Path Integral Optimization for $T \bar{T}$ Deformation

Ghadir Jafari, Ali Nashe, and Hamed Zolfi

School of Particles and Accelerators, Institute for Research in Fundamental Sciences (IPM), P.O.Box 19395-5531, Tehran, Iran

Department of Physics, Sharif University of Technology, P.O. Box 11365-9161, Tehran, Iran

(Dated: September 6, 2019)

We use the path integral optimization approach of Caputa, kundu, Miyaji, Takayanagi and Watanabe to find the time slice of geometries dual to vacuum, primary and thermal states in the $T \bar{T}$ deformed two dimensional CFTs. The obtained optimized geometries actually capture the entire bulk which fits well with the integrability and expected UV-completeness of $T \bar{T}$-deformed CFTs. When deformation parameter is positive, these optimized solutions can be reinterpreted as geometries at finite bulk radius, in agreement with a previous proposal by McGough, Mezei and Verlinde. We also calculate the holographic entanglement entropy and quantum state complexity for these solutions. We show that the complexity of formation for the thermofield double state in the deformed theory is UV finite and it depends to the temperature.

PACS numbers: 03.67.-a, 03.70.+k, 11.10.Gh

INTRODUCTION

Applications of quantum information concepts to gravity and high energy physics recently led to many widespread developments. In particular quantum entanglement helps us to understand how gravity emerges from field theories [1, 2]. One concrete idea to realize this emergent spacetime is to utilize the connection between tensor networks and holographic entanglement surface. It is argued that a time slice of AdS spacetime correspond to a special tensor network called multi-scale entanglement renormalization ansatz (MERA) [3]. However, when one tries to understand the interior of black hole, the entanglement is not enough [4]. This observation has led to significant interest to the information notion of quantum state complexity which can be used as a probe to investigate the growth rate of the Einstein-Rosen bridge [5].

Two holographic conjectures are proposed for quantum complexity: the CV conjecture [6, 7] and CA conjecture [8] which share some similarities but differ in many ways. To understand better the holographic results, recently the computational complexity for quantum field theory states have been studied [8, 10]. In one approach which is based on the idea of Nielsen and collaborators [11], one associates a geometry to the space of unitaries that connect the desired state to a reference state. Then the complexity of desired state is defined as a length of a geodesic which captures the entire bulk, specially the region out side the finite bulk cut-off in [20]. This result is in agreement with recent study [21] where it is shown that the dual gravitational theory to the $T \bar{T}$ deformed 2D CFTs, has mixed boundary conditions for the non-dynamical graviton. It is worth to mention that our result for positive sign of deformation parameter can be interpreted as existence of hard bulk cut-off in agreement with [20]. Moreover, we calculate the holographic entanglement entropy for the obtained solutions and also quantum state complexity for some dual boundary QFT states. These states are $T \bar{T}$ deformation of vacuum, primary and thermal states. The quantum complexity of these states has the same UV structure as corresponding states in 2D CFTs. Moreover, the complexity of formation for thermofield double state is finite and it depends to the temperature [30].
PATH INTEGRAL OPTIMIZATION

For 2D CFTs, the ground state wave functional on \( \mathbb{R}^2 \) is computed by a Euclidean path integral:

\[
\Psi_{\text{CFT}}[\Phi(x)] = \int \prod_x \prod_{-\infty < \tau < \infty} \mathcal{D}\Phi \ e^{-S_{\text{CFT}}(\Phi)} \times \prod_x \delta \left( \Phi(\epsilon, x) - \Phi(x) \right),
\]

where \( \tau \) is Euclidean time and \( \epsilon \) is UV cutoff (i.e. the lattice constant). It is worth noting that to evaluate this discretized path integral in an optimal way, one can omit any unnecessary lattice sites. To systematically quantify such coarse-graining, one might introduce a 2D metric (on which the path integration is performed)

\[
ds^2 = \frac{l^2}{\epsilon^2} (d\tau^2 + dx^2),
\]

such that one lattice site has a unit area. The optimization procedure then can be described by modifying the background metric for the path integration as

\[
ds^2 = g_{\tau\tau}(\tau, x)d\tau^2 + g_{xx}(\tau, x)dx^2 + 2g_{\tau x}(\tau, x)d\tau dx.
\]

Now, the key point is that the optimized wave functional, up to a normalization factor, should be proportional to the correct ground state wavefunction, i.e. the \( \Pi \) for the metric (2). This constraint implies that

\[
g_{\tau\tau}(\epsilon, x) = g_{xx}(\epsilon, x) = \frac{l^2}{\epsilon^2}, \quad g_{\tau x}(\epsilon, x) = 0.
\]

