LONG-TIME BEHAVIOR OF SSEP WITH SLOW BOUNDARY

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Abstract. We consider the symmetric simple exclusion process with slow boundary first introduced in [Baldasso et al., Journal of Statistical Physics, 167(5), 2017]. We prove a law of large number for the empirical measure of the process under a longer time scaling instead of the usual diffusive time scaling.

1. Introduction

Interacting particle systems in contact with reservoirs have been investigated in various literature [5, 7, 8]. We study in this article the symmetric simple exclusion process (SSEP) on a line segment \( \{1, \ldots, N-1\} \) with slow boundary, which is first introduced by Baldasso et al. [1]. Here, \( N \) is the scaling parameter. There is at most one particle per site. In the bulk, a particle jumps to one of its neighbors at rate one provided the target site is empty. Fix parameters \( c > 0, \theta \geq 0 \) and \( \alpha, \beta \in (0, 1) \). At the boundary site 1 (resp. \( N-1 \)), a particle is created at rate \( c\alpha N^{-\theta} \) (resp. \( c\beta N^{-\theta} \)) if site 1 (resp. \( N-1 \)) is empty, and a particle is destroyed at rate \( c(1-\alpha)N^{-\theta} \) (resp. \( c(1-\beta)N^{-\theta} \)) if site 1 (resp. \( N-1 \)) is occupied. Therefore, the particle density of the left (resp. right) reservoir is \( \alpha \) (resp. \( \beta \)), and the interaction strength between the bulk and the reservoirs is \( cN^{-\theta} \). The hydrodynamic equation of the model turns out to be the heat equation with Dirichlet boundary conditions if \( \theta < 1 \), with Robin boundary conditions if \( \theta = 1 \) and with Neumann boundary conditions if \( \theta > 1 \). We refer the readers to [1] for more background of the model.

The hydrodynamic limit of the model is considered under the diffusive time scaling, i.e., with time speeded up by \( N^2 \) and space divided by \( N \). The aim of this article is to consider the behavior of the process under a longer time scaling \( N^{2+\gamma} \), \( \gamma > 0 \). Since the process is irreducible, it has a unique invariant measure. Under the invariant measure, the empirical measure converges in probability to the stationary solution of the corresponding hydrodynamic equations as \( N \to \infty \). This is called hydrostatic limit [1, 13]. The hydrostatic limit could be formally interpreted as taking \( \gamma = \infty \). For \( 0 < \gamma < \infty \), it is natural to expect that the limit of the empirical measure should coincide with the hydrostatic limit. This is indeed true for \( \theta \leq 1 \), since the stationary solution of the corresponding hydrodynamic equation is unique in this case. For \( \theta > 1 \), since the stationary solution is not unique, three regimes appear depending on whether \( \gamma < \theta - 1 \), \( \gamma = \theta - 1 \) or \( \gamma > \theta - 1 \). See Theorem 2.3 for details.

Despite the simple structure of the model, it has attracted a lot of attention since then. The equilibrium/non-equilibrium fluctuations from the hydrodynamic limit are considered in [9, 11]. The large deviation of the SSEP with slow boundary are investigated in [2, 6, 10].

Key words and phrases. Exclusion process; slow boundary; empirical measure, law of large numbers.
The paper is organized as follows. In Section 2 we define the model rigorously via its infinitesimal generator, review the hydrodynamic/hydrostatic limit already proven in [1, 13], and state the main result of the article. In Section 3 we introduce the notation of Dirichlet forms and prove the so-called replacement lemmas under a longer time scaling. The estimates involving the Dirichlet forms are mostly borrowed from [1]. The proof of Theorem 2.3 is presented in Section 4.

2. Notation and Results

2.1. The model. The state space of the process \((\eta_t)_{t \geq 0}\) is \(\Omega_N := \{0, 1\}^{\mathbb{N}},\) where \(I_N := \{1, \ldots, N - 1\}\). Here, \(N\) is the scaling parameter. For a configuration \(\eta \in \Omega_N, \eta(x) = 1\) if and only if there is a particle at site \(x\). Fix parameters \(c > 0, \theta \geq 0\) and \(\alpha, \beta \in (0, 1)\). The parameter \(\theta\) denotes the strength of interaction with reservoirs and \(\alpha, \beta\) are the particle densities of reservoirs. The generator \(L_N\) of the process \((\eta_t)_{t \geq 0}\) is given by

\[
L_N = L_{N,0} + L_{N,b}^\alpha + L_{N,b}^\beta.
\]

Above, the generator \(L_{N,0}\) of the bulk dynamics acting on functions \(f : \Omega_N \to \mathbb{R}\) is

\[
(L_{N,0}f)(\eta) = \sum_{x=1}^{N-2} [f(\eta_{x,x+1}) - f(\eta)],
\]

where \(\eta_{x,y}\) is the configuration obtained from \(\eta\) by exchanging the values of \(\eta(x)\) and \(\eta(y)\), i.e., \(\eta_{x,y}(x) = \eta(y), \eta_{x,y}(y) = \eta(x)\) and \(\eta_{x,y}(z) = \eta(z)\) for \(z \neq x, y\). The generators \(L_{N,b}^\alpha\) and \(L_{N,b}^\beta\) correspond to the boundary effects, and are given by

\[
(L_{N,b}^\alpha f)(\eta) := cN^{-\theta}r_\alpha(\eta) \left[ f(\eta^1) - f(\eta) \right], \quad (L_{N,b}^\beta f)(\eta) := cN^{-\theta}r_\beta(\eta) \left[ f(\eta^{N-1}) - f(\eta) \right],
\]

where \(\eta^x\) is the configuration obtained from \(\eta\) by flipping the value of \(\eta(x)\), i.e., \(\eta^x(x) = 1 - \eta(x)\) and \(\eta^x(z) = \eta(z)\) for \(z \neq x\), and

\[
\begin{align*}
\alpha(\eta) &= \alpha(1 - \eta(1)) + (1 - \alpha)\eta(1), \quad r_\beta(\eta) = \beta(1 - \eta(N - 1)) + (1 - \beta)\eta(N - 1).
\end{align*}
\]

