A general wave-packet evolution method for studying coherence of matter-wave interferometers

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We present a general method for calculating wave-packet evolution in the context of ultracold atoms in matter-wave interferometers. This method provides an efficient tool for analyzing the performance of atomic interferometers based on atom clouds initially prepared in a trap in a Bose-Einstein condensate (BEC) state or in a thermal distribution. Without having to solve explicitly the Gross-Pitaevskii or the Schrödinger equation, this provides a good estimation of dynamic wave-packet properties such as size and phase across the wave-packet, and slows to predict the coherence of the interferometer. We develop a generalized Thomas-Fermi approximation for a BEC in a harmonic trap, which is valid for any atom-atom interaction strength. In particular, this approximation also provides a good estimation of the transition from a three-dimensional to a one-dimensional BEC in an elongated trap. The approximation is followed by a theory of dynamic wave-packet evolution when the trap is fully or partially switched off and the atoms are split and propagate, for example, in free space or in a matter waveguide. The method is applied for studying two examples of interferometric effects: reduction of coherence of two-state interferometers due to imperfect spatial recombination of the two interferometer arms (the “Humpty-Dumpty effect”) and phase diffusion due to number uncertainty in the two interferometer arms, which was previously studied thoroughly only for interferometric schemes where the BECs in the two arms stay trapped (for example, in a double-well potential). For both effects we extend the applicability of the theory to a wide range of interferometric scenarios that were not included in previous theories.

I. INTRODUCTION

Matter-wave interferometry with ultracold atoms has become a wide field of fundamental and applied research using many different techniques for splitting, guiding and probing. Coherent spatial splitting of initially trapped atoms is performed by light pulses (in a Ramsey-Bordé, Raman or Bragg configuration), by optical or magnetic fields that form double-well potentials or by magnetic gradient pulses in free space. In any of these schemes the most crucial factor is coherence, namely keeping and retrieving a well-defined phase difference between the interferometer arms.

A common source of decoherence – a process that randomizes the phase at the interferometer output – is external noise, i.e., coupling of the atoms in the two interferometer arms to classical random forces or entanglement with the quantum states of the environment, both leading to a similar result in the output port. However, coherence may also be limited by internal properties of the system that are not related to quantum or classical fluctuations of the environment. Here we discuss two such examples and derive a wave-packet evolution method that provides an efficient tool allowing for quantitative predictions related to these decoherence effects and for designing the system parameters to optimize the performance.

One effect that limits coherence was first discussed in the framework of a Stern-Gerlach interferometer where a spin-$\frac{1}{2}$ particle with the spin oriented perpendicular to the magnetic field splits into two trajectories in a magnetic field gradient, which are then recombined to form an interference signal. The signal is the spin orientation at the output port, which depends on the relative phase accumulated along the two paths. The contrast of this spin signal with changing phase (spin-coherence) depends on how accurately the two paths are recombined to become indistinguishable in position and momentum at the output port. Theoretical studies of this spin coherence showed that a very high level of accuracy in manipulating the trajectories of the two arms is needed. Coherent magnetic Stern-Gerlach splitting to micrometer distances was realized recently and increasing the splitting distance requires a better understanding of the effect termed “the Humpty-Dumpty effect”. In principle, this effect may occur in any interferometric configuration that involves entanglement between internal and spatial degrees of freedom. In order to quantitatively estimate this effect it is essential to know the overlap between the spatial states of the two interferometer arms in position and momentum at the output port. It is therefore necessary to apply a wave-packet propagation mode that goes beyond a semiclassical picture of well-defined trajectories.

Another kind of interferometric decoherence is a many-body effect related to atom-atom interactions relevant to interferometers using a Bose-Einstein condensate (BEC). Phase diffusion due to atom-atom interactions emerges from the uncertainty of the number of atoms in each arm, which is an essential part of the splitting process and an essential condition for a well-defined relative phase between the paths.

Since the first observation of interference between two BECs and the explanation of phase randomness by the process of phase diffusion due to atom-atom interactions, it is well established that the time of coherent separation of a BEC after splitting is limited...
by this inherent decoherence process to a few tens of milliseconds, unless the BEC is very dilute or number squeezing takes place due to slow separation \[13\, 27\, 28\]. The rate of phase diffusion was previously calculated for cases in which the BEC is kept in a trap or a harmonic potential along the whole interferometric sequence \[25\, 27\]. In contrast, if the interferometer is implemented in free space, where the BEC is freely expanding, then it is expected that phase diffusion due to atom-atom interactions is negligible due to the fact that the atomic density drastically drops during expansion. In order to understand this critical process in cases where the BEC is not completely released, for example, if it continues to propagate in a guiding potential such as a ring that confines the motion to a specific trajectory, it is essential to use a wave-packet propagation model to account for the evolution during propagation.

A fairly accurate way to calculate the dynamics of a BEC under the influence of time-dependent potentials responsible for the splitting, guiding, and recombining of the atomic wave-packets through the interferometer would be to solve the Gross-Pitaevskii (GP) equation, which is a successful mean-field approximation for many practical cases \[30\]. However, for many interferometric scenarios it is impractical to solve the GP equation in three dimensions along the interferometric sequence, especially when such a calculation needs to be iterated many times for the purpose of design and stability prediction. An effective approximate solution of the GP equation in a quadratic potential and in the time-dependent Thomas-Fermi (TF) approximation, where the kinetic energy of the trapped BEC is neglected, was first presented by Castin and Dum \[31\], who also provided an analytical approximate solution for the expansion of a BEC in free space. Later work generalized the dynamics in free space to beyond the TF limit \[32\] and other works have also extended the Castin-Dum theory to any kind of quadratic potential with time-dependent linear and quadratic terms, including the case of rotating traps (see a review in \[33\]).

Here we further generalize the previously developed BEC evolution methods to enable the calculation of the dynamics of initially trapped atomic wave-packets in the presence of any atom-atom interaction strength, from the limit of weak interactions suitable for dilute thermal clouds, to the case of a tightly confined BEC, subject to dynamic potential gradients or guiding potentials. The dependence of wave-packet dynamics on the number of atoms in each interferometer arm after splitting give rise to phase diffusion due to number uncertainty. Our analysis starts with a generalization of the static Thomas-Fermi approximation for the time-independent GP equation in a trap, which we compare to a full three-dimensional solution of the GP equation. We find that it provides a good approximation for the chemical potential and widths of the atomic cloud across the relevant range of interaction strengths. We also compare our splitting dynamics provided by the wave-packet evolution method developed here to the numerical solution of the time-dependent GP equation and find that it provides a good approximation for the dynamics if the splitting time is short.

Based on our wave-packet evolution method, we then provide a detailed theory of the two decoherence effects mentioned above: decoherence of a two-state spatial interferometer due to imperfect recombination and phase diffusion due to atom-atom interactions in an interferometer using BEC wave-packets evolving in trapping or guiding potentials or in free space.

The structure of this paper is as follows: in Section II we present the generalized TF approximation for a trapped BEC and the wave-packet evolution method. In Section III we present the theory of coherence in a two-state spatial interferometer, for the case of a BEC or a thermal cloud. In Section IV we present the theory of phase diffusion for trapped or propagating wave-packets and finally we provide an outlook and conclusions in Section V.

II. WAVE PACKET EVOLUTION

Let us consider atoms of mass \(m\) initially trapped in a harmonic potential \(V(r) = \frac{1}{2} m \sum_{j=1}^{3} \omega_j^2 r_j^2\), where \(r_j = 1, 2, 3\) for \(j = 1, 2, 3\), respectively, are the Cartesian coordinates parallel to the axes of the trap and \(\omega_j\) are the respective trap frequencies. We consider two cases that will be treated on the same footing: the initial atomic wave function may either be any eigenstate of the single-particle Hamiltonian, satisfying the time-independent Schrödinger equation, if atom-atom interactions are negligible, or it may be in the presence of interactions, the lowest energy solution (Bose-Einstein condensate, BEC) of the time-independent Gross-Pitaevskii (GP) mean-field equation

\[
\left[-\frac{\hbar^2}{2m} \nabla^2 + \frac{1}{2} m \sum_{j=1}^{3} \omega_j^2 r_j^2 + g N |\Phi_0|^2 - \mu \right] \Phi_0(r) = 0.
\]

Here the mean-field repulsive potential (third term in the brackets) is proportional to the atom density \(N|\Phi_0|^2\), where \(N\) is the number of atoms and the wave function \(\Phi_0\) is normalized to unity, and the coupling strength is \(g = 4 \pi \hbar^2 a_s/m\) with \(a_s\) being the s-wave scattering length. \(\mu\) in Eq. (1) is the chemical potential or it may be replaced by any eigenstate energy \(\mu \to \hbar \sum_{j=1}^{3} \omega_j (n_j + \frac{1}{2})\), with \(n_j\) being non-negative integers, in the absence of interactions \((gN \to 0)\).

Eq. (1) has two opposite limits: (a) the single-particle limit where the interaction term is negligible and the solutions for \(\Phi_0\) are Hermite-Gaussian functions; (b) the Thomas-Fermi (TF) limit, where the kinetic term is negligible relative to the potential terms, in which case the ground state solution for \(\Phi_0\) has an inverse parabolic shape with a sharp edge. In what follows we discuss
the evolution of $\Phi_0$ when the potential changes in time. If the potential stays smooth enough then one can use an approximate solution for the evolution. Such an approximation was derived by Castin and Dum \cite{21} for the TF limit and later generalized to free expansion with any interaction strength \cite{32}. Here we further generalize previous methods to permit calculations of the propagation of a BEC with any interaction strength or a non-interacting Bose gas at a finite temperature in a time-dependent potential that can be expanded in a quadratic form across the volume occupied by the atomic cloud. We first derive a generalized TF approximation that can be used to parameterize the initial trapped wave function for all interaction strengths. This parameterization will be then used to derive the equations of motion for the dynamical parameters of evolution when the potential changes in time.

\section{Generalized Thomas-Fermi approximation}

The Thomas-Fermi (TF) approximation \cite{30} usually provides successful predictions for the state of a BEC of many atoms in a trap with a relatively low aspect ratio between its axes. It assumes that the kinetic energy is negligible with respect to the interaction energy, so that $gN|\Phi_0(r)|^2 = \mu - \frac{1}{2} m \sum_j \omega_j^2 r_j^2$ in the ellipsoidal volume where $\sum_j \omega_j^2 r_j^2 < 2 \mu$ and is zero elsewhere. Here we adopt a hybrid approach that does not neglect any term in Eq. \eqref{1}: we define an interaction chemical potential $\mu_{\text{int}}$ that defines the edge of the inverse parabolic shape of the wave function and at the same time approximate the wave function as a Gaussian $\Phi_0(r) \propto \exp[-\sum_j r_j^2/4\sigma_j^2]$ for estimating the kinetic term in Eq. \eqref{1}, namely

\begin{equation}
-\frac{\hbar^2}{2m} \nabla^2 \Phi_0 \approx \sum_j \frac{\hbar^2}{4m \sigma_j^2} \left(1 - \frac{r_j^2}{2 \sigma_j^2}\right) \Phi_0. \tag{2}
\end{equation}

This term, which is completely neglected in the standard TF approximation, provides a way to tailor between the strong and weak interaction limits. The rationale of this hybrid assumption where the wave function is taken to be Gaussian for approximating the kinetic term and an inverse parabola for approximating the interaction term is that in the non-interacting limit the wave function is indeed a Gaussian and the interaction term is small compared to the kinetic term, while in the strong interaction limit the kinetic term is negligible and the wave function is indeed an inverse parabola. By taking this approximation in the range of intermediate interaction strength we tailor the two limits. As we demonstrate below, this provides a good approximation for the chemical potential and the wave-packet size, regardless of the exact shape of the wave function.