For 2D \( T\bar{T} \) deformed CFTs, which are described by the action

\[
S = S_{\text{CFT}} + \mu \int d\tau dx T\bar{T}(\tau, x),
\]

the deformation operator is a double-trace operator. According to \( \Pi \), this double-trace deformation of CFT corresponds to imposing mixed boundary condition on the bulk metric. Motivating by this fact and noting that the position dependent coupling \( \mu \) is related to the bulk matter field, in the following we keep the coupling constant \( \mu \) independent from \((\tau, x)\) coordinates and just allow the metric to vary. Accordingly, the optimization is performed by the following ansatz

\[
ds^2 = e^{2\Omega(\tau, x)} (d\tau^2 + dx^2),
\]

where that mixed boundary condition is encoded in two requirements: \( e^{2\Omega(\tau, x)} = \frac{l^2}{\epsilon^2} \) and the energy of optimized solution matches with the energy spectrum of \( T\bar{T} \) deformed CFT.

On the metric \( \hat{g}_{\alpha\beta} = e^{2\Omega(\tau, x)}g_{\alpha\beta} \), the partition function of 2D \( T\bar{T} \) deformed CFTs is given by

\[
Z_\mu[\hat{g}] = \int \mathcal{D}\Phi \ \text{Exp} \left[ -S_{\text{CFT}}[\Phi, \hat{g}] - \int d^2\sigma \sqrt{\hat{g}} \left( \Lambda + \mu T\bar{T}[\Phi, \hat{g}] \right) \right],
\]

where \( d^2\sigma = d\tau dx \) and \( \Lambda \) is related to the cosmological constant. It is worth noting that this counterterm action is responsible for removing the UV divergence in presence of \( T\bar{T} \) deformation. Under change of Weyl factor \( \Omega(\tau, x) \), this partition function changes to

\[
\frac{1}{Z_\mu[\hat{g}]} \frac{\partial Z_\mu[\hat{g}]}{\partial \Omega} = \frac{1}{Z_\mu[\hat{g}]} \int \mathcal{D}\Phi \ e^{-S_{\text{CFT}} - \int d^2\sigma \sqrt{\hat{g}} (\Lambda + \mu T\bar{T})} \times \left( -\frac{\partial S}{\partial \hat{g}_{\alpha\beta}} \frac{\partial \hat{g}_{\alpha\beta}}{\partial \Omega} - \int d^2\sigma \frac{\partial}{\partial \Omega} \sqrt{\hat{g}} (\Lambda + \mu T\bar{T}) \right)
\]

Using the above equation and also assuming that \( \mu \) is small, we get

\[
\frac{\partial \log Z_\mu[\hat{g}]}{\partial \Omega} = \frac{c}{24\pi} \sqrt{\hat{g}} R(\hat{g}) - 2\Lambda \sqrt{\hat{g}} - \mu \partial_\Omega (T\bar{T}),
\]

where constant \( c \) is the central charge of 2D CFT and

\[
\langle T\bar{T} \rangle_{\hat{g}} = \int d^2\sigma \sqrt{\hat{g}} \left( \langle 0|T_{zz}|0 \rangle_{\hat{g}} \langle 0|T_{\bar{z}\bar{z}}|0 \rangle_{\hat{g}} - e^{-4\Omega_{12}} \langle 0|T_{z\bar{z}}|0 \rangle_{\hat{g}}^2 \right)
\]

One can now just treat \( \Omega \) as a differential equation for the partition function \( Z_\mu \) and solve it. This allows us to express the partition function \( Z_\mu[\hat{g}] \), defined on one metric \( \hat{g} \), in terms of \( Z[\hat{g}] \), defined on metric \( g \). The relationship is,

\[
Z_\mu[\hat{g}] = e^{S_{\text{GL}}[\Omega, g] - S_{\text{GL}}[0, g]} Z_\mu[\hat{g}],
\]

where

\[
S_{\text{GL}}[\Omega, g] = \int d^2\sigma \left( \frac{c}{24\pi} \sqrt{\hat{g}} (R(\hat{g})\Omega - \Omega \nabla^2 \Omega - \tilde{\Lambda} e^{2\Omega}) - \mu \langle T\bar{T} \rangle_{\hat{g}} \right),
\]

and \( \tilde{\Lambda} = \frac{28\pi}{c} \Lambda \). The subscript (GL) in the above equation means generalize Liouville since the first term in the above equation is actually the standard Liouville action. For \( g_{\alpha\beta} = \delta_{\alpha\beta} \), the last term in \( [12] \) can be calculated explicitly by noting that in complex coordinates (\( w = \tau + ix \), \( \bar{w} = \tau - ix \)) we have the below relations

\[
\langle 0|T_{zz}|0 \rangle_{e^{2\Omega}g_{\alpha\beta}} = \frac{c}{12} e^{-2\Omega} \partial^2 \Omega,
\]

