The homotopy classification of four-dimensional toric orbifolds

Xin Fu
Department of Mathematics, Ajou University, Suwon 16499, Republic of Korea (xfu87@ajou.ac.kr)

Tseleung So
Department of Mathematics and Statistics, University of Regina, Regina, SK S4S 0A2, Canada (tse.leung.so@uregina.ca)

Jongbaek Song
School of Mathematics, KIAS, Seoul 02455, Republic of Korea (jongbaek@kias.re.kr)

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Let $X$ be a 4-dimensional toric orbifold. If $H^3(X)$ has a non-trivial odd primary torsion, then we show that $X$ is homotopy equivalent to the wedge of a Moore space and a CW-complex. As a corollary, given two 4-dimensional toric orbifolds having no 2-torsion in the cohomology, we prove that they have the same homotopy type if and only their integral cohomology rings are isomorphic.

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1. Introduction

One of the central problems in topology is the rigidity question, namely when a weaker equivalence between two spaces implies a stronger equivalence between them. Freedman’s work [11] on the classification of closed oriented simply connected topological 4-manifolds via the intersection form is a good example of this type of question. In toric topology, a similar type of question was posed in [15], which is now called the cohomological rigidity problem, which asks if homeomorphism/diffeomorphism classes of quasitoric manifolds can be classified by their integral cohomology rings.

Although the problem looks overambitious, it is a sensible question to ask on the following basis. No counter-example has been found since it was formulated. On the contrary, there is a piece of evidence supporting the cohomological rigidity of quasitoric manifolds. Indeed, the classification result in [16] together with the description of the cohomology ring of a quasitoric manifold [9, theorem 4.14]
implies the cohomological rigidity of 4-dimensional quasitoric manifolds. Besides, many affirmative answers have been proved, for instance certain Bott manifolds [5], generalized Bott manifolds [6] and 6-dimensional quasitoric manifolds associated to 3-dimensional Pogorelov polytopes [1].

Being a generalized notion of quasitoric manifold, a toric orbifold [9] is a 2n-dimensional compact orbifold equipped with a locally standard $T^n$-action whose orbit space is a simple polytope. It is known that the cohomology rings fail to classify toric orbifolds up to homeomorphism. For instance, there are weighted projective spaces with isomorphic cohomology rings that are not homeomorphic. Therefore, toric orbifolds do not satisfy cohomological rigidity. However, in the above counterexamples, two weighted projective spaces with isomorphic cohomology rings are homotopy equivalent [2]. Hence, we take a step back and ask a homotopical version of the cohomological rigidity:

**Question 1.1.** Are two toric orbifolds homotopy equivalent if their integral cohomology rings are isomorphic as graded rings?

This paper aims to answer this question for certain 4-dimensional toric orbifolds. We first study certain CW-complexes which model 4-dimensional toric orbifolds and investigate their homotopy theory. In what follows, $H^*(X)$ denotes the cohomology ring with integral coefficients unless otherwise stated, and $P^3(k)$ denotes the 3-dimensional mod-k Moore space for $k > 1$. It is known that $H^3(X)$ is a finite cyclic group for all 4-dimensional toric orbifolds $X$. We refer to [10, 13]. Let $H^3(X) \cong \mathbb{Z}_m$ with $m = 2^s q$ for $q$ odd and $s \geq 0$. When $q > 1$, we show that $X$ decomposes into a wedge of $P^3(q)$ and a recognizable space.

**Theorem 1.2.** Let $X$ be a 4-dimensional toric orbifold such that $H^3(X) \cong \mathbb{Z}_m$. If $m = 2^s q$ for an odd integer $q > 1$ and $s \geq 0$, then $X$ is homotopy equivalent to $\hat{X} \vee P^3(q)$, where $\hat{X}$ is a simply connected 4-dimensional CW-complex with $H^3(\hat{X}) = \mathbb{Z}_{2^s}$ and $H^i(\hat{X}) \cong H^i(X)$ for $i \neq 3$.

If $m$ is odd or equivalently $s = 0$, then Theorem 1.2 implies $X \simeq \hat{X} \vee P^3(m)$ where $H^3(\hat{X}) = 0$. As an application, we can answer question 1.1 for certain 4-dimensional toric orbifolds in the following theorem.

**Theorem 1.3.** Let $X$ and $X'$ be 4-dimensional toric orbifolds such that $H^3(X)$ and $H^3(X')$ have no 2-torsion. Then $X$ is homotopy equivalent to $X'$ if and only if there is a ring isomorphism $H^*(X) \cong H^*(X')$.

This paper is organized as follows. In §2, we review the constructive definition of a 4-dimensional toric orbifold $X$. In particular, it is important to see that $X$ is the mapping cone of a map from a lens space to a wedge of 2-spheres. This phenomenon is motivated by the study of [4] and can also be understood in terms of a $q$-CW complex studied in [3]. In §3, we define a category $\mathcal{C}_{n,m}$ of certain CW-complexes which model 4-dimensional toric orbifolds and study the homotopy theory of $\mathcal{C}_{n,m}$. Section 4 aims to give a necessary and sufficient condition for $X \in \mathcal{C}_{n,m}$ to decompose into a wedge of $P^3(q)$ and a space in $\mathcal{C}_{n,2^s}$. In §5, we study the $p$-local version of the discussion of §4 for some odd prime $p$ and apply this to
4-dimensional toric orbifolds. Combining the equivalent condition (Proposition 4.8) and the $p$-local decomposition (Proposition 5.3), we finally complete the proofs of theorems 1.2 and 1.3 in §6.

2. Toric orbifolds of dimension 4

We begin with a summary of the constructive definition of a toric orbifold. For our purpose, we focus on the 4-dimensional case. For more details on toric orbifolds see [9, §7], [18, §2] and [7, chapters 3, 10].

Let $P$ be an $(n + 2)$-gon on vertices $v_1, \ldots, v_{n+2}$ for some $n \geq 0$. We denote by $E_i$ the edge connecting $v_i$ and $v_{i+1}$ for $i = 1, \ldots, n + 2$, where we take indices modulo $n + 2$. To each edge $E_i$, assign a primitive vector $\xi_i = (a_i, b_i) \in \mathbb{Z}^2$ such that two adjacent vectors $\xi_i$ and $\xi_{i+1}$ are linearly independent. We often describe this combinatorial data as in Figure 1.

Identify $\mathbb{Z}^2$ with $\text{Hom}(S^1, T^2)$. Each $\xi_i$ defines a one-parameter subgroup of $T^2_{S^1}$

$$S^1_{\xi_i} = \{(t^{a_i}, t^{b_i}) \in T^2 : t \in S^1\}.$$  

Now, define the following identification space

$$X = P \times T^2 / \sim$$

(2.1)

where we identify $(p, g)$ and $(p, h)$ for $gh^{-1} \in S^1_{\xi_i}$ if $p$ is in the relative interior of $E_i$, and for all $g, h \in T^2$ if $p$ is a vertex of $P$. Note that there is no identification between $(p, g)$ and $(g, h)$ unless $p = q$. Here, the torus $T^2$ acts on $X$ by the multiplication on the second factor, which yields the orbit map $\pi: X \to P$ by the projection onto the first factor.

We roughly describe the orbifold structure on $X$ following the identification (2.1). First, there is a standard presentation of $\mathbb{C}^2$ given by a homeomorphism $\mathbb{R}^2_\geq \times T^2 / \sim_{\text{std}} \simeq \mathbb{C}^2$ that maps $[(x, y), (t, s)]$ in $\mathbb{R}^2_\geq \times T^2 / \sim_{\text{std}}$ to $(xt, ys)$ in $\mathbb{C}^2$. Here, the standard identification $\sim_{\text{std}}$ is given by $((x, y), g) \sim_{\text{std}} ((x, y), h)$

1. for $gh^{-1} \in 1 \times S^1$ if $x = 0$ and $y \neq 0$;
2. for $gh^{-1} \in S^1 \times 1$ if $x \neq 0$ and $y = 0$;
3. for all $g, h \in T^2$ if $x = y = 0$. 

Figure 1. $(n + 2)$-gon with primitive vectors on facets.
The homotopy classification of four-dimensional toric orbifolds

Note that there is no identification between \(((x_1, y_1), g)\) and \(((x_2, y_2), h)\) in \(\mathbb{R}^2 \times T^2\) unless \((x_1, y_1) = (x_2, y_2)\).

Let \(U_i\) be a neighbourhood of \(v_i\) in \(P\), which is homeomorphic to \(\mathbb{R}^2\) as a manifold with corners. Let \(\psi_i\) be a homeomorphism \(\mathbb{R}^2 \cong U_i\) and let \(\rho_i: T^2 \to T^2\) be an endomorphism of \(T^2\) given by

\[
\rho_i: T^2 \to T^2, \quad (t_1, t_2) \mapsto (t_1^\alpha t_2^{a_{i+1}}, t_1^{b_i} t_2^{b_{i+1}}).
\]

(2.2)

Since \(\xi_i = (a_i, b_i)\) and \(\xi_{i+1} = (a_{i+1}, b_{i+1})\) are linearly independent, the kernel \(K_i = \ker \rho_i\) is a cyclic subgroup of \(T^2\). Then the map \(\psi_i \times \rho_i\) induces a surjection

\[
\mathbb{C}^2 \cong \mathbb{R}^2 \times T^2 / \sim_{std} \xrightarrow{\psi_i \times \rho_i} U_i \times T^2 / \sim.
\]

(2.3)

This shows that \(U_i \times T^2 / \sim\) is homeomorphic to the quotient \(\mathbb{C}^2 / K_i\), where \(K_i\) acts on \(\mathbb{C}^2\) as a subgroup of \(T^2\). Hence, the map (2.3) forms an orbifold chart around the point \([v_i, g] \in X\). The gluing maps among these orbifold charts are determined by the underlying polygon.

