Packing and doubling in metric spaces with curvature bounded above

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Abstract. We study locally compact, locally geodesically complete, locally CAT($\kappa$) spaces (GCBA$^\kappa$-spaces). We prove a Croke-type local volume estimate only depending on the dimension of these spaces. We show that a local doubling condition, with respect to the natural measure, implies pure-dimensionality. Then, we consider GCBA$^\kappa$-spaces satisfying a uniform packing condition at some fixed scale $r_0$ or a doubling condition at arbitrarily small scale, and prove several compactness results with respect to pointed Gromov-Hausdorff convergence. Finally, as a particular case, we study convergence and stability of $M^\kappa$-complexes with bounded geometry.

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1 Introduction

Metric spaces with curvature bounded from above are currently one of the main topics in metric geometry. They have been studied from various points of view during the last decades. In general, these metric spaces can be very wild and the local geometry can be difficult to understand. Under basic additional assumptions (local compactness and local geodesic completeness) it is possible to control much better the local and asymptotic properties of these spaces, as proved by Kleiner and Lytchak-Nagano. In particular, under these assumptions, the topological dimension coincides with the Hausdorff dimension, the local dimension can be detected from the tangent cones, and there exists a decomposition of $X$ in $k$-dimensional subspaces $X^k$ (containing dense open subsets locally bilipschitz equivalent to $\mathbb{R}^k$ and admitting a regular Riemannian metric), and a canonical measure $\mu_X$, coinciding with the restriction of the $k$-dimensional Hausdorff measure on each $X^k$, which is positive and finite on any open relatively compact subset (cp. the foundational works [Kle99], [LN19], [LN18]). Following [LN19], we will call for short GCBA-spaces the locally geodesically complete, locally compact and separable metric spaces satisfying some curvature upper bound, i.e. which are locally CAT($\kappa$) for some $\kappa$. When we want to emphasize in a statement the role of $\kappa$, we will write GCBA$^\kappa$.

GCBA-spaces arise in a natural way as generalizations and limits of Riemannian manifolds with sectional curvature bounded from above. However, many geometric results in the Riemannian setting, such as convergence and finiteness theorems, Margulis’ lemma etc. also require lower bounds on the curvature. For instance, by Bishop-Gromov’s Theorem, a lower bound on the Ricci curvature implies a bound of the complexity of the manifold as a metric space: namely, a lower bound $\operatorname{Ric}_X \geq -a^2$ implies a uniform estimate of the packing function at any fixed scale $r_0$.

We recall that a metric space $(X,d)$ satisfies the $P_0$-packing condition at scale $r_0 > 0$ if all balls of radius $3r_0$ contain at most $P_0$ points that are $2r_0$-separated from each other (this can be equivalently expressed in terms of coverings with balls, see Sec[1]). Also, we will say that a metric-measured space $(X,d,\mu)$ satisfies a $D_0$-doubling condition up to scale $r_0 > 0$ if for any $0 < r \leq r_0$ and for any $x \in X$ it holds

$$\frac{\mu(B(x,2r))}{\mu(B(x,r))} \leq D_0.$$

From a metric-geometry perspective, the original interest in studying metric spaces satisfying a packing condition at arbitrarily small scales is Gromov’s famous Precompactness Theorem [Gro81]. Another major outcome involving packing is Gromov’s celebrated result on groups with polynomial growth, as extended by Breuillard-Green-Tao [BGT11] (cp. also the previous results [Kle10] and [ST10]), which shows that a uniform bound of the packing or
doubling constant for $X$ at arbitrarily large scale (or even at fixed, sufficiently large scale with respect to the diameter) yields an even stronger limitation on the complexity of the fundamental group of $X$, that is almost-nilpotency. We will see soon (cp. Theorem C below) that, for GCBA-spaces, a doubling condition for the canonical measure $\mu_X$ at arbitrarily small scales also has interesting consequences on the local structure of $X$.

The first key-result of the paper is a Croke-type local volume estimate for GCBA-spaces of dimension bounded above, for balls of radius smaller than the almost-convexity radius:

**Theorem A (Theorem 3.1).**

For any complete GCBA space $X$ of dimension $\leq n_0$ and any ball of radius $r < \min\{\rho_{ac}(X), 1\}$ it holds:

$$\mu_X(B(x,r)) \geq c_{n_0} \cdot r^{n_0}$$

where $c_{n_0}$ is a constant depending only on the dimension $n_0$.

The almost-convexity radius $\rho_{ac}(x)$ of a geodesic space $X$ at a point $x$ is defined as the supremum of the radii $r$ such that for any $y, z \in B(x,r)$ and any $t \in [0,1]$ it holds:

$$d(y_t, z_t) \leq 2t \cdot d(y, z)$$

where $y_t, z_t$ denote points along geodesics $[x,y]$ and $[x,z]$ at distance $td(x,y)$ and $td(x,z)$ respectively from $x$. The almost-convexity radius of $X$ is correspondingly defined as $\rho_{ac}(X) = \inf_{x \in X} \rho_{ac}(x)$. It is not difficult to show that every GCBA-space $X$ always has positive almost-convexity radius at every point: namely, if $X$ is locally CAT($\kappa$) and $x \in X$, then $\rho_{ac}(x)$ is always greater than or equal to the CAT($\kappa$)-radius $\rho_{cat}(x)$ (see Section 2 for all details and the relation with the contraction and the logarithmic maps). However, the almost-convexity radius is a more flexible geometric invariant than the CAT($\kappa$)-radius, much alike the injectivity radius for Riemannian manifolds, since a space $X$ might have a large curvature $\kappa$ concentrated in a very small region around $x$, so that it may happen that $\rho_{ac}(x)$ is much larger than the CAT($\kappa$)-radius at $x$.

We stress the fact that no explicit upper bound on the curvature is assumed for the estimate (1); the condition GCBA is only needed to ensure sufficient regularity of the space (and the existence of a natural measure to compute volumes).

For all subsequent results, we will take as standing assumptions a complete, GCBA-space with a uniform upper bound on the packing constant at some fixed scale $r_0$ smaller than the almost-convexity radius, or a doubling condition up to an arbitrary small scale. These classes of metric spaces are large enough to contain many interesting examples besides Riemannian manifolds, and small enough to be, as we will see, compact in the Gromov-Hausdorff sense.
Notice that, for Riemannian manifolds, a local doubling or a packing condition at some scale $r_0 > 0$ are much weaker assumption than a lower bound of the Ricci curvature (see [BCGS17], Sec.3.3, for different examples and a comparison of Ricci, packing and doubling conditions). However, there are a lot of non-manifolds examples in these classes of metric spaces. The simplest ones are simplicial complexes with locally constant curvature (also called $M^\kappa$-complexes, cp. [BH13]) and "bounded geometry" in an appropriate sense: they will be studied in detail in Section 7. Other interesting classes of spaces satisfying a uniform packing condition at fixed scale are the class of Gromov-hyperbolic spaces with bounded entropy, admitting a cocompact group of isometries (as shown in [BCGS17], [BCGS]), or the class of (universal coverings of) compact, non-positively curved manifolds with bounded entropy, admitting acylindrical splittings (see [CSb]). See also [CSa] for applications in the non-cocompact case.

In Sections 3 and 4 we will see how the packing or covering conditions and the upper bound on the curvature interact. While in geodesic metric spaces it is always possible to extend the packing condition at some scale to bigger scales (cp. Lemma 4.7), it is not possible in general to extend the packing condition uniformly to smaller scales, see Example 4.3. Another key-result of the paper is that this extension is possible when the metric space has curvature bounded from above and is locally geodesically complete. In particular, the local geometry at scales smaller than $r_0$ is controlled by the packing condition:

**Theorem B** (Extract from Theorem 4.9).

Let $X$ be a complete, geodesic, GCBA-space with almost-convexity radius $\rho_{ac}(X) \geq \rho_0 > 0$. Then, the following conditions are equivalent:

(a) there exist $P_0$ and $r_0 \leq \rho_0/3$ such that $X$ satisfies the $P_0$-packing condition at scale $r_0$;

(b) there exist $n_0$ and $V_0, R_0 > 0$ such that $X$ has dimension $\leq n_0$ and $\mu_X(B(x, R_0)) \leq V_0$ for all $x \in X$;

(c) there exist two functions $c(r), C(r)$ such that for any $x \in X$ and for any $0 < r < \rho_0$:

\[ 0 < c(r) \leq \mu_X(B(x, r)) \leq C(r) < +\infty. \]

For Riemannian manifolds of dimension $n$, the measure $\mu_X$ coincides with the $n$-dimensional Hausdorff measure, so (b) corresponds simply to a uniform upper bound on the Riemannian volume of balls of some fixed radius $R_0$, a condition that it is sometimes easier to verify than the bounded packing. The proof of Theorem 4.9 is essentially based on universal estimates from below and from above of the volume of small balls of $X$ in terms of dimension and of the packing constants. We will prove these estimates in Section 3. We want to point out that, while the estimate (1) and Theorem 4.9 are
new, many of the ideas behind these results are already implicitly present in [LN19].

In Section 5, we investigate the relation between the local doubling condition\(^1\) with respect to the natural measure \(\mu_X\) and the local structure of GCBA-spaces. It is easy to show that a local doubling condition implies the packing. However, it turns out that the doubling property is much stronger and characterizes GCBA-spaces which are purely dimensional spaces, i.e. those whose points have all the same dimension. Indeed, we prove:

**Theorem C** (Extract from Corollary 5.5 & Theorem 5.2). Let \(X\) be a complete, geodesic, GCBA-space with almost-convexity radius \(\rho_{ac}(X) \geq \rho_0 > 0\). The following conditions are equivalent:

(a) there exists \(D_0 > 0\) such that the natural measure \(\mu_X\) is \(D_0\)-doubling up to some scale \(r_0 > 0\);

(b) \(X\) is purely \(n\)-dimensional for some \(n\), and there exist constants \(P_0\) and \(r_0 \leq \rho_0/3\) such that \(X\) satisfies the \(P_0\)-packing condition at scale \(r_0\).

The families of spaces with uniformly bounded diameter, satisfying a packing condition for some universal function \(P = P(r)\) and all \(0 < r \leq r_0\), are classically called uniformly compact; actually, one can always extract from them convergent subsequences for the Gromov-Hausdorff distance (see [Gro81]). Moreover, it is classical that an upper bound on the curvature is stable under Gromov-Hausdorff convergence, provided that the corresponding CAT(\(\kappa\))-radius is uniformly bounded below. Starting from the results proved above, it is possible to decline Gromov’s Precompactness Theorem for GCBA-spaces as follows. Consider the classes\(^2\)

\[\text{GCBA}_{\text{pack}}^\kappa(P_0; r_0; \rho_0), \quad \text{GCBA}_{\text{vol}}^\kappa(V_0; R_0; \rho_0, n_0)\]

of complete, geodesic, GCBA-spaces with curvature \(\leq \kappa\), almost-convexity radius \(\rho_{ac}(X) \geq \rho_0 > 0\) and satisfying, respectively, condition (a) or (b) of Theorem B. Let also denote by

\[\text{GCBA}_{\text{vol}}^\kappa(V_0; \rho_0, n_0)\]

the class of complete, geodesic, GCBA-spaces with curvature \(\leq \kappa\), total measure \(\mu_X(X) \leq V_0\), almost-convexity radius \(\rho_{ac}(X) \geq \rho_0 > 0\) and dimension precisely equal to \(n_0\). Then:

\(^1\)Beware that the doubling constant which is used in [LN19] is a different notion, which is purely metric and does not depend on the measure.

\(^2\)Mnemonically, we write before the semicolon the parameters which are relative to the packing condition or to the condition on the natural measure \(\mu_X\)
Theorem D (Theorem 6.1, Corollary 6.9 & 6.7).
(a) The classes \( GCBA^\kappa_{\text{pack}}(P_0, r_0; \rho_0) \) and \( GCBA^\kappa_{\text{vol}}(V_0, r_0; \rho_0, n_0) \) are compact with respect to the pointed Gromov-Hausdorff convergence;
(b) the class \( GCBA^\kappa_{\text{vol}}(V_0; \rho_0, n_0) \) is compact with respect to the Gromov-Hausdorff convergence and contains only finitely many homotopy types.

We will also see that a uniform packing at some scale \( r_0 \) is also a necessary condition for compactness (see Theorem 6.4 for the precise statement).

As our spaces are locally \( \text{CAT}(\kappa) \) with \( \text{CAT}(\kappa) \)-radius uniformly bounded below (see inequality (2) in Sec. 2), it is not surprising that the limit space is again locally \( \text{CAT}(\kappa) \). Less trivially, as a part of the proof of the compactness, we need to show that the conditions on the measure, on the almost-convexity radius and on the dimension are stable under Gromov-Hausdorff limits.

So, let us highlight the following results, which are consequence of the estimates in Theorems A and B, and are part of the compactness theorem. They will be proved in Section 6:

Theorem E (Proposition 6.2 & Proposition 6.5).
Let \((X_n, x_n)\) be \( GCBA^\kappa \)-spaces converging to \((X, x)\) with respect to the pointed Gromov-Hausdorff topology. Then:
(a) \( \rho_{\text{ac}}(X) \geq \limsup_{n \to \infty} \rho_{\text{ac}}(X_n) \);
(b) if \( \rho_{\text{ac}}(X_n) \geq \rho_0 > 0 \) for all \( n \), then \( \dim(X) \leq \lim_{n \to +\infty} \dim(X_n) \) and the equality holds if and only if the distance from \( x_n \) to the maximal dimensional subspace \( X_n^{\text{max}} \) of \( X_n \) stays uniformly bounded when \( n \to \infty \).

(The second assertion refines Lemma 2.1 of [Nag18], holding for \( \text{CAT}(\kappa) \)-spaces).

Therefore, GCBA spaces with curvature uniformly bounded from above and almost convexity radius uniformly bounded below can collapse only if the maximal dimensional subspaces go to infinity. We will see such an example in Section 6.

On the other hand, the lower-semicontinuity of the natural measure of balls and of the total volume will follow from [LN19], where it is proved that if \((X_n)_{n \geq 0}\) is a sequence of GCBA-spaces converging to \( X \), then the natural measures \( \mu_{X_n} \) converge weakly to the natural measure \( \mu_X \) (see Lemma 2.7 and the proof of Corollary 5.7 for details). We will see in Section 5 that, under the stronger assumptions that the natural measure is doubling up to some arbitrarily small scale, then the volume of balls is actually continuous (cp. Corollary 5.7).

Once proved that the bound on the total volume is stable under Gromov-Hausdorff convergence and that this implies the uniform boundedness of the spaces in our class, the homotopy finiteness stated in (ii) is a particular
case of Petersen’s finiteness theorem [Pet90]; actually, as the CAT(κ)-radius is uniformly bounded below, these spaces have a common local geometric contractibility function \( LGC(r) = r \) for \( r \leq \rho_0 \).

It is not difficult (see Section 6) to check that also the doubling property is stable under pointed Gromov-Hausdorff convergence and so is the property of being pure dimensional. Namely, let us also consider the classes (with the same conventions as before)

\[
\text{GCBA}^\kappa_{\text{doub}}(D_0, r_0; \rho_0) \quad \text{and} \quad \text{GCBA}^\kappa_{\text{vol}}(V_0; \rho_0, n_0^{\text{pure}})
\]

of complete, geodesic, GCBA-spaces \( X \) with curvature \( \leq \kappa \), almost-convexity radius \( \rho_{ac}(X) \geq \rho_0 > 0 \) and which are, respectively, either \( D_0 \)-doubling up to scale \( r_0 \), or purely \( n_0 \)-dimensional with total measure \( \mu_X(X) \leq V_0 \).

We then deduce the following additional compactness results:

**Theorem F** (Extract from Corollaries 6.9 & 6.7).

The classes \( \text{GCBA}^\kappa_{\text{doub}}(D_0, r_0; \rho_0) \) and \( \text{GCBA}^\kappa_{\text{vol}}(V_0; \rho_0, n_0^{\text{pure}}) \) are compact with respect to, respectively, pointed and unpointed Gromov-Hausdorff convergence. Moreover \( \text{GCBA}^\kappa_{\text{vol}}(V_0; \rho_0, n_0^{\text{pure}}) \) contains only finitely many homotopy types.

The proof of these and other compactness and stability results is presented in Section 6.

Finally, in Section 7 we specialize our results to study the convergence and stability of \( M^\kappa \)-complexes with bounded geometry. We will first establish some basic relations relating the injectivity radius to the size and valency of the complexes. Recall that the valency of a \( M^\kappa \)-complex \( X \) is the maximum number of simplices having a same vertex in common, and the size of the simplices of a \( X \) is defined as the smallest radius \( R > 0 \) such that any simplex contains a ball of radius \( \frac{1}{R} \) and is contained in a ball of radius \( R \); we refer to Sec. 7.1 for further definitions and details. Then, we prove:

**Theorem G** (Proposition 7.12, Sec. 7).

Let \( X \) be a \( M^\kappa \)-complex whose simplices have size bounded by \( R \), with valency at most \( N \) and no free faces. Then the following conditions are equivalent:

(a) \( X \) is a complete GCBA-space with curvature \( \leq \kappa \);
(b) \( X \) satisfies the link condition at all vertices;
(c) \( X \) is locally uniquely geodesic;
(d) \( X \) has positive injectivity radius;
(e) \( X \) has injectivity radius \( \geq \iota_0 \), for some \( \iota_0 \) depending only on \( R \) and \( N \).

The equivalence of the first four conditions is well-known for \( M^\kappa \)-complexes with finite shape (that is, whose geometric simplices, up to isometry, vary in a finite set), see [BH13]. The last condition is new and we will use it to exhibit other examples of compact families of GCBA-spaces. Namely, let

\[
M^\kappa(R_0, N_0), \quad M^\kappa(R_0; V_0, n_0)
\]
be the class of $M^\kappa$-complexes $X$ without free faces, with positive injectivity radius (but nor a-priori uniformly bounded below), simplices of size bounded by $R_0$ and, respectively, valency bounded by $N_0$ or total volume bounded by $V_0$ and $\dim(X) \leq n_0$. It is immediate to check that, for suitable $N_0 = N_0(R_0, V_0, n_0)$, the class $M^\kappa(R_0; V_0, n_0)$ is a subclass of $M^\kappa(R_0, N_0)$, made of compact $M^\kappa$-complexes, namely with a uniformly bounded number of simplices (cp. proof of Theorem 7.16); hence, it contains only finitely many $M^\kappa$-complexes, up to simplicial homeomorphism. On the other hand, we prove:

**Theorem H** (Extract from Theorem 7.14 & Corollary 7.16, Sec.7).
The classes $M^\kappa(R_0, N_0)$ and $M^\kappa(R_0; V_0, n_0)$ are compact, respectively, under pointed and unpointed Gromov-Hausdorff convergence. Moreover, there are only finitely many $M^\kappa$-complexes of diameter $\leq \Delta$ in $M^\kappa(R_0, N_0)$, up to simplicial homeomorphisms.

All the assumptions in this result are necessary. Indeed, we will see how, dropping the bounds on the valency or on the size of the simplices, we do not have neither finiteness nor compactness (see Example 7.17).

We think that Theorems D, F and H mark quite well the advantage of the synthetic condition of curvature $\leq \kappa$ over sectional curvature bounds, by identifying classes which are closed under Gromov-Hausdorff convergence, in contrast with the the classical convergence theorems of Riemannian geometry.

The Appendix is devoted to recall, for the reader’s convenience, some basics of ultrafilters and ultraconvergence of metric spaces, which is a tool heavily used all along the paper.

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## 2 Preliminaries on GCBA-spaces

First of all we fix the notation. The open and the closed ball of radius $R$ centered at $x$ in a metric space $X$ will be denoted by $B_X(x, R)$ and $\overline{B}_X(x, R)$ respectively; if the metric space is clear from the context, we will simply write $B(x, R)$ and $\overline{B}(x, R)$.

The closed annulus with center at $x$ and radii $r_1 < r_2$ will be denoted by $A(x, r_1, r_2)$. If $(X, d)$ is a metric space and $\lambda$ is a positive real number we denote by $\lambda X$ the metric space $(X, \lambda d)$, where $(\lambda d)(x, y) = \lambda d(x, y)$ for any $x, y \in X$, i.e. the rescaled metric space. We denote with $B_{\lambda X}(x, r)$ the ball of center $x$ and radius $r$ with respect to the metric $\lambda d$. The identity map from $(X, d)$ to $(X, \lambda d)$ is denoted by $dil_\lambda$.

A geodesic is a curve $\gamma: I \rightarrow X$, where $I$ is an interval in $\mathbb{R}$, such that for any $t \leq s \in I$ it holds $d(\gamma(t), \gamma(s)) = |t - s|$. If $I = [a, b]$ we say that $\gamma$ is a geodesic joining $x = \gamma(a)$ to $y = \gamma(b)$. A generic geodesic joining two points $x, y \in X$ will be denoted by $[x, y]$, even if there are more geodesics joining $x$
and \( y \). A curve is a local geodesic if it is a geodesic around any point in its interval of definition.

Finally, we stress the fact that we consider pointed Gromov-Hausdorff convergence only for complete metric spaces: so, every time we write \((X_n, x_n) \to (X, x)\) in the pointed Gromov-Hausdorff sense we mean that \(X_n\) and \(X\) are complete. This condition is not restrictive; indeed if \((X_n, x_n)\) converges to \((X, x)\), then it converges also to the completion \((\hat{X}, \hat{x})\). As a consequence if \((X_n, x_n)\) is a sequence of proper metric spaces converging to \((X, x)\), then \(X\) is proper (see Corollary 3.10 of [Her16]).

2.1. CAT(\(\kappa\)) and GCBA-spaces

We recall the definition of locally CAT(\(\kappa\)) metric space. We fix \(\kappa \in \mathbb{R}\). We denote by \(M^n_\kappa\) the unique simply connected, complete, 2-dimensional Riemannian manifold of constant sectional curvature equal to \(\kappa\) and by \(D_\kappa\) the diameter of \(M^n_\kappa\). So \(D_\kappa = +\infty\) if \(\kappa \leq 0\) and \(D_\kappa = \frac{\pi}{\kappa}\) if \(\kappa > 0\).

