A GEOMETRIC CONDITION, NECESSITY OF ENERGY, AND TWO WEIGHT BOUNDEDNESS OF FRACTIONAL RIESZ TRANSFORMS

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Abstract. Let $\sigma$ and $\omega$ be locally finite positive Borel measures on $\mathbb{R}^n$ with no common point masses. We assume that at least one of the two measures $\sigma$ and $\omega$ is supported on a line in $\mathbb{R}^n$. Let $R^{\alpha,n}$ be the $\alpha$-fractional Riesz transform vector on $\mathbb{R}^n$. We prove that the energy conditions in arXiv:1302.5093v7 are implied by the $A_2$ and cube testing conditions for $R^{\alpha,n}$. Then we apply the main theorem there to give a $T1$ theorem for $R^{\alpha,n}$: namely that $R^{\alpha,n}$ is bounded from $L^2(\sigma)$ to $L^2(\omega)$ if and only if the $A_2$ conditions hold, the cube testing conditions for $R^{\alpha,n}$ and its dual both hold, and the weak boundedness property for $R^{\alpha,n}$ holds.

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1. Introduction

In [SaShUr], under a side assumption that certain energy conditions hold, the authors show in particular that the two weight inequality

\[ \| R^{\alpha,n} (f \sigma) \|_{L^2(\omega)} \lesssim \| f \|_{L^2(\sigma)}, \]

for the vector of Riesz transforms $R^{\alpha,n}$ in $\mathbb{R}^n$ (with $0 \leq \alpha < n$) holds if and only if the $A_2$ conditions hold, the cube testing conditions hold, and the weak boundedness property holds. It is not known at the time of this writing whether or not these or any other energy conditions are necessary for any vector $T^{\alpha,n}$ of fractional singular integrals in $\mathbb{R}^n$ with $n \geq 2$, apart from the trivial case of positive operators. In
particularly there are no known counterexamples. We also showed in [SaShUr2] and [SaShUr3] that the technique of reversing energy, typically used to prove energy conditions, fails spectacularly in higher dimension (and we thank M. Lacey for showing us this failure for the Cauchy transform with the circle measure). See also the counterexamples for the fractional Riesz transforms in [LaWi2].

The purpose of this paper is to show that if $\sigma$ and $\omega$ are locally finite positive Borel measures without common point masses, and at least one of the two measures $\sigma$ and $\omega$ is supported on a line in $\mathbb{R}^n$, then the energy conditions are indeed necessary for boundedness of the fractional Riesz transform $R_{\alpha}^{\sigma,n}$, and hence that a T1 theorem holds for $R_{\alpha}^{\sigma,n}$. M. Lacey and B. Wick [LaWi] have independently obtained a similar result for the Cauchy transform in the plane, and the five authors have combined on the paper [LaSaShUrWi]. The vector of $\alpha$-fractional Riesz transforms is given by

$$R_{\alpha}^{\sigma,n} = \{R_{\ell}^{\sigma,n} : 1 \leq \ell \leq n\}, \quad 0 \leq \alpha < n,$$

where the component Riesz transforms $R_{\ell}^{\sigma,n}$ are the convolution fractional singular integrals $R_{\ell}^{\sigma,n} f \equiv K_{\ell}^{\alpha,n} * f$ with odd kernel defined by

$$K_{\ell}^{\alpha,n} (w) \equiv c_{\alpha,n} \frac{w^\ell}{|w|^{n+1-\alpha}}.$$ 

Finally, we remark that the T1 theorem under this geometric condition has application to the weighted discrete Hilbert transform $H_{(\Gamma,v)}$ when the sequence $\Gamma$ is supported on a line in the complex plane. See [BeMeSe] where $H_{(\Gamma,v)}$ is essentially the Cauchy transform with $n = 2$ and $\alpha = 1$.

We now recall a special case of our main two weight theorem from [SaShUr]. Let $Q^n$ denote the collection of all cubes in $\mathbb{R}^n$, and denote by $D^n$ a dyadic grid in $\mathbb{R}^n$. The definitions of the remaining terms used below will be given in the next section.

**Theorem 1.** Suppose that $R_{\alpha}^{\sigma,n}$ is the vector of $\alpha$-fractional Riesz transforms in $\mathbb{R}^n$, and that $\omega$ and $\sigma$ are positive Borel measures on $\mathbb{R}^n$ without common point masses. Set $R_{\sigma}^{\alpha,n} f = R_{\alpha}^{\sigma,n} (f)$ for any smooth truncation of $R_{\alpha}^{\sigma,n}$.

1. Suppose $0 \leq \alpha < n$ and that $\gamma \geq 2$ is given. Then the operator $R_{\sigma}^{\alpha,n}$ is bounded from $L^2(\sigma)$ to $L^2(\omega)$, i.e.

$$(1.2) \quad \|R_{\sigma}^{\alpha,n} f\|_{L^2(\omega)} \leq M_{R_{\sigma}^{\alpha,n}} \|f\|_{L^2(\sigma)},$$

uniformly in smooth truncations of $T^n$, and moreover

$$M_{R_{\sigma}^{\alpha,n}} \leq C_{\alpha} \left( A_{2}^{\sigma} + \mathbb{A}_{2}^{\sigma} \right. + K_{R_{\sigma}^{\alpha,n}} + \mathbb{K}_{R_{\sigma}^{\alpha,n}} + \mathcal{E}_{\alpha} + \mathcal{E}^*_{\alpha} + \mathcal{WBP}_{R_{\sigma}^{\alpha,n}} \left. \right),$$

provided that the two dual $A_{2}$ conditions hold, and the two dual testing conditions for $R_{\sigma}^{\alpha,n}$ hold, the weak boundedness property for $R_{\sigma}^{\alpha,n}$ holds for a sufficiently large constant $C$ depending on the goodness parameter $r$, and provided that the two dual energy conditions $E_{\alpha} + E^*_{\alpha} < \infty$ hold uniformly over all dyadic grids $D^n$, and where the goodness parameters $r$ and $\varepsilon$ implicit in the definition of $M_{r-deep}^{\ell}(K)$ are fixed sufficiently large and small respectively depending on $n$, $\alpha$ and $\gamma$.

2. Conversely, suppose $0 \leq \alpha < n$ and that the Riesz transform vector $R_{\sigma}^{\alpha,n}$ is bounded from $L^2(\sigma)$ to $L^2(\omega)$,

$$\|R_{\sigma}^{\alpha,n} f\|_{L^2(\omega)} \leq M_{R_{\sigma}^{\alpha,n}} \|f\|_{L^2(\sigma)}.$$
Then the testing conditions and weak boundedness property hold for $R_2^\alpha,n$, the fractional $A_2^\alpha$ conditions hold, and moreover,

$$\sqrt{A_2^\alpha + A_2^\alpha + \mathcal{T}_{R_2^\alpha,n} + \mathcal{T}_{R_2^\alpha,n} + \mathcal{WBP}_{R_2^\alpha,n}} \leq C n R_2^\alpha,n.$$  

**Problem 1.** It is an open question whether or not the energy conditions are necessary for boundedness of $R_2^\alpha,n$. See [SaShUr3] for a failure of energy reversal in higher dimensions - such an energy reversal was used in dimension $n = 1$ to prove the necessity of the energy condition for the Hilbert transform.

**Remark 1.** The boundedness of an individual operator $T^\alpha$ cannot in general imply the finiteness of either $A_2^\alpha$ or $\mathcal{E}_\alpha$. For a trivial example, if $\sigma$ and $\omega$ are supported on the $x$-axis in the plane, then the second Riesz transform $R_2$ is the zero operator from $L^2(\sigma)$ to $L^2(\omega)$, simply because the kernel $K_2((x_1,0),(y_1,0)) = \frac{1}{|x_1-y_1|^{\alpha,n}} = 0$.

**Remark 2.** In [LaWi2], M. Lacey and B. Wick use the NTV technique of surgery to show that the weak boundedness property for the Riesz transform vector $R_2^\alpha,n$ is implied by the $A_2^\alpha$ and cube testing conditions, and this has the consequence of eliminating the weak boundedness property as a condition from the statement of Theorem 1.

The next result shows that the energy conditions are in fact necessary for boundedness of the Riesz transform vector $R_2^\alpha,n$ when one of the measures is supported on a line.

**Theorem 2.** Let $\sigma$ and $\omega$ be locally finite positive Borel measures on $\mathbb{R}^n$ with no common point masses. Suppose that $R_2^\alpha,n$ is the fractional Riesz transform with $0 \leq \alpha < n$, and consider the tangent line truncations for $R_2^\alpha,n$ in the testing conditions. If at least one of the measures $\sigma$ and $\omega$ is supported on a line, then

$$\mathcal{E}_\alpha \lesssim \sqrt{A_2^\alpha + \mathcal{T}_{R_2^\alpha,n}} \quad \text{and} \quad \mathcal{E}_\alpha^* \lesssim \sqrt{A_2^\alpha + \mathcal{T}_{R_2^\alpha,n}}.$$  

If we combine Theorems 2 and 1, we obtain the following theorem as a corollary, which generalizes the T1 theorem for the Hilbert transform ([Lac], [LaSaShUr3]). See also related work in the references given at the end of the paper. We use notation as in Theorem 1.

