CLIFFORD ALGEBRAS IN
FINITE QUANTUM FIELD THEORIES

I. Irreducible Yukawa Finiteness Condition

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Abstract

Finite quantum field theories may be constructed from the most general renormalizable quantum
field theory by forbidding, order by order in the perturbative loop expansion, all ultraviolet-divergent
renormalizations of the physical parameters of the theory. The relevant finiteness conditions resulting
from this requirement relate all dimensionless couplings in the theory. At first sight, Yukawa couplings
which are equivalent to the generators of some Clifford algebra with identity element represent a very
promising type of solutions of the condition for one-loop finiteness of the Yukawa couplings. However,
under few reasonable and simplifying assumptions about their particular structure, these Clifford-like
Yukawa couplings prove to be in conflict with the requirements of one- and two-loop finiteness of the
gauge coupling and of the absence of gauge anomalies, at least for all simple gauge groups up to and
including rank 8.

PACS: 11.10.Gh, 11.30.Pb

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1 Introduction

Renormalizable quantum field theories appear to be the appropriate framework for the comprehensive understanding of nature at a rather fundamental level. In particular, the so-called “standard model” of elementary particle physics, a spontaneously broken non-Abelian gauge theory based on the gauge group $SU(3) \times SU(2) \times U(1)$, describes extremely successfully the strong and electroweak interactions. At present, this standard model is jeopardized only by the still unsettled question of the existence of the Higgs boson, required by the mechanism for spontaneous breakdown of a (local) gauge symmetry. Nevertheless, renormalizable quantum field theories exhibit, in general, the not very appealing feature that, in their loopwise perturbative evaluation, there still appear ultraviolet divergences, even though these can be handled by application of the renormalization programme. Therefore, it is, beyond doubt, legitimate to wonder whether among all the renormalizable quantum field theories there are theories which are finite, in the sense that they do not evolve ultraviolet divergences (up to some loop order).

Supersymmetry, by reducing the number of uncorrelated ultraviolet divergences in quantum field theories, represents the first example of a global symmetry which allows to construct finite quantum field theories:

- All one-loop finite $N = 1$ supersymmetric theories are (at least) two-loop finite $\text{[1]}$, even if this $N = 1$ supersymmetry is softly broken (in a well-defined way) $\text{[2]}$. Under certain circumstances, $N = 1$ supersymmetric theories may be finite to all orders of their perturbative expansion $\text{[3]}$.

- All $N = 2$ supersymmetric theories satisfying merely one single “finiteness condition” are finite to all orders of the perturbative expansion $\text{[4]}$, even if one or both supersymmetries are softly broken (in a well-defined way) $\text{[5]}$; these theories have been classified under various aspects $\text{[6]}$.

- In the case of the $N = 4$ supersymmetric Yang–Mills theory, that “$N = 2$ finiteness condition” is trivially fulfilled by the particle content of this theory enforced by $N = 4$ supersymmetry $\text{[7]}$.

Clearly, the next logical step is to impose the requirement of finiteness to arbitrary renormalizable quantum field theories in four space-time dimensions $\text{[8]} \text{[9]} \text{[10]}$; of particular interest here is the question whether every finite theory must indeed be supersymmetric. The inspection of general gauge theories shows immediately that finiteness of some quantum field theory may only be achieved if the particle content of this theory comprises vector bosons, fermions, and scalar bosons $\text{[11]} \text{[12]} \text{[13]} \text{[14]} \text{[15]}$. The complete set of finiteness conditions for general quantum field theories has not yet been solved. Some insights, however, may be gained by analysis of specific (classes of) models. For instance, models being finite in dimensional regularization, at least up to some loop order, may be shown to be plagued by quadratic divergences in cut-off regularization $\text{[16]} \text{[17]}$.

A useful instrument in the search for non-supersymmetric finite theories is the observation $\text{[18]} \text{[19]}$ that, for all finite quantum field theories, a certain group-theoretic quantity turns out to be bounded. In fact, it has even been speculated $\text{[20]}$ that all finite theories might belong to a particular class of models characterized by the circumstance that this group-theoretic quantity takes its maximal value. Within this class—which encompasses all supersymmetric finite models $\text{[21]}$—attempts to construct explicit non-supersymmetric finite theories have been undertaken $\text{[22]}$ and large sets of such candidate models based on the gauge group $SU(N)$ have been excluded $\text{[23]}$.

In the course of analyzing this specific class of models, explicit solutions of the one-loop finiteness condition for the Yukawa couplings which resemble the generators of a Clifford algebra with identity element have been found $\text{[24]}$. The present investigation scrutinizes the relevance of these Clifford-like Yukawa solutions for the construction of new, i.e., non-supersymmetric, finite quantum field theories.

The outline of this paper is as follows: In Sec. 2, we formulate the conditions under which we regard an arbitrary quantum field theory as finite (up to some loop order). For the investigation of the high-energy behaviour of some quantum field theory, only the massless limit of this theory, characterized by the vanishing of all dimensional parameters in this theory, is relevant. Consequently, without loss of generality, we confine ourselves to the discussion of theories involving only dimensionless couplings. In the order of increasing complexity, the first genuine hurdle to be taken is the condition for one-loop finiteness of the Yukawa couplings. Finding corresponding solutions is greatly facilitated by adopting the standard form of this relation, re-derived in Sec. 3. The above-mentioned specific class of models is briefly reviewed in Sec. 4. Stripping off irrelevant ballast, the one-loop Yukawa finiteness condition is reduced, in Sec. 5, to its “hard core” which, under the simplifying assumptions about the structure of the Yukawa couplings specified in Sec. 4, is then carefully investigated along the lines sketched in Sec. 4. Section 6 summarizes our findings, the requirements for their validity, and the way they may be obtained. Several more or less merely technical details are banished to Appendices A through E.
\section{Finiteness of General Quantum Field Theories}