By substituting Eq. \eqref{2} into Eq. \eqref{1} we obtain

\begin{equation}
gN|\Phi_0(r)|^2 \approx \max \left\{ \mu_{\text{int}} - \frac{1}{2} m \sum_j \tilde{\omega}_j^2 r_j^2, 0 \right\}, \tag{3}
\end{equation}

where $\max\{x, 0\} \equiv x \theta(x)$ ($x$ unless $x < 0$) and where

\begin{equation}
\tilde{\omega}_j^2 = \omega_j^2 - \nu_j^2, \quad \nu_j = \frac{\hbar}{2m \sigma_j^2} \tag{4}
\end{equation}

and $\mu_{\text{int}} = \mu - \frac{\hbar}{2} \sum_j \nu_j$ is the total chemical potential minus the kinetic energy at $r_j = 0$.

In order for the Gaussian widths $\sigma_j$ in Eq. \eqref{2} to be meaningful for the inverse parabolic interaction terms of Eq. \eqref{3} we identify them with the square roots of the variances $\langle r_j^2 \rangle$. For the interaction term in Eq. \eqref{3} these variances are given by

\begin{equation}
\sigma_j^2 = \int d^3r r_j^2|\Phi_0(r)|^2 = \frac{1}{7} \langle r_j^2 \rangle_{\text{max}} = \frac{1}{7} \frac{\mu_{\text{int}}}{\bar{\omega}_j^2}. \tag{5}
\end{equation}

In the TF limit, where $\sigma_j \gg \ell_j$, where $\ell_j = \sqrt{\hbar/2m \omega_j}$ are the single-particle harmonic oscillator widths, the kinetic energy terms are indeed small and $\mu_{\text{int}} \rightarrow \mu$, $\tilde{\omega}_j \rightarrow \omega_j$. On the other hand, in the limit of no interaction ($gN|\Phi_0|^2 \rightarrow 0$) the size of the wave-packet is given by $\sigma_j^2 = \hbar/2m \omega_j$, such that $\tilde{\omega}_j^2 \rightarrow 0$ and $\mu_{\text{int}} \rightarrow 0$, while the ratio $\mu_{\text{int}}/\tilde{\omega}_j^2$ becomes constant.

The interaction chemical potential $\mu_{\text{int}}$ and the widths $\sigma_j$ are determined self-consistently by using the normalization condition $\int d^3r |\Phi_0(r)|^2 = 1$. This condition together with Eq. \eqref{3} yields the relation

\begin{equation}
\mu_{\text{int}} = \frac{7}{2} m \bar{\omega}_j^2 \sigma_j^2 = \hbar \bar{\Omega} \left(\frac{15a_s N}{8\ell^3}\right)^{2/5}, \tag{6}
\end{equation}

where $\bar{\Omega} \equiv \left(\prod_i \bar{\omega}_i\right)^{1/3}$ is the geometric mean of the modified frequencies, and $\ell \equiv \sqrt{\hbar/2m \bar{\omega}}$ is the harmonic length associated with this mean. Note that Eq. \eqref{6} should hold self-consistently for each $j = 1, 2, 3$ in its middle term. Eq. \eqref{6} must have a self-consistent solution because it has simple solutions in the two interaction limits. In the standard TF limit $\tilde{\omega}_j \rightarrow \omega_j$ and Eq. \eqref{6} reduces to the usual closed form of the TF formula with $\sqrt{\bar{\Omega} \sigma_j} = r_j, \text{max}$ being the distance of the cloud’s edges from the center. However, in general $\tilde{\omega}_j$ depend on $\sigma_j$ and reduces to zero in the non-interacting limit together with $\mu_{\text{int}}$, while $\sigma_j \rightarrow \ell_j$.

In order to assess the validity of the generalized Thomas-Fermi (GTF) approximation, we present in Fig. \ref{fig:1} a comparison to the numerical solution of the Gross-Pitaevskii equation (GPE). The GTF approximation is in excellent agreement with the GPE solution over the whole range between the standard TF regime (large atom number $N$, dominant interaction energy) and the weak interaction limit in what regards the wave-packet width and the chemical potential. The GTF approximation does not provide a prediction about the shape of the wave function so that we present in the inset a comparison between the two limits of the wave-packet shape compared to the shape obtained from the GPE. In the case of weak interaction ($N = 100$) the shape is closer to
FIG. 1. Generalized Thomas-Fermi (GTF) approximation for the ground state in a harmonic trap: comparison of GTF with a numerical solution of GPE and with standard TF approximation. GTF is shown to be valid over the whole range of atom-atom interaction strengths. (a) Wave function widths $\sigma_{\parallel}$ (blue) and $\sigma_{\perp}$ (red) as a function of atom number for $^{87}\text{Rb}$ atoms (mass $m = 1.44 \times 10^{-25}$ kg, s-wave scattering length $a_s = 5.29$ nm) in a cylindrically symmetric harmonic trap with frequencies $\omega_{\parallel} = 2\pi \times 40$ Hz and $\omega_{\perp} = 2\pi \times 100$ Hz. GTF agrees well with GPE over all the range while standard TF fails at low density. Inset: probability density profiles $|\Phi_0(x, 0, 0)|^2$ along the longitudinal trap axis: GPE result (solid) compared to an inverse parabolic [Eq. (3), dashed] and Gaussian (dashed-dotted) profiles. For $N = 100$ (blue) the GPE profile is closer to the Gaussian, while for $N = 1000$ (red) it lies between the two approximate profiles. (b) chemical potential due to interaction $\mu_{\text{int}}$ and kinetic energy $\mu_{\text{kin}} = \frac{1}{2} \hbar \sum_j \nu_j$. The total chemical potential $\mu_{\text{tot}} = \mu_{\text{int}} + \mu_{\text{kin}}$ according to GTF shows excellent agreement with GPE solution (circles) while the standard TF (dashed line) agrees only at large $N$. Inset: potential isosurface and definition of axes.

FIG. 2. Generalized Thomas-Fermi (GTF) approximation for a BEC in an elongated trap ($\omega_y = \omega_z = \omega_{\perp} = 2\pi \times 10$ kHz, $\omega_x = 2\pi \times 40$ Hz) at zero temperature as a function of atom number $N$ (for $^{87}\text{Rb}$ as in Fig. 1). The longitudinal cloud width (a) and the ground state chemical potential (b) resulting from a numerical solution of the GPE (circles) agree very well with the GTF approximation over the whole range and agree with the 1D TF approximation only for low atom numbers, where the transverse wave function is the single particle Gaussian ground state in the transverse potential. This demonstrates the validity of the GTF approximation in predicting the transition from 3D to 1D for a BEC in elongated traps. The 3D TF approximation is valid only for $N$ beyond the range shown. The condensate approximation is valid throughout the range shown at zero temperature as the $\gamma$ factor for transition into the Tonks-Girardeau regime is small.

An important application of the generalized TF approximation is the transition from a three-dimensional (3D) BEC to a quasi-one-dimensional (1D) BEC in an elongated trap [35, 36]. In the 1D limit the large energy splitting between transverse eigenmodes of the potential allows scattering only along the longitudinal direction and hence the atomic dynamics is limited to one dimension while the wave function in the transverse direction is fixed at the lowest eigenstate of the harmonic potential. The dynamics along the longitudinal axis is then governed by an effective interaction strength $g_{1D} = g/4\pi \ell_{\perp}^2 = 2\hbar \omega_{\perp} a_s$ [36]. As long as the factor $\gamma = 2m\omega_{\perp}/\hbar n$, where $n$ is the 1D atomic density, is small ($\gamma \ll 1$) the condensate assumption for the many-body ground state is valid (otherwise a Tonks-Girardeau gas is formed [36, 37]). As demonstrated in Fig. 2, the GTF approximation allows a fairly accurate prediction of the properties of the BEC over a broad range of parameters starting with a fully 1D BEC at low atom numbers (weak interaction) along the transition to a 3D BEC, where the interaction is strong enough to become dominant in the

the Gaussian approximation of the GTF (dashed-dotted curve) whereas in the intermediate interaction regime ($N = 1000$) the shape obtained from the GPE mediates between the Gaussian shape and the inverse parabolic shape (dashed curve), all having the same widths $\sigma_j$ as defined in Eq. (5).
transverse direction.

Note that a better agreement between the accurate GP solution and the GTF approximation could be reached by refining the self-consistent equation (10). The 7 factor in this equation emerges from the assumption that as far as the repulsive interaction potential is concerned the atomic density has an inverse parabolic shape. However, in the case of a quasi-1D BEC the atomic density is expected to have an inverse parabolic shape only in the longitudinal direction, while its transverse shape is nearly a Gaussian. In this case one would need to replace the factor 7 (coming from a 3D integral on the inverse parabolic shape) to a factor 5 (coming from a 1D integral only along the longitudinal direction). A refinement of the theory that will hold in the intermediate transition range between 1D and 3D requires a bit of sophisticated manipulation that will not be further discussed here.

B. Evolution equations

Suppose that an atomic cloud is initially trapped in a harmonic potential and then at time \( t > 0 \) the potential changes in time. For example, one may consider switching off the trapping potential along one or more axes, changing the harmonic frequencies or applying potential gradients. Here we treat the atoms in terms of single-atom wave-packets: a single wave-packet for a BEC, a mixture of wave-packets for a thermal cloud and two or more wave-packets for the different interferometer arms if the initial cloud is coherently split. We parameterize the wave-packet and derive equations of motion for the parameters, which are valid as long as the potential varies smoothly over the volume occupied by the wave-packet. The wave function of a BEC wave-packet satisfies the time-dependent GP equation

\[
\frac{\partial \psi}{\partial t} = \hat{H}_{\text{GP}}(t, \psi)\psi,
\]

where

\[
\hat{H}_{\text{GP}}(t, \psi) = \frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{r}, t) + \eta(t)|\psi|^2,
\]

and \( \eta(t)N \) is the number of atoms in the wave-packet at any time. This number might change in time in different interferometric situations. For example, if a wave-packet is split into two wave-packets with an equal number of particles then after they spatially separate each one of them has \( N/2 \) particles and hence the strength of the interaction term reduces by a factor \( \eta = 1/2 \). When treating a thermal cloud where atom-atom interactions are negligible, the interaction term in Eq. (8) vanishes and Eq. (7) turns into a Schrödinger equation, while \( \psi(\mathbf{r}, t) \) may represent an arbitrary wave-packet in the mixture of wave-packets contained in the cloud. A partially Bose-condensed atomic cloud at finite temperature is beyond the scope of this work.