A metric space \(X\) is CAT(\(\kappa\)) if any two points at distance less than \(D_\kappa\) can be connected by a geodesic and if the geodesic triangles with perimeter less than \(2D_\kappa\) are thinner than their comparison triangles in the model space \(M^n_\kappa\). This means the following. For any three points \(x, y, z \in X\) such that \(d(x, y) + d(y, z) + d(z, x) < 2D_\kappa\), a geodesic triangle with vertices \(x, y, z\) is the choice of three geodesics \([x, y]\), \([y, z]\) and \([x, z]\), denoted by \(\Delta(x, y, z)\). For any such triangle there exists a unique triangle \(\Delta^\kappa(\bar{x}, \bar{y}, \bar{z})\) in \(M^n_\kappa\), up to isometry, with vertices \(\bar{x}, \bar{y}\) and \(\bar{z}\) satisfying \(d(\bar{x}, \bar{y}) = d(x, y)\), \(d(\bar{y}, \bar{z}) = d(y, z)\) and \(d(\bar{x}, \bar{z}) = d(x, z)\); such a triangle is called the \(\kappa\)-comparison triangle of \(\Delta(x, y, z)\). The comparison point of \(p \in [x, y]\) is the point \(\bar{p} \in [\bar{x}, \bar{y}]\) such that \(d(x, p) = d(\bar{x}, \bar{p})\). The triangle \(\Delta(x, y, z)\) is thinner than \(\Delta^\kappa(\bar{x}, \bar{y}, \bar{z})\) if for any couple of points \(p \in [x, y]\) and \(q \in [x, z]\) we have \(d(p, q) \leq d(\bar{p}, \bar{q})\).

A metric space \(X\) is called locally CAT(\(\kappa\)) if for any \(x \in X\) there exists \(r > 0\) such that \(B(x, r)\) is a CAT(\(\kappa\)) metric space. The supremum among the radii \(r < \frac{D_\kappa}{2}\) satisfying this property is called the CAT(\(\kappa\))-radius at \(x\) and it is denoted by \(\rho_{\text{cat}}(x)\). The infimum of \(\rho_{\text{cat}}(x)\) among the points \(x \in X\) is called the CAT(\(\kappa\))-radius of \(X\) and it is denoted by \(\rho_{\text{cat}}(X)\); therefore, by definition, \(\rho_{\text{cat}}(X) \leq \frac{D_\kappa}{2}\).

A metric space \(X\) is GCBA if there exists a \(\kappa\) such that \(X\) is locally CAT(\(\kappa\)), locally compact, separable and locally geodesically complete. The last property means that any local geodesic in \(X\) defined on an interval \([a, b]\) can be extended, as a local geodesic, to a bigger interval \([a-\varepsilon, b+\varepsilon]\). In some case we will write GCBA\(^\kappa\), if we want to emphasize the role of \(\kappa\). This class of metric spaces is the one studied in [LN19]. A metric space is geodesically complete if any local geodesic can be extended, as a local geodesic, to the whole \(\mathbb{R}\). We recall a well known fact: any complete, locally geodesically complete metric space is geodesically complete.
A tiny ball, according to [LN19], is a metric ball $B(x, r)$ such that $r < \min\{1, \frac{D_0}{100}\}$ and $\overline{B}(x, 10r)$ is compact.

2.2. Contraction maps and almost-convexity radius

We suppose $X$ is a complete, locally geodesically complete, locally CAT($\kappa$), geodesic metric space. If $x, y \in X$ satisfy $d(x, y) \leq \rho_{\text{cat}}(x)$ then there exists a unique geodesic joining them. Hence for any $x \in X$ and $0 < r \leq R < \rho_{\text{cat}}(x)$ it is well defined the contraction map:

$$\varphi^R_r : \overline{B}(x, R) \to \overline{B}(x, r)$$

by sending a point $y \in \overline{B}(x, R)$ to the unique point $y'$ along the geodesic $[x, y]$ satisfying $d(x, y')/r = d(x, y)/R$. Moreover any local geodesic starting at $x$ which is contained in $B(x, \rho_{\text{cat}}(x))$ is a geodesic. This fact, together with the locally geodesically completeness and the completeness of $X$, shows that the map $\varphi^R_r$ is surjective. It is also $\frac{2r}{R}$-Lipschitz as stated in [LN19].

We sketch here the computation.

**Lemma 2.1.** Any contraction map is $\frac{2r}{R}$-Lipschitz.

**Proof.** By the CAT($\kappa$) condition it is enough to prove the thesis on the model space $M^2_\kappa$. The result is clearly true when $\kappa \leq 0$, so we can assume $\kappa = 1$. In this case $M^2_\kappa$ is the standard sphere $S^2$.

**Step 1.** For any $x \in S^2$ and for any $0 \leq R \leq \frac{\pi}{2}$ the inverse of the exponential map, the logarithmic map $\log_x : B(x, R) \to B_{T_xS^2}(O, R)$, is $\frac{R}{\sin R}$-Lipschitz. So, for any $R$ in our range we have that the logarithmic map is 2-Lipschitz.

Thus we can conclude that, for any $y, z \in B(x, \frac{\pi}{2})$,

$$d(y, z) \leq d(\log_x(y), \log_x(z)) \leq 2d(y, z)$$

where the first inequality follows by standard comparison results.

**Step 2.** We fix $0 < r \leq R \leq \frac{\pi}{2}$ and $y, z \in B(x, R)$. Let $y'$ and $z'$ be the contractions of $y$ and $z$. We observe that the contraction of $\log_x(y)$, on the tangent space, from the radius $R$ to $r$ coincides with the point $\log_x(y')$ and the same holds for $z$; this contraction map is a dilation of factor $\frac{r}{R}$. Therefore

$$d(y', z') \leq d(\log_x(y'), \log_x(z')) = \frac{r}{R}d(\log_x(y), \log_x(z)) \leq \frac{2r}{R}d(y, z). \quad \Box$$

The natural set of scales where the contraction map is defined is not bounded from above by the CAT($\kappa$)-radius but rather from the almost-convexity radius. The almost-convexity radius at a point $x \in X$ is defined as the supremum of the radii $r$ such that for any two geodesics $[x, y], [x, z]$ of length at most $r$ and any $t \in [0, 1]$ it holds:

$$d(y_t, z_t) \leq 2td(y, z)$$

where $y_t, z_t$ are respectively the points along $[x, y]$ and $[x, z]$ satisfying $d(x, y_t) = td(x, y)$ and $d(x, z_t) = td(x, z)$. The almost-convexity radius at
$x$ does not depend on $\kappa$ and is denoted by $\rho_{ac}(x)$. Then, by definition, for any point $y \in B(x, \rho_{ac}(x))$ there exists a unique geodesic joining $x$ to $y$ (the existence follows from the assumptions on $X$), so the contraction map is well defined for any $0 < r \leq R < \rho_{ac}(x)$. A straightforward modification of Corollary 8.2.3 of [Pap05] shows that any local geodesic joining $x$ to a point $y$ at distance $d(x, y) < \rho_{ac}(x)$ is actually a geodesic. This fact and the geodesic completeness of $X$ imply again that any contraction map within the almost-convexity radius is surjective and $\frac{2r}{R}$-Lipschitz, by definition.

The (global) almost-convexity radius of the space $X$, denoted by $\rho_{ac}(X)$, is correspondingly defined as the infimum over $x$ of the almost-convexity radius at $x$.

Clearly, we always have $\rho_{ac}(X) \geq \rho_{cat}(X)$. The inequality can be partially reversed when $X$ is proper: indeed, in this case it holds

$$\rho_{cat}(X) \geq \min \left\{ \frac{D_\kappa}{2}, \rho_{ac}(X) \right\},$$

therefore a lower bound on the almost-convexity radius and the knowledge of the upper bound $\kappa$ yield a lower bound on the CAT($\kappa$)-radius. The proof of (2) follows directly from Corollary II.4.12 of [BH13] once observed that any two points of $X$ at distance less than $\rho_{ac}(X)$ are joined by a unique geodesic.

### 2.3. Tangent cone and the logarithmic map

We fix a complete, geodesic, GCBA-space $X$.

Given two local geodesics $\gamma, \eta$ starting at the same point $x \in X$ we can consider the geodesic triangle $\Delta(x, \gamma(t), \eta(t))$ for any small enough $t > 0$. The comparison triangle $\Delta^x(x, \gamma(t), \eta(t))$ has an angle $\alpha_t$ at $\bar{x}$. By the CAT($\kappa$) condition, the angle $\alpha_t$ is decreasing when $t \to 0$, see [BH13]. Hence it is possible to define the angle between $\gamma$ and $\eta$ at $x$ as $\lim_{t \to 0} \alpha_t$: it is denoted by $\angle_x(\gamma, \eta)$ and it takes values in $[0, \pi]$. For any $x \in X$, the space of directions of $X$ at $x$ is defined as

$$\Sigma_x X = \{ \gamma \text{ local geodesic s.t. } \gamma(0) = x \}/\sim$$

where $\sim$ is the equivalence relation $\gamma \sim \eta$ if and only if $\angle_x(\gamma, \eta) = 0$. The function $\angle_x(\cdot, \cdot)$ defines a distance which makes of $\Sigma_x X$ a compact, geodesically complete, CAT(1) metric space with diameter $\pi$ (see [LN19]). The tangent cone of $X$ at the point $x$ is the metric space

$$T_x X = \Sigma_x X \times [0, +\infty)$$

up to the equivalence relation $(v, 0) \sim (w, 0)$ for every $v, w \in \Sigma_x X$. The point corresponding to $t = 0$ is called the vertex of the tangent cone, denoted by $O$. The metric on $T_x X$ is given by the following formula: given two points $V = (v, t)$ and $W = (w, s)$ of $T_x X$ we define $d_T(V, W)$ as the unique positive real number satisfying:

$$d_T(V, W)^2 = t^2 + s^2 - 2ts \cos(\angle_x(v, w)).$$

(3)
In other words, $T_xX$ is the euclidean cone over $\Sigma_xX$. With this metric $T_xX$ is a proper, geodesically complete, CAT(0) metric space ([LN19]).

**Remark 2.2.** Let $Y = \mathbb{S}^{n-1}$ be the euclidean standard sphere of radius 1. Then the euclidean cone over $Y$ is isometric to $\mathbb{R}^n$.

For any point $x \in X$ the logarithmic map at $x$ is defined as:

$$\log_x : B(x, \rho_{ac}(x)) \to T_xX, \quad y \mapsto ([x,y], d(x,y)),$$

where $[x,y]$ is the unique geodesic from $x$ to $y$ (uniqueness is due to the definition of almost-convexity radius).

The logarithmic map can be recovered by the contraction maps as follows. First notice that if $X$ is a GCBA-space and $\lambda > 0$, then the space $\lambda X$ is GCBA.

Now, let the logarithmic map on the space $\lambda X$ at $\text{dil}_\lambda(x)$ be denoted by

$$\log_{\text{dil}_\lambda(x)} : B_{\lambda X}(\text{dil}_\lambda(x), \lambda \rho_{ac}(x)) \to T_{\text{dil}_\lambda(x)}(\lambda X).$$

The spaces $T_{\text{dil}_\lambda(x)}(\lambda X)$ and $T_xX$ are canonically isometric since the respective space of directions are canonically isometric. Let $R < \rho_{ac}(x)$: we consider a sequence of real numbers $r_n \to 0$, we set $\lambda_n = \frac{R}{r_n}$ and we define the maps

$$g_n = \log_{\text{dil}_{\lambda_n}(x)} \circ \text{dil}_{\lambda_n} \circ \varphi_{r_n}^R : B_X(x, R) \to T_xX$$

where we are using the natural identification $T_{\text{dil}_{\lambda_n}(x)}(\lambda_nX) \cong T_xX$. By the CAT($\kappa$) condition, the map $\log_{\text{dil}_{\lambda_n}(x)}$ is $(1 + \varepsilon_n)$-Lipschitz with $\varepsilon_n \to 0$, for $r_n \to 0$. So, by Lemma 2.1 the map $g_n$ is $2(1 + \varepsilon_n)$-Lipschitz and for any non-principal ultrafilter $\omega$ this sequence defines a ultralimit map $g_\omega$ between the ultralimit spaces (cp. Proposition A.3 in the Appendix).

Since $T_xX$ is proper we can apply Proposition A.3 and find that the target space of $g_\omega$ is $T_xX$, i.e.

$$g_\omega : \omega-\lim B_X(x, R) \to T_xX.$$
Lemma 2.3. Let $x \in X$ be a point of a complete, geodesic, GCBA space. Then the logarithmic map $\log_x$ has the following properties:

(a) $\log_x(B(x, r)) = B(O, r)$ for any $r < \rho_{ac}(x)$;
(b) $d(O, \log_x(y)) = d(x, y)$ for any $y \in B(x, \rho_{ac}(x))$;
(c) it is 2-Lipschitz on $B(x, \rho_{ac}(x))$.

Proof. Let $y \in B(x, \rho_{ac}(x))$. By definition, we have $\log_x(y) = ([x, y], d(x, y))$, where $[x, y]$ is the unique geodesic from $x$ to $y$. From (3) we immediately infer that $d_T(\log_x(y), O) = d(y, x)$. This proves (b) and that $\log_x(B(x, r))$ is included in $B(O, r)$ for any $r < \rho_{ac}(x)$. Now let $V = (v, t) \in B(O, r)$, for $r < \rho_{ac}(x)$. We take a geodesic $\gamma$ in the class of $v$. Since $X$ is locally geodesically complete, there exists an extension of $\gamma$ as a geodesic to the interval $[0, r]$ (this follows from the completeness of $X$ and the fact that any local geodesic is a geodesic if it is contained in a ball of radius smaller than the almost-convexity radius). Then, using the definition of the logarithmic map, we deduce that $\log_x(\gamma(r)) = V$. Now, $d(x, \gamma(r)) = r$, which concludes the proof of (a). Finally, we have seen that the logarithmic map is obtained as the restriction of the limit map $g_\omega : \omega \text{-lim} B_X(x, R) \to T_x X$ to $B_X(x, R)$. It is 2-Lipschitz for all $R \leq \rho_{ac}$, therefore it is 2-Lipschitz on $B(x, \rho_{ac}(x))$. \qed

The logarithmic map gives a good local approximation of $X$ by the tangent cone, as expressed in the following result.

Lemma 2.4 ([LN19], Lemma 5.5). Let $x \in X$ be a point of a complete, geodesic, GCBA space. For any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $r < \delta$ and for every $y_1, y_2 \in B(x, r)$ it holds

$$|d(y_1, y_2) - d_T(\log_x(y_1), \log_x(y_2))| \leq \varepsilon r.$$ 

As a consequence of this fact, Lytchak and Nagano proved that the tangent cone at $x$ can be seen as the Gromov-Hausdorff limit of a rescaled tiny ball around $x$. We explicit the proof of this fact because in the following we will need to write who are the maps realizing the Gromov-Hausdorff approximations.

Lemma 2.5 ([LN19], Corollary 5.7). Let $x \in X$ be a point of a complete, geodesic, GCBA space. For any sequence $\lambda_n \to \infty$, consider the sequence of CAT$(\kappa)$, pointed spaces $Y_n = (\lambda_n B(x, r), x)$, for any $r < \rho_{cat}(x)$. Then:

(a) $Y_n \to (T_x X, d_T, O)$ in the pointed Gromov-Hausdorff convergence;
(b) the approximating maps $f_n : Y_n \to T_x X$ are given by $f_n = \log_{dil_{\lambda_n}}(x)$ (using again the natural identification $T_{dil_{\lambda_n}}(x)(\lambda_n X) \cong T_x X$).

Proof. Fix $R > 0$ and any $\varepsilon > 0$. Let $\delta$ be as in Lemma 2.4 and set $r_n = 1/\lambda_n$. We may assume that $r_n \cdot R < \delta$. Then, for all $y_1, y_2 \in B_{Y_n}(x, R)$ we have $y_1, y_2 \in B_X(x, r_n R)$ and we can apply the Lemma 2.4 which yields
\[ |d(y_1, y_2) - d_T(\log_x(y_1), \log_x(y_2))| \leq \varepsilon r_n R. \]

We have \( d_{Y_n}(y_1, y_2) = d(y_1, y_2) \) and, by (3) and by the definition of the logarithmic map,

\[ d_T(f_n(y_1), f_n(y_2)) = \frac{1}{r_n}d_T(\log_x(y_1), \log_x(y_2)). \]

In conclusion, we get

\[ |d_{Y_n}(y_1, y_2) - d_T(f_n(y_1), f_n(y_2))| \leq \varepsilon R. \]

Since this is true for any \( \varepsilon > 0 \), the thesis follows from Lemma 2.3.

Finally, we observe that this characterization of \( T_xX \) has another consequence. Fix any \( v \in \Sigma_xX \), which can be naturally seen as an element of \( T_xX \), and take any geodesic \( \gamma \) starting at \( x \) defining \( v \): then, for any sequence \( r_n \to 0 \) we have that the sequence \( \gamma(r_n) \in Y_n \) defines \( v \) in the limit (indeed, \( f_n(\gamma(r_n)) = v \) for any \( n \)).

2.4. Dimension and natural measure

We recall some fundamental properties of GCBA-spaces proved in [LN19].

For any point \( x \in X \) there exists an integer number \( k \in \mathbb{N} \) such that any sufficiently small ball around \( x \) has Hausdorff dimension \( k \). This number is called the dimension of \( X \) at the point \( x \) and it is denoted by \( \dim(x) \).

It is possible to show that \( \dim(x) \) is equal to the geometric dimension of the tangent cone to \( X \) at \( x \) as defined in [Kle99]. The dimension of \( X \) is the (possibly infinite) quantity \( \dim(X) = \sup_{x \in X} \dim(x) \in [0, +\infty] \).

There exists a natural stratification of \( X \) into disjoint subsets \( X^k \), where \( X^k \) is the set of points of dimension \( k \), for \( k \in \mathbb{N} \). In other words, \( X = \bigsqcup_{k \in \mathbb{N}} X^k \).

Moreover, the \( k \)-dimensional Hausdorff measure \( \mathcal{H}^k \) is locally positive and locally finite on \( X^k \). Hence it is defined a measure on \( X \) as

\[ \mu_X = \sum_{k \in \mathbb{N}} \mathcal{H}^k \upharpoonright X^k. \]

The measure \( \mu_X \) is locally positive and locally finite: we call it the natural measure of \( X \).

**Example 2.6.** If \( X \) is a \( n \)-dimensional Riemannian manifold with sectional curvatures \( \leq \kappa \), then \( X \) is a locally geodesically complete, locally compact, separable, locally CAT(\( \kappa \)) metric space. In this case \( \mu_X \) is the \( n \)-dimensional Hausdorff measure and it coincides with the Riemannian volume measure, up to a multiplicative constant.

This stratification of \( X \) has good local properties, as shown in [LN19]. For any \( k \in \mathbb{N} \) it is possible to define the set of regular points \( \text{Reg}^k(X) \) of the \( k \)-dimensional part \( X^k \) of \( X \). We do not present here the definition of regular
points (they are those points that are \((k,\delta)-\)strained for a suitable small \(\delta\), according to \([LN19]\), Sec. 11.4). Instead, we recall the main properties of the set of \(k\)-dimensional and regular \(k\)-dimensional points we will need. For every \(S \subset X\) we will denote \(S^k = S \cap X^k\) and \(\text{Reg}^k(S) = S^k \cap \text{Reg}^k(X)\). Then:

- the set \(\text{Reg}^k(X)\) is open in \(X\) and dense in \(X^k\) (Cor. 11.8 of \([LN19]\));
- for any tiny ball \(B(x, r)\) there exists \(k\) such that \(B(x, r)\) does not contain points of dimension \(> k\) (Corollary 5.4 of \([LN19]\));
- for any tiny ball \(B(x, r)\) there exists a constant \(C\), only depending on the maximal number of \(r\)-separated points in \(\overline{B}(x, 10r)\), such that:
  \[
  \mathcal{H}^k \left( B(x, r)^k \right) \leq C \cdot r^k \quad (4)
  \]
  \[
  \mathcal{H}^{k-1} \left( B(x, r)^k \setminus \text{Reg}^k(B(x, r)) \right) \leq C \cdot r^{k-1} \quad (5)
  \]
  (Cor.11.8 of \([LN19]\); see Sec.4 for the definition of \(r\)-separated points).

2.5. Gromov-Hausdorff convergence

We recall here some facts about the behaviour of the natural measures and the dimension under pointed Gromov-Hausdorff convergence. Consider a proper GCBA-space \(X\) and its natural measure \(\mu_X = \sum_{k=0}^{n} \mathcal{H}^k X^k\), where \(n = \dim(X)\) is assumed to be finite. The \(k\)-dimensional Hausdorff measure \(\mathcal{H}^k\) restricted to the \(k\)-dimensional part is a Radon measure (indeed it is Borel regular and locally finite on the proper metric space \(X\), so it is \(\mu_X\). In particular for any open subset \(U \subset X\) it holds:

\[
\mu_X(U) = \sup \{ \mu_X(K) \text{ s.t. } K \text{ is a compact subset of } U \}.
\]

Now suppose to have a sequence of proper GCBA-spaces \(X_n\) converging in the pointed Gromov-Hausdorff sense to some (proper) GCBA-space \(X\). Arguing as in the first part of the proof of Theorem 1.5 of \([LN19]\) we deduce that the natural measures \(\mu_{X_n}\) converge in the weak sense to the natural measure of the limit, \(\mu_X\). This means that for any compact subsets \(K_n \subset X_n\) converging to a compact subset \(K \subset X\) it holds:

\[
\lim_{\varepsilon \to 0} \liminf_{n \to +\infty} \mu_{X_n}(B(K_n, \varepsilon)) = \lim_{\varepsilon \to 0} \limsup_{n \to +\infty} \mu_{X_n}(B(K_n, \varepsilon)) = \mu_X(K) \quad (6)
\]

where we denote by \(B(K_n, \varepsilon)\) the \(\varepsilon\)-neighbourhood of \(K_n\). As a consequence:

**Lemma 2.7.** Let \(X_n\) be a sequence of proper, GCBA-spaces converging in the pointed Gromov-Hausdorff sense to a proper, GCBA-space \(X\). Let \(x_n \in X_n\) be a sequence of points converging to \(x \in X\). Then, for any \(R > 0\) it holds:

\[
\mu_X(B(x, R)) \leq \limsup_{n \to +\infty} \mu_{X_n}(B(x_n, R)). \quad (7)
\]
Proof. The natural measure $\mu_X$ is Radon and any compact subset contained in $B(x, R)$ is contained in $\overline{B}(x, R - 2\eta)$ for some $\eta > 0$, therefore

$$\mu_X(B(x, R)) = \sup_{\eta > 0} \mu_X(\overline{B}(x, R - 2\eta)).$$

On the other hand, for any $\eta > 0$ we have by (6)

$$\mu_X(\overline{B}(x, R - 2\eta)) \leq \limsup_{n \to +\infty} \mu_X(B(x_n, R - \eta)) \leq \limsup_{n \to +\infty} \mu_X(B(x_n, R)).$$

The equality in (7) would follow from a uniform estimate on the volumes of the annuli of a given thickness. Indeed this is the case when the metric spaces satisfy a uniform doubling condition, as we will see in Section 5.

