A dissipative logarithmic type evolution equation: asymptotic profile and optimal estimates

Alessandra Piske* and Ruy Coimbra Charão†
Department of Mathematics
Graduate Program on Pure and Applied Mathematics
Federal University of Santa Catarina
88040-270, Florianopolis, Brazil,
and
Ryo Ikehata‡
Department of Mathematics
Division of Educational Sciences
Graduate School of Humanities and Social Sciences
Hiroshima University
Higashi-Hiroshima 739-8524, Japan

Abstract

We introduce a new model of the logarithmic type of wave equation with a nonlocal logarithmic damping mechanism, which is rather weakly effective as compared with frequently studied fractional damping cases. We consider the Cauchy problem for this new model in $\mathbb{R}^n$, and study the asymptotic profile and optimal decay and/or blowup rates of solutions as $t \to \infty$ in $L^2$-sense. The operator $L$ considered in this paper was used to dissipate the solutions of the wave equation in the paper studied by Charão-Ikehata and in the low frequency parameters the principal part of the equation and the damping term is rather weakly effective than that of well-studied power type one such as $(-\Delta)^\theta u_t$ with $\theta \in (0, 1]$.

1 Introduction

We present and consider a new model of evolution equation with a logarithmic damping term:

$$ u_{tt} + Lu + Lu_t = 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}^n, \quad (1.1) $$

$$ u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad x \in \mathbb{R}^n, \quad (1.2) $$

where the linear operator

$$ L : D(L) \subset L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) $$

is defined as follows:

$$ D(L) := \left\{ f \in L^2(\mathbb{R}^n) \mid \int_{\mathbb{R}^n} (\log(1 + |\xi|^2)) |\hat{f}(\xi)|^2 d\xi < +\infty \right\}, $$

$$ (Lf)(x) := \mathcal{F}_{\xi \to x}^{-1} \left( \log(1 + |\xi|^2) \hat{f}(\xi) \right)(x), \quad f \in D(L). $$

* alessandrapiske@gmail.com
† Corresponding author: ruy.charao@ufsc.br
‡ ikehatar@hiroshima-u.ac.jp

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Here, one has just denoted the Fourier transform $\mathcal{F}_{x \to \xi}(f)(\xi)$ of $f(x)$ by

$$\mathcal{F}_{x \to \xi}(f)(\xi) := \hat{f}(\xi) := \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) dx, \quad \xi \in \mathbb{R}^n,$$

as usual with $i := \sqrt{-1}$, and $\mathcal{F}_{\xi \to x}^{-1}$ expresses its inverse Fourier transform. Since the operator $L$ is non-negative and self-adjoint in $L^2(\mathbb{R}^n)$ (see [6]), the square root

$$L^{1/2} : D(L^{1/2}) \subset L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

can be defined, and is also nonnegative and self-adjoint with its domain

$$D(L^{1/2}) = \left\{ f \in L^2(\mathbb{R}^n) \mid \int_{\mathbb{R}^n} \log(1 + |\xi|^2) |\hat{f}(\xi)|^2 d\xi < +\infty \right\}.$$

Note that $D(L^{1/2})$ becomes Hilbert space with its graph norm

$$\|v\|_{D(L^{1/2})} := \left( \|v\|^2_{L^2} + \|L^{1/2}v\|^2_{L^2} \right)^{1/2}.$$

It is easy to check that

$$H^s(\mathbb{R}^n) \hookrightarrow D(L^{1/2}) \hookrightarrow L^2(\mathbb{R}^n)$$

for $s > 0$.

Symbolically writing, one can see

$$L = \log(I - \Delta),$$

where $\Delta$ is the usual Laplace operator defined on $H^2(\mathbb{R}^n)$.

Now, for the time being, we choose the initial data $(u_0, u_1)$ as follows:

$$u_0 \in D(L^{1/2}), \quad u_1 \in L^2(\mathbb{R}^n).$$

The existence of the unique solution to problem (1.1)-(1.2) can be discussed by employing a similar argument to [23, Proposition 2.1] based on Lumer-Phillips Theorem, and one can find that the problem (1.1)-(1.2) has a unique mild solution

$$u \in C([0, \infty); D(L^{1/2})) \cap C^1([0, \infty); L^2(\mathbb{R}^n))$$

satisfying the energy inequality

$$E_u(t) \leq E_u(0), \quad (1.3)$$

where

$$E_u(t) := \frac{1}{2} \left( \|u_t(t, \cdot)\|^2_{L^2} + \|\log^{1/2}(I - \Delta)u(t, \cdot)\|^2_{L^2} \right).$$

The inequality (1.3) implies that the the total energy is a non increasing function in time because of the existence of some kind of dissipative term $L_{ut}$.

A main topic of this paper is to find an asymptotic profile of solutions in the $L^2$ framework to problem (1.1)-(1.2), and to apply it to investigate the optimal rate of decay of solutions in terms of the $L^2$-norm. We study the equation (1.4) only from the pure mathematical point of view.

A motivation of this research has its origin in the study of the strongly damped wave equation:

$$u_{tt} - \Delta u - \Delta u_t = 0. \quad (1.4)$$

An analysis of the dissipative mechanism of (1.4) goes back to the two pioneering works of G. Ponce [29] and Y. Shibata [30], where they studied various $L^p$-$L^q$ estimates of the solution to the Cauchy problem of (1.4). After them, an asymptotic profile and the optimal estimates of the solution can be introduced in the papers [23], [19] and [21]. They investigated a singularity near 0-frequency region of the solution to (1.4) in terms of $L^2$-norm of solutions. In this connection, in [11] [23] and [20] a higher order asymptotic expansion of the solution as $t \to \infty$ to the equation (1.4) is precisely investigated.
On the other hand, the so-called critical exponent problem for semi-linear equations of (1.4) is first developed by D’Abbicco-Reissig [12], and this paper has been the beginning of a series of related papers studying structurally damped wave models with nonlinearity. Unfortunately, at present nobody knows the precise value of the critical exponent $p^*$ of the equation (1.4) with power type nonlinearity $|u|^p$. A study in [12] is based on the $L^p-L^q$-estimates derived in [30].

Recently, the equation (1.4) is generalized to the linear and semi-linear models, respectively:

$$u_{tt} + (-\Delta)^\sigma u + (-\Delta)^\theta u_t = 0,$$  \hspace{1cm} (1.5)

$$u_{tt} + (-\Delta)^\sigma u + (-\Delta)^\theta u_t = f(u, u_t).$$  \hspace{1cm} (1.6)

A study on asymptotic profile and $L^p-L^q$-estimates to the equation (1.5) has been done in the papers [7, 8, 10, 11, 25, 27], and [22], and the corresponding critical exponent problems (mainly) to the equation (1.6) are treated in the papers [13, 14, 9, 15, 24], and [28].

In [20] and [16], the so-called regularity-loss structure of the solution in the high frequency zone can be studied to the equation (1.5) with $\sigma = 1$ and $\theta > 1$, and these researches are strongly inspired from the abstract theory due to [17]. Such a regularity-loss structure has been first discovered by S. Kawashima through the analysis for dissipative Timoshenko system. A more general model than (1.5) is studied by [5]. The aim of that work in [5] is to obtain asymptotic profile and optimal decay rates in case of a super damping (i.e., $\sigma < \theta$). We have much more interesting results about more generalized evolution equations such as memory type of damping, double one, rotational inertia term case, and etc... than (1.6), however, we do not mention them not to spread in vain our topics.

Quite recently Charão-Ikehata [6] introduced a new type of damping term of logarithm type to the wave equation, and it is expressed in the Fourier space as follows:

$$\hat{u}_{tt} + |\xi|^2 \hat{u} + \log(1 + |\xi|^2) \hat{u}_t = 0.$$  \hspace{1cm} (1.7)

Symbolically writing, one sees

$$u_{tt} - \Delta u + \log(I - \Delta) u_t = 0.$$  \hspace{1cm} (1.8)

In [4], (1.7) is more generalized to the equation such that

$$u_{tt} - \Delta u + \log(I - (-\Delta)^\theta) u_t = 0$$

for $\theta > 1/2$. As is easily seen that the characteristic roots $\lambda_{\pm}$ for the characteristic polynomial of (1.7) such that

$$\lambda^2 + \log(1 + |\xi|^2) \lambda + |\xi|^2 = 0$$

are complex-valued for all $\xi \in \mathbb{R}^n$, although the contribution on the decay structure from the high frequency parameters is very small. In this connection, in [32] they study another model with double dispersion for which oscillations appear at both low and high frequencies.

On reconsidering our problem (1.1)-(1.2) in the Fourier space, our equation becomes

$$\hat{u}_{tt} + \log(1 + |\xi|^2) \hat{u} + \log(1 + |\xi|^2) \hat{u}_t = 0.$$  \hspace{1cm} (1.8)

Characteristics roots of (1.8) are complex-valued only for small $\xi \in \mathbb{R}^n$, and in the large frequency zone, the roots are real-valued, and this is similar to the strong damping case (1.4).

To get started, we first investigate the decay rate of the total energy $E_u(t)$ and $L^2$-norm of the solution itself under the $L^1$-framework on the initial data.

**Proposition 1.1** Let $u(t,x)$ be the solution to problem (1.1)-(1.2) with initial data

$$(u_0, u_1) \in \left(D(L^{1/2}) \cap L^1(\mathbb{R}^n)\right) \times \left(L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)\right).$$

Then, the total energy of this system satisfies for $t \gg 0$

$$\left\|u(t,\cdot)\right\|_{L^2}^2 + \left\|L^{1/2}u(t,\cdot)\right\|_{L^2}^2 \leq C_n \left(\|u_0\|_{L^2}^2 t^{-\frac{\theta}{2}} + \|u_0\|^2_{L^1} t^{-\frac{\sigma}{2}}\right) + 2^{-\frac{\theta}{2}} \left(\|u_1\|^2_{L^2} + \|u_0\|^2_{L^2}\right) + 2e^{-\frac{\theta}{2}} E_u(0).$$
Remark 1.1 The above proposition says that the total energy of the system decays as $t^{-n/2}$, that is
$$E_u(t) \leq C_{1,n} \left( E_u(0) + \| u_0 \|_{L^2}^2 + \| u_0 \|_{L^1}^2 + \| u_1 \|_{L^2}^2 \right) t^{-\frac{n}{2}}, \quad t \gg 1,$$
with a constant $C_{1,n} > 0$ depending only on $n$.

