On Equivalence and Computational Efficiency of the Major Relaxation Methods for Minimum Ellipsoid Containing the Intersection of Ellipsoids

Zhiguo Wang, Xiaojing Shen* and Yunmin Zhu

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Abstract

This paper investigates the problem on the minimum ellipsoid containing the intersection of multiple ellipsoids, which has been extensively applied to information science, target tracking and data fusion etc. There are three major relaxation methods involving SDP relaxation, S-procedure relaxation and bounding ellipsoid relaxation, which are derived by different ideas or viewpoints. However, it is unclear for the interrelationships among these methods. This paper reveals the equivalence among the three relaxation methods by three stages. Firstly, the SDP relaxation method can be equivalently simplified to a decoupled SDP relaxation method. Secondly, the equivalence between the SDP relaxation method and the S-procedure relaxation method can be obtained by rigorous analysis. Thirdly, we establish the equivalence between the decoupled SDP relaxation method and the bounding ellipsoid relaxation method. Therefore, the three relaxation methods are unified through the decoupled SDP relaxation method. By analysis of the computational complexity, the decoupled SDP relaxation method has the least computational burden among the three methods. The above results are helpful for the research of set-membership filter and distributed estimation fusion. Finally, the performance of each method is evaluated by some typical numerical examples in information fusion and filtering.

keywords: Semi-definite programming; the intersection of ellipsoids; S-procedure; optimal bounding ellipsoid; distributed estimation fusion.

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1 Introduction

Problems involving the intersection of ellipsoids occur in many practiced fields such as information science, data fusion, automatic control and target tracking [1, 2, 3, 4, 5]. It is well known that the problems of state and parameter estimation are usually solved by a stochastic method with the noise assumed random [6]. However, the information of precise system or the distribution of noise may not be obtained. Thus, uncertain systems are alternatively considered in [7, 8], such as biased dynamic systems or unknown noise distributions [9]. Since only requiring system bias and noise bounds are much easier than a precise system, it is sometimes more likely to assume that only bounds on the possible amplitude of the perturbations. Moreover, it is natural to use an ellipsoid to describe the uncertain set containing the all possible value of the unknown state vector or parameter [4]. Assume each estimated ellipsoid guarantees to contain the true state, then the intersection of these ellipsoids can also contain it. In fact, we want to find a minimum ellipsoid to contain the intersection of these ellipsoids.

The ellipsoidal bounding technique in the problem of state estimation with norm-bounded disturbance has attracted the attention of many researchers [10], [11], [12]. For example, in [11] the authors consider two measures of the size of a minimum ellipsoid involving the approximation of the intersection of ellipsoids. In [13], the authors also provide a suboptimal solution to the problem of finding the ellipsoid with minimum volume containing the intersection of the two ellipsoids defining the convex combination. In distributed fusion setting [14, 15, 16, 17], when the cross-correlation of local sensor estimation errors is unknown or impractical, the covariance intersection (CI) algorithm are derived to deal with this problem in [18, 19, 20]. It provides not only a fused estimate point but also estimate covariance, the results are parameterized as convex combination of the local estimates.

For which the data is not specified exactly and only known to belong to a given uncertainty set, the authors in [21] lay the foundation of robust convex optimization. It is argued that the more reasonable types of uncertainty sets are ellipsoids and intersection of finitely many ellipsoids. When we want to analyze the level of conservativeness of convex approximations to robust counterparts of semi-definite programming, the problem of ellipsoidal approximation of the intersection of ellipsoids is very useful [22]. Although the problem of finding an optimal ellipsoid containing the intersection of ellipsoids exits in many engineering fields, unfortunately, so far, it has not been received enough attention.

In fact, it is a nonconvex optimization problem for determining the minimum ellipsoid containing the intersection of several ellipsoids, since we need to maximize a convex function over a convex set. In [21], the authors demonstrate that a robust optimization problem with uncertainty set such as the intersection of the ellipsoids is a semi-infinite optimization problem, which leads to computationally intractable robust counterpart. The problem of determining a minimum ellipsoid to contain the intersection of the ellipsoids also involves solving nonconvex quadratically constrained quadratic programs (QCQP) [23], it captures many problems that are of interest to the signal processing, for instance, Boolean quadratic program (BQP). The BQP is long-known to be a computationally difficult problem, and it belongs to the class of NP-hard problems.
Moreover, the studied problem can be viewed as a special instance of calculating the minimum ellipsoid that covers a bounded convex set. In [25] the authors formulate the problem of finding the minimum volume ellipsoid that covers a bounded convex set as convex programming problems, but they are tractable only in special cases, such as finite set, polyhedron, and union of ellipsoids. Therefore, we do not expect to solve the problem of determining the minimum ellipsoid containing the intersection of ellipsoids in polynomial time, or find the global solution of this problem.

There are many authors study different methods for the problems involving the intersection of ellipsoids, and they may provide many approximate solutions by different ideas or viewpoints. In [1], the authors use S-procedure to approximate the original problem to a semi-definite programming (SDP), then we can use the efficient interior point methods [26] to obtain the solution. Since the ellipsoidal approximation of the intersection of the ellipsoids involves solving nonconvex QCQP, then the semi-definite relaxation (SDR) technique [24] can be used. If these intersecting ellipsoids have a common center, the problem of quadratic form maximization over the intersection of these ellipsoids is studied in [27], and it also derives a new approximation bound based on the SDR technique. In [11] and [13], the authors use the bounding ellipsoid relaxation method to contain the intersection of ellipsoids. Although these relaxation methods can obtain some ellipsoids to contain the intersection, it is unclear for the interrelationships among these methods and insight to the pros and cons of these methods. In [22], the authors survey various linear matrix inequality relaxation techniques for evaluating the maximum norm vector within the intersection of several ellipsoids, however, which is different from the problem of minimum ellipsoid containing the intersection of multiple ellipsoids. In fact, the described problem in [22] is similar to the Chebyshev center of a convex set [28].

Motivated by the aforementioned analysis, the goal of this paper is to study the problem of determining a minimum ellipsoid containing the intersection of ellipsoids, and establish the equivalence of the three major relaxation methods involving SDP relaxation, S-procedure relaxation and bounding ellipsoid relaxation, meanwhile, we also analyze the computational efficiency of each method. Our contribution has three aspects:

- Firstly, we present a comprehensive overview of some common relaxation techniques to deal with the problem of calculating the minimum ellipsoid containing the intersection of multiple ellipsoids, and the pros and cons among these relaxation methods are discussed.

- Secondly, we establish the equivalence among these relaxation methods though the decoupled SDP relaxation method, which has the least computational complexity among the three relaxation methods. The equivalence result is helpful for the research of set-membership filter and distributed estimation fusion.

- Thirdly, we derive an analytic expression of the shape matrix of the minimum ellipsoid by the decoupled SDP relaxation method. It is similar to that based on the covariance intersection method. However, we show that this minimum ellipsoid based on the decoupled SDP relaxation method is tighter than that based on the covariance intersection method.
The structure of the paper is as follows. In Section 2, we describe the optimization problem on the minimum ellipsoid containing the intersection of multiple ellipsoids. Section 3 presents the SDP relaxation method and the decoupled SDP relaxation method. Section 4 provides the S-procedure relaxation method. The equivalence between the SDP relaxation and S-procedure is established. Section 5 presents the bounding ellipsoid relaxation method. The equivalence between the decoupled SDP relaxation method and the bounding ellipsoid relaxation method is proved. Simulations are given in Section 6. Several conclusions and potential directions for further research are drawn in Section 7.

