A NOTE ON CLIFFORD BUNDLES AND CERTAIN
FINSLER TYPE SPACES

Ricardo Gallego Torromé

Department of Mathematics
Faculty of Mathematics, Natural Sciences and Information Technologies
University of Primorska, Koper, Slovenia

Abstract. We study a relation between certain extensions of the Clifford bundle and Finsler type structures that naturally generalize the standard Clifford relation between (pseudo)-Riemannian metric structures and Dirac matrices. We show for flat metrics that there is a triangle map between Finsler structures constructed from an (pseudo)-Riemannian metric and 1-forms on $M$, the extension of the Clifford bundle and relevant Dirac type operators.

1. Introduction

Let $(M, g)$ be a four dimensional (pseudo)-Riemannian space. The Clifford bundle is constructed canonically from the Clifford algebras $C(T_x M, g_x)$ associated to each fiber $T_x M$ or, due to the isomorphism induced by $g$, from the dual bundle with fibers $T^* M$. After introducing natural coordinates, a generic tangent vector $y = y^i \frac{\partial}{\partial x^i} |_{x} \in T_x M$ is represented by the components $\{y^\mu, \mu = 1, 2, 3, 4\}$. Then for each Clifford algebra $C(T_x M, g_x)$ we have pointwise relations

$$y^\mu e_\mu(x) z^\nu e_\nu(x) + z^\nu e_\nu(x) y^\mu e_\mu(x) = 2 g_x(y, z)$$

for each $y, z \in T_x M$ and where $e_\mu$ for $\mu = 1, 2, 3, 4$ are the generators of the Clifford algebra $C(T_x M, g_x)$. In this work we report a generalization of the Clifford relation to certain sheaves constructed from the geometric data of certain types of Finsler spacetimes in dimension four.

The theory of Finsler spaces of positive signature can be formulated in the context of robust geometric frameworks. The case of Finsler structures with Lorentzian metrics still lacks of a general framework, but such models have found many applications in mathematical physics, ranging from application in emergent quantum mechanics, geometries of extended relativistic gravitational models, dual dispersion relations and as a spacetime geometry compatible with clock hypothesis of relativity. It is

email: rigato39@gmail.com
this ample range of applicability of the Finsler concept of space that partially motivates many studies aiming to extend (pseudo)-Riemannian theory to the Finsler setting.

We show in this work that the Clifford relation (1) can be extended to Randers spaces and other types of Finsler spaces determined by vector fields defined on $M$. This extension is achieved by considering certain sub-algebras of a free algebra constructed from the Clifford algebra $C(T_x M, g_x)$ for each $x \in M$ and the trace operator. We also show that the elements of the algebra associated to Finsler type metrics are related with partial first order differential operators. Such relation is established for flat metrics, but the theory can be systematically extended to arbitrary spin manifolds.

The construction of Clifford algebras and Clifford bundles from Finsler type structures and Lagrange structures has been investigated by several authors [12]. However, The method followed in this work makes use of the standard Clifford bundle of a (pseudo)-Riemannian metric and the trace operator to define a particular sheaf, instead of providing a direct generalization using the Lagrange structure.

Finally, let us remark that a formal point of view has been adopted, discussing a generalization of the relation (1) to certain spaces where formal Finsler structures are defined, but disregarding most of the regularity issues concerning the theory of Finsler structures with Lorentzian signature.

2. On certain algebras obtained from the Clifford algebra in dimension four

Let $(M, g)$ be a four dimensional Lorentzian space. Most of our discussion requires only local methods. Therefore, we will make extensive use of local pointwise, orthonormal frames, where $g_x := g|_x$ for $x \in M$ is diagonal. For the components of geometric objects in local orthonormal frames we use Latin indices $\{y^i, i = 1, 2, 3, 4\}$. From now on we are going to work on orthonormal frames, if anything else is not stated. We also adopt Einstein’s index convention. In an orthonormal frame at $x \in M$ the Clifford relations adopt the form

\begin{equation}
\gamma_i \gamma_j + \gamma_j \gamma_i = 2 \eta_{ij} \mathbb{I}_4, \quad i, j = 1, 2, 3, 4,
\end{equation}