Proposition 1.2 Let $n > 2$ and $u(t, x)$ be the solution to problem (1.1)-(1.2) with initial data $u_0, u_1 \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$. Then
$$\| u(t, \cdot) \|_{L^2} \leq C_n (\| u_0 \|_{L^2} + \| u_1 \|_{L^2} + \| u_0 \|_{L^1} + \| u_1 \|_{L^1}) t^{-\frac{n-2}{4}}, \quad t \gg 1,$$
with a constant $C_n > 0$ depending only on $n$.

Remark 1.2 The decay rate of the quantity $\| u(t, \cdot) \|_{L^2}$ can be derived only for the spatial dimension $n > 2$ under the $L^1$-regularity on the initial data. $n = 1, 2$ cases have a strong singularity near $0$-frequency region. This singularity can be observed in the following main results below.

In order to investigate the optimality of decay rates of the quantity $\| u(t, \cdot) \|_{L^2}$ just obtained in Proposition 1.2 we do study the asymptotic profile of the solution $u(t, x)$ as $t \to \infty$ in $L^2$-sense. Our new result reads as follows. At this stage, it suffices to assume $u_0 = 0$ without loss of generality.

Theorem 1.1 Let $n \geq 1$, and let $u_0 = 0$, and $u_1 \in (L^2(\mathbb{R}^n) \cap L^{1,1}(\mathbb{R}^n))$. Then, the unique solution $u(t, x)$ to problem (1.1)-(1.2) satisfies
$$\left\| u(t, \cdot) - \left( \int_{\mathbb{R}^n} u_1(x) dx \right) F_{\xi \to x}^{-1} \left( (1 + |\xi|^2)^{-\frac{1}{2}} \frac{\sin(\sqrt{\log(1 + |\xi|^2)} \cdot t))}{\sqrt{\log(1 + |\xi|^2)}} \right) \right\|_{L^2} \leq I_0 t^{-\frac{n}{2}}, \quad (t \gg 1),$$
where
$$I_0 := \| u_1 \|_{L^2} + \| (1 + |x|) u_1 \|_{L^1}.$$
Theorem 1.2 Let \( n \geq 1 \), and let \( u_0 = 0 \), and \( u_1 \in (L^2(\mathbb{R}^n) \cap L^{1,1}(\mathbb{R}^n)) \). Then, the unique solution \( u(t, x) \) to problem (1.1)-(1.2) satisfies the following properties:

(i) if \( n = 1 \), then \( C_1 |P_1| \sqrt{t} \leq ||u(t, \cdot)||_{L^2} \leq C_1^{-1} I_0 \sqrt{t} (t \gg 1) \),

(ii) if \( n = 2 \), then \( C_2 |P_1| \sqrt{\log t} \leq ||u(t, \cdot)||_{L^2} \leq C_2^{-1} I_0 \sqrt{\log t} (t \gg 1) \),

(iii) if \( n \geq 3 \), then \( C_n |P_1| t^{\frac{1}{n-2}} \leq ||u(t, \cdot)||_{L^2} \leq C_n^{-1} I_0 t^{\frac{1}{n-2}} (t \gg 1) \).

Here \( I_0 \) is a constant defined in Theorem 1.1, and \( C_n (n \in \mathbb{N}) \) are constants independent from any \( t \) and initial data.

Remark 1.4 As a result, all estimates derived in Theorem 1.2 are overlapped already known results in [21] and/or [6], and this is quite natural because \( \log(1 + ||\xi||^2) \sim ||\xi||^2 \) for small \( \xi \in \mathbb{R}^n \), and the main contribution to the estimates above comes from the low frequency region in \( \xi \in \mathbb{R}^n \). However, by replacing the operator \( A = -\Delta \) to \( L = \log(I - \Delta) \) in the equation (1.9), we encounter a big obstacle when one gets such estimates stated in Theorem 1.2 and this difficulty comes from the way that how we treat the improper integral (1.10). A big technical difficulties occur.

This paper is organized as follows. In section 2 we prepare several important lemmas, which will be used later, and in particular, these lemmas are closely related with hypergeometric functions (see [6]). Propositions 1.1 and 1.2 can be proved in Section 3 based on the energy method due to [31]. In Section 4, we derive the leading term (as \( t \to \infty \)) of the solution to problem (1.1)-(1.2). Section 5 is devoted to the derivation of the optimal decay rate of the \( L^p \)-norm of the solution in case of \( n \geq 3 \), and 1 and 2 dimensional cases for the optimality of the \( L^2 \)-estimates of the solution will be investigated in Sections 6.

Notation. Throughout this paper, \( \| \cdot \|_q \) stands for the usual \( L^q(\mathbb{R}^n) \)-norm. For simplicity of notation, in particular, we use \( \| \cdot \|_2 \) instead of \( \| \cdot \|_2 \). Furthermore, we denote \( \| \cdot \|_{H^1} \) as the usual \( H^1 \)-norm. Furthermore, we define a relation \( f(t) \sim g(t) \) as \( t \to \infty \) by: there exist constant \( C_j > 0 \) \((j = 1, 2)\) such that

\[
C_1 g(t) \leq f(t) \leq C_2 g(t) \quad (t \gg 1).
\]

We also introduce the following weighted functional spaces.

\[
L^{1,\gamma}(\mathbb{R}^n) := \left\{ f \in L^1(\mathbb{R}^n) \mid \| f \|_{1,\gamma} := \int_{\mathbb{R}^n} (1 + |x|^\gamma)|f(x)|dx < +\infty \right\}.
\]

Finally, we denote the surface area of the \( n \)-dimensional unit ball by \( \omega_n := \int_{|x|=1} dx \).

2 General basic results

In this section we shall prepare important lemmas to derive precise estimates of the various quantities related to the solution to problem (1.1)-(1.2). These are already studied and developed in our previous works (see [6], [4]).

The following main estimate for the function

\[
I_p(t) = \int_0^1 (1 + r^2)^{-t} r^p dr
\]

is a direct consequence of the cases \( p \geq 0 \) in Charão-Ikehata [6] and \( -1 < p < 0 \) in Charão-D’Abbicco-Ikehata [4].

Lemma 2.1 Let \( p > -1 \) be a real number. Then

\[
I_p(t) \sim t^{-\frac{p+1}{p+1}}, \quad t \gg 1.
\]

In order to deal with the high frequency part of estimates, one relies on the function again

\[
J_p(t) = \int_1^\infty (1 + r^2)^{-t} r^p dr
\]

for \( p \in \mathbb{R} \).

Then the next lemma is important to get estimates on the zone of high frequency to problem (1.1)-(1.2). The proof appears in Charão-Ikehata [6].
Lemma 2.2 Let \( p \in \mathbb{R} \). Then it holds that
\[
J_p(t) \sim \frac{2^{-t}}{t-1}, \quad t \gg 1.
\]

For later use we prepare the following simple lemma, which implies the exponential decay estimates of the middle frequency part.

Lemma 2.3 Let \( p \in \mathbb{R} \), and \( \eta \in (0, 1] \). Then there is a constant \( C > 0 \) such that
\[
\int_\eta^1 (1 + r^2)^{-t} r^p dr \leq C(1 + \eta^2)^{-t}, \quad t \geq 0.
\]

Remark 2.1 We note that the proof of Lemma 2.1 are proved using simple differential calculus and the theory from hypergeometric functions (see Watson [33]).

3 Asymptotic behavior via multiplier method

In this section, we shall obtain optimal estimates of the total energy of the following Fourier transformed equation together with initial data of the original system (1.1)-(1.2). To do so we employ the so-called energy method in the Fourier space developed in [31]. It seems to be a new development for this type of equation with logarithmic operators.

\[
\hat{u}_{tt} + \log(1 + |\xi|^2) \hat{u} + \log(1 + |\xi|^2) \hat{u}_t = 0, \quad (t, \xi) \in (0, \infty) \times \mathbb{R}^n, \tag{3.1}
\]
\[
\hat{u}(0, \xi) = \hat{u}_0(\xi), \quad \hat{u}_t(0, \xi) = \hat{u}_1(\xi), \quad \xi \in \mathbb{R}^n. \tag{3.2}
\]

Multiplying the equation (3.1) by \( \hat{u}_t \) one can get the following point wise energy identity
\[
\frac{dE_0(t, \xi)}{dt} + \log(1 + |\xi|^2) |\hat{u}_t(t, \xi)|^2 = 0, \tag{3.3}
\]
where
\[
E_0(t, \xi) = \frac{|\hat{u}_t(t, \xi)|^2}{2} + \log(1 + |\xi|^2) |\hat{u}(t, \xi)|^2,
\]
for \( t > 0 \) and \( \xi \in \mathbb{R}^n \), is the total density of energy of the system (3.1)-(3.2). Note from (3.3) that \( E_0(t, \xi) \) is a decreasing function of \( t \) for each \( \xi \).

Now we define the following function of \( \xi \). The way to choose the best \( \rho(\xi) \)-function is showed in the work by Luz-Ikehata-Charão [25]:
\[
\rho(\xi) = \begin{cases} 1/2 \log(1 + |\xi|^2) & \text{if } |\xi| \leq \sqrt{e - 1}, \\ 1/2 & \text{if } |\xi| \geq \sqrt{e - 1}. \end{cases} \tag{3.4}
\]

By multiplying the equation (3.1) by \( \rho(\xi) \hat{u} \) we obtain the identity
\[
\rho(\xi) \frac{d}{dt} (\hat{u}_t \hat{u}) - \rho(\xi) |\hat{u}_t|^2 + \log(1 + |\xi|^2) \rho(\xi) |\hat{u}|^2 + \log(1 + |\xi|^2) \rho(\xi) \frac{d}{dt} |\hat{u}|^2 = 0,
\]
for all \( t > 0 \) and \( \xi \in \mathbb{R}^n \). Taking the real part on the last identity we arrive at
\[
\frac{d}{dt} \left[ \rho(\xi) \Re(\hat{u}_t \hat{u}) + \rho(\xi) \log(1 + |\xi|^2) |\hat{u}|^2 \right] + \rho(\xi) \log(1 + |\xi|^2) |\hat{u}|^2 = \rho(\xi) |\hat{u}_t|^2, \tag{3.5}
\]
which holds for \( t > 0 \) and \( \xi \in \mathbb{R}^n \).

