Local Galois representations of Swan conductor one

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Abstract

We construct the local Galois representations over the complex field whose Swan conductors are one by using etale cohomology of Artin–Schreier sheaves on affine lines over finite fields. Then, we study the Galois representations, and give an explicit description of the local Langlands correspondences for simple supercuspidal representations. We discuss also a more natural realization of the Galois representations in the etale cohomology of Artin–Schreier varieties.

Introduction

Let $K$ be a non-archimedean local field. Let $n$ be a positive integer. The existence of the local Langlands correspondence for $GL_n(K)$, proved in [LRS93] and [HT01], is one of the fundamental results in the Langlands program. However, even in this fundamental case, an explicit construction of the local Langlands correspondence has not yet been obtained. One of the most striking results in this direction is Bushnell–Henniart’s result for essentially tame representations in [BH05a], [BH05b] and [BH10]. On the other hand, we don’t know much about the explicit construction outside essentially tame representations.

We discuss this problem for representations of Swan conductor 1. The irreducible supercuspidal representations of $GL_n(K)$ of Swan conductor 1 are equivalent to the simple supercuspidal representations in the sense of [AL16] (cf. [GR10], [RY14]). Such representations are called “epipelagic” in [BH14].

Let $p$ be the characteristic of the residue field $k$ of $K$. If $n$ is prime to $p$, the simple supercuspidal representations of $GL_n(K)$ are essentially tame. Hence, this case is covered by Bushnell–Henniart’s work. See also [AL16]. It is discussed in [Kal15] to generalize the construction of the local Langlands correspondence for essentially tame epipelagic representations to other reductive groups.

In this paper, we consider the case where $p$ divides $n$. In this case, the simple supercuspidal representations of $GL_n(K)$ are not essentially tame. Moreover, if $n$ is a power of $p$, the irreducible representations of the Weil group $W_K$ of Swan conductor 1, which correspond to the simple supercuspidal representations via the local Langlands correspondence, cannot be induced from any proper subgroup. Such representations are called primitive (cf. [Koc77]). For simple supercuspidal representations, we have a straightforward characterization of the local Langlands correspondence given in [BH14]. Further, Bushnell–Henniart study the restriction to the wild inertia subgroup of the Langlands parameters for the simple supercuspidal representations explicitly. Actually, the restriction to the wild inertia subgroup already determines the original Langlands parameters up to character twists, but we need additional data, which appear in Bushnell–Henniart’s characterization, to pin down the

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correct Langlands parameters. On the other hand, the construction of the irreducible representations of $W_K$ of Swan conductor 1 is a non-trivial problem. What we will do in this paper is

- to construct the irreducible representations of $W_K$ of Swan conductor 1 without appealing to the existence of the local Langlands correspondence, and
- to give a description of the Langlands parameters themselves for the simple supercuspidal representations.

Let $\ell$ be a prime number different from $p$. For the construction of the irreducible representations of $W_K$ of Swan conductor 1, we use etale cohomology of an Artin–Schreier $\ell$-adic sheaf on $A^1_{k_{ac}}$, where $k_{ac}$ is an algebraic closure of $k$. It will be possible to avoid usage of geometry in the construction of the irreducible representations of $W_K$ of Swan conductor 1. However, we prefer this approach, because

- we can use geometric tools such as the Lefschetz trace formula and the product formula of Deligne–Laumon to study the constructed representations, and
- the construction works also for $\ell$-adic integral coefficients and mod $\ell$ coefficients.

A description of the local Langlands correspondence for the simple supercuspidal representations is discussed in [IT22] in the special case where $n = p = 2$. Even in the special case, our method in this paper is totally different from that in [IT22].

We explain the main result. We write $n = p^{e}n'$, where $n'$ is prime to $p$. We fix a uniformizer $\varpi$ of $K$ and an isomorphism $\iota : \overline{\mathbb{Q}}_{\ell} \simeq \mathbb{C}$.

Let $\mathcal{L}_{\psi}$ be the Artin–Schreier $\overline{\mathbb{Q}}_{\ell}$-sheaf on $A^1_{k_{ac}}$ associated to a non-trivial character $\psi$ of $F_p$. Let $\pi : A^1_{k_{ac}} \rightarrow A^1_{k_{ac}}$ be the morphism defined by $\pi(y) = y^{p^{e}+1}$. Let $\zeta \in \mu_{q-1}(K)$, where $q = |k|$. We put $E_{\zeta} = K[X]/(X^{n'} - \zeta \varpi)$. Then we can define a natural action of $W_{E_{\zeta}}$ on $H^1_c(A^1_{k_{ac}}, \pi^*\mathcal{L}_{\psi})$. Using this action, we can associate a primitive representation $\tau_{\zeta,\chi,c}$ of Swan conductor 1 as the induction of $\tau_{n,\zeta,\chi,c}$ to $W_K$. We can associate a simple supercuspidal representation $\pi_{\zeta,\chi,c}$ of $GL_n(K)$ to the same triple $(\zeta, \chi, c)$ by type theory. Any simple supercuspidal representation can be written in this form uniquely (cf. [IT18 Proposition 1.3]).

**Theorem.** The representations $\tau_{\zeta,\chi,c}$ and $\pi_{\zeta,\chi,c}$ correspond via the local Langlands correspondence.

In Section 1, we recall a general fact on representations of a semi-direct product of a Heisenberg group with a cyclic group. In Section 2 we give a construction of the irreducible representations of $W_K$ of Swan conductor 1. To construct a representation of $W_K$ which naturally fits a description of the local Langlands correspondence, we need a subtle character twist. Such a twist appears also in the essentially tame case in [BH10], where it is called a rectifier. Our twist can be considered as an analogue of the rectifier. We construct the representations of $W_K$ using geometry, but we give also a representation theoretic characterization of the constructed representations. In Section 3 we give a construction of the simple supercuspidal representations of $GL_n(K)$ using the type theory.

In Section 4 we state the main theorem and recall a characterization of the local Langlands correspondence for simple supercuspidal representations given in [BH14]. The characterization consists of the three equalities of (i) the determinant and the central character, (ii) the refined Swan conductors, and (iii) the epsilon factors.

2
In Section 5, we recall some general facts on epsilon factors. In Section 6, we recall facts on Stiefel–Whitney classes, multiplicative discriminants and additive discriminants. We use these facts to calculate Langlands constants of wildly ramified extensions. In Section 7, we recall the product formula of Deligne–Laumon. In Section 8, we show the equality of the determinant and the central character using the product formula of Deligne–Laumon.

In Section 9, we construct a field extension $T_u\zeta$ of $E_\zeta$ such that the restriction of $\tau_{n,\zeta,\chi,c}$ to $W_{T_u\zeta}$ is an induction of a character and $p \nmid [T_u : E_\zeta]$, which we call an imprimitive field. In Section 10, we show the equality of the refined Swan conductors. We see also that the constructed representations of $W_K$ are actually of Swan conductor 1.

In Section 11, we show the equality of the epsilon factors. It is difficult to calculate the epsilon factors of irreducible representations of $W_K$ of Swan conductor 1 directly, because primitive representations are involved. However, we know the equality of the epsilon factors up to $p^e$-th roots of unity if $n = p^e$, since we have already checked the conditions (i) and (ii) in the characterization. Using this fact and $p \nmid [T_u : E_\zeta]$, the problem is reduced to study an epsilon factor of a character. Next we reduce the problem to the case where the characteristic of $K$ is $p$ and $k = \mathbb{F}_p$. At this stage, it is possible to calculate the epsilon factor if $p \neq 2$. However, it is still difficult if $p = 2$, because the direct calculation of the epsilon factor involves an explicit study of the Artin reciprocity map for a wildly ramified extension with a non-trivial ramification filtration. This is a special phenomenon in the case where $p = 2$. We will avoid this difficulty by reducing the problem to the case where $e = 1$. In this case, we have already known the equality up to sign. Hence, it suffices to show the equality of non-zero real parts. This is easy, because the difficult study of the Artin reciprocity map involves only the imaginary part of the equality.

In Appendix A, we discuss a construction of irreducible representations of $W_K$ of Swan conductor 1 in the cohomology of Artin–Schreier varieties. This geometric construction incorporates a twist by a “rectifier”. We see that the “rectifier” parts come from the cohomology of Artin–Schreier varieties associated to quadratic forms. The Artin–Schreier varieties which we use have origins in studies of Lubin–Tate spaces in [IT17] and [IT21].

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Notation
For a finite abelian group $A$, let $A^\vee$ denote the character group $\text{Hom}_\mathbb{Z}(A, \mathbb{C}^\times)$. For a non-archimedean local field $K$, let

- $\mathcal{O}_K$ denote the ring of integers of $K$,
- $p_K$ denote the maximal ideal of $\mathcal{O}_K$,
- $v_K$ denote the normalized valuation of $K$ which sends a uniformizer of $K$ to 1,
- $\text{ch} K$ denote the characteristic of $K$,
- $G_K$ denote the absolute Galois group of $K$,
- $W_K$ denote the Weil group of $K$.
• $I_K$ denote the inertia subgroup of $W_K$,
• $P_K$ denote the wild inertia subgroup of $W_K$,
and we put $U^m_K = 1 + p^m_K$ for any positive integer $m$.

1 Representations of finite groups

First, we recall a fact on representations of Heisenberg groups. Let $G$ be a finite group with center $Z$. We assume the following:

(i) The group $G/Z$ is an elementary abelian $p$-group.
(ii) For any $g \in G \setminus Z$, the map $c_g : G \to Z; g' \mapsto [g, g']$ is surjective.

Remark 1.1. The map $c_g$ in (ii) is a group homomorphism. Hence, $Z$ is automatically an elementary abelian $p$-group.

Let $\psi \in Z^\vee$ be a non-trivial character.

Proposition 1.2. There is a unique irreducible representation $\rho_{\psi}$ of $G$ such that $\rho_{\psi}|Z$ contains $\psi$. Moreover, we have $(\dim \rho_{\psi})^2 = [G : Z]$ and we can construct $\rho_{\psi}$ as follow: Take an abelian subgroup $G_1$ of $G$ such that $Z \subset G_1$ and $2 \dim_{\mathbb{F}_p}(G_1/Z) = \dim_{\mathbb{F}_p}(G/Z)$. Extend $\psi$ to a character $\psi_1$ of $G_1$. Then $\rho_{\psi} = \text{Ind}_{G_1}^{G} \psi_1$.

Proof. The claims other than the construction of $\rho_{\psi}$ is [BF83, (8.3.3) Proposition]. Note that if an abelian subgroup $G_1$ of $G$ satisfies the conditions in the claim, then $G_1/Z$ is a maximal totally isotropic subspace of $G/Z$ under the pairing

$$(G/Z) \times (G/Z) \to \mathbb{C}^\times; (gZ, g'Z) \mapsto \psi([g, g']).$$

Hence the construction follows from the proof of [BF83, (8.3.3) Proposition].

Next, we consider representations of a semi-direct product of a Heisenberg group with a cyclic group. Let $A \subset \text{Aut}(G)$ be a cyclic subgroup of order $p^e+1$ where $e = (\log_p[G : Z])/2$. We assume the following:

(iii) The group $A$ acts on $Z$ trivially.
(iv) For any non-trivial element $a \in A$, the action of $a$ on $G/Z$ fixes only the unit element.

We consider the semi-direct product $A \rtimes G$ by the action of $A$ on $G$.

Lemma 1.3. There is a unique irreducible representation $\rho'_{\psi}$ of $A \rtimes G$ such that $\rho'_{\psi}|_G \simeq \rho_{\psi}$ and $\text{tr} \rho'_{\psi}(a) = -1$ for every non-trivial element $a \in A$.

Proof. The claim is proved in the proof of [BH06, 22.2 Lemma] if $Z$ is cyclic and $\psi$ is a faithful character. In fact, the same proof works also in our case.

Corollary 1.4. There exists a unique representation $\rho'_{\psi}$ of $A \rtimes G$ such that $\rho'_{\psi}|_Z \simeq \psi \otimes p^e$ and $\text{tr} \rho'_{\psi}(a) = -1$ for every non-trivial element $a \in A$. Further, the representation $\rho'_{\psi}|_G$ is irreducible.

Proof. First we show the existence. We take the representation $\rho'_{\psi}$ in Lemma 1.3. Then $\rho'_{\psi}$ has a central character and the central character is equal to $\psi$ by Proposition 1.2. This shows the existence.

We show the uniqueness and the irreducibility of $\rho'_{\psi}|_G$. Assume that $\rho'_{\psi}$ satisfies the condition in the claim. Take an irreducible subrepresentation $\rho_{\psi}$ of $\rho'_{\psi}|_G$. Then $\rho_{\psi}$ satisfies the condition of Proposition 1.2. Hence, $\dim \rho_{\psi} = p^e$. Then we have $\rho_{\psi} = \rho'_{\psi}|_G$ and $\rho'_{\psi}|_G$ is irreducible. Such $\rho_{\psi}$ is unique by Lemma 1.3.
2 Galois representations

2A. Swan conductor

Let $K$ be a non-archimedean local field with residue field $k$. Let $p$ be the characteristic of $k$. Let $f$ be the extension degree of $k$ over $\mathbb{F}_p$. We put $q = p^f$.

Let

$$\text{Art}_K : K^\times \overset{\sim}{\longrightarrow} W_K^{ab}$$

be the Artin reciprocity map, which sends a uniformizer to a lift of the geometric Frobenius element.

Let $\tau$ be a finite dimensional irreducible continuous representation of $W_K$ over $\mathbb{C}$. Let $\Psi : K \to \mathbb{C}^\times$ be a non-trivial additive character. Let $\varepsilon(\tau,s,\Psi)$ denote the Deligne–Langlands local constant of $\tau$ with respect to $\Psi$. We simply write $\varepsilon(\tau,1/2,\Psi)$.

We define an unramified character $\omega_s : K^\times \to \mathbb{C}^\times$ by $\omega_s(\varpi) = q^{-s}$ for $s \in \mathbb{R}$, where $\varpi$ is a uniformizer of $K$. We recall that

$$\varepsilon(\tau,s,\Psi) = \varepsilon(\tau \otimes \omega_s,0,\Psi) \quad (2.1)$$

(cf. [Tat79 (3.6.4)]).

We define $\psi_0 \in \mathbb{F}_p^\times$ by $\psi_0(1) = e^{2\pi \sqrt{-1}/p}$. We take an additive character $\psi_K : K \to \mathbb{C}^\times$ such that $\psi_K(x) = \psi_0(\text{Tr}_K/\mathbb{F}_p(x))$ for $x \in \mathcal{O}_K$. By [BH06] 29.4 Proposition, there exists an integer $\text{sw}(\tau)$ such that

$$\varepsilon(\tau,s,\psi_K) = q^{-\text{sw}(\tau)s} \varepsilon(\tau,0,\psi_K).$$

We put $\text{Sw}(\tau) = \max\{\text{sw}(\tau),0\}$, which we call the Swan conductor of $\tau$.

2B. Construction

In this subsection, we construct a group $Q$ which acts on a curve $C$ over an algebraic closure of $k$. By using this action of $Q$ and Frobenius action, we construct a representation of a semi-direct product $Q \rtimes \mathbb{Z}$ in etale cohomology of $C$. Then we use the representation of $Q \rtimes \mathbb{Z}$ to construct a representation of a Weil group.

We fix an algebraic closure $K^{ac}$ of $K$. Let $k^{ac}$ be the residue field of $K^{ac}$. Let $n$ be a positive integer. We write $n = p^n n'$ with $(p,n') = 1$. Throughout this paper, we assume that $e \geq 1$. Let

$$Q = \left\{(a,b,c) \mid a \in \mu_{p^{e+1}}(k^{ac}), \ b,c \in k^{ac}, \ b^{p^{e+1}} + b = 0, \ c^p - c + b^{p^{e+1}} = 0\right\}$$

be the group whose multiplication is given by

$$(a_1,b_1,c_1) \cdot (a_2,b_2,c_2) = \left(a_1 a_2, b_1 + a_1 b_2, c_1 + c_2 + \sum_{i=0}^{e-1} (a_1 b_1^{p^i} b_2^{p^i})^{p^i}\right).$$

Remark 2.1. The construction of the group $Q$ has its origin in a study of the automorphism of a curve $C$ defined below. We can check that the above multiplication gives a group structure on $Q$ directly, but it’s also possible to show this by checking that the inclusion from $Q$ to the automorphism group of $C$ defined below is compatible with the multiplications.

Note that $|Q| = p^{2e+1}(p^e + 1)$. Let $Q \rtimes \mathbb{Z}$ be a semidirect product, where $m \in \mathbb{Z}$ acts on $Q$ by $(a,b,c) \mapsto (a^{p^{-m}}, b^{p^{-m}}, c^{p^{-m}})$. We put

$$\text{Fr}(m) = ((1,0,0),m) \in Q \rtimes \mathbb{Z} \quad (2.2)$$

5
for \( m \in \mathbb{Z} \).

Let \( C \) be the smooth affine curve over \( k_{ac} \) defined by
\[
x^p - x = y^{p^e + 1} \quad \text{in} \quad \mathbb{A}_{k_{ac}}^2.
\]

We define a right action of \( Q \rtimes \mathbb{Z} \) on \( C \) by
\[
(x, y)((a, b, c), 0) = (x + \sum_{i=0}^{e-1} (by)^{p^i} + c, a(y + b^{p^i}))
\]
\[
(x, y)Fr(1) = (x^p, y^p).
\]

We consider the morphisms
\[
h: \mathbb{A}_{k_{ac}}^1 \to \mathbb{A}_{k_{ac}}^1; \quad x \mapsto x^p - x,
\]
\[
\pi: \mathbb{A}_{k_{ac}}^1 \to \mathbb{A}_{k_{ac}}^1; \quad y \mapsto y^{p^e + 1}.
\]

Then we have the fiber product
\[
\begin{array}{ccc}
C & \xrightarrow{h'} & \mathbb{A}_{k_{ac}}^1 \\
\pi' \downarrow & & \downarrow \pi \\
\mathbb{A}_{k_{ac}}^1 & \xrightarrow{h} & \mathbb{A}_{k_{ac}}^1
\end{array}
\]

where \( \pi' \) and \( h' \) are the natural projections to the first and second coordinates respectively. Let \( g = ((a, b, c), m) \in Q \rtimes \mathbb{Z} \). We consider the morphism
\[
g_0: \mathbb{A}_{k_{ac}}^1 \to \mathbb{A}_{k_{ac}}^1; \quad y \mapsto (a(y + b^{p^i}))^{p^m}.
\]

Let \( \ell \) be a prime number different from \( p \). Then we have a natural isomorphism
\[
c_0: g_0^*h_0'\mathbb{Q}_{\ell} \cong h_0'g_0^*\mathbb{Q}_{\ell} \cong h_0'\mathbb{Q}_{\ell}.
\]

We take an isomorphism \( \iota: \mathbb{Q}_\ell \cong \mathbb{C} \). We sometimes view a character over \( \mathbb{C} \) as a character over \( \mathbb{Q}_\ell \) by \( \iota \). Let \( \psi \in \mathbb{F}_p^\times \). We write \( L_\psi \) for the Artin–Schreier \( \mathbb{Q}_\ell \)-sheaf on \( \mathbb{A}_{k_{ac}}^1 \) associated to \( \psi \), which is equal to \( \mathfrak{N}(\psi) \) in the notation of [Del77, Sommes trig. 1.8 (i)]. Then we have a decomposition \( h_0\mathbb{Q}_\ell = \bigoplus_{\psi \in \mathbb{F}_p^\times} L_\psi \). This decomposition gives canonical isomorphisms
\[
h_0'\mathbb{Q}_\ell \cong \psi h_0\mathbb{Q}_\ell \cong \bigoplus_{\psi \in \mathbb{F}_p^\times} \psi^\ast L_\psi.
\]

The isomorphisms \( c_0 \) and (2.3) induce \( c_{g, \psi}: g_0^*\psi^\ast L_\psi \to \psi^\ast L_\psi \). We define a left action of \( Q \rtimes \mathbb{Z} \) on \( H_c^1(\mathbb{A}_{k_{ac}}^1, \psi^\ast L_\psi) \) by
\[
H_c^1(\mathbb{A}_{k_{ac}}^1, \psi^\ast L_\psi) \xrightarrow{g_0} H_c^1(\mathbb{A}_{k_{ac}}^1, g_0^*\psi^\ast L_\psi) \xrightarrow{c_{g, \psi}} H_c^1(\mathbb{A}_{k_{ac}}^1, \psi^\ast L_\psi)
\]
for \( g \in Q \rtimes \mathbb{Z} \). Let \( \tau_\psi \) be the representation of \( Q \rtimes \mathbb{Z} \) over \( \mathbb{C} \) defined by \( H_c^1(\mathbb{A}_{k_{ac}}^1, \psi^\ast L_\psi) \) and \( \iota \). For \( \theta \in \mu_{p^e + 1}(k_{ac})^\vee \), let \( K_\theta \) be the smooth Kummer \( \mathbb{Q}_\ell \)-sheaf on \( \mathbb{G}_{m, k_{ac}} \) associated to \( \theta \). We view \( \mu_{p^e + 1}(k_{ac}) \times \mathbb{F}_p \) as a subgroup of \( Q \) by \( (a, c) \mapsto (a, 0, c) \).

