Distributed Algorithms for Aggregative Games on Graphs

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Abstract

We consider a class of Nash games, termed as aggregative games, being played over a networked system. In an aggregative game, a player’s objective is a function of the aggregate of all the players’ decisions. Every player maintains an estimate of this aggregate, and the players exchange this information with their local neighbors over a connected network. We study distributed synchronous and asynchronous algorithms for information exchange and equilibrium computation over such a network. Under standard conditions, we establish the almost-sure convergence of the obtained sequences to the equilibrium point. We also consider extensions of our schemes to aggregative games where the players’ objectives are coupled through a more general form of aggregate function. Finally, we present numerical results that demonstrate the performance of the proposed schemes.

1 Introduction

An aggregative game is a non-cooperative Nash game in which each player’s payoff depends on its action and an aggregate function of the actions taken by all players [37, 16, 17, 28]. Nash-Cournot games represent an important instance of such games; here, firms make quantity bids that fetch a price based on aggregate quantity sold, implying that the payoff of any player is a function of the aggregate sales [8, 14]. The ubiquity of such games has grown immensely in the last two decades and examples emerge in the form of supply function games [17], common agency games [17], and power and rate control in communication networks [2, 3, 4, 62] (see [1] for more examples). Our work is motivated by the development of distributed algorithms on a range of game-theoretic problems in wired and wireline communication networks where such an aggregate function captures the link-specific congestion [3, 4] or the signal-to-noise ratio [10, 39, 56]. In almost all of the algorithmic research on equilibrium computation, it is assumed that the aggregate of player decisions is observable to all players, allowing every player to evaluate its payoff function without any prior communication.

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1Such games have been shown to be closely related with subclasses of potential games [51, 9] where a potential game refers to a Nash game in which the payoff functions admit a potential function [31]. The potential function of an aggregative game is a special case of the function employed in [29], where distributed algorithms for optimization problems with general separable convex functions are presented.
In this paper, we consider aggregative games wherein the players (referred to as agents) compete over a network. Distributed computation of equilibria in such games is complicated by two crucial challenges. First, the connectivity graphs of the underlying network may evolve over time. Second, agents do not have ready access to aggregate decisions, implying that agents cannot compute their payoffs (or their gradients). Consequently, distributed gradient-based [3, 40, 22, 23] or best-response schemes [52] cannot be directly implemented since agents do not have immediate access to the aggregate. Accordingly, we propose two distributed agreement-based algorithms that overcome this difficulty by allowing agents to build estimates of the aggregate by communicating with their local neighbors and consequently compute an equilibrium of aggregative games. Of these, the first is a synchronous algorithm where all agents update simultaneously, while the second, a gossip-based algorithm, allows for asynchronous computation:

(a) **Synchronous distributed algorithm:** At each epoch, every agent performs a “learning step” to update its estimate of the aggregate using the information obtained through the time-varying states of its neighbors. All agents exchange information and subsequently update their decisions simultaneously via a gradient-based update. This algorithm builds on the ideas of the method developed in [47] for distributed optimization problems.

(b) **Asynchronous distributed algorithm:** In contrast, the asynchronous algorithm uses a gossip-based protocol for information exchange. In the gossip-based scheme, a single pair of randomly selected neighboring agents exchange their information and update their estimates of both the aggregate and their individual decisions. This algorithm combines our synchronous method in (a) with the gossip technique proposed in [7] for the agreement (consensus) problem.

We investigate the convergence behavior of both algorithms under a diminishing stepsize rule, and provide error bounds under a constant steplength regime. Additionally, the results are supported with numerics derived from application of the proposed schemes on a class of networked Nash-Cournot games. The novelty of this work lies in our examination of distributed (neighbor-based) algorithms for computation of a Nash equilibrium point for aggregative Nash games, while the majority of preceding efforts on such algorithms have been spent towards solving feasibility and optimization problems. Before proceeding, a caveat is in order. While the proposed game-theoretic problem can be easily solved via a range of centralized schemes (see [11] for a comprehensive survey), any such approach relies on the centralized availability of all information, a characteristic that does not hold in the present setting. Instead, our interest is not merely in equilibrium computation but in the development of stylized distributed protocols, implementable on networks, and complicated by informational restrictions, local communication access, and a possibly evolving network structure.

Broadly speaking, the present work can be situated in the larger domain of distributed computation of equilibria in networked Nash games. First proposed by Nash in 1950 [32], this equilibrium concept has found application in modeling strategic interactions in oligopolistic problem settings drawn from economics, engineering, and the applied sciences [14, 12, 4]. More recently, game-theoretic models have assumed relevance in the control of a large collection of coupled nonlinear systems, instances of which arise in production planning [15], synchronization of coupled oscillators [60], amongst others. In particular, agents in such settings have conflicting objectives and the centralized control problem is challenging. By allowing agents to compete, the equilibrium behavior may be analyzed exactly or approximately (in large population regimes), allowing for the derivation of distributed control laws. In fact, game-theoretic approaches have been effectively utilized in obtaining distributed control laws in complex engineered systems [49, 50]. Motivated
by the ubiquity of game-theoretic models, arising either naturally or in an engineered form, the distributed computation of equilibria has immense importance.

While equilibrium computation is a well-studied topic [13], our interest lies in networked regimes where agents can only access or observe the decisions of their neighbors. In such contexts, our interest lies in developing distributed gradient-based schemes. While any such algorithmic development is well motivated by protocol design in networked multi-agent systems, best-response schemes, rather than gradient-based methods, are natural choices when players are viewed as fully rational. However, gradient-response schemes assume relevance for several reasons. First, increasingly game-theoretic approaches are being employed for developing distributed control protocols where the choice of schemes lies with the designer (cf. [25] [27]). Given the relatively low complexity of gradient updates, such avenues are attractive for control systems design. Second, when players rule out strategies that are characterized by high computational complexity [41] (referred to as a “bounded-rationality” setting), gradient-based approaches become relevant and have been employed extensively in the context of communication networks [2, 3, 62, 40, 56].

The present work assumes a strict monotonicity property on the mapping corresponding to the associated variational problem. This assumption is weaker than that imposed by related studies on communication networks [2, 3] where strong monotonicity properties are imposed. From a methodological standpoint, we believe that this work represents a first step. By combining a regularization technique, this requirement can be weakened [22] while extensions to stochastic regimes can also be incorporated by examining regularized counterparts of stochastic approximation [23]. However, all of these approaches are under the assumption that agents have access to the decisions of all their competitors.

Finally, it should be emphasized that the distributed algorithms presented in this paper draw inspiration from the seminal work in [57], where a distributed method for optimization has been developed by allowing agents to communicate locally with their neighbors over a time-varying communication network. This idea has attracted a lot of attention recently in an effort to extend the algorithm of [57] to more general and broader range of problems [35, 46, 15, 19, 18, 36, 54, 33, 26, 58, 48, 51, 5]. Much of the aforementioned work focuses on optimizing the sum of local objective function [35, 46, 15, 36, 51, 33, 26] in a multi-agent networks, while a subset of recent work considered the min-max optimization problem [55, 6], where the objective is to minimize the maximum cost incurred by any agent in the network. Notably, extensions of consensus-based algorithms have also been studied in the domain of distributed regression [48], estimation and inference tasks [58, 59]. While much of the aforementioned work focuses on consensus-based algorithms, an alternative distributed messaging protocol for consensus propagation across a network is presented in [30]. The work in this paper extends the realm of consensus-based algorithms (and not consensus propagation) to capture competitive aspect of multi-agent networks.

The remainder of the paper is organized as follows. In section 2, we describe the problem of interest, provide two motivating examples and state our assumptions. A synchronous distributed algorithm is proposed in section 3 and convergence theory is provided. An asynchronous gossip-based variant of this algorithm is described in section 4 and is supported by convergence theory and error analysis. In section 5.2, we present an extension to the problem presented in section 2 and suitably adapt the distributed synchronous and asynchronous algorithm to address this generalization. We present some numerical results in section 6 and, finally, conclude in section 7.

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2This notion is rooted in the influential work by Simon [53] where it is suggested that, when reasoning and computation are costly, agents may not invest in these resources for marginal benefits.
Throughout this paper, we view vectors as columns. We write $x^T$ to denote the transpose of a vector $x$, and $x^T y$ to denote the inner product of vectors $x$ and $y$. We use $\|x\| = \sqrt{x^T x}$ to denote the Euclidean norm of a vector $x$. We use $\Pi_K$ to denote the Euclidean projection operator on a set $K$, i.e. $\Pi_K(x) \triangleq \arg\min_{z \in K} \|x - z\|$. The expectation of a random variable $Y$ is denoted by $E[Y]$ and a.s. denotes almost surely. A matrix $W$ is row-stochastic if $W_{ij} \geq 0$ for all $i, j$, and $\sum_j W_{ij} = 1$. A matrix $W$ is doubly stochastic if both $W$ and $W^T$ are row stochastic.

2 Problem Formulation and Background

In this section we introduce an aggregative game of our interest and provide its sufficient equilibrium conditions. The players in this game are assumed to have local interactions with each other over time, where these interactions are modeled by time-varying connectivity graphs. We also discuss some auxiliary results for the players’ connectivity graphs and present our distributed algorithm for equilibrium computation.

2.1 Formulation and Examples

Consider a set of $N$ players (or agents) indexed by $1, \ldots, N$, and let $\mathcal{N} = \{1, \ldots, N\}$. The $i$th player is characterized by a strategy set $K_i \subseteq \mathbb{R}^n$ and a payoff function $f_i(x_i, \bar{x})$, which depends on player $i$ decision $x_i$ and the aggregate $\bar{x} = \sum_{i=1}^N x_i$ of all player decisions. Furthermore, $\bar{x}_{-i}$ denotes the aggregate of all players excepting player $i$, i.e.,

$$\bar{x}_{-i} = \sum_{j=1, j \neq i}^N x_j.$$ 

To formalize the game, let $\bar{K}$ denote the Minkowski sum of the sets $K_i$, defined as follows:

$$\bar{K} \triangleq \sum_{i=1}^N K_i.$$ 

(1)

In a generic aggregative game, given $\bar{x}_{-i}$, player $i$ faces the following parametrized optimization problem:

\begin{align*}
\text{minimize} & \quad f_i(x_i, \bar{x}) \triangleq f_i(x_i, x_i + \bar{x}_{-i}) \\
\text{subject to} & \quad x_i \in K_i, \\
\end{align*}

(2)

where $K_i \subseteq \mathbb{R}^n$ and $\bar{x}$ is the aggregate of the agent’s decisions $x_i$, i.e.,

$$\bar{x} := \sum_{j=1}^N x_j = x_i + \bar{x}_{-i}, \quad \bar{x} \in \bar{K},$$

(3)

with $\bar{K} \subseteq \mathbb{R}^n$ as given in (1), and $f_i : K_i \times \bar{K} \rightarrow \mathbb{R}$. The set $K_i$ and the function $f_i$ are assumed to be known by agent $i$ only. Next, we motivate our work by providing an example of an aggregative game, whose broad range emphasizes the potential scope of our work.
Example 1 (Nash-Cournot game over a network). A classical example of an aggregative game is a Nash-Cournot played over a network [8, 28, 22]. Suppose a set of $N$ firms compete over $L$ locations. In this situation, the communication network of our interest is formed by the players which are viewed as the nodes in the network. One such instance of connectivity graph is as shown in Figure 1. This graph determines how the firms communicate their production decision over $L$ locations. More specifically, the firm in the center of the graph has access to information from all the other firms, whereas all the other firms have access to the information of the firm in the center only. We consider other instances of connectivity graph in Section 6. To this end, let firm $i$’s production and sales at location $l$ be denoted by $g_{il}$ and $s_{il}$, respectively, while its cost of production at location $l$ is denoted by $c_{il}(g_{il})$. Consequently, goods sold by firm $i$ at location $l$ fetch a revenue $p_l(s_l)s_{il}$ where $p_l(s_l)$ denotes the sales price at location $l$ and $s_l = \sum_{i=1}^{N} s_{il}$ represents the aggregate sales at location $l$. Finally, firm $i$’s production at location $l$ is capacitated by $\text{cap}_{il}$ and its optimization problem is given by the following:

$$\begin{align*}
\text{minimize} & \quad \sum_{l=1}^{L} (c_{il}(g_{il}) - p_l(s_l)s_{il}) \\
\text{subject to} & \quad \sum_{l=1}^{L} g_{il} \geq \sum_{l=1}^{L} s_{il}, \\
& \quad g_{il}, s_{il} \geq 0, \quad g_{il} \leq \text{cap}_{il}, \quad l = 1, \ldots, L. \quad (4)
\end{align*}$$

In effect, firm $i$’s payoff function is parametrized by nodal aggregate sales, thus rendering an aggregative game. Note that, in this example we have two independent networks, the first being used to model the communication of the firms and the second being used to model the physical layout of the firms production unit and locations. We allow the communication network to be dynamic but the layout network is assumed to be static.

2.2 Equilibrium Conditions and Assumptions

To articulate sufficiency conditions, we make the following assumptions on the constraint sets $K_i$ and the functions $f_i$.

Assumption 1. For each $i = 1, \ldots, N$, the set $K_i \subset \mathbb{R}^n$ is compact and convex. Each function $f_i(x_i, y)$ is continuously differentiable in $(x_i, y)$ over some open set containing the set $K_i \times \bar{K}$, while each function $x_i \mapsto f_i(x_i, \bar{y})$ is convex over the set $K_i$.

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Note that the transportation costs are assumed to be zero.
Under Assumption 1, the (sufficient) equilibrium conditions of the Nash game in (2) can be specified as a variational inequality problem VI($K, \phi$) (cf. [11]). Recall that VI($K, \phi$) requires determining a point $x^* \in K$ such that

$$(x - x^*)^T \phi(x^*) \geq 0 \quad \text{for all } x \in K,$$

where

$$\phi(x) \triangleq \begin{pmatrix} \nabla x_1 f_i(x_1, \bar{x}) \\ \vdots \\ \nabla x_N f_N(x_N, \bar{x}) \end{pmatrix}, \quad K = \prod_{i=1}^{N} K_i,$$

(5)

with $x \triangleq (x_1^T, \ldots, x_N^T)$, $x_i \in K_i$ for all $i$, and $\bar{x}$ defined by (3). Note that, by Assumption 1, the set $K$ is a compact and convex set in $\mathbb{R}^{nN}$, and the mapping $\phi : K \to \mathbb{R}^{nN}$ is continuous. To emphasize the particular form of the mapping $\phi$, we define $F_i(x_i, \bar{x})$ as follows:

$$F_i(x_i, \bar{x}) = \nabla x_i f_i(x_i, \bar{x}) \quad \text{for all } i = 1, \ldots, N.$$

(6)

The mapping $F(x, u)$ is given by

$$F(x, u) \triangleq \begin{pmatrix} F_1(x_1, u) \\ \vdots \\ F_N(x_N, u) \end{pmatrix},$$

(7)

where the component maps $F_i : K_i \times \bar{K} \to \mathbb{R}^n$ are given by (6). With this notation, we have

$$\phi(x) = F(x, \bar{x}) \quad \text{for all } x \in K.$$

(8)

Next, we make an assumption on the mapping $\phi(x)$.

**Assumption 2.** The mapping $\phi(x)$ is strictly monotone over $K$, i.e.,

$$(\phi(x) - \phi(x'))^T (x - x') > 0, \quad \text{for all } x, x' \in K, \text{ where } x \neq x'.$$

Assumption 1 allows us to claim the existence of a Nash equilibrium, while Assumption 2 allows us to claim the uniqueness of the equilibrium.

**Proposition 1.** Consider the aggregative Nash game defined in (2). Suppose Assumptions 1 and 2 hold. Then, the game admits a unique Nash equilibrium.

**Proof.** By Assumption 1, the set $K$ is compact and $\phi$ is continuous. It follows from Corollary 2.2.5 [11] that VI($K, \phi$) has a solution. By the strict monotonicity of $\phi(x)$, VI($K, \phi$) has at most one solution based on Theorem 2.3.3 [11] and uniqueness follows.

Strict monotonicity assumptions on the mapping are seen to hold in a range of practical problem settings, including Nash-Cournot games [22], rate allocation problems [2, 3, 61, 62], amongst others. We now state our assumptions on the mappings $F_i$, which are related to the coordinate mappings of $\phi$ in (5).
Assumption 3. Each mapping $F_i(x_i, u)$ is uniformly Lipschitz continuous in $u$ over $\bar{K}$, for every fixed $x_i \in K_i$ i.e., for some $\bar{L}_i > 0$ and for all $z_1, z_2 \in \bar{K}$,

$$\|F_i(x_i, z_1) - F_i(x_i, z_2)\| \leq \bar{L}_i \|z_1 - z_2\|,$$

where $\bar{K}$ is as defined in (1).

One would naturally question whether such assumptions are seen to hold in practical instances of aggregative games. We will show in section 6 that the assumptions are satisfied for the Nash-Cournot game described in Example 1.

Before proceeding, it is worthwhile to reiterate the motivation for the present work. In the context of continuous-strategy Nash games, when the mapping $\phi$ satisfies a suitable monotonicity property over $K$, then a range of distributed projection-based schemes [11, 3, 2, 42, 43] and their regularized variants schemes [61, 62, 21, 22] can be constructed. In all of these instances, every agent should be able to observe the aggregate $\bar{x}$ of the agent decisions. In this paper, we assume that this aggregate cannot be observed and no central entity exists that can globally broadcast this quantity at any time. Yet, when agents are connected in some manner, then a given agent may communicate locally with their neighbors and generate estimates of the aggregate decisions. Under this restriction, we are interested in designing algorithms for computing an equilibrium of an aggregative Nash game (2).

3 Distributed Synchronous Algorithm

In this section we develop a distributed synchronous algorithm for equilibrium computation of the game in (2) that relies on agents constructing an estimate of the aggregate by mixing information drawn from local neighbors and making a subsequent projection step. In Section 3.1 we describe the scheme and provide some preliminary results in Section 3.2. This section concludes in Section 3.3 with an analysis of the convergence of the proposed scheme.

3.1 Outline of Algorithm

Our algorithm equips each agent in the network with a protocol that mandates that every agent exchange information with its neighbors, and subsequently update its decision and the estimate of the aggregate decisions, simultaneously. We employ a synchronous time model which can contend with a time varying connectivity graph. Consequently, in this section we consider a time varying network to model agent’s communications in time. More specifically, let $\mathcal{E}_k$ be the set of underlying undirected edges between agents and let $\mathcal{G}_k = (\mathcal{N}, \mathcal{E}_k)$ denote the connectivity graph at time $k$. Let $\mathcal{N}_i(k)$ denote the set of agents who are immediate neighbors of agent $i$ at time $k$ that can send information to $i$, assuming that $i \in \mathcal{N}_i(k)$ for all $i \in \mathcal{N}$ and all $k \geq 0$. Mathematically, $\mathcal{N}_i(k)$ can be expressed as:

$$\mathcal{N}_i(k) = \{j : \{i, j\} \in \mathcal{E}_k\}.$$

We make the following assumption on the graph $\mathcal{G}_k = (\mathcal{N}, \mathcal{E}_k)$.

Assumption 4. There exists an integer $Q \geq 1$ such that the graph $(\mathcal{N}, \bigcup_{\ell=1}^{Q} \mathcal{E}_{\ell+k})$ is connected for all $k \geq 0.$
This assumption ensures that the intercommunication intervals are bounded for agents that communicate directly; i.e., every agent sends information to each of its neighboring agents at least once every $Q$ time intervals. This assumption has been commonly used in distributed algorithms on networks, starting with [57].

