SHAPE DYNAMICS. An Introduction

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Abstract. Shape dynamics is a completely background-independent universal framework of dynamical theories from which all absolute elements have been eliminated. For particles, only the variables that describe the shapes of the instantaneous particle configurations are dynamical. In the case of Riemannian three-geometries, the only dynamical variables are the parts of the metric that determine angles. The local scale factor plays no role. This leads to a shape-dynamic theory of gravity in which the four-dimensional diffeomorphism invariance of general relativity is replaced by three-dimensional diffeomorphism invariance and three-dimensional conformal invariance. Despite this difference of symmetry groups, it is remarkable that the predictions of the two theories – shape dynamics and general relativity – agree on spacetime foliations by hypersurfaces of constant mean extrinsic curvature. However, the two theories are distinct, with shape dynamics having a much more restrictive set of solutions. There are indications that the symmetry group of shape dynamics makes it more amenable to quantization and thus to the creation of quantum gravity. This introduction presents in simple terms the arguments for shape dynamics, its implementation techniques, and a survey of existing results.

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1 Introduction

One of Einstein’s main aims in creating general relativity was to implement Mach’s idea [1, 2] that dynamics should use only relative quantities and that inertial motion as expressed in Newton’s first law should arise, not as an effect of a background absolute space, but from the dynamical effect of the universe as a whole. Einstein called this Mach’s principle [3]. However, as he explained later [4, 5] (p. 186), he found it impractical to realize Mach’s principle directly and was forced to use coordinate systems. This has obscured the extent to which and how general relativity is a background-independent theory. My aim in this paper is to present a universal framework for the direct and explicit creation of completely background-independent theories.

I shall show that this leads to a theory of gravity, shape dynamics, that is distinct from general relativity because it is based on a different symmetry group, according to which only the local shapes of Riemannian 3-geometries are dynamical. Nevertheless, it is remarkable that the two theories have a nontrivial ‘intersection’, agreeing exactly in spatially closed universes whenever and wherever Einsteinian spacetimes admit foliation by hypersurfaces of constant mean extrinsic curvature. However, many solutions of general relativity that appear manifestly unphysical, such as those with closed time-like curves, are not allowed in shape dynamics. In addition, it appears that the structure of shape dynamics makes it significantly more amenable to quantization than general relativity.

This is not the only reason why I hope the reader will take an interest in shape dynamics. The question of whether motion is absolute or relative has a venerable history [6, 7], going back to long before Newton made it famous when he formulated dynamics in terms of absolute space and time [8]. What is ultimately at stake is the definition of position and, above all, velocity. This has abiding relevance in our restless universe. I shall show that it is possible to eliminate every vestige of Newtonian absolutes except
for just one. But this solitary remnant is hugely important: it allows the universe to expand. Shape dynamics highlights this remarkable fact.

This introduction will be to a large degree heuristic and based on Lagrangian formalism. A more rigorous Hamiltonian formulation of shape dynamics better suited to calculations and quantum-gravity applications was recently discovered in [9] (a simplified treatment is in [10]). Several more papers developing the Hamiltonian formulation in directions that appear promising from the quantum-gravity perspective are in preparation. A dedicated website (shapedynamics.org) is under construction; further background information can be found at platonia.com.

The contents list obviates any further introduction, but a word on terminology will help. Two distinct meanings of *relative* are often confused. Mach regarded inter-particle separations as relative quantities; in Einstein’s theories, the division of spacetime into space and time is made relative to an observer’s coordinate system. To avoid confusion, I use *relational* in lieu of Mach’s notion of relative.

2 The Relational Critique of Newton’s Dynamics

2.1 Elimination of redundant structure

Newton’s First Law states: “Every body continues in its state of rest or uniform motion in a right line unless it is compelled to change that state by forces impressed on it.” Since the (absolute) space in which the body’s motion is said to be straight and the (absolute) time that measures its uniformity are both invisible, this law as stated is clearly problematic. Newton knew this and argued in his Scholium in the *Principia* [8] that his invisible absolute motions could be deduced from visible relative motions. This can be done but requires more relative data than one would expect if only directly observable initial data governed the dynamics. As we shall see, this fact, which is not widely known, indicates how mechanics can be reformulated with less kinematic structure than Newton assumed and simultaneously be made more predictive. It is possible to create a framework that fully resolves the debate about the nature of motion. In this framework, the *fewest possible observable initial data determine the observable evolution*.

I show first that all candidate relational configurations of the universe

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3 The notion of what is observable is not unproblematic. For now it will suffice that inter-particle separations are more readily observed than positions in invisible space.

4 We shall see (Sec. 4) that the foundation of dynamics on instantaneous extended configurations, rather than point events, is perfectly compatible with Einsteinian relativity.
have structures determined by a Lie group, which may be termed their *structure group*. The existence of such a group is decisive. It leads directly to a natural way to achieve the aim just formulated and to a characteristic universal structure of dynamics applicable to which is not widely known, a large class of systems. It is present in modern gauge theories and, in its most perfect form, in general relativity. However, the relational core of these theories is largely hidden because their formulation retains redundant kinematic structure.

To identify the mismatch that shape dynamics aims to eliminate, the first step is to establish the essential structure that Newtonian dynamics employs. It will be sufficient to consider \( N, N \geq 3 \), point particles interacting through Newtonian gravity. In an assumed inertial frame of reference, each particle \( a, a = 1, ..., N, \) has coordinates \( x^i_a(t), i = x, y, z, \) that depend on \( t \), the Newtonian time. The \( x^i_a \)'s and \( t \) are all assumed to be observable. The particles, assumed individually identifiable, also have constant masses \( m_a \). For the purposes of our discussion, they can be assumed known.

Let us now eliminate potentially redundant structure. Newton granted that only the inter-particle separations \( r_{ab} \), assumed to be ‘seen’ all at once, are observable. In fact, this presupposes an external (absolute) ruler. Closer to empirical reality are the dimensionless ratios

\[
\tilde{r}_{ab} := \frac{r_{ab}}{R_{rmh}}, \quad R_{rmh} := \sqrt{\sum_{a<b} r_{ab}^2},
\]

where \( R_{rmh} \) is the root-mean-harmonic separation. It is closely related to the centre-of-mass moment of inertia \( I_{cms} \):

\[
I_{cms} := \sum_a m_a x^a \cdot x^a \equiv \frac{1}{M} \sum_{a<b} m_a m_b r_{ab}^2, \quad M := \sum_a m_a.
\]

The system has the ‘size’ \( \sqrt{I_{cms}} \) if we grant a scale, but we do not and take the instantaneous sets \( \{ \tilde{r}_{ab} \} \) of scale-free ratios \( \tilde{r}_{ab} \) to be our raw data. They are ‘snapshots’ of the *instantaneous shapes* of the system. The time \( t \) too is unobservable. There is no clock hung up in space, just the particles moving relative to each other. All that we have are the sets \( \{ \tilde{r}_{ab} \} \). The totality of such sets is *shape space* \( Q_{ss}^N \), which only exists for \( N \geq 3 \). The number of dimensions of \( Q_{ss}^N \) is \( 3N - 7 \): from the \( 3N \) Cartesian coordinates, six are

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5A single point is not a shape, and the distance between two particles can be scaled to any value, so nothing dimensionless remains to define a shape. Also the configuration in which all particles coincide is not a shape and does not belong to shape space.
subtracted because Euclidean translations and rotations do not change the \( r_{ab} \)'s and the seventh because the \( \{ \tilde{r}_{ab} \} \)'s are scale invariant.

Shape space is our key concept. Mathematically, we reach it through a succession of spaces, the first being the \( 3N \)-dimensional Cartesian configuration space \( Q^N \). In it, all configurations that are carried into each other by translations \( t \) in \( T \), the group of Euclidean translations, belong to a common orbit of \( T \). Thus, \( T \) decomposes \( Q^N \) into its group orbits, which are defined to be the points of the \( 3N - 3 \)-dimensional quotient space \( T^N := Q^N / T \). This first quotienting to \( T^N \) is relatively trivial. More significant is the further quotienting by the rotation group \( R \) to the \( 3N - 6 \)-dimensional relative configuration space \( Q^N_{rcs} := Q^N / TR \) [11]. The final quotienting by the dilatation (scaling) group \( S \) leads to shape space \( Q^N_{ss} := Q^N / TRS \) [12]. The groups \( T \) and \( R \) together form the Euclidean group, while the inclusion of \( S \) yields the similarity group. The orbit of a group is a space with as many dimensions as the number of elements that specify a group element. The orbits of \( S \) thus have seven dimensions (Fig. 1).

The groups \( T, R, S \) are groups of motion, or Lie groups (groups that are simultaneously manifolds, i.e., their elements are parametrized by continuous parameters). If we have a configuration \( q \) of \( N \) particles in Euclidean space, \( q \in Q^N \), we can ‘move it around’ with \( T \) or \( R \) or ‘change its size’ with \( S \). This intuition was the basis of Lie’s work. It formalizes the fundamental geometrical notions of congruence and similarity. Two figures are congruent if they can be brought to exact overlap by a combination of translations and rotations and similar if dilatations are allowed as well.

Relational particle dynamics can be formulated in any of the quotient spaces just considered. Intuition suggests that the dynamics of an ‘island universe’ in Euclidean space should deal solely with its possible shapes. The similarity group is then the fundamental structure group. This leads to particle shape dynamics and by analogy to the conformal geometrodynamics that will be considered in the second part of the paper.

Lie groups and their infinite-dimensional generalizations are fundamental in modern mathematics and theoretical physics. They play a dual role in shape dynamics, first in indicating how potentially redundant structure can be pared away and, second, in providing the tool to create theories that are relationally perfect, i.e., free of the mismatch noted above. Moreover, because Lie groups, as groups that are simultaneously manifolds, have a common underlying structure and are ubiquitous, they permit essentially

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6One might want to go further and consider the general linear group, under which angles are no longer invariant. I will consider this possibility later.
Figure 1. Shape space for the 3-body problem is obtained by decomposing the Newtonian configuration space $Q^3$ into orbits of the similarity group $S$. The points on any given vertical line (any group orbit) correspond to all possible representations in Euclidean space of one of the possible shapes of the triangle formed by the three particles. Each such shape is represented below its orbit as a point in shape space. Each orbit is actually a seven-dimensional space. The effects of rotation and scaling are shown.

identical methods to be applied in many different situations. This is why shape dynamics is a universal framework.

2.2 Newtonian dynamics in shape space

We now identify the role that absolute space and time play in Newtonian dynamics by projection to shape space. We have removed structure from the $q$'s in $Q^N$, reducing them to points $s \in Q^N_{ss}$. This is projection of $q$'s. We can also project complete Newtonian histories $q(t)$. To include time at the start, we adjoin to $Q^N$ the space $T$ of absolute times $t$, obtaining the space $Q^N T$. Newtonian histories are then (monotonically rising) continuous curves in $Q^N T$. However, clocks are parts of the universe; there is no external clock available to provide the reading for the $T$ axis. All the objective information is carried by the successive configurations of the universe. We must therefore remove the $T$ axis and, in the first projection, label the points representing the configurations in $Q^N$ by an arbitrary increasing parameter $\lambda$ and then
Figure 2. In Newtonian dynamics, the history of a system is a monotonically rising curve $q(t)$ in $Q^N T$ or a curve $q(\lambda)$ in $Q^N$ labelled by a monotonic $\lambda$. The objective observable history is the projected curve $s(\lambda)$ in shape space $Q_N^{ss}$.

make the further projection to the shape space $Q_N^{ss}$. The history becomes $s(\lambda)$ (Fig. 2). A history is the next most fundamental concept in shape dynamics. There is no ‘moving now’ in this concept. History is not a spot moving along $s(\lambda)$, lighting up ‘nows’ as it goes. It is the curve; $\lambda$ merely labels its points. Newtonian dynamics being time-reversal invariant, there is no past-to-future direction on curves in $Q_N^{ss}$.

Given a history of shapes $s$, we can define a shape velocity. Suppose first that in fact by some means we can define a distinguished parameter $p$, or independent variable, along a suitably continuous curve in $Q_N^{ss}$. Then at each point along the curve we have a shape $s$ and its (multi-component) velocity $ds/dp$. This is a tangent vector to the curve. If we have no $p$ but only an arbitrary $\lambda$, we can still define shape velocities $ds/d\lambda = s'$, but all we really have is the direction $d$ (in $Q_N^{ss}$) in which $s$ is changing. The difference between tangent vectors and directions associated with curves in shape space will be important later.

