Dynamical Symmetry Breaking in Spaces with Constant Negative Curvature

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Abstract

By using the Nambu-Jona–Lasinio model, we study dynamical symmetry breaking in spaces with constant negative curvature. We show that the physical reason for zero value of critical coupling value $g_c = 0$ in these spaces is connected with the effective reduction of dimension of spacetime $1+D \rightarrow 1+1$ in the infrared region, which takes place for any dimension $1+D$. Since the Laplace–Beltrami operator has a gap in spaces with constant negative curvature, such an effective reduction for scalar fields is absent and there are not problems with radiative corrections due to scalar fields. Therefore, dynamical symmetry breaking with the effective reduction of the dimension of spacetime for fermions in the infrared region is consistent with the Mermin–Wagner–Coleman theorem, which forbids spontaneous symmetry breaking in $(1 + 1)$-dimensional spacetime.

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1 Introduction.

It is well known that dynamical symmetry breaking (DSB) presents an attractive alternative to the Higgs mechanism of electroweak symmetry breaking in Standard Model and in a natural way solves the so called hierarchy problem connected with the quadratic divergence of the mass of the Higgs boson (for a general introduction to DSB see [1]). Moreover, DSB allows at least in principle to deduce all relevant parameters of symmetry breaking. However, usually DSB requires strong coupling ($\alpha_c \geq 1$) that essentially restricts the choice of models that can be used and it makes also quantative studying of DSB a difficult problem. Physically, it is easy to understand why $\alpha_c \geq 1$. For the state with the condensate of fermion-antifermion pairs to have lower energy than the trivial vacuum, it is necessary that energy of the corresponding fermion-antifermion bound state be negative. Then, for example, in QED, in view of the uncertainty principle, it implies $\alpha_c \geq 1$. Therefore, it is very interesting to consider situations when DSB takes place in the regime of weak coupling ($\alpha_c \approx 0$).

Two examples of DSB in the regime of weak coupling are known. The first is symmetry breaking in the presence of the Fermi surface (i.e. chemical potential is nonzero). In this case, as well known from the Bardeen–Cooper–Schrieffer theory of superconductivity [2], a bound state forms for any (however small) attraction between fermions. The effective field theory description of this phenomenon based on the renormalization group was developed in [3], where it was shown that renormalization group scaling takes place only in the direction perpendicular to the Fermi surface, therefore, from the viewpoint of renormalization group scaling the effective dimension of spacetime is $1 + 1$. Since in two-dimensional spacetime a bound state forms even in the case of arbitrary small attraction, we obtain that $\alpha_c = 0$ in this
case. (Note that this is one of the key ideas of QCD color superconductivity at finite density in the regime of weak coupling, which has been actively studied in recent years [4, 5]). The other example of DSB in the regime of weak coupling is DSB in external constant magnetic field. This phenomenon was discovered in [3, 4], where it was shown that chiral symmetry is dynamically broken in the Nambu—Jona-Lasinio (NJL) model [8] and QED in external constant magnetic field $B$ for an arbitrary weak interaction, i.e. the critical coupling constant is zero in this case $^{1}$.

The essence of this strong magnetic catalysis [3, 4] is that electrons are effectively $(1 + 1)$-dimensional when their energy is much less than the Landau gap $\sqrt{|eB|}$. The lowest Landau level plays here the role similar to that of the Fermi surface in the BCS theory of superconductivity, leading to dimensional reduction in dynamics of fermion pairing.

Recently another example of DSB in the regime of weak coupling was discovered. By using the NJL-type models it was shown [11, 12] that critical coupling constant is zero in spaces with constant negative curvature (for an excellent review of DSB in curved spacetime see [13]), i.e. chiral symmetry is always broken for any $g > 0$ (note that the fact of impossibility of keeping chiral invariance for free massless fermions in spaces with negative constant curvature was also noted in [14]). The physical explanation of this very interesting fact is lacking. The authors of these works calculated the effective potential for an order parameter and then showed that it has a nontrivial minimum for any $g > 0$. To find an explanation of this result in more physical terms was one of the main motivations of the present work.

$^{1}$The case of the discrete three-dimensional NJL model was considered in [9] and the effect of enhancement of the chiral condensate in supercritical ($g > g_c > 0$) phase of the four-dimensional NJL model was studied in [10].
2 The model.

