NEW TRANSFORMATIONS FOR ELLIPTIC
HYPERGEOMETRIC SERIES ON THE ROOT SYSTEM $A_n$

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Abstract. Recently, Kajihara gave a Bailey-type transformation relating basic hypergeometric
series on the root system $A_n$, with different dimensions $n$. We give, with a new, elementary, proof,
an elliptic analogue of this transformation. We also obtain further Bailey-type transformations
as consequences of our result, some of which are new also in the case of basic and classical
hypergeometric series.

1. Introduction

Elliptic hypergeometric series form an extension of classical and basic (or $q$-) hyper-
geometric series, which was introduced by Frenkel and Turaev in 1997 [FT]. It was
found that Jackson’s $8W_7$ summation and Bailey’s $10W_9$ transformation admit one-
parameter extensions, roughly speaking obtained by replacing “$1 - x$” by the theta
function $\prod_{j=0}^{\infty} (1 - p^j x) (1 - p^{j+1}/x)$. For elliptic hypergeometric series, the so called
balanced and well-poised conditions on the series appearing in these identities reflect
invariance properties under the modular group [FT, S1].

In the last few years, multivariable elliptic hypergeometric series, in particular
series associated to classical root systems, has received much attention [DS1, DS2,
R1, R4, RS, S1, S2, W]. In the present paper we build upon the work in [R4]
to obtain some new transformation formulas for elliptic hypergeometric series on the
root system $A_n$.

In Theorem 3.1 we give an elliptic analogue of a multivariable Bailey transfor-
mation recently discovered by Kajihara [K2]. In contrast to most known transforma-
tions, Kajihara’s identity relates sums of different dimension; see [GR, K1, R2, R3]
for further results with this property. (We mention that, in view of the analogy
between hypergeometric series and hypergeometric integrals, there may exist related
transformations between integrals of different dimension. The only such result we
are aware of is in the recent paper [LV]; this seems not directly related to series of
the type studied here, but rather to discrete Selberg integrals [A].)

In Section 4 we obtain further Bailey-type transformations, between series of the
same dimension, by iterating Theorem 3.1. Most of these are new also in the case of
basic and classical hypergeometric series.

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series, multiple Bailey transformation.
2. Notation

Elliptic hypergeometric series may be built from the theta function

\[
\theta(x) = \prod_{j=0}^{\infty} (1 - p^j x)(1 - p^{j+1}/x), \quad |p| < 1.
\]

We will often use that \( \theta(1/x) = -\theta(x)/x \).

We denote elliptic Pochhammer symbols by

\[
(a)_k = \theta(a) \theta(aq) \cdots \theta(aq^{k-1}).
\]

The constants \( p \) and \( q \) are fixed throughout the paper and will be suppressed from the notation. The elementary identities

\[
(a)_{n+k} = (a)_n (aq^n)_k,
\]

\[
(a)_{n-k} = (-1)^k q^{(k)} (q^{1-n}/a)_k (a)_n (q^{1-n}/a)_k,
\]

\[
(a)_n = (-1)^n q^{(n)} a^n (q^{1-n}/a)_n
\]

will be used repeatedly and without comment. Occasionally, we use the shorthand notation

\[
(a_1, \ldots, a_n)_k = (a_1)_k \cdots (a_n)_k.
\]

We write

\[
\Delta(z) = \prod_{1 \leq j<k \leq n} z_j \theta(z_k/z_j).
\]

This may be viewed as an elliptic analogue of the Weyl denominator for the root system \( A_n \). Elliptic hypergeometric series on \( A_n \) are characterized by the factor

\[
\frac{\Delta(zq^y)}{\Delta(z)} = \prod_{1 \leq j<k \leq n} q^{y_j} \theta(z_k q^{y_k} / z_j q^{y_j}) \theta(z_k / z_j),
\]

where the \( z_k \) are free parameters and the \( y_k \) summation indices.

