Uniformly moving clocks in special relativity: Time dilatation, but no relativity of simultaneity or length contraction

J.H.Field
Département de Physique Nucléaire et Corpusculaire, Université de Genève
24, quai Ernest-Ansermet CH-1211Genève 4.
E-mail: john.field@cern.ch

Abstract

Time-like and space-like invariant space-time intervals are used to analyse measurements of spatial and temporal distances defined by two spatially-separated clocks in the same inertial frame. The time dilatation effect is confirmed, but not ‘relativity of simultaneity’ or ‘relativistic length contraction’. How these latter, spurious, effects arise from misuse of the Lorentz transformation is also explained.

PACS 03.30.+p

The Lorentz transformation (LT) relates space-time coordinates \((x,y,z,t)\) of an event in one inertial frame, \(S\), to those of the same event \((x',y',z',t')\) in another inertial frame \(S'\). As is conventional, it is assumed that \(S'\) moves with uniform velocity, \(v\), along the common \(x,x'\) axis of the two frames.

In any actual experiment, the times \(t\) and \(t'\) must be measured by clocks at rest in \(S\) and \(S'\) respectively. Consider a clock, \(C_1'\), at rest in \(S'\), observed from the frame \(S\), to register time \(t'_1\). The proper time of a clock at rest in \(S\) is denoted by \(\tau\). The LT may be used to define the time-like invariant interval relation \([1, 2]\) connecting these times:

\[
c^2(\Delta\tau)^2 - (\Delta x)^2 = c^2(\Delta t')^2 - (\Delta x')^2
\]

where \(c\) is the speed of light in vacuum, and \(\Delta\tau \equiv \tau_1 - \tau_2\) etc. Considering two events on the worldline of \(C_1'\), in virtue of the relations \(\Delta x' = 0\) and \(\Delta x = v\Delta\tau\), Eqn(1) gives the time dilatation (TD) formula relating \(\Delta t'\), a time interval registered by \(C_1'\), as viewed from \(S\), to \(\Delta\tau\), the corresponding time interval recorded by a clock, similar to \(C_1'\), but at rest in \(S\):

\[
\Delta\tau = \gamma \Delta t'
\]

where \(\gamma \equiv 1/\sqrt{1-(v/c)^2}\). Introducing such a clock at rest in \(S\), and setting both this clock and \(C_1'\) to zero, when they both have the same \(x\)-coordinate, enables (2) to be written as:

\[
\tau_1 = \gamma t'_1
\]

where \(t'_1\) is the time recorded by \(C_1'\), as viewed from \(S\), and \(\tau_1\) that of the clock in \(S\), as viewed in \(S\). This method for synchronising clocks in different inertial frames has been called ‘system external synchronisation’ by Mansouri and Sexl [3]. Following Einstein [4, 5] it is the standard way to synchronise clocks in different frames in special relativity.
Carrying out exactly the same procedure for a second clock C2', lying on the \( x' \)-axis, at rest in S', at a different position to C1', gives similarly:

\[ \tau_2 = \gamma t'_2 \]  

(4)

where \( t'_2 \) is the time recorded by C2' as viewed from S and \( \tau_2 \) that of a clock at rest in S, as viewed in S. If now the two clocks in S that have been used to ‘externally synchronise’ \[3\] C1' and C2' also happen to be synchronised in S, so that \( \tau_1 = \tau_2 \), it follows from (3) and (4) that the clocks C1' and C2' are also synchronous, \( t'_1 = t'_2 \), in S', —there is no ‘relativity of simultaneity’ (RS) effect.

A space-like invariant interval between arbitrary events on the world lines of C1' and C2' may be defined as:

\[ (\Delta x_{12})^2 - c^2(\Delta \tau_{12})^2 = (\Delta x'_{12})^2 - c^2(\Delta t'_{12})^2 \]  

(5)

where \( \Delta \tau_{12} \equiv \tau_1 - \tau_2 \) etc and the subscripts 1 and 2 denote the clocks C1' and C2' respectively. The spatial separation of C1' and C2' in S is, by definition, the value of \( \Delta x_{12} \) at any definite instant in S, i.e. when \( \tau_1 = \tau_2 = \tau \) or \( \Delta \tau_{12} = 0 \). It follows from (3) and (4), in the case that the S-frame clocks are synchronised, that, at the same instant, \( t'_1 = t'_2 = t' \) or \( \Delta t'_{12} = 0 \). Thus (5) may be written, taking the positive square root of both sides, as:

\[ \Delta x_{12}(\Delta \tau_{12} = 0) \equiv L = \Delta x'_{12}(\Delta t'_{12} = 0) \equiv L' \]  

(6)

Thus the spatial separation of the clocks is a Lorentz invariant quantity \[6\] —there is no ‘length contraction’ (LC).

