Density operator of a system pumped with polaritons: a Jaynes–Cummings-like approach

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Abstract
We investigate the effects of considering two different incoherent excitation mechanisms on microcavity quantum dot systems modeled using the Jaynes–Cummings Hamiltonian. When the system is incoherently pumped with polaritons it is able to sustain a large number of photons inside the cavity with Poisson-like statistics in the stationary limit, and it also leads to a separable exciton–photon state. We also investigate the effects of both types of pumpings (excitonic and polaritonic) in the emission spectrum of the cavity. We show that the polaritonic pumping considered here is unable to modify the dynamical regimes of the system at variance with the excitonic pumping. Finally, we obtain a closed form expression for the negativity of the density matrices that the quantum master equation considered here generates.

(Some figures in this article are in colour only in the electronic version)

1. Introduction
The study of solid-state systems of nanometric dimensions in which light interacts with matter has been the subject of continuing interest during the last decade. The study of excitons interacting with a confined mode of light has made possible the observation of two different coupling regimes [1, 2] (strong and weak coupling), as well as collective phenomena of quasiparticles in semiconductor microcavities (µC) [3–5]. Motivated by very interesting experimental results regarding condensation of polaritons in µC [6, 7] it has been proposed [8, 9] that some of the coherence properties of these systems can be understood in terms of an effective pumping of polaritons. In the latter references, the authors propose a model based on a finite system of electrons and holes confined in a parabolic quantum dot (QD), in which the Coulomb interaction between charge carriers and the light–matter dipole interaction are explicitly included. Subsequently, they obtain the dressed states of the finite system model (using a numerical diagonalization procedure), and finally they solve a zero-temperature quantum master equation that incorporates incoherent pumping of polaritons in addition to relaxation processes due to coherent emission. The proposed model has been able to reproduce the polariton laser threshold reported in [10].

In a recent theoretical and experimental study [11], using an extended Jaynes–Cummings model that includes dissipative processes, a surprisingly good agreement between calculated and measured polariton spectra is obtained. It is also shown that the effective dissipative parameters depend on the nominal excitation power density. They also discuss the difficulties involved in determining the strong coupling (SC) regime since the broadening of the spectral lines hinders the well-known anti-crossing characteristic features. The determination of a clear signature for the strong coupling regime motivated several recent theoretical works [12–14].

The above paragraph highlights the importance of studying the role of incoherent pumping in the determination of the dynamical regimes in the system. In particular, the role of incoherent pumpings of polaritons or excitons must be clarified. In this work we seek to discuss the dynamical effects of the interplay between polaritonic–excitonic pumping, by using the simplest model of quantized light–matter interaction, the Jaynes–Cummings model [15–17]. We also seek to
understand how the entanglement between excitons and photons is affected by the incoherent pumping since we have two strongly coupled interacting systems. Previous works [18–21] have followed this direction.

This paper is organized as follows. In section 2 a description of the master equation of the system is given. The matrix elements of the density operator related to the polariton pumping Lindblad superoperator are obtained. We also obtain the operators and dynamical equations necessary to calculate the emission spectrum using the quantum regression theorem, and we derive a closed form expression for the Peres positive partial transpose criterion in order to quantify the exciton–photon entanglement. In section 3, we compare the effects of the two types of pumping both in the entanglement of the system and in the statistical properties of the steady state density operator, and study how the polariton pumping affects the emission spectrum of the system. We finish with some conclusions and some details are relegated to the appendices.

2. Theoretical background

The quantum states resulting from the electrostatic interaction between holes in the valence band and electrons in the conduction band in a solid-state system are called excitons. These quasiparticles exhibit a discrete or continuum spectrum depending on their confinement. In this work, we are interested in studying an exciton interacting with the lowest energy (frequency) mode of a semiconductor microcavity, and we consider only the lowest energy levels of the system, the ground state, \( |G \rangle \) (electron in the valence band) and excited \( |X \rangle \) state (electron in the conduction band). In this simplified formulation all the complexities related to the many-body problem of considering electrons and holes in a quantum dot are effectively included in the energy separation between the ground and the excited states. The field will be treated as a single electromagnetic quantized mode. This approximation holds due to the existence of well-separated energy modes in the cavity. Both assumptions are usually considered in most theoretical works [11, 12, 17, 18].

