Enumeration of $n$-fold tangent hyperplanes to a surface
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Abstract
For each $1 \leq n \leq 6$ we present formulas for the number of $n$–nodal curves in an $n$–dimensional linear system on a smooth, projective surface. This yields in particular the numbers of rational curves in the system of hyperplane sections of a generic $K3$–surface imbedded in $\mathbb{P}^n$ by a complete system of curves of genus $n$ as well as the number 17,601,000 of rational (singular) plane quintic curves in a generic quintic threefold.

1 Introduction

The purpose of this article is to present formulas for the number of $n$–nodal curves in an $n$–dimensional linear system on a smooth, projective surface for $1 \leq n \leq 6$. The method also yields formulas for the number of multi–tangent planes to a hypersurface. In particular, it enables us to find the number 17,601,000 of rational (singular) plane quintic curves in a generic quintic threefold. We give several examples and discuss the difficulties involved for $n \geq 7$.

Our motivation was in response to a question asked by A. Lopez and C. Ciliberto regarding the number of rational curves in the system of hyperplane sections in a generic $K3$–surface imbedded in $\mathbb{P}^4$ (resp. $\mathbb{P}^5$) as a $(2,3)$ (resp. $(2,2,2)$)–complete intersection. In [6] (joint with Miranda) they study degenerations $K3 \rightarrow$ union of 2 scrolls. According to A. Lopez (priv. comm.), the consideration of limit curves in the scrolls suggests a formula for the number of rational curves in the $K3$–surface. However, the numbers they have found are so far in disagreement with those obtained by the formulas presented here for $n = 4, 5, 6$ (cf. 5.3).

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A similar question communicated by S. Katz concerns the number of plane rational curves of degree 5 contained in a generic quintic 3-fold in $\mathbb{P}^4$.

The subject was raised by Clemens [7] and has received striking contributions from physicists (cf. Morrison [22], Piene [23], Bershadski et al. [5]). The total number of rational curves of degree $\geq 4$ has not been verified so far au goût du jour des mathématiciens. The cases of degrees 1 and 2 were treated by Harris [14] and Katz [15]. As for degree 3, it required a thorough investigation of the Chow ring of the variety of twisted cubics (cf. Ellingsrud and Strømme [10], [11] (see also [12] for a simpler approach). A pleasant byproduct was the development of the computer package SCHUBERT by Katz and Strømme [16].

The work of Coray [9] reduces certain enumerative questions concerning rational curves in $\mathbb{P}^3$ to the question of finding the numbers $\Delta_{\mu,\nu}$ of irreducible rational curves of bidegree $(\mu, \nu)$ passing through $2(\mu + \nu) - 1$ general points on a quadric surface. He computes $\Delta_{2,3}$ and $\Delta_{2,4}$ (in addition to a few trivial cases). We also obtain here $\Delta_{3,3}$ (5.2.1.4), $\Delta_{2,5}$ (5.2.1.5), and $\Delta_{3,4}$ (5.2.3.2).

Counting hyperplanes multi–tangent to a curve is well known as a particular case of the classical formula of De Jonquières [3], [28].

For surfaces, the cases $n \leq 3$ are classical and have been checked with currently standard tools of intersection theory, cf. Kleiman [17], [28]. The degrees of the “Severi varieties” of nodal curves in the plane were computed (in principle) by Ran in [25], [26].

Although we have at our disposal multiple point formulas (Kleiman [18], Ran [24]), they do not give the correct answer for multi–tangencies already for $n = 2$ or 3 due to the presence of cusps. There are also formulas taking into account stationary multiple-points (Colley [8]). However, for $n \geq 4$ the relevant map does not satisfy a required curvilinearity hypothesis. This is due to the existence of curves with a triple point in virtually any linear system of dimension $\geq 4$ on a surface.

Our approach is based on the iteration procedure presented in [27], [28] (also explored in a broader context in [15], [24], [8]).

We obtain, for each $n = 1, \ldots, 6$ a formula for the degree of a zero cycle supported on the set of sequences $(C, y_1, \ldots, y_n)$ such that $C$ is a member of a (sufficiently general) linear system of dimension $n$ and $y_1$ is a singular point of $C$, $y_2$ is a singular point of the blowup of $C$ at $y_1$, and so on (roughly speaking, cf. §2 and [33], [6] for the precise statement).
The main novelty here is, essentially, detecting the contribution to that zero cycle due to singularities worse than nodes (cf. [4.1]). We also sharpen the scope of validity of the formulas, now requiring only that the relevant loci be finite (3.3).

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2 Notation and basic definitions

We recall, for the reader’s benefit, some definitions from [28]. Let \( Y \) be a smooth variety. For each sequence of integers \( m = (m_1, \ldots, m_r) \) we say an effective divisor \( D \) has a singularity of (weak) type \( m \) if the following holds:

- there is a point \( y_1 \) of multiplicity \( \geq m_1 \) in \( D \); next
- blowup \( Y \) at \( y_1 \), let \( E_1 \) denote the exceptional divisor and let \( D_1 \) denote the total transform of \( D \); then
- require that the effective divisor \( D_1 - m_1 E_1 \) have a point \( y_2 \) of multiplicity \( \geq m_2 \), and so on.

The sequence \((y_1, y_2, \ldots)\) thus constructed is called a singularity of type \( m \) of \( D \). We further say the type is strict if all inequalities are equalities and each \( y_i \) lies off the exceptional divisor.

One may also consider nested sequences \((\ldots, m_i (m_i+1, \ldots), \ldots)\) and say a singularity is of such type if \( y_i \) is of multiplicity \( \geq m_i \) and \( y_{i+1} \) is infinitely near to \( y_i \), i.e., lies on the exceptional divisor besides being of multiplicity \( \geq m_{i+1} \), etc. We write \( m^{[k]} \) to indicate \( k \) repetitions of \( m \).

2.1 Example. Let \( Y \) be a surface and \( y_1 \) a triple point on the curve \( C \). Then of course \( C \) has a singularity of strict type (3). However, if the 3 tangents are distinct, \( C \) also has a singularity of weak type (2\(^4\)) due to the intersections of the strict transform of \( C \) and the exceptional line \( E_1 \):
2.2 Example. On the other hand, if $y_1$ is of type $(3(2))$, it follows that $C$ has a singularity of type $(2^{[6]})$!

Indeed, let $y_2$ be the double point infinitely near to the triple point $y_1$, and let $C_1$ denote the total transform of $C$; then $C_1 - 3E_1$ is effective and intersects $E_1$ twice at $y_2$ and once at the (smooth) branch $y_3$. Thus, the divisor $C' := C_1 - 2E_1$ has multiplicity 3 at the point $y_2$. Blowing it up, let $C_2$ be the total transform of $C'$; now $C_2 - 2E_2$ still contains the exceptional line $E_2$ once and therefore has 4 double points: one for the intersection of $E_2$ and the strict transform of $E_1$, two for the branches over $y_2$ and finally one over $y_3$.

2.3 Example. Let $Y$ be a surface and $y_1$ a fourfold point on the curve $C$. Then $C_1 - 2E_1$ is nonreduced, hence $C$ has a singularity $(y_1, \ldots, y_r)$ of type $(2^{[r]})$ for any $r$.

This illustrates a main difficulty in our approach to enumeration of singularities. Formulas for a given type are usually not hard to obtain, at least in principle (cf. (3) below), but the exact contribution of each strict type actually occurring seems less evident. For the case we’re interested in, we have the following description of the possible singularity types.

2.4 Proposition. Let $Y$ be a smooth surface; fix $n \in \{1, \ldots, 6\}$. Let $D$ be an ample divisor on $Y$. Then there exists $r_0$ such that for all $r \geq r_0$ and any sufficiently general linear subsystem $S$ of $|rD|$ of dimension $n$, there are at most finitely many members $C \in S$ with a singularity of type $(2^{[n]})$. Moreover, we have the following list of possible strict types actually occurring in type $(2^{[n]})$:

- $n \leq 3 \Rightarrow (2^{[n]})$ only;
- $n = 4 \Rightarrow (2^{[4]})$ or (3);
- $n = 5 \Rightarrow (2^{[5]})$ or (3, 2) or (2, 3);
- $n = 6 \Rightarrow (2^{[6]})$ or $(3(2))$ or any of $(3, 2, 2), (2, 3, 2), (2, 2, 3)$.
Proof. Set $\mathcal{L} = \mathcal{O}(D)$ and let $\mathcal{M}_y$ be the ideal sheaf of a point $y \in Y$. The members of $|D|$ with an $m$-fold point at $y$ come from $H^0(Y, \mathcal{M}_y^m \otimes \mathcal{L})$. Let $Y^O_n$ denote the complement of the diagonals in $Y^{\times n}$. Given a sequence of positive integers, $(m_1, \ldots, m_n)$, replacing $\mathcal{L}$ by a sufficiently high power, we may assume $H^1(Y, \mathcal{M}_y^{m_1} \mathcal{M}_y^{m_2} \otimes \mathcal{L}) = 0$ for all $(y_1, \ldots, y_n) \in Y^O_n$. It follows that the set

$$\{(C, y_1, \ldots, y_n) \in |D| \times Y^O_n \mid \text{mult}_{y_i} C \geq m_i \}$$

is a projective bundle over $Y^O_n$ with fibre dimension $\dim |D| - \sum m_i (m_i + 1)/2$. Its image in $|D|$ is of codimension $\sum m_i (m_i + 1)/2 - 2$. Therefore no sufficiently general subsystem of dimension $\leq 3$ (resp. $\leq 7$) has a member with a triple (resp. 4-fold) point. It can be easily checked that a singularity of type $(2(2))$ (i.e., a double point with another infinitely near) (resp. $(2(2), 2)$) imposes 3 (resp. 4) independent conditions.

Let $(y_1, \ldots, y_6)$ be a singularity of weak type $(2^6)$ occurring in a general $\infty^6$ linear system. As explained just above, a 4-fold point imposes 8 conditions, so each $y_i$ is at worst a triple point. Moreover, it can be checked that 2 triple points (infinitely near or not) impose at least 8 conditions, thus at most one of the $y_i$ is triple. We claim that $y_i$ cannot be a triple point unless $i \leq 3$. Indeed, the imposition of 3 double points $(y_1, y_2, y_3)$ costs at least 3 parameters, leaving less than the 4 required for the acquisition of an additional triple point.

A similar argument rules out other sequences of double points (with some possibly infinitely near) different from those listed. \hspace{1cm} \Box

3 Basic setup

Let $f : X \to S$ be proper and smooth. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module and let $D \subset X$ be the scheme of zeros of a section of $\mathcal{L}$. As in [28], we construct a scheme $\Sigma(m; D)$ whose fibre over each $s \in S$ consists of the sequences of singularities of type $m$ of the fibre $D_s$.

Set $X_0 = S$, $X_1 = X$, $f_1 = f : X_1 \to X_0$. For $r \geq 1$ denote by

$$b_{r+1} : X_{r+1} \to X_r \times_{f_r} X_r \quad \text{and} \quad p_{r+1,i} : X_r \times_{f_r} X_r \to X_r,$$

respectively the blowup of the diagonal and the projection.
Set $f_{r+1,i} = p_{r+1,i} \circ b_{r+1}$. We think of each $X_r$ as a scheme over $X_{r-1}$ with structure map $f_r = f_{r,1}$.

