Fibonacci Oscillators in the Landau Diamagnetism problem

André A. Marinho\textsuperscript{a}, Francisco A. Brito\textsuperscript{b}, Carlos Chesman\textsuperscript{a}

\textsuperscript{a} Departamento de Física Teórica e Experimental, Universidade Federal do Rio Grande do Norte, 59078-970 Natal, RN, Brazil.

\textsuperscript{b} Departamento de Física, Universidade Federal de Campina Grande, 58109-970 Campina Grande, Paraíba, Brazil

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Abstract

We address the issue of the Landau diamagnetism problem via $q$-deformed algebra of Fibonacci oscillators through its generalized sequence of two real and independent deformation parameters $q_1$ and $q_2$. We obtain $q$-deformed thermodynamic quantities such as internal energy, number of particles, magnetization and magnetic susceptibility which recover their usual form in the limit of $q_1=q_2=1$.

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I. INTRODUCTION

The Landau diamagnetism problem continues to play a role in several issues of many physical systems and has strong relevance today [1–5]. The diamagnetism can be used as an illustrative phenomenon that plays essential role in quantum mechanics on the surface, the perimeter, and the dissipation of statistical mechanics of non-equilibrium.

In this paper, we are interested in investigating this phenomenon in $q$-deformed algebra in order to understand impurities effects in, for example, magnetization and susceptibility. The magnetic susceptibility is an intrinsic characteristic of each material and its identity is related to the atomic and molecular structure. In Ref. [6] it was performed the calculation of susceptibility for electrons moving in a uniform external magnetic field, developing Landau diamagnetism, by applying the nonextensive Tsallis statistics [7–10], which is a strong candidate for solving problems where the standard thermodynamics is not applicable — see also Ref. [11] for a similar study using another method. Of course, other noncommutative deformations can be applied, for example $q$-deformation derived via Jackson (JD) [12].

The study of quantum groups and quantum algebras has attracted great interest in recent years, stimulated intense research in various fields of physics [13, 14], taking into account a range of applications, covering cosmology and condensed matter, e.g. black holes, fractional quantum Hall effect, high-temperature (high-$T_c$) superconductors [15], rational field theories, noncommutative geometry, quantum theory of super-algebras and so on [16]. Furthermore, statistical and thermodynamic properties by studying $q$-deformed physical systems have been intensively investigated in the literature [17–31].

Another important discussion is about the main reasons to consider two deformation parameters in some different physical applications. Starting from the generalization of the $q$-algebra [32], in Ref. [33] it was generalizid the Fibonacci sequence, which is a well-known linear combination where the third number is the sum of two predecessors and so on. Here, the numbers is in that sequence of generalized Fibonacci oscillators, where there new parameters are $(q_1, q_2)$ are introduced [33–35]. They provide a unification of quantum oscillators with quantum groups, keeping the degeneration property of the spectrum invariant under the symmetries of the quantum group. The quantum algebra with two deformation parameters may have a greater flexibility when it comes to application in the concrete phenomenological physical models [36, 37], and may increase interest in physical applications.
The paper is organized as follows. In Sec. (II) we introduce the $q$-deformed algebra. In Sec. (III) we develop the $(q_1, q_2)$-deformed Landau diamagnetism problem and in the Sec. (IV) we make our final comments.

II. FIBONACCI OSCILLATORS ALGEBRA

We consider a system of generalized oscillators now entering two parameters in statistical distribution function, whose energy spectrum may be determined by Fibonacci’s generalized sequence \[33-35\]. This will establish a statistical system depending on the deformation parameters $(q_1, q_2)$, allowing us to calculate the thermodynamics quantities in the limit of high temperatures.

The $q$-deformed quantum oscillator is now defined by the Heisenberg algebra in terms the annihilation and creation operators in $c$, $c_1$, respectively, and the number operator $N$ \[19, 35\], as follows

\[
c_i c_i - K q_1^2 c_i c_i = q_2^{2n_i}, \quad e c_i c_i - K q_2^2 c_i c_i = q_1^{2n_i},
\]

\[
[N, c_i] = c_i, \quad [N, c] = -c,
\]

where $K = \pm 1$, stands for bosons and fermions, respectively. In addition, the operators also obey the relations

\[
c_i c = [N], \quad cc = [1 + KN],
\]

\[
[1 + K n_i, q_1, q_2] = K q_1^2 [n_i] + q_2^{2n_i}, \quad \text{or} \quad [1 + K n_i, q_1, q_2] = K q_2^2 [n_i] + q_1^{2n_i}.
\]

The Fibonacci basic number is defined as \[33\]

\[
[n_i, q_1, q_2] = c_i c_i = \frac{q_2^{2n_i} - q_1^{2n_i}}{q_2^2 - q_1^2}.
\]

