Disorder-assisted graph coloring on quantum annealers

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We are at the verge of a new era, which will be dominated by Noisy Intermediate-Scale Quantum Devices. Prototypical examples for these new technologies are present-day quantum annealers. In the present work, we investigate to what extent static disorder generated by an external source of noise does not have to be detrimental, but can actually assist quantum annealers in achieving better performance. In particular, we analyze the graph coloring problem that can be solved on a sparse topology (i.e. chimera graph) via suitable embedding. We show that specifically tailored disorder can enhance the fidelity of the annealing process and thus increase the overall performance of the annealer.

I. INTRODUCTION

The first concept of quantum computing was formulated several decades ago in an attempt to faithfully simulate many-body quantum systems, which is known to be an impossible feat with classical computers [1, 2]. However, only very recently novel technologies have become available that promise to make quantum computers a practical reality [3]. Quite remarkably, already the first generation of fully operational quantum computers is expected to outperform (for specific tasks) even the most advanced, state-of-the-art classical computers [3, 4]. To be ready for the first physical realizations of such powerful information technology, quantum computer science has been developing a plethora of quantum algorithms for a wide variety of optimization problems [5]. Famous examples include the Deutsch–Jozsa algorithm [6] to evaluate a function, the Grover algorithm [7] for searches of a (possibly large) database, or Shor’s algorithm [8] designed for prime factorization.

In the present work we will focus on adiabatic quantum computation (AQC) [9], which relies on quantum annealing [10]. In comparison to other computational paradigms, AQC is technologically slightly more advanced due to the commercial availability of D-Wave’s quantum annealers [11–13]. Adiabatic quantum computing is a computational paradigm [14] that has the potential to solve many problems that a universal quantum computer can also solve [15]. Although, a polynomial time penalty may be necessary to achieve this, with AQC one can still outperform classical computers in many practical cases [16].

AQC relies on the quantum adiabatic theorem [9]. In this paradigm, a quantum system is prepared in the ground state of an initial (“easy”) Hamiltonian \( H_0 \). Then, the system is let to evolve adiabatically—inﬁnitely slowly—towards the ground state of the ﬁnal Hamiltonian \( H_f \). The latter system encodes the problem of interest and its ground state stores the desired solution (i.e. an answer to the problem). Devices that can realize such evolution are called quantum annealers [10]. Quantum annealers are typically designed with one and only one particular task in mind—namely, to solve combinatorial optimization problems from the \( \text{NP} \) complexity class [17, 18]. These problems are “very hard” to solve with classical computers, however their solutions can still be veriﬁed (in polynomial time).

Several advantages of quantum annealing over other computational paradigms have been identiﬁed [19–22]. However, currently available technology still exhibits hardware issues, of which the most important one is static disorder [23–27]. Rather counter-intuitively, however, it also has been shown that static disorder is not always detrimental, but can rather be a valuable resource in achieving quantum tasks [28, 29].

In the present work, we study the inﬂuence of static disorder on the annealing dynamics and analyze its effect on the performance of near-term quantum annealers. To this end we mainly focus on a selected problem of graph coloring [30]. This a fundamental problem in modern computer science with various applications in many different areas, e.g. in scheduling [31], pattern [32] and frequency [33] matching, or memory allocation [34], to name just a few.

The main objective of the graph coloring problem is to find a minimal number of colors, \( \text{chromatic number} = \chi(G) \), that are required to color a graph \( G \), so that no adjacent sites share the same color. In this context, colors can encode any arbitrary information. Typical examples are shown Fig. 1. Remarkably, we will find that for the graph coloring problem D-Wave like annealers may actually be robust against certain type of noise. Even more importantly, we will see that particular types of disorder can assist the adiabatic computation to achieve better performance.