![Connectivity graph at one instant](image1)

(a) Connectivity graph at one instant

![Connectivity graph at another instant](image2)

(b) Connectivity graph at another instant

Figure 2: A depiction of an undirected communication network.

Due to incomplete information at any point, an agent only has an estimate of $\bar{x}$ in contrast to the actual $\hat{x}$. We describe how an agent may build this estimate. Let $x_i^k$ be the iterate and $\hat{v}_i^k$ be the estimate of the average of the decisions $x_1^k, \ldots, x_N^k$ for agent $i$ at the end of the $k$th iteration. At the beginning of the $(k+1)$st iteration, agent $i$ receives the estimates $\hat{v}_j^k$ from its neighbors $j \in \mathcal{N}_i(k+1)$. Using this information, agent $i$ aligns its intermediate estimate according to the following rule:

$$\hat{v}_i^k = \sum_{j \in \mathcal{N}_i(k)} w_{ij}(k) \hat{v}_j^k,$$

where $w_{ij}(k)$ is the nonnegative weight that agent $i$ assigns to agent $j$’s estimate. By specifying $w_{ij} = 0$ for $j \notin \mathcal{N}_i(k)$ we can write:

$$\hat{v}_i^k = \sum_{j=1}^N w_{ij}(k) v_j^k \quad \text{with} \quad v_j^0 = x_j^0 \text{ for all } j = 1, \ldots, N.$$

Using this aligned average estimate $\hat{v}_i^k$ and its own iterate $x_i^k$, agent $i$ updates its iterate and average estimate as follows:

$$x_i^{k+1} := \Pi_{K_i}[x_i^k - \alpha_k F_i(x_i^k, N\hat{v}_i^k)],$$

$$v_i^{k+1} := \hat{v}_i^k + x_i^{k+1} - x_i^k,$$

where $\alpha_k$ is the stepsize, $\Pi_{K_i}(u)$ denotes the Euclidean projection of a vector $u$ onto the set $K_i$ and $F_i$ is as defined in [55]. The quantity $N\hat{v}_i^k$ in (10) is the aggregate estimate that agent $i$ uses instead of the true estimate $\sum_{i=1}^N x_i^k$ of the agent decisions at time $k$. Under suitable conditions on the agents weights $w_{ij}(k)$ and the stepsize $\alpha_k$, the iterate vector $(x_1^k, \ldots, x_N^k)$ can converge to a Nash equilibrium point $(x_1^*, \ldots, x_N^*)$ and the estimates $N\hat{v}_i^k$ in (10) will converge to the true aggregate value $\sum_{i=1}^N x_i^*$ at the equilibrium. These assumptions are given below.
Assumption 5. Let $W(k)$ be the weight matrix with entries $w_{ij}(k)$. For all $i \in \mathcal{N}$ and all $k \geq 0$, the following hold:

(i) $w_{ij}(k) \geq \delta$ for all $j \in \mathcal{N}_i(k)$ and $w_{ij}(k) = 0$ for $j \not\in \mathcal{N}_i(k)$;
(ii) $\sum_{j=1}^{N} w_{ij}(k) = 1$ for all $i$;
(iii) $\sum_{i=1}^{N} w_{ij}(k) = 1$ for all $j$.

Assumption 5 essentially requires every player to assign a positive weight to the information received from its neighbor. Following Assumption 5 (ii)-(iii), the matrix $W(k)$ is doubly stochastic.

We point the reader to [35] for the examples and a detailed discussion of the weights satisfying the preceding assumption.

Assumption 6. The stepsize $\alpha_k$ is chosen such that the following hold:

(i) The sequence $\{\alpha_k\}$ is monotonically non-increasing i.e., $\alpha_{k+1} \leq \alpha_k$ for all $k$;
(ii) $\sum_{k=0}^{\infty} \alpha_k = \infty$;
(iii) $\sum_{k=0}^{\infty} \alpha_k^2 < \infty$.

Such an assumption is satisfied for a stepsize of the form $\alpha_k = (k+1)^{-b}$ where $0.5 < b \leq 1$.

3.2 Preliminary Results

We next provide some auxiliary results for the weight matrices and the estimates generated by the method. We introduce the transition matrices $\Phi(k, s)$ from time $s$ to $k > s$, as follows:

$$\Phi(k, s) = W(k)W(k-1) \cdots W(s+1)W(s) \quad \text{for } 0 \leq s < k,$$

where $\Phi(k, k) = W(k)$ for all $k$. Let $[\Phi(k, s)]_{ij}$ denote the $(i,j)$th entry of the matrix $\Phi(k, s)$, and let $\mathbf{1} \in \mathbb{R}^N$ be the column vector with all entries equal to 1. We next state a result on the convergence properties of the matrix $\Phi(k, s)$. The result can be found in [34] (Corollary 1).

Lemma 1 ([57] Lemma 5.3.1). Let Assumptions 4 and 5 hold. Then the following hold:

(i) $\lim_{k \to \infty} \Phi(k, s) = \frac{1}{N} \mathbf{1}\mathbf{1}^T$ for all $s \geq 0$.

(ii) The convergence rate of $\Phi(k, s)$ is geometric; specifically, we have $|[\Phi(k, s)]_{ij} - \frac{1}{N}| \leq \theta \beta^{k-s}$ for all $k \geq s \geq 0$ and for all $i$ and $j$, where $\theta = (1 - \frac{\delta}{4N^2})^{-2}$ and $\beta = (1 - \frac{\delta}{4N^2})^{\frac{1}{2}}$.

Next, we state some results which will allow us to claim the convergence of the algorithm. These results involve the average of the estimates $v_i^k$, $i \in \mathcal{N}$, defined by $y^k$:

$$y^k = \frac{1}{N} \sum_{i=1}^{N} v_i^k \quad \text{for all } k \geq 0. \quad (12)$$

As we proceed to show, $y^k$ will play a key role in establishing the convergence of the iterates produced by the algorithm in (10)-(11). One important property of $y^k$ is that we have $y^k = \frac{1}{N} \sum_{j=1}^{N} x_j^k$ for all $k \geq 0$. Thus, $y^k$ not only captures the average belief of the agents in the network but it also represents the true average information. This property of the true average $y^k$ has been shown in [48] within the proof of Lemma 5.2 for a different setting, and it is given in the following lemma for sake of clarity.
Lemma 2. Let \( W(k) \) be such that \( \sum_{j=1}^{N}[W(k)]_{ji} = 1 \) for every \( i \) and \( k \). Then, \( y^k = \frac{1}{N} \sum_{i=1}^{N} x_i^k \) for all \( k \geq 0 \), where \( y^k \) is defined by (12).

Proof. It suffices to show that for all \( k \geq 0 \),

\[
\sum_{j=1}^{N} v_j^k = \sum_{j=1}^{N} x_j^k. \tag{13}
\]

We show this by induction on \( k \). For \( k = 0 \) relation (13) holds trivially, as we have initialized the beliefs with \( v_j^0 = x_j^0 \) for all \( j \). Assuming relation (13) holds for \( k - 1 \), as the induction step, we have

\[
\sum_{j=1}^{N} v_j^k = \sum_{j=1}^{N} \left( \hat{v}_j^{k-1} + x_j^k - x_j^{k-1} \right) \\
= \sum_{j=1}^{N} \sum_{i=1}^{N} [W(k-1)]_{ji} v_i^{k-1} + \sum_{j=1}^{N} \left( x_j^k - x_j^{k-1} \right) \\
= \sum_{i=1}^{N} v_i^{k-1} + \sum_{j=1}^{N} \left( x_j^k - x_j^{k-1} \right),
\]

where the first equality follows from (11), the second inequality is a consequence of the mixing relationship articulated by (9), and the last equality follows from \( \sum_{j=1}^{N}[W(k)]_{ji} = 1 \) for every \( i \) and \( k \). Furthermore, using the induction hypothesis, we have \( \sum_{j=1}^{N} (x_j^k - x_j^{k-1}) = \sum_{j=1}^{N} x_j^k - \sum_{j=1}^{N} v_j^{k-1} \), thus implying that \( \sum_{j=1}^{N} v_j^k = \sum_{j=1}^{N} x_j^k \).

As a consequence of Lemma 2 Assumptions 1 and 3 we have the following result which will be often used in the sequel.

Lemma 3. Let \( W(k) \) be such that \( \sum_{j=1}^{N}[W(k)]_{ji} = 1 \) for every \( i \) and \( k \). Also, let Assumptions 7 and 2 hold. Then, there exists a constant \( C \) such that

\[
\|F_i(x_i^k, Ny^k)\| \leq C, \quad \|F_i(x_i^k, N\hat{v}_i^k)\| \leq C \quad \text{for all } i \text{ and } k \geq 0.
\]

Proof. By Lemma 2 we have \( Ny^k = \sum_{i=1}^{N} x_i^k = \bar{x}_k \in \bar{K} \), where \( \bar{K} \) is compact since each \( K_i \) is compact (Assumption 1). Since each \( F_i \) is continuous over \( K_i \times \bar{K} \), the first inequality follows. To show that \( \{F_i(x_i^k, N\hat{v}_i^k)\} \) is bounded, we write

\[
\|F_i(x_i^k, N\hat{v}_i^k)\| \leq \|F_i(x_i^k, N\hat{v}_i^k) - F_i(x_i^k, Ny^k)\| + \|F_i(x_i^k, Ny^k)\|.
\]

Using the Lipschitz property of \( F_i \) of Assumption 3 we obtain

\[
\|F_i(x_i^k, N\hat{v}_i^k)\| \leq L_i N \|\hat{v}_i^k - y^k\| + \|F_i(x_i^k, Ny^k)\|.
\]

Let \( \hat{K} \) be the convex hull of the union set \( \cup_i K_i \). Note that \( \hat{v}_i^k, y^k \in \hat{K} \) for all \( k \) and that \( \hat{K} \) is compact (since each \( K_i \) is compact)\(^4\). Thus, \( \{\|\hat{v}_i^k - y^k\|\} \) is bounded. As already established, \( \{F_i(x_i^k, Ny^k)\} \) is also bounded, implying that \( \{F_i(x_i^k, N\hat{v}_i^k)\} \) is bounded as well. \( \square \)

\(^4\)Though \( y^k \in \bar{K} \), we cannot claim the same for \( \hat{v}_i^k \).
In the following lemma, we establish an error bound on the norm $\|y^k - \hat{v}_i^k\|$ which plays an important role in our analysis.

**Lemma 4.** Let Assumptions 1–5 hold, and let $y^k$ be defined by (12). Then, we have

$$
\|y^k - \hat{v}_i^k\| \leq \theta \beta^k M + \theta NC \sum_{s=1}^{k} \beta^{k-s} \alpha_{s-1} \quad \text{for all } i \in \mathcal{N} \text{ and all } k \geq 1,
$$

where $\hat{v}_i^k$ is defined in (9), $\theta = (1 - \frac{\delta}{4N^2})^{-2}$, $\beta = (1 - \frac{\delta}{4N^2})^{1/2}$, $M = \sum_{j=1}^{N} \max_{x_j \in K_j} \|x_j\|$ and $C$ denotes the bound in Lemma 3.

**Proof.** Using the definitions of $v_i^{k+1}$ and $\hat{v}_i^k$ given in Eqs. (11) and (9), respectively, we have

$$
v_i^{k+1} = \sum_{j=1}^{N} w_{ij}(k) v_j^k + x_i^{k+1} - x_i^k,
$$

which through an iterative recursion leads to

$$
v_i^{k+1} = \sum_{j=1}^{N} w_{ij}(k) \left( \sum_{\ell=1}^{N} w_{j\ell}(k-1) v_\ell^{k-1} + x_j^{k-1} - x_j^{k-1} \right) + x_i^{k+1} - x_i^k
$$

$$
= \sum_{\ell=1}^{N} [\Phi(k, k-1)]_i^{\ell} v_\ell^{k-1} + \sum_{j=1}^{N} [\Phi(k, k)]_{ij} \left( x_j^k - x_j^{k-1} \right) + x_i^{k+1} - x_i^k
$$

$$
= \cdots
$$

$$
= \sum_{\ell=1}^{N} [\Phi(k, 0)]_i^{\ell} v_\ell^{0} + \sum_{s=1}^{k} \left( \sum_{j=1}^{N} [\Phi(k, s)]_{ij} (x_j^s - x_j^{s-1}) \right) + x_i^{k+1} - x_i^k.
$$

The preceding relation can be rewritten as:

$$
v_i^{k+1} - x_i^{k+1} + x_i = \sum_{\ell=1}^{N} [\Phi(k, 0)]_i^{\ell} v_\ell^{0} + \sum_{s=1}^{k} \left( \sum_{j=1}^{N} [\Phi(k, s)]_{ij} (x_j^s - x_j^{s-1}) \right).
$$

By the definition of $v_i^{k+1}$ in Eq. (11), we have $\hat{v}_i^k = v_i^{k+1} - x_i^{k+1} + x_i^k$, through which we get

$$
\hat{v}_i^k = \sum_{\ell=1}^{N} [\Phi(k, 0)]_i^{\ell} v_\ell^{0} + \sum_{s=1}^{k} \left( \sum_{j=1}^{N} [\Phi(k, s)]_{ij} (x_j^s - x_j^{s-1}) \right).
$$

(14)

Now, consider $y^k$ which may be written as follows:

$$
y^k = y^{k-1} + (y^k - y^{k-1}) = \cdots = y^0 + \sum_{s=1}^{k} (y^s - y^{s-1}).
$$

By Lemma 2 we have $y^s = \frac{1}{N} \sum_{j=1}^{N} x_j^s$ for all $s \geq 0$, which implies

$$
y^k = y^0 + \sum_{s=1}^{k} \sum_{j=1}^{N} \frac{1}{N} (x_j^s - x_j^{s-1}) = \sum_{\ell=1}^{N} \frac{1}{N} v_\ell^{0} + \sum_{s=1}^{k} \sum_{j=1}^{N} \frac{1}{N} (x_j^s - x_j^{s-1}),
$$

(15)
where the last equality follows by the definition of \( y^0 \) (see (12)).

From relations (14) and (15) we have

\[
\|y^k - \hat{v}_i^k\| = \left\| \sum_{\ell=1}^N \left( \frac{1}{N} - [\Phi(k, 0)]_{i\ell} \right) v^0_\ell + \sum_{s=1}^k \sum_{j=1}^N \left( \frac{1}{N} - [\Phi(k, s)]_{ij} \right) \left( x^s_j - x^{s-1}_j \right) \right\|
\]

\[
\leq \sum_{\ell=1}^N \left( \frac{1}{N} - [\Phi(k, 0)]_{i\ell} \right) \|v^0_\ell\| + \sum_{s=1}^k \sum_{j=1}^N \left( \frac{1}{N} - [\Phi(k, s)]_{ij} \right) \|x^s_j - x^{s-1}_j\|
\]

\[
\leq \sum_{\ell=1}^N \beta^k \|v^0_\ell\| + \sum_{s=1}^k \sum_{j=1}^N \beta^{k-s} \|x^s_j - x^{s-1}_j\| \tag{16}
\]

where the last inequality follows from \( \left| \frac{1}{N} - [\Phi(k, s)]_{ij} \right| \leq \theta \beta^{k-s} \) for all \( 0 \leq s \leq k \) (cf. Lemma 1).

Now, we estimate \( \|x^s_i - x^{s-1}_i\| \). From relation (10) we see that for any \( s \geq 1 \),

\[
\|x^s_i - x^{s-1}_i\| = \|\Pi_{K_i}[x^{s-1}_i - \alpha_{s-1} F_i(x^{s-1}_i, N \hat{v}_i^{s-1})] - x^s_i\|
\]

\[
\leq \|x^{s-1}_i - \alpha_{s-1} F_i(x^{s-1}_i, N \hat{v}_i^{s-1}) - x^{s-1}_i\|
\]

\[
= \alpha_{s-1} \|F_i(x^{s-1}_i, N \hat{v}_i^{s-1})\|
\]

\[
\leq C \alpha_{s-1}, \tag{17}
\]

where the first inequality follows by the non-expansive property of projection map, and the second inequality follows by Lemma 3. Combining (17) and (16), we have

\[
\|y^k - \hat{v}_i^k\| \leq \theta \beta^k \sum_{\ell=1}^N \|v^0_\ell\| + \theta N \sum_{s=1}^k \beta^{k-s} \alpha_{s-1} C \leq \theta \beta^k M + \theta NC \sum_{s=1}^k \beta^{k-s} \alpha_{s-1},
\]

where in the last inequality, we use \( v^0_\ell = x^0_\ell \in K_\ell \) and \( M = \sum_{\ell=1}^N \max_{x_i \in K_\ell} \|x^0_\ell\| \), which is finite since each \( K_\ell \) is a compact set (cf. Assumption 1). \( \square \)

From the right hand side of the expression in Lemma 4 it is apparent that the parameter for network connectivity, \( Q \) (cf. Assumption 1) determines the rate of convergence of player’s estimate of the aggregate to the actual aggregate. If the network connectivity is poor, \( Q \) is large implying \( \beta \) is close to 1 resulting in a slower convergence rate.

### 3.3 Convergence theory

In this subsection, under our assumptions, we prove that the sequence produced by the proposed algorithm does indeed converge to the unique Nash equilibrium, which exists by Proposition 1. Our next proposition provides the main convergence result for the algorithm. Prior to providing this result, we state two lemmas that will be employed in proving the required result, the first being a supermartingale convergence result (see for example [44, Lemma 11, Pg. 50]) and the second being [47, Lemma 3.1(b)].

**Lemma 5.** Let \( V_k, u_k, \beta_k \) and \( \gamma_k \) be non-negative random variables adapted to some \( \sigma \)-algebra \( F_k \). If almost surely \( \sum_{k=0}^\infty u_k < \infty \), \( \sum_{k=0}^\infty \beta_k < \infty \), and

\[
\mathbb{E}[V_{k+1} \mid F_k] \leq (1 + u_k)V_k - \gamma_k + \beta_k \quad \text{for all } k \geq 0,
\]

then almost surely \( V_k \) converges and \( \sum_{k=0}^\infty \gamma_k < \infty \).
Lemma 6. [47, Lemma 3.1(b)] Let \( \{\zeta_k\} \) be a non-negative scalar sequence. If \( \sum_{k=0}^{\infty} \zeta_k < \infty \) and \( 0 < \beta < 1 \), then \( \sum_{k=0}^{\infty} \left( \sum_{s=0}^{k} \beta^{k-s} \zeta_s \right) < \infty \).

In what follows, we use \( x^k \) to denote the vector with components \( x^k_i, i = 1, \ldots, N \), i.e., \( x^k = (x^k_1, \ldots, x^k_N) \) and, similarly, we write \( x^* \) for the vector \( (x^*_1, \ldots, x^*_N) \).

Proposition 2. Let Assumptions 1–6 hold. Then, the sequence \( \{x^k\} \) generated by the method (10)–(11) converges to the (unique) solution \( x^* \) of VI(\( K, \phi \)).

