Representations of General Dimension for the Skyrme Model

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Abstract

We construct the representations of general dimension for the soliton solution to the $SU(2)$ Skyrme model, and show that at the classical level the dependence on the dimension of the representation $(2j+1)$ appears only as an overall factor $j(j+1)(2j+1)$ in the Lagrangian density, which may be absorbed by a rescaling of the parameters. Alternate stabilizing terms in the model will in general have a different $j$-dependence and have to be rescaled accordingly in order to achieve representation independent predictions. In contrast the quantum corrections do depend on the dimension of the representation and in general differ from those obtained in the fundamental representation of $SU(2)$.
1. Introduction

Skyrme’s topological soliton model for the baryons [1,2] and its immediate generalizations [3] have proven able to provide a qualitatively remarkably successful description of most of the observed properties of the nucleons, including the hyperons [4,5]. The model is formed by a Lagrangian density for an $SU(2)$ field $U$, and topological baryon current $B^\mu$, which is conserved by the unitarity condition $UU^\dagger = 1$ independently of the form of the Lagrangian.

The soliton solution is obtained by the hedgehog ansatz

$$U_0 = e^{i\vec{\tau} \cdot \hat{F}(r)},$$

(1.1)

where $\vec{\tau}$ is a Pauli-isospin matrix and $F(r)$ a scalar function, which satisfies a second order differential equation that is obtained by the requirement that the solution lead to a stationary energy. The spherical components of the operator $\frac{1}{2}\vec{\tau}$ form the generators of the fundamental representation of the group $SU(2)$, and satisfy the commutation relations

$$[\hat{J}_a, \hat{J}_b] = \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \end{bmatrix} \hat{J}_c,$$

(1.2)

where the factor on the r.h.s. is the Clebsch-Gordan coefficient $(1a1b|1c)$, in a more convenient notation. Here we have used the normalization $\hat{J}_\pm = -J_{\pm 1}/\sqrt{2}$, $\hat{J}_0 = -J_0/\sqrt{2}$.

In this paper we shall form the analogs to the hedgehog solution (1.2) for representations of arbitrary integral dimension and derive the corresponding expressions for the soliton mass. At the classical level we find that the predictions for the static baryon observables can be made independent of the dimension of the representation by a dimension dependent rescaling of the parameters of the model. In the case of the original Skyrme model the dependence on the dimension $(2j + 1)$ of the representation appears only in the form of an overall factor $j(j + 1)/(2j + 1)$. The quantized Hamiltonian for the soliton, will in contrast depend nontrivially on the dimensionality. This opens up a hitherto unexplored phenomenological degree of freedom, which may be of dynamical significance.

In section 2 of this note we express the Skyrme model and its most direct generalizations in a general representation of the group $SU(2)$ and derive the explicit representation dependence of the terms in the Lagrangian density at the
classical level. In section 3 we derive the expression for the hedgehog solution and the corresponding soliton mass for the case of a representation of arbitrary order. In section 4 we consider the quantized version of the model. Section 5 contains a summarizing discussion.

2. The Skyrme model in a general representation

In an irreducible representation of the group $SU(2)$ of dimension $j$ the unitary field $U$ has the form

$$U(\vec{r}, t) = D^j(\vec{\alpha}(\vec{r}, t)),$$

(2.1)

where $\vec{\alpha}(\vec{r}, t)$ is a vector formed of three Euler angles $\vec{\alpha} \equiv (\alpha^1, \alpha^2, \alpha^3)$:

$$0 \leq \alpha^1 < 2\pi, \quad 0 \leq \alpha^2 \leq \pi, \quad 0 \leq \alpha^3 < 4\pi,$$

(2.2)

and $D^j$ is the Wigner $D$-function, that has the explicit form

$$D^j_{m'm}(\vec{\alpha}) = \langle jm| e^{i\sqrt{2}\alpha^1 J_0} e^{-\alpha^2 (J_+ + J_-)} e^{i\sqrt{2}\alpha^3 J_0} |jm'\rangle.$$  (2.3)

