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Two-Level Finite Element Iterative Algorithm Based on Stabilized Method for the Stationary Incompressible Magnetohydrodynamics

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Abstract: In this paper, based on the stabilization technique, the Oseen iterative method and the two-level finite element algorithm are combined to numerically solve the stationary incompressible magnetohydrodynamic (MHD) equations. For the low regularity of the magnetic field, when dealing with the magnetic field sub-problem, the Lagrange multiplier technique is used. The stabilized method is applied to approximate the flow field sub-problem to circumvent the inf-sup condition restrictions. One- and two-level stabilized finite element algorithms are presented, and their stability and convergence analysis is given. The two-level method uses the Oseen iteration to solve the nonlinear MHD equations on a coarse grid of size $H$, and then employs the linearized correction on a fine grid with grid size $h$. The error analysis shows that when the grid sizes satisfy $h = O(H^2)$, the two-level stabilization method has the same convergence order as the one-level one. However, the former saves more computational cost than the latter one. Finally, through some numerical experiments, it has been verified that our proposed method is effective. The two-level stabilized method takes less than half the time of the one-level one when using the second class Nédélec element to approximate magnetic field, and even takes almost a third of the computing time of the one-level one when adopting the first class Nédélec element.

Keywords: finite element method; two-level method; stabilized method; Oseen iteration; stationary incompressible MHD

MSC: 35Q30; 65M60; 65N30; 76D05

1. Introduction

Consider the following stationary incompressible MHD

$$
-R_e^{-1} \Delta u + (u \cdot \nabla)u + \nabla p - S_c \text{curl} b \times b = f, \quad \text{in } \Omega,
R_m^{-1} S_c \text{curl}(\text{curl } b) - S_c \text{curl}(u \times b) - \nabla r = g, \quad \text{in } \Omega,
\text{div } u = 0, \quad \text{div } b = 0, \quad \text{in } \Omega,
u = 0, \quad b \cdot n = 0, \quad n \times \text{curl } b = 0, \quad r = 0, \quad \text{on } \partial \Omega,
$$

(1)

where $\Omega \in \mathbb{R}^d (d = 2, 3)$ is a bounded Lipschitz domain, $R_e$ and $R_m$ are the hydrodynamic and magnetic Reynolds numbers, respectively. $S_c$ is the coupling number, and $f$ and $g$ are source terms with $\nabla \cdot g = 0$. $n$ is the unit outward normal vector on $\partial \Omega$.

Incompressible MHD describes the dynamics of a viscous, incompressible, electrically conducting fluid under an external magnetic field. The MHD (1) is a coupled multi-physical system of the classical Navier–Stokes equations and Maxwell’s equations. MHD modelling has a number of applications in physics and engineering technology, such as radio wave propagation in ionosphere in geophysics, MHD engine, control of MHD boundary layer and liquid-metal MHD electricity generation (see [1]). Since MHD equations are strongly
nonlinear and have many physical quantities, it is needed to find effective numerical methods to solve them.

For the MHD modelling (1) without the Lagrange multiplier \( r \) term, the early study of the exact penalty regularization finite element method on a convex domain is carried out in [2]. Based on this format, the nonconforming mixed finite element methods [3], the Stokes, Newton and Oseen finite element iterative methods [4,5], the penalty based finite element iterative methods [6], and the generalized Arrow–Hurwicz iterative methods [7] are investigated. In view of multi-physical coupling and nonlinearity of system (1), two-level method and finite element iterative algorithms are combined by [8–12] to reduce the computing cost, and local and parallel finite element algorithms based on some iterations are proposed in [13–16]. On the other hand, a number of effective solvers based on the finite element methods are presented in [17–19]. To keep the physical property of the Gauss law of the magnetic field, the constrained transport divergence-free finite element method is designed and analyzed. To deal with the nonlinear term, the stabilized finite element method based on Oseen iteration is studied. Compared to the existing literature, the stabilized scheme with the Lagrange multiplier has not been reported.

and the natural convection problem [30]. However, the stabilized finite element algorithm for MHD with respect to the Lagrange multiplier has not been reported.

In this paper, a two-level finite element iterative algorithm based on the stabilized method is proposed to numerically solve the stationary incompressible MHD equations. Compared to the existing literature, the stabilized scheme with the Lagrange multiplier proposed here have two main benefits. One is that the lowest equal-order finite element pairs in engineering preferred do not satisfy the inf-sup condition. Numerical experiments show that the break of the inf-sup condition often leads to unphysical pressure oscillations. To avoid the instability problem and use the lowest equal-order elements, the popular stabilized methods based on local Gauss integrations are proposed and studied, for example, for the Stokes problem [22,23], the coupled Stokes–Darcy problem [24], the Stokes eigenvalue system [25], the Navier–Stokes equations [26–29] and the natural convection problem [30]. However, the stabilized finite element algorithm for MHD with respect to the Lagrange multiplier has not been reported.

In this paper, a two-level finite element iterative algorithm based on the stabilized method is proposed to numerically solve the stationary incompressible MHD equations. Compared to the existing literature, the stabilized scheme with the Lagrange multiplier proposed here have two main benefits. One is that the lowest equal-order finite element pairs can be used to approximate hydrodynamic subproblem, and the other is that our scheme preserve the physical property of Gauss law weakly for magnetic subproblem by adding the Lagrange multiplier. In the next section, the stabilized finite element discretization based on local Gauss integrations is designed and analyzed. To deal with the nonlinear term, the stabilized finite element method based on Oseen iteration is studied. The two-level stabilized finite element algorithm and its convergence are given in Section 3. In the last section, some numerical experiments are tested to support the theoretical analysis of our proposed method.

2. Stabilized Finite Element Discretization Based on Local Gauss Integrations

We will introduce some Sobolev spaces, and the norms of the product spaces:

\[
\begin{align*}
H^1(\Omega) &= H^1(\Omega)^d, \quad X := H^1_0(\Omega) = \{ \mathbf{u} \in H^1(\Omega) : \mathbf{u}\big|_{\partial\Omega} = \mathbf{0} \}, \\
H(\text{div};\Omega) &= \{ \mathbf{u} \in L^2(\Omega)^d : \text{div} \mathbf{u} \in L^2(\Omega) \}, \\
W :&= \mathcal{H}_0(\text{curl};\Omega) = \{ \mathbf{b} \in H(\text{curl};\Omega) : \mathbf{b} \times \mathbf{n}\big|_{\partial\Omega} = \mathbf{0} \}, \\
Q :=& L^2_0(\Omega) = \{ q \in L^2(\Omega) : \int_{\Omega} q dx = 0 \}, \\
S :=& H^1_0(\Omega), \\
\| (\mathbf{u}, \mathbf{b}) \|_1 &= (\| \nabla \mathbf{u} \|_0^2 + S_c \| \text{curl} \bigr. \mathbf{b} \bigr. \|_0^2)^{\frac{1}{2}}, \\
\| (p, r) \|_1 &= (\| p \|_0^2 + \| \nabla r \|_0^2)^{\frac{1}{2}}, \\
\| (\mathbf{u}, \mathbf{b}) \|_0 &= (\| \mathbf{u} \|_0^2 + S_c \| \mathbf{b} \|_0^2)^{\frac{1}{2}}, \\
\| \mathbf{f} \|_{-1} &= \sup_{\mathbf{v} \in \mathcal{H}_0^1(\Omega)} \frac{(\mathbf{v}, \mathbf{f})}{\| \nabla \mathbf{v} \|_0^2}, \\
\| \mathcal{F} \|_0 &= (\| \mathbf{f} \|_{-1}^2 + S_c^{-1} \| \mathbf{g} \|_0^2)^{\frac{1}{2}}, \\
\| \mathcal{F} \|_0 &= (\| \mathbf{f} \|_0^2 + S_c^{-1} \| \mathbf{g} \|_0^2)^{\frac{1}{2}}.
\end{align*}
\]
For all \( w, u, v \in X, b, c, d \in W, q \in Q, s \in S \), let

\[
\begin{align*}
    a_b(u, v) &= R_c^{-1} \langle \nabla u, \nabla v \rangle, \\
    a_m(b, c) &= R_m^{-1} S_{\text{curl}}(b, \text{curl} c), \\
    c_0(w, u, v) &= \frac{1}{2} \langle w \cdot \nabla u, v \rangle - \frac{1}{2} \langle w \cdot \nabla v, u \rangle, \\
    c_1(d, v, b) &= S_c(b \times d, v), \\
    b_b(q, v) &= \langle q, \nabla \cdot v \rangle, \\
    b_m(s, c) &= \langle \nabla s, c \rangle, \\
    A(u, b; v, c) &= a_b(u, v) + a_m(b, c), \\
    C(w, d; u, b; v, c) &= c_0(w, u, v) - c_1(d, v, b) + c_1(d, u, c), \\
    B(q, s; v, c) &= b_b(q, v) + b_m(s, c), \\
    \langle F, (v, c) \rangle &= \langle f, v \rangle + \langle g, c \rangle.
\end{align*}
\]

By the Lagrange multiplier technique, the variational form of system (1) is [31]: Find \((u, b, p, r) \in X \times W \times Q \times S\), for all \((v, c, q, s) \in X \times W \times Q \times S\) such that

\[
\begin{align*}
    a_b(u, v) - b_b(p, v) + b_b(q, u) + c_0(u, u, v) - c_1(b, v, b) &= \langle f, v \rangle, \\
    a_m(b, c) - b_m(r, c) + b_m(s, b) + c_1(b, u, c) &= \langle g, c \rangle.
\end{align*}
\]

The compact form of (2) and (3) is read as

\[
A(u, b; v, c) + C(u, b; u, b; v, c) - B(p, r; v, c) + B(q, s; u, b) = \langle F, (v, c) \rangle.
\]

The properties of the bilinear and trilinear forms from [32–34] are useful for our analysis. For all \( u, v \in X, b, c \in W, q \in Q, r \in S \), there have

\[
\begin{align*}
    |A(u, b; v, c)| &\leq \beta ||(u, b)||_1||(v, c)||_1, \\
    \|A(u, b; u, b)\| &\geq \underline{\beta} ||(u, b)||_1^2, \\
    B(q, s; v, c) &\leq \sqrt{d} ||(q, s)||_1 ||(v, c)||_1, \\
    C(w, d; u, b; u, b) &\leq 0, \\
    |C(w, d; u, b; v, c)| &\leq \bar{N} ||(w, d)||_1 ||(u, b)||_1 ||(v, c)||_1,
\end{align*}
\]

where \( \bar{N} \) and \( \lambda_0 \) are positive constants that depend only on \( \Omega \). In the next content, we use \( C \) to represent a general positive constant independent of mesh sizes \( H \) and \( h \).

