LOCAL NULL-CONTROLLABILITY OF A SYSTEM COUPLING KURAMOTO-SIVASHINSKY-KDV AND ELLIPTIC EQUATIONS

KUNTAL BHANDARI*, AND SUBRATA MAJUMDAR†

ABSTRACT. This paper deals with the null-controllability of a system of mixed parabolic-elliptic pdes at any given time \( T > 0 \). More precisely, we consider the Kuramoto-Sivashinsky-Korteweg-de Vries equation coupled with a second order elliptic equation posed in the interval \((0,1)\). We first show that the linearized system is globally null-controllable by means of a localized interior control acting on either the KS-KdV or the elliptic equation. Using the Carleman approach, we provide the existence of a control with the explicit cost \( K e^{K/T} \) with some constant \( K > 0 \) independent in \( T \). Then, applying the source term method developed in [44], followed by the Banach fixed point argument, we conclude the small-time local null-controllability result of the nonlinear systems.

Besides, we also established a uniform null-controllability result for an asymptotic two-parabolic system (fourth and second order) that converges to the concerned parabolic-elliptic model when the control is acting on the second order pde.

1. Introduction and main results

1.1. Bibliographic comments and motivations. The controllability of coupled partial differential equations has gained immense interest to the control communities since the past few decades. Due to the significant importance of the parabolic models in physical, chemical and biological sciences, several methods have been developed to study their controllability properties. We first address the pioneer work by Fattorini and Russell [20, 21] where the authors utilized the spectral strategy, i.e., the so-called moments method to study the controllability (distributed or boundary) of 1D linear parabolic equations. Regarding the distributed controllability of parabolic pdes in dimension higher than 1, we refer to the well-known Lebeau-Robbiano spectral method [43] (see also [41]) which is based on a global elliptic Carleman estimate. Next, we must mention that the global Carleman estimate, established by Fursikov and Imanuvilov in [28] has been intensively used to study the controllability of parabolic equations and systems. In this context, we refer the work [25] by E. Fernández-Cara and S. Guerrero where a global Carleman estimate has been established to study the controllability properties of some linear and certain semilinear parabolic equations. Later on, the controllability of a system of two parabolic equations with only one control force has been proved by S. Guerrero [33].

However, in most cases the Carleman approach is inefficient to handle the boundary controllability of the coupled parabolic systems. In fact, the boundary controllability for such systems is no longer equivalent with distributed controllability as it has been observed for instance in [24]. Due to this reason, most available results are in 1D and mainly based on the moments method. Among the fewest, we refer for instance, [1, 2, 3, 7]. For the multi-D situation, we address the work [6] where the boundary controllability results are obtained in the cylindrical geometries.

In the direction of controllability of nonlinear parabolic systems, we mention the work by E. Fernández-Cara and E. Zuazua [27] where the authors proved small-time global null-controllability of semilinear heat equations with the growth of nonlinear function \(|f(r)| \leq |r| \ln^{3/2}(1+|r|)\), see also [4] by V. Barbu. Very recently, the large-time global null-controllability of semilinear heat equations has been obtained in [40] for nonlinearities \( f \) growing slower than \( |r| \ln^2(1+|r|) \) under the sign condition: \( f(r) > 0 \) for \( r > 0 \) and \( f(r) < 0 \) for \( r < 0 \).

Another fascinating area in control theory is the study of controllability properties of the fourth order parabolic equation: \( u_t + \gamma_1 u_{xxxx} + \gamma_2 u_{xx} + uu_x = 0 \) (\( \gamma_1, \gamma_2 > 0 \)), the so-called Kuramoto-Sivashinsky (KS) equation. This has been popularly studied in the past decades, such as [12], [51], [11], [49]. In particular, the authors in [11] proved that the linear KS equation with Neumann boundary conditions is null-controllable by a control acting in some open subset of the domain. The author in [49] proved the boundary local null-controllability of the KS equation by utilizing the source term method (see [44]) followed by the Banach fixed point argument.

\[ \text{Date: September 19, 2022.} \]
\[2010 \text{ Mathematics Subject Classification. 35M33, 93B05, 93B07, 93C20.} \]
\[\text{Key words and phrases. KS-KdV-elliptic system, null-controllability, Carleman estimates, source term method, fixed point argument.} \]
\[\text{†Laboratoire de Mathématiques Blaise Pascal, UMR 6620, Université Clermont Auvergne, CNRS, 63178 Aubière, France; e-mail: kuntal.bhandari1993@gmail.com (corresponding author).} \]
\[\text{‡Indian Institute of Science Education and Research Kolkata, Campus road, Mohanpur, West Bengal 741246, India; e-mail: smi18ts0160@iiserkol.ac.in.} \]
point argument where a suitable control cost $Ce^{C/T}$ of the linearized model plays the crucial role. Similar strategy has been applied in [37] to study the boundary local null-controllability of a simplified stabilized KS system (see below about this system). In this regard, we must mention that the nonlinearity $uu_x$ in the KS system also appears in the Burgers equation: $u_t - u_{xx} + uu_x = 0$, and concerning the controllability results of this equation we address the following works: [26], [32], [31].

To include dispersive and extra dissipative effects in the KS equation, a coupled system containing fourth (Kuramoto-Sivashinsky-Korteweg-de Vries, in short KS-KdV equation) and second order parabolic (heat) equations, under the name of stabilized Kuramoto-Sivashinsky system

$$\begin{align*}
    u_t + \gamma_1 u_{xxxx} + u_{xxx} + \gamma_2 u_{xx} + uu_x &= v_x, \\
v_t - \Gamma v_{xx} + cv_x &= bu_x,
\end{align*}$$

has appeared for instance in [45]. The controllability of such system has been considered in several works, namely [10], [13], [14], [9], [38]. E. Cerpa, A. Mercado and A. F. Pazoto in [14] demonstrated the local null-controllability of (1.1) by means of localized interior control acting in the KS-KdV equation. Then, E. Cerpa along with N. Carreño [9] proved the local null-controllability to the trajectories with a localized interior control exerted in the heat equation. Two different types of Carleman estimates for the linearized model followed by the inverse mapping theorem have been implemented to conclude their controllability results. We also point out that recently, the controllability of KS-KdV-transport model has been discussed in [39]. Finally, it is worth mentioning that an insensitizing control problem for the stabilized KS system has been explored in [8].

Study of the parabolic-elliptic model, namely

$$\begin{align*}
    u_t - u_{xx} &= au + bv, \\
-v_{xx} &= cu + dv,
\end{align*}$$

enlarges the literature of the control of pdes and such type of models significantly appear in several physical, chemical and biological situations: as for example, the prey-predator interaction, tumor growth therapy etc. One may refer to the works [23], [34], [36], [48] concerning the controllability issues of such systems.

In the work [23], the authors proved the local null-controllability of a semilinear parabolic-elliptic equation by means of a localized interior control acting in one of the two equations. Similar question has been addressed in [34] for the chemotaxis system of parabolic-elliptic type which is specially known as Keller-Segel model. Both papers utilized the Kakutani’s fixed-point theorem to handle the nonlinearity. In [48], a local interior null-controllability of such system with local and nonlocal nonlinearity has been established using appropriate Carleman estimates and fixed point argument. The paper [22] dealt with the same issue for a system coupling two parabolic models and an elliptic model with one scalar control force, locally supported in space. These works principally motivate us to study the null-controllability of a parabolic-elliptic system of coupled Kuramoto-Sivashinsky-Korteweg-de Vries equation and a second order elliptic equation; and to the best of our knowledge, this question has not yet been arose in the literature of control theory for pdes.

1.2. The systems under study. Let $T > 0$ be any finite time and define $Q_T := (0, T) \times (0, 1)$. We also take any non-empty open set $\omega \subset (0, 1)$. Then, we consider the following control problems

$$\begin{align*}
    \begin{cases}
        u_t + \gamma_1 u_{xxxx} + u_{xxx} + \gamma_2 u_{xx} + uu_x = av + \chi_\omega h, & \text{in } Q_T, \\
-v_{xx} + cv = bu, & \text{in } Q_T,
    \end{cases}
\end{align*}$$

with the constants $(a, b) \in \mathbb{R}^2$ such that $b \neq 0$; or

$$\begin{align*}
    \begin{cases}
        u_t + \gamma_1 u_{xxxx} + u_{xxx} + \gamma_2 u_{xx} + uu_x = av, & \text{in } Q_T, \\
-v_{xx} + cv = bu + \chi_\omega h, & \text{in } Q_T,
    \end{cases}
\end{align*}$$

with $a \neq 0$. Here $h$ is some control function (to be determined) localized in $\omega$, acting either through the KS-KdV equation or elliptic equation; $\gamma_1, \gamma_2 > 0$ denote the coefficients accounting the long wave instabilities and the short wave dissipation respectively. Finally, we consider the velocity $c > \pi^2$, the first eigenvalue of $\partial_x x$ with homogeneous Dirichlet boundary conditions, (one can find similar condition on $c$ in [23], [15]).

The boundary conditions for the state components are given by

$$\begin{align*}
    \begin{cases}
        u(t, 0) = 0, & \text{in } (0, T), \\
        u_x(t, 0) = 0, & \text{in } (0, T), \\
        v(t, 0) = 0, & \text{in } (0, T), \\
        v_x(t, 1) = 0, & \text{in } (0, T),
    \end{cases}
\end{align*}$$

and the initial condition is

$$u(0, x) = u_0(x), \quad x \in (0, 1).$$

We now prescribe the main results of our present work for given initial data $u_0 \in H^{-1}(0, 1)$ in some ball.
1.2.1. Controllability results of the nonlinear systems.

**Theorem 1.1** (Control on KS-KdV equation). Let be \((a, b) \in \mathbb{R}^2\) with \(b \neq 0\). Then, the system (1.3)–(1.5)–(1.6) is small-time locally null-controllable around the equilibrium \((0, 0)\), that is to say, for any given time \(T > 0\), there is a \(R > 0\) such that for chosen initial data \(u_0 \in H^{-1}(0, 1)\) with \(\|u_0\|_{H^{-1}(0, 1)} \leq R\), there exists a control \(h \in L^2((0, T) \times \omega)\) such that the associated solution \((u, v)\) satisfies

\[
(u(T, x), v(T, x)) = (0, 0), \quad \forall x \in (0, 1).
\]

Our next main result is the following: when a control force is acting only on the elliptic equation.

**Theorem 1.2** (Control on elliptic equation). Let be \((a, b) \in \mathbb{R}^2\) with \(a \neq 0\). Then, the system (1.4)–(1.5)–(1.6) is small-time locally null-controllable around the equilibrium \((0, 0)\), that is, for any given time \(T > 0\), there is a \(R > 0\) such that for chosen initial data \(u_0 \in H^{-1}(0, 1)\) with \(\|u_0\|_{H^{-1}(0, 1)} \leq R\), there exists a control \(h \in L^2((0, T) \times \omega)\) such that the associated solution \((u, v)\) satisfies

\[
u(T, x) = 0, \quad \forall x \in (0, 1), \quad \limsup_{t \to T^-} \|v(t, \cdot)\|_{H^{-1}(0, 1)} = 0.
\]

**Strategy of proofs.**

- First, we shall prove the global interior null-controllability results of the associated linearized models, given by

\[
\begin{align*}
\begin{cases}
u_t + \gamma_1 u_{xxxx} + u_{xxx} + \gamma_2 u_{xx} = au + \chi \omega h, & \text{in } Q_T, \\
v_{xx} + cu = bu, & \text{in } Q_T,
\end{cases}
\end{align*}
\]

or,

\[
\begin{align*}
\begin{cases}
u_t + \gamma_1 u_{xxxx} + u_{xxx} + \gamma_2 u_{xx} = au, & \text{in } Q_T, \\
v_{xx} + cu = bu + \chi \omega h, & \text{in } Q_T,
\end{cases}
\end{align*}
\]

with the boundary and initial conditions as given by (1.5) and (1.6) respectively.

The global Carleman estimate will help us to prove the null-controllability of the above linearized systems with a proper control cost \(K e^{K/T}\). This is crucial to deduce the local controllability results for the non-linear models.

- Next, we shall apply the source term method introduced in [44]; more precisely, we prove the null-controllability of our linearized models with additional source term (in the KS equation) from some suitable Hilbert space which is exponentially decreasing while \(t \to T^-\).

- After that, we use the Banach fixed-point argument to obtain the local null-controllability for the non-linear models.

1.2.2. Controllability results of the associated linearized systems.

**Theorem 1.3.** Let any \(u_0 \in H^{-2}(0, 1)\) and \(T > 0\) be given and \((a, b) \in \mathbb{R}^2\). Then, we have the following.

1. When \(b \neq 0\), there exists a control \(h \in L^2((0, T) \times \omega)\) such that the system (1.7)–(1.5)–(1.6) is null-controllable at time \(T\), that is,

\[
u(T, x) = 0, \quad \forall x \in (0, 1).
\]

2. Similarly when \(a \neq 0\), there exists a control \(h \in L^2((0, T) \times \omega)\) such that the solution of (1.8)–(1.5)–(1.6) satisfies

\[
u(T, x) = 0, \quad \forall x \in (0, 1), \quad \limsup_{t \to T^-} \|v(t, \cdot)\|_{H^{-2}(0, 1)} = 0.
\]

In both cases, the control function \(h\) has the following estimate:

\[
\|h\|_{L^2((0, T) \times \omega)} \leq K e^{K/T\|u_0\|_{H^{-2}(0, 1)}},
\]

where the constant \(K > 0\) depends neither on \(T\) nor on \(u_0\).

1.3. An asymptotic control system. In the present work, we also consider a system of two parabolic equations of the form

\[
\begin{align*}
\begin{cases}
u_t + \gamma_1 u_{xxxx} + u_{xxx} + \gamma_2 u_{xx} + uu_x = av, & \text{in } Q_T, \\
v_{xx} + cu = bu + \chi \omega h, & \text{in } Q_T, \\
u(t, 0) = 0, & \text{in } (0, T), \\
u_x(t, 0) = 0, & t \in (0, T), \\
v(t, 1) = 0, & \text{in } (0, T), \\
v_x(t, 1) = 0, & t \in (0, T), \\
u(0, x) = u_0(x), & v(0, x) = v_0(x), \quad x \in (0, 1),
\end{cases}
\end{align*}
\]

with given initial data \((u_0, v_0)\) from some suitable Hilbert space and a localized control acting only in the heat equation. Here, \(\varepsilon > 0\) is a degenerating parameter and our aim is to show the uniform local null-controllability.
1.1. The system (1.10) is devoted to study the null-controllability of our parabolic-elliptic model where a localized force acts in the elliptic equation only. In this regard, we mention the work [15], [23], [16], [17] for the system of two parabolic (second order) equations.

Remark 1.4. The parabolic-elliptic control system (1.4)–(1.5)–(1.6) (with a control in the elliptic equation) can be viewed as an asymptotic limit of a two parabolic equations, namely the stabilized Kuramoto-Sivashinsky equation with the degenerating parameter $\varepsilon$.

Let us write the main result concerning this case.

Theorem 1.5. Let be $(a, b) \in \mathbb{R}^2$ with $a \neq 0$ and $0 < \varepsilon \leq 1$. Then, the system (1.10) is small-time locally uniformly null-controllable w.r.t. $\varepsilon$ around the equilibrium $(0, 0)$, that is to say, for any given time $T > 0$, there is a $R > 0$ such that for given initial data $(u_0, v_0) \in [H^{-1}(0, 1)]^2$ with $\| (u_0, v_0) \|_{H^{-1}(0, 1)^2} \leq R$, there exists a control $h_\varepsilon \in L^2((0, T) \times \omega)$ such that the associated solution $(u_\varepsilon, v_\varepsilon)$ satisfies

$$u_\varepsilon(T, x) = 0, \quad v_\varepsilon(T, x) = 0, \quad \forall x \in (0, 1).$$

The corresponding linear system to (1.10) reads as

$$\begin{align*}
u_t + \gamma_1 u_{xxx} + u_{xxx} + \gamma_2 u_{xx} &= av, \quad \text{in } Q_T, \\
\varepsilon v_t - v_{xx} + cv &= bu + \chi_0 h, \quad \text{in } Q_T, \\
u(0, x) = 0, \quad u(t, 1) = 0, \quad t \in (0, T), \\
\v(t, 0) = 0, \quad v(t, 1) = 0, \quad t \in (0, T), \\
u(0, x) = u_0(x), \quad v(0, x) = v_0(x), \quad x \in (0, 1),
\end{align*}
$$

(1.11)

with given initial data $(u_0, v_0) \in H^{-2}(0, 1) \times H^{-1}(0, 1)$ and any parameter $\varepsilon > 0$.