We end this preliminary section recalling some facts about the stability of the dimension under Gromov-Hausdorff convergence. In [LN19] (Def. 5.12), Lytchak and Nagano introduce the notion of standard setting of convergence. This means considering a sequence of tiny balls

$$B(x_n, r_0) \subset \overline{B}(x_n, 10r_0)$$

in a sequence of GCBA-spaces $X_n$, satisfying the following assumptions:

- the closed balls $\overline{B}(x_n, 10r_0)$ have uniformly bounded $\frac{\eta}{r_0}$ covering number (i.e. $\exists C_0$ such that the ball $\overline{B}(x_n, 10r_0)$ can be covered by $C_0$ closed balls of radius $\frac{\eta}{r_0}$ with centers in $\overline{B}(x_n, 10r_0)$ for all $n$, cp. Sec[4];
- the balls $\overline{B}(x_n, 10r_0)$ converge to a compact ball $\overline{B}(x, 10r_0)$ of a GCBA-space $X$ in the Gromov-Hausdorff sense;
- the closures $\overline{B}(x_n, r_0)$ converge to the closure $\overline{B}(x, r_0)$ of a tiny ball in $X$.

We then have:

**Lemma 2.8** (Lemma 11.5 & Lemma 11.7 of [LN19]).

Let $B(x_n, r_0)$ be a sequence of tiny balls in the standard setting of convergence.

Let $y_n \in B(x_n, r_0)$ be a sequence converging to $y \in B(x, r_0)$. Then:

(a) $\dim(y) \geq \limsup_{n \to +\infty} \dim(y_n)$;
(b) if $y$ is $k$-regular then $\dim(y) = \dim(y_n)$ for all $n$ large enough.

For non-compact spaces, the following general result is known:

**Lemma 2.9** (Lemma 2.1 of [Nag18]).

Let $(X_n, x_n)$ be a sequence of pointed, proper, geodesically complete CAT($\kappa$) spaces converging to some $(X, x)$ in the pointed Gromov-Hausdorff sense.

Then, $\dim(X) \leq \liminf_{n \to +\infty} \dim(X_n)$.
3 Estimate of volume of balls from below

We fix again a complete, geodesic, GCBA-space $X$.

From (4) & (5) it follows that there exists an upper bound for the measure of any tiny ball $B(x,r)$; moreover, one can find a uniform upper bound of the measure of all balls, independently of the center $x$, provided that $X$ satisfies a uniform packing condition at some scale (see Theorem 4.9 in Section 4 for a precise statement). It is less clear if there exists a lower bound on the measure, and in particular if this lower bound depends only on some universal constant. Indeed, in general the $\mu_X$-volume of balls of a given radius is not uniformly bounded below independently of the space $X$.

For instance, consider the balls of radius $1/2$ inside $\mathbb{R}^n$: when $n$ grows, the measure of these balls tends to 0. The next theorem shows that, if the dimension is bounded from above, then there is a uniform bound from below to the measure of balls of a given (sufficiently small) radius:

**Theorem 3.1.** Let $X$ be a complete, geodesic, GCBA metric space. If $\dim(X) \leq n_0$ then for any $x \in X$ and any $r < \min\{1, \rho_{ac}(x)\}$ it holds

$$\mu_X(B(x,r)) \geq c_{n_0} \cdot r^{n_0},$$

where $c_{n_0}$ is a constant only depending on $n_0$.

The proof of this fact is based on ideas most of which are already present in [LN19]. First of all we have:

**Proposition 3.2.** Let $X$ be a complete, geodesic, GCBA metric space and $x \in X$ be a point of dimension $n$. Then, there exists a 1-Lipschitz, surjective map $P : T_xX \to \mathbb{R}^n$ such that:

(a) $P(O) = 0$;
(b) $P(B(O,r)) = B(0,r)$ for any $r > 0$;
(c) $d_{T}(V,O) = d_{\mathbb{R}^n}(P(V),0)$ for any $V \in T_xX$.

**Proof.** As the point $x$ has dimension $n$, then the geometric dimension of $T_xX$ is $n$. This implies that $\Sigma_xX$ is a space of dimension $n-1$ satisfying the assumptions of Proposition 11.3 of [LN19]. So, there exists a 1-Lipschitz surjective map $P' : \Sigma_xX \to \mathbb{S}^{n-1}$. We extend the map $P'$ to a map $P$ over the tangent cones by sending the point $V = (v,t)$ to the point $(P'(v),t)$. It is immediate to check that $P$ is surjective and that $P(0) = 0$.

Moreover, the tangent cone over $\mathbb{S}^{n-1}$ is $\mathbb{R}^n$, as said in Example 2.2; therefore, the equality $P(\overline{B}(O,R)) = \overline{B}(0,R)$ follows directly from (3). Always by (3), we have $d_T(V,O) = d_{\mathbb{R}^n}(P(V),0)$ for any $V \in C_xX$. Finally, the 1-Lipschitz property of $P$ follows from the same property of $P'$ and from the properties of the cosine function. \qed
Combining this result with the properties of the logarithmic map explained in Section 2.3, we deduce the following:

**Proposition 3.3.** Let $X$ be a complete, geodesic, GCBA metric space and $x \in X$ be a point of dimension $n$. Then, there exists a 2-Lipschitz, surjective map $\Psi_x: B(x, \rho_{ac}(x)) \to \mathbb{R}^n$ such that

(a) $\Psi_x(x) = 0$;
(b) $\Psi_x(B(x, r)) = B(0, r)$ for any $0 < r < \rho_{ac}(x)$;
(c) $d(x, y) = d(0, \Psi_x(y))$ for any $y \in B(x, \rho_{ac}(x))$.

**Proof.** Define $\Psi_x = P \circ \log_x$, where $P$ is the map of the previous proposition and $\log_x$ is the logarithmic map at $x$. Then $\Psi$ satisfies the thesis. \qed

Using the map $\Psi_x$ we can transport metric and measure properties from $\mathbb{R}^n$ to $X$. We denote by $\omega_n$ the $\mathcal{H}^n$-volume of the ball of radius 1 of $\mathbb{R}^n$.

**Corollary 3.4.** Let $X$ be a complete, geodesic, GCBA metric space and $x \in X$ be a point of dimension $n$. Then

$$\mathcal{H}^n(B(x, r)) \geq \frac{1}{2^n} \omega_n r^n$$

for any $0 < r < \rho_{ac}(x)$.

**Proof.** It follows directly from the properties of the map $\Psi_x$ and the behaviour of the Hausdorff measure under Lipschitz maps. \qed

**Proof of Theorem 3.1.** We fix $x \in X$, $0 < r < \min\{1, \rho_{ac}(x)\}$ and $\varepsilon = \frac{r}{2n \omega_n}$. We call $d_0$ the dimension of $x$. We look for the biggest ball around $x$ of Hausdorff dimension exactly $d_0$. In order to do that we define

$$r_1 = \sup\{\rho > 0 \text{ s.t. } \text{HD}(B(x, \rho)) = d_0\}.$$

(where HD denotes the Hausdorff dimension). Notice that $\text{HD}(B(x, \rho))$ is monotone increasing in $\rho$. If $r_1 \geq r$ we stop and we redefine $r_1 = r$.

Otherwise, there exists a point $x_1$ such that $d(x, x_1) \leq r_1 + \varepsilon$ and the dimension of $x_1$ is $d_1 > d_0$, by definition of $r_1$. Now we look for the biggest ball around $x_1$ of Hausdorff dimension $d_1$. We define

$$r_2 = \sup\{\rho > 0 \text{ s.t. } \text{HD}(B(x_1, \rho)) = d_1\}.$$

Arguing as before, if $r_1 + \varepsilon + r_2 \geq r$ we stop the algorithm and we redefine $r_2$ as $r = r_1 + \varepsilon + r_2$. Otherwise we can find again a point $x_2$ such that $d(x_2, x_1) \leq r_2 + \varepsilon$ and whose dimension is $d_2 > d_1$. We continue the algorithm until $r_1 + \varepsilon + \ldots + r_k = r$. It happens in at most $n_0$ steps. At the end we have points $x = x_0, x_1, \ldots x_k$ with $k \leq n_0$ such that $d(x_i, x_j) \leq r_j + \varepsilon.$
Therefore, consider the functions in the case where these numbers can be bounded independently of \( r \). We observe that the \( d_j \)-dimensional parts of the balls \( B(x_j, r_j) \), denoted by \( B_{d_j}(x_j, r_j) \), are disjoint and contained in \( \overline{B}(x, r) \), by construction. Moreover the open ball \( B(x_j, r_j) \) has no point of dimension greater than \( d_j \). So

\[
\mu_X(\overline{B}(x, r)) = \sum_{k=0}^{n_0} H^k \omega^k B(x, r) \geq \sum_j H^{d_j}(B_{d_j}(x_j, r_j)).
\]

The last step is to estimate the last term of the sum. Since \( k \leq n_0 \) and \( r_1 + \varepsilon + \ldots + r_k = r \), then \( r_1 + \ldots + r_k = r - (k-1)\varepsilon \). Hence there exists an index \( j \) such that \( r_j \geq \frac{r}{2n_0} \). By definition, any point of the ball \( B(x_j, r_j) \) is of dimension \( \leq d_j \). Hence by the properties of the Hausdorff measure we get

\[
H^{d_j}(B_{d_j}(x_j, r_j)) = H^{d_j}(B(x_j, r_j)) \geq \frac{1}{2n_0} \omega^{d_j} r_j^{d_j} \geq c_{n_0} r^{n_0},
\]

where the first inequality follows directly from the previous corollary, and the last one holds since \( r \leq 1 \). So we can choose

\[
c_{n_0} = \left( \frac{1}{4n_0} \right)^{n_0} \min_{k=0,\ldots,n_0} \omega_k
\]

that is a constant depending only on \( n_0 \). This concludes the proof. \( \square \)

### 4 Packing in GCBA-spaces

Let \( Y \subset X \) be any subset of a metric space:
- a subset \( S \) of \( Y \) is called \( r \)-dense if \( \forall y \in Y \exists z \in S \) such that \( d(y, z) \leq r \);
- a subset \( S \) of \( Y \) is called \( r \)-separated if \( \forall y, z \in S \) it holds \( d(y, z) > r \).

The \( r \)-packing number of \( Y \) is the maximal cardinality of a \( 2r \)-separated subset of \( Y \) and is denoted by \( \text{Pack}(Y, r) \). The \( r \)-covering number of \( Y \) is the minimal cardinality of a \( r \)-dense subset of \( Y \) and is denoted by \( \text{Cov}(Y, r) \).

These two quantities are classically related by the following relations:

\[
\text{Pack}(Y, 2r) \leq \text{Cov}(Y, 2r) \leq \text{Pack}(Y, r).
\]

On a given space \( X \), the numbers \( \text{Pack}(\overline{B}(x, R), r) \) and \( \text{Cov}(\overline{B}(x, R), r) \), for \( 0 < r \leq R \), depend in general on the chosen point \( x \). We are interested in the case where these numbers can be bounded independently of \( x \in X \). Therefore, consider the functions

\[
\text{Pack}(R, r) = \sup_{x \in X} \text{Pack}(\overline{B}(x, R), r), \quad \text{Cov}(R, r) = \sup_{x \in X} \text{Cov}(\overline{B}(x, R), r)
\]

called, respectively, the packing and covering functions of \( X \). They take values on \([0, +\infty]\); moreover, as an immediate consequence of (8) we have

\[
\text{Pack}(R, 2r) \leq \text{Cov}(R, 2r) \leq \text{Pack}(R, r).
\]

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**Definition 4.1.** Let $X$ be a metric space and let $C_0, P_0, r_0 > 0$. We say that $X$ is $P_0$-packed at scale $r_0$ if $\text{Pack}(3r_0, r_0) \leq P_0$, that is every ball of radius $3r_0$ contains no more than $P_0$ points that are $2r_0$-separated. Analogously, we say that $X$ is $C_0$-covered at scale $r_0$ if $\text{Cov}(3r_0, r_0) \leq C_0$, i.e. every ball of radius $3r_0$ can be covered by at most $C_0$ balls of radius $r_0$.

The next theorem affirms that the packing functions can be well controlled for complete, locally CAT($\kappa$)-spaces which are locally geodesically complete (notice that no local compactness is assumed, since it will follow from the packing condition):

**Theorem 4.2.** Let $X$ be a complete, locally CAT($\kappa$), locally geodesically complete, geodesic metric space with $\rho_{ac}(X) > 0$. Suppose that $X$ satisfies

$$\text{Pack}\left(3r_0, \frac{r_0}{2}\right) \leq P_0 \quad \text{for } 0 < r_0 < \frac{\rho_{ac}(X)}{3}.$$ 

Then $X$ is proper and geodesically complete; so, it is a GCBA metric space. Moreover, for any $0 < r \leq R$ it holds:

$$\text{Pack}(R, r) \leq P_0(1 + P_0)^{\frac{R}{r_0}} - 1, \text{ if } r \leq r_0;$$

$$\text{Pack}(R, r) \leq P_0(1 + P_0)^{\frac{R}{r_0}} - 1, \text{ if } r > r_0.$$ 

We want to remark that, in general, a control of the packing function at some fixed scale does not imply any control at smaller scales, as shown in the following example.

**Example 4.3.** Let $D^n \subset \mathbb{R}^n$ be the closed Euclidean disk of radius 1. Let $X_n$ be the space obtained gluing a Euclidean ray $[0, +\infty)$ to a point of the boundary of $D^n$. Fix $r_0 = 1$. Any $2r_0$-separated subset $S$ of $X_n$ contains at most one point of $D^n$. Hence $\text{Pack}(3r_0, r_0) \leq 2$, in other words $X_n$ is 2-packed at scale 1 for every $n$. However at smaller scales, for example at scale $r = \frac{1}{n}$, we can easily show that $\text{Pack}(3r, r) \to +\infty$ when $n \to +\infty$. Notice that the spaces $X_n$ in this example are complete and CAT(0) but fail to be geodesically complete.

We also remark that a packing condition to scales bigger than the almost-convexity radius does not propagate to smaller scales:

**Example 4.4.** Let $X_n$ be the graph with one vertex and $n$ loops of length 1. For any $n$, we glue an half-line to the vertex obtaining a complete, GCBA, length metric space $Y_n$. As in Example 4.3 it is easy to show that at big scales the spaces $Y_n$ satisfy a uniform packing condition, while at small scales they do not.

The proof of Theorem 4.2 is based on some preliminary lemmas.
Lemma 4.5. Let $X$ be a space satisfying the assumptions of Theorem 4.2. Then, $X$ is $P_0$-packed at scale $r$ for any $r \leq r_0$.

Proof. We fix $x \in X$ and $r \leq r_0$. We take a $2r$-separated subset $\{x_1, \ldots, x_N\}$ of $B(x, 3r)$. We consider the contraction map $\phi_{3r}$ which is surjective and $\frac{2r}{r_0}$-Lipschitz. For any $i$ we fix a preimage $y_i$ of $x_i$ under $\phi_{3r}$. We have

$$2r < d(x_i, x_j) \leq \frac{2r}{r_0} d(y_i, y_j)$$

for any $i \neq j$. This means that the set $\{y_1, \ldots, y_N\}$ is $r_0$-separated in $B(x, 3r_0)$, hence $N \leq P_0$.

Corollary 4.6. Let $X$ be as in Theorem 4.2. Then $X$ is locally compact.

Proof. We fix a point $x \in X$. The ball $B(x, 3r_0)$ is complete since it is closed and $X$ is complete. Moreover for any $\varepsilon > 0$ the maximal cardinality of a $\varepsilon$-separated subset of $B(x, 3r_0)$ is finite, hence this ball is totally bounded. We can conclude it is compact.

As a consequence, since $X$ is a locally compact, complete, geodesic metric space, then by Hopf-Rinow theorem it is proper. Moreover since it is complete and locally geodesically complete then it is also geodesically complete. This proves the first assertion of Theorem 4.2.

We will now prove that the $P_0$-packing condition at every scale $r \leq r_0$ implies the announced estimate of $\text{Pack}(R, r)$ for every $R$. First, we show:

Lemma 4.7. Let $X$ be a geodesic metric space that is $P_0$-packed at scale $r_0$. Then, for any $R \geq 3r_0$, it holds:

$$\text{Pack}(R, r_0) \leq P_0(1 + P_0)^{R/r_0 - 1}.$$ 

Proof. We prove the thesis by induction on $k$, where $k$ is the smallest integer such that $R \leq 3r_0 + kr_0$. The case $k = 0$ clearly holds as for $R = 3r_0$ we have $\text{Pack}(R, r_0) \leq P_0 \leq P_0(1 + P_0)^2$. Let now $k \geq 1$ and $R \geq 3r_0$ such that $R \leq 3r_0 + kr_0$. We consider the sphere $S(x, R - r_0)$ of points at distance exactly $R - r_0$ from $x$. We observe that $R - r_0 \leq 3r_0 + (k - 1)r_0$, so by induction we can find a $2r_0$-separated subset $y_1, \ldots, y_n$ of $S(x, R - r_0)$ of maximal cardinality, where $n \leq P_0(1 + P_0)^{R-r_0 - 1}$. Moreover

$$\bigcup_{i=1}^n B(y_i, 3r_0) \supset A(x, R - r_0, R).$$

Indeed for any $y \in A(x, R - r_0, R)$ we take a geodesic $[x, y]$ and we call $y'$ the point on the geodesic $[x, y]$ at distance $R - r_0$ from $x$. Then $y \in B(y', r_0)$.
Moreover there exists $y_i$ such that $d(y', y_i) \leq 2r_0$, because of the maximality of the set \{ $y_1, \ldots, y_n$ \}. Hence $d(y, y_i) \leq 3r_0$. Therefore we get:

$$\text{Pack}(\overline{B}(x, R), r_0) \leq \text{Pack}(\overline{B}(x, R - r_0), r_0) + \text{Pack}(A(x, R - r_0), R, r_0)$$

$$\leq \text{Pack}(\overline{B}(x, R - r_0), r_0) + \sum_{i=1}^{n} \text{Pack}(\overline{B}(y_i, 3r_0), r_0).$$

Since $\text{Pack}(\overline{B}(y_i, 3r_0), r_0) \leq P_0$, we obtain

$$\text{Pack}(\overline{B}(x, R), r_0) \leq \text{Pack}(\overline{B}(x, R - r_0), r_0) + P_0 \cdot n$$

$$\leq (1 + P_0)P_0(1 + P_0)^{\frac{R - r_0}{r_0} - 1} = P_0(1 + P_0)^{\frac{R}{r_0} - 1}.$$  

We can now prove Theorem 4.2.

**Proof of Theorem 4.2.** We have already shown that $X$ is proper and geodesically complete, and that it is $P_0$-packed at every scale $0 < r \leq r_0$. Therefore, for these values of $r$, Lemma 4.7 yields

$$\text{Pack}(R, r) \leq P_0(1 + P_0)^{\frac{R}{r} - 1}$$

$\forall R \geq 3r$; but this also holds for $R \leq 3r$, since then $\text{Pack}(R, r) \leq \text{Pack}(3r, r)$. On the other hand, if $r \geq r_0$ the thesis follows directly from Lemma 4.7. Indeed, when $R \geq 3r_0$ then $\text{Pack}(R, r) \leq \text{Pack}(R, r_0)$ and Lemma 4.7 concludes. If $R < 3r_0$ we get

$$\text{Pack}(R, r) \leq \text{Pack}(R, r_0) \leq \text{Pack}(3r_0, r_0) \leq P_0$$

and $P_0(1 + P_0)^{\frac{R}{r} - 1} \geq P_0$. \hfill \Box

We can read this result in terms of the covering functions instead of the packing functions using (9).

**Corollary 4.8.** Let $X$ be a complete, locally CAT($\kappa$), locally geodesically complete, geodesic metric space with $\rho_{ac}(X) > 0$. Suppose that $X$ satisfies

$$\text{Cov}(3r_0, \frac{r_0}{2}) \leq C_0 \quad \text{for} \quad r_0 < \rho_{ac}(X)/3.$$  

Then for any $0 < r \leq R$ it holds:

$$\text{Cov}(R, r) \leq C_0(1 + C_0)^{\frac{2R}{r_0} - 1}, \quad \text{if} \quad r \leq 2r_0;$$

$$\text{Cov}(R, r) \leq C_0(1 + C_0)^{\frac{2R}{r_0} - 1}, \quad \text{if} \quad r > 2r_0.$$
Proof. By (10) we have that \( X \) satisfies \( \text{Pack}(3r_0, \frac{r_0}{2}) \leq C_0 \). Hence we can apply the previous proposition to get:

\[
\text{Cov}(R, r) \leq \text{Pack}\left(R, \frac{r}{2}\right) \leq C_0(1 + C_0)^{\frac{2k}{r_0} - 1}, \quad \text{if } \frac{r}{2} \leq r_0
\]

\[
\text{Cov}(R, r) \leq C_0(1 + C_0)^{\frac{4k}{r_0} - 1}, \quad \text{if } \frac{r}{2} > r_0.
\]

We are ready to characterize the packing condition in terms of dimension and measure of a GCBA metric space.

**Theorem 4.9.** Let \( X \) be a complete, geodesic GCBA metric space with \( \rho_{ac}(X) \geq \rho_0 > 0 \). The following facts are equivalent.

a) There exist \( P_0 > 0 \) and \( 0 < r_0 < \frac{\rho_0}{4} \) such that \( \text{Pack}(3r_0, \frac{r_0}{2}) \leq P_0 \);

b) There exist \( n_0, V_0, R_0 > 0 \) such that \( \dim(X) \leq n_0 \) and \( \mu_X(B(x, R_0)) \leq V_0 \) for any \( x \in X \);

c) There exists a measure \( \mu \) on \( X \) and there exist two functions \( c(r), C(r) \) such that for any \( x \in X \) and for any \( 0 < r < \rho_0 \):

\[
0 < c(r) \leq \mu(B(x, r)) \leq C(r) < +\infty.
\]

Moreover the set of constants \( (n_0, V_0, R_0, \rho_0, \kappa) \) can be expressed only in terms of the set of constants \( (P_0, r_0, \rho_0, \kappa) \) and viceversa.

Finally, if any of the above conditions holds then the natural measure \( \mu_X \) satisfies condition (c), and \( X \) is proper and geodesically complete.