\( H \) and \( h(h \ll H) \) are now two real positive parameters that tend to 0. \( T_H \) is a uniformly regular partition of \( \Omega \) into triangular \((d = 2)\) or tetrahedral \((d = 3)\) element \( K \) with diameters bounded by \( H \), and \( T_h \) is the fine mesh generated by a mesh refinement process to \( T_H \). Let \( T_H(K) = H, h \) is a partition. \( P_s(K) \) is the space of polynomials of degree \( s \) (positive integers) over \( K \). \( P_s \) element is utilized to approximate the velocity field, pressure, and Lagrange multiplier, and two kinds of lowest order Nédélec elements are applied to approximate the magnetic field. The subspaces of \( X, W, Q, S \) are

\[
\begin{align*}
    X_\mu &:= \{ u_\mu \in H^1_0(\Omega) : u_\mu|_K \in P_1(K)^d, \forall K \in T_H \}, \\
    W_\mu &:= \{ b_\mu \in H_0(\text{curl}; \Omega) : b_\mu|_K \in \mathcal{N}_1^\text{curl} (K), \forall K \in T_H \}, I = 1, 2, \\
    Q_\mu &:= \{ q_\mu \in L^2_0(\Omega) : q_\mu|_K \in P_1(K), \forall K \in T_H \}, \\
    S_\mu &:= \{ s_\mu \in H_0^1(\Omega) : s_\mu|_K \in P_1(K), \forall K \in T_H \}, \\
    V_\mu &:= \{ u_\mu \in X_\mu : b_b(q_\mu, u_\mu) = 0, \forall q_\mu \in Q_\mu \}, \\
    \mathcal{C}_\mu &:= \{ b_\mu \in W_\mu : b_m(s_\mu, b_\mu) = 0, \forall s_\mu \in S_\mu \}.
\end{align*}
\]
Here, Nédélec elements of the first family and the second one are as follows [35]

\[ \Lambda_k^{(1)}(K) = [p_{k-1}(K)]^d \oplus D_k(K), D_k(K) = \left\{ p \in [p_k(K)]^d \mid p(x) \cdot x = 0, \text{ in } K \right\}, \]

\[ \Lambda_k^{(2)}(K) = [p_k(K)]^d, \]

where \([p_k(K)]^d\) is the homogeneous polynomials of degree \(k\).

\( W_\mu \) and \( S_\mu \) satisfy the discrete inf-sup condition [31]

\[ \sup_{c_\mu \in W_\mu} \frac{b_m(s_\mu, c_\mu)}{\|\text{curl} c_\mu\|_0} \geq \hat{\beta} \|\nabla s_\mu\|_0, \forall s_\mu \in S_\mu, \]  

where the constant \( \hat{\beta} > 0 \) is independent of \( \mu \).

Denote \( P_\mu \) and \( R_{0\mu} \) by the \( L^2 \)-orthogonal projectors

\[ P_\mu : L^2(\Omega) \to V_\mu, \quad R_{0\mu} : L^2(\Omega) \to C_\mu. \]

Define the discrete Stokes operator by \( A_{1\mu} = -P_\mu \Delta_\mu, \) in which \( \Delta_\mu \) is defined by (see [32,33])

\[ -(\Delta_\mu u_\mu, v) = (\nabla u_\mu, \nabla v), \forall u_\mu, v \in X_\mu, \]

and the discrete norm \( \|u_\mu\|_{k,\mu} = \left\| A_{1\mu}^{\frac{1}{2}} u_\mu \right\|_0 \) of the \( k \in R \) order, where

\[ \|u_\mu\|_{1,\mu} = \|\nabla u_\mu\|_0, \quad \|u_\mu\|_{2,\mu} = \|A_{1\mu} u_\mu\|_0, \forall u_\mu \in X_\mu. \]

Meanwhile, \( A_{2\mu} b_\mu = R_{0\mu}(\nabla_\mu \times \nabla \times b_\mu + b_\mu) \) is defined as [33]:

\[ (\nabla_\mu \times \nabla \times b_\mu, c) = (\nabla \times b_\mu, \nabla \times c), \forall c \in W_\mu. \]

It is necessary to introduce some discrete estimates [33,34]

\[ \|\nabla v_\mu\|_{L^3} + \|v_\mu\|_{L^\infty} \leq \|\nabla v_\mu\|_0^{\frac{1}{2}} \|A_{1\mu} v_\mu\|_0^{\frac{1}{2}}, \]

\[ \|\nabla v_\mu\|_{L^8} \leq C \|A_{1\mu} v_\mu\|_0, \forall v_\mu \in X_\mu. \]  

The trilinear form \( C(\cdot, \cdot, \cdot) \) has the properties [34]: for all \( w_\mu, u_\mu, v_\mu \in X_\mu, d_\mu, b_\mu, c_\mu \in W_\mu, \)

\[ |C(w_\mu, d_\mu, u_\mu, b_\mu, v_\mu, c_\mu)| + |C(u_\mu, b_\mu, w_\mu, d_\mu, v_\mu, c_\mu)| \leq \tilde{N} \|A_{1\mu} w_\mu, A_{2\mu} d_\mu\|_0 \|u_\mu, b_\mu\|_0 \|v_\mu, c_\mu\|_1. \]  

It is apparent that the discrete inf-sup condition is not valid to the subspace \( X_\mu \) and \( Q_\mu \). To meet the needs of this property, as in [22,26], a mixed stability term with the universal bilinear form is added:

\[ D_\mu(u_\mu, p_\mu; v_\mu, q_\mu) = a_\mu(u_\mu, v_\mu) - b_\mu(p_\mu, v_\mu) + b_\mu(q_\mu, u_\mu) + G(p_\mu, q_\mu), \]

where

\[ G(p_\mu, q_\mu) = \sum_{k \in T_\mu} \left( \int_{K_k} p_\mu q_\mu \, d\xi - \int_{K_k} p_\mu q_\mu \, d\xi \right), k \geq 2, \]

for all \( p_\mu, q_\mu \in Q_\mu \). \( \int_{K_k} p_\mu q_\mu \, d\xi \) means that makes use of an \( i \)-order \( (i = 1, 2) \) local Gauss integral to calculate it over the element K.

Let \( \Pi_\mu : L^2(\Omega) \to P_0 \) be a \( L^2 \)-projection with the properties as follows [22,26]:

\[ (p, q_\mu) = (\Pi_\mu p, q_\mu), \forall p \in L^2(\Omega), q_\mu \in Q_\mu, \]

\[ ||\Pi_\mu||_0 \leq C \|p\|_0, \forall p \in L^2(\Omega), \]

\[ ||p - \Pi_\mu p||_0 \leq C_{\mu} \|p\|_{\gamma}, \forall p \in H^1(\Omega) \cap H^\gamma(\Omega). \]
As a consequence, the local Gauss integral can be restated as:

\[
G(p_\mu,q) = (p_\mu - \Pi p_\mu,q - \Pi q), \quad \forall p_\mu,q \in Q_\mu.
\]  

(15)

\( \mathcal{D}_\mu(\cdot,\cdot,\cdot) \) satisfies the following important properties (see [22,26]): For all \((u_\mu, p_\mu), (v_\mu, q_\mu) \in X_\mu \times Q_\mu\),

\[
\sup_{(v,q) \in (X_\mu,Q_\mu)} \frac{\|\mathcal{D}_\mu(u_\mu,p_\mu,v,q)\|}{\|\nabla u_\mu\|_{L_2} + \|q\|_{L_2}} \geq \beta (R_\tau^{-1}\|\nabla u_\mu\|_0 + \|p_\mu\|_0).
\]  

(16)

The stabilized discrete scheme reads: Find \((u_\mu, p_\mu, r_\mu) \in X_\mu \times W_\mu \times Q_\mu \times S_\mu\), such that

\[
\begin{align*}
& a_s(u_\mu,v) + c_0(u_\mu,u_\mu,v) - c_1(b_\mu,v,b_\mu) - b_s(p_\mu,v) + b_s(q,u_\mu) + G(p_\mu,q) + \frac{\sigma_\mu}{R_\tau} a_s(u_\mu,v) = \langle f,v \rangle, \\
& a_m(b_\mu,c) + c_1(b_\mu,u_\mu,c) - b_m(r_\mu,c) + b_m(s,b_\mu) = \langle g,c \rangle,
\end{align*}
\]  

(17)

where \(\sigma > 0\) is an artificial viscosity parameter, \(\sigma\) can be rewritten as:

\[
A(u_\mu, b_\mu; v, c) + C(u_\mu, b_\mu; u_\mu, b_\mu; v, c) + \frac{\sigma_\mu}{R_\tau} a_s(u_\mu, v) + G(p_\mu, q) \\
B(p_\mu, r_\mu; v, c) + B(q, s; u_\mu, b_\mu) = \langle f, v \rangle + \langle g, c \rangle.
\]  

(18)

Let

\[
A_\mu(u_\mu, b_\mu; v, c) = A(u_\mu, b_\mu; v, c) + \frac{\sigma_\mu}{R_\tau} a_s(u_\mu, v).
\]

Then the bilinear form \(A_\mu\) satisfies the following coercive and continuous properties:

\[
|A_\mu(u_\mu, b_\mu; v, c)| \leq C_{\text{max}} \|u_\mu\|_1 \|v\|_1, \\
A_\mu(u_\mu, b_\mu; u_\mu, b_\mu) \geq C_{\text{min}} \|u_\mu\|^2_1,
\]  

(19)

(20)

where

\[
C_{\text{max}} = \{R_\tau^{-1} + \sigma_\mu, R_m^{-1}\}, C_{\text{min}} = \{R_\tau^{-1} + \sigma_\mu, R_m^{-1} \lambda_0\}.
\]

Rewrite \(18\) as

\[
A_\mu(u_\mu, b_\mu; v, c) + C(u_\mu, b_\mu; u_\mu, b_\mu; v, c) + G(p_\mu, q) \\
B(p_\mu, r_\mu; v, c) + B(q, s; u_\mu, b_\mu) = \langle f, v \rangle + \langle g, c \rangle.
\]  

(21)

In order to derive error estimates, we introduce two projections. The Stokes projection is defined as follows [26,36]: Find \(R(u, p) \in X_\mu, Q(u, p) \in Q_\mu\) such that

\[
a_s(u - R(u, p), v) - b_s(p - Q(u, p), v) + b_s(q, u - R(u, p)) = 0,
\]  

(22)

for all \((v,q) \in X_\mu \times Q_\mu\). If \(u \in H^{1+\gamma}(\Omega), p \in H^\gamma(\Omega), \gamma > \frac{1}{2}\), there holds [26,36]

\[
\|u - R(u, p)\|_0 + \mu(\|\nabla(u - R(u, p))\|_0 + \|p - Q(u, p)\|_0) \\
\leq C_\mu \min\{\gamma + 1, 2\gamma + 1\} \|u\|_{1+\gamma} + \|p\|_\gamma.
\]  

(23)

The Maxwell’s projection is defined by [37]: Assume that \(b \in H^\gamma(\Omega), \text{curl } b \in H^\gamma(\Omega), \tau > \frac{1}{2}\), find \(\Lambda b \in W_\mu, \Lambda r \in S_\mu\) such that

\[
a_m(b - \Lambda b, c) - b_m(r - \Lambda r, c) + b_m(s, b - \Lambda b) = 0, \quad \forall b \in W, r \in S.
\]  

(24)

By the property of \(\Lambda\), it can be shown that

\[
\|\text{curl } b - \text{curl } \Lambda b\|_0 + \|b - \Lambda b\|_0 + \|\nabla(r - \Lambda r)\|_0 \\
\leq C_\mu \min\{1, \tau\} (\|b\|_\tau + \|\text{curl } b\|_\tau + \|r\|_{\tau+1}).
\]  

(25)
Now, we will give the stability and error estimate for the problem (21).

