Thermodynamics and weak cosmic censorship conjecture of an AdS black hole with a monopole in the extended phase space

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Abstract: The first law of black hole thermodynamics in the extended phase space is prevailing recently. However, the second law as well as the weak cosmic censorship conjecture has not been investigated extensively. In this paper, we investigate the laws of thermodynamics and the weak cosmic censorship conjecture of an AdS black hole with a global monopole in the extended phase space under a charged particle absorption. It is shown that the first law of thermodynamics is valid, while the second law is violated for the extremal and near-extremal black holes. Moreover, for the weak cosmic censorship conjecture, we find it is valid only for the extremal black holes, and it is violable for the near-extremal black hole, which is different from the previous results.

Keywords: weak cosmic censorship conjecture, thermodynamics, global monopole, black holes

PACS: 04.20.Dw 04.70.Dy 04.20.Bw

1 Introduction

In 1974, Stephen Hawking proved that the black hole had the quantum radiation with the temperature \( T = \frac{\kappa}{2\pi} \) [1]. This discovery promoted the investigation on thermodynamics of black holes, and since then, it is believed that a black hole could be viewed as a thermodynamic system [2, 3]. Until now, there have been many work on thermodynamics of black holes, such as the four laws of thermodynamics [4, 5], quantum effect [6, 7], phase transition [8, 9], and so on. In these studies, the AdS space is more popular. In the AdS space-time, the cosmological constant can be considered as a thermodynamic variable [10]. The thermodynamic phase space thus is extended, and some new thermodynamic phenomena emerge. So far there have been many researches concentrating on the extend phase space of black holes. Especially, as the cosmological constant and its conjugate quantity were viewed as the pressure and volume of the black holes, one can construct the extended first law of thermodynamics [11]. In addition, in the extended phase space, one also can discuss the \( P - V \) critical behavior [12, 13]. In this framework, some interesting phenomena, such as Van der Waals phase transition [14, 15], engine cycle [16, 17], can be investigated.

Until now, a lot of researches have been conducted on the first law of thermodynamics and phase transition of black holes in the extended phase space. However, the second law of thermodynamics and the weak cosmic censorship conjecture (WCCC) have rarely been reported yet. On the premise that the first law of thermodynamics is valid, it does not mean that the second law of thermodynamics and the WCCC still holds. Therefore, it is extremely important and necessary to check the validity of the second law of thermodynamics and the WCCC in the extended phase space.

Recently, Ref.[20] investigated the thermodynamics and WCCC in the extended phase space. Their work was based on the Gedanken experiment in Ref.[21]. After a test particle is absorbed by a black hole, they investigated the change of location of the event horizon. As a result, they found that the first law of thermodynamics and the WCCC in the extended phase space were always valid for the charged Reissner-Nordström AdS black hole, and the second law of thermodynamics was not valid for the extremal and near-extremal black holes. Because there is no general method to prove the WCCC in gravity systems, we should check it in different space-time backgrounds one by one. Based on the idea in Ref.[20], thermodynamics and WCCC in the extended...
phase space of a series black holes have been investigated
[22][34].

In this paper, we will investigate the thermodynamics and WCCC of an AdS black hole with a global monopole. On one hand, we intend to discuss the influence of global monopole on the thermodynamics and WCCC of the black hole. On the other hand, we want to explore whether the high order corrections to the mass of the absorbed particle will affect the WCCC. In Ref.[20], the WCCC was found to be valid in the extended phase space for the extremal and near-extremal Reissner Nordström-AdS black holes. However, we found that they considered only the first order correction to the mass. As the high order corrections to the mass is considered, whether the conclusion will be changed is worth exploring. The high order corrections to the mass is considered, whether the high order corrections are important for us to investigate the WCCC. Recently Refs. [35–38] investigated the WCCC were neglected. As a result, we find the global monopole absorption particle will affect the WCCC. In Ref.[20], the WCCC was found to be valid in the extended phase space.