The starting point of our considerations is the most general \[\mathfrak{A}\] renormalizable quantum field theory (for particles up to spin 1 \(\frac{1}{2}\)) invariant with respect to gauge transformations forming some compact simple Lie group \(G\) with corresponding Lie algebra \(\mathfrak{A}\). The particle content of this theory consists of

- gauge vector bosons \(A_\mu(x) = (A_\mu^a(x)) \in \mathfrak{A}\), transforming according to the adjoint representation \(R_{ad}: \mathfrak{A} \rightarrow \mathfrak{A}\) of the gauge group \(G\), of dimension \(d_g := \dim \mathfrak{A}\);
- two-component Weyl fermions \(\psi(x) = (\psi^i(x)) \in V_F\), transforming according to a representation \(R_F: V_F \rightarrow V_F\) of \(G\), of dimension \(d_F := \dim V_F\); and
- real scalar bosons \(\phi(x) = (\phi^\alpha(x)) \in V_B\), transforming according to some real representation \(R_B: V_B \rightarrow V_B\) of \(G\), of dimension \(d_B := \dim V_B\).

Apart from terms involving dimensional parameters, like mass terms and cubic self-interaction terms of scalar bosons, as well as gauge-fixing and ghost terms, the Lagrangian defining this theory is given by

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu} + i \bar{\psi}_i \gamma^i (D_\mu)^F \psi^i + \frac{1}{2} [(D_\mu)^B \phi^\alpha] (D_\mu)^B \phi^\alpha
- \frac{1}{2} \phi^\alpha Y_{\alpha ij} \psi^i \psi^j - \frac{1}{2} \bar{\phi}_\alpha Y^{\alpha ij} \bar{\psi}_i \bar{\psi}_j - \frac{1}{4!} V_{\alpha\beta\gamma\delta} \phi^\alpha \phi^\beta \phi^\gamma \phi^\delta .
\]

Here, we employ the following notation: The Hermitian generators \(T^a_R\), \(R = \text{ad}, F, B\), \(a = 1, 2, \ldots, d_g\), of the Lie algebra \(\mathfrak{A}\) in each of the three representations \(R_{ad}, R_F, \) and \(R_B\) introduced above satisfy the commutation relations

\[
[T^a_R, T^b_R] = i f^{abc} T^c_R , \quad R = \text{ad}, F, B ,
\]

where \(f^{abc}, a, b, c = 1, 2, \ldots, d_g\), denote the structure constants characterizing the Lie algebra \(\mathfrak{A}\). The gauge coupling constant is denoted by \(g\). The (gauge-covariant) field strength tensor \(F^{\mu\nu}_a\) is given by

\[
F^{\mu\nu}_a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c ,
\]

The gauge-covariant derivatives \(D_\mu\) acting on the representation spaces \(\mathfrak{A}, V_F, \) and \(V_B\), respectively, read

\[
(D_\mu)^R := \partial_\mu - ig T^a_R A_\mu^a , \quad R = \text{ad}, F, B .
\]

Finally, the four \(2 \times 2\) matrices \(\sigma^i\) embrace the \(2 \times 2\) unit matrix \(1_2\) and the three Pauli matrices \(\sigma\) according to the definition \(\sigma^i = (1_2, -\sigma)\).

Quite obviously, the Yukawa couplings \(Y_{\alpha ij}\) must be totally symmetric in their fermionic indices \(i\) and \(j\), and the quartic scalar-boson self-couplings \(V_{\alpha\beta\gamma\delta}\) must be totally symmetric under an arbitrary permutation of their indices.

In order to facilitate the formulation of the finiteness conditions below, we would like to introduce some (group-theoretic) quantities. For an arbitrary representation \(R\) of \(G\), we define, in terms of the generators \(T^a_R\) of \(\mathfrak{A}\) in this representation, the corresponding quadratic Casimir operator \(C_R^a\) by

\[
C_R := \sum_{a=1}^{d_g} T^a_R T^a_R
\]

and the corresponding Dynkin index \(S_R\) by

\[
S_R \delta^{ab} := \text{Tr} (T^b_R T^a_R) .
\]

In the adjoint representation \(R_{ad}\), the Casimir eigenvalue \(c_g\) equals the Dynkin index \(S_g\), i.e., \(c_g = S_g\). Moreover, we shall take advantage of the abbreviations

\[
Q_F := \sum_I f_I S_I C_I = \frac{1}{d_g} \text{Tr}(C_F)^2 ,
\]

\[
Q_B := \sum_I b_I S_I C_I = \frac{1}{d_g} \text{Tr}(C_B)^2 ,
\]
where the summation index $I$ runs over all inequivalent irreducible representations $R_I$ of multiplicities $f_I$ and $b_I$ in $R_F$ and $R_B$, respectively. Finally, it proves to be advantageous to introduce the shorthand notation

$$E(Y) := 6 \, g^2 \, \text{Tr}_F \left( C_F \sum_{\beta=1}^{d_B} Y^{\dagger \beta} Y_\beta \right),$$

where by $\text{Tr}_F$ we mean the partial trace over the fermionic indices only.

