BOUNDDED SUBGROUPS OF RELATIVELY FINITELY PRESENTED GROUPS

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Abstract. Given a finitely generated group $G$ that is relatively finitely presented with respect to a collection of peripheral subgroups, we prove that every infinite subgroup $H$ of $G$ that is bounded in the relative Cayley graph of $G$ is conjugate into a peripheral subgroup. As an application, we obtain a trichotomy for subgroups of relatively hyperbolic groups. Moreover we prove the existence of the relative exponential growth rate for all subgroups of limit groups.

1. Introduction

The notion of a group $G$ that is hyperbolic relative to a finite set $H_\Lambda$ of its subgroups was introduced by Gromov \cite{Gro} as a generalization of a word hyperbolic group. In his definition, the groups $H \in H_\Lambda$ appear as stabilizers of points at infinity of a certain hyperbolic space $X$ the group $G$ acts on. Since then, the study of relatively hyperbolic groups remained an active field of research and several characterizations of relative hyperbolicity were introduced by Bowditch \cite{Bow}, Farb \cite{Farb}, and Osin \cite{Osin}. In the latter work, Osin uses the concept of relative presentations in order to define the relative hyperbolicity of a group $G$ with respect to a set $H_\Lambda = \{H_\lambda \mid \lambda \in \Lambda\}$ of its subgroups. To make this more precise, let $X \subseteq G$ be a symmetric subset such that $G$ is generated by $\bigcup_{\lambda \in \Lambda} H_\lambda \cup X$. Then we obtain a canonical epimorphism

$$\varepsilon : F := (*_{\lambda \in \Lambda} \tilde{H}_\lambda) * F(X) \rightarrow G,$$

where the groups $\tilde{H}_\lambda$ are disjoint isomorphic copies of $H_\lambda$ and $F(X)$ denotes the free group over $X$. Consider a subset $R \subseteq F$ whose normal closure is the kernel of $\varepsilon$. Then $R$ gives rise to a so-called relative presentation of $G$ with respect to $H_\Lambda$ of the form

$$\langle X, H \mid S = 1, S \in \bigcup_{\lambda \in \Lambda} S_\lambda, R = 1, R \in R \rangle,$$

where $H := \bigcup_{\lambda \in \Lambda} (\tilde{H}_\lambda \setminus \{1\})$ and $S_\lambda$ is the set of all relations over the alphabet $\tilde{H}_\lambda$. In this framework, $G$ is said to be hyperbolic relative to $H_\Lambda$ if $X$ and $R$ can be chosen to be finite and $\varepsilon$ admits a linear relative Dehn function. That is, there is some $C > 0$ such that for every word $w$ of length at most $\ell$ over $X \cup H$ that represents the identity in $G$, there is an equality of the form

$$w = \prod_{i=1}^{k} f_i^{-1} R_i^{\pm 1} f_i$$

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that holds in $F$, where $k \leq C\ell, f_i \in F$ and $R_i \in R$. Note that in general, there is no reason to expect that for every $\ell \in \mathbb{N}$ and every relation $w$ of length at most $\ell$ there is a uniform upper bound $n \in \mathbb{N}$ such that $w$ can be written as in (2) with $k \leq n$. Even if $X$ and $R$ are finite, in which case we say that (1) is a finite relative presentation for $G$, there are easy examples where there is no such $n$, see [14, Example 1.3].

In this paper we study groups $G$ that admit a finite relative presentation as in (1) whose relative Dehn function $\delta^r_{G,H_\Lambda}$ is well-defined. This means that for every $\ell \in \mathbb{N}$ there is a minimal number $\delta^r_{G,H_\Lambda}(\ell)$, such that for every relation $w$ of length at most $\ell$ there is an expression of the form (2) with $k \leq \delta^r_{G,H_\Lambda}(\ell)$. Examples of relatively finitely presented groups that admit a well-defined, non-linear relative Dehn function were considered in [10]. The study of groups with a well-defined relative Dehn function, typically involves considerations in the so-called relative Cayley graph $\Gamma(G,X \cup H)$ of $G$. Since $X \cup H$ can be (and usually is) infinite, it is natural to ask the following.

**Question 1.1.** Which subgroups of $G$ have bounded diameter in $\Gamma(G,X \cup H)$?

Note that, next to the finite subgroups of $G$, every subgroup of $G$ that can be conjugated into some of the groups $H_\lambda$ has bounded diameter in $\Gamma(G,X \cup H)$. It turns out that for finitely generated $G$, the existence of a well-defined relative Dehn function is enough to deduce that there are no further examples of subgroups of $G$ whose diameter in $\Gamma(G,X \cup H)$ is finite.

**Theorem 1.2.** Let $G$ be a finitely generated group. Suppose that $G$ is relatively finitely presented with respect to a collection $H_\Lambda = \{ H_\lambda \mid \lambda \in \Lambda \}$ of its subgroups and that the relative Dehn function $\delta^r_{G,H_\Lambda}$ is well-defined. Then every subgroup $K \leq G$ satisfies exactly one of the following conditions:

1) $K$ is finite.
2) $K$ is infinite and conjugate to a subgroup of some $H_\lambda$.
3) $K$ is unbounded in $\Gamma(G,X \cup H)$.

Note for example that if some of the subgroups $H_\lambda$ in Theorem 1.2 is infinite, then there is no subgroup $K \leq G$ that contains $H_\lambda$ as a proper subgroup of finite index. This also follows from the fact that each $H_\lambda$ is almost malnormal, which is shown in [14, Proposition 2.36].

If the group $G$ in Theorem 1.2 is relatively hyperbolic with respect to $H_\Lambda$, then it is known that a subgroup $K \leq G$ with infinite diameter in $\Gamma(G,X \cup H)$ contains a loxodromic element (see [13, Theorem 1.1] together with [13, Proposition 5.2]).

Recall that an element $g \in G$ is called loxodromic if the map $\mathbb{Z} \to \Gamma(G,X \cup H), \ n \mapsto g^n$ is a quasiisometrical embedding. We therefore obtain the following classification of subgroups of relatively hyperbolic groups which, to the best of my knowledge, was not recorded before.

**Corollary 1.3.** Let $G$ be a finitely generated group. Suppose that $G$ is relatively hyperbolic with respect to a collection $H_\Lambda = \{ H_\lambda \mid \lambda \in \Lambda \}$ of its subgroups. Then every subgroup $K \leq G$ satisfies exactly one of the following conditions:

1) $K$ is finite.
2) $K$ is infinite and conjugate to a subgroup of some $H_\lambda$.
3) $K$ contains a loxodromic element.

As an application of Corollary 1.3 we consider relative exponential growth rates in finitely generated groups. Recall that for a finitely generated group $G$ and a
finite generating set $X$ of $G$, the growth function $\beta_{X}^{G} : \mathbb{N} \to \mathbb{N}$ of $G$ with respect to $X$ is defined by $\beta_{X}^{G}(n) = |B_{X}^{G}(n)|$, where $B_{X}^{G}(n)$ denotes the set of all elements of $G$ that are represented by words of length at most $n$ in the generators of $X$ and $X^{-1}$.

Using Fekete’s Lemma, it is easy to see that the limit $\lim_{n\to\infty} n^{\frac{1}{2}} \beta_{X}^{G}(n)$, known as the exponential growth rate of $G$ with respect to $X$, always exist (see for example [11]).

Given a subgroup $H \leq G$, a relative analogue of the exponential growth rate is obtained by counting the elements in the relative balls $B_{X}^{H}(n) := B_{X}^{G}(n) \cap H$. The resulting function $\beta_{X}^{H} : \mathbb{N} \to \mathbb{N}, \ n \mapsto |B_{X}^{H}(n)|$ is called the relative growth function of $H$ with respect to $X$. In [12, Remark 3.1] Olshanskii pointed out that, unlike in the non-relative case, the limit $\lim_{n\to\infty} n^{\frac{1}{2}} \beta_{X}^{H}(n)$ does not exist in general. As a consequence, the relative exponential growth rate of $H$ in $G$ with respect to $X$ is typically defined as $\limsup_{n\to\infty} n^{\frac{1}{2}} \beta_{X}^{H}(n)$. Nevertheless, in many cases where the relative exponential growth rate is studied in the literature (see for example [3], [8], [12], [17], [4], [6] where $G$ is free or hyperbolic) the limit $\lim_{n\to\infty} n^{\frac{1}{2}} \beta_{X}^{H}(n)$ is known to exist, in which case we say that the relative exponential growth rate of $H$ in $G$ exists with respect to $X$. In the case where $G$ is a free group, the existence of the relative exponential growth rate was proven by Olshanskii in [12], extending prior results of Cohen [3] and Grigorchuk [8] who have independently proven the existence for normal subgroups of $G$. More recently, these existence results where generalized by the author to the case where $G$ is a finitely generated acylindrically hyperbolic group and $H$ is a subgroup that contains a generalized loxodromic element of $G$, see [15]. By combining this with Corollary 1.3 we will be able conclude the following.

