Probability and entropy in quantum theory

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Abstract
Entropic arguments are shown to play a central role in the foundations of quantum theory. We prove that probabilities are given by the modulus squared of wave functions, and that the time evolution of states is linear and also unitary.

1 Introduction
One of the curious features of quantum mechanics is that it is a theory in which probabilities play a most central role and yet, from a foundational point of view, the concept of entropy is conspicuously absent. Entropy appears only later as an auxiliary quantity to be used only when a problem is sufficiently complicated that clean deductive methods have failed and one is forced to use dirtier inference methods. This is curious indeed because once the use of the notion of probability has been accepted, the issue of whether or not quantum mechanics is a theory of inference has been unequivocally settled. Quantum theory should be regarded as a set of rules for reasoning in situations where even under optimal conditions the information available to predict the outcome of an experiment may still turn out to be insufficient. In such a theory entropy, as the measure of the amount of information, should play a central role. It is difficult to avoid the feeling that perhaps the use of entropic arguments has been inadvertently encoded into the usual postulates of quantum mechanics. The purpose of this paper is to show that this is in fact the case.

This paper is a continuation of previous work in which quantum theory is formulated as the only consistent way to manipulate the amplitudes for quantum processes. The result of this consistent-amplitude approach is the standard quantum theory, in a form that is very close to Feynman’s. The first and most crucial step is a decision about the subject matter. We choose a pragmatic, operational approach: statements about a system are identified with

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those experimental setups designed to test them; the objective is to predict the outcomes of experiments [7]. The observation that if two setups are related in some way then information about one may be relevant to predictions about the other leads one to identify the possible relations among setups and to introduce the means for handling these relations quantitatively. This is the role played by the amplitudes. Amplitudes are tools for reasoning that encode information about how one builds complicated setups by combining more elementary ones. The question of how amplitudes are used to predict the outcomes of experiments is addressed through a single general interpretative rule. A brief summary is given in section 2.

It is quite remarkable that although the interpretative rule does not in itself involve probabilities it can be used to prove the Born statistical “postulate” (or, better, Born “rule”) [8] provided one extra ingredient is added. The point is that the application of the interpretative rule requires a criterion to quantify the change in amplitudes when setups are modified. In ref. [8] the criterion adopted was to use the Hilbert norm as the means to measure the distance between wave functions. This may seem reasonable but such a very technical assumption without an obvious physical basis clearly detracts from the beauty of the argument. In section 3 this blemish is corrected. The realization that the components out of which setups are built, the filters, already supply us with a notion of orthogonality brings us very close to the inner product needed for the Hilbert norm. We close this remaining gap using a symmetry argument. The implication is very interesting: the opinion that quantum probabilities differ in an essential way from ordinary classical probabilities is very widespread; not only are they calculated using apparently different rules but, being given a priori by the Born rule, they also seem to depend less on information, to be more “objective”. Our proof of the Born rule, tracing it to a form of the principle of insufficient reason, supports the opposite point of view, that such differences are an illusion, that there is only one kind of probability.

The constraint that amplitudes be assigned consistently leads to a time evolution that is linear [3] but remains silent about whether it should also be unitary. A common explanation is that imposing unitary time evolution guarantees that probabilities be conserved. This is true but it is also irrelevant; that probabilities should add up to one is a matter of definition [9]; the “non-conservation of probabilities” can always be “fixed” by a suitable reinterpretation. The usual symmetry arguments based on Wigner’s theorem are also inadequate. Why should time evolution be a symmetry in the very technical sense of preserving inner products? Finally, arguments based on von Neumann’s entropy are circular [10], they implicitly assume that one can measure observables other than position [10], an assumption which itself (see section 5) relies on unitary evolution.

The argument we offer in section 4 is based on the idea of array entropy, a concept that was briefly introduced by Jaynes [12] only to be dismissed as an inadequate candidate for the entropy of a quantum system, a quantity which
he rightfully identified with von Neumann’s entropy. From our point of view, however, amplitudes and wave functions are assigned not just to the system but to the whole experimental setup, and this turns the array entropy into a very useful notion. The idea is simple. In situations where the information available for the prediction of experimental outcomes is not spoiled by just waiting entropy should be conserved. Our problem is to identify the appropriate entropy (it is the array entropy). Its conservation implies the conservation of the Hilbert norm and unitary evolution. As claimed above, the postulate that time evolution is unitary is derivable from an entropic argument.

