**K-Theory of Cones of Smooth Varieties**

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Abstract. Let $R$ be the homogeneous coordinate ring of a smooth projective variety $X$ over a field $k$ of characteristic 0. We calculate the $K$-theory of $R$ in terms of the geometry of the projective embedding of $X$. In particular, if $X$ is a curve then we compute $K_0(R)$ and $K_1(R)$, and prove that $K_{-1}(R) = \oplus H^1(C, \mathcal{O}(n))$. The formula for $K_0(R)$ involves the Zariski cohomology of twisted Kähler differentials on the variety.

Let $R = k \oplus R_1 \oplus \cdots$ be the homogeneous coordinate ring of a smooth projective variety $X$ over a field $k$ of characteristic 0. In this paper we compute the lower $K$-theory ($K_i(R)$, $i \leq 1$) in terms of the Zariski cohomology groups $H^*(X, \mathcal{O}(t))$ and $H^*(X, \Omega^*_X(t))$, where $\mathcal{O}(1)$ is the ample line bundle of the embedding and $\Omega^*_X$ denotes the Kähler differentials of $X$ relative to $\mathbb{Q}$. We also obtain computations of the higher $K$-groups $K_n(R)/K_n(k)$, especially for curves. A complete calculation for the conic $xy = z^2$ is given in Theorem 4.3. These calculations have become possible thanks to the new techniques introduced in [1], [2] and [4].

Here, for example, is part of Theorem 2.1; $R^+$ is the seminormalization of $R$.

**Theorem.** Let $R$ be the homogeneous coordinate ring of a smooth $d$-dimensional projective variety $X$ in $\mathbb{P}^N_k$. Then $\text{Pic}(R) \cong (R^+/R)$ and

\[
K_0(R) \cong \mathbb{Z} \oplus \text{Pic}(R) \oplus \bigoplus_{i=1}^d \bigoplus_{t=1}^\infty H^i(X, \Omega^*_X(t)), \quad \text{and}
\]

\[
K_{-m}(R) \cong \bigoplus_{i=0}^{d-m} \bigoplus_{t=1}^\infty H^{m+i}(X, \Omega^*_X(t)), \quad m > 0.
\]

We have $K_{-m}(R) = 0$ for $m > d$, and $K_{-d}(R) = \bigoplus_{t \geq 1} H^d(X, \mathcal{O}(t))$.

If $k$ has finite transcendence degree over $\mathbb{Q}$ then $K_0(R)/\mathbb{Z}$ and each $K_{-m}(R)$ are finite-dimensional $k$-vector spaces.

For example, if $X = \text{Proj}(R)$ is a smooth curve over $k$ which is definable over a number field contained in $k$, we show that $\Omega^*_X(k) \otimes \mathcal{O}(t) \to \Omega^*_X(t)$ induces:

\[
K_0(R) = \mathbb{Z} \oplus \text{Pic}(R) \oplus (\Omega^*_X \otimes K_{-1}(R)), \quad K_{-1}(R) = \bigoplus_{t \geq 1} H^1(X, \mathcal{O}(t)).
\]

We also have $K_n^{(n+2)}(R) \cong \Omega^*_X \otimes K_{-1}(R)$ for all $n \geq 1$. (See Proposition 3.2(d).)

When $R$ is normal, (0.1) implies that $K_0(R) = \mathbb{Z}$ holds if and only if either (a) $k$ is algebraic over $\mathbb{Q}$, or (b) $K_{-1}(R) = 0$. Case (a) was discovered by Krishna and Srinivas [10, 1, 2], while parts of case (b) were discovered in [25]. By Riemann-Roch, the vanishing of $K_{-1}(R)$ is equivalent to the vanishing of the vector spaces $H^0(X, \Omega^*_X/k(-t))$ for $t > 0$, which is a delicate arithmetic question (unless, for
example, the embedding has degree $d \geq 2g - 2).$ Note that case (b) clarifies Srinivas’ theorem in [17] that when $k = \mathbb{C}$ and $H^1(X, \mathcal{O}(1)) \neq 0$ we have $K_0(R) \neq \mathbb{Z}.$

Still assuming that $X$ is a curve, suppose in addition that $k$ is a number field; then $\Omega_k^1 = 0$ and hence $K_0(R) = \mathbb{Z} \oplus (R^+ / R).$ We also establish (in 1.17 and 2.12) the previously unknown calculations that

$$K_1(R) = k^X \oplus \left( \bigoplus_{t=1}^{\infty} H^0(X, \Omega_{X/k}^1(t)) \right) / \Omega_{R/k}^1, \quad K_2(R) = K_2(k) \oplus \text{tors} \Omega_{R/k}^1,$$

$$K_n(R) = K_n(k) \oplus HC_{n-1}(R)/HC_{n-1}(k), \quad n \geq 3.$$ 

The $K_1$ formula (0.2) is a clarification of a result of Srinivas [19]. When $k$ is not algebraic over $\mathbb{Q},$ formulas (0.1), (0.2) and (0.3) need to be altered to involve the arithmetic Gauss-Manin connection; see Proposition 3.5 and Example 3.6.

For any smooth $d$-dimensional variety $X,$ $K_0(R)/\mathbb{Z}$ is the direct sum of the eigenspaces $K^{(i)}_0(R)$ of the Adams operation, $1 \leq i \leq d + 1 = \dim R,$ and we give a formula for these eigenspaces. For example, the top eigenspace, $K^{(d+1)}_1(R),$ may be identified with the Chow group of smooth zero-cycles in $\text{Spec}(R);$ we show that

$$K^{(d+1)}_0(R) \cong \bigoplus_{t=1}^{\infty} H^d(X, \Omega_{X}^1(t)).$$

As pointed out in [10], the normal domain $R_k = k[x, y, z]/(x^n + y^n + z^n)$ has $K_0(R_k) = \mathbb{Z}$ but if $n \geq 4$ then $H^1(X, \mathcal{O}(1))$ is nonzero while $K_0(R_k)/\mathbb{Z}$ is a very big $\mathbb{C}$-vector space; by (0.1), it is the direct sum of the $\Omega_{\mathbb{C}}^1 \otimes H^1(X, \mathcal{O}(t)),$ $t \geq 1.$

We also obtain reasonably nice formulas for the eigenspaces $K^{(i)}_n(R)$ when $n > 0$ and $i \geq n;$ see Theorem 1.13. To illustrate the range of our cohomological results, consider $K_1(R)$ when $X$ is a smooth curve and $R$ is normal; we have $K_1(R) = k^X \oplus K^{(2)}_1(R) \oplus K^{(3)}_1(R),$ where

$$K^{(2)}_1(R) \cong \left( \bigoplus_{t=1}^{\infty} H^0(X, \Omega_{X}^1(t)) \right) / \Omega_{R}^1,$$

$$K^{(3)}_1(R) = \bigoplus_{t=1}^{\infty} \text{coker}\{ \Omega_{k}^1 \otimes H^0(X, \Omega_{X/k}^1(t)) \xrightarrow{\nabla} \Omega_{k}^2 \otimes H^1(X, \mathcal{O}_X(t)) \}.$$ 

The map $\nabla$ in (0.4) is a twisted Gauss-Manin connection (see Lemma 3.4). In Section 3, we prove that if $n \geq 1$ then $K^{(n+1)}_n(R)$ contains $\Omega_{k}^{n-1} \otimes_{\mathbb{Q}} k^{d+g-1}$ as a direct summand provided that either

(a) $X$ has genus $g$ and is embedded in $\mathbb{P}^N_k$ by a complete linear system of degree $d,$ with $d \geq 2g - 1,$ or

(b) $X$ is induced by base change to $k$ from a curve defined over a number field contained in $k.$

(See Theorem 3.8 and Example 3.9.) In particular $K^{(2)}_1(R) \neq 0,$ and in general, $K^{(n+1)}_n(R) \neq 0$ if $n - 1 \leq \text{tr. deg}(k/\mathbb{Q}).$ Observe that the case $n = 1$ improves the result of Srinivas in [19, §1] that there is a surjection from $K_1(R) = K_1(\mathbb{C})/K_1(k)$ to $H^0(X, \Omega_{X/k}^1(1))$ and hence that $K_1(R) \neq 0$ if $d \geq 2g + 1.$

Finally, in Theorem 4.3 we give a complete calculation of the $K$-theory of the homogeneous coordinate ring of the plane conic, $R = k[x, y, z]/(xy - z^2).$

This paper is organized as follows. In Section 1, we reduce the calculation of $K_n(R)$ to a cdh-cohomology computation and knowledge of $HC_{n-1}(R).$ This relies
on the basic observation that cones are $A^1$-contractible, so that the reduced $K$-theory $\tilde{K}_n(R) = K_n(R)/K_n(k)$ can be calculated in terms of $NK_n(R)$, making our previous calculations (see [1], [2], [4]) applicable. Several of the formulas we obtain are valid for general graded algebras of the form $R = k \oplus R_1 \oplus \cdots$. We also specialize these formulas to the case when $\dim R = 2$, and obtain an expression for $K_n(R)$ in terms of $cdh$ cohomology and cyclic homology ($n \geq 1$).

In Section 2 we compute the $cdh$ terms in the formulas of the previous sections for the case when $R$ is the affine cone of a smooth variety. In Section 3, we return to the case when the graded coordinate ring has dimension 2, that is, we investigate cones over smooth projective curves. Finally, in Section 4 we apply the techniques of this paper to completely determine the $K$-theory of $R = k[x, y, z]/(xy - z^2)$.

Notations: Throughout this paper we consider (commutative, unital) algebras over a fixed ground field $k$, which we assume has characteristic zero. Undecorated tensor products $\otimes$ and differential forms $\Omega^*$ are taken over $\mathbb{Q}$; we write $\otimes_k$ and $\Omega^*_k$ for tensor product and forms relative to $k$. Similarly, cyclic homology is always taken over $\mathbb{Q}$. If $F$ is a functor defined on schemes over $k$, we will write $F(R)$ for $F(\text{Spec}(R))$. If $R$ is an augmented $k$-algebra (for example, the homogeneous coordinate ring of a variety), and $F$ is a functor from rings to some abelian category, then we write $\tilde{F}(R)$ for the (split) quotient $F(R)/F(k)$.