Proof. By Proposition 1, VI(\( K, \phi \)) has a unique solution \( x^* \in K \). When \( x^* \) solves the variational inequality problem VI(\( K, \phi \)), the following relation holds \( x^* = \Pi_{K_i}[x^*_i - \alpha_k F_i(x^*_i, \bar{x}^*)] \) (see [11 Proposition 1.5.8, p. 83]). From this relation and the non-expansive property of projection operator, we see that

\[
\|x^k_{i+1} - x^*_i\|^2 = \|\Pi_{K_i}[x^k_i - \alpha_k F_i(x^k_i, N\hat{v}^k_i)] - x^*_i\|^2 \\
= \|\Pi_{K_i}[x^k_i - \alpha_k F_i(x^k_i, N\hat{v}^k_i)] - \Pi_{K_i}[x^*_i - \alpha_k F_i(x^*_i, \bar{x}^*)]\|^2 \\
\leq \|x^k_i - x^*_i - \alpha_k (F_i(x^k_i, N\hat{v}^k_i) - F_i(x^*_i, \bar{x}^*))\|^2.
\]

By expanding the last term, we obtain the following expression:

\[
\|x^k_{i+1} - x^*_i\|^2 \leq \|x^k_i - x^*_i\|^2 + \alpha_k^2 \|F_i(x^k_i, N\hat{v}^k_i) - F_i(x^*_i, \bar{x}^*)\|^2 \\
- 2\alpha_k \left(F_i(x^k_i, N\hat{v}^k_i) - F_i(x^*_i, \bar{x}^*)\right)^T(x^k_i - x^*_i). \tag{18}
\]

To estimate Term 1, we use the triangle inequality and the identity \( (a + b)^2 \leq 2(a^2 + b^2) \), which yields

\[
\text{Term 1} \leq 2\|F_i(x^k_i, N\hat{v}^k_i)\|^2 + 2\|F_i(x^*_i, \bar{x}^*)\|^2 \leq \tilde{C} \quad \text{with} \quad \tilde{C} = 2C^2 + 2 \max_{(x_i, \bar{x}) \in K_i \times K} \|F_i(x_i, \bar{x})\|^2,
\]

where \( C \) is such that \( \|F_i(x^k_i, N\hat{v}^k_i)\| \leq C \) for all \( i \) and \( k \) (cf. Lemma 3) and \( \max_{(x_i, \bar{x}) \in K_i \times \bar{K}} \|F_i(x_i, \bar{x})\| \) is finite by Assumption 1. Next, we consider Term 2. By adding and subtracting \( F_i(x^k_i, Ny^k) \) in

\[
\text{Term 2} = \left(F_i(x^k_i, N\hat{v}^k_i) - F_i(x^k_i, Ny^k)\right)^T(x^k_i - x^*_i) + \left(F_i(x^k_i, Ny^k) - F_i(x^*_i, \bar{x}^*)\right)^T(x^*_i - x^*_i).
\]

By applying the Cauchy-Schwarz inequality, i.e. \( a^T b \geq -\|a\||b| \), to the first term on the right hand side of the preceding relation and the Lipschitz continuity of \( F_i(x_i, u) \) in \( u \) (cf. Assumption 3), we see that

\[
(F_i(x^k_i, N\hat{v}^k_i) - F_i(x^k_i, Ny^k))^T(x^k_i - x^*_i) \geq -\|F_i(x^k_i, N\hat{v}^k_i) - F_i(x^k_i, Ny^k)\| \cdot \|x^k_i - x^*_i\| \\
\geq -L_i N \|\hat{v}^k_i - y^k\| \cdot \|x^k_i - x^*_i\| \\
\geq -2L_i MN \|\hat{v}^k_i - y^k\|,
\]

where

\[
L_i = \max_{(x_i, \bar{x}) \in K_i \times K} \|F_i(x_i, \bar{x})\|,
\]

and

\[
N = \max_{x_i \in K_i} \|x_i\|.
\]
where in the last inequality we use $x_i^k, x_i^* \in K_i$ and the compactness of $K_i$ (cf. Assumption 1) and $M \geq \max_{x_i \in K_i} \|x_i\|$ for all $i$. Therefore, we have

**Term 2**

$$2 \geq -2L_iMN\|\hat{v}_i^k - y\| + \left( F_i(x_i^k, Ny^k) - F_i(x_i^*, \bar{x}) \right)^T (x_i^k - x_i^*).$$

By substituting the preceding estimates of **Term 1** and **Term 2** in (8), we obtain

$$\|x_i^{k+1} - x_i^*\| \leq \|x_i^k - x_i^*\| + N\tilde{C}\alpha_k^2 + 4\alpha_kMN \sum_{i=1}^N \bar{L}_i\|\hat{v}_i^k - y\|$$

$$- 2\alpha_k \left( F_i(x_i^k, Ny^k) - F_i(x_i^*, \bar{x}) \right)^T (x_i^k - x_i^*).$$

Summing over all agents from $i = 1$ to $i = N$, yields

$$\sum_{i=1}^N \|x_i^{k+1} - x_i^*\| \leq \sum_{i=1}^N \|x_i^k - x_i^*\| + N\tilde{C}\alpha_k^2 + 4\alpha_kMN \sum_{i=1}^N \bar{L}_i\|\hat{v}_i^k - y\|$$

$$- 2\alpha_k \left( \phi(x^k) - \phi(x^*) \right)^T (x^k - x^*),$$

where we also use the fact that $F_i(x_i, \bar{x})$ is a coordinate map for the mapping $\phi(x) = F(x, \bar{x})$ (see (7) and (5)). To claim that the sequence $\{x_i\}$ converges to $x^*$, we apply Lemma 1 for the deterministic sequences) to relation (19). To apply this lemma, since $\sum_{k=0}^\infty \alpha_k^2 < \infty$ by Assumption 6, we only need to prove

$$\sum_{k=0}^\infty \alpha_k\|\hat{v}_i^k - y\| < \infty \quad \text{for all } i \in \mathcal{N}. \quad (20)$$

In view of Lemma 4 we have

$$\|y^k - \hat{v}_i^k\| \leq \theta \beta_k M + \theta NC \sum_{s=1}^k \beta^{k-s}\alpha_{s-1} \quad \text{for all } i \in \mathcal{N} \text{ and all } k \geq 1,$$

so it suffices to prove that

$$\sum_{k=1}^\infty \alpha_k \left( \sum_{s=1}^k \beta^{k-s}\alpha_{s-1} \right) < \infty \quad \text{and} \quad \sum_{k=1}^\infty \alpha_k \beta^k < \infty.$$

Using $\alpha_k \leq \alpha_s$ for all $k \geq s$ (Assumption 4), for the series $\sum_{k=1}^\infty \alpha_k \left( \sum_{s=1}^k \beta^{k-s}\alpha_{s-1} \right)$ we have

$$\sum_{k=1}^\infty \alpha_k \left( \sum_{s=1}^k \beta^{k-s}\alpha_{s-1} \right) = \sum_{k=1}^\infty \left( \sum_{s=1}^k \beta^{k-s}\alpha_{s-1} \right) \leq \sum_{k=1}^\infty \left( \sum_{s=1}^k \beta^{k-s}\alpha_{s-1}^2 \right).$$
We now use Lemma 6 from which by letting $\zeta = \alpha^2$ we can see that $\sum_{k=1}^{\infty} \alpha_k \left( \sum_{s=1}^{k} \beta^{k-s} \alpha_{s-1} \right) < \infty$. To establish the convergence of $\sum_{k=0}^{\infty} \alpha_k \beta^k$, we note that $\alpha_k \leq \alpha_0$ (Assumption 6), implying that $\sum_{k=0}^{\infty} \alpha_k \beta^k \leq \alpha_0 \sum_{k=0}^{\infty} \beta^k < \infty$ since $0 < \beta < 1$. Thus, relation (20) is valid.

As relation (19) satisfies the conditions of (the deterministic case of) Lemma 5, it follows that

$$\{\|x^k - x^*\|\} \text{ is a convergent sequence},$$

and

$$\sum_{k=0}^{\infty} \alpha_k (\phi(x^k) - \phi(x^*))^T (x^k - x^*) < \infty.$$  \tag{22}$$

Since $\{x^k\} \subset K$ and $K$ is compact (Assumption 1), $\{x^k\}$ has accumulation points in $K$. By (22) and $\sum_{k=0}^{\infty} \alpha_k = \infty$ it follows that $(\phi(x^k) - \phi(x^*))^T (x^k - x^*) \to 0$ along a subsequence, say $\{x^{k_l}\}$.

This observation, together with the strict monotonicity of $\phi$, implies that $\{x^{k_l}\} \to x^*$ as $l \to \infty$. To claim $\|x^k - x^*\| \to 0$, we proceed by contradiction and assume that $\|x^k - x^*\|$ does not converge to 0. Then there exists an $\epsilon > 0$ and a subsequence $\{x_{n_l}\}$ such that $\|x_{n_l} - x^*\| \geq \epsilon$ for $l$ sufficiently large. But since the set is bounded, there exists a further subsequence which converges to some $\tilde{x}$. By relation (21), the sequence $\{x^k\}$ is convergent and therefore, it must be that $\tilde{x} = x^*$ and the entire sequence $\{x^k\}$ must converge to $x^*$.

Though it is difficult to make a statement on the rate of convergence for the result of Proposition 2, the network connectivity plays an important role in determining the rate as already discussed for the results of Lemma 4. Indeed, if $\beta$ is close to 1, which is the case for a network with poor connectivity, players take longer to converge on their estimate of the true aggregate, thereby taking it longer to converge on their optimal decision.

4 Distributed Asynchronous Algorithm

In this section, we propose a distributed gossip-based algorithm for computing an equilibrium of aggregative Nash game, as defined by (2). In a gossip protocol, the information is propagated by running a round of information exchange between a randomly chosen player who communicates with another player chosen at random. A more detailed description of the algorithm and some preliminary results are provided in section 4.1. The global convergence of the algorithm is examined in section 4.2 while constant steplength error bounds are provided in section 4.3.

4.1 Outline of Algorithm

In the proposed algorithm, agents perform their estimate and iterate updates the same as in the synchronous algorithm (10)–(11), but the updates occur asynchronously. As a mechanism for generating asynchronous updates we employ the gossip model for agent communications [7]. Together with the asynchronous updates, we allow the agents to use uncoordinated stepsize values by letting each agent choose a stepsize based on its own information-update frequency. To accommodate these updates and stepsize selections, we model the agent connectivity structure by an undirected static graph $\mathcal{G}(\mathcal{N}, \mathcal{E})$, with node $i \in \mathcal{N}$ being agent $i$ and $\mathcal{E}$ being the set of undirected edges among the agents. When $\{i, j\} \in \mathcal{E}$, the agents $i$ and $j$ may talk to each other. We let $\mathcal{N}_i$ denote the set of neighbors of agent $i$, i.e., $\mathcal{N}_i = \{j \mid \{i, j\} \in \mathcal{E}\}$. We use the following assumption for the graph $\mathcal{G}(\mathcal{N}, \mathcal{E})$. 
Assumption 7. The undirected graph $G(N, E)$ is connected.

We use a gossip protocol to model agent communication and exchange of the estimates of the aggregate $\bar{x}$. In this model, each agent is assumed to have a local clock which ticks according to a Poisson process with rate 1. At a tick of its clock, an agent $i$ wakes up and contacts its neighbor $j \in N_i$ with probability $p_{ij}$. The agents’ clocks processes can be equivalently modeled as a single (virtual) clock which ticks according to a Poisson process with rate $N$. We assume that only one agent wakes up at each tick of the global clock, and we let $Z^k$ denote $k$th tick time of the global Poisson process. We discretize time so that instant $k$ corresponds to the time-slot $[Z^{k-1}, Z^k)$. At each time $k$, every agent $i$ has its iterate $x^k_i$ and estimate $v^k_i$ of the average of the current aggregate.

We let $I^k$ denote the agent whose clock ticked at time $k$ and we let $J^k$ be the agent contacted by the agent $I^k$, where $J^k$ is a neighbor of agent $I^k$, i.e., $J^k \in N_{I^k}$. At time $k$, agents $I^k$ and $J^k$ exchange their estimates $v^k_{I^k}$ and $v^k_{J^k}$ and compute intermediate estimates:

$$\hat{v}^k_i = \frac{v^k_{I^k} + v^k_{J^k}}{2} \quad \text{for } i \in \{I^k, J^k\},$$

and update their iterates and estimates of the aggregate average, as follows:

$$\begin{align*}
  x^{k+1}_i &:= \Pi_{K_i} [x^k_i - \alpha_{k,i} F_i(x^k_i, N\hat{v}^k_i)] \\
v^{k+1}_i &:= \hat{v}^k_i + x^{k+1}_i - x^k_i
\end{align*} \quad \text{for } i \in \{I^k, J^k\},$$

where $\alpha_{k,i}$ is the stepsize for agent $i$ and $F_i(x, y) = \nabla_x f_i(x, y)$. The other agents do nothing, i.e.,

$$\begin{align*}
  \hat{v}^k_i &= v^k_i, \\
x^{k+1}_i &= x^k_i, \quad \text{and} \quad v^{k+1}_i = v^k_i \quad \text{for } i \notin \{I^k, J^k\}.
\end{align*}$$

As seen from the preceding update relations, the agents perform the same updates as in the synchronous algorithm (10)–(11), but instead of all agents updating, only two randomly selected agents update their estimates and iterates, while the other agents do not update.

We now rewrite the update steps more compactly. To capture the step in (23), we define the weight matrix $W(k)$:

$$W(k) = \mathbb{I} - \frac{1}{2}(e_{I^k} - e_{J^k})(e_{I^k} - e_{J^k})^T,$$

where $\mathbb{I}$ stands for the identity matrix, $e_i$ is $N$-dimensional vector with $i$th entry equal to 1, and the other entries equal to 0. By using $W(k)$ we can rewrite the intermediate estimate update (23),
as follows: for all \( i = 1, \ldots, N \),
\[
\hat{v}_i^k = \sum_{j=1}^{N} [W(k)]_{ij} v_j^k \quad \text{for all } k \geq 1, \quad \text{with } v_i^0 = x_i^0,
\] (27)

where \( x_i^0 \in K_i, i = 1, \ldots, N \), are initial (random) agent decisions. To rewrite the iterate \( x_i^{k+1} \) update (or no update) compactly for all agents, we let \( 1_{\{i \in S\}} \) denote the indicator of the event \( \{i \in S\} \). Then, the update relations in (24) and (25) can be written as:
\[
x_i^{k+1} = \left( \Pi_{K_i} [x_i^k - \alpha_{k,i} F_i(x_i^k, N \hat{v}_i^k)] - x_i^k \right) 1_{\{i \in \{I^k, J^k\}\}} + x_i^k,
\] (28)
\[
v_i^{k+1} = \hat{v}_i^k + x_i^{k+1} - x_i^k,
\] (29)

Note that only agents \( i \in \{I^k, J^k\} \) update since \( 1_{\{i \in \{I^k, J^k\}\}} = 0 \) when \( i \notin \{I^k, J^k\} \) and, hence, \( x_i^{k+1} = x_i^k \) and \( v_i^{k+1} = \hat{v}_i^k \) with \( \hat{v}_i^k = v_i^k \) (by (27)).

We allow agents to use uncoordinated stepsizes that are based on the frequency of the agent updates. Specifically, agent \( i \) uses the stepsize \( \alpha_{k,i} = 1/\Gamma_k(i) \), where \( \Gamma_k(i) \) denotes the number of updates that agent \( i \) has executed up to and including at time \( k \). These stepsizes are of the order of \( 1/k \) in a long run \([35, 33]\). To formalize this result, we need to introduce the probabilities of agents updates. We let \( p_i \) denote the probability of the event that agent \( i \) updates, i.e. \( \{i \in \{I^k, J^k\}\} \), for which we have
\[
p_i = \frac{1}{N} \left( 1 + \sum_{j \in N_i} p_{ji} \right) \quad \text{for all } i \in N,
\]

where \( p_{ji} > 0 \) is the probability that agent \( i \) is contacted by its neighbor \( j \). The long term estimates for \( \alpha_{k,i} \) that we use in our analysis are given in the following lemma (cf. \([33]\), Lemma 3), the proof of which is provided in the appendix for completeness.

**Lemma 7.** Let Assumption \([7]\) hold, and let \( \hat{p} = 1 + \min_{\{i,j\} \in \mathcal{E}} p_{ij} \) and \( \alpha_{k,i} = 1/\Gamma_k(i) \) for all \( k \) and \( i \). Then, for any \( q \in (0, 1/2) \) and for every \( \omega \in \Omega \), there exists a sufficiently large \( \tilde{k}(\omega) = \tilde{k}(q,N) \) such that we have for all \( k \geq \tilde{k}(\omega) \) and \( i \in N \),
\[
\alpha_{k,i} \leq 2/kp_i \quad \text{and} \quad \left| \alpha_{k,i} - \frac{1}{kp_i} \right| \leq \frac{2}{k^{3/2-4q}\hat{p}^2}.
\]

Note that \( \tilde{k}(\omega) \) is contingent on the sample path corresponding to \( \omega \). More precisely, we claim the following:
\[
P \left[ \omega : \alpha_{k,i} \leq 2/kp_i \quad \text{for } k \geq \tilde{k}(\omega) \right] = 1.
\]

Based on Lemma \([7]\) we provide the next corollary without a proof.

**Corollary 1.** Let Assumption \([7]\) hold, and let \( \hat{p} = 1 + \min_{\{i,j\} \in \mathcal{E}} p_{ij} \) and \( \alpha_{k,i} = 1/\Gamma_k(i) \) for all \( k \) and \( i \). Then for all \( i \in N \), the following hold with probability one:
\[
(a) \sum_{k=1}^{\infty} \frac{\alpha_{k,i}}{k} < \infty; \quad (b) \sum_{k=1}^{\infty} \alpha_{k,i}^2 < \infty; \quad (c) \sum_{k=1}^{\infty} \left| \alpha_{k,i} - \frac{1}{kp_i} \right| < \infty.
\]
Another useful result is provided in [43, Lemma 1], and stated below in a suitable form.

**Lemma 8.** Let $G(N, E)$ be a graph that satisfies Assumption 7. Let $W$ be an $N \times N$ random stochastic matrix such that $E[W]$ is doubly stochastic and $E[W]_{ij} > 0$ whenever $\{i, j\} \in E$. Furthermore, let the diagonal elements of $W$ be positive almost surely. Then, for the matrix $D = W - \frac{1}{N}I1^TW$, there exists a scalar $\lambda \in (0, 1)$ such that

$$E[\|Dz\|^2] \leq \lambda \|z\|^2 \quad \text{for all } z \in \mathbb{R}^N.$$  

The random matrices $W(k)$ in (26) are in fact doubly stochastic and thus, $\tilde{W} = E[W(k)]$ is doubly stochastic. Moreover, it can be easily seen that $\tilde{W}_{ij} > 0$ whenever $\{i, j\} \in E$. In addition, $W(k)$ has positive diagonal entries. Hence, Lemma 8 applies to random matrices $W(k)$. However, since each $W(k)$ is in fact doubly stochastic, we have $1^TW(k) = 1^T$, implying that $W(k) - \frac{1}{N}I1^TW(k) = W(k) - \frac{1}{N}I1^T$. Hence, using this observation and Lemma 8, we find that there exists $\lambda \in (0, 1)$ such that for the matrix $D(k) = W(k) - \frac{1}{N}I1^T$ we have

$$E[\|D(k)z\|^2] \leq \lambda \|z\|^2 \quad \text{for all } z \in \mathbb{R}^N. \quad (30)$$

By Jensen’s inequality we have $|E[X]| \leq \sqrt{E[X^2]}$ for any random variable $X$ (with a finite expectation), which when applied to relation (30) yields

$$E[\|D(k)z\|] \leq \sqrt{\lambda} \|z\| \quad \text{for all } z \in \mathbb{R}^N. \quad (31)$$

The value of second largest eigenvalue $\lambda$ controls the rate at which information is dispersed over the network. A network with large $\lambda$ will have players agreeing faster on their estimate of the aggregate than a network with a smaller $\lambda$. In Section 4 we consider a variety of networks to demonstrate the impact of network topology through $\lambda$ on the rate of convergence.