**Theorem 3.** Let $\sigma$ and $\omega$ be locally finite positive Borel measures on $\mathbb{R}^n$ with no common point masses. Suppose that $R_2^\alpha,n$ is the fractional Riesz transform with $0 \leq \alpha < n$. Set $R_2^\alpha,n f = R_2^\alpha,n (f \sigma)$ for any smooth truncation of $R_2^\alpha,n$. If at least one of the measures $\sigma$ and $\omega$ is supported on a line, then the operator norm $\mathfrak{N}_{R_2^\alpha,n}^\alpha$ of $R_2^\alpha,n$ as an operator from $L^2(\sigma)$ to $L^2(\omega)$, uniformly in smooth truncations, satisfies

$$\mathfrak{N}_{R_2^\alpha,n} \approx C_{\alpha} \left( \sqrt{A_2^\alpha + A_2^\alpha + \mathcal{T}_{R_2^\alpha,n} + \mathcal{T}_{R_2^\alpha,n} + \mathcal{WBP}_{R_2^\alpha,n}} \right).$$  

2. **Definitions**

As mentioned above, the $\alpha$-fractional Riesz vector $R_2^\alpha,n = \{ R_\ell^\alpha,n : 1 \leq \ell \leq n \}$ has as components the Riesz transforms $R_\ell^\alpha,n$ with odd kernel $K_{\ell}^\alpha,w (|w|)$ of $R_\ell^\alpha$ and $\mathcal{T}_{R_\ell^\alpha,n}$. The tangent line truncation of the Riesz transform $R_\ell^\alpha,n$ has kernel $\Omega_{\ell} (w) \psi_{\ell,R} (|w|)$.
where \( \psi_{\delta,R}^\alpha \) is continuously differentiable on an interval \((0, S)\) with \(0 < \delta < R < S\), and where \( \psi_{\delta,R}^\alpha (r) = r^{\alpha - n} \) if \( \delta \leq r \leq R \), and has constant derivative on both \((0, \delta)\) and \((R, S)\) where \( \psi_{\delta,R}^\alpha (S) = 0 \). As shown in the one dimensional case in [LaSaShUr3], boundedness of \( R_n^{\alpha,\ell} \) with one set of appropriate truncations together with the \( A_2^\alpha \) condition below, is equivalent to boundedness of \( R_n^{\alpha,\ell} \) with all truncations.

2.1. Cube testing, the weak boundedness property, and the \( A_2^\alpha \) conditions. The following ‘dual’ cube testing conditions are necessary for the boundedness of \( R_n^{\alpha,\ell} \) from \( L^2(\sigma) \) to \( L^2(\omega) \):

\[
\mathcal{T}^2_{R_n^{\alpha,\ell}} = \sup_{Q \in \mathcal{Q}_n} \frac{1}{|Q|_\sigma} \int_Q |R_n^{\alpha,\ell} (1_Q \sigma)|^2 \omega < \infty,
\]

\[
(\mathcal{T}^*_{R_n^{\alpha,\ell}})^2 = \sup_{Q \in \mathcal{Q}_n} \frac{1}{|Q|_\omega} \int_Q |(R_n^{\alpha,\ell})^* (1_Q \omega)|^2 \sigma < \infty.
\]

The weak boundedness property for \( R_n^{\alpha,\ell} \) with constant \( C \) is given by

\[
\int_Q R_n^{\alpha,\ell} (1_Q \sigma) d\omega \leq \text{WBP}_{R_n^{\alpha,\ell}} \sqrt{|Q|_\sigma |Q|_\omega},
\]

for all cubes \( Q, Q' \) with \( \frac{1}{C} \leq \frac{|Q|_\sigma^\frac{1}{\ell}}{|Q'|_\omega^\frac{1}{\ell}} \leq C \),

and either \( Q \subset 3Q' \setminus Q' \) or \( Q' \subset 3Q \setminus Q \).

Now let \( \mu \) be a locally finite positive Borel measure on \( \mathbb{R}^n \), and suppose \( Q \) is a cube in \( \mathbb{R}^n \). The two \( \alpha \)-fractional Poisson integrals of \( \mu \) on a cube \( Q \) are given by:

\[
\mathcal{P}^\alpha (Q, \mu) = \int_{\mathbb{R}^n} \left( \frac{|Q|_\sigma^\frac{1}{\ell}}{|Q|_\sigma^\frac{1}{\ell} + |x - x_Q|} \right)^{n+\alpha} d\mu (x),
\]

\[
\mathcal{P}^\alpha (Q, \mu) = \int_{\mathbb{R}^n} \left( \frac{|Q|_\omega^\frac{1}{\ell}}{|Q|_\omega^\frac{1}{\ell} + |x - x_Q|^2} \right)^{n-\alpha} d\mu (x).
\]

We refer to \( \mathcal{P}^\alpha \) as the standard Poisson integral and to \( \mathcal{P}^\alpha \) as the reproducing Poisson integral. Let \( \sigma \) and \( \omega \) be locally finite positive Borel measures on \( \mathbb{R}^n \) with no common point masses, and suppose \( 0 \leq \alpha < n \). The classical \( A_2^\alpha \) constant is defined by

\[
A_2^\alpha = \sup_{Q \in \mathcal{Q}_n} \frac{|Q|_\sigma}{|Q|_\sigma^{1-\frac{\alpha}{n}}} \frac{|Q|_\omega}{|Q|_\omega^{1-\frac{\alpha}{n}}},
\]

and the one-sided constants \( A_2^{\alpha,*} \) and \( A_2^{\alpha,*,*} \) for the weight pair \((\sigma, \omega)\) are defined by

\[
A_2^{\alpha,*} = \sup_{Q \in \mathcal{Q}_n} \mathcal{P}^\alpha (Q, \sigma) \frac{|Q|_\omega}{|Q|_\omega^{1-\frac{\alpha}{n}}} < \infty,
\]

\[
A_2^{\alpha,*,*} = \sup_{Q \in \mathcal{Q}_n} \mathcal{P}^\alpha (Q, \omega) \frac{|Q|_\sigma}{|Q|_\sigma^{1-\frac{\alpha}{n}}} < \infty.
\]
2.2. Energy conditions. We begin by briefly recalling some of the notation used in [SaShUr]. Given a dyadic cube $K \in \mathcal{D}$ and a positive measure $\mu$ we define the Haar projection $P_K^\mu = \sum_{J \in \mathcal{G}(K)} \Delta^\mu_J$. Now we recall the definition of a *good* dyadic cube - see [NTV4] and [LaSaUr2] for more detail.

**Definition 1.** Let $r \in \mathbb{N}$ and $0 < \varepsilon < 1$. A dyadic cube $J$ is $(r, \varepsilon)$-good, or simply good, if for every dyadic supercube $I$, it is the case that either $J$ has side length at least $2^{-r}$ times that of $I$, or $J \Subset I$ is $(r, \varepsilon)$-deeply embedded in $I$.

Here we say that a dyadic cube $J$ is $(r, \varepsilon)$-deeply embedded in a dyadic cube $K$, or simply $r$-deeply embedded in $K$, which we write as $J \Subset_r K$, when $J \subset K$ and both

\[
|J|^{\frac{1}{r}} \leq 2^{-r} |K|^{\frac{1}{r}},
\]

\[
\text{dist} (J, \partial K) \geq \frac{1}{2} |J|^{\frac{1}{r}} |K|^{-\frac{1}{r}}.
\]

We say that $J$ is $r$-nearby in $K$ when $J \subset K$ and

\[
|J|^{\frac{1}{r}} > 2^{-r} |K|^{\frac{1}{r}}.
\]

We denote the set of such good dyadic cubes by $\mathcal{D}_{\text{good}}$.

Then we define the smaller ‘good’ Haar projection $P^\mu_{\text{good}}$ by

\[
P^\mu_{\text{good}} f = \sum_{J \in \mathcal{G}(K)} \Delta^\mu_J f,
\]

where $\mathcal{G}(K)$ consists of the good subcubes of $K$:

\[
\mathcal{G}(K) \equiv \{ J \in \mathcal{D}_{\text{good}} : J \subset K \},
\]

and also the larger ‘subgood’ Haar projection $P^\mu_{\text{subgood}}$ by

\[
P^\mu_{\text{subgood}} f = \sum_{J \in \mathcal{M}_{\text{good}}(K)} \sum_{J' \subset J} \Delta^\mu_J f,
\]

where $\mathcal{M}_{\text{good}}(K)$ consists of the maximal good subcubes of $K$. We thus have

\[
\| P^\mu_{\text{good}} f \|^2_{L^2(\mu)} \leq \| P^\mu_{\text{subgood}} f \|^2_{L^2(\mu)} \leq \| P^\mu_I f \|^2_{L^2(\mu)} = \int_I \left| \frac{1}{|I|_\mu} \int_I x \, dx \right|^2 d\mu(x), \quad x = (x_1, \ldots, x_n),
\]

where $P^\mu_I f$ is the orthogonal projection of the identity function $x : \mathbb{R}^n \to \mathbb{R}^n$ onto the vector-valued subspace of $\oplus_{k=1}^n L^2(\mu)$ consisting of functions supported in $I$ with $\mu$-mean value zero.

