Analysis of endomorphisms

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Abstract. In this expository article, we discuss the recent progress in the study of endomorphisms and automorphisms of the Cuntz algebras and, more generally graph $\mathcal{C}^*$-algebras (or Cuntz-Krieger algebras). In particular, we discuss the definition and properties of both the full and the restricted Weyl group of such an algebra. Then we outline a powerful combinatorial approach to analysis of endomorphisms arising from permutation unitaries. The restricted Weyl group consists of automorphisms of this type. We also discuss the action of the restricted Weyl group on the diagonal MASA and its relationship with the automorphism group of the full two-sided $n$-shift. Finally, several open problems are presented.

1. Introduction
The purpose of this note is to summarize some recent results obtained in a series of papers concerning endomorphisms and especially automorphisms of the Cuntz algebras and, more generally, graph algebras, [47, 17, 14, 18, 16, 27, 11, 12, 13].

1.1. Cuntz algebras and their subalgebras
The Cuntz algebra $\mathcal{O}_n$ is defined as the universal $\mathcal{C}^*$-algebra generated by $n$ isometries $S_i$, $i = 1, \ldots, n$, whose range projections have sum equal to 1, [19]. It turns out that $\mathcal{O}_n$ is a unital, separable, nuclear, simple, purely infinite $\mathcal{C}^*$-algebra with the $K$-groups $K_0 \cong \mathbb{Z}_{n-1}$ and $K_1 = 0$, [19, 22]. Being simple, $\mathcal{O}_n$ has no ideals, however it has interesting unital subalgebras, namely the canonical UHF subalgebra $\mathcal{F}_n$ (isomorphic to the CAR algebra, [6], for $n = 2$), generated by Wick ordered monomials $S_\alpha^* S_\beta$ where the lengths of the multi-indices $\alpha$ and $\beta$ are the same, and the diagonal $\mathcal{D}_n$ generated by the range projections $P_\alpha = S_\alpha^* S_\alpha^*$, which is a MASA (maximal abelian subalgebra) in $\mathcal{O}_n$.

1.2. Endomorphisms of Cuntz algebras
It is of great interest to study the group of automorphisms of the Cuntz algebra $\text{Aut}(\mathcal{O}_n)$ providing insight into internal symmetries of this complicated $\mathcal{C}^*$-algebra. More generally, it is also of interest to study (unital, $*$-) endomorphisms of $\mathcal{O}_n$, as for instance they provide a direct link to subfactor and sector theory, [38], or wavelets, [5]. One nice feature of the Cuntz algebras is that it is possible to describe all endomorphisms in a fairly simple way, namely they can be written in the form $\lambda_u$, where $u$ is a unitary in $\mathcal{O}_n$, [20]. Here, $\lambda_u$ is the unique endomorphism

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of $\mathcal{O}_n$ which acts on generators as $\lambda_n(S_i) = uS_i$, $i = 1, \ldots, n$. For instance, for the so-called canonical endomorphism $\varphi$ of $\mathcal{O}_n$, defined by $\varphi(x) = \sum_{i=1}^n S_i x S_i^*$, one has $\varphi = \lambda_\theta$, where $\theta$ is the flip unitary $\sum_{i,j=1}^n S_i S_j S_i^* S_j^*$. This $\varphi$ may be considered a noncommutative analogue of the one-sided shift on the space $X_n$ of infinite strings over the alphabet $\{1, \ldots, n\}$.

Notice that for $x \in F_n$ one has

$$\lambda_n(x) = \lim_{k \to \infty} u \varphi(u) \ldots \varphi^{k-1}(u)x \varphi^{k-1}(u^*) \ldots \varphi(u^*)u^*.$$

1.3. State of the arts

Recently, a significant progress in understanding the general structure of the endomorphisms of the Cuntz algebras preserving both the UHF-subalgebra and the diagonal MASA has been achieved. The main motivation for this analysis was to provide an answer to a question posed by Cuntz about thirty years ago, [20], about the structure of certain Weyl groups as explained below. Perhaps one reason for this problem remaining open for such a long time was the high level of complexity of the involved combinatorial structure. At the same time, the solution to this problem provides also an interesting direct link to the theory of shifts as studied in the theory of classical dynamical systems, [34], and this link opens great new perspectives, [12]. It is an exciting possibility that it could even provide some new tools for attacking certain old open problems in symbolic dynamics.