We have the following result regarding the null-controllability of (1.11).

Theorem 1.6. Let any $(u_0, v_0) \in H^{-2}(0, 1) \times H^{-1}(0, 1)$ and $T > 0$ be given. We also take $(a, b) \in \mathbb{R}^2$ with $a \neq 0$ and $0 < \varepsilon \leq 1$. Then, there exists a control $h_\varepsilon \in L^2((0, T) \times \omega)$ such that the system (1.11) is null-controllable at time $T$ and in addition, we have the following estimate

$$\| h_\varepsilon \|_{L^2((0, T) \times \omega)} \leq K_\varepsilon \varepsilon^{\frac{1}{T}} \left( \| u_0 \|_{H^{-2}(0, 1)} + \varepsilon \| v_0 \|_{H^{-1}(0, 1)} \right),$$

(1.12)

where the constant $K > 0$ does not depend on $T$, $(u_0, v_0)$ and $\varepsilon$.

Notations. Throughout the paper, $C$ denotes the generic positive constant which may depend on $T$, $u_0$, $\gamma_1$, $\gamma_2$, $a, b, c$ and change from line to line. Other constant terms will be specified in the places of their appearance. We sometimes use the short notations $H^m(X)$ or $C^0(X)$ to denote the spaces $H^m(0, T; X)$ and $C^0([0, T]; X)$ respectively, for some $m \in \mathbb{Z}$ and $X$ is a Lebesgue space.

Paper organization. The paper is organized as follows:

- In Section 2, we roughly discuss the well-posedness of the linearized control systems.
- Section 3 is devoted to study the null-controllability of our parabolic-elliptic model where a localized control is acting on the KS-KdV equation. Thereafter, in Section 4, we study the case when a control force acts in the elliptic equation only. In both cases, the controllability of the associated linearized models will be established by global Carleman estimates and then using the source term method followed by a fixed point argument, we shall study the nonlinear systems; see Section 3.3 for more details.
- In Section 5, we discuss about the controllability of an asymptotic control system (1.10) and in this case, we will show the existence of a control (acting on the second order pde) with a uniform estimate that allows us to pass to the limit as $\varepsilon \to 0$ to retrieve the associated parabolic-elliptic coupled system, precisely the system (1.8)–(1.5)–(1.6) with a control $h$ (acting in the elliptic equation) which is a weak limit of the sequence $(h_\varepsilon)_\varepsilon$ as $\varepsilon \to 0$.
- Finally, we conclude our paper with some remarks and open questions related to the KS-KdV-elliptic system, given by Section 6.

2. Well-posedness of the linearized systems

This section is devoted to discuss the well-posedness of the linear systems (1.7)/(1.8)–(1.5)–(1.6). Let us first write the system (1.7) in infinite dimensional ODE setup:

$$\begin{align*}
u_t &= Au + Bh, \quad t \in (0, T), \\
u(0) &= u_0,
\end{align*}
$$

(2.1)
where the operators \( A, B \) are given by
\[
\begin{aligned}
Au &= -\gamma_1 u_{xxxx} - u_{xxx} - \gamma_2 u_{xx} + av, \\
-\varepsilon_{xx} + cv &= bu, \text{ with } \nu(0) = 0, \nu(1) = 0,
\end{aligned}
\tag{2.2}
\]
and
\[
D(A) = \{ u \in H^4(0,1) : u(0) = u(1) = u_x(0) = u_x(1) = 0 \}.
\]
The control operator \( B \in \mathcal{L} (\mathbb{R}, L^2(0,1)) \) can be defined as
\[
B^* h(t) = \chi_\omega h(t, \cdot).
\]
Next, one can find the adjoint operator \( A^* \) of \( A \) with \( D(A^*) = D(A) \) as follows:
\[
\begin{aligned}
A^* \sigma &= -\gamma_1 \sigma_{xxxx} + \sigma_{xxx} - \gamma_2 \sigma_{xx} + b \psi, \\
-\psi_{xx} + cv &= a \sigma, \text{ with } \psi(0) = 0, \psi(1) = 0.
\end{aligned}
\tag{2.3}
\]
Therefore, one has the following adjoint system to (1.7)/(1.8)–(1.5)–(1.6), given by
\[
\begin{aligned}
-\sigma_t + \gamma_1 \sigma_{xxxx} - \sigma_{xxx} + \gamma_2 \sigma_{xx} = b \psi + g, & \quad (t, x) \in Q_T, \\
-\psi_{xx} + cv &= a \sigma, & \quad (t, x) \in Q_T, \\
\sigma(t, 0) = 0, & \quad \sigma(t, 1) = 0, \quad t \in (0, T), \\
\sigma_x(t, 0) = 0, & \quad \sigma_x(t, 1) = 0, \quad t \in (0, T), \\
\psi(t, 0) = 0, & \quad \psi(t, 1) = 0, \quad t \in (0, T), \\
\sigma(T, x) = \sigma_T(x), & \quad x \in (0, 1),
\end{aligned}
\tag{2.4}
\]
with given final data \( \sigma_T \in H^2_0(0,1) \) and right hand sides \( g \in L^2(0, T; L^2(0,1)) \).

With these, we have the following proposition.

**Proposition 2.1.** For given \( \sigma_T \in H^2_0(0,1) \) and \( g \in L^2(0, T; L^2(0,1)) \) or \( g \in L^1(0, T; H^2_0(0,1)) \), there exists unique solution \((\sigma, \psi)\) to (2.4) such that
\[
\sigma \in C^0([0,T]; H^2_0(0,1)) \cap L^2(0,T; H^4(0,1)) \cap H^1(0,T; L^2(0,1)),
\]
\[
\psi \in L^2(0,T; H^2(0,1) \cap H^1_0(0,1)),
\]
and in addition, they satisfy the following estimates
\[
\| \sigma \|_{C^0(H^2_0 \cap H^4 \cap L^2(0,1))} \leq C \left( \| \sigma_T \|_{H^2_0(0,1)} + \| g \|_{L^2(L^2)} \right),
\tag{2.5a}
\]
\[
\| \psi \|_{L^2(H^2 \cap L^2)} \leq C \left( \| \sigma_T \|_{H^2_0(0,1)} + \| g \|_{L^2(L^2)} \right),
\tag{2.5b}
\]
or
\[
\| \sigma \|_{C^0(H^2_0 \cap L^2(0,1))} \leq C \left( \| \sigma_T \|_{H^2_0(0,1)} + \| g \|_{L^2(H^2_0)} \right),
\tag{2.6a}
\]
\[
\| \psi \|_{L^2(H^2 \cap H^1_0)} \leq C \left( \| \sigma_T \|_{H^2_0(0,1)} + \| g \|_{L^1(H^2_0)} \right),
\tag{2.6b}
\]
where \( C > 0 \) is some constant that may depend on \( a, b, c, \gamma_1, \gamma_2 \) and \( T \).

**Proof.** We start the proof with the estimate of the elliptic equation. Multiplying the second equation of (2.4) by \( \psi \), then integrating by parts and considering the homogeneous boundary conditions on \( \psi \), we obtain
\[
\int_0^1 \psi^2_t + c \int_0^1 \psi^2 = a \int_0^1 \sigma \psi.
\]
Recall the Poincaré inequality with best constant:
\[
\int_0^1 |\phi|^2 \leq \frac{1}{\pi^2} \int_0^1 |\phi'|^2, \quad \forall \phi \in H^1_0(0,1).
\tag{2.7}
\]
Then, using the Cauchy-Schwartz and Young’s inequality together with (2.7), we deduce
\[
\| \psi(t, \cdot) \|_{H^1_0(0,1)} \leq C \| \sigma(t, \cdot) \|_{L^2(0,1)}.
\]
Next, multiplying the second equation of (2.4) by \( \psi_{xx} \) and performing the same business as above, we get the following estimate:
\[
\| \psi(t, \cdot) \|_{H^2(0,1) \cap H^1_0(0,1)} \leq C \| \sigma(t, \cdot) \|_{L^2(0,1)}^2, \quad \text{for a.e. } t \in [0, T].
\]
Then, integrating on \([0, T]\), we have
\[
\| \psi \|_{L^2(0,T; H^2(0,1) \cap H^1_0(0,1))} \leq C \| \sigma \|_{L^2(0,T; L^2(0,1))}^2.
\]
To obtain the regularity result for \( \sigma \), let us make a change of variable from \( t \) to \( T - t \) so that the first equation of (2.4) becomes forward in time. Then, testing the equation against \( \sigma_{xxxx} \), we get
\[
\int_0^1 \sigma \sigma_{xxxx} + \gamma_1 \int_0^1 |\sigma_{xxxx}|^2 = \int_0^1 (b \psi + \sigma_{xxxx} - \gamma_2 \sigma_{xx} + g) \sigma_{xxxx}.
\] (2.8)

Performing consecutive integration by parts in the first integral and using the Young’s inequality in the right hand side of (2.8), leads to
\[
\frac{d}{dt} \int_0^1 |\sigma_{xx}|^2 + \gamma_1 \int_0^1 |\sigma_{xxxx}|^2 \leq \epsilon \int_0^1 |\sigma_{xxxx}|^2 + C \epsilon \int_0^1 (|\psi|^2 + |\sigma_{xxxx}|^2 + |\sigma_{xx}|^2 + |g|^2).
\]

Using the estimate of \( \psi \) from (2.5b) and choosing \( \epsilon = \gamma_1/2 \), we get
\[
\frac{d}{dt} \int_0^1 |\sigma_{xx}|^2 + \gamma_1 \int_0^1 |\sigma_{xxxx}|^2 \leq C \int_0^1 (|\sigma|^2 + |\sigma_{xxxx}|^2 + |\sigma_{xx}|^2 + |g|^2).
\] (2.9)

Thanks to the Ehrling’s lemma, we have for \( \epsilon > 0 \),
\[
\int_0^1 |\sigma_{xxxx}|^2 \leq \epsilon \int_0^1 |\sigma_{xxxx}|^2 + C(\epsilon) \int_0^1 |\sigma|^2,
\]
and applying this in (2.9) for \( \epsilon > 0 \) small enough and the Poincaré inequality, we have
\[
\frac{d}{dt} \int_0^1 |\sigma_{xx}|^2 + \int_0^1 |\sigma_{xxxx}|^2 \leq C \int_0^1 (|\sigma_{xx}|^2 + |g|^2).
\] (2.10)

Then the Gronwall’s lemma yields to
\[
||\sigma||_{L^\infty(0,T;H_0^2(0,1))} \leq C \left( ||\sigma_T||_{H_0^2(0,1)} + ||g||_{L^2(0,T;L^2(0,1))} \right).
\]

Next, integrating both sides of the inequality (2.10) with respect to \( (0,t) \) and taking supremum on the both sides for \( t \in [0,T] \), we shall obtain the \( L^2(0,T;H^1(0,1)) \) estimate of \( \sigma \) and finally from the equation of \( \sigma \), we obtain \( \sigma_1 \in L^2(0,T;L^2(0,1)) \). The estimate of \( \sigma_1 \) can be obtained using the \( L^\infty(H_0^1) \) and \( L^2(H^1) \) estimations of \( \sigma \). This ensures that \( \sigma \in C^0([0,T];H_0^2(0,1)) \).

On the other hand, assuming \( g \in L^1(0,T;H_0^2(0,1)) \), one can obtain the estimates (2.6a) and (2.6b).

**Remark 2.2.** Note that in (2.4), the component \( \omega \) belongs to the space \( C^0([0,T];H^1(0,1)) \) and \( H_0^2(0,1) \).

Next let us give the well-posedness of the linearized system (1.7)–(1.5)–(1.6) and (1.8)–(1.5)–(1.6). We only write the details for the system (1.7)–(1.5)–(1.6). For a second order parabolic-elliptic control problem, similar well-posedness results have been appeared in [36].

**Definition 2.3.** Let be \( u_0 \in H^{-2}(0,1) \) and \( h \in L^2(0,T;H^{-2}(0,1)) \). Then, \( u,v \in [L^2(0,T;L^2(0,1))]^2 \) is said to be a solution to the system (1.7)–(1.5)–(1.6) if for any \( g \in L^2(0,T;L^2(0,1)) \) the following holds:
\[
\int_0^T 0 1 u(t,x)g(t,x) = \langle u_0, \sigma(0,x) \rangle_{H^{-2}(0,1),H_0^2(0,1)} + \langle h, \sigma \rangle_{L^2(0,T;H^{-2}(0,1)),L^2(0,T;H_0^2(0,1))},
\] (2.11)
where \( (\sigma, \psi) \) is a solution of the adjoint system (2.4) with \( \sigma_T = 0 \).

Moreover, we have the following result.

**Theorem 2.4.** Let \( u_0 \in H^{-2}(0,1) \) and \( h \in L^2(0,T;H^{-2}(0,1)) \). Then the system (1.7)–(1.5)–(1.6) has a unique solution \( (u,v) \in [C^0([0,T];H^{-2}(0,1))]^2 \cap [L^2(0,T;L^2(0,1))]^2 \) with the following estimate:
\[
||u,v||_{[C^0([0,T];H^{-2}(0,1))]^2} + ||(u,v)||_{[L^2(0,T;L^2(0,1))]^2} \leq C \left( ||u_0||_{H^{-2}(0,1)} + ||h||_{L^2(0,T;H^{-2}(0,1))} \right).
\]

**Proof.** Let us define a linear map \( \mathcal{L} : L^2(0,T;L^2(0,1)) \to \mathbb{R} \) by
\[
\mathcal{L}(g) = \langle u_0, \sigma(0,x) \rangle_{H^{-2}(0,1),H_0^2(0,1)} + \langle h, \sigma \rangle_{L^2(0,T;H^{-2}(0,1)),L^2(0,T;H_0^2(0,1))},
\] (2.12)
where \( (\sigma, \psi) \) is the solution of the equation (2.4). Thanks to Proposition 2.1, \( \mathcal{L} \) is a continuous linear functional.

Thus, by the Riesz-representation theorem, we have a unique \( (u,v) \in [L^2(0,T;L^2(0,1))]^2 \) such that (2.11) holds.

Next, if we define the map \( \mathcal{L} : L^1(0,T;H_0^2(0,1)) \to \mathbb{R} \) by (2.12), then using estimate (2.6a), we get the continuity of \( \mathcal{L} \). This yields that \( u \in L^\infty(0,T;H^{-2}(0,1)) \) satisfying (2.11) and consequently, \( v \in L^\infty(0,T;H^{-2}(0,1)) \). Indeed, we have the following estimate
\[
||u,v||_{L^\infty(0,T;H^{-2}(0,1))} \leq C \left( ||u_0||_{H^{-2}(0,1)} + ||h||_{L^2(0,T;H^{-2}(0,1))} \right).
\]
Then, the standard density argument will give the desired result.
Analogous to the previous theorem, one can prove the following result when a control is acting in the elliptic equation.

