**Abstract.** In this paper we study the rank-one convex hull of a differential inclusion associated to entropy solutions of a hyperbolic system of conservation laws. This was introduced in [7, Section 7], and many of its properties have already been shown in [8, 9]. In particular, in [9] it is shown that the differential inclusion does not contain any $T_3$ configurations. Here we continue that study by showing that the differential inclusion does not contain $T_5$ configurations.

1. Introduction

We study compactness properties of weak Lipschitz solutions $\varphi$ to the equation

$$\partial_t \varphi - \partial_x (a(\partial_x \varphi)) = 0,$$

which additionally satisfy

$$\partial_t (\eta(\partial_t \varphi, \partial_x \varphi)) - \partial_x (q(\partial_t \varphi, \partial_x \varphi)) \leq 0,$$

for

$$\eta(x, y) = \frac{1}{2} x^2 + F(y), \quad q(x, y) = a(y)x.$$

Here, $a$ is a $C^1$, strictly convex and monotone increasing function on the real line, and $F(y) = \int_y^0 a(\xi) d\xi$. Inequality (1.2) is called an entropy for the equation (1.1). It can be checked that, if $\varphi$ is smooth, then (1.2) is automatically satisfied (with an equality), but it needs to be imposed for weak solutions. This problem has its origins in the classical paper [3] by R. J. DiPerna. In [7, Section 7], B. Kirchheim, S. Müller and V. Šverák propose the study of (1.1)-(1.2) via differential inclusions. In other words, introducing auxiliary functions

$$u := \partial_t \varphi, \quad v := \partial_x \varphi,$$

and assuming equality in (1.2), we can rewrite (1.1)-(1.2) as the system

$$\begin{cases}
\partial_t u - \partial_x (a(v)) = 0, \\
\partial_t v - \partial_x u = 0, \\
\partial_t (\eta(u, v)) - \partial_x (q(u, v)) = 0.
\end{cases}$$

At least locally, this is equivalent to asking that there exist Lipschitz functions $\varphi_1, \varphi_2$ such that the Lipschitz map $\psi = (\varphi_1, \varphi, \varphi_2)$ solves, a.e. in $\Omega$,

$$D\psi = (\partial_t \varphi, \partial_x \varphi) \in K_a := \left\{ \begin{bmatrix} a(v) & u \\ u & v & \frac{1}{2} u^2 + F(v) \end{bmatrix} : u, v \in \mathbb{R}^2 \right\}.$$

Rewriting a system of PDEs as a differential inclusions allows to prove compactness properties or to use convex integration methods to show unexpected features of the original system. Given a set $K \subset \mathbb{R}^{m \times n}$,
we have compactness for the associated differential inclusion if, given an open and bounded set \( \Omega \subset \mathbb{R}^n \), a sequence \( u_j \in \text{Lip}(\Omega, \mathbb{R}^m) \) with equibounded Lipschitz norm and weakly-* converging to \( u \) satisfying
\[
\text{dist}(Du_j, K) \to 0 \quad \text{in } L^1_{\text{loc}},
\]
then \( u_j \) converges strongly in \( W^{1,1}_{\text{loc}} \) to \( u \). This is related to the classification of gradient Young measures supported in \( K \). In particular, compactness for the differential inclusion associated to \( K \) holds if and only if the only Young measures supported in \( K \) are Dirac deltas. We refer the reader to [16, Section 4] and references therein for additional information. Usually, showing compactness of differential inclusions is a hard task, and few general methods are known, see for instance [13, Section 3].

Conversely, in order to disprove compactness for the differential inclusion, there are some natural steps that need to be taken. First, one needs to study rank-one connections. A rank-one connection in \( K \) is a couple of matrices \( \{A, B\} \) such that \( A, B \in K \) and \( \text{rank}(A - B) = 1 \). If there exists a rank-one connection in \( K \), then it is possible to construct a highly oscillatory approximate solution to the differential inclusion, see for instance [6, Lemma 3.2], thus disproving compactness. For the particular case of \( K_a \), with \( a \) strictly convex and monotone increasing, it has been shown in [9, Proposition 4] that there are can be no rank-one connections inside \( K_a \). The next step is to see whether \( K \) contains \( T_N \) configurations. We will postpone the precise definition of \( T_N \) configuration to Section 2. \( T_N \) configurations in \( \mathbb{R}^{m \times n} \) are ordered sets of \( N \) distinct matrices \( X_1, \ldots, X_N \) such that possibly
\[
\text{rank}(X_i - X_j) \geq 2, \quad \forall i \neq j,
\]
but \( \{X_1, \ldots, X_N\} \) supports a non-trivial gradient Young measure, see [10, Lemma 2.6]. These special sets of matrices, found independently by many authors, see [10, Section 2.5], have been the fundamental building block in the convex integration construction of the highly irregular critical points of quasiconvex functionals [11] and polyconvex functionals [14]. Since then, they never ceased to be a fundamental tool to construct pathological solutions to PDEs, compare [1, 2, 4, 5, 12].

Concerning the particular differential inclusion associated to \( K_a \), Kirchheim, Müller and Šverák proposed in [7] the study of four properties \((P1) - (P4)\) which concern local compactness properties of the differential inclusion. To the best of our knowledge, the study of these properties has started in [8]. Subsequently, A. Lorent and G. Peng started the study of non-local compactness properties of the differential inclusion in [9], showing that the differential inclusion associated to \( K_a \) does not contain rank-one connections and \( T_4 \) configurations. Notice that in \( \mathbb{R}^{n \times n} \) for \( n \neq 2 \) or \( m \neq 2 \) a \( T_N \) configuration does not need to contain\(^1\) a \( T_4 \) configuration. For instance in [14] the author finds a \( T_5 \) configuration in a set \( K \) that, by [7, Proposition 9], cannot contain a \( T_4 \) configuration. Thus, even after the result of [9], it is unclear whether \( K_a \) contains \( T_N \) configurations for \( N \geq 5 \). Aim of this paper is to show that \( K_a \) does not contain \( T_5 \) configurations. Our proof exploits the results of [15] and is close in spirit to those of [2, 5]. Our method yields also a short, independent proof of the non-existence of \( T_4 \) configurations inside \( K_a \), i.e. the main result of [9]. We end this introduction by stating our two main results:

**Theorem 1.1.** Assume \( a \in C^1(\mathbb{R}) \), \( a(0) = 0 \) and \( a \) is strictly convex and increasing. Let
\[
K_a = \left\{ \begin{bmatrix} a(v) & u \\ u & v \\ a(v)u & \frac{1}{2}u^2 + F(v) \end{bmatrix} : u, v \in \mathbb{R}^2 \right\},
\]
where \( F(v) = \int_0^v a(\xi)d\xi \). Then, \( K_a \) does not contain \( T_4 \) configurations.