2.1. Hamiltonian and dressed states

We consider the well-known Jaynes–Cummings Hamiltonian [15] (\( \hbar = 1 \)):

\[
H = (\omega_X - \Delta) a^\dagger a + \omega_X \sigma^\dagger \sigma + g (\sigma a^\dagger + \sigma^\dagger a),
\]

where \( \sigma = |G\rangle \langle X| \), \( \sigma^\dagger = |X\rangle \langle G| \) are the QD transition operators between the ground state \( |G\rangle \) and the exciton state \( |X\rangle \), \( a \) and \( a^\dagger \) are the annihilation and creation field operators, \( \omega_X \) is the energy required to create an exciton, \( \Delta = \omega_X - \omega_0 \) corresponds to the detuning between the exciton and photon frequencies and the constant \( g \) represents the light–matter coupling (Rabi constant). The Jaynes–Cummings Hamiltonian considers both the dipole and the RWA approximations in the light–matter interaction and that the exciton is coupled only to one mode of the \( \mu \mathrm{C} \) [16].

The Hamiltonian \( H \) can be diagonalized in the basis \( \{|G\rangle, |X\rangle\} \otimes \{|n\rangle\}_{n=0}^\infty \) (the Bare basis). It takes a block-diagonal form

\[
\begin{pmatrix}
(n+1)(\omega_X - \Delta) + \omega_X & g \sqrt{n} \\
\sqrt{n} & n(\omega_X - \Delta)
\end{pmatrix},
\]

when written in the \( n \)th excitation manifold basis \( \{|Xn\rangle, |Gn\rangle\} \). The eigenvalues and eigenvectors can be easily obtained and are given by [16]

\[
o_{n\pm} = \frac{1}{2} \left( -2n\Delta + \Delta \pm \sqrt{4n^2g^2 + \Delta^2} \right) + n\omega_X,
\]

\[
\begin{pmatrix}
|n, +\rangle \\
|n, -\rangle
\end{pmatrix} = A \begin{pmatrix}
|Xn - 1\rangle \\
|Gn\rangle
\end{pmatrix},
\]

where \( A \) is a clockwise rotation matrix by the angle \( \phi_n/2 = \tan^{-1}(\sqrt{n}/n) \). The states \( |n, \pm\rangle \) correspond to the dressed states of the Hamiltonian (1). If the states \( |G\rangle, |X\rangle \) are the excitonic states of the QD then the states \( |n, \pm\rangle \) can be considered as the polaritonic states of the system.

The emission spectrum of the system in this idealistic model is given by the transitions that occur between two given dressed states, and the values of the transition energies are precisely the differences of their respective frequencies (times \( \hbar \)).

2.2. Master equation

Since the microcavity quantum dot under consideration is an open quantum system, the effect of the environment must be included. The effect of a weak coupling with the environment is accounted for by using a master equation in the Born–Markov approximation. This master equation represents the dynamics of the density operator for the reduced light–matter system [22] and it includes the following processes:

(i) The continuous and incoherent pumping of the exciton.
(ii) The direct coupling of the exciton to the leaky modes which induces the spontaneous emission process.
(iii) The escape of cavity mode photons out of the microcavity due to incomplete reflectance of the mirrors, the so-called coherent emission, which may be recorded to obtain the emission spectrum of the system.
(iv) The incoherent pumping of polariton (dressed) states.

