The number of configurations in the full shift with a given least period

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Abstract

For any group $G$ and any set $A$, consider the shift action of $G$ on the full shift $A^G$. A configuration $x \in A^G$ has least period $H \leq G$ if the stabiliser of $x$ is precisely $H$. Among other things, the number of such configurations is interesting as it provides an upper bound for the size of the corresponding Aut($A^G$)-orbit. In this paper we show that if $G$ is finitely generated and $H$ is of finite index, then the number of configurations in $A^G$ with least period $H$ may be computed by using the Möbius function of the lattice of subgroups of finite index in $G$. Moreover, when $H$ is a normal subgroup, we classify all situations such that the number of $G$-orbits with least period $H$ is at most 10.

Keywords: Full shift; periodic configurations; subgroup lattice; Möbius function.

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1 Introduction

Let $G$ be a group and let $A$ be a set. Consider the set $A^G$ of all functions from $G$ to $A$ equipped with the shift action of $G$, defined by

$$(g \cdot x)(h) := x(g^{-1}h),$$

for all $g, h \in G$ and $x \in A^G$. Although we shall not focus on this, the set $A^G$ is usually seen as a topological space with the product topology of the discrete topology on $A$.

The $G$-space $A^G$ is a fundamental object in areas such as symbolic dynamics and the theory of cellular automata (e.g. see [4, 10]). Following [4], we call the elements of $A^G$ configurations. For any $x \in A^G$, the stabiliser $G_x$ of $x$ and the $G$-orbit $Gx$ of $x$ are defined as follows:

$$G_x := \{g \in G : g \cdot x = x\} \quad \text{and} \quad Gx := \{g \cdot x \in A^G : g \in G\}.$$

For a subgroup $H$ of $G$, a configuration $x \in A^G$ has period $H$, or is $H$-periodic, if $h \cdot x = x$ for all $h \in H$, or, equivalently, if $H \leq G_x$. Denote by Fix($H$) the subset of $A^G$ consisting of all $H$-periodic configurations. It is known (see [4, Proposition 1.3.3]) that Fix($H$) is in bijection with $A^{H \setminus G}$, where $H \setminus G = \{Hg : g \in G\}$ is the set of rights cosets of $H$ in $G$. Hence, it follows that $|\text{Fix}(H)| = |A|^{[G:H]}$, where $[G : H] := |H \setminus G|$ is the index of $H$ in $G$. In particular, the

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configurations whose period is the trivial subgroup of \( G \) are known as \textit{aperiodic points}, and have been used in \([6]\) as powerful tools to study the dynamics in \( A^G \) and its \textit{subshifts}, or \textit{subflows} (i.e. closed \( G \)-equivariant subsets of \( A^G \)).

We say that \( x \in A^G \) has \textit{least period}, or \textit{fundamental period}, \( H \) if \( G_x = H \) (c.f. [10] Definition 1.1.3.). In this paper we are interested in the number of subshifts of \( A^G \),\( \psi(G;A) \) of configurations with least period \( H \):\[ \psi_H(G;A) := |\{ x \in A^G : G_x = H \}|. \]

If \( x, y \in A^G \) satisfy that \( y = g \cdot x \), then \( G_y = gG_xg^{-1} \); hence, it is sometimes convenient to consider the \( G \)-invariant set \( \{ x \in A^G : [G_x] = [H] \} \), where \([H] := \{ gHg^{-1} : g \in G \} \) is the conjugacy class of \( H \), and its cardinality
\[ \psi_H(G;A) := |\{ x \in A^G : [G_x] = [H] \}|. \]

As \( \psi_H(G;A) = \psi_{gHg^{-1}}(G;A) \) for all \( g \in G \), we have
\[ \psi_H(G;A) = |[H]| \psi_H(G;A). \]

Finally, we also consider the number of \( G \)-orbits whose stabiliser is conjugate to \( H \):
\[ \alpha_H(G;A) := |\{ Gx : [G_x] = [H] \}|. \]

By the Orbit-Stabiliser Theorem ([14 Theorem 7.2.1]), all \( G \)-orbits inside \( \{ x \in A^G : [G_x] = [H] \} \) have size \( |G : H| \); therefore, we have
\[ \alpha_H(G;A) |G : H| = \psi_H(G;A). \]