To proceed further we need to define the following functions on \( (0, \infty) \times \mathbb{R}^n \):
\[
E(t, \xi) = E_0(t, \xi) + \rho(\xi) \Re(\hat{u}_t(t, \xi) \hat{u}(t, \xi)) + \frac{\rho(\xi)}{2} \log(1 + |\xi|^2) |\hat{u}(t, \xi)|^2,
\]
\[
F(t, \xi) = \log(1 + |\xi|^2) |\hat{u}_t(t, \xi)|^2 + \rho(\xi) \log(1 + |\xi|^2) |\hat{u}(t, \xi)|^2, \tag{3.6}
\]
\[
R(t, \xi) = \rho(\xi) |\hat{u}_t(t, \xi)|^2.
\]
Then, adding (3.2) and (3.3), we get the following identity
\[
\frac{d}{dt}E(t, \xi) + F(t, \xi) = R(t, \xi),
\]
which also holds for \( t > 0 \) and \( \xi \in \mathbb{R}^n \). Before continuing our argument, we need the next remark.

**Lemma 3.1** The function \( \rho(\xi) \) defined in (3.3) satisfies the estimates
\[
\rho(\xi) \leq \frac{1}{2}
\]
for \( |\xi| \leq \sqrt{e - 1} \). Moreover,
\[
\rho^2(\xi) \leq \frac{1}{4} \log(1 + |\xi|^2)
\]
for all \( \xi \in \mathbb{R}^n \).

*Proof.* Indeed, for \( |\xi| \leq \sqrt{e - 1} \) we have \( \log(1 + |\xi|^2) \leq 1 \) which implies
\[
\log(1 + |\xi|^2) \leq \log(1 + |\xi|^2).
\]
Thus, \( \rho^2(\xi) \leq \frac{1}{4} \log(1 + |\xi|^2) \) and \( \rho(\xi) \leq \frac{1}{2} \) according to the definition of \( \rho(\xi) \) in (3.3).

For \( |\xi| \geq \sqrt{e - 1} \) one has \( \log(1 + |\xi|^2) \geq 1 \). Thus, \( \rho^2(\xi) = \frac{1}{4} \leq \frac{1}{4} \log(1 + |\xi|^2) \).
\( \square \)

**Lemma 3.2**
\[
\frac{1}{2} E_0(t, \xi) \leq E(t, \xi) \leq 3E_0(t, \xi), \quad t > 0, \ \xi \in \mathbb{R}^n.
\]

*Proof.* Using the inequality \( \rho(\xi) \Re(\tilde{u}\tilde{\bar{u}}) \geq -\frac{|\tilde{u}|^2}{4} - \rho^2(\xi)|\tilde{u}|^2 \) and Lemma 3.1 one has
\[
E(t, \xi) = E_0(t, \xi) + \rho(\xi) \Re(\tilde{u}\tilde{\bar{u}}) + \frac{\rho(\xi)}{2} \log(1 + |\xi|^2)|\tilde{u}|^2 \\
\geq E_0(t, \xi) - \frac{|\tilde{u}|^2}{4} - \rho^2(\xi)|\tilde{u}|^2 \\
= \frac{|\tilde{u}|^2}{2} + \log(1 + |\xi|^2)|\tilde{u}|^2 - \frac{|\tilde{u}|^2}{4} - \rho^2(\xi)|\tilde{u}|^2 \\
= \frac{1}{4} |\tilde{u}|^2 + \log(1 + |\xi|^2)|\tilde{u}|^2 - \rho^2(\xi) \\
\geq \frac{1}{4} |\tilde{u}|^2 + \log(1 + |\xi|^2)|\tilde{u}|^2 = \frac{1}{2} E_0(t, \xi),
\]
which holds for \( t > 0 \) and \( \xi \in \mathbb{R}^n \).

On the other hand, using Lemma 3.1 one has the estimates
\[
E(t, \xi) = E_0(t, \xi) + \rho(\xi) \Re(\tilde{u}\tilde{\bar{u}}) + \frac{\rho(\xi)}{2} \log(1 + |\xi|^2)|\tilde{u}|^2 \\
\leq E_0(t, \xi) + \frac{|\tilde{u}|^2}{2} + \rho^2(\xi)|\tilde{u}|^2 + \frac{|\tilde{u}|^2}{2} \log(1 + |\xi|^2)|\tilde{u}|^2 \\
\leq E_0(t, \xi) + \frac{|\tilde{u}|^2}{2} + \frac{\log(1 + |\xi|^2)}{8} |\tilde{u}|^2 + \frac{1}{4} \log(1 + |\xi|^2)|\tilde{u}|^2 \\
\leq 3E_0(t, \xi),
\]
which also holds for \( t > 0 \) and \( \xi \in \mathbb{R}^n \).
\( \square \)

**Lemma 3.3**
\[
\frac{d}{dt}E(t, \xi) + \frac{\rho(\xi)}{2} E(t, \xi) \leq 0, \quad t > 0, \ \xi \in \mathbb{R}^n.
\]
Proof. (3.7), (3.6) and Lemma 3.2 imply that
\[
\frac{d}{dt}E(t, \xi) + \frac{\rho(\xi)}{2}E(t, \xi) = R(t, \xi) - F(t, \xi) + \frac{\rho(\xi)}{2}E(t, \xi)
\]
\[
\leq R(t, \xi) - F(t, \xi) + \frac{3\rho(\xi)}{2}E_0(t, \xi)
\]
\[
= \rho(\xi)|\hat{u}_t|^2 - \log(1 + |\xi|^2)|\hat{u}_t| - \rho(\xi)\log(1 + |\xi|^2)|\hat{u}|^2 + \frac{3\rho(\xi)}{4}|\hat{u}_t|^2
\]
\[
+ \frac{3\rho(\xi)}{4}\log(1 + |\xi|^2)|\hat{u}|^2
\]
\[
= \left(\frac{7\rho(\xi)}{4} - \log(1 + |\xi|^2)\right)|\hat{u}_t|^2 - \frac{1}{4}\rho(\xi)\log(1 + |\xi|^2)|\hat{u}|^2
\]
\[
\leq 0,
\]
where we have just used the fact that
\[
\frac{7\rho(\xi)}{4} - \log(1 + |\xi|^2) = \begin{cases} \\
\frac{7}{8} - \log(1 + |\xi|^2) \text{ se } |\xi| \leq \sqrt{e - 1}, \\
\frac{7}{8} \log(1 + |\xi|^2) \text{ se } |\xi| > \sqrt{e - 1},
\end{cases}
\]
and the fact that \(\log(1 + |\xi|^2) \geq 1\) for \(|\xi| > \sqrt{e - 1}\). Therefore,
\[
\frac{7}{8} - \log(1 + |\xi|^2) < -\frac{1}{8}
\]
for \(|\xi| > \sqrt{e - 1}\). \(\square\)

Now we may note that Lemma 3.3 implies
\[
E(t, \xi) \leq E(0, \xi)e^{-\frac{\rho(\xi)}{2}t}.
\]
Combining the last estimate with Lemma 3.2 we arrive at the important proposition.
\[
E_0(t, \xi) \leq 6E_0(0, \xi)e^{-\frac{\rho(\xi)}{2}t},
\]
for all \(t > 0\) and \(\xi \in \mathbb{R}^n\).

That is, using the definition of \(E(t, \xi)\) we have obtained the important point wise estimates in the Fourier space.

**Proposition 3.1** It holds that
\[
|\hat{u}_t(t, \xi)|^2 + \log(1 + |\xi|^2)|\hat{u}(t, \xi)|^2 \leq 6 \left(|\hat{u}_1(\xi)|^2 + \log(1 + |\xi|^2)|\hat{u}_0(\xi)|^2\right) e^{-\frac{\rho(\xi)}{2}t},
\]
(3.8)
for all \(t > 0\) and \(\xi \in \mathbb{R}^n\), and
\[
|\hat{u}(t, \xi)|^2 \leq 6 \left(\frac{1}{\log(1 + |\xi|^2)}|\hat{u}_1(\xi)|^2 + |\hat{u}_0(\xi)|^2\right) e^{-\frac{\rho(\xi)}{2}t},
\]
(3.9)
for all \(t > 0\) and \(\xi \in \mathbb{R}^n, \xi \neq 0\).

### 3.1 Proof of Propositions 1.1 and 1.2

In this subsection, let us prove Propositions 1.1 and 1.2 by basing on the results of Proposition 3.1. We first prove Proposition 1.1. To begin with, applying the Plancherel theorem, and integrating the
inequality \((3.10)\) over \(\mathbb{R}^n\) one has

\[
\begin{align*}
\|u(t, \cdot)\|^2_{L^2} + \|L^{1/2}u(t, \cdot)\|^2_{L^2} = & \|\hat{u}(t, \cdot)\|^2_{L^2} + \|\log^{1/2}(1 + |\xi|^2)|\hat{u}(t, \cdot)|\|^2_{L^2} \\
= & \int_{\mathbb{R}^n} (|\tilde{u}|^2 + \log(1 + |\xi|^2)|\tilde{u}|^2) \, d\xi \\
\leq & 6 \int_{\mathbb{R}^n} (|\hat{u}_1|^2 + \log(1 + |\xi|^2)|\hat{u}_0|^2) \, e^{-\frac{\mathcal{L}}{30}t} \, d\xi \\
& \quad + 6 \int_{|\xi| \leq \sqrt{e^{-1}}} |\hat{u}_1|^2 e^{-\frac{\mathcal{L}}{2} t} \, d\xi + 6 \int_{|\xi| > \sqrt{e^{-1}}} \log(1 + |\xi|^2)|\hat{u}_0|^2 e^{-\frac{\mathcal{L}}{2} t} \, d\xi \\
= & 6(A_1 + A_2 + A_3),
\end{align*}
\]

with \(A_i \ (i = 1, 2, 3)\) according to the integrals on low, middle and high frequencies, respectively.

1) Estimate on the zone \(|\xi| \leq 1\)

On this zone we have \(\rho(\xi) = \frac{1}{2} \log(1 + |\xi|^2)\).