For our aims it is enough to consider the NJL model in curved spacetime

\[ S = \int \sqrt{-g} d^4x \left[ \sum_{k=1}^{N} \bar{\psi}_k i \gamma^\mu \nabla_\mu \psi_k + \frac{G}{2N} \left( \left( \sum_{k=1}^{N} \bar{\psi}_k \psi_k \right)^2 - \left( \sum_{k=1}^{N} \bar{\psi}_k \gamma_5 \psi_k \right)^2 \right) \right], \]  

(1)

where \( N \) is the number of flavors, \( g = \det(g_{\mu\nu}) \) the determinant of metric, \( \nabla_\mu = \partial_\mu + i \omega_\mu^{ab} \sigma_{ab} \) the covariant derivative with spin connection \( \omega_\mu^{ab} \), and \( \gamma^\mu \) matrices in curved spacetime are expressed through the Dirac \( \gamma^a \) matrices in flat spacetime with the help of vierbeins \( \gamma^a_\mu \). The action (1) is invariant with respect to chiral transformations \( \psi \rightarrow e^{i\gamma_5 \beta} \psi \). For practical calculations in four-fermion theories it is convenient to use the so called auxiliary field method \[13, 16\], where Lagrangian (1) is represented in the equivalent form

\[ L = \sum_{k=1}^{N} \left( i \bar{\psi}_k \gamma^\mu \nabla_\mu \psi_k + \bar{\psi}_k (\sigma + i \gamma_5 \pi) \psi_k \right) - \frac{N}{2G} \sigma^2, \]  

(2)

where \( \sigma \) and \( \pi \) are auxiliary fields. If we integrate over \( \sigma \) and \( \pi \), we obtain the initial action (1). If the field \( \sigma \) acquires a nonzero vacuum expectation value, then obviously fermions acquire mass and chiral symmetry is broken. To find the effective action for the fields \( \sigma \) and \( \pi \), we integrate over the fermion fields. We obtain (without loss of generality one can set \( \pi = 0 \) because it is always possible to restore the dependence on \( \pi \) by requiring chiral symmetry of the effective action)

\[ \Gamma(\sigma_c) = -N \int \sqrt{-g} d^4x \frac{\sigma_c^2}{2G} - i \text{Ln} \text{ Det}(i\gamma^\mu \nabla_\mu - \sigma_c), \]  

(3)

where \( \sigma_c(x) = <0|\sigma|0> \). The effective potential \( V(\sigma_c) \) (we set \( \sigma_c(x) = \text{const} \)) is given by the expression

\[ V(\sigma_c) = -\frac{\Gamma(\sigma_c)}{\int \sqrt{-g} d^4x}, \]  

(4)
Furthermore,

\[
\ln \det (\gamma^\mu \nabla_\mu - \sigma_c) = \text{Tr} \ln (\gamma^\mu \nabla_\mu - \sigma_c) = \\
\frac{1}{2} \text{Tr} \ln (\gamma^\mu \nabla_\mu - \sigma_c) + \frac{1}{2} \text{Tr} \ln (\gamma_5 (\gamma^\mu \nabla_\mu - \sigma_c) \gamma_5) = \\
\frac{1}{2} \text{Tr} \ln ((\gamma^\mu \nabla_\mu - \sigma_c) (-i \gamma^\nu \nabla_\nu - \sigma_c)).
\]

(5)

By using the Schwinger proper time method [17], we get the effective potential (we also perform the Wick rotation)

\[
V(\sigma_c) = N \left( \frac{\sigma_c^2}{2G} + \frac{1}{2} \int_{\frac{1}{\lambda^2}}^{\infty} \frac{ds}{s} \text{tr} <x|e^{-sH}|x> \right)
\]

(6)

where \( H = -(\gamma_E^\mu \nabla_\mu)^2 + \sigma_c^2 \) (\( \gamma_E^\mu \) are Euclidean \( \gamma \)-matrices). Consequently, the gap equation \( \left( \frac{dV}{d\sigma_c}|_{\sigma_c=m=0} = 0 \right) \) is

\[
1 = G \int_{\frac{1}{\lambda^2}}^{\infty} ds \text{tr} <x|e^{-sH}|x>.
\]

(7)

Thus, we need to find the diagonal heat kernel \( \text{tr} <x|e^{-sH}|x> \) in spaces with constant negative curvature. Before doing it we first describe what these spaces are (for a very good introduction see [18]).

The D-dimensional Riemannian space of constant negative curvature \( H^D \) (hyperbolic space) can be described as a hyperboloid

\[
-x_0^2 + x_1^2 + x_2^2 + \ldots + x_D^2 = -a^2
\]

(8)

embedded in (D+1)-dimensional Minkowski space with metric \( ds^2 = -dx_0^2 + dx_1^2 + \ldots + dx_D^2 \). It is easy to show that the Minkowski metric becomes positive definite on the surface given by Eq.(8) (this is the reason why we have chosen the Minkowski metric in the form \((-,-,+,...,+))\). Obviously by construction hyperbolic space has the group of isometry SO(1, D) and is a homogeneous space because any two points on \( H^D \) can be connected by some isometry (all points of this space are equivalent).

