VOLUME RIGIDITY ON LIMIT SPACES WITH RICCI CURVATURE BOUNDED FROM BELOW

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INTRODUCTION

Let $M(n,\kappa,v)$ be the collection of $n$-dimensional Riemannian manifolds $M$ with Ricci curvature bounded from below by $-(n-1)\kappa$ and $\text{vol}(B_1(p)) \geq v$ for all $p \in M$. By Cheeger-Gromov Compactness Theorem, $M(n,\kappa,v)$ is pre-compact in the pointed Gromov-Hausdorff topology. Let $M_\infty(n,\kappa,v)$ be the closure of $M(n,\kappa,v)$.

We let “vol” denote the $n$-dimensional Hausdorff measure. It has been proved in [9] and [4] that $\text{vol} \cdot$ is a continuous function over balls respect to their Hausdorff distance. T. Colding [8] proved that an $n$-dimensional Riemannian manifold with $\text{Ricci} \geq (n-1)\kappa$ is Hausdorff close to the unit sphere $S^n$ if and only if its volume is close to the volume of $S^n$. In this paper, we prove a geometric rigidity theorem respect to the volume and the existence of a 1-Lipschitz function.

**Theorem A** (Volume Rigidity). Let $X,Y \in M_\infty(n,\kappa,v)$. Suppose that there is a 1-Lipschitz map $f:X \to Y$. If $\text{vol}(X) = \text{vol}(f(X))$, then $f$ is an isometry respect to the intrinsic metrics of $X$ and $f(X)$. In particular, if $f$ is also onto, then $Y$ is isometric to $X$.

Together with the theorem of volume continuity, we get the following.

**Corollary 0.1.** Let $Y \in M_\infty(n,\kappa,v)$. For any $\epsilon > 0$, there is $\delta = \delta(Y) > 0$ such that for any $X \in M_\infty(n,\kappa,v)$, if

1. $|\text{vol}(X) - \text{vol}(Y)| < \delta$,
2. there is a map $\phi:X \to Y$ such that $Y \subset B_\delta(f(X))$ and for all $x,y \in X$,

$$|\phi(x)\phi(y)|_Y \leq |xy|_X + \delta,$$

then $\phi$ is an $\epsilon$-Gromov-Hausdorff approximation. In particular, $d_{GH}(X,Y) < 3\epsilon$.

Theorem A fails for general length metric spaces, partially because a lemma of dimension control (Lemma 1.6) fails in these cases. For example, $Y$ can be the quotient space of $X$ with any lower dimensional subset identified as one point. See [10] for more examples. A special case of Theorem A was proved by Bessières, Besson, Courtois, and Gallot in [2]. Similar results in Alexandrov Geometry [1] have been proved in [10].

**Conventions and notations**

- $d_{GH}(X,Y)$: the Gromov-Hausdorff distance between $X$ and $Y$.
- $\dim_H(X)$: the Hausdorff dimension of $X$.
- $B^n_r(0)$: the $n$-dimensional Euclidean ball with radius $r$. 
• $\mathcal{R}(X)$: the regular set in $X$, see [4].
• $\mathcal{S}(X)$: the singular set in $X$, see [4].
• $[ab]_X$: a length minimizing geodesic connecting $a$ and $b$ in $X$.
• $|ab|_X$: the distance of $a$ and $b$ respect to the intrinsic metric of $X$.
• $\psi(\epsilon)$: a function (could be different even in the same context) depending only on $n, \kappa, v, \epsilon$ that satisfies $\lim_{\epsilon \to 0} \psi(\epsilon) = 0$.

