Oracle inequalities for image denoising with total variation regularization

Francesco Ortelli
Rämistrasse 101
8006 Zürich
e-mail: fortelli@ethz.ch
and
Sara van de Geer
Rämistrasse 101
8006 Zürich
e-mail: geer@ethz.ch

Abstract: We derive oracle results for discrete image denoising with a total variation penalty. We consider the least squares estimator with a penalty on the $\ell_1$-norm of the total discrete derivative of the image. This estimator falls into the class of analysis estimators. A bound on the effective sparsity by means of an interpolating matrix allows us to obtain oracle inequalities with fast rates. The bound is an extension of the bound by Ortelli and van de Geer [2019c] to the two-dimensional case. We also present an oracle inequality with slow rates, which matches, up to a log-term, the rate obtained for the same estimator by Mammen and van de Geer [1997]. The key ingredient for our results are the projection arguments to bound the empirical process due to Dalalyan et al. [2017].

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1. Introduction

1.1. Model

We consider the problem of image denoising. We want to estimate the image $f^0 \in \mathbb{R}^{n_1 \times n_2}$ based on its noisy observation

$$Y = f^0 + \epsilon,$$

where $\epsilon \in \mathbb{R}^{n_1 \times n_2}$ is a noise matrix with i.i.d. Gaussian entries with known variance $\sigma^2$. Without loss of generality we assume throughout the paper that $\sigma = 1$. Note that $Y$, $f^0$, $\epsilon \in \mathbb{R}^{n_1 \times n_2}$ are matrices of dimension $n_1 \times n_2$ with entries the pixel values. Let $n := n_1 n_2$. To estimate the image $f^0$ we use total variation regularized least squares.

For two integers $i_1 \leq i_2$, we use the notation $[i_1 : i_2] = \{i_1, \ldots, i_2\}$. If $i_1 = 1$, we write $[1 : i_2] = : [i_2]$. 

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For \((j, k) \in [n_1] \times [n_2]\), we denote the entries of a matrix \(f \in \mathbb{R}^{n_1 \times n_2}\) by \(f_{j,k}\) (using subscripts) or alternatively by \(f(j, k)\) (using arguments). Indeed, we can also regard a matrix \(f \in \mathbb{R}^{n_1 \times n_2}\) as a function with domain \([n_1] \times [n_2]\) mapping to \(\mathbb{R}\) or a subset thereof.

In this paper, the total variation of an image \(f \in \mathbb{R}^{n_1 \times n_2}\) is defined as

\[
TV(f) := \sum_{j=2}^{n_1} \sum_{k=2}^{n_2} |(\Delta f)_{j,k}|,
\]

where, for \((j, k) \in [2 : n_1] \times [2 : n_2]\),

\[
(\Delta f)_{j,k} := f_{j,k} - f_{j,k-1} - f_{j-1,k} + f_{j-1,k-1}
\]

and \(\Delta : \mathbb{R}^{n_1 \times n_2} \to \mathbb{R}^{(n_1-1) \times (n_2-1)}\) is the two dimensional discrete derivative operator. Note that \(\Delta f = D_1 f D_2^T\), where \(D_1 \in \mathbb{R}^{(n_1-1) \times n_1}\) and \(D_2 \in \mathbb{R}^{(n_2-1) \times n_2}\) are of the form

\[
\begin{pmatrix}
-1 & 1 \\
-1 & 1 \\
\vdots & \vdots \\
-1 & 1 \\
\end{pmatrix}
\]

Thus we see that \(\Delta\) is a linear operator.

### 1.2. ANOVA decomposition

We now present the ANOVA decomposition of an image. This decomposition separates an image into four mutually orthogonal components: the global mean, the two matrices of the main effects - one for each of the two dimensions - and the matrix of interaction terms. We will need the insights brought by the ANOVA decomposition of an image to define the estimator for the interaction terms, which is the main object studied in this paper.

Let \(\psi^{1,1} = \{1\}^{n_1 \times n_2}\) be the matrix having all the entries equal to one.

We decompose an image \(f\) as

\[
f = f(\cdot, \cdot)\psi^{1,1} + f(\cdot, \circ) + f(\circ, \cdot) + \tilde{f},
\]

where \(f(\cdot, \circ) \in \mathbb{R}\) is the global mean, \(f(\cdot, \circ) \in \mathbb{R}^{n_1 \times n_2}\) and \(f(\circ, \cdot) \in \mathbb{R}^{n_1 \times n_2}\) are the matrices of main effects and \(\tilde{f} \in \mathbb{R}^{n_1 \times n_2}\) is the matrix of interaction terms.

The **global mean** \(f(\circ, \circ)\) is defined as

\[
f(\circ, \circ) := \frac{1}{n_1 n_2} \sum_{j=1}^{n_1} \sum_{k=1}^{n_2} f(j, k).
\]

The **main effects** are defined as \(f(\cdot, \circ) = \{f(j, \circ)\}_{(j,k) \in [n_1] \times [n_2]}\) and \(f(\circ, \cdot) = \{f(\circ, k)\}_{(j,k) \in [n_1] \times [n_2]}\), where

\[
f(j, \circ) := \frac{1}{n_2} \sum_{k=1}^{n_2} f(j, k) - f(\circ, \circ), \; j \in [n_1]
\]

\[
f(\circ, j) := \frac{1}{n_1} \sum_{k=1}^{n_1} f(j, k) - f(\circ, \circ), \; j \in [n_2]
\]

\[
\tilde{f} = f - f(\cdot, \cdot)\psi^{1,1} - f(\cdot, \circ) - f(\circ, \cdot).
\]
and

\[ f(o,k) := \frac{1}{n_1} \sum_{j=1}^{n_1} f(j,k) - f(o,o), \quad k \in [n_2]. \]

Note that \( f(\cdot, o) \) has identical columns and \( f(o, \cdot) \) has identical rows.

The interaction terms are defined as

\[ \hat{f}(j,k) = f(j,k) - f(o,o) - f(j,o) - f(o,k), \quad (j,k) \in [n_1] \times [n_2]. \]

We define

\[ \text{TV}_1(f) := \sum_{j=2}^{n_1} |f(j,o) - f(j-1,o)|. \]

and similarly

\[ \text{TV}_2(f) := \sum_{k=2}^{n_2} |f(o,k) - f(o,k-1)|. \]

Note that \( f(o,o)\psi_1^{1,1}, f(\cdot, o), f(o, \cdot) \) and \( \hat{f} \) are mutually orthogonal.

Let \( \|f\|_2 \) be the Frobenius norm

\[ \|f\|_2 := \left( \sum_{j=1}^{n_1} \sum_{k=1}^{n_2} f_{j,k}^2 \right)^{1/2} \]

and let \( \|f\|_1 \) be the sum of the absolute values of the entries of \( f \), i.e.

\[ \|f\|_1 := \sum_{(j,k) \in [n_1] \times [n_2]} |f_{j,k}|. \]

We endow \( \mathbb{R}^{n_1 \times n_2} \) with the trace-inner product. By orthogonality

\[ \|f\|^2_2 = n_1 n_2 f_2(o,o) + \|f(\cdot, o)\|^2_2 + \|f(o, \cdot)\|^2_2 + \|\hat{f}\|^2_2. \]

1.3. Estimator

We consider the estimator

\[ \hat{f} := \arg\min_{f \in \mathbb{R}^{n_1 \times n_2}} \left\{ \|Y - f\|^2_2 / n + 2\lambda_1 \text{TV}_1(f) + 2\lambda_2 \text{TV}_2(f) + 2\lambda \text{TV}(\hat{f}) \right\}, \]

where \( \lambda_1, \lambda_2, \lambda > 0 \) are positive tuning parameters. We call \( \hat{f} \) the two-dimensional total variation regularized least squares estimator. This estimator has the form of a so-called analysis estimator: it approximates the observations under a regularization penalty on the \( \ell_1 \)-norm of a linear operator of the signal \( f \).

