Conditions for realizing one-point interactions from a multi-layer structure model

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Abstract

A heterostructure composed of $N$ parallel homogeneous layers is studied in the limit as their widths $l_1, \ldots, l_N$ shrink to zero. The problem is investigated in one dimension and the piecewise constant potential in the Schrödinger equation is given by the strengths $V_1, \ldots, V_N$ as functions of $l_1, \ldots, l_N$, respectively. The key point is the derivation of the conditions on the functions $V_1(l_1), \ldots, V_N(l_N)$ for realizing a family of one-point interactions as $l_1, \ldots, l_N$ tend to zero along available paths in the $N$-dimensional space. The existence of equations for a squeezed structure, the solution of which determines the system parameter values, under which the non-zero tunneling of quantum particles through a multi-layer structure occurs, is shown to exist and depend on the paths. This tunneling appears as a result of an appropriate cancellation of divergences.

Keywords: one-dimensional quantum systems, multi-layer heterostructures, point interactions, resonance sets

1. Introduction

The differential operators with singular zero-range potentials are frequently used to model different physical systems. The particular class of such models is represented by Schrödinger operators in one dimension with regular realistic potentials, which can be approximated by point (called also contact) interactions (see books [1, 2] for details and references) defined on the sets consisting of isolated points, curves or surfaces. In this asymptotic approach, for a general class of potentials $V$, under certain conditions the functions $\varepsilon^{-1}V(x/\varepsilon)$ and $\varepsilon^{-2}V(x/\varepsilon)$ converge in the sense of distributions as $\varepsilon \to 0$ to the Dirac delta-function $\delta(x)$ and its first derivative $\delta'(x)$, respectively, realizing one-point interactions. Thus, the one-dimensional Schrödinger operator with the $\delta'(x)$ distribution has rigorously been treated in

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works [3–6] and afterwards these studies were extended to the quantum graph theory [7–9]. Some particular cases of the regularization of the $\delta'(x)$ function for realizing one-point interactions have been used in [10]. Using a special class of piecewise constant potentials $V$, it was possible [11–13] to construct explicitly the matrices, which connect the two-sided boundary conditions at the point of singularity.

In a similar way, the Schrödinger operator with the combined potential $a\delta'(x) + b\delta(x)$ was investigated [14–16], including the studies of the discrete spectrum [17, 18]. The one-dimensional quantum Hamiltonians with potentials having a regular unperturbed part of different shape and a singular perturbation of the form $a\delta'(x) + b\delta(x)$ have been examined in publications [19–21].

Various aspects have been studied in numerous works (see, e.g. publications [3, 22–28], a few to mention) concerning the $\delta'$-interaction, which should not be confused with the $\delta'$-potential. It is worth mentioning the works on the three delta approximation of the $\delta'$-interaction [29, 30] as well as publications [21, 31, 32], where the $\delta'$-interaction was considered as a perturbation of regular potentials.

The most simple example of regularizing the $\delta'$-potential is the sequence of piecewise constant functions, which from the physical point of view can be related to planar heterostructures. In particular, for the double-layer case, using a three-scale parametrization of the layer parameters such as their strengths, thickness and the distance between the layers, a whole family of one-point interactions has been constructed, where the scattering data [13, 33, 34] as well as the existence of a bound state [35, 36] have been examined. In general, here it is not required that a piecewise constant potential must have a well-defined limit (for instance, in the sense of distributions) as the thickness of layers and the distance between them shrink to zero. As a result, within the three-scaled squeezing procedure, it was possible to describe both the resonant point interactions [4–6, 10, 11, 14–16] and the point interaction produced by the potential $a\delta'(x)$ potential, that comes from the theory developed by Kurasov and co-workers [1, 37, 38], for which the elements of the diagonal connection matrix are $\theta = (2 + a)/(2 - a)$ and $\theta^{-1}$ [17], within an unique scheme [34].

The present paper is devoted to the investigation of a planar heterostructure composed of an arbitrary number of layers. The layers may be separated by distances and these spaces (if any) are considered as a particular case of layers with zero strength potential. The goal of the paper is to study this multi-layer structure in the limit as both the layer thickness and the distance between the layers shrink to zero. The electron motion in this system is supposed to be confined in the longitudinal direction (say, along the $x$-axis) being perpendicular to the transverse planes where electronic motion is free. Then the three-dimensional Schrödinger equation of such a structure can be separated into longitudinal and transverse parts, resulting in the reduced stationary one-dimensional Schrödinger equation

$$-\psi''(x) + V(x)\psi(x) = E\psi(x)$$

with respect to the longitudinal component of the wave function $\psi(x)$ and the electron energy $E$. Here, $V(x)$ is the potential for quantum particles in the form of a piecewise constant function and the prime stands for the differentiation over $x$.

2. Transmission matrix for a multilayer structure

Consider one-dimensional stationary Schrödinger equation (1), where the potential $V(x)$ is a piecewise constant function defined on the interval $(x_0, x_N)$ with $N$ subsets $(x_{i-1}, x_i)$, where $i = 1, \ldots, N$ and $N = 1, 2, \ldots$. Each strength $V_i \in \mathbb{R}$ defined on this interval is supposed
to depend on the \( i \)th layer width \( l_i \). Therefore, we have the set of functions: \( V_1(l_1), \ldots, V_N(l_N) \).

The transmission matrix \( \Lambda_i(x_{i-1}, x_i) \) for the \( i \)th layer connects the values of the wave function \( \psi_i(x) \) and its derivative \( \psi'_i(x) \) at the boundaries \( x = x_{i-1} \) and \( x = x_i \) according to the matrix equation

\[
\begin{pmatrix}
\psi_i(x_i) \\
\psi'_i(x_i)
\end{pmatrix} = \Lambda_i(x_{i-1}, x_i) \begin{pmatrix}
\psi_{i-1}(x_{i-1}) \\
\psi'_{i-1}(x_{i-1})
\end{pmatrix}, \quad \Lambda_i(x_{i-1}, x_i) = \begin{pmatrix}
\lambda_{i11} & \lambda_{i12} \\
\lambda_{i21} & \lambda_{i22}
\end{pmatrix},
\]

(2)

The transmission matrix for each layer defined on the interval \((x_{i-1}, x_i)\) can be computed through the solutions \( u_i(x) \) and \( v_i(x) \) and their derivatives taken at the boundaries \( x = x_{i-1} \) and \( x = x_i \). Let \( u_i(x) \) and \( v_i(x) \) be linearly independent solutions on the interval \((x_{i-1}, x_i)\) obeying the initial conditions

\[
u_i(x_{i-1}) = 1, \quad u_i(x_{i-1}) = 0, \quad v_i(x_{i-1}) = 0, \quad v'_i(x_{i-1}) = 1.
\]

(3)

Then the representation of the \( \Lambda_i \)-matrix reads

\[
\Lambda_i(x_{i-1}, x_i) = \begin{pmatrix}
u_i(x_i) \\
u'_i(x_i)
\end{pmatrix}, \quad \det \Lambda_i(x_{i-1}, x_i) = 1.
\]

(4)

Particularly, for the piecewise constant function \( V_i(x) \), the solutions \( u_i(x) \) and \( v_i(x) \), \( x_{i-1} \leq x \leq x_i \), obeying conditions (3), read

\[
u_i(x) = \cos[q_i(x - x_{i-1})], \quad v_i(x) = q_i^{-1} \sin[q_i(x - x_{i-1})],
\]

(5)

where

\[
q_i = \sqrt{E - V_i}.
\]

(6)

Therefore the \( \Lambda_i \)-matrix in this particular case is

\[
\Lambda_i(x_{i-1}, x_i) = \begin{pmatrix}
\cos(q_i l_i) & q_i^{-1} \sin(q_i l_i) \\
-q_i \sin(q_i l_i) & \cos(q_i l_i)
\end{pmatrix}, \quad i = 1, N,
\]

(7)

and the full transmission matrix, connecting the boundary conditions at \( x = x_0 \) and \( x = x_N \), is the product

\[
\Lambda(x_0, x_N) = \Lambda_N(x_{N-1}, x_N) \ldots \Lambda_1(x_0, x_1) = \begin{pmatrix}
\lambda_{11} & \lambda_{12} \\
\lambda_{21} & \lambda_{22}
\end{pmatrix}.
\]

(8)

Clearly, due to the form of matrices (7), \( \det \Lambda(x_0, x_N) = 1 \).

3. Structure of the full transmission matrix \( \Lambda(x_0, x_N) \)

Using matrix representation (7) in product (8), by induction one can derive some properties of the \( \Lambda(x_0, x_N) \)-matrix elements. First of all note that each element of this matrix consists of \( 2^{N-1} \) terms, each of which is the \( N \)-multiple product of sines and cosines involved in matrices (7). By lengthy and straightforward computation one can derive by induction the structure of
these terms:

\[
\left\{ \begin{array}{c}
\lambda_{11} \\
\lambda_{12} \\
\lambda_{21} \\
\lambda_{22}
\end{array} \right\} = \prod_{n=1}^{N} \cos(q_n l_n) \times \begin{cases}
1 + Q_{11}, \\
\sum_{i=1}^{N} q_i^{-1} I_i + Q_{12}, \\
- \sum_{i=1}^{N} q_i I_i - Q_{21}, \\
1 + Q_{22},
\end{cases}
\tag{9}
\]

where \( t_i := \tan(q_i l_i) \) and

\[
Q_{11} = \sum_{n=1}^{[N/2]} (-1)^n \sum_{1 \leq i < j < k \leq n} D_{i1j} \cdots D_{inkn},
\tag{10}
\]

\[
Q_{12} = \sum_{n=1}^{[N-1]/2} (-1)^n \sum_{1 \leq i < j < k \leq n} D_{i1j} \cdots D_{inkn} q_k^{-1} h_n,
\tag{11}
\]

\[
Q_{21} = \sum_{n=1}^{[N-1]/2} (-1)^n \sum_{1 \leq i < j < k \leq n} D_{i1j} \cdots D_{inkn} q_ik^{-1} h_n,
\tag{12}
\]

\[
Q_{22} = \sum_{n=1}^{[N/2]} (-1)^n \sum_{1 \leq i < j < k \leq n} D_{i1j} \cdots D_{inkn},
\tag{13}
\]

with

\[
D_{ij} := (q_i/q_j) t_i t_j, \quad i, j = 1, N \ (i \neq j).
\tag{14}
\]

Here, \([N/2]\) and \([N-1]/2\) stand for the integer parts of \(N/2\) and \((N - 1)/2\), respectively.

It is remarkable that all the \(Q\)-series appear in the form of pairwise terms (14), called ‘dyads’ from now on. The \(n\)th group of terms in (10) or (13) consists of \(\binom{N}{n} \frac{N!}{(2n)! (N - 2n)!}\) summands. Thus, for \(n = 1\) there are \(N(N - 1)/2\) dyads (14), where \(i = 1, \ldots, N - 1\) and \(j = 2, \ldots, N\) with the ordering relation \(i < j\). Next, the terms in the group with \(n = 2\) are of the form \(D_{i1j} D_{i2j}\) in \(\lambda_{11}\) and \(D_{i1j} D_{i2j}\) in \(\lambda_{22}\), where \(i_1 = 1, \ldots, N - 3, j_1 = 2, \ldots, N - 2, i_2 = 3, \ldots, N - 1\) and \(j_2 = 4, \ldots, N\) with the ordering relation \(i_1 < j_1 < i_2 < j_2\). In a similar way, one can present the terms in the group with \(n = 3\) and so on.