\[
\langle 0|T_{\bar{z}\bar{z}}|0 \rangle_{e^{2\Omega}g_{\alpha\beta}} = \frac{c}{12} e^{-2\Omega} \partial^2 \Omega,
\]

\[
\langle 0|T_{z\bar{z}}|0 \rangle_{e^{2\Omega}g_{\alpha\beta}} = \frac{c}{6} \partial \partial \Omega.
\]
By substituting (13) in (10), the action (12) simplified to
\[
S_{GL}[\Omega, \delta] = \frac{c}{24\pi} \int d^2x \left[ -4\partial\bar{\partial}\Omega - \Lambda e^{2\Omega} - \tilde{\mu} e^{-2\Omega} \left( \partial^2 \bar{\partial}^2 \Omega - 4(\bar{\partial}\Omega)^2 - \frac{3}{16} \bar{\partial}^2 \eta \Omega \right) \right],
\]
where \(\tilde{\mu} = \mu\pi c/6\). In obtaining the above equation, we used \(\Lambda = \Lambda + \frac{\mu}{\pi} \tilde{\mu} \Lambda^2\) where the reason for that will be clear in the following. Finally, the definition (11) together with (11) imply that the ground-state wave functional \(\Psi_{x_0}^{\delta_a, \delta_b}\) computed from the path integral for the metric (8) is proportional to the one \(\Psi_{\delta_a, \delta_b}\) for the flat metric
\[
\Psi_{x_0}^{\delta_a, \delta_b}(\Phi(x)) = e^{S_{GL}[\Omega, \delta] - S_{GL}[\Omega, 0]} \Psi_{\delta_a, \delta_b}(\Phi(x)).
\]
It is worth to emphasize that these two states describe the same quantum state if at UV cutoff, \(\Omega(\epsilon, x) = \log(1/\epsilon)\).

**OPTIMIZING VARIOUS STATES IN \(TT\) DEFORMED CFTS**

Now, according to (12,13) the optimization is equivalent to minimizing the normalization factor \(e^{S_{GL}[\Omega, \delta]}\) of the wave functional. The intuition behind this proposal comes from the tensor network representation of vacuum wave functional. In that language, a quantum state is defined as a minimal number of the quantum gates (operators) needed to create the state starting from a reference state. Here, the factor \(e^{S_{GL}[\Omega, \delta]}\) actually measures the number of repetitions of the same operation (i.e. the path integral over a cell). In order to find this minimum value, we must vary the action \(S_{GL}[\Omega, \delta]\) (14), which gives following equation of motion:
\[
4\partial\bar{\partial}\Omega - \Lambda e^{2\Omega} + \tilde{\mu} e^{-2\Omega} \left( \frac{3}{16} \Lambda e^{2\Omega} + 3e^{-2\Omega} \partial^2 \bar{\partial}^2 \Omega + e^{-2\Omega} \partial^2 \bar{\partial}^2 \Omega - 2(\partial\Omega)^2 + 3\partial^2 \Omega - 2e^{-2\Omega} \left[ 3\partial\Omega \partial^2 \bar{\partial} \Omega + 6(\bar{\partial}\Omega)^2 + 2\partial(\partial\Omega)^2 + \bar{\partial}(\partial\Omega)^2 \right] \right) = 0
\]
Since finding the analytic solutions of this equation in general is difficult, one can solve it perturbatively in \(\tilde{\mu}\) parameter. As in this letter, we are interested to find the solutions which are only depend to \(2\tau = (\omega + \bar{\omega})\), we consider \(\Omega(\omega + \bar{\omega}) = \Omega_{\text{CFT}}(\omega + \bar{\omega}) + \tilde{\mu} \Omega_{T\bar{T}}(\omega + \bar{\omega})\). According to (13), a vacuum state in two dimensional QFT can be constructed from various co-dimension one surfaces in the 3D gravity dual. For example one of them can be 2 dimensional boundary and another can be time slice of 3D bulk geometry. These two surfaces can be related with the bulk coordinate transformations and till they have the same topology, they give the same state up to the normalization factor. In the following, we represent this bulk co-dimension one surface with \((z, x)\) coordinates.

