No metrics with Positive Scalar Curvatures on Aspherical 5-Manifolds

Misha Gromov

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Abstract

A metric space $X$ is called uniformly acyclic if there exists an acyclicity control function $R = R(r) = R_X(r) \geq r$, $0 \leq r < \infty$, such that the homology inclusion homomorphisms between the balls around all points $x \in X$,

$$H_i(B_x(r)) \rightarrow H_i(B_x(R))$$

vanish for all $i = 1, 2, \ldots$.

We show that if a complete orientable $m$-dimensional manifold $\tilde{X}$ of dimension $m \leq 5$ admits a proper (infinity goes to infinity) distance decreasing map to a complete $m$-dimensional uniformly acyclic manifold, then the scalar curvature of $\tilde{X}$ can’t be uniformly positive,

$$\inf_{x \in \tilde{X}} Sc(\tilde{X}, x) \leq 0.$$
Since the universal coverings $\tilde{X}$ of compact aspherical manifolds $X$ are uniformly acyclic, (in fact, uniformly contractible), these $X$, admit no metrics with $Sc > 0$ for $\dim(X) \leq 5$.

Our argument, that depends on torical symmetrization of stable $\mu$-bubbles, is inspired by the recent paper [C&L 2020] by Otis Chodosh and Chao Li on non-existence of metrics with $Sc > 0$ on aspherical 4-manifolds and is also influenced by the ideas of Jintian Zhu and Thomas Richard from their papers [J.Z. 2019] and [T.R. 2020].

1 $T^*$-Stabilization of Scalar Curvature

Let $X = (X, g(x))$ be a Riemannian manifold, let $\Phi$ be an $N$-tuple of positive smooth functions $\phi_i(x)$, $i = 1, 2, \ldots, N$, on $X$, and let

$$g^* = g_\Phi = g(x) + \phi_1^2(x)dt_1^2 + \phi_2^2(x)dt_2^2 + \ldots + \phi_N^2(x)dt_N^2$$

be the iterated warped product metric on $X^* = X \times \mathbb{R}^N$.

Observe that

- $\bullet_1$ the metric $g^*$ is $\mathbb{R}^N$-invariant under the natural action of $\mathbb{R}^N$ on $X^*$, where $X$ is identified with the quotient space $X^*/\mathbb{R}^N$ for this action;
- $\bullet_2$ the natural (quotient) map $X^* \to X$ is a Riemann submersion: it is isometric on the smooth horizontal curves $C \subset X^*$, i.e. those normal to the $\mathbb{R}^N$-fibers that are $\mathbb{R}^N_x \subset X^*, x \in X$;
- $\bullet_3$ the zero-section embedding $X \to X^*$ is an isometry.

Furthermore, a straightforward computation (compare [F-C&S 1980], [G&L 1983], [J.Z. 2019]) shows that

- $\bullet_4$ the scalar curvature of $X^*$, which, being $\mathbb{R}^N$-invariant, is regarded as a function on $X$, satisfies:

$$Sc(g^*, x) = Sc(g, x) - 2 \sum_{i=1}^{N} \frac{\Delta g(\phi_i(x))}{\phi_i(x)} - 2 \sum_{i<j} (\nabla g(\log \phi_i(x), \nabla g(\log \phi_j(x))).$$

Manifolds $X^*$, which are defined as above by $N$-tuples $\Phi$ are called warped $T^N$-extensions of $X$.

From $X^*$ to $\overline{X^*} = X^*/\mathbb{Z}^N$. In what follows, we prefer to work with $X^*$ divided by the action of the lattice $\mathbb{Z}^N \subset \mathbb{R}^n$, where the Euclidean fibers $\mathbb{R}^N_x \subset X^*$ become torical, that are $T^N_x = \mathbb{R}^N_x/\mathbb{Z}^N$; this explains our $T^*$-terminology.

These $X^*$ and $\overline{X^*}$ are involved in our arguments in two ways,

I. $T^*$-Stabilization Principle: Many geometric properties of manifolds $X$ implied by the inequality $Sc(X) \geq \sigma$ follow (possibly in a weaker form) from the inequality $Sc(X^*) \geq \sigma(x)$ satisfied by some warped $T^N$-extension $X^*$ of $X$.

Example 1: Weakened $T^*$-Stable 2d Bonnet-Myers Diameter Inequality. If a closed connected surface $X$ admits a warped $T^N$-extension $X^*$ with $Sc^*(X) \geq \sigma$, then the diameter of $X$ is bounded as follows.

$$\text{diam}(X) \leq 2\pi \sqrt{\frac{N+1}{(N+2)\sigma}} < \frac{2\pi}{\sqrt{\sigma}}$$
Proof. Given two points $x_1, x_2 \in X$, take two small $\varepsilon$-circles $Y_{-1}$ and $Y_{+1}$ around them, let $X_\varepsilon \subset X$ be the band between them and apply the (elementary) $T^n$-invariant case of the (over)-torical $\frac{2\pi}{n}$-Inequality from [M.G. 2018].

Remarks. (a) This proof is similar to that of theorem 10.2 in [G&L 1983], where the latter concerns stable minimal surfaces $Y$ in 3-manifolds $X$ with $Sc(X) \geq \sigma > 0$, and delivers the following bound on the filling radii of contractible (actually homologous to zero) curves $S \subset X$:

$$\text{fil.rad}(S \subset X) \leq \frac{2\pi}{\sqrt{\sigma}}$$

and where a version of our example 1 for $N = 1$ is implicit in the calculations on pp 178-180.

(The above proof gives a slightly better bound in this case, namely $\text{fil.rad}(S \subset X) \leq \frac{\pi}{\sqrt{2\sqrt{\sigma}}}$.)

(b) It is not hard to adapt the warped product metric on $T^n - 1 \times [-1, +1]$ that was used for showing optimality of the $\frac{2\pi}{n}$ -inequality (see section 2 in [M.G. 2018]), for proving the optimality of the above inequality for closed surfaces.

(c) If $X$ is a complete Riemannian 3-manifold with $H_1(X; \mathbb{Q}) = 0$, e.g. homeomorphic to the sphere $S^3$ or to a connected sum of space forms, the above argument shows that the inequality $Sc(X^*) \geq \sigma > 0$ for a warped $T^N$-extension of $X$ implies that the distances $d$ between pairs of circles in $X$ with non-zero linking numbers are bounded by $2\pi\sqrt{(N+2)/(N+3)}$. Thus, for instance, if $(S^2, g) \to (S^2, h)$ is a non-contractible distance non-increasing map and if $Sc(g) \geq \sigma > 0$, then $\text{diam}_g(S^2) \leq 2\pi\sqrt{2/3}\sigma$.

Example 2: Sharp Stabilization of the Gauss-Bonnet inequality. If a closed surface $X$ admits a warped $T^N$-extension $X^*$ with $Sc^*(X) \geq 2$, then the area of $X$ satisfies

$$\text{area}(X) \leq 4\pi.$$
Commentaries. (a) This kind of symmetrization for \( n = 2 \) and \( N = 1 \) goes back to the 1980-paper by Fisher-Colbrie and Schoen [F-C&S 1980], which was extended to other dimensions \( n \) and \( N \) in [G&L 1983] with a use of the Schoen-Yau descent method by minimal hypersurfaces and applied in [M.G. 2018] to manifolds with boundaries.

(b) Our "submanifold" \( Y \) is, in fact, a disjoint union of several components that may come with integer multiplicities \( \neq \pm 1 \), but this is irrelevant for our applications. Besides, we limit ourselves to those situations, where one can slightly move identical copies of these components without affecting our inequalities, by replacing \( kY \) by \( \cup_{i=1}^{k} Y_i \) with \( Y_i \) being small normal mutually disjoint shifts of \( Y \).