2 The consistent-amplitude approach to quantum theory

The objective of quantum theory is to predict the outcomes of experiments; statements about the quantum system are identified with the experimental setups designed to test them [3][4]. To avoid irrelevant technical distractions we consider a very simple system, a particle that lives on a discrete lattice and has no spin or other internal structure. The generalization to more complex configuration spaces should be straightforward.

The simplest experimental setup, denoted by \([x_f, x_i]\), consists of placing a source that prepares the particle at a space-time point \(x_i = (\vec{x}_i, t_i)\) and placing a detector at \(x_f = (\vec{x}_f, t_f)\). To test a more complex statement such as “the particle goes from \(x_1\) to \(x_f\) and from there to \(x_f\)’’ denoted by \([x_f, x_1, x_f]\), requires a more complex setup involving an idealized device, a “filter” which prevents any motion from \(x_i\) to \(x_f\) except via the intermediate point \(x_1\). This filter is some sort of obstacle or screen that exists only at time \(t_1\), blocking the particle everywhere in space except for a small “hole” around \(\vec{x}_1\). The possibility of introducing many filters each with many holes leads to allowed setups of the general form \(a = [x_f, s_N, s_{N-1}, \ldots, s_2, s_1, x_i]\) where \(s_n = (x_n, x'_n, x''_n, \ldots)\) is a filter at time \(t_n\), intermediate between \(t_i\) and \(t_f\), with holes at \(\vec{x}_n, \vec{x}'_n, \vec{x}''_n, \ldots\).

There are two basic kinds of relations among setups. The first, called \(\text{and}\), arises when two setups \(a\) and \(b\) are placed in immediate succession resulting in a third setup which we denote by \(ab\). It is necessary that the destination point of the earlier setup coincide with the source point of the later one, otherwise the combined \(ab\) is not allowed. The second relation, called \(\text{or}\), arises from the possibility of opening additional holes in any given filter. Specifically, when (and only when) two setups \(a'\) and \(a''\) are identical except on one single filter where none of the holes of \(a'\) overlap any of the holes of \(a''\), then we may form a third setup \(a\), denoted by \(a' \lor a''\), which includes the holes of both \(a'\) and \(a''\). Provided the relevant setups are allowed the basic properties of \(\text{and}\) and \(\text{or}\) are quite obvious: \(\text{or}\) is commutative, but \(\text{and}\) is not; both \(\text{and}\) and \(\text{or}\) are
associative, and finally, and distributes over or.

A quantitative representation of and/or is obtained by assigning a single complex number $\phi(a)$ to each setup $a$ in such a way that relations among setups translate into relations among the corresponding complex numbers. What gives the theory its robustness, its uniqueness, is the requirement that the assignment be consistent: if there are two different ways to compute $\phi(a)$ the two answers must agree. The remarkable consequence of the consistency constraints is the possibility of regrading $\phi(a)$ with a function $\psi$ to switch to an equivalent and particularly convenient representation, $\psi(a) \equiv \psi(\phi(a))$, in which and and or are respectively represented by multiplication and addition. Explicitly, $\psi(ab) = \psi(a) \psi(b)$ and $\psi(a \lor a') = \psi(a) + \psi(a')$. Complex numbers assigned in this way are called “amplitudes”. For a similar (earlier) derivation of the quantum sum and product rules see ref. [13].

The observation that a single filter that is totally covered with holes is equivalent to having no filter at all leads to the fundamental equation of motion. The idea is expressed by writing the relation among setups $[x_f, x_i] = \bigvee_{\text{all } \vec{x}} \psi([x_f, x_t][x_t, x_i])$ in terms of the corresponding amplitudes. Using the sum and product rules, we get $\psi(x_f, x_i) = \sum_{\text{all } \vec{x}} \psi(x_f, x_t) \psi(x_t, x_i)$.

Following Feynman [6], we introduce the wave function $\Psi(\vec{x}, t)$ as the means to represent those features of the setup prior to $t$ that are relevant to time evolution after $t$. Notice that there are many possible combinations of starting points $x_i$ and of interactions prior to the time $t$ that will result in identical evolution after time $t$. What these different possibilities have in common is that they all lead to the same numerical value for the amplitude $\psi(x_t, x_i)$. Therefore we set $\Psi(\vec{x}, t) = \psi(x_t, x_i)$ and all reference to the irrelevant starting point $x_i$ can be omitted. The traditional language is that $\Psi$ describes the state of the particle at time $t$, that the effect of the interactions was to prepare the particle in state $\Psi$. Now we see that the word “state” just refers to a concise means of encoding information about those aspects of the setup prior to the time $t$ that are relevant for evolution into the future.

