Two-Loop Calculation of the Anomalous Exponents in the Kazantsev–Kraichnan Model of Magnetic Hydrodynamics

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Abstract. The problem of anomalous scaling in magnetohydrodynamics turbulence is considered within the framework of the kinematic approximation, in the presence of a large-scale background magnetic field. Field theoretic renormalization group methods are applied to the Kazantsev–Kraichnan model of a passive vector advected by the Gaussian velocity field with zero mean and correlation function $\propto \delta(t - t')/k^{d+\epsilon}$. Inertial-range anomalous scaling for the tensor pair correlators is established as a consequence of the existence in the corresponding operator product expansions of certain “dangerous” composite operators, whose negative critical dimensions determine the anomalous exponents. The main technical result is the calculation of the anomalous exponents in the order $\epsilon^2$ of the $\epsilon$ expansion (two-loop approximation).

Keywords: Turbulence, Renormalization Group, Operator Product Expansion, Anomalous Scaling, Kraichnan’s Rapid-Change Model.

1 Introduction

Much attention has been paid recently to a simple model of the passive advection of a scalar quantity by a Gaussian short-correlated velocity field, introduced first by Obukhov [9] and Kraichnan [7]. The structure functions of the field in this model exhibit anomalous scaling behavior, and the corresponding anomalous exponents can be calculated within regular expansions in a small parameter.

Effects of intermittency and anomalous scaling are even more important for vector fields. In particular, the large-scale intense anisotropic magnetic fields coexist with small-scale turbulent activity in solar wind, see e.g. [10] and references therein.

In this communication, we discuss the anomalous scaling of magnetic fields in the presence of large-scale anisotropy within the framework of the kinematic
Kazantsev-Kraichnan model, using the field theoretic methods of renormalization group and operator product expansion. We extend the one-loop results derived in [5] to the two-loop order of the \( \epsilon \)-expansion.

### 1.1 Kinematic MHD Kazantsev-Kraichnan Model

In the presence of a mean component \( \theta^0 \) (actually supposed to be varying on a very large scale \( L \), the largest one in our problem) the kinematic MHD equations, describing the evolution of the fluctuating part \( \theta = \theta(x) \) of the magnetic field, are

\[
\partial_t \theta_i + \mathbf{v} \cdot \nabla \theta_i = \theta \cdot \nabla v_i + \theta^0 \cdot \nabla v_i + \nu_0 \partial^2 \theta_i, \quad i = 1, \ldots, d, \tag{1}
\]

where the term \( \theta^0 \cdot \nabla v_i \equiv f_i \) effectively plays the same role as an external force, driving the system, with correlator

\[
\langle f_i(x)f_j(x') \rangle = \delta(t - t')C_{ij}(mr), \tag{2}
\]

where \( C \) is some function finite at \( r = 0 \) and decaying for \( r \to \infty \) and \( m = 1/L \) is the reciprocal of the integral turbulence scale. Here and below \( x \equiv \{t, \mathbf{x}\} \), \( \partial \equiv \{\partial_i = \partial/\partial x_i\} \), \( \partial^2 = \partial_0 \partial_i \equiv \Delta \) is the Laplace operator, \( d \) is the dimensionality of \( \mathbf{x} \) space, \( \mathbf{v}(x) \) is the velocity field. Both \( \mathbf{v} \) and \( \theta \) are divergence-free (solenoidal) vector fields: \( \partial_i v_i = \partial_i \theta_i = 0 \). In the real problem, \( \mathbf{v} \) obeys the NS equation with the additional Lorentz force term \( \propto (\partial \times \theta) \times \theta \), which describes the effects of the magnetic field on the velocity field. The framework of our analysis is the kinematic MHD problem, where the reaction of the magnetic field \( \theta \) on the velocity field is neglected. We assume that at the initial stages \( \theta \) is weak and does not affect the motions of the conducting fluid: it becomes then a natural assumption to consider the dynamics linear in the magnetic field strength.