2 Stable \( \mu \)-Bubbles in Riemannian Bands

Let \( X \) be a compact Riemannian band (in the sense of [M.G. 2018]), of dimension \( m \) (also called condenser in [M.G. 2019]), that is a compact Riemannian manifold with a boundary, where this boundary \( \partial X \) is decomposed into two non-empty mutually disjoint parts,

\[
\partial(X) = \partial_- \cup \partial_+,
\]

and where both parts \( \partial_- \subset \partial X \) and \( \partial_+ \subset \partial X \) are unions of connected components of \( \partial X \).

Let \( d_\pm = d_\pm(X) \) denote the width of \( X \), that is

\[
d_\pm = width(X) = dist(\partial_-, \partial_+)
\]

and let the scalar curvature of \( X \) be bounded from below by \( \sigma > 0 \),

\[
Sc(X) \geq \sigma.
\]

**Theorem 2.** Let

\[
\gamma = \sigma d_\pm^2 \geq \frac{4\pi^2(m-1)}{m}.
\]

If \( m = dim(X) \leq 8 \), then there exists a smooth hypersurface \( Y \subset X \), which separates \( \partial_- \) from \( \partial_+ \), and a warped \( T^1 \)-extension \( Y^* \) of \( Y \), such that

\[
Sc(Y^*) \geq \beta_m \sigma,
\]

where \( \beta_m = \beta_m(\gamma) \) is a monotone increasing function in \( \gamma \) for \( \frac{4\pi^2(m-1)}{m} \leq \gamma < \infty \), such that

\[
\beta_m(\frac{4\pi^2(m-1)}{m}) = 0,
\]

\[
\beta_m(\gamma) \to 1 \text{ for } \gamma \to \infty,
\]

\[
\beta_m(\gamma) > 0 \text{ for } \gamma > \frac{4\pi^2(m-1)}{m};
\]

moreover,

\[
[\circ^m] \quad \beta_m(m(m-1)\pi^2) \geq \frac{m-2}{m}.
\]
About the Proof. The required separating hypersurface \( Y \subset X \) is obtained as a minimizing hypersurface for the functional

\[ Y \mapsto \text{vol}_{m-1}(Y) - \int_{X_-} \mu(x)dx, \]

for a suitable function \( \mu(x) \) on \( X \), where \( X_- \subset X \) denotes the band between \( Y \) and \( \partial_- \), see section 5.4 in [M.G. 2019], where the inequality \([\cdot m]\) follows by a comparison argument applied to the unit sphere \( S^m \) minus two opposite points and where the spherical metric \( g_m \) on \( S^m \) is decomposed in polar coordinates as a warped product of \( S^{m-1} \) with \((-\pi/2, \pi/2)\),

\[ g_m = (\cos t)^2 g_{m-1} + dt^2. \]

Toroidal symmetrization (II from the previous section) applied to the above \( \mu \)-bubble yields the following.

**Corollary.** Let a compact oriented Riemannian band \( X \) of dimension \( m = n+N+1 \) admit a continuous band map \( f \) to \( T^N \times [-1,1] \), (i.e. \( f : \partial_\pm \to T^N \times \{ \pm 1 \} \)).

Let

\[ Sc(X) \geq \sigma > 0 \]

and

\[ \sigma d_+^2 \geq m(m-1)\pi^2, \]

where \( d_+ \) denotes the width of the band \( X \) as earlier.

If \( m \leq 8 \), then

the homology class of the \( f \)-pullback of a point \( t \in \mathbb{T}^N \times [-1,1] \) (i.e. dual to \( f^*[\mathbb{T}^N \times [-1,1]] \in H^{N+1}(X,\partial X) \) of the the relative fundamental class of \( \mathbb{T}^N \times [-1,1] \)) can be realized by a closed smooth \( n \)-dimensional submanifold \( Y \subset X \), which admits a warped \( T^{N+1} \)-extension \( Y^* \), such that

\[ Sc(Y^*) \geq \frac{m-2}{m} \sigma. \]

**Commentaries** (a) The above torical symmetrization of \( \mu \)-bubbles is similar to that used by Thomas Richard in [T.R. 2020] used for a bound on the 2-systole of metrics with \( Sc \geq \sigma > 0 \) on \( S^2 \times S^2 \).

(b) Probably, the desingularization result for minimizing hypersurfaces by Joachim Lohkamp [J.L. 2018] would allow the above to hold for \( m \geq 9 \) as well.

(c) Probably comparison of \( X \) with suitable non-spherical warped products (see 5.4 in [M.G. 2019]) would lead to a better estimate of the function \( \beta_m \) and allow sharpening of the above inequalities.

### 3 \( \mu \)-Bubbles in Codimension 2

The following proposition, that is adapted from [R.T. 2020] (compare [J.Z. 2019]), plays a key role in the proof of our main result.

**Richard's Lemma.** Let \( X \) be an oriented \( m \)-dimensional Riemannian manifold (possibly non-compact and non-complete) with compact boundary and \( X_0 \subset X \) be an open subset with smooth boundary such that the complement \( X \setminus X_0 \) is compact.
Let \( h \in H_{m-2}(\partial X) \) and \( h_0 \in H_{m-2}(X_0) \) be homology classes, which have equal images under the homomorphisms induced by the inclusions \( \partial X \to X \leftarrow X_0 \), that are

\[
h \in H_{m-2}(\partial X) \to H_{m-2}(\partial X) \leftarrow H_{m-2}(X_0) \ni h_0.
\]

Let

\[
Sc(X) \geq \sigma > 0,
\]

and

\[
\text{dist}^2(X_0, \partial X) \geq \frac{m(m-1)\pi^2}{\sigma}.
\]

If \( m \leq 8 \), then

the image of the homology class \( h \) in \( H_{m-2}(X) \) can be realized by a closed smooth \( (m-2) \)-dimensional submanifold \( Y \subset X \), which admits a warped \( T^2 \)-extension \( Y^* \), such that

\[
Sc(Y^*) \geq \frac{m-2}{m}\sigma.
\]

Proof. Let \( \overrightarrow{X} = X \setminus X_0 \subset X \) be the band between \( \partial_- = \partial X_0 \) and \( \partial_+ = \partial X \) and let \( \overrightarrow{h} \in H_{m-1}(\overrightarrow{X}, \partial \overrightarrow{X}) \) be the relative class that establish homology equivalence between \( h \in H_{m-2}(\partial X) \) and \( h_0 \in H_{m-2}(X_0) \), where the latter is moved from \( H_{m-2}(X_0) \) to \( H_{m-2}(\partial X_0) \) by the excision property.

Let \( \overrightarrow{h}^* \in H^1(\overrightarrow{X}) \) be the integer cohomology class Poincaré dual to \( \overrightarrow{h} \) and let us induce \( \overrightarrow{h}^* \) from the fundamental class of the circle by a continuous map \( f_1 : \overrightarrow{X} \to T^1 \).

Then, using the distance function \( x \mapsto \text{dist}(x, X_0) \), we construct a band map \( \overrightarrow{f} : \overrightarrow{X} \to [-1, 1] \) and apply the above corollary to the band map defined by the pair \((f_1, \overrightarrow{f})\).

\[
(f_1, \overrightarrow{f}) : \overrightarrow{X} \to T^1 \times [-1, 1].
\]

4 Recollections on Filling Radius, Uryson Width and Uniform Acyclicity

The filling radius \( \text{fil.rad}(X) \), also called the absolute filling radius of an \( n \)-dimensional orientable manifold (or a pseudomanifold) \( X \) with a metric \( \text{dist}_X \) on it is defined as the supremum of the numbers \( r > 0 \), such that if \( X_r \supset X \) is a metric extension of \( X \), that is a metric space that contains \( X \) and such that \( \text{dist}_{X_r}[X = \text{dist}_X \), the fundamental class \( [X] \in H_n(X) \) doesn’t vanish (i.e. \( X \) doesn’t bound) in the \( \tau \)-neighbourhood \( U_\tau(X) \subset X \).