We use the collection $\mathcal{M}_{r-\text{deep}}(K)$ of maximal $r$-deeply embedded dyadic subcubes of a dyadic cube $K$. We let $J^* = \gamma J$ where $\gamma \geq 2$. The goodness parameter $r$ is chosen sufficiently large, depending on $\varepsilon$ and $\gamma$, that the bounded overlap property

\[
\sum_{J \in \mathcal{M}_{r-\text{deep}}(K)} 1_{J^*} \leq \beta 1_K,
\]

holds for some positive constant $\beta$ depending only on $n, \gamma, r$ and $\varepsilon$. We will also need the following refinement of $\mathcal{M}_{r-\text{deep}}(K)$ for each $\ell \geq 0$ that consists of some
of the maximal cubes $Q$, whose $\ell$-fold dyadic parent $\pi^\ell Q$ is $r$-deeply embedded in $K$:

$$\mathcal{M}_{r-deep}^\ell(K) \equiv \{ J \in \mathcal{M}_{r-deep}^\ell(\pi^\ell K) : J \subset L \text{ for some } L \in \mathcal{M}_{\text{deep}}(K) \}.$$  

Since $J \in \mathcal{M}_{r-deep}^\ell(K)$ implies $\gamma J \subset K$, we also have from (2.2) that

$$\sum_{J \in \mathcal{M}_{r-deep}^\ell(K)} 1_J \beta_1 K, \quad \text{for each } \ell \geq 0.$$  

(2.3)

Of course $\mathcal{M}_{r-deep}^0(K) = \mathcal{M}_{r-deep}(K)$, but $\mathcal{M}_{r-deep}^\ell(K)$ is in general a finer subdecomposition of $K$ the larger $\ell$ is, and may in fact be empty. The following definition of the energy constant $\mathcal{E}_\alpha$ is larger than that used in [SaShUr].

**Definition 2.** Suppose $\sigma$ and $\omega$ are positive Borel measures on $\mathbb{R}^n$ without common point masses. Then the energy condition constant $\mathcal{E}_\alpha$ is given by

$$\left( \mathcal{E}_\alpha \right)^2 \equiv \sup_{\ell \geq 0} \sup_{I = \cup I_r} \sum_{\ell = 1}^{\infty} \sum_{J \in \mathcal{M}_{r-deep}^\ell(I_r)} \left( \frac{\mathcal{P}_J^\alpha (J \setminus \gamma J, \sigma)}{|J|^{\frac{1}{2}}} \right)^2 \left\| \mathcal{P}_J^{\text{subgood}, \omega} \right\|_{L^2(\omega)}^2,$$

where $\sup_I = \cup I_r$ above is taken over

1. all dyadic grids $\mathcal{D}$,
2. all $\mathcal{D}$-dyadic cubes $I$,
3. and all subpartitions $\{I_r\}$ of the cube $I$ into $\mathcal{D}$-dyadic subcubes $I_r$.

There is a similar definition for the dual (backward) energy condition that simply interchanges $\sigma$ and $\omega$ everywhere. These definitions of the energy conditions depend on the choice of goodness parameters $r$ and $\varepsilon$. We can ‘plug the $\gamma$-hole’ in the Poisson integral $\mathcal{P}_J^\alpha (J \setminus \gamma J, \sigma)$ using the $A_2^\alpha$ condition and the bounded overlap property $E_{\alpha}$.

Indeed, with

$$\left( \mathcal{E}_{\alpha}^{\text{plug}} \right)^2 \equiv \sup_{\ell \geq 0} \sup_{I = \cup I_r} \sum_{\ell = 1}^{\infty} \sum_{J \in \mathcal{M}_{r-deep}^\ell(I_r)} \left( \frac{\mathcal{P}_J^\alpha (J \setminus \gamma J, \sigma)}{|J|^{\frac{1}{2}}} \right)^2 \left\| \mathcal{P}_J^{\text{subgood}, \omega} \right\|_{L^2(\omega)}^2,$$

we have, as shown in [SaShUr], that

$$\left( \mathcal{E}_{\alpha}^{\text{plug}} \right)^2 \lesssim \left( \mathcal{E}_{\alpha} \right)^2 + \beta A_2^\alpha,$$

upon using (2.3).

2.3. **Energy lemma.** We will need the following elementary special case of the Energy Lemma from [SaShUr].

**Lemma 1 (Energy Lemma).** Let $J$ be a cube in $\mathcal{D}$, let $\Psi_J$ be an $L^2(\omega)$ function supported in $J$ and with $\omega$-integral zero. Let $\nu$ be a positive measure supported in $\mathbb{R}^n \setminus \gamma J$ with $\gamma \geq 2$. Then we have

$$\left| \langle \mathcal{R}^{\alpha,n}(\nu), \Psi_J \rangle_{\mathcal{D}} \right| \lesssim \| \Psi_J \|_{L^2(\omega)} \left( \frac{\mathcal{P}_J^\alpha (J, \nu)}{|J|^{\frac{1}{2}}} \right) \| \mathcal{P}_J^{\omega} x \|_{L^2(\omega)}.$$


3. One measure supported in a line

In this section we prove Theorem 2, i.e. we prove the necessity of the energy conditions for the \( A^*_2 \) conditions and the testing conditions \( \mathcal{T}_{R^{\alpha,n}} \) and \( \mathcal{T}_{R^{\alpha,n}}^{\mathcal{R}} \) associated to the tangent line truncations of the \( \alpha \)-fractional Riesz transform \( R^{\alpha,n} \), when just one of the measures \( \sigma \) or \( \omega \) is supported in a line \( L \), and the other measure is arbitrary. The one-dimensional character of just one of the measures is enough to circumvent the failure of strong reversal of energy as described in [SaShUr2] and [SaShUr3].

Fix a dyadic grid \( D \), and suppose that \( \omega \) is supported in a line \( L \). We will show that both energy conditions hold relative to \( D \). We can suppose that \( L \) is the \( x_1 \)-axis, since using that the Riesz transform vector \( R^{\alpha,n} \) is rotation invariant, one can verify that the argument below does not depend in a critical way on this or any other special relationship between \( D \) and \( L \).

3.1. Backward energy condition. The dual (backward) energy condition \( E^*_\alpha \subseteq \mathcal{T}_{R^{\alpha,n}} + \sqrt{A^{\alpha}_2} \) is the more straightforward of the two to verify, and so we turn to it first. We must show

\[
\sup_{\ell \geq 0} \sum_{r=1}^\infty \sum_{J \in M_{\text{deep}}(I_r)} \left( \frac{P^n(J,1_{I \cap J \omega})}{|J|^\frac{1}{n}} \right)^2 \left( \nu^\text{subgood, } \sigma \right) x \left( \frac{1}{L^2(\sigma)} \right) \leq \left( (\mathcal{T}_{R^{\alpha,n}})^* + A^{\alpha}_2 \right) |I| \omega,
\]

for all partitions of a dyadic cube \( I = \bigcup_{r=1}^\infty I_r \) into dyadic subcubes \( I_r \). We fix \( \ell \geq 0 \) and suppress both \( \ell \) and \( r \) in the notation \( M_{\text{deep}}(I_r) = M_{\ell-\text{deep}}(I_r) \). Recall that \( J^* = \gamma J \), and that the bounded overlap property \( (2.3) \) holds. We may of course assume that \( I \) intersects the \( x_1 \)-axis \( L \). Now we set \( M_{\text{deep}} = \bigcup_{r=1}^\infty M_{\text{deep}}(I_r) \) and write

\[
\sum_{r=1}^\infty \sum_{J \in M_{\text{deep}}(I_r)} \left( \frac{P^n(J,1_{I \cap J \omega})}{|J|^\frac{1}{n}} \right)^2 \left( \nu^\text{subgood, } \sigma \right) x \left( \frac{1}{L^2(\sigma)} \right) = \sum_{J \in M_{\text{deep}}} \left( \frac{P^n(J,1_{I \cap J \omega})}{|J|^\frac{1}{n}} \right)^2 \left( \nu^\text{subgood, } \sigma \right) x \left( \frac{1}{L^2(\sigma)} \right).
\]

Let \( 2 < \gamma' < \gamma \) where both \( \gamma' \) and \( \frac{1}{n-\alpha} \) will be taken sufficiently large for the arguments below to be valid - see both \( (3.6) \) and \( (3.8) \) below. For example taking \( \gamma' = \sqrt{\gamma} \) and \( \gamma \gg (n-\alpha)^{-2} \) works, but is far from optimal. We will consider the cases \( \gamma' J \cap L = \emptyset \) and \( \gamma' J \cap L \neq \emptyset \) separately.