Since \( Y(o,o)\psi^{1,1}_1, Y(\cdot, o), Y(o, \cdot) \) and \( \hat{Y} \) are mutually orthogonal, we may decompose the estimator as

\[ \hat{f} = \hat{f}(o,o)\psi^{1,1}_1 + \hat{f}(\cdot, o) + \hat{f}(o, \cdot) + \hat{f}, \]
where
\[ \hat{f}(\cdot, \cdot) = Y(\cdot, \cdot), \]
\[ \hat{f}(\cdot, \cdot) = \arg\min_{f \in \mathbb{R}^{n_1 \times n_2}} \{ \|Y(\cdot, \cdot) - f\|_2^2/n + 2\lambda_1 \text{TV}_1(f) \}, \]
\[ \hat{f}(\cdot, \cdot) = \arg\min_{f \in \mathbb{R}^{n_1 \times n_2}} \{ \|Y(\cdot, \cdot) - f\|_2^2/n + 2\lambda_2 \text{TV}_2(f) \}, \]
\[ \hat{f} = \arg\min_{f \in \mathbb{R}^{n_1 \times n_2}} \{ \|\hat{Y} - f\|_2^2/n + 2\lambda \text{TV}(f) \}. \]

We are going to focus on the estimator of the interaction terms \( \hat{\tilde{f}} \), which is an analysis estimator as well.

### 1.4. Contributions

We apply to image denoising an extension of the approach developed in Ortelli and van de Geer [2019c] for bounding the the effective sparsity for (higher order) total variation regularized estimators in one dimension. We obtain oracle inequalities with fast and slow rates for image denoising with the total variation penalty.

**Theorem 1.1** (Main result for fast rates - simplified version of Theorem 2.1)

Let \( S \) be an arbitrary subset of size \( s := |S| \) of \([3 : n_1 - 1] \times [3 : n_2 - 1]\) defining a regular grid parallel to the coordinate axes. Choose

\[ \lambda \asymp \sqrt{\frac{\log n}{n \sqrt{s}}}. \]

Then, \( \forall f \in \mathbb{R}^{n_1 \times n_2} \), it holds that, with high probability,

\[ \|\hat{\tilde{f}} - \tilde{f}_0\|_2^2/n \leq \|f - \tilde{f}_0\|_2^2/n + 4\lambda \sum_{(j,k) \in [2:n_1] \times [2:n_2] \setminus S} |(\Delta f)_{j,k}| + O\left(\frac{s^{3/2} \log n}{n}\right). \]

We also show that the projection arguments by Dalalyan et al. [2017] allow us to recover the same rate obtained by entropy calculations for the estimator of the interactions in a simple and constant-friendly way at the price of an additional log term.

**Theorem 1.2** (Main result for slow rates - simplified version of Theorem 3.1)

Consider a square image (i.e. \( n_1 = n_2 \)). Choose a subset \( S \) of \([3 : n_1 - 1] \times [3 : n_2 - 1]\) of size \( s := |S| \) defining a regular grid parallel to the coordinate axes with

\[ s \asymp n^{2/5} \text{TV}(f_0)^{4/5} (\log n)^{2/5}. \]

Choose

\[ \lambda \asymp \sqrt{\frac{\log n}{n^{1/5} s^{1/5}}}. \]

Then

\[ \|\hat{f} - \tilde{f}_0\|_2^2/n = O_P\left(\frac{n^{-3/5} \text{TV}(f_0)^{4/5} (\log n)^{2/5}}{n}\right). \]
1.5. Related literature

Total variation regularization as a method of proximal denoising of images dates back to Rudin et al. [1992]. An early contribution treating total variation regularization on a two dimensional grid of points is the work by Mammen and van de Geer [1997]. Two dimensional total variation regularization has been also in the focus of some recent studies as Sadhanala et al. [2016], Wang et al. [2016], Hütter and Rigollet [2016], Chatterjee and Goswami [2019].

Total variation regularized estimators in two dimensions usually take the form of analysis estimators, for which general oracle inequalities with fast and slow rates are derived in Ortelli and van de Geer [2019b]. Moreover Ortelli and van de Geer [2019a] explain how to transform general analysis estimators into their synthesis form, based on the work by Elad et al. [2007].

A criterion to divide the previous theoretical contributions in the field of image denoising with total variation regularization is the way in which total variation in two dimensions is defined.

Let us in a first moment think of an image as of a (differentiable) function $f : (x, y) \mapsto f(x, y)$, $f : [0, n_2] \times [0, n_1] \mapsto \mathbb{R}$. In the continuous case, we can define the total variation of $f$ in two ways.

- Either we use partial derivatives and define the total variation as
  $$
  \int_0^{n_1} \int_0^{n_2} \left| \frac{\partial f(x, y)}{\partial x} \right| \, dx \, dy + \int_0^{n_1} \int_0^{n_2} \left| \frac{\partial f(x, y)}{\partial y} \right| \, dx \, dy.
  $$

- Or we use the total derivative and define the total variation as
  $$
  \int_0^{n_1} \int_0^{n_2} \left| \frac{\partial^2 f(x, y)}{\partial x \partial y} \right| \, dx \, dy.
  $$

In this paper we are going to consider the discrete case, where an image is a matrix of pixels $f \in \mathbb{R}^{n_1 \times n_2}$. For the discrete case, by replacing derivatives by discrete differences, we find the following counterparts to the definitions of total variation provided above for the continuous case.

- The first approach translates to the discrete case as
  $$
  \frac{1}{n_1} \sum_{j=1}^{n_1} \sum_{k=2}^{n_2} |f_{j,k} - f_{j,k-1}| + \frac{1}{n_2} \sum_{k=1}^{n_2} \sum_{j=2}^{n_1} |f_{j,k} - f_{j-1,k}|.
  $$

It corresponds, up to normalization, to summing up the edge differences of $f$ across the two dimensional grid graph. This approach is used in Sadhanala et al. [2016], Wang et al. [2016], Hütter and Rigollet [2016], Chatterjee and Goswami [2019]. We denote the sum of the edge differences of $f$ across the two dimensional grid graph by

$$
TV_G(f) := \sum_{j=1}^{n_1} \sum_{k=2}^{n_2} |f_{j,k} - f_{j,k-1}| + \sum_{k=1}^{n_2} \sum_{j=2}^{n_1} |f_{j,k} - f_{j-1,k}|.
$$
The parallel to the second approach is given by

$$TV(f) := \sum_{j=2}^{n_1} \sum_{k=2}^{n_2} |f_{j,k} - f_{j,k-1} - f_{j-1,k} + f_{j-1,k-1}|$$

and corresponds to the sum of the interaction terms among groups of 4 pixels. This approach is used in Mammen and van de Geer [1997] and will be used in this paper as well. It can also be found in Fang et al. [2019], who study a constrained version of the estimator, where the constraint is on $TV(f) + \sum_{j=2}^{n_1} |f_{j,1} - f_{j-1,1}| + \sum_{k=2}^{n_2} |f_{1,k} - f_{1,k-1}|$.

For the first approach, we can find numerous papers in the literature. Sadhanala et al. [2016] derive minimax rates, which, for large $n := n_1 n_2$ and under the canonical scaling $TV_G \approx n^{1/2}$, are of order $\sqrt{\log(n)/n}$. Hütter and Rigollet [2016] derive sharp oracle inequalities, that result in the rate $\log n / \sqrt{n}$. The work by Wang et al. [2016] renders a rate of order $n^{-4/5} TV_G^{4/5}$ (neglecting log terms), which under the canonical scaling $TV_G \approx n^{1/2}$ (s. Sadhanala et al. [2016]), would result in an upper bound of order $n^{-2/5}$. Lastly, the very recent work by Chatterjee and Goswami [2019] focuses on the constrained optimization problem and a tuning-free version thereof. For a certain $f_0$ they obtain a rate faster than the minimax rate. Their approach, as the one of the work for higher order total variation regularization (Guntuboyina et al. [2017]), is based on bounding Gaussian widths of tangent cones.

In our latest work (Ortelli and van de Geer [2019c]), we derived oracle inequalities with fast rates by bounding the weighted weak compatibility constant introduced by Dalalyan et al. [2017]. We want to apply the same approach here to obtain an oracle inequality with fast rates for the two dimensional total variation regularized estimator as defined by Mammen and van de Geer [1997].

Using oracle inequalities with slow rates, we also match (up to log terms) the rate obtained in Mammen and van de Geer [1997] by entropy calculations, which is $n^{-3/5} TV(f_0)^{4/5}$. This is the last missing piece in our attempt to derive nonasymptotic counterparts of the theoretical guarantees on the mean squared error of the total variation regularized estimators handled in Mammen and van de Geer [1997]. For the constrained version of the estimation problem with the constraint

$$TV(f) + \sum_{j=2}^{n_1} |f_{j,1} - f_{j-1,1}| + \sum_{k=2}^{n_2} |f_{1,k} - f_{1,k-1}| \leq C$$

for some constant $C > 0$, Fang et al. [2019] show that the minimax rate is of order $n^{-2/3} C^{2/3}$ up to log terms and that the constrained estimator attains it. However, it is not clear if the arguments used there apply to our case as well.

