Statistical Linearization Method for the Piecewise Nonlinear Systems

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Abstract. The Monte-Carlo numerical simulation method is applied to obtain the approximate probability distributions of the response of the piecewise nonlinear systems to random excitation. The expression of the equivalent linearization parameters for the piecewise nonlinear system is derived by the proposed statistical linearization method. To illustrate the accuracy of the linearization method, the present paper has compared the statistical characteristic value obtained by both the Monte Carlo numerical simulation method and the Statistical linearization method, while the time history analysis also shows a good agreement. Linearization of piecewise stiffness nonlinear model is displayed.

Keywords: Piecewise nonlinear systems; Random vibrations; Statistical linearization; Gaussian distribution; Monte Carlo numerical simulation

1. Introduction

It is very important for buildings to control the adverse vibration of structures. Most of the vibration systems in engineering practice are nonlinear random vibration systems. The general methodology of attacking nonlinear random vibration systems includes the perturbation method (e.g. Nayfeh 1990), Fokker-Planck-Kolmogorov (FPK) approach (e.g. Zhao & Ma 1999) and the statistical linearization method (e.g. Roberts & Spanos 2003). A common difficulty of the perturbation method lies in a combination of poor convergence and complex calculation, frequently it is found that the reliable results may be obtained only if the vibratory systems are a very small degree of non-linearity (e.g. Stephen & Crandall 1963). For a particular class of linear or nonlinear system to white noise random excitation, the FPK equation can be used to solve the vibration problems exactly, but it is quite limited. The Monte Carlo simulation method is digitally simulated based on a large amount of calculation and analysis, and it does not be affected by the type of systems or random excitation. Generally, Monte Carlo simulation results are being compared with other means studies, for assessing the accuracy. Until now, the Statistical linearization method is the most widely used approximate solution to the nonlinear systems to random excitation. Stochastic equivalent linearization method was originally proposed by Booton (e.g. Booton 2012) and Kazakov (e.g. Kazakov 1956), and was used to nonlinear stochastic vibration systems by Caughey (e.g. Caughey 1959). In terms of linearization criteria, the linearization parameters can be chosen to minimize the difference of the restoring force or potential energy between the nonlinear system and the equivalent linear system (e.g. Proppe et al. 2003). Atalik (e.g. Atalik 1976) used the first recipe to get the linearization form of a nonlinear system when the Gaussian probability density of response was assumed.

The dampers have been extensively applied to control structural displacement caused by seismic load and wind load, as a device of vibration attenuation and energy dissipation. Some systems can be simplified to the piecewise nonlinear models. The piecewise nonlinear systems have been used...
frequently in control of structure vibration (e.g., Wang and Sun 2018).

In this paper, the probability distribution model of the piecewise nonlinear model under the modulation non-stationary random process model (e.g. Clough & Penzien 2006), the Clough-Penzien model, is studied. It is reliably assumed that the probability density functions of displacement and velocity response are Gaussian distribution. By the proposed method, the equivalent linearization equation is obtained, and the characteristic parameters are compared with the Monte Carlo numerical simulation results obtained (e.g. Liu & Hu 2014) to indicate the feasibility and practicality of the statistical linearization method.

2. Proposed Method

2.1. Stochastic Model of the Seismic Excitation

On account of the non-stationary characteristics of ground acceleration records, the numerical simulation of non-stationary accelerations is made by multiplying the stationary earthquake wave and the modulation function (e.g. Fan 2014), which is defined by the following equation

\[ a(t) = \zeta(t) \alpha(t) \]  

where \( \alpha(t) \) is a stable stochastic process. The Clough-Penzien model is employed in this article (e.g. Li & Zhai 2003), and the power spectral density is given by

\[ s(\omega) = \left( \frac{\omega^2 + \omega_g^2}{\omega^2 - \omega_f^2 + \omega_q^2} \right) \times \left( \frac{\omega_f^2}{\omega^2 - \omega_f^2 + \omega_q^2} \right) s_0 \]  

where \( \omega_g, \omega_f, \omega_q \) are the filter parameters, and \( s_0 \) means the amplitude of the spectral density of the white noise.

