SMALL REPRESENTATIONS FOR AFFINE $q$-SCHUR ALGEBRAS

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Abstract. When the parameter $q \in \mathbb{C}^*$ is not a root of unity, simple modules of affine $q$-Schur algebras have been classified in terms of Frenkel–Mukhin’s dominant Drinfeld polynomials ([6, 4.6.8]). We compute these Drinfeld polynomials associated with the simple modules of an affine $q$-Schur algebra which come from the simple modules of the corresponding $q$-Schur algebra via the evaluation maps.

1. Introduction

Small representations of quantum affine $\mathfrak{sl}_n$ are the representations which are irreducible when regarded as representations of (non-affine) quantum $\mathfrak{sl}_n$. In other words, by the evaluation maps [11] from $\mathcal{U}_\mathbb{C}(\widehat{\mathfrak{sl}}_n)$ to $\mathcal{U}_\mathbb{C}(\mathfrak{gl}_n)$, these representations are obtained from irreducible representations of quantum $\mathfrak{gl}_n$. Small representations have been identified by Chari–Pressley [3] in terms of Drinfeld polynomials whose roots are described explicitly but fairly complicatedly.

When the parameter is not a root of unity, simple representations of affine $q$-Schur algebras have also been classified [6, Ch. 4] in terms of finite dimensional simple polynomial representation for the quantum loop algebras of $\mathfrak{gl}_n$. These representations are labeled by dominant Drinfeld polynomials in the sense of [9]. Using the evaluation maps from affine $q$-Schur algebras to $q$-Schur algebras, every irreducible representation of a $q$-Schur algebra becomes an irreducible representation of the corresponding affine $q$-Schur algebra. Motivated by the work of [3], we will identify these small representations of affine $q$-Schur algebras in this paper by working out precisely their associated dominant Drinfeld polynomials.

Our method uses directly evaluation maps from affine $q$-Schur algebras to $q$-Schur algebras. By a compatibility relation with evaluation maps for quantum $\widehat{\mathfrak{sl}}_n$ and $\mathfrak{gl}_n$ (Proposition 5.4), we will reproduce Chari–Pressley’s result [3, 3.5] with simplified formulas for the roots of Drinfeld polynomials associated with small representations of $\mathcal{U}_\mathbb{C}(\widehat{\mathfrak{sl}}_n)$. In this way, the dominant Drinfeld polynomials for small representations of affine $q$-Schur algebras can be easily described by their roots in segments; see Corollary 7.3.

In a forthcoming paper, we will look at a more general question. Since every simple representations of an affine $q$-Schur algebra can be obtained by a generalized evaluation map from a simple representation of a certain cyclotomic $q$-Schur algebra introduced by Lin–Rui [13], it would be interesting to classify those which are inflated from semisimple cyclotomic $q$-Schur algebras. By

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identifying them, we would be able to construct a completely reducible full subcategory of finite dimensional modules of an affine $q$-Schur algebra.

The paper is organized as follows. The first three sections are preliminary. Starting with the definitions of the double Ringel-Hall algebra $\mathcal{D}_{\Delta, \mathbb{C}}(n)$ and the quantum loop algebra $U_\mathbb{C}(\hat{gl}_n)$ of $\hat{gl}_n$, and an isomorphism between them in §2, we discuss in §3 polynomial representations of $U_\mathbb{C}(\hat{gl}_n)$ and the tensor space representations of $\mathcal{D}_{\Delta, \mathbb{C}}(n)$ and present a classification of simple modules for the affine $q$-Schur algebras. In §4, we look at the classification of simple modules of the affine $q$-Schur algebras arising from representations of affine Hecke algebra. Evaluation maps from affine $q$-Schur algebras to $q$-Schur algebras are defined in §5 and we also prove a certain compatibility identity associated with the evaluation maps for quantum groups. In §6, we reproduce naturally a result of Chari–Pressley by simplifying the formulas for the roots of $\mathfrak{gl}_n$-Schur algebras arising from representations of affine Hecke algebra. Evaluation maps from affine $q$-Schur algebras to $q$-Schur algebras are computed in §7.

We end the paper with an application to representations of affine Hecke algebra.

Throughout the paper, $q \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$ denote a complex number which is not a root of unity. For $m, n \in \mathbb{Z}^+ := \mathbb{N} \setminus \{0\}$, $m \leq n$, let

$$[m, n] = \{m, m+1, \ldots, n\}, \quad [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad \text{and} \quad \begin{bmatrix} n \\ m \end{bmatrix} = \frac{[n]_q[n - 1]_q \cdots [n - m + 1]_q}{[m]_q[m - 1]_q \cdots [1]_q}.$$ 

All algebras are over $\mathbb{C}$.

2. Quantum loop algebras: a double Ringel–Hall algebra interpretation

We define two algebras by their generators and relations and give an explicit isomorphism between them. The first algebra is constructed as a Drinfeld double of two extended Ringel–Hall algebras.

Let $(c_{i,j})$ be the Cartan matrix of affine type $A$ and let $I = \mathbb{Z}/n\mathbb{Z} = \{1, 2, \ldots, n\}$.

**Definition 2.1 ([6, 2.2.3]).** The double Ringel–Hall algebra $\mathcal{D}_{\Delta, \mathbb{C}}(n)$ of the cyclic quiver $\triangle(n)$ is the $\mathbb{C}$-algebra generated by $E_i$, $F_i$, $K_i$, $K_i^{-1}$, $z_s^+$, $z_s^-$, for $i \in I$, $s \in \mathbb{Z}^+$, and relations:

1. $K_i K_j = K_j K_i$, $K_i K_i^{-1} = 1$;
2. $K_i E_j = q^{\delta_{i,j} - \delta_{i,j+1}} E_j K_i$, $K_i F_j = q^{\delta_{i,j} + \delta_{i,j+1}} F_j K_i$;
3. $E_i F_j - F_j E_i = \delta_{i,j} \frac{K_i - K_i^{-1}}{q - q^{-1}}$, where $K_i = K_i K_{i+1}^{-1}$;
4. $\sum_{a + b = 1 - c_{i,j}} (-1)^a \begin{bmatrix} 1 - c_{i,j} \\ a \end{bmatrix}_q E_i^a E_j B_i^b = 0$ for $i \neq j$;
5. $\sum_{a + b = 1 - c_{i,j}} (-1)^a \begin{bmatrix} 1 - c_{i,j} \\ a \end{bmatrix}_q F_i^a F_j B_i^b = 0$ for $i \neq j$;
6. $z_s^k z_s^l = z_s^k z_s^l$, $z_s^k z_s^- = z_s^- z_s^k$;
7. $K_i z_s^k = z_s^k K_i$, $K_i z_s^- = z_s^- K_i$;
8. $E_i z_s^k = z_s^k E_i$, $E_i z_s^- = z_s^- E_i$, $F_i z_s^- = z_s^- F_i$, and $z_s^k F_i = F_i z_s^k$. 

where \( i, j \in I \) and \( s, t \in \mathbb{Z}^+ \). It is a Hopf algebra with comultiplication \( \Delta \), counit \( \varepsilon \), and antipode \( \sigma \) defined by

\[
\Delta(E_i) = E_i \otimes \tilde{K}_i + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + \tilde{K}_i^{-1} \otimes F_i,
\]

\[
\Delta(K_i^{\pm 1}) = K_i^{\pm 1} \otimes K_i^{\pm 1}, \quad \Delta(z_s^{\pm}) = z_s^{\pm} \otimes 1 + 1 \otimes z_s^{\pm};
\]

\[
\varepsilon(E_i) = \varepsilon(F_i) = 0 = \varepsilon(z_s^{\pm}), \quad \varepsilon(K_i) = 1;
\]

\[
\sigma(E_i) = -E_i\tilde{K}_i^{-1}, \quad \sigma(F_i) = -\tilde{K}_iF_i, \quad \sigma(K_i^{\pm 1}) = K_i^{\mp 1},
\]

and \( \sigma(z_s^{\pm}) = -z_s^{\mp} \),

where \( i \in I \) and \( s \in \mathbb{Z}^+ \).

Let \( U_\mathbb{C}(\widehat{\mathfrak{sl}}_n) \) be the subalgebra generated by \( E_i, F_i, \tilde{K}_i, \tilde{K}_i^{-1} \) for \( i \in [1, n] \). This is the quantum affine \( \mathfrak{sl}_n \). Let \( U_\mathbb{C}(\widehat{\mathfrak{gl}}_n) \) (resp., \( U(n)_\mathbb{C} \)) be the subalgebra generated by \( E_i, F_i, K_j, K_j^{-1} \) for \( i, j \in [1, n] \) (resp., \( i \in [1, n], j \in [1, n] \)). This is the (extended) quantum affine \( \mathfrak{sl}_n \) (resp. quantum \( \mathfrak{gl}_n \)).

The second algebra follows Drinfeld [5].

**Definition 2.2.** The quantum loop algebra \( U_\mathbb{C}(\widehat{\mathfrak{gl}}_n) \) (or quantum affine \( \mathfrak{gl}_n \)) is the \( \mathbb{C} \)-algebra generated by \( x_{i,s}^{\pm} \) (\( 1 \leq i < n, s \in \mathbb{Z} \)), \( k_i^{\pm 1} \) and \( g_{i,t} \) (\( 1 \leq i \leq n, t \in \mathbb{Z} \setminus \{0\} \)) with the following relations:

1. \( k_i k_i^{-1} = 1 = k_i^{-1}k_i, [k_i, k_j] = 0; \)
2. \( k_i^{x_{j,s}^{\pm}} = q^{(\delta_{i,j} - \delta_{i,j+1})}x_{j,s}^{\pm}k_i, [k_i, g_{j,s}] = 0; \)
3. \( [g_{i,s}, x_{j,t}^{\pm}] = \begin{cases} 0 & \text{if } i \neq j, j + 1, \\ \pm q^{-is} [k_i, x_{i,s+t}^{\pm}] & \text{if } i = j, \\ \mp q^{-s} [k_i, x_{i-1,s+t}^{\pm}] & \text{if } i = j + 1; \end{cases} \)
4. \( [g_{i,s}, g_{j,t}] = 0; \)
5. \( [x_{i,s}^{\pm}, x_{j,t}^{\pm}] = \delta_{i,j} \frac{\phi_{i,s+t}^{\pm} - \phi_{i,s+t}^{-\pm}}{q - q^{-1}}; \)
6. \( x_{i,s}^{\pm} x_{j,t}^{\pm} = x_{j,t}^{\pm} x_{i,s}^{\pm}, \) for \( |i - j| > 1 \), and \( [x_{i,s+1}^{\pm}, x_{j,t}^{\pm}]_{q^{\pm v_{ij}}} = -[x_{i,s+1}^{\pm}, x_{j,t}^{\pm}]_{q^{v_{ij}}}; \)
7. \( [x_{i,s}^{\pm}, [x_{j,t}^{\pm}, x_{i,s}^{\pm}]]_{q} = -[x_{i,s}^{\pm}, [x_{j,t}^{\pm}, x_{i,s}^{\pm}]]_{q} \) for \( |i - j| = 1, \)

where \( [x, y]_{a} = xy - ayx \) and \( \phi_{i,s}^{\pm} \) are defined by the generating functions in indeterminate \( u \):

\[
\Phi_{i}^{\pm}(u) := \widetilde{k}_i^{\frac{1}{q} - 1} \exp(\pm(q - q^{-1}) \sum_{m \geq 1} h_{i,\pm m} u^{\pm m}) = \sum_{s \geq 0} \phi_{i,\pm s} u^{\pm s}
\]

with \( \widetilde{k}_i = k_i/k_{i+1} \) (\( k_{n+1} = k_1 \)) and \( h_{i,m} = q^{(i-1)m}g_{i,m} - q^{(i+1)m}g_{i+1,m} \) (\( 1 \leq i < n \)).