### 4.2 Convergence Theory

In this section we establish the convergence of the asynchronous algorithm (27)–(29) with the agent specific diminishing stepsize of the form $\alpha_{k,i} = \frac{1}{\lambda(k)}$. To take account of the history, we introduce $F_k$ to denote the $\sigma$–algebra generated by the entire history up to $k$. More precisely

$$F_k = F_0 \cup \{I^l, J^l; 1 \leq l \leq k - 1\} \quad \text{for all } k \geq 2,$$

with $F_1 = F_0 \{x_i^0, i \in N\}$. Thus, given $F_k$, the vectors $v_i^k$ and $x_i^k$ are fully determined. First we state several result which we will use to claim the convergence of the algorithm, as well as to analyze the error bounds.

In what follows, we will use a vector-component based analysis. To this end, we introduce $[z]_\ell$ to denote the $\ell$-th component of a vector $z \in \mathbb{R}^n$, with $\ell = 1, \ldots, n$. A component-wise update of each $v_i^{k+1}$ in (29) is given by: for all $i = 1, \ldots, N$,

$$[v_i^{k+1}]_\ell = \sum_{j=1}^N [W(k)]_{ij}[v_j^k]_\ell + [x_i^k - x_i^k]_\ell \quad \text{for } \ell = 1, \ldots, n.$$

We collect all $\ell$th coordinates of the vectors $v_1^k, \ldots, v_N^k$ and let $v^k(\ell) = ([v_1^k]_\ell, \ldots, [v_N^k]_\ell)^T$. We similarly do for the vectors $x_1^k, \ldots, x_N^k$ and let $x^k(\ell) = ([x_1^k]_\ell, \ldots, [x_N^k]_\ell)^T$. Using the vectors $v^k(\ell)$ and $x^k(\ell)$, we can rewrite the preceding relation as follows:

$$v^{k+1}(\ell) = W(k)v^k(\ell) + \zeta^{k+1}(\ell) \quad \text{with } \zeta^{k+1}(\ell) = x^{k+1}(\ell) - x^k(\ell) \quad \text{for all } \ell = 1, \ldots, n. \quad (32)$$

We have the following result for $v^{k+1}(\ell)$ for any $\ell$. 

Lemma 9. Let Assumptions $\mathbb{A}$ and Assumption $\mathbb{B}$ hold. Then, for all $\ell = 1, \ldots, n$ and $k \geq 0$,

$$\|v^{k+1}(\ell) - [y^{k+1}]_\ell\| \leq \|D(k)(v^k(\ell) - [y^k]_\ell\| + \sqrt{2C} \sum_{i=1}^{N} \alpha_{k,i},$$

where $D(k) = W(k) - \frac{1}{N}11^T$ and $C$ is a constant as in Lemma $\mathbb{B}$.

Proof. We fix an arbitrary coordinate $\ell$. By the decision update rule of (28), the $i$th coordinate of the vector $\zeta^{k+1}(\ell)$ is $[\zeta^{k+1}(\ell)]_i = \left([\Pi_{K_i} [x^k_i - \alpha_{k,i}F_i(x^k_i, N\hat{v}^k_i)] - x^k_i] 1_{i \in \{I_k, J_k^i\}}\right)_\ell$. Since $y^{k+1}$ is the average of the vectors $v_i^{k+1}$, from (32) for the $\ell$th coordinate of this vector we obtain

$$[y^{k+1}]_\ell = \frac{1}{N}v^{k+1}(\ell) = \frac{1}{N} \left(1^TW(k)v^k(\ell) + 1^T\zeta^{k+1}(\ell)\right),$$

which together with (32) leads us to

$$v^{k+1}(\ell) - [y^{k+1}]_\ell 1 = \left(W(k) - \frac{1}{N}11^TW(k)\right)v^k(\ell) + \left(I - \frac{1}{N}11^T\right)\zeta^{k+1}(\ell),$$

where $I$ is the identity matrix. Note that each $W(k)$ is a doubly stochastic matrix i.e., $W(k)1 = 1$ and $1^TW(k) = 1^T$. Thus, $\frac{1}{N}11^TW(k) = \frac{1}{N}11^T$. Furthermore, we have $(W(k) - \frac{1}{N}11^T)1 = 0$, implying that

$$\left(W(k) - \frac{1}{N}11^T\right) [y^k]_\ell 1 = 0.$$

By combining the preceding two relations, using $\frac{1}{N}11^TW(k) = \frac{1}{N}11^T$, and letting $D(k) = W(k) - \frac{1}{N}11^T$, we obtain

$$v^{k+1}(\ell) - [y^{k+1}]_\ell 1 = D(k)(v^k(\ell) - [y^k]_\ell 1) + \left(I - \frac{1}{N}11^T\right)\zeta^{k+1}(\ell).$$

Taking norms, we obtain

$$\|v^{k+1}(\ell) - [y^{k+1}]_\ell 1\| \leq \|D(k)(v^k(\ell) - [y^k]_\ell 1)\| + \left\|\left(I - \frac{1}{N}11^T\right)\zeta^{k+1}(\ell)\right\|.$$  \hspace{1cm} (33)

We next estimate the last term in (33). The matrix $I - \frac{1}{N}11^T$ is a projection matrix (corresponds to the projection on the subspace orthogonal to the vector $1$), so $\|I - \frac{1}{N}11^T\| = 1$, implying that

$$\left\|\left(I - \frac{1}{N}11^T\right)\zeta^{k+1}(\ell)\right\| \leq \|I - \frac{1}{N}11^T\| \|\zeta^{k+1}(\ell)\| = \|\zeta^{k+1}(\ell)\|. \hspace{1cm} (34)$$

From the definition of $\zeta^{k+1}(\ell)$ in (32), we see that

$$\|\zeta^{k+1}(\ell)\|^2 = \sum_{i \in \{I_k, J_k^i\}} \left\|\Pi_{K_i} [x^k_i - \alpha_{k,i}F_i(x^k_i, N\hat{v}^k_i)] - x^k_i\right\|_{\ell}^2.$$  

Using the non-expansive property of the projection operator and

$$\alpha^2_{k,i} \leq \sum_{i \in \{I_k, J_k^i\}} \alpha^2_{k,i} \leq \sum_{i=1}^{N} \alpha^2_{k,i},$$

we have

$$\|v^{k+1}(\ell) - [y^{k+1}]_\ell 1\| \leq \|D(k)(v^k(\ell) - [y^k]_\ell 1)\| + \sqrt{2C} \sum_{i=1}^{N} \alpha_{k,i},$$

which completes the proof.
we have

$$\|\zeta^{k+1}(\ell)\|^2 \leq \sum_{i \in \{I^k, J^k\}} \|\alpha_{k,i}[F_i(x_i^k, N\hat{v}_i^k)]\ell\|^2 \leq \sum_{i=1}^{N} \alpha_{k,i}^2 \sum_{i \in \{I^k, J^k\}} \|F_i(x_i^k, N\hat{v}_i^k)\|^2.$$  

By Lemma 3, \(\|F_i(x_i^k, N\hat{v}_i^k)\| \leq C\) for all \(i, k\) and some \(C > 0\). This and \(|\{I^k, J^k\}| = 2\) imply \(\|\zeta^{k+1}(\ell)\|^2 \leq 2C^2 \sum_{i=1}^{N} \alpha_{k,i}^2\). By taking square roots we obtain

$$\|\zeta^{k+1}(\ell)\| \leq \sqrt{2C} \sqrt{\sum_{i=1}^{N} \alpha_{k,i}^2} \leq \sqrt{2C} \sum_{i=1}^{N} \alpha_{k,i}$$

which when combined with (33) and (34) yields

$$\|v^{k+1}(\ell) - [y^{k+1}]\ell 1\| \leq \|D(k)(v^k(\ell) - [y^k]\ell 1)\| + \sqrt{2C} \sum_{i=1}^{N} \alpha_{k,i}.$$  

\(\square\)

Our result involves the average \(y^k\) of the estimates \(v_i^k, i \in N\), which will be important in establishing the convergence of the algorithm.

**Lemma 10.** Let Assumptions 1-3 and Assumption 4 hold. Let \(v_i^k\) be given by (27) and (29), respectively, and let \(y^k = \frac{1}{N} \sum_{i=1}^{N} v_i^k\). Then, we have

$$\sum_{k=1}^{\infty} \frac{1}{k} \|v_i^k - y^k\|^2 < \infty \quad a.s. \text{ for all } i \in N.$$  

**Proof.** From Lemma 9 we obtain that the following holds for \(k \geq 0\) almost surely,

$$\|v^{k+1}(\ell) - [y^{k+1}]\ell 1\| \leq \|D(k)(v^k(\ell) - [y^k]\ell 1)\| + \sqrt{2C} \sum_{i=1}^{N} \alpha_{k,i}.$$  

By taking conditional expectations with respect to \(F_k\), we obtain that the following holds almost surely for all \(k\):

$$\mathbb{E}[\|v^{k+1}(\ell) - [y^{k+1}]\ell 1\| \mid F_k] \leq \mathbb{E}[\|D(k)(v^k(\ell) - [y^k]\ell 1)\| \mid F_k] + \sqrt{2C} \mathbb{E} \left[ \sum_{i=1}^{N} \alpha_{k,i} \mid F_k \right]. \tag{35}$$

Note that the expectation in the term on the right hand side is taken with respect to the randomness in the matrix \(W(k)\) only. By relation (31), we have

$$\mathbb{E}[\|D(k)(v^k(\ell) - [y^k]\ell 1)\| \mid F_k] \leq \sqrt{\lambda} \|v^k(\ell) - [y^k]\ell 1\|,$$

which combined with (35) yields that the following holds for all \(k\) in an almost sure sense:

$$\mathbb{E}[\|v^{k+1}(\ell) - [y^{k+1}]\ell 1\| \mid F_k] \leq \sqrt{\lambda} \|v^k(\ell) - [y^k]\ell 1\| + \sqrt{2C} \mathbb{E} \left[ \sum_{i=1}^{N} \alpha_{k,i} \mid F_k \right]. \tag{36}$$
By multiplying both sides of (33) with \( \frac{1}{k} \) and by using \( \frac{1}{k+1} < \frac{1}{k} \) we find that almost surely for all \( k \):

\[
\frac{1}{k+1} \mathbb{E}[\|v^{k+1}(\ell) - [y^{k+1}]_{\ell}1\| | \mathcal{F}_k] \leq \frac{\sqrt{\lambda}}{k} \|v^k(\ell) - [y^k]_{\ell}1\| + \frac{\sqrt{2} C \mathbb{E}[\sum_{i=1}^N \alpha_{k,i} | \mathcal{F}_k]}{k} = \frac{1}{k} \|v^k(\ell) - [y^k]_{\ell}1\| - \frac{1 - \sqrt{\lambda}}{k} \|v^k(\ell) - [y^k]_{\ell}1\| + \sqrt{2} C \mathbb{E} \left[ \sum_{i=1}^N \frac{\alpha_{k,i}}{k} | \mathcal{F}_k \right].
\]

Since \( \lambda \in (0, 1) \) we have \( 1 - \sqrt{\lambda} > 0 \) and \( \sum_k \mathbb{E} \left[ \sum_{i=1}^N \frac{\alpha_{k,i}}{k} | \mathcal{F}_k \right] < \infty \) since \( \sum_k \sum_{i=1}^N \frac{2}{k^2 p_i} < \infty \) with probability one from Corollary \( \text{II} \) allowing for invoking the supermartingale convergence lemma (Lemma \( \text{[5]} \)). Therefore, we may conclude that

\[
\sum_{k=1}^{\infty} \frac{1}{k} \|v^k(\ell) - [y^k]_{\ell}1\| < \infty \quad \text{a.s.}
\]

Recalling that \( v^k(\ell) = ([v^k_1]_{\ell}, \ldots, [v^k_N]_{\ell})^T \), the preceding relation implies that

\[
\sum_{k=1}^{\infty} \frac{1}{k} \left[ |v^k_i|_{\ell} - |y^k_i|_{\ell} \right] < \infty \quad \text{for all } i \in \mathcal{N} \text{ a.s.}
\]

The coordinate index \( \ell \) was arbitrary, so the relation is also true for every coordinate index \( \ell = 1, \ldots, n \). In particular, since \( \|v^k_i - y^k_i\| \leq \sum_{\ell=1}^n |[v^k_i]_{\ell} - |y^k_i|_{\ell}| \) we have

\[
\sum_{k=1}^{\infty} \frac{1}{k} \left\| v^k_i - y^k_i \right\| \leq \sum_{k=1}^{\infty} \frac{1}{k} \sum_{\ell=1}^n |[v^k_i]_{\ell} - |y^k_i|_{\ell}| < \infty \quad \text{for all } i \in \mathcal{N} \text{ a.s.}
\]

For the rest of the paper, we use \( x^k \) to denote the vector with components \( x^k_i, \) \( i = 1, \ldots, N, \) i.e., \( x^k = (x^k_1, \ldots, x^k_N) \) and we write \( x^* \) for the vector \( (x^*_1, \ldots, x^*_N) \). We now show the convergence of the algorithm. We have the following result, where \( x^* \) denotes the unique Nash equilibrium of the aggregative game in (2).

**Proposition 3.** Let Assumptions \( \text{[4]} \) and Assumption \( \text{[7]} \) hold. Then, the sequence \( \{x^k\} \) generated by the method (27)–(29) with the stepsize \( \alpha_{k,i} = \frac{1}{1+k(i)} \) converges to the (unique) \( x^* \) of the game almost surely.

**Proof.** Under strict monotonicity of the mapping and the compactness of \( K \), uniqueness of the equilibrium follows from Proposition \( \text{[4]} \). Then, by the definition of \( x^{k+1}_i \) we have

\[
\left\| x^{k+1}_i - x^*_i \right\|^2 = \left\| \Pi_K [x^k_i - \alpha_{k,i} F_i(x^k_i, N \hat{v}^k_i)] - x^*_i \right\|^2 + x^k_i - x^*_i \right\|^2.
\]
Using \( x_i^* = \Pi_{K_i} [x_i^* - \alpha_{k,i} F_i(x_i^*, \bar{x}^*)] \) and the non-expansive property of the projection operator, we have for \( i \in \{ I^k, J^k \} \),

\[
\| x_i^{k+1} - x_i^* \| \leq \| x_i^k - x_i^* - \alpha_{k,i} F_i(x_i^k, \hat{v}_i^k) - x_i^* + \alpha_{k,i} F_i(x_i^*, \bar{x}^*) \| \nn = \| x_i^k - x_i^* \|^2 + \alpha_{k,i}^2 \| F_i(x_i^k, \hat{v}_i^k) - F_i(x_i^*, \bar{x}^*) \|^2 + 2 \alpha_{k,i} \bigl( F_i(x_i^k, \hat{v}_i^k) - F_i(x_i^*, \bar{x}^*) \bigr)^T (x_i^k - x_i^*) \nn - 2 \alpha_{k,i} \bigl( F_i(x_i^k, \hat{v}_i^k) - F_i(x_i^*, \bar{x}^*) \bigr)^T (x_i^k - x_i^*) \nn - 2 \| (F_i(x_i^k, \hat{v}_i^k) - F_i(x_i^*, \bar{x}^*)) \|^2 (x_i^k - x_i^*)\]

By expressing \( \alpha_{k,i} = \alpha_{k,i} - \frac{1}{k_{pi}} \) and \( \alpha_{k,i} = \frac{1}{k_{pi}} \), we have the following for all \( k \):

\[
\| x_i^{k+1} - x_i^* \|^2 = \| x_i^k - x_i^* \|^2 + \alpha_{k,i}^2 \| F_i(x_i^k, \hat{v}_i^k) - F_i(x_i^*, \bar{x}^*) \|^2 \nn - 2 \alpha_{k,i} \bigl( F_i(x_i^k, \hat{v}_i^k) - F_i(x_i^*, \bar{x}^*) \bigr)^T (x_i^k - x_i^*) \nn - \frac{2}{k_{pi}} \| (F_i(x_i^k, \hat{v}_i^k) - F_i(x_i^*, \bar{x}^*)) \|^2 (x_i^k - x_i^*)\]

By Lemma 3 and Assumption 1, we can see that \( \| F_i(x_i^k, \hat{v}_i^k) - F_i(x_i^*, \bar{x}^*) \|^2 \leq C_1 \) for some scalar \( C_1 \), and for all \( i \) and \( k \). Similarly, for the term in (37) involving the absolute value, we can see that \( \| (F_i(x_i^k, \hat{v}_i^k) - F_i(x_i^*, \bar{x}^*))^T (x_i^k - x_i^*) \| \leq C_2 \) for some scalar \( C_2 \), and for all \( i \) and \( k \). Substituting these estimates in (37), we obtain

\[
\| x_i^{k+1} - x_i^* \|^2 \leq \| x_i^k - x_i^* \|^2 + \alpha_{k,i}^2 C_1 + 2C_2 \biggl| \alpha_{k,i} - \frac{1}{k_{pi}} \biggr| \nn - \frac{2}{k_{pi}} \| (F_i(x_i^k, \hat{v}_i^k) - F_i(x_i^*, \bar{x}^*)) \|^2 (x_i^k - x_i^*)\]

For the last term in the preceding relation, by adding and subtracting \( F_i(x_i^k, N \hat{v}_i^k) \) and using \( N y_i = \sum_{i=1}^N x_i^k = \bar{x}^k \) (cf. Lemma 2), we write

\[
(F_i(x_i^k, N \hat{v}_i^k) - F_i(x_i^*, \bar{x}^*))^T (x_i^k - x_i^*) = (F_i(x_i^k, N \hat{v}_i^k) - F_i(x_i^k, N y_i^k))^T (x_i^k - x_i^*) \nn + (F_i(x_i^k, \bar{x}^k) - F_i(x_i^*, \bar{x}^*))^T (x_i^k - x_i^*) \nn \geq - \| F_i(x_i^k, N \hat{v}_i^k) - F_i(x_i^k, N y_i^k) \| \| x_i^k - x_i^* \| + (F_i(x_i^k, \bar{x}^k) - F_i(x_i^*, \bar{x}^*))^T (x_i^k - x_i^*) \nn \geq - L_i \| \hat{v}_i^k - y_i^k \| M + (F_i(x_i^k, \bar{x}^k) - F_i(x_i^*, \bar{x}^*))^T (x_i^k - x_i^*)\]

where we use the Lipschitz property of the mapping \( F_i \) (Assumption 3), while \( M \) is a constant such that \( \max_{x_i, z_i \in K_i} \| x_i - z_i \| \leq M \) for all \( i \). The vector \( \hat{v}_i^k \) is a convex combination of \( \hat{v}_j^k \) over \( j = 1, \ldots, N \) (cf. (27)). Therefore, by the convexity of the norm, we have \( \| \hat{v}_j^k - y_j^k \| \leq \sum_{j=1}^N [W(k)]_{ij} \| v_j^k -
\[ y^k, \text{ which yields} \]
\[
(F_i(x_i^k, Nv_i^k) - F_i(x_i^*, \bar{x}^*))^T (x_i^k - x_i^*) \geq -\bar{L}_i N M \sum_{j=1}^N [W(k)]_{ij} \|v_j^k - y^k\| \\
+ (F_i(x_i^k, \bar{x}^k) - F_i(x_i^*, \bar{x}^*))^T (x_i^k - x_i^*). \tag{39}
\]