The equation of motion can then be written as

$$\Psi(\vec{x}_f, t_f) = \sum_{\text{all } \vec{x}} \psi(\vec{x}_f, t_f; \vec{x}, t) \Psi(\vec{x}, t),$$

(1)

which is equivalent to a linear Schrödinger equation as can easily be seen by differentiating with respect to $t_f$ and evaluating at $t_f = t$.

This result is important because a variety of nonlinear modifications of quantum mechanics have been proposed, either attempting to solve the problems

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1These are physical rather than logical connectives. They represent our idealized ability to construct more complex setups out of simpler ones and they differ substantially from their Boolean and quantum logic counterparts. In Boolean logic not only and distributes over or but or also distributes over and while in quantum logic propositions refer to quantum properties at one time rather than to processes in time.
with macroscopic quantum superpositions, or to explore the possibility that the linear theory might just be a low “intensity” limit of the true nonlinear theory \[14\]. Furthermore, even though experimental bounds have become increasingly stringent \[15\] experimentation alone cannot logically rule out small nonlinearities.

The question of how amplitudes or wave functions are used to predict the outcomes of experiments is addressed through the time evolution equation \([1]\). For example, suppose the preparation procedure is such that \(\Psi(\vec{x},t)\) vanishes at a certain point \(\vec{x}_0\). Then, according to eq. \([1]\), placing an obstacle at the single point \((\vec{x}_0, t)\) (i.e., placing a filter at \(t\) with holes everywhere except at \(\vec{x}_0\)) should have no effect on the subsequent evolution of \(\Psi\). Since relations among amplitudes are meant to reflect corresponding relations among setups, it seems natural to assume that the presence or absence of the filter will have no effect on whether detection at \(x_f\) occurs or not. Therefore when \(\Psi(\vec{x}_0,t) = 0\) the particle will not be detected at \((\vec{x}_0, t)\).

This idea can be generalized to the following general interpretative rule: Suppose the wave function of a setup is \(\Psi(t)\) and at time \(t\) one introduces or removes a filter that blocks out those components of \(\Psi\) characterized by a certain property \(P\). Suppose further that this modification of the setup has a negligible effect on the evolution of \(\Psi\) after \(t\). Then when the wave function is \(\Psi(t)\) property \(P\) will not be detected.

In ref. \([1]\) we showed how this interpretative rule implies the Born statistical postulate provided one uses the Hilbert norm as the means to quantify the change in the wave function as it evolves through a filter. In the next section we show why the choice of the Hilbert norm is the natural one.

3 The Hilbert inner product

In order to justify the use of the Hilbert norm we show how the concepts of distance and angle among states, that is an inner product, can be physically motivated. The argument has three parts.

First, we note that wave functions form a linear space. To illustrate this point suppose that \(\Psi_1(\vec{x},t) = \psi(\vec{x},t; \vec{x}_1, t_0)\) is the wave function at time \(t\) of a particle that at time \(t_0\) was prepared at the point \(\vec{x}_1\), and \(\Psi_2(\vec{x},t) = \psi(\vec{x},t; \vec{x}_2, t_0)\) is the wave function at time \(t\) of a particle that at time \(t_0\) was prepared at the point \(\vec{x}_2\). One way to prepare linear superpositions of \(\Psi_1(\vec{x},t)\) and \(\Psi_2(\vec{x},t)\) is by placing the source at an initial point \((\vec{x}_i, t_i)\) with \(t_i\) earlier than \(t_0\) and letting the particle evolve through a filter at \(t_0\) with holes at \(\vec{x}_1\) and \(\vec{x}_2\). Then the amplitude \(\psi(\vec{x},t; \vec{x}_i, t_i)\) is

\[
\psi(\vec{x}, t; \vec{x}_i, t_i) = \psi(\vec{x}, t; \vec{x}_1, t_0)\psi(\vec{x}_1, t_0; \vec{x}_i, t_i) + \psi(\vec{x}, t; \vec{x}_2, t_0)\psi(\vec{x}_1, t_0; \vec{x}_i, t_i) ,
\]

and, in an obvious notation, the wave function at time \(t\) is given by the superposition \(\Psi(\vec{x},t) = \alpha\Psi_1(\vec{x},t) + \beta\Psi_2(\vec{x},t)\). Notice that the complex numbers \(\alpha\) and
\( \beta \) can be changed at will by changing the starting point \((\vec{x}_i, t_i)\) or by modifying the setup between \(t_i\) and \(t_0\) in any arbitrary way.