Write $E_{1,r}$ for the exceptional divisor of $b_r$.

For $2 \leq j < r$ set $E_{r-j+1,j} = f_{r,2}^* \cdots f_{j+1,2}^* E_{1,j}$. By abuse, still denote by the same symbol pullbacks of $E_{r-j+1,j}$ via compositions of the structure maps $f_3, f_4, \ldots$. Notice the second index in $E_{r-j+1,j}$ indicates where the divisor first appears in the sequence of blowups, whereas $r - j$ keeps track of the number of pullbacks via the $f_{k,2}$.

For each sequence of nonnegative integers $m = (m_1, \ldots, m_r)$ we define the divisor on $X_{r+1}$,

$$mE = m_rE_{1,r+1} + \cdots + m_2E_{r-1,3} + m_1E_{r,2}.$$ 

Let $y_1 \in X_1$ lie over $s \in X_0$. Notice that, by construction, the fibre $X_{2y_1}$ of $f_2$ over $y_1$ is equal to the blowup of the fibre $f_1^{-1}(s)$ at $y_1$. By the same token, a point in $X_r$ lying over $s$ should be thought of as a sequence $(y_1, \ldots, y_r)$ of points in $f_1^{-1}(s)$ each possibly infinitely near to a previous one. Also, the fibre of $mE$ over a point $(y_1, \ldots, y_r) \in X_r$ is equal to $m_rE_{y_r} + \cdots + m_1E_{y_1}$, where $E_{y_i} \subset X_{i+y_i}$ denotes (for $i < r$, the total transform of) the exceptional divisor of the blowup of $X_{iy_i-1}$ at $y_i$. We set

$$\mathcal{L}(m) = f_{r+1,2}^* \cdots f_{2,2}^* \mathcal{L} \otimes \mathcal{O}_{X_{r+1}}(-mE).$$

Pulling back the section of $\mathcal{L}$ defining $D$, we get the diagram of maps of $\mathcal{O}_{X_{r+1}}$-modules,

$$
\begin{array}{ccc}
\mathcal{O}_{X_{r+1}} & \xrightarrow{\sigma^D_m} & \mathcal{O}_{X_{r+1}}^* \\
\downarrow & & \downarrow \\
f_{r+1,2}^* \cdots f_{2,2}^* \mathcal{L} & \rightarrow & f_{r+1,2}^* \cdots f_{2,2}^* \mathcal{L} \otimes \mathcal{O}_{mE}.
\end{array}
$$

By construction, $\sigma^D_m$ vanishes on a fibre $f_{r+1}(y_1, \ldots, y_r)$ iff $y_1, \ldots, y_r$ is a singularity of type $m$ of $D_s$, where $s = f_1(y_1)$.

We define the $m$-contact sheaf as the $\mathcal{O}_{X_r}$-module,

$$\mathcal{E}(m, \mathcal{L}) = f_{r+1}^*(\mathcal{O}_{mE} \otimes f_{r+1,2}^* \cdots f_{2,2}^* \mathcal{L}).$$

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3.1 Lemma. Notation as above, we have:

1. \( \mathcal{E}(m, \mathcal{L}) \) is a locally free \( \mathcal{O}_{X,r} \) module of rank \( \Sigma \left( \frac{d_{mf} + m_i - 1}{d_{mf}} \right) \) and its formation commutes with base change;

2. there are exact sequences,

\[
0 \rightarrow \mathcal{E}(m, \mathcal{L}) \rightarrow \mathcal{E}(m', \mathcal{L}) \rightarrow f^* \mathcal{E}(m', \mathcal{L}) \rightarrow 0,
\]

where \( m' \) denotes the truncated sequence \( (m_1, \ldots, m_{r-1}) \);

3. we have \( \mathcal{E}(1, \mathcal{L}) = \mathcal{L} \) and for \( \mu \geq 2 \) we have an exact sequence,

\[
0 \rightarrow \mathcal{L} \otimes \text{Sym}^{\mu-1} \Omega^1_{X/S} \rightarrow \mathcal{E}(\mu, \mathcal{L}) \rightarrow \mathcal{E}(\mu - 1, \mathcal{L}) \rightarrow 0.
\]

Proof. The inclusion \( f_{r+1,2}^* m'E \subset mE \) yields the exact sequence

\[
0 \rightarrow \mathcal{O}_{mE}(-f_{r+1,2}^* m'E) \rightarrow \mathcal{O}_{mE} \rightarrow \mathcal{O}_{f_{r+1,2} m'E} \rightarrow 0
\]

\[
\mathcal{O}_{m, E_{1,r+1}}(-f_{r+1,2}^* m'E)
\]

Notice \( f_{r+1,2}^* m'E \) and \( m, E_{1,r+1} \) are \( f_{r+1,1} \)-flat. Indeed, for a divisor such as \( E_{2,r} := f_{r+1,2}^* E_{1,r} \) which intersects the blowup center \( \Delta(X_r) \) properly (along \( \Delta(E_{1,r}) \)), the total and strict transforms are one and the same. Thus, to show \( f_{r+1,1} \)-flatness of \( E_{2,r} \) it suffices to verify that each power of the ideal sheaf of \( \Delta(E_{1,r}) \) in \( p_{r+1,2}^* E_{1,r} \) is \( p_{r+1,1} \)-flat. This is a consequence of the following.

3.2 Lemma. Let \( p : X \rightarrow Y \) be a smooth map of smooth varieties. Let \( Z \subset X \) be a smooth subvariety of \( X \) such that the restriction of \( p \) induces an isomorphism \( Z \cong p(Z) \) onto a hypersurface of \( Y \). Let \( \mathcal{I} \) denote the ideal of \( Z \) in \( X \). Then each power \( \mathcal{I}^m \) is \( p \)-flat.

Proof. We assume for simplicity \( \dim p=1 \) (hence \( \text{codim}(Z, X) = 2 \)). There is a local representation of \( p \) by a ring homomorphism \( \mathcal{A} \rightarrow \mathcal{B} \) fitting into a commutative diagram,

\[
\begin{array}{ccc}
\mathcal{A} & \rightarrow & \mathcal{B} \\
\uparrow & & \uparrow \\
\mathcal{C} : = \mathcal{R}[u] & \rightarrow & \mathcal{D} : = \mathcal{R}[u, v]
\end{array}
\]
such that the vertical maps are étale, \( \mathcal{R} \) is regular, \( u, v \) denote indeterminates and the image of \( u \) (resp. \( u, v \)) generates the ideal of \( p(Z) \) (resp. \( Z \)) (cf. [1], p. 128–130). Under these circumstances, let \( \mathcal{M} \) be a \( \mathcal{D} \)-module flat/\( \mathcal{C} \). Then \( \mathcal{B} \otimes_\mathcal{D} \mathcal{M} \) is flat/\( \mathcal{A} \). Indeed, put \( \mathcal{A}' := \mathcal{A} \otimes_\mathcal{C} \mathcal{D} \); clearly \( \mathcal{M}_{\mathcal{A}} := \mathcal{A} \otimes_\mathcal{C} \mathcal{M} \) is an \( \mathcal{A}' \)-module flat/\( \mathcal{A} \). Notice \( \mathcal{A} \to \mathcal{B} \) factors as \( \mathcal{A} \to \mathcal{A}' \to \mathcal{B} \) and \( \mathcal{B} \) is étale, hence flat/\( \mathcal{A}' \). Let \( \mathcal{J} \subset \mathcal{A} \) be an ideal. We have \( 0 \to \mathcal{J} \otimes_\mathcal{A} \mathcal{M} \to \mathcal{M} \) exact. Hence

\[
0 \to \mathcal{B} \otimes_\mathcal{A}' \mathcal{J} \otimes_\mathcal{A} \mathcal{M} \to \mathcal{B} \otimes_\mathcal{A} \mathcal{M}
\]

is exact by flatness of \( \mathcal{B} / \mathcal{A}' \). Apply this to the ideal \( \mathcal{M} = (u, v) \), which is a flat, in fact free \( \mathcal{C} \)-module with basis \( \{ u, \ldots, u^{m-1}, v^m, \ldots \} \). \( \square \)

The same argument applies to all \( E_{j,r-j+1} \). Since a sum of flat divisors is flat, we’ve proved that \( mE \) is \( f_{r+1,1} \)-flat.

Tensoring \( (2) \) with \( f_{r+1,2} \cdots f_{2,2} \mathcal{L} \) and pushing forward by \( f_{r+1} = p_{r+1,1}b_{r+1} \), the assertions follow by a standard base change argument (cf. [28], p. 411).

\( \square \)

3.3 Proposition. Let \( \Sigma(m; D) \subset X_r \) be the scheme of zeros of the section \( \sigma^D_m : \mathcal{O}_{X_r} \to \mathcal{E}(m, \mathcal{L}) \) defined in (1). Then:

1. \( \Sigma(m; D) \) is equal to the scheme of zeros of \( \sigma^D_m \) along the fibres of \( f_{r+1} \), thus parametrizing the singularities of type \( m \) of the fibres of \( D \);

2. with notation as in Lemma 3.1, setting \( D' = f_{r+1,2}(f_{r+2,2} \cdots f_{2,2} D - m'E) \) restricted over \( \Sigma(m'; D) \), we have

\[
\Sigma(m; D) = \Sigma((m_r'; D')); \\
\]

3. each component of \( \Sigma(m; D) \) is of codimension \( \leq \rho = \Sigma \left( \frac{\dim f_{r+1,m} - 1}{\dim f_{r+1}} \right) \);

4. if \( \Sigma(m; D) \) is empty or of the right codimension \( \rho \) then its class in the Chow group of \( X_r \) is given by the formula,

\[
[\Sigma(m; D)] = c_{\rho}(\mathcal{E}(m, \mathcal{L})) \cap [X_r].
\]

(3)
Proof. The 1st assertion follows from [2], Prop.(2.3). The 2nd one derives from the exact sequence in Lemma 3.1(2). The remaining are well known facts (cf. Fulton[13]).

3.3.1 Remark. In practice, the formula (3) may be computed using the exact sequences in 3.1. However, it is only useful to the extent the conditions of (3.3)4 are met; we then say $D$ is $m$-regular. We refer to 28 for sufficient conditions for $m$-regularity.

3.4 Proposition. Let $D \subset X\rightarrow S$ be as in the beginning of §3. Set $S' = \Sigma(2,S)$. Fix $P \in D$. Assume that

1. $S$ is regular at the image of $P$;
2. the “total space” $D$ is smooth at $P$ and
3. the fibre of $D$ through $P$ has an ordinary double point (odp) there.

Then we have that $S'$ is smooth at $P$. Moreover, $D' := f_{2,2}^*D|_{S'} - 2E_{1,2}|_{S'}$ is regular along the inverse image of $P$.

