Phase transitions and continuously variable scaling in a chiral quenched disordered model

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Abstract

We elucidate the effects of chiral quenched disorder on the scaling properties of pure systems by considering a reduced model that is a variant of the quenched disordered cubic anisotropic $O(N)$ model near its second order phase transition. A generic short-ranged Gaussian disorder distribution is considered. For distributions not invariant under spatial inversion (hence chiral), the scaling exponents are found to depend continuously on a model parameter that describes the extent of inversion symmetry breaking. Experimental and phenomenological implications of our results are discussed.

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I. INTRODUCTION

The large-scale, macroscopic effects of disorder in statistical mechanics models and condensed matter systems have been a subject of intense study for a long time now. Very generally, depending upon time-scales, there can be two kinds of structural disorders that can exist in a system, namely, annealed and quenched disorders. In a system with annealed disorder the impurities can diffuse freely in a system until they come to a thermal equilibrium state. The time-scale of the annealed disorder dynamics is comparable or shorter than that for the physical degrees of freedom of the corresponding pure system, and hence the dynamics of the disorder becomes important. In contrast, for a system with a quenched disorder the impurities are fixed in particular configurations and do not evolve in time, and, as a result, the disorder configuration is not in thermodynamic equilibrium. Studies on the effects of random quenched disorder on pure systems, e.g., the classical $O(N)$ spin model \cite{1-3} and self-avoiding walks on random lattices \cite{4-8} are important, particularly because of the modifications in the critical behaviour brought about by the presence of such impurities. Whether or not quenched disorder changes the universal scaling properties of the pure system is given by the heuristic arguments of the well-known Harris criterion \cite{5}. For Gaussian-distributed random impurities, both spatially short-ranged \cite{9} or long-ranged \cite{3, 8} variances have been considered. For instance, perturbative renormalization group (RG) calculations on $O(N)$ symmetric models yield that for short-ranged disorder, the scaling exponents, even when they are affected by the disorder, do not generally depend upon the strength of the disorder \cite{9}. In addition, experiments on liquid crystal systems reveal important (i.e., relevant in a RG sense) effects of quenched disorder \cite{10}. These studies typically considered disordered media with variances invariant under spatial parity inversion (achiral). In contrast, the influence of chiral disordered media on the scaling properties of pure systems are yet to be explored systematically. Chirality is known to affect universal properties of systems; see, e.g., Refs. \cite{11, 12} where effects of chirality is discussed on a class of driven nonequilibrium models. Furthermore, it is now possible to fabricate chiral liquid crystals by embedding nematic liquid crystals into porous chiral film made of deposition of helical columns of MgF$_2$ on glass substrates. This is believed to be of use in practical applications, e.g., optical switching \cite{13}. In view of these examples and considering the fact that chiral disordered substance are now prepared experimentally (e.g., chiral aerogels \cite{14}),
it is pertinent and relevant to theoretically investigate the effects of chiral disordered media on the scaling properties of pure systems in a general setting.

In this article, we propose a reduced minimal model with quenched chiral disorder and analyse it to study the generic effects of such a kind of disorder on the critical scaling of pure systems. While our intention is not to model any specific real system in details, our results should have broad implication in understanding of media with orientational disorder, e.g., the effects of chiral porous media on the nematic-to-smectic A (N-A) transition of liquid crystals or on the scaling properties of smectic-A liquid crystals. Effects of usual (achiral) short-ranged disorder on N-A transitions and smectic A are both theoretically and experimentally well-studied and a variety of results are obtained including possible destruction of translational (smectic) order, but stabilisation of the smectic Bragg glass phase. There are other systems where effects of chiral quenched disorder is likely to be of significance, e.g., disordered cholesteric liquid crystals, smectic A to C (A-C) transition and superfluids in aerogels. The first one would be particularly intriguing, due to the fact that pure cholesteric liquid crystals are themselves chiral. However, the ensuing algebraic manipulations for the specific systems mentioned above in conjunction with chiral quenched disorder and its coupling with pure system variables, will be quite challenging. Thus, studies on a simpler reduced minimal model should be welcome. Such approaches are useful provided it allows one to address questions of basic principles, which in the present case is the effects of chirality or handedness of the disorder medium. To this effect, in this article we consider a variant of the usual quenched disorder $O(N)$ model ($N \geq 2$) with cubic anisotropy in the presence of a disorder distribution that breaks the invariance under spatial reflection or parity inversion and study its scaling properties near the critical point. Our model is essentially a generalisation of the random cubic anisotropic model introduced in Ref. [18]; see also Ref. [19] for related discussions on a problem of unconventional superconductors with quenched impurities. The lack of parity inversion symmetry in our model represents chirality or handedness of the disordered medium. Possible macroscopic effects of the chirality of the disordered medium, e.g., effects on universal scaling properties have not been considered before; nevertheless, formally their existence cannot be ruled out on any general (microscopic) ground, for they represent lack of reflection invariance of the impurity distribution at small scales. By systematically using a perturbative RG framework together with the replica formalism [20, 21], we find (a) for order parameter component $N > 4$,
the disorder is irre
everent (in a RG sense, see below) with the second order phase transition
in the model being described by the corresponding pure system scaling exponents, (b) for
\( N < 4 \) when the disorder distribution is reflection invariant, the system is described by a
random isotropic (RI) fixed point (FP), at which disorder is relevant (in a RG sense) and
which is identical to the scaling behaviour of the isotropic \( O(N) \) symmetric model with
short-ranged quenched disorder \[9\], in agreement with Ref. \[18\], and (c) one may vary this
random isotropic FP continuously by tuning a non-negative model parameter \( N_x \) (see be-
low), that describes the strength of the parity-breaking parts in the variance of the disorder
distribution from zero. The scaling exponents that characterise the second-order phase tran-
sition at this FP vary continuously with \( N_x \). At the technical level, as we shall see below,
\( N_x \) appears as a marginal operator in the ensuing disorder-averaged state of the system. In
addition, for a non-zero \( N_x \), near the critical point the model displays a diverging correla-
tion length \( \xi \) and a fluctuation-corrected renormalized \( T_c \) that depend on \( N_x \), suggesting the
possibility of \( N_x \)-dependent scaling near renormalized critical point in different realizations
of the model system. Our results should be directly testable in disordered classical systems
with carefully chosen distributions for the disorder at dimension \( d = 3 \) \[22\]. The remaining
part of the article is organised as follows: In Sec.II we propose and define our model. Then
we extract the critical scaling exponents from the model in Sec.III. In Sec.IV we summarise
and discuss our results.