Finally, by combining relations (38) and (39) we obtain for \(i \in \{I^k, J^k\}\) and for all \(k \geq 0\),
\[
\|x_i^{k+1} - x_i^*\|^2 \leq \|x_i^k - x_i^*\|^2 + \alpha_{k,i}^2 C_1 + 2C_2 \left| \alpha_{k,i} - \frac{1}{kp_i} \right| + \frac{2}{kp_i} \bar{L}_i N M \sum_{j=1}^N [W(k)]_{ij} \|v_j^k - y^k\| \\
- \frac{2}{kp_i} (F_i(x_i^k, \bar{x}^k) - F_i(x_i^*, \bar{x}^*))^T (x_i^k - x_i^*).
\]

Since \(x_i^{k+1} = x_i^k\) when \(i \not\in \{I^k, J^k\}\), it follows that \(\|x_i^{k+1} - x_i^*\|^2 = \|x_i^k - x_i^*\|^2\) for \(i \not\in \{I^k, J^k\}\). We combine these two cases with the fact that agent \(i\) updates with probability \(p_i\) and, thus obtain almost surely for all \(i \in \mathcal{N}\) and for all \(k\)
\[
\mathbb{E}[\|x_i^{k+1} - x_i^*\|^2 | \mathcal{F}_k] \leq \|x_i^k - x_i^*\|^2 + C_1 \mathbb{E}[\alpha_{k,i}^2 | \mathcal{F}_k] + 2C_2 \mathbb{E} \left[ |\alpha_{k,i} - \frac{1}{kp_i}| | \mathcal{F}_k \right] \\
+ \frac{2}{k} \bar{L}_i N M \sum_{j=1}^N [W(k)]_{ij} \|v_j^k - y^k\| - \frac{2}{k} (F_i(x_i^k, \bar{x}^k) - F_i(x_i^*, \bar{x}^*))^T (x_i^k - x_i^*). \tag{40}
\]

Summing relations (40) over \(i = 1, \ldots, N\), using the fact that \(W(k)\) is doubly stochastic and recalling that \(F_i\) are coordinate maps for \(F\) and \(F(x, \bar{x})\) defines \(\phi\) (cf. (7) and (8)), we further obtain for all \(k\):\[
\mathbb{E}[\|x^{k+1} - x^*\|^2 | \mathcal{F}_k] \leq \|x^k - x^*\|^2 + C_1 \mathbb{E} \left[ \sum_{i=1}^N \alpha_{k,i}^2 | \mathcal{F}_k \right] + 2C_2 \mathbb{E} \left[ \sum_{i=1}^N |\alpha_{k,i} - \frac{1}{kp_i}| | \mathcal{F}_k \right] \\
+ \frac{2}{k} \bar{L}_i N M \sum_{i=1}^N \|v_i^k - y^k\| - \frac{2}{k} (\phi(x^k) - \phi(x^*))^T (x^k - x^*), \tag{41}
\]

where \(p_{\min} = \min_{i} p_i\) and \(p_{\max} = \max_{i} p_i\). We now verify that we can apply the supermartingale convergence result (cf. Lemma 5) to relation (11). From Corollary 11 it follows that
\[
\sum_{k=1}^\infty \left( C_1 \mathbb{E} \left[ \sum_{i=1}^N \alpha_{k,i}^2 | \mathcal{F}_k \right] + 2C_2 \mathbb{E} \left[ \sum_{i=1}^N |\alpha_{k,i} - \frac{1}{kp_i}| | \mathcal{F}_k \right] \right) < \infty.
\]

Further from Lemma 10 it follows that \(\sum_{k=1}^\infty \sum_{i=1}^N \frac{1}{k} \|v_i^k - y^k\| < \infty\) almost surely. Thus, all conditions of Lemma 5 are satisfied and we conclude that
\[
\{\|x^k - x^*\|\} \text{ converges a.s.}, \tag{42}
\]
\[
\sum_{k=1}^\infty \frac{2}{k} (\phi(x) - \phi(x^*))^T (x^k - x^*) < \infty \text{ a.s.} \tag{43}
\]
4.3 Error Bounds for Constant Stepsize

In this section, we investigate the properties of the algorithm when agents employ a deterministic constant, albeit uncoordinated, stepsize. More specifically, our interest lies in establishing error bounds contingent on the deviation of stepsize across agents. Under this setting, the stepsize is \( \alpha_{k,i} = \alpha_i \) in the update rule for agents’ decisions in (28), which reduces to

\[
x_{i}^{k+1} = \left( \Pi_{K_i} \left[ x_i^k - \alpha_i F_i(x_i^k, N \hat{v}_i^k) \right] - x_i^k \right) \mathbb{1}_{\{i \in \{1, \ldots, \ell \}\}} + x_i^k,
\]

where \( \alpha_i \) is a positive constant stepsize for agent \( i \). It is worth mentioning that the rules for mixing estimates (23) and updating estimates (29) are invariant under this modification. Also, we allow agents to independently choose \( \alpha_i \), thereby maintaining the complete decentralization feature of the gossip algorithm. We begin by providing an updated estimate for the disagreement among the agents.

Our result is analogous to that of Lemma 10.

**Lemma 11.** Let Assumptions 1, 3 and 7 hold. Consider \( \{v_i^k\}, i = 1, \ldots, N \), that are generated by algorithm in 27 - 29 with \( \alpha_{k,i} = \alpha_i \). Then, for \( y^k = \frac{1}{N} \sum_{i=1}^N v_i^k \) we have

\[
\limsup_{k \to \infty} \frac{1}{N} \sum_{i=1}^N \mathbb{E}[\|v_i^k - y^k\|^2] \leq \frac{2n \sigma_{\text{max}}^2 C^2}{(1 - \sqrt{\lambda})^2}, \quad \limsup_{k \to \infty} \frac{1}{N} \sum_{i=1}^N \mathbb{E}[\|v_i^k - y^k\|] \leq \frac{\sqrt{2n}N \sigma_{\text{max}} C}{1 - \sqrt{\lambda}}, \quad \text{a.s.,}
\]

where \( \sigma_{\text{max}} = \max_i \{\alpha_i\} \), \( C \) is the constant as in Lemma 12 and \( \lambda \) is as given in (30).

**Proof.** We fix an arbitrary index \( \ell \). By Lemma 9 with \( \alpha_{k,i} = \alpha_i \), we have for all \( k \geq 0 \),

\[
\|v^{k+1}(\ell) - [y^{k+1}]_{\ell}1\| \leq \|D(k)(v^k(\ell) - [y^k]_{\ell}1)\| + \sqrt{2}C \sigma_{\text{max}}, \quad (44)
\]

where \( D(k) = W(k) - \frac{1}{N} \mathbf{1}\mathbf{1}^T \) and \( \sigma_{\text{max}} = \max_i \alpha_i \). Note that by relation (31) we have

\[
\mathbb{E}[\|D(k)(v^k(\ell) - [y^k]_{\ell}1)\|] = \mathbb{E}\left[\|D(k)(v^k(\ell) - [y^k]_{\ell}1)\| \mid \mathcal{F}_k\right] \leq \sqrt{\lambda} \mathbb{E}[\|v^k(\ell) - [y^k]_{\ell}1\|]. \quad (45)
\]

Thus, by taking the expectation of both sides in (44), we obtain

\[
\mathbb{E}\left[\|v^{k+1}(\ell) - [y^{k+1}]_{\ell}1\|\right] \leq \sqrt{\lambda} \mathbb{E}\left[\|v^k(\ell) - [y^k]_{\ell}1\|\right] + \sqrt{2}C \sigma_{\text{max}} \quad \text{for all } k \geq 0,
\]

which by iterative recursion leads to

\[
\mathbb{E}\left[\|v^{k+1}(\ell) - [y^{k+1}]_{\ell}1\|\right] \leq \left( \sqrt{\lambda} \right)^{k+1} \mathbb{E}\left[\|v^0(\ell) - [y^0]_{\ell}1\|\right] + \sqrt{2}C \sigma_{\text{max}} \sum_{s=0}^{k-1} (\sqrt{\lambda})^s \quad \text{for all } k \geq 0.
\]

Thus, by letting \( k \to \infty \), we obtain the following limiting result

\[
\limsup_{k \to \infty} \mathbb{E}[\|v^{k+1}(\ell) - [y^{k+1}]_{\ell}1\|] \leq \frac{\sqrt{2}C \sigma_{\text{max}}}{1 - \sqrt{\lambda}}, \quad (46)
\]
By taking the squares of both sides in relation (44), we find
\[ \|v^{k+1}(\ell) - [y^{k+1}]_\ell \mathbf{1}\|^2 \leq \|D(k)(v^k(\ell) - [y^k]_\ell \mathbf{1})\|^2 + 2\sqrt{2}C\alpha_{\text{max}}\|D(k)(v^k(\ell) - [y^k]_\ell \mathbf{1})\| + 2C^2\alpha_{\text{max}}^2. \]
Taking the expectation on both sides in the preceding relation and using estimate (45), we obtain
\[
\mathbb{E}[\|v^{k+1}(\ell) - [y^{k+1}]_\ell \mathbf{1}\|^2] \leq \mathbb{E}[\|D(k)(v^k(\ell) - [y^k]_\ell \mathbf{1})\|^2] + 2\sqrt{2}C\alpha_{\text{max}}\sqrt{\mathbb{E}[\|v^k(\ell) - [y^k]_\ell \mathbf{1}\|]} + 2C^2\alpha_{\text{max}}^2 \\
\leq \lambda \mathbb{E}[\|v^k(\ell) - [y^k]_\ell \mathbf{1}\|^2] + 2\sqrt{2}C\alpha_{\text{max}}\sqrt{\mathbb{E}[\|v^k(\ell) - [y^k]_\ell \mathbf{1}\|]} - [y^k]_\ell \mathbf{1}\|] + 2C^2\alpha_{\text{max}}^2 \tag{47}
\]
where the last inequality follows by
\[
\mathbb{E}[\|D(k)(v^k(\ell) - [y^k]_\ell \mathbf{1})\|^2] = \mathbb{E} \left[ \mathbb{E}[\|D(k)(v^k(\ell) - [y^k]_\ell \mathbf{1})\|^2 \mid \mathcal{F}_k] \right] \leq \lambda \mathbb{E}[\|v^k(\ell) - [y^k]_\ell \mathbf{1}\|^2],
\]
which is a consequence of relation (30). Since \(v^k\) and \(y^k\) are convex combinations of points in \(K_1, \ldots, K_N\) and each \(K_i\) is compact, the sequence \(\{\|v^{k+1}(\ell) - [y^{k+1}]_\ell \mathbf{1}\|\}\) is bounded, implying that so is the sequence \(\{\mathbb{E}[\|v^{k+1}(\ell) - [y^{k+1}]_\ell \mathbf{1}\|]\}\). Thus, \(\limsup_{k \to \infty} \mathbb{E}[\|v^{k+1}(\ell) - [y^{k+1}]_\ell \mathbf{1}\|^2]\) exists, and let us denote this limit by \(S\). Letting \(k \to \infty\) in relation (47) and using (16), we obtain
\[
S \leq \lambda S + \left(\frac{2\sqrt{\lambda}}{1 - \sqrt{\lambda}} + 1\right) 2C^2\alpha_{\text{max}}^2 = \lambda S + \frac{1 + \sqrt{\lambda}}{1 - \sqrt{\lambda}} 2C^2\alpha_{\text{max}}^2,
\]
which upon solving for \(S\) and recalling the notation yields
\[
\limsup_{k \to \infty} \mathbb{E}[\|v^{k+1}(\ell) - [y^{k+1}]_\ell \mathbf{1}\|^2] \leq \frac{1 + \sqrt{\lambda}}{(1 - \lambda)(1 - \sqrt{\lambda})} 2C^2\alpha_{\text{max}}^2 = \frac{1}{(1 - \sqrt{\lambda})^2} 2C^2\alpha_{\text{max}}^2.
\]
The preceding relation is true for any \(\ell\). Thus, since \(\limsup\) is invariant under the index-shift, we have
\[
\limsup_{k \to \infty} \sum_{\ell = 1}^{n} \mathbb{E}[\|v^k(\ell) - [y^k]_\ell \mathbf{1}\|^2] \leq \sum_{\ell = 1}^{n} \limsup_{k \to \infty} \mathbb{E}[\|v^k(\ell) - [y^k]_\ell \mathbf{1}\|^2] \leq \frac{2nC^2\alpha_{\text{max}}^2}{(1 - \sqrt{\lambda})^2},
\]
and by the linearity of the expectation, it follows
\[
\limsup_{k \to \infty} \mathbb{E} \left[ \sum_{\ell = 1}^{n} \|v^k(\ell) - [y^k]_\ell \mathbf{1}\|^2 \right] \leq \frac{2nC^2\alpha_{\text{max}}^2}{(1 - \sqrt{\lambda})^2}.
\]
Recalling that vector \(v^k(\ell)\) consists of the \(\ell\)th coordinates of the vectors \(v^k_1, \ldots, v^k_N\) (i.e., \(v^k(\ell) = ([v^k_1]\ell, \ldots, [v^k_N]\ell)^T\)), we see that \(\|\sum_{\ell = 1}^{n} \|v^k(\ell) - [y^k]_\ell \mathbf{1}\|^2 = \sum_{i = 1}^{N} \|v^k_i - y^k\|^2\). Hence, the preceding relation is equivalent to
\[
\limsup_{k \to \infty} \sum_{i = 1}^{N} \mathbb{E}[\|v^k_i - y^k\|^2] \leq \frac{2nC^2\alpha_{\text{max}}^2}{(1 - \sqrt{\lambda})^2},
\]
which is the first relation stated in the lemma. In particular, the preceding relation implies that
\[
\limsup_{k \to \infty} \sqrt{\sum_{i = 1}^{N} \mathbb{E}[\|v^k_i - y^k\|^2]} \leq \frac{\sqrt{2n} C\alpha_{\text{max}}}{1 - \sqrt{\lambda}}. \tag{48}
\]
On the other hand, by Holders’ inequality we have

\[
\sum_{i=1}^{N} \mathbb{E}[\|v_i^k - y^k\|] \leq \sqrt{N} \sqrt{\sum_{i=1}^{N} \mathbb{E}[\|v_i^k - y^k\|^2]}.
\]

from which by taking the limit as \( k \to \infty \) and using (48), we obtain

\[
\limsup_{k \to \infty} \sum_{i=1}^{N} \mathbb{E}[\|v_i^k - y^k\|] \leq \frac{\sqrt{2nN C_{\alpha_{\max}}}}{1 - \sqrt{\lambda}}.
\]

\[\square\]

We now estimate the limiting error of the algorithm under the additional assumption of strong monotonicity of the mapping \( \phi \). For this result, we also assume an additional Lipschitz property for the maps \( F_i \), as given below.

**Assumption 8.** Each mapping \( F_i(x_i, u) \) is uniformly Lipschitz continuous in \( x_i \) over \( K_i \), for every fixed \( u \in K \) i.e., for some \( L_i > 0 \) and for all \( x_i, y_i \in K_i \),

\[
\|F_i(x_i, u) - F_i(y_i, u)\| \leq L_i\|x_i - y_i\|.
\]

We have the following result.

**Proposition 4.** Let Assumptions 4, 7, and 8 hold, and let the mapping \( \phi \) be strongly monotone over the set \( K \) with a constant \( \mu > 0 \), in the following sense:

\[(\phi(x) - \phi(y))^T(x - y) \geq \mu\|x - y\|^2 \quad \text{for all } x, y \in K.\]

Consider the sequence \( \{x^k\} \) generated by the method (27) - (29) with \( \alpha_{k,i} = \alpha_i \). Suppose that the stepsizes \( \alpha_i \) are such that

\[
0 < 1 - 2\mu p_{\min} \alpha_{\min} + 2p_{\max}(\max_i \lambda_i)(\alpha_{\max} - \alpha_{\min}) < 1,
\]

where for \( i = 1, \ldots, N \), \( L_i \) are Lipschitz constants from Assumption 8, \( \alpha_{\max} = \max_{i} \alpha_i \), \( \alpha_{\min} = \min_{i} \alpha_i \), \( p_{\max} = \max_{i} p_i \), and \( p_{\min} = \min_{i} p_i \). Then, the following result holds

\[
\limsup_{k \to \infty} \mathbb{E}[\|x^k - x^*\|^2] \leq \frac{p_{\max}^2 \alpha_{\max}^2 (2C^2 N + BC\sqrt{2nN})}{\mu p_{\min} \alpha_{\min} - p_{\max}(\max_i L_i)(\alpha_{\max} - \alpha_{\min})},
\]

where \( x^* \) is the unique solution of \( \text{VI}(K, \phi) \), \( C \) is as in Lemma 3, \( \lambda \) is as in (30), and \( B = (\max_i L_i)NM \) with \( L_i, i = 1, \ldots, N \), being the Lipschitz constants from Assumption 3 and \( M \geq \max_{x_i, z_i \in K_i} \|x_i - z_i\| \) for all \( i \).