**Theorem 2.5.** Let \( u_0 \in H^{-2}(0,1) \) and \( h \in L^2(0,T;H^{-2}(0,1)) \). Then the system (1.8)–(1.5)–(1.6) has a unique solution \((u,v)\) such that \( u \in C^0([0,T];H^{-2}(0,1)) \cap L^2(0,T;L^2(0,1)), v \in L^2(0,T;L^2(0,1)) \), and in addition, one has

\[
\|u\|_{C^0([0,T];H^{-2}(0,1)) \cap L^2(0,T;L^2(0,1))} + \|v\|_{L^2(0,T;L^2(0,1))} \leq C \left( \|u_0\|_{H^{-2}(0,1)} + \|h\|_{L^2(0,T;H^{-2}(0,1))} \right).
\]

3. Control acting in the KS-KdV equation

This section is devoted to study the controllability of our coupled system when a localized control is exerted in the KS-KdV equation. We start by proving the null-controllability of the associated linearized model and as mentioned earlier, using the source term method and a fixed point argument, we will give the required local null-controllability for the nonlinear system.

3.1. A global Carleman estimate. We shall utilize the Carleman technique to deduce the null-controllability of the linearized system.

To do so, let us define the some useful weight functions. Consider a non-empty open set \( \omega_0 \subset \subset \omega \). There exists a function \( \nu \in C^\infty([0,1]) \) such that

\[
\begin{align*}
\nu(x) & > 0 \quad \forall x \in (0,1), \quad \nu(0) = \nu(1) = 0, \\
|\nu'(x)| & \geq \tilde{c} > 0 \quad \forall x \in (0,1) \setminus \omega_0 \quad \text{for some } \tilde{c} > 0.
\end{align*}
\]

(3.1)

It is clear that \( \nu'(0) > 0 \) and \( \nu'(1) < 0 \). We refer [28] where the existence of such constant has been addressed.

Now, for any constant \( k > 1 \) and parameter \( \lambda > 0 \), we define the weight functions

\[
\varphi(t,x) = \frac{e^{2\lambda k\|\nu\|_\infty} - e^{\lambda(k\|\nu\|_\infty + \nu(x))}}{t(T-t)}, \quad \xi(t,x) = \frac{e^{\lambda(k\|\nu\|_\infty + \nu(x))}}{t(T-t)}, \quad \forall (t,x) \in Q_T.
\]

(3.2)

We have defined the above weight functions by means of the works [51] and [33]. It is clear that both \( \varphi \) and \( \xi \) are positive functions in \( Q_T \).

Some immediate results associated with the weight functions are the followings.

- For any \( l \in \mathbb{N}^* \) and \( s > 0 \), there exists some \( C > 0 \) such that
  \[
  |(e^{-2s\varphi}\xi^l)| \leq Cse^{-2s\varphi}\xi^{l+2}.
  \]
  (3.3)

- For any \((l,n) \in \mathbb{N}^* \times \mathbb{N}^* \) and \( s > 0 \), there exists some \( C > 0 \) such that
  \[
  |(e^{-2s\varphi}\xi^l)_{n,x}| \leq Cs^l\lambda^n e^{-2s\varphi}\xi^{l+n}.
  \]
  (3.4)

Some useful notations. We also declare the following notations which will simplify the expressions of our Carleman inequalities.

- For any \( \sigma \in C^2([0,T];C^4([0,1])) \) and positive parameters \( s, \lambda \), we denote
  \[
  I_{KS}(s,\lambda;\sigma) := s^7\lambda^8 \int_{Q_T} e^{-2s\varphi}\xi^3|\sigma|^2 + s^5\lambda^6 \int_{Q_T} e^{-2s\varphi}\xi^5|\sigma_x|^2 + s^3\lambda^4 \int_{Q_T} e^{-2s\varphi}\xi^3|\sigma_{xx}|^2
  + s\lambda^2 \int_{Q_T} e^{-2s\varphi}\xi|\sigma_{xxx}|^2 + s^{-1} \int_{Q_T} e^{-2s\varphi}\xi^{-1}(|\sigma|^2 + |\sigma_{xxx}|^2).
  \]
  (3.5)

- For any function \( \psi \in C^2(Q_T) \) and positive parameters \( s, \lambda \), we denote
  \[
  I_E(s,\lambda;\psi) := s^3\lambda^4 \int_{Q_T} e^{-2s\varphi}\xi^3|\psi|^2 + s\lambda^2 \int_{Q_T} e^{-2s\varphi}\xi|\psi_x|^2 + s^{-1} \int_{Q_T} e^{-2s\varphi}\xi^{-1}|\psi_{xx}|^2.
  \]
  (3.6)

With help of the above notations, we now prescribe the Carleman estimate satisfied by the solution \((\sigma,\psi)\) to the following adjoint system (of (1.7)/(1.8)–(1.5)–(1.6))

\[
\begin{align*}
-\sigma_x + \gamma_1\sigma_{xxx} - \sigma_{xxxx} + \gamma_2\sigma_{xx} &= b\psi, & (t,x) \in Q_T, \\
-\psi_{xx} + c\psi &= a\sigma, & (t,x) \in Q_T, \\
\sigma(t,0) &= 0, & \sigma(t,1) = 0, & t \in (0,T), \\
\sigma_x(t,0) &= 0, & \sigma_x(t,1) = 0, & t \in (0,T), \\
\psi(t,0) &= 0, & \psi(t,1) = 0, & t \in (0,T), \\
\sigma(T,x) &= \sigma_T(x), & x \in (0,1),
\end{align*}
\]

(3.7)

with given final data \( \sigma_T \in H^2_0(0,1) \).
Theorem 3.1 (Carleman estimate: control in KS-KdV eq.). Let the weight functions \( (\varphi, \xi) \) be given by (3.2). Then, there exist positive constants \( \lambda_0, s_0 := \mu_0(T + T^2) \) for some \( \mu_0 > 0 \) and \( C \) which depend on \( \gamma_1, \gamma_2, a, b, c \) and the set \( \omega \), such that we have the following estimate satisfied by the solution to (3.7),

\[
I_{KS}(s, \lambda; \sigma) + I_E(s, \lambda; \psi) \leq C S^{15} \lambda^{16} \int_0^T \int_\omega e^{-2s \varphi} \xi^5 |\sigma|^2,
\]

for all \( \lambda \geq \lambda_0 \) and \( s \geq s_0 \), where \( I_{KS}(s, \lambda; \sigma) \) and \( I_E(s, \lambda; \psi) \) are introduced in (3.5) and (3.6) respectively.

Before going to the proof of the above Carleman inequality, let us write the individual Carleman inequalities for the components \( \sigma \) and \( \psi \).

Proposition 3.2. Let the weight functions \( (\varphi, \xi) \) be given by (3.2). Then, there exist positive constants \( \lambda_1, s_1 := \mu_1(T + T^2) \) for some \( \mu_1 > 0 \) and \( C \) which depend on \( \gamma_1, \gamma_2, a, b, c \) and the set \( \omega \), such that we have the following estimate satisfied by the solution component \( \sigma \) of (3.7),

\[
I_{KS}(s, \lambda; \sigma) \leq C \int \int_{Q_T} e^{-2s \varphi} |\psi|^2 + C s^8 \lambda^8 \int_0^T \int_{\omega_0} e^{-2s \varphi} \xi^5 |\sigma|^2,
\]

for all \( \lambda \geq \lambda_1 \) and \( s \geq s_1 \), and \( I_{KS}(s, \lambda; \sigma) \) is introduced by (3.5).

The proof of above such Carleman estimate is initially established in [51]. We also refer [14] where a similar Carleman estimate has been addressed.

Let us also write the following elliptic Carleman estimate (see for instance [28]) satisfied by \( \psi \).

Proposition 3.3. Let the weight functions \( (\varphi, \xi) \) be given by (3.2). Then, there exist positive constants \( \lambda_2, s_2 := \mu_2(T + T^2) \) with some \( \mu_2 > 0 \) and \( C \) which depend on \( \gamma_1, \gamma_2, a, b, c \) and the set \( \omega \), such that we have the following estimate satisfied by the solution component \( \psi \) of (3.7),

\[
I_E(s, \lambda; \psi) \leq C \int \int_{Q_T} e^{-2s \varphi} |\sigma|^2 + C s^8 \lambda^4 \int_0^T \int_{\omega_0} e^{-2s \varphi} \xi^5 |\psi|^2,
\]

for all \( \lambda \geq \lambda_2 \) and \( s \geq s_2 \), where \( I_E(s, \lambda; \psi) \) is introduced by (3.6).

With help of the above two Carleman estimates (3.9) and (3.10), we are now ready to prove the main Carleman estimate (3.8).

Proof of Theorem 3.1. Let us add both the Carleman estimates (3.9) and (3.10), we have

\[
I_{KS}(s, \lambda; \sigma) + I_E(s, \lambda; \psi) \leq C \left[ \int \int_{Q_T} e^{-2s \varphi} |\psi|^2 + \int \int_{Q_T} e^{-2s \varphi} |\sigma|^2 \right] + s^7 \lambda^8 \int_0^T \int_{\omega_0} e^{-2s \varphi} \xi^5 |\sigma|^2 + s^3 \lambda^4 \int_0^T \int_{\omega_0} e^{-2s \varphi} \xi^5 |\psi|^2,
\]

for all \( \lambda \geq \lambda_0 := \max\{\lambda_1, \lambda_2\} \) and \( s \geq \tilde{\mu}(T + T^2) \) with \( \tilde{\mu} := \max\{\mu_1, \mu_2\} \).

Step 1. Absorbing the lower order integrals. Observe that \( 1 \leq CT^2 \xi \) for some \( C > 0 \). This yields

\[
\int \int_{Q_T} e^{-2s \varphi} |\psi|^2 + \int \int_{Q_T} e^{-2s \varphi} |\sigma|^2 \leq CT^6 \int \int_{Q_T} e^{-2s \varphi} \xi^3 |\psi|^2 + CT^{14} \int \int_{Q_T} e^{-2s \varphi} \xi^7 |\sigma|^2.
\]

Then by choosing \( s \geq \tilde{C} T^2 \) for some constant \( \tilde{C} > 0 \), the integrals in the right hand side of (3.12) can be absorbed by the associated leading integrals appearing in the left hand side of (3.11). This leads to the following

\[
I_{KS}(s, \lambda; \sigma) + I_E(s, \lambda; \psi) \leq C \left[ s^7 \lambda^8 \int_0^T \int_{\omega_0} e^{-2s \varphi} \xi^5 |\sigma|^2 + s^3 \lambda^4 \int_0^T \int_{\omega_0} e^{-2s \varphi} \xi^5 |\psi|^2 \right],
\]

for all \( \lambda \geq \lambda_0 \) and \( s \geq \mu_0(T + T^2) \) for some \( \mu_0 \geq \tilde{\mu} \).

Step 2. Absorbing the observation integral in \( \psi \). Consider \( \omega_0 \) and another non-empty open set \( \omega_1 \) in such a way that \( \omega_0 \subset \omega_1 \subset \omega \). Then, we consider the function

\[
\phi \in C^\infty_0(\omega_1) \quad \text{with} \quad 0 \leq \phi \leq 1, \quad \text{and} \quad \phi = 1 \quad \text{in} \quad \omega_0.
\]

(3.14)

Now, recall the adjoint system (3.7), one has (since \( b \neq 0 \)),

\[
\psi = \frac{1}{b} \left( -\sigma_t + \gamma_1 \sigma_{xxx} - \sigma_{xxx} + \gamma_2 \sigma_{xx} \right), \quad \text{in} \quad Q_T.
\]
Using this, we have
\[ s^3 \lambda^4 \int_0^T \int_{\omega_0} e^{-2s \varphi \xi_3^2} |\psi|^2 \leq s^3 \lambda^4 \int_0^T \int_{\omega_1} \phi e^{-2s \varphi \xi_3^2} |\psi|^2 \]
\[ = \frac{1}{b} s^3 \lambda^4 \int_0^T \int_{\omega_1} \phi e^{-2s \varphi \xi_3^3} \psi \left(-\sigma_t + \gamma_1 \sigma_{xxx} - \sigma_{xx} + \gamma_2 \sigma_{xx} \right) \]
\[ = A_1 + A_2 + A_3 + A_4. \quad (3.15) \]

(i) Let us start with the following.
\[ A_1 := -\frac{1}{b} s^3 \lambda^4 \int_0^T \int_{\omega_1} \phi e^{-2s \varphi \xi_3^3} \psi \sigma_t \]
\[ = \frac{1}{b} s^3 \lambda^4 \int_0^T \int_{\omega_1} \phi e^{-2s \varphi \xi_3^3} \psi \sigma_t + \frac{1}{b} s^3 \lambda^4 \int_0^T \int_{\omega_1} \phi e^{-2s \varphi \xi_3^3} \psi \sigma_t. \quad (3.16) \]

In above, we perform an integration by parts with respect to \( t \). There is no boundary terms since the weight function \( e^{-2s \varphi} \) vanishes near \( t = 0 \) and \( T \).

Now, recall the fact (3.3), so that the first integral in the right hand side of (3.16) gives
\[ \left| \frac{1}{b} s^3 \lambda^4 \int_0^T \int_{\omega_1} \phi (e^{-2s \varphi \xi_3^3}) \sigma_t |\psi| \sigma \right| \leq C s^4 \lambda^4 \int_0^T \int_{\omega_1} e^{-2s \varphi \xi_3^3} \left| \sigma \right|^2, \]
\[ \leq C s^4 \lambda^4 \int_0^T \int_{\omega_1} e^{-2s \varphi \xi_3^3} |\psi|^2 + \frac{C}{\epsilon} s^7 \lambda^8 \int_0^T \int_{\omega_1} e^{-2s \varphi \xi_3^3} |\sigma|^2, \quad (3.17) \]
for any \( \epsilon > 0 \), where we have used the Cauchy-Schwarz and Young’s inequalities.

