Embedding of Time-Delayed Quantum Feedback in a Nonreciprocal Array

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(Dated: April 5, 2022)

Time-delayed quantum feedback is a fast and efficient method to control and stabilize few and many-body quantum systems. However, a proper understanding of such systems stays opaque due to the non-Markovian nature of the feedback protocol. Here, we present a method of encoding time-delayed quantum feedback into a chain of nonreciprocally coupled auxiliary oscillators. Our approach serves as a novel method of introducing time-delayed quantum feedback and provides the advantage of large tunability. Importantly, within our approach the original non-Markovian system is embedded into an enlarged Markovian open quantum system, which can be treated efficiently with quantum many-body methods. As an exemplary illustration, we apply our Markovian embedding approach to the paradigmatic case of a driven-dissipative atom in front of a mirror.

With the rapid advances of various quantum platforms [1–5], time-delayed quantum feedback has been an increasingly important element in designing and controlling quantum devices [6–10], as well as for processing of quantum information in quantum networks [11–13], especially when the time delay between quantum nodes becomes significant [14]. A paradigmatic example for a quantum system with time-delayed feedback is an atom in front of a mirror [Fig. 1(c)] or similarly an atom coupled with a semi-infinite/chiral waveguide (see recent experiments [15–18]). The atom at time t undergoes here a self-driven cycle of absorbing excitations emitted at an earlier time t − ∆t, which turns the dynamics of the system non-Markovian. Such memory effects are a crucial characteristic of the quantum dynamics with time-delayed feedback [19, 20], and are the origin of the underdeveloped understanding of the effects of this sort of autonomous feedback protocols. Current theoretical approaches to time-delayed quantum feedback (e.g. [8, 21–32]) usually become intractable beyond the few-photon regime or the transient regime. To explore previously inaccessible regimes, novel approaches are needed.

In this letter, we introduce a novel method for tackling time-delayed quantum feedback, which embeds the time-delayed memory into a chain of dissipative oscillators whose dynamics is nonreciprocal [Fig. 1(a)]. This approach is inspired by the Markovian embedding approach to time-delayed Langevin equations (see e.g. [33–36] and a brief introduction in [37]), which embeds delayed feedback into auxiliary variables based on the linear chain trick [38, 39]. Here, in the quantum setting, we show that the resulting system is a Markovian open quantum system, which can be analyzed efficiently using quantum many-body methods. The purpose here is twofold: (i) our Markovian embedding approach serves as an efficient method for analyzing and understanding time-delayed quantum feedback. For illustrative purpose, we apply this to the paradigmatic atom-mirror case and characterize its long-time dynamics using exact diagonalization (in the linear case) and matrix product state methods (in the nonlinear case). (ii) the proposed setup represents a novel way of introducing time-delayed feedback, which features large tunability on the memory kernel by tuning parameters of the auxiliary oscillators. It can therefore enable us to explore more feedback-induced effects, considering its solvability.

The model and its dynamics.—To introduce time-delayed feedback on a system of interest at site 0, we couple it with the two ends of a chain of n auxiliary dissipative oscillators as shown in Fig. 1. The key is that the dynamics in the chain is engineered such that signals can only propagate unidirectionally from the oscillator at site 0 to the oscillator at site n. Thus, the system at site 0 at time t is affected by its historical quantum state at time t − ∆t, where ∆t the time of propagation from site 0 to site n. In this letter, we show that this intuition indeed works and the desired time-delayed quantum feedback is realized.

As noticed, the crucial ingredient here is that the oscillators are coupled nonreciprocally, which means that the dynamics of site j depends on site j − 1 but not vice versa, which can
be realized by reservoir engineering using the recipe developed in Ref. [40, 41]. For example, to engineer a nonreciprocal coupling of two oscillators with annihilation operators $a_{1,2}$, the coherent hopping interaction $H_{12} = \frac{\gamma}{2} (a_1 a_2^\dagger + \text{h.c.})$ is combined with the corresponding dissipative hopping process, modeled via the jump operator $J_{12} = \sqrt{\gamma} (a_1 - i a_2)$, in a Lindblad master equation

$$\frac{d}{dt} \rho_{12} = -i[H_{12}, \rho_{12}] + J_{12} \rho_{12} J_{12} - \frac{1}{2} \{ \rho_{12}, J_{12}^\dagger J_{12} \}, \quad (1)$$

which coincides with the master equation for cascaded quantum systems [42]. The corresponding Heisenberg-Langevin equations yield for (details see [37])

$$\frac{d}{dt} a_0 = -\frac{\gamma}{2} a_0 + \eta(t), \quad \frac{d}{dt} a_1 = -\frac{\gamma}{2} a_1 - i \gamma a_1 + i \eta(t), \quad (2)$$

where $\eta(t) = -\sqrt{\gamma/2\pi} \int dk b_k(0)e^{-ikt}$ is a Gaussian white noise with $\langle \eta(t) \eta(t') \rangle = \gamma \delta(t - t')$, $\langle \eta(t) \eta(t') \rangle = 0$, and $\langle \eta(t) \eta(t') \rangle = 0$. $b_k(0)$ is the bosonic annihilation operator at time $t = 0$ for the environmental degrees of freedom of the two dissipative oscillators. Eq. (2) clearly shows that the dynamics is nonreciprocal since $a_2$ depends on $a_1$ but not vice versa. Generally, there is freedom in engineering $H_{12}$ and $J_{12}$ via a complex weight $\chi \in \mathbb{C}$ such that $H_{12} = \frac{\gamma}{2} (\chi a_1 a_2^\dagger + \text{h.c.})$ and $J_{12} = \sqrt{\gamma} (a_1 - i \chi a_2)$, and this method is straightforwardly generalized to a network of oscillators [43].