Filling with Coefficients. The above definition makes sense for the fundamental class of \( X \) in the homology with a given coefficient ring. For instance one may speak of the rational filling radius \( \text{fil.rad}_\mathbb{Q}(X) \) which may be smaller than \( \text{fil.rad} = \text{fil.rad}_\mathbb{Z}(X) \).

Moreover, if \( \overline{X} \to X \) is finite covering map, then \( \text{fil.rad}_\mathbb{Q}(X) \leq \text{fil.rad}_\mathbb{Z}(\overline{X}) \).

\[\text{See [M.G. 1983], [M.K. 2007], [S.W 2007], [L.G. 2010], [L.G. 2016], [P .P . 2020] for details and other properties of the filling radius and other concepts discussed in this section.}\]
(It seems unclear, in general, what possible values of the ratio $\frac{\text{fil.rad}(X)}{\text{fil.rad}(\hat{X})}$ could be for various Riemannian metrics on $X$ and those on $\hat{X}$ induced by the covering $X \to \hat{X}$.)

4.A. Cubic Example If $X$ can be covered by $2n + 2$ subsets $Y_{\pm i} \subset X$, $i = 0, 1, \ldots, n$, such that $\text{dist}_X(Y_{\pm i}, Y_{\mp i}) \geq d$, then $\text{fil.rad}(X) \geq d/2$.

Indeed, map $X_+ \to \mathbb{R}^n$ by $n$ distance functions $x \mapsto \text{dist}_X(x, Y_{\pm i})$ and compose it with the radial projection from $\mathbb{R}^n$ to the $d$-cube $[0, d]^n \subset \mathbb{R}^n$.

The restriction of this composed map $F : X_+ \to [0, d]^n$ sends $X$ to the boundary sphere $\partial[0, d]^n$ with degree $1$.

Hence, if an $(n+1)$-chain $C \subset X_+$ bounds $X$, it is is sent by $F$ onto this cube where all points $x_+ \in C$ from the $F$-pullback of the center of the cube, $(d/2, d/2, \ldots, d/2) \in [0, d]^n$ have $\text{dist}(x_+, X) \geq d/2$. QED.

4.B* Corollary \footnote{This corollary and everything else marked with "*" is not used in the proof of our main theorems.} Let $X$ be a Riemannian manifold, let $Y \subset X$ be the shortest non-contractible closed curve and let $Y' \to Y'$ be a finite covering of this $Y$ (where the curve $Y'$ may be contractible in $X$).

Then the filling radius of $Y$ for the metric on $Y'$ induced by distance function on $X'$ is bounded by the length $l$ of $Y$ as follows

$$\frac{\text{fil.rad}(Y)}{l} \geq \frac{1}{8}.$$  

Moreover, the same remains true for the shortest curve non-trivial in some quotient group of the fundamental group $\pi_1(X)$.

Proof. Divide $Y$ in a square like fashion into four segments $s_{\pm i}$, $i = 1, 2$, of lengths $\frac{l}{4}$.

If $Y$ is the shortest, then the distances $d$ between both pairs of the opposite segments must be $\geq \frac{l}{4}$; otherwise $Y$ could be decomposed into two curves of curves of lengths $l_1 + d$ and $l_2 + d$, where $l_1, l_2 \leq \frac{d}{2}$ (and $l_1 + l_2 = l$). QED.

(This, as shall see in section 6, implies that closed orientable $3$-manifolds $X$ with $\text{Sc}(X) \geq \sigma > 0$ and the fundamental groups of which are non-free, contain closed non-contractible geodesics of length $\leq 100/\sqrt{\sigma}$.)

Uryson’s k-width, denoted $\text{width}_k(X)$, of a metric space $X$ is the infimum of the numbers $w$, such that $X$ admits a covering of multiplicity $\leq k + 1$ by closed subsets $V_i \subset X$ with $\text{diam}(V_i) \leq w$.

Equivalently, – assuming that $X$ is compact for safety sake – this is equal to the infimum of $w$, such that $X$ admits a continuous map to a $k$-dimensional polyhedral space, $f : X \to P$,

such that the diameters of the $f$-pullbacks of all points $p \in P$ satisfy $\text{diam}(f^{-1}(p)) \leq w$.

Indeed, given a covering of $X$ by $V_i$, let $P$ be the nerve of this covering and let $f : X \to P$ be the standard continuous map defined via a partition of unity subordinated to the covering of $X$ by small $\epsilon$-neighbourhoods $U_i \supset V_i$.

\footnote{Never mind vanishing of this metric on the pullbacks of he points by the covering map $X \to \hat{X}$.}
Conversely, given a continuous map $f \to P$, $\text{dim}(P) = k$, take a covering of $P$ by sufficiently small subsets with multiplicity $k + 1$ and then pull it back by $f$ to $X$.

4.C. The filling radius is bounded by the Uryson width as follows.

$$\text{fill.rad}(X) \leq \frac{1}{2} \text{width}_k(X) \text{ for all } k < \text{dim}(X).$$

Indeed, let $X_+$ be the cylinder $C_f$ of the map $f$, obtained by attaching the product $X \times [0, w/2]$ to $P$ by the map $f : X = X \times \{w/2\} \to P$ and observe that

- there is an extension of the metric of $X$ to $C_f$ that keeps the lengths of the segments $\{(x) \times [0, w/2], x \in X\}$, equal to $w/2$.

and that

- if $\text{dim}(P) < \text{dim}(X)$, then $X$ is homologous to zero in $C_f \supset X = X \times \{0\}$.

4.D. Elementary Metric Lemma. Let $X$ be a path metric space, e.g. a Riemannian manifold, possibly non-complete and or with a boundary and let $\phi(x)$ be the distance function to a point $x_0 \in X$,

$$\phi(x) = \text{dist}(x, x_0).$$

If a connected component of a level of $\phi$ has diameter $\geq d$, then the space $X$ contains a closed curve $S$, such that, this curve $S$ itself and all its finite coverings $\tilde{S} \to S$ have their filling radii bounded from below by

$$\text{fill.rad}(\tilde{S}) \geq \frac{\text{diam}(L)}{6}.$$ 

Proof. Let $x_1, x_2 \in \phi^{-1}(r) \in X$ be two points in a connected component of an $r$-level of $\phi$ for some $r > 0$, such that $\text{dist}(x_1, x_2) \geq d$, assume without loss of generality that this component is path connected and proceed as follows.

- Join the points $x_1$ and $x_2$ by a curve $l$ in this $r$-level of $\phi$,
- Join $x_1$ and $x_2$ with $x_0$ by shortest segments $s_1$ and $s_2$ in $X$,
- Let $S = s_1 \cup l \cup s_2 \subset X$ be the closed curve composed of these three (see the figure on page 175 in [G&L 1983]).

Then let $[x_1, \delta] \subset s_1$ and $[x_2, \delta] \subset s_2$ be the subsegments of length $\delta$ attached to the $x_1 \in s_1$ and $x_2 \in s_2$ ends of the segments $s_1$ and $s_2$ for $\delta = \text{dist}(x_1, x_2)/3$.