Note that when \( p = 0 \) we have

\[
(a)_k = (1 - a)(1 - aq) \cdots (1 - aq^{k-1}),
\]

the standard building blocks of basic hypergeometric series. Moreover, in this case,

\[
\frac{\Delta(zq^y)}{\Delta(z)} = \prod_{1 \leq j<k \leq n} z_k q^{y_k} - z_j q^{y_j} / z_k - z_j.
\]

Rescaling and letting \( q \to 1 \) one recovers the classical Pochhammer symbols

\[
a(a + 1) \cdots (a + k - 1)
\]

and double product

\[
\frac{\Delta(z + y)}{\Delta(z)} = \prod_{1 \leq j<k \leq n} \frac{z_k + y_k - z_j - y_j}{z_k - z_j},
\]

characterizing classical hypergeometric series on \( A_n \) [HBL].
For later reference we give some facts about one-variable elliptic hypergeometric series. Let $E$ be the function

$$E(a; q^{-N}, b, c, d, e, f, g) = \sum_{k=0}^{N} \frac{\theta(aq^{2k})}{\theta(a)} \frac{(a, q^{-N}, b, c, d, e, f, g)_k}{(q, aq^{N+1}, aq/b, aq/c, aq/d, aq/e, aq/f, aq/g)_k} q^k.$$ 

This is a $10W_9$ sum when $p = 0$. Frenkel and Turaev \cite{FT} proved the transformation formula

$$E(a; q^{-N}, b, c, d, e, f, g) = \frac{(aq, aq/ef, \lambda q/e, \lambda q/f)_N}{(aq/e, aq/f, \lambda q, \lambda q/ef)_N} E(\lambda; q^{-N}, \lambda b/a, \lambda c/a, \lambda d/a, e, f, g),$$

where $bcdefg = a^3 q^{N+2}$ and $\lambda = qa^2bcd$. When $p = 0$, this is the famous Bailey transformation \cite{B, GR}. Iterating (2) one obtains

$$E(a; q^{-N}, b, c, d, e, f, g) = g^N \frac{(aq/cg, aq/dg, aq/eg, aq/fg, aq/b)_N}{(aq/c, aq/d, aq/e, aq/f, aq/g, b/g)_N} \times E(gq^{-N}/b; q^{-N}, gq^{-N}/a, aq/bc, aq/bd, aq/be, aq/bf, g),$$

still assuming $bcdefg = a^3q^{N+2}$. When $p = 0$, this is \cite{GR, Exercise 2.19}.

### 3. An elliptic Kajihara transformation

The following identity is our main result.

**Theorem 3.1.** Assuming

$$w_1 \cdots w_m = z_1 \cdots z_n a_1 \cdots a_{m+n},$$

the following identity holds:

$$\sum_{y_1, \ldots, y_n \geq 0 \atop y_1 + \cdots + y_n = N} \frac{\Delta(zq^y)}{\Delta(z)} \frac{\prod_{k=1}^{n} (a_j z_k)_{y_k}}{\prod_{j=1}^{m+n} (w_k z_j)_{y_k} \prod_{j=1}^{m+n} (q z_k / z_j)_{y_k}} = \sum_{y_1, \ldots, y_m \geq 0 \atop y_1 + \cdots + y_m = N} \frac{\Delta(wq^y)}{\Delta(w)} \frac{\prod_{k=1}^{m} (w_k/a_j)_{y_k}}{\prod_{j=1}^{m+n} (w_k z_j)_{y_k} \prod_{j=1}^{m+n} (q w_k / w_j)_{y_k}}.$$

The case $m = n = 2$ is easily seen to be equivalent to (3), so Theorem 3.1 is a multivariable generalization of this transformation. On the other hand, if $m = 1$ but $n$ is general we have the summation formula

$$\sum_{y_1, \ldots, y_n \geq 0 \atop y_1 + \cdots + y_n = N} \frac{\Delta(zq^y)}{\Delta(z)} \frac{\prod_{k=1}^{n} (a_j z_k)_{y_k}}{\prod_{j=1}^{m+n} (w z_k)_{y_k} \prod_{j=1}^{m+1} (q z_k / z_j)_{y_k}} = \prod_{j=1}^{n+1} (w/a_j)_N \prod_{j=1}^{n} (w z_j)_{N(q) N},$$

where $w = z_1 \cdots z_n a_1 \cdots a_{n+1}$. This is \cite{R4, Theorem 5.1}, which is an elliptic analogue of Milne’s $A_n$ Jackson summation \cite{M}. See also \cite{S2}, where it was shown that (6) follows from a certain conjectured multiple integral evaluation.
The non-elliptic case, \( p = 0 \), of Theorem 3.1 is equivalent to Proposition 6.2 of \[K2\], where it was derived using Macdonald polynomials. A different proof, based on Gustafson’s \( A_n \) \( \psi_6 \) summation \[G\], was given in \[R2\]. Neither of these proofs is likely to admit a straight-forward elliptic generalization. Here we will use a simple inductive argument, which works also in the elliptic case.