We can now identify the mismatch that, when eliminated, leads to the shape-dynamic ideal. To this end, we recall Laplacian determinism in Newtonian dynamics: given $q$ and $\dot{q}$ at some instant, the evolution of the sys-
Figure 3. The two triangles of slightly different shapes formed by three particles $a, b, c$ define a point $s$ and direction $d$ uniquely in shape space, but changes to an original placing (1) in Euclidean space of the dashed triangle relative to the grey one generated by translations (t), rotations (r), and dilatations (d) give rise to different Newtonian initial velocities $\delta x_a^i / \delta t$.

tem is uniquely determined (the particle masses and the force law assumed known). The question is this: given the corresponding shape projections $s$ and $d$, is the evolution in shape space $Q^N_{ss}$ uniquely determined? The answer is no for a purely geometrical reason. The fact is that certain initial velocities which are objectively significant in Newtonian dynamics can be generated by purely group actions. To be precise, different Newtonian velocities can be generated from identical data in $Q^N_{ss}$. This is illustrated for the 3-body problem (in two dimensions) in Fig. 3.

I will not go into the details of the proof (see [13]), but in a Newtonian N-body system the velocities at any given instant can be uniquely decomposed into parts due to an intrinsic change of shape and three further parts due to the three different group actions – translations, rotations, and dilatations – applied as in Fig. 3. These actions are obviously ‘invisible’ in the shape-space $s$ and $d$, which define only the shape and the way it is changing.

By Galilean relativity, translations of the system have no effect in $Q^N_{ss}$. We can ignore them but not rotations and dilatations. Four dimensionless dynamically effective quantities are associated with them. First, two angles determine the direction in space of a rotation axis. Second, from the kinetic energies associated with rotation, $T_r$, dilatation $T_d$, and change of shape, $T_s$, we can form two dimensionless ratios, which it is natural to take to be $T_r / T_s$ and $T_d / T_s$ (since change of shape, represented by $T_s$, is our ‘gold standard’). Thus, the kinematic action of the Lie groups generates four parameters that affect the histories in shape space without changing the initial $s$ and $d$. This is already so for pure inertial motion. If forces are present, there is a fifth parameter, the ratio $T/V$ of the system’s kinetic energy $T$ to its potential energy $V$, that is dynamically significant but is also invisible in the $s$ and $d$ in shape space.

We now see that although Newtonian dynamics seems wonderfully ra-
tional and transparent when expressed in an inertial frame of reference, it
does not possess perfect Laplacian determinism in shape space. This fail-
ure appears especially odd if $N$ is large. Choose some coordinates $s_i, i = 1, 2, ..., 3N - 7$, in shape space and take one of them, call it $\tau$, as a surrogate for Newton’s $t$. If only shapes had dynamical effect, then by analogy with inertial-frame Newtonian dynamics, the initial values of $\tau, s_i, ds_i/d\tau, i = 1, ..., 3N - 8$ would fix the evolution. They do not.

Five more data are needed and must be taken from among the second
derivatives $d^2s_i/d\tau^2$. Moreover, no matter how large $N$, say a million as
in a globular cluster, we always need just five. They make no sense from
the shape-dynamic perspective. Poincaré, writing as a philosopher deeply
committed to relationalism, found the need for them repugnant [14, 15].
But, in the face of the manifest presence of angular momentum in the solar
system, he resigned himself to the fact that there is more to dynamics than,
literally, meets the eye. In fact, the extra $d^2s_i/d\tau^2$’s are explained by
Newton’s assumption of an all-controlling but invisible frame for dynamics.
They are the evidence, and the sole evidence at that, for absolute space.

For some reason, Poincaré did not consider Mach’s suggestion [1, 2] that
the universe in its totality might somehow determine the structure of the
dynamics observed locally. Indeed, the universe exhibits evidence for angular
momentum in innumerable localized systems but none overall. This suggests
that, regarded as a closed dynamical system, it has no angular momentum
and meets the Poincaré principle: either a point and direction (strong form)
or a point and a tangent vector (weak form) in the universe’s shape space
determine its evolution. The stronger form of the principle will hold if the
universe satisfies a geodesic principle in $Q^N$, since a point and a direction
are the initial conditions for a geodesic. The need for the two options, either
of which may serve as the definition of Mach’s principle [16], will be clarified
in the next section.

To summarize: on the basis of Poincaré’s analysis and intuition, we
would like the universe to satisfy Laplacian determinism in its shape space
and not merely in a special frame of reference that employs kinematic struc-

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7 If $N = 3$ or $4$, there are insufficient $d^2s_i/d\tau^2$’s and we need higher derivatives too.
8 Poincaré’s penetrating analysis, on which this subsection is based, only takes into
account the role of angular momentum in the ‘failure’ of Newtonian dynamics when
expressed in relational quantities. Despite its precision and clarity, it has been almost totally
ignored in the discussion of the absolute vs relative debate in dynamics.
3 The Universal Structure of Shape Dynamics

3.1 The elimination of time

In standard dynamical theory, the time $t$ is an independent variable supplied by an external clock. But any clock is a mechanical system. If we wish to treat the universe as a single system, the issue of what clock, if any, to use becomes critical. In fact, it is not necessary to use any clock.

This can be demonstrated already in $Q^N$. We simply proceed without a clock. Histories of the system are then just curves in $Q^N$, and we seek a law that determines them. An obvious possibility is to define a metric on $Q^N$ and require histories to be geodesics with respect to it.

A metric is readily found because the Euclidean geometry of space that defines $Q^N$ in the first place also defines a natural metric on $Q^N$:

$$ds_{kin} = \sqrt{\sum_a \frac{m_a}{2} dx_a \cdot dx_a}.$$  
(3)

This is called the kinetic metric; division of $dx_a$ by an external $dt$ transforms the radicand into the Newtonian kinetic energy. We may call (3) a supermetric. We shall see how it enables us to exploit structure defined at the level of $Q^N$ at the shape-space level.

We can generate further such supermetrics from (3) by multiplying its radicand by a function on $Q^N$, for example $\sum_{a<b} m_a m_b / r_{ab}$. We obtain a whole family of geodesic principles defined by the variational requirement

$$\delta I = 0, \quad I = 2 \int d\lambda \sqrt{(E - V(q))T_{kin}}, \quad T_{kin} := \frac{1}{2} \sum_a m_a \frac{dx_a}{d\lambda} \cdot \frac{dx_a}{d\lambda},$$  
(4)

where $\lambda$ is a curve parameter, the 2 is for convenience and, since a constant is a function on $Q^N$, the constant $E$ reflects its possible presence.

The Euler–Lagrange equations that follow from (4) are

$$\frac{d}{d\lambda} \left( \sqrt{\frac{E - V}{T_{kin}}} m_a \frac{dx_a}{d\lambda} \right) = -\sqrt{\frac{T_{kin}}{E - V}} \frac{\partial V}{\partial x_a}.$$  
(5)

This equation simplifies if we choose the freely specifiable $\lambda$ such that

$$E - V = T_{kin}.$$  
(6)

If we denote this $\lambda$ by $t$, then (5) becomes Newton’s second law and (6) becomes the energy theorem. However, in our initially timeless context
it becomes the definition of an emergent time, or better duration, created
by a geodesic principle. In fact, the entire objective content of Newtonian
dynamics for a closed system is recovered. It is illuminating to give the
explicit expression for the increment of this emergent duration:

$$\delta t = \sqrt{\frac{\sum a m_a \delta x_a \cdot \delta x_a}{2(E - V)}}.$$  (7)

This is the first example of the holism of relational dynamics: the time
that we take to flow locally everywhere is a distillation of all the changes
everywhere in the universe. Since everything in the universe interacts with
everything else, every difference must be taken into account to obtain the
exact measure of time. The universe is its own clock.

The definition of duration through (7) is unique (up to origin and unit) if
clocks are to have any utility. Since we use them to keep appointments, they
are useless unless they march in step. This leads unambiguously to (7) as
the only sensible definition. For suppose an island universe contains within
it subsystems that are isolated in the Newtonian sense. We want to use the
motions within each to generate a time signal. The resulting signals must all
march in step with each other. Now this will happen if, for each system, the
signal is generated using (7). The reason is important. Suppose we used only
the numerators in (7) to measure time; then subsystems without interactions
would generate time signals that march in step, but with interactions one
system may be sinking into its potential well as another is rising out of its.
Then the ‘time’ generated by the former will pass faster than the latter’s.
However the denominators in (7) correct this automatically since $E - V$
increases or decreases with $T$. Time must be measured by some motion,
but for generic systems only the time label that ensures conservation of
the energy can meet the marching-in-step criterion. Duration is defined as
uniquely as entropy is through the logarithm of probability.

In textbooks, (4) is derived as Jacobi’s principle [17] and used to de-
determine the dynamical orbit of systems in $Q^N$ (as, for example, a planet’s
orbit, which is not to be confused with a group orbit). The speed in orbit is
then determined from (6) regarded as the energy theorem. The derivation
above provides the deeper interpretation of (6) in a closed system. It is the
definition of time. Note that time is eliminated from the initial kinematics
by a square root in the Lagrangian. This pattern will be repeated in more
refined relational settings below, in which we can address the question of
what potentials $V$ are allowed in relational dynamics.

A final comment. Time has always appeared elusive. It is represented
in dynamics as the real line $R^1$. Instants are mere points on the line, each
identical to the other. This violates the principle that things can be distinguished only by differences. There must be variety. In relational dynamics $R^3$ is redundant and there are only configurations, but they double as instants of time. The need for variety is met.

### 3.2 Best matching

The next step is to determine curves in shape space $Q^{ss}_N$ that satisfy the strong or weak form of the Poincaré principle. As already noted, the strong form, with which we begin, will be satisfied by geodesics with respect to a metric defined on $Q^{ss}_N$. For this, given two nearly identical shapes, $s_1, s_2$, i.e., neighbouring points in $Q^N$, we need to define a ‘distance’ between them based on their difference and nothing else. Once again we use the Euclidean geometry that underlies both $Q^N$ and $Q^{ss}_N$.

Shape $s_1$ in $Q^{ss}_N$ has infinitely many representations in $Q^N$: all of the points on its group orbit in $Q^N$. Pick one with coordinates $x^1_a$. Pick a nearby point on the orbit of $s_2$ with coordinates $x^2_a$. In Newtonian dynamics, the coordinate differences $d x_a = x^2_a - x^1_a$ are physical displacements, but in shape dynamics they mix physical difference of shape with spurious difference due to the arbitrary positioning of $s_1$ and $s_2$ on their orbits. To obtain a measure of the shape difference, hold $s_1$ fixed in $Q^N$ and move $s_2$ around in its orbit for the moment using only Euclidean translations and rotations. This changes $d x_a = x^2_a - x^1_a$ and simultaneously

$$d s_{\text{trial}} := \sqrt{(E - V) \sum_a \frac{m_a}{2} d x_a \cdot d x_a}. \quad (8)$$

Since (8) is positive definite and defines a nonsingular metric on $Q^N$, it will be possible to move shape $s_2$ into the unique position in its orbit at which (8) is minimized (for given position of $s_1$). This unique position can be characterized in two equivalent ways: 1) Shape $s_2$ has been moved to the position in which it most closely ‘covers’ $s_1$, i.e., the two shapes, which are incongruent, have been brought as close as possible to congruence, as measured by (8). This is the best-matched position. 2) The $3N$-dimensional vector joining $s_1$ and $s_2$ in their orbits in $Q^N$ is orthogonal to the orbits. This is true in the first place for the kinetic metric, for which $E - V = 1$, but also for all choices of $E - V$. For each, the best-matched position is the

\[ \text{Recall that a group orbit is generically a multi-dimensional space.} \]
Figure 4. a) An arbitrary placing of the dashed triangle relative to the undashed triangle; b) the best-matched placing reached by translational and rotational minimization of (8); c) the two positions of the triangle configurations on their group orbits in $Q^N$. The connecting ‘strut’ is orthogonal with respect to the supermetric on $Q^N$ in the best-matched position. Best matching brings the centres of mass to coincidence and reduces the net rotation to zero.

Same but there is a different best-matched ‘distance’ between $s_1$ and $s_2$:

$$d_{s_{bm}} := \min \sqrt{(E - V) \sum_a \frac{m_a}{2} \mathbf{dx}_a \cdot \mathbf{dx}_a} \text{ between orbits.} \quad (9)$$

Because orthogonality of two vectors can only be established if all components of both vectors are known, best matching introduces a further degree of holism into relational physics. The two ways of conceptualizing best matching are shown in Fig. 4 for the 3-body problem in two dimensions.

