I. INTRODUCTION

Among the many interesting issues of General Relativity one of the most important is the energy-momentum localization. To solve the problem of energy-momentum localization means to develop an unique mathematical formula for energy density. Nowadays, in General Relativity there are some well-known tools for the calculations of the energy-momentum like superenergy tensors [1], quasi-local expressions [2], the energy-momentum complexes of Einstein [3], Landau-Lifshitz [4], Papapetrou [5], Bergmann-Thomson [6], Weinberg [7], Qadir-Sharif [8] and Møller [9] and the tele-parallel theory of gravitation [10]. The tele-parallel theory of gravitation [10] presents the advantage that the calculations can be performed in such a manner that the problem of the coordinate dependence can be avoided. The pseudotensorial definitions [3]-[9] have been used by many authors and have yielded meaningful and interesting results [11]. The Einstein [3], Landau-Lifshitz [4], Papapetrou [5], Bergmann-Thomson [6], Weinberg [7] and Qadir-Sharif [8] definitions are coordinate dependent and the calculations have to be done in Cartesian coordinates. Only the Møller [9] prescription allows to perform the calculations in any coordinate system. We also notice the similarity of some results obtained with the energy-momentum complexes [3]-[9] with the results given by their tele-parallel versions [12]. A great contribution to the rehabilitation of the pseudotensors has been done by Chang, Nester and Chen [13], they demonstrated that different quasi-local definitions correspond to different boundary conditions. Recently, study of noncommutative geometry has emerged. To quantize the spacetime in string/M theory, it is realized that coordinates may become noncommutative operators on a D-brane[14] - [15]. The result is a discretization of spacetime where the spacetime coordinate operators satisfy the relation $[x^\mu, x^\nu] = i\theta^{\mu\nu}$, where $\theta^{\mu\nu}$ is an anti symmetric matrix which determines the fundamental discretization of spacetime. It is shown that the divergences that appear in General Relativity could be avoided if non commutativity replaces point like structures by smeared objects. The smearing effect is mathematically implemented with a substitution of Dirac delta function by a Gaussian distribution of minimal length $\sqrt{\theta}$. Schwarzschild spacetime is changed some what when a noncommutative spacetime is taken into account.

In this paper, we calculate the energy-momentum for a non-commutative radiating Schwarzschild black hole [14]-[15] and study some limiting cases. For our purpose, we use the Einstein and Møller prescriptions. The structure of our article is as follows: in Section II we present the non-commutative radiating Schwarzschild black hole [14]-[15]. In Section III, we present the Einstein and Møller energy-momentum complexes whereas in Section IV we performed the calculations of the energy distributions for the non-commutative radiating Schwarzschild black hole. In Section V we briefly present our concluding remarks. Throughout our work we use for performing the calculations the signature $(1, -1, -1, -1)$ and the geometrized units ($c = 1; G = 1$). Also, Greek (Latin) indices take value from 0 to 3 and 1 to 3, respectively.
II. NON-COMMUTATIVE RADIATING
SCHWARZSCHILD BLACK HOLE

In this section we present the non-commutative radiating Schwarzschild black hole [14]-[15] that is under study. The spacetime is described by the metric given by

\[ ds^2 = \left[ 1 - \frac{4M}{r} \frac{3}{2} \frac{r^2}{\pi^3} \right] dt^2 - \frac{dr^2}{\left[ 1 - \frac{4M}{r} \frac{3}{2} \frac{r^2}{\pi^3} \right]} - r^2 d\Omega^2, \]

(1)

where \( \gamma \) is the lower incomplete gamma function that has the expression \( \gamma(x, a) = \int_0^x \exp(-t) dt \). In flat spacetime noncommutativity eliminates point-like structures in favor of smeared objects [14]. The authors of [14] considered the mass density of a static, spherically symmetric, smeared, particle-like gravitational source given by

\[ \rho_0(r) = \frac{M}{(4\pi \theta)^{3/2}} \exp \left( -\frac{r^2}{4\theta} \right). \]  

(2)

The particle of mass \( M \) is not localized at a point, but is diffused throughout a region of linear size \( \sqrt{\theta} \). This is the results of the intrinsic uncertainty that is encoded in the coordinate commutator.

At presently accessible energies, i.e. \( \sqrt{\theta} < 10^{-16} \) cm the noncommutativity is not visible. We notice that minimal deviations from standard vacuum Schwarzschild black hole are expected at large distances. Also, at the distance \( r \approx \sqrt{\theta} \) some behaviour of new physics is expected, because in this case the mass density is non negligible and present. For balancing the inward gravitational pull and to prevent droplet to collapse into a matter point the radial pressure \( p_r = -\rho_0 \) has to be different by zero. The spacetime noncommutativity produces this important physical effect on matter. Also, this implies the existence of the new physics at the distance \( r \approx \sqrt{\theta} \).

The Einstein equations were solved considering \( \rho_0(r) \) as a matter source and the resulting gravitational background is given by (1).

