta_{\alpha\beta} P_\beta . \tag{54}
\]

Thus in the canonical decomposition, the error operators \( \tilde{A}_\alpha \) act like multiples of unitary operators on \( C \) and map \( C \) to orthogonal subspaces.

It is important for the considerations in Sec. 4 to note that when we go to a canonical decomposition, we are diagonalizing the completely positive superoperator \((39)\),

\[
A_\rho = \frac{1}{\mu^2} \sum_\alpha \tilde{A}_\alpha \sqrt{\rho} \otimes \sqrt{\rho} \tilde{A}_\alpha^\dagger = \sum_\alpha \lambda_\alpha U_\alpha \sqrt{\rho} \otimes \sqrt{\rho} U_\alpha^\dagger = \sum_\alpha \lambda_\alpha \sqrt{\rho_\alpha} U_\alpha \otimes U_\alpha^\dagger \sqrt{\rho_\alpha} , \tag{55}
\]

for the operators \( \tilde{A}_\alpha \sqrt{\rho} \) are orthogonal according to Eq. \((50)\). In the last equality of Eq. \((55)\), we introduce the orthogonal density operators \( \rho_\alpha = U_\alpha \rho U_\alpha^\dagger \) that \( \rho \) is mapped to by the unitaries \( U_\alpha \). Notice that the eigenvalues and normalized eigenoperators of \( A_\rho \) (relative to the left-right action) are \( \lambda_\alpha \) and \( U_\alpha \sqrt{\rho} = \sqrt{\rho_\alpha} U_\alpha \). The equivalence between \( A_\rho \) and \( \rho^{RQ'} \) means that OP’ing the operators \( U_\alpha \sqrt{\rho} \) generates the eigenvectors of \( \rho^{RQ'} \).

For our purposes the only relevant part of the reversal operation is its restriction to the subspace \( N \) that is the direct sum of the unitary images of \( C \) under the unitaries \( U_\alpha \). This restriction, which is unique \[23\], is given by

\[
R_N = \sum_\alpha U_\alpha^\dagger P_\alpha \otimes P_\alpha U_\alpha . \tag{56}
\]

It is easy to verify that this \( R_N \) reverses \( A_C \), and it also satisfies \( R_N^2(1) = \sum_\alpha P_\alpha = P_N \), the appropriate trace-preserving condition for the restriction. It is interesting to note that since

\[
\rho' = \sum_\alpha \lambda_\alpha U_\alpha \rho U_\alpha^\dagger = \sum_\alpha \lambda_\alpha \rho_\alpha \tag{57}
\]

can be written in terms of an ensemble of orthogonal density operators \( \rho_\alpha \), we can give a very compact equation for the relevant part of the reversal operation:

\[
R_{\rho'} = A_{\rho}^\times . \tag{58}
\]

4 Entropy Exchange and Entanglement Fidelity

4.1 Information-theoretic formulation of reversibility

Our starting point in this section is Eq. \((57)\), which gives \( \rho' \) in terms of an ensemble of orthogonal density operators \( \rho_\alpha \). This allows to conclude that

\[
S(\rho') = S(\rho) - \sum_\alpha \lambda_\alpha \log \lambda_\alpha , \tag{59}
\]
where \( S(\rho) = -\text{tr}(\rho \log \rho) \) is the von Neumann entropy of \( \rho \). Since the quantities \( \lambda_\alpha \) are the eigenvalues of \( A_\rho \), the second term on the right is the entropy of the completely positive superoperator \( A_\rho \) or, equivalently, the entropy of \( \rho^{RQ'} \). Schumacher [16] introduced this entropy and dubbed it the entropy exchange

\[
S_e(\rho, A) \equiv S(\rho^{RQ'}) = S(A_\rho) = - \sum \lambda_\alpha \log \lambda_\alpha .
\]  

(60)

It is useful at this point to introduce the superoperator trace

\[
\text{Tr}(S) \equiv \sum \alpha (\tau_\alpha | S | \tau_\alpha) = \text{tr}((I_\mathcal{R} \otimes S)(|\Psi\rangle\langle\Psi|)) = \text{tr}(S(1)) ,
\]  

(61)

where the operators \( \tau_\alpha \) make up an orthonormal operator basis and where we use Eqs. (32) and (35) to reduce the definition to an operator trace. Not surprisingly, \( A_\rho \) has unit trace:

\[
\text{Tr}(A_\rho) = \text{tr}(A_\rho(1)) = \text{tr}(\rho') = 1 .
\]  