Beck [6] proved that the subalgebra, called the quantum loop algebra of \( \mathfrak{sl}_n \), generated by all \( x_{i,s}^{\pm}, \widetilde{k}_i^{\pm 1} \) and \( h_{i,t} \) is isomorphic to \( U_\mathbb{C}(\widehat{\mathfrak{sl}}_n) \).
Set, for each \( s \in \mathbb{Z}^+ \) and each \( i \in [1, n] \),
\[
\theta_{\pm s} = \frac{1}{[s]_q} (g_{1, \pm s} + \cdots + g_{n, \pm s}),
\]
(2.3.1)
\[
\mathcal{X}_i = [x_{i-1,0}, x_{i-2,0}, \ldots, x_{i+1,0}, x_{i,0}, \ldots, x_{1,0}, x_{0,0}, x_{-1,0}, \ldots, x_{-i,0}],
\]
and \( \mathcal{Y}_1 = [\cdots [x_{1,-1}^+, x_{2,0}^+, x_{3,0}^+, \ldots, x_{n,-1}^+] q, \ldots, x_{-n,1} q ] \).

The following isomorphism extends Beck's isomorphism.

**Theorem 2.3** ([E 4.4.1]). There is a Hopf algebra isomorphism
\[ f : \mathcal{D}_{\mathbb{C}}(n) \rightarrow U_{\mathbb{C}}(\widehat{\mathfrak{gl}_n}) \]
such that
\[
K_i^{\pm 1} \mapsto k_i^{\pm 1}, \quad E_j \mapsto x_{j,0}^+, \quad F_j \mapsto x_{j,0}^- (1 \leq i \leq n, 1 \leq j < n),
\]
\[
E_n \mapsto q \mathcal{X}_i \widehat{\mathfrak{gl}}_n, \quad F_n \mapsto q^{-1} \mathcal{K}_i \widehat{\mathfrak{gl}}_n, \quad z_s^\pm \mapsto \mp sq^\pm s \theta_{\pm s} (s \geq 1).
\]

It is known from [5 Rem. 6.1] that, for every \( 1 \leq i \leq n - 1 \),
\[
f(E_n) = q \mathcal{X}_i \widehat{\mathfrak{gl}}_n = (-1)^{i-1} q \mathcal{X}_i \widehat{\mathfrak{gl}}_n.
\]
(2.3.1)

The following elements we define will be used in defining pseudo-highest modules in next section.

For \( 1 \leq j \leq n - 1 \) and \( s \in \mathbb{Z} \), define the elements \( \mathcal{P}_{j,s} \in U_{\mathbb{C}}(\widehat{\mathfrak{gl}_n}) \) through the generating functions
\[
\mathcal{P}_j^\pm (u) := \exp \left( - \sum_{t \geq 1} \frac{1}{[t]_q} h_{j,t}(qu)^\pm t \right) = \sum_{s \geq 0} \mathcal{P}_{j,s} u^{\pm s} \in U_{\mathbb{C}}(\widehat{\mathfrak{gl}_n})[[u, u^{-1}]].
\]
Note that
\[
(2.3.2) \quad \Phi_j^\pm(u) = k_j^{\pm 1} \frac{\mathcal{P}_j^\pm(q^{-2}u)}{\mathcal{P}_j^\pm(u)}.
\]

For \( 1 \leq i \leq n \) and \( s \in \mathbb{Z} \), define the elements \( \mathcal{Q}_{i,s} \in U_{\mathbb{C}}(\widehat{\mathfrak{gl}_n}) \) through the generating functions
\[
(2.3.3) \quad \mathcal{Q}_i^\pm(u) := \exp \left( - \sum_{t \geq 1} \frac{1}{[t]_q} g_{i,t}(qu)^\pm t \right) = \sum_{s \geq 0} \mathcal{Q}_{i,s} u^{\pm s} \in U_{\mathbb{C}}(\widehat{\mathfrak{gl}_n})[[u, u^{-1}]].
\]
Note that
\[
(2.3.4) \quad \mathcal{P}_j^\pm(u) = \frac{\mathcal{Q}_j^\pm(uq^2)}{\mathcal{Q}_{j+1}^\pm(uq^2+1)},
\]
for \( 1 \leq j \leq n - 1 \).

\(^1\)Note that \( f|_{U_{\mathbb{C}}(\widehat{\mathfrak{gl}_n})} \) is the same map as given in [5 Prop. 6.1] as \([x, y]_{q^{-1}} = q^{-1/2}(q^{1/2}xy - q^{-1/2}yx)\). However, the map used in [3 Prop. 2.5] is the \( f \) here followed by an automorphism of the form \( x_{i,s}^\pm \mapsto a^i x_{i,s}^\pm \), \( g_{i,t} \mapsto a^i g_{i,t} \), and \( k_i \mapsto k_i \) for some \( a \in \mathbb{C} \).
3. Simple pseudo-highest weight $U_C(\widehat{\mathfrak{gl}_n})$-modules

Let $V$ be a weight $U_C(\widehat{\mathfrak{sl}_n})$-module (resp., weight $U_C(\widehat{\mathfrak{gl}_n})$-module) of type 1. Then $V = \oplus_{\mu \in \mathbb{Z}^{n-1}} V_{\mu}$ (resp., $V = \oplus_{\lambda \in \mathbb{Z}^n} V_{\lambda}$) where

$$V_{\mu} = \{v \in V \mid \tilde{k}_j w = q^{\mu_j} w, \forall 1 \leq j < n\} \quad \text{(resp.,} \quad V_{\lambda} = \{v \in V \mid k_j w = q^{\lambda_j} w, \forall 1 \leq j \leq n\}\).$$

Nonzero elements of $V_{\lambda}$ are called weight vectors.

Following [9], an nonzero weight vector $w \in V$ is called a pseudo-highest weight vector, if there exist some $P_{j,s} \in \mathbb{C}$ (resp. $Q_{i,s} \in \mathbb{C}$) such that

$$x_{j,s}^+ w = 0, \quad P_{j,s} w = P_{j,s} w, \quad \text{(resp.,} \quad x_{j,s}^+ w = 0, \quad Z_{i,s} w = Q_{i,s} w)\) for all $s \in \mathbb{Z}$. The module $V$ is called a pseudo-highest weight module if $V = U_C(\widehat{\mathfrak{sl}_n}) w$ (resp., $V = U_C(\widehat{\mathfrak{gl}_n}) w$) for some pseudo-highest weight vector $w$.

Following [2], an $n$-tuple of polynomials $Q = (Q_1(u), \ldots, Q_n(u))$ with constant terms 1 is called dominant if, for $1 \leq i \leq n - 1$, the ratios $Q_i(q^{-1}u)/Q_{i+1}(q^{i+1}u)$ is a polynomial. Let $Q(n)$ be the set of dominant $n$-tuples of polynomials.

For $Q = (Q_1(u), \ldots, Q_n(u)) \in Q(n)$, define $Q_{i,s} \in \mathbb{C}$ ($1 \leq i \leq n, s \in \mathbb{Z}$) by the following formula

$$Q_{i}^\pm(u) = \sum_{s \geq 0} Q_{i,\pm s} u^\pm s.$$

Here $f^+(u) = \prod_{1 \leq i \leq m} (1 - a_i u) \iff f^-(u) = \prod_{1 \leq i \leq m} (1 - a_i^{-1} u^{-1})$.

Let $I(Q)$ be the left ideal of $U_C(\widehat{\mathfrak{gl}_n})$ generated by $x_{j,s}^+, Z_{i,s} - Q_{i,s}$, and $k_i - q^{\lambda_i}$, where $1 \leq j \leq n - 1, 1 \leq i \leq n, s \in \mathbb{Z}$, and $\lambda_i = \text{deg}Q_i(u)$ Define Verma type module

$$M(Q) = U_C(\widehat{\mathfrak{gl}_n})/I(Q).$$

Then $M(Q)$ has a unique (finite dimensional) simple quotient, denoted by $L(Q)$. The polynomials $Q_i(u)$ are called Drinfeld polynomials associated with $L(Q)$.

Similarly, for an $(n-1)$-tuples $P = (P_1(u), \ldots, P_{n-1}(u)) \in \mathcal{P}(n)$ of polynomials with constant terms 1, define $P_{j,s} \in \mathbb{C}$ ($1 \leq j \leq n - 1, s \in \mathbb{Z}$) as in $P_{j}^\pm(u) = \sum_{s \geq 0} P_{j,\pm s} u^\pm s$ and let $\mu_j = \text{deg}P_j(u)$. Replacing $Z_{i,s} - Q_{i,s}, k_i - q^{\lambda_i}$ by $P_{i,s} - P_{i,s}, \tilde{k}_i - q^{\mu_i}$ in the above construction defines a simple $U_C(\widehat{\mathfrak{sl}_n})$-module $\tilde{L}(P)$. The polynomials $P_i(u)$ are called Drinfeld polynomials associated with $\tilde{L}(P)$.

**Theorem 3.1.** (1)[3] The $U_C(\widehat{\mathfrak{gl}_n})$-modules $L(Q)$ with $Q \in Q(n)$ are all nonisomorphic finite dimensional simple polynomial representations of $U_C(\widehat{\mathfrak{gl}_n})$. Moreover,

$$L(Q)|_{U_C(\widehat{\mathfrak{sl}_n})} \cong \tilde{L}(P)$$

where $P = (P_1(u), \ldots, P_{n-1}(u))$ with $P_i(u) = Q_i(q^{-1}u)/Q_{i+1}(q^{i+1}u)$.

(2)[2] Let $\mathcal{P}(n)$ be the set of $(n - 1)$-tuples of polynomials with constant terms 1. The modules $\tilde{L}(P)$ with $P \in \mathcal{P}(n)$ are all nonisomorphic finite dimensional simple $U_C(\widehat{\mathfrak{sl}_n})$-modules of type 1.
Let $\Omega_\mathbb{C}$ (resp., $\Omega_{n,\mathbb{C}}$) be a vector space over $\mathbb{C}$ with basis \{\(\omega_i \mid i \in \mathbb{Z}\}\} (resp., \{\omega_i \mid 1 \leq i \leq n\}). It is a natural $\mathcal{D}_{\triangledown,\mathbb{C}}(n)$-module with the action
\[
E_i \cdot \omega_s = \delta_{i+1,s} \omega_{s-1}, \quad F_i \cdot \omega_s = \delta_{i,s} \omega_{s+1}, \quad K_i^{\pm 1} \cdot \omega_s = q^{\pm \delta_{i,s}} \omega_s,
\]
(3.1.1) \(z^+_i \cdot \omega_s = \omega_{s-t_n},\) and \(z^-_i \cdot \omega_s = \omega_{s+t_n}.
\)
Hence, $\Omega_{n,\mathbb{C}}$ is a $U(n)_\mathbb{C}$-module.

Note that there is no weight vector in $\Omega_\mathbb{C}$ which is vanished by all $E_i$. Hence, this is not a highest weight module in the sense of [14].