II. DISORDER GRAPH COLORING PROBLEM

The dynamics of quantum annealers is typically described by the following Hamiltonian,

\[
\hat{H}(s) = f(s)\hat{H}_f + [1-f(s)]\hat{H}_0, \quad s \in [-1,1],
\]

where \( f(s) \in [0,1] \) could be an arbitrary function such that \( f(-1) = 1 \) and \( f(1) = 0 \) [35]. Typically, \( f(s) = s+1 \) where \( s(t) = t/\tau \) and \( \tau \) is the annealing time [26]. For the present purposes, initial and final Hamiltonian are instances of the Ising
spin-glass [36], where in particular,

$$\hat{H}_I = \sum_{(i,j) \in E} J_{ij} S_i^x S_j^x + \sum_{i \in V} h_i S_i^z, \quad \hat{H}_S = 4 \sum_{i \in V} S_i^z, \quad (2)$$

Here, the problem Hamiltonian, $\hat{H}_I$, is defined on a graph, $G = (E, V)$, specified by its edges, $E$, and vertices, $V$. This simple model can already be realized with present-day quantum annealers [20], where the graph $G$ is set to reflect the chimera [37, 38] or pegasus topology [39]. The programmable input parameters [40] are the elements of the coupling matrix, $J_{ij}$, and the onsite magnetic fields, $h_i$. Spin operators are denoted by $S_i^x, S_i^z$ and they describe spins in the $z, x$ directions respectively.

All Ising variables can admit only two values ($s_i = \pm 1$). Since there are, however, typically more than two colors necessary to solve a graph coloring problem, one cannot map it directly onto the Ising Hamiltonian. Thus, graph coloring problems are first expressed as spin-lattices, where the spins can take more than two values. These so-called Potts models [41, 42] can then be mapped onto the Ising Hamiltonian using a suitable embedding (i.e. with the help of auxiliary variables).

When designing quantum algorithms, it is often convenient to work with the Quadratic Unconstrained Binary Optimization framework or QUBO [43]. Here, we introduce a binary variable $X_{ic} = 1$ if a vertex $i \in \{1, 2, \ldots, N\}$ is colored with a color $c \in \{1, 2, \ldots, K\}$ and we set $X_{ic} = 0$ otherwise. Then the graph coloring problem can be formulated in the following simple terms (cf. Fig. 1)

$$\hat{H}_Q = \sum_{i=1}^N \left( 1 - \frac{K}{2} \sum_{c=1}^K X_{ic} \right)^2 + \sum_{(i,j) \in E} \sum_{c=1}^K X_{ic} X_{jc}, \quad (3)$$

where $(i, j)$ indicates summation over all connected vertices. If the ground state of the Hamiltonian in Eq. (3), corresponding to the energy $E = 0$, exists then the graph $G$ can be properly colored with at least $K$ colors. The purpose of the first term in the above Hamiltonian is to assure that each vertex $i$ is colored with only one specific color $c$, as only then $\sum_{c=1}^K X_{ic} = 1$. The second term introduces an energy penalty whenever neighboring vertices have the same color $c$. Similar encoding strategies have also been discussed in the context of quantum error correcting codes for quantum annealers [44].

Having formulated the graph coloring problem in terms of binary variables, one can convert it back into the Ising Hamiltonian, which is more common for quantum annealers. Namely,

$$\hat{H}_I = \sum_{i=1}^N J_{ii} \sum_{c_1 < c_2} S_{ic_1}^z S_{ic_2}^z + \sum_{(i,j) \in E} \sum_{c=1}^K S_{ic}^z S_{jc}^z \ + \sum_{i=1}^N h_i \sum_{c=1}^K S_{ic}^z + C, \quad (4)$$

where $S_{ic}^z = X_{ic} - 1/2$ is the spin $z$-operator indexed by two variables $(i, c)$; $C = [1 + K(K - 3)/4]N + K|E|/4$ is a constant, and $|E|$ denotes the total number of edges. The coefficients $h_i$ are given by

$$h_i = K + \frac{1}{2} \deg(i) - 2 \quad \text{and} \quad J_{ij} = \begin{cases} 2 & i = j, \\ 1 & i \neq j, \end{cases} \quad (5)$$

where $\deg(i)$ is the number of edges at vertex $i$.