More precisely, we shall consider a simplified model, in which \( \mathbf{v}(x) \) is a Gaussian random field, homogeneous, isotropic and \( \delta \)-correlated in time, with zero mean and covariance

\[
\langle v_i(x)v_j(x') \rangle = D_0 \delta(t - t') \frac{1}{(2\pi)^d} \int dk P_{ij}(k) k^{-d-\epsilon} e^{ik(x-x')} , \tag{3}
\]

where \( P_{ij}(k) = \delta_{ij} - k_i k_j / k^2 \) is the transverse projector, \( k \) is the momentum, \( k = |k| \), \( D_0 \) is an amplitude factor, \( d \) is dimensionality of the \( x \) space and \( \epsilon \) is a free parameter. The IR regularization is provided by the cutoff in the integral from below at \( k \approx m \propto 1/L \). The case of anisotropic velocity ensemble was studied in [6].

### 2 Field Theoretic Formulation

This stochastic problem is equivalent to the field theoretic model of the set of three fields \( \Phi = \{\theta, \theta', \mathbf{v}\} \) with action functional [11]:

\[
S(\Phi) = \theta' D_0 \theta' / 2 + \theta' \left[ -\partial_0 \theta_i + \nu_0 \Delta \theta_i - \partial_k (v_k \theta_i - v_i \theta_k) \right] - \mathbf{v} D^{-1}_\epsilon \mathbf{v} / 2 , \tag{4}
\]
where the first four terms represent the De Dominicis–Janssen-type action for the stochastic problem \( D_\theta = \langle ff \rangle \) and \( D_v = \langle vv \rangle \) are the correlators (2) and (3) respectively, the required integrations over \( x = \{ t, x \} \) and summations over the vector indices are understood.

The diagrams of the perturbation theory are constructed of the four elements. In the \( \omega, k \) representation the factor \( i[k_a \delta_{bc} - p_b \delta_{ac}] \) corresponds to the vertex, and the lines \( vv, \theta \theta \) correspond to the bare propagators

\[
\gamma = \frac{P_{ij}(k)}{k^{d+1}}, \quad \frac{P_{ij}(k)}{-\omega + i k^2}, \quad \frac{C_{ij}(k)}{\omega^2 + \nu k^2},
\]

(5)

where \( C_{ij}(k) \) is the Fourier transform of the function \( C_{ij} \) from Eq. (2).

The UV divergences manifest themselves as poles in \( \epsilon \) in the diagrams. For the complete elimination of these divergences it is sufficient to perform the multiplicative renormalization of the parameters \( \nu_0 \) and \( g_0 \) with the only independent renormalization constant \( Z_\nu \) (see Ref. [1]):

\[
\nu_0 = \nu Z_\nu, \quad g_0 = g \epsilon Z_g, \quad Z_g^{-1} = Z_\nu.
\]

(6)

The exact response function \( G_{ij} = \langle \theta_i \theta'_j \rangle \) satisfies the standard Dyson equation with just one self-energy diagram, and therefore one can obtain an exact expression for the renormalization constant \( Z_\nu \) in the MS scheme:

\[
Z_\nu = 1 - u \frac{d - 1}{2d} \frac{1}{\epsilon}, \quad u = g S_d / (2\pi)^d,
\]

(7)

where \( S_d \) is the area of the unit sphere in \( d \)-dimensional space.

### 2.1 RG Equations for Composite Operators

The basic RG equation for a multiplicatively renormalizable quantity \( F = Z_F \cdot F_R \) (correlation function, composite operator etc) has the form

\[
[D_{\text{RG}} + \gamma_F] F_R = 0, \quad D_{\text{RG}} = D_u + \beta \partial_u - \gamma_u D_u, \quad D_x = x \partial / \partial x,
\]

(8)

where RG functions \( \beta \) and \( \gamma \) (anomalous dimension) are defined as

\[
\beta = \tilde{D}_\mu u, \quad \gamma_F = \tilde{D}_\mu \ln Z_F \forall Z_F,
\]

(9)

where \( \tilde{D}_\mu \) is the operation \( D_\mu \) at fixed bare parameters.

From the analysis of RG functions it follows, that the RG equations (8) possess an IR stable positive fixed point \( u_* \):

\[
u_* = \frac{2d}{d+1} \epsilon, \quad \beta(u_*) = 0, \quad \beta'(u_*) > 0.
\]

(10)

The value of anomalous dimension \( \gamma_\nu(u) \) at fixed point \( u_* \) is

\[
\gamma_\nu^* \equiv \gamma_\nu(u_*) = \epsilon.
\]

(11)
This fact implies that correlation functions of this model exhibit scaling behavior; the corresponding critical dimensions $\Delta [F] \equiv \Delta_F$ can be calculated as series in $\epsilon$. For the basic fields and quantities the dimensions are found exactly\[4]:

$$\Delta_\theta = -1 + \epsilon/2, \quad \Delta_{\theta'} = d + 1 - \epsilon/2, \quad \Delta_\omega = 1 - \epsilon$$ \hspace{1cm} (12)

(there is no corrections of order $\epsilon^2$ and higher, this is a consequence of the exact equality $\gamma(u^*) = \epsilon$).