**Lemma 1.** Suppose the condition $\sigma_1 := \frac{N_1 \|F\|}{C_{\min}} \leq 1$ holds, the solution of the problem (21) satisfies
\[
C_{\min} \| (u_{\mu}, b_{\mu}) \|_1 \leq \| F \|_s,
\]
\[
C_{\min} (1 - \sigma_1) \| (A_{1\mu} u_{\mu}, A_{2\mu} b_{\mu}) \|_0 \leq \| F \|_0.
\]

**Proof.** For (21), taking $(v, b, q, s) = (u_{\mu}, b_{\mu}, p_{\mu}, r_{\mu}) \in X_{\mu} \times W_{\mu} \times Q_{\mu} \times S_{\mu}$, then by (8), we have
\[
A_{\mu} (u_{\mu}, b_{\mu}, p_{\mu}, q_{\mu}) + G(p_{\mu}, p_{\mu}) = (F, (u_{\mu}, b_{\mu})).
\]

Using (20) and (15), we can easily receive (26).
Replacing $(v, c) = (A_{1\mu} u_{\mu}, A_{2\mu} b_{\mu})$, $q = 0, s = 0$ in (21), and applying (12) to have

\[
C_{\min} \| (A_{1\mu} u_{\mu}, A_{2\mu} b_{\mu}) \|_0 \leq \| F \|_0 + \| \hat{N} (A_{1\mu} u_{\mu}, A_{2\mu} b_{\mu}) \|_1 \| (u_{\mu}, b_{\mu}) \|_1
\]
\[
\leq \| F \|_0 + \| \hat{N} (A_{1\mu} u_{\mu}, A_{2\mu} b_{\mu}) \|_1 \leq \frac{C_{\min} \| F \|_0}{1 - \sigma_1}.
\]

Furthermore, we arrive at (27). \( \square \)

**Theorem 1.** Let $(u, b, p, r)$ be the solution of the problem (4) satisfying $u \in H^{1+\gamma}(\Omega), p \in H^\gamma(\Omega), b \in H^\gamma(\Omega), \text{curl } b \in H^{1+\gamma}(\Omega), r \in H^{1+\gamma}(\Omega), \gamma > \frac{1}{2}$. Then, the error estimate $(u - u_{\mu}, b - b_{\mu})$ and $(p - p_{\mu}, r - r_{\mu})$ of the solution (18) satisfying the upper bound
\[
\| (u - u_{\mu}, b - b_{\mu}) \|_1 \leq C_{\mu}^\min(1, 1 + \gamma) \| b \|_\tau + \| \text{curl } b \|_\tau + \| p \|_\gamma + \| r \|_{1+\gamma}
\]
\[
+ C\sigma_1 \| F \|_s.
\]
\[
\| (p - p_{\mu}, r - r_{\mu}) \|_0 \leq C_{\mu}^\min(1, 1 + \gamma) \| b \|_\tau + \| \text{curl } b \|_\tau + \| p \|_\gamma + \| r \|_{1+\gamma} + C\mu^2.
\]

The proof of Theorem 1 is shown in the section of Appendix A.

In the following, the Oseen iteration is used to linearize the stabilized finite element discrete form (17). The stability and convergence is proven. The stabilized finite element algorithm based on Oseen iteration is stated as follows: Given $(u_{\mu}^{n-1}, b_{\mu}^{n-1}, p_{\mu}^{n-1}, r_{\mu}^{n-1})$, find $(u_{\mu}^n, b_{\mu}^n, p_{\mu}^n, r_{\mu}^n) \in X_{\mu} \times W_{\mu} \times Q_{\mu} \times S_{\mu}$ such that
\[
a_s(u_{\mu}^n, v) + c_0(u_{\mu}^{n-1}, u_{\mu}^n, v) - c_1(b_{\mu}^{n-1}, v, b_{\mu}^n) - b_s(q, u_{\mu}^n) + b_s(q, u_{\mu}^n) + G(p_{\mu}^n, q) + \frac{\sigma_1}{R^\gamma} a_s(u_{\mu}^n, v) = (f, v),
\]
\[
a_m(b_{\mu}^n, c) + c_1(b_{\mu}^{n-1}, u_{\mu}^n, c) - b_m(r_{\mu}^n, c) + b_m(s, b_{\mu}^n) = (g, c).
\]

Here, the initial value $(u_{\mu}^0, b_{\mu}^0, p_{\mu}^0, r_{\mu}^0) \in X_{\mu} \times W_{\mu} \times Q_{\mu} \times S_{\mu}$ is given by
\[
a_s(u_{\mu}^0, v) - b_s(p_{\mu}^0, v) + b_s(q, u_{\mu}^0) + G(p_{\mu}^0, q) + \frac{\sigma_1}{R^\gamma} a_s(u_{\mu}^0, v) = (f, v),
\]
\[
a_m(b_{\mu}^0, c) - b_m(r_{\mu}^0, c) + b_m(s, b_{\mu}^0) = (g, c).
\]

Rewrite (31) and (32) in compact form as
\[
A_{\mu}(u_{\mu}^n, b_{\mu}^n, v, c) + C(u_{\mu}^{n-1}, b_{\mu}^{n-1}; u_{\mu}^n, b_{\mu}^n, v, c) + G(p_{\mu}^n, q) - B(p_{\mu}^n, r_{\mu}^n; v, c)
\]
\[
+ B(q, s; u_{\mu}^n, b_{\mu}^n) = (F, (v, c)).
\]
for \( n = 1, 2, \cdots \), for all \((v, c, q, s) \in X_\mu \times W_\mu \times Q_\mu \times S_\mu\).

**Lemma 2.** If the condition \( 0 < \sigma_1 < 1 \) holds, for all \( m \geq 0 \), then the solution \((u^m_\mu, b^m_\mu, p^m_\mu, r^m_\mu)\) of (35) satisfies

\[
C_{\min} \| (u^m_\mu, b^m_\mu) \|_1 + \| (p^m_\mu, r^m_\mu) \| \leq C \| F \|_s, \tag{36}
\]

\[
\| (A_{1\mu} u^m_\mu, A_{2\mu} b^m_\mu) \|_0 \leq C \| F \|_0. \tag{37}
\]

**Proof.** For \( m = 0 \), taking \((v, b, q, s) = (u^0_\mu, b^0_\mu, p^0_\mu, r^0_\mu) \in X_\mu \times W_\mu \times Q_\mu \times S_\mu\) in (33) and (34), we can access

\[
a_s(u^0_\mu, u^0_\mu) + a_m(b^0_\mu, b^0_\mu) + G(p^0_\mu, p^0_\mu) + \frac{\sigma_\mu}{R_\mu} a_s(u^0_\mu, u^0_\mu) = (f, u^0_\mu) + (g, b^0_\mu),
\]

Using (20), we have

\[
C_{\min} \| (u^0_\mu, b^0_\mu) \|_1 \leq \| F \|_s.
\]

For \( m = f \), assuming that (36) holds, it is sufficient to prove that it also holds for \( m = f + 1 \). Let \( m = f + 1 \) take \((v, c, q, s) = (u^{f+1}_\mu, b^{f+1}_\mu, p^{f+1}_\mu, r^{f+1}_\mu) \in X_\mu \times W_\mu \times Q_\mu \times S_\mu\) in (33) and (34), using (8), we get

\[
a_s(u^{f+1}_\mu, u^{f+1}_\mu) + a_m(b^{f+1}_\mu, b^{f+1}_\mu) + G(p^{f+1}_\mu, p^{f+1}_\mu) + \frac{\sigma_\mu}{R_\mu} a_s(u^{f+1}_\mu, u^{f+1}_\mu) = (f, u^{f+1}_\mu) + (g, b^{f+1}_\mu).
\]

and by applying (20) and (8), we derive that

\[
C_{\min} \| (u^{f+1}_\mu, b^{f+1}_\mu) \|_1 \leq \| F \|_s.
\]

Thus, the proof of the first part of (36) has been finished.

Applying (16) to (31), and using (10) in (32) with \( s = 0 \), we have

\[
\hat{\beta} \left( R_\mu^{-1} \| \nabla u^m_\mu \|_0 + \| p^m_\mu \|_0 \right) + \hat{\beta} \| \nabla r^m_\mu \|_0 \leq C \| u^m_\mu - b^m_\mu \|_1 \| (u^m_\mu, b^m_\mu) \|_1 + \| F \|_s.
\]

Combining with the stability of \( \| (u^m_\mu, b^m_\mu) \|_1 \), we can complete the proof of (36). Similar to (27), (37) can be derived. The proof ends. \( \square \)

Next, we will establish the upper bound of error \((u_\mu - u^m_\mu, b_\mu - b^m_\mu, p_\mu - p^m_\mu, r_\mu - r^m_\mu)\). For convenience, let \((e^m, b^m, \eta^m, \tau^m) = (u_\mu - u^m_\mu, b_\mu - b^m_\mu, p_\mu - p^m_\mu, r_\mu - r^m_\mu)\).

**Theorem 2.** Suppose that \( 0 < \sigma_1 < 1 \), for all \( m \geq 0 \), \((e^m, b^m, \eta^m, \tau^m)\) satisfies

\[
C_{\min} \| (e^m, b^m) \|_1 \leq \sigma_1 \| F \|_s, \tag{38}
\]

\[
\| (u - u^m_\mu, b - b^m_\mu) \|_1 + \| (p - p^m_\mu, r - r^m_\mu) \| \leq C \| \min(1, \sigma_1) \| F \|_s + C \| \eta^m \|_1. \tag{39}
\]

**Proof.** Subtracting (35) from (21), there holds

\[
A_\mu(e^m, b^m; v, c) + C(e^{m-1}, b^{m-1}; u, b; v, c) + C(u^{m-1}, b^{m-1}; e^m, b^m; v, c) + G(\eta^m, q) - B(\eta^m, \tau^m; v, c) + B(q, s; e^m, b^m) = 0.
\]

Choose \((v, c, q, s) = (e^m, b^m, \eta^m, \tau^m)\) to obtain

\[
A_\mu(e^m, b^m; e^m, b^m) + C(e^{m-1}, b^{m-1}; u, b; e^m, b^m) + G(\eta^m, \eta^m) = 0. \tag{40}
\]
It follows from (20), (9) and Lemma 2 that
\[
C_{\text{min}} \| (e^m, b^m) \|_1 \leq \hat{N} \| (e^{m-1}, b^{m-1}) \|_1 \| (u^*_m, b^*_m) \|_1 \\
\leq c_1 C_{\text{min}} \| (e^{m-1}, b^{m-1}) \|_1 \\
\leq \sigma_m C_{\text{min}} \| (e^0, b^0) \|_1 \leq \sigma_m^m \| \mathcal{F} \|_s.
\]
Subtracting (31) from (16), we have
\[
D_{\mu}(e^m, \eta^m; v, q) + c_0(e^{m-1}, u_\mu, v) + c_1(b^{m-1}, e^m, v) - c_1(b^{m-1}, v, b_\mu) + c_1(b^{m-1}_\mu, v, b^m) + \frac{c_m}{R\gamma} a_\delta(e^m, v) = 0.
\]
Using the second equation of (17) minus equation (32) and choosing s = 0, we get
\[
b_m(\tau^m, c) = a_m(b^m, c) + c_1(b^{m-1}, u_\mu, c) + c_1(b^{m-1}_\mu, e^m, c).
\]
Applying (16) and (10) to the above two equations, respectively, there holds
\[
\hat{\beta}(R\gamma^{-1} \| \nabla e^m \|_0 + \| \eta^m \|_0) + \hat{\beta} \| \eta^m \|_0 \\
\leq \hat{N} \| (e^{m-1}, b^{m-1}) \|_1 \| (u^*_m, b^*_m) \|_1 \\
+ \hat{N} \| (u^{m-1}_\mu, b^{m-1}_\mu - b^{m-1}) \|_1 \| (e^m, b^m) \|_1 \\
+ \sigma_m \| (e^m, b^m) \|_1 \\
\leq C \sigma_m \| \mathcal{F} \|_s.
\]
The result (39) can be obtained by (38) and Theorem 1. The proof ends. \qed

3. Two-Level Stabilized Finite Element Algorithm
In this section, motivated by [38], two-level stabilized finite element algorithm for incompressible MHD equations is presented. The stability analysis and the optimal error estimation with respect to the mesh size H and h and the iterative step m are obtained.