At this stage we assume that the initial data \(u_0, u_1 \in L^1(\mathbb{R}^n)\) . Then \(\hat{u}_0, \hat{u}_1 \in L^\infty(\mathbb{R}^n)\) and

\[
\|\hat{u}_0\|_\infty \leq \|u_0\|_1 \text{ and } \|\hat{u}_1\|_\infty \leq \|u_1\|_1.
\]

Then, using the definition of \(\rho(\xi)\) we may estimate the integrals on the low frequency region as follows.

\[
\begin{align*}
A_1 = & \int_{|\xi| \leq 1} |\hat{u}_1|^2 e^{-\frac{\mathcal{L}}{2} t} \, d\xi + \int_{|\xi| \leq 1} \log(1 + |\xi|^2)|\hat{u}_0|^2 e^{-\frac{\mathcal{L}}{2} (1 + |\xi|^2)} \, d\xi \\
= & \int_{|\xi| \leq 1} |\hat{u}_1|^2 (1 + |\xi|^2)^{-\frac{1}{2}} d\xi + \int_{|\xi| \leq 1} \log(1 + |\xi|^2)|\hat{u}_0|^2 (1 + |\xi|^2)^{-\frac{1}{2}} d\xi \\
\leq & \|\hat{u}_1\|^2 \int_{|\xi| \leq 1} (1 + |\xi|^2)^{-\frac{1}{2}} d\xi + \|\hat{u}_0\|^2 \int_{|\xi| \leq 1} \log(1 + |\xi|^2)(1 + |\xi|^2)^{-\frac{1}{2}} d\xi \\
\leq & \|u_1\|^2 \int_{|\xi| \leq 1} |1 + |\xi|^2|^{-\frac{1}{2}} d\xi + \|u_0\|^2 \int_{|\xi| \leq 1} \log(1 + |\xi|^2)(1 + |\xi|^2)^{-\frac{1}{2}} d\xi \\
= & \|u_1\|^2 \omega_n \int_0^1 (1 + r^2)^{-\frac{1}{2}} r^n-1 dr + \|u_0\|^2 \omega_n \int_0^1 \log(1 + r^2)(1 + r^2)^{-\frac{1}{2}} r^n-1 dr \\
\leq & \|u_1\|^2 \omega_n I_{n-1}(t/4) + \|u_0\|^2 \omega_n I_{n+1}(t/4) \\
\leq & C_n \left( \|u_1\|^2 t^{-\frac{n}{2}} + \|u_0\|^2 t^{-\frac{n+2}{2}} \right), \quad t \gg 1,
\end{align*}
\]

because of the fact that \(\log(1 + r^2) \leq r^2\) for all \(r \geq 0\), and Lemma 2.1 with \(2.1\), where \(C_n\) is a positive constant depending only on \(n\).

2) Estimate on the middle frequency zone \(1 \leq |\xi| \leq \sqrt{e^{-1}}\)

In this middle region we also have \(\rho(\xi) = \frac{1}{2} \log(1 + |\xi|^2)\) and we may estimate \(\log(1 + |\xi|^2)\) by

\[
\log 2 \leq \log(1 + |\xi|^2) \leq 1.
\]
Thus, one has for all $t > 0$
\[
A_2 = \int_{1 \leq |\xi| \leq ve^{-1}} |\hat{u}_1|^2 e^{-\frac{4}{n}t} d\xi + \int_{|\xi| \leq ve^{-1}} \log(1 + |\xi|^2) |\hat{u}_0|^2 e^{-\frac{4}{n}t} d\xi
\]
\[
\leq \int_{1 \leq |\xi| \leq ve^{-1}} |\hat{u}_1|^2 e^{-\frac{4}{n}t} d\xi + \int_{|\xi| \leq ve^{-1}} |\hat{u}_0|^2 e^{-\frac{4}{n}t} d\xi
\]
\[
\leq 2^{-\frac{4}{n}} \|\hat{u}_1\|_2^2 + 2^{-\frac{4}{n}} \|\hat{u}_0\|_2^2 = 2^{-\frac{4}{n}} \left(\|u_1\|_2^2 + \|u_0\|_2^2\right) \quad (t \geq 0).
\]

3) Estimate on the high frequency zone $|\xi| \geq \sqrt{e - 1}$

On this region we have $\rho(\xi) = \frac{1}{\xi}$. Thus we obtain the estimate
\[
A_3 = \int_{|\xi| \geq \sqrt{e - 1}} |\hat{u}_1|^2 e^{\frac{4}{n}t} d\xi + \int_{|\xi| \geq \sqrt{e - 1}} \log(1 + |\xi|^2) |\hat{u}_0|^2 e^{\frac{4}{n}t} d\xi
\]
\[
\leq e^{\frac{4}{n}} \|\hat{u}_1\|_2^2 + e^{\frac{4}{n}} \int_{\mathbb{R}^n} \log(1 + |\xi|^2) |\hat{u}_0|^2 d\xi
\]
\[
= e^{\frac{4}{n}} \left(\|u_1\|_2^2 + \|L^2 u_0\|_2^2\right) = 2e^{\frac{4}{n}} E_0(0), \quad t > 0.
\]

Under these preparations obtained in 1), 2) and 3) above, one can prove Proposition 1.1.

Proof of Proposition 1.1 By combining the estimates for $A_1, A_2, A_3$ with (3.10) the proof is now completed. \qed

The above proposition says that the total energy of the system decays as $t^{-n/2}$, that is, To estimate the $L^2$-norm of $u(t, x)$ we first observe that
\[
\lim_{r \to 0} \frac{r^2}{\log(1 + r^2)} = 1.
\]

Thus, there exists a small $\delta \in (0, 1)$ such that
\[
\frac{1}{2} \leq \frac{r^2}{\log(1 + r^2)} \leq \frac{3}{2}
\]
for $0 < r \leq \delta$.

By integrating the inequality (3.10) on $\mathbb{R}^n$ and using the Plancherel theorem we obtain
\[
\|u(t, \cdot)\|_2^2 \leq 6 \int_{\mathbb{R}^n} \left(\frac{1}{\log(1 + |\xi|^2)} |\hat{u}_1|^2 + |\hat{u}_0|^2\right) e^{-\frac{4}{n}t} d\xi
\]
\[
= 6 \int_{|\xi| \leq \delta} \left(\frac{1}{\log(1 + |\xi|^2)} |\hat{u}_1|^2 + |\hat{u}_0|^2\right) e^{-\frac{4}{n}t} d\xi + 6 \int_{|\xi| > \delta} \left(\frac{1}{\log(1 + |\xi|^2)} |\hat{u}_1|^2 + |\hat{u}_0|^2\right) e^{-\frac{4}{n}t} d\xi
\]
\[
=: 6(B_1 + B_2).
\]

Analogous to the estimates for the energy we may obtain exponential decay to the integral $B_2$ on the high frequency zone $|\xi| > \delta$, that is,
\[
B_2 \leq C \left(\|u_0\|_2^2 + \|u_1\|_2^2\right) e^{-\frac{\delta}{4}} t > 0.
\]
On the low frequency region $|\xi| \leq \delta$, by using Lemma 2.1 together with 2.1, one has

$$B_1 = \int_{|\xi| \leq \delta} \frac{1}{\log(1 + |\xi|^2)} |\hat{u}|^2 e^{-\frac{\log(1 + |\xi|^2)}{4}} d\xi + \int_{|\xi| \leq \delta} \frac{1}{2} \hat{u}_0^2 e^{-\frac{\log(1 + |\xi|^2)}{4}} d\xi$$

$$\leq \|u_1\|_1^2 \int_{|\xi| \leq \delta} \frac{1}{\log(1 + |\xi|^2)} e^{-\frac{\log(1 + |\xi|^2)}{4}} d\xi + \|u_0\|_1^2 \int_{|\xi| \leq \delta} (1 + |\xi|^2)^{-\frac{3}{4}} d\xi$$

$$\leq \|u_1\|_1^2 \int_{0}^{1} \frac{1}{\log(1 + r^2)} (1 + r^2)^{-\frac{3}{4}} r^n-1 dr + \|u_0\|_1^2 \int_{0}^{1} (1 + r^2)^{-\frac{3}{4}} r^n-1 dr$$

$$\leq \|u_1\|_1^2 \int_{0}^{1} (1 + r^2)^{-\frac{3}{4}} r^n-1 dr + \|u_0\|_1^2 \int_{0}^{1} (1 + r^2)^{-\frac{3}{4}} r^n-1 dr$$

$$= \|u_1\|_1^2 \int_{0}^{1} (1 + r^2)^{-\frac{3}{4}} r^n-1 dr + \|u_0\|_1^2 \int_{0}^{1} (1 + r^2)^{-\frac{3}{4}} r^n-1 dr$$

$$\leq C_n \left( \|u_1\|_1^2 (t^{-\frac{3}{4}} + \|u_0\|_1^2 t^{-\frac{3}{4}}) \right), \quad t \gg 1$$

for $n > 2$, where $C_n > 0$ depends only on $n$.

**Proof of Proposition 4.2** By combining estimates for $B_1, B_2$ with 3.1, we have just proved Proposition 1.2.

4. **Asymptotic profile of solutions**

To obtain an asymptotic profile we consider, without loss of generality, the case of initial amplitude $u_0 = 0$. Then, the corresponding Cauchy problem to problem (1.1), (1.2) in the Fourier space is given by

$$\hat{u}_{tt}(t, \xi) + \log(1 + |\xi|^2)\hat{u}_t(t, \xi) + \log(1 + |\xi|^2)\hat{u}(t, \xi) = 0, \quad t > 0, \quad \xi \in \mathbb{R}^n,$$

$$\hat{u}(0, \xi) = 0, \quad \hat{u}_t(0, \xi) = u_1(\xi), \quad \xi \in \mathbb{R}^n. \quad (4.1)$$

The characteristics roots $\lambda_+$ and $\lambda_-$ of the characteristic polynomial

$$\lambda^2 + \log(1 + |\xi|^2)\lambda + \log(1 + |\xi|^2) = 0, \quad \xi \in \mathbb{R}^n$$

associated to the equation (4.1) are given by

$$\lambda_{\pm} = \frac{-\log(1 + |\xi|^2) \pm i \sqrt{4 \log(1 + |\xi|^2) - \log^2(1 + |\xi|^2)}}{2}, \quad (4.2)$$

for $\xi \leq \sqrt{e^4 - 1}$. The solution formula can be expressed by

$$\hat{u}(t, \xi) = \frac{\hat{u}_1}{b(\xi)} e^{-a(\xi)t} \sin(b(\xi)t) \quad (4.3)$$

for small frequency region such that $|\xi| \leq \sqrt{e^4 - 1}$, where $a(\xi)$ and $b(\xi)$ are the real and imaginary parts of the characteristics roots, that is

$$a(\xi) = \frac{\log(1 + |\xi|^2)}{2} \quad \text{and} \quad b(\xi) = \sqrt{\frac{4 \log(1 + |\xi|^2) - \log^2(1 + |\xi|^2)}{2}}. \quad (4.4)$$

We note that $a(\xi)$ and $b(\xi)$ are well defined for $|\xi| \leq 1$. In fact, it is easy to see that

$$4 \log(1 + |\xi|^2) - \log^2(1 + |\xi|^2) > 0$$

for $0 \leq |\xi| < \sqrt{e^4 - 1}$.

**Remark 4.1** It holds that

$$\sqrt{\log(1 + |\xi|^2)} \leq 2b(\xi) \leq 2\sqrt{\log(1 + |\xi|^2)}$$

for $|\xi| \leq 1$.  