Suppose \( \gamma' J \cap L = \emptyset \). There is \( c > 0 \) and a finite sequence \( \{\xi_k\}_{k=1}^N \in \mathbb{N}^{n-1} \) (actually of the form \( \xi_k = (0, \xi_k^1, \ldots, \xi_k^n) \)) with the following property. For each \( J \in M_{\text{deep}} \) with \( \gamma' J \cap L = \emptyset \), there is \( 1 \leq k = k(J) \leq N \) such that for \( y \in J \) and \( x \in I \cap L \), the linear combination \( \xi_k \cdot K^{n,n}(y,x) \) is positive and satisfies

\[
\xi_k \cdot K^{n,n}(y,x) = \frac{\xi_k \cdot (y-x)}{|y-x|^{n+1-\alpha}} \geq c \frac{|J|^{\frac{1}{n}}}{|y-x|^{n+1-\alpha}}.
\]

For example, in the plane \( n = 2 \), if \( J \) lies above the \( x_1 \)-axis \( L \), then for \( y \in J \) and \( x \in L \) we have \( y_2 \geq (\gamma' - 1)|J|^{\frac{1}{n}} > |J|^{\frac{1}{n}} \) and \( x_2 = 0 \), hence the estimate

\[
(0,1) \cdot K^{n,n}(y,x) = \frac{y_2 - x_2}{|y-x|^{n+1-\alpha}} \geq \frac{|J|^{\frac{1}{n}}}{|y-x|^{n+1-\alpha}}.
\]
For $J$ below $L$ we take the unit vector $(0, -1)$ in place of $(0, 1)$. Thus for $y \in J \in \mathcal{M}_{\text{deep}}$ and $k = k(J)$ we have the following ‘weak reversal’ of energy,

\[
(3.1) \quad |R_{\alpha,n}^{\alpha,n}(1_{I \cap L} \omega)(y)| = \left| \int_{I \cap L} K^{\alpha,n}(y, x) \, d\omega(x) \right| \\
\geq \left| \int_{I \cap L} \xi_k \cdot K^{\alpha,n}(y, x) \, d\omega(x) \right| \\
\geq c \int_{I \cap L} |y - x|^{\frac{1}{n} + 1 - \alpha} \, d\omega(x) \approx cP^\alpha(J, 1_J \omega).
\]

Thus from (3.1) and the pairwise disjointedness of $J \in \mathcal{M}_{\text{deep}}$, we have

\[
\sum_{J \in \mathcal{M}_{\text{deep}}, \gamma'J \cap L = \emptyset} \left( \frac{P^\alpha(J, 1_J \omega)}{|J|^\frac{1}{n} + 1 - \alpha} \right)^2 \| P_{\text{subgood}, \sigma} x \|_{L^2(\sigma)}^2 \leq \sum_{J \in \mathcal{M}_{\text{deep}}, \gamma'J \cap L = \emptyset} P^\alpha(J, 1_J \omega)^2 |J|_\sigma
\]

\[
\leq \int_I |R_{\alpha,n}^{\alpha,n}(1_J \omega)(y)|^2 \, d\sigma(y) \leq (\mathcal{S}_{R_{\alpha,n}}^\alpha)^2 |I|_\omega.
\]

Now we turn to estimating the sum over those cubes $J \in \mathcal{M}_{\text{deep}}$ for which $\gamma'J \cap L \neq \emptyset$. In this case we use the one-dimensional nature of $\omega$ to obtain a strong reversal of one of the partial energies. Recall the Hilbert transform inequality for intervals $J$ and $I$ with $2J \subset I$ and $\text{supp} \, \mu \subset \mathbb{R} \setminus I$:

\[
(3.2) \quad \sup_{y, z \in J} \frac{H_\mu(y) - H_\mu(z)}{y - z} = \left\{ \begin{array}{ll} 1 & \frac{1}{x} - \frac{1}{y} \\ 0 & x \neq y \end{array} \right\} d\mu(x) \\
= \int_{\mathbb{R} \setminus I} \frac{1}{(x - y)(y - z)} d\mu(x) \approx \frac{P(J, \mu)}{|J|}.
\]

We wish to obtain a similar control in the situation at hand, but the matter is now complicated by the extra dimensions. Fix $y = (y^1, y')$, $z = (z^1, z') \in J$ and $x = (x^1, 0) \in L \setminus \gamma J$. We consider first the case

\[
|y' - z'| \leq C_0 |y^1 - z^1|,
\]

where $C_0$ is a positive constant satisfying (3.3) below. Now the first component $R_{1,\gamma}^{\alpha,n}$ is ‘positive’ in the direction of the $x^1$-axis $L$, and so for $(y^1, y'), (z^1, z') \in J$, we write

\[
R_{1,\gamma}^{\alpha,n} 1_{I \cap \gamma J \omega}(y^1, y') - R_{1,\gamma}^{\alpha,n} 1_{I \cap \gamma J \omega}(z^1, z') \\
= \int_{I \cap \gamma J} \left\{ K_{1,\gamma}^{\alpha,n}((y^1, y'), x) - K_{1,\gamma}^{\alpha,n}((z^1, z'), x) \right\} \, d\omega(x) \\
= \int_{I \cap \gamma J} \left\{ \frac{y^1 - z^1}{|y^1 - z^1|^\alpha} \right\} d\omega(x).
\]
For $0 \leq t \leq 1$ define
\[
\begin{align*}
w_t & \equiv ty + (1-t)z = z + t(y-z), \\
w_t - x & = t(y-x) + (1-t)(z-x),
\end{align*}
\]
and
\[
\Phi(t) \equiv \frac{w_t^1 - x^1}{|w_t - x|}^{n+1-\alpha},
\]
so that
\[
\frac{y^1 - x^1}{|y - x|} = \frac{z^1 - x^1}{|z - x|} = \Phi(1) - \Phi(0) = \int_0^1 \Phi'(t) \, dt.
\]
Then using $\nabla |\xi|^\gamma = \gamma |\xi|^{\gamma-2} \xi$ we compute that
\[
\frac{d}{dt} \Phi(t) = \frac{(y^1 - z^1)}{|w_t - x|^{n+1-\alpha}} + (w_t^1 - x^1)(y - z) \cdot \nabla |w_t - x|^{n+1-\alpha}
\]
\[
= \frac{(y^1 - z^1)}{|w_t - x|^{n+1-\alpha}} - (n + 1 - \alpha) (w_t^1 - x^1) \frac{\nabla |w_t - x|^{n+1-\alpha}}{|w_t - x|^{n+3-\alpha}}
\]
\[
= \frac{(y^1 - z^1)}{|w_t - x|^{n+1-\alpha}} - (n + 1 - \alpha) (w_t^1 - x^1) \frac{(w_t^1 - x^1)}{|w_t - x|^{n+3-\alpha}} (y - z)
\]
\[
\approx (y^1 - z^1) \left\{ \frac{|w_t - x|^2}{|w_t - x|^{n+3-\alpha}} - (n + 1 - \alpha) \frac{|w_t^1 - x^1|^2}{|w_t - x|^{n+3-\alpha}} \right\}
\]
\[
= (y^1 - z^1) \left\{ (w_t^1 - x^1) \cdot (y' - z') \right\}
\]
\[
= (y^1 - z^1) \left\{ A(t) + B(t) \right\}.
\]
Now $|w_t^1 - x^1| \approx |y - x|$ and $|w_t^1 - x^1| = |y'| \approx |y'| \leq \gamma' \frac{|y - x|}{\gamma}$ because $\gamma' J \cap L = \emptyset$, and so if $\gamma \gg \gamma'$ we obtain using $|y - x| \approx |w_t^1 - x^1|$ that
\[
|w_t^1 - x^1| \leq \frac{n - \alpha}{2} |w_t^1 - x^1|,
\]
and hence that
\[
-A(t) = -\frac{|w_t - x|^2}{|w_t - x|^{n+3-\alpha}} + (n + 1 - \alpha) \frac{|w_t^1 - x^1|^2}{|w_t - x|^{n+3-\alpha}}
\]
\[
= -|w_t - x|^2 + (n + 1 - \alpha) |w_t^1 - x^1|^2
\]
\[
= -|w_t^1 - x^1|^2 + (n - \alpha) \frac{(w_t^1 - x^1)^2}{|w_t - x|^{n+3-\alpha}}
\]
\[
\approx (n - \alpha) \frac{(w_t^1 - x^1)^2}{|w_t - x|^{n+3-\alpha}}.
\]
Now from our assumption (3.3) we have

$$|B(t)| = \left| (n + 1 - \alpha) \left( \frac{w_i^1 - x^1}{y^1 - z^1} \right) \frac{(w_i' - x') \cdot (y' - z')}{|w_i - x|^{n+3-\alpha}} \right|$$

$$\leq (n + 1 - \alpha) \frac{|w_i^1 - x^1|}{|y^1 - z^1|} \frac{|w_i' - x'| |y' - z'|}{|w_i - x|^{n+3-\alpha}}$$