First we consider the classical motion of the center of the wave-packet \( \mathbf{R}(t) \), which evolves according to Newton’s equations of motion \( m\ddot{\mathbf{R}} = -\nabla V(\mathbf{R}, t) \). As is well known, as long as the external potential \( V(\mathbf{r}, t) \) can be represented by a quadratic form, the evolution of the center-of-mass coordinates of a many-particle system can be separated from the evolution of the internal degrees of freedom of the system. We can write the wave function as

\[
\psi(\mathbf{r}, t) = e^{i|\mathbf{P}(\mathbf{r} - \mathbf{R}) + S(t)/\hbar}\Phi(\mathbf{r} - \mathbf{R}, t),
\]

where \( \mathbf{P} = m\dot{\mathbf{R}} \). By substituting this form in the evolution equation (7) we obtain the usual expression for the action \( s \) as an integral over the local Lagrangian

\[
S = \int_0^t dt' \left[ \frac{1}{2m} \mathbf{P}(t')^2 - V(\mathbf{R}(t'), t') \right],
\]

and the equation for \( \Phi(\mathbf{r} - \mathbf{R}) \) in the center-of-mass coordinates becomes

\[
i\hbar \frac{\partial \Phi}{\partial t} = [H(t, \Phi) - V(\mathbf{R}) - (\mathbf{r} - \mathbf{R}) \cdot \nabla V(\mathbf{R})]\Phi,
\]

such that the values of the potential and its gradient at the central point \( \mathbf{r} = \mathbf{R} \) are eliminated. In this moving frame of reference we approximate the potential as quadratic (the next order in the Taylor series around \( \mathbf{r} = \mathbf{R} \)) in a volume occupied by the wave-packet

\[
V_c(\mathbf{r} - \mathbf{R}, t) \approx \frac{1}{2} \sum_j Q_j(t)(r_j - R_j)^2,
\]

where the quadratic potential has its axes aligned along the same axes of the initial trap, while the more general case of rotating axes is left for another work (see Ref. 33). From here on we transform into the center-of-mass coordinate system \( \mathbf{r} = \mathbf{R} \) in a volume occupied by the wave-packet

\[
\Phi(\mathbf{r}, t) = \frac{1}{\sqrt{\lambda_1 \lambda_2 \lambda_3}} \Phi_0 \left( \frac{x}{\lambda_1}, \frac{y}{\lambda_2}, \frac{z}{\lambda_3} \right) e^{i(\sum_j \alpha_j r_j^2 + \varphi)},
\]

where \( \Phi_0 \) is the wave function at time \( t = 0 \) that satisfied Eq. (10), and \( \alpha_j, \lambda_j \) and \( \varphi \) are time dependent and will be found below. The ansatz (13) was first used for the evolution of a BEC by Castin and Dum 31 and used by many authors since 33. Here we will derive the equations for the parameters that are valid for all the range of interaction strengths discussed above in Section II A including also higher modes of the harmonic potential in the initial trap in the non-interacting case. Such solutions for the latter case can serve as a part of the estimation of interferometric performance of an initial thermal distribution rather than a BEC.

By substituting the ansatz (13) into the left-hand side
of Eq. (7) and in the kinetic term we obtain
\[ \frac{i}{\Phi} \frac{\partial \Phi}{\partial t} = -\sum_j \left( \frac{i}{2\lambda_j} \dot{\lambda}_j + \frac{i\hbar}{\lambda_j} \frac{\partial^2 \Phi_0}{\partial \phi^2} + \alpha_j r_j^2 \right) - \dot{\phi} \] (14)
\[ -\frac{\hbar}{2m} \nabla^2 \Phi = -\frac{\hbar}{2m} \sum_j \left\{ \frac{1}{\lambda_j^2} \frac{\partial^2 \Phi_0}{\partial \phi^2} + \frac{4i\alpha_j r_j \partial \Phi_0}{\lambda_j} \right\} + 2(\alpha_j - 2\alpha_j r_j^2) \] (15),
where \( \partial_j \) denotes differentiation with respect to the argument \( r_j / \lambda_j \) of the function \( \Phi_0 \). By equating the terms proportional to \( r_j \partial \Phi_0 \) in the two equations we obtain
\[ \alpha_j = \frac{m \dot{\lambda}_j}{2h \lambda_j}. \] (16)

In the absence of interactions, where the initial state is a harmonic oscillator eigenstate separable into its Cartesian components, we may replace \( -(\hbar^2/2m)\partial^2 \Phi_0/\Phi_0 = \hbar \omega_j (n_j + 1/2) - \frac{1}{2} \hbar \omega_j^2 (r_j / \lambda_j)^2 \), where \( n_j \) is the eigenstate number. In the case of a BEC with atom-atom interactions we use the generalized TF approach and replace the first term in Eq. (14) by the expression in Eq. (2) with \( r_j \to r_j / \lambda_j \) (which coincides with the expression for non-interacting atoms in the ground state \( n_j = 0 \)). Similarly, we replace the interaction term in Eq. (5) by the expression in Eq. (3). By collecting the terms proportional to \( r_j^2 \) we obtain
\[ \dot{\lambda}_j = \frac{\hbar^2}{8m \sigma_\lambda^3 \lambda_j^3} + \frac{m \dot{\omega}_j^2}{2h \lambda_1 \lambda_2 \lambda_3 \lambda_j^3} - \frac{2\hbar}{m} \alpha_j^2 - \frac{1}{2h} Q_j \]
As \( \dot{\lambda}_j = (m/2h) \dot{\lambda}_j / \lambda_j - (2\hbar^2/m) \alpha_j^2 \) according to Eq. (16), we obtain a differential equation for \( \dot{\lambda}_j \)
\[ \ddot{\lambda}_j = \frac{\nu_j^2}{\lambda_j} + \eta \frac{\dot{\omega}_j^2}{\lambda_j} - \frac{Q_j}{m} \lambda_j \] (17)
where the coefficients \( \nu_j \) and \( \dot{\omega}_j \) are defined in Eq. (4) above. The first term in the right-hand side of Eq. (17) is responsible for wave-packet expansion due to position-momentum uncertainty, the second term is responsible for expansion due to the collisional repulsive force, and the third term is the external harmonic force (negative for \( Q_j > 0 \) and positive for \( Q_j < 0 \)). In the absence of interactions \( n_j \to \omega_j \) and \( \dot{\omega}_j \to 0 \), such that the second term vanishes. In the opposite TF limit \( \dot{\omega}_j \to \omega_j \) and \( \nu_j \to 0 \), such that the first term in Eq. (17) vanishes and we reproduce the result of the time-dependent TF approximation [31].

Finally by collecting the remaining terms in Eqs. (14) and (15), which do not depend on the coordinates, together with the coordinate independent of the interaction term in Eq. (8) we find
\[ \phi = \frac{1}{\hbar} \left[ \eta \mu_{\text{int}} \lambda_1 \lambda_2 \lambda_3 + \sum_j \hbar \nu_j \left( n_j + \frac{1}{2} \right) \right], \] (18)
where the first term represents the mean-field effective chemical potential of a single atom under the influence of all other atoms, whose density is decreasing with the \( \lambda_j \)'s, while the second term represents the kinetic energy, with \( n_j \) corresponding to the mode numbers in the case where the initial state is an eigenstate of a harmonic oscillator, and \( n_j = 0 \) in the case where the initial state is the ground state of either an interacting or non-interacting system of atoms.

C. Solutions for some simple situations

One of the simplest examples of wave-packet dynamics is free expansion \( [Q_j = 0 \text{ for } t > 0 \text{ in Eq. (17)]} \), when either the interactions are negligible \( (\dot{\omega}_j \to 0 \text{ and } \nu_j \to \omega_j) \), or when the interactions are dominant but the initial trap is cylindrically symmetric and elongated \( (\omega_x \ll \omega_\perp) \), where \( \omega_\perp \equiv \omega_y = \omega_z \). In the latter case for short enough times the expansion along the slow axis is negligible \( (\lambda_1 \approx 1) \) and the first and second terms in Eq. (17) merge to \( (\nu_j^2 + \dot{\omega}_j^2)/\lambda_j^3 = \omega_x^2 / \lambda_j^3 \). Both for free expansion of a non-interacting cloud (at any temperature) and for the expansion along the fast axis \( (j = 2, 3) \) of a BEC we obtain for the relevant axes
\[ \lambda_j(t) = \sqrt{1 + \omega_j^2 t^2}. \] (19)

A more specific scenario that may occur in atomic interferometry is an initially trapped BEC in a cylindrical trap (transverse frequency \( \omega_\perp \)) that is released from the trap and allowed to freely expand for some time \( t_0 \) until its size is scaled by \( \lambda_\perp(t_0) \equiv \lambda_0 \) and its rate of expansion is \( \dot{\lambda}_\perp(t_0) = \omega_\perp^2 t_0 / \lambda_0 \equiv \dot{\lambda}_0 \). Then the BEC is split into two wave-packets, each of them having \( \eta = 1/2 \) of the initial number of particles. In this case the expansion of each of the wave-packets continues with a reduced rate, \( \omega_j^2 \to \eta \omega_j^2 + \nu_j^2 \). The solution for the expansion as a function of the time \( t \) since the splitting time \( t_0 \) is then given by
\[ \lambda_\perp(t) = \sqrt{(\lambda_0 + \lambda_0 t)^2 + \lambda_0^{-2}(\eta \omega_j^2 + \nu_j^2) t^2}. \] (20)

A simple case of this expansion, where we take the TF limit \( \dot{\omega} \to \omega \) and ignore the initial expansion time \( t_0 \), so that \( \lambda_1 \approx \sqrt{1 + \eta \omega_j^2 t^2} \), will be taken as a test case for phase diffusion in Section IV.

Next, we study the process of momentum splitting into a waveguide and compare the results of the wave-packet propagation model to the results of a numerical GP calculation. We start by studying the splitting process itself (Fig. 3) and then examine the long time evolution after splitting (Fig. 4). We consider a BEC prepared in a cylindrical trap with the same parameters of Fig. 1. The longitudinal potential is ramped down quickly to form a waveguide potential. Then quick Bragg pulses imprint a sinusoidal density grating that represents a superposition of two opposite momenta.
\[ \pm 2\hbar k, \text{ where } k \text{ is the wave-vector of the Bragg laser.} \]

Here we ignore the specific atom-light interaction scheme and take it as a black box generating a transformation \( \psi_0(r) \rightarrow \psi_0(r)\left[e^{2ikx} + e^{-2ikx}\right]/\sqrt{2} \). This causes a separation of the two wave-packets, which propagate with velocities \( \pm 2\hbar k/m \). The atomic density within the interference fringe pattern that exists at the overlap region between the two wave-packets before they separate is responsible for an enhanced collisional repulsion force along the transverse directions, which is larger than what would be expected if the density was uniform along \( \hat{x} \). We therefore model the time-dependent repulsion strength in the transverse and longitudinal directions by two different time-dependent atomic fraction factors \( \eta \) in Eq. (17), such that the transverse effective fraction \( \eta_\perp \) just after the splitting is larger than 1, while the effective factor \( \eta_x \) in the longitudinal direction is smaller than 1. Then during the separation the effective factors reduce until they reach \( \eta_\perp = \eta_{\text{sep}} = 1/2 \) when the two wave-packets completely separate so that each of them acts as an independent wave-packet with half the original number of atoms. Our simple model for these factors \( \eta_j(t) = \eta_{\text{sep}} + \delta\eta_j e^{-t/t_j} \) is shown in Fig. 3 to provide a good agreement with the GP calculation (see caption for the parameter values). The transverse size of the BEC first grows due to the enhanced repulsive force in this direction but then it shrinks due to the reduced effective repulsive potential relative to the initial repulsive force (with \( \eta = 1 \)) that balanced the confining harmonic potential.