With all the above preliminaries, we are now in the position to formulate the finiteness conditions we are interested in. We adhere to the notion of “finiteness” for general renormalizable quantum field theories as advocated and investigated first in Refs. [34, 35]. Hence, any such theory will be regarded as “finite” if it does not require divergent renormalizations of its physical parameters, that is, masses and coupling constants. This is equivalent to demanding finiteness of the resulting $S$-matrix elements (not of the Green’s functions) without divergent renormalizations of the involved coupling constants. Consequently, our finiteness conditions may be found by requiring the beta functions of these physical parameters to vanish. Evidently, within a perturbative evaluation of the quantum field theory under consideration, the vanishing of all beta functions must take place order by order in the loop expansion. By application of the standard renormalization procedure with the help of dimensional regularization in the minimal-subtraction scheme, the relevant finiteness conditions may be easily extracted [19, 20], see also Refs. [34, 35]; they read, for one-loop finiteness of the gauge coupling constant $g$,

$$22 \, c_g - 4 \, S_F - S_B = 0,$$

for two-loop finiteness of the gauge coupling constant $g$,

$$E(Y) - 12 \, g^4 \, d_g \, [Q_F + Q_B + c_g (S_B - 2 \, c_g)] = 0,$$

and, for one-loop finiteness of the Yukawa couplings $Y_{\alpha ij}$,

$$\sum_{\beta=1}^{d_B} \{ 4 \, Y_\beta \, Y^{\dagger \alpha} \, Y_\beta + Y_\alpha \, Y^{\dagger \beta} \, Y_\beta + Y_\beta \, Y^{\dagger \beta} \, Y_\alpha + Y_\beta \, \text{Tr} \{ Y^{\dagger \alpha} \, Y_\beta + Y^{\dagger \beta} \, Y_\alpha \} \} - 6 \, g^2 \, \left[ Y_\alpha \, C_F + (C_F)^T \, Y_\alpha \right] = 0 .$$

In the following, we call Eq. (11), our main concern, for short, “Yukawa finiteness condition” (YFC). It has been noticed at several occasions in the literature [12, 17] that the above lowest-order finiteness conditions for gauge and Yukawa couplings, i.e., Eqs. (10) and (11), constitute the central part of the whole set of finiteness conditions, in the sense that the inspection of the finiteness conditions for the quartic scalar-boson self-couplings $V_{\alpha \beta \gamma \delta}$ or of higher order in the loop expansion makes sense only after this central part has been solved.

### 3 The Standard Form of the Yukawa Finiteness Condition

Let $B_F = \{ e_i \}$ be some basis of the “fermionic” representation space $V_F$ and let $B_B = \{ f_\alpha \}$ be some basis of the “bosonic” representation space $V_B$; in terms of these bases, we may write

$$\psi = \sum_{i=1}^{d_F} \psi^i e_i,$$

$$\phi = \sum_{\alpha=1}^{d_B} \phi^\alpha f_\alpha .$$

Then $Y_{\alpha ij}$ may be interpreted as the components of a **Yukawa coupling tensor** $Y$ in the corresponding tensor basis $\{ f_\alpha \otimes e_i \otimes e_j \}$ of the product space $V_B \times V_F \times V_F$. Gauge invariance of the Lagrangian $\mathcal{L}$ requires the invariance of $Y$ under the contragredient representation $R^c$ of $R = R_B \otimes R_F \otimes R_F$.

$$(R^c_B \otimes R^c_F \otimes R^c_F) \, Y = Y .$$

This statement expresses, of course, nothing else but the (trivial) fact that the Yukawa coupling strength for any fixed irreducible representations $R_I, R_J \subset R_F$ and $R_A \subset R_B$ is not affected by gauge transformations.
Let us now introduce a quantity \( x = (x^{i\alpha}_{j\beta}) \), which transforms like an operator on the product space \( V_F \times V_B \), by defining
\[
2 x^{i\alpha}_{j\beta} = (Y^{i\alpha} Y_{\beta} + Y^{i\beta} Y_{\alpha})^j_i.
\]

**Proposition 1:** The operator \( x \) is gauge invariant and diagonalizable on \( V_F \times V_B \).

**Proof:** Since \( x \) is normal it is diagonalizable. The gauge invariance of \( x \) is shown in Appendix A. \( \blacksquare \)

A system \( \Sigma \ni M \): \( V \rightarrow V \) of matrices is called reducible if there exists an invariant subspace of \( V \) under the action of \( \Sigma \), otherwise \( \Sigma \) is called irreducible. The commutant of such a system \( \Sigma \), defined by \( \text{Comm}(\Sigma) := \{ N: V \rightarrow V \mid [M, N] = 0, \forall M \in \Sigma \} \), forms a matrix algebra \([2]\). Now, suppose that \( \Sigma \) is completely reducible, i.e., that \( \Sigma \) is the direct sum of irreducible systems. In this case \( \text{Comm}(\Sigma) \) is isomorphic to the direct sum of matrix rings \([2]\). Let \( M \in \Sigma \) and \( N \in \text{Comm}(\Sigma) \). We may write
\[
M = \bigoplus_i 1_{r_i} \times M_i,
\]
\[
N = \bigoplus_i N_i \times 1_{n_i},
\]
where \( i \) labels the inequivalent irreducible components \( M_i \) of \( M \), of dimension \( n_i \) and multiplicity \( r_i \), respectively, \( 1_d \) represents the \( d \)-dimensional unit matrix, and \( N_i \) denotes an arbitrary \( r_i \times r_i \) matrix.

In Ref. [13] it was shown that the YFC is invariant under an arbitrary \( U(d_F) \otimes O(d_B) \) transformation. We may take advantage of the \( U(d_F) \) symmetry by choosing \( B_F \) such that \( R_F \) becomes block diagonal in each irreducible representation \( R^I_F \). The \( O(d_B) \) symmetry may transform \( R_B \) into a direct sum of real orthogonal blocks \( R^{\mu}_B = R^{\mu*}_B \),
\[
R_F = \bigoplus_I R^I_F,
\]
\[
R_B = \bigoplus_\mu R^{\mu}_B.
\]