It is important that the orthogonal separation (9) is the same at all points within the orbits of either $s_1$ or $s_2$. This is because the metric (8) on $Q^N$ is equivariant: if the same group transformations are applied to the configurations in $Q^N$ that represent $s_1$ and $s_2$, the value of (8) is unchanged. In differential-geometric terms, equivariance is present because the translation and rotation group orbits are Killing vectors of the kinetic metric in $Q^N$. The equivariance property only holds if $E - V$ satisfies definite conditions, which I have tacitly assumed so far but shall spell out soon.
In fact, it is already lost if we attempt to include dilational best matching with respect to the kinetic metric in order to determine a ‘distance’ between shapes rather than only relative configurations as hitherto. For suppose we represent two shapes by configurations of given sizes in $Q^N$ and find their best-matched separation $d_{bm}$ using Euclidean translations and rotations. We obtain some value for $d_{bm}$. If we now change the scale of one of the shapes, $d_{bm}$ must change because the kinetic metric has dimensions $m^{1/2}l$ and scales too. To correct for this in a natural way, we can divide the kinetic metric by the square root of $I_{cms}$, the centre-of-mass moment of inertia (2), and then best match to get the inter-shape distance

$$d_{ssbm} := \min \sqrt{I_{cms}^{-1} \sum_a m_a d_{xa} \cdot d_{xa}}$$

between orbits. \hspace{1cm} (10)

As it must be, $d_{ssbm}$ is dimensionless and defines a metric on $Q^{Nss}$. It is precisely such a metric that we need in order to implement Poincaré’s principle.

Terminologically, it will be convenient to call directions in $Q^N$ that lie entirely in group orbits \textit{vertical} and the best-matched orthogonal directions \textit{horizontal}. Readers familiar with fibre bundles will recognize this terminology. A paper presenting best-matching theory \textit{ab initio} in terms of fibre bundles is in preparation.

### 3.3 The best-matched action principle

We can now implement the strong Poincaré principle. We calculate in $Q^N$, but the reality unfolds in $Q^{Nss}$. The task is this: given two shapes $s_a$ and $s_b$ in $Q^{Nss}$, find the geodesic that joins them. The distance along the trial curves between $s_a$ and $s_b$ is to be calculated using the best-matched metric (10) found in $Q^N$ and then ‘projected’ down to $Q^{Nss}$. The projected metric is unique because the best-matching metric in $Q^N$ is equivariant.

The action principle in $Q^N$ has the form

$$\delta I_{bm} = 0, \ I_{bm} = 2 \int d\lambda \sqrt{W T_{bm}}, \ T_{bm} = \frac{1}{2} \sum_a d_{xa} \cdot \frac{d_{xa}}{d\lambda},$$

where $d_{xa}/d\lambda$ is the limit of $\delta x_{a}^{bm}/\delta \lambda$ as $d\lambda \to 0$, and the potential-type term $W$ must be such that equivariance holds. In writing the action in this way, I have taken a short cut. Expressed properly (11), $I_{bm}$ contains the generators of the various group transformations, and the variation with respect to them leads to the best-matched velocities $d_{xa}^{bm}/d\lambda$. It is assumed in (11) that this variation has already been done.
The action \( (11) \) is interpreted as follows. One first fixes a trial curve in \( Q_{ss}^N \) between \( s_a \) and \( s_b \) and represents it by a trial curve in \( Q^N \) through the orbits of the shapes in the \( Q_{ss}^N \) trial curve. The \( Q^N \) trial curve must never ‘run vertically’. It may run orthogonally to the orbits, and this is just what we want. For if it does, the \( \delta x_a \) that connect the orbits are best matched. It is these \( \delta x_{bn} \), dependent only on the shape differences, that are to determine the action.

To make the trial curve in \( Q^N \) orthogonal, we divide it into infinitesimal segments between adjacent orbits 1, 2, 3, ..., \( m \) (orbits 1 and \( m \) are \( s_a \) and \( s_b \), respectively). We hold the initial point of segment 1–2 fixed and move the other end into the horizontal best-matched position on orbit 2. We then move the original 2–3 segment into the horizontal with its end 2 coincident with the end of the adjusted 1–2 segment. We do this all the way to the \( s_b \) orbit. Making the segment lengths tend to zero, we obtain a smooth horizontal curve. Because the \( Q^N \) metric is equivariant, this curve is not unique – its initial point can be moved ‘vertically’ to any other point on the initial orbit; all the other points on the curve are then moved vertically by the same amount. If \( M \) is the dimension of the best-matching group (\( M = 7 \) for the similarity group), we obtain an \( M \)-parameter family of horizontal best-matched curves that all yield a common unique value for the action along the trial curve in \( Q_{ss}^N \). This is illustrated in Fig. 5.

In this way we obtain the action for all trial curves in shape space between \( s_a \) and \( s_b \). The best-matching construction ensures that the action depends only on the shapes that are explored by the trial curves and nothing else. It remains to find which trial curve yields the shortest distance between \( s_a \) and \( s_b \). This requires us to vary the trial curve in \( Q_{ss}^N \), which of course changes the associated trial curves in \( Q^N \), which, when best matched, give different values for the best-matched action. When we find the (in general unique) curve for which the shape-space action is stationary, we have found the solution that satisfies the strong Poincaré principle. Theories satisfying only the weak principle arise when the equivariance condition imposed on \( W \) in \( (11) \) is somewhat relaxed, as we shall now see.

### 3.4 Best-matching constraints and consistency

To obtain a definite representation in the above picture of best matching, we must refer the initial shape \( s_1 \) to a particular Cartesian coordinate system with a definite choice of scale. This ‘places’ shape \( s_1 \) at some position on its group orbit in \( Q^N \). If we now place the next, nearly identical shape \( s_2 \) on its orbit close to the position chosen for \( s_1 \) on its orbit but not in
Figure 5. The action associated with each trial curve between shapes $s_a$ and $s_b$ in shape space is calculated by finding a best-matched curve in $Q^N$ that runs through the group orbits ‘above’ the trial curve. The best-matched curve is determined uniquely apart from ‘vertical lifting’ by the same amount in each orbit, which does not change the best-matched action $I_{bm}$. The trial curve in shape space for which $I_{bm}$ is extremal is the desired curve in shape space.

the best-matched position, we obtain certain coordinate differences $\delta x_a = x^2_a - x^1_a$. Dividing these by a nominal $\delta t$, we obtain velocities from which, in Newtonian terms, we can calculate a total momentum $P = \sum_a m_a \dot{x}_a$, angular momentum $L = \sum_a m_a x_a \times \dot{x}_a$ and rate of change of $I$: $\dot{I} = D = 2\sum_a m_a x_a \cdot \dot{x}_a$. We can change their values by acting on $s_2$ with translations, rotations, and dilatations respectively. Indeed, it is intuitively obvious that by choosing these group transformations appropriately we can ensure that

$$P = 0, \quad \text{(12)}$$

$$L = 0, \quad \text{(13)}$$

$$D = 0. \quad \text{(14)}$$

It is also intuitively obvious that the fulfilment of these conditions is precisely the indication that the best-matching position has been reached.

Let us now stand back and take an overall view. The reality in shape dynamics is simply a curve in $Q^N_{ss}$, which we can imagine traversed in either
direction. There is no rate of change of shapes, just their succession. The
only convenient way to represent this succession is in $Q^N$. However, any one
curve in $Q^N_{ss}$, denote it $C_{ss}$, is represented by an infinite set $\{C^{Q^N}_{ss}\}$ of curves
in $Q^N$. They all pass through the orbits of the shapes in $C_{ss}$, within which the $\{C^{Q^N}_{ss}\}$ curves can run anywhere. Prior to the introduction of the best-matching dynamics, all the curves $\{C^{Q^N}_{ss}\}$ are equivalent representations of $C_{ss}$ and no curve parametrization is privileged.

Best matching changes this by singling out curves in the set $\{C^{Q^N}_{ss}\}$ that ‘run horizontally’. They are distinguished representations, uniquely determined by the best matching up to a seven-parameter freedom of position in one nominally chosen initial shape in its orbit. There is also a distinguished curve parametrization (Sec. 3.1), uniquely fixed up to its origin and unit. When speaking of the distinguished representation, I shall henceforth mean that the curves in $Q^N$ and their parametrization have both been chosen in the distinguished form (modulo the residual freedoms).

Let us now consider how the dynamics that actually unfolds in $Q^N_{ss}$ is seen to unfold in the distinguished representation. From the form of the action (11), knowing that Newton’s second law can be recovered from Jacobi’s principle by choosing the distinguished curve parameter using (7), we see that we shall recover Newton’s second law exactly. We derive not only Newton’s dynamics but also the frame and time in which it holds (Fig. 6). There is a further bonus, for the best-matching dynamics is more predictive: the conditions (12), (13) and (14) must hold at any initial point that we choose and be maintained subsequently. Such conditions that depend only on the initial data (but not accelerations) and must be maintained (propagated) are called constraints. This is the important topic treated by Dirac [18].

Since the dynamics in the distinguished representation is governed by Newton’s second law, we need to establish the conditions under which it will propagate the constraints (12), (13) and (14). In fact, we have to impose conditions on the potential term $W$ in (11). If (12) is to propagate, $W$ must be a function of the coordinate differences $x_a - x_b$; if (13) is to propagate, $W$ can depend on only the inter-particle separations $r_{ab}$. These are both standard conditions in Newtonian dynamics, in which they are usually attributed to the homogeneity and isotropy of space. Here they ensure consistency of best matching wrt the Euclidean group. Propagation of (14) introduces a novel element. It requires $W$ to be homogeneous with length dimension $l^{-2}$. This requirement is immediately obvious in (11) from the length dimension $l^2$ of the kinetic term, which the potential must balance out. Note that in this case a constant $E$, corresponding to a nonzero energy
Figure 6. The distinguished representation of best-matched shape dynamics for the 3-body problem. For the initial shape, one chooses an arbitrary position in Euclidean space. Each successive shape is placed on its predecessor in the best-matched position (‘horizontal stacking’). The ‘vertical’ separation is chosen in accordance with distinguished curve parameter $t$ determined by the condition (7). In the framework thus created, the particles behave exactly as Newtonian particles in an inertial frame of reference with total momentum and angular momentum zero.

of the system, cannot appear in $W$. The system must, in Newtonian terms, have total energy zero. However, potentials with dimension $l^{-2}$ are virtually never considered in Newtonian dynamics because they do not appear to be realized in nature. I shall discuss this issue in the next subsection after some general remarks.

Best matching is a process that determines a metric on $Q^N_{ss}$. For this, three things are needed: a supermetric on $Q^N$, best matching to find the orthogonal inter-orbit separations determined by it, and the equivariance property that ensures identity of them at all positions on the orbits. Nature gives us the metric of Euclidean space, and hence the supermetric on $Q^N$; the

\footnote{It is in fact possible to recover Newtonian gravitational and electrostatic forces exactly from $l^{-2}$ potentials by dividing the $l^{-1}$ Newtonian potentials by the square root of the moment of inertia $I_{cma}$. This is because $I_{cma}$ is dynamically conserved and is effectively absorbed into the gravitational constant $G$ and charge values. However, the presence of $I_{cma}$ in the action leads to an additional force that has the form of a time-dependent ‘cosmological constant’ and ensures that $I_{cma}$ remains constant. See [12] for details.}
second and third requirements arise from the desire to implement Poincaré type dynamics in $Q^N_{ss}$. The orbit orthogonality, leading to the constraints (12), (13), and (14), distinguishes best-matched dynamics from Newtonian theory, which imposes no such requirements. Moreover, the constraint propagation needed for consistency of best matching enforces symmetries of the potential that in Newtonian theory have to be taken as facts additional to the basic structure of the theory.

It is important that the constraints (12), (13), and (14) apply only to the ‘island universe’ of the complete $N$-body system. Subsystems within it that are isolated from each other, i.e., exert negligible forces on each other, can perfectly well have nonvanishing values of $P, L, D$. It is merely necessary that their values for all of the subsystems add up to zero. However, the consistency conditions imposed on the form of the potential must be maintained at the level of the subsystems.

We see that shape dynamics has several advantages over Newtonian dynamics. The two forms of dynamics have Euclidean space in common, but shape dynamics derives all of Newton’s additional kinematic structure: absolute space (inertial frame of reference), the metric of time (duration), and the symmetries of the potential. Besides these qualitative advantages, shape dynamics is more powerful: fewer initial data predict the evolution.

This subsection has primarily been concerned with the problem of defining change of position. Newton clearly understood that this requires one to know when one can say a body is at the same place at different instants of time. Formally at least he solved this problem by the notion of absolute space. Best matching is the relational alternative to absolute space. For when one configuration has been placed relative to another in the best-matched position, every position in one configuration is uniquely paired with a position in the other. If a body is at these paired positions at the two instants, one can say it is at the same place. The two positions are equilocal. The image of ‘placing’ one configuration on another in the best-matched position is clearly more intuitive than the notion of inter-orbit orthogonality. It makes the achievement of relational equilocality manifest. It is also worth noting that the very thing that creates the problem of defining change of position – the action of the similarity group – is used to resolve it in best matching.