**Proof.** We prove Theorem 3.1 by induction on \( n \). As a starting point we need the case \( n = 1 \), or equivalently \( m = 1 \), that is, the identity \[4\].

Assume that Theorem 3.1 holds for fixed \( n \) but general \( m \), and denote the left-hand side of \[3\] by \( S_{nm}(z; w; a) \). Consider the sum \( S_{n+1,m}(z; w; a) \). We replace the index set \((y_1, \ldots, y_{n+1})\) by \((y_1, \ldots, y_n, s)\) and rewrite part of the summand as

\[
\frac{\Delta(z q^{(y, s)})}{\Delta(z)} \prod_{j=1}^{n+1} \frac{1}{(q z_{n+1}/z_j)_s} = \frac{1}{(q)_s} \frac{\Delta(z q^y)}{\Delta(z)} \prod_{j=1}^{n} \frac{q^{-y_j} \theta(z_{n+1} q^s / z_j q^{y_j})}{\theta(z_{n+1}/z_j) (q z_{n+1}/z_j)_s} = \frac{1}{(q)_s} \frac{\Delta(z q^y)}{\Delta(z)} \prod_{j=1}^{n} \frac{\theta(z_j q^{-y_j} / z_{n+1})}{\theta(z_j q^{-s} / z_{n+1})},
\]

where \( \tilde{z} = (z_1, \ldots, z_n) \). This gives

\[
S_{n+1,m}(z; w; a) = \sum_{s=0}^{N} \frac{\prod_{j=1}^{m+n+1} (a_j z_{n+1})_s}{(q)_s \prod_{j=1}^{m} (w_j z_{n+1})_s \prod_{j=1}^{n} (z_{n+1}/z_j)_s} \times \sum_{y_1 + \cdots + y_{m+1} = N-s} \frac{\Delta(\tilde{z} q^y)}{\Delta(\tilde{z})} \prod_{k=1}^{n} \frac{(q^{-y_k} z_{n+1})_y (q^{-y_k} q^{-s} / z_{n+1})_y}{(q^{-y_k} / z_{n+1})_y (q^{-y_k} / z_{n+1})_y}.
\]

We observe that the inner sum is of the form \( S_{n,m+2}(\tilde{z}; \tilde{w}; \tilde{a}) \), with

\[
\tilde{w} = (w_1, \ldots, w_m, q/z_{n+1}, q^{-s} / z_{n+1}),
\]

\[
\tilde{a} = (a_1, \ldots, a_{m+n+1}, q^{-1-s} / z_{n+1}).
\]

We apply the induction hypothesis to this sum, writing \( y_{m+1} = t \), \( y_{m+2} = u \), and then change the order of summation according to

\[
\sum_{s=0}^{N} \sum_{y_1 + \cdots + y_{m+1} + u = N-s} \sum_{t=0}^{N} \sum_{y_1 + \cdots + y_{m} \leq N-t} \sum_{s + u = N-t - |y|} (\cdots).
\]
We consider first the inner sum, collecting all factors involving $s$ and $u$:

\[
\prod_{j=1}^{m+n+1} (a_j z_{n+1})_s \frac{\prod_{j=1}^{m} q^{u_j} \theta((\bar{w}_{m+2}q^u/w_j q^u) (w_j/\bar{a}_{m+2} y_j))}{\prod_{j=1}^{m+n+2} (\bar{w}_{m+2}/\bar{a}_{j} u)} \\
\times q^{u_j} \frac{\theta(\bar{w}_{m+2} q^u/\bar{w}_{m+1} q^t)}{\theta(w_{m+2}/w_{m+1})} \prod_{j=1}^{m+n+2} (\bar{w}_{m+2}/\bar{a}_{j} u) \\
\times \prod_{j=1}^{m} \frac{q^{u_j} \theta(q^{u-s-u_j}/z_{n+1} w_j)}{\theta(q^{s}/z_{n+1} w_j)} \frac{(q^{s-1} w_j z_{n+1}) y_j}{(w_j z_{n+1}) s (q^{1+s} w_j z_{n+1} y_j) (q^{1-s}/z_{n+1} w_j) u}.
\]