Now we are going to solve (14). If we set \(\tilde{\mu} = 0\), the solution which minimize \(S_{GL}[\Omega, \delta]\) (13) is
\[
\Omega_{\text{CFT}} = -\frac{1}{2} \log \left( \Lambda z^2 \right)
\]
which describes the time slice of AdS\(_3\) geometry in Poincare coordinate. Using this solution for unperturbed CFT\(_2\), the first order perturbation \(\Omega_{T\bar{T}}\) becomes
\[
\Omega_{T\bar{T}}(z) = c_1 \frac{z^2}{l^2} + c_2 \frac{l}{z},
\]
where \(c_1\) and \(c_2\) are arbitrary dimensionless integration constants. By choosing \(\Lambda = 1/l^2\), the optimized metric (6) becomes
\[
ds^2 = \frac{l^2}{z^2} \left( 1 + 8c_1 \tilde{\mu} \frac{z^2}{l^2} + c_2 \tilde{\mu} \frac{l}{z} \right)(dz^2 + dx^2).
\]
Imposing the UV condition \(g_{zz}(z = \epsilon, x) \sim 1/\epsilon^2\) and IR condition \(g_{zz}(z = \infty, x) = 0\) respectively imply that \(c_2\) and \(c_1\) should be zero which it means that the Poincare AdS\(_3\) remains undeformed. Another interesting solutions for \(\tilde{\mu} = 0\) are excited states created by acting a primary operator \(O_\alpha\) with the conformal dimension \(h_\alpha = \hat{h}_\alpha\) which its behavior under the Weyl re-scaling is expressed as
\[
O_\alpha \sim e^{-2h_\alpha} O_\alpha.
\]
It is shown that in absence of \(\tilde{\mu}\), the geometry dual to this state is given by
\[
\Omega_{\text{CFT}} = \frac{1}{2} \log \left( \frac{a^2}{\sinh^2(\frac{az}{l})} \right), \quad a = 1 - \frac{12h_\alpha}{c}
\]
For \(a = 1\), this describes the time slice of global AdS\(_3\). Using this unperturbed solution in (16), the perturbed solution becomes
\[
ds^2 = a^2 \text{csch}^2(\frac{az}{l}) \left( 1 + 2c_1 \frac{az}{l} \left( 1 - \frac{az}{l} \coth(\frac{az}{l}) \right) \right)(dz^2 + dx^2),
\]
where \(c_1\) is a dimensionless constant of integration and also a coefficient of a term which violates the condition \(g_{zz}(\epsilon, x) \sim 1/\epsilon^2\) is set to zero. Using the coordinate transformation
\[
z = \frac{2l}{a} \tanh^{-1}(\frac{\sqrt{3}}{2}(1 - \frac{\tilde{\mu} c_1}{l^2})),
\]
the metric (22) takes below Fefferman-Graham expansion
\[
ds^2 = \frac{l^2 a^2}{4\rho^2} d\rho^2 + \frac{l^2 a^2}{\rho} (1 + 2\frac{\tilde{\mu} c_1}{l^2})(1 - \frac{1}{2} \rho + \frac{1}{16} \rho^3) dx^2.
\]
The gravitational energy of the this solution matches with correspondent energy of deformed primary state \[18\] for \( c_1 = -1/32\pi^2 \).

All we have done up to now was for a \( I\bar{I} \) deformation of a CFT at zero temperature. To extend the analysis to a finite temperature \( T = 1/\beta' \) case, one can use the thermofield double representation of wave functional. In this representation, the wave functional is computed from a path integral on a cylinder with a finite width \( -\frac{\beta'}{4} < z < \frac{\mu}{4} \) accordingly

\[
\Psi_{\beta'}[\Phi_2(x), \Phi_1(x)] = \int \left( \prod_x \prod_{\frac{\beta'}{4} \leq z < \frac{\beta'}{4}'} \mathcal{D}\Phi \right) e^{-S(\Phi)} \times \prod_x \delta \left( \Phi \left( -\frac{\beta'}{4}, x \right) - \Phi_1(x) \right) \delta \left( \Phi \left( \frac{\beta'}{4}, x \right) - \Phi_2(x) \right),
\]

(25)

which is actually describe time slice of BTZ black hole. It is worth noting that the temperature for the solution \[26\] is assumed to be different with the temperature \( 1/\beta' \). The reason for this will be clear in the following. Substituting this solution in (16) gives following equation for \( \Omega_{\text{CFT}}(z) \),

\[
\Omega_{\text{CFT}}(z) = \frac{1}{2} \log \left( \frac{4\pi l^2 \sec^2 \left( \frac{2\pi z}{\beta} \right)}{\beta^2} \right), \tag{26}
\]

which is actually describe time slice of BTZ black hole. It is worth noting that the temperature for the solution \[26\] is assumed to be different with the temperature \( 1/\beta' \). The reason for this will be clear in the following. Substituting this solution in (16) gives following equation for \( \Omega_{\tau} \),

\[
\frac{\beta^2}{4} \cos^2 \left( \frac{2\pi z}{\beta} \right) \beta^2 \Omega_{\tau}(z) - 2\pi^2 \Omega_{\tau}(z) = 0, \tag{27}
\]

which its solution is

\[
\Omega_{\tau}(z) = \left( \frac{2\pi c_1}{\beta} \right) \tan \left( \frac{2\pi z}{\beta} \right) + c_1. \tag{28}
\]