\textbf{Theorem 1.4.} Let $G$ be a finitely generated group that is relatively hyperbolic with respect to a collection $H_{\Lambda} = \{ H_{\lambda} \mid \lambda \in \Lambda \}$ of its subgroups. Suppose that each of the groups $H_{\lambda}$ has subexponential growth. Then the relative exponential growth rate of every subgroup $H \leq G$ exists with respect to every finite generating set of $G$.

By Osin [14, Theorem 1.1], each of the groups $H_{\lambda}$ in Theorem 1.4 is finitely generated so that the assumption on subexponential growth indeed makes sense. Relatively hyperbolic groups $G$ as in Theorem 1.4 include many naturally occurring examples of groups. A particularly interesting such class is given by limit groups, which were introduced by Zela in his solution of the Tarski problems [16] and naturally generalize the class of free groups. By work of Dahmani [5] and Alibegovic [1], limit groups are known to be relatively hyperbolic with respect to a system of representatives for the conjugacy classes of its maximal abelian non-cyclic subgroups. As a consequence, we obtain the following generalization of Olshanskii’s existence result.

\textbf{Corollary 1.5.} Let $G$ be a limit group. Then the relative exponential growth rate of every subgroup $H \leq G$ exists with respect to every finite generating set of $G$.

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2. Preliminaries

In this section we introduce some definitions and properties that will be relevant for our study of relatively finitely presented groups. More information about these groups can be found in [14].
2.1. Relative presentations. Let us fix a group $G$ and a collection $H_\Lambda = \{ H_\lambda \mid \lambda \in \Lambda \}$ of so-called peripheral subgroups of $G$. Let $X \subseteq G$ be a symmetric subset such that $G$ is generated by $\bigcup_{\lambda \in \Lambda} H_\lambda \cup X$. Such $X$ will be referred to as a relative generating of $G$ with respect to $H_\Lambda$. Note that this gives us a canonical epimorphism

$$\varepsilon : F := \langle \bigcup_{\lambda \in \Lambda} \tilde{H}_\lambda \rangle \ast F(X) \to G,$$

where the groups $\tilde{H}_\lambda$ are pairwise disjoint isomorphic copies of $H_\lambda$ and $F(X)$ denotes the free group over $X$. Let us also assume that $\tilde{H}_\lambda \cap X = \emptyset$ for every $\lambda \in \Lambda$. Let $N$ denote the kernel of $\varepsilon$ and let $R \subseteq N$ be a subset whose normal closure in $F$ coincides with $N$. For each $\lambda \in \Lambda$ let $S_\lambda$ be the set of words over $\tilde{H}_\lambda \setminus \{1\}$ that represents the identity in $G$.

Definition 2.1. With the notation above, we say that a relative presentation of $G$ with respect to $H_\Lambda$ is a presentation of the form (3)

$$(X, H \mid S = 1, S \in \bigcup_{\lambda \in \Lambda} S_\lambda, R = 1, R \in R),$$

where $H := \bigcup_{\lambda \in \Lambda} (\tilde{H}_\lambda \setminus \{1\})$. The relative presentation (3) is called finite if $X$ and $R$ are finite. In this case $G$ is said to be relatively finitely presented with respect to $H_\Lambda$.

The following result will be crucial for us. It can be found in [13, Theorem 1.1].

Theorem 2.2. Let $G$ be a finitely generated group and let $H_\Lambda = \{ H_\lambda \mid \lambda \in \Lambda \}$ be a collection of its subgroups. Suppose that $G$ is finitely presented with respect to $H_\Lambda$. Then the following conditions hold.

1. The collection $H_\Lambda$ is finite, i.e. $|\Lambda| < \infty$.
2. Each subgroup $H_\lambda$ is finitely generated.

2.2. Relative Dehn functions. Let $G$ be a relatively finitely presented group with a finite relative presentation as in Definition 2.1. For each $\ell \in \mathbb{N}$, let $N_\ell$ denote the set of words of length at most $\ell$ over $X \cup H$ that represent the identity in $G$. Given $w \in N_\ell$, let $\text{vol}(w) \in \mathbb{N}$ be minimal with the property that there is an expression of the form

$$w = F \prod_{i=1}^{\text{vol}(w)} f_i^{-1} R_i^\pm 1 f_i,$$

where the equality is taken in $F$ and $f_i \in F$, $R_i \in R$ for every $1 \leq i \leq \text{vol}(w)$.

Definition 2.3. The relative Dehn function for the finite relative presentation (3) of $G$ is defined by

$$\delta_{G,H_\Lambda}^{\text{rel}} : \mathbb{N} \to \mathbb{N} \cup \{\infty\}, \ell \mapsto \sup\{ \text{vol}(w) \mid w \in N_\ell \}.$$

We say that $\delta_{G,H_\Lambda}^{\text{rel}}$ is well-defined if $\delta_{G,H_\Lambda}^{\text{rel}}(\ell) < \infty$ for every $\ell \in \mathbb{N}$.

An important class of relatively finitely presented groups with a well-defined Dehn function consists of relatively hyperbolic groups, which can be defined in terms of the relative Dehn function.

Definition 2.4. A relatively finitely presented group $G$ with a relative presentation (3) is called relatively hyperbolic with respect to $H_\Lambda$ if there is some $C > 0$ such the relative Dehn function satisfies $\delta_{G,H_\Lambda}^{\text{rel}}(\ell) \leq C\ell$ for every $\ell \in \mathbb{N}$. 

2.3. Geometry of relative Cayley graph. Let us again consider a relatively finitely presented group $G$ with a finite relative presentation as in Definition 2.4. The Cayley graph of $G$ with respect to $X \cup \mathcal{H}$ is called the relative Cayley graph of $G$ and will be denoted by $\Gamma(G, X \cup \mathcal{H})$. In the following, we will study the local geometry of $\Gamma(G, X \cup \mathcal{H})$. In order to do so, we fix some terminology. Given an edge $e$ of $\Gamma(G, X \cup \mathcal{H})$, we write $\partial_0(e)$ to denote the initial vertex of $e$ and $\partial_1(e)$ to denote the terminal vertex of $e$. A sequence $p = (e_1, \ldots, e_n)$ of edges in $\Gamma(G, X \cup \mathcal{H})$ is called a path if $\partial_1(e_i) = \partial_0(e_{i+1})$ for $1 \leq i < n$. If moreover $\partial_0(e_1) = \partial_1(e_n)$, then $p$ is said to be cyclic. The label of a path $p$ will be denoted by $\text{Lab}(p)$. Sometimes it is useful to forget about the initial vertex of a cyclic path $p = (e_1, \ldots, e_n)$. To make this precise, we define the loop associated to $p$ as the set $[p]$ of all paths of the form $(e_1, \ldots, e_n, e_1, \ldots, e_{i+1})$ for $1 \leq i \leq n$. A subpath of a loop $[p]$ is a subpath of some representative $p' \in [p]$. The algebraic counterpart of a loop is the set $[w]$ of all cyclic conjugates of a word $w$ over $X \cup \mathcal{H}$, which will be referred to as a cyclic word. Accordingly, the label of a loop $[p]$ is defined as $\text{Lab}([p]) := [\text{Lab}(p)]$. Up to minor notational differences, the following definitions can be found in [14].

