Uncertainty estimates and $L_2$ bounds for the Kuramoto–Sivashinsky equation

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Abstract

We consider the Kuramoto–Sivashinsky (KS) equation in one dimension with periodic boundary conditions. We apply a Lyapunov function argument similar to the one first introduced by Nicolaenko et al (1985 Physica D 16 155–83) and later improved by Collet et al (1993 Commun. Math. Phys. 152 203–14) and Goodman (1994 Commun. Pure Appl. Math. 47 293–306) to prove that $\limsup_{t \to \infty} \|u\|_2 \leq CL^{3/2}$. This result is slightly weaker than a related result by Giacomelli and Otto (2005 Commun. Pure Appl. Math. 58 297–318), but applies in the presence of an additional linear destabilizing term. We further show that for a large class of functions $\phi_x$ the exponent $3/2$ is the best possible from this line of argument. Finally, we mention several related results from the literature on equations related to the KS equation that can be improved using these ideas.

Mathematics Subject Classification: 37L30, 35P15, 35K30

1. Introduction

1.1. Background

The Kuramoto–Sivashinsky (KS) equation,

$$
\begin{align*}
&u_t = -u_{xxxx} - u_{xx} - uu_x, \\
&\frac{d^j u}{dx^j}(L) = \frac{d^j u}{dx^j}(-L) \quad j = 0, \ldots, 3, \\
&u(x, 0) = u_0(x) \quad x \in (-L, L), \\
&\int_{-L}^{L} u_0(x)dx = 0,
\end{align*}
$$

(1)
arises as a model of certain hydrodynamic problems, most notably the propagation of flame fronts [29]. The KS equation is interesting mathematically because the linearization about the zero state has a large number (\(O(L/\pi)\)) of exponentially growing modes. The growth of these modes corresponds, in the combustion problem, to the development of nontrivial structures. In addition to its importance as a model for flame fronts [29], phase turbulence [22] and plasmas [23] the KS equation has become one of the canonical models for spatio-temporal chaos in 1+1 dimensions [13, 18, 19, 24].

Nicolaenko et al [26] gave the first long-time boundedness result for the KS equation, showing that \(\limsup_{t \to \infty} \|u\|_2 \leq CL^{5/2}\) for odd initial data, as well as showing that bounds on the \(L_2\) norm imply bounds on the dimension of the attractor (also see [8, 9, 31]). The \(L_2\) estimate was improved by Collet et al [1] who extended it to any mean-zero initial data and improved the exponent from \(\frac{5}{2}\) to \(\frac{5}{3}\) and by Goodman [12], who extended it to any mean-zero initial data (with exponent \(\frac{3}{2}\)). All these papers use some variation of the original argument of Nicolaenko et al, namely to establish that the function \(\|u - \phi\|_2^2\) is a Lyapunov function for an appropriately chosen \(\phi\) and \(\|u\|_2\) sufficiently large. There are also two bounds which do not fit into this Lyapunov function framework, those of Ilyashenko [17] and of Giacomelli and Otto [11]. The latter, which uses a deep result of DeLellis et al [5] to treat the KS equation as a perturbation of Burgers’ equation, is currently the best estimate, establishing that

\[
\limsup_{t \to \infty} \|u\|_2 = o(L^{3/2}).
\]

In this paper we give an elementary argument of the Lyapunov function type which establishes the weaker result

\[
\limsup_{t \to \infty} \|u\|_2 = O(L^{3/2}).
\]

Our proof applies equally to the destabilized Kuramoto–Sivashinsky (dKS) equation:

\[
\begin{align*}
    u_t &= -u_{xxxx} + uu_x + \gamma u, \\
    \frac{d^j u}{dx^j} (L) &= \frac{d^j u}{dx^j} (-L), \quad j = 0, \ldots, 3, \\
    u(x, 0) &= u_0(x), \quad x \in (-L, L), \\
    \int_{-L}^{L} u_0(x) dx &= 0.
\end{align*}
\]

This equation arises, for instance, as a model in plasma physics [23]. It was shown by Wittenberg [32] via a matched asymptotic expansion that this equation has stationary solutions which satisfy \(\|u\|_2 \propto L^{3/2}\). Since a Lyapunov function argument for the KS equation also applies to the dKS equation (for sufficiently small \(\gamma\)) Wittenberg argued that \(\frac{3}{2}\) is the best exponent that one can expect from the Lyapunov function approach. This paper completes this circle of ideas, by showing that this exponent can actually be achieved by a Lyapunov function approach. Note that the proof of Giacomelli and Otto avoids the obstruction presented by this example by requiring the vanishing of the Fourier symbol of the linear part of the evolution at long wave-lengths. For this reason the method of Giacomelli and Otto is currently the only one which could potentially be extended to smaller exponents: \(\frac{3}{2}\) is believed to be the best possible exponent.

We also give an independent scaling argument that motivates the choice of \(\phi_x\) and shows that for a large class of potentials \(\frac{3}{2}\) is the best exponent possible from this line of argument. This argument makes clear the physical basis of the scaling and is potentially applicable to other equations.
1.2. Notation

Throughout this paper we will use the following notation. We will use \( \| \cdot \|_2 \) to denote the usual \( L_2 \) norm:

\[
\| \phi \|_2^2 = \int_{-L}^{L} \phi^2(x)\,dx,
\]

and \( \| \cdot \|_{H_k} \) to denote the usual Sobolev norm:

\[
\| \phi \|_{H_k}^2 = \| \phi \|_2^2 + \| \phi_x \|_2^2 + \cdots + \| \frac{d^k \phi}{dx^k} \|_2^2.
\]

We will use \( H_2^{\text{per}}[-L, L] \) to denote the periodic Sobolev space obtained by taking the completion with respect to the norm \( \| \cdot \|_{H_2} \) of smooth functions satisfying periodic boundary conditions

\[
\frac{d^j \phi}{dx^j}(L) = \frac{d^j \phi}{dx^j}(-L) = 0, \quad j = 0, \ldots, 3.
\]

Since the KS equation is typically posed for initial data with mean zero (which is preserved under the flow) it is often convenient to consider subspaces of functions with zero mean. Following the notation introduced in [26] we take a dot above any space to denote the subspace of functions of zero mean. Thus, \( \phi \in H_2^{\text{per}}[-L, L] \) if and only if \( \phi \in H_2^{\text{per}}[-L, L] \) and

\[
\int_{-L}^{L} \phi(x)\,dx = 0.
\]

We will also have occasion to consider Sobolev spaces without any additional boundary conditions. We will use \( H^{1}[-L, L] \) to denote the completion with respect to the norm \( \| \cdot \|_{H^1} \) of smooth functions on \((-L, L)\).