A certain cofibration construction of \(X\) is studied in [4] based on the orbifold structure on \(X\). Pick a vertex \(v_i\) of \(P\) and \(U_i\) is its neighbourhood as above. Consider a line segment \(\ell_i\) in \(P\) connecting two points lying in the relative interior of \(E_i\) and \(E_{i+1}\), respectively. The restriction of identification (2.1) to \(\ell_i\) gives rise to a subspace of \(X\)

\[
L_i = \ell_i \times T^2 / \sim.
\]

By assuming that the homeomorphism \(\psi_i: \mathbb{R}^2 \to U_i\) sends the arc \(S_1^1 = S^1 \cap \mathbb{R}^2\) to \(\ell_i\), the restriction of (2.3) to \(S_1^1 \times T^2 / \sim_{std}\) induces a homeomorphism \((S_1^1 \times T^2 / \sim_{std}) / K_i \cong L_i\). Here, we notice that \(K_i\) is isomorphic to \(\mathbb{Z}_{m_{i,i+1}}\), where

\[
m_{i,j} = |\det [\xi_i^j \xi_j^i]|.
\]

(2.4)

As \(S_1^1 \times T^2 / \sim_{std}\) is homeomorphic to \(S^3\), we conclude that \(L_i\) is homeomorphic to \(S^3 / \mathbb{Z}_{m_{i,i+1}}\), which is \(S^3\) if \(m_{i,i+1} = 1\) and is a lens space otherwise. This description can be found in [19, Proposition 2.3] including higher dimensional cases.

Moreover, the subspace \(U_i \times T^2 / \sim\) is homeomorphic to a tubular neighbourhood of the cone on \(L_i\). Let \(B\) be the union of all edges \(E_j\) where \(j \neq i, i + 1\). The subspace \(B \times T^2 / \sim\) is homotopic to a wedge of \(n\) copies of 2-spheres and the subspace \(P - \{v_i\} \times T^2 / \sim\) retracts to \(B \times T^2 / \sim\). As \(X\) is a union of \((P - \{v_i\}) \times T^2 / \sim\) and \(U_i \times T^2 / \sim\), it implies a homotopy cofibration

\[
L_i \xrightarrow{f_i} \bigvee_{j=1}^n S^2 \to X
\]

(2.5)

where the map \(f_i\) is induced by the composition of the inclusion \(\iota\) and the retraction \(r\)

\[
\ell_i \times T^2 / \sim \xrightarrow{\iota} (P - \{v_i\}) \times T^2 / \sim \xrightarrow{r} B \times T^2 / \sim.
\]

See Figure 2 for a pictorial illustration of (2.5).

Applying the cohomology functor to the cofibre sequence (2.5) and referring to [3, theorem 1.1], we can compute the free part of \(H^*(X)\). The cohomology of \(X\) has
been discussed using various tools in the studies [13, theorem 2.5.5], [10, theorem 2.3] and [14, corollary 5.1] which can be summarized as follows.

**Proposition 2.1.** Let $X$ be a toric orbifold of dimension 4. Then we have

$$
\begin{array}{c|cccccc}
  i & 0 & 1 & 2 & 3 & 4 & \geq 5 \\
  \text{H}^i(X) & \mathbb{Z} & 0 & \mathbb{Z}^n & \mathbb{Z}_m & \mathbb{Z} & 0 \\
\end{array}
$$

(2.6)

where $m$ is the greatest common divisor of \(\{m_{i,j} \mid 1 \leq i < j \leq n + 2\}\) for $m_{i,j}$’s defined in (2.4). We set $\mathbb{Z}_m = 0$ if $m = 1$.

**Remark 2.2.** The way of realizing $X$ as a cofibre in (2.5) can be understood in a more general framework of a $q$-CW complex. A $q$-CW complex is defined inductively starting from a discrete set $X_0$ of points. Then, $X_i$ is defined by the pushout

$$
\begin{array}{c}
\bigsqcup_{\alpha} S^{i-1}/K_{\alpha} \\
\downarrow \phi_{\alpha} \\
X_{i-1}
\end{array} \xleftarrow{i} \begin{array}{c}
\bigsqcup_{\alpha} e^i/K_{\alpha} \\
\downarrow \\
X_i
\end{array}
$$

where $e^i$ and $S^{i-1}$ are $i$-dimensional cell and its boundary, respectively, and $K_{\alpha}$ is a finite group acting linearly on $e_i$. Every toric orbifold is a $q$-CW complex. We refer to [3] for more details.

3. Cohomology of 4-dimensional CW-complexes

3.1. A category of 4-dimensional CW-complexes

Suppose that $X$ is a simply connected CW-complex satisfying (2.6). By [12, Proposition 4H.3] it is homotopy equivalent to a CW-complex

$$
\left( \bigsqcup_{i=1}^{n} S^2 \vee P^3(m) \right) \cup_f e^4
$$

(3.1)

where $f : S^3 \to \bigsqcup_{i=1}^{n} S^2 \vee P^3(m)$ is the attaching map of the 4-cell. In this section, we study the homotopy theory of CW-complexes in this form.

Define $\mathcal{C}_{n,m}$ to be the full subcategory of $\text{Top}_*$ consisting of mapping cones as in (3.1). Here, the orientation of the 4-cell $e^4$ is the induced orientation of the upper
hemisphere in $S^5$. We label the $i$th copy of 2-spheres in $\bigvee_{i=1}^n S^2_i$ by $S^2_i$ for $1 \leq i \leq n$ and write
\[
Y = \bigvee_{i=1}^n S^2_i \vee P^3(m)
\]
for short. Let $\mu_i, \nu \in H_2(Y)$ be homology classes representing $S^2_i$ and the 2-cell of $P^3(m)$ respectively. Then, we have
\[
H_2(Y) \cong \mathbb{Z}\langle \mu_1, \ldots, \mu_n \rangle \oplus \mathbb{Z}_m\langle \nu \rangle.
\] (3.2)

Let $g: Y \to Y$ be a map. Then the induced homology map $g_*: H_2(Y) \to H_2(Y)$ is given by $g_*(\mu_i) = \sum_{j=1}^n x_{ij} \mu_j + y_i \nu$ and $g_*(\nu) = z\nu$ for some integers $x_{ij}$ and mod-$m$ congruence classes $y_i$ and $z$. Conversely, we have the following lemma.

**Lemma 3.1.** Given a vector $(y_1, \ldots, y_n, z) \in (\mathbb{Z}_m)^{n+1}$ and an $(n \times n)$-integral matrix
\[
\begin{pmatrix}
x_{11} & \cdots & x_{1n} \\
\vdots & \ddots & \vdots \\
x_{n1} & \cdots & x_{nn}
\end{pmatrix}
\in \text{Mat}_n(\mathbb{Z}),
\]
there exists a map $g: Y \to Y$ such that $g_*(\mu_i) = \sum_{j=1}^n x_{ij} \mu_j + y_i \nu$ and $g_*(\nu) = z\nu$.

**Proof.** First, consider the string of isomorphisms
\[
\bigvee_{i=1}^n S^2_i \vee \bigvee_{j=1}^n S^2_j \vee P^3(m) \cong \bigoplus_{i=1}^n S^2_i \vee \bigvee_{j=1}^n S^2_j \vee P^3(m)
\]
\[
\cong \bigoplus_{i=1}^n \pi_2 \left( \bigvee_{j=1}^n S^2_j \vee P^3(m) \right)
\]
\[
\cong \bigoplus_{i=1}^n H_2 \left( \bigvee_{j=1}^n S^2_j \vee P^3(m) \right)
\]
\[
\cong \bigoplus_{i=1}^n \left( \bigoplus_{j=1}^n \mathbb{Z} \oplus \mathbb{Z}_m \right)
\]
where the third isomorphism is due to the Hurewicz theorem. Under these isomorphisms, take $g': \bigvee_{i=1}^n S^2_i \to Y$ to be the map corresponding to
\[
\begin{pmatrix}
x_{11} & \cdots & x_{1n} \\
\vdots & \ddots & \vdots \\
x_{n1} & \cdots & x_{nn}
\end{pmatrix} \oplus (y_1, \ldots, y_n) \in \left( \bigoplus_{i=1}^n \bigoplus_{j=1}^n \mathbb{Z} \right) \oplus \left( \bigoplus_{i=1}^n \mathbb{Z}_m \right).
\]
Then $g'_*(\mu_i) = \sum_{j=1}^n x_{ij} \mu_j + y_i \nu$. 
Next, for $z \in \mathbb{Z}_m$ let \( g'' : P^3(m) \to Y \) be the composition
\[
g'' : P^3(m) \xrightarrow{z} P^3(m) \hookrightarrow Y,
\]
where \( z : P^3(m) \to P^3(m) \) is the degree-\( z \) map. Let \( g : Y \to Y \) be the wedge sum \( g = g' \vee g'' \). Then \( g_*(\mu_i) = \sum_{j=1}^n x_{ij}^r + y_i \nu \) and \( g_*(\nu) = z \nu \). \( \square \)

3.2. Cellular cup product representation

Let \( C_f \in \mathcal{C}_{n,m} \) be the mapping cone of a map \( f : S^3 \to Y \). As the inclusion \( Y \hookrightarrow C_f \) induces an isomorphism \( H_2(Y) \to H_2(C_f) \), we do not distinguish \( \mu_i, \nu \in H_2(Y) \) and their images in \( H_2(C_f) \). Let \( u_i \in H^2(C_f) \) and \( e \in H^4(C_f) \) be cohomology classes dual to \( \mu_i \) and the homology class represented by the 4-cell in \( C_f \) respectively. Let \( v \in H^3(C_f) \) be the Ext image of \( \nu \). Then
\[
H^2(C_f) \cong \mathbb{Z}\langle u_1, \ldots, u_n \rangle, \quad H^3(C_f) \cong \mathbb{Z}_m\langle v \rangle, \quad H^4(C_f) \cong \mathbb{Z}\langle e \rangle. \tag{3.3}
\]
We call the set \{\( u_1, \ldots, u_n, v, e \)\} the cellular basis of \( H^*(C_f) \).