**Definition 2.5.** Let $w$ be a word over $X \cup \mathcal{H}$. A subword $v$ of $w$ is a $\lambda$-subword if it consists of letters of $\mathcal{H}_A$. If a $\lambda$-subword $v$ of $w$ is not properly contained in any other $\lambda$-subword of $w$, then $v$ is called a $\lambda$-syllable of $w$. Similarly, we say that a word $v$ over $X \cup \mathcal{H}$ is a $\lambda$-subword of a cyclic word $[w]$ if it is a $\lambda$-subword of some cyclic conjugate of $w$. If a $\lambda$-subword $v$ of $[w]$ is not properly contained in any other $\lambda$-subword of $[w]$, then $v$ is called a $\lambda$-syllable of $[w]$.

Let us now translate Definition 2.5 into conditions for paths in $\Gamma(G, X \cup \mathcal{H})$.

**Definition 2.6.** Let $q$ be a path in $\Gamma(G, X \cup \mathcal{H})$. A subpath $p$ of $q$ is a $\lambda$-subpath if $\text{Lab}(p)$ is a $\lambda$-subword of $\text{Lab}(q)$. A $\lambda$-subpath $p$ of $q$ is called a $\lambda$-component of $q$ if $\text{Lab}(p)$ is a $\lambda$-syllable of $\text{Lab}(q)$. Suppose now that $q$ is cyclic, and consider the loop $[q]$ associated to $q$. We say that a subpath $p$ of $[q]$ is a $\lambda$-subpath of $[q]$ if $\text{Lab}(p)$ is a $\lambda$-subword of $\text{Lab}([q])$. If moreover $\text{Lab}(p)$ is a $\lambda$-syllable of $\text{Lab}([q])$, then $p$ is called a $\lambda$-component of $[q]$.

**Definition 2.7.** Let $p_1$ and $p_2$ be $\lambda$-components of a path $p$, respectively a loop $[q]$, in $\Gamma(G, X \cup \mathcal{H})$. We say that $p_1$ and $p_2$ are connected, if there is a path $c$ in $\Gamma(G, X \cup \mathcal{H})$ that connects a vertex of $p_1$ with a vertex of $p_2$ and $\text{Lab}(c)$ consists of letters of $\mathcal{H}_A$. We say that $p_1$ is isolated in $p$, respectively $[q]$, if there are no further $\lambda$-components of $p$, respectively $[q]$, that are connected to $p_1$.

Let us now translate the notion of an isolated component of a path (loop) in a corresponding notion for syllables in (cyclic) words.

**Definition 2.8.** Let $w$ be a word over $X \cup \mathcal{H}$ and let $p$ be any path in $\Gamma(G, X \cup \mathcal{H})$ with $\text{Lab}(p) = w$. We say that two $\lambda$-syllables $v_1, v_2$ of $w$ are connected, respectively isolated, if the corresponding $\lambda$-components $p_1, p_2$ of $p$ are connected, respectively isolated. If $w$ represents the identity in $G$ and $v_1, v_2$ are $\lambda$-syllables of the cyclic word $[w]$, then $v_1, v_2$ are connected, respectively isolated, if the corresponding $\lambda$-components $p_1, p_2$ of the loop $[p]$ are connected, respectively isolated.
The following lemma is a direct consequence of Lemma [13] Lemma 2.27. It will help us to study the local structure of $\Gamma(G, X \cup \mathcal{H})$ and often lets us switch between the word metrics $d_X$ and $d_{X \cup \mathcal{H}}$.

**Lemma 2.9.** Let $G$ be a finitely generated group with a finite generating set $X$. Suppose that $G$ is relatively finitely presented with respect to a collection $H_\Lambda = \{ H_\lambda \mid \lambda \in \Lambda \}$ of its subgroups and that the relative Dehn function $\delta_{G,H_\Lambda}^{rel}$ is well-defined. Then for every $n \in \mathbb{N}$ there is a finite subset $\Omega_n \subseteq G$ with the property that for every cyclic path $\eta$ in $\Gamma(G, X \cup \mathcal{H})$ of length at most $n$ and every isolated component $p$ of the loop $[\eta]$, the label $\text{Lab}(p)$ represents an element in $\Omega_n$.

### 3. The alternating growth condition

In this section we introduce the alternating growth condition, which will play a central role in our proof of Theorem [12].

#### 3.1. Regular neighbourhoods

Let us start by defining a condition for paths in graphs that can be thought of as a strong form of having no self-intersections.

**Definition 3.1.** Let $\Gamma$ be a graph and let $p$ be a path in $\Gamma$ that consecutively traverses the sequence $v_0, \ldots, v_n$ of vertices in $\Gamma$. We say that $p$ has a regular neighbourhood in $\Gamma$ if every two vertices $v_i, v_j$ that can be joined by an edge in $\Gamma$ satisfy $|i - j| \leq 1$.

**Example 3.2.** If $p$ is a geodesic path in a graph $\Gamma$, then $p$ has a regular neighbourhood in $\Gamma$.

**Example 3.3.** If $p$ is a non-trivial cyclic path in a graph $\Gamma$, then $p$ does not have a regular neighbourhood $\Gamma$.

**Remark 3.4.** Note that every path $p$ that has a regular neighbourhood in a graph $\Gamma$ is locally 2-geodesic, i.e. the restriction of $p$ to each subpath of length at most 2 is geodesic.

It will be useful for us to translate the concept of regular neighbourhoods to words over some generating set of a group.

**Definition 3.5.** Let $G$ be a group and let $X$ be a generating set of $G$. A word $w$ over $X$ is called regular (with respect to $X$) if some path $p$ in $\Gamma(G, X)$ with $\text{Lab}(p) = w$ has a regular neighbourhood in $\Gamma(G, X)$.

**Remark 3.6.** Let $G$ be a group and let $X$ be a generating set of $G$. From the definitions if directly follows that a word $w$ over $X$ is regular if and only if every subword $v$ of $w$ of length at least 2 satisfies $|v|_X \geq 2$, where $| \cdot |_X$ denotes the word metric corresponding to $X$.

#### 3.2. Sequences of alternating growth

We want to study sequences of regular words in the context of finitely generated, relatively finitely presented groups. Let us therefore fix a finitely generated group $G$, a finite generating set $X$ of $G$, and a collection $H_\Lambda = \{ H_\lambda \mid \lambda \in \Lambda \}$ of peripheral subgroups of $G$. Suppose that $G$ is relatively finitely presented with respect to $H_\Lambda$ and that the relative Dehn function $\delta_{G,H_\Lambda}^{rel}$ is well-defined. As in Section 2 we write $\hat{H}_\Lambda$ to denote pairwise disjoint isomorphic copies of $H_\Lambda$ that also intersect trivially with $X$. Let us fix some notation in order to avoid ambiguities concerning the length and the evaluation of a word over $X \cup \mathcal{H}$, where as always $\mathcal{H} = \bigcup_{\lambda \in \Lambda} (\hat{H}_\lambda \setminus \{1\})$.

**Notation 3.7.** Let $w = w_1 \ldots w_\ell$ be a word over $X \cup \mathcal{H}$. We write $|w|$ = $\ell$ for the word length of $w$. The image of $w$ in $G$ will be denoted by $\overline{w}$. For any subset $Y \subseteq G$ we write $|\overline{w}|_Y$ for the length of a shortest word over $Y$ that represents $\overline{w}$. If there is no such word, then we set $|\overline{w}|_Y = \infty$. 

Definition 3.8. A sequence of words \((w_1^{(n)} \ldots w_{\ell}^{(n)})_{n \in \mathbb{N}}\) of fixed length \(\ell \geq 2\) over \(X \cup \mathcal{H}\) satisfies the alternating growth condition if the following conditions are satisfied:

I) If \(w_i^{(n)} = x\) for some \(1 \leq i \leq \ell\), \(n \in \mathbb{N}\) and \(x \in X\), then \(w_m^{(n)} = x\) for every \(m \in \mathbb{N}\). In this case we say that \(i\) is an index of type \(X\).

II) If \(w_i^{(n)} \in \tilde{H}_\lambda\) for some \(1 \leq i \leq \ell\), \(n \in \mathbb{N}\) and \(\lambda \in \Lambda\), then \(w_m^{(n)} \in \tilde{H}_\lambda\) for every \(m \in \mathbb{N}\). In this case we say that \(i\) is an index of type \(\lambda\).

III) The index 1 is not of type \(X\).