where $\eta_{ij}$ are the components of the metric $g$ in the orthonormal frame and $\gamma_i = 1, 2, 3, 4$ are $4 \times 4$ Dirac gamma matrices. In this setting, let us consider the following linear combinations:

\begin{align*}
\mathcal{M} &= \frac{\gamma_i y^i}{|\eta(y, y)|^{1/2}} - \mathbb{I}_4, \\
\tilde{\mathcal{M}} &= \frac{\gamma_i y^i}{|\eta(y, y)|^{1/2}} + \mathbb{I}_4,
\end{align*}

where $\eta(y, y) := \eta_{ij} y^i y^j$ for each $y \in T_x M$ such that $\eta(y, y) \neq 0$. One can also form the linear combinations

\begin{align*}
\mathcal{F}_A &= \gamma_i y^i - A_i y^i \mathbb{I}_4, \\
\tilde{\mathcal{F}}_A &= \gamma_i y^i + A_i y^i \mathbb{I}_4,
\end{align*}
\[ F_{AB} = \gamma_i y^i \gamma_j y^j - (A_i y^i)(B_j y^j) \mathbb{I}_4, \quad \tilde{F}_{AB} = \gamma_i y^i \gamma_j y^j + (A_i y^i)(B_j y^j) \mathbb{I}_4 \]
and similarly for \( F_{ABC} \), etc..., where \( A, B, C, \ldots \in \Lambda^1 M \). The combinations \( \mathcal{M}, \tilde{\mathcal{M}}, \mathcal{F}_A, \tilde{\mathcal{F}}_A, \mathcal{F}_{AB}, \tilde{\mathcal{F}}_{AB}, \text{etc...} \) generate the complex free algebra

\[
\mathcal{A}_C(x, y) := \{ \lambda_M \mathcal{M} + \lambda_{\tilde{\mathcal{M}}} \tilde{\mathcal{M}} + \lambda_{\mathcal{F}_A} \mathcal{F}_A + \lambda_{\tilde{\mathcal{F}}_A} \tilde{\mathcal{F}}_A + \lambda_{\tilde{\mathcal{M}}.\tilde{\mathcal{M}}} \mathcal{M} \cdot \tilde{\mathcal{F}}_A \\
+ \lambda_{\tilde{\mathcal{M}}.\mathcal{F}_A} \mathcal{M} \cdot \tilde{\mathcal{F}}_A + \lambda_{\tilde{\mathcal{F}}.\mathcal{F}_A} \mathcal{F}_A + \lambda_{\tilde{\mathcal{F}}.\tilde{\mathcal{F}}_A} \mathcal{F}_A + \lambda_{\tilde{\mathcal{M}}.\tilde{\mathcal{F}}_A} \tilde{\mathcal{M}} \cdot \mathcal{F}_A + \lambda_{\tilde{\mathcal{M}}.\tilde{\mathcal{F}}_A} \tilde{\mathcal{M}} \cdot \tilde{\mathcal{F}}_A + \lambda_{\tilde{\mathcal{F}}.\tilde{\mathcal{F}}_A} \tilde{\mathcal{F}}_A + \ldots \}
\]

for each \( y \in T_x M \) and \( x \in M \). Note that the algebra \( \mathcal{A}_C(x, y) \) is generated by \( y \)-homogeneous elements. Indeed, the algebra can be decomposed as

\[ \mathcal{A}_C(x, y) = \oplus_{k=0}^{+\infty} A^k_C(x, y), \]

with

\[
A^k_C(x, y) = \left\{ \lambda_M \mathcal{M} + \lambda_{\tilde{\mathcal{M}}} \tilde{\mathcal{M}}, \lambda_M, \lambda_{\tilde{\mathcal{M}}} \in \mathbb{C} \right\},
\]