With coefficient \( \mathbb{Z}_m \), let \( \bar{u}_i \in H^2(C_f; \mathbb{Z}_m) \) and \( \bar{e} \in H^4(C_f; \mathbb{Z}_m) \) be the mod-\( m \) images of \( u_i \) and \( e \), and let \( \bar{v} \in H^2(C_f; \mathbb{Z}_m) \) be the cohomology class dual to \( \nu \). Then
\[
H^2(C_f; \mathbb{Z}_m) \cong \mathbb{Z}_m\langle \bar{u}_1, \ldots, \bar{u}_n, \bar{v} \rangle, \quad H^3(C_f; \mathbb{Z}_m) \cong \mathbb{Z}_m\langle \beta(\bar{v}) \rangle, \quad H^4(C_f; \mathbb{Z}_m) \cong \mathbb{Z}_m\langle \bar{e} \rangle,
\]
where \( \beta \) is the Bockstein homomorphism. We call the set \{\( \bar{u}_1, \ldots, \bar{u}_n, \bar{v}; \bar{e} \)\} the mod-\( m \) cellular basis of \( H^*(C_f; \mathbb{Z}_m) \).

**Definition 3.2.** Let \( C_f \) be a mapping cone in \( \mathcal{C}_{n,m} \). Then the cellular cup product representation \( M_{\text{cup}}(C_f) \) of \( C_f \) is \( A \in \text{Mat}_n(\mathbb{Z}) \) if \( m = 1 \), and is a triple \((A, b, c) \in \text{Mat}_n(\mathbb{Z}) \oplus (\mathbb{Z}_m)^n \oplus \mathbb{Z}_m \) if \( m > 1 \), where
\[
A = \begin{pmatrix}
a_{11} & \cdots & a_{1n} \\
\vdots & \ddots & \vdots \\
a_{n1} & \cdots & a_{nn}
\end{pmatrix}
\quad \text{and} \quad b = (b_1, \ldots, b_n)
\]
are given by \( u_i \cup u_j = a_{ij} e, \quad \bar{u}_i \cup \bar{v} = b_i \bar{e} \) and \( \bar{v} \cup \bar{v} = c \bar{e} \).

Here, \( A \) is a symmetric matrix since it is the matrix representation of the bilinear form
\[
(-\cup-) : H^2(C_f; \mathbb{Z}) \otimes H^2(C_f; \mathbb{Z}) \to H^4(C_f; \mathbb{Z})
\]
with respect to the cellular basis \{\( u_1, \ldots, u_n; e \)\}. Furthermore, universal coefficient theorem implies \( \bar{u}_i \cup \bar{u}_j = a_{ij} \bar{e} \) (mod \( m \)). So, \((-\cup-) : H^2(C_f; \mathbb{Z}_m) \otimes H^2(C_f; \mathbb{Z}_m) \to H^4(C_f; \mathbb{Z}_m) \) can be recovered from \( M_{\text{cup}}(C_f) \) as well.
REMARK 3.3. When $X$ is an oriented compact smooth 4-manifold, the intersection form $I(X)$ is the bilinear form given by cup products of degree 2 cohomology classes modulo torsion

$$I(X): H^2(X)/\text{Tor} \otimes H^2(X)/\text{Tor} \to \mathbb{Z}, \quad x \otimes y \mapsto \langle x \cup y, [X] \rangle,$$

where $[X] \in H_4(X)$ is the fundamental class. Although defined in a similar fashion, $I(X)$ and $M_{\text{cup}}(X)$ are different. First, $I(X)$ only concerns cup products of free elements in $H^2(X)$ and its matrix representation is a symmetric matrix, while $M_{\text{cup}}(X)$ concerns cup products of cohomology with integral and $\mathbb{Z}_m$-coefficients and is a triple consisting of a matrix, a mod-$m$ vector and a mod-$m$ congruence class that record all data. Second, a matrix representation of $I(X)$ depends on the choice of generators of $H^2(X)$, whereas we define $M_{\text{cup}}(X)$ using a fixed CW-complex structure of $X$. In the following section, we will discuss the transformation between cellular map representations of two CW-complex structures of the same $X$. It is similar to matrix congruence but is slightly more complicated, as cup products of cohomology with $\mathbb{Z}_m$-coefficient are involved.

Let $g: S^3 \to Y$ be another map and let $C_g \in \mathcal{C}_{n,m}$ be its mapping cone. Recall that $f + g$ is the composition

$$f + g: S^3 \xrightarrow{\text{comult}} S^3 \vee S^3 \xrightarrow{f \vee g} Y \vee Y \xrightarrow{\text{fold}} Y.$$

Denote its mapping cone by $C_{f+g}$.

LEMMA 3.4. Let $Y$ be (1) $S^1_1 \vee S^2_2$ or (2) $S^1_2 \vee P^3(m)$ and let $f, g: S^3 \to Y$ be two maps. Then $M_{\text{cup}}(C_{f+g}) = M_{\text{cup}}(C_f) + M_{\text{cup}}(C_g)$.

Proof. In the following, we only prove case (2). The proof also works for case (1) but is simpler. Let

- $\{u, v; e\}, \{u_1, v_1; e_1\}$ and $\{u_2, v_2; e_2\}$ be the cellular bases of $H^*(C_{f+g}), H^*(C_f)$ and $H^*(C_g)$, respectively;
- $\{\bar{u}, \bar{v}; \bar{e}\}, \{\bar{u}_1, \bar{v}_1; \bar{e}_1\}$ and $\{\bar{u}_2, \bar{v}_2; \bar{e}_2\}$ be the mod-$m$ cellular bases of $H^*(C_{f+g}; \mathbb{Z}_m), H^*(C_f; \mathbb{Z}_m)$ and $H^*(C_g; \mathbb{Z}_m)$, respectively;
- the cellular cup product representations of $C_{f+g}, C_f$ and $C_g$ be $M_{\text{cup}}(C_{f+g}) = (A, b, c), M_{\text{cup}}(C_f) = (A_1, b_1, c_1)$ and $M_{\text{cup}}(C_g) = (A_2, b_2, c_2)$, respectively.

Here, $A, A_1, A_2$ are integers and $b, b_1, b_2$ are mod-$m$ congruence classes. We claim that

$$A = A_1 + A_2, \quad b = b_1 + b_2 \quad \text{and} \quad c = c_1 + c_2.$$

Consider the mapping cone $C' = Y \cup_{f \vee g} (e_1^4 \vee e_2^4)$ of $g \vee h: S^3 \vee S^3 \to Y$. Let

- $u' \in H^2(C'), e'_1, e'_2 \in H^4(C')$ be cohomology classes dual to $S^2, e_1^4$ and $e_2^4$;
- $\bar{u}' \in H^2(C'; \mathbb{Z}_m), \bar{e}'_1, \bar{e}'_2 \in H^4(C'; \mathbb{Z}_m)$ be the mod-$m$ images of $u, e'_1$ and $e'_2$;
- $\bar{v}' \in H^2(C'; \mathbb{Z}_m)$ be the cohomology class dual to the 2-cell of $P^3(m)$.
Observe that $C_f$ and $C_g$ are subcomplexes of $C'$. Let $\iota_1 : C_f \to C'$ and $\iota_2 : C_g \to C'$ be natural inclusions and let $q : C_{f+g} \to C'$ be the map collapsing the equatorial disk of the 4-cell in $C_{f+g}$ to a point. Then

$$q^*(u') = u, \quad q^*(\bar{v}) = \bar{v}, \quad q^*(e'_1) = q^*(e'_2) = e,$$

$$i_j^*(u') = u_j, \quad i_j^*(\bar{v}) = \bar{v}_j, \quad i_j^*(e'_k) = \delta_{jk} e_j$$

for $j, k \in \{1, 2\}$, where $\delta_{jk}$ is the Kronecker symbol. On the one hand, $u' \cup u' = \alpha_1 e'_1 + \alpha_2 e'_2$ for some integers $\alpha_1$ and $\alpha_2$. Now the naturality of cup products implies

$$i_j^*(u' \cup u') = i_j^*(\alpha_1 e'_1 + \alpha_2 e'_2)$$

$$i_j^*(u') \cup i_j^*(u') = \alpha_1 i_j^*(e'_1) + \alpha_2 i_j^*(e'_2)$$

$$u_j \cup u_j = \alpha_j e_j$$

for $j \in \{1, 2\}$. So, $\alpha_j = A_j$. On the other hand,

$$u \cup u = q^*(u') \cup q^*(u')$$

$$= q^*(u' \cup u')$$

$$= q^*(A_1 e'_1 + A_2 e'_2)$$

$$= A_1 q^*(e'_1) + A_2 q^*(e'_2)$$

$$= (A_1 + A_2)e.$$

So, $A = A_1 + A_2$. Similarly we can show $b = b_1 + b_2$ and $c = c_1 + c_2$. Therefore, we have

$$M_{cup}(C_{f+g}) = M_{cup}(C_f) + M_{cup}(C_g). \quad \Box$$

### 3.3. Cellular map representations

Let $f, f' : S^5 \to Y$ be two maps and $C_f, C_{f'} \in \mathcal{C}_{n,m}$ be their mapping cones. Let

- $\{u_1, \ldots, u_n, v, e\}$ and $\{u'_1, \ldots, u'_n, v', e'\}$ be the cellular bases of $H^*(C_f)$ and $H^*(C_{f'})$,

- $\{\bar{u}_1, \ldots, \bar{u}_m, \bar{v}, \bar{e}\}$ and $\{\bar{u}'_1, \ldots, \bar{u}'_m, \bar{v}', \bar{e}'\}$ be the mod-$m$ cellular bases of $H^*(C_f; \mathbb{Z}_m)$ and $H^*(C_{f'}; \mathbb{Z}_m)$.