The mass distribution is

\[ m(r) \equiv \frac{2M}{\sqrt{\theta}} \frac{3}{2} \frac{r^2}{4\theta}, \]

(3)

with \( M \) being the total mass of the source. Analogous to the General Relativity we have

\[ m'(r) = 4\pi r^2 \rho_0(r). \]

(4)

In the limit \( r/\sqrt{\theta} \to \infty \) the classical Schwarzschild black hole solution is recovered. The metric (1) can give useful insights about possible noncommutative effects on Hawking radiation [14].

III. EINSTEIN AND MØLLER
ENERGY-MOMENTUM COMPLEXES

In this section we present the Einstein and Møller energy-momentum complexes.

The Einstein energy-momentum complex [3] in a four-dimensional gravitational background is given by

\[ \theta_{\mu}^\nu = \frac{1}{16\pi} h_{\mu}^{\nu, \lambda}. \]

(5)

The Einstein superpotentials \( h_{\mu}^{\nu, \lambda} \) have the expression

\[ h_{\mu}^{\nu, \lambda} = \frac{1}{\sqrt{-g}} \partial_{\mu} \left[ -g_{\nu, \lambda} g_{\rho, \sigma} g_{\sigma, \rho} - g_{\nu, \lambda} g_{\rho, \sigma} \right] \]

(6)

and obey the antisymmetry property

\[ h_{\mu}^{\nu, \lambda} = -h_{\nu}^{\mu, \lambda}. \]

(7)

\( \theta_0^0 \) and \( \theta_i^0 \) represent the energy and momentum density components, respectively. The Einstein energy-momentum complex observes the local conservation law

\[ \theta_{\nu, \mu}^\mu = 0. \]

(8)

The energy and momentum in Einstein’s definition are given by

\[ P_{\nu} = \int \int \int \theta_{\mu}^{\alpha} dx^1 dx^2 dx^3 \]

(9)

and applying Gauss’ theorem the energy-momentum is

\[ P_{\nu} = \frac{1}{16\pi} \int \int h_{\nu}^{\mu} n_{\mu} dS, \]

(10)

where \( n_{\mu} \) represents the outward unit normal vector over the surface \( dS \). Here \( P_0 = E \) is the energy.

In (9) and (10) \( P_i, i = 1, 2, 3 \), represent the momentum components.

The definition of the Møller energy-momentum complex [9] is given by

\[ J_{\nu}^\mu = \frac{1}{8\pi} M_{\nu}^{\mu, \lambda}, \]

(12)

where we have the Møller superpotentials \( M_{\nu}^{\mu, \lambda} \) given as below

\[ M_{\nu}^{\mu, \lambda} = \sqrt{-g} \left( \frac{\partial g_{\nu, \sigma}}{\partial x^\sigma} - \frac{\partial g_{\nu, \kappa}}{\partial x^\kappa} \right) g^{\mu, \rho} g^{\sigma, \rho}. \]

(13)
The Møller superpotentials \( M^\mu_\nu \) present the antisymmetric property
\[
M^\mu_\nu = -M^\lambda_\nu. \tag{14}
\]

Very important is that Møller’s energy-momentum complex observes the local conservation law
\[
\partial_\mu J^\mu = 0, \tag{15}
\]
where \( J^0_\mu \) is the energy density and \( J^i_\mu \) represents the momentum density components.

In the Møller definition the energy and momentum are given by
\[
P_\mu = \int \int \int J^0_\mu dx^1 dx^2 dx^3. \tag{16}
\]
The energy distribution is obtained with the expression
\[
E = \int \int \int J^0_0 dx^1 dx^2 dx^3. \tag{17}
\]
With the aid of Gauss’ theorem we obtain
\[
P_\mu = \frac{1}{8\pi} \int \int M^0_\mu v_0 dS. \tag{18}
\]

In their important works Cooperstock [16] and Lessner [17] stressed the importance of the Møller energy-momentum complex. In addition, we notice the good results obtained with the Einstein and Møller definitions for the energy-momentum in the case of various geometries [11].

### IV. ENERGY DISTRIBUTION OF THE NON-COMMUTATIVE RADIATING SCHWARZSCHILD BLACK HOLE

The Einstein definition required Cartesian coordinates for performing the calculations. We transform the gravitational background given by (1) in Schwarzschild Cartesian coordinates, as given by
\[
ds^2 = B(r)dt^2 - (dx^2 + dy^2 + dz^2) - \frac{\gamma(r)^{-1}}{r^2} (x dx + y dy + z dz)^2,
\]
with
\[
B(r) = 1 - \frac{4Mr}{r^3} \gamma(\frac{3}{2}, \frac{r^2}{3M}), \quad A(r) = \frac{1}{1 - \frac{4Mr}{r^3} \gamma(\frac{3}{2}, \frac{r^2}{3M})}.
\]

We use Maple program with the GRTensor II attached package to calculate the energy distribution and momenta and to make plots.