1. Volume Rigidity Theorem

Not losing generality, we assume that $f$ is onto and $\text{diam}(Y) \leq \text{diam}(X) \leq D$, since our proof only relies on the local structures. For simplicity, we only consider $X \in \mathcal{M}_{\infty}(n, -1, v)$. By the assumption, it’s not hard to see that for any subset $A \subset Y$, $\text{vol}(A) = \text{vol}(f^{-1}(A))$. One of the key step in our proof is showing that $f$ is injective (Lemma 1.7). We first prove this with $f$ restricted to the regular part. Let $\mathcal{R}_{\epsilon, \delta}(X) = \{p \in X : d_{GH}(B_r(p), B_r^n(0)) < \epsilon r \text{ for all } 0 < r < \delta\}$ and $\mathcal{R}_\epsilon(X) = \bigcup_{\delta} \mathcal{R}_{\epsilon, \delta}(X)$ be the $\epsilon$-regular set. By the volume continuity, we know that for any $x \in \mathcal{R}_{\epsilon, \delta}$ and $r < \delta$,

$$(1 + \psi(\epsilon)) \cdot \text{vol}(B_r(x)) = \text{vol}(B_r^n(0)) = \text{vol}(S_{1}^{n-1}) \int_{0}^{\delta} t^{n-1} \, dt$$

$$= 2r \cdot \text{vol}(B_r^{n-1}(0)) \int_{0}^{\frac{\theta}{2}} \sin^n(t) \, dt.$$

Lemma 1.1. $f(\mathcal{R}_{\epsilon, \delta}(X)) \subset \mathcal{R}_{\psi(\epsilon), \delta/10}(Y)$. Consequently, $f(\mathcal{R}_\epsilon(X)) \subset \mathcal{R}_{\psi(\epsilon)}(Y)$ and $f(\mathcal{R}(X)) \subset \mathcal{R}(Y)$.

Proof. Let $x \in \mathcal{R}_{\epsilon, \delta}(X)$ and $y = f(x)$. Apply the volume formula for $B_{\delta/10}(x)$. Because $f$ is volume preserving and $f^{-1}((B_{\delta/10}(y)) \supset B_{\delta/10}(x)$, we have the following volume comparison:

$$\text{vol}(B_{\delta/10}(y)) = \text{vol}(f^{-1}(B_{\delta/10}(y)))$$

$$\geq \text{vol}(B_{\delta/10}(x)) = (1 + \psi(\epsilon)) \cdot \text{vol}(B_{\delta/10}^{n}(0)).$$

By the almost maximum volume theorem [7], $y \in R_{\psi(\epsilon), \delta/10}(Y)$. \hfill $\square$

Let $G_Y = \{y \in Y : f^{-1}(y) \text{ has a cardinality of more than 1}\}$ and $G_X = f^{-1}(G_Y) \subset X$.

Lemma 1.2. There is an $\epsilon = \epsilon(n, v) > 0$ such that $\mathcal{R}_{\epsilon}(X) \cap G_X = \emptyset$.

Proof. Argue by contradiction. Assume $x_1 \in R_{\epsilon, \delta_0}(X)$, $x_2 \in X$ and $f(x_1) = f(x_2) = y$. By Lemma 1.1 $y \in R_{\psi(\epsilon), \delta_0/10}(Y)$. Let $0 < \delta < \delta_0/10$ be small such that $B_\delta(x_1) \cap B_\delta(x_2) = \emptyset$. By the volume continuity, Bishop-Gromov Relative Volume Comparison holds on $X$ and $Y$. Thus
we have

\[
1 = \frac{\text{vol} \left( f^{-1}(B_\delta(y)) \right)}{\text{vol}(B_\delta(y))} \geq \frac{\text{vol}(B_\delta(x_1)) + \text{vol}(B_\delta(x_2))}{\text{vol}(B_\delta(y))} \\
\geq \frac{\text{vol}(B_\delta(x_1)) + v \cdot \int_0^\delta \sinh^{n-1}(t) \, dt}{\text{vol}(B_\delta(y))} \\
\geq \frac{(1 + \psi(\epsilon)) \cdot \text{vol} \left( S_1^{n-1} \right) \cdot \int_0^\delta t^{n-1} \, dt + v \cdot \int_0^\delta \sinh^{n-1}(t) \, dt}{(1 + \psi(\epsilon)) \cdot \text{vol} \left( S_1^{n-1} \right)}.
\]

Let \( \delta \to 0 \), we get

\[
1 \geq \frac{(1 + \psi(\epsilon)) \cdot \text{vol} \left( S_1^{n-1} \right) + \int_0^\delta t^{n-1} \, dt}{(1 + \psi(\epsilon)) \cdot \text{vol} \left( S_1^{n-1} \right)}.
\]

This is a contradiction for \( \epsilon > 0 \) sufficiently small. \( \square \)

In the next step, we prove that \( f \) is almost isometry when restricted to the regular part. We need a volume formula for the union of two balls, which follows by the volume continuity and direct computations in Euclidean space.