IV) Two consecutive indices are never of the same type.

V) If \(i\) is of type \(\lambda\), then \(\overline{w}_i^{(n)} \notin H_\mu\) for every \(\mu \in \Lambda \setminus \{\lambda\}\) and every \(n \in \mathbb{N}\). If \(\overline{w}_i^{(n)} \in \langle X\rangle\) for some \(n \in \mathbb{N}\), then \(|\overline{w}_i^{(n)}|_{X} \geq n\).

VI) Each word \(w_1^{(n)} \ldots w_{\ell}^{(n)}\) is regular with respect to \(X \cup \mathcal{H}\).

The following observation will be used frequently.

Remark 3.9. Given a regular word \(w\) over \(X \cup \mathcal{H}\), it directly follows from the definitions that every syllable \(v\) in \(w\) is isolated and consists of a single edge.

3.3. Concatenating sequences of alternating growth. In what follows we need to construct certain sequences \((w_1^{(n)} \ldots w_{\ell}^{(n)})_{n \in \mathbb{N}}\) of words over \(X \cup \mathcal{H}\) that satisfy the alternating growth condition such that \(\ell\) can be chosen arbitrarily large. In order to do so, we will use the following lemma which allows us to “concatenate” two sequences of words that satisfy the alternating growth condition such that the resulting sequence also satisfies the alternating growth condition.

Lemma 3.10. With the notation above, suppose that there are two sequences \((v_1^{(n)} \ldots v_M^{(n)})_{n \in \mathbb{N}}\) and \((w_1^{(n)} \ldots w_N^{(n)})_{n \in \mathbb{N}}\) of words over \(X \cup \mathcal{H}\) that satisfy the alternating growth condition. Let \(\lambda \in \Lambda\) be such that \(w_1^{(n)} \in \tilde{H}_\lambda\) for some \(n \in \mathbb{N}\).

1) Suppose that \(\overline{v}_M^{(n)} \notin H_\lambda\) for every \(n \in \mathbb{N}\). Then there is a strictly increasing sequence of natural numbers \((s_n)_{n \in \mathbb{N}}\) such that
\[
(v_1^{(s_1)} \ldots v_M^{(s_n)} w_1^{(s_n)} \ldots w_N^{(s_n)})_{n \in \mathbb{N}}
\]
satisfies the alternating growth condition.

2) Suppose that \(\overline{v}_M^{(n)} \in H_\lambda\) for every \(n \in \mathbb{N}\). Then there are strictly increasing sequences of natural numbers \((s_n)_{n \in \mathbb{N}}\), \((t_n)_{n \in \mathbb{N}}\) such that the sequence
\[
(v_1^{(s_1)} \ldots v_M^{(s_n)} z_n^{(s_n)} w_2^{(t_n)} \ldots w_N^{(t_n)})_{n \in \mathbb{N}},
\]
where \(z_n^{(s_n)} \in \tilde{H}_\lambda\) is the element representing \(v_M^{(s_n)} w_1^{(t_n)} \in H_\lambda\), satisfies the alternating growth condition.

Proof. Let us first prove 1). Suppose that there is no sequence \((s_i)_{i \in \mathbb{N}}\). Then there are infinitely many \(n \in \mathbb{N}\) such that \(v_1^{(n)} \ldots v_M^{(n)} w_1^{(n)} \ldots w_N^{(n)}\) does not satisfy some of the conditions of Definition 3.8. Since I) - V) are clearly satisfied, it follows that \(v_1^{(n)} \ldots v_M^{(n)} w_1^{(n)} \ldots w_N^{(n)}\) is not regular (with respect to \(X \cup \mathcal{H}\)) for infinitely many \(n \in \mathbb{N}\). By restriction to a subsequence if necessary, we can assume that none of the words \(v_1^{(n)} \ldots v_M^{(n)} w_1^{(n)} \ldots w_N^{(n)}\) is regular. Since \(\overline{v}_M^{(n)} \notin H_\lambda\) for every \(n \in \mathbb{N}\), none of the subwords \(v_M^{(n)} w_1^{(n)}\) represents the trivial element in \(G\). Together with the assumption that \(v_1^{(n)} \ldots v_M^{(n)}\) and \(w_1^{(n)} \ldots w_N^{(n)}\) are regular, it follows that there is a maximal index \(a_n\) such that
\[
|v_{a_n}^{(n)} \ldots v_M^{(n)} w_1^{(n)} \ldots w_{b_n}^{(n)}|_{X \cup \mathcal{H}} = 1
\]
for some index $b_n$. Suppose that each $b_n$ is chosen to be minimal with respect to $a_n$. Then there are generators $u_n \in X \cup H$ such that
\[ q_n = v_{a_n}^{(n)} \cdots v_{M}^{(n)} w_1^{(n)} \cdots w_{b_n}^{(n)} u_n \]
represents the identity in $G$ for every $n \in \mathbb{N}$. We want to argue that $w_1^{(n)}$ is an isolated $\lambda$-syllable in the cyclic word $[q_n]$. Suppose that this is not the case. Then there are 3 cases to consider.

**Case 1:** $v_i^{(n)} \cdots v_M^{(n)} w_1^{(n)} \in H_\lambda$ for some $a_n \leq i \leq N$. Then $v_i^{(n)} \cdots v_M^{(n)} \in H_\lambda$ and since $v_1^{(n)} \cdots v_M^{(n)}$ is regular, we obtain $i = M$. Thus $v_1^{(n)} \in H_\lambda$, in contrast to our assumption that $v_1^{(n)} \notin H_\lambda$.

**Case 2:** $w_1^{(n)} \cdots w_{b_n}^{(n)} \in H_\lambda$ for some $2 \leq i \leq b_n$. This is a contradiction since $w_1^{(n)} \cdots w_{b_n}^{(n)}$ is regular.

**Case 3:** $v_1^{(n)} \cdots v_M^{(n)} u_n \in H_\lambda$. In this case we also have $v_1^{(n)} \cdots v_M^{(n)} \in H_\lambda$.

Using again the assumption that $v_1^{(n)} \cdots v_M^{(n)}$ is regular, it follows that $a_n = M$ and $v_1^{(n)} \in H_\lambda$, which contradicts our assumption that $v_1^{(n)} \notin H_\lambda$.

Thus we see that $w_1^{(n)}$ is indeed an isolated $\lambda$-syllable in $[q_n]$. Moreover we have $\|q_n\| \leq M + N + 1$ for every $n \in \mathbb{N}$. From Lemma 2.9 it therefore follows that $\{ w_1^{(n)} | n \in \mathbb{N} \}$ is a finite subset of $G$. On the other hand, the alternating growth condition ensures that $|w_1^{(n)}|_X \geq n$ for every $n \in \mathbb{N}$. This finally gives us the contradiction that arose from our assumption that there is no sequence $(s_i)_{i \in \mathbb{N}}$ as in the first case of the lemma.

Let us now prove case 2) of the Lemma. From the alternating growth condition we know that $|w_1^{(n)}|_X \geq n$ for every $n \in \mathbb{N}$. Thus we can choose strictly increasing sequences of natural numbers $(s_n)_{n \in \mathbb{N}}, (t_n)_{n \in \mathbb{N}}$ such that $|w_1^{(n)}| \leq s_n \leq n$ for every $n \in \mathbb{N}$. Note that the conditions (1) - (V) of Definition 2.8 are clearly satisfied for
\[ (v_1^{(s_n)} \cdots v_M^{(s_n)} z^{(n)} w_1^{(t_n)} \cdots w_{b_n}^{(t_n)})_{n \in \mathbb{N}}, \]
where $z^{(n)} \in \tilde{H}_\lambda$ is the element representing $v_1^{(s_n)} w_1^{(t_n)}$. In order to prove the lemma it therefore suffices to show that $v_1^{(s_n)} \cdots v_M^{(s_n)} z^{(n)} w_1^{(t_n)} \cdots w_{b_n}^{(t_n)}$ is regular for all but finitely many $n \in \mathbb{N}$. To see this, let us first consider the subwords $v_1^{(s_n)} \cdots v_M^{(s_n)} z^{(n)}$ and $z^{(n)} w_1^{(t_n)} \cdots w_{b_n}^{(t_n)}$. Suppose that there is some $1 \leq i \leq M - 1$ with $|v_1^{(s_n)} \cdots v_M^{(s_n)} z^{(n)}|_{X \cup H} \leq 1$. Then there are two cases to consider.