The Euler angles $\vec{\alpha}$ then form the dynamical variables of the theory. Note that the trace of a bilinear combination of two generators of the groups depends on the dimension of the representation ($j$) as

$$Tr\langle jm| J_a J_b |jm'\rangle = (-)^a \frac{1}{6} j(j+1)(2j+1) \delta_{a,-b}.$$  (2.4)

The Skyrme model is the chirally symmetric Lagrangian density

$$\mathcal{L} = -\frac{f^2}{4} Tr\{R_\mu R^\mu\} + \frac{1}{32\epsilon^2} Tr\{[R_\mu, R_\nu]^2\},$$

(2.5)

where the ”right” current $R_\mu$ is defined as

$$R_\mu = (\partial_\mu U)U^\dagger.$$  (2.6)

and $f_\pi$ (the pion decay constant) and $\epsilon$ are parameters. For the purpose of expressing the Lagrangian density in terms of the Euler angles $\{\vec{\alpha}\}$ it is convenient to note that

$$\frac{\partial}{\partial \alpha_i} D^j_{mn}(\vec{\alpha}) = C_i^{(a)}(\vec{\alpha}) \langle jm| J_a |jm'\rangle D^j_{m'n}(\vec{\alpha}).$$

(2.7)
Here the coefficients \( C_i^{(a)}(\vec{\alpha}) \) have the explicit form

\[
\begin{align*}
C_1^{(+)}(\vec{\alpha}) &= 0, & C_2^{(+)}(\vec{\alpha}) &= -e^{-i\alpha_1}, & C_3^{(+)}(\vec{\alpha}) &= -i \sin \alpha^2 e^{-i\alpha_1}, \\
C_1^{(0)}(\vec{\alpha}) &= i\sqrt{2}, & C_2^{(0)}(\vec{\alpha}) &= 0, & C_3^{(0)}(\vec{\alpha}) &= i\sqrt{2} \cos \alpha^2, \\
C_1^{(-)}(\vec{\alpha}) &= 0, & C_2^{(-)}(\vec{\alpha}) &= -e^{i\alpha_1}, & C_3^{(-)}(\vec{\alpha}) &= i \sin \alpha^2 e^{i\alpha_1}.
\end{align*}
\]  
\tag{2.8}

The right current \( R_\mu \) then takes the form

\[
(R_\mu)_{mnm'} = \partial_\mu \alpha^i C_i^{(a)}(\vec{\alpha}) \langle jm|\hat{J}_a|jm'\rangle,
\]  
\tag{2.9}

where summation over the indices \( i \) and \( a \) is understood.

The chiral invariance of the theory is the invariance under the transformation

\[
U(\vec{r}, t) \rightarrow VU(\vec{r}, t)W^\dagger,
\]  
\tag{2.10}

in the \( (j, j) \) representation of \( SU(2) \times SU(2) \), where the left and right transformation matrices \( V \) and \( W^\dagger \) belong to the irreducible representations of the product groups. Under the left chiral transformation the right current \( R_\mu \) transforms as

\[
R'_\mu = VR_\mu V^\dagger = \partial_\mu \alpha^i C_i^{(a)}(\vec{\alpha}) \langle j|\hat{J}_a'|j\rangle D_{a'a}^{1}(\vec{\beta}).
\]  
\tag{2.11}

Here \( \vec{\beta} \) are the Euler angles that define the transformation matrix \( V \).

When reexpressed in terms of the Euler angles \( \vec{\alpha} \) the Lagrangian density (2.4) takes the form

\[
\mathcal{L} = \frac{1}{3}j(j+1)(2j+1)\left\{ \frac{f_\pi^2}{4} \left[ \partial_\mu \alpha^i \partial^\mu \alpha^i + 2 \cos \alpha^2 \partial_\mu \alpha^1 \partial^\mu \alpha^3 \right] \\
- \frac{1}{16\epsilon^2} \left[ \partial_\mu \alpha^2 \partial^\mu \alpha^2 (\partial_\nu \alpha^1 \partial^\nu \alpha^1 + \partial_\nu \alpha^3 \partial^\nu \alpha^3) - (\partial_\mu \alpha^1 \partial^\mu \alpha^2)^2 \\
- (\partial_\mu \alpha^2 \partial^\mu \alpha^3)^2 + \sin^2 \alpha^2 \partial_\mu \alpha^1 \partial^\mu \alpha^1 \partial_\nu \alpha^3 \partial^\nu \alpha^3 - (\partial_\mu \alpha^1 \partial^\mu \alpha^3)^2 \right] \\
+ 2 \cos \alpha^2 [\partial_\mu \alpha^2 \partial^\mu \alpha^2 \partial_\nu \alpha^1 \partial^\nu \alpha^3 - \partial_\mu \alpha^1 \partial^\mu \alpha^2 \partial_\nu \alpha^2 \partial^\nu \alpha^3] \right\}.
\]  
\tag{2.12}