1. $K$-theory of graded algebras

Throughout this section, we let $R = R_0 \oplus R_1 \oplus \cdots$ be a finitely generated graded algebra over a field $k$ of characteristic 0 such that $R_0$ is a local, artinian $k$-algebra whose residue field is isomorphic to $k$ as a $k$-algebra. These conditions ensure that the map $K_n(R) \to K_n(k)$ induced by the composition of $R \to R_0 \to k$ is a split surjection. For example, $R_0$ might be $k$ itself, and indeed for most of the calculations in this paper, one may as well assume $R_0 = k$. Let $m_R$ denote the unique graded maximal ideal of $R$; that is, $m_R$ is the kernel of the split surjection $R \to k$.

We let $R_{\text{red}}$ denote the reduced ring associated to $R$. It is a graded ring whose degree 0 piece is the field $k$. We let $\tilde{R}$ denote the normalization of $R_{\text{red}}$ (i.e., the integral closure of $R_{\text{red}}$ in its ring of total quotients). It is well known that $\tilde{R} = \tilde{R}_0 \oplus \tilde{R}_1 \oplus \cdots$ is graded, that $\tilde{R}_0$ is a product of fields, and that $\text{Pic}(\tilde{R}) = 0$.

We let $R^+$ denote the semi-normalization of $R_{\text{red}}$, that is, the maximal extension of $R_{\text{red}}$ inside its total quotient ring $Q$ such that for all $x \in Q$, $x^2, x^3 \in R^+$ implies $x \in R^+$; see [20]. Alternatively, $\text{Spec}(R^+) \to \text{Spec}(R_{\text{red}})$ is a universal homeomorphism.

We are interested in computing the kernel $\tilde{K}_n(R)$ of the split surjection $K_n(R) \to K_n(k)$, for $n = 1, 0, -1, \ldots, 1 - d$. (By [1], $K_n(R) = NK_n(R) = 0$ for $n \leq -d$.) In general, for any graded ring $R = R_0 \oplus R_1 \oplus R_2 \oplus \cdots$, the groups $\tilde{K}_n(R)$ are known to be $R_0$-modules (see [22]), and hence (since $R_0$ contains $\mathbb{Q}$) they are uniquely divisible as abelian groups. Thus there is a decomposition $\tilde{K}_n(R) \cong \bigoplus_i \tilde{K}_n^{(i)}(R)$ according to the eigenvalues $k^i$ of the Adams operations $\psi^k$.

Remark 1.1. Suppose that the punctured spectrum, $\text{Spec}(R_{\text{red}}) \setminus \{m_R\}$, is nonsingular. Then the conductor $\mathfrak{c}$ to the normalization $\tilde{R}$ of $R_{\text{red}}$ is $m_R$-primary. An easy calculation shows that the seminormalization of $R_{\text{red}}$ is

$$R^+ = k \oplus \tilde{R}_1 \oplus \tilde{R}_2 \oplus \cdots,$$
In the exceptional case, let
\[ \frac{\mathcal{R}/R^+}{\mathcal{R}_0/k} = \mathcal{R}_0 + R_{red} = \mathcal{R}/(\mathcal{R}_0 + R_{red}). \]
Then \( K_n(i) = K_n(i)(R) \) for \( n \leq 1 \), with two exceptions: \( K_0(0)(R) = 0 \), and \( K_1(1)(R) \cong \text{nil}(R)/\text{nil}(R_0) \). The problem of computing \( \mathcal{R}/R_{red} \) (and hence \( R^+/R_{red} \)) is hard.

The main results of this section, Theorems 1.2 and 1.13, are formulated in terms of the cdh cohomology groups \( H^*_{cdh}(R, \Omega^i) \) introduced in [1] and [2], where the Kähler differentials, \( \Omega^i = \Omega^i_{R/\mathbb{Q}} \), are taken relative to the base field \( \mathbb{Q} \). By [4, 2, 5], we have
\[ H^m_{cdh}(R, \Omega^i)/dH^m_{cdh}(R, \Omega^{i-1}) \]
for the cokernel of the map \( d : H^m_{cdh}(R, \Omega^{i-1}) \to H^m_{cdh}(R, \Omega^i) \) induced by the Kähler differential. Theorem 1.2 will follow from Proposition 1.5 and Theorem 1.12 below.

**Theorem 1.2.** Let \( R = R_0 \oplus R_1 \oplus \cdots \) be a finitely generated graded algebra over a field \( k \) of characteristic 0. Assume \( R_0 \) is local artinian with residue field \( k \). Then the Adams operations induce an eigenspace decomposition:
\[ K_0(R) = \mathbb{Z} \oplus R^+/R_{red} \oplus \bigoplus_{i=1}^{\dim R-1} H^i_{cdh}(R, \Omega^i)/dH^i_{cdh}(R, \Omega^{i-1}). \]
The negative \( K \)-groups are given by
\[ K_{-m}(R) = H^m_{cdh}(R, \mathcal{O}) \oplus \bigoplus_{i=1}^{\dim R-m-1} H^m_{cdh}(R, \Omega^i)/dH^m_{cdh}(R, \Omega^{i-1}). \]
for \( m > 0 \). Here, \( K_0(0)(R) = \mathbb{Z}, K_0(1)(R) = R^+/R_{red}, K_{-m}(R) = H^m_{cdh}(R, \mathcal{O}) \) and the groups indexed by \( i \) are \( K_{n+i}(R) \) and \( K_{-m}(R) \), respectively.

By [23, 1.2 and 2.3] we have \( KH_*(R) \cong KH_*(R_0) \cong K_*(k) \), and thus by [2, 1.6], we have
\[ (1.3) \quad \tilde{K}_n(R) \cong \pi_nF_K(R) \cong \pi_{n-1}F_{HC}(R) \quad \text{for all } n. \]
Here, \( F_{HC}(R) = F_{HC}(R/\mathbb{Q}) \) is the homotopy fiber of \( HC(R) \to \mathbb{H}_{cdh}(R, HC) \), with cyclic homology taken relative to the subfield \( \mathbb{Q} \) of \( k \), so that there is a long exact sequence
\[ \cdots \to HC_n(R) \to \mathbb{H}_{cdh}^{-n}(R, HC) \to \tilde{K}_n(R) \to HC_{n-1}(R) \to \cdots. \]
These groups all have \( \lambda \)-decompositions and the maps in these decompositions are compatible with these decompositions (see [3]), but there is a weight shift in that \( \tilde{K}_n(i)(R) \) maps to \( HC_{n-i}(R) \). We have \( \tilde{K}_n(0)(R) = 0 \) for all \( n \) because \( F_{HC}^{-1}(R) \simeq 0 \). Moreover, by [2, 2.2] we have \( \mathbb{H}_{cdh}^m(R, HC^{(i)}) \cong \mathbb{H}_{cdh}^{2i+m}(R, \mathbb{Q}^{\mathbb{Z}^i}) \), so the long exact sequence becomes
\[ (1.4) \quad \cdots \to HC_{n-i}(R) \to \mathbb{H}_{cdh}^{2i-n}(R, \mathbb{Q}^{\mathbb{Z}^i}) \to \tilde{K}_n(i)(R) \to HC_{n-1-i}(R) \to \cdots. \]
The general picture is given by the following proposition.

**Proposition 1.5.** Let \( R = R_0 \oplus R_1 \oplus \cdots \) be as in Theorem 1.2. Then \( \tilde{K}_n(0)(R) = 0 \) for all \( n \). For \( n \leq 0 \), or for \( n > 0 \) and \( i \geq n+2 \), we have
\[ \tilde{K}_n(i)(R) \cong \mathbb{H}_{cdh}^{2i-n-2}(R, \mathbb{Q}^{\mathbb{Z}^i}), \quad \text{except for } (n, i) = (0, 1), \]
In the exceptional case, \( \tilde{K}_0(1)(R) = \text{Pic}(R) = R^+/R_{red} \).
Proof. The group $HC_n(R)$ vanishes for $n < 0$ and is $R$ for $n = 0$. Similarly, $HC_n^{(i)}(R)$ vanishes for $i > n > 0$. The proposition now follows from (1.4) and the fact that $H^0_{cdh}(R, O) = R^+$ by [4, 2.5].

To go further, it is useful to invoke the following trick, using the standard $\mathbb{A}^1$-contraction of a cone to its vertex.

**Standard Trick 1.6.** If $R$ is a positively graded algebra, there is an algebra map $\nu : R \to R[t]$ sending $r \in R_n$ to $rt^n$. If $F$ is a functor on algebras, then the composition of $\nu$ with evaluation at $t = 0$ factors as $R \to R_0 \to R$, so $F(R) \to F(R[\nu]) \to F(R)$ is zero on the kernel $\bar{F}(R)$ of $F(R) \to F(R_0)$. Similarly, the composition of $\nu$ with evaluation at $t = 1$ is the identity. That is, $\nu$ maps $\bar{F}(R)$ isomorphically onto a summand of $NF(R)$, and $\bar{F}(R)$ is in the image of the map $(t = 1) : NF(R) \to F(R)$.

The following technical result is crucial for our calculations; it asserts that many SBI sequences decompose into split short exact sequences. We write $\mathcal{F}_{HH}$ and $\mathcal{F}_{HC}$ for the homotopy fibers of $HH(R) \to \mathbb{H}_{cdh}(R, HH)$ and $HC(R) \to \mathbb{H}_{cdh}(R, HC)$, respectively. Then we have distinguished cohomological triangles

$$\mathcal{F}_{HC}[-1] \to \mathcal{F}_{HC}[1] \to \mathcal{F}_{HH} \to \mathcal{F}_{HC},$$

$$\mathbb{H}_{cdh}(R, HC)[-1] \to \mathbb{H}_{cdh}(R, HC)[1] \to \mathbb{H}_{cdh}(R, HH) \to \mathbb{H}_{cdh}(R, HC).$$

**Lemma 1.7.** If $R = R_0 \oplus R_1 \oplus \cdots$ is a graded algebra then for each $m$ the map $\pi_m F_{HC}(R) \to \pi_{m-1} F_{HC}(R)$ is zero, and there is a split short exact sequence:

$$0 \to \pi_{m-1} F_{HC}(R) \to \pi_m F_{HC}(R) \to \pi_m F_{HC}(R) \to 0.$$

Similarly, there are split short exact sequences:

$$0 \to \tilde{\mathbb{H}}_{cdh}^{m+1}(R, HC) \to \tilde{\mathbb{H}}_{cdh}^m(R, HH) \to \tilde{\mathbb{H}}_{cdh}^m(R, HC) \to 0,$$

and

$$0 \to \tilde{\mathbb{H}}_{cdh}^{m+1}(R, \Omega^{<i}) \to \tilde{\mathbb{H}}_{cdh}^{m-i}(R, \Omega^{<i}) \to \tilde{\mathbb{H}}_{cdh}^m(R, \Omega^{<i}) \to 0.$$

Proof. The third sequence is obtained from the second one by taking the $i$th component in the Hodge decomposition, described in [2, 2.2], and setting $n = 2i + m$. For the first two sequences to split, it suffices to show that $I$ is onto and split.