**Case 1:** $v_1^{(s_n)} \cdots v_{M-1}^{(s_n)} z^{(n)} \in H_\lambda$. Then we also have $v_1^{(s_n)} \cdots v_{M-1}^{(s_n)} \in H_\lambda$, and since $v_1^{(s_n)} \cdots v_{M-1}^{(s_n)}$ is regular, it follows that $M - 1 = 1$. But then $v_1^{(s_n)}$ and $v_1^{(s_n)} \cdots v_{M-1}^{(s_n)}$ both represent elements of $H_\lambda$, which in turn contradicts the regularity of $v_1^{(s_n)} \cdots v_{M-1}^{(s_n)}$.

**Case 2:** $v_1^{(s_n)} \cdots v_{M-1}^{(s_n)} z^{(n)} \notin H_\lambda$. Then there is some $u_n \in X \cup H$ that does not lie in $H_\lambda$ such that $q_n := v_1^{(s_n)} \cdots v_{M-1}^{(s_n)} z^{(n)} u_n$ represents the identity in $G$. We claim that $z^{(n)}$ is an isolated syllable in the cyclic word $[q_n]$. Indeed, otherwise there would be some $i \leq j \leq M - 1$ with $v_1^{(s_n)} \cdots v_{M-1}^{(s_n)} z^{(n)} \in H_\lambda$, which is impossible as we have seen in Case 1. Moreover we have $\|q_n\| \leq M$. From Lemma 2.9 it therefore follows that $\{ z^{(n)} | n \in \mathbb{N} \}$ is a finite subset of $G$. Since $|z^{(n)}|_X \geq n$, we see that there are only finitely many $n \in \mathbb{N}$ such that $|v_1^{(s_n)} \cdots v_{M-1}^{(s_n)} z^{(n)}|_{X \cup H} \leq 1$ for some $1 \leq i \leq M - 1$. Thus $v_1^{(s_n)} \cdots v_{M-1}^{(s_n)} z^{(n)}$ is regular for all but finitely many $n \in \mathbb{N}$. Symmetric argumentation shows that $z^{(n)} w_1^{(t_n)} \cdots w_{b_n}^{(t_n)}$ is regular for
all but finitely many \( n \in \mathbb{N} \). By restriction to a subsequence if necessary, we can therefore assume that the words \( v_1(s_n) \ldots v_{M-1}^{(s_n)} z(n) w_2^{(t_n)} \ldots w_N^{(t_n)} \) are regular for every \( n \).

Suppose now that \( v_1^{(s_n)} \ldots v_{M-1}^{(s_n)} z(n) w_2^{(t_n)} \ldots w_N^{(t_n)} \) is not regular. Then we can choose \( 1 \leq a_n \leq M - 1 \) and \( 2 \leq b_n \leq N \) such that \( v_1^{(s_n)} \ldots v_{M-1}^{(s_n)} z(n) w_2^{(t_n)} \ldots w_{b_n}^{(t_n)} \) is a minimal subword of \( v_1^{(s_n)} \ldots v_{M-1}^{(s_n)} z(n) w_2^{(t_n)} \ldots w_N^{(t_n)} \) with

\[
|v_1^{(s_n)} \ldots v_{M-1}^{(s_n)} z(n) w_2^{(t_n)} \ldots w_{b_n}^{(t_n)}|_{X \cup \hat{H}} \leq 1.
\]

**Case 1:** The word \( q_n := v_1^{(s_n)} \ldots v_{M-1}^{(s_n)} z(n) w_2^{(t_n)} \ldots w_{b_n}^{(t_n)} \) represents the identity in \( G \).

Since \( v_1^{(s_n)} \ldots v_{M-1}^{(s_n)} z(n) w_2^{(t_n)} \ldots w_N^{(t_n)} \) are regular, it follows that \( z(n) \) is an isolated syllable in the cyclic word \([q_n] \). In view of Lemma 2.9 we see that there are only finitely many such \( n \).

**Case 2:** The word \( v_1^{(s_n)} \ldots v_{M-1}^{(s_n)} z(n) w_2^{(t_n)} \ldots w_{b_n}^{(t_n)} \) does not represent an element of \( H_\lambda \). Then there is some \( u_n \in \bigcup_{\mu \in \Lambda \setminus \{\Lambda\}} (\hat{H}_\lambda \setminus \{1\}) \cup X \) such that

\[
q_n := v_1^{(s_n)} \ldots v_{M-1}^{(s_n)} z(n) w_2^{(t_n)} \ldots w_{b_n}^{(t_n)} u_n
\]

represents the trivial element in \( G \). In particular, \( u_n \) is not part of a \( \lambda \)-syllable in the cyclic word \([q_n] \). Another application of Lemma 2.9 now reveals that there are only finitely many \( n \in \mathbb{N} \) such that \( v_1^{(s_n)} \ldots v_{M-1}^{(s_n)} z(n) w_2^{(t_n)} \ldots w_{b_n}^{(t_n)} \) does not represent an element of \( \hat{H}\Lambda \).

**Case 3:** The word \( v_1^{(s_n)} \ldots v_{M-1}^{(s_n)} z(n) w_2^{(t_n)} \ldots w_{b_n}^{(t_n)} \) represents a non-trivial element in \( H_\lambda \). Then there is some \( u_n \in \overline{H}_\lambda \) such that

\[
q_n := v_1^{(s_n)} \ldots v_{M-1}^{(s_n)} z(n) w_2^{(t_n)} \ldots w_{b_n}^{(t_n)} u_n
\]

represents the identity in \( G \). Suppose that \( z(n) \) is connected to some further \( \lambda \)-syllable in the cyclic word \([q_n] \). Since \( v_1^{(s_n)} \ldots v_{M-1}^{(s_n)} z(n) \) and \( z(n) w_2^{(t_n)} \ldots w_{b_n}^{(t_n)} \) are regular, \( z(n) \) has to be connected to \( u_n \). Hence we obtain \( z(n) w_2^{(t_n)} \ldots w_{b_n}^{(t_n)} u_n \in H_\lambda \), which implies \( w_2^{(t_n)} \ldots w_{b_n}^{(t_n)} \in H_\lambda \). From the regularity of \( z(n) w_2^{(t_n)} \ldots w_{b_n}^{(t_n)} \) it therefore follows that \( N = 2 \). But then \( w_2^{(t_n)} \in H_\lambda \), which contradicts the regularity of \( w_2^{(t_n)} \ldots w_{b_n}^{(t_n)} \). Thus \( u_n \) is an isolated syllable in \([q_n] \) and a final application of Lemma 2.9 proves that Case 3 can only occur finitely many times.

Altogether we have shown that \( v_1^{(s_n)} \ldots v_{M-1}^{(s_n)} z(n) w_2^{(t_n)} \ldots w_{N}^{(t_n)} \) is regular for all but finitely many \( n \in \mathbb{N} \), which proves the lemma.

**Corollary 3.11.** Let \( G \) be a finitely generated group with a finite generating set \( X \). Suppose that \( G \) is relatively finitely presented with respect to a collection of peripheral subgroups \( H_\lambda = \{ H_\lambda | \lambda \in \Lambda \} \) and that the relative Dehn function \( \delta^G_{rel} \) is well-defined. Let \( (w^{(n)})_{n \in \mathbb{N}} \) be a sequence of words over \( X \cup \hat{H} \) that satisfies the alternating growth condition and let \( K \) be the subgroup of \( G \) generated by \( \{ w^{(n)} | n \in \mathbb{N} \} \). Then there is some \( C \in \mathbb{N} \) that satisfies the following. For every \( L \in \mathbb{N} \) there is a sequence of words \( (v_n)_{n \in \mathbb{N}} \) over \( X \cup \hat{H} \) such that:

1. \( (v_n)_{n \in \mathbb{N}} \) satisfies the alternating growth condition.
2. The length of every word \( v_n \) is bounded by \( L \leq \| v_n \| \leq L + C \).
3. Every word \( v_n \) represents an element of \( K \).