\[ A^1_C(x, y) = \{ \lambda_{\mathcal{M}_A} \mathcal{M} \cdot \mathcal{F}_A + \lambda_{\mathcal{F}_A} \mathcal{F}_A + \lambda_{\tilde{\mathcal{M}}.\mathcal{F}_A} \mathcal{M} \cdot \tilde{\mathcal{F}}_A + \mathcal{M} \cdot \mathcal{F}_A + \lambda_{\mathcal{F}.\mathcal{F}_A} \mathcal{F}_A + \lambda_{\tilde{\mathcal{M}}.\mathcal{F}_A} \mathcal{M} \cdot \mathcal{F}_A + \lambda_{\tilde{\mathcal{M}}.\tilde{\mathcal{F}}_A} \mathcal{M} \cdot \tilde{\mathcal{F}}_A + \lambda_{\tilde{\mathcal{M}}.\mathcal{F}_A} \tilde{\mathcal{M}} \cdot \mathcal{F}_A + \lambda_{\tilde{\mathcal{M}}.\tilde{\mathcal{F}}_A} \tilde{\mathcal{M}} \cdot \tilde{\mathcal{F}}_A + \ldots \}
\]

Each \( A^k_C(x, y) \) is an \( y \)-homogeneous of degree \( k \), complex vector space. Furthermore, \( \mathcal{A}_C(x, y) \) is a commutative algebra,

**Proposition 2.1.** \( \mathcal{A}_C(x, y) \) is a graded, commutative algebra:

\[
\sum_{k=1}^{+\infty} \alpha_k g_k \sum_{l=1}^{+\infty} \beta_l h_l = 0,
\]

for any pair of finite linear combinations of the form

\[
\sum_{k=1}^{+\infty} \alpha_k g_k = \alpha_M \mathcal{M} + \alpha_{\tilde{\mathcal{M}}} \tilde{\mathcal{M}} + \alpha_{\mathcal{F}_A} \mathcal{F}_A + \alpha_{\tilde{\mathcal{F}}_A} \tilde{\mathcal{F}}_A + \ldots
\]

\[
\sum_{l=1}^{+\infty} \beta_l h_l = \beta_M \mathcal{M} + \beta_{\tilde{\mathcal{M}}} \tilde{\mathcal{M}} + \beta_{\mathcal{F}_A} \mathcal{F}_A + \beta_{\tilde{\mathcal{F}}_A} \tilde{\mathcal{F}}_A + \ldots
\]

with only a number of coefficients in \( \{ \alpha_k, k = 1, 2, \ldots \} \) and \( \{ \beta_k, k = 1, 2, \ldots \} \) non-zero.

**Proof.** The following relations follow straightforward,

\[
[\mathcal{M}, \tilde{\mathcal{M}}] = 0, \quad [\mathcal{M}, \mathcal{F}_A] = 0, \quad [\tilde{\mathcal{M}}, \mathcal{F}_A] = 0, \quad [\mathcal{F}_A, \mathcal{F}_B] = 0, \quad [\mathcal{M}, \mathcal{F}_{AB}] = 0, \quad \ldots
\]
for $A, B, \ldots \in \Lambda^1 M$, from which the result follows. \hfill \square

The collection of algebras $A_C := \{A_C(x, y), (x, y) \in TM \setminus NC\}$ determines a sheaf over bundle $TM \setminus NC \to M$,

$$\pi_A: A_C \to TM \setminus NC.$$ 

The stalk over $(x, y)$ is the algebra $A_C(x, y)$. We call $A_C$ the Clifford sheaf over $TM \setminus NC$.

The trace operation $Tr: A_C \to \mathbb{C}$ satisfies the usual rules as for Dirac matrices. Specifically, the trace operator is a cyclic operator such that in an orthonormal basis the generators $\{\gamma_i, i = 1, 2, 3, 4\}$ satisfies usual Dirac type relations,

$$Tr(\gamma_i) = 0, \quad i = 1, 2, 3, 4,$$

$$Tr(\gamma_i \gamma_j) = 4 \eta_{ij}, \quad i, j = 1, 2, 3, 4,$$

$$Tr(\gamma_i \gamma_j \gamma_k) = 0, \quad i, j, k = 1, 2, 3, 4,$$

$$Tr(\gamma_i \gamma_j \gamma_k \gamma_l) = 4 (\eta_{ij} \eta_{kl} - \eta_{ik} \eta_{jl} + \eta_{il} \eta_{jk}), \quad i, j, k, l = 1, 2, 3, 4,$$

... These relations will play a relevant role in our considerations.