By using the parametrization

\[
x_0 = a \cosh \sigma, \quad x_1 = a \sinh \sigma \cos \theta_1, \\
x_2 = a \sinh \sigma \sin \theta_1 \cos \theta_2, \quad x_3 = a \sinh \sigma \sin \theta_1 \sin \theta_2, \ldots,
\]

(9)

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the line element $ds^2 = -dx_0^2 + dx_1^2 + \ldots + dx_D^2$ becomes

$$ds^2 = a^2(d\sigma^2 + \sinh^2 \sigma d\Omega_{D-1}), \quad (10)$$

where $d\Omega_{D-1}$ is the metric on unit $(D-1)$-dimensional sphere and the curvature is equal to

$$R = -\frac{D(D-1)}{a^2}. \quad (11)$$

Recall that for Euclidean space the linear element in spherical coordinates is

$$ds^2 = (dr^2 + r^2 d\Omega_{D-1}). \quad (12)$$

By comparing Eq.(10) and Eq.(12), we see that the difference between flat and hyperbolic space is that the volume of sphere in hyperbolic space grows with radius $r$ as $a^2 \sinh^2 \frac{r}{a}$ instead $r^2$ as in flat space. In the present work we actually consider the $(D+1)$-dimensional ultrastatic spacetime $R \times H^D$, where the components of metric are time independent and the conditions $g_{00} = 1$ and $g_{0i} = 0$ are true in an appropriate system of coordinates (thus, time coordinate describes evolution of fields on $H^D$).

3 The effective reduction.

3.1 Heat kernels.

Since the metric on $R \times H^D$ is time independent, the heat kernel in Eq.(6) trivially factorizes and we are left with the problem of calculation of the heat kernel $\text{tr} \langle x|e^{-sH}|x \rangle$ on the hyperbolic space $H^D$. As we mentioned in Introduction
it was shown that \( g_c = 0 \) in spaces with constant negative curvature. The authors of these works \cite{11, 12} calculated heat kernel in the closed form either using the Schwinger method for calculation of \( \langle x|e^{-sH}|x \rangle \) \cite{17} or expressing it through the spinor Green function, which was obtained as a solution of the corresponding differential equation in \cite{19}. Heat kernel is in a certain sense an integral characteristic. To reveal the underlying dynamics which gives \( g_c = 0 \), we need more detailed information about the system. For this we calculate heat kernel by summing over the eigenfunctions of the Dirac operator on \( H^D \) that allows us to investigate what dynamics is responsible for \( g_c = 0 \) in spaces with negative curvature\footnote{We would like to thank V.P. Gusynin for suggesting this approach.}. To calculate heat kernel in the form of sum over eigenfunctions, we use the method and the results of \cite{20}. To illustrate the method, we first calculate the heat kernel for scalar field \( h_{\text{scalar}} = \langle x|e^{-sA}|x \rangle \), where \( A = -\Delta + m^2a^2 \) and \( -\Delta \) is the Laplace–Beltrami operator on \( H^D \) (we will use this heat kernel when we discuss the role of Goldstone bosons), which is given by

\[
\Delta = \frac{\partial^2}{\partial \sigma^2} + (D - 1) \coth \sigma \frac{\partial}{\partial \sigma} + (\sinh \sigma)^{-2} \Delta_{S^{D-1}}, \tag{13}
\]

where the last term denotes the Laplace–Beltrami operator on the unit sphere \( S^{D-1} \). If the eigenfunctions of the operator (13) are known, then we can insert their complete set in the matrix element for the heat kernel. Then the heat kernel is represented in the form of sum over eigenfunctions

\[
h_{\text{scalar}} = \langle x|e^{-sA^{-1}}|x \rangle = \sum_{\lambda} e^{-s(\lambda + m^2a^2)}|\phi_\lambda(x)|^2, \tag{14}
\]

where \( \phi_\lambda \) are eigenfunctions \((-\Delta \phi_\lambda = \lambda \phi_\lambda\)). It is obvious from Eq.(13) that the equation for eigenfunctions admits the separation of variables, therefore, we seek them in the form \( \phi = f_\lambda(\sigma)Y_{lm} \), where \( Y_{lm} \) are the spherical harmonics on \( S^{D-1} \)

\[
\Delta_{S^{D-1}}Y_{lm} = -l(l + D - 2)Y_{lm}.
\]
Thus, the radial wave functions satisfy the ordinary differential equation

\[ f''_{\lambda} + (D - 1) \coth \sigma f'_{\lambda} + \left[ \lambda - \frac{l(l + D - 2)^2}{\sinh^2 \sigma} \right] f_{\lambda} = 0. \tag{15} \]

