Most binary matrices have no small defining set

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Abstract

Consider a matrix $M$ chosen uniformly at random from a class of $m \times n$ matrices of zeros and ones with prescribed row and column sums. A partially filled matrix $D$ is a defining set for $M$ if $M$ is the unique member of its class that contains the entries in $D$. The size of a defining set is the number of filled entries. A critical set is a defining set for which the removal of any entry stops it being a defining set.

For some small fixed $\varepsilon > 0$, we assume that $n \leq m = o(n^{1+\varepsilon})$, and that $\lambda \leq 1/2$, where $\lambda$ is the proportion of entries of $M$ that equal 1. We also assume that the row sums of $M$ do not vary by more than $O(n^{1/2+\varepsilon})$, and that the column sums do not vary by more than $O(m^{1/2+\varepsilon})$. Under these assumptions we show that $M$ almost surely has no defining set of size less than $\lambda mn - O(m^{7/4+\varepsilon})$. It follows that $M$ almost surely has no critical set of size more than $(1-\lambda) mn + O(m^{7/4+\varepsilon})$. Our results generalise a theorem of Cavenagh and Ramadurai, who examined the case when $\lambda = 1/2$ and $n = m = 2^k$ for an integer $k$.

1 Introduction

Let $m$ and $n$ be integers, and let $s = (s_1, s_2, \ldots, s_m)$ and $t = (t_1, t_2, \ldots, t_n)$ be vectors of non-negative integers. Then $A(s, t)$ is defined to be the set of all $m \times n$ binary matrices with $s_i$ ones in row $i$ and $t_j$ ones in column $j$, where $1 \leq i \leq m$ and $1 \leq j \leq n$. We say almost all matrices in $A(s, t)$ have a property if the probability that a matrix chosen uniformly at random from $A(s, t)$ has the property tends to 1 as $m, n \rightarrow \infty$.

A partial binary matrix is a matrix $M$ with entries 0, 1 or $\star$, where we call a cell empty if its entry is $\star$. Let $A'(s, t)$ denote the set of all $m \times n$ partial binary matrices with at most $s_i$ ones in row $i$ and $n - s_i$ zeros in row $i$, and at most $t_j$ ones and $m - t_j$ zeros in column $j$. Given $M \in A(s, t)$ and $D = [D_{ij}] \in A'(s, t)$ we write $D \subseteq M$ if $D_{ij} \in \{M_{ij}, \star\}$, for all $1 \leq i \leq m$ and $1 \leq j \leq n$.

Suppose $D \in A'(s, t)$ and $M \in A(s, t)$. Then we say $D$ is a defining set for $M$ if $M$ is the unique member of $A(s, t)$ such that $D \subseteq M$. Furthermore, for $D \subseteq M$ we define the
partial matrix $M \setminus D \in \mathcal{A}'(s, t)$ by
\[(M \setminus D)_{ij} = \begin{cases} M_{ij} & \text{if } D_{ij} = * \\ * & \text{otherwise.} \end{cases} \]

The size of a partial binary matrix $D$, denoted $|D|$, is the number of nonempty cells. We define
\[
sds(M) = \min\{|D| : D \text{ is a defining set of } M\}, \\
\maxsds(s, t) = \max\{sds(M) : M \in \mathcal{A}(s, t)\}.
\]

Also, define $\maxsds(m, n)$ to be the maximum of $sds$ amongst all $m \times n$ binary matrices.

For integers $k$ and $n$, let $\Lambda^k_n$ be the set of all $n \times n$ binary matrices with constant row and column sum $k$. In [3], Cavenagh and Ramadurai construct a matrix in $\Lambda^k_n$ with no defining set of size less than $2k^2 - \mathcal{O}(k^{7/4})$ whenever $k$ is a power of 2. In [3] we prove a similar result for almost all matrices $M \in \mathcal{A}(s, t)$, for every pair of integers $m$ and $n$, provided $M$ is not too far from the average values and column does not stray too far from the average values $s$ and $t$. Our result is this:

**Theorem 1.** Fix a sufficiently small $\varepsilon > 0$. For integers $m, n \to \infty$ with $n \leq m = o(n^{1+\varepsilon})$, let $s = (s_1, s_2, \ldots, s_m)$ and $t = (t_1, t_2, \ldots, t_n)$ be vectors of positive integers such that $\sum_{i=1}^m s_i = \sum_{j=1}^n t_j$. Define $s = m^{-1} \sum_{i=1}^m s_i$ and $t = n^{-1} \sum_{j=1}^n t_j$ and suppose that $|s_i - s| = \mathcal{O}(n^{1/2+\varepsilon})$ uniformly for $1 \leq i \leq m$, and $|t_j - t| = \mathcal{O}(m^{1/2+\varepsilon})$ uniformly for $1 \leq j \leq n$. Suppose $\lambda = s/n = t/m \leq 1/2$ and that $\lambda$ is bounded away from zero. Also suppose that
\[
\frac{(1 - 2\lambda)^2}{4\lambda(1 - \lambda)} \left(1 + \frac{5m}{6n} + \frac{5n}{6m}\right) \leq \frac{\log m}{3}.
\]

Then almost all matrices in $\mathcal{A}(s, t)$ have no defining set of size less than $\lambda mn - \mathcal{O}(m^{7/4+\varepsilon})$.

This result significantly generalises the theorem of Cavenagh and Ramadurai mentioned above, albeit with a slightly worse error term. Taking $m = n = 2k$ and $s_i = t_j = k$ for all $i, j$, Theorem 1 implies the following corollary.

**Corollary 2.** For any integer $k$, almost all matrices $M \in \Lambda^k_{2k}$ have no defining set of size smaller than $2k^2 - \mathcal{O}(k^{7/4+\varepsilon})$.

Since every matrix in $\mathcal{A}(s, t)$ has a defining set of size $\lambda mn$, another way to state the conclusion of Theorem 1 is that $sds(M) = \lambda mn - \mathcal{O}(m^{7/4+\varepsilon})$ for almost all $M \in \mathcal{A}(s, t)$. It also follows that $\maxsds(s, t) = \lambda mn - \mathcal{O}(m^{7/4+\varepsilon})$. These results are limited to the case when $m, n, s, t$ satisfy the hypotheses of our theorem. Since every $m \times n$ binary matrix has a defining set of size at most $mn/2$, we can also say:

**Corollary 3.** For $n \leq m = o(n^{1+\varepsilon})$, we have $\maxsds(n, m) = mn/2 - \mathcal{O}(m^{7/4+\varepsilon})$.

Cavenagh and Wright [4] studied critical sets, that is, defining sets which are minimal in the sense that the removal of any element destroys the property of being a defining set. They showed that the complement of a critical set is itself a defining set. Therefore Theorem 1 implies that almost all binary matrices contain no large critical set. More specifically:

**Corollary 4.** Under the hypotheses of Theorem 1, almost all elements of $\mathcal{A}(s, t)$ have no critical set of size more than $(1 - \lambda)mn + \mathcal{O}(m^{7/4+\varepsilon})$. 
We refer to the parameter $\lambda$ in Theorem 1 as the density of $M \in \mathcal{A}(s, t)$. It is the proportion of entries in $M$ which equal one. Throughout this paper we require $\lambda$ to be bounded away from zero. Our approach relies on an asymptotic formula from [1] for the number of bipartite graphs with a given degree sequence. Similar enumeration results do exist for the very sparse range [5], but the intermediate range is not yet covered. This is why we decided to not consider the case when $\lambda \rightarrow 0$. Furthermore, we will assume that $\lambda \leq 1/2$. Without that assumption, our problem has symmetry between zeros and ones in the sense that we may switch zeros and ones without changing the size of the smallest defining set. We can easily form a defining set for any matrix $M$ by taking either all the ones or all the zeros in $M$. The smaller of these two options turns out to provide a good upper bound on the size of the smallest defining set. The justification for legislating that $\lambda \leq 1/2$ is that it simplifies the exposition if we know that the number of ones does not exceed the number of zeros. We then get good estimates by observing that the minimum size of a defining set in $M \in \mathcal{A}(s, t)$ cannot exceed $\lambda mn$, and $\maxsd(s, t) \leq \lambda mn$. We note that the case where $\lambda > 1/2$ would be easily handled by replacing $\lambda$ with $1 - \lambda$ in the appropriate places, but our statements are simpler if we do not need to say this each time. For similar reasons, we assume throughout that $n \leq m$.