The Hopf algebra structure induces a $\mathcal{D}_{\triangledown,\mathbb{C}}(n)$-module $\Omega^{\otimes r}_\mathbb{C}$, and hence, an algebra homomorphism
\[
(3.1.2) \quad \zeta_{\triangledown,r} : \mathcal{D}_{\triangledown,\mathbb{C}}(n) \to \text{End}(\Omega^{\otimes r}_\mathbb{C}).
\]
Similarly, there is an algebra homomorphism
\[
(3.1.3) \quad \zeta_r : U(n)_\mathbb{C} \to \text{End}(\Omega^{\otimes r}_{n,\mathbb{C}}).
\]
The images $\mathcal{S}_\triangledown(n,r)_\mathbb{C} = \text{im}(\zeta_{\triangledown,r})$ and $\mathcal{S}(n,r)_\mathbb{C} = \text{im}(\zeta_r)$ are called an affine $q$-Schur algebra and a $q$-Schur algebra, respectively.

The study of representations of affine $q$-Schur algebras [6, Ch. 4] shows that all finite dimensional simple $U_\mathbb{C}(\widehat{\mathfrak{gl}}_n)$-modules $L(Q)$ constructed above are simple $\mathcal{S}_\triangledown(n,r)_\mathbb{C}$-modules. They are nothing but the composition factors of all tensor spaces.

**Theorem 3.2.** Let $Q(n)_r = \{Q \in Q(n) \mid r = \sum_{i=1}^n \deg Q_i(u)\}$ and let $\text{Irr}(\mathcal{S}_\triangledown(n,r)_\mathbb{C})$ denote the set of isoclasses of simple $\mathcal{S}_\triangledown(n,r)_\mathbb{C}$-modules (and hence, simple $U_\mathbb{C}(\widehat{\mathfrak{gl}}_n)$-modules under $\zeta_{\triangledown,r}$). Then
\[
\bigcup_{r \geq 0} \text{Irr}(\mathcal{S}_\triangledown(n,r)_\mathbb{C}) = \{[L(Q)] \mid Q \in Q(n)_r\},
\]
and $\{[L(Q)] \mid Q \in Q(n)_r\}$ is a complete set of isoclasses of simple $\mathcal{S}_\triangledown(n,r)_\mathbb{C}$-modules. Moreover, every $L(Q)$ with $Q \in Q(n)_r$ is a quotient (equivalently, subquotient) module of $\Omega^{\otimes r}_\mathbb{C}$, and every composition factor of $\Omega^{\otimes r}_\mathbb{C}$ is isomorphic to $L(Q)$ for some $Q \in Q(n)_r$.

**Proof.** The first assertion is the classification theorem of simple $\mathcal{S}_\triangledown(n,r)_\mathbb{C}$-modules (see [6] 4.5.8). The second assertion follows from the fact that every finite dimensional $\mathcal{S}_\triangledown(n,r)_\mathbb{C}$-module is a homomorphic image of $\Omega^{\otimes r}_\mathbb{C}$ ([6] 4.6.2]). The last assertion is clear since every composition factor of $\Omega^{\otimes r}_\mathbb{C}$ is an $\mathcal{S}_\triangledown(n,r)_\mathbb{C}$-module and every simple $\mathcal{S}_\triangledown(n,r)_\mathbb{C}$-module is finite dimensional [6] 4.1.6].

4. AFFINE $q$-SCHUR ALGEBRAS AND THEIR SIMPLE REPRESENTATIONS

Let $\mathfrak{S}_{\triangledown,r}$ be the affine symmetric group consisting of all permutations $w : \mathbb{Z} \to \mathbb{Z}$ such that $w(i+r) = w(i) + r$ for $i \in \mathbb{Z}$. Then, $\mathfrak{S}_{\triangledown,r} \cong \mathfrak{S}_r \ltimes \mathbb{Z}^r$, where $\mathfrak{S}_r$ is the symmetric group on $r$ letters. Let $\Lambda(n,r) = \{\lambda \in \mathbb{N}^n \mid r = |\lambda|\}$ be the set of compositions of $r$ into $n$ parts and, for $\lambda \in \Lambda(n,r)$, let $\mathfrak{S}_{\lambda}$ be the Young subgroup of $\mathfrak{S}_r$ (or of $\mathfrak{S}_{\triangledown,r}$).
Let $\Lambda^+(n, r)$ be the subset of partitions (i.e. weakly decreasing compositions) in $\Lambda(n, n)$. Thus, $\Lambda^+(r) = \Lambda^+(r, r)$ is the set of all partitions of $r$. For partition $\lambda$, let $\lambda'$ be the dual partition of $\lambda$ (so $\lambda'_i = \# \{ j \mid \lambda_j \geq i \}$).

Let $H_{\Delta}(r)_C$ be the Hecke algebra associated with $S_{\Delta, r}$. Thus, $H_{\Delta}(r)_C$ has a presentation with generators $T_i, X_j$ ($i = 1, \ldots, r - 1, j = 1, \ldots, r$) and relations

$$(T_i + 1)(T_i - q^2) = 0,$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad T_i T_j = T_j T_i (|i - j| > 1),$$

$$X_i X_i^{-1} = 1 = X_i^{-1} X_i, \quad X_i X_j = X_j X_i,$$

$$T_i X_j T_i = q^2 X_j, \quad X_j T_i = T_i X_j (j \neq i, i + 1).$$

Let $H(r)_C$ be the subalgebra generated by all $T_i$. This is the Hecke algebra of $S_r$.

Following [17], the tensor space $\Omega_{n, C}^{\otimes r}$ admits a right $H_{\Delta}(r)_C$-module structure defined by

$$\omega_i \cdot X_i^{-1} = \omega_i \cdots \omega_{i-t} \omega_{i-t+1} \cdots \omega_{i-r}, \quad \omega_i \cdot T_k = \begin{cases} q^2 \omega_i, & \text{if } i_k = i_{k+1}; \\
q \omega_{i_k}, & \text{if } i_k < i_{k+1}; \\
q \omega_{i_k} + (q^2 - 1) \omega_i, & \text{if } i_{k+1} < i_k,
\end{cases} \text{ for all } i \in \mathbb{Z}^r,$$

(4.0.1)

where $1 \leq k \leq r - 1$, $1 \leq t \leq r$, and the action of $S_r$ on $I(n, r) := [1, n]^r$ is the place permutation. Apparently, this also defines an action of $H(r)_C$ on $\Omega_{n, C}^{\otimes r}$.

The formulas of the comultiplication on $z_i^+$ (2.1.1) and the first relation in (4.0.1) implies immediately the following.

**Lemma 4.1 ([6 (3.5.5.2)])**. For any $i = (i_1, \ldots, i_r) \in \mathbb{Z}^r$,

$$z_i^+ \cdot \omega_i = \sum_{s=1}^r \omega_{i_1} \otimes \cdots \otimes \omega_{i_s-1} \otimes \omega_{i_s-1-n} \otimes \omega_{i_s+1} \otimes \cdots \otimes \omega_{i_r} = \omega_i \sum_{i=1}^r X_i^+,$$

and

$$z_i^- \cdot \omega_i = \sum_{s=1}^r \omega_{i_1} \otimes \cdots \otimes \omega_{i_s-1} \otimes \omega_{i_s+1+n} \otimes \omega_{i_{s+1}} \otimes \cdots \otimes \omega_{i_r} = \omega_i \sum_{i=1}^r X_i^-.$$

We have the following generalization of the fact $S(n, r)_C = \text{End}_{H(r)_C}(\Omega_{n, C}^{\otimes r})$ for $q$-Schur algebras.

**Theorem 4.2 ([6 (3.8.1)])**. The actions of $D_{\Delta, C}(n)$ and $H_{\Delta}(r)_C$ commute and

$$S_{\Delta}(n, r)_C = \text{End}_{H_{\Delta}(r)_C}(\Omega_{C}^{\otimes r}) \quad (n \geq 2, r \geq 1) \quad \text{and} \quad H_{\Delta}(r)_C \cong \text{End}_{S_{\Delta}(n, r)_C}(\Omega_{C}^{\otimes r})^{\text{op}} \quad (n \geq r).$$

Let

$$x_{\lambda} := \sum_{w \in S_\lambda} T_w.$$

Then, there are $H_{\Delta}(r)_C$-module and $H(r)_C$-module isomorphisms: $\Omega_{C}^{\otimes r} \cong \oplus_{\lambda \in \Lambda(n, r)} x_{\lambda} H_{\Delta}(r)_C$ and $\Omega_{n, C}^{\otimes r} \cong \oplus_{\lambda \in \Lambda(n, r)} x_{\lambda} H(r)_C$, which induce algebra isomorphisms:

$$S_{\Delta}(n, r)_C \cong \text{End}_{H_{\Delta}(r)_C}(\oplus_{\lambda \in \Lambda(n, r)} x_{\lambda} H_{\Delta}(r)_C), \quad S(n, r)_C \cong \text{End}_{H(r)_C}(\oplus_{\lambda \in \Lambda(n, r)} x_{\lambda} H(r)_C).$$
Like the $q$-Schur algebras, Theorem 4.12 implies that affine $q$-Schur algebras play a bridging role between representations of quantum affine $\mathfrak{gl}_n$ and affine Hecke algebras. This becomes possible since we have established the isomorphism $f$ in Theorem 2.3 between double Ringel-Hall algebras $D_{\omega, \mathbb{C}}(n)$ and the quantum loop algebra $U_{\mathbb{C}}(\hat{\mathfrak{gl}}_n)$. We now describe how simple $S_r(n, r)_C$-modules arise from simple $H_\omega(r)_C$-modules.

A segment with center $a \in \mathbb{C}^*$ and length $k$ is by definition an ordered sequence

$$[a; k] = (aq^{-k+1}, aq^{-k+3}, \ldots, aq^{k-1}) \in (\mathbb{C}^*)^k.$$ 

A multisegment is an unordered collection of segments, denoted by formal sum

$$\sum_{i=1}^{p} [a_i; \nu_i] = [a_1; \nu_1] + \cdots + [a_p; \nu_p],$$

where, possibly, $[a_i; \nu_i] = [a_j; \nu_j]$ for $i \neq j$.

Let $\mathcal{S}_r$ be the set of all multisegments of total length $r$:

$$\mathcal{S}_r = \{[a_1; \nu_1] + \cdots + [a_p; \nu_p] \mid a_i \in \mathbb{C}^*, p, \nu_i \geq 1, r = \sum \nu_i\}.$$ 

For $s = \sum_{i=1}^{p} [a_i; \nu_i] \in \mathcal{S}_r$, let

$$(s_1, s_2, \ldots, s_r) \in (\mathbb{C}^*)^r$$

be the $r$-tuple obtained by juxtaposing the segments in $s$ and let $J_s$ be the left ideal of $H_\omega(r)_C$ generated by $X_j - s_j$ for $1 \leq j \leq r$. Then $M_s = H_\omega(r)_C/J_s$ is a left $H_\omega(r)_C$-module which as an $H(r)_C$-module is isomorphic to the regular representation of $H(r)_C$.

Let $\nu = (\nu_1, \ldots, \nu_p)$. After reordering, we may assume that $\nu$ is a partition. Then, the element

$$y_\nu = \sum_{w \in \mathfrak{S}_\nu} (-q^2)^{-\ell(w)} T_w \in H_\omega(r)_C$$

generates the submodule $H_\omega(r)_C y_\nu$ of $M_\nu$ which, as an $H(r)_C$-module, is isomorphic to $H(r)_C y_\nu$.

For each partition $\lambda$ of $r$, let $E_\lambda$ be the left cell module defined by the Kazhdan–Lusztig’s C-basis [12] associated with the left cell containing $w_{0, \lambda}$, the longest element of the Young subgroup $\mathfrak{S}_\lambda$. Then, as an $H(r)_C$-module,

$$H(r)_C y_\nu \cong E_\nu \bigoplus_{\mu \vdash r, \mu \geq \nu} m_{\mu, \nu} E_\mu. \tag{4.2.1}$$

Let $V_s$ be the unique composition factor of the $H_\omega(r)_C$-module $H_\omega(r)_C y_\nu$ such that the multiplicity of $E_\nu$ in $V_s$ as an $H(r)_C$-module is nonzero. Note that, if $V_s$ is $H(r)_C$-irreducible, then $V_s = E_\nu$. We will use this fact in the last section.