Proof. Assume first that \( X \) satisfies \( \text{Pack}(3r_0, \frac{r_0}{2}) \leq P_0 \). First of all it follows that \( \dim(X) \) is bounded. Indeed, we fix any \( x \in X \) and we denote by \( n \) its dimension. We consider the map \( \Psi_x : B(x, 2r_0) \to \mathbb{R}^n \) given by Proposition 3.3. Let \( x_1, \ldots, x_k \) be a \( 2r_0 \)-separated subset of \( B_{2r_0}(0, 2r_0) \). Since \( \Psi_x \) is surjective we can take preimages \( y_i \) of \( x_i \) under \( \Psi_x \). Moreover \( d(y_i, x) = d(\Psi_x(y_i), 0) \), hence \( y_i \in \overline{B}(x, 2r_0) \). As \( \Psi_x \) is 2-Lipschitz the set \( \{y_1, \ldots, y_k\} \) is a \( r_0 \)-separated subset of \( \overline{B}(x, 2r_0) \). Then

\[
k \leq \text{Pack}(2r_0, \frac{r_0}{2}) \leq \text{Pack}(3r_0, \frac{r_0}{2}) \leq P_0
\]

by Theorem 4.2. But it is easy to show that \( k \geq 2n \). Therefore \( 2n \leq P_0 \) is the bound on the dimension we were looking for. We observe that this bound is expressed only in terms of \( P_0 \). We fix now \( x \in X \) and any \( R > 0 \). Let \( r = \min\{1, \frac{1}{4}r_0, \frac{1}{10}D_x\} \). We take a covering of \( \overline{B}(x, R) \) with balls of radius \( r \). By Theorem 4.2 it is possible to do that with \( k \) balls, where \( k \) can be estimated in the following way:

\[
k = \text{Cov}(\overline{B}(x, R), r) \leq \text{Pack}\left(\overline{B}(x, R), \frac{r}{2}\right) \leq P_0(1 + P_0)^{\frac{2k}{r_0} - 1}.
\]

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We call $y_1, \ldots, y_k$ the centers of these balls. By Theorem 4.2 the space $X$ is proper, then from the choice of $r$ we get that $B(y_i, r)$ is a tiny ball for any $i$, as follows from (2). Moreover the maximal number of $r$-separated points inside $\overline{B}(y_i, 10r)$ is bounded by $\text{Pack}(10r, \frac{r}{2}) \leq P_0 (1 + P_0)^{19}$, as follows again by Theorem 4.2. Hence by (4) we have

$$\mathcal{H}^j(\overline{B}(y_i, r)^j) \leq C(P_0)r^j,$$

where $C(P_0)$ is a constant depending only on $P_0$. Therefore, using the fact that the dimension of $X$ is bounded above by $n_0 = \frac{P_0}{2}$ and $r \leq 1$, we get:

$$\mu_X(B(y_i, r)) = \sum_{j=0}^{n_0} \mathcal{H}^j(\overline{B}(y_i, r)^j) \leq \frac{P_0}{2} \cdot C(P_0)$$

for any $i$. Finally,

$$\mu_X(B(x, R)) \leq P_0 (1 + P_0)^{\frac{R_0}{2r}} \cdot \frac{P_0}{2} \cdot C(P_0) = V(P_0, r_0, R, \kappa). \quad (10)$$

This shows that for any $x \in X$ and any $R_0$ we can find the desired uniform bound on the volume of the ball $B(x, R_0)$. This ends the proof of the implication (a) $\Rightarrow$ (b). Moreover this part of the proof, together with Theorem 3.1 shows that if (a) holds then the measure $\mu_X$ is a measure that satisfies condition (c) of the theorem.

Assume now that has dimension bounded above by $n_0$ and that the volume of the balls of radius $R_0$ are uniformly bounded above by $V_0$. We set $r_0 = \min\{\frac{R_0}{n_0}, 1, \frac{\beta_0}{2}\}$. The claim is that $X$ satisfies $\text{Pack}(3r_0, \frac{r_0}{2}) \leq P_0$ for some $P_0$ depending only on $V_0, R_0, n_0$ and $\rho_0$. We consider the ball of radius $\frac{R_0}{2}$ centered at a point $x \in X$. We take a $r_0$-separated subset of $\overline{B}(x, \frac{R_0}{2})$ and we suppose its cardinality is bigger than some $k$. It means that there are $k$ points $y_1, \ldots, y_k \in \overline{B}(x, \frac{R_0}{2})$ such that $d(y_i, y_j) > r_0$ for any $i \neq j$. Hence the balls centered at $y_1$ of radius $\frac{r_0}{2}$ are pairwise disjoint and satisfy $\overline{B}(y_i, \frac{r_0}{2}) \subset \overline{B}(x, \frac{R_0}{2} + \frac{r_0}{2}) \subset B(x, R_0)$, since $\frac{R_0}{2} + \frac{r_0}{2} \leq \frac{R_0}{n_0} + \frac{r_0}{2} < R_0$. We can apply Theorem 3.1 to get $\mu_X(\overline{B}(y_i, \frac{r_0}{2})) \geq c_{n_0} \left(\frac{r_0}{2}\right)^{n_0}$ for any $i$. Thus

$$V_0 \geq \mu_X(B(x, R_0)) \geq \sum_{i=1}^{k} \mu_X(\overline{B}(y_i, \frac{r_0}{2})) \geq k \cdot c_{n_0} \left(\frac{r_0}{2}\right)^{n_0},$$

then

$$k \leq \frac{2^{n_0} V_0}{c_{n_0} r_0} = \frac{2^{n_0} V_0}{c_{n_0}} \cdot \min\left\{1, \left(\frac{6}{\rho_0}\right)^{n_0}, \left(\frac{6}{R_0}\right)^{n_0}\right\} = P_0.$$

It means that $\text{Pack}(\overline{B}(x, \frac{R_0}{2}), \frac{r_0}{2}) \leq P_0$. Since $R_0 \geq 6r_0$ we can conclude that $\text{Pack}(3r_0, \frac{r_0}{2}) \leq P_0$ that is what claimed.
Finally, assume that there exists a measure $\mu$ such that for any $x \in X$ and for any $0 < r < \rho_0$ it holds
\[ 0 < c(r) \leq \mu(B(x, r)) \leq C(r) < +\infty. \]

We take any $r_0 < \frac{\rho_0}{3}$ and we fix any point $x \in X$. Let $k$ be the maximal cardinality of a $r_0$-separated subset of $\overline{B}(x, 3r_0)$. Then, arguing as before, we can find $k$ disjoint balls of radius $\frac{r_0}{2}$ contained in $B(x, 4r_0)$. Since
\[ C(4r_0) \geq \mu(B(x, 4r_0)) \geq k \cdot c\left(\frac{r_0}{2}\right) \text{ then } k \leq \frac{C(4r_0)}{c\left(\frac{r_0}{2}\right)} = P_0. \]
This shows that $X$ satisfies (a) with these choices of $r_0$ and $P_0$. \hfill \Box

5 The doubling condition in GCBA-spaces

In this section $X$ will be a complete, geodesic GCBA-space. We say that $X$ is purely $n$-dimensional if $\dim(x) = n$ for any $x \in X$. Moreover, we say that a measure $\mu$ on $X$ is:

- $D$-doubling up to the scale $t$ at $x \in X$ if there exists a constant $D > 0$ such that for any $0 < t' \leq t$ it holds
  \[ \frac{\mu(B(x, 2t'))}{\mu(B(x, t'))} \leq D; \]

- $D$-doubling up to scale $t$ if it is $D$-doubling up to scale $t$ at any point $x \in X$ (for a uniform doubling constant $D$).

When uniformity of the constant and of the scale is not an issue, we will simply say that $\mu$ is locally doubling on $X$: that is, if for any $x \in X$ there exist $t_x > 0$ and $D_x > 0$ such that $\mu$ is $D_x$-doubling up to scale $t_x$ at any point of $B(x, t_x)$.

Remark 5.1. Notice that any metric measured space $(X, \mu)$ satisfying a $D_0$-doubling condition up to scale $t_0$ is $P_0$-packed at scale $r_0 = \frac{t_0}{4}$ for $P_0 = D_0^4$ (provided that the measure gives positive mass to the balls of positive radius). Actually, let $x \in X$ and take any $r_0$-separated subset $\{y_1, \ldots, y_k\}$ of $\overline{B}(x, 3r_0)$. So, the balls $B(y_i, \frac{r_0}{2})$ are pairwise disjoint. From the doubling property we get:
\[ \mu_X(B(x, 3r_0)) \geq \sum_{i=1}^{k} \mu(B(y_i, r_0/2)) \geq \sum_{i=1}^{k} \frac{1}{D_0^4} \mu(B(y_i, 8r_0)) \]
and since $B(y_i, 8r_0) \supset B(x, 3r_0)$ we deduce that $k \leq D_0^4$.

The next result characterizes GCBA-spaces whose natural measure is locally doubling:

Theorem 5.2. Let $X$ be a proper, geodesic GCBA metric space. Suppose $\mu_X$ is locally doubling: then $X$ is purely $n$-dimensional for some $n$. 25
We begin the proof of Theorem 5.2 with the following two preliminary results.

**Lemma 5.3.** Let $X$ be a proper, geodesic GCBA metric space and $x \in X$. Let $v \in \Sigma_x X$, and assume that every point of $B((v, 1), \varepsilon)$ is a $k$-regular point of $T_x X$, for some $\varepsilon > 0$. Then, there exists $r > 0$ such that all points of the set

$$A_{v, \varepsilon}(r) = \{ y \in X \text{ s.t. } d_T(\log_x(y), (v, d(x, y))) \leq \varepsilon d(x, y) \} \cap B(x, r)$$

have dimension $k$.

We recall that, since $T_x X$ is a GCBA-space and since the set of $k$-regular points is open in $T_x X$, if $(v, 1)$ is $k$-regular point in $T_x X$ then it is always possible to find $\varepsilon$ satisfying the assumptions of the lemma.

**Proof.** Suppose the thesis is false. Then, there exists a sequence of points $y_n$ of dimension different from $k$ at distance $r_n \to 0$ from $x$ such that

$$d_T(\log_x(y_n), (v, r_n)) \leq \varepsilon r_n.$$

We consider rescaled tiny balls $Y_n = \frac{1}{r_n} B(x, r_0)$ as in Lemma 2.5, together with the approximating maps $f_n$; so, for all $n$ we have:

$$d_T(f_n(y_n), (v, 1)) \leq \varepsilon.$$

Moreover, we are in the standard setting of convergence. Indeed, the GCBA-space $X$ is geodesic and complete, so the contraction maps $\varphi^R_r$ are well-defined for any $R < \rho_{\text{cat}}(x)$, and they are surjective and $\frac{2r}{R}$-Lipschitz; therefore, by applying the same proof as in Lemma 4.5, we conclude that the rescaled balls are uniformly packed (the other properties follow from the discussion in Section 2). Moreover, the sequence $y_n \in Y_n$ converges to some point $y_\infty \in \overline{B}((v, 1), \varepsilon)$. So, $y_\infty$ is $k$-regular by assumption. But, by Lemma 5.3, the points $y_n$ must be $k$-dimensional for $n$ large enough, which is a contradiction.

**Lemma 5.4.** Let $v \in \Sigma_x X$ and let $\gamma$ be a geodesic starting at $x$ defining $v$. For any $0 < \varepsilon < 1$ we have, for all $r > 0$ small enough:

$$B\left(\gamma\left(\frac{r}{2}\right), \frac{\varepsilon r}{8}\right) \subset A_{v, \varepsilon}(r)$$

**Proof.** Actually, as the logarithm map is 2-Lipschitz we have

$$d_T(\log_x(y), (v, d(x, y))) \leq d_T(\log_x(y), \log_x(\gamma(T_2))) + d_T\left((v, T_2), (v, d(x, y))\right)$$

$$\leq 2d\left(y, \gamma\left(\frac{r}{2}\right)\right) + \left|\frac{r}{2} - d(x, y)\right|$$

$$\leq 3d\left(y, \gamma\left(\frac{r}{2}\right)\right) \leq \frac{3\varepsilon r}{8} \leq \varepsilon d(x, y)$$

since $d(x, y) \geq \frac{r}{2} - \frac{\varepsilon r}{8}$. On the other hand, if $y \in B(\gamma(\frac{r}{2}), \frac{\varepsilon r}{8})$ we have $d(x, y) \leq \frac{r}{2} + \frac{\varepsilon r}{8} < r$, so the ball $B(\gamma(\frac{r}{2}), \frac{\varepsilon r}{8})$ is included in $A_{v, \varepsilon}(r)$. \hfill \Box
Proof of Theorem 5.2. Let us suppose $X$ is not pure dimensional. We take a point $x_0 \in X$ of minimal dimension $d_0$. Then, we have by assumption

$$r_0 = \sup\{\rho > 0 \text{ s.t. } \text{HD}(B(x_0, \rho)) = d_0\} < +\infty.$$ 

We can find a point $x \in X$ with dimension $d > d_0$ such that $d(x_0, x) = r_0$. Indeed, for any $n$ we can find a point $x_n$ such that $d(x_0, x_n) < r_0 + \frac{1}{n}$ and $\dim(x_n) > d_0$. The sequence of points $x_n$ converge, as the space is proper, to a point $x$ at distance exactly $r_0$ from $x_0$. Assume that $\dim(x) = d_0$: then, there would exist a small radius $\rho$ such that the Hausdorff dimensions of $B(x, \rho)$ is exactly $d_0$. But $x_n$ belongs to $B(x, \rho)$ for $n \gg 0$, and any open ball around $x_n$ has Hausdorff dimension strictly greater than $d_0$; therefore $\text{HD}(B(x, \rho)) > d_0$, a contradiction.

Now, the tangent cone $T_x X$ at $x$ has dimension $d$. Hence, there exists a point $v \in \Sigma_x X$ and $\varepsilon > 0$ such that any point of the ball $B((v, 1), \varepsilon)$ is regular and of dimension $d$. We take any geodesic $\gamma$ starting at $x$ and defining $v$, and we set $y_r = \gamma(\frac{r}{2})$. Applying the two lemmas above we have that, for all $r$ small enough, any point of the ball $B(y_r, \frac{r}{8})$ is $d$-dimensional.

Consider now the ball $B(y_r, r)$: notice that there exists a ball of radius at least $\frac{r}{2}$ contained in $B(y_r, r) \cap B(x_0, r_0)$, so made only of $d_0$-dimensional points. In particular by Corollary (3.4) we have

$$\mu_X(B(y_r, \frac{r}{8})) \geq c_{d_0}(\frac{r}{8})^{d_0},$$

where $c_{d_0}$ is a constant depending only on $d_0$. Thus

$$\frac{\mu_X(B(y_r, r))}{\mu_X(B(y_r, \frac{r}{8}))} \geq C' r^{d_0-d},$$

where $C'$ is a constant that does not depend on $r$. Since this is true for any $r$ small enough and $d_0 < d$, this inequality contradicts the doubling assumption at $y_r$, when $r$ goes to 0.

As a consequence of what proved in Section 4 we obtain the following:
Corollary 5.5. Let $X$ be a complete, geodesic $\text{GCBA}^\kappa$ metric space with $\rho_{ac}(X) \geq \rho_0 > 0$. The following facts are equivalent:

(a) there exist $D_0 > 0$ and $t_0 > 0$ such that the natural measure $\mu_X$ is $D_0$-doubling up to scale $t_0$;

(b) $X$ is purely dimensional and there exist $P_0 > 0$ and $0 < r_0 < \rho_0/3$ such that $\text{Pack}(3r_0, \frac{r_0}{3}) \leq P_0$;

(c) there exist $n_0, V_0, R_0 > 0$ such that $X$ is purely $n_0$-dimensional and $\mu_X(B(x, R_0)) \leq V_0$ for any $x \in X$.

Moreover each of the three sets of constants $(D_0, t_0, \rho_0, \kappa)$, $(P_0, r_0, \rho_0, \kappa)$, $(n_0, V_0, R_0, \rho_0, \kappa)$ can be expressed in terms of the others.

Finally if the conditions hold then $X$ is proper and geodesically complete.

Proof of Corollary 5.5

The implication $(a) \Rightarrow (b)$ follows from Theorem 5.2 and from Remark 5.1 together with Theorem 4.2.

Assume now $X$ purely $n$-dimensional and $\text{Pack}(3r_0, \frac{r_0}{3}) \leq P_0$. We recall that by Theorem 4.9 $n$ can be bounded from above in terms of $P_0$. We fix $t_0 < \min\{1, R, \frac{10}{100}D_\kappa\}$ as in the proof of Theorem 4.9. By Theorem 4.2 we know $X$ is proper, so it is easy to check that $\rho_{cat}(X) \geq t_0$ by (2). Therefore by Theorem 3.1 we have $\mu_X(B(x, t)) \geq c_n t^n = c(P_0) t^n$

for any $t \leq t_0$. Moreover, by the same estimate used in the proof of Theorem 4.9 and using the fact that $\mu_X$ is just the $n$-dimensional Hausdorff measure, we get $\mu_X(B(x, 2t)) \leq P_0(1 + P_0) \cdot \frac{P_0}{2} \cdot C(P_0) t^n$

for any $t \leq t_0$. Hence

$$\frac{\mu_X(B(x, 2t))}{\mu_X(B(x, t))} \leq \frac{P_0(1 + P_0) \cdot \frac{P_0}{2} \cdot C(P_0)}{c(P_0)} = D_0$$

which shows the implication $(b) \Rightarrow (a)$.

The equivalence between $(b)$ and $(c)$ is proved in Theorem 4.9 \qed

Finally, the doubling condition also implies the uniform continuity of the natural measure of annuli:

Lemma 5.6. Let $X$ be a complete, geodesic, $\text{GCBA}^\kappa$-space which is $D_0$-doubling up to scale $t_0$ and satisfies $\rho_{ac}(X) \geq \rho_0$. There exists $\beta > 0$, only depending on $D_0$, such that for every $R > 0$ and for every positive $\varepsilon < \min\{\frac{t_0}{24R}, \frac{1}{9}\}$ it holds :

$$\mu_X(A(x, R, (1 - \varepsilon)R)) \leq \left(\max\left\{\frac{24R}{t_0}, 9\right\}\right)^\beta \cdot \varepsilon^\beta \cdot \mu_X(B(x, R)).$$
Proof. The proof is exactly the same as in Proposition 11.5.3 of [HKST15], with a minor modification due to the fact that we assume that \( \mu_X \) is doubling only up to scale \( t_0 \). Actually, arguing as in the first part of the proof of Proposition 11.5.3 of [HKST15] one deduces that

\[
\mu_X(A(x, R, R - t)) \leq D_0^4 \cdot \mu_X(A(x, R - t, R - 3t))
\]

for all \( x \in X \) and all positive \( t \leq \min \left\{ \frac{4}{5}, \frac{R}{9} \right\} \) := \( t_R \). From (11), we deduce that for all \( t \leq t_R \) it holds

\[
\mu_X(A(x, R, R - t)) \leq D_0^4 \left( \mu_X(B(x, R)) - \mu_X(A(x, R, R - t)) \right)
\]

hence

\[
\mu_X(A(x, R, (1 - t_m)R)) \leq \left( \frac{D_0^4}{1 + D_0^4} \right)^{m+1-m_0} \cdot \mu_X(B(x, R))
\]

for all \( m \geq m_0 = \left\lceil \log_3 \left( \frac{R}{2t_R} \right) \right\rceil \). Our claim then follows for \( \varepsilon \leq \min \left\{ \frac{t_0}{2t_R}, \frac{1}{9} \right\} \) by choosing \( \beta = \log_3 \left( \frac{1 + D_0^4}{D_0^4} \right) \). Indeed for every such \( \varepsilon \) we choose the unique integer \( m \geq m_0 \) such that \( t_{m+1} \leq \varepsilon \leq t_m \). Therefore we have

\[
\mu_X(A(x, R, (1 - \varepsilon)R)) \leq \mu_X(A(x, R, (1 - t_m)R)) \leq \left( \frac{D_0^4}{1 + D_0^4} \right)^{m+1-m_0} \cdot \mu_X(B(x, R)).
\]

Using the fact that \( m + 1 \geq - \log_3 2 \varepsilon \) we get

\[
\mu_X(A(x, R, (1 - \varepsilon)R)) \leq (2 \cdot 3^{m_0})^{1/\beta} \cdot \varepsilon^{\beta} \cdot \mu_X(B(x, R)).
\]

Since \( m_0 \leq \log_3 \left( \frac{R}{2t_R} \right) + 1 \) the thesis follows.

As a consequence, we deduce that for \( D \)-doubling GCBA-spaces the measure of balls is continuous under the Gromov-Hausdorff convergence, which sharpens Lemma 2.7:

**Corollary 5.7.** Let \( X_n \) be a sequence of geodesic, GCBA\( ^\kappa \)-spaces which are \( D_0 \)-doubling up to scale \( t_0 \) and satisfying \( \rho_{ac}(X) \geq \rho_0 \). Assume that the \( X_n \) converge in the pointed Gromov-Hausdorff sense to some GCBA-space \( X \) and let \( x_n \in X_n \) be a sequence of points converging to \( x \in X \). Then for any \( R \geq 0 \) it holds

\[
\mu_X(B(x, R)) = \lim_{n \to +\infty} \mu_{X_n}(B(x_n, R)).
\]
We denote by $\text{GCBA}^\kappa$ the class of complete, geodesic GCBA$^\kappa$ metric spaces $X$ with $\rho_{ac}(X) \geq \rho_0$ and $\text{Pack}(3r_0, \frac{\rho_0}{2}) \leq P_0$. Then, we have the following result which is strictly related to Gromov’s Precompactness Theorem, see [Gro81]:

**Theorem 6.1.** The class $\text{GCBA}^\kappa_{\text{pack}}(P_0, r_0; \rho_0)$ is closed under ultralimits and compact under pointed Gromov-Hausdorff convergence.

*Proof.* Any space $X \in \text{GCBA}^\kappa_{\text{pack}}(P_0, r_0; \rho_0)$ is proper by Theorem 4.2. geodesic and geodesically complete. Consider any sequence $(X_n, x_n)$ of elements of $\text{GCBA}^\kappa_{\text{pack}}(P_0, r_0; \rho_0)$ and any non-principal ultrafilter $\omega$. For any $n$ we have $\rho_{cat}(X_n) \geq \min\{\frac{P_0}{2}, \rho_0\} = \rho_0' > 0$ from (2). Then, by Corollary A.10 we have that $X_\omega$ is a complete, locally geodesically complete, locally CAT($\kappa$), geodesic metric space with again $\rho_{cat}(X_\omega) \geq \rho_0'$. We want to prove now that $\text{Pack}(3r_0, \frac{\rho_0}{2}) \leq P_0$ holds on $X_\omega$. We fix a point $y = (y_n) \in X_\omega$: by Lemma A.8 we have $\overline{B}(y, 3r_0) = \omega$-lim $\overline{B}(y_n, 3r_0)$. Let $z^i = (z^i_n)$, $i = 1, \ldots, N$ be a $r_0$-separated subset of $\overline{B}(y, 3r_0)$, that is $d(z^i, z^j) > r_0$ for all $i \neq j$. For any couple $i \neq j$ we have $d(z^i_n, z^j_n) > r_0$, $\omega$-a.s. Since there are a finite number of couples, $d(z^i_n, z^j_n) > r_0$ for any $i \neq j$, $\omega$-a.s. Moreover the points $z_n^i$ belong to $\overline{B}(y_n, 3r_0)$ for any $i$. So, $\omega$-a.s., there is a $r_0$-separated subset of $\overline{B}(y_n, 3r_0)$ of cardinality $N$. Therefore $N \leq P_0$ and in particular $\text{Pack}(3r_0, \frac{\rho_0}{2}) \leq P_0$ on $X_\omega$. We can now apply again Theorem 4.2 to conclude that $X_\omega$ is proper, hence a GCBA$^\kappa$ metric space. To finish the first part of the proof we need to show that $\rho_{ac}(X_\omega) \geq \rho_0$. This is the object of the following:
Proposition 6.2. Let \((X_n, x_n)\) be GCBA\(^n\)-spaces converging to \((X, x)\) with respect to the pointed Gromov-Hausdorff topology. Then:

\[
\rho_{ac}(X) \geq \limsup_{n \to \infty} \rho_{ac}(X_n)
\]

We postpone the proof of this proposition, to end the proof of Theorem 6.1. In order to prove the compactness under pointed Gromov-Hausdorff convergence we take a sequence of spaces \((X_n, x_n) \in \text{GCBA}\_\text{pack}^n(P_0, r_0; \rho_0)\) and we fix any non-principal ultrafilter \(\omega\). Let \((X_\omega, x_\omega) \in \text{GCBA}\_\text{pack}^n(P_0, r_0; \rho_0)\) be the ultralimit. Since the limit is proper we can apply Proposition A.11 to find a subsequence \((X_{n_k}, x_{n_k})\) that converges in the pointed Gromov-Hausdorff sense to \((X_\omega, x_\omega)\), showing the compactness part of the statement.  