Applying the dissipation engineering protocol to the ring lattice as shown in Fig. 1(a), we obtain clockwise nonreciprocal dynamics (site 0 → 1 → ... → $n$ → 0), captured in the Heisenberg-Langevin equations (for details see [37])

$$\frac{d}{dt} a_0 = -\frac{1}{2}(\chi_{n,0}^2 \gamma_{n,0} + \gamma_{1,0})a_0 - i \chi_{n,0} \gamma_{n,0} a_n + \xi_0(t) + i[H_0, a_0],$$

$$\frac{d}{dt} a_1 = -\frac{1}{2}(\chi_{0,1}^2 \gamma_{0,1} + \gamma_{0,1})a_1 - i \chi_{0,1} \gamma_{0,1} a_1 + \xi_1(t),$$

$$\frac{d}{dt} a_j = -\gamma a_j - i \gamma a_{j-1} + \xi_j(t), \quad \text{for } 2 \leq j \leq (n-1),$$

$$\frac{d}{dt} a_n = -\frac{1}{2}(\gamma + \gamma_{n,0})a_n - i \gamma a_{n-1} + \xi_n(t),$$

where we denote the decay rate and weight at bond $j \rightarrow j + 1$ (with $n + 1 \equiv 0$) as $\gamma_{j,j+1}$ and $\chi_{j,j+1}$. Note that we account for a local Hamiltonian $H_0$ at site 0 which can be nonlinear. $\xi_j(t)$ are noise operators due to dissipation and we have chosen the weights $\chi$ to be real. Because each site $j$ has noises from two neighboring dissipators, $\xi_j(t)$ is given by a summation of two Gaussian white noises

$$\xi_j(t) = -\sqrt{\gamma/2\pi} \int dk \left( b_k^{(j+1,j)}(0) + b_k^{(j+1,j+1)}(0) \right) e^{-ikt}, \quad (4)$$

where $b_k^{(j,j+1)}$ denotes bosonic operator for the bath of the bond $j \rightarrow j + 1$.

Like the case with two oscillators, the dynamics in this lattice is nonreciprocal, with site $j$ depends only on itself and site $j - 1$ but not on site $j + 1$. For simplification of analysis, in the current study, we let bonds $0 \rightarrow 1$ and $n \rightarrow 0$ have free parameters while other bonds are homogeneous with decay rate $\gamma$ and weight $\chi = 1$ [Fig. 1(a)]. Though this choice is sufficient for the current study, in general, keeping more free parameters gives more tunability, which can be crucial in studies of other models.

**Time-delayed feedback mediated by the nonreciprocal chain.**—From Eq. (3), we see that the dynamics for site $1, \ldots, n$ is linear, which means that we can analytically integrate out $a_{j>0}$ of the auxiliary oscillators. As shown in the following, with proper choice of parameters, we get $a_n(t) \propto a_0(t - \Delta t)$. After substituting it into the equation of motion (EOM) of $a_0$ in Eq. (3), we obtain a time-delayed feedback for $a_0$ with discrete delay time $\Delta t$.

The linear EOM for $a_{j>0}$ in Eq. (3) can be solved with a Laplace transformation. In what follows we consider the case that all auxiliary oscillators are exposed to the same local damping $\gamma$, i.e., we set $\gamma_{n,0} = \gamma$ and $\gamma_{0,1} = \gamma/\chi_{0,1}$. For the general case please see [37]. We obtain for the auxiliary oscillator operators in Laplace space

$$\tilde{a}_j(s) = -\frac{i \gamma}{s + \gamma} \tilde{a}_{j-1}(s) + \frac{1}{s + \gamma} \left( \tilde{\xi}_j(s) + a_j(0) \right), \quad (5)$$

with $\gamma_1 = 1/\chi_{0,1}$ and $\gamma_j = \gamma$ for $2 \leq j \leq n$. Then iteratively, we get a relation between $\tilde{a}_n(s)$ and $a_0(s)$

$$\tilde{a}_n(s) = \frac{1}{\chi_{0,1}} \left( \frac{-i \gamma}{s + \gamma} \right)^n \tilde{a}_0(s) + \sum_{j=1}^{n-1} \left( \frac{-i \gamma}{s + \gamma} \right)^{n-1-j} \tilde{\xi}_j(s), \quad (6)$$

where we have defined $\tilde{\xi}_j(s) = \tilde{\xi}_j(s) + a_j(0)$. Performing the inverse Laplace transformation of (6), we obtain

$$a_n(t) = \frac{1}{\chi_{0,1}} \int_0^t d\tau K_n(t - \tau) a_0(\tau + \nu(t)), \quad (7)$$

with the memory kernel $K_n(t - \tau)$ defined as

$$K_n(t - \tau) = \left( \frac{-i \gamma}{n-1} \right)^n (t - \tau)^{n-1} e^{-\gamma(t-\tau)}, \quad (8)$$

and $\nu(t)$ denotes the noise operator which is a function of $b_k^{(j,j+1)}(0)$ and $a_j(0)$. The memory Kernel corresponds to the probability density function of the Gamma distribution, and it is noteworthy that the memory kernel $K_n(t - \tau)$ peaks at $t - \tau = (n - 1)/\gamma$. Thus, by setting $\gamma = (n - 1)/\Delta t$, we obtain a memory kernel which peaks at $\Delta t$. Furthermore, the variance of $(t - \tau)$ can be shown to be $\sqrt{\langle (t - \tau - \Delta t)^2 \rangle} = \sqrt{\pi/(n-1)} \Delta t$, which vanishes as $\Delta t/\sqrt{n}$ in the large $n$ limit. Crucially, the memory Kernel becomes in this limit a delta function

$$K_n(t - \tau) \rightarrow (-i)^n \delta(t - \tau - \Delta t), \quad \text{as } n \rightarrow \infty, \quad (9)$$
which is the discrete-time memory kernel desired. In the following, we work with system size $n \gg 1$, which renders (9) hold approximately. This approximation works as long as the width of $K_n(t - \tau)$ is negligible compared to the other timescale of site 0.