Throughout this paper we will frequently rescale the original domain \((-L, L)\). There are essentially two scalings of interest. The first is the physical domain, \( x \in (-L, L) \). The second is an inner scaling that is determined by a competition between the highest order term in the dissipation and the nonlinearity. We will consistently use the variable \( x \) as the coordinate in the original domain \((-L, L)\) and \( y \) as the coordinate in the rescaled domain. Note that in every instance except one the rescaling is \( y = L^{1/3}x \) and the rescaled domain is \( y \in (-L^{4/3}, L^{4/3}) \).

The exception is section 2.1, where we discuss the Lyapunov functions constructed in earlier papers. In this subsection the rescaling is the more general \( y = Mx/L \), where \( M \) is the high-wavenumber cut-off and the domain is \((-M, M)\).

All constants which appear should be taken to be independent of \( L \) except for \( M \), the high-wavenumber cut-off parameter, which will always be chosen to scale with \( L \).

1.3. Fundamental lemmas

We begin by stating two basic lemmas which form the core of the Lyapunov function argument. These lemmas are basically equivalent to equations (2.11), (12) and (3.10)–(3.12) in [26] or analogous results in [1, 12]. It is worth noting that similar ideas of considerably greater generality have been used by Constantin and Doering to establish bounds on energy dissipation in fluids and generally go by the name ‘background flow method’ [3, 4].

**Lemma 1.** Given \( u = u(x, t) \in L_2[-L, L] \) and \( \phi = \phi(x) \in L_2[-L, L] \) satisfying the following inequality:

\[
\frac{d}{dt} \| u - \phi \|_2^2 \leq -\lambda_0 \| u \|_2^2 + P^2
\]

(6)
for some constants $\lambda_0 > 0$ and $P$, then $B(0, R^*)$, the ball of radius $R^*$ centred about the origin, is an attracting region, where the radius $R^*$ is given by

$$R^* = \sqrt{\frac{2\|\phi\|_2^2 + \frac{2P^2}{\lambda_0}}{\lambda_0}} + \|\phi\|_2. \quad (7)$$

**Proof.** The parallelogram law implies

$$-\lambda_0\|u - \phi\|_2^2 \geq -2\lambda_0\|u\|_2^2 - 2\lambda_0\|\phi\|_2^2,$$

which in turn gives

$$\frac{d}{dt}\|u - \phi\|_2^2 \leq \lambda_0\|\phi\|_2^2 + P^2 - \frac{\lambda_0}{2}\|u\|_2^2.$$

If we apply the obvious Gronwall estimate to the above inequality it is apparent that $B(\phi, R^*)$, the ball of radius $R^*$ centred about $\phi$, is exponentially attracting, with $R^* = 2\|\phi\|_2^2 + (2P^2/\lambda_0)$. The triangle inequality implies $B(\phi, R^*) \subset B(0, R^*)$. \hfill \Box

**Lemma 2.** For any $\phi \in H^3_{\text{per}}[-L, L]$ and $u(x, t)$ solving the Kuramoto–Sivashinsky equation we have (after some rescaling) the inequality

$$\frac{1}{4} \frac{d}{dt} \int_{-2L}^{2L} (u - 8\phi)^2 \leq 4\left( \int_{-2L}^{2L} u_x^2 - u_{xx}^2 - \dot{\phi}\phi_x u^2 \right) + \int_{-2L}^{2L} 32\phi_x^2 + 256\phi_{xx}^2.$$

**Proof.** A straightforward calculation gives

$$\frac{1}{2} \frac{d}{dt}\|u - \phi\|_2^2 = \int_{-L}^{L} u_x(u - \phi) = \int_{-L}^{L} (-u_{xx} - u_{xxxx} - uu_x)(u - \phi).$$

After integrating by parts and applying periodic boundary conditions this becomes

$$\frac{1}{2} \frac{d}{dt}\|u - \phi\|_2^2 = \int_{-L}^{L} u_x^2 - u_{x}^2 - \dot{\phi}\phi_x + \phi_xu_x + \phi_xu_{xx} = \frac{1}{2}\phi_xu^2.$$

Applying the Cauchy–Schwarz inequality in the form $\langle f, g \rangle \leq p/2\langle f, f \rangle + 1/2p\langle g, g \rangle$ gives

$$\frac{1}{2} \frac{d}{dt}\|u - \phi\|_2^2 \leq \int_{-L}^{L} \left( 1 + \frac{1}{2p} \right) u_x^2 + \left( \frac{1}{2q} - 1 \right) u_{xx}^2 + \frac{p}{2}\phi_x^2 + \frac{q}{2}\phi_{xx}^2 - \frac{1}{2}\phi_xu^2.$$

If we then make the substitution $\phi = \gamma\tilde{\phi}$, $\tilde{x} = \beta x$, we find that

$$\int_{-\beta L}^{\beta L} (u - \gamma\tilde{\phi})^2 d\tilde{x} \leq \frac{1}{2p} \frac{1 + 2p}{2p} \beta u_x^2 + \frac{1}{2q} \beta^2 u_{xx}^2 + \frac{p}{2} \beta \gamma^2 \phi_{xx}^2 + \frac{q}{2} \beta^3 \gamma^2 \phi_{xx}^2 - \frac{1}{2} \gamma \phi_x u^2 d\tilde{x}.$$