Proof. We assume for simplicity $\dim X/S = 2$ and $\dim S \geq 1$. The question is local analytic. Let $A$ be a regular local ring and $\mathcal{M}$ its maximal ideal, let $h \in B = A[[x_1,x_2]]$ and set $\mathcal{N} = \mathcal{M}B + (x_1,x_2)B$. Assume that $B/(h)$ is regular and $h = x_1x_2 \mod (x_1,x_2)^3 + \mathcal{M}B$. Then $\bar{B} := B/(h, h_{x_1}, h_{x_2})$ is regular. Indeed, we may write $h = t + m_1x_1 + m_2x_2 + x_1x_2 + \cdots$, with $t, m_1, m_2 \in \mathcal{M}$. Notice that, since $h \in \mathcal{N} - \mathcal{N}^2$, we have in fact $t \in \mathcal{M} - \mathcal{M}^2$. From $h_{x_i} = m_i + x_j + \cdots (i,j) = (1,2)$, it follows that $h, h_{x_1}, h_{x_2}$ are linearly independent mod $\mathcal{N}^2$, as desired for the regularity of $\bar{B}$.

Let $t_1 = t, \ldots, t_n$ generate $\mathcal{M}$ minimally. We may replace $S$ by the germ of curve defined by $t_2, \ldots, t_n$. Thus $t$ is a uniformizing parameter of $A$.

Since the map germ of $D\rightarrow S$ has an ordinary quadratic singularity at $P$, there are regular parameters $\tilde{x}_1, \tilde{x}_2$ of $D$ around $P$ such that $t \mapsto \tilde{x}_1\tilde{x}_2$.

So now we have reduced to the following. The completion of the local ring of $S$ at the image of $P$ may be written as $A[[t]]$ for some power series ring $A$. The completion of the local ring of $X$ (resp. $D$) at $P$ is of the
form \( B = A[[t, x_1, x_2]] \) (resp. \( B/(t - x_1x_2) \)). Hence \( S' \) is represented by the ideal \((t, x_1, x_2) \subset B\). The diagonal and the fibre product \( X \times_S X \) are represented by \((x_1 - x'_1, x_2 - x'_2) \subset A[[t, x_1, x_2, x'_1, x'_2]]\). The blowup \( X_2 \to X \times_S X \) is given by the inclusion \( A[[t, x_1, x_2, x'_1, x'_2, u]] \subset A[[t, x_1, x_2, x'_1, x'_2, u]] \) defined by \( x'_2 = x_2 + u(x'_1 - x_1) \). Restriction over \( S' \) therefore takes on the form, \( A \to A[[x'_1, x'_2]] \subset A[[x'_1, u]] \), with \( D' \) defined by \( u \).

3.5 Proposition. Let \( Y \) be a smooth, projective surface and let \( D \) be an ample divisor on \( Y \). Fix \( n \in \{1, \ldots, 6\} \). Then there exists an integer \( r_0 \) such that, for all \( r \geq r_0 \), for all linear subsystems \( S \) of \( |rD| \) of dimension \( n \) in an open dense subset of the appropriate grassmannian, the following holds:

\[ \Sigma((2^n); S) \text{ is finite, reduced, and for } (C, y_1, \ldots, y_n) \in \Sigma((2^n); S) \text{ we have that } (y_1, \ldots, y_n) \text{ is a singularity of one of the strict types described in Prop.2.4}. \]

Proof. As observed in the proof of Prop.2.4, ampleness ensures that for any fixed sequence \( m = (m_1, \ldots, m_n) \) of positive integers there exists \( r_0 \) such that, for all \( r \geq r_0 \), and for any sequence \((y_1, \ldots, y_n)\) of distinct points in \( Y \), the sheaf \( M^{m_1}_{y_1} \cdots M^{m_n}_{y_n} \otimes O(rD) \) is generated by global sections. It follows that distinct \( y_i \)'s impose independent conditions to be a singularity of strict type \( m \) on the system \( |rD| \) and in fact, \( \Sigma(m, rD) \) restricted to the complement of the union of the exceptional divisors in \( Y_n \) is a projective bundle. In [28] ((9.1),p. 417) it is shown the same is true over all of \( Y_n \) provided \( m \) satisfies the relaxed proximity inequalities \( m_i \geq -1 + m_{i+1} + \cdots + m_n \) for \( i = 1, \ldots, n - 1 \).

As this is no longer the case for \( m = (2^n), \ n \geq 3 \), a direct verification of smoothness is required.

At any rate, \( \Sigma((2); S) \) and \( \Sigma((2, 2); S) \) are smooth for all sufficiently ample complete system \( S \) and remain smooth upon replacing \( S \) by a general subsystem by transversality of a general translate [19].

For \( n \geq 3 \) we proceed by the following iteration argument. Recall from Prop.3.3 that for any \( D \subset X \to S \) as in 13, we have

\[ \Sigma((2^3); D) = \Sigma((2); D'), \]

where \( D' = (f_{3,2}^*f_{2,2}^*D - 2E_{1,2} - 2E_{1,3}|_{\Sigma((2, 2); D)}). \)
If $S$ is a sufficiently ample complete system, one checks that $D'$ is regular. In fact, it is the total space of a family of basepoint–free divisors in the fibres of $Y_3 \to Y_2$. Indeed, let $Y' \to Y$ be the blowup at $y_1 \in Y$ and let $Y'' \to Y'$ be the blowup at $y_2 \in Y'$. Let $y_3 \in Y''$. Let $L$ be an ample line bundle over $Y$. Then

$$H^1(Y'', L^\otimes r \otimes O_{Y''}(-2E_{y_2}) \otimes O_{Y'}(2E_{y_1}) \otimes M_{y_3}) = 0$$

for $r >> 0$ because the sequence $(2, 2, 1)$ satisfies the relaxed proximity inequalities. Hence Prop. 3.4 implies that $\Sigma((2^{[3]}, D))$ is regular at any $(C, y_1, y_2, y_3)$ such that $y_3$ is an odp of $C - 2E_{y_1} - 2E_{y_2}$. Now, if $y_3$ were a triple point (allowed if $n = 6$), then we would certainly have $y_3$ not infinitely near $y_2$. Let $\pi$ be the involution of $X_2 \times_X X_2$ (so that $p_{3,2}\pi = p_{3,1}$). It lifts to an involution of $X_3$ that leaves $\Sigma((2^{[3]}, D))$ invariant. Since $\pi$ maps $(C, y_1, y_2, y_3)$ to $(C, y_1, y_3, y_2)$, we get regularity at the latter as well. The same argument yields regularity of $\Sigma((2^{[n]}, D))$ for $n = 4, 5, 6$ and $S$ generic, $\infty$. For instance, to show $\Sigma((2^{[6]}, D))$ is regular at $(C, y_1, \ldots, y_6)$ such that $y_1$ is of strict type $(3(2))$ and $y_2$ is the double point infinitely near $(cf 2.2)$, we argue by regularity of $\Sigma((2^{[3]}, D))$ at $(y_1, y_2)$ and apply iteration, observing that $(y_3, \ldots, y_6)$ are all ordinary quadratic singularities. If $y_2$ were the intersection of the exceptional line and the smooth branch, then $y_3$ must be the double point infinitely near to $y_1$. In this case apply first a permutation and argue as before. \hfill \Box

4 Applications

Here are two situations we may apply the above constructions to.

1: Linear systems. Let $Y$ be a smooth projective surface, let $\mathcal{M}$ be an invertible $\mathcal{O}_Y$–module and let $V \subset H^0(Y, \mathcal{M})$ be a subspace. Set $S = \mathbb{P}(V^*)$, $X = S \times Y$ and let $f : X \to S$ be the projection. Then $\mathcal{L} = \mathcal{M} \otimes O(1)$ has a section defining the universal divisor $D$ of the linear system parametrized by $S$. We also write in this case, $\Sigma(m; S) := \Sigma(m; D)$.

2: Hypersurfaces. Let $S = Gr(2, N)$ be the Grassmann variety of planes in $\mathbb{P}^N$, with tautological quotient sheaf $O^{N+1} \to Q$, where $\text{rank} Q = 3$. Let $X = \mathbb{P}(Q) \subset S \times \mathbb{P}^N$ be the universal plane in $\mathbb{P}^N$. Set $\mathcal{L} = O_{\mathbb{P}(Q)}(d)$ and let $D \subset X$ be defined by a form of degree $d$. Thus the fibre of $D$ over $s \in S$ is the intersection of a fixed hypersurface with the plane $s$ represents.
Using Prop. 3.5 we get the following formulas for the number $t_{g_n}$ of $n$-nodal curves in an $\infty^n$ family of curves, for $n \in \{1, \ldots, 6\}$.

4.1 Proposition. Fix $n \in \{1, \ldots, 6\}$. Let $D \subset X \to S$ be a family of curves in a smooth family of surfaces of dimension $n$. Assume $\Sigma((2^n); D)$ is reduced and receives contributions only from the strict types described in Prop. 2.4. Then we have:

\begin{align*}
t_{g_n} &= (\# \Sigma((2^n); S))/n! \quad \text{for } n \in \{1, 2, 3\}; \\
t_{g_4} &= (\# \Sigma((2^4); S) - 6\# \Sigma((3); S))/4!; \\
t_{g_5} &= (\# \Sigma((2^5); S) - 30\# \Sigma((3, 2); S))/5!; \\
t_{g_6} &= (\# \Sigma((2^6); S) - 30\# \Sigma((3(2)); S) - 90\# \Sigma((3, 2, 2); S))/6!.
\end{align*}

Proof. Let us explain for instance the coefficient 90 appearing in the formula for $t_{g_6}$. Pick $(C, z_1, z_2, z_3)$ in $\Sigma((3, 2); S)$. Here $C$ is a curve in the system $S$ and $(z_1, z_2, z_3)$ is a singularity of strict type $(3, 2, 2)$. Let $z_{11}, z_{12}, z_{13}$ be the branches over $z_1$. It gives rise to the following list of singularities $(y_1, \ldots, y_6)$ of weak type $2^6$ on $C$:

\begin{align*}
y_1 &= z_1 \quad \text{and } (y_2, \ldots, y_6) = \text{any permutation of } \{z_2, z_3, z_{11}, z_{12}, z_{13}\} \\
&\quad \text{SUBTOTAL: 120.}
\end{align*}

\begin{align*}
y_1 &= z_i, \quad y_2 = z_1 \quad \text{and } (y_3, \ldots, y_6) = \text{any permutation of } z_j, z_{11}, z_{12}, z_{13} \quad \text{with } \{i, j\} = \{2, 3\} \\
&\quad \text{SUBTOTAL: 48.}
\end{align*}

\begin{align*}
y_1 &= z_i, \quad y_2 = z_j, \quad y_3 = z_1 \quad \text{and } (y_4, y_5, y_6) = \text{any permutation of } z_{11}, z_{12}, z_{13} \quad \text{with } \{i, j\} = \{2, 3\} \\
&\quad \text{SUBTOTAL: 12.}
\end{align*}

The factor $180/2$ comes from the fact that $(C, z_1, z_2, z_3)$ and $(C, z_1, z_3, z_2)$ yield the same contributions to $\Sigma((2^6); S)$.

