Rotation of a rigid satellite with a fluid component
A new light onto Titan’s obliquity

Gwenaël BOUÉ · Nicolas RAMBAUX · Andy RICHARD

Abstract We revisit the rotation dynamics of a rigid satellite with either a liquid core or a global sub-surface ocean. In both problems, the flow of the fluid component is assumed inviscid. The study of a hollow satellite with a liquid core is based on the Poincaré-Hough model which provides exact equations of motion. We introduce an approximation when the ellipticity of the cavity is low. This simplification allows to model both types of satellite in the same manner. The analysis of their rotation is done in a non-canonical Hamiltonian formalism closely related to Poincaré’s “forme nouvelle des équations de la mécanique”. In the case of a satellite with a global ocean, we obtain a seven-degree of freedom system. Six of them account for the motion of the two rigid components, and the last one is associated with the fluid layer. We apply our model to Titan for which the origin of the obliquity is still a debated question. We show that the observed value is compatible with Titan slightly departing from the hydrostatic equilibrium and being in a Cassini equilibrium state.

Keywords multi-layered body · spin-orbit coupling · Cassini state · synchronous rotation · analytical method · Io · Titan

1 Introduction
The spin pole of Titan, Saturn’s largest moon, is lying close to the plane defined by its orbit pole and the Laplace pole (Stiles et al., 2008, 2010). This observation,
made by the RADAR instrument of the Cassini mission, suggests that Titan is in
(or very close to) a Cassini state (Colombo, 1966; Peale, 1969). For a rigid body,
the equilibrium obliquity is a function of its moments of inertia. Those of Titan
have been deduced from its Stokes coefficients $J_2 = (33.599 \pm 0.332) \times 10^{-6}$ and
$C_{22} = (10.121 \pm 0.029) \times 10^{-6}$ and from the hydrostatic equilibrium hypothesis
implying a mean moment of inertia $I/(mR^2) = 0.3431$ (Iess et al., 2012, SOL1a),
where $m$ and $R$ are the mass and radius of Titan, respectively. The assumed
hydrostatic equilibrium is suggested by the ratio $J_2/C_{22} \approx 10/3$ which is precisely
the expected value for a hydrostatic body (e.g., Rappaport et al., 1997). Assuming
these values, if Titan were rigid and in a Cassini equilibrium state, its obliquity
would be 0.113 deg (Bills and Nimmo, 2011), i.e. about one third of the radiometric
value 0.32 deg (Stiles et al., 2008, 2010; Meriggiola et al., 2016). To match the
observations, the frequency of the free libration in latitude must be reduced by
a factor 0.526 (Bills and Nimmo, 2011). In particular, this would be the case
if $I/(mR^2)$ were increased to 0.45 (ibid.), a value exceeding 2/5 obtained for a
homogeneous body, as if the mass of the satellite was concentrated toward the
surface. This result leads to think that the observed obliquity is that of a thin
shell partially decoupled from the interior by, e.g., a global ocean (ibid.).

The idea that the ice-covered satellites of the outer planets hold a global under-
neath ocean has already been proposed based on models of their internal structures
(e.g., Lewis, 1971). Even the dwarf planet Pluto is suspected to harbour a subsur-
fase ocean (Nimmo et al., 2016). In the case of Titan, the presence of the ocean
is also revealed by laboratory experiments on the behaviour of water-ammonia
compounds at high pressure and low temperature (Grasset and Sotin, 1996), by
the detection of electromagnetic waves in its atmosphere (Béghin et al., 2012) and
by the high value of its Love number $k_2$ (Iess et al., 2012).

A dynamical problem closely related to the present one is that of a hollow
satellite with a liquid core as described by the Poincaré-Hough model (Poincaré,
1910; Hough, 1895). For this specific problem, Poincaré (1901) developed a new
Lagrangian formalism, based on the properties of the Lie group acting on the
configuration space, which allows to derive the equations of motion in a very
simple and elegant manner. Such a system is characterised by four degrees of
freedom, three of them being associated with the rotation of the rigid mantle and
the last one being due to the motion of the liquid core (e.g., Henrard, 2008).
Applying this model to Jupiter’s satellite Io, Henrard (2008) observed that the
frequency of the additional degree of freedom is close to the orbital frequency and
should thus multiply the possibility of resonances. For Titan, we shall expect the
same conclusion due to the presence of the ocean, but unfortunately, Poincaré’s
model relies on the concept of a fluid simple motion which cannot be rigorously
transposed to the case of a satellite with a global subsurface ocean.

In the case of Titan, the effect of an ocean on the rotation dynamics has
been studied numerically using Euler’s rotation equations taking into account
the gravitational interaction of Saturn on each layer, the pressure torques at the two
fluid-solid boundaries, and the gravitational coupling between the interior and the
shell (Baland et al., 2011, 2014; Noyelles and Nimmo, 2014). The elastic deforma-
tion of the solid layers and the atmospheric pressure have also been included in a
modelling of the libration in longitude (Richard et al., 2014) and in a modelling
of the Chandler polar motion (Coyette et al., 2016). Despite several arguments in
favour of an ocean, this model does not easily explain the tilt of Titan’s spin-axis.
Indeed, under the hydrostatic equilibrium hypothesis, Baland et al. (2011) and Noyelles and Nimmo (2014) found that the obliquity of the Cassini state remains bounded below 0.15 deg, i.e. about one half of the observed value. There thus seemed to be a need for a significant resonant amplification to bring the system out of the Cassini equilibrium (Baland et al., 2011; Noyelles and Nimmo, 2014). However these studies do not invoke the same mode as the origin of the resonant amplification. In addition, this solution does not agree with extended observations of the spin-axis orientation (Meriggiola and Iess, 2012). The model has then been amended to allow the Cassini state obliquity to reach the observed 0.32 deg, but this has only been made possible after releasing the hydrostatic shape assumption leaving the ratio $J_2/C_{22} \approx 10/3$ unexplained (Baland et al., 2014).

It should be stressed that models developed thus far discard the rotation of the ocean relative to the inertial frame. This is a valid assumption to reproduce librations in longitude (e.g., Richard, 2014), but not anymore for precession motion. By consequence, the associated dynamical system only has 6 degrees of freedom equally shared by the rigid interior and the shell (Noyelles and Nimmo, 2014). Yet, a comparison of this problem with that of a satellite with a liquid core strongly suggests that a three layered body must have 7 degrees of freedom, one of which being brought by the ocean. Here, we aim at building a new dynamical model accounting for the rotation of the liquid layer as done by Mathews et al. (1991) for the Earth. More recently, the latter model has been adapted to the study of the Moon (Dumberry and Wieczorek, 2016) and of Mercury (Peale et al., 2016). Here we reconsider the problem with a Hamiltonian approach. In that scope, we first extend the Lagrangian formalism described in (Poincaré, 1901) to a non-canonical Hamiltonian formalism allowing to study relative equilibria in a very efficient manner as in (Maddocks, 1991; Beck and Hall, 1998). The method has proven its efficiency in the context of a rigid satellite in circular orbit (Beck and Hall, 1998), in the analysis of the two rigid body problem (Maciejewski, 1995), and in several studies of the attitude of a satellite with a gyrostaf (e.g., Hall and Beck, 2007; Wang and Xu, 2012, and references therein). The approach is described in Sect. 2 and illustrated in the case of a rigid satellite in Sect. 3. We revisit the problem of a moon with a fluid core with this approach and we propose a simplification straightforwardly transposable to a three layered body in Sect. 4. The rotation dynamics of a satellite with a subsurface ocean is presented in Sect. 5. In the subsequent section 6, we test our model and our simplification on Io, a satellite with a liquid core, verifying that the derived eigenfrequencies are in very good agreement with those obtained in previous studies of the same problem made by Noyelles (2013, 2014). In this section, we also analyse the case of Titan showing that the additional degree of freedom makes the system highly sensitive to the internal structure and that the observed obliquity can be easily reproduced. Finally, we discuss our model and conclude in Sect. 7. The notation used in this paper is explained in Tab. 1.

1 Dumberry and Wieczorek (2016) could only highlight 5 degrees of freedom because their model of the Moon is axisymmetric and not triaxial.
2 Non-canonical Hamiltonian formalism

2.1 Equations of motion

2.1.1 General case

Let a dynamical system with \( n \) degrees of freedom described by a Lagrangian \( L \). We denote by \( Q \) the configuration space and each point \( q \in Q \) is represented by a set of \( m \geq n \) coordinates \( (q_1, \ldots, q_m) \). The number of coordinates is purposely allowed to be greater than the actual dimension of the manifold \( Q \). As in (Poincaré, 1901), we assume that there exists a transitive Lie group \( G \) acting on \( Q \). The transitivity of \( G \) means that for all \( q, q' \in Q \), there exists an element \( g \in G \) such that \( q' = g q \). In particular, given an initial condition \( q_0 \), there exists \( g_t \in G \) such that the configuration \( q(t) \) at time \( t \) reads \( q(t) = g_t q_0 \). In this work, \( G \) will be the rotation group \( SO(3) \), the translation group \( T(3) \), or some combinations of both.

Let \( g \) be the Lie algebra of \( G \). By definition, there exists \( X \in g \) such that the generalised velocity reads \( \dot{q} = X(q) \). Since the action of \( G \) on \( Q \) is transitive, the dimension of \( g \) is equal to the number \( n \) of degrees of freedom. Let \( B = (X_1, \ldots, X_n) \) be a basis of \( g \) and \((X_{ij})_{1 \leq i \leq n, 1 \leq j \leq m}\) be the \( n \times m \) functions of \( q \) defined as

\[
X_i = \sum_{j=1}^{m} X_{ij} \frac{\partial}{\partial q_j}.
\]

We denote by \( \eta = (\eta_1,\ldots,\eta_n) \in \mathbb{R}^n \) the coordinates of \( X \) in \( B \) such that

\[
\dot{q} = \sum_{i=1}^{n} \eta_i X_i(q).
\]

Because the term “generalised velocity” is already attributed to \( \dot{q} \), hereafter we call \( \eta \) the Lie velocity of the system. Given two configurations \( q \) and \( q' \) infinitely close to each other, we also define the \( n \)-tuple \( \delta \xi = (\delta \xi_1, \ldots, \delta \xi_n) \) such that

\[
\delta q := q' - q = \sum_{i=1}^{n} X_i(q) \delta \xi_i.
\]

Poincaré considers the Lagrangian as a function of \((\eta, q)\) and writes its infinitesimal variation as

\[
\delta L = \sum_{i=1}^{n} \frac{\partial L}{\partial \eta_i} \delta \eta_i + X_i(L) \delta \xi_i.
\]

The resulting equations of motion are (Poincaré, 1901)

\[
\frac{d}{dt} \frac{\partial L}{\partial \eta_i} = \sum_{j,k} c_{ijk}^k \frac{\partial L}{\partial \eta_j} + X_i(L),
\]

where \( c_{ij}^k \), defined as

\[
[X_i, X_j] := X_i X_j - X_j X_i = \sum_{k=1}^{n} c_{ij}^k X_k,
\]
are the structure constants of $\mathfrak{g}$ with respect to the chosen basis $\mathcal{B}$.

To get the Hamiltonian equations equivalent to Eq. (5), we introduce a momentum $\pi$ associated with the Lie velocity $\eta$, and defined as

$$
\pi := \frac{\partial L}{\partial \eta}.
$$

(7)

Following the same nomenclature as for $\eta$, we call this momentum $\pi$ the *Lie momentum* of the system. The Hamiltonian $H$ is constructed by means of a Legendre transformation as

$$
H(\pi, \eta) := \pi \cdot \eta - L(\eta, \eta).
$$

(8)

Using Eqs. (4) and (7), the infinitesimal variation of $H$ (Eq. 8) reads

$$
\delta H = \sum_{i=1}^{n} \eta_i \delta \pi_i - X_i(L) \delta \xi_i.
$$

(9)

But since $H$ is a function of $\pi$ and $\eta$, we also have, as in Eq. (4),

$$
\delta H = \sum_{i=1}^{n} \frac{\partial H}{\partial \pi_i} \delta \pi_i + X_i(H) \delta \eta_i.
$$

(10)

The identification of Eqs. (9) and (10) gives

$$
\eta_i = \frac{\partial H}{\partial \pi_i} \quad \text{and} \quad X_i(H) = -X_i(L).
$$

(11)

Using these identifications, the expression of $\dot{q}$ (Eq. 2), and Poincaré’s equation (5) where $\partial L/\partial \eta_i$ is replaced by $\pi_i$ (Eq. 7), we get the non-canonical equations of motion associated with $H$, viz.,

$$
\dot{q}_i = \sum_{j=1}^{n} \frac{\partial H}{\partial \pi_j} X_{ij}(q_i) \quad \text{and} \quad \dot{\pi}_i = \sum_{j,k} c_{ij} \frac{\partial H}{\partial \pi_j} \pi_k - X_i(H).
$$

(12)

Let us denote the state vector by $y = (\pi, \eta) \in \mathbb{R}^{n+m}$. The equations of motion (12) written in matrix form read

$$
\dot{y} = -B(y) \nabla_y H.
$$

(13)

The so-called Poisson matrix $B(y)$ is

$$
B(y) = \begin{bmatrix} -X & \pi \\ -X^T & 0 \end{bmatrix}
$$

(14)

where $(\cdot)^T$ means the transpose of a vector or of a matrix. $X$ is an $n \times m$ matrix and $C$ an $n \times n$ matrix whose elements are

$$
[X]_{ij} = X_{ij} \quad \text{and} \quad [C]_{ij} = -\sum_k c_{ij} \pi_k.
$$

(15)
2.1.2 Translation group

The simplest illustration of the above formalism is the case where \(G\) is the translation group. In that case, \(\eta\) is the usual velocity vector \(v\) and \(\pi\) is the standard linear momentum, commonly denoted \(p\). The vector fields of the tangent configuration space are \(X_i = \frac{\partial}{\partial q_i}\). The associated structure constants \(c^k_{ij}\) are all nil. The Poisson matrix is then

\[
B(y) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}
\]  

(16)

and we retrieve the canonical equations of motion

\[
\dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad \dot{q}_i = \frac{\partial H}{\partial p_i}.
\]  

(17)