Let $G(r) = \langle F_1(x)F_2(x') \rangle$ be a single-time two-point quantity; for example, the pair correlation function of the primary fields $\Phi = \{\theta, \theta', v\}$ or some multiplicatively renormalizable composite operators. The solution of the RG equation gives:

$$G(r) \sim \nu^{-\Delta_G} G_0 \Lambda^{-\omega} G(\Lambda r) - \Delta G \xi(mr),$$ \hspace{1cm} (13)

where the canonical dimensions $d_\omega, d_G$ and the critical dimension $\Delta_G$ of the function $G(r)$ are equal to the sums of the corresponding dimensions of the quantities $F_i$.

This representation describes the behavior of the correlation functions for $\Lambda r \gg 1$ and any fixed value of $mr$. The inertial range $\Lambda^{-1} = l \ll r \ll L = m^{-1}$ corresponds to the additional condition $mr \ll 1$, the form of the functions $\xi(mr)$ in the interval $mr \ll 1$ is studied using the operator product expansion (OPE).

### 3 Operator Product Expansion

According to the OPE, the single-time product $F_1(x)F_2(x')$ of two renormalized operators has the form

$$F_1(x)F_2(x') = \sum_{\alpha} C_{\alpha}(r)F_{\alpha}(x, t),$$ \hspace{1cm} (14)

where $x \equiv (x + x')/2 = \text{const}$, $r \equiv x - x' \rightarrow 0$, the functions $C_{\alpha}$ are the Wilson coefficients regular in $m^2$ and $F_{\alpha}$ are all possible renormalized local composite operators allowed by symmetry, with definite critical dimensions $\Delta_\alpha$.

The renormalized correlator $\langle F_1(x)F_2(x') \rangle$ is obtained by averaging Eq. (14) with the weight exp $S_R$; hence the desired asymptotics for the correlator $\langle F_1(x)F_2(x') \rangle$ is the sum, in which the operator possessing the minimal dimension gives the leading term:

$$\xi(mr) \cong \text{const} \cdot (mr)^{\Delta_{\text{min}}}. $$ \hspace{1cm} (15)

The feature typical to the models describing turbulence is the existence of composite operators with negative critical dimensions, for example, critical dimension of the field $\theta_i$ is $\Delta_\theta = (-1 + \epsilon/2)$; their contributions in the OPE lead to singular behavior of the scaling functions at $mr \rightarrow 0$, that is, to the anomalous scaling. The operators with minimal $\Delta_F$ are those involving the maximal possible number of fields $\theta$ and the minimal possible number of derivatives. Therefore the needed operators are tensors, constructed from the fields $\theta_i$ themselves:

$$F_{nl} = \theta_{i_1} \cdots \theta_{i_l} (\theta_i \theta_i)^p, \quad n = l + 2p.$$ \hspace{1cm} (16)
The critical dimension of any multiplicatively renormalizable quantity \( F = Z_F \cdot F_R \) is \( \Delta F = d_F^k + \Delta d_F + \gamma_F^* \). Then for the operator \( F_{nl} \) we obtain
\[
\Delta F_{nl} = n(-1 + \epsilon/2) + \gamma_F^* F_{nl},
\]
and the leading asymptotic term of the correlator \( \langle F_{nl}(x)F_{pq}(x') \rangle \) in the \( j \)th anisotropic sector has the form
\[
\langle F_{nl}(x)F_{pq}(x') \rangle \propto (Ar)^{-\Delta_F^j} \Delta_F F_{nl} + \Delta_F F_{pq}. \tag{17}
\]
Thus one has to calculate critical dimensions \( \Delta F_{nl} \) of the operators \( F_{nl} \).