**Theorem 1.2.** Assume \( a \in C^1(\mathbb{R}) \), \( a(0) = 0 \) and \( a \) is strictly convex and increasing. Let \( K_a \) be defined as in Theorem 1.1. Then, \( K_a \) does not contain \( T_5 \) configurations.

\(^1\)The situation is different in \( \mathbb{R}^{2 \times 2} \), where every \( T_N \) configuration contains a \( T_4 \) configuration by [15].
The paper is organized as follows. In Section 2, we recall the definition of $T_N$ configuration and a characterization coming from [2,15]. In Section 3, we show some useful inequalities on minors of elements of $K_a$, that will be used in Section 4 to show our two main Theorems 1.1-1.2.

1.1. Notation. The set of matrices with $m$ rows and $n$ columns will be denoted by $\mathbb{R}^{m \times n}$. The Hilbert-Schmidt scalar product in $\mathbb{R}^{m \times n}$ is denoted by $\langle \cdot, \cdot \rangle$. For any matrix $A \in \mathbb{R}^{m \times m}$, we define $\text{cof} A$ as the $m \times m$ matrix whose value in the $i$-th row and $j$-th column is

$$(\text{cof} A)_{ij} = (-1)^{i+j} \det(A^{i,j})$$

where $A^{i,j}$ denotes the $(m-1) \times (m-1)$ matrix obtained by removing the $i$-th column and the $j$-th row from $A$. The set of symmetric matrices in $\mathbb{R}^{m \times m}$ is denoted by $\mathbb{R}^{m \times m}_{\text{sym}}$. For two symmetric matrices $A$ and $B$, we will write $A \preceq B$ if $B - A$ is positive semi-definite. Finally, given vectors $a_1, \ldots, a_n$ for some $n \geq 1$, we set

$$\sum_{k=i}^{j} a_k := 0 \quad \text{whenever } j < i.$$

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2. Preliminaries on $T_N$ configurations

In this section we recall a useful characterization of $T_N$ configurations. This was first discovered by L. Székelyhidi in [15] for $T_N$ configurations contained in $\mathbb{R}^{2 \times 2}$ and generalised by C. De Lellis, G. De Philippis, B. Kirchheim and the second named author in [2] to study $T_N$ configurations contained in $\mathbb{R}^{m \times n}$ for arbitrary integers $m$ and $n$.

Definition 2.1. An ordered set of $N$ distinct matrices $\{X_1, \ldots, X_N\}$ is said to induce a $T_N$ configuration if there exist matrices $Q, D_1, \ldots, D_N \in \mathbb{R}^{m \times n}$ and real numbers $k_i > 1$ such that

(i) $\text{rank}(D_i) \leq 1$ for each $i$;
(ii) the following closing condition is fulfilled

$$\sum_{i=1}^{N} D_i = 0;$$

(iii) it holds

$$X_i = Q + \sum_{j=1}^{i-1} D_j + k_i D_i, \quad \forall 1 \leq i \leq N. \quad (2.1)$$

Let us introduce some notation. Let $I$ and $J$ be multi-indices. More precisely, we will use $I$ to denote multi-indices referring to ordered sets of rows of matrices and $J$ for multi-indices referring to ordered sets of columns of matrices. In this section, we deal with matrices in $\mathbb{R}^{m \times n}$ and we will therefore have

$$I = (i_1, \ldots, i_r) \in \mathbb{N}^r, 1 \leq i_1 < \ldots < i_r \leq m$$

$$J = (j_1, \ldots, j_s) \in \mathbb{N}^s, 1 \leq i_1 < \ldots < i_s \leq n$$

We will use the notation $|I| = r$ and $|J| = s$. In this paper, we will always have $r = s$. We define

$$\mathcal{A}_r = \{(I, J) : I \text{ and } J \text{ are ordered sets and } |I| = |J| = r\}.$$
For any matrix $M \in \mathbb{R}^{m \times n}$ and any $Z \in A_r$, we write $M^Z$ to denote the $r \times r$ square matrix obtained by considering just the elements $M_{ij}$ where $(i, j) \in I \times J$. Given a set of matrices $\{X_1, \ldots, X_N\} \subset \mathbb{R}^{m \times n}$, $\mu \in \mathbb{R}$ and $Z \in A_r$, we introduce the matrix

$$A_Z^\mu = \begin{bmatrix}
0 & \mu \det(X_1^Z - X_1^Z) & \det(X_2^Z - X_1^Z) & \cdots & \det(X_N^Z - X_1^Z) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\mu \det(X_1^Z - X_N^Z) & \mu \det(X_2^Z - X_N^Z) & \mu \det(X_3^Z - X_N^Z) & \cdots & 0
\end{bmatrix}.$$  

The following proposition provides us with a characterisation of $T_N$ configurations.

**Proposition 2.2.** A set $\{X_1, \ldots, X_N\} \subset \mathbb{R}^{m \times n}$ induces a $T_N$ configuration if and only if there is a real $\mu > 1$ and a vector $\lambda \in \mathbb{R}^N$ with positive components such that

$$A_Z^\mu \lambda = 0 \quad \forall Z \in A_2.$$

**Corollary 2.3.** Let $\{X_1, \ldots, X_N\} \subset \mathbb{R}^{m \times n}$ induce a $T_N$ configuration and let $\lambda \in \mathbb{R}^N$ be a vector with positive components given by Proposition 2.2 such that

$$A_Z^\mu \lambda = 0 \quad \forall Z \in A_2.$$

Define the vectors $t^i$ as

$$t^i = \frac{1}{\xi_i}(\mu \lambda_1, \ldots, \mu \lambda_{i-1}, \lambda_i, \ldots, \lambda_N) \quad \forall i = 1, \ldots, N$$  

where $\xi_i > 0$ is a normalising constant such that $\|t^i\|_1 = 1$. Define the coefficients $k_i$ through

$$k_i = \frac{\mu \lambda_1 + \ldots + \mu \lambda_i + \lambda_{i+1} + \cdots + \lambda_N}{(\mu - 1)\lambda_i} \quad i = 1, \ldots, N.$$  

Then define the matrices $D_j$ with $j = 1, \ldots, N - 1$ and $Q$ by solving recursively

$$\sum_{j=1}^N t_j^i X_j = Q + D_1 + \ldots + D_{i-1}, \quad \forall i = 1, \ldots, N.$$  

and setting $D_N := -D_1 - \ldots - D_{N-1}$. Then $P, D_1, \ldots, D_N, k_1, \ldots, k_N, X_1, \ldots, X_N$ solve equations (2.1).