(62)

The entropy exchange (60) can now be written as

\[
S(A_\rho) = - \text{Tr}(A_\rho \log A_\rho) .
\]  

(63)

Equations (46) and (59) are consequences of the reversibility of \( A \) on \( C \). It turns out that they are also sufficient to ensure reversibility [23]. A quantum operation \( A \) is reversible on the code subspace \( C \) if and only if the following two conditions are satisfied:

**Condition 1.**
\[
\text{tr}(A(\rho)) = \mu^2
\]  

(64)

for all \( \rho \) whose support is confined to \( C \), where \( \mu \) is a real constant satisfying \( 0 < \mu \leq 1 \);

**Condition 2.**
\[
S(\rho) = S(\rho') - S(A_\rho)
\]  

(65)

for any one \( \rho \) whose support is the entirety of \( C \) (and then for all \( \rho \) whose support is confined to \( C \)). A proof of this theorem is given in [23]. Here I give a very simple proof of sufficiency, necessity already having been demonstrated.

We start with the orthogonal decomposition of

\[
A_\rho = \sum \alpha \lambda_\alpha \tau_\alpha \otimes \tau_\alpha^\dagger ,
\]  

(66)

where the eigenoperators are orthonormal, i.e., \((\tau_\alpha | \tau_\beta) = \delta_{\alpha\beta}\). The polar-decomposition theorem [24] guarantees that there exists a unitary operator \( U_\alpha \) such that

\[
\tau_\alpha = U_\alpha \sqrt{\sigma_\alpha} = \sqrt{\rho_\alpha} U_\alpha ,
\]  

(67)

where \( \sigma_\alpha = \tau_\alpha^\dagger \tau_\alpha \) and \( \rho_\alpha = \tau_\alpha \tau_\alpha^\dagger \) are normalized density operators that are unitarily equivalent, i.e., \( \rho_\alpha = U_\alpha \sigma_\alpha U_\alpha^\dagger \). The unitary equivalence implies that \( S(\rho_\alpha) = S(\sigma_\alpha) \). We have from Eqs. (44) and (41) that

\[
\rho' = A_\rho(1) = \sum \alpha \lambda_\alpha \rho_\alpha ,
\]  

(68)

\[
\sigma = A_\rho^\times(1) = \sum \alpha \lambda_\alpha \sigma_\alpha .
\]  

(69)
These ensembles for \( \rho \) and \( \sigma \) give several inequalities [23, 26]:

\[
S(\mathcal{A}_\rho) \geq S(\rho') - \sum_\alpha \lambda_\alpha S(\rho_\alpha) \geq 0 ,
\]

(70)

\[
S(\mathcal{A}_\rho) \geq S(\sigma) - \sum_\alpha \lambda_\alpha S(\sigma_\alpha) \geq 0 .
\]

(71)

Equality holds on the left if and only if the density operators \( \rho_\alpha \) (\( \sigma_\alpha \)) are orthogonal, whereas equality holds on the right if and only if the density operators \( \rho_\alpha = \rho' \) (\( \sigma_\alpha = \sigma \)) for all \( \alpha \).

We have not yet used the two Conditions. If we write Condition 1 as \( \mu^2 = \text{tr}(\mathcal{A}(\rho)) = \text{tr}(\mathcal{A}^\times(1)\rho) \) for all \( \rho \) whose support lies in \( C \), we see that \( P_C\mathcal{A}^\times(1)P_C = \mu^2P_C \), which implies

\[
\sigma = \mathcal{A}^\times(1) = \rho .
\]

(72)

Stringing together the left inequality in Eq. (70) and the right inequality in Eq. (71) gives

\[
S(\mathcal{A}_\rho) \geq S(\rho') - \sum_\alpha \lambda_\alpha S(\rho_\alpha) = S(\rho') - \sum_\alpha \lambda_\alpha S(\sigma_\alpha) \geq S(\rho') - S(\rho) .
\]

(73)

Condition 2 dictates equality all the way across here: equality on the left implies that the density operators \( \rho_\alpha \) are orthogonal, and equality on the right implies that the density operators \( \sigma_\alpha \) are all equal to \( \rho \). Thus we have that \( \mathcal{A}_\rho \) has the form (55), with orthogonal density operators \( \rho_\alpha \). Since \( \rho \) has support on the entirety of \( C \), we can put \( \mathcal{A}_C \) in the reversible form (53), thus completing the proof.