The subsequent evolution after full separation is characterized by expansion in the longitudinal direction along the waveguide and oscillations of the cloud size in the transverse direction, as demonstrated in Fig. 4. To understand the oscillations of \( \lambda_x \) in the waveguide potential, let us examine Eq. (17) in the case where evolution along the longitudinal axis is much slower than the evolution along the radial axis. In this case the equations of motion for \( \lambda_\perp \) can be written as

\[
\ddot{\lambda}_\perp = \left( \nu_\perp^2 + \omega_\perp^2 \frac{\eta}{\lambda_x} \right) \frac{1}{\lambda_\perp^3} - \omega_\perp^2 \lambda_\perp, \tag{21}
\]

where \( \lambda_x(t) \) is assumed to vary on a time scale that is much longer than the time scale determined by the frequency \( \omega_\perp \). In this case Eq. (21) is equivalent to the classical equation of motion for a massive particle in a potential \( V(q) = a/q^2 + \frac{1}{2} ma^2 q^2 \). This potential has a minimum at \( q_0^2 = 2a/m\omega_x^2 \) and the frequency at the bottom of the potential is \( \omega_0^2 = \partial^2 V/\partial q^2 \big|_{q=q_0} = 6a/mq_0^2 + \omega_x^2 = 4a^2 \). It follows that the oscillations of \( \lambda_\perp \) have a frequency that is twice the trap frequency and their center is given by

\[
\ddot{\lambda}_\perp = \left[ \left( \frac{\nu_\perp^2}{\omega_\perp} \right)^2 + \left( \frac{\omega_\perp}{\nu_\perp} \right)^2 \frac{\eta}{\lambda_x} \right]^{1/4} \approx \left( \frac{\eta}{\lambda_x} \right)^{1/4}, \tag{22}
\]

with the right expression valid in the case where the guiding frequency is the same as before release and splitting and when the TF approximation is valid for the radial axis. In this last case the equation for the longitudinal scaling becomes \( \lambda_x \approx \eta_\perp \omega_\perp (0) \lambda_x^2 = \sqrt{\eta_\perp} \omega_\perp (0) \lambda_x^2 \)

By analogy to a classical mass in a potential \( V(q) = 2a/\sqrt{\bar{q}} \), where \( a = 2\sqrt{\eta_\perp} \omega_\perp(0) \), we find that after a long time \( t \gg \omega_\perp(0)^{-1} \) the longitudinal cloud size expands with a constant rate \( \dot{\lambda}_x = 2\eta^{1/4} \omega_\perp(0) \). The shrinking of the cloud size in the transverse direction continues until it reaches the uncertainty limit where \( \sigma_\perp(t) = \sigma_\perp(0) \lambda_\perp \rightarrow \hbar \pi /2m\omega_\perp \), as can be verified from Eq. (22).

The long-time dynamics of split wave-packet expansion into a waveguide is demonstrated in Fig. 3 for the same parameters that were used in Fig. 2 except that here we ignore the details of the splitting process at short times and set \( \eta = 1/2 \) for \( t > 0 \). These details are found to have a small effect on the evolution at long times \( t \gg m\sigma_\perp/2\hbar k \). We also examine the effect of trap release
expected to be set them to zero (τ value for abrupt release (λ by Eq. (22) (dashed line). (b) Longitudinal scaling factor of the release time of the longitudinal trapping potential is ramped down gradually as ω⊥(t) = ω⊥(0)e−t/τ⊥ and the splitting is performed at t0 = 2τ⊥. (a) The transverse scaling factor λ⊥(t) = σ⊥(t)/σ⊥(0) (for τ⊥ = 1 ms) oscillates due to the sudden decrease of the atom-atom repulsive force by a factor of 2 after splitting. The oscillation with frequency 2ω⊥ is around λ⊥ given by Eq. (22) (dashed line). (b) Longitudinal scaling factor λ⊥(t) = σ⊥(t)/σ⊥(0) for different release times. The asymptotic expansion rate is inversely proportional to the initial cloud size and therefore it reduces when the cloud is allowed to expand slowly within a time τ⊥ before full release. For abrupt release (ω⊥τ⊥ ≪ 1) the asymptotic expansion rate is expected to be λ⊥ ~ 0.42 ms−1 and found to be 80% of this value for τ⊥ = 1 ms and less for longer release times, approaching λ⊥ ~ 1/τ⊥ for ω⊥τ⊥ ≫ 1. The long-time evolution is not sensitive to the splitting parameters δη⊥ of Fig. 3 and Fig. 4 so we have set them to zero (i.e., η⊥ = 1/2 for t > t0). See section IV and Fig. 4 for implications of this evolution.

FIG. 4. Long-time cloud size evolution during release, splitting and expansion into a matter-waveguide. The procedure and parameters are as in Fig. 3 except that the longitudinal trapping potential is ramped down gradually as ω⊥(t) = ω⊥(0)e−t/τ⊥ and the splitting is performed at t0 = 2τ⊥. (a) The transverse scaling factor λ⊥(t) = σ⊥(t)/σ⊥(0) (for τ⊥ = 1 ms) oscillates due to the sudden decrease of the atom-atom repulsive force by a factor of 2 after splitting. The oscillation with frequency 2ω⊥ is around λ⊥ given by Eq. (22) (dashed line). (b) Longitudinal scaling factor λ⊥(t) = σ⊥(t)/σ⊥(0) for different release times. The asymptotic expansion rate is inversely proportional to the initial cloud size and therefore it reduces when the cloud is allowed to expand slowly within a time τ⊥ before full release. For abrupt release (ω⊥τ⊥ ≪ 1) the asymptotic expansion rate is expected to be λ⊥ ~ 0.42 ms−1 and found to be 80% of this value for τ⊥ = 1 ms and less for longer release times, approaching λ⊥ ~ 1/τ⊥ for ω⊥τ⊥ ≫ 1. The long-time evolution is not sensitive to the splitting parameters δη⊥ of Fig. 3 and Fig. 4 so we have set them to zero (i.e., η⊥ = 1/2 for t > t0). See section IV and Fig. 4 for implications of this evolution.

time τ⊥ on the expansion dynamics. The longitudinal frequency is ramped down as ω⊥(t) = ω⊥(0)e−t/τ⊥ and the splitting is performed at t = t0 = 2τ⊥. The asymptotic longitudinal expansion rate is proportional to the initial longitudinal trap frequency if τ⊥ < ω⊥−1 and to the inverse of the release time τ⊥−1 if the release time is longer than the initial trap period. These results have significant implications on phase diffusion determining the coherence of the interferometry, as we discuss in Section IV.

Before concluding this section we note that calculations of BEC splitting in a waveguide have already been done in the past [20–22]. These calculations, which involved a comparison between a GP calculation and two limits of atom-atom interaction: TF approximation and perturbation theory, aimed at understanding an experiment where the longitudinal potential was not turned off so that the atomic clouds moved in a harmonic potential [6–7]. Our method is suitable for efficient calculations of dynamics in an interferometer in a broad range of possible applications such as a Sagnac interferometer in a ring waveguide in all interaction regimes including a non-interacting thermal cloud or a BEC in either the TF limit or weak interactions.

III. COHERENCE OF A SPATIAL TWO-STATE INTERFEROMETER

In many interferometric schemes the particles traveling along the two arms have two orthogonal internal states. For example, the beam splitters in the Ramsey-Bordé and Kasevich-Chu configurations are based on entanglement between the internal state of the atoms and the momentum transferred to the atoms by the laser beams. An archetype of such an interferometric scheme, which was envisioned in the early days of quantum mechanics is based on the Stern-Gerlach effect and uses a superposition of an atom in two spin states for splitting it into two paths [14–17]. In contrast to interferometers based on the double-slit scheme, where spatial interference fringes are observed directly, this kind of interferometer has the advantage that it may reveal the phase difference accumulated along the two paths even if the interference fringes cannot be observed (for example, due to limited optical imaging resolution) and the signal is the internal state: spin orientation or population if an appropriate internal rotation is applied, such as in a Ramsey scheme. The challenge of such interferometry, and specifically that of the final recombining beam splitter, is to bring the two wave-packets at the recombination beam-splitter accurately to the same position and momentum so that the two arms are distinguishable only by the internal state. In the framework of the Stern-Gerlach interferometer (SGI), erasing the entanglement between the spatial degrees of freedom and the internal degrees of freedom was considered to be a very difficult task that requires accurate manipulation of magnetic fields that can be hardly achieved by macroscopic experimental devices [20–22] (so called the “Humpty-Dumpty effect”). While matter-wave interferometers based on Raman or Bragg momentum transitions utilize the high momentum accuracy of laser pulses, which is independent of the laser intensity and pulse duration, it is still interesting to investigate the performance of Stern-Gerlach interferometers utilizing current technologies such as atom chips [14–17]. While the original theoretical works [20–22] that investigated the required accuracy of such interferometers used simplified models for the calculation, a more recent work has investigated this effect more thoroughly in the context of light-pulse interferometers [23]. Here we develop a more general theory based on the general method of wave-packet evolution presented in Sec. II. This theory will be relevant to non-interacting thermal atomic clouds as well as BEC clouds with any strength of atom-atom interaction, provided that the interactions during splitting and recombination can be absorbed into parameters of the theory as in the example given in Fig. 5. The theory will enable practical calculation of interferometric perfor-
mance in systems such as the SG interferometers recently realized [15][17], but also for other spatial interferometers involving transfer into different internal atomic states or momentum transfer within the same internal state, such as in Bragg light pulses, as we show below.