II. THE MODEL

Let us begin by discussing possible requirements of a simple model that will be able to
capture the effects of chiral quenched disordered media. First of all, if the chirality of the
disordered medium has any discernible effect on the system, we expect two-point correlation
functions of the physical variables that describe the corresponding pure system to display
chirality (i.e., have parts that are odd parity under spatial inversion). This evidently requires
two or more variables describing the disorder in the system, since only a cross-correlation
function of two different variables can have a part that is odd under spatial inversion.
Secondly, for an impurity distribution that displays lack of symmetry under inversion, it
is expected to be described by more than one (frozen) field. Keeping these features in
mind, and considering systems that are purely achiral in the absence of any disorder, we
consider a disordered version of the well-known pure cubic anisotropic $O(N)$ model \[23\] as a reduced minimal model for chiral quenched disorder systems. The corresponding free energy functional is

$$F = \int d^d x f$$

with

$$f = \sum_{i=1}^{N} \frac{1}{2} \left[ r_0 (\phi_i^2) + r_i(x) \phi_i^2 + (\nabla \phi_i)^2 + u (\phi_i^2)^2 + v \phi_i^4 \right],$$

where $r_0 = (T - T_c)/T_c$ with $T$ and $T_c$ referring to the temperature and the mean-field critical temperature, respectively. Stochastic function $r_i(x)$ represents the coupling with the disorder, such that $T_c^L = T_c + r_i(x)$ is the local fluctuating critical temperature for $\phi_i$. Further, $u > 0$ and $v > 0$ are the bare nonlinear coupling constants in the model. In the limit of $v = 0$ and for all $r_i(x) = \tilde{r}(x)$, the microscopic rotational invariance in the order parameter space is restored and we get back the usual $O(N)$ model with quenched disorder. On the other hand, if all $r_i = 0$ and $v \neq 0$, it reduces to the well-known cubic anisotropic model \[23\]. Furthermore, with all $r_i(x) = \tilde{r}(x)$ and both $u, v \neq 0$, it is identical to the model of Ref \[18\]. We have kept the gradient term of the free energy functional spatially isotropic for simplicity. This, though admittedly an idealisation, simplifies the ensuing algebra considerably without destroying the effects of chirality of the disorder \[24\].

The cubic anisotropic terms reflect any possible breakdown of rotational invariance of the order parameter. The relative relevance (in a RG sense) between $u$ and $v$ has been addressed by perturbative RG calculations, see, e.g., \[23\]: For $N < 4$, the scaling properties are described by the stable isotropic FP $v_R = 0$, $u_R > 0$, with suffix $R$ referring to renormalized quantities, and the associated scaling exponents at the critical points are identical to that for the usual $O(N)$ model (with bare $v = 0$) \[23\]. In contrast, for $N > 4$, the cubic anisotropy is a relevant perturbation on the isotropic FP and the system cross over to the cubic anisotropic FP with $u_R > 0$, $v_R > 0$. In addition, the system can display fluctuation induced first-order transition due to the Coleman-Weinberg mechanism for a range of (bare) values of the coupling constants \[25\]. With this background and in order to include the effects of spatial reflection symmetry breaking in the disorder distribution, we assume the fluctuations in $T_c^L$, $r_i(x)$ in our model to be Gaussian distributed with variances

$$\langle r_i(x) r_j(0) \rangle_{av} = 2D_\delta \delta(x) + a_{ij} [\tilde{D} \delta(x) + \tilde{D}(x)]$$