### 4.2 Bounds on entropy exchange and entanglement fidelity

Thus far we have dealt with exact exact error correction. Both more difficult and more important is approximate error correction, for which we need a measure of the fidelity of a reversal. Schumacher [16] has introduced a suitable measure,

\[
F_e(\rho, \mathcal{A}) \equiv \langle \Phi_{\sqrt{\rho}}|\rho RQ^\times|\Phi_{\sqrt{\rho}}\rangle ,
\]

(74)

called the entanglement fidelity; it measures the fidelity with which \( \mathcal{A} \) preserves the entanglement of the primary system with the reference system. Using the superoperator formalism of Sec. 3.2, we can write the entanglement fidelity as a superoperator matrix element:

\[
F_e(\rho, \mathcal{A}) = (\sqrt{\rho} | \mathcal{A}_\rho | \sqrt{\rho}) = (\rho | \mathcal{A} | \rho) .
\]

(75)

Since \( \mathcal{A}_\rho \) is positive and has unit trace relative to the left-right action, \( F_e(\rho, \mathcal{A}) = 1 \) if and only if \( \sqrt{\rho} \) is a (normalized) eigenoperator of \( \mathcal{A}_\rho \) with eigenvalue 1, i.e., \( \mathcal{A}_\rho = \sqrt{\rho} \otimes \sqrt{\rho} \). In this case \( \mathcal{A}_\rho \) is a pure completely positive superoperator; both the von Neumann entropy (or entropy exchange), \( S(\mathcal{A}_\rho) = 0 \), and the quadratic entropy, \( \text{Tr}(\mathcal{A}_\rho^2) = 1 \), faithfully report this purity. To analyze approximate reversal, we need relations between these two measures of purity and the entanglement fidelity. Obtaining such relations is the task of this subsection.

We look first at ways to use the purity measures to bound the entanglement fidelity away from 1. Let the operators \( \eta_\alpha \), \( \alpha = 1, \ldots, D^2 \), be an orthonormal operator basis chosen so that \( \eta_1 = \sqrt{\rho} \), which means that \( F_e = (\eta_1 | \mathcal{A}_\rho | \eta_1) \). Then we have

\[
S(\mathcal{A}_\rho) \leq -\sum_{\alpha=1}^{D^2} (\eta_\alpha | \mathcal{A}_\rho | \eta_\alpha) \log(\eta_\alpha | \mathcal{A}_\rho | \eta_\alpha) = -F_e \log F_e - \sum_{\alpha=2}^{D^2} (\eta_\alpha | \mathcal{A}_\rho | \eta_\alpha) \log(\eta_\alpha | \mathcal{A}_\rho | \eta_\alpha) .
\]

(76)
The inequality here is the standard result that the von Neumann entropy is never greater than the entropy calculated in any orthonormal basis [25, 26]. Renormalizing the distribution remaining in the sum, we can write

\[ S(A_\rho) \leq h(F_e) + (1 - F_e) \left( - \sum_{\alpha=2}^{D^2} \frac{\langle \eta_\alpha | A_\rho | \eta_\alpha \rangle}{1 - F_e} \log \frac{\langle \eta_\alpha | A_\rho | \eta_\alpha \rangle}{1 - F_e} \right), \]

(77)

where \( h(x) \equiv -x \log x - (1 - x) \log(1 - x) \) is the binary entropy. Using the fact that the entropy within the large parentheses is bounded above by \( \log(D^2 - 1) \), we get Schumacher’s quantum Fano inequality,

\[ S(A_\rho) \leq h(F_e) + (1 - F_e) \log(D^2 - 1), \]

(78)

which bounds the entanglement fidelity away from one. Schumacher obtained the quantum Fano inequality by applying the same reasoning to \( \rho_{RQ}' \).