We now can state the following classification theorem due to Zelevinsky and Rogawski. The construction above follows [16].

---

2 Strictly speaking, the module $M_s$ depends on the order of the segments in $s$. However, the module $V_s$ below does not.
Theorem 4.3. Let $\text{Irr}(\mathcal{H}_\lambda(r)_\mathbb{C})$ be the set of isoclasses of all simple $\mathcal{H}_\lambda(r)_\mathbb{C}$-modules. Then the correspondence $s \mapsto [V_s]$ defines a bijection from $\mathcal{S}_r$ to $\text{Irr}(\mathcal{H}_\lambda(r)_\mathbb{C})$.

Suppose $n > r$, we define a map

$$\partial : \mathcal{S}_r \rightarrow Q(n)_r, \quad s \mapsto Q_s = (Q_1(u), \ldots, Q_n(u))$$

as follows: for $s = \sum_{i=1}^{p}[a_i; \nu_i] \in \mathcal{S}_r$, let $Q_i(u) = 1$ and, for $1 \leq i \leq n - 1$, define

$$Q_i(u) = P_i(uq^{-i+1})P_{i+1}(uq^{-i+2}) \cdots P_{n-1}(uq^{n-2i}),$$

where $P_i(u) = \prod_{1 \leq j \leq s}(1 - a_ju)$.

Here $\nu = (\nu_1, \ldots, \nu_p)$ is a partition of $r$. If $\mu_i := \deg P_i(u) = \#\{j \in [1, p] \mid \nu_j = i\}$ and $\lambda_i := \deg Q_i(u) = \#\{j \in [1, p] \mid \nu_j \geq i\}$, then $\lambda = (\lambda_1, \ldots, \lambda_n) = \nu'$, the dual partition of $\nu$, and $\lambda_i - \lambda_{i+1} = \mu_i$ for all $1 \leq i < n$.

This map is a bijection which gives the following identification theorem; see [6, §§4.3-5].

Theorem 4.4. Assume $n > r$. Then we have $S_\lambda(n, r)_\mathbb{C}$-module isomorphisms $\Omega^\otimes(r)_\mathbb{C} \otimes \mathcal{H}_\lambda(r)_\mathbb{C} V_s \cong L(Q_s)$ for all $s \in \mathcal{S}_r$. Furthermore, for any $n$ and $r$, the set

$$\{\Omega^\otimes(r)_\mathbb{C} \otimes \mathcal{H}_\lambda(r)_\mathbb{C} V_s \mid s = [a_1; \nu_1] + \cdots + [a_p; \nu_p] \in \mathcal{S}_r, p \geq 1, \nu_i \leq n, \forall i\}$$

forms a complete set of nonisomorphic simple $S_\lambda(n, r)_\mathbb{C}$-modules.

5. Compatibility of evaluation maps

Following [11] Rem. 2, every $a \in \mathbb{C}^*$ defines a surjective algebra homomorphism

$$\text{Ev}_a : U_\lambda(n)_\mathbb{C} \rightarrow U(n)_\mathbb{C}, \text{ the quantum } \mathfrak{gl}_n$$

such that, for all $1 \leq i \leq n - 1$ and $1 \leq j \leq n$,

\begin{align*}
(1) \quad & \text{Ev}_a(E_i) = E_i, \quad \text{Ev}_a(F_i) = F_i, \quad \text{Ev}_a(K_j) = K_j, \\
(2) \quad & \text{Ev}_a(E_n) = aq^{-1}[F_{n-1}[F_{n-2}, \cdots, [F_2, F_1]q^{-1}\cdots]q^{-1}]K_1K_n, \\
(3) \quad & \text{Ev}_a(F_n) = a^{-1}q[[E_1, E_2]q, E_3q, \cdots, E_{n-1}]q(K_1K_n)^{-1}.
\end{align*}

This is called the evaluation map at $a$ for quantum affine $\mathfrak{sl}_n$. Note that our definition here is exactly the same as given in [5] p.316 or [3] Prop.3.4] (cf. footnote 1).

For any $a \in \mathbb{C}^*$, there is also an evaluation map $\text{ev}_a : \mathcal{H}_\lambda(r)_\mathbb{C} \rightarrow \mathcal{H}(r)_\mathbb{C}$ (see, e.g., [5, 5.1]) such that

$$\text{ev}_a(T_i) = T_i, \quad 1 \leq i \leq r - 1, \quad \text{and}$$

$$\text{ev}_a(X_j) = aq^{-2(j-1)}T_{j-1} \cdots T_1T_1T_2 \cdots T_{j-1}, 1 \leq j \leq r.$$

Following the notation used in [10] 2.1 with $r = 1$ and $T_0 = a$ (in the notation there), we will write $L_j := \text{ev}_a(X_j)$. Note that the elements $a^{-1}L_j = (q - q^{-1})L_j + 1$ where $L_j = \tilde{T}_{(1,j)} + \tilde{T}_{(2,j)} + \cdots + \tilde{T}_{(j-1,j)}$ ($\tilde{T}_w = q^{-l(w)}T_w$) are the usual Murphy operators.
We now use the evaluation map $ev_a$ to induce an evaluation map $\tilde{ev}_a$ from the affine $q$-Schur algebra to the $q$-Schur algebra.

First, there is a right $\mathcal{H}_\delta(r)_C$-module isomorphism $\Omega_C^{\otimes r} \cong \Omega_n^{\otimes r} \otimes \mathcal{H}(r)_C \mathcal{H}_\delta(r)_C$. Second, the evaluation map $ev_a : \mathcal{H}_\delta(r)_C \rightarrow \mathcal{H}(r)_C$ induces a natural $\mathcal{H}(r)_C$-module homomorphism
\begin{equation}
(5.0.3) \quad \varepsilon_a : \Omega_C^{\otimes r} \rightarrow \Omega_n^{\otimes r}, \quad xh \mapsto x \cdot ev_a(h),
\end{equation}
for all $x \in \Omega_n^{\otimes r}$ and $h \in \mathcal{H}_\delta(r)_C$.

**Proposition 5.1.** The linear map
\begin{equation}
(5.1.1) \quad \tilde{ev}_a : \mathcal{S}_\delta(n, r)_C \rightarrow \mathcal{S}(n, r)_C
\end{equation}
defined by $(\tilde{ev}_a(\varphi))(x) = \varepsilon_a(\varphi(x))$, for any $\varphi \in \mathcal{S}_\delta(n, r)_C$ and $x \in \Omega_n^{\otimes r}$, is an algebra homomorphism.

**Proof.** By regarding $\mathcal{H}(r)_C$ as a subalgebra of $\mathcal{H}_\delta(r)_C$, one sees easily $ev_a |_{\mathcal{H}(r)_C}$ is the identity map on $\mathcal{H}(r)_C$. This fact implies that, for any $\varphi \in \mathcal{S}_\delta(n, r)_C$, $\tilde{ev}_a(\varphi)$ is an $\mathcal{H}(r)_C$-module homomorphism. To quickly see $\tilde{ev}_a(\psi \varphi) = \tilde{ev}_a(\psi)\tilde{ev}_a(\varphi)$, we may identify $\Omega_C^{\otimes r}$ as the direct sum
$$\mathfrak{T}_\delta(n, r) = \bigoplus_{\lambda \in \Lambda(n, r)} x_{\lambda} \mathcal{H}_\delta(r)_C$$
of $q$-permutation modules [17 Lem. 8.3], where $x_{\lambda} = T_{\mathfrak{e}_{\lambda}} = \sum_{w \in \mathfrak{e}_{\lambda}} T_w$, and take two double coset basis elements $\varphi = \varphi^x_{\mu \lambda}$ and $\psi = \varphi^y_{\nu \mu}$ so that $\varphi(x_{\lambda}) = T_{\mathfrak{e}_{\mu \lambda}} = x_{\mu} T_{\lambda} h$ and $\psi(x_{\mu}) = T_{\mathfrak{e}_{\nu \mu}} = x_{\nu} T_{\lambda} h'$ for some $h, h' \in \mathcal{H}(r)_C$, we have with $\bar{T}_z = ev_a(T_z)$
$$\tilde{ev}_a(\psi \varphi)(x_{\lambda}) = \varepsilon_a(x_{\nu} T_{\lambda} h' T_{\lambda} h) = x_{\nu} T_{\lambda} T_{\lambda} h = \tilde{ev}_a(\psi)(x_{\mu} T_{\lambda} h) = \tilde{ev}_a(\psi)(\tilde{ev}_a(\varphi)(x_{\lambda})) = (\tilde{ev}_a(\psi)\tilde{ev}_a(\varphi))(x_{\lambda}),$$
as desired. \qed

In order to make a comparison of representations, we need a certain compatibility relation between the evaluation maps $\tilde{ev}_a$ and $Ev_a$ and the natural homomorphisms $\zeta_{\delta, r}$ and $\zeta_r$ given in (3.1.2) and (3.1.3). In the rest of the section, we establish such a result.

For notational simplicity, we will write the elements in $\Omega_C^{\otimes r}$ by omitting the tensor sign $\otimes$. For $i = (i_1, i_2, \ldots, i_r) \in \mathbb{Z}^r$, $\lambda \in \Lambda(n, r)$, and $1 \leq j \leq \lambda_1$, let
$$\omega_1 := \omega_{i_1} \omega_{i_2} \cdots \omega_{i_r}, \quad u_{\lambda, j} = \omega_1^{j-1} \omega_2^{\lambda_1-j} \omega_3^{\lambda_2-f} \cdots \omega_n^{\lambda_n}.$$ 

**Lemma 5.2.** Maintain the notation above. The action of $T_1 T_2 \cdots T_k$ on $u_{\lambda, 1}$ is given by the formula:
$$u_{\lambda, 1} T_1 T_2 \cdots T_k = q^k u_{\lambda, k+1} + (q^2 - 1) \sum_{1 \leq s \leq k} q^{2k-s-1} u_{\lambda, s},$$
for all $1 \leq k \leq \lambda_1 - 1$. 
Lemma 5.3. Recall that $(5.3.1)$

\[ u_{\lambda,1}T_1T_2\cdots T_k = q^{k-1}u_{\lambda,k}T_k + (q^2 - 1) \sum_{1 \leq s \leq k-1} q^{2k-s-3}u_{\lambda,s}T_k \]

\[ = q^{k-1}(qu_{\lambda,k+1} + (q^2 - 1)u_{\lambda,k}) + (q^2 - 1) \sum_{1 \leq s \leq k-1} q^{2k-s-1}u_{\lambda,s} \]

proving the formula. \hfill \Box

For $3 \leq k \leq n$, let

\[ f_k = [F_{k-1}F_{k-2}, \cdots, [F_2, F_1]_{q^{-1}}]_{q^{-1}} \]

and let $f_2 = F_1$. Then $E_{\nu}(E_n) = aq^{-1}f_nK_1K_n$. For $\lambda \in \Lambda(n, r)$ and $2 \leq k \leq n$, let

\[ \lambda_{\{1,k\}} = \lambda_1 + \lambda_2 + \cdots + \lambda_{k-1}. \]