### 3.5 Two forms of scale invariance

We now return to the reasons for the failure of Laplacian determinism of Newtonian dynamics when considered in shape space. This will explain why it is desirable to keep open the option of the weaker form of the Poincaré...
principle. It will be helpful to consider Newtonian dynamics once more in the form of Jacobi’s principle:

$$\delta I = 0, \quad I = 2 \int d\lambda \sqrt{(E - V(q))T_{\text{kin}}}, \quad T_{\text{kin}} := \frac{1}{2} \sum_a m_a \frac{dx_a}{d\lambda} \cdot \frac{dx_a}{d\lambda}.$$  \hspace{1cm} (15)

Typically $V(q)$ is a sum of terms with different, usually integer homogeneity degrees: gravitational and electrostatic potentials have $l^{-1}$, harmonic-oscillator potentials are $l^2$. Moreover, since (15) is timeless and only the dimensionless mass ratios have objective meaning, length is the sole significant dimension. Because all terms in $V$ must have the same dimension, dimensionful coupling constants must appear. One can be set to unity because an overall factor multiplying the action has no effect on its extremals. If we take $G=1$, a fairly general action will have

$$W = E - V = E + \sum_{i<j} \frac{m_im_j}{r_{ij}} - g_iV_i,$$  \hspace{1cm} (16)

where the $V_i$’s have different homogeneity degrees, some perhaps the same (as for gravity and electrostatics). Now the crux: different $E$ and $g_i$ values lead to different curves in shape space, but not to any differences that can be expressed through an initial point and direction in $Q_{\text{ss}}^N$, which cannot encode dimensionful information. Thus, each such $E$ and $g_i$ present in (16) adds a one-parameter degree of uncertainty into the evolution from an initial point and direction in $Q_{\text{ss}}^N$. If the strong Poincaré principle holds, this unpredictability is eliminated. There may still be several different terms in $V$ but they must all have the same homogeneity degree $-2$ and dimensionless coupling constants; in particular, the constant $E$ cannot be present. Note also that any best matching enhances predictability and eliminates potentially redundant structure. But other factors may count. Nature may have reasons not to best match with respect to all conceivable symmetries.

Indeed, the foundation of particle shape dynamics on the similarity group precluded consideration of the larger general linear group. I suspect that this group would leave too little structure to construct dynamics at all easily and that angles are the irreducible minimum needed. Another factor, possibly more relevant, is the difference between velocities (and momenta), which are vectors, and directions, which are not (since multiplication of them by a number is meaningless). Vectors and vector spaces have mathematically desirable properties. In quantum mechanics, the vector nature of momenta ensures that the momentum and configuration spaces have the same number of dimensions, which is important for the equivalence of the position and momentum representations (transformation theory). If we insist on the strong
Poincaré principle, the equivalence will be lost for a closed system regarded as an island universe. There are then two possibilities: either equivalence is lost, and transformation theory only arises for subsystems (just as inertial frames of reference arise from shape dynamics), or the strong Poincaré principle is relaxed just enough to maintain equivalence.

There is an interesting way to do this. In the generic $N$-body problem the energy $E$ and angular momentum $L$, as dimensionful quantities, are not scale invariant. But they are if $E = L = 0$. Then the behaviour is scale invariant. Further [12], there is a famous qualitative result in the $N$-body problem, first proved by Lagrange, which is that $\dot{I} > 0$ if $E \geq 0$. Then the curve of $I$ as a function of time is concave upwards and its time derivative, which is $2D$ (defined just before [14]), is strictly monotonic, increasing from $-\infty$ to $\infty$ (if the evolution is taken nominally to begin at $D = -\infty$).

Now suppose that, as I conjecture, in its classical limit the quantum mechanics of the universe does require there to be velocities (and with them momenta) in shape space and its geometrodynamical generalization, to which we come soon. Then there must at the least be a one-parameter family of solutions that emanate from a point and a direction in $Q^N_{ss}$. We will certainly want rotational best matching to enforce $L = 0$. We will then have to relax dilatational best matching in such a way that a one-parameter freedom is introduced. In the $N$-body problem we can do this, without having a best-matching symmetry argument that enforces it, by requiring $E = 0$. The corresponding one-parameter freedom in effect converts a direction in $Q^N_{ss}$ into a vector. The interesting thing is now that, by Lagrange’s result, $D$ is monotonic when $E = 0$. This means that the shape-space dynamics can be monotonically parametrized by the dimensionless ratio $D_c/D_0$, where $D_0$ is an initial value of $D$ and $D_c$ is the current value. Thus, $D_c/D_0$ provides an objective ‘time’ difference between shapes $s_1$ and $s_2$. The scare quotes are used because it does not march in step with the time defined by [6].

An alternative dimensionless parametrization of the shape-space curves in this case is by means of the (not necessarily monotonic) ratio $I_c/I_0$. Because the moment of inertia measures the ‘size’ of the universe, this ratio measures ‘the expansion of the universe’ from an initial size to its current size. One might question whether in this case one should say that the dynamics unfolds on shape space. Size still has some meaning, though not at any one instant but only as a ratio at two instants. Moreover, on shape space this ratio plays the role of ‘time’ or ‘independent variable’. It does not appear as a dependent dynamical variable. This is related to the cosmological puzzle that I highlighted at the end of the introduction: from the shape-dynamic perspective, the expansion of the universe seems to be made
possible by a last vestige of Newton’s absolute space. I shall return to this after presenting the dynamics of geometry in terms of best matching.

To conclude the particle dynamics, the strong form of the Poincaré principle does almost everything that one could ask. It cannot entirely fix the potential term $W$ but does require all of its terms to be homogeneous of degree $l^{-2}$ with dimensionless coefficients, one of which can always be set to unity. If the strong Poincaré principle fails, the most interesting way the weak form can hold in the $N$-body problem is if $E = 0$. In this case a one-parameter freedom in the shape-space initial data for given $s$ and $d$ is associated with the ratio $T_s/T$ in $Q^N$.

4 Conformal Geometrodynamics

Although limited to particle dynamics, the previous section has identified the two universal elements of shape dynamics: derivation of time from difference and best matching to obviate the introduction of absolute (nondynamical) structure. However, nothing can come of nothing. The bedrock on which dynamics has been derived is the geometrical structure of individual configurations of the universe. We began with configurations in Euclidean space and removed from them more and more structure by group quotienting. We left open the question of how far such quotienting should be taken, noting that nature must decide that. In this section, we shall see that, with two significant additions, the two basic principles of shape dynamics can be directly applied to the dynamics of geometry, or geometrodynamics. This will lead to a novel derivation of, first, general relativity, then special relativity (and gauge theory) and after that to the remarkable possibility that gravitational theory introduces a dynamical standard of rest in a closed universe.

In this connection, let me address a likely worry of the reader, anticipated in footnote 49 about the fundamental role given to instantaneous configurations of the universe. Does this not flagrantly contradict the relativity of simultaneity, which is confirmed by countless experiments? In response, let me mention some possibly relevant facts.

When Einstein and Minkowski created special relativity, they did not ask how it is that inertial frames of reference come into existence. They took them as given. Even when creating general relativity, Einstein did not directly address the origin of local inertial frames of reference. Moreover, although he gave a definition of simultaneity at spatially separated points, he never asked how temporally separated durations are to be compared. What does it mean to say that a second today is the same as a second yesterday?
Shape dynamics directly addresses both of these omissions of Einstein, to which may be added his adoption of length as fundamental, which Weyl questioned in 1918 [19, 20]. Finally, it is a pure historical accident that Einstein, as he himself said, created general relativity so early, a decade before quantum mechanics was discovered. Now it is an architectonic feature of quantum mechanics that the Schrödinger wave function is defined on configuration space, not (much to Einstein’s dismay) on spacetime.

This all suggests that instantaneous spatial configurations of the universe could at the least be considered as the building blocks of gravitational theory. Indeed, they are in the Hamiltonian dynamical form of general relativity introduced by Dirac [21] and Arnowitt, Deser and Misner [22]. However, many relativists regard that formulation as less fundamental than Einstein’s original one. In contrast, I shall argue that the shape-dynamical approach might be more fundamental and that the geometrical theory of gravity could have been found rather naturally using it. I ask the reader to keep an open mind.

4.1 Superspace and conformal superspace

Differential geometry begins with the idea of continuity, encapsulated in the notion of a manifold, the rigorous definition of which takes much care. I assume that the reader is familiar with the essentials and also with diffeomorphisms; if not, [23] is an excellent introduction. To model a closed universe, we need to consider closed manifolds. The simplest possibility that matches our direct experience of space is $S^3$, which can be pictured as the three-dimensional surface of a four-dimensional sphere.

Now suppose that on $S^3$ we define a Riemannian 3-metric $g_{ij}(x)$. As a $3 \times 3$ symmetric matrix at each space point, it can always be transformed at a given point to diagonal form with 1, 1, 1 on the diagonal. Such a metric does three things. First, it defines the length $ds$ of the line element $dx^i$ connecting neighbouring points of the manifold: $ds = \sqrt{g_{ij}dx^i dx^j}$. This is well known. However, for shape dynamics it is more important that $g_{ij}(x)$ determines angles. Let two curves at $x$ be tangent to the line elements $dx^i$ and $dy^j$ and $\theta$ be the angle between them. Then

$$\cos \theta = \frac{g_{ij}dx^i dy^j}{\sqrt{g_{kl}dx^k dx^l} g_{mn} dy^m dy^n}. \quad (17)$$

The third thing that the metric does (implicitly) is give information about the coordinates employed to express the metric relations.
We see here an immediate analogy between a 3-metric and an $N$-body configuration of particles in Euclidean space. Coordinate information is mixed up with geometrical information, which itself comes in two different forms: distances and angles. Let us take this analogy further and introduce corresponding spaces and structure groups.

$\text{Riem}(S^3)$ is the (infinite-dimensional) space of all suitably continuous Riemannian 3-metrics $g_{ij}$ on $S^3$. Thus, each point in Riem is a 3-metric. However, many of these 3-metrics express identical distance relationships on the manifold that are simply expressed by means of different coordinates, or labels. They can therefore be carried into each other by three-dimensional diffeomorphisms without these distance relations being changed. They form a diffeomorphism equivalence class $\{g_{ij}\}_{\text{diff}}$, and the 3-diffeomorphisms form a structure group that will play a role analogous to the Euclidean group in particle dynamics. Each such equivalence class is an orbit of the 3-diffeomorphism group in Riem and is defined as a three-geometry. All such 3-geometries forms superspace. This is a familiar concept in geometrodynamics [24]. Less known is conformal superspace, which is obtained from superspace by the further quotienting by conformal transformations:

$$g_{ij}(x) \rightarrow \phi(x)^4 g_{ij}(x), \ \phi(x) > 0.$$  \hfill(18)

Here, the fourth power of the position-dependent function $\phi$ is chosen for convenience, since it makes the transformation of the scalar curvature $R$ simple (in four dimensions, the corresponding power is 2); the condition $\phi > 0$ is imposed to stop the metric being transformed to the zero matrix.

The transformations (18) change the distance relations on the manifold but not the angles between curves. Moreover, distances are not directly observable. To measure an interval, we must lay a ruler adjacent to it. If the interval and the ruler subtend the same angle at our eye, we say that they have the same length. This is one reason for thinking that angles are more fundamental than distances; another is that they are dimensionless. We also have the intuition that shape is more basic than size; we generally speak of the, not an, equilateral triangle. It is therefore natural to make the combination of the group of 3-diffeomorphisms and the conformal transformations (18) the structure group of conformal geometrodynamics.

Before continuing, I want to mention the subgroup of the transformations (18) that simply multiply the 3-metric by a constant $C$:

$$g_{ij}(x) \rightarrow C g_{ij}(x), \ C > 0.$$ \hfill(19)

One can say that the transformations (19) either ‘change the size of the universe’ or change the unit of distance. Like similarity transformations,
they leave all length ratios unchanged, and are conceptually distinct from the general transformations (18), which change the ratios of the geodesic lengths \(d(a, b)\) and \(d(c, d)\) between point pairs \(a, b\) and \(c, d\). As a result, general conformal transformations open up a vastly richer field for study than similarity transformations. Another subgroup consists of the \textit{volume-preserving conformal transformations} (18). They leave the total volume \(V = \int \sqrt{g}d^3x\) of the universe unchanged. We shall see that these transformations play an important role in cosmology. The seemingly minor restriction of the transformations (18) to be volume preserving is the mysterious last vestige of absolute space that I mentioned in the introduction.

The idea of geometrodynamics is nearly 150 years old. Clifford, the translator of Riemann’s 1854 paper on the foundations of geometry, conjectured in 1870 that material bodies in motion might be nothing more than regions of empty but differently curved three-dimensional space moving relative to each other [24], p. 1202. This idea is realized in Einstein’s general relativity in the vacuum (matter-free) case in the geometrodynamic interpretation advocated by Wheeler [24]. I shall briefly describe his superspace-based picture, before taking it further to conformal superspace.