We will assume throughout the paper that we have i.i.d. Gaussian errors with unit variance. For the case of unknown variance, Ortelli and van de Geer [2019b] show how to simultaneously estimate the signal and the variance of the noise by extending the idea of the square root lasso (Belloni et al. [2011]) to analysis estimators.
1.6. Synthesis formulation

The synthesis formulation of the two dimensional total variation regularized estimator expresses a matrix $f$ as linear combination of dictionary matrices, such that the penalty is the $\ell_1$-norm of (a subset of) the coefficients of the dictionary matrices. Here the notation with arguments instead of subscripts comes in handy. Consider some $f \in \mathbb{R}^{n_1 \times n_2}$. We may write for $j \in [n_1]$ and $k \in [n_2]$,

$$f(j, k) = \sum_{j'=1}^{n_1} \sum_{k'=1}^{n_2} \beta_{j', k'} \psi_{j', k'}(j, k)$$

where for $(j', k') \in [n_1] \times [n_2]$ the dictionary matrices are $\psi_{j', k'}$ with

$$\psi_{j', k'}(j, k) = 1_{(j \geq j', k \geq k')} \cdot (j, k) \in [n_1] \times [n_2].$$

We call the collection of matrices $\{ \psi_{j', k'} \}_{(j', k') \in [n_1] \times [n_2]}$ the dictionary. Moreover,

$$\beta_{1, 1} = f(1, 1),$$
$$\beta_{j', 1} = f(j', 1) - f(j' - 1, 1), \quad j' \in [2 : n_1],$$
$$\beta_{1, k'} = f(1, k') - f(1, k' - 1), \quad k' \in [2 : n_2],$$
$$\beta_{j', k'} = (\Delta f)_{j', k'}, \quad (j', k') \in [2 : n_1] \times [2 : n_2].$$

Define

$$\tilde{\psi}_{1, 1} = \psi_{1, 1},$$
$$\tilde{\psi}_{1, k} = \psi_{1, k} - \psi_{1, k}(0, 0), \quad k \in [2 : n_2],$$
$$\tilde{\psi}_{j, k} = \psi_{j, k} - \psi_{j, k}(0, 0) - \psi_{j, k}(0, *) - \psi_{j, k}(*, 0), \quad (j, k) \in [2 : n_1] \times [2 : n_2].$$

Remark 1.1

For $W \subset \mathbb{R}^{n_1 \times n_2}$ a linear space, let let $P_W$ denote the projection operator on $W$ and $A_W := I - P_W$ be the anti-projection operator. Let $V \subset \mathbb{R}^{n_1 \times n_2}$ be another linear space. Then $P_{W \cap V} = P_W + P_V - P_{W \cup V}$.

We thus note that

$$\tilde{\psi}_{j, 1} = A_{\text{span}(\psi_{1, j})} \psi_{j, 1}, \quad j \in [2 : n_1],$$
$$\tilde{\psi}_{1, k} = A_{\text{span}(\psi_{1, 1})} \psi_{1, k}, \quad k \in [2 : n_2],$$
$$\tilde{\psi}_{j, k} = A_{\text{span}(\psi_{1, j}) \cap (\psi_{1, k})_{k \in [n_2]}} \psi_{j, k}, \quad (j, k) \in [2 : n_1] \times [2 : n_2].$$

Thus, the four linear spaces $\text{span}(\tilde{\psi}_{1, 1})$, $\text{span}(\{ \tilde{\psi}_{j, 1} \}_{j \in [2 : n_1]})$, $\text{span}(\{ \tilde{\psi}_{1, k} \}_{k \in [2 : n_2]})$, $\text{span}(\{ \tilde{\psi}_{j, k} \}_{(j, k) \in [2 : n_1] \times [2 : n_2]})$ are mutually orthogonal.
The following lemma shows how a matrix $f$ can be expressed as a linear combination of the elements of the dictionary $\{\tilde{\psi}^j,k\}_{(j,k)\in[n_1] \times [n_2]}$ and how in that case the meaning of the coefficients for the dictionary atoms $\{\tilde{\psi}^j,k\}_{(j,k)\in[n_1] \times [n_2]}$ has to be adapted accordingly. In particular, the new dictionary $\{\tilde{\psi}^j,k\}_{(j,k)\in[n_1] \times [n_2]}$ can be partitioned into four mutually orthogonal parts corresponding to the sets $\{\tilde{\psi}^{1,1}\}_{j\in[2: n_1]}$, $\{\tilde{\psi}^{1,1}\}_{k\in[2: n_2]}$, and $\{\tilde{\psi}^{1,1}\}_{(j,k)\in[2: n_1] \times [2: n_2]}$.

**Lemma 1.3**

It holds that
\[
  f = \sum_{j=1}^{n_1} \sum_{k=1}^{n_2} \tilde{\beta}_{j,k} \tilde{\psi}^{j,k},
\]
where
\[
  \tilde{\beta}_{1,1} = f(\cdot, \cdot),
  \tilde{\beta}_{j,1} = f(j, \cdot) - f((j-1), \cdot), j \in [2 : n_1],
  \tilde{\beta}_{1,k} = f(\cdot, k) - f(\cdot, k-1), k \in [2 : n_1],
  \tilde{\beta}_{j,k} = (\Delta f)_{j,k}, (j, k) \in [2 : n_1] \times [2 : n_2].
\]

**Proof of Lemma 1.3.** See Appendix A.

The next lemma, which is based on Lemma 1.3, gives us a synthesis form of the estimator for the interaction terms $\hat{f}$.

**Lemma 1.4**

We have
\[
  \hat{f} = \sum_{j=2}^{n_1} \sum_{k=2}^{n_2} \hat{\beta}_{j,k} \hat{\psi}^{j,k},
\]
where
\[
  \hat{\beta}_{j,k} = \arg\min_{\{b_{j,k}\}_{(j,k)\in[2: n_1] \times [2: n_2]}} \left\{ \|Y - \sum_{j=2}^{n_1} \sum_{k=2}^{n_2} b_{j,k} \tilde{\psi}^{j,k}\|^2 / n + 2\lambda \sum_{j=2}^{n_1} \sum_{k=2}^{n_2} |b_{j,k}| \right\}.
\]

**Proof of Lemma 1.4.** See Appendix A.

### 1.7. Notation

Fix some set $S \subseteq [3 : n_1 - 1] \times [3 : n_2 - 1] := \{t_1, \ldots, t_s\}$.

One may think of $S$ as being the active set of the interactions of some “oracle” approximation of $f^0$. We refer to the elements of $S$ as (interaction) jump locations. The cardinality of $S$ is denoted by $s := |S|$ and the coordinates of a jump location $t_m$ are denoted by $(t_{1, m}, t_{2, m})$, $m = 1, \ldots, s$. Note that we require that $2 < t_{1, m} < n_1$ and $2 < t_{2, m} < n_2$. This assumption ensures that we have no boundary effects when doing partial integration, see Corollary 2.7.
Given $S$, we can partition $[2 : n_1] \times [2 : n_2]$ into $s$ subsets, consisting of the points closest to $t_m$, $m = 1, \ldots, s$. This corresponds to a Voronoi tessellation. The natural metric here would be the city block metric. However, a Voronoi tessellation typically has subsets of relatively irregular shape. We will require that the partition consists of rectangles. Indeed, a partition consisting of rectangles will ease the construction of an interpolating matrix, which is our tool to bound the effective sparsity (Theorem 2.1) and to obtain an oracle inequality with fast rates. The concepts of interpolating matrix and effective sparsity are presented in Section 2.2.

**Definition 1.5**

We call $\{R_m\}_{m=1}^s$ a rectangular tessellation of $[2 : n_1] \times [2 : n_2]$ if it satisfies the following conditions:

- each $R_m \subseteq [2 : n_1] \times [2 : n_2]$ is a rectangle ($m = 1, \ldots, s$),
- $\cup_{m=1}^s R_m = [2 : n_1] \times [2 : n_2]$,
- for all $m$ and $m' \neq m$, the rectangles $R_m$ and $R_{m'}$ ($m \neq m'$) possibly share boundary points, but not interior points,
- for all $m$ the jump location $t_m$ is an interior point of $R_m$.