$$\leq C_0 (n + 1 - \alpha) \frac{|w_i^1 - x^1|}{|w_i - x|^{n+3-\alpha}}$$

$$\leq C_0 (n + 1 - \alpha) \gamma \frac{|w_i^1 - x^1|^2}{|w_i - x|^{n+3-\alpha}} \leq \frac{1}{2} (n - \alpha) \frac{(w_i^1 - x^1)^2}{|w_i - x|^{n+3-\alpha}}$$

if

$$C_0 \leq \frac{\gamma}{2\gamma n + 1 - \alpha}.$$ 

Thus altogether in case (3.3) we have

$$|R_1^{\alpha,n} 1_{I \setminus \gamma,j} \omega \left( y^1, y' \right) - R_1^{\alpha,n} 1_{I \setminus \gamma,j} \omega \left( z^1, z' \right)|$$

$$\approx |y^1 - z^1| \int_{I \setminus \gamma,j} \left| \int_0^1 \frac{1}{x} d\Phi (t) dt \right| d\omega (x)$$

$$\approx |y^1 - z^1| \int_{I \setminus \gamma,j} \int_0^1 \left\{ A(t) + B(t) \right\} dtd\omega (x)$$

$$\approx |y^1 - z^1| \int_{I \setminus \gamma,j} \int_0^1 \left\{ (n - \alpha) \frac{(w_i^1 - x^1)^2}{|w_i - x|^{n+3-\alpha}} \right\} dtd\omega (x)$$

$$\approx |y^1 - z^1| \int_{I \setminus \gamma,j} \frac{(c_j^1 - x^1)^2}{|c_j - x|^{n+3-\alpha}} d\omega (x)$$

$$\approx |y^1 - z^1| \frac{P^\alpha (J, 1_{I \setminus \gamma,j} \omega)}{|J|^{\frac{1}{\alpha}}}.$$ 

On the other hand, in the case that

$$|y' - z'| > C_0 |y^1 - z^1|,$$

we write

$$(R_1^{\alpha,n})' = (R_2^{\alpha,n}, ..., R_n^{\alpha,n}),$$

$$\Phi (t) = \frac{w_i' - x'}{|w_i - x|^{n+1-\alpha}},$$

with $w_i = ty + (1 - t) z$ as before. Then as above we obtain

$$\frac{y' - x'}{|y - x|^{n+1-\alpha}} - \frac{z' - x'}{|z - x|^{n+1-\alpha}} = \Phi (1) - \Phi (0) = \int_0^1 \frac{d}{dt} \Phi (t) dt,$$
where if we write \( \tilde{y}^k \equiv (y^1, \ldots, y^{k-1}, 0, y^{k+1}, \ldots, y^n) \), we have

\[
\frac{d}{dt} \Phi(t) = \left\{ \frac{d}{dt} \Phi_k(t) \right\}_{k=2}^n
\]

\[
= \left\{ (y^k - z^k) \frac{|w_t - x|^2}{|w_t - x|^{n+3-\alpha}} - (n + 1 - \alpha) \frac{|w_t^k - x|^2}{|w_t - x|^{n+3-\alpha}} \right\}_{k=2}^n
\]

\[
- \left\{ (n + 1 - \alpha) (w_t^k - x^k) \left( \frac{w_t^k - x^k}{|w_t - x|^{n+3-\alpha}} \right) \right\}_{k=2}^n
\]

\[
\equiv \left\{ (y^k - z^k) A_k(t) \right\}_{k=2}^n + \left\{ V_k(t) \right\}_{k=2}^n \equiv U(t) + V(t).
\]

Now for \( 2 \leq k \leq n \) we have \( x^k = 0 \) and so

\[
A_k(t) = \frac{|w_t - x|^2}{|w_t - x|^{n+3-\alpha}} - (n + 1 - \alpha) \frac{|w_t^k|^2}{|w_t - x|^{n+3-\alpha}}
\]

\[
= \frac{|w_t - x|^2 - (n + 1 - \alpha) |w_t^k|^2}{|w_t - x|^{n+3-\alpha}}
\]

\[
= \frac{|w_t^k - x^k|^2 + \sum_{j \neq 1, k} |w_t^j|^2 - (n - \alpha) (w_t^k)^2}{|w_t - x|^{n+3-\alpha}}
\]

\[
\approx (n - \alpha) \frac{|w_t^k - x^k|^2}{|w_t - x|^{n+3-\alpha}} \approx (n - \alpha) \frac{1}{|c_j - x|^{n+1-\alpha}}.
\]

Thus we have

\[
\int_{I \setminus J} A_k(t) \, d\omega(x) \approx (n - \alpha) \int_{I \setminus J} \frac{1}{|c_j - x|^{n+1-\alpha}} \, d\omega(x) \approx (n - \alpha) \frac{\alpha^n (J, 1_{I \setminus J}, \omega)}{|J|^\frac{n}{2}}
\]

and hence

\[
\left| \int_{I \setminus J} \int_0^1 U(t) \, dt \, d\omega(x) \right|^2 = \left| \int_{I \setminus J} \int_0^1 \left\{ (y^k - z^k) A_k(t) \right\}_{k=2}^n \, dt \, d\omega(x) \right|^2
\]

\[
= \sum_{k=2}^n (y^k - z^k)^2 \left| \int_{I \setminus J} \int_0^1 A_k(t) \, dt \, d\omega(x) \right|^2
\]

\[
\approx \sum_{k=2}^n (y^k - z^k)^2 (n - \alpha)^2 \left( \frac{\alpha^n (J, 1_{I \setminus J}, \omega)}{|J|^\frac{n}{2}} \right)^2
\]

\[
\approx (n - \alpha)^2 |y' - z'|^2 \left( \frac{\alpha^n (J, 1_{I \setminus J}, \omega)}{|J|^\frac{n}{2}} \right)^2.
\]
For $2 \leq k \leq n$ we also have using (3.5) that

$$|V_k(t)| = \left| (n + 1 - \alpha) \left( w_k^t - x^k \right) \frac{(y^k - z^k)}{|w_t - x|^{n+3-\alpha}} \right|$$

$$\leq (n + 1 - \alpha) \frac{|w_k^t| |y^k - z^k|}{|w_t - x|^{n+3-\alpha}} + \sum_{j \neq 1, k} |w_j^t| |y_j - z_j|$$

$$\leq (n + 1 - \alpha) \left\{ \frac{|y^k - z^k|}{|w_t - x|^{n+2-\alpha}} + \sum_{j \neq 1, k} \frac{|w_j^t| |y_j - z_j|}{|w_t - x|^{n+3-\alpha}} \right\}$$

$$\leq (n + 1 - \alpha) \left\{ \gamma' \frac{|y^k - z^k|}{|w_t - x|^{n+1-\alpha}} + (\gamma')^2 \frac{|y' - z'|}{|c_j - x|^{n+1-\alpha}} \right\}$$

Thus

$$\left| \int_{I \cap \gamma J} \int_0^1 V(t) \, dt \, d\omega(x) \right| \lesssim (n + 1 - \alpha) \left\{ \frac{\gamma'}{\gamma C_0} + \left( \frac{\gamma'}{\gamma} \right)^2 \right\} \int_{I \cap \gamma J} \frac{|y' - z'|}{|c_j - x|^{n+1-\alpha}} d\omega(x)$$

$$\lesssim (n + 1 - \alpha) \left\{ \frac{\gamma'}{\gamma C_0} + \left( \frac{\gamma'}{\gamma} \right)^2 \right\} |y' - z'| \frac{P^\alpha (J, 1_{I \cap \gamma J \omega})}{|J|^\frac{1}{2}}$$

and so

$$\left| \int_{I \cap \gamma J} \int_0^1 V(t) \, dt \, d\omega(x) \right| \leq \frac{1}{2} \left| \int_{I \cap \gamma J} \int_0^1 U(t) \, dt \, d\omega(x) \right|,$$

provided

$$\left( n + 1 - \alpha \right) \left\{ \frac{\gamma'}{\gamma C_0} + \left( \frac{\gamma'}{\gamma} \right)^2 \right\} \ll n - \alpha,$$

i.e. $$\left( \frac{n + 1 - \alpha}{n - \alpha} \right)^2 \left( \frac{\gamma'}{\gamma} \right)^2 \ll 1,$$

$$\left( 3.6 \right)$$
where we have used (3.4) with an optimal $C_0$. Then if both (3.5) and (3.6) hold we have

$$\begin{align*}
&\left| (R^{\alpha,n}) J \right| (y, y') - (R^{\alpha,n}) J (z, z') \\
&= \left| \int_{J \setminus G} \left\{ \frac{y^k - x^k}{y - x} - \frac{z^k - x^k}{z - x} \right\}_1^n d\omega(x) \right| \\
&= \left| \int_{J \setminus G} \Phi(t) dt d\omega(x) \right| \\
&\geq \left| \int_{J \setminus G} U(t) dt d\omega(x) \right| - \left| \int_{J \setminus G} V(t) dt d\omega(x) \right| \\
&\geq \frac{1}{2} \left| \int_{J \setminus G} U(t) dt d\omega(x) \right| \\
&\geq C_0 \int_{J \setminus G} \left| (y - z)^2 \right| P^\alpha \left( J, 1_{J \setminus G} \right) \\
&\approx C_0 |y - z|^2 \frac{P^\alpha (J, 1_{J \setminus G})}{|J|^\frac{1}{2}}.
\end{align*}$$