**Lemma 1.3.** For any \( x_1, x_2 \in \mathcal{R}_{\epsilon, \delta} \) with \( |x_1 x_2| \leq 2r < \delta/5 \),

\[
(1 + \psi(\epsilon)) \cdot \text{vol}(B_r(x_1) \cup B_r(x_2)) = \text{vol}(B_r^n(0)) + 2r \cdot \text{vol}(B_r^{n-1}(0)) \int_0^{\theta} \sin^n(t) \, dt,
\]

where \( \theta = \cos^{-1} \left( \frac{|x_1 x_2|}{2r} \right) \).

Now we can prove that \( f|_{\mathcal{R}_{\epsilon, \delta}} \) is locally almost isometry and the proof is similar as in [10]. We include it here for the convenience to the readers.

**Lemma 1.4.** There are \( \epsilon, \delta > 0 \) sufficiently small so that if \( y_1, y_2 \in f(\mathcal{R}_{\epsilon, \delta}(X)) \) with \( |y_1 y_2| < \delta/20 \), then

\[
|f^{-1}(y_1)f^{-1}(y_2)|_X < (1 + \psi(\epsilon)) \cdot |y_1 y_2|_Y.
\]

**Proof.** Let \( |f^{-1}(y_1)f^{-1}(y_2)|_X = \lambda \cdot |y_1 y_2|_Y \). Consider the metric balls \( B_r(y_1) \) and \( B_r(y_2) \). Take \( r = \frac{1}{2} \lambda \cdot |y_1 y_2|_Y \) and assume that \( r < \delta/10 \). By the volume formula in Lemma 1.3,

\[
(1 + \psi(\epsilon)) \cdot \text{vol}(B_r(y_1) \cup B_r(y_2)) = \text{vol}(B_r^n(0)) + 2r \cdot \text{vol}(B_r^{n-1}(0)) \int_0^{\pi/2} \sin^n(t) \, dt = 2r \cdot \text{vol}(B_r^{n-1}(0)) \int_0^{\pi/2} \sin^n(t) \, dt + 2r \cdot \text{vol}(B_r^{n-1}(0)) \int_0^{\pi/2} \sin^n(t) \, dt,
\]
where $\theta = \cos^{-1}\left(\frac{|y_1y_2|}{2r}\right) = \cos^{-1}(1/\lambda)$. Note that $B_r(f^{-1}(y_1)) \cap B_r(f^{-1}(y_2)) = \emptyset$. We have

$$(1 + \psi(\epsilon)) \cdot \text{vol} \left( B_r(f^{-1}(y_1)) \cup B_r(f^{-1}(y_2)) \right) = 2 \text{vol} (B^n_r(0)) = 4r \cdot \text{vol} \left( B^{n-1}_r(0) \right) \int_0^{\pi/2} \sin^n(t) \, dt.$$  

Because $f$ is 1-Lipschitz, we have $f^{-1}(B_r(y_1) \cup B_r(y_2)) \supseteq B_r(f^{-1}(y_1)) \cup B_r(f^{-1}(y_2))$. Together with that $f$ is volume preserving, we get

$$1 = \frac{\text{vol} \left( f^{-1}(B_r(y_1) \cup B_r(y_2)) \right)}{\text{vol} (B_r(y_1) \cup B_r(y_2))} \geq \frac{\text{vol} \left( B_r(f^{-1}(y_1)) \cup B_r(f^{-1}(y_2)) \right)}{\text{vol} (B_r(y_1) \cup B_r(y_2))}$$

$$(1.1) \quad = (1 - \psi(\epsilon)) \frac{2 \int_0^{\pi/2} \sin^n(t) \, dt}{\int_0^{\pi/2} \sin^n(t) \, dt + \int_0^{\pi/2} \sin^n(t) \, dt}. $$