2 Preliminary results

In this section we provide some preliminary results used in the proof of Theorem 1. We utilise the following elegant characterisation of defining sets from [3]. It uses the idea of a South-East walk tracing through a matrix using steps to the right or downward. Such a walk separates the entries of the matrix into two classes: those above (and to the right of) the walk and those below (and to the left of) the walk. In particular, no entry lies on the walk itself. We say a partial matrix $M \in \mathcal{A}'(s, t)$ is in good form if whenever $M_{i,j} = 1$ and $M_{i',j'} = 0$ then $j < j'$ and whenever $M_{i,j} = 0$ and $M_{i',j} = 1$ then $i < i'$. In other words, a partial matrix $M \in \mathcal{A}'(s, t)$ is in good form if a South-East walk in $M$ exists with only ones (or empty cells) below the walk and only zeros (or empty cells) above it.

**Theorem 5.** Let $M \in \mathcal{A}(s, t)$ and let $D \in \mathcal{A}'(s, t)$. Then $D$ is a defining set of $M$ if and only if $D \subseteq M$ and the rows and columns of the partial matrix $M \setminus D$ can be permuted to be in good form.

The family of matrices constructed in [3] have the special property that within any rectangular subarray the difference between the number of ones and zeros is small. This property, combined with Theorem 5 guarantees no small defining set. In our more general setting, we are interested in the property that the difference between the number of ones and the expected number of ones in any subarray is small. Here, and henceforward, when we refer to the expected number of ones occupying a particular set of cells, the underlying distribution involves a matrix being chosen uniformly at random from all binary matrices with given dimensions and density.

Let $R$ and $C$ be any subsets of the rows and columns, respectively, of $M \in \mathcal{A}(s, t)$. Let $\lambda$ be the density of $M$. Then the subarray $M[R, C]$ is the $|R| \times |C|$ array of $M$ induced by $R$ and $C$ and $\delta(M[R, C])$ denotes the number of ones in $M[R, C]$ minus $\lambda|R||C|$, which is the expected number of ones in $M[R, C]$.

**Lemma 6.** Fix $\varepsilon' > 0$ and let $\Delta$ be a function of integers $m$ and $n$. Let $M \in \mathcal{A}(s, t)$ have a density $\lambda$ satisfying $\varepsilon' \leq \lambda \leq 1/2$. Let $D$ be a defining set for $M$. If for every $R$ and $C$, subsets of the rows and columns of $M$, respectively, we have

$$|\delta(M[R, C])| \leq \Delta(m, n),$$

(1)
then \(|D| \geq \lambda mn - \mathcal{O}(m^{7/4} + m^{1/4} \Delta(m, n))\).

**Proof.** Let \(M \in \mathcal{A}(s, t)\) be such that \(|\delta(M[R, C])| \leq \Delta(m, n)\) for any subsets \(R\) and \(C\) of the rows and columns, respectively. Let \(D \subseteq M\) be a minimal defining set of \(M\). We show that the size of \(D\) cannot be less than \(\lambda mn - \mathcal{O}(m^{7/4} + m^{1/4} \Delta(m, n))\). By Theorem 5 we can assume that the rows and columns of \(M\) have been permuted so that \(M \setminus D\) is in good form. That is, we can draw a South-East walk \(W\) in the matrix \(M \setminus D\) so that all non-empty cells above \(W\) are zeros and all non-empty cells below \(W\) are ones. Since \(D\) is minimal, \(M \setminus D\) must contain every one that occurs in \(M\) below \(W\) and every zero that occurs in \(M\) above \(W\).