It is worth pointing out that in this research theoretical progress often goes hand-in-hand with massive computer calculations. Due to the large scale of the involved data, pen and paper calculations albeit possible in principle are utterly impractical. On the other hand, symbolic computer calculation (performed on Magma software, [4]) allowed for locating interesting examples illuminating the abstract theory and for complete classification of automorphisms in lower dimensional cases, [17, 14].

1.4. Graph algebras

By now, this program has been partially extended to the much wider class of graph algebras. Graph $C^*$-algebras are generated by partial isometries with mutually commuting domain and range projections, satisfying certain natural relations, [36, 35, 43]. For finite graphs, their $C^*$-algebras coincide with the class of Cuntz-Krieger algebras, [23, 21, 44].

Recently, both the full and the restricted Weyl groups of a graph algebra $C^*(E)$ have been defined, [13], at least in the case of a finite graph $E$. A combinatorial approach to permutative endomorphisms of graph algebras has been developed as well. In this approach, we see many similarities to the simpler case of the Cuntz algebras. Nevertheless, a number of new conceptual phenomena and technical difficulties arise.

2. Cuntz algebras and their Weyl groups

2.1. The structure of the Weyl group of a Cuntz algebra

For algebras $A \subseteq B$ we denote by $N_B(A) = \{a \in \mathcal{U}(B) : uAu^* = A\}$ the normalizer of $A$ in $B$, and by $A' \cap B = \{b \in B : (\forall a \in A) ab = ba\}$ the relative commutant of $A$ in $B$. For algebras $A \subseteq B$, we denote by $\text{Aut}(B, A)$ the collection of all those automorphisms $\alpha$ of $B$ such that $\alpha(A) = A$, and by $\text{Aut}_A(B)$ those automorphisms of $B$ which fix $A$ point-wise.

One can show that $\lambda$ establishes an isomorphism between the group $\mathcal{U}(\mathcal{D}_n)$ and the maximal abelian subgroup $\text{Aut}_{\mathcal{D}_n}(\mathcal{O}_n)$ of $\text{Aut}(\mathcal{O}_n)$, which is the limit of higher dimensional tori, [20]. Furthermore, $\mathcal{O}_n \cap \text{Aut}_{\mathcal{D}_n}(\mathcal{O}_n) = \mathcal{D}_n$. The idea is now to define a Weyl group as the quotient of the normalizer of $\text{Aut}_{\mathcal{D}_n}(\mathcal{O}_n)$ in $\text{Aut}(\mathcal{O}_n)$.

**Proposition 2.1.** Let $A \subseteq B$ be a unital inclusion of $C^*$-algebras such that $B^\text{Aut}_A(B) = A$. Then

$$N_{\text{Aut}_A(B)}(\text{Aut}(B)) = \text{Aut}(B, A).$$

(1)
One then defines the Weyl group $\mathfrak{W}_n$ for $\text{Aut}(O_n)$ as

$$\mathfrak{W}_n := \text{Aut}(O_n, D_n)/\text{Aut}_{D_n}(O_n),$$

as well as its restricted version

$$\mathfrak{W}_n := \text{Aut}(O_n, D_n) \cap \text{Aut}(O_n, F_n)/\text{Aut}_{D_n}(O_n).$$

The Weyl group is countable, [20].

Next, it turns out that $\text{Aut}(O_n, D_n) = \lambda(N_{O_n}(D_n))^{-1}$ and $\text{Aut}(O_n, D_n) \cap \text{Aut}(O_n, F_n) = \lambda(N_{F_n}(D_n))^{-1}$, where for a subset $E \subset U(O_n)$ we set $\lambda(E)^{-1} = \{\lambda_u \mid u \in E\} \cap \text{Aut}(O_n)$.