Let us consider the state of the interferometer just before the final recombination step (π/2, Raman or Bragg pulse) as given by a superposition of two wave-packets ψ1(r,t) and ψ2(r,t) with corresponding central phases φ1 and φ2 accumulated during the propagation along the two arms, such that the two wave-packets are distinguishable by their corresponding internal states |1⟩ and |2⟩. The total wave function is then

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}} [\psi_1(r,t)e^{i\phi_1}|1\rangle + \psi_2(r,t)e^{i\phi_2}|2\rangle]. \quad (23)$$

In the case of a SGI the states |1⟩ and |2⟩ represent two spin states, while in the case of a Ramsey-Bordé or Raman interferometer they may represent two hyperfine states attached to an accurate momentum state induced by the two arms. The final recombination is done by projecting the state into a superposition (|1⟩ + |2⟩)/√2, which may be implemented by performing a π/2 pulse and then probing the population in an internal eigenstate. The result of this operation is the population (or probability)

$$P = \frac{1}{2} [1 + V \cos(\phi_1 - \phi_2 + \delta \varphi)], \quad (24)$$

where we have assumed that the wave-packets are normalized (∫d³r |ψ⟩² = 1). The visibility V and additional phase is given by

$$V e^{-i\delta \varphi} = \langle \psi_1 | \psi_2 \rangle \equiv \int d³r \, \psi_1^*(r,t)\psi_2(r,t). \quad (25)$$

Note that the same result is obtained for an interferometer that involves Bragg momentum transfer within a single internal atomic state, as the wave function before the final Bragg pulse can be written as ψ₁(r,t) = e^[i(k₁·r + φ₁)] + ψ₂(r,t) and the Bragg pulse performs a momentum transition ∆k = ±(k₁ - k₂) which mixes between the two momentum states, followed by a selection of one final momentum state, which is equivalent to the internal state projection considered here.

The reduction of visibility due to imperfect overlap [Eq. (25)] occurs if the two wave-packets are displaced in position or momentum or if they differ in size or in the internal spatially dependent phase. Here we will provide an explicit expression for this overlap integral in the special case where the wave-packets at the recombination time have the same size in position and momentum space but differ in their central position or momentum. The expressions that were derived in previous theoretical studies [20][22] were based on a Gaussian wave-packet approximation and neglected wave-packet expansion and assumed that the target output wave-packets are similar to the wave-packet at the input port. Here we generalize the treatment to BEC or thermal clouds with an arbitrary target wave function. Within the wave-packet evolution theory of Section II one can also treat the situation where the two wave-packets have different sizes, but here we do not explicitly derive expressions for this case, which are expected to be more cumbersome.

Note that the overlap integral in Eq. (25) is time-independent as long as the wave-packets evolve in free space or under the influence of internal state-independent potential. This kind of evolution may be represented by the action of a state-independent unitary evolution operator $\hat{U}$, such that $\psi_1(r,t+\tau) = \hat{U}(\tau)\psi_1(r,t)$, so that in Eq. (25)

$$\langle \psi_1^*(t+\tau)|\psi_2(t+\tau)\rangle = \langle \psi_1^*(t)|\hat{U}^†\hat{U}|\psi_2(t)\rangle = \langle \psi_1^*(t)|\psi_2(t)\rangle. \quad (26)$$

This implies that in such an interferometer, where the two paths are combined in position and momentum before internal state projection, the result of measurement does not depend on the time of this projection and the overlap integral may be evaluated at any time. This does not apply when the projection operation itself involves the spatial degrees of freedom as in beam splitters involving Raman or Bragg transitions.

Let us consider the overlap integral between two wave functions having the form of Eqs. (9) and (13) with different positions of the center coordinates (differing by $\delta R = R_1 - R_2$) and momenta (differing by $\delta P = P_1 - P_2$) but with the same initial wave function $\Phi_0$ at $t = 0$ and the same phase curvatures $\alpha_j(t)$ and scaling factors $\lambda_j(t)$

$$\langle \psi_1|\psi_2\rangle = e^{-i\delta \varphi} \int d³x \, e^{-i\delta P \cdot x/h} \Phi_0(x - \frac{1}{2}\delta X)\Phi_0(x + \frac{1}{2}\delta X), \quad (27)$$

where we have transformed the integration coordinates to the scaled coordinates $x_j = (r_j - R_j)/\lambda_j$ centered at the center of mass $R = \frac{1}{2}(R_1 + R_2)$ of the two wave-packets. The normalized center-to-center displacement and the scaled effective momentum difference are

$$\delta X_j = \delta R_j/\lambda_j \quad \delta P_j = \lambda_j \delta P_j - m\lambda_j \delta R_j. \quad (28)$$

Note that Eq. (27) has the same form as Eq. (12) of Ref. [23], except that here we give the explicit scaling of the effective momentum and position displacement with the scaling factors $\lambda_j$, which are relevant to wave-packet dynamics with or without non-linear atom-atom interactions, as derived in Section II above.

The additional phase difference in Eq. (27), beyond the phases $\phi_1$ and $\phi_2$ that are related to the wave-packet trajectories ($\phi_a = S_a/h$ for $a = 1, 2$), is

$$\delta \varphi = \varphi_1 - \varphi_2 - P \cdot \delta R/h, \quad (29)$$

where $\varphi_a$ (for $a = 1, 2$) is given by Eq. (15) for each wave-packet and $P = \frac{1}{2}(P_1 + P_2)$ is the center-of-mass momentum of the two wave-packets. It contains the propagation phase difference due to the motion of the wave-packets’ center coordinates, the internal phase difference, and a
phase difference due to the displacement of the two wave-packet centers. The last term of the phase difference satisfies the requirement of invariance under free-space evolution, where \(\delta S(\tau) = (P^2 - P_0^2)\tau/2m = P \cdot \delta v\), is exactly opposite to the change of \(-P \cdot \delta R\) over the time \(\tau\).

If the two wave-packets are displaced only by momentum (\(\delta R = 0\)) then the visibility in Eq. (27) is nothing but a Fourier transform of the probability density of the initial wave function \(|\Phi_0(x)|^2\), such that if the widths of the spatial density \(|\Phi_0(x)|^2\) are \(\sigma_j\) we have \(V \approx \exp\left(-\frac{1}{2} \sum_j (\lambda_j \sigma_j^2 P_j/h)^2\right)\). If, on the other hand, the two wave-packets are only displaced in position and have a minimal wave-packet size at the time of recombination (\(\delta P = 0, \lambda_j = 0\)), then the visibility is \(V \approx \exp\left(-\frac{1}{2} \sum_j (\delta R_j/2\sigma_j \lambda_j)^2\right)\).

In the more general case of both position and momentum displacement we apply the principle of conservation of overlap of Eq. (26) and operate on both wave functions \(\Phi_{\pm}(x) = e^{\pm iP^2x/2\hbar} \Phi_0(x \pm \delta X/2)\) with a unitary transformation \(U(\theta) = \prod_j \exp\left(-i\theta_j [(\sigma_j P_j/h)^2 + (\delta_j/2\sigma_j)^2]\right)\), where \(\theta_j = 0\) are the momentum operators conjugate to the position operators \(\delta_j\). The operation of \(U\) is a phase space rotation induced by a Hamiltonian of a harmonic oscillator with frequencies \(\nu_j = \hbar / 2m \lambda_j^2\) [see Eq. (1)]. As a unitary operation this rotation should not change the overlap integral, as shown in Eq. (26). We then choose \(\theta_j\) such that \(\delta X_j = \cos \theta_j \delta X_j - \sin \theta_j \delta P_j/m \nu_j = 0\), while \(\delta P_j = \cos \theta_j \delta P_j + m \nu_j \sin \theta_j \delta X_j = \sqrt{\delta P_j^2 + (\hbar \delta X_j/2\sigma_j)^2}\).

After this transformation the visibility in Eq. (27) reduces to a Fourier transform of the initial density, but this time with the transformed scaled momentum. This is a general result that is exact for any initial wave-packet shape that satisfies the scaling approximation of Section III. In the Gaussian approximation we therefore obtain

\[
V = e^{-\frac{1}{2} \sum_j[(\sigma_j \delta P_j/h)^2+(\delta X_j/2\sigma_j)^2]} \quad (30)
\]

with \(\delta X_j\) and \(\delta P_j\) defined above in Eq. (28).

Now let us consider a more general initial state: a mixed state that can be represented by a density matrix

\[
\rho_0(r, r') = \sum_n W_n \Phi_n(r) \Phi_n^*(r') \quad (31)
\]

where \(W_n\) are weights of the different wave functions. The interferometric process splits each wave function into a superposition \(\Phi_n \rightarrow \frac{1}{\sqrt{2}} (\psi_n^{(1)} + \psi_n^{(2)})\), where \(\psi_n^{(l)}(r, t = 0) = \Phi_n(r)\) for \((l = 1, 2)\) but then change in time independently for some time until the recombination of the two beams. The final density is then given by

\[
n_f(r) = \rho_f(r, r) = \frac{1}{2}(\rho_f^{11} + \rho_f^{22} + \rho_f^{12} + \rho_f^{21}) \quad (32)
\]

where the visibility is given by the magnitude of the two last terms, namely

\[
V = \left| \int d^2r \rho_f^{12}(r, r) \right| = \sum_n W_n V_n e^{i\delta \varphi_n} \quad (33)
\]

where

\[
V_n e^{i\delta \varphi_n} = \int d^2r \psi_n^{(1)}(r, t_f) \psi_n^{(2)*}(r, t_f). \quad (34)
\]

As above, we assume that the initial wave functions \(\Phi_n\) are the eigenstates of an initial harmonic oscillator potential (where \(n\) is to be understood as a vector \((n_1, n_2, n_3)\) along the potential axes). This assumption, together with the simplifying assumption that the evolution of the scaling factors \(\lambda_j\) is the same for both interferometer arms, leads to the simple result that the phases \(\delta \varphi_n\) do not depend on \(n\) and the visibility can be taken simply as \(V = \sum_n W_n V_n\). Under these assumptions we can generalize Eq. (27) together with the phase space transformation leading to Eq. (30) to obtain

\[
V = \int d^2x e^{i\sum_j \sqrt{\delta P_j^2 + (\hbar \delta X_j/2\sigma_j)^2}} \rho_0(x, x) \quad (35)
\]

Then if we take the initial atomic cloud in the trap to have a Gaussian shape with widths \(\Delta_j\) along the three trap axes we obtain by making the Fourier transform of the cloud density \(\rho_0(x, x)\)

\[
V = \exp \left[-\frac{1}{8} \sum_{j=1}^3 \left(\sum_{j=1}^3 (\Delta_j \delta P_j/h)^2 + (\delta X_j/\lambda_j)^2\right)\right], \quad (36)
\]

where \(\lambda_j = 2\sigma_j^2/\Delta_j = \hbar/\Delta p_j\) is a coherence length equal to the inverse of the momentum width of the atomic cloud in the trap.

In Fig. 5 we use the above equations to calculate the spin coherence (visibility) of a Stern-Gerlach interferometer when the recombination of the wave-packets that were initially split by a gradient pulse is not accurate. The interferometer sequence consists of four consecutive gradient pulses of the same duration (see caption for parameters) and inaccurate recombination is simulated by changing the duration of the last two pulses symmetrically or antisymmetrically. For this interferometer sequence, the ratio between the contributions of position mismatch and momentum mismatch to the reduction of visibility is \((\sigma_z \delta P_z/\hbar)/\Delta Z/2\sigma_z \sim \omega_z T/2\), where \(\omega_z\) is the trap frequency in the \(z\) (splitting) direction and \(T\) is the pulse duration. For the parameters of this example \(\omega_z T < 1\) such that the interferometer is much sensitive to momentum mismatch.

