Statistical Inference of Constrained Stochastic Optimization via Sketched Sequential Quadratic Programming

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Abstract

We consider statistical inference of equality-constrained stochastic nonlinear optimization problems. We develop a fully online stochastic sequential quadratic programming (StoSQP) method to solve the problems, which can be regarded as applying Newton’s method to the first-order optimality conditions (i.e., the KKT conditions). Motivated by recent designs of numerical second-order methods, we allow StoSQP to adaptively select any random stepsize $\tilde{\alpha}_t$, as long as $\beta_t \leq \tilde{\alpha}_t \leq \beta_t + \chi_t$, for some control sequences $\beta_t$ and $\chi_t = o(\beta_t)$. To reduce the dominant computational cost of second-order methods, we additionally allow StoSQP to inexactly solve quadratic programs via efficient randomized iterative solvers that utilize sketching techniques. Notably, we do not require the approximation error to diminish as iteration proceeds. For the developed method, we show that under mild assumptions (i) computationally, it can take at most $O(1/\epsilon^4)$ iterations (same as samples) to attain $\epsilon$-stationarity; (ii) statistically, its primal-dual sequence $1/\sqrt{\beta_t} \cdot (x_t - x^*, \lambda_t - \lambda^*)$ converges to a mean-zero Gaussian distribution with a nontrivial covariance matrix depending on the underlying sketching distribution. Additionally, we establish the almost-sure convergence rate of the iterate $(x_t, \lambda_t)$ along with the Berry-Esseen bound; the latter quantitatively measures the convergence rate of the distribution function. We analyze a plug-in limiting covariance matrix estimator, and demonstrate the performance of the method both on benchmark nonlinear problems in CUTEst test set and on linearly/nonlinearly constrained regression problems.

1 Introduction

We study constrained stochastic nonlinear optimization problems of the form

$$
\min_{x \in \mathbb{R}^d} f(x) = \mathbb{E}_P[f(x; \xi)],
$$

s.t. $c(x) = 0,$

where $f : \mathbb{R}^d \to \mathbb{R}$ is a stochastic objective function that involves a random variable $\xi \sim P$, and $c : \mathbb{R}^d \to \mathbb{R}^m$ provides deterministic equality constraints. Problems of this form appear widely in a variety of applications in statistics and machine learning, including constrained $M$-estimation (Geyer, 1991, 1994; Wets, 1999), multi-stage stochastic optimization (Dantzig and Infanger, 1993; Veliz et al., 2014), physics-informed neural networks (Karniadakis et al., 2021; Cuomo et al., 2022), and algorithmic fairness (Zafar et al., 2019). In practice, the random variable $\xi$ corresponds to a data sample; $f(x; \xi)$ is the loss occurred at sample $\xi$ when using parameter $x$ to fit the model; and $f(x)$
is the expected loss. Deterministic constraints are prevalent in real examples, which can encode prior model information, address identifiability issue, and/or reduce searching complexity.

To infer statistical properties of a (local) primal-dual solution \((x^*, \lambda^*)\) of Problem (1), statisticians often generate \(N\) samples \(\xi_1, \ldots, \xi_N \sim \mathcal{P}, \text{iid}\), and consider the corresponding empirical risk minimization (ERM) problem:

\[
\min_{x \in \mathbb{R}^d} \tilde{f}(x) = \frac{1}{N} \sum_{i=1}^{N} f(x; \xi_i),
\]

\[
\text{s.t. } c(x) = 0.
\]

Under certain regularity conditions, we can establish the asymptotic consistency and normality of the minimizer \((\hat{x}_N, \hat{\lambda}_N)\) of (2), also called \(M\)-estimator, given by

\[
\sqrt{N} \left( \frac{\bar{x}_N - x^*}{\hat{\lambda}_N - \lambda^*} \right) \xrightarrow{d} \mathcal{N} \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \nabla^2 L^* & (G^*)^T \\ G^* & 0 \end{pmatrix}^{-1} \begin{pmatrix} \text{cov}(\nabla f(x^*; \xi)) \\ 0 \end{pmatrix} \begin{pmatrix} \nabla^2 L^* & (G^*)^T \\ G^* & 0 \end{pmatrix}^{-1} \right),
\]

where \(L(x, \lambda) = f(x) + \lambda^T c(x)\) is the Lagrangian function with \(\lambda \in \mathbb{R}^m\) being the dual variables associated with the constraints, \(\nabla^2 L^*\) is the Lagrangian Hessian with respect to \(x\) evaluated at \((x^*, \lambda^*)\), and \(G^* = \nabla c(x^*) \in \mathbb{R}^{m \times d}\) is the constraints Jacobian. See (Shapiro et al., 2014, Chapter 5) for the result of (3). Numerous deterministic methods can then be applied to constrained ERM (2), including penalty methods, augmented Lagrangian methods, and sequential quadratic programming (SQP) methods (Nocedal and Wright, 2006).

Given the prevalence of streaming datasets in modern problems, deterministic methods that require processing a large batch set in each step are less attractive. It is desirable to develop computationally efficient online stochastic methods, where only a single sample is used in each step. Without constraints, one can apply stochastic gradient descent (SGD) and its many variates, whose properties have been comprehensively studied from different aspects (Robbins and Monro, 1951; Kiefer and Wolfowitz, 1952). However, unlike solving constrained deterministic programs and unconstrained stochastic programs, significantly fewer methods have been proposed for constrained stochastic programs (1). Due to equality constraints, (1) is highly nonlinear and certainly nonconvex as long as \(c(x)\) is not affine, regardless of the landscape of \(f(x)\). It is noted that projection-based methods (e.g., projected SGD) are inapplicable because the projection onto the manifold \(\{x \in \mathbb{R}^d : c(x) = 0\}\) is generally intractable.

Only recently, a growing body of literature in optimization has utilized optimization techniques to develop fully online methods for Problem (1). Among them, stochastic SQP (StoSQP) is considered one of the most effective methods for both small- and large-scale problems. SQP methods can be viewed as Newton’s method applied to the first-order optimality conditions, i.e., the Karush-Kuhn-Tucker (KKT) conditions. We refer to Na et al. (2022a, 2023); Berahas et al. (2021a,b); Curtis et al. (2021); Fang et al. (2022) for recent numerical StoSQP designs and their promising performance on benchmark nonlinear problems. We will review these optimization literature in Section 1.2. These literature established global convergence of various StoSQP methods — starting from any initial point, the KKT residual \(\|\nabla L(x_t, \lambda_t)\|\) converges to zero almost surely or in expectation. However, they overlooked the recursive sampling nature and did not quantify the uncertainty inherent in StoSQP methods, which are yet crucial for statisticians to apply the methods and perform statistical inference on \((x^*, \lambda^*)\). Inspired by the success of StoSQP, we pose the following question:
Can we perform online inference on \((x^*, \lambda^*)\) based on the StoSQP iterates, while preserving (or even improving) the computational cost of existing StoSQP methods?

In this paper, we answer this question by connecting statistical and optimization communities. We investigate the asymptotic behavior of the iterates generated by an Adaptive Inexact StoSQP method, referred to as AI-StoSQP. By adaptive we mean our method inherits the merits of existing numerical StoSQP designs, and allows for adaptive selection of a random stepsize \(\bar{\alpha}_t\), as long as \(\beta_t \leq \bar{\alpha}_t \leq \beta_t + \chi_t =: \eta_t\). Here, \(\beta_t, \chi_t = o(\beta_t)\) are some control sequences. Existing numerical experiments have demonstrated the superiority of random stepsizes over deterministic stepsizes, \(\alpha_t = \alpha_0 / t^p\) for \(0.5 < p < 1\) (Berahas et al., 2021a,b; Curtis et al., 2021). By inexact we mean our method significantly reduces the computational cost of solving Newton systems (linearly constrained quadratic programs), and allows approximately solving the systems via randomized solvers that utilize sketching techniques (Strohmer and Vershynin, 2008; Gower and Richtárik, 2015; Pilanci and Wainwright, 2017). Solving Newton systems is regarded as the most computationally expensive step of second-order methods; and randomized solvers have advantages over deterministic solvers by requiring less flops and memory when equipped with proper sketching matrices (e.g., sparse sketching). Notably, we only perform a constant number of sketching steps (see (7)); thus, the approximation error does not vanish and the per-iteration computational cost remains fixed even near stationarity.

We quantify the uncertainty of StoSQP and perform statistical inference on the local solution \((x^*, \lambda^*)\). Our uncertainty quantification is challenging since the method involves three sources of randomness: random sampling, random sketching, and random stepsize. We show two main results informally summarized as follows.

(a) **Computationally**, the method can take at most \(O(\epsilon^{-4})\) iterations (same as samples) to reach an \(\epsilon\)-stationary point. Furthermore, the method exhibits a local primal-dual convergence rate of \(\| (x_t - x^*, \lambda_t - \lambda^*) \| = O(\sqrt{\beta_t \log(1/\beta_t)}) + O(\chi_t / \beta_t)\) almost surely.

(b) **Statistically**, we show \(1 / \sqrt{T_t} : (x_t - x^*, \lambda_t - \lambda^*) \overset{d}{\longrightarrow} N(0, \Xi^*)\), where the limiting covariance \(\Xi^*\) solves a Lyapunov equation that depends on the underlying sketching distribution. Additionally, we provide the Berry-Esseen bound to measure the convergence rate of the distribution function of \((x_t, \lambda_t)\), along with the error bound of a plug-in limiting covariance estimator.

The above results contribute to the literature in both optimization and statistics. As previously mentioned, the optimization literature (Na et al., 2022a, 2023; Berahas et al., 2021a,b; Curtis et al., 2021; Fang et al., 2022) has showed the global convergence of various StoSQP schemes. However, none of those studies provided a statistical perspective. In a recent work, Curtis et al. (2023) showed an \(O(\epsilon^{-4})\) iteration (and sample) complexity for an StoSQP scheme with Newton systems being exactly solved. Our results match theirs while allowing for the use of randomized solvers to approximately solve the Newton systems. On the other hand, as we will review in Section 1.2, there exists a long sequence of statistical literature that considered the online inference of the solutions to optimization problems. The majority of the studies have focused on unconstrained problems and developed first-order methods (Robbins and Monro, 1951; Polyak and Juditsky, 1992; Toulis and Airoldi, 2017; Li et al., 2018; Liang and Su, 2019). In contrast, we focus on constrained problems and develop a second-order method, overcoming its computational bottleneck by sketching techniques. It is worth mentioning that, unlike solving unconstrained problems, second-order methods are the dominant approach for solving constrained nonlinear problems (Bertsekas, 1982; Nocedal and Wright, 2006).
By our analysis, if one degrades AI-StoSQP by solving Newton systems exactly and applying deterministic stepsizes $\alpha_t = \frac{(1 + \delta)}{t}$ for any $\delta > 0$ (i.e., remove adaptive and inexact), then the limiting covariance of StoSQP can be arbitrarily close to that of $M$-estimators (see (3)), which is known to be locally asymptotic minimax optimal (Duchi and Ruan, 2021).

In the rest of this section, we first present some motivating examples of (1) in Section 1.1, and then review related literature in Section 1.2. Notation is presented in Section 1.3.

### 1.1 Motivating examples

Many statistical and machine learning problems can be cast into the form of Problem (1).

**Example 1.1 (Constrained regression problems).** Under classical regression setup, we let $\xi_t = (\xi_{a,t}, \xi_{b,t})$ be the $t$-th sample, where $\xi_{a,t} \in \mathbb{R}^d$ is the feature vector independently drawn from some multivariate distribution, and $\xi_{b,t}$ is the response. For linear models, we assume

$$
\xi_{b,t} = \xi_{a,t}^T \mathbf{x}^* + \epsilon_t
$$

where $\mathbf{x}^* \in \mathbb{R}^d$ is the true parameters and $\{\epsilon_t\}$ are i.i.d noises. For logistic models, we assume

$$
P(\xi_{b,t} | \xi_{a,t}) = \frac{\exp(\xi_{b,t} \cdot \xi_{a,t}^T \mathbf{x}^*)}{1 + \exp(\xi_{b,t} \cdot \xi_{a,t}^T \mathbf{x}^*)}
$$

with $\xi_{b,t} \in \{-1, 1\}$.

With the above models, we define the corresponding loss functions at $\mathbf{x}$:

- **Linear models:** $f(\mathbf{x}; \xi_t) = \frac{1}{2}(\xi_{a,t}^T \mathbf{x} - \xi_{b,t})^2$,
- **Logistic models:** $f(\mathbf{x}; \xi_t) = \log \left( 1 + \exp(-\xi_{b,t} \cdot \xi_{a,t}^T \mathbf{x}) \right)$.

Then, we can verify that $\mathbf{x}^* = \arg\min_{\mathbf{x}} f(\mathbf{x}) = \arg\min_{\mathbf{x}} E[f(\mathbf{x}; \xi)]$. In many cases, we access prior information for the model parameters, which is encoded as constraints. For example, in portfolio selection, $\mathbf{x}$ represents the portfolio allocation vector that should satisfy $\mathbf{x}^T \mathbf{1} = 1$ (total allocation is 100%). We may also hope to fix the target percentage on each sector or each region, which is translated into the constraint $\mathbf{A} \mathbf{x} = \mathbf{d}$. See (Fan, 2007, (4.3), (4.4)), (Fan et al., 2012, (2.1)), (Du et al., 2022, (1)) and references therein for such applications. In principle component analysis and semiparametric single/multiple index regressions, we enforce $\mathbf{x}$ to have a unit norm to address the identifiability issue, leading to a nonlinear constraint $||\mathbf{x}||^2 = 1$. We point the reader to Kirkegaard and Eldrup (1972); Kaufman and Pereyra (1978); Sen (1979); Nagaraj and Fuller (1991); Dupacova and Wets (1988); Shapiro (2000); Na et al. (2019); Na and Kolar (2021) for various examples of linearly/nonlinearly constrained $M$-estimations and semiparametric estimations.

**Example 1.2 (Physics-informed machine learning).** Recent decades have seen machine learning (ML) making significant inroads into science. The major task in ML is to learn an unknown mapping $\mathbf{z}(\cdot) : \mathcal{A} \rightarrow \mathcal{B}$ from data that can perform well in the downstream tasks. Since $\mathbf{z}(\cdot)$ is infinite-dimensional, one key step in ML is to use neural networks (NNs) to parameterize $\mathbf{z}(\cdot)$ as $\mathbf{z}(\mathbf{x}, \cdot)$, and learn the optimal weight parameters $\mathbf{x} \in \mathbb{R}^d$ instead (called function approximation). One of the trending topics in ML now is physics-informed ML, where one requires $\mathbf{z}(\cdot)$ to obey some physical principles that are often characterized by partial differential equations (PDEs) (Karniadakis et al., 2021; Cuomo...
et al., 2022). In such applications, we use the squared loss function, defined for the $t$-th sample $\xi_t = (\xi_a_t, \xi_b_t) \in A \times B$ as

$$f(x; \xi_t) = \frac{1}{2} (z_x(\xi_a_t) - \xi_b_t)^2.$$ 

Here, $\xi_a$ is NN inputs that can be spatial and/or temporal coordinates; $\xi_b$ is measurements that can be speed, velocity, and temperature, etc; and $z_x \in C^\infty(A, B)$ is NN architecture. Let $F : C^\infty(A, B) \to C^\infty(A, B)$ be the PDE operator, which encodes the underlying physical law (e.g., energy conservation law). We hope to find optimal weights $x^*$ that not only minimize the mean squared error of observed data, but also satisfy the constraints $F(z_x) = 0$. To this end, we select some leverage points $\{\xi_a_i\}_{i=1}^m$ in $A$ and impose deterministic constraints:

$$F(z_x)(\xi_a_i) = 0, \quad \forall i = 1, 2, \ldots, m.$$ 

Here, we abuse the notation $0$ to denote either a zero mapping of $C^\infty(A, B)$ or a zero element of $B$. The Jacobian of the above constraints with respect to the parameters $x$ can be computed by automatic differentiation in ML. For more background on this problem formulation, see (Lu et al., 2021, (2.3)), (Krishnapriyan et al., 2021, (2)), and references therein.

1.2 Related literature

This paper relates to the literature on both optimization and statistics.

- **Optimization.** There exist numerous methods for solving constrained deterministic optimization problems, such as projection-based methods, penalty methods, augmented Lagrangian methods, and sequential quadratic programming (SQP) methods. Compared to other methods, SQP preserves the problem structure in computation, is robust to initialization, does not suffer from ill-conditioning issues, and does not require a projection step that is intractable for general equality constraints (Nocedal and Wright, 2006). This paper particularly considers solving constrained stochastic optimization problems in online fashion. Berahas et al. (2021b) designed the very first online StoSQP scheme. At each step, the method selects a suitable penalty parameter of an $\ell_1$-penalized objective; ensures the Newton direction produces a sufficient reduction on the penalized objective; and then selects a random stepsize $\bar{\alpha}_t$ based on the sequences $\beta_t$ and $\chi_t = O(\beta_t^2)$ such that $\beta_t \leq \bar{\alpha}_t \leq \eta_t = \beta_t + \chi_t$. An alternative StoSQP scheme was then reported in Na et al. (2022a), where $\bar{\alpha}_t$ is selected by performing stochastic line search on the augmented Lagrangian with batch sizes increasing as iteration proceeds. Subsequently, Na et al. (2023); Berahas et al. (2021a); Curtis et al. (2021); Fang et al. (2022) proposed different variates of StoSQP to cope with inequality constraints, degenerate constraints, etc. These works all proved, under mild assumptions, that StoSQP methods enjoy global convergence, and illustrated promising empirical performance on benchmark problems. However, they fall short of uncertainty quantification and statistical inference goals.

- **Statistics.** With the celebrated asymptotic normality of $M$-estimators, a large body of literature utilizes optimization procedures to facilitate online inference, starting with Robbins and Monro (1951); Kiefer and Wolfowitz (1952) and continuing through Venter (1967); Robbins and Siegmund (1971); Fabian (1973); Ljung (1977); Ermoliev (1983); Walk (1983). To study the asymptotic distribution of stochastic gradient descent (SGD), Ruppert (1988) and Polyak and Juditsky (1992) averaged SGD iterates and established the optimal central limit theorem rate. More recently, Toulis and Airoldi (2017) designed an implicit SGD method and showed the asymptotics of averaged implicit
SGD iterates. Li et al. (2018) designed an inference procedure for constant-stepsize SGD by averaging the iterates with recurrent burn-in periods. Mou et al. (2020) further showed the asymptotic covariance of constant-stepsize SGD with Polyak-Ruppert averaging. Liang and Su (2019) designed a moment-adjusted SGD method and provided non-asymptotic results that characterize the statistical distribution as the batch size of each step tends to infinity. Chen et al. (2020) and Zhu et al. (2021) proposed different covariance matrix estimators constructed by grouping SGD iterates. Additionally, Chen et al. (2021) designed a distributed method for the inference of non-differentiable convex problems, and Duchi and Ruan (2021) designed a projected Riemannian SGD method for the inference of inequality-constrained convex problems. The aforementioned literature all studied first-order methods with deterministic stepsizes.

The asymptotics of second-order Newton’s methods for unconstrained problems have recently been investigated. Bercu et al. (2020) designed an online Newton’s method for logistic regression, and Boyer and Godichon-Baggioni (2023) generalized that method to general regression problems. Compared to first-order methods that often consider averaged iterates and/or exclude the stepsize $O(1/t)$ due to technical challenges, both works showed the normality of the last iterate with $O(1/t)$ stepsize. However, those analyses are not applicable to our study for several reasons. First, they only studied unconstrained regression problems with objectives of the form $f(x^T \xi)$, resulting in the objective Hessians owning rank-one updates. Second, they solved Newton systems exactly and employed $O(1/t)$ deterministic stepsize. In contrast, we use a randomized solver—sketching—to approximately solve Newton systems, with the approximation error not vanishing, along with a random stepsize. Both components affect the uncertainty quantification. Third, the Berry-Esseen bound and the limiting covariance matrix estimation remain open in their studies.

1.3 Organization and notation

We introduce AI-StoSQP in Section 2 and establish the global almost sure convergence with iteration complexity in Section 3. We conduct statistical inference by showing the asymptotic convergence rate and normality of AI-StoSQP in Section 4. Experiments and conclusions are presented in Sections 5 and 6, respectively. We defer all the proofs to appendices.

Throughout the paper, we use $\| \cdot \|$ to denote $\ell_2$ norm for vectors and spectral norm for matrices. For scalars $a$, $b$, $a \vee b = \max(a,b)$ and $a \wedge b = \min(a,b)$. We use $O(\cdot)$ (or $o(\cdot)$) to denote big (or small) $O$ notation in the usual almost sure sense. For a sequence of compatible matrices $\{A_i\}_i$, we let $\prod_{k=1}^i A_k = A_i A_{i-1} \cdots A_1$ if $j \geq i$ and $I$ (the identity matrix) if $j < i$. We use bar notation, $\bar{\cdot}$, to denote algorithmic quantities that are random (i.e., depending on a realized sample), except for the iterates. We reserve the notation $G(x)$ to denote the constraints Jacobian, i.e., $G(x) = \nabla c(x) = (\nabla c_1(x), \ldots, \nabla c_m(x))^T \in \mathbb{R}^{m \times d}$.

2 Adaptive Inexact StoSQP Method

Let $\mathcal{L}(x, \lambda) = f(x) + \lambda^T c(x)$ be the Lagrangian function of (1), where $\lambda \in \mathbb{R}^m$ is the dual vector. Under certain constraint qualifications (introduced later), a necessary condition for $(x^*, \lambda^*)$ being a local solution to (1) is the KKT conditions: $\nabla \mathcal{L}^* = (\nabla_x \mathcal{L}^*, \nabla_\lambda \mathcal{L}^*) = (0, 0)$.

AI-StoSQP applies Newton’s method to the equation $\nabla \mathcal{L}(x, \lambda) = 0$, involving three steps: estimating the objective gradient and Hessian, (approximately) solving Newton’s system (i.e., a quadratic program), and updating the primal-dual iterate. We detail each step as follows. For simplicity, we
denote $c_t = c(x_t)$ (similarly, $G_t = G(x_t) = \nabla c(x_t)$, $\nabla L_t = \nabla L(x_t, \lambda_t)$, etc.).

- **Step 1: Estimate the gradient and Hessian.** We realize a sample $\xi_t \sim P$ and estimate 

$$\bar{g}_t = \nabla f(x_t; \xi_t) \quad \text{and} \quad \bar{H}_t = \nabla^2 f(x_t; \xi_t).$$

Then, we compute three quantities:

$$\bar{\nabla}_x L_t = \bar{g}_t + G_t^T \lambda_t, \quad \bar{\nabla}^2 ax L_t = \bar{H}_t + \sum_{i=1}^{m} (\lambda_t)_i \nabla^2 c_i(x_t), \quad B_t = \frac{1}{t} \sum_{i=0}^{t-1} \bar{\nabla}^2 ax L_i + \Delta_t.$$ 

Here, $\bar{\nabla}_x L_t$ and $\bar{\nabla}^2 ax L_t$ are the estimates of the Lagrangian gradient and Hessian with respect to $x$, respectively; and $B_t$ is a regularized averaged Hessian used in the quadratic program (4). We let $\Delta_t = \Delta(x_t, \lambda_t)$ be any regularization term ensuring $B_t$ to be positive definite in the null space $\{x \in \mathbb{R}^d : G_t x = 0\}$. The average $\sum_{i=0}^{t-1} \bar{\nabla}^2 ax L_i / t$ can be updated online.

**Remark 2.1.** We note that $B_t$ and $\Delta_t$ are deterministic given $(x_t, \lambda_t)$, and $\Delta_t$ can simply be Levenberg-Marquardt type regularization of the form $\delta_t I$ with suitably large $\delta_t > 0$ (Nocedal and Wright, 2006). The Hessian regularization is standard for nonlinear problems, together with linear independence constraint qualification (LICQ, Assumption 3.1), ensuring that the quadratic program (4) is bounded below. For convex problems, one can set $\Delta_t = 0$, $\forall t$. See Bertsekas (1982); Nocedal and Wright (2006) for more details and other numerical regularization approaches. Furthermore, we note that Na et al. (2022b) adopted Hessian averaging for solving unconstrained deterministic problems, and empirically showed that Hessian averaging exhibits superior performance over using a single noisy Hessian.

- **Step 2: Solve the quadratic program.** We solve the quadratic program (QP):

$$\min_{\Delta x_t \in \mathbb{R}^d} \frac{1}{2} \bar{\Delta} x_t^T B_t \bar{\Delta} x_t + \bar{g}_t^T \bar{\Delta} x_t, \quad \text{s.t.} \quad c_t + G_t \bar{\Delta} x_t = 0. \quad (4)$$

Here, the objective can be seen as a quadratic approximation of $f(x; \xi)$ at $(x_t; \xi_t)$, and the constraint can be seen as a linear approximation of $c(x)$ at $x_t$. It is easy to verify that solving the above QP is equivalent to solving the following Newton system

$$\begin{pmatrix} B_t & G_t^T \\ G_t & 0 \end{pmatrix} \begin{pmatrix} \bar{\Delta} x_t \\ \bar{\Delta} \lambda_t \end{pmatrix} = - \begin{pmatrix} \bar{\nabla}_x L_t \\ \bar{\nabla} \lambda_t \end{pmatrix}, \quad (5)$$

where $K_t$, $\bar{\nabla}_x L_t$ are the Lagrangian Hessian and gradient, and $\bar{z}_t$ is the exact Newton direction.

Instead of solving QP (4) exactly, we approximately solve it by a randomized iterative solver utilizing the sketching technique, which is more efficient than deterministic solvers when equipped with suitable sketching matrices (Strohmer and Vershynin, 2008; Gower and Richtárik, 2015; Pilanci and Wainwright, 2017). In particular, we generate a random sketching matrix/vector $S \in \mathbb{R}^{(d+m) \times q}$, whose column dimension $q \geq 1$ can also be random, and transform the original large-scale linear system to the sketched, small-scale system as

$$K_t z_t = -\bar{\nabla}_x L_t \quad \Rightarrow \quad S^T K_t z_t = -S^T \bar{\nabla}_x L_t.$$
Clearly, there are multiple solutions to the sketched system, and \( z_t = \tilde{z}_t \) is one of them. We prefer the solution that is closest to the current solution approximation. That is, the \( j \)-th iteration of the sketching solver has the form (\( z_{t,0} = 0 \))

\[
z_{t,j+1} = \text{arg min}_{z} \| z - z_{t,j} \|^2 \quad \text{s.t.} \quad S_{t,j}^T K_t z = -S_{t,j}^T \nabla L_t,
\]

where \( S_{t,j} \sim S, \forall j \) are independent and identically distributed and are also independent of \( \xi_t \). An explicit recursion of (6) is given by

\[
z_{t,j+1} = z_{t,j} - K_t S_{t,j} (S_{t,j}^T K_t S_{t,j})^\dagger S_{t,j}^T (K_t z_{t,j} + \nabla L_t),
\]

where \((\cdot)^\dagger\) denotes the Moore–Penrose pseudoinverse. One can set \( q = 1 \) (i.e., using sketching vectors) so that \( S_{t,j}^T K_t S_{t,j} \) reduces to a scalar and the pseudoinverse reduces to the reciprocal. With different sketching distributions, we recover various randomized methods. For a concrete example, when \( S \sim \{e_1, \ldots, e_{d+m}\} \) with equal probability, where \( e_i \) is the \( i \)-th canonical basis of \( \mathbb{R}^{d+m} \), (7) is widely-used randomized Kaczmarz method (Strohmer and Vershynin, 2008).

Our method performs \( \tau \geq 1 \) iterations of (7) and uses

\[
(\Delta x_t, \Delta \lambda_t) := z_{t,\tau}
\]
as the approximate Newton direction. We emphasize that \( \tau \) is independent of \( t \); thus, we do not require a vanishing approximation error and blow up the computational cost as \( t \to \infty \).