Lemma 5.3. For $\lambda \in \Lambda(n, r)$ and $2 \leq k \leq n$. The action of $f_k$ on a tensor of the form $\omega_1^1\cdots \omega_{k-1}^1\omega_j$ with $j \in [k, n]^r-\lambda_{\{1,k\}}$ is given by the formula

\[ f_k \cdot \omega_1^1\cdots \omega_{k-1}^1\omega_j = \sum_{1 \leq \omega_1^1 \leq \lambda_1} q^{1-s_{\lambda_1}-1}\omega_1^1\omega_1^1\omega_1^2\cdots \omega_{k-1}^1\omega_j. \]

Proof. Recall that $F_i$ acts on $\Omega^{\overline{\omega}_r}_C$ via $\Delta^r(F_i) = \sum_{s=1}^{r} \tilde{K}_{i}^{-1} \otimes \cdots \otimes \tilde{K}_{i}^{-1} \otimes F_i \otimes 1 \otimes \cdots \otimes 1$. If $k = 2$, the action $F_1 \cdot \omega_1^1\omega_1^1 = \sum_{1 \leq s \leq \lambda_1} q^{1-s_{\lambda_1}-1}\omega_1^1\omega_1^1\omega_1^2\cdots \omega_{k-1}^1\omega_j$ follows from $(3.1.1)$. Assume now $k > 2$ and $j \in [k, n]^r-\lambda_{\{1,k\}}$. Since $f_k = [F_{k-1}, f_{k-1}]_{q^{-1}}$, we have

\[ f_k \cdot \omega_1^1\cdots \omega_{k-1}^1\omega_j = F_{k-1}^2 f_{k-1} \cdot \omega_1^1\cdots \omega_{k-1}^1\omega_j - q^{-1}f_{k-1} F_{k-1} \cdot \omega_1^1\cdots \omega_{k-1}^1\omega_j. \]

Since

\[ F_{k-1} \cdot \omega_1^1\cdots \omega_{k-1}^1\omega_j = \sum_{1 \leq \omega_1^1 \leq \lambda_1} q^{1-t_{\omega_1^1}}\omega_1^1\cdots \omega_{k-2}^1\omega_{k-1}^1\omega_{k-1}^1\omega_j, \]

where $j' \in [k - 1, n]^r-\lambda_{\{1,k-1\}}$, by induction,

\[ f_{k-1} F_{k-1} \cdot \omega_1^1\cdots \omega_{k-1}^1\omega_j = \sum_{1 \leq \omega_1^1 \leq \lambda_1} q^{2-t_{\omega_1^1}}\omega_1^1\cdots \omega_{k-2}^1\omega_{k-1}^1\omega_{k-1}^1\omega_j. \]

On the other hand, by induction again,

\[ F_{k-1} f_{k-1} \cdot \omega_1^1\cdots \omega_{k-1}^1\omega_j = \sum_{1 \leq \omega_1^1 \leq \lambda_1} F_{k-1} \cdot q^{1-s_{\lambda_1}-1}\omega_{k-1}\omega_1^1\omega_1^1\omega_1^2\cdots \omega_{k-1}^1\omega_j \]

(noting the extra $\omega_{k-1}$) =

\[ \sum_{1 \leq \omega_1^1 \leq \lambda_1} q^{1-s_{\lambda_1}-1}\omega_{k-1}\omega_1^1\omega_1^1\omega_1^2\cdots \omega_{k-2}^1\omega_{k-1}^1\omega_{k-1}^1\omega_j \]

\[ + \sum_{1 \leq \omega_1^1 \leq \lambda_1} q^{1-s_{\lambda_1}-1}\omega_{k-1}\omega_1^1\omega_1^1\omega_1^2\cdots \omega_{k-2}^1\omega_{k-1}^1\omega_{k-1}^1\omega_j. \]
Substituting into (5.3.1) cancels the double indexed sum and yields the desired formula. \( \square \)

**Proposition 5.4.** For \( a \in \mathbb{C}^* \), we have \((\zeta_r \circ \mathbf{E}v_a)(E_n) = (\tilde{\mathbf{e}}v_a \circ \zeta_{\alpha,r})(E_n)\).

**Proof.** We need the check that the images of both sides at \( \omega_{1,\lambda} \) are equal for all \( \lambda \in \Lambda(n, r) \). Since \( \Delta^{(r)}(E_i) = \sum_{s=1}^{r} \prod_{s=1}^{r} E_i \),

\[
(\tilde{\mathbf{e}}v_a \circ \zeta_{\alpha,r})(E_n) \cdot \omega_{1,\lambda} = \varepsilon_a \left( E_n \cdot \omega_{1,\lambda} \right) = \varepsilon_a \left( q^{\lambda_n - \lambda_1 + j} \sum_{1 \leq j \leq \lambda_1} \omega_1^{j-1} \omega_1^{\lambda_n - j} \omega_2^{\lambda_2} \cdots \omega_n^{\lambda_n} \right)
\]

\[
= \sum_{1 \leq j \leq \lambda_1} q^{\lambda_n - \lambda_1 + j} \left( \omega_1^{j-1} \omega_1^{\lambda_n - j} \omega_2^{\lambda_2} \cdots \omega_n^{\lambda_n} \cdot \mathbf{e}v_a(X_j) \right)
\]

\[
= aq^{\lambda_n - \lambda_1 + 1} \sum_{1 \leq j \leq \lambda_1} \omega_n \omega_1^{\lambda_n - 1} \omega_2^{\lambda_2} \cdots \omega_n^{\lambda_n} T_1 T_2 \cdots T_{j-1} = aq^{\lambda_n - \lambda_1 + 1} \sum_{1 \leq j \leq \lambda_1} u_{\lambda,1} T_1 T_2 \cdots T_{j-1}.
\]

Now applying Lemma 5.2 yields (noting that the second sum is zero if \( j = 1 \))

\[
(\tilde{\mathbf{e}}v_a \circ \zeta_{\alpha,r})(E_n) \cdot \omega_{1,\lambda} = aq^{\lambda_n - \lambda_1 + 1} \left( \sum_{1 \leq j \leq \lambda_1} q^{j-1} u_{\lambda,j} + \sum_{1 \leq j \leq \lambda_1} q^{2j-3} u_{\lambda,s} \right)
\]

\[
= aq^{\lambda_n - \lambda_1 + 1} \left( q^{j-1} u_{\lambda,1} \lambda_1 + \sum_{1 \leq j \leq \lambda_1} q^{j-1} u_{\lambda,j} + (q^2 - 1) \sum_{1 \leq s \leq \lambda_1} q^{2j-3} u_{\lambda,s} \right).
\]

Since \( (q^2 - 1) \sum_{s+1 \leq j \leq \lambda_1} q^{2j-3} = q^{2\lambda_1 - s - 1} - q^{-1} \), it follows that

\[
(\tilde{\mathbf{e}}v_a \circ \zeta_{\alpha,r})(E_n) \cdot \omega_{1,\lambda} = aq^{\lambda_n - \lambda_1 + 1} \left( q^{\lambda_1 - 1} u_{\lambda,1} \lambda_1 + \sum_{1 \leq s \leq \lambda_1} q^{2\lambda_1 - s - 1} u_{\lambda,s} \right)
\]

\[
= aq^{\lambda_n} \sum_{1 \leq s \leq \lambda_1} q^{\lambda_1 - s} u_{\lambda,s}.
\]

On the other hand, applying Lemma 5.3 yields

\[
(\zeta_r \circ \mathbf{E}v_a)(E_n) \cdot \omega_{1,\lambda} = aq^{\lambda_1 + \lambda_n - 1} f_n \cdot \omega_{1,\lambda} = aq^{\lambda_n} \sum_{1 \leq s \leq \lambda_1} q^{\lambda_1 - s} u_{\lambda,s}.
\]

Hence, \((\tilde{\mathbf{e}}v_a \circ \zeta_{\alpha,r})(E_n) \cdot \omega_{1,\lambda} = (\zeta_r \circ \mathbf{E}v_a(E_n)) \cdot \omega_{1,\lambda}\) for all \( \lambda \in \Lambda(n, r) \). \( \square \)

**Remark 5.5.** We believe that the equation \((\zeta_r \circ \mathbf{E}v_a)(F_n) = (\tilde{\mathbf{e}}v_a \circ \zeta_{\alpha,r})(F_n)\) holds as well. Thus, the following diagram is commutative:

\[
\begin{array}{ccc}
U(n)_C & \xrightarrow{\mathbf{E}v_a} & U(n)_C \\
\downarrow \zeta_{\alpha,r} & & \downarrow \zeta_r \\
S(n, r)_C & \xrightarrow{\tilde{\mathbf{e}}v_a} & \mathbf{S}(n, r)_C
\end{array}
\]

However, the proof is much more complicated than the \( E_n \) case. Fortunately, the Identification Theorem we will establish requires only the compatibility for the \( E_n \) case.
6. A result of Chari–Pressley [3 3.5]. There are two differences in our approach. First, the isomorphism given in [3 2.5] and used in the proof of [3 3.5(2)(3)] has been replaced by the isomorphism $f$ given in [2.3] (see footnote 1). Second, the proof of [3 3.5] used a category equivalence [3 3.2] which turns a simple $U_C(\mathfrak{sl}_n)$-module into a simple $U_C(\mathfrak{gl}_n)$-module and, hence, a $U_C(\widehat{\mathfrak{gl}}_n)$-module under the evaluation map. We directly start with a simple $q$-Schur algebra module regarded as a simple $U_C(\mathfrak{gl}_n)$-module with a partition as the highest weight. It turns out that the description of the centers of the segments, which consist of the roots of a Drinfeld polynomial, is considerably simpler under this approach.

For $\lambda \in \Lambda^+(n, r)$, let $L(\lambda)$ be the simple $S(n, r)_C$-module with highest weight $\lambda$.

**Theorem 6.1.** For $a \in \mathbb{C}^*$ and $\lambda \in \Lambda^+(n, r)$, let $L(\lambda)_a$ be the inflated $U_C(\widehat{\mathfrak{gl}}_n)$-module by the evaluation map $\tilde{\mathcal{E}}_a$ in (5.1.1). If $L(\lambda)_a|_{U_C(\mathfrak{gl}_n)} \cong \tilde{L}(P)$ with $P = (P_1(u), P_2(u), \ldots, P_{n-1}(u))$, then

$$P_j(u) = \prod_{\lambda_{j+1}+1 \leq s \leq \lambda_j} (1 - aq^{2s-1-j}u)$$

for $1 \leq j \leq n - 1$. In particular, the roots of $P_j(u)$ forms the segment $[a^{-1}q^{j-\lambda_j-\lambda_{j+1}}; \lambda_j - \lambda_{j+1}]$ with center $a^{-1}q^{j-\lambda_j-\lambda_{j+1}}$ and length $\lambda_j - \lambda_{j+1}$. (Note that $\lambda_j = \lambda_{j+1} \implies P_j(u) = 1$.)

**Proof.** The proof here follows that of [3 3.5]. By Lemma 2.3 we identify $\mathcal{D}_{\partial, \mathfrak{c}}(n)$ with $U_C(\widehat{\mathfrak{gl}}_n)$ under the isomorphism $f$. Thus, $F_j = x_{j,0}^-, E_j = x_{j,0}^+$, and $K_i = \kappa_i$, for all $1 \leq j \leq n - 1$, $1 \leq i \leq n$. With the notation in (5.2.1), we have $\mathcal{E}_a(E_n) = aq^{-1}f_nK_iK_i$.