The only bounded solutions of Eq.(15) are

\[ f_{\lambda}(\sigma) = C \Gamma \left( \frac{D}{2} \right) \left( \frac{\sinh \sigma}{2} \right)^{1-D/2} P_{-1/2+ir}^{\mu}(\cosh \sigma), \tag{16} \]

where \( P_{\nu}^{\mu}(z) \) are the associated Legendre functions of the first kind \cite{21}, \( r = (\lambda - \rho_D^2)^{1/2} \) is used as a label for the continuum spectrum, \( \rho_D = \frac{D-1}{2} \), and \( C \) is the normalization constant. The asymptotic behaviour of \( P_{\nu}^{\mu}(z) \) for \( |z| \gg 1 \) is

\[ P_{\nu}^{\mu}(z) \approx \frac{2^\nu \Gamma(\nu + 1/2)}{\pi^{1/2} \Gamma(\nu - \mu + 1)} z^\nu + \frac{\Gamma(-\nu - 1/2)}{2^{\nu+1} \pi^{1/2} \Gamma(-\nu - \mu)} z^{-\nu-1}, \tag{17} \]

from which we obtain the asymptotic behavior of the eigenfunctions

\[ f_{\lambda}(\sigma) \simeq C \frac{2^D \Gamma(D/2) \Gamma(ir)}{4\pi^{1/2} \Gamma(\rho_D + ir)} e^{-\rho_D \sigma + ir \sigma} + h.c.. \tag{18} \]

The radial functions are bounded at infinity provided the parameter \( r \) is real, which is equivalent to the condition \( \lambda \geq \rho_D^2 \). Thus, the spectrum of the Laplace–Beltrami operator has a gap which is determined by the curvature and depends on \( D \). Since \( H^D \) is a homogeneous space (i.e. all points are equivalent), the heat kernel does not depend on \( x \) and one can use any point to calculate the heat kernel. In the spherical coordinates it is very convenient to use the origin because as follows from the explicit solutions (Eq.(16)) only modes with \( l = 0 \) are not equal to zero at this point. We normalized eigenfunctions so that \( f_{\lambda}(0) = C \) at \( x = 0 \). The invariant measure defining the scalar product between eigenfunctions is

\[ (f_{\lambda}, f_{\lambda'}) = \Omega_{D-1} \int_0^\infty f_{\lambda}^* f_{\lambda'} \sinh \sigma^{D-1} d\sigma, \tag{19} \]

where \( \Omega_{D-1} \) is the volume of the (D-1)-dimensional sphere and the factor \( \sinh^{D-1} \sigma \) follows from the square root of the determinant of metric. The normalization constant is determined from the usual condition of normalization of eigenfunctions of
The easiest way to calculate this scalar product and determine the normalization constant for eigenfunctions (16) is to use the fact that the scalar product of two eigenfunctions is expressed through the derivative of their Wronskian $W[\cdot, \cdot]$ at an arbitrary point. For us it is most convenient to calculate the Wronskian at infinity because we know the asymptotic behavior of eigenfunctions there. Thus, we obtain

$$
(f_\lambda, f_{\lambda'}) = 2\pi^{D-1} \pi^2 \frac{\Gamma(N)}{2} \lim_{\sigma \to \infty} (1 - \cosh^2 \sigma) W[P^\mu_{\frac{1}{2} - ir}(\cosh \sigma), P^\mu_{\frac{1}{2} + ir}(\cosh \sigma)],
$$

where the limit is taken in the sense of distributions. Thus, we find

$$
|C(r)|^2 = \frac{2}{(4\pi)^{D/2} \Gamma(D/2)} \frac{\Gamma(ir + l + \rho_D)}{\Gamma(ir)} |2^n|^2
$$

for the square of the normalization constant. Since we normalized eigenfunctions as $f_\lambda(0) = C$, the heat kernel is given by

$$
\frac{1}{a^D} \int_0^\infty e^{-\frac{a^2}{2}(r^2 + m^2 a^2)} |C^2(r)| dr,
$$

where we have made the change of variables $r = \sqrt{\lambda - m^2 a^2}$. Note that $C(r)dr$ is the measure of eigenfunctions. It defines the number of states per unit volume in the range $dr$.