To estimate the last integral term in (3.16), we need to differentiate the second equation of (3.7) with respect to \( t \) and that yields
\[ \begin{cases} 
- (\psi_{\xi})_{xx} + \psi_{\xi} = a \sigma_t, & \text{in } QT, \\
\psi_{t}(t,0) = \psi_{t}(t,1) = 0, & t \in (0,T). \end{cases} \quad (3.18) \]

Let us estimate the last integral in (3.16) as
\[ \left| \frac{1}{b} s^3 \lambda^4 \int_0^T \int_{\omega_1} \phi e^{-2s \varphi \xi_3^3} \psi \sigma_t \right| \leq C s^3 \lambda^4 \int_0^T \int_{\omega_1} e^{-2s \varphi \xi_3^3} \left| (-\partial_{xx} + cI)^{-1} \sigma_t \right| |\sigma| \]
\[ \leq C s^8 \lambda^4 \int_0^T \int_{\omega_1} e^{-2s \varphi \xi_3^3} |\sigma_t|^2 + \frac{C}{\epsilon} s^7 \lambda^8 \int_0^T \int_{\omega_1} e^{-2s \varphi \xi_3^3} |\sigma|^2, \quad (3.19) \]
for any \( \epsilon > 0 \), where we have used the Cauchy-Schwarz and Young’s inequalities and the fact that the operator \((-\partial_{xx} + cI)^{-1}\) is bounded. Thus, we eventually have
\[ |A_1| \leq C s^3 \lambda^4 \int_0^T \int_{\omega_1} e^{-2s \varphi \xi_3^3} \sigma_t^2 + \epsilon s^{-1} \int_0^T \int_{\omega_1} e^{-2s \varphi \xi_3^3} \sigma_t^2 + \frac{C}{\epsilon} s^7 \lambda^8 \int_0^T \int_{\omega_1} e^{-2s \varphi \xi_3^3} |\sigma|^2. \quad (3.20) \]

(ii) Next, after integrating by parts twice with respect to \( x \), we get
\[ A_2 := \frac{1}{b} \gamma_1 s^3 \lambda^4 \int_0^T \int_{\omega_1} \phi e^{-2s \varphi \xi_3^3} \psi \sigma_{xxxx} \]
\[ = -\frac{1}{b} \gamma_1 s^3 \lambda^4 \left[ \int_0^T \int_{\omega_1} \phi \sigma \left( e^{-2s \varphi \xi_3^3} \right)^3 \psi \sigma_{xxxx} + \int_0^T \int_{\omega_1} \phi \left( e^{-2s \varphi \xi_3^3} \right)^3 \psi \sigma_{xxxx} \right] \]
\[ + \int_0^T \int_{\omega_1} \phi \left( e^{-2s \varphi \xi_3^3} \right)^3 \psi \sigma_{xxxx} \]
\[ = \frac{1}{b} \gamma_1 s^3 \lambda^4 \int_0^T \int_{\omega_1} \left[ \phi \sigma \left( e^{-2s \varphi \xi_3^3} \right)^3 \psi \sigma_{xxxx} + \phi \left( e^{-2s \varphi \xi_3^3} \right)^3 \psi \sigma_{xxxx} \right] \]
\[ + \frac{2}{b} \gamma_1 s^3 \lambda^4 \int_0^T \int_{\omega_1} \left[ \phi \left( e^{-2s \varphi \xi_3^3} \right)^3 \psi \sigma_{xxxx} + \phi \left( e^{-2s \varphi \xi_3^3} \right)^3 \psi \sigma_{xxxx} \right] \quad (3.21) \]

Using (3.4) and Cauchy-Schwarz inequality, we have from (3.21),
\[ |A_2| \leq C s^3 \lambda^4 \int_0^T \int_{Q_T} e^{-2s \varphi \xi_3^3} |\psi|^2 + \epsilon s^8 \lambda^2 \int_0^T \int_{Q_T} e^{-2s \varphi \xi_3^3} |\psi|^2 + \epsilon s^{-1} \int_0^T \int_{Q_T} e^{-2s \varphi \xi_3^3} |\psi|^2 \]
\[ + \frac{C}{\epsilon} s^7 \lambda^8 \int_0^T \int_{\omega_1} e^{-2s \varphi \xi_3^3} |\sigma_{xx}|^2. \quad (3.22) \]
(iii) Let us estimate the third term of (3.15) in the following way

\[ A_3 = \frac{1}{b} s^3 \lambda^4 \int_0^T \int_\omega \dot{\phi} e^{-2s \varphi} \xi^3 \sigma_{xxx} \]

\[ = \frac{1}{b} s^3 \lambda^4 \int_0^T \int_\omega \left( \phi_x e^{-2s \varphi} \xi^3 \sigma_{xx} + \phi (e^{-2s \varphi} \xi)_x \psi \sigma_{xx} + \phi e^{-2s \varphi} \xi \psi \sigma_{xx} \right). \]

Therefore, for \( \epsilon > 0 \) we have, using (3.4) and the Cauchy-Schwarz inequality, that

\[ |A_3| \leq 2\epsilon s^3 \lambda^4 \int_{Q_T} e^{-2s \varphi} \xi^3 |\psi|^2 + \epsilon s^2 \lambda^2 \int_{Q_T} e^{-2s \varphi} \xi |\psi_x|^2 + \frac{C}{\epsilon} s^3 \lambda^6 \int_0^T \int_\omega e^{-2s \varphi} \xi_5 |\sigma_{xx}|^2. \]  

(3.23)

(iv) Next, it is easy to see that

\[ |A_4| \leq \epsilon s^3 \lambda^4 \int_{Q_T} e^{-2s \varphi} \xi^3 |\psi|^2 + \frac{C}{\epsilon} s^3 \lambda^4 \int_0^T \int_\omega e^{-2s \varphi} \xi^3 |\sigma_{xx}|^2. \]  

(3.24)

Combining the estimates (3.20), (3.22), (3.23), (3.24), and applying in (3.15) we get

\[ s^3 \lambda^4 \int_0^T \int_{\omega_0} e^{-2s \varphi} \xi^3 |\psi|^2 \leq 3\epsilon s^3 \lambda^4 \int_{Q_T} e^{-2s \varphi} \xi^3 |\psi|^2 + 3 \epsilon s \lambda^2 \int_{Q_T} e^{-2s \varphi} \xi |\psi_x|^2 + \epsilon s^{-1} \int_{Q_T} e^{-2s \varphi} \xi |\psi_{xx}|^2 + \epsilon s^{-1} \int_{Q_T} e^{-2s \varphi} \xi |\sigma_x|^2 + \frac{C}{\epsilon} s^3 \lambda^4 \int_0^T \int_\omega e^{-2s \varphi} \xi^3 |\sigma_{xx}|^2. \]  

(3.25)

Now, fix \( \epsilon > 0 \) small enough in (3.25), so that all the terms with coefficient \( \epsilon \) can be absorbed by the associated integrals in the left hand side of (3.13). As a consequence, the estimate (3.13) boils down to

\[ I_{KS}(s, \lambda; \sigma) + I_E(s, \lambda; \psi) \leq C s^7 \lambda^8 \int_0^T \int_\omega e^{-2s \varphi} \xi^7 |\sigma_{xx}|^2 + C s^7 \lambda^8 \int_0^T \int_\omega e^{-2s \varphi} \xi^7 |\sigma_{xx}|^2, \]  

(3.26)

for all \( \lambda \geq \lambda_0 \) and \( s \geq \mu_0(T + T^2) \).

**Step 3. Absorbing the observation integral in \( \sigma_{xx} \).** We need to estimate the last integral of (3.26). Consider a function (recall that \( \omega_1 \subset \subset \omega \))

\[ \hat{\phi} \in C^\infty_c(\omega) \quad \text{with} \quad 0 \leq \hat{\phi} \leq 1, \quad \text{and} \quad \hat{\phi} = 1 \quad \text{in} \ \omega_1. \]  

(3.27)

With this, we have

\[ s^7 \lambda^8 \int_0^T \int_{\omega_1} e^{-2s \varphi} \xi^7 |\sigma_{xx}|^2 \leq s^7 \lambda^8 \int_0^T \int_{\omega} \hat{\phi} e^{-2s \varphi} \xi^7 |\sigma_{xx}|^2. \]

Successive integrating by parts yields to

\[ s^7 \lambda^8 \int_0^T \int_{\omega} \hat{\phi} e^{-2s \varphi} \xi^7 \sigma_{xx} \sigma_{xx} \]

\[ = s^7 \lambda^8 \int_0^T \int_{\omega} \left[ \hat{\phi} e^{-2s \varphi} \xi^7 \sigma_{xx} \sigma_x + \phi (e^{-2s \varphi} \xi^7)_x \sigma_{xx} \sigma_x + \phi_x e^{-2s \varphi} \xi^7 \sigma_{xx} \sigma_x \right] \]

\[ = s^7 \lambda^8 \int_0^T \int_{\omega} \left[ \hat{\phi} e^{-2s \varphi} \xi^7 \sigma_{xx} \sigma_x + \phi (e^{-2s \varphi} \xi^7)_x \sigma_{xx} \sigma_x + \phi_x e^{-2s \varphi} \xi^7 \sigma_{xx} \sigma_x \right] \]

\[ + 2 s^7 \lambda^8 \int_0^T \int_{\omega} \left[ \hat{\phi} e^{-2s \varphi} \xi^7 \sigma_{xx} \sigma_x + \phi (e^{-2s \varphi} \xi^7)_x \sigma_{xx} \sigma_x + \phi_x e^{-2s \varphi} \xi^7 \sigma_{xx} \sigma_x \right]. \]

Again by using the information (3.4) and Cauchy-Schwarz inequality, we get for some \( \epsilon > 0 \), that

\[ s^7 \lambda^8 \int_0^T \int_{\omega_1} e^{-2s \varphi} \xi^7 |\sigma_{xx}|^2 \leq \epsilon s^{-1} \int_{Q_T} e^{-2s \varphi} \xi^7 |\sigma_{xx}|^2 + 2 \epsilon s \lambda^2 \int_{Q_T} e^{-2s \varphi} \xi |\sigma_{xx}|^2 \]

\[ + 3 \epsilon s^3 \lambda^4 \int_{Q_T} e^{-2s \varphi} \xi^3 |\sigma_{xx}|^2 + \frac{C}{\epsilon} s^{15} \lambda^{16} \int_0^T \int_{\omega} e^{-2s \varphi} \xi^{15} |\sigma|^2. \]  

(3.28)

We fix small enough \( \epsilon > 0 \) in (3.28) so that the integrals with coefficient \( \epsilon \) can be absorbed by the leading integrals in the left hand of (3.26) and as a result we obtain

\[ I_{KS}(s, \lambda; \sigma) + I_E(s, \lambda; \psi) \leq C s^{15} \lambda^{16} \int_0^T \int_{\omega} e^{-2s \varphi} \xi^{15} |\sigma|^2, \]
for all $\lambda \geq \lambda_0$ and $s \geq \mu_0(T + T^2)$.

This is the required joint Carleman estimate (3.8) of our theorem. The proof is finished. \hfill \Box

3.2. Observability inequality and null-controllability of the linearized model. With the Carleman estimate (3.8), it is difficult to obtain the desired observability inequality with norm of the components $\sigma, \psi$ at $x = 0$ in the left hand side. The reason behind is that the weight function $e^{-2\kappa \varphi}$ is vanishing near $t = 0$.

To avoid this obstacle, we consider

$$\ell(t) = \begin{cases} T^2/4, & 0 \leq t \leq T/2, \\ t(T-t), & T/2 \leq t \leq T, \end{cases} \tag{3.29}$$

and the following modified weight functions

$$\mathcal{G}(t, x) = \frac{e^{2\lambda k \|\varphi\|_{L^\infty}} - e^{\lambda(k\|\varphi\|_{L^\infty} + \nu(x))}}{\ell(t)}, \quad \tilde{G}(t, x) = \frac{e^{\lambda(k\|\varphi\|_{L^\infty} + \nu(x))}}{\ell(t)}, \quad \forall(t, x) \in Q_T. \tag{3.30}$$

for any constants $\lambda > 1$ and $k > 1$.

Further, we denote

$$\overline{M} = e^{2\lambda k \|\varphi\|_{L^\infty}} - e^{\lambda k \|\varphi\|_{L^\infty}}, \tag{3.31}$$

$$M_* = e^{2\lambda k \|\varphi\|_{L^\infty}} - e^{\lambda(k+1)\|\varphi\|_{L^\infty}}. \tag{3.32}$$

**Proposition 3.4** (Observability inequality: control in KS-KdV eq.). There exists a constant $K > 0$ depending on $\omega, a, b, c, \gamma_1, \gamma_2$ but not $T$, such that we have the following observability inequality

$$\|\sigma(0, \cdot)\|^2_{H^2_\omega[0, 1]} + \|\psi(0, \cdot)\|^2_{H^2_\omega[0, 1]} \leq K e^{\frac{M}{10}} \int_0^T \int_\omega |\sigma|^2, \tag{3.33}$$

where $(\sigma, \psi)$ is the solution to the adjoint system (3.7).

**Proof.** By construction we have that $\varphi = \mathcal{G}$ and $\xi = \tilde{G}$ in $(T/2, T) \times (0, 1)$, hence

$$\int_\omega \int_0^1 e^{-2s\Theta} 3^7 |\sigma|^2 = \int_\omega \int_0^1 e^{-2s\varphi} 3^7 |\sigma|^2.$$

Therefore, from the Carleman estimate (3.8) we readily get

$$s^7 \lambda^8 \int_\omega \int_0^T \int_0 e^{-2s\Theta} 3^7 |\sigma|^2 \leq C s^{15} \lambda^{16} \int_0^T \int_\omega e^{-2s\Theta} 3^{15} |\sigma|^2, \tag{3.34}$$

for all $\lambda \geq \lambda_0$ and $s \geq s_0$ and the constant $C > 0$ does not depend on $T$.

Let us introduce a function $\eta \in C^1([0, T])$ such that

$\eta = 1$ in $[0, T/2], \quad \eta = 0$ in $[3T/4, T]$.

It is clear that $\text{Supp}(\eta') \subset [T/2, 3T/4]$.

Recall the adjoint system (3.7). Then, the pair $(\tilde{\sigma}, \tilde{\psi})$ with $\tilde{\sigma} = \eta \sigma, \tilde{\psi} = \eta \psi$ satisfies the following set of equations

$$\begin{cases} -\tilde{\sigma}_x + \gamma_1 \tilde{\sigma}_{xxxx} - \tilde{\sigma}_{xxx} + \gamma_3 \tilde{\sigma}_{xx} = b \tilde{\psi} - \eta' \sigma, & \text{in } Q_T, \\ -\tilde{\psi}_x + c \tilde{\psi} = a \tilde{\sigma}, & \text{in } Q_T, \\ \tilde{\sigma}(t, 0) = \tilde{\sigma}(t, 1) = 0, & \text{in } (0, T), \\ \tilde{\sigma}(0, t) = \tilde{\sigma}(1, t) = 0, & \text{in } (0, T), \\ \tilde{\psi}(t, 0) = \tilde{\psi}(t, 1) = 0, & \text{in } (0, T), \\ \tilde{\sigma}(T) = 0, & \text{in } (0, 1). \end{cases}$$

Thanks to the **Proposition 2.1**, we have the following energy estimate for $\tilde{\sigma}$,

$$\|\eta \sigma\|^2_{L^\infty(0, T; H^2_\omega[0, 1])} \leq C \|\eta' \sigma\|^2_{L^2(Q_T)} \leq C \|\sigma\|^2_{L^2((T/2, 3T/4) \times (0, 1))}, \tag{3.35}$$

for some constant $C > 0$ which does not depend on $T$. Now, observe that

$$\|\eta \sigma\|^2_{L^\infty(0, T; H^2_\omega[0, 1])} \leq C s^7 \lambda^8 \int_{T/2}^T \int_0 e^{2s\Theta} 3^{-7} e^{-2s\Theta} 3^7 |\sigma|^2$$

$$\leq C \max_{[T/2, 3T/4]} \left| e^{2s\Theta} 3^{-7} s^7 \lambda^8 \int_{T/2}^T \int_0 e^{-2s\Theta} 3^7 |\sigma|^2 \right|$$

$$\leq C T^{14} e^{\frac{M}{10}} s^{15} \lambda^{16} \int_0^T \int_\omega e^{-2s\Theta} 3^{15} |\sigma|^2, \tag{3.36}$$
where we have used that the maximum of $e^{2s\xi}3^{-7}$ w.r.t. time occurs at $t = T/4$ or at $3T/4$ (the quantity $\hat{M}$ is defined in (3.31)) and we also used the Carleman estimate (3.34) to obtain the last inclusion.

Hereinafter, we fix $\lambda = \lambda_0$ and $s = s_0 = \mu_0(T + T^2)$. Then, observe that the maximum of $e^{2s\xi}3^{-7}$ w.r.t. time occurs at $t = T/2$. Using this in (3.36), we get

$$
\|\eta\sigma\|^2_{L^\infty(0,T;H^2_x(0,1))} \leq C T^{14} e^{2\beta_0 \xi_0} e^{-\frac{s \mu_0}{2}} \frac{1}{T^{30}} \int_0^T \int_\omega |\sigma|^2 \\
\leq \frac{C}{T^{10}} e^{\beta_0 \xi_0} \int_0^T \int_\omega |\sigma|^2 \\
\leq \langle K e^{\hat{\psi}} \int_0^T \int_\omega |\sigma|^2, \rangle
$$

(3.37)

where $M_\star$ is defined in (3.32) and $K > 0$ is some constant that does not depend on $T$.

Next, from the equation of $\hat{\psi}$, we get

$$
\|\eta\psi(t,\cdot)\|_{H^2(0,1)} \leq |\sigma|\|\eta\sigma\|_{L^\infty(0,T;L^2(0,1))}.
$$

This, combining with the inequality (3.37), we deduce the required observability estimate (3.33).

Below, we sketch the proof of null-controllability for the corresponding linearized system when a control acts on the KS-KdV equation.

