A Proof of the Front-Door Adjustment Formula

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Abstract

We provide a proof of the Front-Door adjustment formula using the do-calculus.

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1. Introduction

In (Pearl, 2009), a formula for computing the causal effect of $X$ on $Y$ in the causal model of figure 1 is derived and used to motivate the definition of front-door criterion. Pearl then states, without proof, the Front-Door Adjustment Theorem (Pearl [2009] Theorem 3.3.4). In section 3.4.3, he provides a symbolic derivation of the front door adjustment formula for the same example from the do-calculus. In this short technical report, we provide a proof of Theorem 3.3.4 using the do-calculus. The next section consists of the proof of the front-door adjustment formula; the theorem is restated for the reader’s convenience. The do-calculus rules, the back-door criterion, the back-door adjustment formula, and the front-door criterion are in the slide set provided as an ancillary document.

2. Front-Door Adjustment Theorem

Theorem 1 (Front-Door Adjustment) If a set of variables $Z$ satisfies the front-door criterion relative to $(X, Y)$ and if $P(x, z) > 0$, then the causal effect of $X$ on $Y$ is identifiable and is given by the formula

$$P(y|\hat{x}) = \sum_z P(z|x) \sum_{x'} P(y|x', z) P(x')$$

Proof By well known probability identities (for example, the Fundamental Rule and the Theorem of Total Probability), $P(y|\hat{x}) = \sum_z P(y|z, \hat{x}) P(z|\hat{x})$. In Step 1, below, we show how to compute $P(z|\hat{x})$ using only observed quantities. In Steps 2 and 3, we show how to compute $P(y|z, \hat{x})$ using only observed quantities; this part of the proof is by far the hardest.
• Step 1: Compute $P(z|\hat{x})$
  - $X \perp Z$ in $G_X$ because there is no outgoing edge from $X$ in $G_X$, and also by condition (ii) of the definition of the front-door criterion, all back-door paths from $X$ to $Z$ are blocked.

  - $G$ satisfies the applicability condition for Rule 2:
    
    $$P(y|\hat{x}, \hat{z}, w) = P(y|\hat{x}, z, w) \quad \text{if} \quad (Y \perp Z|X, W)_{G_{XZ}}.$$

    - In Rule 2, set $y = z$, $x = \emptyset$, $z = x$, $w = \emptyset$:
      $$P(z|\hat{x}) = P(z|x)$$
      because $(Z \perp X)_{G_X}$

• Step 2: $P(y|\hat{z})$
  - $P(y|\hat{z}) = \sum_x P(y|x, \hat{z})P(x|\hat{z})$.
  - $X \perp Z$ in $G_Z$ because there is no incoming edge to $Z$ in $G_Z$, and also all paths from $X$ to $Z$ either by condition (ii) of the definition of the front-door criterion (blue-type paths), or because of existence of a collider node on the path (green-type paths) are blocked.

  - $G$ satisfies the applicability condition for Rule 3:
    $$P(y|\hat{x}, \hat{z}, w) = P(y|\hat{x}, w) \quad \text{if} \quad (Y \perp Z|X, W)_{G_{X,\hat{z}W}}.$$
– In Rule 3, set \( y = x, x = \emptyset, z = z, w = \emptyset \):

\[
P(x|\hat{z}) = P(x) \quad \text{because} \quad (Z \perp\!\!\!\perp X)_{G_{\bar{z}}}.
\]

– \((Z \perp\!\!\!\perp Y|X)_{G_{\bar{z}}}\) because there is no outgoing edge from \( Z \) in \( G_{\bar{z}} \), and also by condition (iii) of the definition of the front-door criterion, all back-door paths from \( Z \) to \( Y \) are blocked by \( X \).

\[
P(y|x, \hat{z}) = P(y|x, z) \quad \text{because} \quad (Z \perp\!\!\!\perp Y|X)_{G_{\bar{z}}}.
\]

\[
P(y|\hat{z}) = \sum_x P(y|x, \hat{z})P(x|\hat{z}) = \sum_x P(y|x, z)P(x) \quad (3)
\]

This formula is a special case of the back-door formula.

• Step 3: Compute \( P(y|\hat{x}) \)

As already noted at the beginning of the proof, \( P(y|\hat{x}) = \sum_{\hat{z}} P(y|z, \hat{x})P(z|\hat{x}) \).

– \( P(z|\hat{x}) = P(z|x) \), as shown in Step 1 (see equation (2))

There is no rule of the do-calculus that allows the elimination of the hat from \( P(y|z, \hat{x}) \), so we take a circuitous route: we first replace an observation \( z \) with an intervention \( \hat{z} \) using Rule 2, and then remove an intervention variable \( \hat{z} \) using Rule 3.

– \((Y \perp\!\!\!\perp Z|X)_{G_{\bar{Z}}}\) because there is no outgoing edge from \( Z \) in \( G_{\bar{Z}} \), and also by condition (iii) of the definition of the front-door criterion, all back-door paths from \( Z \) to \( Y \) are blocked by \( X \).

– \( G \) satisfies the applicability condition for Rule 2: \( P(y|\hat{x}, \hat{z}, w) = P(y|\hat{x}, z, w) \) if \( (Y \perp\!\!\!\perp Z|X,W)_{G_{\bar{Z}}} \).

– In Rule 2, set \( y = y, x = \emptyset, z = z, w = x \):

\[
P(y|x, \hat{z}) = P(y|x, \hat{z}) \quad \text{because} \quad (Z \perp\!\!\!\perp Y|X)_{G_{\bar{z}}}.
\]
- \( (Y \independent X|Z)_{G_{XZ}} \) because there is no incoming edge to \( X \) in \( G_{XZ} \), and also all paths from \( X \) to \( Y \) are blocked either because of condition (i) of the definition of the front-door criterion (blue-type paths)[directed paths from \( X \) to \( Y \)], or because of the existence of a collider on the path (green-type paths) (note that the case \( T \in Z \) cannot happen because there is no incoming edge to \( Z \) in \( G_{XZ} \)).

- \( G \) satisfies the applicability condition for Rule 3:

\[
P(y|x', z, w) = P(y|z, w) \quad \text{if} \quad (Y \independent Z|X, W)_{G_{XZ}}.
\]

- In Rule 3, set \( y = y, x = z, z = x, w = \emptyset \):

\[
P(y|z, x) = P(y|z) \quad \text{because} \quad (Y \independent Z|X)_{G_{XZ}}.
\]

Now, by equations (2) and (3),

\[
P(y|x) = \sum_z P(y|z, x)P(z|x) = \sum_z P(z|x) \sum_{x'} P(y|x', z)P(x').
\]

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References

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