We now consider the heat kernel for the Dirac operator on $R \times H^D$. Before doing it we first discuss what we mean by chiral symmetry in spacetimes of arbitrary dimension (we consider again spacetimes whose metric has Minkowski spacetime signature). In Section 2, we described the chirally invariant NJL model in four-dimensional spacetime. As well known chiral symmetry is connected with properties of representations of the Clifford algebra (for a good description of spinors in n-dimensional spacetime see, e.g., [22]).
the $2^{n/2}$-dimensional spinor space. These spinors are reducible with respect to the even subalgebra (generated by products of an even number of Dirac matrices) and split in a pair of $2^{n/2-1}$-component irreducible Weyl spinors ($\gamma_{n+1} = \gamma_0...\gamma_{n-1}$ is an analog of the $\gamma_5$ matrix in n-dimensional spacetime and $\frac{1+\gamma_{n+1}}{2}$ are the corresponding chiral projectors). In odd-dimensional spacetimes, there are two different representations of the Clifford algebra (they differ by the sign of the $\gamma$-matrices) and chiral symmetry is not defined because $\gamma_{n+1}$ is proportional to the unity. In order to define chiral symmetry in odd-dimensional spacetimes, it is the usual practice to assume that fermion fields are in a reducible representation of the Clifford algebra so that we can define an analog of chiral symmetry (for an explicit example in $(2 + 1)$-dimensional spacetime see, e.g., [23]). In what follows we understand chiral symmetry in odd-dimensional spacetimes in this sense.

In the scalar case it is easy to factorize the part of heat kernel, which contains time derivatives. It is a little bit more elaborated for spinors because the time derivative is multiplied by the $\gamma_0$-matrix. By using (see Eq.(5))

$$(i\gamma^\mu \nabla_\mu + m)(i\gamma^\nu \nabla_\nu - m) = (i\nabla_0 + i\vec{\alpha}\gamma_0 \vec{\nabla} + m\gamma_0)\gamma_0\gamma_0(i\nabla_0 + i\gamma_0 \vec{\nabla} - m\gamma_0) =$$

$$(i\nabla_0 - i\vec{\alpha}\vec{\nabla} + m\gamma_0)(i\nabla_0 + i\vec{\alpha}\vec{\nabla} - m\gamma_0) = (i\nabla_0)^2 - (-i\vec{\alpha}\vec{\nabla} + m\gamma_0)^2, (24)$$

where $\vec{\alpha} = \gamma_0 \vec{\gamma}$, we get rid of the $\gamma_0$-matrix. It is no wonder why such an operator for the heat kernel for spatial coordinates appears after the separation of time derivatives. Indeed, the Dirac equation can be written in the form of a Schrödinger equation

$$i\frac{\partial \psi}{\partial t} = H\psi$$

with the Hamiltonian $H = -i\vec{\alpha}\vec{\nabla} + \beta m$, where $\vec{\alpha} = \gamma_0 \vec{\gamma}$ and $\beta = \gamma_0$. Therefore, we immediately recognize our operator $(-i\vec{\alpha}\vec{\nabla} + m\gamma_0)^2$ as the square of the Hamiltonian, which is obviously a positive definite operator. Thus, the gap equation (see Eq.(7))
on $R \times H^D$ is

$$1 = G \int_{\frac{1}{x^2}}^\infty ds \, \text{tr} \, \langle t, x | e^{-s \left( -\nabla_0^2 + (-i\vec{\alpha} \vec{\nabla} + m\gamma_0)^2 \right)} | t, x \rangle,$$

where we again performed the Wick rotation. The contribution of time derivatives to the heat kernel is easy to be found. Therefore, we are left with the problem of calculation of heat kernel on $H^D$ for the operator $(-i\vec{\alpha} \vec{\nabla} + m\gamma_0)^2$.

To calculate the heat kernel for the operator $(-i\vec{\alpha} \vec{\nabla} + m\gamma_0)^2$ we use as in the scalar case expansion in eigenfunctions. The equation for eigenfunctions is

$$(-i\vec{\alpha} \vec{\nabla} + m\gamma_0)\psi_\lambda = \lambda \psi_\lambda.$$

The covariant derivative of a spinor field on $H^D$ can be decomposed in a radial part plus the covariant derivative along the unit $S^{(D-1)}$-sphere. Furthermore, making the decomposition of $2^{D+1}$-dimensional representation in a Dirac-like representation of $\gamma$-matrices, the equation for eigenfunctions takes the form of a coupled system (for more details see [20])

$$i\gamma_1 (\partial_\sigma + \rho_D \coth \sigma) \psi_1 + \frac{1}{\sinh \sigma} i \, \vec{\nabla}_s \psi_1 = -a(\lambda + m)\psi_2,$$  \hspace{1cm} (26)

$$i\gamma_1 (\partial_\sigma + \rho_D \coth \sigma) \psi_2 + \frac{1}{\sinh \sigma} i \, \vec{\nabla}_s \psi_2 = -a(\lambda - m)\psi_1,$$  \hspace{1cm} (27)