**Proof.** Let us write \( w^{(n)} = v_1^{(n)} \ldots v_1^{(n)} \) for every \( n \in \mathbb{N} \). From the properties II) and III) of the alternating growth condition we know that there is some \( \lambda \in \Lambda \) such
that \(w_1^{(n)} \in \overline{H}_\lambda\) for every \(n \in \mathbb{N}\). By restriction to a subsequence if necessary, we may assume that \((w_n)_{n \in \mathbb{N}}\) satisfies one of the following two conditions:

1) \(\overline{w}_k^{(n)} \notin H_\lambda\) for every \(n \in \mathbb{N}\).

2) \(\overline{w}_k^{(n)} \in H_\lambda\) for every \(n \in \mathbb{N}\).

Suppose that the first case is satisfied and let \(k \in \mathbb{N}\). Then an inductive application of the first case of Lemma 3.10 provides us with subsequences

\[w_1^{(s_1,n)} \cdots w_k^{(s_k,n)}\]

of \(w(n)\) for each \(1 \leq i \leq k\) such that the sequence of concatenated words

\[w_n := (w_1^{(s_1,n)} \cdots w_k^{(s_k,n)}), (w_1^{(s_2,n)} \cdots w_k^{(s_k,n)}) \cdots (w_1^{(s_k,n)} \cdots w_k^{(s_k,n)})\]

has length \(k\ell\) and satisfies the alternating growth condition. Thus the corollary is clearly satisfied for \(C = \ell\).

Let us now consider case 2) and let \(k \in \mathbb{N}\). Then an inductive application of the second case of Lemma 3.10 provides us with subsequences

\[w_1^{(s_1,n)} \cdots w_k^{(s_k,n)}\]

of \(w(n)\) for each \(1 \leq i \leq k\) such that the sequence of words \(v_n\) given by

\[(w_1^{(s_1,n)} \cdots w_k^{(s_k,n)}) \cdot \cdots \cdot (w_1^{(s_k,n)} \cdots w_k^{(s_k,n)})\]

where \(z(t_i,n) \in \overline{H}_\lambda\) is the element representing \(w_k^{(s_k,n)}w_1^{(s_1,n)} \in H_\lambda\), satisfies the alternating growth condition. In this case \(v_n\) has length \(k\ell - 1\) + 1 and we see that the corollary is satisfied for \(C = \ell\). \(\square\)

4. Dichotomy of infinite subgroups

Endowed with Corollary 3.11 we are now ready to study the subgroup of a relatively finitely presented group \(G\) that is generated by all the elements \(\overline{w}_n\), where \((w_n)_{n \in \mathbb{N}}\) is a sequence that satisfies the alternating growth condition.

**Lemma 4.1.** Let \(G\) be a finitely generated group with a finite generating set \(X\). Suppose that \(G\) is relatively finitely presented with respect to a collection of peripheral subgroups \(H_\lambda = \{ H_\lambda \mid \lambda \in \Lambda \}\) and that the relative Dehn function \(\delta_{G/H_\lambda}\) is well-defined. Suppose that \((w_n)_{n \in \mathbb{N}}\) is a sequence of words over \(X \cup H\) that satisfies the alternating growth condition. Then the subgroup \(K \leq G\) generated by \(\{ \overline{w}_n \in G \mid n \in \mathbb{N} \}\) is unbounded with respect to \(d_{X \cup H}\).

**Proof.** Suppose that \(K\) is bounded with respect to \(d_{X \cup H}\), i.e. that there is some \(N \in \mathbb{N}\) with \(|k|_{X \cup H} \leq N\) for every \(k \in K\). Due to Corollary 3.11 there is a number \(L \geq 4N\) and a sequence \((v_n)_{n \in \mathbb{N}}\) of words \(v_n = v_1^{(n)} \cdots v_L^{(n)}\) over \(X \cup H\) that satisfies the alternating growth condition such that each \(v_n\) represents an element of \(K\). By restriction to a subsequence, we can assume that there is some \(M \in \mathbb{N}\) with \(|v_n|_{X \cup H} = M \leq N\) for every \(n \in \mathbb{N}\). Let \(u_1^{(n)} \cdots u_M^{(n)}\) be a shortest word over \(X \cup H\) representing \(\overline{w}_n^{-1}\). Then each word \(g_n := v_1^{(n)} \cdots v_L^{(n)} u_1^{(n)} \cdots u_M^{(n)}\) represents the identity in \(G\). Recall that the alternating growth condition ensures that \(v_1^{(n)} \cdots v_L^{(n)}\) is regular and that two consecutive letters of \(v_n\) do not lie in \(X\). It therefore follows that at least every second of its letters is an isolated syllable in \(v_n\). Thus there are at least \(2N\) isolated syllables in \(v_n = v_1^{(n)} \cdots v_L^{(n)}\). Note that for every \(\lambda \in \Lambda\) and every \(\lambda\)-syllable of \(u_1^{(n)} \cdots u_M^{(n)}\) which necessarily consists of a single letter \(u_i^{(n)}\), there is at most one \(\lambda\)-syllable \(v_j^{(n)}\) in \(v_1^{(n)} \cdots v_L^{(n)}\) that is connected to \(u_i^{(n)}\) in the cyclic word \([g_n]\). Indeed, otherwise there would be a connection between two different isolated \(\lambda\)-syllables of \(v_1^{(n)} \cdots v_L^{(n)}\) by a \(\lambda\)-word. This implies that there
are at least $2N - M \geq N$ isolated syllables in $[g_n]$ that become arbitrarily large
with respect to $X$ as $n$ goes to $\infty$. But this is a contradiction to Lemma 2.20 since
$\|g_n\| \leq M + L$ for every $n \in \mathbb{N}$. Thus it follows that $K$ is an unbounded subset of
$\Gamma(G, X \cup H)$.

\[ \square \]

**Lemma 4.2.** Let $G$ be a finitely generated group with a finite generating set $X$.
Suppose that $G$ is relatively finitely presented with respect to a collection of
peripheral subgroups $H_\Lambda = \{ H_\lambda \mid \lambda \in \Lambda \}$ and that the relative Dehn function
$\delta_{G,H_\Lambda}^{rel}$ is well-defined. Let $K \leq G$ be an infinite subgroup that is bounded with
respect to $d_{X \cup H}$. Then there is an element $g \in G$ and an index $\eta \in \Lambda$ such that
$|gKg^{-1} \cap H_\eta| = \infty$.

**Proof.** Since $K$ is bounded with respect to $d_{X \cup H}$, each of its conjugates $gKg^{-1}$
is a bounded subset of $\Gamma(G, X \cup H)$. Let $m \in \mathbb{N}$ be minimal with the following property:

\[ (*) \] There is a conjugate $H := gKg^{-1}$ of $K$, a finite relative generating set $Y$ of
$G$, and an infinite sequence $(k_n)_{n \in \mathbb{N}}$ of pairwise distinct elements $k_n \in H$ with
$|k_n Y \cup H| = m$ for every $n \in \mathbb{N}$.

Let $g, Y$ and $(k_n)_{n \in \mathbb{N}}$ be as in $(*)$. For each $n$ let $u^{(n)} = u_1^{(n)} \cdots u_m^{(n)}$ be a
(shortest) word over $Y \cup H$ that represents $k_n$. Due to the minimality of $m$, we can
extend $Y$ to any finite relative generating set $Y'$ of $G$ such that $(*)$ is still satisfied
for an appropriate subsequence of $(k_n)_{n \in \mathbb{N}}$. Since $G$ is finitely generated, we can
therefore assume that $Y$ is a symmetric generating set of $G$.

Suppose first that $m = 1$. Then $u_1^{(n)} \in H = \bigcup \{ H_\lambda \mid \lambda \in \Lambda \}$ for all but finitely
many $n \in \mathbb{N}$. Since $\Lambda$ is finite by Theorem 2.24, there is some $\eta \in \Lambda$ such that
infinitely many pairwise distinct letters $u_1^{(n)}$ lie in $H_\eta$. It therefore follows that
$|gKg^{-1} \cap H_\eta| = \infty$.

Let us now consider the case $m \geq 2$. We want to modify $Y$ and $u^{(n)}$ in such a
way that some subsequence of $(u^{(n)})_{n \in \mathbb{N}}$ satisfies the alternating growth condition.
This will be done inductively by going through the letters $u_1^{(n)}$ of $u^{(n)}$.