with the matrix $a_{ij}$ having a structure of the form $a_{11} = 0 = a_{22}$ and $a_{12} = 1 = a_{21}$; function $\tilde{D}(x) = -\tilde{D}(-x)$, $D$ is a positive constant and $\tilde{D}$ is a constant that can be both positive
and negative. The symbol $\langle \ldots \rangle_{av}$ represents averages over the chosen Gaussian disorder distribution. A non-zero $\tilde{D}(x)$ thus introduces breakdown of parity in the system. Since $\tilde{D}(x)$ is an odd function of $x$, its Fourier transform must be imaginary and odd in Fourier wavevector $q$, such that the Fourier transform of $\tilde{D}(x)$ is given by $iD_x(q)$, where $D_x(q)$ is a real odd function of $q$: $D_x(q) = -D_x(-q)$. In order for $\tilde{D}(x)$ to be equally relevant (in a RG sense) with $D\delta(x)$ and $\hat{D}\delta(x)$, i.e., for them to have the same physical dimensions, $D_x(q)$ should have the same $q$-dependence (in a power counting sense) as $D$ or $\hat{D}$. Since, $D$ and $\hat{D}$ are just constants, independent of $q$, $D_x(q)$ must not depend on the magnitude of $q$. Therefore, we further define an amplitude $D_x^2 = D_x(q)D_x(q)$ and a dimensionless ratio $N_x = (D_x/\hat{D})^2 \geq 0$. Clearly, $\tilde{D}(x)$ is an odd function of $x$ that has the same dimension $1/L^d$ ($L$ is a length) as the $\delta$-function $\delta^d(x)$ in $d$-dimension. It is easy to work out an explicit representation of $\tilde{D}(x)$ in 1d: Writing $\tilde{D}(x)$ as the inverse Fourier Transform of $D_x(q)$ where $q$ is a one-dimensional Fourier wavevector, we find

$$\tilde{D}(x) = i \int_0^{\infty} dk D_x \left[ \exp(ikx) - \exp(-ikx) \right] \sim D_x/x. \tag{3}$$

Although $1/x$ has a range longer than $\delta(x)$, it has the same physical dimension and hence scales the same way as $\delta(x)$ under rescaling of space, i.e., $x$. This paves the way for different elements in (2) to compete in an RG sense. The $d$-dimensional analogue of the 1d form of $\tilde{D}(x)$ above is rather complicated. Nevertheless, it should generally be of the form $1/r^d$ on general dimensional ground in one hemisphere, with a change in sign in the other hemisphere.

We then ask: What are the scaling properties of our model? Our heuristic arguments and detailed calculations below reveal that $N_x$ may be varied to tune the emerging critical scaling behaviour continuously. Before we discuss our results in details below, a few words about the interpretation of the structure of our model in the context of possible physically realizable examples are in order: The mixing of order parameter indices and spatial dependence in (2), although not allowed in the usual spin models, is consistent with the Frank free energy for nematic liquid crystals [24], where the director field is defined in the coordinate space. A chiral disordered material has both positional and orientational disorder. For instance, in a chiral aerogel the positional disorder is related to the local density fluctuations of the aerogel pores, where as the orientational disorder reflects the randomness in the orientation of the pores. The latter one, represented formally by a quenched vector field [15], may in general have a parity breaking variance, and should couple with the fields, e.g., the nematic
director field in the N-A or the displacement fields in the A-C transitions. While detailed form of such couplings are model-specific and are not necessarily as simple as we have in (2), these, in-principle, should generate disorder distributions that breaks symmetry under inversion. A simple choice as (2), despite its limitations, suffices for our purposes here. In general, the distributions of the two types of disorders may not have any simple relation as they may occur independently.

III. CRITICAL SCALING EXPONENTS

Our model (2) without any disorder (all \( r_i = 0 \) identically) displays standard order-disorder transition through a second order critical point (in addition to a fluctuation induced first order transition) as discussed above. To get an idea about the possible macroscopic effects of disorder on the pure system properties, it is instructive to first consider the prediction of the Harris criteria [5] for the present model. In order to retain the effects of the parity breaking part of the disorder variance while constructing the Harris criterion, we formulate it as given below. While here we closely follow the derivation of the standard Harris criterion and the notations as in Ref. [3], we nevertheless rephrase the details here again for the sake of completeness. To this end, we divide the system into subsystems of linear dimension \( \xi \), where \( \xi \) is the correlation length at that temperature of the corresponding pure system. The idea is to find out if the variation of the critical temperature of these regions of size \( \sim \xi \) becomes negligible as \( T \rightarrow T_c \). Since the spins are expected to be correlated and on average aligned for up to a distance \( \sim \xi \), the transition temperature for the \( i \)th component \( \phi_i \) of the order parameter field of a region of size \( \xi \) may be defined as the average of \( T_{ci}(x) \) over that region. We define reduced temperature \( t = (T - T_c)/T_c \) and local reduced temperature \( t_i(x) = \tau_i(x)/T_c \), \( \tau_i(x) = r_0 + r_i(x) \) for the \( i \)-th component of the order parameter field, we have \( \langle t_i(x) \rangle_{av} = t \). Further, as defined in Ref. [3]

\[
 t_{iV} = \frac{1}{V} \int d^d x t_i(x)
\]

is the effective reduced temperature of a region \( V = \xi^d \). Note that we have formally allowed an effective reduced temperature \( t_{iV} \) for the \( i \)-th component \( \phi_i \) of the order parameter field. This is consistent with the fact that our model allows for order parameter component dependent effective reduced temperature in the free energy functional \( \Pi \). The variance \( \Delta_{ij} \)
of $t_{iV}$ is defined as

$$\Delta_{ij} = \langle t_{iV}t_{jV} \rangle_{av}^c$$

$$= \frac{1}{V^2} \int_V d^d x \int_V d^d y \langle t_i(x)t_j(y) \rangle_{av}^c$$

$$= \frac{1}{T_c^2} \frac{1}{V^2} \int_V d^d x \int_V d^d y g_{ij}(x-y),$$  \tag{5}

where $g_{ij}(x-y) = \langle r_i(x)r_j(y) \rangle_{av}$, as given in Eq. (2). Here, a superscript $c$ refers to the connected part of the variance. Thus

$$\Delta_{ij} = \frac{1}{T_c^2} \frac{1}{V^2} \int_V d^d x \int_V d^d y \left(2D \delta_{ij}\delta(x-y) + a_{ij}[\hat{D}\delta(x-y) + \tilde{D}(x-y)]\right).$$  \tag{6}