Besides being interesting for their own right, the above numbers have connections with the structure of the automorphism group of \( A^G \). Recall that a map \( \tau : A^G \to A^G \) is \textit{G-equivariant} if \( \tau(g \cdot x) = g \cdot \tau(x) \), for all \( g \in G \), \( x \in A^G \). Let \( \text{Aut}(A^G) \) the group of all \( G \)-equivariant homeomorphisms of \( A^G \). By the Curtis-Heldund Theorem ([4 Theorem 1.8.1]), \( \text{Aut}(A^G) \) is the same as the group of invertible cellular automata of \( A^G \). It follows by \( G \)-equivariance that for every \( \tau \in \text{Aut}(A^G) \), \( x \in A^G \), we have \( G_x = G_{\tau(x)} \). Thus, \( \psi_{G_x}(G;A) \) is an upper bound for the cardinality of the \( \text{Aut}(A^G) \)-orbit of \( x \). Moreover, if the group \( G \) is finite, the structure of \( \text{Aut}(A^G) \) was described in \([3\) Theorem 3\] as
\[
\text{Aut}(A^G) \cong \prod_{i=1}^{r}((N_G(H_i)/H_i) \wr \text{Sym}_{\alpha_i}),
\] (1)

where \([H_1], \ldots, [H_r] \) is the list of all different conjugacy classes of subgroups of \( G \), and \( \alpha_i = \alpha_{[H_i]}(G;A) \), as defined above. Hence, the structure of \( \text{Aut}(A^G) \) completely depends on the quotient groups \( N_G(H_i)/H_i \), which may be easily calculated by knowing the group \( G \), and the integers \( \alpha_i(G;A) \), which depend on \( \psi_H(G;A) \). Finally, in [11] [2], the sets of points of a given least period were shown in [3], in [3] [1], a fundamental tool of the automorphism groups of shifts of finite type, which include the group \( \text{Aut}(A^G) \).

As \( \psi_{G_x}(G;A) \) is finite if and only if \( |G : H| \) is finite (see Lemma [3] below), we shall focus on finite index subgroups of \( G \). In the first part of this paper, we prove that, when \( G \) is finitely generated, the poset \( L(G) \) of finite index subgroups of \( G \) is a locally finite lattice, so we use M"obius inversion to show that
\[
\psi_H(G;A) = \sum_{H \leq K \leq G} \mu(H,K)|A^[G,K],
\] (2)

where \( \mu \) is the M"obius function of \( L(G) \). In the second part of this paper, we note that if \( H \) is a normal subgroup, then \( \psi_H(G;A) = \psi_1(G/H;A) \) and \( \alpha_H(G;A) = \alpha_1(G/H;A) \). Hence, by
computing the M"obius function of the subgroup lattice of all finite groups of size up to 7, we classify under which situations we have $\alpha_H(G; A) \leq 10$.

Our work generalises previous results known in the literature. When $G = \mathbb{Z}_n$ is a cyclic group and $H = 1$ is the trivial subgroup, $\alpha_H(\mathbb{Z}_n; A)$ is equivalent to the number of aperiodic necklaces of length $n$, and equation (2) gives the so-called Moreau's necklace-counting function [12]. Moreover, $\alpha_H(\mathbb{Z}_n; A)$ is also equivalent to the number of Lyndon words of length $n$ (see Sec. 5.1. in [11]). For a finite group $G$, this equation may be derived using the result of Sec. 4 in [9]. However, as far as we know, equation (2) had not been derived when $G$ is an arbitrary finitely generated group.

2 Periodic configurations when $G$ is finitely generated

For the rest of the paper, let $A$ be a set with at least two elements and assume that $\{0, 1\} \subseteq A$. We begin by justifying our claim that $\psi_H(G; A)$ is finite if and only if $[G : H]$ is finite.

Lemma 1. Let $G$ be a group and let $H$ be a subgroup of $G$. Then $\psi_H(G; A)$ is finite if and only if $[G : H]$ is finite.