Note that the only dependence on the dimension of the representation is in the overall factor \( j(j+1)(2j+1) \) in the Lagrangian density. This implies that the equation of motion for the dynamical variable \( \vec{\alpha} \) is independent of the dimension of the representation as the common factor can be absorbed into the two
parameters of the model.

The conserved topological current in the Skyrme model is the baryon current

\[ B^\mu = N \epsilon^{\mu\nu\beta\gamma} \text{Tr} R_\nu R_\beta R_\gamma, \]  

(2.13)

where the normalization factor \( N \) depends on the dimension of the representation and has the value \( 1/24\pi^2 \) in the fundamental representation. The baryon number \( B \) is obtained as the spatial integral of the time component \( B^0 \). In terms of the Euler angle variables \( \vec{\alpha} \) the baryon current takes the form

\[ B^\mu = -\frac{N}{6} j(j+1)(2j+1) \sin \alpha^2 \epsilon^{\mu\nu\beta\gamma} \epsilon_{ikl} \partial_\nu \alpha^i \partial_\beta \alpha^k \partial_\gamma \alpha^l. \]  

(2.14)

As the dimensionality of the representation appears in this expression in the same overall factor as in the Lagrangian density (2.11) it follows that all calculated dynamical observables will be independent of the dimension of the representation at the classical level, if this factor \( j(j+1)(2j+1) \) is absorbed into the parameters of the model.

There exists an infinite class of alternate stabilizing terms for the Lagrangian density (2.5), combinations of which can be used in place of Skyrme’s quartic stabilizing term or be added to it [3]. An alternate term of quartic order, which leads to identical results as the Skyrme term in the fundamental representation, is [6]

\[ \mathcal{L}_4' = \frac{1}{16\epsilon^2} \{(\text{Tr} R_\mu R_\nu)^2 - (\text{Tr} R_\mu R^\mu)^2\}. \]  

(2.15)

When this term is expressed in terms of the Euler angles \( \{\vec{\alpha}\} \) (2.1), the resulting Lagrangian density has the form (2.12), with the exception that stabilizing term that is proportional to \( \epsilon^{-2} \) has an additional factor \( \frac{2}{\pi} j(j+1)(2j+1) \). Hence invariance of the physical predictions requires that the parameter \( 1/\epsilon^2 \) of the stabilizing term (2.15) be taken to be proportional to \( [j(j+1)(2j+1)]^{-2} \), and the parameter \( (f_\pi) \) of the quadratic term to be proportional to \( [j(j+1)(2j+1)]^{-1} \), when a representation of dimension \( j \) is employed.

Consider then the sixth order stabilizing term [3,7]

\[ \mathcal{L}_6 = e_6 \text{Tr} \{[R_\mu, R^\nu][R_\nu, R^\lambda][R_\lambda, R_\mu]\}. \]  

(2.16)
In terms of the Euler angles \{\vec{\alpha}\} this Lagrangian density takes the form

$$L_6 = -\frac{j(2j+1)}{6}e_6 \epsilon_{i_1i_2i_5} \epsilon_{i_3i_4i_6} \sin^2 \alpha^2$$

$$\partial_\mu \alpha^{i_1} \partial_\nu \alpha^{i_2} \partial_\rho \alpha^{i_3} \partial_\lambda \alpha^{i_4} \partial_\sigma \alpha^{i_5} \partial_\tau \alpha^{i_6}.$$  \hfill (2.17)

This result reveals that the dependence on the dimension of the representation of this term is contained in the same overall factor \(j(2j+1)\) as the Skyrme model Lagrangian (2.12). Hence addition of the term \(L_6\) maintains the simple overall dimension dependent factor of the original Skyrme model.