Consider a matter-free spacetime that is globally hyperbolic. This means that one can slice it by nowhere intersecting spacelike hypersurfaces identified by a monotonic time label \(t\) (Fig. 7). Each hypersurface carries a 3-geometry, which can be represented by many different 3-metrics \(g_{ij}\). At any point \(x\) on one hypersurface labelled by \(t\) one can move in spacetime orthogonally to the \(t + \delta t\) hypersurface, reaching it after the proper time \(\delta \tau = N\delta t\), where \(N\) is called the \textit{lapse}. If the time labelling is changed, \(N\) is rescaled in such a way that \(N\delta t\) is invariant. In general, the coordinates on successive 3-geometries will be chosen arbitrarily, so that the point with coordinate \(x\) on hypersurface \(t + \delta t\) will not lie at the point at which the normal erected at point \(x\) on hypersurface \(t\) pierces hypersurface \(t + \delta t\). There will be a lateral displacement of magnitude \(\delta x^i = N^i\delta t\). The vector \(N^i\) is called the \textit{shift}. The lapse and shift encode the \(g_{00}\) and \(g_{0i}\) components respectively of the 4-metric: \(g_{00} = N_iN^i - N^2, g_{0i} = N_i\).

Each 3-metric \(g_{ij}\) on the successive hypersurfaces is a point in Riem, and the one-parameter family of successive \(g_{ij}\)'s is represented as a curve in Riem parametrized by \(t\). This is just one representation of the spacetime. First, one can change the time label freely on the curve (respecting monotonicity). This leaves the curve in Riem unchanged and merely changes its parametrization. Second, by changing the spatial coordinates on each hypersurface one can change the successive 3-metrics and move the curve around to a considerable degree in Riem. However, each of these curves cor-
responds to one and the same curve in superspace. But, third, one and the same spacetime can be sliced in many different ways because the definition of simultaneity in general relativity is to a high degree arbitrary (Fig. 8). Thus, an infinity of curves in superspace, and an even greater infinity of curves in Riem, represent the same spacetime. In addition, they can all carry infinitely many different parametrizations by time labels. This huge freedom corresponds to the possibility of making arbitrary four-dimensional coordinate transformations, or equivalently 4-diffeomorphisms, on spacetime.

As long as one insists on the equal status of all different slicings by spacelike hypersurfaces – on slicing or foliation invariance – it is not possible to represent the evolution of 3-geometry by a unique curve in a geometrical configuration space. This is the widely accepted view of virtually all relativists. Shape dynamics questions this. I shall now sketch the argument.

Purely geometrically, distinguished foliations in spacetime do exist. The flat intrinsic (two-dimensional) geometry of a sheet of paper is unchanged when it is rolled into a tube and acquires extrinsic curvature. By analogy, just as a 3-metric $g_{ij}$ describes intrinsic geometry, a second fundamental form, also a $3 \times 3$ symmetric tensor $K^{ij}$, describes extrinsic curvature. Its trace $K = g_{ij}K^{ij}$ is the mean extrinsic curvature. A constant-mean-curvature (CMC) hypersurface is one embedded in spacetime in such a way
Slicings of spacetime Curves in superspace

Figure 8. Because there is no distinguished definition of simultaneity in general relativity, a spacetime can be sliced in many different ways. This slicing, or foliation, freedom leads to many different representations of the spacetime by curves in superspace. Two slicings and corresponding curves in superspace are shown.

that $K$ is everywhere constant. In three-dimensional Euclidean space two-dimensional soap bubbles have CMC surfaces. Such surfaces are extremal and are therefore associated with ‘good’ mathematics. At least geometrically, they are clearly distinguished.

A complete understanding of the possibilities for slicing a spatially closed vacuum Einsteinian spacetime, i.e., one that satisfies Einstein’s field equations $G_{\mu\nu} = 0$, by CMC hypersurfaces does not yet exist. However, as we shall see, there exists a very effective and reliable way to generate ‘patches’ of CMC-foliated Einsteinian spacetimes. In such a patch CMC-foliated spacetime exists in an open neighbourhood either side of some CMC hypersurface labelled by $t = 0$. A noteworthy property of CMC foliations is that $K$, which is necessarily a spatial constant on each hypersurface, must change monotonically in the spatially closed case. Moreover, $K$ measures the rate of change of the spatial volume $V = \int \sqrt{g} \, d^3x$ in unit proper time. In both these respects, $K$ is closely analogous to the quantity $D$ in particle mechanics. We recall that it is the rate of increase of the moment of inertia

\footnote{There certainly exist spacetimes that satisfy Einstein’s field equations and do not admit CMC foliation. However, shape dynamics does not have to yield all solutions allowed by general relativity but only those relevant for the description of the universe. The ability to reproduce nature, not general relativity, is what counts.}

\footnote{The value of $K$ is only defined up to its sign.}
I, which, like V, characterizes the size of the universe.

Let us now suppose that we do have a vacuum Einsteinian spacetime that is CMC foliated either in its entirety or in some patch. On each leaf (slice) of the foliation there will be some 3-geometry and a uniquely determined conformal 3-geometry, i.e., that part of the 3-geometry that relates only to angle measurements. We can take the successive conformal 3-geometries and plot them as a curve in conformal superspace (CS). Having done this, we could change the slicing in the spacetime, obtaining a different curve of 3-geometries in superspace. They too would have associated conformal 3-geometries, and each different curve of 3-geometries in superspace would generate a different curve of conformal 3-geometries in CS. According to the standard interpretation of general relativity, all these different curves in superspace and in CS are to be regarded as physically equivalent. I believe there are grounds to at least question this.

If we go back to Clifford’s original inspiration, note that only angles are observable, and insist on either the strong or weak form of the Poincaré principle, we are led naturally to the desire to create a dynamical theory of conformal geometry in which either a point and a direction in CS or a point and a tangent vector in CS suffice to determine a unique evolution in CS. This will be exactly analogous to the aim of particle shape dynamics and will be implemented in the following subsections. What makes this shape-dynamic approach interesting is that the successions of conformal 3-geometries generated in the weak case correspond exactly to the successions of conformal 3-geometries obtained on CMC foliated Einsteinian spacetimes. Moreover, the best matching by which the dynamic curves in CS are obtained simultaneously generates the complete spacetime as the distinguished representation of the conformal dynamics. Once this spacetime has been generated in the CMC foliation, one can go over to an arbitrary foliation within it and recover all of the familiar results of general relativity. Three distinct ingredients create conformal dynamics. I shall present them one by one.

4.2 The elimination of time

It is easy (Sec. 3.1) to remove time from the kinematics of particle dynamics and recover it as a distinguished parameter from geodesic dynamics. It will help now to look at the structure of the canonical momenta in relational particle dynamics. Given a Lagrangian \( L(q_a, q'_a) \) that depends on dynamical variables \( q_a \) and their velocities \( q'_a = dq_a/d\lambda \), the canonical momentum of
where $q_a$ is $p^a := \partial L / \partial q'_a$. For the best-matched action \(1\),

\[
p^a := \frac{\partial L^{bm}}{\partial \dot{x}^a_b} = \sqrt{\frac{W}{T_{bm}}} m_a \frac{dx^{bm}_a}{d\lambda}, \quad L^{bm} = \sqrt{WT^{bm}}.
\]  

(20)

The distinguished time label $t$ is obtained by choosing $\lambda$ such that always $W = T_{bm}$ so that the cofactor of $m_a dx^{bm}_a / d\lambda$ is unity. The definition is holistic for two reasons. First, the $dx^{bm}_a / d\lambda$ are obtained by global best matching and are therefore determined by all the changes of the relative separations of the particles. Second, the denominator of the factor $\sqrt{W/T_{bm}}$ is a sum over the displacements of all the particles in the universe. This is seen explicitly in the expression \(3\). The $p^a$ have a further important property: they are reparametrization invariant. If one rescales $\lambda, \lambda \to \bar{\lambda}(\lambda)$, the velocities $\dot{q}_a'$ scale, but because velocities occur linearly in the numerator and denominator of \(20\) there is no change in $p^a$, which is in essence is a direction. (It is in fact a direction cosine wrt to the conformal metric obtained by multiplying the kinetic metric by $W$.)

However, one could take the view that, at any instant, one should obtain a local measure of time derived from purely local differences. This would still yield an holistic notion of time if the local differences were obtained by best matching. However, in the case of particle dynamics, a local derived time of this kind cannot be obtained for the simple reason that particles are, by definition, structureless. The situation is quite different in field dynamics because fields have several components at each space point. This opens up the possibility of a local measure of time, as I shall now show (deferring the conceptually distinct issue of best matching until later).

Let the action on Riem, the configuration space in which calculations are of necessity made, have the form

\[
I = \int d\lambda L, \quad L = \int d^3x \sqrt{gWT},
\]

(21)

where $g = \sqrt{\text{det} g_{ij}}$ is introduced explicitly to make the integrand a tensor density, the scalar $W$ is a local functional of $g_{ij}$ (that is, it depends on $g_{ij}$ and its spatial derivatives up to some finite order, which will in fact be the second), and $T$ depends quadratically on the metric velocities $g_{ij}' := d g_{ij} / d\lambda$ and also quadratically on $g^{ij}$. It will actually have the form

\[
T = G_A^{ijkl} g_{ij} g_{kl}, \quad G_A^{ijkl} = g^{ik} g^{jl} + A g^{ij} g^{kl}.
\]

(22)

Here, $G_A^{ijkl}$, in which $A$ is an as yet arbitrary constant, is a supermetric (cf \(3\)) and appears because one needs to construct from the velocities $g_{ij}'$ a
quantity that is a scalar under 3-diffeomorphisms. Because $g'_{ij}$ is, like the 3-metric, a symmetric tensor, there are only two independent scalars that one can form from it by contraction using the inverse metric.\footnote{The most general supermetric formed from $g^{ij}$ and acting on a general tensor has three terms. For $A = -1$, we obtain the DeWitt supermetric, which will appear later. In principle, one could also consider supermetrics formed with spatial derivatives of $g^{ij}$, but these would lead to very complicated theories. As Einstein always recommended, it is advisable to look first for the simplest nontrivial realizations of an idea.}

The key thing about (21) is that one first forms a quantity quadratic in the velocities at each space point, takes the square root at each space point, and only then integrates over space. This is a local square root and can be justified as follows. First, a square root must be introduced in some way to create a theory without any external time variable. Next, there are two ways in which this can be done. The first is by direct analogy with Jacobi’s principle (4) or (15), and would lead to an action with ‘global’ square roots of the form

$$I = \int d\lambda \sqrt{\int d^3x \sqrt{g} W} \sqrt{\int d^3x \sqrt{g} T}. \quad (23)$$

Besides being a direct generalization, the action (23) is on the face of it mathematically more correct than (21) since it defines a proper metric on Riem, which is not the case if the square root is taken, as in (21), before the integration over space. Nevertheless, it turns out that an action of the form (21) does lead to a consistent theory. This will be shown below, but we can already see that in such a case we obtain a theory with a local emergent time. For this, we merely need to calculate the form of the canonical momenta of the 3-metric $g_{ij}$ that follow from (21):

$$p^{ij} := \frac{\partial L}{\partial g'_{ij}} = \sqrt{\frac{W}{T}} G^{ijkl} g_{kl}'. \quad (24)$$

The similarity of the $p^{ij}$ to the particle canonical momenta $p^a$ (20) is obvious. First, under $\lambda \rightarrow \bar{\lambda}(\lambda)$, the momenta $p^{ij}$ are, like $x^a$, unchanged. Second, the complex of bare velocities $G^{ijkl} g_{kl}'$ is multiplied by the Jacobi-type factor $\sqrt{W/T}$. However, the key difference is that this factor is no longer a global but a position-dependent local quantity. I will not go into further details yet except to say that when the theory is fully worked out it leads to the appearance of a local increment of proper time given by $\delta \tau = N \delta \lambda$, where $N = \sqrt{T/4R}$ can be identified with the lapse in general relativity.
Whereas the elimination of time in Jacobi’s principle and for an action like (23) with global square roots is, at the classical level at least, a trivial matter with no impact on the best matching (and vice versa), the elimination of external time by the local square root has a huge effect and its consequences become intimately interconnected with those of the best matching. Perhaps the most important effect is that it drastically reduces the number of consistent actions that one can construct. This was first recognized by my collaborator Niall Ó Murchadha, and its consequences were explored in [28], about which I shall say something after the description of geometrodynamic best matching.

4.3 Geometrodynamic best matching

The basic idea of geometrodynamic best matching is exactly as for particles but leads to a vastly richer theory because a 3-geometry, either Riemannian or conformal, is infinitely more structured than a configuration of particles in Euclidean space. However, the core idea is the same: to ‘minimize the incongruence’ of two intrinsically distinct configurations. This is done by using the spatial structure groups of the configurations to bring one configuration into the position in which it most closely overlaps the other.