**Remark 2.2.** A significant difference between randomized solvers and deterministic solvers is that the approximation error \( \| z_{t,j} - \tilde{z}_t \| \) of randomized solvers may not be monotonically decreasing as \( j \) increases. This subtlety challenges both inference and convergence analysis. In classical optimization world, it is unanimously agreed that if the search direction is asymptotically close to the exact Newton direction (here \( \tilde{z}_t \)), then the algorithm will locally behave just like Newton’s method, with a similar convergence rate. A precise characterization is called the Dennis-Moré condition (Dennis and Moré, 1974). In our study, \( z_{t,\tau} \) can be far from \( \tilde{z}_t \) for any \( t \) and any large \( \tau \).

**• Step 3: Update the iterate with a random stepsize.** With the direction \((\Delta x_t, \Delta \lambda_t) = z_{t,\tau}\) from Step 2, we update the iterate \((x_t, \lambda_t)\) by a random stepsize \( \tilde{\alpha}_t \):

\[
(x_{t+1}, \lambda_{t+1}) = (x_t, \lambda_t) + \tilde{\alpha}_t \cdot (\Delta x_t, \Delta \lambda_t).
\]

In principle, the stepsize \( \tilde{\alpha}_t \) should rely on the random direction \((\Delta x_t, \Delta \lambda_t)\), so it is also random. We allow for any adaptive schemes for selecting \( \tilde{\alpha}_t \), but require a sandwich condition:

\[
0 < \beta_t \leq \tilde{\alpha}_t \leq \eta_t \quad \text{with} \quad \eta_t = \beta_t + \chi_t.
\]

Here, \( \{\beta_t, \eta_t\} \) are upper and lower bound sequences and \( \chi_t \) is the gap. The adaptive stepsize selection schemes designed in existing StoSQP methods (Berahas et al., 2021b,a; Curtis et al., 2021) all satisfy the condition (9); see (Berahas et al., 2021b, Lemma 3.6) and (Curtis et al., 2021, (25),(28)). As suggested by their numerical experiments, adaptive random stepsizes have promising empirical benefits over non-adaptive deterministic stepsizes (i.e., \( \chi_t = 0 \)).

For completeness, we briefly introduce a stepsize selection scheme based on the penalized objective in the next example.
Algorithm 1 Adaptive Inexact StoSQP Method

1: **Input:** initial iterate \((x_0, \lambda_0)\), positive sequences \(\{\beta_t, \eta_t\}\), an integer \(\tau > 0\), \(B_0 = I\);
2: for \(t = 0, 1, 2, \ldots\) do
3: Realize \(\xi_t\) and compute \(\bar{g}_t = \nabla f(x_t; \xi_t), \bar{H}_t = \nabla^2 f(x_t; \xi_t), \text{ and } \nabla^2_L \lambda_t;\)
4: Compute the regularized averaged Hessian \(B_t = \frac{1}{t} \sum_{i=0}^{t-1} \nabla^2_L \lambda_i + \Delta_t;\)
5: Generate \(\zeta_t = \{\zeta_{t,j}\}_{j=0}^{\tau-1}\) from sketching distribution and iterate (7) for \(\tau\) times;
6: Select any random stepsize \(\bar{\alpha}_t\) with \(\beta_t \leq \bar{\alpha}_t \leq \eta_t\), and update the iterate as (8);
7: end for

**Example 2.3.** Let us select a stepsize to decrease the penalized objective \(\phi_{\nu}(x; \xi) := \nu f(x; \xi) + \|c(x)\|\). Note that decreasing the objective \(f(x; \xi)\) only is not reasonable for constrained problems, since we may violate constraints arbitrarily. The local linear approximation of \(\phi_{\nu}(x; \xi)\) at \((x_t; \xi_t)\) along the direction \(\Delta x_t\) is \(\phi_{\nu,loc}(x_t; \xi_t, \Delta x_t) := \nu g^T T \Delta x_t + \|c_t + G_t \Delta x_t\|\). Then, we can further define the local model reduction, a negative quantity for sufficiently small \(\nu > 0\) and approximation error, as

\[
\Delta \phi_{\nu,loc}(x_t; \xi_t, \Delta x_t) := \phi_{\nu,loc}(x_t; \xi_t, \Delta x_t) - \phi_{\nu,loc}(x_t; \xi_t, 0) = \nu g^T \Delta x_t + \|c_t + G_t \Delta x_t\| - \|c_t\|. \tag{10}
\]

For a given scalar \(\kappa_t \in (0, 1)\), we select \(\bar{\alpha}_t\) such that \(\phi_{\nu}(x_t + \bar{\alpha}_t \Delta x_t; \xi_t)\) decreases \(\phi_{\nu}(x_t; \xi_t)\) by at least a factor of \(\kappa_t\bar{\alpha}_t\) of the local model reduction \(\Delta \phi_{\nu,loc}\) (called the Armijo condition). Specifically, we require

\[
\phi_{\nu}(x_t + \bar{\alpha}_t \Delta x_t; \xi_t) \leq \phi_{\nu}(x_t; \xi_t) + \kappa_t \bar{\alpha}_t \cdot \Delta \phi_{\nu,loc}(x_t; \xi_t, \Delta x_t). \tag{11}
\]

To satisfy (11), we suppose \(\nabla f(x; \xi_t)\) and \(G(x)\) are local Lipschitz continuous around \(x_t\), and suppose \(\bar{\alpha}_t \leq 1\). Then, for a constant \(\Upsilon_{\nu,t} > 0\), we have

\[
\begin{align*}
\phi_{\nu}(x_t + \bar{\alpha}_t \Delta x_t; \xi_t) &= \nu f(x_t + \bar{\alpha}_t \Delta x_t; \xi_t) + \|c(x_t + \bar{\alpha}_t \Delta x_t)\| \\
&\leq \phi_{\nu}(x_t; \xi_t) + \nu \bar{\alpha}_t g^T \Delta x_t + \|c_t + \bar{\alpha}_t G_t \Delta x_t\| - \|c_t\| + \Upsilon_{\nu,t} \bar{\alpha}_t^2 \|\Delta x_t\|^2 \\
&\leq \phi_{\nu}(x_t; \xi_t) + \nu \bar{\alpha}_t g^T \Delta x_t + \bar{\alpha}_t \|c_t + G_t \Delta x_t\| - \bar{\alpha}_t \|c_t\| + \Upsilon_{\nu,t} \bar{\alpha}_t^2 \|\Delta x_t\|^2 \quad \text{(since } \bar{\alpha}_t \leq 1) \\
&\overset{(10)}{=} \phi_{\nu}(x_t; \xi_t) + \bar{\alpha}_t \cdot \Delta \phi_{\nu,loc}(x_t; \xi_t, \Delta x_t) + \Upsilon_{\nu,t} \bar{\alpha}_t^2 \|\Delta x_t\|^2.
\end{align*}
\]

Therefore, (11) is satisfied as long as

\[
\bar{\alpha}_t \leq \frac{(\kappa_t - 1) \cdot \Delta \phi_{\nu,loc}(x_t; \xi_t, \Delta x_t)}{\Upsilon_{\nu,t} \|\Delta x_t\|^2} \wedge 1 =: \bar{\alpha}_{t,thres}. \tag{12}
\]

The Lipschitz constant \(\Upsilon_{\nu,t}\) can be estimated around \(x_t\) (Curtis and Robinson, 2018) or simply pre-specified as a large constant. The condition (12) leads us to propose \(\bar{\alpha}_t := \text{Proj}_{[\beta_t, \eta_t]}(\bar{\alpha}_{t,thres})\). See Berahas et al. (2021b,a); Curtis et al. (2021) for similar random projections.

We combine the above three steps and summarize AI-StoSQP method in Algorithm 1. To end this section, we introduce some additional notation of Algorithm 1 that we will use later.

For notational consistency, we use the variables \(\zeta_t = \{\zeta_{t,j}\}_{j}\) to denote the randomness of \(\{S_{t,j}\}_{j}\) at Step 2. We also allow the stepsize \(\bar{\alpha}_t\) at Step 3 to depend on another source of randomness \(\psi_t\) in addition to \(\xi_t\) and \(\zeta_t\). For the generated sequence \(\{(\xi_t, \zeta_t, \psi_t)\}_t, \mathcal{F}_t = \sigma(\{\xi_i, \zeta_i, \psi_i\}_{i=0}^t), \forall t \geq 0\) is its
adapted filtration. Moreover, we let \( F_{t-2/3} = \sigma(\{\xi_i, \zeta_i, \psi_i\}_{i=0}^{t-1} \cup \xi_t) \), \( F_{t-1/3} = \sigma(\{\xi_i, \zeta_i, \psi_i\}_{i=0}^{t-1} \cup \xi_t \cup \zeta_t) \), and have \( F_{t-1} \subseteq F_{t-2/3} \subseteq F_{t-1/3} \subseteq \mathcal{F}_t \). For consistency, \( \mathcal{F}_{-1} \) is the trivial \( \sigma \)-algebra.

With these notation, Algorithm 1 has a generating process as follows: given \((x_t, \lambda_t)\), we first realize \( \xi_t \) to estimate the gradient \( \check{g}_t \) and Hessian \( \check{H}_t \) and derive \( \mathcal{F}_{t-2/3} \); then we generate \( \zeta_t \) to obtain the approximate Newton direction and derive \( \mathcal{F}_{t-1/3} \); then we may (i.e., not necessarily) generate \( \psi_t \) to select the stepsize \( \bar{\alpha}_t \) and derive \( \mathcal{F}_t \). It is easy to see that the random quantities in Algorithm 1 have the following recursion:

\[
\sigma(x_t, \lambda_t) \cup \sigma(B_t) \subseteq F_{t-1}, \quad \sigma(g_t) \cup \sigma(H_t) \cup \sigma(\Delta x_t, \Delta \lambda_t) \subseteq \mathcal{F}_{t-2/3}, \quad \sigma(\Delta x_t, \Delta \lambda_t) \subseteq \mathcal{F}_{t-1/3}, \quad \sigma(\alpha_t) \cup \sigma(x_{t+1}, \lambda_{t+1}) \cup \sigma(B_{t+1}) \subseteq F_t.
\]

We also let \((\Delta x_t, \Delta \lambda_t)\) be the exact solution of (5) with \( \nabla_x \mathcal{L}_t \) being replaced by \( \nabla_x \mathcal{L}_t \).

3 Global Almost Sure Convergence

We establish global almost sure convergence of our online StoSQP method. We show that under standard assumptions, the KKT residual \( \|\nabla \mathcal{L}_t\| \) converges to zero almost surely from any initial point. The almost sure convergence of StoSQP methods was established in Na et al. (2022a, 2023); however, those methods gradually increase the batch size as iteration proceeds. Duchi and Ruan (2021) showed almost sure convergence of an online projected first-order method, a variant of Nesterov’s dual averaging, while we study an online second-order method.

We use an adapted augmented Lagrangian function as the Lyapunov function to show the convergence, which has two penalty terms of the form

\[ \mathcal{L}_{\mu, \nu}(x, \lambda) = \mathcal{L}(x, \lambda) + \frac{\mu}{2} \|c(x)\|^2 + \frac{\nu}{2} \|\nabla_x \mathcal{L}(x, \lambda)\|^2, \quad \text{with } \mu, \nu > 0. \]

The first penalty term biases the feasibility error; while, in contrast to the standard augmented Lagrangian (\( \nu = 0 \)), the extra second penalty term biases the optimality error. By direct calculation, we have (the evaluation point is suppressed for simplicity)

\[
\begin{pmatrix}
\nabla_x \mathcal{L}_{\mu, \nu} \\
\nabla_x \mathcal{L}_{\mu, \nu}
\end{pmatrix}
\begin{pmatrix}
I + \nu \nabla^2 \mathcal{L} \\
\nu G
\end{pmatrix}
\begin{pmatrix}
\mu G^T \\
I
\end{pmatrix}
\begin{pmatrix}
\nabla_x \mathcal{L} \\
c
\end{pmatrix}.
\]

We will first show that the inner product between the exact Newton direction \((\Delta x_t, \Delta \lambda_t)\) and the augmented Lagrangian gradient \( \nabla \mathcal{L}_{\mu, \nu} \), with proper parameters \( \mu \) and \( \nu \), is sufficiently negative (Lemma 3.5). Thus, the Newton direction is a descent direction of \( \mathcal{L}_{\mu, \nu} \). Then, we will show that the augmented Lagrangian \( \mathcal{L}_{\mu, \nu} \) decreases at each step even with an approximate Newton direction (Lemma 3.6), so that the residual \( \|\nabla \mathcal{L}_t\| \) finally vanishes (Theorem 3.7).

3.1 Assumptions and preliminary results

We state the following assumptions that are standard and proposed in the optimization literature (Kushner and Clark, 1978; Bertsekas, 1982; Nocedal and Wright, 2006; Na et al., 2022a).

**Assumption 3.1.** We assume the existence of a closed, bounded, convex set \( \mathcal{X} \times \Lambda \) containing the iterates \( \{(x_t, \lambda_t)\}_t \), such that \( f \) and \( c \) are twice continuously differentiable over \( \mathcal{X} \). We also assume that the Hessian \( \nabla^2 \mathcal{L} \) is \( \mathcal{Y}_L \)-Lipschitz continuous over \( \mathcal{X} \times \Lambda \). In other words,

\[
\|\nabla^2 \mathcal{L}(x, \lambda) - \nabla^2 \mathcal{L}(x', \lambda')\| \leq \mathcal{Y}_L \| (x - x', \lambda - \lambda') \|, \quad \forall (x, \lambda), (x', \lambda') \in \mathcal{X} \times \Lambda.
\]
Furthermore, we assume that the constraints Jacobian $G_t$ has full row rank with $G_t G_t^T \succeq \gamma_G I$ for a constant $\gamma_G > 0$. Additionally, the regularization $\Delta_t$ ensures that $B_t$ satisfies $\|B_t\| \leq Y_B$ and $x^T B_t x \geq \gamma_{RH} \|x\|^2$ for any $x \in \{x \in \mathbb{R}^d : G_t x = 0\}$, for some constants $\gamma_{RH}, Y_B > 0$.

Assumption 3.1 assumes that $G_t$ has full row rank, which is referred to as the linear independence constraint qualification (LICQ). LICQ is a common constraint qualification ensuring the uniqueness of the dual solution, and is also necessary for the inference analysis (see (Duchi and Ruan, 2021, Assumption B)). LICQ and conditions on $B_t$ are critical for SQP methods; they imply that QP (4) has a unique solution (Nocedal and Wright, 2006, Lemma 16.1).

We also impose bounded moment conditions on the stochastic estimates $\bar{g}_t$ and $\bar{H}_t$.

**Assumption 3.2.** We assume $\mathbb{E}[\bar{g}_t \mid x_t] = \nabla f_t, \mathbb{E}[\bar{H}_t \mid x_t] = \nabla^2 f_t$, and assume the following moment conditions when needed: for a constant $Y_m > 0$,

$$
\begin{align*}
\text{gradient} & \quad \text{(bounded 2nd moment)} : \quad \mathbb{E}[\|\bar{g}_t - \nabla f_t\|^2 \mid x_t] \leq Y_m, \quad (15a) \\
& \quad \text{(bounded 3rd moment)} : \quad \mathbb{E}[\|\bar{g}_t - \nabla f_t\|^3 \mid x_t] \leq Y_m, \quad (15b) \\
& \quad \text{(bounded 4th moment)} : \quad \mathbb{E}[\|\bar{g}_t - \nabla f_t\|^4 \mid x_t] \leq Y_m, \quad (15c)
\end{align*}
$$

and

$$
\begin{align*}
\text{Hessian} & \quad \text{(bounded 2nd moment)} : \quad \mathbb{E}[\|\bar{H}_t - \nabla^2 f_t\|^2 \mid x_t] \leq Y_m, \quad (15d) \\
& \quad \text{(bounded 2nd moment)} : \quad \mathbb{E}[\sup_{x \in X} \|\nabla^2 f(x;\xi)\|^2] \leq Y_m. \quad (15e)
\end{align*}
$$

We write $\mathbb{E}[\cdot \mid x_t]$ to express the conditional variable. It can also be written as $\mathbb{E}[\cdot \mid \mathcal{F}_{t-1}]$, meaning the expectation is taken over the randomness of sample $\xi_t$. For conditions (15), we do not impose all of them at once, but impose them step by step. In this section, we only require (15a) to show the convergence of $\nabla L_t$. In the next section, we require higher-order moments for inference. In fact, (15c) implies (15b), which implies (15a), and (15e) implies (15d).

Assumption 3.2 is standard for uncertainty quantification of stochastic methods. We would like to mention that (15e) is also required for the asymptotic analysis of averaged SGD (Chen et al., 2020), which ensures the Lipschitz continuity of the mapping $x \to \mathbb{E}[\nabla f(x;\xi)\nabla^T f(x;\xi)]$, as proved in (D.11). Please refer to (Chen et al., 2020, Assumption 3.2(2) and Lemma 3.1) for further discussions. Moreover, (15e) is satisfied by various objectives, such as logistic and least squares losses in Example 1.1, as long as the feature variable $\xi_a$ has a bounded 4-th moment.

In terms of the sketching matrices, we need the following assumption.

**Assumption 3.3.** For $t \geq 0$, we assume that the sketching matrices $S_{t,j} \sim S$, iid, satisfy

$$
\mathbb{E}[K_t S(S^T K_t^2 S)^T S^T K_t \mid x_t, \lambda_t] \succeq \gamma_S I \quad \text{for some} \; \gamma_S > 0.
$$

Assumption 3.3 is required by sketching solvers to converge in expectation (Gower and Richtárik, 2015, Theorem 4.6). This assumption can be easily verified for different sketching matrices. For example, for randomized Kaczmarz method where $S \sim \{e_1, \ldots, e_{d+m}\}$ with equal probability (Strohmer and Vershynin, 2008), we have

$$
\mathbb{E}[K_t S(S^T K_t^2 S)^T S^T K_t \mid x_t, \lambda_t] \succeq \frac{\mathbb{E}[K_t S^T K_t \mid x_t, \lambda_t]}{\max_j [K_t^2]_{jj,j}} = \frac{K_t^2}{(d + m) \cdot \max_j [K_t^2]_{jj,j}} \succeq \frac{I}{(d + m) \kappa(K_t^2)},
$$

where $[K_t^2]_{jj,j}$ denotes the $(j,j)$-entry of $K_t^2$ and $\kappa(K_t^2)$ denotes the condition number of $K_t^2$ (it is independent of $t$ by Assumption 3.1). Assumption 3.3 directly leads to the following result.
Lemma 3.4 (Guarantees of sketching solvers). Under Assumption 3.3, for all \( t \geq 0 \):

(a): Let \( \rho = 1 - \gamma_S \). We have \( 0 \leq \rho < 1 \).

(b): \( \mathbb{E}[z_{t,\tau} - \bar{z}_t \mid \mathcal{F}_{t-2/3}] = -(I - \mathbb{E}[K_t S(S^T K_t^2 S)^T S^T K_t \mid \mathcal{F}_{t-1}])^T \bar{z}_t =: C_t \bar{z}_t \), and \( \|C_t\| \leq \rho^\tau \).

(c): \( \mathbb{E}[\|z_{t,\tau} - \bar{z}_t\|^2 \mid \mathcal{F}_{t-2/3}] \leq \rho^\tau \|\bar{z}_t\|^2 \).

3.2 Almost sure convergence

We now set the stage to show global almost sure convergence. The first result shows that the exact Newton direction \( (\Delta x_t, \Delta \lambda_t) \) is a descent direction of \( L_{\mu,\nu}^t = L_{\mu,\nu}(x_t, \lambda_t) \) if \( \mu \) is sufficiently large and \( \nu \) is sufficiently small.

Lemma 3.5. Under Assumption 3.1, there exists a deterministic constant \( \Upsilon_1 > 0 \), depending only on \( (\gamma_G, \gamma_{RH}, \Upsilon_B) \), such that

\[
\begin{pmatrix}
\nabla_x L_{\mu,\nu}^t \\
\nabla_\lambda L_{\mu,\nu}^t
\end{pmatrix}^T \begin{pmatrix}
\Delta x_t \\
\Delta \lambda_t
\end{pmatrix} \leq -\frac{\nu}{\Upsilon_1} \left\{ \|\Delta x_t\|^2 + \|\nabla_x L_{\mu,\nu}^t\|^2 \right\},
\]

provided \( \mu \nu \geq \Upsilon_1 \) and \( \nu \leq 1/\Upsilon_1 \).

With Lemmas 3.4 and 3.5, we are able to show the following one-step recursion of \( L_{\mu,\nu}^t \).

Lemma 3.6. Under Assumptions 3.1, 3.2(15a), 3.3, we suppose that the pair \( (\mu, \nu) \) satisfies the condition in Lemma 3.5 and \( \tau \) satisfies \( \rho^\tau \leq \nu/(\mu \Upsilon_1) \). Then, there exists a deterministic constant \( \Upsilon_2 > 0 \), depending on \( (\mu, \nu, \gamma_G, \gamma_{RH}, \Upsilon_B, \Upsilon_L, \Upsilon_m) \), such that

\[
\mathbb{E}[L_{\mu,\nu}^{t+1} \mid \mathcal{F}_{t-1}] \leq L_{\mu,\nu}^t - \frac{\nu}{2\Upsilon_1} \cdot \beta_t \|\nabla L_t\|^2 + \Upsilon_2(\chi_t + \eta_t^2).
\]

With Lemma 3.6, we can apply Robbins-Siegmund theorem (Robbins and Siegmund, 1971) to establish the convergence of the KKT residual \( \|\nabla L_t\| \).

Theorem 3.7 (Global convergence of AI-StoSQP). Consider Algorithm 1 under Assumptions 3.1, 3.2(15a), 3.3. Suppose we perform the sketching solver (7) for \( \tau \) steps with \( \tau \geq 4 \log \Upsilon_1 / \log \{1/(1 - \gamma_S)\} \), where \( \Upsilon_1 \) is from Lemma 3.5. Also, we let \( \{\beta_t, \eta_t = \beta_t + \chi_t\} \) satisfy

\[
\sum_{t=0}^{\infty} \beta_t = \infty, \quad \sum_{t=0}^{\infty} \beta_t^2 < \infty, \quad \sum_{t=0}^{\infty} \chi_t < \infty.
\]

Then, we have \( \|x_{t+1} - x_t, \lambda_{t+1} - \lambda_t\| \to 0 \) and \( \|\nabla L_t\| \to 0 \) as \( t \to \infty \) almost surely.

Theorem 3.7 indicates that all limiting points of the primal-dual iteration sequence \( (x_t, \lambda_t) \) are stationary. This guarantee aligns with the standard guarantee of deterministic SQP methods (Nocedal and Wright, 2006, Theorem 18.3), despite the fact that we possess different sources of randomness at each step of the SQP methods. The convergence of \( (x_t, \lambda_t) \) is equivalent to the convergence of \( \nabla L_t \) if Problem (1) is convex. Furthermore, the convergence results \( \|\nabla L_t\| \to 0 \) imply the existence of an attraction neighborhood around the local solution \( (x^*, \lambda^*) \). Once \( (x_t, \lambda_t) \) lies in the neighborhood, all subsequent iterates will stay in the neighborhood and \( (x_t, \lambda_t) \to (x^*, \lambda^*) \) (Bertsekas, 1982, Chapter 4.4).
Based on Theorem 3.7, we can now provide the worst-case iteration complexity of Algorithm 1 in the following corollary. Due to the online nature of the method, the iteration complexity is equivalent to the sample complexity, as we observe one sample in each iteration.

Corollary 3.8. Consider Algorithm 1 under Assumptions 3.1, 3.2(15a), 3.3. Suppose \( \tau \) satisfies the condition in Theorem 3.7, and let \( \tau = (t + 1)^{-a}, \chi = (t + 1)^{-b} \) where \( a \in (0, 1) \) and \( a < b \). Also, define \( T_\epsilon = \inf \{ t \geq 1 : \mathbb{E}[\|\nabla L_t\|] \leq \epsilon \} \). Then, we have

\[
T_\epsilon = O\left( \epsilon^{-\frac{2}{a(1-a)\chi(1-b)}} \right).
\]

In particular, \( T_\epsilon \) attains the minimum \( O(\epsilon^{-4}) \) with \( a = 1/2 \) and \( b = 1 \).

Remark 3.9. We highlight that Corollary 3.8 is based on a non-asymptotic convergence rate of the averaged expected KKT residual \( \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}[\|\nabla L_t\|] \). This non-asymptotic result is in contrast to the inference analysis in Section 4, where the results hold asymptotically.

4 Statistical Inference via StoSQP

We perform online statistical inference for Problem (1) using the developed StoSQP method. To segue into our inference analysis, we suppose in this section that the method converges to a local solution \((x^*, \lambda^*)\) of (1); specifically, \( G^* = \nabla c^* \) has full row rank and \( \nabla^2 x L^* \) is positive definite in the null space \( \{x \in \mathbb{R}^d : G^* x = 0 \} \). These optimality conditions ensure that the Lagrangian Hessian \( K^* = \nabla^2 L^* \) is non-singular, as necessary for \( M \)-estimators in (3).

4.1 Iteration recursion

From a high-level view, our method generates a stochastic sequence

\[
\begin{align*}
(x_{t+1} - x^*) &= (1 - \bar{\alpha}_t) (x_t - x^*) + \bar{\alpha}_t \left( \theta^t_x \theta^t_\lambda \right) + \bar{\alpha}_t \left( \delta^t_x \delta^t_\lambda \right),
\end{align*}
\]

where \( (\theta^t_x, \theta^t_\lambda) \) is a martingale difference with \( \mathbb{E}[\theta^t_x, \theta^t_\lambda | \mathcal{F}_{t-1}] = 0 \), and \( (\delta^t_x, \delta^t_\lambda) \) is the remaining error term. Compared to existing stochastic first- and second-order methods (Polyak and Juditsky, 1992; Chen et al., 2020; Duchi and Ruan, 2021; Bercu et al., 2020; Boyer and Godichon-Baggioni, 2023), the randomness brought by the adaptivity and inexactness (AI) of our method affects all the terms in (17). This includes the random stepsize \( \bar{\alpha}_t \), as well as the random approximation errors in \( (\theta^t_x, \theta^t_\lambda) \) and \( (\delta^t_x, \delta^t_\lambda) \) associated with the sketching solver.

We formalize the recursion (17) in the following lemma.

Lemma 4.1. Let \( \varphi_t = (\beta_t + \eta_t)/2 \). The iteration sequence of Algorithm 1 can be expressed as

\[
\begin{align*}
(x_{t+1} - x^*) &= \mathcal{I}_1, t + \mathcal{I}_2, t + \mathcal{I}_3, t
\end{align*}
\]
Assumption 4.3. We assume that
where

The asymptotic rate of each term is provided in Appendix D.4.

Under Assumption 4.3, Corollary 4.4.

Lemma 4.2.

Next, we establish a continuity property for the projection matrix $K_t$, with a random factor scaling with the condition number of the sketching matrix $K^*$.

Under Assumptions 3.2, 3.3, $\theta^i = (\theta_x^i, \theta_\lambda^i)$ is a martingale difference with $\mathbb{E}[\theta^i | \mathcal{F}_{i-1}] = 0$.

From Lemma 4.1, we observe that the recursion consists of three terms. $I_{1,t}$ is a martingale that accounts for the randomness of sampling $\xi_t$ to estimate $\nabla f_t$ and the randomness of sketching $\zeta_t$ to solve QP (4). $I_{2,t}$ captures the randomness of the stepsize $\bar{\alpha}_t$. $I_{3,t}$ contains all the remainder terms. The asymptotic rate of each term is provided in Appendix D.4.

Next, we establish a continuity property for the projection matrix $K_tS(S^T(K^*)^2S)^\dagger S^T K_t$, a critical quantity of the sketching solver appeared in $C_t$ and $C^*$ (cf. Lemma 3.4(b) and (19a)).

**Lemma 4.2.** Suppose $K_t, K^* \in \mathbb{R}^{(d+m) \times (d+m)}$ are non-singular. For any $S \in \mathbb{R}^{(d+m) \times q}$,

$$||K_tS(S^T(K^*)^2S)^\dagger S^T K_t - K^*S(S^T(K^*)^2S)^\dagger S^T K^*|| \leq \frac{2\|K_t - K^*\|}{\sigma_{\text{min}}(K^*)} \cdot ||S|| ||S^\dagger||,$$

where $\sigma_{\text{min}}(\cdot)$ denotes the least singular value.