**Remark 2.4.** With the vectors $t^i$ defined as in (2.2), Proposition 2.2 implies that

$$\sum_{j=1}^N t_j^i \det(X_j^Z - X_i^Z) = 0, \quad \forall i \in \{1, \ldots, N\}, \forall Z \in A_2.$$  

### 3. Inclusion sets in $K_a$

In this section we start our study of subsets $\{X_1, \ldots, X_N\} \subset \mathbb{R}^{3 \times 2}$ of $K_a$, for $a$ a strictly convex and increasing function with $a(0) = 0$ as in Theorem 1.1. These properties will then be used to prove our main Theorems 1.1 and 1.2 in the next section. For future reference, we define $Z_{12}, Z_{13} \in A_2$ as

$$Z_{12} = ((1, 2), (1, 2)) \in A_2 \quad \text{and} \quad Z_{13} = ((1, 3), (1, 2)) \in A_2.$$  

3.1. **General inclusion sets.** We begin by studying general inclusion sets and find that a number of inequalities must be fulfilled in order to have $\{X_1, \ldots, X_N\} \subset K_a$.

**Proposition 3.1.** If a set $\{X_1, \ldots, X_N\} \subset \mathbb{R}^{3 \times 2}$ is contained in $K_a$ and

$$X_i = \begin{bmatrix}
a(v_i) & u_i \\
u_i & v_i \\
u_i a(v_i) & \frac{1}{2}u_i^2 + F(v_i)
\end{bmatrix} \quad \forall i = 1, \ldots, N,$$
then for all $i,j \in \{1,\ldots,N\}$
\[
(F(v_j) - F(v_i))(a(v_j) - a(v_i)) < \frac{(a(v_j) + a(v_i))(v_j - v_i)(a(v_j) - a(v_i))}{2}
\] (3.1)
whenever $v_j \neq v_i$.

Proof. Let $\{X_1,\ldots,X_N\} \subset K_a$. Let $v_i \neq v_j$. Then, since $a$ is monotone increasing and strictly convex,
\[
(F(v_j) - F(v_i))(a(v_j) - a(v_i)) = (a(v_j) - a(v_i)) \int_{v_i}^{v_j} a(t) \, dt
\]
\[
= (a(v_j) - a(v_i))(v_j - v_i) \int_0^1 a(v_i + s(v_j - v_i)) \, ds
\]
\[
= (a(v_j) - a(v_i))(v_j - v_i) \int_0^1 a(sv_j + (1 - s)v_i) \, ds
\]
\[
< (a(v_j) - a(v_i))(v_j - v_i) \int_0^1 [sa(v_j) + (1 - s)a(v_i)] \, ds
\]
\[
= \frac{(a(v_j) + a(v_i))(v_j - v_i)(a(v_j) - a(v_i))}{2}.
\]

\[\square\]

3.2. Properties of $T_N$ configurations contained in $K_a$. We now turn our attention to the particular case in which $\{X_1,\ldots,X_N\}$ induces a $T_N$ configuration.

Proposition 3.2. If a set $\{X_1,\ldots,X_N\} \subset K_a$ induces a $T_N$ configuration with
\[
X_i = \begin{bmatrix} a(v_i) \\ u_i \\ v_i \\ u_i a(v_i) \end{bmatrix} \quad \forall i = 1,\ldots,N.
\]
Then there must exist $i \neq j$ such that $v_i \neq v_j$.

Proof. For a contradiction, assume that all $v_i$ take the same value. Remark 2.4 yields
\[
0 = \sum_{j=1}^N t_j^N \det(X_j^{Z_{12}} - X_i^{Z_{12}}) = \sum_{j=1}^N t_j^N \left[ (a(v_j) - a(v_i))(v_j - v_i) - (u_j - u_i)^2 \right]_{j=0}^N
\]
\[
= -\sum_{j=1}^N t_j^N (u_j - u_i)^2.
\]
This means that all $u_i$ take the same value and therefore all $X_i$ are identical. This contradicts our definition of $T_N$ configurations.

Before going further, we introduce the following notation. If $\{X_1,\ldots,X_N\}$ induces a $T_N$ configuration with
\[
X_i = Q + D_1 + \cdots + D_{i-1} + k_i D_i, \quad \text{rank}(D_i) \leq 1, \quad \sum D_i = 0,
\] (3.2)
then we define $\{Y_1,\ldots,Y_N\}$ as
\[
Y_j = X_j^{Z_{12}}, C_j = D_j^{Z_{12}} \quad \forall j = 1,\ldots,N \quad \text{and} \quad P = Q^{Z_{12}}.
\] (3.3)
We will write
\[
P_0 = P_1 = P, P_\ell := P + C_1 + \cdots + C_{\ell-1}, \quad \forall \ell \geq 1.
\] (3.4)
In other words, this operation simply consists in restricting our attention to the first two lines of the $3 \times 2$ matrices $X_i$. Notice that, by (2.4), it follows that $\{C_1,\ldots,C_N\} \subset \mathbb{R}_{sym}^{2 \times 2}$. We use the above notation throughout the remainder of this paper.
Lemma 3.3. Let \{X_1, \ldots, X_N\} ⊂ K_n induce a \(T_N\) configuration of the form (3.2). Then, in the notation above and for \(\xi_i\) as in (2.2), we have

\[
(*)_i := \xi_i \sum_{j=1}^{N} t_j^i (X_{j1})_{11} \det(X_j^{Z_{i12}} - X_i^{Z_{i12}}) = \xi_i \sum_{j=1}^{N} t_j^i (Y_j)_{11} \det(Y_j - Y_i) > 0 \quad \forall i = 1, \ldots, N.
\]

Let us point out that the reason for which we define the quantity \((*)_i\) with a factor \(\xi_i > 0\) is purely practical. Indeed, it is only useful to obtain a cleaner expression in the forthcoming (3.7).