**Proposition 2.2.** For $|g(y, y)|^{1/2} \neq 0$, the following relation holds:

$$(3) \quad Tr(M \cdot F_A) = 4 \left( \mp |g(y, y)|^{1/2} + A_j y^j \right),$$

where the negative sign corresponds to the case when $\eta(y, y) < 0$ and the positive sign corresponds to the case $\eta(y, y) > 0$.

**Proof.** We consider orthonormal coordinates, where the metric $g$ has components $\eta_{ij}$. By direct computation, using the standard properties of the trace of the Dirac matrices and since $g(x, y)$ is numerically equal to $\eta_{ij} y^i y^j$,

we have

$$Tr(M \mathcal{F}_A) = Tr \left( \left( \frac{\gamma_i y^i}{|g(y, y)|^{1/2}} + I_4 \right) \cdot (\gamma_j y^j - A_j y^j I_4) \right)$$

$$= 4 \left( \frac{\eta_{ij} y^i y^j}{|g(y, y)|^{1/2}} + A_j y^j \right)$$

$$= 4 \left( \mp |g(y, y)|^{1/2} + A_j y^j \right),$$

where the negative sign corresponds to $\eta_{ij} y^i y^j < 0$ and the positive sign corresponds to $\eta_{ij} y^i y^j > 0$. Then the result follows. \hfill \square

Similarly, one obtains the relation

$$(4) \quad Tr(M \cdot \tilde{F}_A) = 4 \left( \mp |g(y, y)|^{1/2} - A_j y^j \right),$$

for $g(y, y) < 0$. 

3. Examples

We discuss in this section examples of metric structures that can be described within the framework of the algebra $\mathcal{A}_C$. In all these examples, it is assumed that $(M, g)$ is either a Riemannian or a Lorentzian space. Hence we use the Clifford relation in an orthonormal basis in all our calculations.

3.1. Lorentzian manifold. The element $\mathcal{M} \in \mathcal{A}_C$ provides examples of Riemannian norms,

$$Tr(\mathcal{M} \cdot \tilde{\mathcal{M}}) + 4 = \pm 4|g(y, y)|^{1/2},$$

the sign depending on the temporal (−) or spacelike (+) character of $y \in T_xM$. Similarly, we have

$$Tr(\mathcal{M} \cdot \mathcal{M}) - 4 = \pm 4|g(y, y)|^{1/2},$$

For null vectors $g(y, y)$, one defines the limit

$$\lim_{g(y, y) \to 0} \frac{1}{2} Tr(\mathcal{M} \cdot \mathcal{M}) - 1 = 0.$$  

Therefore, the metric structure $(M, g)$ is encoded in $\mathcal{M} \in \mathcal{A}_C$.

3.2. Angular metric type structure. Given a vector field $A$, the trace $Tr(\mathcal{F}_A \cdot \tilde{\mathcal{F}}_A)$ formally provides a physical relevant metric,

$$Tr(\mathcal{F}_A \cdot \tilde{\mathcal{F}}_A) = 4 \left( g(y, y) - (A \cdot y)^2 \right),$$

where for the 1-form $A \in \Lambda M$ and each tangent vector $x \in T_xM$, one has $A \cdot y := A_i(x) y^i$. In order that $(M, g)$ is a non-degenerate spacetime structure, further constrains on $A$ must be imposed. In particular, it is required that the Hessian of $L_A$

$$L_A(x, y) = g(y, y) - (A \cdot y)^2$$

to be non-degenerated. Since the fundamental tensor of $L_A$ is of the form

$$g_{ij} := \frac{1}{2} \frac{\partial^2 L_A(x, y)}{\partial y^i \partial y^j} = \eta_{ij}(x) - A_i(x)A_j(x),$$

the sufficient condition for this regularity is that $|g(y, y)| < 1$.

Similarly,

$$Tr(\mathcal{F}_A \cdot \mathcal{F}_A) = 4 \left( g(y, y) + (A \cdot y)^2 \right).$$

The Hessian of the corresponding Finsler structure is regular if $|g^*(A, A)| < 1$, where $g^*$ is the dual metric of $g$. 