It is evident that in Steps 2–5 of Algorithm 1 the iteration is controlled by \( \| (u^m_H - u^m_{H^1}, b^m_H - b^m_{H^1}) \|_1 \leq C \| (u^m_H - u^{m-1}_H, b^m_H - b^{m-1}_H) \|_0 \) (see Theorem 5 of [4]), which provides an operable way to acquire the desired solution \((u^m_H, b^m_H)\).

**Algorithm 1** Two-level stabilized finite element algorithm
1. Give the initial value \((u^0_H, b^0_H, p^0_H, r^0_H) \in X_H \times W_H \times Q_H \times S_H\) by (33) and (34) with \( \mu = H \).
2. while \( \| (u^m_H - u^{m-1}_H, b^m_H - b^{m-1}_H) \|_1 > \epsilon \) do
3. Solve MHD on the coarse grid: Find \((u^m_H, b^m_H, p^m_H, r^m_H) \in X_H \times W_H \times Q_H \times S_H\) by (35) with \( \mu = H \).
4. \( (u^{m-1}_H, b^{m-1}_H, p^{m-1}_H, r^{m-1}_H) = (u^m_H, b^m_H, p^m_H, r^m_H) \).
5. end while
6. Find \((u_{mH}, b_{mH}, p_{mH}, r_{mH}) \in X_h \times W_h \times Q_h \times S_h\) on the fine grid, for any \((v, c, q, s) \in X_h \times W_h \times Q_h \times S_h\), such that
\[
a_s(u_{mH}, v) + c_0(u^m_H, u_{mH}, v) - c_1(b^m_H, v, b_{mH}) - b_s(p_{mH}, v) + b_s(q, u_{mH}) + G(p_{mH}, q) + \frac{c_m}{R\gamma} a_\delta(u_{mH}, v) = (f, v),
\]
\[
a_m(b_{mH}, c) + c_1(b^m_H, u_{mH}, c) - b_m(r_{mH}, c) + b_m(s, b_{mH}) = (g, c).
\]

**Theorem 3.** Under the assumption of Theorem 2, the solutions \((u_{mH}, b_{mH})\) obtained from (41) and (42) satisfy
\[
C_{\text{min}} \| (u_{mH}, b_{mH}) \|_1 \leq \| \mathcal{F} \|_s,
\]
\[
\| (u - u_{mH}, b - b_{mH}) \|_1 + \| (p - p_{mH}, r - r_{mH}) \|_1 \\
\leq C (H^2 + H^{2\min(1,\gamma, r)} + H^{\min(2,\gamma+1, r+1)} + h + h^{\min(1,\gamma, r)} + c^m_m).
\]
Proof. Taking \((v, c, q, s) = (u_{mhr}, b_{mhr}, p_{mhr}, r_{mhr})\) \(\in X_h \times W_h \times Q_h \times S_h\) in (41) and (42), by using of (8) and (20), we can easily derive (43).

Subtracting (41) and (42) from (2) and (3), respectively, we have

\[
\begin{aligned}
A(u - u_{mhr}, b - b_{mhr}; v, c) + C(u - u_H^m, b - b_H^m; u, b; v, c) \\
+ B(q, s; u - u_{mhr}, b - b_{mhr}) - G(p_{mhr}, q) = \frac{\alpha_h}{R^2} d_k(u_{mhr}, v).
\end{aligned}
\]

(45)

By the two projection operators of (22) and (24) with \(\mu = h\), letting \(e_{ab} = R(u, p) - u_{mhr}, e_{ph} = \Lambda b - b_{mhr}, e_{ph} = Q(u, p) - p_{mhr}, e_{ph} = \Lambda r - r_{mhr}\), and then taking \(v = e_{ab}, c = e_{bh}, q = e_{ph}, s = e_{ph}\), there holds

\[
A_p(e_{ab}, e_{bh}; e_{ab}, e_{bh}) + G(e_{ph}, e_{ph}) \\
= C(u_H^m - u, b_H^m - b; u - u_H^m, b - b_H^m; e_{ab}, e_{bh}) \\
+ C(u_H^m - u, b_H^m = b; u_H^m, b_H^m; e_{ab}, e_{bh}) \\
+ C(u_H^m, b_H^m; R(u, p) - u, \Lambda b - b; e_{ab}, e_{bh}) \\
+ \frac{\alpha_h}{R^2} a_e(u, e_{ab}) + \frac{\alpha_h}{R^2} a_e(R(u, p) - u, e_{ab}) \\
+ G(p, e_{ph}) + G(Q(u, p) - p, e_{ph}).
\]

(46)

The left-hand side of (46) can be estimated as

\[
l.h.s. \geq C_{min}\left(\|e_{ab}, e_{bh}\|_2^2 + \|e_{ph} - \Pi_h e_{ph}\|_0^2\right) \\
\geq \min\{C_{min}, 1\}\left(\|e_{ab}, e_{bh}\|_2^2 + \|e_{ph} - \Pi_h e_{ph}\|_0^2\right).
\]

(47)

Making use of (14), the right-hand side of (46) for \(G(\cdot, \cdot)\) can be estimated as

\[
G(p, e_{ph}) = (p - \Pi_h p, e_{ph} - \Pi_h e_{ph}) \\
\leq C_{\Pi_h} \|p\|_\gamma \|e_{ph} - \Pi_h e_{ph}\|_0,
\]

(48)

\[
G(Q(u, p) - p, e_{ph}) \leq C \|Q(u, p) - p\|_\gamma \|e_{ph} - \Pi_h e_{ph}\|_0 \\
\leq C_{\Pi_h} \|Q(u, p) - p\|_\gamma \|e_{ph} - \Pi_h e_{ph}\|_0.
\]

(49)

Using (9), (12) and (37), as well as Theorems 1, 2 and Lemma 2, the right-hand side of (46) can be estimated that

\[
r.h.s. \leq N(||u - u_H^m, b - b_H^m||_2^2 + ||e_{ab}, e_{bh}||_1) \\
+ N(||u - u_H^m, b - b_H^m||_1 \|A_{11}H u_H^m, A_{21}H b_H^m||_0 \|e_{ab}, e_{bh}||_1 \\
+ N(||u_H^m - u_H^m, b_H^m - b_H^m||_1 \|u_H^m, b_H^m||_1 \|e_{ab}, e_{bh}||_1 \\
+ N(||u_H^m, b_H^m||_1 \|u - R(u, p), b - \Lambda b||_1 \|e_{ab}, e_{bh}||_1 \\
+ \gamma h \|u, b||_1 \|e_{ab}, e_{bh}||_1 \\
+ \gamma h \|u - R(u, p), b - \Lambda b||_1 \|e_{ab}, e_{bh}||_1 \\
+ C_{\Pi_h} \|u\|_1 + \|p\|_\gamma \|e_{ph} - \Pi_h e_{ph}\|_0 \\
\leq C (H^{2\min(1, \gamma, \tau)} + \gamma^{2m} + C_{H}^{\min(2, \gamma + 1, \tau + 1)} + H^2 + \gamma^{2m} + h + h^{\min(1, \gamma, \tau)}) \\
\cdot \left(\|e_{ab}, e_{bh}\|_2^2 + \|e_{ph} - \Pi_h e_{ph}\|_0^2\right)^{\frac{1}{2}},
\]

(50)

Combining (23), (25) and (47) with (50), we can get the first part of (44).

To estimate the pressure, we rewrite (45) with \(s = 0\) as

\[
D_p(u - u_{mhr}, p - p_{mhr}; v, q) + c_0(u - u_H^m, u, v) + c_0(u_H^m, u - u_{mhr}, v) \\
- c_1(b - b_H^m, v, b) - c_1(b_H^m, b - b_{mhr}) - \frac{\alpha_h}{R^2} d_k(u_{mhr}, v) = G(p, q),
\]

(45)
\[ b_m(r - r_{mb}, c) = a_m(b - b_{mb}, c) + c_1(b - b_{pb}^m, u - u_{pb}^m, c) + c_1(b - b_{pb}^m, u - u_{pb}^m, c). \]

Applying (16), (10) and the standard technique to the above two equations, we can derive the second part of (44). We complete the proof. \(\square\)

4. Numerical Examples

In this section, some numerical experiments are shown to verify the correctness and effectiveness of the one-level stabilized finite element method and the two-level stabilized one. Here, the velocity, pressure and quasi-pressure are approximated by \(P_1\) and the magnetic field by the first (or second) class Nédélec edge element. SFEM denotes by the stabilized finite element method (31) and (32). The software FEAIPy 1.0 [39] created by Huayi Wei, Yangtze University, Yangtze, China is used in the numerical examples.

Smooth solution in 2D: Set \(\Omega = [0, 1]^2\) and \(R_c = R_m = S = 1, \sigma = 0.01\). Given the source terms \(f, g\) such that the exact solution is

\[
\begin{align*}
 u_1 &= 10x_1^2(x_1 - 1)^2x_2(x_2 - 1)(2x_2 - 1), \\
 u_2 &= -10x_1(x_1 - 1)(2x_1 - 1)x_2^2(x_2 - 1)^2, \\
 b_1 &= \cos(\pi x_1) \sin(\pi x_2), \\
 b_2 &= -\sin(\pi x_1) \cos(\pi x_2), \\
 p &= 10(2x_1 - 1)(2x_2 - 1), \\
 r &= 0.
\end{align*}
\]

Tables 1 and 2 display the errors of SFEM and two-level SFEM for 2D MHD Equation (1). It is shown that the corresponding errors are smaller and smaller along with the grid getting finer and finer, the convergence order is optimal. When \(h = O(H^2)\), the error accuracy of the two methods is almost the same. From CPU time, compared to SFEM, two-level SFEM save much computational cost.