Denote by \(\mu_N\) the initial measure of the process. For any positive integer \(k\), let \(C^k[0, 1]\) be the family of functions on \([0, 1]\) such that the \(m\)-th derivative is uniformly continuous in \((0, 1)\) for any \(m \leq k\).

2.2. Hydrodynamic limit. It has been proven in [1] that phase transitions occur for the SSEP with slow boundary, depending on whether \(\theta < 1, \theta = 1\) or \(\theta > 1\). To state the hydrodynamic limit, we impose the following assumptions on the initial measure \(\mu_N\): there exists a measurable initial density profile \(\rho_0 : [0, 1] \to [0, 1]\) such that for any \(G \in C[0, 1]\),

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{x=1}^{N-1} \eta(x)G(\frac{x}{N}) = \int_0^1 \rho_0(u)G(u) \, du
\]

in probability with respect to \(\mu_N\). The following result characterize the macroscopic density profile under the diffusive time scaling.
Theorem 2.1 (Cf. [1, Theorem 2.8]). For any \( t \geq 0 \) and for any \( G \in C[0,1] \),
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{x=1}^{N-1} \eta_N(x)G\left(\frac{x}{N}\right) = \int_{0}^{1} \rho(t,u)G(u) \, du
\]
in probability, where

(i) if \( 0 \leq \theta < 1 \), then \( \rho(t,u) \) is the unique weak solution to the heat equation with Dirichlet boundary
\[
\begin{align*}
\partial_t \rho(t,u) &= \Delta \rho(t,u), \quad u \in (0,1), \quad t \geq 0 \\
\rho(t,0) &= \alpha, \rho(t,1) = \beta, \quad t \geq 0, \\
\rho(0,u) &= \rho_0(u), \quad u \in [0,1]
\end{align*}
\] (2.1)

(ii) if \( \theta = 1 \), then \( \rho(t,u) \) is the unique weak solution to the heat equation with Robin boundary
\[
\begin{align*}
\partial_t \rho(t,u) &= \Delta \rho(t,u), \quad u \in (0,1), \quad t \geq 0 \\
\partial_u \rho(t,0) &= c(\rho(t,0) - \alpha), \quad t \geq 0, \\
\partial_u \rho(t,1) &= c(\beta - \rho(t,1)), \quad t \geq 0 \\
\rho(0,u) &= \rho_0(u), \quad u \in [0,1]
\end{align*}
\] (2.2)

(iii) if \( \theta > 1 \), then \( \rho(t,u) \) is the unique weak solution to the heat equation with Neumann boundary
\[
\begin{align*}
\partial_t \rho(t,u) &= \Delta \rho(t,u), \quad u \in (0,1), \quad t \geq 0 \\
\partial_u \rho(t,0) = 0, \partial_u \rho(t,1) = 0, \quad t \geq 0 \\
\rho(0,u) &= \rho_0(u), \quad u \in [0,1]
\end{align*}
\] (2.3)

We refer the readers to [1] for rigorous definitions of weak solutions the above PDEs.

2.3. Hydrostatic limit. Since the process \((\eta_t)_{t \geq 0}\) is irreducible, it has a unique invariant measure denoted by \(\mu_{N}^{ss}\). The following result characterize the macroscopic density profile under the invariant measure \(\mu_{N}^{ss}\):

Theorem 2.2 (Cf. [1, Theorem 2.2] and [13]). For any \( G \in C[0,1] \),
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{x=1}^{N-1} \eta(x)G\left(\frac{x}{N}\right) = \int_{0}^{1} \bar{\rho}(u)G(u) \, du
\]
in probability with respect to \(\mu_{N}^{ss}\), where
\[
\bar{\rho}(u) = \begin{cases} 
(\beta - \alpha)u + \alpha, & \text{if } \theta \in [0,1) \\
\frac{c(\beta - \alpha)}{2}u + \alpha + \frac{\beta - \alpha}{2+c}, & \text{if } \theta = 1 \\
\frac{\beta + \alpha}{2}, & \text{if } \theta \in (1,\infty)
\end{cases}
\]