Current quantum annealers, such as the D-Wave machine, are imperfect due to a variety of factors, chief among them is static disorder originating in the limited control at the hardware level [23, 45, 46]. Therefore, our objective is to investigate what happens to the quantum annealing when all couplings $J_{ij}$ and magnetic fields $h_i$ are slightly perturbed. To be
more specific, we introduce static disorder,

\[ h_i \to h_i + \delta h_i, \quad J_{ij} \to J_{ij} + \delta J_{ij}. \] (6)

where perturbations \( \delta h_i \) and \( \delta J_{ij} \) are random variables with flat distributions and symmetric amplitudes, e.g. \( \delta J_{ij} \in [-W_j, W_j] \) and \( \delta h_i \in [-W_h, W_h] \).

For the sake of simplicity and without any loss of generality we focus in particular on the disorder generator where \( \delta J_{ij} = 0 \) and moreover (cf. Fig. 5)

\[ h_i \to \begin{cases} h_i + \delta h_i, & \text{for } h_i + \delta h_i < \max\{h_i\}; \\ \max\{h_i\}, & \text{otherwise}. \end{cases} \] (7)

Such disorder (6) mimics to some extent a situation, in which the actual values of interaction strengths at the hardware level differ from the input parameters provided by the programmer operating at the software level.

### III. RESULTS

To investigate the dynamics/annealing of the graph coloring problem formulated in Eq. (4), we focus on all non-isomorphic graphs, \( G(E, V) \), having \( |V| = 3, 4, 5 \) vertices and for which the chromatic number \( \chi(G) = K > 2 \). We omit the \( K = 2 \) case as one can reduce its problem Hamiltonian to the antiferromagnetic Ising model.

The quality of a quantum computation/annealing can be measured in various ways [47]. For instance, one may try to count defects [26], estimate fluctuations [27], calculate the fidelity between the final state, \( |\psi(\tau)\rangle \), and the true ground state of the problem Hamiltonian [48], \( |\phi\rangle \), or simply determine the difference between their corresponding energies, \( \delta E = \langle \psi(\tau)|\hat{H}|\psi(\tau)\rangle - \langle \phi|\hat{H}|\phi\rangle \) [24].

In the present work we calculate the probability to observe the correct final result,

\[ P = \sum_{i \in \mathcal{S}} |\langle \psi(\tau)|\phi_i\rangle|^2. \] (8)

Here, \( \mathcal{S} \) is a set that labels all possible solutions, \( |\phi_i\rangle \), of the disorder-free problem encoded in the Hamiltonian (4). The final state \( |\psi(\tau)\rangle \) is obtained by solving the time dependent Schrödinger equation, \( i\partial_t |\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle \), numerically [49, 50]. The total Hamiltonian \( \hat{H}(t) \) is defined in Eq. (1) with the objective Hamiltonian (encoding the graph coloring problem) given by Eq. (4) where all couplings, \( J_{ij} \), and biases, \( h_i \), are redefined according to Eq. (6).

A priori, the disorder amplitudes \( W_h, W_J \) could be arbitrarily large. However, to ensure that the ground state of the disordered problem matches at least one solution to the disorder-free problem at all, both \( W_h, W_J \) need to be carefully chosen. For instance, picking \( W_J = W_h = 0.5 \) guarantees 0.99 probability of this event to occur (cf. Fig. 2). For the sake of simplicity, we choose a simple annealing protocol such that \( f(t) = t/\tau \).

Moreover, we assume without loss of generality that \( W_J = 0 \).

### A. Disordered energy spectrum

As depicted in Fig. 3, introducing the disorder to the Hamiltonian (4) removes the degeneracy of its ground state. As a result, a solution to the graph coloring problem can be found not only in the degenerate ground state (as in the disorder-free case) but also in low energy spectrum consisting of \( M \ll 2^K \) states. In principle, this effect has the potential to increase the overall chances of finding a correct solution, in particular close to the adiabatic limit, e.g. on a time scale \( \tau \sim 1/\Delta \). Here, \( \Delta := E_0 - E_1 \) is an effective gap. That is, the difference
between the ground state energy $E_0$ and the energy of the first accessible state, $E_i$, which does not encode a solution.