Finally, taking $p = \frac{1}{2}$, $q = 1$, $\beta = 2$, $\gamma = 8$ we get

$$\frac{1}{4} \frac{d}{dt} \int_{-2L}^{2L} (u(\tilde{x}, t) - 8\tilde{\phi}(\tilde{x}))^2 d\tilde{x} \leq 4 \int_{-2L}^{2L} (u_x^2 - u_{xx}^2 - \dot{\phi}\phi_x u^2) d\tilde{x} + \int_{-2L}^{2L} 32\tilde{\phi}_x^2 + 256\tilde{\phi}_{xx}^2 d\tilde{x},$$

as claimed. \hfill \Box
Remark 1. Since we are only concerned with the scaling of the estimates with $L$, and not with the actual constants, we will henceforth drop the tildes and replace $\beta L$ with $L$ and $\tilde{x}$ with $x$. The preceding lemmas show that if we can construct $\phi \in \dot{H}^2[-L, L]$ such that the coercivity estimate
\[ \langle u, Ku \rangle = \int_{-L}^{L} u_{xx}^2 - u_x^2 + \phi_x u^2 \, dx > \lambda_0 \|u\|_2^2 > 0 \] holds for some $\lambda_0$ independent of $L$ then we get an estimate of the form
\[ \limsup_{t \to \infty} \|u\|_2 \leq R^{**} = \sqrt{c_1 \|\phi\|_{H^2}^2 + c_2 \|\phi_x\|_{H^2}^2 + c_3 \|\phi_{xx}\|_{H^2}^2 + c_4 \|\phi\|_2^2} \]
Since it is clear that that $R^{**}$ is comparable to the $H^2$ norm, $c \|\phi\|_{H^2} \leq R^{**} \leq C \|\phi\|_{H^2}$, we will write this in the form
\[ \limsup_{t \to \infty} \|u\|_2 \leq c \|\phi\|_{H^2}. \]

While we will not present any more details of the arguments of previous papers we would like to make some comments on the form of the Lyapunov functions considered in previous papers, since this will help to motivate the choice that we make in this paper. The Lyapunov functions constructed in [26] and [1] are of the same basic form, namely
\[ \phi_x = \sum_{k=0}^{\infty} a_k \cos \left( \frac{2\pi k x}{L} \right), \]
where the coefficients $a_k$ are given by
\[ a_k = \begin{cases} 0, & k = 0, \\ \tilde{\psi} \left( \frac{k}{M} \right), & k > 0, \end{cases} \]
where $\tilde{\psi}$ is some even rapidly decaying function on $R$ and $M$ is some large parameter that is chosen to scale with $L$ and should be thought of as a kind of high-wavenumber cut-off. In [26] the function $\tilde{\psi}$ is chosen to be the characteristic function of the interval $[0, 1]$ and $M \propto L^2$, while in [1] the function $\tilde{\psi}$ is identically constant in an interval and a $C^1$ positive monotonically decaying function outside this interval, with $M \propto L^{7/5}$. In each case the Fourier coefficients are chosen to be constant in some interval to take advantage of some cancellation in the quadratic form. The salient feature in both cases is that the potential is constructed by taking Fourier coefficients which can be considered to be a discrete ‘sampling’ of a relatively nice function on the real line. This suggests that we apply the Poisson summation formula [20]. In the obvious way we take $\psi(x)$ to be the inverse Fourier transform of $\tilde{\psi}(\eta)$ on the real line, $\psi(x) = \int \exp(-2\pi i \eta x) \tilde{\psi}(\eta) \, d\eta$. After adding and subtracting the term $\psi(0)$ an application of the Poisson summation formula gives
\[ M \sum_{j=-\infty}^{\infty} \psi \left( \frac{M x}{L} - 2jL \right) - \int_{-\infty}^{\infty} \psi(x) \, dx = \sum_{k=0}^{\infty} \tilde{\psi} \left( \frac{k}{M} \right) \cos \left( \frac{2\pi k x}{L} \right) \]
whenever the left-hand side is absolutely convergent. Thus, the real-space picture for the construction of these Lyapunov functions is as follows: one takes as a basic unit an $L^2$ function of width $L/M$ and amplitude $M$. This is then extended to a periodic, mean-zero function by

\[ \text{Note that this formula does not hold pointwise for the Lyapunov function in [26] due to slow decay caused by the jump discontinuity in $\tilde{\psi}$. However, this jump can be smoothed without changing the argument at all.} \]
summing over translates and subtracting the mean. It is this interpretation of the original idea that will motivate our choice of Lyapunov function.

Our argument proceeds along the same lines as that of previous papers, but with a real-space construction of the Lyapunov function based on the above observations. We first construct a \( \phi_x \) such that the operator \( K \) defined above is positive definite for \( u \) satisfying a Dirichlet boundary condition at the origin. This establishes an \( L_2 \) bound for odd solutions of the KS equation, which are preserved under the flow. This result can be easily extended to all mean-zero data by allowing a time-dependent \( \phi \), which translates under a kind of gradient-flow dynamics, as was first done by Collet et al. We give a sketch of how this works: for more details the reader is referred to the original paper. The related idea of Goodman, that of looking at the distance to the set of all translates of the Lyapunov function, would also go through essentially unchanged, but we do not do this here.

2. Main results

2.1. Scaling, uncertainty and bounds

In this section we present our main results. We begin with a discussion of the role of scaling and uncertainty in determining exponents in the Lyapunov function approach to proving boundedness of the KS equation. The construction of a suitable function \( \phi_x \) can be viewed as a competition between kinetic energy and potential energy terms in the operator, and relatively simple scaling arguments make it clear how the function should scale with \( L \), the length of the interval. With this as motivation, we proceed to prove that a suitable Lyapunov function with the critical scaling exponents can be constructed. There will be two main technical tools. The first will be an inequality of Hardy–Rellich type, which will allow us to derive a lower bound on a second order kinetic energy term with a Dirichlet boundary condition in terms of a standard first-order kinetic energy term. The second will be a lower bound on a Schrödinger operator in terms of a finite-dimensional quadratic form.

As outlined in the previous section the basic strategy is to choose a periodic function \( \phi_x \) of zero mean such that the following quadratic form is coercive,

\[
\langle u, Ku \rangle = \int_{-L}^{L} u_{xx}^2 - u_x^2 + \phi_x u^2 \geq \lambda_0 \|u\|^2,
\]

for all \( u \) satisfying a Dirichlet boundary condition at the origin and some positive \( \lambda_0 \) independent of \( L \). Doing so gives a bound on the radius of the attracting ball in \( L_2 \) (for odd solutions) which scales like \( \|\phi\|_{H^2} \). As noted in the preceding section previous authors have constructed potentials \( \phi \) that are large and negative (roughly \( O(M) \)) on a small interval (roughly \( O(L/M) \)) near the origin, and positive on the rest of the interval, in such a way that the potential is mean zero. The uncertainty principle and Dirichlet boundary condition together imply that little of the mass of the ground state can be concentrated in the small region where the potential is negative, so that the net effect is to produce a positive ground state eigenvalue. In this section we present a scaling argument which suggests the best possible estimate one can expect from a potential of this form. In the next section we show that this estimate can actually be achieved.