3.3. Randers type structures. An Asanov-Randers space \([9]\) is characterized by a Finsler function \(F_A : \tilde{N} \to ]-\infty, 0]\) of the form

\[
F_A(x, y) = (-g(y, y))^{1/2} + A \cdot y, \quad g(y, y) \leq 0,
\]

where \(g\) is a metric of Lorentzian signature. As a consequence of Proposition 2.2, we have that for \(y \in T_x M\) such that \(g(y, y) < 0\), the relation

\[
-\frac{1}{4} \text{Tr}(\mathcal{M} \cdot F_A) = F_A(x, y)
\]

holds good. Similarly, if \(g(y, y) > 0\), then the expression

\[
\frac{1}{4} \text{Tr}(\mathcal{M} \cdot \tilde{F}_A) = F_A(x, y)
\]

holds good. Also note that

\[
\lim_{g(y, y) \to 0} -\frac{1}{4} \text{Tr}(\mathcal{M} \cdot F_A) = \lim_{g(y, y) \to 0} \frac{1}{4} \text{Tr}(\mathcal{M} \cdot \tilde{F}_A) = A_j y^j.
\]

Therefore, it is natural to define

\[
F_A(x, y) = A_j y^j, \quad \text{if} \quad g(y, y) = 0.
\]

Gluing together the relations (11), (12) and (13) we have an example of Randers spacetime as introduced in the sheave-theoretical approach to Finsler spacetimes introduced in [9]. To avoid degeneracies of the Finsler structure associated with Lorentz transformations, it is convenient to consider local 1-forms \(A\) defined on open subsets of \(M\). For each of the local representatives \(A \in [\tilde{A}]\) one can define in an open set \(U \subset M\) the function \(F_A : TM \setminus \{0\} \to \mathbb{R}\) as

\[
F_A(x, y) := \begin{cases} 
-\frac{1}{4} \text{Tr}(\mathcal{M} \cdot F_A) & \text{for } g(y, y) < 0, \\
-\lim_{n \to +\infty} \frac{1}{4} \text{Tr}(\mathcal{M}(y_n) \cdot F_A(y_n)) & \text{for } \{y_n\} \to y \text{ with } g(y, y) = 0, \\
\frac{1}{4} \text{Tr}(\mathcal{M} \cdot \tilde{F}_A) & \text{for } g(y, y) > 0.
\end{cases}
\]

Note that

\[
\lim_{n \to +\infty} \text{Tr}(\mathcal{M}(y_n) \cdot F_A(y_n)) = \lim_{n \to +\infty} \text{Tr}(\mathcal{M}(y_n) \cdot \tilde{F}_A(y_n))
\]

for the sequence \(\{y_n\} \to y\) with \(g(y, y) = 0\).

Let us consider now the second order homogeneous formulation of Randers spaces as discussed in [10]. The Lagrangian function is of the form

\[
\bar{L}_A = (|g(y, y)|^{1/2} + A \cdot y)^2.
\]
Then it is easy to show that

\[ \tilde{L}_A(x, y) := \begin{cases} \frac{1}{4} Tr^2(M \cdot \tilde{F}_A) & \text{for } g(y, y) < 0, \\ \lim_{n \to +\infty} \frac{1}{4} Tr^2(M(y_n) \cdot F_A(y_n)) & \text{for } \{y_n\} \to y \text{ with } g(y, y) = 0, \\ \frac{1}{4} Tr^2(M \cdot F_A) & \text{for } g(y, y) > 0. \end{cases} \] (16)

The reason of the difference between (14) and (16) is the relative minus sign between the two terms \( |g(y, y)|^{1/2} \) and \( A \cdot y \) in the definitions of \( F_A \) and \( \tilde{L}_A \). Also, note that because of the properties of the trace of Dirac matrices,

\[ Tr^2(M \cdot F_A) = 4 Tr(M \cdot M \cdot F_A \cdot F_A), \]
\[ Tr^2(M \cdot \tilde{F}_A) = 4 Tr(M \cdot M \cdot \tilde{F}_A \cdot \tilde{F}_A). \]

Thus \( \tilde{L} \) can be re-written directly in terms of trace of elements of the algebra \( A_C \).