For any operator acting on the product space \( V_F \times V_B \), we define, with respect to some corresponding tensor basis \( \{ e_i \otimes f_\alpha \} \), partial traces \( \text{Tr}_B \) and \( \text{Tr}_F \) over bosonic and over fermionic indices, respectively. For \( x^{i\alpha}_{j\beta} \), the contraction of either the two bosonic or the two fermionic indices yields
\[
(y_F)^i_j := \sum_{\beta=1}^{d_B} x^{i\beta}_{j\beta} = \sum_{\beta=1}^{d_B} \sum_{k=1}^{d_F} Y^{i\beta k} Y_{\beta k}^j,
\]
\[
2(y_B)^\alpha_\beta := \sum_{i=1}^{d_F} x^{i\alpha}_{i\beta} = \sum_{i,j=1}^{d_F} (Y^{i\alpha i j} Y_{\beta j} + Y^{i\beta i j} Y_{\alpha j}) = \text{Tr}_F \left( Y^{i\alpha} Y_{\beta} + Y^{i\beta} Y_{\alpha} \right). \tag{16}
\]
\( y_F = \text{Tr}_B x \) and \( y_B = \text{Tr}_F x \) transform as invariant operators on \( V_F \) and \( V_B \), respectively. By choosing, for every type of mutually equivalent blocks in \( R_F \) and \( R_B \) a representative \( R^I \) and \( R^{\mu} \), respectively, we have, in the notation \([14]\),
\[
R_F = \bigoplus_I 1_{f_I} \times R^I,
\]
\[
R_B = \bigoplus_\mu 1_{b_\mu} \times R^{\mu}, \tag{17}
\]
where the direct sums in \( R_F \) and \( R_B \) extend over all inequivalent irreducible representations \( R^I \subset R_F \), of dimensions \( d_I \) and multiplicities \( f_I \), as well as all inequivalent orthogonal representations \( R^{\mu} \subset R_B \).

To be more precise, \( x^{i\alpha}_{j\beta} \) may be interpreted as the components of the operator \( x \) with respect to the tensor basis \( \{ e_i \otimes f_\alpha \} \).

Every non-orthogonal irreducible representation \( R^A_B \subset R_B \) has to find a mutually contragredient companion \( (R^A_B)^c \subset R_B \) in order to be able to form a real orthogonal block: \( R^A_B \simeq R^A_B \oplus (R^A_B)^c \).
of dimensions $d_\mu$ and multiplicities $b_\mu$, respectively. The invariance of $y_F$ and $y_B$ under $R_F$ and $R_B$, respectively, implies $y_F \in \text{Comm}(R_F)$ and $y_B \in \text{Comm}(R_B)$. According to Eq. (14), these operators may be represented in the form

$$y_F = \bigoplus_l W_l \times 1_{d_l} ,$$

$$y_B = \bigoplus_\mu Z_\mu \times 1_{d_\mu} ,$$

for arbitrary systems of $f_l \times f_l$ matrices $W_l$ and $b_\mu \times b_\mu$ matrices $Z_\mu$. The invariance of the operators $y_F$ and $y_B$ under $R_F$ and $R_B$, respectively, guarantees the vanishing of their commutators with the corresponding Casimir operators $C_F$ and $C_B$:

$$[y_F, C_F] = 0 ,$$

$$[y_B, C_B] = 0 .$$

Now, for a finite or infinite system of diagonalizable matrices acting on some finite-dimensional linear space, there exists always a basis such that all members of this system are diagonal in this very basis if and only if they commute with each other. Consequently, there must exist unitary and orthogonal transformations $U_F$ and $O_B$ on the representation spaces $V_F$ and $V_B$, respectively, such that both $y_F$ and $C_F$, on the one hand, as well as $y_B$ and $C_B$, on the other hand, are diagonalizable simultaneously. This property of diagonalizability is, of course, transferred to the matrices $W_l$ and $Z_\mu$ introduced in Eq. (18); in the course of this, the transformations $U_F$ and $O_B$ become explicitly

$$U_F = \bigoplus_l T_F^l \times 1_{d_l} ,$$

$$O_B = \bigoplus_\mu T_B^\mu \times 1_{d_\mu} ,$$

where each of the transformations $T_F^l$ serves to diagonalize a certain isotypical block $W_l$ in $y_F$ while each of the transformations $T_B^\mu$ serves to diagonalize a certain isotypical block $Z_\mu$ in $y_B$. By applying this diagonalization procedure to the operators $y_F$ and $y_B$, we thus obtain, in the fermionic sector,

$$(y_F)^i_j = \delta^i_j y_F^i = \sum_{\beta=1}^{d_\alpha} (Y^{1\beta} Y_\beta)^i_j ,$$

$$(C_F)^i_j = \delta^i_j C_F^i ,$$

and, in the bosonic sector,

$$2 (y_B)^\alpha_\beta = 2 \delta^\alpha_\beta y_B^\beta = \text{Tr}_F (Y^{1\alpha} Y_\beta + Y^{1\beta} Y_\alpha) ,$$

$$(C_B)^\alpha_\beta = \delta^\alpha_\beta C_B^3 .$$

As already mentioned, the YFC is invariant under all our unitary and orthogonal transformations. Furthermore, the relations $U_F R_F U_F^\dagger = R_F$ and $O_B R_B O_B^\dagger = R_B$ guarantee that upon application of the above diagonalization procedure $R_F$ and $R_B$ remain blockdiagonal with respect to each irreducible representation $R_F^l \subset R_F$ as well as with respect to each orthogonal block $R_B^\mu \subset R_B$. With the above sets of decompositions (21) and (22), the YFC (1) assumes what is usually called its standard form:

$$4 \sum_{\beta=1}^{d_\alpha} (Y_\beta Y^{1\alpha} Y_\beta)_{ij} + Y_{\alpha ij} \left( 2 y_B^\alpha + y_F^i + y_F^j - 6 g^2 C_F^i - 6 g^2 C_F^j \right) = 0 .$$

We conclude that this standard form of the YFC is quite naturally related to a basis where both $R_F$ and $R_B$ are blockdiagonal.