By [2, 2.4], $F_{HH}(k) = \mathcal{F}_{HC}(k) = 0$, so $\mathcal{F}_{HH} \subset \mathcal{F}_{HC}$ and $\mathcal{F}_{HC} = \mathcal{F}_{HC}$. By the standard trick 1.6, it suffices to show that the maps $N\pi_m F_{HH}(R) \to N\pi_m F_{HC}(R)$ and $N\mathbb{H}_{cdh}^m(R, HH) \to N\mathbb{H}_{cdh}^m(R, HC)$ are onto and split. But they are split surjections, as is evident from the respective decompositions of their terms in [4, 3.2] and [4, 2.2]: $\mathbb{H}_{cdh}(R, NHH^{(i)}) \simeq \mathbb{H}_{cdh}(R, NHC^{(i)}) \oplus \mathbb{H}_{cdh}(R, NHC^{(i+1)})$ and $N\mathcal{F}_{HC}^{(i)}(R) \simeq N\mathcal{F}_{HC}^{(i-1)}(R) \oplus N\mathcal{F}_{HC}^{(i-1)}(R)$.

Splicing the final sequences of Lemma 1.7 together, we see that the de Rham complexes are exact in $cdh$-cohomology:

**Proposition 1.8.** The following sequences are exact:

$$(1.8a) \quad 0 \to k \to R^+ \to \tilde{H}_{cdh}^0(R, \Omega^1) \to \tilde{H}_{cdh}^0(R, \Omega^2) \to \cdots$$

$$(1.8b) \quad 0 \to H_{cdh}^m(R, O) \to H_{cdh}^m(R, \Omega^1) \to H_{cdh}^m(R, \Omega^2) \to \cdots, \quad m > 0.$$ 

Note that the first complex is the $cdh$ reduced de Rham complex.
An analogous exact sequence
\[ \cdots \to \pi_{m-1}^\infty \mathcal{F}_{HH}(R) \to \pi_m^\infty \mathcal{F}_{HH}(R) \to \pi_{m+1}^\infty \mathcal{F}_{HH}(R) \to \cdots \]
is obtained by splicing the other sequences in 1.7. Using the interpretation of their Hodge components, described in [4, 3.4], produces two more exact sequences:

**Proposition 1.9.** The following sequences are exact:

(1.9a) \[ 0 \to \text{nil}(R) \to \text{tors} \Omega^1_R \to \text{tors} \Omega^2_R \to \text{tors} \Omega^3_R \to \cdots \]

(1.9b) \[ 0 \to (R^+/R)/\Omega^1_{cdh}(R) \to \Omega^2_{cdh}(R)/\Omega^1_{cdh}(R) \to \cdots . \]

Here we have used the following notation

(1.10) \[ \Omega^i_{cdh}(R) = H^i_{cdh}(R, \Omega^i) \]

(1.11) \[ \text{tors} \Omega^i_R = \ker(\Omega^i_R \to \Omega^i_{cdh}(R)) \]

If \( R \) is reduced then \( \text{tors} \Omega^i_R \) is the usual torsion submodule, by [4, 5.6.1].

We can now make the calculations necessary to deduce Theorem 1.2.

**Theorem 1.12.** Let \( R = R_0 \oplus R_1 \oplus \cdots \) be a graded algebra, finitely generated over a field \( k \) of characteristic 0. Assume \( R_0 \) is local artinian with residue field \( k \). Then we have

\[ \mathbb{H}^{q+i}_{cdh}(R, \Omega^\leq i) = \begin{cases} H^q_{dR}(k), & q < 0; \\ \text{coker} \{ H^q_{cdh}(R, \Omega^{i-1}) \to H^q_{cdh}(R, \Omega^i) \}, & q \geq 0; \\ 0, & q \geq \dim(R). \end{cases} \]

**Proof.** The Cartan-Eilenberg spectral sequence for \( \Omega^\leq i \) is

\[ E^{p,q}_1 = H^q_{cdh}(R, \Omega^p) \Rightarrow \mathbb{H}^{p+q}_{cdh}(R, \Omega^\leq i) \quad (0 \leq p \leq i, \ q \geq 0). \]

(See [24, 5.7.9].) Since \( H^q_{cdh}(R, \Omega^p) = \Omega^p_k \oplus \tilde{H}^q_{cdh}(R, \Omega^p) \), the row \( q = 0 \) is the brutal truncation of the direct sum of the de Rham complex of \( k \) over \( Q \) and the complex (1.8a), which is acyclic by Proposition 1.8. Since \( H^q_{cdh}(R, \Omega^p) = H^q_{cdh}(R, \Omega^p) \) for \( q > 0 \), the other rows on the \( E_1 \)-page are the truncations of the complex (1.8b), which is also acyclic by 1.8. Hence the spectral sequence degenerates at \( E_2 \), yielding the calculation. Note that the last possible nonzero group is \( \mathbb{H}^{i+\dim R^+_{cdh} R, \Omega^\leq i} = H^{\dim R^+_{cdh} R, \Omega^i} \) by the cohomological bound in [2, 2.6].

**Proof of Theorem 1.2.** Simply plug the calculations of Theorem 1.12 into those of Proposition 1.5 to get the asserted result.

We conclude the section with a calculation of the higher \( K \)-theory of \( R \) in terms of Kähler differentials, the cyclic homology of \( R \) and the cdh-cohomology of Spec(\( R \)).

**Theorem 1.13.** Let \( R = R_0 \oplus R_1 \oplus \cdots \) be a finitely generated graded algebra over a field \( k \) of characteristic 0. Assume \( R_0 \) is local artinian with residue field \( k \). Then for \( n \geq 1 \) we have:

(a) \( \widetilde{K}^{(i)}_n(R) \cong HC^{(i-1)}_{n-1}(R) \) whenever \( 0 < i < n \);

(b) \( \widetilde{K}^{(n)}_n(R) \cong \text{tors} \Omega^{n-1}_R / d \text{tors} \Omega^{n-2}_R \). In particular, \( \widetilde{K}^{(1)}_1(R) \cong \text{nil}(R) \) and \( \widetilde{K}^{(2)}_2(R) \cong \text{tors} \Omega^1_R / d \text{nil}(R) \).
(c) \( K_n^{(n+1)}(R) \cong \text{coker}\{\Omega^{n-1}_{\text{cdh}}(R) \to \Omega^n_{\text{cdh}}(R)/\Omega^n_R\} \)

(d) \( K_n^{(i)}(R) \cong \text{coker}\{H^{i-(n+1)}_{\text{cdh}}(R, \Omega^{-2}) \to H^{i-(n+1)}_{\text{cdh}}(R, \Omega^{i-1})\} \) when \( i \geq n+2 \).

**Proof.** By Theorem 1.12, we have \( \mathbb{H}^m_{\text{cdh}}(R, \Omega^\geq i) = 0 \) whenever \( m < i \) (i.e., \( q < 0 \)). Substituting this into (1.4) gives assertion (a), because \( HC_n^{(i)}(k) \xrightarrow{\sim} H^{2i-n}_d(k/\mathbb{Q}) \) also holds. Taking \( m = i \), it also gives exactness of the top row in the diagram:

\[
\begin{array}{cccc}
0 & \longrightarrow & K_n^{(n)}(R) & \longrightarrow & \mathbb{H}^{n-1}_{\text{cdh}}(R) & \longrightarrow & \Omega^{n-1}_{\text{cdh}}(R)/d\Omega^{n-2}_{\text{cdh}}(R) \\
& & \downarrow B & & \downarrow B & & \downarrow B \\
0 & \longrightarrow & \text{tors} \Omega^n_R & \longrightarrow & \Omega^n_R/\Omega^n_k & \longrightarrow & \Omega^n_{\text{cdh}}(R)/\Omega^n_k \\
& & \downarrow d & & \downarrow d & & \downarrow d \\
0 & \longrightarrow & \text{tors} \Omega^{n+1}_R & \longrightarrow & \Omega^{n+1}_R/\Omega^{n+1}_k & \longrightarrow & \Omega^{n+1}_{\text{cdh}}(R)/\Omega^{n+1}_k.
\end{array}
\]

The other two rows are exact by definition, see (1.11). The two right columns are exact by [24, 9.9.1] and (1.8a), respectively. By a diagram chase, \( K_n^{(n)}(R) \) is the kernel of \( \text{tors} \Omega^n_R \to \text{tors} \Omega^{n+1}_R \). Part (b) now follows from (1.9a). Part (c) is immediate from (1.4), given the following information: \( \mathbb{H}^m_{\text{cdh}}(R, \Omega^\geq n) \) is the cokernel of \( d : \Omega^{n-1}_{\text{cdh}}(R) \to \Omega^{n}_{\text{cdh}}(R) \) by Theorem 1.12, \( HC_n^{(n)}(R) = \Omega^n_R/d\Omega^n_R \) and \( HC_n^{(n-1)}(R) = 0 \). Part (d) follows from Theorem 1.12 and the formula \( K_n^{(i)}(R) \cong \mathbb{H}^{2i-n-2}(R, \Omega^{<i}) \) for \( i \geq n + 2 \), which is Proposition 1.5. \( \square \)

**Corollary 1.14.** If \( i > n \) and \((n, i) \neq (0, 1)\), the map \( K_n^{(i)}(R) \to K_n^{(i)}(R^+) \) is an isomorphism.

**Proof.** For \( n \geq 1 \), it follows from Theorem 1.13, and for \( n = 0 \), it follows from Proposition 1.5. \( \square \)

If the dimension of \( R \) is 2 (for example, if \( R \) is the cone over a projective curve), then the calculations of Theorem 1.13 apply to compute the higher \( K \)-groups of \( R \), but here the more dominant role is played by Kähler differentials. As in (1.10), we write \( \Omega^*_\text{cdh}(R) \) for \( H^0_{\text{cdh}}(R, \Omega^1) \).