Finally, we can insert the relation between $a_n$ and $a_0$ back to the EOM of $a_0$, and obtain

$$
\frac{d}{dt} a_0(t) = -\kappa a_0(t) + i[H_0(t), a_0(t)] + \eta\Theta(t - \Delta t) a_0(t - \Delta t) + \nu_0'(t),
$$

where the decay rate $\kappa = (\chi_{n,0}^2 + 1/\chi_{0,0}^2)\gamma/2$ [44], the feedback strength $\eta = (-1)^n+1\gamma(\chi_{n,0}/\chi_{0,1})$, and the noise operator $\nu_0'(t)$ is a function of $b_{k(j,j+1)}(0)$ and $a_{j>0}(0)$.

Note that (10) is the same as that in conventionally studied models with time-discrete feedback, such as the atom-mirror case (see e.g. [26, 27]), except the noise operator $\nu_0'(t)$. Notice that $\nu_0'(t)$ is a function of annihilation operators $b_{k(j,j+1)}(0)$ and $a_{j>0}(0)$. When we start from vacuum initial state for the bath and the auxiliary oscillator $|\psi_{1,\ldots,n,b}(t = 0)\rangle = |0\rangle_1 \cdots |0\rangle_n |0\rangle_b$, we have $\nu_0'(t) |\psi_{1,\ldots,n,b}(t = 0)\rangle = 0$. So, when we consider expectation values of normally-ordered operators like $<a_0^\dagger(t_1) a_0(t_2)>$, their EOMs necessarily become independent of $\nu_0'(t)$ under this initial condition [45]. Similar conclusion holds for the atom-mirror case. That is, although these two cases have different noise operators, they give the same EOMs for the normally-ordered operators. Therefore, when considering the normally-ordered operators, we have provided a Markovian embedding of the paradigmatic atom-mirror case [46]. The discrepancy in noise makes a difference when anti-normally-ordered operators are considered, which is left for future research.

In the following, we apply this Markovian embedding approach to a linear (a harmonic oscillator) and a nonlinear system (a qubit) with time-delayed feedback. The purpose is to give a better understanding of our approach and to show that it is a useful technique for dealing with non-Markovian dynamics efficiently.

**Linear case: single photon decay.—**To give an exemplary illustration of our approach, we apply it to the simplest case with a linear oscillator at site 0. Here we study decaying dynamics of a single photon at site 0 i.e. the initial state is $|\Psi_{1,\ldots,n}(t = 0)\rangle = |1\rangle_0 |0\rangle_1 \cdots |0\rangle_n$. By letting the oscillator be on-resonant with the auxiliary oscillators, we have $H_0 = 0$. Since there is only one excitation in the system, this case is the same as decay of a qubit in its excited state.

In the time-delayed EOM (10), decay rate $\kappa$ and the feedback strength $\eta$ can be tuned with parameters $\{(\chi_{0,1}, \chi_{n,0}, \gamma)\}$, which is a minimal set of parameters we found to keep $\kappa$ and $\eta$ of order $1/\Delta t$ even in presence of the large decay rate $\gamma \sim n/\Delta t$. Here, without loss of generality, we set $\eta = \kappa$.

The phase of $\eta$ can be set to be 1 by letting $n = 4m + 3$ with $m \in \mathbb{Z}_+$. Note that phases $\{\pm, \pm i\}$ can be tuned by simply varying $n$. In general, $\eta$ can have arbitrary phase by making $\chi_{i,j+1}$ complex. Then by using the expression of $\eta$, we get

![FIG. 2. Decay of a single photon with time-delayed feedback.](image)

(a) Photon population at site 0 for $n$ auxiliary oscillators ($n = 23, 83, 203, 403$). (b) The memory profile function for different size $n$. It converges to the step function in (S28) as $n$ increases. (c) Population of the auxiliary oscillators in the case with $n = 83$. (d) Slices of panel (c) at site $j = 1, 20, 40, 60, 83$. (values of $j$ refer curves from left to right). (Parameters: $\eta = \kappa, \kappa = 1, \Delta t = 1$.)

our parameters in terms of $\kappa$ and $\Delta t$:

$$
\chi_{n,0} = \sqrt{\frac{\kappa}{\gamma}}, \ \chi_{0,1} = \sqrt{\frac{1}{\kappa}}, \ \gamma = \frac{(n - 1)}{\Delta t}.
$$

Since the EOMs (3) are linear and time-local in this case, we can calculate $<a_0^\dagger(t)a_0(t)>$ by writing out the EOMs of $<a_j^\dagger(t)a_j(t)>$. Then the matrix EOMs of these $(n + 1)^2$ variables can be solved with exact diagonalization. The results are shown in Fig. 2 for $\kappa = 1$ and $\Delta t = 1$. Fig. 2(a) shows time evolution of the 0-th site’s photon population $<a_0^\dagger(t)a_0(t)>$ for system size $n = 23, 83, 203, 403$. It can be seen that results converges quickly in $n$, and even a fairly small size $n = 83$ gives a good approximation to the time-discrete feedback as predicted by our theory above. It is also easy to see the residual photon occupation that does not decay, which is exactly the well-known photon bound state in atom-mirror case (see e.g. [26, 27]). Here, the residual photon occupation can be understood as a result of self-driving due to feedback, which gives a nonequilibrium steady state with non-zero occupation.