Combining the inequalities from each case (3.3) and (3.5) above, and assuming (3.6), we conclude that for all $y, z \in J$ we have the following ‘strong reversal’ of the $1$-partial energy,

$$|y - z| \left( \frac{P^\alpha (J, 1_{J \setminus G})}{|J|^\frac{1}{2}} \right)^2 \leq (R^{\alpha,n}) J (y, y') - (R^{\alpha,n}) J (z, z').$$

Thus we have

$$\begin{align*}
&\sum_{J \in \mathcal{M}_{deep} \setminus J \cap L \neq \emptyset} \left( \frac{P^\alpha (J, 1_{J \setminus G})}{|J|^\frac{1}{2}} \right)^2 \int_J |y - \mathbb{E} y|^2 d\sigma(y) \\
&= \frac{1}{2} \sum_{J \in \mathcal{M}_{deep} \setminus J \cap L \neq \emptyset} \left( \frac{P^\alpha (J, 1_{J \setminus G})}{|J|^\frac{1}{2}} \right)^2 \frac{1}{|J|^2} \int_J \int_J (y - z)^2 d\sigma(y) d\sigma(z) \\
&\leq \sum_{J \in \mathcal{M}_{deep} \setminus J \cap L \neq \emptyset} \int_J \int_J (R^{\alpha,n} J \omega (y, y') - R^{\alpha,n} J \omega (z, z'))^2 d\sigma(y) d\sigma(z) \\
&\leq \sum_{J \in \mathcal{M}_{deep} \setminus J \cap L \neq \emptyset} \int_J (R^{\alpha,n} J \omega (y, y'))^2 d\sigma(y) + \sum_{J \in \mathcal{M}_{deep} \setminus J \cap L \neq \emptyset} \int_J (R^{\alpha,n} J \omega (y, y'))^2 d\sigma(y),
\end{align*}$$

and now we obtain in the usual way that this is bounded by

$$\begin{align*}
&\int_J |R^{\alpha,n} J \omega (y, y')|^2 d\sigma(y) + \sum_{J \in \mathcal{M}} \left( \frac{T^\ast R^{\alpha,n}}{|J|^\frac{1}{2}} \right)^2 |\gamma J| \omega \\
&\leq \left( \frac{T^\ast R^{\alpha,n}}{\mathcal{R}^{\alpha,n}} \right)^2 |I| \omega + \beta \left( \frac{T^\ast R^{\alpha,n}}{\mathcal{R}^{\alpha,n}} \right)^2 |I| \omega \lesssim \left( \frac{T^\ast R^{\alpha,n}}{\mathcal{R}^{\alpha,n}} \right)^2 |I| \omega.
\end{align*}$$
energy condition

Now we turn to the other partial energies and begin with the estimate that for $2 \leq j \leq n$, we have the following ‘weak reversal’ of energy,

$$|R^\alpha_j n 1_{I_{\gamma,j} \omega}(y)| = \left| \int_{I_{\gamma,j}} \frac{y^j - 0}{|y - x|^{n+1-\alpha}} d\omega(x_1, 0, \ldots, 0) \right| \approx \frac{|y^j|}{|J|^\frac{1}{n}} \int_{I_{\gamma,j}} \frac{|J|^\frac{1}{n}}{|y - x|^{n+1-\alpha}} d\omega(x_1, 0, \ldots, 0) \approx |y^j| \frac{P^\alpha_j (J, 1_{I_{\gamma,j} \omega})}{|J|^\frac{1}{n}}.$$

Thus for $2 \leq j \leq n$, we use $\int_j |y^j - E^\sigma_j y^j|^2 d\sigma(y) \leq \int_j |y^j|^2 d\sigma(y)$ to obtain

$$\sum_{J \in \mathcal{M}_{\text{deep}} \gamma' \setminus L \neq \emptyset} \left( \frac{P^\alpha_j (J, 1_{I_{\gamma,j} \omega})}{|J|^\frac{1}{n}} \right)^2 \int_j |y^j - E^\sigma_j y^j|^2 d\sigma(y) \leq \sum_{J \in \mathcal{M}_{\text{deep}} \gamma' \setminus L \neq \emptyset} \left( \frac{P^\alpha_j (J, 1_{I_{\gamma,j} \omega})}{|J|^\frac{1}{n}} \right)^2 \int_j |y^j|^2 d\sigma(y) \leq \sum_{J \in \mathcal{M}_{\text{deep}} \gamma' \setminus L \neq \emptyset} \int_j |R^\alpha_j n 1_{I_{\gamma,j} \omega}(y)|^2 d\sigma(y) \lesssim \int_j |R^\alpha_j n 1_{I_{\gamma,j} \omega}(y)|^2 d\sigma(y) \lesssim \left( \sum_{R \in \mathcal{R}_{\mathcal{R}, \alpha,n}^2} |I| \right)^2 |I|.$$

Summing these estimates for $j = 1$ and $2 \leq j \leq n$ completes the proof of the dual energy condition $\mathcal{E}_\alpha^* \lesssim \mathcal{E}_{\mathcal{T}, \mathcal{R}, \alpha,n} + A_2^\infty$.

3.2. Forward energy condition. Now we turn to proving the (forward) energy condition $\mathcal{E}_\alpha \lesssim \mathcal{E}_{\mathcal{T}, \mathcal{R}, \alpha,n} + A_2^\infty$. We must show

$$\sup_{\ell \geq 0} \sum_{r=1}^{\infty} \sum_{J \in \mathcal{M}_{\text{deep}}(I_r)} \left( \frac{P^\alpha_j (J, 1_{I_{\gamma,j} \omega})}{|J|^\frac{1}{n}} \right)^2 \|p_{\text{subgood}, \omega}\|_{L^2(\omega)}^2 \leq \left( \sum_{R \in \mathcal{R}_{\mathcal{R}, \alpha,n}^2} |I| \right)^2 |I|_\sigma,$$

for all partitions of a dyadic cube $I = \bigcup_{r=1}^{\infty} I_r$ into dyadic subcubes $I_r$. We again fix $\ell \geq 0$ and suppress both $\ell$ and $r$ in the notation $\mathcal{M}_{\text{deep}}(I_r) = \mathcal{M}_{r=\text{deep}}(I_r)$. We may assume that all the cubes $J$ intersect $\text{supp} \omega$, hence that all the cubes $I_r$ and $J$ intersect $L$, which contains $\text{supp} \omega$. Let $I_r = I_r \cap L$ and $J = J \cap L$ for these cubes. We must show

$$\sum_{r=1}^{\infty} \sum_{J \in \mathcal{M}_{\text{deep}}(I_r)} \left( \frac{P^\alpha_j (J, 1_{I_{\gamma,j} \omega})}{|J|^\frac{1}{n}} \right)^2 \|p_{\text{subgood}, \omega}\|_{L^2(\omega)}^2 \leq \left( \sum_{R \in \mathcal{R}_{\mathcal{R}, \alpha,n}^2} |I| \right)^2 |I|_\sigma.$$

Let $\mathcal{M}_{\text{deep}} = \bigcup_{r=1}^{\infty} \mathcal{M}_{\text{deep}}(I_r)$ as above, and for each $J \in \mathcal{M}_{\text{deep}}$, make the decomposition

$$I \setminus J^* = E(J^*) \cup S(J^*)$$
of $I \setminus J^*$ into end $E(J^*)$ and side $S(J^*)$ disjoint pieces defined by

$$
E(J^*) \equiv I \cap \left\{ (y^1, y') : |y^1 - c^1_j| \geq \frac{\gamma}{2} |J|^\frac{1}{n} \text{ and } |y' - c'_j| \leq \frac{1}{\gamma} |y^1 - c^1_j| \right\};
$$

$$
S(J^*) \equiv (I \setminus J^*) \setminus E(J^*).
$$

Then it suffices to show both

$$
A \equiv \sum_{J \in \mathcal{M}_{\text{deep}}} \left( \frac{P_{\alpha} (J, 1_{E(J^*)} \sigma)}{|J|^{\frac{n}{p}}} \right)^2 \left\| p_{\text{subgood}, \omega} \mathbf{x} \right\|_{L^2(\omega)}^2 \leq \left( \mathfrak{T}^2_{\mathcal{R}_{\alpha, n}} + \mathcal{A}_2^0 \right) |I|_{\sigma},
$$

$$
B \equiv \sum_{J \in \mathcal{M}_{\text{deep}}} \left( \frac{P_{\alpha} (J, 1_{E(J^*)} \sigma)}{|J|^{\frac{n}{p}}} \right)^2 \left\| p_{\text{subgood}, \omega} \mathbf{x} \right\|_{L^2(\omega)}^2 \leq \left( \mathfrak{T}^2_{\mathcal{R}_{\alpha, n}} + \mathcal{A}_2^0 \right) |I|_{\sigma}.
$$

Term $A$ is estimated in analogy with the Hilbert transform estimate (3.2), while term $B$ is estimated by summing Poisson tails. Both estimates rely heavily on the one-dimensional nature of $\alpha$.