It is clear from (6) that the contribution from the odd parity part of the disorder variance to the variance $\Delta_{ij}$ vanishes owing to the odd parity of $\tilde{D}(x)$. In order to capture the effect of the parity breaking part of the disorder variance (i.e., non-zero $D_x$), we modify Eq. (6) to

$$\Delta_{ij} = \frac{1}{T_c^2} \frac{1}{V^2} \int_V d^d x \int_V d^d y \left(2D \delta_{ij}\delta(x-y) + a_{ij}[\hat{D}\delta(x-y) + |\tilde{D}(x-y)|]\right).$$  \tag{7}

Alternatively, one may restrict the domain of integrations above to hemisphere having a single signature of the parity breaking part in the disorder variance. While the above modification, in terms of considering the absolute value of $\tilde{D}(x-y)$, is admittedly apriori designed to capture non-zero contributions from the parity breaking part of the disorder distribution, this does not alter the power counting in the integral in (6). For $D_x = 0$ this modification still leads to the well-known Harris criterion \[3, 5\]. With our modification, therefore, $\Delta_{ij}$ is expected to have a contribution proportional to $|D_x|$ or $D_x^2$. From (7) we note that, $\Delta_{ij}$ will have a part $\sim \xi^{-d}$ coming from $D$ and $\hat{D}$ in (2). In addition, there should a part $\sim \xi^{-d} \ln \xi$ (this may be shown explicitly in 1d with $\tilde{D}(x) \sim 1/x$) coming from the parity breaking part in (2). Proceeding as in Refs. \[3, 5\], we then conclude that disorder is relevant as long as the specific heat exponent of the corresponding pure system $\alpha > 0$. This condition is same as the usual Harris criteria \[3, 5\]. Consider now the fact that the borderline of relative relevance (in a RG sense) between the short-ranged and long-ranged disorder (with a variance $|x|^a$) is determined by the condition $a = d$ \[3\], which yields a logarithmic contribution to the analogue of $\Delta_{ij}$ in Ref. \[3\]. Thus looking at the logarithmic dependence associated with the variance $\Delta_{ij}$ above, it appears that the present model is at
the borderline between (δ-correlated) short-ranged and long-ranged disorder. Furthermore, for relevant long-ranged disorder the scaling exponents depend explicitly on $a$. Since we can write $|x|^a \sim |x|^{d+\delta} \sim \delta|x|^d \ln x$ with $a = d - \delta$, $\delta \to 0$, $\delta$ should appear as a control parameter in the scaling exponents. Hence, drawing on the analogy between $\Delta_{ij}$ as above and the corresponding expression in Ref. [3], and comparing with (2), amplitude $D_x$ (equivalently $N_x$) should appear as a tuning parameter. Thus, any correction to the critical exponents due to $D_x$ must be at least $O(D_x)^2$ [and hence $O(N_x)$], since our perturbative calculations given below should be analytic in $D_x$ (or $N_x$), where $N_x$ appears as an expansion parameter.

While our arguments above are of heuristic nature and do not constitute a rigorous proof, they are indicative of non-trivial behaviour with finite $D_x$; our detailed RG calculations below confirm this qualitative physical picture.

In order to systematically investigate the properties of systems with quenched disorder it is required to average the free energy over the disorder distribution. This can be conveniently done using the well-known replica method [21]. We start with the partition function for the free energy functional (1)

$$Z = \prod_{i=1}^{N} \text{Tr} \phi_i \exp[-\beta \mathcal{F}\{\phi_i\}].$$

Then the free energy averaged over the disorder distribution can be written as

$$F \equiv -\langle \ln Z \rangle_{\text{av}} = \lim_{m \to 0} \left[ \frac{\langle Z_m \rangle_{\text{av}} - 1}{m} \right]$$

$$= \lim_{m \to 0} \left[ \frac{\prod_{i=1}^{m} \prod_{i=1}^{N} \text{Tr} \{\phi_i^\alpha\} \exp[-\beta \mathcal{F}\{\phi_i^\alpha\}] - 1}{m} \right]_{\text{av}}.$$  \hspace{1cm} (9)