The remainder of this paper is outlined as follows. In section 2, we briefly review the motion of a charged particle in an AdS black holes with a global monopole. In section 3, we investigate the first and second law of thermodynamics in the extended phase space. In section 4, we investigate the WCCC in the extended phase space. Especially, the second order correction to the mass of the particle has been considered. Section 5 is devoted to our conclusions. In this paper, we will adopt $G = c = \hbar = 1$.

2 The motion of a charged particle in an AdS black holes with a global monopole

The spherically symmetric AdS black hole solution with a global monopole can be expressed as [39]

$$ds^2 = -F(\tilde{r})dt^2 + \frac{1}{F(\tilde{r})}d\tilde{r}^2 + \tilde{r}^2 (d\theta^2 + \sin^2\theta d\phi^2),$$  \hspace{1cm} (1)

where

$$F(\tilde{r}) = 1 - 8\pi \eta_0^2 - \frac{2\tilde{m}}{\tilde{r}} + \frac{\tilde{q}^2}{\tilde{r}^2} + \frac{\tilde{r}^2}{l^2},$$  \hspace{1cm} (2)

in which, $\eta_0$ is a parameter related to symmetry breaking, $\tilde{m}$ and $\tilde{q}$ correspond to mass and charge, $l$ is the radius of AdS space, which relates to the cosmological parameters with $\Lambda = -\frac{3}{l^2}$. In this gravitational system, the non-zero electromagnetic four-vector component is

$$\tilde{A}_i = -\frac{\tilde{q}}{\tilde{r}}.$$  \hspace{1cm} (3)

In order to study various properties of black holes more easily, we will introduce the following coordinate transformation

$$\tilde{t} = (1 - 8\pi \eta_0^2)^{-1/2}t, \tilde{r} = (1 - 8\pi \eta_0^2)^{1/2}r,$$  \hspace{1cm} (4)

and redefine the following physical quantities

$$m = (1 - 8\pi \eta_0^2)^{-3/2}\tilde{m}, q = (1 - 8\pi \eta_0^2)^{-1}\tilde{q}, \eta = 8\pi \eta_0^2.$$  \hspace{1cm} (5)

In this way, the metric in Eq.(1) can be written as

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + (1 - \eta^2) r^2 (d\theta^2 + \sin^2\theta d\phi^2),$$  \hspace{1cm} (6)

with

$$f(r) = 1 - \frac{2m}{r} + \frac{q^2}{r^2} + \frac{r^2}{l^2}.$$  \hspace{1cm} (7)

At the same time, the potential of the black hole is

$$A_\nu = -\frac{q}{r}.$$  \hspace{1cm} (8)

In this case, when $\eta = 0$, the black hole returns to a four-dimensional Reissner-Nordström AdS black hole. The existence of the global monopole will also affect the ADM mass and charge of the black hole, namely

$$M = (1 - \eta^2) m, Q = (1 - \eta^2) q.$$  \hspace{1cm} (9)

In this section, we will consider the energy momentum relation of the charged particle as it is absorbed by the black hole. In the potential field $A_\mu$, the Hamilton-Jacobi equation can be expressed as

$$g^{\mu\nu}(p_\mu - eA_\mu)(p_\nu - eA_\nu) + u^2 = 0,$$  \hspace{1cm} (10)

where $u$ is the mass, $e$ is the charge, and $p_\mu$ is the momentum of the particle which can be expressed as

$$p_\mu = \partial_\mu I,$$  \hspace{1cm} (11)

in which $I$ is the Hamiltonian action of the particle. Considering the symmetry of the gravitational system, this action can be separated into

$$I = -(Et + I_0(r) + I_0(\theta) + L\phi),$$  \hspace{1cm} (12)

where, $E$ and $L$ are the energy and angular momentum of the particle. Form Eq.(6), we can get the inverse metric of the black hole

$$g^{\mu\nu}\partial_\mu\partial_\nu = -f(r)^{-1}(\partial_t)^2 + f(r)(\partial_r)^2$$

$$+ [(1 - \eta^2) r^2]^{-1}(\partial_\theta^2 + \sin^2\theta \partial_\phi^2).$$  \hspace{1cm} (13)
In this case, the Hamilton-Jacobi equation can be rewritten as

$$u^2 = \left(\frac{E + eA_t}{f(r)}\right)^2 + f(r)(\partial_r I_r(r))^2 + \left(\frac{(\partial_r I_\theta(r))^2 + \sin^{-2}(\theta)L^2}{(1 - \eta^2)r^2}\right) = 0. \quad (14)$$