**Lemma 2.2.** We have a natural isomorphism
\[
H_c^1(\mathbb{A}_{k_{ac}}^1, \psi^\ast L_\psi) \cong \bigoplus_{\theta \in \mu_{p^e + 1}(k_{ac})^\vee \setminus \{1\}} H_c^1(\mathbb{G}_{m, k_{ac}}, L_\psi \otimes K_\theta).
\]
which is compatible with the actions of $\mu_{p^{e+1}}(k^{ac}) \times \mathbb{F}_p$ where $(a, c) \in \mu_{p^{e+1}}(k^{ac}) \times \mathbb{F}_p$ acts on $H^1_c(\mathbb{G}_{m,k^{ac}}, \mathcal{L}_\psi \otimes \mathcal{K}_\theta)$ by $\theta(a)\psi(c)$. Further, we have

$$\dim H^1_c(\mathbb{G}_{m,k^{ac}}, \mathcal{L}_\psi \otimes \mathcal{K}_\theta) = 1$$

for any $\theta \in \mu_{p^{e+1}}(k^{ac})^\vee \setminus \{1\}$.

**Proof.** By the projection formula, we have natural isomorphisms

$$\pi_*\pi^*\mathcal{L}_\psi \simeq \pi_*\pi^*\mathcal{L}_\psi \otimes \mathcal{Q}_\ell \simeq \mathcal{L}_\psi \otimes \pi_*\mathcal{Q}_\ell$$
on $\mathcal{A}_{k^{ac}}$. Further, we have

$$\pi_*\mathcal{Q}_\ell \simeq \bigoplus_{\theta \in \mu_{p^{e+1}}(k^{ac})^\vee} \mathcal{K}_\theta$$
on $\mathbb{G}_{m,k^{ac}}$, since $\pi$ is a finite etale $\mu_{p^{e+1}}(k^{ac})$-covering over $\mathbb{G}_{m,k^{ac}}$. Therefore, we have

$$\pi_*\pi^*\mathcal{L}_\psi \simeq \mathcal{L}_\psi \otimes \pi_*\mathcal{Q}_\ell \simeq \bigoplus_{\theta \in \mu_{p^{e+1}}(k^{ac})^\vee} \mathcal{L}_\psi \otimes \mathcal{K}_\theta \quad (2.4)$$
on $\mathbb{G}_{m,k^{ac}}$. Let $\{0\}$ denote the origin of $\mathcal{A}_{k^{ac}}$. Let $i : \{0\} \to \mathcal{A}_{k^{ac}}$ and $j : \mathbb{G}_{m,k^{ac}} \to \mathcal{A}_{k^{ac}}$ be the natural immersions. From the exact sequence

$$0 \to j_!j^*\pi^*\mathcal{L}_\psi \to \pi^*\mathcal{L}_\psi \to i_*i^*\pi^*\mathcal{L}_\psi \to 0,$$

we have the exact sequence

$$0 \to H^0(\{0\}, i^*\pi^*\mathcal{L}_\psi) \to H^1(\mathbb{G}_{m,k^{ac}}, \pi^*\mathcal{L}_\psi) \to H^1_c(\mathcal{A}_{k^{ac}}, \pi^*\mathcal{L}_\psi) \to 0, \quad (2.5)$$

since $H^0(\mathcal{A}_{k^{ac}}, \pi^*\mathcal{L}_\psi) = 0$ and $H^1(\{0\}, i^*\pi^*\mathcal{L}_\psi) = 0$. Note that $H^0(\{0\}, i^*\pi^*\mathcal{L}_\psi) \simeq \psi$. By (2.4), we have isomorphisms

$$H^1_c(\mathbb{G}_{m,k^{ac}}, \pi^*\mathcal{L}_\psi) \simeq H^1_c(\mathbb{G}_{m,k^{ac}}, \pi_*\pi^*\mathcal{L}_\psi) \simeq \bigoplus_{\theta \in \mu_{p^{e+1}}(k^{ac})^\vee} H^1_c(\mathbb{G}_{m,k^{ac}}, \mathcal{L}_\psi \otimes \mathcal{K}_\theta). \quad (2.6)$$

We know that

$$\dim H^1_c(\mathbb{G}_{m,k^{ac}}, \mathcal{L}_\psi \otimes \mathcal{K}_\theta) = 1 \quad (2.7)$$

for any $\theta \in \mu_{p^{e+1}}(k^{ac})^\vee$ by the proof of [IT17 Lemma 7.1] (cf. [IT23 (2.3)]). Since the composition of

$$H^0(\{0\}, i^*\pi^*\mathcal{L}_\psi) \to H^1_c(\mathbb{G}_{m,k^{ac}}, \pi^*\mathcal{L}_\psi)$$

and (2.6) is compatible with the actions of $\mu_{p^{e+1}}(k^{ac}) \times \mathbb{F}_p$, it factors through an isomorphism $H^0(\{0\}, i^*\pi^*\mathcal{L}_\psi) \simeq H^1_c(\mathbb{G}_{m,k^{ac}}, \mathcal{L}_\psi)$ by (2.7). Then the claim follows from (2.5), (2.6) and (2.7).

Let $\varphi : \mu_2(k) \to \mathbb{C}^\times$ be the non-trivial group homomorphism, if $p \neq 2$. We define a character $\theta_0 \in \mu_{p^{e+1}}(k^{ac})^\vee$ by

$$\theta_0(a) = \begin{cases} \varphi(a^{p^{e+1}/2}) & \text{if } p \neq 2, \\ 1 & \text{if } p = 2 \end{cases} \quad (2.8)$$
for $a \in \mu_{p^e+1}(k^{ac})$. For an integer $m$ and a positive odd integer $m'$, let $(\frac{m}{m'})$ denote the Jacobi symbol. For an odd prime $p$, we set

$$
\epsilon(p) = \begin{cases} 
1 & \text{if } p \equiv 1 \mod 4, \\
\sqrt{-1} & \text{if } p \equiv 3 \mod 4.
\end{cases}
$$

We have $\epsilon(p)^2 = (\frac{-1}{p})$. We define a representation $\tau_n$ of $Q \rtimes \mathbb{Z}$ as the twist of $\tau_{\psi_0}$ by the character

$$
Q \rtimes \mathbb{Z} \rightarrow \mathbb{C}^\times; \ ((a, b, c), m) \mapsto \begin{cases} 
\theta_0(a)^n((-\epsilon(p)\left(\frac{-2n}{p}\right))^n p^{-\frac{1}{2}})^m & \text{if } p \neq 2, \\
((\sqrt{-1})^{\frac{n(n-2)}{8}} p^{-\frac{1}{2}})^m & \text{if } p = 2.
\end{cases}
$$

(2.9)

The value of this character is related to a quadratic Gauss sum. A geometric origin of this character is given in [A.3]. Let $(\zeta, \chi, c) \in \mu_{q-1}(k) \times (k^\times) \times \mathbb{C}^\times$. We take a uniformizer $\varpi$ of $K$. We choose an element $\varphi'_\zeta \in K^{ac}$ such that $\varphi'^{-1}_\zeta = \zeta \varpi$ and set $E_\zeta = K(\varphi'_\zeta)$. We choose elements $\alpha_\zeta, \beta_\zeta, \gamma_\zeta \in K^{ac}$ such that

$$
\alpha_\zeta^{p+1} = -\varphi'_\zeta, \quad \beta_\zeta^{p+1} + \beta_\zeta = -\alpha_\zeta^{-1}, \quad \gamma_\zeta^p - \gamma_\zeta = \beta_\zeta^{p+1}.
$$

(2.10)

For $\sigma \in W_{E_\zeta}$, we set

$$
a_\sigma = \sigma(\alpha_\zeta)/(\alpha_\zeta), \quad b_\sigma = a_\sigma \sigma(\beta_\zeta) - \beta_\zeta, \quad c_\sigma = \sigma(\gamma_\zeta) - \gamma_\zeta + \sum_{i=0}^{\epsilon-1} (b_\sigma^i (\beta_\zeta + b_\sigma))^p.
$$

(2.11)

Then we have $a_\sigma, b_\sigma, c_\sigma \in \mathcal{O}_{K^\times}$. For $\sigma \in W_{E_\zeta}$, we put $n_\sigma = v_{E_\zeta}(\text{Art}_{E_\zeta}(\sigma))$. We have the homomorphism

$$
\Theta_\zeta: W_{E_\zeta} \rightarrow Q \rtimes \mathbb{Z}; \ \sigma \mapsto ((\tilde{a}_\sigma, \tilde{b}_\sigma, \tilde{c}_\sigma), f n_\sigma).
$$

(2.12)

**Lemma 2.3.** The image of the homomorphism $\Theta_\zeta$ is $Q \rtimes (f\mathbb{Z})$.

**Proof.** It suffices to show that the image of $I_{E_\zeta} \subset W_{E_\zeta}$ under $\Theta_\zeta$ is equal to $Q \subset Q \rtimes \mathbb{Z}$, since the homomorphism $W_{E_\zeta} \rightarrow f\mathbb{Z}; \ \sigma \mapsto f n_\sigma$ is surjective. We put $N_\zeta = E_\zeta(\alpha_\zeta, \beta_\zeta, \gamma_\zeta)$. Then the kernel of $\Theta_\zeta$ is equal to $I_{N_\zeta}$ by the definition. Hence we have an injection $I_{E_\zeta}/I_{N_\zeta} \rightarrow Q$. This injection is actually a bijection, since $N_\zeta$ is a totally ramified extension over $E_\zeta$ of degree $p^2+1(p^e+1)$, which equals to $|Q|$. Therefore, we obtain the claim. \qed

We write $\tau_{n_\zeta}$ for the representation of $W_{E_\zeta}$ given by $\Theta_\zeta$ and $\tau_n$. Recall that $c$ is an element of $\mathbb{C}^\times$. Let $\phi_\zeta: W_{E_\zeta} \rightarrow \mathbb{C}^\times$ be the character defined by $\phi_\zeta(\sigma) = c^{n_\sigma}$. We have the isomorphism $\phi^{\mathfrak{G}}_\zeta \times \mathcal{O}_{E_\zeta}^\times \simeq E_\zeta^\times$ given by the multiplication. Let $\text{Frob}_p: k^\times \rightarrow k^\times$ be the inverse of the $p$-th power map. We consider the following composition:

$$
\lambda_\zeta: W_{E_\zeta}^{ab} \simeq E_\zeta^\times \simeq \phi^{\mathfrak{G}}_\zeta \times \mathcal{O}_{E_\zeta}^\times \xrightarrow{\text{pr}_2} \mathcal{O}_{E_\zeta}^\times \xrightarrow{\text{can}} k^\times \xrightarrow{\text{Frob}_p} k^\times.
$$

We put

$$
\tau_{n_\zeta, c} = \tau_{n_\zeta} \otimes (\chi \circ \lambda_\zeta) \otimes \phi_\zeta \quad \text{and} \quad \tau_{c, \chi, c} = \text{Ind}_{E_\zeta/K} \tau_{n_\zeta, c}.
$$

(2.13)

We will see that $\tau_{c, \chi, c}$ is an irreducible representation of Swan conductor 1 in Proposition 10.8. This Galois representation $\tau_{c, \chi, c}$ is our main object in this paper. We will study several invariants associated to this, for example its determinant and epsilon factor.
2C. Characterization

We put
\[ Q_0 = \{(1, b, c) \in Q \}, \quad F = \{(1, 0, c) \in Q \mid c \in \mathbb{F}_p \}. \]

We identify \( \mathbb{F}_p \) with \( F \) by \( c \mapsto (1, 0, c) \).

**Lemma 2.4.** For any \( g = (1, b, c) \in Q_0 \) with \( b \neq 0 \), the map \( Q_0 \to F; \ g' \mapsto [g, g'] \) is surjective.

**Proof.** For \( (1, b_1, c_1), (1, b_2, c_2) \in Q_0 \), we have
\[ [(1, b_1, c_1), (1, b_2, c_2)] = \left( 1, 0, \sum_{i=0}^{s-1} (b_1^p b_2^p - b_1 b_2^p) i \right). \]

If \( b_1 \neq 0 \), then
\[ \{ b \in k^c \mid b^{p^e} + b = 0 \} \to \mathbb{F}_p; \ b_2 \to b_1^p b_2 - b_1 b_2^p \]
is surjective. The claim follows from the surjectivity of \( \text{Tr}_{\mathbb{F}_p/\mathbb{F}_p} \). \( \square \)

By this lemma, we can apply the results from Section 1 to our situation with \( G = Q_0 \), \( Z = F \) and \( A = \mu_{p^e+1}(k^c) \), where the action of \( \mu_{p^e+1}(k^c) \) on \( Q_0 \) is given by the embedding
\[ \mu_{p^e+1}(k^c) \to Q; \ a \mapsto (a, 0, 0) \]
and the conjugation. Let \( \tau_0 \) denote the unique representation of \( Q \) characterized by
\[ \tau_0|_F \simeq \psi_0^{\oplus p^e}, \quad \text{Tr} \tau_0((a, 0, 0)) = -1 \]
for \( a \in \mu_{p^e+1}(k^c) \setminus \{1\} \) (cf. Corollary 1.4).

We have a decomposition
\[ \tau_0 = \bigoplus_{\theta \in \mu_{p^e+1}(k^c) \setminus \{1\}} L_{\theta} \]
(2.15)
such that \( a \in \mu_{p^e+1}(k^c) \) acts on \( L_{\theta} \) by \( \theta(a) \), since the both sides of (2.15) have the same character as representations of \( \mu_{p^e+1}(k^c) \). For a positive integer \( m \) dividing \( p^e + 1 \), we consider \( \mu_m(k^c) \) as a subset of \( \mu_{p^e+1}(k^c) \) by the dual of the surjection
\[ \mu_{p^e+1}(k^c) \to \mu_m(k^c); \ x \mapsto x^{p^e+1} \]
We simply write \( Q \) for the subgroup \( Q \times \{0\} \subset Q \times Z \).

**Lemma 2.5.** We have \( \tau_0|_Q \simeq \tau_0 \).

**Proof.** The representation \( \tau_0|_Q \) satisfies the characterization (2.14) by Lemma 2.2. Hence \( \tau_0|_Q \) is isomorphic to \( \tau_0 \). \( \square \)

**Corollary 2.6.** The representation \( \tau_0|_{Q_0} \) is irreducible.

**Proof.** This follows from Corollary 1.4 (2.14) and Lemma 2.5. \( \square \)

For any odd prime \( p \), we have
\[ \sum_{x \in \mathbb{F}_p^e} \psi_0(x^2) = \sum_{x \in \mathbb{F}_p^e} \left( \frac{x}{p} \right) \psi_0(x) = \epsilon(p) \sqrt{p} \]
(2.16)
by Gauss.
Lemma 2.7. We have
\[ \text{Tr} \tau_{\psi_0} (\text{Fr}(1)) = \begin{cases} -\epsilon(p) \sqrt{p} & \text{if } p \neq 2, \\ 0 & \text{if } p = 2. \end{cases} \]

Proof. By the Lefschetz trace formula, we have
\[ \sum_{x \in A^1(\mathbb{F}_p)} \text{Tr}(\text{Fr}_p, (\pi^* \mathcal{L}_{\psi_0})_x) = \sum_{i=0}^{2} (-1)^i \text{Tr}(\text{Fr}_p, H^i_c(\mathbb{A}^1_{k^{ac}}, \pi^* \mathcal{L}_{\psi})) \]
where Fr denotes the geometric p-th power Frobenius morphism. Since \( H^i_c(\mathbb{A}^1_{k^{ac}}, \pi^* \mathcal{L}_{\psi}) \) vanishes for \( i = 0, 2 \), we have
\[ \text{Tr} \tau_{\psi_0} (\text{Fr}(1)) = - \sum_{x \in \mathbb{F}_p} \text{Tr}(\text{Fr}_p, (\pi^* \mathcal{L}_{\psi_0})_x) \]
\[ = - \sum_{x \in \mathbb{F}_p} \psi_0(x^{p+1}) - \sum_{x \in \mathbb{F}_p} \psi_0(x^2) = \begin{cases} -\epsilon(p) \sqrt{p} & \text{if } p \neq 2, \\ 0 & \text{if } p = 2, \end{cases} \]
where we use (2.16) in the last equality.

We assume \( p = 2 \) in this paragraph. We take \( b_0 \in \mathbb{F}_{2e} \) such that \( \text{Tr}_{\mathbb{F}_{2e}/\mathbb{F}_2}(b_0) = 1 \). Further, we put
\[ c_0 = b_0^{2e} + \sum_{0 \leq i < j \leq e-1} b_0^{2e+i+2j}. \]
Then we have
\[ c_0^2 - c_0 = b_0^{2e+1} + b_0^{2e} + \sum_{0 \leq i < j \leq e-1} b_0^{2e+i+2j+1} + \sum_{0 \leq i < j \leq e-1} b_0^{2e+i+2j} \]
\[ = b_0^{2e+1} + b_0^{2e} + \sum_{i=0}^{e-2} b_0^{2e+i+1+2e} + \sum_{j=1}^{e-1} b_0^{2e+2j} \]
\[ = b_0^{2e+1} + b_0^{2e} + b_0^{2e}(1 + b_0 + b_0^{2e}) = b_0^{2e+1}, \]
where we use \( \text{Tr}_{\mathbb{F}_{2e}/\mathbb{F}_2}(b_0) = 1 \) at the third equality. We put
\[ g = ((1, b_0, c_0), -1) \in Q \times \mathbb{Z}. \]

Lemma 2.8. We assume that \( p = 2 \). Then we have \( \text{Tr} \tau_{\psi_0}(g^{-1}) = -2 \).

Proof. We note that
\[ g^{-1} = \text{Fr}(1) \left( \left( 1, b_0, c_0 + \sum_{i=0}^{e-1} (b_0^{2e+1})^2 \right), 0 \right). \]
For \( y \in k^{ac} \) satisfying \( y^2 + b_0^{2e} = y \), we take \( x_y \in k^{ac} \) such that \( x_y^2 - x_y = y^{2e+1} \). We take \( y_0 \in k^{ac} \) such that \( y_0^2 + b_0^{2e} = y_0 \). Then, by the Lefschetz trace formula and (2.19), we have
\[ \text{Tr} \tau_{\psi_0}(g^{-1}) = - \sum_{y^2 + b_0^{2e} = y} \text{Tr}(g^{-1}, (\pi^* \mathcal{L}_{\psi_0})_y) \]
\[ = - \sum_{y^2 + b_0^{2e} = y} \psi_0(y^2 - x_y + \sum_{i=0}^{e-1} (b_0 y^2)^2 + c_0 + \sum_{i=0}^{e-1} (b_0^{2e+1})^2) \]
\[ = - \sum_{y \in \mathbb{F}_2} \psi_0((y_0 + z)^{2e+1} + \sum_{i=0}^{e-1} (b_0(y_0 + z))^2 + c_0) = -2, \]
where we change a variable by $y = y_0 + z$ at the second equality, and use

$$y_0^{2^e + 1} + \sum_{i=0}^{e-1} (y_0 b_0)^{2^i} = y_0 \left( y_0 + \sum_{i=0}^{e-1} b_0^{2^{e+i}} \right) + \sum_{i=0}^{e-1} b_0^{2^i} \left( y_0 + \sum_{j=0}^{i-1} b_0^{2^{e+j}} \right) = c_0,$$

$$y_0^{2^e} + y_0 + \sum_{i=0}^{e-1} b_0^{2^i} = \sum_{i=0}^{e-1} (y_0^2 + y_0)^{2^i} + \sum_{i=0}^{e-1} b_0^{2^i} = \text{Tr}_{F_2/F_2}(b_0) = 1$$

at the last equality.