The second part of the argument is to point out that the basic components of setups, the filters, already supply us, without any additional assumptions, with a concept of orthogonality. The action of a filter \(P\) at time \(t\) with holes at a set of points \(\vec{x}\) is to turn the wave function \(\Psi(\vec{x})\) into the wave function \(P\Psi(\vec{x}) = \sum_p \delta_\vec{x},\vec{x}_p \Psi(\vec{x})\), and since filters \(P\) act as projectors, \(P^2 = P\), any given filter defines two special classes of wave functions. One is the subspace of those wave functions such as \(\Psi_P \equiv P\Psi\) that are unaffected by the filter, \(P\Psi_P = \Psi_P\). The other is the subspace of those that are totally blocked by the filter, such as \(\Psi_{1-P} \equiv (1-P)\Psi\), for which \(P\Psi_{1-P} = 0\). We will say that these two subspaces are orthogonal to each other.

Any wave function can be decomposed into orthogonal components, \(\Psi = \Psi_P + \Psi_{1-P}\). A particularly convenient expansion in orthogonal components is that defined by a complete set of elementary filters. A filter \(P_i\) is elementary if it has a single hole at \(\vec{x}_i\); it acts by multiplying \(\Psi(\vec{x})\) by \(\delta_\vec{x},\vec{x}_i\); the set is complete if \(\sum_i P_i = 1\). Then \(\Psi(\vec{x}) = \sum_i A_i \delta_\vec{x},\vec{x}_i\), where \(A_i = \Psi(\vec{x}_i)\).

In the third and last step of our argument, as a matter of convenience, we switch to the familiar Dirac notation. Instead of writing \(\Psi_1(\vec{x}, \vec{x}_0)\) and \(\delta_\vec{x},\vec{x}_i\), we shall write \(|\Psi\rangle\) and \(|i\rangle\), so that \(\Psi = \sum_i A_i |i\rangle\). The question is what else, in addition to the notion of orthogonality described above, is needed to determine a unique inner product. Recall that an inner product satisfies three conditions:

(a) \(|\Psi|\Psi\rangle \geq 0\) with \(|\Psi|\Psi\rangle = 0\) if and only if \(|\Psi\rangle = 0\), (b) linearity in the second factor \(\langle \Phi | \Psi_1 + \Psi_2 | \Phi \rangle = \langle \Phi | \Psi_1 | \Phi \rangle + \langle \Phi | \Psi_2 | \Phi \rangle\), and (c) antilinearity in the first factor, \(\langle \Phi | \Psi \rangle = \langle \Psi | \Phi \rangle^*\). Conditions (b) and (c) determine the product of the state \(|\Psi\rangle = \sum_j B_j |j\rangle\) with \(|\Psi\rangle = \sum_i A_i |i\rangle\) in terms of the product of \(|j\rangle\) with \(|i\rangle\), \(\langle \Phi | \Psi \rangle = \sum_j B_j^* A_j \langle j| i \rangle\). The orthogonality of the basis functions \(\delta_\vec{x},\vec{x}_i\) is encoded into the inner product by setting \(\langle j| i \rangle = 0\) for \(i \neq j\), but the case \(i = j\) remains undetermined, constrained only by condition (a) to be real and positive. Clearly an additional ingredient is needed. What could be more natural than the symmetry argument that if space itself is homogeneous then there is a priori no reason to favor one location over another? We therefore choose \(\langle i|i \rangle\) equal to a constant which, without losing generality, we set equal to one. Thus the principle of insufficient reason enters quantum theory through the inner product, \(\langle i|j \rangle = \delta_{ij} \Rightarrow \langle \Phi | \Psi \rangle = \sum_i B_i^* A_i\), and this leads to the Hilbert norm \(||\Psi||^2 = \sum_i |A_i|^2\).