After variable separation, Eq.(14) can be separated into the following angular and radial equations

$$K = (\partial_r I_\theta(r))^2 + \frac{1}{\sin^2 \theta} L^2, \quad (15)$$

and

$$K = -(1 - \eta^2)u^2r^2 + \frac{1}{f(r)} (-E - eA_t)^2 - (1 - \eta^2)r^2 f(r)(\partial_r I_r(r))^2. \quad (16)$$

In this way, we can rewrite Eq.(12) as

$$I_r = \frac{1}{2}m^2 \lambda - Et + \int dr \sqrt{R} + \int d\theta \sqrt{\Theta} + L\phi, \quad (17)$$

where

$$I_r = \int dr \sqrt{R}, \quad I_\theta = \int d\theta \sqrt{\Theta}, \quad \Theta = K - \frac{1}{\sin^2 \theta} L^2, \quad \Theta = \frac{K}{f(r)} + \frac{(1 - \eta^2)}{(1 - \eta^2)r^2} \frac{f(r)^2}{f(r)}, \quad (18)$$

Based on Eq.(17), the radial and angular momentum can be expressed as

$$p_r = f(r) \sqrt{\frac{1}{f^2(r)} (E + eA_t)^2 - \frac{K}{f(r)} + \frac{(1 - \eta^2)(u^2r^2)}{f(r)}), \quad (19)$$

$$p_\theta = \frac{1}{(1 - \eta^2)r^2} \sqrt{K - \frac{1}{\sin^2 \theta} L^2}. \quad (20)$$

We are most interested in the radial momentum of the particle. At the event horizon, we can get

$$E = \frac{q}{r_h} e + |p^r_h|. \quad (21)$$

In the front of $|p^r_h|$, we choose the positive sign. That is, we want to ensure that particle drops in the positive direction of time. In this case, the energy $E$ and momentum $p^r_h$ of the particle are positive too.

### 3 Thermodynamic laws in the extended phase space

In order to study the thermodynamics of black holes, we need to obtain some physical quantities at the event horizon of the black hole. The event horizon of the black hole is defined at $r = r_h$, and its concrete form can be obtained by $f(r_h) = 0$. At the event horizon, the potential energy of the black hole is

$$\Phi = \frac{q}{r_h}. \quad (22)$$

According to the definition of surface gravity, the temperature of the black hole can be expressed as

$$T = \frac{k}{2\pi} = \frac{1}{4\pi r_h} \left[1 + \frac{3r_h^2}{r_h^2} - \frac{Q^2}{(1 - \eta^2)r_h^2}\right]. \quad (23)$$

And the entropy of the black hole can be obtained as

$$S = \frac{A_{bh}}{4} = \pi (1 - \eta^2) r_h^2. \quad (24)$$

Recently, some researches have shown that the cosmological parameter could be regarded as the pressure in the thermodynamic system of the black hole, and the corresponding conjugate quantity was viewed as the volume of the system. In this case, the first law of thermodynamics still holds [13], that is,

$$dM = TdS + \Phi dQ + VdP, \quad (25)$$

where

$$P = \frac{A}{8\pi} = \frac{3}{8\pi r_h^2}, \quad (26)$$

$$V = \frac{4}{3} \pi (1 - \eta^2) r_h^3. \quad (27)$$

Accordingly, the Smarr relation in the extended phase space can be obtained as

$$M = 2(TS - VP) + \Phi Q. \quad (28)$$

In the above equation, $M$ is not the internal energy but enthalpy. The relation between the enthalpy and internal energy is

$$M = U + PV. \quad (29)$$

When the black hole absorbs charged particles, it is assumed that energy and charge are conserved. According to the first law of thermodynamics, the energy and charge of the black hole will increase accordingly, which implies

$$E = dU = d(M - PV), \quad e = dQ. \quad (30)$$
Combining the Eq.(30) and Eq.(21), we can get
\[
dU = \frac{q}{r_h} dQ + |p'_h|.
\] (31)