(9)

Note that, because of the factor $(q^{-1})_u$, (9) vanishes unless $u \in \{0, 1\}$. (If $s = 0$ and $u = 1$, the factor $1/(q^{-s})_u$ gives an apparent singularity, but this is removed by $(q^s)_t$ if $t > 0$ and by $\theta(q^{u-s-t-1})$ if $t = 0$.) This leads to considerable simplification. For instance,

\[
(a_j z_{n+1})_s (q^{-s}/z_{n+1} a_j)_u = (-1)^u q^{u} (q^{s}/z_{n+1} a_j) (a_j z_{n+1})_s (q^{1-u+a_j z_{n+1}}) u
\]

if $u \in \{0, 1\}$. Similar computations, eventually using (10), reveal that (9) equals

\[
(-1)^u (q^{s+u})_t \prod_{j=1}^{m+n+1} (a_j z_{n+1})_{s+u} \prod_{j=1}^{m} (q^{s+u-1} w_j z_{n+1}) y_j \\
\quad \prod_{j=1}^{m+n+2} (w_j z_{n+1} a_j)_y \\
\times \prod_{j=1}^{m} \frac{q^{u_j} \theta(q^{u-s-u_j}/z_{n+1} w_j)}{\theta(q^{s}/z_{n+1} w_j)} \frac{(q^{s-1} w_j z_{n+1}) y_j}{(w_j z_{n+1}) s (q^{1+s} w_j z_{n+1} y_j) (q^{1-s}/z_{n+1} w_j) u}.
\]

(10)

if $u \in \{0, 1\}$. Thus, the inner sum in (8) is proportional to

\[
\min(1, N-t-|y|) \\
\sum_{u=0} (-1)^u = \delta_{N-t-|y|, 0},
\]

so we may assume $t + |y| = N$, $s = u = 0$. But then the factor $(q^{s+u})_t$ in (10) equals zero unless $t = 0$, so (8) is reduced to the sum

\[
\sum_{y_1+\cdots+y_m=N} \frac{\Delta(\bar{w}_q(y,0,0))}{\Delta(\bar{w})} \prod_{k=1}^{m} \prod_{j=1}^{m+n+2} (w_k z_j y_k) \prod_{j=1}^{m+n+2} (q w_k \bar{w}_j y_k),
\]

where $s = 0$ in (7). It is easily verified that this simplifies to

\[
\sum_{y_1+\cdots+y_m=N} \frac{\Delta(w^y)}{\Delta(w)} \prod_{k=1}^{m} \prod_{j=1}^{m+n+1} (w_k a_j) y_k \prod_{j=1}^{m+n+1} (w_k z_j y_k) \prod_{j=1}^{m} (q w_k \bar{w}_j y_k).
\]

This completes the proof.

\[\square\]
4. Further multiple Bailey transformations

When \( m = 2 \), the right-hand side of (5) has some additional symmetry which allows us to obtain further transformations by iterating Theorem 3.1. This was observed in [K1], although the idea was not fully exploited there.

We first rewrite the case \( m = 2 \) of (5) with \( k = y_1 = N - y_2 \) as summation index on the right-hand side, giving

\[
\sum_{y_1 + \cdots + y_n = N} \frac{\Delta(z q^y)}{\Delta(z)} \prod_{k=1}^{n} \frac{\prod_{j=1}^{n+2} (a_j z_k)^{y_k}}{(w_1 z_k)_{y_k} \prod_{j=1}^{n} (q z_k / z_j)_{y_k}}
\]