\footnote{This result coincides with the well-known standard form of the YFC but, in contrast to Ref. [15], the simultaneous diagonal form of the operators $y_F$, $C_F$, $y_B$, and $C_B$ was derived here without making use of the YFC.}
4 \( F^2 = 1 \) Theories

In Ref. [13] a certain—upon application of the two-loop gauge-coupling finiteness condition, Eq. (11), purely group-theoretic—quantity called \( F \), defined by

\[
F^2 := \frac{E(Y)}{36 g^4 d_B Q_F} = Q_F + Q_B + c_g (S_F - 2 c_g),
\]

has been introduced. Remarkably, all theories which satisfy the central part of finiteness conditions as represented by Eqs. (8), (10), and (11) also satisfy the inequality \( F \leq 1 \). In particular, the extremum \( F = 1 \) seems to play a decisive rôle in the analysis of these finiteness conditions [13]:

- If and only if this quantity \( F \) is restricted to the value \( F = 1 \), the (cubic) YFC (11) is equivalent to the (quadratic) “\( F = 1 \) system”

\[
\sum_{\beta=1}^{d_B} (Y_{\beta i j} Y_{\beta k l} + Y_{\beta j k} Y_{\beta i l} + Y_{\beta l i} Y_{\beta j k}) = 0 \quad \forall \ i, j, k, l,
\]

\[
\sum_{\alpha=1}^{d_B} Y^{i \alpha} Y_{\alpha} = 6 g^2 C_F,
\]

\[
\text{Tr}_F (Y^{i \alpha} Y_{\beta}) = \text{Tr}_F (Y^{\gamma \beta} Y_{\alpha}) \quad \forall \ \alpha, \beta.
\]  

- All \( N = 1 \) supersymmetric finite theories have \( F = 1 \) and are thus solutions to the system (23).

- The incorporation of all supersymmetric finite theories, numerical checks, and the fact that, in contrast to the YFC (11) which is cubic in \( Y \), the system (23) is only quadratic in \( Y \) led to the conjecture that all finite theories satisfy \( F = 1 \) and belong to the solutions of the system (23).

By exploiting the highly symmetric structure of the \( F = 1 \) system but ignoring the requirements imposed by gauge invariance, a class of explicit solutions of this system has been found: all members of this class are characterized by the fact that \( R_F \) is the direct sum of merely one type of irreducible representation while the involved Yukawa couplings are isomorphic to generators of (a representation of) a Clifford algebra with identity element [13]. In this class of theories, the ratio of the “bosonic” dimension \( d_B \) and the “fermionic” dimension \( d_F \) is restricted to values like \( d_B / d_F = \frac{3}{2} \), as is realized, for instance, in all \( N = 4 \) supersymmetric theories (which, in fact, also exhibit a certain Clifford-like structure in their Yukawa couplings [10]).

However, the construction of all these particular Clifford-like solutions of the YFC (14) takes into account neither the one-loop gauge-coupling finiteness condition (8) nor the restrictions (12) on the Yukawa couplings due to gauge invariance of the theory. The present analysis aims at the systematic investigation of the consequences of a Clifford-like structure of the Yukawa couplings \( Y \) for finiteness of general gauge theories.

5 Reducibility of the Yukawa Finiteness Condition

Let us now focus our attention to the standard form (23) of the YFC, obtained under the constraints (21) and (24). We notice that \( y_{B}^i \) is nothing else but the Hilbert–Schmidt norm of the matrix \( Y_{\alpha} = Y_{\alpha}^T \) and that \( y_{F}^i \) may be interpreted as the Hilbert–Schmidt norm of some \( d_F \times d_B \) matrix, say \( A^i \), formed by \( Y_{\alpha} \). Thus, \( y_{B}^i = 0 \) implies that \( Y_{\alpha} \) is the null matrix, and \( y_{F}^i = 0 \) implies that \( A^i \) is the null matrix. Consequently, for a vanishing \( y_{B}^i \), there cannot arise any contributions to the YFC from \( Y_{\alpha} \) for all \( i, j \in \{1, \ldots, d_F\} \), and, for a vanishing \( y_{F}^i \), there cannot arise any contributions to the YFC from \( Y_{\alpha} \) for all \( \alpha \in \{1, \ldots, d_B\} \) and for all \( j \in \{1, \ldots, d_F\} \). For precisely this reason, we find it very convenient to re-order the two bases \( B_F \) and \( B_B \) of the representation spaces \( V_F \) and \( V_B \), respectively, according to the following

**Definition 1:**

\[
i \in \{1, \ldots, n\} \iff y_{F}^i \neq 0,
\]

\[
i \in \{n+1, \ldots, d_F\} \iff y_{F}^i = 0,
\]

\[
\alpha \in \{1, \ldots, m\} \iff y_{B}^\alpha \neq 0,
\]

\[
\alpha \in \{m+1, \ldots, d_B\} \iff y_{B}^\alpha = 0.
\]
Due to Schur’s lemma, this rearrangement of indices does not affect the block structure of $R_F$ or $R_B$ because, as expressed by Eq. \(9\), both $y_{F}$ and $y_{B}$ are proportional to unity on each of the irreducible blocks given in Eq. \(7\) since, according to Eq. \(10\), they form invariant operators acting on $V_{F}$ and $V_{B}$, respectively. This rearrangement procedure reduces the YFC \(11\) to a new system of equations. For $\alpha \in \{m+1, \ldots, d_{B}\}$, the couplings $Y_{\alpha}$ do not contribute to this new system. For $\alpha \in \{1, \ldots, m\}$, all $Y_{\alpha}$ are of the form $Y_{\alpha} = E Y_{\alpha}$, with projectors $E$ onto the subspace of $V_{F}$ with non-vanishing $y_{F}^{\alpha}$. Therefore, the YFC will involve only quantities with indices which correspond to $y_{F}^{\alpha} \neq 0$ and $y_{B}^{\beta} \neq 0$:

\[
4 \sum_{\beta=1}^{m} (Y_{\beta} Y^{\dagger \alpha} Y_{\beta})_{ij} + Y_{\alpha ij}(2 y_{F}^{\alpha} + y_{F}^{i} + y_{F}^{j} - 6 g^{2} C_{F}^{i} - 6 g^{2} C_{F}^{j}) = 0,
\]

\[
\delta^{ij} y_{F}^{j} = \sum_{\beta=1}^{m} (Y^{\dagger \beta} Y_{\beta})_{ij},
\]

\[
2 \delta^{\alpha \beta} y_{B}^{\beta} = T_{F} (Y^{\dagger \alpha} Y_{\beta} + Y^{\dagger \beta} Y_{\alpha}),
\]