For a rectangular tessellation $\{R_m\}_{m=1}^s$ we let $d_{m}^-$ be the area of the rectangle $R_m$ North-West of $t_m$, $d_{m}^+$ the area to the South-West, $d_{m}^+$ the area to the South-East and $d_{m}^-$ the area to the North-East. In other words, if $(t_{1,m}^-, t_{2,m}^-), (t_{1,m}^+, t_{2,m}^+), (t_{1,m}^+, t_{2,m}^-), (t_{1,m}^-, t_{2,m}^-)$ are the four corners of the rectangle $R_m$, starting with the top-left corner and going clockwise along the boundary, then

$$
\begin{align*}
   d_{m}^- &= d_{1,m}^- d_{2,m}, & d_{m}^+ &= d_{1,m}^+ d_{2,m}, \\
   d_{m}^- &= d_{1,m}^+ d_{2,m}, & d_{m}^+ &= d_{1,m}^- d_{2,m},
\end{align*}
$$

where

$$
\begin{align*}
   d_{1,m}^- &= (t_{1,m}^+ - t_{1,m}^-), & d_{2,m}^- &= (t_{2,m}^- - t_{2,m}^-), \\
   d_{1,m}^+ &= (t_{1,m}^- - t_{1,m}^-), & d_{2,m}^+ &= (t_{2,m}^+ - t_{2,m}^-).
\end{align*}
$$

For two matrices $a = \{a_{j,k} : (j, k) \in [2 : n_1] \times [2 : n_2]\} \in \mathbb{R}^{(n_1-1) \times (n_2-1)}$ and $b = \{b_{j,k} : (j, k) \in [2 : n_1] \times [2 : n_2]\} \in \mathbb{R}^{(n_1-1) \times (n_2-1)}$ we use the symbol $\odot$ for entry-wise multiplication:

$$(a \odot b)_{j,k} := a_{j,k} b_{j,k}, \quad (j, k) \in [2 : n_1] \times [2 : n_2].$$

Moreover

$$
a_S := \{a_{j,k} : (j, k) \in S\}, \quad a_{-S} := \{a_{j,k} : (j, k) \notin S\}.
$$

We will use the same notation $a_S \in \mathbb{R}^{(n_1-1) \times (n_2-1)}$ for the matrix which shares its entries with $a$ for $(j, k) \in S$ and has all its other entries equal to zero. Similarly, $a_{-S} \in \mathbb{R}^{(n_1-1) \times (n_2-1)}$ shares its entries with $a$ for $(j, k) \notin S$ and has its other entries equal to zero.
2. Fast rates

Our objective is now to establish that $\hat{f}$ can adapt to the number of jumps in the main effects and the interactions. The main effects can be dealt with by using the results for the one-dimensional total variation regularized estimator (see Ortelli and van de Geer [2018] and Ortelli and van de Geer [2019b]). Thus, our main result will be the adaptation for the estimator of the interactions. It will be derived invoking a general result from Ortelli and van de Geer [2019b] for analysis problems. For that purpose, we need to establish a bound for the so-called effective sparsity (Definition 2.3), which in turn can be done by using interpolating matrices (see Lemma 2.5). This way of bounding the effective sparsity is an extension to the two-dimensional case of the bound on the effective sparsity for one-dimensional total variation regularized estimators based on interpolating vectors exposed in Ortelli and van de Geer [2019c].

2.1. Main result

In this section we present the main result: an oracle inequality for the estimator $\hat{f}$ of the interactions $\tilde{f}_0$.

Fix a set $S \subseteq [3 : n_1 - 1] \times [3 : n_2 - 1]$. The set $S$ could be the jump locations of some oracle approximation of $f^0$, but we do not insist on this. Let

$$d_{1, \text{max}}(S) := \max_{m \in [1 : s]} \max\{d_{1,m}^-, d_{1,m}^+\}$$

and

$$d_{2, \text{max}}(S) := \max_{m \in [1 : s]} \max\{d_{2,m}^-, d_{2,m}^+\}.$$ 

For all $t > 0$ we define

$$\lambda_0(t) := \sqrt{\frac{2 \log(2n) + 2t}{n}}.$$ 

**Theorem 2.1**

Let $x, t > 0$. Choose

$$\lambda \geq 2\lambda_0(t) \sqrt{\frac{d_{1, \text{max}}(S)/n_1 + d_{2, \text{max}}(S)/n_2}{n}}.$$ 

For all $f \in \mathbb{R}^{n_1 \times n_2}$, we have with probability at least $1 - e^{-x} - e^{-t}$

$$\|\hat{f} - \hat{f}_0\|_2^2/n \leq \|f - \hat{f}_0\|_2^2/n + 4\lambda \|\Delta f\|_{S^{-1}} + \left(\sqrt{\frac{s}{n}} + \sqrt{\frac{2x}{n}} + \lambda \Gamma(S, v-S)\right)^2,$$

where

$$\Gamma^2(S, v-S) \leq \frac{1}{2} \left(\log(en_1) + \log(en_2)\right) \sum_{m=1}^s \left(\frac{n}{\Delta^-} + \frac{n}{\Delta^+} + \frac{n}{\Delta^-} + \frac{n}{\Delta^+}\right).$$

\[1\]
Theorem 2.1 is an application of Theorem 4.4 in Ortelli and van de Geer [2019c]. The work goes mainly into establishing a bound for quantity $\Gamma^2(S, v_{-S}, q_S)$, the so-called “effective sparsity” which we define in Section 2.2. It depends on $S$, on the weights $v_{-S}$ and on a sign pattern $q_S$. The weights are given in (2).

Corollary 2.2
Let us take $f = f^0$ and $S$ the jump locations of $f^0$. Suppose these jump locations lie on a regular grid. We have $s = s_1 s_2$, $d^-_{1, m} = d^+_{1, m} = (n_1 - 1)/(2s_1)$ and $d^-_{2, m} = d^+_{2, m} = (n_2 - 1)/(2s_2)$ for all $m \in [1 : s]$. Then
\[
\Gamma^2(S, v_{-S}) \leq 8 \frac{(\log(en_1) + \log(en_2))n_1n_2}{(n_1 - 1)(n_2 - 1)} s^2
\]
and we may take
\[
\lambda = \sqrt{2} \lambda_0(t) \sqrt{1/s_1 + 1/s_2}.
\]
Then
\[
\lambda^2 \Gamma^2(S, v_S, q_S) \leq 16 \lambda_0^2(t) s(s_1 + s_2) n \frac{\log(en_1) + \log(en_2)}{(n_1 - 1)(n_2 - 1)}.
\]
If $s_1 = s_2$ say, we see that $s(s_1 + s_2) = 2s\sqrt{s}$. In other words, the oracle bound is then of order $s^{3/2}/n$, up to log-terms.

2.2. Interpolating matrix and partial integration

The definition for effective sparsity is as in Ortelli and van de Geer [2019c], but formulated in matrix (instead of vector) form.

Definition 2.3
Let $v \in [0, 1]^{(n_1 - 1) \times (n_2 - 1)}$ be a matrix of weights. Let $q \in [-1, 1]^{(n_1 - 1) \times (n_2 - 1)}$ be a matrix s.t.
\[
q_{j,k} \begin{cases} 
\in \{-1, +1\}, & (j,k) \in S, \\
\in [-1, 1], & (j,k) \notin S.
\end{cases}
\]
We call the matrix $q_S \in \{-1, 0, 1\}^{(n_1-1) \times (n_2-1)}$ a sign configuration. The effective sparsity is
\[
\Gamma(S, v_{-S}, q_S) = 
\max \left\{ \text{trace}(q_S^T (D_1 f D^T_2)_{-S}) - \| (1 - v)_{-S} \odot (D_1 f D^T_2)_{-S} \|_1 : \|f\|_2/n = 1 \right\}.
\]
Moreover we write
\[
\Gamma(S, v_{-S}) := \max_{q_S} \Gamma(S, v_{-S}, q_S).
\]

Definition 2.4
Consider some sign configuration $q_S \in \{-1, 0, 1\}^{(n_1-1) \times (n_2-1)}$ and a matrix $w(q_S) = \{w_{j,k}(q_S) : (j,k) \in [2 : n_1] \times [2 : n_2]\} \in \mathbb{R}^{(n_1-1) \times (n_2-1)}$. We call $w(q_S)$ an interpolating matrix for the weights $v$ if it has the following properties:
\begin{itemize}
  \item $w_{m,m}(q_S) = q_m$, $\forall m \in [1 : s],$
  \item $|w_{j,k}(q_S)| \leq 1 - v_{j,k}, \forall (j,k) \notin S$.
\end{itemize}
For completeness, we give the matrix version of Lemma 4.2 in Ortelli and van de Geer [2019c].