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To see this, we observe that
\[
\begin{align*}
b(\xi) &= \sqrt{\frac{4 \log(1 + |\xi|^2) - \log^2(1 + |\xi|^2)}{2}} \\&\leq \frac{\sqrt{4 \log(1 + |\xi|^2)}}{2} \\
&= \sqrt{\log(1 + |\xi|^2)},
\end{align*}
\]
and for $|\xi| \leq \sqrt{e^3 - 1}$, we have
\[
1 \leq |\xi|^2 + 1 \leq e^3 \iff 0 \leq \log(1 + |\xi|^2) \leq 3 \iff \log^2(1 + \xi^2) - 3 \log(1 + \xi^2) \leq 0 \\
\iff \log(1 + \xi^2) \leq 4 \log(1 + \xi^2) - \log^2(1 + \xi^2),
\]
thus
\[
\frac{\sqrt{\log(1 + \xi^2)}}{2} \leq b(\xi).
\]

In order to study an asymptotic profile of the solution to problem (1.1–1.2) we consider a decomposition of the Fourier transformed initial data.

**Remark 4.2** Using the Fourier transform we can get a decomposition of the initial data $\hat{u}_1$ as follows:
\[
\hat{u}_1(\xi) = A(\xi) - iB(\xi) + P_1, \quad \xi \in \mathbb{R}^n,
\]
where $P_1, A, B$ are defined by
\[
P_1 = \int_{\mathbb{R}^n} u_1(x)dx, \quad A(\xi) = \int_{\mathbb{R}^n} u_1(x)(1 - \cos(\xi x))dx, \quad B(\xi) = \int_{\mathbb{R}^n} u_1(x)\sin(\xi x)dx.
\]

According to the above decomposition we can know the following lemma (see [18]).

**Lemma 4.1** Let $\kappa \in [0, 1]$. For $u_1 \in L^{1,\kappa}(\mathbb{R}^n)$ and $\xi \in \mathbb{R}^n$ it holds that
\[
|A(\xi)| \leq K|\xi|^{-\kappa}\|u_1\|_{L^{1,\kappa}} \quad \text{and} \quad \|B(\xi)\| \leq M|\xi|^{-\kappa}\|u_1\|_{L^{1,\kappa}},
\]
with positive constants $K$ and $M$ depending only on $n$.

Let us capture a leading term of the solution based on (4.3) and Remark 4.2. First, we apply the mean value theorem to get
\[
\sin(b(\xi)t) - \sin\left(t\sqrt{\log(1 + |\xi|^2)}\right) = t\cos(\mu(\xi)t)\left[b(\xi) - \sqrt{\log(1 + |\xi|^2)}\right], \quad (4.5)
\]
where $\mu(\xi) = \theta b(\xi) + (1 - \theta)\sqrt{\log(1 + |\xi|^2)}$ for some $0 < \theta < 1$. By this reason, we can rewrite the solution formula (4.3) as
\[
\hat{u}(t, \xi) = \frac{A(\xi) - iB(\xi)}{b(\xi)}e^{-a(\xi)t}\sin(b(\xi)t) + \frac{P_1}{b(\xi)}e^{-a(\xi)t}\sin\left(t\sqrt{\log(1 + |\xi|^2)}\right) \\
+ \frac{P_1}{b(\xi)} \frac{-\sqrt{\log(1 + |\xi|^2)}}{b(\xi)}e^{-a(\xi)t}t\cos(\mu(\xi)t).
\]

Our goal in this section is to get decay estimates in time to the remainder terms defined above. To proceed with that we define the next 3 functions
\[
F_1(t, \xi) = \frac{A(\xi) - iB(\xi)}{b(\xi)}e^{-a(\xi)t}\sin(b(\xi)t),
\]
\[
F_2(t, \xi) = \frac{P_1}{b(\xi)} \frac{-\sqrt{\log(1 + |\xi|^2)}}{b(\xi)}e^{-a(\xi)t}t\cos(\mu(\xi)t),
\]
\[
F_3(t, \xi) = \frac{P_1}{b(\xi)}e^{-a(\xi)t}\sin\left(t\sqrt{\log(1 + |\xi|^2)}\right).
\]
Then, we get
\[ \dot{u}(t, \xi) - F_3(t, \xi) = F_1(t, \xi) + F_2(t, \xi). \]
We know that
\[ \lim_{r \to 0} \frac{r^2}{\log(1 + r^2)} = 1, \]
so, there exists \( 0 < \delta_1 < 1 \) such that
\[ \frac{r^2}{\log(1 + r^2)} < 2 \]
for all \( 0 < r < \delta_1 \). By using this fact, Remark 4.1, Lemma 4.1 with \( \kappa = 1 \) and Lemma 2.1, we obtain
\[
\int_{|\xi| \leq \delta_1} |F_1(t, \xi)|^2 d\xi \leq 4 \int_{|\xi| \leq \delta_1} \frac{|A(\xi) - iB(\xi)|^2}{\log(1 + |\xi|^2)} e^{-2a(\xi)t} \sin^2(b(\xi)t) d\xi
\]
\[
\leq 4 \int_{|\xi| \leq \delta_1} \frac{(|A(\xi)| + |B(\xi)|)^2}{\log(1 + |\xi|^2)} e^{-2a(\xi)t} d\xi
\]
\[
\leq 4 \int_{r \leq \delta_1} \frac{(K + M)^2 \xi^2 \|u_1\|^2_1}{\log(1 + |\xi|^2)} (1 + |\xi|^2)^{-\epsilon} d\xi
\]
\[
= 4\omega_n (K + M)^2 \|u_1\|^2_1 \int_{r \leq \delta_1} \frac{r^{n+1}}{\log(1 + r^2)} (1 + r^2)^{-\epsilon} dr
\]
\[
\leq 4\omega_n (K + M)^2 \|u_1\|^2_1 \int_{r \leq \delta_1} (1 + r^2)^{-\epsilon} r^{n-1} dr
\]
\[
\leq 8\omega_n (K + M)^2 \|u_1\|^2_1 \int_{r \leq 1} (1 + r^2)^{-\epsilon} r^{n-1} dr
\]
\[
= 8\omega_n \|u_1\|^2_1 (K + M)^2 I_{n-1}(t)
\]
\[
\leq C_{1,n} \|u_1\|^2_1 t^{-\frac{\epsilon}{2}}.
\]
Now, we observe that for \( r := |\xi| < \sqrt{e^4 - 1} \), we have
\[
b(r) - \sqrt{\log(1 + r^2)} = \frac{\sqrt{4 \log(1 + r^2) - \log^2(1 + r^2)} - \sqrt{\log(1 + r^2)}}{2}
\]
\[
= \sqrt{\log(1 + r^2)} \left( \sqrt{1 - \frac{\log^2(1 + r^2)}{4 \log(1 + r^2)}} - 1 \right)
\]
\[
= \sqrt{\log(1 + r^2)} \left( \sqrt{1 - \frac{\log^2(1 + r^2)}{4 \log(1 + r^2)}} - 1 \right) \left( \frac{\sqrt{1 - \frac{\log^2(1 + r^2)}{4 \log(1 + r^2)}} + 1}{\sqrt{1 - \frac{\log^2(1 + r^2)}{4 \log(1 + r^2)}} + 1} \right)
\]
\[
= -\sqrt{\log(1 + r^2)} \frac{\log^2(1 + r^2)}{4 \log(1 + r^2)} \left( 1 + \sqrt{1 - \frac{\log^2(1 + r^2)}{4 \log(1 + r^2)}} \right)
\]
Thus
\[
|b(r) - \sqrt{\log(1 + r^2)}| = \left| \sqrt{\log(1 + r^2)} \frac{\log^2(1 + r^2)}{4 \log(1 + r^2)} \right| \left( 1 + \sqrt{1 - \frac{\log^2(1 + r^2)}{4 \log(1 + r^2)}} \right)
\]
\[
\leq \sqrt{\log(1 + r^2)} \frac{\log^2(1 + r^2)}{4 \log(1 + r^2)}, \quad (\forall r < \sqrt{e^4 - 1}).
\]
By combining this fact with Remark 4.1, we obtain
\[
\frac{|b(r) - \sqrt{\log(1 + r^2)}|^2}{|b(r)|^2} \leq \log^2(1 + r^2), \quad (\forall r \leq 1).
\]
Also, we know that
\[
\lim_{r \to 0^+} \frac{\log^2 (1 + r^2)}{r^4} = 1,
\]
thus there exists \(0 < \delta < \delta_1\) such that
\[
\frac{1}{2} \leq \frac{\log^2 (1 + r^2)}{r^4} \leq \frac{3}{2}
\]
for all \(0 \leq r \leq \delta\). Hence (see (2.1))
\[
\int_{|\xi| \leq \delta} |F_2(t, \xi)|^2 d\xi = \int_{|\xi| \leq \delta} |P_1|^2 \frac{|b(\xi) - \sqrt{\log(1 + |\xi|^2)}|}{|b(\xi)|^2} e^{-\alpha(\xi)t^2} \cos(\mu(\xi)t) d\xi
\]
\[
\leq |P_1|^2 \int_{|\xi| \leq \delta} \frac{|b(\xi) - \sqrt{\log(1 + |\xi|^2)}|}{|b(\xi)|^2} (1 + |\xi|^2)^{-4} d\xi
\]
\[
= |P_1|^2 t^2 \omega_n \int_0^\delta (1 + r^2)^{-\frac{3}{2}} (1 + r^2)^{-t} r_n d r
\]
\[
\leq 3\omega_n |P_1|^2 t^2 \int_0^\delta (1 + r^2)^{-t} r_n^3 d r
\]
\[
\leq 3\omega_n |P_1|^2 t^2 \int_0^\delta (1 + r^2)^{-t} r_n^3 d r
\]
\[
\leq C_{2,n} |P_1|^2 t^{-\frac{3}{2}}.
\]

Here, note that
\[
|\hat{u}(t, \xi) - F_3(t, \xi)| = |F_1(t, \xi) + F_2(t, \xi)|^2 \leq 2 \left( |F_1(t, \xi)|^2 + |F_2(t, \xi)|^2 \right).
\]

Now, we define the following function
\[
\varphi(t, \xi) = \frac{P_1}{\sqrt{\log(1 + |\xi|^2)}} e^{-\alpha(\xi)t} \sin \left( t \sqrt{\log(1 + |\xi|^2)} \right),
\]
which is equivalent to \(F_3(t, \xi)\) according to the Remark 4.1.

Then we have the following result, which implies that the leading term of the Fourier transformed solution in the low frequency region is the very \(\varphi(t, \xi)\). The result holds for all \(n \geq 1\).