The Einstein superpotentials that we use for the evaluation of the energy distribution \( h^0_{\nu} \) are given by
\[
h^0_{0} = \frac{2r}{r^2 r\sqrt{\pi}} \left( \frac{3}{2} \frac{r^2}{4\theta} \right), \tag{20}
\]
\[
h^0_{\theta} = \frac{2y}{r^2 r\sqrt{\pi}} \left( \frac{3}{2} \frac{r^2}{4\theta} \right), \tag{21}
\]
\[
h^0_{z} = \frac{2z}{r^2 r\sqrt{\pi}} \left( \frac{3}{2} \frac{r^2}{4\theta} \right). \tag{22}
\]

Using (10) and (20)-(22) the expression for the energy distribution in the Einstein definition is given
\[
E_E = M - \frac{Mr}{\sqrt{\pi} \sqrt{\theta}} \exp \left( -\frac{r^2}{4\theta} \right) - \text{Merfc} \left( \frac{1}{2} \sqrt{\frac{r}{\theta}} \right). \tag{23}
\]
The energy depends on the mass \( M \), \( \theta \) parameter and radial coordinate. In the limit case \( r/\sqrt{\theta} \rightarrow \infty \) we obtain the energy of the classical Schwarzschild black hole solution \( E_E = M \).

The Møller superpotential involved in the calculation of the energy \( M^0_\nu \) is
\[
M^0_\nu = \left[ 2M - \frac{2Mr}{\sqrt{\pi} \sqrt{\theta}} \exp(-\frac{r^2}{4\theta}) - \frac{Mr^3}{2\sqrt{\theta} \sqrt{\pi}} \exp(-\frac{r^2}{4\theta}) \right. \\
\left. - 2\text{Merfc} \left( \frac{1}{2} \sqrt{\frac{r}{\theta}} \right) \right] \sin \theta.
\]
\[
\tag{24}
\]

The energy distribution in the Møller prescription is obtained combining (18) with (24)
\[
E_M = M - \frac{Mr}{\sqrt{\pi} \sqrt{\theta}} \exp \left( -\frac{r^2}{4\theta} \right) - \frac{Mr^3}{2\sqrt{\theta} \sqrt{\pi}} \exp \left( -\frac{r^2}{4\theta} \right) - \text{Merfc} \left( \frac{1}{2} \sqrt{\frac{r}{\theta}} \right). \tag{25}
\]

In this case, the energy distribution also presents a dependence on the mass \( M \), \( \theta \) parameter and radial coordinate. In the Møller definition also for the limit \( r/\sqrt{\theta} \rightarrow \infty \), we recovered the energy of the classical Schwarzschild black hole solution \( E_M = M \).
FIG. 1: The plot for the energy $E$ prescribed by Einstein vs. $x = \frac{r}{\sqrt{\theta}}$. Solid and dotted curves cut the x at the horizons of NCBH and Schwarzschild black hole.

FIG. 2: The plot for the energy $E$ prescribed by Møller vs. $x = \frac{r}{\sqrt{\theta}}$. Solid and dotted curves cut the x at the horizons of NCBH and Schwarzschild black hole.

FIG. 3: Comparison of the energy $E$ prescribed by Møller and Einstein vs. $x = \frac{r}{\sqrt{\theta}}$. Solid curve cuts the x at the horizon.

FIG. 4: Variation of the energy $E$ prescribed by Møller with respect to $r$ and $\theta$.

FIG. 5: Variation of the energy $E$ prescribed by Einstein with respect to $r$ and $\theta$.

FIG. 6: Variation of the energy $E$ prescribed by Einstein and Møller with respect to $r$ for a fixed $\theta = .01$.

V. FINAL REMARKS

In our paper, we study the energy distribution of a non-commutative radiating Schwarzschild black hole in the Einstein and Møller prescriptions. In both prescriptions the expressions for energy depend on the mass $M$, $\theta$ parameter as well as radial coordinate. One can observe from the figures 1-3 that the energy in Einstein’s prescription is always positive whereas the energy in Møller’s prescription assumes positive values only after a certain distance from the horizon. Interestingly, we note that the energy in Einstein’s prescription adopts some real values within the horizon. However, in the limiting case $r/\sqrt{\theta} \to \infty$ both yield the same expression for energy as $E_E = E_M = M$ that corresponds to the case of the classical Schwarzschild black hole solution. This also represents the ADM mass. The above limit can be achieved in two ways: either, $\theta \to 0$ i.e. when the noncommutativity is not visible or $r \to \infty$ i.e. at large distance. When $r \approx \sqrt{\theta}$, then $E_M \neq E_E$. In fact $E_E > 0$ and $E_M < 0$. Our results show that the Einstein prescription is a powerful concept than Møller’s prescription. This is also sustained by the meaningful results obtained with the Einstein definition. One can mention the work of Virbhadra [18], where he emphasized the importance of the Einstein prescription.
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