Given a map $\psi : C_{f'} \to C_f$ and a coefficient ring $R$, let

$$\psi_R^* : H^2(C_f; R) \to H^2(C_{f'}; R)$$

be the induced morphism on the second cohomology with coefficient $R$.

**Definition 3.5.** Let $\psi : C_{f'} \to C_f$ be a map. Then the cellular map representation $M(\psi)$ of $\psi$ is $W \in \text{Mat}_n(\mathbb{Z})$ if $m = 1$, and is the triple $(W, y, z) \in \text{Mat}_n(\mathbb{Z}) \oplus \text{Mat}_n(\mathbb{Z}) \oplus \cdots \oplus \text{Mat}_n(\mathbb{Z})$.
Lemma 3.7. Following lemma. With respect to different bases of \(W\) is an invertible matrix and \(\psi\) is dual to \(\psi\). Equivalence \(\psi\) (we have \(\psi\), \(\psi\)).

Proof. Since \(C_f\) and \(C_{f'}\) are simply connected, universal coefficient theorem implies that

\[ H^2(C_f; R) \cong \text{Hom}(H_2(C_f), R), H^2(C_{f'}; R) \cong \text{Hom}(H_2(C_{f'}), R) \]

is dual to \(\psi : H_2(C_{f'}) \to H_2(C_f)\). So, \(\psi^*_{Z_m}(\bar{u}_j)\) is the mod-\(m\) image of \(\psi^*_Z(u_j)\) and the first part of the lemma follows.

If \(\psi\) is a homotopy equivalence, then \(W \in \text{Mat}_n(Z)\) and

\[
\begin{pmatrix}
\bar{x}_{11} & \cdots & \bar{x}_{1n} & y_1 \\
\vdots & \ddots & \vdots & \vdots \\
\bar{x}_{n1} & \cdots & \bar{x}_{nn} & y_n \\
0 & \cdots & 0 & z
\end{pmatrix}
\]

are invertible matrices. So, the second part follows.

The cellular map representation records the data of \(\psi^*_Z\) and \(\psi^*_{Z_m}\). The square matrix \(W\) in (3.4) is the map representation of \(\psi^*_Z\) with respect to bases \(\{u_1, \ldots, u_n\}\) and \(\{u'_1, \ldots, u'_n\}\). Lemma 3.6 implies that

\[
\begin{pmatrix}
\bar{W} & y^t \\
0 & z
\end{pmatrix}
\]

is the matrix representation of \(\psi^*_{Z_m}\) with respect to bases \(\{\bar{u}_1, \ldots, \bar{u}_n, \bar{v}\}\) and \(\{\bar{u}'_1, \ldots, \bar{u}'_n, \bar{v}'\}\), where \(\bar{W}\) is the mod-\(m\) image of \(W\) and \(0 = (0, \ldots, 0)\).

Recall that in linear algebra, matrix representations of a bilinear form \(V \otimes V \to Z\) with respect to different bases of \(V\) are congruent to each other. So, we have the following lemma.

Lemma 3.7. For \(C_f, C_{f'} \in \mathcal{C}_{n,m}\), let \(M_{\text{cup}}(C_f) = (A, b, c)\) and \(M_{\text{cup}}(C_{f'}) = (A', b', c')\) be their cellular cup product representations. If there is a homotopy equivalence \(\psi : C_f \to C_{f'}\) with \(M(\psi) = (W, y, z) \in GL_n(Z) \oplus (Z_m)^n \oplus Z_m\), then

\[
A' = W^tAW, \quad b' = yAW + zBW, \quad c' = yAg + 2zby + z^2c.
\]
In particular, if two maps $f$ and $f': S^3 \to Y$ are homotopic, then the matrix cup product representations of their mapping cones are the same.

**Lemma 3.8.** If $f$ is homotopic to $f'$, then $M_{\text{cup}}(C_f) = M_{\text{cup}}(C_{f'})$.

**Proof.** Take a homotopy $\phi: S^3 \times I \to Y$ between $f$ and $f'$. It induces a homotopy equivalence $\Phi: C_f \to C_{f'}$ such that its restriction to $Y$ is the identity map. So, $M(\Phi) = (I_n, 0, 1)$. Then the lemma follows from lemma 3.7. $\square$

**Lemma 3.9.** Let $C_f \in \mathcal{C}_{n,m}$ and let $(W, y, z)$ be a triple in $GL_n(\mathbb{Z}) \oplus (\mathbb{Z}_m)^n \oplus \mathbb{Z}_m^*$. Then there exist a CW-complex $C_{f'} \in \mathcal{C}_{n,m}$ and a homotopy equivalence $\psi: C_f \to C_{f'}$ such that the cellular map representation $M(\psi)$ is $(W, y, z)$.

**Proof.** Let $W = \left( \begin{array}{ccc} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{array} \right)$ and $y = (y_1, \ldots, y_n)$.

By lemma 3.1, there exists a map $\tilde{\psi}: Y \to Y$ such that $\tilde{\psi}_*(\mu_i) = \sum_{j=1}^n x_{ij} \mu_j + y_i \nu$ and $\tilde{\psi}_*(\nu) = z \nu$, where $\mu_1, \ldots, \mu_n$ and $\nu$ are elements in $H_2(Y)$ as in (3.2). Thus, we have $\tilde{\psi}_*^{-1}(u_i) = \sum_{j=1}^n x_{ij} u_j$ and $\tilde{\psi}_*^{-1}(v) = \sum_{i=1}^n y_i u_i + z v$.

Let $f' = \tilde{\psi} \circ f$ and let $C_{f'}$ be its mapping cone. Then there is a diagram of cofibration sequences

\[
\begin{array}{ccc}
S^3 & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
S^3 & \xrightarrow{f'} & Y \\
\end{array}
\]

\[
\begin{array}{ccc}
& & \xrightarrow{\psi} \\
\downarrow & & \downarrow \\
& & \xrightarrow{\psi} \\
\end{array}
\]

where $\psi$ is an induced map. Since $W \in GL_n(\mathbb{Z})$ and $z \in \mathbb{Z}_m^*$, the middle vertical arrow $\tilde{\psi}$ induces an isomorphism in homology. By five lemma, $\psi_*$ is an isomorphism, which implies that $\psi$ is a homotopy equivalence. Finally, we have $M(\psi) = (W, y, z)$ by the construction. $\square$

4. The homotopy theory of complexes in $\mathcal{C}_{n,m}$

4.1. The $\mathcal{C}_{n,1}$ case

When $m = 1$, the mapping cone $C_f \in \mathcal{C}_{n,1}$ is in the form $\bigvee_{i=1}^n S^2_i \cup_f e^4$ where $f: S^3 \to \bigvee_{i=1}^n S^2_i$ is the attaching map of the 4-cell. The Hilton–Milnor theorem (see for instance [20, theorem 7.9.4]) implies that $f$ is homotopic to a wedge sum

\[
\sum_{i=1}^n a_i \eta_i + \sum_{1 \leq j < k \leq n} a_{jk} \omega_{jk},
\]
for some integers $a_i$’s and $a_{jk}$’s. Here $\eta_i$’s and $\omega_{jk}$’s are compositions

$$\eta_i : S^3 \xrightarrow{\eta} S^2_i \hookrightarrow \bigvee_{\ell=1}^n S^2_\ell$$

$$\omega_{jk} : S^3 \xrightarrow{[t_1,t_2]} S^2_j \vee S^2_k \hookrightarrow \bigvee_{\ell=1}^n S^2_\ell$$

of Hopf map $\eta$, Whitehead product $[t_1,t_2]$ and canonical inclusions of $S^2_i$ and $S^2_j \vee S^2_k$ into $\bigvee_{\ell=1}^n S^2_\ell$. The lemma below shows that the coefficients $a_i$ and $a_{jk}$ are determined by $M_{\text{cup}}(C_f)$.

**Lemma 4.1.** Let $C_f \in C_{n,1}$ be the mapping cone of $f \simeq \sum_{i=1}^n a_i\eta_i + \sum_{1 \leq j < k \leq n} a_{jk}\omega_{jk}$. If

$$M_{\text{cup}}(C_f) = \begin{pmatrix} a'_1 & a'_{12} & \cdots & a'_{1n} \\ a'_{12} & a'_2 & \cdots & a'_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a'_{1n} & a'_{2n} & \cdots & a'_n \end{pmatrix},$$

then $a_i = a'_i$ and $a_{jk} = a'_{jk}$ for all $i$, $j$ and $k$.