The first three processes have already been discussed in detail [11, 14, 17, 18], hence we omit the details here. An schematic representation of the processes involved in the dynamics of the system is given in figure 1(a). The fourth process is intended to cause incoherent transitions among dressed states of two consecutive excitation manifolds \( |n, \pm\rangle \rightarrow |n+1, \pm\rangle \) as shown in figure 1(b). This prescription is equivalent to the polariton pumping considered in [8, 9], in which the dressed states of the light–matter Hamiltonian are used to define raising and lowering operators between two consecutive excitation manifolds of polaritonic
states. In complete analogy with \cite{8,9} we introduce the following lowering operators between polaritonic states:

\[ P_{++} = |n, +\rangle \langle n + 1, +|, \quad P_{--} = |n, -\rangle \langle n + 1, -|, \quad P_{-+} = |n, -\rangle \langle n + 1, +|, \quad P_{+-} = |n, +\rangle \langle n + 1, -|, \]

(5)

With the previous considerations the master equation for the density operator of the system \( \rho \) takes the form

\[
\frac{d}{dt} \rho = i [\rho, H] + \frac{k}{2}(2\alpha \rho a^\dagger a - a^\dagger a \rho - \rho a^\dagger a) \\
+ \frac{\gamma}{2}(2\delta a^\dagger \rho - \sigma^\dagger \sigma \rho - \rho \sigma^\dagger \sigma) \\
+ \frac{P}{2}(2\sigma^\dagger \rho - \sigma^\dagger \sigma \rho - \rho \sigma^\dagger \sigma) \\
+ \frac{P_R}{2} \sum_{n,j} \left[ 2P_{ij,n}^+ \rho P_{ij,n} - P_{ij,n} \rho P_{ij,n}^+ - \rho P_{ij,n} P_{ij,n}^+ \right],
\]

(6)

where \( k \) is the decay rate of the cavity photons due to the incomplete reflectance of the cavity mirrors, \( \gamma \) is the exciton decay rate due to spontaneous emission, \( P \) is the rate at which excitons are being pumped and \( P_R \) is the rate at which polaritons are being pumped. The indices \( i \) and \( j \) take the values \([+,-]\) and \( n \in \mathbb{N} \). The polaritonic bath associated with the polariton pumping term in the last equation can be thought of as a resonant coupling between the undressed electron states and the intersubband cavity polariton excitations \cite{23}.

To study the dynamics of the system we write the equations of motion of the density operator matrix elements taken in the bare basis. The evaluation of these matrix elements is straightforward except for the term \( L_{Pi}[\rho] = \frac{1}{2} \sum_{n,j} \left[ 2P_{ij,n}^+ \rho P_{ij,n} - P_{ij,n} \rho P_{ij,n}^+ - \rho P_{ij,n} P_{ij,n}^+ \right] \). Using the fact that \( \{ n, + | n, + \} = |Gn\rangle \langle Gn| + |Xn - 1\rangle \langle Xn - 1| \) and that the trace of an operator is invariant under unitary transformations, it is easy to arrive at

\[
(i, n | L_{Pi}[\rho] | j, m) = \delta_{G,G'_i} \delta_{G,j} \delta_{m,n} \rho_{Gn-1,Gm-1} \\
+ \delta_{i, j} \delta_{m,n} \rho_{Gn} + \delta_{G,G'_i} \delta_{m,n} \rho_{Xm-2,Xm-2} \\
+ \delta_{i, j} \delta_{m,n} \rho_{Xn-1,Xn-1} - \delta_{G,G'_i} \rho_{Gn,jm} - \delta_{i, j} \rho_{Xn,jm} \\
- \delta_{G,G'_i} \rho_{Gn,m} - \delta_{i, j} \rho_{Xn,m}.
\]

(7)

which is independent of both \( g \) and \( \Delta \). The dynamical equations for the populations and coherences in the bare basis are given in appendix A. For simplicity in what follows we have taken as the unit of frequency (equivalently energy) the Rabi constant \( g = 1 \) meV, which is a typical value of the light-matter coupling constant in semiconductors. All the quantities in what follows are given in units of \( g \).