2.1.3 Group SO(3) in the body-fixed frame

The group SO(3) naturally appears in studies of the rotation motion of solid bodies. For this problem, two choices can be made: vectors are expressed either in the body-fixed frame or in the “laboratory” frame. Here, we consider the first option where vectors are written in the body-fixed frame. The Lie velocity is the rotation vector designated by \(\omega\) and the orientation of the body is parametrised by the coordinates in the body-fixed frame of the laboratory base vectors, i.e., \(q = (i, j, k)\). For any function \(f(i, j, k)\), we have

\[
\frac{d}{dt} f(i, j, k) = - (\omega \times i) \cdot \frac{\partial f}{\partial i} - (\omega \times j) \cdot \frac{\partial f}{\partial j} - (\omega \times k) \cdot \frac{\partial f}{\partial k} = -\omega \cdot \left( i \times \frac{\partial f}{\partial i} + j \times \frac{\partial f}{\partial j} + k \times \frac{\partial f}{\partial k} \right).
\]  

(18)

Thus, the vector field \(X = (X_1, X_2, X_3)\) is

\[
X = -i \times \frac{\partial}{\partial i} - j \times \frac{\partial}{\partial j} - k \times \frac{\partial}{\partial k},
\]  

(19)

with structure constants \(c^k_{ij} = -\epsilon_{ijk}\) where \(\epsilon_{ijk} = 1\) when \((i, j, k)\) is a cyclic permutation of \((1, 2, 3)\), -1 when \((i, j, k)\) is a cyclic permutation of \((3, 2, 1)\), 0 otherwise. Hence, the Poisson matrix reads

\[
B = \begin{bmatrix}
\pi & i & j & k \\
0 & 0 & 0 & 0 \\
i & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
k & 0 & 0 & 0
\end{bmatrix}
\]  

(20)

where for any vector \(v\), we have defined

\[
\dot{v} = \begin{bmatrix} 0 & -v_z & v_y \\
v_z & 0 & -v_x \\
-v_y & v_x & 0 \end{bmatrix}.
\]  

(21)
The corresponding equations of motion are

\[
\frac{d\pi}{dt} = \pi \times \frac{\partial H}{\partial \pi} + i \times \frac{\partial H}{\partial i} + j \times \frac{\partial H}{\partial j} + k \times \frac{\partial H}{\partial k}, \quad (22)
\]

\[
\frac{di}{dt} = i \times \frac{\partial H}{\partial \pi}, \quad (23)
\]

\[
\frac{dj}{dt} = j \times \frac{\partial H}{\partial \pi}, \quad (24)
\]

\[
\frac{dk}{dt} = k \times \frac{\partial H}{\partial \pi} \quad (25)
\]

with \(\partial H/\partial \pi = \omega\).

\[2.1.4\] Group SO(3) in the laboratory frame

Here we again consider the rotation motion of a solid body but now vector coordinates are written in the laboratory frame. The latter is the frame with respect to which the motion of the spinning body is described. Note that it does not have to be inertial. The generalised coordinates are the base vectors of the rotated frame \(q = (I, J, K)\) and the Lie momentum associated with the rotation vector is denoted \(\Pi\). Applying the same method as above, we get

\[
X = I \times \frac{\partial}{\partial I} + J \times \frac{\partial}{\partial J} + K \times \frac{\partial}{\partial K}. \quad (26)
\]

For this basis, the structure constants are \(c_{ij}^k = \epsilon_{ijk}\) and thus, the Poisson matrix is

\[
B = \begin{bmatrix}
\Pi & i & j & k \\
i & 0 & 0 & 0 \\
j & 0 & 0 & 0 \\
k & 0 & 0 & 0
\end{bmatrix}. \quad (27)
\]

The associated equations of motion are

\[
\frac{d\Pi}{dt} = \frac{\partial H}{\partial \Pi} \times \Pi + \frac{\partial H}{\partial i} \times I + \frac{\partial H}{\partial j} \times J + \frac{\partial H}{\partial k} \times K, \quad (28)
\]

\[
\frac{di}{dt} = \frac{\partial H}{\partial \Pi} \times I, \quad (29)
\]

\[
\frac{dj}{dt} = \frac{\partial H}{\partial \Pi} \times J, \quad (30)
\]

\[
\frac{dk}{dt} = \frac{\partial H}{\partial \Pi} \times K \quad (31)
\]

where \(\partial H/\partial \Pi\) still is the rotation vector, although expressed in the laboratory frame.
2.2 Linearisation and driven solution

For the sake of completeness, we here recall the general method leading to the linearisation of the equations of motion in the non-canonical Hamiltonian formalism (Maddocks, 1991; Beck and Hall, 1998). We also present the criterion of nonlinear stability as described in ibid.

Let a non-autonomous Hamiltonian $H(y, t)$ associated with an $n$ degrees of freedom system expressed as a function of non-canonical variables $y \in \mathbb{R}^p$ with $p \geq 2n$. We assume that $H(y, t)$ can be split as follows

$$H(y, t) = H_0(y) + H_1(y, t),$$

(32)

where $H_0(y)$ is the autonomous part of $H(y, t)$ and $H_1(y, t)$ a small perturbation.

Let us skip the perturbation $H_1$ for a moment. The equations of motion associated with $H_0$ are of the form

$$\dot{y} = -B(y) \nabla_y H_0(y).$$

(33)

The system has $n$ degrees of freedom, its phase space $\Sigma$ is thus a manifold of dimension $2n$. Since $y \in \mathbb{R}^p$, there exists $s = p - 2n$ Casimir functions $C_i(y)$ and $s$ constants $c_i$, $1 \leq i \leq s$, such that

$$\Sigma = \{ y \in \mathbb{R}^p : C_1(y) = c_1, \ldots, C_s(y) = c_s \}.$$  

(34)

We recall that Casimir functions are constants of the motion for any Hamiltonian because their gradients constitute a basis of the kernel of the Poisson matrix:

$$\ker B(y) = \text{span} \{ \nabla_y C_1(y), \ldots, \nabla_y C_s(y) \},$$

(35)

and thus

$$\dot{C}_i(y) = (\nabla_y H_0)^T B(y) \nabla_y C_i = 0$$

(36)

for all Hamiltonian $H_0$.

Let $y_e$ be an equilibrium, i.e., a fixed point of $H_0$. According to Eq. (33), $\dot{y}_e = 0$ implies $\nabla_y H_0(y_e) \in \ker B(y_e)$. Thus, there exists $s$ coefficients $(\mu_i)_{1 \leq i \leq s}$ such that

$$\nabla_y H_0(y_e) = \sum_{i=1}^{s} \mu_i \nabla_y C_i(y_e).$$

(37)

Let

$$F(y) = H_0(y) - \sum_{i=1}^{s} \mu_i C_i(y).$$

(38)

By construction, $F$ satisfies $\nabla_y F(y_e) = 0$. Coefficients $\mu_i$ can be seen as Lagrange multipliers and functions $C_i(y)$ as constraints since we search for an extremum of $H_0(y)$ under the conditions $C_i(y) = c_i$. The $p + s$ equations $\nabla_y F(y_e) = 0$ and $C_i(y_e) = c_i$ allow to determine $y_e$ and the coefficients $\mu_i$.

Once $y_e$ and coefficients $\mu_i$ are known, the linearisation of the equations of motion (Eq. 33) are given by

$$\delta \dot{y} = A(y_e) \delta y$$

(39)
Rotation of a rigid satellite with a fluid component

Fig. 1 Rigid satellites are characterised by their basis vectors $(I, J, K)$ and their rotation vector $\omega$ with respect to the laboratory frame. The same vectors are used for satellites with a liquid core, but the angular speed $\omega'_c$ of the core with respect to the mantle is also specified. In the case of a satellite with a global ocean, all vectors are expressed in the laboratory frame. These are the basis vectors of the shell $(I_s, J_s, K_s)$ and of the interior $(I_c, J_c, K_c)$, and the rotation vectors $\omega_c, \omega_o, \omega_s$ associated with the central region, the ocean and the shell, respectively.

with $\delta y = y - y_e$ and (Maddocks, 1991)

$$A(y_e) = -B(y_e) \nabla^2_y F(y_e). \quad (40)$$

In a last step, the perturbation $H_1(y, t)$ is taken into account and the equations of motion become

$$\delta \dot{y} - A(y_e) \delta y = z(t), \quad (41)$$

with

$$z(t) = -B(y_e) \nabla_y H_1(y_e, t). \quad (42)$$

Equation (41) is then solved using standard techniques.

The relative equilibria $y = y_e$ is said to be nonlinearly stable if the quadratic form (or Lyapunov function) $N(y) = y^T N y$, defined on the phase space $\Sigma$ by its Hessian (below), is a strictly convex function (Beck and Hall, 1998). The Hessian of $N(y)$ is given by (see ibid.)

$$N := \nabla^2 N = Q(y_e) \nabla^2 y F(y_e) Q(y_e), \quad (43)$$

where $Q(y)$ is the orthogonal projection matrix onto the range of $A(y)$,

$$Q(y) = 1 - K(y)(K^T(y)K(y))^{-1}K^T(y), \quad (44)$$

and where $K(y)$ is a $p \times s$ matrix given by

$$K(y) = [\nabla C_1(y) \cdots \nabla C_s(y)]. \quad (45)$$
Table 1 Notations

| symbol | definition |
|--------|------------|
| $c_i, o, m, s$ | indices standing for Core, Ocean, Mantle, and Shell, respectively |
| $rs, fc, go$ | indices standing for Rigid Satellite, Fluid Core, and Global Ocean |
| $\mathcal{F}_{\text{in}} = (i_0, j_0, k_0)$ | inertial frame |
| $\mathcal{F}_{\text{lab}} = (i, j, k)$ | laboratory frame |
| $\mathcal{F}_i = (I_i, J_i, K_i)$ | frame associated with the layer i |
| $\Omega$ | rotation vector of $\mathcal{F}_{\text{lab}}$ with respect to $\mathcal{F}_{\text{in}}$ expressed in $\mathcal{F}_{\text{lab}}$ |
| $\omega_i$ | rotation vector of $\mathcal{F}_i$ with respect to $\mathcal{F}_{\text{lab}}$ expressed in $\mathcal{F}_{\text{lab}}$ |
| $\omega'_c$ | rotation vector of $\mathcal{F}_c$ with respect to $\mathcal{F}_m$ expressed in $\mathcal{F}_m$ |
| $\Pi_i$ | Lie momentum associated with $\omega_i$ |
| $\Pi'_c$ | Lie momentum associated with $\omega'_c$ |
| $I_i, J_i, K_i$ | basis vectors of $\mathcal{F}_i$ expressed in $\mathcal{F}_{\text{lab}}$ |
| $y_i = (\Pi_i, I_i, J_i, K_i)$ | state vector of the layer i |
| $y$ | state vector of the whole system |
| $T(y)$ | kinetic energy |
| $U(y, t)$ | potential energy |
| $L_i(y, t)$ | Lagrangian |
| $C_i(y)$ | Casimir functions |
| $\mu_i$ | Lagrange multipliers |
| $F(y)$ | Lagrangian associated with the minimisation of $H_0$ with constraints |
| $B(y)$ | Poisson matrix |
| $A(y)$ | matrix of the linearised system |
| $U_0(y)$ | constant part of $U(y, t)$ |
| $U_1(y, t)$ | perturbation $U(y, t) - U_0(y)$ |
| $U_{\text{self}}(y)$ | self gravitational energy of the satellite |
| $U_{\text{self}}(y)$ | constant parameters of $U_{\text{self}}$ |
| $H_0(y)$ | autonomous part of $H(y, t)$ |
| $H_1(y, t)$ | perturbation $H(y, t) - H_0(y)$ |
| $r(t)$ | radius vector connecting the satellite barycenter to the planet |
| $S(t)$ | $GM_p T^3/r^5$ |
| $S_0$ | constant part of $S(t)$ |
| $S_1(t)$ | $S(t) - S_0$ |
| $(\sigma^{0}_{uv})_{u,v \in \{x,y,z\}}$ | elements of the matrix $S_0$ |
| $(\sigma^{1}_{uv}(t))_{u,v \in \{x,y,z\}}$ | elements of the matrix $S_1(t)$ |
| $\gamma$ | gravitational constant |
| $M_p$ | mass of the central planet |
| $\alpha_i, \beta_i, \gamma_i$ | radii of the outer boundary of the layer i |
| $\rho_i$ | density of the layer i |
| $\zeta$ | equatorial flattening $(a - b)/a$ |
| $I_i$ | inertia tensor of the layer i expressed in $\mathcal{F}_{\text{lab}}$ |
| $V'$ | ancillary inertia tensor |
| $A_i, B_i, C_i$ | principal moments of inertia of the layer i |
| $A', B', C'$ | ancillary moments of inertia |
| $\omega_u$ | frequency of libration in longitude |
| $\omega_v$ | frequency of libration in latitude |
| $\omega_w$ | wobble frequency |
3 Rigid satellite

Let a rigid satellite whose rotation is close to the synchronous state, i.e., whose mean rotation rate is equal to the orbital mean motion. The goal of this section is to compute the frequencies associated with the free modes of rotation, to evaluate the forced obliquity driven by the orbital precession, and eventually to check the nonlinear stability of the system in the vicinity of the equilibrium. The analysis is performed using the non-canonical Hamiltonian formalism described in Sect. 2. It turns out to be convenient to describe the problem in a laboratory frame rotating at constant angular speed \( \Omega \) with respect to the inertial frame. \( \Omega \) will then be chosen equal to the mean orbital motion. We denote by \( \omega \) the rotation vector of the satellite with respect to the laboratory frame \( F_{lab} \) and by \((I, J, K)\) its principal axes of inertia such that the matrix of inertia reads

\[
I = R \text{diag}(A, B, C)R^T,
\]

where \( R = [I, J, K] \) is the rotation matrix of the satellite with respect to the laboratory frame and where \((.)^T \) denotes the transpose operator. Note that the matrix of inertia can also be written in an equivalent form facilitating the computation of the gradient of the forthcoming Hamiltonian

\[
I = AII^T + BJJ^T + CKK^T. \tag{47}
\]