We will assume for simplicity of discussion that for \( x \in (-L, L) \) the function \( \phi_x \) takes the following form (compare equation (11)):

\[
\phi_x = \gamma L^{c_2 - c_1 - 1} + L^{c_1} \tilde{q}(x L^{c_1}) \quad \gamma, c_{1,2} > 0,
\]

where \( \gamma \) is a constant and \( \tilde{q} \) is a compactly supported \( C^2 \) function, with \( \phi_x \) extended to a \( 2L \) periodic function in the usual way. We also assume that \( L \) is sufficiently large that the support of \( \tilde{q}(x L^{c_1}) \) lies completely in \((-L, L)\). For reference the functions constructed by Nicolaenko
et al and by Goodman have scaling exponents $c_1 = 1, c_2 = 2$, while the function constructed by Collet et al has scaling exponents $c_1 = \frac{5}{7}, c_2 = \frac{7}{5}$, as noted in the previous section.

Our first observation is that the operator $K = \partial_{xxxx} + \partial_{xx} + \phi_x$ with $\phi_x$ defined as above cannot be positive for $c_2 - c_1 - 1 < 0$. This follows from a straightforward test-function argument using a delocalized test function such as $u = L^{-1/2} \sin(k \pi x / L)$, for suitably chosen $k$ (say $k = [\pi L / 2]$). Next, if one makes the rescaling $y = L^c x$, the quadratic form becomes

$$L^{3c_1} \left( \int_{-L^{c_1}}^{L^{c_1}} u_y^2 - u_x^2 - L^{-2c_1} u_x^2 + L^{2 - 4c_1} \tilde{q}(y) u_x^2 (y) dy \right) + \gamma L^{2 - c_1 - 1} \|u\|_2^2.$$

Note the prefactor of $L^{c_2 - 4c_1}$ in front of the potential $\tilde{q}$. Motivated by this, we refer to potentials for which $c_2 - 4c_1 < 0$ as weak potentials, those for which $c_2 - 4c_1 > 0$ as strong potentials and those for which $c_2 = 4c_1$ as critical potentials. Strong potentials are those for which the potential energy term dominates the low-lying eigenvalues in the limit $L \to \infty$ (essentially the WKB limit), while weak potentials are those for which the kinetic energy term dominates. All the potentials constructed in previous papers are weak potentials, with the potential in [1] being closest to critical.

In the case of a strong potential $c_2 - 4c_1 > 0$ it is again clear from a test-function argument that the operator $K$ again cannot be positive. Taking a compactly supported test function whose support is contained in a region where $\tilde{q} < 0$ gives an estimate of the following form

$$\lambda_0(K) \leq -CL^{c_2 - c_1} + O(L^{3c_1}, L^{c_1}, L^{c_2 - c_1 - 1}),$$

where the leading order term is due to the potential and the three error terms come from the $u_{xx}^2, u_x^2$ and $u^2$ terms, respectively. A simple calculation shows that the $H^2$ norm of $\phi$ is bounded below by

$$\|\phi\|_{H^2} \geq \|\phi_{xx}\|_2 = O(L^{c_2 + c_1/2}).$$

Thus, the best estimate possible for a $\phi_x$ of the form given by (12) is given by the solution to the constrained minimization problem

$$\text{minimize} \quad c_2 + \frac{c_1}{2} \quad \text{subject to},$$

$$c_2 \geq c_1 + 1,$$

$$c_2 \leq 4c_1.$$

(15)

It is easy to check that the solution to this constrained minimization problem is given by

$$c_2 = \frac{1}{3} \quad c_1 = \frac{1}{5}.$$

(16)

A diagram of such a potential is shown in figure 1. A potential of this form would give an estimate of the radius of the attractor that scales like $L^{\frac{1}{3} + (1/2)} (1/3) = L^{3/2}$. We show in this paper that these critical exponents can actually be achieved.

It is worth noting that the scaling given above for the potential function $\phi_x$ is exactly the same as that for the viscous shock profile for the dKS equation constructed by Wittenberg [32]. This is not too surprising, since the same Lyapunov function argument applies to the destabilized KS equation, and one might reasonably expect that the Lyapunov function should look something like the steady state. In fact, it is easy to check that if $\phi$ is a stationary solution to the destabilized KS equation, and the linearized operator is negative semi-definite, then $\|u - \phi\|^2$ is a Lyapunov function for sufficiently large $\|u\|$. We will comment more on the relationship between Lyapunov functions and viscous shocks later.

This calculation makes clear the construction of a Lyapunov function for this problem involves a competition between the kinetic energy terms in the functional (which scale with
the width of the potential) and the potential energy terms, which scale with the height of the potential. In particular, it should be clear from this calculation that the scaling depends crucially on the order of the operator. In particular, one expects different critical exponents for a second order operator, since the kinetic energy term is less effective at small scales than the analogous term for a fourth order operator. Along these lines we make a couple of other comments, directed specifically at the papers of Nicolaenko et al and of Goodman. In [26] and [12] the authors make an additional simplification by using the inequality \( \partial_{xxxx} + \partial_{xx} \geq -\partial_{xx} - 1 \) to bound the operator \( K \) from below by a standard second order Schrödinger operator \( \tilde{K} \):

\[
\langle u, \tilde{K}u \rangle = \int_{-L}^{L} u_x^2 - u^2 + \phi_x u^2 \leq \langle u, Ku \rangle = \int_{-L}^{L} u_{xx}^2 - u_x^2 + \phi_x u^2.
\]  

(17)

If one carries out the same scaling analysis presented above for this quadratic form one finds that the exponents \( c_1, c_2 \) defined above must satisfy the inequalities

\[
c_2 \geq c_1 + 1, \tag{18}
\]

\[
c_2 \leq 2c_1, \tag{19}
\]

along with the same estimate of the penalty term due to the potential

\[
\|\phi\|_{H^2} \geq \|\phi_{xx}\|_2 = O(L^{c_2 + (c_1/2)}).
\]