Using the formula (2) in Prop. 3.3 the rhs can be computed in terms of Chern classes for each of the two situations envisaged above. We’ve made extensive use of MAPLE [21] & SCHUBERT [16]). See the appendix for the computations.
5 Surfaces

For the case of linear systems on a surface \( Y \), setting for short,

\[
c_2 = \text{degree}(c_2 \Omega_Y^1), \quad k_1 = \text{degree}(c_1 \Omega_Y^1 \cdot c_1 \mathcal{L}),
\]
\[
k_2 = \text{degree}((c_1 \Omega_Y^1)^2), \quad d = \text{degree}(c_1 \mathcal{L}^2).
\]

we get from (3.1), (1.1) and (3),

\[
tg_1 := 3d+2k_1+c_2;
\]
\[
tg_2 := (tg_1(7+3d+2k_1+c_2) - 6k_2-25k_1-21d)/2;
\]
\[
tg_3 := (2tg_2(-14+3d+2k_1+c_2)+tg_1(-6k_2-25k_1-21d)2-63d+40) - (-6k_2-25k_1-21d)2d-63d^2 + (-18k_2-117k_1+672)2d - 50k_2^2 + (-12k_2+894k_1+950)k_1 + 292k_2)/6;
\]
\[
tg_4 := (81d^4 + (216k_1+108c_2-2268)d^2 + (54c_2^2 + (216k_1-1890)c_2 - 324k_2 + 21852 - 5130k_1 + 216k_1^2)d^2 + (12c_2^2 + (-504+72k_1)c_2^2 + (-216k_2+8940+144k_1^2 - 2916k_1)c_2 - 3816k_2^2 + 39780k_1 + 96k_1^2 + 6024k_2^2 - 72360 - 432k_2k_3)d^2 + c_2^2 + (-42+8k_1)c_2^2 + (-402k_1-36k_2+24k_1^2 + 699)c_2^2 - (3888 - 144k_1k_2 + 756k_2 + 904k_1 - 1104k_1^2 + 32k_3^2)c_2 - 144k_1k_2^2 + 16k_1^4 + 108k_2^2 + 441k_1k_2 - 936k_1^2 + 1711k_2^2 - 28842k_2 - 95670k_1)/24;
\]
\[
tg_5 := (81/40d^3 + (27/8c_2 + 27/4k_1 - 189/2)d^2 + (9/4c_2^2 + (441/4+9k_1)c_2 + 9k_2^2 - 27/2k_2 - 1107/4k_1 + 3393/2)d^3 + (3/4c_2^2 + (-189/4+9/2k_1)c_2^2 + (9k_1^2 - 981/4k_1 + 2469/2 - 27/2k_2)c_2 - 27k_1k_2 + 6k_1^2 - 603/2k_1^2 - 13875 + 471k_2 + 8463/2k_1)d^4 + (1/8c_2^2 + (-35/4 + k_1)c_2^2 + (3k_1^2 - 285/4k_1 + 2207/8 - 9/2k_2)c_2^2 + (4k_1^3 - 4789 - 180k_1k_2 - 180k_1^2 + 565/2k_2 + 8589/4k_1)c_2 - 145k_1k_2^2 - 22445k_2 + 27403/8k_1^2 + 2k_1^4 + 27/2k_1^2 + 1355/2k_1k_2 - 111959/4k_1 + 217728/5 - 18k_1^2k_2)d + (10 + 1/12k_1 - 7/12)c_2^2 + (141/8 + 1/3k_2 - 27/4k_1)c_2^3 + (251/6k_2 - 53/2k_1^2 - 3k_1k_2 + 2/3k_3^2 - 455/2 + 1547/6k_1)c_2^4 + (-1781/12k_1 + 3516/5 + 1299/6k_2 - 6817/12k_1 - 131/3k_1^3 + 9/2k_1^2 + 2151/24k_1^2 + 3k_1^3 - 6k_1^2k_2)c_2^5 + 727/3k_1^2k_2 - 188k_2^2 - 8827/2k_1k_2 + 321882/5k_1 + 9k_2k_1 + 226952k_2 + 10867/12k_1 - 26189/24k_1^2 - 4/15k_1^3 - 6k_1^2k_2 + 4/15k_1^3; \]
\[
tg_6 := (81/80d^6 + (81/40c_2 + 567/8 + 81/20k_1)d^5 + (27/16c_2^2 + (27/4k_1 - 1701/16)c_2 - 81/8k_2 + 8109/4 + 27/4k_1 - 4077/16k_1)d^4 + (3/4c_2^2 + (9/2k_1 - 63)c_2^2 + (8523/4 - 27/2k_2 + 9k_1^2 - 1233/4k_1)c_2 + 1131/2k_2 + 6k_1^2 - 29601 - 27k_1k_2 - 729/2k_1^2 + 25671/4k_1)d^3 + (3/16c_2^2 + (3/2k_1 - 147/8)c_2^2 + (12909/16 - 27 - 4k_1^2 + 9/2k_2^4 - 1107/8k_1)c_2^3 + (2073/4k_2 - 76959/4 - 27k_1k_2 + 4149/8k_3 + 8k_1^3 - 333k_1^2)c_2^3 + 3k_1^4 + 81/4k_1^2 - 27k_1^2k_2 - 96699/8k_2 - 519/2k_1 + 1102009/5 + 119961/16k_1^2 - 639927/8k_1 + 4821/4k_1k_2)d^2 + (1/4c_2^2 + (1/4k_1 - 21/8)c_2^2 + (-3/2k_2 + 3071/24 - 109/4k_1 + k_1^2)c_2^2 + (-201/2k_1^2 + 15k_2k_1 + 2k_1^2 - 29213/8 - 9k_1k_2 + 5421/4k_1)c_2^2 + (-26787/4k_2 + 6489097/10 - 74149/2k_1 - 159k_1^3 + 1481/2k_1k_2 + 27/2k_1^2 + 32959/8k_1^2 + 2k_1^4 - 18k_1k_2^2)c_2 + 853k_1k_2 - 18481k_1k_2 - 1317/2k_1 + 27k_2^2k_1 + 1401361/12k_2 + 28988249/60k_1 + 46109/12k_2^2 - 12k_1^2k_2 + 4/5k_1^2 - 668388 - 554465/8k_1^2 - 92k_1^4)d + (720e_6 - (7/48 + 1/60k_1)c_2^2 + (-1/8k_2 + 1/12k_1^2 - 95/48k_1 + 331/48)c_2^2 + (-k_1k_2 - 10k_1^2 + 8147/72k_1 - 8095/48 + 565/36k_2 + 2/9k_1^3)c_2^2 + (-145/6k_1^3 + 15347/10 + 1355/12k_1k_2 - 3k_1^2k_2 + 1355/12k_1k_2) + 8147/12k_1k_2 - 1355/12k_1k_2) d + ...
Again setting 8 general points: the configurations must consist of 4 lines.

\[ 1/3k_1^3 + 9/4k_2^2 - 190339/48k_1 + 26519/48k_2^2 - 10891/12k_2 \] \[ c_2^2 + (-4k_1^3k_2 - 85/3k_1^4 + 4291/4k_1^3 + 9k_2^2k_1 + 10998 - 815/4k_2^2 - 807341/48k_2^2 + 790/3k_2^2k_2 + 4/15k_2^5 - 62339/12k_1k_2 + 691883/24k_2 + 10672201/120k_1k_2 - 311237/16k_1^3 - 9/2k_2^2 + 4/45k_1^3 + 7001519/72k_1k_2 - 2k_1^2k_2 - 1855/4k_2^2k_1 + 9k_1^2k_2^2 + 1805/9k_1^3k_2 - 108064/6k_1 + 186753363/360k_1^2 + 200477/36k_2^2 + 26297/36k_1^4 - 13k_1^2 - 55951/8k_1^2k_2 - 2567321/6k_2. \]

5.1 Example. \( Y = \mathbb{P}^2 \). We make the substitutions,

\[ c_2 = 3, d = m^2, k_1 = -3m, k_2 = 9. \]

5.1.1 \( n = 4 \). The expression for \( tg_4 \) above reduces to

\[ tg_4(m) = -8865 + 18057/4m + 37881/8m^2 - 2529m^3 - 642m^4 + 1809/4m^5 - 27m^7 + 27/8m^8. \]

Setting \( m = 4 \) we get \( 666 = 126 + 540 \) for the number of 4–nodal quartics through 10 general points. Indeed, a plane quartic with 4 nodes splits as a union of 2 conics, 126 of which pass through 10 points, or of a singular cubic and a line through 10 points.

5.1.2 \( n = 5 \). We find,

\[ tg_5(m) = 81/40m^{10} - 81/4m^9 - 27/8m^8 + 2349/4m^7 - 1044m^6 - 127071/20m^5 + 128859/8m^4 + 59097/2m^3 - 3528381/40m^2 - 946929/20m + 153513. \]

Setting \( m = 4 \) and picking a system of quartics through 9 general points, we do get the right answer, \( 378 = (\binom{9}{5}) \times 3 \). Indeed, a plane quartic with 5 nodes can only be a union of a conic and line pair: hence \( (\binom{9}{5}) \) for the choice of 5 points determining a conic, times the number 3 of line pairs through the 4 remaining points...

5.1.3 \( n = 6 \). We have,

\[ tg_6(m) = 81/80m^{12} - 243/20m^{11} - 81/20m^{10} + 8667/16m^9 - 9297/8m^8 - 47727/5m^7 + 2458629/80m^6 + 3243249/40m^5 - 6577679/20m^4 - 25387481/80m^3 + 6352577/4m^2 + 8290623/20m - 2699706. \]

Again setting \( m = 4 \), we find 105 for the number of 6–nodal quartics through 8 general points: the configurations must consist of 4 lines.

5.1.3.1 Setting \( m = 5 \), we can find the number of irreducible rational plane quintic curves through 14 general points. This is \( tg_6(\mathbb{P}^2)(5) - (\binom{14}{5})tg_2(\mathbb{P}^2)(4) - (\binom{14}{5}) = 109781 - 20475 - 2002 = 87304. \) The corrections are due to the reducible 6–nodal quintics: either line+binodal quartic or conic+cubic.
5.2 Example. $Y = \mathbb{P}^1 \times \mathbb{P}^1$. For a system of curves of type $(m_1, m_2)$, we set
\[
c_2 = 4, k_2 = 8, k_1 = -2(m_1 + m_2), d = 2m_1m_2.
\]

5.2.1 $n = 4$. We get,
\[
tg_{4, \mathbb{P}^1 \times \mathbb{P}^1}(m_1, m_2) = (32/3 - 64m_2 + 144m_2^2 - 144m_2^3 + 54m_2^4)m_1^2 + (808/6 - 3112/3m_2 + 1230m_2^2 - 324m_2^3 - 144m_2^4)m_1^3 + (11987/6 - 3494m_2 - 2m_2^2 + 1230m_2^3 + 144m_2^4)m_1^4 + (17359/6 + 11333/3m_2 - 3494m_2^2 - 3112/3m_2^3 - 64m_2^4)m_1 - 7460 + 17359/6m_2 + 11987/6m_2^2 + 808/3m_2^3 + 32/3m_2^4.
\]

5.2.1.1 If $m_1 = m_2 = 2$, it checks with the number 6 of configurations of 4 lines in the system $(2, 2)$ through 4 general points on a quadric. Indeed, since $p_a = 1$, the curve splits in one of the types: $(1, 1) + (1, 1), (2, 0) + (0, 2)$, $(2, 1) + (0, 1)$ or $(1, 2) + (1, 0)$). The latter two cases consist of the union of a twisted cubic and a bi-secant line, hence get for free two nodes due to the intersections. In order to present 4 nodes, the twisted cubic must split further. One easily sees that the only possibility is indeed a configuration $(2, 0) + (0, 2)$ of 4 lines. We may assume no 2 of the 4 points are on a ruling. Label the points 1, 2 so that the lines composing the curve $(2, 2)$ through them are both of system $(1, 0)$; this forces the other 2 lines to be of the opposite system $(0, 1)$. Thus, the choice of 1, 2 completely determines the solution, hence $(\frac{4}{2})$.