Proof. If $[G : H]$ is finite, then $\psi_H(G; A)$ is clearly finite, as every configuration with least period $H$ is contained in $\text{Fix}(H)$ and $|\text{Fix}(H)| = |A|^{[G : H]} < \infty$.

Conversely, suppose that $[G : H]$ is infinite. Let $T \subseteq G$ be a transversal for the set of right cosets of $H$ in $G$, i.e., $T$ contains exactly one element from each right coset of $H$ in $G$. It is clear that $|T| = [G : H]$. For each $s \in T$, consider the configuration $x_s \in A^G$ defined by

$$x_s(g) = \begin{cases} 1 & \text{if } g \in Hs \\ 0 & \text{otherwise} \end{cases},$$

for any $g \in G$. Given $h \in H$, then $h \cdot x_s(g) = x_s(h^{-1}g) = x_s(g)$, as $h^{-1}g \in Hs$ if and only if $g \in Hs$. Hence, $H \leq G_{x_s}$. On the other hand, if $k \in G_{x_s}$, then $k \cdot x_s = x_s$; in particular we have $(k \cdot x_s)(s) = x_s(k^{-1}s) = x_s(s) = 1$, which implies that $k^{-1}s \in Hs$. Therefore, $k \in H$, which shows that $G_{x_s} = H$. As $|T| = [G : H]$ is infinite, we have constructed infinitely many different configurations with least period $H$, which establishes that $\psi_H(G; A)$ is infinite. \qed

We shall recall some basic definitions on posets; for further details see [15, Ch. 3]. Recall that a partially ordered set, or a poset, is a set $P$ equipped with a partial order relation $\leq$. Given $s, t \in P$ with $s \leq t$, define the closed interval $[s, t] := \{u \in P : s \leq u \leq t\}$. We say that $P$ is locally finite if every closed interval of $P$ is finite. A chain of $P$ is a subposet $S$ of $P$ that is totally ordered, i.e., any two elements of $S$ are comparable. For $t \in P$, the principal order ideal generated by $t$ is $\Lambda_t := \{s \in P : s \leq t\}$, and the principal dual order ideal generated by $t$ is $V_t := \{s \in P : s \geq t\}$.

A lattice is a poset $L$ for which every pair of elements $s, t \in L$ has a least upper bound, denoted by $s \lor t$ and read $s$ join $t$, and a greatest lower bound, denoted by $s \land t$ and read $s$ meet $t$.

The M"obius function of a locally finite poset $P$ is the map $\mu : P \times P \rightarrow \mathbb{Z}$ defined inductively by the following equations:

$$\mu(a, a) = 1, \; \forall a \in P,$$

$$\mu(a, b) = 0, \; \forall a \not\leq b,$$

$$\sum_{a \leq c \leq b} \mu(a, c) = 0, \; \forall a < b.$$

The M"obius function is the inverse of the zeta function of a locally finite poset, and it importantly satisfies the so-called M"obius inversion formula (see [15, Sec. 3.7]). In this section we shall use the dual form of the M"obius inversion formula [15, Proposition 3.7.2].
Theorem 1 (Möbius inversion formula, dual form). Let \( P \) be a poset for which every principal dual order ideal \( V_t \) is finite. Consider functions \( f, g : P \to K \), where \( K \) is a field. Then

\[
g(t) = \sum_{s \geq t} f(s), \quad \forall t \in P,
\]

if and only if

\[
f(t) = \sum_{s \geq t} g(s)\mu(t, s), \quad \forall t \in P.
\]

For any group \( G \), it is standard to consider the poset of all subgroups of \( G \) ordered by inclusion. Here, we shall consider the poset \( L(G) \) of all subgroups of \( G \) of finite index ordered by inclusion. The following is a key observation for this section.

Lemma 2. The poset \( L(G) \) is a lattice. Furthermore, if \( G \) is finitely generated, then for every \( H \in L(G) \), the principal dual order ideal \( V_H = \{ K \leq G : H \leq K \} \) is finite, so \( L(G) \) is a locally finite lattice.

Proof. We shall show that \( L(G) \) is a sublattice of the subgroup lattice of \( G \) by showing that it is closed under the join, given by \( H \vee J = \langle H \cup J \rangle \), and the meet, given by \( H \wedge J = H \cap J \).