**Proof of Proposition 6.2.** Assume that \(\rho_{ac}(X_n) \geq \rho_0 > 0\) for infinitely many \(n\). Take any non-principal ultrafilter \(\omega\); since by definition \(X\) is proper, then by Proposition A.11 we have \(X = \omega\)-lim \(X_n\). If \(\rho_0 \leq \frac{D}{\omega}\) we have \(\rho_{cat}(X_n) \geq \rho_0\) for all \(n\), so by Corollary A.10 we conclude immediately that \(\rho_{ac}(X_\omega) \geq \rho_{cat}(X_\omega) \geq \rho_0\).

Assume now that \(\rho_0 > \frac{D}{\omega}\); in particular, as before we deduce \(\rho_{cat}(X_\omega) = \frac{D}{\omega}\). The strategy is the following: we claim that for any \(y = (y_n) \in X_\omega\) and for any point \(z = (z_n)\) at distance \(\rho_0\) from \(y\) there exists a unique geodesic joining \(y\) to \(z\). In particular this geodesic must coincide with the ultralimit of the geodesics \([y_n, z_n]\) of length \(\rho_0\). If this is true, then for any two points \(z = (z_n), w = (w_n)\) of \(X_\omega\) at distance \(\rho_0\) from \(y\) and any \(t \in [0, 1]\) we get

\[
d(z, w) = \omega\text{-lim} d((z_n)_t, (w_n)_t) \leq \omega\text{-lim} 2\ell d(z_n, w_n) = 2\ell d(z, w)
\]

which implies that \(\rho_{ac}(y) \geq \rho_0\) for any \(y \in X_\omega\).

So, suppose our claim is not true: that is, assume that there exists a point \(y = (y_n) \in X_\omega\), a radius \(\rho_1 \in (\frac{D}{\omega}, \rho_0)\) such that any point at distance \(\rho_1\) from \(y\) is joined to \(y\) by a unique geodesic, while for arbitrarily small values \(\epsilon > 0\) there exist two different geodesics \(\gamma, \gamma'\) joining \(y\) to the same point \(z_\epsilon = (z_\epsilon, n)\) with \(d(y, z_\epsilon) = \rho_1 + \epsilon\).

We consider the points \(w_\epsilon = \gamma_\epsilon(\rho_1 - \epsilon), w'_\epsilon = \gamma'_\epsilon(\rho_1 - \epsilon)\) and set \(\ell = d(w_\epsilon, w'_\epsilon)\). We observe we have \(\ell \leq 4\epsilon\) and \(\ell > 0\) since the ball of radius \(\frac{D}{\omega}\) around \(z_\epsilon\) is CAT(\(\kappa\)) by assumption, so uniquely geodesic. Similarly, we consider the points \(u_\epsilon = \gamma_\epsilon(\rho_1 + \epsilon - \frac{D}{\omega}), u'_\epsilon = \gamma'_\epsilon(\rho_1 + \epsilon - \frac{D}{\omega})\) and we set \(L = d(u_\epsilon, u'_\epsilon)\). Our first step is to prove that

\[
L = d(u_\epsilon, u'_\epsilon) \geq \frac{D_\epsilon}{8} \cdot \frac{\ell}{2\epsilon} =: \delta.
\]  

(12)

So, suppose by contradiction that (12) does not hold. First of all we remark that \(\delta \leq \frac{D_\epsilon}{8}\), since \(\ell \leq 4\epsilon\). Then, as the ball \(B(z_\epsilon, \frac{D}{\omega})\) is CAT(\(\kappa\)), we can consider the \(\kappa\)-comparison triangle \(\Delta^\kappa(z_\epsilon, u_\epsilon, u'_\epsilon)\). As usual we denote by
that there is no subsequence.

We have therefore proved that $\text{GCBA}_\varepsilon$.

Then $X$ is a proper space. Notice that, in general, the ultralimit of a sequence of proper spaces is not proper, even if the spaces are really mild. For instance, let $(X_n, x_n) = (\mathbb{R}^n, 0)$ and $\omega$ be any non-principal ultrafilter. Then $X_\omega$ is isometric to $\ell^2(\mathbb{R})$, the spaces of sequences $\{a_n\}$ of real numbers such that $\sum a_n^2 < +\infty$. This is a non-proper space of infinite dimension.

The compactness of a class of proper metric spaces $C$ is hard to achieve since properness and dimension are in general not stable under limits. In the following theorem we characterize the classes of proper, $\text{GCBA}_\varepsilon$, geodesic metric spaces with almost-convexity radius uniformly bounded from below that are precompact and compact under pointed Gromov-Hausdorff convergence:

Remark 6.3. In particular, for any sequence of metric spaces $X_n$ in $\text{GCBA}_\varepsilon\text{pack}(P_0, r_0; \rho_0)$ and for any non-principal ultrafilter $\omega$ the ultralimit $X_\omega$ is a proper space. Notice that, in general, the ultralimit of a sequence of proper spaces is not proper, even if the spaces are really mild.

For instance, let $(X_n, x_n) = (\mathbb{R}^n, 0)$ and $\omega$ be any non-principal ultrafilter. Then $X_\omega$ is isometric to $\ell^2(\mathbb{R})$, the spaces of sequences $\{a_n\}$ of real numbers such that $\sum a_n^2 < +\infty$. This is a non-proper space of infinite dimension.

The compactness of a class of proper metric spaces $C$ is hard to achieve since properness and dimension are in general not stable under limits. In the following theorem we characterize the classes of proper, $\text{GCBA}_\varepsilon$, geodesic metric spaces with almost-convexity radius uniformly bounded from below that are precompact and compact under pointed Gromov-Hausdorff convergence:
Theorem 6.4. Let $\mathcal{C}$ be a class of proper, GCBA$^\kappa$, geodesic metric spaces $X$ with $\rho_{ac}(X) \geq \rho_0 > 0$. Then, $\mathcal{C}$ is precompact under the pointed Gromov-Hausdorff convergence if and only if there exist $P_0, r_0 > 0$ such that

$$\mathcal{C} \subset \text{GCBA}^\kappa_{\text{pack}}(P_0, r_0; \rho_0).$$

Moreover, $\mathcal{C}$ is compact if and only if it is precompact and closed under ultralimits.

We stress the “only if” part in the above statement: for GCBA$^\kappa$ spaces, a uniform packing assumption at some fixed scale is a necessary and sufficient condition in order to have precompactness (we recall that, in the general Gromov’s Precompactness Theorem, one needs to have a uniform control of the packing function at every scale in order to achieve precompactness).

Proof of Theorem 6.4. Let $\mathcal{C}$ be a class of proper, GCBA$^\kappa$, geodesic spaces $X$ with $\rho_{ac}(X) \geq \rho_0 > 0$. Let us prove the first equivalence stated in 6.4.

So, assume that it is precompact in the pointed Gromov-Hausdorff sense, i.e. the closure $\overline{C}$ is compact under pointed Gromov-Hausdorff convergence. Suppose $\mathcal{C}$ is not contained in $\text{GCBA}^\kappa_{\text{pack}}(P_0, r_0; \rho_0)$ for any choice of $P_0$ and $r_0$. Hence there exists $r_0 < \frac{\rho_0}{3}$ such that for any $n$ there is a space $X_n \in \mathcal{C}$ and a point $x_n \in X_n$ with a set of $r_0$-separated points inside $\overline{B}(x_n, 3r_0)$ of cardinality at least $n$. By assumption, there exists a subsequence, denoted again $(X_n, x_n)$, converging in the pointed Gromov-Hausdorff sense to $(X, x)$. The space $X$ is proper, see Section 2. Fix now any non-principal ultrafilter $\omega$. Then $(X_\omega, x_\omega)$ is isometric to $(X, x)$ by Proposition A.11. We are going to prove that inside $\overline{B}(x, 3r_0)$ there are infinitely many points that are at distance at least $r_0$ one from the other: therefore, $X$ cannot be proper and this is a contradiction. For any $n$ we denote the set of $r_0$-separated points of cardinality $n$ inside $\overline{B}(x_n, 3r_0)$ by $\{z_n^1, \ldots, z_n^n\}$. Then, for any fixed $k \in \mathbb{N}$, we consider the admissible sequence $z^k_n = (z^k_n) \in X_\omega$ (notice that $z^k_n$ is defined only for $n \geq k$, but this suffices to define a point $z^k$ in the ultralimit). Clearly, $z^k \in \overline{B}(x_\omega, 3r_0)$ for all $k$. Moreover if $k \neq l$ then $d(z^k_n, z^l_n) > r_0$ for all $n$, hence $d(z^k, z^l) \geq r_0$.

This shows that $\overline{C}$ is a subclass of $\text{GCBA}^\kappa_{\text{pack}}(P_0, r_0; \rho_0)$ for some $P_0$ and $r_0$.

Viceversa, if $\mathcal{C} \subset \text{GCBA}^\kappa_{\text{pack}}(P_0, r_0; \rho_0)$ then its closure $\overline{C}$ is contained in the compact space $\text{GCBA}^\kappa_{\text{pack}}(P_0, r_0; \rho_0)$ by Theorem 6.1, so $\overline{C}$ is compact. Let us show now the second equivalence. Suppose that $\mathcal{C}$ is precompact and closed under ultralimits. Applying the same proof of the second part of Theorem 6.1 we get that $\mathcal{C}$ is compact under pointed Gromov-Hausdorff convergence. Viceversa, if $\mathcal{C}$ is compact under Gromov-Hausdorff convergence then it is contained in $\text{GCBA}^\kappa_{\text{pack}}(P_0, r_0; \rho_0)$ for some $P_0, r_0$. In particular for any non-principal ultrafilter $\omega$ and any sequence of spaces $(X_n, x_n) \in \mathcal{C}$ we have that $X_\omega$ is a proper metric space. By Proposition A.11 there exists a subsequence that converges in the pointed Gromov-Hausdorff sense to $X_\omega$, hence $X_\omega \in \mathcal{C}$ since $\mathcal{C}$ is compact. 

\[\square\]
As a consequence of Theorem 6.4 and of the estimates on volumes and packing proved in Sections 3 & 4, we deduce that the dimension is almost stable under pointed Gromov-Hausdorff convergence, in the following sense:

**Proposition 6.5.** Let $(X_n, x_n)$ be a sequence of GCBA$^\kappa$-spaces with almost convexity radius $\rho_{ac}(X_n) \geq \rho_0 > 0$, converging to $(X, x)$ in the pointed Gromov-Hausdorff sense. Let $X_n^{\text{max}}$ be the maximal dimensional subspace of $X_n$. Then,

$$\dim(X) \leq \liminf_{n \to +\infty} \dim(X_n)$$

and the equality $\dim(X) = \lim_{n \to +\infty} \dim(X_n)$ holds if and only if the distance $d(x_n, X_n^{\text{max}})$ stays uniformly bounded when $n \to \infty$.

**Proof.** As the spaces $(X_n, x_n)$ converge to $(X, x)$, they form a precompact family and so they belong to GCBA$^\kappa_{\text{pack}}(P_0, r_0, \rho_0)$, for some constants $P_0$ and $r_0$, by Theorem 6.4. Let us first show that we always have

$$\dim(X) \leq \liminf_{n \to +\infty} \dim(X_n) \quad (14)$$

Actually, consider a subsequence, we we still denote $(X_n)$, whose dimensions equal the limit inferior, denoted $d_0$. Now suppose that there exists a point $y \in X$ with $\dim(y) = d > d_0$. We may assume that $y$ is $d$-regular, since $\text{Reg}^d(X)$ is dense in $X^d$. The point $y$ is the limit of a sequence of points $y_n \in X_n$ and for any $r > 0$ the volume of the ball $B(y, r)$ is bigger than or equal to the limit of the volumes of the balls $B(y_n, \frac{r}{2})$, by (10).

By Theorem 3.1, we have for all $n$:

$$\mu_X\left(B\left(y_n, \frac{r}{2}\right)\right) \geq c_{d_0} \cdot \left(\frac{r}{2}\right)^{d_0}$$

where $c_{d_0}$ is a constant depending only on $d_0$. Moreover, since $y$ is $d$-regular, then for any $r$ small enough the ball $B(y, r)$ contains only $d$-dimensional points. We conclude by (11) & (12) that

$$\mu_X(B(y, r)) \leq C \cdot r^d,$$

where $C$ is a constant depending only on $y$ and not on $r$. Therefore, as $d_0 < d$, we have a contradiction if $r$ is small enough, and (14) is proved.

Assume now that $d(x_n, X_n^{\text{max}}) < D$ for all $n$. Since the almost convexity radius is bounded below by $\rho_0$ both for $X_n$ and for $X$, also the CAT$(\kappa)$-radius is bounded below by (2). So we can consider tiny balls $B(y_n, r_0)$ centered at regular points $y_n$ of maximal dimension, all with the same radius $r_0$, such that the closed ball $\overline{B}(y_n, 10r_0)$ converge to some ball $\overline{B}(y, 10r_0)$ of $X$ and satisfy the condition Pack$(P_0, \frac{\rho_0}{2})$ for some constant $P_0$ for all $n$, by 4.2.

We are then in the standard setting of convergence, which implies by 2.8 that

$$\dim(X) \geq \dim(y) \geq \limsup_{n \to +\infty} \dim(y_n) = \limsup_{n \to +\infty} \dim(X_n).$$

Conversely, if we assume that $\dim(X) = \lim_{n \to +\infty} \dim(X_n)$, then in particular $\dim(X_n)$ is constant for $n \gg 0$ and equal to $d_0 = \dim(X)$. Consider a
regular point \( y = (y_n) \in X \) of dimension \( d_0 \): then, the points \( y_n \) are admissible by definition (that is, \( d(x_n, y_n) \) stays uniformly bounded); moreover, as we can choose as before uniformly packed tiny balls with \( B(y_n, 10r_0) \) converging to \( B(y, 10r_0) \), then the points \( y_n \) necessarily belong to \( X^\text{max}_n \), again by Lemma 2.8 (b).

Example 6.6. Let \( (X, x) \in \text{GCBA}^\kappa_{\text{pack}}(P_0, r_0, \rho_0) \) be any space. We consider the space \( Y \) obtained by gluing the half-line \([0, +\infty)\) to \( X \) at the point \( x \). Clearly \( Y \) belongs to \( \text{GCBA}^\kappa_{\text{pack}}(P_0', r_0', \rho_0') \). The pointed Gromov-Hausdorff limit of the sequence \((Y, n)\), where \( n \in [0, +\infty) \), is the real line. This is an example where the maximal dimension part escapes to infinity and the dimension is not preserved.

We are going now to explore some variations of Theorem 6.4. We fix constants \( \kappa \in \mathbb{R} \) and \( P_0, r_0, V_0, R_0, D_0, t_0, \rho_0, n_0 > 0 \), with \( r_0 \leq \rho_0/3 \), and consider the following classes of complete, geodesic GCBA\(^\kappa\) spaces \( X \):

- the class \( \text{GCBA}^\kappa_{\text{pack}}(P_0, r_0; \rho_0, n_0^\text{pure}) \) of spaces which are \( P_0 \)-packed at scale \( r_0 \), with almost-convexity radius \( \rho_{\text{ac}}(X) \geq \rho_0 \) and dimension equal to \( n_0 \);

- the class \( \text{GCBA}^\kappa_{\text{pack}}(P_0, r_0; \rho_0) \) of spaces \( P_0 \)-packed at scale \( r_0 \), with almost-convexity radius \( \rho_{\text{ac}}(X) \geq \rho_0 \) and of pure dimension \( n_0 \);

- the classes \( \text{GCBA}^\kappa_{\text{vol}}(V_0, R_0; \rho_0, n_0^\text{pure}) \), \( \text{GCBA}^\kappa_{\text{vol}}(V_0, R_0; \rho_0, n_0^\text{pure}, n_0^\text{max}) \) of those satisfying \( \mu_X(B(x, R_0)) \leq V_0 \), \( \rho_{\text{ac}}(X) \geq \rho_0 \) and which have, respectively, dimension equal to \( n_0 \) and pure dimension \( n_0 \);

- the class \( \text{GCBA}^\kappa_{\text{doub}}(D_0, t_0; \rho_0) \) of spaces \( D_0 \)-doubled up to scale \( t_0 \), with \( \rho_{\text{ac}}(X) \geq \rho_0 \).

Then:

Corollary 6.7. All the above classes are compact with respect to the pointed Gromov-Hausdorff convergence.

Proof. By Theorem 4.9 and Corollary 5.5, the above are all subclasses of \( \text{GCBA}^\kappa_{\text{pack}}(P_0, r_0; \rho_0) \), for suitable \( P_0 \) and \( r_0 \). By the compactness Theorem 6.4 the proof then reduces to show that the additional conditions on the dimension, on the measure of balls of given radius or on the doubling constant are stable under Gromov-Hausdorff limits. By Lemma 2.7 if a sequence \( X_n \) in \( \text{GCBA}^\kappa_{\text{vol}}(V_0, R_0; \rho_0, n_0^\text{pure}) \) converges to \( X \), then \( \mu_X(B(y, R_0)) \leq V_0 \) for any \( y \in X \). On the other hand, from Corollary 5.7 it follows that the doubling condition is preserved to the limit. The stability of the dimension is proved in Proposition 6.5. To conclude, we need to show that pure-dimensionality is stable under Gromov-Hausdorff limits: this is the object of the Proposition which follows.

Proposition 6.8. Let \( (X_n, x_n) \) be a sequence of GCBA\(^\kappa\)-spaces with almost convexity radius \( \rho_{\text{ac}}(X_n) \geq \rho_0 > 0 \), converging to \((X, x)\) in the pointed Gromov-Hausdorff sense. Assume that \( X_n \) is pure-dimensional for all \( n \): then, \( X \) is pure-dimensional of dimension \( \text{dim}(X) = \lim_{n \to +\infty} \text{dim}(X_n) \).
Proof. The spaces \((X_n, x_n)\) form a precompact family and so, by Theorem 6.4 they belong to \(\text{GCBA}^\kappa_\text{pack}(P_0, r_0, \rho_0)\), for suitable constants \(P_0\) and \(r_0\). Then, by Theorem 4.9 the numbers \(\dim(X_n)\) belong to the finite set \([0, n_0]\). Suppose to have two integers \(d_1 \neq d_2\) and two infinite subsequences \(X_{n_{i_1}}, X_{n_{i_2}}\) such that \(\dim(X_{n_{i_1}}) = d_1\) for any \(i_1\) and \(\dim(X_{n_{i_2}}) = d_2\) for any \(i_2\). We consider the sequences \(x_{n_{i_1}}\) and \(x_{n_{i_2}}\): for any \(r > 0\) the volumes of the balls of radius \(r\) around these points converge to the volume of the ball of radius \(r\) around \(x\), by Corollary 5.7 and Corollary 5.5. By (4), (5) and Theorem 3.1 we have

\[
\frac{1}{C} r^{d_1} \leq \mu(x_{n_{i_1}}(B(x_{n_{i_1}}, r))) \leq Cr^{d_1},
\]

\[
\frac{1}{C} r^{d_2} \leq \mu(x_{n_{i_2}}(B(x_{n_{i_2}}, r))) \leq Cr^{d_2},
\]

where \(C\) is a constant depending only on \(P_0\) and \(r_0\). Since this is true for any arbitrarily small \(r\), we deduce that \(d_1 = d_2\). Therefore \(\lim_{n \to +\infty} \dim(X_n)\) exists and we denote it by \(d_0\). We again apply the same estimate as before to conclude that for any \(y \in X\) and for any small \(r > 0\) we have

\[
\frac{1}{C} r^{d_0} \leq \mu(B(y, r)) \leq Cr^{d_0}.
\]

Therefore the dimension of \(y\) is \(d_0\), which concludes the proof.

Finally, we can specialize these theorems to subclasses of compact spaces. Clearly, the subclasses of the above classes made of spaces with diameter less than or equal to some constant \(\Delta\) will be compact with respect to the usual Gromov-Hausdorff distance. We state here just two particularly interesting cases, which are reminiscent of the classical finiteness theorems of Riemannian geometry. Consider the classes:

\[
\text{GCBA}^\kappa_\text{vol}(V_0; \rho_0, n_0)\quad \text{and} \quad \text{GCBA}^\kappa_\text{vol}(V_0; \rho_0, n_0)_{\text{pure}}.
\]

of complete, geodesic \(\text{GCBA}^\kappa\) with total measure \(\mu(X) \leq V_0\), almost convexity radius \(\rho_{\text{ac}}(X) \geq \rho_0\) and which are, respectively, \(n_0\)-dimensional and purely \(n_0\)-dimensional.

Corollary 6.9. The classes \(\text{GCBA}^\kappa_\text{vol}(V_0; \rho_0, n_0)\) and \(\text{GCBA}^\kappa_\text{vol}(V_0; \rho_0, n_0)_{\text{pure}}\) are compact under Gromov-Hausdorff convergence and contain only finitely many homotopy types.

Proof. First, we show that the diameter is uniformly bounded in both classes. Actually, consider \(X \in \text{GCBA}^\kappa_\text{vol}(V_0; \rho_0, n_0)\) and take any two points \(y, y' \in X\) such that \(d(y, y') = \Delta > \rho := \min\{\rho_0, 2\}\). Let \(\gamma\) be a geodesic joining \(y\) to \(y'\). Along \(\gamma\) we take points at distance \(\rho\) one from the other: they are at least \(\frac{\Delta - 1}{\rho}\), and the balls of radius \(\frac{\rho}{2}\) around these points are disjoint. Then, by Theorem 3.1 we get
\[ V_0 \geq \mu_X(X) \geq \frac{c_{n_0}}{2^{n_0}} \rho^{n_0} \left( \frac{\Delta}{\rho} - 1 \right) \]

so the diameter of \( X \) is bounded from above in terms of \( n_0, \rho_0 \) and \( V_0 \) only.