Suppose that $u_1^{(n)} \in Y$ for infinitely many $n \in \mathbb{N}$. Then we can choose some
$x_1 \in Y$ and a subsequence $(k_{j_n})_{n \in \mathbb{N}}$ of $(k_n)_{n \in \mathbb{N}}$ with $u_1^{(j_n)} = x_1$ for every $n$. In this
case we replace $(k_n)_{n \in \mathbb{N}}$ by $(k_{j_n})_{n \in \mathbb{N}}$.

Suppose next that $u_1^{(n)} \in H$ for all but finitely many $n \in \mathbb{N}$. Since $\Lambda$ is finite,
there is some $\lambda_1 \in \Lambda$ with $u_1^{(n)} \in H_{\lambda_1}$ for infinitely many $n \in \mathbb{N}$. We have to
consider 2 cases.

\textbf{Case 1:} There is some $\tilde{h}_1 \in H_{\lambda_1}$ with $u_1^{(n)} = \tilde{h}_1$ for infinitely many $n \in \mathbb{N}$.
Then we restrict $(k_n)_{n \in \mathbb{N}}$ to a subsequence $(k_{j_n})_{n \in \mathbb{N}}$ such that $u_1^{(j_n)} = \tilde{h}_1$ for every
$n \in \mathbb{N}$. Moreover we add $\tilde{h}_1$ and $\tilde{h}_1^{-1}$ to $Y$ and replace the letter $u_1^{(j_n)} \in H_{\lambda_1}$ in
$u^{(j_n)}$ by $\tilde{h}_1 \in Y$ for every $n \in \mathbb{N}$. Next we replace the resulting sequence by a
subsequence that satisfies $(*)$, which is possible by the choice of $m$.

\textbf{Case 2:} There is no $\tilde{h}_1 \in H_{\lambda_1}$ with $u_1^{(n)} = \tilde{h}_1$ for infinitely many $n \in \mathbb{N}$. In this case we replace $(u^{(n)})_{n \in \mathbb{N}}$ by a subsequence $(u^{(j_n)})_{n \in \mathbb{N}}$ such that $|u_1^{(j_n)}| > n$ for every
$n \in \mathbb{N}$.

We proceed analogously with the other indices $i \in \{2, \ldots, m\}$. The resulting
sequence of words over $Y \cup H$ will be denoted by $(v_1^{(n)} \cdots v_m^{(n)})_{n \in \mathbb{N}}$. Let $g_n \in H$ be
the element represented by $v_1^{(n)} \cdots v_m^{(n)}$. 

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Suppose that either two consecutive letters \( v_i^{(n)}, v_{i+1}^{(n)} \) or the letters \( v_1^{(n)}, v_m^{(n)} \) both lie in \( Y \). Then we could add \( v_i^{(n)} v_{i+1}^{(n)} \) and \( (v_i^{(n)} v_{i+1}^{(n)})^{-1} \) (respectively \( v_m^{(n)} v_1^{(n)} \) and \( (v_m^{(n)} v_1^{(n)})^{-1} \)) to \( Y \) in order to obtain a shorter sequence of infinitely many pairwise distinct elements of \( H \) (respectively of \( v_i^{(n)} H v_1^{(n)} \)) with respect to \( d_Y \). But this is a contradiction to the choice of \( m \). Thus it follows that neither \( v_i^{(n)}, v_{i+1}^{(n)} \) nor \( v_1^{(n)}, v_m^{(n)} \) both lie in \( Y \). In particular we see that we can replace \( v_1^{(n)} \cdots v_m^{(n)} \) by its inverse \( (v_m^{(n)})^{-1} \cdots (v_1^{(n)})^{-1} \) to ensure that the first letter does not lie in \( Y \).

Let us therefore assume that \( v_1^{(n)} \) is never contained in \( Y \). In order to prove that \( (v_1^{(n)} \cdots v_m^{(n)})_{n \in \mathbb{N}} \) satisfies the alternating growth condition, it remains to show that each \( v_1^{(n)} \cdots v_m^{(n)} \) is regular and that two consecutive letters \( v_1^{(n)}, v_1^{(n)} \) cannot lie in the same group \( H \). But these properties are direct consequences of the condition \( [g_n : Y \cup H] = m \) from (1), where \( k_n \) plays the role of \( g_n \). Altogether we have shown that there is a conjugate \( H \) of \( K \) and a sequence \( (g_n)_{n \in \mathbb{N}} \) of elements in \( H \), that can be represented by a sequence \( (v_1^{(n)} \cdots v_m^{(n)})_{n \in \mathbb{N}} \) of words over \( Y \cup H \) that satisfies the alternating growth condition. In this case Lemma 4.1 tells us that \( H \) is an unbounded subset of \( \Gamma(G, Y \cup H) \), which clearly contradicts our assumption that \( K \) is a bounded subset of \( \Gamma(G, X \cup H) \). Hence \( m = 1 \), in which case we have already proven the lemma.

We are now ready to prove our main theorem.

**Theorem 4.3.** Let \( G \) be a finitely generated group and let \( X \) be a finite generating set of \( G \). Suppose that \( G \) is relatively finitely presented with respect to a collection of peripheral subgroups \( H_\lambda = \{ H_\lambda \mid \lambda \in \Lambda \} \) and that the relative Dehn function \( \delta_G^r : H_\Lambda \) is well-defined. Then every subgroup \( K \leq G \) satisfies exactly one of the following conditions:

1) \( K \) is finite.
2) \( K \) is infinite and conjugated to a subgroup of a peripheral subgroup.
3) \( K \) is unbounded in \( \Gamma(G, X \cup H) \).

**Proof.** Suppose that \( K \) is infinite and bounded as a subset of \( \Gamma(G, X \cup H) \). From Lemma 4.2 we know that there is an index \( \eta \in \Lambda \) and an element \( g \in G \) such that the \( gKg^{-1} \cap H_\eta \) is infinite. We can therefore choose a sequence \( (h_n)_{n \in \mathbb{N}} \) of distinct, non-trivial elements \( h_n \in gKg^{-1} \cap H_\eta \). Suppose that \( gKg^{-1} \) is not a subgroup of \( H_\eta \) and let \( a \in gKg^{-1} \setminus H_\eta \). Let \( h_n \in H_\eta \) be the element representing \( h_n \). Then, after adding \( \{a, a^{-1}\} \) to \( X \) if necessary, we can consider the sequence of words \( (h_n a)_{n \in \mathbb{N}} \) over \( X \cup H \). We claim that \( (h_n a)_{n \in \mathbb{N}} \) has a subsequence that satisfies the alternating growth condition. The only property that is not directly evident is that \( (h_n a)_{n \in \mathbb{N}} \) has a subsequence consisting of regular words. Suppose that this is not the case. Since \( \Lambda \) is finite by Theorem 2.2 it then follows that there is some \( \mu \in \Lambda \) such that \( h_n a \) represents an element in \( H_\mu \) for infinitely many \( n \in \mathbb{N} \). Then \( h_n a (h_n a)^{-1} = \tilde{h}_n \tilde{h}_n^{-1} \) represents an element in \( H_\mu \cap H_\eta \) whenever \( \tilde{h}_n \) and \( \tilde{h}_n a \) both represent element of \( H_\mu \). Since \( a \) was chosen outside of \( H_\eta \), it moreover follows that \( \tilde{h}_n a \) can never represent an element of \( H_\eta \). In particular we see that \( \eta \neq \mu \). But this is a contradiction to Proposition 2.36 which says that \( H_\mu \cap H_\eta \) is finite for \( \mu \neq \eta \). Thus \( (h_n a)_{n \in \mathbb{N}} \) has a subsequence that satisfies the alternating growth condition. In this case Lemma 4.1 tells us that the subgroup \( \{a h_n \mid n \in \mathbb{N}\} \) of \( gKg^{-1} \) is an unbounded in \( \Gamma(G, X \cup H) \), which is contradicts our assumption that \( K \) is bounded in \( \Gamma(G, X \cup H) \). Finally this proves that \( gKg^{-1} \) is a subgroup of \( H_\eta \).
Let us now consider the important special case of Theorem 1.2, where $G$ is relatively hyperbolic with respect to $H$. Recall that an element $g \in G$ is called loxodromic if the map $\mathbb{Z} \to \Gamma(G, X \cup \mathcal{H})$, $n \mapsto g^n$ is a quasisisometrical embedding. It is known that a subgroup $K \leq G$ with infinite diameter in $\Gamma(G, X \cup \mathcal{H})$ contains a loxodromic element. This follows from a corresponding result for acylindrically hyperbolic groups [13, Theorem 1.1] and the fact that relatively hyperbolic groups act acylindrically on the (hyperbolic) graph $\Gamma(G, X \cup \mathcal{H})$ [13, Proposition 5.2].