**Lemma 2.5**
We have
\[
\Gamma^2(S,v-S,q_S) \leq n \min_w \|D_1^T w(q_S) D_2\|_2^2
\]
where the minimum is over all interpolating matrices \(w(q_S)\) for the sign configuration \(q_S\).

**Proof of Lemma 2.5.** Let \(f \in \mathbb{R}^{n_1 \times n_2}\) be arbitrary and let \(q_S\) be a sign configuration. Then
\[
\begin{align*}
\text{trace}(q_S^T \odot (D_1 f D_2^T)_S) &- \|(1-v)_S \odot (D_1 f D_2^T)_S\|_1 \\
\leq \text{trace}(q_S^T \odot (D_1 f D_2^T)_S) - \|w_S(q_S) \odot (D_1 f D_2^T)_S\|_1 \\
\leq \text{trace}(q_S^T \odot (D_1 f D_2^T)_S) + \text{trace}(w_S(q_S)^T \odot (D_1 f D_2^T)_S) \\
= \text{trace}(w(q_S)^T D_1 f D_2^T) \\
= \text{trace}(D_2^T w(q_S)^T D_1 f) \\
\leq \sqrt{n} \|D_1^T w(q_S) D_2\|_2 \|f\|_2 / \sqrt{n}.
\end{align*}
\]

In the proof of Lemma 2.5 we used the equation
\[
\text{trace}(w^T D_1 f D_2^T) = \text{trace}(D_2^T w^T D_1 f).
\]
When \(D_1 f D_2^T = \Delta f\) this equality is called partial integration. We study it further in the next lemma.

**Lemma 2.6**
For any matrix \(w = \{w_{j,k} : (j,k) \in [2 : n_1] \times [2 : n_2]\}\) it holds that
\[
\begin{align*}
\text{trace}(w^T \Delta f) &= \sum_{k=2}^{n_2} \sum_{j=2}^{n_1} w_{j,k}(\Delta f)_{j,k} \\
= &w_{n_1,n_2} f_{n_1,n_2} - w_{n_1,2} f_{n_1,1} - w_{2,n_2} f_{1,n_2} + w_{2,2} f_{1,1} \\
- &\left\{ \sum_{j=2}^{n_1-1} (w_{j+1,n_2} - w_{j,n_2}) f_{j,n_2} - \sum_{j=2}^{n_1-1} (w_{j+1,2} - w_{j,2}) f_{j,1} \right\} \\
- &\left\{ \sum_{k=2}^{n_2-1} (w_{n_1,k+1} - w_{n_1,k}) f_{n_1,k} - \sum_{k=2}^{n_2-1} (w_{2,k+1} - w_{2,k}) f_{1,k} \right\} \\
+ &\sum_{j=2}^{n_1-1} \sum_{k=2}^{n_2-1} (\Delta w)_{j+1,k+1} f_{j,k}.
\end{align*}
\]
Proof of Lemma 2.6. Partial integration in one dimension gives, for \( N \in \mathbb{N} \) and sequences \( \{a_j\}_{j=2}^N \) and \( \{b_j\}_{j=1}^N \) of real numbers,

\[
\sum_{j=2}^N a_j(b_j - b_{j-1}) = a_N b_N - a_2 b_1 - \sum_{j=2}^{N-1} (a_{j+1} - a_j)b_j.
\]

Apply this to the inner sum and then apply it again to each of the terms. \( \square \)

Corollary 2.7

Suppose \( w \) has its boundary entries equal to zero:

\[
w_{j,2} = w_{j,n_2} = 0, \quad \forall \ j \in [2 : n_1] \quad \text{and} \quad w_{2,k} = w_{n_1,k} = 0, \quad \forall \ k \in [2 : n_2].
\]

Then

\[
\sum_{k=2}^{n_2} \sum_{j=2}^{n_1} w_{j,k}(\Delta f)_{j,k} = \sum_{j=2}^{n_1-1} \sum_{k=2}^{n_2-1} (\Delta w)_{j+1,k+1} f_{j,k}.
\]

2.3. A bound for the effective sparsity

Given a set \( S \subseteq [3, n_1-1] \times [3, n_2-1] \), let \( \{R_m\}_{m=1}^s \) be a rectangular tessellation. The rectangles \( \{R_m\}_{m=1}^s \) are not mutually disjoint because they may share their corners and share their borders parallel to the coordinate axes. Each jump location \( t_m \) is an interior point of \( R_m \). The rectangle \( R_m \) thus consists of a North-West rectangle \( R_m^- \), a North-East rectangle \( R_m^+ \), a South-East rectangle \( R_m^{++} \) and a South-West rectangle \( R_m^{+-} \). Thus

\[
R_m^- := \{(j,k) : t_{1,m}^- \leq j \leq t_{1,m}^+, t_{2,m}^- \leq k \leq t_{2,m}^+\},
\]

\[
R_m^+ := \{(j,k) : t_{1,m}^- \leq j \leq t_{1,m}^+, t_{2,m}^- \leq k \leq t_{2,m}^+\},
\]

\[
R_m^{++} := \{(j,k) : t_{1,m}^- \leq j \leq t_{1,m}^+, t_{2,m}^- \leq k \leq t_{2,m}^+\},
\]

\[
R_m^{+-} := \{(j,k) : t_{1,m}^- \leq j \leq t_{1,m}^+, t_{2,m}^- \leq k \leq t_{2,m}^+\}.
\]

For \( m = 1, \ldots, s \), the weights will be

\[
u_{j,k} = \begin{cases} 
1 - \frac{1}{2} \left( 1 - \sqrt{\frac{|j - t_{1,m}^-|}{d_{1,m}^+}} \right) \left( 1 - \frac{|k - t_{2,m}^+|}{d_{2,m}^-} \right), & (j,k) \in R_m^-,
\end{cases}
\]

\[
-\frac{1}{2} \left( 1 - \sqrt{\frac{|j - t_{1,m}^-|}{d_{1,m}^+}} \right) \left( 1 - \frac{|k - t_{2,m}^+|}{d_{2,m}^-} \right), & (j,k) \in R_m^+,
\]

\[
1 - \frac{1}{2} \left( 1 - \sqrt{\frac{|j - t_{1,m}^-|}{d_{1,m}^+}} \right) \left( 1 - \frac{|k - t_{2,m}^+|}{d_{2,m}^-} \right), & (j,k) \in R_m^{++},
\]

\[
-\frac{1}{2} \left( 1 - \sqrt{\frac{|j - t_{1,m}^-|}{d_{1,m}^+}} \right) \left( 1 - \frac{|k - t_{2,m}^+|}{d_{2,m}^-} \right), & (j,k) \in R_m^{+-},
\]

\[
1 - \frac{1}{2} \left( 1 - \sqrt{\frac{|j - t_{1,m}^-|}{d_{1,m}^+}} \right) \left( 1 - \frac{|k - t_{2,m}^+|}{d_{2,m}^-} \right), & (j,k) \in R_m^{+-}.
\]

(2)
where \((d_{1,m}^{-}, d_{1,m}^{+}, d_{2,m}^{-}, d_{2,m}^{+})\) are given in Subsection 1.7.

For \(q_S \in \{-1, 0, 1\}^{(n_1-1) \times (n_2-1)},\) take

\[
w_{j,k}(q_S) = \begin{cases} +1 - v_{j,k}, & q_m = +1, \quad (j, k) \in R_m, \quad m \in [s]. \\
-1 + v_{j,k}, & q_m = -1, \quad (j, k) \in R_m, \quad m \in [s].
\end{cases}
\]

Then \(w(q_S)\) is an interpolating matrix for \(q_S.\) It moreover has the property that \(w_{j,k}(z) = 0\) as soon as \((j, k)\) is at the boundary of \(R_m\) for some \(m \in [s].\)

**Remark 2.1**

This section is based on the weights given in \((2)\) and the interpolating matrix given in \((3)\). Both of them depend on the rectangular tessellation \(\{R_m\}_{m \in [s]}\) chosen given the set \(S.\) As a consequence, also the bound on the effective sparsity derived in the next lemma depends on \(\{R_m\}_{m \in [s]}\) and can be interpreted to hold for an arbitrary rectangular tessellation \(\{R_m\}_{m \in [s]},\) given a set \(S.\) However we do not stress this dependence any further.