**Proposition 2.9.** The representation $\tau_{\psi_0}$ is characterized by $\tau_{\psi_0}|_Q \simeq \tau_0$ and

$$\begin{cases} 
\text{Tr} \tau_{\psi_0}(\text{Fr}(1)) = -\epsilon(p)\sqrt{p} & \text{if } p \neq 2, \\
\text{Tr} \tau_{\psi_0}(g^{-1}) = -2 & \text{if } p = 2.
\end{cases}$$

In particular, $\tau_{\psi_0}$ does not depend on the choice of $\ell$ and $t$.

**Proof.** This follows from Lemma 2.5, Lemma 2.7 and Lemma 2.8. □

## 3 Representations of general linear algebraic groups

### 3A. Simple supercuspidal representation

Let $\pi$ be an irreducible supercuspidal representation of $\text{GL}_n(K)$ over $\mathbb{C}$. Let $\varepsilon(\pi, s, \Psi)$ denote the Godement–Jacquet local constant of $\pi$ with respect to the non-trivial character $\Psi: K \to \mathbb{C}^\times$. We simply write $\varepsilon(\pi, \Psi)$ for $\varepsilon(\pi, 1/2, \Psi)$. By [GJ72, Theorem 3.3 (4)], there exists an integer $\text{sw}(\pi)$ such that

$$\varepsilon(\pi, s, \psi_K) = q^{-\text{sw}(\pi)s} \varepsilon(\pi, 0, \psi_K).$$

We put $\text{Sw}(\pi) = \max\{\text{sw}(\pi), 0\}$, which we call the Swan conductor of $\pi$.

**Definition 3.1.** An irreducible supercuspidal representation $\pi$ of $\text{GL}_n(K)$ over $\mathbb{C}$ is called simple supercuspidal if $\text{Sw}(\pi) = 1$.

This definition is equivalent to [IT18, Definition 1.1] by [IT18, Proposition 1.3].

### 3B. Construction

In the following, we construct a smooth representation $\pi_{\zeta, \chi, c}$ of $\text{GL}_n(K)$ for each triple $(\zeta, \chi, c) \in \mu_{q-1}(K) \times (k^\times)^\vee \times \mathbb{C}^\times$.

Let $B \subset M_n(k)$ be the subring consisting of upper triangular matrices. Let $\mathfrak{J} \subset M_n(\mathcal{O}_K)$ be the inverse image of $B$ under the reduction map $M_n(\mathcal{O}_K) \to M_n(k)$. Then $\mathfrak{J}$ is a hereditary $\mathcal{O}_K$-order (cf. [BK93, (1.1)])]. Let $\Psi$ denote the Jacobson radical of the order $\mathfrak{J}$. We put $U_1 = 1 + \Psi \subset \text{GL}_n(\mathcal{O}_K)$. We set

$$\phi_\zeta = \begin{pmatrix} 0 & I_{n-1} \\ \zeta \infty & 0 \end{pmatrix} \in M_n(K) \quad \text{and} \quad L_\zeta = K(\phi_\zeta).$$

Then, $L_\zeta$ is a totally ramified extension of $K$ of degree $n$. 

11
We put $\varphi_{\zeta,n} = n' \varphi_{\zeta}$ and

$$\epsilon_0 = \begin{cases} (n' + 1)/2 & \text{if } p^e = 2, \\ 0 & \text{if } p^e \neq 2. \end{cases}$$

We define a character $\Lambda_{\zeta,\chi,c}: L_\zeta^2 U_1^3 \to \mathbb{C}^\times$ by

$$\Lambda_{\zeta,\chi,c}(\varphi_{\zeta}) = (-1)^{n-1+\epsilon_0} f, \quad \Lambda_{\zeta,\chi,c}(x) = \chi(x) \text{ for } x \in \mathcal{O}_K^\times, \quad \Lambda_{\zeta,\chi,c}(x) = (\psi_K \circ \text{tr})(\varphi_{\zeta,n}^{-1}(x - 1)) \text{ for } x \in U_1^3,$$

where $\text{tr}$ means the trace as an element of $M_n(K)$. We put

$$\pi_{\zeta,\chi,c} = c\text{-Ind}_{GL_n(K)}^{L_\zeta^2 U_1^3} \Lambda_{\zeta,\chi,c}.$$

Then, $\pi_{\zeta,\chi,c}$ is a simple supercuspidal representation of $GL_n(K)$, and every simple supercuspidal representation is isomorphic to $\pi_{\zeta,\chi,c}$ for a uniquely determined $(\zeta, \chi, c) \in \mu_{q-1}(K) \times (k^\times)^\vee \times \mathbb{C}^\times$ by [IT18, Proposition 1.3]. The representation $\pi_{\zeta,\chi,c}$ contains the $m$-simple stratum $[I, 1, 0, \varphi_{\zeta,n}^{-1}]$ in the sense of [BH14, 2.1].

**Proposition 3.2.** $\varepsilon(\pi_{\zeta,\chi,c}, \psi_K) = (-1)^{n-1+\epsilon_0} f \chi(n') c$.

**Proof.** This follows from [BH99, 6.1 Lemma 2 and 6.3 Proposition 1]. $\square$

## 4 Local Langlands correspondence

Our main theorem is the following:

**Theorem 4.1.** The representations $\pi_{\zeta,\chi,c}$ and $\tau_{\zeta,\chi,c}$ correspond via the local Langlands correspondence.

To prove this theorem, we recall a characterization of the local Langlands correspondence for epipelagic representations due to Bushnell–Henniart. Recall that $\Psi: K \to \mathbb{C}^\times$ is a non-trivial character. The following lemma is a special case of [DHS1, Proposition 4.13].

**Lemma 4.2.** ([BH14, 2.3 Lemma]) Let $\tau$ be an irreducible smooth representation of $W_K$ such that $\text{sw}(\tau) \geq 1$. Then, there exists $\gamma_{\tau,\Psi} \in K^\times$ such that

$$\varepsilon(\chi \otimes \tau, s, \Psi) = \chi(\gamma_{\tau,\Psi})^{-1} \varepsilon(\tau, s, \Psi)$$

for any tamely ramified character $\chi$ of $W_K$. This property determines the coset $\gamma_{\tau,\Psi} U_K^1$ uniquely.

**Definition 4.3.** Let $\tau$ be an irreducible smooth representation of $W_K$ such that $\text{sw}(\tau) \geq 1$. We take $\gamma_{\tau,\Psi}$ as in Lemma 4.2. We put

$$\text{rsw}(\tau, \Psi) = \gamma_{\tau,\Psi}^{-1} \in K^\times / U_K^1,$$

which we call the refined Swan conductor of $\tau$ with respect to $\Psi$.

**Remark 4.4.** By (2.1), we have $v_K(\text{rsw}(\tau, \psi_K)) = \text{Sw}(\tau)$ in Definition 4.3.
Lemma 4.5. Let $\pi$ be an irreducible supercuspidal representation of $GL_n(K)$ such that $sw(\pi) \geq 1$.
1. There exists $\gamma_{\pi,\Psi} \in K^\times$ such that
   
   $\varepsilon(\chi \otimes \pi, s, \Psi) = \chi(\gamma_{\pi,\Psi})^{-1}\varepsilon(\pi, s, \Psi)$

   for any tamely ramified character $\chi$ of $K^\times$. This property determines the coset $\gamma_{\pi,\Psi}U^1_K$ uniquely.
2. Let $[A, m, 0, \alpha]$ be a simple stratum contained in $\pi$. Then we have $\gamma_{\pi,\Psi} \equiv \det \alpha \mod U^1_K$.

Proof. The first statement is [BH99, 1.4 Theorem (i)]. The second statement follows from [BH99, 1.4 Remark].

Definition 4.6. Let $\pi$ be an irreducible supercuspidal representation of $GL_n(K)$ such that $sw(\pi) \geq 1$. We take $\gamma_{\pi,\Psi}$ as in Lemma 4.5. Then we put

   $rsw(\pi, \Psi) = \gamma_{\pi,\Psi}^{-1} \in K^\times/U^1_K,$

which we call the refined Swan conductor of $\pi$ with respect to $\Psi$.

Remark 4.7. We have $v_K(rsw(\pi, \psi_K)) = Sw(\pi)$ in Definition 4.6.

For an irreducible supercuspidal representation $\pi$ of $GL_n(K)$, let $\omega_\pi$ denote the central character of $\pi$.

Proposition 4.8. ([BH14, 2.3 Proposition]) Let $\pi$ be a simple supercuspidal representation of $GL_n(K)$. The Langlands parameter for $\pi$ is characterized as the $n$-dimensional irreducible smooth representation $\tau$ of $W_K$ satisfying

(i) $\det \tau = \omega_\pi$,

(ii) $rsw(\tau, \psi_K) = rsw(\pi, \psi_K)$,

(iii) $\varepsilon(\tau, \psi_K) = \varepsilon(\pi, \psi_K)$.

We will show that $\tau_{\xi,\chi,c}$ and $\pi_{\xi,\chi,c}$ satisfy the conditions of Proposition 4.8 in Proposition 8.6, Proposition 10.5, Lemma 10.7 and Proposition 11.6.

5 General facts on epsilon factors

In this section, we recall some general facts on epsilon factors.

For a finite separable extension $L$ over $K$, we put $\Psi_L = \Psi \circ \Tr_{L/K}$ and let

\[ \lambda(L/K, \Psi) = \frac{\varepsilon(\Ind_{L/K}1, s, \Psi)}{\varepsilon(1, s, \Psi_L)} \]

denote the Langlands constant which is independent of $s$, where $1$ is the trivial representation of $W_L$ (cf. [BH06, 30.4]).

Proposition 5.1. Let $\tau$ be a finite dimensional smooth representation of $W_K$ such that $\tau|_{PK}$ is irreducible and non-trivial. Let $L$ be a tamely ramified finite extension of $K$. Then we have

\[ \varepsilon(\tau|_{W_L}, \Psi_L) = \lambda(L/K, \Psi)^{-\dim \delta_{L/K}(rsw(\tau, \Psi))}\varepsilon(\tau, \Psi)^{[L:K]}. \]
Proposition 5.2. Let $\tau$ be a finite dimensional smooth representation of $W_K$ such that $\tau|_{P_K}$ does not contain the trivial character.
1. If $\phi$ is a tamely ramified character of $W_K$, then we have $rsw(\tau \otimes \phi, \Psi) = rsw(\tau, \Psi)$.
2. Let $L$ be a tamely ramified finite extension of $K$. Then we have
$$rsw(\tau|_{W_L}, \Psi_L) = rsw(\tau, \Psi) \mod U_L^{1}.$$

Proof. This is [BH06, 48.1 Theorem (2) and (3)].

Proposition 5.3. Let $\xi$ be a character of $K^\times$ of level $m \geq 1$. Assume that $\gamma \in K^\times$ satisfies
$$\xi(1 + x) = \Psi(\gamma x)$$
for $x \in p_{K}^{[m/2]+1}$.
1. We have $rsw(\xi, \Psi) = \gamma^{-1}$.
2. We have
$$\varepsilon(\xi, \Psi) = q^{[(m+1)/2]-(m+1)/2} \sum_{y \in U_{K}^{[m+1/2]/[U_{K}^{m/2}]+1}} \xi(\gamma y)^{-1}\Psi(\gamma y).$$

Proof. The claim 1 follows from [BH06, 23.8 Stability theorem]. The claim 2 follows from [BH06 23.5 Lemma 1, (23.6.2) and 23.6 Proposition].

Lemma 5.4. Let $m$ be a positive integer dividing $f$. Let $h$ be a positive integer that is prime to $p$ and less than $p^m v_K(p)/(p^m - 1)$. Let $L$ be a Galois extension of $K$ defined by $x^{p^m} - x = 1/\varpi^h$. Then we have
$$Gal(L/K)_i = \begin{cases} Gal(L/K) & \text{if } i \leq h, \\ \{1\} & \text{if } i > h \end{cases}$$
and
$$\psi_{L/K}(v) = \begin{cases} v & \text{if } v \leq h, \\ p^m(v - h) + h & \text{if } v > h. \end{cases}$$

Proof. Take an integer $l$ such that $lh \equiv 1 \mod p^m$. Then we have
$$v_L\left(\frac{1}{x^l \varpi^{(l-1)/p^m}}\right) = 1.$$ 
Hence, for $\sigma \in Gal(L/K)$ and $i \geq 0$, we have $\sigma \in Gal(L/K)_i$ if and only if
$$i + 1 \leq v_L\left(\sigma\left(\frac{1}{x^l \varpi^{(l-1)/p^m}}\right) - \frac{1}{x^l \varpi^{(l-1)/p^m}}\right) = v_L(\sigma(x)^l - x^l) + hl + 1. \quad (5.1)$$
The right hand side of (5.1) is $h + 1$ if $\sigma \neq 1$. Hence the first claim follows. The second claim follows from the first claim.
Lemma 5.5. Let $L$ be a totally ramified finite abelian extension of $K$. Let $m \geq 1$.

1. We have

$$\text{Nr}_{L/K}(U^\psi_{L/K}(m)) \subset U^m_K, \quad \text{Nr}_{L/K}(U^\psi_{L/K}(m) + 1) \subset U^{m+1}_K,$$

$$\text{Art}_K(U^m_K) \subset \text{Gal}(L/K)\psi_{L/K}(m).$$

2. We take $\alpha \in K$ and $\beta \in L$ such that $v_K(\alpha) = m$ and $v_L(\beta) = \psi_{L/K}(m)$. We put $P(z) = z^p - z$ for $z \in k$. Assume that

$$U^\psi_{L/K}(m) \rightarrow U^m_K, \quad p_{L,\beta}: U^\psi_{L/K}(m) \rightarrow k; \quad 1 + \beta x \mapsto \bar{x},$$

is commutative, where

$$p_{K,\alpha}: U^m_K \rightarrow k; \quad 1 + \alpha x \mapsto \bar{x},$$

Let $\varpi_L$ be a uniformizer of $L$. Then we have

$$p_{L,\beta}\left(\text{Art}_K(1 + \alpha x)(\varpi_L)\right) = \text{Tr}_{k/F}(\bar{x})$$

for $x \in \mathcal{O}_K$.

Proof. The first claim follows from [Ser68, V, §3, Proposition 4 and XV, §2, Corollaire 3 of Théorème 1]. We note that our normalization of the Artin reciprocity map is inverse to that in [Ser68, XIII, §4]. Let $x \in \mathcal{O}_K$. By [Ser68, XV, §3, Proposition 4] and the construction of the isomorphism of [Ser68, XV, §2, Proposition 3], we have

$$p_{L,\beta}\left(\text{Art}_K(1 + \alpha x)(\varpi_L)\right) = z^q_x - z_x,$$

where we take $z_x \in k^{ac}$ such that $z^p_x - z_x = \bar{x}$. Then we have the second claim, since

$$z^q_x - z_x = \text{Tr}_{k/F}(z^p_x - z_x) = \text{Tr}_{k/F}(\bar{x})$$

for such $z_x$. \qed

6 Stiefel–Whitney class and discriminant

6A. Stiefel–Whitney class

Let $R(W_K, \mathbb{R})$ be the Grothendieck group of finite-dimensional representations of $W_K$ over $\mathbb{R}$ with finite images. For $V \in R(W_K, \mathbb{R})$, we put $V_\mathbb{C} = V \otimes \mathbb{R} \mathbb{C}$ and define $\varepsilon(V_\mathbb{C}, \Psi)$ by the additivity using the epsilon factors in subsection 2A. For $V \in R(W_K, \mathbb{R})$, we define the $i$-th Stiefel–Whitney class $w_i(V) \in H^i(G_K, \mathbb{Z}/2\mathbb{Z})$ for $i \geq 0$ as in [Del76, (1.3)]. Let

$$\text{cl}: H^2(G_K, \mathbb{Z}/2\mathbb{Z}) \rightarrow H^2(G_K, K^{ac, x}) \iso \mathbb{Q}/\mathbb{Z},$$

where the first map is induced by $\mathbb{Z}/2\mathbb{Z} \rightarrow K^{ac, x}; m \mapsto (-1)^m$ and the second isomorphism is the invariant map.
**Theorem 6.1.** ([Del76, (1.5) Théorème]) Assume that $V \in R(W_K, \mathbb{R})$ has dimension 0 and determinant 1. Then we have

$$\varepsilon(V_C, \Psi) = \exp\left(2\pi\sqrt{-1}\text{cl}(w_2(V))\right).$$

In particular, we have $\varepsilon(V_C, \Psi) = 1$ if $\text{ch } K = 2$.

### 6B. Discriminant

Let $L$ be a finite separable extension of $K$. We put

$$\delta_{L/K} = \det(\text{Ind}_{L/K} 1).$$

#### 6B1. Multiplicative discriminant

We assume that $\text{ch } K \neq 2$ in this subsubsection. We define $d_{L/K} \in \mathbb{K}/(\mathbb{K})^2$ as the discriminant of the quadratic form $\text{Tr}_{L/K}(x^2)$ on $L$. For $a \in K^\times/(K^\times)^2$, let $\{a\} \in H^1(G_K, \mathbb{Z}/2\mathbb{Z})$ and $\kappa_a \in \text{Hom}(W_K, \{\pm 1\})$ be the elements corresponding to $a$ under the natural isomorphisms

$$K^\times/(K^\times)^2 \simeq H^1(G_K, \mathbb{Z}/2\mathbb{Z}) \simeq \text{Hom}(W_K, \{\pm 1\}).$$

We have

$$\delta_{L/K} = \kappa_{d_{L/K}}$$

by [Bou81, V, §10, 2 Example 6]) (cf. [Ser84, 1.4]). For $a, b \in K^\times/(K^\times)^2$, we put

$$\{a, b\} = \{a\} \cup \{b\} \in H^2(G_K, \mathbb{Z}/2\mathbb{Z}).$$

**Proposition 6.2.** ([AS10, Proposition 6.5]) Let $m$ be the extension degree of $L$ over $K$. We take a generator $a$ of $L$ over $K$. Let $f(x) \in K[x]$ be the minimal polynomial of $a$. We put $D = f'(a) \in L$. Then we have

$$d_{L/K} = (-1)^{\binom{m}{2}} \text{Nr}_{L/K}(D) \in K^\times/(K^\times)^2,$$

$$w_2(\text{Ind}_{L/K} \kappa_D) = \left(\begin{array}{c} m \\ 4 \end{array}\right)\{-1, -1\} + \{d_{L/K}, 2\} \in H^2(G_K, \mathbb{Z}/2\mathbb{Z}).$$

#### 6B2. Additive discriminant

We put $P_m(x) = x^m - x$ for any positive integer $m$. We assume that $\text{ch } K = 2$ in this subsubsection.

**Definition 6.3.** ([BM85, Définition 2.7]) Let $m$ be the extension degree of $L$ over $K$. Let $f(x) \in K[x]$ be the minimal polynomial of a generator of $L$ over $K$. We have a decomposition $f(x) = \prod_{1 \leq i \leq m} (x - a_i)$ over the Galois closure of $L$ over $K$. We put

$$d^+_{L/K} = \sum_{1 \leq i < j \leq m} \frac{a_i a_j}{(a_i + a_j)^2} \in K/P_2(K),$$

which we call the additive discriminant of $L$ over $K$.

**Theorem 6.4.** ([BM85, Théorème 2.7]) Let $L'$ be the subextension of $K^\text{ac}$ over $K$ corresponding to $\ker \delta_{L/K}$. Then the extension $L'$ over $K$ corresponds to $d^+_{L/K} \in K/P_2(K)$ by the Artin–Schreier theory.
7 Product formula of Deligne–Laumon

We recall a statement of the product formula of Deligne–Laumon. In this paper, we need only the rank one case, which is proved in [Del73, Proposition 10.12.1], but we follow the notation in [Lau87].

7A. Local factor

We consider a triple \((T, F, \omega)\) which consists of the following:

- The affine scheme \(T = \text{Spec} \, \mathcal{O}_{K_T}\) where \(\mathcal{O}_{K_T}\) is the ring of integers in a local field \(K_T\) of characteristic \(p\) whose residue field contains \(k\).
- A constructible \(\mathbb{Q}_\ell\)-sheaf \(F\) on \(T\).
- A non-zero meromorphic 1-form \(\omega\) on \(T\).

Then we can associate \(\varepsilon_{\psi_0}(T, F, \omega) \in \mathbb{C}^\times\) to the triple \((T, F, \omega)\) as in [Lau87 Théorème (3.1.5.4)] using \(\iota\).