**Proof.** By lemma 3.8, we may assume $f = \sum_{i=1}^n a_i\eta_i + \sum_{1 \leq j < k \leq n} a_{jk}\omega_{jk}$. For $n = 2$, let $C_1, C_2$ and $C_{12}$ be the mapping cones of $a_1\eta_1, a_2\eta_2$ and $a_{12}\omega_{12}$. Then their cellular cup product representations are

$$M_{\text{cup}}(C_1) = \begin{pmatrix} a'_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M_{\text{cup}}(C_2) = \begin{pmatrix} 0 & 0 \\ 0 & a_2 \end{pmatrix},$$

$$M_{\text{cup}}(C_{12}) = \begin{pmatrix} 0 & a_{12} \\ a_{12} & 0 \end{pmatrix}.$$ 

By lemma 3.4, we have

$$M_{\text{cup}}(C_f) = \begin{pmatrix} a'_1 & a'_{12} \\ a'_{12} & a'_2 \end{pmatrix}.$$ 

So, the lemma holds.

For $n \geq 3$, let $\{u_1, \ldots, u_n, e\}$ be the cellular basis of $H^*(C_f)$. We claim that

$$u_i \cup u_i = a_i e \quad \text{and} \quad u_j \cup u_k = a_{jk} e,$$

for each $1 \leq i \leq n$ and $1 \leq j < k \leq n$. The composition

$$f_{jk} : S^3 \xrightarrow{f} \bigvee_{l=1}^n S^2_l \xrightarrow{\text{pinch}} S^2_j \vee S^2_k$$

is homotopic to $a_j\eta'_1 + a_k\eta'_2 + a_{jk}\omega'_{12}$, where $\eta'_1 : S^3 \xrightarrow{\eta} S^2_j \hookrightarrow S^2_j \vee S^2_k$ and $\eta'_2 : S^3 \xrightarrow{\eta} S^2_k \hookrightarrow S^2_j \vee S^2_k$ are compositions of Hopf map $\eta$ and canonical inclusions and
$\omega_{12}': S^3 \to S^2_j \lor S^2_k$ is the Whitehead product. Let $C_{jk}$ be the mapping cone of $f_{jk}$. By lemma 3.8 and the above argument, we have

$$M_{\text{cup}}(C_{jk}) = \begin{pmatrix} a_j & a_{jk} \\ a_{jk} & a_k \end{pmatrix}.$$ 

Let $\{u'_j, u'_k; e'\}$ be the cellular basis of $H^*(C_{jk})$ and let $\alpha: C_f \to C_{jk}$ be the map which pinches all 2-spheres in $C_f$ to the basepoint except for $S^2_j$ and $S^2_k$. Then

$$\alpha^*(u'_j) = u_j, \quad \alpha^*(u'_k) = u_k \quad \text{and} \quad \alpha^*(e') = e.$$ 

By the naturality of cup products, we have

$$\alpha^*(u'_j) \cup \alpha^*(u'_k) = \alpha^*(u'_j \cup u'_k) = a_{jk}e.$$ 

So, $a'_{jk} = a_{jk}$. Similarly we can show $a'_i = a_i$. Hence, the lemma follows. \qed

Now we classify the homotopy types of CW-complexes in $\mathcal{C}_{n,1}$ by their integral cohomology rings in the next statement.

**Proposition 4.2.** Let $f, f': S^3 \to \bigvee_{i=1}^n S^2_i$ be two maps and let $C_f, C_{f'} \in \mathcal{C}_{n,1}$ be their mapping cones. Then $C_f \simeq C_{f'}$ if and only if there is a ring isomorphism $H^*(C_f) \cong H^*(C_{f'})$.

**Proof.** The ‘only if’ part is trivial. Assume that $H^*(C_f) \cong H^*(C_{f'})$. Then there is an invertible matrix $W \in \text{GL}_n(\mathbb{Z})$ such that

$$W^t \cdot M_{\text{cup}}(C_f) \cdot W = \varepsilon M_{\text{cup}}(C_{f'}).$$

where $\varepsilon$ is either 1 or $-1$. Suppose first $\varepsilon = 1$. By lemma 3.9, there is a CW-complex $\tilde{C} \in \mathcal{C}_{n,1}$ together with a homotopy equivalence $\psi: \tilde{C} \to C_f$ such that $M(\psi) = W$. We claim that $\tilde{C} \simeq C_{f'}$. By lemma 3.7, we have

$$M_{\text{cup}}(\tilde{C}) = W^t \cdot M_{\text{cup}}(C_f) \cdot W = M_{\text{cup}}(C_{f'}).$$

Let $\tilde{f}$ be the attaching map of the 4-cell in $\tilde{C}$ and let

$$M_{\text{cup}}(\tilde{C}) = M_{\text{cup}}(C_{f'}) = \begin{pmatrix} a_1 & a_{12} & \cdots & a_{1n} \\ a_{12} & a_2 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_n \end{pmatrix}.$$ 

Then lemma 4.1 implies that $f'$ and $\tilde{f}$ are homotopic to the wedge sum

$$\sum_{i=1}^n a_i\eta_i + \sum_{1 \leq i < j \leq n} a_{ij}\omega_{ij},$$

which means $C_{f'} \simeq \tilde{C}$. Therefore, $C_f \simeq C_{f'}$. 

Suppose $\varepsilon = -1$. Let $C_{-f'}$ be the mapping cone of $-f' : S^3 \rightarrow S^3 \rightarrow \bigvee_{i=1}^n S_i^2$.
Then
$$W^f \cdot M_{cup}(C_f) \cdot W = M_{cup}(C_{-f'})$$
and $C_{-f'} \simeq C_{f'}$. The above argument implies $C_f \simeq C_{-f'}$, so $C_f \simeq C_{f'}$. Hence, we have established the claim. $\square$

**Corollary 4.3.** Two 4-dimensional toric orbifolds without torsion in (co)homology are homotopy equivalent if and only if their integral cohomology rings are isomorphic.

As 4-dimensional quasitoric manifolds always have torsion-free (co)homology, corollary 4.3 implies that the homotopy types of 4-dimensional quasitoric manifolds are classified by their cohomology rings. As we mentioned in Introduction, the homeomorphism types of 4-dimensional toric manifolds are cohomologically rigid. One can deduce the conclusion from the topological classification of 4-dimensional smooth manifolds with $T^2$-action studied in [16] together with the cohomology formula [9, theorem 4.14].

We note that the method in Proposition 4.2 applies to CW-complexes in $\mathcal{C}_{n,1}$ which are not necessarily manifolds. We also refer to [8, §5] for the computation of the cohomology ring of toric orbifolds considered in corollary 4.3.

### 4.2. The $\mathcal{C}_{n,m}$ case

From now on, we assume $m = 2^s q$, where $q > 1$ is odd and $s \geq 0$. Recall from (3.3) that $H^3(C_f) \cong \mathbb{Z}_m$ for $C_f \in \mathcal{C}_{n,m}$. In this subsection, we discuss the homotopy type of $C_f$. To be more precise, we study a necessary and sufficient condition for a wedge decomposition

$$C_f \simeq \hat{C} \vee P^3(q)$$  \hspace{1cm} (4.1)

where $\hat{C}$ is a complex in $\mathcal{C}_{n,2}$, so that $H^i(C_f) \cong H^i(\hat{C})$ for $i \neq 3$ and $H^3(\hat{C}) \cong \mathbb{Z}_2^*$. 

**Lemma 4.4.** Let $q$ be odd and greater than 1. Consider

(i) a map $g_1 : S^3 \rightarrow P^3(q)$ and its mapping cone $C_1$,

(ii) a map $g_2 : S^3 \rightarrow P^4(q)$ and the mapping cone $C_2$ of the composition

$$S^3 \xrightarrow{g_2} P^4(q) \xrightarrow{[\kappa_1, \kappa_2]} S^2 \vee P^3(q),$$

where $[\kappa_1, \kappa_2]$ is the Whitehead product of inclusions

$$\kappa_1 : S^2 \rightarrow S^2 \vee P^3(q) \text{ and } \kappa_2 : P^3(q) \rightarrow S^2 \vee P^3(q).$$

For $i = 1$ or 2, if $H^i(C_i; \mathbb{Z}_m)$ has trivial cup products, then $g_i$ is null homotopic.
Lemma 4.5. Let \( q = p_1^r_1 \ldots p_s^r_s \) be a primary factorization of \( q \) such that \( p_j \)'s are different odd primes and all \( r_j \)'s are at least 1. By the Hurewicz theorem, \( \pi_3(P^4(q)) \cong \mathbb{Z}_q \).

By [17, theorem 4] and [21, lemma 2.1],

\[
\pi_3(P^3(q)) \cong \bigoplus_{j=1}^\ell \pi_3(P^3(p_j^{r_j})) \cong \bigoplus_{j=1}^\ell \mathbb{Z}_{p_j^{r_j}} \cong \mathbb{Z}_q.
\]

It suffices to prove the two cases after localization at \( p_j \).

For the \( i = 1 \) case, the lemma is a special case of [21, Proposition 4.4]. For the \( i = 2 \) case, it can be proved by the argument of [21, Proposition 3.2] and replacing \( P^3(p^4) \) by \( S^2 \) and the index \( t \) by \( \infty \), respectively.

\[\square\]

Lemma 4.5. Let \( m = 2^s q \), where \( q \) is odd and greater than 1. Let \( f : S^3 \to \bigvee_{i=1}^n S^2 \lor P^3(m) \) be the attaching map of the 4-cell in \( C_f \) and let \( M_{\text{cup}}(C_f) = (A, b, c) \). If \( b \equiv (0, \ldots, 0) \) (mod \( q \)) and \( c \equiv 0 \) (mod \( q \)), then there is a CW-complex \( \hat{C} \in \mathcal{C}_{n,2} \), such that \( C_f \simeq \hat{C} \lor P^3(q) \).