Let \( R_0 \) such an upper bound. Then these classes are included in \( \text{GCBA}^{\kappa}_{\text{vol}}(V_0, R_0; \rho_0, n_0) \), whose compactness we have just proved. The conclusion follows from Propositions 6.5 and 6.8.

Finally, notice that any element of both classes has local geometric contractibility function \( \text{LGC}(r) = r \) for \( r \leq \rho_0 \) (see [Pet90] for the definition).

Moreover the covering dimension of any space in both classes coincides with the Hausdorff dimension, so it is uniformly bounded from above. We can then apply Corollary B of [Pet90] to conclude that there are only finitely many homotopy types inside any of the two classes. \( \square \)

7 Examples: \( M^\kappa \)-complexes

Beyond Riemannian manifolds with uniform upper bounds on the sectional curvature and injectivity radius bounded below, an important class of \( \text{GCBA}^\kappa \) spaces is provided by \( M^\kappa \)-complexes with \textit{bounded geometry}, in a sense we are going to explain. We will prove that the metric spaces in this class are uniformly packed and we will show that this class is compact under pointed Gromov-Hausdorff convergence. Other finiteness results will be presented.

7.1. Geometry of \( M^\kappa \)-complexes

First of all we recall briefly the definitions and the properties of the class of simplicial complexes we are interested in. A \( \kappa \)-simplex \( S \) is the convex set generated by \( n + 1 \) points \( v_0, \ldots, v_n \) of \( M^\kappa_n \) in general position, where \( M^\kappa_n \) is the unique \( n \)-dimensional space-form with constant sectional curvature \( \kappa \). If \( \kappa > 0 \) the points \( v_0, \ldots, v_n \) are required to belong to an open hemisphere. We say that \( S \) has dimension \( n \). Each \( v_i \) is called a vertex. A \( d \)-dimensional face \( T \) of \( S \) is the convex hull of a subset \( \{v_{i_0}, \ldots, v_{i_d}\} \) of \( (d + 1) \) vertices. The interior of \( S \), denoted \( \mathring{S} \), is defined as \( S \) minus the union of its lower dimensional faces; the boundary \( \partial S \) is the union of its codimension 1 faces. Let \( \Lambda \) be any set and \( E = \bigsqcup_{\lambda \in \Lambda} S_\lambda \), where any \( S_\lambda \) is a \( \kappa \)-simplex. Let \( \sim \) be an equivalence relation on \( E \) satisfying:

(i) for any \( \lambda \in \Lambda \) the projection map \( p: S_\lambda \to E/\sim \) is injective;

(ii) for any \( \lambda, \lambda' \in \Lambda \) such that \( p(S_\lambda) \cap p(S_{\lambda'}) \neq \emptyset \) there exists an isometry \( h_{\lambda, \lambda'} \) from a face \( T \subset S_\lambda \) onto a face \( T' \subset S_{\lambda'} \) such that \( p(x) = p(x') \), for \( x \in S_\lambda \) and \( x' \in S_{\lambda'} \), if and only if \( x' = h_{\lambda, \lambda'}(x) \).

The quotient space \( K = E/\sim \) is called a \( M^\kappa \)-simplicial complex or simply \( M^\kappa \)-complex; the set \( E \) is the total space. A subset \( S \subset K \) is called an \( m \)-simplex of \( K \) if it is the image under \( p \) of an \( m \)-dimensional face of some \( S_\lambda \);
its interior and its boundary are, respectively, the image under \( p \) of the interior and the boundary of \( S_\lambda \). The support of a point \( x \in K \), denoted \( \text{supp}(x) \), is the unique simplex containing \( x \) in its interior (notice that \( \text{supp}(v) = v \) when \( v \) is a vertex). The open star around a vertex \( v \) is the union of the interior of all simplices having \( v \) as a vertex.

Metrically, \( K \) is equipped with the quotient pseudometric. By Lemma I.7.5 of [BH13] the pseudometric can be expressed using strings. A \( m \)-string in \( K \) from \( x \) to \( y \) is a sequence \( \Sigma = (x_0, \ldots, x_m) \) of points of \( K \) such that \( x = x_0 \), \( y = x_m \) and for each \( i = 0, \ldots, m-1 \) there exists a simplex \( S_i \) containing \( x_i \) and \( x_{i+1} \). Moreover, a \( m \)-string \( \Sigma = (x_0, \ldots, x_m) \) from \( x \) to \( y \) is taut if

- there is no simplex containing \( \{x_{i-1}, x_i, x_{i+1}\} \);
- if \( x_{i-1}, x_i \in S_i \) and \( x_i, x_{i+1} \in S_{i+1} \) then the concatenation of the segments \( [x_{i-1}, x_i] \) and \( [x_i, x_{i+1}] \) is geodesic in the subcomplex \( S_i \cup S_{i+1} \).

The length of \( \Sigma \) is defined as:

\[
\ell(\Sigma) = \sum_{i=0}^{m-1} d_{S_i}(x_i, x_{i+1})
\]

where \( d_S \) denotes the standard \( M^\kappa \)-metric on a geodesic simplex \( S \) of \( M^\kappa \). Then, any string can be identified to a path in \( K \), and the natural quotient pseudometric on \( K \) coincides with the following ([BH13], Lemma I.7.21):

\[
d_K(x, y) = \inf\{\ell(\Sigma) \text{ s.t. } \Sigma \text{ is a taut string from } x \text{ to } y\}.
\]

Moreover, for any \( x \in K \) one can define the number

\[
\varepsilon(x) = \inf_{S \text{ simplex of } K} \left( \inf_{x \in S} \inf_{T \text{ face of } S} d_S(x, T) \right)
\]

which has the following fundamental property:

**Lemma 7.1** (Lemma I.7.9 and Corollary I.7.10 of [BH13]). If \( \varepsilon(x) > 0 \) for any \( x \) and \( K \) is connected then \( d_K \) is a metric and \((K, d_K)\) is a length space. Moreover if \( y \in K \) satisfies \( d_K(x, y) < \varepsilon(x) \) then any simplex \( S \) containing \( y \) contains also \( x \) and \( d_K(x, y) = d_S(x, y) \).

For any vertex \( v \in K \) it is possible to define the link \( \text{Lk}(v, K) \) of \( K \) at \( v \) as follows. We fix any \( \lambda \in \Lambda \) such that \( v = p(\lambda) \), where \( \lambda \) is a vertex of \( S_\lambda \). The set of unit vectors \( w \) of \( T_v M^n_\kappa \) such that the geodesic starting in direction \( w \) stays inside \( S_\lambda \) for a small time is a geodesic simplex of \( M^n_{\kappa-1} = S^{n-1} \), denoted \( \text{Lk}(\lambda, S_\lambda) \). Consider the equivalence relation on the disjoint union \( \bigsqcup_{p(S_\lambda) \geq v} S_\lambda \), given by \( w_\lambda \sim w_{\lambda'} \) if and only if \( p(S_\lambda) \cap p(S_{\lambda'}) \neq \emptyset \) and
We say that $K$ has valency at most $N$ if for all $v \in K$ the number of simplices having $v$ as a vertex is bounded above by $N$. Notice that if the valency is at most $N$, then the maximal dimension of a simplex of $K$ is at most $N$ too. We say that a simplex $S$ has size bounded by $R > 0$ if it contains a ball of radius $\frac{1}{R}$ and it is contained in a ball of radius $R$; accordingly, we say the simplicial complex $K$ has size bounded by $R$ if all the simplices $S_{\lambda}$ defining $K$ have size bounded by $R$.

**Lemma 7.2.** Let $S$ be a $M^n$-simplex of dimension $n$ and size bounded by $R$. Then any face of $S$ of dimension $d$ has size bounded by $2^{n-d}R$.

**Proof.** We prove the lemma by induction on the dimension $n$. If $n = 0, 1$ there is nothing to prove. Assume now that the bounds hold for all faces of $M^n$-simplices of dimension $\leq n-1$, and consider a $n$-dimensional $M^n$-simplex $S = \text{Conv}(v_0, \ldots, v_n)$ of size bounded by $R$. Let $S' = \text{Conv}(v_0, \ldots, v_{n-1})$ be the face of $S$ opposite to $v_n$, and identify $M^n_{n-1}$ with the $\kappa$-model space containing $S'$. It is clear that $S'$ is contained in a ball $B_{M^n_{n-1}}(x, 2R)$ of $M^n_{n-1}$. On the other hand, let $B_{M^n_n}(x, \frac{R}{2})$ be the ball of $M^n_n$ which is contained in $S$. Call $\psi : S \to S'$ the map sending every point $z$ of $S$ to the intersection of the extension of the geodesic $[v_n, z]$ after $z$ with $S'$, and let $y = \psi(x)$; moreover, let $\varphi$ be the contraction map centered at $v_n$ sending $y$ to $x$. Notice that $\psi \circ \varphi(z) = z$ for all $z \in M^n_{n-1}$. The map $\varphi$ is at most $2$-Lipschitz, so any point of $B_{M^n_n}(y, \frac{R}{2})$ is sent to $B_{M^n_n}(x, \frac{R}{2})$ under $\varphi$. Therefore,

$$B_{M^n_{n-1}}(y, \frac{R}{2}) = B \left( y, \frac{R}{2} \right) \cap M^n_{n-1} \subset \psi \left( B_{M^n_n}(x, \frac{R}{2}) \right) \subset S'$$

which proves the induction step. \[ \square \]

**Proposition 7.3.** The class of $n$-dimensional $\kappa$-simplices of size bounded by $R$ and having a fixed point $o$ as a vertex is compact under the Hausdorff distance on $M^n_n$. Moreover, under this convergence, any face of the limit space is limit of faces of the simplices in the sequence. Finally, the same class is closed under ultralimits.

**Proof.** We take a sequence of simplices $S_l$ as in the assumption. We denote by $v^0_l = o, v^1_l, \ldots, v^n_l$ the vertices of $S_l$. All the sequences $(v^i_l)$ are contained in a compact subset of $M^n_n$, so up to subsequence they converge to $v_i$ for all $i = 0, \ldots, n$, in particular $v_0 = o$. Then, the $\varepsilon$-neighbourhood $\text{Conv}(v_0, \ldots, v_n)_\varepsilon$ of $\text{Conv}(v_0, \ldots, v_n)$ is a convex subset of $M^n_n$ which definitely contains $v^0_l = o, v^1_l, \ldots, v^n_l$, hence $\text{Conv}(v_0, \ldots, v_n)_\varepsilon \supset \text{Conv}(v^0_l = o, v^1_l, \ldots, v^n_l)$. Analogously, $\text{Conv}(v_0, \ldots, v_n) \subset \text{Conv}(v^0_l = o, v^1_l, \ldots, v^n_l)_\varepsilon$ definitely, hence
Conv\((v_0, \ldots, v_n) \rightarrow Conv(v_0^l = a, v_1^l, \ldots, v_n^l)\) for the Hausdorff distance. Similarly, any face of \(S\) is limit of corresponding faces of \(S_l\). We now claim that \(v_0, \ldots, v_n\) are in general position. If not, then there are three vertices, say \(v_0, v_1, v_2\), belonging to the same 1-dimensional space. This means the faces Conv\((v_0^l, v_1^l, v_2^l)\) tend to a 1-dimensional face, therefore they cannot have size bounded below uniformly, which contradicts Lemma 7.2. Therefore \(S\) is a \(n\)-dimensional simplex. Moreover it is clear it is contained in a ball of radius \(R\) and it contains a ball of radius \(\frac{1}{\kappa}\). Fix now any non-principal ultrafilter \(\omega\) and a sequence \(S_l\) as above. Each \(S_l\) is proper and the sequence converges in the Gromov-Hausdorff sense to the proper space \(S\). Then by Proposition A.11 we get that the ultralimit \(S_\omega\) is isometric to \(S\).

Clearly the same conclusion holds for the class of simplices of dimension at most \(n\) and size bounded by \(R\) since it is the finite union of compact classes. From this compactness result we get useful uniform estimates.

**Lemma 7.4.** Let \(K\) be a \(M^s\)-complex of size bounded by \(R\) and \(\dim(K) \leq n\). Then there exists a constant \(\varepsilon_0(R, n) > 0\) depending only on \(R\) and \(n\) such that for all vertices \(v, w\) of \(K\) it holds \(\varepsilon(v) > \varepsilon_0(R, n)\) and \(d_K(v, w) \geq \varepsilon_0(R, n)\).

**Proof.** The class of simplices with size bounded by \(2^{n-d}R\) and dimension exactly \(d\) is compact with respect to the Hausdorff distance of \(M^s\) by 7.3. Moreover the map \(\text{Conv}(v_0, \ldots, v_d) \mapsto d_{M^s}(v_0, \text{Conv}(v_1, \ldots, v_d))\) is continuous with respect to the Hausdorff distance and it is positive. Therefore it attains a global minimum \(\varepsilon_d > 0\). Setting \(\varepsilon_0(R, n) = \min_{d=0, \ldots, n} \varepsilon_d\), we have \(\varepsilon(v) \geq \varepsilon_0(R, n) > 0\) for every vertex \(v \in K\). Therefore, every two vertices \(v, w\) of \(K\) satisfy \(d_K(v, w) \geq \varepsilon_0(R, n)\) (or, by Lemma 7.1, there would exist a simplex \(S\) of \(K\) such that \(d_K(v, w) = d_S(v, w) < \varepsilon_0(R, n)\), a contradiction).

**Lemma 7.5.** Let \(S\) be a \(K\)-simplex of size bounded by \(R\) and \(\dim(S) \leq n\). Let \(\partial T_\varepsilon\) denote the \(\varepsilon\)-neighbourhood of the boundary of any face \(T\) of \(S\). For any positive \(\tau\) there exists \(\varepsilon(R, n, \tau) > 0\) such that for all faces \(T\) of \(S\), for all \(x \in T \setminus \partial T_\tau\) and all the faces \(T'\) of \(S\) which do not contain \(x\) it holds:

\[
d(x, T') \geq \varepsilon(R, n, \tau)
\]

Moreover, for any integer \(d \geq 0\) there exist \(\eta_d = \eta_d(R, n), \varepsilon_d = \varepsilon_d(R, n) > 0\), where \(\varepsilon_0 = \varepsilon_0(R, n)\) is the function given by Lemma 7.4 and \(\eta_0 = \frac{\varepsilon_0}{8(n+1)^7}\), satisfying the following conditions:

(a) for all \(d\)-dimensional faces \(T\) of \(S\), for every \(x \in T \setminus \partial T_{\eta_d-1}\) and every face \(T'\) of \(S\) not containing \(x\) it holds: \(d(x, T') \geq \varepsilon_d\);

(b) \(\eta_k + \eta_{k+1} + \cdots + \eta_m \leq \frac{\varepsilon_0}{8},\) for all \(0 \leq k \leq m \leq n\).
Proof. The proof follows same arguments of Lemma 7.4. Indeed it is sufficient to consider the positive, lower semicontinuous map

\[ h(S) = \min_{T \text{ face of } S} \inf_{x \in T \setminus \partial T} \min_{x \neq T'} d(x, T') \]

on the compact set of \( \kappa \)-simplices of size bounded by \( R \) and dimension at most \( n \), and take as \( \varepsilon(R, n, \tau) \) its positive minimum.

To prove the second part of the Lemma, we define \( \varepsilon_1(R, n) \) as \( \varepsilon(R, n, \eta_0) \), where this is the number given by the first statement with \( \tau = \eta_0 \). Then, we choose \( 0 < \eta_1 = \min\{\varepsilon_1, \frac{\varepsilon_1}{2}\} \) and again we define \( \varepsilon_2 > 0 \) as \( \varepsilon(R, n, \eta_1) \).

We can continue choosing \( 0 < \eta_2 = \min\{\varepsilon_2, \frac{\varepsilon_2}{2}\} \) and so on. This process produces the announced \( \varepsilon_i, \eta_i \), which clearly satisfy (b). \( \square \)

As a consequence, we get the following useful estimates (the second of which is similar to Lemma 1.7.54 of [BH13]):

**Lemma 7.6.** Let \( K \) be a \( M^n \)-complex of size bounded by \( R \) and \( \text{dim}(K) \leq n \). For all \( \tau > 0 \) there exists \( \kappa \) such that there exists \( \varepsilon(R, n, \tau) > 0 \) with the following property: for all \( x \in K \) whose support is \( S \) satisfying \( d_S(x, \partial S) \geq \tau \) we have \( \varepsilon(x) \geq \varepsilon(R, n, \tau) \).

In particular, if \( K \) is connected then \( (K, d_K) \) is a length metric space.

**Proof.** Let \( x \in K \). Any simplex containing \( x \) must contain \( \text{supp}(x) \) as a face. It is then enough to apply the first claim of Lemma 7.5 to get the estimate on \( \varepsilon(x) \). The second part follows immediately from Lemma 7.4. \( \square \)

**Lemma 7.7.** Let \( K \) be a \( M^n \)-complex of size bounded by \( R \) and \( \text{dim}(K) \leq n \). Then, there exists \( \delta = \delta(R, n) > 0 \) depending only on \( R \) and \( n \) such that:

(a) if two simplices \( S, S' \) of \( K \) are at distance \( \leq \delta(R, N) \), they share a face;

(b) moreover, for every \( x \in K \) the ball \( \overline{B}(x, \delta) \) is contained in the open star of some vertex;

(c) finally, for every \( x \in K \) there exists \( y \in K \) such that \( \overline{B}(x, \delta) \subset \overline{B}(y, \varepsilon(y)) \) (where \( \varepsilon(y) \) is the function defined in (15)).

**Proof.** We start proving (c). Consider the numbers \( \varepsilon_d, \eta_d \) given by Lemma 7.5

The claim is that \( \delta = \min_{d=0,\ldots,n} \eta_d \) satisfies the thesis of (c). Actually, take any \( x \in K \) and consider the \( d \)-dimensional simplex \( S = \text{supp}(x) \). There are two possibilities: either \( x \in S \setminus \partial S_{d-1} \) or there exists a point \( y_1 \in \partial S \) such that \( d(x, y_1) \leq \eta_{d-1} \). In the first case we observe that any simplex \( S' \) containing \( x \) must have \( S \) has a face and by Lemma 7.5 we can conclude that \( \varepsilon(x) \geq \varepsilon_d \). Hence, in this case \( \overline{B}(x, \delta) \subset \overline{B}(x, \varepsilon_d) \subset \overline{B}(x, \varepsilon(x)) \) as follows by Lemma 7.3 (b). Otherwise, let \( S_1 = \text{supp}(y_1) \) and call \( 0 \leq d_1 \leq 41 \).
Arguing as before, we find that either \( \eta(y_1) \geq \varepsilon d_1 \), or there exists again a point \( y_2 \) whose support \( S_2 \) has dimension \( 0 \leq d_2 < d_1 \) such that \( d(y_1, y_2) \leq \eta d_1 - 1 \). In the first case we have
\[
\overline{B}(x, \delta) \subset \overline{B}(y_1, \eta d_1) \subset \overline{B} \left( y_1, \frac{\varepsilon d_1}{4} \right) \subset \overline{B} \left( y_1, \frac{\varepsilon(y_1)}{4} \right),
\]
otherwise we continue the procedure inductively. Then either at some step we have the thesis, or we find a vertex \( v \) of \( K \) such that
\[
d(x, v) \leq \eta d_1 + \eta d_2 + \ldots + \eta_0 \leq \frac{\varepsilon_0}{8}.
\]
Therefore \( \overline{B}(x, \delta) \subset \overline{B} \left( v, \frac{\varepsilon_0(R, n)}{4} \right) \subset \overline{B} \left( v, \frac{\varepsilon(v)}{4} \right) \), which proves (c).

In order to prove (b) we fix \( x \in K \) and we find the corresponding \( y \) given by (c).

Then for all point \( z \in \overline{B}(x, \delta) \) we can apply Lemma 7.1 and find that any simplex \( S \) containing \( z \) must contain also \( y \). This means that any such \( S \) has the vertices of \( \text{supp}(y) \) as vertices. This concludes the proof of (b).

Finally, the proof of (a) is an easy consequence: suppose to have two points \( x \) and \( x' \), belonging to two simplices \( S, S' \) respectively, such that \( d(x, x') \leq \delta \); then, they belong to the open star of a same vertex by (b). In particular \( S \) and \( S' \) share a vertex.

Another straightforward application of compactness and continuity yields the following, whose proof is omitted:

**Lemma 7.8.** Let \( K \) be a \( M^\kappa \)-complex of size bounded by \( R \) and \( \dim(K) \leq n \). Then, there exists \( R' = R'(R, n) \) depending only on \( R \) and \( n \) such that for every vertex \( v \) of \( K \) the \( M^1 \)-complex \( \text{Lk}(v, K) \) has size bounded by \( R' \).

We start now considering \( M^\kappa \)-complexes with bounded size and valency:

**Proposition 7.9.** Let \( K \) be a connected \( M^\kappa \)-complex of size bounded by \( R \) and valency at most \( N \). Then \( K \) is locally finite (i.e. for all \( x \in K \) there are a finite number of simplices containing \( x \)) and \((K, d_K)\) is a proper, geodesic metric space.

**Proof.** Any simplex \( S \) containing a point \( x \) must have \( \text{supp}(x) \) as a face; in particular, if \( v \) is a vertex of \( \text{supp}(x) \), then it is also a vertex of \( S \). So the number of simplices containing \( x \) is bounded by the number of simplices containing \( v \), which is bounded by \( N \) by assumption. By Lemma 7.6, we know that \((K, d_K)\) is a length metric space. Finally, by Lemma 7.7 for all \( y \in K \) the ball \( \overline{B}(y, \delta) \) belongs to the open star of a vertex, which is the union of a finite number of simplices, hence \( K \) is locally compact and complete. Then, as \( K \) is a complete, locally compact, length metric space, it is proper and geodesic by Hopf-Rinow’s Theorem.
The following is the analogue of Theorem I.7.28 of [BH13]:

**Proposition 7.10.** Let $K$ be a connected $M^\kappa$-complex of size bounded by $R$ and valency at most $N$. Then for any $\ell > 0$ there exists $m_0 = m_0(\ell, R, N)$ depending only on $\ell, R$ and $N$ such that any $m$-taut string of length $\leq \ell$ satisfies $m \leq m_0$.

**Proof.** We use the same proof of Theorem I.7.28 of [BH13] (which is for $M^\kappa$-complexes of finite shape), proceeding by induction on the dimension of $K$. The first step is to prove that if a $m$-string $\Sigma$ is included in the open star of a vertex $v$, then $m$ is bounded by a function $m_0'(\ell, R, N)$. This is clear with $m_0' = 3$ if the geodesic associated to $\Sigma$ passes through $v$, otherwise it follows by the inductive hypothesis by projecting radially $\Sigma$ to $Lk(v, K)$ (which has lower dimension), using Lemma 7.8.