Here, $\alpha = 1, 2, ..., m$ are the replica indices and $\{\phi_i^\alpha\}$ represents $m$ replications of the order parameters $\phi_i$. The corresponding $m$-replicated disorder averaged partition function is given
by (we set $k_B T = 1$, where $k_B$ is the Boltzmann constant)

$$
\langle Z^m \rangle_{av} = \prod_{\alpha=1}^m \prod_{i=1}^N \int D\phi_{i\alpha}
$$

$$
\exp \left[ \int d^d x \left\{ \frac{T_0}{2} \sum_{i=1}^N \sum_{\alpha=1}^m \phi_{i,\alpha}^2 + \frac{1}{2} \sum_{i,\alpha} (\nabla \phi_{i\alpha})^2 \right\} \right]
\times \exp \left[ \int d^d x \left\{ u \sum_{\alpha=1}^m \sum_{i=1}^N \phi_{i\alpha}^2 + v \sum_{i=1}^N \sum_{\alpha=1}^m \phi_{i\alpha}^4 \right\} \right]
\times \exp \left[ -D \int \sum_{i=1}^N \sum_{\alpha,\beta=1}^m \frac{\phi_{i\alpha}^2 \phi_{i\beta}^2}{\alpha \neq \beta} - \hat{D} \int \sum_{i,j=1}^N \sum_{\alpha,\beta=1}^m \frac{\phi_{i\alpha}^2 \phi_{j\beta}^2}{\alpha \neq \beta} \right]
\times \exp \left[ -\int \sum_{i,j=1}^N \sum_{\alpha,\beta=1}^m \frac{\phi_{i\alpha}^2 (x) \tilde{D} (x-x') \phi_{j\beta}^2 (x')}{} \right].
$$

(10)

Here $\alpha, \beta = 1, 2, ..., n$ are the replica indices. Nonlinear terms with coupling constants $D$, $\hat{D}$ and $\tilde{D}$ involve fields with different replica indices; thus, these terms lead to mixing of replica indices. These terms in Eq. (10) originate due to the averaging over the disorder distribution. Our purpose is to calculate the scaling exponents $\eta$ (anomalous dimension) and $\nu$ (correlation length exponent) near the second order phase transition, which are formally defined through the relation $\langle \phi_i(r) \phi_j(0) \rangle \sim r^{2-d-\eta} f_{ij}^\phi (r/\xi)$ where $\xi \sim |T - T_c|^{-\nu}$ is the correlation length; $f_{ij}^\phi$ is a dimensionless scaling function and averages $\langle .. \rangle$ are thermal averages to be obtained from the disorder averaged free energy functional. Anomalous dimension $\eta$ describes the spatial scaling of the correlation function at $T_c$, whereas $\nu$ describes the divergence of $\xi$ as $T$ approaches renormalized $T_c$. If we take only the terms quadratic in $\phi_{i\alpha}$ in Eq. (10), the model can be solved exactly. When the nonlinear terms are present, exact solutions are ruled out. The RG framework provides a systematic method to extract scaling behaviour in a nonlinear theory. The detailed methods of the RG framework are well-documented in literature, see, e.g., Ref. [25, 26]; see also Ref. [9] for applications of RG in a disordered system. The presence of the nonlinear terms requires expanding the different vertex or correlation functions in powers of the coupling constants. Near the critical point the perturbative corrections diverge leading to failure of na"ive perturbation expansion. The perturbative corrections are represented by using the standard Feynmann diagrams [26]. We use a minimal subtraction scheme together with the dimensional regularisation scheme to evaluate the diagrams, where only the
diverging parts of the diagrammatic corrections are obtained in inverse powers of \( \epsilon = d_c - d \). Here \( d_c \) is the upper critical dimension, at which the relevant coupling constants become dimensionless. For the present model, \( d_c = 4 \) for all the nonlinearities \[^{27} \]. Thus all of them are equally relevant. We define the renormalized coupling constants via the renormalization \( Z \)-factors

\[
\begin{align*}
Z_{u_R} &= u_R, \\
Z_{v_R} &= v_R, \\
Z_{D_R} &= D_R, \\
Z_{\hat{D}_R} &= \hat{D}_R,
\end{align*}
\]

where \( \mu \) is an arbitrary momentum scale and \( A_d = \frac{1}{2\pi^{d/2}} \). The \( Z \)-factors, defined to absorb the divergences, are used to obtain the RG equation for the correlation or vertex functions, which yield the scaling exponents at the different RG FPs. The FPs are formally given by the zeros of the RG \( \beta \)-functions

\[
\beta_a = \mu \frac{\partial}{\partial \mu} a, \quad a = u, v, D, \hat{D},
\]

(11)

The RG FPs are given by the zeros of the \( \beta \)-functions \[^{11} \]. It is useful to first consider the case with \( D_\times = 0 \). The FPs are

- **Gaussian FP**: \( u_R = 0, v_R = 0, D_R = 0, \hat{D}_R = 0, \)

- **Heisenberg FP**: \( u_R = \frac{\epsilon}{8(N+8)}, v_R = 0, D_R = 0, \hat{D}_R = 0, \)

- **Cubic anisotropic (CA) FP**: \( u_R = -\frac{\epsilon}{24N}, v_R = \frac{(N-4)\epsilon}{72N}, D_R = 0, \hat{D}_R = 0, \)

- **Random isotropic (RI) FP**: \( u_R = \frac{\epsilon}{32(N-1)}, v_R = 0, D_R = \frac{(4-N)\epsilon}{128(N-1)}, \hat{D}_R = \frac{(4-N)\epsilon}{64(N-1)}, \)