Let \(\alpha_0\) and \(\alpha_1\) denote the number of zeros and ones (respectively) in \(M\) above \(W\), and let \(\beta_0\) and \(\beta_1\) denote the number of zeros and ones (respectively) in \(M\) below \(W\). Hence, we have

\[
\alpha_1 + \beta_1 = \lambda mn, \quad \text{and} \quad |D| = \alpha_1 + \beta_0.
\]

We now find an upper bound on \(|\beta_1 - \lambda(\beta_1 + \beta_0)|\), which is the number of ones minus the expected number of ones below \(W\). For \(1 \leq i \leq m\), define \(f(i)\) to be the number of cells in row \(i\) to the left of \(W\) and let \(f(0) = 0\). By definition, the sequence \(f(0), \ldots, f(m)\) is weakly increasing. Let \(h = \lceil m^{3/4}\rceil\). For \(1 \leq i \leq \lceil m^{1/4}\rceil\), define a block \(B_i = M[R_i, C_i]\) where \(R_i = \{ih, \ldots, m\}\) and \(C_i = \{f((i-1)h) + 1, \ldots, f(ih)\}\). Note that each block \(B_i\) lies entirely below \(W\) and is disjoint from \(B_j\) for \(j \neq i\). Moreover, in any column there are at most \(h\) cells that are below \(W\) but are not in any of the \(B_i\). For these cells, the difference between the number of ones and the expected number of ones cannot exceed \(nh\), the total number of cells involved. For each block \(B_i\), we then employ the bound (1) to give

\[
|\beta_1 - \lambda(\beta_1 + \beta_0)| = nh + \lceil m^{1/4}\rceil \Delta(m, n) = \mathcal{O}(nm^{3/4}) + \mathcal{O}(m^{1/4} \Delta(m, n)).
\]

Now \(n \leq m\) and \(\lambda \leq 1/2\) with \(1/\lambda = \mathcal{O}(1)\), so

\[
\beta_0 = \frac{1 - \lambda}{\lambda} \beta_1 - \mathcal{O}(m^{7/4} + m^{1/4} \Delta(m, n)) \geq \beta_1 - \mathcal{O}(m^{7/4} + m^{1/4} \Delta(m, n)).
\]

It follows that

\[
|D| = \beta_0 + \alpha_1 \geq \beta_1 + \alpha_1 - \mathcal{O}(m^{7/4} + m^{1/4} \Delta(m, n)) = \lambda mn - \mathcal{O}(m^{7/4} + m^{1/4} \Delta(m, n))
\]

as claimed. \(\square\)

Let \(\mathcal{N}(s, t)\) be the number of labelled bipartite graphs with \(m\) vertices on one side of the bipartition with degrees given by \(s\), and \(n\) vertices on the other side with degrees given by \(t\). We utilise the following well-known results called the Chernoff bounds [6].

**Theorem 7.** Let \(m, n, s, t, \lambda, A, \text{ and } \varepsilon\) be defined as in Theorem [7]. Then we have

\[
\mathcal{N}(s, t) = \left(\frac{mn}{\lambda mn}\right)^{-1} \prod_{i=1}^{m} \left(\frac{n}{s_i}\right)^{n} \prod_{j=1}^{n} \left(\frac{m}{t_j}\right)^{m} \exp\left(-\mathcal{O}(\lambda mn^2\varepsilon)\right).
\]

Lastly, we need the following well-known results called the Chernoff bounds [6].

**Theorem 8.** Let \(X_1, \ldots, X_n\) be independent Bernoulli random variables where \(X_i = 1\) with probability \(p_i\) and \(X_i = 0\) with probability \(1 - p_i\). Let \(X = \sum_{i=1}^{n} X_i\) and \(\mu = \mathbb{E}(X) = \sum_{i=1}^{n} p_i\).

Then

(i) \(\mathbb{P}(X \geq (1 + \gamma)\mu) \leq \exp\left(-\frac{\gamma^2}{2 + \gamma} \mu\right)\) for all \(\gamma > 0\),

(ii) \(\mathbb{P}(|X - \mu| \geq \gamma \mu) \leq 2 \exp\left(-\mu\gamma^2 / 3\right)\) for all \(0 < \gamma < 1\).
3 Proof of the main result

An element of $A(s, t)$ is the bi-adjacency matrix of a bipartite graph with $m$ vertices on one side of the bipartition with degrees given by $s$, and $n$ vertices on the other side with degrees given by $t$. We define the density of a bipartite graph to be the density of its bi-adjacency matrix.

Let $A$ and $B$ be subsets of the vertices of $G$, each from a different side and denote the number of edges between $A$ and $B$ by $e(A, B)$. The property (1) is equivalent to the difference between the number of edges and the expected number of edges between $A$ and $B$ being at most $\Delta(m, n)$. Therefore, by Lemma 6, the following theorem implies our main result, Theorem 4.