2.3. Quantum regression theorem and the emission spectrum

One of the few things that can be measured directly from a quantum system is its spectrum. To obtain the emission spectrum of a system we need to take the Fourier transform of the first-order correlation function \( \langle a^\dagger(t + \tau)a(t)\rangle \), which requires the knowledge of the expectation value of two operators at different times. To obtain the dynamical equation of such a correlation function we take advantage of the quantum regression theorem (QRT) \cite{24} which states that, given a set of operators \( O_i \) satisfying, \( \frac{1}{i} \langle O_i(t + \tau) \rangle = \sum \sum_l L_{jk}(t + \tau) \langle O_k \rangle \), then \( \frac{1}{i} \langle O_i(t + \tau) O(t) \rangle = \sum \sum_l L_{jk}(t + \tau) O(t) \) for any operator \( O \). We follow Tejedor and co-workers \cite{17} and write \( \langle a^\dagger(t + \tau)a(t)\rangle = \sum \sqrt{n + 1}\langle a^\dagger_{Gn}(t + \tau)a(t)\rangle + \langle a^\dagger_{Xn}(t + \tau)a(t)\rangle \), where the following definitions have been used:

\[
a^\dagger_{Gn} = |Gn + 1\rangle \langle Gn|, \\
a^\dagger_{Xn} = |Xn + 1\rangle \langle Xn|, \\
\sigma^\dagger_n = |Xn\rangle \langle Gn|, \\
\zeta_n = |Gn + 1\rangle \langle Xn - 1|.
\]

(8)

Note that these operators act between two consecutive excitation manifolds. It turns out that the last operators satisfy the following set of closed differential equations:

\[
\frac{d}{dt} \langle a^\dagger_{Gn}(\tau) \rangle = \left( -P - 2P_{R} - i\Delta - nk - \frac{k}{2} + i\omega_X \right) \langle a^\dagger_{Gn}(\tau) \rangle \\
+ \kappa \sqrt{(n + 1)(n + 2)} \langle a^\dagger_{Gn+1}(\tau) \rangle \\
+ \gamma \langle a^\dagger_{Xn}(\tau) \rangle - i\sqrt{n} \langle \zeta_n(\tau) \rangle + i\sqrt{n + 1} \langle \sigma^\dagger_n(\tau) \rangle,
\]

Figure 1. (a) Ladder of bare states for a two-level quantum dot coupled to a cavity mode. The double headed solid green arrow depicts the radiation–matter coupling \( g \), dashed yellow lines the exciton pumping rate \( P \), dotted blue lines the spontaneous emission rate \( \gamma \), dashed–dotted red lines the emission of the cavity mode \( \kappa \) and wavy gray lines the polariton pumping process \( P_p \). (b) Transitions due to the polariton pumping term in the master equation (6) in the ladder of dressed states.
Here we consider the emission of excitonic radiation in the stationary limit so that the asymptotic limit \( t \to \infty \) is applied in (10). The role of the parameters \( \phi, \Delta, g, \kappa, \gamma, P, P_p \) is twofold. On the one hand they determine the dynamics of the two-time correlation functions via (9), and, on the other hand, they set the initial conditions (10) that are propagated according to the dynamical equations (9). Here we consider the emission of radiation in the stationary limit so that the asymptotic limit \( t \to \infty \) is applied in (10).

The eigenvalues of the last matrix are then easily obtained. The ones corresponding to the left upper and right lower entries (blocks) are \( \rho_{X_0, X_0} \) and \( \rho_{G_m, G_n} \) respectively, and they are always positive or zero. Those corresponding to the \( 2 \times 2 \) blocks are \( \frac{1}{2} [ \rho_{G_m, G_n} + \rho_{X_0+1, X_{m+1}} \pm \sqrt{\rho_{G_m, G_n} \rho_{X_0+1, X_{m+1}}^* + \frac{1}{4} | \rho_{X_0, X_0} |^2} ] \). In order to have a negative eigenvalue (and an entangled state) the following condition must be met for some \( n \):

\[
| \rho_{X_n, X_n} | > \sqrt{\rho_{G_m, G_n} \rho_{X_0+1, X_{m+1}}}. \tag{13}
\]

Notice that the above inequality can also be obtained by using the criterion recently derived in [26]. Then, we can quantify the entanglement by using the following function, which is equivalent to the Peres criterion:

\[
E(\rho) = 4 \sum_n \left( \max \left\{ \langle 0 | \rho_{X_n, G_n} \rho_{X_{m+1}, G_m} \rangle - \frac{1}{2} \rho_{G_m, G_n} \right\} \right)^2. \tag{14}
\]

For Bell-like states \( \rho_{\text{Bell}} = |\psi\rangle \langle \psi|, |\psi\rangle = \frac{1}{\sqrt{2}}((G_m + 1) + e^{i\phi}|X_n\rangle), \phi \in \mathbb{R}, E(\rho) \) will be unity. In particular, the polaritonic states \( |n, \pm\rangle \) in resonance have \( E(\rho) = 1 \).