We can do the same thing with the quadratic entropy. The analogue of Eqs. (76) and (77) is

\[ \text{Tr}(A_\rho^2) = \sum_{\alpha, \beta} |\langle \eta_\alpha | A_\rho | \eta_\beta \rangle|^2 \geq \sum_{\alpha=1}^{D^2} \langle \eta_\alpha | A_\rho | \eta_\alpha \rangle^2 = F_e^2 + (1 - F_e)^2 \sum_{\alpha=2}^{D^2} \left( \frac{\langle \eta_\alpha | A_\rho | \eta_\alpha \rangle}{1 - F_e} \right)^2. \]

(79)

Since the remaining sum is bounded below by \( (D^2 - 1)^{-1} \), we get a quadratic quantum Fano inequality,

\[ \text{Tr}(A_\rho^2) \geq F_e^2 + \frac{(1 - F_e)^2}{D^2 - 1}, \]

(80)

which like the entropy version, bounds the entanglement fidelity away from one.

We now turn to the opposite task: using the purity measures to place lower bounds on the entanglement fidelity. That this might work is suggested by the case \( S(A_\rho) = 0 \) or, equivalently, \( \text{Tr}(A_\rho^2) = 1 \). Then \( A_\rho = \tau \otimes \tau^\dagger \) is pure, the normalized eigenoperator \( \tau \) having eigenvalue 1. If \( A \) is trace preserving, we have that \( \rho = A_\rho^\times(1) = \tau^\dagger \tau \). Then, by the polar-decomposition theorem, there exists a unitary operator \( U \) such that \( \tau = U \sqrt{\rho} \). Defining a new operation by

\[ A' \equiv U^\dagger \otimes U \circ A, \]

(81)

we have that \( A'_\rho = \sqrt{\rho} \otimes \sqrt{\rho} \), so that \( F_e(\rho, A') = 1 \). What we have shown is that \( A \) is within a unitary of an operation that has unity entanglement fidelity for input density operator \( \rho \).

We now mimic this construction when \( A_\rho \) is not pure. In doing so, we assume that \( A \) is trace preserving. Suppose that \( A_\rho \) has the eigenvalue decomposition (66). Let \( \lambda_1 \) be the largest eigenvalue, and let

\[ \tau_1 = U \sqrt{\sigma_1} \tau_1 = U \sqrt{\sigma_1}, \]

(82)

as in Eq. (67). Define the new operation (81), with the result that

\[ A_\rho' = \lambda_1 \sqrt{\sigma_1} \otimes \sqrt{\sigma_1} + \sum_{\alpha \neq 1} \lambda_\alpha U^\dagger \tau_\alpha \otimes \tau_\alpha^\dagger U. \]

(83)

Now we use the fact that \( A \) is trace preserving to write

\[ \rho = A_\rho^\times(1) = \lambda_1 \sigma_1 + \sum_{\alpha \neq 1} \lambda_\alpha \tau_\alpha^\dagger \tau_\alpha, \]

(84)
which implies that \( \rho \geq \lambda_1 \sigma_1 \). Because the operator square-root function is an operator-monotone function [27], this operator inequality remains true upon taking the square root of both sides:

\[
\sqrt{\rho} \geq \sqrt{\lambda_1} \sqrt{\sigma_1} .
\]  

(85)

The operator-monotone property of the square root is proved in the Appendix. By writing \( \sigma_1 \) in terms of its eigendecomposition, we see that the operator inequality (85) implies that

\[
\text{tr}(\sqrt{\rho} \sqrt{\sigma_1}) \geq \sqrt{\lambda_1} \text{tr}(\sigma_1) = \sqrt{\lambda_1} .
\]  

(86)

Now we notice that

\[
F_e(\rho, A') = (\sqrt{\rho} | A'_\rho \rangle \langle \rho |) \geq \lambda_1 (\text{tr}(\sqrt{\rho} \sqrt{\sigma_1}))^2 \geq \lambda_1^2 .
\]  

(87)

This is our key result. It says that if the largest eigenvalue of \( A_\rho \) is close to 1, then \( A \) can be corrected by a unitary so that the entanglement fidelity is close to 1.