Proof. By Remark 2.4 we have

\[
0 = \sum_{j=1}^{N} t_j^i \det(X_j^{Z_{i13}} - X_i^{Z_{i13}})
= \sum_{j=1}^{N} t_j^i \left[ (a(v_j) - a(v_i)) \left( \frac{1}{2} u_j^2 - \frac{1}{2} u_i^2 + F(v_j) - F(v_i) \right) - (u_j - u_i)(u_ja(v_j) - u_i a(v_i)) \right]
= \sum_{j=1}^{N} t_j^i (a(v_j) - a(v_i)) (F(v_j) - F(v_i))
+ \sum_{j=1}^{N} t_j^i \left[ (a(v_j) - a(v_i)) \left( \frac{1}{2} u_j^2 - \frac{1}{2} u_i^2 \right) - (u_j - u_i)(u_ja(v_j) - u_i a(v_i)) \right]
= : \Phi_{i,j}
\]

Firstly, by (3.1) combined with the fact that all \(v_j\) are not identical, see Proposition 3.2, we deduce

\[
\sum_{j=1}^{N} t_j^i (a(v_j) - a(v_i)) (F(v_j) - F(v_i)) < \sum_{j=1}^{N} t_j^i (a(v_j) + a(v_i))(v_j - v_i)(a(v_j) - a(v_i)) < \frac{1}{2} \sum_{j=1}^{N} t_j^i (a(v_j) + a(v_i))(v_j - v_i)(a(v_j) - a(v_i))
\]

(3.5)

Secondly, we compute the quantities \(\Phi_{i,j}\) for all \(i\) and \(j\). We have, by direct computation,

\[
\Phi_{i,j} = -\frac{1}{2} (u_j - u_i)^2 (a(v_j) + a(v_i)).
\]

(3.6)

Then by combining (3.5) and (3.6), we find

\[
0 = \sum_{j=1}^{N} t_j^i \det(X_j^{Z_{i13}} - X_i^{Z_{i13}})
< \frac{1}{2} \sum_{j=1}^{N} t_j^i (a(v_j) + a(v_i))(v_j - v_i)(a(v_j) - a(v_i)) - \frac{1}{2} \sum_{j=1}^{N} t_j^i (u_j - u_i)^2 (a(v_j) + a(v_i))
= \frac{1}{2} \sum_{j=1}^{N} t_j^i (a(v_j) + a(v_i))(v_j - v_i)(a(v_j) - a(v_i)) - (u_j - u_i)^2
= \frac{1}{2} \sum_{j=1}^{N} t_j^i (a(v_j) + a(v_i)) \det(X_j^{Z_{i12}} - X_i^{Z_{i12}}).
\]

Since by Remark 2.4 we have

\[
\sum_{j=1}^{N} t_j^i \det(X_j^{Z_{i12}} - X_i^{Z_{i12}}) = 0,
\]

we have

\[
0 = \sum_{j=1}^{N} t_j^i \det(X_j^{Z_{i12}} - X_i^{Z_{i12}})
= \sum_{j=1}^{N} t_j^i (X_{j1})_{11} \det(X_j^{Z_{i12}} - X_i^{Z_{i12}})
= \sum_{j=1}^{N} t_j^i (Y_j)_{11} \det(Y_j - Y_i).
\]
we deduce that
\[ \sum_{j=1}^{N} t_j^i a(v_j) \det(X_j^{Z_{j12}} - X_i^{Z_{i12}}) > 0. \]
Equivalently, the latter can be rewritten as
\[ 0 < \sum_{j=1}^{N} t_j^i (X_j)_{11} \det(X_j^{Z_{j12}}) = \sum_{j=1}^{N} t_j^i (Y_j)_{11} \det(Y_j - Y_i), \]
which concludes the proof.

The next proposition, whose proof is postponed to the end of the section, gives us a new formulation of \((*)_i\).

**Lemma 3.4.** With the notation introduced in (3.3) and (3.4), we have
\[ (*)_i = \xi \sum_{\alpha=1}^{N} k_\alpha(k_\alpha - 1) t_\alpha^i (C_\alpha)_{11} \cof C_\alpha, P_\alpha - Y_i \quad \forall i = 1, \ldots, N. \]

Before stating the final result of this section, we introduce some more notation. Let \(\{X_1, \ldots, X_N\}\) be a \(T_N\) configuration. As above, let \(\{Y_1, \ldots, Y_N\}\) be the \(T_N\) configuration defined by (3.3) and \(\{C_1, \ldots, C_N\}\) the rank-one matrices of this \(T_N\) configuration. Recall also the definitions of \(k_i\) of Definition 2.1 and of \(\lambda_i\) and \(\mu\) of Proposition 2.2. We define \(S_\beta\) and \(T_\beta\) for all \(\beta = 1, \ldots, N\) as
\[ S_1 := 0 \quad \text{and} \quad S_\beta = \sum_{\alpha=1}^{\beta-1} k_\alpha(k_\alpha - 1)\lambda_\alpha(C_\alpha)_{11} \cof C_\alpha, \quad \forall 2 \leq \beta \leq N \]
and
\[ T_N := 0 \quad \text{and} \quad T_\beta = \sum_{\alpha=\beta+1}^{N} k_\alpha(k_\alpha - 1)\lambda_\alpha(C_\alpha)_{11} \cof C_\alpha, \quad \forall 1 \leq \beta \leq N - 1. \]

We can now prove the following proposition, which is the main result of this section.

**Proposition 3.5.** Let \(\{X_1, \ldots, X_N\} \subset \mathbb{R}^{3 \times 2}\). In the notation above, we have
\[ (*)_i = -\mu \sum_{\beta=1}^{i-1} \langle S_\beta, C_\beta \rangle + \sum_{\beta=i+1}^{N} \langle T_\beta, C_\beta \rangle - \mu k_i \langle S_i, C_i \rangle + (1 - k_i) \langle T_i, C_i \rangle. \quad (3.7) \]

**Proof.** By Lemma 3.4,
\[ (*)_i = \xi \sum_{\alpha=1}^{N} k_\alpha(k_\alpha - 1) t_\alpha^i (C_\alpha)_{11} \cof C_\alpha, P_\alpha - Y_i \quad \forall i = 1, \ldots, N. \]
By computing the quantities \(P_\alpha - Y_i\) in terms of rank-one matrices \(C_1, \ldots, C_N\) and exploiting the fact that, for all \(1 \leq \alpha \leq N,\)
\[ \langle \cof C_\alpha, C_\alpha \rangle = 2 \det(C_\alpha) = 0, \]
we find
\[ (*)_i = -\mu \sum_{\alpha=1}^{i-1} \sum_{\beta=\alpha+1}^{i-1} k_\alpha(k_\alpha - 1)\lambda_\alpha(C_\alpha)_{11} \langle \cof C_\alpha, C_\beta \rangle \]
\[ - \mu k_i \sum_{\alpha=1}^{i-1} k_\alpha(k_\alpha - 1)\lambda_\alpha(C_\alpha)_{11} \langle \cof C_\alpha, C_i \rangle \]
\[ + (1 - k_i) \sum_{\alpha = i+1}^{N} k_{\alpha}(k_{\alpha} - 1)\lambda_{\alpha}(C_{\alpha})_{11}\langle \text{cof } C_{\alpha}, C_{i} \rangle \]
\[ + \sum_{\alpha = i+1}^{N} \sum_{\beta = i+1}^{N} k_{\alpha}(k_{\alpha} - 1)\lambda_{\alpha}(C_{\alpha})_{11}\langle \text{cof } C_{\alpha}, C_{\beta} \rangle. \]

By permuting the sums, the quantities \( S_\beta \) and \( T_\beta \) appear and we find (3.7).

