Symbolic Exact Inference for Discrete Probabilistic Programs

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Abstract
The computational burden of probabilistic inference remains a hurdle for applying probabilistic programming languages to practical problems of interest. In this work, we provide a semantic and algorithmic foundation for efficient exact inference on discrete-valued finite-domain imperative probabilistic programs. We leverage and generalize efficient inference procedures for Bayesian networks, which exploit the structure of the network to decompose the inference task, thereby avoiding full path enumeration. To do this, we first compile probabilistic programs to a symbolic representation. Then we adapt techniques from the probabilistic logic programming and artificial intelligence communities in order to perform inference on the symbolic representation. We formalize our approach, prove it sound, and experimentally validate it against existing exact and approximate inference techniques. We show that our inference approach is competitive with inference procedures specialized for Bayesian networks, thereby expanding the class of probabilistic programs which can be practically analyzed.

1 Introduction
When it is computationally feasible, exact probabilistic inference is vastly preferable to approximation techniques. Exact inference methods are deterministic and reliable, so they can be trusted for making high-consequence decisions and do not propagate errors to subsequent analyses. Ideally, one would use exact inference whenever possible, only resorting to approximation when exact inference strategies become infeasible. Even when approximating inference, one often performs exact inference in an approximate model. This is the case for a wide range of approximation schemes, including message passing [9, 40], sampling [18, 20], and variational inference [43].

Existing probabilistic programming systems lag behind state-of-the-art techniques for performing exact probabilistic inference in other domains such as graphical models. Fundamentally, inference – both exact and approximate – is theoretically hard [37]. However, exact inference is routinely performed in practice. This is because many interesting inference problems have structure: there are underlying repetitions and decompositions that can be exploited to perform inference more efficiently than the worst case. Existing efficient exact inference procedures – notably techniques from the graphical models inference community – systematically find and exploit the underlying structure of the problem in order to mitigate the inherent combinatorial explosion problem of exact probabilistic inference [5, 23, 32].

We seek to close the performance gap between exact inference in discrete graphical models and discrete-valued finite-domain probabilistic programs. The key idea behind existing state-of-the-art inference procedures in discrete graphical models is to compile the graphical model into a representation known as a weighted Boolean formula (WBF), which is a symbolic representation of the joint probability distribution over the graphical model’s random variables. This symbolic representation exposes key structural elements of the distribution, such as independences between random variables. Then, inference is performed via a weighted sum of the models of the WBF, a process known as weighted model counting (WMC). This WMC process exploits the independences present in the WBF, and is competitive with state-of-the-art inference techniques in other domains, such as probabilistic logic programming, Bayesian networks, and probabilistic databases [8, 17, 41].

Thus, we propose a symbolic exact inference procedure for discrete imperative finite-domain probabilistic programs. To arrive at an efficient exact inference algorithm, we first spend significant effort formalizing a particular choice of denotational semantics. With these semantics in place, we find that an exact inference algorithm simply falls out from representing the semantic objects with the appropriate data structure. Our approach adapts and generalizes inference via WMC to the context of probabilistic programs by combining it with symbolic reasoning techniques from the model checking and verification communities.

Once we formalize our semantics and develop our compilation procedure, we experimentally validate our approach against (1) existing exact and approximate inference techniques for probabilistic programs; and (2) specialized inference procedures for graphical models. We show that our method can perform exact inference on challenging graphical models which are encoded as probabilistic programs.

Contributions We make the following contributions:

1. We develop a novel denotational semantics for a discrete-valued finite-domain loopy probabilistic programming language called DIPPL.
2. We present an exact inference strategy for DIPPL based on state-of-the-art inference techniques from the probabilistic logic programming and graphical models communities, and prove it is correct.
3. We experimentally validate our approach against well-known benchmarks and show that our technique is competitive with specialized graphical models inference techniques.

Outline Section 2 gives a motivating example which shows the strengths of our technique. Section 3 introduces the syntax and semantics of DIPPL. Section 4 describes the procedure for compiling DIPPL programs into weighted Boolean formulae. Section 5 describes how our compilation procedure exploits nuanced program properties in order to perform exact inference in an efficient manner. Section 6 adds bounded loops to the syntax and semantics DIPPL, and describes the symbolic compilation of these loops. Section 7 describes our experimental validation of our compilation rules in comparison with existing exact and approximate inference procedures. Finally, Section 8 discusses related work, and Section 9
A simple probabilistic program. The notation \( x \sim \text{flip}(\theta) \) denotes drawing a sample from a Bernoulli(\( \theta \)) distribution and assigning the outcome to the variable \( x \). The label \( f \) is not actually part of the syntax but is used so we can refer to each \( \text{flip} \) uniquely.

(a) A simple probabilistic program. The notation \( x \sim \text{flip}_p(0.5) \);
    \[ 
    \begin{align*}
    &x \sim \text{flip}_p(0.5); \\
    &\text{if}(x) \{ y \sim \text{flip}_p(0.6) \} \\
    &\text{else} \{ y \sim \text{flip}_p(0.4) \}; \\
    &\text{if}(y) \{ z \sim \text{flip}_p(0.6) \} \\
    &\text{else} \{ z \sim \text{flip}_p(0.9) \}
    \end{align*} 
    \]

(b) A binary decision diagram representing the Boolean formula compiled from the program in Figure 1a; a low edge is denoted a dashed line, and a high edge is denoted with a solid line. The variables \( f_x, f_i, f_2, f_3, \) and \( f_4 \) correspond to annotations in Figure 1a.

Figure 1. Probabilistic program and its symbolic representation.

concludes. The appendix in the supplementary material contains detailed proofs for all of the theorems included in this text.

2 Exact Symbolic Inference

In this section we present a motivating example which highlights key elements of our approach. Figure 1a shows a simple probabilistic program that encodes a linear Bayesian network, a structure known as a Markov chain [23]. In order to perform inference efficiently on a Markov chain – or any Bayesian network – it is necessary to exploit the independence structure of the model. Exploiting independence is one of the key techniques for efficient graphical model inference procedures. Markov chains encoded as probabilistic programs have \( 2^n \) paths, where \( n \) is the length of the chain. Thus, inference methods which rely on exhaustively exploring the paths in a program – a strategy we refer to as path-based inference methods – will require exponential time in the length of the Markov chains; see our experiments in Figure 6.

However, it is well known that Markov chains support linear-time inference in the length of the chain [23]. The reason for this is that the structure of a Markov chain ensures a strong form of conditional independence: each node in the chain depends only on the directly preceding node in the chain. In the program of Figure 1a, for example, the probability distribution for \( z \) is independent of \( x \) given \( y \), i.e., if \( y \) is fixed to a particular value, then the probability distribution over \( z \) can be computed without considering the distribution over \( x \). Therefore inference can be factorized: the probability distribution for \( y \) can be determined as a function of that for \( x \), and then the probability distribution for \( z \) can be determined as a function of that for \( y \). More generally, inference for a chain of length \( n \) can be reduced to inference on \( n - 1 \) separate chains, each of length two.

To close this performance gap between Bayesian networks and exact PPL inference, we leverage and generalize state-of-the-art techniques for Bayesian inference, which represent the distribution symbolically [7, 17]. In this style, the Bayesian network is compiled to a Boolean function and represented using a binary decision diagram (BDD) or related data structure [13]. The BDD structure directly exploits conditional independences – as well as other forms of independence – by caching and re-using duplicate sub-functions during compilation [1].

In this paper we describe an algorithm for compiling a probabilistic program to a Boolean formula, which can then be represented by a BDD. As an example, Figure 1b shows a BDD representation of the program in Figure 1a. The outcome of each \( \text{flip}_p(\theta) \) expression in the program is encoded as a Boolean variable labeled \( f_i \). A model of the BDD is a truth assignment to all the variables in the BDD that causes the BDD to return \( T \), and each model of the BDD in Figure 1b represents a possible execution of the original program.

The exploitation of the conditional independence structure of the program is clearly visible in the BDD. For example, any feasible execution in which \( y \) is true has the same sub-function for \( z \) – the subtree rooted at \( f_3 \) – regardless of the value of \( x \). The same is true for any feasible execution in which \( y \) is false. Moreover, the BDD for a Markov chain has size linear in the length of the chain, despite the exponential number of possible execution paths.

To perform inference on this BDD, we first associate a weight with each truth assignment to each variable: the variables \( x, y, \) and \( z \) are given a weight of 1 for both the true and false assignments, and the \( \text{flip}(\theta) \) variables are given a weight of \( \theta \) and \( 1 - \theta \) for their true and false assignments respectively. The Boolean formula together with these weights is called a weighted Boolean formula.

Finally, we can perform inference on the original probabilistic program relative to a given inference query (e.g., “What is the probability that \( z \) is false?”) via weighted model counting (WMC). The weight of a model of the BDD is defined as the product of the weights of each variable assignment in the model, and the WMC of a set of models is the sum of the weights of the models. Then the answer to a given inference query \( Q \) is simply the WMC of all models of the BDD that satisfy the query. WMC is a well-studied general-purpose technique for performing probabilistic inference and is currently the state-of-the-art technique for inference in discrete Bayesian networks, probabilistic logic programs, and probabilistic databases [7, 17, 41]. BDDs support linear-time weighted model counting by performing a single bottom-up pass of the diagram [13]; thus, we can compile a single BDD for a probabilistic program which can be used to exactly answer many inference queries.

3 The DIPPL Language

Here we formally define the syntax and semantics of our discrete finite-domain imperative probabilistic programming language DIPPL language. First we will introduce and discuss the syntax. Then, we will describe the semantics and its basic properties. A notable property of our semantics is its semantic domain, which will later be required for formalizing our symbolic compilation procedure. Finally, in order to contextualize our language, we will compare with existing semantic frameworks for probabilistic programs.
3.1 Syntax

Figure 2 gives the syntax of our probabilistic programming language DIPPL. Metavariable $x$ ranges over variable names, and metavariables $\theta$ ranges over rational numbers in the interval $[0, 1]$. All data is Boolean-valued, and expressions include the usual Boolean operations, though it is straightforward to extend the language to other finite-domain datatypes. In addition to the standard loop-free imperative statements, there are two probabilistic statements. The statement $x \sim \text{flip}(\theta)$ samples a value from the Bernoulli distribution defined by parameter $\theta$ (i.e., $\top$ with probability $\theta$ and $\bot$ with probability $1 - \theta$). The statement $\text{observe}(e)$ conditions the current distribution of the program on the event that $e$ evaluates to true.

3.2 Semantics

The goal of the semantics of any probabilistic programming language is to define the distribution over which one wishes to perform inference. In this section, we introduce a denotational semantics that directly produces this distribution of interest, and it is defined over program states. A state $\sigma$ is a finite map from variables to Boolean values, and $\Sigma$ is the set of all possible states. We will be interested in probability distributions on $\Sigma$, defined formally as follows:

**Definition 3.1** (Discrete probability distribution). Let $\Omega$ be a set called the sample space. Then, a discrete probability distribution on $\Omega$ is a function $\Pr : 2^\Omega \rightarrow [0, 1]$ such that (1) $\Pr(\Omega) = 1$; (2) for any $\omega \in \Omega$, $\Pr(\omega) \geq 0$; (3) for any countable set of disjoint elements $\{A_i\}$ where $A_i \subseteq 2^\Omega$, we have that $\Pr(\bigcup_i A_i) = \sum_i \Pr(A_i)$.

We denote the set of all possible discrete probability distributions with $\Sigma$ as the sample space as $\text{Dist} \Sigma$. We add a special element to $\text{Dist} \Sigma$, denoted $\bot$, which is the function that assigns a probability of zero to all states (this will be necessary to represent situations where an observed expression is false).

