Bumpless Pipe Dream RSK, Growth diagrams, and Schubert Structure Constants

AlCoVE 2022

Daoji Huang (UMN)
joint w/ Paulo Pylyavskyy
Bumpless Pipe Dreams (Lam-Lee-Shimozono '18)

- **Defn:** A Schubert polynomial for $\omega \in S_n$ is defined as

$$S_\omega = \sum_{\text{DEBD(T)}} \prod_{(\text{row}, \text{col}) \in \text{blank(D)}} x_{\text{row}}$$

- Allowed tiles:

$$\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 \\
1 & 2 & 3 & 4 & 5 \\
1 & 2 & 3 & 4 & 5 \\
1 & 2 & 3 & 4 & 5 \\
1 & 2 & 3 & 4 & 5 \\
\end{array}$$

$$S_{31524} = x_1^2 x_3^2 + x_1^2 x_2 x_3 + x_1^2 x_2^2 + x_1^3 x_3 + x_1^3 x_2$$
Schubert Polynomials and Structure Constants

- Schubert polynomials form a basis of the polynomial ring in $x_1, x_2, ...$

\[ S_\pi S_\rho = \sum_s C^s_{\pi \rho} S_s \]

These $C^s_{\pi \rho}$ are called the Schubert structure constants.

- When $\pi$ and $\rho$ are both permutations with a single descent at position $k$, $C^s_{\pi \rho}$ are the well-known Littlewood-Richardson coefficients.
Classical RSK Correspondence

\{\text{words in } 1, 2, \ldots, k\} \overset{\text{bij}}{\longleftrightarrow} \{(P, Q) : \begin{array}{c} P \in \text{SSYT} \\ Q \in \text{SYT} \\ P, Q \text{ same shape} \end{array}\}

Example 322121 \rightarrow \begin{array}{cc}
1 & 12 \\
2 & 2 \\
3 & \\
\end{array} \begin{array}{c}
135 \\
26 \\
4 \\
\end{array}
Right (row) and left (column) insertion

Row and column insertions commute in classical RSK.
Bumpless pipe dream RSK

- Monk's rule for Schubert polynomials

\[ \sum_{\pi \in BPD(\pi), \tau \in BPD(\tau)} S_{\pi \tau} = \sum_{\pi, \tau, \tau' \in BPD(\pi), \tau' \neq \tau} S_{\pi \tau} t_{\tau \tau'} \]

bijectively,

\[ (x_i, D) \leftrightarrow D' \]

\[ \text{for } i \leq k \leq m \in BPD(\pi) \text{, left insert } x_i \text{ for } k=3 \]

\[ \text{for } i \leq k \leq m \in BPD(\tau) \cup BPD(\tau') \text{, right insert } x_i \text{ for } k=3 \]

- Pylyavskyy'22 give two insertion algorithms that generalize left and right insertions on SSYT to BPDs
**Bumpless pipe dream RSK**

**Def** A biletter is a pair of positive integers \((a,k)\) with \(a \leq k\). We write it as \(ak\). A biword is a word of biletters.

**Def** Let \(W\) and \(V\) be permutations. Define \(W <_k V\) iff \(V = W \text{tab} \text{ where } l(V) = l(W) + 1\) and \(a \leq k < b\).

**Thm** [BPD RSK] (H.-Pylyavskyy)

\[
\{ \text{biwords of length } m \} \xleftarrow{\text{left/right}} \xrightarrow{\text{insertion}} \bigcup_{\pi \text{ set}} \left\{ (D, \leq) : D \in \text{BPD}(\pi) \text{ mixed } k\text{-chain for } \pi \right\}
\]

**Example:** \(23122_2\) \leftarrow \text{right insert} \rightarrow \left(\begin{array}{cccc}
1 & 4 & 3 & 2 \\
4 & 3 & 2 & 1
\end{array}\right)

\(23122_2\) \leftarrow \text{left insert} \rightarrow \left(\begin{array}{cccc}
2 & 4 & 3 & 1 \\
4 & 3 & 1 & 2
\end{array}\right)
Lenart's growth diagram of permutations

Lenart's growth diagram of permutations is a matrix of permutations subject to a local condition on each square:

\[
\begin{array}{cccccc}
\pi_{0,n} & \pi_{1,n} & \pi_{2,n} & \cdots & \pi_{m,n} \\
\downarrow & \downarrow & \downarrow & \cdots & \downarrow \\
\pi_{0,2} & \pi_{1,2} & \pi_{2,2} & \cdots & \pi_{m,2} \\
\downarrow & \downarrow & \downarrow & \cdots & \downarrow \\
\pi_{0,1} & \pi_{1,1} & \pi_{2,1} & \cdots & \pi_{m,1} \\
\downarrow & \downarrow & \downarrow & \cdots & \downarrow \\
\text{id} = \pi_{0,0} & \pi_{1,0} & \pi_{2,0} & \cdots & \pi_{m,0}
\end{array}
\]
Lenart's growth diagram of permutations

Lenart's growth diagram of permutations is a matrix of permutations subject to a local condition on each square:

\[
\begin{array}{c}
\pi_{m,n} & \text{Given} & \delta \\
\vdots & \vdots & \vdots \\
\delta_k & \delta & \sigma \\
\end{array}
\]

\[\exists \delta \text{ that completes the Bruhat diamond.}\]

\[\text{if } \delta \text{ works} \quad \text{if } \delta \text{ does not work}\]

Example:

\[
\begin{array}{ccc}
132 & 312 & 132 \\
213 & 213 & 132 \\
123 & 123 & 123
\end{array}
\]
Growth diagrams and Schubert multiplication

Want to find
* chain for \( \text{id} \to w \)
* chain for \( \text{id} \to v \)
such that \( \forall u \in S_v \),

\[
C_{v,w} = \# \text{ chains } w \to u \text{ that }
\]
“rectify” to the given chain of \( u \)

- Lenart '17 gave a rule for multiplying \( S_v \) and \( S_w \) where
  \( v \) has a single descent at position \( K \), and \( w \) has all descents \( > k \)
or all descents \( \leq k \).

- H.-Pylyavskyy '22* generalize this to \( S_v \cdot S_w \) where \( v \)
  has descents \( \leq k \), and \( w \) has descents \( \geq k \). ( \( v \) and \( w \) have
  separated descent)
Connecting insertions and growth diagrams

**Thm (H.-Pylyavskyy)** Let $D \in \text{BPDC}(\Pi)$. Suppose $\Pi$'s first descent position is $d_1$ and last descent position is $d_2$ when $\Pi \neq \text{id}$. Suppose $l \leq d_1 \leq d_2 \leq k$. Then left insertion of $ak$ and right insertion of $bl$ commute:

$$(ak \rightarrow D) \leftarrow bl = a \varepsilon^2(D \leftarrow bl)$$

**Proposition (H.-Pylyavskyy)** When two permutations $w$ and $v$ have separated descents, it is possible to construct explicit chains in mixed $k$-Bruhat order such that every square in the growth diagram $\delta$ of $\sigma$ satisfies $\left\{ \begin{array}{ll} k \geq \text{last descent of } \Pi \\ l \leq \text{first descent of } \Pi \end{array} \right.$
Separated descent Schubert calculus

Thm (H.-pylyavskyy)

Let $w$ and $v$ be permutation s.t.
- last descent of $v$ is $\leq k$
- first descent of $w$ is $\geq k$

Then there exist explicit chains $ch_{\Psi}(w)$ and $ch_{\Phi}(v)$ s.t.

$\forall u \in S_n,$

$c_{w,v}^u = \# \text{ growth diagrams of the form}$