We can translate Eq. (87) into weaker bounds that involve the purity measures. For the entropy exchange we have that

\[
S(A_\rho) = -\sum_\alpha \lambda_\alpha \log \lambda_\alpha \geq -\sum_\alpha \lambda_\alpha \log \lambda_1 = -\log \lambda_1 .
\]  

(88)

Using Eq. (87), we obtain a quantum anti-Fano inequality

\[
F_e(\rho, A') \geq \exp\left(-2S(A_\rho)\right) .
\]  

(89)

For the quadratic entropy we can write

\[
\text{Tr} (A^2_\rho) = \sum_\alpha \lambda_\alpha^2 \leq \sum_\alpha \lambda_\alpha \lambda_1 = \lambda_1 .
\]  

(90)

Again using Eq. (87), we obtain a quadratic quantum anti-Fano inequality

\[
F_e(\rho, A') \geq \left(\text{Tr} (A^2_\rho)\right)^2 .
\]  

(91)

Both of the anti-Fano inequalities place lower bounds on the entanglement fidelity as we had hoped. It should be remembered that they apply only to trace-preserving operations.

Appendix

Let \( A \) and \( B \) be positive operators such that \( A \geq B \). Define the Hermitian operator

\[
F \equiv \sqrt{A} - \sqrt{B} = \sum_k F_k | f_k \rangle \langle f_k | ,
\]  

(92)

where the sum is the eigenvalue decomposition of \( F \). The eigenvalues of \( F \) satisfy

\[
F_k = \langle f_k | \sqrt{A} | f_k \rangle - \langle f_k | \sqrt{B} | f_k \rangle \geq -\langle f_k | \sqrt{B} | f_k \rangle .
\]  

(93)

Now define the Hermitian operator

\[
G \equiv A - B = \sqrt{B} F + F \sqrt{B} + F^2
\]  

(94)

Since \( G \geq 0 \), we have

\[
0 \leq \langle f_k | G | f_k \rangle = 2F_k \langle f_k | \sqrt{B} | f_k \rangle + F_k^2 .
\]  

(95)

Equations (92) and (95) together imply that \( F_k \geq 0 \), which means that \( \sqrt{A} \geq \sqrt{B} \).
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Table 1: How the 27 single-qubit error operators affect an arbitrary superposition of the logical zero $|0\rangle_L = |\uparrow\uparrow\uparrow\rangle$ and the logical one $|1\rangle_L = |\downarrow\downarrow\downarrow\rangle$ in Shor’s nine-bit code. The first line shows the superposition state with no error.