One should emphasize that the symmetry argument invoked here differs from the usual symmetry arguments leading to conservation laws through Noether’s theorem. The latter depends strongly on the particular form of the Hamiltonian; the former does not.

The deduction of the Born statistical rule now proceeds as in ref. 4. Briefly the idea is as follows. We want to predict the outcome of an experiment in which a detector is placed at a certain \(\vec{x}_k\) when the system is in state \(|\Psi\rangle = \sum_i A_i |i\rangle\).
In [4] we showed that the state for an ensemble of $N$ identically prepared, independent replicas of our particle is the product $|\Psi_N\rangle = \prod_{i=1}^{N} |\Psi_i\rangle$. Now we apply the interpretative rule. Suppose that in the $N$-particle configuration space we place a special filter, denoted by $P_{f,\varepsilon}$, the action of which is to block all components of $|\Psi_N\rangle$ except those for which the fraction $n/N$ of replicas at $\vec{x}_k$ lies in the range from $f - \varepsilon$ to $f + \varepsilon$. The difference between the states $P_{f,\varepsilon}|\Psi_N\rangle$ and $|\Psi_N\rangle$ is measured by the relative Hilbert distance, $\|P_{f,\varepsilon}|\Psi_N\rangle - |\Psi_N\rangle\|^2/\langle\Psi_N|\Psi_N\rangle$. The result of this calculation is [4]

$$\lim_{N \to \infty} \|P_{f,\varepsilon}|\Psi_N\rangle - |\Psi_N\rangle\|^2 = 1 - \int_{f-\varepsilon}^{f+\varepsilon} \delta\left(f' - |A_k|^2\right) df',$$  

where we have normalized $\langle\Psi|\Psi\rangle = \langle\Psi_N|\Psi_N\rangle = 1$. We see that for large $N$ the filter $P_{f,\varepsilon}$ has a negligible effect on the state $|\Psi_N\rangle$ provided $f$ lies in a range $2\varepsilon$ about $|A_k|^2$. Therefore the state $|\Psi_N\rangle$ does not contain any fractions outside this range. On choosing stricter filters with $\varepsilon \to 0$ we conclude that detection at $\vec{x}_k$ will certainly occur for a fraction $|A_k|^2$ and that it will not occur for a fraction $1 - |A_k|^2$. For any one of the identical individual replicas there is no such certainty; the best one can do is to say that detection will occur with a certain probability $\Pr(k)$. In order to be consistent with the law of large numbers the assigned value must agree with the Born rule,

$$\Pr(k) = |A_k|^2.$$  

Had we weighted the $|i\rangle$'s differently and chosen a different normalization $\langle i|i\rangle = w_i$, the probability would be given by $\Pr(i) = w_i|A_i|^2$ rather than by eq. (3). It is instructive to explore this issue further particularly in the continuum limit. Let us weight each cell of the discrete lattice by its own volume, call it $g_i^{1/2}\Delta x$, and let $\Delta x \to dx$. Replacing $w_i^{-1}|i\rangle = (g_i^{1/2}\Delta x)^{-1}|i\rangle$ by $\delta\vec{x}$ the completeness condition $\sum_i P_i = \sum_i w_i^{-1}|i\rangle\langle i|$ becomes $1 = \int g^{1/2}dx |\vec{x}\rangle\langle \vec{x}|$. Next, replace $\delta_{ij}/\Delta x$ by $\delta(\vec{x} - \vec{x'})$ and the inner product $\langle i|j\rangle = w_i\delta_{ij}$ becomes $\langle \vec{x}|\vec{x'}\rangle = g^{-1/2}\delta(\vec{x} - \vec{x'})$. Finally, replace $A_i$ by $A(\vec{x})$ and the state $|\Psi\rangle = \sum_i A_i|i\rangle$ becomes $|\Psi\rangle = \int g^{1/2}dx A(\vec{x})|\vec{x}\rangle$. The Born rule, eq. (3), becomes $\Pr(dx) = g^{1/2}dx |A(\vec{x})|^2$; $|A(\vec{x})|^2$ is the probability density. These results apply both to situations in which the choice of coordinates is such that the homogeneity of space is not obvious, and also to intrinsically inhomogeneous, curved spaces.