We define a denotational semantics for DIPPL, which we call its transition semantics and denote $\llbracket \cdot \rrbracket_T$. These semantics are summarized in Figure 3a. The transition semantics will be the primary semantic object of interest for DIPPL and will directly produce the distribution over which we wish to perform inference. For some statement $s$, the transition semantics is written $\llbracket s \rrbracket_T(\sigma' | \sigma)$, and it computes the conditional probability upon executing $s$ of transitioning to state $\sigma'$ given that the start state is $\sigma$ and no observe statements are violated. The transition semantics have the following type signature:

$$\llbracket s \rrbracket_T : \Sigma \rightarrow \text{Dist} \Sigma.$$

The transition semantics of DIPPL is shown in Figure 3a. The semantics of skip, assignment, and conditionals are straightforward. The semantics of sampling from a Bernoulli distribution is analogous to that for assignment, except that there are two possible output states depending on the value that was sampled. An observe statement has no effect if the associated expression is true in $\sigma$; otherwise the semantics has the effect of mapping $\sigma$ to the special $\bot$ distribution.
The Role of Observe in Sequencing

The transition semantics of DIPPL require that each statement be interpreted as a conditional probability. Ideally, we would like this conditional probability to be sufficient to describe the semantics of compositions. Perhaps surprisingly, the conditional probability distribution of transitioning from one state to another alone is insufficient for capturing the behavior of compositions in the presence of observations. We will illustrate this principle with an example. Consider the following two DIPPL statements:

\[
\begin{align*}
\text{bar}_1 &= \begin{cases} 
\text{if}(x) \ (y \sim \text{flip}(1/4)) & \text{if } x \text{ is true}, \\
\text{else} \ (y \sim \text{flip}(1/2)) & \text{if } x \text{ is false}, 
\end{cases} \\
\text{bar}_2 &= \begin{cases} 
\text{if}(y) \ (y \sim \text{flip}(1/2)) & \text{if } y \text{ is true}, \\
\text{observe}(x \lor y); & \text{if } y \text{ is false}, \\
\text{else} \ (y := F) & \text{otherwise}.
\end{cases}
\end{align*}
\]

Both statements represent exactly the same conditional probability distribution from input to output states:

\[
\begin{align*}
\|\text{bar}_1\|_T(\sigma' \mid \sigma) &= \|\text{bar}_2\|_T(\sigma' \mid \sigma) \\
&= \begin{cases} 
1/2 & \text{if } x[\sigma] = x[\sigma'] = \text{true}, \\
1/4 & \text{if } x[\sigma] = x[\sigma'] = \text{true} \text{ and } y[\sigma'] = \text{true}, \\
3/4 & \text{if } x[\sigma] = x[\sigma'] = \text{true} \text{ and } y[\sigma'] = \text{false}, \\
0 & \text{otherwise}.
\end{cases}
\end{align*}
\]

This is easy to see for \text{bar}_1, which encodes these probabilities directly. For \text{bar}_2, intuitively, when \(y = \text{true}\) in the output, both \text{flip} statements must return \(\text{true}\), which happens with probability 1/4. When \(x = \text{false}\) in the input, \text{bar}_2 uses an observe statement to disallow executions where the first \text{flip} returned \(\text{false}\). Given this observation, the \text{then} branch is always taken, so output \(y = \text{true}\) has probability 1/2.

Because the purpose of probabilistic programming is often to represent a conditional probability distribution, one is easily foiled into believing that these programs are equivalent. This is not the case: \text{bar}_1 and \text{bar}_2 behave differently when sequenced with other statements. For example, consider the sequences (\text{foo; bar}_1) and (\text{foo; bar}_2) where

\[
\text{foo} = \{ \ x \sim \text{flip}(1/3) \ \}.
\]

Let \(\sigma'_{\text{ex}}\) be an output state where \(x = \text{false}, y = \text{true}\), and let \(\sigma_{\text{ex}}\) be an arbitrary state. The first sequence’s transition semantics behave naturally for this output state:

\[
\|\text{foo; bar}_1\|_T(\sigma'_{\text{ex}} \mid \sigma_{\text{ex}}) = 2/3 \cdot 1/2 = 1/3
\]

However, (\text{foo; bar}_2) represents a different distribution: the observe statement in \text{bar}_2 will disallow half of the execution paths where \text{foo} set \(x = \text{true}\). After the observe statement is executed in \text{bar}_2, \text{Pr}(x = \text{true}) = 1/2: the observation has increased the probability of \(x\) being true in \text{foo}, which was 1/3. Thus, it is clear \text{foo} and \text{bar}_2 cannot be reasoned about solely as conditional probability distributions: observe statements in \text{bar}_2 affect the conditional probability of \text{foo}. Thus, the semantics of sequencing requires information beyond solely the conditional probability of each of the sub-statements, as we discuss next.

Sequencing Semantics

The most interesting case in the semantics is sequencing. We compute the transition semantics of sequencing \(\|s_1; s_2\|_T(\sigma' \mid \sigma)\) using the rules of probability. To do this we require the ability to compute the probability that a particular statement will not violate an observation when beginning in state \(\sigma\). Thus we introduce a helper relation that we call the accepting semantics (denoted \(\|s\|_A\)), which provides the probability that a given statement will be accepted (i.e., that no observations will fail) when executed from a given initial state:

\[
\|s\|_A : \Sigma \rightarrow [0,1]
\]

The accepting semantics is defined in Figure 3b. The first three rules in the figure are trivial. An observe statement accepts with probability 1 if the associated expression is true in the given state, and otherwise with probability 0. A sequence of two statements accepts if both statements accept, so the rule simply calculates that probability by summing over all possible intermediate states. Last, the accepting probability of an if statement in state \(\sigma\) is simply the accepting probability of whichever branch will be taken from that state.

Now we can use the accepting semantics to give the transition semantics for sequencing. First, we can compute the probability of both transitioning from some initial state \(\sigma\) to some final state \(\sigma'\) and the fact that no observations are violated in \(s_1\) or \(s_2\):

\[
\alpha = \sum_{\tau \in \Sigma} \|s_1\|_A(\tau \mid \sigma) \times \|s_2\|_A(\sigma' \mid \tau) \times \|s_1\|_A(\sigma) \times \|s_2\|_A(\tau).
\]

In order to obtain the distribution of transitioning between states \(\sigma\) and \(\sigma'\) given that no observations are violated, we must re-normalize this distribution by the probability that no observations are violated:

\[
\beta = \frac{\|s_1\|_A(\sigma) \times \sum_{\tau \in \Sigma} \|s_1\|_T(\tau \mid \sigma) \times \|s_2\|_A(\tau)}{\|s_1; s_2\|_T(\sigma' \mid \sigma)}.
\]

Thus, our conditional probability is \(\|s_1; s_2\|_T(\sigma' \mid \sigma) = \frac{\beta}{\alpha}\). For completeness, we define the 0/0 case to be equal to 0.

3.3 Semantic Properties

Here we briefly describe some properties of these semantics and compare them against existing probabilistic program semantics in the literature. Detailed proofs for all the theorems in this section can be found in Appendix A.1.

One of the most intuitive properties in an imperative program-language is that sequencing and the \text{skip} statement should form a monoid in terms of their semantics. This is equivalent to sequencing being associative and \text{skip} is both the left and right unit of sequencing. This property serves as a sanity check for our proposed semantics. Formally:

\[
\text{Theorem 3.2.} \quad \text{Sequencing is associative, that is, for any statements } s_1, s_2, s_3, \text{ we have } \|s_1; (s_2; s_3)\|_T = \|s_1; (s_2; s_3)\|_T = \|s_1; s_2; s_3\|_A = \|s_1; s_2; s_3\|_A.
\]

\[
\text{Proof.} \quad \text{By structural induction in Appendix A.2.} \quad \Box
\]

\[
\text{Theorem 3.3.} \quad \text{The statement \text{skip} is both a left and right unit of sequencing, that is, for any statement } s, \text{ we have } \|s; \text{skip}\|_T = \|s\|_T \text{ and } \|\text{skip}; s\|_T = \|s\|_T. \quad \text{Similarly for the accepting semantics, } \|\text{skip}; s\|_A = \|s\|_A \text{ and } \|s; \text{skip}\|_A = \|s\|_A.
\]

\[
\text{Proof.} \quad \text{By structural induction in Appendix A.3.} \quad \Box
\]

An additional useful property of our semantics is that it is fully complete with respect to the semantic domain. Formally, this means...
that any element of the semantic domain of the accepting or transition semantics can be expressed as a DIPPL program:

**Theorem 3.4** (Full completeness). Let \( f \in \Sigma \rightarrow \text{Dist} \Sigma \) and let \( g \in \Sigma \rightarrow [0, 1] \). Then, there exists a program \( s \) such that \([s]_A = g\) and \([s]_T = f\).

**Proof.** By structural induction in Appendix A.1. Intuitively, this theorem holds because it is possible to inspect an arbitrary discrete distribution and construct an equivalent DIPPL program. \( \Box \)

### 3.4 Comparison with existing semantics

We include here a brief discussion of alternative probabilistic programming semantics to contextualize our semantics. Probabilistic program semantics are primarily distinguished by their choice of semantic domain. Here we briefly discuss alternative choices to our semantic domain. The first choice is the probability kernel approach, introduced by Kozen [24], where the denotation of statements would have the following signature:

\[
[s] : (\Sigma \rightarrow [0, 1]) \rightarrow (\Sigma \rightarrow [0, 1])
\]

Intuitively, under these semantics, distributions on program states are mapped to other distributions on program states via the application of a measurable function. This semantic domain is used by Claret et al. [10] and Borgström et al. [4]. One consequence of this choice of semantics domains is that it is easier to define the semantics of second-order constructs such as observe, as the expense of making the semantics of first-order constructs like assignments more challenging.

Another arguably more important consequence of this choice of semantic domain is that it is typically not fully complete for a particular choice of syntax. For instance, a natural operation (or, in technical terms, a morphism) on the semantic domain \((\Sigma \rightarrow [0, 1])\) is to test whether or not the distribution on a particular element of \( \Sigma \) is uniform. This is not a facility typically present in a probabilistic programming language as it is infeasible to implement. Thus this semantic domain does not support full-completeness, and leaves an undesirable gap between what programs and semantic domains can do.

Another choice of semantic domain is that of un-normalized probabilities. This choice of domain was used by Hur et al. [21] and Hur et al. [22]. Under these semantics, statements in the language are of the following form:

\[
[s] : \Sigma \rightarrow (\Sigma \rightarrow [0, 1])
\]

**Un-normalized distribution**

The reason that these semantics are un-normalized is to handle the arbitrary placement of observe statements throughout the program. The simplest way to handle such arbitrary observes is to disregard the probability mass which violates the observation condition. Then, at query-time, one simply renormalizes to again recover a valid probability distribution. We utilize the accepting probability in order to re-normalize after the execution of each statement.

In comparison with unnormalized semantics, our semantics more directly express the semantic object of interest: a conditional probability distribution. In Section 5.3 we will show that one can still maintain an unnormalized representation for efficient inference, even with these stronger normalized semantics.

### 4 Symbolic Compilation for Inference

Existing approaches to exact inference for imperative PPLs perform path enumeration: each execution path is individually analyzed to determine the probability mass along the path, and the probability masses of all paths are summed. Indeed, this approach is similar to what our semantics in the previous section does. As argued earlier, such approaches are inefficient due to the need to enumerate complete paths and the inability to take advantage of key properties of the probability distribution across paths, notably forms of independence.

In this section we present an alternative approach to exact inference for PPLs, which is inspired by state-of-the-art techniques for exact inference in Bayesian networks [7]. We describe how to compile a probabilistic program to a weighted Boolean formula, which symbolically represents the program as a relation between input and output states. This process is similar to model checking a Boolean program, as in SLAM [3]. Inference is then reduced to performing a weighted model count (WMC) on this formula, which can be performed efficiently using BDDs and related data structures.

#### 4.1 Weighted Model Counting

Weighted model counting is a well-known general-purpose technique for performing probabilistic inference in the artificial intelligence and probabilistic logic programming communities, and it is currently the state of the art technique for performing inference in certain classes of Bayesian networks and probabilistic logic programs [7, 17, 38, 41]. There exist a variety of general-purpose black-box tools for performing weighted model counting, similar to satisfiability solvers [28, 30, 31].

First, we give basic definitions from propositional logic. A **literal** is either a Boolean variable or its negation. For a formula \( \phi \) over variables \( V \), a sentence \( \omega \) is a **model** of \( \phi \) if it is a conjunction of literals, contains every variable in \( V \), and \( \omega \models \phi \). We denote the set of all models of \( \phi \) as \( \text{Mod}(\phi) \). Now we are ready to define a weighted Boolean formula:

**Definition 4.1** (Weighted Boolean Formula). Let \( \phi \) be a Boolean formula, \( L \) be the set of all literals for variables which occur in \( \phi \), and \( w : L \rightarrow \mathbb{R}^+ \) be a function that associates a real-valued positive weight with each literal \( l \in L \). The pair \((\phi, w)\) is a weighted Boolean formula (WBF).

Next, we define the weighted model counting task, which computes a weighted sum over the models of a weighted Boolean formula:

**Definition 4.2** (Weighted Model Count). Let \((\phi, w)\) be a weighted Boolean formula. Then, the weighted model count (WMC) of \((\phi, w)\) is defined as:

\[
\text{WMC}(\phi, w) \triangleq \sum_{\omega \in \text{Mod}(\phi)} \prod_{l \in \omega} w(l)
\]

where the set \( l \in \omega \) is the set of all literals in the model \( \omega \).