We say \( \rho : [0,1] \to [0,1] \) is a stationary solution to (2.1) if
\[
\Delta \rho(u) = 0, \quad u \in (0,1), \quad \rho(0) = \alpha, \quad \rho(1) = \beta.
\]
Stationary solutions to (2.2) and (2.3) could be defined in the same way. Note that the hydrostatic limits \(\bar{\rho}\) are stationary solutions to the corresponding hydrodynamic equations as stated in Theorem 2.1. We underline that the stationary solution to (2.1) and (2.2) is unique, while the stationary solution to (2.3) is not unique. Indeed, any constant function is a stationary solution to (2.3). The above theorem tells us that the correct choice is \((\alpha + \beta)/2\).
2.4. Long-time limit. In this subsection, we state the main result of the article. Fix $\gamma > 0$. We shall consider the process speeded up by $N^{2+\gamma}$. Denote by $(\eta_t^N)_{t \geq 0}$ the process with generator $N^{2+\gamma}L_N$. Then $(\eta_t^N)_{t \geq 0}$ and $(\eta_{t,N^{2+\gamma}})_{t \geq 0}$ have the same distribution. We are interested in the long time behavior of the empirical measure $\pi_t^N$ of the process defined as

$$\pi_t^N(du) = \frac{1}{N} \sum_{x=1}^{N-1} \eta_t^N(x) \delta_{x/N}(du),$$

where $\delta_{x/N}(du)$ is the Dirac measure on the point $x/N$. Whence, $\pi_t^N$ is a random measure on $[0, 1]$ with total mass bounded by one. With this notation, for any $G \in C[0, 1]$,

$$\langle \pi_t^N, G \rangle = \frac{1}{N} \sum_{x=1}^{N-1} \eta_t^N(x) G\left(\frac{x}{N}\right).$$

For $\theta > 1$ and $0 < \gamma \leq \theta - 1$, we assume that the average number of particles converges in the following sense: there exists $m_0 \in [0, 1]$ such that

$$\lim_{N \to \infty} E_{\mu_N} \left[ \frac{1}{N-1} \sum_{x=1}^{N-1} \eta(x) - m_0 \right] = 0. \quad (2.4)$$

We underline that we impose no restrictions on the initial measure $\mu_N$ in the rest of the cases.

Denote by $\mathbb{P}_{\mu_N}^N$ the probability measure on $D([0, \infty), \Omega_N)$ associated to the process $(\eta_t^N)_{t \geq 0}$ and the initial measure $\mu_N$, and by $E_{\mu_N}^N$ the corresponding expectation.

We now state the law of large numbers for the empirical measure $\pi_t^N$.

**Theorem 2.3.** For any $t > 0$ and for any $G \in C[0, 1]$,

$$\lim_{N \to \infty} E_{\mu_N}^N \left[ \left| \int_0^t \left\{ \langle \pi_s^N, G \rangle - \int_0^1 \rho_{\theta, \gamma}(s, u) G(u) du \right\} ds \right| \right] = 0,$$

where

$$\rho_{\theta, \gamma}(t, u) = \begin{cases} 
(\beta - \alpha)u + \alpha, & \text{if } 0 \leq \theta < 1, \\
\frac{c(\beta - \alpha)}{2}u + \alpha + \beta - \alpha, & \text{if } \theta = 1, \\
m_0, & \text{if } \theta > 1, 0 < \gamma < \theta - 1, \\
\frac{\beta + \alpha}{2}, & \text{if } \theta > 1, \gamma > \theta - 1, \\
\frac{\beta + \alpha}{2} + \left( m_0 - \frac{\alpha + \beta}{2} \right) e^{-2ct}, & \text{if } \theta > 1, \gamma = \theta - 1.
\end{cases} \quad (2.5)$$

**Remark 2.4.** Note that $\rho_{\theta, \infty} = \bar{\rho}$. This is not surprising since the hydrostatic limit stated in Theorem 2.2 could be formally interpreted as taking $\gamma = \infty$. Compared with Theorem 2.2, the above theorem states that phase transition occurs even in the supercritical case $\theta > 1$.

**Remark 2.5.** The result should also hold if the density reservoirs vary slowly with time, i.e., if replacing $\alpha$ (resp. $\beta$) with some smooth function $\rho_-(t) : \mathbb{R}_+ \to (0, 1)$ (resp. $\rho_+(t) : \mathbb{R}_+ \to (0, 1)$). This is called quasi-static hydrodynamic limit [3, 4].

3. Preliminary results

In this section, we introduce the notion of Dirichlet forms and prove several replacement lemmas in different regimes, which are crucial in the proof of Theorem 2.3.
3.1. Dirichlet form. For a probability measure \( \mu \) on \( \Omega_N \) and a function \( g : \Omega_N \to \mathbb{R} \), the Dirichlet forms corresponding to the bulk/boundary dynamics are defined as

\[
D_{N,0}(g, \mu) := \frac{1}{2} \sum_{x=1}^{N-2} \sum_{\eta \in \Omega_N} \left( g(\eta^{x,x+1}) - g(\eta) \right)^2 \mu(\eta),
\]

\[
D_{N,b}^\alpha(g, \mu) := \frac{1}{2} \sum_{\eta \in \Omega_N} cN^{-\theta} r_\alpha(\eta) \left( g(\eta^1) - g(\eta) \right)^2 \mu(\eta),
\]

\[
D_{N,b}^\beta(g, \mu) := \frac{1}{2} \sum_{\eta \in \Omega_N} cN^{-\theta} r_\beta(\eta) \left( g(\eta^{N-1}) - g(\eta) \right)^2 \mu(\eta).
\]

For any two functions \( f, g : \Omega_N \to \mathbb{R} \), denote

\[
\langle f, g \rangle_\mu = \sum_{\eta \in \Omega_N} f(\eta)g(\eta)\mu(\eta).
\]

For any density profile \( \lambda : [0, 1] \to [0, 1] \), let \( \nu^N_{\lambda(\cdot)} \) be the product measure on \( \Omega_N \) with marginals given by

\[
\nu^N_{\lambda(\cdot)} \{ \eta : \eta(x) = 1 \} = \lambda \left( \frac{x}{N} \right), \quad x \in I_N.
\]

In particular, if \( \lambda(\cdot) \equiv \rho \) for some \( \rho \in [0, 1] \), we simply write \( \nu^N_\rho \).