Let \( H \) and \( K \) be subgroups of \( G \) such that \( H \leq K \). It is well-known (see, for instance [14, Theorem 3.1.3]) that the indices of \( H \) and \( K \) in \( G \) satisfy, as cardinal numbers, that

\[
[G : H] = [G : K][K : H].
\]

Hence, if \([G : H]\) is finite, then \([G : K]\) must be finite. This implies that for any \( H, J \in L(G) \), then \( \langle H \cup J \rangle \in L(G) \). On the other hand, it is also well-known (see, for instance [14, Theorem 3.1.6]) that the intersection of subgroups of finite index has finite index, so \( H \cap J \in L(G) \), and the first part of the lemma follows.

For the second part, for any \( H \in L(G) \) and \( K \in V_H \), the index of \( K \) in \( G \) must be a divisor of \([G : H]\). The result follows as in a finitely generated group there are only finitely many subgroups of a given finite index (this is a well-known theorem by M. Hall [7]; see also [14, Theorem 4.20]).

The previous lemma allows us to use the Möbius inversion formula for the poset \( L(G) \) when \( G \) is finitely generated. Let \( \mu \) be the Möbius function of \( L(G) \).

Theorem 2. Let \( G \) be a finitely generated group, let \( H \) be a subgroup of \( G \) of finite index, and let \( A \) be a finite set. Then,

\[
\psi_H(G; A) = \sum_{H \leq K \leq G} \mu(H, K)|A|^{[G : K]}.
\]

Proof. It follows from the definitions that

\[
|\text{Fix}(H)| = \sum_{K \geq H} \psi(K; G; A).
\]

By Lemma 2 this summation is finite and we may use Theorem 1 with \( g(H) = |\text{Fix}(H)| \) and \( f(K) = \psi_K(G; A) \). Therefore, we obtain

\[
\psi_H(G; A) = \sum_{K \geq H} \mu(H, K)|\text{Fix}(K)|.
\]

The result follows as \( |\text{Fix}(K)| = |A|^{[G : K]} \) by [4, Proposition 1.3.3].
Remark 1. Note that, for any $H, J \in L(G)$, the value of $\mu(H, J)$ only depends on the on the interval $[H, J]$. Hence, $\psi_H(G; A)$ may be calculated by only knowing the subposet $[H, G]$.

**Corollary 1.** With the notation of Theorem 2, suppose that the interval from $H$ to $G$ consists of a chain $H = H_0 < H_1 < \cdots < H_k = G$. Then,

$$\psi_H(G; A) = |A[G : H]| - |A[H_1 : H]|.$$

In particular, if $H$ is a maximal subgroup of $G$, then

$$\psi_H(G; A) = |A[G : H]| - |A|.$$

**Proof.** By Theorem 2

$$\psi_H(G; A) = \sum_{i=0}^{k} \mu(H, H_i) |A[H_i : H]|.$$

Now, by the definition of the Möbius function,

$$\mu(H, H_0) = 1,$$
$$\mu(H, H_1) = -1,$$
$$\mu(H, H_i) = 0, \quad \forall i = 2, 3, \ldots, k.$$

The result follows.

**Corollary 2.** With the notation of Theorem 2

$$\psi_{[H]}(G; A) = |[H]| \sum_{H \leq K \leq G} \mu(H, K) |A[K : G]|,$$

$$\alpha_{[H]}(G; A) = \frac{|[H]|}{[G : H]} \sum_{H \leq K \leq G} \mu(H, K) |A[K : G]|.$$

### 3 Configurations with normal period

In this section we shall specialise on the case when $H$ is a normal subgroup of $G$ of finite index. In this case, the conjugacy class of $H$ just contains $H$ itself, so

$$\psi_H(G; A) = \psi_{[H]}(G; A).$$

Denote by 1 the trivial subgroup. The following result has been noted in [3, Lemma 6].