**Theorem 4.1** Let \(n \geq 1\) and let \(u_1 \in L^2(\mathbb{R}^n) \cap L^{1,1}(\mathbb{R}^n)\). Then, there exists \(0 < \delta < 1\) such that
\[
\int_{|\xi| \leq \delta} |\hat{u}(t, \xi) - \varphi(t, \xi)|^2 d\xi \leq C_{1,n} \|u_1\|_{1,1}^2 t^{-\frac{3}{2}} + C_{2,n} |P_1|^2 t^{-\frac{3}{2}},
\]
for \(t \gg 1\) with positive constants \(C_{1,n}\) and \(C_{2,n}\) depending only on \(n \in \mathbb{N}\).

On the other hand, in the zone of high frequency \(\{ |\xi| \geq \delta \}\) we have the following estimates.

**Theorem 4.2** Let \(n \geq 1\) and let \(u_1 \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)\). Then,
\[
\int_{|\xi| \geq \delta} |\hat{u}(t, \xi) - \varphi(t, \xi)|^2 d\xi \leq C(\|u_1\|^2 + |P_1|^2) e^{-\eta t},
\]
for \(t \gg 1\) with positive constant \(C\) and \(\eta\) depending on \(n \in \mathbb{N}\).
Proof. It follows from Proposition 3.1 that
\[
|\dot{u}(t, \xi)|^2 \leq 6 \frac{|\dot{u}_1(\xi)|^2}{\log(1 + |\xi|^2)} e^{-\frac{2}{\log(1 + |\xi|^2)} t} \\
\leq 6 |\dot{u}_1(\xi)|^2 e^{-\frac{2}{\log(1 + |\xi|^2)} t} \quad (|\xi| \geq \sqrt{e - 1}).
\] (4.7)

Additionally,
\[
|\varphi(t, \xi)|^2 \leq |P_1|^2 \frac{1}{\log(1 + |\xi|^2)} (1 + |\xi|^2)^{-t} \sin^2(t \sqrt{\log(1 + |\xi|^2)}) \\
\leq |P_1|^2 (1 + |\xi|^2)^{-t} \quad (|\xi| \geq \sqrt{e - 1}).
\] (4.8)

Thus, (2.2), (4.7), (4.8) and Lemma 2.2 imply
\[
\int_{|\xi| \geq \sqrt{e - 1}} |\dot{u}(t, \xi) - \varphi(t, \xi)|^2 d\xi \leq 2 \int_{|\xi| \geq \sqrt{e - 1}} (|\dot{u}(t, \xi)|^2 + |\varphi(t, \xi)|^2) d\xi \\
\leq 12 e^{-\frac{2}{\log(1 + |\xi|^2)} t} \|u_1\|^2 + 2 |P_1|^2 \omega_n \int_1^\infty (1 + s^2)^{-t} s^{n-1} ds \\
= 12 e^{-\frac{2}{\log(1 + |\xi|^2)} t} \|u_1\|^2 + 2 |P_1|^2 \omega_n \frac{2^{-t}}{t - 1}, \quad (t \gg 1).
\] (4.9)

On the other hand, similarly to the derivation of (4.9), if \( \delta \leq |\xi| \leq \sqrt{e - 1} \), then
\[
\log(1 + \delta^2) \leq \log(1 + \delta^2) \leq 1,
\]
so that from Proposition 3.1 one can get
\[
|\dot{u}(t, \xi)|^2 \leq 6 \frac{|\dot{u}_1(\xi)|^2}{\log(1 + |\xi|^2)} e^{-\frac{2}{\log(1 + \delta^2)} t} \\
\leq 6 \frac{|\dot{u}_1(\xi)|^2}{\log(1 + \delta^2)} e^{-\frac{2}{\log(1 + \delta^2)} t},
\] (4.10)

and
\[
|\varphi(t, \xi)|^2 \leq |P_1|^2 \frac{1}{\log(1 + |\xi|^2)} (1 + |\xi|^2)^{-t} \\
\leq \frac{|P_1|^2}{\log(1 + \delta^2)} (1 + \delta^2)^{-t} \quad (\delta \leq |\xi| \leq \sqrt{e - 1}).
\] (4.11)

Finally, these estimates (4.9), (4.10) and (4.11) in the high and middle frequency zones \( |\xi| \geq \sqrt{e - 1} \) and \( \delta \leq |\xi| \leq \sqrt{e - 1} \) imply the desired exponential decay estimates.

The validity of Theorem 1.1 is a direct consequence of Theorems 4.1 and 4.2, and so we shall omit its detail.

5 Optimal decay rate of \( L^2 \)-norm for \( n \geq 3 \)

In this section, we investigate the precise rate of decay of the leading term (4.6) in \( L^2 \)-sense as \( t \to \infty \). The case of \( n \geq 3 \) is first treated.

Proposition 5.1 Let \( n \geq 3 \). Then there exists \( t_0 > 0 \) such that for \( t \geq t_0 \) it holds that
\[
C_{1,n} t^{-\frac{n - 2}{2}} \leq \int_{\mathbb{R}^n} \frac{(1 + |\xi|^2)^{-t} \sin^2(t \sqrt{\log(1 + |\xi|^2)})}{\log(1 + |\xi|^2)} d\xi \leq C_{2,n} t^{-\frac{n - 2}{2}},
\]
where \( C_{1,n} \) and \( C_{2,n} \) are positive constants depending only on \( n \).
Proof. We first observe that
\[
\lim_{r \to 0} \frac{r^2}{\log(1 + r^2)} = 1.
\]
Then, we can obtain \(0 < \delta < 1\) such that
\[
\frac{1}{2} \leq \frac{r^2}{\log(1 + r^2)} \leq \frac{3}{2}
\]
for \(0 < r \leq \delta\). Thus, for \(t > 1\) based on Lemmas 2.1 and 2.2 one has
\[
\int_0^\delta (1 + r^2)^{-t} \frac{\sin^2(t \sqrt{\log(1 + r^2)})}{\log(1 + r^2)} r^{n-1} dr \leq \int_0^\delta \frac{r^2}{\log(1 + r^2)} (1 + r^2)^{-t} r^{n-3} dr \\
\leq \frac{3}{2} \int_0^\delta (1 + r^2)^{-t} r^{n-3} dr \\
\leq \int_0^1 (1 + r^2)^{-t} r^{n-3} dr \\
\leq C_{1,n} t^{-\frac{n-2}{2}},
\]
\[
\int_{\delta}^1 (1 + r^2)^{-t} \frac{\sin^2(t \sqrt{\log(1 + r^2)})}{\log(1 + r^2)} r^{n-1} dr \leq \frac{1}{\log(1 + \delta^2)} \int_{\delta}^1 (1 + r^2)^{-t} \sin^2(t \sqrt{\log(1 + r^2)}) r^{n-1} dr \\
\leq \frac{1}{\log(1 + \delta^2)} \int_{\delta}^1 (1 + r^2)^{-t} r^{n-1} dr \\
\leq \frac{1}{\log(1 + \delta^2)} \int_0^1 (1 + r^2)^{-t} r^{n-1} dr \\
\leq C_{2,n,\delta} t^{-\frac{n}{2}},
\]
\[
\int_1^\infty (1 + r^2)^{-t} \frac{\sin^2(t \sqrt{\log(1 + r^2)})}{\log(1 + r^2)} r^{n-1} dr \leq \frac{1}{\log 2} \int_1^\infty (1 + r^2)^{-t} r^{n-1} \sin^2(t \sqrt{\log(1 + r^2)}) dr \\
\leq \frac{1}{\log 2} \int_1^\infty (1 + r^2)^{-t} r^{n-1} dr \\
\leq C_{3,n} 2^{-t} t^{n-1},
\]
These three estimates above imply that there exists \(t_0 > 0\) such that
\[
\int_0^\infty (1 + r^2)^{-t} \frac{\sin^2(t \sqrt{\log(1 + r^2)})}{\log(1 + r^2)} r^{n-1} dr \leq C_{4,n} t^{-\frac{n-2}{2}}
\]
for all \(t \geq t_0\).

On the other hand, one notices the following computation such that
\[
M(t) := \int_0^\infty (1 + r^2)^{-t} \frac{\sin^2(t \sqrt{\log(1 + r^2)})}{\log(1 + r^2)} r^{n-1} dr \\
= \int_0^\infty (1 + r^2)^{-t} (1 + r^2) \frac{\sin^2(t \sqrt{\log(1 + r^2)})}{\sqrt{\log(1 + r^2)}(1 + r^2) \sqrt{\log(1 + r^2)}} r^{n-2} dr \\
\geq \int_0^\infty (1 + r^2)^{-t} \frac{\sin^2(t \sqrt{\log(1 + r^2)})}{\sqrt{\log(1 + r^2)}(1 + r^2) \sqrt{\log(1 + r^2)}} r^{n-2} dr.
\]
And also, it is known that \( r^2 \geq \log(1+r^2) \), so that by using the change of variable \( y = \sqrt{t} \log(1+r^2) \) we have

\[
M(t) \geq \int_0^\infty \frac{(1 + r^2)^{-t} \sin^2(t \sqrt{\log(1+r^2)}) (\log(1 + r^2))}{\sqrt{\log(1 + r^2)}(1 + r^2) \sqrt{\log(1 + r^2)}} dr
\]

\[
= \int_0^\infty e^{-y^2} \sin^2(\sqrt{t} y) y^{-2} dr
\]

\[
= \frac{t^{-3/2}}{2} \int_0^\infty e^{-y^2} y^{-2} dy - \frac{t^{-3/2}}{2} \int_0^\infty e^{-y^2} y^{-3} \cos(2\sqrt{t}y) dy
\]

\[
= \frac{t^{-3/2}}{2} (A_n - F_n(t)),
\]

where

\[
A_n := \int_0^\infty e^{-y^2} y^{-2} dy \quad \text{and} \quad F_n(t) := \int_0^\infty e^{-y^2} y^{-3} \cos(2\sqrt{t}y) dy.
\]

Due to the fact \( e^{-y^2} y^{-3} \in L^1(\mathbb{R}) \) for \( n \geq 3 \), we can apply the Riemann-Lebesgue Lemma to get

\[
F_n(t) \to 0, \quad (t \to \infty).
\]

Then there exists \( t_1 > t_0 \) such that \( F_n(t) \leq \frac{A_n}{2} \) for all \( t \geq t_1 \), that is

\[
A_n - F_n(t) \geq \frac{A_n}{2} \quad \text{for all} \quad t \geq t_1.
\]

Thus, one has

\[
\int_0^\infty \frac{(1 + r^2)^{-t} \sin^2(t \sqrt{\log(1+r^2)})}{\log(1 + r^2)} r^{-1} dr \geq \frac{A_n}{4} t^{-3/2}
\]

(5.2)

for \( t \geq t_1 \).