Assume that \(K_T = k((t))\). Let \(\eta = \text{Spec} \, k((t))\) be the generic point of \(T\) with the natural inclusion \(j: \eta \to T\). We define a character \(\Psi_\omega: k((t)) \to \mathbb{C}^\times\) by

\[
\Psi_\omega(a) = (\psi_0 \circ \text{Tr}_{k/F_p})(\text{Res}(a\omega))
\]

for \(a \in k((t))\). Let \(l(\Psi_\omega)\) be the level of \(\Psi_\omega\) in the sense of [BH06 1.7 Definition]. We fix an algebraic closure \(k((t))^\text{ac}\) of \(k((t))\). For a rank 1 smooth \(\mathbb{Q}_\ell\)-sheaf \(V\) on \(\eta\) corresponding to a character \(\chi: G_{k((t))} \to \mathbb{C}^\times\) via \(\iota\), we have

\[
\varepsilon_{\psi_0}(T, j_*V, \omega) = q^{-l(\Psi_\omega)}/2 \varepsilon(\chi \omega^{-\frac{1}{2}}, \Psi_\omega)
\]

by [Lau87 Théorème (3.1.5.4)(v)], [Tat79 (3.6.2)] and [BH06 23.1 Proposition (3)].

7B. Product formula

Let \(X\) be a geometrically connected proper smooth curve over \(k\) of genus \(g\). Let \(F\) be a constructible \(\mathbb{Q}_\ell\)-sheaf on \(X\). Let \(\text{Frob}_q \in G_k\) be the geometric Frobenius element. We put

\[
\varepsilon(X, F) = \iota \left( \prod_{i=0}^2 \det \left( -\text{Frob}_q; H^i(X \otimes_k k^{\text{ac}}, F) \right)^{(-1)^{i-1}} \right).
\]

Let \(\text{rk}(F)\) be the generic rank of \(F\).

Theorem 7.1. ([Lau87 Théorème (3.2.1.1)]) Let \(\omega\) be a non-zero meromorphic 1-form on \(X\). Then we have

\[
\varepsilon(X, F) = q^{\text{rk}(F)(1-g)} \prod_{x \in |X|} \varepsilon_{\psi_0}(X(x), F|_{X(x)}, \omega|_{X(x)}),
\]

where \(|X|\) is the set of closed points of \(X\), and \(X(x)\) is the completion of \(X\) at \(x\).
8 Determinant

In this section, we study \( \det \tau_{\psi_0} \) to show the equality \( \omega_{\xi,\chi,c} = \det \tau_{\xi,\chi,c} \) of the central character and the determinant. We use the product formula of Deligne–Laumon to study \( \det \tau_{\psi_0}(\text{Fr}(1)) \), where \( \text{Fr}(1) \) is defined in (2.2).

**Lemma 8.1.** We have \( Q^{ab} = Q/Q_0 \).

**Proof.** By Lemma 2.4, we have \( Q^{ab} = (Q/F)^{ab} \). For \((a, b, c) \in Q\), let \((a, b)\) be the image of \((a, b, c)\) in \(Q/F\). Then we have
\[
(a, 0)(a, 0)^{-1}(a, b)^{-1} = (1, (a - 1)b).
\]
Hence, we obtain the claim.

We view \( \theta_0 \) defined in (2.8) as a character of \( Q \) by \((a, b, c) \mapsto \theta_0((a, b, c))\). Recall that \( \tau_0 \) is the representation of \( Q \) defined in (2.14).

**Lemma 8.2.** We have \( \det \tau_0 = \theta_0 \).

**Proof.** By Lemma 8.1, it suffices to show \( \det \tau_0 = \theta_0 \) on \( \mu_{p^e+1}(k^{ac}) \). By Lemma 2.2 and Lemma 2.5, we have
\[
\det \tau_0(a) = \prod_{\chi \in \mu_{p^e+1}(k^{ac})^{\times} \setminus \{1\}} \chi(a)
\]
for \( a \in \mu_{p^e+1}(k^{ac}) \). Hence, the claim follows.

For \( a \in k^{\times} \), let \( \left( \frac{a}{k} \right) \) denote the quadratic residue symbol of \( k \) defined by
\[
\left( \frac{a}{k} \right) = \begin{cases} 
1 & \text{if } a \text{ is square in } k, \\
-1 & \text{if } a \text{ is not square in } k.
\end{cases}
\]

**Lemma 8.3.** Let \( m \) be a positive integer that is prime to \( p \). We take an \( m \)-th root \( \overline{\omega}^{1/m} \) of \( \overline{\omega} \), and put \( L = K(\overline{\omega}^{1/m}) \). Let \( \chi \in \mu_{p^e+1}(k^{ac}) \). Then we have
\[
\delta_{L/K}(\overline{\omega}) = \left( \frac{q}{m} \right) \quad \text{and} \quad \delta_{L/K}(\chi) = \left( \frac{\chi}{k} \right) \quad \text{for } \chi \in O_K^{\times}.
\]

**Proof.** These are proved in [BF83, (10.1.6)] if \( \text{ch } K = 0 \). Actually, the same proof works also in the positive characteristic case.

**Lemma 8.4.** Let \( m, m' \) be positive integers that are prime to \( p \). We take an \( m \)-th root \( \overline{\omega}^{1/m} \) of \( \overline{\omega} \), and put \( L = K(\overline{\omega}^{1/m}) \). Let \( \psi_K : K \to \mathbb{C}^\times \) be a character such that \( \psi'_K(x) = \psi_0(\text{Tr}_{K/F_p}(x)) \) for \( x \in O_K^{\times} \). Then we have
\[
\lambda(L/K, \psi'_K) = \begin{cases} 
\left( \frac{q}{m} \right) & \text{if } m \text{ is odd}, \\
-\epsilon(p) \left( \frac{2mm'}{p} \right) \left( \frac{-1}{p} \right)^{m-1} \text{ if } m \text{ is even}.
\end{cases}
\]

**Proof.** If \( m \) is odd, we have
\[
\lambda(L/K, \psi'_K) = \epsilon(\delta_{L/K}, \psi'_K) = \left( \frac{q}{m} \right)
\]
by [Hen84, Proposition 2] and Lemma 8.3.1.
Assume that $m$ is even. Note that $p \neq 2$ in this case. Then we have
\[
d_{L/K} = (-1)^{m/2} \text{Nr}_{L/K}(m(\mathfrak{o}^{1/m})^{m-1}) = -(-1)^{m/2} \mathfrak{o} \in K^\times/(K^\times)^2
\] (8.1)
by Proposition 6.2. For $\chi \in (\mathbb{F}_q^\times)^\omega$ and $\psi \in \mathbb{F}_q^\times \setminus \{1\}$, we set
\[
\tau(\chi, \psi) = -\sum_{x \in \mathbb{F}_q^\times} \chi^{-1}(x)\psi(x)
\]
and have the Hasse–Davenport formula
\[
\tau(\chi \circ \text{Nr}_{F_{q^n}/F_q}, \psi \circ \text{Tr}_{F_{q^n}/F_q}) = \tau(\chi, \psi)^n.
\] (8.2)
Let $(\ , \ )_K: K^\times/(K^\times)^2 \times K^\times/(K^\times)^2 \to \{\pm 1\}$ denote the Hilbert symbol. By (6.1) and (8.1), we have
\[
\delta_{L/K}(x) = \kappa_{d_{L/K}}(x) = (x, d_{L/K})_K = (x, \mathfrak{o})_K = \left(\frac{x}{k}\right)
\]
for $x \in \mathcal{O}_K^\times$. By [BH06] 23.5 Theorem, we have
\[
\varepsilon(\delta_{L/K}, \psi'_K) = q^{-\frac{1}{2}} \sum_{x \in \mathcal{O}_K^\times/U_k^1} \delta_{L/K}(x)\psi'_K(x) = q^{-\frac{1}{2}} \sum_{x \in k^\times} \left(\frac{x}{k}\right) \psi_0(\text{Tr}_{k/F_p}(m'x)).
\]
By applying (8.2) to the extension $k$ over $F_p$ and using (2.16), we have
\[
q^{-\frac{1}{2}} \sum_{x \in k^\times} \left(\frac{x}{k}\right) \psi_0(\text{Tr}_{k/F_p}(m'x)) = -\left(-\varepsilon(p)\left(\frac{m'}{p}\right)\right)^f.
\]
Hence, we have
\[
\lambda(L/K, \psi'_K) = \varepsilon(\delta_{L/K}, \psi'_K) \left(\frac{m}{q}\right) \left(-\frac{1}{q}\right) \left(\frac{d_{L/K}, 2)_K}{q}\right) = -\left(-\varepsilon(p)\left(\frac{2mm'}{p}\right)\left(-1\right)\left(\frac{1}{p}\right)^{f-1}\right)^f
\]
by [San95] Theorem II.2B and [Tat79] (3.6.1).

Lemma 8.5. We have
\[
\det \tau_{\psi_0}(\text{Fr}(1)) = \begin{cases} (-\varepsilon(p)(\frac{2}{p}))^f q^{\frac{m}{2}e} & \text{if } p \neq 2, \\ q^{\frac{m}{2}e-1} & \text{if } p = 2. \end{cases}
\]

Proof. Let $x$ be the standard coordinate of $A_k^1$. Let $j$ be the open immersion $A_k^1 \hookrightarrow \mathbb{P}_k^1$. We put $t = 1/x$. As in Subsection 7A, we put $T = \text{Spec } k[[t]]$ and $\eta = \text{Spec } k((t))$ with the open immersion $j: \eta \to T$.

We consider $k((s))$ as a subfield of $k((t))$ by $s = v^{p+1}$. Let $\tilde{\xi}: G_{k((s))} \to \mathbb{C}^\times$ be the Artin–Schreier character associated to $y^p - y = 1/s$ and $\psi_0$, which means the composite of
\[
G_{k((s))} \to \mathbb{F}_p; \sigma \mapsto \sigma(y) - y
\]
and $\psi_0^{-1}$ where $y$ is an element of $k((t))^\times$ such that $y^p - y = 1/s$. 

19
We use the notation in Lemma 5.5. Note that \( \psi_{k((s))(y)/k((s))}(1) = 1 \) by Lemma 5.4. We can check that

\[
\text{Nr}_{k((s))(y)/k((s))}(1 + y^{-1}x) = 1 + s(x^p - x)
\]

for \( x \in k \). For \( x \in \mathcal{O}_{k((s))} \), we have

\[
\check{\xi}(1 + sx) = \psi_0^{-1}\left(\text{Art}_{k((s))}(1 + sx)(y) - y\right)
\]

\[
= \psi_0^{-1}\left(-p_{k((s))(y),y}^{-1}\left(\frac{\text{Art}_{k((s))}(1 + sx)(y^{-1})}{y^{-1}}\right)\right) = \psi_0(\text{Fr}_{k/\mathcal{O}_p}(x)),
\]

where we use Lemma 5.5 with \( \alpha = s, \beta = \omega_{k((s))} = y^{-1} \). Hence, we have \( \text{rsw}(\check{\xi}, \Psi_{s^{-1}dx}) = s \) by Proposition 5.3.1.

Let \( \xi : G_{k((t))} \to \mathbb{C}^\times \) be the restriction of \( \check{\xi} \) to \( G_{k((t))} \). Then \( \xi \) is the Artin–Schreier character associated to \( y^p - y = 1/t^{p^r+1} \) and \( \psi_0 \).

Let \( \check{V}_\xi \) be the smooth \( \mathbb{Q}_r \)-sheaf on \( \eta \) corresponding to \( \xi \) via \( \iota \). Then we have \( \check{V}_\xi \cong \mathcal{L}_{\psi_0}|_\eta \) by [Del77, Déninition 1.7 in Sommes trig.]. Let the notation be as in Lemma 2.2. We write \( \omega \) for the meromorphic 1-form \( dx \) on \( \mathbb{P}^1_k \). By [Lau87, Théorème (3.1.5.4)(v)], we have

\[
\varepsilon_{\psi_0}(X(x), (j_!\pi^*\mathcal{L}_{\psi_0})|_{X(x)}, \omega|_{X(x)}) = 1
\]

for any \( x \in |A_1| \times \mathbb{P}^1_k \) in the notation of Theorem 7.1. We simply write \( \omega \) for \( \omega|_T \).

Then we have

\[
\det \tau_{\psi_0}(\text{Fr}(1)) = (-1)^{p^r}\varepsilon(\mathbb{P}^1_k, j_!\pi^*\mathcal{L}_{\psi_0}) = (-1)^p q\varepsilon_{\psi_0}(T, j_!V_\xi, \omega)
\]

by Theorem 7.1. Since \( \xi \) is a ramified character, we have \( j_!V_\xi \cong j_*V_\xi \). Hence, we obtain

\[
\varepsilon_{\psi_0}(T, j_!V_\xi, \omega) = \varepsilon_{\psi_0}(T, j_!V_\xi, \omega) = q^{-1}\varepsilon(\xi_{\omega_{-\frac{1}{2}}}, \Psi_{1,\text{t-1}})
\]

by (7.1). Since \( \omega = -t^{-2}dt \) on \( T \), we have

\[
\varepsilon(\xi_{\omega_{-\frac{1}{2}}}, \Psi_{\omega}) = (\xi_{\omega_{-\frac{1}{2}}})(-t^{-1})\varepsilon(\xi_{\omega_{-\frac{1}{2}}}, \Psi_{1,\text{t-1}})
\]

by [BH06, 23.5 Lemma 1]. We have

\[
(\xi_{\omega_{-\frac{1}{2}}})(-t^{-1}) = \xi(-t^{p^r}) = \xi(-t)^{p^r} = 1,
\]

since \( \text{Nr}_{k((t))(y)/k((t))}(y) = 1/t^{p^r+1} \). Hence we obtain

\[
(\xi_{\omega_{-\frac{1}{2}}})(-t^{-1})\varepsilon(\xi_{\omega_{-\frac{1}{2}}}, \Psi_{1,\text{t-1}}) = q^{\frac{p^r}{2}}\varepsilon(\xi, \Psi_{1,\text{t-1}})
\]

by Lemma 4.2 since \( \text{rsw}(\xi, \Psi_{1,\text{t-1}}) = s \) by \( \text{rsw}(\check{\xi}, \Psi_{s^{-1}dx}) = s \) and Proposition 5.2.

By Proposition 5.3, we have \( \varepsilon(\xi, \Psi_{s^{-1}dx}) = \xi(s) = 1 \), since the level of \( \xi \) is 1 and \( \text{Nr}_{k((s))(y)/k((s))}(y^{-1}) = s \). Hence, we obtain

\[
\varepsilon(\xi, \Psi_{1,\text{t-1}}) = \lambda(k((t))/k((s)), \Psi_{s^{-1}dx})^{-1}\delta_{k((t))/k((s))}(\text{rsw}(\check{\xi}, \Psi_{s^{-1}dx}))
\]

by Proposition 5.1. We have

\[
\lambda(k((t))/k((s)), \Psi_{s^{-1}dx}) = \begin{cases} 
-\left(-\varepsilon(p)\left(\frac{2}{p}\right)\left(-1\right)^{\frac{p^r-1}{2}}\right)^f & \text{if } p \neq 2, \\
\left(-\frac{q}{p^r+1}\right)^{\frac{p^r-1}{2}} & \text{if } p = 2,
\end{cases}
\]

\[
\delta_{k((t))/k((s))}(\text{rsw}(\check{\xi}, \Psi_{s^{-1}dx})) = \begin{cases} 
\left(-\frac{1}{q}\right)^{\frac{p^r-1}{2}} & \text{if } p \neq 2, \\
\left(-\frac{q}{p^r+1}\right)^{\frac{p^r-1}{2}} & \text{if } p = 2
\end{cases}
\]

by Lemma 8.4 and Lemma 8.3 respectively. The claim follows from the above equalities.
We simply write $\tau_\zeta$ for $\tau_{\zeta,1,1}$.

**Proposition 8.6.** We have $\omega_{\pi_{\zeta,\chi,c}} = \det \tau_{\zeta,\chi,c}$.

**Proof.** By (2.13) and [Gal65 (1)], we have

$$\det \tau_{\zeta,\chi,c} = \delta_{E_\zeta/K}^p(\det \tau_{n,\zeta,\chi,c}|_{K^\times}), \quad (8.3)$$

since $\delta_{E_\zeta/K} = \det(\text{Ind}_{E_\zeta/K} 1)$ and the transfer homomorphism $W_{E_\zeta}^{\text{ab}} \to W_{E_\zeta}^{\text{ab}}$ is compatible with the natural inclusion $K^\times \to E_\zeta^\times$ under the Artin reciprocity maps. Hence, we may assume $\chi = 1$ and $c = 1$ by twist (cf. (2.13)). Then it suffices to show $\det \tau_\zeta = 1$. We see that $\det \tau_\zeta$ is unramified by (2.9), Lemma 2.5, Lemma 8.2, Lemma 8.5. We see that $\det \tau_{\zeta,\chi,c}$.

If $p$ and $n'$ are odd, we have

$$\det \tau_\zeta(\varpi) = \left(\frac{q}{n'}\right)^{p^e} \left(-\epsilon(p)\left(\frac{2}{p}\right) p^{\varphi(e)}\right)^{fn'} \left(\left(-\epsilon(p)\left(-\frac{2n'}{p}\right)\right)^{n-1} p^{\frac{1}{2}}\right)^{fn'} = \left(\frac{p}{n'}\right) \left(\frac{n'}{p}\right) \epsilon(p)^{p^{n-1}} = 1$$

by (8.3), Lemma 8.3 and Lemma 8.5. We see that $\det \tau_\zeta(\varpi) = 1$ similarly also in the other case using (8.3), Lemma 8.3 and Lemma 8.5.

\[\square\]

## 9 Imprimitive field

In this section, we construct a field extension $T_\zeta^n$ of $E_\zeta$ such that $\tau_{n,\zeta}|_{W_{T_\zeta^n}}$ is an induction of a character. We call $T_\zeta^n$ an imprimitive field of $\tau_{n,\zeta}$, since $\tau_{n,\zeta}|_{W_{T_\zeta^n}}$ is not primitive.

### 9.A. Construction of character

In this subsection, we construct subgroups $R \subset Q' \subset Q \times Z$ and a character $\phi_n$ of $R$. In the next subsection, we will see that $\tau_n|_{Q'} \simeq \text{Ind}_R^{Q'} \phi_n$. Our imprimitive field $T_\zeta^n$ will correspond to the subgroup $Q' \subset Q \times Z$.

Let $e_0$ be the positive integer such that $e_0 \in 2^N$ and $e/e_0$ is odd.

**Lemma 9.1.** Assume $p \neq 2$. Then we have $\text{Tr} \tau_{\psi_0}(\text{Fr}(2e_0)) = p^{e_0}$.

**Proof.** For $a \in k^{ac}$ and $b \in \mathbb{F}_{p^{e_0}}$ such that $a^p - a = b^{p^{e+1}}$, we have that

$$a^{p^{2e_0}} - a = \text{Tr}_{\mathbb{F}_{p^{e_0}}/\mathbb{F}_p}(b^{p^{e+1}}). \quad (9.1)$$

By (9.1) and the Lefschetz trace formula, we see that

$$\text{Tr} \tau_{\psi_0}(\text{Fr}(2e_0)) = -\sum_{b \in \mathbb{F}_{p^{2e_0}}} (\psi_0 \circ \text{Tr}_{\mathbb{F}_{p^{e_0}}/\mathbb{F}_p})(\text{Tr}_{\mathbb{F}_{p^{e_0}}/\mathbb{F}_p}(b^{p^{e_0}+1}))$$

$$= -1 - (p^{e_0} + 1) \sum_{x \in \mathbb{F}_p^\times} (\psi_0 \circ \text{Tr}_{\mathbb{F}_{p^{e_0}}/\mathbb{F}_p})(x) = p^{e_0}$$

using $(p^r + 1, p^{2e_0} - 1) = p^{e_0} + 1$.

\[\square\]

**Corollary 9.2.** Assume $p \neq 2$. Then we have $\text{Tr} \tau_n(\text{Fr}(2e_0)) = (-1)^{ne_0(p-1)/2}$.
Proof. This follows from (2.9) and Lemma 9.1.

Let \( n_0 \) be the biggest integer such that \( 2^{n_0} \) divides \( p^{-e_0} + 1 \). We take \( r \in k^{ac} \) such that \( r^{2^{n_0}} = -1 \). We define a subgroup \( R_0 \) of \( Q_0 \) by

\[
R_0 = \{(1, b, c) \in Q_0 \mid b^{p^e} - rb = 0\}.
\]

Lemma 9.3. 1. If \( p \neq 2 \), the action of \( 2e_0Z \subset Z \) on \( Q \) stabilizes \( R_0 \).
2. If \( p = 2 \), the action of \( g \) on \( Q \rtimes Z \) by conjugation stabilizes \( R_0 \).