As in the case of the quartic term one can construct an alternate sixth order term, which is equivalent to (2.16) in the case of the fundamental representation, but which differs in its dependence on the dimension \(j\):

$$L'_6 = e'_6 \epsilon^{\mu_1\nu_2\nu_3} \epsilon_{\mu_1\eta_2\eta_3} \text{Tr} \{R_{\nu_1} R_{\nu_2} R_{\nu_3} R_{\eta_1} R_{\eta_2} R_{\eta_3}\}.$$ \hfill (2.18)

In terms of the Euler angles \{\vec{\alpha}\} this term also reduces to the expression (2.17), with the exception of an additional factor \(j(2j+1)\epsilon'_6/e_6\). Its dependence on \(j\) is thus different from (2.16), although by adjusting the values of the parameters \(e_6\) and \(e'_6\) differently in each representation equivalent dynamical predictions can be maintained.

3. The hedgehog solution in a general representation

The hedgehog field (1.1) represents the soliton solution in the fundamental representation of \(SU(2)\). In order to find its generalizations in representations of higher dimension one may compare it to the matrix elements \(D_{mm'}^{1/2}(\vec{\alpha})\), and thus obtain the explicit expressions for the Euler angles \(\vec{\alpha}\) in terms of the chiral angle \(F(r)\). The result is

$$\alpha^1 = \varphi - \arctan(\cos \vartheta \tan F(r)) - \pi/2,$$

$$\alpha^2 = -2 \arcsin(\sin \vartheta \sin F(r)),$$

$$\alpha^3 = -\varphi - \arctan(\cos \vartheta \tan F) + \pi/2.$$ \hfill (3.1)

Here the angles \(\varphi, \vartheta\) are the polar angles that define the direction of the unit vector \(\hat{r}\) in spherical coordinates.
Substitution of the expressions (3.1) into the general expression (2.1) for the unitary field \( U \) then gives the hedgehog field in a representation with arbitrary \( j \). As an example the hedgehog field in the representation \( j = 1 \) has the form

\[
U_0 = \sin^2 F \begin{pmatrix}
G^2 & i\sqrt{2}G \sin \vartheta e^{-i\varphi} & -\sin^2 \vartheta e^{-2i\varphi} \\
n\sqrt{2}G \sin \vartheta e^{i\varphi} & \cot^2 F + \cos 2\vartheta & i\sqrt{2} \sin \vartheta G^* e^{-i\varphi} \\
\sin^2 \vartheta e^{2i\varphi} & i\sqrt{2}G^* \sin \vartheta e^{i\varphi} & G^*e^2
\end{pmatrix}.
\] (3.2)

Here we have used the abbreviations

\[ G = \cot F + i \cos \vartheta. \] (3.3)

The Lagrangian density (2.12) reduces to the following simple form, when the hedgehog ansatz (3.1) is employed:

\[
\mathcal{L} = -\frac{4}{3} j(j+1)(2j+1) \left\{ \frac{f_\pi^2}{4} \left( F''^2 + \frac{2}{r^2} \sin^2 F \right) + \frac{1}{16e^2} \frac{\sin^2 F}{r^2} \left( 2F'^2 + \frac{\sin^2 F}{r^2} \right) \right\}. \] (3.4)

For \( j = 1/2 \) this reduces to the result of ref. [2]. The corresponding mass density is obtained by reversing the sign of \( \mathcal{L} \).

The requirement that the soliton mass be stationary yields the equation of motion for the chiral angle \( F \) [2,8]:

\[
f_\pi^2 \left( F'' + \frac{2}{r} F' - \frac{\sin 2F}{r^2} \right) - \frac{1}{e^2} \left( \frac{1}{r^4} \sin^2 F \sin 2F \right.
\]

\[
\left. \frac{1}{r^2} (F'^2 \sin 2F + 2F'' \sin^2 F) \right) = 0,
\] (3.5)

which is independent of the dimension of the representation. Note that this differential equation is nonsingular only if \( F(0) \) is an integer multiple of \( \pi \).