Carrying out this minimization problem gives the critical exponents \( c_1 = 1, c_2 = 2 \), giving an estimate of the radius of the attracting ball of \( R = CL^{5/2} \). Note that this scaling is exactly that governing the transition region for the viscous shock solution to the Burgers–Sivashinsky (BS) equation derived by Goodman [12]. Thus, the scaling analysis shows that the estimate in [12] gives the best possible exponent for Lyapunov functions of the form given by equation (12) and the second order operator \( \tilde{K} \). There is an analogous calculation in the Fourier domain that shows that the estimate in [26] gives the best possible exponent for potentials \( \phi_x \) given by a Fourier series whose Fourier coefficients are of the form \( a_k = \hat{f}(k/M) \), where \( \hat{f} \) is a function of bounded variation with \( \int k^2 \hat{f}^2(k) dk \) finite and certain flatness properties in a neighbourhood of the origin. It is possible (though we believe it unlikely) that a potential could be constructed
which is not of one of these forms and leads to a better estimate. Such a construction would be extremely interesting.

This calculation is, in a sense, complementary to Lieb–Thirring type inequalities. In Lieb–Thirring inequalities one attempts to maximize some measure of the negative part of the spectrum of an operator over all potentials with a fixed norm. In the Lyapunov function method one would like to minimize the negative part of the spectrum to obtain a positive operator. Unfortunately, there appear to be only few results of this sort for operators of higher order than second (see, however, the work of Tadjbakhsh and Keller [30] and Cox and Overton [6] on the optimal shape of columns).

As a sidenote we note that the BS equation, being second order, provides an amusing application of this point of view. An analysis of the BS equation
\[ u_t = u_{xx} + u + uu_x \]
by the Lyapunov function method leads to a bound of the form
\[ \frac{d}{dt} \| u - \phi \|^2 \leq -\lambda_0 \| u \|^2 + c \| \phi_x \|^2, \]
where \( \lambda_0 \) is the smallest eigenvalue of the operator
\[ Hu = -u_{xx} + \phi_x u. \]
One can choose the potential \( \phi_x \) in an ‘optimal’ way by maximizing the ground state eigenvalue \( \lambda_0(\phi_x) \) subject to the constraints \( \int \phi_x^2 \ dx = \text{constant} \).

2.2. Proof of main results

In this section we show that the critical exponents \( c_1 = \frac{1}{3}, c_2 = \frac{4}{3} \) given by the solution to minimization problem equation (15) can actually be achieved. Our main techniques are a Hardy–Rellich inequality together with an elementary uncertainty estimate, which allow us to bound the quadratic form from below by a finite-dimensional one. Our main result can be stated as follows

**Theorem 1.** There exists a 2L periodic potential function \( \phi_x \) such that
\[ \int_{-L}^{L} \left( u_{xx}^2 - u_x^2 + \phi_x u^2 \right) \ dx \geq \frac{1}{4} \int_{-L}^{L} \left( u_{xx}^2 + u^2 \right) \ dx \]
for all \( u \in C^1[-L, L] \) with \( u(0) = 0 \). Further we have the estimate \( \| \phi \|_{L^2} \leq cL^{3/2} \).
Note. The above theorem requires only a single Dirichlet boundary condition at the origin, and thus potentially applies to many different boundary conditions on \( u \). We will primarily be interested in the domain of odd, periodic flows, which is preserved under the KS flow. Note that the solutions of the KS equation are known to be of Gevrey class \([2]\) (also see \([15,21]\) for related results on time analyticity), and thus in \( C^3[−L, L] \), for any positive time, the above theorem applies to any odd solution of KS at positive time.

The proof of this theorem is presented as a series of simple lemmas. The first lemma is an uncertainty inequality that allows us to estimate the second order \( H^2 \) type kinetic energy term by a first order \( H^1 \) type kinetic energy which is more analytically tractable. This lemma has an advantage over the standard Poincaré inequality, since it is in some sense ‘local’ and does not scale badly with large intervals. The downside is that we ‘use up’ the Dirichlet boundary condition, so we are forced to consider a larger set of admissible functions \( v \).

**Lemma 3.** Suppose that \( u \in C^3[−a, a] \) with \( u(0) = 0 \). Then, if \( v(y) = u(y)/y \) we have the inequality
\[
\int_{−a}^{a} \frac{1}{4} u_{yy}^2(y) \geq \int_{−a}^{a} \frac{1}{2} \left( \frac{u(y)}{y} \right)_y^2 = \int_{−a}^{a} \frac{1}{2} v_y^2 \, dy.
\]

**Proof.** Let \( u(y) = yv(y) \). Obviously \( u \in C^3[−a, a] \) and \( u(0) = 0 \) implies that \( v \in C^2[−a, a] \), and we have \( u_{yy} = yv_{yy} + 2v_y \). Upon substituting into the above integral we find
\[
\int_{−a}^{a} u_{yy}^2 \, dy = \int_{−a}^{a} (yv_{yy} + 2v_y)^2 \, dy
\]
\[
= \int_{−a}^{a} y^2 v_{yy}^2 \, dy + 2 \int_{−a}^{a} v_y^2 \, dy + 2a(v_y^2(a) + v_y^2(−a)) \geq 2 \int_{−a}^{a} v_y^2 \, dy.
\]
Here we have integrated by parts once and used the fact that the term \( 2yv_y^2|_{−a}^{a} = 2a(v_y^2(a) + v_y^2(−a)) \) is positive. □

**Remark 2.** This is essentially a higher order analogue of the Hardy inequality, which says that for \( F \in C^1[−a, a] \), \( F(0) = 0 \) one has the estimate
\[
\int_{−a}^{a} |F_x|^2 \, dx \geq \frac{1}{4} \int_{−a}^{a} \frac{|F(x)|^2}{x^2} \, dx.
\]
While the inequality in lemma 3 seems to be new there are many similar inequalities, typically in a larger number of spatial dimensions, that are generally referred to as Rellich or Hardy–Rellich inequalities. For a review see the paper of Owen \([27]\).

In the next lemma we show the positivity of an operator which we will later use to bound then operator \( K \) from below.