Proof. The first claim follows from \( r^{p^{2n_0} - 1} = 1 \). We can see the second claim easily using (2.19).

We put

\[
Q' = \begin{cases} 
Q_0 \rtimes (2e_0Z) & \text{if } p \neq 2, \\
Q_0 \rtimes Z & \text{if } p = 2,
\end{cases} \quad R = \begin{cases} 
R_0 \rtimes (2e_0Z) & \text{if } p \neq 2, \\
R_0 \cdot \langle g \rangle & \text{if } p = 2.
\end{cases}
\]

as subgroups of \( Q \rtimes Z \), which are well-defined by Lemma 9.3. We are going to construct a character \( \phi_n \) of \( R \) in this subsection. Then, we will show that \( \tau_n|_{Q'} \simeq \text{Ind}_{R}^{Q} \phi_n \) in the next subsection.

First, we consider the case where \( p \) is odd. We define a homomorphism \( \phi_n : R \to \mathbb{C}^\times \) by

\[
\phi_n((1, b, c)) = \psi_0 \left(c - \frac{1}{2} \sum_{i=0}^{e-1} (rb^2)^p \right) \quad \text{for } (1, b, c) \in R_0,
\]

\[
\phi_n(\text{Fr}(2e_0)) = (-1)^{n_0 \frac{e_0}{2}}.
\]

Then \( \phi_n \) extends the character \( \psi_0 \) of \( F \).

Next, we consider the case where \( p = 2 \). We define an abelian group \( R'_0 \) as

\[
R'_0 = \{(b, c) \mid b \in F_2, c \in F_{2e}, c^{2^e} - c = b\}
\]

with the multiplication given by

\[(b_1, c_1) \cdot (b_2, c_2) = (b_1 + b_2, c_1 + c_2 + b_1b_2).\]

We define \( \phi : R_0 \to R'_0 \) by

\[
\phi((1, b, c)) = \left(\text{Tr}_{F_{2^e}/F_2}(b), c + \sum_{0 \leq i < j \leq e-1} b^{2^i}b^{2^j} \right) \quad \text{for } (1, b, c) \in R_0,
\]

which is a homomorphism by

\[
\text{Tr}_{F_{2^e}/F_2}(b) \text{Tr}_{F_{2^e}/F_2}(b') = \text{Tr}_{F_{2^e}/F_2}(bb') + \sum_{0 \leq i < j \leq e-1} (b^{2^i}b^{2^j} + b^{2^j}b^{2^i}) \quad (9.3)
\]

for \( b, b' \in F_{2^e} \). Let \( b_0 \in F_{2^{2e}} \) be as before Lemma 2.8. Let \( F' \) be the kernel of the homomorphism

\[
F_{2^e} \to F_2; \quad c \mapsto \text{Tr}_{F_{2^e}/F_2}(b_0 + b_0^{2^e}c).
\]
We put \( R'_0 = R'_0/F' \), where we consider \( F' \) as a subgroup of \( R'_0 \) by \( c \mapsto (0, c) \). Then \( R'_0 \) is a cyclic group of order 4. We write \( \tilde{g}(b, c) \) for the image of \( (b, c) \in R'_0 \) under the projection \( R'_0 \to R'_0' \). Let \( \phi' : R_0 \to R'_0' \) be the composite of \( \phi \) and the projection \( R'_0 \to R'_0' \). We put

\[
s = \sum_{i=0}^{e-1} b_0^2, \quad t = \text{Tr}_{\mathbb{F}_2e/\mathbb{F}_2}(b_0).
\]

\[\text{(9.4)}\]

We have \( s^2 + s = t \) and \( \text{Tr}_{\mathbb{F}_2e/\mathbb{F}_2}(t) = \text{Tr}_{\mathbb{F}_2e/\mathbb{F}_2}(b_0) = 1 \). We have

\[
\left(1, s^2 + \sum_{0 \leq i < j \leq e-1} t^{2^{2^i-2}}\right) \in R'_0,
\]

which is of order 4. The element \( \tilde{g}(1, s^2 + \sum_{0 \leq i < j \leq e-1} t^{2^{2^i-2}}) \) is a generator of \( R'_0' \), because \( 2\tilde{g}(1, s^2 + \sum_{0 \leq i < j \leq e-1} t^{2^{2^i-2}}) = \tilde{g}(0, 1) \neq 0 \). Let \( \psi_0 : R'_0' \to \mathbb{C}^\times \) be the faithful character satisfying

\[
\tilde{\psi}_0 \left( \tilde{g} \left( 1, s^2 + \sum_{0 \leq i < j \leq e-1} t^{2^{2^i-2}} \right) \right) = -\sqrt{-1}.
\]

We define a homomorphism \( \phi_n : R \to \mathbb{C}^\times \) by

\[
\phi_n((b, c, 0)) = (\tilde{\psi}_0 \circ \phi')(1, b, c) \quad \text{for} \quad (1, b, c) \in R_0,
\]

\[
\phi_n(g) = (-1)^{\frac{n(n-2)}{8}} \frac{1 + \sqrt{-1}}{\sqrt{2}},
\]

which is a character of order 8. Then \( \phi_n \) extends the character \( \psi_0 \) of \( F \).

9B. Induction of character

Lemma 9.4. We have \( \tau_n|_{Q'} \simeq \text{Ind}_{R}^{Q'} \phi_n \).

Proof. We write \( \tilde{\psi}_n \) for \( \phi_n|_{R_0} \). We know that \( \tau_n|_{Q_0} \cong \text{Ind}_{R_0}^{Q_0} \tilde{\psi}_n \) by Proposition \[1.2\] since \( R_0 \) is an abelian group such that \( 2\dim_{\mathbb{F}_p}(R_0/F) = \dim_{\mathbb{F}_p}(Q_0/F) \).

First, we consider the case where \( p \) is odd. The claim for general \( f \) follows from the claim for \( f = 1 \) by the restriction. Hence, we may assume that \( f = 1 \).

If \( \psi \in R'_0 \) satisfies \( \psi|_F = \psi_0 \), then we have \( \tau_n|_{Q_0} \cong \text{Ind}_{R_0}^{Q_0} \psi \) by Proposition \[1.2\] and obtain an injective homomorphism \( \tilde{\psi} \to \tau_n|_{R_0} \) as representations of \( R_0 \) by Frobenius reciprocity. Hence we have a decomposition

\[
\tau_n|_{R_0} = \bigoplus_{\tilde{\psi} \in R'_0, \tilde{\psi}|_F = \psi_0} \tilde{\psi},
\]

\[\text{(9.6)}\]

since the number of \( \tilde{\psi} \in R'_0 \) such that \( \tilde{\psi}|_F = \psi_0 \) is \( p^e \).

We put \( \overline{R}'_0 = \{ b \in k^{ac} \mid b^e - rb = 0 \} \). The \( \psi_n \)-component in \( \overline{R}'_0 \) is the unique component that is stable by the action of \( ((1, 0, 0), 2c_0) \), since the homomorphism

\[
\overline{R}'_0 \to \overline{R}'_0; \quad b \mapsto b^{2c_0} - b
\]

is an isomorphism. Hence, we have a non-trivial homomorphism \( \phi_n \to \tau_n|_R \) by Corollary \[9.2\]. Then we have a non-trivial homomorphism \( \text{Ind}_R^{Q'} \phi_n \to \tau_n|_{Q'} \) by Frobenius reciprocity.
The representation \( \tau_n|_{Q'} \) is irreducible by Corollary 2.6. Then we obtain the claim, since \([Q' : R] = p^e\).

Next we consider the case where \( p = 2 \). Then it suffices to show that

\[
\text{Tr}(\text{Ind}^Q_R \phi_n)(g^{-1}) = -(1 - \frac{n(n-2)}{2}) \sqrt{2}
\]

by (2.9) and Proposition 2.9. We have a decomposition

\[
(\text{Ind}^Q_R \phi_n)|_{R_0} = \bigoplus_{\phi \in R_0^*, \phi|_F = \psi_0} \phi.
\]

Let \( \tilde{\psi}'_n \) be the twist of \( \tilde{\psi}_n \) by the character

\[
R_0 \to \mathbb{T}_t^2; (1, b, c) \mapsto \psi_0(\text{Tr}_{F_{2e}/F_2}(b)).
\]

Then only the \( \tilde{\psi}_n \)-component and the \( \tilde{\psi}'_n \)-component in (9.6) are stable by the action of \(((1, b_0, c_0), 1)\), since the image of the homomorphism

\[
F_{2e} \to F_{2e}; b \mapsto b^2 - b
\]

is equal to \( \text{Ker} \text{Tr}_{F_{2e}/F_2} \). The action of \( \text{Fr}(e) \) permutes the \( \tilde{\psi}_n \)-component and the \( \tilde{\psi}'_n \)-component. Hence, \( g \) acts on the \( \tilde{\psi}'_n \)-component by \( \phi_n(g) \) times

\[
\phi_n\left(\text{Fr}(e)^{-1}g \text{Fr}(e)g^{-1}\right) = \phi_n\left(\left(1, t, c_0, t + c_0, \sum_{i=0}^{e-1} (b_0^{2i+1} + b_0^{2i+1})^2\right), 0\right) = \sqrt{-1}.
\]

Hence we have

\[
\text{Tr}(\text{Ind}^Q_R \phi_n)(g^{-1}) = \left(1 - \sqrt{-1}\right) \phi_n(g^{-1}) = -(1 - \frac{n(n-2)}{2}) \sqrt{2}.
\]

We use the notations in (2.10). We set \( T_\zeta = E_\zeta(\alpha_\zeta), M_\zeta = T_\zeta(\beta_\zeta) \) and \( N_\zeta = M_\zeta(\gamma_\zeta) \). Let \( f_0 \) be the positive integer such that \( f_0 \in 2^3 \) and \( f/f_0 \) is odd. We put

\[
N = \begin{cases} 
2e_0/f_0 & \text{if } p \neq 2 \text{ and } f_0 \mid 2e_0, \\
1 & \text{otherwise}.
\end{cases}
\]

Let \( K^{ur} \) be the maximal unramified extension of \( K \) in \( K^{ac} \). Let \( K^u \subset K^{ur} \) be the unramified extension of degree \( N \) over \( K \). Let \( k_N \) be the residue field of \( K^u \). For a finite field extension \( L \) of \( K \) in \( K^{ac} \), we write \( L^u \) for the composite field of \( L \) and \( K^u \) in \( K^{ac} \). For \( a \in k^{ac} \), we write \( \hat{a} \in \mathcal{O}_{K^{ur}} \) for the Teichmüller lift of \( a \). We put

\[
\delta'_\zeta = \begin{cases} 
\beta_{2e}^e - \hat{\beta}_\zeta & \text{if } p \neq 2, \\
\beta_{2e}^e - \beta_\zeta + \sum_{i=0}^{e-1} \hat{b}_0^2 & \text{if } p = 2,
\end{cases}
\]

\[
e_1 = \begin{cases} 
0 & \text{if } p \neq 2, \\
1 & \text{if } p = 2,
\end{cases}
\]

Then we have \( \delta_{T_\zeta}^{\phi,u} = \hat{\tau}^{-1} \delta'_\zeta \equiv -\alpha_\zeta^{-1} + e_1 \mod p_T^{\phi,u}(\delta'_\zeta) \). We take \( \delta_\zeta \in T_\zeta(\delta'_\zeta) \) such that

\[
\delta_{T_\zeta}^{\phi,u} = \hat{\tau}^{-1} \delta_\zeta \equiv -\alpha_\zeta^{-1} + e_1 \mod p_T^{\phi,u}(\delta'_\zeta),
\]

(9.9)

We put \( M^{au}_\zeta = T_\zeta(\delta_\zeta) \). The image of \( \Theta_\zeta|_{W_{M^{au}_\zeta}} \) is contained in \( R \). Let \( \xi_{n,\zeta} : W_{M^{au}_\zeta} \to \mathbb{C}^\times \) be the composite of the restrictions \( \Theta_\zeta|_{W_{M^{au}_\zeta}} \) and \( \phi_n|_R \). By the local class field theory, we regard \( \xi_{n,\zeta} \) as a character of \( M^{au}_\zeta \).
Proposition 9.5. We have \( \tau_{n,\zeta}|_{W_{T\zeta}} \simeq \text{Ind}_{M_{\zeta}^n/T\zeta}^n \xi_{n,\zeta} \).

Proof. This follows from Lemma 9.4.

Remark 9.6. Our imprimitive field is different from that in [BH14, 5.1]. In our case, \( T\zeta \) need not be normal over \( K \). This choice is technically important in our proof of the main result.

9C. Study of character

In this subsection, we study the character \( \xi_{n,\zeta} \) in detail.

Assume that \( \text{ch} K = p \) and \( f = 1 \) in this subsection. We will use results in this subsection to compute the epsilon factor of \( \xi_{n,\zeta} \) later after a reduction to the case where \( \text{ch} K = p \) and \( f = 1 \). By (2.10), (9.8), (9.9) and \( \text{ch} K = p \), we have that \( \delta_{\zeta} = \delta'_{\zeta} \).

9C. Odd case

Assume \( p \neq 2 \) in this subsubsection. We put

\[
\theta_{\zeta} = \gamma_{\zeta} + \frac{1}{2} \sum_{i=0}^{e-1} (r\beta_{\zeta}^2)^{p^i}.
\]

(9.10)

Since \( rp^{e+1} = -1 \) and \((p^e + 1)/(p^e + 1)\) is an odd integer, we have \( rp^{e+1} = -1 \). Then we have

\[
\theta_{\zeta}^p - \theta_{\zeta} = \beta_{\zeta}^{p^{e+1}} - \frac{1}{2r}(2\beta_{\zeta}^{2p^e} + r^2\beta_{\zeta}^2) = -\frac{1}{2r}(2\beta_{\zeta}^{2p^e} - 2r\beta_{\zeta}^{p^{e+1}} + r^2\beta_{\zeta}^2) = -\frac{1}{2r}\delta_{\zeta}^2.
\]

(9.11)

We put \( N_{\zeta}^n = M_{\zeta}^n(\theta_{\zeta}) \). Let \( \xi_{n,\zeta}' \) be the twist of \( \xi_{n,\zeta} \) by the unramified character

\[
W_{M_{\zeta}^n} \rightarrow \mathbb{C}^\times; \sigma \mapsto \sqrt{-1}^{n_{\sigma}^{\frac{p-1}{2}}},
\]

where \( n_{\sigma} \) is as before (2.12).

Lemma 9.7. If \( p \neq 2 \), then \( \xi_{n,\zeta}' \) factors through \( \text{Gal}(N_{\zeta}^n/M_{\zeta}^n) \).

Proof. Let \( \sigma \in \text{Ker} \xi_{n,\zeta}' \). Recall that \( a_{\sigma}, b_{\sigma}, c_{\sigma} \) are defined in (2.11). Then we have \((\bar{a}_{\sigma}, \bar{b}_{\sigma}, \bar{c}_{\sigma}) \in R_0 \) and

\[
\bar{c}_{\sigma} - \frac{1}{2} \sum_{i=0}^{e-1} (r\bar{b}_{\sigma}^2)^{p^i} = 0
\]

by (9.2). Hence, we see that

\[
\sigma(\theta_{\zeta}) - \theta_{\zeta} = c_{\sigma} - \frac{1}{2} \sum_{i=0}^{e-1} (r\bar{b}_{\sigma}(\beta_{\zeta} + b_{\sigma}))^{p^i} + \frac{1}{2} \sum_{i=0}^{e-1} (r((\beta_{\zeta} + b_{\sigma})^2 - \beta_{\zeta}^2))^{p^i}
\]

\[
= c_{\sigma} - \frac{1}{2} \sum_{i=0}^{e-1} (r\bar{b}_{\sigma}^2)^{p^i} \equiv 0 \mod pN_{\zeta}^n
\]

by (2.11). Therefore, we obtain the claim by \( \sigma(\delta_{\zeta}) = \delta_{\zeta} \) and (9.11).
9C2. Even case

Assume $p = 2$ in this subsubsection. Let $\xi_{n, \zeta}$ be the twist of $\xi_{n, \zeta}$ by the character

$$W_{M_n} \to \mathbb{C}^\times; \sigma \mapsto \left( -1 \right)^{\frac{n(4 - 2)}{s} - 1 + \frac{\sqrt{-1}}{2}}^{n_s}.$$  \hspace{1cm} (9.12)

We take $b_1; b_2 \in k^{ac}$ such that

$$b_1^2 - b_1 = s, \quad b_2^2 - b_2 = t \left( b_1^2 + \sum_{i=0}^{e-1} (b_1 s)^{2i} \right).$$  \hspace{1cm} (9.13)

We put

$$\eta_\zeta = \sum_{i=0}^{e-1} \beta_\zeta^{2i} + b_1, \quad \eta'_\zeta = \eta_\zeta + \sum_{0 \leq i < j \leq e-1} \beta_\zeta^{2i+2j},$$  \hspace{1cm} (9.14)

$$\theta'_\zeta = \sum_{i=0}^{e-1} (t \gamma_\zeta)^{2i} + \sum_{0 \leq i < j \leq e-2} t^{2i} (\delta_\zeta \eta_\zeta)^{2j} + \sum_{0 \leq j < i \leq e-1} t^{2i} (b_1 \delta_\zeta + s \eta_\zeta)^{2j} + b_1^2 \eta_\zeta + b_2.$$  \hspace{1cm} (9.15)

**Lemma 9.8.** We have $\eta_\zeta^2 - \eta_\zeta = \delta_\zeta$ and $\theta'_\zeta - \theta'_\zeta = (\delta_\zeta \eta_\zeta)^{2e-1}$.

**Proof.** We can check the first claim easily. We show the second claim. We use $P_m$ in Subsubsection 6B2. We have

$$P_2(\gamma_\zeta') = (\beta_\zeta^{2e} - \beta_\zeta) \sum_{i=0}^{e-1} \beta_\zeta^{2i} + \beta_\zeta^2 = (\delta_\zeta - s)(\eta_\zeta - b_1) + \beta_\zeta^2.$$  \hspace{1cm} (9.16)

Hence, we have

$$P_{2e}(\gamma_\zeta') = \sum_{i=0}^{e-1} ((\delta_\zeta - s)(\eta_\zeta - b_1))^{2i} + (\eta_\zeta - b_1)^2.$$  \hspace{1cm} (9.17)

By $b_1^2 + b_1 = s^2 + s = t$ and $\eta_\zeta^2 - \eta_\zeta = \delta_\zeta$, we have

$$(b_1^2 \eta_\zeta)^2 + b_1^2 \eta_\zeta = t \eta_\zeta^2 + b_1 \eta_\zeta^2 + b_1^2 \eta_\zeta = t \eta_\zeta^2 + b_1 (\eta_\zeta^2 + \eta_\zeta) + s \eta_\zeta = t \eta_\zeta^2 + b_1 \delta_\zeta + s \eta_\zeta.$$

Hence, by using $\sum_{i=1}^{e-1} t^{2i} = 1 - t$ and $t \in \mathbb{F}_{2e}$, we have

$$\theta_\zeta^2 - \theta'_\zeta = t P_{2e}(\gamma_\zeta') + t \sum_{i=0}^{e-1} (\delta_\zeta \eta_\zeta + b_1 \delta_\zeta + s \eta_\zeta)^{2i} + (\delta_\zeta \eta_\zeta)^{2e-1} + t \eta_\zeta^2 + b_1^2 - b_2 = t \left( \sum_{i=0}^{e-1} (b_1 s)^{2i} + \eta_\zeta^2 + b_1^2 \right) + (\delta_\zeta \eta_\zeta)^{2e-1} + t \eta_\zeta^2 + b_1^2 - b_2 = (\delta_\zeta \eta_\zeta)^{2e-1},$$

where we use (9.17) at the second equality and (9.13) at the third one. \hfill \Box

We take $\theta_\zeta \in K^{ac}$ such that $\theta'_\zeta = \theta_\zeta^{2e-1}$. Then we have $\theta_\zeta^2 - \theta_\zeta = \delta_\zeta \eta_\zeta$. We put $N^{ac}_\zeta = M^{ac}_\zeta(\eta_\zeta, \theta_\zeta)$, which is a cyclic extension of $M^{ac}_\zeta$ of order 4 by Lemma 9.8.