The example presented in Fig. 5 does not demonstrate the full novelty of the theory presented here and similar results could be obtained by methods presented in Ref. [23], which treats either Gaussian wave-packets or BEC in the time-dependent TF approximation. These
approximations for the thermal state or a BEC, respectively, are quite suitable for the present example. However, we emphasize that our treatment has the advantage that it unifies both cases into the same formalism and enables practical and easy predictions for the intermediate case where the atomic cloud does not satisfy the TF approximation. In addition, we provide a general simple expression [Eq. (36)] that permits an estimation of interferometric contrast based on phenomenological lengths: cloud size and coherence length.

At the end of this section let us note that the law of conservation of visibility [Eq. (26)] can be explicitly verified for the visibility in Eqs. (30) and (36) by checking the time derivative \( \frac{d}{dt}[\delta P^2_m + (m\nu_j\delta X_j)^2] \) vanishes when the dynamics of the evolution of \( \delta P_j, \delta R_j \) and \( \lambda_j \) is taken into account. We find that this conservation explicitly holds for the case of free evolution and evolution in a quadratic potential, i.e., when \( \delta R_j = \delta P_j/m, \delta P_j = -m\omega^2 \delta R_j \) and \( \lambda_j \) satisfies Eq. (17) with \( \eta = 0 \). However, the conservation of overlap is not explicitly satisfied in the presence of collisional interactions (\( \eta \neq 0 \)), as our approximation in Section IV does not take into account the interaction between different wave-packets. This means that the effective potential is different for the two wave-packets, being proportional to the atomic density that is different for the two wave-packets.

IV. PHASE DIFFUSION OF PROPAGATING WAVE-PACKETS

The question of whether a BEC has a well-defined phase and how long can its coherence be maintained was controversial since the realization of atomic BECs and the first observation of its coherence signal [24, 26, 39]. The main stream of theoretical study of intrinsic static and dynamic properties of a split BEC in the presence of atom-atom interactions concentrates on a two-mode quantized model, which is often called the “Two-mode Bose-Hubbard model”, which is equivalent under some assumptions to a model of a Josephson junction. Many theoretical and experimental works used such a configuration to study tunneling oscillations of two condensates through a potential barrier [40–48], while some others also dealt with coherence and its dynamics [25, 29, 49, 50]. While most of these works introduce phenomenological parameters into the two-mode model, only a few of them attempt to calculate these parameters from first principles [25, 49, 51]. However, all of these models use the adiabatic approximation, in which the spatial wave function of the BEC is in the ground state, namely, the steady-state solution of the Gross-Pitaevskii equation.

The dynamics of splitting an initially trapped BEC into two completely non-overlapping parts has two different aspects. From the spatial aspect the splitting may be adiabatic, such that the two BEC parts after splitting occupy the lowest energy state (or a steady-state) of two traps formed after splitting, or it may be non-adiabatic, such that the two parts continue to evolve or even propagate in the new potential. From the aspect of the internal many-body state, the dynamics also depends on the duration of the splitting process, but with different time scales. In the limit of very slow splitting the many-body state follows an eigenstate of the many-particle Hamiltonian, whose eigenstates after full separation are Fock states (number states) with a fixed number of particles in each side and no phase relation between the two BECs. In the other limit of fast splitting the number of particles in each side after splitting has a bi-
nominal (or approximately a Poissonian) distribution. As each number state has a different repulsive energy, each component in the superposition of many number states accumulates a different phase so that the total phase becomes uncertain and the coherence is lost after some time of evolution. This effect, which is termed phase diffusion due to atom-atom interaction, is crucial for any kind of interferometry with a BEC and limits the coherence time \[ \text{[13 25] 26 29 38 49]}. When splitting with intermediate duration both phase diffusion and number-squeezing effects take place \[ \text{[13 45] 48]}. 

In this section we develop a theory of phase diffusion in a split BEC for a general case that includes non-adiabatic evolution during and after splitting. We start from the basic many-body theory and develop a practical method with time-dependent parameters based on the wave-packet evolution method presented in Section \[ \text{[11]}. \]

We take into account not only the central phase of each BEC wave-packet, which is given above in Eq. \[ \text{[18]} \] but also spatial features of the wave-packets and find that these features may also affect the rate of phase diffusion. We then apply the theory to a free-space interferometer as well as an interferometric scenario of splitting a BEC into a waveguide where it propagates and expands along one axis while being confined in the transverse axis.

A. Evolution of the many-particle state

We consider coherent splitting of an N-particle BEC into two parts: “left” and “right”. We assume that the splitting is gentle enough such that the two parts are represented by smooth single-particle wave functions \( \phi_L(r,t) \) and \( \phi_R(r,t) \). If the splitting time is short enough the atom-atom interactions would not have time to affect the spatial shape of the wave function beyond the usual mean-field effect that can be handled by the GP equation. However, during subsequent long-time evolution the wave-packet shape and phase may depend on the exact number of particles in each side, which is not exactly \( N/2 \) due to number uncertainty. In general, the many-body wave function has the form

\[
\Psi(r_1, \ldots, r_N, t) = \sum_{k=0}^{N} c_k \Psi_{k,N-k}(r_1, \ldots, r_N, t), \tag{37}
\]

where \( c_k \) the amplitudes for configurations with \( k \) particles in the left and \( N-k \) particles in the right side, and the many-particle wave function of each configuration \( \Psi_{k,N-k} \) with is in general a solution of the many-particle problem. If the particle-particle interactions are not too strong we can use a mean-field approximation where the configuration wave function is a direct product of \( k \) single particle “left” wave functions and \( N-k \) “right” wave functions

\[
\Psi_{k,N-k}(r_1, \ldots, r_N, t) \approx e^{-i\chi_{k,N-k}(t)} \times \\
\times \mathcal{S}\left\{ \prod_{i=1}^{k} \prod_{r=k+1}^{N} \phi_{L}^{(k)}(r_i, t) \phi_{R}^{(N-k)}(r_r, t) \right\}. \tag{38}
\]

Here \( \chi_{k,N-k}(t) \) is a global phase of the configuration, \( \mathcal{S} \) is a symmetrization operator for bosons and the wave functions \( \phi_{L}(r,t) \) and \( \phi_{R}(N-k)(r,t) \) are the solutions of the time dependent Gross-Pitaevskii (GP) mean-field equations for the left and right wave-packets, each being a BEC with \( k \) particles and \( N-k \) particles, respectively. If the right and left wave-packets do not overlap in space the GP equations read

\[
\frac{i\hbar}{\partial t} \phi^{(n)}_{a} = \hat{H}_{GP}^{(n)} \phi^{(n)}_{a}(r,t), \\
\hat{H}_{GP}^{(n)} = -\frac{\hbar^2}{2m} \nabla^2 + V_a(r,t) + gn|\phi^{(n)}_{a}(r,t)|^2, \tag{39}
\]

where here and in what follows \( n \) and \( a \) are dummy indices representing the number of particles and the interferometer arms (side) label, respectively. Here \( n = k \) for \( a = L \) is the number of particles in the left and \( n = N-k \) for \( a = R \) is the number of particles in the right. In the GP Hamiltonian \(-\hbar^2 \nabla^2/2m + V_a \equiv H_0^{(n)} \) are the single-particle kinetic and potential energies, which may be different for left and right particles, and the last term is due to the mean-field repulsive atom-atom potential, as in Eq. \[ \text{[6]} \]. At time \( t = 0 \), just after splitting, we assume that \( \Psi^{(n)}_{a}(r,t=0) = \Psi^{(N/2)}_{a}(r,0) \) does not depend on the number of particles and the wave functions for different \( n \) start to differ only after a long time of propagation. For example, if the net effect of splitting is to give the initial wave function \( \Psi_0 \) opposite momentum kicks \( \pm h \mathbf{K} \), then just after the splitting \( \Psi_{\pm}(r,0) = e^{\pm i\mathbf{K} \cdot r} \Psi_0 \). Dependence on the number of particles in the left and right (corresponding to \( -\mathbf{K} \) and \( +\mathbf{K} \)) starts to develop at long times.

During the splitting and before the two wave-packets are completely separate one may need to add another term to the GP equations \[ \text{[39]} \] to account for the mutual interaction between the two wave-packets. This part of the evolution must be treated separately for each specific situation and may be neglected only in the case of a quick wave-packet separation or in the other limit of adiabatic splitting in a double well, where the details of the splitting process are unimportant as far as the spatial shape of the wave function is concerned (but are important in the many-body aspect \[ \text{[13 25]} \] ). In the latter case the wave functions \( \phi_{L}^{(k)}(r,t) \) and \( \phi_{R}^{(N-k)}(r,t) \) for each configuration are the single-particle ground states in each well, which are the steady-state lowest energy solutions of the GP equation for any given numbers \( k, N-k \). In this case the initial state and the history of the splitting process are not important for the evolution of each configuration after separation.
In contrast to the spatial shape of the wave-packets, the structure of the many-body state, represented by the coefficients $c_k$ in Eq. (37), is sensitive to the details of the splitting process. In quick splitting these coefficients represent a binomial number distribution

$$P_k = |c_k|^2 = \frac{1}{2^N} \binom{N}{k}$$

(40)

of width $\Delta_n \approx \sqrt{N/2}$ around $k = N/2$. In the opposite limit of very slow splitting, the many-body state follows adiabatically the ground state and ends up after splitting with a state where exactly $N/2$ particles occupy each arm ($c_k = \delta_{k,N/2}$ if $N$ is even). In this case no atom is in a superposition of the two arms and the phase between the two arms is not defined. In the intermediate case of non-adiabatic splitting with large $N$ we may approximate the number distribution as Gaussian such that

$$c_k \approx (\sqrt{2\pi}\Delta_n)^{-1/2}e^{-(k-N/2)^2/4\Delta_n^2}.$$ 

(41)

The configuration phase $\chi_{k,N-K}(t)$ in Eq. (38) may be derived from the Schrödinger equation for the many-body state $\hbar \partial \Psi / \partial t = \mathcal{H}_N \Psi$, where the many-particle Hamiltonian is

$$\mathcal{H}_N = \sum_{\pi=1}^{N} \left[ -\frac{\hbar^2}{2m} \nabla^2 + \hat{V}(\mathbf{r}_\pi, t) + \sum_{q<p} U(\mathbf{r}_q - \mathbf{r}_p) \right],$$

(42)

where the external potential $\hat{V}$ may depend on internal degrees of freedom, if attached to the interferometer arms, and $U(\mathbf{r} - \mathbf{r}')$ is the inter-particle potential, which we usually approximate for slow collisions by $U(\mathbf{r} - \mathbf{r}') = g\delta(\mathbf{r} - \mathbf{r}')$. If we assume no overlap between $\Phi_L$ and $\Phi_R$ then the Schrödinger equation yields, when using Eq. (39) and integrating over all coordinates,

$$\frac{\hbar}{i} \frac{\partial \chi_{k,N-k}}{\partial t} = (\langle \mathcal{H}_N \rangle)_{(k,N-k)} - k\mu_L^{(k)} - (N-k)\mu_R^{(N-k)},$$