**Lemma 2.8**

With the weights \(v\) given in \((2)\) we have

\[
\Gamma^2(S, v, S, q_S) \leq \frac{1}{2} \left( \log(en_1) + \log(en_2) \right) \sum_{m=1}^{s} \left( \frac{n}{d_m} + \frac{n}{d_m^{+}} + \frac{n}{d_m^{-}} \right).
\]

**Proof of Lemma 2.8.** We say that \(\{w(j, k), (j, k) \in \{2 : n_1 \times \{2 : n_2\}\}\}\) has product structure if it is of the form \(w(j, k) = w_1(j)w_2(k)\) for all \((j, k).\) Clearly, if it has this structure, then

\[
(\Delta w)_{j,k} = (\Delta_1 w_1)_j(\Delta_2 w_2)_k.
\]

We examine now a prototype rectangle \([-d_1^{-} : d_1^{+}] \times [-d_2^{-} : d_2^{+}].\) Consider, starting from top left and proceeding clockwise, the four rectangles

\[
\begin{align*}
R^{-} & := [-d_1^{-} : 0] \times [-d_2^{-} : 0], \\
R^{+} & := [-d_1^{-} : 0] \times [0 : d_2^{+}], \\
R^{++} & := [0 : d_1^{+}] \times [0 : d_2^{+}], \\
R^{+^{-}} & := [0 : d_1^{+}] \times [-d_2^{-} : 0],
\end{align*}
\]

and let \(R := R^{-} \cup R^{+} \cup R^{++} \cup R^{+^{-}} = [-d_1^{-} : d_1^{+}] \times [-d_2^{-} : d_2^{+}]\) be their union. Thus \(R\) is a rectangle surrounding the origin \((0, 0).\) Take

\[
w_{j,k} := \begin{cases} \frac{1}{2} \left( 1 - \sqrt{\frac{d_1'}{d_1}} \right) \left( 1 - \frac{|k|}{d_1'} \right) + \frac{1}{2} \left( 1 - \frac{|j|}{d_1'} \right) \left( 1 - \sqrt{\frac{d_1'}{d_1}} \right), & (j, k) \in R^{-}, \\
\frac{1}{2} \left( 1 - \frac{|k|}{d_1} \right) \left( 1 - \sqrt{\frac{d_1}{d_1'}} \right) + \frac{1}{2} \left( 1 - \frac{|j|}{d_1} \right) \left( 1 - \sqrt{\frac{d_1}{d_1'}} \right), & (j, k) \in R^{+}, \\
\frac{1}{2} \left( 1 - \frac{|k|}{d_2} \right) \left( 1 - \sqrt{\frac{d_2}{d_2'}} \right) + \frac{1}{2} \left( 1 - \frac{|j|}{d_2} \right) \left( 1 - \sqrt{\frac{d_2}{d_2'}} \right), & (j, k) \in R^{++}, \\
\frac{1}{2} \left( 1 - \sqrt{\frac{d_1}{d_1'}} \right) \left( 1 - \frac{|k|}{d_2'} \right) + \frac{1}{2} \left( 1 - \frac{|j|}{d_2'} \right) \left( 1 - \sqrt{\frac{d_1}{d_1'}} \right), & (j, k) \in R^{+^{-}}.
\end{cases}
\]
Then \( w_{0,0} = 1 \) and \( w_{j,k} = 0 \) for all \((j, k)\) at the border of \( R \).

Because \( R^-, R^+, R^{++} \) and \( R^+ \) are rectangles aligned with the coordinate axes, \( \{w_{j,k}, (j, k) \in [2 : n_1] \times [2 : n_2]\} \) is the sum of two terms with product structure. We see that

\[
|\Delta w_{j,k}| \leq \begin{cases}
\frac{1}{2} \frac{1}{d_1} \frac{1}{d_2} + \frac{1}{2} \frac{1}{d_1} \frac{1}{d_2} \sqrt{|k|}, & (j, k) \in R^-, \\
\frac{1}{2} \frac{1}{d_1} \frac{1}{d_2} + \frac{1}{2} \frac{1}{d_1} \frac{1}{d_2} \sqrt{|k|}, & (j, k) \in R^+, \\
\frac{1}{2} \frac{1}{d_1} \frac{1}{d_2} + \frac{1}{2} \frac{1}{d_1} \frac{1}{d_2} \sqrt{|k|}, & (j, k) \in R^{++}, \\
\frac{1}{2} \frac{1}{d_1} \frac{1}{d_2} + \frac{1}{2} \frac{1}{d_1} \frac{1}{d_2} \sqrt{|k|}, & (j, k) \in R^+.
\end{cases}
\]

Invoking the inequality \((a+b)^2 \leq 2a^2+2b^2\) for real numbers \(a\) and \(b\), we conclude that

\[
\sum_{(j,k) \in R}(\Delta w_{j,k})^2 \leq \frac{1}{2} \frac{1}{d_1} \frac{1}{d_2} \left( \sum_{j=1}^{d_1^-} \sum_{k=1}^{d_2^-} \frac{1}{j} \frac{1}{k} + \sum_{j=1}^{d_1^+} \sum_{k=1}^{d_2^+} \frac{1}{j} \frac{1}{k} \right) \\
+ \frac{1}{2} \frac{1}{d_1} \frac{1}{d_2} \left( \sum_{j=1}^{d_1^-} \sum_{k=1}^{d_2^+} \frac{1}{j} \frac{1}{k} + \sum_{j=1}^{d_1^+} \sum_{k=1}^{d_2^-} \frac{1}{j} \frac{1}{k} \right) \\
+ \frac{1}{2} \frac{1}{d_1} \frac{1}{d_2} \left( \sum_{j=1}^{d_1^+} \sum_{k=1}^{d_2^+} \frac{1}{j} \frac{1}{k} \right) \leq \frac{1}{2} \frac{1}{d_1} \frac{1}{d_2} \left( \log(ed_1^-) + \log(ed_2^-) \right) \\
+ \frac{1}{2} \frac{1}{d_1} \frac{1}{d_2} \left( \log(ed_1^+) + \log(ed_2^+) \right) \\
+ \frac{1}{2} \frac{1}{d_1} \frac{1}{d_2} \left( \log(ed_1^+) + \log(ed_2^-) \right). 
\]

The interpolating matrices \( w(qs) \) given by (3) are of the above form on each of the surrounding rectangles \( R_m \) and are equal to zero on their common borders. The final result follows from gluing the \( \{R_m\}_{m=1}^n \) together. \( \Box \)

### 2.4. Dealing with the noise

We start with an auxiliary lemma.
Lemma 2.9
For all \((x, y) \in [0, 1]^2\)
\[
1 - \frac{1}{2}(1 - \sqrt{x})(1 - y) - \frac{1}{2}(1 - x)(1 - \sqrt{y}) \geq (\sqrt{x} + \sqrt{y})/2
\]
or in other words
\[
\frac{1}{2}(1 - \sqrt{x})(1 - y) + \frac{1}{2}(1 - x)(1 - \sqrt{y}) \leq 1 - (\sqrt{x} + \sqrt{y})/2
\]
Proof of Lemma 2.9. By direct calculation
\[
1 - \frac{1}{2}(1 - \sqrt{x})(1 - y) - \frac{1}{2}(1 - x)(1 - \sqrt{y}) = \frac{1}{2}\sqrt{x} + \frac{1}{2}(1 - \sqrt{x})y + \frac{1}{2}\sqrt{y} + \frac{1}{2}(1 - \sqrt{y})x \geq (\sqrt{x} + \sqrt{y})/2
\]
where we used that \((x, y) \in [0, 1]^2\) so that \((1 - \sqrt{x})y\) and \(x(1 - \sqrt{y})\) are non-negative.

Lemma 2.10
For any \(((t_1, t_2), (d_1, d_2)) \in \mathbb{N}^4\) and for \(j \in \{t_1, \ldots, t_1 + d_1\}\) and \(k \in \{t_2, \ldots, t_2 + d_2\}\)
\[
\sqrt{\frac{j - t_1}{n_1} + \frac{k - t_2}{n_2}} \leq \frac{1}{2} \left( \sqrt{\frac{j - t_1}{d_1}} + \sqrt{\frac{k - t_2}{d_2}} \right) 2\sqrt{\frac{d_1}{n_1} + \frac{d_2}{n_2}}.
\]
Proof of Lemma 2.10. For \(j \in \{t_1, \ldots, t_1 + d_1\}\) and \(k \in \{t_2, \ldots, t_2 + d_2\}\)
\[
\sqrt{\frac{j - t_1}{n_1} + \frac{k - t_2}{n_2}} \leq \sqrt{\frac{j - t_1}{d_1}} + \sqrt{\frac{k - t_2}{d_2}}\sqrt{\frac{d_1}{n_1} + \frac{d_2}{n_2}}
\]
The function
\[
\tilde{f} = \sum_{j=2}^{n_1} \sum_{k=2}^{n_2} b_{j,k} \tilde{\psi}_{j,k}
\]
automatically has zero margins: we have no restrictions on the coefficients \(\{b_{j,k}, (j, k) \in [2 : n_1] \times [2 : n_2]\}\). Moreover, \(\sum_{j=2}^{n_1} \sum_{k=2}^{n_2} |b_{j,k}| = \text{TV}(\tilde{f}) =: \|D_1 \tilde{f} D_2^T\|_1\). So we can carry out the general theory for the analysis problem.
For \( W \subset \mathbb{R}^{n_1 \times n_2} \) a linear space, let let \( P_W \) denote the projection operator on \( W \) and \( A_W := I - P_W \) be the anti-projection operator. Invoking Ortelli and van de Geer [2019b], we know that the weights can be bounded using the squared distance of the inactive variables on the plane spanned by the active ones. In this case it concerns projections of \( \{ \tilde{\psi}_{j,k} \}_{(j,k) \notin S} \) on \( \{ \tilde{\psi}_{j,k} \}_{(j,k) \in S} \). But projections reduce the length of the vector involved. So we may also look at the projections of the original variables. This is a consequence of the next lemma.