2 Problem Statement

In many practical fields, we often need to deal with the problem of determining the minimum ellipsoid containing the intersection of multiple ellipsoids. For example, the measurement update step of set-membership filter always involves the intersection of predicted ellipsoid and measurement ellipsoid [4]. In the multi-sensor estimation fusion setting, each sensor sends the local estimated ellipsoid to the fusion center and cross-correlation is unknown, then we hope to derive an optimal ellipsoid to contain the intersection of local ellipsoids [29], which can improve the accuracy of estimation.

In order to derive some highly effective algorithms to solve this important problem, we describe it as the following optimization problem.

\[
\min_{P_0, x_0} f(P_0) \quad (1)
\]

\[
\text{s.t. } (x - x_0)^T P_0^{-1} (x - x_0) \leq 1, \quad \forall x \in \mathcal{F}, \quad (2)
\]

where the vector \(x_0 \in \mathbb{R}^n\) and the symmetric positive-definite matrix \(P_0 \in \mathbb{R}^{n \times n}\) are the decision variables, and they are also the center and the shape matrix of the optimal ellipsoid \(E_0\), respectively. The set \(\mathcal{F}\) denotes the intersection of \(m\) ellipsoids, which is defined as follows

\[
\mathcal{F} = \bigcap_{i=1}^{m} E_i, \quad (3)
\]

\[
E_i = \{ x \in \mathbb{R}^n : (x - x_i)^T P_i^{-1} (x - x_i) \leq 1 \}, \quad i = 1, \ldots, m, \quad (4)
\]

where \(x_i\) and \(P_i\) are the known center and the shape matrix of ellipsoid \(E_i\), respectively, and \(P_i\) is a symmetric positive-definite matrix, \(i = 1, \ldots, m\). The objective function \(f(P_0)\) is the “size” of the optimized ellipsoid, which is a convex function and aimed at minimizing the shape matrix \(P_0\). The common “size” of the ellipsoid is \(\text{trace}(P_0)\) or \(\logdet(P_0)\), which means the sum of squares of semiaxes lengths or the volume of the ellipsoid \(E_0\), respectively. For the two criterions, in [11], the authors have proved this optimal ellipsoid exists and is unique. Actually, the constraint (2) has infinite inequalities, which enforces that the intersection of these ellipsoids is contained in the optimized ellipsoid. Thus, the optimization problem (1)-(2) is also called semi-infinite optimization problem [25].
Some authors would like to describe a general ellipsoid $E_i$ as

$$E_i = \{ x : \| A_i x - b_i \| \leq 1, \ i = 0, 1, \ldots, m \},$$

(5)
i.e., the inverse image of the Euclidean unit ball under an affine mapping, where $A_i^T A_i = P_i^{-1}$, $b_i = A_i x_i$.

Then the optimization problem (1)-(2) is equivalent to the following problem in [25],

$$\min_{A_0, b_0} f(A_0^{-1})$$

(6)

s.t. $$\| A_0 x - b_0 \| \leq 1, \quad \forall x \in \mathcal{F},$$

(7)

where the optimization variables are the positive-definite matrix $A_0$ and the vector $b_0$.

In fact, an alternative description of the feasible set in (2) is given by

$$\varphi(x_0, P_0) \leq 0$$

(8)

with the function

$$\varphi(x_0, P_0) \triangleq \max_{x \in \mathcal{F}} (x - x_0)^T P_0^{-1} (x - x_0) - 1.$$  

(9)

Obtaining the optimal value $\varphi(x_0, P_0)$ is very hard, since the objective function in (9) is a convex in $x$, when $P_0$ is a positive definite matrix. In other words, it needs to calculate the maximum of a convex function, which is a non-convex optimization problem [30]. Therefore, the computational complexity of the optimization problem (1)-(2) is NP-hard. Meanwhile, simply verifying that $E_0 \supset \bigcap_{i=1}^m E_i$ holds, given $E_1, \ldots, E_m$, is NP-complete [1].

In next sections, we do not expect to solve the problem (1)-(2) to obtain the optimized ellipsoid in polynomial time, but the efficient relaxation methods can be provided for approximation solutions. We concentrate on deriving the equivalence of typical relaxation methods and their interrelationships. Meanwhile, we also analyze the computational complexity of each method.

3 SDP Relaxation

In this section, the optimization problem (1)-(2) can be relaxed to an SDP problem by extending the relaxed Chebyshev center algorithm [28]. Moreover, we derive an analytic expression of the shape matrix of the minimum ellipsoid based on a decoupled technique, and the SDP problem can be simplified to another convex problem, which has lower computational complexity.

3.1 Convex Relaxation of $X = xx^T$

Denoting $X = xx^T$, (9) can be written equivalently as

$$\max_{X, x} f_0(X, x)$$

(10)

s.t. $$(X, x) \in \mathcal{F},$$

(11)
where
\[ \mathcal{F} = \{(X, x) : f_i(X, x) \leq 0, 1 \leq i \leq m, X = xx^T\}, \] (12)

and
\[ f_i(X, x) = \text{tr}(P_i^{-1}X) - 2x_i^TP_i^{-1}x + x_i^TP_i^{-1}x_i - 1, 0 \leq i \leq m. \] (13)

The objective function (10) is linear in decision variable \((X, x)\), but the set \(\mathcal{F}\) is not convex due to \(X = xx^T\).

In order to obtain a relaxation of the optimization problem (10)-(11), usually \(\mathcal{F}\) is replaced by the following convex set
\[ \mathcal{T} = \{(X, x) : f_i(X, x) \leq 0, 1 \leq i \leq m, xx^T \preceq X\}. \] (14)

Based on the relaxed convex set \(\mathcal{T}\), we can obtain a suboptimal ellipsoid to contain the intersection \(\mathcal{F}\) by the SDP relaxation method as follows.

**Lemma 3.1. (SDP relaxation) [25]:** The optimization problem (1)-(2) can be relaxed to an SDP problem as follows:

\[
\begin{align*}
\min & \quad f(P_0) \\
\text{s.t.} & \quad \lambda_i \geq 0, i = 1, \ldots, m, \\
& \begin{bmatrix}
P_0^{-1} - \sum_{i=1}^{m} \lambda_i P_i^{-1} & -\bar{x}_0 + \sum_{i=1}^{m} \lambda_i P_i^{-1} x_i & 0 \\
-\bar{x}_0 + \sum_{i=1}^{m} \lambda_i P_i^{-1} x_i^T & -1 - \sum_{i=1}^{m} \lambda_i (x_i^T P_i^{-1} x_i - 1) & \bar{x}_0^T \\
0 & \bar{x}_0 & -P_0^{-1}
\end{bmatrix} \preceq 0,
\end{align*}
\] (17)

where \(\bar{x}_0 = P_0^{-1}x_0\). Moreover, the relaxed optimization problem (15)-(17) is convex in the variables \(P_0^{-1}\), \(\bar{x}_0\), \(\lambda_1, \ldots, \lambda_m\).

**Proof.** In order to see clearly the equivalence of these relaxation methods, we present a proof for our specific formulation of Lemma 3.1 in Appendix.