Now, if the bound stated in the proposition did not hold, there would exist tout $m$-strings $\Sigma_i$ in $M^\kappa$-complexes $K_i$ with length $\leq \ell$ and arbitrary large $m$. Then, there would exist also tout $m'$-substrings $\Sigma'_i$ of the $\Sigma_i$, with $m' > m_0'(\ell, R, N)$, included in some ball $B(x_i, \delta) \subset K_i$, for $\delta = \delta(R, N)$ defined in Lemma 7.7. By the same Lemma, $\Sigma'_i$ would be included in the open star of some vertex, which by step one implies that $m' \leq m_0'(\ell, R, N)$, a contradiction.

**Corollary 7.11.** Let $K$ be a connected $M^\kappa$-complex of size bounded by $R$ and valency at most $N$. Let $x, y \in K$ such that $d_K(x, y) \leq \ell$. Then, there exists a geodesic joining $x$ to $y$ realized as the concatenation of at most $m_0(\ell, R, N)$ geodesic segments, each contained in a simplex of $K$.

**Proof.** Immediate from the fact that $K$ is a geodesic space (by 7.9), the characterization of $d_K$ in terms of tout strings and Proposition 7.10.

In order to establish if a $M^\kappa$-complex is a locally CAT($\kappa$) space we use the following improvement of a well-known criteria. We recall that the injectivity radius of a complex $K$, denoted $\rho_{m_0}(K)$, is defined as the supremum of the $r \geq 0$ such that any two points of $K$ that are at distance at most $r$ are joined by a unique geodesic.

**Proposition 7.12.** Let $K$ be a connected $M^\kappa$-complex of size bounded by $R$ and valency at most $N$. The following facts are equivalent:

(a) $(K, d_K)$ is locally CAT($\kappa$);

(b) $K$ satisfies the link condition, i.e. the link at any vertex is CAT(1);

(c) $(K, d_K)$ is locally uniquely geodesic;

(d) $(K, d_K)$ has positive injectivity radius;

(e) $\rho_{m_0}(K) \geq \delta(R, N)$, where $\delta(R, N)$ is the function defined in Lemma 7.7.
Moreover if \( K \) satisfies one of the equivalent conditions above, then for any \( x \in K \) the ball \( B(x, \delta(R, N)) \) is a CAT(\( \kappa \)) space, i.e. the CAT(\( \kappa \))-radius of \( K \) is at least \( \delta(R, N) \).

The equivalences between (a), (b) and (c) are quite standard. The equivalence of these conditions with (d) is known for simplicial complexes with finite shapes, see [BH13]. The main point of Proposition 7.12 is that the last equivalence continues to hold in our setting, and moreover we can bound from below the injectivity radius of \( K \) in terms of \( R \) and \( N \) only.

Proof of Proposition 7.12. The equivalence between (a) and (b) follows from Theorem II.5.2 and Remark II.5.3 of [BH13], while (a) \( \Rightarrow \) (c) is straightforward. The implication (c) \( \Rightarrow \) (e) follows as in Proposition I.7.55 of [BH13]. Actually, by Proposition 7.9 we have \( \varepsilon(x) > 0 \) for every \( x \in K \), so the ball \( B(x, \varepsilon(x)/2) \) is isometric to the open ball \( B(O, \varepsilon(x)/2) \) of the \( \kappa \)-cone \( C_\kappa(Lk(v, K)) \) centred at the cone point \( O \) (cp. Theorem I.7.39 in [BH13]). Moreover by assumption a neighbourhood of \( O \) of the cone \( C_\kappa(Lk(v, K)) \) is uniquely geodesic, which implies that the whole \( C_\kappa(Lk(v, K)) \) is uniquely geodesic (cp. Corollary I.5.11, [BH13]), and this in turns implies that \( B(x, \varepsilon(x)/2) \) is.

By Lemma 7.7(c), we conclude that the injectivity radius is bounded below by \( \delta(R, N) \) (recall that the dimension of \( K \) is bounded above by \( N \)). The implication (e) \( \Rightarrow \) (d) is obvious, while (c) \( \Rightarrow \) (b) follows exactly as in Theorem II.5.4 of [BH13]. Finally, the last remark follows from Theorem I.7.39 & Theorem II.3.14 of [BH13] together with Lemma 7.7(c).

We recall that a locally compact, locally CAT(\( \kappa \)) \( M^\kappa \)-complex is locally geodesically complete if and only if it has no free faces (see II.5.9 and II.5.10 of [BH13] for the definition of having free faces and the proof of this fact). We can finally show that the class of metric spaces we are studying in this section is uniformly packed.

**Proposition 7.13.** Let \( K \) be a connected \( M^\kappa \)-complex without free faces, of size bounded by \( R \), valency at most \( N \) and positive injectivity radius. Then, \( K \) is a proper, geodesic GCBA\(^\kappa\)-space with \( \rho_{\text{cat}}(K) \geq \rho_0 \) and satisfying \( \text{Pack}(3r_0, \frac{\rho_0}{3}) \leq P_0 \), for constants \( \rho_0, P_0, r_0 \) depending only on \( R, N \) and \( \kappa \), and \( r_0 \leq \rho_0/3 \).

**Proof.** By the proof of Proposition 7.10 we know that \( K \) is proper and geodesic. Moreover since the injectivity radius is positive then \( K \) is locally CAT(\( \kappa \)), and by Proposition 7.12 the CAT(\( \kappa \))-radius of \( K \) is at least \( \rho_0 = \delta(N, R) \). Since \( K \) has no free faces then it is locally geodesically complete. This shows that \( K \) is also a GCBA\(^\kappa\)-space. We remark that clearly \( H^k(K) = 0 \) if \( k > N \) since the projection map from a simplex to \( K \) is 1-Lipschitz; this shows that there are no points of dimension greater than \( N \), i.e. \( \dim(K) \leq N \). We now use Lemma 7.7 to estimate the number of simplices intersecting a

[118x704] Moreover if \( K \) satisfies one of the equivalent conditions above, then for any \( x \in K \) the ball \( B(x, \delta(R, N)) \) is a CAT(\( \kappa \)) space, i.e. the CAT(\( \kappa \))-radius of \( K \) is at least \( \delta(R, N) \).

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Proof of Proposition 7.12. The equivalence between (a) and (b) follows from Theorem II.5.2 and Remark II.5.3 of [BH13], while (a) \( \Rightarrow \) (c) is straightforward. The implication (c) \( \Rightarrow \) (e) follows as in Proposition I.7.55 of [BH13]. Actually, by Proposition 7.9 we have \( \varepsilon(x) > 0 \) for every \( x \in K \), so the ball \( B(x, \varepsilon(x)/2) \) is isometric to the open ball \( B(O, \varepsilon(x)/2) \) of the \( \kappa \)-cone \( C_\kappa(Lk(v, K)) \) centred at the cone point \( O \) (cp. Theorem I.7.39 in [BH13]). Moreover by assumption a neighbourhood of \( O \) of the cone \( C_\kappa(Lk(v, K)) \) is uniquely geodesic, which implies that the whole \( C_\kappa(Lk(v, K)) \) is uniquely geodesic (cp. Corollary I.5.11, [BH13]), and this in turns implies that \( B(x, \varepsilon(x)/2) \) is.

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**Proof.** By the proof of Proposition 7.10 we know that \( K \) is proper and geodesic. Moreover since the injectivity radius is positive then \( K \) is locally CAT(\( \kappa \)), and by Proposition 7.12 the CAT(\( \kappa \))-radius of \( K \) is at least \( \rho_0 = \delta(N, R) \). Since \( K \) has no free faces then it is locally geodesically complete. This shows that \( K \) is also a GCBA\(^\kappa\)-space. We remark that clearly \( H^k(K) = 0 \) if \( k > N \) since the projection map from a simplex to \( K \) is 1-Lipschitz; this shows that there are no points of dimension greater than \( N \), i.e. \( \dim(K) \leq N \). We now use Lemma 7.7 to estimate the number of simplices intersecting a
ball around any point \(x \in K\). Any simplex \(S\) which intersect \(\mathcal{B}(x, \delta(R, N))\) intersects the open star around some vertex \(v\), by Lemma 7.7.(b). Therefore \(v\) must be a vertex of \(S\). If follows that the number of simplices intersecting \(\mathcal{B}(x, \delta(R, N))\) intersects the open star around some vertex \(v\), by Lemma 7.7.(b). Therefore \(v\) must be a vertex of \(S\). If follows that the number of simplices intersecting \(\mathcal{B}(x, \delta(R, N))\) is bounded by \(N\). Therefore, for any \(x \in K\) we have

\[
\mu_K(B(x, \delta(R, N))) \leq \sum_{d=0}^{N} N \cdot \mathcal{H}^d\left(B_{M^\kappa}(o, \delta(R, N))\right) \leq V_0,
\]

where \(V_0\) depends just on \(R, N\) and \(\kappa\) (here \(o\) is any point of \(M^\kappa\)).

The conclusion follows from Theorem 4.9.

7.2. Compactness of \(M^\kappa\)-complexes

The aim of this section is to provide compactness and finiteness results for simplicial complexes. We denote by \(M^\kappa(R, N)\) the class of \(M^\kappa\)-complexes without free faces, of size bounded by \(R\), valency at most \(N\) and positive injectivity radius.

**Theorem 7.14.** The class \(M^\kappa(R, N)\) is compact under pointed Gromov-Hausdorff convergence.

By Proposition 7.13 there exist \(P_0, r_0, \rho_0\) such that any \(K \in M^\kappa(R, N)\) belongs to \(\text{GCBA}^\kappa_{\text{pack}}(P_0, r_0; \rho_0)\). So, by Theorem 6.4 the class \(M^\kappa(R, N)\) is precompact, and it is compact if and only if it is closed under ultralimits.

We are going now to show that \(M^\kappa(R, N)\) is closed under ultralimits. We fix a non-principal ultrafilter \(\omega\) and we take any sequence \((K_n, o_n)\) in \(M^\kappa(R, N)\). We denote by \(K_\omega\) the ultralimit of this sequence. Our aim is to prove that \(K_\omega\) is isometric to a \(M^\kappa\)-complex \(\hat{K}_\omega\) satisfying the same conditions as the \(K_n\)’s.

**Step 1: construction of the simplicial complex \(\hat{K}_\omega\).**

Let us start defining who are the simplices of \(\hat{K}_\omega\). Let \((x_n)\) be any admissible sequence of points, with \(x_n \in K_n\), and consider the unique simplex \(\text{sup}(x_n)\) of \(K_n\) containing \(x_n\) in its interior: we define \(S(x_n) = \omega\text{-limsup}(x_n)\).

The metric space \(S(x_n)\) is a \(\kappa\)-complex with size bounded by \(R\) by 7.3. Notice that, a priori, if \(y_n\) is another sequence defining the same point as \(x_n\) in \(K_\omega\), then \(S(y_n)\) might be different from \(S(x_n)\).

Now we define \(\hat{K}_\omega\) as follows. Let \(p_n : S \to K_n\) denote the projection of any simplex of the total space of \(K_n\) to \(K_n\). The total space of \(\hat{K}_\omega\) will be

\[
\bigsqcup_{(x_n) \text{ admissible}} S(x_n)
\]

where \((x_n)\) is any admissible sequence of points in \(K_n\), and the equivalence relation is: if \(z_\omega = \omega\text{-lim } z_n \in S(x_n)\) and \(z'_\omega = \omega\text{-lim } z'_n \in S(x'_n)\) (i.e. \((z_n), (z'_n)\)
are admissible sequences of points respectively in \( \text{supp}(x_n) \) and \( \text{supp}(x'_n) \), we say that \( z_\omega \sim z'_\omega \) if and only if \( \omega \text{-}\lim d_{K_n}(p_n(z_n), p_n(z'_n)) = 0 \). That is, we compare the points \( z_n \) and \( z'_n \) in the common space \( K_n \) where they live. For simplicity we will abbreviate \( d_{K_n}(p_n(z_n), p_n(z'_n)) \) with \( d_{K_n}(z_n, z'_n) \). First of all we need to check that the relation is well defined: given other admissible sequences \( (w_n), (w'_n) \) with \( w_n \in \text{supp}(x_n) \) and \( w'_n \in \text{supp}(x'_n) \) such that \( z_\omega = \omega \text{-}\lim w_n \) and \( z'_\omega = \omega \text{-}\lim w'_n \), we have

\[
d_{K_n}(w_n, w'_n) \leq d_{\text{supp}(x_n)}(w_n, z_n) + d_{K_n}(z_n, z'_n) + d_{\text{supp}(x'_n)}(z'_n, w'_n)
\]
hence \( \omega \text{-}\lim d_{K_n}(w_n, w'_n) = 0 \). Once proved it is well defined it is easy to show it is an equivalence relation. We call \( \hat{K}_\omega \) the quotient space, and denote \( p_\omega : S(x_n) \to \hat{K}_\omega \) the projections.

**Step 2:** \( \hat{K}_\omega \) satisfies axiom (i) of \( M^\varepsilon \)-complexes.

We fix an admissible sequence \( (x_n) \) and the corresponding simplex \( S(x_n) \). We need to prove that the map \( p_\omega : S(x_n) \to \hat{K}_\omega \) is injective. For this, consider points \( z_\omega = \omega \text{-}\lim z_n \) and \( z'_\omega = \omega \text{-}\lim z'_n \) in \( S(x_n) \), with \( z_n, z'_n \in \text{supp}(x_n) \) for all \( n \); then there exists \( \varepsilon_0 > 0 \) such that \( \omega \text{-}\lim d_{\text{supp}(x_n)}(z_n, z'_n) > \varepsilon_0 > 0 \). In particular \( d_{\text{supp}(x_n)}(z_n, z'_n) > \varepsilon_0 \) \( \omega \text{-a.e.}(n) \). Now, for any point \( z \) of a \( M^\varepsilon \)-complex define \( \dim(z) \) as the dimension of \( \text{supp}(z) \). The strategy to prove the injectivity is by induction on

\[
d = \max\{\omega \text{-}\lim \dim(z_n), \omega \text{-}\lim \dim(z'_n)\}.
\]

Observe that if \( \omega \text{-}\lim \dim(z_n) = k \) then we have \( \dim(z_n) = k \ \omega \text{-a.e.}(n) \) because the possible dimensions belong to a finite set. For \( d = 0 \), we have that \( z_n, z'_n \) are both vertices of \( \text{supp}(x_n) \), \( \omega \text{-a.e.}(n) \). If \( p_\omega \) is not injective then for every \( \varepsilon > 0 \) we have \( d_{K_n}(z_n, z'_n) \leq \varepsilon \ \omega \text{-a.e.}(n) \). By Lemma 7.3 we know that if \( d_{K_n}(z_n, z'_n) \leq \varepsilon_0(R, N) \) then \( z_n = z'_n \) as points of \( \text{supp}(x_n) \). We consider now the inductive step. We denote by \( T_n, T'_n \) the faces of \( S_n \) containing \( z_n \) and \( z'_n \) in their interior, respectively. We suppose there exists \( \tau > 0 \) such that for \( \omega \text{-a.e.}(n) \) it holds \( z_n \in T_n \setminus (\partial T_n)_\tau \). By Lemma 7.4 we have \( \varepsilon(z_n) \geq \varepsilon(R, N, \tau) \ \omega \text{-a.e.}(n) \), and similarly for \( z'_n \). Once again this fact implies the injectivity. Consider now the case where for all \( \tau > 0 \) the set

\[
\{n \in \mathbb{N} \text{ s.t. } d(z_n, \partial T_n) \leq \tau \text{ and } d(z'_n, \partial T'_n) \leq \tau\}
\]
belongs to \( \omega \). Therefore \( \omega \text{-}\lim d(z_n, \partial T_n) = \omega \text{-}\lim d(z'_n, \partial T'_n) = 0 \). This means that \( z_\omega \) belongs to \( \partial T_\omega \) and \( z'_\omega \) belongs to \( \partial T'_\omega \), by Proposition 7.3. Hence \( z_\omega = \omega \text{-}\lim w_n \) and \( z'_\omega = \omega \text{-}\lim w'_n \), where \( w_n \) and \( w'_n \) belong to a lower dimensional face of \( T_n \) and \( T'_n \) respectively. We then apply the inductive assumption to get the thesis.

**Step 3:** \( \hat{K}_\omega \) satisfies axiom (ii) of \( M^\varepsilon \)-complexes.

Consider two simplices \( S(x_n), S(x'_n) \) and suppose \( p_\omega(S(x_n)) \cap p_\omega(S(x'_n)) \neq \emptyset \). This means that for any \( \varepsilon > 0 \) there exist \( y_\omega = \omega \text{-}\lim y_n \) and \( y'_\omega = \omega \text{-}\lim y'_n \).
with $y_n \in \text{supp}(x_n)$ and $y'_n \in \text{supp}(x'_n)$ such that $d_{K_n}(y_n, y'_n) < \varepsilon$, \(\omega\)-a.e.\((n)\).

If \(\varepsilon < \delta(R, N)\) then by Lemma \[7.7\] \((a)\) we know that $\text{supp}(x_n)$ and $\text{supp}(x'_n)$ share a face in $K_n$. Let then $T_n \subset \text{supp}(x_n)$ and $T'_n \subset \text{supp}(x'_n)$ such faces and $h_n: T_n \to T'_n$ an isometry such that $p_n(z) = p_n(z')$ for $z \in T_n$, $z' \in T'_n$ if and only if $z' = h_n(z)$. By assumption this holds $\omega$-a.e.\((n)\). By Proposition \[7.3\] it is easy to see that the metric spaces $T_\omega = \omega\text{-lim} T_n$ and $T'_\omega = \omega\text{-lim} T'_n$ are, respectively, faces of $S(x_n)$ and $S(x'_n)$. Moreover the sequence of maps $(h_n)$ defines a limit map $h_\omega: T_\omega \to T'_\omega$ which is an isometry, by Proposition \[A.3\] It remains to show that $p_\omega(z_\omega) = p_\omega(z'_\omega)$, for $z_\omega \in T_\omega$ and $z'_\omega \in T'_\omega$, if and only if $h_\omega(z_\omega) = z'_\omega$. But given $z_\omega = \omega\text{-lim} z_n$ and $z'_\omega = \omega\text{-lim} z'_n$ with $z_n \in \text{supp}(x_n)$, $z'_n \in \text{supp}(x'_n)$ we have $p_\omega(z_\omega) = p_\omega(z'_\omega)$ by definition if and only if $\omega\text{-lim} d_{K_n}(p_n(z_n), p_n(z'_n)) = 0$. This happens if and only if for any $\varepsilon > 0$ the inequality

$$d_{K_n}(p_n(z_n), p_n(z'_n)) < \varepsilon$$

holds $\omega$-a.e.\((n)\). This means that $d_{K_n}(p_n(h_n(z_n)), p_n(z'_n)) < \varepsilon$ holds $\omega$-a.e.\((n)\), in particular $p_\omega(h_\omega(z_\omega)) = p_\omega(z'_\omega)$. By the injectivity of the projection map $p_\omega$ we then obtain $h_\omega(z_\omega) = z'_\omega$, which is the thesis.

**Step 4:** $\dot{K}_\omega$ belongs to $M^\omega(R, N)$.

It is clear that $\dot{K}_\omega$ has size bounded by $R$ by construction.

We want to show it has valency at most $N$. Fix a vertex $v$ of $\dot{K}_\omega$ and parameterize by $\alpha \in A$ the set of simplices $S(x_n(\alpha))$ of $\dot{K}_\omega$ having $v$ as a vertex. For any fixed $\alpha \in A$ there is a vertex $v_n(\alpha)$ of $\text{supp}(x_n(\alpha))$ such that the sequence $(v_n(\alpha))$ converges $\omega$-a.e.\((n)\) to $v$, by Proposition \[7.3\] In particular for all $\alpha, \alpha' \in A$ we get $d_{K_n}(v_n(\alpha), v_n(\alpha')) < \varepsilon(R, N)$ $\omega$-a.e.\((n)\), and then $v_n(\alpha) = v_n(\alpha')$ by Lemma \[7.4\] Let now $S(x_n(\alpha)) \neq S(x_n(\alpha'))$ be distinct elements of $\dot{K}_\omega$, for $\alpha, \alpha' \in A$. Then, there exist a vertex of the first simplex $u = \omega\text{-lim} u_n$, with $u_n \in \text{supp}(x_n(\alpha))$, which does not belong to the second one. So, $d_{K_n}(u_n, \text{supp}(x_n(\alpha'))) > 0$ $\omega$-a.e.\((n)\), hence $\text{supp}(x_n(\alpha)) \neq \text{supp}(x_n(\alpha'))$ $\omega$-a.e.\((n)\). Therefore, if $\dot{K}_\omega$ has $m$ different simplices $S(x_n(\alpha))$ sharing the vertex $v$, there also exist $m$ different simplices $\text{supp}(x_n(\alpha))$ of $K_n$ sharing the same vertex $v_n(\alpha)$, $\omega$-a.e.\((n)\). This contradicts our assumptions if $m > N$.

Finally, the fact that $\dot{K}_\omega$ has positive injectivity radius and has no free faces will follow from the last step, where we prove that $\dot{K}_\omega$ and $K_\omega$ are isometric.

In fact, $\dot{K}_\omega$ is geodesically complete and locally CAT($\kappa$), as ultralimit of complete, geodesically complete, locally CAT($\kappa$) spaces with CAT($\kappa$)-radius uniform bounded below; hence, $\dot{K}_\omega$ (and in turns $K_\omega$) has positive injectivity radius and no free faces, by Proposition \[7.12\] and II.5.9&II.5.10 of \[BH13\].

**Step 5:** $\dot{K}_\omega$ is isometric to $K_\omega$.

We define a map $\Phi: K_\omega \to \dot{K}_\omega$ as follows. Let $y_\omega = \omega\text{-lim} y_n$ the $\omega$-limit of $y_n \in \text{supp}(y_n)$ of $K_n$. Any $y_n$ belongs to $\text{supp}(y_n)$: we
will denote by \((y_n)_{\text{supp}(y_n)}\) the point, in the ultralimit of the sequence of simplices \(\text{supp}(y_n)\), which is defined by the admissible sequence of points \((y_n)\). We then define \(\Phi\) as
\[
\Phi(y_\omega) = p_\omega((y_n)_{\text{supp}(y_n)}).
\]

It is easy to see it is well defined and surjective.