After the charged particles are absorbed by black holes, the enthalpy, charge, pressure and volume of the black holes will change accordingly, which are labelled as \((dM, dQ, dl, dV)\). The other variables can be represented by these variables. Our goal is to obtain the first law of thermodynamics based on Eq.(31). Therefore, we will discuss the change in enthalpy of the black hole under the absorption of a charged particle. Form Eq.(24), we have
\[
dS = 2\pi (1 - \eta^2) r_h dr_h.
\] (32)

The change of the event horizon is determined by the parameters of the absorbed particle \((e, p'_h)\). Due to the change of the horizon, the function \(f(r)\) determining the location of the horizon will also change accordingly, namely
\[
df_h = \frac{\partial f_h}{\partial M} dM + \frac{\partial f_h}{\partial Q} dQ + \frac{\partial f_h}{\partial l} dl + \frac{\partial f_h}{\partial r_h} dr_h = 0,
\] (33)
where we have used the relation \(f(r_h) = f(r_h + dr_h) = 0\), and
\[
\begin{align*}
\frac{\partial f_h}{\partial M} &= \frac{2}{r_h (\eta^2 - 1)}, \\
\frac{\partial f_h}{\partial Q} &= \frac{2Q}{(\eta^2 - 1)^2 r_h^2}, \\
\frac{\partial f_h}{\partial l} &= -\frac{2r_h^2}{l^2}, \\
\frac{\partial f_h}{\partial r_h} &= \frac{2r_h - 2((\eta^2 - 1) M r_h + Q^2)}{(\eta^2 - 1)^2 r_h^3}.
\end{align*}
\] (34)

In addition, due to \(M\) is enthalpy, the Eq.(31) can be rewritten as
\[
dM - d(PV) = \frac{q}{r_h} dQ + p'_h.
\] (35)

Combining Eq.(33) and Eq.(35), we can remove the \(dl\) term. Interestingly, we find \(dQ\) and \(dM\) are also eliminated. Therefore, we finally get
\[
dr_h = \frac{2l^2 p'_h r_h}{2l^2 ((1 - \eta^2) r_h - M) + (1 - \eta^2) r_h^3},
\] (36)

Form Eqs.(24), (27) and (36), the variation of entropy and volume of the black hole can be written as the function of particle momentum, which are
\[
\begin{align*}
dS &= \frac{4(1 - \eta^2) \pi l^2 p'_h r_h^2}{2l^2 ((1 - \eta^2) r_h - M) + (1 - \eta^2) r_h^3},
\end{align*}
\] (37)
\[
\begin{align*}
dV &= \frac{8(1 - \eta^2) \pi l^2 p'_h r_h^3}{2l^2 ((1 - \eta^2) r_h - M) + (1 - \eta^2) r_h^3},
\end{align*}
\] (38)

Combine Eqs.(38), (37), (23) and (24), we find
\[
TdS - PdV = p'_h.
\] (39)

In this case, the energy momentum relation in Eq.(31) becomes as
\[
dU = \Phi dQ + TdS - PdV.
\] (40)

In the extended phase space, the mass of the black hole has already been defined as enthalpy. We can express the internal energy of above equation as enthalpy through Eq.(29), namely
\[
dM = dU + PdV + V dP.
\] (41)

Substituting Eq.(41) into Eq.(40), we can obtain
\[
dM = TdS + \Phi dQ + V dP,
\] (42)

which is the same as Eq.(25). Obviously, the first law of thermodynamics holds after the charged particle is absorbed by the AdS black hole with a global monopole.

Now, we discuss the second law of thermodynamics of the black hole. It is well known that the black hole entropy never decreases in the clockwise direction. In other words, the black hole absorbs the charged particle should be a process of entropy incasement. Thereafter, we will check whether this is true in the extended phase space by Eq.(37).