#### 4.2 Symbolic Compilation

In this section we formally define our symbolic compilation of a DIPPL program to a weighted Boolean formula, denoted \( s \rightsquigarrow (\phi, w) \). Intuitively, the formula \( \phi \) produced by the compilation represents the program \( s \) as a relation between initial states and final states, where initial states are represented by unprimed Boolean variables \( \{x_i\} \) and final states are represented by primed Boolean variables.
fine a version that relabels state variables to their primed versions,\( F \)ing the appropriate weighted model count:
\[ (x' \Leftrightarrow f) \land y(V \setminus \{x\}), \delta(V)[f \Rightarrow \theta, \neg f \Rightarrow 1 - \theta] \]
\[ x := e \Leftrightarrow ((x' \Leftrightarrow e) \land y(V \setminus \{x\}), \delta(V)) \]
\[ s_1 \Rightarrow (p_1, w_1) \quad s_2 \Rightarrow (p_2, w_2) \]
\[ \phi'_2 = \phi_2[x_i \mapsto x'_i, x'_j \mapsto x''_j] \]
\[ s_1; s_2 \Rightarrow ((\exists x'_i, \phi_1 \land \phi'_2)[x'_i \mapsto x''_i], w_1 \uplus w_2) \]
\[ \text{observe}(e) \Leftrightarrow (e \land y(V), \delta(V)) \]
\[ s_1 \Rightarrow (p_1, w_1) \quad s_2 \Rightarrow (p_2, w_2) \]
\[ \text{if } \{s_1\} \text{else } \{s_2\} \Rightarrow ((e \land \phi_1) \lor (\neg e \land \phi_2), w_1 \uplus w_2) \]

**Figure 4.** Symbolic compilation rules.

\{x'_i\}. This is similar to a standard encoding for model checking Boolean program, except we include auxiliary variables in the encoding which are neither initial nor final state variables [3]. These compiled weighted Boolean formula will have a probabilistic semantics which allow them to be interpreted as either an accepting or transition probability for the original statement.

Our goal is to ultimately give a correspondence between the compiled weighted Boolean formula and the original denotational semantics of the statement. First we define the translation of a state \( \sigma \) to a logical formula:

**Definition 4.3** (Boolean state). Let \( \sigma = \{(x_1, b_1), \ldots, (x_n, b_n)\} \). We define the Boolean state \( F(\sigma) \) as \( l_1 \land \ldots \land l_n \) where for each \( i \), \( l_i \) is \( x_i \) if \( \sigma(x_i) = 1 \) and \( \neg x_i \) if \( \sigma(x_i) = 0 \). For convenience, we also define a version that relabels state variables to their primed versions, \( F'(\sigma) \triangleq F(\sigma)[x_i \mapsto x'_i] \).

Now, we formally describe how every compiled weighted Boolean formula can be interpreted as a conditional probability by computing the appropriate model weight count:

**Definition 4.4** (Transition and accepting semantics). Let \( (\varphi, w) \) be a weighted Boolean formula, and let \( \sigma \) and \( \sigma' \) be states. Then, the transition semantics of \( (\varphi, w) \) is defined:

\[
\| (\varphi, w) \|_{T}(\sigma' \mid \sigma) \triangleq \frac{\text{WMC}(\varphi \land F(\sigma) \land F'(\sigma'), w)}{\text{WMC}(\varphi \land F(\sigma), w)}
\]

In addition the accepting semantics of \( (\varphi, w) \) is defined:

\[
\| (\varphi, w) \|_{A}(\sigma) \triangleq \frac{\text{WMC}(\varphi \land F(\sigma), w)}{w}
\]

Moreover, the transition semantics of Definition 4.4 allow for more general queries to be phrased as WMC tasks as well. For example, the probability of some event \( \sigma \) being true in the output state \( \sigma' \) can be computed by replacing \( F'(\sigma') \) in Equation 3 by a Boolean formula for \( \sigma \).

Finally, we state our correctness theorem, which describes the relation between the accepting and transition semantics of the compiled WBF to the denotational semantics of DIPPL:

**Theorem 4.5** (Correctness of Compilation Procedure). Let \( s \) be a DIPPL program, \( V \) be the set of all variables in \( s \), and \( s \leadsto (\varphi, w) \). Then for all states \( \sigma \) and \( \sigma' \) over the variables in \( V \), we have:

\[
\| (\varphi, w) \|_{T}(\sigma' \mid \sigma) = \| (\varphi, w) \|_{T}(\sigma' \mid \sigma)
\]

and

\[
\| (\varphi, w) \|_{A}(\sigma) = \| (\varphi, w) \|_{A}(\sigma).
\]

**Proof.** A complete proof can be found in Appendix A.4. \( \square \)

Theorem 4.5 allows us to perform inference via weighted model counting on the compiled WBF for a DIPPL program. Next we give a description of the symbolic compilation rules that satisfy this theorem.

**4.2.1 Symbolic Compilation Rules**

In this section we describe the symbolic compilation which satisfies Theorem 4.5 for each DIPPL statement. The rules for symbolic compilation are defined in Figure 4. They rely on several conventions. We denote by \( V \) the set of all variables in the entire program being compiled. If \( V = \{x_1, \ldots, x_n\} \) then we use \( y(V) \) to denote the formula \( x_1 \Leftrightarrow x'_1 \land \ldots \land (x_n \Leftrightarrow x'_n) \), and we use \( \delta(V) \) to denote the weight function that maps each literal over \( \{x_1, x'_1, \ldots, x_n, x'_n\} \) to 1.

The WBF for \text{skip} requires that the input and output states are equal and provides a weight of 1 to each literal. The WBF for an assignment \( x := e \) requires that \( x' \) be logically equivalent to \( e \) and all other variables’ values are unchanged. Note that \( e \) is already a Boolean formula by the syntax of DIPPL so expressions simply compile to themselves. The WBF for drawing a sample from a Bernoulli distribution, \( x \sim \text{Flip}(\theta) \), is similar to that for an assignment, except that we introduce a (globally) fresh variable \( f \) to represent the sample and weight its true and false literals respectively with the probability of drawing the corresponding value.

The WBF for an \text{observe} statement requires the corresponding expression to be true and that the state remains unchanged. The WBF for an \text{if} statement compiles the two branches to formulas and then uses the standard logical semantics of conditionals. The weight function \( w_1 \uplus w_2 \) is a shadowing union of the two functions, favoring \( w_2 \). However, by construction whenever two weight functions created by the rules have the same literal in their domain, the corresponding weights are equal. Finally, the WBF for a sequence composes the WBEs for the two sub-statements via a combination of variable renaming and existential quantification.

In the following section, we delineate the advantages of utilizing WMC for inference, and describe how WMC exploits program structure in order to perform inference efficiently.

**5 Efficient Inference**

Inference is theoretically hard [37]. Exploiting the structure of the problem – and in particular, exploiting various forms of independence – are essential for scalable and practical inference procedures [5, 23, 32]. In this section, we will represent the compiled weighted Boolean formula as a binary decision diagram (BDD). We will show how BDDs implicitly exploit the problem structure in a manner which is competitive with state of the art techniques from the Bayesian networks [7, 38] and probabilistic logic programming languages [16].
5.1 BDD Representation

BDDs are a popular choice for representing the set of reachable states in the symbolic model checking community [3]. BDDs support a variety of useful properties which make them suitable for this task: they support an array of composition operations, including conjunction, disjunction, existential quantification and variable relabeling. These composition operators are efficient, i.e. performing them requires time polynomial in the size of the two BDDs that are being composed together.

In addition to supporting efficient compositional operators, BDDs also support a variety of efficient queries, including satisfiability and weighted model counting [13]. Thus BDDs support the compositional operators and queries necessary for representing the compilation rules and performing probabilistic inference following Definition 4.4.

BDD Variable Ordering The variable order is an essential component to determine the size of the BDD representation [1]. In this work, will utilize a statically-determined variable ordering which is driven by the order in which variables and flip statements appear in the data-flow graph of the program. There are three adjacent BDD variables introduced for each program variable: $x$, $x'$, and $x''$ (although note that variables $x''$ are not included in any final WMC computation). Each flip statement in the program is represented by a single fresh BDD variable in the order.

5.2 Exploiting Program Structure

Compilation to BDDs – and related representations – is currently the state-of-the-art approach to inference in certain kinds of discrete Bayesian networks, probabilistic logic programs, and probabilistic databases [7, 17, 41]. The fundamental reason is that BDDs exploit duplicate sub-functions if there is a sub-function that is constructed more than once in the symbolic compilation, that duplicate sub-function is cached and re-used. This sub-function deduplication is critical for efficient inference. In this section, we explore how BDDs exploit specific properties of the program, and discuss when a program will have a small BDD.

Independence Exploiting independence is essential for efficient inference, and is the backbone of existing state-of-the-art inference algorithms. There are three kinds of independence structure which we seek to exploit. The first is the strongest form:

**Definition 5.1** (Independence). Let $Pr(X, Y)$ be a joint probability distribution over sets of random variables $X$ and $Y$. Then, we say that $X$ is independent of $Y$, written $X \perp Y$, if $Pr(X, Y) = Pr(X) \times Pr(Y)$. In this case, we say that this distribution factorizes over the variables $X$ and $Y$.

Figure 5a shows a probabilistic program with two independent random variables $x$ and $y$. The corresponding BDD generated in Figure 5b exploits the independence between the variables $x$ and $y$. In particular, we see that node $f_2$ does not depend on the particular value of $x$. Thus, the BDD factorizes the distribution over $x$ and $y$. As a consequence, the size of the BDD grows linearly with the number of independent random variables.

Conditional independence The next form of independence we consider is conditional independence:

**Definition 5.2** (Conditional independence). Let $Pr(X, Y, Z)$ be a joint probability distribution over sets of random variables $X, Y,$ and $Z$. Then, we say $X$ is independent of $Z$ given $Y$, written $X \perp Z \mid Y$, if $Pr(X, Z \mid Y) = Pr(X \mid Y) \times Pr(Z \mid Y)$.

Figure 1 gave an example probabilistic program that exhibits conditional independence. In this program, the variables $x$ and $z$ are correlated unless $y$ is fixed to a particular value: thus, $x$ and $z$ are conditionally independent given $y$. Figure 1b shows how this conditional independence is exploited by the BDD; thus, Markov...
chains have BDD representations which are linear in size to the length of chain.

Conditional independence is exploited by specialized inference algorithms for Bayesian networks like the join-tree algorithm [23]. However, conditional independence is not exploited by path-based or enumerative - probabilistic program inference procedures, such as the method utilized by Psi [19].

**Context-specific independence** The final form of independence we will discuss is context-specific independence. Context-specific independence is a weakening of conditional independence that occurs when two sets of random variables are independent only when a third set of variables all take on a particular value [5]:

**Definition 5.3** (Context-specific independence). Consider a joint probability distribution \( \Pr(X, Y, Z) \) over sets of random variables \( X, Y, \) and \( Z, \) and let \( c \) be an assignment to variables in \( Z. \) Then, we say \( X \) is contextually independent of \( Y \) given \( Z = c, \) written \( X \perp Y \mid Z = c, \) if \( \Pr(X, Y \mid Z = c) = \Pr(X \mid Z = c) \times \Pr(Y \mid Z = c). \)

An example program that exhibits context-specific independence is limited to bounded loops in existing PPLs. Here we show how to handle such loops in the context of our symbolic compilation approach to inference.

**Syntax** We update the DIPPL syntax with loops as follows:

```plaintext
: while (e) { s }
```

**Denotational semantics** To define the semantics of bounded loops, we first define a syntactic rewrite to unroll the loop a given number \( n \) of times:

\[
\text{unroll}(\text{while}(e)(s), n) \equiv \begin{cases} \text{skip} & \text{if } n = 0 \\ \text{if}(e); \text{unroll}(\text{while}(e)(s), n - 1); \text{else}(\text{skip}) & \text{if } n > 0 \end{cases}
\]

Since we assume that the loop is bounded, there exists an integer \( n \) such that no execution of the loop requires more than \( n \) iterations. Given a program over variables \( V, \) there are at most \( 2^{|V|} \) reachable states, so it suffices to unroll any bounded loop that many times. Therefore we can define the semantics of a while loop as follows:

\[
\begin{align*}
\text{while}(e)(s) & \Rightarrow (\phi, w) \\
\text{while}(e)(s) & \Rightarrow (\phi, w) \\
\end{align*}
\]

This denotational semantics of loops as a finite unrolling is similar to the semantics of bounded loops in several existing PPLs [10, 19, 21, 29, 39].

**Symbolic Compilation** Now we show how to associate a weighted Boolean formula with a bounded loop. Again we exploit the fact that the loop can be unrolled a finite number of times. However, we do not want to have to unroll the loop \( 2^{|V|} \) times! Fortunately, it is easy to test during symbolic compilation whether a given number \( n \) of unrollings is sufficient - the current symbolic state implies that the loop guard must be false. Formally:

\[
\begin{align*}
\text{unroll}(\text{while}(e)(s), n) & \Rightarrow (\phi, w) \\
\text{while}(e)(s) & \Rightarrow (\phi, w) \\
\end{align*}
\]

Loops can be compiled more efficiently than naively implementing the above rule by utilizing the symbolic state. In particular, rather than "guessing" \( n \) as in the rule above, we can compute the smallest such \( n \) by iteratively unrolling the loop and checking whether the current unrolling level is sufficient.