The Lie velocity of the system is thus \( \omega \) while \((I, J, K)\) are the generalised coordinates. We also denote by \((i, j, k)\) the basis vectors associated with the laboratory frame. The radius vector connecting the planet and the satellite barycenter is assumed to be a known function of time and is denoted either by \( r(t) \) or simply by \( r \). \( G \) and \( M_p \) are the gravitational constant and the mass of the planet, respectively. With these notations, the (non-autonomous) Lagrangian \( L_{rs}(\omega, I, J, K, t) \) governing the rotation of the rigid satellite is

\[
L_{rs}(\omega, I, J, K, t) = \frac{(\omega + \Omega)^T I (\omega + \Omega)}{2} - \frac{3GM_p}{2} \frac{r^T r}{r^5}. \tag{48}
\]

The Lie momentum \( \Pi \) associated with \( \omega \) reads

\[
\Pi = \frac{\partial L_{rs}}{\partial \dot{\omega}} = I(\omega + \Omega). \tag{49}
\]

We recognise the spin angular momentum of the satellite with respect to the inertial frame and expressed in the laboratory frame. The Hamiltonian \( H_{rs}(\Pi, I, J, K, t) \) resulting from the Legendre transformation applied to \( L_{rs}(\omega, I, J, K, t) \) reads

\[
H_{rs}(\Pi, I, J, K, t) = \frac{\Pi^T \Pi^{-1} \Pi}{2} - \Omega^T \Pi + \frac{3GM_p}{2} \frac{r^T r}{r^5} \tag{50}
\]

with

\[
\Pi^{-1} = \frac{\Pi^T}{A} + \frac{JJ^T}{B} + \frac{KK^T}{C}. \tag{51}
\]
As a result, the Hamiltonian \( H \) with \( S \) Eq. (27). The gradient of the Hamiltonian reads

\[
\frac{\partial H_{rs}}{\partial \Pi} = \Omega^{-1} \Pi - \Omega = \omega, \tag{52}
\]

\[
\frac{\partial H_{rs}}{\partial \Pi} = \frac{(\mathbf{I} \cdot \Pi)}{\mathbf{A}} \Pi + 3 \frac{\mathcal{G} M_p}{r^5} A(r \cdot \mathbf{I}) \mathbf{r}, \tag{53}
\]

\[
\frac{\partial H_{rs}}{\partial \mathbf{J}} = \frac{(\mathbf{J} \cdot \Pi)}{\mathbf{B}} \Pi + 3 \frac{\mathcal{G} M_p}{r^5} B(r \cdot \mathbf{J}) \mathbf{r}, \tag{54}
\]

\[
\frac{\partial H_{rs}}{\partial \mathbf{K}} = \frac{(\mathbf{K} \cdot \Pi)}{\mathbf{C}} \Pi + 3 \frac{\mathcal{G} M_p}{r^5} C(r \cdot \mathbf{K}) \mathbf{r}, \tag{55}
\]

and thus the equations of motion are

\[
\dot{\Pi} = \Pi \times \Omega - 3 \frac{\mathcal{G} M_p}{r^5} (\mathbf{I} \times \mathbf{r}), \tag{56}
\]

\[
\dot{\mathbf{I}} = \omega \times \mathbf{I}, \tag{57}
\]

\[
\dot{\mathbf{J}} = \omega \times \mathbf{J}, \tag{58}
\]

\[
\dot{\mathbf{K}} = \omega \times \mathbf{K}. \tag{59}
\]

Equations of motion (Eqs. 56-59) are those of the full Hamiltonian. Because \( \mathbf{r}(t) \) is a function of time, the set of equations (56-59) has no fixed point. To proceed, we set \( \Omega = \Omega \hat{\mathbf{k}} \) with \( \Omega \) equal to the mean orbital motion such that, in the laboratory frame \((i,j,k)\),

\[
\mathbf{S}(t) := \mathcal{G} M_p \frac{\mathbf{r} \mathbf{r}^T}{r^5} = \mathbf{S}_0 + \mathbf{S}_1(t) \tag{60}
\]

where \( \mathbf{S}_0 \) is a constant matrix and \( \mathbf{S}_1(t) \) a small perturbation. Furthermore, the initial angle of the rotation is chosen such that \( \mathbf{S}_0 \) is diagonal with components \((\sigma^0_{xx}, \sigma^0_{yy}, \sigma^0_{zz})\). Similarly, we denote by \( \sigma^u_{uv} \), where \( u, v \in \{x, y, z\} \), the elements of \( \mathbf{S}_1(t) \). The gravitational potential energy \( U(y,t) \) is then split into \( U_0(y) + U_1(y,t) \) with

\[
U_0(y) = \frac{3}{2} \left( A \mathbf{I}^T \mathbf{S}_0 \mathbf{I} + B \mathbf{J}^T \mathbf{S}_0 \mathbf{J} + C \mathbf{K}^T \mathbf{S}_0 \mathbf{K} \right), \tag{61}
\]

\[
U_1(y,t) = \frac{3}{2} \left( A \mathbf{I}^T \mathbf{S}_1(t) \mathbf{I} + B \mathbf{J}^T \mathbf{S}_1(t) \mathbf{J} + C \mathbf{K}^T \mathbf{S}_1(t) \mathbf{K} \right). \tag{62}
\]

As a result, the Hamiltonian \( H_{rs}(y,t) \) also get split into \( H_{rs}^0(y) + H_{rs}^1(y,t) \) with

\[
H_{rs}^0(y) = \frac{\Pi^T \Omega^{-1} \Pi}{2} - \Omega^T \Pi + U_0(y), \tag{63}
\]

\[
H_{rs}^1(y,t) = U_1(y,t). \tag{64}
\]

In the case of a Keplerian orbit with eccentricity \( e \) and inclination \( i \) with respect to the reference frame,

\[
\sigma^0_{xx} = \frac{\mathcal{G} M_p}{a^3} \left( X_0^{-3,0}(e) + X_2^{-3,2}(e) \cos^4 \left( \frac{i}{2} \right) + X_0^{-3,0}(e) \sin^4 \left( \frac{i}{2} \right) \right), \tag{65}
\]

\[
\sigma^0_{yy} = \frac{\mathcal{G} M_p}{a^3} \left( X_0^{-3,0}(e) + X_2^{-3,2}(e) \cos^4 \left( \frac{i}{2} \right) + X_0^{-3,0}(e) \sin^4 \left( \frac{i}{2} \right) \right), \tag{66}
\]

\[
\sigma^0_{zz} = \frac{\mathcal{G} M_p}{a^3} X_0^{-3,0}(e) \sin^2 i, \tag{67}
\]
where $X_{n,m}^k(e)$ are Hansen coefficients (Hansen, 1855) defined as Fourier coefficients of the series

$$
\left( \frac{L}{a} \right)^n e^{im\varpi} = \sum_{k=-\infty}^{\infty} X_{n,m}^k(e) e^{ikM} (68)
$$

with $a, \varpi, M$ being the semimajor axis, the true anomaly and the mean anomaly, respectively. Besides, in this study a single element of the matrix $S_1(t)$ plays a role in the tilting of the Cassini state, this is the term in $\sigma_{xz}^1(t) = \sigma_{zx}^1(t)$ corresponding to the first harmonic of the orbital precession in inclination whose expression is

$$
\sigma_{xz}^1(t) = \frac{GM_P}{a^3} \left( \frac{X_{0}^{-3,0}(e)}{2} \cos i + \frac{X_{2}^{-3,2}(e)}{2} \cos^2 \frac{i}{2} \right) \sin i \sin(\Omega t - \Phi) (69)
$$

where $\Phi$ is the longitude of the ascending node. The expression of the Hansen coefficients involved in $S_0$ and $S_1(t)$ are

$$
X_{0}^{-3,0}(e) = (1 - e^2)^{-3/2}, \quad (70)
$$

$$
X_{2}^{-3,2}(e) = 1 - \frac{5}{2} e^2 + \frac{13}{16} e^4 - \frac{35}{288} e^6 + O(e^8). \quad (71)
$$

Following the steps recalled in the previous section 2.2, we now skip the perturbation $S_1(t)$ for a while and only retain the autonomous part of the Hamiltonian $H_0^{rs}(y)$. The gradient of the Hamiltonian reads

$$
\frac{\partial H_0^{rs}}{\partial \Pi} = \Pi^{-1} \Pi - \Omega = \omega, \quad (72)
$$

$$
\frac{\partial H_0^{rs}}{\partial I} = \frac{(I \cdot \Pi)}{A} \Pi + 3A S_0 I, \quad (73)
$$

$$
\frac{\partial H_0^{rs}}{\partial J} = \frac{(J \cdot \Pi)}{B} \Pi + 3B S_0 J, \quad (74)
$$

$$
\frac{\partial H_0^{rs}}{\partial K} = \frac{(K \cdot \Pi)}{C} \Pi + 3C S_0 K. \quad (75)
$$

Only $\tilde{\Pi}$ (Eq. 56) is affected by the averaging process. Its new equation of motion reads

$$
\tilde{\Pi} = \Pi \times \Omega + 3A (S_0 I) \times I + 3B (S_0 J) \times J + 3C (S_0 K) \times K. \quad (76)
$$

3.1 Linearisation

To perform the linearisation of Eqs. (76, 57-59), we note that the phase space $\Sigma_{rs}$ of the system is a manifold of dimension 6 (associated with the 3 degrees of freedom of the group $SO(3)$) defined as

$$
\Sigma_{rs} = \{ y \in \mathbb{R}^{12} : C_{rs}^1(y) = C_{rs}^2(y) = C_{rs}^3(y) = 1/2, \quad C_{rs}^4(y) = C_{rs}^5(y) = C_{rs}^6(y) = 0 \}, \quad (77)
$$

where the Casimir functions are

$$
C_{rs}^1(y) = \frac{1}{2} I \cdot I, \quad C_{rs}^2(y) = \frac{1}{2} J \cdot J, \quad C_{rs}^3(y) = \frac{1}{2} K \cdot K,
$$

$$
C_{rs}^4(y) = J \cdot K, \quad C_{rs}^5(y) = K \cdot I, \quad C_{rs}^6(y) = I \cdot J. \quad (78)
$$
Indeed, it can be checked that

\[
\ker B_{rs}(y) = \text{span} \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}.
\] (79)

Let \( F_{rs}(y) = H_{rs}^0(y) - \sum_i \mu_i C_{rs}(y) \). The condition \( \nabla_y F_{rs}(y_e) = 0 \) leads to

\[
E^{-1} \Pi_e - \Omega = \omega_e = 0,
\] (80)

\[
\frac{1}{A} \Pi_e + 3A \delta \Pi_e = \mu_1 \delta I_e - \mu_2 \delta K_e - \mu_3 \delta J_e = 0,
\] (81)

\[
\frac{1}{B} \Pi_e + 3B \delta \Pi_e = \mu_2 \delta J_e - \mu_3 \delta K_e - \mu_4 \delta I_e = 0,
\] (82)

\[
\frac{1}{C} \Pi_e + 3C \delta \Pi_e = \mu_3 \delta K_e - \mu_4 \delta J_e - \mu_5 \delta I_e = 0,
\] (83)

whose a solution is

\[
\omega_e = 0, \quad \Pi_e = C \Omega k, \quad I_e = I, \quad J_e = j, \quad K_e = k,
\]

\[
\mu_1 = 3A \sigma_{xx}, \quad \mu_2 = 3B \sigma_{yy}, \quad \mu_3 = 3C \sigma_{zz} + C \Omega^2, \quad \mu_4 = \mu_5 = 0. \tag{84}
\]

The other solutions are equivalent to this one but with a permutation of the moments of inertia \( A, B, C \). The matrix \( A_{rs}(y_e) \) of the linearised system is given by Eq. (40). To simplify the result, we perform the change of variables \( \delta y = P \delta y^* \) with

\[
\delta y^* = \begin{pmatrix} \delta I_x, \delta I_y, \delta I_z, \delta J_x, \delta J_y, \delta J_z, \delta K_x, \\
\delta I_y + \delta J_z, \delta I_z + \delta K_x, \delta J_x + \delta K_y \end{pmatrix}^T.
\] (85)

The first two components of \( \delta y^* \) are associated with the libration in longitude, the next four components describe the wobble and the libration in latitude, and finally, the last six coordinates being in the kernel of \( B_{rs}(y_e) \) remain identically equal to zero. Let \( A_{rs}^*(y_e) \) be the matrix of the linear system in the new variables \( \delta y^* \), i.e., \( A_{rs}^* = P^{-1} A_{rs} P \), and let \( A_{rs}^1 \) and \( A_{rs}^2 \) be the respective \( 2 \times 2 \) and \( 4 \times 4 \) matrices such that

\[
A_{rs}^*(y_e) = \begin{bmatrix} A_{rs}^1 & 0 \\ 0 & A_{rs}^2 \end{bmatrix},
\] (86)

where the dots \( \cdot \) represent arbitrary matrices not influencing the motion. We have

\[
A_{rs}^1 = \begin{bmatrix} 0 & -3(B - A)(\sigma_{xx}^0 - \sigma_{yy}^0) \\ 1/C & 0 \end{bmatrix},
\] (87)

and

\[
A_{rs}^2 = \begin{bmatrix} 0 & \Omega & 0 & 3(C - B)(\sigma_{xx}^0 - \sigma_{yy}^0) \\ -\Omega & 0 & 3(C - A)(\sigma_{xx}^0 - \sigma_{zz}^0) & 0 \\ 0 & -1/C & 0 & 0 \\ 1/A & 0 & \frac{C - A}{A} \Omega & 0 \end{bmatrix}.
\] (88)
Hence, the frequency of libration in longitude $\omega_{rs,u}$, which is the eigenvalue of $A_{rs}^1$, reads
\[ \omega_{rs,u} = \sqrt{3\gamma (\kappa_1 - \kappa_2)}, \] (89)
and the frequencies associated with the wobble $\omega_{rs,w}$ and the libration in latitude $\omega_{rs,v}$, the eigenvalues of $A_{rs}^2$, are given by
\[ \omega_{rs,w} = \left( \frac{p - \sqrt{p^2 - 4q}}{2} \right)^{1/2}, \quad \omega_{rs,v} = \left( \frac{p + \sqrt{p^2 - 4q}}{2} \right)^{1/2} \] (90)
with
\[ p = (1 + \alpha\beta) \Omega^2 + 3(\beta\kappa_1 + \alpha\kappa_2), \] (91)
\[ q = \alpha\beta \left( \Omega^4 + 3(\kappa_1 + \kappa_2) \Omega^2 + 9\kappa_1\kappa_2 \right), \] (92)
\[ \kappa_1 = \sigma^0_{xx} - \sigma^0_{zz}, \] (93)
\[ \kappa_2 = \sigma^0_{yy} - \sigma^0_{zz}, \] (94)
and
\[ \alpha = \frac{C - B}{A}, \quad \beta = \frac{C - A}{B}, \quad \gamma = \frac{B - A}{C}. \] (95)
Here we retrieve the well-known eigenfrequencies of a rigid satellite close to the synchronous equilibrium state (e.g., Rambaux et al., 2012). Let us nevertheless stress that Eqs. (89) and (90) are associated with the motion of the three vectors $\mathbf{I}, \mathbf{J}, \mathbf{K}$ in the rotating frame. By consequence, if we denote by $\tilde{\omega}_{rs,v} \approx 3\beta\Omega/2$ the frequency of libration in latitude associated with the motion of the sole vector $\mathbf{K}$ with respect to the inertial frame (as it is commonly defined for an axisymmetric body), we have $\omega_{rs,v} = \tilde{\omega}_{rs,v} + \Omega$.