5.2.1.2 For $(m_1, m_2) = (2, 3)$, we find $tg_{4, \mathbb{P}^1 \times \mathbb{P}^1}(2, 3) = 133$. As $p_a = 2$, we obtain again reducible configurations. Notice the system $|(2, 3)|$ is $\infty^{11}$. Let the $\infty^4$ subsystem be defined by imposing 7 points. Possible splitting types? (i)$(2, 0) + (0, 3)$ is $\infty^5$, too small. (ii)$(2, 1) + (0, 2)$ is $\infty^7$; 4 nodes due to intersection, $(\frac{7}{2}) = 21$ choices for configuration consisting of twisted cubic $|(2, 1)|$ through 5 points and line pairs $|(0, 2)|$ through 2 points. SUBTOTAL: 21. (iii)$(1, 1) + (1, 2)$ is $\infty^{3+5}$; 3 nodes due to intersection, hence need additional node for either $(1, 1)$ or $(1, 2)$ component. If the new node is on $(1, 1)$, this curve must be a line pair; make it pass through 2 of the points $(\frac{7}{2})$ choices for these) $\times$ 2 (number of such line pairs for each choice of 2 points), total 42. One takes the $(1, 2)$ component through the remaining 5 points, unique choice. SUBTOTAL: 42. If the new node is on
a $(1,2)$-curve, this must split as $(0,1) + (1,1)$, so the actual solutions are of the form $(1,1) + (0,1) + (1,1)$; if the $7$th point is on the line, the remaining 6 will be on $(\frac{6}{3})/2$ conic pairs. SUBTOTAL: 70. (iv) $(2,2) + (0,1)$ has 2 nodes due to intersection, hence need two additional nodes for $(2,2)$-component; now if a $(2,2)$-curve acquires 2 double points, it splits as $(2,1) + (0,1)$ or $(1,2) + (1,0)$ or $(1,1) + (1,1)$; these have already been accounted for! Thus it all happily adds up to the right TOTAL: 133.

5.2.1.3 For $(m_1, m_2) = (2, 4)$, we find $tg_{4,\bf{P}^1 \times \bf{P}^1} (2, 4) = 1261$. The system $|((2,4))|$ is $\infty^{14}$. We impose 10 points to select an $\infty^4$ subsystem. Possible splitting types? (i) $(2,3) + (0,1)$ is $\infty^{12}$; 2 nodes due to intersection. Impose 2 new nodes for $(2,3)$-component; there are $tg_{2,\bf{P}^1 \times \bf{P}^1} (2,3) = 105$ through each of the 10 choices of 9 points. Notice that among these 1050 curves there are 90 in $|((2,2) + (0,2))|$. These will be accounted for separately below. SUBTOTAL: 960. (ii) $(2,2) + (0,2)$ is $\infty^{10}$; 4 nodes due to intersection; $\binom{10}{2} = 45$ choices for 2 points determining a line pair in the system $|((0,2))|$, the remaining 8 points singling out a member in $|((2,2))|$. SUBTOTAL: 45. (iii) $(2,1) + (0,3)$ is $\infty^8$: too small! Similarly for $(2,0) + (0,4)$. (iv) $(1,4) + (1,0)$ is $\infty^{10}$; 4 nodes due to intersection; 10 choices for the point determining the component $(1,0)$. SUBTOTAL: 10. (v) $(1,3) + (1,1)$ is $\infty^{10}$; 4 nodes due to intersection; $\binom{10}{3}$ choices for 3 points determining a conic while the 7 other points determine the component $(1,3)$. SUBTOTAL: 120. (vi) $(1,2) + (1,2)$ is $\infty^{10}$; 4 nodes due to intersection; $\binom{10}{5}$ choices for 5 points determining a twisted cubic $(1,2)$ SUBTOTAL: 126. It gives the expected TOTAL: 1261.

5.2.1.4 Irreducible rational curves with $p_a = 4$ on $\bf{P}^1 \times \bf{P}^1$. We may compute the number 3510 of irreducible rational curves of type $|((3,3))|$ passing through $11 = 15 - 4$ general points. We subtract from $tg_{4,\bf{P}^1 \times \bf{P}^1} (3,3) = 4115$, the contributions given by: (i) (nodal $(3,2)$ through 10 points + $(0,1)$ through the $11\bf{P}^1$): $20 \times 11 = 220$; (ii) (nodal $(2,3)$ through 10 points + $(1,0)$): 220; (iii) $(2,2)$ through 8 points + $(1,1)$ through 3 others): $\binom{11}{8}$=165. (Note that $(3,1) + (0,2)$ is $\infty^9$—too small.)
5.2.1.5 Reasoning as above, we also find the number 3684 of irreducible rational curves in the system (2, 5) passing through 13 = 17 – 4 general points. This is \( tg_4 \mathbb{P}^1 \times \mathbb{P}^1 (2, 5) = 7038 \) minus \( (tg_2 \mathbb{P}^1 \times \mathbb{P}^1 (2, 4) = 252) \times 13 \) due to binodal \((2, 4) + (0, 1))\) minus \((\frac{13}{11}) = 78\) due to curves \((2, 3)\) through 11 points + \((0, 2)\) through 2 others).

5.2.2 \( n = 5 \). The first interesting check is provided by the system \(|(3, 3)|\) on \( \mathbb{P}^1 \times \mathbb{P}^1 \). We find \( tg_5 \mathbb{P}^1 \times \mathbb{P}^1 (3, 3) = 3702 \). Here we have \( p_a = 4 \), hence imposing 5 nodes will force again reducible curves. Fix 10 points in general position to define an \( \infty^5 \) subsystem of \(|(3, 3)|\). Possible splitting types? (i)(3, 1) + 2(0, 1) is \( \infty^{7+1} \) and \((3, 1) + (0, 2)\) is \( \infty^{7+2} \), both too small. (ii)(3, 2) + (0, 1): \( \infty^{12} \); there are 3 nodes due to intersection. Look at members of \(|(3, 2)|\) through 9 points and with 2 additional nodes: we find \( tg_2 \mathbb{P}^1 \times \mathbb{P}^1 (3, 2) = 105 \). Among these, 9 split further as \((2, 2) + (1, 0)\) and will be accounted for separately in (iv). Since there are 10 choices for the 9 points, we have the SUBTOTAL: 960. (iii)(2, 3) + (1, 0): just as in (ii), SUBTOTAL: 960. (iv)(2, 2) + (1, 0) + (0, 1): there are \( \binom{10}{2} = 45 \) times 2 for choices of points and system of line through them. SUBTOTAL: 90. (v)(2, 2) + (1, 1): we have 4 nodes due to intersection. When the additional node is on the (2, 2) component which passes through 7 points, we find \( tg_2 \mathbb{P}^1 \times \mathbb{P}^1 (2, 2) = 12 \), times \( \binom{10}{3} \) obtaining the SUBTOTAL: 1440. If the additional node be on \((1, 1)\), the type becomes \((2, 2) + (1, 0) + (0, 1)\), already accounted for in (iv) above. (vi)(2, 1) + (1, 2): there are 5 nodes due to intersection; contributes \( \binom{10}{5} \), SUBTOTAL: 252, fortunately totaling 3702.

5.2.2.1 How about the irreducible rational curves with \( p_a = 5 \) on \( \mathbb{P}^1 \times \mathbb{P}^1 \)? The possible bidegrees are \((2, 6), (6, 2)\). One expects finitely many of these passing through 15 points. However we notice that any subsystem \( S \subset \|(2, 6)\|\) of codimension 15 meets the family of curves of type \((2, 4) + 2(0, 1)\). Since these present a nonreduced component, therefore \( \Sigma((2^{[5]}); S)\) contains components of wrong dimension (cf.[2, 3]), so that the formula is not applicable to the present case. It would be nice to compute the equivalence of these bad components.

5.2.3 \( n = 6 \).
5.2.3.1 We look again at the system $|(3, 3)|$ on $\mathbb{P}^1 \times \mathbb{P}^1$. We find $tg_{6, \mathbb{P}^1 \times \mathbb{P}^1}(3, 3) = 2224$. Fix 9 points in general position to define an $\infty^6$ subsystem. Possible splitting types? (i)$(3, 1) + 2(0, 1): \infty^{7+1}$, too small. (ii)$(\text{nodal}(2, 2)) + (1, 0) + (0, 1): 12 \times \binom{9}{7} \times 2$. SUBTOTAL: 864. (iii)$\infty^7 + 1$; too small. (iv)$\text{(nodal}(2, 3) + (1, 0) + (0, 1): \infty^{5+3+1}$; contributes $\binom{9}{3} \times \binom{6}{3}$. SUBTOTAL: 864. (v)$\text{(nodal}(2, 1) + (1, 1) + (0, 1): \infty^5 + 3 + 1$; contributes $\binom{9}{5} \times \binom{4}{3}$. SUBTOTAL: 504. For several days, we had found only these 1944. The 280 then missing were pointed out to me (after a lunch break at the MSRI) by Enrique Arrondo: $\binom{9}{3} \times \binom{6}{3}/6 = 280$ curves of the form $(1, 1) + (1, 1) + (1, 1)$!!

5.2.3.2 Irreducible rational curves of bidegree $(3, 4)$ passing through $13 = 19 - 6$ general points: $90508$. We subtract from $tg_{6, \mathbb{P}^1 \times \mathbb{P}^1}(3, 4) = 122865$, the contributions given by: (i)$(\text{trinodal}(3, 3)$ through 12 points) + $((0, 1)$ through the $13 \mathbb{L}$): $1944 \times 13 = 25722$; (ii)$(\text{nodal}(2, 3)$ through 10 points) + $((1, 1)$ through 3 others): $20 \times \binom{13}{3} = 5720$; (iii)$(\text{(2, 2)$ through 8 points} + $((1, 2)$ through 5 others): $\binom{13}{8} = 1287$; (iv)$((3, 2)$ through 11 points) + $((0, 2)$ through 2 others): $\binom{13}{8} = 78$.

5.3 Example. Del-Pezzo surface: $Y = \mathbb{P}^2$ blown up at 5 points, imbedded in $\mathbb{P}^4$ as a $(2, 2)$ intersection by the system of plane cubics through the 5 points. There are 40 fourfold tangent hyperplanes. Indeed, label the points $\{1, \ldots, 5\}$; draw the lines $\overline{12}$, $\overline{15}$, $\overline{34}$; let $a = \overline{12} \cap \overline{34}$, $b = \overline{15} \cap \overline{34}$. Note 1 is double on $\overline{12} + \overline{15} + \overline{34}$. After blowing up, the hyperplane system $|3L - e_1 - \cdots - e_5|$ will contain the curve $c_1 + \overline{12}' + \overline{15}' + \overline{34}'$ (the $'$ denoting strict transform). It presents the 2 double points $a'$, $b'$ and two others on $e_1$. The number of such configurations can be counted as 5 choices for the point labeled 1, times $\binom{4}{2}$ choices for $\overline{12}$, $\overline{15}$, totaling 30. In addition to these configurations of lines, we may also take the conic $c$ and a line through a pair of the points, say $\overline{12}$; then we get the hyperplane section $c' + \overline{12}' + e_1 + e_2$. This gives 10 more, totaling 40, as predicted by the formula.