Finally, the desired statement can be obtained by \([5.1]\) and \([5.2]\). \( \square \)

### 6 \hspace{1mm} Blow-up on infinite time for \( n = 1 \) and \( n = 2 \)

In this section we study the optimal blowup rate in the sense of \( L^2 \)-norm of the solution to problem (1.1)-(1.2).

We first derive the following proposition to the case of dimension \( n = 1 \).

**Proposition 6.1** There exists \( T > 2 \) such that

\[
\frac{(64 + 49\pi^2)t}{196\pi^2} \leq \int_{\mathbb{R}} \frac{(1 + \xi^2)^{-t} \sin^2(t \sqrt{\log(1 + |\xi|^2)})}{\log(1 + |\xi|^2)} d\xi \leq 12t.
\]

for all \( t \geq T \).

**Proof.** We have

\[
\frac{1}{2} \int_{\mathbb{R}} \frac{(1 + |\xi|^2)^{-t} \sin^2(t \sqrt{\log(1 + |\xi|^2)})}{\log(1 + |\xi|^2)} d\xi = \int_0^\infty \frac{(1 + r^2)^{-t} \sin^2(t \sqrt{\log(1 + r^2)})}{\log(1 + r^2)} dr.
\]

Initially, we will obtain a lower bound for this integral. Set

\[
Q_l(t) := \int_0^t \frac{(1 + r^2)^{-t} \sin^2(t \sqrt{\log(1 + r^2)})}{\log(1 + r^2)} dr,
\]

\[
Q_h(t) := \int_t^\infty \frac{(1 + r^2)^{-t} \sin^2(t \sqrt{\log(1 + r^2)})}{\log(1 + r^2)} dr.
\]

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This implies
\[ \frac{1}{2} T_1(t) = Q_1(t) + Q_h(t). \]

By using the mean value theorem, for \( 0 \leq r \leq \frac{1}{t} \) we obtain
\[
\sin(t \sqrt{\log(1 + r^2)}) \geq \frac{t}{2} \sqrt{\log(1 + r^2)}.
\]

\[
Q_1(t) = \int_0^t \frac{(1 + r^2)^{-t} \sin^2(t \sqrt{\log(1 + r^2)})}{\log(1 + r^2)} \, dr
\]
\[
\geq \frac{t^2}{4} \int_0^t (1 + r^2)^{-t} \log(1 + r^2) \, dr
\]
\[
= \frac{t^2}{4} \int_0^t (1 + r^2)^{-t} \, dr \geq \frac{t^2}{4} \left( 1 + \frac{1}{t^2} \right)^{-t} \int_0^t \, dr
\]
\[
= \frac{t}{4} \left( 1 + \frac{1}{t^2} \right)^{-t}.
\]

Now, since
\[
\lim_{t \to \infty} \left( 1 + \frac{1}{t^2} \right)^{-t} = 1,
\]
there exist a constant \( t_1 \geq 1 \) such that
\[
\left( 1 + \frac{1}{t^2} \right)^{-t} \geq \frac{1}{2} \text{ for all } t > t_1.
\]

Thus for \( t \geq t_1 \),
\[
Q_1(t) \geq \frac{t}{8}. \quad (6.1)
\]

To deal with the integral \( Q_h(t) \), we consider
\[
\nu_1 := \sqrt{e^\frac{25\pi^2}{16-t^2}} - 1 \text{ and } \nu_2 := \sqrt{e^\frac{49\pi^2}{16-t^2}} - 1.
\]

Note that for \( \nu_1 \leq r \leq \nu_2 \) it holds that
\[
|\sin(t \sqrt{\log(1 + r^2)})| \geq \frac{1}{\sqrt{2}}.
\]

Then, we can estimate
\[
Q_h(t) \geq \int_{\nu_1}^{\nu_2} \frac{(1 + r^2)^{-t} \sin^2(t \sqrt{\log(1 + r^2)})}{\log(1 + r^2)} \, dr
\]
\[
\geq \frac{1}{2} \int_{\nu_1}^{\nu_2} (1 + r^2)^{-t} \log(1 + r^2) \, dr
\]
\[
\geq \frac{8t^2}{49\pi^2} \int_{\nu_1}^{\nu_2} (1 + r^2)^{-t} \, dr
\]
\[
\geq \frac{8t^2}{49\pi^2} e^{-\frac{49\pi^2}{16-t^2}} \int_{\nu_1}^{\nu_2} \, dr
\]
\[
= \frac{8t^2}{49\pi^2} e^{-\frac{49\pi^2}{16-t^2}} \left( \sqrt{e^{\frac{49\pi^2}{16-t^2}}} - 1 - \sqrt{e^{\frac{25\pi^2}{16-t^2}}} - 1 \right).
\]

Note that one knows the fact that
\[
\lim_{t \to \infty} t \sqrt{e^{\pi^2} - 1} = \sqrt{\gamma} \quad (\gamma > 0).
\]

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Therefore, since one can get
\[
\lim_{t \to \infty} te^{-\frac{49\pi^2}{16t^2}} \left( \sqrt{e^{\frac{49\pi^2}{16t^2}}} - 1 - \sqrt{e^{\frac{25\pi^2}{16t^2}}} - 1 \right) = \frac{\pi}{2},
\]
there exist \( t_2 \geq 1 \) such that
\[
te^{-\frac{49\pi^2}{16t^2}} \left( \sqrt{e^{\frac{49\pi^2}{16t^2}}} - 1 - \sqrt{e^{\frac{25\pi^2}{16t^2}}} - 1 \right) \geq 1 \text{ for } t \geq t_2.
\]
Therefore,
\[
Q_h(t) \geq \frac{8t}{49\pi^2} \text{ for } t \geq t_2.
\]
(6.2)

By adding (6.1) and (6.2), we conclude that
\[
\frac{1}{2}I_1(t) = \int_R (1 + |\xi|^2)^{-t} \sin^2(t \sqrt{\log(1 + |\xi|^2)}) \frac{d\xi}{\log(1 + |\xi|^2)} \geq \frac{8t}{49\pi^2} + \frac{16t}{49\pi^2},
\]
for all \( t \geq \max\{t_1, t_2\} \). This estimate concludes the proof of lower bound of proposition.

In order to obtain the upper bound, we separate the integral into three parts as follows:
\[
R_l(t) := \int_0^1 (1 + r^2)^{-t} \sin^2(t \sqrt{\log(1 + r^2)}) \frac{dr}{\log(1 + r^2)},
\]
\[
R_m(t) := \int_1^{\sqrt{t}} (1 + r^2)^{-t} \sin^2(t \sqrt{\log(1 + r^2)}) \frac{dr}{\log(1 + r^2)},
\]
\[
R_h(t) := \int_{\sqrt{t}}^{\infty} (1 + r^2)^{-t} \sin^2(t \sqrt{\log(1 + r^2)}) \frac{dr}{\log(1 + r^2)}.
\]

This implies
\[
\frac{1}{2}I_1(t) = R_l(t) + R_m(t) + R_h(t).
\]

Now, using the fact \( \frac{\sin x}{x} \leq 1 \) for all \( x > 0 \), for \( t > 0 \) one has
\[
R_l(t) \leq \int_0^1 (1 + r^2)^{-t} r^2 dr
\leq t^2 \int_0^1 (1 + r^2)^{-t} dr
\leq t^2 \int_0^1 dr
= t.
\]
(6.4)

In order to estimate the middle part, we first observe that
\[
\lim_{\sigma \to 0} \frac{\sigma}{\log(1 + \sigma)} = 1.
\]
So, there exists \( \delta_0 > 0 \) such that
\[
\frac{\sigma}{\log(1 + \sigma)} < 2
\]
for all \( 0 < \sigma < \delta_0 \). Therefore, if \( \frac{1}{t} < r < \frac{1}{\sqrt{t}} \), then \( \frac{1}{r^2} < r^2 < \frac{1}{t} \) and for \( t > \frac{1}{\delta_0} \), we have
\[
\frac{1}{\log(1 + r^2)} < \frac{2}{r^2}.
\]
Therewith, using integration by parts we can get

\[ R_m(t) = \int_1^t \frac{(1 + r^2)^{-t} \sin^2(t \sqrt{\log(1 + r^2)})}{\log(1 + r^2)} dr \]

\[ \leq 2 \int_1^t \frac{(1 + r^2)^{-t}}{r^2} dr \]

\[ = 2t \left( 1 + \frac{1}{t^2} \right)^{-t} - 2\sqrt{t} \left( 1 + \frac{1}{t} \right)^{-t} - 4t \int_1^t (1 + r^2)^{-t -1} dr \]

\[ \leq 2t \left( 1 + \frac{1}{t^2} \right)^{-t} . \]

Since

\[ \lim_{t \to \infty} \left( 1 + \frac{1}{t^2} \right)^{-t} = 1, \]

there exists \( t_3 \geq 1 \) such that for all \( t \geq t_3 \)

\[ \left( 1 + \frac{1}{t^2} \right)^{-t} \leq 2, \]

which implies

\[ R_m(t) \leq 4t, \quad t \geq t_3. \quad (6.5) \]

Now, for \( t \geq 2 \) we estimate \( R_l(t) \) as follows:

\[ R_h(t) = \int_1^\infty \frac{(1 + r^2)^{-t} \sin^2(t \sqrt{\log(1 + r^2)})}{\log(1 + r^2)} dr \]

\[ \leq \int_1^\infty \frac{(1 + r^2)^{-t}}{\log(1 + r^2)} dr \]

\[ \leq \int_1^\infty \frac{(1 + r^2)^{-t+1}}{(1 + r^2) \log(1 + r^2)} dr \]

\[ \leq \left( 1 + \frac{1}{t} \right)^{-t+1} \frac{1}{\log(1 + \frac{1}{t})} \int_1^\infty \frac{1}{1 + r^2} dr \]

\[ = \left( 1 + \frac{1}{t} \right)^{-t+1} \frac{1}{\log(1 + \frac{1}{t})} \left( \frac{\pi}{2} - \tan^{-1}(t^{-\frac{1}{2}}) \right) . \]

Due to the fact

\[ \lim_{t \to \infty} \frac{1}{t} \left[ \left( 1 + \frac{1}{t} \right)^{-t+1} \frac{1}{\log(1 + \frac{1}{t})} \left( \frac{\pi}{2} - \tan^{-1}(t^{-\frac{1}{2}}) \right) \right] = \frac{\pi}{2e}, \]

there exist \( t_4 \geq \max\{2, t_3\} \) such that

\[ \left( 1 + \frac{1}{t} \right)^{-t+1} \frac{1}{\log(1 + \frac{1}{t})} \left( \frac{\pi}{2} - \tan^{-1}(t^{-\frac{1}{2}}) \right) \leq t \]

for all \( t \geq t_4 \), where one has just used the fact that

\[ \lim_{t \to \infty} t \log(1 + \frac{1}{t}) = 1. \]

Thus one has

\[ R_h(t) \leq t \quad (6.6) \]

for all \( t \geq t_4 \).
Finally, by adding (6.4), (6.5) and (6.6) one can obtain the desired upper bound:

$$I_1(t) = \int_0^\infty \frac{(1 + r^2)^{-t} \sin^2(t \sqrt{\log(1 + r^2)})}{\log(1 + r^2)} dr \leq 6t$$

for all $t \geq t_4$.