**Lemma 4.** Define a piecewise constant compactly supported function \( Q(y) \) as follows:
\[
Q(y) = \begin{cases} 
-q_0, & \text{when } 0 \leq |y| \leq \frac{a}{2}, \\
q_1, & \text{when } \frac{a}{2} < |y| \leq a, \\
0, & \text{when } a < |y|.
\end{cases}
\]
with $a$, $q_0$, $q_1$ positive constants satisfying the inequalities

$$q_0 a^2 < 1,$$  
(24)

$$q_1 > \frac{q_0}{1 - a^2 q_0},$$  
(25)

$$a < L^{4/3}.$$  
(26)

Then for all $v \in H^1[-L^{4/3}, L^{4/3}]$ we have

$$\int_{-L^{4/3}}^{L^{4/3}} \frac{1}{2} v_y^2 + Q(y)v^2 \, dy \geq \int_{-a}^{a} \frac{1}{2} v_y^2 + Q(y)v^2 \, dy \geq 0.$$  
(27)

**Proof.** We will show that $\int_{-a}^{a} \frac{1}{2} v_y^2 + Q(y)v^2 \, dy \geq 0$ for any $v \in H^1[L^{-4/3}, L^{4/3}]$. Since $Q$ is even the same argument holds for $\int_{0}^{a} \frac{1}{2} v_y^2 + Q(y)v^2 \, dy$. Obviously the integral over $|y| > a$ is positive, since $Q$ is zero, and the other part of the inequality follows. Note that we are not assuming any particular boundary conditions on $v$.

For any $v \in H^1[-L^{4/3}, L^{4/3}]$ and any two points $y_1$ and $y_2$ we have the elementary uncertainty inequality

$$\int_{y_1}^{y_2} v_x^2 \geq \frac{(v(y_1) - v(y_2))^2}{y_2 - y_1}.$$  

Since $v \in H^1$ is continuous we can ‘sample’ $v(y)$ at three locations $y_0 = 0$, $y_1 \in [0, a/2]$, $y_2 \in [a/2, a]$ defined as follows

$$v(y_0) = v_0 = v(0),$$

$$v(y_1) = v_1 = \max_{y \in [0, a/2]} |v(y)|,$$

$$v(y_2) = v_2 = \min_{y \in [a/2, a]} |v(y)|.$$  

In the case where there is not a unique point in $[0, a/2]$ at which $|v|$ attains its maximum $y_1$ can be chosen to be any point at which the maximum is achieved, and similarly for $y_2$.

One has the obvious lower bound on the kinetic energy in terms of $v_i, y_i$:

$$\int_0^a \frac{1}{2} v_y^2 \geq \int_0^{y_1} \frac{1}{2} v_y^2 + \int_{y_1}^{y_2} \frac{1}{2} v_y^2$$

$$\geq \frac{(v_1 - v_0)^2}{a} + \frac{(v_2 - v_1)^2}{2a},$$

as well as a bound on the potential energy term,

$$\int_0^a Q(y)v^2 = \int_0^{a/2} Q(y)v^2 + \int_{a/2}^a Q(y)v^2$$

$$\geq -q_0 v_0^2 a + \frac{q_1 v_2^2 a}{2}.$$  

The kinetic energy bound is clearly not sharp, and can be improved, but it suffices to prove the lemma. Combining these two lower bounds we find that the functional is bounded below by a quadratic form in three unknowns $v_0, v_1, v_2$:

$$\int_{-a}^{a} \frac{1}{2} v_y^2 + Q(y)v^2 \geq \frac{(v_1 - v_0)^2}{a} + \frac{(v_2 - v_1)^2}{2a} - \frac{q_0 v_0^2 a}{2} + \frac{q_1 v_2^2 a}{2}.$$  
(28)
The quadratic form is given by $v^T A v$, where $A$ is defined by

$$A \equiv \begin{bmatrix}
\frac{1}{a} & -\frac{1}{a} & 0 \\
-\frac{1}{a} & \frac{3}{2a} & -\frac{aq_0}{2} \\
0 & -\frac{1}{2a} & \frac{1}{2a} + \frac{aq_1}{2}
\end{bmatrix}.$$

A symmetric matrix $A$ is positive definite if and only if all of the principal minors are positive. Positivity of the $1 \times 1$ principal minor is equivalent to positivity of $a$, which is assumed. Positivity of the $2 \times 2$ minor is then equivalent to the condition

$$q_0 < \frac{1}{a^2}.$$  \hfill (29)

Physically, this is a requirement that the negative part of the potential well not be too deep compared with its width. This is natural, since a sufficiently deep well will always create a bound state. Finally, the positivity of the $3 \times 3$ minor is equivalent to

$$q_1 > \frac{q_0}{1 - a^2 q_0},$$  \hfill (30)

implying a minimum size of the positive part of the potential. Again this condition is very natural, since a strictly negative potential will always support a bound state. For a fixed $a > 0$ the above inequalities always have a solution in the positive quadrant of the $(q_0, q_1)$ plane above the hyperbola defined by $q_1 - q_0 - a^2 q_0 q_1 = 0$. For purposes of studying the large $L$ behaviour we can choose any $O(1)$ constants $q_0, q_1, a$ satisfying the above conditions, and clearly we will have $a < L^{4/3}$ for sufficiently large $L$. □

In the previous lemma we constructed a piecewise constant potential such that the quadratic form $\int_{-L/3}^{L/3} \frac{1}{2} v^2 y + Q(y) v^2$ is positive definite. The construction of the Lyapunov functional requires the potential $\phi_e$ to be in $H^1[-L, L]$. Since $\phi_e$ is (up to scaling and an additive constant) given by $Q(y)/y^2$ it is clearly necessary that $Q(y)$ vanish at least quadratically at the origin. In the next lemma we show that we can construct a modified potential $\tilde{Q}$ such that $\tilde{q}(y) = \tilde{Q}(y)/y^2$ is smooth and the quadratic form does not decrease.

**Lemma 5.** Given any constant $\mu$, and $a, q_0, q_1$ from the previous lemma for some $\delta$ sufficiently small there exists a potential function $\tilde{Q}$ such that

- $\tilde{q} = \tilde{Q}(y)/y^2 \in C_0^\infty$ and
- $\int \tilde{q} \leq -\mu$,
- $\int_{-\delta}^{\delta} \frac{1}{2} v^2 + \tilde{Q} v^2 \geq 0 \quad \forall v \in H^1[-L^{4/3}, L^{4/3}]$.