**Proposition 3.1.** For the Randers type function (16) one has that

\[ \tilde{L}_A(x, y) := \begin{cases} \frac{1}{4} Tr(M \cdot F_A \cdot \tilde{F}_A)(x, y) & \text{for } g(y, y) < 0, \\ \lim_{n \to +\infty} \frac{1}{4} Tr(M \cdot M \cdot \tilde{F}_A \cdot \tilde{F}_A)(x, y)(x, y) & \text{for } \{y_n\} \to y \text{ with } g(y, y) = 0, \\ \frac{1}{4} Tr(M \cdot F_A \cdot \tilde{F}_A)(x, y) & \text{for } g(y, y) > 0. \end{cases} \] (17)

Let us remark that the relations (8), (9), (11), (12) describe only formal Finsler spacetimes. It is necessary to adopt further restrictions on the 1-form \( A \), in order to satisfy regularity conditions of the associated metric.

The theory sketched above can also be applied to Euclidean signature spaces. If \( g \) is a Riemannian metric, instead than a Lorentzian structure, then the same algebra contains elements related with positive definite Finsler spacetimes of the form \( Tr(M \cdot \tilde{F}_A) \) and \( Tr(F_A \cdot F_A) \). There are also elements of \( A_C \) associated with relevant norm functions of Lorentzian and Riemannian metrics, namely, \( Tr(M \cdot \tilde{M} - I_4) \) and \( Tr(M \cdot M - I_4) \). Furthermore, the methodology discussed above suggests that any metric spacetime structure determined by a Lorentzian metric \( (M, g) \) and a finite set of 1-forms on \( M \) can be expressed in terms of the trace of elements of \( A_C \).

### 3.4. Trace operator and Finsler type structures.

The elements that we have used in the extension are the (pseudo)-Riemannian metric \( g \), differential forms defined on \( M \) and the trace operator \( Tr : A_C \to \mathbb{C} \). Recall that the trace operator has in general the following properties:

- \( Tr(M_1 M_2) = tr(M_2 M_1) \), for \( M_1, M_2 \in A_C \),
- It is linear, \( Tr(M_1 + \lambda M_2) = Tr(M_1) + \lambda Tr(M_2) \).
• It is invariant under unitary transformations,
\[ Tr \left( U M_1 U^{-1} \right) = Tr \left( M_1 \right) \]

These are just the properties required for the construction of our map from elements of \( \mathcal{A}_\mathbb{C} \) to \( \mathbb{C} \). Therefore, the trace operator provides the natural map from \( \mathcal{A}_\mathbb{C} \) to \( \mathcal{A}_\mathbb{C} \) to the Finsler type space structures.

4. FIRST ORDER DIFFERENTIAL OPERATORS AND FINSLER NORMS

The above construction determines several interesting Finsler type space-time structures. Moreover, the sheaf \( \mathcal{A}_\mathbb{C} \) is related to geometric first order operators. In order to simplify the treatment from a geometric point of view, let us consider the case when \( M \) is four dimensional and the metric \( g \) is flat. If the velocity vector \( y \) is expressed in terms of canonical momentum as \( y = p^*/m \), where \( p \) is the 4-momenta of a single particle system and \( p^* \) is the dual vector to \( p \) via \( g \), then one finds easily a relation between the operator \( M = \frac{\gamma_i p^i}{m} - A_4 \), where \( p^i = m y^i \) and the Dirac type operator \( \hat{D} = \gamma_i \hat{p}^i - m \mathbb{I}_4 \):

\[ M = \frac{\gamma_i p^i}{m} - A_4 \quad \Leftrightarrow \quad \hat{D} = i \gamma^i \partial_i - m \mathbb{I}_4, \]

where we have assumed that \( m = \sqrt{|g(p,p)|} \) and \( \hat{p}^i = -i \eta^{ij} \partial_j \). Therefore, the element \( M \) of the algebra \( \mathcal{A}_\mathbb{C} \) is related with the Dirac type operator associated to a Dirac particle of mass \( m \).