In the above expression \( c_1 \) is arbitrary integration constant. Substituting (27) and (28) in (16), up to first order in \( \tilde{\mu} \), we have

\[
ds^2 = \frac{4\pi l^2}{\beta^2} \sec^2 \left( \frac{2\pi z}{\beta} \right) \left( 1 + \frac{2\mu c_1}{l^2} \left[ 1 + \frac{2\pi z}{\beta} \tan \left( \frac{2\pi z}{\beta} \right) \right] \right) \times (dz^2 + dx^2).
\]

(29)

To determine the constant \( c_1 \), it is better to transform this metric in the global coordinate. This can be achievable according the below coordinate transformation

\[
z = \frac{\beta}{2\pi} \left( 1 - \frac{\tilde{\mu} c_1}{l^2} \right) \cos^{-1} \left( \frac{2\pi l^2}{\beta r} \right) - \frac{\tilde{\mu} c_1}{r \sqrt{1 - \frac{4\pi l^2}{\beta^2 r^2}}}.
\]

(30)

which by that the metric \[29\] changes to

\[
ds^2 = \frac{dr^2}{f(r)} + r^2 dx^2, \quad f(r) = \frac{r^2}{l^2} - \frac{4\pi l^2}{\beta^2} (1 + 2\tilde{\mu}/l^2 c_1).
\]

(31)

The temperature of this black hole is

\[
1/\beta' \equiv T_{\tau} = \frac{1 + \tilde{\mu}/l^2 c_1}{\beta}.
\]

(32)

Let us remind that the time slice of standard BTZ black hole with the ADM mass \( M \) is

\[
ds^2 = \frac{dr^2}{f(r)} + r^2 dx^2, \quad f(r) = \frac{r^2}{l^2} - 8GM_{\text{BTZ}}, \tag{33}
\]

where the mass and temperature of it is related according to following

\[
M_{\text{BTZ}} = \frac{\pi l^2}{2G} \frac{1}{\beta'}. \tag{34}
\]

For \( c_1 = GM_{\text{BTZ}}/4\pi^2 \) and considering the \( T_{\tau} \) in the right hand side of (33) instead of \( 1/\beta \), for \( M_{\tau} \) up to first order in \( \mu \), we find

\[
M_{\tau} = M_{\text{BTZ}}(1 + \frac{M_{\text{BTZ}} l^2}{8\pi^4 \mu}), \tag{35}
\]

which exactly matches with the energy spectrum of \( I\bar{I} \) deformed CFT \[18, 19\] for small \( \mu \). By using

\[
r(\rho) = -\frac{l(\beta^2 + \pi^2 l^2 \rho)}{\beta^2 \sqrt{\rho}} \left( 1 + c_1 \tilde{\mu}/l^2 \right), \tag{36}
\]

the metric \[31\] changes to

\[
ds^2 = \frac{l^2 dr^2}{4\rho^2} + \frac{l^2 (1 + \frac{2\mu c_1}{l^2})}{\rho} \left( 1 + \frac{2\pi l^2}{\beta^2} \rho + \frac{\pi^4 l^2}{\beta^4} \rho^2 \right) dx^2.
\]

(37)

To understand better this result, let us remind that The most general solution of 3 dimensional Einsteins equations with a negative cosmological constant can be expressed in FeffermanGraham gauge by

\[
ds^2 = \frac{l^2 dr^2}{4\rho^2} + \frac{l^2}{\rho} g_{\alpha\beta}(\rho, x) dx^\alpha dx^\beta,
\]

\[
g_{\alpha\beta}(\rho, x^\alpha) = g^{(0)}(x^\alpha) + g^{(2)}(x^\alpha) \rho + g^{(4)}(x^\alpha) \rho^2,
\]

(38)

which \( g^{(4)} \) and \( g^{(2)} \) are determined algebraically in terms of \( g^{(0)} \) \[24\]. Based on variational principle approach to the \( I\bar{I} \) deformation of CFTs in \[21\] (actually with mixed boundary condition as we discussed below Eq.(5)), it is argued that in the deformed theory the source for the stress tensor is given by a non-linear combination of the metric and stress tensor expectation value in the original CFT as following \[31\]

\[
g^{(\mu)} \equiv g^{(0)} + \frac{\mu}{32\pi Gl} g^{(2)} + O(\mu^2).
\]

(39)
For the BTZ black hole $g_{xx}^{(0)} = 1$ and $g_{xx}^{(2)} = \frac{2a^2 \ell^2}{\beta^2}$, which together with (34) implies that

$$g[\mu] = 1 + \frac{M_{\text{BTZ}}}{8\pi \ell} \mu + O(\mu^2).$$

Remarkably, it is the conformal boundary of (37) by setting $c_1 = GM_{\text{BTZ}}/4\pi^2$ same as previous analysis for the energy of black hole solution (31). Another interpretation for the geometry (37) is that its conformal boundary corresponds to fixing the induced metric on a constant $\rho = \rho_c$ surface, with

$$\rho_c = \frac{\mu}{32\pi G l},$$

in agreement with the earlier proposal of [21]. It is worth noting that the path integral optimization solution (31) shows that the entire spacetime should be kept especially if $c_1$ becomes which again it vanishes in the UV limit, $R \to 0$, same as the result in [21]. It is worth noting that from this latter result we can not conclude that the theory flows to a trivial theory since we have assumed $\mu$ is small.