Lemma 4.2 indicates that the difference between the projection matrices $K_tS(S^T(K^*)^2S)^\dagger S^T K_t$ and $K^*S(S^T(K^*)^2S)^\dagger S^T K^*$ is proportional to the difference between the Hessian matrices $K_t$ and $K^*$, with a random factor scaling with the condition number of the sketching matrix $S$.

In practice, using sketching vectors ($q = 1$), such as in randomized Kaczmarz method, can reduce computational cost and result in a unit condition number $||S|| ||S^\dagger|| = 1$. Lemma 4.2 leads to the following condition to ensure the convergence of $C_t$, which is the expectation of the product of projection matrices.

**Assumption 4.3.** We assume that $S$ satisfies $\mathbb{E}[||S|| ||S^\dagger||] \leq \Upsilon_S$ for a constant $\Upsilon_S > 0$.

**Corollary 4.4.** Under Assumption 4.3, $||C_t - C^*|| \leq 2\tau \Upsilon_S ||K_t - K^*|| / \sigma_{\text{min}}(K^*)$. 

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4.2 Asymptotic rate and normality

We are now ready to state inference theory. Let \( S_1, \ldots, S_n \sim S, \text{iid} \), and define a random matrix:

\[
\tilde{C}^* = -\prod_{j=1}^{t}(I - K^*S_j(S_j^T(K^*)^2S_j)^\dagger S_j^T K^*).
\]

Clearly, \( \mathbb{E}[^{\tilde{C}}^*] = C^* \). Also, define the sandwich matrix that appears as the limiting covariance of \( M \)-estimators in (3):

\[
\Omega^* = (K^*)^{-1}\text{cov}(\nabla L^*)(K^*)^{-1} = \begin{pmatrix} \nabla^2 L^* & (G^*)^T \\ G^* & 0 \end{pmatrix}^{-1} \begin{pmatrix} \text{cov}(\nabla f(x^*; \xi)) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \nabla^2 L^* & (G^*)^T \\ G^* & 0 \end{pmatrix}^{-1}.
\]

To allow for general stepsize control sequences \( \{\beta_t, \chi_t = o(\beta_t)\} \), we define three quantities:

\[
\beta := \lim_{t \to \infty} t \left( 1 - \frac{\beta_{t-1}}{\beta_t} \right), \quad \tilde{\beta} := \lim_{t \to \infty} t \beta_t, \quad \chi := \lim_{t \to \infty} t \left( 1 - \frac{\chi_{t-1}}{\chi_t} \right).
\]

The polynomial sequences \( 1/t^p \) are specialized in Lemma 4.9.

**Theorem 4.5** (Asymptotic rate of AI-StoSQP). Under Assumptions 3.1, 3.2(15a, 15e), 3.3, we suppose \( \{\beta_t, \chi_t = o(\beta_t)\} \) satisfy

\[
\beta < 0, \quad \tilde{\beta} \in (0, \infty), \quad 1 - \rho^\tau + \beta/\tilde{\beta} > 0, \quad 2(1 - \rho^\tau) + (2\chi - \beta)/\tilde{\beta} > 0.
\]

For any \( \upsilon > 0 \), we have

\[
\| (x_t - x^*, \lambda_t - \lambda^*) \| = o\left( \sqrt{\beta_t \log(1/\beta_t)} \right) + O(\chi_t/\beta_t) \quad \text{a.s.}
\]

Furthermore, if (15a) is strengthen to (15b), then

\[
\| (x_t - x^*, \lambda_t - \lambda^*) \| = O\left( \sqrt{\beta_t \log(1/\beta_t)} \right) + O(\chi_t/\beta_t) \quad \text{a.s.}
\]

The asymptotic convergence rate consists of two terms. The first term \( o(\sqrt{\beta_t \log(1/\beta_t)}) \) comes from the strong law of large number for the martingale \( I_{1,t} \), which can be strengthened to \( O(\sqrt{\beta_t \log(1/\beta_t)}) \) if the stochastic gradient estimate \( \tilde{g}_t \) has a bounded moment of order higher than two (Duflo, 1997, Theorem 1.3.15). That is, the bounded 3rd moment in (15b) can be directly replaced by a bounded 2+δ moment. The second term \( O(\chi_t/\beta_t) \) comes from \( I_{2,t} \), characterizing the adaptivity of random stepsize \( \tilde{\alpha}_t \). This term is suppressed if we degrade the method to a non-adaptive one \( \chi_t = 0 \), or contributes to a higher-order term if we set \( \chi_t = O(\beta_t^{3/2}) \). We should mention that the StoSQP methods in Berahas et al. (2021b,a); Curtis et al. (2021) set \( \chi_t = O(\beta_t^2) \), while we allow for a larger \( \chi_t \), leading to a wider interval, for stepsize adaptivity to have the first term dominate.

We will investigate the condition (22) in Lemma 4.9 and demonstrate that it is weak enough to allow for different setups of sequences \( \{\beta_t, \chi_t\} \). In fact, (22) reveals a relationship between the inexactness of the sketching solver (i.e., the parameter \( \tau \)) and the setup of the stepsize. When \( \tilde{\beta} = \infty \) (e.g., \( 1/t^p \) for \( p < 1 \)), the third and fourth conditions in (22) hold trivially.
Theorem 4.6 (Asymptotic normality and Berry-Esseen bound). Under the conditions in Theorem 4.5 with (15b), let Assumption 4.3 and \( \chi < 1.5 \beta \) hold. Then, we have

\[
\sqrt{1/\beta_t} \cdot (x_t - x^*, \lambda_t - \lambda^*) \xrightarrow{d} N(0, \Xi^*),
\]

where \( \Xi^* \) is the solution of the following Lyapunov equation:

\[
(\{1 + \beta/(2\beta)\}I + C^*)\Xi^* + \Xi^*(\{1 + \beta/(2\beta)\}I + C^*) = \mathbb{E}[(I + \tilde{C}^*)\Omega^*(I + \tilde{C}^*)^T].
\]  

(23)

Furthermore, for any vector \( w = (w_x, w_\lambda) \in \mathbb{R}^{d+m} \) such that \( w^T\Xi^*w \neq 0 \) and any \( v > 0 \),

\[
\sup_{z \in \mathbb{R}} \left| P \left( \frac{\sqrt{1/\beta_t} \cdot w^T(x_t - x^*, \lambda_t - \lambda^*)}{\sqrt{w^T\Xi^*w}} \leq z \right) - P(N(0,1) \leq z) \right| = \begin{cases} o((\log t)^{1+v}/\sqrt{t}) + O(\chi_t/\beta_t^{3/2}) & \text{if } \tilde{\beta} < \infty, \\ O(\sqrt{\beta_t} \log(1/\beta_t)) + O(\chi_t/\beta_t^{3/2}) & \text{if } \tilde{\beta} = \infty. \end{cases}
\]  

(24)

The explicit form of \( \Xi^* \) is given by

\[
\Xi^* = U(\Theta \circ U^T \mathbb{E}[(I + \tilde{C}^*)\Omega^*(I + \tilde{C}^*)^T])U^{T} \quad \text{with} \quad [\Theta]_{k,l} = 1/(\sigma_k + \sigma_l + \beta/\tilde{\beta}),
\]  

(25)

where \( I + C^* = U\Sigma U^T \) with \( \Sigma = \text{diag}(\sigma_1, \ldots, \sigma_{d+m}) \) is the eigenvalue decomposition of \( I + C^* \), and \( \circ \) denotes the matrix Hadamard product. From Theorem 4.6, we see that the limiting covariance \( \Xi^* \) depends on the sandwich matrix \( \Omega^* \) in (21), which is the same as the one for \( M \)-estimators, but it also depends on the underlying sketching distribution. The sketching matrices affect both the left- and right-hand sides of the Lyapunov equation (23). If we degrade the method and solve QPs exactly, we have \( \tau = \infty, C^* = \tilde{C}^* = 0 \), and

\[
\Xi^* = \frac{\Omega^*}{2 + \beta/\tilde{\beta}}.
\]  

(26)

In this case, we can let \( \beta_t = (1 + \delta)/t \) and \( \chi_t = 1/t^p \) for any \( \delta > 0 \) and \( p \in (1.5, 1.5 + \delta) \) (or simply let \( \chi_t = 0 \)), and obtain

\[
\sqrt{t}(x_t - x^*, \lambda_t - \lambda^*) \xrightarrow{d} N(0, (1 + \delta)^2/(1 + 2\delta)\Omega^*).
\]  

(27)

Comparing (27) with (3), we see that the limiting covariance of StoSQP can be arbitrarily close to the asymptotic minimax optimum \( \Omega^* \), up to an \( O(\delta^2) \) term. However, \( M \)-estimators are computed by offline methods, while we achieve this result via an online method.

The Berry-Esseen bound in (24) consists of two terms. The first term is due to the random sample and random sketching, while the second term is due to the random stepsize. Note that the second term \( O(\chi_t/\beta_t^{3/2}) \) is negligible when \( \chi_t = O(\beta_t^2) \). Furthermore, similar to Theorem 4.5, if we assume the stochastic Hessian estimate \( \tilde{H}_t \) has a bounded moment of order higher than two, the first term can be strengthened to \( O(\log t/\sqrt{t}) \) for \( \tilde{\beta} < \infty \). This additional analysis causes no special difficulty in the proofs; hence, we omit it.
4.3 An estimator of the covariance matrix

We analyze a plug-in covariance estimator. Since estimating $\mathbb{E}[(I + \tilde{C}x)\Omega^* (I + \tilde{C}^*)^T]$ is very challenging in practice, we neglect the sketching matrices in (25) and estimate (26) instead. We show that such negligence only leads to an $O(p^\tau)$ error term, which is generally small even for a moderate $\tau$. Particularly, our estimator of $\Xi^*$ is defined as

$$\Omega_t = K_t^{-1} \begin{pmatrix} \text{sample_cov} & 0 \\ 0 & 0 \end{pmatrix} K_t^{-1} \quad \text{and} \quad \Xi_t = \frac{\Omega_t}{2 + \beta/\beta},$$

where

$$\text{sample_cov}(\{\tilde{g}_i\}_{i=0}^{t-1}) = \frac{1}{t} \sum_{i=0}^{t-1} \tilde{g}_i \tilde{g}_i^T - \left( \frac{1}{t} \sum_{i=0}^{t-1} \tilde{g}_i \right) \left( \frac{1}{t} \sum_{i=0}^{t-1} \tilde{g}_i \right)^T$$

is the sample covariance.

**Theorem 4.7.** Consider (28) under the conditions of Theorem 4.6 with (15c). For any $\nu > 0$,

$$\|\Xi_t - \Xi^*\| = \begin{cases} o(\sqrt{(\log t)^{1+\nu}/t}) + O(\chi_t/\beta_t) + O(p^\tau) & \text{if } \bar{\beta} < \infty, \\ O(\sqrt{\beta_t \log (1/\beta_t)}) + O(\chi_t/\beta_t) + O(p^\tau) & \text{if } \bar{\beta} = \infty. \end{cases}$$

Furthermore, for any vector $w = (w_x, w_\lambda) \in \mathbb{R}^{d+m}$ such that $w^T \Xi_t w \neq 0$ and any $\nu > 0$,

$$\sup_{z \in \mathbb{R}} \left| P \left( \frac{\sqrt{1/\beta_t} \cdot w^T (x_t - x^*, \lambda_t - \lambda^*)}{\sqrt{w^T \Xi_t w}} \leq z \right) - P(\mathcal{N}(0, 1) \leq z) \right| = \begin{cases} o((\log t)^{1+\nu}/\sqrt{t}) + O(\chi_t/\beta_t^{3/2}) + O(p^\tau) & \text{if } \bar{\beta} < \infty, \\ O(\sqrt{\beta_t \log (1/\beta_t)}) + O(\chi_t/\beta_t^{3/2}) + O(p^\tau) & \text{if } \bar{\beta} = \infty. \end{cases}$$

We need (15c) to ensure the convergence of the sample covariance of $\{\tilde{g}_i\}_{i=0}^{t-1}$. The bounded 4-th moment of $\tilde{g}_t$ is standard in the literature on covariance estimation. It is required for analyzing different covariance estimators for SGD (Chen et al., 2020; Zhu et al., 2021).

**Remark 4.8 (Discussion on $\{\beta_t, \chi_t = \eta_t - \beta_t\}$).** We summarize all the conditions on the sequences $\{\beta_t, \chi_t\}$. The global convergence requires (16) (Theorem 3.7); the local convergence requires (22) (Theorem 4.5); and the inference additionally requires $\chi < 1.5\beta$ (Theorem 4.6). In fact, by Raabe’s test, (16) also relates to the quantities $\beta$ and $\chi$: (16) holds if $-1 \leq \beta < -0.5$ and $\chi < -1$. We now specialize $\beta_t$ and $\chi_t$ to be polynomial in $t$, and demonstrate that all the conditions can be easily satisfied simultaneously.

**Lemma 4.9.** Suppose $\beta_t = c_1/t^{c_2}$ and $\chi_t = \beta_t^{c_3}$. Then,

(a): (16) holds if $c_1 > 0$, $c_2 \in (0.5, 1]$, and $c_3 > 1/c_2$.

(b): (16) and (22) hold if $c_2 = 1$, $c_3 > 1$, and $c_1 > \frac{1/(c_3-0.5)}{1-p^\tau}$ OR $c_2 \in (0.5, 1)$, $c_1 > 0$, and $c_3 > 1/c_2$.

(c): (16) and (22) hold with $\chi < 1.5\beta$ if $c_2 = 1$, $c_3 > 1.5$, and $c_1 > \frac{c_3-0.5}{1-p^\tau}$ OR $c_2 \in (0.5, 1)$, $c_1 > 0$, and $c_3 > 1.5 \lor 1/c_2$.

The proof of the above lemma is immediate by noting that $\beta = -c_2$, $\chi = -c_2c_3$, and $\bar{\beta} = c_1$ if $c_2 = 1$ and $\infty$ if $c_2 < 1$. Thus, we omit it.
Table 1: The coverage rate and average length of 95% confidence intervals for five CUTEst problems. The standard errors of the interval length are also reported.

| Prob      | $\sigma^2$       | Cov Rate (%) | Avg Len ($10^{-2}$) |
|-----------|------------------|-------------|---------------------|
|           | $10^{-4}$ | $10^{-2}$ | $10^{-1}$ | $1$ | $10^{-4}$ | $10^{-2}$ | $10^{-1}$ | $1$ |
| MARATOS   | 97.50    | 94.50    | 96.00    | 92.50 | 0.05(0.00) | 0.55(0.00) | 1.73(0.02) | 5.47(0.17) |
| ORTHREGB  | 95.50    | 93.50    | 96.50    | 96.00 | 0.05(0.01) | 0.53(0.15) | 1.81(0.23) | 5.26(0.67) |
| HS7       | 95.00    | 92.50    | 95.00    | 94.50 | 0.06(0.00) | 0.58(0.00) | 1.82(0.01) | 6.20(0.16) |
| HS48      | 99.00    | 96.00    | 97.50    | 96.00 | 0.15(0.05) | 0.91(0.01) | 2.84(0.01) | 8.96(0.02) |
| HS78      | 94.00    | 96.50    | 98.50    | 95.50 | 0.03(0.00) | 0.19(0.00) | 0.58(0.00) | 2.09(0.35) |

5 Numerical Experiments

We provide experimental results of AI-StoSQP. We apply AI-StoSQP to both benchmark constrained nonlinear optimization problems in CUTEst set (Gould et al., 2014), and to linearly/nonlinearly constrained regression problems. For regression problems, we investigate both squared loss and logistic loss. Due to the space limit, full implementation details and results are deferred to Appendix E.

Our code is available at https://github.com/senna1128/Inf-StoSQP.

5.1 Benchmark constrained problems

The CUTEst test set collects a number of nonlinear optimization problems with and without constraints. We implement five equality-constrained problems: MARATOS, ORTHREGB, HS7, HS48, HS78. The solution of each problem is solved by IPOPT solver (Wächter and Biegler, 2006).

For our method, we perform $10^5$ iterations and, at each step, we perform $\tau = 20$ randomized Kaczmarz steps to approximately solve QPs. Given the iterate $x_t$, we generate $\bar{g}_t \sim \mathcal{N}(\nabla f_t + \sigma^2(I + 11^T))$, where $1 \in \mathbb{R}^d$ is an all-one vector. We also generate the $(i, j)$ and $(j, i)$ entries of $\bar{H}_t$ from $\mathcal{N}((\nabla^2 f_t)_{i,j}, \sigma^2)$. We vary $\sigma^2 \in \{10^{-4}, 10^{-2}, 10^{-1}, 1\}$ and let $\beta_t = 1/t^{0.501}$ (power slightly larger than 0.5) and $\chi_t = \beta_t^2$. We randomly choose $\alpha_t \sim \text{Uniform}([\beta_t, \eta_t])$ with $\eta_t = \beta_t + \chi_t$. For each problem, we aim to estimate $x_t^* + \lambda_t$, and set the nominal coverage probability to 95%. The confidence intervals are constructed by estimating the limiting covariance using Theorem 4.7. The performance of the method is measured by the coverage rate (Cov Rate) of the confidence intervals and their average length (Avg Len) over 200 runs.

The results are summarized in Table 1. From Table 1, we observe that, for the majority of cases, our constructed confidence intervals cover the true solution with probability of at least 95%, and the coverage rate is robust to the sampling variance $\sigma^2$. This observation suggests that neglecting the sketching randomness in the estimation of the limiting covariance does not obviously deteriorate the coverage rate. However, using (sparse) sketching vectors to solve QPs as in (7) is computationally more efficient than exact second-order methods.

From Table 1, we also observe that the average length of the confidence intervals gradually increases as $\sigma^2$ increases. When $\sigma^2$ increases from $10^{-4}$ to 1, the length increases from $10^{-4}$ to $10^{-2}$. This is as expected, as the asymptotic covariance $\Xi^*$ depends on $\text{cov}(\nabla f(x^*; \xi))$ in (23). We testify the almost sure convergence rate of our method (Theorem 4.5), and visualize online construction of confidence intervals in Appendix E.1.
5.2 Constrained regression problems

We implement our method on constrained regression problems, considering both linear regression and logistic regression (see Example 1.1). We also allow for either linear constraints $Ax = d$ or nonlinear constraints $\|x\|^2 = b$. Therefore, there are four cases in total. For each case, we vary the parameter dimension $d \in \{5, 20, 40, 60\}$, and the true solution $x^*$ is linearly spaced between 0 and 1. For each $d$, we randomly sample a covariate $\xi_a \sim \mathcal{N}(0, \Sigma_a)$ at each step, with three different choices of $\Sigma_a$ (also considered in Chen et al. (2020)). (i) Identity: $\Sigma_a = I$. (ii) Toeplitz: $[\Sigma_a]_{i,j} = r|i-j|$. (iii) Equi-correlation: $[\Sigma_a]_{i,j} = r$ for $i \neq j$ and $[\Sigma_a]_{i,i} = 1$ for all $i$. For linear constraints, we let $A \in \mathbb{R}^{m \times d}$ with $m = \lceil \sqrt{d} \rceil$ and entries being independently generated from standard normal distribution. For logistic models, we regularize the loss by a quadratic penalty. We estimate $\frac{1}{d+m} (\sum_{i=1}^{d} x_i^* + \sum_{j=1}^{m} \lambda_j^*)$ by constructing 95% confidence interval. We follow Section 5.1 to implement the method, including the setup of the stepsize and the sketching solver. We report the results only for $r = 0.5$ for Toeplitz and $r = 0.2$ for Equi-correlation. The comprehensive comparisons between inexact and exact methods with varying $d, r$ and $\tau$ are reported in Section E.2.

We summarize the results in Tables 2 and 3. From Table 2, we observe that even if nonlinearly constrained regressions are more difficult to estimate than linearly constrained regressions (more evident for linear models with Toeplitz $\Sigma_a$ and logistic models with identity $\Sigma_a$), our method still achieves a promising coverage rate with nonlinear constraints. From Table 3, we see that the length of confidence intervals gradually decreases when the dimension $d$ increases. In our case, $X^* = 0$ so that the true solution $\sum_{i=1}^{d} x_i^*/(d+m)$ decreases to 0 as $d$ increases. We note that the lengths are all within the order $10^{-2}$, which is comparable to the (unconstrained) first-order methods under similar setups (Chen et al., 2020).

Table 2: The coverage rate (%) for constrained regression problems.

| Obj | Cons | Dimension | Identity $\Sigma_a$ | Toeplitz $\Sigma_a$ (r=0.5) | Equi-correlation $\Sigma_a$ (r=0.2) |
|-----|------|-----------|---------------------|-------------------------------|------------------------------------|
|     |      | 5 20 40 60 | 5 20 40 60          | 5 20 40 60                  | 5 20 40 60                        |
| Lin | Lin  | 93.50 95.00 95.50 94.00 | 94.50 92.00 95.00 96.50 | 94.50 92.00 89.00 87.00          |
| Lin | Non  | 94.50 93.50 89.50 95.50 | 91.50 93.50 90.00 88.50 | 96.00 92.50 90.50 88.50          |
| Logit | Lin | 94.50 94.50 96.50 97.50 | 92.50 96.50 96.00 95.50 | 97.50 96.00 97.00 95.00          |
| Logit | Non | 88.00 88.00 91.50 93.50 | 90.00 93.50 96.00 99.00 | 95.50 95.00 96.00 95.50          |

Table 3: The average length ($10^{-2}$) for regression problems. For visibility, we do not provide standard deviations, which are mostly around 0.02. See Appendix E.2 for the complete tables.
6 Conclusion and Future Work

We proposed a fully online second-order method for solving nonlinearly constrained stochastic optimization problems, called AI-StoSQP. We allow the scheme to adaptively employ random stepsizes and inexactly solve Newton systems (i.e., quadratic programs) via sketching. Thus, our method is more adaptive and computationally efficient than existing exact second-order methods. For our proposed method, we established an almost sure convergence rate and iteration complexity, and showed asymptotic normality property for the iterates. We also provided and analyzed a plug-in covariance estimator. Our analysis precisely quantified the uncertainty of the stochastic process generated by StoSQP, which consists of the randomness of sampling, the randomness of sketching, and the randomness of stepsize. With our results, one can apply AI-StoSQP to perform online inference for constrained estimation problems.

As for future directions, it is of interests to establish a non-asymptotic analysis for StoSQP. Such a result would complement our analysis by bounding the distance between the distribution of $(x_t, \lambda_t)$ and the normal distribution for any given $t$. Furthermore, while we show that StoSQP can be arbitrarily close to the local asymptotic minimax optimality, whether it is possible to close the gap without compromising the efficiency of the method remains an interesting open question. Finally, incorporating nonlinear inequality constraints into the problems and developing second-order methods without projections also deserves studying in the future work.

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References

A. S. Berahas, F. E. Curtis, M. J. O’Neill, and D. P. Robinson. A stochastic sequential quadratic optimization algorithm for nonlinear equality constrained optimization with rank-deficient jacobians. arXiv preprint arXiv:2106.13015, 2021a.

A. S. Berahas, F. E. Curtis, D. Robinson, and B. Zhou. Sequential quadratic optimization for nonlinear equality constrained stochastic optimization. SIAM Journal on Optimization, 31(2): 1352–1379, 2021b.

B. Bercu, A. Godichon, and B. Portier. An efficient stochastic Newton algorithm for parameter estimation in logistic regressions. SIAM Journal on Control and Optimization, 58(1):348–367, 2020.

D. Bertsekas. Constrained Optimization and Lagrange Multiplier Methods. Elsevier, Belmont, Mass, 1982.
C. Boyer and A. Godichon-Baggioni. On the asymptotic rate of convergence of stochastic newton algorithms and their weighted averaged versions. *Computational Optimization and Applications*, 84(3):921–972, 2023.

X. Chen, J. D. Lee, X. T. Tong, and Y. Zhang. Statistical inference for model parameters in stochastic gradient descent. *The Annals of Statistics*, 48(1):251–273, 2020.

X. Chen, W. Liu, and Y. Zhang. First-order Newton-type estimator for distributed estimation and inference. *Journal of the American Statistical Association*, pages 1–17, 2021.

S. Cuomo, V. S. D. Cola, F. Giampaolo, G. Rozza, M. Raissi, and F. Piccialli. Scientific machine learning through physics–informed neural networks: Where we are and what’s next. *Journal of Scientific Computing*, 92(3):88, 2022.

F. E. Curtis and D. P. Robinson. Exploiting negative curvature in deterministic and stochastic optimization. *Mathematical Programming*, 176(1-2):69–94, 2018.

F. E. Curtis, D. P. Robinson, and B. Zhou. Inexact sequential quadratic optimization for minimizing a stochastic objective function subject to deterministic nonlinear equality constraints. *arXiv preprint arXiv:2107.03512*, 2021.

F. E. Curtis, M. J. O’Neill, and D. P. Robinson. Worst-case complexity of an SQP method for nonlinear equality constrained stochastic optimization. *Mathematical Programming*, 2023.

G. B. Dantzig and G. Infanger. Multi-stage stochastic linear programs for portfolio optimization. *Annals of Operations Research*, 45(1):59–76, 1993.

C. Davis and W. M. Kahan. The rotation of eigenvectors by a perturbation. III. *SIAM Journal on Numerical Analysis*, 7(1):1–46, 1970.

J. E. Dennis and J. J. Moré. A characterization of superlinear convergence and its application to quasi-newton methods. *Mathematics of Computation*, 28(126):549–560, 1974.

J.-H. Du, Y. Guo, and X. Wang. High-dimensional portfolio selection with cardinality constraints. *Journal of the American Statistical Association*, 118(542):779–791, 2022.

J. C. Duchi and F. Ruan. Asymptotic optimality in stochastic optimization. *The Annals of Statistics*, 49(1), 2021.

M. Duflo. *Random iterative models*, volume 34. Springer, Berlin New York, 1997.

J. Dupacova and R. Wets. Asymptotic behavior of statistical estimators and of optimal solutions of stochastic optimization problems. *The Annals of Statistics*, 16(4):1517–1549, 1988.

R. Durrett. *Probability*, volume 49. Cambridge University Press, 2019.

Y. Ermoliev. Stochastic quasigradient methods and their application to system optimization. *Stochastics*, 9(1-2):1–36, 1983.

V. Fabian. Asymptotically efficient stochastic approximation; the RM case. *The Annals of Statistics*, 1(3):486–495, 1973.
J. Fan. Variable screening in high-dimensional feature space. In Proceedings of the 4th international congress of chinese mathematicians, volume 2, pages 735–747, 2007.

J. Fan, J. Zhang, and K. Yu. Vast portfolio selection with gross-exposure constraints. Journal of the American Statistical Association, 107(498):592–606, 2012.

X. Fan. Exact rates of convergence in some martingale central limit theorems. Journal of Mathematical Analysis and Applications, 469(2):1028–1044, 2019.

Y. Fang, S. Na, M. W. Mahoney, and M. Kolar. Fully stochastic trust-region sequential quadratic programming for equality-constrained optimization problems. arXiv preprint arXiv:2211.15943, 2022.

C. J. Geyer. Constrained maximum likelihood exemplified by isotonic convex logistic regression. Journal of the American Statistical Association, 86(415):717–724, 1991.

C. J. Geyer. On the asymptotics of constrained $m$-estimation. The Annals of Statistics, 22(4):1993–2010, 1994.

N. I. M. Gould, D. Orban, and P. L. Toint. CUTEst: a constrained and unconstrained testing environment with safe threads for mathematical optimization. Computational Optimization and Applications, 60(3):545–557, 2014.

R. M. Gower and P. Richtárik. Randomized iterative methods for linear systems. SIAM Journal on Matrix Analysis and Applications, 36(4):1660–1690, 2015.

G. E. Karniadakis, I. G. Kevrekidis, L. Lu, P. Perdikaris, S. Wang, and L. Yang. Physics-informed machine learning. Nature Reviews Physics, 3(6):422–440, 2021.