Proof. Since \( 2^s \) and \( q \) are coprime, we have \( P^3(m) \cong P^3(2^s) \lor P^3(q) \). By the Hilton–Milnor theorem, \( f \) is homotopic to a wedge sum

\[ f \simeq \sum_{i=1}^{n} a_i \eta_i + \sum_{1 \leq j < k \leq n} a_{jk} \omega_{jk} + \eta' + \sum_{i=1}^{n} \omega'_i + \eta_q + \sum_{i=1}^{n} \omega_{iq} \]

for some integers \( a_i \)'s and \( a_{jk} \)'s. Here, \( \eta' \), \( \omega'_i \), \( \eta_q \) and \( \omega_{iq} \) are compositions

\[
\eta' : S^3 \xrightarrow{b'} P^3(2^s) \hookrightarrow \bigvee_{j=1}^{n} S^2_j \lor P^3(2^s) \lor P^3(q) \\
\omega'_i : S^3 \xrightarrow{b'_i} P^4(2^s) \xrightarrow{[\kappa'_1, \kappa'_2]} S^2_i \lor P^3(2^s) \hookrightarrow \bigvee_{j=1}^{n} S^2_j \lor P^3(2^s) \lor P^3(q) \\
\eta_q : S^3 \xrightarrow{b_q} P^3(q) \hookrightarrow \bigvee_{j=1}^{n} S^2_j \lor P^3(2^s) \lor P^3(q) \\
\omega_{iq} : S^3 \xrightarrow{b_{iq}} P^4(q) \xrightarrow{[\kappa_1, \kappa_2]} S^2_i \lor P^3(q) \hookrightarrow \bigvee_{j=1}^{n} S^2_j \lor P^3(2^s) \lor P^3(q)
\]

for some maps \( b' \), \( b'_i \) and \( b_q \), \( b_{iq} \). Here

\[ [\kappa'_1, \kappa'_2] : P^4(2^s) \to S^2_i \lor P^3(2^s) \]

is the Whitehead product of inclusions \( \kappa'_1 : S^2_i \to S^2_i \lor P^3(2^s) \) and \( \kappa'_2 : P^3(2^s) \to S^2_i \lor P^3(2^s) \), and

\[ [\kappa_1, \kappa_2] : P^4(q) \to S^2_i \lor P^3(q) \]

is the Whitehead product of inclusions \( \kappa_1 : S^2_i \to S^2_i \lor P^3(q) \) and \( \kappa_2 : P^3(q) \to S^2_i \lor P^3(q) \). If \( \eta_q \) and \( \omega_{iq} \)'s are null homotopic, then \( f \) factors through a map \( \hat{f} : S^3 \to \bigvee_{i=1}^{n} S^2_i \lor P^3(2^s) \). Let \( \hat{C} \) be the mapping cone of \( \hat{f} \). Then \( C_f \simeq \hat{C} \lor P^3(q) \).
Hence, it suffices to show that $\eta_q$ and $\omega_{\ell q}$ are null homotopic for any $\ell$ with $1 \leq \ell \leq n$. After localization away from 2, $P^3(2^*)$ becomes contractible and the composition

$$f_{\ell q}: S^3 \xrightarrow{f} \bigvee_{j=1}^n S^2_j \vee P^3(q) \xrightarrow{\text{pinch}} S^2_\ell \vee P^3(q)$$

is homotopic to the wedge sum $a_\ell \tilde{\eta}_\ell + j \circ b_q + [\kappa_1, \kappa_2] \circ b_{\ell q}$, where $\tilde{\eta}$ is the composition of Hopf map $\eta$ and inclusion $S^2_\ell \to S^2_\ell \vee P^3(q)$, and $j: P^3(q) \to S^2_\ell \vee P^3(q)$ is the inclusion.

Consider the diagram of homotopy cofibration sequences

$$
\begin{array}{ccc}
S^3 & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \psi \\
S^3 & \xrightarrow{f'} & C_f \\
\end{array}
\begin{array}{ccc}
Y & \longrightarrow & C_f \\
\psi & & \psi \\
C_f & \longrightarrow & C_{f'}
\end{array}
$$

where $C_{\ell q}$ is the mapping cone of $f_{\ell q}$ and $\pi$ is an induced map. Let $\{\tilde{u}_1, \ldots, \tilde{u}_n, \tilde{v}; \tilde{e}\}$ and $\{\tilde{u}', \tilde{v}'; \tilde{e}'\}$ be the mod-$q$ cellular bases of $H^*(C_f; \mathbb{Z}_q)$ and $H^*(C_{\ell q}; \mathbb{Z}_q)$. Then

$$\pi^*(\tilde{u}') = \tilde{u}_\ell, \quad \pi^*(\tilde{v}') = \tilde{v}, \quad \pi^*(\tilde{e}') = \tilde{e}.$$ 

By the hypothesis, $\pi^*(\tilde{u}' \cup \tilde{v}') = \tilde{u}_\ell \cup \tilde{v} = 0$. Since $\pi^*: H^4(C_{\ell q}; \mathbb{Z}_q) \to H^4(C_f; \mathbb{Z}_q)$ is isomorphic, we have $\tilde{u}' \cup \tilde{v}' = 0$. Similarly, we have $\tilde{v}' \cup \tilde{v}' = 0$ and $\tilde{u}' \cup \tilde{u}' = a_\ell \tilde{e}$ so that $M_{\text{cup}}(C_{\ell q}) = (a_\ell, 0, 0)$. Let $C_1$ and $C_2$ be the mapping cones of $[\kappa_1, \kappa_2] \circ b_{\ell q}$ and $j \circ b_q$. By lemma 3.4,

$$M_{\text{cup}}(C_1) = M_{\text{cup}}(C_2) = (0, 0, 0)$$

so $H^*(C_1; \mathbb{Z}_q)$ and $H^*(C_2; \mathbb{Z}_q)$ have trivial cup products. By lemma 4.4, $b_{\ell q}$ is null homotopic and so is $\omega_{\ell q}$. Also, notice that $C_2 \simeq S^2_\ell \vee C'$ where $C'$ is the mapping cone of $b_q$. So $H^*(C'; \mathbb{Z}_q)$ has trivial cup products. By lemma 4.4, $b_q$ is null homotopic and so is $\eta_q$. \hfill \Box

**Remark 4.6.** In general, $\hat{C}$ cannot be further decomposed into a wedge of non-contractible spaces, for example $\hat{C} = \Sigma \mathbb{R}P^3$.

Notice that $\hat{C} \vee P^3(q)$ is not contained in $\mathcal{C}_{n,m}$, but it is homotopic to a mapping cone in $\mathcal{C}_{n,m}$ as follows. Since $2^*$ and $q$ are coprime to each other, there exist integers $\alpha$ and $\beta$ such that $2^*\alpha + q\beta = 1$, where the mod-$q$ congruence class of $\alpha$ and the mod-2$^*$ congruence class of $\beta$ are unique. Identify $\mathbb{Z}_{2^*} \oplus \mathbb{Z}_q$ with $\mathbb{Z}_m$ via the isomorphism

$$\rho: \mathbb{Z}_{2^*} \oplus \mathbb{Z}_q \to \mathbb{Z}_m, \quad (x, y) \mapsto q\beta x + 2^*\alpha y.$$  \hfill (4.2)
It induces a homotopy equivalence $\rho: P^3(2s) \vee P^3(q) \to P^3(m)$. Consider the diagram of homotopy cofibrations

\[
\begin{array}{ccc}
* & \to & \bigvee_{i \neq \ell}^n S_i^2 \\
\downarrow & & \downarrow \\
S^3 & \xrightarrow{f} & \bigvee_{i=1}^n S_i^2 \vee P^3(q) \\
\downarrow & \quad \downarrow \text{pinch} & \quad \downarrow \\
S^3 & \xrightarrow{f_{\ell q}} & S_{\ell}^2 \vee P^3(q) \\
\end{array}
\]

(4.3)

where $C'$ is the mapping cone of $(id \vee \rho) \circ (\hat{f} \vee *)$ and $\tilde{\rho}$ is an induced homotopy equivalence. Then $C' \simeq \hat{C} \vee P^3(q)$ via $\tilde{\rho}$.

**Lemma 4.7.** Let $M_{\cup P}(C') = (A, b, c)$. Then $b \equiv (0, \ldots, 0)$ and $c \equiv 0 \pmod{q}$.

**Proof.** We prove $b_i \equiv 0 \pmod{m}$. Let

- $\bar{u}_i \in H^2(\hat{C}; \mathbb{Z}_m)$ and $\bar{e} \in H^4(\hat{C}; \mathbb{Z}_m)$ be the mod-$m$ cohomology classes dual to homology classes representing $S_i^2$ and the 4-cell in $\hat{C}$, respectively;
- $\mu_i, \omega_{2s}, \omega_q \in H_2(\hat{C} \vee P^3(q); \mathbb{Z})$ be the homology classes representing $S_i^2$, the bottom cells of $P^3(2s)$ and $P^3(q)$;
- $\bar{w}_{2s}, \bar{w}_q \in H^2(\hat{C} \vee P^3(q); \mathbb{Z}_m)$ be the cohomology classes such that
  \[
  \bar{w}_{2s}(\omega_{2s}) \equiv q\beta \quad \bar{w}_{2s}(\omega_q) \equiv 0 \quad \bar{w}_{2s}(\mu_i) \equiv 0 \\
  \bar{w}_q(\omega_{2s}) \equiv 0 \quad \bar{w}_q(\omega_q) \equiv 2^s \alpha \quad \bar{w}_q(\mu_i) \equiv 0 \pmod{m}.
  \]

Denote $\bar{v} = \bar{w}_{2s} + \bar{w}_q$. Then $\bar{u}_1, \ldots, \bar{u}_n$ and $\bar{v}$ form a basis of $H^2(\hat{C} \vee P^3(q); \mathbb{Z}_m)$. The right square of (4.3) implies

\[
\tilde{\rho}^*(\bar{u}_i^\prime) = \bar{u}_i, \quad \tilde{\rho}^*(\bar{v}^\prime) = \bar{v} = \bar{w}_{2s} + \bar{w}_q.
\]