(26)

for all $i, j \in \{1, \ldots, n\}$ and for all $\alpha, \beta \in \{1, \ldots, m\}$. This new system of equations is, of course, of the same structure as the one derived in Sec. \(3\); however, here the Yukawa couplings $Y_{\alpha ij}$ contribute only for $\alpha \in \{1, \ldots, m\}$ and $i, j \in \{1, \ldots, n\}$. Similarly, for the bounds \(13\) on the quantity $E(Y)$ of Eq. \(3\), the same restricted range of fermionic indices as for the system \(23\) is relevant:

\[
\sum_{i=1}^{n} (y_{F}^{\alpha})^{2} \leq E(Y) = 6 g^{2} \sum_{i=1}^{n} C_{F}^{i} y_{F}^{\alpha} \leq 36 g^{4} \sum_{i=1}^{n} (C_{F}^{i})^{2}.
\]

(27)

Hence, we encounter some fundamental difference between, on the one hand, the full particle content of the Lagrangian \(1\), which enters in all group-theoretic quantities like $S_{F}$, $S_{B}$, $Q_{F}$, or $Q_{B}$, and, on the other hand, the subset of only those particles which also have a non-vanishing Yukawa coupling. Just as the constraint $F = 1$ can be expressed by requiring $y_{F}^{i} = 6 g^{2} C_{F}^{i}$ for all $i \in \{1, \ldots, n = d_{F}\}$, we may set $y_{B}^{\beta} = 6 g^{2} C_{F}^{\beta}$ for all $\beta \in \{1, \ldots, n = d_{B}\}$ and get a system of the form \(25\) with $F < 1$. For this, the existence of potentially finite theories solving Eqs. \(3\) and \(10\) may be shown numerically.

In order to construct invariant tensors for the Yukawa couplings, we decompose both the bosonic index $\alpha$ and the fermionic index $i$ into pairs of indices, say $\alpha = (A, \alpha_{A})$ and $i = (I, i_{I})$, where the indices $A$ and $I$ serve to distinguish irreducible representations $R_{A}^{B} \subset R_{B}$ and $R_{I}^{B} \subset R_{F}$, respectively, while the indices $\alpha_{A} = 1, \ldots, d_{A}$ and $i_{I} = 1, \ldots, d_{I}$ label the components of $R_{A}^{B}$ and $R_{I}^{F}$, respectively. Let $R_{A}^{B} \subset R_{B}$, $R_{I}^{B} \subset R_{F}$, and $R_{I}^{F} \subset R_{F}$ be three irreducible representations of $G$. If and only if their product $R_{A}^{B} \otimes R_{I}^{B} \otimes R_{I}^{F}$ contains the trivial representation, $1$, $N(A, I, J)$ times, there exist $N(A, I, J)$ invariant tensors $(\Lambda^{(k)})_{\alpha_{A} i_{I} j_{I}}$. In terms of the latter, the expansion of $Y$, with coefficients $p_{A I J}^{(k)} \in \mathbb{C}$, reads

\[
Y_{\alpha ij} = Y_{(A, \alpha_{A})(I, i_{I})(J, j_{I})} = \sum_{k=1}^{N(A, I, J)} p_{A I J}^{(k)} (\Lambda^{(k)})_{\alpha_{A} i_{I} j_{I}} .
\]

(28)

We realize the naturalness of $n < d_{F}$ and $m < d_{B}$ in the YFC \(26\): not all combinations of irreducible representations contained in $R_{F}$ and $R_{B}$ allow to build invariant tensors \(4\) every $R_{F}^{B}$ without partners to form invariants reduces $n$ by $d_{I}$, every $R_{A}^{B}$ without partners to form invariants reduces $m$ by $d_{A}$. Now, let $M_{1} = \{(R_{A}^{\mu_{1}}, R_{I}^{\mu_{1}}, R_{I}^{\mu_{1}})\}$ and $M_{2} = \{(R_{I}^{\mu_{2}}, R_{I}^{\mu_{2}}, R_{I}^{\mu_{2}})\}$ be two sets of combinations of real bosonic blocks $R_{A}^{\mu_{1}} \subset R_{B}$ and irreducible fermionic representations $R_{I}^{\mu_{1}}, R_{I}^{\mu_{2}}, R_{I}^{\mu_{2}} \subset R_{F}$ in the Yukawa couplings $Y_{(\mu, \alpha_{\mu})(I, i_{I})(J, j_{I})}$. We define any two sets $M_{1}$ and $M_{2}$ to be disjoint if and only if $\{R_{A}^{\mu_{1}}\} \cap \{R_{I}^{\mu_{2}}\} = \{R_{I}^{\mu_{1}}\} \cap \{R_{I}^{\mu_{2}}\} = \emptyset$.