**Algorithm 3.2. SDP relaxation**

1. Solve the optimization problem (15)-(17) for the optimal solutions \(P_0^{-1}\) and \(\bar{x}_0\).
2. Compute the shape matrix \(P_0\) and the center \(x_0 = P_0\bar{x}_0\).

As we know, this SDP problem (15)-(17) can be efficiently solved by interior point methods [31]. Meanwhile, we can use the convex optimization toolbox CVX [32] to solve (15)-(17) in MATLAB, and Algorithm 3.2 gives us a suboptimal ellipsoid that contains the intersection of ellipsoids \(E_1, \ldots, E_m\).
3.2 Decoupled SDP relaxation

In this subsection, we use a decoupled technique [12] to make further efforts to improve the complexity of this SDP problem (15)-(17). The specific method can be seen in the following proposition.

Proposition 3.3. When the objective function is trace or logdet function, the optimization problem (15)-(17) based on SDP relaxation technique can be equivalently decoupled to

\[
\min f \left( \sum_{i=1}^{m} \lambda_i P_i^{-1} \right)^{-1}
\]

s.t. \( \lambda_i \geq 0, \ i = 1, \ldots, m, \)

\[
1 - \sum_{i=1}^{m} \lambda_i + \sum_{i=1}^{m} \lambda_i x_i^T P_i^{-1} x_i \geq 0,
\]

where \( \lambda_i, \ i = 1, \ldots, m \) are the optimization variables of this problem. If the above problem is feasible, then there exists an optimal solution for optimization problem (15)-(17). In this case, calling \( \lambda_i^* \) optimal values of the problem variable \( \lambda_i, \ i = 1, \ldots, m \), then the optimal shape matrix and the center of problem (15)-(17) satisfy

\[
P_0^{-1} = \sum_{i=1}^{m} \lambda_i^* P_i^{-1}
\]

\[
x_0 = P_0 \sum_{i=1}^{m} \lambda_i^* P_i^{-1} x_i.
\]

Proof. Using Schur complement, (17) is feasible if and only if

\[
\left[ \sum_{i=1}^{m} \lambda_i P_i^{-1} - P_0^{-1} (\bar{x}_0 - \sum_{i=1}^{m} \lambda_i P_i^{-1} x_i)^T 1 + \sum_{i=1}^{m} \lambda_i (x_i^T P_i^{-1} x_i - 1) - \bar{x}_0 \bar{x} \right] \preceq 0.
\]

where \( \bar{x}_0 = P_0^{-1} x_0 \). Thus, we only need to prove that (22) is feasible if and only if (19) is feasible. It is given from the following two aspects:

“ \( \Leftarrow \) ” Based on the definition of the shape matrix \( P_0 \) and the center \( x_0 \) in (20)-(21),

\[
P_0^{-1} = \sum_{i=1}^{m} \lambda_i P_i^{-1}
\]

\[
x_0 = P_0 \sum_{i=1}^{m} \lambda_i P_i^{-1} x_i.
\]

If the optimal values \( \lambda_i, \ i = 1, \ldots, m \), satisfy the constraint (19), then the optimal variables \( P_0, x_0, \lambda_i, \ i = 1, \ldots, m \) are also feasible for the constraint (22) by Schur complement. That is, that (19) is feasible implies (22) is feasible.
Based on Schur complement, (22) is equivalent to

\[
\sum_{i=1}^{m} \lambda_i P_i^{-1} - P_0^{-1} - \left( \bar{x}_0 - \sum_{i=1}^{m} \lambda_i P_i^{-1} x_i \right) \cdot \left( 1 + \sum_{i=1}^{m} \lambda_i (x_i^T P_i^{-1} x_i - 1) - \bar{x}_0^T P_0 \bar{x}_0 \right)^{-1} \cdot \left( \bar{x}_0 - \sum_{i=1}^{m} \lambda_i P_i^{-1} x_i \right)^T \succeq 0.
\] (23)

Both in the case of trace and logdet function, \( f(X_1) \geq f(X_2) \) whenever \( X_1 \succeq X_2 \). Then, according to the decoupled technique in [12], the minimum of \( f(P_0) \) is achieved, when \( P_0 = \left( \sum_{i=1}^{m} \lambda_i P_i^{-1} \right)^{-1} \) and \( \bar{x}_0 = \sum_{i=1}^{m} \lambda_i P_i^{-1} x_i \). Thus, (22) is feasible implies (19) is feasible.

All in all, the problem (18)-(19) is equivalent to the optimization problem (15)-(17).

Algorithm 3.4. Decoupled SDP relaxation

1. Solve the optimization problem (18)-(19) for the optimal solutions \( \lambda_i, i = 1, \ldots, m \).
2. Compute the shape matrix \( P_0 \) and the center \( x_0 \) by (20)-(21).

Remark 3.5.

- Proposition 3.3 shows that the shape matrix \( P_0 \) and the center \( x_0 \) (20)-(21) of the minimum ellipsoid derived by the decoupled relaxation method are the weighted combination of those of multiple ellipsoids. The optimization problem (18)-(21) is also a significant bridge for deriving the equivalence among three typical relaxation methods in next section.

- From Proposition 3.3, we can calculate a suboptimal ellipsoid to contain the intersection \( \mathcal{F} \) by (18)-(21), which is described by Algorithm 3.4. Comparing the optimization problem (18)-(19) with (15)-(17), it is easy to see that the dimension of the constraint matrix (19) of Algorithm 3.4 is \( n + 1 \), which is smaller than that of matrix (17). Meanwhile, the number of the decision variables of the optimization problem (18)-(19) is \( m \), but that of the optimization problem (15)-(17) is \( m + \frac{n(n+1)}{2} + n \). Therefore, if we use Proposition 3.3 to calculate the ellipsoid containing the intersection of the ellipsoids, the computing time may be reduced much more, which can be clearly seen in Table 1 in the numerical section.

4 S-Procedure Relaxation

In system and control theory, one often encounters the constraint that a quadratic form is negative when other quadratic forms are all negative. In some cases, this constraint can be relaxed as a linear matrix inequality
(LMI) based on S-procedure method. Actually, the ellipsoids can be written as quadratic forms in (5), therefore, [1] uses the S-procedure relaxation technique to deal with the constrain condition (2).

### 4.1 The Equivalence Between S-Procedure Relaxation and SDP Relaxation

**Lemma 4.1.** (S-procedure)[33]: Let \( F_0(\xi), F_1(\xi), \ldots, F_m(\xi) \), be quadratic functions in variable \( \xi \in \mathbb{R}^n \)

\[
F_i(\xi) = \xi^T T_i \xi, \quad i = 0, \ldots, m
\]  

with \( T_i = T_i^T \). Then the implication

\[
F_1(\xi) \leq 0, \ldots, F_m(\xi) \leq 0 \Rightarrow F_0(\xi) \leq 0
\]  

holds if there exist \( \tau_1, \ldots, \tau_m \geq 0 \) such that

\[
F_0(\xi) - \sum_{i=1}^{m} \tau_i F_i(\xi) \leq 0, \text{ for all } \xi,
\]  

or

\[
T_0 - \sum_{i=1}^{m} \tau_i T_i \preceq 0.
\]  

It is a nontrivial fact that when \( m = 1 \), the converse holds [11].