We claim that $\lambda \leq 2$. If this is not true, we repeat the above calculation with $r = |y_1y_2|$. In this case $\theta = \frac{\pi}{4}$, which yields a contraction when $\epsilon$ is small. Once the claim is proved, the assumption $r < \delta/10$ automatically holds and then inequality (1.1) holds for all $|y_1y_2| < \delta/20$. This implies that $0 < \theta < \psi(\epsilon)$ and thus $\lambda = \frac{1}{\cos \theta} < 1 + \psi(\epsilon)$. \hfill $\Box$

To prove that $f$ almost preserves the length of path for any curve $\gamma \subset R_\epsilon(X)$, we need the existence of $\delta_0 > 0$ so that $\gamma \subset R_{\epsilon,\delta_0}(X)$. Note that $R_{\epsilon,\delta}(X)$ may not be open, but by the continuity of volume and the rigidity of almost maximal volume, we get that for any $\epsilon, \delta > 0$ small, there is $\epsilon_1 = \psi(\epsilon_1) < \epsilon$ so that $R_{\epsilon,\delta}(X) \subset \overset{\circ}{R}_{\epsilon_1,\delta/2}(X)$. Thus

$$R_{\epsilon}(X) = \cup_\delta R_{\epsilon,\delta}(X) \subset \cup_\delta \overset{\circ}{R}_{\epsilon_1,\delta/2}(X).$$

If a compact set $A \subset R_\epsilon(X)$, then there is $\delta_0 > 0$ such that $A \subset \overset{\circ}{R}_{\psi(\epsilon),\delta_0}(X)$. The following are direct consequences of Lemma 1.1–1.4

**Lemma 1.5 (Almost Isometry over $R_\epsilon(X)$).** There is $\epsilon > 0$ small so that the following holds.

1. If $[pq]_Y \subset f(R_\epsilon(X))$, then $\gamma = f^{-1}([ab]_Y)$, parameterized by arc length, is a Lipschitz curve with

$$L(\gamma_{|t_1,t_2}) < (1 + \psi(\epsilon)) \cdot |\gamma(t_1)\gamma(t_2)|_{R_\epsilon(X)}.$$

2. $f|_{R_\epsilon(X)}$ is $(1 + \psi(\epsilon))$-Lipschitz. In particular, if geodesic $[f(a)f(b)]_Y \subset f(R_\epsilon(X))$, then

$$1 \leq \frac{|ab|_X}{|f(a)f(b)|_Y} < 1 + \psi(\epsilon).$$

3. $f(R_\epsilon(X)) \subset Y^{\psi(\epsilon)}$ is open and dense in $Y$.

Now we can prove that $f$ is injective with the following Dimension Control Lemma (compare to [10] for the Alexandrov case), which is a direct consequence of volume convergence and Lemma 3.1 in [5], which substantially relies on the volume comparison.

**Lemma 1.6 (Dimension Control).** Let $\Omega_0 \subset X \in M_\infty(n,\kappa,v)$ be a subset with $\text{vol}(\Omega_0) > 0$ and $p \in X$ be a fixed point. For each point $x \in \Omega_0$, select one point $\bar{x}$ on a geodesic $[px]_X$. Let $\Omega$ be the collection of the $\bar{x}$s for all $x \in \Omega_0$. If $d(p,\Omega) > 0$, then

$$\dim_H(\Omega) \geq n - 1.$$
Lemma 1.7. \( f : X \rightarrow Y \) is injective.

Proof. Assume \( G_X \neq \emptyset \). Let \( p, q \in G_X \) such that \( f(p) = f(q) = a \in G_Y \). We will show that there exists \( \epsilon > 0 \) such that \( \dim_H(f(X \setminus R_\epsilon(X))) \geq n - 1 \). Then

\[
\dim_H(S(X)) \geq \dim_H(X \setminus R_\epsilon(X)) \geq \dim_H(f(X \setminus R_\epsilon(X))) \geq n - 1,
\]

This contradicts to the fact that \( \dim_H(S(X)) \leq n - 2 \) proved in [4].