### 3. Results and discussion

#### 3.1. Exciton pumping versus polariton pumping

In this section, we compare the evolution of some observables in the stationary limit as a function of the detuning (\( \Delta \)) and the pumping rates of both excitons (\( P \)) and polaritons (\( P_p \)).

In figure 2, we present the evolution of the average number of photons, \( \langle n \rangle = \langle a^\dagger a \rangle \), in the stationary limit for the case of strong coupling \( (| \kappa - \gamma | / 4 \ll g) \) as a function of \( \Delta \) and either \( P \) or \( P_p \). From this figure several conclusions can be drawn. First the pumping of polaritons is more efficient in accumulating photons in the cavity. This effect can be understood by looking at the transitions that both types of pumpings cause in the bare basis ladder of states, as is seen in figure 1. \( P \) causes diagonal transitions, whereas \( P_p \) induces vertical (and diagonal) transitions that can rapidly populate states with a high number of photons. Second, and rather surprisingly, the average number of photons and the observables we have monitored are not sensitive to the detuning when the term \( P_p > \kappa = 0.1 \).

We have also analyzed the effect of both mechanisms on the population inversion \( \langle \sigma_+ \rangle \) and the second-order coherence function at zero delay \( g^2(\tau = 0) = \langle a^\dagger a^\dagger a a \rangle / \langle a^\dagger a \rangle^2 \). In figure 2 we present the contour plots for \( g^2(\tau = 0) \) and \( \langle \sigma_+ \rangle \). It is clearly seen that one can explore different field statistics as \( P \) and \( \Delta \) are varied and that the population inversion grows as a function of \( P \). The corresponding results for the polariton pumping mechanism can be summarized as follows:
Figure 2. Average number of photons, second-order coherence function at zero delay and population inversion as a function of the detuning $\Delta$ and the exciton pumping rate $P$ or the polariton pumping $P_p$. Parameters: $\kappa = 0.1$, $g = 1$, $\omega_X = 1000$, $\gamma = 0$, for the upper panel $P_p = 0$ and for the lower panel $P = 0$. In most experimental situations $\gamma$ is at least two orders of magnitude smaller than the rest of the parameters [27]. For these calculations the Fock space was truncated in $n_{\text{max}} = 40$.

- The second-order coherence function is almost equal to one and is independent of $\Delta$ for values of $P_p > \kappa = 0.1$.
- The population inversion $\langle \sigma_z \rangle$ presents a similar behavior to $g^2(\tau = 0)$. It approaches 0 from below as $P_p$ is increased and it is nearly independent of $\Delta$ for values of $P_p > \kappa = 0.1$.

Comparing the results of both types of pumping, we conclude that, firstly, the polariton pumping mechanism is unable to cause a positive population inversion; and, secondly, it can greatly increase the intensity of the light stored in the cavity, and the light stored in the cavity may have Poisson statistics since the variance of the number of photons equals its mean value (although in general it will not be a coherent state since the reduced density matrix of the photons will be a mixed state). The first observation is understandable when one inspects the action of the term $P_p$ upon the bare basis. The term $P_p$ causes transitions with equal intensity from $|X\rangle \rightarrow |G\rangle$ and from $|G\rangle \rightarrow |X\rangle$ and this explains why $\langle \sigma_z \rangle$ is close to zero. The fact of being always slightly negative is related to the asymmetry that the vacuum state $|G0\rangle$ introduces between the states $G$ and $X$, in such a way that state vectors with no exciton have a slightly larger statistical weight. The second observation is related to the possibility of having an inversionless polaritonic laser [9, 10].