Now, a suboptimal ellipsoid for the intersection \( \mathcal{F} \) can be derived by the S-procedure relaxation technique. Denoting \( \xi = [x^T 1]^T \), then (4) can be equivalent to

\[
f_i(X, x) = \xi^T A_i \xi \leq 0, \quad 0 \leq i \leq m,
\]  

where

\[
A_i = \begin{bmatrix}
P_i^{-1} & -P_i^{-1} x_i \\-x_i^T P_i^{-1} & x_i^T P_i^{-1} x_i - 1
\end{bmatrix}, \quad 0 \leq i \leq m.
\]  

Thus, the problem \( \mathcal{E}_0 \supset \bigcap_{i=1}^{m} \mathcal{E}_i \) is equivalent to

\[
f_0(X, x) = \xi^T A_0 \xi \leq 0
\]  

whenever

\[
f_i(X, x) = \xi^T A_i \xi \leq 0, \quad 1 \leq i \leq m.
\]  

By S-procedure, we can obtain \( \mathcal{E}_0 \) containing \( \bigcap_{i=1}^{m} \mathcal{E}_i \) if there exist \( \tau_1, \ldots, \tau_m \geq 0 \) such that

\[
A_0 \leq \sum_{i=1}^{m} \tau_i A_i.
\]
Using the definition of $A_i$ in (29), so we can write the equation (32) as an LMI
\[
\begin{bmatrix}
P_0^{-1} & -P_0^{-1}x_0 \\
-x_0^T P_0^{-1} & x_0^T P_0^{-1} x_0 - 1
\end{bmatrix} \leq \sum_{i=1}^{m} \tau_i \begin{bmatrix}
P_i^{-1} & -P_i^{-1}x_i \\
-x_i^T P_i^{-1} & x_i^T P_i^{-1} x_i - 1
\end{bmatrix}.
\]
(33)

Then, based on the S-procedure relaxation technique, the best such outer ellipsoid of the intersection $\mathcal{F}$ can be derived by solving the following optimization problem
\[
\begin{align*}
\min & \quad f(P_0) \\
\text{s.t.} & \quad \tau_i \geq 0, i = 1, \ldots, m, \\
& \quad \begin{bmatrix}
P_0^{-1} & -P_0^{-1}x_0 \\
-x_0^T P_0^{-1} & x_0^T P_0^{-1} x_0 - 1
\end{bmatrix} \leq \sum_{i=1}^{m} \tau_i \begin{bmatrix}
P_i^{-1} & -P_i^{-1}x_i \\
-x_i^T P_i^{-1} & x_i^T P_i^{-1} x_i - 1
\end{bmatrix}.
\end{align*}
\]
(35)

with variables $P_0, x_0, \tau_i, i = 1, \ldots, m$. Interestingly, replacing the variable $\tilde{x}_0$ by $\tilde{x}_0 = P_0^{-1} x_0$, (35) is equivalent to (17) by Schur complement (see the proof of Lemma 3.1). Therefore, we can obtain the following proposition.

**Proposition 4.2.** The optimization problem (15)-(17) based on the SDP relaxation technique is equivalent to the optimization problem (34)-(35) by the S-procedure relaxation method, i.e., they can derive the same minimum ellipsoid to contain the intersection of these ellipsoids.

### 4.2 S-Procedure Relaxation with Maximum Volume Inscribed Ellipsoid

Another method that can derive a suboptimal solution for the original problem (1)-(2) is to first calculate the maximum volume ellipsoid that is contained in the intersection, which can be also cast as convex problem with LMI constrains [25]. If we use Lemma 4.3 to shrink this ellipsoid by a factor of $n$ about its center, then it is guaranteed to contain the intersection.

**Lemma 4.3.** [25] Let $\mathcal{E}_{ij}$ be the Löwner-John ellipsoid\(^1\) of the convex set $C \subseteq \mathbb{R}^n$, which is a compact set with a nonempty interior, and let $x_{ij}$ be its center. Then,
\[
x_{ij} + \frac{1}{n}(\mathcal{E}_{ij} - x_{ij}) \subseteq C \subseteq \mathcal{E}_{ij}.
\]
(36)

Now, we consider the problem of finding a maximum volume ellipsoid that lies inside the intersection $\mathcal{F}$. Let ellipsoid $\mathcal{E}_0$ is contained in the intersection of the ellipsoids $\mathcal{E}_1, \ldots, \mathcal{E}_m$, and it is defined as follows
\[
\mathcal{E}_0 = \{x \in \mathbb{R}^n : (x - \tilde{x}_0)^T P_0^{-1} (x - \tilde{x}_0) \leq 1\}
\]
(37)
\[
\mathcal{E}_0 = \{x \in \mathbb{R}^n : \xi^T \tilde{A}_0 \xi \leq 0, \xi = [x^T 1]^T\}.
\]
(38)

\(^1\)The minimum volume ellipsoid that contains a set $C$ is called the Löwner-John ellipsoid of the set $C$.\]
where
\[
\tilde{A}_0 = \begin{bmatrix}
\tilde{P}_0^{-1} & -\tilde{P}_0^{-1}x_0 \\
-x_0^T\tilde{P}_0^{-1} & x_0^T\tilde{P}_0^{-1}x_0 - 1
\end{bmatrix}.
\] (39)

Since \(\tilde{E}_0 \subseteq E_i\) if and only if for every \(x\) satisfying
\[
(x - \bar{x}_0)^T\tilde{P}_0^{-1}(x - \bar{x}_0) \leq 1,
\] (40)
we have
\[
(x - x_i)^T\tilde{P}_i^{-1}(x - x_i) \leq 1, \; i = 1, \ldots, m.
\] (41)
Then, by Lemma 4.1, this is equivalent to that there exist nonnegative scalars \(\tau_1, \ldots, \tau_m\) satisfying
\[
(x - x_i)^T\tilde{P}_i^{-1}(x - x_i) - \tau_i(x - \bar{x}_0)^T\tilde{P}_0^{-1}(x - \bar{x}_0) \leq 1 - \tau_i,
\] (42)
\[
\tau_i \geq 0, \; i = 1, \ldots, m, \; \forall x \in \mathbb{R}^n.
\] (43)
Using Schur Complements, \(1\) has proved that (42) is equivalent to the following LMI
\[
\begin{bmatrix}
-P_i & x_i - \bar{x}_0 & \tilde{E}_0 \\
(x_i - \bar{x}_0)^T & \tau_i - 1 & 0 \\
\tilde{E}_0^T & 0 & -\tau_i I
\end{bmatrix} \preceq 0, \; i = 1, \ldots, m,
\] (43)
where \(\tilde{P}_0 = \tilde{E}_0^T\tilde{E}_0\), \(I\) is identity matrix. Therefore, we can obtain the optimal maximum volume ellipsoid contained in the intersection of the ellipsoids \(E_1, \ldots, E_m\) by solving the problem
\[
\max \log \det(\tilde{E}_0)
\] (44)
\[
s.t. \; \tau_i \geq 0, \; i = 1, \ldots, m, \; (45)
\]
with variables \(\tilde{E}_0, \; \bar{x}_0, \; \tau_i, \; i = 1, \ldots, m\). Next, we can use Lemma 4.3 to achieve an ellipsoid containing the intersection of the ellipsoids. Algorithm 4.4 gives us the specific process to calculate an outer ellipsoid, indeed, it is a rather conservative approximation \(34\).