**Correctness** The denotational semantics of the loop and its semantics of its compiled WBF are equivalent:

**Theorem 6.1** (Correctness of Loop Compilation). Assume that \( \text{while}(e)(s) \Rightarrow (\phi, w), \) and that the loop is bounded. Then, \( \text{while}(e)(s) \Rightarrow (\phi, w) \) if \( \text{while}(e)(s) \Rightarrow (\phi, w) \) and \( \text{while}(e)(s) \Rightarrow (\phi, w) \).

Proof. Let \( \text{unroll}(\text{while}(e)(s), n) \Rightarrow (\phi, w), \) and let \( n \) be an integer such that the loop is guaranteed to terminate after \( n \) iterations. Then we must have that \( \phi \Rightarrow \neg c[x \leftarrow x'] \), so \( (\phi, w) \) is a valid symbolic compilation of the loop. But also by the definition of \( n \) we
We elaborate on the motivating example from Section 2, and clearly demonstrate how our symbolic compilation can exploit context-specific independence to perform inference on a synthetic grids dataset. All experiments were conducted on a 2.3GHz Intel i5 processor with 16GB of RAM.

7.1 Experiments

In this section we experimentally validate the effectiveness of our symbolic compilation procedure for performing inference on DIPPL programs. We directly implemented the compilation procedure described in Section 4 in Scala. We used the JavaBDD library in order to create and manipulate binary decision diagrams.

7.1.1 Baselines

First, we discuss a collection of simple baseline inference tasks that match the accepting and transition semantics of unrolling, which is simply a series of nested if-statements. Hence by Theorem 4.5, the accepting and transition semantics of $(\phi, w)$ match the accepting and transition semantics of unrolling, and is simply a series of nested if-statements.

7.1.2 Markov Chain

Section 2 discussed Markov chains and demonstrated that a compact BDD can be compiled which exploits the conditional independence of the network. In particular, a Markov chain of length $n$ can be compiled to a linear-sized BDD in $n$.

Figure 6 shows how two exact probabilistic programming inference tools compare against our symbolic inference technique for inference on Markov chains. WebPPL [44] and Psi [19] rely on enumerative concrete exact inference, which is exponential in the length of the chain. To compare against Storm, we compiled these models directly into discrete-time Markov chains. As the length of the Markov chain grows, the size of the encoded discrete-time Markov chain grows exponentially. Symbolic inference exploits the conditional independence of each variable in the chain, and is thus linear time in the length of the chain. In general, our symbolic inference technique can achieve state-of-the-art performance for any discrete-valued Bayesian network encoded as a probabilistic program.

7.1.3 Bayesian Network Encodings

In this section we demonstrate the power of our symbolic representation by performing exact inference on Bayesian networks encoded as probabilistic programs. We compared the performance of our symbolic compilation procedure against an exact inference technique that can achieve performance which is competitive with specialized Bayesian networks inference techniques. Finally, we demonstrate how our symbolic compilation can exploit context-specific independence to scale to large Markov models. Next, we show how our technique can achieve performance which is competitive with specialized Bayesian networks inference techniques. Finally, we demonstrate how our symbolic compilation can exploit context-specific independence to perform inference on a synthetic grids dataset. All experiments were conducted on a 2.3GHz Intel i5 processor with 16GB of RAM.

7.1.3 Bayesian Network Encodings

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\[ \text{unroll}(\text{while}(e)(s), n) = \text{unroll}(\text{while}(e)(s), 2^{\lceil \log_2 n \rceil}) \]

7 Implementation & Experiments

In this section we experimentally validate the effectiveness of our symbolic compilation procedure for performing inference on DIPPL programs. We directly implemented the compilation procedure described in Section 4 in Scala. We used the JavaBDD library in order to create and manipulate binary decision diagrams.

\[ \phi \\
\text{Symbolic (This Work)} \\
\text{Psi} \\
\text{Storm} \\
\text{WebPPL} \]

\[ \text{Time (s)} \]

\[ \text{Length of Markov Chain} \]

\[ \text{Time (ms)} \]

\[ \text{Alarm} \\
\text{Two Coins} \\
\text{Noisy Or} \\
\text{Grass} \]

Figure 6. Experimental comparison between techniques for performing exact inference on a Markov chain.

Figure 7. Performance comparison for discrete inference between Psi [19], R2 [29], and symbolic inference. The y-axis gives the time in milliseconds to perform inference.

\[ \text{Symbolic} \\
\text{Psi} \\
\text{R2} \]

\[ \text{Time (ms)} \]

\[ \text{Alarm} \\
\text{Two Coins} \\
\text{Noisy Or} \\
\text{Grass} \]

\[ \text{Symbolic (This Work)} \\
\text{Psi} \\
\text{Storm} \\
\text{WebPPL} \]

\[ \text{Time (s)} \]

\[ \text{Length of Markov Chain} \]

\[ \text{Time (ms)} \]

\[ \text{Alarm} \\
\text{Two Coins} \\
\text{Noisy Or} \\
\text{Grass} \]

\[ \text{Symbolic (This Work)} \\
\text{Psi} \\
\text{Storm} \\
\text{WebPPL} \]

\[ \text{Time (s)} \]

\[ \text{Length of Markov Chain} \]

\[ \text{Time (ms)} \]

\[ \text{Alarm} \\
\text{Two Coins} \\
\text{Noisy Or} \\
\text{Grass} \]

\[ \text{Symbolic (This Work)} \\
\text{Psi} \\
\text{Storm} \\
\text{WebPPL} \]

\[ \text{Time (s)} \]

\[ \text{Length of Markov Chain} \]

\[ \text{Time (ms)} \]

\[ \text{Alarm} \\
\text{Two Coins} \\
\text{Noisy Or} \\
\text{Grass} \]

\[ \text{Symbolic (This Work)} \\
\text{Psi} \\
\text{Storm} \\
\text{WebPPL} \]

\[ \text{Time (s)} \]

\[ \text{Length of Markov Chain} \]

\[ \text{Time (ms)} \]

\[ \text{Alarm} \\
\text{Two Coins} \\
\text{Noisy Or} \\
\text{Grass} \]

\[ \text{Symbolic (This Work)} \\
\text{Psi} \\
\text{Storm} \\
\text{WebPPL} \]

\[ \text{Time (s)} \]

\[ \text{Length of Markov Chain} \]

\[ \text{Time (ms)} \]

\[ \text{Alarm} \\
\text{Two Coins} \\
\text{Noisy Or} \\
\text{Grass} \]

\[ \text{Symbolic (This Work)} \\
\text{Psi} \\
\text{Storm} \\
\text{WebPPL} \]

\[ \text{Time (s)} \]

\[ \text{Length of Markov Chain} \]

\[ \text{Time (ms)} \]

\[ \text{Alarm} \\
\text{Two Coins} \\
\text{Noisy Or} \\
\text{Grass} \]

\[ \text{Symbolic (This Work)} \\
\text{Psi} \\
\text{Storm} \\
\text{WebPPL} \]

\[ \text{Time (s)} \]

\[ \text{Length of Markov Chain} \]

\[ \text{Time (ms)} \]

\[ \text{Alarm} \\
\text{Two Coins} \\
\text{Noisy Or} \\
\text{Grass} \]

\[ \text{Symbolic (This Work)} \\
\text{Psi} \\
\text{Storm} \\
\text{WebPPL} \]

\[ \text{Time (s)} \]

\[ \text{Length of Markov Chain} \]

\[ \text{Time (ms)} \]

\[ \text{Alarm} \\
\text{Two Coins} \\
\text{Noisy Or} \\
\text{Grass} \]

\[ \text{Symbolic (This Work)} \\
\text{Psi} \\
\text{Storm} \\
\text{WebPPL} \]

\[ \text{Time (s)} \]

\[ \text{Length of Markov Chain} \]

\[ \text{Time (ms)} \]

\[ \text{Alarm} \\
\text{Two Coins} \\
\text{Noisy Or} \\
\text{Grass} \]

\[ \text{Symbolic (This Work)} \\
\text{Psi} \\
\text{Storm} \\
We encode these multi-valued nodes as Boolean program variables. Each of these Bayesian networks is from Chavira and Darwiche [7], labeled as “BN Time”. In addition, we report the final size of our compiled BDD.

| Model     | Us (s) | BN Time (s) [7] | Size of BDD |
|-----------|--------|----------------|-------------|
| Alarm [7] | 1.872  | 0.21           | 52k         |
| Half finder | 12.652 | 1.37           | 157k        |
| Hepar2    | 7.834  | Not reported   | 139k        |
| pathfinder| 62.034 | 14.94          | 392k        |

Table 1. Experimental results for Bayesian networks encoded as probabilistic programs. We report the time it took to perform exact inference in seconds for our method compared against the Bayesian network inference algorithm from Chavira and Darwiche [7], labeled as "BN Time". In addition, we report the final size of our compiled BDD.

procedure for Bayesian networks [7]. Each of these Bayesian networks is from Chavira and Darwiche [7]. Table 1 shows the experimental results: our symbolic approach is competitive with specialized Bayesian network inference.

These Bayesian networks are not necessarily Boolean valued: they may contain multi-valued nodes. For instance, the Alarm network has three values that the StrokeVolume variable may take. We encode these multi-valued nodes as Boolean program variables using a one-hot encoding in a style similar to Sang et al. [38]. The generated dPPL files are quite large: the pathfinder program has over ten thousand lines of code. Furthermore, neither ProbLog [14, 16] nor Psi could perform inference within 300 seconds on the alarm example, the smallest of the above examples, thus demonstrating the power of our encoding over probabilistic logic programs and enumerative inference on this example.

7.1.4 Grids

This experiment showcases how our method exploits context-sensitive independence to perform inference more efficiently in the presence of determinism. Grids were originally introduced by Sang et al. [38] to demonstrate the effectiveness exploiting determinism during Bayesian network inference. A 3-grid is Boolean-valued Bayesian network arranged in a three by three grid:

For these experiments we encoded grid Bayesian networks into probabilistic programs. Grids are typically hard inference challenges even for specialized Bayesian network inference algorithms. However, in the presence of determinism, the grid inference task can become vastly easier. A grid is n%-deterministic if n% of the f1ips in the program are replaced with assignments to constants. Figure 8 shows how our symbolic compilation exploits the context-sensitive independence induced by the determinism of the program in order to perform inference more efficiently.

8 Related Work

First we discuss two closely related individual works on exact inference for PPLs; then we discuss larger categories of related work.

Claret et al. [10] compiles imperative probabilistic programs to algebraic decision diagrams (ADDs) via a form of data-flow analysis [10]. This approach is fundamentally different from our approach, as the ADD cannot represent the distribution in a factorized way. An ADD must contain the probability of each model of the Boolean formula as a leaf node. Thus, it cannot exploit the independence structure required to compactly represent joint probability distributions with independence structure efficiently. In addition, Claret et al. [10] uses a different semantic domain — a map between distributions — which is incompatible with our composable compilation approach.

Also closely related is the work of Pfeffer et al. [35], which seeks to decompose the probabilistic program inference task at specific program points where the distribution is known to factorize due to conditional independence. This line of work only considers conditional independence — not context-specific independence — and requires hand-annotated program constructs in order to expose and exploit the independences.

Path-based Program Inference Many techniques for performing inference in current probabilistic programming languages are enumerative or path-based: they perform inference by integrating or approximating the probability mass along each path of the probabilistic program [2, 19, 39, 44]. The complexity of inference for path-based inference algorithms scales with the number of paths through the program. The main weakness with these inference strategies is that they do cannot exploit common structure across paths — such as independence — and thus scale poorly on examples with many paths.

Probabilistic Logic Programs Most prior work on exact inference for probabilistic programs was developed for probabilistic logic programs [14, 16, 36, 42]. Similar to our work, these techniques compile a probabilistic logic program into a weighted Boolean formula, and utilize state-of-the-art WMC solvers to compile the WBF into a representation that supports efficient WMC evaluation, such as a binary decision diagram (BDD) [6], sentential decision diagram (SDD) [12], or d-DNNF circuit [13]. Currently, WMC-based inference remains the state-of-the-art inference strategy for probabilistic logic programs. These techniques are not directly applicable to imperative probabilistic programs such as dPPL due to the
presence of sequencing, arbitrary observation, and other imperative programming constructs.

**Probabilistic Model Checkers** Probabilistic model checkers such as Storm [15] and Prism [25] can be used to perform Bayesian inference on probabilistic systems. These methods work by compiling programs to a representation such as a discrete-time Markov chain or Markov decision process, for which there exist well-known inference strategies. These representations allow probabilistic model checkers to reason about loops and non-termination. In comparison with this work, probabilistic model checkers suffer from a state-space explosion similar to path-based inference methods due to the fact that they devote a node to each possible configuration of variables in the program.