Similarly, the \(n\)th group of terms in (11) and (12) consists of \(\binom{N}{n+1} \frac{N!}{(2n+1)! (N - 2n - 1)!}\) summands. Thus, for \(n = 1\) we have the sum \(N(N - 1)(N - 2)/6\) terms in real-valued functions \(V_i(l_i)\) the form of ‘triads’ \(D_{ij} q_i^{-1} l_k = (q_i/q_j) t_i t_j t_k \lambda_{12}\) and \(D_{ij} q_i t_k = (q_i/q_j) t_i t_k t_k \lambda_{21}\), where \(i = 1, \ldots, N - 2, j = 2, \ldots, N - 1\) and \(k = 3, \ldots, N (i < j < k)\). In a similar way, one can present the terms in the group with \(n = 2\) and so on.

4. Conditions for realizing one-point interactions with finite diagonal elements in the connection matrix

A family of one-point interactions can be realized from equation (1) with a piecewise constant potential \(V(x)\) defined on the interval \(x_0 \leq x \leq x_N\) if all the \(N\) layer widths \(l_i, \ldots, l_N\) shrink to
zero, setting then \( x_0 \to -0 \) and \( x_N \to +0 \). Any finite configuration of the layer widths can be associated with a point (vector) \( l := \text{col}(l_1, \ldots, l_N) \) in the orthogonal angle of dimension \( N \) with the vertex at the origin \( l = 0 \), so that the squeezing of the full layer thickness \( x_N - x_0 \) to zero can be described as a path approaching the origin \( l = 0 \). At the same time, for the realization of a non-trivial point interaction, at least one of the potential strengths \( V_1, \ldots, V_N, \ i = 1, \ldots, N \), must tend to infinity with some rate. Otherwise, the transmission through the squeezed structure will be trivial, i.e., the limit transmission (connection) matrix will be just the identity. In general, under the squeezing procedure, the behavior of the functions \( V_i(l_1), \ldots, V_N(l_N) \) versus the paths projected onto the \((N-1)/2 (l_i, l_j)\)-faces of the \(N\)-dimensional orthogonal angle.

4.1. A \( \delta \)-like squeezing limit

Consider first the case with the asymptotic behavior \( V_i(l_i)l_i \to c \in \mathbb{R} \) for all the functions \( V_i(l_i) \) as \( l_i \to 0 \). Here and in the following \( c \) stands for an arbitrary constant. Assume that the functions \( V_i(l_i) \) belong to the sets

\[
G^0_i := \left\{ V_i \in \mathbb{R} \mid \lim_{l_i \to 0} V_i(l_i) l_i = \alpha_i \in \mathbb{R} \right\}, \quad i = 1, \ldots, N. \tag{15}
\]

Note that the free space \((x_{i-1}, x_i)\) with \( V_i \equiv 0 \), is also treated as a ‘layer’ with strength zero.

For any two variables \( \beta_1 \) and \( \beta_2 \), we write \( \beta_1 \sim \beta_2 \) if they are of the same order. Let us now estimate series (10)–(13) in the case as all the functions \( V_i \) belong to \( G^0_i \) \((i = 1, \ldots, N)\). In the limit as \( l \to 0 \), we have \( D_{ij} \sim \alpha_i l_j \to 0 \) and \( D_{jk} \sim \alpha_j l_i \to 0 \) as well as \( D_{ik} q_k^{-1} l_k \sim \alpha_i l_k \to 0 \) and \( D_{jk} q_k^{-1} l_k \sim \alpha_j l_k \to 0 \). Moreover, the shrinking of all the widths \( l_i \)'s to zero is available to be arranged in a repeated manner (not simultaneously). For instance, the shrinking may pass along some edges of the orthogonal angle or be found in its \((l_i, l_j)\)-faces. Hence, all the paths with \( l \to 0 \) constitute in this case a pencil that coincides with the \(N\)-dimensional orthogonal angle. Thus, as follows from equations (10)–(13), the \(Q\)-series in equation (9) vanish, resulting in the limit transmission (connection) matrix of the form

\[
\Lambda_0 := \lim_{l_i \to 0} \Lambda(x_0, x_N) = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}, \quad \alpha := \sum_{i=1}^{N} \alpha_i. \tag{16}
\]

In the case if \( \alpha > 0 \), we have the point interaction specified by a \( \delta \)-potential barrier, while for \( \alpha < 0 \), we are dealing with an effective \( \delta \)-well potential. Using the general equation for bound state levels \( \kappa \ (k = i\kappa, \ \kappa > 0) \), written in terms of the \( \Lambda \)-matrix elements as (see, e.g. [36, 39])

\[
\lambda_{11}(\kappa) + \lambda_{22}(\kappa) + \kappa \lambda_{12}(\kappa) + \kappa^{-1} \lambda_{21}(\kappa) = 0, \tag{17}
\]

we obtain \( \kappa = -\alpha/2 \). Thus, one can formulate.

**Conclusion 1.** Let all the functions \( V_i(l_i), i = 1, \ldots, N, \) belong to the \( G^0_i \)-sets defined by condition (15) and the vector \( l = \text{col}(l_1, \ldots, l_N) \) tends to the origin \( l = 0 \) along any path.
within the pencil formed by the $N$-dimensional orthogonal angle with the vertex at the origin $l = 0$. Then, in the squeezing limit $l \to 0$, the point interactions are realized as the family of $\delta$-potentials with the connection matrix given by equation (16).

4.2. A singular squeezing limit

The set of functions $V_i(l_i) \in G_i^0$, used for materializing $\delta$-potentials, can be extended to include the functions with a more singular behavior like $|V_i(l_i)|l_i \to \infty$ as $l_i \to 0$. In the case if at least one of the functions $V_i(l_i), i = 1, \ldots, N$, possesses such an infinite behavior, the shrinking of the full system to zero dimension will be referred to as a singular squeezing limit from here on.

The sufficient requirement for realizing a point interaction is that the arguments $q_i l_i$ of the trigonometric functions in matrix expressions (7) must be finite in the limit as $l_i \to 0$ even if the product $V_i(l_i)l_i$ diverges as $l_i \to 0$. In order to derive the conditions under which the $l_i \to 0$ limit is finite, we define the sets of real-valued functions $V_i(l_i), i = 1, \ldots, N,$ by the conditions

$$G_i := \left\{ V_i(l_i) \in \mathbb{R} \mid \lim_{{l_i \to 0}} |V_i(l_i)|^{1/2} l_i = 0 \right\},$$

with the corresponding complements

$$G'_i := \left\{ V_i(l_i) \in \mathbb{R} \mid \lim_{{l_i \to 0}} |V_i(l_i)|^{1/2} l_i = c > 0 \right\}.$$  

Hence, the trigonometric functions in expressions (9)–(14) make sense if each function $V_i(l_i)$ belongs to the union $G_i := G_i \cup G'_i$.

The condition $V_i(l_i) \in \bar{G}_i, i = 1, \ldots, N,$ is not sufficient for the existence of finite squeezed limits of the diagonal elements $\lambda_{11}$ and $\lambda_{22}$ because of the presence of the factors $q_i$ with $|q_i| \to \infty$ in series (10) and (13). However, this divergence can be suppressed due to the presence of the ‘neighboring’ factors $q_j (j \neq i)$, leading to the ambiguity of type $\propto \propto$. Contrary to the $\delta$-potential case, here the finite limits are possible only if all the thickness parameters $l_i, i = 1, \ldots, N$, converge to the origin simultaneously. Thus, for the existence of finite squeezed limits of the elements $\lambda_{11}$ and $\lambda_{22}$, it is sufficient to estimate both the dyads $D_{ij}$ and $D_{ji} (i < j)$. Due to definition (14), the asymptotic expressions for these dyads, written in terms of the functions $V_i(l_i)$ and $V_j(l_j)$ as well as the parameters $l_i$ and $l_j$, read

$$|D_{ij}| \sim \begin{cases} |V_i(l_i)/V_j(l_j)|^{1/2} \\ |V_i(l_i)/|V_j(l_j)|^{1/2} l_j \\ |V_j(l_j)|^{1/2} l_j \\ |V_i(l_i)/l_i| l_j \\ |V_j(l_j)/l_j| \\ V_i \in G_i, V_j \in G'_j \\ V_i \in G'_i, V_j \in G'_j \\ V_i \in G'_i, V_j \in G'_i \\ V_i \in G'_i, V_j \in G'_i \end{cases}$$

for $V_i \in G'_i, V_j \in G'_j$.

in series (10) for $\lambda_{11}$ and

$$|D_{ji}| \sim \begin{cases} |V_j(l_j)/V_i(l_i)|^{1/2} \\ |V_j(l_j)/|V_i(l_i)|^{1/2} l_i \\ |V_i(l_i)/|V_j(l_j)|^{1/2} l_j \\ |V_j(l_j)/l_j| l_i \\ |V_i(l_i)/l_i| l_j \\ V_i \in G'_i, V_j \in G'_j \\ V_i \in G'_i, V_j \in G'_j \\ V_i \in G'_i, V_j \in G'_j \\ V_i \in G'_i, V_j \in G'_i \end{cases}$$

for $V_i \in G'_i, V_j \in G'_j$. 


in series (13) for $\lambda_{22}$. Both the terms $D_{ij}$ and $D_{ji}$ ($i < j$) will be finite in the limit as $l_i \to 0$ and $l_j \to 0$ if

\[
\begin{align*}
|V_i(l_i)/V_j(l_j)| & \to c > 0 \\
|V_i(l_i)/l_i| & \to c > 0, \quad l_i/l_j \to c \geq 0 \\
|V_j(l_j)/l_j| & \to c > 0, \quad l_j/l_i \to c \geq 0 \\
l_i/l_j & \to c \geq 0 \\
l_j/l_i & \to c \geq 0
\end{align*}
\]