**Theorem 1.15.** Assume \( \dim(R) = 2 \) and that \( R \) is reduced. Then we have:

1. \( K_1(R) = k^\times \oplus K_1^{(2)}(R) \oplus K_1^{(3)}(R) \) with \( K_1^{(i)}(R) = 0 \) for all \( i \geq 4 \), with:
   \[
   K_1^{(2)}(R) \cong \Omega^1_{\text{cdh}}(R)/\Omega^1_R + d(R^+), \quad \text{and}
   K_1^{(3)}(R) \cong \mathbb{H}^3_{\text{cdh}}(R, \Omega^{\leq 2}) \cong \text{coker}\{H^1_{\text{cdh}}(R, \Omega^1) \to H^1_{\text{cdh}}(R, \Omega^2)\};
   \]

2. \( K_2(R) \cong K_2(k) \oplus \text{tors} \Omega^1_R \oplus K_2^{(3)}(R) \oplus K_2^{(4)}(R) \) with
   \[
   K_2^{(3)}(R) \cong \Omega^2_{\text{cdh}}(R)/\Omega^2_R + d\Omega^1_{\text{cdh}}(R) \quad \text{and}
   K_2^{(4)}(R) \cong \text{coker}\{H^1_{\text{cdh}}(R, \Omega^2) \to H^1_{\text{cdh}}(R, \Omega^3)\};
   \]
(3) For all \(n \geq 3\), \(K_n(R) \cong K_n(k) \oplus \bigoplus_{i=2}^{n+2} K_n^{(i)}(R)\), where

\[
\tilde{K}_n^{(i)}(R) = \begin{cases} 
HC_{n-1}(R), & i < n, \\
\text{tors } \Omega_{R}^{n-1}/d \text{tors } \Omega_{R}^{n-2}, & i = n, \\
\text{coker} \{ \Omega_{cdh}^{n-1}(R) \to \Omega_{cdh}^{n}(R)/\Omega_{R}^{n} \}, & i = n + 1, \\
\text{coker} \{ H_{cdh}^{1}(R, \Omega^{n}) \to H_{cdh}^{1}(R, \Omega^{n+1}) \}, & i = n + 2.
\end{cases}
\]

Proof. For \(n = 1\) we see from Remark 1.1 that \(\tilde{K}_1^{(1)}(R) = \text{nil}(R) = 0\), and from Theorem 1.13(c) that \(K_1^{(2)}(R)\) is the cokernel of \(d : R^{+} \to \Omega_{cdh}^{1}(R)/\Omega_{R}^{1}\). Since \(R^{+} \to \Omega_{cdh}^{1}(R)\) factors through \(\Omega_{R}^{1}\), the description of \(K_1^{(2)}(R)\) follows. From (1.4), we have \(K_1^{(3)}(R) \cong H_{cdh}^{3}(R, \Omega^{\leq 2})\), which is described by 1.12, and \(K_1^{(i)}(R) = H_{cdh}^{2i-3}(R, \Omega^{<i})\) for \(i \geq 4\), which vanishes because \(H_{cdh}^{m}(R, \Omega^{<i}) = 0\) for \(m \geq 1 + i\) by Theorem 1.12.

For \(n \geq 2\), \(K_n^{(1)}(R)\) was described in Proposition 1.5 and Theorem 1.13. \(\square\)

Lemma 1.16. Assume that \(R = k \oplus R_1 \oplus \cdots\) is graded and \(\text{dim}(R) = 2\). Then for all \(i \geq 2\):

\[
\Omega_{R/k}^{i}/d(\Omega_{R/k}^{i-1}) \cong \text{tors } \Omega_{R/k}^{i}/d(\text{tors } \Omega_{R/k}^{i-1}).
\]

Proof. For \(i \geq 3\) the \(R\)-module \(\Omega_{R/k}^{i}\) is torsion because \(\Omega_{cdh}^{1}(R/k) = 0\). For \(i = 2\) we simply chase the diagram

\[
\begin{array}{cccccc}
tors \Omega_{R/k}^{1} & \to & tors \Omega_{R/k}^{2} & \to & tors \Omega_{R/k}^{3} & \to & tors \Omega_{R/k}^{4} \\
| & \downarrow \text{into} & | & \downarrow \text{into} & | & \downarrow \text{into} & | \\
\Omega_{R/k}^{1} & \to & \Omega_{R/k}^{2} & \to & \Omega_{R/k}^{3} & \to & \Omega_{R/k}^{4},
\end{array}
\]

comparing the exact sequence for tors \(\Omega_{R/k}^{i}\), analogous to (1.9a), to the de Rham sequence for \(\Omega_{R/k}^{i}\) (which is exact by [24, 9.9.3]). \(\square\)

Proposition 1.17. If \(k\) is algebraic over \(Q\) and \(R = k \oplus R_1 \oplus \cdots\) is seminormal of dimension 2, then:

a) \(K_1(R) \cong k^{\times} \oplus \Omega_{R}^{1}\); 

b) \(K_2(R) \cong K_2(k) \oplus \text{tors } \Omega_{R}^{1}\); 

c) \(K_n(R) \cong K_n(k) \oplus HC_{n-1}(R), \quad n \geq 3\).

Proof. These assertions are special cases of Theorem 1.15. Using Lemma 1.16 for \(n \geq 3\) we have

\[
\tilde{K}_n^{(n)}(R) \cong tors \Omega_{R}^{n-1}/d \text{tors } \Omega_{R}^{n-2} \cong \Omega_{R}^{n-1}/d \Omega_{R}^{n-2} = HC_{n-1}(R).
\]

By (1.8a), \(K_n^{(n+1)}(R)\) is a subquotient of \(\Omega_{cdh}^{n+1}(R)\) and vanishes for \(n \geq 2\); by (1.8b), \(K_n^{(n+2)}(R)\) is a subgroup of \(H_{cdh}^{1}(R, \Omega^{n+2})\) and vanishes for \(n \geq 1\). \(\square\)

We conclude this section with two classical examples for which \(\text{Spec}(R)\) has a smooth affine \(cdh\) cover, so that \(\Omega_{cdh}^{*}\) is easy to determine.

Example 1.18. The cusp \(R = k[t^{2}, t^{3}]\) has \(R^{+} = k[t]\) and \(K_1^{(2)}(R) = \Omega_{cdh}^{1}/d(R^{+}) = \Omega_{k}^{1}\) (cf. [12, 12.1]). The computation of \(K_n(R)\) for \(n \geq 2\) is also easily derived from Theorem 1.13, and stated explicitly in [6, 6.7].
Example 1.19. The seminormal ring $R = k[x_1, x_2, y_1, y_2]/(x_iy_j : 1 \leq i, j \leq 2)$ is the homogeneous coordinate ring of a pair of skew lines in $\mathbb{P}^3_k$. Its normalization is $\tilde{R} = k[x_1, x_2] \times k[y_1, y_2]$, and Spec(\(\tilde{R}\)) \to Spec(R) is a cdh cover. It is easy to see that $H^i_{\text{cdh}}(R, \mathcal{O}^i) = 0$, and $\mathcal{O}_R^i \to \mathcal{O}_{\text{cdh}}^i(R)$ is onto for $i \neq 0$. Applying Theorem 1.15, we see that $K_0(R) = \mathbb{Z}$, $K_1(R) = k^\times$ and $K_{-1}(R) = 0$. This recovers a classic result of Murthy in [15]. If $k$ is algebraic over $\mathbb{Q}$ then we also have tors $\mathcal{O}_R^i \cong k^4$ (on the $x_iy_j$), tors $\mathcal{O}_R^2 \cong k^4$ (on the $dx_i dy_j$) and $\mathcal{O}_R^3 = 0$, so by Proposition 1.17 we have

$$K_2(R) = K_2(k) \oplus k^4,$$
while $\tilde{K}_n(R) = \tilde{H}C_{n-1}(R)$ for all $n \geq 3$.

2. AFFINE CONES OF SMOOTH VARIETIES

Let $X$ be a smooth projective variety in $\mathbb{P}^N_k$, and let $R = k \oplus R_1 \oplus R_2 \oplus \cdots$ be the associated homogeneous coordinate ring. We will write $L$ for the pullback to $X$ of the ample bundle $\mathcal{O}(1)$ on $\mathbb{P}^N_k$, and if $\mathcal{F}$ is a quasi-coherent sheaf on $X$, we write $\mathcal{F}(t)$ for $\mathcal{F} \otimes_{\mathcal{O}_X} L^t$. In this section we compute the cdh cohomology of Spec(R) and use it to compute the $K$-theory of $R$, via Proposition 1.5. The main result is the theorem below, computing the non-positive $K$-groups of $R$. Later in this section, we give partial calculations of the positive $K$-groups.

Recall from Proposition 1.5 that $K_{-m}^{(0)}(R) = 0$ for all $m > 0$ and $\tilde{K}_n^{(0)}(R) = 0$ for $n \geq 0$. Thus we are interested in $K_{-m}^{(i+1)}(R)$ for $i \geq 0$.

**Theorem 2.1.** Let $X$ be a smooth projective variety in $\mathbb{P}^N_k$ with homogeneous coordinate ring $R$. Then

$$K_0^{(1)}(R) \cong R^+ / R = \bigoplus_{t=1}^\infty H^0(X, \mathcal{O}_X(t))/R_t,$$
while

$$K_0^{(i+1)}(R) \cong \bigoplus_{t=1}^\infty H^i(X, \mathcal{O}_X^i(t)),$$ for all $i \geq 1$.

For any $m > 0$, and all $i \geq 0$, we have:

$$K_{-m}^{(i+1)}(R) \cong \bigoplus_{t=1}^\infty H^{m+i}(X, \mathcal{O}_X^i(t)).$$

If $k$ has finite transcendence degree over $\mathbb{Q}$ then each vector space $K_0(R)/\mathbb{Z}$ and $K_{-m}(R)$ is finite-dimensional.

A few parts of Theorem 2.1 are easy to prove. The formula $K_0^{(1)}(R) = R^+ / R$ is given in Proposition 1.5. Since Spec(R) \setminus \{m_R\} is regular, we see from Remark 1.1 that $R^+$ agrees with the normalization $\tilde{R}$ of $R$ in degrees $t > 0$, and it is well known that $\tilde{R} = \bigoplus_{t=0}^\infty H^0(X, \mathcal{O}(t))$; see [7, Theorem 7.16] and [26, Ch VII, §2, Remark at the bottom of page 159]. This yields the first display. The final assertion, when $\text{tr. deg.}(k/\mathbb{Q}) < \infty$, follows from the fact that each $\mathcal{O}_X^i$ is a coherent sheaf; for each $q > 0$ the $H^q(X, \mathcal{O}_X^i(t))$ are finite-dimensional, and only finitely many are nonzero, by Serre’s Theorem B ([5, III.5.2]).