\[
= \frac{\prod_{j=1}^{n+2} (w_2 / a_j)_N}{(w_2 / w_1)_N(q)_N \prod_{j=1}^{n} (w_2 z_j)_N} \sum_{k=0}^{N} \left( \frac{\theta(q^{2k-N} w_1 / w_2)}{\theta(q^{-N} w_1 / w_2)} \right) \times \frac{(q^{-N} w_1 / w_2)_k (q^{-N})_k \prod_{j=1}^{n+2} (w_1 / a_j)_k \prod_{j=1}^{n} (q^{-N} / w_2 z_j)_k}{(q)_k (q w_1 / w_2)_k \prod_{j=1}^{n+2} (q^{-N} a_j / w_2)_k \prod_{j=1}^{n} (w_1 z_j)_kq^k},
\]

which is consistent with condition (12), we obtain the same right-hand side up to the multiplier

\[
\frac{\prod_{j=1}^{n+2} (w_2 / a_j)_N \prod_{j=1}^{n} (w_2 x_j)_N}{\prod_{j=1}^{n+2} (w_2 z_j)_N \prod_{j=1}^{n+2} (w_2 / b_j)_N} = \prod_{j=1}^{n-m} \frac{(w_2 / a_j)_N (q^{-N} a_j / w_1)_N}{(w_2 z_j)_N (q^{-N} / w_1 z_j)_N}
\]

\[
= \prod_{j=1}^{n-m} \frac{(a_j z_j)^N (w_1 / a_j)_N (w_2 / a_j)_N}{(w_1 z_j)_N (w_2 z_j)_N}.
\]

This proves the following result.
Corollary 4.1. Assuming \( w_1 w_2 = z_1 \cdots z_n a_1 \cdots a_{n+2} \) and \( 0 \leq m \leq n \), we have

\[
\sum_{y_1, \ldots, y_n \geq 0 \atop y_1 + \cdots + y_n = N} \frac{\Delta(zq^y)}{\Delta(z)} \prod_{j=1}^{n+2} (a_j z_j)_{y_j} \prod_{j=1}^{n+2} (w_j z_j)_{y_j} \prod_{j=1}^{n+2} (q z_j / z_j)_{y_j} \\
= \prod_{j=1}^{n-m} (a_j z_j)^N (w_1 / a_j)_N (w_2 / a_j)_N (w_1 z_j)_N (w_2 z_j)_N \prod_{j=1}^{n+2} (b_j x_j)_{y_j} \prod_{j=1}^{n+2} (q x_j / x_j)_{y_j},
\]

where \( x \) and \( b \) are given by \([13]\).

In the one-variable case, \( n = 2 \), there are three choices of \( m \): \( m = 2 \), which is trivial, \( m = 1 \), which gives \([2]\) and \( m = 0 \), which gives \([3]\). For general \( n \) we have a sequence of non-trivial identities: \( m = 0, 1, \ldots, n-1 \). The case \( m = n-1 \) is equivalent to \([13]\) Corollary 8.2], which is an elliptic analogue of a multiple Bailey transformation of Milne and Newcomb \([MN]\) (a closely related identity was obtained in \([DG]\)). The remaining identities, with \( m \leq n-2 \), appear to be new also in the non-elliptic case. The extreme case \( m = 0 \) is particularly elegant, so we write it out explicitly. It gives a multivariable generalization of \([3]\) that is different from Theorem \([3]\). We have made the replacements \( a_{n+1} = b, a_{n+2} = c, w_1 = d, w_2 = e \).

Corollary 4.2. Assuming \( de = a_1 \cdots a_n b c z_1 \cdots z_n \), we have

\[
\sum_{y_1, \ldots, y_n \geq 0 \atop y_1 + \cdots + y_n = N} \frac{\Delta(zq^y)}{\Delta(z)} \prod_{j,k=1}^{n} \frac{(a_j z_j)_{y_j} (b_j x_j)_{y_k}}{(q z_k / z_j)_{y_k} (d z_k)_{y_k} (e z_k)_{y_k}} \prod_{j=1}^{n} \frac{(q_{a_j} z_j)_{y_j}}{(d z_j)_{y_j} (e z_j)_{y_j}} \prod_{j=1}^{n} (q^{1-N} a_k b / d)_{y_k} (q^{1-N} a_k c / d)_{y_k} \\
= \prod_{j=1}^{n} (a_j z_j)^N (d / a_j)_N (e / a_j)_N (d z_j)_{N} (z_j)_{N} \prod_{j=1}^{n} (q z_k / z_j)_{y_k} (q^{1-N} a_k b / d)_{y_k} (q^{1-N} a_k c / d)_{y_k}
\]

Finally, we give a companion identity to Corollary \([4]\) with the sum supported on a hyper-rectangle rather than a simplex. There are similar companions to the other cases of Corollary \([4]\) but these are more complicated to write down.