**Lemma 2.11**

Let \( U = \text{span}(\{ u_j \}) \) and \( W \) be linear spaces. Let \( \tilde{U} := \text{span}(\{ \tilde{u}_j \}) \) where \( \tilde{u}_j = P_W u_j \) for all \( j \). Then for any \( z \), we have for \( \tilde{z} = P_W z \),

\[
\| \tilde{z} - P_{\tilde{U}} \tilde{z} \|_2 \leq \| z - P_U z \|_2.
\]

**Proof of Lemma 2.11.** Clearly

\[
\| P_W (z - P_U z) \|_2 \leq \| z - P_U z \|_2.
\]

We moreover have for some vector \( \gamma \)

\[
P_{\tilde{U}} z = \sum_j \gamma_j u_j
\]

so that

\[
P_W (z - P_U z) = \tilde{z} - \sum_j \gamma_j \tilde{u}_j.
\]

Thus

\[
\| \tilde{z} - P_{\tilde{U}} \tilde{z} \|_2 = \min_e \| \tilde{z} - \sum_j c_j \tilde{u}_j \|_2 \\
\leq \| \tilde{z} - \sum_j \gamma_j \tilde{u}_j \|_2 = \| P_W (z - P_U z) \|_2 \\
\leq \| z - P_U z \|_2.
\]

Let

\[
\tilde{U} := \text{span}(\{ \tilde{\psi}_{m} \}_{m=1}^{r}).
\]

**Lemma 2.12**

For \( m \in [1:s] \) and all \( (j,k) \in R_m \)

\[
\| A_U \tilde{\psi}_{j,k} \|_2^2 \leq \frac{|j - t_{1,m}|}{n_1} + \frac{|k - t_{2,m}|}{n_2}.
\]

**Proof of Lemma 2.12.** This follows from the fact that for \( (j,k) \in R_m \),

\[
\| A_U \tilde{\psi}_{j,k} \|_2^2 \leq \| \tilde{\psi}_{j,k} - \psi_{m} \|_2^2 \leq |j - t_{1,m}| n_2 + |k - t_{2,m}| n_1.
\]
2.5. Proof of Theorem 2.1

We have shown that the estimator of the interactions can be studied separately from the main effects and overall effects (Lemma 1.4). We moreover presented a bound for the effective sparsity in Lemma 2.8 with given weights $v$ and the combination of Lemmas 2.12 and 2.10 shows that these weights are upper-bounding the length of the anti-projections after dividing the latter by $2 \sqrt{d_{1, \max}(S)/n_1 + d_{2, \max}(S)/n_2}$. Thus all ingredients for application of Theorem 4.4 in Ortelli and van de Geer [2019c] were established.

Proof of Theorem 2.1. Theorem 2.1 follows from the results of Subsections 2.2-2.4 combined with the following theorem, which is a reformulation of Theorem 4.4 in Ortelli and van de Geer [2019c].

**Theorem 2.13** (Theorem 4.4 in Ortelli and van de Geer [2019c], with background)

Let $S \subseteq [3 : n_1 - 1] \times [3 : n_2 - 1]$ be arbitrary. Let

$$
\mathcal{U} := \text{span} \left( \{ \psi^m \}_{m \in [s]} \right),
$$

$$
\hat{\mathcal{U}} := \text{span} \left( \{ \hat{\psi}^m \}_{m \in [s]} \right)
$$

and

$$
\mathcal{W} := \text{span} \left( \{ \psi^{j,k} \}_{(j,k) \in \{1\} \times [n_2] \cup [n_1] \times \{1\}} \right).
$$

Let $\tilde{v}_{j,k}, (j, k) \in [2 : n_1] \times [2 : n_2] \in \mathbb{R}^{(n_1-1) \times (n_2-1)}$ be a matrix s.t.

$$
\tilde{v}_{j,k} \geq \| A_{\mathcal{U}} \tilde{\psi}^{j,k} \|_2 / \sqrt{n}, \quad \forall (j, k) \in [2 : n_1] \times [2 : n_2]
$$

and $v \in [0, 1]^{(n_1-1) \times (n_2-1)}$ be a matrix s.t.

$$
v_{j,k} \geq \tilde{v}_{j,k} / \bar{\gamma}, \quad \forall (j, k) \in [2 : n_1] \times [2 : n_2],
$$

where $\bar{\gamma}$ is chosen s.t. $\max_{(j,k) \in [2 : n_1] \times [2 : n_2]} v_{j,k} = 1$. Let $q_S$ be a sign configuration and $w(q_S)$ be an interpolating matrix for $v$. Then, $\forall f \in \mathbb{R}^{n_1 \times n_2}$, for $t > 0$ and for $\lambda \geq \bar{\gamma} \lambda_0(t)$, it holds that, with probability at least $1 - e^{-x} - e^{-t}$,

$$
\| \hat{f} - \tilde{f}^0 \|_2^2 / n \leq \| f - \tilde{f}^0 \|_2^2 / n + 4 \lambda \| (\Delta f)_{-S} \|_1 + \left( \sqrt{s/n} + \sqrt{2\pi/n} + \lambda \Gamma(S, v_{-S}) \right)^2.
$$

To prove Theorem 2.1 we need to find suitable $\tilde{v}$, $\gamma$, $v$, $w$ and in a second step we need to find and upper bound for the effective sparsity.

- By Lemma 2.11 we have that

  $$
\| A_{\mathcal{U}} \tilde{\psi}^{j,k} \|_2 / \sqrt{n} \leq \| A_{\mathcal{U}} \psi^{j,k} \|_2 / \sqrt{n},
$$

  since $\tilde{\psi}^{j,k} = P_{\mathcal{W}} \psi^{j,k}, (j, k) \in [2 : n_1] \times [2 : n_2]$. 


• Upper bounds on the values of $\|A_U \psi^{j,k}\|_2/\sqrt{n}$ are given by Lemma 2.12.

• By applying Lemma 2.10 and Lemma 2.9 to the results of Lemma 2.12 we see that for $v$ as defined in (2) we have

$$v_{j,k} \geq \tilde{v}_{j,k}/\tilde{\gamma}, \forall (j,k) \in [2 : n_1] \times [2 : n_2],$$

where

$$\tilde{\gamma} = 2 \sqrt{\frac{d_{1,\max}(S)}{n_1} + \frac{d_{2,\max}(S)}{n_2}}$$

and $v$ reaches its maximal values of 1 on the boundaries of the rectangles of the rectangular tessellation $\{R_n\}_{n=1}^n$.

• As a consequence of the above point $w(q_S)$ as defined in (3) is an interpolating matrix for the sign configuration $q_S$.

Lemma 2.8 gives us a bound for the effective sparsity by using the interpolating matrix $w(q_S)$ given in (3) and thus the claim of Theorem 2.1 follows. $\square$

3. Slow rates

As we previously saw, we can decompose the estimator $\hat{f}$ as

$$\hat{f} = \hat{f}(\cdot, o) \psi^{1,1} + \hat{f}(, o) + \hat{f}(o, \cdot) + \hat{f}.$$

We can now derive oracle inequalities with slow rates for $\hat{f}(\cdot, o)$, $\hat{f}(o, \cdot)$, $\hat{f}$ distinctly, since they are mutually orthogonal. To derive oracle inequalities with fast and slow rates for $\hat{f}(\cdot, o)$ and $\hat{f}(o, \cdot)$, we refer to the examples exposed in Ortelli and van de Geer [2019b]. We now derive an oracle inequality with slow rates for $\hat{f}$ and show that it recovers the rate found by Mammen and van de Geer [1997] up to a log term. This is another confirmation, in addition to the ones found in Ortelli and van de Geer [2019b] and Ortelli and van de Geer [2019c], of the fact that the projection arguments by Dalalyan et al. [2017] can catch in a simple way the rate obtained by complicated entropy calculations, up to a log term.