L. Kaufman and V. Pereyra. A method for separable nonlinear least squares problems with separable nonlinear equality constraints. SIAM Journal on Numerical Analysis, 15(1):12–20, 1978.

J. Kiefer and J. Wolfowitz. Stochastic estimation of the maximum of a regression function. The Annals of Mathematical Statistics, 23(3):462–466, 1952.

P. Kirkegaard and M. Eldrup. POSITRONFIT: A versatile program for analysing positron lifetime spectra. Computer Physics Communications, 3(3):240–255, 1972.

A. Krishnapriyan, A. Gholami, S. Zhe, R. Kirby, and M. W. Mahoney. Characterizing possible failure modes in physics-informed neural networks. Advances in Neural Information Processing Systems, 34:26548–26560, 2021.

H. J. Kushner and D. S. Clark. Stochastic Approximation Methods for Constrained and Unconstrained Systems, volume 26. Springer New York, 1978.

T. Li, L. Liu, A. Kyrillidis, and C. Caramanis. Statistical inference using SGD. In Proceedings of the AAAI Conference on Artificial Intelligence, volume 32. Association for the Advancement of Artificial Intelligence (AAAI), 2018.

T. Liang and W. J. Su. Statistical inference for the population landscape via moment-adjusted stochastic gradients. Journal of the Royal Statistical Society: Series B (Statistical Methodology), 81(2):431–456, 2019.
L. Ljung. Analysis of recursive stochastic algorithms. *IEEE Transactions on Automatic Control*, 22(4):551–575, 1977.

L. Lu, R. Pestourie, W. Yao, Z. Wang, F. Verdugo, and S. G. Johnson. Physics-informed neural networks with hard constraints for inverse design. *SIAM Journal on Scientific Computing*, 43(6):B1105–B1132, 2021.

W. Mou, C. J. Li, M. J. Wainwright, P. L. Bartlett, and M. I. Jordan. On linear stochastic approximation: Fine-grained polyak-ruppert and non-asymptotic concentration. In *Conference on Learning Theory*, pages 2947–2997. PMLR, 2020.

S. Na and M. Kolar. High-dimensional index volatility models via stein’s identity. *Bernoulli*, 27(2), 2021.

S. Na, Z. Yang, Z. Wang, and M. Kolar. High-dimensional varying index coefficient models via stein’s identity. *J. Mach. Learn. Res.*, 20:152–1, 2019.

S. Na, M. Anitescu, and M. Kolar. An adaptive stochastic sequential quadratic programming with differentiable exact augmented lagrangians. *Mathematical Programming*, 2022a.

S. Na, M. Dereziński, and M. W. Mahoney. Hessian averaging in stochastic Newton methods achieves superlinear convergence. *Mathematical Programming*, 2022b.

S. Na, M. Anitescu, and M. Kolar. Inequality constrained stochastic nonlinear optimization via active-set sequential quadratic programming. *Mathematical Programming*, 2023.

N. K. Nagaraj and W. A. Fuller. Estimation of the parameters of linear time series models subject to nonlinear restrictions. *The Annals of Statistics*, 19(3):1143–1154, 1991.

J. Nocedal and S. J. Wright. *Numerical Optimization*. Springer Series in Operations Research and Financial Engineering. Springer New York, 2nd edition, 2006.

M. Pilanci and M. J. Wainwright. Newton sketch: A near linear-time optimization algorithm with linear-quadratic convergence. *SIAM Journal on Optimization*, 27(1):205–245, 2017.

B. T. Polyak and A. B. Juditsky. Acceleration of stochastic approximation by averaging. *SIAM Journal on Control and Optimization*, 30(4):838–855, 1992.

H. Robbins and D. Siegmund. A convergence theorem for non negative almost supermartingales and some applications. In *Optimizing Methods in Statistics*, pages 233–257. Elsevier, 1971.

H. Robbins and S. Monro. A stochastic approximation method. *Ann. Math. Statistics*, 22:400–407, 1951.

D. Ruppert. Efficient estimations from a slowly convergent robbins-monro process. Technical report, 1988.

P. K. Sen. Asymptotic properties of maximum likelihood estimators based on conditional specification. *The Annals of Statistics*, 7(5):1019–1033, 1979.

A. Shapiro. On the asymptotics of constrained local $\text{sm}$-estimators. *The Annals of Statistics*, 28(3):948–960, 2000.
A. Shapiro, D. Dentcheva, and A. Ruszczyński. Lectures on Stochastic Programming: Modeling and Theory, Second Edition. Society for Industrial and Applied Mathematics, 2014.

T. Strohmer and R. Vershynin. A randomized kaczmarz algorithm with exponential convergence. Journal of Fourier Analysis and Applications, 15(2):262–278, 2008.

P. Toulis and E. M. Airoldi. Asymptotic and finite-sample properties of estimators based on stochastic gradients. The Annals of Statistics, 45(4):1694–1727, 2017.

F. B. Veliz, J.-P. Watson, A. Weintraub, R. J.-B. Wets, and D. L. Woodruff. Stochastic optimization models in forest planning: a progressive hedging solution approach. Annals of Operations Research, 232:259–274, 2014.

J. H. Venter. An extension of the robbins-monro procedure. The Annals of Mathematical Statistics, 38(1):181–190, 1967.

A. Wächter and L. T. Biegler. On the implementation of an interior-point filter line-search algorithm for large-scale nonlinear programming. Math. Program., 106(1, Ser. A):25–57, 2006.

H. Walk. Stochastic iteration for a constrained optimization problem. Communications in Statistics. Part C: Sequential Analysis, 2(4):369–385, 1983.

J.-G. Wang. The asymptotic behavior of locally square integrable martingales. The Annals of Probability, 23(2):552–585, 1995.

P.-Å. Wedin. Perturbation bounds in connection with singular value decomposition. BIT, 12(1):99–111, 1972.

R. J.-B. Wets. Statistical estimation from an optimization viewpoint. Annals of Operations Research, 85(0):79–101, 1999.

M. B. Zafar, I. Valera, M. Gomez-Rodriguez, and K. P. Gummadi. Fairness constraints: A flexible approach for fair classification. The Journal of Machine Learning Research, 20(1):2737–2778, 2019.

W. Zhu, X. Chen, and W. B. Wu. Online covariance matrix estimation in stochastic gradient descent. Journal of the American Statistical Association, pages 1–30, 2021.
A Preparation Lemmas

Lemma A.1. Suppose \( \{\varphi_i\}_i \) is a positive sequence that satisfies \( \lim_{i \to \infty} i (1 - \varphi_{i-1}/\varphi_i) = \varphi \). Then, for any \( p \geq 0 \), we have \( \lim_{i \to \infty} i (1 - \varphi_{i-1}/\varphi_i^p) = p \cdot \varphi \).

Lemma A.2. Let \( \{\varphi_i\}_i \) be a positive sequence. If \( \lim_{i \to \infty} i (1 - \varphi_{i-1}/\varphi_i) = \varphi < 0 \), then \( \lim_{i \to \infty} \varphi_i = 0 \).

Lemma A.3. Let \( \{\phi_i\}_i \), \( \{\varphi_i\}_i \), \( \{\sigma_i\}_i \) be three positive sequences. Suppose
\[
\lim_{i \to \infty} i (1 - \phi_{i-1}/\phi_i) = \phi, \quad \lim_{i \to \infty} \varphi_i = 0, \quad \lim_{i \to \infty} i \varphi_i = \bar{\varphi} \tag{A.1}
\]
for a constant \( \phi \) and a (possibly infinite) constant \( \bar{\varphi} \in (0, \infty] \). For any \( l \geq 1 \), if we further have \( \sum_{k=1}^{l} \sigma_k + \phi/\bar{\varphi} > 0 \), then the following results hold as \( t \to \infty \)
\[
\frac{1}{\phi_t} \sum_{i=0}^{t} \prod_{j=i+1}^{l} (1 - \varphi_j \sigma_k) \varphi_i \phi_i \to \frac{1}{\sum_{k=1}^{l} \sigma_k + \phi/\bar{\varphi}},
\]
\[
\frac{1}{\phi_t} \left\{ \sum_{i=0}^{t} \prod_{j=i+1}^{l} (1 - \varphi_j \sigma_k) \varphi_i \phi_i a_i + b \prod_{j=0}^{t} \prod_{k=1}^{l} (1 - \varphi_j \sigma_k) \right\} \to 0,
\tag{A.2}
\]
where the second result holds for any constant \( b \) and any sequence \( \{a_i\}_i \) such that \( a_i \to 0 \).

Lemma A.4. For any scalars \( a, b \), we have \( P (a < \mathcal{N}(0, 1) \leq b) \leq b - a \). Furthermore, if \( 0 < a \leq b \), then \( P (a < \mathcal{N}(0, 1) \leq b) \leq b/a - 1 \).

Lemma A.5. Let \( A_t, B_t, C_t \) be three variables depending on the index \( t \); also let \( \Phi(z) = P(\mathcal{N}(0, 1) \leq z) \) be the cumulative distribution function of standard Gaussian variable. Suppose for the index \( t \),
\[
\sup_{z \in \mathbb{R}} |P (A_t \leq z) - \Phi(z)| \leq a_t, \quad |B_t| \leq b_t, \quad |C_t| \leq c_t \quad \text{almost surely} \tag{A.3}
\]
where \( a_t, b_t \geq 0 \) and \( 0 \leq c_t < 1 \). Then, we have
\[
\sup_{z \in \mathbb{R}} \left| P \left( \frac{A_t + B_t}{\sqrt{1 + C_t}} \leq z \right) - \Phi(z) \right| \leq a_t + b_t + \frac{c_t}{\sqrt{1 - c_t}}.
\]

B Proofs of Preparation Lemmas

B.1 Proof of Lemma A.1

By the condition, we know \( \varphi_{i-1}/\varphi_i = 1 - \varphi/i + o(1/i) \). Thus, we have
\[
i (1 - \varphi_{i-1}^p/\varphi_i^p) = i (1 - (1 - \varphi/i + o(1/i))^p) = p \varphi + o(1).
\]
This completes the proof.

B.2 Proof of Lemma A.2

By Lemma A.1, we know for any positive constant \( p \), \( \lim_{i \to \infty} i (1 - \varphi_{i-1}^p/\varphi_i^p) = p \varphi \). Choosing \( p \) large enough such that \( p \varphi < -1 \), the Raabe’s test indicates that \( \sum_{i=0}^{\infty} \varphi_i^p < \infty \). This implies \( \varphi_i \to 0 \) and we complete the proof.
B.3 Proof of Lemma A.3

For any scalar $A$, we have

$$
\frac{1}{\phi_t} \sum_{i=0}^{t} \prod_{j=i+1}^{t} \prod_{k=1}^{l} (1 - \varphi_j \sigma_k) \varphi_i \phi_i - A
$$

$$= \frac{1}{\phi_t} \prod_{j=0}^{t} \prod_{k=1}^{l} (1 - \varphi_j \sigma_k) \left\{ \sum_{i=0}^{t} \prod_{j=0}^{t} \prod_{k=1}^{l} (1 - \varphi_j \sigma_k)^{-1} \varphi_i \phi_i - A \phi_t \prod_{j=0}^{t} \prod_{k=1}^{l} (1 - \varphi_j \sigma_k)^{-1} \right\}.
$$

For the last term, we have

$$A \phi_t \prod_{j=0}^{t} \prod_{k=1}^{l} (1 - \varphi_j \sigma_k)^{-1}
$$

$$= \sum_{i=1}^{t} \left( A \phi_i \prod_{j=0}^{i-1} \prod_{k=1}^{l} (1 - \varphi_j \sigma_k)^{-1} - A \phi_{i-1} \prod_{j=0}^{i-1} \prod_{k=1}^{l} (1 - \varphi_j \sigma_k)^{-1} \right) + A \phi_0 \prod_{k=1}^{l} (1 - \varphi_0 \sigma_k)^{-1}
$$

$$= \sum_{i=1}^{t} A \phi_i \prod_{j=0}^{t} \prod_{k=1}^{l} (1 - \varphi_j \sigma_k)^{-1} \left\{ 1 - \frac{\phi_{i-1}}{\phi_i} \prod_{k=1}^{l} (1 - \varphi_i \sigma_k) \right\} + A \phi_0 \prod_{k=1}^{l} (1 - \varphi_0 \sigma_k)^{-1}.
$$

Combining the above two displays, we obtain

$$\frac{1}{\phi_t} \sum_{i=0}^{t} \prod_{j=i+1}^{t} \prod_{k=1}^{l} (1 - \varphi_j \sigma_k) \varphi_i \phi_i - A
$$

$$= \frac{1}{\phi_t} \prod_{j=0}^{t} \prod_{k=1}^{l} (1 - \varphi_j \sigma_k) \left\{ \sum_{i=0}^{t} \prod_{j=0}^{t} \prod_{k=1}^{l} (1 - \varphi_j \sigma_k)^{-1} \varphi_i \phi_i \right\} - A \phi_t \prod_{j=0}^{t} \prod_{k=1}^{l} (1 - \varphi_j \sigma_k)^{-1}
$$

$$+ \phi_0 \prod_{k=1}^{l} (1 - \varphi_0 \sigma_k)^{-1} \left( \varphi_0 - A \right) \right\}.
$$

(B.1)

We aim to select $A$ such that the middle term in (B.1) is small. By (A.1), we know $\phi_{i-1}/\phi_i = 1 - \phi/i + o(1/i) = 1 - \phi \cdot \varphi_i / \overline{\varphi} + o(\varphi_i)$, where the second equality is due to $1/(i \varphi_i) = 1/\overline{\varphi} + o(1)$ (which is true even if $\overline{\varphi} = \infty$). Furthermore, we know $\prod_{k=1}^{l} (1 - \varphi_i \sigma_k) = 1 - \varphi_i \sum_{k=1}^{l} \sigma_k + o(\varphi_i)$. With these two facts, we have

$$\varphi_i - A \left( 1 - \frac{\phi_{i-1}}{\phi_i} \prod_{k=1}^{l} (1 - \varphi_i \sigma_k) \right) = \varphi_i - A \left\{ 1 - \left( 1 - \frac{\phi}{\overline{\varphi}} \cdot \varphi_i + o(\varphi_i) \right) \left( 1 - \varphi_i \sum_{k=1}^{l} \sigma_k + o(\varphi_i) \right) \right\}
$$

$$= \varphi_i - A \left( \frac{\phi}{\overline{\varphi}} + \sum_{k=1}^{l} \sigma_k \right) \varphi_i + o(\varphi_i).
$$

(B.2)
Thus, we let $A = 1/(\sum_{k=1}^l \sigma_k + \phi/\varphi)$ and (B.1) leads to
\[
\frac{1}{\phi_t} \sum_{i=0}^t \prod_{j=i+1}^t \prod_{k=1}^l (1 - \varphi_j \sigma_k) \varphi_i \phi_i - \frac{1}{\sum_{k=1}^l \sigma_k + \phi/\varphi} = \frac{1}{\phi_t} \prod_{j=0}^t \prod_{k=1}^l (1 - \varphi_j \sigma_k) \left\{ \sum_{i=1}^t \prod_{j=0}^t \prod_{k=1}^l (1 - \varphi_j \sigma_k)^{-1} \varphi_i \cdot o(\varphi_i) + \phi_0 \prod_{k=1}^l (1 - \varphi_0 \sigma_k)^{-1} \left( \varphi_0 - \frac{1}{\sum_{k=1}^l \sigma_k + \phi/\varphi} \right) \right\}.
\]
Comparing the above display with (A.2), we note that the first result in (A.2) is implied by the second result. Thus, it suffices to prove the second result. We define
\[
\Psi_t = \frac{1}{\phi_t} \left\{ \sum_{i=0}^t \prod_{j=i+1}^t \prod_{k=1}^l (1 - \varphi_j \sigma_k) \varphi_i \phi_i a_t + b \prod_{j=0}^t \prod_{k=1}^l (1 - \varphi_j \sigma_k) \right\}, \tag{B.3}
\]
then
\[
\Psi_t = \frac{1}{\phi_t} \left\{ \varphi_t \phi_t a_t + \prod_{k=1}^l (1 - \varphi_t \sigma_k) \left( \sum_{i=0}^{t-1} \prod_{j=0}^{i+1} \prod_{k=1}^l (1 - \varphi_j \sigma_k) \varphi_i \phi_i a_i + b \prod_{j=0}^{t-1} \prod_{k=1}^l (1 - \varphi_j \sigma_k) \right) \right\} \tag{B.3}
\]
\[
\overset{(B.2)}{=} \frac{\phi_{t-1}}{\phi_t} \prod_{k=1}^l (1 - \varphi_t \sigma_k) \Psi_{t-1} + \varphi_t a_t.
\]
By (B.2), we know that
\[
\frac{\phi_{t-1}}{\phi_t} \prod_{k=1}^l (1 - \varphi_t \sigma_k) = 1 - \left( \frac{\phi}{\varphi} + \sum_{k=1}^l \sigma_k \right) \cdot \varphi_t + o(\varphi_t).
\]
Since $\sum_{k=1}^l \sigma_k + \phi/\varphi > 0$, we immediately conclude that for a constant $c > 0$ and for all large enough $t$, $|\Psi_t| \leq (1 - c\varphi_t) |\Psi_{t-1}| + \varphi_t |a_t|$. Let $t_1$ be a fixed integer. We apply this inequality recursively and have for any $t \geq t_1 + 1$,
\[
|\Psi_t| \leq \prod_{i=t_1+1}^t (1 - c\varphi_i) |\Psi_{t_1}| + \sum_{i=t_1+1}^t \prod_{j=i+1}^t (1 - c\varphi_j) \varphi_i |a_i|.
\]
For any $\epsilon > 0$, since $a_i \rightarrow 0$, we select $t_1$ such that $|a_i| \leq \epsilon$, for all $i \geq t_1$. Then, the above inequality leads to
\[
|\Psi_t| \leq \prod_{i=t_1+1}^t (1 - c\varphi_i) |\Psi_{t_1}| + \epsilon \sum_{i=t_1+1}^t \prod_{j=i+1}^t (1 - c\varphi_j) \varphi_i
\]
\[
= \prod_{i=t_1+1}^t (1 - c\varphi_i) |\Psi_{t_1}| + \epsilon \left\{ 1 - \prod_{j=t_1+1}^t (1 - c\varphi_j) \right\} \leq |\Psi_{t_1}| \exp \left( -c \sum_{i=t_1+1}^t \varphi_i \right) + \frac{\epsilon}{c}.
\]
Since $n\varphi_i \rightarrow \bar{\varphi} \in (0, \infty)$, we know $\sum_i \varphi_i \rightarrow \infty$. Thus, for the above $\epsilon > 0$, there exists $t_2 \geq t_1$ such that $|\Psi_{t_1}| \exp \left( -c \sum_{i=t_1+1}^t \varphi_i \right) \leq \epsilon/c$, $\forall t \geq t_2$, which implies $|\Psi_t| \leq 2\epsilon/c$. This means $|\Psi_t| \rightarrow 0$ and we complete the proof.
B.4 Proof of Lemma A.4

The first part of statement holds naturally due to the fact that the density of the standard Gaussian satisfies $\exp(-t^2/2)/\sqrt{2\pi} \leq 1$ for any $t \in \mathbb{R}$. Moreover, for $0 < a \leq b$, we have

$$P(a < N(0, 1) \leq b) = \int_a^b \frac{1}{\sqrt{2\pi}} \exp(-t^2/2) \, dt \leq \frac{b - a}{\sqrt{2\pi}} \exp(-a^2/2) = \left(\frac{b}{a} - 1\right) \frac{a}{\sqrt{2\pi}} \exp(-a^2/2) \leq \frac{b}{a} - 1,$$

where the last inequality uses $a \exp(-a^2/2) \leq 1$ for all $a$. This completes the proof.

B.5 Proof of Lemma A.5

We only prove the result for $z > 0$. The result of $z \leq 0$ can be shown in the same way. We know from (A.3) that $\frac{A_t - b_t}{\sqrt{1+c_t}} \leq \frac{A_t + B_t}{\sqrt{1+c_t}} \leq \frac{A_t + b_t}{\sqrt{1-c_t}}$, almost surely. Therefore, we have

$$P\left(\frac{A_t + B_t}{\sqrt{1+C_t}} \leq z\right) \geq P\left(\frac{A_t + b_t}{\sqrt{1-C_t}} \leq z\right) = P(A_t \leq z(1-c_t)^{1/2} - b_t) \geq \Phi(z(1-c_t)^{1/2} - b_t) - a_t$$

(by Lemma A.4)

$$\geq \Phi(z) - b_t - \frac{c_t}{\sqrt{1-c_t}} - a_t.$$

On the other hand, we have

$$P\left(\frac{A_t + B_t}{\sqrt{1+C_t}} \leq z\right) \leq P\left(\frac{A_t - b_t}{\sqrt{1+C_t}} \leq z\right) = P(A_t \leq z(1+c_t)^{1/2} + b_t) \leq \Phi(z(1+c_t)^{1/2} + b_t) + a_t$$

(by Lemma A.4)

Combining the above two displays completes the proof.

C Proofs of Section 3

C.1 Proof of Lemma 3.4

We note that $\gamma_S \leq \|\mathbb{E}[K_t S(S^T K_t^2 S)^t S^T K_t] \mid x_t, \lambda_t]\| \leq \mathbb{E}\|\mathbb{E}[K_t S(S^T K_t^2 S)^t S^T K_t] \mid x_t, \lambda_t]\| \leq 1$, where the second inequality is by Jensen’s inequality; the third inequality is because $K_t S(S^T K_t^2 S)^t S^T K_t$ is a projection matrix. This shows (a). Let us define for $j = 1, \ldots, \tau$, $C_{t,j} = I - K_t S_{t,j} (S_{t,j} K_t^2 S_{t,j})^t S_{t,j}^T K_t$. Then, we obtain from (7) that

$$z_{t,\tau} - \tilde{z}_t = C_{t,\tau-1}(z_{t,\tau-1} - \tilde{z}_t) = \left(\prod_{j=0}^{\tau-1} C_{t,j}\right) (z_{t,0} - \tilde{z}_t) = -\left(\prod_{j=0}^{\tau-1} C_{t,j}\right) \tilde{z}_t. \quad \text{(C.1)}$$
Thus, (b) follows from (C.1) and the independence among \{\zeta_{t,j}\}. Moreover, (c) is proved by (Gower and Richtárik, 2015, Theorem 4.6).

### C.2 Proof of Lemma 3.5

By Assumption 3.1, there exists a constant \(Y_u \geq 1\) such that

\[
\|\nabla^2 \mathcal{L}(x, \lambda)\| \vee \|\nabla \mathcal{L}(x, \lambda)\| \leq Y_u, \quad \forall (x, \lambda) \in \mathcal{X} \times \Lambda. \tag{C.2}
\]

Using (13) and the definition of \((\Delta x_t, \Delta \lambda_t)\), we have

\[
\begin{align*}
\left(\nabla_x \mathcal{L}_{t, \mu, \nu}^T \right) \left(\nabla_{\lambda} \mathcal{L}_{t, \mu, \nu}^T \right)^T \left(\Delta x_t \right) & \leq -\Delta x_t^T B_t \Delta x_t + \nu Y_B Y_u \|\Delta x_t\|^2 - \mu \|G_t \Delta x_t\|^2 - 2 \Delta \lambda_t^T G_t \Delta x_t + \nu (Y_u + Y_B) \|\Delta x_t\| \|G_t^T \Delta \lambda_t\| - \nu \|G_t^T \Delta \lambda_t\|^2 \\
& \leq -\Delta x_t^T B_t \Delta x_t + \nu Y_B Y_u \|\Delta x_t\|^2 - \mu \|c_t\|^2 + \nu \|\Delta \lambda_t\|^2 - \frac{\nu(Y_u + Y_B)^2}{2} \|\Delta x_t\|^2 - \frac{\nu}{2} \|G_t^T \Delta \lambda_t\|^2 \\
& \leq -\Delta x_t^T B_t \Delta x_t + \nu Y_B Y_u \|\Delta x_t\|^2 - \mu \|c_t\|^2 + \frac{8}{\nu \gamma G} \|\Delta \lambda_t\|^2 \\
& \quad - \frac{\nu \gamma G}{4} \|\Delta \lambda_t\|^2 - \frac{\nu}{4} \|G_t^T \Delta \lambda_t\|^2 \quad \text{(Young’s inequality and Assumption 3.1)}
\end{align*}
\]

\[
\begin{align*}
& \leq -\Delta x_t^T B_t \Delta x_t + \nu (Y_B + Y_u)^2 \|\Delta x_t\|^2 - \left(\mu - \frac{8}{\nu \gamma G}\right) \|c_t\|^2 - \nu \frac{\gamma G}{8} \|\Delta \lambda_t\|^2 - \frac{\nu}{4} \|B_t \Delta x_t + \nabla x \mathcal{L}_t\|^2 \\
& \leq -\Delta x_t^T B_t \Delta x_t + \nu (Y_B + Y_u)^2 \|\Delta x_t\|^2 - \left(\mu - \frac{8}{\nu \gamma G}\right) \|c_t\|^2 - \nu \frac{\gamma G}{8} \|\Delta \lambda_t\|^2 - \frac{\nu}{8} \|\nabla x \mathcal{L}_t\|^2 + \frac{\nu Y_B^2}{4} \|\Delta x_t\|^2 \\
& \leq -\Delta x_t^T B_t \Delta x_t + 2 \nu (Y_B + Y_u)^2 \|\Delta x_t\|^2 - \left(\mu - \frac{8}{\nu \gamma G}\right) \|c_t\|^2 - \nu \frac{\gamma G}{8} \|\Delta \lambda_t\|^2 - \frac{\nu}{8} \|\nabla x \mathcal{L}_t\|^2, \tag{C.3}
\end{align*}
\]

where the second last inequality uses \(\|B_t \Delta x_t + \nabla x \mathcal{L}_t\|^2 \geq \|\nabla x \mathcal{L}_t\|^2 \vee \|B_t \Delta x_t\|^2 \geq \|\nabla x \mathcal{L}_t\|^2 \vee \nu \|\Delta \lambda_t\|^2 \vee \nu \|\nabla x \mathcal{L}_t\|^2\). To further simplify (C.3), we decompose the step \(\Delta x_t\) as

\[
\Delta x_t = \Delta u_t + \Delta v_t, \quad \text{where } \Delta u_t \in \text{span}(G_t^T) \text{ and } G_t \Delta v_t = 0.
\]

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Then, the first two terms of (C.3) can be simplified as

\[-\Delta x_t^T B_t \Delta x_t + 2\nu(B_t + Y_u)^2 \|\Delta x_t\|^2\]

\[= -\Delta u_t^T B_t \Delta u_t - 2\Delta u_t^T B_t \Delta v_t - \Delta v_t^T B_t \Delta v_t + 2\nu (B_t + Y_u)^2 \|\Delta x_t\|^2\]

\[\leq Y_B \|\Delta u_t\|^2 + 2Y_B \|\Delta u_t\| \|\Delta v_t\| - \gamma_{RH} \|\Delta v_t\|^2 + 2\nu Y_B \|\Delta x_t\|^2 \quad \text{(Assumption 3.1)}\]

\[\leq \left( Y_B + \frac{2\nu}{\gamma_{RH}} \right) \|\Delta u_t\|^2 - \frac{\gamma_{RH}}{2} \|\Delta v_t\|^2 + 2\nu Y_B \|\Delta u_t\|^2 \quad \text{(Young’s inequality)}\]

\[= \left( Y_B + \frac{2\nu}{\gamma_{RH}} + \frac{\gamma_{RH}}{2} \right) \|\Delta u_t\|^2 - \frac{\gamma_{RH}}{2} - 2\nu (Y_B + Y_u)^2 \|\Delta x_t\|^2\]

where the last inequality uses the fact that \(\|c_t\|^2 = \|G_t \Delta x_t\|^2 = \|G_t \Delta u_t\|^2 \geq Y_G \|\Delta u_t\|^2\). Here, the inequality is due to Assumption 3.1. Combining the above display with (C.3), we have

\[
\left( \frac{\nabla L_{\mu,\nu}^t}{\nabla \lambda L_{\mu,\nu}^t} \right)^T \left( \frac{\Delta x_t}{\Delta \lambda_t} \right) \leq -\frac{\nu \gamma_G}{8} \left( \|\Delta x_t\|^2 - \frac{\nu}{8} \left( \|\nabla L_{\mu,\nu}^t\|_c^2 \right) - \frac{\gamma_{RH}}{2} - 2\nu (Y_B + Y_u)^2 - \frac{\nu \gamma_G}{8} \right) \|\Delta x_t\|^2
\]

\[\leq \left\{ \mu - \frac{8}{\nu \gamma_G} - \left( Y_B + \frac{2\nu}{\gamma_{RH}} + \frac{\gamma_{RH}}{2} \right) \frac{1}{\gamma_G} - \frac{\nu}{8} \right\} \|c_t\|^2.
\]

Thus, choosing \(Y_1\) large enough (depending only on \(\gamma_G, \gamma_{RH}, Y_B\)), we complete the proof.