For the hedgehog form the baryon density reduces to the expression

\[ B^0 = -8Nj(j+1)(2j+1) \frac{\sin^2 F}{r^2} F'. \] (3.6)

The corresponding baryon number is

\[ B = \int d^3r B^0 = 16N\pi j(j+1)(2j+1) [F(0) - \frac{1}{2} \sin 2F(0)]. \] (3.7)
Combining the requirement that $F(0)$ be an integer multiple of $\pi$ with the requirement that the lowest nonvanishing baryon number be 1 gives the general expression for the normalization factor $N$ as

$$N = \frac{1}{16\pi^2 j(j + 1)(2j + 1)}, \quad (3.8)$$

which reduces to the usual result $1/24\pi^2$ for $j = 1/2$.

4. The quantized Lagrangian density

In the quantized version of the Lagrangian density (2.5) of the Skyrme model (and the generalizations of it considered in section 2 above) the analogs of the Euler angles $\{\vec{\alpha}\}$ that define the $SU(2)$ matrices $U$ are a set of three real parameters $\{\vec{q}\}$, which satisfy the general commutation relations [9]

$$[q^a, q^b] = -if^{ab}(\vec{q}). \quad (4.1)$$

Here the tensor $f^{ab}$ is a function of the generalized coordinates $\{\vec{q}\}$ only, the explicit form of which is determined after the quantization condition has been imposed. Explicit expressions for $f^{ab}$ have given in ref.[9] for the case of the fundamental representation using the method of collective coordinates in the quantization. The tensor $f^{ab}$ is symmetric with respect to interchange of the indices $a$ and $b$ as a consequence of the commutation relation $[q^a, q^b] = 0$. The commutator between a generalized velocity component $\dot{q}^a$ and a function $F$ of the coordinates is given by

$$[\dot{q}^a, F(\vec{q})] = -i \sum_r f^{ar}(\vec{q}) \frac{\partial}{\partial q^r} F(\vec{q}) = -i \sum_r f^{ar}(\vec{q}) \nabla_r F(\vec{q}). \quad (4.2)$$

The commutation relation (4.1) leads to complicated expressions for the time derivatives of the $SU(2)$ matrices $U$, $U^\dagger$, which appear in the time components of the right invariant current $R$ (2.6). In a representation of the group $SU(2)$ of dimension $j$ we find the expression

$$\begin{align*}
(R_0)^j_{\ell m n} &= \frac{1}{2} \{\dot{q}^i, C_{(a)}^{(i)}(\vec{q})\} \langle jm|\vec{J}_a|jn\rangle \\
&+ \frac{1}{2} \sum_{\ell=0,2} i f^{kd}(\vec{q}) C^{(a)}(\vec{q}) C^{(b)}(\vec{q}) \sum_{l=0,2} \left[ \begin{array}{cc} 1 & 1 \\ a & b \\ u & \end{array} \right] \langle jm|\vec{J}_a^l|jn\rangle. \quad (4.3)
\end{align*}$$

Here \{, \} denotes an anticommutator. Whereas in the classical case the currents form an \(SU(2)\) algebra, the time components of the currents in the quantum mechanical case belong to the product space of two generators, the elements of which can be decomposed into sums of tensors of rank 0, 1 and 2 as indicated in the r.h.s. of (4.3).

The expression (4.3) implies that the fundamental representation represents a special case, because in it the generators can be expressed as Pauli matrices, the algebra of which is exceptionally simple, and for which

\[
\langle \frac{1}{2}m|\hat{J}_u^2|\frac{1}{2}n\rangle = 0, \quad (4.4)
\]

so that the rank two tensors absent in this representation.