**Theorem 3.1**

Let $S \subseteq [3 : n_1 - 1] \times [3 : n_2 - 1]$ be arbitrary and $x, t > 0$. Choose

$$\lambda \geq \lambda_0(t) \sqrt{\frac{d_{1,\max}(S)}{n_1} + \frac{d_{2,\max}(S)}{n_2}},$$

For the estimator

$$\hat{f} := \arg \min_{f \in \mathbb{R}^{n_1 \times n_2}} \left\{ \|\hat{Y} - f\|_2^2/n + 2\lambda TV(f) \right\}, \lambda > 0,$$

it holds that, $\forall f \in \mathbb{R}^{n_1 \times n_2}$, with probability at least $1 - e^{-x} - e^{-t}$,

$$\|\hat{f} - \hat{f}^0\|_2^2/n \leq \|f - \hat{f}^0\|_2^2/n + \frac{1}{n}(\sqrt{2x} + \sqrt{s})^2 + 4\lambda TV(f).$$
Proof of Theorem 3.1. The theorem follows directly from Theorem 2.2 in Ortelli and van de Geer [2019b]. The quantity
\[
\gamma := \max_{m \in S} \max_{(j,k) \in \mathbb{R}_m} \| A_U \psi_{j,k} \|_2 / \sqrt{n} = \sqrt{\frac{d_{1,\text{max}}(S)}{n_1} + \frac{d_{2,\text{max}}(S)}{n_2}}
\]
is obtained by Lemma 2.12 and corresponds to the normalized inverse scaling factor as defined in Ortelli and van de Geer [2019b].

Remark 3.1

Note that in the proof of Theorem 2.1, we take \( \tilde{\gamma} = 2\gamma \). This is due to the fact that we need a factor \( 1/2 \) in the right hand side of the claim of Lemma 2.10 to be able to apply Lemma 2.9 to it. Thus in Theorem 3.1 we can choose \( \lambda \geq \gamma \lambda_0(t) \), while in Theorem 2.1 we have to choose \( \lambda \geq 2\gamma \lambda_0(t) \).

Let us now define the class
\[
\mathcal{F}(C) = \{ f \in \mathbb{R}^n: \text{TV}(f) \leq C \} , \quad C > 0.
\]

Corollary 3.2

Assume that \( f^0 \in \mathcal{F}(C) \). Consider a square image (i.e. \( n_1 = n_2 \)). Choose \( S \) to be a regular grid parallel to the coordinate axes with
\[
s \asymp n^{2/5} \text{TV}(f^0)^{4/5} (\log n)^{2/5}.
\]
Choose
\[
\lambda \geq \sqrt{2s^{-1/4}} \lambda_0(\log(2n)).
\]
Then
\[
\| \hat{f} - f^0 \|_2^2 / n = \mathcal{O}_p \left( n^{-3/5} C^{4/5} (\log n)^{2/5} \right).
\]

Proof of Corollary 3.2. The result follows by setting \( f = \hat{f} \) and \( n_1 = n_2, s_1 = s_2 \) in the above theorem.

Remark 3.2

The rate obtained corresponds, up to a log-term, to the rate found by Mammen and van de Geer [1997] using entropy.

4. Conclusion

We have shown how the argument presented in Ortelli and van de Geer [2019c] used to bound the version of the compatibility constant proposed by Dalalyan et al. [2017] can be applied to total variation denoising in two dimensions. As a result, we can show that the estimator for the interactions satisfies an oracle inequality with fast rates and can adapt to the number and the locations of the unknown “jumps” to optimally trade off approximation and estimation error as if it would know the true image \( f^0 \). This is encouraging, since arguments of the same flavour could be used to prove similar results for other instances of analysis estimators.
In addition to that, by looking at slow rates, we found another evidence for how the projection arguments by Dalalyan et al. [2017] are able to catch the same rates of the results by Mammen and van de Geer [1997] following from complicated entropy calculations. We therefore suspect that there might be a connection between entropy arguments and projection arguments.

A question remaining open is what would be the counterpart of the minimax rate obtained by Sadhanala et al. [2016] for the total variation regularized estimator in two dimensions defined as in Mammen and van de Geer [1997].

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Appendix A: Proofs

Proof of Lemma 1.3. We note that
\[
\psi^{j,k}(o, o) = \psi^{1,k}(o, o)\psi^{j,1}(o, o),
\]
\[
\psi^{j,k}(\cdot, o) = \psi^{1,k}(o, o)\psi^{j,1} - \psi^{j,k}(o, o)\psi^{1,1},
\]
\[
\psi^{j,k}(o, \cdot) = \psi^{j,1}(o, o)\psi^{1,k} - \psi^{j,k}(o, o)\psi^{1,1}.
\]
Thus
\[
\psi^{j,k} = \tilde{\psi}^{j,k} + \psi^{1,k}(o, o)\tilde{\psi}^{j,1} + \psi^{j,1}(o, o)\tilde{\psi}^{1,k} + \psi^{j,k}(o, o)\psi^{1,1}.
\]
From the definitions of \(\tilde{\psi}^{j,k}, (j, k) \in [n_1] \times [n_2]\) it follows that
\[
f = \beta^{1,1} + \sum_{j=2}^{n_1} \tilde{\beta}_{j,1} \tilde{\psi}^{j,1} + \sum_{k=2}^{n_2} \tilde{\beta}_{1,k} \tilde{\psi}^{1,k} + \sum_{j=2}^{n_1} \sum_{k=2}^{n_2} \tilde{\beta}_{j,k} \tilde{\psi}^{j,k},
\]
where
\[
\tilde{\beta}_{1,1} = \beta_{1,1} + \sum_{j=2}^{n_1} \beta_{j,1} \psi^{j,1}(o, o) + \sum_{k=2}^{n_2} \beta_{1,k} \psi^{1,k}(o, o) + \sum_{j=2}^{n_1} \sum_{k=2}^{n_2} \beta_{j,k} \psi^{j,k}(o, o),
\]
\[
\tilde{\beta}_{j,1} = \beta_{j,1} + \sum_{k=2}^{n_2} \beta_{j,k} \psi^{1,k}(o, o), \quad j \in [2 : n_1],
\]
\[
\tilde{\beta}_{1,k} = \beta_{1,k} + \sum_{j=2}^{n_1} \beta_{j,k} \psi^{j,1}(o, o), \quad k \in [2 : n_2],
\]
\[
\tilde{\beta}_{j,k} = \beta_{j,k}, \quad (j, k) \in [2 : n_1] \times [2 : n_2].
\]
Note that \(\psi^{1,k}(o, o) = 1 - (k - 1)/n_2, \psi^{j,1}(o, o) = 1 - (j - 1)/n_1\) and
\[
\sum_{j=2}^{n_1} \beta_{j,1} \psi^{j,1}(o, o) = -f(1, 1) + \frac{1}{n_1} \sum_{j=1}^{n_1} f(j, 1).
\]
Analogously it holds that
\[
\sum_{k=2}^{n_2} \beta_{j,k} \psi^{1,k}(o, o) = -f(1, 1) + \frac{1}{n_2} \sum_{k=1}^{n_2} f(1, k).
\]
By plugging in the expressions for \(\tilde{\beta}_{j,k}, (j, k) \in [n_1] \times [n_2]\) and by using the above equations the result follows. \(\square\)
Proof of Lemma 1.4. Since $f - \tilde{f}$ and $\tilde{f}$ are orthogonal, $\text{trace}(\tilde{f}^T(f - \tilde{f})) = 0$. We have

$$
\|Y - f\|_2^2 - \|Y\|_2^2 = -2\text{trace}(Y^Tf) + \|f\|_2^2
$$

$$
= -2\text{trace}(Y^T(f - \tilde{f})) - 2\text{trace}(Y^T\tilde{f}) + \|f - \tilde{f}\|_2^2 + \|\tilde{f}\|_2^2
$$

$$
= \|Y - (f - \tilde{f})\|_2^2 + \|Y - \tilde{f}\|_2^2 - 2\|Y\|_2^2.
$$

The result now follows from Lemma 1.3. \hfill \square