**Proof.** Since the map is strongly monotone, there is a unique solution \( x^* \in K \) to \( \text{VI}(K, \phi) \) (see Theorem 2.3.3. in [11]). Then, by the definition of \( x_{i}^{k+1} \) we have

\[
\|x_{i}^{k+1} - x^*\|^2 = \left\| (\Pi_{K_i}[x_i^k - \alpha_i F_i(x_i^k, N\nu_i^k)] - x_i^k) 1_{\{i \in \{k, k\} \}} + x_i^k - x_i^k \right\|^2.
\]
Using $x_i^* = \Pi_{K_i}[x_i^* - \alpha_i F_i(x_i^*, \bar{x}^*)]$ and the non-expansive property of the projection operator, we have for $i \in \{I^k, J^k\}$,

$$
\|x_i^{k+1} - x_i^*\|^2 \leq \|x_i^k - \alpha_i F_i(x_i^k, N\bar{v}^k) - x_i^k + \alpha_i F_i(x_i^*, \bar{x}^*)\|^2.
$$

By subtracting $\alpha_i F_i(x_i^*, \bar{x}^*)$ we obtain

$$
\|x_i^k - x_i^*\|^2 + \alpha_i^2 \|F_i(x_i^k, N\bar{v}^k) - F_i(x_i^*, \bar{x}^*)\|^2 - 2\alpha_i (F_i(x_i^k, N\bar{v}^k) - F_i(x_i^*, \bar{x}^*))^T(x_i^k - x_i^*).
$$

(50)

We now approximate the inner product term by adding and subtracting $F_i(x_i^k, N\bar{y}^k)$ and using $y^k = \sum_{i=1}^N x_i^k = \bar{x}^k$ (see Lemma [2]), to obtain

$$
(F_i(x_i^k, N\bar{v}^k) - F_i(x_i^*, \bar{x}^*))^T(x_i^k - x_i^*) \geq - \|(F_i(x_i^k, N\bar{v}^k) - F_i(x_i^k, N\bar{y}^k))^T(x_i^k - x_i^*)
$$

$$
+ (F_i(x_i^k, \bar{x}^k) - F_i(x_i^*, \bar{x}^*))^T(x_i^k - x_i^*).
$$

By the Lipshitz property of the mapping $F_i$ in Assumption [3] we have

$$
|F_i(x_i^k, N\bar{v}^k) - F_i(x_i^k, N\bar{y}^k)| \leq L_i N \|\bar{v}^k - \bar{y}^k\| \|x_i^k - x_i^*\| \leq L_i N M \|\bar{v}^k - \bar{y}^k\|
$$

where $M \geq \max_{ix_i, i} \|x_i - z_i\|$ for all $i$, which exists by compactness of each $K_i$. Upon combining the preceding estimates with (50), we obtain

$$
\|x_i^{k+1} - x_i^*\|^2 \leq \|x_i^k - x_i^*\|^2 + 4\alpha_i^2 C^2 + 2\alpha_i L_i N M \|\bar{v}^k - \bar{y}^k\|
$$

$$
- 2\alpha_i (F_i(x_i^k, \bar{x}^k) - F_i(x_i^*, \bar{x}^*))^T(x_i^k - x_i^*).
$$

(51)

Now, we work with the last term in (51), by letting $\alpha_{\text{min}} = \min_i \alpha_i$, and by adding and subtracting $2\alpha_{\text{min}} (F_i(x_i^k, \bar{x}^k) - F_i(x_i^*, \bar{x}^*))^T(x_i^k - x_i^*)$, we can see that

$$
\|x_i^{k+1} - x_i^*\|^2 \leq \|x_i^k - x_i^*\|^2 + 4\alpha_i^2 C^2 + 2\alpha_i L_i N M \|\bar{v}^k - \bar{y}^k\|
$$

$$
+ 2(\alpha_i - \alpha_{\text{min}})(F_i(x_i^k, \bar{x}^k) - F_i(x_i^*, \bar{x}^*))^T(x_i^k - x_i^*)
$$

$$
- 2\alpha_{\text{min}} (F_i(x_i^k, \bar{x}^k) - F_i(x_i^*, \bar{x}^*))^T(x_i^k - x_i^*).
$$

(52)

By using the Cauchy-Schwarz inequality and the Lipshitz property of $F_i$ given in Assumption [8], we obtain

$$
|F_i(x_i^k, \bar{x}^k) - F_i(x_i^*, \bar{x}^*)|^T(x_i^k - x_i^*) \leq L_i \|x_i^k - x_i^*\|^2.
$$

(53)

Further, by letting $\alpha_{\text{max}} = \max_i \alpha_i$, from (52) and (53) by collecting the common terms we have for $i \in \{I^k, J^k\}$,

$$
\|x_i^{k+1} - x_i^*\|^2 \leq (1 + 2L_i(\alpha_{\text{max}} - \alpha_{\text{min}})) \|x_i^k - x_i^*\|^2 + 4\alpha_i^2 C^2 + 2\alpha_i L_i N M \|\bar{v}^k - \bar{y}^k\|
$$

$$
- 2\alpha_{\text{min}} (F_i(x_i^k, \bar{x}^k) - F_i(x_i^*, \bar{x}^*))^T(x_i^k - x_i^*).
$$
The fact that \( x_i^{k+1} = x_i^k \) when \( i \not\in \{I^k, J^k\} \) implies that \( \|x_i^{k+1} - x_i^*\|^2 = \|x_i^k - x_i^*\|^2 \) for \( i \not\in \{I^k, J^k\} \). Next, we take the expectation in (54), whereby we combine the preceding two cases and take into account that agent \( i \) updates with probability \( p_i \), and obtain for all \( i \in N \),

\[
\mathbb{E}[\|x_i^{k+1} - x_i^*\|^2 \mid \mathcal{F}_k] \leq (1 + 2p_i L_i(\alpha_{\max} - \alpha_{\min})) \|x_i^k - x_i^*\|^2 + 4p_i \alpha_i^2 C^2
\]

\[
+ 2p_i \alpha_i L_i N M \mathbb{E}[\|\hat{v}_k^i - y^k\| \mid \mathcal{F}_k] - 2p_i \alpha_{\min}(F_i(x_i^k, \bar{x}^k) - F_i(x_i^*, \bar{x}^*))^T (x_i^k - x_i^*).
\]

Since \( \hat{v}_k^i \) is a convex combination of \( v_j^k \), \( j \in N \) and the norm is a convex function, we have

\[
\|\hat{v}_k^i - y^k\| \leq \sum_{j=1}^N [W(k)]_{ij} \|v_j^k - y^k\|.
\]

Thus, \( \mathbb{E}[\|\hat{v}_k^i - y^k\| \mid \mathcal{F}_k] \leq \sum_{j=1}^N \mathbb{E}[W(k)]_{ij} \|v_j^k - y^k\| \). By using the preceding relation, \( \min_i p_i = p_{\min}, \ p_{\max} = \max_i p_i \), and \( \alpha_i \leq \alpha_{\max} \), we arrive at the following relation for all \( i \in N \),

\[
\mathbb{E}[\|x_i^{k+1} - x_i^*\|^2 \mid \mathcal{F}_k] \leq (1 + 2p_{\max} \max_i L_i(\alpha_{\max} - \alpha_{\min})) \|x_i^k - x_i^*\|^2 + 4p_{\max} \alpha_{\max}^2 C^2
\]

\[
+ 2p_{\max} \alpha_{\max} B \sum_{j=1}^N \mathbb{E}[W(k)]_{ij} \|v_j^k - y^k\| - 2p_{\min} \alpha_{\min}(F_i(x_i^k, \bar{x}^k) - F_i(x_i^*, \bar{x}^*))^T (x_i^k - x_i^*), \quad (54)
\]

with \( B = (\max_i L_i) N M \).

Summing the relations in (54) over all \( i = 1, \ldots, N \), recalling that \( F_i, \ i = 1, \ldots, N \) are coordinate maps for the map \( F \) (see (7)), which in turn defines the mapping \( \phi \) through (8), we further obtain

\[
\mathbb{E}[\|x^{k+1} - x^*\|^2 \mid \mathcal{F}_k] \leq (1 + 2p_{\max} \max_i L_i(\alpha_{\max} - \alpha_{\min})) \|x^k - x^*\|^2 + 4p_{\max} \alpha_{\max}^2 C^2 N
\]

\[
+ 2p_{\max} \alpha_{\max} B \sum_{i=1}^N \sum_{j=1}^N \mathbb{E}[W(k)]_{ij} \|v_j^k - y^k\| - 2p_{\min} \alpha_{\min}(\phi(x^k) - \phi(x^*))^T (x^k - x^*),
\]

The matrix \( \mathbb{E}[W(k)] \) is doubly stochastic, so we have

\[
\sum_{i=1}^N \sum_{j=1}^N \mathbb{E}[W(k)]_{ij} \|v_j^k - y^k\| = \sum_{j=1}^N \sum_{i=1}^N \mathbb{E}[W(k)]_{ij} \|v_j^k - y^k\| = \sum_{j=1}^N \|v_j^k - y^k\|.
\]

Using this relation and the strong monotonicity of the mapping \( \phi \) with a constant \( \mu \), gathering the common terms, and taking the total expectation, we obtain for all \( k \geq 0 \),

\[
\mathbb{E}[\|x^{k+1} - x^*\|^2] \leq q \mathbb{E}[\|x^k - x^*\|^2] + 4p_{\max} \alpha_{\max}^2 C^2 N + 2p_{\max} \alpha_{\max} B \sum_{j=1}^N \mathbb{E}[\|v_j^k - y^k\|], \quad (55)
\]

where \( q = 1 - 2\mu p_{\min} \alpha_{\min} + 2p_{\max} \max_i L_i(\alpha_{\max} - \alpha_{\min}) \). Note that by the condition

\[
0 < 1 - 2\mu p_{\min} \alpha_{\min} + 2p_{\max} \max_i L_i(\alpha_{\max} - \alpha_{\min}) < 1,
\]

we have \( 0 < q < 1 \). We further have \( \{x^k\} \subseteq K \) for a compact set \( K \), so the limit superior of \( \mathbb{E}[\|x^k - x^*\|^2] \) exists. Thus, by taking the limit as \( k \to \infty \) in relation (55) and using Lemma 11, we obtain

\[
\lim_{k \to \infty} \mathbb{E}[\|x^{k+1} - x^*\|^2] \leq q \lim_{k \to \infty} \mathbb{E}[\|x^k - x^*\|^2] + 4p_{\max} \alpha_{\max}^2 C^2 N + 2p_{\max} \alpha_{\max} B \frac{\sqrt{2nN} C \alpha_{\max}}{1 - \sqrt{\lambda}},
\]

which implies the stated result. \( \square \)
We have few comments on the result of Proposition 4 as follows. The error bound depends on the dimension $n$ of the decision variables, the number $N$ of players, the frequency with which players update their decisions (captured by $p_{\text{min}}$ and $p_{\text{max}}$), and the network properties including the connectivity time bound $B$ and the ability to propagate the information (captured by the value $1 - \sqrt{\lambda}$). When the network parameters $B$ and $\lambda$, and the players’ update probabilities ($p_{\text{min}}$ and $p_{\text{max}}$) do not depend on $N$, the error bound grows linearly with the number $N$ of players.

As a special case, consider the case when the agents employ an equal stepsize, i.e., $\alpha_{\text{min}} = \alpha_{\text{max}} = \alpha$ and $\alpha$ satisfies the following condition $0 < \alpha < \frac{1}{2\mu\rho_{\text{min}}}$. Then, the result of Proposition 4 reduces to

$$\limsup_{k \to \infty} E[\|x^k - x^*\|^2] \leq \frac{p_{\text{max}}\alpha \left( 2C^2N + BC\sqrt{2nN} \right)}{\mu\rho_{\text{min}}}.$$  

As another special case, consider the case when all players have equal probabilities of updating, i.e., $p_{\text{min}} = p_{\text{max}} = p$. Then, we have the following result:

$$\limsup_{k \to \infty} E[\|x^k - x^*\|^2] \leq \frac{\alpha^2 \max \left( 2C^2N + BC\sqrt{2nN} \right)}{\mu\rho_{\text{min}} - (\max_i L_i)(\alpha_{\text{max}} - \alpha_{\text{min}})},$$  

When all players have equal probabilities of updating and all use equal stepsizes, i.e., $p_{\text{min}} = p_{\text{max}} = p$ and $\alpha_{\text{min}} = \alpha_{\text{max}} = \alpha$, then the condition of Proposition 4 reduces to $0 < \alpha < \frac{1}{2\mu\rho}$, and the bound further simplifies to:

$$\limsup_{k \to \infty} E[\|x^k - x^*\|^2] \leq \frac{\alpha \left( 2C^2N + BC\sqrt{2nN} \right)}{\mu(1 - \sqrt{\lambda})}.$$  

5 Generalizations and Extensions

In prior sections, we have developed two algorithms for addressing a class of Nash games. To recap, our prescribed class of equilibrium computation schemes may accommodate a specific subclass of noncooperative $N$-person Nash games, qualified as aggregative. Specifically, in such games, given $\bar{x}_{-i}$, the $i$th player solves the deterministic convex program given by (2). Unlike in much of prior work, players cannot observe $\bar{x}_{-i}$ but may learn it through the exchange of information with their local neighbors based on an underlying graph. This underlying graph may either be time-varying with a connectivity requirement or be fixed. In the case of the former, we present a synchronous scheme that mandates that every agent synchronizes its updates and information exchanges while in the latter case, we develop a an asynchronous gossip protocol for communication. Under a strict monotonicity and suitably defined Lipschitzian requirements on the map $F$ and compactness of the set $K$, we show that both algorithms produce sequences that are guaranteed to converge to the unique equilibrium in an almost sure sense. More succinctly, under appropriate communication requirements, any deterministic convex aggregative Nash game may be addressed through such synchronous/asynchronous techniques as long as the requirements on the map and strategy sets hold.

Naturally, one may rightly question whether the presented schemes (and their variants) can accommodate weakening some of the assumptions, both on the map and more generally on the model. Motivated by this concern, in Section 5.1 we begin by discussing how extensions of the
algorithm can allow for accommodating weaker assumptions on the problem setting. Subsequently, in Section 5.2 we discuss how the very nature of the coupling across agents can also be generalized.

5.1 Weakening assumptions on problem parameters

We consider three extensions to our prescribed class of games:

**Strict monotonicity:** The a.s. convergence theory for both the synchronous and asynchronous schemes rely on the strict monotonicity of the map as asserted by Assumption 2. There are several avenues for weakening such a requirement. For instance, one approach relies on using a regularized variant of (10), given by

\[
x_{i}^{k+1} = \Pi_{K_i} \left[ x_{i}^{k} - \alpha_k \left( F_i(x_{i}^{k}, N_{i}^{k}) + \epsilon_{k,i} x_{i}^{k} \right) \right],
\]

\[
u_{i}^{k+1} = \bar{v}_{i}^{k} + x_{i}^{k+1} - x_{i}^{k},
\]

where \(\{\epsilon_{k,i}\}\) denotes a sequence that diminishes to zero at a prescribed rate. Such an approach has been employed in prior work and allow for the solution of both deterministic [22] and stochastic [24] monotone variational inequality problems. An alternative approach may lie in the usage of an extragradient framework that requires taking two, rather than one, gradient step. Via such approaches, it has been shown [20] that the gap function associated with a monotone stochastic variational inequality problem tends to zero in mean. We believe distributed counterparts of extragradient schemes hold significant and represent a generalization of the schemes presented in this paper.

**Lipschitz continuity:** A second assumption employed in deriving convergence and rate statements is the (uniform) Lipschitzian assumption as articulated in Assumption 3. We believe that there are at least two avenues that can be adopted in weakening this requirement. First, there has been significant recent work that integrates the use of local or randomized smoothing (also called Steklov-Sobolev smoothing) to address stochastic variational inequalities in which the maps are not necessarily Lipschitz continuous (cf. [63]). It may well be possible to extend such techniques to address settings where the uniformly Lipschitzian assumption does not hold. An alternate approach may lie in developing convergence in mean of the gap function, as adopted in [20] where the either Lipschitz continuity or boundedness of the map is necessary.

**Stochastic payoff functions:** Presently, the main source of uncertainty arises either from the evolution of the connectivity graph (synchronous scheme) or the randomness in the choice of players that communicate as per the gossip protocol (asynchronous) scheme. Yet, the player objectives could also be expectation-valued. Consequently, the equilibrium conditions are given by a stochastic variational inequality. In such instances, one may articulate suitably defined distributed stochastic approximation counterparts of (10) to cope with such a challenge (cf. [24]). We believe convergence analysis of such schemes, while more complicated, is likely to carry through under suitable assumptions.

5.2 Extensions to model

It may have been observed that the proposed developments in the earlier two sections required that the agent decisions be of the same dimension. In this subsection, we extend the realm of (2) and
generalize the algorithms presented in section 3 and section 4. To this end, consider the following aggregative game

\[
\begin{align*}
\text{minimize} & \quad f_i \left( x_i, \sum_{i=1}^{N} h_i(x_i) \right) \\
\text{subject to} & \quad x_i \in K_i,
\end{align*}
\]

where \( K_i \subseteq \mathbb{R}^{n_i} \), \( h_i : K_i \to \mathbb{R}^n \). The mappings \( f_i \) and \( h_i \) are considered to be information private to player \( i \). Such an extension allows players decisions to have different dimensionality. To recover the problem articulated by (2), we set \( h_i(x_i) = x_i \) with \( K_i \subseteq \mathbb{R}^{n_i} \) for all \( i \). Next, we discuss the generalization of the proposed distributed algorithms to solve problem (58).

**Synchronous Algorithm:** To make the synchronous algorithm suitable for the generalized problem in (58), the mixing step in (9) remains the same, but with a different initial condition. Namely, the mixing in (9) is initiated with

\[
v_i^0 = h_i(x_i^0) \quad \text{for all } i = 1, \ldots, N,
\]

where \( x_i^0 \in K_i \) are initial players’ decisions. The iterate update of (10) and update of the average estimate (57) are modified, leading to the following:

\[
\begin{align*}
x_i^{k+1} &= \Pi_{K_i} \left[ x_i^k - \alpha_{k,i} F_i(x_i^k, N\hat{v}_i^k) \right] / BD_{\{i \in \mathcal{I}_k, J_k\}} + x_i^k, \\
v_i^{k+1} &= \hat{v}_i^k + h_i(x_i^{k+1}) - h_i(x_i^k),
\end{align*}
\]

where \( \alpha_{k,i} \) is the stepsize, the mapping \( F_i \) is as defined in (62). The following result establishes the convergence of the extended synchronous algorithm.

**Proposition 5.** Let Assumptions 1–6 hold for the mapping \( \phi(x) = (\phi_1(x), \ldots, \phi_N(x))^T \) with coordinates \( \phi_i(x) = \nabla_{x_i} f_i \left( x_i, \sum_{i=1}^{N} h_i(x_i) \right) \) and \( x = (x_1^T, \ldots, x_N^T)^T \). Then, the sequence \( \{x^k\} \) generated by the method (60)–(61) converges to the (unique) solution \( x^* \) of the game in (58).

**Proof.** The proof mimics the proof of Proposition 2. \( \square \)

**Asynchronous Algorithm:** We now discuss how the gossip algorithm in section 4 may be modified. While the estimate mixing in (23) remains unchanged, the initial condition is replaced by one given in (59). The iterate update of (28) and average estimate update of (29) are modified, as follows:

\[
\begin{align*}
x_i^{k+1} &= \Pi_{K_i} \left[ x_i^k - \alpha_{k,i} F_i(x_i^k, N\hat{v}_i^k) \right] - x_i^k 1_{\{i \in \mathcal{I}_k, J_k\}} + x_i^k, \\
v_i^{k+1} &= \hat{v}_i^k + h_i(x_i^{k+1}) - h_i(x_i^k),
\end{align*}
\]

where \( \alpha_{k,i} \) is the stepsize for user \( i \) and the mapping \( F_i \) is as defined in (62). The following result establishes the convergence of the extended asynchronous algorithm.
Proposition 6. Let Assumptions 1, 2, and Assumption 7 hold. Then, the sequence \( \{x^k\} \) generated by the method (63)–(64) with stepsize \( \alpha_{k,i} = \frac{1}{k(i)} \) converges to the (unique) \( x^* \) of the game almost surely.

Proof. With the initial condition specified by (59), the proof follows in a fashion similar to that of Proposition 3. \qed

6 Numerics

In this section, we examine the performance of the proposed algorithms on a class of Nash-Cournot games. Such games represent an instance of aggregative Nash games and in section 6.1 we describe the player payoffs and strategy sets as well as verify that they satisfy the necessary assumptions. In section 6.2, we discuss the synchronous setting and present the results arising from applying our algorithms. In section 6.3, we turn our attention to asynchronous regime where we present our numerical experience of applying the gossip algorithm.

6.1 Nash-Cournot Game

We consider a networked Nash-Cournot games which is possibly amongst the best known examples of an aggregative game. Specifically, the aggregate in such games is the total sales which is the sum of production over all the players. The market price is set in accord with an inverse demand function which depends on the aggregate of the network. A formal description of such a game over a network is provided in Example 1. Before proceeding to describe our experimental setup, we show that Nash-Cournot games do indeed satisfy Assumptions 3 and 8, respectively, under some mild assumptions on the cost and price functions in Nash-Cournot games, as shown next.