**Algorithm 4.4.** S-procedure relaxation with maximum volume inscribed ellipsoid

1. Solve the optimization problem (44)-(45) of determining the maximum volume ellipsoid for the optimal solutions \(\tilde{E}_0\) and \(\bar{x}_0\).

2. Compute the shape matrix \(\tilde{P}_0\) and the center \(x_0\) of the outer ellipsoid by (36).

### 5 Bounding Ellipsoid Relaxation

The authors \(11\) divide the problem of computing the minimum bounding ellipsoid of the intersection of ellipsoids into two steps. First, one needs to find a parameterized ellipsoid to contain the intersection. Second, the optimal ellipsoid can be derived by optimizing the parameter.
5.1 The Equivalence Between Bounding Ellipsoid Relaxation and Decoupled SDP Relaxation

Assume that the intersection \( \mathcal{F} \) of \( m \) ellipsoids is a nonempty bounded region, which is defined by (3), i.e.,

\[
\mathcal{F} = \bigcap_{i=1}^{m} \{ x : (x - x_i)^T P_i^{-1} (x - x_i) \leq 1, \ i = 1, \ldots, m \}. \tag{46}
\]

Let \( \mathcal{D}^+ \) be the convex set of all vectors \( t = (t_1, \ldots, t_m) \in \mathbb{R}^m \), with \( t_i \geq 0, i = 1, \ldots, m \), and \( \sum_{i=1}^{m} t_i = 1 \). If \( x \in \mathcal{F} \), then

\[
\sum_{i=1}^{m} t_i (x - x_i)^T P_i^{-1} (x - x_i) \leq 1, \ \forall t \in \mathcal{D}^+. \tag{47}
\]

After simple calculations and transformations, we can obtain the following equation,

\[
\sum_{i=1}^{m} t_i (x - x_i)^T P_i^{-1} (x - x_i) = (x - x_t)^T P_t^{-1} (x - x_t) + \delta_t, \tag{48}
\]

where

\[
P_t^{-1} = \sum_{i=1}^{m} t_i P_i^{-1} \tag{49}
\]

\[
x_t = P_t \sum_{i=1}^{m} t_i P_i^{-1} x_i \tag{50}
\]

\[
\delta_t = \sum_{i=1}^{m} t_i x_i^T P_i^{-1} x_i - x_t^T P_t^{-1} x_t. \tag{51}
\]

Then, \( \mathcal{F} \) can be rewritten as follows [11, 13],

\[
\mathcal{F} = \{ x : (x - x_t)^T P_t^{-1} (x - x_t) \leq 1 - \delta_t, \ \forall t \in \mathcal{D}^+ \}. \tag{52}
\]

Obviously, (52) may not be an ellipsoid. Since \( \mathcal{F} \) is a nonempty set, then \( \delta_t \leq 1 \). Let the ellipsoid \( \mathcal{E}_t \) denoted as \( \mathcal{E}_t = \{ x : (x - x_t)^T ((1 - \delta_t) P_t)^{-1} (x - x_t) \leq 1 \} \), then \( \mathcal{F} \subseteq \mathcal{E}_t \). Therefore, we have found an ellipsoid to contain the intersection of multiple ellipsoids, but it depends on the parameter \( t_i, i = 1, \ldots, m \). In order to derive an optimal ellipsoid \( \mathcal{E}_t \), we need to minimize the function of the shape matrix \( (1 - \delta_t) P_t \). The optimization problem can be written as follows

\[
\min \ f((1 - \delta_t) P_t) \tag{53}
\]

s.t. \((t_1, \ldots, t_m) \in \mathcal{D}^+ \), \tag{54}
where the vector \( t = (t_1, \ldots, t_m) \) is the optimization variable of this problem.

If we can obtain the optimal solution \( t^* \) of the optimization problem (53)-(54), then according to (49)-(50), an outer suboptimal ellipsoid for intersection \( F \) can be derived. In Section 3.2, we also derive the analytic expressions (20)-(21) of the shape matrix and the center of the minimum ellipsoid by the decoupled SDP relaxation method, interestingly, they are similar to the equations (49)-(50). In fact, the optimization problems (53)-(54) and (18)-(19) are equivalent for calculating the weight of the each ellipsoid but not in form.

**Proposition 5.1.** When the objective function is logdet or trace function, then the optimization problem (53)-(54) based on the bounding ellipsoid relaxation technique is equivalent to the optimization problem (18)-(19) based on the decoupled SDP relaxation method, i.e., they have the same minimum ellipsoid to contain the intersection \( F \).

**Proof.** The proof has three steps. Firstly, when the objective function is logdet function, we continue to simplify the optimization problem (18)-(19). Let \( \eta = \sum_{i=1}^m \lambda_i \) and \( t_i = \frac{\lambda_i}{\eta}, i = 1, \ldots, m \), then \( \sum_{i=1}^m t_i = 1 \). Since \( \eta > 0 \), if the constraint (19) is scaled by a factor of \( \frac{1}{\eta} \), then the optimization problem (18)-(19) can be rewritten as

\[
\begin{align*}
\min & \quad - \log \det \left( \sum_{i=1}^m t_i P_i^{-1} \right) - n \cdot \log(\eta) \\
\text{s.t.} & \quad \frac{1}{\eta} - \sum_{i=1}^m t_i + \sum_{i=1}^m t_i x_i^T P_i^{-1} x_i - \sum_{i=1}^m t_i x_i^T P_i^{-1} x_i \geq 0 \\
& \quad \sum_{i=1}^m t_i = 1, \quad t_i \geq 0, \quad i = 1, \ldots, m.
\end{align*}
\]  

Let \( \xi = \frac{1}{\eta} \), and use Schur Complements, the above optimization problem (55)-(57) is equivalent to the following problem

\[
\begin{align*}
\min & \quad - \log \det \left( \sum_{i=1}^m t_i P_i^{-1} \right) + n \cdot \log(\xi) \\
\text{s.t.} & \quad \xi - \sum_{i=1}^m t_i + \sum_{i=1}^m t_i x_i^T P_i^{-1} x_i \\
& \quad - \sum_{i=1}^m t_i x_i^T P_i^{-1} \left( \sum_{i=1}^m t_i P_i^{-1} \right)^{-1} \sum_{i=1}^m t_i P_i^{-1} x_i \geq 0 \\
& \quad \sum_{i=1}^m t_i = 1, \quad t_i \geq 0, \quad i = 1, \ldots, m.
\end{align*}
\]  

Secondly, because of the monotonicity of the log function, we can introduce an optimization variable \( \xi \), then
The optimization problem (15)-(17) based on SDP relaxation

Proposition 2

Proposition 3

The optimization problem (18)-(21) based on decoupled SDP relaxation

The optimization problem (34)-(35) based on S-procedure relaxation

The optimization problem (53)-(54) based on bounding ellipsoid relaxation

Figure 1: The equivalence among these relaxation techniques.

the optimization problem (53)-(64)

\[
\begin{align*}
\min & \quad n \cdot \log(1 - \delta_t) - \log \det \left( \sum_{i=1}^{m} t_i P_i^{-1} \right) \\
\text{s.t.} & \quad (t_1, \ldots, t_m) \in \mathcal{D}^+ \quad (61)
\end{align*}
\]

is equivalent to

\[
\begin{align*}
\min & \quad n \cdot \log(\xi) - \log \det \left( \sum_{i=1}^{m} t_i P_i^{-1} \right) \\
\text{s.t.} & \quad \xi - 1 + \delta_t \geq 0, \quad (t_1, \ldots, t_m) \in \mathcal{D}^+. \quad (64)
\end{align*}
\]

Thirdly, according to the definition of \(\delta_t\) in (51), it is shown that the optimization problems (63)-(65) and (58)-(60) have the same optimal solution. When the objective function is trace function, the proof is similar. Thus the proof of Proposition 5.1 is finished.