Let us first consider 3-diffeomorphisms. If we make an infinitesimal coordinate transformation on a given 3-metric \( g_{ij}(x) \), obtaining new functions of new coordinates, \( g_{ij}(x) \rightarrow \tilde{g}_{ij}(\tilde{x}) \), and then consider \( \tilde{g}_{ij}(\tilde{x}) \) at the old \( x \) values, the resulting 3-metric \( \tilde{g}_{ij}(x) \) is what one obtains by a 3-diffeomorphism generated by some 3-vector field \( \xi^i(x) \): 

\[
\tilde{g}_{ij}(x) = g_{ij}(x) + \xi_{(i;j)}
\]

(semicolon denotes the covariant derivative wrt to \( g_{ij} \) and the round parentheses symmetrization). The two 3-metrics \( g_{ij}(x) \) and \( \tilde{g}_{ij}(x) \) are diffeomorphically related representations of one and the same 3-geometry. This is analogous to changing the Cartesian coordinates of a particle configuration.

Now suppose that \( g_{ij}(x) + \delta g_{ij}(x) \) represents a 3-geometry genuinely distinct from \( g_{ij}(x) \), i.e., \( \delta g_{ij} \) cannot be represented in the form \( \xi_{(i;j)} \), which would indicate a spurious diffeomorphically-induced change. The difficulty that we now face is that, because we are considering intrinsically different 3-geometries, mere identity of the coordinate values \( x_i \) does not mean that they specify ‘the same point’ in the two different 3-geometries. In fact, the problem is nothing to do with coordinates. Given an apple and a pear, there does not appear to be any way to establish a 1-to-1 pairing of all the points.

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\(^{14}\)In quantum mechanics, the effect is dramatic, since the quantization of Jacobi’s principle leads to a time-independent, and not time-dependent Schrödinger equation. This is one aspect of the famous ‘problem of time’ in canonical quantum gravity [23, 24, 27].
on the apple’s surface with all those on the pear’s. However, best matching does just that if the compared objects differ only infinitesimally. To apply the technique rigourously, one must use rates of change rather than finite differences.

Mathematically, we can always specify a 3-metric $g_{ij}(x)$ and its velocity $g'_{ij} = dg_{ij}/d\lambda$. The problem is that $g'_{ij} = dg_{ij}/d\lambda$ mixes information about the intrinsic change of the described 3-geometry with arbitrary information about the way in which the coordinates are laid down as the 3-geometry changes. There is an equivalence class of velocities $\{g'_{ij} - \xi'_{(ij)}\}$ that all represent the same intrinsic change. The task of best matching is to select a unique one among them that can be said to measure the true change.

We note first that there is no objection to fixing coordinates on the original 3-geometry, giving $g_{ij}(x)$, just as we chose an initial Cartesian representation for the particle configurations. To fix the way the coordinates are then laid down, we consider the effect of $\lambda$-dependent diffeomorphisms on (21). It becomes

$$I = \int d\lambda L, \quad L = \int d^3x \sqrt{g} W T, \quad T = G^{ijkl}_{\lambda}(g'_{ij} - \xi'_{(ij)})(g'_{kl} - \xi'_{(kl)}).$$

(25)

The possibility of constructing consistent geometrodynamical theories is considered in [28], to which I refer the reader for details, since I only wish to indicate what the results are.

The basic theoretical structure obtained in geometrodynamics is broadly the same as in particle dynamics. One obtains constraints and conditions under which they propagate consistently. These conditions strongly restrict the set of consistent theories. I shall first identify the constraints and then indicate how they act as ‘theory selectors’.

First, there are constraints because of the local square root in (25). Before giving them, I need to draw attention to a similar constraint, or rather identity, in the particle model. It follows from the form (20) of the canonical momenta $p^a$ that

$$\sum_a P^a \cdot P^a = W.$$

(26)

This is a square-root identity, since it follows directly from the square root in the Lagrangian and means that the $p^a$ are in essence direction cosines.

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15 Some of the conclusions reached in [28] are too strong, being based on tacit simplicity assumptions that Anderson identified [29, 30]. I shall report here the most interesting results that are obtained when the suitable caveats are made.
In the Hamiltonian formalism, (26) becomes a constraint and is a single global relation. In contrast, the geometrodynamic action contains an infinity of square roots, one at each space point. Correspondingly, the canonical momenta satisfy infinitely many identities (or Hamiltonian constraints):

$$p_{ij}p^{ij} - \frac{2A}{3A-1}p^2 \equiv gW, \quad p = g_{ij}p^{ij}. \quad (27)$$

Second, constraints arise through the best matching wrt diffeomorphisms. This is implemented by variation of (25) with respect to $\xi_i'$, treated as a Lagrange multiplier. This also leads to a constraint at each space point:

$$p_{ij}^\prime j = 0. \quad (28)$$

These linear constraints are closely analogous to the linear constraints (12) and (13) in particle dynamics. For the form (25) of the action, they propagate automatically and do not lead to restrictions. This is because the action (21) was chosen in advance in a form invariant under $\lambda$-independent 3-diffeomorphisms, which in turn ensured that (25) is invariant under $\lambda$-dependent 3-diffeomorphisms. Had we chosen a more general functional of $g_{ij}$ and its spatial and $\lambda$ derivatives, propagation of the constraints (28) would have forced us to specialize the general form to (25). This is another manifestation of the power of combining the structure group of the 3-metrics (the 3-diffeomorphism group) with the Poincaré requirement.

There is no analogous control over the quadratic constraints (27) that arise from the local square root in (21) and (25). As is shown in [28], the only actions that consistently propagate both the quadratic and linear constraints has the form

$$I_{BSW} = \int d\lambda \int d^3x \sqrt{(\Lambda + dR)T_{\lambda=-1}}, \quad (29)$$

where the subscript $\lambda = -1$ of $T$ indicates that the undetermined coefficient in the supermetric is forced to take the DeWitt value. More impressive is the drastic restriction on the possible form of the potential term $W$, which is restricted to be $\Lambda + dR, d = 0$ or $\pm 1$. The action (29) is in fact the Baierlein–Sharp–Wheeler action [31], which is dynamically equivalent to the Einstein–Hilbert action for globally hyperbolic spacetimes. The only freedom is in the choice of the constant $\Lambda$, which corresponds to the cosmological constant, and the three options for $d$. The case $d = 0$ yields so-called strong gravity and is analogous to pure inertial motion for particles. The case $d$ corresponds to a Lorentzian spacetime and hence to the standard form of general relativity, while $d = -1$ gives Euclidean general relativity.
When translated into spacetime terms, the constraints (27) and (28) are respectively the 00 and 0i, i = 1, 2, 3, Einstein field equations $G_{\mu\nu} = 0$, $\mu, \nu = 0, 1, 2, 3$. Whereas the particle dynamics associated with the global Euclidean group leads to global relations, implementation of the Poincaré principle in geometrodynamics by the local elimination of time and best matching wrt local 3-diffeomorphisms leads to local constraints, the propagation of which directly determines the simplest nontrivial realization of the whole idea: general relativity. Of course, the immense power of local symmetry requirements was one of the great discoveries of 20th-century physics. It first became apparent with Einstein’s creation of general relativity. If shape dynamics has value, it is not so much in the locality of the symmetries as in their choice and in the treatment of time. I shall compare the shape-dynamic approach with Einstein’s at the end of the paper. Here I want to continue with the results of [28].

So far, we have considered pure geometrodynamics. The assumption that the structure of spacetime always reduces locally to the Minkowski-space form of special relativity (a key element in Einstein’s approach) played no role in the derivation. The manner in which special relativity arises in [28] is striking. In field theory, the essence of special relativity is a universal light cone: all fields must have the same limiting signal propagation velocity. Now vacuum general relativity has a ‘light cone’. What happens if we attempt, as the simplest possibility, to couple a scalar field $\varphi$ to vacuum geometrodynamics described by the action (25)?

The propagation speed of such a field, with action containing the field velocities $d\varphi/d\lambda$ and first spatial derivatives $\partial_i\varphi$ quadratically, is determined by a single coefficient $C$, which fixes the ratio of the contributions of $d\varphi/d\lambda$ and $\partial_i\varphi$ to the action. When the scalar field is added to the action (for details see [28]), the constraints (27) and (28) acquire additional terms, and one must verify that the modified constraints are propagated by the equations of motion. It is shown in [28] that propagation of the modified quadratic constraint fixes the coefficient $C$ to be exactly such that it shares the geometrodynamic light cone. Otherwise, the scalar field can have a term in its action corresponding to a mass and other self-interactions.

The effect of attempting to couple a single 3-vector field $A$ to the geometry is even more remarkable. In this case, there are three possible terms that can be formed from the first spatial (necessarily covariant) derivatives of $A$. Each may enter in principle with an arbitrary coefficient. The requirement that the modified quadratic constraint propagate not only fixes

\[^{16}\text{The ‘construction of spacetime’ will be described later.}\]
all three coefficients in such a way that the 3-vector field has the same light cone as the geometry but also imposes the requirement that the canonical momenta $\mathbf{P}$ of $\mathbf{A}$ satisfy the constraint $\text{div} \mathbf{P} = 0$. In fact, the resulting field is none other than the Maxwell field interacting with gravity. The constraint $\text{div} \mathbf{P} = 0$ is the famous Gauss constraint. This can be taken further. If one attempts to construct a theory of several 3-vector fields that interact with gravity and with each other, they have to be Yang–Mills gauge fields. Unlike the scalar field, all the gauge fields must be massless.

To conclude this subsection, let us indulge in some ‘what-might-have-been’ history. Clifford’s ‘dream’ of explaining all motion and matter in terms of dynamical Riemannian 3-geometry was in essence a proposal for a new ontology of the world. The history of science shows that new, reasonably clearly defined ontologies almost always precede major advances. A good example is Descartes’s formulation of the mechanical world view; it led within a few decades to Newton’s dynamics (Chaps. 8–10). Clifford died tragically young; he could have lived to interact with both Mach and Poincaré. Between them, they had the ideas and ability needed to create a relational theory of dynamical geometry (and other fields) along the lines described above. In this way, well before 1905, they could have discovered, first, general relativity in the form of the Baierlein–Sharp–Wheeler action (29), next special relativity through a universal light cone, and even, third, gauge theory. All of this could have happened as part of a programme to realize Clifford’s original inspiration in the simplest nontrivial way.

I want to emphasize the role that the concept of time would have played in such a scenario. In 1905, Einstein transformed physics by insisting that the description of motion has no meaning “unless we are quite clear as to what we understand by ‘time’ ”. He had in mind the problem of defining simultaneity at spatially separated points. Resolution of this issue in 1905 was perhaps the single most important thing that then led on to general relativity. However, in 1898, Poincaré had noted the existence of two fundamental problems related to time: the definition of simultaneity and the older problem of defining duration: What does it mean to say that a second today is the same as a second tomorrow? Even earlier, in 1883, Mach had said: “It is utterly beyond our power to measure the changes of things by time. Quite the contrary, time is an abstraction at which we arrive by means of the changes of things.” Both Mach and Poincaré had clearly recognized the need for a theory of duration along the lines of Sec. 3.1.

\[\text{17}\] Despite a careful search through his papers, I have been unable to find any evidence that Einstein ever seriously considered the definition of duration. As we saw in Sec. 3.1.
There is now an intriguing fact. The structure of dynamics so far presented in this paper has been based on two things: best matching and a theory of duration. Both were initially realized globally, after which a local treatment was introduced. Moreover, entirely different schemes were used to achieve the desired aims of a relational treatment of displacement and of duration (best matching and a square root in the action respectively). Remarkably, Einstein’s theory of simultaneity appeared as a consequence of these relational inputs. In line with my comments at the end of Sec. 3.1, I believe that the concept of duration as a measure of difference is more fundamental than the definition of simultaneity, so it is reassuring that Einstein’s well confirmed results can be recovered starting from what may be deeper foundations. In this connection, there is another factor to consider. In the standard representation of general relativity, spacetime is a four-dimensional block. One is not supposed to think that the Riemannian 3-geometry on the leaves of a 3+1 foliation is more fundamental than the lapse and shift, which tell one how the 3-geometries on the leaves ‘fit together’ (Fig. 7). The lapse is particularly important: it tells you the orthogonal separation (in spacetime) between the 3-geometries that are the leaves of a 3+1 foliation. However, the $G^{00}$ Einstein field equation enables one to solve algebraically for the lapse in terms of the other variables. It is precisely this step that led Baierlein, Sharp and Wheeler to the BSW action \(^{(29)}\). It contains no lapse, but, as we have seen, is exactly the kind of action that one would write down into to implement (locally) Mach’s requirement that time (duration) be derived from differences. Thus, there is an exactly right theory of duration at the heart of general relativity, but it is hidden in the standard representation.

However, this is not the end of the story. Quite apart from the implications of the two aspects of time – duration and simultaneity – for the quantum theory of the universe, there is also what Weyl \(^{(37)}\) called the “disrupting question of length”: Why does nature seem to violate the principle that size should be relative? We shall now see that a possible answer to this question may add yet another twist to the theory of time.