**Theorem 9.** Let $G(s, t)$ be chosen uniformly at random from the bipartite graphs with one side of the bipartition of size $m$ with degrees from $s$ and the other side of size $n$ with degrees from $t$. Let $\lambda$ be the density of $G(s, t)$ and suppose that $m$, $n$, $s$, $t$ and $\lambda$ satisfy the hypotheses of Theorem 4. Then there is some constant $c > 0$ such that, with probability $1 - o(1)$,

$$|e(A, B) - \lambda|A||B|| \leq c(mn^{1/2+\varepsilon} + nm^{1/2+\varepsilon}),$$

for any two subsets $A$ and $B$ of the vertices, each from a different side.

Let $G(n, m, \lambda)$ be a random bipartite graph with sides of size $m$ and $n$, in which each of the $mn$ possible edges occurs independently with probability $\lambda$. Note that with respect to this graph, the expectation of $e(A, B)$ is $\lambda|A||B|$, for any two subsets $A$ and $B$ of the vertices of $G(n, m, \lambda)$, each from a different side.

**Proof of Theorem 9.** Fix a positive constant $c$. We say a bipartite graph has property $\mathcal{P}$, if there exist subsets $A$ and $B$, each from a different side, such that

$$|e(A, B) - \lambda|A||B|| > c(mn^{1/2+\varepsilon} + nm^{1/2+\varepsilon}).$$

Let $P_{s,t}(\mathcal{P})$ denote the probability that $G(s, t)$ has property $\mathcal{P}$ and let $P_\lambda(\mathcal{P})$ be the probability that $G(n, m, \lambda)$ has property $\mathcal{P}$. We define $E_{s,t}$ to be the event that $G(m, n, \lambda)$ has degree sequence $(s, t)$. Then

$$P_{s,t}(\mathcal{P}) = P_\lambda(\mathcal{P} | E_{s,t}) \leq \frac{P_\lambda(\mathcal{P})}{P_\lambda(E_{s,t})}.$$

We claim that $P_{s,t}(\mathcal{P})$ goes to zero as $m, n \to \infty$. Firstly we find a lower bound on $P_\lambda(E_{s,t})$. To simplify our calculations we let $\lambda' = (1 - \lambda)$. Applying Stirling’s formula to the binomials given in Theorem 7, we have the following approximations, provided $m, n, s$ and $t$ satisfy the hypotheses of Theorem 4:

$$\binom{mn}{\lambda mn} = \exp \left( - (\lambda \log \lambda + \lambda' \log \lambda') mn - O(\log(mn)) \right),$$

$$\prod_{i=1}^{m} \binom{n}{s_i} = \exp \left( mn \log n - \sum_{i=1}^{m} s_i \log s_i - \sum_{i=1}^{m} (n - s_i) \log(n - s_i) - O(m \log n) \right).$$

By assumption, for $1 \leq i \leq m$ we have $s_i = \lambda n + s'_i$ where $s'_i = O(n^{1/2+\varepsilon})$ and $\sum_{i=1}^{m} s'_i = 0$. Hence we have

$$\sum_{i=1}^{m} s_i \log s_i = \lambda mn \log(\lambda n) + \sum_{i=1}^{m} (\lambda n + s'_i) \log \left( 1 + \frac{s'_i}{\lambda n} \right) = \lambda mn \log(\lambda n) + O(mn^{2\varepsilon}).$$
Similarly,
\[ \sum_{i=1}^{m} (n - s_i) \log(n - s_i) = \lambda' mn \log(\lambda'n) + \mathcal{O}(mn^{2\epsilon}). \]

Hence, we have
\[ \prod_{i=1}^{m} \left( \frac{n}{s_i} \right) = \exp \left( -(\lambda \log \lambda + \lambda' \log \lambda') mn - \mathcal{O}(mn^{2\epsilon}) \right). \]

A similar argument yields
\[ \prod_{j=1}^{n} \left( \frac{m}{t_j} \right) = \exp \left( -(\lambda \log \lambda + \lambda' \log \lambda') mn - \mathcal{O}(mn^{2\epsilon}) \right). \]