4 Scalarization of the Diagrams

The operator \( F_{nl} = \theta_1 \cdots \theta_n (\theta \theta')^p, n = l + 2p \) is renormalized multiplicatively, \( F_{nl} = Z_{nl} \cdot F_{nl}^R \), and the renormalization constants \( Z_{nl} = Z_{nl}(g, \epsilon, d) \) are determined by the requirement that the 1-irreducible correlation function
\[
\langle F_{nl}^R(x_1) \cdots \theta(x_n) \rangle_{1-irr} = Z_{nl}^{-1} \langle F_{nl}(x_1) \cdots \theta(x_n) \rangle_{1-irr} \equiv Z_{nl}^{-1} \Gamma_{nl}(x_1, \ldots, x_n) \tag{19}
\]
be UV finite in renormalized theory, i.e., have no poles in \( \epsilon \) when expressed in renormalized variables \( \tilde{n} \).

Below we present, along with respective symmetry coefficients, all the diagrams needed for the two-loop calculation of the function \( \Gamma_{nl} \), except for those with the self-energy insertions in the \( \theta \theta' \) lines.

\[
\Gamma^{(1)} = \frac{1}{2}, \quad \Gamma^{(2)} = \frac{1}{2} + \frac{1}{8} \quad \text{(D1)} \quad \text{(D2)} \quad \text{(D3)} \quad \text{(D4)}
\]

The contribution of a specific diagram into the functional \( \Gamma_{nl} \) has the form
\[
\dot{\Gamma} = \sum_i p_i A_i, \tag{20}
\]
where \( p_i \) are known combinatorial coefficients and \( A_i \) are certain scalar quantities \( \text{[2], [3]} \).

5 Calculation of Anomalous Exponents

In this section we present the two-loop calculation of the critical dimensions \( \gamma_F^* \) of the composite operators \( F_{nl} \), which determine the anomalous exponents in expression \( \text{[18]} \). We need to extract from the diagrams only the singular parts that contain the first-order poles in \( \epsilon \).
Calculation of Diagram $D_2$. The diagram $D_2$ is represented by the integral

$$I = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{+\infty} \frac{d\omega'}{2\pi} \int_{m}^{\infty} \frac{dk}{(2\pi)^d} \int_{m}^{\infty} \frac{dq}{(2\pi)^d} \cdot \frac{P_{b\gamma}(k + q)}{i(\omega + \omega') + (k + q)^2} \cdot \frac{P_{\alpha\alpha}(k)}{-i\omega + k^2} \cdot \frac{P_{\alpha\beta}(k)}{i\omega + k^2} \cdot \frac{P_{\rho\sigma}(q)}{q^2 + i\varepsilon} \cdot \frac{P_{\rho\sigma}(k)}{k^2 + \varepsilon} \cdot \varepsilon \frac{\delta}{\delta \varepsilon}.$$

The use of transversality of the vertex greatly simplifies our calculations:

$$p_1 V_{123} = p_1(p_2\delta_{13} - p_3\delta_{12}) \equiv 0, \Rightarrow P_{mn}(p + k) \rightarrow \delta_{mn}. \quad (22)$$

Therefore after contraction of the vector indices with standard symmetric structures, constructed from $\delta$-symbols, and differentiation over $m$, which allows to single out the first-order pole $1/\varepsilon$ explicitly, one obtains:

$$A_1 = \frac{1}{2} \frac{1}{8} \int_{1}^{+\infty} dx \int_{0}^{\pi} d\theta \left[ \frac{x^2 + 1}{(x^2 + 2x \cos \theta + 1)} - 1 \right] \frac{\sin^5 \theta}{x} = \frac{1}{150}; \quad (23)$$

$$A_2 = \frac{1}{2} \frac{1}{8} \int_{1}^{+\infty} dx \int_{0}^{\pi} d\theta \left[ \frac{x^2 + 1}{(x^2 + 2x \cos \theta + 1)} - 1 \right] \frac{\sin^3 \theta}{x} = \frac{1}{36}, \quad (24)$$

where $x = q/m$. All such integrals were calculated analytically for the most important physical case $d = 3$.