| Error                          | Error operator | State after error |
|-------------------------------|---------------|------------------|
| no error                      | $1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1$ | $\alpha |\uparrow\uparrow\uparrow\rangle + \beta |\downarrow\downarrow\downarrow\rangle$ |
| bit flip on 1st qubit         | $\sigma_1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1$ | $\alpha |\uparrow\uparrow\uparrow\rangle + \beta |\downarrow\downarrow\downarrow\rangle$ |
| bit flip on 2nd qubit         | $1 \otimes \sigma_1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1$ | $\alpha |\uparrow\uparrow\uparrow\rangle + \beta |\downarrow\downarrow\downarrow\rangle$ |
| bit flip on 3rd qubit         | $1 \otimes 1 \otimes \sigma_1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1$ | $\alpha |\uparrow\uparrow\uparrow\rangle + \beta |\downarrow\downarrow\downarrow\rangle$ |
| bit flip on 4th qubit         | $1 \otimes 1 \otimes 1 \otimes \sigma_1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1$ | $\alpha |\uparrow\uparrow\uparrow\rangle + \beta |\downarrow\downarrow\downarrow\rangle$ |
| bit flip on 5th qubit         | $1 \otimes 1 \otimes 1 \otimes 1 \otimes \sigma_1 \otimes 1 \otimes 1 \otimes 1 \otimes 1$ | $\alpha |\uparrow\uparrow\uparrow\rangle + \beta |\downarrow\downarrow\downarrow\rangle$ |
| bit flip on 6th qubit         | $1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes \sigma_1 \otimes 1 \otimes 1 \otimes 1$ | $\alpha |\uparrow\uparrow\uparrow\rangle + \beta |\downarrow\downarrow\downarrow\rangle$ |
| bit flip on 7th qubit         | $1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes \sigma_1 \otimes 1 \otimes 1$ | $\alpha |\uparrow\uparrow\uparrow\rangle + \beta |\downarrow\downarrow\downarrow\rangle$ |
| bit flip on 8th qubit         | $1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes \sigma_1 \otimes 1$ | $\alpha |\uparrow\uparrow\uparrow\rangle + \beta |\downarrow\downarrow\downarrow\rangle$ |
| bit flip on 9th qubit         | $1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes \sigma_1$ | $\alpha |\uparrow\uparrow\uparrow\rangle + \beta |\downarrow\downarrow\downarrow\rangle$ |
| phase flip on 1st qubit      | $\sigma_3 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1$ | $\alpha |\downarrow\uparrow\uparrow\rangle + \beta |\uparrow\downarrow\downarrow\rangle$ |
| phase flip on 2nd qubit      | $1 \otimes \sigma_3 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1$ | $\alpha |\downarrow\uparrow\uparrow\rangle + \beta |\uparrow\downarrow\downarrow\rangle$ |
| phase flip on 3rd qubit      | $1 \otimes 1 \otimes \sigma_3 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1$ | $\alpha |\downarrow\uparrow\uparrow\rangle + \beta |\uparrow\downarrow\downarrow\rangle$ |
| phase flip on 4th qubit      | $1 \otimes 1 \otimes 1 \otimes \sigma_3 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1$ | $\alpha |\downarrow\uparrow\uparrow\rangle + \beta |\uparrow\downarrow\downarrow\rangle$ |
| phase flip on 5th qubit      | $1 \otimes 1 \otimes 1 \otimes 1 \otimes \sigma_3 \otimes 1 \otimes 1 \otimes 1 \otimes 1$ | $\alpha |\downarrow\uparrow\uparrow\rangle + \beta |\uparrow\downarrow\downarrow\rangle$ |
| phase flip on 6th qubit      | $1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes \sigma_3 \otimes 1 \otimes 1 \otimes 1$ | $\alpha |\downarrow\uparrow\uparrow\rangle + \beta |\uparrow\downarrow\downarrow\rangle$ |
| phase flip on 7th qubit      | $1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes \sigma_3 \otimes 1 \otimes 1$ | $\alpha |\downarrow\uparrow\uparrow\rangle + \beta |\uparrow\downarrow\downarrow\rangle$ |
| phase flip on 8th qubit      | $1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes \sigma_3 \otimes 1$ | $\alpha |\downarrow\uparrow\uparrow\rangle + \beta |\uparrow\downarrow\downarrow\rangle$ |
| phase flip on 9th qubit      | $1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes \sigma_3$ | $\alpha |\downarrow\uparrow\uparrow\rangle + \beta |\uparrow\downarrow\downarrow\rangle$ |
| phase-bit flip on 1st qubit  | $-i\sigma_2 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1$ | $\alpha |\down\uparrow\uparrow\rangle + \beta |\up\down\down\down\rangle$ |
| phase-bit flip on 2nd qubit  | $1 \otimes -i\sigma_2 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1$ | $\alpha |\down\uparrow\uparrow\rangle + \beta |\up\down\down\down\rangle$ |
| phase-bit flip on 3rd qubit  | $1 \otimes 1 \otimes -i\sigma_2 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1$ | $\alpha |\down\uparrow\uparrow\rangle + \beta |\up\down\down\down\rangle$ |
| phase-bit flip on 4th qubit  | $1 \otimes 1 \otimes 1 \otimes -i\sigma_2 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1$ | $\alpha |\down\uparrow\uparrow\rangle + \beta |\up\down\down\down\rangle$ |
| phase-bit flip on 5th qubit  | $1 \otimes 1 \otimes 1 \otimes 1 \otimes -i\sigma_2 \otimes 1 \otimes 1 \otimes 1 \otimes 1$ | $\alpha |\down\uparrow\uparrow\rangle + \beta |\up\down\down\down\rangle$ |
| phase-bit flip on 6th qubit  | $1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes -i\sigma_2 \otimes 1 \otimes 1 \otimes 1$ | $\alpha |\down\uparrow\uparrow\rangle + \beta |\up\down\down\down\rangle$ |
| phase-bit flip on 7th qubit  | $1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes -i\sigma_2 \otimes 1 \otimes 1$ | $\alpha |\down\uparrow\uparrow\rangle + \beta |\up\down\down\down\rangle$ |
| phase-bit flip on 8th qubit  | $1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes -i\sigma_2 \otimes 1$ | $\alpha |\down\uparrow\uparrow\rangle + \beta |\up\down\down\down\rangle$ |
| phase-bit flip on 9th qubit  | $1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes -i\sigma_2$ | $\alpha |\down\uparrow\uparrow\rangle + \beta |\up\down\down\down\rangle$ |