We first replace \( n \) by \( n + 1 \) in Corollary \([4]\) and assume that \( a_j = q^{-m_j} / z_j \), \( 1 \leq j \leq n \), where \( m_j \) are non-negative integers such that \( |m| \leq N \). Then all terms
with $y_k > m_k$ for some $k$ vanish. Eliminating $y_{n+1}$ from both summations we obtain

$$
\sum_{y_1, \ldots, y_n=0}^{m_1, \ldots, m_n} \frac{\Delta(zq^y)}{\Delta(z)} \prod_{k=1}^n \frac{q^{N+y} \theta(z_k q^{y_k} / z_{n+1} q^{N+y})}{\theta(z_k / z_{n+1})} \prod_{j=k}^n \frac{(q^{-m_j} z_k / z_j)_{y_k}}{(q z_k / z_j)_{y_k}}
\times \prod_{j=1}^n \frac{\left(\prod_{k=1}^n (a_{n+1} z_k, b_{n+1}, c_{n+1})_{y_k}\right)}{\left(\prod_{k=1}^n (q z_k / z_{n+1}, d_{n+1}, e_{n+1})_{y_k}\right)} \prod_{j=1}^n \frac{(q^{-m_j} z_k / z_j)_{y_k}}{(q z_k / z_j)_{y_k}} =
\prod_{k=1}^n \left(\prod_{j=1}^n \frac{\left(\prod_{k=1}^n (a_{n+1} z_k, b_{n+1}, c_{n+1})_{y_k}\right)}{\left(\prod_{k=1}^n (q z_k / z_{n+1}, d_{n+1}, e_{n+1})_{y_k}\right)} \prod_{j=1}^n \frac{(q^{-m_j} z_k / z_j)_{y_k}}{(q z_k / z_j)_{y_k}} \right),
$$

where $deq^{|m|} = a_{n+1}bcz_{n+1}$.

We observe that the right-hand side will look nicer after the change of variables $m_k \mapsto y_k - m_k$. We do that and also manipulate the Pochhammer symbols so that $N$ never appears as a subscript. After a tedious but straight-forward computation we arrive at

$$
\sum_{y_1, \ldots, y_n=0}^{m_1, \ldots, m_n} \frac{\Delta(zq^y)}{\Delta(z)} \prod_{k=1}^n \frac{\theta(z_k q^{y_k} / z_{n+1} q^{N+y})}{\theta(z_k q^{-N+y} / z_{n+1})} \prod_{j=k}^n \frac{(q^{-m_j} z_k / z_j)_{y_k}}{(q z_k / z_j)_{y_k}} \prod_{k=1}^n \frac{\left(\prod_{j=1}^n (a_{n+1} z_k, b_{n+1}, c_{n+1})_{y_k}\right)}{\left(\prod_{j=1}^n (q z_k / z_{n+1}, d_{n+1}, e_{n+1})_{y_k}\right)} \prod_{j=1}^n \frac{(q^{-m_j} z_k / z_j)_{y_k}}{(q z_k / z_j)_{y_k}} =
\prod_{k=1}^n \left(\prod_{j=1}^n \frac{\left(\prod_{k=1}^n (a_{n+1} z_k, b_{n+1}, c_{n+1})_{y_k}\right)}{\left(\prod_{k=1}^n (q z_k / z_{n+1}, d_{n+1}, e_{n+1})_{y_k}\right)} \prod_{j=1}^n \frac{(q^{-m_j} z_k / z_j)_{y_k}}{(q z_k / z_j)_{y_k}} \right).
$$
In this computation the following identity, which is equivalent to [R4 Equation (3.8)], is useful:
\[
\frac{\Delta(1/z)}{\Delta(q^{-m}/z)} \prod_{j,k=1}^{n} \frac{(q^{-m_k}z_j/z_k)^{m_k}}{(q^{1-m_k-m_j}z_j/z_k)^{m_k}} = (-1)^{|m|} q^{-|m|-(\frac{|m|}{2})}.
\]