where $\psi_{1,2}$ are the $2^{D+1}$-components Weyl spinors and $i \, \vec{\nabla}_s$ is the Dirac operator on $S^{D-1}$. The spinors $\psi_{1,2}$ transform irreducibly under $SO(D)$ so that we can put $\psi_{1,2} = f_{1,2}(\sigma)\chi_{1,2}$, where $\chi_{1,2}$ are spinors on $S^{D-1}$. The eigenvalues of the Dirac operator on $S^{D-1}$ are known to be $\kappa = \pm(l + \rho_D)$, $l = 0, 1, 2, ...$ [24]. The solutions of Eqs.(26) and (27) are given in terms of hypergeometric functions as follows:

$$f_1^+(\sigma) = A \lambda + m \frac{l + N/2}{4\lambda} \left( \frac{\lambda - m}{4\lambda} \right)^\frac{1}{2} (1 + z)^\frac{1}{2}(z - 1)^\frac{1}{4} F \left( \alpha, \alpha^*; l + \rho_N + \frac{3}{2}, \frac{1 - z}{2} \right),$$

$$f_2^+(\sigma) = A \left( \frac{\lambda - m}{4\lambda} \right)^\frac{1}{2} (1 + z)^\frac{1}{4} (z - 1)^\frac{1}{4} F \left( \alpha, \alpha^*; l + \rho_N + \frac{1}{2}, \frac{1 - z}{2} \right),$$

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\[ f_1^-(\sigma) = A \left( \frac{\lambda + m}{4\lambda} \right)^{\frac{1}{2}} (1 + z) \frac{\Gamma(l + \rho_N + 1, 1 - z)}{\Gamma(2l + 1, 1 - z)} \],

\[ f_2^-(\sigma) = A_{ia} \left( \frac{\lambda + m}{4\lambda} \right)^{\frac{1}{2}} (1 + z) \frac{\Gamma(l + 1, 1 - z)}{\Gamma(2l + 1, 1 - z)} \],

where \( f_{1,2}^\pm(r) \) are the solutions with \( \kappa = \pm(l + \rho_N) \), \( z = \cosh \sigma \), \( \alpha = l + D/2 + i\sigma \), \( r = a\sqrt{\lambda^2 - m^2} \), and \( A \) is the normalization constant. As in the scalar case the normalization constant \( A \) is determined from the usual \( \delta \)-function condition of normalization of eigenfunctions of continuous spectrum

\[ |A(r)|^2 = \frac{\left[ \frac{D}{2} \right]}{\pi^{D/2 + 1} \int_0^\infty e^{-s} \left( r^2 + m^2 \right) A(r) |A(r)|^2 d\sigma} \].

Note that the solutions remain bounded at infinity if \( r \) is real. Hence, the spectrum of the Dirac operator on \( H_D \) is \( |\lambda| \geq m \). Thus, unlike the scalar case, there is no gap for fermions on \( H_D \) (this fact is very important for what follows). Nevertheless, the solutions are exponentially vanishing at infinity. Having determined the normalization constant, we immediately get the heat kernel for spinors on \( H_D \) (again as in the scalar case only modes with \( l = 0 \) are not equal to zero at the origin)

\[ h_{H_D} = 2^{\frac{D+1}{2}} a^D \int_0^\infty e^{-\frac{1}{4} (r^2 + m^2 a^2)} |A(r)|^2 dr \],

where \( \left[ \frac{D+1}{2} \right] \) denotes the integer part of \( \frac{D+1}{2} \), which results from the trace over the spinor indices. (Note that this heat kernel calculated by 'brute force' through summation over eigenfunctions coincides with the heat kernel calculated in [19], which is expressed through the spinor Green function obtained as a solution of the corresponding differential equation).
3.2 Analysis.

To interpret the heat kernel obtained, we remind the results of the corresponding calculations in flat spacetime. The gap equation in flat spacetime is

$$1 = G \int_{1/\Lambda}^{\infty} ds \, h_{\text{flat}},$$

(31)

where $h_{\text{flat}} = tr < x | e^{-s(-\gamma_{E}^{\mu} \nabla_{\mu})^2 + m^2} | x >$. The eigenfunctions of the Dirac operator in flat spacetime are just plane waves, therefore, the corresponding heat kernel is

$$h_{\text{flat}} = 2^n k^{n-1} \int \frac{d^m k}{(2\pi)^n} e^{-s(k^2 + m^2)}. \quad (32)$$

By integrating over angular variables, we obtain

$$h_{\text{flat}} = 2^n \int_0^{\infty} \frac{2 dk k^{n-1}}{(4\pi)^{\frac{n}{2}} \Gamma(n/2)} e^{-s(k^2 + m^2)} = \frac{2^n e^{-sm^2}}{(4\pi s)^{\frac{n}{2}}} \quad (33)$$

Thus, we see that the function $k^{n-1}$ defines a measure in space of eigenfunctions and for $m = 0$ determines the asymptotic behavior of the heat kernel at large $s$. Obviously, every new dimension gives an additional factor $s^{-\frac{1}{2}}$ to the asymptotic behavior of the heat kernel. Furthermore, we see from the gap equation (31) that in two-dimensional spacetime $G \to 0$ if $m \to 0$ because the integral over $s$ is divergent on the upper limit in this case. Thus, the critical value of coupling constant is zero.