(43)

so that the rate of change of the configuration phase $\chi_{k,N-k}$ is the difference between the many-body energy expectation value $\langle \mathcal{H}_N \rangle$ (involving integration over all $N$ sets of coordinates) for the configuration and the sum of all single-particle energy expectation values

$$\mu_a^{(n)} = \int d^3r |\Phi_a^{(n)}(\mathbf{r})|^2 \hat{H}_{GP}^{(a,n)} \Phi_a^{(n)}(\mathbf{r}).$$

(44)

Here we have assumed that the two wave-packets do not overlap in space so that integrals involving a different number of left and right wave functions vanish. Note that in Eq. (43) the single-particle parts of the many-body Hamiltonian in Eq. (41) cancel with the collisionless terms following from $\hat{H}_{GP}^{(a,n)}$ of Eq. (39) and we are left with

$$\dot{\chi}_{k,N-k} = -\frac{g}{\hbar} \int d^3r \left[ (k(k-1)/2) \left| \Phi_L^{(k)} \right|^4 + \frac{(N-k)(N-k-1)}{2} \left| \Phi_R^{(N-k)} \right|^4 \right]$$

$$= -\frac{1}{2\hbar} k\mu_L^{(L,k)} + (N-k)\mu_R^{(R,N-k)}$$

(45)

where $\mu_{\text{int}}^{(a,n)}(t)$ are the interaction potentials of the two arms [given in terms of the initial chemical potential as $\mu_{\text{int}}(t) = \eta \mu_{\text{int}}(0)/\lambda_1 \lambda_2 \lambda_3$ in Eq. (18)]. When comparing this to the first term (interaction term) of Eq. (18) we obtain

$$\chi_{k,N-k} = -\frac{1}{2} \left[ k\varphi_{\text{int}}^{(L,k)} + (N-k)\varphi_{\text{int}}^{(R,N-k)} \right],$$

(46)

where $\varphi_{\text{int}}^{(a,n)}$ is the interaction part of the single-particle wave-packet center phase of atoms in the two arms with corresponding particle numbers. When the global configuration phase is added to the single wave-packet phases the result is a contribution of a number-dependent phase $\frac{1}{2} \varphi_{\text{int}}^{(k)}$ and $\frac{1}{2} \varphi_{\text{int}}^{(N-k)}$ from each of the wave-packets in the configuration.

The global configuration phase in Eqs. (45) and (46) together with the phase due to the solution of the mean-field equations for the single-particle wave functions $\Phi_a^{(n)}$ in the configuration sums up to an interaction phase proportional to $n/2$ — half of the number of particles in each arm times the mean field interaction of each particle $g(n-1)\Phi_a^{(n)}|^2$. This reproduces correctly the interaction energy of the configuration, which is proportional to the number of particle pairs $a(n-1)/2$ in the two arms.

**B. Interferometric visibility**

We now consider an interferometric scheme where the two arms of the interferometer are brought back to spatial overlap after propagation. Here we consider an interferometer whose interference signal is based on a spatial interference fringe pattern that may be probed either by direct imaging or by momentum transfer (for example, by Bragg pulses) that transforms the phase between the momentum components of the fringe into relative populations of output components. The spatial interference fringe is formed when the left and right wave-packets overlap in position, while their overlap integral vanishes $\int d^3r \Phi_L^{(a,n)}(\mathbf{r}) \Phi_R^{(a,n)}(\mathbf{r}) = 0$ due to lack of overlap in momentum. We therefore analyze the atomic density just before probing and calculate the interference visibility of the fringe pattern.

The atomic density follows from the many-particle...
wave function in Eqs. (37) and (38):
\[
\rho(\mathbf{r}, t) = \int d^3 r_2 \cdots \int d^3 r_N |\Psi(\mathbf{r}, r_2, \ldots, r_N, t)|^2
= \sum_{k=0}^{N} |c_k|^2 \left( \frac{k}{N} \Phi_L^k \right)^2 + \frac{N - k}{N} \left| \Phi_R^{(N-k)} \right|^2
+ \sum_{k=1}^{N} \left( c_k^* c_{k-1} - \frac{k}{N} A_L^{(k)} \right) \left[ A_R^{(N-k+1)} \Phi_L^{(k)} \right]^* \times \Phi_R^{(N-k+1)} e^{i \chi_{k-1,N-k-1,N-k+1}} + \text{c.c.} \right) \tag{47}
\]
where
\[
A_a^{(n)} = \left[ \int d^3 r |\Phi_a^{(n)}(\mathbf{r})|^2 \right]^{n-1} \tag{48}
\]
is the product of the overlap integrals of the single-atom wave functions of the same arm \(a = L, R\) with different mean-field potential due to different numbers of particles \(n\) and \(n-1\).

The overlap integrals \(A_a^{(n)}\) in Eq. (48) have two effects on the interference term (bottom lines) in Eq. (47). They add phase and may also modify the amplitude of the interference term if \(|A_a^{(n)}| < 1\). In Appendix A we show that the latter effect is negligible, and the interference term in Eq. (47) may be approximated by
\[
\rho_{\text{intef}} = e^{i [\mathbf{S}_a - \mathbf{S}_b]/\hbar} e^{i [\mathbf{P}_R \cdot (\mathbf{r} - \mathbf{R}_L) - \mathbf{P}_L \cdot (\mathbf{r} - \mathbf{R}_L)]/\hbar}
\times \frac{N}{k=1} c_k^* c_{k-1} \left| \Phi_L^{(k)} \right| \left| \Phi_R^{(N-k+1)} \right| e^{i \left( \phi_L^{(k)} - \phi_R^{(N-k)} \right)}
\times e^{i \sum_{j=1}^{3} \left( \alpha_{L,k}^\lambda \sigma_{a,n,j}^2 - \alpha_{R,N-k}^\lambda \sigma_{a,n,j}^2 \right)^2}, \tag{49}
\]
where the first line contains phases due to the motion of the wave-packet centers, which are independent of the number of particles in each of them, the second line involves phases due to internal wave-packet dynamics, and the third line contains coordinate-dependent phases along each wave-packet, with the number-dependent coefficients \(\alpha_{a,n,j}^\lambda\) given by Eq. (16). The coordinate-independent phases in the second line are given by
\[
\phi_a^{(n)} = \phi_a^{(n)} + (n - 1) \sum_{j=1}^{3} \sigma_{a,n,j}^2 \frac{\partial \alpha_{a,n,j}^\lambda}{\partial n} \tag{50}
\]
where \(\phi_a^{(n)}\) (for each arms \(a\) and particle number \(n\)) is given in Eq. (18), \(\alpha_{a,n,j}^\lambda\) is the solution of Eq. (17) for each arm \(a\) and particle number \(n\), and \(\partial \alpha_{a,n,j}^\lambda/\partial n = \alpha_{a,n-1,j}^\lambda - \alpha_{a,n,j}^\lambda\).

In an interferometer where the wave-packets are stopped at a distance \(d = |\mathbf{d}| = |\mathbf{R}_R - \mathbf{R}_L|\) from each other and then allowed for expand for a long time \(t_f\) the interference fringes are formed by the expression in the second line of Eq. (49), their period is \(2\pi (a \cdot \mathbf{d})^{-1} \approx 2\pi \hbar t_f/m\mathbf{d}\), where \(\alpha_j\) are averaged over particle numbers and assumed to be equal for the left and right wave-packets. Variations of \(\alpha_j\) due to particle numbers in the second line of Eq. (49) and in Eq. (50) may give rise to the decay of visibility but they may be regarded as a secondary effect compared to the main phase-diffusion effect caused directly by the internal wave-packet phases \(\varphi\) in Eq. (50).

We now expand the phase in the last line of Eq. (49) in the particle number and obtain for the contrast \(C\):
\[
\phi_L^{(k)} - \phi_R^{(N-k)} \approx \delta_{L,R}^{(n/2)} - \phi_R^{(N/2)} + \frac{\partial \phi}{\partial n}(k - N/2), \tag{51}
\]
where
\[
\frac{\partial \phi}{\partial n} = \frac{\partial}{\partial n} \left( \phi_L^{(n)} + \phi_R^{(n)} \right)_{n=N/2}. \tag{52}
\]

The contrast of the interference fringes is determined by the overlap between the wave-packet envelopes in the first line of Eq. (49) and by the variation of the phase over the number distribution. We assume that this distribution is relatively broad and smooth so that it can be approximated as a Gaussian distribution of width \(\Delta_n\) as in Eq. (11), in which case \(c_k^* c_{k-1} \approx P_k \exp\{-(k - N/2)/2\Delta_n^2\}\). Ignoring the loss of visibility due to lack of envelope overlap we then obtain
\[
C = \sum_k P_k e^{-(k - N/2)/2\Delta_n^2} e^{i(k - N/2)\partial \phi/\partial n}, \tag{53}
\]
where \(\Delta_n = 1/2\Delta_n\) is the initial phase uncertainty due to the number uncertainty \(\Delta_n\) just after splitting.

The expression for the internal wave-packet phase in Eq. (18) contains a direct dependence on particle number \(n\) through the parameter \(\eta = n/N\) and an indirect dependence through the scaling factors \(\lambda_j\). The same two kinds of dependence apply to the second term in Eq. (50). We can therefore write the phase uncertainty as
\[
\Delta \phi(t) = \Delta \phi_0^2 + \Delta \phi_t^2, \tag{54}
\]
where
\[
\Delta \phi_t = \frac{\Delta_n}{N} \sum_{a=L,R} \frac{d}{d \eta} \left[ -\frac{\eta}{\hbar} \int_0^t d\tau \frac{\mu_{\text{int}}}{\hbar N A_a(\tau)} \right]
+ \sum_{j=1}^{3} \left( \frac{\nu_j}{2} \int_0^t \frac{d\tau}{\lambda_j^2 a_{a,n,j}} + \eta \sigma_j^2 a_{a,n,j}^2 \frac{\partial \alpha_j^{(a,n/2)}}{\partial n} \right), \tag{55}
\]
where \(\Delta_\nu = \lambda_{a,1}^2 \lambda_{a,2}^2 \lambda_{a,3}^2\) is the scaling factor for the volume of wave-packet \(a = L, R\) and the derivative \(d/d\eta\) with respect to \(\eta = n/N\) involves both explicit appearance of \(\eta\) in Eq. (55) and the implicit dependence of the scaling factors \(\lambda_{a,j}\) on \(\eta\) through their evolution according to Eq. (17).

Finally we note that Eq. (55) generalizes the standard theory of phase diffusion in two aspects. First, it provides a parametric expression for the phase uncertainty,
which includes wave-packet propagation beyond the TF approximation. In this sense it simplifies an equivalent treatment that would use a numerical solution of the GP equation with the top line of Eq. (55) being equivalent to defining the interaction phase as \( gN \int d^3r |\Phi|^4 \). Second, it includes two additional terms due (bottom line) due to the kinetic energy or the phase along the wave-packets due to their expansion. The significance of these additional terms in different circumstances is yet to be revealed.