The proof of the rest of the theorem will be given in Corollary 2.5 and Proposition 2.11, building upon several intermediate results.

To compute the cdh cohomology of Spec(R), we will use the blowup $Y$ of Spec(R) at the origin (i.e., at $m_R$). The following description of $Y$ is well known.

**Lemma 2.2.** The exceptional fiber of $\pi : Y \to \text{Spec}(R)$ is isomorphic to $X$ and there is a projection $p : Y \to X$ identifying $Y$ with the geometric line bundle
Spec\(_X(\text{Sym}(L))\) over \(X\), with sheaf of sections \(L^*\). Moreover, the inclusion of the exceptional fiber \(X\) into \(Y\) is the zero section of the bundle \(p: Y \to X\).

**Proof.** The exceptional fiber is \(\text{Proj}\) of the Rees algebra \(\bigoplus \mathfrak{m}^i / \mathfrak{m}^{i+1}\), which is just \(R\), and \(X = \text{Proj}(R)\) by construction. For each \(x \in R_1\), the affine open \(D_+(x)\) of \(X\) is \(\text{Spec}(A)\), where \(R[1/x] = A[x, 1/x]\), and the line bundle \(L^n\) restricts to the \(A\)-submodule \(x^n A\) of \(R[1/x]\).

We now consider \(Y = \text{Proj}(R[mt])\). For \(x \in R_1\), and \(xt \in R_1 t\), the affine open \(D_+(xt)\) in \(Y\) is \(\text{Spec}(B)\), where \(R[mt][1/xt] = B[xt, 1/xt]\). The graded map \(R \cong \oplus R_i t^i \to R[mt]\) induces a projection \(Y \to X\) as well as an inclusion of \(A[x]\) in \(B\). This is onto, since \(B\) is generated by elements of the form \(rt^m/(xt)^m = (r/x^n)x^{n-m}\) for \(r \in R_n, n \geq m\). Hence \(B = A[x]\). This shows that \(Y\) is the geometric line bundle over \(X\), associated to the locally free sheaf \(L\) (see [5, Ex. II.5.18]). \(\square\)

By [1] and [2], we have split exact sequences

\[
0 \to H^0_{\text{cdh}}(R, \mathcal{F}) \to H^0_{\text{zar}}(Y, \mathcal{F}) \oplus \mathcal{F}(k) \to H^0_{\text{zar}}(X, \mathcal{F}) \to 0,
\]

\[
0 \to H^m_{\text{cdh}}(R, \mathcal{F}) \to H^m_{\text{zar}}(Y, \mathcal{F}) \to H^m_{\text{zar}}(X, \mathcal{F}) \to 0, \quad \text{for } m > 0,
\]

when \(\mathcal{F}\) is one of the cdh sheaves \(\mathcal{O}\) or \(\mathcal{O}^!\), or a complex of cdh sheaves of the form \(\Omega^\leq i\). Thus the calculation of \(H^m_{\text{cdh}}(R, \mathcal{F})\) is reduced to the calculation of \(H^m_{\text{zar}}(Y, \mathcal{F})\).

**Lemma 2.4.** We have \(H^0_{\text{cdh}}(R, \mathcal{O}) = R^+\) and \(H^m_{\text{cdh}}(R, \mathcal{O}) = \bigoplus_{i=1}^\infty H^m(X, \mathcal{O}_X(t))\) for \(m > 0\).

**Proof.** Since \(p\) is affine, \(H^*_{\text{zar}}(Y, \mathcal{O}_Y) = H^*_{\text{zar}}(X, p_* \mathcal{O}_Y)\), and \(p_* \mathcal{O}_Y = \text{Sym}(L)\) by Lemma 2.2. Hence \(H^m(Y, \mathcal{O}) = \bigoplus_{i=0}^\infty H^m(X, \mathcal{O}_X(t))\) for all \(m\); if \(m = 0\), this equals \(R^+\). Now apply (2.3). \(\square\)

From Proposition 1.5 and 2.4 we deduce the case \(K_*^{(1)}\) of Theorem 2.1. For comparison, recall that \(K_*^{(1)}(R) = \text{Pic}(R), K_*^{(1)}(R) = R^\times = k^\times\) and \(K_n^{(1)}(R) = 0\) for all \(n \geq 2\) by Soulé [16].

**Corollary 2.5.** For \(m > 0\) we have

\[
K^{(1)}_{-m} = H^m_{\text{cdh}}(R, \mathcal{O}) = \bigoplus_{i=1}^\infty H^m(X, \mathcal{O}_X(t)).
\]

**Remark 2.6.** This clarifies results of Srinivas in [17, Thm. 3], [18] and Weibel [25], which observed (when \(X\) is a curve) that the right side of the display in Corollary 2.5 is an obstruction to the vanishing of \(K_0(R)\) and \(K_{-1}(R)\).

There is an exact sequence \(0 \to p^* \Omega^i_X \to \Omega^i_Y \to \Omega^i_{Y/X} \to 0\) of sheaves on \(Y\). The relative sheaf \(\Omega^i_{Y/X}\) is the line bundle \(p^* L\), and so we deduce exact sequences for all \(i \geq 1\):

\[
0 \to p^* \Omega^i_X \to \Omega^i_Y \to p^* (\Omega^{i-1}_X \otimes L) \to 0.
\]

Since \(p_* p^* \mathcal{F} = \mathcal{F} \otimes \text{Sym}(L)\), applying \(p_*\) yields (graded) exact sequences of sheaves on \(X\) for all \(i \geq 1\):

\[
0 \to \Omega^i_X \otimes \text{Sym}(L) \to p_* \Omega^i_Y \to \Omega^{i-1}_X \otimes L \otimes \text{Sym}(L) \to 0.
\]
Lemma 2.8. The sequence (2.7) determines a graded split exact sequence
\[ 0 \to \bigoplus_{t=0}^{\infty} H^*_\text{zar}(X, \Omega^i_X(t)) \to H^*_\text{zar}(Y, \Omega^i_Y) \to \bigoplus_{t=1}^{\infty} H^*_\text{zar}(X, \Omega^{i-1}_X(t)) \to 0 \]
for each \( i \geq 1 \). The left-hand map is an isomorphism in degree 0, and in degrees \( t \geq 1 \), its splitting is a consequence of the fact that the composition
\[ H^*_\text{zar}(X, \Omega^i_X(t)) \to H^*_\text{zar}(Y, \Omega^i_Y) \xrightarrow{d} H^*_\text{zar}(Y, \Omega^{i+1}_Y) \to H^*_\text{zar}(X, \Omega^i_X(t)) \]
is an isomorphism.

Proof. It follows from (2.7) that we have a (graded) exact sequence
\[ \cdots \xrightarrow{d} \bigoplus_{t=0}^{\infty} H^*_\text{zar}(X, \Omega^i_X(t)) \xrightarrow{p^*} H^*_\text{zar}(Y, \Omega^i_Y) \to \bigoplus_{t=1}^{\infty} H^*_\text{zar}(X, \Omega^{i-1}_X(t)) \xrightarrow{d} \cdots \]
Therefore, the assertion that (2.9) is an isomorphism implies the first assertion. Referring to the maps of (2.7), it suffices to show that the composition
\[ \Omega^i_X \otimes \text{Sym}(L) \to p_* \Omega^i_Y \xrightarrow{d} p_* \Omega^{i+1}_Y \to \Omega^i_X \otimes L \otimes \text{Sym}(L) \]
is the evident graded surjection, with kernel \( \Omega^i_X \). But, in the notation of the proof of Lemma 2.2, it suffices to look on the affine \( D_+(x) = \text{Spec}(A) \) of \( X \), and here this is the map \( \Omega^i_A \otimes_A A[x] \to \Omega^i_A \otimes_A \Omega^{i+1}_A[x]/A \) sending \( \omega \otimes x^n \) to \( \omega \otimes nx^{n-1} \, dx \).

Example 2.9.1. In particular, \( 0 \to H^0(X, \Omega^1_X(t)) \to H^0(Y, \Omega^1_Y(t)) \to R_t \to 0 \) is exact for \( t \geq 1 \), and the composition \( R_t \xrightarrow{d} H^0(Y, \Omega^1_Y(t)) \to R_t \) is an isomorphism.

Corollary 2.10. For \( i \geq 1 \) and \( m \geq 1 \) we have:
\[ \Omega^i_{\text{cdh}}(R) \cong \Omega^i_k \oplus \bigoplus_{t=1}^{\infty} H^0_{\text{zar}}(X, \Omega^i(t)) \oplus H^0_{\text{zar}}(X, \Omega^{i-1}(t)); \]
\[ H^m_{\text{cdh}}(R, \Omega^i) \cong \bigoplus_{t=1}^{\infty} H^m_{\text{zar}}(X, \Omega^i(t)) \oplus H^m_{\text{zar}}(X, \Omega^{i-1}(t)). \]

The cokernel of \( \Omega^i_{\text{cdh}}(R) \xrightarrow{d} \Omega^i_{\text{cdh}}(R) \) is \( \Omega^i_k / d\Omega^i_{k-1} \oplus \bigoplus_{t=1}^{\infty} H^0_{\text{zar}}(X, \Omega^i_X(t)) \), and the cokernel of \( H^m_{\text{cdh}}(R, \Omega^i) \xrightarrow{d} H^m_{\text{cdh}}(R, \Omega^i) \) is the summand \( \bigoplus_{t=1}^{\infty} H^m_{\text{zar}}(X, \Omega^i_X(t)) \).

Proof. The first assertions follow from Lemma 2.8 and (2.3). The cokernel assertions follow from this using (1.8a), (1.8b) and induction on \( i \).

We may now deduce the remaining cases of Theorem 2.1, the main theorem of this section. Recall that \( K^{(1)}_{-m}(R) \) is \( \oplus_t H^m(X, O(t)) \) by Corollary 2.5.