**Compilation-based PPLs** There exists a large number of PPLs that perform inference by converting the program into a probabilistic graphical model [26, 27, 33, 34], assuming a fixed set of random variables. There are two primary shortcomings of these techniques in relation to ours. The first is that these techniques cannot exploit the context-specific independence present in the program structure, since the topology of the graph – either a Bayesian network or factor graph – does not make this information explicit. Second, these techniques restrict the space of programs to those which can be compiled. Thus, they require constraints on the space of programs, such as requiring a statically-determined number of variables, or requiring that loops can be statically unrolled. In comparison, our technique works directly on the core program, and can thus inspect and utilize the program structure in order to perform inference more efficiently.

9 Conclusion & Future Work

In conclusion, we developed a semantics and symbolic compilation procedure for exact inference in a discrete imperative probabilistic programming language called dPPL. In doing so, we have drawn connections between the probabilistic logic programming, symbolic model checking, and artificial intelligence communities. We theoretically proved our symbolic compilation procedure correct, and experimentally validated it against existing probabilistic systems. Finally, we showed that our method is competitive with state-of-the-art Bayesian network inference tasks, showing that our compilation procedures scale to large complex probabilistic models.

We anticipate much future work in this direction. First, we plan to extend our symbolic compilation procedure to handle richer classes of programs. For instance, we would like to support almost-surely terminating loops and procedures, as well as enrich the class of datatypes supported by the language. Second, we would like to quantify precisely the complexity of inference for discrete probabilistic programs. The graphical models community has metrics such as tree-width which provides precise notions of complexity of inference; we believe such notions may exist for probabilistic programs as well [11, 23]. Finally, we anticipate that techniques from the symbolic model checking community – such as Bebop [3] – may be applicable here, and applying these techniques is also promising future work.

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References

[1] Sheldon B. Alers. 1978. Binary Decision Diagrams. IEEE Trans. Comput. C-27 (1978), 509–516.

[2] Aws Albarghouthi, Loris D’Antoni, Samuel Drews, and Aditya V. Nori. 2017. FairSquare: Probabilistic Verification of Program Fairness. Proc. ACM Program. Lang. 1, OOPSLA, Article 80 (Oct. 2017), 30 pages. https://doi.org/10.1145/3133904

[3] Thomas Ball and Siriram K. Rajamani. 2000. Bebop: A Symbolic Model Checker for Boolean Programs. In SPIN Model Checking and Software Verification. 113–130.

[4] Johannes Borgström, Andrew D Gordon, Michael Greenberg, James Margison, and Jurgen Van Gool. 2013. Measure Transformer Semantics for Bayesian Machine Learning. Proc. of ESOF 6602 (2013), 77–96. https://doi.org/10.2168/LMCS-9(3:11)2013 arXiv:1308.0689

[5] Craig Boutilier, Nir Friedman, Noises Goldszmidt, and Daphne Koller. 1996. Context-specific independence in Bayesian networks. In Proceedings of the Twelfth international conference on Uncertainty in artificial intelligence. Morgan Kaufmann Publishers Inc., 115–123.

[6] R. Bryant. 1986. Graph-based algorithms for Boolean function manipulation. IEEE TC C-35 (1986), 677–691.

[7] Mark Chavira and Adnan Darwiche. 2008. On Probabilistic Inference by Weighted Model Counting. J. Artificial Intelligence 172, 6–7 (April 2008), 782–799. https://doi.org/10.1016/j.artint.2007.11.002

[8] Mark Chavira, Adnan Darwiche, and Manfred Jaeger. 2006. Compiling relational Bayesian networks for exact inference. International Journal of Approximate Reasoning 42, 1 (2006), 4–20.

[9] Arthur Choi and Adnan Darwiche. 2011. Relax, Compensate and then Recover. In New Frontiers in Artificial Intelligence, Takashi Onada, Daiuuki Bekki, and Eric McCreedy (Eds.). Lecture Notes in Computer Science, Vol. 6797. Springer Berlin / Heidelberg, 167–180.

[10] Guillaume ClarEt, Siriram K. Rajamani, Aditya V Nori, Andrew D. Gordon, and Johannes Borgström. 2013. Bayesian inference using data flow analysis. Proceedings of the 2013 9th Joint Meeting on Foundations of Software Engineering - ESEC/FSE 2013 (2013), 92. https://doi.org/10.1145/2491424.2491423

[11] A. Darwiche 2009. Modeling and Reasoning with Bayesian Networks. Cambridge University Press.

[12] Adnan Darwiche. 2011. SDD: A new canonical representation of propositional knowledge bases. In IJCAI Proceeding International Joint Conference on Artificial Intelligence. 819.

[13] A. Darwiche and P. Markou. 2002. A Knowledge Compilation Map. Journal of Artificial Intelligence Research 17 (2002), 229–264.

[14] Luc De Raedt, Angelika Kimmig, and Hannu Tovonen. 2007. ProbLog: A Probabilistic Prolog and Its Application in Link Discovery. In Proceedings of IJCAI Vol. 7. 2462–2467.

[15] Christian Dehnert, Sebastian Junges, Joost-Pieter Katoen, and Matthias Volk. 2017. A storm is coming: A modern probabilistic model checker. In International Conference on Computer Aided Verification. Springer, 592–600.

[16] Daan Fierens, Guy Van den Broeck, Joris Renkens, Dimitar Shetleriov, Bernd Gutmann, Ingo Thon, Gerda Janssens, and Luc De Raedt. 2013. Inference and learning in probabilistic logic programs using weighted Boolean formulas. J. Theory and Practice of Logic Programming 15(3) (2013), 358 – 401.

[17] Daan Fierens, Guy Van den Broeck, Ingo Thon, Bernd Gutmann, and Luc De Raedt. 2011. Inference in probabilistic logic programs using weighted CNFs. In Proceedings of UAI. 211–220.

[18] Tal Friedman and Guy Van den Broeck. 2018. Approximate Knowledge Compil- iation by Online Collapsed Importance Sampling. In Advances in Neural Information Processing Systems 31 (NIPS).

[19] Timon Gehr, Sasa Misailovic, and Martin Vechev. 2016. PSI: Exact symbolic inference for probabilistic programs. Proc. of ESOP ETAPS 9779 (2016), 62–83.

[20] V Gogate and R. Dechter. 2011. SampleSearch: Importance sampling in presence of determinism. Artificial Intelligence 175, 2 (2011), 694–729.

[21] Chung-kil Hur, Aditya V. Nori, Sriram K. Rajamani, and Selva Samuel. 2015. A Provably Correct Sampler for Probabilistic Programming. FSTTCS, 9773 (2015), 1–14. https://doi.org/10.4230/LIPIcs.FSTTCS.2015.475

[22] Chung-Ki Hur, Aditya V. Nori, Siriram K. Rajamani, and Selva Samuel. 2014. Slicing probabilistic programs. Proc. of PLDI. 2014, 133–144. https://doi.org/10.1145/2594291.2594303

[23] D. Koller and N. Friedman. 2009. Probabilistic graphical models: principles and techniques. MIT press.
[24] Dexter Kozen. 1979. Semantics of Probabilistic Programs. In Proceedings of the 20th Annual Symposium on Foundations of Computer Science (SFCS ’79). IEEE Computer Society, Washington, DC, USA, 181–114. https://doi.org/10.1109/SFCS.1979.38

[25] Marta Kwiatkowska, Gethin Norman, and David Parker. 2011. PRISM 4.0: Verification of Probabilistic Real-time Systems. In Proceedings of the 23rd International Conference on Computer Aided Verification (CAV’11). Springer-Verlag, Berlin, Heidelberg, 585–591.

[26] a McCallum, K Schultz, and S Singh. 2009. Factorie: Probabilistic programming via imperatively defined factor graphs. Proc. of NIPS 22 (2009), 1249–1257.

[27] T. Minka, J.M. Winn, J.P. Guiver, S. Webster, Y. Zaykov, B. Yang, A. Sprenger, and J. Bronskill. 2014. Infer.NET 2.6. (2014). Microsoft Research Cambridge. http://research.microsoft.com/infernet.

[28] C. Musc, S. McBeth, J.C. Beck, and E. Hsu. 2010. Fast d-DNNF Compilation with sharpSAT. In Workshops at the Twenty-Fourth AAAI Conference on Artificial Intelligence.

[29] Aditya V Nori, Sriram K Rajamani, and Selva Samuel. 2014. R2 : An Efficient MCMC Sampler for Probabilistic Programs. Aaai (2014), 2476–2482.

[30] Umut Oztop and Adnan Darwiche. 2014. On Compiling CNF into Decision-DNNF. In Proceedings of the 20th International Conference on Principles and Practice of Constraint Programming (CP). 42–57.

[31] U. Oztok and A. Darwiche. 2015. A Top-Down Compiler for Sentential Decision Diagrams. In Proceedings of the 24th International Joint Conference on Artificial Intelligence (IJCAI).

[32] Judea Pearl. 1988. Probabilistic Reasoning in Intelligent Systems: Networks of Plausible Inference. Morgan Kaufmann.

[33] A. Pfeffer. 2001. IBAL: A probabilistic rational programming language. In International Joint Conference on Artificial Intelligence, Vol. 17. 733–740.

[34] Avi Pfeffer. 2009. Figaro: An object-oriented probabilistic programming language. Charles River Analytics Technical Report 137 (2009).

[35] Avi Pfeffer, Brian Ruttenberg, William Kretschmer, and Alison OConnor. 2018. Structured Factored Inference for Probabilistic Programming. In International Conference on Artificial Intelligence and Statistics. 1224–1232.

[36] Fabrizio Riguzzi and Terrance Swift. 2011. The PITA System: Tabling and Answer Subsumption for Reasoning under Uncertainty. Theory and Practice of Logic Programming 11, 4–5 (2011), 433–449.

[37] Dan Roth. 1996. On the hardness of approximate reasoning. Artificial Intelligence 82, 1 (1996), 273–302.

[38] Tian Sang, Paul Beame, and Henry A Kautz. 2005. Performing Bayesian inference by weighted model counting. In AAAI, Vol. 5. 475–481.

[39] Sriram Sankaranarayanan, Aleksandar Chakarov, and Sunut Gulwani. 2013. Static Analysis for Probabilistic Programs: Inferring Whole Program Properties from Finitely Many Paths. SICPLAN Not. 48, 6 (June 2013), 447–458. https://doi.org/10.1145/2409370.2462179

[40] David Sontag, Amir Globerson, and Tommi Jaakkola. 2011. Introduction to dual composition for inference. In Optimization for Machine Learning. MIT Press.

[41] Guy Van den Broeck and Dan Suciu. 2017. Query Processing on Probabilistic Data: A Survey. Now Publishers. https://doi.org/10.1561/970000052

[42] Jonas Vlasselaer, Guy Van den Broeck, Angelika Kimmig, Wannes Meert, and Luc De Raedt. 2015. Anytime inference in probabilistic logic programs with Tp-compilation. In Proceedings of 24th International Joint Conference on Artificial Intelligence (IJCAI).

[43] Martin J Wainwright, Michael I Jordan, et al. 2008. Graphical models, exponential families, and variational inference. Foundations and Trends® in Machine Learning 1, 1–2 (2008), 1–305.

[44] David Wingate and Theophane Weber. 2013. Automated variational inference in probabilistic programming. arXiv preprint arXiv:1301.1299 (2013).
A Appendix

A.1 Semantic Properties of DIPPL

Here we elaborate on the semantic properties of DIPPL. Intuitively, the accepting semantics gives the probability that no observations are violated. Then if we already know that an observation has been violated, the result should be zero. This is captured in the following lemma.

Lemma A.1. If an observation has failed in state \( s \) given initial state \( \sigma \), then \([s]_A(\sigma) = 0\).

Proof. Structural induction. (For full proof, see appendix).

Similarly, the transition semantics gives the conditional probability of transitioning to a state given that no observations are violated. If an observe statement has in fact been violated, the result is the special distribution \( \bot \). This is captured in the following lemma.

Lemma A.2. For any state \( \sigma \) and any statement \( s \), if at least one observe inside statement \( s \) fails given initial state \( \sigma \), we have

\[
\sum_{\tau \in \Sigma} [s]_T(\tau | \sigma) = 0.
\]

Proof. Structural induction. (For full proof, see appendix).

Furthermore, since the transition semantics maps program states to distributions, it then makes sense to require all probabilities to sum to one.

Lemma A.3. For any state \( \sigma \) and any statement \( s \), if none of the observes inside statement \( s \) fail, we have

\[
\sum_{\tau \in \Sigma} [s]_T(\tau | \sigma) = 1.
\]

Proof. Structural induction. (For full proof, see appendix).