These finite limits can be fulfilled if $l_i$ and $l_j$ tend to zero simultaneously, forming a pencil of paths in the $N$-dimensional orthogonal angle with the vertex at $l = 0$. Therefore we need to describe these paths projected onto each $(l_i, l_j)$-face. Thus, let us introduce the following pencil projections onto a given $(l_i, l_j)$-face:

\[
\begin{align*}
\Gamma_{ij} & := \{ (l_i, l_j) \to 0 \mid l_i/l_j \to 0 \} \\
\Gamma'_{ij} & := \{ (l_i, l_j) \to 0 \mid l_i/l_j \to c > 0 \} = \Gamma'_j, \\
\Gamma_{Vij} & := \{ V_i(l_i) \in G_j \setminus G_j^0, \quad (l_i, l_j) \in \Gamma_{ij} \mid |V_i(l_i)/l_i| \to 0 \} \\
\Gamma'_{Vij} & := \{ V_i(l_i) \in G_j \setminus G_j^0, \quad (l_i, l_j) \in \Gamma_{ij} \mid |V_i(l_i)/l_i| \to c > 0 \}
\end{align*}
\]

where $i, j = 1, \ldots, N$ and $i \neq j$. Here, the linear $(l_i, l_j)$-projective paths $\Gamma_{ij}' = \Gamma_{ij}$ determine the pencil composed of all the straight lines filling in the interior of the $N$-dimensional orthogonal angle. They are boundary (limit) cluster sets of both the pencils $\Gamma_{ij}$ and $\Gamma_{ji}$, which consist of curved lines in this interior. The projective nonlinear paths $\Gamma_{Vij}$ and $\Gamma'_{Vij}$ may be termed as ‘adjoint’ $(l_i, l_j)$-curves of the strength $V_i(l_i)$. More precisely, the paths $\Gamma_{Vij}$ mean that $l_i/l_j$ tends to zero more slowly than $|V_i(l_i)/l_i|^2$ (notice that $V_i \in G_j \setminus G_j^0$, i.e. $|V_i(l_i)/l_i|^2(l_i/l_j) \to 0$), while for the limit sets $\Gamma'_{Vij}$ this ratio is non-zero: $l_i/l_j \sim |V_i(l_i)/l_i|^2 \to 0$. The unions

\[
\begin{align*}
\tilde{\Gamma}_{ij} & := \Gamma_{ij} \cup \Gamma'_{ij} = \{ (l_i, l_j) \to 0 \mid l_i/l_j \to c \geq 0 \} \\
\tilde{\Gamma}_{Vij} & := \Gamma_{Vij} \cup \Gamma'_{Vij} = \{ (l_i, l_j) \to 0 \mid l_i/l_j \to 0, \quad \mid V_i(l_i)/l_i| \to c > 0 \}
\end{align*}
\]

where $\tilde{\Gamma}_{Vij} \subset \Gamma_{ij}$, will also be used from now on. The full family of the $(l_i, l_j)$-projective paths, under which all conditions (22) hold true, can be collected together as follows
Thus, under any of nine conditions (25) fulfilled on all the \((l_i, l_j)\)-faces, the \(l \to 0\) limit of the \(\Lambda\)-matrix elements \(\lambda_{11}\) and \(\lambda_{22}\) is finite. Next, under these conditions, we also have \(\lambda_{12} \to 0\) and this immediately follows from formulas (9) and (11) because \(q_i^{-1}t_i\)'s and therefore all the terms in the series \(Q_{1}\) tend to zero. Contrary, in expressions (9) and (12), at least one of the summands \(q_i t_i\) and possibly some terms in the series \(Q_{22}\) in general diverge as \(l \to 0\), except for some cases as all appearing divergences will mutually canceled out and this situation will be analyzed below. Hence, under the conditions ensuring the finite squeezed limits of \(\lambda_{11}\) and \(\lambda_{22}\), the resulting point interactions are separated. Thus, the above results can be summarized in the form of

**Conclusion 2.** Let all the functions \(V_i(l_i), i = 1, \ldots, N\), belong to any of the subsets \(G_i[0]\), \(G_i \backslash G_i[0]\) or \(G'_i\) of the \(G_i\)-set defined by conditions (15), (18) and (19). Assume that for at least one layer, the function \(V_i(l_i)\) belongs to \(G'_i \backslash G_i[0]\) or \(G'_i\). Then the element \(\lambda_{21}\) diverges in the limit as \(l \to 0\) and under any of conditions (25) fulfilled on all the \((i, l_i)\)-faces, \(i, j = 1, \ldots, N\) \((i < j)\), the family of one-point interactions is realized in the squeezing limit with the two-sided boundary conditions on the wave function \(\psi(x)\) being of the Dirichlet type: \(\psi(\pm 0) = 0\).

**5. Resonant tunneling one-point interactions**

We have derived above the conditions, under which the \(l \to 0\) limit of the matrix elements \(\lambda_{11}\) and \(\lambda_{22}\) is finite as well as the limit relation \(\lambda_{12} \to 0\). Here, on the basis of expressions (9) and (12), we will analyze the element \(\lambda_{21}\), which turns out to be the most singular \(\Lambda\)-matrix element as \(l \to 0\). The behavior of each term in this element depends on the paths, along which the squeezing limit is implemented.

As shown above, the squeezed \(\delta\)-potentials have been materialized for all the strengths \(V_i(l_i)\) obeying the condition \(|V_i(l_i)|l_i \to c \geq 0\) and the corresponding \(l \to 0\) paths were arbitrary, including various repeated limits \(l_i \to 0\), \(i = 1, \ldots, N\). In this (regular) case, the connection matrix is of form (16). In the following for this particular case, the summands in \(\sum_{i=1}^{N} q_i t_i\) will be referred to as regular terms. Since no additional restrictions on this behavior were imposed, the transmission through this point potential can be referred to as a non-resonant tunneling.

The situation crucially changes if at least one of the terms \(q_i\), for which \(V_i(l_i) \in G'_i \backslash G_i[0]\), is present in series (9) and (12). In this (singular) case, \(|V_i(l_i)|l_i \to \infty\) as \(l \to 0\) and in spite of the validity of conditions (25), the divergent terms in series (9) and (12) in general will be present.
We refer these terms to as singular ones. One can write \( \lambda_{21} = \lambda_{21}^L + \lambda_{21}^\alpha \), where \( \lambda_{21}^\alpha \to \alpha \in \mathbb{R} \) and each term from \( \lambda_{21}^\alpha \) diverges as \( l \to 0 \). Under some conditions on the functions \( V_i(l_i) \) and the paths converging to the origin \( l = 0 \), the cancellation of divergent terms in the group \( \lambda_{21}^\alpha \) can be materialized, resulting in a finite limit value of \( \lambda_{21} \). However, this is possible if all the divergent terms are of the same order on certain paths. Therefore it is necessary to ‘measure’ the divergence rate of these terms by the multiplication them by an appropriate factor \( L \to 0 \), chosen in such a way that all the resulting products are finite non-zero quantities in the limit as \( l \to 0 \). In other words, a one-to-one correspondence between these quantities and the divergent terms must be provided in the squeezed limit. As a result, the equation for the term \( \lambda_{21}^\alpha \) becomes

\[
\exists \alpha \in \mathbb{R}, \quad \lim_{l \to 0} \frac{q_i l_i}{l} = \alpha \quad \text{and} \quad \lim_{l \to 0} \frac{q_j l_j}{l} = \alpha,
\]

\[
\lambda_{21}^\alpha = \lim_{l \to 0} \frac{q_i l_i}{l} \quad \text{and} \quad \lambda_{21}^\alpha = \lim_{l \to 0} \frac{q_j l_j}{l}.
\]

We consider the configuration \( (\lambda_{21}^\alpha) \) as a limiting case of \( \lambda_{21} \). The configuration \( (\lambda_{21}^\alpha) \) sets the conditions on these limit parameters, which in turn depend only on the divergent terms. In case (i), the cancellation equation reads

\[
\eta_{ij} := \lim_{l_i, l_j \to 0} \frac{(l_i/l_j)}{\eta_{ij}} = \chi_{ij}^{-1}
\]

for \( V_i \in G_i^\prime \), where \( i, j = 1, \ldots, N \) (\( i \neq j \)). Clearly, if \( V_i \in G_i^0 \), then \( \eta_{ij} = 0 \). Thus, the cancellation of divergences imposes some constraints on the configurations \( V_i(l_i) \), \( V_j(l_j) \) and the path pencils, which have to be derived properly in the limit as \( l \to 0 \).

Let us consider two simple examples illustrating how the cancellation procedure does work. Namely the pairwise configurations (i) \( V_i \in G_i^\prime \) and \( V_j \in G_j^\prime \), and (ii) \( V_i \in G_i^\prime \cap G_i^\prime \) and \( V_j \in G_j^\prime \), whereas the rest of strengths are regular terms. In case (i), the cancellation equation reads

\[
\lambda_{21}^\alpha \sim q_i l_i + q_j l_j = 0
\]

and both the divergent terms in this equation are of the same rate in the squeezing limit if the \((l, l)\)-paths belong to the pencil \( \Gamma_{ij} \). Therefore this equation can be multiplied by either \( l_i \) or \( l_j \). Thus, multiplying it by \( l_i \), in the limit as \( l_i \to 0 \) and \( l_j \to 0 \), we arrive at the resonance equation

\[
\eta_{ij} + \chi_{ij} l_j = 0,
\]

written in terms of limit parameters (26) and (27), which in their turn depend only on the asymptotic behavior of the layer parameters \( V_i(l_i) \), \( V_j(l_j) \), \( l_i \) and \( l_j \). In particular, on the paths with \( l_i = l_j \) and for the configuration \( V_i(l_i) = -V_j(l_j) \), equation (28) reduces to

\[
\tan s_i = \tanh s_i, \quad s_i = \lim_{l_i \to 0} \frac{|V_i(l_i)|^{1/2} l_i}.
\]
In the multi-layer case described above, the solutions \( \{ s_i \}_{n=-\infty} \) of this equation form the discrete resonance set of the point interaction that corresponds to the potential \( V(x) = a\delta(x) + b\delta(x) \). At the values \( a = s_n \), this interaction is partially transparent, whereas beyond these values the transmission is zero.

Similarly, in the other case (ii), it would be possible to multiply the equation \( \lambda_{21} = q_i^2 l_i + q_j t_j = 0 \), where both the summands diverge as \((l_i, l_j) \to 0 \) (\( V_i \in \mathcal{G}_0 \) and \( V_j \in \mathcal{G}_j \)), by \( l_i \). However, in this case, \( |q_i^2 l_i^2| \to 0 \), while the other term \( \chi_l t_j \) is non-zero. This means that the divergence of the terms \( q_i^2 l_i \) and \( q_j t_j \) is of different order on the pencil \( \Gamma_{ij} \), where \( l_i \) and \( l_j \) approach the origin linearly. Therefore this pencil is not appropriate for the cancellation of divergences. Instead, let us consider the cancellation of divergences on the pencil \( \Gamma_{ij} \) of curved paths, for which \( l_i / l_j \sim |V_i(l_i)| l_i^2 \to 0 \), where the divergence rate of both terms is of the same order: \( |q_i^2 l_i| \sim |V_i(l_i)| l_i^2 \sim l_i^{-1} \) and \( |q_j t_j| \sim |V_j(l_j)| \sim l_j^{-1} \). In this case, the parameter \( L = l_i \) can be used as a multiplier for the comparison of the divergence of \( q_i^2 l_i \) and \( q_j t_j \). Then the resonance equation is well-defined on the \( \Gamma_{ij} \)-paths and it can be written in terms of finite quantities as follows

\[
\lim_{(l_i, l_j) \to \Gamma_{ij}} (q_i^2 l_i + q_j t_j) l_j = \eta_{ij} + \tau_j = 0. \tag{30}
\]

For a negative strength \( V_j \) \((s_j > 0)\), this equation (with respect to the variable \( s_j \)) admits a countable set of solutions. The configuration with some \( V_i \in \mathcal{G}_0 \) can also be considered and then in equation (30) we have \( \eta_{ij} = 0 \). This situation is the limiting case as the divergence vanishes due to the solution of the equation \( \tau_j = 0 \), forming the resonance set consisting of points \( s_j = n\pi \) with integers \( n_i \).