Then the distances between these two subsegments, as well as between the pair of the two complementary segments in $S$, that are $l$ and that the part of $S$ within distance $\leq r - \delta$ from $a_0$, are, clearly, both $\geq \delta$; thus, according to the above cubical example,

$$\text{fill.rad}_3(\tilde{S}) \geq \frac{1}{2} \delta \geq \frac{\text{diam}(L)}{6}.$$ 

QED.

Remarks. The above is borrowed from the proof of corollary 10.11 in [G&L 1980] claiming a bound on Uryson 1-width of 3-manifolds $X$ with $\text{Sc}(X) \geq \sigma > 0$.

Since the inequality $\text{fill.rad} \leq \text{const}/\sqrt{\sigma}$ for 3-manifolds $X$ is crucial for our proof of non-asphericity of 5-manifolds with positive scalar curvature and since the argument following, 10.11 in [G&L 1980] contained a flaw, we furnish all details needed for the proof of this inequality here and in section 6.
(b) There are similar bounds on the diameters of connected components of the "annuli" \( \phi^{-1}[r,R] \subset X \), where, moreover, instead of the distance function to a points \( x_0 \in X \), one may use distance to a connected subset \( X_0 \subset X \).

4.E. Corollary. If all closed curves \( Y \subset X \) admit finite covering \( \hat{Y} \to Y \), such that

\[ \text{fil.rad}(\hat{Y}) \leq \delta, \]

then \( X \) admits a continuous map onto a 1-dimensional simplicial space, i.e. a graph \( \Gamma \),

\[ f : X \to \Gamma, \]

such that the diameters of the pullbacks \( f^{-1}(\gamma) \subset X \), \( \gamma \in \Gamma \), are bounded by \( 6\delta \), that is

\[ \text{width}_1(X) \leq 6\delta. \]

In fact, such a \( \Gamma \) is obtained as the quotient of \( X \) by the relation, where two points are declared equivalent if they lie in the same connected component of a level set of some distance function on \( X \). As it is explained in argument following corollary 10.11 in [G&L 1983].

Question. Let \( X \) be a path metric space, where all closed curves \( S \) homologous to zero bounds within distance \( d \) from \( S \). Does then \( X \) admit a map to a graph, such that the pullbacks of all points have diameters \( \leq \text{const} \cdot d \)?

A metric space \( X \) is called uniformly contractible or geometrically controlled contractible if there exists a contractibility control function \( R(r) = R_X(r) \geq r \), such that all \( r \)-balls \( B_x(r) \subset X \) are contractible in the concentric \( R \)-balls \( B_x(R) \), for \( R \geq R(r) \) and all \( x \in X \).

A metric space \( X \) is called uniformly acyclic if the inclusion homomorphisms

\[ H_i(B_x(r)) \to H_i(B_x(R)) \]

vanish for \( i = 1, 2, ... \), where the corresponding \( R(r) \) is called the acyclicity control function.

Similarly, one defines uniform \( Q \)-acyclicity, where the homomorphisms

\[ H_i(B_x(r); Q) \to H_i(B_x(R); Q) \]

are required to vanish.

Our main examples of such spaces come from covering of compact spaces via the following obvious

Proposition. If a \( X \) is a contractible, acyclic or rationally acyclic manifold; such that the action of the isometry group is cocompact on \( X \), then \( X \) is uniformly contractible or, respectively, uniformly acyclic or uniformly rationally acyclic.

In particular,

universal coverings of compact aspherical manifolds are uniformly contractible.

4.F. Bounds on Relative Filling Radii of Cycles in Uniformly Acyclic Spaces.

If \( X \) is uniformly acyclic, then the (relative) filling radii of all submanifolds (and subpsedomanifolds) in \( X \) are bounded in terms of their absolute filling radii.

More generally, let \( Y \) be a closed orientable manifold (or psedomanifold) of dimension \( n \) endowed with a metric and denote \( r = \text{fil.rad}(Y) \).

Let \( X \) be a uniformly acyclic space and let \( \phi : Y \to X \) be a distance non-increasing map.

\[ \text{This is called "geometrically contractible" in [M.G. 1983].} \]
Then the image of the fundamental cycle \([Y] \in H_n(Y)\) in \(X\) is homologous to zero in the \(R^*(r)\)-neighbourhood of the image \(\phi(Y) \subset X\), where the function \(R^*(r)\) depends only on the acyclicity control function \(R_X(r)\) and \(n = \dim(Y)\).

**Proof.** To clarify, let first \(X\) be uniformly contractible with control function \(R(r)\) and extend the map \(\phi : Y \rightarrow X\) to a map \(\Phi : Y_+ \rightarrow X\) for a metric extension \(Y_+ \supset Y\) of \(Y\), where \(Y\) is homologous to zero and where \(\text{dist}(y_+, Y) \leq r\) for all \(y_+ \in Y_+\).

Assume without loss of generality that \(Y_+\) is a polyhedral space and divide it into \(\varepsilon\)-small simplices \(\Delta\).

Start with the map \(\Phi_0\) on the 0-skeleton, of \(Y_+\) obtained by sending all vertices from \(Y_+\) to the nearest points in \(Y \subset Y_+\) and then applying \(\phi\).

Next, extend \(\Phi_0\) to a map \(\Phi_1\) from the 1-skeleton of \(Y_+\) to \(X\), extend this to the 2-skeleton etc.

At every stage, an extension of \(\Phi_{i-1}\) from the boundary of an \(i\)-simplex \(\delta \subset Y_+\) is possible in the \(R(2r_{i-1})\)-neighbourhood of the \(\Phi_{i-1}\)-image of \(\partial \delta\) for \(r_{i-1}\) denoting the diameter of this image. Thus, eventually, the extension will take place in the \(R^*(r)\)-neighbourhood of \(\phi(X)\), where this \(R^*\) is the value of the \((n+1)\)-th iteration of the function \(2R(2r)\).

Now, in the general uniformly acyclic case, instead of extending maps from \(\partial \Delta^i\) to \(\Delta^i\) we fill in \(\partial \Delta^i\) at every stage by an \(i\)-chain and thus eventually fill-in all of \(\phi_+[X]\) in the \(R^*\) neighbourhood of \(\phi(Y) \subset X\). QED.

**Remark.** The above argument also yields a similar inequality for rational filling radii in uniformly \(\mathbb{Q}\)-acyclic spaces.

It seems, the universal coverings \(\tilde{X}\) of closed aspherical \(m\)-manifolds in all known examples contain \(i\)-cycles \(Y_i \subset \tilde{X}\), for all \(i = 0, 1, 2, m - 1\), with arbitrary large filling radii,

\[
\text{fil.rad}(Y_i \subset \tilde{X}) \geq d \rightarrow \infty,
\]

i.e. these \(Y_i\) don’t bound in their \(d\)-neighbourhoods in \(\tilde{X}\), that is equivalent to the existence of \((m - i - 1)\)-cycles \(Y^{\phi} \subset \tilde{X}\) which have non-zero linking numbers with \(Y_i\) and

\[
\text{dist}(Y_i, Y^{\phi}) \geq d.
\]

The existence of such cycles is obvious for \(i = 0, m - 1\), quite easy for \(i = 1\), while the case \(i = m - 2\) follows by the Alexander duality from that for \(i = 1\) as it is shown in the lemma below.

On the other hand, it seems to be unknown for \(n \geq 5\)

if the universal coverings of all compact aspherical \(n\)-manifolds contain \(2\)-cycles with arbitrarily large filling radii.

