Effective Search Templates for a Primordial Stochastic Gravitational Wave Background

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Abstract

We calculate the signal-to-noise ratio (SNR) of the stochastic gravitational-wave background in an extreme case that its spectrum has a sharp falloff with its amplitude close to the detection threshold. Such a spectral feature is a characteristic imprint of the change in the number of relativistic degrees of freedom on the stochastic background generated during inflation in the early Universe. We find that, although SNR is maximal with the correct template which is proportional to the assumed real spectrum, its sensitivity to the shape of template is fairly weak indicating that a simple power-law template is sufficient to detect the signature.

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I. INTRODUCTION

The detection of a stochastic background of primordial gravitational waves (GWs) is an exciting challenge. Since the GWs are decoupled from other ingredients of the Universe after the Planck time, their detection enables us to probe the early Universe long before the recombination, providing much information of the early Universe and high energy physics at the epoch when they are generated [1]. For example, we can probe the equation of state $w = p/\rho$ in the early Universe using the spectrum of a stochastic GW background [2], and lepton asymmetry can be evaluated by investigating effects of neutrino free streaming on the power spectrum of primordial GWs [3].

One of the most probable sources of such primordial GWs is inflation. Inflation was proposed as the most natural solution to the difficulties of the standard big bang cosmology, such as the horizon problem and the flatness problem [4]. It also generates primordial gravitational waves (tensor modes) [5] as well as the primordial density fluctuations (scalar modes) [6]. The latter has been observed as the cosmic microwave background anisotropies by the cosmic background explorer (COBE) satellite [7], the Wilkinson microwave anisotropy probe (WMAP) satellite [8, 9], and so on. On the other hand, the tensor fluctuations have not yet been detected and only the upper limit on the ratio of the amplitude of tensor fluctuations to scalar fluctuations is obtained as $r < 0.55$ (95% C.L.) at $k = 0.002\text{Mpc}^{-1}$ [9].

The detection of tensor fluctuations (GWs) generated during inflation is very important in that their amplitudes can determine the energy scale of inflation directly, while scalar fluctuations are sensitive to a combination of the potential energy and its derivative [10]. The energy scale of the inflation strongly constrains inflation models proposed so far. Furthermore, the consistency relation [10] which relates the ratio of the amplitude of tensor fluctuations to scalar fluctuations $r$ to the spectral index of tensor fluctuations $n_T$, if confirmed, would be an extremely important signature of single-field inflation and could help to discriminate from other mechanisms for the generation of the spectra such as a curvaton mechanism [11] or a cyclic universe [12].

The observational programs for detecting such GWs have been proposed as the next generation projects. For example, DECIGO is proposed in Japan [13] and the big bang observer (BBO) at NASA [14]. Since the typical amplitude of a stochastic background of primordial GWs is extremely small, the signal to noise ratio (SNR) is not so large even if noises are suppressed below the quantum level (ultimate DECIGO). Thus, in order to enhance the SNR, search templates as well as cross-correlation analysis are necessary for observations of a stochastic GW background [15].

Although the spectrum of GWs generated during inflation may well be approximated by power-law shape, the present-day spectrum is different from the original one. The amplitude of GWs is redshifted by the cosmic expansion and the expansion rate depends on the matter content of the Universe. The spectrum is roughly proportional to $f^{-2}$, $f^{0}$, $f$ for the modes which reenter the horizon in the matter, the radiation, and the kinetic-energy dominated phases, respectively, where $f$ is the frequency [1, 16]. Therefore, templates with (broken) power-law shapes are considered and their improvements of SNR have been discussed [17]. In reality, however, the present-day spectrum is never power law nor smooth even if the initial spectrum is power law. The effective number of relativistic degrees of freedom changes with temperature and these changes leave characteristic features in the spectrum [18].

\footnote{The effect of quark gluon plasma phase transition is discussed in [19].}
changes of the effective number of degrees of freedom depending on the mass thresholds induce relatively rapid changes of the amplitude of the spectrum, rather than the change of the power-law index. Therefore, it is important to assess the validity of the use of the templates of power-law shape as search templates for a stochastic gravitational-wave background.