**Remark 5.2.** Based on Proposition 3.3, Proposition 4.2 and Proposition 5.1, we can derive that the SDP relaxation method, S-procedure method and bounding ellipsoid relaxation method can achieve the same solution for the optimization problem (1)-(2). The equivalence among these important relaxation techniques is showed in Fig. 1.

### 5.2 The Significance of the Equivalence

Actually, in [11], the authors have proved \(\delta_t \geq 0\), then the ellipsoid \(\mathcal{E}_t' = \{ x : (x - x_t)^T P_t^{-1} (x - x_t) \leq 1 \} \supseteq \mathcal{E}_t\). As the optimization problem (53)-(54) may lead to high computational complexity, then they advise to find the optimal value of vector \(t\) by deleting \((1 - \delta_t)\) in (53), which yields the following optimization problem

\[
\begin{align*}
\min & \quad f(P_t) \\
\text{s.t.} & \quad (t_1, \ldots, t_m) \in \mathcal{D}^+. \quad (67)
\end{align*}
\]
Here the vector $t = (t_1, \ldots, t_m)$ is the optimization variable. If the optimal value $t^*$ of $t$ takes place within the ellipsoid $\mathcal{E}'$, the final ellipsoidal approximation is improved by taking

$$\mathcal{E}_{t^*} = \{ x : (x - x_{t^*})^T((1 - \delta_{t^*})P_{t^*})^{-1}(x - x_{t^*}) \leq 1 \}. \quad (68)$$

Finally, we can use Algorithm 5.3 to obtain a suboptimal ellipsoid containing the intersection of these ellipsoids $\mathcal{E}_1, \ldots, \mathcal{E}_m$.

Algorithm 5.3. Bounding ellipsoid relaxation without optimizing $1 - \delta_t$

1. Solve the optimization problem (66)-(67) for the optimal solutions $t_i$, $i = 1, \ldots, m$.

2. Compute the shape matrix $P_t$ and center $x_t$ of the outer ellipsoid by (49)-(50). Meanwhile, calculate $\delta_t$ by (51).

3. Obtain the suboptimal ellipsoid with the shape matrix $P_0 = (1 - \delta_t)P_t$ and the center $x_0 = x_t$.

Corollary 5.4. When the objective function is logdet or trace function, compared with Algorithm 5.3, Algorithm 3.4 based on the decoupled SDP relaxation method can derive a tighter ellipsoid to contain the intersection.

Proof. Since the optimal solution of the optimization problem (66)- (68) is feasible for the problem (53)-(54), according to Proposition 5.1, we can obtain the result of Corollary 5.4.

Remark 5.5. Algorithm 5.3 is widely applied to set-membership filter for target tracking [4], since it has less computing time, which can be seen in Table 1. However, based on Corollary 5.4, the minimum ellipsoid calculated by Algorithm 3.4 can be tighter than that by Algorithm 5.3. Therefore, if Algorithm 3.4 is used in set-membership filter, it can achieve a better estimation performance (see Figs. 3-4).

In the distributed estimation fusion setting, if the cross-correlation of local sensor estimation errors is unknown or impractical, the covariance intersection (CI) algorithm [18, 19, 17] is widely used to deal with this problem. Notice the definition of $P_t$ in (49), interestingly, the steps 1-2 of Algorithm 5.3 is same as the CI algorithm. Based on Proposition 5.1 and $0 \leq \delta_t \leq 1$, we can obtain the following corollary.

Corollary 5.6. Compared with CI algorithm, Algorithm 3.4 based on the decoupled SDP relaxation method can derive a tighter ellipsoid containing the intersection of ellipsoids.

Therefore, if we use the decoupled SDP relaxation method to solve the distributed estimation fusion with unavailable cross-correlation among multiple sensors, the estimation performance may be further improved (see Figs. 5-6).

A recursive version of the problems in (66)-(68) is also considered by [11]. Let $\mathcal{E}^k$ be the approximate ellipsoid obtained after processing the first $k$ ellipsoids $\mathcal{E}_1, \ldots, \mathcal{E}_k$. The center and the shape matrix of
ellipsoid $E^k$ are denoted by $x^k$ and $P^k$. The next approximation is to find $E^{k+1}$ containing $E^k \cap E_{k+1}$. The recursive algorithm is formulated in Algorithm 5.7.

**Algorithm 5.7.** Recursive bounding ellipsoid relaxation without optimizing $1 - \delta_t$

1. **Initialized**, $x^1 = x_1$ and $P^1 = P_1$.

2. Calculate the shape matrix $P_t^{k+1} = (t(P_k)^{-1} + (1 - t)P_{k+1}^{-1})^{-1}$ based on (49).

3. Compute the center $x_t^{k+1} = P_t^{k+1}(t(P_k)^{-1}x_k + (1 - t)P_{k+1}^{-1}x_{k+1})$ and $\delta_t$ based on (50) and (51), respectively.

4. Minimize the objective function $f(P_t^{k+1})$ in (66)-(67), and obtain the optimal value $t^*$.

5. Achieve the shape matrix $P^{k+1} = (1 - \delta_t)P_t^{k+1}$ and the center $x^{k+1} = x_t^{k+1}$ of the ellipsoid $E^{k+1}$.

6. Go to step 2 until $k = m$.

**Remark 5.8.**

- If the steps 4-5 are replaced by solving the optimization problem (18)-(21), then Algorithm 5.7 may get a tighter ellipsoid to contain the intersection.

- Algorithm 5.7 may be applied to the distributed estimation fusion for delay systems [35], but it generates a more pessimistic final ellipsoid than the nonrecursive algorithm. If we modify the shape matrix to $P^{k+1} = P_t^{k+1}$ in the step 5, then it is similar to the sequential CI Kalman fusion [20].

6 Simulation Results

In this section, in order to show the analytic results of the equivalence and the computational complexity, we consider two cases: static case and dynamic case. The following simulation results are under Matlab R2015a with CVX.

1) **Static Case:** Since the optimization problem (15)-(17) by the SDP relaxation method is same as (34)-(35) by the S-procedure relaxation method, they have the same computational complexity. For the optimization problem (53)-(54) by the bounding ellipsoid relaxation method, Durieu, Walter and Polyak [11] recognized that the direct acquisition of the optimal value of the problem (53)-(54) becomes marginal in most examples treated so far, since it is at cost of more computation. Therefore, we focus on comparing the computational complexity of the SDP relaxation method, the decoupled SDP relaxation method and the other three algorithms.
Suppose that there exist three local estimated ellipsoids $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$, the shape matrix and the center are denoted as follows

$$P_1 = \begin{bmatrix} 6 & -5 \\ -5 & 12 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 10 & 1 \\ 1 & 3 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 5 & 5 \\ 5 & 9 \end{bmatrix}$$

$$x_1 = [12 \ 11]^T, \quad x_2 = [12 \ 10], \quad x_3 = [12 \ \xi].$$

Here, $\xi$ is a random variable with uniform distribution in interval $[9 \ 10]$. The goal is determining an ellipsoid $\mathcal{E}$ containing the intersection of the three ellipsoids. The matrix $P$ is denoted by the shape matrix ellipsoid $\mathcal{E}$. The objective function $f(P)$ is $\log\det(P)$, which means the volume of the ellipsoid $\mathcal{E}$.