4.G. **Codim 2 Linking Lemma.** Let \(\tilde{X}\) be a complete uniformly acyclic Riemannian manifold of dimension \(m \geq 2\) (e.g. the universal covering of an aspherical manifold). Then

for all \(d > 0\) there exist a pair of domains with smooth boundaries, \(X = X_d \subset \tilde{X}\) and \(X_0 \subset X\), such that

- \(\text{dist}(X_d, \partial \tilde{X}) \geq d\);
- there exists a homology class \(h_0 \in H_{m-2}(X_0)\) such that its image \(h \in H_{m-2}(X)\) under the inclusion homomorphism for \(X_d \hookrightarrow X\) is indivisible\(^\text{5}\), hence non-zero and non-torsion.

\(^5\) That is \(h \neq kh'\) unless \(k = \pm 1\).
Proof. Since $\tilde{X}$ is uniformly $(m-1)$-acyclic, i.e. all $(m-1)$-cycles with diameter $r$ bound in their $R$-neighbourhood for some control function $R = R_{\tilde{m}-1}(r)$, the manifold $\tilde{X}$ is also uniformly connected at infinity for $m \geq 2$.

It follows that, for all $r > 0$,

there exists a distance minimizing geodesic segment $\gamma \subset \tilde{X}$ of length $\geq 100r$ and a curve $C \subset \tilde{X}$ that joins the ends $x_1$ and $x_2$ of $\gamma$ and doesn’t intersect the $r$-ball around the central point $x \in \gamma$.

Let $F : X \to \mathbb{R}^2_+$ be the map defined by the pair $(f_1, f_2)$ of the distance functions $f_1(x) = dist(x, x_1)$ and $f_2(x) = dist(x, \gamma)$.

Let $D^2 \subset \mathbb{R}^2_+$ be the disc with center at the point $(r/2, r/2) \in \mathbb{R}^2_+$ and and with radius $r/3$.

Let

$$\Phi : X \to D^2$$

be the composition of $F$ with the normal projection (retraction) $\mathbb{R}^2_+ \to D^2$.

Since the $f$-image of the closed curve $S = \gamma \cup C$ doesn’t intersect $D^2 \subset \mathbb{R}^2_+$, the map $\Phi$ sends $S$ to the boundary circle $\partial D^2$, where the resulting map $S \to \partial D^2$ has degree one (or minus one depending on how we orient the two circles), because this map is locally one-to-one over the point $(r/2, r/2) \in \partial D^2$.

It follows that

the pullback $Z_r = F^{-1}(r/2, r/2) \subset \tilde{X}$ or rather, this pullback for a generic smooth approximation of $F$, has linking number one with $S$.

Since $X$ is acyclic, the (transversal) intersections of $Y_0$ with (smooth approximations of ) the boundaries $R$-balls around $x$,

$$Z_r \cap \partial B_x(R)$$

are homologous to zero in the complements to the balls $B_x(\rho) \subset B_x(R)$, where $\rho \to \infty$ for $R \to \infty$.

Let $Y_x \subset \tilde{X} \setminus B_x(R)$ be the $(m-2)$-cycle obtained by attaching a $(m-2)$-chain $Z' \subset \tilde{X} \setminus B_x(\rho)$ with $\partial Z' = -\partial Z_r$ to $Z_r$.

Finally, let $X_d \subset \tilde{X}$ be a small smooth neighbourhood of $Y_x$, where $R_s \geq R$, and let $\tilde{X} = X_d \cup \tilde{X}$ be a smoothed $d$-neighbourhood of $X_0$.

Since the cycle $Y_x$ linked with the curve $S$ in $\tilde{X}$ with linking number one, the homology class $[\tilde{Z}] \in H_{m-2}(\tilde{X})$ represented by $Y_x \subset \tilde{X} = X_d$ is indivisible, where $d \to \infty$ for $r \to \infty$ and provided that $R$ as well as $\rho$ are much larger than $r$. QED.

5 Non-compact Version of the Chodosh-Li Theorem

Let $\tilde{X}$ be a complete orientable Riemannian 4-manifold, which admits a proper distance non-increasing map to a complete uniformly acyclic Riemannian 4-manifold.

\footnote{Acyclicity (e.g. contractibility) of $\tilde{X}$ implies that the inclusions homomorphisms $H_i(\partial B_2(R)) \to H_i(\tilde{X} \setminus B_2(r))$, $i = 1, 2, ..., m-2$, vanish for $\rho = \rho(R) = R_{\tilde{m}-2}(R) \to \infty$.}

\footnote{Small* means that it is contained in the $\varepsilon$ neighbourhood, where, eventually, $\varepsilon \to 0$.}
Let $\tilde{X} \overset{f}{\to} \tilde{X}$ and let the degree of this map be non-zero. Then the lower bound $\sigma$ on the scalar curvature of $\tilde{X}$ is non-positive, \[ \sigma = \inf_{x \in \tilde{X}} Sc(\tilde{X}, x) \leq 0. \]

Consequently, compact aspherical 4-manifolds admit no metrics with $Sc > 0$.

**Proof.** Let $(\tilde{X}_d, X_0)$, \[ \tilde{X} \supset X_d \supset X_0, \]
be a pair delivered by the above codim 2 linking Lemma and let $(X, X_0)$ be the $f$-pullback of this pair, \[ X = X_d = f^{-1}(\tilde{X}_d) \subset \tilde{X} \quad \text{and} \quad X_0 = f^{-1}(\tilde{X}_0) \subset \tilde{X}. \]

Since $k = \deg(f) \neq 0$, there exists a homology class $h_0 \in H_{m-2}(X_0)$, namely (Gysin’s) pullback $Y_d = f^{-1}(\tilde{Y}_d)$ of the cycle $\tilde{Y}_d$, such that the class $f_*(h_0) = \tilde{h}_0 \in H_{m-2}(\tilde{X}_0)$ doesn’t vanish in $\tilde{X} \supset \tilde{X}_0$.

Since $f$ is distance non-increasing, \[ \text{dist}(X_0, \partial X) \geq \text{dist}(X_0, \partial X_d) \]
and, also, \[ f \] doesn’t increase the filling radii of cycles in $\tilde{X}$. It follows that the filling radii of all cycles in $X_0$ in the class $h_0$, i.e. homologous to $Y_d$, are greater than or equal to $d$.

On the other hand, by Richard’s lemma, there exists a closed surface $Y \subset X$, homologous to $Y_d$, such that a warped $T^2$-extension $Y^*$ of $Y$ has $Sc(Y^*) \geq \sigma/2$.

According to Example 1, all connected components $Y_i$ of this $Y$ have diameters $\delta \leq \frac{4\pi}{\sqrt{\sigma}}$ and since the map is distance non-increasing, the $f$-images $Y_i \subset X_0$ of these components have diameters $\tilde{\delta} \leq \delta \leq \frac{4\pi}{\sqrt{\sigma}}$ as well.

Since the ambient manifold $\tilde{X} \supset \tilde{X}_0$ is uniformly acyclic, the filling radii of all $Y_i$, hence of $Y = f(Y)$, are bounded by $R(\tilde{\delta})$, where this quantity remains bounded for $\sigma > 0$, while $d \to \infty$; hence $\sigma = 0$. QED.

**Remark on Non-complete Manifolds.** All of the above remains true for compact manifolds with boundaries, in-so-far we operate sufficiently far from these boundaries.

It follows, for instance, that if $X$ is complete uniformly acyclic 4-manifold, then for all $\sigma > 0$, there exists $R = R_X(\sigma)$, such that no (possibly non-complete) 4-manifold $X$ with $Sc(X) \geq \sigma$ admit a proper distance decreasing map of positive degree onto an (open) $R$-ball in $X$.