3.2 Stability

For this problem, the Lyapunov function $N_{rs}(y)$, as defined in Eq. (43), is
\[
N_{rs}(y) = \frac{1}{2A} \left( \Pi_z + \frac{1}{2}(C - A)\Omega(I_z - K_z) \right)^2 + \frac{1}{2B} \left( \Pi_y + \frac{1}{2}(C - B)\Omega(J_z - K_y) \right)^2 + \frac{1}{2C} \Pi_z^2 + \frac{1}{2}n_1(I_y - J_z)^2 + \frac{1}{2}n_2(I_z - K_z)^2 + \frac{1}{2}n_3(J_z - K_y)^2 \] (96)
with
\[ n_1 = \frac{3}{4}(B - A)(\kappa_1 - \kappa_2), \quad n_2 = \frac{1}{8}(C - A)(\Omega^2 + 3\kappa_1), \quad n_3 = \frac{1}{8}(C - B)(\Omega^2 + 3\kappa_2). \] (97)
We recall that the system is nonlinearly stable if $N_{rs}(y)$ is a strictly convex function. Coefficients $A$, $B$, and $C$ are positive, as required. The nonlinear stability is then achieved when $n_1$, $n_2$, and $n_3$ are all positive. Given that $\kappa_1 > \kappa_2 > 0$ at low inclination $i$, the criterion implies $C > B > A$, which is the well-known stability condition for this classical equilibrium where the longest axis points towards the parent planet (e.g., Beck and Hall, 1998).
3.3 Driven solution

Here we look for the forced solution when the time-dependent perturbation $H_{rs}^1(t)$ is taken into account. In the variables $\delta y^*$ (Eq. 85), and with the notation of Eq. (41), the perturbation $\delta z_{rs}^*(t)$ is given by

$$\delta z_{rs}^*(t) = -P^{-1}B_{rs}(y_e)\nabla_y H_{rs}^1(y_e, t).$$  \hspace{1cm} (98)

To match the notation of the matrix $A_{rs}^*$, let $\delta y_1$ and $\delta y_2$ be the first 2 and the next 4 components of $\delta y^*$, idem for $\delta z_{rs}^*(t)$, such that the linear problem with perturbation reads

$$\delta y^k - A_{rs}^k \delta y^k = \delta z_{rs}^k(t), \quad k = 1, 2.$$  \hspace{1cm} (99)

By definition,

$$\delta y^1 = (\delta II_x, \delta I_y)^T, \quad \delta y^2 = (\delta II_x, \delta II_y, \delta I_z, \delta J_z)^T,$$  \hspace{1cm} (100)

and Eq. (98) implies

$$\delta z_{rs}^1(t) = \begin{pmatrix} 3(B - A)\sigma_{xy}^1(t) \\ 0 \end{pmatrix}, \quad \delta z_{rs}^2(t) = \begin{pmatrix} 3(C - B)\sigma_{yz}^1(t) \\ -3(C - A)\sigma_{xz}^1(t) \\ 0 \end{pmatrix}.$$  \hspace{1cm} (101)

Note that the term $\sigma_{yz}^1(t)$ is present in the perturbation $\delta z_{rs}^2(t)$ but its effect on the orientation of the spin axis is very weak. For instance, according to the ephemeris of Titan in TASS1.6 (Vienne and Duriez, 1995), the amplitude associated with the angle $(\Omega t - \Phi)$ in $\sigma_{yz}^1(t)$ is about 500 times lower than that in $\sigma_{xz}^1(t)$. In the numerical applications (Sect. 6), $\sigma_{yz}^1(t)$ is simply discarded.

4 Satellite with a liquid core

In this section we consider a satellite with a rigid mantle/crust layer surrounding a liquid core. In a first step, we analyse the problem using the Poincaré-Hough model which is valid for all eccentricities of the ellipsoidal cavity containing the fluid core (Poincaré, 1910; Hough, 1895). In a second one, we truncate the problem at the first order with respect to the equatorial and polar flattening of the cavity. The same simplification will be used again in Sect. 5 where the case of a satellite with a subsurface ocean is treated. Here, the two models of the same problem are used to estimate the error made by the approximation.

4.1 Poincaré-Hough model

As in the previous model, $A, B, C$ designate the principal moments of inertia of the whole satellite. Those of the liquid core are denote by $A_c, B_c, C_c$. We assume that the axes of the core/mantle ellipsoidal boundary are aligned to those of the satellite surface. Hence, the principal axes $(I_c, J_c, K_c)$ of the core are aligned to those of the mantle denoted $(I_m, J_m, K_m)$ which are also aligned to those of the whole satellite.
The vector \( \omega \) still represents the rotation vector of \((\mathbf{I}, \mathbf{J}, \mathbf{K})\) with respect to the laboratory frame expressed in the laboratory frame. We add the rotation vector \( \omega'_c \) associated with the simple motion of the liquid core with respect to the mantle fixed in the laboratory frame (Poincaré, 1910). As in the rigid case, the laboratory frame rotates with respect to the inertial frame at the speed \( \Omega \). Let \( \mathbf{I}, \mathbf{I}'_c \) and \( \mathbf{I}' \) be the inertia matrices defined as

\[
\mathbf{I} = \mathbf{R} \text{diag}(A, B, C) \mathbf{R}^T, \\
\mathbf{I}'_c = \text{diag}(A_c, B_c, C_c), \\
\mathbf{I}' = \text{diag}(A', B', C') \mathbf{R}^T,
\]

where \( \mathbf{R} = [\mathbf{I}, \mathbf{J}, \mathbf{K}] \) is the rotation matrix of the mantle relative to the laboratory frame. Furthermore, we have defined the Lagrangian \( L \)

\[
\text{the potential energy is the same as in the rigid satellite case (see sect. 3). Thus,} \\
\text{rotation of the satellite is (Poincaré, 1910; Hough, 1895)} \\
T_{lc}(\eta, \mathbf{q}) = \frac{(\omega + \Omega)^T \mathbf{I}(\omega + \Omega)}{2} + \frac{\omega'_c^T \mathbf{I}'_c \omega'_c}{2} + \omega'_c^T \mathbf{I}'(\omega + \Omega),
\]

The potential energy is the same as in the rigid satellite case (see sect. 3). Thus, the Lagrangian \( L_{lc}(\eta, \mathbf{q}) \) reads

\[
L_{lc}(\eta, \mathbf{q}) = \frac{(\omega + \Omega)^T \mathbf{I}(\omega + \Omega)}{2} + \frac{\omega'_c^T \mathbf{I}'_c \omega'_c}{2} + \omega'_c^T \mathbf{I}'(\omega + \Omega) - \frac{3GM_p}{2} \frac{r^T \mathbf{r}}{r^2}.
\]

The Lie momenta associated with \( \omega \) and \( \omega'_c \) are respectively

\[
\mathbf{\Pi} = \frac{\partial L_{lc}}{\partial \omega} = \mathbf{I}(\omega + \Omega) + \mathbf{I}' \omega'_c, \\
\mathbf{\Pi}'_c = \frac{\partial L_{lc}}{\partial \omega'_c} = \mathbf{I}'_c \omega'_c + \mathbf{I}'(\omega + \Omega),
\]

with the inverse transformation,

\[
\omega = \mathbf{Q} \mathbf{\Pi} - \mathbf{Q}'^T \mathbf{\Pi}'_c - \Omega, \\
\omega'_c = \mathbf{Q}' \mathbf{\Pi}_c - \mathbf{Q}'^T \mathbf{\Pi}.
\]

where

\[
\mathbf{Q} = \mathbf{R} \text{diag} \left( \frac{A_c}{AA_c - A'^2}, \frac{B_c}{BB_c - B'^2}, \frac{C_c}{CC_c - C'^2} \right) \mathbf{R}^T, \\
\mathbf{Q}' = \text{diag} \left( \frac{A}{AA_c - A'^2}, \frac{B}{BB_c - B'^2}, \frac{C}{CC_c - C'^2} \right), \\
\mathbf{Q}' = \text{diag} \left( \frac{A'}{AA_c - A'^2}, \frac{B'}{BB_c - B'^2}, \frac{C'}{CC_c - C'^2} \right) \mathbf{R}^T.
\]
The Hamiltonian of the problem is then

\[
H_{fc}(y, t) = \frac{\Pi^T Q \Pi}{2} + \frac{\Pi'^T Q' \Pi'}{2} - \frac{\Pi'^T Q' \Pi - \Omega^T \Pi}{2} + \frac{3GM_p r^T \mathbf{r}}{r^5},
\]

(116)

with the state vector \( y = (\Pi', \Pi, I, J, K) \). In these variables, the Poisson matrix reads

\[
B_{fc}(y) = \begin{bmatrix}
\Pi' & 0 & 0 & 0 \\
0 & \Pi & I & J \\
0 & J & 0 & 0 \\
0 & K & 0 & 0 \\
\end{bmatrix}
\]

(117)

and the equations of motion are

\[
\dot{\Pi}'_c = \omega'_c \times \Pi'_c, \quad \dot{\Pi} = \Pi \times \Omega - 3 \frac{G M_p}{r^5}(\mathbf{I} \mathbf{r}) \times \mathbf{r},
\]

\[
\dot{I} = \omega \times \mathbf{I}, \quad \dot{J} = \omega \times \mathbf{J}, \quad \dot{K} = \omega \times \mathbf{K}.
\]

(118) - (122)

As in the rigid case (Sect. 3), we now split the Hamiltonian \( H_{fc}(y, t) \) into its autonomous part \( H_{fc}^R(y) \) and a perturbation \( H_{fc}^P(y, t) \) using the decomposition of the gravitational potential energy \( U_0(y) \) and \( U_1(y, t) \), Eqs. (61-62). There are seven Casimir functions given by

\[
C_i^0(y) = \frac{1}{2} \Pi'_c \cdot \Pi'_c,
\]

(123)

\[
C_i^1(y) = \frac{1}{2} I_1 \cdot I, \quad C_i^2(y) = \frac{1}{2} J_1 \cdot J, \quad C_i^3(y) = \frac{1}{2} K_1 \cdot K,
\]

\[
C_i^4(y) = J_1 \cdot K, \quad C_i^5(y) = K_1 \cdot I, \quad C_i^6(y) = I_1 \cdot J.
\]

The equilibrium \( y_e \) of \( H_{fc}^0(y) \) is solution of

\[
\begin{align*}
\omega'_{e,c} - \mu_0 \Pi'_{e,c} &= 0, \\
\omega_e &= 0, \\
\frac{A_e(I_e \cdot \Pi_e) - A'(I_e \cdot \Pi'_e)}{AA_e - A'^2} \Pi_e + 3A S_0 I_e - \mu_1 I_e - \mu_5 K_e - \mu_6 J_e &= 0, \\
\frac{B_e(J_e \cdot \Pi_e) - B'(J_e \cdot \Pi'_e)}{BB_e - B'^2} \Pi_e + 3B S_0 J_e - \mu_2 J_e - \mu_4 K_e - \mu_6 I_e &= 0, \\
\frac{C_e(K_e \cdot \Pi_e) - C'(K_e \cdot \Pi'_e)}{CC_e - C'^2} \Pi_e + 3C S_0 K_e - \mu_3 K_e - \mu_4 J_e - \mu_5 I_e &= 0.
\end{align*}
\]

(124) - (128)

We stress that \( \Pi \) is written in the laboratory frame while \( \Pi'_e \) is expressed in the mantle-fixed frame. Thus, in Eq. (126), \( (I \cdot \Pi) = I_z \Pi_z + I_y \Pi_y + I_x \Pi_x \) whereas \( (I \cdot \Pi'_e) = \Pi'_{e,x} \). The same reasoning holds in Eqs. (127,128). The norm of the angular velocity \( \omega'_{e,c} \) can be arbitrarily chosen. This is due to the conservation of the Casimir \( C_{i,fc}^0(y) \). Here, we assume that the fluid core has no mean angular
velocity with respect to the mantle and thus $\omega'_c = 0$. Under this hypothesis, we get
\[
\omega_e = 0, \quad \Pi'_c = C'k, \quad \Pi_e = Ck, \quad I_e = i, \quad J_e = j, \quad K_e = k,
\]
\[
\mu_0 = 0, \quad \mu_1 = 3A\sigma^0_{xx}, \quad \mu_2 = 3B\sigma^0_{yy}, \quad \mu_3 = 3C\sigma^0_{zz} + C\Omega^2, \mu_4 = \mu_5 = \mu_6 = 0.
\]

(129)

The linear system is expressed in the coordinates
\[
\delta y^* = \left(\delta \Pi_z, \delta I_j, \delta \Pi'_c, \delta I_j, \delta I_z, \delta J_j, \delta I'_c, \delta I_j, \delta J_z, \delta K_z, \delta I_y + \delta J_x, \delta I_z + \delta K_x, \delta J_y + \delta K_y\right)^T.
\]

(130)