In this paper, we consider the frequency ranges of $10^{-4} - 10$ Hz suitable for proposed future detectors of GWs (LISA, DECIGO, BBO). These frequency ranges correspond to the modes which reentered into the horizon at the temperature $100 \text{GeV} - 10^4 \text{TeV}$ and the spectrum is damped due to the electroweak phase transition. In the next section, we briefly review primordial GWs produced during inflation and search templates to detect them. In Sec. III, we compare SNRs for templates with a simple power-law type to those for templates with a rapid change of the amplitude and discuss how effective the former templates are for not only detection of such GWs but also probing the change of the amplitude of their spectrum.

We give discussions and summary in the final section.

II. DETECTING A STOCHASTIC BACKGROUND OF GRAVITATIONAL WAVES

A. Property of a stochastic background of GWs produced during inflation

In order to discuss the property of an isotropic and homogeneous stochastic background of GWs, one introduces the dimensionless quantity $\Omega_{gw}(f)$ that is the energy density of a GW stored in a logarithmic frequency interval around $f$ divided by the critical energy density. In terms of the characteristic amplitude of a GW, $h_c$, this quantity is expressed as

$$\Omega_{gw}(f) = \frac{2\pi^2}{3H_0^2} f^2 |h_c|^2,$$

where $H_0 \cong 72 \text{km/sec/Mpc} = 2.3 \times 10^{-18} \text{/sec}$ is the present Hubble parameter. The spectrum of stochastic background of GWs generated quantum mechanically during inflation is calculated by the quantum field theory of a massless minimally coupled field in inflationary background [5]. However, we obtain some insights into the spectrum shape without detailed calculation. Until a mode reenters the Hubble radius, the characteristic amplitude $h_c$ takes a constant value proportional to the Hubble parameter, $H_{\text{inf}}$, when the mode left the Hubble radius during inflation. On the other hand, the amplitude damps as $1/a$ after the mode reenters the Hubble radius. Therefore, the present characteristic amplitude is given by

$$|h_c| \simeq \sqrt{\frac{8}{\pi} \frac{H_{\text{inf}} a(t_k)}{M_{\text{Pl}} a(t_0)}},$$

Here, $t_0$ is the present time and $t_k$ is the epoch when the mode (with wave number $k$) reentered the Hubble radius, $2\pi f = k = a(t_k)H(t_k)$. If at $t_k$ the Universe is dominated by an ingredient with the equation of state $w$, then (naively) from the Friedmann equation we obtain $H(t_k) \propto a(t_k)^{-3(1+w)/2}$ and thus $f \propto a(t_k)^{-(1+3w)/2}$. Therefore the shape of the present density parameter $\Omega_{gw}(f)$ is given by

$$\Omega_{gw}(f) \propto a(t_k)^{1-3w} \propto f^{2(1-3w)/(1+3w)}.$
Thus we recover $\Omega_{gw} \propto f^0, f^{-2}, f^1$ for $w = 1/3, 0, 1$, respectively.

This naive estimate neglects the effect of the change of relativistic degrees of freedom during the radiation dominated epoch [18]. During the radiation dominated epoch, the energy density of relativistic particle $\rho_{\text{rad}} = (\pi^2/30)g_\ast T^4$ does not scale as $a^{-4}$. From the entropy conservation [20],

$$\frac{\pi^2}{45}g_\ast S T^3 a^3 = \text{constant}, \quad (4)$$

we obtain

$$\rho_{\text{rad}} \propto g_\ast g_\ast S^{-4/3} a^{-4}. \quad (5)$$

Here $g_\ast$ and $g_\ast S$ account for the total number of effective relativistic degrees of freedom. As long as the Universe is fully thermalized, these two coincide with each other. Taking account of the effective relativistic degrees of freedom, for the mode entering into the horizon during the radiation era, we have

$$\Omega_{gw} \propto f^2 a(t_k)^2 \propto a(t_k)^4 H(t_k)^2 \propto g_\ast(t_k)g_\ast S(t_k)^{-4/3}. \quad (6)$$

Since $g_\ast$ and $g_\ast S$ coincide for $100\text{GeV} < T < 10^3\text{TeV}$, we find $\Omega_{gw} \propto g_\ast(t_k)^{-1/3}$.