We use the above five algorithms to compute the ellipsoid $\mathcal{E}$ containing the intersection of the three ellipsoids. Table 1 shows the performance of the above five algorithms. Meanwhile, the values of $\log\det(P)$ and the computing time are calculated by the average of 100 Monte Carlo runs. It is easy to see that Algorithm 3.2 and Algorithm 3.4 do derive the same ellipsoid to contain the intersection, which is consistent with the result of Proposition 3.3. But the computing time of Algorithm 3.4 is less than that of Algorithm 3.2, the reason is that Algorithm 3.4 has less optimization variables (see Remark 3.5).

Fig. 2 shows that the minimum ellipsoid derived by Algorithm 3.2 can contain the intersection of the three ellipsoids. Moreover, by Fig. 1 Table 1 Remark 5.5 and Remark 5.8 Algorithm 3.2 and 3.4 may provide the smallest volume ellipsoid compared with the other three algorithms. Although Algorithm 4.4 can produce a maximum volume ellipsoid inscribed in the intersection of the three ellipsoids, this ellipsoid is shrunk by a factor of $n$ about its center, then it is only guaranteed to contain the intersection. If we consider the computing time, the Algorithm 5.3 might be the appropriate choice. By Table 1 and Remark 5.8 Algorithm 5.7 generates a more pessimistic ellipsoid than Algorithm 5.3 but this recursive ellipsoid algorithm can be applied to the distributed estimation fusion for delay systems.

2) Dynamic Case: Consider a constant-velocity moving target with three local sensors, and the dynamic...
Figure 2: Using Algorithm 3.4 to calculate a best outer ellipsoid containing the intersection of three ellipsoids.

The model is as follows:

\[
\begin{align*}
x_{k+1} &= F_k x_k + w_k \\
y_{k+1}^i &= x_{k+1} + v_{k}^i, \quad i = 1, 2, 3,
\end{align*}
\]

where \(F_k = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}\) and sampling time \(T = 1\). The process noise and measurement noise are assumed to be confined to specified ellipsoidal sets

\[
\begin{align*}
W_k &= \{ w_k : w_k^T Q_k^{-1} w_k \leq 1 \} \\
V_k &= \{ v_k : v_k^T R_k^{-1} v_k \leq 1 \}, \quad i = 1, 2, 3,
\end{align*}
\]

where

\[
\begin{align*}
Q_k &= \begin{bmatrix} T^3 & T^2 \\ T^2 & T \end{bmatrix} \\
R_k^1 &= \begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix}, \quad R_k^2 = \begin{bmatrix} 18 & 0 \\ 0 & 22 \end{bmatrix}, \quad R_k^3 = \begin{bmatrix} 22 & 0 \\ 0 & 18 \end{bmatrix}.
\end{align*}
\]

The goal of this example is to find a minimum ellipsoid to contain the true state by the set-membership filter and the distributed estimation fusion, respectively. In general, the set-membership filter contains two steps: predicted step and updated step.

The predicted step of a single sensor is to compute a predicted ellipsoid \(\mathcal{E}_{k+1|^k}^i\) containing the true state by the state function and an updated ellipsoid \(\mathcal{E}_{k|^k}^i\), \(i = 1, 2, 3\). Meanwhile, the shape matrix \(P_{k+1|^k}^i\) and the
center $x_{k+1|k}^i$ of the predicted ellipsoid are calculated as follows [11]:

$$P_{k+1|k}^i = \frac{1}{\tau_1} F_k P_{k|k}^i F_k^T + \frac{1}{\tau_2} Q_k$$

$$x_{k+1|k}^i = F_k x_{k|k}^i,$$

where

$$\tau_1^i = \frac{\sqrt{\text{trace}(F_k P_{k|k}^i F_k^T)}}{\sqrt{\text{trace}(F_k P_{k|k}^i F_k^T) + \text{trace}(Q_k)}}$$

$$\tau_2^i = \frac{\sqrt{\text{trace}(Q_k)}}{\sqrt{\text{trace}(F_k P_{k|k}^i F_k^T) + \text{trace}(Q_k)}},$$

$x_{k|k}^i$ and $P_{k|k}^i$ are the center and the shape matrix of the updated ellipsoid $E_{k|k}^i$ at time $k$, respectively.

The updated step is to determine an updated ellipsoid $E_{k+1|k+1}^i$ to contain the intersection of the predicted ellipsoid $E_{k+1|k}^i$ and the measurement ellipsoid by each sensor, where the measurement ellipsoid is defined by $V_{k+1}^i = \{ x_{k+1} : (y_{k+1}^i - x_{k+1})^T R_{k+1}^{-1} (y_{k+1}^i - x_{k+1}) \leq 1 \}$. In this example, the target starts with $x_0 = [1 1]^T$. Assume that the center and the shape matrix of the initial bounding ellipsoid are $\hat{x}_0 = [2 2]^T$ and $P_0 = \text{diag}(50, 50)$, respectively. The following simulation results are calculated by the average of 100 Monte Carlo runs.

Firstly, we compare the estimation precision of Algorithm 3.4 with that of Algorithm 5.3 by a single sensor. Since Algorithm 5.3 is widely applied to set-membership filter in updated step, we use the measurement of sensor 1 to calculate the updated ellipsoid $E_{k+1|k+1}^1$ by Algorithm 3.4 and Algorithm 5.3, respectively. Figs. 3-4 show the root mean square error (RMSE) of the state estimation and the volume of the updated ellipsoid $E_{k+1|k+1}^1$. It is clear to see that Algorithm 3.4 based on the decoupled SDP method performs better than Algorithm 5.3 which is consistent with the result of Corollary 5.4. Therefore, if we want to obtain the higher estimation precision, Algorithm 3.4 is a better choice.

Secondly, in the fusion center, we compare the estimation precision of Algorithm 3.4 with that of CI method. Assume that each sensor can obtain an updated ellipsoid $E_{k+1|k+1}^i$ by the set-membership filter with Algorithm 3.4, $i = 1, 2, 3$, and these local updated ellipsoids are sent to the fusion center. The goal of the fusion center is to calculate an optimal ellipsoid containing the intersection of $E_{k+1|k+1}^i$, $i = 1, 2, 3$. Based on Algorithm 3.4 and CI method, we can derive the fused ellipsoid, respectively. The results are plotted in Figs. 5-6. From Figs. 5-6 we can see that the estimation precisions of Algorithm 3.4 and CI method are higher than that of each sensor. The Algorithm 3.4 performs the best, which is also consistent with the result of Corollary 5.6.
Conclusions and Future Work

This paper has investigated various solving techniques for the optimization problem that determining a minimum ellipsoid containing the intersection of multiple ellipsoids. In fact, this optimization problem is difficult to solve due to that there exist the infinite number of constraints. Moreover, it is also a non-convex optimization problem, which needs to compute the maximum of the quadratic function. Therefore, there are many researchers address itself to relax this hard optimization problem to a convex optimization problem, which can be solved effectively by interior point methods. There are three major relaxation methods involving SDP relaxation, S-procedure relaxation and bounding ellipsoid relaxation, which are derived by different ideas or viewpoints. However, it is unclear for the interrelationships among these methods and insight to the pros and cons of these methods. This paper has revealed the equivalence among the three relaxation methods by three stages. Firstly, the SDP relaxation method can be equivalently simplified to a decoupled SDP relax-
Figure 5: Comparison of the RMSE of state estimation in the fusion center

Figure 6: Comparison of the volume of fused ellipsoid in the fusion center

We believe that this important problem, determining a minimum ellipsoid containing the intersection of multiple ellipsoids, is worth studying in future work, since it can be widely applied to many practiced fields. The future researches may involve:
• Since this optimization problem is NP-hard, we may aim at finding a better relaxation technique to derive a tighter ellipsoid that containing the intersection of the ellipsoids.