To see how good a finite-sized system approximates the time-discrete feedback, we define a memory profile function $f(t) \propto <a_j^\dagger(t)a_j(t)> / <a_0^\dagger(t - \Delta t)a_0(t - \Delta t)>$ [47] according to (10). With proper normalization, we know that $f(t) \to \Theta(t - \Delta t)$ as $n \to \infty$. Indeed, as shown in Fig. 2(b), $f(t)$ converges to the step function as $n$ increases.

To understand the roles played by the auxiliary oscillators, their populations are shown in Fig. 2(c) and (d) for the case with $n = 83$. We can see that the photon travels unidirectionally from site 1 to site $n$ with an almost linear speed. At time $t = \Delta t$, the photon reaches the last site $n$, which turns on the time-delayed feedback. As $n$ gets larger, the turning-on process gets sharper. Finite size $n$ gives to a finite bandwidth in
where $\Theta$ is the auxiliary chain and therefore determines how sharp is the turning-on process for $\langle a_j^\dagger a_n \rangle$, which is the origin of non-zero width in the memory kernel $K_{n-2}(t - \tau)$.

**Nonlinear case: a coherently driven qubit.**—As a showcase of applying our approach to nonlinear systems, here we study a driven-dissipative qubit at site 0. Then $H_0$ in (10) has the form $H_0 = (\Omega a_0^\dagger + \text{h.c.}) + U a_0^\dagger a_0 a_0 + U \rightarrow \infty$, where $\Omega$ is the driving strength and $U$ is the nonlinearity to ensure site 0 acts as a qubit. This is equivalent to replacing the bosonic operator $a_0$ by the Pauli operator $\sigma_0^x$ in the master equation, i.e. treating $a_0$ as hard-core boson. Then by going through the process from the master equation to (10), we see that, compared with bosonic representation, an additional factor $[\sigma_0^+, \sigma_0^-]$ is present in the EOM of $\sigma_0^x$ [37], which corresponds exactly to the EOM of a qubit in front of a mirror (see e.g. [26, 27]).

The many-body Lindblad master equation corresponding to (10) can be solved numerically using matrix product state (MPS) methods. Here we write the density matrix as a matrix product density operator (MPDO), and then perform time evolution by applying Trotter gates onto it (see e.g. [48]). It can be a problem for MPS simulation if the local Hilbert space dimension required by the auxiliary oscillators is large. Fortunately, the auxiliary oscillators here can be treated as qubits as shown in the following argument. We can write down an approximate relation between $a_j$ and $a_0$:

$$a_j(t) \propto \Theta(t - \Delta t_j) a_0(t - \Delta t_j) + \text{noise},$$

where $\Delta t_j \approx (j/n)\Delta t$. Then we know $\langle a_j^\dagger(t) a_j(t) \rangle \propto \Theta(t - \Delta t_j) \langle a_0^\dagger(t - \Delta t_j) a_0(t - \Delta t_j) \rangle$ for large $n$. As shown in Fig. 3(b), their discrepancies decreases as $n$ increases. Because the total system is an open system, the bond dimension required in MPS simulation stops growing after a while (similar to that in Ref. [27]), as shown in Fig. 3(c). It is in stark contrast to the generally exponential increase of bond dimension in time evolution of closed systems (see e.g. [48]). Thus, the computational cost in our approach is basically linear in time in long-time evolution. This enables us to evolve the system for much longer times, even to approach its steady state. Comparison with other methods like the MPS simulation of stochastic Schrödinger equation [27] needs to be explored in future study, especially when time delay or driving strength gets larger. Another noteworthy thing, in both Fig. 2(a) and Fig. 3(a), is that the steady state seems to be insensitive to finite size effect, which can be an advantage in future study of steady state properties.

**Conclusion and outlook.**—In summary, we show how to perform Markovian embedding of time-delayed quantum feedback, using a chain of nonreciprocally coupled dissipative oscillators (see Fig. 1). The original non-Markovian system, which is notoriously difficult to analyze, is thus turned into a Markovian open quantum system, which can be tackled efficiently using many-body methods. As an example, we apply this approach to the paradigmatic case with a qubit in front of a mirror, using MPS simulation. The complexity is found to be linear in time for long-time evolution, which enables us to go beyond transient regime and even to reach the nonequilibrium steady state.

For future research, vast interesting applications of our approach await to be explored. For example, using the tunability of auxiliary oscillators, the memory kernel can be tuned for quantum control purpose. Besides the currently used reservoir engineering approach, it would be interesting to explore Markovian embedding using other approaches to nonreciproc-
ity [50]. Though a qubit is used as an example in the current study, our approach is general, with no requirement on the local Hamiltonian $H_0$ at site 0. Therefore, feedback on various other models, such as nonlinear oscillators or even many-body models, can be studied, using readily available methods for open quantum many-body systems [51]. From a more general perspective, we have put forward an approach originally created to treat non-Markovian effects in classical stochastic systems, including feedback-controlled and active systems (see e.g. [36, 52]), towards the quantum regime, which could lead to more interdisciplinary research between these fields.

Numerical data for figures are available at https://doi.org/10.5281/zenodo.6380671. This work is supported by the Deutsche Forschungsgemeinschaft through the Emmy Noether program (Grant No. ME 4863/1-1) and the project 163436311-SFB 910.

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6

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In this Supplemental Material, we present (i) the Markovian embedding for classical Langevin equation, (ii) Heisenberg-Langevin equation for two oscillators, (iii) Heisenberg-Langevin equation for a chain of oscillators, (iv) derivation of the memory kernel for a0, (v) matrix EOMs in the linear case, (vi) qubit EOM in the nonlinear case, and (vii) more information on MPS simulation of the qubit case.