We first discuss the case of the extremal black hole. A typical feature of the extremal black hole is that its temperature vanishes. Based on this fact and Eq.(23), we can get the mass of the extremal black hole. Substituting this mass into Eq.(37), we obtain
\[
dS_{\text{extreme}} = -\frac{4\pi l^2 p'_h}{3r_h} < 0.
\] (43)

Obviously, the increase in the entropy of the black hole is negative, implying the second law of thermodynamics is invalid in this case.

Next, we will discuss the near-extremal black hole. After the values of mass \(M\) and the monopole parameter \(\eta\) are given, we find the value of charge \(Q\) that satisfies extremal conditions by numerical approximation method. The condition that the black hole can be established is \(Q \leq Q_e\). When the charge and mass are given, we can find the event horizon \(r_h\) of the black hole by the equation \(f(r_h) = 0\). In this way, based on Eq.(37), we can get the variation of entropy \(dS\). In this paper, we fix the parameters \(M = 0.5, l = 1\). For the case \(\eta = 0.1\), we find that the extremal charge is \(Q_e = 0.462988\). For black holes with different charges, the location of event horizon and the variation of entropy are listed in Table 1.

}\]
Table 1. The relation between $dS$, $Q$ and $r_h$ while $\eta = 0.1$. 

| $Q$   | $r_h$   | $dS$  |
|-------|---------|-------|
| 0.462988 | 0.389008 | -10.9783 |
| 0.46   | 0.425981 | -28.2074 |
| 0.455  | 0.449251 | -121.19  |
| 0.44   | 0.489249 | 35.1871  |
| 0.42   | 0.522924 | 19.2248  |
| 0.4    | 0.547731 | 15.0991  |
| 0.3    | 0.62191  | 10.2483  |
| 0.2    | 0.660824 | 9.15501  |
| 0.1    | 0.680281 | 8.74138  |

It is obvious that as the black hole charge decreases, the horizon radius gradually increases. And the amount of change in charge and entropy is not a simple monotonic relationship. When the charge approaches to the extremal charge, the value of change in entropy is negative. When the charge is far from the extremal charge, the change of entropy is positive. In the positive region and the negative region, the entropy decreases as the charge decreases. The near-extreme black hole violates the second law of thermodynamics, and the far-extreme black hole follows the second law of thermodynamics. In Figure 1, we directly give the relationship between the change of the entropy and the event horizon.

![Fig. 1. The relation between $dS$ and $r_h$ with $M = 0.5$, $l = p_h^*$ and $\eta = 0.1$.](image)

Table 2. The relation between $dS$, $Q$ and $r_h$ while $\eta = 0.1$. 

| $Q$   | $r_h$   | $dS$  |
|-------|---------|-------|
| 0.448134 | 0.466004 | -9.09252 |
| 0.44   | 0.537818 | -35.5889 |
| 0.43   | 0.571502 | -111.876 |
| 0.42   | 0.595534 | 64.6394 |
| 0.41   | 0.61485  | 36.8817 |
| 0.4    | 0.631209 | 27.7267 |
| 0.3    | 0.72788  | 13.1042 |
| 0.2    | 0.775407 | 11.0511 |
| 0.1    | 0.799693 | 10.3366 |

Form Table 2, we also find that the change in entropy is negative for the case of the near-extremal black hole, and the second law of thermodynamics is violated. For far-extremal black holes, the amount of shift in entropy is positive, and the second law of thermodynamics holds. The relation between the change in entropy and the event horizon is shown in Figure 2. Similarly, we can find a phase transition point at $r_h = 0.5727$, and the result shown in Figure 2 is consistent with the result in Table 2.

![Fig. 2. The relation between $dS$ and $r_h$ with $M = 0.5$, $l = p_h^*$ and $\eta = 0.5$.](image)

4 The weak cosmic censorship conjecture in the extended phase space

In the extended phase space with consideration of the thermodynamic volume, we find that the second law of thermodynamics is not valid for the case of the extremal black hole and the near-extremal black hole. Therefore, we want to further check whether the WCCC is valid in the extended phase space.