However, even if the pencils of paths have been chosen in the proper way as described above, the conditions imposed by resonance equations (28) and (30) as constraints on the system parameters, are not sufficient for the desired convergence of the element \( \lambda_{21} \). This is because only the most singular terms in the equations \( q_i l_i + q_j t_j = 0 \) or \( q_i^2 l_i + q_j t_j = 0 \) are achieved to be of the same order on the chosen paths. Therefore if there are lower order divergent terms, they will be thrown out after the multiplication of these equations by \( l_i \) or \( l_j \). Thus, resonance equations (28) and (30) can be used as necessary, but not sufficient conditions to provide the finite convergence of the \( \Lambda \)-matrix elements. In order to illustrate this situation more clearly, let us consider an example with \( V_i \in \mathcal{G}_i \setminus \mathcal{G}_0 \) and \( V_j \in \mathcal{G}_j \), where both the terms \( q_i l_i \sim q_i^2 l_i \) and \( q_j t_j \) diverge in the squeezing limit. More precisely, one can write

\[
V_i = \varepsilon^{-2-a} h_i + \varepsilon^{-3/b}, \quad l_i = \varepsilon^{1+a} d_i, \quad V_j = \varepsilon^{-2} h_j, \quad l_j = \varepsilon d_j, \tag{31}
\]

where \( a \in (0, 1/2) \), \( b, h_i, h_j \in \mathbb{R} \), \( d_i, d_j > 0 \) and \( \varepsilon \to 0 \) is a squeezing parameter. Then on the pencil \( \Gamma_{ij} \) where \( l_i / l_j \sim |V_i| l_i^2 \sim \varepsilon^a \to 0 \), the leading divergent terms are of the same order \( \varepsilon^{-1} \) and equation (30) reduces to

\[
h_i d_i = \sqrt{-h_i} \tan(\sqrt{-h_i} d_j), \tag{32}
\]

admitting a countable set of solutions if \( h_i < 0 \). However, the divergence in the original equation \( q_i^2 l_i + q_j t_j = 0 \), rewritten explicitly in the form

\[
\varepsilon^{-1} \left[ h_i d_i - \sqrt{-h_i} \tan(\sqrt{-h_i} d_j) \right] + \varepsilon^{-1/2 + \delta} b d_i = 0, \tag{33}
\]

is not canceled out if \( b \neq 0 \), even if equation (32) is fulfilled. Thus, equation (32), as a particular example of (30), is not sufficient to obtain the desired convergence of the element \( \lambda_{21} \).
Since, in the limit as \( l \to 0 \), the dyads \( D_{ij} \) are finite under conditions (25), for the systems with \( N \geq 3 \), the divergence rate of the terms in series (12) does not exceed that of \( q_t \)'s. Moreover, due to the presence of the triads
\[
T_{ijk} := D_{ij}q_{tk} = (q_t q_t /q_t)_{ij}T_{tk} \quad (1 \leq i < j < k \leq N) \tag{34}
\]
in the \( Q_{21} \)-series, rewritten in the form
\[
Q_{21} = \sum_{n=1}^{(N-1)/2} (-1)^n \sum_{1=i<j<k} D_{1i1} \ldots D_{mn,jn} \ldots D_{n,jn} T_{mn,jn}, \tag{35}
\]
where the products \( D_{mn,jn} q_{tk} \) have been replaced by the triads \( T_{mn,jn} \) and the symbol \( \sum \) stands for the absence of an indicated term (here the factor \( D_{mn,jn} \)), it is possible that the \( l \to 0 \) limit of these products will be of type \( 0 \cdot \infty \sim c \neq 0 \) if \( D_{ij} \to 0 \). Hence, depending on the paths, there are two possibilities: (i) the triads \( T_{ijk} \) are divergent and the order of their divergence is the same as that of the terms \( q_t \)'s and (ii) \( |T_{ijk}| \to c \neq 0 \), leading to the cancellation of divergences only in the sum \( \sum_{n=1}^{N} q_{tk} \). Clearly, some terms in the latter sum as well as in the series \( Q_{21} \) may be finite belonging to the group \( \lambda_{21}^k \).

Let us analyze the asymptotic behavior of a \( T_{ijk} - \)triad, where all the strengths \( V_i, V_j \) and \( V_k \) belong to the corresponding \( G^\prime \)-sets. In this situation, we have the following asymptotic relation:
\[
T_{ijk} \sim (l_j/l_k) \frac{\sigma \tau \tau /g_k /s_j}{s_j} \sim \Gamma_{ij}^{-1} \chi_{ij} \chi_{ij} \frac{\tau \tau /g_k /s_j}{s_j}. \tag{36}
\]
Hence, in the linear squeezing limit \( (l_j \sim l_k \sim l_i) \), \( T_{ijk} \) diverges as \( l_j^{-1} \sim l_k^{-1} \sim l_i^{-1} \). Therefore, if one would like to have the limit \( |T_{ijk}| \to c \neq 0 \), it is necessary that at least \( l_i/l_j \to 0 \) or \( l_j/l_k \to 0 \) sufficiently fast. This nonlinear behavior could be realized on the adjoint paths \( \Gamma_{ij}^\prime \) or \( \Gamma_{ij} \) but the strength \( V_j \) in this case must belong to \( G_j \setminus G_j' \), that contradicts with the assumption \( V_j \in G_j' \). Therefore the only possibility for materializing a finite limit of \( T_{ijk} \) is a nonlinear extension of the linear squeezing. To this end, we introduce a parameter \( \sigma \in [1, \infty) \) and extend the notions of the boundary pencils \( \Gamma_{ij}^\sigma \) and \( \Gamma_{ij}^\sigma \) given in equation (23) to
\[
\Gamma_{ij}^\sigma := \left\{(l_i, l_j) : 0 < l_i^{1/\sigma} / l_j \to c > 0\right\},
\Gamma_{ij}^\sigma := \left\{(V_i(l)) \in G_i \setminus G_i^0, (l_i, l_j) \in \Gamma_{ij} ||V_i(l)|| l_i^{1/\sigma} \to c > 0\right\}, \tag{37}
\]
so that \( \Gamma_{ij}^\sigma |_{\sigma=1} \equiv \Gamma_{ij}^\prime \) and \( \Gamma_{ij}^\sigma |_{\sigma=1} \equiv \Gamma_{ij} \). Since \( \sigma \geq 1 \), we have \( \Gamma_{ij}^\sigma \subset \Gamma_{ij}^\prime \) and \( \Gamma_{ij}^\prime \subset \Gamma_{ij} \). Therefore conditions (25) are not violated on paths (37). Note that for the paths \( \Gamma_{ij}^\sigma \), as follows from definition (37), we have the relation \( l_i^{1/\sigma} / l_j \sim |V_i(l_i)||l_i^{1/\sigma} \to 0 \). Accordingly, for each \( \sigma \) we introduce a new set of the functions \( V_i(l_i) \) as
\[
G_i^\sigma := \left\{V_i(l) \in G_i \setminus G_i^0 \mid \lim_{l_i \to 0} |V_i(l)||l_i^{1+1/\sigma} = c > 0\right\}, \quad 1 \leq \sigma < \infty, \tag{38}
\]
where \( G_i^\sigma |_{\sigma=1} \equiv G_i^0 \). Next, for \( \sigma \in (1, \infty) \), we have \( G_i^\sigma \subset G_i \setminus G_i^0 \) and \( \lim_{\sigma \to \infty} G_i^\sigma = G_i^0 \), so that the pencil \( G_i^\sigma \) is found between the sets \( G_i^0 \) and \( G_i^0 \).

Next, we have to extend the definition of the limit quantities \( \chi_{ij} \) and \( \eta_{ij} \) given by equation (27) for all values \( \sigma \in [1, \infty) \). Thus, we define (\( i,j = 1, \ldots, N \))
\[
\chi_{ij}^\sigma := \lim_{(l_i, l_j) \in \Gamma_{ij}^\sigma} (l_i^{1/\sigma} / l_j), i \neq j, \quad \text{and} \quad \eta_{ij}^\sigma := \lim_{(l_i, l_j) \in \Gamma_{ij}^\sigma} (q_{ij}^\sigma l_i^{1/\sigma}) \quad \text{if} \ V_i \in G_i, \tag{39}
\]
where the definition of the pencils $\Gamma_{ij}^\sigma$ and $\Gamma_{ij}^\sigma_\delta$ is given in equation (37). According to notations (27), $\chi_{ij}^{\sigma}\mid_{\sigma=1} = \chi_{ij} = \chi_{ij}^1$, $\eta_{ij}^{\sigma}\mid_{\sigma=1} = \eta_{ij}$ and $\eta_{ij}^0 \equiv 0$ if $V_i \in G_i^0$. Contrary to the case $\sigma=1$, for $\sigma > 1$ it is possible to consider the particular case $j=i$. From definition (38), we have $|V_i| \sim l_i^{-\left(1+1/\sigma\right)}$ for $V_i \in G_i^0$, $i=1, \ldots, N$.

Thus, asymptotic relation (36) can be generalized to

$$T_{ijk} \sim \left(\frac{l_i^{\sigma}}{l_j} / l_k\right) T_{ijk} \sim l_j^{-2/\sigma} T_{ij} \tau_j \chi_j / s_j^2 \sim l_i^{-2/\sigma} T_{ik} \chi_j \tau_j \chi_k / s_j^2,$$

being valid on the paths, where $l_j^{\sigma} \sim l_j$ and $l_i^{\sigma} \sim l_i$, i.e. on the pencil projections $\Gamma_{ij}^\sigma$, $\Gamma_{ij}^\sigma_\delta$ and $\Gamma_{ij}^\sigma$. Hence, representation (40) coincides with (36) at $\sigma=1$ and at the same time it also holds true on the whole interval $2 \leq \sigma < \infty$. Exactly, at $\sigma=2$, it was possible to prove the existence of a bound state for the pure $\delta$-potential realized on the resonance set [36].

In addition to (40), where $V_j \in G_j^0$, one can write the asymptotic representation for the other ‘lateral’ configurations $V_i \in G_i^0$ and $V_k \in G_k^0$ as follows

$$T_{ijk} \sim l_j^{-2/\sigma}$$

and

$$T_{ijk} \sim l_j^{-1/\sigma}$$

Equations (36) and (40)–(42) can also be used for the case as $V_j \in G_j^0$, just setting in these formulas $\tau_j / s_j^2 = 1$. The asymptotic representation given by relations (41) holds true for $\sigma=1$ and $\sigma \in [2, \infty)$, whereas representation (42) is valid on the whole interval $1 \leq \sigma < \infty$. Below the cases $\sigma=1$ and $1 < \sigma < \infty$ will be considered separately.

5.1. Necessary conditions for the existence of resonance sets on the paths with $\sigma=1$

The cancellation procedure can particularly be performed on the paths with $\sigma=1$, which are defined by the pencil projections $\Gamma_{ij}^\sigma$ and $\Gamma_{ij}^\sigma_\delta$ (see relations (23)). Consider first the case $N=3$, where series (35) reduces to the single summand $Q_{21} = T_{ijk} (i=1, j=2, k=3)$. Assume that all the strengths belong to either $G_i^\sigma$ or $G_j^\sigma$-$G_k^\sigma$-sets. Then, according to the asymptotic representation given by relations (40) and (41) with $\sigma=1$, the $T_{ijk}$-triad diverges as $l_j^{-1} \sim l_j^{-1}$. Hence, all the terms in $\lambda_{21}$ are divergent and the condition for the cancellation of divergences yields the equation

$$q_i l_i + q_j l_j + q_k l_k = T_{ijk}.$$

For the comparison of the divergence rate of all the terms on the paths with $\sigma=1$, this equation can be multiplied by one of the widths $l_i, l_j$ or $l_k$, but $V_i, V_j$ or $V_k$ must belong to the corresponding $G_i^\sigma$-sets. More precisely, if for instance, the strength $V_j$ belongs to $G_j^\sigma$-$G_k^\sigma$, resulting in the
behavior \(|q_j l_j| \sim |q_j l_j|^2 \), the multiplication of this term by \(l_j\) leads to its disappearance as \(l_j \rightarrow 0\) because \(|q_j l_j| \rightarrow 0\). Therefore the width \(l_j\) cannot be used here for ‘measuring’ the rate of the divergence of \(|q_j l_j|^2\) and in this case another parameter, say \(l_l\) or \(l_k\) should be chosen as a multiplier if \(V_l \in \mathcal{G}_l'\) or \(V_k \in \mathcal{G}_k'\), respectively. Thus, at least the strength in one of the sums of the left-hand side of equation (43), where all the terms are singular, has to belong to the corresponding \(G\)-set.