- **Random cubic (RC) FP**: \( u_R = \frac{\epsilon}{48(N-2)}), v_R = \frac{\epsilon}{144 N-2}, D_R = \frac{\epsilon}{192 N-2}, \hat{D}_R = 2D_R = \frac{\epsilon}{96 N-2}. \)
Among the coupling constants, only $D_R$ must be non-negative in order to be physically meaningful. Thus, both RI and RC FPs may exist only for $4 \geq N$. For $N > 4$ the disorder will not be relevant any more and the system will be described by the pure system FPs. Although even with $D_\times = 0$ microscopically our model here is different from (and a slight generalisation of) that in Ref. [18] in having two parameters $D$ and $\hat{D}$ denoting variances of the disorder distributions, note that at the RI FP, $\hat{D}_R = 2D_R$. This holds at the RC FP as well, although the overall system is no longer isotropic (due to a non-zero $v_R$) in the order parameter space. Thus, generically, $\hat{D}_R = 2D_R$ at the FPs that depend upon the disorder. This shows the redundancy of having two independent parameters $D$ and $\hat{D}$ in a RG sense, and demonstrates the robustness of the results of Ref. [18]. We, however, shall see below that independent $D$ and $\hat{D}_R$ are required when $D_\times \neq 0$. Unsurprisingly, our results on the FP values of the coupling constants at the RC FP above (with $\hat{D}(x) = 0$) match with those of Ref. [18]. In fact, with $\hat{D}_R = 2D_R$ and $\hat{D}(x) = 0$, the $\beta$-functions (11) exactly correspond to the recursion relations for the coupling constants in Ref. [18]. At the Gaussian, Heisenberg and CA FPs, the exponents are well-known [9]. With the RG FPs available, the different scaling exponents may now be obtained by using standard procedures: One begins by calculating the the RG $Z$-factors $Z_r$ and $Z_\phi$, defined via renormalised $r_R = Z_r r_0$ and $\phi_R = Z_\phi \phi$. The $Z$-factors $Z_r$ and $Z_\phi$ are obtained from the one- and two-loop Feynmann diagrams by using a minimal subtraction scheme together with dimensional regularisation. The corresponding Wilson flow functions $\gamma_r = \ln Z_r$ and $\gamma_\phi = \ln Z_\phi$ then immediately yield the exponents $\nu$ and $\eta$ respectively. For further details we refer the reader to Refs. [25, 26].

The scaling exponents at the RI FP are modified by the quenched disorder and are given by

$$\eta = \frac{(N + 2)\epsilon^2}{32(N - 1)^2} - \frac{(4 - N)(N + 2)\epsilon^2}{64(N - 1)^2} + \frac{(N - 4)^2\epsilon^2}{256(N - 1)^2},$$

$$\frac{1}{\nu} = 2 - \frac{3N\epsilon}{8(N - 1)}.$$  \hspace{1cm} (12)

These are identical to those at the random FP in Ref. [9]. The remaining critical exponents may be obtained from the exponent expressions (12). We perform linear stability analyses around the RI and RC FPs; we consider only $N < 4$, for which these FPs are physically meaningful [29]. Since $2D_R = \hat{D}_R$ is maintained at both RC and RI FPs, we are concerned with linear stability in the $(u_R, v_R, D_R)$ space. Further, we concern ourselves with second order phase transitions only. First, the RI FP: We find for the eigenvalues of the linear
stability matrix
\[
\Lambda = \epsilon, \frac{(4 - N)\epsilon}{4(N - 1)}, \frac{(4 - N)\epsilon}{4(N - 1)}, \frac{(4 - N)\epsilon}{6(N - 2)}. \tag{13}
\]
Thus, for \( N < 4 \) all the eigenvalues are positive, and hence stable in all the three directions. The discussions of the stability of the RC FP in Ref. [18] directly apply here as well. For the sake of completeness, we show the eigenvalues up to \( O(\epsilon) \) at the RC FP
\[
\Lambda = \epsilon, \frac{(N - 4)\epsilon}{4(N - 2)}, \frac{(4 - N)\epsilon}{6(N - 2)}. \tag{14}
\]
Thus, two of them diverge at \( N = 2 \). It is thus generally expected that flow lines from near the RC FP flow to the (stable) RI FP. See Ref. [18] for detailed discussions on this.