**Lemma 3.** Let $G$ be any group and let $H$ be a normal subgroup of $G$ of finite index. Then,

$$\psi_H(G; A) = \psi_{[1]}(G/H; A) \quad \text{and} \quad \alpha_{[H]}(G; A) = \alpha_{[1]}(G/H; A).$$

**Proof.** By [4, Proposition 1.3.7.], there is a $G/H$-equivariant bijection between $A^{G/H}$ and $\text{Fix}(H)$. Hence, configurations in $A^{G/H}$ with trivial stabiliser are in bijection with the configurations in $A^G$ with stabiliser equal to $H$.

The previous lemma allows to apply the machinery of Möbius functions of subgroup lattices which has been developed for a variety of finite groups (e.g. see [5, 8, 13]).

Recall that the classical Möbius function $\mu$ of the poset of natural numbers $\mathbb{N}$ ordered by divisibility is given by

$$\mu(d) = \begin{cases} 
0 & \text{if $d$ has a squared prime factor} \\
1 & \text{if $d$ is square-free with an even number of prime factors} \\
-1 & \text{if $d$ is square-free with an odd number of prime factors.}
\end{cases}$$

Using Lemma 3, the following result gives the values of $\psi_H(G; A)$ in some particular cases when $H$ is a normal subgroup of $G$. 

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**Remark 1.** Note that, for any $H, J \in L(G)$, the value of $\mu(H, J)$ only depends on the on the interval $[H, J]$. Hence, $\psi_H(G; A)$ may be calculated by only knowing the subposet $[H, G]$.

**Corollary 1.** With the notation of Theorem 2, suppose that the interval from $H$ to $G$ consists of a chain $H = H_0 < H_1 < \cdots < H_k = G$. Then,

$$\psi_H(G; A) = |A[G : H]| - |A[H_1 : H]|.$$

In particular, if $H$ is a maximal subgroup of $G$, then

$$\psi_H(G; A) = |A[G : H]| - |A|.$$

**Proof.** By Theorem 2

$$\psi_H(G; A) = \sum_{i=0}^{k} \mu(H, H_i) |A[H_i : H]|.$$

Now, by the definition of the Möbius function,

$$\mu(H, H_0) = 1,$$
$$\mu(H, H_1) = -1,$$
$$\mu(H, H_i) = 0, \quad \forall i = 2, 3, \ldots, k.$$

The result follows.

**Corollary 2.** With the notation of Theorem 2

$$\psi_{[H]}(G; A) = |[H]| \sum_{H \leq K \leq G} \mu(H, K) |A[K : G]|,$$

$$\alpha_{[H]}(G; A) = \frac{|[H]|}{[G : H]} \sum_{H \leq K \leq G} \mu(H, K) |A[K : G]|.$$
Lemma 4. Let $G$ be a finitely generated group, let $H$ be a normal subgroup of $G$ of finite index, and let $A$ be a finite set. Let $n \in \mathbb{N}$, and let $p$ and $p'$ be two distinct primes.

1. If $G/H \cong \mathbb{Z}_n$, then $\psi_H(G; A) = \sum_{d|n} \mu(d)|A|^{n/d}$.

2. If $G/H \cong \mathbb{Z}_{p^k}$, then $\psi_H(G; A) = |A|^{p^k} - |A|^{p^k-1}$.

3. If $G/H \cong \mathbb{Z}_{pp'}$, then $\psi_H(G; A) = |A|^{pp'} - |A|^p - |A|^{p'} + |A|$.

4. If $G/H \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$, then $\psi_H(G; A) = |A|^{p^2} - (p+1)|A|^p + p|A|$.

Proof. Parts (1), (2) and (3) follow as it is well-known that $\mu(1, \mathbb{Z}_n) = \tilde{\mu}(n)$ (as the subgroup lattice of $\mathbb{Z}_n$ is isomorphic to the divisibility lattice of $n$). For part (4), just observe that the group $\mathbb{Z}_p \oplus \mathbb{Z}_p$ has $\frac{p^2-1}{p-1} = p+1$ subgroups isomorphic to $\mathbb{Z}_p$ (as each of the $p^2-1$ nontrivial elements of $\mathbb{Z}_p \oplus \mathbb{Z}_p$ generates a subgroup with $p-1$ nontrivial elements), which account for all its proper nontrivial subgroups.