**B. Detection method of a stochastic background of GWs**

In general, the GW background signal is expected to be very week and is usually masked by the detector noises. To detect such tiny signals, it is practically impossible to detect the signal from the single-detector measurement. Thus, we cross correlate the two outputs obtained from the different detectors and seek a common signal. We denote the detector outputs by $s_i$ with

$$s_i(t) = h_i(t) + n_i(t), \quad (7)$$

where $i = 1, 2$ corresponds to the $i$-th detector, and $h_i(t)$ is the gravitational-wave signal and $n_i(t)$ is the noise. Then, the cross-correlation signal $S$ is given by multiplying the outputs of the two detectors and integrating over the observational time:

$$S \equiv \int_{-T/2}^{T/2} dt \int_{-T/2}^{T/2} dt' s_1(t)s_2(t')Q(t - t'), \quad (8)$$

where the filter function $Q$ is introduced to enhance the detectability of the GW signals and we take the observation time $T$ to be large enough. We also assume that the statistical property of both the GW signal and the noise is stationary, which implies that the argument of $Q$ depends only on the time difference $t - t'$.

The detectability in the context of stochastic background searches is quantified by the signal-to-noise ratio for the cross-correlation signal $S$

$$\text{SNR} = \frac{\langle S \rangle}{\sqrt{\langle S^2 \rangle - \langle S \rangle^2}}. \quad (9)$$
Under the assumption that the two different detectors (or output data stream) have no correlation of noise in the weak limit ($h_i \ll n_i$), the mean and variance are given by

$$\langle S \rangle = \frac{3H_0^2}{20\pi^2} T \int_{-\infty}^{\infty} df |f|^{-3}\Omega_{gw}(|f|)\gamma(|f|)\tilde{Q}(f),$$  \hspace{1cm} (10)$$

$$\langle S^2 \rangle - \langle S \rangle^2 \approx \frac{T}{4} \int_{-\infty}^{\infty} df P_1(|f|)P_2(|f|)|\tilde{Q}(f)|^2;$$  \hspace{1cm} (11)$$

where $\tilde{Q}(f)$ is the Fourier transform of the filter function $Q(t - t')$ in the frequency domain. $P_i(|f|)$ is the noise power spectrum of $i$-th detector defined by

$$\langle n_i(t)n_i(t') \rangle = \frac{1}{2} \int_{-\infty}^{\infty} df e^{2\pi i f(t-t')}P_i(|f|).$$  \hspace{1cm} (12)$$

$\gamma(|f|)$ is the overlap reduction function which characterizes the reduction in sensitivity to a stochastic background arising from the separation time delay and relative orientation of the two detectors [21]. If their orientations are coincident and coaligned without any systematic noise correlation between them, the overlap reduction function $\gamma(|f|)$ becomes constant for all frequencies $f$. When the arms of a detector are separated by 90 degrees, $\gamma(|f|) = 1$.

We would like to determine the functional form of $\tilde{Q}(f)$ which maximizes SNR. In order to find such a function, we introduce an inner product $(A|B)$ for any pair of complex functions $A(f)$ and $B(f)$ with weight functions $P_1(|f|) \cdot P_2(|f|)$ [15]:

$$(A|B) \equiv \int_{-\infty}^{\infty} df A^*(f)B(f)P_1(|f|)P_2(|f|).$$  \hspace{1cm} (13)$$