Fix $i \in [1, n - 1]$. Recall from (2.2.1) the notation $\mathcal{X}_i$. Then (2.3.1) becomes $E_n = (-1)^i-1q^{\lambda_n-\lambda_i+1}\mathcal{X}_i w_0 = \mathcal{E}_a \circ \mathcal{C}_{\partial, \mathfrak{c}}(E_n) \cdot w_0 = \mathcal{C}_i \circ \mathcal{E}_a(E_n) \cdot w_0 = aq^{\lambda_i+\lambda_n-1}f_n w_0$.

Thus, (6.1.1)

$$\mathcal{X}_i w_0 = (-1)^{i-1}aq^{2(\lambda_i-1)}f_n w_0.$$

Let $\mu = (\mu_1, \cdots, \mu_{n-1})$ with $\mu_i = \lambda_i - \lambda_{i+1}$ and define recursively

$$\mathcal{X}_{i,j} = \begin{cases} [F_1, [F_2, \cdots, [F_{i-1}, x_{i,1}^-[q^{-1}]q^{-1} \cdots q^{-1}]q^{-1}], & \text{if } j = i; \\ [F_j, \mathcal{X}_{i,j-1}]q^{-1}, & \text{if } i + 1 \leq j \leq n - 1. \end{cases}$$

Then, $\mathcal{X}_{i,n-1} = \mathcal{X}_i$ and

$$E_{n-1} \mathcal{X}_i = E_{n-1}F_{n-1} \mathcal{X}_{i,n-2} - q^{-1} \mathcal{X}_{i,n-2}E_{n-1}F_{n-1}$$

$$= (F_{n-1}E_{n-1} + \frac{\tilde{k}_{n-1} - \tilde{k}_{n-1}^{-1}}{q - q^{-1}}) \mathcal{X}_{i,n-2} - q^{-1} \mathcal{X}_{i,n-2}(F_{n-1}E_{n-1} + \frac{\tilde{k}_{n-1} - \tilde{k}_{n-1}^{-1}}{q - q^{-1}}).$$

By (QGL2), we see that $\frac{\tilde{k}_{n-1} - \tilde{k}_{n-1}^{-1}}{q - q^{-1}} \mathcal{X}_{i,n-2} = \mathcal{X}_{i,n-2} \frac{qk_{n-1}^{-1} - q^{-1}k_{n-1}^{-1}}{q - q^{-1}}$. Hence,

$$E_{n-1} \mathcal{X}_i w_0 = ((\mu_{n-1} + 1) - q^{-1}[\mu_{n-1}]) \mathcal{X}_{i,n-2} w_0.$$
Inductively, we obtain
\[ E_{i+1}E_{i+2} \cdots E_{n-1}X_i w_0 = \prod_{i+1 \leq s \leq n-1} ([\mu_s + 1] - q^{-1}[\mu_s])X_i w_0 \]
and, similarly,
\[ E_{i-1}E_{i-2} \cdots E_1X_{i-1} w_0 = E_{i-1}E_{i-2} \cdots E_1[F_1, F_2, \ldots, [F_{i-1}, X_{i-1}]_{q^{-1}} \cdots]_{q^{-1}}w_0 \]
\[ = \prod_{1 \leq s \leq i-1} ([\mu_s + 1] - q^{-1}[\mu_s])X_{i-1} w_0. \]

Hence,
\[ E_{i-1} \cdots E_2 E_1 E_{i+1} E_{i+2} \cdots E_{n-1} X_i w_0 = \prod_{1 \leq s \leq n-1} ([\mu_s + 1] - q^{-1}[\mu_s])X_{i-1} w_0. \] (6.1.2)

On the other hand,
\[ E_{i-1} \cdots E_2 E_1 E_{i+1} E_{i+2} \cdots E_{n-1} x_n w_0 = \prod_{i+1 \leq s \leq n-1} ([\mu_s + 1] - q^{-1}[\mu_s])E_{i-1} \cdots E_2 E_1 x_{i+1} w_0 \]
\[ = \prod_{i+1 \leq s \leq n-1} ([\mu_s + 1] - q^{-1}[\mu_s]) \times \prod_{1 \leq s \leq i-1} ([\mu_s] - q^{-1}[\mu_s + 1])F_i w_0. \]

This together with (6.1.1) and the fact that
\[ \prod_{1 \leq s \leq i-1} \frac{[\mu_s] - q^{-1}[\mu_s + 1]}{[\mu_s + 1] - q^{-1}[\mu_s]} = (-1)^{i-1} q^{2(\lambda_i - \lambda_1) - i+1} \]
implies that
\[ X_{i-1} w_0 = aq^{2\lambda_i - i-1}F_i w_0. \]

Applying $E_i$ to the above equation and noting $\phi_{i,1}^- = 0$ yields
\[ \phi_{i,1}^+ w_0 = aq^{2\lambda_i - i-1}(q^{\mu_i} - q^{-\mu_i})w_0. \] (6.1.3)

By the corollary in [3, 3.5], we may assume
\[ P_1(u) = \prod_{1 \leq j \leq \mu_i} (1 - b_j q^{2j-\mu_i-1} u). \]

Thus,
\[ q^{\mu_i} P_1(u) = q^{\mu_i} u + q^{\mu_i} b_i (q^{\mu_i-1} - q^{-\mu_i-1})u + O(u^2). \]

This together with (2.3.2) and (6.1.3) implies that
\[ aq^{2\lambda_i - i-1}(q^{\mu_i} - q^{-\mu_i}) = q^{\mu_i} b_i (q^{\mu_i-1} - q^{-\mu_i-1}). \]

Hence, $b_i = aq^{\lambda_i + \lambda_{i+1} - i}$ and
\[ P_i(u) = \prod_{1 \leq j \leq \mu_i} (1 - aq^{2(\lambda_{i+1} + j) - i-1}) = \prod_{\lambda_{i+1} + 1 \leq s \leq \lambda_i} (1 - aq^{2s-1-i} u), \]
as required. \qed
7. An Identification Theorem

We now compute the dominant Drinfeld polynomials \( Q = (Q_1(u), \ldots, Q_n(u)) \) such that the \( U_\mathbb{C}(\mathfrak{gl}_n) \)-module \( L(\lambda)_a \cong L(Q) \). By Theorem [6.4], it remains to compute \( Q_n(u) \). This will be done by the action of the central elements \( z_t^\pm \) on a highest weight vector \( w_0 \in L(\lambda) \).

We first apply a result of James–Mathas to compute the action of \( z_t^\pm \) on the simple \( S(n, r)_\mathbb{C} \)-module \( L(\lambda) \) via the evaluation map \( \tilde{e}_\lambda \) in [5.11].

The Hecke algebra \( H(r)_\mathbb{C} \) of the symmetric groups \( S_r \) admits a so-called Murphy’s basis [15]

\[
\{ x_{s, t} := T_{d(s)}^* x_{\lambda} T_{d(t)} \mid s, t \in T^*(\lambda), \lambda \in \Lambda^+(r, r) \},
\]

where \( T^*(\lambda) \) is the set of all standard \( \lambda \)-tableaux, \( * \) is the anti-involution satisfying \( T_i^* = T_i \), and \( d(t) \) is the permutation mapping the standard \( \lambda \)-tableau \( t^\lambda \) (obtained by filling \( 1, 2, \ldots, r \) from left to right down successive row) to \( t \). The subspace \( H^{\lambda \lambda} \) of \( H(r)_\mathbb{C} \) spanned by \( \{ x_{st} \mid s, t \in T^*(\mu), \mu \triangleright \lambda \} \) is a two sided ideal of \( H(r)_\mathbb{C} \). Let \( S^\lambda = x_{\lambda} H(r)_\mathbb{C} / (x_{\lambda} H(r)_\mathbb{C} \cap H^{\lambda \lambda}) \). Then \( S^\lambda \cong \bigoplus_{\nu} \bigotimes_{\mu} x_{\lambda} H(r)_\mathbb{C} \) (see, e. g., [11 §3]).

Similarly, for partition \( \lambda \) of \( r \), let \( T^{ss}(\lambda, n) \) (resp. \( T^{ss}(\lambda, \mu) \)) be the set of all semistandard \( \lambda \)-tableaux with content in \([1, n]\) (resp., of content \( \mu \)). Then, the tensor space \( \Omega_{n, \mathbb{C}}^{\otimes r} \) can be identified with

\[
\mathfrak{S}(n, r) = \bigoplus_{\mu \in \Lambda(n, r)} x_{\lambda} H(r)_\mathbb{C},
\]

and the Murphy basis induces a basis (see, e.g., [10])

\[
\{ m_{s, t} \mid (s, t) \in T^{ss}(\lambda, n) \times T^*(\lambda), \forall \lambda \in \Lambda^+(n, r) \}.
\]

Fix a linear ordering on \( \Lambda^+(n, r) = \{ \lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(N)} \} \) which refines the dominance ordering \( \triangleright \), i.e., \( \lambda^{(i)} \triangleright \lambda^{(j)} \) implies \( i < j \). For each \( 1 \leq i \leq N \), let \( \mathfrak{T}_i \) denote the subspace of \( \mathfrak{S} \) spanned by all \( m_{s, t} \) such that \( (s, t) \in T^{ss}(\lambda, n) \times T^*(\lambda) \) for some \( \lambda \in \{ \lambda^{(1)}, \ldots, \lambda^{(i)} \} \). Then we obtain a filtration by \( S(n, r)_\mathbb{C} \cdot H(r)_\mathbb{C} \)-subbimodules:

\[
0 = \mathfrak{T}_0 \subseteq \mathfrak{T}_1 \subseteq \cdots \subseteq \mathfrak{T}_N = \mathfrak{S}(n, r)
\]

such that \( \mathfrak{T}_i / \mathfrak{T}_{i-1} \cong L(\lambda^{(i)}) \otimes S^{\lambda^{(i)}} \) as \( S(n, r)_\mathbb{C} \cdot H(r)_\mathbb{C} \)-bimodules.

For \( \lambda = \lambda^{(i)} \), \( S \in T^{ss}(\lambda, n) \), \( s \in T^*(\lambda) \), \( T^{\lambda} \in T^{ss}(\lambda, \lambda) \), and \( t^\lambda \) as above, let \( \varphi_S = m_{s, t^\lambda} + \mathfrak{T}_{i-1} \) and \( \psi_s = m_{T^{\lambda} \cdot s} + \mathfrak{T}_{i-1} \). Then

\[
W^\lambda := S(n, r)_\mathbb{C} \varphi_{T^\lambda}
\]

is a simple \( S(n, r)_\mathbb{C} \)-module, isomorphic to \( L(\lambda) \), and \( \{ \varphi_S \}_{S \in T^{ss}(\lambda, n)} \) forms a basis for \( W^\lambda \). Since \( m_{T^{\lambda} \cdot t^\lambda} = x_{\lambda} \varphi_{T^\lambda} H(r)_\mathbb{C} = \psi_t H(r)_\mathbb{C} \cong S^\lambda \) with basis \( \{ \psi_s \}_{s \in T^*(\lambda)} \).