### PATH INTEGRAL COMPLEXITY

It seems that the similarity between tensor network representation and Path integral representation of vacuum state, can be utilize to define computational complexity of ground state [11] as the minimum value of the $S_{\text{GL}}[\Omega, g]$ action. But $S_{\text{GL}}[\Omega, g]$ [14], not only depends to the final metric $e^{2\Omega}g$ but also it depends to the reference metric $g$ which means that this action does not provide us with an absolute quantity which measures the complexity of the optimized state. According to [13], it is convenient to look at the relative quantity $I_{\text{GL}}[g_2, g_1]$ which satisfy the following identity

$$I_{\text{GL}}[g_1, g_2] + I_{\text{GL}}[g_2, g_3] = I_{\text{L}}[g_1, g_3].$$

The above relation implies that

$$I_{\text{GL}}[e^{2\Omega}g, g] = I_{\text{GL}}[e^{2\Omega}g, g] - I_{\text{GL}}[\tilde{g}, g],$$

which means that $I_{\text{GL}}[g_2, g_1]$ actually measures the difference of complexity between the path-integral in $g_2$ and $g_1$. In order to find this quantity, motivating by the form of $S_{\text{GL}}$ [14], we assume the general following form

$$I_{\text{GL}}[e^{2\Omega}g, g] = \int d^2 \sigma \sqrt{g} \left( R[g] \Omega + \nabla_\alpha \Omega \nabla^\alpha \Omega + b_1(\Omega) + \mu \nabla_\alpha \Omega \nabla^\alpha \Omega + \mu \nabla_\alpha \Omega \nabla^\alpha \Omega + b_2(\Omega) \right),$$

where $b_i$'s are arbitrary function of $\Omega$. Now, the constraint equation (45) implies that, the general function (47) reduces to

$$I_{\text{GL}}[\Omega, g] = \int d^2 \sigma \sqrt{g} \left( R[g] \Omega + \nabla_\alpha \Omega \nabla^\alpha \Omega + b_1(\Omega) + \mu \nabla_\alpha \Omega \nabla^\alpha \Omega + \mu \nabla_\alpha \Omega \nabla^\alpha \Omega + b_2(\Omega) \right).$$

Comparing this action with (14) results in $b_1 = -\tilde{A}$, $b_2 = -3\tilde{A}^2/16$, $b_3 = 3/16$. To have a well defined variational principle, we should add proper Generalized Gibbons- Hawking term to the above action which is given by

$$I_{\text{GGH}}[\Omega, \gamma] = 2 \int dx \sqrt{g} \left( K[\gamma] \Omega - \frac{3}{16} \tilde{A} e^{2\Omega} \left[ K[\gamma] \nabla^\gamma + \frac{1}{2} K[\gamma] R[g] (e^{2\Omega} - 1) - n^\alpha \nabla_\alpha (R[g] \nabla^\gamma) \right] \right),$$

(49)
where \( \gamma \) and \( K[\gamma] \) are respectively induce metric and its extrinsic curvature. In the above we have set \( \Lambda = 1/l^2 \). Now we have all ingredients to calculate the path integral complexity

\[
C_{T} = C_{GL}^T + I_{GH}^T, \tag{50}
\]

for the solutions which we have found in previous section. The final results for solutions (22) and (29) are respectively

\[
C^{AdS}_{T} = \frac{c}{6} \left( 1 + \frac{\pi c \mu}{64 l^2} \right) \frac{1}{\epsilon} - \frac{c}{6} \left( 1 + \frac{\pi c \mu}{32 l^2} \left( \frac{1}{3\pi^2} + \frac{1}{2} \right) \right) - \frac{\pi c^2 \mu}{192 l^2} \left( \frac{1}{3\pi^2} + \frac{1}{2} \right) z_{\infty},
\]

\[
C^{BTZ}_{T} = \frac{c}{3} \left( 1 + \frac{\pi c \mu}{64 l^2} \right) \frac{1}{\epsilon} - \frac{\beta c}{24 l} - \frac{\pi^2 c l}{6} \left( \frac{\beta c}{128 l^3} + \frac{\pi^2 c l}{48 l^3} - \frac{c}{192 l^3} \right), \tag{51}
\]

where \( z_{\infty} (\rightarrow \infty) \) is the IR cut off in the \( z \) integral. This IR divergence might be related to the non-local nature of \( T \) deformed CFTs [29]. Remarkably, the complexity of formation, \( C_{BTZ}^{AdS} - 2C^{AdS} \), in these theories is UV finite same as with 2D CFTs. This result that the UV divergence of \( C_{BTZ}^{AdS} \) is independent from its mass is in agreement with CA holographic complexity proposal [34].