As long as $P_i(|f|)$ is positive for all frequencies, $(A|B)$ satisfies the same properties of ordinary inner product in three-dimensional Euclidean space. Using Eqs.(10) and (11) and the inner product defined in Eq.(13), the SNR (9) is rewritten as [15]

$$\text{SNR}^2 \simeq \left( \frac{3H_0^2}{10\pi^2} \right)^2 \frac{T}{\langle \tilde{Q}^2 \rangle} \left( \frac{\gamma(|f|)\Omega_{gw}(|f|)}{|f|^3P_1(|f|)P_2(|f|)} \right)^2.$$  \hspace{1cm} (14)$$

Then, regarding functions $\tilde{Q}(f)$ and $\gamma(|f|)\Omega_{gw}(|f|)/|f|^3P_1(|f|)P_2(|f|)$ as three-dimensional vectors, we find that the filter function maximizing the SNR is given by

$$\tilde{Q}(f) = \frac{\gamma(|f|)\Omega_{gw}(|f|)}{|f|^3P_1(|f|)P_2(|f|)}. $$  \hspace{1cm} (15)$$

Here we have neglected an overall normalization constant because the SNR (9) is independent of it. Note that $\tilde{Q}(f)$ also becomes an even function of a frequency $f$. Hence, in the following we assume that $f$ is positive definite.

It should be noted that the resultant filter function $\tilde{Q}(f)$ depends on not only known functions such as the overlap reduction function and the noise spectrum but also an unknown function, that is, the spectrum of GWs $\Omega_{gw}(f)$. Then, it is not until we assume the spectrum of GWs that we can determine the maximized SNR and the corresponding filter function, which requires us to arrange search templates of the spectrum of GWs. However, since the number of templates is limited in practice, such a template does not necessarily coincide
with the real spectrum of the GWs background. Thus, it is very important to estimate SNRs for such templates when the real spectrum of the GWs background is given. More concretely, as explained in the Introduction, the spectrum of GWs generated during inflation is in fact neither power law nor smooth. Instead, the amplitude of the spectrum changes rapidly due to the changes of the effective number of degrees of freedom depending on the mass thresholds. Then, we wonder how much it is justified to use templates with a simple power-law type, which have been often considered. If such simple templates have large enough SNRs, the number of templates is significantly reduced. On the contrary, in case their SNRs are small, we would be able to find a sudden change of the amplitude of the spectrum if we use templates which take a sudden change of the amplitude into account, albeit the number of the templates becomes much larger. Therefore, we compare the SNRs of templates with a simple power-law type to those of templates with a realistic form of the spectrum.

### III. Effectiveness of Search Templates

In this section, we compare SNRs for templates with a simple power-law type to those for templates with the change of the amplitude for the LIGO II and the next-generation space interferometers such as FP-DECIGO and LISA. Then, we discuss how effective the former templates are for not only detection of such GWs but also probing the change of the amplitude of their spectrum.

As the spectrum of GW background, taking into account the change of the effective number of relativistic degrees of freedom, we adopt the following simple model in which \( \Omega_{gw}(f) \) is characterized by the step function:

\[
\Omega_{gw}(f) = \Omega_N \left[ \Theta \left( 1 - \frac{f}{f_c} \right) + d \cdot \Theta \left( \frac{f}{f_c} - 1 \right) \right],
\]

where \( f_c \) is the critical frequency corresponding to the electroweak phase transition (\( \sim 10^{-4} \) Hz) and \( d \) is the damping factor that reflects the change of the effective number of relativistic degrees of freedom \( g_* \), and \( \Omega_N \) is a constant. Since \( \Omega_{gw} \) is proportional to \( g_*^{-1/3} \) (Eq. (14)), the damping factor due to electroweak phase transition in the standard model of elementary particles \( d_{SM} \) is \( d_{SM} \approx [g_*(> \text{TeV})/g_*(1\text{GeV})]^{-1/3} \approx 0.9 \). Even in the supersymmetric extension of the standard model, only \( g_*(> \text{TeV}) \) is doubled and the damping factor becomes \( d \approx 2^{-1/3}d_{SM} \approx 0.7 \).