**Remark 3.6.** We make some observations about the quantities \( S_\beta \) and \( T_\beta \). We see that all the matrices in the sums defining \( S_\beta \) and \( T_\beta \) are symmetric positive semi-definite. Moreover, by the definitions of \( S_\beta \) and \( T_\beta \) we have

\[ S_\beta \leq S_{\beta+1} \quad \text{and} \quad T_\beta \geq T_{\beta+1}, \quad \forall \beta \in \{1, \ldots, N-1\}. \]  

Finally, we also notice that \( S_1 = 0 \), \( T_N = 0 \) and since \( \langle \text{cof } C_j, C_j \rangle = 2\det(C_j) = 0 \) for all \( j \), we deduce that

\[ \langle S_\beta, C_\beta \rangle = \langle S_{\beta+1}, C_\beta \rangle \quad \text{and} \quad \langle T_\beta, C_\beta \rangle = \langle T_{\beta+1}, C_\beta \rangle, \quad \forall \beta \in \{1, \ldots, N-1\}. \]  

We finish this section by proving Lemma 3.4. In the proof we will need the following technical result.

**Lemma 3.7.** Assume the real numbers \( \mu > 1 \), \( \lambda_1, \ldots, \lambda_N > 0 \) and \( k_1, \ldots, k_N > 1 \) are linked by the formulas (2.3). Assume \( p, x_j, c_j \) are numbers satisfying the relations

\[ x_i = p + c_1 + \ldots + c_{i-1} + k_ic_i \]
\[ 0 = c_1 + \ldots + c_N. \]

If we define the vectors \( t^i \) as in (2.2), then, for any \( 1 \leq i \leq N \),

\[ \sum_j t^i_jx_j = p + c_1 + \ldots + c_{i-1}. \]  

Furthermore, if \( q, y_j, d_j \) are numbers again fulfilling, for all \( 1 \leq i \leq N \),

\[ y_i = q + d_1 + \ldots + d_{i-1} + k_id_i \]
\[ 0 = d_1 + \ldots + d_N, \]

then

\[ \sum_j t^i_jx_jy_j = \sum_j k_j(k_j - 1)t^i_jc_jd_j + \sum_{j,k} c_jd_k + pq + \sum_{j=1}^{i-1} d_j + q \sum_{j=1}^{i-1} c_j. \]  

**Proof.** We refer the reader to [2, Lemma 3.10] for a proof of (3.10). Let us then fix \( i \in \{1, \ldots, N\} \) and show (3.11). By writing

\[ \sum_j t^i_jx_jy_j = \sum_j t^i_j(x_j - p)y_j + p \left( \sum_j t^i_jy_j \right) = \sum_j t^i_j(x_j - p)(y_j - q) + p \left( \sum_j t^i_jy_j \right) + q \sum_j t^i_j(x_j - p), \]

we see that the last two terms can be computed using (3.10). Thus, we add the assumption \( p = q = 0 \) and compute (3.10) in this simplified setting. The next computation is entirely analogous to the one of [2, Theorem 1.2], and we repeat it for the reader’s convenience. We start by computing the sum for \( i = 1 \), \( \sum_j \lambda_jx_jy_j \). We rewrite it in the following way:

\[ \sum_j \lambda_jx_jy_j = \sum_{j=1}^{N} \lambda_j \left( \sum_{1 \leq a, b \leq j-1} c_ad_b + k_j \sum_{1 \leq a \leq j-1} c_ad_j + k_j \sum_{1 \leq b \leq j-1} c_bd_j + k_j^2c_jd_j \right) = \sum_{i,j} g_{ij}c_id_j. \]
where

\[ g_{ij} = \begin{cases} 
\lambda_i k_i + \sum_{r=i+1}^{N} \lambda_r, & \text{if } i > j, \\
\lambda_j k_j + \sum_{r=j+1}^{N} \lambda_r, & \text{if } i < j, \\
\lambda_i k_i^2 + \sum_{r=i+1}^{N} \lambda_r, & \text{if } i = j.
\end{cases} \]

Using (2.3), we compute:

\[ g_{ij} = g_{ji} = \lambda_i k_i + \sum_{r=i+1}^{N} \lambda_r = \frac{\mu}{\mu - 1} \cdot g_{ii} = k_i^2 \lambda_i + \sum_{r=i+1}^{N} \lambda_r = k_i (k_i - 1) \lambda_i + \frac{\mu}{\mu - 1}. \]

Since \( \sum \ell c_\ell = 0 = \sum \ell d_\ell \), then also \( \sum_{i,j} c_i d_j = 0 \), and so \( \sum_{i\neq j} c_i d_j = - \sum_i c_i d_i \). Hence, (3.12) becomes

\[ \sum_{i,j} g_{ij} c_i d_j = \frac{\mu}{\mu - 1} \sum_{i\neq j} c_i d_j + \sum_i \left( k_i (k_i - 1) \lambda_i + \frac{\mu}{\mu - 1} \right) c_i d_i = \sum_i k_i (k_i - 1) \lambda_i c_i d_i. \]

We just proved that

\[ \sum_j \lambda_j x_j y_j = \sum_i k_i (k_i - 1) \lambda_i c_i d_i, \tag{3.13} \]

that is (3.11) in the case \( i = 1 \) and \( p = q = 0 \). Now we show it for \( i \). Recall that

\[ t^i = \frac{1}{\xi_i} (\mu \lambda_1, \ldots, \mu \lambda_{i-1}, \lambda_i, \ldots, \lambda_N) \]

and hence

\[ \sum_j t_j^i x_j y_j = \frac{1}{\xi_i} \left( (\mu - 1) \sum_{j=1}^{i-1} \lambda_j x_j y_j + \sum_{j=1}^{N} \lambda_j x_j y_j \right) = \frac{1}{\xi_i} \left( (\mu - 1) \sum_{j=1}^{i-1} \lambda_j x_j y_j + \sum_{j} k_j (k_j - 1) \lambda_j c_j d_j \right) \]

We again express the sum up to \( i - 1 \) in the following way:

\[ \sum_{j=1}^{i-1} \lambda_j x_j y_j = \sum_{k,j} s_{k,j} c_k d_j, \]

where

\[ s_{\alpha,\beta} = \begin{cases} 
k_\alpha \lambda_\alpha + \cdots + \lambda_{i-1}, & \text{if } \alpha > \beta \\
k_\beta \lambda_\beta + \cdots + \lambda_{i-1}, & \text{if } \alpha < \beta \\
k_\alpha^2 \lambda_\alpha + \cdots + \lambda_{i-1}, & \text{if } \alpha = \beta.
\end{cases} \]