Consider now the situation as \(V_l \in \mathcal{G}_l'\) and \(V_j \in \mathcal{G}_j \setminus \mathcal{G}_j^0\). According to the above arguments, in this case, equation (43) can be multiplied by \(l_l\). For \(V_k \in \mathcal{G}_k'\), the resulting products make sense on the paths, where \(l_j / l_l \sim |V_j|^2 l_l^2 \rightarrow 0, l_k \sim l_l\) and \(l_j \sim l_l\) (see representation (36)). However, the last two relations contradict with the first one. Similarly, for \(V_k \in \mathcal{G}_k \setminus \mathcal{G}_k^0\), we have \(l_j / l_l \sim |V_j|^2 l_l^2 \rightarrow 0, l_k / l_l \sim |V_k|^2 l_l^2 \rightarrow 0\). From the last two relations, one obtains that \(l_l \sim l_j\) and this again contradicts with the first relation. Thus, the configuration of the singular strengths \(V_l, V_j\) and \(V_k\) in equation (43), where the ‘middle’ strength \(V_j\) belongs to \(\mathcal{G}_j \setminus \mathcal{G}_j^0\) cannot be used for deriving resonance equations in the squeezing limit. In other words, for the strength configurations with \(\mathcal{G}_j \setminus \mathcal{G}_j^0\), it is impossible to find appropriate paths, on which the divergent terms are of the same order. However, if \(V_j \in \mathcal{G}_j'\), multiplying then equation (43) by \(l_j\) and using asymptotic relations (36) and (41) with \(\sigma = 1\), one obtains the correct resonance equations, which are collected in table 1. Note that these equations serve as necessary conditions, but not sufficient, for the convergence of the element \(\lambda_{2j}\). Here, all the four terms in equation (43) are divergent being of the same order, so that all the terms in the resonance equations are finite and non-zero. Note that for the configurations pointed out in the second, the third and the fourth lines in table 1, it is impossible to derive resonance equations on the pencils \(\{\Gamma^r_{ij}, \Gamma^s_{ik}\}\) because the divergent terms of equation (43) in this situation are of different order.

Next, we have to consider the configurations \(V_l, V_j\) and \(V_k\) \((N = 3)\), where one or two strengths belongs to \(\mathcal{G}_j^0\)-sets. Thus, setting in table 1 \(\tau_l = 0\) and \(\tau_j / s_j^2 = 1\), one obtains the resonance equations for the configuration with \(V_j \in \mathcal{G}_j^0\). The general form of these equations reads

\[
A_{ij} + A_{kj} = A_{ij} A_{kj}, \quad (44)
\]

where \((i \neq j)\)

\[
A_{ij} := \begin{cases} 
\lim_{(l_j, l_l) \in \Gamma^r_{ij}} (q_j l_j l_l) = \chi_j \tau_j & \text{if } V_l \in \mathcal{G}_j', \\
\lim_{(l_j, l_l) \in \Gamma^s_{ij}} (q_j l_j l_l) = \eta_j & \text{if } V_l \in \mathcal{G}_j \setminus \mathcal{G}_j^0.
\end{cases} \quad (45)
\]

The solution of equation (44) are curves on the \(\{A_{ij}, A_{kj}\}\)-plane. The corresponding resonance sets have been analyzed in detail for various situations earlier in works [34, 36]. Thus, we
conclude that equation (43) with the singular lateral terms \(q_l t_l\) and \(q_l t_k\) multiplied by \(l_l\) makes sense only if the middle strength \(V_j \in \mathcal{G}_j^{o} \cup \mathcal{G}_j^{0}\).

Finally, for the case \(N = 3\), where one of the lateral strengths \(V_i\) or \(V_k\) (with the \(\delta\)-like shrinking) belongs to \(\mathcal{G}_j^{0}\) or \(\mathcal{G}_j^{o}\), respectively, we have to use asymptotic representation (42), where \(|T_{ik}| \to c > 0\). For instance, for the structure with \(V_i \in \mathcal{G}_j^{o}\), \(V_j \in \mathcal{G}_j^{0}\) and \(V_k \in \mathcal{G}_j^{o}\), multiplying equation (43) by \(l_l\), we arrive at equation (28). Similarly, for \(V_i \in \mathcal{G}_j^{o}\), \(V_j \in \mathcal{G}_j^{0}\) and \(V_k \in \mathcal{G}_j^{o}\), one obtains \(\tau_l + \eta_l = 0\), in fact, equation (30). In both these situations, the cancellation of divergences occurs only between two terms in equation (43) as in a double-layer system.

The above arguments can be extended to the general case with \(N > 3\). Consider the configuration \(V_1(l_1), \ldots, V_N(l_N)\), for which at least three terms \(q_l t_l\), \(q_l t_j\) and \(q_l t_k\) \((1 \leq i < j < k \leq N)\) diverge in the squeezing limit. Then, according to asymptotic relations (40) and (41), the corresponding triad \(T_{ijk}\) will also be a divergent term at \(\sigma = 1\). Let the lateral strengths \(V_j\) and \(V_k\) belong to the corresponding \(\mathcal{G}_j^{0}\) or \(\mathcal{G}_j^{o}\)-sets, while \(V_i \in \mathcal{G}_j^{0}\). For the comparison of the divergence rate of these four terms, one can multiply them by some \(l_m\), \(m = 1, \ldots, N (m \neq i, j, k)\). Thus, multiplying the sum \(q_l t_l + q_l t_j + q_l t_k\) \((V_i \in \mathcal{G}_j^{0}\) and \(V_j \in \mathcal{G}_j^{0}\)) and \(T_{ijk} = q_l t_l q_l t_j q_l t_k\) by \(l_m\), one obtains that the resulting products will be finite and non-zero quantities on the paths characterized by the relations \(l_i \sim l_m\), \(l_j / l_m \sim |V_j|^2 \to 0\), \(l_k \sim l_m\) and \(l_l \sim l_k\). From the last two relations we have \(l_l \sim l_m\), but this contradicts with \(l_l / l_m \to 0\). Next, for instance, in the case with \(V_i \in \mathcal{G}_j^{0}\) and \(V_k \in \mathcal{G}_j^{o}\), one obtains the relations \(l_j / l_m \sim |V_j|^2 \to 0\), \(l_j / l_m \sim |V_k|^2 \to 0\), \(l_k / l_m \sim |V_i|^2 \to 0\) and \(l_l / l_j \sim |V_l|^2 \to 0\). Again, from the last two relations we obtain \(l_l \sim l_j\) that contradicts with \(l_l / l_m \to 0\). This zero convergence is replaced by \(l_l \sim l_m\) if \(V_j \in \mathcal{G}_j^{0}\), so that the contradiction vanishes. In the case as \(V_j \in \mathcal{G}_j^{0}\), the restriction \(l_l / l_m \to 0\) is absent. Hence, the ‘internal’ strengths \(V_j \in \mathcal{G}_j^{0}\) with \(j = 2, \ldots, N - 1\) have to be excluded in the procedure of deriving resonance equations and, as a result, each strength \(V_j\) in the sum \(\sum_{i=1}^{N+1} q_l t_l\) must belong to either \(\mathcal{G}_j^{0}\) or \(\mathcal{G}_j^{o}\). Note that the lateral terms in the sum \(\sum_{i=1}^{N+1} q_l t_l\) (from one to several ones) may belong to \(\mathcal{G}_j^{0}\)-sets, so that the next neighboring strengths appear to be singular ones. Then, the applying of relations (42) results in the same result: for the existence of resonance equations, the internal strengths must belong to \(\mathcal{G}_j^{0} \cup \mathcal{G}_j^{o}\)-sets.

Thus, multiplying the element \(\lambda_{21}\) given by series (9) and (12) by one of \(l_j\) with \(j = 2, \ldots, N - 1\), from the condition \(\lim_{\varepsilon \to 0}(\lambda_{21} x_j) = \lim_{\varepsilon \to 0}(\lambda_{21} x_j) = 0\), we obtain the limit equation

\[
\sum_{i=1}^{N} A_{ij} + \sum_{n=1}^{(N-1)/2} (-1)^n \sum_{i_1 < i_2 < \ldots < i_n \leq N} (A_{i_1 x_i j / s_{i_1}^2} \ldots A_{i_n x_i j / s_{i_n}^2}) \lambda_{n,j} = 0,
\]

where we have used the limits

\[
\begin{align*}
&\lim_{(l_i, l_j) \in \Gamma_{ij}} D_{ij} = \begin{cases} 
A_{ij} & \text{if } V_i \in \mathcal{G}_j^{o}, \ V_j \in \mathcal{G}_j^{o}, \\
V_i \in \mathcal{G}_j^{o}, \ V_j \in \mathcal{G}_j^{o}, \\
V_i \in \mathcal{G}_j^{o}, \ V_j \in \mathcal{G}_j^{o}, \\
V_i \in \mathcal{G}_j^{0}, \ V_j \in \mathcal{G}_j^{0}.
\end{cases}
\end{align*}
\]

\[
(46)
\]
Here, the limit quantities $A_{ij}$ are defined by equation (45). In addition to these equations, notice that $A_{ij} = 0$ if $V_i \in G'_j (i \neq j)$. Next, in equation (46), $A_{ij} = \tau_j$ if $V_i \in G'_j$, while $A_{ij} = 0$ and $\tau_j / \delta_j^2 = 1$ if $V_j \in G^0_i$. Clearly, all the terms from the group $\lambda^2_1$ disappear in equation (46). All the equations obtained from the condition $\lim_{\epsilon \to 0} (\lambda^2_1) = 0$, where $j = 2, \ldots, N - 1$, are equivalent, so that the resonance sets do not depend on the subscript $j$ in equation (46).

Thus, the solutions of resonance equation (46) determine the conditions on the limit parameters $\chi_{ij}$, $\delta_i$ and $\eta_j$ under which a non-zero (resonant) transmission through the squeezed structure could be possible. These conditions form one or several hypersurfaces in the $\{\ldots, \chi_{ij}, \delta_i, \eta_j, \ldots\}$-space. The number and structure of resonance sets (hypersurfaces) depend on the configuration $V_i(l_1), \ldots, V_N(l_N)$ and the corresponding paths converging to the origin $l = 0$. For a fixed number of layers $N$, several resonance sets can exist and its number increases with $N$.

Finally, consider the particular case as all the functions $V_i(l_i)$ are non-negative. For realizing a non-trivial point interaction in the singular squeezing limit, assume that some of these functions $V_i(l_i) \in G'_i \cup G^0_i$ are positive (barriers). Then $\delta_i \in \mathbb{R}$, so that $\tau_i < 0 (\delta_i^2 < 0)$ and $\eta_j < 0$. As a result, all the terms in equation (46) become negative, and consequently, the total left-hand expression in this equation will also be negative. Hence, in the case of the absence of well-like potential strengths $V_i$, there are no cancellations of divergences resulting in a finite limit of $\lambda^2_1$.

