Algebraic Geometry

On Euler characteristics for large Kronecker quivers

Sur la caractéristique d'Euler de l'espace des représentations stables d'un grand carquois de Kronecker

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A B S T R A C T
We study Euler characteristics of moduli spaces of stable representations of m-Kronecker quivers for \( m \gg 0 \). In particular, we study an asymptotic log formula of Euler characteristics and a normalized asymptotic log formula of Euler characteristic, motivated by so-called Douglas conjecture.

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R É S U M É
Nous étudions la caractéristique d'Euler des espaces de modules de représentations stables des \( m \)-carquois de Kronecker pour \( m \) grand. En particulier, nous étudions une formule log asymptotique pour la caractéristique d'Euler et une formule asymptotique normalisée, motivées par la conjecture de Douglas.

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1. Introduction

For each positive integer \( m \), let \( K^m \) be the \( m \)-Kronecker quiver which consists of two vertices and \( m \) arrows from one to the other. For generic non-trivial stability conditions [1] on the category of representations of \( K^m \) and moduli spaces of stable representations \( M(K^m(a, b)) \) of coprime dimension vectors \( (a, b) \) [5], we study Euler characteristics \( \chi(K^m(a, b)) \).

We give some more details in the later section and we go on as follows. Notice that for the Euler form \( \langle \cdot , \cdot \rangle \) and a symplectic form \( \{\cdot , \cdot \} \), which is an anti-symmetrization of the Euler form, we may take a non-trivial stability condition on the category of representations of \( K^m \) such that for representations \( E, F \) of \( K^m \) and the slope function \( \mu \), we have \( \mu(E) > \mu(F) \) if and only if \( \{E, F\} > 0 \).

For objects to study in terms of wall-crossings, stability conditions such that the positivity of the difference of slopes coincides with that of symplectic forms on the Grothendieck group have been commonly called Denef's stability conditions in physics [2]. We employ these special stability conditions and the terminology.

Euler characteristics \( \chi(K^m(a, b)) \) have been studied extensively. In particular, formulas of Kontsevich–Soibelman and Reineke [6,10,12] have been known. In this article, we would like to study quantitative questions for \( m \gg 0 \).

To analyze further, for each coprime \( a, b \) and \( m > 0 \), let us define the bipartite quiver \( Q^m(a, b) \) which consists of \( a \) source vertices and \( b \) terminal vertices with \( m \) arrows from each source vertex to each terminal vertex. On representations of \( Q^m(a, b) \), we have Denef's stability conditions (see Section 2).
We denote $M(Q^m(a,b))$ to be the moduli space of stable representations of dimension vectors being one on each vertex of $Q^m(a,b)$ and $\chi(Q^m(a,b))$ to be the corresponding Euler characteristic. We have the following:

**Theorem 1.** For each coprime $a$, $b$, and $m \gg 0$, we have

$$\chi(Q^1(a,b)) \sim \frac{a!b!}{m^{a+b-1}} \chi(K^m(a,b)).$$

We would like to mention that in Theorem 1, Euler characteristics in the left-hand and right-hand sides are discussed in terms of black hole counting in supergravity [7] and Witten index in superstring theory [3] (see also [15]).

Key tools to obtain Theorem 1 are the recently obtained formula in Theorem 3 on $\chi(K^m(a,b))$ by Manschot, Pioline and Sen [7] (MPS formula for short, see also [8,9,14]) and our Lemma 2.1. We realize that by taking $m$ to be a variable, MPS formula provides the polynomial expansion of $\chi(K^m(a,b))$ whose coefficients involve Euler characteristics of bipartite quivers such as $Q^1(a,b)$. Indeed, we are dealing with nothing but the first-order approximation of $\chi(K^m(a,b))$ for $m \gg 0$.

By Theorem 1, to compute $\chi(Q^1(a,b))$, we can take the advantage of $\chi(K^m(a,b))$. Since the explicit formula of $\chi(K^m(a,b+1))$ has been provided in [16], we can obtain $\chi(Q^1(a,b+1))$ as in Corollary 5. Let us mention that for the cases of $a = 1$ and arbitrary $b$, we see that Stirling formula explains Theorem 1.