Now

\[ k_i \lambda_i + \cdots + \lambda_{i-1} = \frac{\mu (\sum_{j=1}^{i-1} \lambda_j) + \sum_{j=i}^{N} \lambda_j}{\mu - 1} \]

and so

\[ k_i \lambda_i + \cdots + \lambda_{i-1} = \frac{(\mu - 1)(\sum_{j=1}^{i-1} \lambda_j) + 1}{\mu - 1} = \frac{\xi_i}{\mu - 1} =: b_{i-1} \]

Hence

\[ \sum_{j=1}^{i-1} \lambda_j x_j y_j = \sum_{k,j} s_{k,j} c_k d_j = b_{i-1} \sum_{k,j} c_k d_j + \sum_{\alpha=1}^{i-1} k_\alpha (k_\alpha - 1) \lambda_\alpha c_\alpha d_\alpha \]

and we conclude

\[ \sum_j t_j^i x_j y_j = \frac{1}{\xi_i} \left[ (\mu - 1) \sum_{j=1}^{i-1} \lambda_j x_j y_j + \sum_j k_j (k_j - 1) \lambda_j c_j d_j \right] = \sum_j k_j (k_j - 1) t_j^i c_j d_j + \sum_{j,k} c_j d_k. \]
Proof of Lemma 3.4. Fix $i \in \{1, \ldots, N\}$. Notice that, for any matrix $A \in \mathbb{R}^{2 \times 2}$,

$$(\ast)_i = \xi_i \sum_{j=1}^{N} t_j^i (Y_j)_{11} \det(Y_j - Y_i) = (\ast)_i = \xi_i \sum_{j=1}^{N} t_j^i (Y_j - A)_{11} \det(Y_j - A - (Y_i - A)),$$

since by Proposition 2.2,

$$\sum_{j=1}^{N} t_j^i \det(Y_j - Y_i) = 0.$$

Thus, even though $P$ of (3.3) may not, in principle, be 0, we can shift $\{Y_1, \ldots, Y_N\}$ by $-P$ to add the additional assumption that

$$Y_i = C_1 + \cdots + C_{i-1} + k_i C_i, \text{ for } C_\ell \in \mathbb{R}_{\text{sym}}^{2 \times 2}, \det(C_\ell) = 0, \sum_{\ell}^{N} C_\ell = 0,$$  \hspace{1cm} (3.14)

without changing the value of $(\ast)_i$. Since

$$\det(A + B) = \det(A) + \langle (\text{cof } A)^T, B \rangle + \det(B)$$  \hspace{1cm} (3.15)

for all $A, B \in \mathbb{R}^{2 \times 2}$ and $(\text{cof } A)^T = \text{cof } A$ for all $A \in \mathbb{R}^{2 \times 2}$, we can write

$$(\ast)_i = \xi_i \sum_{j=1}^{N} t_j^i (Y_j)_{11} \det(Y_j - Y_i)$$

$$= \xi_i \sum_{j=1}^{N} t_j^i (Y_j)_{11} \det(Y_j) - \xi_i \sum_{j=1}^{N} t_j^i (Y_j)_{11} \langle \text{cof } (Y_j), Y_i \rangle + \xi_i \det(Y_i) \sum_{j=1}^{N} t_j^i (Y_j)_{11}.$$

We start by computing $I$. Recalling (3.4), let

$$d_\ell := \langle (\text{cof } C_\ell)^T, P_\ell \rangle = \langle \text{cof } C_\ell, P_\ell \rangle.$$

For $\{Y_1, \ldots, Y_N\}$ given by (3.14), we have

$$\det(Y_i) = d_1 + \cdots + k_i d_\ell,$$  \hspace{1cm} (3.16)

which can be shown inductively using (3.14) and (3.15). Moreover, again using (3.15), we rewrite

$$d_\ell = \det(P_{\ell+1}) - \det(P_\ell).$$

It follows that

$$\sum_{\ell=1}^{N} d_\ell = \det(P_{N+1}) - \det(P_0) = 0.$$  \hspace{1cm} (3.17)

Equations (3.14)-(3.16)-(3.17) allow us to use (3.11) of Lemma 3.7 to compute

$$I = \sum_{j=1}^{N} t_j^i (Y_j)_{11} \det(Y_j) = \sum_{j=1}^{N} k_j (k_j - 1) t_j^i (C_j)_{11} \langle \text{cof } C_j, P_j \rangle + \sum_{j,k}^{i-1} (C_j)_{11} \langle \text{cof } C_k, P_k \rangle.$$  \hspace{1cm} (3.18)

Again (3.11) of Lemma 3.7 (applied componentwise) implies

$$II = \sum_{j=1}^{N} t_j^i (Y_j)_{11} \langle \text{cof } (Y_j), Y_i \rangle = \sum_{j=1}^{N} k_j (k_j - 1) t_j^i (C_j)_{11} \langle \text{cof } C_j, Y_i \rangle + \sum_{j,k}^{i-1} (C_j)_{11} \langle \text{cof } C_k, Y_i \rangle$$  \hspace{1cm} (3.19)

Finally, (3.10) yields

$$III = \det(Y_i) \sum_{j=1}^{N} t_j^i (Y_j)_{11} = \det(Y_i) \sum_{j=1}^{i-1} (C_j)_{11}.$$  \hspace{1cm} (3.20)
Thus, combining (3.18)-(3.19)-(3.20), we obtain

\[ I - II + III = \sum_j k_j (k_j - 1) t_j (C_j)_{11} (\text{cof } C_j, P_j - Y_i) + \sum_{j,k}^{i-1} (C_j)_{11} (\text{cof } C_k, P_k) \]

\[ - \sum_{j,k}^{i-1} (C_j)_{11} (\text{cof } C_k, Y_i) + \det(Y_i) \sum_{j=1}^{i-1} (C_j)_{11}. \]

The proof is thus concluded if we show

\[ \sum_{j, k}^{i-1} (C_j)_{11} (\text{cof } C_k, P_k) - \sum_{j, k}^{i-1} (C_j)_{11} (\text{cof } C_k, Y_i) + \det(Y_i) \sum_{j=1}^{i-1} (C_j)_{11} = 0. \] (3.21)

To see this, we compute

\[ \sum_{k=1}^{i-1} (\text{cof } C_k, P_k - Y_i) \overset{(3.15)}{=} \sum_{k=1}^{i-1} [\det(P_{k+1} - Y_i) - \det(P_k - Y_i)] \]

\[ = \det(P_i - Y_i) - \det(P_1 - Y_i) = \det(k_i C_i) - \det(Y_i) = -\det(Y_i) \]

This shows (3.21) and concludes the proof. \( \square \)