As previously discussed in detail elsewhere \[7, 8, 9, 10\] the spurious RS and LC effects of conventional special relativity are the result of an incorrect use of the space-time LT to analyse space and time measurements. With arbitrary clock synchronisation, the LT relating events in the frames S and S’ is:

\[
x' = \gamma [x - v(\tau - \tau_0)]
\]

(7)

\[
t' - t'_0 = \gamma [\tau - \tau_0 - \frac{vx}{c^2}]
\]

(8)

The time offsets \( \tau_0 \) and \( t'_0 \) are specific to a particular synchronised clock and must be chosen in such a way as to correctly describe the times registered by such clocks at different spatial locations \[8\]. The necessity to introduce such additive constants was already pointed out in Ref. \[4\], but never (to the present writer’s best knowledge) implemented by Einstein, or any later author, before the work presented in Ref. \[7\]. Placing the clocks C1' and C2' at \( x'_1 = -L/2 \) and \( x'_2 = L/2 \) respectively, in the case when the clocks in S are synchronised, so that \( \tau_1 = \tau_2 = \tau \) and \( t'_1 = t'_2 = t' \), the clock C1’ is described by the LT:

\[
x'_1 + \frac{L}{2} = \gamma [x_1 + \frac{L}{2} - v\tau] = 0
\]

(9)

\[
t' = \gamma [\tau - \frac{v(x_1 + \frac{L}{2})}{c^2}]
\]

(10)

and C2' by the LT:

\[
x'_2 - \frac{L}{2} = \gamma [x_2 - \frac{L}{2} - v\tau] = 0
\]

(11)

\[
t' = \gamma [\tau - \frac{v(x_2 - \frac{L}{2})}{c^2}]
\]

(12)
According to Eqns(10)-(12) C1’ and C2’ are externally synchronised with the synchronised clocks in S at the time \( \tau = t' = 0 \) when \( x_1 = -L/2 \) and \( x_2 = L/2 \). Using (9) to eliminate \( x_1 \) from (10) or (11) to eliminate \( x_2 \) from (12) gives:

\[
\tau = \gamma t'
\]

This is the time dilatation formula that is valid for all synchronised clocks in the frame S’ when they are viewed from the frame S. The equation (13) contains no spatial coordinates and so is valid for synchronised clocks, one at rest in S the other at rest in S’, as viewed from S, at any spatial locations whatever. A corollary is that simultaneity and the sign of a time interval are both absolute —they are the same in all reference frames whether inertial or accelerated. The TD relation (13) describes the only way in which the space-time geometry of special relativity differs from that of Galilean relativity. The equations (9) and (11), describing the motion of the clocks in S, are the same as the corresponding Galilean formulae.

The spurious ‘RC’ and ‘LC’ effects result from the use of an incorrect LT to describe a moving clock, i.e. using wrong values for the time offsets \( \tau_0 \) and \( t'_0 \) in (7) and (8). For example, setting \( \tau = 0 \) and making the substitutions \( x'_1 \rightarrow x'_2 \), \( x_1 \rightarrow x_2 \) and \( t' \rightarrow t'_2 \) in (9) and (10). That is, substituting the coordinates of C2’ into the LT appropriate for C1’. This gives:

\[
x'_2 + \frac{L}{2} = \gamma \left( x_2 + \frac{L}{2} \right)
\]

\[
t'_2 = -\gamma \frac{v(x_2 + \frac{L}{2})}{c^2}
\]

Since setting \( \tau = 0 \) in (9) and (10) gives \( t' = t'_1 = 0 \) and \( x_1 = x'_1 = -L/2 \), (14) and (15) may be written as:

\[
x'_2 - x'_1 = \gamma (x_2 - x_1)
\]

\[
t'_2 - t'_1 = -\frac{v(x'_2 - x'_1)}{c^2} = -\frac{vL}{c^2}
\]

(16) is the LC effect —the spatial separation of the clocks is reduced by the factor 1/\( \gamma \) in S as compared to S’, and (17) is the RS effect —simultaneous events in S: \( \tau_1 = \tau_2 = 0 \) are not so in S’: \( t'_1 > t'_2 \) and the sign of \( t'_2 - t'_1 \) depends on that of \( v \). These spurious predictions result from the use of wrong values of \( \tau_0 \) and \( t'_0 \) for the clock C2’. This is what has, hitherto, been universally done, following Einstein [4, 5], when applying the LT to space-time measurements.