Since \( z_t^\pm \) are central elements in \( U_\mathbb{C}(\widehat{\mathfrak{gl}}_n) \), it follows that \( (\tilde{e}_\lambda \circ \zeta_{\lambda, r})(z_t^\pm) \) are central in \( S(n, r)_\mathbb{C} \). By Schur’s Lemma, \( (\tilde{e}_\lambda \circ \zeta_{\lambda, r})(z_t^\pm) \) acts on \( W^\lambda \) by a scalar \( c^\pm_t(\lambda) \). We now compute this scalar.
Lemma 7.1. Let $\lambda \in \Lambda^+(n, r)$ and $t$ a positive integer. If $(\tilde{e}_a \circ \zeta_{\lambda, r})(z_t^\pm)$ acts on $W^\lambda$ by $c^\pm_t(\lambda) \in \mathbb{C}$, then

$$c^\pm_t(\lambda) = a^{\pm t} \sum_{1 \leq i \leq n, 1 \leq j \leq \lambda_i} q^{\pm 2t(j-i)}.$$  

Proof. Under the $\mathcal{H}_\lambda(r)_\mathbb{C}$-module isomorphism $\Omega^o_{\mathbb{C}} \cong \mathfrak{T}_\lambda(n, r)$ and its restriction giving an $\mathcal{H}(r)_\mathbb{C}$-module isomorphism $\Omega^o_{\mathbb{C}, \mathcal{H}} \cong \mathfrak{T}(n, r)$, we identify the two $\mathcal{H}_\lambda(r)_\mathbb{C}$-modules and the two $\mathcal{H}(r)_\mathbb{C}$-modules. In particular, the tensor $\omega_i$ identifies $q^{-\ell(w_0, \lambda)}x_{\tau\lambda, t\lambda} = q^{-\ell(w_0, \lambda)}x_{\lambda}$, where $i_\lambda = (1, \ldots, 1, n, \ldots, n)$ and $w_0, \lambda$ is the longest element in $\mathfrak{S}_\lambda$.

Suppose $(\tilde{e}_a \circ \zeta_{\lambda, r})(z_t^\pm) \cdot \varphi_{\tau\lambda} = c_t^\pm(\lambda)\varphi_{\tau\lambda}$. Then

$$\tag{7.1.1} (\tilde{e}_a \circ \zeta_{\lambda, r})(z_t^\pm)(x_\lambda) \equiv c^\pm_t(\lambda)x_\lambda \mod \mathfrak{T}_i,$$

On the other hand, by Lemma 4.1

$$(\tilde{e}_a \circ \zeta_{\lambda, r})(z_t^\pm)(x_\lambda) = \tilde{e}_a(\zeta_{\lambda, r}(z_t^\pm))(x_\lambda) = \varepsilon_a(z_t^\pm \cdot x_\lambda) = \varepsilon_a(x_\lambda \cdot \sum_{s=1}^r X_t^{s\pm t}) = x_\lambda \cdot \sum_{s=1}^r L_t^{s\pm t},$$

where $L_s = ev_a(X_s) \in \mathcal{H}(r)_\mathbb{C}$ for all $1 \leq s \leq r$. By [10] 3.7, we have

$$x_\lambda \cdot L_s \equiv \text{res}_{\tau\lambda}(s)x_{\lambda} \mod x_\lambda \mathcal{H}(r)_\mathbb{C} \cap \mathcal{H}_{\lambda, t\lambda},$$

where, if $s$ in $t_{\lambda}$ is at row $i$ and column $j$, then $\text{res}_{\tau\lambda}(s) = aq^{2t(j-i)}$ is the residue at $s$. Thus,

$$(\tilde{e}_a \circ \zeta_{\lambda, r})(z_t^\pm)(x_\lambda) = \sum_{1 \leq s \leq r} x_\lambda \cdot L_s^{s\pm t} \equiv \sum_{1 \leq s \leq r} (\text{res}_{\tau\lambda}(s))^{s\pm t} x_\lambda \mod \mathfrak{T}_i.$$

Comparing this with (7.1.1) yields

$$c^\pm_t(\lambda) = \sum_{1 \leq s \leq r} (\text{res}_{\tau\lambda}(s))^{s\pm t} = a^{\pm t} \sum_{1 \leq i \leq n, 1 \leq j \leq \lambda_i} q^{\pm 2t(j-i)},$$

as desired. \hfill $\square$

Theorem 7.2. For $a \in \mathbb{C}^*$ and $\lambda \in \Lambda^+(n, r)$, suppose $L(\lambda)_a \cong L(Q)$ for some $Q \in \mathfrak{D}(n, r)$. Then $Q = (Q_1(u), \ldots, Q_m(u), 1, \ldots, 1)$, where $m$ is the number of parts of $\lambda$,

$$Q_i(uq^{i-1}) = \prod_{\lambda_{i+1}+1 \leq s \leq \lambda_i} (1 - aq^{2s-1-i}u),$$

for all $1 \leq i \leq m - 1$, and $Q_m(u) = \prod_{1 \leq s \leq \lambda_m} (1 - aq^{2(s-m)}u)$.

Proof. Since $\tilde{e}_a$ is surjective, $L(\lambda)_a$ is an irreducible $\mathfrak{S}_\lambda(n, r)_\mathbb{C}$-module. Thus, by [6] 4.5.8, there exists $Q = (Q_1(u), \ldots, Q_n(u)) \in \mathfrak{Q}(n, r)$ such that $L(\lambda)_a \cong L(Q)$. For $1 \leq j \leq n - 1$, let

$$P_j(u) = \frac{Q_j(uq^{j-1})}{Q_{j+1}(uq^{j+1})}.$$

3The $q, Q_1$ in [10] are $q^2, a$ here.
By Theorems 3.1 and 6.1 we have, for $1 \leq j \leq n - 1$,
\[ P_j(u) = \prod_{\lambda_{j+1} \leq \lambda_j \leq \lambda} \left( 1 - aq^{2s - j}u \right). \]

Thus, if $m < n$, then $P_j(u) = 1$, for all $m + 1 \leq j < n$. Hence, $Q_i(u) = 1$ for all $m < i \leq n$ and
\[ Q_1(u) = P_1(u)P_2(uq) \cdots P_{m-1}(uq^{m-2})P_m(uq^{m-1}), \]
\[ Q_2(u) = P_2(uq^{-1}) \cdots P_{m-1}(uq^{-m-4})P_m(uq^{-m-3}), \]
\[ Q_{m-1}(u) = P_{m-1}(uq^{-m+2})P_m(uq^{-m+3}), \]
\[ Q_m(u) = P_m(uq^{-m+1}). \]

In particular, $Q_m(u) = P_m(uq^{-m+1}) = \prod_{1 \leq s \leq \lambda_m} (1 - aq^{2(s-m)}u)$, as required in this case.

We now assume $m = n$. Then we have recursively,
\[ Q_i(u) = P_i(uq^{-i+1})P_{i+1}(uq^{-i+2}) \cdots P_{n-1}(uq^{-n+2})P_n(uq^{2(n-i)}), \]
for all $1 \leq i \leq n$. We now compute $Q_n(u)$.

Let $w_0$ be a nonzero vector in $L(\lambda)_\chi$. Since $z_t^+ = \frac{tq^t}{[t]_q} \sum_{1 \leq i \leq n} g_{i,t}$ under the isomorphism in Lemma 2.3, Lemma 7.1 implies
\[ \frac{tq^t}{[t]_q} \sum_{1 \leq i \leq n} g_{i,t}w_0 = c_t^+ (\lambda) w_0 = (e_{\lambda_n} \circ \zeta_{\lambda,t}(z_t^+))w_0 = a^t \sum_{1 \leq i \leq n} q^{2t(j-i)}w_0. \]

Thus, by (3.3.3), $\mathcal{D}_i^+(u) \cdot w_0 = Q_{1,s}w_0$ for all $s \geq 0$ give an identity in $L(\lambda)[[u]]$:
\[ \prod_{1 \leq i \leq n} Q_i(u)w_0 = \prod_{1 \leq i \leq n} \mathcal{D}_i^+(u) \cdot w_0. \]

However, by (2.3.3),
\[ \prod_{1 \leq i \leq n} \mathcal{D}_i^+(u) \cdot w_0 = \exp \left( - \sum_{t \geq 1} \frac{1}{[t]_q} \left( \sum_{1 \leq i \leq n} g_{i,t} \right)(uq)^t \right)w_0 \]
\[ = \exp \left( - \sum_{1 \leq i \leq n} \sum_{t \geq 1} \frac{1}{t} (auq^{2(j-i)}t)^t \right)w_0 \]
\[ = \prod_{1 \leq i \leq n} \exp \left( - \sum_{t \geq 1} \frac{1}{t} (auq^{2(j-i)}t)^t \right)w_0 \]
\[ = \prod_{1 \leq i \leq n} \left( 1 - auq^{2(j-i)} \right)w_0. \]

Hence,
\[ (7.2.2) \prod_{1 \leq i \leq n} Q_i(u) = \prod_{1 \leq i \leq n} \prod_{1 \leq j \leq \lambda_i} \left( 1 - auq^{2(j-i)}u \right). \]
Since \( Q_i(u) = P_i(uq^{-i+1})P_{i+1}(uq^{-i+2}) \cdots P_{n-1}(uq^{n-2i})Q_n(uq^{2(n-i)}) \), we have

\[
(7.2.3) \prod_{1 \leq i \leq n} Q_i(u) = \prod_{1 \leq k \leq n-1} \left( P_k(uq^{-k+1})P_k(uq^{-k+3}) \cdots P_k(uq^{k-1}) \right) \prod_{0 \leq l \leq n-1} Q_n(uq^{2l}).
\]

Now,

\[
\prod_{1 \leq k \leq n-1} \left( P_k(uq^{-k+1})P_k(uq^{-k+3}) \cdots P_k(uq^{k-1}) \right) = \prod_{1 \leq k \leq n-1} \prod_{1 \leq i < k, \lambda_k+1 \leq j \leq \lambda_k} (1 - aq^{2(j-i)}u) \\
= \prod_{1 \leq i \leq n-1} \prod_{1 \leq k \leq n-1, \lambda_k+1 \leq j \leq \lambda_k} (1 - aq^{2(j-i)}u) \\
= \prod_{1 \leq i \leq n-1} \prod_{\lambda_n+1 \leq j \leq \lambda_i} (1 - aq^{2(j-i)}u).
\]

This together with (7.2.2) and (7.2.3) implies that

\[
\prod_{0 \leq l \leq n-1} Q_n(uq^{2l}) = \prod_{1 \leq i \leq n} \prod_{1 \leq j \leq \lambda_i} (1 - aq^{2(j-i)}u).
\]

Hence, we have in \( L(\lambda)[[u]] \):

\[
\prod_{0 \leq l \leq n-1} \mathcal{D}_n(uq^{2l}) \cdot w_0 = \prod_{0 \leq l \leq n-1} Q_n(uq^{2l})w_0 = \prod_{1 \leq i \leq n} \prod_{1 \leq j \leq \lambda_i} (1 - aq^{2(j-i)}u)w_0.
\]

On the other hand, by (2.3.3) again,

\[
(7.2.4) \prod_{0 \leq l \leq n-1} \mathcal{D}_n(uq^{2l}) \cdot w_0 = \exp \left( -\sum_{t \geq 1} \frac{1}{[t]_q} g_n,t \left( \sum_{0 \leq l \leq n-1} q^{2lt}(uq)^{t} \right) \right) \cdot w_0.
\]

Applying \( \ln(\ ) \) formerly yields,

\[
-\sum_{t \geq 1} \frac{1}{[t]_q} g_n,t \left( \sum_{0 \leq l \leq n-1} q^{2lt+1} \right) u^t \cdot w_0 = \sum_{1 \leq i \leq n} \sum_{1 \leq j \leq \lambda_i} \ln(1 - aq^{2(j-i)}u)w_0 \\
= -\sum_{t \geq 1} \left( \sum_{1 \leq i \leq n} \frac{1}{t} (aq^{2(j-i)})^t \right) u^t w_0.
\]

Equating coefficients of \( u^t \) gives

\[
\frac{1}{[t]_q} g_n,t \sum_{0 \leq l \leq n-1} q^{2l+1} \cdot w_0 = \frac{1}{t} \sum_{1 \leq i \leq n} \sum_{1 \leq j \leq \lambda_i} (aq^{2(j-i)})^t w_0.
\]

Since

\[
\sum_{1 \leq j \leq \lambda_i} a^t q^{2(j-i)}t = \sum_{1 \leq j \leq \lambda_i} a^t q^{2(j-i)}(1 - q^{-2tn})
\]

and \( \sum_{0 \leq l \leq n-1} q^{2l+1} = \frac{q^{2n+1} - q}{q - q^{-1}} \), it follows that

\[
g_n,t \cdot w_0 = \frac{[t]_q}{t} \sum_{1 \leq j \leq \lambda_i} (aq^{2j-2n-1})^t w_0.
\]
Substituting it in (7.2.4) gives
\[ Q_n(u) \cdot w_0 = \exp \left( -\sum_{i \geq 1} \frac{1}{i} \sum_{1 \leq j \leq \lambda_i} (aq^{2(j-n)}u)^i \right) w_0 \]
\[ = \prod_{1 \leq j \leq \lambda_n} (1 - aq^{2(j-n)}u)w_0. \]
Hence, we obtain \( Q_n(u) = \prod_{1 \leq j \leq \lambda_n} (1 - aq^{2(j-n)}u) \).