To make the connection with [B] transparent we make the change of parameters

\[
(q^{-N}, a_{n+1}, b, c, d, e, z_{n+1}) \mapsto (b, g, e, f, aq/c, aq/d, b/a).
\]

We then obtain the following transformation, in the special case when \( b = q^{-N} \) with \( N \geq |m| \) a non-negative integer.

**Corollary 4.3.** Assuming \( a^3q^{|m|+2} = bcdefg \), the following identity holds:
\[
\sum_{j_1,\ldots,j_n=0}^{m_1,\ldots,m_n} \frac{\Delta(zq^{y_j})}{\Delta(z)} \prod_{k=1}^{n} \frac{\theta(a z_k q^{m+|y|})}{\theta(a z_k)} \prod_{j=1}^{n} \frac{(a z_j)|_{|y|}}{(q^{1+|m|} a z_j)|_{|y|}} \frac{(b, c, d)|_{|y|}}{(aq/e, aq/f, aq/g)|_{|y|}} q^{|y|} \times \prod_{j,k=1}^{n} \frac{(q^{-m_j} z_k/z_j)_{y_k}}{(q z_k/z_j)_{y_k}} \prod_{k=1}^{n} \frac{(e z_k, f z_k, g z_k)_{y_k}}{(aq z_k/b, a q z_k/c, a q z_k/d)_{y_k}}
\]
\[
= g^{|m|} q^{-\sum_{j<k} m_j m_k} \frac{(b, aq/c, aq/d, g)_{|m|}}{(aq/e, aq/f, aq/g)_{|m|}} \times \prod_{k=1}^{n} \frac{z_k^m (aq z_k, q^{-1+|m|-m_k} a/z_k e g, q^{1+|m|-m_k} a/z_k f g)_{m_k}}{(aq z_k/c, aq z_k/d, q^{-m_k} b/g z_k)_{m_k}}
\]
\[
\times \sum_{j_1,\ldots,j_n=0}^{m_1,\ldots,m_n} \frac{\Delta(zq^{y_j})}{\Delta(z)} \prod_{k=1}^{n} \frac{\theta(g z_k q^{y_k+|y|-|m|}/b)}{\theta(g z_k q^{-|m|}/b)} \times \prod_{j=1}^{n} \frac{(g z_j q^{-m_j}/b)_{|y|}}{(g z_j q^{m_j+1-|m|}/b)_{|y|}} \frac{(q^{-|m|} g/a, aq/b, aq/bf)_{|y|}}{(q^{-|m|} c g/a, q^{-|m|} d g/a, q^{1-|m|}/b)_{|y|}} q^{|y|} \times \prod_{j,k=1}^{n} \frac{(q^{-m_j} z_k/z_j)_{y_k}}{(q z_k/z_j)_{y_k}} \prod_{k=1}^{n} \frac{(aq z_k/bc, aq z_k/bd, g z_k)_{y_k}}{(aq z_k/b, aq z_k/b, q^{-m_k} eg z_k/a, q^{-m_k} f g z_k/a)_{y_k}}.
\]

To complete the proof we must extend the result from the case \( b = q^{-N} \) to generic \( b \). This may be done exactly as in the proof of Corollary 5.3 of [R4]. That is, one considers the function \( f(b) = L - R \), where \( L \) and \( R \) are the left- and right-hand side of the identity we want to prove, and where \( c = a^3 q^{|m|+2}/bcdefg \) is viewed as depending on \( b \) while the other parameters are fixed. It is then straightforward to check that \( f(pb) = f(b) \), where \( p \) is the elliptic nome as in [I]. Thus, \( f(p^k q^{-N}) = 0 \) for \( k \in \mathbb{Z} \) and \( N \in \mathbb{Z}_{\geq |m|} \). This is enough to conclude, by analytic continuation, that \( f \) is identically zero.

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