One of the most intuitive properties in an imperative programming language is that sequencing and the \texttt{skip} statement should form a monoid in terms of their semantics. This is equivalent to sequencing being associative and \texttt{skip} is both the left and right unit of sequencing. This property can serve as one of the sanity checks for the proposed semantics.

Theorem A.4. Sequencing is associative, that is, for any statements \( s_1, s_2, s_3 \), we have \([((s_1; s_2); s_3)]_T = [s_1; (s_2; s_3)]_T\) and \([s_1; s_2; s_3]_A = [s_1; (s_2; s_3)]_A\).

Proof. Substitute the definition followed by algebraic manipulation. (For full proof, see appendix).

Henceforth, we may now omit parentheses when we have multiple sequencing operations in a row.

The next part of the theorem is that \texttt{skip} is both the left and right unit of sequencing; this precisely characterizes the \texttt{skip} statement as a no-op.

Theorem A.5. The statement \texttt{skip} is both a left unit and right unit of sequencing, that is, for any statement \( s \), we have \([\text{skip}; s]_T = [s]_T\) and \([s; \text{skip}]_T = [s]_T\). Similarly for the accepting semantics, \([\text{skip}; s]_A = [s]_A\) and \([s; \text{skip}]_A = [s]_A\).

To prove this, we need Lemma A.2 and Lemma A.3.

Proof. Substitute the definition followed by algebraic manipulation. (For full proof, see appendix).

Another desirable property of any semantics is full completeness. Full completeness states that for any arbitrary object in the semantic domain, there exists a program whose semantics is that object.

Theorem A.6 (Full completeness). Let \( f \in \Sigma \rightarrow \text{Dist} \Sigma \) and let \( g \in \Sigma \rightarrow [0, 1] \). Then, there exists a program \( s \) such that \([s]_A = g\) and \([s]_T = f\).

Proof. We will prove by constructing a program that will have the desired property. We will define some auxiliary functions to help this construction.

First, we define a function \( \text{SetState} : \Sigma \rightarrow s \); it maps a program state to a statement that performs a sequence of deterministic assignments. It is defined inductively:

\[
\text{SetState}(0) = \text{skip} \\
\text{SetState}(\langle u = b \rangle \cup \sigma') = u := b; \text{SetState}(\sigma')
\]

where \( u \) is a metavariable for a variable in the program, and \( b \) is a boolean literal (either \( \text{T} \) or \( \text{F} \)). This \text{SetState} function simply sets the program state to the one provided.

Next, we define a function \( \text{SetDist} : \text{Dist} \Sigma 	imes [0, 1] \rightarrow s \) whose value is the statement with the desired transition probability. It is defined inductively as follows:

\[
\text{SetDist}(0, r) = \text{skip}
\]
where \( \sigma \) is any variable from the set of variables. Intuitively, the \( \text{SetDist} \) function attempts to perform a series of non-deterministic assignments such that the resulting distribution of the program state matches that of the given distribution.

In a similar vain, it is easy to define a function that will result in the correct acceptance \( \text{MakeAccept} : [0, 1] \rightarrow s \):

\[
\text{MakeAccept}(\mu) = \mu \sim \text{flip}(\mu); \text{observe}(\mu)
\]

where \( \nu \) is a metavariable denoting any variable in the program.

We can define a function \( \text{TestState} : \Sigma \rightarrow e \) that takes a particular program state and returns an expression testing for this program state:

\[
\text{TestState}(\emptyset) = \top
\]
\[
\text{TestState}(\{ \nu = \top \} \cup \sigma') = \nu \wedge \text{TestState}(\sigma')
\]
\[
\text{TestState}(\{ \nu = \bot \} \cup \sigma') = \neg \nu \wedge \text{TestState}(\sigma')
\]

where \( \nu \) is a metavariable for a variable in the program.

Finally, we can define the function to construct a statement given \( f \) and \( g \):

\[
\text{Construct}(f, g) = ; \sigma \in \Sigma
\]

where the \( ; \sigma \in \Sigma \) notation means a series of statements connected by the sequencing operator.

\[\square\]

### A.1.1 Detailed Proof of Lemma A.1

**Proof.** Use structural induction. Case analysis on the form of \( s \):

- If \( s \) is \( \text{skip} \), contradiction.
- If \( s \) is \( x \sim \text{flip}(\theta) \), contradiction.
- If \( s \) is \( x = e \), contradiction.
- If \( s \) is \( \text{observe}(e) \), then by hypothesis we know that this observation did fail, that is \( \|e\|_A = F \), and therefore
  
  \[\|\text{observe}(e)\|_A(\sigma) = 0.\]

- If \( s \) is \( s_1; s_2 \), then either an observation failed in \( s_1 \) or \( s_2 \).
  
  - If an observation failed in \( s_1 \), then by induction hypothesis we know that
    
    \[\|s_1\|_A(\sigma) = 0.\]
    
    therefore
    
    \[
    \|s_1; s_2\|_A(\sigma) = \|s_1\|_A(\sigma) \times \sum_{\tau \in \Sigma} (\|s_1\|_T(\tau | \sigma) \times \|s_2\|_A(\tau))
    \]
    
    \[
    = 0 \times \sum_{\tau \in \Sigma} (\|s_1\|_T(\tau | \sigma) \times \|s_2\|_A(\tau))
    \]
    
    \[
    = 0.
    \]
  
  - The meaning of a failed observation in \( s_2 \) is that for all intermediate state \( \tau \), if there is nonzero probability of reaching state \( \tau \) after \( s_1 \), then there exists some \( \text{observe}(e) \) statement inside \( s_2 \) for which the expression \( e \) evaluates to \( F \). In other words,
    
    \[\|s_1\|_T(\tau | \sigma) \neq 0 \quad \text{implies} \quad \|s_2\|_A(\tau) = 0.\]
Proof. A.1.2 Detailed Proof of Lemma A.2

If an observation failed in \( \tau \), then by hypothesis we know that this observation did fail, that is \( \llbracket e \rrbracket = F \), and therefore

\[
\sum_{\tau \in \Sigma} \llbracket \text{observe}(e) \rrbracket \llbracket \tau \mid \sigma \rrbracket = 0.
\]

If \( s \) is \( s_1; s_2 \), then either an observation failed in \( s_1 \) or \( s_2 \).
- If an observation failed in \( s_1 \), then by induction hypothesis we know that
  \[
  \sum_{\tau \in \Sigma} \llbracket s_1 \rrbracket \llbracket \tau \mid \sigma \rrbracket = 0,
  \]
  that is
  \[
  \forall \tau \in \Sigma, \llbracket s_1 \rrbracket \llbracket \tau \mid \sigma \rrbracket = 0,
  \]
  therefore
  \[
  \sum_{\tau \in \Sigma} \llbracket s_1; s_2 \rrbracket \llbracket \tau \mid \sigma \rrbracket = \sum_{\tau \in \Sigma} \frac{\sum_{\tau' \in \Sigma} \llbracket s_1 \rrbracket \llbracket \tau \mid \sigma \rrbracket \llbracket s_2 \rrbracket \llbracket \tau' \mid \sigma \rrbracket \times \llbracket s_2 \rrbracket \llbracket A \rrbracket (\tau)}{\sum_{\tau' \in \Sigma} \llbracket s_1 \rrbracket \llbracket \tau \mid \sigma \rrbracket \llbracket s_2 \rrbracket \llbracket A \rrbracket (\tau)}
  = \sum_{\tau \in \Sigma} \frac{\sum_{\tau' \in \Sigma} 0 \times \llbracket s_2 \rrbracket \llbracket \tau' \mid \sigma \rrbracket \times \llbracket s_2 \rrbracket \llbracket A \rrbracket (\tau)}{\sum_{\tau' \in \Sigma} 0 \times \llbracket s_2 \rrbracket \llbracket A \rrbracket (\tau)}
  = 0
  = 0.
  
  If an observation failed in \( s_2 \), the precise meaning is that for all intermediate state \( \tau \), if there is nonzero probability of reaching state \( \tau \) after \( s_1 \), then there exists some \( \text{observe}(e) \) statement inside \( s_2 \) for which the expression \( e \) evaluates to \( F \). In other words, for all intermediate state \( \tau \), if there is nonzero probability of reaching state \( \tau \) after \( s_1 \), then \( \llbracket s_2 \rrbracket \llbracket A \rrbracket (\tau) = 0 \).

So therefore we have, for all \( \tau \),

\[
\llbracket s_1 \rrbracket \llbracket \tau \mid \sigma \rrbracket \times \llbracket s_2 \rrbracket \llbracket A \rrbracket (\tau) = 0.
\]

Therefore,

\[
\sum_{\tau \in \Sigma} \llbracket s_1; s_2 \rrbracket \llbracket \tau \mid \sigma \rrbracket = \sum_{\tau \in \Sigma} \frac{\sum_{\tau' \in \Sigma} \llbracket s_1 \rrbracket \llbracket \tau \mid \sigma \rrbracket \llbracket s_2 \rrbracket \llbracket \tau' \mid \sigma \rrbracket \times \llbracket s_2 \rrbracket \llbracket A \rrbracket (\tau)}{\sum_{\tau' \in \Sigma} \llbracket s_1 \rrbracket \llbracket \tau \mid \sigma \rrbracket \times \llbracket s_2 \rrbracket \llbracket A \rrbracket (\tau)}
  = \sum_{\tau \in \Sigma} \frac{\sum_{\tau' \in \Sigma} 0 \times \llbracket s_2 \rrbracket \llbracket \tau' \mid \sigma \rrbracket \times \llbracket s_2 \rrbracket \llbracket A \rrbracket (\tau)}{\sum_{\tau' \in \Sigma} 0}
  = 0
  = 0.
\]
• If \( s \) is if \( \{ s_1 \} \) else \( \{ s_2 \} \), we note that there is an observation that must have failed in the branch that was taken. If \( s_1 \) was taken, an observation in \( s_1 \) has failed, and
\[
\sum_{\tau \in \Sigma} \| s \|_T(\tau \mid \sigma) = \sum_{\tau \in \Sigma} \| s_1 \|_T(\tau \mid \sigma) = 0
\]
by induction hypothesis; and if \( s_2 \) was taken, an observation in \( s_2 \) has failed, and
\[
\sum_{\tau \in \Sigma} \| s \|_T(\tau \mid \sigma) = \sum_{\tau \in \Sigma} \| s_2 \|_T(\tau \mid \sigma) = 0
\]
by induction hypothesis.

\[\square\]

A.1.3 Detailed Proof of Lemma A.3

Proof. Use structural induction. Let \( \sigma \) be an arbitrary program state, and \( s \) be an arbitrary statement. For the first part of the lemma, assume that none of the observations inside \( s \) fail. Case analysis on the form of \( s \):

• If \( s \) is skip, then
\[
\sum_{\tau \in \Sigma} \| \text{skip} \|_T(\tau \mid \sigma) = \| \text{skip} \|_T(\sigma \mid \sigma) = 1.
\]

• If \( s \) is \( x \sim \text{flip}(\theta) \), then
\[
\sum_{\tau \in \Sigma} \| x \sim \text{flip}(\theta) \|_T(\tau \mid \sigma) = \| x \sim \text{flip}(\theta) \|_T(\sigma[x \mapsto T] \mid \sigma) + \| x \sim \text{flip}(\theta) \|_T(\sigma[x \mapsto F] \mid \sigma) = \theta + (1 - \theta) = 1.
\]

• If \( s \) is \( x := e \), then
\[
\sum_{\tau \in \Sigma} \| x := e \|_T(\tau \mid \sigma) = \| \text{skip} \|_T(\sigma[x \mapsto \| e \|_T(\sigma)]) \mid \sigma) = 1.
\]

• If \( s \) is observe(\( e \)), then by hypothesis we know that this observation did not fail, that is \( \| e \|_T = T \), and therefore
\[
\sum_{\tau \in \Sigma} \| \text{observe}(e) \|_T(\tau \mid \sigma) = \| \text{observe}(e) \|_T(\sigma \mid \sigma) = 1.
\]

• If \( s \) is \( s_1; s_2 \), we have
\[
\sum_{\tau \in \Sigma} \| s_1; s_2 \|_T(\tau \mid \sigma) = \sum_{\tau \in \Sigma} \sum_{\tau \in \Sigma} \| s_1 \|_T(\tau \mid \sigma) \times \| s_2 \|_T(\tau \mid \sigma) \times \| s_2 \|_T(\tau \mid \sigma)
\]
\[
\sum_{\tau \in \Sigma} \sum_{\tau \in \Sigma} \| s_1 \|_T(\tau \mid \sigma) \times \| s_2 \|_T(\tau \mid \sigma)
\]
\[
\sum_{\tau \in \Sigma} \| s_1 \|_T(\tau \mid \sigma) \times \sum_{\tau \in \Sigma} \| s_2 \|_T(\tau \mid \sigma)
\]
\[
\sum_{\tau \in \Sigma} \| s_1 \|_T(\tau \mid \sigma)
\]
\[
= 1
\]
by the hypothesis that observations did not fail
by induction hypothesis
exchange summation
factorization
by induction hypothesis
by induction hypothesis

• If \( s \) is if \( \{ s_1 \} \) else \( \{ s_2 \} \), we note that either \( \| e \|_T(\sigma) = T \) or \( \| e \|_T(\sigma) = F \). In case of former,
\[
\sum_{\tau \in \Sigma} \| s \|_T(\tau \mid \sigma) = \sum_{\tau \in \Sigma} \| s_1 \|_T(\tau \mid \sigma) = 1
\]
by induction hypothesis; and in case of latter,
\[
\sum_{\tau \in \Sigma} \| s \|_T(\tau \mid \sigma) = \sum_{\tau \in \Sigma} \| s_2 \|_T(\tau \mid \sigma) = 1
\]
by induction hypothesis.

\[\square\]
A.2 Proof of Theorem A.4

Proof. From the left, we have

\[
\| (s_1; s_2) \|_{\text{A}_\tau} (\sigma' | \sigma) = \sum_{\tau' \in \Sigma} \| s_1 \|_{\text{A}_\tau} (\tau | \sigma) \times \| s_2 \|_{\text{A}_\tau} (\tau' | \tau) \times \| s_3 \|_{\text{A}_\tau} (\tau' | \tau)
\]

which is what we have on the right.