The following lemma compares \( \langle L_N g, g \rangle_\mu \) with \( D_N(g, \mu) \).

**Lemma 3.1.** (i) Let \( \lambda : [0, 1] \to (0, 1) \) be a smooth density profile such that there exists a neighborhood of 0 where \( \gamma(\cdot) = \alpha \), and a neighborhood of 1 where \( \gamma(\cdot) = \beta \). Let \( f \) be a \( \nu^N_{\lambda(\cdot)} \)-density,

\[
f \geq 0, \quad \sum_{\eta \in \Omega_N} f(\eta)\nu^N_{\lambda(\cdot)}(\eta) = 1.
\]

Then

\[
\langle L_N \sqrt{f}, \sqrt{f} \rangle_{\nu^N_{\lambda(\cdot)}} = -(1/2)D_{N,0}(\sqrt{f}, \nu^N_{\lambda(\cdot)}) - D^\alpha_{N,b}(\sqrt{f}, \nu^N_{\lambda(\cdot)}) - D^\beta_{N,b}(\sqrt{f}, \nu^N_{\lambda(\cdot)}) + O(N^{-1}),
\]

where \( |O(N^{-1})| \leq CN^{-1} \) for some finite constant \( C \).

(ii) Let \( \rho \in (0, 1) \) be a constant. Let \( f \) be a \( \nu^N_\rho \)-density. Then

\[
\langle L_N \sqrt{f}, \sqrt{f} \rangle_{\nu^N_\rho} = -D_{N,0}(\sqrt{f}, \nu^N_\rho) - D^\alpha_{N,b}(\sqrt{f}, \nu^N_\rho) - D^\beta_{N,b}(\sqrt{f}, \nu^N_\rho) + O(N^{-\theta}).
\]

The first statement (i) is a direct consequence of [1, Lemma 5.1 (ii) and Lemma 5.2]. The second statement (ii) follows directly from [1, Lemma 5.1 (i) and Corollary 5.3]. For this reason, we omit the proof here.

The following lemma bound the occupation variables at the boundary sites by the corresponding Dirichlet forms.

**Lemma 3.2.** Let \( \lambda : [0, 1] \to (0, 1) \) be a smooth density profile such that there exists a neighborhood of 0 where \( \gamma(\cdot) = \alpha \), and a neighborhood of 1 where \( \gamma(\cdot) = \beta \). Let \( f \) be a \( \nu^N_{\lambda(\cdot)} \)-density. Then there exists a finite constant \( C_\alpha \) such that for any \( B > 0 \),

\[
\left| \sum_{\eta \in \Omega_N} (\eta(1) - \alpha)f(\eta)\nu^N_{\lambda(\cdot)}(\eta) \right| \leq C_\alpha B + C_\alpha N^\theta B^{-1}D^\alpha_{N,b}(\sqrt{f}, \nu^N_{\lambda(\cdot)}).
\]
The same result holds at the right boundary: there exists a finite constant $C_β$ such that for any $B > 0$,
\[
\left| \sum_{η ∈ Ω_ν} (η(N - 1) - β) f(η) ν^N_{ξ(η)}(η) \right| ≤ C_β B + C_β N^θ B^{-1} D^θ_{N,b}(\sqrt{f}, ν^N_{ξ(η)}).
\]

The above lemma is a direct consequence of [1, Lemmas 5.6 and 5.7]. For that reason, we omit the proof here.

3.2. Replacement lemmas. In this subsection, we prove several replacement lemmas under the longer time scaling $N^{2+γ}$, $γ > 0$.

The following lemma states that in the subcritical regime $0 ≤ θ < 1$, we could replace the occupation variable at the boundary sites with the corresponding particle density of reservoirs.

**Lemma 3.3** (Replacement lemma for the case $0 ≤ θ < 1$). Suppose $0 ≤ θ < 1$. Then for any $t > 0$,
\[
\lim_{N → ∞} \mathbb{E}^N_{μ_N} \left[ \left| \int_0^t (η_s^N(1) - α) ds \right| \right] = 0.
\]
The same result holds with $η_s^N(1)$ replaced with $η_s^N(N - 1)$ and $α$ with $β$.