As mentioned in the previous section, the optimal filter is determined by the possible spectrum of the stochastic background of GWs. Given the noise power spectrum of the detector, we compute the SNRs of the GW spectrum \( \Omega_{gw} \) using Eq. (14) with

\[
\tilde{Q}(f) = \frac{\gamma(|f|) \Omega_{\text{filter}}(|f|)}{|f|^3 P_1(|f|) P_2(|f|)}.
\]

Here, we consider templates with a simple power-law type for the optimal filters

\[
\Omega_{\text{filter}}(f) \propto f^\alpha,
\]

\( \ddagger \) In reality, the change of the spectrum is much milder than the step function [18]. Here we consider the extreme case so that the SNR might change most drastically.
where \( \alpha \) is a constant. We also consider templates with the change of the amplitude \( f \), that is, \( \Omega_{\text{filter}}(f) = \Omega_{\text{gw}}(f) \) given in Eq. (16) for another optimal filter. Then, we compare both results and examine whether the template Eq. (16) is effective not only for the detection of such GWs but also for probing the change of the amplitude of their spectrum.

Before going into the detailed computation, we may have a qualitative understanding of SNR for a flat \( \Omega_{\text{filter}} \). From Eq. (17) for a flat \( \Omega_{\text{filter}} \) and constant \( \gamma(f) \), a “V”-shaped noise density \( P_i(f) \) becomes a sharper “Λ”-shaped optimal filter \( \tilde{Q}(f) \) and hence the “bandwidth” of \( \tilde{Q}(f) \) is very narrow. SNR gets its significant contributions from this bandwidth and does not care about the change of the filter beyond that frequency band. So the question is how much SNR is improved when \( f_c \) falls into the bandwidth of \( \tilde{Q}(f) \).

A. Sensitivity of LIGO II to stochastic background of GWs

First we give the results of SNRs for several templates in the setup of LIGO II. For ground based detectors like LIGO II, the sources of noise consist of seismic, thermal, and photon shot noises. Then, we use the noise power spectrum of the detectors giving the Figure 1 in [22]. The fitting function of this noise power spectrum is given by [23]:

\[
P_i(f) = \begin{cases} 
\text{Max}[10^{-44}(f/10\text{Hz})^{-4} + 10^{-47.25}(f/100\text{Hz})^{-1.7}, 10^{-46}(f/10^3\text{Hz})^3]; & 10 < f < 3000\text{Hz}, \\
\infty; & \text{otherwise}.
\end{cases}
\]

The overlap reduction function \( \gamma(f) \) is calculated by giving each location for the detector pair. Fig. 4 shows the overlap reduction function for the Hanford and Livingston LIGO detector pair [15]. Then we compute the SNRs of the GW spectrum \( \Omega_{\text{gw}} \) using Eq. (14). We take \( T = 10^7 \) sec and \( \Omega_N = 10^{-10} \). The results are shown in Figure 2 and Table I. We note that SNR is proportional to \( \Omega_N \).
Fig. 2: The value (top) and ratio (bottom) of SNRs for $\Omega_{\text{filter}} = \Omega_{\text{gw}}$ (solid line in top panel) and $\Omega_{\text{filter}} \propto f^0$ (dotted line in top panel) for LIGO II. We set $d = 0.7$.

| damping rate: $d$ | 0.9  | 0.7  | 0.5  | 0.1  | $\Omega_{\text{filter}}$ |
|------------------|------|------|------|------|--------------------------|
| SNR_{max}        | 0.465| 0.361| 0.258| 0.0516| $\Omega_{\text{gw}}$    |
| SNR_{flat}       | 0.465| 0.361| 0.258| 0.0516| power-law with $\alpha = 0$ |
| SNR_{PL-1}       | 0.429| 0.333| 0.238| 0.0476| power-law with $\alpha = -1$ |
| SNR_{PL+1}       | 0.423| 0.329| 0.235| 0.0470| power-law with $\alpha = +1$ |