**Corollary 4.4.** Let $G$ be a finitely generated group. Suppose that $G$ is relatively hyperbolic with respect to a collection $\mathcal{H} = \{ H_\lambda \mid \lambda \in \Lambda \}$ of its subgroups. Then every subgroup $K \leq G$ satisfies exactly one of the following conditions:

1) $K$ is finite.

2) $K$ is infinite and conjugate to a subgroup of some $H_\lambda$.

3) $K$ contains a loxodromic element.

5. Applications

As an application of the classification of subgroups of relatively hyperbolic groups given in Corollary 4.4, we prove the existence of the relative exponential growth rate for all subgroups of a large variety of relatively hyperbolic groups.

**Definition 5.1.** Let $G$ be a finitely generated group and let $X$ be a finite generating set of $G$. Given a subgroup $H \leq G$, we define the relative growth function of $H$ in $G$ with respect to $X$ by

$$\beta^X_H : \mathbb{N} \to \mathbb{N}, \ n \mapsto |B^X_H(n)|,$$

where $B^X_H(n)$ denotes the set of elements in $H$ that are represented by words of length at most $n$ over $X \cup X^{-1}$. The relative exponential growth rate of $H$ in $G$ with respect to $X$ is defined by $\limsup_{n \to \infty} \sqrt[n]{\beta^X_H(n)}$.

It is natural to ask whether $\limsup$ can be replaced by $\lim$, i.e. whether the limit $\lim_{n \to \infty} \sqrt[n]{\beta^X_H(n)}$ exists. Unlike in the important special case $H = G$, in which this limit is well-known to exist (see e.g. [11]), it does now exist in general (see [12, Remark 3.1]). In the case where the limit $\lim_{n \to \infty} \sqrt[n]{\beta^X_H(n)}$ does exist, we will say that the relative exponential growth rate of $H$ in $G$ exists with respect to $X$.

The following result provides us with a large variety of finitely generated relatively hyperbolic groups $G$ for which the relative exponential growth rate exists for every of its subgroups and generating sets.

**Theorem 5.2.** Let $G$ be a finitely generated group that is relatively hyperbolic with respect to a collection $\mathcal{H} = \{ H_\lambda \mid \lambda \in \Lambda \}$ of its subgroups. Suppose that each of the groups $H_\lambda$ has subexponential growth. Then the relative exponential growth rate of every subgroup $K \leq G$ exists with respect to every finite generating set of $G$.

**Proof.** Let $X$ be a finite generating set of $G$. We go through the 3 cases of Corollary 4.4.

Suppose first that $K$ is finite. Then $\beta^X_K$ is eventually constant and it trivially follows that $\lim_{n \to \infty} \sqrt[n]{\beta^X_K(n)}$ exists and is equal to 1.

Let us next consider the case where $K$ contains a loxodromic element $k$. By [13, Proposition 5.2], $G$ acts acylindrically on the (hyperbolic) graph $\Gamma(G, X \cup \mathcal{H})$. It this case [13, Theorem 1.1] tells us that either $G$ is virtually cyclic, in which case
the claim follows trivially, or $G$ is acylindrically hyperbolic, in which case the claim is covered by [15] Theorem 5.8.

Consider now the remaining case, where $K$ is infinite and conjugated to a subgroup of some peripheral subgroup. Thus there is some $g \in G$ and some $\lambda \in \Lambda$ such that $K \leq gH_\lambda g^{-1}$. By Theorem 2.8 each $H_\lambda$, and hence $gH_\lambda g^{-1}$, is finitely generated. We can therefore choose a finite generating set $Y$ of $gH_\lambda g^{-1}$. Moreover it follows from [14, Lemma 5.4] that each peripheral subgroup, and hence $gH_\lambda g^{-1}$, is undistorted in $G$. We can therefore choose a constant $C > 0$ such that

$$
\beta_K^X(n) \leq \beta_Y^{gH_\lambda g^{-1}}(Cn)
$$

for every $n \in \mathbb{N}$. By assumption each $H_\lambda$, and therefore $gH_\lambda g^{-1}$, has subexponential growth. Thus we have $\lim_{n \to \infty} \frac{\beta_K^X(n)}{a^n} = 0$ for every $a > 1$. In view of (6), this implies that

$$
\lim_{n \to \infty} \frac{\beta_K^X(n)}{a^n} = 0.
$$

Since $\beta_K^X(n) \leq \beta_Y^{gH_\lambda g^{-1}}(n)$ for $n \in \mathbb{N}$, we see that $\lim_{n \to \infty} \sqrt[n]{\beta_K^X(n)} = 1$ and in particular that the limit exists. □

References

1. Emina Alibegović, A combination theorem for relatively hyperbolic groups, Bull. London Math. Soc. 37 (2005), no. 3, 459–466. MR 2131490
2. B. H. Bowditch, Relatively hyperbolic groups, Internat. J. Algebra Comput. 22 (2012), no. 3, 1250016, 66. MR 2922380
3. Joel M. Cohen, Cogrowth and amenability of discrete groups, J. Funct. Anal. 48 (1982), no. 3, 301–309. MR 678175
4. Rémi Coulon, Françoise Dal’Bo, and Andrea Sambusetti, Growth gap in hyperbolic groups and amenability, Geom. Funct. Anal. 28 (2018), no. 5, 1260–1320. MR 3856793
5. François Dahmani, Combination of convergence groups, Geom. Topol. 7 (2003), 933–963. MR 2026551
6. François Dahmani, David Futer, and Daniel T. Wise, Growth of quasiconvex subgroups, Math. Proc. Cambridge Philos. Soc. 167 (2019), no. 3, 505–530. MR 4015649
7. B. Farb, Relatively hyperbolic groups, Geom. Funct. Anal. 8 (1998), no. 5, 810–840. MR 1650094
8. R. I. Grigorchuk, Symmetrical random walks on discrete groups, Multicomponent random systems, Adv. Probab. Related Topics, vol. 6, Dekker, New York, 1980, pp. 285–325. MR 599539
9. M. Gromov, Hyperbolic groups, Essays in group theory, Math. Sci. Res. Inst. Publ., vol. 8, Springer, New York, 1987, pp. 75–263. MR 919829
10. Sam Hughes, Eduardo Martínes-Pedroza, and Luis Jorge Sánchez Saldaña, Quasi-isometry invariance of relative filling functions, arXiv preprint arXiv:2107.03355 (2021).
11. J. Milnor, A note on curvature and fundamental group, J. Differential Geometry 2 (1968), 1–7. MR 0232311
12. A. Yu. Olshanskii, Subnormal subgroups in free groups, their growth and cogrowth, Math. Proc. Cambridge Philos. Soc. 163 (2017), no. 3, 499–531. MR 3708520
13. D. Osin, Acylindrically hyperbolic groups, Trans. Amer. Math. Soc. 368 (2016), no. 2, 851–888. MR 3493052
14. Denis V. Osin, Relatively hyperbolic groups: intrinsic geometry, algebraic properties, and algorithmic problems, Mem. Amer. Math. Soc. 179 (2006), no. 843, vi+100. MR 2182266
15. Eduard Schesler, The relative exponential growth rate of subgroups of acylindrically hyperbolic groups, J. Group Theory 25 (2022), no. 2, 293–326. MR 4388363
16. Zlil Sela, Diophantine geometry over groups i: Makanin-razborov diagrams, Publications Mathématiques de l’IHÉS 93 (2001), 31–105 (en). MR 1863735
17. Richard Sharp, Relative growth series in some hyperbolic groups, Math. Ann. 312 (1998), no. 1, 125–132. MR 1645953

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