It remains to prove it is an isometry. Let \(y_n, z_n \in K_n\) define admissible sequences. So, the distances \(d_{K_n}(y_n, z_n)\) are uniformly bounded by some constant \(L\). Therefore by Proposition \([\ldots]\) for any \(n\) there exists a geodesic between \(y_n\) and \(z_n\) which is the concatenation of at most \(m_0(L, R, N)\) segments, each of them contained in a simplex. Since the number of segments is uniformly bounded, we can define a path in \(\hat{K}_\omega\) which is the concatenation of geodesic segments, each contained in a simplex of \(\hat{K}_\omega\), and whose length is the limit of the lengths of the segments in \(K_n\). This shows that
\[
d_{\hat{K}_\omega}(p_\omega((y_n)_{\text{supp}(y_n)}), p_\omega((z_n)_{\text{supp}(z_n)})) \leq \omega^- \lim d_{K_n}(y_n, z_n).
\]

In order to prove the other inequality, we fix two points \(y = p_\omega((y_n)_{\text{supp}(y_n)})\) and \(z = p_\omega((z_n)_{\text{supp}(z_n)})\) of \(\hat{K}_\omega\). Notice that from the inequality above we deduce that \(\hat{K}_\omega\) is path-connected. Hence, by Proposition \([\ldots]\) we know that there exists a geodesic between \(y\) and \(z\) which is the concatenation of at most \(m_0(\ell, R, N)\) geodesic segments, each of them contained in a simplex, where \(\ell = d_{\hat{K}_\omega}(x, y)\). These segments cross finitely many simplices, each of which can be seen as the \(\omega^\pm\)-limit of a sequence of simplices in \(K_n\). Since the number is finite we can see the union of these simplices of \(\hat{K}_\omega\) as the ultralimit of the union of the corresponding simplices in \(K_n\). We can therefore approximate the geodesic in \(\hat{K}_\omega\) with paths in \(K_n\) between \(y_n\) and \(z_n\), whose total length tend to \(\ell\). So
\[
d_{\hat{K}_\omega}(p_\omega((y_n)_{\text{supp}(y_n)}), p_\omega((z_n)_{\text{supp}(z_n)})) \geq \omega^- \lim d_{K_n}(y_n, z_n),
\]
which ends the proof of Theorem \([\ldots]\).

We can specialize this compactness theorem to other families of \(M^\infty\)-complexes, as done for GCBA\textsubscript{pack}\((P_0, r_0; \rho_0)\). Namely, consider:

- the subclass \(M^\infty(R, N; \Delta) \subset M^\infty(R, N)\) of complexes with diameter \(\Delta\);
- the class \(M^\infty(R, V, n)\) of \(M^\infty\)-complexes without free faces, with size bounded by \(R\), total volume \(\leq V\), dimension bounded above by \(n\) and positive injectivity radius.

**Remark 7.15.** We should specify the measure on the complexes \(K\) of the class \(M^\infty(R, V, n)\) under consideration. Any such space is stratified in subspaces of different dimension, so it is natural to consider the measure which

\[3\text{The notation stresses the fact that we see } (y_n)_{\text{supp}(y_n)} \text{ as limit of points in the abstract simplices } \text{supp}(y_n) \text{ (not in } K_n)\). Namely, \((y_n)_{\text{supp}(y_n)}\) belongs to the total space of \(\hat{K}_\omega\), while \(y_\omega \in K_\omega\).
is the sum over \( k = 0, \ldots, n \) of the \( k \)-dimensional Hausdorff measure on each \( k \)-dimensional part. This clearly coincides with the natural measure \( \mu_K \) of \( K \) seen as GCBA-space.

**Corollary 7.16.** For any choice of \( R, n, V, N \) and \( \Delta \), the above classes are compact under Gromov-Hausdorff convergence and contain only finitely many simplicial complexes up to simplicial homeomorphisms.

**Proof.** The compactness of \( M^\kappa(R, N; \Delta) \) is clear from the one of \( M^\kappa(R, N) \). Moreover, by Proposition 7.13 we know that any \( K \in M^\kappa(R, N; \Delta) \) satisfies the condition \( \text{Pack}(3r_0, \frac{\rho_0}{2}) \leq P_0 \) for constants \( P_0, r_0 \) only depending on \( R \) and \( N \). Moreover, by Lemma 7.4 we know that any two vertices of \( K \) are \( \eta(R) \)-separated: in particular, the number of vertices of \( K \) is bounded above by \( \text{Pack}(\frac{\Delta}{2}, \eta(R)) \) which is a number depending only on \( R, N, \kappa \) and \( \Delta \). Since the valency is bounded and the total number of vertices is bounded, we have only finitely many possible simplicial complexes up to simplicial homeomorphisms.

On the other hand, it is straightforward to show that any \( K \in M^\kappa(R; V, n) \) has valency bounded from above by a function depending only on \( R, V, n \) and \( \kappa \), because any simplex of locally maximal dimension contributes to the total volume with a quantity greater than a universal function \( v(R, n, \kappa) > 0 \).

This also shows also that the total number of simplices of \( K \) is uniformly bounded in terms of \( R, V \) and \( n \), hence the combinatorial finiteness of \( M^\kappa(R; V, n) \). Moreover, since any simplex has uniformly bounded size, also the diameters of complexes in this class are uniformly bounded. Therefore, \( M^\kappa(R; V, n) \subset M^\kappa(R, N) \) for a suitable \( N \) and, as the class is made of compact metric spaces, it is actually precompact under (unpointed) Gromov-Hausdorff convergence. It remains to show that \( M^\kappa(R; V, n) \) is closed.

By the proof of Theorem 7.14 it is clear that the upper bound on the dimension of the simplices is preserved under limits. The stability of the upper bound on the total volume is proved as for the class \( \text{GCBA}_\kappa^\text{vol}(V_0, R_0; \rho_0, n_0^\kappa) \) in Corollary 6.7.

Finally, we want to point out that the assumptions on size and diameter in the above compactness results are essential:

**Examples 7.17.** Non-compact families of \( M^\kappa \)-complexes.

1. Let \( X_n \) be a wedge of \( n \) circles of radius 1. The family of \( M^0 \)-complexes \( \{X_n\} \) has uniformly bounded size and uniformly bounded diameter, but the valency is not bounded. Notice that this family is neither finite nor uniformly packed. In particular, it is not precompact.
Let $X_n$ be obtained from a circle of radius 1, then choosing $n$ equidistant points on the circle and gluing $n$ circles of radius 1 to them. The $X_n$’s admit $M^1$-complex structures with uniformly bounded valency and uniformly bounded diameter, but the size of the simplices is not bounded. Again, this family is neither finite nor uniformly packed, hence not pre-compact.

A Ultralimits

An ultrafilter on $\mathbb{N}$ is a subset $\omega$ of $\mathcal{P}(\mathbb{N})$ such that:

1) $\emptyset \notin \omega$;

2) if $A,B \in \omega$ then $A \cap B \in \omega$;

3) if $A \in \omega$ and $A \subset B$ then $B \in \omega$;

4) for any $A \subset \mathbb{N}$ then either $A \in \omega$ or $A^c \in \omega$.

We recall that there is a one-to-one correspondence between the ultrafilters $\omega$ on $\mathbb{N}$ and the finitely-additive measures defined on the whole $\mathcal{P}(\mathbb{N})$ with values on $\{0,1\}$ such that $\omega(\mathbb{N}) = 1$. Indeed given an ultrafilter $\omega$ we define the measure $\omega(A) = 1$ if and only if $A \in \omega$; conversely, given a measure $\omega$ as before we define the ultrafilter as the set $\omega = \{A \subset \mathbb{N} \text{ s.t. } \omega(A) = 1\}$ (it is easy to show it actually is an ultrafilter). In the following, $\omega$ will denote both an ultrafilter and the measure that it defines. Therefore we will write that a property $P(n)$ holds $\omega$-a.s. when the set $\{n \in \mathbb{N} \text{ s.t. } P(n)\} \in \omega$.

There is an easy example of ultrafilter: fix $n \in \mathbb{N}$ and consider the set $\omega$ of subsets of $\mathbb{N}$ containing $n$. An ultrafilter of this type is called principal. The interesting ultrafilters are the non-principal ones; it turns out that an ultrafilter is non-principal if and only if it does not contain any finite set. The existence of non-principal ultrafilters follows from Zorn’s lemma.

The interest on non-principal ultrafilters is due to the fact that they can define a notion of limit of a bounded sequence of real numbers:

**Lemma A.1.** Let $a_n \in [a,b]$ be a bounded sequence of real numbers. Let $\omega$ be a non-principal ultrafilter. Then, there exists a unique point $x$ in $[a,b]$ such that for all $\eta > 0$ the set $\{n \in \mathbb{N} \text{ s.t. } |a_n - x| < \eta\}$ belongs to $\omega$. The real number $x$ is said the $\omega$-limit of the sequence $(a_n)$ and it is denoted by $x = \omega$-lim $a_n$. Moreover, if $a_n$ and $b_n$ are two bounded sequence of real numbers, it holds:

(a) $\omega$-lim$(a_n + b_n) = \omega$-lim $a_n + \omega$-lim $b_n$;

(b) if $\lambda \in \mathbb{R}$ then $\omega$-lim$(\lambda a_n) = \lambda \cdot \omega$-lim $a_n$.
(c) if $a_n \leq b_n$ then $\omega$-lim $a_n \leq \omega$-lim $b_n$;

(d) if $a=\omega$-lim $a_n$ and $f$ is continuous at $a$, then $\omega$-lim $f(a_n)=f(\omega$-lim $a_n)$.

(The proof of the main part can be found in [DK18], Lemma 7.23, while properties (a)-(d) are trivial.)

The ultralimit of unbounded sequences of real numbers can be defined in the following way. Given an unbounded sequence of real numbers $a_n$ the following mutually exclusive situations can occur:

- there exists $L > 0$ such that $a_n \in [-L,L]$ for $\omega$-a.e. $n$.
  In this case the ultralimit of $(a_n)$ can be defined using Lemma A.1.

- for any $L > 0$ the set $\{n \in \mathbb{N} \text{ s.t. } a_n \geq L\}$ belongs to $\omega$.
  In this case we set $\omega$-lim $a_n = +\infty$.

- for any $L < 0$ the set $\{n \in \mathbb{N} \text{ s.t. } a_n \leq -L\}$ belongs to $\omega$.
  In this case we set $\omega$-lim $a_n = -\infty$.

We remark that the limit depends strongly on the non-principal ultrafilter $\omega$. The ultralimit of a sequence of metric spaces is defined as follows.

**Definition A.2.** Let $(X_n, x_n)$ be a sequence of pointed metric spaces and $\omega$ be a non-principal ultrafilter. We set:

$$X = \{(y_n) : y_n \in X_n \text{ and } \exists L > 0 \text{ s.t. } d(y_n, x_n) \leq L \text{ for any } n\}.$$ 

and, for $(y_n), (z_n) \in X$, we define the distance as:

$$d((y_n), (z_n)) = \omega\text{-}\lim d(y_n, z_n).$$

The space $X_\omega = (X, d)/d=0$ is a metric space and it is called the $\omega$-limit of the sequence of spaces $(X_n, x_n)$. The fact that $(X, d)$ is a metric space follows immediately from the properties of the ultralimit of a sequence of real numbers and from the fact that $d_n$ is a distance for any $n$. In general the limit depends on the non-principal ultrafilter $\omega$ and on the basepoints.

A basic example is provided by the ultralimit of a constant sequence.

**Proposition A.3.** Let $(X, x)$ be a metric space and $\omega$ a non-principal ultrafilter.

Consider the constant sequence $(X, x)$ and the corresponding ultralimit $(X_\omega, x_\omega)$, where $x_\omega$ is the constant sequence of points $(x)$. Then

(a) The map $\iota: (X, x) \rightarrow (X_\omega, x_\omega)$ that sends $y$ to the constant sequence $(y_n = y)$ is an isometric embedding;

(b) if $X$ is proper then $\iota$ is surjective, and $(X_\omega, x_\omega)$ is isometric to $(X, x)$. 

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Proof. The first part is obvious by the definitions. If $X$ is proper and $(y_n)$ is an admissible sequence defining a point of $X_\omega$, then it is contained in a closed ball of $X$, that is compact. By Lemma 7.23 of [DK18] we find $y \in X$ such that for all $\varepsilon > 0$ the set

$$\{ n \in \mathbb{N} \text{ s.t. } d(y, y_n) < \varepsilon \}$$

belongs to $\omega$. Therefore it is clear that the constant sequence $(y_n = y)$ defines the same point as the sequence $(y_n)$ in $X_\omega$, which proves (b). \qed

An interesting consequence of the definition is that the ultralimit of pointed metric spaces is always complete (the proof is given in [DK18], Proposition 7.44):

**Proposition A.4.** Let $(X_n, x_n)$ be a sequence of pointed metric spaces and let $\omega$ be a non-principal ultrafilter. Then $X_\omega$ is a complete metric space.

Once defined the limit of pointed metric spaces it is useful to define limit of maps. We take two sequences of pointed metric spaces $(X_n, x_n)$ and $(Y_n, y_n)$. A sequence of maps $f_n: X_n \to Y_n$ is said admissible if there exists $M \in \mathbb{R}$ such that $d(f_n(x_n), y_n) \leq M$ for any $n \in \mathbb{N}$. In general an admissible sequence of maps does not define a limit map, but it is the case if the maps are equi-Lipschitz. A sequence of maps $f_n: X_n \to Y_n$ is equi-Lipschitz if there exists $\lambda \geq 0$ such that $f_n$ is $\lambda$-Lipschitz for any $n$.

**Proposition A.5.** Let $(X_n, x_n), (Y_n, y_n)$ be two sequences of pointed metric spaces. Let $f_n: X_n \to Y_n$ be an admissible sequence of equi-Lipschitz maps. Let $\omega$ be a non-principal ultrafilter. Let $X_\omega$ and $Y_\omega$ be the $\omega$-limits of $(X_n, x_n)$ and $(Y_n, y_n)$ respectively. Define $f = f_\omega: X_\omega \to Y_\omega$ as $f((z_n)) = (f_n(z_n))$. Then:

a) $f$ is well defined;

b) $f$ is Lipschitz with the same constants of the sequence $f_n$.

In particular if for any $n$ the map $f_n$ is an isometry then $f$ is an isometry, while if $f_n$ is an isometric embedding for any $n$ then $f$ is again an isometric embedding.

The map $f = f_\omega$ is called the $\omega$-limit of the sequence of maps $f_n$ and we denote it by $f_\omega = \omega\text{-lim } f_n$. The proof in case of isometric embeddings is given in [DK18], Lemma 7.47; the general case is analogous.

This result can be applied to the special case of geodesic segments, since they are isometric embeddings of an interval into a metric space $X$. However, we first need to explain what is the ultralimit of a sequence of intervals:
Lemma A.6. Let $I_n = [a_n, b_n] \subset \mathbb{R}$ be a sequence of intervals containing 0 (possibly with $a_n = -\infty$ or $b_n = +\infty$). Let $\omega$ be a non-principal ultrafilter. Then $\omega$-lim$(I_n, 0)$ is isometric to $I$, where $I = [\omega$-lim$a_n, \omega$-lim$b_n]$ (possibly with $a = -\infty$ or $b = +\infty$) contains 0.

Proof. We define a map from $I_\omega$ to $I$ as follows. Given an admissible sequence $(x_n)$ such that $x_n \in I_n$ then $x_n$ is $\omega$-a.s. bounded, so it is defined $\omega$-lim$x_n$ by Lemma A.1. We define the map as $(x_n) \mapsto \omega$-lim$x_n$. It is easy to check it is surjective. Moreover it is an isometry, indeed:

$$|\omega$-lim$x_n - \omega$-lim$y_n| = \omega$-lim$|x_n - y_n| = d((x_n), (y_n)).$$

In particular, the limit of geodesic segments is a geodesic segment.

Lemma A.7. Let $(X_n, x_n)$ be a sequence of pointed metric spaces and let $\omega$ be a non-principal ultrafilter. Let $X_\omega$ be the ultralimit of $(X_n, x_n)$, and let $z = (z_n), w = (w_n) \in X_\omega$. Suppose that for all $n$ there exists a geodesic $\gamma_n : [0, d(z_n, w_n)] \to X_n$ joining $z_n$ and $w_n$: then, there exists a geodesic joining $z$ and $w$ in $X_\omega$. In particular, if $X_n$ is a geodesic space for all $n$, then the ultralimit $X_\omega$ is a geodesic space.

Proof. We denote by $I_n$ the interval $[0, d(z_n, w_n)]$. Since $z$ and $w$ belongs to $X_\omega$ then the distance between them is uniformly bounded. Hence from the previous lemma it follows that the ultralimit of the spaces $(I_n, 0)$ is $I_\omega = [0, \omega$-lim$ d(z_n, w_n)] = [0, d(z, w)]$. The maps $\gamma_n$ define an admissible sequence of isometric embedding, so in particular they define a limit isometric embedding $\gamma_\omega : I_\omega \to X$. So $\gamma_\omega$ is a geodesic and clearly $\gamma_\omega(0) = (\gamma_n(0)) = (z_n) = z$ and $\gamma_\omega(d(z, w)) = w$.

In order to prove stability results for classes of metric spaces we also need to establish the convergence of balls under ultralimits:

Lemma A.8. Let $(X_n, x_n)$ be a sequence of geodesic metric spaces and $\omega$ be a non-principal ultrafilter. Let $X_\omega$ be the ultralimit of the sequence $(X_n, x_n)$. Let $y = (y_n)$ be a point of $X_\omega$. Then for any $R \geq 0$ it holds

$$B(y, R) = \omega$-lim B(y_n, R).$$

Proof. First of all $\omega$-lim$B(y_n, R) \subset B(y, R)$. Indeed $z = (z_n)$ belongs to $\omega$-lim $B(y_n, R)$ if and only if $d(z_n, y_n) \leq R$ for all $n$. Then $d(z, y) \leq R$, i.e. $z \in B(y, R)$. The next step is to show that the set $\omega$-lim$B(y_n, R)$ is closed. We take a sequence $z^k = (z^k_n)$ of points of $\omega$-lim$B(y_n, R)$ that converges to some point $z = (z_n)$ of $X_\omega$. This implies that $d(y, z) \leq R$. We consider a geodesic segment of $X_n$ between $y_n$ and $z_n$ and we denote by $w_n$ the point along this geodesic at distance exactly $R$ from $y_n$, if it exists. Otherwise $z_n \in B(y_n, R)$ and in this case we set $w_n = z_n$. We observe that $w = (w_n) \in \omega$-lim$B(y_n, R)$ by definition. We claim that $w = z$. In order to
prove that we fix \( \varepsilon > 0 \). Then \( \omega \)-a.s. \( d(y_n, z_n) < R + \varepsilon \). This implies that \( d(y_n, w_n) < \varepsilon \). Since it holds \( \omega \)-a.s. then \( d(w, z) < \varepsilon \). From the arbitrariness of \( \varepsilon \) the claim is proved. The last step is to show that the open ball \( B(y, R) \) is contained in \( \omega \)-limit \( B(y_n, R) \). Indeed, given \( z = (z_n) \in B(y, R) \), then there exists \( \varepsilon > 0 \) such that \( d(z, y) < R - \varepsilon \). The set of indices \( n \) such that \( d(z_n, y_n) < d(z, y) + \varepsilon < R \) belongs to \( \omega \), hence \( z \in \omega \)-limit \( B(y_n, R) \). Since \( X_\omega \) is geodesic and in any length space the closed ball is the closure of the open ball the proof is concluded. \( \square \)

In general, even if every space \( X_n \) is uniquely geodesic, the ultralimit \( X_\omega \) may be not uniquely geodesic. This is because, in general, it is not true that all the geodesics of \( X_\omega \) are limit of sequences of geodesics of \( X_n \). The fact that the geodesics of \( X_\omega \) are actually limit of geodesics of the spaces \( X_n \) is true when all the \( X_n \) are CAT(\( \kappa \)). We recall the following fact which is well known (see \cite{BH13} or \cite{DKN18} for instance):

**Proposition A.9.** Let \((X_n, x_n)\) be a sequence of CAT(\( \kappa \)) pointed metric spaces and \( \omega \) be a non-principal ultrafilter. Then any geodesic of length \( < D_\kappa \) in \( X_\omega \) is limit of a sequence of geodesics of \( X_n \). As a consequence \( X_\omega \) is a CAT(\( \kappa \)) metric space.

The main result of the appendix is the following stability property for the CAT(\( \kappa \))-radius:

**Corollary A.10.** Let \((X_n, x_n)\) be a sequence of complete, locally geodesically complete, locally CAT(\( \kappa \)), geodesic metric spaces with \( \rho_{\text{cat}}(X_n) \geq \rho_0 > 0 \). Let \( \omega \) be a non-principal ultrafilter. Then \( X_\omega \) is a complete, locally geodesically complete, locally CAT(\( \kappa \)), geodesic metric space with \( \rho_{\text{cat}}(X_\omega) \geq \rho_0 \).

**Proof.** Let \( y = (y_n) \) be a point of \( X_\omega \). For any \( r < \rho_0 \) and for any \( n \) the ball \( B(y_n, r) \) is a CAT(\( \kappa \)) metric space. Moreover by Lemma A.8 we have that \( B(y, r) \) is the ultralimit of a sequence of CAT(\( \kappa \)) metric spaces, hence it is CAT(\( \kappa \)) by Proposition A.9. This shows that \( X_\omega \) is locally CAT(\( \kappa \)) and \( \rho_{\text{cat}}(X_\omega) \geq \rho_0 \) by the arbitrariness of \( r \). Moreover \( X_\omega \) is geodesic by Corollary A.7. We fix now a geodesic segment \( \gamma \) of \( X_\omega \) defined on \([a, b]\]. We look at the ball \( B(\gamma(a), \rho_0) \) which is CAT(\( \kappa \)) and we take a sequence of points \( z_n \) such that \( (z_n) = \gamma(a) \). The subsegment of \( \gamma \) inside this ball, defined on \([a, a + \rho_0]\) is the limit of a sequence of geodesics \( \gamma_n \) inside the corresponding balls \( B(z_n, \rho_0) \), by Proposition A.9 Each \( \gamma_n \) can be extended to a geodesic segment \( \tilde{\gamma}_n \) on the interval \((a - \rho_0, a + \rho_0)\) since each \( X_n \) is locally geodesically complete and complete. The ultralimit of the maps \( \tilde{\gamma}_n \) is a geodesic segment defined on \([a - \rho_0, a + \rho_0]\) which extends \( \gamma \). We can do the same around \( \gamma(b) \). This proves that \( X_\omega \) is locally geodesically complete. \( \square \)

We conclude the appendix recalling the relations between ultralimits and pointed Gromov-Hausdorff convergence, which we will use in Section 5.
Proposition A.11 (see [Jan17]). Let \((X_n, x_n)\) be a sequence of proper, length metric spaces and \(\omega\) be a non-principal ultrafilter. Then:

(a) if the ultralimit \((X_\omega, x_\omega)\) is proper then it is the limit of a convergent subsequence in the pointed Gromov-Hausdorff sense;

(b) reciprocally, if \((X_n, x_n)\) converges to \((X, x)\) in the pointed Gromov-Hausdorff sense then for any non-principal ultrafilter \(\omega\) the ultralimit \(X_\omega\) is isometric to \((X, x)\) (we recall that, in this case, \((X, x)\) is proper by definition of Gromov-Hausdorff convergence).
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