Let $A^*_c(y_e)$ be the matrix of the linear system evaluated at the equilibrium point and expressed in the coordinates $\delta y^*$. As in the rigid case, we define the matrices $A^1_c$ and $A^2_c$ such that
\[
A^*_c(y_e) = \begin{bmatrix} A^1_c & 0 \\ 0 & A^2_c \end{bmatrix},
\]

where the dots $\cdot$ still denote arbitrary matrices. We get
\[
A^1_c = \begin{bmatrix} 0 & -3(B - A)(\sigma^0_{xx} - \sigma^0_{yy}) \\ Cc & 0 \end{bmatrix},
\]

(132)

and
\[
A^2_c = \begin{bmatrix} 0 & C' \Omega & 0 & -C' \Omega & 0 & -C' \Omega C \Omega^2 \\ -C' \Omega & 0 & -C' \Omega & 0 & C' \Omega & 0 \\ \frac{1}{A} & 0 & 0 & \frac{1}{A} & 0 & \frac{1}{A} \\ 0 & 0 & -\Omega & 0 & 3(C - A)K_2 & 0 \\ 0 & 0 & \frac{1}{B} & 0 & 0 & \left(1 - \frac{C}{A}\right) \Omega \\ -\frac{1}{A} & 0 & 0 & \frac{1}{A} & 0 & -\left(1 - \frac{C}{A}\right) \Omega \end{bmatrix},
\]

(133)

with
\[
\begin{align*}
1 \frac{A}{c} &= AA - A'^2, & 1 \frac{A}{c} &= AA - A'^2, & 1 \frac{A}{c} &= AA - A'^2, \\
1 \frac{B}{c} &= BB - B'^2, & 1 \frac{B}{c} &= BB - B'^2, & 1 \frac{B}{c} &= BB - B'^2, \\
1 \frac{C}{c} &= CC - C'^2, & 1 \frac{C}{c} &= CC - C'^2, & 1 \frac{C}{c} &= CC - C'^2.
\end{align*}
\]

(134)
The eigenfrequencies are
\[ \omega_{fc,u} = \left( \frac{CC_c}{CC_c - C^2} \right)^{1/2} \omega_{rs,u}, \] (135)
\[ \omega_{fc,v} = \omega_{rs,v} + O(\epsilon), \] (136)
\[ \omega_{fc,w} = \omega_{rs,w} + O(\epsilon), \] (137)
\[ \omega_{fc,z} = \frac{C'}{\sqrt{A_c B_c}} \Omega + O(\epsilon) \] (138)

with \( \epsilon \) being the mass of the core divided by the total mass of the satellite. \( \omega_{rs,u}, \omega_{rs,v}, \) and \( \omega_{rs,w} \) are the frequencies obtained in the rigid case (Eqs. 89, 90). \( \omega_{fc,z} \) is the frequency of the additional degree of freedom induced by the presence of the liquid core. In the case where the fluid core represents a significant fraction of the total mass of the satellite, Eqs. (136-138) are no longer valid and eigenfrequencies should be directly computed from the matrix \( \mathbf{A}^2_{fc} \) (Eq. 133).

The Lyapunov function (Eq. 43) associated with this problem is
\[
N_{fc}(y) = \frac{1}{2A_c} \left( \Pi_x - \frac{A_c'}{A_c} \Pi_{c,x} + \frac{1}{2} (C - A_c) (I_z - K_x) \right)^2 + \frac{1}{2B_c} \left( \Pi_y - \frac{B_c'}{B_c} \Pi_{c,y} + \frac{1}{2} (C - B_c) (J_z - K_y) \right)^2 + \frac{1}{2C_c} \Pi_z^2 \\
+ \frac{1}{2A_c} \left( \Pi_{c,x} - \frac{A_c'}{A_c} \Omega (I_z - K_x) \right)^2 + \frac{1}{2B_c} \left( \Pi_{c,y} - \frac{B_c'}{B_c} \Omega (J_z - K_y) \right)^2 \\
+ \frac{1}{2} n_1 (I_y - J_x)^2 + \frac{1}{2} n_2 (I_z - K_x)^2 + \frac{1}{2} n_3 (J_z - K_y)^2,
\] (139)

where \( n_1, n_2, \) and \( n_3 \) are the same as in the rigid case (see Eq. 97). Given that \( A_c, B_c, C_c, A_c', \) and \( B_c' \) are all positive, the nonlinear stability criterion is identical to that of a rigid satellite, namely \( C > B > A \). In particular, there is no restriction on the moments of inertia of the core \( (A_c, B_c, C_c) \).

The driven equations of motion of the satellite with a liquid core in the vicinity of the relative equilibrium \( y_e \) are of the form
\[
\delta \dot{y}^k - \mathbf{A}_{fc}^k \delta y^k = \delta z_{fc}^k(t), \quad k = 1, 2,
\] (140)

with
\[
\delta y^1 = (\delta \Pi_z, \delta I_y)^T, \quad \delta y^2 = (\delta \Pi_{c,x}, \delta \Pi_{c,y}, \delta \Pi_x, \delta \Pi_y, \delta I_z, \delta J_z)^T,
\] (141)

and
\[
\delta z_{fc}^1(t) = \begin{pmatrix} 3(B - A)\sigma_{xy}(t) \\ 0 \end{pmatrix}, \quad \delta z_{fc}^2(t) = \begin{pmatrix} 3(C - B)\sigma_{yz}(t) \\ -3(C - A)\sigma_{xz}(t) \\ 0 \\ 0 \\ 0 \end{pmatrix}.
\] (142)
4.2 Quasi-spherical approximation

In this section, we reconsider the case of a satellite with a liquid core, but we assimilate $A'$, $B'$, and $C'$ to the moments of inertia of the core, i.e., we assume

$$A' \approx A_c, \quad B' \approx B_c, \quad C' \approx C_c.$$  \hfill (143)

According to Eq. (105), this is equivalent to a first order approximation in $\alpha_c$, $\beta_c$ and $\gamma_c$. With this simplification, the kinetic energy (Eq. 107) can be rewritten as follows

$$T_{fc}'(\eta, q) = \frac{1}{2} (\omega' + \Omega)^T I_m (\omega' + \Omega) + \frac{1}{2} (\omega_c' + R^T (\omega + \Omega))^T I_c' (\omega_c' + R^T (\omega + \Omega)).$$  \hfill (144)

where

$$I_m = I - R I_c' R^T = R \text{diag}(A_m, B_m, C_m) R^T$$  \hfill (145)

is the inertia tensor of the mantle written in the laboratory frame ($A_m = A - A_c$, $B_m = B - B_c$, and $C_m = C - C_c$). According to the expression (144), the problem behaves as if the liquid core were rotating rigidly relative to the mantle at the angular velocity $\omega_c'$ with a matrix of inertia $I_c'$ constant in the mantle-fixed frame. Indeed, $\omega_c' + R^T (\omega + \Omega)$ is the rotation speed of the core with respect to the inertial frame written in the mantle-fixed frame. We here retrieve the approximation made by Mathews et al. (1991) who neglected the small departure of the fluid velocity field from a pure solid rotation. Following the same procedure as in Sect. 4.1, the two submatrices of the linearised system written in the set of variables $\delta y^*$ (Eq. 130) become

$$A^1_{fc} = \begin{bmatrix} 0 & -3(B - A)(\sigma_{xx}^0 - \sigma_{yy}^0) \\ 1 & 0 \end{bmatrix},$$  \hfill (146)

and

$$A^2_{fc} = \begin{bmatrix} 0 & BC_c & 0 & -C_c B_m & 0 & -CC_c B_m^2 \\ -AC_c A_m B_c \Omega & 0 & -C_c A_m & 0 & CC_c A_m^2 & 0 \\ 0 & 0 & -C_c A_m \Omega & 0 & 3(C - A)K_1 & -3(C - B)K_2 \\ 0 & 0 & -1 & B_m & 0 & 0 \end{bmatrix}.$$

Although we retrieve the eigenfrequencies obtained in section 4.1 within the approximation (Eq. 143) only, the second member $\delta z_{fc}(t)$ of the driven system is exactly the same as $\delta z_{fc}(t)$ (Eq. 142).
5 Satellite with a subsurface ocean

Here, we consider a satellite with a rigid central part \( c \) (also called interior) and a rigid shell \( s \) separated by a global ocean \( o \). By assumption, the shell is ellipsoidal with inner radii \( a_s, b_s, c_s \) and outer radii \( a_o, b_o, c_o \). The interior, an ellipsoid of radii \( a_c, b_c, c_c \), might be differentiated, i.e., it can be made of a succession of \( N \) concentric ellipsoidal layers with different densities \( \rho_i \) and outer radii \( a_i, b_i, c_i \). We have thus \( a_N = a_c, b_N = b_c \) and \( c_N = c_c \). The ocean and the shell are assumed to be homogeneous with respective density \( \rho_o \) and \( \rho_s \). Nevertheless, the results can easily be extended to the case of a stratified rigid shell. Because the simple motion introduced by Poincaré (1910) for a satellite with a liquid core cannot be applied in this case, we use the approximation described in Sect. 4.2. We could describe the evolution of the central region and of the ocean in the shell-fixed frame to remain close to the study made on the satellite with a liquid core, but equations are more symmetrical if all coordinates are given with respect to a same given frame which we chose to be the laboratory frame. In this frame, the configuration of the system is given by the coordinates of the principal axes of the interior and the shell, i.e., the generalised coordinates are \( q = (I_c, J_s, K_c, I_s, J_o, K_o) \). The Lie velocities are the rotation vectors of the three layers with respect to the laboratory frame \( \eta = (\omega_o, \omega_c, \omega_s) \). Within the approximation of Sect. 4.2, the kinetic energy of the satellite with a global ocean reads

\[
T_{EO}(\eta, q) = \frac{(\omega_c + \Omega)^T I_c (\omega_c + \Omega)}{2} + \frac{(\omega_s + \Omega)^T I_s (\omega_s + \Omega)}{2} + \frac{(\omega_o + \Omega)^T I_o (\omega_o + \Omega)}{2},
\]

with the inertia tensors

\[
I_c = R_c \text{ diag}(A_c, B_c, C_c) R_c^T, \quad I_s = R_s \text{ diag}(A_s, B_s, C_s) R_s^T, \quad I_o = R_o \text{ diag}(A'_o, B'_o, C'_o) R_o^T - R_c \text{ diag}(A'_c, B'_c, C'_c) R_c^T,
\]

where \( R_c = [I_c, J_c, K_c] \), \( R_s = [I_s, J_s, K_s] \), and

\[
A_c = \sum_{i=1}^N 4\pi \rho_i \left( a_i b_i c_i (b_i^2 + c_i^2) - a_{i-1} b_{i-1} c_{i-1} (b_{i-1}^2 + c_{i-1}^2) \right),
\]

\[
A_s = 4\pi \rho_s \left( a_s b_s c_s (b_s^2 + c_s^2) - a_o b_o c_o (b_o^2 + c_o^2) \right),
\]

\[
A'_c = 4\pi \rho_o a_o b_o c_o (b_o^2 + c_o^2),
\]

\[
A'_s = 4\pi \rho_o a_o b_o c_o (b_o^2 + c_o^2).
\]

In Eq. (152), we apply the convention \( a_0 = b_0 = c_0 = 0 \). The other quantities \( B, C \) are deduced from Eqs. (152-155) by circular permutation of \( a, b, c \). Let us stress that the matrix of inertia of the whole satellite is simply

\[
I = I_c + I_s + I_o.
\]
In addition to the gravitational potential energy $U(y, t)$ between the planet point mass and the extended satellite, to get the Lagrangian we also need to include the self gravitational potential energy $U_{\text{self}}(q)$ of the satellite as it is a function of the relative orientation of the interior and the shell. This potential energy reads (Laplace, 1798)

$$
U_{\text{self}}(q) = \frac{u_{xx}}{2} (\mathbf{I}_c \cdot \mathbf{L}_c)^2 + \frac{u_{xy}}{2} (\mathbf{I}_c \cdot \mathbf{J}_c)^2 + \frac{u_{xz}}{2} (\mathbf{K}_c \cdot \mathbf{J}_c)^2
$$

with

$$
u_{xx} = \frac{8\pi}{15} G (\rho_s f_s + (\rho_o - \rho_s) f_o) \sum_{i=1}^{N} (\rho_i - \rho_{i+1}) a_{i}^{3} b_{i} c_{i},
$$

$$
u_{xy} = \frac{8\pi}{15} G (\rho_s g_s + (\rho_o - \rho_s) g_o) \sum_{i=1}^{N} (\rho_i - \rho_{i+1}) a_{i}^{3} b_{i} c_{i} c_{i},
$$

$$
u_{xz} = \frac{8\pi}{15} G (\rho_s h_s + (\rho_o - \rho_s) h_o) \sum_{i=1}^{N} (\rho_i - \rho_{i+1}) a_{i} b_{i}^{2} c_{i},
$$

$$
u_{yy} = \frac{8\pi}{15} G (\rho_s g_s + (\rho_o - \rho_s) g_o) \sum_{i=1}^{N} (\rho_i - \rho_{i+1}) a_{i} b_{i}^{2} c_{i} c_{i},
$$

$$
u_{yz} = \frac{8\pi}{15} G (\rho_s h_s + (\rho_o - \rho_s) h_o) \sum_{i=1}^{N} (\rho_i - \rho_{i+1}) a_{i} b_{i}^{2} c_{i},
$$

$$
u_{zz} = \frac{8\pi}{15} G (\rho_s h_s + (\rho_o - \rho_s) h_o) \sum_{i=1}^{N} (\rho_i - \rho_{i+1}) a_{i} b_{i}^{2} c_{i} c_{i},
$$

where $\rho_{N+1} := \rho_o$ and for $* \in \{s, o\}$.

$$
f_s = 2\pi \frac{a_s b_s}{c_s^2} \int_0^1 \left(1 + \frac{a_s^2 - c_s^2 t^2}{c_s^2}ight)^{-3/2} \left(1 + \frac{b_s^2 - c_s^2 t^2}{c_s^2}ight)^{-1/2} t^2 dt,
$$

$$
g_s = 2\pi \frac{a_s b_s}{c_s^2} \int_0^1 \left(1 + \frac{a_s^2 - c_s^2 t^2}{c_s^2}ight)^{-1/2} \left(1 + \frac{b_s^2 - c_s^2 t^2}{c_s^2}ight)^{-3/2} t^2 dt,
$$

$$
h_s = 2\pi \frac{a_s b_s}{c_s^2} \int_0^1 \left(1 + \frac{a_s^2 - c_s^2 t^2}{c_s^2}ight)^{-1/2} \left(1 + \frac{b_s^2 - c_s^2 t^2}{c_s^2}ight)^{-1/2} t^2 dt.
$$