The standard LT derived by Einstein is given by setting \( L = 0 \) in (9) and (10) or (11) and (12). This transformation externally synchronises clocks in S and S’ at \( \tau = t' = 0 \) when \( x = x' = 0 \). The usual application of this LT (e.g. the discussion of LC in Ch XII of Ref. [5]) assumes, incorrectly, that it also valid for a synchronised clock in S’ at \( x' = L \). This gives the same LC prediction as (16), and RS prediction as (17). The close connection between LC and RS in (16) and (17), discussed in [11], was not mentioned by Einstein, and is typically not mentioned either in text books on special relativity. An exception is the book by Stephenson and Kilmister [12].

The essential point made in this letter is that physics, either that underlying the clock mechanism, or the relativistic TD effect of Eqn(2), predicts only the rate of a clock, not
its setting. The spurious RS and LC effects arise because a fixed clock setting, built into the ‘standard’ LT ((9) and (10) with \(L = 0\)) is misinterpreted as a physical time interval.

A pendulum constitutes a clock. Its motion is completely defined by its period (determined by physics) and a single geometrical constant, say the angular displacement from equilibrium at which its kinetic energy vanishes, which is analogous to the clock offset \(t'_0\). This constant is arbitrary and not predictable. The RS effect of conventional special relativity is tantamount to predicting the angular displacement of a pendulum, in a moving frame, knowing only its length and the acceleration of gravity — the essential physical parameters of the problem (c.f. Eqn(17), where the essential physical parameters are \(L\), \(v\) and \(c\)) — an evidently nonsensical procedure.

Experiments have recently been proposed to search for the existence (or not) of the RS effect [13].

**Added Note**

The time intervals in the space-like invariant interval relation (5) do not correspond, as assumed in the derivation of (6), to the case when \(C_1'\) and \(C_2'\) are synchronised. Equation (5) is derived on the assumption that the usual LT, given by setting \(t'_0 = \tau_0 = 0\) in (7) and (8), is valid for both clocks. This implies that the clocks are not synchronised. Using the correct LT for each clock, (9) and (10) for \(C_1'\) and (11) and (12) for \(C_2'\), which synchronise the clocks so that \(t' = \tau = 0\) when \(x(C_1') = -\frac{L}{2}\) and \(x(C_2') = \frac{L}{2}\), to calculate the intervals \(\Delta x'_{12}\) and \(\Delta t'_{12}\) gives, on replacing \(L\) by \(L'\) on the left sides of (9) and (11):

\[
(\Delta x'_{12})^2 - c^2(\Delta t'_{12})^2 = (\Delta x_{12})^2 - c^2(\Delta \tau_{12})^2 + 2\gamma L'(\Delta x_{12} - L) - 2\Delta x_{12}L - 2\gamma v \Delta \tau_{12} + L^2 + (L')^2 \quad (18)
\]

Setting \(\Delta t'_{12} = \Delta \tau_{12} = 0\) and \(\Delta x_{12} = L\) in this equation leads not to the relation \(L = L'\) but to the identity

\[
\Delta x'_{12} \equiv L' = L' \quad (19)
\]

In fact the equality of \(L\) and \(L'\) is established in a straightforward manner from, for example, (9) on replacing \(L\) by \(L'\) on the left side:

\[
x_1' + \frac{L'}{2} = \gamma [x_1 + \frac{L}{2} - v\tau] = 0 \quad (20)
\]

The parameters \(L\) and \(L'\) are constants fixed by the choice of spatial coordinate systems in \(S\) and \(S'\) respectively, that are independent both of time and the velocity \(v\):

\[
x_1'(t') = -\frac{L'}{2} \quad (\text{all } t') \quad (21)
\]

\[
x_1(\tau = 0) = -\frac{L}{2} \quad (22)
\]

Since \(L\) and \(L'\) in (20) are independent of \(v\), the equation is valid for all values of \(v\), in particular for \(v = 0, \gamma = 1\) and \(x \to x'\):

\[
x_1' + \frac{L'}{2} = x_1' + \frac{L}{2} \quad (23)
\]

or

\[
L' = L \quad (24)
\]
Acknowledgements

I gratefully acknowledge discussions with, or correspondence from: G.Boas, B.Echenard, M.Grünewald, Y.Keilman, M.Kloster, B.Rothenstein and D.Utterback, whose tenacious defence of the conventional interpretation of the LT forced me to sharpen and condense the logical structure of my arguments in order to write this letter.

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