Calculation of Diagram $D_3$. In the similar manner, for the diagram $D_3$ one obtains:

$$A_1 = \frac{1}{48} \int_{1}^{+\infty} dx \int_{0}^{\pi} d\theta \left[ \frac{x^2 + 1}{(x^2 + x \cos \theta + 1)} - 1 \right] \frac{\sin^5 \theta}{x} = \frac{1}{48} \left( -\frac{8\sqrt{3}}{5} \pi + \frac{656}{75} \right); \quad (25)$$

$$A_2 = -\frac{1}{24} \left[ \int_{1}^{+\infty} dx \int_{0}^{\pi} d\theta \frac{4 \cos \theta \sin^3 \theta}{x^2 + x \cos \theta + 1} + \right], \quad (26)$$

$$+ \frac{1}{2} \int_{1}^{+\infty} dx \int_{0}^{\pi} d\theta \left[ \frac{x^2 + 1}{(x^2 + x \cos \theta + 1)} - 1 \right] \frac{\cos^2 \theta \sin^3 \theta}{x} = \frac{1}{24} \left( -\frac{\sqrt{3}}{5} \pi + \frac{24}{25} \right). \quad (27)$$

Calculation of Diagram $D_1$. For the simplest diagram $D_1$ one obtains

$$A_1 = 0, A_2 = -\frac{1}{\varepsilon}. \quad (27)$$

Diagram $D_4$. The factorized four-ray diagram $D_4$ contains only a second-order pole in $\varepsilon$ and therefore is not needed for the calculation of $\gamma^p$. 
Anomalous Dimension $\gamma^*_{F_{nl}}$. The value of anomalous dimension $\gamma^*_{F}$ is

$$\gamma^*_F = \sum_i (\bar{q}_i \bar{A}_i u_* + 2\tilde{q}_i \tilde{A}_i u_*)^2),$$  \hspace{1cm} (28)$$

where the quantities with bars and with tildes correspond to the one-loop and two-loop contributions, respectively.

Finally, combining (28) with (23), (24), (25), (26) and (27), for the anomalous dimension of the operator $F_{nl}$ with arbitrary $n$ and $l$ one obtains:

$$\gamma^*_{F_{nl}} = -\left\{ \frac{1}{10} [n(n+3) - 2l(l+1)] \cdot \epsilon + \epsilon^2 \cdot \left( \frac{2n(n-2)}{125} - \frac{22l(l+1)}{375} \right) + \frac{n(n+3)}{30} + \frac{3}{35} \left(-\frac{\sqrt{3}}{5} \pi + \frac{82}{75}\right)(n-2) [2n(n-4) + 3l(l+1)] - \frac{9}{140} \left(-\frac{\sqrt{3}}{5} \pi + \frac{24}{25}\right)(n-2) [n(n+3) - 2l(l+1)] \right\}. \hspace{1cm} (29)$$

6 Comparison with the Exact Solution

The exact solution for the pair correlator of the problem (1) was derived in [8]; see also [5] for a more detailed discussion. In particular, the exponents $\zeta_0$ and $\zeta_2$, describing the scaling behavior in the isotropic and leading anisotropic sectors, were derived exactly for any $d$. Expanding those expressions to the second order in $\epsilon$ and setting $d = 3$ gives

$$\zeta_0 = -\epsilon - \frac{1}{3} \epsilon^2, \quad \zeta_2 = \frac{1}{5} \epsilon + \frac{7}{375} \epsilon^2. \hspace{1cm} (30)$$

In the RG approach, these exponents should be identified with the anomalous dimensions of the operators $\theta_i \theta_i$ and $\theta_i \theta_j$, that is, with $\gamma^*_{F_{20}}$ and $\gamma^*_{F_{22}}$. It is easily checked that our expression (29) is in agreement with (30).

7 Conclusion

We have applied the RG and OPE methods to the simple Kazantsev–Kraichnan model, which describes the advection of a passive vector by the Gaussian velocity field, decorrelated in time and self-similar in space.

We have shown that the correlation functions of the vector field in the convective range exhibit anomalous scaling behavior, what is closely related with existence in this model of composite operators with negative dimensions. The corresponding anomalous exponents have been calculated to the second order of the $\epsilon$-expansion (the two-loop approximation).

It is worth noting that the hierarchy relations between the anisotropic exponents [5] persist in the two-loop contributions. It is also worth noting that,
in contrast to the scalar case, the two-loop contributions for scalar operators have the same sign as the first-order ones, see e.g. (30) for $\gamma_{F_{20}}^* = \zeta_0$. Thus the anomalous scaling and the anisotropic hierarchy become even more strongly pronounced due to the higher-order contributions of the $\epsilon$-expansion.

The agreement between the exact exponents for the pair correlation function is also established. This fact strongly supports the applicability of the RG technique and the $\epsilon$-expansion to the problem of anomalous scaling for finite values of $\epsilon$, at least for low-order correlation functions.

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