We now return to the heat kernel on $R \times H^D$. It is

$$h_{R\times H^D} = \frac{2^{(D+1)/2}}{D^D} \int_0^{\infty} e^{-\frac{a^2}{4\pi s}(r^2 + m^2 a^2)} |A(r)|^2 dr, \quad (34)$$

where the factor $(4\pi s)^{\frac{D}{2}}$ is the contribution to the heat kernel from time coordinate and the rest is the heat kernel on $H^D$ (see Eq.(30)). Let us explicitly calculate the measure $|A(r)|^2$, which is given by Eq.(29) with $l = 0$. By using the formulas [21]

$$|\Gamma(iy)|^2 = \frac{\pi}{y \sinh(\pi y)},$$

$$|\Gamma(\frac{1}{2} + iy)|^2 = \frac{\pi}{\cosh(\pi y)}, \quad (35)$$

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we get

$$|A(r)|^2 = \frac{r \coth(\pi r) \prod_{j=1}^{D-2} (r^2 + j^2)}{\pi^{\frac{D-2}{2}} 2^{D-1} \Gamma\left(\frac{D}{2}\right)}$$

(36)

for even D and

$$|A(r)|^2 = \frac{\prod_{j=\frac{D-2}{2}}^{D-1} (r^2 + j^2)}{\pi^{\frac{D-2}{2}} 2^{D-1} \Gamma\left(\frac{D}{2}\right)}$$

(37)

for odd D.

The asymptotic behavior of the heat kernel for large s in the case of critical coupling constant ($m = 0$) is determined by the behavior of the integrand at small $r$. As follows from Eqs. (36) and (37), for small $r$, the measure $|A(r)|^2$ tends to a constant for any D. Consequently, we obtain that the leading term of the heat kernel (34) is $h_{R \times H^D} \sim \frac{1}{s}$. Thus, the fermion dynamics on $R \times H^D$ in the infrared region corresponds to the dynamics of $(1 + 1)$-dimensional theory. (Note that in the opposite limit of small s (large energies that corresponds to large $r$) the measure $|A(r)|^2$ tends to $\frac{2^{D-1}}{(4\pi)^{\frac{D}{2}} \Gamma\left(\frac{D}{2}\right)}$. Therefore, the leading term of the heat kernel on $R \times H^D$ at $s \to 0$ is $\frac{1}{(4\pi s)^{\frac{D-1}{2}}}$ that corresponds to the behavior of $(D+1)$-dimensional theory as expected). Consequently, we can say that the effective reduction of the dimension of spacetime $1 + D \to 1 + 1$ takes place in the infrared region for fermion fields for any $1 + D$. This explains why $g_c = 0$ in spaces with constant negative curvature (it immediately follows from the gap equation if $h \sim \frac{1}{s}$ for large s). For completeness we present the corresponding results of the effective reduction for the case of the NJL model in four-dimensional spacetime in external magnetic field [3, 4]. The heat kernel for the Dirac operator in constant magnetic field is

$$h_{\text{magnetic}} = \frac{e^{-sm^2} e^B \cot(eBs)}{16\pi^2 s},$$

(38)

where $e$ is the charge of the electron and $B$ is magnetic field. The heat kernel (38) evidently also corresponds to $(1 + 1)$-dimensional theory in the infrared because $\coth(eBs)$ tends to 1 for large s.
In the preceding subsection we found that the dynamics of fermions in the infrared is effectively $(1 + 1)$-dimensional. Potentially, it may present a problem for dynamical symmetry breaking of a continuous symmetry because, according to the Coleman–Mermin–Wagner theorem [25], spontaneous symmetry breaking of a continuous symmetry is not possible in $1 + 1$ due to strong infrared divergences connected with massless Nambu–Goldstone bosons (the existence of this potential problem in theories with the effective reduction of dimension of spacetime was indicated in [6, 7]). For example, in the NJL model the following diagram of the next-to-leading (in $1/N$) correction to vacuum energy is infrared divergent in $1 + 1$:

![Diagram](image)

Fig. 1. The next-to-leading order correction to vacuum energy in the NJL model. The fermion propagators are denoted by solid lines. A dashed line denotes the propagators of $\sigma$ and $\pi$ in the leading order in $1/N$.