### C.3 Proof of Lemma 3.6

Let use denote \(z_t = (\Delta x_t, \Delta \lambda_t)\) and recall that \(z_{t,\tau} = (\Delta x_t, \Delta \lambda_t)\) and \(\bar{z}_t = (\Delta x_t, \Delta \lambda_t)\). By Assumption 3.1 and the expression (13), it is straightforward to see that \(\nabla L_{\mu,\nu}^t\) is Lipschitz continuous with a constant \(Y_{AL} > 0\) depending on \((\mu, \nu, \bar{Y}_L)\). Thus, using (8) we have

\[L_{\mu,\nu}^{t+1} \leq L_{\mu,\nu}^t + \tilde{\alpha}_t (\nabla L_{\mu,\nu}^t)^T z_{t,\tau} + \frac{Y_{AL} \tilde{\alpha}_t^2}{2} \|z_{t,\tau}\|^2 \]

\[= L_{\mu,\nu}^t + \tilde{\alpha}_t (\nabla L_{\mu,\nu}^t)^T (I + C_t) z_t + \tilde{\alpha}_t (\nabla L_{\mu,\nu}^t)^T \{ z_{t,\tau} - (I + C_t) z_t \} + \frac{Y_{AL} \tilde{\alpha}_t^2}{2} \|z_{t,\tau}\|^2, \quad \text{(C.4)}\]

where \(C_t\) is from Lemma 3.4(b). By Lemmas 3.5 and 3.4, the second term can be bounded as

\[(\nabla L_{\mu,\nu}^t)^T (I + C_t) z_t \leq -\frac{\nu}{Y_1} \left( \|z_t\|^2 + \|\nabla L_t\|^2 \right) + \|C_t\| \|\nabla L_{\mu,\nu}^t\| \|z_t\| \]

\[\leq -\frac{\nu}{Y_1} \left( \|z_t\|^2 + \|\nabla L_t\|^2 \right) + \rho^* (1 + (2\nu + \mu) Y_u) \|\nabla L_t\| \|z_t\|
\]

\[\leq -\frac{\nu}{Y_1} \left( \|z_t\|^2 + \|\nabla L_t\|^2 \right) + 2\rho^* \mu Y_u \|\nabla L_t\| \|z_t\|
\]

\[\leq - \left( \frac{\nu}{Y_1} - \rho^* \mu Y_u \right) \left( \|z_t\|^2 + \|\nabla L_t\|^2 \right), \]

where the third inequality uses the facts that \(Y_u \geq 1\) and \(2\nu \leq \mu\) (as long as \(Y_1 \geq 2\)). Thus, we can re-define \(Y_1\) as \(\bar{Y}_1 = 2Y_1 Y_u\). If \(\rho^* \leq \nu/(\mu Y_1)\), then we have

\[(\nabla L_{\mu,\nu}^t)^T (I + C_t) z_t \leq -\frac{\nu}{2Y_1} \left( \|z_t\|^2 + \|\nabla L_t\|^2 \right). \quad \text{(C.5)}\]
Now, we deal with the last two terms of (C.4). By Lemma 3.4(b), we have

\[
E[z_{t,\tau} \mid F_{t-1}] = E\left[ E \left[ z_{t,\tau} \mid F_{t-2/3} \right] \mid F_{t-1} \right] = E \left[ (I + C_t)z_t \mid F_{t-1} \right]
\]

\[
\overset{(5)}{=} -(I + C_t)K_t^{-1}E \left[ \bar{\nabla}L_t \mid F_{t-1} \right] = -(I + C_t)K_t^{-1}\nabla L_t \quad \text{(Assumption 3.2)}
\]

\[
\overset{(5)}{=} (I + C_t)z_t.
\]  

(C.6)

By Lemma 3.4(b, c), we also have

\[
E \left[ \left\| z_{t,\tau} - (I + C_t)z_t \right\|^2 \mid F_{t-1} \right]
\]

\[
\leq 3E \left[ \left\| z_{t,\tau} - z_t \right\|^2 \mid F_{t-1} \right] + 3E \left[ \left\| \bar{z}_t - z_t \right\|^2 \mid F_{t-1} \right] + 3\|C_t\|^2 \left\| z_t \right\|^2
\]

\[
\leq 3\rho^t E \left[ \left\| \bar{z}_t \right\|^2 \mid F_{t-1} \right] + 3E \left[ \left\| \bar{z}_t - z_t \right\|^2 \mid F_{t-1} \right] + 3\rho^{2t} \left\| z_t \right\|^2
\]

\[
= 3(\rho^t + \rho^{2t}) \left\| z_t \right\|^2 + 3(1 + \rho^t)E \left[ \left\| \bar{z}_t - z_t \right\|^2 \mid F_{t-1} \right]
\]  

(bias-variance decomposition).

By Assumption 3.1 and (Na et al., 2022a, Lemma 1), there exists a constant $Y_K \geq 1$ depending on $(\gamma_G, \gamma_{RH}, \bar{T}_H)$ such that $\|K_t^{-1}\| \leq Y_K$. Thus, we apply (5) and (C.2), and obtain

\[
E \left[ \left\| z_{t,\tau} - (I + C_t)z_t \right\|^2 \mid F_{t-1} \right] \leq 3(\rho^t + \rho^{2t})Y_K^2 \tau^2 u^2 + 3(1 + \rho^t)Y_K^2 \bar{E} \left[ \left\| \bar{g}_t - \nabla f_t \right\|^2 \mid F_{t-1} \right]
\]

\[
\leq 3(1 + \rho^t)Y_K^2 (\rho^t Y_u^2 + Y_m) \quad \text{(Assumption 3.2(15a)).} \quad \text{(C.7)}
\]

Thus, using (C.6) and (C.7), we have

\[
E \left[ \bar{\alpha}_t (\nabla L_{t,\mu,\nu})^T \{ z_{t,\tau} - (I + C_t)z_t \} \mid F_{t-1} \right]
\]

\[
\overset{(C.6)}{=} E \left[ \{ \alpha_t - (\beta_t + \mu) \}/2 \right] \cdot (\nabla L_{t,\mu,\nu})^T \{ z_{t,\tau} - (I + C_t)z_t \} \mid F_{t-1} \right]
\]

\[
\overset{(9)}{=} \frac{\eta_t - \beta_t}{2} E \left[ \left\| \nabla L_{t,\mu,\nu} \right\| \left\| z_{t,\tau} - (I + C_t)z_t \right\| \mid F_{t-1} \right]
\]

\[
\overset{(C.2)}{\leq} \frac{\eta_t - \beta_t}{2} (1 + (2\nu + \mu)Y_u)Y_u \bar{E} \left[ \left\| z_{t,\tau} - (I + C_t)z_t \right\| \mid F_{t-1} \right]
\]

\[
\leq (\eta_t - \beta_t)\mu Y_u^2 \sqrt{\bar{E} \left[ \left\| z_{t,\tau} - (I + C_t)z_t \right\|^2 \mid F_{t-1} \right]} \quad (1 \leq Y_u \text{ and } 1 + 2\nu \leq \mu)
\]

\[
\overset{(C.7)}{\leq} 2\mu Y_K Y_u^2 (1 + \rho^t)(\sqrt{Y_u} \vee Y_u)(\eta_t - \beta_t) \leq 4\mu Y_K Y_u^2 (\sqrt{Y_u} \vee Y_u)(\eta_t - \beta_t), \quad \text{(C.8)}
\]

and

\[
E[\left\| z_{t,\tau} \right\|^2 \mid F_{t-1}] \overset{(6)}{=} \left\| (I + C_t)z_t \right\|^2 + E \left[ \left\| z_{t,\tau} - (I + C_t)z_t \right\|^2 \mid F_{t-1} \right]
\]

\[
\overset{(5),(C.2)}{\leq} (1 + \rho^{2t})^2 Y_K^2 Y_u^2 + E \left[ \left\| z_{t,\tau} - (I + C_t)z_t \right\|^2 \mid F_{t-1} \right] \quad \text{(also use Lemma 3.4(b))}
\]

\[
\overset{(C.7)}{\leq} (1 + \rho^{2t})^2 Y_K^2 Y_u^2 + 3(1 + \rho^t)Y_K^2 (\rho^t Y_u^2 + Y_m) \leq 16Y_K^2 (Y_u^2 \vee Y_m). \quad \text{(C.9)}
\]

Combining (C.9) with (C.8) and (C.5), plugging into (C.4), and using (9), we obtain

\[
E[L_{t,\mu,\nu}^{t+1} \mid F_{t-1}] \leq L_{t,\mu,\nu}^t - \frac{\nu \beta_t}{2 Y_1} \left( \left\| z_t \right\|^2 + \left\| \nabla L_t \right\|^2 \right)
\]

\[
+ 4\mu Y_K Y_u^2 (\sqrt{Y_u} \vee Y_u)(\eta_t - \beta_t) + 8Y_{AC} Y_K^2 (Y_u^2 \vee Y_m)\eta_t^2.
\]
Choosing $\Upsilon_2$ large enough, depending on $(\mu, \nu, \gamma_G, \gamma_{RH}, \Upsilon_B, \Upsilon_m, \Upsilon_L)$, we complete the proof.

### C.4 Proof of Theorem 3.7

Note that the condition of $\tau$ in the statement implies that we can select $(\mu, \nu)$ to satisfy the condition in Lemma 3.5 and have $\rho^2 \leq \nu/(\mu \Upsilon_1)$ with $\rho = 1 - \gamma_S$. Thus, Lemma 3.6 leads to

$$
\mathbb{E}[\mathcal{L}_{\mu, \nu}^{t+1} - \min_{\mathcal{X} \times \Lambda} \mathcal{L}_{\mu, \nu} | \mathcal{F}_{t-1}] \leq \mathcal{L}_{\mu, \nu}^{t} - \min_{\mathcal{X} \times \Lambda} \mathcal{L}_{\mu, \nu} - \frac{\nu^3}{2\Upsilon_1^2} \|\nabla \mathcal{L}_t\|^2 + 2(\chi_t + \eta_t^2).
$$

By Robbins-Siegmund theorem (see Robbins and Siegmund (1971) or (Duflo, 1997, Theorem 1.3.12)), we conclude that $\sum_i \beta_t \|\nabla \mathcal{L}_t\|^2 < \infty$. Since $\sum_i \beta_t = \infty$ from (16), we have that $\lim_{t \to \infty} \|\nabla \mathcal{L}_t\| = 0$.

Furthermore, we note that

$$
\rho \tau \leq L \min(\Upsilon_{\mu, \nu}^2, \Upsilon_m^2) \cdot \eta_t^2.
$$

Summing over $t = 1$ to $\infty$, exchanging the expectation and summation by applying Fubini’s theorem (Durrett, 2019, Theorem 1.7.2), and noting that $\sum_j \eta_j^2 < \infty$, we obtain

$$
\mathbb{E} \left[ \sum_{t=1}^{\infty} \|\mathbf{x}_{t+1} - \mathbf{x}_t, \lambda_{t+1} - \lambda_t\|^2 \right] < \infty.
$$

This implies $\sum_{t=1}^{\infty} \|\mathbf{x}_{t+1} - \mathbf{x}_t, \lambda_{t+1} - \lambda_t\| < \infty$ almost surely and, thus, $\|\mathbf{x}_{t+1} - \mathbf{x}_t, \lambda_{t+1} - \lambda_t\| \to 0$ as $t \to \infty$ almost surely. Suppose for any run of the algorithm $\lim_{t \to \infty} \|\nabla \mathcal{L}_t\| = 0$, then we have $\limsup_{t \to \infty} \|\nabla \mathcal{L}_t\| = \epsilon > 0$. Then, there exist two index sequences $\{t_{1,i}\}, \{t_{2,i}\}$ with $t_{1,i+1} > t_{2,i} > t_{1,i}$ such that, for all $i = 1, 2, \ldots$,

$$
\|\nabla \mathcal{L}_{t_{1,i}}\| \geq \epsilon/2, \quad \|\nabla \mathcal{L}_j\| \geq \epsilon/3 \quad \text{for} \quad j = t_{1,i} + 1, \ldots, t_{2,i} - 1, \quad \|\nabla \mathcal{L}_{t_{2,i}}\| < \epsilon/3. \quad (C.11)
$$

Since $\sum_t \beta_t \|\nabla \mathcal{L}_t\|^2 < \infty$, we know

$$
\infty \geq \sum_{i=1}^{\infty} \sum_{j=t_{1,i}}^{t_{2,i}-1} \beta_j \|\nabla \mathcal{L}_j\|^2 \geq \frac{\epsilon^2}{9} \sum_{i=1}^{\infty} \sum_{j=t_{1,i}}^{t_{2,i}-1} \beta_j. \quad (C.12)
$$

Furthermore, by (C.10), we have

$$
\mathbb{E} \left[ \|\mathbf{x}_{t_{2,i}} - \mathbf{x}_{t_{1,i}}, \lambda_{t_{2,i}} - \lambda_{t_{1,i}}\| \right] \leq 4\Upsilon_K(\Upsilon_u \lor \Upsilon_m) \sum_{j=t_{1,i}}^{t_{2,i}-1} \eta_j 
$$

$$
\equiv 4\Upsilon_K(\Upsilon_u \lor \Upsilon_m) \left\{ \sum_{j=t_{1,i}}^{t_{2,i}-1} \beta_j + \sum_{j=t_{1,i}}^{t_{2,i}-1} \chi_j \right\}. \quad (9)
$$

Summing over $i = 1$ to $\infty$, and noting that $\sum_i \sum_{j=t_{1,i}}^{t_{2,i}-1} \beta_j < \infty$ by (C.12) and $\sum_i \sum_{j=t_{1,i}}^{t_{2,i}-1} \chi_j \leq \sum_{j=1}^{\infty} \chi_j < \infty$, we exchange the expectation and summation by applying Fubini’s theorem again. We know that the sequence $\{(\mathbf{x}_{t_{2,i}} - \mathbf{x}_{t_{1,i}}, \lambda_{t_{2,i}} - \lambda_{t_{1,i}})\}$ converges to zero as $t \to \infty$ with probability one. This contradicts with $\|\nabla \mathcal{L}_{t_{1,i}}\| \geq \epsilon/2$ and $\|\nabla \mathcal{L}_{t_{2,i}}\| < \epsilon/3$ in (C.11). We complete the proof.
C.5 Proof of Corollary 3.8

Applying Lemma 3.6 and taking full expectation, we know for some constants $h_1, h_2 > 0$,

$$\mathbb{E}[\mathcal{L}^{t+1}_{\mu, \nu}] \leq \mathbb{E}[\mathcal{L}^t_{\mu, \nu}] - h_1 \beta_t \mathbb{E}[\|\nabla \mathcal{L}_t\|^2] + h_2 (\chi_t + \eta_t^2), \quad \forall t \geq 0.$$  

Rearranging the inequality and summing over $t = 0$ to $T_\epsilon - 1$, we obtain

$$h_1 \sum_{t=0}^{T_\epsilon-1} \mathbb{E}[\|\nabla \mathcal{L}_t\|^2] \leq \sum_{t=0}^{T_\epsilon-1} \frac{1}{\beta_t} \left( (\mathbb{E}[\mathcal{L}^t_{\mu, \nu}] - \min_{\chi \times \Lambda} \mathcal{L}_{\mu, \nu}) - (\mathbb{E}[\mathcal{L}^{t+1}_{\mu, \nu}] - \min_{\chi \times \Lambda} \mathcal{L}_{\mu, \nu}) \right) + h_2 \sum_{t=0}^{T_\epsilon-1} \frac{\chi_t + \eta_t^2}{\beta_t}.$$

Denoting $\Delta \mathcal{L}_{\mu, \nu} = \max_{\chi \times \Lambda} \mathcal{L}_{\mu, \nu} - \min_{\chi \times \Lambda} \mathcal{L}_{\mu, \nu}$, we further have

$$h_1 \sum_{t=0}^{T_\epsilon-1} \mathbb{E}[\|\nabla \mathcal{L}_t\|^2] \leq (\Delta \mathcal{L}_{\mu, \nu} \lor h_2) \left\{ \frac{1}{\beta_0} + \sum_{t=1}^{T_\epsilon-1} \left( \frac{1}{\beta_t} - \frac{1}{\beta_{t-1}} \right) + \sum_{t=0}^{T_\epsilon-1} \frac{\chi_t + \eta_t^2}{\beta_t} \right\}$$

For the last term on the right hand side, we have

$$\sum_{t=0}^{T_\epsilon-1} \frac{\chi_t + \eta_t^2}{\beta_t} = \sum_{t=0}^{T_\epsilon-1} \left\{ (t+1)^{a+b} + (t+1)^a \left( (t+1)^{-2a} + 2(t+1)^{-(a+b)} + (t+1)^{-2b} \right) \right\}$$

$$\leq \sum_{t=0}^{T_\epsilon-1} (t+1)^{a+b} + 4 \sum_{t=1}^{T_\epsilon-1} (t+1)^{-a} = 5 + \sum_{t=1}^{T_\epsilon-1} (t+1)^{a+b} + 4(t+1)^{-a}$$

$$\leq 5 + \int_0^{T_\epsilon-1} (t+1)^{a+b} + 4(t+1)^{-a} \, dt \quad \text{(by the convexity of } x^p \text{ with } p < 0)$$

$$\leq \begin{cases} 
5 + \frac{T_\epsilon^{1+a+b}}{1+a-b} + \frac{4T_\epsilon^{1-a}}{1-a} & \text{if } 1 + a > b, \\
5 + \log(T_\epsilon) + \frac{4T_\epsilon}{1-a} & \text{if } 1 + a = b, \\
5 + \frac{1}{1-a} + \frac{4T_\epsilon}{1-a} & \text{if } 1 + a < b. 
\end{cases}$$

Combining the above two displays, dividing $T_\epsilon$ on both sides, and using “$\lesssim$” to neglect constant factors (i.e., not depending on $T_\epsilon$), we complete the proof by noting:

$$\epsilon^2 \leq \left( \frac{1}{T_\epsilon} \sum_{t=0}^{T_\epsilon-1} \mathbb{E}[\|\nabla \mathcal{L}_t\|] \right)^2 \leq \frac{1}{T_\epsilon} \sum_{t=0}^{T_\epsilon-1} \mathbb{E}[\|\nabla \mathcal{L}_t\|]^2 \leq \frac{1}{T_\epsilon} \sum_{t=0}^{T_\epsilon-1} \mathbb{E}[\|\nabla \mathcal{L}_t\|^2]$$

$$\lesssim \begin{cases} 
\frac{1}{T_\epsilon^{1+a}} + \frac{1}{T_\epsilon^{1-b}} + \frac{1}{T_\epsilon^a} & \text{if } 1 + a > b, \\
\frac{1}{T_\epsilon^{1+a}} + \frac{1}{T_\epsilon^a} & \text{if } 1 + a = b, \\
\frac{1}{T_\epsilon^{1-a}} + \frac{1}{T_\epsilon^a} & \text{if } 1 + a < b, 
\end{cases}$$

$$\lesssim \begin{cases} 
\frac{1}{T_\epsilon^{(1+b)-a}} + \frac{1}{T_\epsilon^a} \lesssim \frac{1}{T_\epsilon^{a \wedge (1-b) \vee (b-a)}}. 
\end{cases}$$

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D Proofs of Section 4

D.1 Proof of Lemma 4.1

For notational brevity, we let \( \omega_t = (x_t - x^*, \lambda_t - \lambda^*) \). By the scheme of Algorithm 1, we have

\[
\omega_{t+1} = \omega_t + \alpha_t z_{t, \tau} = \omega_t + \varphi_t z_{t, \tau} + (\alpha_t - \varphi_t) z_{t, \tau}
\]

\[
= \omega_t + \varphi_t (I + C_t) \tilde{z}_t + \varphi_t \{ z_{t, \tau} - (I + C_t) \tilde{z}_t \} + (\alpha_t - \varphi_t) z_{t, \tau}
\]

\[
= \omega_t - \varphi_t (I + C_t) K_t^{-1} \nabla L_t + \varphi_t \{ z_{t, \tau} - (I + C_t) \tilde{z}_t \} + (\alpha_t - \varphi_t) z_{t, \tau}
\]

\[
= \omega_t - \varphi_t (I + C_t) K_t^{-1} \nabla L_t - \varphi_t (I + C_t) K_t^{-1} (\nabla L_t - \nabla \tilde{L}_t) + \varphi_t \{ z_{t, \tau} - (I + C_t) \tilde{z}_t \}
\]

\[
+ (\alpha_t - \varphi_t) z_{t, \tau}.
\]

We apply the above equation recursively and show the result. Moreover, under Assumptions 3.2 and 3.3, we know \( \mathbb{E}[\tilde{g}_t \mid \mathcal{F}_{t-1}] = 0 \) and \( \mathbb{E}[z_{t, \tau} - (I + C_t) \tilde{z}_t \mid \mathcal{F}_{t-1}] = 0 \). Thus, \( \mathbb{E}[\theta^i \mid \mathcal{F}_{i-1}] = 0 \) and \( \theta^i \) is a martingale difference.

D.2 Proof of Lemma 4.2

Let us denote \( \text{rank}(S) = r \). Since \( K_t, K^* \) have full rank, \( \text{rank}(K_t S) = \text{rank}(K^* S) = r \). Let \( K^* = E D F^T \) be the truncated singular value decomposition of \( K^* S \). We have

\[
E \in \mathbb{R}^{(d+m) \times r}, \quad F \in \mathbb{R}^{q \times r}, \quad E^T E = F^T F = I, \quad D = \text{diag}(D_1, \ldots, D_r) \quad \text{with} \quad D_1 \geq \ldots \geq D_r > 0.
\]

Similarly, we let \( K_t S = E^t D^t (E^t)^T \). By direct calculation, we have

\[
\| K_t S (S^T K_t^2 S)^\dagger S^T K_t - K^* S (S^T (K^*)^2 S)^\dagger S^T K^* \| = \| EE^T - E^t (E^t)^T \|.
\]

Define the principle angles \( \theta_p \) between \( \text{span}(E) \) and \( \text{span}(E^t) \) to be \( \theta_p = (\theta_{p,1}, \ldots, \theta_{p,r}) \), so that \( E^T E^t \) has the singular value decomposition \( E^T E^t = P \cos(\theta_p) Q^T \), where \( P, Q \in \mathbb{R}^{r \times r} \) are orthonormal matrices and \( \cos(\theta_{p,1}), \ldots, \cos(\theta_{p,r}) \) (similar for \( \sin(\theta_p) \)). We further let \( E^\perp \in \mathbb{R}^{(d+m) \times (d+m-r)} \) be the complement of \( E \), and express \( E^t \) as

\[
E^t = EA + E^\perp B.
\]

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Then, \( E^T E' = A = P \cos(\theta_p)Q^T \) and \( I = (E')^T E' = A^T A + B^T B \). By the above formulation,

\[
\| E^T - E'(E')^T \| = 2 \left\| (E, E^\perp) \left(I - AA^T - AB^T \right) \left(E^T \right) \right\| = \left\| \left(I - AA^T - AB^T \right) \right\| \\
\leq \left\| \left(I - AA^T \right) \left(0 \quad 0 \right) \right\| + \left\| \left(0 \quad AB^T \right) \right\|
\leq \max\{ \| I - AA^T \|, \| BB^T \| \} + \| AB^T \|
= \max\{ \| I - AA^T \|, \| I - A^T A \| \} + \| AB^T \|
= \| \sin(\theta_p) \|^2 + \sqrt{P \cos(\theta_p) \sin^2(\theta_p) \cos(\theta_p) P^T} \|
= \| \sin(\theta_p) \|^2 + \| \sin(\theta_p) \cos(\theta_p) \| \leq 2 \| \sin(\theta_p) \|. \tag{D.3}
\]

On the other hand, by Wedin’s sin(\( \Theta \)) theorem (Wedin, 1972, (3.1)), we know

\[
\| \sin(\theta_p) \| \leq \frac{\| (K^* - K_t) S \|}{D_r}. \tag{D.4}
\]

We let \( F_r \) be the \( r \)-th column of \( F \) and have \( D_r^2 = F_r^T S^T (K^*)^2 S F_r \geq (\sigma_{\min}(K^*))^2 F_r^T S^T S F_r \). Since \( \ker(K^* S) = \ker(S) \) and \( F_r \in \ker^\perp(K^* S) \), we know \( F_r \in \ker^\perp(S) = \text{span}(S^T) \). Thus, \( F_r^T S^T S F_r \geq \lambda_{\min}^+(S^T S) \), where \( \lambda_{\min}^+(S^T S) = (\sigma_{\min}(S))^2 \) is the least positive eigenvalue of \( S^T S \). Therefore, we have

\[
D_r \geq \sigma_{\min}(K^*) \sigma_{\min}^+(S). \tag{D.5}
\]

Combining all above derivations, we obtain

\[
\| K_t S (S^T K_t^2 S)^\dagger S^T K_t - K^* S (S^T (K^*)^2 S)^\dagger S^T K^* \| \overset{(D.1)}{=} \| E^T - E'(E')^T \| \overset{(D.3)}{\leq} 2 \| \sin(\theta_p) \| \overset{(D.4)}{\leq} \frac{2 \| K_t - K^* \| \cdot \| S \|}{D_r} \overset{(D.5)}{\leq} \frac{2 \| K_t - K^* \| \cdot \| S \|}{\sigma_{\min}(K^*) \cdot \sigma_{\min}^+(S)}. \]

This completes the proof.

### D.3 Proof of Corollary 4.4

Denote \( A_t = I - \mathbb{E}[K_t S (S^T K_t^2 S)^\dagger S^T K_t \mid x_t, \lambda_t] \) and \( A^* = I - \mathbb{E}[K^* S (S^T (K^*)^2 S)^\dagger S^T K^*] \). We have

\[
\| C_t - C^* \| = \| A_t - (A^*)^\perp \| \leq \| A_t^{-1} - (A_t - A^*) \| + \| (A_t^{-1} - (A^*)^{-1}) A^* \|
\leq \| A_t - A^* \| + \| A_t^{-1} - (A^*)^{-1} \| \ (\| A_t \| \lor \| A^* \| \leq 1)
\leq \tau \| A_t - A^* \| \leq \tau \mathbb{E} \left[ \| K_t S (S^T K_t^2 S)^\dagger S^T K_t - K^* S (S^T (K^*)^2 S)^\dagger S^T K^* \| \mid x_t, \lambda_t \right]
\leq \frac{2\tau \| K_t - K^* \| \mathbb{E} \| S \| \| S^\dagger \|}{\sigma_{\min}(K^*) \cdot \sigma_{\min}^+(S)} \leq \frac{2\tau \gamma_S}{\sigma_{\min}(K^*) \cdot \sigma_{\min}^+(S)} \| K_t - K^* \| \ (\text{by Assumption 4.3}).
\]

This completes the proof.
D.4 Proof of Theorem 4.5

We present some lemmas that bound $I_{1,t}$, $I_{2,t}$, $I_{3,t}$, $K_t$ in (18a), (18b), (18c), (5), respectively. The proofs of these lemmas are presented in Appendix D.4.1 – D.4.4.