Let $S_n$ denote the group of unitaries that can be written as finite words in the $S_i$'s and their adjoints and $P_n := S_n \cap F_n$. Then $P_n = \cup P^k_n$, with $P^k_n$ isomorphic to the symmetric group on $n^k$ elements. Using the fact that $N_{O_n}(D_n) = U(D_n) \cdot S_n$ and, similarly, $N_{F_n}(D_n) = U(D_n) \cdot P_n$, [42], one can show that $\text{Aut}(O_n, D_n) = \text{Aut}_{D_n}(O_n) \times \lambda(S_n)^{-1}$ and $\text{Aut}(O_n, P_n) \cap \text{Aut}(O_n, F_n) = \text{Aut}_{D_n}(O_n) \times \lambda(P_n)^{-1}$, [17]. It follows that there is an isomorphism $\mathfrak{W}_n \cong \lambda(P_n)^{-1}$. Furthermore there exists a short exact sequence

$$1 \to P_n \overset{\text{ad}}\to \lambda(P_n)^{-1} \to G_n \to 1$$

where $G_n$ is the image of $\lambda(P_n)^{-1}$ through the quotient map $q : \text{Aut}(O_n) \to \text{Out}(O_n)$. This is a first step towards a deeper understanding of the detailed structure of the automorphism group of $O_n$. Several challenging questions remain open about the properties of these groups. For instance, is $\lambda(P_n)^{-1}$ isomorphic to $\lambda(P_m)^{-1}$ when $n \neq m$? It can be shown that these groups are non-amenable, [17, 12], but we do not know if they are exact or possess any other weak amenability-type property or what kind of operator algebras they generate. We will come back to this issue again later.

2.2. Combinatorial approach to permutative endomorphisms

In this line of research, along with investigating global properties of the aforementioned groups, it is important to be able to analyze endomorphisms (and automorphisms) corresponding to specific elements of the unitary group of $O_n$. Results of this type have been appearing right from the beginnings of the theory, [1, 41, 25, 24, 32, 33]. To this end, efficient combinatorial algorithms have been developed for deciding invertibility of endomorphisms related to permutation unitaries in the matrix algebras of the canonical UHF subalgebra $F_n$, [17]. Somewhat surprisingly, labeled rooted trees play an important role in this scheme and allow some significant simplification of the computational complexity of the problem. It is worth pointing out that (finite) trees continue to be used in modern analysis in a number of different contexts, sometimes closely related to operator algebras, see e.g. [2, 3, 37, 10, 26] for a small sample.

We have obtained a complete classification of permutative endomorphisms of $O_n$ associated to unitaries in $P^k_n$ for small values of $n$ and $k$, namely $n + k \leq 6$, [17, 14]. This was achieved with help of large scale symbolic computer calculations (in principle, these could be performed by hand, save for the high impracticality of this approach). Larger values of the parameters require capabilities which are not available at present, even for fast modern computers. It appears useful and interesting to carry out a complete analysis in this vein of certain well-defined subclasses of permutative endomorphisms, for example of the induced permutations considered in [27]. Closely related to this classification is the problem of determining the subgroup of the outer automorphism group of $O_n$ generated by the image of $\lambda(P^k_n)^{-1}$, at least for some small values of the parameters $n$ and $k$. So far this has been achieved only for $k = 1$ (and arbitrary $n$), and $n = 2$ and $k \leq 3$, [17]. Partial results in the case $n = 2$ and $k = 4$ have been obtained in [17].
2.3. Interaction with symbolic dynamics
The restriction map yields an embedding of the Weyl group $\mathfrak{M}_n$ of $O_n$ into the automorphism group of the canonical diagonal MASA $D_n$. By Gelfand’s theorem, the latter is isomorphic to the algebra of complex-valued, continuous functions on the full one-sided shift space on $n$-letters $X_n := \{1, \ldots, n\}^\mathbb{N}$. A deeper analysis of the action of $\mathfrak{M}_n$ on $X_n$ reveals a striking relation between Cuntz algebras and classical symbolic dynamical systems. Namely, it turns out that, modulo inner automorphisms of $O_n$, the restricted Weyl group $\mathfrak{R} \mathfrak{M}_n$ admits a natural embedding into the group of automorphisms of the full two-sided $n$-shift $\Sigma_n$ modulo its center $\langle \sigma \rangle$, generated by the shift itself, [12]. That is, we have the following:

$$\mathfrak{R} \mathfrak{M}_n \cong \lambda(\mathcal{P}_n)^{-1} \rightarrow G_n \hookrightarrow \text{Aut}(\Sigma_n)/\langle \sigma \rangle.$$ 

A key step in arriving at this relationship is the observation that elements of the restricted Weyl group may be identified (by restriction to the diagonal) with those homeomorphisms of $X_n$ which eventually commute with the one-sided shift, [12].