Having known the FP values of the coupling constants for \( D_x = 0 \) or \( N_x = 0 \), we now obtain the corresponding values when \( N_x > 0 \). For simplicity of the ensuing algebraic manipulations, we find corrections to the \((N_x = 0)\) FP values of the coupling constants up to \( O(N_x) \), assuming a small \( N_x \). Thus the finite \( N_x \) FP values, as we write down below, are not going to be quantitatively accurate for \( N_x \sim O(1) \). For \( D_x = 0 \), the zeros of the \( \beta \)-functions \( \beta_D \) and \( \beta_{D_R} \), as given in Eqs. (11), yield \( \hat{D}_R = 2D_R \) generically at the RG FPs. This then, in turn, leads to \( \hat{D}_R = 0 = D_R \) as a FP solution, regardless of its linear stability properties. Thus, pure system properties may be restored in the large scale provided \( \hat{D}_R = 0 = D_R \) is a stable FP. However, when \( D_x \neq 0 \), two observations may be made immediately: (i) \( \hat{D}_R \neq 2D_R \) and (ii) \( \hat{D}_R = 0 \) and \( D_R = 0 \) are no longer solutions of \( \beta_D = 0 \) and \( \beta_{D_R} = 0 \). Thus, pure system behaviour is not expected to be observed. We obtain modifications to both RI and RC FPs up to \( O(N_x) \) separately. First the modified RI FP: After some straightforward algebra, we find
\[
u^*_R = \frac{\epsilon}{32(N - 1)} \left[ 1 + \frac{(4 - N)N_x}{2(4 + N)} \right], \tag{15}
\]
\[
v^*_R = \frac{\epsilon(4 - N)N_x}{16(4 + N)(N - 1)}, \tag{16}
\]
\[
D_R = \frac{\epsilon(4 - N)}{128(N - 1)} \left[ 1 + \frac{(16 - N)N_x}{2(4 + N)} \right], \tag{17}
\]
\[
\hat{D}_R = \frac{\epsilon(4 - N)}{64(N - 1)} \left[ 1 + \frac{(N + 8)N_x}{2(4 + N)} \right]. \tag{18}
\]
Here, \( N_x = \frac{D_x^2}{D_x^2} \geq 0 \) is a dimensionless parameter. Notice that: (i) \( \nu_R \) picks up a small non-zero value \( O(N_x) \) and (ii) \( \hat{D}_R \neq 2D_R \). Thus the modified RI is no longer isotropic; however, the departure from isotropicity is small, \( O(N_x) \). It is clear from Eqs. (15-18) that
for $1 < N < 4$ the FP values are physically meaningful, e.g., $D_R$ is positive definite, for any values of (small) $N_x$; we, therefore, consider only the range $1 < N < 4$ below. Corrections to $O(N_x)$ for the RC FP may also be obtained, which we do not show here explicitly; see Ref. [18] for the details in this context. For small $N_x$ and $2 < N < 4$ this modified RC FP remains unstable. The general picture of the second order phase transition remains unchanged at such finite but small $N_x$. When $N_x \sim O(1)$, the FPs are expected to be substantially modified, which we do not discuss here. We are unable to comment about the physical picture for $N > 4$ on the basis of the FPs given by (15-18), since $D_R < 0$ for $N > 4$ from Eq. (17).

The critical exponents $\eta$ and $\nu$, which may be evaluated following the procedure outlined above, at the modified RI FP are:

$$\eta = \frac{(N + 2)\epsilon^2}{32(N - 1)^2} - \frac{(4 - N)(N + 2)\epsilon^2}{64(N - 1)^2} + \frac{(N - 4)^2\epsilon^2}{256(N - 1)^2} - \frac{(4 - N)^2\epsilon^2}{256(N - 1)^2} N_x, \quad (19)$$

$$\frac{1}{\nu} = 2 - \frac{3N\epsilon}{8(N - 1)} - \frac{3(4 - N)\epsilon}{16(N - 1)^2} N_x, \quad (20)$$

As usual, the critical exponent expressions (19) and (20) yield all other critical exponents. The scaling exponents given in (19) and (20) constitute the principal results of this work. From Eq. (19), for $N_x = 0$, $\eta$ is always positive (we consider $1 < N < 4$). For small $N_x \neq 0$, $\eta$ remains positive, but its value decreases. Thus the correlation function for $\phi_i$ decays more rapidly with spatial separation for $N_x = 0$ than for $N_x \neq 0$. It apparently suggests that $\eta$ can be brought to zero for sufficiently large $N_x$; however, since our expression (19) are valid for small $N_x$ only, we cannot say anything conclusively when $N_x$ is large $\sim O(1)$. Similarly, from Eq. (20), $\nu^{-1}$ remains positive for small value of $N_x$ and it has a larger value when $N_x > 0$ than for $N_x = 0$. Since $\nu$ is an explicit function of $N_x$, the measured diverging correlation length $\xi \sim |T - T_c|^{-\nu}$ will depend on the value of $N_x$; $\xi$ for $N_x > 0$ is larger than that for $N_x = 0$. Furthermore, notice that $\nu$ picks up an $N_x$-dependent part at $O(\epsilon)$, where as the same for $\eta$ appears at $O(\epsilon^2)$. Thus, experimental detection of any $N_x$ dependence will be revealed much more clearly in measurements of the correlation length near renormalized $T_c$.

What could be an upper bound of $N_x$? By demanding positivity of the eigenvalues $\Lambda_e$ of the disorder variance matrix, we can enforce a bound on $D_x$ (or, equivalently, an upper
limit on $N_x$). Eigenvalues $\Lambda_e$ in general depend both on $N$, the number of order parameter components, and $D, \hat{D}, D_x$. We take $N = 2$ as a specific example. The variance matrix $M$ of the disorder distribution is given by

$$M = \begin{pmatrix} 2D & \hat{D} + iD_x \\ \hat{D} - iD_x & 2D \end{pmatrix}. \tag{21}$$

The corresponding eigenvalues are $\Lambda_e = 2D \pm \sqrt{(\hat{D}^2 + D_x^2)} \geq 0$. This gives a bound on the off-diagonal elements of the variance matrix $M$. This is consistent with the limit $D_x = 0$, where $\hat{D}_R = 2D_R$ at the FP. Similar exercises may be under taken for higher values of $N$, which we do not discuss here. Finally, our claim of continuously varying universal properties rests on the possible continuous variation of $D_x$, and hence of $N_x$. We have shown, in our low order (two-loop) renormalized perturbation theory, that there are no fluctuation corrections to $D_x$. We believe this holds to any order in the perturbative expansion. To see this notice that any non-zero fluctuation corrections to $D_x(q)$ must be an odd function of the its wavevector argument. In order to have such a non-zero correction one must have an odd number of $D_x(q)$ vertex in the diagram. Since all internal wavevectors are integrated over, such a contribution will vanish in the limit of vanishing external wavevector. Thus any putative diagrammatic corrections to $D_x(q)$ vanishes and hence $N_x$ should appear as a dimensionless marginal operator to any order in perturbation. This is technically similar to a marginal operator that exists in the models of Refs. [11, 12]. Numerical verification of our results on equivalent lattice-gas models requires generation of $N$ stochastic functions having variances as given by (2). This may be conveniently done by following the method outlined in Ref. [12] (see also Ref. [30] for a general discussion on related issues).