Combining all of the above approximations with Theorem 7, we find that
\[ \mathcal{N}(s, t) = \exp \left( -(\lambda \log \lambda + \lambda' \log \lambda') mn - \mathcal{O}(mn^{2\epsilon} + nm^{2\epsilon}) \right). \]

There are \( \binom{mn}{\lambda mn} \) labelled bipartite graphs with sides of size \( m \) and \( n \) and density \( \lambda \), so
\[ P_{\lambda}(E_{s,t}) = \frac{\mathcal{N}(s, t)}{\binom{mn}{\lambda mn}} = \exp \left( -\mathcal{O}(mn^{2\epsilon} + nm^{2\epsilon}) \right). \tag{2} \]

We now need to find an upper bound on \( P_{\lambda}(\mathcal{P}) \). Let \( N = \lfloor (c/\lambda)(mn^{1/2+\epsilon} + nm^{1/2+\epsilon}) \rfloor \). Then we have
\[ P_{\lambda}(\mathcal{P}) \leq P \left( \exists A, B \text{ such that } |e(A, B) - \lambda |A||B| > \lambda N \right) \]
\[ \leq \sum_{A, B} P \left( |e(A, B) - \lambda |A||B| > \lambda N \right) \]
\[ = \sum_{|A||B|>N} P \left( |e(A, B) - \lambda |A||B| > \lambda N \right) + \sum_{|A||B|\leq N} P \left( |e(A, B) - \lambda |A||B| > \lambda N \right), \tag{3} \]

where the first inequality follows from the union bound and this sum is over all pairs of subsets \( A \) and \( B \) of the vertices of \( G(n, m, \lambda) \), each from a different side. By Theorem 8 (ii),
\[ \sum_{|A||B|>N} P \left( |e(A, B) - \lambda |A||B| > \lambda N \right) = \sum_{k=N+1}^{mn} \sum_{|A||B|=k} P \left( |e(A, B) - \lambda |A||B| > \lambda N \right) \]
\[ \leq \sum_{k=N+1}^{mn} \sum_{|A||B|=k} 2 \exp \left( -\lambda N^2 / (3k) \right) \]
\[ \leq \exp \left( -\lambda N^2 / (3mn) \right) \sum_{k=N+1}^{mn} \sum_{|A||B|=k} 2 \]
\[ \leq \exp \left( -\lambda N^2 / (3mn) \right) 2^{m+n+1}, \tag{4} \]

where the last inequality is due to the fact that the number of pairs \( A, B \) is bounded above
by $2^{m+n}$. We now move on to those subsets satisfying $|A||B| \leq N$. By Theorem 8 (i),

$$
\sum_{|A||B|\leq N} P \left( \left| e(A, B) - \lambda |A||B| \right| > \lambda N \right) = \sum_{k=1}^{N} \sum_{|A||B|=k} P \left( \left| e(A, B) - \lambda |A||B| \right| > \lambda N \right) \\
\leq \sum_{k=1}^{N} \sum_{|A||B|=k} \exp \left( -\frac{\lambda N^2}{2k + N} \right) \\
\leq \exp \left( -\lambda N^2/(3N) \right) \sum_{k=1}^{N} \sum_{|A||B|=k} 1 \\
\leq \exp \left( -\lambda N/3 \right) 2^{m+n}.
$$

Combining (3), (4) and (5), we have

$$
P_\lambda(P) \leq \exp \left( -\frac{\lambda N^2}{3mn}(1 + o(1)) \right).
$$

We can choose $c$ large enough that $\lambda N^2/(3mn)$ exceeds any fixed multiple of $(mn^{2\epsilon} + nm^{2\epsilon})$. In comparison, (2) is independent of $c$. So for an appropriately large $c$,

$$
\frac{P_\lambda(P)}{P_\lambda(E_{s,t})} \leq \exp \left( O(mn^{2\epsilon} + nm^{2\epsilon}) - \lambda N^2/(3mn) \right) = o(1).
$$

Hence $P_{s,t}(P)$ tends to zero as $m, n \to \infty$ and we are done.

A topic for future research might be to try to identify which matrices have the largest sds and what structure those defining sets have. Our proofs do not give much insight into these questions. However, we do at least know that $\lambda$ must be very close to $1/2$ in order to achieve $\text{maxsds}(m, n)$.

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