Now, (7.2.1) together with Theorem 6.1 implies immediately the following.

**Corollary 7.3.** Suppose \( a \in \mathbb{C}^* \) and \( \lambda \in \Lambda^+(n, r) \) has \( m \) parts. If \( L(\lambda) \cong L(Q) \) for some \( Q \in \mathcal{B}(n)_r \), then, for all \( 1 \leq i \leq m \), \( Q_i(u) \) is the polynomial with degree \( \lambda_i \), constant 1, and roots forming the segment \([a^{-1}q^{-\lambda_i+2i-1}; \lambda_i] \).

8. **Application to affine Hecke algebras**

By the evaluation map \( \text{ev}_a : \mathcal{H}_\delta(r)_C \to \mathcal{H}(r)_C \) defined in (5.0.2), every \( \mathcal{H}(r)_C \)-module \( N \) defines a \( \mathcal{H}_\delta(r)_C \)-module \( N_\lambda \). We now identify the simple \( \mathcal{H}_\delta(r)_C \)-modules \( (E_\mu)_\lambda \) for every partition \( \mu \) and \( a \in \mathbb{C}^* \) in terms of multisegments in \( \mathcal{S}_r \). Recall from [12, Th. 1.4] that the left cell modules \( E_\lambda \) \((\lambda \in \Lambda^+(r)) \) defined in (4.2.1) form a complete set of simple \( \mathcal{H}(r)_C \)-modules.

By Theorem 7.2 we may define a map
\[ \tilde{\partial} : \Lambda^+(n, r) \times \mathbb{C}^* \longrightarrow \mathcal{Q}(n)_r, \quad (\lambda, a) \mapsto \mathcal{Q}(\lambda, a) = (Q_1(u), \ldots, Q_n(u)), \]
where \( Q_n(u) = \prod_{1 \leq k \leq \lambda_n} (1 - aq^{2(k-\lambda_n)}u) \) and \( P_i(u) = \frac{Q_i(uq^{i-1})}{Q_{i+1}(uq^{i-1})} = \prod_{\lambda_i+1+1 \leq k \leq \lambda_i} (1 - aq^{2k-1-i}u) \)
for \( 1 \leq i \leq n-1 \). If \( n > r \) and \( \lambda = (\lambda_1, \ldots, \lambda_m) \) has \( m(\leq r) \) parts, then \( \mathcal{Q}(\lambda, a) \) has the form \((Q_1(u), \ldots, Q_m(u), 1, \ldots, 1)\) as in (7.2.1). We now compute \( \partial^{-1}(\mathcal{Q}(\lambda, a)) \).

**Lemma 8.1.** Let \( n > r \). Suppose the map \( \partial^{-1} \circ \tilde{\partial} : \Lambda^+(r) \times \mathbb{C}^* \longrightarrow \mathcal{S}_r \) is given by \( (\lambda, a) \mapsto s(\lambda, a) \) and \( \lambda \) has \( m \) parts. Then
\[ s(\lambda, a) = \sum_{i=1}^{m} \sum_{\lambda_i+1 \leq k \leq \lambda_i} [aq^{2k-1-i}; i], \]
and the partition associated with \( s(\lambda, a) \) is \( \lambda' \).

**Proof.** Since the (inverses of the) roots of \( P_i(u) \) are
\[ \{aq^{2k-1-i} | \lambda_i+1 \leq k \leq \lambda_i \}, \]
by the definition of \( \partial \), \( s(\lambda, a) \) consists of \( \lambda_i - \lambda_{i+1} \) segments with length \( i \) and centers
\[ aq^{2k-1-i}, \lambda_i+1 \leq k \leq \lambda_i, \]
for all \( 1 \leq i \leq m \).

Recall from (4.2.1) that, for each \( \mu \in \Lambda^+(r) \), \( E_\mu \) is the left cell module for \( \mathcal{H}(r)_C \) defined by Kazhdan–Lusztig’s C-basis [12] associated with the left cell containing \( w_{0,\mu} \), where \( w_{0,\mu} \) is the longest element in \( \mathcal{S}_\mu \). If \( S_\mu \) denotes the Specht module contained in \( \mathcal{H}(r)_C x_\mu \) (so that \( S_\mu = \mathcal{H}(r)_C y_\mu T_{w_\mu} x_\mu \)), then \( S_{\lambda'} \cong E_\lambda \).
Proposition 8.2. For $a \in \mathbb{C}^*$ and $\lambda \in \Lambda^+(r)$. Let $s = s(\lambda, a)$ as above. Then $(S_\lambda)_a \cong V_s$.

Proof. Assume $n > r$. Then there is an idempotent $e \in S(n, r)_\mathbb{C}$ such that $eS(n, r)_\mathbb{C}e \cong \mathcal{H}(r)_\mathbb{C}$. This algebra isomorphism gives rise to the so-called Schur functor from the category of finite dimensional $S(n, r)_\mathbb{C}$-modules to the category of finite dimensional $\mathcal{H}(r)_\mathbb{C}$-modules by sending $N$ to $eN$. By Theorems [4,7] and [7,2] we have

$$L(\lambda)_a \cong L(Q) \cong \Omega^\otimes_{n,\mathbb{C}} \otimes_{\mathcal{H}(r)_{\mathbb{C}}} V_s,$$

where $Q = Q(\lambda, a)$. Restriction gives an $S(n, r)_\mathbb{C}$-module isomorphism $L(\lambda) \cong \Omega^\otimes_{n,\mathbb{C}} \otimes_{\mathcal{H}(r)_{\mathbb{C}}} V_s$. By applying Schur’s functor, we see that $\mathcal{H}(r)_\mathbb{C}$-module $V_s$ is irreducible. Hence, $V_s = E_\lambda$. Therefore, $(S_\lambda)_a \cong V_s$. □

Corollary 8.3. (1) For any partition $\lambda \in \Lambda^+(r)$ and $n \geq r$, if $L(\lambda)$ is the simple $S(n, r)_\mathbb{C}$-module with highest weight $\lambda$, then $eL(\lambda) \cong S_\lambda$.

(2) If $Q \in Q(n)_r$ and $\lambda = (\deg Q_1, \ldots, \deg Q_n)$, and $\mu$ is a weight of $L(Q)$, then $\mu \preceq \lambda$ under the dominance order $\preceq$.

Proof. Statement (1) follows the proof above. It remains to prove the second statement. By applying another type of Schur functor, we may assume $n > r$. Thus, $L(Q) \cong \Omega^\otimes_{n,\mathbb{C}} \otimes_{\mathcal{H}(r)_{\mathbb{C}}} V_s$, for some $s \in S'_r$ such that $Q = Q_s$, and $\lambda'$ is the partition associated with $s$. Since, as a $\mathcal{H}(r)_\mathbb{C}$-module, $V_s \cong E_{\lambda'} \oplus (\oplus_{\nu > \lambda} n_{\nu, \lambda} E_{\nu})$ by [4,2,1]. Thus, as a $U(\mathfrak{gl}_n)$-module, the first assertion implies

$$L(Q) \cong L(\lambda) \oplus (\oplus_{\nu > \lambda} n_{\nu, \lambda} \Omega^\otimes_{n,\mathbb{C}} \otimes S_{\nu'}) \cong L(\lambda) \oplus (\oplus_{\nu > \lambda} n_{\nu, \lambda} L(\nu')),$$

where the second sum is over $\nu$ with $\Omega^\otimes_{n,\mathbb{C}} \otimes S_{\nu'} \neq 0$. Now, since $\nu > \lambda'$ implies $\nu' < \lambda$ and the weight spaces of $L(Q)$ as $U(\mathfrak{gl}_n)$-module or as $U(\mathfrak{gl}_n)$ are the same, our assertion follows. □

Part (2) of the result above is [6, Lem. 4.5.1]. The proof here is different.

References

[1] J. Beck, Braid group action and quantum affine algebras, Comm. Math. Phys. 165 (1994), 655–668.
[2] V. Chari and A. Pressley, Quantum affine algebras, Comm. Math. Phys. 142 (1991), 261–283.
[3] V. Chari and A. Pressley, Small representations of quantum affine algebras, Lett. Math. Phys. 30 (1994), 131–145.
[4] V. Chari and A. Pressley, A Guide to Quantum Groups, Cambridge University Press, Cambridge, 1994.
[5] V. Chari and A. Pressley, Quantum affine algebras and affine Hecke algebras, Pacific J. Math. 174 (1996), 295–326.
[6] B. Deng, J. Du and Q. Fu, A double Hall algebra approach to affine quantum Schur–Weyl theory, LMS Lecture Notes Series (to appear).
[7] R. Dipper and G. James, Blocks and idempotents of Hecke algebras of general linear groups, Proc. London Math. Soc 53 (1987), 57–82.
[8] V. G. Drinfeld, A new realization of Yangians and quantized affine algebras, Soviet Math. Dokl. 32 (1988), 212–216.
[9] E. Frenkel and E. Mukhin, The Hopf algebra Rep $U_q(\hat{\mathfrak{gl}}_\infty)$, Sel. math., New Ser. 8 (2002), 537–635.
[10] G. James and A. Mathas, *The Jantzen sum formula for cyclotomic q-Schur algebras* Trans. Amer. Math. Soc. **352** (2000), 5381–5404.

[11] M. Jimbo, *A q-analogue of U(gl(N + 1)), Hecke algebra, and the Yang-Baxter equation*, Letters in Math. Physics **11** (1986), 247–252.

[12] D. Kazhdan and G. Lusztig, *Representations of Coxeter groups and Hecke Algebras*, Invent. Math. **53** (1979), 165–184.

[13] Z. Lin, and H. Rui, *Cyclotomic q-Schur algebras and Schur-Weyl duality*, Representations of algebraic groups, quantum groups, and Lie algebras, 133–155, Contemp. Math., **413**, Amer. Math. Soc., Providence, RI, 2006.

[14] G. Lusztig, *Introduction to quantum groups*, Progress in Math. **110**, Birkhäuser, 1993.

[15] G. E. Murphy, *On the representation theory of the symmetric groups and associated Hecke algebras*, J. Algebra **152** (1992) 492–513.

[16] J. D. Rogawski, *On modules over the Hecke algebra of a p-adic group*, Invent. Math. **79** (1985), 443–465.

[17] M. Varagnolo and E. Vasserot, *On the decomposition matrices of the quantized Schur algebra*, Duke Math. J. **100** (1999), 267–297.

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