**Proof of Theorem 1.3—Item 1.** Once we have the observability inequality (3.33), then by the Hilbert Uniqueness Method, one can prove the existence of a null-control $h \in L^2((0, T) \times \omega)$ (in terms of the adjoint component $\sigma$ localized in $\omega$) for the linearized system (1.7)–(1.5)–(1.6). This method is standard and has been applied in several works; see for instance the pioneer works [28] or [25, Section 1.3]. The estimation of the control cost $Ke^{K/T}\|u_0\|_{H^{-2}(0,1)}$ is followed from the sharp observability inequality in Proposition 3.4.

3.3. Null controllability of the nonlinear system: control in KS-KdV equation. In this section, by using the control cost of the linear system, given by Theorem 1.3—Item 1, we shall prove the local null-controllability of our nonlinear model (1.3)–(1.5)–(1.6). The proof will be based on the so-called source term method developed in [44] followed by a Banach fixed point argument.

3.3.1. The source term method. Let us discuss the source term method (see [44]) for our case. We assume the constants $p > 0$, $q > 1$ in such a way that

$$
1 < q < \sqrt{2}, \quad \text{and} \quad p > \frac{q^2}{2 - q^2}.
$$

(3.38)

We also recall the constant $K$ appearing in the control estimate (1.9) of the linearized models, more precisely the control cost is given by $Ke^{K/T}$. Now define the functions

$$
\begin{cases}
\rho_0(t) = e^{-\frac{2qK}{(q+1)p^2K}}, \\
\rho_{\mathcal{F}}(t) = e^{-\frac{(2q^2+q^2p^2)}{(q+1)p^2K}},
\end{cases}
\quad \forall t \in \left[T \left(1 - \frac{1}{q^2}\right), T\right],
$$

(3.39)

extended in $[0, T(1 - 1/q^2)]$ in a constant way such that the functions $\rho_0$ and $\rho_{\mathcal{F}}$ are continuous and non-increasing in $[0, T]$ with $\rho_0(T) = \rho_{\mathcal{F}}(T) = 0$.

**Remark 3.5.** We compute that

$$
\frac{\rho_0^2(t)}{\rho_{\mathcal{F}}(t)} = e^{\frac{q^2K + q^2p^2}{(q+1)p^2K}}, \quad \forall t \in \left[T \left(1 - \frac{1}{q^2}\right), T\right],
$$

Due to the choices of $p, q$ in (3.38), we have $K(q^2 + p(q^2 - 2)) < 0$, $(q - 1) > 0$ and therefore we can conclude that

$$
\frac{\rho_0^2(t)}{\rho_{\mathcal{F}}(t)} \leq 1, \quad \forall t \in [0, T].
$$

Let us write the following Gelfand triple

$$
H^3(0,1) \cap H^0_0(0,1) \subset H^{-1}(0,1) \subset \left(H^3(0,1) \cap H^0_0(0,1)\right)',
$$

(3.40)
and denote \( X := [L^2(0, 1)]^2 \). Then, we define the following weighted spaces:
\[
\mathcal{F} := \left\{ f \in L^2(0, T; (H^3(0, 1) \cap H^2_0(0, 1))') \left| \frac{\rho f}{|\rho|^2} \in L^2(0, T; (H^3(0, 1) \cap H^2_0(0, 1))') \right. \right\},
\]
\[
\mathcal{Y} := \left\{ (u, v) \in L^2(0, T; X) \left| \frac{\rho}{\rho_0} (u, v) \in L^2(0, T; X) \right. \right\},
\]
\[
\mathcal{V} := \left\{ h \in L^2((0, T) \times \omega) \left| \frac{h}{\rho_0^2} \in L^2((0, T) \times \omega) \right. \right\}.
\]

It is clear that the aforementioned spaces are Hilbert spaces. In particular, the inner products in \( \mathcal{F} \) and \( \mathcal{V} \) are respectively defined by
\[
\langle f_1, f_2 \rangle_\mathcal{F} := \int_0^T \rho_0^{-2} \langle f_1, f_2 \rangle_0 \quad \text{and} \quad \langle h_1, h_2 \rangle_\mathcal{V} := \int_0^T \int_\omega \rho_0^{-2} h_1 h_2,
\]
and consequently, the norms by
\[
\|f\|_\mathcal{F} := \left( \int_0^T \rho_0^{-2} \|f\|_0^2 \right)^{1/2} \quad \text{and} \quad \|h\|_\mathcal{V} := \left( \int_0^T \int_\omega \rho_0^{-2} |h|^2 \right)^{1/2}.
\]

Let us now consider the following system (still linear) with a source term \( f \) in the right hand side of the KS-KdV equation:
\[
\begin{aligned}
&u_t + \gamma_1 u_{xxxx} + u_{xxx} + \gamma_2 u_{xx} = f + av + \chi \omega h, \quad \text{in } Q_T, \\
&v_t + cv = bu, \quad \text{in } Q_T, \\
&u(t, 0) = 0, \quad u(t, 1) = 0, \quad \text{for } t \in (0, T), \\
&u_x(t, 0) = 0, \quad u_x(t, 1) = 0, \quad \text{for } t \in (0, T), \\
&v(t, 0) = 0, \quad v(t, 1) = 0, \quad \text{for } t \in (0, T), \\
&u(0, x) = u_0(x), \quad x \in (0, 1).
\end{aligned}
\]

The classical parabolic regularity gives the following result (see [19, Chapter 3] or [50, Section 3, Chapter II] for more details).

**Proposition 3.6.** For any given initial data \( u_0 \in H^{-1}(0, 1) \), source term \( f \in L^2(0, T; (H^3(0, 1) \cap H^2_0(0, 1))') \) and \( h \in L^2((0, T) \times \omega) \), there exists unique solution \((u, v)\) of (3.42) such that
\[
\begin{aligned}
u &\in C^0([0, T]; H^{-1}(0, 1)) \cap L^2(0, T; H^1_0(0, 1)) \cap H^1(0, T; (H^3(0, 1) \cap H^2_0(0, 1))'), \\
v &\in C^0([0, T]; H^{-1}(0, 1)) \cap L^2(0, T; H^2(0, 1) \cap H^3_0(0, 1)).
\end{aligned}
\]

In addition, we have the following estimate
\[
\|u\|_{C^0([H^{-1}(1)] \cap L^2(H^1_0) \cap H^1((H^3 \cap H^2_0)'))} + \|v\|_{C^0([H^{-1}(1)] \cap L^2(H^2 \cap H^3_0))} \leq C \left( \|u_0\|_{H^{-1}(0, 1)} + \|f\|_{L^2((H^3 \cap H^2_0)')} + \|h\|_{L^2((0, T) \times \omega)} \right),
\]
for some constant \( C > 0 \).

Using the above proposition and the embedding
\[
\left\{ u \in L^2(0, T; H^1_0(0, 1)) \left| u_x \in L^2(0, T; (H^3(0, 1) \cap H^2_0(0, 1))') \right. \right\} \hookrightarrow L^4(0, T; L^2(0, 1)),
\]
we deduce that \( u \in L^4(0, T; L^2(0, 1)) \).

Next, by the following proposition we shall obtain the existence of a control \( h \in \mathcal{V} \) for the system (3.42) with given source term \( f \in \mathcal{F} \) and initial data \( u_0 \in H^{-1}(0, 1) \). Observe that, we choose here slightly higher regular initial data than the linear system (see Theorem 1.3). The reason is to handle the nonlinear term \( uu_x \) in our system (1.3)—(1.5)—(1.6).

Let us prove the following result.

**Proposition 3.7.** Let \( T > 0 \), \( u_0 \in H^{-1}(0, 1) \) and \( f \in \mathcal{F} \) be given. We also take any \((a, b) \in \mathbb{R}^2 \) with \( b \neq 0 \). Then, there exists a linear map
\[
[u_0, f] \in H^{-1}(0, 1) \times \mathcal{F} \mapsto [(u, v), h] \in \mathcal{Y} \times \mathcal{V},
\]
(3.46)
such that \([u, v, h]\) solves (3.42). Moreover, we have the following regularity estimate
\[
\left\| \frac{u}{\rho_0} \right\|_{C^0(H^{-1}) \cap L^2(H_0^1) \cap H^1((H^3 \cap H_0^2)(\gamma))} + \left\| \frac{v}{\rho_0} \right\|_{C^0(H^{-1}) \cap L^2(H_0^2 \cap H_0^1)} + \left\| \frac{h}{\rho_F} \right\|_{L^2((0,T) \times \omega)} \leq C_1 \left( \|u_0\|_{H^{-1}(0,1)} + \left\| \frac{f}{\rho_F} \right\|_{L^2((H^3 \cap H_0^2)(\gamma))} \right),
\]
where the constant \(C_1 > 0\) does not depend on \(u_0\), \(f\) or \(h\).

**Proof.** Let any \(T > 0\) be given. Then, we define the sequence \((T_k)_{k \geq 0}\) as
\[
T_k = T - \frac{T}{q^k}, \quad \forall k \geq 0,
\]
where \(q\) is given by (3.38). With this \(T_k\), it can be shown that the weight functions \(\rho_0\) and \(\rho_F\) enjoy the following relation:
\[
\rho_0(T_{k+2}) = \rho_F(T_k) e^{\frac{k}{k+2}}, \quad \forall k \geq 0.
\]
We also define a sequence \((m_k)_{k \geq 0}\) with
\[
m_0 = u_0 \in H^{-1}(0,1), \quad m_{k+1} = \tilde{u}(T_{k+1}^-), \quad \forall k \geq 0,
\]
where \((\tilde{u}, \tilde{v})\) is such that
\[
\begin{align*}
\tilde{u} \in C^0([T_k, T_{k+1}; H^{-1}(0,1)] \cap L^2(T_k, T_{k+1}; H_0^1(0,1)) \cap H^1(T_k, T_{k+1}; (H^3(0,1) \cap H_0^2(0,1))'),
\tilde{v} \in C^0([T_k, T_{k+1}; H^{-1}(0,1)] \cap L^2(T_k, T_{k+1}; H^2(0,1) \cap H_0^4(0,1)),
\end{align*}
\]
and that \((\tilde{u}, \tilde{v})\) uniquely satisfies the following set of equations
\[
\begin{align*}
\tilde{u}_t + \gamma_1 \tilde{u}_{xxxx} + \tilde{u}_{xxx} + \gamma_2 \tilde{u}_{xx} &= f + a\tilde{v} \quad \text{in } (T_k, T_{k+1}) \times (0,1), \\
-\tilde{v}_{xx} + c\tilde{v} &= b\tilde{u}, \\
\tilde{u}(t,0) &= 0, \quad \tilde{u}(t,1) = 0, \\
\tilde{u}_x(t,0) &= 0, \quad \tilde{u}_x(t,1) = 0 & \text{if } t \in (T_k, T_{k+1}), \\
\tilde{v}(t,0) &= 0, \quad \tilde{v}(t,1) = 0 & \text{if } t \in (T_k, T_{k+1}), \\
\tilde{u}(T_{k+1}^+, x) &= m_k & \text{if } x \in (0,1).
\end{align*}
\]
Moreover, it satisfies the following regularity estimate (using Proposition 3.6)
\[
\left\| \tilde{u} \right\|_{C^0([T_k, T_{k+1}; H^{-1}] \cap L^2(T_k, T_{k+1}; H_0^1) \cap H^1(T_k, T_{k+1}; (H^3 \cap H_0^2)(\gamma))} + \left\| \tilde{v} \right\|_{C^0([T_k, T_{k+1}; H^{-1}] \cap L^2(T_k, T_{k+1}; H^2 \cap H_0^4(\gamma))} \leq C \left\| f \right\|_{L^2(T_k, T_{k+1}; (H^3 \cap H_0^2)(\gamma))},
\]
for all \(k \geq 0\). In particular, we have
\[
\left\| m_{k+1} \right\|_{H^{-1}(0,1)} \leq C \left\| f \right\|_{L^2(T_k, T_{k+1}; (H^3 \cap H_0^2)(\gamma))}, \quad \forall k \geq 0.
\]

\textbf{• Weighted estimate of the control.} Let us consider the following control system in the time interval \((T_k, T_{k+1})\) for all \(k \geq 0\),
\[
\begin{align*}
\tilde{u}_t + \gamma_1 \tilde{u}_{xxxx} + \tilde{u}_{xxx} + \gamma_2 \tilde{u}_{xx} &= \chi h_k + a\tilde{v} \quad \text{in } (T_k, T_{k+1}) \times (0,1), \\
-\tilde{v}_{xx} + c\tilde{v} &= b\tilde{u}, \\
\tilde{u}(t,0) &= 0, \quad \tilde{u}(t,1) = 0, \\
\tilde{u}_x(t,0) &= 0, \quad \tilde{u}_x(t,1) = 0 & \text{if } t \in (T_k, T_{k+1}), \\
\tilde{v}(t,0) &= 0, \quad \tilde{v}(t,1) = 0 & \text{if } t \in (T_k, T_{k+1}), \\
\tilde{u}(T_{k+1}^+, x) &= m_k & \text{if } x \in (0,1).
\end{align*}
\]
By using Theorem 1.3–Item 1, we have the existence of a control \(h_k\) that satisfies
\[
\left\| h_k \right\|_{L^2((T_k, T_{k+1}) \times \omega)} \leq K e^{K/(T_{k+1} - T_k)} \left\| m_k \right\|_{H^{-1}(0,1)}, \quad \forall k \geq 0,
\]
where the constant \(K > 0\) (appearing in Theorem 1.3) neither depends on \(T\) nor on \(u_0\). As a result, the solution to (3.52) satisfies
\[
\begin{align*}
(\tilde{u}(T_{k+1}^-, x), \tilde{v}(T_{k+1}^-, x)) &= (0,0), \quad \forall x \in (0,1), \quad \forall k \geq 0.
\end{align*}
\]
Now, combining (3.51) and (3.53), we have for all \( k \geq 0 \),
\[
\|h_{k+1}\|_{L^2(T_{k+1}, T_{k+2}) \times \omega} \leq Ke^{K(T_{k+2}-T_{k+1})} \|m_{k+1}\|_{H^{-1}(0,1)} \\
\leq K_1e^{K(T_{k+2}-T_{k+1})} \|f\|_{L^2(T_k, T_{k+1}; (H^3 \cap H_y^2)' )} \\
\leq K_1e^{K(T_{k+2}-T_{k+1})} \rho_F(T_k) \|f\|_{L^2(T_k, T_{k+1}; (H^3 \cap H_y^2)' )},
\]
(3.55)

since \( \rho_F \) is a non-increasing function. Then, using the relation (3.48), one can write
\[
\|h_{k+1}\|_{L^2((T_{k+1}, T_{k+2}) \times \omega)} \leq K_1\rho_0(T_{k+2}) \left\| \frac{f}{\rho_F} \left\|_{L^2(T_k, T_{k+1}; (H^3 \cap H_y^2)' )} \right\|, \forall k \geq 0.
\]
(3.56)

Again, since \( \rho_0 \) is also non-increasing, we deduce that
\[
\left\| \frac{h_{k+1}}{\rho_0} \right\|_{L^2((T_{k+1}, T_{k+2}) \times \omega)} \leq \frac{1}{\rho_0(T_{k+2})} \left\| h_{k+1} \right\|_{L^2((T_{k+1}, T_{k+2}) \times \omega)} \\
\leq K_1 \left\| \frac{f}{\rho_F} \right\|_{L^2(T_k, T_{k+1}; (H^3 \cap H_y^2)' )}, \text{ for all } k \geq 0.
\]
(3.57)

We now define the control function \( h \) as
\[
h := \sum_{k \geq 0} h_k \chi(T_k, T_{k+1}), \text{ in } (0, T) \times \omega,
\]
(3.58)

where \( \chi \) denotes the characteristic function.

Recall that, we have the \( L^2 \)- estimates of \( \frac{h_{k+1}}{\rho_0} \) for all \( k \geq 0 \). It remains to find the \( L^2 \)- estimate of \( \frac{h_0}{\rho_0} \).