It is sometimes argued that while there is an element of subjectivity in the nature of classical probabilities that quantum probabilities are different, that they are totally objective because they are given by $|A|^2$. We have just shown that this assignment is neither more nor less subjective than say, assigning probabilities to each face of a die. Just like one assigns probability 1/6 when there is no reason to favor one face of a die over another, the Born rule follows, even in curved spaces, from giving the same a priori weight, the same preference, to spatial volume elements that are equal. (Perhaps it should be the other way
around: equally preferred spatial regions are assigned equal volumes. This would explain what a physical volume is: just a measure of a priori preference.) We have thus uncovered an interesting connection between quantum theory and the geometry of space. The full implications of this connection remain to be explored.

4 Array entropy and unitary time evolution

When we know everything that is relevant about the experimental setup prior to time $t = 0$ we know $\Psi(\vec{x}, 0)$; this situation is one of optimal information. But if less information is available perhaps the best we can do is conclude that the actual preparation procedure is one among several possibilities $\alpha = 1, 2, 3, ...$ each one with probability $p_\alpha$. (For simplicity we initially assume these possibilities form a discrete set.) The usual linguistic trap is to say the system is in state $\Psi_\alpha(\vec{x}, 0)$ with probability $p_\alpha$, but it is better to say that the preparation procedure is $\Psi_\alpha(\vec{x}, 0)$ with probability $p_\alpha$. To this state of knowledge, which one may represent as a set of weighted points in Hilbert space, and which Jaynes referred to as an array\footnote{If the states are normalized the points of the array lie on the surface of a unit sphere, but normalization is not necessary for our argument.}, one may associate the entropy

$$S_A = - \sum_\alpha p_\alpha \log p_\alpha .$$

Jaynes’ objection to using this quantity as the entropy of the quantum system is that if the $\Psi_\alpha(\vec{x}, 0)$ are not orthogonal then the $p_\alpha$ are not the probabilities of mutually exclusive events. When regarded as a property or an attribute of the quantum system the various $\Psi_\alpha(\vec{x}, 0)$ need not, in fact, be mutually exclusive; if $\langle \Psi_\alpha | \Psi_\beta \rangle \neq 0$, knowing that the system is in $\Psi_\alpha(\vec{x}, 0)$ does not exclude the possibility that it will be found in $\Psi_\beta(\vec{x}, 0)$. However, if the $\Psi_\alpha(\vec{x}, 0)$ are attributes of the preparation procedure then they are mutually exclusive because the preparation devices are macroscopic! $S_A$ is the entropy of the preparation procedure not the entropy of the quantum system.

The importance of this conceptual point cannot be overemphasized and a more explicit illustration may clarify it further. Consider a spin $1/2$ particle prepared either with spin along the $z$ direction or with spin along the $x$ direction. These states are not orthogonal and by looking at the particle there is no sure way to tell which of the two alternatives holds, and yet the slightest glimpse at the Stern-Gerlach magnets will reveal which of the two mutually exclusive orientations was used. One can distinguish non-orthogonal states by looking at the devices that prepared the system rather than by looking at the system itself.

Turning to the issue of time evolution, we consider situations where those parts of the setup after time 0 are known and no further uncertainty is intro-
duced. Under these conditions the points of the new array are shifted from $\Psi_\alpha(\vec{x}, 0)$ to $\Psi_\alpha(\vec{x}, t)$ but their probabilities $p_\alpha$ and the corresponding array entropy $S_A$ remain unchanged.

So far our uncertainty about the preparation procedure was of a rather simple nature, it led to a probability distribution defined over a discrete array. But in general there is no such restriction and we may deal with a continuous array. The simplest continuous array is one dimensional, a weighted curve $C$ in Hilbert space. We could consider higher dimensional arrays but this would unnecessarily obscure the argument. The reparametrization-invariant entropy of this continuous array is

$$S_A = -\int_C d\alpha p(\alpha) \log \frac{p(\alpha)}{\ell(\alpha)},$$

(6)

where $p(\alpha)d\alpha$ is the probability that the preparation procedure lies in the interval between $\alpha$ and $\alpha + d\alpha$ and $\ell(\alpha)d\alpha$ is a measure of the distance in Hilbert space between $\Psi_\alpha(\vec{x}, 0)$ and $\Psi_{\alpha+d\alpha}(\vec{x}, 0)$. As discussed in the last section the Hilbert norm is the uniquely natural choice of distance, thus $\ell(\alpha)d\alpha = \|\Psi_{\alpha+d\alpha}\rangle - |\Psi_\alpha\rangle\|.