(i) the Markovian embedding for classical Langevin equation

Here, we give a short pedagogical summary of the Markovian embedding for classical Langevin equation (for detailed studies, see e.g. [35, 36, 38, 39]).

Let’s have the variable \( x_0 \) of interest, which has a non-Markovian dynamics i.e. a memory kernel in the equation of motion. Then let \( x_j, j = 1, 2, \ldots n \) be the auxiliary variables that can act as a memory. The couplings are nonreciprocal: \( 0 \to 1 \to 2 \to \ldots \to n \to 0 \).

Starting from the Langevin equation of variable \( x_0 \), we can have

\[
\lambda_0 \frac{d}{dt} x_0 = v_{0,0} x_0 + f(x_0) + \eta_0 + w_{0,n} x_n, \tag{S1}
\]

where \( v_{0,0} < 0 \) is a linear force, \( f(x_0) \) is a higher-order force, \( \eta_0 \) is a Gaussian white noise, and \( w_{0,n} \) denotes coupling with \( x_n \).

For the linear chain acting as a memory, we have

\[
\lambda_j \frac{d}{dt} x_j = v_{j,j} x_j + \eta_j + w_{j,j-1} x_{j-1}, \tag{S2}
\]

where the last term represents the nonreciprocal interaction from \( j-1 \) to \( j \).

For simplicity, we can assume the linear chain is homogeneous i.e. \( v_{j,j} = v \), \( w_{j,j-1} = w \), and \( \lambda_j = \lambda \). Then we have \( j > 0 \)

\[
\lambda \frac{d}{dt} x_j = v x_j + \eta_j + w x_{j-1}. \tag{S3}
\]

Using Laplace transformation, we know

\[
\tilde{x}_j(s) = \frac{w}{\lambda s - v} \tilde{x}_{j-1}(s) + \frac{1}{\lambda s - v} \tilde{\eta}_j(s), \tag{S4}
\]

where \( x_j(0) = 0 \) is assumed. Iteratively, we get

\[
\tilde{x}_n(s) = \left( \frac{w}{\lambda s - v} \right)^n \tilde{x}_0(s) + \sum_{k=1}^{n} \left( \frac{w}{\lambda s - v} \right)^{n-k} \frac{1}{\lambda s - v} \tilde{\eta}_k(s). \tag{S5}
\]

We know the inverse Laplace transformation \( \mathcal{L}^{-1}(\frac{1}{(s+n)^a}) = \frac{1}{(n-1)!} \int_0^t \tau^{n-1} e^{-\alpha \tau} \Theta(t) \, d\tau \), where \( \Theta(t) \) is a step function. We also know \( \mathcal{L}^{-1}(f(s)g(s)) = \int_0^t f(\tau)g(t-\tau) \, d\tau \). Then from the first term of (S5), we get the memory kernel

\[
K_n(t-\tau) = \left( \frac{w}{\lambda} \right)^n \frac{1}{(n-1)!} (t-\tau)^{n-1} e^{\frac{w}{\lambda} (t-\tau)} \tag{S6}
\]

and the memory kernel for the noise \( \eta_k \)

\[
K_{\eta_k}(t-\tau) = \frac{1}{w} K_{n-k+1}(t-\tau). \tag{S7}
\]
We can see that $K_n(t - \tau)$ peaks at $(t - \tau) = (n - 1)\lambda/(-\nu)$. To get delay time $\Delta t$, we need to choose

$$v = -\frac{(n - 1)\lambda}{\Delta t}. \quad (S8)$$

To make the memory kernel a distribution, we require normalization $\int_0^\infty d\tau K_n(\tau) = 1$. That is,

$$w = \frac{(n - 1)\lambda}{\Delta t} = -v. \quad (S9)$$

In summary, we now have

$$K_n(t - \tau) = \left(\frac{n - 1}{\Delta t}\right)^n \frac{1}{(n - 1)!} (t - \tau)^{n-1} e^{-\frac{n-1}{\Delta t}(t-\tau)}. \quad (S10)$$

It centers at $\Delta t$ and the variance is given by

$$\sqrt{\langle (t - \tau - \Delta t)^2 \rangle} = \frac{\sqrt{n+1}}{n-1} \Delta t \sim \frac{1}{\sqrt{n}} \Delta t,\quad (S11)$$

which means a sharp peak for large $n$. That is, $K_n(t - \tau) \approx \delta(t - \tau - \Delta t)$ for $n \gg 1$.

Finally, (S5) gives

$$x_n(t) = \int_0^t d\tau K_n(t - \tau)x_0(\tau) + \nu(t),$$

$$\approx x_0(t - \Delta t) + \nu(t), \quad (S12)$$

where the noise $\nu(t)$ from the memory chain is given by

$$\nu(t) = \frac{\Delta t}{(n - 1)\lambda} \sum_{k=1}^n \int_0^t d\tau \eta_k(\tau)K_{n-k+1}(t - \tau). \quad (S13)$$

Putting (S12) back to (S1), we get the time-delayed Langevin equation

$$\lambda_0 \frac{d}{dt} x_0 = v_{0,0}x_0 + f(x_0) + \eta_0 + w_{0,n}x_0(t - \Delta t) + w_{0,n}\nu(t), \quad (S14)$$