From the control estimate (3.53) and since \( \rho_0(T_1) = e^{-\frac{1}{2} \beta(T_1-T_0)} \), we get
\[
\|h_0\|_{L^2((0, T_1) \times \omega)} \leq Ke^{K_2} \|u_0\|_{H^{-1}(0,1)} \\
= Ke^{K_2} \|u_0\|_{H^{-1}(0,1)} \\
\leq Ke^{K_2} \rho_0(T_1) \|u_0\|_{H^{-1}(0,1)},
\]
(3.59)

where \( K_2 := \frac{q(1+p)K}{(q-1)} > K \). But \( \rho_0 \) is non-increasing function in \((0, T_1)\) which yields
\[
\left\| \frac{h_0}{\rho_0} \right\|_{L^2((0, T_1) \times \omega)} \leq Ke^{K_2} \|u_0\|_{H^{-1}(0,1)}.
\]
(3.60)

Combining the estimates (3.60) and (3.57), we obtain
\[
\left\| \frac{h}{\rho_0} \right\|_{L^2((0, T) \times \omega)} \leq \sum_{k \geq 0} \left\| \frac{h_k}{\rho_0} \right\|_{L^2((T_k, T_{k+1}) \times \omega)} \leq Ke^{K_2} \|u_0\|_{H^{-1}(0,1)} + K_1 \sum_{k \geq 0} \left\| \frac{f}{\rho_F} \right\|_{L^2(T_k, T_{k+1}; (H^3 \cap H_y^2)' )} \\
\leq Ke^{K_2} \left\| u_0 \right\|_{H^{-1}(0,1)} + \left\| \frac{f}{\rho_F} \right\|_{L^2(0, T; (H^3 \cap H_y^2)' )},
\]
(3.61)

for some constant \( K_3 > 0 \).

\textbf{Weighted estimate of the solution.} Let us set
\[
(u, v) = (\tilde{u}, \tilde{v}) + (\hat{u}, \hat{v}),
\]
(3.62)

which satisfies the following control system with the source term \( f \), given by
\[
\begin{align*}
\left\{ 
\begin{array}{ll}
 u_t + \gamma_1 u_{xxxx} + u_{xxx} + \gamma_2 u_{xx} = f + \alpha v + \chi_h h_k, & \text{in } (T_k, T_{k+1}) \times (0,1), \\
 -v_{xx} + cv = bu, & \text{in } (T_k, T_{k+1}) \times (0,1), \\
u(t, 0) = 0, u(t, 1) = 0, & \text{in } (T_k, T_{k+1}), \\
u_x(t, 0) = 0, u_x(t, 1) = 0, & \text{in } (T_k, T_{k+1}), \\
v(t, 0) = 0, v(t, 1) = 0, & \text{in } (T_k, T_{k+1}), \\
u(T_k, x) = m_k, & \text{in } (0,1).
\end{array}
\right\
\end{align*}
\]
(3.63)

for all \( k \geq 0 \). Note that, the solution \((u, v)\) satisfies
\[
u(T_0) = m_0
\]
and, using (3.54) one has for all \(k \geq 0\), that
\[
\begin{align*}
    u(T_{k+1}^-) &= \tilde{u}(T_{k+1}^-) + \bar{u}(T_{k+1}^-) = m_{k+1}, \\
    u(T_{k+1}^+) &= \tilde{u}(T_{k+1}^+) + \bar{u}(T_{k+1}^+) = m_{k+1+1}.
\end{align*}
\]
This gives the continuity of the component \(u\) at each \(T_{k+1}\) for \(k \geq 0\), more precisely
\[
u \in C^0([T_k, T_{k+1}]; H^{-1}(0, 1)).
\]
Once we have this, the elliptic part \(v\) enjoys
\[
v(T_k) = b(-\partial_{xx} + cI)^{-1} u(T_k) \in H^{-1}(0, 1), \quad \forall k \geq 0,
\]
which yields the continuity of \(v\) at time \(T_k\) \((k \geq 0)\). Moreover, we have the following estimate (thanks to Proposition 3.6) for all \(k \geq 0\):
\[
\begin{align*}
    \|u\|_{C^0([T_k, T_{k+1}]; H^{-1}) \cap L^2(T_k, T_{k+1}; \mathbb{H}_4^1)} &+ \|v\|_{C^0([T_k, T_{k+1}]; H^{-1}) \cap L^2(T_k, T_{k+1}; \mathbb{H}_4^1)} \\
    \leq C \left( \|m_k\|_{H^{-1}(0, 1)} + \|h_k\|_{L^2(T_k, T_{k+1} \times \omega)} + \|f\|_{L^2(T_k, T_{k+1}; \mathbb{H}_3^1)} \right).
\end{align*}
\] (3.64)
We start with \(k \geq 1\); using the estimates of \(m_k\) and \(h_k\) from (3.51) and (3.53) (resp.), we get
\[
\begin{align*}
    \|u\|_{C^0([T_k, T_{k+1}]; H^{-1}) \cap L^2(T_k, T_{k+1}; \mathbb{H}_4^1)} &+ \|v\|_{C^0([T_k, T_{k+1}]; H^{-1}) \cap L^2(T_k, T_{k+1}; \mathbb{H}_4^1)} \\
    \leq K_4 \rho_F(T_k) e^{\frac{b}{1 - \rho_F} - 1} \left( \|f\|_{L^2(T_k, T_{k+1}; \mathbb{H}_3^1)} \right) + K_4 \|f\|_{L^2(T_k, T_{k+1}; \mathbb{H}_3^1)},
\end{align*}
\]
for some constant \(K_4 > 0\). But \(\rho_0(T_{k+1}) = e^{\frac{k}{k+1}} \rho_F(T_{k+1})\) for all \(k \geq 1\) and therefore
\[
\begin{align*}
    \left\| \frac{u}{\rho_0} \right\|_{C^0([T_k, T_{k+1}]; H^{-1}) \cap L^2(T_k, T_{k+1}; \mathbb{H}_4^1)} &+ \left\| \frac{v}{\rho_0} \right\|_{C^0([T_k, T_{k+1}]; H^{-1}) \cap L^2(T_k, T_{k+1}; \mathbb{H}_4^1)} \\
    \leq K_4 \left\| \frac{f}{\rho_F} \right\|_{L^2(T_k, T_{k+1}; \mathbb{H}_3^1)}, \quad \forall k \geq 1, \quad \text{since } \rho_0 \text{ is also non-increasing in } (0, T).
\end{align*}
\] (3.65)
Finally, using the estimate of \(h_0\) from (3.59), one can deduce that
\[
\begin{align*}
    \|u\|_{C^0([0, T_1]; H^{-1}) \cap L^2(0, T_1; \mathbb{H}_4^1)} &+ \|v\|_{C^0([0, T_1]; H^{-1}) \cap L^2(0, T_1; \mathbb{H}_4^1)} \\
    \leq K_5 \rho_0(T_1) e^{\frac{b}{1 - \rho_0}} \left( \left\| u_0 \right\|_{H^{-1}(0, 1)} + \left\| f \right\|_{L^2(0, T_1; \mathbb{H}_3^1)} \right),
\end{align*}
\]
for some constant \(K_5 > 0\). But \(\rho_0\) is non-increasing function and it is easy to observe that
\[
\|f\|_{L^2(0, T_1; \mathbb{H}_3^1)} \leq \left\| \frac{f}{\rho_F} \right\|_{L^2(0, T_1; \mathbb{H}_3^1)},
\]
which leads to the following:
\[
\begin{align*}
    \left\| \frac{u}{\rho_0} \right\|_{C^0([0, T_1]; H^{-1}) \cap L^2(0, T_1; \mathbb{H}_4^1)} &+ \left\| \frac{v}{\rho_0} \right\|_{C^0([0, T_1]; H^{-1}) \cap L^2(0, T_1; \mathbb{H}_4^1)} \\
    \leq K_6 e^{\frac{b}{1 - \rho_0}} \left( \left\| u_0 \right\|_{H^{-1}(0, 1)} + \left\| f \right\|_{L^2(0, T_1; \mathbb{H}_3^1)} \right).
\end{align*}
\] (3.66)
Combining the estimates (3.65) and (3.66), we obtain
\[
\begin{align*}
    \left\| \frac{u}{\rho_0} \right\|_{C^0([0, T_1]; H^{-1}) \cap L^2(0, T_1; \mathbb{H}_4^1)} &+ \left\| \frac{v}{\rho_0} \right\|_{C^0([0, T_1]; H^{-1}) \cap L^2(0, T_1; \mathbb{H}_4^1)} \\
    \leq K_6 e^{\frac{b}{1 - \rho_0}} \left( \left\| u_0 \right\|_{H^{-1}(0, 1)} + \left\| f \right\|_{L^2(0, T_1; \mathbb{H}_3^1)} \right),
\end{align*}
\] (3.67)
for some constant \(K_6 > 0\). This, along with the control estimate (3.61), we get the mentioned weighted estimate (3.47). The proof is complete.
3.3.2. Application of Banach fixed point argument. Recall that our nonlinear system when a control acts in KS equation is:

\[
\begin{align*}
&u_t + \gamma_1 u_{xxx} + u_{xx} + \gamma_2 u_{xx} + uu_x = av + \chi_\omega b, \quad \text{in } Q_T, \\
&-v_{xx} + cv = bu, \quad \text{in } Q_T, \\
u(t, 0) = 0, \quad u(t, 1) = 0, \quad t \in (0, T), \\
u_x(t, 0) = 0, \quad u_x(t, 1) = 0, \quad t \in (0, T), \\
v(t, 0) = 0, \quad v(t, 1) = 0, \quad t \in (0, T), \\
u(0, x) = u_0(x), \quad x \in (0, 1).
\end{align*}
\]

(3.68)

Let us formally test the equation of \((u, v)\) in (3.68) by any \((\varphi, \theta) \in [H^4(0, 1) \cap H^2_0(0, 1)] \times [H^2(0, 1) \cap H^1_0(0, 1)]\), leading to

\[
\frac{d}{dt} \int_0^1 u \varphi + \gamma_1 \int_0^1 u \varphi_{xxx} - \frac{1}{2} \int_0^1 u^2 \varphi_x - \int_0^1 v \varphi_{xx} + c \int_0^1 v \theta = a \int_0^1 v \varphi + b \int_0^1 u \theta + \int_0^1 h \varphi. 
\]

(3.69)

We now define the nonlinearity \(uu_x\) in terms of the following function: \(F : L^2(0, 1) \to H^{-2}(0, 1)\) such that

\[
\langle F(u), \varphi \rangle_{H^{-2}, H^0_0} = -\frac{1}{2} \int_0^1 u^2 \varphi_x, \quad \forall \varphi \in H^0_0(0, 1),
\]

(3.70)

Now, recall that the system (3.42) has unique solution \((u, v)\) given by (3.43)–(3.44) and moreover, \(u \in L^4(0, T; L^2(0, 1))\) (see (3.45)). Therefore, one has

\[
\left| \langle F(u), \varphi \rangle_{H^{-2}, H^0_0} \right| \leq \frac{1}{2} \| \varphi \|_{L^\infty} \| u(t) \|_{L^2}^2 \leq C_2 \| \varphi \|_{H^0_0} \| u(t) \|_{H^1}^2,
\]

for some constant \(C_2 > 0\), since \(H^0_0(0, 1) \hookrightarrow L^\infty(0, 1)\). This yields

\[
\| F \|_{L^2(0, T; H^{-2}(0, 1))} \leq C_2 \| u \|_{L^4(0, T; L^2(0, 1))}^2,
\]

(3.71)

that is the function \(F\) is well-defined. In what follows, we define the map

\[
\Lambda : F \rightarrow F
\]

\[
f \mapsto -F(u),
\]

(3.72)

where \(u\) is the solution of the system (3.42). We also consider an open ball \(B(0, R)\) in \(F\) with center 0 and radius \(R > 0\). We begin with the following lemma.

Lemma 3.8 (Stability). There exists some \(R > 0\) such that \(B(0, R) \subset F\) is stable under the map \(\Lambda\). In other words, \(\Lambda(B(0, R)) \subset B(0, R)\).

Proof. Using the definition (3.70), we have \(\forall \varphi \in H^0_0(0, 1),\)

\[
\left| \langle F(u), \varphi \rangle_{H^{-2}, H^0_0} \right| \leq \frac{1}{2} \| \varphi \|_{L^\infty} \| u(t) \|_{L^2}^2 \leq C_2 \| \varphi \|_{H^0_0} \| u(t) \|_{H^1}^2.
\]

But \(\rho^2_0(t)/\|F\| \leq 1\) for all \(t \in [0, T]\) and thus there exists some constant \(C_2 > 0\) such that

\[
\| F(u) \|_{H^{-2}(0, 1)} \leq C_2 \| u \|_{H^1} \| u \|_{H^{-1}(0, 1)}.
\]

(3.73)

Hence, for any source term \(f \in \overline{B(0, R)}\), one has

\[
\left| \langle \Lambda(f), \varphi \rangle_{H^{-2}, H^0_0} \right| \leq \left| \langle F(u), \varphi \rangle_{H^{-2}, H^0_0} \right| \leq C_2 \| u \|_{H^1} \| u \|_{H^{-1}}.
\]

(3.74)

But \(C_2^2 \leq 4C_2^2 R^2\),

\[
\left| \langle \Lambda(f), \varphi \rangle_{H^{-2}, H^0_0} \right| \leq 4C_2^2 R^2,
\]

(3.75)
where we have used the estimate (3.47) to obtain the equation (3.75) from (3.74), and then in (3.75) we consider the initial data $u_0$ such that $\|u_0\|_{H^{-1}(0,1)} \leq R$. Now, consider

$$R = \frac{1}{8C_1^2 C_2},$$

(3.77)

so that, from (3.76), we have

$$\left\| \frac{\Lambda(f)}{\rho_F} \right\|_{L^2(0,T;H^{-2}(0,1))} \leq R/A \leq R,$$

yielding that $B(0,R)$ is invariant under the map $\Lambda$, defined in (3.72).

\[ \text{□} \]

**Lemma 3.9 (Contraction).** The map $\Lambda$ defined by (3.72) is a contraction map on the closed ball $B(0,R)$.

**Proof.** Consider $f_1, f_2 \in B(0,R)$. Then in view of Proposition 3.7, there exists control $h_j \in \mathcal{V}$ and solution $(u_j, v_j)$ to the system (3.42) associated with the source term $f_j$ for $j = 1, 2$.