**Quadratic Decay Corollary.** Let $X$ and $\tilde{X}$ be complete orientable Riemannian 4-manifolds, where $Sc(X) > 0$ and $\tilde{X}$ is uniformly acyclic.

Let $f : X \to \tilde{X}$ be a proper smooth map of non-zero degree. Then
the ratio \( \frac{Sc(X,x)}{|df(x)|^2} \), where \( df : T(X) \to T(X) \) stands for the differential of \( f \) decays at least quadratically on \( X \), that is the quantity

\[
\inf_{x \in B_\infty(R)} \frac{R^2 \cdot Sc(X,x)}{|df(x)|^2}
\]

remains bounded for \( R \to \infty \), where \( B_{x_0}(R) \subset X \) denotes the \( R \)-ball around a fixed point \( x_0 \in X \).

6 Bounds on Uryson’s Widths and Filling Radii of 3-Manifolds with Stabilized Scalar Curvatures \( \geq \sigma > 0 \)

In order to extend the above 4D-argument to 5-dimensional manifolds we need to show that

closed 3-manifolds \( X \), which admit warped \( \mathbb{T}^2 \)-extensions \( X^* \) with \( Sc(X^*) \geq \sigma > 0 \), have their (absolute) filling radii bounded by

\[
\text{fil.rad}(X) \leq \text{const}_3 \sqrt{\sigma}.
\]

This is proven in this section below, where, in fact, we show that that Uryson’s 2-width of such an \( X \) is bounded by \( \text{const}'_3 \sqrt{\sigma} \).

Namely we prove the following

6.A. 3d-Covering Theorem. Let a closed orientable Riemannin 3-manifold \( X \) admit a warped \( \mathbb{T}^N \)-extension \( X^* \), such that

\[
Sc(X^*) \geq \sigma.
\]

Then there exists a finite collection of closed subsets \( V_i \) that cover \( X \),

\[
X = \bigcup_i V_i,
\]

such that

- the multiplicity of this covering is \( \leq 3 \), i.e. no four of these subsets intersect;
- the diameters of all these subsets are bounded by a constant \( D \) depending only on \( \sigma \), namely,

\[
diam(V_i) \leq D \leq \frac{12\pi}{\sqrt{\sigma}} + \varepsilon.
\]

6.B. Filling Corollary.

\[
\text{fil.rad}(X) \leq \frac{6\pi}{\sqrt{\sigma}} + \varepsilon.
\]

Proof. We shall obtain the required covering of \( X \) by firstly cutting \( X \) into submanifolds \( X_j \) with boundaries, such that the homology groups \( H_1(X_j) \) are

---

\( ^8 \) It is sufficient to have such a bound for a finite, or even amenable, covering \( \hat{X} \) of \( X \), which may be useful in some cases.
pure torsion and then applying the argument following corollary 10.11 in [G&L 1983] to these $X_j$.

To clarify the idea, let us start with the case, where the manifold $X$ itself has $Sc(X) \geq \sigma$ and recall that the fundamental group of such an $X$ is a free product of finite and infinite cyclic groups, e.g. section 8 in [G&L 1983].

Let $\{\Sigma_k^2\}$ be a maximal collection of disjoint connected stable minimal surfaces in $X$, where "maximal" means that every such connected surface in the complement $X \setminus \cup_k \Sigma_k^2$ is isotopic to one of $\Sigma_k^2$.

Since all these $\Sigma_k^2$ are topological spheres by the Schoen-Yau argument from [S&Y 1979] and since all 2-dimensional homology classes in manifolds with mean convex, e.g. minimal, boundaries are realizable by stable minimal surfaces, all connected components $X_j$ of $X \setminus \cup_k \Sigma_k^2$ have their 1-dimensional homology pure torsion, i.e. $H_1(X_j; \mathbb{Q}) = 0$. In fact, the fundamental groups of $X_i$, are free products of finite groups.\footnote{If one feels uncomfortable with torsion, one may take a finite covering $\tilde{X} \to X$ with a free fundamental group $\pi_1(\tilde{X})$ and apply what follows to $\tilde{X}$. This will result in a lower bound on the rational filling radius of $\tilde{X}$, i.e. defined with the rational homology, $H_3(\tilde{X}; \mathbb{Q})$ rather than $H_3(X) = H_3(X; \mathbb{Z})$. This causes no problem, since the class $\tilde{h}$ in the codim 2 linking lemma in section 4 is indivisible; hence, remains non-zero after tensoring with $\mathbb{Q}$.}

Also, since the boundaries of all connected components $X_j$ of $X \setminus \cup_k \Sigma_k^2$ are mean convex, a multiple of every closed connected curve $S^1$ in $X$, which, according to theorem 10.7 in [G&L 1983], lies within distance $\leq 2\pi/\sqrt{\sigma}$ from $S$.\footnote{The argument indicated for the proof of example 1 in section 1 yields the bound $\text{dist}(y, \partial S) \leq \sqrt{2}\sqrt{\sigma}$, $y \in Y$.}

Now it follows from 4.E that each of the above 3-manifolds $X_j$ with $\Sigma$-boundaries admits a map to a finite 1-dimensional simplicial space, i.e. a graph $\Gamma_j$, $f_j: X_j \to \Gamma_j$,

such that the diameters of the pullbacks $f^{-1}(\gamma) \subset X_j$, $\gamma \in \Gamma_j$ are bounded by $\frac{12\pi}{\sqrt{\sigma}}$.

Cover the graphs $\Gamma_j$ by $\epsilon$-small subsets $\Gamma_{jl}$ with multiplicities 2 and then cover $X$

- by $\epsilon$-neighbourhoods $U_{ke} \subset X$ of $\Sigma_k^2$
- and the closures of the differences

$X_{j,k,l} = f^{-1}(\Gamma_{j,l}) \setminus U_{ke} \subset X_j \subset X$ for all $j, k$ and $l$.

The resulting covering of $X$, call it $\{V_{j,l}\}$, clearly, has multiplicity $\leq 3$, where the diameters of $U_{ke}$ are bounded by $2\pi/\sqrt{\sigma} + 2\epsilon$ (Example 1) and the pullbacks $f^{-1}(\Gamma_{j,l})$ satisfy

$$\text{diam}(f^{-1}(\Gamma_{j,l})) \leq 12\pi\sqrt{\sigma} + \epsilon,$$

where $\epsilon = \epsilon(\epsilon) \to 0$.

This furnishes the proof in the case $Sc(X) \geq \sigma$.

Now, let us turn to the general case, where a warped $\mathbb{T}^N$-torical extension $X^*$ of $X$ has $Sc(X^*) \geq \sigma > 0$.\footnote{The argument indicated for the proof of example 1 in section 1 yields the bound $\text{dist}(y, \partial S) \leq \sqrt{2}\sqrt{\sigma}$, $y \in Y$.}
It is convenient at this stage to work with the quotient manifold $X^\ast = X^\ast / \mathbb{Z}^N$ (see section 1), where, observe, stable minimal $T^N$-invariant hypersurfaces in $X^\ast / \mathbb{Z}^N$ correspond to local minima of the following functional on closed surfaces $Y \subset X$,

$$Y \mapsto \int_Y \text{vol}(T^N_{y}) dy,$$

where $T^N_{y}$ are the torical fibers $\mathbb{R}^N / \mathbb{Z}^n$ (see section 1).

Thus the above existence theorems for stable minimal surfaces in $X = (X, g)$ apply to $X$ where the original metric $g$ is modified by the conformal factor,

$$g \sim \psi(x)g(x) \text{ for } \psi(x) = \text{vol}(T^N_x), \ x \in X.$$

On the other hand, the necessary bounds on distances in these surfaces with respect to the original metric $g$ follow by the argument used in example 1.