However, in the case if at least one well-like potential strength ($V_i < 0$) is present, equation (46) admits a countable set of non-trivial solutions due to the functions $\tan \delta_i$ with $\delta_i \in \mathbb{R}$. This leads to the realization of non-separated point interactions in the squeezing limit. For instance, one of the particular cases illustrating the existence of solutions to equation (46) is the configuration, where each divergent potential strength $V_i$ belongs to the set $G'_i$ and all these potential strengths are defined on the paths $\Gamma'_i$. In this case, all the divergent terms in (46) are of the same order. Hence, in the presence of at least one well-like potential strength, the cancellation of divergences provides the appearance of non-separated point interactions in the squeezing limit. Thus, from the above arguments we single out the results, which can be formulated as

**Conclusion 3.** (a) The resonance sets for $\sigma = 1$ are obtained as solutions to limit equation (46) defined for the configurations $V_1(l_1), \ldots, V_N(l_N), i = 1, \ldots, N$, where the lateral strengths $V_i (i = 1, N)$ are singular, i.e. they belong to either the $G'_i$- or $G^0_i$-sets, whereas the internal strengths $V_i (i = 2, \ldots, N - 1)$ are from the $G'_i \cup G^0_i$-sets. (b) In the case as some lateral strengths (from one to several) belong to the corresponding $G'_i$-sets, more precisely, $V_1 \in G'_1, \ldots, V_p \in G'_p$ (left layers) and $V_{N-p} \in G'_p, \ldots, V_N \in G^0_N$ (right layers), so that the strengths $V_{p+1}$ and $V_{N-p-1}$ are singular, now the strengths $V_{p+2}, \ldots, V_{N-p-2}$ must belong to either the $G'_i$- or $G^0_i$-sets. If one of the lateral strengths or both these are regular, i.e. $V_1 \in G'_1$ and $V_N \in G^0_N$, then only the $\delta$-like squeezing of these strengths makes sense. (c) The presence in the configuration $V_1, \ldots, V_N$ at least one strength of a well-like form, say, $V_i < 0$, implies the existence of a countable set of solutions to the resonance equations.

5.2. **Necessary conditions for the existence of resonance sets in the case $\sigma > 1$**

Similarly to the case with $\sigma = 1$, consider first the situation as $N = 3$ and analyze in detail all possible cancellations of divergences in $\lambda^2_1$ on the paths with $\sigma > 1$. Here, the cancellation can also be possible within equation (43), where $i = 1, j = 2$ and $k = 3$. Assume that all the terms $q_i t_i, q_j t_j$ and $q_k t_k$ diverge in the squeezing limit. The configuration $V_i \in G'_i, V_j \in G'_j$ and $V_k \in G'_k$ is the most singular case. The divergence rate of all the terms in this case could be compared
by multiplying equation (43) by one of the factors \( t_i^{1/\sigma} \), \( t_k^{1/\sigma} \) or \( t_k^{1/\sigma} \) with \( \sigma > 1 \). However, the multiplication by any of these factors does not lead to finite values. For instance, the resulting term \( t_i^{1/\sigma} q_{it_i} \sim t_i^{1/\sigma-1} \) diverges for \( \sigma > 1 \) as \( l_i \to 0 \). Hence, in the three-layer structure, the cancellation of divergences on the paths with \( \sigma > 1 \) is impossible if all the three strengths \( V_i \), \( V_j \) and \( V_k \) belong to the corresponding \( G' \)-sets. Therefore at least one of the strengths must belong to the \( G \)-set, which can be either lateral (\( V_l \) or \( V_r \)) or middle (\( V_c \)).

In addition to relations (40) and (41), for the case \( V_j \in G_j' \), we also need the asymptotic representation

\[
T_{ijk} \sim \begin{cases} 
\frac{1}{l_i^{1/\sigma}} \chi_{ij} \eta_{ii}^r \tau_i / \mathcal{S}_i^2 & \text{for } V_i \in G_i^\sigma, \ V_k \in G_k^\sigma, \\
\frac{1}{l_k^{1/\sigma}} \chi_{jk} \eta_{jk}^r \tau_j / \mathcal{S}_j^2 & \text{for } V_i \in G_i^\sigma, \ V_k \in G_k^\sigma, \\
\frac{1}{l_i^{1/\sigma}} \eta_{li}^r \eta_{jk}^r \tau_j / \mathcal{S}_j^2 & \text{for } V_i \in G_i^\sigma, \ V_k \in G_k^\sigma, \\
\frac{1}{l_k^{1/\sigma}} \eta_{li}^r \eta_{jk}^r \tau_j / \mathcal{S}_j^2 & \text{for } V_i \in G_i^\sigma, \ V_k \in G_k^\sigma. 
\end{cases}
\]

Using next these relations after the multiplication of equation (43) by \( t_i^{1/\sigma} \) and \( t_k^{1/\sigma} \), respectively, we get the four resonance equations for \( V_j \in G_j' \) and \( \sigma > 1 \), which are collected in table 2, and necessary for the convergence of the element \( \lambda_{21} \). Here, the cancellation of divergences occurs between all the four terms if \( V_i \in G_i \setminus G_i^\sigma \) or \( V_k \in G_k \setminus G_k^\sigma \). In the case as \( V_i \in G_i^\sigma \) and \( V_k \in G_k^\sigma \), we have accordingly \( \eta_{ii}^r = 0 \) and \( \eta_{kk}^r = 0 \) in table 2 and therefore only two terms take place in the cancellation.

In the ‘middle’ case as \( V_j \in G_j' \), we have to apply the asymptotic representation given by equations (40) and (41), where \( \sigma \geq 2 \) resulting in the limit \( |T_{ijk}| \to c \geq 0 \). Multiplying equation (43) by \( t_i^{1/\sigma} \), we obtain the four resonance equations as necessary conditions for the convergence of \( \lambda_{21} \), which are present in table 3. Note that the paths given in this table and those derived for the existence of relations (40) and (41) are the same. The general form of the resonance equations in table 3 reads

\[
A_{ij}^\sigma + A_{kl}^\sigma = 0,
\]

where

\[
A_{ij}^\sigma := \begin{cases} 
\lim_{(l \to j) \in \Gamma_{ij}} (t_i^{1/\sigma} q_{it_i}) = \chi_{ij}^r \tau_i & \text{if } V_i \in G_i', \ i \neq j, \\
\lim_{(l \to j) \in \Gamma_{ij}} (t_l^{1/\sigma} t_i^{1/\sigma} q_{it_i}) = \eta_{ij}^r & \text{if } V_i \in G_i \setminus G_i^\sigma. 
\end{cases}
\]

### Table 2. Resonance equations and available paths for a three-layer structure, resulting from the multiplication of equation (43) by \( t_i^{1/\sigma} \) or \( t_k^{1/\sigma} \), where \( V_j \in G_j' \) and \( \sigma > 1 \).

| \( V_i(l_i), V_k(l_k) \) | Resonance equations | Paths |
|-------------------------|---------------------|-------|
| \( V_i \in G_i^\sigma, V_k \in G_k^\sigma \) | \( \eta_{ii}^r + \chi_{ij}^r \tau_i / \mathcal{S}_i^2 \) | \( \Gamma_{ij}, \Gamma_{ij}^\sigma, \Gamma_{kk}^\sigma \) |
| \( V_i \in G_i^\sigma, V_k \in G_k^\sigma \) | \( \eta_{ii}^r + \chi_{ij}^r \tau_i / \mathcal{S}_i^2 \) | \( \Gamma_{ij}, \Gamma_{ij}^\sigma, \Gamma_{kk}^\sigma \) |
| \( V_i \in G_i^\sigma, V_k \in G_k^\sigma \) | \( \eta_{ii}^r + \chi_{ij}^r \tau_i / \mathcal{S}_i^2 \) | \( \Gamma_{ij}, \Gamma_{ij}^\sigma, \Gamma_{kk}^\sigma \) |
The solution of equation (49) are curves on the \( \{A^s_{jl}, A^s_{jk}\} \)-plane, which depend on the parameter \( \sigma \geq 2 \). The corresponding resonance sets for the structure composed of two separated layers (\( N = 3, V_l = 0 \)) have been analyzed in works [34, 36].

Thus, we have established in the case \( N = 3 \) that for the existence of resonance equations, at least one of the strengths among \( V_i, V_j \) and \( V_k \) must belong to the corresponding \( G \)-set. For illustration of this rule for \( N > 3 \), let us write a couple of resonance equations if \( N = 4 \). For this particular case, series (35) reduces to the sum

\[
Q_{21} = T_{ik} + T_{jm} + T_{km} + T_{jm} \quad (i = 1, j = 2, k = 3, m = 4)
\]

and the equation for the cancellation of divergences \( \lambda_{21} = 0 \) becomes \( \sum_{\ell=1}^{4} q_{i\ell} = Q_{21} \) if all the terms \( q_{i\ell} \)'s are divergent.

Let us consider first the configuration as one of the lateral strengths, say \( V_m, \) belongs to the \( G_m \)-set. Assume that \( V_i \in G^s_i, V_j \in G^s_j, V_k \in G^s_k \) and \( V_m \in G_m \). Then the equation for the cancellation of divergences takes the form

\[
q_{ih} + q_{j\ell} + q_{k\ell} + q_{m\ell} = \frac{q_{ij} q_{kl}}{q_k} h_{j\ell} + \frac{q_{jm} q_{kl}}{q_k} h_{i\ell} + \frac{q_{km} q_{ij}}{q_k} h_{j\ell} + \frac{q_{km} q_{ij}}{q_k} h_{i\ell}.
\]

The multiplication of this equation by \( l_{m}^{1/\sigma} \) yields the resonance equation

\[
\chi_{ij}^s \tau_{ij} + \chi_{mj}^s \tau_{mj} + \chi_{mk}^s \tau_{mk} + \eta_{km}^s = \chi_{ij}^s \sum_{j}^{2} \chi_{mk}^s \tau_{mk} + \left( \chi_{ij}^s \sum_{j}^{2} \chi_{mk}^s \tau_{mk} + \chi_{mj}^s \sum_{j}^{2} \chi_{mk}^s \tau_{mk} + \chi_{mk}^s \sum_{j}^{2} \chi_{mk}^s \tau_{mk} \right) \eta_{km}^s.
\]

Here, in each of the four triads (see equation (51)), the middle strength is from the corresponding \( G^s \)-set and therefore representation (48) has been applied, similarly as for deriving the first equation in table 2. In equation (53), the available paths are characterized by the relations \( l_i \sim l_j \sim l_k \sim l_{m}^{1/\sigma} \) forming the paths \( \Gamma_{ij}, \Gamma_{jk}, \Gamma_{mk}, \Gamma_{mjk}, \Gamma_{mkj} \) with \( \sigma \in (1, \infty) \). Besides these conditions, an additional restriction appears on the strength \( V_m \), namely, this strength must belong to \( G_m \subset G_m \setminus G_m^0 \). In the case as \( V_m \in G_m^0 \), in equation (53) we have \( \eta_{km}^s = 0 \) and therefore only the four terms participate in removing the leading divergences, instead of all the eight terms.