By the linearity of the solution map (thanks to Proposition 3.7), we have by means of the estimate (3.47),

$$\left\| \frac{u_1 - u_2}{\rho_0} \right\|_{C_0(H^{-1}) \cap L^1(0,T;H^1(\Omega \subset H^2 \cap H^2_{\gamma})))} + \left\| \frac{v_1 - v_2}{\rho_0} \right\|_{L^1(0,T;H^1_\Omega)} + \left\| \frac{h_1 - h_2}{\rho_0} \right\|_{L^2((0,T) \times \Omega)} \leq C_1 \left\| \frac{f_1 - f_2}{\rho_F} \right\|_{L^2((H^3 \cap H^3_{\gamma})))}.$$  

(3.78)

Now, using (3.70), we have $\forall \phi \in H^3_{\gamma}(0,1)$,

$$\left\langle \frac{F(u_1) - F(u_2)}{\rho_F}, \phi \right\rangle_{H^{-2}, H^2_{\gamma}} \leq \frac{1}{2} \left\| \frac{\rho_0^3(t)}{\rho_F(t)} \right\| \left\| \int_{0}^{1} \left( \frac{u_1^2 - u_2^2}{\rho_0^3} \right) \phi_x \right\|_{L^2((0,T) \times \Omega)},$$

and so,

$$\left\| \frac{F(u_1) - F(u_2)}{\rho_F} \right\|_{H^{-2}(0,1)} \leq C_2 \left( \left\| \frac{u_1}{\rho_0} \right\|_{H^1_{\gamma}(0,1)} + \left\| \frac{u_2}{\rho_0} \right\|_{H^1_{\gamma}(0,1)} \right),$$

where the constant $C_2$ is the one which has appeared in (3.73). Using this, we have

$$\left\| \frac{\Lambda(f_1) - \Lambda(f_2)}{\rho_F} \right\|_{L^2(0,T;H^{-2}(0,1))} \leq C_2 \left( \left\| \frac{u_1}{\rho_0} \right\|_{L^2(0,T;H^1_{\gamma}(0,1))} + \left\| \frac{u_2}{\rho_0} \right\|_{L^2(0,T;H^1_{\gamma}(0,1))} \right) \left\| \frac{u_1 - u_2}{\rho_0} \right\|_{C_0([0,T];H^{-1}(0,1))},$$

(3.79)

$$\leq C_2 \left( 2C_1 \left\| u_0 \right\|_{H^{-1}(0,1)} + C_1 \left\| \frac{f_1}{\rho_F} \right\|_{L^1(H^3 \cap H^3_{\gamma})))} + C_1 \left\| \frac{f_2}{\rho_F} \right\|_{L^1(H^3 \cap H^3_{\gamma}))) \right\| \frac{f_1 - f_2}{\rho_F} \right\|_{L^2((H^3 \cap H^3_{\gamma})))},$$

(3.80)

$$\leq 4C_1^2 C_2 R \left\| \frac{f_1 - f_2}{\rho_F} \right\|_{L^2((H^3 \cap H^3_{\gamma})))},$$

(3.81)

$$\leq \frac{1}{2} \left\| \frac{f_1 - f_2}{\rho_F} \right\|_{L^2((H^3 \cap H^3_{\gamma})))},$$

(3.82)

where we make use of the estimate (3.47) to deduce the inequality (3.80) from (3.79) and then in (3.80), we used $f_1, f_2 \in B(0,R)$ and $\left\| u_0 \right\|_{H^{-1}(0,1)} \leq R$. Finally, thanks to the choice of $R$ in (3.77), we get (3.82) leading to the fact that the map $\Lambda$ is a contraction map on $B(0,R)$.

\[ \text{□} \]
3.3.3. Proof of Theorem 1.1.

Proof. The proof is followed by the Lemma 3.8 and 3.9. Thanks to those results, we can apply the Banach fixed point argument which ensures the existence of a unique fixed point of the map $A$ in $B(0, R) \subset F$. Denote the fixed point by $f_0 \in F$.

Now, in terms of to Proposition 3.7, with the above $f_0 \in B(0, R)$ and initial data $\|u_0\|_{H^{-1}(0,1)} \leq R$, there exists a control $h \in V$ that satisfies the estimate (3.47). Then, the property $\lim_{t \to T^-} \rho_0(t) = 0$ forces that

$$(u(T, x), v(T, x)) = (0, 0), \quad \forall x \in (0, 1).$$

This proves the local null-controllability of the nonlinear system (1.3)–(1.5)–(1.6). \hfill \Box

4. Control acting in the elliptic equation

In this section, we study the controllability property of the system (1.4)–(1.5)–(1.6), that is when a control locally acts only in the elliptic equation. As earlier, a suitable Carleman estimate is required to get an observability inequality for the linearized model (1.8)–(1.5)–(1.6).

4.1. A global Carleman estimate. We have the following estimate.

Theorem 4.1 (Carleman estimate: control in elliptic eq.). Let the weight functions $(\varphi, \xi)$ be given by (3.2). Then, there exist positive constants $\lambda_0, s_0 := \mu_0(T + T^2)$ for some $\mu_0$ and $C$ which depend on $\gamma_1, \gamma_2, a, b, c$ and the set $\omega$, such that we have the following estimate satisfied by the solution to (3.7),

$$I_{KS}(s, \lambda; \sigma) + I_E(s, \lambda; \psi) \leq Cs^{11} \lambda^{12} \int_0^T \int_\omega e^{-2s\varphi}\xi^{11}\psi|\psi|^2,$$

for all $\lambda \geq \lambda_0$ and $s \geq s_0$, where $I_{KS}(s, \lambda; \sigma)$ and $I_E(s, \lambda; \psi)$ are introduced by (3.5) and (3.6) respectively.

Proof. Adding the two individual Carleman estimates for KS and elliptic parts, we have the following estimate (as obtained in (3.13)),

$$I_{KS}(s, \lambda; \sigma) + I_E(s, \lambda; \psi) \leq C \left[ s^7 \lambda^8 \int_0^T \int_{\omega_0} e^{-2s\varphi} \xi^7|\sigma|^2 + s^3 \lambda^4 \int_0^T \int_{\omega_0} e^{-2s\varphi} \xi^3|\psi|^2 \right],$$

for all $\lambda \geq \lambda_0$ and $s \geq s_0(T + T^2)$ where $\lambda_0$ and $\mu_0$ are as introduced in the Step 1 of the proof of Theorem 3.1.

Thus, we just need to absorb the integral related to $\sigma$ from the right hand side of (4.2).

Absorbing the observation integral in $\psi$. Recall the adjoint system (3.7), one has (since $a \neq 0$),

$$\sigma = \frac{1}{a} (-\psi_{xx} + cv), \quad \text{in } Q_T.$$

We also recall the smooth function $\phi \in C^{\infty}_c(\omega_1)$ as given by (3.14).

Using these, we have

$$s^7 \lambda^8 \int_0^T \int_{\omega_0} e^{-2s\varphi} \xi^7|\sigma|^2 \leq s^7 \lambda^8 \int_0^T \int_{\omega_1} \phi e^{-2s\varphi} \xi^7|\sigma|^2$$

$$= \frac{1}{a} s^7 \lambda^8 \int_0^T \int_{\omega_1} \phi e^{-2s\varphi} \xi^7 \sigma (-\psi_{xx} + cv)$$

$$= B_1 + B_2.$$

(i) Let us first calculate $B_1$.

$$B_1 = \frac{1}{a} s^7 \lambda^8 \int_0^T \int_{\omega_1} \left[ \phi_x e^{-2s\varphi} \xi^7 \sigma_x + \phi (e^{-2s\varphi} \xi^7)_x \sigma_x + \phi_x e^{-2s\varphi} \xi^7 \sigma_x \psi_x \right]$$

$$= -\frac{1}{a} s^7 \lambda^8 \int_0^T \int_{\omega_1} \left[ \phi_x e^{-2s\varphi} \xi^7 + 2\phi_x (e^{-2s\varphi} \xi^7)_x + \phi (e^{-2s\varphi} \xi^7)_x \right] \sigma \psi$$

$$+ \phi e^{-2s\varphi} \xi^7 \sigma_x \psi + (2\phi_x e^{-2s\varphi} \xi^7 + 2\phi (e^{-2s\varphi} \xi^7)_x) \sigma_x \psi.$$
Next, using the bounds (3.4) and applying the Cauchy-Schwarz and Young’s inequalities successively, we get

\[ |B_1| \leq C s^7 \lambda^8 \int_0^T \int_{\omega_1} (e^{-2s\varphi \xi^7})|\sigma||\psi| + C s^8 \lambda^9 \int_0^T \int_{\omega_1} (e^{-2s\varphi \xi^8})|\sigma||\psi| \]

\[ + C s^9 \lambda^{10} \int_0^T \int_{\omega_1} (e^{-2s\varphi \xi^9})|\sigma||\psi| + C s^7 \lambda^8 \int_0^T \int_{\omega_1} (e^{-2s\varphi \xi^7})|\sigma_{xx}||\psi| \]

\[ + C s^7 \lambda^8 \int_0^T \int_{\omega_1} (e^{-2s\varphi \xi^7})|\sigma_x||\psi| + C s^8 \lambda^9 \int_0^T \int_{\omega_1} (e^{-2s\varphi \xi^8})|\sigma_x||\psi| \]

\[ \leq 3\epsilon s^7 \lambda^8 \int_0^T \int_{\omega_1} e^{-2s\varphi \xi^7}|\sigma|^2 + \epsilon s^8 \lambda^4 \int_0^T \int_{\omega_1} e^{-2s\varphi \xi^3}|\sigma_{xx}|^2 + 2\epsilon s^5 \lambda^6 \int_0^T \int_{\omega_1} e^{-2s\varphi \xi^5}|\sigma_x|^2 \]

\[ + \frac{C}{\epsilon} \int_0^T \int_{\omega_1} e^{-2s\varphi} (s^7 \lambda^8 \xi^7 + s^9 \lambda^{10} \xi^9 + s^{11} \lambda^{12} \xi^{11}) |\psi|^2. \]

(ii) Estimate of $B_2$ can be taken care in a similar manner. We then fix a small $\epsilon > 0$ such that there is a positive constant $C$ with the following:

\[ I_{KS}(s, \lambda; \sigma) + I_E(s, \lambda; \psi) \leq C s^{11} \lambda^{12} \int_0^T \int_{\omega} (e^{-2s\varphi \xi^{11}})|\psi|^2, \]

for all $\lambda \geq \lambda_0$ and $s \geq \mu_0(T + T^2)$.

This ends the proof. \qed

4.2. Observability inequality and null-controllability of the linearized model. Using the Carleman estimate (4.1), one can prove the following observability inequality.

**Proposition 4.2** (Observability inequality: control in elliptic eq.). There exists a positive constant $K$ depending on $\omega$, $a$, $b$, $c$, $\gamma_1$, $\gamma_2$ but not $T$, such that we have the following observability inequality

\[ \|\sigma(0, \cdot)\|^2_{H^2_0(\omega, 1)} + \|\psi(0, \cdot)\|^2_{H^2(\omega, 1)} \leq Ke^{\Phi} \int_0^T \int_{\omega} |\psi|^2, \]

where $(\sigma, \psi)$ is the solution to the adjoint system (3.7).

**Proof.** The proof of above proposition will be done in the same spirit of the proof of Proposition 3.4. We skip the details. \qed

Once we have the above observability inequality, the proof of Item 2 of Theorem 1.3 will be followed as given below.

**Proof of Theorem 1.3–Item 2.** This proof is in the same spirit of Theorem 1.2–Item 1. In fact, one can arrive that there exist positive constants $K$, $K'$ independent on $T$, such that the following observability inequality holds

\[ \|\sigma(0, \cdot)\|^2_{H^2_0(\omega, 1)} + \|\psi(0, \cdot)\|^2_{H^2(\omega, 1)} \leq K e^{\Phi} \int_0^T \int_{\omega} e^{-\frac{K}{\epsilon^2}} |\sigma|^2 \]

where $(\sigma, \psi)$ is the solution to the adjoint system (3.7). Then, following a similar technique as [23, Theorem 3.2], we can conclude the desired null-controllability result. \qed

4.3. Null-controllability of the nonlinear system: proof of Theorem 1.2. To prove the null-controllability of the system (1.4)–(1.5)–(1.6), we will follow the same procedure as described in Section 3.3.

In what follows, recall the weight functions $\rho_0$, $\rho_\mathcal{F}$ and the spaces $\mathcal{F}$, $\mathcal{Y}$ from Section 3.3. Then, we consider the following system with a source term $f \in \mathcal{F}$,

\[
\begin{aligned}
\begin{cases}
u_t + \gamma_1 u_{xxx} + u_{xxx} + \gamma_2 u_{xx} = f + av & \text{in } Q_T, \\
-\nu_{xx} + c v = bu + \chi \omega h, & \text{in } Q_T, \\
u(t, 0) = 0, \ u(t, 1) = 0, & t \in (0, T), \\
u_x(t, 0) = 0, \ u_x(t, 1) = 0 & t \in (0, T) \\
v(t, 0) = 0, \ v(t, 1) = 0 & t \in (0, T), \\
u(0,x) = u_0(x), & x \in (0, 1) \end{cases}
\end{aligned}
\]

with $u_0 \in H^{-1}(0, 1)$.

Similar to Proposition 3.7, one can prove the following results.
Theorem 1.1

1.1. We omit the proof here.

5.1. We skip the details here since it can be done in a similar fashion as in the previous case.

5. Uniform null-controllability of a coupled parabolic system

This section is devoted to study the null-controllability of a two parabolic system with a degenerating parameter, namely the system (1.10).

First, our objective is to study the uniform (w.r.t. \( \varepsilon \)) null-controllability of the associated linearized model (1.11) by means of single localized interior control \( h \) which ensures the convergence of the system to the concerned parabolic-elliptic control system.

Let us write the adjoint to the system (1.11):

\[
\begin{aligned}
-\sigma_t + \gamma_1 \sigma_{xxxx} - \varepsilon \psi_{xx} + \psi_{xx} &= b\psi + g_1, & (t, x) \in Q_T, \\
-\varepsilon \psi_t - \psi_{xx} + c\psi &= a\psi + g_2, & (t, x) \in Q_T, \\
\sigma(t, 0) &= 0, & t \in (0, T), \\
\sigma(x, t) &= 0, & t \in (0, T), \\
\psi(t, 0) &= 0, & t \in (0, T), \\
\psi(T, x) &= \sigma(T, x), & x \in (0, 1), \\
\psi(T, x) &= \psi_T(x), & x \in (0, 1).
\end{aligned}
\]

with given final data \((\sigma_T, \psi_T) \in H_0^2(0, 1) \times H_0^1(0, 1)\) and source term \((g_1, g_2) \in L^2(Q_T)\).

We have the following energy estimate.

Proposition 5.1. Let \((\sigma_T, \psi_T) \in H_0^2(0, 1) \times H_0^1(0, 1), (g_1, g_2) \in [L^2(0, T; L^2(0, 1))]^2\) and \(\varepsilon \in (0, 1]\). Then the above system (5.1) possesses a unique solution \((\sigma, \psi) \in C^3(0, T; H_0^2(0, 1) \times H_0^1(0, 1)) \cap L^2(0, T; H_0^2(0, 1) \times H_0^1(0, 1))\). Furthermore, there exists a positive constant \(C\) independent of \(\varepsilon\) such that the solution of (5.1) satisfies the following estimates:

\[
\|\sigma\|_{L^\infty(0, T; H_0^2(0, 1))} + \|\psi\|_{L^\infty(0, T; H_0^1(0, 1))} \leq C \left( \|\sigma_T\|_{H_0^2(0, 1)}^2 + \|\psi_T\|_{H_0^1(0, 1)}^2 + \|(g_1, g_2)\|_{L^2(0, T; L^2(0, 1))}^2 \right).
\]

Proof. This proof is standard and can be done following Proposition 2.1. We omit the proof here. \(\square\)

5.1. A global Carleman estimate. Before going to prescribe a Carleman estimate for the solution \((\sigma, \psi)\) to (5.1) with \(g_1 = g_2 = 0\), we introduce the following notation: for any function \(\psi \in C^2(\overline{Q_T})\), we denote

\[
I_{H(\gamma)}(s, \lambda; \psi) := \varepsilon s^{-1} \int_{Q_T} e^{-2s \gamma^1 \xi^{-1}} |\psi_t|^2 + s^3 \lambda^4 \int_{Q_T} e^{-2s \gamma^3 \xi^{-1}} |\psi|^2 \\
+ s \lambda^2 \int_{Q_T} e^{-2s \gamma^2 \xi^{-1}} |\psi_x|^2 + s^{-1} \int_{Q_T} e^{-2s \gamma^1 \xi^{-1}} |\psi_{xx}|^2.
\]

Let us write the following joint Carleman estimate.