### 3.2. Emission spectrum

Since the effects of the excitonic pumping have been extensively studied in [12–14] we only study the effects of the polaritonic pumping in the emission spectrum of the system. To do so we rewrite the equations of motion for the variables in equations (9) in compact form as

$$\frac{d}{dt} \mathbf{v}(t) = \mathbf{A}(P_p, P, g, \kappa, \gamma, \omega_X, \Delta) \mathbf{v}(t), \quad (15)$$

where $\mathbf{v}(t) = \{|\sigma_0^+\rangle, |\sigma_G^+\rangle, \ldots, |\xi_n\rangle, |\sigma_0^-\rangle, |\sigma_G^-\rangle, \ldots\}$. The formal solution of equation (15) is given by

$$\mathbf{v}(t + \tau) = \exp(\mathbf{A}(P_p, P, g, \kappa, \gamma, \omega_X, \Delta) \tau) \mathbf{v}(t). \quad (16)$$

From equations (9) one sees that $\mathbf{A}(P_p, P, g, \kappa, \gamma, \omega_X, \Delta) = -2P_p \mathbf{I} + \mathbf{B}(P, g, \kappa, \gamma, \omega_X, \Delta)$ ($\mathbf{I}$ is the identity matrix), i.e. the polariton pumping term is diagonal in equations (9). Since $-2P_p \mathbf{I}$ commutes with the matrix $\mathbf{B}(P, g, \kappa, \gamma, \omega_X, \Delta)$ it can be factored out in equation (16), which now is $\mathbf{v}(t + \tau) = \exp(-2P_p \tau) \exp(\mathbf{B}(P, g, \kappa, \gamma, \omega_X, \Delta) \tau) \mathbf{v}(t)$.

The last equation implies that the pumping rate $P_p$ cannot modify the oscillation frequencies of the first-order correlation
Figure 3. Emission spectrum of the system with different rates of polaritonic pump $P_p$ for the parameters $\omega_X = 1000, \kappa = 0.1, \Delta = \gamma = P = 0$. For these calculations we truncated equations (9) considering a maximum number of photons $n_{\text{max}} = 20$.

Figure 4. In the left and central panels we present the fraction of the population of the density matrix that has more than two photons as a function of the exciton or polariton pumping rates and the detuning $\Delta$ for the parameters $\kappa = 0.1, \gamma = 0, g = 1$. The results do not change significantly if $\gamma = 0.005$. The right panel examines the entanglement measure $E(\rho)$ as a function of the exciton pumping rate $P$ and the detuning $\Delta$. (The other parameters are the same as in the other two panels.) It is seen that even in the stationary state one may have entangled states, which is not the case when one only pumps polaritons. The Fock space was truncated to $n_{\text{max}} = 40$.

function since it acts as a common global decay rate for all the operators involved in equations (9). It can only redistribute the statistical weights of the different frequencies by modifying the initial values of the two-time operators that are related to the populations and coherences in equations (10). For instance, in the case where one considers transitions between the ground state $|G0\rangle$ and the states $|X0\rangle$ and $|G1\rangle$ only two operators appear in the equations of the QRT, $\langle a_{G0}(t) \rangle$ and $\langle a_{G0}(t) \rangle$. This approximation is valid as long as the pumping rates ($P$ or $P_p$) are small enough compared with the losses ($\gamma$ and $\kappa$) so as not to have an average photon number of more than one [18]. In this case the equations of the QRT are reduced to
\[ \frac{d}{dt} \left( \begin{pmatrix} a_G(t) \\ \sigma_1(t) \end{pmatrix} \right) = \left( \begin{pmatrix} P - 2P_p - \frac{\gamma}{2} + i\omega_x & ig \\ ig & -P - 2P_p - i\Delta - \frac{\gamma}{2} + i\omega_x \end{pmatrix} \right) \times \left( \begin{pmatrix} a_G(t) \\ \sigma_1(t) \end{pmatrix} \right). \]  

(17)

The eigenvalues \( \lambda_{\pm} \) of the square matrix in the last equation are related to the positions \( \omega_\pm \) and widths \( \Gamma_\pm \) of the emission spectrum \( \omega_\pm = \omega_0 + i\Gamma_\pm \) and are given by:

\[ \lambda_{\pm} = \frac{1}{2}(-P - 2P_p - \gamma - 2i\Delta - \kappa + 4i\omega_x) \pm i\sqrt{16g^2 - (P - \gamma + 2i\Delta + \kappa)^2}. \]  

(18)

As expected, the term \( P_p \) only enters as a decay rate that broadens both emission peaks in the same way.