The modulation function is (e.g. Li & Zhai 2003)

\[ \zeta(t) = \begin{cases} \frac{c}{t_t} & 0 \leq t < t_t \\ 1 & t_t \leq t < t_2 \\ e^{-c(t-t_2)} & t \geq t_2 \end{cases} \]  

where \( c \) is the decay factor, and \( t_t, t_2 \) represent the main shock duration.

2.2. Statistical Linearization

The differential equation of the single degree-of-freedom nonlinear vibration system (e.g. Ou & Wang 1998)

\[ m\ddot{x} + g(x, \dot{x}) = f(t) \]  

where the three parameters \( \ddot{x}, \dot{x} \) and \( x \) represent the relative acceleration, velocity, and displacement, respectively. \( m \) is the mass of the system, and \( g(x, \dot{x}) \) is a nonlinear function, where

\[ g(x, \dot{x}) = c\dot{x} + g_1(x) \]  

the structural is stiff nonlinear. And where

\[ g(x, \dot{x}) = g_2(\dot{x}) + kx \]  

the structural is damper nonlinear.

where \( g(x, \dot{x}) \) is a bivariate nonlinear function about \( x \) and \( \dot{x} \), the structural is stiff nonlinear and damper nonlinear. To apply the statistical linearization procedure, the nonlinear system (4) is replaced by the equivalent linear system
\[ m\ddot{x} + c_e \dot{x} + k_e x = f(t) \]  \hspace{1cm} (6)

where the coefficients \( c_e \) and \( k_e \) are the equivalent linear stiffness and damping, respectively. They make the equivalent linear equation constructed approximate to the original nonlinear equation as much as possible, namely, the error of primary system and the equivalent linear system should be as small as possible. Now the difference is the discrepancy between the non-linear system and the equivalent linear system, i.e.

\[ \varepsilon = g(x, \dot{x}) - c_e \dot{x} - k_e x \]  \hspace{1cm} (7)

A common minimization principle is to minimize the expectation of the mean square of the error term (e.g. Foliente 1993), given by

\[ E[\varepsilon^2] \rightarrow \text{minimum} \]  \hspace{1cm} (8)

Assuming that the \( \dot{x}(t) \) and \( x(t) \) are Gauss processes with zero mean, then the linearization coefficients \( c_e \) and \( k_e \) may be obtained as

\[
\frac{\partial}{\partial c_e} E\{\varepsilon^2\} = \frac{\partial}{\partial k_e} E\{\varepsilon^2\} = 0
\]  \hspace{1cm} (9)

For the calculation of the equation (6), a vector \( y \) is defined as

\[ y = [x, \dot{x}]^T \]  \hspace{1cm} (10)

The equivalent linear equations can be written as the first order linear differential equation

\[
\frac{d}{dt} y = gy + F
\]  \hspace{1cm} (11)

With

\[
g = \begin{bmatrix} 0 & 1 \\ -k_e / m & -c_e / m \end{bmatrix}
\]  \hspace{1cm} (12)

And

\[
F = \begin{bmatrix} 0 \\ f(t) \end{bmatrix}
\]  \hspace{1cm} (13)

Now the covariance matrix \( \Gamma \) is defined as

\[ \Gamma_y = E[y, y]^T \]  \hspace{1cm} (14)

When Eq. (11) is post-multiplied by the vector \( y^T \), the problem can be transformed into solving the equation (e.g. Zhao 2012)

\[
\frac{d}{dt} \Gamma = g\Gamma + \Gamma g^T + B
\]  \hspace{1cm} (15)

In the B matrix, all the elements are zero except \( B_{22} = I(t) \)

That is
\[ B = \begin{bmatrix} 0 & 0 \\ 0 & 2\pi S_0 \zeta^2(t) \end{bmatrix} \]  

(16)

where \( I(t) \) is a zero mean Gaussian white noise stochastic process with the constant power density \( S_0 \).