IV. SUMMARY AND OUTLOOK

In summary, thus, we have proposed and studied a variant of the classical cubic anisotropic $O(N)$ model with short ranged quenched disorder having a parity breaking part, with a strength parametrised by $D_x$ (equivalently by $N_x$), in its variance, as a reduced minimal model to study the effects of quenched chiral disorder on the scaling properties of pure systems. For our work, we use a generalisation of the model used in Ref. [18]. The truly novel result from our reduced model is that the explicit and continuous dependence of
the scaling exponents on $N_\times \neq 0$, with our results reducing to those of Ref. 18 in the limit $N_\times = 0$. These $N_\times$-dependent results on the scaling exponents are qualitatively new contributions made in the present work. We have worked out the dependences of the scaling exponents up to $O(N_\times)$ for simplicity using two-loop $\epsilon$-expansions. If the disorder is long-ranged 3, i.e., if the magnitudes of $D, \tilde{D}, D_\times(\mathbf{k})$ have $\mathbf{k}$-dependent parts $\sim k^{-y}, y > 0$, in addition to constant parts, preliminary calculations analogous to that here reveal that a parameter analogue to $N_\times$ will appear in the scaling exponents. Thus, the effects of a parity breaking part appears to be quite robust. Assuming the disorder in our model results from a variety of microscopic sources, the tuning parameter $N_\times$ may be interpreted as a measure of the relative concentration of the microscopic impurities that causes a non-zero parity breaking variance in the resulting impurity distribution. Hence, our results are illustrations of dependences of the scaling exponents on the relative concentration. Our reduced model and the results that follow are not directly applicable to any specific system we discussed (e.g., the N-A transition), due to the underlying simplicity and idealisation of the model. Nevertheless, the broad qualitative picture that chirality of impurity distributions may be relevant in determining the universal properties of disordered systems is sufficiently general and expected to be observed in more realistic and specific experimentally accessible systems. From a general point of view, our results open up the possibility of a new paradigm in the scaling properties of quenched chiral disorder systems. Notice that our results do not contradict the recent results that critical exponents in the disordered Ising model are independent of impurity concentration along the transition line between paramagnetic and ferromagnetic phases 31, since $D_\times = 0$ (hence $N_\times = 0$) identically for the Ising model with $N = 1$. Many experiments tend to suggest smearing of phase transition in a variety of disorder systems 32. Our results on $N_\times$-dependent correlation length as $T \to T_c$ and, similarly, $N_\times$-dependent renormalized $T_c$ are reminiscent of a system with a continuous spectrum of critical points and relevant diverging length scales (near $T_c$) and critical points, and thus loosely resemble a smeared transition (although there are no formal connections). Measurements on model systems (for $N \geq 2$) with different realizations of the disorder having different $N_\times$, corresponding to a given impurity distribution, naturally leading to a broadening of the measured values of the scaling exponents. Although at present no experiments on chiral disordered systems are available to our knowledge, we expect such experiments (e.g., N-A or A-C transitions) should be performed in the near future by using,
say, chiral aerogels [14]. Numerical simulations of our reduced model with appropriately chosen disorder distributions should be useful. From the point of view of the notion of universality in both equilibrium and nonequilibrium systems, our results are yet another demonstration of the important role that breakdown of spatial parity invariance may play in determining the universal properties. Effects of parity breakdown in pure nonequilibrium systems have already been elucidated in several examples in Refs. [11, 12], where breakdown of microscopic parity invariance is introduced by means of a reflection invariance breaking stochastic noise correlator. In each of those cases, universal properties depend explicitly on a model parameter analogous to \( N \) here. Noting that the disorder variance in the present study [2] is symmetric under the exchange of \((i, j)\), our model may be generalised further by allowing for terms in the disorder variance that are antisymmetric under exchange of \(i, j\), which may separately have even and odd parity parts. We expect the coefficient of the new odd parity part should also appear as a tuning parameter in the critical scaling exponents. Calculations analogous to the above may be performed for detailed study. Lastly, it will be interesting to see how the predictions of Ref. [33] for the disordered classical XY model in 2\(d\) are modified for the cubic anisotropic \(O(2)\) model at 2\(d\) with an impurity distribution as here. In addition, effects of quenched chiral disorder of the type discussed here on systems similar to Ref. [34] should be investigated. We hope our studies here will stimulate further theoretical and experimental studies.

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The upper critical dimension $d_c$ of a field theory is the dimension at which the relevant coupling constant becomes dimensionless; see, e.g., [25, 26] for more detailed technical discussions about $d_c$ for any model. In the present model, all the coupling constants $u, v, D, \hat{D}$ of the effective disordered averaged theory, as given by (10), are dimensionless at dimension $d = 4$. Hence, $d_c = 4$ for all the nonlinearities.

The RC FP diverges at $N = 2$. Our calculation fails to suggest any physical interpretation or consequence for it.

The Gaussian, Heisenberg and CA FPs are all unstable for $N < 4$.

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