The quantal analog of the chiral transformation law (2.11) is

\[
(P_0')^j_{mn} = \frac{1}{2}\{\hat{q}^2, C_i^{(a)}(\hat{q})\} \langle jm|\hat{J}_a'|jn\rangle D_{a'a}(\hat{\beta})
\]

\[
+ \frac{i}{2} f^{kd}(\hat{q}) C_i^{(a)}(\hat{q}) C_d^{(b)}(\hat{q}) \sum_{l=0,2} \left[ \begin{array}{ccc} 1 & 1 & l \\ a & b & u \end{array} \right] \langle jm|\hat{J}_u^l|jn\rangle D_{u'u}(\hat{\beta}), \quad (4.5)
\]

where \(\hat{\beta}\) are the Euler angles that define the transformation matrix.

In the quantum mechanical case the Lagrangian density (2.5) (as well as the additional terms (2.15)-(2.17)) remain invariant under chiral transformations (2.10), (4.5). The explicit expressions will in this case depend on the tensor \(f^{ab}(\hat{q})\) that determines the commutation relation (4.1). The quantum mechanical expression for the quadratic term which depends on time derivatives in the Lagrangian density is

\[
\mathcal{L}'_2 = -\frac{f^2}{24} j(j+1)(2j+1) \left\{ \hat{q}^a g_{ab} \hat{q}^b ight\}
\]

\[
- \frac{1}{4} f^{a'b'} \nabla_a'(f^{b'c'} g_{ab}) - \frac{j(j+1)}{24(2j+1)} f^{ab} g_{ab} f^{a'b'} g_{a'b'}
\]

\[
- \frac{1}{80} (2j-1)(2j+3) f^{ab} C_a^{(k)} C_b^{(l)} \left[ \begin{array}{ccc} 1 & 1 & 2 \\ k & l & k+l \end{array} \right]
\]

\[
\times f^{a'b'} C_a^{(m)} C_b^{(-k-l-m)} \left[ \begin{array}{ccc} 1 & 1 & 2 \\ m & -k-m & -k-l \end{array} \right] \right\}. \quad (4.6)
\]
Here we have defined the $3 \times 3$ tensor $g_{ab}$ as the scalar product

$$g_{ab} = \sum_{m} (-)^{m} C_{a}^{(m)} C_{b}^{(-m)} = -2 \delta_{ab} - 2(\delta_{a1}\delta_{b3} + \delta_{a3}\delta_{b1}) \cos q^2. \quad (4.7)$$

In terms of the quantum mechanical variables $\{\tilde{q}\}$ the quartic term of the Lagrangian density (2.5) is

$$\mathcal{L}' = \frac{1}{192e^2} j(j+1)(2j+1) \left\{ \tilde{q}^a \tilde{g}_{ab} \tilde{q}^b - \frac{1}{4} f^{b\mu} \nabla_{\nu} (f^{a\mu} \nabla_{\nu} \tilde{g}_{ab}) - \frac{j(j+1)}{6(2j+1)} (-)^{k+m} \partial_{\mu} q^a \partial_{\mu} q^b C_{a}^{(k)} C_{b}^{(m)} f^{a\mu'} \nabla_{\nu'} C_{a'}^{(-k)} f^{b\mu'} \nabla_{\nu'} C_{b'}^{(-m)} + \frac{(-)^{1+k}}{20} (2j-1)(2j+3) \partial_{\mu} q^a \partial_{\mu} q^b \right\} \times \left( \sqrt{6} C_{a'}^{(k')} f^{a\mu'} C_{a}^{(k-k'-k'')} C_{a'}^{(k'')} \left[ \begin{array}{ccc} 1 & 1 & 2 \\ k-k' & k'' & k-k' \\ k-k' & k & k \end{array} \right] \left[ \begin{array}{ccc} 2 & 1 & 2 \\ k-k' & k & k \end{array} \right] \right) \times \left( \sqrt{6} C_{b'}^{(m')} f^{b\mu'} C_{b}^{(-k-m-m')} C_{b'}^{(m'')} \left[ \begin{array}{ccc} 1 & 1 & 2 \\ -k-m' & m'' & -k-m' \\ -k-m' & m' & -k \end{array} \right] \right) \times \left[ \begin{array}{ccc} 2 & 1 & 2 \\ k-m' & m' & -k \end{array} \right] - 2C_{b}^{(-k-m')} f^{b\mu'} \nabla_{\nu'} C_{b'}^{(m')} \left[ \begin{array}{ccc} 1 & 1 & 2 \\ -k-m' & m' & -k \end{array} \right] \right). \quad (4.8)$$