By the naturality of cup products, we have

\[
\tilde{\rho}^*(\bar{u}_i^\prime \cup \bar{v}^\prime) = \tilde{\rho}^*(b_i \bar{e}^\prime)
\]

\[
\bar{u}_i \cup (\bar{w}_{2s} + \bar{w}_q) = b_i \bar{e}
\]

\[
\bar{u} \cup \bar{w}_{2s} = b_i \bar{e}.
\]

Since $\bar{w}_{2s}$ is a multiple of $q$ and $\bar{e}$ is a generator, $b_i \equiv 0 \pmod{q}$. Similarly we can show that $c \equiv 0 \pmod{q}$. \qed
Proposition 4.8. Let $m = 2^s q$ as before. For $C_f \in \mathcal{C}_{n,m}$, let $M_{cu}(C_f) = (A, b, c)$ where

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \quad \text{and} \quad b = (b_1, \ldots, b_n).$$

Then $C_f \simeq \hat{C} \vee P^3(q)$ for some $\hat{C} \in \mathcal{C}_{n,2^s}$ if and only if the system of mod-$q$ linear equations

$$\begin{align*}
 a_{11}y_1 + \cdots + a_{1n}y_n & \equiv -b_1 \\
 \vdots & \\
 a_{n1}y_1 + \cdots + a_{nn}y_n & \equiv -b_n \\
 b_1y_1 + \cdots + b_ny_n & \equiv -c
\end{align*} \quad (\text{mod } q) \quad (4.4)$$

has a solution $(y_1, \ldots, y_n) \in (\mathbb{Z}_q)^n$.

Proof. Suppose $g : C' \simeq \hat{C} \vee P^3(q) \to C_f$ is a homotopy equivalence. Let $M(g) = (W, y, z)$ and let $M_{cu}(C') = (A', b', c')$. By lemma 3.7, we have

$$A' = W^tAW, \quad b' = yAW + zbw \quad \text{and} \quad c' = yAy + 2zyb + z^2c. \quad (4.5)$$

Lemma 4.7 implies $b' \equiv (0, \ldots, 0)$ and $c' \equiv 0$ modulo $q$. Since $W$ and $z$ are invertible in $\mathbb{Z}_q$, we can rewrite equations in (4.5) as

$$z^{-1}yA \equiv -b \quad \text{and} \quad z^{-1}yb^t \equiv -c \quad (\text{mod } q).$$

Therefore, $z^{-1}y$ is a solution of (4.4).

Conversely, suppose there is a solution $y = (y_1, \ldots, y_n) \in (\mathbb{Z}_q)^n$ of (4.4). By lemma 3.9, there exist $C'' \in \mathcal{C}_{n,m}$ and a homotopy equivalence $g : C'' \to C_f$ such that $M(g) = (I, y', 1)$ and

$$M_{cu}(C'') = A, \ y'\bar{A} + b, \ y'\bar{A}(y')^t + 2b(y')^t + c,$$

where $y' = (\rho(y_1, 0), \ldots, \rho(y_n, 0)) \in (\mathbb{Z}_m)^n$ for $\rho$ defined in (4.2) and $\bar{A}$ is the mod-$m$ image of $A$. Then

$$y'\bar{A} + b \equiv 0 \quad \text{and} \quad y'\bar{A}(y')^t + 2b(y')^t + c \equiv 0 \quad (\text{mod } q).$$

Note that lemma 4.5 implies $C'' \simeq \hat{C} \vee P^3(q)$ for some $\hat{C} \in \mathcal{C}_{n,2^s}$. Consequently, we have $C_f \simeq \hat{C} \vee P^3(q)$. \hfill $\square$

5. Odd primary local decomposition of toric orbifolds

Let $X = P \times T^2/\sim$ be a 4-dimensional toric orbifold associated with the combinatorial data described in § 2. Since $X$ is simply connected and $H^*(X)$ satisfies (2.6), [12, Proposition 4H.3] implies that $X$ is in $\mathcal{C}_{n,m}$ up to homotopy. Let $m = 2^s q,$
where \( q \) is odd and \( s \geq 0 \). In this section, we show that for any odd prime \( p \), there is a \( p \)-local equivalence

\[
X \simeq_{(p)} \hat{X} \vee P^3(q) \tag{5.1}
\]

for a CW-complex \( \hat{X} \) in \( \mathcal{C}_{n,2} \) and \( P^3(q) \) denotes a point if \( q = 1 \).

The \( \mathfrak{q} \)-CW complex structure of \( X \) with respect to a vertex \( v_i \) (see remark 2.2) implies that \( X \) is homotopy equivalent to the mapping cone of a map

\[
f : L_i \rightarrow \bigvee_{j=1}^n S^2
\]

where \( L_i \) is the quotient \( S^3/\mathbb{Z}_{m_i,i+1} \) and \( m_{i,i+1} = |\det [\xi_i^t, \xi_{i+1}^t]| \). Recall that \( \mathbb{Z}_{m_i,i+1} \) is isomorphic to a subgroup \( \ker \rho_i \) of \( T^2 \), where \( \rho_i \) is defined in (2.2). The \( \mathbb{Z}_{m_i,i+1} \)-action on \( S^3 \) is given by the inclusion \( \ker \rho_i \hookrightarrow T^2 \) and the standard \( T^2 \)-action on \( S^3 \). If \( m_{i,i+1} = 1 \), then \( L_i \cong S^3 \) and \( X \) is in \( \mathcal{C}_{n,1} \). So, the equivalence (5.1) holds. If \( m_{i,i+1} > 1 \), then \( L_i \) is a lens space \( L(m_{i,i+1}; k_i) \) for some \( k_i \) coprime to \( m_{i,i+1} \).

In the following, the \( p \)-component \( \nu_p(t) \) of a number \( t \) is defined to be the \( p \)-power \( p^r \) such that \( p^r \) divides \( t \) but \( p^{r+1} \) does not.

**Lemma 5.1.** For odd prime \( p \), let \( \nu_p(m_{i,i+1}) = p^r \) and let \( L_i = L(m_{i,i+1}; k_i) \) be a lens space. Then there is a map \( \alpha_p : \Sigma L_i \rightarrow S^4 \vee P^3(p^r) \) that is a \( p \)-local equivalence.

**Proof.** Let \( m_{i,i+1} = p^r t \) where \( p \) and \( t \) are coprime. Then \( P^3(m_{i,i+1}) \simeq P^3(p^r) \vee P^3(t) \). Consider the diagram of homotopy cofibration sequences

\[
\begin{array}{ccc}
\ast & \rightarrow & P^3(t) \\
\downarrow & & \downarrow \\
S^3 & \phi & \rightarrow P^3(m_{i,i+1}) \\
\downarrow & & \downarrow \text{pinch} \\
S^3 & \phi' & \rightarrow P^3(p^r) \\
\end{array} \rightarrow C
\]

where \( \phi \) is the attaching map of the 4-cell in \( \Sigma L_i \), \( \phi' \) is the composition of \( \phi \) and the pinch map, \( C \) is the mapping cone of \( \phi' \) and \( \alpha_p \) is an induced map. The right column induces an exact sequence

\[
\cdots \rightarrow \tilde{H}^{i-1}(P^3(t); \mathbb{Z}_{p^r}) \rightarrow \tilde{H}^{i}(C; \mathbb{Z}_{p^r}) \xrightarrow{\alpha^*_p} \tilde{H}^{i}(\Sigma L_i; \mathbb{Z}_{p^r}) \rightarrow \tilde{H}^{i}(P^3(t); \mathbb{Z}_{p^r}) \rightarrow \cdots
\]

Since \( \tilde{H}^*(P^3(t); \mathbb{Z}_{p^r}) = 0 \), the map \( \alpha^*_p : \tilde{H}^*(C; \mathbb{Z}_{p^r}) \rightarrow \tilde{H}^*(\Sigma L_i; \mathbb{Z}_{p^r}) \) is an isomorphism. Moreover, \( \tilde{H}^*(C; \mathbb{Z}_{p^r}) \) has trivial cup products because \( \Sigma L_i \) is a suspension. Now, lemma 4.4 shows that \( \phi' \) is null homotopic, which means \( C \cong S^4 \vee P^3(p^r) \). Therefore, we consider \( \alpha_p \) as a map from \( \Sigma L_i \) to \( S^4 \vee P^3(p^r) \). Since \( P^3(t) \) is contractible after \( p \)-localization, the right column implies that \( \alpha_p \) is a \( p \)-local equivalence.
Lemma 5.2. Let $p$ be a prime and let $H^3(X) \cong \mathbb{Z}_m$. Then there exists $i \in \{1, \ldots, n + 2\}$ such that $\nu_p(m_{i,i+1}) = \nu_p(m)$.

Proof. By [14, corollary 5.1] and [10, lemma 3.1], $m = \gcd\{m_{i,j} | 1 \leq i < j \leq n + 2\}$. If $n = 1$, the lemma is trivial. So, we prove the lemma for $n \geq 2$.

Without loss of generality, suppose $\nu_p(m_{1,j}) = \nu_p(m) = p^r$ for some $j \in \{3, \ldots, n + 1\}$. Let $\xi_1 = (a, b)$. Since $a$ and $b$ are coprime, there exist $u$ and $v$ such that

$$\begin{pmatrix} u & -b \\ v & a \end{pmatrix} \in SL_2(\mathbb{Z}).$$

Changing the basis of $\mathbb{Z}^2$ if necessary, we may assume $\xi_1 = (1, 0)$.