**Table 1.** Convergence of \(u_h\) and \(p_h\) (first Nédélec element).

| \(h\)     | \(H\)   | \(||u - u_h||_0\) | order | \(||\nabla (u - u_h)||_0\) | order | \(||p - p_h||_0\) | order | CPU(s) |
|-----------|---------|-----------------|-------|--------------------------|-------|-----------------|-------|-------|
| SFEM      | 1/16    | 4.31 \times 10^{-3} |       | 7.14 \times 10^{-2}   |       | 1.19 \times 10^{-1} |       | 0.49   |
|           | 1/16    | 4.28 \times 10^{-3} |       | 7.15 \times 10^{-2}   |       | 1.28 \times 10^{-1} |       | 0.36   |
| SFEM      | 1/36    | 8.80 \times 10^{-4} | 1.96  | 2.62 \times 10^{-2}   | 1.24  | 2.93 \times 10^{-2} | 1.73  | 2.79   |
|           | 1/36    | 8.76 \times 10^{-4} | 1.96  | 2.64 \times 10^{-2}   | 1.23  | 3.36 \times 10^{-2} | 1.65  | 1.70   |
| SFEM      | 1/64    | 2.81 \times 10^{-4} | 1.98  | 1.36 \times 10^{-2}   | 1.13  | 1.10 \times 10^{-2} | 1.71  | 11.13  |
|           | 1/64    | 2.81 \times 10^{-4} | 1.98  | 1.38 \times 10^{-2}   | 1.13  | 1.36 \times 10^{-2} | 1.57  | 6.16   |
| SFEM      | 1/100   | 1.15 \times 10^{-4} | 1.99  | 8.43 \times 10^{-3}   | 1.08  | 5.21 \times 10^{-3} | 1.67  | 35.79  |
|           | 1/100   | 1.16 \times 10^{-4} | 1.97  | 8.53 \times 10^{-3}   | 1.08  | 6.99 \times 10^{-3} | 1.49  | 17.78  |

**Table 2.** Convergence of \(b_h\) and \(r_h\) (first Nédélec element).

| \(h\)     | \(H\)   | \(||b - b_h||_0\) | order | \(||b - b_h||_{\text{curl}}\) | order | \(||r - r_h||_0\) | CPU(s) |
|-----------|---------|-----------------|-------|--------------------------|-------|-----------------|-------|
| SFEM      | 1/16    | 4.01 \times 10^{-2} |       | 2.09 \times 10^{-1}   |       | 1.22 \times 10^{-14} |       | 0.49   |
|           | 1/16    | 4.01 \times 10^{-2} |       | 2.09 \times 10^{-1}   |       | 1.42 \times 10^{-14} |       | 0.36   |
| SFEM      | 1/36    | 1.78 \times 10^{-2} | 1.00  | 9.30 \times 10^{-2}   | 1.00  | 5.20 \times 10^{-14} | 2.79  |
|           | 1/36    | 1.78 \times 10^{-2} | 1.00  | 9.31 \times 10^{-2}   | 1.00  | 5.03 \times 10^{-14} | 1.70  |
| SFEM      | 1/64    | 1.00 \times 10^{-2} | 1.00  | 5.23 \times 10^{-2}   | 1.00  | 1.33 \times 10^{-13} | 11.13 |
|           | 1/64    | 1.00 \times 10^{-2} | 1.00  | 5.24 \times 10^{-2}   | 1.00  | 1.43 \times 10^{-13} | 6.16  |
| SFEM      | 1/100   | 6.41 \times 10^{-3} | 1.00  | 3.35 \times 10^{-2}   | 1.00  | 2.28 \times 10^{-13} | 35.79 |
|           | 1/100   | 6.41 \times 10^{-3} | 1.00  | 3.35 \times 10^{-2}   | 1.00  | 2.27 \times 10^{-13} | 17.78 |
The numerical results are listed in Tables 3 and 4 when the magnetic field is approximated by the second class Nédélec element. Clearly, the convergence order of $\|b - b_h\|_0$ is one higher than that in Tables 1 and 2, which is consistent with the general theoretical analysis results of the Nédélec element.

**Table 3.** Convergence of $u_h$ and $p_h$ (second class Nédélec element).

| $h$    | $H$    | $\|u - u_h\|_0$ | order | $\|\nabla (u - u_h)\|_0$ | order | $\|p - p_h\|_0$ | order | CPU(s) |
|--------|--------|-----------------|-------|--------------------------|-------|----------------|-------|--------|
| SFEM   | 1/16   | $4.31 \times 10^{-3}$ |   | $7.14 \times 10^{-2}$ |   | $1.20 \times 10^{-1}$ |   | 0.78   |
| 1/16   | $4.28 \times 10^{-3}$ |   | $7.15 \times 10^{-2}$ |   | $1.28 \times 10^{-1}$ |   | 0.47   |
| SFEM   | 1/36   | $8.80 \times 10^{-4}$ | 1.96 | $2.62 \times 10^{-2}$ | 1.24 | $2.94 \times 10^{-2}$ | 1.73 | 7.63   |
| 1/36   | 1/6    | $8.76 \times 10^{-4}$ | 1.96 | $2.64 \times 10^{-2}$ | 1.23 | $3.20 \times 10^{-2}$ | 1.65 | 3.33   |
| SFEM   | 1/64   | $2.81 \times 10^{-4}$ | 1.98 | $1.36 \times 10^{-2}$ | 1.13 | $1.10 \times 10^{-2}$ | 1.70 | 48.23  |
| 1/64   | 1/8    | $2.81 \times 10^{-4}$ | 1.98 | $1.38 \times 10^{-2}$ | 1.13 | $1.16 \times 10^{-2}$ | 1.57 | 18.00  |
| SFEM   | 1/100  | $1.15 \times 10^{-4}$ | 1.99 | $8.43 \times 10^{-3}$ | 1.08 | $5.59 \times 10^{-3}$ | 1.53 | 205.89 |
| 1/100  | 1/10   | $1.16 \times 10^{-4}$ | 1.98 | $8.53 \times 10^{-3}$ | 1.08 | $6.99 \times 10^{-3}$ | 1.49 | 71.25  |

**Table 4.** Convergence of $b_h$ and $r_h$ (second class Nédélec element).

| $h$    | $H$    | $\|b - b_h\|_0$ | order | $\|b_h\|_{\text{curl}}$ | order | $\|r - r_h\|_0$ | order | CPU(s) |
|--------|--------|-----------------|-------|--------------------------|-------|----------------|-------|--------|
| SFEM   | 1/16   | $4.19 \times 10^{-3}$ |   | $2.05 \times 10^{-1}$ |   | $1.44 \times 10^{-14}$ |   | 0.78   |
| 1/16   | $4.10 \times 10^{-3}$ |   | $2.05 \times 10^{-1}$ |   | $1.78 \times 10^{-14}$ |   | 0.47   |
| SFEM   | 1/36   | $8.33 \times 10^{-4}$ | 1.99 | $9.13 \times 10^{-2}$ | 1.00 | $6.62 \times 10^{-14}$ |   | 7.63   |
| 1/36   | 1/6    | $8.39 \times 10^{-4}$ | 1.98 | $9.31 \times 10^{-2}$ | 1.00 | $7.00 \times 10^{-14}$ |   | 3.33   |
| SFEM   | 1/64   | $2.63 \times 10^{-4}$ | 2.00 | $5.14 \times 10^{-2}$ | 1.00 | $1.49 \times 10^{-13}$ |   | 48.23  |
| 1/64   | 1/8    | $2.71 \times 10^{-4}$ | 1.96 | $5.14 \times 10^{-2}$ | 1.00 | $1.74 \times 10^{-13}$ |   | 18.00  |
| SFEM   | 1/100  | $1.08 \times 10^{-4}$ | 1.99 | $3.28 \times 10^{-2}$ | 1.00 | $2.71 \times 10^{-13}$ |   | 205.89 |
| 1/100  | 1/10   | $1.15 \times 10^{-4}$ | 1.91 | $3.29 \times 10^{-2}$ | 1.00 | $3.85 \times 10^{-13}$ |   | 71.25  |

**Smooth solution in 3D:** Set $\Omega = [0, 1]^3$ and $R_e = R_m = S = 1, \sigma = 0.01$. Given $f, g$ such that the exact solution is:

\[
\begin{align*}
u_1 &= 0.5 \sin(\pi x_1) \cos(\pi x_2) \cos(\pi x_3), \\
u_2 &= 0.5 \cos(\pi x_1) \sin(\pi x_2) \cos(\pi x_3), \\
u_3 &= -\cos(\pi x_1) \cos(\pi x_2) \sin(\pi x_3), \\
b_1 &= 0.5 \cos(\pi x_1) \sin(\pi x_2) \sin(\pi x_3), \\
b_2 &= -\sin(\pi x_1) \cos(\pi x_2) \sin(\pi x_3), \\
b_3 &= 0.5 \sin(\pi x_1) \sin(\pi x_2) \cos(\pi x_3), \\
p &= \cos(\pi x_1) \cos(\pi x_2) \cos(\pi x_3), \\
r &= 0.
\end{align*}
\]

In Tables 5 and 6, the variable quantity $(u, b, p, r)$ is approximated by $P_1$, the first class Nédélec element, $P_1$ and $P_3$, respectively. Tables 7 and 8 list the results when $(u, b, p, r)$ is approximated by $P_1$, the second class Nédélec element, $P_1$ and $P_3$. It is observed that the numerical results agree well with the theoretical results of Theorems 1–3. On the other hand, from Tables 7 and 8, we find that SFEM (35) does not work when $H = 1/16$, however two-level SFEM (41) and (42) is valid in the current computing environment for our computer. On the other hand, the stability results of (43) are checked by Figure 1.
Table 5. Convergence of $\mathbf{u}_h$ and $p_h$ (first class Nédélec element).

| $h$ | $H$   | $\| \mathbf{u} - \mathbf{u}_h \|_0$ | order | $\| \nabla (\mathbf{u} - \mathbf{u}_h) \|_0$ | order | $\| p - p_h \|_0$ | order | CPU(s) |
|-----|-------|----------------------------------|-------|----------------------------------|-------|----------------|-------|--------|
| SFEM | 1/4   | $7.86 \times 10^{-2}$           |       | 1.17                             |       | 1.39           |       | 2.03    |
|      | 1/2   | $7.86 \times 10^{-2}$           |       | 1.17                             |       | 1.39           |       | 1.44    |
| SFEM | 1/9   | $1.67 \times 10^{-2}$           | 1.91  | $5.33 \times 10^{-1}$            | 0.97  | $3.26 \times 10^{-1}$ | 1.79  | 28.05  |
|      | 1/3   | $1.67 \times 10^{-2}$           | 1.91  | $5.34 \times 10^{-1}$            | 0.97  | $3.29 \times 10^{-1}$ | 1.78  | 16.26  |
| SFEM | 1/16  | $5.34 \times 10^{-3}$           | 1.98  | $2.99 \times 10^{-1}$            | 1.00  | $1.08 \times 10^{-1}$ | 1.91  | 577.84 |
|      | 1/4   | $5.38 \times 10^{-3}$           | 1.98  | $2.99 \times 10^{-1}$            | 1.00  | $1.12 \times 10^{-1}$ | 1.86  | 189.50 |

Table 6. Convergence of $\mathbf{b}_h$ and $r_h$ (first class Nédélec element).