**Lemma D.1.** Under Assumptions 3.1, 3.2(15a, 15e), 3.3, and $(x_t, \lambda_t) \to (x^*, \lambda^*)$, suppose

$$
\lim_{t \to \infty} t (1 - \varphi_{t-1}/\varphi_t) = \varphi < 0, \quad \lim_{t \to \infty} t \varphi_t = \bar{\varphi} \in (0, \infty], \quad 1 - \rho^* + \varphi/\bar{\varphi} > 0. \tag{D.6}
$$

Then, for any $\nu > 0$,

$$
I_{1,t} = o\left(\sqrt{\varphi_t \{\log (1/\varphi_t)\}^{1+\nu}}\right) \quad \text{a.s.} \tag{D.7}
$$

Furthermore, if (15a) is strengthened to (15b), then we have

(a): (asymptotic rate) $I_{1,t} = O(\sqrt{\varphi_t \log (1/\varphi_t)})$ a.s.

(b): (asymptotic normality) $\sqrt{1/\varphi_t} \cdot I_{1,t} \overset{d}{\to} N(0, \Xi^*)$ where $\Xi^*$ is from (25).

(c): (Berry-Esseen bound) For any vector $w = (w_x, w_\lambda) \in \mathbb{R}^{d+m}$ such that $w^T \Xi^* w \neq 0$,

$$
\sup_{z \in \mathbb{R}} \left| P \left( \frac{\sqrt{1/\varphi_t} \cdot w^T I_{1,t}}{\sqrt{w^T \Xi^* w}} \leq z \right) - P (N(0,1) \leq z) \right| = O(\sqrt{\varphi_t \log (1/\varphi_t)}). \tag{D.8}
$$

**Lemma D.2.** Under the conditions of Lemma D.1 with (15a) and assume for some $\omega > 0$,

$$
\lim_{t \to \infty} t (1 - \chi_{t-1}/\chi_t) = \chi, \quad 2(1 - \rho^*) + (2\chi - \varphi)/\bar{\varphi} > 0, \quad \{\log (1/\chi_t)\}^{1+\omega} = O(1/\varphi_t). \tag{D.9}
$$

Then, $I_{2,t} = O(\chi_t/\varphi_t)$, a.s.

**Lemma D.3.** Under the conditions of Lemma D.2, we have for any $\nu > 0$,

$$
I_{3,t} = o(\sqrt{\varphi_t \{\log (1/\varphi_t)\}^{1+\nu}}) + o(\chi_t/\varphi_t) = o(I_{1,t} + I_{2,t}) \quad \text{a.s.} \tag{D.10}
$$

If (15a) is strengthened to (15b), the above result holds with $\nu = 0$.

We apply the above results. We first check the conditions (D.6) and (D.8). Since $\varphi_t = (\beta_t + \eta_t)/2 = \beta_t + \chi_t/2$ and $\chi_t = o(\beta_t)$, we know $\beta_t \leq \varphi_t \leq \beta_t + o(\beta_t)$ and $\lim t \varphi_t = \lim t \beta_t = \bar{\beta}$. Furthermore, we have

$$
\lim_{t \to \infty} t \left( 1 - \frac{\varphi_{t-1}}{\varphi_t} \right) = \lim_{t \to \infty} t \left( 1 - \frac{\beta_{t-1}}{\beta_t} + \frac{\beta_{t-1}}{\beta_t} \left \{ 1 - \frac{2 + \chi_{t-1}/\beta_{t-1}}{2 + \chi_t/\beta_t} \right \} \right)
$$

$$
= \beta + \lim_{t \to \infty} t \left( 1 - \frac{2 + \chi_{t-1}/\beta_{t-1}}{2 + \chi_t/\beta_t} \right) = \beta + \frac{1}{2} \lim_{t \to \infty} t \left( \frac{\chi_t}{\beta_t} - \frac{\chi_{t-1}}{\beta_{t-1}} \right) \quad \text{since} \frac{\chi_t}{\beta_t} \to 0.
$$

$$
= \beta + \frac{1}{2} \lim_{t \to \infty} \frac{\chi_t}{\beta_t} \cdot t \left( 1 - \frac{\chi_{t-1}}{\chi_t} \cdot \frac{\beta_t}{\beta_{t-1}} \right) = \beta + \frac{\chi - \beta}{2} \lim_{t \to \infty} \frac{\chi_t}{\beta_t} = \beta,
$$

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and for any $\omega \geq 0$,
\[
\lim_{t \to \infty} t \left(1 - \frac{\varphi_{t-1}\{\log(1/\chi_{t-1})\}^{1+\omega}}{\varphi_t\{\log(1/\chi_t)\}^{1+\omega}}\right) \overset{\text{Lem. A.1}}{=} \beta + (1 + \omega) \lim_{t \to \infty} t \left(1 - \frac{\log(1/\chi_{t-1})}{\log(1/\chi_t)}\right)
\]
\[
= \beta + (1 + \omega) \lim_{t \to \infty} t \left(\frac{\log(\chi_{t-1}/\chi_t)}{\log(1/\chi_t)}\right) = \beta + (1 + \omega) \lim_{t \to \infty} t \left(\frac{\chi_{t-1}/\chi_t + O\left(\frac{(\chi_{t-1}/\chi_t)^2}{\chi_t^2}\right)}{\log(1/\chi_t)}\right)
\]
\[
= \beta - (1 + \omega) \chi \lim_{t \to \infty} 1/\log(1/\chi_t) = \beta.
\]

The above derivations show that $\varphi = \beta$, $\bar{\varphi} = \bar{\beta}$, and $\{\log(1/\chi_t)\}^{1+\omega} = o(1/\varphi_t)$. Thus, (22) implies (D.6) and (D.8). The convergence rate of $(x_t, \lambda_t)$ comes from Lemmas 4.1, 4.1, 4.2, 4.3 and the fact that $\beta_t \leq \varphi_t \leq 2\beta_t$. This completes the proof.

D.4.1 Proof of Lemma D.1

We need a preparation lemma. Recall that we suppose $(x^*, \lambda^*)$ is a local solution of (1) with $G^*$ being full row rank and $\nabla_x^2 G^*$ being positive definite in the null space $\{x \in \mathbb{R}^d : G^* x = 0\}$.

**Lemma D.4.** Under Assumptions 3.1, 3.2(15d) and $(x_t, \lambda_t) \to (x^*, \lambda^*)$, we have $\frac{1}{t} \sum_{i=0}^{t-1} \nabla^2 L_i \to \nabla_x^2 G^*$ as $t \to \infty$. Further, with a small $\gamma_{R\hat{\Pi}}$ and a large $\Gamma_B$, $\Delta_t = 0$ for all large enough $t$.

Let $I + C^* = U \Sigma U^T$ with $\Sigma = \text{diag}(\sigma_1, \ldots, \sigma_d)$. Be the eigenvalue decomposition. Then,
\[
\mathcal{I}_{1,t} \overset{(18a)}{=} \sum_{i=0}^{t} \prod_{j=i+1}^{t} \{I - \varphi_j(I + C^*)\} \varphi_j^i \mathbf{1} = U \sum_{i=0}^{t} \prod_{j=i+1}^{t} \{I - \varphi_j \Sigma\} \varphi_i U^T \mathbf{1}.
\]

Since $\mathbb{E}[\mathbf{1} | F_{i-1}] = 0$, $\mathcal{I}_{1,t}$ is a martingale. Therefore, we aim to apply the strong law of large number (Duflo, 1997, Theorem 1.3.15), the central limit theorem (Duflo, 1997, Corollary 2.1.10), and the Berry-Esseen inequality (Fan, 2019, Theorem 2.1) for martingales to show each result. We compute the conditional covariance of $\mathcal{I}_{1,t}$, which is defined as (Duflo, 1997, Proposition 1.3.7)
\[
\langle \mathcal{I}_{1,t} \rangle := U \sum_{i=0}^{t} \prod_{j=i+1}^{t} \{I - \varphi_j \Sigma\} \varphi_i^2 U^T \mathbb{E}[\mathbf{1}^{(\mathbf{1})}]^T | F_{i-1} | U ( \prod_{j=i+1}^{t} \{I - \varphi_j \Sigma\} )^T U^T. \tag{D.9}
\]

For the term $\mathbb{E}[\mathbf{1}^{(\mathbf{1})}]^T | F_{i-1}]$, we note that
\[
\mathbb{E}[\mathbf{1}^{(\mathbf{1})}]^T | F_{i-1}] \overset{(19b)}{=} \mathbb{E}\left[ \{(I + C_i) K_i^{-1} (\nabla L_i - \nabla L_i) - \{z_{i,\tau} - (I + C_i) \bar{z}_i\}\} \right] = (I + C_i) K_i^{-1} (\nabla L_i - \nabla L_i) - (I + C_i) \bar{z}_i.
\]

(C.6)
\[
\mathbb{E}[\mathbf{1}^{(\mathbf{1})}]^T | F_{i-1}] = (I + C_i) K_i^{-1} \mathbb{E}\left[(\nabla L_i - \nabla L_i)(\nabla L_i - \nabla L_i)^T | F_{i-1} \right] K_i^{-1} (I + C_i) + \mathbb{E}\left[\{z_{i,\tau} - (I + C_i) \bar{z}_i\} \{z_{i,\tau} - (I + C_i) \bar{z}_i\}^T | F_{i-1} \right] = \mathcal{J}_{1,i} + \mathcal{J}_{2,i}.
\]

For the term $\mathcal{J}_{1,i}$, we apply Assumption 3.2 and have $\mathbb{E}\left[(\bar{g}_i - \nabla f_i)(\bar{g}_i - \nabla f_i)^T | F_{i-1} \right] = \mathbb{E}[\bar{g}_i \bar{g}_i^T | F_{i-1} | - \nabla f_i \nabla f_i^T]).$ We also note that
\[
\|\mathbb{E}[\bar{g}_i \bar{g}_i^T - \nabla f(x^*; \xi) \nabla f(x^*; \xi) | F_{i-1}]\|
\leq 2\mathbb{E}\left[\|\bar{g}_i - \nabla f(x^*; \xi)\| \cdot \|\bar{g}_i\| | F_{i-1} \right] + \mathbb{E}\left[\|\bar{g}_i - \nabla f(x^*; \xi)\|^2 | F_{i-1} \right]
\]
\[
\leq 2\sqrt{\mathbb{E}\left[\|\bar{g}_i - \nabla f(x^*; \xi)\|^2 | F_{i-1} \right]} \mathbb{E}\left[\|\bar{g}_i\|^2 | F_{i-1} \right] + \mathbb{E}\left[\|\bar{g}_i - \nabla f(x^*; \xi)\|^2 | F_{i-1} \right],
\]

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and
\[
\mathbb{E} \left[ \| \bar{g}_i - \nabla f(x^*; \xi) \|^2 \mid \mathcal{F}_{i-1} \right] \leq \mathbb{E} \left[ \sup_{x \in X} \| \nabla^2 f(x; \xi) \|^2 \right] \cdot \| x_i - x^* \|^2 \quad \text{(15e)}
\]

By Assumptions 3.1, 3.2(15a), we suppose \( \| \nabla f_i \| \leq \Upsilon_i \) (we abuse \( \Upsilon_i \) from (C.2)), and obtain
\[
\mathbb{E}[\| \bar{g}_i \|^2 \mid \mathcal{F}_{i-1}] = \| \nabla f_i \|^2 + \mathbb{E}[\| \bar{g}_i - \nabla f_i \|^2 \mid \mathcal{F}_{i-1}] \leq \Upsilon_i^2 + \Upsilon_m \leq 2(\Upsilon_i^2 \vee \Upsilon_m).
\]

Combining the above three displays, we have
\[
\mathbb{E}[\| \bar{g}_i g_i^T - \nabla f(x^*; \xi) \nabla^T f(x^*; \xi) \mid \mathcal{F}_{i-1}] \leq 2\sqrt{2\Upsilon_m(\Upsilon_i \vee \sqrt{\Upsilon_m})} (\| x_i - x^* \| + \| x_i - x^* \|^2) \to 0. \quad \text{(D.11)}
\]

This implies that
\[
\lim_{i \to \infty} \mathbb{E}[(\bar{g}_i - \nabla f_i)(\bar{g}_i - \nabla f_i)^T \mid \mathcal{F}_{i-1}] = \mathbb{E}[\nabla f(x^*; \xi) \nabla^T f(x^*; \xi) - \nabla f(x^*) \nabla^T f(x^*)]. \quad \text{(D.12)}
\]

Furthermore, by Lemma D.4 we know \( K_i \to K^* \) as \( i \to \infty \). Since \( \| K_i S(S^T K_i^2 S)^1 S^T K_i \| \leq 1 \), we apply dominated convergence theorem (Durrett, 2019, Theorem 1.6.7) and Lemma 4.2, and have \( \mathbb{E}[K_i S(S^T K_i^2 S)^1 S^T K_i \mid x_i, \lambda_i] \to \mathbb{E}[K^* S(S^T (K^*)^2 S)^1 S^T K^*] \) as \( i \to \infty \). Here, the expectation is taken over randomness of \( S \). Thus, \( C_i \to C^* \). By the definition (21), we obtain
\[
\mathcal{J}_{1,i} = (I + C^*) \Omega^*(I + C^*) + O(\mathcal{K}_{1,i}) \quad \text{(D.13)}
\]

with \( \mathcal{K}_{1,i} \to 0 \) as \( i \to \infty \) almost surely. For the term \( \mathcal{J}_{2,i} \), we apply (C.1) and define \( \tilde{C}_i := - \prod_{j=0}^{i-1} C_{i,j} \).

Then, we have
\[
\mathcal{J}_{2,i} = \mathbb{E}[(\tilde{C}_i - C_i) \tilde{z}_i \tilde{z}_i^T (\tilde{C}_i^T - C_i^T) \mid \mathcal{F}_{i-1}] \overset{(a)}{=} \mathbb{E}[(\tilde{C}_i - C_i) K_i^{-1} \nabla L_i \nabla^T L_i K_i^{-1} (\tilde{C}_i^T - C_i^T) \mid \mathcal{F}_{i-1}]
\]

For the last two terms, we apply the tower property of conditional expectation by first conditioning on the randomness of \( \zeta_i \) to take expectation over the randomness of \( \xi_i \), and then taking expectation over the randomness of \( \zeta_i \). In particular, we have (similar for the second last term in (D.14))
\[
\mathbb{E}[(\tilde{C}_i - C_i) K_i^{-1} \nabla L_i \nabla^T L_i K_i^{-1} (\tilde{C}_i^T - C_i^T) \mid \mathcal{F}_{i-1}]
\]

For the second term in (D.14), it converges to zero almost surely as \( i \to \infty \) since \( \| \tilde{C}_i \| \vee \| C_i \| \leq 1, \| K_i^{-1} \| \leq \Upsilon_K \), and \( \nabla L_i \to 0 \). For the first term in (D.14), we have
\[
\begin{align*}
\mathbb{E}[(\tilde{C}_i - C_i) K_i^{-1} (\nabla L_i - \nabla L_i) (\nabla L_i - \nabla L_i)^T K_i^{-1} (\tilde{C}_i^T - C_i^T) \mid \mathcal{F}_{i-1}] \\
= \mathbb{E}[(\tilde{C}_i - C_i) K_i^{-1} \nabla L_i] \mathbb{E}[(\nabla L_i - \nabla L_i)^T K_i^{-1} (\tilde{C}_i^T - C_i^T) \mid \mathcal{F}_{i-1}] \\
= \mathbb{E}[(\tilde{C}_i - C_i) K_i^{-1} \nabla L_i] \mathbb{E}[(\nabla L_i - \nabla L_i)^T K_i^{-1} (\tilde{C}_i^T - C_i^T) \mid \mathcal{F}_{i-1}] \to 0. \quad \text{(20)}
\end{align*}
\]
Again, the convergence here is due to the dominated convergence theorem, (D.12), and $K_i \to K^*$; and the expectation is taken over the randomness of $\tau$ sketch matrices $S_1, \ldots, S_\tau$ only. Thus, combining the above two displays with (D.14), we have

$$J_{2,i} = \mathbb{E}[C^\ast \Omega^\ast (\bar{C}^\ast)^T] - C^\ast \Omega^\ast C^\ast + O(K_{2,i})$$

with $K_{2,i} \to 0$ as $i \to \infty$ almost surely. Combining (D.15), (D.13), and (D.10), we obtain

$$\mathbb{E}[\theta^i(\theta^i) \mid \mathcal{F}_{i-1}] = \mathbb{E}[(I + \bar{C}^\ast)^\ast (I + \bar{C}^\ast)^T] + O(K_{1,i} + K_{2,i}).$$

By the definition of $\langle \mathcal{I}_i \rangle_t$ in (D.9), let us denote $\Gamma := U^T \mathbb{E}[(I + \bar{C}^\ast)^\ast (I + \bar{C}^\ast)^T]U$. For any $k, l \in \{1, \ldots, d + m\}$, the $(k, l)$ entry of the matrix $U^T \langle \mathcal{I}_i \rangle_t U$ can be written as

$$[U^T \langle \mathcal{I}_i \rangle_t U]_{k,l} = \sum_{i=0}^{t} \prod_{j=i+1}^{t} (1 - \varphi_j \sigma_k)(1 - \varphi_j \sigma_l) \varphi_i^2 (\Gamma_{kl} + r_{i,kl}),$$

where $r_{i,kl} \to 0$ as $i \to \infty$ almost surely. By Lemma 3.4(b) and the fact that $C_i \to C^\ast$ as $i \to \infty$, we know $\|C^\ast\| \leq \rho^\ast$. Since $C^\ast \preceq 0$, we have $0 < 1 - \rho^\ast \leq \sigma_i \leq 1$ for $i = 1, \ldots, d + m$, which implies $\sigma_k + \sigma_l \geq 2(1 - \rho^\ast)$. Using the condition (D.6), Lemmas A.2 and A.3, we obtain $[U^T \langle \mathcal{I}_i \rangle_t U]_{k,l}/\varphi_i \to \Gamma_{kl}/(\sigma_k + \sigma_l + \varphi/\varphi_i)$ as $t \to \infty$ almost surely. Thus, by (25), we have

$$\langle \mathcal{I}_i \rangle_t / \varphi_i \overset{a.s.}{\rightarrow} U(\Theta \circ \Gamma)U^T = \Xi^\ast.$$  \tag{D.16}

Then, (Duflo, 1997, Theorem 1.3.15) indicates (D.7) holds. This shows the first part of the results. For the second part of the results, we assume the condition (15b) and have

$$\mathbb{E}[\|\Theta^i\|^3 \mid \mathcal{F}_{i-1}] \overset{(19b)}{\leq} 4 \mathbb{E}[\|(I + C_i)K_i^{-1}(\nabla \mathcal{L}_i - \nabla \mathcal{L}_i)\|^3 \mid \mathcal{F}_{i-1}] + \mathbb{E}[\|z_{i,\tau} - (I + C_i)\bar{z}_i\|^3 \mid \mathcal{F}_{i-1}]
\overset{(C.1)}{\leq} 4 \left(8Y^3_K \mathbb{E}[\|g_i - \nabla f_i\|^3 \mid \mathcal{F}_{i-1}] + \mathbb{E}[\|(\bar{C}_i - C_i)\bar{z}_i\|^3 \mid \mathcal{F}_{i-1}]\right) \ (\|C_i\| \leq 1, \|K_i^{-1}\| \leq Y_K)
\overset{(15b)}{\leq} 4 \left(8Y^3_K \gamma_m + 8\mathbb{E}[\|\bar{z}_i\|^3 \mid \mathcal{F}_{i-1}]\right) \ (\|\bar{C}_i\| \vee \|C_i\| \leq 1)
\overset{(5)}{\leq} 4 \left(8Y^3_K \gamma_m + 8Y^3_K \mathbb{E}[\|\nabla \mathcal{L}_i\|^3 \mid \mathcal{F}_{i-1}]\right) \ (\|K_i^{-1}\| \leq Y_K)
\overset{(5)}{\leq} 4 \left(8Y^3_K \gamma_m + 8y^3_K \{4\|\nabla \mathcal{L}_i\|^3 + 4\mathbb{E}[\|g_i - \nabla f_i\|^3 \mid \mathcal{F}_{i-1}]\}\right) \ (\text{also use (C.2)).}
\overset{(15b)}{\leq} 4 \left(8Y^3_K \gamma_m + 8Y^3_K \{4\gamma^3_m + 4\gamma_m\}\right).$$  \tag{D.17}

Thus, $\theta^i$ has bounded third moment; and (Wang, 1995, pp. 554) together with (D.16) give the result (a). For (b), we verify the Lindeberg’s condition. For any $\varepsilon > 0$, we have

$$\frac{1}{\varphi_i} \sum_{i=0}^{t} \mathbb{E}[\| \prod_{j=i+1}^{t} \{I - \varphi_j (I + C^\ast)\} \varphi_i \Theta^i \|^2 \cdot 1_{\|\prod_{j=i+1}^{t} (I - \varphi_j (I + C^\ast)) \varphi_i \Theta^i \| \geq \varepsilon \sqrt{\varphi_i} \mid \mathcal{F}_{i-1}}$$

$$\leq \frac{1}{\varepsilon \varphi_i^{3/2}} \sum_{i=0}^{t} \mathbb{E}[\| \prod_{j=i+1}^{t} \{I - \varphi_j (I + C^\ast)\} \varphi_i \Theta^i \|^3 \mid \mathcal{F}_{i-1}]
= \frac{1}{\varepsilon \varphi_i^{3/2}} \sum_{i=0}^{t} \mathbb{E}[\| \prod_{j=i+1}^{t} \{I - \varphi_j \Delta \} \varphi_i U^T \Theta^i \|^3 \mid \mathcal{F}_{i-1}].$$

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To show the right hand side converges to zero, it suffices to show that each entry of the vector on the right hand side converges to zero. In particular, we show for any $1 \leq k \leq d + m,$

$$
\frac{1}{\varepsilon \varphi_t^{3/2}} \sum_{i=0}^{t} \prod_{j=i+1}^{t} |1 - \varphi_j \sigma_k|^3 \varphi_i^3 \mathbb{E}[|U^T \theta_i|^3 | \mathcal{F}_{i-1}] \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty.
$$

By (D.17) and $\mathbb{E}[|U^T \theta_i|^3 | \mathcal{F}_{i-1}] \leq \mathbb{E}[|\theta_i|^3 | \mathcal{F}_{i-1}]$, we only show $\sum_{i=0}^{t} \prod_{j=i+1}^{t} |1 - \varphi_j \sigma_k|^3 \varphi_i^3 = o(\varphi_t^{3/2})$. Without loss of generality, we suppose $1 - \varphi_j \sigma_k \geq 0$ for all $j \geq 1$ and show

$$
\sum_{i=0}^{t} \prod_{j=i+1}^{t} (1 - \varphi_j \sigma_k)^3 \varphi_i^3 = o(\varphi_t^{3/2}). \quad (D.18)
$$

Otherwise, since $\varphi < 0$ from (D.6), Lemma A.2 shows that $\varphi_i \rightarrow 0$. Thus, there exists $\bar{t}$ such that $1 - \varphi_j \sigma_k \geq 0$, $\forall j \geq \bar{t}$. Then,

$$
\sum_{i=0}^{t} \prod_{j=i+1}^{t} |1 - \varphi_j \sigma_k|^3 \varphi_i^3 = \sum_{i=0}^{\bar{t} - 2} \prod_{j=i+1}^{t} |1 - \varphi_j \sigma_k|^3 \varphi_i^3 + \sum_{i=\bar{t}-1}^{t} \prod_{j=i+1}^{t} (1 - \varphi_j \sigma_k)^3 \varphi_i^3
$$

$$
= \prod_{j=\bar{t}}^{t} (1 - \varphi_j \sigma_k)^3 \sum_{i=0}^{\bar{t}-2} \prod_{j=i+1}^{\bar{t}-1} |1 - \varphi_j \sigma_k|^3 \varphi_i^3 + \sum_{i=\bar{t}-1}^{t} \prod_{j=i+1}^{t} (1 - \varphi_j \sigma_k)^3 \varphi_i^3
$$

$$
= \sum_{i=\bar{t}-1}^{t} \prod_{j=i+1}^{t} (1 - \varphi_j \sigma_k)^3 (\varphi_i')^3, \quad (D.19)
$$

where

$$
\varphi_i' = \left( \sum_{i=0}^{\bar{t}-1} \prod_{j=i+1}^{\bar{t}-1} |1 - \varphi_j \sigma_k|^3 \varphi_i^3 + \varphi_{\bar{t}-1}^3 \right)^{1/3}, \quad \text{and} \quad \varphi_i' = \varphi_i, \quad \forall i \geq \bar{t}.
$$

Note that (D.19) has the same form as (D.18), and $\varphi_i'$ differs from $\varphi_i$ only at $i = \bar{t} - 1$. Thus, (D.19) and (D.18) have the same limit. For (D.18), we apply Lemma A.1 and observe that

$$
\lim_{i \rightarrow \infty} i \left( 1 - \varphi_i^2 / \varphi_i^2 \right) \overset{(D.6)}{=} 2 \varphi \quad \text{and} \quad 3 \sigma_k + 2 \varphi / \sqrt{\varphi} \overset{(D.6)}{>} 0.
$$

Thus, Lemma A.3 suggests that

$$
\sum_{i=0}^{t} \prod_{j=i+1}^{t} (1 - \varphi_j \sigma_k)^3 \varphi_i^3 = O(\varphi_t^2).
$$

This verifies (D.18) and further verifies the Lindeberg’s condition. Thus, the central limit theorem of martingale in (Duflo, 1997, Corollary 2.1.10) leads to (b). For (c), we apply (Fan, 2019, Theorem 2.1) with $\epsilon = \sqrt{\varphi_t}$, $\delta = 0$, $\rho = 1$ (in their notation), as proved for verifying the Lindeberg’s condition above, and obtain the result immediately. This completes the proof.
D.4.2 Proof of Lemma D.2

We have

\[ I_{2,t} = \sum_{i=0}^{t} \prod_{j=i+1}^{t} \{ I - \varphi_j (I + C^*) \} (\bar{\alpha}_i - \varphi_i) z_{i,t} = U \sum_{i=0}^{t} \prod_{j=i+1}^{t} \{ I - \varphi_j \Sigma \} (\bar{\alpha}_i - \varphi_i) U^T z_{i,t}. \]

Thus, for any \( 1 \leq k \leq d + m \), we have \( [U^T I_{2,t}]_k = \sum_{i=0}^{t} \prod_{j=i+1}^{t} (1 - \varphi_j \sigma_k)(\bar{\alpha}_i - \varphi_i)[U^T z_{i,t}]_k \). For the same reason as (D.18) and (D.19), we suppose for any \( j \geq 0 \) that \( 1 - \varphi_j \sigma_k \geq 0 \). Then,

\[ \left| [U^T I_{2,t}]_k \right| \leq \frac{1}{2} \sum_{i=0}^{t} \prod_{j=i+1}^{t} |1 - \varphi_j \sigma_k| \chi_i \left| [U^T z_{i,t}]_k \right| = \frac{1}{2} \sum_{i=0}^{t} \prod_{j=i+1}^{t} (1 - \varphi_j \sigma_k) \chi_i \left| [U^T z_{i,t}]_k \right| \]

\[ = \frac{1}{2} \sum_{i=0}^{t} \prod_{j=i+1}^{t} (1 - \varphi_j \sigma_k) \chi_i E \left[ \left| [U^T z_{i,t}]_k \right| \right| F_{i-1} \right] + \frac{1}{2} \sum_{i=0}^{t} \prod_{j=i+1}^{t} (1 - \varphi_j \sigma_k) \chi_i \left| [U^T z_{i,t}]_k \right| \]

\[ - E \left[ \left| [U^T z_{i,t}]_k \right| \right| F_{i-1} \right] \right\} =: J_{3,t,k} + J_{4,t,k}. \] (D.20)

Intuitively, \( J_{3,t,k} \) dominates \( J_{4,t,k} \) since the latter term measures the error to the mean. We precisely show this result as follows. We first show \( \left| [U^T z_{i,t}]_k \right| \) has bounded variance. We have

\[ E \left[ \left\{ \left| [U^T z_{i,t}]_k \right| - E \left[ \left| [U^T z_{i,t}]_k \right| \right| F_{i-1} \right] \right\}^2 \left| F_{i-1} \right| \right] \]

\[ \leq E \left[ \left| [U^T z_{i,t}]_k \right|^2 \left| F_{i-1} \right| \right] \leq E \left[ \left| [z_{i,t}]_k \right|^2 \left| F_{i-1} \right| \right] \leq 16 \gamma_n^2 (\gamma_n^2 \vee Y_m). \] (D.21)

Thus, \( J_{4,t,k} \) is a square integrable martingale. Its variance is bounded by

\[ \langle J_{4,k} \rangle_t := \frac{1}{4} \sum_{i=0}^{t} \prod_{j=i+1}^{t} (1 - \varphi_j \sigma_k)^2 \chi_i^2 E \left[ \left\{ \left| [U^T z_{i,t}]_k \right| - E \left[ \left| [U^T z_{i,t}]_k \right| \right| F_{i-1} \right] \right\}^2 \left| F_{i-1} \right| \right] \]

\[ \leq 4 \gamma_n^2 (\gamma_n^2 \vee Y_m) \sum_{i=0}^{t} \prod_{j=i+1}^{t} (1 - \varphi_j \sigma_k)^2 \chi_i^2. \] (D.21)

Using (D.6) and (D.8), we know

\[ \lim_{i \to \infty} \left( 1 - \frac{\chi_i}{\varphi_i} \frac{1}{\chi_i} \right) = \lim_{i \to \infty} \left( 1 - \frac{\chi_i}{\chi_i^2} + \frac{\chi_i}{\chi_i^2} \left( 1 - \frac{\varphi_i}{\varphi_i - 1} \right) \right) ] \]

\[ = \lim_{i \to \infty} i \left\{ \left( 1 - \frac{\chi_i}{\chi_i} \right) \left( 1 + \frac{\chi_i}{\chi_i - 1} \right) \right\} = \frac{2 \chi}{\varphi_i}. \] (D.22)

Further, (D.8) implies \( 2 \sigma_k + (2 \chi - \varphi)/\bar{\varphi} > 0 \). Thus, Lemma A.3 leads to \( \langle J_{4,k} \rangle_t = O(\gamma_n^2/\varphi_t) \); and the strong law of large number (Duflo, 1997, Theorem 1.3.15) suggests that

\[ J_{4,t,k} = o(\sqrt{\gamma_n^2/\varphi_t \cdot \left\{ \log(\varphi_t/\chi_t^2) \right\}^{1+\omega}}) \equiv o(\gamma_n/\varphi_t). \] (D.23)
For the term $J_{3,t,k}$, we have
\[
J_{3,t,k} \leq \frac{1}{2} \sum_{i=0}^{t} \prod_{j=i+1}^{t} (1 - \varphi_j \sigma_k) \chi_i \sqrt{\mathbb{E}[||U^T z_i,\tau||^2 | F_{i-1}]} \tag{D.21}
\]
\[
\leq 2 \Upsilon_K (\Upsilon_u \sqrt{\Upsilon_m}) \sum_{i=0}^{t} \prod_{j=i+1}^{t} (1 - \varphi_j \sigma_k) \chi_i.
\]

Using (D.6), (D.8), and the facts that $\lim_{i \to \infty} (1 - \frac{\chi_i}{\chi_i \varphi_i}) = \chi - \varphi$ and $\sigma_k + (\chi - \varphi)/\varphi > 0$ (as implied by (D.8)), we apply Lemma A.3 and obtain $J_{3,t,k} = O(\chi t / \varphi t)$. Together with (D.23) and (D.20), we complete the proof.