The possibility of continuous arrays adds a new twist to our considerations about time evolution. Again we consider setups for which no further uncertainty is introduced between times 0 and $t$. We find that points $\Psi_\alpha(\vec{x}, 0)$ of the old line array at $t = 0$ will move to points $\Psi_\alpha(\vec{x}, t)$ to form a new line array at time $t$. Since no information was lost between times 0 and $t$ we expect that, just as in the discrete case, the probabilities $p(\alpha)d\alpha$ remain unchanged and the corresponding array entropy $S_A$ is conserved. But entropy conservation,

$$\frac{\partial S_A}{\partial t} = \int_C d\alpha \frac{p(\alpha)}{\ell(\alpha)} \frac{\partial \ell(\alpha)}{\partial t} = 0,$$

(7)

should hold for any curve $C$ and any function $p(\alpha)$, therefore $\partial \ell(\alpha)/\partial t = 0$. Thus the conservation of the array entropy leads to the conservation of Hilbert space distances. Time evolution must be unitary; the Hamiltonian must be Hermitian.

5 Observables other than position

We have only discussed the measurement of position. How about other observables, uncertainty relations, and so many other notions that are central in standard quantum theory? Our brief answer (a more detailed discussion will appear in [18]) is that other observables are useful concepts in that they facilitate the description of complex experiments but, from our point of view, they are of only secondary importance and play no role at the foundational level.

One can effectively build more complex detectors by modifying the setup (by introducing, e.g., magnetic fields or diffraction gratings) just prior to the final
position detection at $x_f$. The skill of an experimentalist consists of arranging the interactions between time $t$ and the time of detection $t_f$ in such a way that each state $\Phi_n(x,t)$ of an orthogonal set evolves to a corresponding state $\phi_n(x,t_f) = \delta_{x,x_n}$ which also form an orthogonal set. Then, when the particle is finally detected at time $t_f$ we say that “at time $t_f$ the particle was found at $x_n$,“ or alternatively, we convey the same information by saying, somewhat inappropriately, that “the particle was found to be in state $\Phi_n(x,t)$ at time $t$.”

What this particular complex detector “measures” is all observables of the form $Q = \sum_n f_n |\Phi_n\rangle \langle \Phi_n|$. It is noteworthy that the eigenvalues $f_n$ need not be real; the observables $Q$ are diagonalizable, i.e., normal ($[Q,Q^\dagger] = 0$), but not necessarily Hermitian. Clearly, this notion of observables other than position can only be introduced after one understands that time evolution must be unitary.

6 Final remarks

For over a century now an enormous effort has been directed towards deriving the second law of thermodynamics from the laws of mechanics. The successive realization by Gibbs, by Shannon [1], and even more so by Jaynes [2] that the validity of entropic arguments rests on elements that are foreign to mechanics opens the way to inverting the logic and deriving the laws of mechanics from those same principles of inference which lie at the heart of thermodynamics. In fact, according to Jaynes’ beautiful explanation [17], the validity of the second law hinges on the conservation of Gibbs’ or von Neumann’s entropies. Could this conservation also be used to deduce unitary time evolution? No. Such arguments would be circular because these entropies rely for their very definition on having singled out certain measures (phase space volumes, and Hilbert norms respectively) as being privileged and the only reason they are special is precisely that they are conserved under unitary time evolution.

In this work, however, we have given an argument that singles out the Hilbert norm without appealing to unitarity; this clears the road to defining an entropy, the array entropy, from the conservation of which one can deduce the unitarity of time evolution.

The mystery of why complex numbers are sufficient to encode information about relations between setups remains. It seems that one could use other mathematical objects with the required associativity and distributivity, for example matrices or other Clifford numbers [19]. The recent work by Rodríguez [20] may contain important steps in exploring this possibility from a rather different point of view. My own belief is that the connection between the quantum inner product and spatial measures of volume strongly suggests that the reason for complex numbers will be found in the geometry of space. Perhaps eventually even the geometry of space itself will be determined by entropic considerations as well.
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References

[1] C. E. Shannon, Bell Systems Tech. Jour. 27, 379, 623 (1948), reprinted in C. E. Shannon and W. Weaver “The Mathematical Theory of Communication” (Univ. of Illinois Press, Urbana, 1949); see also

[2] “E. T. Jaynes: Papers on Probability, Statistics and Statistical Physics,” edited by R. D. Rosenkrantz (Reidel, Dordrecht, 1983).