Proposition 2.11. For \( i \geq 1 \), we have
\[ K^{(i+1)}_{-m}(R) \cong \bigoplus_{t=1}^{\infty} H^{m+i}_{\text{cdh}}(R, \Omega^i_X(t)), \quad m \geq 0. \]

Proof. The first isomorphism is Proposition 1.5. The second isomorphism is established in Lemma 2.8, using the isomorphism
\[ H^{m+i}_{\text{cdh}}(R, \Omega^i_X(t)) \cong \text{coker} \left\{ H^{m+i}_{\text{cdh}}(R, \Omega^{i-1}) \xrightarrow{d} H^{m+i}_{\text{cdh}}(R, \Omega^i) \right\} \]
of Theorem 1.12.
The proof of Theorem 2.1 is now complete. We next deduce partial information about the groups $K_n(R)$ for $n \geq 1$.

**Proposition 2.12.** Let $X$ be a smooth projective variety in $\mathbb{P}^N_k$ with homogeneous coordinate ring $R$. Then for all $n \geq 1$ we have graded isomorphisms:

\[
K_n^{(n+1)}(R) \cong \text{coker} \left\{ \Omega^n_R / d\Omega^{n-1}_R \to \bigoplus_{i=1}^{\infty} H^0(X, \Omega^n_i(t)) \right\};
\]

\[
K_n^{(1)}(R) \cong \bigoplus_{i=1}^{\infty} H^{i-n-1}(X, \Omega^n_i(t)), \quad i \geq n + 2.
\]

The graded decomposition of $K_n^{(n+1)}(R) = \bigoplus_{i=1}^{\infty} K_n^{(n+1)}(R)_i$ is:

\[
K_n^{(n+1)}(R)_i \cong \text{coker} \left\{ (\Omega^n_R / d\Omega^{n-1}_R)_i \to H^0(X, \Omega^n_i(t)) \right\}.
\]

**Proof.** By Theorem 1.13(c),

\[
K_n^{(n+1)}(R) \cong \Omega^n_{cdh}(R) / (\Omega^n_R + d\Omega^{n-1}_{cdh}(R))
\]

\[
= \text{coker} \left\{ (\Omega^n_R / d\Omega^{n-1}_R) \to \Omega^n_{cdh}(R) / d\Omega^{n-1}_{cdh}(R) \right\}.
\]

Since $\widetilde{H}^0_{cdh}(R, \Omega^n) = \Omega^n_{cdh}(R) / \Omega^n_R$ and $\Omega^n_R \subseteq \Omega^n_R$, we see from Corollary 2.10 that this is the cokernel of $\Omega^n_R / d\Omega^{n-1}_R \to \bigoplus_{i=1}^{\infty} H^0(X, \Omega^n_i(t))$, as claimed. \(\square\)

**Remark 2.13.** When $X = \mathbb{P}^1_k$ is embedded in $\mathbb{P}^N_k$ as a subvariety of degree $d > r$, our $L^t = \mathcal{O}_X(t)$ agrees with $\mathcal{O}_{\mathbb{P}^N_k}(d \cdot t)$, because it is the pullback of $\mathcal{O}_{\mathbb{P}^N_k}(t)$ to $X = \mathbb{P}^1_k$. Similarly, the terms written as $\Omega^n_X(t)$ in Proposition 2.12 should be read as $\Omega^n_{\mathbb{P}^N_k}(t) \otimes \mathcal{O}_{\mathbb{P}^N_k}(d \cdot t)$.

3. Cones over smooth curves

In this section, we focus on the case when $X$ is a curve (i.e., a smooth projective variety of dimension one, embedded in $\mathbb{P}^N_k$), and apply the results of Sections 1 and 2 in this case. Recall from Theorem 2.1 that $K_{-m}(R) = 0$ for $m > 1$.

The simplest case is when $k$ is algebraic over $\mathbb{Q}$. In this case, we know from Proposition 1.17 that $K_2(R) \cong \text{tors} \Omega^1_R$ and if $n \geq 3$ then $K_n(R) \cong \widetilde{HC}_{n-1}(R)$. It remains to describe the situation when $-1 \leq n \leq 1$.

**Lemma 3.1.** Suppose that $k$ is algebraic over $\mathbb{Q}$ and that $R$ is the homogeneous coordinate ring of a smooth curve $X$ over $k$. Then $K_{-1}(R) = \bigoplus_{i=1}^{\infty} H^1(X, \mathcal{O}(t))$, $K_0(R) = \mathbb{Z} \oplus (\mathcal{O}^+ / R)$ and $K_1(R) = \bigoplus_{i=1}^{\infty}(H^0(X, \Omega^1_X(t))) / \Omega^1_R$.

**Proof.** By Theorem 2.1, $K_0^{(1)}(R) = 0$ for $i \geq 3$ and $K_{i-1}^{(1)}(R)$ is zero for $i \geq 2$, while $K_{-1}^{(1)}(R)$ is the sum of the $H^1(X, \mathcal{O}(t))$ by 2.5. By Serre Duality, $K_0^{(1)}(R)$ is the sum of the $H^1(X, \Omega^1_X(t)) = H^0(X, \mathcal{O}_X(-t))^*$, which are zero for all $t > 0$.

The formula for $K_1(R)$ is immediate from Propositions 1.17 and 2.12. \(\square\)

**Proposition 3.2.** Suppose that $R$ is the homogeneous coordinate ring of a smooth curve $X$ over a number field $F$ contained in $k$. Then for $R_k = R \otimes_F k$:

(a) For $i < n$ we have $K_n^{(i)}(R_k) \cong \bigoplus_{p=0}^{i} \Omega^n_k \otimes_F \widetilde{K}_n^{(i-p)}(R_k)$.

(b) For all $n \geq 2$, $\widetilde{K}_n^{(n)}(R_k) \cong \bigoplus_{p=0}^{n-2} \Omega^n_k \otimes_F \widetilde{K}_n^{(n-p)}(R_k)$.

(c) For all $n \geq 1$, $K_n^{(n+1)}(R_k) \cong \Omega^{n-1}_k \otimes_F K_1^{(2)}(R_k)$. 
(d) For all \( n \geq 0 \), \( K_n^{(n+1)}(R_k) \cong \Omega_k^{n+1} \otimes_F K_{-1}(R) \cong \Omega_k^{n+1} \otimes_k K_{-1}(R_k) \).

Proof. Write \( \otimes \) for \( \otimes_F \). Part (a) is immediate from Theorem 1.13(a) and Kassel’s base change formula \( HC_*(R_k) \cong \Omega_k \otimes HC_*(R) \). (See \([8\), (3.2)\].)

For (b), recall that \( R_n^*(R_k) \cong \text{tors} \Omega_R^{n-1}/d \text{tors} \Omega_R^{n-2} \) by Theorem 1.13(b). By the Künneth formula, \( \text{tors} \Omega_R^{n_k} = \oplus_{p+q=n} \Omega_k^p \otimes \text{tors} \Omega_k^q \). Filtering by \( p \geq 0 \) yields a 2-diagonal spectral sequence computing the kernel and cokernel of \( d : \text{tors} \Omega_R^{n_k} \rightarrow \text{tors} \Omega_R^{n_k-1} \). \( E_0^{p,-p} = \Omega_k^p \otimes \tilde{K}_{n+1}(R) \) and \( E_1^{p,-1-p} = \Omega_k^p \otimes \text{tors} \Omega_R^{n_k-2} \). Given \( \alpha \) in \( \Omega_k^p \) and \( \text{d} \tau \) in \( d \text{tors} \Omega_R^{n_k-2} \), \( d(\alpha \otimes \text{d} \tau) = \text{d} \alpha \otimes \text{d} \tau = \text{d}(\text{d} \alpha \otimes \tau) \) in \( \text{tors} \Omega_R^{n_k} \), which shows that \( d^2 = 0 \) and establishes (b).

By the Künneth formula and Proposition 2.12, \( K_n^{(n+1)}(R_k) \) is the direct sum over \( p + q = n \) of the cokernels of the maps

\[
\Omega_k^p \otimes \Omega_k^q \rightarrow \Omega_k^p \otimes H^0(Y, \Omega_Y^1) \rightarrow \Omega_k^p \otimes \oplus \nu H^0(X, \Omega_X^1(t)).
\]

For \( q = 0 \), the composite is the identity map of \( \Omega_k^p \otimes R \). For \( q = 1 \), the composite is \( \Omega_k^{n-1} \) tensored with the map \( \Omega_R^1 \rightarrow \oplus \nu H^0(X, \Omega_X^1(t)) \) defining \( K_1^{(2)}(R) \). For \( q \geq 2 \), the right side is zero. This establishes part (c).

Since \( K_{-1}(R) \cong H^1(X, \mathcal{O}(t)) \), part (d) is just Proposition 2.12, together with the Künneth formula that \( H^1(X, \Omega_X^1(t)) \) is the direct sum of \( \Omega_k^p \otimes H^1(X, \mathcal{O}(t)) \) and \( \Omega_k^{n-1} \otimes H^1(X, \Omega_X^1(t)) \), which is zero for \( t > 0 \) by Serre Duality. \( \square \)

When \( k/Q \) is transcendental, we will use a variant of the arithmetic Gauss-Manin connection \( H_{\text{dR}}^1(X/k) \rightarrow \Omega_k^1 \otimes H_{\text{dR}}^1(X/k) \), or rather its \( (k\text{-linear}) \) filtered piece

\[
\nabla : H^0(X, \Omega_X^{1/k}) \rightarrow \Omega_k^1 \otimes H^1(X, \mathcal{O}_X)
\]

as described in \([9\), Thm. 2\] and \([13\), 3.2\]. When \( k = \mathbb{C} \), this can be interpreted in terms of the Hodge filtration as a map \( H^{1,0}(X, \mathbb{C}) \rightarrow \Omega_{C/k}^1 \otimes H^{1,0}(X, \mathbb{C}) \).

It is known (see \([9\]) that \( \nabla \) is the cohomology boundary map associated to the fundamental short exact sequence \( 0 \rightarrow \Omega_k^1 \otimes \mathcal{O}_X \rightarrow \Omega_X^1 \rightarrow \Omega_X^1/X/k \rightarrow 0 \). Twisting this short exact sequence by \( \mathcal{O}(t) \) yields a twisted version \( \nabla_t : H^0(X, \Omega_X^1(X/k(t))) \rightarrow \Omega_k^1 \otimes H^1(X, \mathcal{O}(t)) \). We see from Lemma 2.8 that the direct sum of the \( \nabla_t \) is a component of the cohomology boundary map associated to \( 0 \rightarrow \Omega_k^1 \otimes \mathcal{O}_Y \rightarrow \Omega_Y^1 \rightarrow \Omega_Y^1/X/k \rightarrow 0 \); it follows that \( \oplus \nabla_t \) is \( R \)-linear.