**Proof.** Let $λ : [0, 1] → (0, 1)$ be a smooth density profile such that there exists a neighborhood of 0 where $γ(·) = α$, and a neighborhood of 1 where $γ(·) = β$. By the relative entropy inequality (cf. [12, A.1.8]), for any $A > 0$, the expectation in the lemma could be bounded from above by
\[
\frac{H(μ_N|ν^N_{ξ(·)})}{AN} + \frac{1}{AN} \log \mathbb{E}^N_{ν^N_{ξ(·)}} \left[ \exp \left\{ AN \left| \int_0^t (η_s^N(1) - α) ds \right| \right\} \right],
\]
where for any probability measures $μ, ν$ on $Ω_N$ such that $μ$ is absolutely continuous with respect to $ν$, $H(μ|ν)$ is the relative entropy of $μ$ with respect to $ν$ defined as
\[
H(μ|ν) = \sum_{η ∈ Ω_N} μ(η) \log \frac{μ(η)}{ν(η)}.
\]
It is not hard to prove that $H(μ_N|ν^N_{ξ(·)}) ≤ C_ξ N$ for some finite constant $C_ξ$. Therefore, the first term in (3.1) is bounded by $C_ξ/A$. In the sequel, we shall take $A = A(N) → ∞$ as $N → ∞$. Whence the first term in (3.1) vanishes in the limit. Since
\[
\lim_{N → ∞} r^{-1}_N \log(a_N + b_N) = \max \{ \lim_{N → ∞} r^{-1}_N \log a_N, \lim_{N → ∞} r^{-1}_N \log b_N \}
\]
for any positive sequences $\{a_N\}_{N ≥ 1}, \{b_N\}_{N ≥ 1}$ and $\{r_N\}_{N ≥ 1}$ such that $\lim_{N → ∞} r_N = ∞$, we could remove the modulus inside the exponential for the second term in (3.1). By the Feynman-Kac formula (cf. [12, Lemma A.1.7.2]), the second term in (3.1) is bounded by
\[
t \sup_{f \text{ density}} \left\{ \sum_{η ∈ Ω_N} (η(1) - α) f(η) ν^N_{ξ(η)}(η) + \frac{N^{1+γ}}{A} \langle L_N \sqrt{f}, \sqrt{f} \rangle ν^N_{ξ(η)} \right\}.
\]
By Lemma 3.1 (i),
\[
\langle L_N \sqrt{f}, \sqrt{f} \rangle ν^N_{ξ(η)} ≤ -D^θ_{N,b}(\sqrt{f}, ν^N_{ξ(η)}) + O(N^{-1}).
\]
Together with Lemma 3.2, for any $B > 0$, we may bound (3.2) by

$$t \sup_{f: \nu_N^{(1)}\text{-density}} \left\{ C_\alpha B + C_\alpha N^\theta B^{-1} D_{N,b}^\alpha(\sqrt{f}, \nu_N^{(1)}) - \frac{N^{1+\gamma}}{A} D_{N,b}^\alpha(\sqrt{f}, \nu_N^{(1)}) + O(N^\gamma/A) \right\}$$

for some finite constant $C_\alpha$. Taking $B = C_\alpha A N^{\theta-1-\gamma}$ and $A = N^\gamma \log N$, the above term is bounded by $t(C_\alpha^2 N^\theta-1 \log N + O(1/\log N))$, which converges to zero as $N \to \infty$ since $\theta < 1$. This concludes the proof. □

Let $m_N(\eta)$ be the average number of particles in the system

$$m_N(\eta) = \frac{1}{N-1} \sum_{x=1}^{N-1} \eta(x).$$

Denote $m_N^1 = m_N(\eta^N_1)$. The next result states that in the supercritical regime $\theta > 1$, we could replace the occupation variables at the boundary sites with the average number of particles in the system.

**Lemma 3.4** (Replacement lemma for the case $\theta > 1$.) Suppose $\theta > 1$. Then for any $t > 0$,

$$\lim_{N \to \infty} \mathbb{E}_{\nu_N}^{\eta^N} \left| \int_0^t (\eta^N_s(1) - m_s^N) \, ds \right| = 0.$$

The same result holds with $\eta^N_s(1)$ replaced with $\eta^N_s(N - 1)$

**Proof.** The proof is similar to that of Lemma 3.3, and we only sketch the proof here. Fix a constant $\rho \in (0, 1)$. By the relative entropy inequality (cf. [12, A.1.8]), for any $A > 0$, the expectation in the lemma could be bounded from above by

$$\frac{H(\mu_N | \nu_\rho^N)}{AN} + \frac{1}{AN} \log \mathbb{E}_{\nu_\rho}^{\eta^N} \left[ \exp \left\{ AN \left| \int_0^t (\eta^N_s(1) - m_s^N) \, ds \right| \right\} \right]. \quad (3.3)$$

As in Lemma 3.3, the first term is bounded by $C/A$ for some finite constant $C$. By the Feynman-Kac formula (cf. [12, Lemma A.1.7.2]), the second term in (3.3) is bounded by

$$t \sup_{f: \nu_\rho^{\eta^N}\text{-density}} \left\{ \sum_{\eta \in \Omega_N} (\eta(1) - m_N(\eta)) f(\eta) \nu_\rho^N(\eta) \right\}.$$ \hspace{1cm} (3.4)

We may rewrite $\eta(1) - m_N(\eta)$ as a telescope sum

$$\frac{1}{N-1} \sum_{x=1}^{N-1} \sum_{y=1}^{x-1} (\eta(y) - \eta(y+1)).$$

Making the change of variables $\eta \mapsto \eta^{y,y+1}$,

$$\sum_{\eta \in \Omega_N} (\eta(1)-m_N(\eta)) f(\eta) \nu_\rho^N(\eta) = \frac{1}{2(N-1)} \sum_{x=1}^{N-1} \sum_{y=1}^{x-1} \sum_{\eta \in \Omega_N} (\eta(y) - \eta(y+1)) (f(\eta) - f(\eta^{y,y+1})) \nu_\rho^N(\eta).$$