5.4 Example. Surfaces of degree 9 in $\mathbb{P}^4$. Substituting

$$[d = 9, k_1 = 2p_a - 11, k_2 = 6\chi - 5p_a + 23, c_2 = 12\chi - k_2]$$

18
in $tg_4$ with the list of possible pairs (cf. [4]) $[p_n = \text{sectional genus}; \chi = (c_2 + k_2)/12]$ yields the table,

|  6:1 |  7:1 |  7:2 |  8:2 |  8:3 |  9:4 | 10:5 | 12:9 |
|------|------|------|------|------|------|------|------|
| 15645|  57162| 107646| 248671| 388846| 1022595| 2222868| 10957224|

5.5 Example. $K3$–surfaces. Let $Y$ be embedded by a complete system $|C|$ of curves of genus $n \in \{3, 4, 5, 6\}$. We have $2n - 2 = C \cdot (C + K_Y)$. Substituting $[d = 2n - 2, k_1 = 0, k_2 = 0, c_2 = 24]$ in $tg_n$ we find

\[
\begin{align*}
n & : \quad 3 \quad 4 \quad 5 \quad 6 \\
tg_n & : \quad 3200 \quad 25650 \quad 176256 \quad 1073720
\end{align*}
\]

For $n \in \{4, 5, 6\}$, the values given above for $tg_n$ are smaller than those predicted by a formula Ciliberto and Lopez (priv. communication) obtained by a degeneration argument.

A related development is the work of Manoil [20], where he addresses the question of existence of rational points on $K3$–surfaces defined over a number field. He proves the existence of curves of geometric genus $\leq 1$ for a certain class of surfaces by counting singular curves.

5.6 Example. Abelian surfaces $Y \subset \mathbb{P}^4$. Here we find the number 150 of 4–fold tangent hyperplanes. It might be more than just a coincidence the fact that the contribution from $\#\Sigma(3; S)$ is also $= 150$, suspiciously a factor of the number 15,000 of symmetries of the Horrocks-Mumford bundle, a generic section of which is known to vanish precisely on $Y$ . . .

The following comments were kindly communicated by Chad Schoen.

Let $Y$ be an Abelian surface with a polarization of type $(1, 5)$. Any Horrocks-Mumford Abelian surface is of this type. The converse is almost true. I believe that any simple Abelian surface with a $(1, 5)$ polarization is a Horrocks-Mumford Abelian surface. Let $N$ be an invertible sheaf giving the $(1, 5)$ polarization. A curve in $|N|$ has self-intersection 10. This is the degree of the normal sheaf which is also the dualizing sheaf. Thus the arithmetic genus is 6. If the curve is irreducible and has 4 nodes it’s normalization has genus 2. If $Y$ is “general” its Picard number is 1 and any hyperplane section must be irreducible. Let $C$ be such a 4–nodal curve and $\tilde{C}$ its normalization. There is an isogeny $Jac(\tilde{C}) \to Y$ taking $\tilde{C}$ to $C$. Again if $Y$ has Picard number
1, there is no choice but for this map to have degree 5. Now the degree 5 unramified covers of \( Y \) are parametrized by the subgroups of order 5 in the fundamental group of \( Y \). Write \( L \) for this lattice and \( L' \subset L \) for the index 5 subgroup. Assuming that \( Y \) has Picard number 1, the 5 fold cover \( f : J \rightarrow Y \) will be the Jacobian of a genus 2 curve if and only if \( J \) is principally polarized. This will occur if and only if the pull back of the \((1, 5)\) polarization on \( Y \) is 5 times a polarization on \( J \). In terms of lattices and the Riemann form associated to the polarization we have:

\[
A : (1/5L)/L \times L/5L \rightarrow (1/5Z)/Z = Z/5.
\]

This alternating form on the 5 torsion of \( Y \) has a two dimensional radical—call it \( K \). \((K=\text{vectors in } (1/5)L/L \text{ which are orthogonal to the whole space})\). Now the pull back to \( J \) is divisible by 5 if and only if the restriction of \( A \) to \((1/5)L'/L \times L'/5L \rightarrow (1/5)Z/Z\) is identically zero. This occurs exactly when \( K \) lies in \((1/5)L'/L\). We can count all such \( L' \). They are hyperplanes in \( \mathbb{P}^3 \) containing a fixed \( \mathbb{P}^1 \) all over the field \( Z/5 \). Thus the \( L' \) 's are parametrized by \( \mathbb{P}^1(Z/5) \). There are 6 possible \( L' \) 's. Thus 6 possible \( J \) 's. Finally we note that translation by elements of \( K = Z/5 \times Z/5 \) give automorphisms of \( Y \) preserving the \((1, 5)\) polarization. This gives \( 6 \times 25 = 150 \) four-nodal hyperplane sections. There are only 6 different isomorphism classes of genus 2 curve which occur as normalizations.

**Question:** Inversion in the Abelian variety should also preserve the polarization (I \( (C. Schoen) \) think). How does this permute the 4–nodal hyperplane sections?

### 6 Threefolds

The same method yields the formula,

\[
t_{0, m} = (m^{18} - 12m^{17} + 24m^{16} + 155m^{15} - 405m^{14} + 1082m^{13} - 18469m^{12} + 66446m^{11} - 192307m^{10} + 1242535m^9 - 4049006m^8 + 11129818m^7 - 53664614m^6 + 166756120m^5 - 415820104m^4 + 1293514896m^3 - 2517392160m^2 + 1781049600m)/6!
\]

for the number of planes in \( \mathbb{P}^4 \) that are 6-fold tangent to a hypersurface of degree \( m \).
6.1 Quartics.

For $m = 4$, the formula above gives $5600$. This can be verified by the following direct calculation via the Fano variety $F$ (cf. [2]) of $\infty$ lines contained in a $4$-fold $T$. Presently the counting refers to the set

$$\{(\ell_1, \ldots, \ell_4) \in F^4 | \exists \text{ plane } \pi \ s.t. \ell_1 + \cdots + \ell_4 = \pi \cap T\}$$

of $4$–tuples of coplanar lines in that family.

Let $S_i \to O^{\oplus 5} \to Q_i$ (rank $Q_i = i + 1$) denote the tautological sequence over the Grassmann variety $G_i := Gr(i, 4)$ of $i$–dimensional subspaces of $\mathbb{P}^4$. Go to the incidence variety $I := \{((\ell, \pi) \in G_1 \times G_2 | \ell \subset \pi\}$. It carries the diagram of locally free sheaves, (omitting pullbacks)

\begin{align*}
S_2 & \to S_1 \to \mathcal{M} \\
\| & \downarrow \quad \downarrow \\
S_2 & \to O^{\oplus 5} \to Q_2 \\
\downarrow & \quad \downarrow \\
Q_1 &= Q_1
\end{align*}

(4)

The universal plane $\mathbb{P}(Q_2)$ contains the total space $D$ of the family of intersections with the fixed $4$-fold hypersurface. Our goal is to compute the intersection class supported by

$$I_3 := \{((\ell_1, \ell_2, \ell_3, \pi) \in I \times_{G_2} I \times_{G_2} I | D_\pi \geq \ell_1 + \ell_2 + \ell_3\}.$$

Set $I_1 = \{((\ell, \pi) | \ell \subset \pi \cap D_\pi\}$. This is expressible as zeros of a section of a suitable bundle. Indeed, up on $\mathbb{P}(Q_2)|_I$, we have the Cartier divisors $D_\pi$ and $L_1 := \mathbb{P}(Q_1)|_I$. One checks that $I_1$ is exactly the locus in $I$ where “$L_1 \subset D$” holds along fibers. Studying the natural diagram of $\mathcal{O}_{\mathbb{P}(Q_2)|I}$–modules,

$$\begin{array}{ccc}
\mathcal{O} & \to & 0 \\
\downarrow & & \downarrow \\
\mathcal{O}(D) & \to & \mathcal{O}_{L_1}(D)
\end{array}$$

one sees that the slant arrow $s$ vanishes on the fiber over $(\ell, \pi) \in I$ iff $\ell \subset \pi \cap D_\pi$. Let $p : \mathbb{P}(Q_2)|_I \to I$ denote the structure map; it follows that $I_1$ is the scheme of zeros of the section $p_\ast s$ of the direct image $sym_4 Q_1$ of $\mathcal{O}_{L_1}(D) = \mathcal{O}_{L_1}(4)$. We obtain $[I_1] = c_5 sym_4 Q_1$. Pulling back $D$ to $I_1$ (and
abusing notation), it splits as \( D = D_1 + L_1 \), thus defining \( D_1 \). Moreover, since \( \mathbb{P}(Q_2) \) is the divisor of zeros of a section of \( O_{Q_2}(1) \otimes M^* \), we have \( O(D_1) = O_{Q_2}(4) \otimes O_{Q_2}(-1) \otimes M \). We may ask when does \( D_1 \) split further.

Go to \( I_1 \times_{G_2} I \). Set \( L_2 = I_1 \times_{G_2} L_1 \) and define \( I_2 \) by imposing the fibers of \( D_1 \) to contain a 2nd line. As before, \( I_2 \) is given by the vanishing of a section of the pushforward of \( O_{L_2}(D_1) \). Denoting by \((i)\) the pullback to \( I \times_{G_2} I \) via \( i_{j_{23}} \) projection, we find \([I_2] = c_4(M_{(2)} \otimes \text{sym}_3 Q_{1(2)})\). Similarly, pulling back \( D_1 \) over \( I_2 \) yields \( D_1 = D_2 + L_2 \) and we get \([I_3] = c_3(M_{(3)} \otimes \text{sym}_2 Q_{1(3)})\). See in the Appendix a script for the actual computation using SCHUBERT [16]. Observing that a 6-fold tangent plane \( \pi \) to a 4ic hypersurface cuts 4 lines, the computation gives \( 134400/24=5600 \) as asserted.

6.2 Quintics

Recall that a general 5ic threefold \( T \subset \mathbb{P}^4 \) contains 2,875 lines and 609,250 conics (cf. [14], [15]).

The plane through a conic counts as a 6-fold tangent since its intersection with \( T \) splits as a conic + cubic, thereby presenting 6 nodes.

Through each line, there are \( \infty^2 \) planes in \( \mathbb{P}^4 \). The intersection of any such plane with \( T \) splits as line + quartic thereby counting as a 4-fold tangent. The plane is a 6-fold tangent iff the residual plane quartic is binoodal.