**Definition 2:** Let $M = \{(R^{P}, R^{I}, R^{I}) \mid R^{P} \subset R_{B}, R^{I} \subset R_{F}\}$ be the set of all combinations of real bosonic blocks and irreducible fermionic representations in the YFC \(26\). If $M$ is the union of pairwise disjoint non-empty subsets $M_{k}, k = 1, 2, \ldots$, we call the YFC reducible else irreducible.

\[
y_{B}^{\alpha} \text{ is proportional to unity on whole orthogonal blocks } R_{A}^{\alpha} \simeq R_{A}^{B} \oplus (R_{A}^{B})^{c}. \text{ Thus, the norm of } Y_{\alpha} \text{ on } R_{A}^{B} \text{ equals its norm on } (R_{A}^{B})^{c}.
\]

\(5\)

\(6\)

\(7\)

For more details on the relation of the expansion \(28\) and the real form of $R_{B}$, see Appendix \(3\).

Note that, for every index of $Y_{(\mu, \alpha_{\mu})(I, i_{I})(J, j_{I})}$, the splitting takes place between the irreducible representations in $R_{F}$ and real blocks in $R_{B}$. This is the finest conceivable splitting of the YFC since any finer one would decompose $\Lambda^{(k)}$, in contradiction to $\Lambda^{(k)}$ being a fundamental invariant tensor.
6 Clifford Algebra Representations for Irreducible Yukawa Finiteness Conditions

For the sake of conceptual simplicity, we would like to begin the present investigations of finiteness with the special case of an irreducible YFC. The by far more delicate case of a reducible YFC as well as a more rigorous treatment of the notion of reducibility of systems will be covered in Refs. [22, 23].

Generalizing the ansatz which entails solutions of the YFC equivalent to representations of some Clifford algebra [13], we start with

Definition 3: Let the ranges of indices n and m be as specified in Def. 4. Let the YFC be irreducible in the sense of Def. 3. We assume the invariant diagonalizable operator x defined by Eq. (13) to be of the form

\[ x = u \otimes v, \]
\[ x^{i\alpha}_{j\beta} = u^{\alpha}_\beta v^i_j, \]

where v and u act on \( V_F \) and \( V_B \), respectively.

Recalling \( \text{Tr}_F x = y_F \) and \( \text{Tr}_B x = y_B \) as well as the outcome (21) and (22) of diagonalization entails, for all \( i, j \in \{1, \ldots, n\} \) and for all \( \alpha, \beta \in \{1, \ldots, m\} \),

\[ (\text{Tr}_B x)^i_j = (y_F)^i_j = \delta^i_j y_F^i = v^i_j \text{Tr}(u) = \delta^i_j v^j \text{Tr}(u), \]
\[ (\text{Tr}_F x)^{\alpha\beta} = (y_B)^{\alpha\beta} = \delta^{\alpha\beta} y_B^{\alpha\beta} = u^{\alpha\beta} \text{Tr}(v) = \delta^{\alpha\beta} u^{\beta} \text{Tr}(v). \] (29)

Let us rewrite the quantities \( u \) and \( v \) as well as their traces in polar decomposition:

\[ \text{Tr}(u) = |\text{Tr}(u)| \exp(i \eta), \]
\[ \text{Tr}(v) = |\text{Tr}(v)| \exp(i \varphi), \]
\[ u^\alpha = |u^\alpha| \exp(i \eta^\alpha) \quad \forall \alpha \in \{1, \ldots, m\}, \]
\[ v^i = |v^i| \exp(i \varphi^i) \quad \forall i \in \{1, \ldots, n\}. \]

Substitution of these polar decompositions into the relations \( y_F^i = v^i \text{Tr}(u) \in \mathbb{R} \) for all \( i \in \{1, \ldots, n\} \) and \( y_B^{\alpha} = u^{\alpha} \text{Tr}(v) \in \mathbb{R} \) for all \( \alpha \in \{1, \ldots, m\} \) resulting from Eqs. (23) yields

\[ \varphi^i = -\eta \quad \forall i \in \{1, \ldots, n\}, \]
\[ \eta^\alpha = -\varphi \quad \forall \alpha \in \{1, \ldots, m\}. \]

Moreover, because all \( y_F^i \) and all \( y_B^{\alpha} \) are real, i.e., \( y_F^i \in \mathbb{R} \) and \( y_B^{\alpha} \in \mathbb{R} \), we also have

\[ \sum_{i=1}^{n} y_F^i = \text{Tr}(v) \text{Tr}(u) \in \mathbb{R}, \]
\[ \sum_{\alpha=1}^{m} y_B^{\alpha} = \text{Tr}(u) \text{Tr}(v) \in \mathbb{R}, \]

which, in turn, implies \( \varphi \equiv -\eta \). Therefore, we end up with

\[ u^\alpha = |u^\alpha| \exp(-i \varphi), \]
\[ v^i = |v^i| \exp(i \varphi), \]
\[ x^{i\alpha}_{j\beta} = \delta^\alpha_\beta \delta^i_j v^\beta u^i =: \delta^\alpha_\beta \delta^i_j x^{i\beta}, \] (30)

which demonstrates that \( x \) is diagonal if both \( y_F \) and \( y_B \) are diagonal. The above diagonalization of \( x \) leaves the YFC unchanged; we are thus still allowed to use the standard form of the YFC, Eq. (24).

We conclude that \( x = u \otimes v \) is a member of those solutions of the YFC where \( x \) is diagonalizable by some transformation of the form \( S = U(n) \otimes O(m) \). (This class of solutions will be characterized in more detail in Ref. [13].)

With the result (30) for \( u^\alpha \) and \( v^i \), we are able to prove

\[ \text{Finiteness Conditions} \]

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8 These transformations \( S \) correspond precisely to the \( U(d_F) \otimes O(d_B) \) symmetry of the YFC found in Ref. [13] and mentioned explicitly in Ref. [14].
Proposition 2: The tensorial structure of the ansatz \( x = u \otimes v \) enforces a block structure, determined by \( y_{F}^{i} \), upon \( Y_{\alpha} \) for all \( \alpha \in \{1, \ldots, m\} \):

\[
Y_{\alpha ij} \left( y_{F}^{i} - y_{F}^{j} \right) = 0 \quad \forall \ i, j \in \{1, \ldots, n\} .
\]

Proof: We shall take repeatedly advantage of the symmetry of \( Y_{\alpha} \) in its fermionic indices, \( Y_{\alpha} = Y_{\alpha}^{T} \).