For the second,

\[
\| (s_1; s_2) \|_{\text{A}_\tau} (\sigma | \sigma) = \sum_{\tau' \in \Sigma} \| s_1 \|_{\text{A}_\tau} (\tau' | \sigma) \times \| s_2 \|_{\text{A}_\tau} (\tau | \tau') \times \| s_3 \|_{\text{A}_\tau} (\tau' | \tau)
\]

which is what we have on the right.

For the second,

\[
\| (s_1; s_2; s_3) \|_{\text{A}_\tau} (\sigma) = \| s_1 \|_{\text{A}_\tau} (\sigma) \times \sum_{\tau \in \Sigma} \| s_2 \|_{\text{A}_\tau} (\tau | \sigma) \times \| s_3 \|_{\text{A}_\tau} (\tau)
\]

which is what we have on the right.

\[
\| (s_1; s_2) \|_{\text{A}_\tau} (\sigma) = \| s_1 \|_{\text{A}_\tau} (\sigma) \times \sum_{\tau' \in \Sigma} \| s_2 \|_{\text{A}_\tau} (\tau | \sigma)
\]

which is what we have on the right.

A.3 Detailed Proof of Theorem A.5

Proof. For skip on the left, in transition semantics we have

\[
\| \text{skip} : s \|_{\text{A}_\tau} (\sigma' | \sigma) = \sum_{\tau \in \Sigma} \| \text{skip} \|_{\text{A}_\tau} (\tau | \sigma) \times \| s \|_{\text{A}_\tau} (\sigma' | \tau) \times \| s \|_{\text{A}_\tau} (\sigma)
\]

which is what we have on the right.

For accepting semantics, we have

\[
\| \text{skip} : s \|_{\text{A}_\tau} (\sigma) = \| \text{skip} \|_{\text{A}_\tau} (\sigma) \times \sum_{\tau \in \Sigma} \| \text{skip} \|_{\text{A}_\tau} (\tau | \sigma) \times \| s \|_{\text{A}_\tau} (\tau)
\]

which is what we have on the right.
For skip on the right, in both transition and accepting semantics, we need to make use of Lemma A.3. Here, there is an additional complication of whether or not the statement on the left contains a failed observation.

If no observations fail, for transition semantics we have
\[
\| s; \text{skip} \|_T(\sigma' \mid \sigma) = \sum_{\tau \in \Sigma} [\| s; \text{skip} \|_T(\tau \mid \sigma) \times \| \text{skip} \|_A(\tau)]
\]
\[
= \sum_{\tau \in \Sigma} \| s; \text{skip} \|_T(\tau \mid \sigma) \times \| \text{skip} \|_A(\tau)
\]
\[
= \| s \|_T(\sigma' \mid \sigma).
\]

And for accepting semantics, we have
\[
\| s; \text{skip} \|_A(\sigma)
\]
\[
= \| s \|_A(\sigma) \times \sum_{\tau \in \Sigma} \| s; \text{skip} \|_T(\tau \mid \sigma) \times \| \text{skip} \|_A(\tau)
\]
\[
= \| s \|_A(\sigma) \times \sum_{\tau \in \Sigma} \| s; \text{skip} \|_T(\tau \mid \sigma)
\]
\[
= \| s \|_A(\sigma).
\]

If some observation in s fails, by Lemma A.3, we note that \( \| s \|_T(\sigma' \mid \sigma) = 0 \), and we have
\[
\| s; \text{skip} \|_T(\sigma' \mid \sigma)
\]
\[
= \sum_{\tau \in \Sigma} \| s; \text{skip} \|_T(\tau \mid \sigma) \times \| \text{skip} \|_A(\tau)
\]
\[
= \sum_{\tau \in \Sigma} \| s \|_T(\tau \mid \sigma) \times \| \text{skip} \|_A(\tau)
\]
\[
= 0.
\]

If some observation in s fails, by Lemma A.1, we have \( \| s \|_A(\sigma) = 0 \), and we have
\[
\| s; \text{skip} \|_A(\sigma)
\]
\[
= \| s \|_A(\sigma) \times \sum_{\tau \in \Sigma} \| s; \text{skip} \|_T(\tau \mid \sigma)
\]
\[
= 0 \times \sum_{\tau \in \Sigma} \| s \|_T(\tau \mid \sigma)
\]
\[
= 0.
\]

\( \square \)

A.4 Proof of Theorem 4.5

A.4.1 Properties of WMC

We begin with some important lemmas about weighted model counting:

Lemma A.7 (Independent Conjunction). Let \( \alpha \) and \( \beta \) be Boolean sentences which share no variables. Then, for any weight function \( w \),
\[
WMC(\alpha \land \beta, w) = WMC(\alpha, w) \times WMC(\beta, w).
\]

Proof: The proof relies on the fact that, if two sentences \( \alpha \) and \( \beta \) share no variables, then any model \( \omega \) of \( \alpha \land \beta \) can be split into two components, \( \omega_\alpha \) and \( \omega_\beta \), such that \( \omega = \omega_\alpha \land \omega_\beta \), \( \omega_\alpha \Rightarrow \alpha \), and \( \omega_\beta \Rightarrow \beta \), and \( \omega_\alpha \) and \( \omega_\beta \) share no variables. Then:
\[
WMC(\alpha \land \beta, w) = \sum_{\omega \in \text{Mod}(\alpha \land \beta)} \prod_{l \in \omega} w(l)
\]
\[
= \sum_{\omega_\alpha \in \text{Mod}(\alpha)} \sum_{\omega_\beta \in \text{Mod}(\beta)} \prod_{a \in \omega_\alpha} w(a) \times \prod_{b \in \omega_\beta} w(b)
\]
\[
= \left[ \sum_{\omega_\alpha \in \text{Mod}(\alpha)} \prod_{a \in \omega_\alpha} w(a) \right] \times \left[ \sum_{\omega_\beta \in \text{Mod}(\beta)} \prod_{b \in \omega_\beta} w(b) \right]
\]
\[
=WMC(\alpha, w) \times WMC(\beta, w).
\]
Lemma A.8. Let \( \alpha \) be a Boolean sentence and \( x \) be a conjunction of literals. For any weight function \( w \), \( \text{WMC}(\alpha, w) = \text{WMC}(\alpha \mid x, w) \times \text{WMC}(x, w) \).

Proof. Follows from Lemma A.7 and the fact that \( \alpha \mid x \) and \( x \) share no variables by definition:

\[
\text{WMC}(\alpha \mid x) \times \text{WMC}(x, w) = \text{WMC}((\alpha \mid x) \land x, w) = \text{WMC}(\alpha, w).
\]

Lemma A.9. Let \( \alpha \) be a sentence, \( x \) be a conjunction of literals, and \( w \) be some weight function. If for all \( l \in x \) we have that \( w(l) = 1 \), then \( \text{WMC}(\alpha \mid x, w) = \text{WMC}(\alpha \land x, w) \).

Proof. The proof relies on the fact that, if two sentences \( \alpha \) and \( \beta \) are mutually exclusive, then any model \( \omega \) of \( \alpha \lor \beta \) either entails \( \alpha \) or entails \( \beta \). We denote the set of models which entail \( \alpha \) as \( \Omega_{\alpha} \), and the set of models which entail \( \beta \) as \( \Omega_{\beta} \). Then,

\[
\text{WMC}(\alpha \lor \beta, w) = \sum_{\omega_{\alpha} \in \Omega_{\alpha}} \sum_{\omega_{\beta} \in \Omega_{\beta}} \prod_{l \in \omega_{\alpha}} w(l) \prod_{l \in \omega_{\beta}} w(l) = \text{WMC}(\alpha, w) + \text{WMC}(\beta, w).
\]

Definition A.11 (Functionally dependent WBF). Let \( (\alpha, w) \) be a WBF, and let \( X \) and \( Y \) be two variable sets which partition the variables in \( \alpha \). Then we say that \( X \) is functionally dependent on \( Y \) for \( \alpha \) if for any total assignment to variables in \( Y \), labeled \( y \), there is at most one total assignment to variables in \( X \), labeled \( x \), such that \( x \land y \models \alpha \).

Lemma A.12 (Functionally Dependent Existential Quantification). Let \( (\alpha, w) \) be a WBF with variable partition \( X \) and \( Y \) such that \( X \) is functionally dependent on \( Y \) for \( \alpha \). Furthermore, assume that for any conjunction of literals \( x \) formed from \( X \), \( \text{WMC}(x) = 1 \). Then, \( \text{WMC}(\alpha) = \text{WMC}(\exists x. \alpha) \).

Proof. The proof follows from Lemma A.9 and Lemma A.10. First, let \( X = x \) be a single variable, and assume all weighted model counts are performed with the weight function \( w \). Then,

\[
\text{WMC}(\exists x. \alpha) = \text{WMC}((\alpha \mid x) \lor (\alpha \mid \neg x)) = \text{WMC}(\alpha \mid x) + \text{WMC}(\alpha \mid \neg x) \]

By mutual exclusion

\[
= \frac{1}{\text{WMC}(x)} \text{WMC}(\alpha \land x) + \frac{1}{\text{WMC}(\neg x)} \text{WMC}(\alpha \land \neg x)
\]

\[
= \text{WMC}(\alpha \land x) + \text{WMC}(\alpha \land \neg x)
\]

\[
= \text{WMC}(\alpha) \quad \text{By mutual exclusion}
\]

This technique easily generalizes to when \( X \) is a set of variables instead of a single variable.

A.4.2 Main Proof
Let \( \sigma \) and \( \sigma' \) be an input and output state, let \( V \) be the set of variables in the entire program. The proof will proceed by induction on terms. We prove the following inductive base cases for terms which are not defined inductively.

\footnote{The notation "\( \alpha \mid x \)" means condition \( \alpha \) on \( x \).}
A.4.3 Base Cases

Skip  First, we show that the accepting semantics correspond. For any $\sigma$, we have that $\llbracket \text{skip} \rrbracket_A(\sigma) = \text{WMC}(\gamma(V) \land F(\sigma)) = 1$, since there is only a single satisfying assignment, which has weight 1. Now, we show that the transition semantics correspond;

- Assume $\sigma' = \sigma$. Then,
  \[
  \llbracket s \rrbracket_T(\sigma' \mid \sigma) = \frac{\text{WMC}(\gamma(V) \land F(\sigma) \land F'(\sigma'), \delta(V))}{\text{WMC}(\gamma(V) \land F(\sigma))} = 1
  \]
  since we have a single model in both numerator and denominator, both having weight 1.
- Assume $\sigma \neq \sigma'$. Then:
  \[
  \llbracket s \rrbracket_T(\sigma' \mid \sigma) = \frac{\text{WMC}(\gamma(V) \land F(\sigma) \land F'(\sigma'), \delta(V))}{\text{WMC}(\gamma(V) \land F(\sigma))} = 0
  \]
  since the numerator counts models of an unsatisfiable sentence.