This accomplishes the proof of the 3d covering theorem.

3d Geometric Decomposition Theorem*. Let $X$ be a compact orientable Riemannian 3-manifold with $Sc(x) > \sigma > 0$. Then there exists a a collection of disjoint smooth embedded spheres $\Sigma_\mu \subset X$ with the following properties.

(i) The diameters of all $\Sigma_\mu$ in the induced Riemannin metrics satisfy

$$\text{diam}(\Sigma_\mu) \leq \frac{3\pi \sqrt{2}}{\sqrt{\sigma}},$$

(ii) the diameters of the connected components $X_\nu$ of the complement $X \setminus \cup_\mu \Sigma_\mu$ satisfy

$$\text{diam}(X_\nu) \leq \frac{(2\sqrt{6} + 3\sqrt{2})\pi}{\sqrt{\sigma}},$$

(iii) all $X_\nu$ have finite fundamental groups; hence, they are diffeomorphic to the spherical 3-forms with finitely many punctures.

Outline of the Proof. We start with cutting $X$ by minimal spheres to pieces $X_j$ as earlier which satisfy (iii) but not necessarily (ii).

Next we cut each $X_j$ by discs with punctures, call them $\Upsilon_{jk} \subset X_j$, into pieces with small diameters by using a version of theorem 2 for manifolds with boundary.

Then we move the boundaries $\partial \Upsilon_{jk} \subset \partial X_j$ inside $X_j$ without creating new intersections.

(Such a "movement" only insignificantly increases the diameters of $\Upsilon_{jk}$, but it may uncontrollably enlarge their areas and it remains unclear if one can have the areas of the spheres $\Sigma_\mu$ bounded by $\text{const}/\sigma$.)

Since we don’t use this in the sequel we leave the actual proof to the reader.

Problem 1. Can one use the Hamilton Ricci flow to canonically decompose complete 3-manifolds $X$ with $Sc(X) \geq \sigma \geq 0$, and more generally, for $Sc(X^\ast) \geq \sigma > 0$, into $X_\nu$-like pieces?

(This would imply, for instance that compact aspherical Riemannin manifolds of all dimensions $n \geq 3$ admit no 3-dimensional foliations with leaves with positive scalar curvatures.)

Problem 2. Is there a counterpart to the 3d-covering and decomposition theorems for 4-manifolds?
7  Proof of the Main Theorem and an Alternative Proof of Chodosh-Li theorem

Let us reformulate the theorem stated in the summary in the the $\mathbb{T}^n$-stable quadratic decay form as in the corollary to the non-compact Chodosh-Li theorem.

Main Theorem. Let $\tilde{X}$ and $\tilde{X}$ be complete orientable Riemannian 5-manifolds, let $\tilde{X}^*$ be a warped $\mathbb{T}^N$-extension of $\tilde{X}$, such that $Sc(\tilde{X}^*, x) > 0$, $x \in \tilde{X}$ and let $\tilde{X}$ be uniformly acyclic.

Let $f : X \to \tilde{X}$ be a proper smooth map of non-zero degree.

Then the ratio $\frac{Sc(\tilde{X}^*, x)}{|df (x)|^2}$, where $df : T(\tilde{X}) \to T(\tilde{X})$ stands for the differential of $f$, decays at least quadratically on $\tilde{X}$, that is the quantity

$$\inf_{x \in B_{x_0} (R)} \frac{R^2 \cdot Sc(\tilde{X}^*, x)}{|df (x)|^2}$$

remains bounded for $R \to \infty$, where $B_{x_0} (R) \subset \tilde{X}$ denotes the R-ball around a fixed point $x_0 \in \tilde{X}$.

Main Corollary. let $\tilde{X}$ be the universal covering of a compact manifold $X$, let $N = 0$, i.e. $\tilde{X}^*$, let also $\tilde{X} = \tilde{X}$ and let the map $f$ be the identity. Then the scalar curvature of $\tilde{X}$ can’t be everywhere positive. This means that compact aspherical 5-manifolds can’t have $Sc > 0$.

(A posteriori, if $Sc(\tilde{X}) \geq 0$, then $\tilde{X}$ is isometric to the Euclidean space $\mathbb{R}^5$.)

Proof. We argue as in the proof of the non-compact Chodosh-Li theorem, in section 5 except that instead of a surface $Y \subset \tilde{X}$ with large filling radius, obtained via a curve $S \subset \tilde{X}$, we have (again by codim 2 linking Lemma 4.G) a closed orientable 3-manifold, now called $X \subset \tilde{X}$, with $Sc(X^*) \geq \frac{2}{3} \sigma$, the $f$-image of (the fundamental class of) which in $\tilde{X}$ is linked to such an $S$.

Besides, instead of directly filling-in $f(X) \subset \tilde{X}$ as we were doing it with $f(Y)$, which had diameter bounded by $\text{const}/\sqrt{\sigma}$, we apply the bound on the absolute filling radius of $X$ in terms of $\sigma = \inf Sc(X^*)$ (filling corollary 6.B from the previous section) combined with the bound 4.F on the relative filling radii of cycles in uniformly acyclic spaces.

Thus we conclude to the vanishing of the fundamental class $[X]$ of $X$ in its $d$-neighbourhood in $\tilde{X}$. QED.

Codim 1 Proof of the Chodosh-Li Theorem. Let $\tilde{X}$ be a complete non-compact Riemannian manifold of with $Sc(\tilde{X}) \geq \sigma > 0$.

If $m = \text{dim} (\tilde{X}) \leq 8$ then

the manifold $\tilde{X}$ can be exhausted by compact domains with smooth boundaries,$X_1 \subset X_2 \subset ... \subset X_i \subset ... \cup X_i = X$,

such that the boundaries $Y_i = \partial X_i$ of all $X_i$ admit warped $\mathbb{T}^N$-extensions $Y_i^*$ with $Sc(Y_i^*) \geq \frac{m^2}{m} \sigma$.

$11$Recall that the function $Sc(\tilde{X}^*)$, on $\tilde{X}^*$ being $\mathbb{R}^N$-invariant, is regarded as a function on $\tilde{X}$. 

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In fact, by theorem 2, one may even choose $X_i$, such that

$$\text{dist}(\partial X_i, X_{i-1}) \leq \frac{2\pi m(m-1)}{\sqrt{\sigma}}.$$ 

On the other hand, if $\tilde{X}$ is uniformly acyclic, then, obviously, the filling radii of the fundamental classes of $Y_i$ tend to infinity, which, for $n = m-1 = 3$ implies that $\sigma \leq 0$ by the filling corollary 6.B to 3d covering theorem in the previous section. QED.

8 Filling radii of Submanifolds in manifolds $X$ with $\text{Sc}(X) \geq \sigma > 0$

The geometric inequalities used in the above arguments can be regarded as generalizations of the bounds on the filling radii of closed curves in 3-manifolds.

In fact, our proofs of these inequalities applied to minimizing submanifolds with given boundaries $Y \subset X$ imply the following.

Proposition. Let $X$ be a compact orientable Riemannian manifold with $\text{Sc}(X) \geq \sigma > 0$ of dimension $m \leq 5$ and let $Y \subset X$ be codimension 2 submanifold that is homologous to zero. Let $Y$ be endowed with the metric, where the distance equals to that measured in $X$.

$$\text{dist}_Y(y_1, y_2) = \text{dist}_X(y_1, y_2), \ y_1, y_2 \in Y \subset X.$$ \(^{12}\)

Then the filling radius of $Y$ is bounded by $\text{const}/\sqrt{\sigma}$ for $\text{const} \leq 100$. More generally, if $X$ is a manifold with boundary, where $\text{dist}(Y, \partial X) \geq 100/\sqrt{\sigma}$, then the filling radius of $Y$ satisfies the same bound, provided the homology class $[Y] \in H_{m-2}(X)$ is contained in the image of the inclusion homomorphism $H_{m-2}(\partial X) \to H_{m-2}(X)$.