Next we study the optimal blow-up order as $t \to \infty$ of $I_2(t)$ given by

$$I_2(t) = \int_{\mathbb{R}^2} \frac{(1 + |\xi|^2)^{-t} \sin^2(t \sqrt{\log(1 + |\xi|^2)})}{\log(1 + |\xi|^2)} d\xi.$$

In order to do this we need the following elementary lemma.

**Lemma 6.1** The inequalities

$$-1 \leq \int_2^{2\sqrt{t}} \frac{\cos y}{y} dy \leq 1$$

hold for all $t > 1$.

**Proof.** Using integration by parts we obtain for $t > 1$

$$\left| \int_2^{2\sqrt{t}} \frac{\cos y}{y} dy \right| = \left| \frac{\sin y}{y} \right|_2^{2\sqrt{t}} + \int_2^{2\sqrt{t}} \frac{1}{y^2} \sin y dy$$

$$\leq \frac{\sin(2\sqrt{t})}{2\sqrt{t}} + \frac{\sin 2}{2} + \int_2^{2\sqrt{t}} \frac{1}{y^2} \sin y dy$$

$$\leq \frac{1}{2\sqrt{t}} + \frac{1}{2} + \int_2^{2\sqrt{t}} \frac{1}{y^2} dy$$

$$= \frac{1}{\sqrt{t}} + \frac{1}{2} + 1 - \frac{1}{2\sqrt{t}} = 1,$$

which implies the desired estimate. \[\square\]

**Remark 6.1** We note that a more precise estimate than that in Lemma 6.1 is

$$-1 < \int_2^{2\sqrt{t}} \frac{\cos y}{y} dy < 0, \quad t > 1.$$

However, it is a little more difficult to prove. For our propose in this paper, it is sufficient to use the rough estimate of Lemma 6.1.

**Proposition 6.2** There exists $T > 1$ such that

$$\frac{\pi}{4e} \log t \leq \int_{\mathbb{R}^2} \frac{(1 + |\xi|^2)^{-t} \sin^2(t \sqrt{\log(1 + |\xi|^2)})}{\log(1 + |\xi|^2)} d\xi \leq 6\pi \log t$$

for all $t \geq T$.

**Proof.** By considering the polar co-ordinate transform, we set

$$\frac{1}{2\pi} I_2(t) = \int_0^\infty \frac{(1 + r^2)^{-t} \sin^2(t \sqrt{\log(1 + r^2)})}{\log(1 + r^2)} r dr.$$

In order to obtain a lower bound for $I_2(t)$, by using the change of variable $w = \sqrt{t \log(1 + r^2)}$ and integration by parts, we observe that

$$\frac{1}{2\pi} I_2(t) = \int_0^\infty \frac{(1 + r^2)^{-t} (1 + r^2) \sin^2(t \sqrt{\log(1 + r^2)})}{\sqrt{t \log(1 + r^2)}(1 + r^2) \sqrt{\log(1 + r^2)}} dr$$

$$\geq \int_0^\infty \frac{(1 + r^2)^{-t} \sin^2(t \sqrt{\log(1 + r^2)})}{\sqrt{t \log(1 + r^2)}(1 + r^2) \sqrt{\log(1 + r^2)}} dr$$

$$= \int_0^\infty e^{-w^2} \sin^2(\sqrt{t}w)dw.$$
Then one have
\[
\frac{1}{2\pi} \mathcal{I}_2(t) \geq e^{-\int_1^t \sin^2(\sqrt{r} w) \frac{dw}{w}}
\]
\[
= e^{-\int_1^t \frac{dw}{w}} e^{-\int_1^t \frac{\cos(2\sqrt{r} w) dw}{w}}
\]
\[
= e^{-\frac{1}{4} \log t} - e^{-\frac{1}{2} \int_1^t \frac{\cos(2\sqrt{r} w) dw}{w}}
\]
\[
\geq e^{-\frac{1}{4} \log t} - e^{-\frac{1}{2} \int_2^\sqrt{t} \frac{\cos y dy}{y}}
\]
\[
\geq e^{-\frac{1}{8} \log t}, \quad t \geq e^4.
\]

The penultimate inequality above is due to Lemma 6.1.

Thus, for \( t \gg 1 \), one has the optimal lower bound

\[
\mathcal{I}_2(t) = \int_0^\infty \frac{(1 + r^2)^{-t} \sin^2(t \sqrt{\log(1 + r^2)}) r dr}{\log(1 + r^2)} \geq \frac{\pi}{4e} \log t. \tag{6.7}
\]

The estimate (6.7) implies the desired estimate from below of Proposition 6.2.

Next, in order to get the upper bound for \( \mathcal{I}_2(t) \) we set

\[
Q_l(t) := \int_0^1 \frac{(1 + r^2)^{-t} \sin^2(t \sqrt{\log(1 + r^2)}) r dr}{\log(1 + r^2)}
\]
\[
Q_m(t) := \int_1^\sqrt{t} \frac{(1 + r^2)^{-t} \sin^2(t \sqrt{\log(1 + r^2)}) r dr}{\log(1 + r^2)}
\]
\[
Q_h(t) := \int_\sqrt{t}^\infty \frac{(1 + r^2)^{-t} \sin^2(t \sqrt{\log(1 + r^2)}) r dr}{\log(1 + r^2)}
\]

This implies

\[
\frac{1}{2\pi} \mathcal{I}_2(t) = Q_l(t) + Q_m(t) + Q_h(t).
\]

For \( t > 1 \), we first have

\[
Q_l(t) = \int_0^1 \frac{(1 + r^2)^{-t} \sin^2(t \sqrt{\log(1 + r^2)}) r dr}{\log(1 + r^2)}
\]
\[
\leq \int_0^1 \frac{(1 + r^2)^{-t} \log(1 + r^2)}{r^2} r dr
\]
\[
\leq t^2 \int_0^1 (1 + r^2)^{-t} r dr
\]
\[
= \frac{t^2}{2(t - 1)} \left[ 1 - \left(1 + \frac{1}{t^2}\right)^{1-t} \right].
\]

Since

\[
\lim_{t \to \infty} \frac{t^2}{t - 1} \left[ 1 - \left(1 + \frac{1}{t^2}\right)^{1-t} \right] = 1,
\]

there exists \( t_2 \geq 1 \) such that

\[
\frac{t^2}{2(t - 1)} \left[ 1 - \left(1 + \frac{1}{t^2}\right)^{1-t} \right] \leq 1
\]
for all $t \geq t_2$. Therefore, it holds that

$$Q_l(t) \leq 1 \quad (6.8)$$

for $t \geq t_2$.

Furthermore, for $t > 1$ one can get the estimate

$$Q_m(t) = \int_1^\infty \frac{(1 + r^2)^{-t} \sin^2(t \sqrt{\log(1 + r^2)})}{\log(1 + r^2)} r dr$$

$$\leq \int_1^\infty \frac{r(1 + r^2)^{-t}}{\log(1 + r^2)} dr$$

$$\leq \int_1^\infty \frac{r(1 + r^2)^{-1}}{\log(1 + r^2)} dr$$

$$= \frac{1}{2} \left[ \log \left( \log \left( 1 + \frac{1}{t} \right) \right) - \log \left( \log \left( 1 + \frac{1}{t^2} \right) \right) \right].$$

Now, since we have

$$\lim_{t \to \infty} \frac{1}{\log t} \left[ \log \left( \log \left( 1 + \frac{1}{t} \right) \right) - \log \left( \log \left( 1 + \frac{1}{t^2} \right) \right) \right] = 1,$$

then there exists $t_3 \geq t_2$ such that

$$\frac{1}{2} \left[ \log \left( \log \left( 1 + \frac{1}{t} \right) \right) - \log \left( \log \left( 1 + \frac{1}{t^2} \right) \right) \right] \leq \log t$$

for all $t > t_3$, where one has just used the facts that

$$\lim_{\sigma \to 0} \frac{\log(\log(1 + \sigma^2))}{\log \sigma} = 2,$$

$$\lim_{\sigma \to 0} \frac{\log(\log(1 + \sigma))}{\log \sigma} = 1.$$

Therefore, one has just arrived at the estimate:

$$Q_m(t) \leq \log t \quad (t \geq t_3). \quad (6.9)$$

Similarly, for $t > 1$ it follows that

$$Q_h(t) = \int_1^\infty \frac{(1 + r^2)^{-t} \sin^2(t \sqrt{\log(1 + r^2)})}{\log(1 + r^2)} r dr$$

$$\leq \int_1^\infty \frac{(1 + r^2)^{-t}}{\log(1 + r^2)} dr$$

$$\leq \frac{1}{\log \left( 1 + \frac{1}{t} \right)} \int_1^\infty (1 + r^2)^{-t} r dr$$

$$= \frac{1}{2(t - 1) \log \left( 1 + \frac{1}{t} \right)} \left( 1 + \frac{1}{t} \right)^{1-t}.$$
for all $t \geq t_4$. This implies
\[ Q_h(t) \leq 1 \quad (t > t_4). \quad (6.10) \]

By combining (6.8), (6.9) and (6.10), one can derive the crucial estimate
\[ \frac{1}{2\pi} \mathcal{I}_2(t) = \int_0^\infty \frac{(1 + r^2)^{-t} \sin^2(t\sqrt{\log(1 + r^2)})}{\log(1 + r^2)} r dr \leq 3 \log t \quad (6.11) \]
for large $t \geq t_4$.

The statement of Proposition 6.2 is now proved from (6.7) and (6.11). \hfill \Box

**Remark 6.2** The proof of Theorem 1.2 is standard, and is a direct consequence of Theorem 1.1, Propositions 5.1, 6.1 and 6.2. We omit its detail (see e.g., [19]).

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