The Lagrangian $L_{go}(\eta, q)$ of the problem is then

$$
L_{go}(\eta, q) = \frac{(\omega_o + \Omega)^T \mathbf{I}_c (\omega_c + \Omega) + (\omega_s + \Omega)^T \mathbf{I}_s (\omega_s + \Omega)}{2} - \frac{3GM_p}{2} \frac{r_T r_T}{r^5}
$$

$$
- \frac{1}{2} U_{\text{self}}(q).
$$

The Lie momenta associated with $\eta = (\omega_o, \omega_c, \omega_s)$ are

$$
\Pi_0 = \frac{\partial L_{go}}{\partial \omega_o} = \mathbf{I}_o (\omega_o + \Omega),
$$

$$
\Pi_c = \frac{\partial L_{go}}{\partial \omega_c} = \mathbf{I}_c (\omega_c + \Omega),
$$

$$
\Pi_s = \frac{\partial L_{go}}{\partial \omega_s} = \mathbf{I}_s (\omega_s + \Omega).
$$
from which we deduce the Hamiltonian

\[
H_{go}(y) = \frac{\Pi_{g0}^T(I_o)^{-1}\Pi_o}{2} + \frac{\Pi_{s0}^T(I_o)^{-1}\Pi_o}{2} + \frac{\Pi_{s1}^T(I_o)^{-1}\Pi_s}{2} - \Omega^T(I_c + \Pi_o + \Pi_s) + \frac{3G M_e r_f^T r_f}{r^5} + U_{scf}(q),
\]

(174)

which is a function of \( y = (\Pi_o, y_c, y_s) \) with \( y_i = (\Pi_i, I_i, J_i, K_i) \). The Poisson matrix \( B_{go}(y) \) associated with this set of variables is

\[
B_{go}(y) = \begin{bmatrix}
\Pi_o & 0 & 0 \\
0 & b(y_c) & 0 \\
0 & 0 & b(y_s)
\end{bmatrix}, \quad b(y_i) = \begin{bmatrix}
\Pi_i & I_i & J_i & K_i \\
I_i & 0 & 0 & 0 \\
J_i & 0 & 0 & 0 \\
K_i & 0 & 0 & 0
\end{bmatrix}, \quad i = c, s.
\]

(175)

Although \( y \) has 27 components, the system evolves in a phase space \( \Sigma_{go} \) of dimension 14 = 2 \times 7 \) whose degrees of freedom are the three rotations of the central region, the three rotation of the shell and an additional degree of freedom associated with the ocean:

\[
\Sigma_{go} = \{ y \in \mathbb{R}^{27} : C_{go}^i(y) = c, 0 \leq i \leq 12 \}
\]

(176)

where the thirteen Casimir functions are

\[
C_{go}^i(y) = \frac{1}{2} \Pi_{x}^j \Pi_{x_i}, \quad C_{go}^i(y) = \frac{1}{2} \Pi_{y}^j I_{x_i}, \quad C_{go}^i(y) = \frac{1}{2} \Pi_{z}^j J_{x_i}, \quad C_{go}^i(y) = \frac{1}{2} \Pi_{y}^j K_{x_i},
\]

\[
C_{go}^i(y) = \frac{1}{2} \Pi_{x}^j \Pi_{x_i}, \quad C_{go}^i(y) = \frac{1}{2} \Pi_{y}^j I_{x_i}, \quad C_{go}^i(y) = \frac{1}{2} \Pi_{z}^j J_{x_i}, \quad C_{go}^i(y) = \frac{1}{2} \Pi_{y}^j K_{x_i},
\]

\[
C_{go}^{12}(y) = \Pi_{x}^j J_{x_i}, \quad C_{go}^{13}(y) = \frac{1}{2} \Pi_{x}^j I_{x_i}, \quad C_{go}^{14}(y) = \frac{1}{2} \Pi_{x}^j K_{x_i}, \quad C_{go}^{15}(y) = \frac{1}{2} \Pi_{x}^j J_{x_i}, \quad C_{go}^{16}(y) = \frac{1}{2} \Pi_{x}^j K_{x_i}, \quad C_{go}^{17}(y) = \frac{1}{2} \Pi_{x}^j I_{x_i},
\]

(177)

In order to proceed, we have to compute the inverse of the inertia matrix of the ocean \((I_o)^{-1}\) for which we are missing the principal basis. The other terms of the Hamiltonian \( H_{go} \) (Eq. 174) are fully explicit and do not cause any problem. To make the computation analytical, we anticipate the equilibrium point solution

\[
\Pi_{o,c} = C_o \Omega k, \quad \Pi_{s,c} = C_s \Omega k, \quad I_{s,c} = i, \quad J_{s,c} = j, \quad K_{s,c} = k,
\]

\[
\Pi_{e,c} = C_e \Omega k, \quad I_{e,c} = i, \quad J_{e,c} = j, \quad K_{e,c} = k,
\]

(178)

where \( C_o = C'_o - C'_c \). We further define \( A_o = A'_o - A'_c \) and \( B_o = B'_o - B'_c \). We then expand \((I_o)^{-1}\) in Taylor series up to the second order in \( y - y_c \). This is sufficient to get the equations of motion of the linearised system. We verify that \( y_c \) (Eq. 178) actually is a solution of \( \nabla_y H_{go}(y_c) = \sum_{i} \mu_i \nabla_y C_{go}^i(y_c) \) where the Lagrange multipliers are

\[
\mu_0 = 0, \quad \mu_1 = 3A'_o \sigma_{xx} + u_{xx}, \quad \mu_2 = 3B'_o \sigma_{yy} + u_{yy},
\]

\[
\mu_3 = 3C'_o \sigma_{zz} + (C_c + C'_c) \Omega^2 + u_{zz}, \quad \mu_4 = \mu_5 = \mu_6 = 0, \quad \mu_7 = 3A'_o \sigma_{xx} + u_{xx},
\]

\[
\mu_8 = 3B'_o \sigma_{yy} + u_{yy}, \quad \mu_9 = 3C'_o \sigma_{zz} + (C_c - C'_c) \Omega^2 + u_{zz},
\]

\[
\mu_{10} = \mu_{11} = \mu_{12} = 0.
\]

(179)
such that, with the driving perturbation, the system reads

\[
\delta \dot{y}^* - A_{g0}^*(y_c)\delta y^* = \delta z_{g0}(t), \quad A_{g0}^*(y_c) := \begin{bmatrix}
0 & -A_{g0}^{12} & 0 & 0 \\
A_{g0}^{21} & 0 & 0 & 0 \\
0 & 0 & -A_{g0}^{34} & 0 \\
0 & 0 & 0 & 0 
\end{bmatrix},
\]

(182)

with

\[
A_{g0}^{12} = \begin{bmatrix}
3(B_c^o - A_c^o)(\kappa_1 - \kappa_2) + U_{xy} & -U_{xy} \\
-U_{xy} & 3(B_c^o - A_c^o)(\kappa_1 - \kappa_2) + U_{xy} 
\end{bmatrix},
\]

(183)

\[
A_{g0}^{21} = \begin{bmatrix}
1 \frac{1}{C_s} \\
0 \frac{1}{C_c} 
\end{bmatrix},
\]

(184)

and

\[
A_{g0}^{34} = \begin{bmatrix}
-\Omega & M^B_x + U_{yx} + F_{1,s}^B & 0 & -U_{yz} - F_{3,s}^B & F_{2,s}^B \\
\frac{B_s}{C_s} & -B_s & 0 & 0 & 0 \\
\frac{B_s}{C_s} & 0 & -U_{yz} - F_{3,s}^B & -\Omega & M^B_x + U_{yx} + F_{1,c}^B & -F_{2,c}^B \\
0 & 0 & \frac{1}{B_c} & \frac{B_c}{C_c} & -\Omega & 0 \\
0 & -F_{2,s}^B C_s & 0 & F_{2,c}^B C_c & -F_{4} & -\Omega 
\end{bmatrix},
\]

(185)

\[
A_{g0}^{43} = \begin{bmatrix}
-\Omega & M^A_x + U_{zx} + F_{1,s}^A & 0 & -U_{xz} - F_{3,s}^A & F_{2,s}^A \\
\frac{B_s}{C_s} & \frac{A_s}{A_c} & 0 & 0 & 0 \\
\frac{A_s}{A_c} & 0 & -U_{xz} - F_{3,s}^A & -\Omega & M^A_x + U_{zx} + F_{1,c}^A & -F_{2,c}^A \\
0 & 0 & \frac{1}{A_c} & \frac{A_c}{A_c} & -\Omega & 0 \\
0 & -F_{2,s}^A C_s & 0 & F_{2,c}^A C_c & -F_{4} & -\Omega 
\end{bmatrix},
\]

(186)

In matrices \(A_{g0}^{34}\) and \(A_{g0}^{43}\) (Eqs. 185,186), the interaction with the central planet is represented by the terms

\[
M^A_i = 3(C_i^0 - A_i^0)\kappa_1, \quad M^B_i = 3(C_i^0 - B_i^0)\kappa_2, \quad i = s, c,
\]

(187)
the core/shell gravitational coupling through the ocean interface is given by

\[
U_{xy} := u_{xy} + u_{yx} - u_{xx} - u_{yy}, 
\]

(188)

\[
U_{xz} := u_{xz} + u_{zx} - u_{xx} - u_{zz}, 
\]

(189)

\[
U_{yz} := u_{yz} + u_{zy} - u_{yy} - u_{zz}, 
\]

(190)

From the expressions of \((u_{ab})_{a,b \in \{x,y,z\}}\) given in Eqs. (158-166), we get

\[
U_{xy} = 2G(B_0^c - A_0^c)(\rho_s(g_s - f_s) + (\rho_o - \rho_s)(g_o - f_o)),
\]

(191)

\[
U_{xz} = 2G(C_0^c - A_0^c)(\rho_s(h_s - f_s) + (\rho_o - \rho_s)(h_o - f_o)),
\]

(192)

\[
U_{yz} = 2G(C_0^c - B_0^c)(\rho_s(h_s - g_s) + (\rho_o - \rho_s)(h_o - g_o)).
\]

(193)

Finally, the remaining terms

\[
F_{K}^1, s = \frac{(C'_s - K'_s)(C'_s - K'_c)}{K_o} \Omega^2, \quad K = A, B
\]

(194)

\[
F_{K}^1, c = \frac{(C'_c - K'_c)(C'_c - K'_c)}{K_o} \Omega^2, \quad K = A, B
\]

(195)

\[
F_{K}^2, i = \frac{C'_i - K'_i}{K_o} \Omega, \quad i = s, c, \quad K = A, B
\]

(196)

\[
F_{K}^3 = \frac{(C'_s - K'_s)(C'_s - K'_c)}{K_o} \Omega^2, \quad K = A, B
\]

(197)

\[
F_{K}^4 = \frac{C_o - K_o}{K_o} \Omega, \quad K = A, B
\]

(198)

are only present in the linearised system because of the rotation of the ocean. If the Casimir \(C_0(y) = \Pi_o \cdot \Pi_o/2\) were set equal to zero, i.e., if the ocean were not rotating with respect to the inertial frame, all \(F_{K}^1, s, F_{K}^1, c, F_{K}^2, i\), and \(F_{K}^3\), with \(K = A, B\) and \(i = s, c\), would be nil. The same conclusion would hold if the kinetic energy of the ocean \(\Pi_o^T(I_o)^{-1}\Pi_o/2\) were skipped from the Hamiltonian \(H_{go}\). We thus interpret these terms as due to the centrifugal force felt by the ocean and responsible for an additional pressure on the interfaces with the interior and the shell. In that case -- i.e., if the kinetic energy of the ocean were dropped --, the ocean angular momentum \(\Pi_o\) would be decoupled from the rest of the system. A quick inspection of the last row and column of the matrices \(A_{go}^{43}\) and \(A_{go}^{44}\) indeed shows that a perturbation of \(\Pi_o\) would rotate at the eigenfrequency \(\Omega\) with respect to the laboratory frame, and would thus be fixed in the inertial frame.