| $h$ | $H$   | $\| \mathbf{b} - \mathbf{b}_h \|_0$ | order | $\| \mathbf{b} - \mathbf{b}_h \|_{\text{curl}}$ | order | $\| r - r_h \|_0$ | CPU(s) |
|-----|-------|----------------------------------|-------|----------------------------------|-------|----------------|--------|
| SFEM | 1/4   | $1.38 \times 10^{-1}$           |       | $8.38 \times 10^{-1}$            |       | 7.86 $10^{-16}$ | 2.03    |
|      | 1/2   | $1.38 \times 10^{-1}$           |       | $8.39 \times 10^{-1}$            |       | 9.82 $10^{-16}$ | 1.44    |
| SFEM | 1/9   | $6.16 \times 10^{-2}$           | 1.00  | $3.81 \times 10^{-1}$            | 0.97  | $4.24 \times 10^{-15}$ | 28.05  |
|      | 1/3   | $6.17 \times 10^{-2}$           | 0.99  | $3.83 \times 10^{-1}$            | 0.96  | $4.71 \times 10^{-15}$ | 16.26  |
| SFEM | 1/16  | $3.47 \times 10^{-2}$           | 1.00  | $2.14 \times 10^{-1}$            | 1.00  | $1.59 \times 10^{-14}$ | 577.84 |
|      | 1/4   | $3.47 \times 10^{-2}$           | 1.00  | $2.18 \times 10^{-1}$            | 0.98  | $1.42 \times 10^{-14}$ | 189.50 |

Table 7. Convergence of $\mathbf{u}_h$ and $p_h$ (second class Nédélec element).

| $h$ | $H$   | $\| \mathbf{u} - \mathbf{u}_h \|_0$ | order | $\| \nabla (\mathbf{u} - \mathbf{u}_h) \|_0$ | order | $\| p - p_h \|_0$ | order | CPU(s) |
|-----|-------|----------------------------------|-------|----------------------------------|-------|----------------|-------|--------|
| SFEM | 1/4   | $7.86 \times 10^{-2}$           |       | 1.17                             |       | 1.39           |       | 2.39    |
|      | 1/2   | $7.86 \times 10^{-2}$           |       | 1.17                             |       | 1.40           |       | 1.85    |
| SFEM | 1/9   | $1.67 \times 10^{-2}$           | 1.91  | $5.33 \times 10^{-1}$            | 0.97  | $3.26 \times 10^{-1}$ | 1.79  | 99.42  |
|      | 1/3   | $1.67 \times 10^{-2}$           | 1.91  | $5.34 \times 10^{-1}$            | 0.97  | $3.30 \times 10^{-1}$ | 1.78  | 42.96  |
| SFEM | 1/16  | $5.35 \times 10^{-3}$           | 1.98  | $2.99 \times 10^{-1}$            | 1.00  | $1.11 \times 10^{-1}$ | 1.86  | 2877.53 |

Table 8. Convergence of $\mathbf{b}_h$ and $r_h$ (second class Nédélec element).

| $h$ | $H$   | $\| \mathbf{b} - \mathbf{b}_h \|_0$ | order | $\| \mathbf{b} - \mathbf{b}_h \|_{\text{curl}}$ | order | $\| r - r_h \|_0$ | CPU(s) |
|-----|-------|----------------------------------|-------|----------------------------------|-------|----------------|--------|
| SFEM | 1/4   | $6.64 \times 10^{-2}$           |       | $8.29 \times 10^{-1}$            |       | $8.89 \times 10^{-16}$ | 2.39    |
|      | 1/2   | $6.64 \times 10^{-2}$           |       | $8.30 \times 10^{-1}$            |       | $1.10 \times 10^{-15}$ | 1.85    |
| SFEM | 1/9   | $1.40 \times 10^{-2}$           | 1.91  | $3.76 \times 10^{-1}$            | 0.98  | $5.60 \times 10^{-15}$ | 99.42  |
|      | 1/3   | $1.41 \times 10^{-2}$           | 1.91  | $3.76 \times 10^{-1}$            | 0.97  | $6.19 \times 10^{-15}$ | 42.96  |
| SFEM | 1/16  | \text{\textbackslash{}}         | \text{\textbackslash{}} | \text{\textbackslash{}} | \text{\textbackslash{}} | \text{\textbackslash{}} | \text{\textbackslash{}} | \text{\textbackslash{}} |
|      | 1/4   | $4.61 \times 10^{-3}$           | 1.95  | $2.12 \times 10^{-1}$            | 1.00  | $2.26 \times 10^{-14}$ | 2877.53 |

Figure 1. Stability of 2D (left) and 3D (right) problems.
2D MHD problem with a singular solution: We consider 2D MHD system (1) in the L-type domain Ω := [−1, 1]²\{(0, 0) \times [−1, 0]\}. Set Re = S = 1, Rm = 0.001, the analytical solution in polar coordinates (ρ, φ) is given by [40]

\[
\begin{align*}
\mathbf{u}(\rho, \varphi) &= \begin{bmatrix}
\rho^\lambda((1 + \lambda) \sin(\varphi)\psi(\varphi) + \cos(\varphi)\psi'(\varphi)) \\
\rho^\lambda(-(1 + \lambda) \cos(\varphi)\psi(\varphi) + \sin(\varphi)\psi'(\varphi))
\end{bmatrix}, \\
\mathbf{b}(\rho, \varphi) &= \nabla \left(\rho^{2/3} \sin(2/3\varphi)\right), \\
p(\rho, \varphi) &= -\rho^{\lambda-1}\left((1 + \lambda)^2 \psi'(\varphi) + \psi''(\varphi)\right)/(1 - \lambda), \\
r &= 0, \\
\psi(\varphi) &= \sin((1 + \lambda)\varphi) \cos(\lambda\omega)/(1 + \lambda) - \cos((1 + \lambda)\varphi) - \sin((1 - \lambda)\varphi) \cos(\lambda\omega)/(1 - \lambda) + \cos((1 - \lambda)\varphi), \\
\omega &= \frac{3}{5} \pi, \\
\lambda &\approx 0.5444837678246.
\end{align*}
\]

In the 2D case, there holds the regularity \( \mathbf{u} \in H^{1,\lambda}(\Omega), \mathbf{b} \in H^{\frac{3}{2}}(\Omega) \) and \( p \in H^{1}(\Omega) \).

In Tables 9 and 10, \( (\mathbf{u}, \mathbf{b}, p, r) \) is approximated by \( P_1 \), the second class Nédelec element, \( P_1 \) and \( P_1 \). Because the regularity of velocity, magnetic field and pressure is low, the convergence of \( ||\mathbf{u} - \mathbf{u}_h||_0, ||\nabla(\mathbf{u} - \mathbf{u}_h)||_0, ||p - p_h||_0, ||\mathbf{b} - \mathbf{b}_h||_0, ||\mathbf{b} - \mathbf{b}_h||_{curl} \) keep the rate of 1.4, 0.54, 0.59, 0.66, 0.66, respectively, which verify the correctness of the theoretical analysis (Theorems 2 and 3) results. In Figure 2 we display the streamlines of the velocity field and magnetic field, and the contours of the pressure, which are consistent with the numerical results in the literature [40].

### Table 9. Convergence of \( \mathbf{u}_h \) and \( p_h \).

| \( h \) | \( H \) | \( ||\mathbf{u} - \mathbf{u}_h||_0 \) | order | \( ||\nabla(\mathbf{u} - \mathbf{u}_h)||_0 \) | order | \( ||p - p_h||_0 \) | order | CPU(s) |
|-------|-------|----------------|-------|----------------|-------|----------------|-------|-------|
| SFEM  | 1/4   | 9.97 × 10^{-2} | 1.55  | 1.74           | 0.16  |
|       | 1/2   | 9.96 × 10^{-2} | 1.55  | 1.76           | 0.16  |
| SFEM  | 1/16  | 1.27 × 10^{-2} | 1.48  | 7.41 × 10^{-1} | 0.54  | 7.74 × 10^{-1} | 0.59  | 1.89  |
| SFEM  | 1/36  | 3.95 × 10^{-3} | 1.45  | 4.78 × 10^{-1} | 0.54  | 5.00 × 10^{-1} | 0.54  | 12.03 |
| SFEM  | 1/64  | 1.77 × 10^{-3} | 1.39  | 3.50 × 10^{-1} | 0.54  | 3.76 × 10^{-1} | 0.49  | 49.61 |
|       | 1/8   | 1.84 × 10^{-3} | 1.37  | 3.50 × 10^{-1} | 0.54  | 3.93 × 10^{-1} | 0.49  | 22.80 |

### Table 10. Convergence of \( \mathbf{b}_h \) and \( r_h \).

| \( h \) | \( H \) | \( ||\mathbf{b} - \mathbf{b}_h||_0 \) | order | \( ||\mathbf{b} - \mathbf{b}_h||_{curl} \) | order | \( ||r - r_h||_0 \) | CPU(s) |
|-------|-------|----------------|-------|----------------|-------|----------------|-------|
| SFEM  | 1/4   | 1.91 × 10^{-1} | 0.64  | 1.91 × 10^{-1} | 0.64  | 5.98 × 10^{-4} | 0.16  |
|       | 1/2   | 1.90 × 10^{-1} | 0.64  | 1.91 × 10^{-1} | 0.64  | 5.98 × 10^{-4} | 0.16  |
| SFEM  | 1/16  | 7.88 × 10^{-2} | 0.64  | 7.88 × 10^{-2} | 0.64  | 1.47 × 10^{-4} | 1.89  |
| SFEM  | 1/36  | 4.63 × 10^{-2} | 0.65  | 4.63 × 10^{-2} | 0.65  | 5.84 × 10^{-5} | 12.03 |
| SFEM  | 1/64  | 3.17 × 10^{-2} | 0.66  | 3.17 × 10^{-2} | 0.66  | 2.97 × 10^{-5} | 49.61 |
|       | 1/8   | 3.17 × 10^{-2} | 0.66  | 3.17 × 10^{-2} | 0.66  | 2.97 × 10^{-5} | 22.80 |
5. Conclusions

Based on the stabilization and the Lagrange multiplier techniques, the stabilized finite element algorithm is designed for the stationary incompressible MHD. The Lagrange multiplier technique idea helps us in dealing with the low regular magnetic field sub-problem by $H(\text{curl}; \Omega)$-conforming element. The stabilized one by using local Gauss integration allow us to adopt the lowest equal-order elements to approximate the flow field sub-problem. The stability and optimal convergence analysis are given. Furthermore, the two-level stabilized finite element algorithms are presented. In the first step we combine the stabilized finite element method with the Oseen iteration for the nonlinear MHD equations on a coarse grid. For the second step, we employ the linearized correction on a fine grid. We give the optimal error analysis, which shows that when the grid sizes satisfy $h = O(H^2)$, the two-level stabilization method not only has the optimal convergence order, but also can save more computational cost than the one-level method. These theoretical analysis results have been verified by some numerical experiments.

Author Contributions: Conceptualization, Q.T., M.H. and L.Y.; methodology, M.H. and L.Y.; software, M.H.; validation, Q.T., M.H., Y.X. and L.Y.; formal analysis, Q.T. and M.H.; investigation, M.H., Y.X. and L.Y.; resources, Q.T. and M.H.; data curation, M.H.; writing—original draft preparation, Q.T., M.H. and Y.X.; writing—review and editing, Q.T. and Y.X.; visualization, M.H.; supervision, Q.T.; project administration, Q.T. and M.H.; funding acquisition, Q.T. and M.H. All authors have read and agreed to the published version of the manuscript.