After a particle is absorbed by the black hole, the change of the black hole is reflected in $M, Q, l$. However, these physical quantities are ultimately related to the metric function $f(M, Q, l, r)$. Therefore, we mainly discuss the changes in $f(M, Q, l, r)$. For the function $f(M, Q, l, r)$, there is always a minimum value $r_{\min}$ which satisfies

$$f(M, Q, l, r)|_{r=r_{\min}} = f_{\min} = \delta \leq 0,$$  \hspace{1cm} (44)

$$\partial_r f(M, Q, l, r)|_{r=r_{\min}} \equiv f'_{\min} = 0,$$  \hspace{1cm} (45)
\[(\partial_r)^2 f(M, Q, l, r)|_{r=r_{\text{min}}} > 0. \quad (46)\]

For the case of the extremal black hole, \(\delta = 0\), and for the near-extremal black hole, \(\delta\) is a very small negative value. The inner and outer horizons of the black hole are distributed on both sides of \(r_{\text{min}}\). After the absorption of particles, \((M, Q, l)\) change into \((M + dM, Q + dQ, l + dl)\). Due to these changes, the position of the minimum value of the function \(f(M, Q, l, r)\) and the position of the event horizon will move to \(r_{\text{min}} \rightarrow r_{\text{min}} + dr_{\text{min}}\), \(r_h \rightarrow r_h + dr_h\). Then, the metric function \(f(M, Q, l, r)\) will also have a change, which is labelled as \(df_{\text{min}}\). At \(r_{\text{min}} + dr_{\text{min}}\), we have

\[
\partial_r f|_{r=r_{\text{min}}+dr_{\text{min}}} = f'_{\text{min}} + df'_{\text{min}} = 0. \quad (47)
\]

That is

\[
df'_{\text{min}} = \frac{\partial f'_{\text{min}}}{\partial M} dM + \frac{\partial f'_{\text{min}}}{\partial Q} dQ + \frac{\partial f'_{\text{min}}}{\partial l} dl + \frac{\partial f'_{\text{min}}}{\partial r_{\text{min}}} dr_{\text{min}} = 0, \quad (48)
\]

where

\[
\frac{\partial f'_{\text{min}}}{\partial M} = \frac{2}{r_{\text{min}}^2 (1 - \eta^2)}, \quad \frac{\partial f'_{\text{min}}}{\partial Q} = -\frac{4M}{(\eta^2 - 1)^2 r_{\text{min}}^4}, \quad \frac{\partial f'_{\text{min}}}{\partial l} = -\frac{4r_{\text{min}}}{l^3}, \quad \frac{\partial f'_{\text{min}}}{\partial r_{\text{min}}} = \frac{2}{l^2} - \frac{4M}{(1 - \eta^2) r_{\text{min}}^4} + \frac{6Q^2}{(\eta^2 - 1)^2 r_{\text{min}}^4}. \quad (49)
\]

At \(r_{\text{min}} + dr_{\text{min}}\), the function \(f(M, Q, l, r)\) becomes as

\[
f(M + dM, Q + dQ, l + dl, r)|_{r=r_{\text{min}}+dr_{\text{min}}} = f_{\text{min}} + df_{\text{min}} = \delta + \left(\frac{\partial f_{\text{min}}}{\partial M} dM + \frac{\partial f_{\text{min}}}{\partial Q} dQ + \frac{\partial f_{\text{min}}}{\partial l} dl\right). \quad (50)
\]

In the above equation, the condition \(f'_{\text{min}} = 0\) has been used. The most important step is how to give the specific form of the Eq.\((50)\). For the extremal black hole, we have \(r_h = r_{\text{min}}\), which means that the Eq.\((35)\) can be used. According to the condition \(f'_{\text{min}} = 0\), we can get an equation about \(M\). So that we can obtain lastly

\[
dM = \frac{l^2Q^2 + 3(\eta^2 - 1)^2 r_{\text{min}}^4 dr_{\text{min}}}{(\eta^2 - 1)^2 l^2 r_{\text{min}}^2} - \frac{2(l \eta^2 - 1)^{\frac{5}{2}} r_{\text{min}}^4 + l^2 Q r_{\text{min}}^2 dQ}{(\eta^2 - 1)^2 l^2 r_{\text{min}}^2}. \quad (51)
\]

Form Eqs.\((35)\) and \((54)\), we find

\[
df_{\text{min}} = 0. \quad (52)
\]

Therefore, we finally get

\[
f(M + dM, Q + dQ, l + dl, r)|_{r=r_{\text{min}}+dr_{\text{min}}} = 0. \quad (53)
\]

The result shows that the minimum value of function \(f(r)\) has not been changed in the extended phase space. That is, as a charged particle is absorbed by the extremal black hole, the configuration of the black hole does not change. In other words, the extremal black hole is still an extremal black hole.