We note that given the structure of the matrix \(A_{go}^*\), the linearised system is characterised by two libration frequencies in longitude and five frequencies associated with libration in latitude and wobble.
For this problem, the Lyapunov function reads

\[ N_{\rho \Omega}(y) = \]
\[ \frac{1}{2A_c} \left( H_{o,x} - \frac{1}{2}(C'_c - A'_c)\Omega(I_{c,z} - K_{c,x}) + \frac{1}{2}(C'_s - A'_s)\Omega(I_{s,z} - K_{s,x}) \right)^2 \]
\[ + \frac{1}{2B_c} \left( H_{o,y} - \frac{1}{2}(C'_c - B'_c)\Omega(J_{c,z} - K_{c,y}) + \frac{1}{2}(C'_s - B'_s)\Omega(J_{s,z} - K_{s,y}) \right)^2 \]
\[ + \frac{1}{2A_c} \left( H_{c,x} + \frac{1}{2}(C_c - A_c)\Omega(I_{c,z} - K_{c,x}) \right)^2 \]
\[ + \frac{1}{2B_c} \left( H_{c,y} + \frac{1}{2}(C_c - B_c)\Omega(J_{c,z} - K_{c,y}) \right)^2 + \frac{1}{2C_c} \Pi_{cz}^2 \]
\[ + \frac{1}{2A_c} \left( H_{s,x} + \frac{1}{2}(C_s - A_s)\Omega(I_{s,z} - K_{s,x}) \right)^2 \]
\[ + \frac{1}{2B_c} \left( H_{s,y} + \frac{1}{2}(C_s - B_s)\Omega(J_{s,z} - K_{s,y}) \right)^2 + \frac{1}{2C_s} \Pi_{sz}^2 \]
\[ + \frac{U_{xy}}{4} ((I_{c,y} - J_{c,x}) - (I_{s,y} - J_{s,x}))^2 + \frac{U_{xz}}{4} ((I_{c,z} - K_{c,x}) - (I_{s,z} - K_{s,x}))^2 \]
\[ + \frac{U_{yz}}{4} ((J_{c,z} - K_{c,y}) - (J_{s,z} - K_{s,y}))^2 \]
\[ + \frac{n_1}{2} (J_{s,z} - I_{s,y})^2 + \frac{n_2}{2} (J_{s,z} - K_{s,x})^2 + \frac{n_3}{2} (J_{s,z} - K_{s,y})^2 \]
\[ + \frac{n_1}{2} (J_{c,x} - I_{c,y})^2 + \frac{n_2}{2} (J_{c,z} - K_{c,x})^2 + \frac{n_3}{2} (J_{c,z} - K_{c,y})^2 , \] (199)

with

\[ n_1 = \frac{3}{4} (B'_c - A'_c)(\kappa_1 - \kappa_2), \]
\[ n_2 = \frac{1}{4} (C'_s - A'_s)(\Omega^2 + 3\kappa_1), \]
\[ n_3 = \frac{1}{4} (C'_c - B'_c)(\Omega^2 + 3\kappa_2), \] (200)

and where \( * = s, c. \) We deduce that the system is nonlinearly stable if the following conditions are met

\[ U_{xy} > 0, \quad U_{zz} > 0, \quad U_{yz} > 0, \quad C^0_* > B^0_* > A^0_* \quad \text{with} \quad * = s, c. \] (201)

Using the expressions of \( U_{xy}, U_{xz}, \) and \( U_{yz} \) (Eqs. 191-193) expanded at first order in the equatorial and polar flatness, the conditions (201) are equivalent to

\[ \begin{cases} 
\rho_1 \frac{a_k - c_k}{a_s} + (\rho_o - \rho_o) \frac{a_o - c_o}{a_o} > \rho_1 \frac{a_s - b_s}{a_s} + (\rho_o - \rho_o) \frac{a_o - b_o}{a_o} > 0, \\
C^0_* > B^0_* > A^0_*, \quad * = s, c. \end{cases} \] (202)
Table 2  Orbital and physical parameters of Io taken from (Noyelles, 2014).

| Parameter       | Value | Units |
|-----------------|-------|-------|
| $GM_p$ (Jupiter) | 126 712 765 | km$^3$/s$^2$ |
| $a$             | 422 029.958 | km |
| $e$             | 0.00415 | |
| $i$             | 2.16 | arcmin |
| $\Omega$        | 1297.204 472 527 9755 | rad/a |
| $A/(mR^2)$      | 0.375 127 | |
| $B/(mR^2)$      | 0.377 342 | |
| $C/(mR^2)$      | 0.378 080 | |
| $A_c/(mR^2)$    | 0.006 283 9600 | |
| $B_c/(mR^2)$    | 0.006 253 4432 | |
| $C_c/(mR^2)$    | 0.006 007 5578 | |

*a Moments of inertia of the core computed from the internal model 1 of (Noyelles, 2014).

Table 3  Eigenperiods of Io’s rotational motion (Eq. 204).

| Source         | $T_u$ (day) | $T_v$ (day) | $T_w$ (day) | $T_z$ (day) |
|----------------|-------------|-------------|-------------|-------------|
| Noyelles (2014)| 13.2322     | 166.3520    | 225.0927    | 1.7382      |
| This work: model fc (Sect. 4.1) | 13.2504 | 157.2780 | 224.5395 | 1.7385 |
| This work: model fc’ (Sect. 4.2) | 13.2502 | 156.5653 | 224.5402 | 1.7368 |

Finally, as in the previous section, to get the forced solution, we decompose the driving excitation $\delta z_{go}(t)$ as $(\delta z_{go}^1, \delta z_{go}^2, 0)$, with

$$ \delta z_{go}^1(t) = \begin{pmatrix} 3(B_o^c - A_o^c)\sigma_{z y}^1 \\ 3(B_o^c - A_o^c)\sigma_{z y}^1 \\ 0 \end{pmatrix}, \quad \delta z_{go}^2(t) = \begin{pmatrix} 3(C_o^c - B_o^c)\sigma_{y z}^1 \\ 3(C_o^c - B_o^c)\sigma_{y z}^1 \\ 0 \end{pmatrix}. $$

(203)

6 Application

6.1 Io’s libration modes

Io, one of the Galilean satellite of Jupiter, is assumed to have a liquid core (Anderson et al., 1996). Its rotation motion has already been studied within the Poincaré-Hough paradigm using a Hamiltonian formalism (Henrard, 2008). This analysis has then been extended using the same method in (Noyelles, 2013, 2014). Although the approach in *ibid.* is Hamiltonian, it differs from that described in Sect. 4 which is expressed in non-canonical variables. Here, we revisit the problem with the aim of validating our method and, more specifically, the quasi-spherical approximation (Sect. 4.2).
The orbital and physical parameters of Io, which are summarised in Tab. 2, are taken from (Noyelles, 2013, 2014). The eigenfrequencies $\omega_u$, $\omega_v$, $\omega_w$, and $\omega_z$ are directly computed from the matrix $A^e$ (Eqs. 132, 133) for the Poincaré-Hough model (Sect. 4.1), and from the matrix $A^e_{fc}$ (Eqs. 146, 147) for the quasi-spherical approximation (Sect. 4.2). Hereafter, the two models are referred to as “model fc” and “model fc′”, respectively. The eigenfrequencies are then converted into periods for a direct comparison with (Noyelles, 2014). The correspondence between the eigenperiods of ibid. and the eigenfrequencies of this work is

$$T_u = \frac{2\pi}{\omega_u}, \quad T_v = \frac{2\pi}{\omega_v - \Omega}, \quad T_w = \frac{2\pi}{\omega_w}, \quad T_z = \frac{2\pi}{\omega_z} \quad (204)$$

The results are gathered in Tab. 3. We observe a good match between model fc and that of (Noyelles, 2014) for $T_u$, $T_w$, and $T_z$ with a maximal error of about 0.2%. There is a larger discrepancy between the two approaches in the case of $T_v$ with a deviation of almost 6%, but this eigenmode is more sensitive due to the small denominator $\omega_v - \Omega$ (Eq. 204). It is also very sensitive to the polar flattening of the core (Noyelles, 2012). Nevertheless, the agreement is satisfactory given that the methods to compute the eigenperiods in both studies are very different. The eigenfrequencies given by models fc and fc′ are also very close to each other. Once again, the largest discrepancy occurs for $T_v$, but here it does not exceed 0.5%. We thus conclude that the quasi-spherical approximation is justified.

Figure 2 represents the trajectories of the principal axes $\mathbf{I}$, $\mathbf{J}$, and $\mathbf{K}$ in the laboratory frame ($\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$) while the system stands in each of the eigenmodes. The corresponding eigenfrequencies are recalled below each subfigure. We recognise the libration motions of a rigid satellite which the name of the eigenmodes have been taken from. In (Henrard, 2008) and in (Noyelles, 2013, 2014), the eigenmode associated with $\omega_z$ is referred to as the free libration of the core. Nevertheless, given the strong similarity between the motions associated with $\omega_v$ and $\omega_z$, we chose to attribute the same name “libration in latitude” for both of them. Furthermore, from the observation of the surface only it is hardly possible to distinguish one from the other. Actually, the distinction between the two modes lies in the relative position of $\Pi_c$ and $\Pi$, as shown in Fig. 3. When the satellite is in the eigenmode associated with $\omega_v$, the two vectors are on the same side from the origin, while in the eigenmode of frequency $\omega_z$ they are on opposite side.

6.2 Titan’s equilibrium obliquity

In this section, we analyse the rotation of Titan orbiting Saturn. Several hints suggest that this satellite holds a global ocean under its surface (Coyette et al., 2016, and references therein). Among these clues, an important one for our purpose is Titan’s “high” obliquity of $0.32^\circ$ which could not be explained if the satellite were solid (Bills and Nimmo, 2011). Nevertheless, a discrepancy still persists between the observations and the expected obliquity associated with the Cassini state, the latter remaining below $0.15^\circ$ for a large class of interior models (e.g., Baland et al., 2011). Therefore, it has been proposed that Titan’s current obliquity is amplified.

---

Here and throughout the paper, we follow the IAU recommendations which state that the symbol for a Julian year is “a”. Hence, radian per year is written “rad/a”.
Fig. 2 Eigenmodes of Io’s rotation motion computed with the parameters of Tab. 2. Positions at constant time intervals of the principal axes (I, J, K) are depicted by black dots. Open circles indicate the initial condition. Intersections of the dotted great circles of the unit sphere represent the laboratory frame (i, j, k). Jupiter is in the direction of the vector i. The associated eigenfrequencies are recalled below each figure.

by a resonance with one of the remaining orbital forcing frequencies (Baland et al., 2011; Noyelles and Nimmo, 2014).

In his abstract, Henrard (2008) wrote about Io that “the addition of a degree of freedom (the spin of the core) with a frequency close to the orbital frequency multiplies the possibility of resonances”. In the case of Titan, we also have an additional degree of freedom in comparison to the previous studies quoted above. We thus expect our model to be able to tilt Titan’s axis more easily.

The orbital elements of Titan are taken from the ephemeris TASS1.6 (Vienne and Duriez, 1995). From the full solution, we only retain the keplerian motion and the nodal precession of the orbit with respect to the Laplace plane\(^3\). These parameters are summarised in Tab. 4. Regarding Titan internal structure, we

\(^3\) Here, we define Titan’s Laplace plane as the plane whose orientation is given by the constant part of the inclination solution of TASS1.6. (Vienne and Duriez, 1995)
select two models proposed by Fortes (2012), hereafter referred to as model F1 and F2. They assume a global ocean with extreme densities equal to 1023 kg/m$^3$ and 1281 kg/m$^3$, respectively. In model F1, the ocean is a mixture of water and methanol, while in model F2, the ocean is made of water and ammonia. Parameters of these interior models are summarised in Tab. 5. In both models, the average density is 1881 kg/m$^3$ and the mean moment of inertia $I/(mR^2)$ remains within the errorbars provided by Iess et al. (2012). The equatorial flattening $\zeta$ is obtained by integration of Clairaut’s equation (Clairaut, 1743) assuming an hydrostatic equilibrium (same as Richard, 2014). The boundary semi-axes at volumetric mean radius $R$ between two layers are given by (e.g., Rambaux and Castillo-Rogez, 2013)

$$a = R \left(1 + \frac{7}{9}\zeta\right), \quad b = R \left(1 - \frac{2}{9}\zeta\right), \quad c = R \left(1 - \frac{5}{9}\zeta\right).$$

The values of the derived parameters involved in the Hamiltonian $H_{lo}(y)$ (Eq. 174) are listed in Tab. 6.

The eigenfrequencies computed for the two interior models F1 and F2 are shown in Tab. 7. For each model, we assume either a rotating or a static ocean with respect to the inertial frame (see Sect. 5). For reference, we also provide the eigenfrequencies assuming a fully rigid satellite. To interpret these eigenfrequencies, the associated trajectories of the vectors ($I_c$, $J_c$, $K_c$) and ($I_s$, $J_s$, $K_s$) are displayed in Fig. 4. We recognise librations in longitude at $\omega_{u_1}$ and $\omega_{u_2}$, librations in latitude at $\omega_{v_1}$, $\omega_{v_2}$, and $\omega_{v_3}$, and wobbles at $\omega_{w_1}$ and $\omega_{w_2}$. From Tab. 7, we observe that each eigenmode has a specific range of frequencies. Libration frequencies in latitude are close to the mean motion $\Omega \approx 143.9240 \text{rad}/\text{a}$. Frequencies of libration in longitude are between 2 and 8 rad/a, and the wobble is the slowest motion with frequencies ranging between 0.01 and 0.2 rad/a.
Fig. 4 Eigenmodes of Titan’s rotation motion computed with the interior model F1. Positions at constant time intervals of the shell principal axes (I_s, J_s, K_s) are depicted by black dots on the unit sphere. Those of the interior (I_c, J_c, K_c) are plotted at half the radius of the unit sphere. The white dots indicate the initial condition. Intersections of the dotted great circles of the unit sphere represent the laboratory frame (i, j, k). Saturn is in the direction of the vector i. The associated eigenfrequencies are recalled below each figure.
Table 4  Orbital parameters of Titan used in this study.

| Parameter | value | units | reference |
|-----------|-------|-------|-----------|
| $\mu M_P$ (Saturn) | 37,931,272 | km$^3$/s$^2$ | (Campbell and Anderson, 1989) |
| $a$ | 1,221,729 | km | computed$^a$ |
| $\epsilon$ | 0.028 | | (Vienne and Duriez, 1995) |
| $i^b$ | 0.320 | deg | (Vienne and Duriez, 1995) |
| $\Omega$ | 143.924 047 85 | rad/a | (Vienne and Duriez, 1995) |
| $d\Phi/dt$ | -0.008 931 24 | rad/a | (Vienne and Duriez, 1995) |

$^a$ The semimajor axis has been computed from the masses of Saturn and Titan given by Campbell and Anderson (1989) and the orbital parameters $N_6$ and $p_{06}$ provided by Vienne and Duriez (1995).

$^b$ Inclination with respect to the Laplace plane given by the amplitude of the second harmonic of $\zeta_{06}$ in the notation of Vienne and Duriez (1995).

Table 5 Physical parameters of the two interior models of Titan considered in this study taken from (Fortes, 2012).

| Layer | $\rho$ (kg/m$^3$) | $R$ (km) | $\zeta$ (10$^{-5}$) | $\rho$ (kg/m$^3$) | $R$ (km) | $\zeta$ (10$^{-5}$) |
|-------|-----------------|--------|-----------------|-----------------|--------|-----------------|
| Ice   | 930.9           | 2575   | 12.068          | 930.9           | 2575   | 12.080          |
| Ocean | 1023.5          | 2475   | 11.878          | 1281.3          | 2475   | 11.887          |
| Ice V | 1272.7          | 2225   | 11.552          | 1350.9          | 2225   | 11.488          |
| Ice VI| 1338.9          | 2163   | 11.521          | -               | -      | -               |
| Silicate | 2542.3         | 2116   | 11.514          | 2650.4          | 1984   | 11.310          |

For each layer, $\rho$ is the density and $R$ and $\zeta$ respectively denote the mean radius and the equatorial flattening of the upper boundary.