Since \( \Omega_X^1/X/k = 0 \), we have fundamental exact sequences for each \( i \):

\[
0 \rightarrow \Omega_k^i \otimes \mathcal{O}_X(t) \rightarrow \Omega_X^i(t) \rightarrow \Omega_k^{i-1} \otimes \Omega_X^{1/k}(t) \rightarrow 0.
\]

The cohomology boundary maps are the \( k \)-linear homomorphisms

\[
\Omega_k^{i-1} \otimes H^0(X, \Omega_X^{1/k}(t)) \rightarrow \Omega_k^i \otimes H^1(X, \mathcal{O}(t)).
\]

The sum of the \( \nabla_t \) is again \( R \)-linear, as the sum of the sequences (3.3) is \( R \)-linear. Alternatively, we can use the fact that the arithmetic Gauss-Manin connection can be extended via the usual formula \( \nabla_t(\omega \otimes x) = d\omega \otimes x + (-1)^{i-1}(i-1)! \omega \wedge \nabla_t(x) \), and the first term vanishes because it is in a lower part of the Hodge filtration.
Lemma 3.4. If $X$ is a smooth curve and $i \geq 1$, there is a graded exact sequence of $R$-modules, the sum over $t > 0$ of the exact sequences

$$0 \to \Omega_k^i \otimes R_t \to H^0(X, \Omega_X^i(t)) \to \Omega_k^{i-1} \otimes H^0(X, \Omega_X^1(t)) \xrightarrow{\nabla_{\delta}} \Omega_k^i \otimes H^1(X, \Omega_X^i(t)) \to H^1(X, \Omega_X^i(t)) \to 0.$$ 

Moreover, we have the identity

$$\nabla(x) = \omega \wedge \nabla(x), \quad \text{for } \omega \in \Omega_k^{i-1} \text{ and } x \in H^0(X, \Omega_X^1(t)).$$

Proof. This is just the cohomology exact sequence for (3.3), together with Serre Duality, which says that $H^1(X, \Omega_X^1(t)) = H^0(X, \Omega_X^1(-t)) = 0$ for all $t > 0$.

To prove that the boundary map is $\nabla$, let $\mathcal{U}$ be a cover of $X$ by affine open subschemes and consider the exact sequence of Čech complexes associated to (3.3). We have $\check{C}(\mathcal{U}, \Omega_X^i \otimes \mathcal{O}_X(t)) = \Omega_k^i \otimes \check{C}(\mathcal{U}, \mathcal{O}_X(t))$ and $\check{C}(\mathcal{U}, \Omega_k^{i-1} \otimes \Omega_k^1(t)) = \Omega_k^{i-1} \otimes \check{C}(\mathcal{U}, \Omega_X^1(t))$.

Let $\omega \in \Omega_k^{i-1}$ and $x \in H^0(X, \Omega_X^1(t)) = H^0(\check{C}(\mathcal{U}, \Omega_X^1(t)))$. If $y \in \check{C}(\mathcal{U}, \Omega_X^1(t))$ maps to $x$, then $\delta(y)$ is in $\Omega_k^i \otimes \check{C}(\mathcal{U}, \mathcal{O}(t))$ and represents $\nabla(x)$. Since $\omega \wedge y$ lifts $\omega \otimes x$, $\nabla(\omega \otimes x)$ is the class of $\delta(\omega \wedge y)$ in $\Omega_k^i \otimes H^1(X, \Omega_X^1(t))$. Since $\omega$ is globally defined, we have $\delta(\omega \wedge y) = \omega \wedge \delta(y)$. \qed

Proposition 3.5. If $X$ is a smooth curve, we have graded exact sequences

$$0 \to K_1^{(2)}(R) \to \frac{\text{image } \Omega_k^1(t)}{\Omega_k^1 \otimes (\oplus_i H^1(X, \mathcal{O}(t)))} \xrightarrow{\nabla} K_0^{(2)}(R) \to 0;$$

$$0 \to K_{n+1}^{(n+2)}(R) \to \frac{\Omega_k^{n+1} \otimes (\oplus_i H^0(X, \Omega_X^{i+1}(t)))}{\text{image } \Omega_k^{n+1}} \xrightarrow{\nabla} \Omega_k^{n+1} \otimes (\oplus_i H^1(X, \mathcal{O}(t))) \to K_n^{(n+2)}(R) \to 0, \quad n \geq 1.$$ 

The direct sums are taken from $t = 1$ to $\infty$.

Proof. This follows from the exact sequence of Lemma 3.4, using the formulas $K_n^{(n+2)}(R) \cong H^1(X, \Omega_X^{n+1}(t))$ and $K_{n+1}^{(n+2)}(R) \cong H^0(X, \Omega_X^{n+1}(t))/\text{im}(\Omega_k^{n+1})$, of Propositions 2.11 and 2.12, once we observe that the first map of Lemma 3.4 factors through $\Omega_R$. This is because it is a quotient of $\Omega_k^1 \otimes R \to \pi_*(\Omega_V^i) = H^n(Y, \Omega_V^i)$, which factors as $\Omega_k^1 \otimes R \to \Omega_R \to \pi_*(\Omega_V^i)$. \qed

Example 3.6. If $X$ is a curve definable over a number field contained in $k$, then the Fundamental Sequence (3.3) (with $i = 1$ and $t = 0$) splits as $\Omega_X^1 \cong \Omega_X^{1/k} \otimes \Omega_X$, by the Künneth formula. This implies that $\Omega_X^1 \cong \left(\Omega_k^{i-1} \otimes \Omega_X^{1/k}\right) \oplus \left(\Omega_k^1 \otimes \mathcal{O}_X\right)$, so the Gauss-Manin connection $\nabla$ of Lemma 3.4 vanishes and therefore:

$$K_n^{(n+1)}(R) = \frac{\Omega_k^{n-1} \otimes (\oplus_i H^0(X, \Omega_X^{i+1}(t)))}{\text{image } \Omega_k^{n+1}}, \quad n \geq 1;$$

$$K_n^{(n+2)}(R) = \Omega_k^{n+1} \otimes \left[\oplus_i H^1(X, \mathcal{O}(t))\right] \cong \Omega_k^{n+1} \otimes K_{-1}(R), \quad n \geq 0.$$ 

Of course, the formula for $K_n^{(n+1)}(R)$ reduces to that of Proposition 3.2(c).
The formula for $K_0^{(2)}(R)$ clarifies the examples given by Srinivas in [17]. There it was shown that if $X$ is definable over a number field, then $K_0(R)$ maps onto $\Omega_k^1 \otimes H^1(X, \mathcal{O}_X(1))$ (see page 264). From this Srinivas deduced that if $k = \mathbb{C}$ and $H^1(X, \mathcal{O}_X(1)) \neq 0$ then $K_0(R) \neq 0$.

The description of $K_0(R) = \mathbb{Z} \oplus K_0^{(2)}(R)$ in this special case was independently discovered by Krishna and Srinivas [11].

**Lemma 3.7.** For any graded algebra $R = k \oplus R_1 \oplus \cdots$, the degree 1 part of $\Omega^1_R$ decomposes as

$$\langle\Omega^1_R\rangle_1 \cong (R_1 \otimes \Omega^1_k) \oplus (\Omega^{1-1}_k \otimes R_1).$$

The inclusions of $R_1 \otimes \Omega^1_k$ and $\Omega^{1-1}_k \otimes R_1$ are given by $r \otimes \omega \mapsto r\omega$ and $\omega \otimes r \mapsto \omega \wedge dr$, respectively.

**Proof.** We may suppose for simplicity that $N = \dim(R_1)$ is finite, so that the polynomial ring $S = k[x_1, \ldots, x_N]$ maps to $R$, and $S \to R$ is an isomorphism in degree 1. For every subfield $\ell$ of $k$, $\Omega^1_{R/\ell}$ is the cokernel of the Hochschild boundary $R^\otimes 1 \to R \otimes 1 R$; thus the map $\Omega^1_{S/\ell} \to \Omega^1_{R/\ell}$ is an isomorphism in degree 1, and therefore so is $\Omega^1_{S/k} \to \Omega^1_{R/k}$. Since $\Omega^1_k \cong (\Omega^1_k \otimes S) \oplus \Omega^{1-1}_k$, it is easy to check that the degree 1 part of $\Omega^1_k$ is $(\Omega^1_k \otimes S_1) \oplus \Omega^{1-1}_k \otimes S_1$, via the given formulas. $\square$

**Theorem 3.8.** Let $X$ be a curve of genus $g$, embedded in $\mathbb{P}^N_k$ by a complete linear system of degree $d > 1$. Assume that the twisted Gauss-Manin connection $\nabla : H^0(X, \Omega^1_{X/k}(1)) \to \Omega^1_k \otimes H^1(X, \mathcal{O}_X(1))$ is zero. Then $K_1^{(2)}(R_1) \cong k^{d+g-1} \neq 0$, and

$$K_1^{(n+1)}(R_1) = \Omega^{n-1}_k \otimes \Omega^{n+1}_k \cong k^{d+g-1} \quad (n \geq 1).$$

In particular, $K_1^{(n+1)}(R) \neq 0$ for all $n$ with $1 \leq n < \text{tr. deg.}(k/\mathbb{Q})$.