Douglas has conjectured the following [4,11,16]. For coprime $a$, $b \gg 0$ such that $b \gg a$ and each $m$, we have that

$$\log \frac{\chi(K^m(a,b))}{a} \sim (1 + r) \log(m).$$

In particular, for $a, b \gg 0$ such that $\frac{b}{a} \sim r$ and large enough $m$ depending on $a, b$, we have

$$\log \frac{\chi(K^m(a,b))}{a} \sim (1 + r) \log(m).$$

**2. Proofs**

Let us expand and introduce notions. For each $a$, let $\bar{a}$ denote a partition of a such that for non-negative integers $a_l$ of $l \geq 1$, we have $\sum_l l a_l = a$. We put $S_a = \sum_j a_j$ for our convenience. When $a_1 = a$, we simply write $a$ for $\bar{a}$. For a quiver $Q$ and representations $E, F$ of $Q$, on the Grothendieck group of the category of representations of $Q$, let $(E,F)Q$ be the Euler form and $\{E,F\}Q$ be the symplectic form $(E,F)Q - (F,E)Q$. For a dimension vector $d$, we call a partition $d^1, \ldots, d^s$ of $d$ such that $\sum_{p=1}^s d^p = d$ and $\{d^p\}Q > 0$ for each $b = 1, \ldots, s - 1$ to be admissible.

For each $m > 0$ and partitions $\bar{a}, \bar{b}$ of $a$ and $b$, we define the bipartite quiver $Q^m(\bar{a}, \bar{b})$ as follows. It consists of $S_a$ source vertices such that for each $l$, we have $a_l$ vertices $v$; for our convenience, we say $a_l$ is the label of $v$ and we put $w(v) = l$. It consists of $S_b$ terminal vertices with labels and $w(\cdot)$ being defined by the same manner. We put $mw(v)w(v')$ arrows from each source vertex $v$ to each terminal vertex $v'$.

Let us explain Denef’s stability conditions in use. For the $m$-Kronecker quiver $K^m$, the source vertex $(1,0)$, and the terminal vertex $(0,1)$, the slope function $\mu$ satisfies $\mu(1,0) > \mu(0,1)$. For $Q^m(\bar{a}, \bar{b})$ and vertices $v$ and $v'$ with the labels being $a_l$ and $b_r$, central charges $Z(v) = Z(v') = 0$. Since the explicit formula of $\chi(K^m(\bar{a}, \bar{b}))$ coincides with those of the vertices $(1,0)$ and $(0,1)$, we write $\chi(Q^m(\bar{a}, \bar{b})) = \chi(K^m(\bar{a}, \bar{b}))$.

We write $(\bar{a}, \bar{b})$ for the dimension vector which has one on each vertex of the quiver $Q^m(\bar{a}, \bar{b})$. We let $M(Q^m(\bar{a}, \bar{b}))$ be the moduli space of stable representations of the dimension vector $(\bar{a}, \bar{b})$ of $Q^m(\bar{a}, \bar{b})$. We denote $P(Q^m(\bar{a}, \bar{b}), y)$ to be the Poincaré polynomial and we put $\chi(Q^m(\bar{a}, \bar{b})) = P(Q^m(\bar{a}, \bar{b}), 1)$. For the $m$-Kronecker quiver $K^m$, we have the following MPS formula by specializing the Poincaré polynomial formula in [7, Appendix D]:

**Theorem 3 (MPS formula).** For each coprime $a$, $b$ and $m > 0$, we have

$$\chi(K^m(a,b)) = \sum_{\bar{a}, \bar{b}} \chi(Q^m(\bar{a}, \bar{b})). \prod_l \frac{1}{a_l!} (-\bar{a}_l(l-\bar{a}_l)}{\bar{b}_l!} \cdot \prod_l \frac{1}{\bar{b}_l!} (-\bar{a}_l(l-1)) \cdot \frac{1}{\bar{b}_l!} (l-\bar{b}_l)}{\bar{b}_l!}.$$

Notice that $M(Q^m(\bar{a}, \bar{b}))$ is a non-trivial smooth projective variety, since we have stable representations including ones with invertible maps on every arrow. We have the following:

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1 In [7], they give their formula in terms of Poincaré polynomials for Denef’s stabilities on quivers without oriented loops. We use its Euler characteristic version on Kronecker quivers. In [13], their formula has been motivically generalized and, for complete bipartite quivers and Euler characteristics, identified with a degeneration formula of Gromow–Witten theory.
Lemma 2.1.
\[ \chi (Q^m(\bar{a}, \bar{b})) = \chi (Q^1(\bar{a}, \bar{b})). \]

\textbf{Proof.} We consider the Poincaré polynomial \( P(Q^m(\bar{a}, \bar{b}), y) \) with Reineke’s formula \cite[Corollary 6.8]{10}. For the dimension vector \((\bar{a}, \bar{b})\), we take an admissible partition \( d^1, \ldots, d^t \) and \((-1)^{i-1} y^{2 \sum_{k \in S} d^i_k d^j_k} \). We notice that \( \{ \cdot \} Q^m(\bar{a}, \bar{b}) = m^{1,1} Q^1(\bar{a}, \bar{b}) \). The set of admissible partitions is invariant under choices of \( m \). For each admissible partition, the power of \( y \) above is the \( m \) times of that for \( P(Q^1(\bar{a}, \bar{b}), y) \). We have that \( P(Q^1(\bar{a}, \bar{b}), y) \) is a non-zero polynomial. Ignoring an overall factor of a power of \( y \) and writing \( y^2 \) as \( \chi \) for simplicity, for some non-trivial and non-negative integers \( \alpha_i \) and \( \beta_i \), we have \( P(Q^1(\bar{a}, \bar{b}), q) = (q - 1)^{1 - 5z - 2z} (\sum_{i \geq 0} \alpha_i (q - 1)^{z^i} q^m) \). For admissible partitions, the second factor is the sum of terms above. So we have \( P(Q^m(\bar{a}, \bar{b}), q) = (q - 1)^{1 - 5z - 2z} (\sum_{i \geq 0} \alpha_i (q^m - 1)^{z^i} q^{m\beta_i}) \). \( \square \)

We give a proof of Theorem 1.

\textbf{Proof.} By Lemma 2.1, \( \chi (Q^m(a, b)) \) carries the highest power of \( m \) among \( \chi (Q^m(\bar{a}, \bar{b})) \) in Theorem 3. \( \square \)

We give a proof of Corollary 2.

\textbf{Proof.} When \( a + b = 1, M(K^m(a, b)) \) is a point. For \( a + b \neq 1 \) and large enough \( m \) so that
\[
\left| \frac{\ln(\chi (Q^1(a, b)))}{(a + b - 1) \ln(m)} \right| < 1,
\]
the first assertion follows. For the second assertion, with \( a_i, b_i, m_i \) such that \( \frac{b_i}{a_i} \to r, \frac{1}{a_i} \to 0 \), and \( \lim (\ln(\chi (K^m(a_i, b_i))) / \ln(m_i a_i)) \to 1 \) for \( i \to \infty \), we use a standard argument. \( \square \)

Let us compute \( \chi (Q^1(a, a + 1)) \) as in the introduction. From \cite{16}, we recall the following:

\textbf{Theorem 4.} (See \cite[Theorem 6.6]{16}.)
\[
\chi (K^m(a, a + 1)) = \frac{m}{(a + 1)((m - 1)a + m)} \left( (m - 1)^2a + (m - 1)m \right). \]

By Theorem 1, we have the following:

\textbf{Corollary 5.}
\[
\chi (Q^1(a, a + 1)) = \lim_{m \to \infty} \chi (K^m(a, a + 1)) a! (a + 1)! \frac{m^{2a}}{m^a} = (a + 1)! (a + 1)^{-2+a}.
\]

\textbf{Remark 1.} With the formula of \( \chi (K^m(2, 2a + 1)) \) in \cite{10}, Manschot has proved
\[
\chi (Q^1(2, 2a + 1)) = \frac{(2a + 1)!}{a^2}.
\]
This sequence and the one in Corollary 5 coincide with A002457 and A066319 at oeis.org.

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