For $(x^1, 0')$, $(z^1, 0') \in J$ in term $A$ we claim the following ‘strong reversal’ of energy,

$$
\left| R^{\alpha, n}_1 1_{E(J^*)} \sigma (x^1, 0') - R^{\alpha, n}_1 1_{E(J^*)} \sigma (z^1, 0') \right| \approx \left| \int_{E(J^*)} \left\{ \frac{x^1 - y^1}{x^1 - z^1} - \frac{z^1 - y^1}{z^1 - z^1} \right\} \frac{(x^1 - y^1)^2 + |y'|^2}{(x^1 - y^1)^2 + |y'|^2} \right| ds (y). 
$$

Indeed, if we set $a = |y'|$ and $s = x^1 - y^1$ and $t = z^1 - y^1$, then the term in braces in (3.7) is

$$
\left( \frac{x^1 - y^1}{(x^1 - y^1)^2 + |y'|^2} \right) - \left( \frac{z^1 - y^1}{(z^1 - y^1)^2 + |y'|^2} \right) = \frac{s}{s^2 + a^2} \frac{s - t}{s - t} = \frac{\varphi (s) - \varphi (t)}{s - t},
$$

where $\varphi (t) = t (t^2 + a^2)^{-\frac{n+1-n}{2}}$. Now the derivative of $\varphi (t)$ is

$$
\frac{d}{dt} \varphi (t) = \left( t^2 + a^2 \right)^{-\frac{n+1-n}{2} - 1} \left( t^2 + a^2 \right)^{-\frac{n+1-n}{2} - 1} 2t^2 = \left( t^2 + a^2 \right)^{-\frac{n+1-n}{2} - 1} \left( (t^2 + a^2) - (n + 1 - \alpha) t^2 \right) = \left( t^2 + a^2 \right)^{-\frac{n+1-n}{2} - 1} \left( a^2 - (n - \alpha) t^2 \right),
$$
and since $|t| \geq \gamma |J|^\frac{d}{2} \geq \gamma a$, we have $(n - \alpha) t^2 \geq (n - \alpha) \gamma^2 a^2 \geq 2a^2$ provided we choose

$$
\gamma \geq \sqrt{\frac{2}{n - \alpha}}.
$$

Thus if (3.8) holds we get

$$
-\frac{d}{dt} \varphi(t) \approx t^2 (t^2 + a^2)^{-\frac{n+2}{n-2}}.
$$

Finally, since $|s - t| \leq a \leq \frac{1}{2} |t| \ll |t|$, the derivative $\frac{d \varphi}{dt}$ is essentially constant on the small interval $(s, t)$, and we can apply the tangent line approximation to $\varphi(t)$ to obtain $\varphi(s) - \varphi(t) \approx \frac{d \varphi}{dt}(t)(s - t)$, and conclude that for $(x^1, 0')$, $(z^1, 0') \in J$, we have $(\sigma, 1_E(J^*) \sigma)$, and conclude that for $(x^1, 0')$, $(z^1, 0') \in J$, our $|z^1 - y^1|^2 \leq |z^1 - y^1|^2 + |y^2|^2$.

Thus we have

$$
\sum_{J \in \mathcal{M}_{\text{deep}}} \left( \frac{P^n(J, 1_{E(J^*)})}{|J|^\frac{n}{2}} \right)^2 \int_{J \cap L} |x^1 - E_J x^1|^2 d\sigma(y)
$$

$$
= \frac{1}{2} \sum_{J \in \mathcal{M}_{\text{deep}}} \left( \frac{P^n(J, 1_{E(J^*)})}{|J|^\frac{n}{2}} \right)^2 \frac{1}{|J \cap L|} \int_{J \cap L} (x^1 - z^1)^2 d\sigma(x) d\sigma(z)
$$

$$
\approx \sum_{J \in \mathcal{M}_{\text{deep}}} \frac{1}{|J|^\frac{n}{2}} \int_{J \cap L} \int_{J \cap L} \left( R_1^{n, 1} 1_{E(J^*)} \sigma \left( x^1, 0' \right) - R_1^{n, 1} 1_{E(J^*)} \sigma \left( z^1, 0' \right) \right)^2 d\sigma(x) d\sigma(z)
$$

$$
+ \sum_{J \in \mathcal{M}_{\text{deep}}} \frac{1}{|J|^\frac{n}{2}} \int_{J \cap L} \int_{J \cap L} \left( R_1^{n, 1} 1_{J^*} \sigma \left( x^1, 0' \right) - R_1^{n, 1} 1_{J^*} \sigma \left( z^1, 0' \right) \right)^2 d\sigma(x) d\sigma(z)
$$

$$
+ \sum_{J \in \mathcal{M}_{\text{deep}}} \frac{1}{|J|^\frac{n}{2}} \int_{J \cap L} \int_{J \cap L} \left( R_1^{n, 1} 1_{S(J^*)} \sigma \left( x^1, 0' \right) - R_1^{n, 1} 1_{S(J^*)} \sigma \left( z^1, 0' \right) \right)^2 d\sigma(x) d\sigma(z)
$$

$$
\equiv A_1 + A_2 + A_3,
$$

since $I = J^* \cup (I^c \setminus J^*) = J^* \cup E(J^*) \cup S(J^*)$. Now we can discard the difference in term $A_1$ by writing

$$
| R_1^{n, 1} 1_{I} \sigma \left( x^1, 0' \right) - R_1^{n, 1} 1_{I} \sigma \left( z^1, 0' \right) | \leq | R_1^{n, 1} 1_{I} \sigma \left( x^1, 0' \right) | + | R_1^{n, 1} 1_{I} \sigma \left( z^1, 0' \right) |
$$

to obtain from pairwise disjointedness of $J \in \mathcal{M}_{\text{deep}},$

$$
A_1 \leq \sum_{J \in \mathcal{M}_{\text{deep}}} \int_{J \cap L} | R_1^{n, 1} 1_{I} \sigma \left( x^1, 0' \right) |^2 d\sigma(x) \leq \int_{I} | R_1^{n, 1} 1_{I} \sigma |^2 d\sigma \leq \mathbb{F}_{R_1^{n, 1}}^2 |J| \sigma,
$$
and similarly we can discard the difference in term $A_2$, and use the bounded overlap property (2.2), to obtain

$$A_2 \lesssim \sum_{J \in \mathcal{M}_{\text{deep}}} \int_{J \cap L} |R_1^{n,1} \mathbf{1}_{J^*} (x^1, 0')|^2 \, d\omega (x) \leq \sum_{J \in \mathcal{M}_{\text{deep}}} \mathcal{T}_{R_1^{n,1}}^2 |J^*|_{\sigma}$$

$$= \mathcal{T}_{R_1^{n,1}}^2 \sum_{r=1}^{\infty} \sum_{J \in \mathcal{M}_{\text{deep}}(I_r)} |J^*|_{\sigma} \leq \mathcal{T}_{R_1^{n,1}}^2 \sum_{r=1}^{\infty} |I_{r}|_{\sigma} \leq \beta \mathcal{T}_{R_1^{n,1}}^2 |I|_{\sigma} .$$

**Remark 3.** The above estimate fails for the nearby cubes $J$ in $I_r$, and so it is important to use the definition of the energy condition as in Definition 2 above.

This leaves us to consider the term

$$A_3 = \sum_{J \in \mathcal{M}_{\text{deep}}} \frac{1}{|J|_{\omega}} \int_{J \cap L} \{ R_1^{n,1} \mathbf{1}_{S(J^*)} (x^1, 0') - R_1^{n,1} \mathbf{1}_{S(J^*)} (z^1, 0') \}^2 \, d\omega (x) \, d\omega (z)$$

$$= 2 \sum_{J \in \mathcal{M}_{\text{deep}}} \int_{J \cap L} \{ R_1^{n,1} \mathbf{1}_{S(J^*)} (x^1, 0') - \mathcal{E}_{J \cap L} \{ R_1^{n,1} \mathbf{1}_{S(J^*)} (z^1, 0') \} \}^2 \, d\omega (x),$$

in which we do not discard the difference. However, because the average is subtracted off, we can apply the Energy Lemma 1 to each term in this sum to dominate it by,

$$B = \sum_{J \in \mathcal{M}_{\text{deep}}} \left( \frac{P^\alpha (J, 1_{S(J^*)})}{|J|^{\frac{3}{2}}} \right)^2 \left\| P^\omega x \right\|_{L^2(\omega)}^2 .$$