(ii) Heisenberg-Langevin equation for two oscillators

The dynamics of two oscillators is given by the Hamiltonian

$$H_{12} = \frac{\gamma}{2} (a_1 a_2^\dagger + \text{h.c.}) \quad (S15)$$

and the jump operator

$$J_{12} = \sqrt{\gamma}(a_1 - ia_2). \quad (S16)$$

Effectively, we can have a microscopic Hamiltonian for this Lindbladian, with an auxiliary strongly dissipative oscillator coupled with a bosonic bath:

$$H = H_{12} + \frac{\sqrt{\kappa}}{2} (J_{12} c^\dagger + \text{h.c.}) + \int dk |kb_k\rangle \langle b_k| + \frac{\sqrt{\kappa}}{2\pi} \int dk |b_k c^\dagger + \text{h.c.}, \quad (S17)$$

where $c$ is the operator for the strongly dissipative oscillator, and $b_k$ are the operators for the modes of the bosonic bath. For 1, 2, and c, we then have a master equation

$$\frac{d}{dt} \rho_{12,c} = -i[H_{12} + \frac{\sqrt{\kappa}}{2} (J_{12} c^\dagger + \text{h.c.}), \rho_{12,c}] + \kappa D[c] \rho_{12,c}, \quad (S18)$$
which gives our master equation for $\rho_{12}$ in the large $\kappa$ limit (see [40, 42, 53]):

$$\frac{d}{dt}\rho_{12} = -i[H_{12}, \rho_{12}] + D[J_{12}]\rho_{12}. \tag{S19}$$

Using the microscopic Hamiltonian (S17), we have the Heisenberg EOM

$$\frac{d}{dt}a_1 = -\frac{\gamma}{2}a_1 - i\frac{\sqrt{\kappa}}{2}c, \tag{S20}$$

$$\frac{d}{dt}a_2 = -\frac{\gamma}{2}a_1 + \frac{\sqrt{\kappa}}{2}c,$n

$$\frac{d}{dt}c = -i\frac{\sqrt{\kappa}}{2}J_{12} - \frac{\kappa}{2}c + F(t),$$

where $F(t) = -i\sqrt{\frac{2}{\pi}}\int dk b_k(0)e^{-ikt}$ is a noise operator and $\langle F(t)F(t')\rangle = \kappa\delta(t - t')$ while other correlations vanish (i.e. zero-temperature bath). By formally integrating out the last one (with $\kappa \to \infty$), we get the Heisenberg-Langevin equation for $a_1$ and $a_2$:

$$\frac{d}{dt}a_1 = -\frac{\gamma}{2}a_1 + \eta(t), \tag{S21}$$

$$\frac{d}{dt}a_2 = -\frac{\gamma}{2}a_2 - i\gamma a_1 + i\eta(t),$$

where $\eta(t) = -i\sqrt{\kappa/\kappa}F(t)$ is a Gaussian white noise with $\langle \eta(t)\eta(t')\rangle = \kappa\delta(t - t')$. Notice that the noise operator $\eta(t)$ is a function of annihilation operators $b_k(0)$.

(iii) Heisenberg-Langevin equation for a chain of oscillators

The dynamics for a chain of oscillators can be described with Lindblad master equation

$$\frac{d}{dt}\rho = -i\left[H_0 + \sum_{j=0}^{n} H_{j,j+1}, \rho\right]$$

$$+ \sum_{j=0}^{n} \left(J_{j,j+1}\rho J_{j,j+1}^\dagger - \frac{1}{2}\{\rho, J_{j,j+1}^\dagger J_{j,j+1}\}\right), \tag{S22}$$

where $H_0$ is a local Hamiltonian at site 0, hopping Hamiltonian

$$H_{j,j+1} = \frac{\gamma_{j,j+1}}{2}(\chi_{j,j+1}a_ja_{j+1}^\dagger + \text{h.c.}), \tag{S23}$$

and the engineered jump operator

$$J_{j,j+1} = \sqrt{\gamma_{j,j+1}}(a_j - i\chi_{j,j+1}a_{j+1}). \tag{S24}$$

Note that although $H_{j,j+1}$ are linear, $H_0$ can be nonlinear.

Thus, for a chain of oscillators, we have the Heisenberg-Langevin equation

$$\frac{d}{dt}a_j = -\frac{\gamma_{j-1,j}}{2}a_j - \frac{\gamma_{j,j+1}}{2}a_j - i\gamma_{j-1,j}a_{j-1} + i\eta_{j-1,j}(t) + \eta_{j,j+1}(t)$$

$$= -\frac{\gamma_{j-1,j} + \gamma_{j,j+1}}{2}a_j - i\gamma_{j-1,j}a_{j-1} + \xi_j(t), \tag{S25}$$

where we have denoted noise operator

$$\xi_j(t) = i\eta_{j-1,j}(t) + \eta_{j,j+1}(t). \tag{S26}$$

We then have the following correlation relations (others vanish)

$$\langle \xi_j(t)\xi_j(t')\rangle = (\gamma_{j-1,j} + \gamma_{j,j+1})\delta(t - t'),$$

$$\langle \xi_j(t)\xi_{j-1}(t')\rangle = i\gamma_{j-1,j}\delta(t - t'), \tag{S27}$$

$$\langle \xi_{j-1}(t)\xi_j(t')\rangle = -i\gamma_{j-1,j}\delta(t - t').$$
(iv) derivation of the memory kernel for \(a_0\)

Using the fact that \(\gamma = (n - 3)/\Delta t\) being a large decay rate, we get \(a_1(t) \approx -2i\chi_01\gamma_01a_0(t)/\chi_01^2\gamma_01 + \gamma\) + noise and \(a_n(t) \approx -2i\gamma a_{n-1}(t)/\gamma n0 + \gamma\) + noise. Combining these relations with (5), we get

\[
a_n(t) \approx \frac{(-i)^n 4\gamma \chi_01\gamma_01}{\chi_01^2\gamma_01 + \gamma}(\gamma n0 + \gamma)\Theta(t - \Delta t)a_0(t - \Delta t) + \text{noise.} \tag{S28}
\]