Similarly, the element \( F_A = (\gamma_i - m A_i \mathbb{I}_4) p^i \in \mathcal{A}_\mathbb{C} \) has associated the differential operator

\[ \hat{D}_A = i \left( \gamma^i - A^i \mathbb{I}_4 \right) \partial_i. \]

Conversely, one can consider differential operators and assign a Finsler type structures. For instance, we can consider the \( U(1) \) covariant derivative acting on fermions,

\[ \hat{\Gamma}_A := \gamma^i \left( i \partial_i - A_i \mathbb{I}_4 \right), \]

where here \( A_i \) are the components of the potential 1-form. The corresponding element of the algebra \( \mathcal{A}_\mathbb{C} \)

\[ \Gamma_{A,m} := \gamma_i y^i - m \gamma_i A^i \mathbb{I}_4. \]

Applying the trace operator one can see that the corresponding norm function is

\[ Tr(\mathcal{M} \cdot \Gamma_{A,m}) = |g(y,y)|^{1/2} - \frac{y_i A^i}{|g(y,y)|^{1/2}}. \]

This is not an \( y \)-homogeneous norm function. The non-homogeneity due to the fact that \( \Gamma_A \) is not \( y \)-homogeneous. This example shows that the theory can be extended to non-homogeneous Finsler type structures in order to catch significant geometric objects.
5. Conclusion

In this work we have observed that the fundamental functions $F$ of certain metric Randers type structures can be re-written in terms of a trace operator acting on an extension of Clifford bundle. The construction relies on the existence of an underlying Lorentzian structure $(M, g)$ and 1-forms defined on $M$. Furthermore, in the case when $g$ is flat, we have shown a relation between first order operators of Dirac type and non-homogeneous Finsler metrics.

There are two limitations in our theory that we should mention. The first is that the theory does not applies to general Finsler spacetimes, since it requires the underlying Lorentzian structure and 1-forms. Currently, this appears as an essential limitation to the scope of the theory. Second, the relation between first order linear operators and non-homogeneous Finsler spacetimes is limited to the case when $g$ is flat. However, this constraint does not appear as a fundamental one. One of the research lines to be developed consists to have a general formulation of the result for generic Lorentzian underlying metric structure.

The relations discussed in this paper can be developed further in several further directions. From a mathematical point of view, the study of bundles $\mathcal{A}_C$ could reveal interesting geometric facts. From a more physical point of view, the interpretation of Randers type spaces in terms of elements of $\mathcal{A}_C$ opens the possibility to extend spinorial methods to field theories based on Randers spacetimes. This is of particular interest to explore generalizations of Einstein gravity coupled to matter.

References

[1] I. M. Benn and R. W. Tucker, An introduction to spinors and geometry with applications in Physics, Adam Hilger (1988).
[2] L. Berwald, Untersuchung der Krümmung allgemeiner metrischer Räume auf Grund in der ihnen herrschenden Parallellum, Math. Zeitschrift 25, 40 (1926).
[3] Élie Cartan, Exposés de géométrie. II Les espaces de Finsler, Hermann, Paris (1971).
[4] S. S. Chern, Local equivalence and Euclidean connections in Finsler spaces, Sci. Rep. Nat- Tsing Hua Uni. Ser. A 5, 95 (1948).
[5] R. Gallego Torromé, A Finslerian version of 't Hooft Deterministic Quantum Models, J. Math. Phys. 47, 072101 (2006).
[6] R. Gallego Torromé, P. Piccione and H. Vitório, On Fermat’s principle for causal curves in time oriented Finsler spacetimes, J. Math. Phys. 53, 123511 (2012).
[7] R. Gallego Torromé, Foundations for a theory of emergent quantum mechanics and emergent classical gravity, arXiv:1402.5070 [math-ph].
[8] R. Gallego Torromé, On singular generalized Berwald spacetimes and the equivalence principle, International Journal of Geometric Methods in Modern Physics, 14(06):1750091, 2017.
[9] R. Gallego Torromé, On the interplay between the notions of Randers space and gauge invariance, arXiv:0906.1940 [math-ph].
[10] M. Hohmann, C. Pfeifer, N. Voicu, *Finsler gravity action from variational completion*, Phys. Rev. D **100**, 064035 (2019).

[11] C. Pfeifer, *Finsler spacetime geometry in physics*, International J. of Geom. Methods in Mod. Phys. Vol. **16**, No. supp02, 1941004 (2019).

[12] S. Vacaru, P. Stavrinos, E. Gaburov, D. Gonta, *Clifford and Riemann-Finsler Structures in Geometric Mechanics and Gravity*, Differential Geometry-Dynamical Systems, Monograph n. 7, Geometry Balkan Press, (2006).