TABLE I: SNRs for several templates in the setup of LIGO II. We take $T = 10^7$ sec, $\Omega_N = 10^{-9}$, and $f_c = 10^{-4}$ Hz. SNR_{max} represents SNR for $\Omega_{\text{filter}} = \Omega_{\text{gw}}$ with $f_c = 10^{-4}$ Hz in Eq.(16). SNR_{flat} for $\Omega_{\text{filter}} \propto f^0$, SNR_{BPL\pm1} for $\Omega_{\text{filter}} \propto f^\pm1$. Note that SNR is proportional to $\Omega_N$.

Fig. 2 shows the dependence of the value and ratio of SNR_{max} and SNR_{flat} on the critical frequency $f_c$ with damping factor $d = 0.7$. Here SNR_{max} and SNR_{flat} are the value of SNR calculated in Eqs. (14) and (17) with $\Omega_{\text{filter}} = \Omega_{\text{gw}}$ and $\Omega_{\text{filter}} \propto f^0$, respectively. We find that SNR_{max} can differ from SNR_{flat} around the range $10 - 50$ Hz which is below the typical frequency range of the noise spectrum of LIGO II due to $f^{-3}$ factor in Eq.(17). However, the difference is quite small. Comparing with the SNR by a flat spectrum, the improvement using the “true” filter $\Omega_{\text{gw}}$ is at most 2% for $d = 0.7$. In Table I we give results for other $d$ for $f_c = 10^{-4}$Hz. We find no improvement. The bandwidth of $\tilde{Q}(f)$ is very narrow ($10\text{Hz} \lesssim f \lesssim 100\text{Hz}$). SNR gets its significant contributions from this bandwidth and does not care about the change of the filter beyond that frequency band.

We thus conclude that search templates with a power-law form are sufficient in practice for the detection of GWs generated during inflation, even though the real spectrum is never power law.
B. Sensitivity of next-generation space interferometers to stochastic background of GWs

Finally, we also give the results of SNRs for several templates with different forms in the setup of next-generation space interferometers such as FP(Fabry-Perot)-DECIGO [24] and LISA [25].

We consider two detectors forming a starlike constellation. In the pre-conceptual design, DECIGO is formed by three drag-free spacecraft, 1000km apart from one another, with observation frequency band of around 0.1 – 1.0Hz. As a result, this starlike configuration is identical to the pair of detectors whose arms are separated by 60 degrees with the separation $2/\sqrt{3} \times 10^6$m in the flat ground. Because the typical wave length of FP-DECIGO is around $10^8 – 10^9$m, much greater than the separation between detectors, the overlap reduction function is almost constant in observation frequency band. Then, we find that $\gamma(f) \sim 0.75$ in the low frequency limit under the starlike configuration in flat ground.

The signal processing of FP-DECIGO may adopt the same technique as used in the ground detectors. The essential requirement is that the relative displacement between the spacecrafts be constant during an observation. Adopting the Fabry-Perot configuration, while the arm-length of the detector can be greatly reduced without changing the observed frequency range, no flexible combination of time-delayed signal is possible anymore. We assume that the output data which is available for data analysis is only one for each set of detectors.