In the rest of this section, we shall focus on the exact determination of the small values of $\alpha_{|H|}(G; A)$. The inspiration for this question is Lemma 5 in [3], which established, without using the Möbius function, that $\alpha_{|H|}(G; A) = 1$ if and only if $[G : H] = 2$ and $|A| = 2$. In general, the classification of small values of $\alpha_{|H|}(G; A)$ is relevant as it classifies configurations with small $\text{Aut}(A^G)$-orbits, and, when $G$ is finite, it classifies the small degrees of the symmetric groups appearing in the decomposition $\mathcal{C}$ of $\text{Aut}(A^G)$.

For $x \in A^G$, we have $G_x = G$ if and only if $x$ is a constant configuration. As we have precisely $|A|$ constant configurations in $A^G$, then $\alpha_{|G|}(G; A) = |A|$. Hence, we shall exclude the case $H = G$ in the following theorem. Moreover, we exclude the degenerate case $|A| = 1$.

\[
\begin{array}{c|cccc}
G/H & |A| & 2 & 3 & 4 & 5 \\
\hline
\mathbb{Z}_2 & 1 & 3 & 6 & 10 \\
\mathbb{Z}_3 & 2 & 8 & 20 & 40 \\
\mathbb{Z}_2^2 & 2 & 15 & 54 & 140 \\
\mathbb{Z}_4 & 3 & 18 & 60 & 150 \\
\mathbb{Z}_5 & 6 & 48 & 204 & 624 \\
S_3 & 7 & 108 & 650 & 2540 \\
\mathbb{Z}_6 & 9 & 116 & 670 & 2580 \\
\mathbb{Z}_7 & 18 & 312 & 2340 & 11160 \\
\end{array}
\]

Table 1: Small values for $\alpha_{|H|}(G; A)$ with $H$ normal in $G$.

Theorem 3. Let $G$ be a finitely generated group, let $H$ be a proper normal subgroup of $G$ of finite index, and let $A$ a finite set with at least two elements.

1. $\alpha_{|H|}(G; A) = 1$ if and only if $|A| = 2$ and $[G : H] = 2$.

2. $\alpha_{|H|}(G; A) = 2$ if and only if $|A| = 2$ and $[G : H] = 3$, or $|A| = 2$ and $G/H \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. 
Figure 1: Subgroup lattice of $S_3$.

3. $\alpha_{[H]}(G; A) = 3$ if and only if $|A| = 3$ and $|G : H| = 2$, or $|A| = 2$ and $G/H \cong \mathbb{Z}_4$.
4. $\alpha_{[H]}(G; A) = 6$ if and only if $|A| = 2$ and $|G : H| = 5$, or $|A| = 4$ and $|G : H| = 2$.
5. $\alpha_{[H]}(G; A) = 7$ if and only if $|A| = 2$ and $G/H \cong S_3$.
6. $\alpha_{[H]}(G; A) = 8$ if and only if $|A| = 3$ and $|G : H| = 3$.
7. $\alpha_{[H]}(G; A) = 9$ if and only if $|A| = 2$ and $G/H \cong \mathbb{Z}_6$.
8. $\alpha_{[H]}(G; A) = 10$ if and only if $|A| = 5$ and $|G : H| = 2$.
9. $\alpha_{[H]}(G; A) \neq 4$ and $\alpha_{[H]}(G; A) \neq 5$.

Proof. By Corollary 1.7.2 in [6],

$$|A|^{-1} |H|^{-1} \leq \alpha_{[H]}(G/H, A) = \alpha_{[H]}(G; A).$$

(This lower bound has been improved in Theorem 5 in [3], but the above is enough for this proof). Hence, we see that $\alpha_{[H]}(G/H, A)$ is a strictly increasing function on both $|G : H|$ and $|A|$. Table 1 shows all values of $\alpha_{[H]}(G/H, A)$ with $|G : H| \leq 7$ and $|A| \leq 5$. Most of these values may be calculated by using the formulas of Lemma 1; the only exception is the case $G/H \cong S_3$, which may be directly computed using the Möbius function of the subgroup lattice of $S_3$ (see Figure 1). The result follows by inspection of Table 1.

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