By Lemma 1.2, let \( \epsilon > 0 \) be such that \( C(\epsilon) \): For any \( \eta > 0 \), there exists \( \epsilon > 0 \) such that \( \dim_H(f(X \setminus R_\epsilon(X))) \geq n - 1 \). Clearly \( \Omega \) is volume preserving, we will show that

\( \dim_H(Y \setminus f(R_\epsilon(X))) \geq n - 1 \).

For any \( \eta > 0 \) small, there is \( q_1 \in R(X) \) with \( |q_1|_X < \eta \). Let \( \Omega_0 = f(B_{\eta}(p) \cap R(X)) \). By Theorem A.1.5 in [4], \( d(a_1, \Omega_0) \geq c(\epsilon) > 0 \). Together with (1.3) and Lemma 1.6, we get that

\( \dim_H(Y \setminus f(R_\epsilon(X))) \geq \dim_H(\Omega) \geq n - 1. \)

\( \square \)

Proof of Theorem A. We first show that for any \( a \in f(R_\epsilon(X)) \), \( y \in Y \) and any \( r > 0 \), there is \( y' \in B_r(y) \) such that \( [ay']_Y \subset f(R_{2r}(X)) \). If this is not true, then \( [ay']_Y \setminus f(R_{2r}(X)) \neq \emptyset \) for any \( y' \in B_r(y) \). Let \( y \in [ay']_Y \setminus f(R_{2r}(X)) \) and \( \Omega \) be the collection of these \( y \)'s for all \( y' \in B_r(y) \). Clearly \( \Omega \subset Y \setminus f(R_\epsilon(X)) \) and \( \dim_H(\Omega) \leq n - 2 \). By the volume convergence of small Euclidean balls, we see that there is a constant \( c = c(\epsilon) \) such that \( |ay| \geq c(\epsilon) \). By Lemma 1.6, we get \( \dim_H(\Omega) \geq n - 1 \), a contradiction.

Let \( \gamma_X : [0,1] \rightarrow X \) be a Lipschitz curve and \( \gamma_Y = f(\gamma_X) \). It's sufficient to show that \( L(\gamma_Y) \geq L(\gamma_X) \). Let \( \{y_i\}_{i=1}^N \) be a partition of \( \gamma_Y \) with \( \max_i \{ |y_i, y_{i+1}| \} \rightarrow 0 \) as \( N \rightarrow \infty \). For any \( \epsilon > 0 \), let \( \epsilon_1 = \psi(\epsilon) > 0 \) be selected as in Lemma 1.5. Take \( y_0' / B_{\eta_0}(\epsilon/N) \setminus \Omega \) to be \( \emptyset \). Select \( y_i' \in B_{\eta_1}(\epsilon/N) \) recursively such that \( [y_i, y_{i+1}'] \subset f(R_{\epsilon_1/2^{N-i+1}}(X)) \) for \( i = 1, 2, \ldots, N-1 \).
Let $x_i = f^{-1}(y'_i)$, $i = 0, 1, \ldots, N$. Applying Lemma 1.5 to $[y'_i y'_{i+1}]Y \subset f(R_{\epsilon/2N-i+1}(X)) \subseteq f(R_{\epsilon}(X))$ we get

$$L(\gamma_Y) \geq \sum_{i=0}^{N-1} |y_i y_{i+1}]Y \geq \sum_{i=0}^{N-1} \left( |y'_i y'_{i+1}]Y - \frac{2\epsilon}{N} \right) \geq \sum_{i=0}^{N-1} |y'_i y'_{i+1}]Y - 2\epsilon$$

$$\geq (1 - \psi(\epsilon)) \sum_{i=0}^{N-1} |x_i x_{i+1}]X - 2\epsilon.$$ 

Let $N \to \infty$ with that $\cup_i [y_i y_{i+1}]Y$ converges to $\gamma_Y$, because $f$ is injective, $\cup_i [x_i x_{i+1}]X$ converges to $\gamma_X$. Thus taking $N \to \infty$ and $\epsilon \to 0$, we get

$$L(\gamma_Y) \geq \liminf_{N \to \infty} \sum_{\epsilon \to 0}^{N-1} |x_i x_{i+1}]X \geq L(\gamma_X).$$

□

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