The sources of noise in FP-DECIGO consist of photon shot noise in the photo detector and acceleration noise from the drag-free system and radiation pressure noise. Each noise spectrum of FP-DECIGO is given in [24]. Photon shot noise is $N_{\text{shot}} = 4.8 \times 10^{-42} (L / \text{km})^{-2} f^{-2}(1 + (f/f_0)^2)$ Hz$^{-1}$, acceleration noise is $N_{\text{accl}} = 4.0 \times 10^{-46} (L / \text{km})^{-2} f^{-4} \text{Hz}^{-1}$, radiation pressure noise is $N_{\text{rad}} = 3.6 \times 10^{-51} f^{-4}[1 + (f/f_0)^2]^{-1} \text{Hz}^{-1}$. Here $f_0$ is the characteristic frequency given by $f_0 = 1/4\mathcal{F} L$ with the fineness of $\mathcal{F} = 10$ and $L$ is the arm-length and we assume $L = 1000$km, so that $f_0 = 7.5$Hz. The noise spectral density is given by the sum: $P_i(f) = N_{\text{shot}} + N_{\text{accl}} + N_{\text{rad}}$. The detailed discussion of noise and instrumental parameters of FP-DECIGO is given in [24]. The noise spectrum of LISA (including white-dwarf binaries background [26]) is taken from [27]. The results for FP-DECIGO are given in Fig. 3. Tables III and IV. We take $T = 10^7$ sec, $\Omega_N = 10^{-15}$ and $f_0 = 7.5$ Hz and assume $\gamma(f) = 0.75$ as mentioned above. The results for LISA are given in Fig. 4 and Table IV. There we take $T = 10^7$ sec, $\Omega_N = 10^{-12}$, $f_c = 10^{-4}$Hz and assume $\gamma(f) = 0.75$ with $f_c \ll 0.1 – 1$ Hz ($10^{-3} – 10^{-2}$ Hz). From Table IV, we find that the value of SNR in each filter is independent of the damping factor $d$. This is because the contribution of the integration for calculation of SNR Eq. (14) is the largest around 0.1 – 1Hz in all frequency ranges. On the others hand, Table III and Table IV show that all SNR is sensitive to the damping factor but is insensitive to the change of the spectrum. This is because $\Omega_{gw}$ is regarded as a flat spectrum with the damping factor $d$ around 0.1 – 1Hz ($10^{-3} – 10^{-2}$ Hz for LISA) in the case of $f_c \ll 0.1 – 1$ Hz ($10^{-3} – 10^{-2}$

\footnote{Using more realistic $\gamma(f)$ only reduces the SNR and the bandwidth of $\tilde{Q}(f)$ and does not affect the conclusion.}
FIG. 3: The value (top) and ratio (bottom) of SNRs for $\Omega_{\text{filter}} = \Omega_{\text{gw}}$ (solid line in top panel) and $\Omega_{\text{filter}} \propto f^0$ (dotted line in top panel) for FP-DECIGO. We set $d = 0.7$.

| damping rate: $d$ | 0.9 | 0.7 | 0.5 | 0.1 | $\Omega_{\text{filter}}$ |
|------------------|-----|-----|-----|-----|------------------|
| $\text{SNR}_{\text{max}}$ | 5.10 | 5.10 | 5.10 | 5.10 | $\Omega_{\text{gw}}$ |
| $\text{SNR}_{\text{flat}}$ | 5.10 | 5.10 | 5.10 | 5.10 | $\Omega_{\text{filter}}$ with $\alpha = 0$ |
| $\text{SNR}_{\text{PL}-1}$ | 4.27 | 4.27 | 4.27 | 4.27 | $\Omega_{\text{filter}}$ with $\alpha = -1$ |
| $\text{SNR}_{\text{PL}+1}$ | 4.60 | 4.60 | 4.60 | 4.60 | $\Omega_{\text{filter}}$ with $\alpha = +1$ |

TABLE II: SNRs for several templates in the setup of FP-DECIGO. We take $T = 10^7$ sec, $\Omega_{N} = 10^{-15}$, and $f_0 = 7.5$ Hz. $\text{SNR}_{\text{max}}$ represents SNR for $\Omega_{\text{filter}} = \Omega_{\text{gw}}$ with $f_c = 7.5$ Hz ($= f_0$) in Eq. (16). $\text{SNR}_{\text{flat}}$ for $\Omega_{\text{filter}} \propto f^0$, $\text{SNR}_{\text{PL} \pm 1}$ for $\Omega_{\text{filter}} \propto f^{\pm 1}$.

Hz for LISA). As a result, although SNRs themselves are enhanced because of much smaller noise power spectrum, we find no significant improvement in the use of the filter $\Omega_{\text{gw}}$.