Fix a line \( \ell \subset T \). Let us find, among these \( \infty^2 \) residual plane quartic curves the number of those with 2 double points. This requires the computation of \( \Sigma((2, 2); D) \) for the family \( D \subset X \rightarrow S \) of residual plane quartic we now describe. Notation as in the previous example, let \( S_2 \rightarrow O^{35} \rightarrow Q_2 \) (rank \( Q_2 = 3 \)) denote the tautological sequence over the Grassmann variety \( G_2 \) of planes in \( \mathbb{P}^4 \). Let \( G_{2, \ell} \) be the Schubert subvariety of all 2-planes through a fixed line \( \ell \). Let \( X = \mathbb{P}(Q_2)|_{G_{2, \ell}} \subset G_{2, \ell} \times \mathbb{P}^4 \) be the restriction over \( G_{2, \ell} \) of the universal plane in \( \mathbb{P}^4 \). Restricting the sequence over \( G_{2, \ell} \) yields an exact sequence, (cf. top sequence in [13]) \( S_2 \rightarrow O^{35} \rightarrow S_{1|\ell} \rightarrow M \), where \( M \) is a line subbundle of \( Q_2 \) with Chern class \( x := c_1 M = -c_1 S = c_1 Q_2 \). Over \( X \), we have the natural commutative diagram of maps of locally free sheaves,

\[
\begin{array}{ccc}
\mathcal{M} & \rightarrow & \mathcal{O}_{Q_2}(1) \\
\downarrow & & \\
Q_2 & \rightarrow & \mathcal{O}_{Q_2}(1)
\end{array}
\]
where the bottom line is the tautological 1–quotient on the projective bundle $\text{Proj}(\text{Sym}(Q_2))$. One checks that $\ell' := G_{2,\ell} \times \ell$ is the divisor in $\mathbb{P}(Q_2)_{\ell}$ of zeros of the slant arrow $\mathcal{M} \to \mathcal{O}(1)$. Therefore, setting $y = c_1\mathcal{O}(1)$ we have $\mathcal{O}(\ell') = \mathcal{O}(y-x)$.

Now let $D_T \subset \mathbb{P}(Q_2)$ be the divisor defined by intersection with $T$, so that $\mathcal{O}(D_T) = \mathcal{O}(5 \cdot y)$. Restriction over $G_{2,\ell}$ splits $D_T = D + \ell'$. By construction, $D$ is the total space of the family of plane quartic curves residual to $\ell$. Finally, we have $L := \mathcal{O}(D) = \mathcal{O}(5 \cdot y - (y-x)) = \mathcal{O}(4y+x)$.

Using Schubert\cite{16} we may compute $I_{G_{2,\ell}}(c_6\mathcal{E}((2,2),L))/2 = 1,185$ (see the appendix) and find the number

$$17,601,000 = tg_{6,5} - 609250 - 1185 \times 2875$$

of irreducible plane rational quintic curves contained in a generic 5ic three-fold. The $1^{\text{st}}$ correction is due to conic + cubic and the $2^{\text{nd}}$ to line + binodal quartic.

7 Final comments

An additional difficulty appears for the case of 7–fold tangent hyperplanes. Indeed, for a general 7–dimensional linear system, we’d expect $\Sigma(2^{[7]}; S)$ to receive contributions from $\Sigma(3(2), 2; S)$, $\Sigma(3, 2^{[3]}; S)$, $\Sigma(3(2)^{\prime}; S)$, so that a naïve count would predict

$$tg_7 := (\#\Sigma(2^{[7]}; S) - 210\#\Sigma(3(2), 2; S) - 1260\#\Sigma(3, 2^{[3]}; S)/6 - 30\#\Sigma(3(2)^{\prime}; S))/7!,$$

where $\Sigma(3(2)^{\prime}; S)$ denotes a cycle supported on the set of $(C, y_1, \ldots, y_7)$ such that $C \in S$ has a triple point $y_1$ with the infinitely near double point $y_2$ presenting a branch tangent to the exceptional line over $y_1$. However, barring some computational error, in fact the rhs did not yield an integer for any of the examples we’ve experimented with. This seems to indicate that $\Sigma(2^{[7]}; S)$ may not be reduced at some of the points involving singularities worse than nodes. In fact, the argument of Prop.3.3 does not apply. This would imply that the coefficients 210, 1260 and 30, postulated by the naïve count of permutations, must be modified.

For $n \geq 8$, we face the intrusion of a component of wrong dimension in $\Sigma(2^{[n]}; S)$ due to 4–fold points. In this case, the technique of residual intersections might shed some light.
8 Appendix: computations

### Cut here for Maple

```maple
with(schubert): with(SF):

# Principal parts of order n, 
# f = cotg, d = line bundle 
princ := proc(n, f, d) local i: d &* sum('symm(i, f)', i=0..n); end:

whichmon := proc(f, vars) local i, v, z:
  z := expand(f):
  if type(z, '+') or type(z, '-') or type(z, 'name') then
    v := seq(vars[i] = 1, i = 1..nops(vars)):
    RETURN(f / subs(v, f)):
  else ERROR('invalid arg')
  fi:end:

# Subs exact monomial relations 
submonpol := proc(f, vars, rels) local z, i, j, term, mono, temp:
  z := expand(f):
  if type(z, '+') then
    for i to nops(z) do
      term := op(i, z):
      mono := whichmon(term, vars):
      for j to nops(rels) while mono <> lhs(rels[j]) do od:
      if j < nops(rels) then
        temp := temp + term / mono * rhs(rels[j])
      else temp := temp + term
      fi:
    od:
  elif type(z, '*') or type(z, '^') or type(z, 'name') then
    term := z:
    mono := whichmon(term, vars):
    for j to nops(rels) while mono <> lhs(rels[j]) do od:
    if j < nops(rels) then
      temp := temp + term / mono * rhs(rels[j])
    else temp := temp + term
    fi:
  fi:
  RETURN(temp) end:

# Kill terms in vars of totdeg > dim 
dimsimpl := proc(x, vars, degs, dim) local i, j, temp, par, n:
  temp := expand(x):
```
if type(temp,'+')then
   par:=0: n:=nops(temp):
   for i to n do
      op(i,temp):
      degree(collect(subs([seq(vars[j]=t^degs[j]*vars[j],
                         j=1..nops(vars))]),t_),t_):
      if "<=dim then par:=par+"":fi:
      od:
      temp:=par:
   else
      degree(collect(subs([seq(vars[j]=t^degs[j]*vars[j],
                          j=1..nops(vars))]),t_),t_):
      if ">dim then temp:=0 fi:
   fi:
RETURN(temp):end:

simplification:=proc () local i, j, n, z, zz:
   n:= args[1]: z:=args[2]:
   if nargs=3 and type(args[3],set) then
      zz:=args[3] else zz:={n}
   fi:
   for i from n by -1 to 2 do for j to n+1-i do
      if 2 < degree(collect(z,e[j,i]),e[j,i]) then
         z:=rem(collect(z,e[j,i]),relexc.i.j,e[j,i]):
         zz:=zz union{i+j-1}:
      fi:
      od:od:
   if opt_=5 then
      for i in zz do
         if 2 < degree(collect(z,y.i),y.i) then
            z:=rem(z,rely.i,y.i):
         fi:
      od:
   else
      for i in zz do z:=dimsimpl(z,var0.(i),deg1,2):od:
   fi:
RETURN(z)end:

#MAIN PROCEDURE FOR PUSHFORWARD {n}->{n-1}
push:= proc(n,f)
   local z,z0,z2,zz,mons,i,j,j1,j1, temp,varn,var0,degn, dd:
   option remember:
   if opt_=5 and type(relpush_5,set)=false then relpush_5:={}:fi:
   if opt_<>5 and type(relpush_,set)=false then relpush_:={}:fi:
if n=1 then
if opt_=5 then
rem(f,rely1,y1):z:=coeff(collect("",y1),y1,2):
else
subs([seq(var1[i]=0,i=1..nops(var1))],f):
f-":dimsimpl("",var0.n,deg1,2):
submonpol("",var1,{c[1,2]=chi,c[1,1]^2=k2,h[1]^2=d,
h[1]*c[1,1]=hk}):
z:=submonpol("",var1,{c[1,1]=0,h[1]=0})
fi:
RETURN(z):
else
convert(var.(n-1),set) minus convert(var.(n-2),set):
var0:=[op(")]:
degn:=[seq(1,i=1..nops(var0))]:
if opt_<>5 then
var0:=[op(var0),c[n-1,2]]:degn:=[op(degn),2]
fi:
varn:=[seq(p.n.2 &^* var0[i],i=1..nops(var0))]:
subs([e[1,n]=0,seq(varn[i]=0,i=1..nops(varn))],f):
z:=collect(f="#",e[1,n]):
if 2<degree(z,e[1,n]) then z:=rem(z,relexc.n.1,e[1,n]):
fi:
z:=collect(z,e[1,n]):
z:=collect(z-e[1,n]*coeff(z,e[1,n],1),e[1,n]):
if z<>0 then
z0:=coeff(z,e[1,n],0):
if z0<>0 then
simplification(n,z0):z0:=collect("",e[2,n-1]):
z0:=z0-e[2,n-1]*coeff(z0,e[2,n-1],1)
fi:
zz:=collect(subs([seq(varn[i]=t_*varn[i],
i=1..nops(varn))],z0),t_):
dd:=degree(zz,t_):
temp:=0:
for i from dd by -1 to 1 do
z0:=expand(coeff(zz,t_,i)):
if z0 <> 0 then
if (opt_=5 and type(relpush_5.n.i,list)=false
or (opt_<>5 and type(relpush.n.i,list)=false)) then
if opt_=5 then
print('BUILD RELPUSH_5'.n.i):
elif opt_<>5 then
print('BUILD RELPUSH'.n.i):
end if:
end if:
end for:
fi:
fi:
fi:
fi:
mons := monomials(i, var0, degn):
z2 := {}:
for j to nops(mons) do
dimsimpl(mons[j], var0...(n-1), deg1, 2):
    if degree(collect("e[1..n-1]", e[1..n-1])) <> 1 and
member(true, seq(type("e[j1..n-j1]^3, polynom), j1 = 1..n-2)) = false then
z2 := z2 union {mons[j]}
fi:
od:
mons := [seq(p.n.2 * z2[j] = push(n-1, z2[j]), j = 1..nops(z2))]:
if opt = 5 then relpush_5.n.i := mons:
else relpush.n.i := mons:
fi:
elif opt = 5 and member([n, i], relpush_5) = false then
print('USING RELPUSH_5.n.i. ' BUILT BEFORE'):
relpush_5 := relpush_5 union{[n, i]}:
elif opt <> 5 and member([n, i], relpush) = false then
print('USING RELPUSH.n.i. ' BUILT BEFORE'):
relpush := relpush union{[n, i]}
fi:
if opt = 5 then mons := relpush_5.n.i:
else mons := relpush.n.i:
fi:
z0 := submonpol(z0, varn, mons)
fi:
temp := temp + z0:
od:
z0 := temp:
z2 := -coeff(z, e[1..n], 2):
for i to nops(varn) while z2 <> 0 do
    z2 := collect(z2, varn[i]):
    if degree(z2, varn[i]) <> 0 then
        z2 := rem(z2, varn[i] - var0[i], varn[i]):
    fi:
od:
    if z2 <> 0 then z2 := simplification(n-1, z2) fi:
z := z0 + z2:
fi:
RETURN(z)
fi:
end: #of push
# CALCULATIONS
for opt_ in [5, 0] do
  if opt_ = 5 then
    grass(3, 5, x, all):
    Grass(g, 1, Qx, y, all):
    omega1 := dual(g[tangentbundle_]):
    rely1 := chern(3, "): var1 := [y1]: deg1 := [1]:
    variety(S1, dim=8, vars=var1, degs=deg1):
  else
    var1 := [c[1, 1], h[1], c[1, 2]]: deg1 := [1, 1, 2]:
    variety(S1, dim=2, vars=var1, degs=deg1):
  fi:
  var01 := var1:
  for n to 6 do
    if n=1 then
      if opt_ = 5 then
        DIM := 3: # ONLY FOR THE SAKE OF RANKS...
        L := o(4*y1+x1): # FOR BINODAL 4ICS
        princ(1, omega1, L): chern(3, "): FB1 := rem(" , rely1, y1):
        print('done FB1'): DIM := 6: M := o(m*y1): DIM := 3:
      else
        opt_ := 0: goto(S1): bundle(2, c):
        omega1 := subs([c1=c[1, 1], c2=c[1, 2]]): # COTANGENT BUNDLE
        M := o(h[1]):
      fi:
      princ(1, omega1, M):
      if opt_ = 5 then chern(3, "): else chern("):
      fi:
      F1 := simplification(n, "): print('done F1'):
      if opt_ = 5 then
        DIM := 6: # ONLY FOR THE SAKE OF RANKS...
      fi:
      princ(2, bundle(2, c, M):
      subs([seq(c.i = chern(1, omega1), i=1..2)], "):
      if opt_ = 5 then chern(6, "): else chern("):
      fi:
      E_31 := simplification(1, "): print('done E_31'):
      elif n >= 2 then
        var0.n := [y.n]:
        var.n := [seq(y.j, j=1..n),
        seq(seq(e[j, k], j=1..n-k+1), k=2..n-1), e[1, n]]:
        deg.n := [seq(1, j=1..n),