We also can discuss the case of the near-extremal black hole. For the near-extremal black hole, Eq.\((35)\) cannot be used since it is valid only at the horizon. But we can expand it at \(r_{\text{min}}\) with the relation \(r_h = r_{\text{min}} + \epsilon\), that is

\[
dM = \frac{(r_{\text{min}}^5 (\eta^2 - 1)^2 dl - l^3 Q r_{\text{min}}^2 dQ)}{r_{\text{min}}^2 (\eta^2 - 1)} + \frac{l^2 (Q^2 - r_{\text{min}}^2 (\eta^2 - 1)^2 - 3r_{\text{min}}^4 (\eta^2 - 1)^2)}{r_{\text{min}}^2 (\eta^2 - 1)} + \frac{l^3 Q r_{\text{min}}^2 dQ + 3 l^3 r_{\text{min}}^5 dl + 6 l^3 r_{\text{min}}^4 \eta^2 dr_{\text{min}} + 3 l^3 r_{\text{min}}^5 \eta^4 dl}{r_{\text{min}}^2 (\eta^2 - 1)^2} + \frac{l^3 Q^2 dr_{\text{min}} + 3 l^3 r_{\text{min}}^6 + 6 l^3 r_{\text{min}}^4 \eta^2 dl + 3 l^3 r_{\text{min}}^4 \eta^4 dr_{\text{min}}}{r_{\text{min}}^2 (\eta^2 - 1)^2} + \mathcal{O}(\epsilon^2). \quad (54)
\]

Substituting \((54)\) into \((50)\), we have

\[
df_{\text{min}} = \left(\frac{l^2 (Q^2 - r_{\text{min}}^2 (\eta^2 - 1)^2 - 3r_{\text{min}}^4 (\eta^2 - 1)^2)}{r_{\text{min}}^2 (\eta^2 - 1)^2} + \frac{2(dQ) l^3 Q r_{\text{min}}^2 + 3dl r_{\text{min}}^5 (\eta^2 - 1)^2)}{r_{\text{min}}^2 (\eta^2 - 1)^2} - \frac{2dl r_{\text{min}} (l^2 Q^2 + 3 l^3 r_{\text{min}}^4 (\eta^2 - 1)^2)}{r_{\text{min}}^2 (\eta^2 - 1)^2} + \mathcal{O}(\epsilon^2). \quad (55)
\]

At \(r_h = r_{\text{min}} + \epsilon\), we also can get

\[
Q = r_{\text{min}} \sqrt{l^2 + 3 r_{\text{min}}^2 (\eta^2 - 1)}, \quad (56)
\]

and

\[
dQ = \frac{((l^3 + 6 l r_{\text{min}}^2) dr_{\text{min}} - 3 r_{\text{min}}^3 dl (\eta^2 - 1))}{(l^2 + 3 r_{\text{min}}^2)} \quad (57)
\]

Substituting Eqs.\((56)\) and \((57)\) into the Eq.\((55)\), we find

\[
df_{\text{min}} = \mathcal{O}(\epsilon^2), \quad (58)
\]

then, we can get

\[
f_{\text{min}} + df_{\text{min}} = \delta + \mathcal{O}(\epsilon^2). \quad (59)
\]