**Proof.** By Proposition 2.12, the degree 1 part of $K_1^{(n+1)}(R)$ is

$$K_1^{(n+1)}(R_1) = \text{coker}((\Omega^1_k / d\Omega^1_R)^{n+1})_1 \to H^0(X, \Omega^1_k(1)).$$

By Lemmas 3.7 and 3.4, and our hypothesis, we have morphisms of exact sequences

$$0 \to \Omega^1_k \otimes R_1 \to (\Omega^1_k)_1 \xrightarrow{\omega \wedge dr + \omega \otimes r} \Omega^{n-1}_k \otimes R_1 \to 0$$

and

$$0 \to \Omega^1_k \otimes R_1 \to H^0(Y, \Omega^1_{Y/k})_1 \to \Omega^{n-1}_k \otimes H^0(Y, \Omega^1_{Y/k})_1 \xrightarrow{\partial} 0$$

where the bottom vertical maps are given in Lemma 2.8 as the quotients by $dH^0(X, \Omega^{n-1}_k(1))$ and $\Omega^{n-1}_k \otimes dR_1$. It follows that the right vertical composite is zero. Hence $K_1^{(n+1)}(R_1)$, which is the cokernel of the middle vertical composite, is isomorphic to $\Omega^{n-1}_k \otimes H^0(X, \Omega^1_{X/k}(1))$. Finally, $\dim H^0(X, \Omega^1_{X/k}(1)) = d + g - 1$ by Riemann-Roch. $\square$

**Example 3.9.** Here are two cases in which the hypotheses of Theorem 3.8 above are satisfied:
Lemma 4.2. For also follows from our Theorem 2.1. Srinivas proved in [19] that \( \tilde{\kappa} \) and zero otherwise. For \( R \) coordinate ring of \( \mathbb{Q} \) formula of [8, (3.2)]. □

Remark 4.1. For \( R = k[x, y, z]/(z^2 - xy) \), \( \Omega^1_{R/k} \) is a torsionfree \( R \)-module.

Proof. As \( R \) is a normal complete intersection, a theorem of Vasconcelos ([21, 2.4]) says that \( \Omega^1_R \) is a torsionfree \( R \)-module. As such, it is a graded submodule of \( \Omega^1_{R[1/z]} \).

From the factorization \( \text{Spec}(R[1/z]) \to Y \to \text{Spec}(R) \), we see that the graded map \( \Omega^1_{R/k} \to H^0(Y, \Omega^1_{Y/k}) \) is an injection. Since \( R/k \overset{d}{\to} \Omega^1_{R/k} \to H^0(Y, \Omega^1_{Y/k}) \) is an injection with cokernel \( \oplus \text{H}^0(Y, \Omega^1_{Y/k}(t)) \) by Lemma 2.8, we are reduced to comparing the Hilbert functions of both sides.

It is easy to show that \( \dim(R_t) = 2t + 1 \) for all \( t \geq 0 \). From the resolution \( 0 \to R(-2) \overset{d^t}{\to} R(-1)^3 \to \Omega^1_{R/k} \to 0 \), we compute that \( \dim(\Omega^1_{R/k}) = 3 \) for \( t = 1 \) and \( 4t \) for \( t \geq 2 \). By Riemann-Roch, we have \( \dim H^0(Y, \Omega^1_{Y/k}(t)) = 2t - 1 \) for \( t > 0 \). By Lemma 2.8, this yields:

\[
\dim H^0(Y, \Omega^1_{Y/k}) = \dim H^0(Y, \Omega^1_{Y/k}(t)) + \dim R_t = (2t - 1) + (2t + 1) = 4t.
\]

This shows that \( (\Omega^1_{R/k})_t \cong R_t \oplus H^0(Y, \Omega^1_{Y/k}(t)) \) when \( t \geq 2 \), as desired. □

Remark 4.1.1. Since \( \Omega^1_R \) is torsionfree, the exact sequence (1.9a) shows that \( d : \text{tors} \Omega^2_{R/k} \cong \Omega^1_{R/k} \cong k \). In fact, the 2-form \( \tau = z \, dx \wedge dy + 2y \, dx \wedge dz \) has \( x \tau = y \tau = \pi \tau = 0 \) and \( dx = dx \wedge dy \wedge dz \).

Lemma 4.2. For \( R = \mathbb{Q}[x, y, z]/(z^2 - xy) \) and \( n \geq 2 \), \( HC_n^{(i)}(R) \) is \( \mathbb{Q} \) if \( n = 2i - 2 \) and zero otherwise. For \( R_k = R \otimes k \), \( HC_n^{(i)}(R_k) \) is \( \Omega^1_k \), where \( p = 2i - n - 2 \).

Proof. The calculation of \( HC_n^{(i)}(R) \) is taken from [14, Thms. 2–3], using the elementary calculation that \( \Omega^1_R \cong \mathbb{Q} \) for \( n > 3 \) and exactness of the augmented Poincaré complex \( \mathbb{Q} \to \Omega^*_{R} \) for \( n = 2, 3 \). The second sentence follows using the base change formula of [8, (3.2)]. □

Theorem 4.3. For \( R_k = k[x, y, z]/(z^2 - xy) \) and all \( n \), we have

\[
\widetilde{K}_n(R_k) \cong \Omega^{n-1}_k \oplus \Omega^{n-3}_k \oplus \Omega^{n-5}_k \oplus \cdots\text{.}
\]

In particular, \( K_1(R_k) \cong K_1(k) \oplus k \) and \( K_2(R_k) \cong K_2(k) \oplus \Omega^1_k \).
Proof. By Proposition 3.2(a) and Lemma 4.2, we see that \( \tilde{K}_n^{(n-j)}(R) \) is \( \Omega_k^{j-2j-3} \) for all \( j > 0 \). By Theorem 1.15 and Remark 4.1.1, we have \( \tilde{K}_3^{(3)}(R) \cong k \) and \( \tilde{K}_n^{(n)}(R) = 0 \) for \( n \neq 3 \). By Proposition 3.2(b) this implies that \( \tilde{K}_n^{(n)}(R) \cong \Omega_k^{n-3} \) for all \( n \neq 3 \). By Proposition 3.5 and Lemma 4.1, we have \( K_1^{(2)}(R) = k \). By Proposition 3.2, this implies that \( K_1^{(n+1)}(R) \cong \Omega_k^{n-1} \) for all \( n \geq 1 \). Finally, by Proposition 2.12 we have \( K_1^{(n+2)}(R) = H^1(X, \Omega_{X_k}^{n+1}(t)) \), which vanishes for all \( n, t \geq 1 \) as it is the sum of \( \Omega_k^t \otimes H^1(X, \Omega_X^t(t)) \), which vanishes by Serre Duality, and \( \Omega_k^{n+1} \otimes H^1(X, \mathcal{O}_X(t)) \), which vanishes as \( X = \mathbb{P}_k^1 \).

Remark 4.3.1. When \( k \) is algebraic over \( \mathbb{Q} \), the formulas in Theorem 4.3 reduce to: \( \tilde{K}_n(R) = \mathbb{Q} \) for \( n \geq 1 \) odd, and \( \tilde{K}_n(R) = 0 \) otherwise.

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References

[1] G. Cortiñas, C. Haesemeyer, M. Schlichting and C. Weibel, Cyclic homology, cdh-cohomology and negative K-theory, *Annals of Math.* 167 (2008), 549–563.
[2] G. Cortiñas, C. Haesemeyer and C. Weibel, K-regularity, cdh-fibrant Hochschild homology, and a conjecture of Vorst, *J. AMS* 21 (2008), 547–561.
[3] G. Cortiñas, C. Haesemeyer and C. Weibel, Infinitesimal cohomology and the Chern character to negative cyclic homology, *Math. Ann.* 345, published electronically DOI: 10.1007/s00208-009-0333-9. Printed version to appear.
[4] G. Cortiñas, C. Haesemeyer, M. Walker and C. Weibel, Bass’ NK groups and cdh-fibrant Hochschild homology, *Preprint*, 2008. Available at http://www.math.uiuc.edu/K-theory/0883/.
[5] R. Hartshorne, *Algebraic Geometry*, Springer Verlag, 1977.
[6] S. Geller, L. Reid and C. Weibel, Cyclic homology and K-theory of curves, *J. reine angew. Math.* 393 (1989), 39–90.
[7] S. Iitaka, *Algebraic geometry*, volume 76 of *Graduate Texts in Mathematics*, Springer-Verlag, New York, 1982. An introduction to birational geometry of algebraic varieties, North-Holland Mathematical Library, 24.
[8] C. Kassel, Cyclic homology, comodules, and mixed complexes, *J. Algebra* 107 (1987), 195–216.
[9] N. Katz and T. Oda, On the differentiation of de Rham cohomology classes with respect to parameters, *J. Math. Kyoto Univ.* 8-2 (1968), 199-213.
[10] A. Krishna and V. Srinivas, Zero-cycles and K-theory on normal surfaces, *Annals Math.* 156 (2002), 155–195.
[11] A. Krishna and V. Srinivas, in preparation.
[12] M. Krusnetsev, Fundamental groups, algebraic K-theory, and a problem of Abhyankar, *Invent. Math.* 19 (1973), 15–47.
[13] J. Lewis and S. Saito, *Algebraic cycles and Mumford-Griffiths invariants*, Amer. J. Math. 129 (2007), 1449–1499.
[14] R. Michler, Hodge-components of cyclic homology for affine quasi-homogeneous hypersurfaces, *Astérisque* 226 (1994), 321–333.
[15] M. P. Murthy, Vector bundles over affine surfaces birationally equivalent to a ruled surface, *Annals of Math.* 89 (1969), 242–253.
[16] C. Soulé, Opérations en K-théorie algébrique, *Canad. J. Math.* 37 (1985), 488–550.
[17] V. Srinivas, Vector bundles on the cone over a curve, *Compositio Math.* 47 (1982), 249–269.
[18] V. Srinivas, Grothendieck groups of polynomial and Laurent polynomial rings, *Duke Math. J.* 53 (1986), 595–633.
[19] V. Srinivas, \( K_1 \) of the cone over a curve, *J. reine angew. Math.* 381 (1987), 37–50.
[20] R. G. Swan, On Seminormality, *J. Algebra* 67 (1980), 210–229.
[21] C. Weibel, \( K_2 \), \( K_3 \) and nilpotent ideals, *J. Pure Appl. Alg.* 18 (1980), 333–345.
[22] C. Weibel, Mayer-Vietoris sequences and module structures on NK∗, , pp. 466–493 in *Lecture Notes in Math.*, volume 854, Springer-Verlag, 1981.
[23] C. Weibel, Homotopy Algebraic $K$-theory, pp. 461–488 in *AMS Contemp. Math.* 83, AMS, 1989.
[24] C. Weibel, *An introduction to homological algebra*, Cambridge Univ. Press, 1994.
[25] C. Weibel, The negative $K$-theory of normal surfaces. *Duke Math. J.* 108 (2001), 1–35.
[26] O. Zariski and P. Samuel. *Commutative algebra. Vol. II*. The University Series in Higher Mathematics. D. Van Nostrand Co., Inc., Princeton, N. J.-Toronto-London-New York, 1960.

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