The differential equations (15) can be rearranged by sorting them out after expansion

\[
\begin{align*}
\Gamma_{xx} &= 2\Gamma_{xl} \\
\Gamma_{xl} &= \Gamma_{xl} - \frac{k_e}{m} \Gamma_{xt} - \frac{c_e}{m} \Gamma_{xi} \\
\Gamma_{xi} &= -\frac{2k_e}{m} \Gamma_{xl} - \frac{2c_e}{m} \Gamma_{xi} + 2\pi S_0 \zeta^2(t)
\end{align*}
\]

(17)

The covariance matrix can be solved by iteration.

3. **Numerical Applications**

Consider a piecewise damped nonlinear model of the form

\[ m\ddot{x} + c\dot{x} + kx + g(x\dot{x}) = f(t) \] 

(18)

With

\[ g(x\dot{x}) = \frac{1}{2} c_g (1 + \text{sgn}(x - x_b)) \dot{x} \] 

(19)

Here, \( x_b \) is the critical displacement of structural response. This model shows that the damping of the system will be larger when the displacement exceeds the critical displacement, otherwise the damping will be relatively small. For the calculating example, the following characteristic parameters are considered: \( \omega_0 = 16.42 \text{ rad/s}, \ c = 5.45 \text{ N} \cdot \text{s} / \text{m} \). A stochastic excitation was used with the amplitude of the spectral density of the white noise \( s_0 = 0.22 \text{ m}^2/\text{s} \) and the filter parameters are \( \omega_x = 7.2, \phi_x = 0.1, \omega_f = 2.53 \) and \( \zeta_f = 0.2 \).

For simulation, the nonlinear system was subjected to non-stationary random excitations, the statistical eigenvalues relevant to the response process are obtained by the usual methods. According to the results of Monte Carlo simulation, the probability distribution image of the response is graphed, (e.g. Wang & Song 2017), shown in Figure 1 and Figure 2. On the basis of the simulation, the response of displacement and velocity of the system obey Gaussian distribution.
Figure 1. Time evolution of the probability density function of the displacement response of the piecewise damped nonlinear model

Figure 2. Time evolution of the probability density function of the velocity response of the piecewise damped nonlinear model

An equivalent linear equation of the Eq. (18) is defined in the form

\[ m\ddot{x} + (c + c_\varepsilon)\dot{x} + (k + k_\varepsilon)x = f(t) \]  

Then the equivalent linear parameters are calculated by using the method mentioned above. The results of these calculations are summarized in the following equations.

\[ c_\varepsilon = \frac{1}{2}c_a \frac{1}{2}c_a\text{erf}(\frac{x_a}{\sqrt{2}\sigma_x}) \]  

\[ k_\varepsilon = 0 \]

where \( \sigma_x \) is the standard deviation of displacement response.

Figure 3 and 4 show a comparison between the statistical eigenvalues of structural responses postulated by the proposed approach and obtained by Monte Carlo simulation.
Figure 3. Standard deviation comparison of displacement response of the piecewise damped nonlinear model

Figure 4. Standard deviation comparison of velocity response of the piecewise damped nonlinear model

For the displacement standard deviation, there is a deviation between the linearization results and simulation results. From the figure, the linearization results are about 10% larger than the simulation results. As for the standard deviation of velocity, the results show a good coincidence.

Time-history comparison of structural responses of nonlinear structure and linearized structure are shown in figure 5 and 6. As can be seen from the figures, the response of linearization model indicates good fit of the reality model.

Figure 5. Time-history comparison of displacement response of the piecewise damped nonlinear model
Figure 6. Time-history comparison of velocity response of the piecewise damped nonlinear model

4. Conclusion
For the piecewise damped nonlinear model commonly used in structural vibration control, based on the assumption of Gaussian distribution of response, statistical linearization is carried out, and the equivalent linear form of the nonlinear model is obtained. By comparing the historic curves of the piecewise nonlinear model and its equivalent linear form under seismic waves, it is found that they are in good agreement. However, there are still some errors of characteristic parameters between the results of equivalent linearization and Monte Carlo simulation. This indicates that the method in this paper needs further improvement.

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