To estimate $B$, we first assume that $n - 1 \leq \alpha < n$ so that $P^\alpha (J, 1_{S(J^*)}) \leq P^\alpha (J, 1_{S(J^*)})$, and then use $\left\| P^\omega x \right\|_{L^2(\omega)}^2 \leq |J|^{\frac{3}{2}} |J|_{\omega}$ and apply the $A_2^\alpha$ condition to obtain the following 'pivotal reversal' of energy,

$$B \leq \sum_{J \in \mathcal{M}_{\text{deep}}} P^\alpha (J, 1_{S(J^*)})^2 |J|_{\omega} \leq \sum_{J \in \mathcal{M}_{\text{deep}}} P^\alpha (J, 1_{S(J^*)}) \{ P^\alpha (J, 1_{S(J^*)}) |J|_{\omega} \}$$

$$\leq A_2^\alpha \sum_{J \in \mathcal{M}_{\text{deep}}} P^\alpha (J, 1_{S(J^*)}) |J|^{1-\frac{\alpha}{n}} = A_2^\alpha \sum_{J \in \mathcal{M}_{\text{deep}}} \int_{S(J^*)} \frac{|J|^{\frac{3}{2}} |J|^{1-\frac{\alpha}{n}}}{|J|^{\frac{3}{2}} + |y - c_J|}^{n+1-\alpha} \, d\sigma (y)$$

$$= A_2^\alpha \sum_{J \in \mathcal{M}_{\text{deep}}} \int_{S(J^*)} \left( \frac{|J|^{\frac{3}{2}}}{|J|^{\frac{3}{2}} + |y - c_J|} \right)^{n+1-\alpha} \, d\sigma (y)$$

$$= A_2^\alpha \int_{S(J^*)} \left\{ \sum_{J \in \mathcal{M}_{\text{deep}}} \left( \frac{|J|^{\frac{3}{2}}}{|J|^{\frac{3}{2}} + |y - c_J|} \right)^{n+1-\alpha} 1_{S(J^*)} (y) \right\} \, d\sigma (y)$$

$$= A_2^\alpha \int F (y) \, d\sigma (y) .$$

At this point we claim that $F (y) \leq C$ with a constant $C$ independent of the decomposition $\mathcal{M}_{\text{deep}} = \bigcup_{r \geq 1} \mathcal{M}_{\text{deep}} (I_r)$. Indeed, if $y$ is fixed, then the only cubes $J \in \mathcal{M}_{\text{deep}}$ for which $y \in S(J^*)$ are those $J$ satisfying

$$J \cap (y; \gamma) \neq \emptyset,$$
where Sh(y; γ) is the Carleson shadow of the point y onto the x₁-axis L with sides of slope \( \frac{1}{2} \), i.e., Sh(y; γ) is interval on L with length 2γ dist (y, L) and center equal to the point on L that is closest to y. Now there can be at most two cubes J whose side length exceeds 2γ dist (y, L), and for these cubes we simply use \( \frac{|J|^\frac{1}{n}}{|J|^{\frac{1}{n}} + |y - c_J|} \leq 1 \).

As for the remaining cubes J, they are all contained inside the triple 3 Sh(y; γ) of the shadow, and the distance \( |y - c_J| \) is essentially dist (y, L) (up to a factor of γ) for all of these cubes. Thus we have the estimate

\[
\sum_{J \in M_{\text{deep}}, J \subset 3 \text{Sh}(y; \gamma)} \left( \frac{|J|^\frac{1}{n}}{|J|^{\frac{1}{n}} + |y - c_J|} \right)^{n+1-\alpha} \lesssim \sum_{J \in M_{\text{deep}}, J \subset 3 \text{Sh}(y; \gamma)} \left( \frac{|J|^\frac{1}{n}}{\text{dist} (y, L)} \right)^{n+1-\alpha} \\
\lesssim \frac{1}{\text{dist} (y, L)^{n+1-\alpha}} \sum_{J \in M_{\text{deep}}, J \subset 3 \text{Sh}(y; \gamma)} |J \cap L|^{n+1-\alpha} \\
\lesssim \frac{1}{\text{dist} (y, L)^{n+1-\alpha}} \sum_{J \in M_{\text{deep}}, J \subset 3 \text{Sh}(y; \gamma)} \text{dist} (y, L)^{n-\alpha} |J \cap L| \\
\lesssim \frac{\text{dist} (y, L)^{n-\alpha}}{\text{dist} (y, L)^{n+1-\alpha}} |3 \text{Sh}(y; \gamma)| \lesssim 1,
\]

because the intervals \( \{ J \cap L \} \) \( J \in M_{\text{deep}}, J \subset 3 \text{Sh}(y; \gamma) \) are pairwise disjoint in 3 Sh(y; γ), and

\( |J \cap L|^{\frac{1}{n}} \) is the length of \( J \cap L \), and since \( n + 1 - \alpha > 1 = \text{dim} L \). It is here that the one-dimensional nature of \( \omega \) delivers the boundedness of this sum of Poisson tails. Thus we have

\[
B \leq A_2^2 \int F(y) \, d\sigma(y) \leq CA_2^2 \, |I|_\sigma,
\]

which is the desired estimate in the case that \( n - 1 \leq \alpha < n \).

Now we suppose that \( 0 \leq \alpha < n - 1 \) and use Cauchy-Schwarz to obtain

\[
P^\alpha \left( J, 1_{S(J^\ast)} \sigma \right) = \int_{S(J^\ast)} \frac{|J|^\frac{1}{n}}{|J|^\frac{1}{n} + |y - c_J|}^{n+1-\alpha} \, d\sigma(y) \\
\leq \left\{ \int_{S(J^\ast)} \frac{|J|^\frac{1}{n}}{|J|^\frac{1}{n} + |y - c_J|}^{n+1-\alpha} \left( \frac{|J|^\frac{1}{n}}{|J|^\frac{1}{n} + |y - c_J|} \right)^{n-1-\alpha} \, d\sigma(y) \right\}^{\frac{1}{2}} \\
\times \left\{ \int_{S(J^\ast)} \frac{|J|^\frac{1}{n}}{|J|^\frac{1}{n} + |y - c_J|}^{n+1-\alpha} \left( \frac{|J|^\frac{1}{n}}{|J|^\frac{1}{n} + |y - c_J|} \right)^{\alpha+1-n} \, d\sigma(y) \right\}^{\frac{1}{2}} \\
= P^\alpha \left( J, 1_{S(J^\ast)} \sigma \right)^\frac{1}{2} \\
\times \left\{ \int_{S(J^\ast)} \left( \frac{|J|^\frac{1}{n}}{|J|^\frac{1}{n} + |y - c_J|} \right)^{\alpha+2-n} \, d\sigma(y) \right\}^{\frac{1}{2}}.
\]
Then arguing as above we have
\[ B \leq \sum_{J \in \mathcal{M}_{\text{deep}}} P^{\alpha} \left( J^*, 1_S(J^*) \sigma \right)^2 |J|_\omega \]
\[ \leq \sum_{J \in \mathcal{M}_{\text{deep}}} \left\{ P^{\alpha} \left( J^*, 1_S(J^*) \sigma \right) |J|_\omega \right\} \int_{S(J^*)} \frac{(|J|^\frac{\alpha}{n} + |y - c_J|)^{\alpha + 2 - n}}{|J|^\frac{\alpha}{n}} \, d\sigma (y) \]
\[ \leq A^2 \sum_{J \in \mathcal{M}_{\text{deep}}} |J|^{1 - \frac{\alpha}{n}} \int_{S(J^*)} \frac{(|J|^\frac{\alpha}{n} + |y - c_J|)^{\alpha + 2 - n}}{|J|^\frac{\alpha}{n}} \, d\sigma (y) = A^2 \sum_{J \in \mathcal{M}_{\text{deep}}} \int_{S(J^*)} \frac{|J|^{1 - \frac{\alpha}{n}}}{|J|^\frac{\alpha}{n}} \, d\sigma (y) \]
\[ = A^2 \int_{I} \left\{ \sum_{J \in \mathcal{M}_{\text{deep}}} \left( \frac{|J|^{1 - \frac{\alpha}{n}}}{|J|^\frac{\alpha}{n} + |y - c_J|} \right)^2 \right\} \, d\sigma (y) = A^2 \int_{I} F (y) \, d\sigma (y), \]
and again \( F (y) \leq C \) because
\[ \sum_{J \in \mathcal{M}_{\text{deep}}} \left( \frac{|J|^{1 - \frac{\alpha}{n}}}{|J|^\frac{\alpha}{n} + |y - c_J|} \right)^2 \leq \frac{1}{\text{dist} (y, L)^2} \sum_{J \in \mathcal{M}_{\text{deep}}} \frac{|J|^{1 - \frac{\alpha}{n}}}{|J|^\frac{\alpha}{n} + |y - c_J|} \]
\[ \leq \frac{1}{\text{dist} (y, L)^2} \sum_{J \subseteq 3 \text{Sh}(y; \gamma)} |J \cap L| \]
\[ \leq \frac{\text{dist} (y, L)}{\text{dist} (y, L)^2} |3 \text{Sh}(y; \gamma)| \lesssim 1. \]

Thus we again have
\[ B \leq A^2 \int_{I} F (y) \, d\sigma (y) \leq CA^2 |I|_\sigma , \]
and this completes the proof of necessity of the energy conditions when one of the measures is supported on a line.

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