IV. DISCUSSIONS AND SUMMARY

Motivated by the fact that the spectrum of gravitational waves has fine structures due to the change of the number of relativistic degrees of freedom, we have investigated the validity of the use of the templates of power-law shape as search templates for a stochastic gravitational-wave background. Comparing the SNR using the template of power-law shape and the SNR using the template of step function shape, we find that the resulting SNR is insensitive to the change of the amplitude.

Although we have focused on the change of the spectrum associated with electroweak phase transition, our analysis is not limited to it and is easily extended to other frequency
damping rate: $d$

| $\Omega_{\text{filter}}$ | $\Omega_{\text{gw}}$ | $\Omega_{\text{filter}}$ with $\alpha = 0$ | $\Omega_{\text{filter}}$ with $\alpha = -1$ | $\Omega_{\text{filter}}$ with $\alpha = +1$ |
|--------------------------|----------------------|---------------------------------------------|---------------------------------------------|---------------------------------------------|
| $\text{SNR}_{\text{max}}$ | 4.59                | 3.57                                        | 2.55                                        | 0.510                                        |
| $\text{SNR}_{\text{flat}}$ | 4.59                | 3.57                                        | 2.55                                        | 0.510                                        |
| $\text{SNR}_{\text{PL} - 1}$ | 3.75                | 2.92                                        | 2.08                                        | 0.417                                        |
| $\text{SNR}_{\text{PL} + 1}$ | 4.14                | 3.22                                        | 2.30                                        | 0.460                                        |

TABLE III: The same as Table I except $f_c = 10^{-4}$Hz in Eq.(16).

FIG. 4: The value (top) and ratio (bottom) of SNRs for $\Omega_{\text{filter}} = \Omega_{\text{gw}}$ (solid line in top panel) and $\Omega_{\text{filter}} \propto f^0$ (dotted line in top panel) for LISA. We set $d = 0.7$.

Our results have both bad news and good news. The bad news is that gravitational wave measurements with the amplitude close to their detection threshold are insensitive to the detailed structure of the spectrum. Although the spectrum beyond $10^{-3}$Hz depends on the particle physics beyond 1 TeV, gravitational waves measurements themselves do not

| $\Omega_{\text{filter}}$ | $\Omega_{\text{gw}}$ | $\Omega_{\text{filter}}$ with $\alpha = 0$ | $\Omega_{\text{filter}}$ with $\alpha = -1$ | $\Omega_{\text{filter}}$ with $\alpha = +1$ |
|--------------------------|----------------------|---------------------------------------------|---------------------------------------------|---------------------------------------------|
| $\text{SNR}_{\text{max}}$ | 2.44                | 1.90                                        | 1.36                                        | 0.276                                        |
| $\text{SNR}_{\text{flat}}$ | 2.44                | 1.90                                        | 1.36                                        | 0.272                                        |
| $\text{SNR}_{\text{PL} - 1}$ | 1.46                | 1.40                                        | 0.822                                       | 0.187                                        |
| $\text{SNR}_{\text{PL} + 1}$ | 2.32                | 1.80                                        | 1.29                                        | 0.258                                        |

TABLE IV: SNRs for several templates for LISA. We take $T = 10^7$ sec, $\Omega_N = 10^{-12}$ and $f_c = 10^{-4}$ Hz.
discern the difference: probing SUSY via gravitational-wave observations is not feasible if its amplitude is so small that a long time observation is required to achieve sufficient SNR as discussed here.

The spectrum of a stochastic gravitational wave background may not be determined by single observations. In order to determine the spectrum (in particular, the spectral index) of gravitational waves, multiple observations at different frequencies are required. If such observations are realized in the future, the change of the number of relativistic degrees of freedom may be detected assuming that the power of the spectrum is known independently.

The bad news becomes good news at the same time. That is, the templates of simple power-law shape are sufficient as search templates for stochastic gravitational waves: no detailed transfer function, which is dependent on particle physics models for $f > 10^{-3}\text{Hz}$, is necessary.

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