Theorem 5.2. Let the weight functions \((\varphi, \xi)\) be given by (3.2) and \(\varepsilon \in (0, 1]\). Then, there exist positive constants \(\tilde{\lambda}_0, \tilde{s}_0 := \mu_0(T + T^2)\) for some \(\mu_0 > 0\) and \(C\) which depend on \(\gamma_1, \gamma_2, a, b, c\) and the set \(\omega\) but independent in \(\varepsilon\), such that we have the following estimate satisfied by the solution to (5.1) with \(g_1 = g_2 = 0\),

\[
I_{KS}(s, \lambda; \sigma) + I_{H(\varepsilon)}(s, \lambda; \psi) \leq C s^{11} \lambda^{12} \int_{0}^{T} e^{-2s \gamma^1 \xi^{-1}} |\psi|^2,
\]

for all \(\lambda > \tilde{\lambda}_0\) and \(s \geq \tilde{s}_0\), where \(I_{KS}(s, \lambda; \sigma)\) and \(I_{H(\varepsilon)}(s, \lambda; \psi)\) are introduced in (3.5) and (5.2) respectively.
Proof. Recall the Carleman estimate (3.9) from Proposition 3.2:

$$I_{K_S}(s, \lambda; \sigma) \leq C \int_Q e^{-2s\varphi} |\psi|^2 + Cs^2 \lambda^2 \int_0^T \int_{\omega} e^{-2s\varphi} |\sigma|^2,$$

(5.4)

for all $\lambda \geq \lambda_1$ and $s \geq s_1$.

One can also prove the following Carleman estimate:

$$I_{H(\varepsilon)}(s, \lambda; \psi) \leq C \int_Q e^{-2s\varphi} |\sigma|^2 + Cs^2 \lambda^2 \int_0^T \int_{\omega} e^{-2s\varphi} |\psi|^2,$$

(5.5)

for all $\lambda \geq \tilde{\lambda}_1$ and $s \geq \tilde{s}_1 := \tilde{\mu}_1(T + T^2)$ for some $\tilde{\mu}_1 > 0$, where $I_{H(\varepsilon)}(s, \lambda; \psi)$ is introduced by (5.2); we refer [15, Lemma 3.2] for rigorous computations.

Thereafter, adding both the Carleman estimates (5.4) and (5.5) and using similar techniques as used in the proofs of Theorem 3.1 and 4.1, one can obtain the required estimate (5.3).

5.2. Observability inequality and null-controllability. In this section, we shall present the proof of uniform null-controllability of the system (1.11). As usual, to prove Theorem 1.6, it is enough to prove a suitable observability inequality. We write it below.

**Proposition 5.3** (Observability inequality). Let $\varepsilon \in (0, 1]$. There exists a positive constant $K$ independent in $\varepsilon$ and $T$ such that the following inequality holds

$$\|\sigma(0, \cdot)\|_{H^2_0(0, 1)}^2 + \varepsilon \|\psi(0, \cdot)\|_{H^1_0(0, 1)}^2 \leq Ke^{K T} \int_0^T \int_\omega |\psi|^2,$$

(5.6)

where $(\sigma, \psi)$ is the solution of the adjoint system (5.1).

**Proof.** Recall the modified weight functions $\mathfrak{G}$ and $\mathfrak{S}$ defined by (3.30).

By construction we have that $\varphi = \mathfrak{G}$ and $\xi = \mathfrak{S}$ in $(T/2, T) \times (0, 1)$, using what we observe that

$$\int_{T/2}^T \int_0^1 e^{-2s\varphi} \mathfrak{G}^2 |\sigma|^2 = \int_{T/2}^T \int_0^1 e^{-2s\varphi} \mathfrak{S}^2 |\sigma|^2, \quad \int_{T/2}^T \int_0^1 e^{-2s\varphi} \mathfrak{G}^2 |\psi|^2 = \int_{T/2}^T \int_0^1 e^{-2s\varphi} \mathfrak{S}^2 |\psi|^2$$

Therefore, using the Carleman estimate (5.3) one has

$$s^2 \lambda^4 \int_{T/2}^T \int_0^1 e^{-2s\varphi} \mathfrak{G}^2 |\sigma|^2 + s^4 \lambda^4 \int_{T/2}^T \int_0^1 e^{-2s\varphi} \mathfrak{G}^2 |\psi|^2 \leq Cs^{11} \lambda^{12} \int_0^T \int_\omega e^{-2s\varphi} \mathfrak{S}^2 |\psi|^2,$$

(5.7)

for all $\lambda \geq \lambda_0$ and $s \geq s_0$ where the constant $C > 0$ does not depend on $T$.

We also recall the function $\eta \in C^1([0, T])$ such that

$$\eta = 1 \text{ in } [0, T/2], \quad \eta = 0 \text{ in } [3T/4, T],$$

and that $\text{Supp } (\eta') \subset [T/2, 3T/4]$. Then, using the equations of $\sigma$ and $\psi$ from the adjoint system (5.1) with $g_1 = g_2 = 0$, it is clear that the pair $(\tilde{\sigma}, \tilde{\psi})$ with $\tilde{\sigma} = \eta \sigma$, $\tilde{\psi} = \eta \psi$ satisfies the following set of equations

$$\begin{cases}
-\tilde{\sigma}_t + \gamma_1 \tilde{\sigma}_{xxx} - \tilde{\sigma}_{xx} + \gamma_2 \tilde{\sigma}_{xx} = b \tilde{\psi} - \eta' \sigma, & \text{in } Q_T, \\
-\varepsilon \tilde{\psi}_t - \tilde{\psi}_{xx} + c \tilde{\psi} = \tilde{a} \tilde{\sigma} - \varepsilon \eta' \psi, & \text{in } Q_T, \\
\tilde{\sigma}(t, 0) = \tilde{\sigma}(t, 1) = 0, & \text{in } (0, T), \\
\tilde{\sigma}_t(t, 0) = \tilde{\sigma}_x(t, 1) = 0, & \text{in } (0, T), \\
\tilde{\psi}_t(t, 0) = \tilde{\psi}_x(t, 1) = 0, & \text{in } (0, T), \\
\tilde{\sigma}(T) = 0, \tilde{\psi}(T) = 0 & \text{in } (0, 1).
\end{cases}$$

Thanks to Proposition 5.1, we have the following energy estimate for $(\tilde{\sigma}, \tilde{\psi})$:

$$\|\eta \sigma\|^2_{L^\infty(0, T; H^1_0(0, 1))} + \varepsilon \|\eta \psi\|^2_{L^\infty(0, T; H^1_0(0, 1))} \leq C \left( \|\eta \sigma\|^2_{L^1((Q_T)} + \varepsilon \|\eta \psi\|^2_{L^1(Q_T)} \right)$$

$$\leq C \left( \|\sigma\|^2_{L^1((T/2, 3T/4) \times (0, 1))} + \|\psi\|^2_{L^1((T/2, 3T/4) \times (0, 1))} \right),$$

(5.8)

for some constant $C > 0$ that neither depends on $T$ nor on $\varepsilon$.

Then, proceeding as similar as the proof of Proposition 3.4, one can prove the required observability estimate (5.6).
Null-controllability. The above observability inequality gives the existence of a control \( h_\varepsilon \in L^2((0,T) \times \omega) \) for the linear system (1.11) and one can show that \( h_\varepsilon \) satisfies
\[
\|h_\varepsilon\|_{L^2((0,T)\times\omega)} \leq K_{\varepsilon}^{K/T} \left( \|u_0\|_{H^{-2}(0,1)} + \varepsilon \|v_0\|_{H^{-1}(0,1)} \right),
\]
for some constant \( K > 0 \) independent in \( T, (u_0, v_0) \) and \( \varepsilon \) (as declared in Theorem 1.6). Then, using the source term method and fixed point argument as previous, we can also handle the local uniform null-controllability for the nonlinear system (1.10) that is Theorem 1.5; we skip the details.

Remark 5.4. From the uniform estimate of the control \( h_\varepsilon \), it is clear that there exists some \( h \in L^2((0,T) \times \omega) \) such that
\[
h_\varepsilon \rightharpoonup h \quad \text{weakly in } L^2((0,T) \times \omega) \quad \text{as } \varepsilon \to 0.
\]
Moreover, it can be shown that the above function \( h \) is indeed a control function for the associated parabolic-elliptic system, namely (1.8)–(1.5)–(1.6). Study of the convergence analysis of the controls \( (h_\varepsilon)_\varepsilon \) as well as the associated solutions \( (u_\varepsilon, v_\varepsilon)_\varepsilon \) towards the control, solution pair of the system (1.8)–(1.5)–(1.6) can be done by following a similar approach as described in [16, Section 5] (for second order parabolic-elliptic system). We skip the details here.

6. Conclusions and open questions

Let us briefly recall the whole study of the present article. We have considered a coupled parabolic-elliptic system containing a fourth-order parabolic equation, namely KS-KdV and a second-order elliptic equation. Local distributed null-controllability of the aforementioned model has been proved by means of a localized interior control acting only in one equation where the nonlinearity is \( uu \).

In both cases, we have established global Carleman estimate which gives rise to some suitable observability inequality and then the usual duality argument provides the null-controllability for the associated linearized models. To this end, we employed the source term method and Banach fixed point argument to obtain the local-null controllability for the nonlinear systems.

We conclude our article by studying an asymptotic behaviour of a control system containing two parabolic equations with a degenerating parameter \( \varepsilon \in (0,1] \). We have shown uniform null-controllability of the concerned system with respect to the parameter \( \varepsilon \). This ensures the null-controllability of the limiting system which is essentially a parabolic-elliptic model, namely, the system (1.8)–(1.5)–(1.6) of our paper.

An immediate question indicates about the uniform null-controllability of a similar type of system when a localized interior control acts in the KS-KdV equation. We refer to the work [15], where the authors have utilized the Carleman estimate in a delicate way to answer this issue for the second order two-parabolic system. We shall consider this matter in the near future for the coupled system of fourth and second order parabolic pdes.

Apart from this, one can think of some other issues which has been studied throughout years for various model. Let us mention some of those which can lead some future research direction.

- Boundary null-controllability. In [36], the authors studied the boundary local null-controllability of some nonlinear parabolic-elliptic system by means of a single control force acting at the left end of the Dirichlet boundary of parabolic component. They handled the linear model by using a boundary Carleman estimate and then the local inversion technique has been applied to conclude the nonlinear system.

Coming to the KS equation, it is known that to deal with the boundary controllability with single control force, one needs to put some restriction on the anti-diffusion parameter \( \gamma_2 \). This issues can be overcome by introducing another control in the dynamics (see [10], [11] for more details). Moreover, in [13], the authors studied the local null-controllability of stabilized KS by means of three boundary controls. In the spirit of these aforementioned works, it will be interesting to consider the system with less number of control:
\[
\begin{align*}
  u_t + \gamma_1 uxxxx + uxxxx + \gamma_2uxx + uu_x = \varepsilon v, & \quad \text{in } Q_T, \\
  -v_x + cv = bu, & \quad \text{in } Q_T, \\
  u(0,x) = u_0(x), & \quad \text{in } (0,1),
\end{align*}
\]
(6.1)
with any of the following boundary control data
\[
\begin{align*}
  & \quad \begin{cases}
    u(t,0) = q_0(t), & \quad t \in (0,T), \\
    u_x(t,0) = 0, & \quad t \in (0,T), \\
  \end{cases} \quad \text{or} \quad \begin{cases}
    u(t,0) = 0, & \quad t \in (0,T), \\
    u_x(t,0) = 0, & \quad t \in (0,T), \\
  \end{cases} \\
  & \quad \begin{cases}
    v(t,0) = 0, & \quad t \in (0,T), \\
  \end{cases} \quad \text{or} \quad \begin{cases}
    v(t,0) = p_0(t), & \quad t \in (0,T),
  \end{cases}
\end{align*}
\]
(6.2)

- Null-controllability in 2D setting. Controllability of KS-KdV-elliptic equation in higher dimension (in particular 2D) would be an interesting problem to study. For the case of Chemotaxis model (second-order nonlinear parabolic-elliptic equation) [34], the authors obtained the local null-controllability result in bounded domain \( \Omega \subset \mathbb{R}^N, N \geq 1 \). For the KS equation, the author in [49] studied the local null-controllability in 2D by means of
a boundary control by assuming some conditions on the boundary. Thanks to the recent result [42] concerning the spectral inequality for the bi-Laplace operator, the null-controllability of the following system

\[
\begin{aligned}
&u_t + \Delta^2 u = \chi \omega h, \quad (t, x) \in (0, T) \times \Omega, \\
&u = 0, \quad \frac{\partial u}{\partial \nu} = 0, \quad (t, x) \in (0, T) \times \partial \Omega, \\
&u(0, x) = u_0(x), \quad x \in \Omega,
\end{aligned}
\]  

(6.3)
can be established. On the light of this discussion, it is reasonable to pose a local controllability problem regarding KS-elliptic model in 2D, for instance:

\[
\begin{aligned}
&u_t + \Delta^2 u + \nu \Delta u + \frac{1}{2} |\nabla u|^2 = v + \chi \omega g, \quad (t, x) \in (0, T) \times \Omega, \\
&-\Delta v + \gamma v = \delta u, \quad (t, x) \in (0, T) \times \Omega,
\end{aligned}
\]  

(6.4)

with the boundary and initial conditions as in (6.3).

- **Boundary feedback stabilization.** Beside the controllability of the parabolic-elliptic system, there are some works which addresses the stabilization issues of the same. In [47], [46], boundary feedback stabilization of a coupled parabolic-elliptic system

\[
\begin{aligned}
u_{tx} - \alpha \nu + \lambda \nu = \alpha v, & \quad (t, x) \in (0, T) \times (0, L), \\
-u_{xx} + \lambda u = \beta u, & \quad (t, x) \in (0, T) \times (0, L), \\
u(t, 0) = 0, \quad u(t, 1) = h(t), & \quad t \in (0, T), \\
v(t, 0) = 0, \quad v(t, 1) = 0, & \quad t \in (0, T), \\
u(0, x) = u_0(x), & \quad x \in (0, 1)
\end{aligned}
\]  

(6.5)
has been explored using the well-known backstepping method (taking some conditions on the system parameters). In the paper [18], the authors utilized a new backstepping technique to answer the exponential stabilization problem for the Kuramoto-Sivashinsky equation with Dirichlet boundary data. Similar question can be treated for the case of KS-KdV-elliptic equation.

- **Inverse problem on KS-elliptic model.** Global Carleman estimate is not only a crucial technique for controllability aspects but also it has immense importance on the various kind of inverse problem. In [5], [35], the authors have utilized this approach to retrieve the diffusion ($\gamma_1 = \gamma_1(x)$) and anti-diffusion ($\gamma_2 = \gamma_2(x)$) coefficients of the KS equation

\[
\begin{aligned}
u_{tx} + (\gamma_1(x)u_{xx})_{xx} + \gamma_2(x)u_{xx} + uu_x = g, & \quad (t, x) \in (0, T) \times (0, L), \\
u(t, 0) = h_1(t), \quad u(t, 1) = h_2(t), & \quad t \in (0, T), \\
u_x(t, 0) = h_3(t), \quad u_x(t, 1) = h_4(t), & \quad t \in (0, T), \\
u(0, x) = u_0(x), & \quad x \in (0, L)
\end{aligned}
\]  

(6.6)
from the measurements of the traces ($u_{xx}(\cdot, 0), u_{xx}(\cdot, 0)$) of the solution at one end of the domain and from the information of state $u(T_0, \cdot), T_0 \in (0, T)$ of the corresponding system at some positive time. Recently for linear KS model [29], using an optimization-based approach, the authors explored the stability result for the anti-diffusion coefficients by means of the measurement of the state at final time $T$. For more information on this topic we refer to the work [30]. Due to the importance of inverse problem in various fields in applied science and biology, showing the uniqueness of diffusion coefficients of KS-KdV-elliptic model would be an interesting topic of research.

**Acknowledgements**

The work of the first author is partially supported by the French government research program “Investissements d’Avenir” through the IDEX-ISITE initiative 16-IDEX-0001 (CAP 20-25). The second author acknowledges the supports from Department of Atomic Energy and NBHM Fellowship, Grant No. 0203/16(21)/2018-R&D-II/10708.

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