Actually one can go further in the analytical calculation by considering the case where there is no exciton pumping, \( P = 0 \). In this case one can obtain analytically all the eigenvalues of the matrix \( \mathbf{A} = \mathbf{A}(P_p, P = 0, g, \kappa, \gamma, \omega_x, \Delta) \). To this end notice that the only term that couples the group of operators \( \{a_G(t), \sigma_1(t)\} \) and \( \{a_{\pm n}(t), \sigma_{\pm n}(t)\} \) is given by the matrix \( \mathbf{P} \) which is only a sufficient condition for having an entangled state. When the pumping \( P_p \) increases new lines appear and these are associated with the energetic transition between the states \( |2, \pm \rangle \) and \( |1, \pm \rangle \), which have frequencies \( \omega_\pm \pm g \pm \sqrt{2\kappa} \) (for the parameters used they are approximately \( \omega_\pm \approx 1002.4, 1000.4, 999.6, 997.6 \)). When the pumping is further increased the lines associated with the transition to the vacuum become completely shadowed by the broad lines associated with the transition \( |2, \pm \rangle \rightarrow |1, \pm \rangle \).

3.3. Entanglement in the stationary state

In this section, we use equation (14) to study the entanglement in the stationary state. It is important to remember that the Peres criterion is a necessary and sufficient condition for having entanglement when the dimensions of the Hilbert spaces \( \{h_i\} \) of the subsystems considered are \( \dim(h_1) = 2 \) and \( \dim(h_2) = 2 \) or 3 [28]. For larger dimensions it is only a sufficient condition for having an entangled state. Under certain conditions the Fock space of our system can be truncated in one or two photons. These conditions are met when we consider small enough pumping rates \( P \) and \( P_p \) as compared with the loss rates \( (\kappa, \gamma) \). This hypothesis is numerically confirmed by figure 4 where it is clearly seen that for \( P < P_p < 0.1 = \kappa + \gamma \) the population corresponding to states with more than two photons is very small or zero. Then in the blue regions of figure 4 we can extract the stationary state of the system considering only a Fock space of two photons. The exact expressions for a maximum of one photon are presented in appendix B. One can also find exact expressions for a maximum of two photons, but they are very cumbersome and are not presented here. In cases where only the term \( P_p \) is present \( P = \gamma = 0 \), one may find analytically that the state is always separable (truncating in one or two photons). It is also numerically confirmed using a larger basis \( (n_{\max} = 40) \) for the parameter region \( 10^{-4} < P_p < 1, 0 < \Delta < 2, \kappa = 0.1 \) and \( \gamma = 0, 0.005 \). When the effect of the excitonic pumping is considered, we find a region of parameters where entangled states exist (see the right panel of figure 4). These results stress the very special role of the excitonic pumping term \( P \); in fact, its variation may enhance or reduce the degree of entanglement between the exciton and the field.
4. Conclusions

In this work a mechanism for incoherent pumping of polaritons is proposed. The form of its matrix elements is derived and used in section 2.2 to obtain the dynamical equations necessary to propagate the density operator of the system and to obtain its emission spectrum. The effects of the new excitation mechanism were compared with the effects of the excitonic pumping. The physical origin of the new mechanism is still controversial, and further work in this direction is required. An effective pumping of excitons and photons has already been considered [11], and the good agreement between theory and experiment shows that both pumpings are able to account for a significant amount of the physics in such systems. Nevertheless, by using off-resonance excitation in quantum wells, the group of Bloch obtained a polariton laser [10], that has been successfully modeled in [9], including an effective polariton pumping term, that is essentially equivalent to our model. Additionally, it has been shown [6] that for quantum wells an effective resonant pumping to the lower polariton branch is a condition for thermalization of a Bose–Einstein condensate of polaritons.