4.4 Conformal best matching

In best matching wrt 3-diffeomorphisms, we are in effect looking at all possible ways in which all points on one 3-geometry can be mapped bijectively to the points of an intrinsically different 3-geometry and selecting the bijec-

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this is intimately related to the theory of clocks, which Einstein did grant had not been properly included in general relativity. He called the omission a ‘sin’ \(^{(36)}\).
tion that extremalizes\textsuperscript{18} the quantity chosen to measure the incongruence of the two. So far, we have not considered changing the local scale factor of the 3-metrics in accordance with the conformal transformations (18). But given that only angles are directly observable, we have good grounds for supposing that lengths should not occur as genuine dynamical degrees of freedom in the dynamics of geometry. If we best match wrt conformal transformations, only the angle-determining part of 3-metrics can play a dynamical role. Moreover, we have already noted (Sec. 4.1) the possibility that we might wish to best match only wrt volume-preserving conformal transformations.

At this point, it is helpful to recall the geometrical description of best matching in the particle model. It relies on a supermetric on the ‘large’ configuration space $Q^N$, which is foliated by the orbits of whatever group one is considering. Each orbit represents the intrinsic physical configuration of the system. Hitherto the supermetric chosen on the ‘large’ space ($Q^N$ or Riem) has been equivariant, so that the orthogonal separation $d_s$ between neighbouring orbits is the same at all points on the orbits. This made it possible to calculate the orthogonal $d_s$ anywhere between the orbits and, knowing that the same value would always be obtained, project any such $d_s$ down to the physical quotient space. This met the key aim – to define a metric on the physical space.

Now there is in principle a different way in which this aim can be met. It arises if the orthogonal separation between the orbits is not constant but has a unique extremum at some point between any two considered orbits\textsuperscript{19}. This unique extremal value can then be taken to define the required distance on the physical quotient space. I shall now indicate how this possibility can be implemented. Since the equations become rather complicated, I shall not attempt to give them in detail but merely outline what happens.

We start with the BSW action (29), since our choices have already been restricted to it by the local square root and the diffeomorphism best matching (neither of which we wish to sacrifice, though we will set $\Lambda = 0$ for simplicity). As just anticipated, we immediately encounter a significant dif-

\textsuperscript{18}We have to extremalize rather than minimize because the DeWitt supermetric ($G^{ijkl}_{A=-1}$ in (22)) is indefinite. Einsteinian gravity is unique among all known physical fields in that its kinetic energy is not positive definite. The part associated with expansion of space – the second term in (25) – enters with the opposite sign to the part associated with the change of the conformal part of the 3-metric, i.e., its shape.

\textsuperscript{19}To the best of my knowledge, this possibility (which certainly does not occur in gauge theory) was first considered by Ó Murchadha, who suggested it as a way to implement conformal best matching in [38].
ference from the best-matching w.r.t 3-diffeomorphisms, for which we noted that (21) is invariant under $\lambda$-independent diffeomorphisms. In the language of gauge theory, (21) has a global (w.r.t $\lambda$) symmetry that is subsequently gauged by replacing the bare velocity $g'_{ij}$ by the corrected velocity $g'_{ij} - \xi'_{(ij)}$. It is the global symmetry which ensures that the inter-orbit separation in Riem is everywhere constant (equivariance). In contrast to the invariance of (21) under $\lambda$-independent diffeomorphisms, there is no invariance of (21) under $\lambda$-independent conformal transformations of the form (18). The kinetic term by itself is invariant, but $\sqrt{gR}$ is not. Indeed,

$$\sqrt{gR} \rightarrow \sqrt{g\phi^4} \sqrt{R - 8 \nabla^2 \phi}. \quad (30)$$

It should however be stressed that when (21) is ‘conformalized’ in accordance with (18) the resulting action is invariant under the combined gauge-type transformation

$$g_{ij} \rightarrow \omega^4 g_{ij}, \quad \phi \rightarrow \frac{\phi}{\omega}, \quad (31)$$

where $\omega = \omega(x, \lambda)$ is an arbitrary function. This exactly matches the invariance of (25) under 3-diffeomorphisms that arises because the transformation of $g'_{ij}$ is offset by a compensating transformation of the best-matching correction $\xi'_{(ij)}$. The only difference is that under the diffeomorphisms the velocities alone are transformed because of the prior choice of an action that is invariant under $\lambda$-independent transformations, whereas (31) generates transformations of both the dynamical variables and their velocities.

We now note that if we best match (25) w.r.t unrestricted conformal transformations, we run into a problem since we can make the action ever smaller by taking the value of $\phi$ ever smaller. Thus, we have no chance of finding an extremum of the action. There are two ways in which this difficulty can be resolved. The first mimics what we did in particle dynamics in order to implement the strong Poincaré principle on shape space, namely use a Lagrangian that overall has length dimension zero.

In the particle model we did this by dividing the kinetic metric $ds$ by $\sqrt{I_{\text{cms}}}$, where $I_{\text{cms}}$ is the cms moment of inertia. The analog of $I_{\text{cms}}$ in geometrodynamics is $V$, the total volume of the universe, and division of the Lagrangian in (21) by $V^{\frac{2}{3}}$ achieves the desired result. This route is explored in [39]. It leads to a theory on conformal superspace that satisfies the strong Poincaré principle and is very similar to general relativity, except for an epoch-dependent emergent cosmological constant. This has the effect of
enforcing $V = \text{constant}$, with the consequence that the theory is incapable of explaining the diverse cosmological phenomena that are all so well explained by the theory of the expanding universe. The theory is not viable.

An alternative is to satisfy the weak Poincaré principle by restricting the conformal transformations (18) to be such that they leave the total volume unchanged. At the end of Sec. 4.1 I briefly described the consequences. Let me now give more details; for the full theory, see [40]. The physical space is initially chosen to be conformal superspace (CS), to which the space $V$ of possible volumes $V$ of the universe is adjoined, giving the space CS+$V$. One obtains a theory that in principle yields a unique curve between any two points in CS+$V$. These two points are specified by giving two conformal geometries $c_1$ and $c_2$, i.e., two points in CS, and associated volumes $V_1$ and $V_2$. However, there are two caveats. First, one cannot guarantee monotonicity of $V$. This difficulty can be avoided by passing from $V$ to its canonically conjugate variable; in spacetime terms, this turns out to be $K$, the constant mean curvature of CMC hypersurfaces. Second, both $V$ and $K$ have dimensions and as such have no direct physical significance. Only the curves projected from CS+$V$ to CS correspond to objective reality. In fact, a two-parameter family of curves in CS+$V$ projects to a single-parameter family of curves in CS labelled by the dimensionless values of $V_2/V_1$ or, better, the monotonic $K_2/K_1$.

A comparison with the standard variational principle for the $N$-body problem is here helpful. In it one specifies initial and final configurations in $Q^N$, i.e., $2 \times 3N$ numbers, together with a time difference $t_2 - t_1$. Thus, the variational problem is defined by $6N + 1$ numbers. However, the initial value problem requires only $6N$ numbers: a point in $Q^N$ and the $3N$ numbers required to specify the (unconstrained) velocities at that point. In a geodesic problem, one requires respectively $6N$ and $6N - 1$ numbers in the two different but essentially equivalent formulations. In the conformal theory, we thus have something very like a monotonic ‘time’, but it does not enter as a difference $t_2 - t_1$ but as the ratio $K_2/K_1$. This result seems to me highly significant because it shows (as just noted in the footnote) that in the shape-dynamic description of gravity one can interpret the local shapes of space as the true degrees of freedom and $K_2/K_1$ as an independent variable. As $K_2/K_1$ varies, the shapes interact with each other. This mirrors the in-

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20This fact escaped notice in [40]. Its detection led to [41], which shows that a point and a tangent vector in CS are sufficient to determine the evolution in CS. In turn, this means that the evolution is determined by exactly four local Hamiltonian shape degrees of freedom per space point. The paper [40] was written in the mistaken belief that one extra global degree of freedom, the value of $V$, also plays a true dynamical role.
teraction of particle positions in Newtonian dynamics as time, or, as we saw earlier, $T/V$ changes. However, the closer analogy in Newtonian dynamics is with the system’s change of the shape as $D_2/D_1$ changes.

The only input data in this form of the shape-dynamic conformal theory are the initial point and tangent vector in CS. There is no trace of local inertial frames of reference, local proper time, or local proper distance. In the standard derivation of general relativity these are all presupposed in the requirement that locally spacetime can be approximated in a sufficiently small region by Minkowski space. In contrast, in the conformal approach, this entire structure emerges from specification of a point and direction in conformal superspace.

One or two points may be made in this connection. First, as the reader can see in [40], the manner in which the theory selects a distinguished 3-geometry in a theory in which only conformal 3-geometry is presupposed relies on intimate interplay of the theory’s ingredients. These are the local square root and the two different best matchings: wrt to diffeomorphisms and conformal transformations. Second, the construction of spacetime in a CMC foliation is fixed to the minutest detail from input that can in no way be reduced. Expressed in terms of two infinitesimally differing conformal 3-geometries $C_1$ and $C_2$, the outcome of the best matchings fixes the local scale factor $\sqrt{g}$ on $C_1$ and $C_2$, making them into 3-geometries $G_1$ and $G_2$. Thus, it takes one to a definite position in the conformal orbits. This is the big difference from the best matching with respect to diffeomorphisms alone and what happens in the particle model and gauge theory. In these cases the position in the orbit is not fixed. Next, the best matching procedure pairs each point on $G_1$ with a unique point on $G_2$ and determines a duration between them. In the spacetime that the theory ‘constructs’ the paired points are connected by spacetime vectors orthogonal to $G_1$ and $G_2$ and with lengths equal to definite (position-dependent) proper times. These are determined on the basis of the expression (24) for the canonical momenta, in which $W = gR$. The lapse $N$ is $N = \sqrt{\frac{T}{4R}}$ and the amount of proper time $\delta \tau$ between the paired points is $\delta \tau = N \delta \lambda$. It is obvious that $\delta \tau$ is the outcome of a huge holistic process: the two best matchings together determine not only which points are to be paired but also the values at the paired points of all the quantities that occur in the expression $N = \sqrt{\frac{T}{4R}}$.

We can now see that there are two very different ways of interpreting

\[21\] The 4-metric $g_{\mu\nu}$ has 10 components, of which four correspond to coordinate freedom. If one takes the view, dictated by general covariance, that all the remaining six are equally physical, then the entire theory rests on Minkowski space. One merely allows it to be bent, as is captured in the ‘comma goes to semicolon’ rule.
general relativity. In the standard picture, spacetime is assumed from the beginning and it must locally have precisely the structure of Minkowski space. From the structural point of view, this is almost identical to an amalgam of Newton’s absolute space and time. This near identity is reflected in the essential identity locally of Newton’s first law and Einstein’s geodesic law for the motion of an idealized point particle. In both cases, it must move in a straight line at a uniform speed. As I already mentioned, this very rigid initial structure is barely changed by Einstein’s theory in its standard form. In Wheeler’s aphorism [24], “Space tells matter how to move, matter tells space how to bend.” But what we find at the heart of this picture is Newton’s first law barely changed. No explanation for the law of inertia is given: it is a – one is tempted to say the – first principle of the theory. The wonderful structure of Einstein’s theory as he constructed it rests upon it as a pedestal. I hope that the reader will at least see that there is another way of looking at the law of inertia: it is not the point of departure but the destination reached after a journey that takes into account all possible ways in which the configuration of the universe could change.

This bears on the debate about reductionism vs holism. I believe that the standard spacetime representation of general relativity helps to maintain the plausibility of a reductionist approach. Because Minkowski’s spacetime seems to be left essentially intact in local regions, I think many people (including those working in quantum field theory in external spacetimes) unconsciously assume that the effect of the rest of the universe can be ignored. Well, for some things it largely can. However, I feel strongly that the creation of quantum gravity will force us to grasp the nettle. What happens locally is the outcome of everything in the universe. We already have a strong hint of this from the classical theory, which shows that the ‘reassuring’ local Minkowskian framework is determined – through elliptic equations in fact – by every last structural detail in the remotest part of the universe.

4.5 Shape dynamics or general relativity?

There is no question that general relativity has been a wonderful success and as yet has passed every experimental test. The fact that it predicts singularities is not so much a failure of the theory as an indication that quantum gravity must at some stage come into play and ‘take over’. A more serious criticism often made of general relativity is that its field equations $G_{\mu\nu} = T_{\mu\nu}$ allow innumerable solutions that strike one as manifestly unphysical, for example, the ones containing closed timelike curves. There is
a good case for seeking a way to limit the number of solutions. Perhaps the least controversial is the route chosen by Dirac \[21\] and Arnowittt, Deser, and Misner (ADM) \[22\]. The main justification for their 3+1 dynamical approach is the assumption that gravity can be described in the Hamiltonian framework, which is known to be extremely effective in other branches of physics and especially in quantum mechanics.

If a Hamiltonian framework is adopted, it then becomes especially attractive to assume that the universe is spatially closed. This obviates the need for arbitrary boundary conditions, and, as Einstein put it when discussing Mach’s principle (\[42\], p. 62), “the series of causes of mechanical phenomena [is] closed”.