In the sequel, within the context of Example 1, we let \( x_{il} = (g_{il}, s_{il}) \) for all \( l = 1, \ldots, L \), \( x_i = (x_{i1}, \ldots, x_{iL}) \) and \( x = (x_1, \ldots, x_N)^T \). Further, we define coordinate maps \( F_i(x_i, u) \), as follows:

\[
F_i(x_i, u) = \begin{pmatrix}
F_{i1}(x_{i1}, u_1) \\
\vdots \\
F_{iL}(x_{iL}, u_L)
\end{pmatrix},
\]

\[
F_{il}(x_{il}, u_l) = \left( -\frac{c_{il}(g_{il})}{p_i(u_l)}, \frac{c_{il}(g_{il})}{p_i(u_l)}s_{il} \right),
\]

where the prime denotes the first derivative. We let \( F(x, u) = (F_1(x_1, u)^T, \ldots, F_N(x_N, u)^T)^T \), and \( K_i \) denote the constraint set on player \( i \) decision, \( x_i \), as given in Example 1.

We note that the Nash-Cournot game under the consideration satisfies Assumption 1 as long as the cost functions \( c_{il} \) are convex and the price functions \( p_i(u_l) \) are concave for all \( i \) and \( l \). Furthermore, the strict convexity condition of Assumption 2 is satisfied when, for example, all price functions \( p_i \) are strictly concave. This can be seen by observing that

\[
(F(x, u) - F(\bar{x}, u))^T (x - \bar{x}) = \sum_{l=1}^{L} \sum_{i=1}^{N} \left( c_{il}'(g_{il}) - c_{il}'(\bar{g}_{il}) \right) (g_{il} - \bar{g}_{il}) - \sum_{l=1}^{L} \sum_{i=1}^{N} p_i'(u_l) (s_{il} - \bar{s}_{il})^2.
\]

Next, we show that the Lipschitzian requirements on the maps \( F_i \) of Assumption 3 holds under some mild assumptions on the cost and price functions in Nash-Cournot games, as shown next.
Lemma 12. Consider the Nash-Cournot game described in Example 1. Suppose that each \( p_l(u_l) \) is concave and has Lipschitz continuous derivatives with a constant \( M_l \) (over a coordinate projection of \( K \) on the \( l \)th coordinate axis). Then, the following relation holds:

\[
\| F_i(x_i, u) - F_i(x_i, z) \| \leq \sqrt{2} \sum_{l=1}^{L} (C_i^2 + M_i^2 \text{cap}_l^2) \| u - z \| \quad \text{for all } u, z \in \bar{K}.
\]

Proof. This result follows directly from the definition of the coordinate maps \( F_{il}(x_{il}, u_l) \) and recalling that \( x_{il} = (g_{il}, s_{il}) \). In particular, for each \( i, l \) we have

\[
\| F_{il}(x_{il}, u_l) - F_{il}(x_{il}, z_l) \| = \sqrt{\left| c_{il}'(g_{il}) - c_{il}'(g_{il}) \right|^2 + \left| p_i(u_i) + p_i'(u_i) s_{il} - p_i(z_i) - p_i'(z_i) s_{il} \right|^2}
\]

\[
= \sqrt{\left| (p_i(u_i) - p_i(z_i)) + (p_i'(u_i) - p_i'(z_i)) s_{il} \right|^2}
\]

\[
\leq \sqrt{2} \sum_{l=1}^{L} \| p_i(u_i) - p_i(z_i) \|^2 + \| p_i'(u_i) - p_i'(z_i) \|^2 s_{il}^2,
\]

where the inequality follows from \((a + b)^2 \leq 2(a^2 + b^2)\). Since \( \bar{K} \) is compact and each \( p_l \) has continuous derivatives, it follows that there exists a constant \( C_i \) for every \( l \) such that

\[ |p_i'(u_l)| \leq C_i \quad \text{for all } u_l \text{ with } u = (u_1, \ldots, u_L)^T \in \bar{K}. \]

Then, by using concavity of \( p_i \), we can see that \( |p_i(u_i) - p_i(z_i)\) \( \leq C_i|u_l - z_i| \) implying that

\[
\| F_{il}(x_{il}, u_l) - F_{il}(x_{il}, z_l) \| \leq \sqrt{2} \sum_{l=1}^{L} (C_i^2 |u_l - z_l|^2 + M_i^2 |s_{il}|^2)
\]

\[
\leq \sqrt{2} \sum_{l=1}^{L} (C_i^2 + M_i^2 s_{il}^2) |u_l - z_l|,
\]

where the last inequality is obtained by using the Lipschitz property of the derivative \( p_i'(u_l) \). From the structure of constraints we have \( s_{il} \leq \text{cap}_l \) yielding

\[
\| F_{il}(x_{il}, u_l) - F_{il}(x_{il}, z_l) \| \leq \sqrt{2} \sum_{l=1}^{L} (C_i^2 + M_i^2 \text{cap}_l^2) |u_l - z_l| \quad \text{for all } l.
\]

Further, by using Hölder’s inequality, and recalling that \( x_i = (x_{i1}, \ldots, x_{iL}) \) and \( u = (u_1, \ldots, u_L) \), from the preceding relation we obtain

\[
\| F_i(x_i, u) - F_i(x_i, z) \| = \sqrt{\sum_{l=1}^{L} \| F_{il}(x_{il}, u_l) - F_{il}(x_{il}, z_l) \|^2} \leq \sqrt{2} \sqrt{\sum_{l=1}^{L} (C_i^2 + M_i^2 \text{cap}_l^2) \| u - z \|}.
\]

\[ \Box \]

We now show that \( F_i(x, u) \) is Lipschitz continuous in \( x \) for every \( u \).

Lemma 13. Consider the Nash-Cournot game described in Example 1. Suppose that each \( c_{il}' \) is Lipschitz continuous with a constant \( L_{il} \) and \( |p_i'(u)| \leq \bar{p}_l \) for some scalar \( \bar{p}_l \) and for all \( u \in \bar{K} \). Then, the following relation holds for all \( i \),

\[
\| F_i(x_i, u) - F_i(\tilde{x}_i, u) \| \leq \sqrt{\sum_{l=1}^{L} (L_{il}^2 + \bar{p}_l^2) \| x_i - \tilde{x}_i \|} \quad \text{for all } x_i, \tilde{x}_i \in K_i.
\]
Proof. First, we note that for each \( i, l \),

\[
\| F_{il}(x_{il}, u_l) - F_{il}(\bar{x}_{il}, u_l) \| = \sqrt{c_{il}'(g_{il}) - c_{il}'(\tilde{g}_{il})^2 + |p_{il}'(u_l)(s_{il} - \bar{s}_{il})|^2}
\]

\[
\leq \sqrt{L_{il}^2 |g_{il} - \tilde{g}_{il}|^2 + \bar{p}_{il}^2 |s_{il} - \bar{s}_{il}|^2}.
\]

Recalling our notation \( x_{il} = (g_{il}, s_{il}) \) and \( \bar{x}_{il} = (\tilde{g}_{il}, \bar{s}_{il}) \), and using Hölder’s inequality, we find that

\[
\| F_{il}(x_{il}, u_l) - F_{il}(\bar{x}_{il}, u_l) \| \leq \sqrt{L_{il}^2 + \bar{p}_{il}^2} \| x_{il} - \bar{x}_{il} \|.
\]

Further, recalling that \( x_i = (x_{i1}, \ldots, x_{iL}) \), \( \bar{x}_i = (\bar{x}_{i1}, \ldots, \bar{x}_{iL}) \), and \( u = (u_1, \ldots, u_L) \), the desired result follows from the preceding relation by using Hölders inequality. \( \square \)

In our numerical study, we consider a Nash-Cournot game played over ten locations, i.e. \( \mathcal{L} = 10 \), in which all players have cost functions of a similar structure and the \( i \)th player’s optimization problem may be expressed as

\[
\begin{align*}
\text{minimize} & \quad \sum_{l=1}^{10} (c_{il}(g_{il}) - p_{il}(\bar{s}_{il}) s_{il}) \\
\text{subject to} & \quad \sum_{l=1}^{10} g_{il} = \sum_{l=1}^{10} s_{il}, \\
& \quad g_{il}, s_{il} \geq 0, \quad g_{il} \leq \text{cap}_{il}, \quad l = 1, \ldots, 10, \quad (66)
\end{align*}
\]

where \( g_{il} \) and \( s_{il} \) denote player \( i \)’s production and sales at location \( l \), respectively, and \( \bar{s}_{l} \) denotes the aggregate of all the players’ decisions \( \bar{s}_{l} = \sum_{i=1}^{N} s_{il} \) at location \( l \). The function \( c_{il}(g_{il}) \) denotes the cost of production for \( i \)th player at location \( l \) and has the following form:

\[
c_{il}(g_{il}) = a_{il} g_{il} + b_{il} g_{il}^2,
\]

where \( a_{il} \) and \( b_{il} \) are scaling parameters for agent \( i \). In our experiments, we draw \( a_{il} \) and \( b_{il} \) from a uniform distribution and fix them over the course of the entire simulation. More precisely, for \( i = 1, \ldots, N \), and \( l = 1, \ldots, 10 \), we have \( a_{il} \sim U(2, 12) \) and \( b_{il} \sim U(2, 3) \), where \( U(t, \tau) \) denotes the uniform distribution over an interval \([t, \tau]\) with \( t < \tau \). The term \( p_{il}(\bar{s}_{il}) \) captures the inverse demand function and takes the following form:

\[
p_{il}(\bar{s}_{il}) = d_{il} - \bar{s}_{il},
\]

where \( d_{il} \) is a parameter for location \( l \). The parameters \( d_{il} \) are also drawn randomly with a uniform distribution, \( d_{il} \sim U(90, 100) \) for all \( l = 1, \ldots, 10 \). Furthermore, we use \( \text{cap}_{il} = 500 \) for all \( i = 1, \ldots, N \) and for all \( l = 1, \ldots, 10 \). The affine price function gives rise to a strongly monotone map \( \phi = F(x, \bar{x}) \), which together with the compactness of the sets \( K_i \), implies that this game has a unique Nash equilibrium. Note that in our setup, we have \( \text{cap}_{il} > d_{il} \) indicating that at a particular location, a player may produce more than the overall demand at that particular location. Such a scenario can arise when it might be more efficient to produce at a location to meet the demand(s) of another location(s) assuming the transportation costs are zero.
6.2 Synchronous Algorithm

In this section, we investigate the performance of synchronous algorithm of section 3 for the computation of the equilibrium of aggregative game (66). We begin by describing our setting for the connectivity graph of the network of players, where each player is seen as a node in a graph. At each iteration $k$, we generate a symmetric $N \times N$ adjacency matrix $A$ such that the underlying graph is connected. The entries of $A$ are generated by performing the following steps:

(0) Let $I$ denote the set of nodes that have already been generated;

(1) For each newly generated node $j$, select a node randomly $i \in I$ to establish an edge $\{i, j\}$ and set $[A]_{ij} = [A]_{ji} = 1$;

(2) Repeat step 1 until $I = \{1, \ldots, N\}$.

Given such an adjacency matrix $A$, we define a doubly stochastic symmetric weight matrix $W$ such that

$$[W]_{ij} = \begin{cases} 
0 & \text{if } A_{ij} = 0 \\
\delta & \text{if } A_{ij} = 1 \text{ and } i \neq j \\
1 - \delta d(i) & \text{if } i = j,
\end{cases}$$

where $d(i)$ represents the number of players communicating with player $i$, and

$$\delta = \frac{0.5}{\max_i \{d(i)\}}.$$ 

Using the adjacency matrix $A$ and the weight matrix $W$, players update their decision and their estimate of the average using (9)–(11). The stepsize rule for agent update is as follows:

$$\alpha_{k,i} = \frac{1}{k} \text{ for all } i = 1, \ldots, N.$$ 

The algorithm is initiated at a random starting point, and terminated after a fixed number of iterations, denoted by $\tilde{k}$, for each sample path. We use a set of 50 sample paths for each simulation setting, and we report the empirical mean of the sample error $\tilde{e}$, defined as:

$$\text{error}_{\tilde{k}} = \max_{i \in \{1, \ldots, N\}} \max_{l \in \{1, \ldots, 10\}} \left\{ \left| g^\tilde{k}_{il} - g^*_{il} \right|, \left| s^\tilde{k}_{il} - s^*_{il} \right| \right\}.$$

(67)

where $g^*_{il}$ and $s^*_{il}$ are the decisions of agent $i$ at the Nash equilibrium. The Nash equilibrium decisions $g^*_{il}$ and $s^*_{il}$ are computed using a constant steplength gradient projection algorithm assuming each agent has true information of the aggregate. Note that such an algorithm is guaranteed to converge under the strict convexity of the players’ costs.

We investigate cases with 20 and 50 players in the network. In Table 1 and Table 2, we report the empirical mean of the error and 90% confidence interval attained for different levels of $\tilde{k}$, (the simulation length), respectively. Some insights that can be drawn from the simulations are provided next:

• Expectedly, as seen in Table 1 and Table 2, the empirical mean of the error upon termination and the width of the confidence interval decrease with increasing $\tilde{k}$ and increase with network size.
The impact of the time-varying nature of the connectivity graph is explored by considering a static complete graph as a basis for comparison. In Table 3 and Table 4, we report the mean error and the confidence interval when the network is static. Under this setting, the agents have access to the true aggregate information throughout the run of the algorithm. Naturally, the performance of the algorithm on a static complete network is orders of magnitude better than that on a dynamic network. This deterioration in performance may be interpreted as the price of information from the standpoint of convergence.

Table 1: Dynamic network: Mean error on termination vs network size for various thresholds

| N  | k = 5e3  | k = 1e4 |
|----|---------|---------|
| 20 | 9.22e-5 | 3.66e-5 |
| 50 | 8.38e-2 | 2.65e-3 |

Table 2: Dynamic network: Width of 90% confidence interval of mean error

| N  | k = 5e3  | k = 1e4 |
|----|---------|---------|
| 20 | 2.147e-4 | 9.33e-5 |
| 50 | 1.24e-1  | 2.78e-2 |

Table 3: Static network: Mean error on termination

| N  | k = 5e3  | k = 1e4 |
|----|---------|---------|
| 20 | 3.66e-5 | 3.66e-5 |
| 50 | 6.23e-5 | 6.23e-5 |

Table 4: Static network: Width of 90% confidence interval of mean error

| N  | k = 5e3  | k = 1e4 |
|----|---------|---------|
| 20 | 6.99e-9 | 6.99e-9 |
| 50 | 1.61e-8 | 1.61e-8 |

6.3 Asynchronous Algorithm

We now demonstrate the performance of the asynchronous algorithm of section 4. We consider four instances of connectivity graphs which we describe next and, also, depict these graphs in Figure 4.5.

- **Cycle**: Every player has two neighbors;
- **Wheel**: There is one central player that is connected to every other player;
- **Grid**: Players on the vertex have two neighbors, players on the edge have three and everyone else has four neighbors. Each row in the grid consists of five players and there are \( N/5 \) rows where \( N \) is the size of the network;
- **Complete graph**: Every player has an edge connecting it to every other player.

Note that, in each connectivity graph, players can only communicate with their immediate neighbors.

\( ^5 \)The network topology shown in Figure 4 is for demo purposes only. For instance, Figure 4(a) is an example of a cycle network with 5 players.
For every type of connectivity graph, we initiate the algorithm from a random starting point and terminate it after $\tilde{k}$ iterations. A 95% confidence interval of the mean sample error at the termination is computed for a sample of size 50, where the sample-path error is defined as in (67). The players’ stepsize rules that we use are:

$$\alpha_{k,i} = \begin{cases} \frac{9}{\Gamma_k(i)} & \text{for a diminishing stepsize} \\ \alpha_i & \text{for a constant stepsize} \end{cases}$$

where $\alpha_i$ is randomly drawn from a uniform distribution, $\alpha_i \sim U(5e^{-3}, 1e^{-2})$. We again investigate cases when there are 20 and 50 players in the network and derive the following insights:

- In Tables 5–6, we report the mean error and in Tables 7–8, we report the width of the confidence interval for various levels of $\tilde{k}$. The results are consistent with our theoretical findings, and they indicate a decrease in the mean error and the width of the confidence interval with increasing $\tilde{k}$. As expected, the mean error at the termination increases with the size of the network. It is worth mentioning the discrepancy in the value of $\tilde{k}$ across synchronous and asynchronous algorithm. Note that in the asynchronous algorithm, only two agents are performing updates and thus, for a network of size $N$, $\tilde{k}$ global iterations translates to $2\tilde{k}/N$ iterations per agent, approximately.

- On comparing the performance of the synchronous algorithm (cf. Tables 1–4) to that of the asynchronous algorithm (cf. Tables 5–8), we observe that the synchronous algorithm performs better than its asynchronous counterpart in terms of mean error and the confidence width at termination. This is expected as in the synchronous setting, the players’ communicate more frequently and the network diffuses information faster than in the asynchronous setting.

- The nature of the connectivity graph plays an important role in the performance of the synchronous algorithm. However, such an influence in the asynchronous setting is less pronounced.

- In an effort to better understand the impact of connectivity, in Table 9, we compare the number of iterations required for the player’s to concur on the aggregate $\bar{g}^*$ within a threshold of $1e^{-3}$ when the network consists of $N = 20$ players. We also present a metric of connectivity density given by $p_{\text{min}}/p_{\text{max}}$ as well as the square root of the second largest eigenvalue of the expected weight matrix, i.e., $\sqrt{\lambda}$, which in effect determines the rate of information dissemination in the network. We note that the number of iterations needed to achieve

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The iteration number is the mean for 50 sample rounded to the smallest integer over-estimate.
the threshold error correlates with the value of $\sqrt{\lambda}$ and this prompts us to arrive at the following conclusion: Having a well-informed up-to-date neighbor is more important than having a denser connectivity. For instance, a wheel network has a poor connectivity of all the network type based on the $p_{\text{min}}/p_{\text{max}}$ criterion yet it has superior aggregate convergence to all but the complete network. In part, this is because the central agent in such a network updates throughout the course of the algorithm, allowing for good mixing of network wide information. In contrast, the cycle network though better connected yet cannot ensure good mixing of information, given that no agent has access to “good information.” Similarly, a complete network provides each agent with an opportunity to communicate with every other agent and thus ensures good mixing of information. The grid network falls between the wheel and the cycle network in terms of availability of well-informed neighbors and thus the performance.