Furthermore, if $n$ is prime then this embedding of $G_n$ is surjective and hence an isomorphism. Immediate consequences of this embedding include residual finiteness of the restricted Weyl group of $O_n$ (modulo inner automorphisms) and its non-amenability.

It is expected that this relation between Cuntz algebras and symbolic dynamics can be exploited further, in both ways. Indeed, despite the many available results, the precise structure of the automorphism group of the two-sided shift remains unknown and one of the most outstanding problems in this theory is the question of the isomorphism of these groups corresponding to different primes. It would also be interesting to analyze in greater detail the properties of those elements in the Weyl group corresponding to certain specific automorphisms of the shift spaces. For example, marker automorphisms provide a convenient set of generators for $\text{Aut}(X_n)$ and it would be interesting to know exactly the types of trees arising from the associated permutative automorphisms of $O_n$.

Currently, investigations of the action of the full Weyl group $\mathfrak{M}_n$ on the (spectrum of) the diagonal MASA $D_n$ are under way.

2.4. Exotic endomorphisms
This circle of ideas eventually contributed also to the construction of endomorphisms of $O_n$ with quite unexpected properties, [16]. These exotic endomorphisms are associated to unitaries $u$ which are not in $F_n$ but nonetheless they satisfy $\lambda_u(F_n) \subset F_n$. As one can show that automorphisms with these properties do not exist, this results was rather surprising.

2.5. Index and entropy
There is a complementary aspect to the study of automorphisms of $O_n$. Since its birth, the theory of endomorphisms of $O_n$ has revealed a close relationship with the theory of subfactors (and its parallel $C^*$-algebraic version), [31, 15]. For instance, let $\omega = \tau \circ \epsilon$ be the state on $O_n$ given by composition of the unique trace on $F_n$ with the canonical conditional expectation $\epsilon : O_n \rightarrow F_n$ obtained by averaging over the gauge action of $T$. Moreover, let $u$ be a unitary in $M_n \otimes M_n \otimes \cdots \otimes M_n$ ($k$ factors) inside $F_n$. Then it is well-known that $\pi_\omega(\lambda_u(O_n))'' \subset \pi_\omega(O_n)''$ is an inclusion of type $III_{1/n}$ factors with index bounded by $n^{2(k-1)}$. It is then a challenging problem to compute the index values obtained in this way and to study the properties of the associated inclusions of factors, in terms of the unitary $u$, [15]. Inclusions corresponding to endomorphisms $\lambda_u$ are also important as a tool for realizing specific examples of subfactors with preassigned fusion rules (or principal graphs) and other combinatorial invariants, [28, 29, 30].

Similarly, one can compute the topological entropy of suitable classes of endomorphisms of $O_n$ and, in the case of permutative endomorphisms, also the entropy of their restriction to $D_n$, [9, 46, 45].
3. Graph algebras and their endomorphisms

Recently, a large part of the theory of localized endomorphisms of the Cuntz algebras has been extended to a much wider class of graph $C^*$-algebras (Cuntz-Krieger algebras), [13]. Among basic examples of such endomorphisms there are the quasi-free automorphisms, [48]. For most of the results, some mild hypotheses on the underlying graphs were imposed. In this way, the key concept of the Weyl group and the restricted Weyl group has been introduced in the context of graph algebras. The two Weyl groups were shown to possess a number of properties very similar to those which hold in the case of $O_n$. Furthermore, criteria for deciding invertibility of localized endomorphisms have been found, and a combinatorial approach to the analysis of permutative endomorphisms has been developed.

The aforementioned progress notwithstanding, a number of interesting challenging problems remain open. Firstly, it would be useful to try and extend these results to include larger classes of graphs (infinite, not row-finite, with sinks, etc.). In particular, it appears natural to ask how these results would be affected by relaxing the hypothesis that the canonical diagonal subalgebra be a MASA (cf. [8, Theorem 6]). Another natural question arises with regard to the combinatorics involved in the analysis of permutative endomorphisms. At this stage, it is not clear if labelled trees are sufficient in this process, and perhaps one should search for some objects of greater complexity. Another natural line of future investigations would be finding a closer relationship between the restricted Weyl group of a graph algebra and the automorphism groups (both one- and two-sided) of the corresponding subshift of finite type, cf. [39, 40, 7]. Certainly, building a larger collection of well understood examples will help in these investigations.

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