**Proof.** Let $Q(y)$ be a piece-wise constant potential as constructed in lemma 6:

$$Q(y) = \begin{cases}
-q_0 & \text{when } 0 \leq |y| \leq \frac{a}{2}, \\
q_1 & \text{when } \frac{a}{2} < |y| \leq a, \\
0 & \text{when } a < |y|,
\end{cases}$$

with $\int_{-a}^{a} \frac{1}{2} v^2 + Q(y) v^2 \, dy > 0$. Define $f(y)$ to be an nondecreasing $C^\infty$ function satisfying

$$f(y) = \begin{cases}
0 & \text{when } y \leq 0, \\
1 & \text{when } y \geq 1.
\end{cases}$$
For instance one could choose
\[ g(y) = \begin{cases} 
  e^{-(1/y + y^2/2)} & \text{if } y \in (0, 1), \\
  0 & \text{if } y \notin (0, 1),
\end{cases} \]
and thence
\[ f(y) = \frac{\int_0^y g(s)\,ds}{\int_0^1 g(s)\,ds}. \]
Clearly we have \( f(0) = 0 \), \( f(1) = 1 \) as well as \( \lim_{y \to 0} f^{(n)}(y) = 0 \), \( \lim_{y \to 1} y^{-k} f(y) = 0 \) for all \( k, n \geq 1 \). Define \( \tilde{Q} \) to be a mollified approximation to \( Q \) in the following way: let \( \tilde{Q} \) be even and defined for \( y \geq 0 \) by
\[ \tilde{Q}(y) = \begin{cases} 
  -q_0 f \left( \frac{y}{\delta} \right) & \text{if } y \in (0, \delta), \\
  -q_0 & \text{if } y \in (\delta, \frac{a}{2} - \delta), \\
  -q_0 + (q_0 + q_1) f \left( \frac{y - (a/2) + \delta}{\delta} \right) & \text{if } y \in (\frac{a}{2} - \delta, \frac{a}{2}), \\
  q_1 & \text{if } y \in (\frac{a}{2}, a), \\
  q_1 f \left( 1 + \frac{a - y}{\delta} \right) & \text{if } y \in (a, a + \delta),
\end{cases} \]
where \( \delta < a/4 \) is small and will be chosen later. Clearly \( \tilde{q} = \tilde{Q}/y^2 \) is in \( C^\infty_0 \), and \( \tilde{q}(0) = 0 \).

Note that since \( a \) is assumed to be \( O(1) \) and \( L \) large then we can assume \( (-(a + \delta), a + \delta) \subset (\frac{-L}{4}, \frac{L}{4}) \).

We are now in a position to prove our main theorem, in which we construct our potential \( \phi_x \). This follows fairly easily from the preceding string of lemmas.

**Proof of theorem 1.** We would like to show that one can construct a potential function \( \phi_x \) such that the inequality claimed in equation (23)
\[ \int_{-L}^{L} (u_x^2 - u^2 + \phi_x u^2)\,dx \geq \frac{1}{4} \int_{-L}^{L} (u_x^2 + u^2)\,dx \]
holds for all \( u \in C^3[\![-L, L]\!] \) with \( u(0) = 0 \). The Cauchy–Schwarz inequality implies that
\[ -\int u_x^2 \geq -\frac{1}{4} \int u_x^2 - \frac{1}{2} \int u^2, \]
so it clearly suffices to show that
\[ \int \frac{1}{4} u_x^2 + \left( \phi_x - \frac{3}{4} \right) u^2 \geq 0. \]
We write the mean zero function \( \phi_x \) in the form
\[ \phi_x = q(x) - \langle q \rangle \]
where \( \langle \cdot \rangle \) denotes the mean value on \([-L, L]\):

\[
\langle q \rangle = \frac{1}{2L} \int_{-L}^{L} q(x) \, dx.
\]

If we can choose \( q \) such that \( \langle q \rangle \leq -\frac{3}{4} \) and

\[
\int_{-L}^{L} \frac{1}{4} u_{xx}^2 + q(x)u^2 \geq 0
\]

then we are done since

\[
\int_{-L}^{L} \frac{1}{4} u_{xx}^2 + q(x)u^2 \geq 0.
\]

Based on the scaling arguments of the previous section we define \( q(x) \) to be

\[
q(x) = L^{4/3} \tilde{q}(x L^{1/3}) \quad x \in [-L, L],
\]

where \( \tilde{q}(y) = \tilde{Q}(y)/y^2 \) is the compactly supported potential function constructed in lemma 5. Obviously this can be extended to a periodic function in the standard way. After the rescaling \( y = L^{1/3} x \) the quadratic form in equation (37) becomes

\[
L \int_{-L^{1/3}}^{L^{1/3}} \frac{1}{4} u_{yy}^2 + \tilde{q}(y)u^2 \, dy.
\]

Letting \( u(y) = yv(y) \) and applying lemma 3 gives the inequality

\[
\int_{-L^{1/3}}^{L^{1/3}} \left( \frac{1}{4} u_{yy}^2 + \tilde{q}(y)u^2 \right) \, dy \geq \int_{-L^{1/3}}^{L^{1/3}} \left( \frac{1}{2} v_y^2 + y^2 \tilde{q}(y)v^2 \right) \, dy
\]

From the results of lemma 5 the above quadratic form is positive for all \( u \in C^3[-L, L] \) with \( u(0) = 0 \). Again the Gevrey regularity of solutions to the KS equation guarantees that solutions are in \( C^3[-L, L] \) for any positive time.

We have constructed a potential function \( \phi_x \) such that the operator \( K \) is positive definite on the space of functions satisfying a Dirichlet boundary condition at the origin. All that remains to be checked is that the \( H^2 \) norm of the potential scales correctly with \( L \). This is the content of the next lemma.