We would like close this section by pointing the interesting consistency of the action [48]. We show that the stress tensor obtained from the first law of entanglement entropy is same as the one obtained from the action [48]. Let us remind that the change in entanglement entropy under a small variation of a quantum state is captured by the variation in the Weyl factor field, \( \Omega(z) \rightarrow \Omega_0(z) + \delta \Omega(z) \). This fact implies that for a small entangling region \( A = [-R/2, R/2] \), the change in entanglement entropy becomes

\[
\Delta S_A \simeq \frac{c}{6} \int ds e^{\Omega(z)} \delta \Omega(z) = \frac{c R^2}{24} \partial_z^2 \delta \Omega(z), \tag{52}
\]

where for the thermofield double solution (29) we have

\[
\delta \Omega(z) = \frac{2\pi^2}{3\beta^2} z^2 + O(z^4), \tag{53}
\]

with \( \beta' \) is given by [32]. Therefore, according to the first law of entanglement entropy [28]

\[
\Delta S_A \simeq \frac{\pi R^2}{3} T_{tt}, \tag{54}
\]

the energy density becomes

\[
T_{tt} = \frac{\pi c}{6y^2}. \tag{55}
\]

Independently, by taking the metric variation from the action [48] and noting that \( b_1 = -\Lambda, b_2 = -3\Lambda^2/16, b_3 = 3/16 \), the corresponding stress tensor components are given by

\[
\begin{align*}
\frac{12\pi c}{e} T_{ww} &= \partial^2 \Omega - (\partial \Omega)^2 - \frac{3}{4} \beta e^{-2V} \left( \partial \partial \Omega - 6\partial \Omega \partial^2 \Omega + 2 \left[ (\partial \Omega)^2 - \partial^2 \Omega \right] \partial \partial \Omega \right), \\
\frac{12\pi c}{e} T_{ww} &= \partial^2 \Omega - (\partial \Omega)^2 - \frac{3}{4} \beta e^{-2V} \left( \partial \partial \Omega - 6\partial \Omega \partial^2 \Omega + 2 \left[ (\partial \Omega)^2 - \partial^2 \Omega \right] \partial \partial \Omega \right), \\
\frac{12\pi c}{e} T_{ww} &= \frac{e^{2\Omega} - 1}{4l^2} - \partial \partial \Omega + \frac{3}{64} \left( \frac{e^{2\Omega} - 1}{l^4} + 16e^{-2V} \right) \\
&+ 4\partial \partial \Omega \partial \partial \Omega, \tag{56}
\end{align*}
\]

which for the solution [29], they give together exactly the energy density [50]. One can also see that for deformed primary states [22], the energy density obtained from the action [48] becomes

\[
T_{tt} = - \frac{a^2 c}{24 \pi l^2} \left( 1 - \frac{\bar{\mu}}{8\pi^2 l^2} \right), \tag{57}
\]

which matches with the one obtained from the first law of entanglement entropy [54]. Last but not least, the energy densities [51] and [57] are in complete agreement with the variational principle analysis of [21]. The authors of [21] have shown that if a 2D CFT lives on the conformal boundary \( g_{\alpha \beta} \) in [48], the deformed stress tensor becomes

\[
T_{\mu \nu}^{(2)} = \frac{1}{8\pi G} g^{(2)}_{\alpha \beta} + \frac{\mu}{64\pi G^2 l^2} g^{(4)}_{\alpha \beta}. \tag{58}
\]

For example for the BTZ solution, \( g^{(2)}_{tt} = 2\pi^2 l^2 / \beta^2 \), \( g^{(4)}_{tt} = \pi^4 l^4 / \beta^4 \) and \( c = 3l / 2G \) which they together imply that the deformed energy density [48] exactly matches [50]. More intriguingly, the deformed energy density [55], for positive value of \( \mu \), is nothing just the tt-component of Brown-York stress tensor on the \( \rho = \rho_c \) surface with \( \rho_c \) is given by [41]. But without any need to this bulk cut off interpretation, the stress tensor [50] is conserved and it satisfies the Zamolodchikov’s flow equation.

**SUMMARY AND OUTLOOK**

We have shown that the path integral optimization approach, for \( T^2 \) deformation of 2D CFTs, implies that
the time slice of optimized geometries indeed capture the entire of spacetime. For positive deformation coupling, these optimized solutions can be reinterpreted as a geometry at finite cut-off radius. In this letter we studied the entanglement entropy and quantum complexity for those optimized solutions. Another interesting quantities which should be studied are correlation functions. Investigating the relation between entanglement of purification and holographic entanglement wedge cross section in this context is also an intriguing future problem.