Sample  Let $\varphi$ and $w$ be defined as in the symbolic compilation rules. First we show that the accepting semantics correspond.

\[
\llbracket x \sim \text{flip}(\emptyset) \rrbracket_A(\sigma) = \text{WMC}(\gamma(x' \Rightarrow f) \land \gamma(V \setminus \{x\}) \land F(\sigma), w) = 1,
\]

By Lemma A.9

Now we show that the transition semantics correspond:

\[
\llbracket x \sim \text{flip}(\emptyset) \rrbracket_T(\sigma' \mid \sigma) = \text{WMC}(\varphi \land F(\sigma) \land F'(\sigma'), w) \times \frac{1}{\text{WMC}(\varphi \land F(\sigma))} = 1,
\]

We can observe the following about $\alpha$:

- If $\sigma = \sigma'[x \mapsto \top]$, then $\text{WMC}(\varphi \land F(\sigma) \land F'(\sigma'), w) = \emptyset$.
- If $\sigma = \sigma'[x \mapsto \bot]$, then $\text{WMC}(\varphi \land F(\sigma) \land F'(\sigma'), w) = 1 - \emptyset$.
- If $\sigma \neq \sigma'[x \mapsto \top]$ and $\sigma \neq \sigma'[x \mapsto \bot]$, $\alpha = 1$, so the weighted model count is 0.

Assignment  First we show that the accepting semantics correspond:

\[
\llbracket x := e \rrbracket_A(\sigma) = \text{WMC}(\gamma(x' \Rightarrow e_S) \land \gamma(V \setminus \{x\}) \land F(\sigma), w) = 1,
\]

since there is exactly a single model and its weight is 1. Now, we show that the transition semantics correspond:

\[
\llbracket x := e \rrbracket_T(\sigma' \mid \sigma) = \text{WMC}(\gamma(x' \Rightarrow e_S) \land \gamma(V \setminus \{x\}) \land F(\sigma) \land F'(\sigma'), w) \times \frac{1}{\text{WMC}(\gamma(x' \Rightarrow e_S) \land \gamma(V \setminus \{x\}) \land F(\sigma))} = 1
\]

- Assume $\sigma = \sigma'[x \mapsto e_S]$. Then, $\alpha$ has a single model, and the weight of that model is 1, so $\text{WMC}(\alpha, w) = 1$.
- Assume $\sigma \neq \sigma'[x \mapsto e_S]$. Then, $\alpha$ is unsatisfiable, so $\text{WMC}(\alpha, w) = 0$.

Observe  First we prove the transition semantics correspond:

\[
\llbracket \text{observe}(e) \rrbracket_A(\sigma) = \text{WMC}(e_S \land \gamma(V) \land F(\sigma), w) = \begin{cases} 1 & \text{if } F(\sigma) \models \llbracket s \rrbracket_S \\ 0 & \text{otherwise} \end{cases}
\]

Now, we can prove that the transition semantics correspond:

\[
\llbracket \text{observe}(e) \rrbracket_T(\sigma' \mid \sigma) = \text{WMC}(e_S \land \gamma(V) \land F(\sigma) \land F'(\sigma'), w) \times \frac{1}{\text{WMC}(e_S \land \gamma(V) \land F(\sigma), w)} = \frac{\text{WMC}(e_S \land \gamma(V) \land F(\sigma) \land F'(\sigma'), w)}{\text{WMC}(e_S \land \gamma(V) \land F(\sigma), w)} \times \frac{1}{\text{WMC}(e_S \land \gamma(V) \land F(\sigma), w)}
\]

We treat a fraction $\frac{\beta}{\beta}$ as 0. Then, we can apply case analysis:

- Assume $\sigma = \sigma'$ and $e_S \models \gamma(V)$. Then, both $\alpha$ and $\beta$ have a single model with weight 1, so $\llbracket \text{observe}(e) \rrbracket_T(\sigma' \mid \sigma) = 1$.
- Assume $\sigma \neq \sigma'$ or $e_S \not\models \gamma(V)$. Then, either $F(\sigma) \land e_S \models \gamma(V) \land F(\sigma) \land F'(\sigma') \models \gamma(V)$; in either case, $\llbracket \text{observe}(e) \rrbracket_T(\sigma' \mid \sigma) = 0$. 


A.4.4 Inductive Step
Now, we utilize the inductive hypothesis to prove the theorem for the inductively-defined terms. Formally, let \( s \) be a \( \text{DIPPL} \) term, let \( \{ s_i \} \) be sub-terms of \( s \). Then, our inductive hypothesis states that for each sub-term \( s_i \) of \( s \), where \( s_i \sim (\phi, w) \), we have that for any two states \( \sigma, \sigma' \), \( \llbracket s_i \rrbracket_T(\sigma' \mid \sigma) = \llbracket (\phi, w) \rrbracket_T(\sigma' \mid \sigma) \) and \( \llbracket s_i \rrbracket_A(\sigma) = \llbracket (\phi, w) \rrbracket_A(\sigma) \). Then, we must show that the theorem holds for \( s \) using this hypothesis.

Remark 1. For the inductively defined compilation semantics, the weight function \( w = w_1 \cup w_2 \) is a unique and well-defined weight function, since the only source of weighted variables is from a \( \text{Flip} \) term, which only assigns a weight to fresh variables; thus, there can never be a disagreement between the two weight functions \( w_1 \) and \( w_2 \) about the weight of a particular variable.

Composition Let \( \varphi, w, \varphi_1, \varphi_2, \varphi_3, w_1, \) and \( w_2 \) be defined as in the symbolic compilation rules. By the inductive hypothesis, we have that the theorem holds for \( (\varphi_1, w_1) \) and \( (\varphi_2, w_2) \). We observe that the weighted model counts of \( \varphi_2 \) are invariant under relabelings. I.e., for any states \( \sigma, \sigma', \sigma'' \):

\[
\text{WMC}(\varphi_2 \land F(\sigma)) = \text{WMC}(\varphi_2' \land F'(\sigma))
\]

\[
\text{WMC}(\varphi_2 \land F(\sigma) \land F'(\sigma')) = \text{WMC}(\varphi_2' \land F'(\sigma') \land F''(\sigma''))
\]

where \( F'() \) generates double-primed state variables. Now we show that the WBF compilation has the correct accepting semantics, where each weighted model count implicitly utilizes the weight function \( w \):

\[
\llbracket s_1; s_2 \rrbracket_A(\sigma) = \llbracket s_1 \rrbracket_A(\sigma) \times \prod_{\tau \in \Sigma} \left( \llbracket s_1 \rrbracket_T(\tau \mid \sigma) \times \llbracket s_2 \rrbracket_A(\tau) \right)
\]

\[
\text{WMC}(\varphi_1 \land F(\sigma) \land F'(\tau)) \times \text{WMC}(\varphi_2' \land F'(\tau))
\]

\[
\text{WMC}(\varphi_1 \land F(\sigma) \land \varphi_2 \land F'(\tau)) \times \text{WMC}(\varphi_2' \land F'(\tau))
\]

\[
\text{WMC}(\varphi_1 \land F(\sigma) \land \varphi_2' \land F'(\tau)) \times \text{WMC}(\varphi_2 \land F'(\tau))
\]

\[
\text{WMC}(\varphi_1 \land F(\sigma) \land \varphi_2 \land F'(\tau)) \times \text{WMC}(\varphi_2' \land F'(\tau))
\]

\[
\text{WMC}(\varphi_1 \land \varphi_2 \land F(\sigma) \land F'(\tau))
\]

\[
\text{WMC}(\text{exists} \{ x_i \}, \varphi_1 \land \varphi_2 \land F(\sigma))
\]

Now, we can prove the transition semantics correspond for composition, where all model counts are implicitly utilizing the weight function \( w \):

\[
\llbracket s_1; s_2 \rrbracket_T(\sigma' \mid \sigma) = \frac{\sum_{\tau \in \Sigma} \llbracket s_1 \rrbracket_T(\tau \mid \sigma) \times \llbracket s_2 \rrbracket_T(\sigma' \mid \tau) \times \llbracket s_2 \rrbracket_A(\tau)}{\llbracket s_1 \rrbracket_A(\sigma) \times \sum_{\tau \in \Sigma} \llbracket s_1 \rrbracket_T(\tau \mid \sigma) \times \llbracket s_2 \rrbracket_A(\tau)}
\]

\[
= \llbracket s_1; s_2 \rrbracket_A(\sigma)
\]
where \( \vdash \) if statements

\[
\sum_{\tau \in \Sigma} \frac{\text{WMC}(\phi \land F(\sigma) \land F(\tau))}{\text{WMC}(\phi \land F(\sigma))} \times \text{WMC}(\phi' \land F'(\sigma') \land F'(\tau))
\]

\[
= \frac{1}{\text{WMC}(\phi \land F(\sigma))} \times \sum_{\tau \in \Sigma} \text{WMC}(\phi_1 \land F(\sigma) \land F'(\tau)) \times \text{WMC}(\phi_2' \land F'(\sigma') \land F'(\tau))
\]

\[
= \frac{1}{\text{WMC}(\phi \land F(\sigma))} \times \sum_{\tau \in \Sigma} \text{WMC}(\phi_1 \land F(\sigma) \mid F'(\tau)) \times \text{WMC}(\phi_2' \land F'(\sigma') \mid F'(\tau)) \times \text{WMC}(F'(\tau))^2 = 1
\]

(\text{By Lemma A.9})

\[
\frac{1}{\text{WMC}(\phi \land F(\sigma))} \times \text{WMC}(\phi_1 \land F(\sigma) \land F'(\sigma') \mid F'(\tau)) \times \text{WMC}(F'(\tau))
\]

(\text{By Lemma A.8})

\[
\frac{1}{\text{WMC}(\phi \land F(\sigma))} \times \text{WMC}
\left(\bigvee_{\tau} \phi_1 \land F(\sigma) \land F'(\sigma') \land \left[\bigvee_{\tau} F'(\tau)\right]\right)
\]

(\text{By Lemma A.10})

\[
\frac{1}{\text{WMC}(\phi \land F(\sigma))} \times \text{WMC}
\left(\phi_1 \land F(\sigma) \land F'(\sigma') \land \left[\bigvee_{\tau} F'(\tau)\right]\right)
\]

\[
= \frac{1}{\text{WMC}(\phi \land F(\sigma))} \times \text{WMC}(\exists x'_1. \phi_1 \land F(\sigma) \land F'(\sigma'))
\]

(\text{By Lemma A.12})

\textbf{if-statements}

Let \( \phi_1, \phi_2, w, \varphi \) be defined as in the compilation rules. First, we prove that the accepting semantics correspond:

\[
\llbracket \text{if}(e) \{s_1\} \text{ else } \{s_2\}\rrbracket_A(\sigma) = \begin{cases} \llbracket s_1 \rrbracket_A(\sigma) & \text{if } \llbracket e \rrbracket(\sigma) = T \\ \llbracket s_2 \rrbracket_A(\sigma) & \text{if } \llbracket e \rrbracket(\sigma) = F \end{cases}
\]

\[
= \begin{cases} \text{WMC}(\phi_1 \land F(\sigma)) & \text{if } \llbracket e \rrbracket_S(\sigma) \land F(\sigma) \models T \\ \text{WMC}(\phi_2 \land F(\sigma)) & \text{otherwise} \end{cases}
\]

(\text{By Inductive Hyp.})

\[
= \text{WMC}(\left(\llbracket e \rrbracket_S \land F(\sigma) \rangle \lor \langle \neg \llbracket e \rrbracket_S \land F(\sigma) \rangle \right) \land F(\sigma))
\]

(\ddagger)

Where (\ddagger) follows from Lemma A.10 and the mutual exclusivity of \( \llbracket e \rrbracket_S \) and \( \neg \llbracket e \rrbracket_S \). Now we can prove the transition semantics correspond:

\[
\llbracket \text{if}(e) \{s_1\} \text{ else } \{s_2\}\rrbracket_T(\sigma' \mid \sigma) = \begin{cases} \llbracket s_1 \rrbracket_T(\sigma' \mid \sigma) & \text{if } \llbracket e \rrbracket(\sigma) = T \\ \llbracket s_2 \rrbracket_T(\sigma' \mid \sigma) & \text{if } \llbracket e \rrbracket(\sigma) = F \end{cases}
\]

\[
= \begin{cases} \text{WMC}(\phi_1 \land F(\sigma) \land F'(\sigma')) & \text{if } \llbracket e \rrbracket_S(\sigma) \land F(\sigma) \models T \\ \text{WMC}(\phi_2 \land F(\sigma) \land F'(\sigma')) & \text{otherwise} \end{cases}
\]

(\text{By Inductive Hyp.})

\[
= \text{WMC}(\left(\llbracket e \rrbracket_S \land \phi_1 \land F(\sigma) \land F'(\sigma') \rangle \lor \langle \neg \llbracket e \rrbracket_S \land \phi_2 \land F(\sigma) \land F'(\sigma') \rangle \right) \land F(\sigma))
\]

\[
= \text{WMC}(\left(\llbracket e \rrbracket_S \land \phi_1 \rangle \lor \langle \neg \llbracket e \rrbracket_S \land \phi_2 \rangle \right) \land F(\sigma))
\]

This concludes the proof.