Consider now a four-layer system with the configuration as one of the strengths belonging to a \( G \)-set is localized between the strengths from \( G^s \)-sets, e.g. \( V_i \in G^s_i, V_j \in G^s_j, V_k \in G^s_k \) and \( V_m \in G_m \). Here, due to representation (40), the squeezing limits of the triads \( T_{ij} = q_{ij} h_{j\ell} q_{kl} \) and \( T_{jm} = q_{jm} h_{j\ell} q_{kl} \) are finite on the paths, where \( l_i \sim l_k \sim l_m \sim l_{m}^{1/\sigma} \). Therefore these triads do not participate in the cancellation of divergences and they have to be omitted because their

| \( V_i(l_i) \), \( V_k(l_k) \) | Resonance equations | Paths |
|-----------------------------|---------------------|-------|
| \( V_i \in G^s_i \), \( V_k \in G^s_k \) | \( \chi_{ij}^s \tau_{ij} + \eta_{ij}^s + \chi_{mk}^s \tau_{mk} = 0 \) | \( \Gamma_{ijk}^s, \Gamma_{ijk}^s \) |
| \( V_i \in G^s_i \), \( V_k \in G^s_k \) | \( \chi_{ij}^s \tau_{ij} + \eta_{ij}^s + \eta_{jk}^s = 0 \) | \( \Gamma_{ijk}^s, \Gamma_{ijk}^s \) |
| \( V_i \in G^s_i \), \( V_k \in G^s_k \) | \( \eta_{ij}^s + \eta_{jk}^s + \chi_{mk}^s \tau_{mk} = 0 \) | \( \Gamma_{ijk}^s, \Gamma_{ijk}^s \) |
| \( V_i \in G^s_i \), \( V_k \in G^s_k \) | \( \eta_{ij}^s + \eta_{jk}^s + \eta_{jk}^s = 0 \) | \( \Gamma_{ijk}^s, \Gamma_{ijk}^s \) |
middle strength $V_j$ belongs to the $G_j$-set. As a result, the divergent terms are contained in the equation

\[ q_{l_t} + q_{l_j}^2 l_j + q_{l_k} + q_m l_m = \frac{q_{lm} q_{m} l_m}{q_{k}} + q_{l_j}^2 l_j q_{m} l_m. \]  

(54)

Multiplying this equation by $l_j^{1/\sigma}$ and using relations (48), we arrive at the resonance equation

\[ \chi_{ji}^{\sigma} \Gamma_{ji} + \gamma_{ji}^{\sigma} + \chi_{jk}^{\sigma} \Gamma_{jk} + \chi_{jm}^{\sigma} \Gamma_{jm} = \chi_{ji}^{\sigma} \Gamma_{ji} + \gamma_{ji}^{\sigma} + \chi_{jm}^{\sigma} \Gamma_{jm}, \]  

(55)

which is similar to the second equation in Table 3. The multiplication by $l_j^{1/\sigma}$ and representation (48) to $\tau_{ji}$ and applying the representation given by equations (40) and (41) to the triads $T_{ijk}$ and $T_{ijm}$, we get the resonance equation in the form

\[ \chi_{ji}^{\sigma} \Gamma_{ji} + \gamma_{ji}^{\sigma} + \chi_{jm}^{\sigma} \Gamma_{jm} = \chi_{ji}^{\sigma} \Gamma_{ji} + \gamma_{ji}^{\sigma} + \chi_{jm}^{\sigma} \Gamma_{jm}. \]  

(56)

valid under the conditions $l_t \sim l_k \sim l_m \sim l_j^{1/\sigma}$, resulting in the paths $\Gamma_{ik}, \Gamma_{im}, \Gamma_{jm}, \Gamma_{jk}, \Gamma_{km}$.

Since representation (40) has been used here, we have to choose $\sigma \in [2, \infty)$, instead of the interval $1 < \sigma < \infty$.

The similar situation takes place as $V_i \in G_i^l, V_j \in G_j^l, V_k \in G_k^l$ and $V_m \in G_m^l$. Here, again the same two triads $T_{ijk}$ and $T_{ijm}$ have the middle strength from $G_j^l$ and for the other two triads $T_{ikm}$ and $T_{jkm}$, the middle strength $V_k$ belongs to $G_j^l$. As a result, again the right-hand part of the resonance equation consists of two summands. Indeed, multiplying the equation

\[ q_{l_t} + q_{l_j}^2 l_j + q_{l_k} + q_{l_m}^2 l_m = q_{l_t} \frac{l_k}{q_{k}} q_{l_m}^2 l_m + q_{l_j}^2 l_j \frac{l_k}{q_{k}} q_{l_m}^2 l_m \]  

(56)

by $l_j^{1/\sigma}$ and applying the representation given by equations (40) and (41) to the triads $T_{ijk}$ and $T_{ijm}$, and representation (48) to $T_{ikm}$ and $T_{jkm}$, we get the resonance equation in the form

\[ \chi_{ji}^{\sigma} \Gamma_{ji} + \gamma_{ji}^{\sigma} + \chi_{jm}^{\sigma} \Gamma_{jm} = \chi_{ji}^{\sigma} \Gamma_{ji} + \gamma_{ji}^{\sigma} + \chi_{jm}^{\sigma} \Gamma_{jm}. \]  

(57)

In this equation, the relations $l_t \sim l_k \sim l_j^{1/\sigma}$ and $|V_m| l_m \sim l_j^{1/\sigma}$ take place, so that the available paths, on which equation (57) is well-defined, are $\Gamma_{ik}, \Gamma_{im}, \Gamma_{jm}, \Gamma_{jk}, \Gamma_{km}$ with $\sigma \in [2, \infty)$. The form of equations (55) and (57) resembles a ‘mixture’ of the equations collected in Tables 2 and 3.

Consider now the configuration $V_i \in G_i^l, V_j \in G_j^l, V_k \in G_k^l$ and $V_m \in G_m^l$. Here, all the triads contain the middle strengths $V_j$ and $V_k$, belonging to the $G_j$- and $G_k$-sets. Therefore only asymptotic formulas (40) and (41) have to be applied on the paths determined by the relations $l_t \sim l_j^{1/\sigma}$, $|V_k| l_k \sim l_j^{1/\sigma}$, $|V_m| l_m \sim l_j^{1/\sigma}$, $l_t \sim l_j^{1/\sigma}$, $|V_m| l_m \sim l_j^{1/\sigma}$, $|V_j| l_j \sim l_j^{1/\sigma}$. There are no contradictions in these relations, because, e.g. from the comparison of the relations with $|V_m| l_m$, one finds that $l_j \sim l_k$, so that $l_k/l_j^{1/\sigma}$ and $l_j/l_m^{1/\sigma}$ tend here to zero as required. Thus, all the triads in the cancellation equation have to be omitted and the multiplication of the equation

\[ q_{l_t} + q_{l_j}^2 l_j + q_{l_k} l_k + q_{l_m}^2 l_m = 0 \]  

(58)

yields the resonance equation

\[ \chi_{ji}^{\sigma} \Gamma_{ji} + \gamma_{ji}^{\sigma} + \gamma_{ik}^{\sigma} + \gamma_{im}^{\sigma} = 0, \]  

(58)

which is similar to the second equation in Table 3. The multiplication by $l_j^{1/\sigma}$ imposes the restriction on $V_j$, namely $V_j \in G_j^l$. If some strength belongs to the corresponding $G_j^l$-set, say $V_m$, then in equation (58), we have to set $\gamma_{im} = 0$. Finally, in a similar way, one can treat the situation as all the strengths are from the $G$-sets. The multiplication of the equation

\[ q_{l_t} + q_{l_j}^2 l_j + q_{l_k} l_k + q_{l_m}^2 l_m = 0, \]  

(59)
which is similar to the last equation in table 3. Thus, even if all the strengths belong to the $\mathcal{G}$-sets, the existence of resonance equations holds true on the corresponding paths with $\sigma \in [2, \infty)$. Concerning the double-layer structures ($N = 2$), where triads (34) are absent, instead of resonance equation (30), which is valid for $\sigma = 1$, one can write the equation for the whole interval $1 < \sigma < \infty$. Indeed, multiplying the equation $q_i^2 h_i + q_{ij} t_{ij} = 0$ by $t_i^{\phi \sigma}$, we get the resonance equation

$$\lim_{(i,j) \in \Gamma^\phi} (q_i^2 h_i + q_{ij} t_{ij}) t_i^{\phi \sigma} = \eta''_i + \chi''_{ij} t_j = 0,$$  \hspace{1em} (60)

where $V_i \in \mathcal{G}_i^\sigma$ and $t_i^{\phi \sigma} \sim l_i$. Both equation (30) for $\sigma = 1$ and (60) for $\sigma > 1$, which are necessary conditions for the desired convergence of the $\Lambda$-matrix, admit countable sets of solutions if $V_j < 0$, that determine the corresponding resonance sets.

Thus, from the above arguments, one can deduce.

**Conclusion 4.** For the existence of resonance sets, at least one of the strengths from the configuration $V_i(l_i), \ldots, V_N(l_N)$, say $V_j$, must belong to the $\mathcal{G}_j$-set. In this case, for deriving a resonance equation, the equation for the cancellation of divergences $\lambda_{11}^j = 0$ can be multiplied by $t_i^{\phi \sigma}$. During this procedure, the asymptotic representation given by equations (40), (41) and (48) has to be adopted and this application depends on whether the middle strength $V_j$ in the $T_{ijk}$-triads belongs to the $\mathcal{G}_j$- or $\mathcal{G}'_j$-set. In the former case, equation (48) are to be used, while in the latter case, due to relations (40) and (41), the $T_{ijk}$-triads are finite in the squeeze limit if $\sigma \in [2, \infty)$. In the case as one of the lateral strengths or both these are from the $\mathcal{G}$-sets ($V_i \in \mathcal{G}_i$ and $V_N \in \mathcal{G}_N$) and all the internal strengths belong to the $\mathcal{G}'$-sets ($V_i \in \mathcal{G}'_i$, $i = 2, \ldots, N - 1$), the resonance equations can exist on the paths with the parameter $\sigma$ from the interval $1 < \sigma < \infty$. If one of the lateral strengths or both these are regular, i.e. $V_i \in \mathcal{G}_{10}$ and $V_N \in \mathcal{G}_{N0}$ then the cases $V_1 \equiv 0$ and $V_N \equiv 0$ have to be excluded from the consideration. Next, in a double-layer structure ($N = 2$), no more than one strength belonging to the corresponding $\mathcal{G}'$-set is available. For a single-layer structure ($N = 1$), for the existence of resonance sets it is necessary that the potential must be negative with the strength belonging to the $\mathcal{G}'$-set.