The $q$-Fock space spanned by the orthonormalized eigenstates $|n\rangle$ is constructed according to

\[
|n\rangle = \frac{(c\dagger)^n}{\sqrt{[n]!}} |0\rangle, \quad c|0\rangle = 0,
\]
The actions of $c$ and $c^\dagger$ and $N$ on the states $|n\rangle$ in the $q$-Fock space are known to be

\begin{align}
    c^\dagger |n\rangle &= [n + 1]^{1/2} |n + 1\rangle, \\
    c |n\rangle &= [n]^{1/2} |n - 1\rangle, \\
    N |n\rangle &= n |n\rangle.
\end{align}

(7) (8) (9)

To calculate the $q$-deformation statistical occupation number, we begin with the Hamiltonian of $q$-deformed noninteracting oscillators (bosons or fermions) [16],

$$
    \mathcal{H}_{q_1,q_2} = \sum_i \left( \epsilon_i - \mu_{q_1,q_2} \right) N_i,
$$

(10)

where $\mu_{q_1,q_2}$ is the $(q_1,q_2)$-deformed chemical potential. It should be noted that this Hamiltonian is a two-parameter deformed Hamiltonian and depends implicitly on the deformation parameters $q_1$ and $q_2$, since the number operator is deformed via Eq. (5).

The mean value of the $(q_1,q_2)$-deformed occupation number can be calculated by

$$
    [n_i] \equiv \langle n_i \rangle = \frac{\text{tr}(\exp(-\beta \mathcal{H}) c_i^\dagger c_i)}{\Xi},
$$

(11)

$$
    [n_{i,q_1,q_2}] = \frac{z' (\exp(\beta \epsilon_i) - z')}{(\exp(\beta \epsilon_i) - q_2^2 z') (\exp(\beta \epsilon_i) - q_1^2 z')},
$$

(12)

where $z_{q_1,q_2} = \exp(\beta \mu_{q_1,q_2})$ is the fugacity of the system, and we shall use the notation $z_{q_1,q_2} = z'$. When $q_1 = q_2 = 1$, we find the usual form

$$
    n_i = \frac{1}{z^{-1} \exp(\beta \epsilon_i) - 1}.
$$

(13)

In the present application of the Fibonacci oscillators, we are interested in obtain new $(q_1,q_2)$-deformed thermodynamics quantities such as internal energy, magnetization, and magnetic susceptibility for the high-temperature case, i.e. the limit ($z \ll 1$).
III. FIBONACCI OSCILLATOR IN THE LANDAU DIAMAGNETISM

To explain the phenomenon of diamagnetism, we have to take into account the interaction between the external magnetic field and the orbital motion of electrons. Disregarding the spin, the Hamiltonian of a particle of mass $m$ and charge $e$ in the presence of a magnetic field $H$ is given by the expression

$$H = \frac{1}{2m} \left(p - \frac{e}{c} A\right)^2,$$

(14)

where $A$ is the vector potential associated with the magnetic field $H$ and $c$ is the speed of light in unit ($CGS$). Let us start to formalize the statistical mechanical problem by using the grand partition function with the parameters $q_1$ and $q_2$ inserted through Eq. (12), in the form

$$\ln \Xi = -\frac{2eHL^2}{hc} \sum_{n}^{\infty} \frac{L}{2\pi} \int_{-\infty}^{\infty} dk_z \frac{1}{(q_1^2 - q_2^2)^2} \left\{ \ln \left[ 1 - Kz'q_1^2 \exp(-\beta \epsilon) \right] (q_1^2 - 1) + \ln \left[ 1 - Kz'q_2^2 \exp(-\beta \epsilon) \right] (1 - q_2^{-2}) \right\},$$

(15)

where $k_z = -\infty, \cdots, \infty$, $\epsilon = \hbar k_z^2 / 2m + \hbar \omega (n + \frac{1}{2})$, $\omega = eH / mc$. However, our study is focused on the analysis of diamagnetism in the limit of high temperatures $(z' \ll 1)$. Thus, performing the sum and integrals, we find the partition function is written as follows

$$\ln \Xi = z' K^2 H C_1 \frac{2}{\sinh(\gamma)} + z'^2 K^3 H C_2 Q,$$

(16)

where $C_1 = eL^3 / 2\pi hc \lambda$, $C_2 = eL^3 / 2\pi \sqrt{2hc\lambda}$.

We note that Eq. (16) shows the $(q_1, q_2)$-deformation in the second term. In the first order does not appear $q$-deformation. It appears after considering at least the second order. Notice, however, as $(q_1 = q_2 = 1)$ the deformation ceases. In the following we shall calculate the $q$-deformed thermodynamic quantities.