Here the tensor $\tilde{g}_{ab}$ has been defined as

$$\tilde{g}_{ab} = (-)^{k} \partial_{\mu} q^a \partial_{\mu} q^b (\nabla_{\mu} C_{a}^{(k)} - \nabla_{\mu} C_{a'}^{(k)}) (\nabla_{\nu} C_{b'}^{(-k)} - \nabla_{\nu} C_{b}^{(-k)}).$$

The terms without time derivatives are the same as in classical case (2.12). In the fundamental representation ($j = 1/2$) the last two terms in the Lagrangian densities (4.6) and (4.8), which contain the factor $(2j-1)$ vanish. In this case using the method of collective coordinates the quantized Lagrangian of the Skyrme model reduces to the case considered in ref. [9]. It is the presence of those terms which leads to the essential inequivalence of the quantized Skyrme...
model Lagrangians that appear in representations of higher dimensionality and that in the fundamental representation. In the case of higher symmetry groups than $SU(2)$ the Wess-Zumino action

$$S = i \frac{N_c}{240 \pi^2} \int d^5 x \, \epsilon^{\mu \nu \alpha \beta \gamma} \, Tr \, R_\mu R_\nu R_\alpha R_\beta R_\gamma$$

(here written with the normalization appropriate for the fundamental representation [2]) also contributes to the energy. This has to be added to the Skyrme model Lagrangian in order to break its discrete reflection symmetry, which is not a symmetry of QCD. While the proof that it vanishes in a general representation of $SU(2)$ is somewhat involved it is straightforward in the case of the fundamental representation for $SU(2)$ because all combinations of its generators (Pauli matrices) can be reduced to linear combinations of the three generators and a scalar, and therefore at least two of the 5 current operators $R$ will have equal space-time indices, and consequently the expression (4.10) vanishes by antisymmetry. In the quantum case one of current operators $R$ can be chosen in the form (4.3). Also the expression (4.10) vanishes by antisymmetry of three Euler angles.

5. Discussion

All phenomenological applications of the Skyrme model and its extensions to the description of the structure of baryons and nuclei have hitherto relied on the fundamental representation of $SU(2)$. Above we have shown that substitution of a higher dimensional irreducible representation in place of the fundamental one does not change the phenomenological predictions obtained with the classical version of the model provided the parameters of the Lagrangian density are rescaled in an appropriate dimension dependent way.

In the quantum mechanical version of the model the representation dependence cannot however be eliminated by a simple rescaling of the parameters. The difference between the fundamental representation and those of higher dimension is the appearance of tensors of rank 2 in the transformation law for the currents $U \partial^\mu U^\dagger$. Thus the baryon spectra that are obtained in the fundamental representation with the collective coordinate quantization method depend in an essential way on the exceptionally simple algebraic properties of the generators of that representation.

We have derived explicit expressions for the Lagrangian density and the currents of the Skyrme model for representations of arbitrary integral dimension $2j + 1$. The derivation of the baryon spectra in the case of an arbitrary
representation in addition requires a choice of quantization method. It shall be an interesting question to determine the particle spectra in the case of the higher dimensional representation using the quantization method of collective variables.

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References

1. T.H.R. Skyrme, Proc. Roy. Soc. A260 (1961) 127
2. G.S. Adkins, C.R. Nappi and E. Witten, Nucl. Phys. B228 (1983) 552
3. L. Marleau, Phys. Rev. D45 (1992) 1776
4. C.G. Callan, K. Hornbostel and I. Klebanov, Phys. Lett. B202 (1988) 269
5. M. Rho, D.O. Riska and N.N. Scoccola, Z. Phys. A341 (1992) 343
6. G. Pari, Phys. Lett. B261 (1991) 347
7. A. Jackson et al., Phys. Lett. 154B (1985) 101
8. E.M. Nyman and D.O. Riska, Rept. Prog. Phys. 53 (1990) 1137
9. K. Fujii, A. Kobushkin, K. Sato and N. Toyota, Phys. Rev. D35 (1987) 1896