seq(seq(1,j=1..n-k+1),k=2..n-1),1]:

if opt_<>5 then
  var0.n:=[c[n,1],h[n],c[n,2]]:
  var.n:=[seq(c[j,1],j=1..n),
    seq(c[j,2],j=1..n),op(subs([seq(y.j=h[j],j=1..n)],var.n))]:
  deg.n:=[seq(1,j=1..n),seq(2,j=1..n),op(deg.n)]
fi:

if opt_=5 then rely.n:=subs(y1=y.n,rely1): fi:

variety(S.n,dim=6/5*opt_+2*n,vars=var.n,degs=deg.n):
morphism(p.(n).2,S.n,S.(n-1),subs([seq(var0.(n-1)[k]=var0.n[k],
    k=1..nops(var.01)),seq(e[n-k,k]=e[n-k+1,k],k=2..n-1)],
  var.(n-1))):print('built S'.n):

DIM:=3: #OK since ranks<=3
omega.n:=((p.n.2)&^*(omega.(n-1)))&*o(e[1,n])+o(-e[1,n])-1:
chern(3,omega.n): #Will set=0 since rk.omega=2
print('DONE OMEGA'.n):
relexc.n.1:=rem("",rely.n,y.n):
for i from n-1 by -1 to 2 do
  relexc.i.(n+1-i):=(p.n.2)&^*(relexc.i.(n-i)):
  if degree(collect(relexc.n.1,e[n+1-i,i]),e[n+1-i,i])>2 then
    relexc.n.1:=rem(relexc.n.1,"e[n+1-i,i])
  fi:
  od:
M:=collect((p.n.2)&^*M&*o(-2*e[1,n]),t): #Adjust M
princ(1,omega.n,M):
if opt_=5 then chern(3,"):else chern("): fi:
F.n:=simplification(n,"):print('done F'.n):
if n=2 then #E_32
  collect(M&*o(-e[1,2]),t):princ(1,omega2,"):
  if opt_=5 then
    E_32:=chern(3,"):
    L:=collect((p.n.2)&^*L&*o(-2*e[1,n]),t): #Adjust L
    princ(1,omega.n,L):chern(3,"):
    FB.n:=simplification(n,"):print('done FB'.n):
    else E_32:=chern("):
    fi:
    E_32:=simplification(n,E_32):print('done E_32'):
    E_3_2:=rem(collect("*e[1,2],e[1,2]),relexc21,e[1,2]):
    print('done E_3_2'):
    fi:
  fi:#n=2
if n=3 then #E_33
  collect(M&*o(-e[2,2]),t):princ(1,omega3,"):
  if opt_=5 then chern(3,"):else chern("): fi:
  E_33:=simplification(n,"):print('done E_33'):
fi: #E_33
fi:
od:
if opt_=5 then
i:
for i from 6 by -1 to 1 do F.i:=push(i,"*F.i):print(i)
od:
E_33:=push(3,E_33):E_32:=push(2,E_32*E_33):
E_3_2:=push(1,E_3_2):e_322:=push(1,E_31*E_32):
etg.6:=integral(Gx,F1-30*e_3_2-90 *e_322)/6!:
lprint('#4-coplanar lines in 4ic 3fild: ',subs(m=4,tg.6)):
lprint('#6-nodal plane sections of 5ic 3fild: ',subs(m=5,tg.6)):
lprint('#binodal plane 4ic residul to line in 5ic 3fild: ‘,
1/2*integral(Gx,(x1^2-x2)^2*push(1,FB1*push(2,FB2))):
lprint('#6-nodal IRREDUCIBLE plane sections of 5ic 3fild: ‘,
subs(m=5,tg.6) - 609250 - 1185*2875):
#4-COPLANAR LINES VIA FANO
DIM:=3:
for i to 3 do relm.i.1:=chern(3,5-bundle(2,z.i)-o(m.i.1)):
for j to 2 do
z.i.j:=chern(j,bundle(3,x)-o(m.i.1)):
od od:
chern(3,symm(2,bundle(2,z3))&*o(2*x1-z11-z21)):
rem("",relm31,m31):
I_3:=coeff("",m31,2):
DIM:=4:
chern(4,symm(3,bundle(2,z2))&*o(x1-z11))*I_3:
rem("",relm21,m21):
I_2:=coeff("",m21,2):
DIM:=5:
I_1:=chern(5,symm(4,bundle(2,z1)))*I_2:
I_1:=rem(I_1,relm11,m11):
I_1:=coeff(I_1,m11,2):
integral(Gx,")/4!:
else
for j to 6 do for i from 0 to 3 do
F.j.i:=coeff(collect(F.j,t),t,i):
if j<3 then E_3.j.i:=coeff(collect(E_3.j,t),t,i):fi:
if j=2 then E_3_2.(i):=coeff(collect(E_3.2,t),t,i)*e[1,2]:fi:
od:
for i from 1 to 3 do F6.i:=push(6,F6.i):od:
ftg6:=ftg5*F62:
for j5 from 1 to 3 do print('j5=‘,j5):

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a5:=push(5, 'F'.5, j5*F63):
  #dim 10-j5-1=9-j5<=8 ok

for j4 from 3-j5 to 3 do
  a4:=push(4, 'F'.4, j4*a5):
    # 6>=dim 9-j5-j4 >=0

    for j3 from max(0, 5-j5-j4) to 3 do
      a3:=push(3, 'F'.3, j3*a4):
        #4>=dim 9-j5-j4-j3 >=0

        for j2 from max(0, 7-j5-j4-j3) to min(3, 9-j5-j4-j3) do
          a2:=push(2, 'F'.2, j2*a3):
            #2>=dim 9-j5-j4-j3-j2 >=0

            j1 := 9-j5-j4-j3-j2 :
            lprint('j5='..j5,' j4='..j4,' j3='..j3,' j2='..j2,' j1='..j1):
            ftg6:=ftg6+push(1, 'F'.1, j1*a2):
        od:od:od:
      od:
    od:
  od:

for i from 1 to 3 do F5.i:=push(5, F5.i):od:
ftg5:=ftg5*F52:
for j4 from 1 to 3 do
  a4:=push(4, 'F'.4, j4*F53):
    #dim 8-j4-1=7-j4<=6 ok

    for j3 from 3-j4 to 3 do
      a3:=push(3, 'F'.3, j3*a4):
        #4>=dim 7-j4-j3 >=0

        for j2 from max(0, 5-j4-j3) to min(3, 7-j4-j3) do
          a2:=push(2, 'F'.2, j2*a3):
            #2>=dim 7-j4-j3-j2 >=0

            j1 := 7-j4-j3-j2 :
            lprint('j4='..j4,' j3='..j3,' j2='..j2,' j1='..j1):
            ftg5:=ftg5+push(1, 'F'.1, j1*a2):
        od:od:od:
      od:
    od:
  od:

for i from 1 to 3 do F4.i:=push(4, F4.i):od:
ftg4:=ftg4*F42:
for j3 from 1 to 3 do
  a3:=push(3, 'F'.3, j3*F43):
    #4>=dim 5-j3 >=0

    for j2 from max(0, 3-j3) to min(3, 5-j3) do
a2:=push(2,'F'.2.j2*a3):
#2>=dim 5-j3-j2=0
j1 :=5-j3-j2 :
lprint('j3='j3,' j2='j2,' j1='j1):
ftg4:=ftg4+push(1,'F'.1.j1*a2):
od:od:

##########
for i from 1 to 3 do F3.i:=push(3,F3.i):E_33.i:=push(3,E_33.i):
od:
ftg3:=ftg2*F32:
for j2 from 1 to 3 do
  a2:=push(2,'F'.2.j2*F33):
  #2>=dim 3-j2 >=0
  j1 :=3-j2 :
lprint(' j2='j2,' j1='j1):
  ftg3:=ftg3+push(1,'F'.1.j1*a2):
od:
e_322:=e_32*E_332:
for j2 from 1 to 3 do
  a2:=push(2,'E_3'.2.j2*E_333):
  #2>=dim 3-j2 >=0
  j1 :=3-j2:
lprint('j2='j2,' j1='j1):
e_322:=e_322+push(1,'E_3'.1.j1*a2):
od:

##########
for i from 1 to 3 do F2.i:=push(2,F2.i):
  E_32.i:=push(2,E_32.i):
  E_3_2.(i):=push(2,E_3_2.(i)):
od:
ftg2:=ftg1*F22+push(1,'F'.1.1*F23):
e_32:=e_3*E_322+push(1,'E_3'.1.1*E_323):
e_3_2:=e_3*E_3_21+push(1,'E_3'.1.1*E_3_22)+
         push(1,'E_3'.1.0*E_3_23):
ftg1:=push(1,F12):e_3:=push(1,E_312):
for n to 3 do tg.n:=ftg.n /n! :od:
tg.4:=(ftg4-6*e_3)/4!:
tg.5:=(ftg5-30*e_32)/5!:
tg.6:=(ftg6-30*e_3_2-90*e_322)/6!:
p2:=proc(m,tg)subs([chi=9,d=m^2,hk=-3*m,k2=9], tg):end:
plxp1:=proc(m1,m2,tg)subs([chi=4,k2=8,hk=-2*m2-2*m1, d=m1*m2+2],tg):end:
K_3:=proc(g)subs([chi=24,k2=0,hk=0,d=2*g-2],'tg'.g):end:
for i from 3 to 6 do
lprint('# ' '.i.'-nodal hyperplane sections of K3-sfce in P'.
i.': ',K_3(i)):
od:
for i from 4 to 6 do lprint('# ' '.i.'-nodal plane quartics through '.
(14-i).' general points: ',p2(4,tg.i)):
od:
fi:
od:
#CUT HERE FOR MAPLE
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