From EOM of \(a_1\) and \(a_n\) in (3), we can get

\[
a_1(t) = -i\chi_01\gamma_01 \int_0^t d\tau e^{-\frac{(\chi_01\gamma_01 + \gamma)}{2}(t-\tau)}a_0(\tau) + \int_0^t d\tau e^{-\frac{(\chi_01\gamma_01 + \gamma)}{2}(t-\tau)}\xi_1^j(\tau)
\]

\[
a_n(t) = -i\gamma \int_0^t d\tau e^{-\frac{(\gamma n0 + \gamma)}{2}(t-\tau)}a_{n-1}(\tau) + \int_0^t d\tau e^{-\frac{(\gamma n0 + \gamma)}{2}(t-\tau)}\xi_n^j(\tau), \tag{S29}
\]

where the noise operator

\[
\xi_j^j(t) = \xi_j(t) + a_j(0)\delta(t) \tag{S30}
\]

comes from inverse Laplace transformation of \(\xi_j^j(s) = \tilde{\xi}_j(s) + a_j(0)\). Note that the exponential decay inside integral has a large decay rate \(\sim \gamma/2\), which is much larger than the typical rate in site 0. We can therefore apply Markov approximation here and have the relations

\[
a_1(t) \approx -i\chi_01\gamma_01 \frac{2}{\chi_01^2\gamma_01 + \gamma}a_0(t) + \frac{2}{\chi_01^2\gamma_01 + \gamma}\xi_1^j(t)
\]

\[
a_n(t) \approx -i\gamma \frac{2}{\gamma n0 + \gamma}a_{n-1}(t) + \frac{2}{\gamma n0 + \gamma}\xi_n^j(t), \tag{S31}
\]

which complements the relation between \(a_{n-1}\) and \(a_1\) to give us the relation between \(a_n\) and \(a_0\) in (S28).

It is also straightforward to write down the expression of the noise operator \(\nu'(t)\). From (6), we know

\[
\nu(t) = \frac{i}{\gamma} \sum_{j=2}^{n-1} \int d\tau K_{n-j}(t - \tau)\xi_j^j(\tau). \tag{S32}
\]

Then using (S31), we see

\[
\nu'(t) \propto \sum_{j=1}^{n} \int d\tau M_j(t - \tau)\xi_j^j(\tau) + \xi_0(t). \tag{S33}
\]

where \(M_j(t - \tau)\) is their corresponding memory kernel. Notice that \(\xi_0(\tau)\) only depends on \(b_k^{(0,1)}(0)\) and \(b_k^{(n,0)}(0)\) unlike (S30).

Now we see that \(\nu'(t)\) is a function of \(\xi_j^{(j+1)}(0)\) and \(a_j>0(0)\).

Interestingly, in the specific case with \(\eta = \kappa\), the parameter choice (11) happens to give exact results without the above Markov approximation. In this case, (11) gives (S37) below. Then, we get

\[
\tilde{a}_1(s) = \frac{-i\sqrt{\kappa_\gamma}}{s + \gamma}\tilde{a}_0(s) + \frac{1}{s + \gamma}\left(\tilde{\xi}_1(s) + a_1(0)\right), \tag{S34}
\]

\[
\tilde{a}_{j,2 \leq j \leq n}(s) = \frac{-i\gamma}{s + \gamma}\tilde{a}_{j-1}(s) + \frac{1}{s + \gamma}\left(\tilde{\xi}_j(s) + a_j(0)\right).
\]

Following the same procedure in the main text, we get

\[
a_n(t) = \sqrt{\frac{\kappa}{\gamma}} \int_0^t K_n(t - \tau)a_0(\tau) + \text{noise}. \tag{S35}
\]

Then the memory term is exactly

\[
- i\sqrt{\kappa_\gamma} a_n(t) = -i\kappa \int_0^t K_n(t - \tau)a_0(\tau) + \text{noise}, \tag{S36}
\]

which is consistent with the general case (S28), since \(K_n \rightarrow (-i)^2 K_{n-2}\) for large \(n\).
(v) matrix EOMs in the linear case

With the parameters chosen in the linear case, (3) has the compact form
\[
\begin{align*}
\frac{d}{dt}a_0 &= -\kappa a_0 - i\sqrt{\kappa\gamma}a_n + \xi_0(t) \\
\frac{d}{dt}a_1 &= -\gamma a_1 - i\sqrt{\kappa\gamma}a_0 + \xi_1(t) \\
\frac{d}{dt}a_{j,2\leq j\leq n} &= -\gamma a_j - i\gamma a_{j-1} + \xi_j(t).
\end{align*}
\] (S37)

Denote (S37) as
\[
\frac{d}{dt}\vec{a} = \mathcal{M}\vec{a} + \vec{\xi},
\] (S38)

with \(\vec{a} = (a_0, a_1, \ldots, a_n)\). To get the population, we then can solve the EOM
\[
\frac{d}{dt}(a^\dagger \otimes \vec{a}) = (\mathcal{M}^* \otimes 1 + 1 \otimes \mathcal{M})(a^\dagger \otimes \vec{a}),
\] (S39)

using exact diagonalization.

(vi) qubit EOM in the nonlinear case

In the qubit case, we have \(a_0 (= \sigma_-)\) being a hard core boson operator. For now, consider the bond \(0 \rightarrow 1\) only. From (S22), we have
\[
H_{01} = \frac{\gamma_{01}\chi_{01}}{2}(a_0a_1^\dagger + a_1a_0^\dagger)
\] (S40)

and the jump operator
\[
J_{01} = \sqrt{\gamma_{01}}(a_0 - i\chi_{01}a_1).
\] (S41)

We can then write down the Heisenberg-Langevin equation
\[
\begin{align*}
\frac{d}{dt}a_0 &= [a_0, a_0^\dagger](-\frac{\gamma_{01}}{2}a_0 + \eta_{01}), \\
\frac{d}{dt}a_1 &= -i\gamma \chi_{01}a_0 - \frac{\gamma_{01}}{2} \chi_{01}^2 a_1 + i\chi_{01}\eta_{01},
\end{align*}
\] (S42)

where nonlinear nature is shown as the commutation relation \([a_0, a_0^\dagger]\) in the EOM of \(a_0\).