**D.4.3 Proof of Lemma D.3**

Based on the definition of $I_{3,t}$ in (18c), we have the recursion
\[
I_{3,t+1} = \{I - \varphi_{t+1}(I + C^*)\} I_{3,t} + \varphi_{t+1} \delta^{t+1}.
\tag{D.24}
\]
By Assumption 3.1 and the fact that $\|C_t\| \leq 1$, we have
\[
\|\delta^t\| \leq 2 \left( \| (K^*)^{-1} \| \| \psi^t \| + \| K_t^{-1} - (K^*)^{-1} \| \cdot \| \nabla L_t \| \right) + \| C_t - C^* \| \cdot \left( \frac{x_t - x^*}{\lambda_t - \lambda^*} \right)
\]
\[
\leq 2 \Upsilon_K \Upsilon_L \left( \| x_t - x^* \| \right) + \left( 2 \Upsilon_K^2 \Upsilon_u \| K_t - K^* \| + \| C_t - C^* \| \right) \left( \frac{x_t - x^*}{\lambda_t - \lambda^*} \right). \tag{D.25}
\]
Since $K_t \to K^*$ (Lemma D.4) and $C_t \to C^*$, we know
\[
\delta^t = o(\|(x_t - x^*, \lambda_t - \lambda^*)\|). \tag{D.26}
\]
Using $\|C^*\| \leq \rho^*$, we know for any $a \in (0, 1)$, there exists an integer $t_1$ such that for any $t \geq t_1$,
\[
\|I_{3,t+1}\| \leq \{1 - \varphi_{t+1}(1 - \rho^*)\} \|I_{3,t}\| + \varphi_{t+1} \cdot o(\|(x_{t+1} - x^*, \lambda_{t+1} - \lambda^*)\|)
\]
\[
\leq \{1 - \varphi_{t+1}(1 - \rho^*) + o(\varphi_{t+1})\} \|I_{3,t}\| + \varphi_{t+1} \cdot o(\|I_{1,t}\| + \|I_{2,t}\|) \quad \text{(by Lemma 4.1)}
\]
\[
\leq \{1 - a(1 - \rho^*)\varphi_{t+1}\} \|I_{3,t}\| + \varphi_{t+1} \cdot o(\|I_{1,t}\| + \|I_{2,t}\|).
\]
We apply the above inequality recursively and obtain
\[
\|I_{3,t+1}\| \leq \prod_{j=t_1+1}^{t+1} \{1 - a(1 - \rho^*)\varphi_j\} \|I_{3,t_1}\|
\]
\[
+ \sum_{i=t_1+1}^{t+1} \prod_{j=i+1}^{t+1} \{1 - a(1 - \rho^*)\varphi_j\} \varphi_i o(\|I_{1,i-1}\| + \|I_{2,i-1}\|). \tag{D.27}
\]

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We apply Lemmas D.1 and D.2 for bounding \( \|\mathcal{I}_{1, i-1}\| \) and \( \|\mathcal{I}_{2, i-1}\| \). In particular, we note that for any \( \nu \geq 0 \),
\[
\lim_{i \to \infty} i \left( 1 - \sqrt[1+
u]{\phi_{i-2} \{\log(1/\phi_{i-2})\}} \right) \leq \lim_{i \to \infty} i \left( 1 - \frac{\sqrt{\phi_{i-2}}}{\sqrt[1+
u]{\phi_{i-1} \{\log(1/\phi_{i-1})\}}} \right) \quad \text{(D.6)}
\]
\[
= \frac{\phi}{2} + \lim_{i \to \infty} i \left( 1 - \frac{\log(1/\phi_{i-2})}{\log(1/\phi_{i-1})} \right) \quad \text{(Lemma A.1)}.
\]
Furthermore, we have
\[
\lim_{i \to \infty} i \left( 1 - \log(1/\phi_{i-2}) \right) = \lim_{i \to \infty} i \log(\phi_{i-2}/\phi_{i-1}) = \lim_{i \to \infty} \frac{i \log(1 + (\phi_{i-2} - \phi_{i-1})/\phi_{i-1})}{\log(1/\phi_{i-1})} = \lim_{i \to \infty} \frac{i \phi_{i-2} - \phi_{i-1}}{\log(1/\phi_{i-1})} + O \left( \frac{(\phi_{i-2} - \phi_{i-1})^2}{\phi_{i-1}^2} \right) = \lim_{i \to \infty} \frac{-\varphi}{\log(1/\phi_{i-1})} = 0,
\]
where the last equality uses the fact that \( \phi_i \to 0 \), as implied by Lemma A.2. Combining the above two displays with Lemma A.1, we have
\[
\lim_{i \to \infty} i \left( 1 - \sqrt[1+
u]{\phi_{i-2} \{\log(1/\phi_{i-2})\}} \right) = \varphi \quad \text{for any } \nu \geq 0.
\]
Moreover, we have
\[
\lim_{i \to \infty} i \left( 1 - \frac{\chi_{i-2}/\phi_{i-2}}{\chi_{i-1}/\phi_{i-1}} \right) \quad \text{(D.22)}
\]
\[
\equiv \chi - \varphi.
\]
Letting \( a \) be any scalar such that
\[
0 < \frac{-\varphi}{2(1 - \rho^\tau)} \sqrt{\frac{-\chi - \varphi}{\rho^\tau}} < a < 1,
\]
which is guaranteed to exist due to (D.6) and (D.8), we obtain
\[
a(1 - \rho^\tau) + \frac{\varphi}{2\varphi} > 0 \quad \text{and} \quad a(1 - \rho^\tau) + \frac{\chi - \varphi}{\varphi} > 0.
\]
Thus, combining (D.27), (D.28), and (D.29) with Lemma A.3, we obtain the results under either (15a) or (15b). This completes the proof.

### D.4.4 Proof of Lemma D.4

We note that
\[
\left\| \frac{1}{t} \sum_{i=0}^{t-1} \nabla_x^2 \mathcal{L}_i - \nabla_x^2 \mathcal{L}^* \right\| \leq \left\| \frac{1}{t} \sum_{i=0}^{t-1} \mathcal{H}_i - \nabla^2 f_i \right\| + \frac{1}{t} \sum_{i=0}^{t-1} \left\| \nabla_x^2 \mathcal{L}_i - \nabla_x^2 \mathcal{L}^* \right\|
\]
\[
\leq \left\| \frac{1}{t} \sum_{i=0}^{t-1} \mathcal{H}_i - \nabla^2 f_i \right\| + \frac{\gamma_L}{t} \sum_{i=0}^{t-1} \left\| (\mathbf{x}_i - \mathbf{x}^*) \right\|.
\]

(D.30)
Since \((x_t-x^*, \lambda_t-\lambda^*) \to 0\), by the fact that \(a_t \to a\) implies \(\frac{1}{t} \sum_{i=0}^{t-1} a_i \to a\) (also known as Stolz–Cesàro theorem), it suffices to show \(\sum_{i=0}^{t-1} H_i - \nabla^2 f_t / t\) converges to zero. In fact, by Assumption 3.2(15d) that \(E[H_i | F_{i-1}] = \nabla^2 f_t\) and \(E[\|H_i - \nabla^2 f_t\|^2 | F_{i-1}] \leq \gamma_m\), we notice \(\sum_{i=0}^{t-1} H_i - \nabla^2 f_i / t\) is a square integrable martingale. Thus, (Duflo, 1997, Theorem 1.3.15) suggests that for any \(v > 0\),
\[
\left\| \frac{1}{t} \sum_{i=0}^{t-1} H_i - \nabla^2 f_i \right\| = o\left(\sqrt{\log t} \cdot \frac{1}{t^{1+v}}\right).
\] (D.31)

Combining (D.30) and (D.31), we obtain \(\frac{1}{t} \sum_{i=0}^{t-1} \nabla^2 \mathcal{L}_i \to \nabla^2 \mathcal{L}^*\) as \(t \to \infty\). For the second result, we suppose \(\|\nabla^2 \mathcal{L}^*\| \leq \gamma_B^*\) and \(x^T \nabla^2 \mathcal{L}^* x \geq \gamma_B^* \|x\|^2\) in the space \(\{x \in \mathbb{R}^d : G^* x = 0\}\). Whenever \(\gamma_{RH} < \gamma_B^*\) and \(\gamma_B > \gamma_B^*\), we know \(\|\frac{1}{t} \sum_{i=0}^{t-1} \nabla^2 \mathcal{L}_i\| \leq \gamma_B\) for large enough \(t\). In addition, we let \(Z_t, Z^* \in \mathbb{R}^{d \times (d-m)}\) be the matrices whose columns are orthonormal and span the spaces of \(\ker(G_t)\), \(\ker(G^*)\), respectively. Since \(G_t \to G^*\), Davis-Kahan \(\sin(\theta)\) theorem suggests that \(Z_t Z_t^T \to Z^* (Z^*)^T\), implying \(\inf_Q \|Z_t - Z^* Q\| \to 0\) with \(Q\) chosen over all \((d-m) \times (d-m)\) orthogonal matrices (Davis and Kahan, 1970). Thus, we have
\[
\lambda_{\min}\left(\sum_{i=0}^{t-1} \nabla^2 \mathcal{L}_i / t\right) Z_t) = \lambda_{\min}\left(Q Z_t^T \left(\sum_{i=0}^{t-1} \nabla^2 \mathcal{L}_i / t\right) Z_t Q^T\right) \to \lambda_{\min}\left((Z^*)^T \nabla^2 \mathcal{L}^* Z^*\right),
\]
which implies \(\lambda_{\min}\left(\sum_{i=0}^{t-1} \nabla^2 \mathcal{L}_i / t\right) \to \gamma_{RH}\) for large enough \(t\). This completes the proof.

D.5 Proof of Theorem 4.6

We first improve the rate of \(\mathcal{I}_{3,t}\). Lemma D.5 differs from Lemma D.3 in the bound of \(\delta^t\). We propose a more precise bound on \(\delta^t\) compared to (D.26). The new bound relies on the convergence rate of the Hessian \(K_t\) in Lemma D.6 and Assumption 4.3 for applying Corollary 4.4.

Lemma D.5. Under the conditions of Theorem 4.6, for any \(v > 0\),
\[
\mathcal{I}_{3,t} = O\left(\varphi_t \log(1/\varphi_t) + o\left(\sqrt{\varphi_t} \log(1/\varphi_t) \cdot \sqrt{(\log t)^{1+v} / t}\right)\right) \text{ a.s.}
\]

By Lemmas D.2 and D.5, we know
\[
\sqrt{1/\varphi_t} \mathcal{I}_{2,t} = O\left(\chi_t / \varphi_t^{3/2}\right), \quad \sqrt{1/\varphi_t} \mathcal{I}_{3,t} = O\left(\sqrt{\varphi_t} \log(1/\varphi_t) + o\left(\log(1/\varphi_t) (\log t)^{1+v} / t\right)\right).
\]

By Lemma A.1 and the facts that
\[
\lim_{t \to \infty} \left(1 - \frac{\sqrt{\varphi_t^{-1}} \log(1/\varphi_t^{-1})}{\sqrt{\varphi_t} \log(1/\varphi_t)}\right) = \frac{\varphi}{2} < 0,
\]
\[
\lim_{t \to \infty} \left(1 - \frac{\sqrt{\log(1/\varphi_t^{-1}) (\log(t - 1))^{1+v} / t^{-1}}}{\sqrt{\log(1/\varphi_t) (\log t)^{1+v} / t}}\right) = -\frac{1}{2} < 0,
\]
we know \(\sqrt{1/\varphi_t} \mathcal{I}_{2,t} = o(1)\) and \(\sqrt{1/\varphi_t} \mathcal{I}_{3,t} = o(1)\) almost surely. Thus, the Slutsky’s theorem together with Lemma D.1 leads to the asymptotic normality. Furthermore, Lemma A.5 with \(C_t = 0, B_t = \mathcal{I}_{2,t} + \mathcal{I}_{3,t}\) leads to the Berry-Esseen bound. This completes the proof.
D.5.1 Proof of Lemma D.5

We need the following lemma to establish the convergence rate of $K_t$. The conditions are the same as those for showing the convergence rate of $(x_t - x^*, \lambda_t - \lambda^*)$, which are weaker than Theorem 4.6. The proof is provided in Appendix D.5.2.

Lemma D.6. Under the conditions Theorem 4.5 with (15a), we have for any $\nu > 0$,

$$\|K_t - K^*\| = o\left(\sqrt{\varphi_t \{\log(1/\varphi_t)\}^{1+\nu}}\right) + O(\chi_t/\varphi_t) \text{ a.s.}$$

If (15a) is strengthened to (15b), we have

$$\|K_t - K^*\| = O\left(\sqrt{\varphi_t \log(1/\varphi_t)}\right) + O(\chi_t/\varphi_t) + o\left((\log t)^{1+\nu}/t\right) \text{ a.s.}$$

Combining (D.25), Theorem 4.5, Lemma D.6, and Corollary 4.4, we have for any $\nu > 0$,

$$\|\delta_t\| = O\left(\varphi_t \log(1/\varphi_t)\right) + O\left(\frac{\chi^2_t}{\varphi_t} + \frac{(\log t)^{1+\nu}}{t}\right) + O\left(\frac{\chi_t}{\varphi_t} \sqrt{(\log t)^{1+\nu}/t}\right)$$

We plug the above bound into the recursion (D.24) and apply Lemma A.3. We note that

$$\lim_{t \to \infty} t \left(1 - \frac{\varphi_{t-1} \log(1/\varphi_{t-1})}{\varphi_t \log(1/\varphi_t)}\right) = \varphi,$$

$$\lim_{t \to \infty} t \left(1 - \frac{\sqrt{\varphi_{t-1} \log(1/\varphi_{t-1}) (\log(t-1))^{1+\nu}/(t-1)}}{\sqrt{\varphi_t \log(1/\varphi_t) (\log t)^{1+\nu}/t}}\right) = \frac{\varphi}{2} - \frac{1}{2^\nu}.$$ 

Thus, Lemma A.3 suggests that $I_{3,t}$ has the same order as $\delta_t$, provided

$$1 - \rho^* + \frac{\varphi}{\varphi} > 0 \quad \text{and} \quad 1 - \rho^* + \frac{\varphi - 1}{2\varphi} > 0.$$

We note from the proof of Theorem 4.5 that $\varphi = \beta$ and $\bar{\varphi} = \bar{\beta}$. Thus, the former condition is implied by (22). For the latter condition, if $\bar{\beta} = \bar{\varphi} = \infty$, it holds trivially. Otherwise, (22) suggests that $\bar{\beta} \in (0, \infty)$, which implies $\beta = \varphi = -1$. In fact, if $-1 < \beta < 0$, then

$$\lim_{t \to \infty} t \left(1 - \frac{t\beta_t}{(t-1)\beta_{t-1}}\right) = \lim_{t \to \infty} t \left(1 - \frac{t}{t-1} + \frac{t}{t-1} \left(1 - \frac{\beta_t}{\beta_{t-1}}\right)\right) = -(1 + \beta) < 0.$$

By Lemma A.2, we have $\bar{\beta} = \infty$. Similarly, if $\beta < -1$, we have $\bar{\beta} = 0$. Plugging $\varphi = -1$ into the latter condition, we see the latter condition is implied by the former condition. We complete the proof.
D.5.2 Proof of Lemma D.6

We apply (14), combine (D.30) and (D.31), and have for any \( v > 0 \) and large enough \( t \) that

\[
\|K_t - K^*\| \leq \frac{\gamma_L}{t} \sum_{i=0}^{t-1} \left\| \frac{x_i - x^*}{\lambda_i - \lambda^*} \right\| + \gamma_L \|x_t - x^*\| + o \left( \frac{\sqrt{(\log t)^{1+v}}}{t} \right)
\]

\[
= \frac{\gamma_L}{t} \left\| \frac{x_0 - x^*}{\lambda_0 - \lambda^*} \right\| + \gamma_L \sum_{i=1}^{t-1} \prod_{j=i+1}^{t} \left( 1 - \frac{1}{j} \right) \left\| \frac{x_i - x^*}{\lambda_i - \lambda^*} \right\| + \gamma_L \|x_t - x^*\| + o \left( \frac{\sqrt{(\log t)^{1+v}}}{t} \right)
\]

\[
= \gamma_L \sum_{i=1}^{t-1} \prod_{j=i+1}^{t} \left( 1 - \frac{1}{j} \right) \frac{a_i}{t} + O \left( \frac{\chi_i}{\varphi_i} \right) + \gamma_L \left( a_t + O \left( \frac{\chi_t}{\varphi_t} \right) \right) + o \left( \frac{\sqrt{(\log t)^{1+v}}}{t} \right), \tag{D.32}
\]

where, by Theorem 4.5, \( a_i = o(\sqrt{\varphi_i} \{\log(1/\varphi_i)\}^{1+v/2}) \) under (15a) and \( a_i = O(\sqrt{\varphi_i} \log(1/\varphi_i)) \) under (15b). We claim that \( \varphi > -2 \). Otherwise, \( \varphi + 1.5 \leq -0.5 < 0 \). We apply Lemma A.1 and have

\[
\lim_{t \to \infty} \left( 1 - \frac{\varphi_{t-1}(t-1)^{1.5}}{\varphi_{t^{1.5}}} \right) = \lim_{t \to \infty} \left( 1 - \frac{\varphi_{t-1}}{\varphi_t} + \frac{\varphi_{t-1}}{\varphi_t} \left( 1 - \frac{(t-1)^{1.5}}{t^{1.5}} \right) \right) = \varphi + 1.5 < 0.
\]

Then, Lemma A.2 suggests that \( \varphi_t^{1.5} \to 0 \), which cannot hold under (D.6). Thus, \( \varphi > -2 \). Using (D.28) and Lemma A.3, and noting that \( 1 + \varphi/2 > 0 \), we obtain

\[
\sum_{i=1}^{t-1} \prod_{j=i+1}^{t} \left( 1 - \frac{1}{j} \right) \frac{a_i}{t} = O(a_t). \tag{D.33}
\]

Furthermore, we deal with the term that involves \( \chi_i/\varphi_i \) in (D.32). Without loss of generality, we suppose \( \chi \geq 3\varphi/2 \). Otherwise, we know

\[
\lim_{t \to \infty} \left( 1 - \frac{\chi_{t-1}/\varphi_{t-1}^{3/2}}{\chi_t/\varphi_t^{3/2}} \right) = \chi - \frac{3\varphi}{2} < 0.
\]

This implies \( \chi_t/\varphi_t = o(\sqrt{\varphi_t}) = o(a_t) \) and, thus, all \( O(\chi_i/\varphi_i) \) terms in (D.32) are negligible and the argument of the lemma holds immediately. Suppose \( \chi \geq 3\varphi/2 \) and noting that

\[
\lim_{t \to \infty} \left( 1 - \frac{\chi_{t-1}/\varphi_{t-1}}{\chi_t/\varphi_t} \right) = \chi - \varphi \quad \text{and} \quad 1 + \chi - \varphi \geq 1 + \frac{\varphi}{2} > 0,
\]

we obtain from Lemma A.3 that

\[
\sum_{i=1}^{t-1} \prod_{j=i+1}^{t} \left( 1 - \frac{1}{j} \right) \frac{1}{t} \cdot O \left( \frac{\chi_i}{\varphi_i} \right) = O \left( \frac{\chi_t}{\varphi_t} \right). \tag{D.34}
\]

Combining (D.32), (D.33), (D.34) together, and noting that \( o(\sqrt{(\log t)^{1+v}}/t) = o(a_t) \) under (15a) (as implied by the fact that \( t\varphi_t \to \bar{\varphi} \in (0, \infty) \)), we complete the proof.
D.6 Proof of Theorem 4.7

We have

\[
\|\Xi - \Xi^*\| \leq \|\Xi^* - \mathbb{E}[(I + \widetilde{C}^*)(\Omega^* (I + \widetilde{C}^*)^T)/(2 + \beta/\widetilde{\beta})]\|
\] 
\[+ \|\mathbb{E}[(I + \widetilde{C}^*)\Omega^* (I + \widetilde{C}^*)^T] - \Omega^*\|/(2 + \beta/\widetilde{\beta}) + \|\Omega^* - \Omega_t\|/(2 + \beta/\widetilde{\beta}). \tag{D.35}
\]

For the first term in (D.35), we have

\[
\|\Xi^* - \mathbb{E}[(I + \widetilde{C}^*)\Omega^* (I + \widetilde{C}^*)^T]/(2 + \beta/\widetilde{\beta})\|
\] 
\[\overset{(25)}{=} \|((\Theta - \mathbf{1}^T/(2 + \beta/\widetilde{\beta})) \circ U^T\mathbb{E}[(I + \widetilde{C}^*)\Omega^* (I + \widetilde{C}^*)^T]U\|
\] 
\[\leq \|\Theta - \mathbf{1}^T/(2 + \beta/\widetilde{\beta})\| \cdot \|\mathbb{E}[(I + \widetilde{C}^*)\Omega^* (I + \widetilde{C}^*)^T]\| \quad (\|A \circ B\| \leq \|A\| \cdot \|B\|)
\] 
\[\leq 4\|\Theta - \mathbf{1}^T/(2 + \beta/\widetilde{\beta})\| \cdot \|\Omega^*\| \quad (\|\widetilde{C}^*\| \leq 1),
\]
and for any \(1 \leq k, l \leq d + m,

\[
|\Theta_{k,l} - 1/(2 + \beta/\widetilde{\beta})| = |1/(\sigma_k + \sigma_l + \beta/\widetilde{\beta}) - 1/(2 + \beta/\widetilde{\beta})| = \frac{|2 - \sigma_k - \sigma_l|}{(\sigma_k + \sigma_l + \beta/\widetilde{\beta})(2 + \beta/\widetilde{\beta})}
\] 
\[
\overset{(22)}{\leq} \frac{2\rho^r}{(2 - 2\rho^r + \beta/\widetilde{\beta})(2 + \beta/\widetilde{\beta})} \leq \frac{2\rho^r}{(2 - 2(1 + \beta/\widetilde{\beta} + \beta/\widetilde{\beta})(2 + \beta/\widetilde{\beta})} = \frac{2\rho^r}{-\beta/(2 + \beta/\widetilde{\beta})}.
\]

Therefore, the above two displays lead to

\[
\|\Xi^* - \mathbb{E}[(I + \widetilde{C}^*)\Omega^* (I + \widetilde{C}^*)^T]/(2 + \beta/\widetilde{\beta})\| = O(\rho^r). \tag{D.36}
\]

For the second term in (D.35), we have

\[
\|\mathbb{E}[(I + \widetilde{C}^*)\Omega^* (I + \widetilde{C}^*)^T] - \Omega^*\| \leq \|C^*\Omega^*\| + \|\Omega^*C^*\| + \|\mathbb{E}[\widetilde{C}^*\Omega^* (\widetilde{C}^*)^T]\| \quad (\mathbb{E}[\widetilde{C}^*] = C^*)
\] 
\[\leq O(\rho^r) + \|\mathbb{E}[\widetilde{C}^*\Omega^* (\widetilde{C}^*)^T]\| \quad (\|C^*\| \leq \rho^r).
\]

Furthermore, since \(\Omega^* \preceq \gamma_K^2 \gamma_m \cdot I\), we obtain

\[
0 \preceq \mathbb{E}[\widetilde{C}^*\Omega^* (\widetilde{C}^*)^T] \preceq \gamma_K^2 \gamma_m \mathbb{E}[\widetilde{C}^* (\widetilde{C}^*)^T]
\]
\[
= \gamma_K^2 \gamma_m \mathbb{E} \left[ \prod_{j=1}^\tau (I - K^* S_j (S_j^T (K^*) S_j)^\dagger S_j^T K^*) \right] 
\] 
\[
= \gamma_K^2 \gamma_m \mathbb{E} \left[ \prod_{j=2}^\tau (I - K^* S_j (S_j^T (K^*) S_j)^\dagger S_j^T K^*) \bigg| S_2: r \right] 
\] 
\[
\left[ \prod_{j=2}^\tau (I - K^* S_j (S_j^T (K^*) S_j)^\dagger S_j^T K^*) \right]^T 
\] 
\[
\leq \gamma_K^2 \gamma_m \rho^r \mathbb{E} \left[ \prod_{j=1}^\tau (I - K^* S_j (S_j^T (K^*) S_j)^\dagger S_j^T K^*) \right] 
\] 
\[
\leq \gamma_K^2 \gamma_m \rho^r \cdot I,
\]

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where the second last inequality is from Assumption 3.3 and Corollary 4.4; and the last inequality applies the same reason for sketch matrices $S_{2,r}$. Combining the above two displays,

\[ \|\mathbb{E}[(I + \tilde{C}^\ast)\Omega^\ast (I + \tilde{C}^\ast)^T] - \Omega^\ast\| \leq O(\rho^\ast). \]  