A. $(q_1, q_2)$-deformed thermodynamical quantities

We obtain the number of particles $N$ by setting,

$$N = z' \frac{\partial}{\partial z'} \ln \Xi = \frac{z' K^2 H C_1}{\sinh(\gamma)} + \frac{z'^2 K^3 H C_2 Q}{2 \sinh(2\gamma)}.$$ 

(17)
We determine the internal energy, and we can write it in terms of \( N \), in the form

\[
U = -\frac{\partial}{\partial \beta} \ln \Xi = \frac{N \mu_B H \left[ C_1 \coth(\gamma) \sinh(2\gamma) + z' K C_2 \sinh(\gamma) \coth(2\gamma) Q \right]}{C_1 \sinh(2\gamma) + z' K C_2 \sinh(\gamma) Q}.
\] (18)

In Fig. 1 we have the behavior of internal energy \( U \) as a function of the magnetic field \( H \) and for some values of \( q_1 \) and \( q_2 \) - see caption. We note that all the curves have different maximum peaks (depending on the values adopted for \( q_1 \) and \( q_2 \)), and as the field magnetic \( H \) increases the curves exhibit the same behavior. We also have proven the symmetry between the oscillators, i.e. when \( q_1 = 1 \) and \( q_2 = 2 \) (black curve) and when \( q_1 = 2 \) and \( q_2 = 1 \) (red curve), they overlap.

The grand potential \( \phi \) is determined as

\[
\phi = -\frac{1}{\beta} \ln \Xi = - \left( \frac{z' K^2 H C_1}{\beta \sinh(\gamma)} + \frac{z'^2 K^3 H C_2 Q}{2 \sinh(2\gamma)} \right).
\] (19)

To determine the magnetization, we carried out the thermodynamical derivative by using Eq. (19), that gives

\[
M = -\frac{\partial \phi}{\partial H} = \frac{z' C_1 K^2 \left( 1 - \gamma \coth(\gamma) \right)}{\beta \sinh(\gamma)} - \frac{z' C_2 K^3 Q \left( 1 - \gamma \coth(2\gamma) \right)}{2\beta \sinh(2\gamma)}.
\] (20)
We can also eliminate the chemical potential through the number of particles $N$ and insert the Langevin functions

$$L(\gamma) = \coth(\gamma) - \frac{1}{\gamma}, \quad L(2\gamma) = \coth(2\gamma) - \frac{1}{2\gamma},$$

(21)

to rewrite the magnetization as

$$M = -\frac{N\mu_B \left[ C_1 \sinh(2\gamma)L(\gamma) + z'KC_2\gamma \sinh(\gamma)L(2\gamma) \right]}{C_1 \sinh(2\gamma) + z'KC_2 \sinh(\gamma)Q}.$$  

(22)

The results obtained for the deformed magnetization are very interesting, because we can compare it with experimental results obtained for superconducting materials (which are perfect diamagnetic materials) as a function of temperature variation [42], in order to strengthen the understanding of the $q$-deformation as a factor of impurity.

In Fig. 2 we have the magnetization curves ($M$) versus magnetic field ($H$) for some values of $q_1$ and $q_2$, and we note that some observations made for internal energy such as oscillators symmetry are also valid for the magnetization.

![Figure 2: $(q_1, q_2)$-deformed magnetization as a function of magnetic field $H$ for several choices of $q_1$ and $q_2$.](image-url)
Now, computing the susceptibility reads,

\[ \chi = \frac{\partial M}{\partial H} = \frac{N \beta \mu_B^2}{C_1 \sinh(2\gamma) + z'KC_2 \sinh(\gamma)Q} \left[ C_1 \sinh(2\gamma) \left( 2 \coth(\gamma) \mathcal{L}(\gamma) - 1 \right) + 
\right. \\
\left. + 2z'C_2 \sinh(\gamma)Q \left( 2 \coth(2\gamma) \mathcal{L}(2\gamma) - 1 \right) \right]. \quad (23) \]

In the limit of weak fields \( \gamma \ll 1 \) we have the leading term

\[ M = -\frac{2N \mu_B \sinh(\gamma) \cosh(\gamma)(C_1 + z'C_2 KQ)}{3(2C_1 \cosh(\gamma) + z'KC_2Q)}, \quad (24) \]

and thus, we have the susceptibility in zero field

\[ \chi_0 = -\frac{2\mu_B z'K^2(C_1 + z'C_2 KQ)}{3(2C_1 + z'KC_2Q)}. \quad (25) \]

IV. CONCLUSIONS

As in our previous works \([12, 38, 39]\), which we have shown that the \( q \)-parameter is associated with impurities in a sample, in particular diamagnetic materials, as in the present study, we put forward new results to strength this interpretation of the \( q \)-deformation.

In this work, we expand the application of \( q \)-calculation through two deformation parameters \((q_1, q_2)\), known as Fibonacci oscillators. We work in the limit of high temperatures (‘dilute gas’ \( z \ll 1 \)), and a \((q_1, q_2)\)-deformed partition function. In first order the results reported in the literature \([40, 41]\), are recovered. However, the \( q \)-deformation takes place at second order.

We note that the in the obtained results were found several interesting behaviors by just varying the values of \( q_1 \) and \( q_2 \). Of course, we performed a theoretical application, and it allows various assumptions. By comparing these results with similar experimental curves, one could understand how impurities could be entering into a material that affects, e.g., superconductivity such its critical temperature increases, which would be of great interest to whom that works with high \( T_c \) superconductors — see \([43]\) for a recent alternative theoretical investigation on these type of superconductors whose structure can be extended via \( q \)-deformation in order to introduce impurities.
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