4. Proof of Theorem 1.1 and 1.2

In this section, we prove Theorem 1.1 and 1.2. Both proofs will be split into different cases depending on the sign of the rank-one matrices \( C_j \). By considering all possible cases, we will see that there always is an index \( i \) such that the \((*)_i \leq 0\), thus contradicting Lemma 3.3 and hence excluding the existence of a \( T_N \) configuration contained in \( K_a \). Throughout this section, we assume that \( \{X_1, \ldots, X_N\} \) is a \( T_N \) configuration. We let \( \{Y_1, \ldots, Y_N\} \) be the \( T_N \) configuration given by (3.3) and \( \{C_1, \ldots, C_N\} \) the rank-one matrices of this \( T_N \) configuration. Since \( C_j \in \mathbb{R}^{2 \times 2} \) for all \( 1 \leq j \leq N \) is a rank-one matrix, \( C_j \) is either positive semi-definite or negative semi-definite. As usual for symmetric matrices, we write \( C_j \leq 0 \) when \( C_j \) is negative semi-definite and \( C_j \geq 0 \) when \( C_j \) is positive semi-definite. In particular, we will say that \( C_j \) has + sign when \( C_j \leq 0 \) and has + sign when \( C_j \geq 0 \). Let us introduce the following notation to denote the cases that we consider. Let \( s_1, \ldots, s_N \) be signs, i.e. \( s_i = + \) or \(-\). When saying that we consider the case

\[ (s_1 \ldots s_N), \]

we mean that we are reducing ourselves to the case where each \( C_j \) has sign \( s_j \). For example for \( N = 4 \), \((+++-)\) denotes the case where \( C_1, C_2, C_3 \geq 0 \) and \( C_4 \leq 0 \). Before reducing ourselves to the cases \( N = 4 \) and \( N = 5 \), we can exclude some cases for general \( N \).

**Proposition 4.1.** *All the rank-one matrices \( C_j \) cannot have the same sign.*

*Proof.* This follows immediately from the fact that \( \sum_{i=1}^{N} C_i = 0 \) and our definition of \( T_N \) configurations. \( \square \)

**Proposition 4.2.** *If there is \( j_0 \) such that \( C_{j_0} \) has different sign than all the other rank-one matrices \( C_j \), \( j \neq j_0 \), then there is \( i \) such that \((*)_i \leq 0 \).*

*Proof.* The proof is divided into two cases.

**Case 1:** \( \exists j_0 \) such that \( C_{j_0} \geq 0 \) and \( C_j \leq 0, \forall j \neq j_0 \). We have

\[ (*)_{j_0} = -\mu(\langle S_1, C_1 \rangle + \ldots + \langle S_{j_0-1}, C_{j_0-1} \rangle + k_{j_0} \langle S_{j_0}, C_{j_0} \rangle) \]

\[ + (1 - k_{j_0}) \langle T_{j_0}, C_{j_0} \rangle + \langle T_{j_0+1}, C_{j_0+1} \rangle + \ldots + \langle T_N, C_N \rangle \]

\[ \leq -\mu(\langle S_1, C_1 \rangle + \ldots + \langle S_{j_0-1}, C_{j_0-1} \rangle + k_{j_0} \langle S_{j_0}, C_{j_0} \rangle) \]
In this case, if (ii) 

\[ 4.1 \]

(iii) 

\[ \exists j_0 \text{ such that } C_{j_0} \leq 0 \text{ and } C_j \geq 0, \forall j \neq j_0 \]

In this case, if \( j_0 > 1 \)

\[ (\star)_{j_0-1} = -\mu(\langle S_1, C_1 \rangle + \ldots + \langle S_{j_0-1}, C_{j_0-1} \rangle + \langle S_{j_0}, C_{j_0} \rangle) + (1 - k_{j_0-1}) \langle T_{j_0-1}, C_{j_0-1} \rangle + \langle T_{j_0}, C_{j_0} \rangle + \ldots + \langle T_N, C_N \rangle \]

\[ \leq \langle T_{j_0}, C_{j_0} \rangle + \ldots + \langle T_N, C_N \rangle \]

On the other hand, if \( j_0 = 1 \), the proof follows immediately considering \((\star)_N\) and using that \( \langle S_1, C_1 \rangle = 0 \).

**Proposition 4.3.** If there is \( j_0 \in \{2, \ldots, N - 1\} \) such that \( C_j \geq 0 \) for all \( 2 \leq j \leq j_0 \) and \( C_j \leq 0 \) for all \( j_0 + 1 \leq j \leq N - 1 \), then there exists \( i \) such that \((\ast)_i \leq 0\).

**Proof.** We have

\[ (\ast)_{j_0} = -\mu \sum_{\beta=1}^{j_0-1} \langle S_\beta, C_\beta \rangle + \sum_{\beta=j_0+1}^{N} \langle T_\beta, C_\beta \rangle - \mu k_{j_0} \sum_{\geq 0}^{\geq 0} \langle S_{j_0}, C_{j_0} \rangle + (1 - k_{j_0}) \langle T_{j_0}, C_{j_0} \rangle \leq 0 \]

as desired. This finishes the proof.

We can now prove Theorem 1.1.

**Proof of Theorem 1.1.** Let \( \{X_1, \ldots, X_4\} \) be a \( T_4 \) configuration and assume by contradiction that

\[ \{X_1, \ldots, X_4\} \subset K_a \]

for some \( a \) which is convex and increasing. Then let \( \{Y_1, \ldots, Y_4\} \) be the \( T_4 \) configuration given by (3.3).

We will prove that there exists an \( i \) such that \((\ast)_i \leq 0\). To this end, we consider all possible cases of combinations of signs of the matrices \( C_j \). In the case \( N = 4 \) that we are considering there are 16 cases. The cases \((++++)\) and \((-+-+-)\) can already be excluded by Proposition 4.1. Due to Proposition 4.2 and Proposition 4.3, we can exclude 11 more cases and the only cases that remain are

(i) \((+-++)\),

(ii) \((+-+-)\) and

(iii) \((-+++)\).

We now prove that in all of these three cases there is an index \( i \) such that \((\ast)_i \leq 0\).