ACKNOWLEDGMENTS

We wish to thank Mohsen Alishahiha, Mehregan Doroudian, Thomas Hartman, Tadashi Takayanagi and Alexander B. Zamolodchikov for fruitful discussions. We are also very grateful to Pawel Caputa for carefully reading the draft and for his valuable comments. This article is part of the PhD project of Hamed Zolfi under the joint supervision of Vahid Karimipour from Sharif University of Technology and Ali Naseh from IPM.

* Electronic address: ghjafari@ipm.ir
† Electronic address: naseh@ipm.ir
‡ Electronic address: hamed.Zolfi@physics.sharif.edu

[1] S. Ryu and T. Takayanagi, Phys. Rev. Lett. 96, 181602 (2006)
[2] M. Van Raamsdonk, Gen. Rel. Grav. 42, 2323 (2010) [Int. J. Mod. Phys. D 19, 2429 (2010)]
[3] B. Swingle, Phys. Rev. D 86, 065007 (2012)
[4] L. Susskind, Fortsch. Phys. 64, 49 (2016)
[5] J. Maldacena and L. Susskind, Fortsch. Phys. 61, 781 (2013)
[6] L. Susskind, Fortsch. Phys. 64, 24 (2016)
[7] M. Alishahiha, Phys. Rev. D 92, no. 12, 126009 (2015)
[8] A. R. Brown, D. A. Roberts, L. Susskind, B. Swingle and Y. Zhao, Phys. Rev. Lett. 116, no. 19, 191301 (2016)
[9] R. Jefferson and R. C. Myers, JHEP 1710, 107 (2017)
[10] S. Chapman, M.P. Heller, H. Marrochio and F. Pastawski, Phys. Rev. Lett. 120, no. 12, 121602 (2018)
[11] Michael A. Nielsen, Mark R. Dowling, Mile Gu, Andrew C. Doherty Science.1121541
[12] P. Caputa, N. Kundu, M. Miyaji, T. Takayanagi and K. Watanabe, Phys. Rev. Lett. 119, no. 7, 071602 (2017)
[13] P. Caputa, N. Kundu, M. Miyaji, T. Takayanagi and K. Watanabe, JHEP 1711, 097 (2017)
[14] A. Bhattacharyya, P. Caputa, S. R. Das, N. Kundu, M. Miyaji and T. Takayanagi, JHEP 1807, 086 (2018)
[15] T. Takayanagi, JHEP 1812, 048 (2018)
[16] P. Caputa and J. M. Magan, Phys. Rev. Lett. 122, no. 23, 231302 (2019)
[17] H. A. Camargo, M. P. Heller, R. Jefferson and J. Knaute, Phys. Rev. Lett. 123, no. 1, 011601 (2019)
[18] A. B. Zamolodchikov, hep-th/0401146
[19] F. A. Smirnov and A. B. Zamolodchikov, Nucl. Phys. B 915, 363 (2017)
[20] L. McGough, M. Mezei and H. Verlinde, JHEP 1804, 010 (2018)
[21] M. Guica and R. Monten, arXiv:1906.11251 [hep-th].
[22] A. Akhavan, M. Alishahiha, A. Naseh and H. Zolfi, JHEP 1812, 090 (2018)
[23] E. Witten, hep-th/0112258
[24] S. de Haro, S. N. Solodukhin and K. Skenderis, Commun. Math. Phys. 217, 595 (2001)
[25] B. Chen, L. Chen and P. X. Hao, Phys. Rev. D 98, no. 8, 086025 (2018)
[26] W. Donnelly and V. Shyam, Phys. Rev. Lett. 121, no. 13, 131602 (2018)
[27] S. Chapman, H. Marrochio and R. C. Myers, JHEP 1701, 062 (2017)
[28] J. Bhattacharya, M. Nozaki, T. Takayanagi and T. Uga-jin, Phys. Rev. Lett. 110, no. 9, 091602 (2013)
[29] J. Cardy, arXiv:1907.03394.
[30] Based on bulk cut-off interpretation, the holographic complexity (CA) of thermofield double state in TT deformed CFTs is studied in [22].
[31] We have considered our different convention in defining the TT operator with [21] in coefficient of g(2) term.
[32] An easy way to see this result is substituting MBTZ with −1/8G in (43) together with using (34).
[33] We would like to mention here that for calculating CBTZ we have used 1/β′, [52], not 1/β. If one uses the undeformed temperature, [52], the complexity contains new UV divergence at order O(1/ε2) which is clearly in contradiction with holographic complexity of TT deformed CFTs.
[34] The holographic complexity of formation for BTZ black hole [53] can be found in [27].