In Ref.\([20]\), it was claimed that \(df_{\text{min}}\) can be neglected for \(\mathcal{O}(\epsilon^2)\) is a small quantity. However, \(\delta\) is also a small negative quantity, we can not directly determine which is smaller. Therefore, for the near-extremal black hole
in the extended phase space, we should find the relationship between δ and O(ε²). Expanding f(rₜₜ) to the second order, we have

\[ f(r_{min} + \epsilon) = \delta + \frac{(l^2 Q^2 + 3r_{min}^4 (\eta^2 - 1)^2)}{l^2 r_{min}^4 (\eta^2 - 1)^2} \epsilon^2 + O(\epsilon^3) = 0, \]

in which, we have used the relation \( r_{h} = r_{min} + \epsilon \), so we can obtain

\[ \delta = -\frac{(l^2 Q^2 + 3r_{min}^4 (\eta^2 - 1)^2)}{l^2 r_{min}^4 (\eta^2 - 1)^2} \epsilon^2 - O(\epsilon^3). \]

Similarly, expanding Eq. (35) to the second order, and inserting the expanded result into Eq. (60), we can obtain

\[ df_{min} = \left( \frac{6(1+dl)}{l^3} + \frac{2dr_{min}}{r_{min}^3} \right) \epsilon^2 + O(\epsilon^3). \]

For simplicity, we define.

\[ \Delta_E = \frac{\delta + df_{min}}{\epsilon^2}. \]

Combining the Eqs. (61), (62) and (63), we have

\[ \Delta_E = \frac{6(1+dl)}{l^3} - \frac{3}{l^2} + \frac{2 r_{min} dr_{min} - Q^2 (\eta^2 - 1)^{-2}}{r_{min}^4}. \]

Obviously, the value of \( \Delta_E \) is directly related to the values of \((Q, \eta, r_{min}, l, dl)\). In order to make it is easier to judge the positive or negative value of \( \Delta_E \), we have plotted Figure 3 and Figure 4 for different values of \((Q, \eta, r_{min}, l, dl)\). For the case \( \Delta_E \) is positive, there does not exist horizon since the equation \( f(r) = 0 \) does not have solutions. And for the \( \Delta_E \) is negative, there exist horizons always. In other words, the WCCC is violated for the case \( \Delta_E > 0 \).

Fig. 3. The value of \( \Delta_E \) for \( l = 1, Q = 2, \eta = 0.1, dl = 0.1 \).

From Figure 3 and Figure 4, we find for different values of \((Q, \eta, r_{min}, l, dl)\), \( \Delta_E \) may be positive or negative. That is, in the extended phase space, the WCCC is violable for the near-extremal black hole. This result is different with that in Ref. [20], where the WCCC is valid always. Our results is more precise and comprehensive since we consider the high order corrections to the mass of the absorbed particles.

5 Discussion and conclusions

The thermodynamics of black holes provides an effective means for studying the relationship between the gravity, thermodynamics and quantum theory. And in-depth study of the thermodynamics of black holes is helpful for us to further understand the nature of the gravity. In the extended phase space, the laws of thermodynamics and WCCC were researched under charged particles absorption. Based on the Hamiltonian-Jacobian equation, we first got the relation between particle momentum and energy. Then, from this relation, we obtained the first law of thermodynamics in the extended phase space, which was found to be valid. Through the methods of the numerical analysis, we got the value of the variation of the entropy in the case of the extremal black hole, near-extremal black holes, and far-extremal black hole. The results shown that for the extremal black hole and the near-extremal black hole, the change of the black hole entropy was negative, while for the far-extremal black hole, it was positive. In other words, the second law of thermodynamics of the black hole was only valid for the far-extremal black hole.

In the extended phase space, we also checked the WCCC. We mainly studied the change of the minimum value of the function \( f(r) \) which determines location of the event horizon. For the case of the extremal black hole, the result shown that \( f(r_{min}) \) did not change as a charged particle is absorbed. Therefore, the extremal black hole was always extremal. The WCCC hence was still valid for the extremal black hole. However, we found
that \( f(r_{\text{min}} + dr_{\text{min}}) > 0 \) could occur for the case of near-extremal black hole. Thus, the WCCC is invalid for the near-extremal black hole. This result was quite different from the result in Ref. \[20\]. Because we did not neglect the contribution of \( \delta \) and \( \mathcal{O}(\epsilon^2) \) by considering the second order correction to the mass of the particle. It seems that our conclusion is more precise and comprehensive.

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