### C. Application to specific schemes

Let us now examine the application of the above results to some important interferometric schemes, starting from the well-known example of spatially adiabatic splitting into a double-well potential or another harmonic potential and continuing with schemes where the wave-packets are propagating.

#### 1. Spatially adiabatic potential

Let us consider spatially adiabatic splitting, where a BEC wave function follows the instantaneous stationary solution of the time independent GP equation [Eq. (1)]. In this case, the chemical potential and the instantaneous BEC size for each arm with particle number \( n = \eta N (n_L + n_R = N) \) is given in Eq. (6), where \( \mu_{\text{int}}(t) \propto t^{-6/5} n^{2/5} \), while \( \sigma_j(t) = \sigma_j(0) \lambda_n(t) \propto \sqrt{\mu_{\text{int}}(t)/\tilde{\omega}_j(t)} \), so that \( \lambda_n \propto t^{-6/5} n^{3/5} \). It follows that the instantaneous chemical potential satisfies \( \mu_{\text{int}}^{(a,n)}(t) \propto t^{-6/5} n^{2/5} \), where \( \mu_{\text{int}}(0) \) is the initial chemical potential and therefore does not depend on the instantaneous trap frequencies and the number of atoms so that it is independent of time. We can therefore conclude that the term \( \eta \mu_{\text{int}}(0)/\lambda_n(t) \) in the upper line of Eq. (55) is the instantaneous chemical potential of each arm. Then we have \( d\mu_{\text{int}}^{(a,n)}(t)/d\eta = 2\mu_{\text{int}}^{(a,n)}(0)/5 \). In the spatially adiabatic case the phase over the BEC wave function is always flat, such that \( \alpha_j \to 0 \). If we ignore the dependence of the kinetic term [first term in the second line of Eq. (55)] on the number of particles we are then left with the first line \( \Delta \phi_t \approx 2 \Delta \phi_0 + \sum j \int_0^t d\tau \mu_{\text{int}}^{(a,N/2)}(\tau) \). Assuming that the two arms are symmetric, we obtain the well-known result that the contrast is expected to drop like \( C(t) \propto \exp \left[ -\frac{1}{2} \left( \int_0^t d\tau \Gamma(\tau) \right)^2 \right] \), where the phase diffusion rate \( \Gamma(t) \) is given by

\[
\Gamma(t) = \frac{4\Delta_n \mu_{\text{int}}(N/2)}{5N},
\]

where \( \mu_{\text{int}}(N/2) \) is the steady-state solution of the GP equation in the final trapping potential with \( N/2 \) atoms, as can be calculated within our generalized TF approximation from Eq. (6). This result is well known from the literature [25] and here we have only generalized it to beyond the TF approximation (while still the contribution of the number-dependence of the kinetic energy).

#### 2. Free expansion

We now consider a BEC initially in a cylindrical trap with \( \omega_\perp \gg \omega_\parallel \). For simplicity we assume that the trapping potential is turned off quickly and that a splitting pulse is applied right after switching off, before the BEC starts to expand. The evolution of the scaling factors obeys Eq. (17) with \( \eta = 1/2 \) with \( \lambda_0 = 1 \) and \( \lambda_\parallel = 0 \) initial conditions of Eq. (20). In the TF limit this gives \( \lambda_\perp = \sqrt{1 + \eta \omega_\perp^2 t^2} \). Here we ignore the slow evolution of the longitudinal scaling factor \( \lambda_\parallel \) over relatively short times where the atom-atom interaction is still significant. In this case we can show that the last term in the second line of Eq. (55) is

\[
N\eta \sigma_\perp^2 \lambda_\perp^2 \frac{\partial \sigma_\perp}{\partial n} = \frac{\eta \mu_{\text{int}}(t)}{2\hbar \lambda_\perp^2} \sigma_\perp^2 \omega_\perp^2 t = \frac{\eta \mu_{\text{int}} t}{7\hbar \lambda_\perp^3}. \tag{57}
\]

It then follows from Eq. (55), when we ignore the term \( v_j/2\lambda_j^2 \)

\[
\Delta \phi_t = \frac{\Delta \phi_0}{\eta N \hbar \omega_\perp} \left[ \frac{\omega_\perp^2 t}{\lambda_\perp^2} - \sqrt{2} \int \tan(\omega_\perp t/\sqrt{2} + \frac{4\omega_\perp^2 t}{\lambda_\perp^2} \right]. \tag{58}
\]

At times \( t \gg \omega_\perp^{-1} \) the phase uncertainty becomes stationary at

\[
|\Delta \phi_t(t \to \infty)| = \frac{\Delta \phi_0}{\sqrt{N}} \frac{\pi \mu_{\text{int}}}{2\hbar \omega_\perp}. \tag{59}
\]

This value scales like \( \Delta \phi_0 \sim (a_s/\ell)^{2/5} N^{-1/10} \) [see Eq. (6)] and is therefore very small. We may conclude that under usual circumstances phase diffusion is not a concern when the BEC is expanding in free space.

#### 3. Expansion in a waveguide

The main importance of the derivation in this section is that it enables the estimation of phase diffusion in cases where this effect is crucial for interferometry. While our result in Eq. (56) for the rate of phase diffusion in adiabatic splitting is already known from the literature and our result for the phase diffusion of freely expanding wave-packets [Eq. (59)] shows that in such circumstances phase diffusion is not crucial, we now provide an example of phase diffusion in a matter-waveguide, where this effect may be of major concern in attempting to design and realize guided matter-wave interferometry.

In Fig. 6 we show the results of a calculation of the phase diffusion process in a matterwaveguide with the
to calculate the phase uncertainty $\Delta \phi$. Interactions in the Thomas-Fermi regime. Our theory of negligible atom-atom interactions to a BEC with strong interactions on the coherence of interferometers based on atomic clouds, where a numerical solution of the GP equations in three dimensions would be very difficult or impractical. We support our theory with a few examples of the evolution of a BEC, which is initially trapped and then split into a waveguide potential, where we compare the results of the theory to full numerical solutions of the GP equation and find excellent agreement.

Although our theory of wave-packet evolution is based on a mean-field approximation for the atom-atom interaction, we apply it to the theory of phase diffusion of a split BEC, where a mean-field approximation is embedded into a many-body theory of the dynamics. Our theory of phase diffusion provides some new analytical results for situations in which the two split wave-packets propagate in free space or in a waveguide, where the simple two-mode model does not apply.

Our wave-packet propagation theory goes beyond some other works that generalized the usual time-dependent TF method [22]. We provide a generalized Thomas-Fermi theory for the BEC in a harmonic trap, which is appropriate to the whole continuous range between a single atom or a non-interacting BEC to the TF regime of a strongly interacting BEC. It also applies to the case of a BEC in an elongated trap, where the transverse direction is dominated by kinetic energy while the longitudinal direction may be dominated by atom-atom interactions. We then provide a semi-analytical approximation for the transition between a three-dimensional BEC and a quasi-one-dimensional BEC [33, 34] at zero temperature. In addition, we provide a description of the evolution of a BEC in such circumstances when the potential is changed in time, as long as the potential may be approximated by a quadratic form.

One example of the application of our wave-packet evolution method to the question of interferometric coherence and demonstrates the versatility of the method is provided in Section III. We generalize the the old theory of spin-coherence in a Stern-Gerlach interferometer, which is relevant to any interferometer where internal levels are entangled with interferometer paths, where the contrast is limited by the precision of the spatial recombination (the “Humpty-Dumpty effect”) [20,22]. This theory was already generalized and applied later to modern light-pulse interferometers [24]. Here we provide practical expressions for the estimation of the coherence that are suitable for cases where the wave-packet evolution is governed by atom-atom interactions of any strength.

Finally, our theory of the effect of atom-atom interactions on the coherence of interferometers based on a BEC is expected to provide useful estimation for the feasibility of new kinds of interferometers, in particular interferometers where the atoms are guided in a dynamic potential [32,33] or waveguide potentials [34,35].
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Appendix A: Derivation of the number-dependent phase

For estimating the overlap integral in Eq. (48) we use a Gaussian approximation for the initial wave-packet \( \Phi_0(\mathbf{r}) \propto \exp[-\sum_j r_j^2/(4\sigma_j^2)] \). We can therefore separate the variables \( r_j = 1, 2, 3 \), so that

\[
A_n^{(a)} = \prod_j A_{a,j}^{(n)} e^{-i(n-1)(\varphi_a^{(n)} - \varphi_a^{(n-1)})},
\]

(A1)

where \( \varphi_a \) are the coordinate independent phases and

\[
A_{a,j}^{(n)} = C_{j,n} \left[ \int dx \exp \left( -\frac{x^2}{4} \left( \frac{1}{\lambda_j^{(n)} + 1} \right) - i(\alpha_j^{(n)} - \alpha_j^{(n-1)})x^2 \right) \right]^{n-1},
\]

(A2)

where \( C_{j,n} \) is a normalization constant \( C_{j,n} = (2\pi\lambda_{j,n}\lambda_{j,n-1}\sigma_j^2)^{-1/2} \). The integral yields

\[
A_{a,j}^{(n)} = \left[ \frac{\lambda_j^{(n)} + \lambda_j^{(n-1)}}{2\lambda_{j,n}\lambda_{j,n-1}} + 2\sigma_j^2\lambda_{j,n-1}(\alpha_j^{(n)} - \alpha_j^{(n-1)}) \right]^{-(n-1)} \delta(x_j - a) = L, R.
\]

(A3)

One can get easily convinced that if \( \lambda_{j,n-1}/\lambda_{j,n} = 1 + \delta \) and \( \sigma_j^2(\alpha_j^{(n)} - \alpha_j^{(n-1)}) = \epsilon \), where \( \delta, \epsilon \ll 1 \), then the absolute value of \( A_{a,j}^{(n)} \) in Eq. (A3) is of quadratic order in the small parameters, so that \( |A_{a,j}^{(n)}| \approx 1 \), while the phase factor contains the first order in \( \epsilon \). We then obtain

\[
A_n^{(a)} \approx e^{i(n-1)\sum_j \sigma_j^2\lambda_{a,j,n}(\alpha_j^{(n)} - \alpha_j^{(n-1)})} e^{-i(n-1)(\varphi_a^{(n)} - \varphi_a^{(n-1)})},
\]

(A4)

The second exponent in Eq. (A4) cancels with the corresponding term in the exponent in Eq. (47) containing the global configuration phases \( \chi_k, N-k \) and \( \chi_{k-1,N-k+1} \). This can be verified by using Eq. (A6) for the configuration phase, which contains the differences

\[
\frac{1}{2}(n\varphi_a^{(n)} - (n-1)\varphi_a^{(n-1)}) = (n-1)\frac{\partial\varphi_a}{\partial n},
\]

(A5)

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