Funding: The research was funded by the National Natural Science Foundation of China (Nos: 12071404,11701151), Young Elite Scientist Sponsorship Program by Cast of CAST (No: 2020QNRC001), China Postdoctoral Science Foundation (Nos: 2018T110073), the Natural Science Foundation of Hunan Province (No: 2019JJ40279), Excellent Youth Program of Scientific Research Project of Hunan Provincial Department of Education (Nos: 20B564), and International Scientific and Technological Innovation Cooperation Base of Hunan Province for Computational Science (No: 2018WK4006).

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.
Appendix A. Proof of Theorem 1

In this part, we will give the detail proof of Theorem 1.

Proof. Let $e_u = R(u, p) - u, e_b = \Lambda b - b, e_p = Q(u, p) - p, e_r = \Lambda r - r$. Subtract (18) from (4) and choose $v = e_u, c = e_p, q = e_p, s = e_r$, we get

$$A_\sigma(e_u, e_b; e_u, e_b) + C(e_u, e_b; u, b; e_u, e_b) + G(e_p, e_p) = C(R(u, p) - u, \Lambda b - b, e_u, e_b)$$

$$+ C(u, b_p; R(u, p) - u, \Lambda b - b, e_u, e_b)$$

$$+ \frac{\epsilon_p}{R_e} a_s(u, e_u) + \frac{\epsilon_b}{R_e} a_s(R(u, p) - u, e_u)$$

$$+ G(p, e_p) + G(Q(u, p) - p, e_p).$$  \hspace{1cm} (A1)

Using (9) and (20) and Cauchy-Schwarz inequality, the left-hand side of (A1) can be bounded by

$$l.h.s \geq \left( C_{\min} - \frac{\tilde{N} \|F\|_s}{\rho} \right) \|e_u, e_b\|^2 + \|e_p - \Pi e_p\|^2_0$$

$$\geq g(1 - \sigma_0) \|e_u, e_b\|^2 + \|e_p - \Pi e_p\|^2_0$$

$$\geq \min\{g(1 - \sigma_0), 1\} \left( \|e_u, e_b\|^2 + \|e_p - \Pi e_p\|^2_0 \right),$$  \hspace{1cm} (A2)

where $\sigma_0 = \frac{\tilde{N} \|F\|_s}{\rho}$.

By (14), the two terms of the right-hand side of (A1) with respect to $G(\cdot, \cdot)$ can be estimated as

$$G(p, e_p) \leq \|p - \Pi p\|_0 \|e_p - \Pi e_p\|_0$$

$$\leq C\mu_{\min} \|p\|_s \|e_p - \Pi e_p\|_0,$$  \hspace{1cm} (A3)

$$G(Q(u, p) - p, e_p) \leq C\|Q(u, p) - p\|_0 \|e_p - \Pi e_p\|_0$$

$$\leq C\mu_{\min} \|u\|_{1 + \gamma} \|p\|_s \|e_p - \Pi e_p\|_0.$$  \hspace{1cm} (A4)

Thus, from (5), (9), (23), (25) and Lemma 1, the right-hand side of (A1) can be estimated as

$$r.h.s \leq \tilde{N} \left( \|u, b\|_1 + \|u, b\|_1 \right) \|u - R(u, p), b - \Lambda b\|_1 \|e_u, e_b\|_1$$

$$+ \sigma \mu \|u, b\|_1 + \|u - R(u, p), b - \Lambda b\|_1 \|e_u, e_b\|_1$$

$$+ C\mu_{\min} \|u\|_{1 + \gamma} + \|p\|_s \|e_p - \Pi e_p\|_0$$

$$\leq \tilde{N} \left( \|F\|_s + \|F\|_c_{\min} \right) \|u - R(u, p), b - \Lambda b\|_1 \|e_u, e_b\|_1$$

$$+ \sigma \mu \left( \frac{\|F\|_c}{\rho} \right) + \|u - R(u, p), b - \Lambda b\|_1 \|e_u, e_b\|_1$$

$$+ C\mu_{\min} \|u\|_{1 + \gamma} + \|p\|_s \|e_p - \Pi e_p\|_0$$

$$\leq C \left( \mu_{\min}^{1 + \gamma - \tau_1} \right) \|u\|_{1 + \gamma} + \|p\|_s + \|b\|_s + \|\text{curl} b\|_s + \|r\|_{s + 1}$$

$$+ \sigma \mu^{-1} \|F\|_s \|e_u, e_b\|^2 + \|e_p - \Pi e_p\|^2_0 \right)^2.$$  \hspace{1cm} (A5)

It follows from (A2)–(A5) that

$$\|e_u, e_b\|_1 \leq C\mu_{\min} \|u\|_{1 + \gamma} + \|p\|_s + \|b\|_s$$

$$+ \|\text{curl} b\|_s + \|r\|_{s + 1} + C\sigma \mu^{-1} \|F\|_s.$$  \hspace{1cm} (A6)

Using triangle inequality, (23), (25) and (A6) to obtain (28).

From (2) and (17), we have

$$a_b(u - u_{\mu}, v) - b_b(p - p_{\mu}, v) + b_b(q, u - u_{\mu}) + G(p - p_{\mu}, q)$$

$$= c_0(u_{\mu} - u, u, v) + c_0(u_{\mu}, u_{\mu} - u, v) + c_1(b - b_{\mu}, v, b)$$

$$+ c_1(b_{\mu}, v, b - b_{\mu}) + \frac{\epsilon_{\mu}}{R_e} a_s(u_{\mu}, v) + G(p, q).$$  \hspace{1cm} (A7)
Using (16), (5), (9), (14) and Lemma 1 to derive
\[ \beta \left( R_{c}^{-1} \| \nabla (u - u_{\mu}) \|_{0} + \| p - p_{\mu} \|_{0} \right) \]
\[ \leq \tilde{N} \left( \| (u, b) \|_{1} + \| (u_{\mu}, b_{\mu}) \|_{1} \right) \| (u - u_{\mu}, b - b_{\mu}) \|_{1} \]
\[ + C_{\max} \sigma \mu \| (u_{\mu}, b_{\mu}) \|_{1} + C \| p - \Pi p_{\mu} \|_{0} \]
\[ \leq \tilde{N} \left( \frac{\| F \|_{*}}{\| u \|_{0}} + \frac{\| F \|_{*}}{\| u \|_{0}} \right) \| (u - u_{\mu}, b - b_{\mu}) \|_{1} \]
\[ + \sigma \mu \frac{C_{\max}}{C_{\min}} \| F \|_{*} + c_{2} \mu^{\min(1, \gamma)} \| p \|_{\gamma} \quad (A8) \]
\[ \leq 2 \gamma_{0} \| (u - u_{\mu}, b - b_{\mu}) \|_{1} + C \gamma \mu \| F \|_{*} + C \mu^{\min(1, \gamma)} \| p \|_{\gamma}. \]

From (3) and (17), and taking \( s = 0 \) to get
\[ b_{m}(r - r_{\mu}, c) = a_{m}(b - b_{\mu}, c) + c_{1}(b - b_{\mu}, u, c) + c_{1}(b_{\mu}, u - u_{\mu}, c). \quad (A9) \]

Applying (10), (5), (9) and Lemma 1, we have
\[ \beta \| \nabla (r - r_{\mu}) \|_{0} \leq C_{\max} \| (u - u_{\mu}, b - b_{\mu}) \|_{1} + \tilde{N} \| (u - u_{\mu}, b - b_{\mu}) \|_{1} \| (u, b) \|_{1} \]
\[ + \tilde{N} \| (u, b) \|_{1} \| (u - u_{\mu}, b - b_{\mu}) \|_{1} \]
\[ \leq C_{\max} \| (u - u_{\mu}, b - b_{\mu}) \|_{1} + 2 \gamma_{0} \| (u - u_{\mu}, b - b_{\mu}) \|_{1} \]
\[ + C_{\min} (1 - r_{1}) \| (u - u_{\mu}, b - b_{\mu}) \|_{1} \quad (A10) \]

Combining (28) and (A8) with (A10), we can obtain (29).

Further, to obtain (30), we need to introduce the dual problem [11]: For all \((v, c, q, l) \in X \times W \times Q \times S\), find \((w, \Phi, s, t) \in X \times W \times Q \times S\) such that
\[ \mathcal{A}(v, c; w, \Phi) + C(u_{\mu}, b_{\mu}; v, c; w, \Phi) + C(v, c; u, b; w, \Phi) = B(s, t; v, c) \]
\[ + B(q, l; w, \Phi) = (u - u_{\mu}, b - b_{\mu}; v, c). \quad (A11) \]

Taking \((v, c, q, l) = (w, \Phi; s, t) \) in (A11) and using (9), then
\[ C_{\min} (1 - r_{1}) \| (w, \Phi) \|_{1} \leq \| (u - u_{\mu}, b - b_{\mu}) \|_{0}. \quad (A12) \]

Assuming that the region \( \Omega \) is sufficiently smooth (see Theorem 2 of [11]), then the following regularity error estimates are available
\[ \| (w, \Phi) \|_{2} + \| (s, t) \|_{1} \]
\[ \leq C \| (u - u_{\mu}, b - b_{\mu}) \|_{0} \]
\[ + \tilde{N} \| (u, b) \|_{1} + \| (u_{\mu}, b_{\mu}) \|_{1} \]
\[ \leq C \| (u - u_{\mu}, b - b_{\mu}) \|_{0} + 2 C_{\min} \| (w, \Phi) \|_{2} \]
\[ \| (w, \Phi) \|_{2} \]
\[ C \| (u - u_{\mu}, b - b_{\mu}) \|_{0} + C_{\min} \| (w, \Phi) \|_{2} \]
thereby
\[ \| (w, \Phi) \|_{2} + \| (s, t) \|_{1} \leq C \| (u - u_{\mu}, b - b_{\mu}) \|_{0}. \quad (A13) \]

Finding \( v = u - u_{\mu}, c = b - b_{\mu}, q = l = 0 \) such that
\[ \mathcal{A}(u - u_{\mu}, b - b_{\mu}; w, \Phi) + C(u_{\mu}, b_{\mu}; u - u_{\mu}, b - b_{\mu}; w, \Phi) \]
\[ + C(u - u_{\mu}, b - b_{\mu}; u - u_{\mu}, b - b_{\mu}; w, \Phi) = (u - u_{\mu}, b - b_{\mu}; u - u_{\mu}, b - b_{\mu}). \quad (A14) \]
Subtracting (2) and (3) from (18), and taking $q = l = 0$, for all $(v, c) \in X_\mu \times W_\mu$, we have
\[ A(u - u_\mu, b - b_\mu, w - v, \Phi - c)
\]
\[ + C(u - u_\mu, b - b_\mu, u, b; w - v, \Phi - c)
\]
\[ + B(q_\mu' - s, I_\mu' - t; u - u_\mu, b - b_\mu)
\]
\[ - B(p - p_\mu, r - r_\mu; w - v, \Phi - c)
\]
\[ + \frac{\sigma_\mu}{R_\mu} a_\mu(u_\mu, v) = 0.
\]
(A15)

Subtract (A15) from (A14) to get
\[ A(u - u_\mu, b - b_\mu, w - v, \Phi - c)
\]
\[ + C(u - u_\mu, b - b_\mu, u, b; w - v, \Phi - c)
\]
\[ + B(q_\mu' - s, I_\mu' - t; u - u_\mu, b - b_\mu)
\]
\[ - B(p - p_\mu, r - r_\mu; w - v, \Phi - c)
\]
\[ + \frac{\sigma_\mu}{R_\mu} a_\mu(u_\mu, v) = \left\| (u - u_\mu, b - b_\mu) \right\|_\mu^2.
\]
(A16)

where $q_\mu' \in Q_\mu$ and $I_\mu' \in S_\mu$ are the approximation of $s$ and $t$, respectively.