(D.37)

For the third term in (D.35), we have

\[ \|\Omega_t - \Omega^\ast\| \overset{(21)}{=} O(\|K_t - K^\ast\|) \]

\[ + O \left( \left\| \frac{1}{t} \sum_{i=0}^{t-1} g_i \tilde{g}_i^T - \left( \frac{1}{t} \sum_{i=0}^{t-1} \tilde{g}_i \right) \left( \frac{1}{t} \sum_{i=0}^{t-1} \tilde{g}_i \right)^T \right\| \right). \]

Furthermore, we have

\[ \left\| \frac{1}{t} \sum_{i=0}^{t-1} g_i \tilde{g}_i^T - \left( \frac{1}{t} \sum_{i=0}^{t-1} \tilde{g}_i \right) \left( \frac{1}{t} \sum_{i=0}^{t-1} \tilde{g}_i \right)^T \right\| \leq \left\| \frac{1}{t} \sum_{i=0}^{t-1} g_i \tilde{g}_i^T - \mathbb{E}[\nabla f(x^\ast; \xi)\nabla^T f(x^\ast; \xi)] \right\| + \left\| \left( \frac{1}{t} \sum_{i=0}^{t-1} \tilde{g}_i \right) \left( \frac{1}{t} \sum_{i=0}^{t-1} \tilde{g}_i \right)^T - \nabla f(x^\ast)\nabla^T f(x^\ast) \right\|. \]

We take the first term as an example, while the second term has the same guarantee following the same derivations. We note that

\[ \left\| \frac{1}{t} \sum_{i=0}^{t-1} g_i \tilde{g}_i^T - \mathbb{E}[\nabla f(x^\ast; \xi)\nabla^T f(x^\ast; \xi)] \right\| \leq \left\| \frac{1}{t} \sum_{i=0}^{t-1} (g_i \tilde{g}_i^T) - \mathbb{E}[g_i \tilde{g}_i^T | F_{i-1}] \right\| + \left\| \frac{1}{t} \sum_{i=0}^{t-1} \mathbb{E}[g_i \tilde{g}_i^T | F_{i-1}] - \mathbb{E}[\nabla f(x^\ast; \xi)\nabla^T f(x^\ast; \xi)] \right\|. \]

By (15c), we know the first term on the right hand side is a square integrable martingale. The strong law of large number (Duflo, 1997, Theorem 1.3.15) suggests that

\[ \left\| \frac{1}{t} \sum_{i=0}^{t-1} (g_i \tilde{g}_i^T) - \mathbb{E}[g_i \tilde{g}_i^T | F_{i-1}] \right\| = o(\sqrt{(\log t)^{1+v}/t}). \]

By (D.11), (D.32), (D.33), (D.34), the second term on the right hand side can be bounded by

\[ \left\| \frac{1}{t} \sum_{i=0}^{t-1} \mathbb{E}[g_i \tilde{g}_i^T | F_{i-1}] - \mathbb{E}[\nabla f(x^\ast; \xi)\nabla^T f(x^\ast; \xi)] \right\| = O \left( \sqrt{\varphi_t \log(1/\varphi_t)} \right) + O(\chi_t/\varphi_t). \]

Combining the above five displays with Lemma D.6, we have

\[ \|\Omega_t - \Omega^\ast\| = O \left( \sqrt{\varphi_t \log(1/\varphi_t)} \right) + O(\chi_t/\varphi_t) + o(\sqrt{(\log t)^{1+v}/t}). \]  

(D.38)

Combining (D.35), (D.36), (D.37), and (D.38), we complete the first part of the proof. For the Berry-Esseen inequality, we simply note that

\[ \frac{w^T(x_t - x^\ast, \lambda_t - \lambda^\ast)}{\sqrt{w^T \Xi_t w}} = \frac{w^T(x_t - x^\ast, \lambda_t - \lambda^\ast)}{\sqrt{w^T \Xi^\ast w} \cdot \sqrt{1 + \frac{w^T \Xi_t w - w^T \Xi^\ast w}{w^T \Xi^\ast w}}}. \]

Thus, we apply Theorem 4.6 and Lemma A.5, and complete the proof.
E Additional Experimental Results

In this section, we provide more implementation details and show additional results. We follow the introduction in Section 5 and implement the method on five problems in CUTEst test set, and on linearly/nonlinearly constrained regression problems. For both implementation, we run $10^5$ iterations, set $\beta_t = 1/t^{0.501}$, $\chi_t = \beta^2_t$, and $\alpha_t \sim \text{Uniform}([\beta_t, \eta_t])$ with $\eta_t = \beta_t + \chi_t$. Regarding the Hessian regularization $\Delta_t$, we let $\lambda_{\min}(\cdot)$ denotes the least eigenvalue

$$
\Delta_t := (\lambda_{\min}(Z_t^T \sum_{i=0}^{t-1} \nabla^2_x L_i Z_t)/t + 0.1) \cdot I
$$

whenever $\lambda_{\min}(Z_t^T \sum_{i=0}^{t-1} \nabla^2_x L_i Z_t) < 0$.

Here, $Z_t \in \mathbb{R}^{d \times (d-m)}$ has orthonormal columns that span the space $\{x \in \mathbb{R}^d : G_t x = 0\}$, which is obtained from the QR decomposition.

E.1 CUTEst problems

We testify the convergence rate in Theorem 4.5 and visualize online confidence intervals.

- Convergence behavior. We randomly pick one run across 200 runs, and show the convergence plots of the KKT residual $\|\nabla L_t\|$, the iteration error $\|(x_t - x^*, \lambda_t - \lambda^*)\|$, and the Hessian error $\|K_t - K^*\|$. By Theorem 4.5, Lemma D.6, and the Lipschitz continuity of the Hessian, the theoretical convergence rate for these three quantities is $O(\beta_t \log(1/\beta_t))$.

The convergence plots are shown in Figures 1 and 2. From the figures, we observe that the method converges faster for small sampling variance $\sigma^2$ and converges slower for large $\sigma^2$. Specifically, our theoretical convergence rate precisely characterizes the asymptotic behavior of the method, and $\sigma^2$ only affects the rate as the constant factor.

- Confidence interval. We take MARATOS, HS7, HS48 as examples to visualize the online construction of the confidence interval. Figure 3 shows the 95% confidence intervals of the last 100 iterations. We see that the intervals consistently cover the true solution, with $\sigma^2$ varying from $10^{-4}$ to 1. This illustrates the effectiveness of our method.

E.2 Constrained regression problems

We follow the experiments in Section 5.2, and provide comprehensive comparisons between inexact and exact second-order methods on linearly/nonlinearly constrained regression problems. The coefficient matrix $A$ in linear constraints is sampled from standard normal; the objective of logistic loss is regularized by a quadratic penalty term. We vary the parameters $d, r$ and $\tau$. In particular, we let $d \in \{5, 20, 40, 60\}$, $r \in \{0.4, 0.5, 0.6\}$ for Toeplitz $\Sigma_a$ and $r \in \{0.1, 0.2, 0.3\}$ for Equi-correlation $\Sigma_a$, and $\tau \in \{\infty, 20, 40, 60\}$. We mention that $\tau = \infty$ corresponds to the exact method. For each setup, we perform 200 independent runs.

The results of the coverage rate and the average length of confidence intervals are reported in Tables 4-11. As an example of coverage rate, we illustrate Table 4 for linearly constrained linear regression problems. From Table 4, we observe that the inexact methods exhibit similar coverage rates to the exact method for almost all setups. One potential exception is when $\Sigma_a$ is Equi-correlation and $d$ is large. In that case, the exact method has a better coverage rate around 95%, while the inexact methods only have an 87%-89% coverage rate. Within the inexact methods, the coverage rates do not
Figure 1: Convergence plots of CUTEst problems. Each row corresponds to one problem and has three figures in the log scale. From the left to the right, they correspond to $\|\nabla L_t\|$ v.s. $t$, $\|(x_t - x^*, \lambda_t - \lambda^*)\|$ v.s. $t$, and $\|K_t - K^*\|$ v.s. $t$. Each figure has five lines; four lines correspond to four setups of $\sigma^2$, and the red line corresponds to $\sqrt{\beta_t \log(1/\beta_t)}$ v.s. $t$, which is the theoretical asymptotic rate.
Problem HS48

Problem HS78

$\sigma^2 = 10^{-4}$ $\sigma^2 = 10^{-2}$ $\sigma^2 = 10^{-1}$ $\sigma^2 = 1$ Theory

Figure 2: Convergence plots of CUTEst problems. See Figure 1 for the interpretation.
Figure 3: 95% confidence intervals of $\mathbf{x}^\dagger + \mathbf{\lambda}^\dagger$ for CUTEst problems. Each row has four figures, from the left to the right, corresponding to $\sigma^2 \in \{10^{-4}, 10^{-2}, 10^{-1}, 1 \}$. Each figure has three lines: the blue and green lines correspond to the lower and upper interval boundaries, respectively; and the red line corresponds to the true value.
vary evidently between \( \tau = 20, 40, 60 \), meaning \( \tau = 20 \) is sufficient in practice. Furthermore, when increasing \( r \) for Toeplitz \( \Sigma_a \), we observe that the coverage rates remain similar for small \( d \), while slightly decreasing for large \( d \). In contrast, the methods have similar rates when increasing \( r \) for Equi-correlation \( \Sigma_a \). Table 5 shares the same pattern as Table 4. Comparing between Tables 4 and 5, we clearly see that nonlinear constraints bring significant difficulties to the regression problems, particular for Toeplitz \( \Sigma_a \) in large dimensions. We also observe that the exact method is indeed more robust to nonlinear constraints, indicating that nonlinear constraints require higher quality search directions. Tables 6 and 7 demonstrate that our method performs equally well on logistic regression with linear and nonlinear constraints.

As an example of the average length of confidence intervals, we illustrate Table 8. From Table 8, we observe that for all setups, the inexact methods report very similar average length to the exact method, suggesting that the inexact methods do not lead to wider conservative intervals. Furthermore, for all types of covariance matrices and all methods, the average length decreases as \( d \) increases, which is particularly evident for the identity \( \Sigma_a \) and Equi-correlation \( \Sigma_a \). This phenomenon is consistent with Chen et al. (2020). For Toeplitz and Equi-correlation \( \Sigma_a \), the average length decreases as \( r \) increases. Table 9 shares the same pattern as Table 8. Comparing between Tables 8 and 9, we see that all methods with all types of covariance provide slightly shorter confidence intervals with nonlinear constraints. Similar comparisons for linearly/nonlinearly constrained logistic regression are displayed in Tables 10 and 11.
Table 4: The coverage rate (%) for **linearly constrained linear regression** problems.

| Dim |  |  | Toeplitz $\Sigma_a (r)$ | Equi-corr $\Sigma_a (r)$ |
|-----|---|---|------------------------|------------------------|
|     |  |  | 0.4 | 0.5 | 0.6 | 0.1 | 0.2 | 0.3 |
| 5   | $\infty$ | 93.00 | 95.50 | 94.00 | 96.00 | 96.50 | 93.00 | 97.50 |
|     | 20  | 93.50 | 97.00 | 94.50 | 94.00 | 94.00 | 94.50 | 94.50 |
|     | 40  | 94.50 | 97.50 | 90.00 | 95.00 | 94.00 | 96.50 | 96.00 |
|     | 60  | 97.00 | 93.50 | 95.50 | 94.50 | 94.00 | 95.50 | 94.00 |

Table 5: The coverage rate (%) for **nonlinearly constrained linear regression** problems.

| Dim |  |  | Toeplitz $\Sigma_a (r)$ | Equi-corr $\Sigma_a (r)$ |
|-----|---|---|------------------------|------------------------|
|     |  |  | 0.4 | 0.5 | 0.6 | 0.1 | 0.2 | 0.3 |
| 5   | $\infty$ | 92.00 | 94.00 | 95.00 | 98.00 | 95.00 | 93.00 | 96.00 |
|     | 20  | 95.00 | 94.50 | 22.00 | 91.00 | 94.00 | 92.00 | 87.00 |
|     | 40  | 95.50 | 95.50 | 92.50 | 94.50 | 91.50 | 88.00 | 90.50 |
|     | 60  | 98.00 | 97.00 | 94.50 | 93.00 | 90.50 | 92.50 | 89.00 |

| Dim |  |  | Toeplitz $\Sigma_a (r)$ | Equi-corr $\Sigma_a (r)$ |
|-----|---|---|------------------------|------------------------|
|     |  |  | 0.4 | 0.5 | 0.6 | 0.1 | 0.2 | 0.3 |
| 20  | $\infty$ | 95.00 | 96.00 | 92.50 | 92.00 | 95.00 | 95.00 | 97.00 |
|     | 20  | 95.00 | 97.50 | 95.00 | 96.00 | 90.00 | 89.00 | 87.50 |
|     | 40  | 99.50 | 98.00 | 96.50 | 91.50 | 87.00 | 91.00 | 89.50 |
|     | 60  | 97.00 | 97.00 | 94.00 | 94.50 | 90.00 | 91.00 | 90.50 |

| Dim |  |  | Toeplitz $\Sigma_a (r)$ | Equi-corr $\Sigma_a (r)$ |
|-----|---|---|------------------------|------------------------|
|     |  |  | 0.4 | 0.5 | 0.6 | 0.1 | 0.2 | 0.3 |
| 40  | $\infty$ | 91.50 | 94.00 | 96.50 | 97.50 | 89.00 | 87.00 | 89.00 |
|     | 20  | 96.50 | 99.00 | 97.00 | 95.00 | 87.00 | 87.00 | 86.00 |
|     | 40  | 99.50 | 96.00 | 94.50 | 96.00 | 96.00 | 94.00 | 96.00 |

| Dim |  |  | Toeplitz $\Sigma_a (r)$ | Equi-corr $\Sigma_a (r)$ |
|-----|---|---|------------------------|------------------------|
|     |  |  | 0.4 | 0.5 | 0.6 | 0.1 | 0.2 | 0.3 |
| 60  | $\infty$ | 92.50 | 94.00 | 95.00 | 96.00 | 95.50 | 95.00 | 92.00 |
|     | 20  | 94.00 | 97.00 | 96.50 | 97.50 | 89.00 | 87.00 | 89.00 |
|     | 40  | 96.50 | 99.00 | 97.00 | 95.00 | 87.00 | 87.00 | 86.00 |
|     | 60  | 99.50 | 96.00 | 94.50 | 94.00 | 88.50 | 89.00 | 87.00 |
| Dim | τ   | Id. Σ_a | Toeplitz Σ_a (r) | Equi-corr Σ_a (r) |
|-----|-----|---------|------------------|-------------------|
|     |     |         | 0.4  | 0.5  | 0.6  | 0.1  | 0.2  | 0.3  |
| 5   | ∞   | 95.50   | 94.00 | 94.00 | 94.50 | 95.50 | 93.50 | 94.00 |
|     | 20  | 94.50   | 95.50 | 92.50 | 93.50 | 97.00 | 97.50 | 96.00 |
|     | 40  | 95.00   | 94.50 | 94.50 | 95.00 | 95.00 | 94.50 | 92.00 |
|     | 60  | 97.00   | 94.00 | 92.50 | 96.00 | 95.00 | 93.00 | 95.50 |
| 20  | ∞   | 94.50   | 95.50 | 95.00 | 93.00 | 95.50 | 90.00 | 95.00 |
|     | 20  | 94.50   | 97.00 | 96.50 | 95.00 | 96.50 | 96.00 | 98.00 |
|     | 40  | 97.50   | 97.50 | 96.00 | 95.50 | 99.50 | 97.50 | 98.00 |
|     | 60  | 96.50   | 95.50 | 96.50 | 97.50 | 95.00 | 94.50 | 95.50 |
| 40  | ∞   | 92.00   | 94.50 | 95.00 | 93.00 | 96.50 | 94.50 | 93.50 |
|     | 20  | 96.50   | 96.50 | 96.00 | 98.00 | 96.00 | 97.00 | 95.50 |
|     | 40  | 97.00   | 99.00 | 99.00 | 95.50 | 95.00 | 95.00 | 99.00 |
|     | 60  | 98.50   | 97.00 | 97.00 | 96.00 | 99.00 | 94.00 | 94.50 |
| 60  | ∞   | 96.00   | 95.50 | 96.00 | 94.00 | 93.00 | 94.50 | 94.50 |
|     | 20  | 97.50   | 96.50 | 95.00 | 94.00 | 96.50 | 94.00 | 98.00 |
|     | 40  | 95.50   | 98.50 | 97.00 | 96.00 | 97.00 | 94.00 | 94.50 |
|     | 60  | 96.00   | 98.00 | 95.00 | 96.00 | 99.50 | 99.00 | 99.00 |

Table 6: The coverage rate (%) for linearly constrained logistic regression problems.

| Dim | τ   | Id. Σ_a | Toeplitz Σ_a (r) | Equi-corr Σ_a (r) |
|-----|-----|---------|------------------|-------------------|
|     |     |         | 0.4  | 0.5  | 0.6  | 0.1  | 0.2  | 0.3  |
| 5   | ∞   | 93.00   | 93.00 | 91.00 | 94.00 | 91.50 | 94.00 | 95.50 |
|     | 20  | 88.00   | 93.50 | 97.00 | 93.50 | 94.00 | 95.50 | 93.50 |
|     | 40  | 94.50   | 94.00 | 91.50 | 92.00 | 92.50 | 91.50 | 93.50 |
|     | 60  | 95.00   | 91.00 | 93.00 | 94.00 | 95.50 | 96.00 | 94.00 |
| 20  | ∞   | 95.50   | 93.00 | 89.00 | 92.50 | 92.50 | 96.50 | 95.50 |
|     | 20  | 88.00   | 95.50 | 93.50 | 93.00 | 93.00 | 95.00 | 93.50 |
|     | 40  | 93.50   | 91.50 | 86.50 | 88.50 | 93.50 | 95.00 | 96.50 |
|     | 60  | 94.50   | 94.00 | 95.50 | 92.50 | 93.50 | 92.50 | 95.50 |
| 40  | ∞   | 92.50   | 94.00 | 92.50 | 86.50 | 95.00 | 95.00 | 94.00 |
|     | 20  | 91.50   | 97.00 | 96.00 | 97.50 | 95.50 | 96.00 | 97.00 |
|     | 40  | 89.50   | 94.00 | 96.00 | 94.00 | 94.00 | 94.50 | 96.00 |
|     | 60  | 93.50   | 94.00 | 94.00 | 93.50 | 93.00 | 94.50 | 96.50 |
| 60  | ∞   | 94.50   | 93.50 | 95.00 | 90.50 | 94.00 | 97.00 | 96.00 |
|     | 20  | 93.50   | 97.00 | 99.00 | 96.50 | 96.00 | 95.50 | 94.00 |
|     | 40  | 97.50   | 98.00 | 93.50 | 97.50 | 96.00 | 95.00 | 99.00 |
|     | 60  | 94.00   | 97.00 | 96.50 | 93.00 | 95.00 | 96.50 | 92.00 |

Table 7: The coverage rate (%) for nonlinearly constrained logistic regression problems.
### Table 8: The average length ($10^{-2}$) for linearly constrained linear regression problems.

| Dim | $\tau$ | Id. $\Sigma_a$ | Toeplitz $\Sigma_a (r)$ | Equi-corr $\Sigma_a (r)$ |
|-----|--------|---------------|-------------------------|--------------------------|
|     |        |               | 0.4 0.5 0.6             | 0.1 0.2 0.3              |
| 5   | $\infty$ | 5.05(0.04)   | 3.84(0.03) 3.57(0.03) 3.29(0.02) | 4.57(0.03) 4.20(0.03) 3.89(0.03) |
|     | 20     | 5.05(0.04)   | 3.84(0.03) 3.56(0.03) 3.29(0.03) | 4.57(0.03) 4.19(0.03) 3.89(0.03) |
|     | 40     | 5.05(0.04)   | 3.84(0.03) 3.56(0.03) 3.29(0.02) | 4.57(0.03) 4.20(0.03) 3.89(0.03) |
|     | 60     | 5.05(0.04)   | 3.84(0.03) 3.56(0.03) 3.29(0.03) | 4.57(0.04) 4.20(0.03) 3.89(0.03) |

### Table 9: The average length ($10^{-2}$) for nonlinearly constrained linear regression problems.

| Dim | $\tau$ | Id. $\Sigma_a$ | Toeplitz $\Sigma_a (r)$ | Equi-corr $\Sigma_a (r)$ |
|-----|--------|---------------|-------------------------|--------------------------|
|     |        |               | 0.4 0.5 0.6             | 0.1 0.2 0.3              |
| 5   | $\infty$ | 3.85(0.12)   | 3.69(0.07) 3.68(0.06) 3.65(0.06) | 3.85(0.09) 3.81(0.06) 3.73(0.04) |
|     | 20     | 3.86(0.12)   | 3.69(0.07) 3.67(0.06) 3.66(0.05) | 3.85(0.09) 3.80(0.06) 3.73(0.05) |
|     | 40     | 3.85(0.14)   | 3.70(0.07) 3.68(0.06) 3.66(0.05) | 3.85(0.09) 3.80(0.06) 3.74(0.04) |
|     | 60     | 3.85(0.13)   | 3.68(0.07) 3.68(0.06) 3.67(0.05) | 3.84(0.09) 3.80(0.06) 3.74(0.05) |
| 20  | $\infty$ | 1.78(0.04)   | 1.33(0.03) 1.26(0.03) 1.22(0.03) | 1.51(0.02) 1.34(0.01) 1.22(0.01) |
|     | 20     | 1.77(0.04)   | 1.31(0.03) 1.22(0.02) 1.15(0.02) | 1.51(0.02) 1.33(0.01) 1.21(0.01) |
|     | 40     | 1.78(0.05)   | 1.31(0.03) 1.22(0.02) 1.15(0.02) | 1.50(0.02) 1.33(0.01) 1.21(0.01) |
|     | 60     | 1.77(0.04)   | 1.31(0.02) 1.23(0.02) 1.16(0.02) | 1.51(0.02) 1.34(0.01) 1.21(0.01) |
| 40  | $\infty$ | 1.28(0.03)   | 0.90(0.01) 0.84(0.02) 0.80(0.02) | 0.93(0.01) 0.77(0.01) 0.67(0.01) |
|     | 20     | 1.27(0.02)   | 0.88(0.02) 0.80(0.01) 0.72(0.01) | 0.92(0.01) 0.75(0.01) 0.65(0.01) |
|     | 40     | 1.27(0.02)   | 0.88(0.02) 0.79(0.01) 0.72(0.01) | 0.92(0.01) 0.75(0.01) 0.65(0.01) |
|     | 60     | 1.27(0.03)   | 0.88(0.01) 0.80(0.01) 0.72(0.01) | 0.92(0.01) 0.75(0.01) 0.65(0.01) |
| 60  | $\infty$ | 1.06(0.02)   | 0.74(0.01) 0.69(0.01) 0.65(0.02) | 0.68(0.01) 0.55(0.00) 0.47(0.00) |
|     | 20     | 1.06(0.02)   | 0.71(0.01) 0.63(0.01) 0.56(0.01) | 0.67(0.01) 0.53(0.00) 0.45(0.00) |
|     | 40     | 1.05(0.02)   | 0.71(0.01) 0.64(0.01) 0.57(0.01) | 0.67(0.01) 0.53(0.00) 0.45(0.00) |
|     | 60     | 1.06 (0.02)  | 0.71(0.01) 0.64(0.01) 0.56(0.01) | 0.67(0.01) 0.53(0.00) 0.45(0.00) |

Table 9: The average length ($10^{-2}$) for nonlinearly constrained linear regression problems.
Table 10: The average length \((10^{-2})\) for linearly constrained logistic regression problems.

| Dim | \(\tau\) | Id. \(\Sigma_a\) | Toeplitz \(\Sigma_a (r)\) | Equi-corr \(\Sigma_a (r)\) |
|-----|-----|-----------------|------------------|------------------|
|     |     |                 | 0.4 0.5 0.6       | 0.1 0.2 0.3       |
| 5   | \(\infty\) | 1.70(0.01) | 1.64(0.02) 1.60(0.02) 1.57(0.02) | 1.68(0.01) 1.65(0.02) 1.62(0.02) |
|     | 20   | 1.70(0.01) | 1.64(0.02) 1.61(0.02) 1.57(0.02) | 1.68(0.01) 1.65(0.02) 1.63(0.02) |
|     | 40   | 1.70(0.01) | 1.64(0.02) 1.60(0.02) 1.57(0.02) | 1.68(0.02) 1.65(0.02) 1.63(0.02) |
|     | 60   | 1.70(0.01) | 1.64(0.02) 1.60(0.02) 1.57(0.02) | 1.68(0.02) 1.65(0.02) 1.63(0.02) |

Table 11: The average length \((10^{-2})\) for nonlinearly constrained logistic regression problems.

| Dim | \(\tau\) | Id. \(\Sigma_a\) | Toeplitz \(\Sigma_a (r)\) | Equi-corr \(\Sigma_a (r)\) |
|-----|-----|-----------------|------------------|------------------|
|     |     |                 | 0.4 0.5 0.6       | 0.1 0.2 0.3       |
| 5   | \(\infty\) | 1.35(0.02) | 1.52(0.02) 1.55(0.02) 1.57(0.02) | 1.43(0.02) 1.48(0.02) 1.51(0.02) |
|     | 20   | 1.35(0.02) | 1.52(0.02) 1.55(0.02) 1.57(0.02) | 1.43(0.02) 1.47(0.02) 1.51(0.02) |
|     | 40   | 1.35(0.02) | 1.52(0.02) 1.55(0.02) 1.57(0.02) | 1.43(0.02) 1.48(0.02) 1.50(0.02) |
|     | 60   | 1.35(0.02) | 1.52(0.02) 1.55(0.02) 1.57(0.02) | 1.43(0.02) 1.48(0.02) 1.51(0.02) |
| 20  | \(\infty\) | 0.54(0.01) | 0.63(0.01) 0.65(0.01) 0.67(0.01) | 0.51(0.01) 0.47(0.01) 0.43(0.01) |
|     | 20   | 0.54(0.01) | 0.63(0.01) 0.65(0.01) 0.67(0.01) | 0.52(0.01) 0.47(0.01) 0.43(0.01) |
|     | 40   | 0.54(0.01) | 0.63(0.01) 0.65(0.01) 0.67(0.01) | 0.51(0.01) 0.47(0.01) 0.43(0.01) |
|     | 60   | 0.54(0.01) | 0.63(0.01) 0.65(0.01) 0.67(0.01) | 0.51(0.01) 0.47(0.01) 0.43(0.01) |
| 40  | \(\infty\) | 0.35(0.00) | 0.41(0.01) 0.43(0.01) 0.45(0.01) | 0.29(0.00) 0.25(0.00) 0.22(0.00) |
|     | 20   | 0.35(0.00) | 0.41(0.01) 0.43(0.01) 0.45(0.01) | 0.29(0.00) 0.25(0.00) 0.22(0.00) |
|     | 40   | 0.35(0.00) | 0.41(0.00) 0.43(0.01) 0.45(0.01) | 0.29(0.00) 0.25(0.00) 0.22(0.00) |
|     | 60   | 0.35(0.00) | 0.41(0.01) 0.43(0.01) 0.45(0.01) | 0.29(0.00) 0.25(0.00) 0.22(0.00) |
| 60  | \(\infty\) | 0.26(0.00) | 0.32(0.00) 0.33(0.01) 0.35(0.01) | 0.20(0.00) 0.17(0.00) 0.15(0.00) |
|     | 20   | 0.26(0.00) | 0.32(0.00) 0.33(0.00) 0.35(0.01) | 0.20(0.00) 0.17(0.00) 0.15(0.00) |
|     | 40   | 0.26(0.00) | 0.32(0.00) 0.33(0.01) 0.35(0.01) | 0.20(0.00) 0.17(0.00) 0.15(0.00) |
|     | 60   | 0.26(0.00) | 0.31(0.00) 0.33(0.00) 0.35(0.00) | 0.20(0.00) 0.17(0.00) 0.15(0.00) |