**Lemma 9.9.** The character $\xi'_{n, \zeta}$ factors through $\text{Gal}(N^{ac}_\zeta/M^{ac}_\zeta)$.
Proof. Let $\sigma \in \text{Ker } \mathcal{E}_{n, \zeta'}$. We take $\sigma_1, \sigma_2 \in \text{Ker } \mathcal{E}_{n, \zeta'}$ such that $\sigma = \sigma_1 \sigma_2^{-1}$, $\sigma_1 \in I_{M_{n, \zeta'}}$ and $\Theta_\zeta(\sigma_2) = ((1, b_0, c_0), -1)$. Then we have $(\vec{a}_{\sigma_1}, \vec{b}_{\sigma_1}, \vec{c}_{\sigma_1}) \in R_0$, $\text{Tr}_{F_{2^n}/F_2}(\vec{b}_{\sigma_1}) = 0$ and

$$\text{Tr}_{F_{2^n}/F_2}\left(t\left(\vec{c}_{\sigma_1} + \sum_{0 \leq i < j \leq \epsilon - 1} \vec{b}_{\sigma_1}^{2i+2j}\right)\right) = 0$$

(9.18)

by (9.5). It suffices to show that $\sigma_i(\eta_\zeta) = \eta_\zeta$ and $\sigma_i(\theta'_\zeta) = \theta'_\zeta$ for $i = 1, 2$.

We have

$$\sigma_1(\eta_\zeta) - \eta_\zeta \equiv \sum_{i=0}^{\epsilon-1} b_{\sigma_1}^{2i} \equiv 0 \pmod{p_{N_{\zeta^n}}},$$

$$\sigma_2(\eta_\zeta) - \eta_\zeta \equiv \sum_{i=0}^{\epsilon-1} b_0^{2i} + b_1^{2i} - b_1 \equiv 0 \pmod{p_{N_{\zeta^n}}}$$

by $\text{Tr}_{F_{2^n}/F_2}(\vec{b}_{\sigma_1}) = 0$ and $b_0^{2i} - b_1 = s$. By Lemma 9.8, we have $\sigma_i(\eta_\zeta) - \eta_\zeta \in F_2$ for $i = 1, 2$. Hence, we have $\sigma_i(\eta_\zeta) = \eta_\zeta$ for $i = 1, 2$. We have

$$\sigma_1(\theta'_\zeta) - \theta'_\zeta = \sum_{i=0}^{\epsilon-1} (t(\sigma_1(\gamma'_\zeta) - \gamma'_\zeta))^{2i}.$$  

Further, we have

$$\sigma_1(\gamma'_\zeta) - \gamma'_\zeta \equiv c_{\sigma_1} + \sum_{i=0}^{\epsilon-1} (b_{\sigma_1})^{2i+1} + \sum_{i=0}^{\epsilon-1} b_{\sigma_1}^{2i} \sum_{i=0}^{\epsilon-1} \beta_{\zeta}^{2i} + \sum_{0 \leq i < j \leq \epsilon - 1} b_{\sigma_1}^{2i+2j} \mod p_{N_{\zeta^n}},$$

where we use (2.4.1) and $\vec{b}_{\sigma_1} \in F_{2^n}$ at the first equality, and use $\text{Tr}_{F_{2^n}/F_2}(\vec{b}_{\sigma_1}) = 0$ at the second one. This implies $\sigma_1(\theta'_\zeta) \equiv \theta'_\zeta \pmod{p_{N_{\zeta^n}}}$ by (9.18). By a similar argument as above using Lemma 9.8, we obtain $\sigma_1(\theta'_\zeta) = \theta'_\zeta$.

It remains to show $\sigma_2(\theta'_\zeta) = \theta'_\zeta$. Using (9.16) and $\text{Tr}_{F_{2^n}/F_2}(t) = 1$, we see that

$$\sum_{i=0}^{\epsilon-1} (t(\gamma'_\zeta)^{2i} = \gamma'_\zeta + \sum_{1 \leq i \leq j \leq \epsilon - 1} t^{2i} \beta_{\zeta}^{2i} + \sum_{0 \leq i < j \leq \epsilon - 1} t^{2i} ((\delta_{\zeta} - s)(\eta_\zeta - b_1))^{2i}.$$  

(9.19)

We put

$$\gamma''_\zeta = \gamma'_\zeta + \sum_{1 \leq i \leq j \leq \epsilon - 1} t^{2i} \beta_{\zeta}^{2i}.$$  

By $c_0^2 + c_0 = b_0^{2i+1}$ and $t = b_0 + b_0^{2i}$ (cf. (2.18), (9.4)), we have

$$\sigma_2(\gamma_\zeta) - \gamma_\zeta \equiv c_0 + \sum_{i=0}^{\epsilon-1} (b_0^{2i}(\beta_{\zeta} + b_0))^{2i} \equiv c_0^{2i} + \sum_{i=0}^{\epsilon-1} (b_0 + t)\beta_{\zeta}^{2i} \equiv \sigma_2(\gamma''_\zeta) \equiv \gamma''_\zeta \equiv \gamma'_\zeta \equiv \gamma_{\zeta} \equiv \eta_{\zeta} - b_1 + \sum_{i=0}^{\epsilon-1} (t\beta_{\zeta})^{2i} + \sum_{0 \leq i < j \leq \epsilon - 1} b_{0}^{2i+2j} \pmod{p_{N_{\zeta^n}}}.$$

Then we have

$$\sigma_2(\gamma'_\zeta) - \gamma'_\zeta \equiv c_0^{2i} + s(\eta_\zeta - b_1) + \sum_{i=0}^{\epsilon-1} (t\beta_{\zeta})^{2i} + \sum_{0 \leq i < j \leq \epsilon - 1} b_{0}^{2i+2j} \pmod{p_{N_{\zeta^n}}}.$$
by (9.4) and (9.14). Hence, we have

\[
\sigma_2(\gamma''_\zeta) - \gamma''_\zeta \equiv \sigma_2(\gamma'_\zeta) - \gamma'_\zeta + \sum_{1 \leq i \leq j < e} t^{2i+1}(\beta_\zeta + b_0)^{2i} - \sum_{1 \leq i \leq j < e} t^{2j}\beta_\zeta^{2j}
\]

\[
\equiv \sigma_2(\gamma'_\zeta) - \gamma'_\zeta + t(\eta_\zeta - b_1) + \sum_{i=0}^{e-1} (t^i \beta_\zeta)^{2i} + \sum_{1 \leq i < j \leq e} b_0^{2i+t} t^{2i} + \sum_{0 \leq i < j < e} b_0^{2i} t^{2j}
\]

\[
\equiv c_0^{2e} + s^2(\eta_\zeta - b_1) + \sum_{0 \leq i < j < e} b_0^{2i+2j} + \sum_{1 \leq i < j \leq e} b_0^{2i} t^{2j} \mod p_{N_\zeta}^a
\]

where we use (9.14) and \( t \in \mathbb{F}_{2e} \) at the second equality and \( s^2 + s = t \) at the last equality. We can check that

\[
c_0^{2e} + \sum_{0 \leq i < j < e} b_0^{2i+2j} + \sum_{1 \leq i < j \leq e} b_0^{2i} t^{2j} = st
\]

by (2.17), (9.4) and \( \text{Tr}_{\mathbb{F}_{2e}/\mathbb{F}_2}(b_0) = 1 \). As a result, we obtain

\[
\sigma_2(\gamma''_\zeta) - \gamma''_\zeta \equiv s^2 \eta_\zeta + b_1 s^2 + st \mod p_{N_\zeta}^a.
\]

Hence, by (9.15) and (9.19), we have

\[
\sigma_2(\theta'_\zeta) - \theta'_\zeta \equiv \sum_{i=0}^{2e-1+2e-2} d_i \eta_\zeta^i \mod p_{N_\zeta}^a
\]

for some \( d_i \in k^{ac} \). We have

\[
d_0 = b_1 s^2 + st + t \sum_{j=1}^{e-1} (b_1 s)^{2j} + b_1 s \sum_{i=1}^{e-1} t^{2i} + b_2 - b_2 = 0.
\]

This implies \( \sigma_2(\theta'_\zeta) = \theta'_\zeta \), since we know that \( \sigma_2(\theta'_\zeta) - \theta'_\zeta \in \mathbb{F}_2^e \) by Lemma 9.8

\[\square\]

## 10 Refined Swan conductor

Let \( \tilde{K} \subset K^{ur} \) be the unramified extension of \( K^{ur} \) generated by \( \mu_{p^{p^e-1}}(K^{ur}) \). For a finite field extension \( L \) of \( K \) in \( K^{ac} \), we write \( \tilde{L} \) for the composite field of \( L \) and \( \tilde{K} \) in \( K^{ac} \). We write \( \tilde{M}'_\zeta \) for \( M'_{\zeta} \). Then \( \tilde{N}_\zeta \) is a Galois extension of \( \tilde{M}'_\zeta \). By (9.8) and (9.9), we can take \( \beta'_\zeta \in \tilde{M}'_\zeta \) such that

\[
\beta''_\zeta - \tilde{\beta}_\zeta = \delta_\zeta, \quad \beta'_\zeta \equiv \beta_\zeta \mod p_{\tilde{M}'_\zeta},
\]

since there is \( x \in \mathbb{F}_{2e}^e \) such that \( x^{2e} - x = \sum_{i=0}^{e-1} b_0^{2i} \) if \( p = 2 \). Then we have \( \tilde{M}_\zeta = \tilde{M}'_\zeta(\beta_\zeta) \) by Krasner's lemma.

**Lemma 10.1.** 1. We have

\[
\psi_{\tilde{N}_\zeta/\tilde{M}'_\zeta}(v) = \begin{cases} 
  v & \text{if } v \leq 1, \\
  p^v(v - 1) + 1 & \text{if } 1 < v \leq 2, \\
  p^{e+1}(v - 2) + p^e + 1 & \text{if } 2 < v.
\end{cases}
\]

2. We have

\[
\text{Gal}(\tilde{N}_\zeta/\tilde{M}'_\zeta)_i = \begin{cases} 
  \text{Gal}(\tilde{N}_\zeta/\tilde{M}'_\zeta) & \text{if } i \leq 1, \\
  \text{Gal}(\tilde{N}_\zeta/\tilde{M}'_\zeta) & \text{if } 2 \leq i \leq p^e + 1, \\
  \{1\} & \text{if } p^e + 2 \leq i.
\end{cases}
\]

28
Proof. We have
\[
\psi_{\tilde{M}_{\zeta}/\tilde{M}'_{\zeta}}(v) = \begin{cases} v & \text{if } v \leq 1, \\ p^e(v - 1) + 1 & \text{if } v > 1, \end{cases}
\]
\[
\psi_{\tilde{N}_{\zeta}/\tilde{M}_{\zeta}}(v) = \begin{cases} v & \text{if } v \leq p^e + 1, \\ p(v - p^e + 1) + p^e + 1 & \text{if } v > p^e + 1 \end{cases}
\]
by (2.10), (10.1) and Lemma 5.4 noting that \( \hat{r} \) has a \((p^e - 1)\)-st root in \( \tilde{M}'_{\zeta} \). Hence, the claim 1 follows from \( \psi_{\tilde{N}_{\zeta}/\tilde{M}_{\zeta}} = \psi_{\tilde{N}_{\zeta}/\tilde{M}_{\zeta}} \circ \psi_{\tilde{M}_{\zeta}/\tilde{M}'_{\zeta}} \). The claim 2 follows from the claim 1 and \( \text{Gal}(\tilde{N}_{\zeta}/\tilde{M}'_{\zeta})_{p^e+1} \supset \text{Gal}(\tilde{N}_{\zeta}/\tilde{M}_{\zeta})_{p^e+1} = \text{Gal}(\tilde{N}_{\zeta}/\tilde{M}_{\zeta}) \).

We set \( \varpi_{\tilde{M}'_{\zeta}} = \delta_{\zeta}^{-1}, \varpi_{\tilde{M}_{\zeta}} = \beta_{\zeta}^{-1} \) and \( \varpi_{\tilde{N}_{\zeta}} = (\gamma_{\zeta} \varpi_{\tilde{M}'_{\zeta}})^{-1} \). Then the elements \( \varpi_{\tilde{M}'_{\zeta}}, \varpi_{\tilde{M}_{\zeta}} \) and \( \varpi_{\tilde{N}_{\zeta}} \) are uniformizers of \( \tilde{M}'_{\zeta}, \tilde{M}_{\zeta} \) and \( \tilde{N}_{\zeta} \) respectively. Let \( \tilde{k} \) be the residue field of \( \tilde{K} \).

Lemma 10.2. We have a commutative diagram
\[
\begin{array}{ccc}
U_{\tilde{N}_{\zeta}/\tilde{M}'_{\zeta}}^{1+p^e} & \xrightarrow{\text{Nr}_{\tilde{N}_{\zeta}/\tilde{M}'_{\zeta}}} & U_{\tilde{M}'_{\zeta}}^{2} \\
\uparrow & & \uparrow \\
\tilde{k} & \xrightarrow{P} & \tilde{k}
\end{array}
\]
where the map \( P \) is given by \( x \mapsto x^p - x \) and the vertical maps are given by
\[
p_{\tilde{N}_{\zeta}/\tilde{M}'_{\zeta}}: U_{\tilde{N}_{\zeta}}^{1+p^e} \rightarrow \tilde{k}; \ 1 - x \gamma_{\zeta}^{-1} \mapsto \bar{x},
\]
\[
p_{\tilde{M}'_{\zeta}/\tilde{M}_{\zeta}}: U_{\tilde{M}'_{\zeta}}^{2} \rightarrow \tilde{k}; \ 1 + x \bar{r} \varpi_{\tilde{M}'_{\zeta}}^2 \mapsto \bar{x}.
\]
Proof. The norm maps \( \text{Nr}_{\tilde{N}_{\zeta}/\tilde{M}'_{\zeta}} \) and \( \text{Nr}_{\tilde{M}'_{\zeta}/\tilde{M}_{\zeta}} \) induce
\[
U_{\tilde{N}_{\zeta}}^{1+p^e}/U_{\tilde{N}_{\zeta}}^{1} \twoheadrightarrow U_{\tilde{M}'_{\zeta}}^{1+p^e}/U_{\tilde{M}'_{\zeta}}^{1}; \ 1 - u \gamma_{\zeta}^{-1} \mapsto 1 - (u^p - u) \varpi_{\tilde{M}'_{\zeta}},
\]
\[
U_{\tilde{M}'_{\zeta}}^{1+p^e}/U_{\tilde{M}'_{\zeta}}^{1} \twoheadrightarrow U_{\tilde{M}_{\zeta}}^{2}/U_{\tilde{M}_{\zeta}}^{3}; \ 1 - u \varpi_{\tilde{M}'_{\zeta}}^{p^e+1} = 1 - u \beta_{\zeta}^{-1} \varpi_{\tilde{M}_{\zeta}}^{-1} \mapsto 1 + u \bar{r} \varpi_{\tilde{M}'_{\zeta}}^2,
\]
respectively by Lemma 5.5.1 and calculations of the norms. Hence, the claim follows.

For any finite extension \( M \) of \( K \), we write \( \psi_M \) for the composite \( \psi_K \circ \text{Tr}_{M/K} \).

Lemma 10.3. We have \( \text{rs} (\xi_{n,\zeta}|_{W_{\tilde{M}'_{\zeta}}}, \psi_{M_{\zeta}}^n) = -n' \delta_{\zeta}^{-1}(p^e+1) \mod U_{M_{\zeta}}^{1} \).

Proof. We put \( \tilde{\xi}_{n,\zeta} = \xi_{n,\zeta}|_{W_{\tilde{M}'_{\zeta}}} \) and regard it as a character of \( \tilde{M}'_{\zeta} \). By (2.12), Lemma 5.5.1 and Lemma 10.1, the restriction of \( \tilde{\xi}_{n,\zeta} \) to \( U_{\tilde{M}'_{\zeta}}^{2} \) is given by the composition
\[
U_{\tilde{M}'_{\zeta}}^{2} \xrightarrow{\text{Art}_{\tilde{M}'_{\zeta}}} \text{Gal}(\tilde{N}_{\zeta}/\tilde{M}_{\zeta}) \simeq \mathbb{F}_p \xrightarrow{\psi_{\tilde{\xi}_{n,\zeta}}} \mathbb{Q}_{\ell}^\times,
\]

29
where the isomorphism $\text{Gal}(\tilde{N}/\tilde{M}) \simeq \mathbb{F}_p$ is given by $\sigma \mapsto \sigma(\gamma_\zeta) - \gamma_\zeta$. We define $p\tilde{N}_\zeta - \gamma_\zeta$ as in Lemma 10.2. For $u \in \mathcal{O}_{\tilde{M}_\zeta}$, we put $\sigma_u = \text{Art}_{\tilde{M}_\zeta}(1 + u\tilde{\varphi}_{\tilde{M}_\zeta})$ and then have

$$
\tilde{\xi}_{n,\zeta}(1 + u\tilde{\varphi}_{\tilde{M}_\zeta}) = \psi_0\left(\frac{\sigma_u(\gamma_\zeta) - \gamma_\zeta}{\gamma_\zeta}ight) = \psi_0\left(p\tilde{N}_\zeta - \gamma_\zeta \left(\frac{\gamma_\zeta}{\sigma_u(\gamma_\zeta)}\right)\right)
$$

where we use Lemma 5.32 and Lemma 10.2 at the last equality. Since we have

$$
\tilde{\xi}_{n,\zeta}(1 + u\tilde{\varphi}_{\tilde{M}_\zeta}) = \psi_0\left(p\tilde{N}_\zeta - \gamma_\zeta \left(\frac{\gamma_\zeta}{\sigma_u(\gamma_\zeta)}\right)\right) = \psi_0 \circ \text{Tr}_{\tilde{k}/\mathbb{F}_p}(\tilde{u}),
$$

we obtain

$$
\tilde{\xi}_{n,\zeta}(1 + x) = \psi_{\tilde{M}_\zeta}(-n'\delta^x\tilde{u})
$$

for $x \in p^2_{\tilde{M}_\zeta}$ by (10.3). This implies

$$
\tilde{\xi}_{n,\zeta}(1 + x) = \psi_{\tilde{M}_\zeta}(-n'\delta^x\tilde{u})
$$

for $x \in p^2_{\tilde{M}_\zeta}$ since $\text{Tr}_{\tilde{k}/\mathbb{F}_p}: \tilde{k} \to k_N$ is surjective. The claim follows from (10.4) and Proposition 5.31.

**Lemma 10.4.** We have $\text{rsw}(\tau_{n,\zeta,\chi,c}, \psi_{E_\zeta}) = n'\varphi'_{\zeta} \mod U_{1, E_\zeta}$.

**Proof.** By Proposition 5.21, we may assume that $\chi = 1$ and $c = 1$. By Proposition 9.5 and Lemma 10.3, we have

$$
\text{rsw}(\tau_{n,\zeta}, \psi_{T_\zeta}) = \text{Nr}_{M_{\zeta}}(\text{rsw}(\xi_{n,\zeta}, \psi_{M_{\zeta}})) = n'\varphi'_{\zeta} \mod U_{1, T_\zeta}.
$$

Since $T_\zeta$ is a tamely ramified extension of $E_\zeta$, we have

$$
\text{rsw}(\tau_{n,\zeta}, \psi_{E_\zeta}) = \text{rsw}(\tau_{n,\zeta}, \psi_{T_\zeta}) \mod U_{1, T_\zeta}
$$

by Proposition 5.22. The claim follows from (10.5) and (10.6).

**Proposition 10.5.** We have $\text{rsw}(\tau_{\zeta,\chi,c}, \psi_K) = \text{rsw}(\pi_{\zeta,\chi,c}, \psi_K)$.

**Proof.** By $\tau_{\zeta,\chi,c} = \text{Ind}_{E_\zeta/K}(\tau_{n,\zeta,\chi,c})$, we have

$$
\text{rsw}(\tau_{\zeta,\chi,c}, \psi_K) = \text{Nr}_{E_\zeta/K}(\text{rsw}(\tau_{n,\zeta,\chi,c}, \psi_{E_\zeta})).
$$

Hence, the claim follows from Lemma 4.3 and Lemma 10.4.