### B. Disorder-assisted dynamics

In Fig. 4 we depict the probability to find the correct answer (8) as a function of the annealing time $\tau$ for the disordered and disorder-free systems. In the adiabatic limit where $\tau \gg 1/\Delta$, the disorder-free system is more likely to reach the ground state than the disordered one. Nevertheless, introducing disorder into the system does not significantly affect the final probability.

On the other hand, for small and moderate $\tau$ we observe that the probability to find the correct solutions is typically larger for the disordered Hamiltonian than in the disorder-free situation. Thus, it is not far-fetched to realize that one can always try to find $\tau_0$ such that $P_{\text{free}}(\tau_0) < P_{\text{disorder}}(\tau_0)$. This suggests a different strategy to perform computation with noisy near-term quantum annealers. Rather then trying to operate the annealer as adiabatically as possible, one identifies the “sweet spot”, $\tau_0$, at which the quantum annealer has optimal performance, even better than in the ideal, disorder free case, despite the inevitable noise in the system. For instance, Fig. 5(c) indicates a clear maximum. Quite remarkably, we also notice that this is truly a finite-time effect. In Fig. 5(d) we plot the optimal value of the noise amplitude as a function of the anneal time. We observe that in the adiabatic limit the disorder-free case is the only “good” realization.

However, the impact of the disorder on the success probability is still relatively small. This is illustrated Fig. 5(e). Even at optimal noise strength $P$ is significantly larger for slower processes. Thus, we must ask whether the noise can be modified to make it more “useful”.

### C. Optimizing disorder

Note that so far we have assumed that noise in the qubit-qubit couplings is uniformly distributed. However, we have also already realized that at intermediate anneal times the presence of noise actually assists the quantum annealer in
finding the correct solution. The natural question then is, whether the disorder in the system can be engineered to further enhance this effect—in other words, how to modify the distribution of the noise in our favor. It is then instructive to analyze the energy diagram and dynamics of single realizations of the disordered problem.

To this end, inspect again Fig. 3. We observe that in the disorder-free case due to the presence of the degeneracy in the ground state the effective gap $\Delta$ never actually closes, cf.

Fig. 3(a). The same holds true for “good” realizations. Except, that the effective gap opens even wider due to the lack of degeneracy, compare Fig. 3(b). On the contrary, for the all the cases we identify as “bad”, we see some mixture of correct and incorrect solutions that basically behave like impurities causing the effective gap to shrink [cf. Fig. 3(c)]. Thus, removing those impurities increases the effective gap which causes the adiabatic threshold to decrease.

Thus, minimizing the influence of the remaining, “bad” realizations may decrease the total time necessary to find a correct solution substantially. This is also demonstrated in Fig. 4 where the averages dynamics is computed over only those realizations that correlate with corrects solutions. This clearly demonstrates the advantage of disordered dynamics over the “ideal”, disorder-free situation.

IV. CONCLUSIONS

It is still a commonly accepted creed that noise and disorder in computing hardware have exclusively negative consequences. In the present work, we have shown that this is not always the case, and that static disorder can actually assist quantum annealers in successfully performing their tasks. More specifically, we have studied the graph coloring problem [24] on disorder-free and disordered quantum annealers.

On a more practical note, our results may suggest an an-
answer to a conundrum about existing hardware. Systems like the D-Wave machine are known to be subject to electrode noise, which can lead to severe disorder in the on-site fields and qubit couplings. Nevertheless, in particular graph coloring problems have been shown to be solved rather accurately [52–54]. A conjecture that can be drawn now is that the D-Wave machine may be operating exactly in such a disorder-assisted regime.

Of course, further characterization of the D-Wave machine appears necessary to verify our hypothesis. However, if this is indeed the case, then the performance of the machine could be dramatically enhanced by post-selecting the answers on the noise distribution (which will need to be measured independently).

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