Table 5: Mean error after $\tilde{k} = 5e4$ iterations for gossip algorithm

| N  | Cycle   | Wheel   | Grid    | Complete | Cycle   | Wheel   | Grid    | Complete |
|----|---------|---------|---------|----------|---------|---------|---------|----------|
| 20 | 2.29e-3 | 3.66e-5 | 3.66e-5 | 3.66e-5  | 2.51e-2 | 1.01e-4 | 1.93e-3 | 4.64e-5  |
| 50 | 2.80e-1 | 6.76e-2 | 1.76e-1 | 1.26e-3  | 1.22    | 2.33e-2 | 8.41e-3 | 3.68e-3  |

Table 6: Mean error after $\tilde{k} = 1e5$ iterations for gossip algorithm

| N  | Cycle   | Wheel   | Grid    | Complete | Cycle   | Wheel   | Grid    | Complete |
|----|---------|---------|---------|----------|---------|---------|---------|----------|
| 20 | 3.78e-5 | 3.66e-5 | 3.66e-5 | 3.66e-5  | 3.93e-3 | 3.65e-5 | 1.69e-4 | 3.67e-5  |
| 50 | 1.65e-1 | 1.09e-3 | 9.19e-2 | 6.23e-5  | 7.63e-1 | 1.99e-3 | 4.57e-1 | 2.83e-4  |

Table 7: Width of 90% confidence interval after $\tilde{k} = 5e4$ iterations for gossip algorithm

| N  | Cycle   | Wheel   | Grid    | Complete | Cycle   | Wheel   | Grid    | Complete |
|----|---------|---------|---------|----------|---------|---------|---------|----------|
| 20 | 1.87e-4 | 8.22e-7 | 5.34e-7 | 0e0      | 7.51e-2 | 4.76e-3 | 2.08e-2 | 3.23e-3  |
| 50 | 1.68e-2 | 6.33e-3 | 1.89e-2 | 1.77e-4  | 5.23e-1 | 7.23e-2 | 4.35e-1 | 2.87e-2  |

Table 8: Width of 90% confidence interval after $\tilde{k} = 1e5$ iterations for gossip algorithm

| N  | Cycle   | Wheel   | Grid    | Complete | Cycle   | Wheel   | Grid    | Complete |
|----|---------|---------|---------|----------|---------|---------|---------|----------|
| 20 | 2.4e-6  | 4.74e-10| 4.74e-10| 4.74e-10 | 4.40e-4 | 7.23e-7 | 2.07e-5 | 1.56e-7  |
| 50 | 1.35e-2 | 1.33e-4 | 1.40e-2 | 1.78e-8  | 6.91e-2 | 1.81e-4 | 4.15e-2 | 2.22e-5  |
Table 9: Number of iteration for concurrence of player’s aggregate within an error of 1e-3

| Network | $p_{\text{min}}/p_{\text{max}}$ | $\lambda$ | Iterations |
|---------|-----------------|--------|-----------|
| Cycle   | 1               | 0.9994 | 48818     |
| Wheel   | 1/19            | 0.1622 | 8324      |
| Grid    | 5/7             | 0.3151 | 17950     |
| Complete| 1               | 1.0888e-08 | 5842     |

7 Summary and Conclusions

This paper focuses on a class of Nash games in which player interactions are seen through the aggregate sum of all players’ actions. The players and their interactions are modeled as a network with limited connectivity which only allows for restricted local communication. We propose two types of algorithms, namely, a synchronous (consensus-based) and an asynchronous (gossip-based) distributed algorithm, both of which abide by an information exchange restriction for computation of an equilibrium point. Our synchronous algorithm allows for implementation in a dynamic network with a time-varying connectivity graph. In contrast, our asynchronous algorithm allows for an implementation in a static network. We establish error bounds on the deviation of players’s decision from the equilibrium decision when a constant, yet player specific, stepsize is employed in the asynchronous algorithm. Our extensions allow the players’ decisions to be coupled in a more general form of “aggregates”. The contribution of our work can broadly be summarized as: (1) the development of synchronous and asynchronous distributed algorithms for aggregative games over graphs; (2) the establishment of the convergence of the algorithms to an equilibrium point, including the case with player specific stepsizes; and (3) an extension to a more general classes of aggregative games. We also provide illustrative numerical results that support our theoretical findings.

Appendix

Proof of Lemma 7

Proof. Note that $\Gamma_{k,i} = \sum_{t=1}^{k} 1_{S_{i,t}}$ where $S_{i,t}$ is the event that agent $i$ updates at time $t$ and $1_S$ is the indicator function of an event $S$. Since the events $S_{i,t}, t = 1, 2, \ldots$, are i.i.d. with mean $p_t$ for each $i \in \mathcal{N}$, by the law of iterated logarithms (cf. [10], pages 476–479), we have for any $q > 0$, with probability 1,

$$\lim_{k \to \infty} \frac{\left| \Gamma_{k}(i) - kp_t \right|}{k^{3/2} + q} = 0 \text{ for all } i \in \mathcal{N}. \quad (68)$$

Thus, for almost every $\omega \in \Omega$, there exists a sufficiently large $\tilde{k}(\omega)$ (depending on $q$ and $\mathcal{N}$), such that

$$\frac{\left| \Gamma_{k}(i) - kp_t \right|}{k^{3/2} + q} \leq \frac{1}{\mathcal{N}^2} \text{ for all } k \geq \tilde{k}(\omega) \text{ and } i \in \mathcal{N}. \quad (69)$$

This implies that for almost every $\omega \in \Omega$ and for all $i \in \mathcal{N}$ and $k \geq \tilde{k}(\omega)$,

$$\Gamma_{k}(i) \geq kp_t - \frac{1}{\mathcal{N}^2}k^{3/2} + q = \left( p_t k^{3/2} - \frac{1}{\mathcal{N}^2} \right)k^{3/2} + q. \quad (69)$$
Now, let $q \in (0, 1/2)$, implying that the term $k^{1/2-q} p_i$ tends to infinity as $k$ tends to infinity. Thus, we can choose a larger $\tilde{k}(\omega)$ (if needed) so that with probability 1, we have
\[ p_i k^{1/2-q} - \frac{1}{N^2} \geq \frac{1}{2} p_i k^{1/2-q} \quad \text{for all } k \geq \tilde{k} \text{ and } i \in \mathcal{N}, \tag{70} \]
or equivalently
\[ \mathbb{P} \left[ \omega : p_i k^{1/2-q} - \frac{1}{N^2} \geq \frac{1}{2} p_i k^{1/2-q} \quad \text{for all } k \geq \tilde{k}(\omega) \text{ and } i \in \mathcal{N} \right] = 1. \]

From (69) and (70), we obtain that for almost every $\omega \in \Omega$, $\Gamma_k(i) \geq \frac{1}{2} k p_i$ for all $k \geq \tilde{k}(\omega)$ and all $i \in \mathcal{N}$, implying that
\[ \mathbb{P} \left[ \omega \cap \left. \Gamma_k(i) \leq \frac{2}{k p_i} \quad \text{for all } k \geq \tilde{k}(\omega) \text{ and } i \in \mathcal{N} \right] = 1, \tag{71} \]
thus showing the first relation of the lemma holds in view of $\alpha_{k,i} = \frac{1}{\Gamma_k(i)}$.

We now consider $|\alpha_{k,i} - \frac{1}{k p_i}|$. For almost every $\omega \in \Omega$, we have for all $k \geq \tilde{k}(\omega)$ and $i \in \mathcal{N}$,
\[ |\alpha_{k,i} - \frac{1}{k p_i}| \leq \frac{1}{k p_i} \frac{1}{\Gamma_k(i)} |k p_i - \Gamma_k(i)| \leq \frac{2}{k^2 p_i^2} |k p_i - \Gamma_k(i)|, \]
where the inequality follows by (71). By using relation (68), it follows that for almost every $\omega \in \Omega$ and for all $k \geq \tilde{k}(\omega)$ and $i \in \mathcal{N}$,
\[ |\alpha_{k,i} - \frac{1}{k p_i}| \leq \frac{2}{k^2 p_i^2} N^2 = \frac{2}{k^{4-q} p_i^2 N^2}. \]

Since agent $i$ updates with probability $p_i = \frac{1}{N} (1 + \sum_{j \in \mathcal{N}, \{i,j\} \in \mathcal{E}} p_{ij})$, it follows that
\[ p_i \geq \frac{1}{N} \left( 1 + \min_{\{i,j\} \in \mathcal{E}} p_{ij} \right) |\mathcal{N}| \geq \frac{1}{N} \left( 1 + \min_{\{i,j\} \in \mathcal{E}} p_{ij} \right), \tag{72} \]
where the last inequality follows from $|\mathcal{N}| \geq 1$ (since the graph $(\mathcal{N}, \mathcal{E})$ has no isolated node). Hence, for almost every $\omega \in \Omega$ and for all $k \geq \tilde{k}(\omega)$ and all $i$,
\[ |\alpha_{k,i} - \frac{1}{k p_i}| \leq \frac{2 N^2}{k^{4-q} \left( 1 + \min_{\{i,j\} \in \mathcal{E}} p_{ij} \right)^2} = \frac{2}{k^{4-q} \left( 1 + \min_{\{i,j\} \in \mathcal{E}} p_{ij} \right)^2}. \]

The desired relation follows by letting $\hat{p} = 1 + \min_{\{i,j\} \in \mathcal{E}} p_{ij}$:
\[ \mathbb{P} \left[ \omega : \left| \alpha_{k,i} - \frac{1}{k p_i} \right| \leq \frac{2}{k^{4-q} \left( 1 + \min_{\{i,j\} \in \mathcal{E}} p_{ij} \right)} \quad \text{for } k \geq \tilde{k}(\omega) \right] = 1. \]
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References

[1] C. Alos-Ferrer and A. B. Ania. The evolutionary stability of perfectly competitive behavior. Economic Theory, 26(3):497–516, 2005.

[2] T. Alpcan and T. Başar. A game-theoretic framework for congestion control in general topology networks. In Proceedings of the 41st IEEE Conference on Decision and Control, pages 1218 – 1224 vol.2, 2002.

[3] T. Alpcan and T. Başar. Distributed algorithms for Nash equilibria of flow control games. In Advances in Dynamic Games, volume 7 of Annals of the International Society of Dynamic Games, pages 473–498. Birkhäuser Boston, 2003.

[4] T. Başar. Control and game-theoretic tools for communication networks. Appl. Comput. Math., 6(2):104–125, 2007.

[5] P. Bianchi and J. Jakubowicz. Convergence of a multi-agent projected stochastic gradient algorithm. http://arxiv.org/abs/1107.2526, 2012.

[6] H. Boche, M. Wiczanowski, and S. Stanczak. Unifying view on min-max fairness, max-min fairness, and utility optimization in cellular networks. EURASIP J. Wirel. Commun. Netw., 2007(1):5–5, Jan. 2007.

[7] S. Boyd, A. Ghosh, B. Prabhakar, and D. Shah. Randomized gossip algorithms. IEEE Transactions on Information Theory, 52(6):2508 – 2530, 2006.

[8] A. Cournot. Recherches sur les principes mathematiques de la theorie des richesses. Paris: L. Hachette, 1838.

[9] P. Dubey, O. Haimanko, and A. Zapechelnyuk. Strategic complements and substitutes, and potential games. Games and Economic Behavior, 54(1):77–94, January 2006.

[10] R. Dudley. Real Analysis and Probability. Cambridge University Press, 2002.

[11] F. Facchinei and J.-S. Pang, Finite-Dimensional Variational Inequalities and Complementarity Problems, volume I and II. Springer-Verlag New York, 2003.

[12] F. Facchinei and J.-S. Pang. Nash equilibria: The variational approach. Convex Optimization in Signal Processing and Communication, Cambridge University Press, to appear in 2009.

[13] D. Fudenberg and D. K. Levine. The theory of learning in games, volume 2 of MIT Press Series on Economic Learning and Social Evolution. MIT Press, Cambridge, MA, 1998.

[14] D. Fudenberg and J. Tirole. Game Theory. MIT Press, 1991.
[15] M. Huang, P. E. Caines, and R. P. Malham’e. Large-population cost-coupled LQG problems with nonuniform agents: Individual-mass behavior and decentralized $\epsilon$-nash equilibria. *IEEE Transactions on Automatic Control*, 52(9):1560–1571, 2007.

[16] M. Jensen. Aggregative games. Discussion Papers 06-10, Department of Economics, University of Birmingham, 2006.

[17] M. Jensen. Aggregative games and best-reply potentials. *Econom. Theory*, 43(1):45–66, 2010.

[18] B. Johansson. *On Distributed Optimization in Networked Systems*. PhD thesis, Royal Institute of Technology (KTH), Dec. 2008. TRITA-EE 2008:065.

[19] B. Johansson, M. Rabi, and M. Johansson. A randomized incremental subgradient method for distributed optimization in networked systems. *SIAM Journal on Optimization*, 20(3):1157–1170, Aug 2009.

[20] A. Juditsky, A. Nemirovski, and C. Tauvel. Solving variational inequalities with stochastic mirror-prox algorithm. *Stochastic Systems*, 1(1):17–58, 2011.

[21] A. Kannan and U. V. Shanbhag. Distributed iterative regularization algorithms for monotone Nash games. *Proceedings of the IEEE Conference on Decision and Control*, pages 1963–1968, 2010.

[22] A. Kannan and U. V. Shanbhag. Distributed computation of equilibria in monotone Nash games via iterative regularization techniques. *SIAM Journal on Optimization*, 22(4):1177–1205, 2012.

[23] J. Koshal, A. Nedić, and U. V. Shanbhag. Single timescale regularized stochastic approximation schemes for monotone Nash games under uncertainty. *Proceedings of the IEEE Conference on Decision and Control*, pages 231–236, 2010.

[24] J. Koshal, A. Nedić, and U. V. Shanbhag. Regularized iterative stochastic approximation methods for stochastic variational inequality problems. *IEEE Trans. Automat. Contr.*, 58(3):594–609, 2013.

[25] N. Li and J. R. Marden. Designing games for distributed optimization. *J. Sel. Topics Signal Processing*, 7(2):230–242, 2013.

[26] I. Lobel and A. E. Ozdaglar. Distributed subgradient methods for convex optimization over random networks. *IEEE Trans. Automat. Contr.*, 56(6):1291–1306, 2011.

[27] J. R. Marden, S. D. Ruben, and L. Y. Pao. A model-free approach to wind farm control using game theoretic methods. *IEEE Trans. Contr. Sys. Techn.*, 21(4):1207–1214, 2013.

[28] D. Martimort and L. Stole. Aggregate representations of aggregate games. MPRA Paper 32871, University Library of Munich, Germany, 2011.

[29] C. Moallemi and B. Van Roy. Consensus propagation. *Information Theory, IEEE Transactions on*, 52(11):4753–4766, Nov 2006.
[30] C. Moallemi and B. Van Roy. Convergence of min-sum message-passing for convex optimization. *Information Theory, IEEE Transactions on*, 56(4):2041–2050, April 2010.

[31] D. Monderer and L. Shapley. Potential games. *Games and Economic Behavior*, 14:124–143, 1996.

[32] J. F. Nash, Jr. Equilibrium points in n-person games. *Proc. Nat. Acad. Sci. U. S. A.*, 36:48–49, 1950.

[33] A. Nedić. Asynchronous broadcast-based convex optimization over a network. *IEEE Transactions on Automatic Control*, 56(6):1337–1351, 2011.

[34] A. Nedić, A. Olshevsky, A. Ozdaglar, and J. Tsitsiklis. On distributed averaging algorithms and quantization effects. In *Decision and Control, 2008. CDC 2008. 47th IEEE Conference on*, pages 4825 –4830, 2008.

[35] A. Nedić and A. E. Ozdaglar. Distributed subgradient methods for multi-agent optimization. *IEEE Trans. Automat. Contr.*, 54(1):48–61, 2009.

[36] A. Nedić, A. E. Ozdaglar, and P. A. Parrilo. Constrained consensus and optimization in multi-agent networks. *IEEE Trans. Automat. Contr.*, 55(4):922–938, 2010.

[37] W. Novshek. On the existence of cournot equilibrium. *Review of Economic Studies*, 52(1):85–98, 1985.

[38] K. Okuguchi and F. Szidarovszky. *The theory of oligopoly with multi-product firms*, volume 342 of *Lecture Notes in Economics and Mathematical Systems*. Springer-Verlag, Berlin, 1990.

[39] Y. Pan, T. Alpcan, and L. Pavel. A system performance approach to osnr optimization in optical networks. *IEEE Transactions on Communications*, 58(4):1193–1200, 2010.

[40] Y. Pan and L. Pavel. Games with coupled propagated constraints in optical network with multi-link topologies. *Automatica*, 45:871–880, 2009.

[41] C. Papadimitriou and M. Yannakakis. On bounded rationality and computational complexity. Technical report, Indiana University, 1994.

[42] L. Pavel. A noncooperative game approach to OSNR optimization in optical networks. *IEEE Transactions on Automatic Control*, 51(5):848–852, 2006.

[43] L. Pavel. An extension of duality to a game-theoretic framework. *Automatica*, 43:226–237, 2007.

[44] B. Polyak. *Introduction to Optimisation*. Optimization Software, Inc., New York, 1987.

[45] S. Ram, A. Nedić, and V. Veeravalli. Asynchronous gossip algorithms for stochastic optimization. In *Proceedings of the 48th IEEE Conference on Decision and Control*, pages 3581–3586, 2009.
[46] S. Ram, A. Nedić, and V. Veeravalli. Asynchronous gossip algorithms for stochastic optimization: Constant stepsize analysis. In M. Diehl, F. Glineur, E. Jarlebring, and W. Michiels, editors, Recent Advances in Optimization and its Applications in Engineering, pages 51–60. Volume of the 14th Belgian-French-German Conference on Optimization (BFG), 2010.

[47] S. Ram, A. Nedić, and V. Veeravalli. Distributed stochastic subgradient projection algorithms for convex optimization. Journal of Optimization Theory and Applications, 147:516–545, 2010.

[48] S. Ram, A. Nedić, and V. Veeravalli. A new class of distributed optimization algorithms: Application to regression of distributed data. Optimization Methods and Software, 27(1):71–88, 2012.

[49] A. Rantzer. Using game theory for distributed control engineering. Technical report, Department of Automatic Control, Lund University, Sweden, July 2008.

[50] A. Rantzer. Dynamic dual decomposition for distributed control. In Proceedings of American Control Conference, St. Louis, June 2009.

[51] B. Schipper. Pseudo-potential games, 2004.

[52] G. Scutari and J. S. Pang. Joint sensing and power allocation in nonconvex cognitive radio games: Nash equilibria and distributed algorithms. IEEE Transactions on Information Theory, 59(7):4626–4661, 2013.

[53] H. A. Simon. The sciences of the artificial (3rd ed.). MIT Press, Cambridge, MA, USA, 1996.

[54] K. Srivastava and A. Nedić. Distributed asynchronous constrained stochastic optimization. IEEE Journal of Selected Topics in Signal Processing, 5(4):772–790, 2011.

[55] K. Srivastava, A. Nedić, and D. Stipanović. Distributed bregman-distance algorithms for min-max optimization. To appear in the book Agent-Based Optimization I. Czarnowski, P. Jedrzejowicz and J. Kacprzyk (Eds.), Springer Studies in Computational Intelligence (SCI), 2012.

[56] N. Stefanovic and L. Pavel. A lyapunov-krasovskii stability analysis for game-theoretic based power control in optical links. Telecommunication Systems, 47(1-2):19–33, 2011.

[57] J. Tsitsiklis. Problems in decentralized decision making and computation. PhD thesis, Massachusetts Institute of Technology, 1984.

[58] S.-Y. Tu and A. H. Sayed. Diffusion strategies outperform consensus strategies for distributed estimation over adaptive networks. CoRR, abs/1205.3993, 2012.

[59] S.-Y. Tu and A. H. Sayed. On the influence of informed agents on learning and adaptation over networks. CoRR, abs/1203.1524, 2012.

[60] H. Yin, P. G. Mehta, S. P. Meyn, and U. V. Shanbhag. Synchronization of coupled oscillators is a game. IEEE Trans. Automat. Contr., 57(4):920–935, 2012.

[61] H. Yin, U. V. Shanbhag, and P. G. Mehta. Nash equilibrium problems with congestion costs and shared constraints. Proceedings of the IEEE Conference on Decision and Control, pages 4649–4654, 2009.
[62] H. Yin, U. V. Shanbhag, and P. G. Mehta. Nash equilibrium problems with scaled congestion costs and shared constraints. *IEEE Transactions on Automatic Control*, 56(7):1702–1708, 2011.

[63] F. Yousefian, A. Nedić, and U. V. Shanbhag. A regularized smoothing stochastic approximation (RSSA) algorithm for stochastic variational inequality problems. In *Winter Simulations Conference: Simulation Making Decisions in a Complex World, WSC 2013, Washington, DC, USA, December 8-11, 2013*, pages 933–944. IEEE, 2013.