**Lemma 10.6.** We have $\text{Sw}(\tau_{\zeta,\chi,c}) = 1$.

**Proof.** This follows from Lemma 10.4 and (10.7).

**Lemma 10.7.** The representation $\tau_{\zeta,\chi,c}$ is irreducible.

**Proof.** We know that the restriction of $\tau_{n,\zeta,\chi,c}$ to the wild inertia subgroup of $W_{E_\zeta}$ is irreducible by Corollary 2.6. Assume that $\tau_{\zeta,\chi,c}$ is not irreducible. Then we have an irreducible factor $\tau'$ of $\tau_{\zeta,\chi,c}$ such that $\text{Sw}(\tau') = 0$, by Lemma 10.6 and the additivity of Sw. Then, the restriction of $\tau'$ to the wild inertia subgroup of $W_K$ is trivial by $\text{Sw}(\tau') = 0$. On the other hand, we have an injective homomorphism $\tau_{n,\zeta,\chi,c} \to \tau'_{W_{E_\zeta}}$ by Frobenius reciprocity. This is a contradiction.

**Proposition 10.8.** The representation $\tau_{\zeta,\chi,c}$ is irreducible of Swan conductor 1.

**Proof.** This follows from Lemma 10.6 and Lemma 10.7.
11 Epsilon factor

11A. Reduction to special cases

In this subsection, we show the equality $\varepsilon(\tau_{\xi,\psi}, \psi_K) = \varepsilon(\tau_{\xi,\psi}, \psi_K)$ of epsilon factors assuming some results in the special case where $n = p^{e}$, $\text{ch } K = p$ and $f = 1$. The results in the special case will be proved in the next subsection.

Lemma 11.1. We have

\[
\lambda(E_{\xi}/K, \psi_K) = \left\{ \begin{array}{ll}
\frac{n}{p} & \text{if } n' \text{ is odd}, \\
-\left(-\varepsilon(p) \left(\frac{2n'}{p}\right) \left(\frac{1}{p}\right)^{f-1}\right) & \text{if } n' \text{ is even},
\end{array} \right.
\]

\[
\lambda(T_{\xi}^u/E_{\xi}, \psi_{E_{\xi}}) = \left\{ \begin{array}{ll}
(-1)^{\frac{(p-1)fN}{4}} & \text{if } p \neq 2,
\frac{q}{p^{r+1}} & \text{if } p = 2.
\end{array} \right.
\]

Proof. We have

\[
\lambda(T_{\xi}^u/E_{\xi}, \psi_{E_{\xi}}) = \lambda(T_{\xi}^u/E_{\xi}, \psi_{E_{\xi}})\lambda(E_{\xi}^u/E_{\xi}, \psi_{E_{\xi}})^{p^{r}+1} = \lambda(T_{\xi}^u/E_{\xi}, \psi_{E_{\xi}}).
\]

If $p \neq 2$, we have

\[
\lambda(T_{\xi}^u/E_{\xi}, \psi_{E_{\xi}}) = -\left(-\varepsilon(p) \left(\frac{2n'}{p}\right) \left(\frac{1}{p}\right)^{f-1}\right)^{fN} = (-1)^{\frac{(p-1)fN}{4}}
\]

by Lemma 8.4, since $fN$ is even. The other assertions immediately follow from Lemma 8.4.

Lemma 11.2. We have

\[
\lambda(M_{\xi}^u/T_{\xi}^u, \psi_{T_{\xi}^u}) = \left\{ \begin{array}{ll}
(-1)^{f} & \text{if } p = 2 \text{ and } e \leq 2,
\frac{r}{kN} & \text{otherwise}.
\end{array} \right.
\]

Proof. Let $K_{(0)}$ and $K_{(p)}$ be non-archimedean local fields of characteristic 0 and $p$ respectively. Assume that the residue fields of $K_{(0)}$ and $K_{(p)}$ are isomorphic to $k$. We take uniformizers $\varpi_{(0)}$ and $\varpi_{(p)}$ of $K_{(0)}$ and $K_{(p)}$ respectively. We define $T_{\xi_{(0)}}^u$ similarly as $T_{\xi}^u$ starting from $K_{(0)}$. We use similar notations also for other objects in the characteristic zero side and the positive characteristic side. We have the isomorphism

\[
\mathcal{O}_{T_{\xi_{(p)}}^u}/p_{T_{\xi_{(p)}}^u} \cong \mathcal{O}_{T_{\xi_{(0)}}^u}/p_{T_{\xi_{(0)}}^u}; \quad \xi_0 + \xi_1 \varpi_{T_{\xi_{(p)}}^u} \mapsto \xi_0 + \xi_1 \varpi_{T_{\xi_{(p)}}^u},
\]

of algebras, where $\xi_0, \xi_1 \in k$. Hence, it suffices to show the claim in one of the characteristic zero side and the positive characteristic side by [Del84 Proposition 3.7.1], since $\text{Gal}(M_{\xi_{(p)}}^u/T_{\xi_{(p)}}^u)^2 = 1$ and $\text{Gal}(M_{\xi_{(0)}}^u/T_{\xi_{(0)}}^u)^2 = 1$, where we use upper numbering filtration of Galois groups.

First, we consider the case where $p \neq 2$ and $\text{ch } K = p$. Then, we have $d_{M_{\xi}^u/T_{\xi}^u} = \delta$ by Proposition 6.2 and the fact that $fN$ is even. Hence, $\delta_{M_{\xi}^u/T_{\xi}^u}$ is unramified by (6.1). Hence, we have

\[
\lambda(M_{\xi}^u/T_{\xi}^u, \psi_{T_{\xi}^u}) = \varepsilon(\delta_{M_{\xi}^u/T_{\xi}^u}, \psi_{T_{\xi}^u})^{p^r} = \left(\frac{r}{kN}\right)
\]

by [Hen84 Proposition 2], [BH06 23.5 Proposition] and (6.1).
We consider the case where $p = 2$. Assume that $e \geq 3$ and $\text{ch } K = 0$. We have $D = 2^e \delta_{\xi}^{-1} + 1$ in the notation of Proposition 5.2 with $(L, K, a) = (M_{\xi}^n, T_{\xi}^n, \delta_{\xi})$. Then, we have $D \in (M_{\xi}^n)^2$. Hence, we have $\kappa_D = 1$, $d_{M_{\xi}^n/T_{\xi}^n} = 1$ and $w_2(\text{Ind}_{M_{\xi}^n/T_{\xi}^n} 1) = 1$ by Proposition 6.2 and $(t_{\xi}^n) \equiv 0 \mod 2$. Therefore we have

$$\lambda(M_{\xi}^n/T_{\xi}^n, \psi_{T_{\xi}^n}) = \varepsilon(\text{Ind}_{M_{\xi}^n/T_{\xi}^n} 1, \psi_{T_{\xi}^n}) = \varepsilon(1^{\oplus p}, \psi_{T_{\xi}^n}) = 1$$

by Theorem 6.1.

Assume that $e = 2$ and $\text{ch } K = 2$. Then we see that $d_{M_{\xi}^n/T_{\xi}^n} = 1$ by Definition 6.3. Hence, $\delta_{M_{\xi}^n/T_{\xi}^n}$ is the unramified character satisfying $\delta_{M_{\xi}^n/T_{\xi}^n} = (-1)^f$ by Theorem 6.4. Then we see that

$$\lambda(M_{\xi}^n/T_{\xi}^n, \psi_{T_{\xi}^n}) = \varepsilon(\text{Ind}_{M_{\xi}^n/T_{\xi}^n} 1, \psi_{T_{\xi}^n}) = \varepsilon(\delta_{M_{\xi}^n/T_{\xi}^n} \oplus 1^{\oplus 3}, \psi_{T_{\xi}^n}) = (-1)^f,$$

where we use Theorem 6.1 at the second equality.

Assume that $e = 1$ and $\text{ch } K = 2$. Let $\kappa_{M_{\xi}^n/T_{\xi}^n}$ be the quadratic character associated to the extension $M_{\xi}^n$ over $T_{\xi}^n$. Then we have

$$\lambda(M_{\xi}^n/T_{\xi}^n, \psi_{T_{\xi}^n}) = \varepsilon(\kappa_{M_{\xi}^n/T_{\xi}^n}, \psi_{T_{\xi}^n})$$

by Theorem 6.1 similarly as above. We can check that the norm map $N_{T_{\xi}^n/T_{\xi}^n}$ induces

$$U_{M_{\xi}^n/T_{\xi}^n}^1/U_{M_{\xi}^n/T_{\xi}^n}^2 \rightarrow U_{T_{\xi}^n}^1/U_{T_{\xi}^n}^2; 1 + u\delta^{-1} \mapsto 1 + (u^2 - u)\alpha_{\xi}.$$

Then, by Lemma 6.3 we have

$$\kappa_{M_{\xi}^n/T_{\xi}^n}(1 + \alpha_{\xi} x) = \psi_0 \left( \text{Art}_{T_{\xi}^n}(1 + \alpha_{\xi} x)(\delta_{\xi}) - \delta_{\xi} \right)$$

$$= \psi_0 \left( p_{M_{\xi}^n} \delta_{\xi}^{-1} \left( \text{Art}_{T_{\xi}^n}(1 + \alpha_{\xi} x)(\delta_{\xi})^{-1} \right) \right) = \psi_0(\text{Tr}_{K/F_p}(\bar{\alpha})).$$

(11.1)

for $x \in O_{T_{\xi}^n}$ noting that $k_N = k$. Hence, we have $\text{rsw}(\kappa_{M_{\xi}^n/T_{\xi}^n}, \psi_{T_{\xi}^n}) = \alpha_{\xi}$ by Proposition 5.3.1. By Proposition 5.3.2, we have

$$\varepsilon(\kappa_{M_{\xi}^n/T_{\xi}^n}, \psi_{T_{\xi}^n}) = \kappa_{M_{\xi}^n/T_{\xi}^n}(\alpha_{\xi}) = \kappa_{M_{\xi}^n/T_{\xi}^n}(1 + \alpha_{\xi}) = (-1)^f,$$

where we use $N_{T_{\xi}^n/T_{\xi}^n}(\delta_{\xi}) = \alpha_{\xi}^{-1} + 1$ and (11.1) at the last equality.

\[\square\]

**Lemma 11.3.** We have

$$\text{Tr}_{M_{\xi}^n/T_{\xi}^n}(\delta_{\xi}^i) = \begin{cases} 0 & \text{if } 1 \leq i \leq p^e - 2, \\ \hat{i}^{-1}(p^e - 1) & \text{if } i = p^e - 1. \end{cases}$$

\[\text{Proof.}\] Vanishing for $1 \leq i \leq p^e - 2$ follows from (9.9). We have also

$$\text{Tr}_{M_{\xi}^n/T_{\xi}^n}(\delta_{\xi}^{p^e - 1}) = \text{Tr}_{M_{\xi}^n/T_{\xi}^n}(\hat{i}^{-1} + \delta_{\xi}^{-1}(-\alpha_{\xi}^{-1} + \epsilon_1)) = \hat{i}^{-1}(p^e - 1)$$

by (9.9).

\[\square\]
Lemma 11.4. We have
\[ \delta_{T^u/\zeta}(\text{rs}w(\tau_{n,\zeta}, \psi_{E_{\zeta}})) = \begin{cases} 1 & \text{if } p \neq 2, \\ \left(\frac{q}{p^u+1}\right) & \text{if } p = 2. \end{cases} \]

Proof. If \( p = 2 \), the claim follows from Lemma 8.3.1 and Lemma 10.4 since \( T^u \) is totally ramified over \( E_{\zeta} \).

Assume that \( p \neq 2 \). Then we have \( d_{T^u/E_{\zeta}} = (-1)^{(p^u+1)/2} \varphi_{E_{\zeta}} \) by Proposition 6.2. Hence, we have \( \delta_{T^u/E_{\zeta}}((-1)^{(p^u-1)/2} \varphi_{E_{\zeta}}) = 1 \) by Lemma 8.3.2. Therefore, we have
\[ \delta_{T^u/E_{\zeta}}(\text{rs}w(\tau_{n,\zeta}, \psi_{E_{\zeta}})) = \delta_{T^u/E_{\zeta}}(n' \varphi_{E_{\zeta}}) = \delta_{T^u/E_{\zeta}}(n'(-1)^{\frac{p^u-1}{2}}) = \left(\frac{n'(-1)^{\frac{p^u-1}{2}}}{q^N}\right) = 1 \]
by \([\text{Gal65} (1)], \text{Lemma 8.3.2}, \text{Lemma 10.4} \) and the fact that \( fN \) is even. \( \square \)

Lemma 11.5. Assume that \( n = p^\varepsilon \). Then we have \( \varepsilon(\tau_{\zeta,\chi,e}, \psi_K) \equiv \varepsilon(\pi_{\zeta,\chi,e}, \psi_K) \mod \mu_{p^\varepsilon}(\mathbb{C}) \).

Proof. Let \( \pi \) be the representation of \( GL_n(K) \) corresponding to \( \tau_{\zeta,\chi,e} \) by the local Langlands correspondence. By the proof of \( \text{[BH14, 2.2 Proposition]} \), Proposition 8.6 and Proposition 10.5 we have \( \pi \simeq c\text{-Ind}_{L_{\zeta}U_3}^K \Lambda \) for a character \( \Lambda : L_{\zeta}^\times U_3^\times \rightarrow \mathbb{C} \times \) which coincides with \( \Lambda_{\zeta,\chi,e} \) on \( K^\times U_3^\times \). Then, the claim follows from \( \text{[BH14, 2.2 Lemma (1)]} \), because \( L_{\zeta}^\times U_3^\times (K^\times U_3^\times) \) is the cyclic group of order \( p^\varepsilon \). \( \square \)

Proposition 11.6. We have \( \varepsilon(\tau_{\zeta,\chi,e}, \psi_K) = \varepsilon(\pi_{\zeta,\chi,e}, \psi_K) \).

Proof. By Proposition 3.2 and \( \tau_{\zeta,\chi,e} \simeq \text{Ind}_{\zeta,\chi,e} \zeta_{\tau_{\zeta,\chi,e}, \psi_{E_{\zeta}}} \), it suffices to show that
\[ \lambda(E_{\zeta}/K, \psi_K)^{p^\varepsilon} \varepsilon(\tau_{n,\zeta,\chi,e}, \psi_{E_{\zeta}}) = (-1)^{n-1+e_0} \chi(n')c. \]

By Lemma 10.4 we may assume \( \chi = 1 \) and \( c = 1 \). Hence, it suffices to show
\[ \lambda(E_{\zeta}/K, \psi_K)^{p^\varepsilon} \varepsilon(\tau_{n,\zeta,\chi,e}, \psi_{E_{\zeta}}) = (-1)^{n-1+e_0}f. \quad (11.2) \]

Assuming that \( (11.2) \) is proved for \( n = p^e \), we show \( (11.2) \) for general \( n \). Let \( \tau'_{n,\zeta} \) denote the representation of \( W_{E_{\zeta}} \) given by \( \Theta_{\zeta} \) in \( (2.12) \) and \( \tau_{p^e} \). We put \( \psi'_{E_{\zeta}} = n'^{-1} \psi_{E_{\zeta}} \). Applying the result for \( n = p^e \) to \( E_{\zeta}, \varphi_{E_{\zeta}} \) in place of \( K, \varpi \), we have \( \varepsilon(\tau'_{n,\zeta}, \psi'_{E_{\zeta}}) = (-1)^{p^e-1+\epsilon_0f} \), where \( \epsilon_0 \) denotes \( e_0 \) for \( n = p^e \). Since \( \det \tau'_{n,\zeta} \) is unramified as in the proof of Proposition 8.6, we have
\[ \varepsilon(\tau'_{n,\zeta}, \psi_{E_{\zeta}}) = \det \tau'_{n,\zeta}(n') \varepsilon(\tau'_{n,\zeta}, \psi'_{E_{\zeta}}) = (-1)^{p^e-1+\epsilon_0f}. \quad (11.3) \]

We note that the inflation of the character in \( (2.9) \) by \( \Theta_{\zeta} \) factors through
\[ W_{E_{\zeta}} \rightarrow \{ \pm 1 \} \times \mathbb{Z}; \; \sigma \mapsto \left( \frac{e_{\sigma}+1}{2}, fn_{\sigma} \right). \]

If \( p \neq 2 \), we have \( (n' \varphi_{E_{\zeta}}, \varphi_{E_{\zeta}})_{E_{\zeta}} = \left( \frac{n}{q} \right) \), where
\[ \left( \frac{,}{}, E_{\zeta}^\times / (E_{\zeta}^\times)^2 \times E_{\zeta}^\times / (E_{\zeta}^\times)^2 \rightarrow \{ \pm 1 \} \right) \]
denotes the Hilbert symbol. Hence, we have
\[ \frac{\varepsilon(\tau_{n,\zeta}, \psi_{E_{\zeta}})}{\varepsilon(\tau'_{n,\zeta}, \psi_{E_{\zeta}})} = \begin{cases} \left(\frac{n}{q}\right)^{n-p^e} \left(\frac{n}{p}\right)^n \left(-\varepsilon(p)\left(\frac{2}{p}\right)\right)^{n-p^e} f & \text{if } p \neq 2, \\ (-1)^{n(n-2)/8} \left(\frac{2^{(p^e-2)}}{8}\right)^{n-p^e} & \text{if } p = 2. \end{cases} \quad (11.4) \]
by (2.9), Lemma 4.2 and Lemma 10.4. Then we have (11.2) by Lemma 11.1, (11.3) and (11.4).

Therefore, we may assume that \( n = p^e \). By Lemma 11.1 and Lemma 11.5, it suffices to show that

\[
\varepsilon(\tau_{n, \zeta}, \psi_{E_{\zeta}})^{N(p^e+1)} = \begin{cases} 
1 & \text{if } p \neq 2, \\
(-1)^{1+\alpha f} & \text{if } p = 2.
\end{cases}
\]

By Proposition 5.1, we have

\[
\varepsilon(\tau_{n, \zeta}, \psi_{E_{\zeta}})^{N(p^e+1)} = \delta_{\mathcal{T}^n_{E_{\zeta}} \big(\text{rsw}(\tau_{n, \zeta}, \psi_{E_{\zeta}})\big)}^{-1} \lambda(T^n_{\zeta}/E_{\zeta}, \psi_{E_{\zeta}})^p \varepsilon(\tau_{n, \zeta}|_{W_{\mathcal{T}^n_{\zeta}}}, \psi_{\mathcal{T}^n_{\zeta}}).
\]

By this, Lemma 11.1 and Lemma 11.4, it suffices to show that

\[
\varepsilon(\tau_{n, \zeta}|_{W_{\mathcal{T}^n_{\zeta}}}, \psi_{\mathcal{T}^n_{\zeta}}) = \begin{cases} 
(-1)^{\frac{(p-1)(p^e)}{2}}(\frac{e}{\mathsf{f}}) & \text{if } p \neq 2, \\
(-1)^{1+\alpha f}(\frac{q}{p^e+1}) & \text{if } p = 2.
\end{cases}
\]

This follows from Lemma 11.2 and Proposition 11.7.

We set \( \varpi_{M^n_{\zeta}} = \delta^{-1}_\zeta \).

**Proposition 11.7.** Assume that \( n = p^e \). Then we have

\[
\varepsilon(\xi_{n, \zeta}, \psi_{M^n_{\zeta}}) = \begin{cases} 
(-1)^{\frac{(p-1)(p^e)}{4}}(\frac{e}{\mathsf{f}}) & \text{if } p \neq 2, \\
(-1)^{1+\alpha f} & \text{if } p = 2.
\end{cases}
\]

**Proof.** First, we reduce the problem to the positive characteristic case. Assume that \( \text{ch } K = 0 \). Take a positive characteristic local field \( K_{(p)} \) whose residue field is isomorphic to \( k \). We define \( M^n_{\zeta_{(p)}} \) similarly as \( M^n_{\zeta} \) starting from \( K_{(p)} \). We use similar notations also for other objects in the positive characteristic side. Then we have the isomorphism

\[
\mathcal{O}^{n}_{M^n_{\zeta_{(p)}}}/p^n_{M^n_{\zeta_{(p)}}} \cong \mathcal{O}^{n}_{M^n_{\zeta}}/p^n_{M^n_{\zeta}}; \quad \xi_0 + \xi_1 \varpi_{M^n_{\zeta_{(p)}}} + \xi_2 \varpi^2_{M^n_{\zeta_{(p)}}} \mapsto \hat{\xi}_0 + \hat{\xi}_1 \varpi_{M^n_{\zeta}} + \hat{\xi}_2 \varpi^2_{M^n_{\zeta}}
\]

of algebras, where \( \xi_1, \xi_2, \xi_3 \in k \). Hence, the problem is reduced to the positive characteristic case by [Del83, Proposition 3.7.1].