Furthermore if $m = \text{dim}(X) \leq 4$, similar inequalities hold for codimension one submanifolds $Y \subset X$.

However, unlike the 3-dimensional case, these inequalities don’t yield bounds on $\text{fil.rad}(X)$, where the difficulty resides with "elementary metric lemma" 4.D, which has no apparent higher dimensional counterpart.

9 Perspectives for $m > 5$

The main theorem, combined with theorems 1 and 2, implies non-existence of metrics with $\text{Sc} \geq \sigma > 0$ on special manifolds of dimensions $m > 5$, at least for $m \leq 8$.

For instance, if a complete Riemannian aspherical manifold $X$ is homeomorphic to $X_0 \times \mathbb{R}^2 \times T^1$, where $X_0$ is a compact 5-manifold, then $\inf_x \text{Sc}(X, x) \leq 0$.

Also, as it is observed in [J.W. 2019],

^{12}This $\text{dist}_Y$ may be significantly smaller than the distance associated to the Riemannian metric in $Y$ induced from $X$. 

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images of the rational fundamental classes of compact 3-manifolds with $Sc > 0$ vanish under continuous maps into aspherical spaces.\footnote{This is unknown for $m$-manifolds for $m \geq 4$.}

Thus for instance,

if a closed aspherical manifold $X$ of dimension $m = 3, 4, \ldots, 8$, admits a continuous map $f$ to the torus $\mathbb{T}^{m-3}$, such that the (homology class of the) point-pullback $f^{-1}(t) \subset X$, $t \in \mathbb{T}^{m-3}$, doesn’t vanish in $H_{m-3}(X; \mathbb{Q})$, then $X$ carries no metric with $Sc > 0$. (This is unknown for such maps $f: X \rightarrow \mathbb{T}^{m-n}$ for $m \geq 6$ and $n \geq 4$.)

In general, there are two problems of quite different kinds for proving non-existence of metrics with $Sc > 0$ on aspherical manifolds of high dimensions that we formulate below for $m = 6$ in the form of two conjectures.

A. "The Universal Coverings of Compact acyclic Manifolds are Uniformly Acyclic in Codimension One": Let $\tilde{X}$ be the universal covering of a compact aspherical 6-manifold. Then, for every $d > 0$ there exits a (Riemannian) band $Z \subset \tilde{X}$ bounded by a pair of (finite) hypersurfaces $\partial_\pm$ and $\partial_\ast$, such that $\text{width}(Z) = \text{dist}(\partial_\pm, \partial_\ast) \geq d$ and such that all (finite) hypersurfaces $Y \subset Z$ which separate $\partial_\pm$ from $\partial_\ast$ admit distance non-increasing\footnote{Maps of degrees $\geq \sigma > 0$ to uniformly rationally acyclic 5-manifolds.\footnote{This is because these manifolds are connected sums of $S^2 \times S^4$ and spherical space forms.}} maps of degrees $\neq 0$ to uniformly rationally acyclic 5-manifolds.\footnote{If a compact $m$-manifold $X$ admits, say piecewise linear, map $\phi$ to a $(m-1)$-dimensional polyhedral space $P$, such that the images of the fundamental groups of the connected components of the pullbacks $\phi^{-1}(p) \subset X$ in the fundamental group of $X$ under the inclusion homomorphisms,

$$\pi_1(\phi^{-1}(p)) \rightarrow \pi_1(X),$$

are finite for all $p \in P$, then all continuous maps from $X$ to aspherical spaces send the fundamental rational homology class $[X]_\mathbb{Q}$ to zero. But it is unlikely(?) that all compact Riemannian manifolds with $Sc > 0$ admit $\phi$ with this property for $m = \dim(X) \geq 3$.}

Validity of either A or B would imply that compact aspherical 6-manifolds carry no metrics with $Sc > 0$.

B. "The Fundamental Classes of Manifolds with Large Scalar Curvatures are Small": Compact aspherical 4-manifolds $Y$ with $Sc(Y) \geq \sigma > 0$ have their filling radii bounded

$$\text{fil.rad}_Q(Y) \leq \frac{\text{const}_4}{\sqrt{\sigma}}.$$\footnote{Here as well as everywhere earlier, "distance non-increasing" conditions on maps $f$, may be replaced by $\text{dist}(f(x_1), f(x_2)) \leq \theta(\text{dist}(x_1, x_2))$, where $\theta(d) = \theta_f(d)$ is an arbitrary continuous (hence locally bounded) function in $d \in [0, \infty)$.}

Remark. The solution of B doesn’t seem sufficient for ruling out maps from $Y$ to aspherical spaces with non-zero images of their fundamental classes $[Y] \in H_4(Y; \mathbb{Q})$, since the following following seems unsettled.

Question. Let let $Z$ be an aspherical space, $Y$ be an orientable $n$-dimensional manifold or pseudomanifold, such that all covering spaces $\tilde{Y} \rightarrow Y$ have their filling radii (defined below) bounded by

$$\text{fill.rad}(\tilde{Y}) \leq \text{const} = \text{const}(Y),$$

and let $f: X \rightarrow Z$ be a continuous map.

\footnote{Examples of such bands are those between pairs of parallel hyperplanes in the Euclidean space $\mathbb{R}^6$ and, more generally, between concentric horospheres in (complete simply connected) manifolds with non-positive sectional curvatures.}
Does then the rational homology image $f_*[Y]_\mathbb{Q} \in H_n(Z;\mathbb{Q})$ vanish?

Definition of filling radii of manifolds with boundaries and non-compact manifolds. (Compare with 4.4.C in [M.G. 1983]). Let $Y$ be an orientable manifold or a pseudomanifold of dimension $n$ with boundary.

By definition, the inequality $\text{fil.rad}(Y) \leq R$ for a given metric on $Y$ signifies that there exists a metric extension $Y_+ \supset Y$, such that $\text{dist}(y_+, Y) \leq R$, $y_+ \in Y_+$, and such that the relative fundamental class $[Y] \in H_n(Y, \partial Y)$ vanishes under the inclusion homomorphism

$$H_n(Y, \partial Y) \to H_n(Y_+, U_R(\partial Y)),$$

where $U_R(\partial Y) \subset Y_+$ denotes the $R$-neighbourhood of $\partial Y \subset Y \subset Y_+$.

Then the inequality $\text{fil.rad}(\hat{Y}) \leq R$ for a non-compact (pseudo) manifold is understood as the existence of an exhaustion of $\hat{Y}$ by compact $Y_1 \subset Y_2 \subset \ldots \subset Y_i \subset \ldots \subset \hat{Y}$ with boundaries, such that $\text{fil.rad}(Y_i) \leq R$ for all $i$.

For instance, $\text{fil.rad}(\hat{Y}) \leq \frac{1}{2}\text{width}_{n-2}(\hat{Y})$.

(This applies to complete 3-manifolds $\hat{Y}$ with $Sc(\hat{Y}) \geq \sigma > 0$ and shows that their filling radii are bounded by $\text{const}/\sqrt{\sigma}$.)

Exercise. Show that the above question has positive answer if the fundamental group of $Z$ is residually finite, or, more generally, residually amenable.

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