If the effective reduction of dimension of spacetime in the infrared region took place for scalars, then we would have a problem connected with infrared divergent radiative corrections due to massless Nambu–Goldstone bosons. For the case of the effective reduction in external magnetic field, Gusynin, Miransky, and Shovkovy [8, 9] presented an elegant solution of this potential problem. They indicated that...
since in the case of chiral symmetry breaking the condensate $\langle 0 | \bar{\psi} \psi | 0 \rangle$ is neutral and the Nambu–Goldstone bosons are neutral particles, the effective dimensional reduction (which for fermions reflects the fact that the motion of charged particles is restricted in the directions perpendicular to the magnetic field) does not affect the dynamics of the center of mass of neutral excitations. Therefore, as they showed by explicit calculations the propagators of Nambu–Goldstone bosons have $(3 + 1)$-dimensional form in the infrared region. Evidently such a solution cannot be used in the case of gravitational field because gravity is universal and all particles including Nambu–Goldstone bosons directly interact with gravitational field. Therefore, we should seek another solution. For this end we consider the propagator of massless scalar field. This propagator can be expressed through the nondiagonal heat kernel of the Laplace–Beltrami operator. Time dependence is trivially factorized and we are left with the problem of calculating heat kernel on $H^D$. In Subsection 3.1 we have calculated the diagonal heat kernel $h_{\text{scalar}} = \langle x | e^{-\frac{s}{a} (\Delta)} | x \rangle$ (see Eq.(23)). The nondiagonal heat kernel was calculated in [26]

\[ \langle x | e^{-\frac{s}{a} (\Delta)} | y \rangle = \frac{1}{a^D} \int_0^\infty \phi_\tau (\tau) | C(r) |^2 e^{-\frac{s}{4 \pi} (r^2 + \rho^2)} dr, \] (39)

where $\phi_\tau (\tau) = F(\tau + \rho D, -i \tau + \rho D, \frac{D}{2}; - \sinh^2 \frac{\tau}{2a})$ ($\tau$ is the geodesic distance between points $x$ and $y$) and $| C(r) |^2$ is given by Eq.(22). Thus, we obtain the propagator for scalar massless particles

\[ G(t - t', \tau) = \frac{1}{a^D} \int_{1/\Lambda^2}^{\infty} ds \frac{e^{-\frac{s}{4 \pi} (t - t')^2}}{(4 \pi s)^{\frac{D}{2}}} \int_0^\infty \phi_\tau (\tau) | C(r) |^2 e^{-\frac{s}{4 \pi} (r^2 + \rho^2)} dr, \] (40)

where we have performed the Wick rotation in time coordinate.

Obviously, since there is a gap in the spectrum of the Laplace–Beltrami operator, there are not any problems with infrared behavior of massless scalar particles. Indeed, in the proper time method infrared divergences are connected with the divergence of the integral over $s$ on the upper limit of integration. Since there is
a gap in the spectrum, the integrand has the factor $e^{-\frac{1}{2} \sigma^\rho}. Therefore, infrared divergences are absent. Thus, we conclude that the effective reduction of dimension of spacetime in the infrared region for fermion fields does not contradict the Mermin–Wagner–Coleman theorem. Note that our calculations show that there are not gapless bosonic modes. Consequently, there are not gapless Nambu–Goldstone bosons in this model. However, this does not contradict the Goldstone theorem: this theorem has been proved only for Minkowski space. The problem of a possible extending the theorem to the case of curved spacetime will be considered elsewhere.

4 Conclusion.

In the present paper we studied chiral symmetry breaking in the NJL model in spaces with constant negative curvature. We showed that zero value of critical coupling constant $g_c = 0$ is connected with the effective reduction of dimension of spacetime $1 + D \rightarrow 1 + 1$ for fermions in the infrared region. Note that this effective reduction has a universal character in the sense that the initial theory reduces in the infrared region to two-dimensional one in the fermion sector for any dimension $1 + D$. In this respect this is similar to the effective reduction in the presence of the Fermi surface when the net fermion charge is not equal to zero.

By analysing the scalar propagator, we showed that such an effective reduction is absent in the scalar sector, therefore, the effective reduction of the dimension of spacetime for fermions and symmetry breaking are consistent and there is not a contradiction with the Coleman–Mermin–Wagner theorem, which states that spontaneous symmetry breaking is not possible in $1 + 1$.

Finally let us mention that the hyperbolic space $H^D$ is an Euclidean analog of
anti-de Sitter space (the Wick rotation of AdS gives $H^D$). Recently the dynamics of quantum fields on AdS has received a lot of attention in view of the conjectured CFT/AdS correspondence [27]. Therefore, it is a natural problem to study what the dynamics we discuss here means in the context of this correspondence. The results of this study will be published elsewhere.

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