We may assume \( K = \mathbb{F}_q((t)) \). We put \( K_{(1)} = \mathbb{F}_p((t)) \). We define \( M^n_{\zeta_{(1)}} \) similarly as \( M^n_{\zeta} \) starting from \( K_{(1)} \). We use similar notations also for other objects in the \( K_{(1)} \)-case. We put \( f' = [M^n_{\zeta} : M^n_{\zeta_{(1)}}] \). We have

\[
\delta_{M^n_{\zeta}/M^n_{\zeta_{(1)}}}(\text{rsw}(\xi_{n, \zeta_{(1)}}, \psi_{M^n_{\zeta_{(1)}}})) = (-1)^{f'-1}
\]

by Lemma 11.3. We have \( \lambda(M^n_{\zeta}/M^n_{\zeta_{(1)}}, \psi_{M^n_{\zeta_{(1)}}}) = 1 \), since the level of \( \psi_{M^n_{\zeta}} \) is \( 2 - p^e \) by Lemma 11.3. Then, we obtain

\[
\varepsilon(\xi_{n, \zeta}, \psi_{M^n_{\zeta}}) = (-1)^{f'-1} \varepsilon(\xi_{n, \zeta_{(1)}}, \psi_{M^n_{\zeta_{(1)}}})^{f'}
\]

by Proposition 5.1. By (11.5), the problem is reduced to the case where \( f = 1 \). In this case, the claim follows from Lemma 11.1 and Lemma 11.16.

**11B. Special cases**

We assume that \( n = p^e \), \( \text{ch } K = p \) and \( f = 1 \) in this subsection.
11B1. Odd case

Assume that \( p \neq 2 \) in this subsection.

**Lemma 11.8.** We have \( \psi_M(\delta^r + 1 + x\omega_M^n) = 1 \) for \( x \in k_N \).

**Proof.** For \( x \in k_N \), we have

\[
\psi_M(\delta^r + 1 + x\omega_M^n) = \psi_M(\delta^r + 1 - x\omega_M^n) = \psi_M(\delta^r + 1 - x\omega_M^n) = 1.
\]

because \( \text{Tr}_{M^n/T^n}(\delta^r) = 0 \) and \([M^n : T^n] = p^r\). If \( p^r \neq 3 \), then we have the claim, because \( \text{Tr}_{M^n/T^n}(\delta^r) = 0 \).

We assume that \( p^r = 3 \). Then we have

\[
\psi_M(\delta^r + 1 + x\omega_M^n) = \psi_M(\delta^r + 1 - x\omega_M^n) = \psi_M(\delta^r + 1 - x\omega_M^n) = 1
\]

by \( \text{Tr}_{M^n/T^n}(\delta^r) = 2r-1 \) and \( r^4 = -1 \).

Let \( \theta \) be as in (9.11).

**Lemma 11.9.** We have

\[
\text{N}_{M^n/M^n}(1 + x\theta^{\frac{p^r-1}{2}}\omega_M^n) \equiv 1 + (-2r)^{\frac{1}{2}}x^p\omega_M^n + \frac{x^2}{2}\omega_M^n \mod p^3\omega_M^n
\]

for \( x \in k_N \).

**Proof.** We put \( T = 1 + x\theta^{\frac{p^r-1}{2}}\omega_M^n \). By \( \theta^p - \theta = (-2r)^{-1}\delta^2 \) in (9.11), we have

\[
\theta = \frac{1}{2r}\delta^2 \left((x^{-1}(T-1)\delta^2)^{-1} - 1\right)
\]

Substituting this to \( x^{-1}(T-1)\delta^2 = \theta^{\frac{p^r-1}{2}} \), we have

\[
(T^2 - 2T + 1 - x^2\omega_M^n) \equiv 1 + (-2r)^{\frac{1}{2}}x^p\omega_M^n = 0.
\]

The claim follows from this.

**Lemma 11.10.** We have

\[
\sum_{x \in k_N} \xi_{n,\alpha}(1 + x\omega_M^n)^{-1} = -((-1)^{\frac{n^2}{2}} p)^{\alpha}(r_{k_N}).
\]

**Proof.** Let \( \xi'_{n,\alpha} \) be as in subsection 9C. We note that the left hand side of the claim does not change even if we replace \( \xi_{n,\alpha} \) by \( \xi'_{n,\alpha} \). We have

\[
\sum_{x \in k_N} \xi'_{n,\alpha}(1 + x\omega_M^n)^{-1} = \sum_{x \in k_N} \xi'_{n,\alpha}(1 + (-2r)^{\frac{1}{2}}x^p\omega_M^n)^{-1}
\]

\[
= \sum_{x \in k_N} \xi'_{n,\alpha}(1 - \frac{x^2}{2}\omega_M^n)^{-1} = \sum_{x \in k_N} \psi_M(1 - \frac{x^2}{2}\omega_M^n)^{-1},
\]

where we use Lemma 9.7 and Lemma 11.13 at the second equality and (10.4) at the last equality. The last expression in (11.6) is equal to

\[
\sum_{x \in k_N} \psi_M(-2r)^{-1}(p^r - 1)x^2 = \sum_{x \in k_N} \psi_M(\text{Tr}_{k_N/P}(rx^2)) = -((-1)^{\frac{n^2}{2}} p)^{\alpha}(r_{k_N})
\]

by (2.16), (8.2), Lemma 11.3 and \( N = 2e_0 \).
Lemma 11.11. We have \( \varepsilon(\xi_{n,\zeta}, \psi_{M_n^u}) = -(1) \frac{(n-1)\log}{\log K} \). 

Proof. We have

\[
\varepsilon(\xi_{n,\zeta}, \psi_{M_n^u}) = p^{-e_0} \sum_{x \in k_N} \xi_{n,\zeta}(-\delta_\zeta^{p+1}(1 + x M_n^u))^{-1} \psi_{M_n^u}(-\delta_\zeta^{p+1}(1 + x M_n^u)) = -(1) \frac{\log \log K}{\log K} \xi_{n,\zeta}(-\delta_\zeta^{p+1})^{-1}
\]

by Proposition [5.32], Lemma [11.8] and Lemma [11.10]. We have

\[
\xi_{n,\zeta}(-\delta_\zeta^{p+1}) = \xi'_{n,\zeta}(-\delta_\zeta^{p+1})(-1) \frac{p^{-1} - \delta_\zeta^{p+1}}{N} = \xi'_{n,\zeta}(-\delta_\zeta^{p+1}) = \xi''_{n,\zeta}(-(2r\delta_\zeta^{p+1}) \frac{1}{r^2}) = 1,
\]

where we use

\[
\text{Nr}_{N_n^u/M_n^u}(\theta_\zeta^{p-1} \psi_{M_n^u}) = (-2r) \frac{1}{r^2} \psi_{M_n^u}
\]

at the third equality and \( k_N^\times \subset \text{Nr}_{N_n^u/M_n^u}((N_n^u)^\times) \) at the last equality. Thus, we have the claim. \( \square \)

11B2. Even case

Assume that \( p = 2 \) in this subsection.

Lemma 11.12. We have \( \text{Tr}_{M_n^u/K}(\delta_\zeta^{2e+1}) = 0 \) and

\[
\text{Tr}_{M_n^u/K}(\delta_\zeta^{2e}) = \begin{cases} 1 & \text{if } e = 1, \\ 0 & \text{if } e \geq 2. \end{cases}
\]

Proof. These follow from \( \delta_\zeta^{2e} - \delta_\zeta = \alpha_\zeta^{-1} + 1 \). \( \square \)

Lemma 11.13. We have \( \text{Nr}_{N_n^u/M_n^u}(\theta_\zeta \delta_\zeta^{-1}) = \delta_\zeta^{-1} \).

Proof. We have \( \text{Nr}_{N_n^u/M_n^u}(\theta_\zeta) = \delta_\zeta^3 \) by \( \theta_\zeta^2 - \theta_\zeta = \delta_\zeta \eta_\zeta \) and \( \eta_\zeta^2 - \eta_\zeta = \delta_\zeta \). The claim follows from this. \( \square \)

Let \( \sigma_0 \in \text{Gal}(N_n^u/M_n^u) \) be a generator of \( \text{Gal}(N_n^u/M_n^u) \) determined by \( \sigma_0(\eta_\zeta) - \eta_\zeta = 1 \) and \( \sigma_0(\theta_\zeta) - \theta_\zeta = \eta_\zeta \).

Lemma 11.14. Let \( \iota_{n,\zeta}: \text{Gal}(N_n^u/M_n^u) \to \mathbb{C}^\times \) be the homomorphism induced by \( \xi'_{n,\zeta} \) (cf. Lemma [9.2]). Then we have \( \iota_{n,\zeta}(\sigma_0) = -\sqrt{-1} \).

Proof. Let \( s, t \) be as in [9.3]. We take \( \sigma \in I_{M_n^u} \) such that \( \Theta_\zeta(\sigma) = ((1, t, s^2), 0) \). Recall that

\[
\phi'(((1, t, s^2)) = \tilde{g} \left( 1, s^2 + \sum_{0 \leq i < j \leq e-1} t^{2i+2j} \right) \in R_0^n
\]

is a generator. Then it suffices to show that \( \sigma(\eta_\zeta) - \eta_\zeta = 1 \) and \( \sigma(\theta_\zeta) - \theta_\zeta = \eta_\zeta \). We can check the first equality easily. To show the second equality, it suffices to show that \( \sigma(\theta'_\zeta) - \theta'_\zeta = \eta_\zeta^{\omega-1} \). By (2.11), we have

\[
\sigma(\gamma_\zeta) - \gamma_\zeta \equiv s^2 + \sum_{i=0}^{e-1} (t_\beta_\zeta + t^2)^2 \mod p_{N_n^u}.
\]

36
By \( t = \sigma(\beta_{\xi}) - \beta_{\zeta}, \) Tr\(_{F_{2}/F_{2}}(t) = 1 \) and (9.14), we have
\[
\sigma(\gamma'_{\xi}) - \gamma_{\xi} = \eta_{\xi} - b_{1} + s^{2} + \sum_{0 \leq i,j \leq e-1} t^{2i+2j} \mod p_{N_{\xi}}.
\]
Hence, by (9.15) and (9.19), we have
\[
\sigma(\gamma_{\xi}) - \gamma_{\xi} = \sum_{i=0}^{2e-1} d_{i} \eta_{\xi}^{i} \mod p_{N_{\xi}}
\]
with some \( d_{i} \in k^{ac}. \) By (9.19), we have
\[
\sum_{i=0}^{e-1} (t(\sigma(\gamma_{\xi}) - \gamma_{\xi})^{2}) = \sigma(\gamma_{\xi}) - \gamma_{\xi} + \sum_{1 \leq i,j \leq e-1} t^{2i+2j} + \sum_{0 \leq i,j \leq e-1} t^{2i}(\delta_{\xi} - s)^{2i}.
\]
Therefore, again by (9.15) and (9.19), we have
\[
d_{0} = b_{1} + s^{2} + \sum_{0 \leq i,j \leq e-1} t^{2i+2j} + \sum_{1 \leq i,j \leq e-1} t^{2i+2j} + b_{1}^{2} = s + s^{2} + t = 0.
\]
This implies \( \sigma(\theta'_{\xi}) - \theta_{\xi} = \eta_{\xi}^{2e-1}, \) since we know that \( \sigma(\theta'_{\xi}) - \theta'_{\xi} - \eta_{\xi}^{2e-1} \in F_{2} \) by Lemma 9.8 and \( \sigma(\eta_{\xi}) - \eta_{\xi} = 1. \)

**Lemma 11.15.** We have
\[
\varepsilon(\xi_{n,\zeta}, \psi_{M_{\xi}}) = \begin{cases}
\frac{1+\sqrt{-1}}{2} & \text{if } e = 1, \\
\frac{1-\sqrt{-1}}{2} & \text{if } e \geq 2.
\end{cases}
\]

**Proof.** By Proposition 5.3, (10.4), Lemma 11.3, Lemma 11.11 and Lemma 11.12 we have
\[
\varepsilon(\xi_{n,\zeta}, \psi_{M_{\xi}}) = 2^{-\frac{1}{2}} \sum_{x \in F_{2}} \xi'_{n,\zeta}(\delta_{\xi}^{2x+1}(1+x\delta_{\xi}^{-1}))^{-1} \psi_{M_{\xi}}(\delta_{\xi}^{2x+1}(1+x\delta_{\xi}^{-1}))
\]
\[
= \begin{cases}
2^{-\frac{1}{2}}(1 - \xi'_{n,\zeta}(1 + \delta_{\xi}^{-1})^{-1}) & \text{if } e = 1, \\
2^{-\frac{1}{2}}(1 + \xi'_{n,\zeta}(1 + \delta_{\xi}^{-1})^{-1}) & \text{if } e \geq 2.
\end{cases}
\]
First assume that \( e = 1. \) Then we know the equality in the claim modulo \( \mu_{2}(C) \) by Lemma 11.11. Hence it suffices to show the equality of the real parts. This follows from (11.7). In particular, we have \( \xi'_{2,1}(1 + \delta_{\zeta}^{-1}) = \sqrt{-1}. \)

Next, we consider the general case. We put \( \alpha'_{1} = 1/(\delta_{\xi}^{2} - \delta_{\xi} + 1) \) and \( \varpi' = \alpha'_{1}. \) Let \( \xi'_{2,1,\zeta} \) denote \( \xi'_{2,1} \) in the case where \( K \) and \( \varpi \) are replaced by \( F_{2}(\varpi') \) and \( \varpi'. \) By applying Lemma 11.11 to \( \xi'_{n,\zeta} \) and \( \xi'_{2,1,\zeta}, \) we have \( \xi'_{n,\zeta} = \xi'_{2,1,\zeta}. \) We know that \( \xi'_{2,1,\zeta}(1 + \delta_{\zeta}^{-1}) = \sqrt{-1} \) by the result in the case \( e = 1. \) Hence, we have \( \xi'_{n,\zeta}(1 + \delta_{\zeta}^{-1}) = \sqrt{-1}, \) which shows the claim. \[ \square \]

**Lemma 11.16.** We have
\[
\varepsilon(\xi_{n,\zeta}, \psi_{M_{\xi}}) = (-1)^{1+e_{0}}.
\]

**Proof.** The epsilon factor \( \varepsilon(\xi_{n,\zeta}, \psi_{M_{\xi}}) \) equals \( \varepsilon(\xi'_{n,\zeta}, \psi_{M_{\xi}}) \) times
\[
\begin{cases}
(\frac{1+\sqrt{-1}}{2})^{-3(2^{e}+1)} & \text{if } e \neq 2, \\
-\frac{1+\sqrt{-1}}{2}^{-3(2^{e}+1)} & \text{if } e = 2
\end{cases}
\]
by Lemma 4.2, (9.12), Lemma 10.3. Hence, the claim follows from Lemma 11.15. \[ \square \]
A Realization in cohomology of Artin–Schreier variety

We realize $\tau_n$ in the cohomology of an Artin–Schreier variety. Let $\nu_{n-2}$ be the quadratic form on $\mathbb{A}^n_{k^{ac}}$ defined by

$$\nu_{n-2}(y_i)_{1 \leq i \leq n-2} = -\frac{1}{n} \sum_{1 \leq i \leq j \leq n-2} y_i y_j.$$  

Let $X$ be the smooth affine variety over $k^{ac}$ defined by

$$x^p - x = y^{p+1} + \nu_{n-2}(y_i)_{1 \leq i \leq n-2} \text{ in } \mathbb{A}^n_{k^{ac}}.$$  

We define a right action of $Q \rtimes \mathbb{Z}$ on $X$ by

$$(x, y, (y_i)_{1 \leq i \leq n-2})((a, b, c), 0) = \left( x + \sum_{i=0}^{e-1} (-by)^p + c, a(y + by^p), (2^{p+1}y_i)_{1 \leq i \leq n-2} \right),$$

$$(x, y, (y_i)_{1 \leq i \leq n-2})^p = (x^p, y^p, (y^p_i)_{1 \leq i \leq n-2} \text{ Fr}(1).$$

We consider the morphism

$$\pi_{n-2} : \mathbb{A}^{n-1}_{k^{ac}} \to \mathbb{A}^1_{k^{ac}}; \ (y, (y_i)_{1 \leq i \leq n-2}) \mapsto y^{p+1} + \nu_{n-2}(y_i)_{1 \leq i \leq n-2}.$$  

Then we have a decomposition

$$H^{n-1}_c(X, \mathbb{Q}_{\ell}) \cong \bigoplus_{\psi \in \mathbb{F}_p^\times \setminus \{1\}} H^{n-1}_c(\mathbb{A}^n_{k^{ac}}, \pi_{n-2}^* \mathcal{L}_\psi)$$  

as $Q \rtimes \mathbb{Z}$ representations. Let $\rho_n$ be the representation over $\mathbb{C}$ of $Q \rtimes \mathbb{Z}$ defined by

$$H^{n-1}_c(\mathbb{A}^{n-1}_{k^{ac}}, \pi_{n-2}^* \mathcal{L}_\psi) \left( \frac{n-1}{2} \right)$$

and $\iota$, where $(\frac{n-1}{2})$ means the twist by the character $((a, b, c), m) \mapsto p^{m(n-1)/2}$.

**Lemma A.1.** If $p \neq 2$, then we have $\det \nu_{n-2} = -(-2n!)^n \in \mathbb{F}_p^\times / (\mathbb{F}_p^\times)^2$.

**Proof.** This is an easy calculation. $\square$

**Proposition A.2.** We have $\tau_n \simeq \rho_n$.

**Proof.** Let $Y$ be the smooth affine variety over $k^{ac}$ defined by

$$x^p - x = \nu_{n-2}(y_i)_{1 \leq i \leq n-2} \text{ in } \mathbb{A}^n_{k^{ac}}.$$  

We define a right action of $Q \rtimes \mathbb{Z}$ on $Y$ by

$$(x, (y_i)_{1 \leq i \leq n-2})((a, b, c), 0) = \left( x, (2^{p+1}y_i)_{1 \leq i \leq n-2} \right),$$

$$(x, (y_i)_{1 \leq i \leq n-2})^p = (x^p, (y^p_i)_{1 \leq i \leq n-2} \text{ Fr}(1).$$

Using the action of $Q \rtimes \mathbb{Z}$ on $Y$, we can define an action of $Q \rtimes \mathbb{Z}$ on $H^{n-2}_c(\mathbb{A}^{n-2}_{k^{ac}}, \pi_{n-2}^* \mathcal{L}_\psi)$. Then we have

$$H^{n-1}_c(\mathbb{A}^{n-1}_{k^{ac}}, \pi_{n-2}^* \mathcal{L}_\psi) \cong H^{1}_c(\mathbb{A}^1_{k^{ac}}, \pi^* \mathcal{L}_\psi) \otimes H^{n-2}_c(\mathbb{A}^{n-2}_{k^{ac}}, \nu_{n-2}^* \mathcal{L}_\psi)$$  

(A.2)
by the Künneth formula, where the isomorphism is compatible with the actions of $Q \rtimes \mathbb{Z}$.

By (A.2), it suffices to show the action of $Q \rtimes \mathbb{Z}$ on

$$H_c^{n-2}(\mathbb{A}^{n-2}_\mathbb{K}, \nu_{n-2}^* \mathcal{L}_{\psi_0}) \left( \frac{n-1}{2} \right)$$  \hspace{1cm} (A.3)

is equal to the character (2.9) via $\iota$.

First, consider the case where $p \neq 2$. The equality of the actions of $Q$ follows from [DL98, Lemma 2.2.3]. We have

$$(-1)^{n-2} \sum_{y \in \mathbb{F}_p^{n-2}} \psi_0(\nu_{n-2}(y)) = \left( \frac{-1}{p} \right) \left( -\frac{-2n'}{p} \right)^n (\epsilon(p)\sqrt{p})^{n-2}$$

$$= \left( -\epsilon(p) \frac{-2n'}{p} \right)^n \sqrt{p}^{n-2}$$  \hspace{1cm} (A.4)

by Lemma [A.1]. The equality of the actions of $\text{Fr}(1) \in Q \rtimes \mathbb{Z}$ follows from [Del77, Sommes trig. Scholie 1.9] and (A.4).

If $p = 2$, the equality follows from [IT20, Proposition 4.5] and $\left( \frac{2}{n-1} \right) = (-1)^{n(n-2)/8}$. \hfill \Box

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