### 5.3. Examples

**Example 1.** As a particular example of the functions $V_i(l_i)$ and $V_j(l_j)$, used for the realization of the singular squeezing limit, one can consider the following asymptotic expressions:

$$V_i(l_i) \sim l_i^{-\mu}, \quad V_j(l_j) \sim l_j^{-\nu/(1-\mu-\nu)}, \quad \mu, \nu > 0, \quad 1 - \mu - \nu > 0,$$  \hspace{1em} (61)

used in several works [12–14, 34, 36, 39]. By a proper choosing of the positive parameters $\mu$ and $\nu$ in (61), all the four singular cases: (i) $V_i \in \mathcal{G}_i^{0j}$, $V_j \in \mathcal{G}_j^0$; (ii) $V_i \in \mathcal{G}_i^{0j}$, $V_j \in \mathcal{G}_j^0$; (iii) $V_i \in \mathcal{G}_i^j$, $V_j \in \mathcal{G}_j \setminus \mathcal{G}_j^0$; (iv) $V_i \in \mathcal{G}_i \setminus \mathcal{G}_i^0$, $V_j \in \mathcal{G}_j \setminus \mathcal{G}_j^0$, which were considered above, can be materialized on the $(\mu, \nu)$-sets

\begin{align*}
P &:= \{ \mu = \nu = 2 \}, & V_i \in \mathcal{G}_i^j, & V_j \in \mathcal{G}_j^0, \\
L_1 &:= \{ 1 < \mu < 2, \nu = 2(\mu - 1) \}, & V_i \in \mathcal{G}_i \setminus \mathcal{G}_i^0, & V_j \in \mathcal{G}_j^0, \\
L_2 &:= \{ \mu = 2, 2 < \nu < \infty \}, & V_i \in \mathcal{G}_i, & V_j \in \mathcal{G}_j \setminus \mathcal{G}_j^0, \\
S &:= \{ 1 < \mu < 2, 2(\mu - 1) < \nu < \infty \}, & V_i \in \mathcal{G}_i \setminus \mathcal{G}_i^0, & V_j \in \mathcal{G}_j \setminus \mathcal{G}_j^0. \hspace{1em} (62)
\end{align*}
These sets form the triangle on the \((\mu, \nu)\)-plane with the vertex at the point \(P\), where \(L_1\) and \(L_2\) are the edges, and \(S\) the interior of this angle. One can check that limits (22), namely the first, the second, the third and the sixth inequalities, are fulfilled accordingly on the paths: \(\Gamma_{ij}^r\) (at point \(P\)), \(\Gamma_{ij}^r \cup \bar{\Gamma}_{ij}^r\) (on line \(L_1\)), \(\Gamma_{ij}^r \cup \bar{\Gamma}_{ij}^r\) (on line \(L_2\)) and \(\Gamma_{ij}^r \cup \bar{\Gamma}_{ij}^r\) (on plane \(S\)), where

\[
\bar{\Gamma}_{ij}^r = \left\{(l_i, l_j) \to 0 \mid l_i/l_j \to 0, l_i^{(1-\mu)/(1-\nu)} \to c > 0 \right\},
\]

\[
\bar{\Gamma}_{ij}^r = \left\{(l_i, l_j) \to 0 \mid l_i/l_j \to 0, l_i^{(1-\mu)/(1-\nu)} \to c \geq 0 \right\}.
\]

In other words, inequalities (22) are fulfilled on the paths with \(l_i/l_j \to c > 0\) at the point \(P\), while on the lines \(L_1\) and \(L_2\) as well as on the plane \(S\), the available paths are bounded by the limit pencils defined by the asymptotic relations: \(l_i/l_j \to c > 0\) and \(l_i/l_j \sim l_i^{\nu} \to 0\) (on \(L_1\)); \(l_i/l_j \sim l_i^{(2(1-\mu)+\nu)/(1-\nu)} \to 0\) and \(l_i/l_j \to c > 0\) (on \(L_2\)); \(l_i/l_j \sim l_i^{2(1-\mu)/(1-\nu)} \to 0\) and \(l_i/l_j \sim l_i^{\nu} \to 0\) (on \(S\)).

**Example 2.** Another particular example can be considered if the widths \(l_i\) and the strengths \(V_i, i = 1, \ldots, N\), are specified by applying a power-connecting parametrization

\[
l_i = \varepsilon^{\mu_i}d_i, \quad V_i = \varepsilon^{\nu_i}h_i, \quad h_i \in \mathbb{R}, \quad d_i, \mu_i, \nu_i > 0, \quad i = 1, N,
\]

with a dimensionless squeezing parameter \(\varepsilon \to 0\). Inserting then the above formulas into definitions (15), (18), (19) and (38) for the \(G\)-sets, we get the restrictions on the parameters \(\mu_i\) and \(\nu_i\) as follows

\[
G_i^\mu = \left\{\mu_i \geq \nu_i\right\}, \quad G_i^\nu = \left\{2\mu_i > \nu_i\right\}, \quad G_i^\sigma = \left\{2\mu_i = \nu_i\right\},
\]

\[
G_i^\mu = \left\{\mu_i < \nu_i < 2\mu_i\right\}, \quad G_i^\sigma = \left\{(1 + 1/\sigma)\mu_i = \nu_i\right\}, \quad 1 \leq \sigma < \infty.
\]

Similarly, inserting expressions (64) into definitions (23) and (37), we obtain the additional restrictions on the parameters \(\mu_i\) and \(\nu_i\):

\[
\Gamma_{ij} = \left\{\mu_i > \mu_j\right\}, \quad \Gamma_{ij}^\mu = \left\{\mu_i = \mu_j\right\},
\]

\[
\Gamma_{ij} = \left\{\mu_i + \mu_j > \nu_i\right\}, \quad \Gamma_{ij}^\mu = \left\{\mu_i + \mu_j = \nu_i\right\},
\]

\[
\Gamma_{ij}^\sigma = \left\{\mu_i = \sigma\mu_j\right\}, \quad \Gamma_{ij}^\sigma = \left\{\mu_i + \mu_j/\sigma = \nu_i\right\}, \quad 1 \leq \sigma < \infty.
\]

The system of restrictions (65) and (66) determines the hyperplanes in the \(2N\)-dimensional space \(\{\mu_1, \ldots, \mu_N\} \times \{\nu_1, \ldots, \nu_N\}\), on which the point interactions can be materialized and the corresponding resonance equations are given in terms of the system parameters \(d_i\) and \(h_i, i = 1, \ldots, N\). For instance, the equation \(q_2^\nu l_i + q_1^\nu l_j = 0\), where both the terms diverge in the squeezing limit, is fulfilled on the intersection of the planes \(\mu_i = \nu_i + \nu_j/2 = 0\) and \(\mu_j = \nu_j/2\), resulting in the condition \(\mu_i + \mu_j = \nu_j\), defining the pencil \(\Gamma_{ij}^\nu\). In this particular case, the resonance equation is given by (32), being both the necessary and sufficient condition for the desired convergence of the element \(\lambda_{21}\).

**6. Final remarks**

Thus, we have formulated above the necessary conditions on the strength configuration \(V_1(l_1), \ldots, V_N(l_N)\) and the paths \(l = \{l_1, \ldots, l_N\} \to 0\), under which the squeezed multi-layer
structure can be described as a non-trivial one-point interaction model. These conditions are of two types. The first of them is formulated by conclusion 2, realizing the finite and non-zero squeezed limits of the diagonal elements $\lambda_{11}$ and $\lambda_{22}$ of the transmission matrix $\Lambda$.

The conditions of the second type described in conclusions 3 and 4 are necessary, but not sufficient for the full cancellation of divergences in the singular element $\lambda_{21}$ of the squeezed transmission matrix $\Lambda$. These conditions provide the cancellation of only the most singular terms, while the lower divergent terms may not be canceled out. In the case if the lower singular terms are absent, the cancellation imposes the constraints on the system parameters, resulting in the appearance of resonance sets. More precisely, using the equation $\det \Lambda(x_0, x_N) = 1$ and $\lambda_{12} \to 0$, valid for any strength configuration, one can assert that $\lambda_{11} \to \theta \in \mathbb{R} \setminus \{0\}$ and $\lambda_{22} \to \theta^{-1}$. As a result, the resulting connection matrix is of the form

$$\Lambda_0 := \lim_{l \to 0} \Lambda(x_0, x_N) = \begin{pmatrix} \theta & 0 \\ \alpha \theta^{-1} & \theta \end{pmatrix}, \quad \alpha \in \mathbb{R}.$$  \hspace{1cm} (67)

As follows from equation (17), the bound state level is $\kappa = -\alpha/(\theta + \theta^{-1})$.

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Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

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References

[1] Albeverio S, Gesztesy F, Høegh-Krohn R and Holden H 2005 *Solvable Models in Quantum Mechanics (With an Appendix by Pavel Exner)* 2nd revised edn (Providence: RI: American Mathematical Society)
[2] Albeverio S and Kurasov P 1999 *Singular Perturbations of Differential Operators: Solvable Schrödinger-Type Operators* (Cambridge: Cambridge University Press)
[3] Šeba P 1986 Rep. Math. Phys. 24 111
[4] Golovaty Y D and Man’ko S S 2009 Ukr. Math. Bull. 6 169
[5] Golovaty Y D and Hryniv R O 2010 J. Phys. A: Math. Theor. 43 155204
[6] Golovaty Y D and Hryniv R O 2011 J. Phys. A: Math. Theor. 44 049802
[7] Man’ko S S 2010 J. Phys. A: Math. Theor. 43 445304
[8] Man’ko S S 2012 J. Math. Phys. 53 123521
[9] Exner P and Manko S S 2013 J. Phys. A: Math. Theor. 46 345202
[10] Toyama F M and Nogami Y 2007 J. Phys. A: Math. Theor. 40 F685
[11] Christiansen P L, Arnbak H C, Zolotaryuk A V, Ermakov V N and Gaididei Y B 2003 J. Phys. A: Math. Gen. 36 7589
[12] Zolotaryuk A V, Christiansen P L and Iermakova S V 2006 J. Phys. A: Math. Gen. 39 9329
[13] Zolotaryuk A V 2008 J. Comput. Theor. Nanosci. 1 187
[14] Zolotaryuk A V and Zolotaryuk Y 2011 J. Phys. A: Math. Theor. 44 375305
[15] Zolotaryuk A V and Zolotaryuk Y 2012 J. Phys. A: Math. Theor. 45 119501
[16] Golovaty Y 2012 Methods Funct. Anal. Topology 18 243
[17] Golovaty Y 2013 Integr. Equ. Oper. Theory 75 341
[18] Gadella M, Negro J and Nieto L M 2009 Phys. Lett. A 373 1310
[19] Gadella M, Mateos-Guijarre J, Muñoz-Castañeda J M and Nieto L M 2016 J. Phys. A: Math. Theor. 49 015204
[20] Gadella M, Glasser M L and Nieto L M 2011 Int. J. Theor. Phys. 50 2144
[21] Gadella M, García-Ferrero M A, González-Martín S and Maldonado-Villamizar F H 2014 Int. J. Theor. Phys. 53 1614
[22] Fassari S, Gadella M, Glasser M L and Nieto L M 2018 Ann. Phys., NY 389 48
[23] Albeverio S, Dabrowski L and Kurasov P 1998 Lett. Math. Phys. 45 33
[24] Albeverio S and Nizhnik L 2007 J. Math. Anal. Appl. 332 884
[25] Lange R-J 2012 J. High Energy Phys. JHEP11(2012)032
[26] Brasche J F and Nizhnik L P 2013 Methods Funct. Anal. Topology 19 4
[27] Albeverio S and Nizhnik L 2013 Methods Funct. Anal. Topology 19 199
[28] Lange R-J 2015 J. Math. Phys. 56 122105
[29] Golovaty Y 2018 J. Phys. A: Math. Theor. 51 255202
[30] Cheon T and Shigeo T 1998 Phys. Lett. A 243 111
[31] Exner P, Neidhardt H and Zagrebnov V A 2001 Commun. Math. Phys. 224 593
[32] Albeverio S, Fassari S and Rinaldi F 2013 J. Phys. A: Math. Theor. 46 385305
[33] Albeverio S, Fassari S and Rinaldi F 2016 J. Phys. A: Math. Theor. 49 025302
[34] Zolotaryuk A V 2018 Physica E 103 81
[35] Zolotaryuk A V 2018 Ann. Phys., NY 396 479
[36] Zolotaryuk A V and Zolotaryuk Y 2020 Low Temp. Phys. 46 927
[37] Zolotaryuk A V and Zolotaryuk Y 2021 J. Phys. A: Math. Theor. 54 035201
[38] Kurasov P 1996 J. Math. Anal. Appl. 201 297
[39] Coutinho F A B, Nogami Y and Pérez J F 1997 J. Phys. A: Math. Gen. 30 3937
[40] Zolotaryuk A V and Zolotaryuk Y 2014 Int. J. Mod. Phys. B 28 1350203