By Cauchy-Schwarz inequality, for any $B > 0$, we may bound the last term by

$$\frac{B}{4(N-1)} \sum_{x=1}^{N-1} \sum_{y=1}^{x-1} \sum_{\eta \in \Omega_N} \left( \sqrt{f(\eta)} - \sqrt{f(\eta^{y,y+1})} \right)^2 \nu_\rho^N(\eta).$$
\[
+ \frac{1}{4B(N-1)} \sum_{x=1}^{N-1} \sum_{y=1}^{x-1} \sum_{\eta \in \Omega_N} (\sqrt{f}(\eta) + \sqrt{f(\eta^{y+1})})^2 \nu_N^N(\eta)
\]
\[
\leq \frac{B}{2} D_{N,0} (\sqrt{f}, \nu_N^N) + \frac{N}{B}.
\]
The last inequality follows from the basic inequality \((a + b)^2 \leq 2(a^2 + b^2)\) and the fact that \(f\) is a density with respect to \(\nu_N^N\). By Lemma 3.1 (ii),
\[
\langle L_N \sqrt{f}, \sqrt{f} \rangle_{\nu_N^N} \leq -D_{N,0} (\sqrt{f}, \nu_N^N) + O(N^{-\theta}).
\]
Whence, (3.4) is bounded by
\[
t \sup_{\nu_N^N-\text{density}} \left\{ \frac{B}{2} D_{N,0} (\sqrt{f}, \nu_N^N) + \frac{N}{B} - \frac{N^{1+\gamma}}{A} D_{N,0} (\sqrt{f}, \nu_N^N) + O(N^{1+\gamma-\theta}/A) \right\}
\]
Taking \(B = 2N^{1+\gamma}/A\) and \(A = N^\gamma/(\log N)\), the above term is bounded by \(1/(2 \log N) + O(N^\gamma \log N)\), which converges to zero as \(N \to \infty\) since \(\theta > 1\). This concludes the proof. \(\square\)

The next lemma concerns about the long time behavior of the average number of particles in the supercritical case.

**Lemma 3.5** (Replacement lemma for the average particle number.). Suppose \(\theta > 1\). Recall \(m_0\) defined in (2.4) is the average number of particles at the initial time. For any \(t > 0\),

(i) if \(0 \leq \gamma < \theta - 1\), then
\[
\lim_{N \to \infty} \mathbb{E}_{\mu_N}^N \left[ \left| \int_0^t (m_s^N - m_0) \, ds \right| \right] = 0,
\]
(ii) if \(\gamma = \theta - 1\), then
\[
\lim_{N \to \infty} \mathbb{E}_{\mu_N}^N \left[ \left| \int_0^t (m_s^N - m_s) \, ds \right| \right] = 0,
\]
where
\[
m_s = \frac{\alpha + \beta}{2} + \left( m_0 - \frac{\alpha + \beta}{2} \right) e^{-2cs}.
\]
(iii) if \(\gamma > \theta - 1\), then
\[
\lim_{N \to \infty} \mathbb{E}_{\mu_N}^N \left[ \left| \int_0^t (m_s^N - (\alpha + \beta)/2) \, ds \right| \right] = 0.
\]

**Proof.** The statement (ii) is a direct consequence of [13, Proposition 4.5]. For the rest of the statements, consider the martingale \(m_t^N\) defined as
\[
m_t^N := m_t^N - m_0^N - \int_0^t N^{2+\gamma} L_N m_s^N \, ds,
\]
whose quadratic variation at time \(t\) is given by
\[
\int_0^t \{ N^{2+\gamma} L_N (m_s^N)^2 - 2m_s^N N^{2+\gamma} L_N m_s^N \} \, ds.
\]
A simple calculation shows that the quadratic variation of \( m_t^N \) is bounded by \( CN^{\gamma - \theta} \) for some finite constant \( C \), and that the integral term in (3.5) equals

\[
\frac{CN^{2+\gamma-\theta}}{N-1} \int_0^t (\alpha - \eta_s^N(1) + \beta - \eta_s^N(N-1)) \, ds.
\]

If \( 0 \leq \gamma < \theta - 1 \), by Doob’s inequality, for any \( T > 0 \),

\[
\lim_{N \to \infty} \mathbb{E}^N_{\mu_N} \left[ \sup_{0 \leq t \leq T} (m_t^N)^2 \right] = 0.
\]

The integral term in (3.5) is of order \( N^{1+\gamma - \theta} \), which converges to zero as \( N \to \infty \) uniformly in a bounded time interval. Therefore,

\[
\lim_{N \to \infty} \mathbb{E}^N_{\mu_N} \left[ \sup_{0 \leq t \leq T} |m_t^N - m_0| \right] = 0
\]

This proves the first statement (i).

If \( \gamma > \theta - 1 \), divided by \( N^{1+\gamma - \theta} \) in (3.5), we have

\[
\lim_{N \to \infty} \mathbb{E}^N_{\mu_N} \left[ \sup_{0 \leq t \leq T} (N^{\theta-\gamma-1}m_t^N)^2 \right] = 0.
\]

Since \( m_t^N \leq 1 \), by (3.5),

\[
\lim_{N \to \infty} \mathbb{E}^N_{\mu_N} \left[ \int_0^t (\alpha - \eta_s^N(1) + \beta - \eta_s^N(N-1)) \, ds \right] = 0.
\]

By Lemma 3.4, we could replace \( \eta_s^N(1) \) and \( \eta_s^N(N-1) \) in the time integral with \( m_s^N \). This concludes the proof. \( \square \)