Similar to the derivation in the literature [41] (see the proof of Theorem 3.2), we get
\[ \left\| (w, \Phi) \right\|_1 \leq C \mu \left( \left\| (w, \Phi) \right\|_2, w \in H^2(\Omega), \Phi \in H^2(\Omega). \right. \]

Taking $v = R(w, s), c = \Lambda \Phi$ in (A16), Applying (5), (7), (9), (23), (25), (A13) to get
\[ \left\| (u - u_\mu, b - b_\mu) \right\|_\mu^2
\]
\[ \leq C \mu \left( \left\| (u - u_\mu, b - b_\mu) \right\|_1 \left\| (w - R(w, s), \Phi - \Lambda \Phi) \right\|_1
\]
\[ + N \left( \left\| (u, b) \right\|_1 \left\| (u - u_\mu, b - b_\mu) \right\|_1 \left\| (w - R(w, s), \Phi - \Lambda \Phi) \right\|_1
\]
\[ + N \left( \left\| (u_\mu, b_\mu) \right\|_1 \left\| (u - u_\mu, b - b_\mu) \right\|_1 \left\| (w - R(w, s), \Phi - \Lambda \Phi) \right\|_1
\]
\[ + \left\| (u - u_\mu, b - b_\mu) \right\|_1 \left\| (s - q_\mu', t - I_\mu') \right\|_1
\]
\[ + \left\| (w - R(w, s), \Phi - \Lambda \Phi) \right\|_1 \left\| (p - p_\mu, r - r_\mu) \right\|_1
\]
\[ + C \mu \left( \left\| (u - u_\mu, b - b_\mu) \right\|_1 \left\| (w - R(w, s), \Phi - \Lambda \Phi) \right\|_1 \right.
\]
\[ + \left\| (w, \Phi) \right\|_1
\]
\[ \leq C \mu \left( \left\| (u - u_\mu, b - b_\mu) \right\|_1 \left\| (u - u_\mu, b - b_\mu) \right\|_0
\]
\[ + C \mu \left( \left\| (p - p_\mu, r - r_\mu) \right\|_1 \left\| (u - u_\mu, b - b_\mu) \right\|_0 \right.
\]
\[ + C \mu \left( \left\| (w, \Phi) \right\|_1 \right.
\]
\[ \left. \left\| (u - u_\mu, b - b_\mu) \right\|_0 \right).
\]

which together with (28) and (29) yield the desired result (30). The proof ends. \(\square\)

References
1. Gerbeau, J.; Le Bris, C.; Lelièvre, T. Mathematical Methods for the Magnetohydrodynamics of Liquid Metals; Clarendon Press: Oxford, UK, 2006.
2. Gunzburger, M.; Meir, A.; Peterson, J. On the existence, uniqueness, and finite element approximation of solutions of the equations of stationary, incompressible magnetohydrodynamics. *Math. Comput.* 1991, 56, 523–563. [CrossRef]
3. Shi, D.; Yu, Z. Nonconforming mixed finite element methods for stationary incompressible magnetohydrodynamics. *Int. J. Numer. Anal. Model.* 2013, 10, 904–919.
4. Dong, X.; He, Y.; Zhang, Y. Convergence analysis of three finite element iterative methods for the 2D/3D stationary incompressible magnetohydrodynamics. *Comput. Meth. Appl. Math. Eng.* 2014, 276, 287–311. [CrossRef]
5. Dong, X.; He, Y. The Oseen type finite element iterative method for the stationary incompressible magnetohydrodynamics. *Adv. Appl. Math. Mech.* 2017, 9, 775–794. [CrossRef]
6. Mao, S.; Feng, X.; Su, H. Optimal error estimates of penalty based iterative methods for steady incompressible magnetohydrodynamics equations with different viscosities. *J. Sci. Comput.* 2019, 79, 1078–1110.
7. Du, B.; Huang, J. The generalized Arrow-Hurwicz method with applications to fluid computation. *Commun. Comput. Phys.* 2019, 25, 752–780. [CrossRef]
8. Layton, W.; Meir, A.; Schmidt, P. A two-level discretization method for the stationary MHD equations. *Electron. Trans. Numer. Anal.* 1997, 6, 198–210.
9. Su, H.; Xu, J.; Feng, X. Optimal convergence analysis of two-level nonconforming finite element iterative methods for 2D/3D MHD equations. *Entropy* 2022, 24, S87. [CrossRef]
10. Su, H.; Feng, X.; Zhao, J. On two-level Oseen penalty iteration methods for the 2D/3D stationary incompressible magnetohydrodynamics. *J. Sci. Comput.* 2020, 83, 11. [CrossRef]
11. Dong, X.; He, Y. Convergence of some finite element iterative methods related to different Reynolds numbers for the 2D/3D stationary incompressible magnetohydrodynamics. Sci. China Math. 2016, 59, 589–608. [CrossRef]

12. Dong, X.; He, Y. Two-level Newton iterative method for the 2D/3D stationary incompressible magnetohydrodynamics. J. Sci. Comput. 2015, 63, 426–451. [CrossRef]

13. Tang, Q.; Huang, Y. Local and parallel finite element algorithm based on Oseen-type iteration for the stationary incompressible MHD flow. J. Sci. Comput. 2017, 70, 149–174.

14. Dong, X.; He, Y.; Wei, H.; Zhang, Y. Local and parallel finite element algorithm based on the partition of unity method for the incompressible MHD flow. Adv. Comput. Math. 2018, 44, 1295–1319. [CrossRef]

15. Tang, Q.; Huang, Y. Analysis of local and parallel algorithm for incompressible magnetohydrodynamics flows by finite element iterative method. Commun. Comput. Phys. 2019, 25, 729–751. [CrossRef]

16. Dong, X.; He, Y. A parallel finite element method for incompressible magnetohydrodynamics equations. Appl. Math. Lett. 2020, 102, 106076. [CrossRef]

17. Li, L.; Zheng, W. A robust solver for the finite element approximation of stationary incompressible MHD equations in 3D. J. Comput. Phys. 2017, 351, 254–270. [CrossRef]

18. Ma, Y.; Hu, K.; Hu, X.; Xu, J. Robust preconditioners for incompressible MHD models. J. Comput. Phys. 2016, 316, 721–746. [CrossRef]

19. Zhang, G.; Yang, M.; He, Y. Block preconditioners for energy stable schemes of magnetohydrodynamics equations. Numer. Methods Partial Differ. Eq. 2022. [CrossRef]

20. Li, L.; Zhang, D.; Zheng, W. A constrained transport divergence-free finite element method for incompressible MHD equations. J. Comput. Phys. 2021, 428, 109980. [CrossRef]

21. Zheng, G.; He, Y.; Yang, D. Analysis of coupling iterations based on the finite element method for stationary magnetohydrodynamics on a general domain. Comput. Math. Appl. 2014, 68, 770–788. [CrossRef]

22. Li, J.; He, Y. A stabilized finite element method based on two local Gauss integrations for the Stokes equations. J. Comput. Appl. Math. 2008, 214, 58–65. [CrossRef]

23. Zheng, H.; Shan, L.; Hou, Y. A quadratic equal-order stabilized method for Stokes problem based on two local Gauss integrations. Numer. Methods Partial Differ. Eq. 2010, 26, 1180–1190. [CrossRef]

24. Li, R.; Li, J.; Chen, Z. A stabilized finite element method based on two local Gauss integrations for a coupled Stokes-Darcy problem. J. Comput. Appl. Math. 2016, 292, 92–104. [CrossRef]

25. Huang, P.; He, Y.; Feng, X. Two-level stabilized finite element method for Stokes eigenvalue problem. Appl. Math. Mech. 2012, 33, 621–630. [CrossRef]

26. Li, J. Investigations on two kinds of two-level stabilized finite element methods for the stationary Navier-Stokes equations. Appl. Math. Comput. 2006, 182, 1470–1481. [CrossRef]

27. Li, J.; He, Y.; Chen, Z. A new stabilized finite element method for the transient Navier-Stokes equations. Comput. Meth. Appl. Math. Eng. 2007, 197, 22–35. [CrossRef]

28. He, Y.; Li, J. A stabilized finite element method based on local polynomial pressure projection for the stationary Navier-Stokes equations. Appl. Numer. Math. 2008, 58, 1503–1514. [CrossRef]

29. Jiang, Y.; Mei, L.; Wei, H. A stabilized finite element method for transient Navier-Stokes equations based on two local Gauss integrations. Int. J. Numer. Meth. Fluids 2012, 70, 713–723. [CrossRef]

30. Huang, P.; Zhao, J.; Feng, X. Highly efficient and local projection-based stabilized finite element method for natural convection problem. Int. J. Heat Mass Tran. 2015, 83, 357–365. [CrossRef]

31. Schötzau, D. Mixed finite element methods for stationary incompressible magnetohydrodynamics. Numer. Math. 2004, 96, 771–800. [CrossRef]

32. Heywood, J.; Rannacher, R. Finite-element approximation of the nonstationary Navier-Stokes problem. Part IV: Error analysis for second-order time discretization. SIAM J. Numer. Anal. 1990, 27, 353–384. [CrossRef]

33. Sermange, M.; Temam, R. Some mathematical questions related to the MHD equations. Comm. Pure Appl. Math. 1984, 34, 635–664. [CrossRef]

34. He, Y.; Zhang, G.; Zou, J. Fully discrete finite element approximation of the MHD flow. Comput. Methods Appl. Math. 2022, 22, 357–388. [CrossRef]

35. Nédélec, J. A new family of mixed finite elements in R3. Numer. Math. 1986, 50, 57–81. [CrossRef]

36. Girault, V.; Raviart, P. Finite Element Approximation of the Navier-Stokes Equations; Springer: Berlin/Heidelberg, Germany, 1979.

37. Monk, P. Finite Element Methods for Maxwell’s Equations; Oxford University Press: Oxford, UK, 2003.

38. Huang, P.; Feng, X.; He, Y. Two-level defect-correction Oseen iterative stabilized finite element methods for the stationary Navier-Stokes equations. Appl. Math. Model. 2013, 37, 728–741. [CrossRef]

39. FEALPy Help and Installation. Available online: https://www.weihuayi.cn/fly/fealpy.html (accessed on 24 March 2022).

40. Greif, C.; Li, D.; Schötzau, D.; Wei, X. A mixed finite element method with exactly divergence-free velocities for incompressible magnetohydrodynamics. Comput. Meth. Appl. Math. Eng. 2010, 199, 2840–2855. [CrossRef]