The condition for Titan to have a significant (shell) obliquity is that one of the libration frequencies in latitude gets close to the excitation frequency of the perturbation $\sigma_{1xz}^3(t)$ (Eq. 69), namely, $\omega_{1xz} = \dot{\Omega} - \dot{\Phi} \approx 143.9330$ rad/a. In the case of a rigid satellite there is no lever arm. The libration frequency only depends on the total moments of inertia which are constrained by observations. This frequency, equal to 143.9582 rad/a, leads to an obliquity of $0.113^\circ$ which is about one third of the actual value $\varepsilon_{obs} = 0.32^\circ$.

When the ocean is taken into account, the system has three distinct frequencies of libration in latitude which can potentially be in resonance with the orbital precession rate. It should nevertheless be stressed that when the rotation of the ocean is set to zero, the frequency $\omega_{w3}$ in Tab. 7 is just the mean rotation of \( \Pi \) which is not involved in the tilting of the shell axis. Titan’s obliquities $\varepsilon$ computed with the different models are gathered in Tab. 8. Note that we allow the obliquity to be negative as explained in Fig. 5. As expected, within the “static ocean” hypothesis the ocean is not affected by the perturbation $\sigma_{1xz}^3$. Its obliquity is $\varepsilon_o = -i$, meaning that $\Pi_o$ remains aligned with the Laplace pole $k$ which is the third axis of our laboratory reference frame. The last two eigenfrequencies $\omega_{v1}$ and $\omega_{v2}$ are further away from $\omega_{1xz}^3$ than $\omega_{w3}^3$. They only produce a shell obliquity of $\varepsilon_s \approx 0.06^\circ$ which is much lower than the observed one. Furthermore, this result does not significantly vary from model F1 to model F2.

If the rotation of the ocean is set equal to the mean rotation of the satellite, $\omega_{w3}$ is the eigenfrequency responsible for the tilt of Titan’s shell spin pole. With the two models F1 and F2 considered here, the results are still very low: $\varepsilon_s = 0.004^\circ$
Table 6 Derived parameters for Titan’s model.

| Parameter      | model F1       | model F2       | units          |
|----------------|----------------|----------------|----------------|
| $A_c/(mR^2)$   | 0.232 133 9588 | 0.213 354 6838 |                |
| $B_c/(mR^2)$   | 0.232 160 7420 | 0.213 379 654  |                |
| $C_c/(mR^2)$   | 0.232 169 6677 | 0.213 387 1908 |                |
| $A_m/(mR^2)$   | 0.035 565 0464 | 0.035 556 8942 |                |
| $B_m/(mR^2)$   | 0.035 569 7420 | 0.035 571 835  |                |
| $C_m/(mR^2)$   | 0.035 571 6677 | 0.035 571 8408 |                |
| $A'_c/(mR^2)$  | 0.104 835 1592 | 0.131 211 1289 |                |
| $B'_c/(mR^2)$  | 0.104 847 2721 | 0.131 226 5042 |                |
| $C'_c/(mR^2)$  | 0.104 851 3089 | 0.131 231 2299 |                |
| $A'_m/(mR^2)$  | 0.178 538 4650 | 0.223 457 6674 |                |
| $B'_m/(mR^2)$  | 0.178 559 6760 | 0.223 484 2365 |                |
| $C'_m/(mR^2)$  | 0.178 566 7448 | 0.223 493 0909 |                |
| $\mathbf{u}_{xx}/(mR^2)$ | 135.969 642 03 109.837 900 34 | 1/day$^2$      |
| $\mathbf{u}_{xy}/(mR^2)$ | 135.989 307 39 109.837 900 34 | 1/day$^2$      |
| $\mathbf{u}_{xz}/(mR^2)$ | 135.989 307 39 109.837 900 34 | 1/day$^2$      |
| $\mathbf{u}_{yx}/(mR^2)$ | 135.989 307 39 109.837 900 34 | 1/day$^2$      |
| $\mathbf{u}_{yy}/(mR^2)$ | 135.989 307 39 109.837 900 34 | 1/day$^2$      |
| $\mathbf{u}_{yz}/(mR^2)$ | 135.989 307 39 109.837 900 34 | 1/day$^2$      |
| $\mathbf{u}_{zx}/(mR^2)$ | 135.989 307 39 109.837 900 34 | 1/day$^2$      |
| $\mathbf{u}_{zy}/(mR^2)$ | 135.989 307 39 109.837 900 34 | 1/day$^2$      |

Note that the number of digits provided in this table is required to recover the values presented in Tabs. 7 and 8.

Table 7 Eigenfrequencies of Titan’s rotation in rad/a.

|                    | rotating ocean | static ocean | rigid   | type of motion |
|--------------------|----------------|--------------|---------|----------------|
| $\omega_{u1}$ | 7.9237         | 2.3950       | 7.9237  | 2.3950         | libration in longitude |
| $\omega_{u2}$ | 144.3272       | 144.3641     | 144.2507| 144.2683       | libration in latitude  |
| $\omega_{v1}$ | 143.9494       | 143.9445     | 143.9528| 143.9472       | libration in latitude  |
| $\omega_{v2}$ | 143.9307       | 143.9266     | 143.924  | 143.924        | wobble                   |

$a$ In the case where the ocean is assumed static, $\omega_{v2} = 143.9240$ rad/a is the mean motion $\Omega$.

with model F1 and $\epsilon_s = 0.108^\circ$ with model F2. However, the two values vary by a factor 27. A modification of Titan’s interior is thus more likely to produce the observed obliquity if the rotation of the ocean is taken into account.

To illustrate this statement, we generate a series of interior models of Titan based on the model F1. To simulate inhomogeneities in the shell, we slightly modify the equatorial flattening $\zeta_s$ of the surface from $11.890 \times 10^{-5}$ to the hydrostatic value $12.068 \times 10^{-5}$ given in Tab. 5. These numbers should be compared to the
Fig. 5 Definition of Titan’s inclination \(i\) and obliquity \(\varepsilon\). In a Cassini state of the averaged problem, the Laplace pole, the orbit pole and the spin pole are in a same plane. We define the orientation of this plane by the inclination measured from the Laplace pole to the orbit pole which by convention is positive. This allows to defined the obliquity as a signed angle measured from the orbit pole to the spin axis. In this figure, \(\varepsilon\) is positive.

Table 8 Obliquity of Titan’s layers in degree.

|       | rotating ocean | static ocean | rigid |
|-------|----------------|--------------|-------|
|       | \(F_1\) | \(F_2\) | \(F_1\) | \(F_2\) | \(F_1/F_2\) |
| core  | 0.294 | 0.272 | 0.149 | 0.207 | 0.113 |
| ocean | -0.479 | 0.208 | -0.320 | -0.320 | 0.113 |
| shell | 0.004 | 0.108 | 0.062 | 0.064 | 0.113 |

The meaning of the sign of the obliquity is explained in Fig. 5.

equatorial flattenings computed with the two models provided by Iess et al. (2012), i.e., \(11.911 \times 10^{-5}\) (SOL2) and \(12.005 \times 10^{-5}\) (SOL1a). To keep the global moments of inertia constant, the equatorial flattening of all the other layers are refitted using Clairaut’s equation. It has been checked that all these models are nonlinearly stable according to the condition Eq. (202). Figure 6 displays the evolution of the libration frequencies in latitude \(\omega_{v2}\) and \(\omega_{v3}\) as a function of the surface equatorial flattening \(\zeta_s\). When the rotation of the ocean is considered (left plots), \(\omega_{v3}\) varies sufficiently to cross the resonant frequency \(\omega_{xz}\) at \(\zeta_s \approx 11.97 \times 10^{-5}\) where, in the linear approximation, the shell obliquity diverges. More interestingly, for \(\zeta_s \approx 11.94 \times 10^{-5}\), the driven shell obliquity \(\varepsilon_s\) is equal to the observed value \(\varepsilon_{obs} = 0.32^\circ\). In comparison, when the ocean is assumed to be static (right plots of Fig. 6), \(\omega_{v3}\) remains strictly equal to \(\Omega\) and \(\omega_{v2}\) barely evolves. As a consequence, the equilibrium shell obliquity remains practically constant close to \(0.062^\circ\).

7 Conclusion

This paper provides a general method for analysing the rotation dynamics of a rigid body with a fluid internal layer. The study is performed in a non-canonical
Hamiltonian formalism well adapted to systems near relative equilibria such as synchronous satellites in a Cassini state. The Poisson structure of the non-canonical Hamiltonian is here obtained by a Legendre transformation of the corresponding Lagrangian written using Poincaré’s formalism which makes use of the properties of the Lie group acting on the configuration space.

With this approach, we have been able to treat the case of a satellite with a liquid core or with a global underneath ocean in the exact same manner as that of a rigid satellite. All the difficulty is in the calculation of the Lagrangian function – and more specifically, of the kinetic energy of the fluid layer – in terms of generalised coordinates and Lie velocities. For a satellite with a liquid core, Poincaré introduced the concept of a fluid simple motion which cannot be rigorously transposed to a satellite with an ocean. Nevertheless, at first order this fluid layer behaves like a rigid body for which the kinetic energy is known. Tests on a satellite with a liquid core, assuming Io’s physical and orbital parameters, have shown that the errors induced by this approximation do not exceed 0.5% on the eigenfrequencies.

The analysis of a hollow satellite with a fluid core leads to a four degree of freedom dynamical model. The linearised problem in the vicinity of the synchronous

Fig. 6 Obliquity and libration frequencies as a function of the surface equatorial flattening assuming a rotating ocean (left) or a static ocean (right). The vertical dashed line indicates the location of the resonance $\omega_{v3} = \omega_{1xz}$. In the upper plots, $\varepsilon_s$ represents the obliquity of the shell at the Cassini state and $\varepsilon_{obs}$ the observed value.
equilibrium state is thus characterised by four eigenmodes. These are a libration in longitude, a wobble and two librations in latitude. To this solved problem, we have provided an analytical expression of the linearised equations written in terms of intuitive variables, namely the components of the angular momenta and of the base frame vectors. We also have clearly identified the fourth eigenmode as a libration in latitude.

The rotation dynamics of a satellite with a global subsurface ocean is governed by seven eigenmodes associated with the seven degrees of freedom of the problem, six of which being equally shared by the interior and the outer shell and the last one being brought by the ocean. Near the synchronous equilibrium state, these eigenmodes are identified as two librations in longitude, two wobbles and three librations in latitude. The amplitude of the third libration in latitude would only vanish if the ocean were static with respect to the inertial frame.

Our study has been motivated by Titan’s obliquity measured by the Cassini-Huygens mission. Thus far, dynamical models struggle to explain its high value under the hydrostatic shape hypothesis suggested by the ratio of its Stokes coefficients $J_2/C_{22} \approx 10/3$. Here, we show that the rotation of the ocean makes the dynamical model much more sensitive to small perturbations of the interior model than when the ocean is assumed static. As an example, starting from a body in perfect hydrostatic equilibrium, we slightly modified the equatorial flattening of the shell by about 1% of the nominal value. This was enough to bring the obliquity of the Cassini state even beyond the radiometric value with the seven degree of freedom model while the same quantity computed with the static ocean hypothesis remained practically constant scarcely reaching a 0.1% increase.

This work is intended to demonstrate the capability of the seven degree of freedom dynamical model to explain the observed high obliquity of Titan. The problem has therefore been intentionally simplified. Tidal deformations, atmospheric torques, and all orbital perturbations but the main precession relative to the Laplace plane have been discarded. These additions would be required for an exhaustive search of the interior models compatible with the measurements made by the Cassini-Huygens mission: the rotation state, the gravity field coefficients, the shape, the tidal Love number, and the electric field. But this is beyond the scope of the present paper and shall be discussed elsewhere.

Acknowledgements We thank Benoît Noyelles and the anonymous referee for their constructive comments. G.B. also thank Philippe Robutel for the useful conversations on the theoretical parts of this work and Rose-Marie Baland and Marie Yseboodt for our instructive discussion on this problem during the 2017 DDA meeting in London.

References

Anderson J. D., Sjogren W. L., Schubert G.: Galileo Gravity Results and the Internal Structure of Io. Science 272, 709–712 (1996)

Baland R.-M., van Hoolst T., Yseboodt M., Karatekin Ö.: Titan’s obliquity as evidence of a subsurface ocean? Astron. Astrophys. 530: A141 (2011)

Baland R.-M., Tobie G., Lefèvre A., Van Hoolst T.: Titan’s internal structure inferred from its gravity field, shape, and rotation state. Icarus 237, 29–41 (2014)

Beck J. A., Hall C. D.: Relative Equilibria of a Rigid Satellite in a Circular Keplerian Orbit. The Journal of the Astronautical Sciences 46(3), 215–247 (1998)
(1910)
Rambaux N., Castillo-Rogez J.: Tides on Satellites of Giant Planets. In: Souchay J., Mathis S., Tokieda T. (eds) Lecture Notes in Physics, Berlin Springer Verlag, Lecture Notes in Physics, Berlin Springer Verlag, vol 861, p 167 (2013)
Rambaux N., Castillo-Rogez J. C., Le Maistre S., Rosenblatt P.: Rotational motion of Phobos. Astron. Astrophys. 548:A14 (2012)
Rappaport N., Bertotti B., Giampieri G., Anderson J. D.: Doppler Measurements of the Quadrupole Moments of Titan. Icarus 126, 313–323 (1997)
Richard A.: Modèle de satellite à trois couches élastiques : application à la libration en longitude de Titan et Mimas (in French). PhD thesis, Observatoire de Paris (2014)
Richard A., Rambaux N., Charnay B.: Librational response of a deformed 3-layer Titan perturbed by non-Keplerian orbit and atmospheric couplings. Planetary and Space Science93, 22–34 (2014)
Stiles B. W., Kirk R. L., Lorenz R. D., et al.: Determining Titan’s Spin State from Cassini RADAR Images. Astronom. J. 135, 1669–1680 (2008)
Stiles B. W., Kirk R. L., Lorenz R. D., et al.: Erratum: ”Determining Titan’s Spin State from Cassini Radar Images”. Astronom. J. 139, 311 (2010)
Vienne A., Duriez L.: TASS1.6: Ephemerides of the major Saturnian satellites. Astron. Astrophys. 297, 588 (1995)
Wang Y., Xu S.: Hamiltonian structures of dynamics of a gyrostat in a gravitational field. Nonlinear Dynamics 70(1), 231–247 (2012)