### 4. Proof of Theorem 2.3

In this section, we prove Theorem 2.3 depending on whether \( 0 \leq \theta < 1 \), \( \theta = 1 \) or \( \theta > 1 \). For \( H \in C^2[0,1] \), consider the martingale defined as

\[
M_t^N(H) = \langle \pi_t^N, H \rangle - \langle \pi_0^N, H \rangle - \int_0^t N^{2+\gamma}L_N\langle \pi_s^N, H \rangle \, ds,
\]

whose quadratic variation at time \( t \) is given by

\[
\int_0^t \left\{ N^{2+\gamma}L_N\langle \pi_s^N, H \rangle^2 - 2\langle \pi_s^N, H \rangle N^{2+\gamma}L_N\langle \pi_s^N, H \rangle \right\} \, ds.
\]

Direct calculations show that the quadratic variation of \( M_t^N(H) \) is bounded by \( C_H(N^{\gamma-1} + N^{\gamma-\theta}) \) for some finite constant \( C_H \). Therefore, for any \( T > 0 \),

\[
\lim_{N \to \infty} \mathbb{E}^N_{\mu_N} \left[ \sup_{0 \leq t \leq T} (N^{\gamma}M_t^N(H))^2 \right] = 0.
\]

Since there is at most one particle per site, \( |\langle \pi_s^N, H \rangle| \leq ||H||_{\infty} \) uniformly in \( t \), where \( ||H||_{\infty} := \max_{u \in [0,1]} |H(u)| \) is the uniform norm. Divided by \( N^{\gamma} \) in (4.1),

\[
\lim_{N \to \infty} \mathbb{E}^N_{\mu_N} \left[ \int_0^t N^2L_N\langle \pi_s^N, H \rangle \, ds \right] = 0.
\]
Direct calculations yield that
\[
\begin{align*}
N^2 L_N \langle \pi_N^N, H \rangle & = \langle \pi_N^N, H'' \rangle - \eta_s^N (N - 1) H'(1) + \eta_s^N (1) H'(0) \\
+ c N^{1 - \theta} (\alpha - \eta_s^N (1)) H(0) & + c N^{1 - \theta} (\beta - \eta_s^N (N - 1)) H(1) + \mathcal{O}(N^{-\theta} + N^{-1}).
\end{align*}
\]

Whence,
\[
\lim_{N \to \infty} \mathbb{E}^N_{\gamma_N} \left[ \left| \int_0^t \left\{ \langle \pi_s^N, H'' \rangle - \eta_s^N (N - 1) H'(1) + \eta_s^N (1) H'(0) \\
+ c N^{1 - \theta} (\alpha - \eta_s^N (1)) H(0) + c N^{1 - \theta} (\beta - \eta_s^N (N - 1)) H(1) \right\} ds \right| \right] = 0. \tag{4.2}
\]

4.1. **The case** \(0 \leq \theta < 1\). In this subsection, we prove Theorem 2.3 for the case \(0 \leq \theta < 1\). Fix \(G \in C[0, 1]\). The main technique here is to find an appropriate function \(H\) such that \(H'' = G\) on \((0,1)\) and that \(H(0) = H(1) = 0\). With such a function \(H\), the second line in (4.2) vanishes and the result follows by the corresponding replacement lemmas.

**Proof of Theorem 2.3 in the case** \(0 \leq \theta < 1\). For \(G \in C[0, 1]\), let
\[
H(u) = H_G(u) = \int_0^u \int_0^v G(w) \, dw \, dv + u \int_0^1 (v - 1) G(v) \, dv. \tag{4.3}
\]
It is easy to check that \(H \in C^2[0, 1]\), \(H'' = G\) on \((0,1)\), and that
\[
H(0) = H(1) = 0, \quad H'(0) = \int_0^1 (u - 1) G(u) \, du, \quad H'(1) = \int_0^1 u G(u) \, du.
\]
Substituting the function \(H\) into (4.2),
\[
\lim_{N \to \infty} \mathbb{E}^N_{\gamma_N} \left[ \left| \int_0^t \{ \langle \pi_s^N, G \rangle - \int_0^1 \left[ (\eta_s^N (N - 1) - \eta_s^N (1)) u + \eta_s^N (1) \right] G(u) \, du \} ds \right| \right] = 0.
\]
By Lemma 3.3, we could replace \(\eta_s^N (1)\) (resp. \(\eta_s^N (N - 1)\)) with \(\alpha\) (resp. \(\beta\)). This concludes the proof for the case \(0 \leq \theta < 1\). \(\square\)

4.2. **The case** \(\theta = 1\). In this subsection, we prove Theorem 2.3 for the case \(\theta = 1\). Fix \(G \in C[0, 1]\). We need to find an appropriate function \(H\) such that \(H'' = G\) on \((0,1)\) and that the coefficients of \(\eta(1)\) and \(\eta(N - 1)\) vanish in (4.2). Note that in this case we do not need any replacement lemma.