• There are many randomized methods to deal with a lot of complex problems [36, 37], then whether these randomized methods can be applied to this optimization problem.

• For large-scale problems, the interior point method may not be suitable for solving SDP. Then we hope to derive a first order method to solve these relaxed optimization problems.

8 Appendix

Lemma 8.1. (Schur Complements): Given constant matrices $A$, $B$, $C$, where $C = C^T$ and $A = A^T$, then the condition

$$\begin{bmatrix}
A & B \\
B^T & C
\end{bmatrix} \succeq 0$$

is equivalent to

$$A \succeq 0, C - B^T A^+ B \succeq 0, (I - A^+ A)B = 0,$$

and also to

$$C \succeq 0, A - B C^+ B^T \succeq 0, (I - C^+ C)B^T = 0,$$

where $A^+$ and $C^+$ denote the Moore-Penrose pseudoinverse of $A$ and $C$, respectively.

Proof. [Proof of Lemma 8.1]: The Lagrangian of (10)-(11) is

$$L(x, X, \lambda_1, \ldots, \lambda_m, G) = \text{tr} \left( \left( P_0^{-1} - \sum_{i=1}^{m} \lambda_i P_i^{-1} + G \right) X \right) - x^T G x$$

$$-2 x_0^T P_0^{-1} x + 2 \sum_{i=1}^{m} \lambda_i x_i^T P_i^{-1} x$$

$$+ x_0^T P_0^{-1} x_0 - 1 - \sum_{i=1}^{m} \lambda_i (x_i^T P_i^{-1} x_i - 1).$$

(72)
Using a Schur complement, we can express the (75)-(76) as following forms based (73),

\[
\begin{pmatrix}
P_0^{-1} x_0 - \sum_{i=1}^{m} \lambda_i P_i^{-1} x_i \\
(-P_0^{-1} x_0 + \sum_{i=1}^{m} \lambda_i P_i^{-1} x_i)^T 
\end{pmatrix}
\begin{pmatrix}
P_0^{-1} - \sum_{i=1}^{m} \lambda_i P_i^{-1} \\
(-P_0^{-1} x_0 + \sum_{i=1}^{m} \lambda_i P_i^{-1} x_i)^T 
\end{pmatrix}^{+}
\begin{pmatrix}
P_0^{-1} x_0 \\
(-P_0^{-1} x_0 + \sum_{i=1}^{m} \lambda_i P_i^{-1} x_i)^T 
\end{pmatrix}
+ x_0^T P_0^{-1} x_0

-1 - \sum_{i=1}^{m} \lambda_i (x_i^T P_i^{-1} x_i - 1),
\]

and Lagrange multipliers \(\lambda_1, \ldots, \lambda_m\) need to satisfy the following constrains

\[
\begin{align*}
\lambda_i &\geq 0, \ i = 1, \ldots, m \\
\sum_{i=1}^{m} \lambda_i P_i^{-1} - P_0^{-1} &\succeq 0.
\end{align*}
\]

From (7), (72)-(73), we can obtain the following inequality

\[
\varphi(x_0, P_0) \leq g(\lambda_1, \ldots, \lambda_m, G).
\]

If \(g(\lambda_1, \ldots, \lambda_m, G) \leq 0\), then \(\varphi(x_0, P_0) \leq 0\), in other words, the feasible set in (7) can be relaxed to the following forms based (73),

\[
\begin{pmatrix}
P_0^{-1} x_0 - \sum_{i=1}^{m} \lambda_i P_i^{-1} x_i \\
(-P_0^{-1} x_0 + \sum_{i=1}^{m} \lambda_i P_i^{-1} x_i)^T 
\end{pmatrix}
\begin{pmatrix}
P_0^{-1} - \sum_{i=1}^{m} \lambda_i P_i^{-1} \\
(-P_0^{-1} x_0 + \sum_{i=1}^{m} \lambda_i P_i^{-1} x_i)^T 
\end{pmatrix}^{+}
\begin{pmatrix}
P_0^{-1} x_0 \\
(-P_0^{-1} x_0 + \sum_{i=1}^{m} \lambda_i P_i^{-1} x_i)^T 
\end{pmatrix}
+ x_0^T P_0^{-1} x_0

-1 - \sum_{i=1}^{m} \lambda_i (x_i^T P_i^{-1} x_i - 1) \leq 0,
\]

\[
\sum_{i=1}^{m} \lambda_i P_i^{-1} - P_0^{-1} \succeq 0,
\]

\[
\lambda_i \geq 0, i = 1, \ldots, m.
\]

Using a Schur complement, we can express the (75)-(76) as

\[
\begin{pmatrix}
P_0^{-1} - \sum_{i=1}^{m} \lambda_i P_i^{-1} \\
(-P_0^{-1} x_0 + \sum_{i=1}^{m} \lambda_i P_i^{-1} x_i)^T 
\end{pmatrix}
\begin{pmatrix}
P_0^{-1} x_0 - \sum_{i=1}^{m} \lambda_i P_i^{-1} x_i \\
(-P_0^{-1} x_0 + \sum_{i=1}^{m} \lambda_i P_i^{-1} x_i)^T 
\end{pmatrix}^{+}
\begin{pmatrix}
P_0^{-1} x_0 \\
(-P_0^{-1} x_0 + \sum_{i=1}^{m} \lambda_i P_i^{-1} x_i)^T 
\end{pmatrix}
\leq 0
\]

or, replacing the variable \(\tilde{x}_0\) by \(\tilde{x}_0 = P_0^{-1} x_0\),

\[
\begin{pmatrix}
P_0^{-1} - \sum_{i=1}^{m} \lambda_i P_i^{-1} \\
(-P_0^{-1} x_0 + \sum_{i=1}^{m} \lambda_i P_i^{-1} x_i)^T 
\end{pmatrix}
\begin{pmatrix}
P_0^{-1} x_0 - \sum_{i=1}^{m} \lambda_i P_i^{-1} x_i \\
(-P_0^{-1} x_0 + \sum_{i=1}^{m} \lambda_i P_i^{-1} x_i)^T 
\end{pmatrix}^{+}
\begin{pmatrix}
P_0^{-1} x_0 \\
(-P_0^{-1} x_0 + \sum_{i=1}^{m} \lambda_i P_i^{-1} x_i)^T 
\end{pmatrix}
\leq 0.
\]

Combine (77) and (78), we can obtain the optimization problem (15)-(17) in Lemma 3.1.
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