When considering other bonds, we still get \(a_n(t) \propto \Theta(t - \Delta t)a_0(t - \Delta t)\) as the linear case, because EOM for \(a_{n=1,\ldots,n}\) are the same. For \(a_0\), we then have
\[
\frac{d}{dt}a_0 = [a_0, a_0^\dagger](\frac{1}{2}(\chi_{n0}^2 \gamma_{n0} + \gamma_{01})a_0 - i\chi_{n0}\gamma_{n0}a_n + \text{noise}).
\] (S43)

We then see that the term contains time-delayed feedback have the form \(\Theta(t - \Delta t)[a_0(t), a_0^\dagger(t)]a_0(t - \Delta t)\), which is exactly the memory for qubit in front of a mirror. Note that if \(a_0\) is a bosonic operator, we get back to the linear case as expected.

(vii) more information on MPS simulation of the qubit case

Starting from the Hamiltonian and jump operator, we have the Lindbladian superoperator at bond \((j \rightarrow j + 1)\)
\[
\mathcal{L}_{j,j+1} = -i[H_{j,j+1}, \cdot] + J_{j,j+1} \cdot J_{j,j+1}^\dagger - \frac{1}{2}(J_{j,j+1}^\dagger J_{j,j+1}, \cdot).
\] (S44)
The total Lindbladian for our system is then $\mathcal{L} = \sum_{j=0}^{n} \mathcal{L}_{j,j+1}$, which is simply a one-dimensional lattice model with periodic boundary condition (PBC).

For the time evolution, we decompose each time step $\exp(\mathcal{L} dt)$ using the 4-th order Trotter decomposition (see e.g. [54]):

$$e^{(\mathcal{L}_t + \mathcal{L}_c) dt} = e^{\mathcal{L}_t dt_1} e^{\mathcal{L}_c dt_2} e^{\mathcal{L}_t dt_3} e^{\mathcal{L}_c dt_4} e^{\mathcal{L}_t dt_5} e^{\mathcal{L}_c dt_6} e^{\mathcal{L}_t dt_7} \cdots + O(dt^5),$$

(S45)

where $dt_1 = \omega dt/2$, $dt_2 = \omega dt$, $dt_3 = (1 - w) dt/2$, $dt_4 = (1 - 2w) dt$ with $w = (2 - 2^{1/3})^{-1}$. Here $\mathcal{L}_c$ denotes bonds with $j$ being even while $\mathcal{L}_t$ denotes bonds with $j$ being odd.

Notice that $dt_1 > 0$ and $dt_2 > 0$ make $\exp(\mathcal{L} dt_{1(2)})$ a completely positive and trace-preserving (CPTP) map, while $dt_3 < 0$ and $dt_4 < 0$ make $\exp(\mathcal{L} dt_{3(4)})$ stop being a CPTP map but just a linear trace-preserving map. That is, we can perform Kraus decomposition of $\exp(\mathcal{L} dt_{1(2)}) \rho$:

$$e^{\mathcal{L} dt_{1(2)}} \rho = \sum_k A_k \rho A_k^\dagger,$$

(S46)

with $\sum_k A_k^\dagger A_k = 1$. We then apply operator $A_k$ and $A_k^\dagger$ on the MPDO to apply a Trotter gate. As for the linear map $\exp(\mathcal{L} dt_{3(4)})$, we can have an operator-sum representation [55]

$$e^{\mathcal{L} dt_{3(4)}} = \sum_k L_k \rho R_k^\dagger,$$

(S47)

which is used for applying a Trotter gate. These decompositions are carried out using the to_kraus and to_stinespring functions of the QuTiP library [56].

As mentioned in the main text, small time step means requirement for smaller cutoff error $\epsilon$ in MPS simulation. The density matrix with $\epsilon = 0$ (≡ $\tilde{\rho}$) can be written as

$$\tilde{\rho} = \rho + \mathcal{E}(\epsilon) \rho_x,$$

(S48)

where $\rho$ is the MPDO we have in simulation, $\rho_x$ (already normalized) is the part that got truncated away, and $\mathcal{E}(\epsilon)$ is an unknown function of the cutoff $\epsilon$. Because $\mathcal{E}(\epsilon)$ is non-zero for finite $\epsilon$. That is, we have trace error $1 - \text{Tr}(\rho) = \mathcal{E}(\epsilon)$ [see Fig. S1(c)]. In our computation, we compute $\langle a_0^\dagger a_0 \rangle$ using $\rho$ instead of the normalized $\rho' = \rho / \text{Tr}(\rho)$ based on the following observation. As shown in Fig. S1(a) and (b), as $\epsilon$ gets smaller, the unnormalized values changes little compared with the normalized ones. We attribute this to the fact that, beyond a cutoff $\epsilon > 10^{-10}$ here, the truncated part $\rho_x$ almost does not contribute to the computed observable $\langle a_0^\dagger a_0 \rangle$. That is, $\text{Tr}(a_0^\dagger a_0 \rho_x) \approx 0$. Based on this property, we use the unnormalized $\rho$ in our computation of $\langle a_0^\dagger a_0 \rangle$. Of course, systematic error analysis will be needed in future studies.