[3] A. Caticha, Phys. Lett. A244, 13 (1998) (quant-ph/9803086).

[4] A. Caticha, Phys. Rev. A57, 1572 (1998) (quant-ph/9804012).

[5] P. A. M. Dirac, “The Principles of Quantum Mechanics,” (Oxford, 1958).

[6] R. P. Feynman, Rev. Mod. Phys. 20, 267 (1948); R. P. Feynman and A. R. Hibbs, “Quantum Mechanics and Path Integrals,” (McGraw-Hill, 1965).

[7] H. Stapp, Am. J. Phys. 40, 1098 (1972).

[8] The fact that Born’s postulate is actually a theorem has been independently discovered several times: A. M. Gleason, J. Rat. Mech. Anal. 6, 885 (1957); D. Finkelstein, Trans. NY Acad. Sci. 25, 621 (1963); J. B. Hartle, Am. J. Phys. 36, 704 (1968); N. Graham, in “The Many-Worlds Interpretation of Quantum Mechanics” edited by B. S. DeWitt and N. Graham (Princeton, 1973). The limit $N \to \infty$ where $N$ is the number of replicas of the system is further discussed in E. Farhi, J. Goldstone and S. Gutman, Ann. Phys. 192, 368 (1989) and in ref. 20.

[9] R. T. Cox, Am. J. Phys. 14, 1 (1946).

[10] S. Weinberg, Phys. Rev. Lett. 63, 1115 (1989).

[11] R. Blankenbecler and M. H. Partovi, Phys. Rev. Lett. 54, 373 (1985).

[12] E. T. Jaynes, Phys. Rev. 108, 171 (1957), reprinted in 2.

[13] Y. Tikochinsky, Int. J. Theor. Phys. 27, 543 (1988) and J. Math. Phys. (1988).
[14] See for example: L. de Broglie “Non-Linear Wave Mechanics-A Causal Interpretation,” (Elsevier, Amsterdam, 1950); P. Pearle, Phys. Rev. D13, 857 (1976); I. Bialynicki-Birula and J. Mycielski, Ann. Phys. (NY) 100, 62 (1976); A. Shimony, Phys. Rev. A20, 394 (1979); S. Weinberg, Phys. Rev. Let. 62, 485 (1989), and Ann. Phys. (NY) 194, 336 (1989); N. Gisin, Helv. Phys. Acta 62, 363 (1989) and Phys. Lett. 143, 1 (1990); J. Polchinski, Phys. Rev. Lett. 66, 397 (1991); H. Scherer and P. Busch, Phys. Rev. 47, 1647 (1993).

[15] C. G. Shull, D. K. Atwood, J. Arthur, and M. A. Horne, Phys. Rev. Let. 44, 765 (1980); R. Gahler, A. G. Klein, and A. Zeilinger, Phys. Rev. A23, 1611 (1981); J. J. Bollinger et al., Phys. Rev. Let. 63, 1031 (1989).

[16] E. T. Jaynes, in “Statistical Physics”, Vol. 3, K. W. Ford, ed., p. 182 (Benjamin, NY 1963) and IEEE Trans. Syst. Sci. Cybern. Vol. SSC-4, 227 (1968), both reprinted in [2]; J. E. Shore and R. W. Johnson, IEEE Trans. Inf. Th. Vol. IT-26, 26 (1980).

[17] E. T. Jaynes, Am. J. Phys. 33, 391 (1965).

[18] A. Caticha, to be submitted to Phys. Rev. A.

[19] See e.g., D. Finkelstein, J. M. Jauch, S. Schiminovich and D. Speiser, J. Math. Phys. 3, 207 (1962) and 4, 788 (1963); S. L. Adler, Phys. Rev. Lett. 55, 783 (1985) and Comm. Math. Phys. 104, 611 (1986); D. Hestenes, “Spacetime Algebra” (Gordon and Breach, 1966); W. E. Baylis (ed.) “Clifford (Geometric) Algebras” (Birkhäuser, Boston, 1996).

[20] C. C. Rodríguez, “Are we cruising hypothesis space?” and “Unreal probabilities–partial truth with Clifford numbers”, both in these Proceedings.