The main difficulty in suggesting that the spacetime picture should be replaced by the more restrictive Hamiltonian framework arises from the relativity principle, i.e., the denial of any distinguished definition of simultaneity. In the ideal form of Hamiltonian theory, one seeks to have the dynamics represented by a unique curve in a phase space of true Hamiltonian degrees of freedom. This is equivalent to having a unique curve in a corresponding configuration space of true geometrical degrees of freedom even if mathematical tractability means that the calculations must always be made in Riem. Dirac and ADM showed that dynamics in Riem could be interpreted in superspace, thereby reducing the six degrees of freedom per space point in a 3-metric to the three in a 3-geometry. But the slicing freedom within spacetime means that a single spacetime still corresponds to an infinity of curves in superspace. The Hamiltonian ideal is not achieved. The failure is tantalizing, because much evidence suggests that gravity has only two degrees of freedom per space point, hinting at a configuration space smaller than superspace.

As long as relativity of simultaneity is held to be sacrosanct, there is no way forward to the Hamiltonian ideal. York and Wheeler came close to suggesting that it was to be found in conformal superspace, but ultimately balked at jettisoning the relativity principle. In this connection, it is worth pointing out that Einstein’s route to general relativity occurred at a partic-

\[22\] York’s highly important work on the initial-value problem of general relativity \[43, 44\] is intimately related to the shape-dynamic programme and was one of its inspirations. For a discussion of the connections, see \[40\]. One of the arguments for the shape-dynamic approach is that it provides a first-principles derivation of York’s method, which in its original form was found by trial and error. It may also be noted here that York’s methods, which were initially developed for the vacuum (matter-free) case, can be extended to include matter \[45, 46\]. This suggests that the principles of shape dynamics will extend to the case in which matter is present.
ular point in history and things could have been approached differently. I think it entirely possible that Einstein’s discovery of his theory of gravitation in spacetime form could be seen as a glorious historical accident. In particular, Einstein could easily have looked differently at certain fundamental issues related to the nature of space, time, and motion. Let me end this introduction to shape dynamics with some related observations on each.

**Space.** Riemann based his generalization of Euclidean geometry on length as fundamental. It was only in 1918, three years after the creation of general relativity, that Weyl [19, 20] challenged this and identified – in a four-dimensional context – angles as more fundamental. I will argue elsewhere that Weyl’s attempt to generalize general relativity to eliminate the correctly perceived weakness of Riemann’s foundations failed because it was not sufficiently radical – instead of eliminating length completely from the foundations, Weyl retained it in a less questionable form.

**Time.** As I noted earlier, in 1898 Poincaré [34, 35] identified two equally fundamental problems related to time: how is one to define duration and how is one to define simultaneity at spatially separated points? Einstein attacked the second problem brilliantly but made no attempt to put a solution to the second into the foundations of general relativity.

**Motion.** In the critique of Newtonian mechanics that was such a stimulus to general relativity, Mach argued that only relative velocities should occur in dynamics. Einstein accepted this aspiration, but did not attempt to put it directly into the foundations of general relativity, arguing that it was impractical ([3, 5], p. 186). Instead it was necessary to use coordinate systems and achieve Mach’s ideal by putting them all on an equal footing (general covariance).

All three alternatives in approach listed above are put directly into shape dynamics. I think that this has been made adequately clear with regard to the treatment of space and motion. I wish to conclude with a comment on the treatment of time, which is rather more subtle.

It is well known that Einstein regarded special relativity as a principles theory like thermodynamics, which was based on human experience: heat energy never flows spontaneously from a cold to a hot body. Similarly, uniform motion was always found to be indistinguishable – within a closed system – from rest. Einstein took this fact as the basis of relativity and never attempted to explain effects like time dilatation at a microscopic level in the way Maxwell and Boltzmann developed the atomic statistical theory of thermodynamics. Since rods and clocks are ultimately quantum objects, I do not think such a programme can be attempted before we have a better idea of the basic structure of quantum gravity. However, I find it interesting
and encouraging that a microscopic theory of duration is built in at a very basic level in shape dynamics. This is achieved in particle dynamics using Jacobi’s principle, which leads to a global definition of duration, and in conformal dynamics using the local-square-root action (21). I also find it striking that, as already noted, the simple device of eliminating the lapse from Einstein’s spacetime theory immediately transforms his theory from one created without any thought of a microscopic theory of duration into one (based on the Baierlein–Sharp–Wheeler action (29)) that has such a theory at its heart. A theory of duration was there all along. It merely had to be uncovered by removing some of the structure that Einstein originally employed – truly a case of less is more.

The effect of the local square root is remarkable. At the level of theory creation in superspace, in which length is taken as fundamental, the local square root acts as an extremely powerful selector of consistent theories and, as we have seen, enforces the appearance of the slicing freedom, universality of the light cone, and gauge fields as the simplest bosonic fields that couple to dynamic geometry. As I have just noted, it also leads to a microscopic theory of local duration (local proper time). Thus, the mere inclusion of the local square root goes a long way to establishing a constructive theory of special-relativistic effects. It is not the whole way, because quantum mechanics must ultimately explain why physically realized clocks measure the local proper time created by the local theory of duration.

The effect of the local square root is even more striking when applied in theory creation in conformal superspace. It still enforces universality of the light cone and the appearance of gauge fields but does two further things. First, it leads to a microscopic theory of length. For the conformal best matching, in conjunction with the constraints that follow from the local square root, fixes a distinguished scale factor of the 3-metric. Second, it introduces the distinguished CMC foliation within spacetime without changing any of the classical predictions of general relativity. It leads to a theory of simultaneity.

Thus, the conformal approach to geometrodynamics suggests that there are two candidate theories of gravity that can be derived from different first principles. Einstein’s general relativity is based on the idea that spacetime is the basic ontology; its symmetry group is four-dimensional diffeomorphism invariance. But there is also an alternative dual theory based on three-dimensional diffeomorphism invariance and conformal best matching [9, 10, 40, 41]. The set of allowed solutions of the conformal theory is significantly smaller than the general relativity set. In principle, this is a good feature, since it makes the conformal theory more predictive, but it cannot be ruled
out that, being tied to CMC foliations, the conformal theory will be unable to describe physically observable situations that are correctly described by general relativity.

I will end with two comments. First, shape dynamics in conformal superspace is a new and mathematically well-defined framework of dynamics. Second, its physical applications are most likely to be in quantum gravity.

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References

[1] E Mach. *Die Mechanik in ihrer Entwicklung Historisch-Kritisch Dargestellt*. 1883.

[2] E Mach. *The Science of Mechanics*. Open Court, 1960.

[3] A Einstein. Prinzipielles zur allgemeinen Relativitätstheorie. *Annalen der Physik*, 55:241–244, 1918.

[4] A Einstein. Dialog über Einwände gegen die Relativitätstheorie. *Die Naturwissenschaften*, 6:697–702, 1918.

[5] J Barbour and H Pfister, editors. *Mach’s Principle: From Newton’s Bucket to Quantum Gravity*, volume 6 of *Einstein Studies*. Birkhäuser, Boston, 1995.

[6] J Barbour. *Absolute or Relative Motion? Volume 1. The Discovery of Dynamics*. Cambridge University Press, 1989.

[7] J Barbour. *The Discovery of Dynamics*. Oxford University Press, 2001.

[8] I Newton. *Sir Isaac Newton’s Mathematical Principles of Natural Philosophy*. University of California Press, 1962.
[9] H Gomes, S Gryb, and T Koslowski. Einstein gravity as a 3D conformally invariant theory (arXiv:1010.2481). Class. Quant. Grav., 28:045005, 2011.

[10] H Gomes and T Koslowski. The link between general relativity and shape dynamics. arXiv:1101.5974.

[11] J Barbour and B Bertotti. Mach’s principle and the structure of dynamical theories (downloadable from platonia.com). Proceedings of the Royal Society London A, 382:295–306, 1982.

[12] J Barbour. Scale-invariant gravity: particle dynamics. Class. Quantum Grav., 20:1543–1570, 2003, gr-qc/0211021.

[13] D Saari. Collisions, Rings, and Other Newtonian N-Body Problems. American Mathematical Society, Providence, Rhode Island, 2005.

[14] Henri Poincaré. Science et Hypothèse. Paris, 1902.

[15] H Poincaré. Science and Hypothesis. Walter Scott, London, 1905.

[16] J Barbour. The definition of Mach’s principle (arXiv:1007.3368). Found. Phys., 40:1263–1284, 2010.

[17] C Lanczos. The Variational Principles of Mechanics. University of Toronto Press, 1949.

[18] P A M Dirac. Lectures on Quantum Mechanics. Belfer Graduate School of Science, Yeshiva University, New York, 1964.

[19] H Weyl. Gravitation und Elektrizität. Sitzungsber. Preuss. Akad. Berlin, pages 465–480, 1918.

[20] H Weyl. Gravitation and electricity. In Ó Raifeartaigh, editor, The dawning of gauge theory, pages 24–37. Princeton University Press, 1997.

[21] P A M Dirac. Generalized Hamiltonian dynamics. Proc. R. Soc. (London), A246:326–343, 1958.

[22] R Arnowitt, S Deser, and C W Misner. The dynamics of general relativity. In L Witten, editor, Gravitation: An Introduction to Current Research, pages 227–265. Wiley, New York, 1962.

[23] B F Schutz. Geometrical Methods of Mathematical Physics. Cambridge University Press, Cambridge, 1980.
[24] C W Misner, K S Thorne, and J A Wheeler. *Gravitation*. W H Freeman and Company, San Francisco, 1973.

[25] Karel Kuchař. Time and interpretations of quantum gravity. In G Kunstatter, D Vincent, and J Williams, editors, *Proceedings 4th Canadian Conf. General Relativity and Relativistic Astrophysics*, pages 211–314. World Scientific, Singapore, 1992.

[26] C J Isham. Canonical quantum gravity and the problem of time. In L A Ibort and M A Rodríguez, editors, *Integrable Systems, Quantum Groups, and Quantum Field Theory*, pages 157–287. Kluwer, Dordrecht, 1993.

[27] J Barbour. *The End of Time*. Weidenfeld and Nicolson, London; Oxford University Press, New York, 1999.

[28] J Barbour, B Z Foster, and N Ó Murchadha. Relativity without relativity. *Class. Quantum Grav.*, 19:3217–3248, 2002, gr-qc/0012089.

[29] E Anderson. On the recovery of geometrodynamics from two different sets of first principles. *Stud. Hist. Philos. Mod. Phys.*, 38:15, 2007. arXiv:gr-qc/0511070.

[30] E Anderson. Does relationalism alone control geometrodynamics with sources? 2007. arXiv:0711.0285.

[31] R Baierlein, D Sharp, and J Wheeler. Three-dimensional geometry as a carrier of information about time. *Phys. Rev.*, 126:1864–1865, 1962.

[32] E Anderson and J Barbour. Interacting vector fields in relativity without relativity. *Class. Quantum Grav.*, 19:3249–3262, 2002, gr-qc/0201092.

[33] A Einstein. Zur Elektrodynamik der bewegter Körper. *Ann. Phys.*, 17:891–921, 1905.

[34] H Poincaré. La mesure du temps. *Rev. Métaphys. Morale*, 6:1, 1898.

[35] H Poincaré. The measure of time. In *The Value of Science*. 1904.

[36] A Einstein. Autobiographical notes. In P Schilpp, editor, *Albert Einstein: Philosopher–Scientist*. Harper and Row, New York, 1949.

[37] H Weyl. *Symmetry*. Princeton University Press, 1952.
[38] J Barbour and N Ó Murchadha. Classical and quantum gravity on conformal superspace. 1999, gr-qc/9911071.

[39] E Anderson, J Barbour, B Z Foster, and N Ó Murchadha. Scale-invariant gravity: geometrodynamics. *Class. Quantum Grav.* 20:1571, 2003, gr-qc/0211022.

[40] E Anderson, J Barbour, B Z Foster, B Kelleher, and N Ó Murchadha. The physical gravitational degrees of freedom. *Class. Quantum Grav.*, 22:1795–1802, 2005, gr-qc/0407104.

[41] J Barbour and N Ó Murchadha. Conformal Superspace: the configuration space of general relativity, arXiv:1009.3559.

[42] A Einstein. *The Meaning of Relativity*. Methuen and Co Ltd, London, 1922.

[43] J W York. Gravitational degrees of freedom and the initial-value problem. *Phys. Rev. Letters*, 26:1656–1658, 1971.

[44] J W York. The role of conformal 3-geometry in the dynamics of gravitation. *Phys. Rev. Letters*, 28:1082–1085, 1972.

[45] J Isenberg, N Ó Murchadha, and J W York. Initial-value problem of general relativity. III. *Phys. Rev. D*, 12:1532–1537, 1976.

[46] J Isenberg and J Nester. Extension of the York field decomposition to general gravitationally coupled fields. *Ann. Phys.*, 108:368–386, 1977.