**Proof of Theorem 2.3 in the case** \(\theta = 1\). For \(G \in C[0, 1]\), let
\[
H(u) = \int_0^1 \left( \frac{v}{2 + c} - \frac{1 + c}{c(2 + c)} \right) G(v) \, dv + u \int_0^1 \left( \frac{cv}{2 + c} - \frac{1 + c}{2 + c} \right) G(v) \, dv + \int_0^u \int_0^v G(w) \, dw \, dv.
\]
It is easy to check that \(H \in C^2[0, 1]\), \(H'' = G\) on \((0,1)\), and that
\[
H(0) = \int_0^1 \left( \frac{v}{2 + c} - \frac{1 + c}{c(2 + c)} \right) G(v) \, dv, \quad H(1) = \int_0^1 \left( - \frac{v}{2 + c} - \frac{1}{c(2 + c)} \right) G(v) \, dv,
\]
\[
H'(0) = \int_0^1 \left( \frac{cv}{2 + c} - \frac{1 + c}{2 + c} \right) G(v) \, dv, \quad H'(1) = \int_0^1 \left( \frac{cv}{2 + c} + \frac{1}{2 + c} \right) G(v) \, dv.
\]
Taking the function $H$ into (4.2) and calculating the coefficients of $\eta(1)$ and $\eta(N - 1)$, we have

$$-H'(1) - cH(1) = 0, \quad H'(0) - cH(0) = 0.$$  

The constant term in (4.2) is given by

$$c\alpha H(0) + c\beta H(1) = \int_0^1 \left( \frac{c(\alpha - \beta)}{2 + c} v - \alpha - \frac{\beta - \alpha}{2 + c} \right) G(v) dv.$$  

Whence, the integrand in (4.2) is equal to

$$\langle \pi^N_s, G \rangle - \int_0^1 \left( \frac{c(\beta - \alpha)}{2 + c} v + \alpha + \frac{\beta - \alpha}{2 + c} \right) G(v) dv.$$  

By (4.2), the time integral of the above term converges in $L^1(\mu_N)$ to zero as $N \to \infty$. This concludes the proof for the case $\theta = 1$. \hfill \Box

4.3. The case $\theta > 1$. In this subsection, we prove Theorem 2.3 for the case $\theta > 1$. In this case, the last line in (4.2) converges to zero as $N \to \infty$. Whence, we do not need special properties of the function $H$.

**Proof of Theorem 2.3 in the case $\theta > 1$.** For $G \in C[0,1]$, let

$$H(u) = \int_0^u \int_0^v G(w) dw dv.$$  

Taking the function $H$ into (4.2) and by Lemma 3.4,

$$\lim_{N \to \infty} \mathbb{E}_{\mu_N}^N \left[ \left| \int_0^t \left\{ \langle \pi^N_s, G \rangle - m^N_s (H'(1) - H'(0)) \right\} ds \right| \right] = 0.$$  

It is easy to check

$$H'(1) - H'(0) = \int_0^1 G(u) du.$$  

Therefore,

$$\lim_{N \to \infty} \mathbb{E}_{\mu_N}^N \left[ \left| \int_0^t \left\{ \langle \pi^N_s, G \rangle - m^N_s \int_0^1 G(u) du \right\} ds \right| \right] = 0.$$  

By Lemma 3.5, we conclude the proof for the case $\theta > 1$. \hfill \Box

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**References**

[1] R. Baldasso, O. Menezes, A. Neumann, and R. Souza. Exclusion process with slow boundary. *Journal of Statistical Physics*, 167(5):1112–1142, 2017.

[2] A Bouley, C Erignoux, and C Landim. Steady state large deviations for one-dimensional, symmetric exclusion processes in weak contact with reservoirs. *arXiv preprint arXiv:2107.06606*, 2021.

[3] A. De Masi, S. Marchesani, S. Olla, and L. Xu. Quasi-static limit for the asymmetric simple exclusion. *arXiv preprint arXiv:2103.08019*, 2021.
[4] A. De Masi and S. Olla. Quasi-static hydrodynamic limits. *Journal of Statistical Physics*, 161(5):1037–1058, 2015.

[5] A. De Masi, E. Presutti, D. Tsagkarogiannis, and M. E. Vares. Current reservoirs in the simple exclusion process. *Journal of Statistical Physics*, 144(6):1151–1170, 2011.

[6] B. Derrida, O. Hirschberg, and T. Sadhu. Large deviations in the symmetric simple exclusion process with slow boundaries. *Journal of Statistical Physics*, 182(1):1–13, 2021.

[7] C. Erignoux. Hydrodynamic limit of boundary driven exclusion processes with nonreversible boundary dynamics. *Journal of Statistical Physics*, 172:1327–1357, 2018.

[8] G. Eyink, J. L. Lebowitz, and H. Spohn. Hydrodynamics of stationary non-equilibrium states for some stochastic lattice gas models. *Communications in mathematical physics*, 132(1):253–283, 1990.

[9] T. Franco, P. Gonçalves, and A. Neumann. Non-equilibrium and stationary fluctuations of a slowed boundary symmetric exclusion. *Stochastic Processes and their Applications*, 129(4):1413–1442, 2019.

[10] T. Franco, P. Gonçalves, and A. Neumann. Large deviations for the SSEP with slow boundary: the non-critical case. *arXiv preprint arXiv:2107.06998*, 2021.

[11] P. Gonçalves, M. Jara, O. Menezes, and A. Neumann. Non-equilibrium and stationary fluctuations for the SSEP with slow boundary. *Stochastic Processes and their Applications*, 130(7):4326–4357, 2020.

[12] C. Kipnis and C. Landim. *Scaling limits of interacting particle systems*, volume 320. Springer Science & Business Media, 2013.

[13] K. Tsunoda. Hydrostatic limit for exclusion process with slow boundary revisited (stochastic analysis on large scale interacting systems). *RIMS Kokyuroku Bessatsu*, 79:149–162, 2020.