**Case (i).** Since \( \langle S_1, C_1 \rangle = 0 \) and \( \langle S_3, C_3 \rangle, \langle T_3, C_3 \rangle \geq 0 \), we have

\[ (\ast)_3 = -\mu(\langle S_2, C_2 \rangle - \mu k_3(\langle S_3, C_3 \rangle + (1 - k_3)\langle T_3, C_3 \rangle) \leq -\mu(\langle S_2, C_2 \rangle + \langle S_3, C_3 \rangle) \]

\[ = -\mu(\langle S_2, C_1 \rangle + \langle S_2, C_2 \rangle + \langle S_2, C_3 \rangle) \leq -\mu(\langle S_2, C_1 \rangle + \langle S_2, C_2 \rangle + \langle S_2, C_3 \rangle) = \mu(\langle S_2, C_4 \rangle \leq 0. \]

**Case (ii).** We have

\[ (\ast)_1 = (1 - k_1)\langle T_1, C_1 \rangle + \langle T_2, C_2 \rangle + \langle T_3, C_3 \rangle \leq 0. \]

**Case (iii).** Since \( \langle S_4, C_4 \rangle \geq 0 \) and \( \langle S_4, C_4 \rangle \geq \langle S_3, C_4 \rangle \)

\[ (\ast)_4 = -\mu(\langle S_2, C_2 \rangle - \mu(\langle S_3, C_3 \rangle - \mu k_4(\langle S_4, C_4 \rangle \leq -\mu(\langle S_2, C_2 \rangle + \langle S_3, C_3 \rangle + \langle S_4, C_4 \rangle)) \]

\[ = -\mu(\langle S_2, C_1 \rangle + \langle S_2, C_2 \rangle + \langle S_2, C_3 \rangle) \leq -\mu(\langle S_2, C_2 \rangle + \langle S_3, C_3 \rangle + \langle S_4, C_4 \rangle) \]

\[ \leq 0. \]
Proof of Theorem 1.2. Let \( \{X_1, \ldots, X_5\} \) be a \( T_5 \) configuration and assume by contradiction that 
\[ \{X_1, \ldots, X_5\} \subset K_a \]
for some \( a \) which is convex and increasing. Then let \( \{Y_1, \ldots, Y_5\} \) be the \( T_5 \) configuration given by (3.3). We will prove that there exists an \( i \) such that \((\ast)_i \leq 0\). This finishes the proof.

We now prove Theorem 1.2.

\[ \begin{align*}
\ast_1 &= -(S_2, C_2) + k_3(S_3, C_3) + (1 - k_3)(T_3, C_3) + (T_4, C_4) \leq -\mu((S_2, C_2) + (S_3, C_3)) \\
&= -\mu((S_1, C_1) + (S_2, C_2) + (S_3, C_3)) \leq -\mu(S_2, C_2 + C_3) = \mu(S_2, C_4 + C_5) \leq 0 \\
\ast_2 &= -(T_2, C_2) + (T_3, C_3) + (T_4, C_4) + (T_5, C_5) \\
&= -(T_4, C_1) \leq 0 \\
\ast_3 &= (1 - k_1)(T_1, C_1) + (T_2, C_2) + (T_3, C_3) + (T_4, C_4) \leq (T_4, C_4) \\
&= -(T_4, C_1) \leq 0 \\
\ast_4 &= -(S_2, C_2) + (S_3, C_3) + k_4(S_4, C_4) \leq -(S_3, C_2) + (S_3, C_3) + (S_4, C_4) \\
&= -(S_3, C_2 + C_3 + C_4) \leq 0 \\
\ast_5 &= -(S_2, C_2) + (S_3, C_3) + k_5(S_4, C_4) \leq -(S_3, C_2) + (S_3, C_3) + (S_4, C_4) \\
&= -(S_3, C_2 + C_3 + C_4) \leq 0
\end{align*} \]
Case (vi).

\[
(*)_5 = -\mu(\langle S_2, C_2 \rangle + \langle S_3, C_3 \rangle + \langle S_4, C_4 \rangle + k_5 \langle S_5, C_5 \rangle) \\
\leq -\mu(S_4, C_2 + C_3 + C_4 + C_5) = \mu(S_4, C_1) \leq 0.
\]

Case (vii).

\[
(*)_5 = -\mu(\langle S_2, C_2 \rangle + \langle S_3, C_3 \rangle + \langle S_4, C_4 \rangle + k_5 \langle S_5, C_5 \rangle) \\
\leq -\mu(S_4, C_2 + C_3 + C_4 + C_5) = \mu(S_5, C_1) \leq 0.
\]

Case (viii).

\[
(*)_5 = -\mu(\langle S_2, C_2 \rangle + \langle S_3, C_3 \rangle + \langle S_4, C_4 \rangle + k_5 \langle S_5, C_5 \rangle) \\
\leq -\mu(S_3, C_2 + C_3 + C_4 + C_5) = \mu(S_3, C_1) \leq 0.
\]

Case (ix).

\[
(*)_5 = -\mu(\langle S_2, C_2 \rangle + \langle S_3, C_3 \rangle + \langle S_4, C_4 \rangle + k_5 \langle S_5, C_5 \rangle) \\
\leq -\mu(S_3, C_2 + C_3 + C_4 + C_5) = \mu(S_3, C_1) \leq 0.
\]

Case (x).

\[
(*)_1 = -\mu k_1(\langle S_1, C_1 \rangle + (1 - k_1)(T_1, C_1) + \langle T_2, C_2 \rangle + \langle T_3, C_3 \rangle + \langle T_4, C_4 \rangle) \\
\leq \langle T_4, C_2 \rangle + \langle T_4, C_3 \rangle + \langle T_4, C_4 \rangle = \langle T_4, C_2 + C_3 + C_4 \rangle = -\langle T_4, C_1 + C_5 \rangle \leq 0.
\]

Case (xi).

\[
(*)_1 = -\mu k_1(\langle S_1, C_1 \rangle + (1 - k_1)(T_1, C_1) + \langle T_2, C_2 \rangle + \langle T_3, C_3 \rangle + \langle T_4, C_4 \rangle + \langle T_5, C_5 \rangle) \\
\leq \langle T_3, C_2 \rangle + \langle T_3, C_3 \rangle + \langle T_3, C_4 \rangle = \langle T_3, C_2 + C_3 + C_4 \rangle = -\langle T_3, C_1 + C_5 \rangle \leq 0.
\]

Case (xii).

\[
(*)_4 \leq -\mu(\langle S_2, C_2 \rangle + \langle S_3, C_3 \rangle + k_4 \langle S_4, C_4 \rangle) \leq -\mu(\langle S_2, C_2 + C_3 + C_4 \rangle) \\
= -\mu(\langle S_2, C_1 + C_2 + C_3 + C_4 \rangle) \leq \mu(\langle S_2, C_5 \rangle) \leq 0
\]

Case (xiii).

\[
(*)_2 = -\mu k_2(\langle S_2, C_2 \rangle + (1 - k_2)(T_2, C_2) + \langle T_3, C_3 \rangle + \langle T_4, C_4 \rangle + \langle T_5, C_5 \rangle) \\
\leq \langle T_4, C_3 + C_4 + C_5 \rangle = -\langle T_4, C_1 + C_2 \rangle \leq 0.
\]

\[\Box\]

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