In this work we have shown that the polariton pumping term \( P_p \) is not able to cause a positive population inversion, and that above a given threshold it drives the field to a Poisson-like statistics where \( \rho^2(\tau = 0) = 1 \). We have shown that the term \( P_p \) is unable to change the dynamical regimes of the system, and that its effect on the photon emission spectrum is twofold: it causes a homogeneous broadening of the emission peaks and an increase in the integrated photon emission. A useful rule-of-thumb based on the Peres criterion has been obtained to determine whether the state of the exciton and the photons is entangled or not. Consequently, we have analyzed how the incoherent pumping mechanisms affect the entanglement of the exciton and the cavity photon in the stationary state. It has been shown that the excitonic pumping term does not completely destroy the entanglement as the polariton pumping does.

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Appendix A. Dynamics of the density matrix elements

The dynamical equations for the populations and coherences in the bare basis are given by
\[
\frac{d}{dt} \rho_{Gn,Gn} = -P_p \rho_{Gn,Gn} + \kappa \left( (n+1) \rho_{Gn+1,Gn+1} - n \rho_{Gn,Gn} \right) + P_p \left( \rho_{Gn-1,Gn-1} - 2 \rho_{Gn,Gn} + \rho_{Gn-2,Gn-2} \right) + i g \sqrt{n} \left( \rho_{Gn,Xn-1} - \rho_{Xn-1,Gn} \right) + \gamma \rho_{Gn,Xn-1,
\]
\[
\frac{d}{dt} \rho_{Gn-1,Xn-1} = P_p \rho_{Gn-1,Gn-1} + \kappa \left( n \rho_{Xn,Xn} - (n-1) \rho_{Xn-1,Xn-1} \right) + \gamma \rho_{Xn-1,Xn-1} - \gamma \rho_{Gn-1,Xn-1}.
\]

Appendix B. Stationary state truncating in one photon

The populations and coherences of the stationary density operator, truncating the Fock space in one photon, are given by
\[
\rho_{X0,G1} = \left( 2 i g (P + 4 P_p + \gamma - 2 \Delta + \kappa) \right) / (\Delta + \kappa) \times (|P_p\rangle - \kappa / \gamma),
\]
\[
\rho_{G0,G0} = \left( 4 (P + 2 P_p) (\gamma + \kappa) (P + 4 P_p + \gamma + \kappa) g^2 \right) / \Delta + \kappa \times \left( 4 \Delta^2 + (P + 4 P_p + \gamma + \kappa)^2 \right) / C,
\]
\[
\rho_{X0,X0} = \left( 4 (P + 2 P_p) (\gamma + \kappa) (P + 4 P_p + \gamma + \kappa) g^2 \right) / \Delta + \kappa \times \left( 4 \Delta^2 + (P + 4 P_p + \gamma + \kappa)^2 \right) / C,
\]
\[
\rho_{G1,G1} = \left( 4 (P + 2 P_p) (\gamma + \kappa) (P + 4 P_p + \gamma + \kappa) g^2 \right) / \Delta + \kappa \times \left( 4 \Delta^2 + (P + 4 P_p + \gamma + \kappa)^2 \right) / C,
\]
\[
\rho_{X1,X1} = \left( 4 g^2 (P + 4 P_p + \gamma + \kappa) (P + 4 P_p)^2 \right) / \Delta + \kappa \times \left( 4 \Delta^2 + (P + 4 P_p + \gamma + \kappa)^2 \right) / C,
\]
\[
N = 4 (P + 2 P_p + \gamma + \kappa) (P + 4 P_p + \gamma + \kappa) g^2 \right) / \Delta + \kappa \times \left( 4 \Delta^2 + (P + 4 P_p + \gamma + \kappa)^2 \right) / C,
\]
\[
C = \left( P + 2 P_p + \gamma + \kappa \right).
\]

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