**Lemma 6.** The potential \( \phi \) satisfies \( \| \phi \|_{H^2} \leq CL^{3/2} \).

**Proof.** From the definition

\[
\phi_x = q(x) - \langle q \rangle = L^{4/3} \tilde{q}(x L^{1/3}) - \langle q \rangle
\]

it is clear on rescaling that \( \| \phi_x \|_2^2 = O(L^{7/3}) \) and that \( \| \phi_{xx} \|_2^2 = O(L^3) \). Thus, we only need to estimate \( \| \phi \|_2^2 \). From the definition of \( \phi_x \) we have

\[
\phi(x) = \int_0^x \phi_s \, ds = \int_0^1 q(s) - \langle q \rangle \, ds,
\]

and after the substitution \( y = s L^{1/3} \) this becomes

\[
\phi(x) = L \int_{0}^{x L^{1/3}} \tilde{q} \, dy - \langle q \rangle x.
\]
We have the obvious estimate

$$|\phi(x)| \leq L \int_0^L |\tilde{q}(y)| \, dy + (q) L.$$  

Since \( \tilde{q} \) is bounded (independently of \( L \)) and supported on \([-a, a]\) we have

$$|\phi(x)| \leq caL + (q) L = O(L).$$

The \( L_2 \) bound now follows since

$$\|\phi\|_2^2 \leq (2L)\|\phi\|_2^2 = O(L^3).$$  

**Remark 3.** The fact that \( \tilde{q} \) can be chosen to have arbitrarily negative mean and still generate a positive operator \( K \) implies that \( \phi \), can be chosen such that \( (u, Ku) \geq \lambda_0\|u\|^2 \) for any constant \( \lambda_0 \) independent of \( L \) with a bound \( \|\phi\|_{H^2} \leq C(\lambda_0)L^{3/2} \), where \( C \) depends on \( \lambda_0 \) but is independent of \( L \). Thus, the above argument may be used to obtain a bound on the \( L_2 \) norm of the destabilized KS equation \( 2 \) which scales like \( C(\gamma)L^{3/2} \) for any fixed \( \gamma \). We have not computed the scaling of \( C(\gamma) \) with \( \gamma \) but the calculation is straightforward.

Finally, we mention some applications of this result to KS-like equations which have appeared in the literature. In most cases these papers use a variation of the results of Collet et al \([1]\) to produce bounds for the relevant equations, and these results can all be improved by the arguments presented here. In some cases (Duan and Ervin \([7]\) and possibly Molinet \([25]\)) we believe that a modification of the div-curl lemma in \([11]\) might be applicable as well, but we have not verified this.

One result along this direction is by Molinet \([25]\), which improves on earlier work of Sell and Taboada \([28]\), giving \( L_2 \) boundedness of the KS equation in two spatial dimensions

$$\tilde{u}_t = -\Delta \tilde{u} - \Delta \tilde{u} + \tilde{\nabla} (\tilde{u} \cdot \tilde{u})$$

for sufficiently thin rectangular domains. The results of Molinet require the construction of a Lyapunov function for the problem in one spatial dimension. Molinet uses the Lyapunov function constructed by Collet et al to show boundedness in \( L_2((0, L) \times (0, L)) \) assuming that the widths satisfy the inequality \( L_y \leq CL_y^{-13/5} \). When this condition holds \( ||\tilde{u}||_2 \leq CL_y^{8/5}L_y^{1/2} \). Using the Lyapunov function constructed here and otherwise applying Molinet’s results verbatim gives \( L_2 \) boundedness assuming the aspect ratio satisfies

$$L_y \leq CL_y^{-13/7}.$$

This, in turn, leads to a bound on the \( L_2 \) norm of the form

$$\limsup_{y \to \infty} ||\tilde{u}||_2 \leq CL_y^{3/2}L_y^{1/2}.$$

Hilhorst et al \([16]\) consider a nonlocal variant of the KS equation

$$u_t = -u_{xxx} - u_{xx} - uu_x + \gamma I(u) - axu_x - 2au \quad u(\pm L) = 0 \quad u_{xx}(\pm L) = 0,$$

where \( I(u) \) is a linear operator with Fourier symbol \( \hat{I}(k) = Lk \). The authors show that the Lyapunov function constructed by Collet et al is a Lyapunov function for the equation given above as well, under the assumption that the coefficients satisfy \( \gamma L \leq (1 + a)^{3/2} \), which leads to an estimate of the form \( ||u||_2 \leq CL^{11/5} \). Again the construction outlined here applies, with the same improvement in the main estimate as for the dKS, leading to an estimate of the form \( ||u||_2 \leq CL^{13/6} \). Similarly, Duan and Ervin consider a different nonlocal variant of the KS equation, incorporating an additional term such as \( H(u_{xxx}) \), where \( H \) is the periodic Hilbert transform. The construction outlined here gives the same bound as for the standard KS: \( ||u||_2 \leq CL^{3/2} \).
Finally we mention the work of Grauer [14] on a variant of the Hasegawa–Mima equation. Grauer shows that this equation in $2 + 1$ dimensions admits a Lyapunov-type energy argument which is very similar to the one for the KS equation. It would be interesting to see if arguments similar to the ones presented here could be applied to this equation.

2.3. Extension to arbitrary initial data

The theorem proved above, together with the lemmas proved in the first section, show that for odd initial data the (destabilized) KS equation remains bounded in $L_2$ for all time. These results can be extended to arbitrary mean-zero initial data in the manner first done by Collet et al [1] or by Goodman [12]. This paper was written in such a way as to be compatible with the results of Collet et al, where one allows the potential $\phi_x$ to translate via a gradient-flow type dynamics. In particular theorem 1 in this paper is stated in such a way as to be compatible with lemma 5.1 in [1]. From this the results of proposition 4.3 in [1] follow, and the $L_2$ boundedness result extends to arbitrary mean-zero initial data. One could equally well employ the related idea of Goodman and look at the rate of change in the distance of $u$ from the set of all translates of $\phi$.

3. Conclusions

In this paper we have constructed a function $\phi$ such that the ball $B(\phi, cL^{3/2})$ is a global attracting set for the (destabilized) KS equation (2). This result has the best possible scaling with $L$, the size of the domain, since the above equation has stationary solutions with $L_2$ norm which scales like $L^{3/2}$. While the argument of Giacomelli and Otto gives a stronger result, the Lyapunov function argument outlined here is still interesting for a number of reasons. The first is that it gives the optimal scaling for $L_2$ boundedness of the destabilized KS equation. Secondly, this calculation shows that the critical exponents for an argument of this type can be achieved and is interesting as a demonstration of the limits of this kind of Lyapunov function argument. Finally, scaling arguments similar to those presented here should be applicable to other equations.

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