Let $\xi_2 = (x, y)$ and $\xi_j = (z, w)$. Then

$$m_{1,2} = \left| \det \begin{pmatrix} 1 & x \\ 0 & y \end{pmatrix} \right| = |y|, \quad m_{1,j} = \left| \det \begin{pmatrix} 1 & z \\ 0 & w \end{pmatrix} \right| = |w|.$$

Write $w = cx^r$ and $y = c'y^s$, where $c$ and $c'$ are integers coprime to $p$ and $s \geq r$. If $s = r$, then $\nu_p(m_{1,2}) = \nu_p(m)$ and consequently the lemma holds. If $s > r$, then

$$m_{2,j} = |xw - yz| = |cx^r - c'y^s| = p^r|cx - c'y^{s-r}|.$$

Since $x$ is coprime to $y$, $x$ is coprime to $p$. So, $cx - c'y^{s-r}$ is coprime to $p$ and $\nu_p(m_{2,j}) = p^r$. If $j = 3$, then we are done. If not, iterate this argument to $m_{2,j}$, $\xi_3$ and $\xi_j$. Then we can conclude that $\nu_p(m_{j-1,j}) = p^r$. \hfill \Box

For any odd prime $p$, let $\nu_p(m) = p^r$. By lemma 5.2, there is an $i \in \{1, \ldots, n + 2\}$ such that $\nu_p(m_{i,i+1}) = \nu_p(m) = p^r$. Pick the vertex $v_i$ and construct the $q$-CW-complex structure with respect to $v_i$. Then there is a homotopy cofibration sequence

$$L_i \xrightarrow{f} \bigvee_{j=1}^n S^2 \to X$$

with a coaction $c : X \to X \vee \Sigma L_i$. Furthermore, the 3-skeleton of $X$ is $\bigvee_{j=1}^n S^2 \vee P^3(m)$ for $m = 2^aq$. Let $\hat{X}$ be the quotient $X/P^3(q)$ and let $\phi_p$ be the composition

$$\phi_p : X \xrightarrow{c} X \vee \Sigma L \xrightarrow{\phi_p} \hat{X} \vee P^3(p^r) \vee S^4 \xrightarrow{\text{pinch}} \hat{X} \vee P^3(p^r)$$

(5.2)

where $\alpha$ is the map in lemma 5.1 and $j : X \to \hat{X}$ is the quotient map.

Proposition 5.3 ($p$-local version of main theorem). Let $p$ be an odd prime. If $\nu_p(m) = p^r$ for some $r \geq 1$, then $\phi_p : X \to \hat{X} \vee P^3(p^r)$ is a $p$-local equivalence.

Proof. We claim that the map $\phi_p$ in (5.2) induces an isomorphism on $\mathbb{Z}_p$-cohomology

$$\phi_p^* : H^*(\hat{X} \vee P^3(p^r); \mathbb{Z}_p) \to H^*(X; \mathbb{Z}_p)$$

(5.3)

where $\mathbb{Z}_p$ is the ring of $p$-local integers.
The cofibration sequence $P^3(q) \hookrightarrow X \xrightarrow{\jmath} \hat{X}$ induces an exact sequence

$$\cdots \rightarrow \tilde{H}^{i-1}(P^3(q)) \rightarrow \tilde{H}^i(\hat{X}) \xrightarrow{j^*} \tilde{H}^i(X) \rightarrow \tilde{H}^i(P^3(q)) \rightarrow \tilde{H}^{i+1}(\hat{X}) \rightarrow \cdots$$

For $i \neq 3$, since $\tilde{H}^i(P^3(p^r)) = 0$ and $j^*: H^i(\hat{X}) \to H^i(X)$ is an isomorphism, the map (5.3) is an isomorphism.

Next, consider the cofibration sequence $L_i \xrightarrow{j} \bigvee_{i=1}^n S^2 \xrightarrow{\delta} \Sigma L_i$, where $i$ is the inclusion and $\delta$ is the coboundary map. It induces an exact sequence

$$\cdots \rightarrow \tilde{H}^i(\Sigma L_i; \mathbb{Z}(p)) \xrightarrow{\delta^*} \tilde{H}^i(X; \mathbb{Z}(p)) \rightarrow \tilde{H}^i\left(\bigvee_{i=1}^n S^2; \mathbb{Z}(p)\right) \rightarrow \tilde{H}^{i+1}(\Sigma L; \mathbb{Z}(p)) \rightarrow \cdots$$

Since $i^*: H^2(X; \mathbb{Z}(p)) \to H^2\left(\bigvee_{i=1}^n S^2; \mathbb{Z}(p)\right)$ is an isomorphism, $\delta^*: \tilde{H}^3(\Sigma L; \mathbb{Z}(p)) \to H^3(X; \mathbb{Z}(p))$ is an isomorphism. Consider the following commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{j^*} & X \vee \Sigma L_i \\
\downarrow & & \downarrow \\
X & \xrightarrow{\delta} & \Sigma L_i \\
\end{array}
\quad
\begin{array}{ccc}
X \vee P^3(p^r) & \xrightarrow{j^*} & \hat{X} \vee P^3(p^r) \\
\downarrow & & \downarrow \\
\Sigma L_i & \xrightarrow{\alpha_p} & P^3(p^r) \\
\end{array}
\quad
\begin{array}{ccc}
\hat{X} \vee P^3(p^r) & \xrightarrow{j^*} & \hat{X} \vee P^3(p^r) \\
\downarrow & & \downarrow \\
P^3(p^r) & \xrightarrow{j^*} & P^3(p^r) \\
\end{array}
$$

where the composite in the upper row is $\phi_p$ in (5.2) and the unnamed arrows are pinch maps. The left square commutes due to the property of the coaction map. By lemma 5.1, the map $\alpha_p^*: H^3(P^3(p^r) \vee S^4; \mathbb{Z}(p)) \to H^3(\Sigma L; \mathbb{Z}(p))$ is isomorphic, so the composite in the lower row induces an isomorphism $H^3(P^3(p^r); \mathbb{Z}(p)) \to H^3(X; \mathbb{Z}(p))$. Since $H^3(\hat{X}; \mathbb{Z}(p)) = 0$, the map (5.3) is an isomorphism for $i = 3$. Therefore, $\phi_p^*: H^3(\hat{X} \vee P^3(p^r); \mathbb{Z}(p)) \to H^3(X; \mathbb{Z}(p))$ is an isomorphism and $\phi_p$ is a $p$-local equivalence. \hfill $\square$

**Lemma 5.4.** Let $X$ be a 4-dimensional toric orbifold with $H^3(X) \cong \mathbb{Z}_m$, and let $\nu_p(m) = p^r$ for some odd prime $p$ and $r \geq 1$. If $M_{\text{cup}}(X) = (A, b, c)$ where

$$A = \begin{pmatrix}
a_{11} & \cdots & a_{1n} \\
\vdots & \ddots & \vdots \\
a_{n1} & \cdots & a_{nn}
\end{pmatrix} \quad \text{and} \quad b = (b_1, \ldots, b_n),$$

then the system of mod-$p^r$ linear equations

$$\begin{cases}
a_{11}y_1 + \cdots + a_{1n}y_n & \equiv -b_1 \\
\vdots & \\
a_{n1}y_1 + \cdots + a_{nn}y_n & \equiv -b_n \\
b_1y_1 + \cdots + b_ny_n & \equiv -c
\end{cases} \pmod{p^r}$$

has a solution in $(\mathbb{Z}_{p^r})^n$. 
Proof. By Proposition 5.3 there is a map $\phi_p : X \to \hat{X} \vee P^3(p^r)$ that becomes a homotopy equivalence after localized at $p$, where $\hat{X} \in \mathcal{C}_{n,2}$ is the quotient $X/P^3(q)$. Let
\[ M(\phi_p) = (W, y, z) \in \text{Mat}_n(\mathbb{Z}) \oplus (\mathbb{Z}_m)^n \oplus \mathbb{Z}_m \]
be the cellular map representation of $\phi_p$. After $p$-localization, $W$ is an invertible matrix and $z$ is a unit. The lemma follows from Proposition 4.8.

6. Proof of the main theorems

Proof of theorem 1.2. Let $q = p_1^{r_1} \ldots p_k^{r_k}$ be the primary factorization where $p_i$’s are different odd primes and $r_i \geq 1$. For each prime $p_i$, lemma 5.4 implies that the mod-$p_i^{r_i}$ version of (4.4) has a solution. By Chinese Remainder theorem, they give a mod-$q$ solution for (4.4). By Proposition 4.8, $X$ is homotopy equivalent to $\hat{X} \vee P^3(q)$. □

Proof of theorem 1.3. The ‘only if’ part is trivial. To prove the ‘if’ part, let $X$ and $X'$ be 4-dimensional toric orbifolds such that $H^3(X) \cong \mathbb{Z}_m$ and $H^3(X') \cong \mathbb{Z}_{m'}$ for $m$ and $m'$ odd. The hypothesis implies that $H^3(X) \cong H^3(X')$, hence we have $m = m'$. By theorem 1.2, we have $X \simeq \hat{X} \vee P^3(m)$ and $X' \simeq \hat{X'} \vee P^3(m)$ for some $\hat{X}, \hat{X'} \in \mathcal{C}_{n,1}$. Since $H^i(X) \cong H^i(\hat{X})$ and $H^i(X') \cong H^i(\hat{X'})$ for $i \neq 3$, we have $H^3(\hat{X}) \cong H^3(\hat{X'})$. Then Proposition 4.2 implies that $\hat{X} \simeq \hat{X'}$, which yields $X \simeq X'$. □

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