We thus have

\[
\sum_{\beta=1}^{m} \left( Y_{\beta} x^{\alpha}_{\beta} \right)_{ij} = \sum_{\beta=1}^{m} \left( Y_{\beta} x^{\alpha}_{\beta} \right)_{ji},
\]

\[
\sum_{\beta=1}^{m} \left( Y_{\beta} x^{\alpha}_{\beta} \right)_{ij} = \sum_{\beta=1}^{m} \left( Y_{\alpha} x^{\beta}_{\beta} \right)_{ji}.
\]

With this and the definition (13) of \( x \), we find

\[
2 \sum_{\beta=1}^{m} \left[ \left( Y_{\beta} x^{\alpha}_{\beta} \right)_{ij} - \left( Y_{\beta} x^{\alpha}_{\beta} \right)_{ji} \right] = Y_{\alpha ij} \left( y_{F}^{i} - y_{F}^{j} \right) .
\] (31)

With the help of Eq. (30), the two sums on the left-hand side of Eq. (31) may be cast into the form

\[
\sum_{\beta=1}^{m} \left( Y_{\beta} x^{\alpha}_{\beta} \right)_{ij} = Y_{\alpha ij} \, u^{\alpha} \, v^{j} ,
\]

while, with \( y_{F}^{i} = v^{i} \, \text{Tr}(u) \), we have

\[
Y_{\alpha ij} \left( y_{F}^{i} - y_{F}^{j} \right) = Y_{\alpha ij} \, \text{Tr}(u) \, (v^{i} - v^{j}) .
\]

Taking into account that

\[
2 \, u^{\alpha} + \text{Tr}(u) = (2 \, |u^{\alpha}| + |\text{Tr}(u)|) \exp(-i \varphi) \neq 0 ,
\]

we obtain

\[
Y_{\alpha ij} \left( v^{i} - v^{j} \right) = 0
\]

and, therefore,

\[
Y_{\alpha ij} \left( y_{F}^{i} - y_{F}^{j} \right) = 0 .
\]

Prop. 2 may be interpreted as the alignment of \( (y_{F}^{i} = y_{F}^{j}) \)-blocks to a blockdiagonal structure for \( Y_{\alpha} \).

This structure is carried over to the YFC \([23]\); it can be inserted there to give a quasi-linear YFC:

\[
Y_{\alpha ij} \left( 8 \, x^{\alpha}_{\alpha} - 2 \, y_{F}^{i} + 2 \, y_{B}^{2} - 6 \, g^{2} \, C_{F}^{i} - 6 \, g^{2} \, C_{F}^{j} \right) = 0 .
\] (32)

Since

\[
x^{\alpha}_{\alpha j} = \left( Y^{\alpha}_{\alpha} \right)_{ij} = \frac{y_{F}^{i} \, y_{F}^{j}}{n} \, \delta^{i}_{j} \sum_{i=1}^{n} y_{F}^{j}
\]

holds, \( Y_{\alpha} \) is invertible for all \( \alpha \in \{1, \ldots, m\} \).

Remark 1: Restricting \( y_{F}^{i} \) by the two requirements \( y_{F}^{i} = 6 \, g^{2} \, C_{F}^{i} \) and \( n = d_{F} \), we recover the \( F = 1 \) theories. In this case, the commutator in Prop. 2 is carried over to \( Y_{\alpha ij} \left( C_{F}^{i} - C_{F}^{j} \right) = 0 \), and we obtain

\[
4 + d_{F} = 2 \, m \quad \text{and} \quad y_{F}^{i} = 6 \, g^{2} \, C_{F}^{i} = y \quad \forall \ i \in \{1, \ldots, d_{F}\} ,
\]

that is, one common value for all fermionic Casimir eigenvalues.

\footnote{In the context of finite quantum field theories, the notion of “quasi-linearity” was mentioned for the first time in Ref. \([15]\).}
In principle, it is now straightforward to solve the YFC in the form (32) for arbitrary values of $F$. The only quantity in Eq. (32) which does not depend on the Yukawa couplings $y_{\alpha ij}$ is the expression $6g^2C_F$, which is also independent of $\alpha$. Furthermore, because of the (highly welcome) quasi-linearity of the YFC (32), for this set of equations to be solvable at all, the quantities depending on the particular value of the constant $a$, where $a$ is the gauge coupling constant $g$. Beyond doubt, the ansatz (33) for $x^{\alpha a}$ which comes first to one’s mind reads
\[ x^{\alpha a} = 6g^2a C_F + b \quad \forall \, i \in \{1, \ldots, n\} \text{,} \tag{34} \]
with arbitrary constants $a, b \in \mathbb{C}$. After elimination of the constant $b$, this ansatz specifies $y_F^i$ and $y_B^i$ to
\[ y_F^i = \sum_{j=1}^{m} x^{ij} = m a \left( 6g^2 C_F^i - \frac{6g^2}{n} \sum_{k=1}^{n} C_F^k \right) + \frac{1}{n} \sum_{k=1}^{n} y_B^k + \frac{1}{m} \sum_{k=1}^{m} \sum_{l=1}^{n} y_F^l \text{,} \]
\[ y_B^i = \sum_{k=1}^{n} x^{k\alpha} = \frac{1}{m} \sum_{k=1}^{m} \sum_{l=1}^{n} y_F^l \text{.} \tag{35} \]
Substitution of these expressions into the quasi-linear YFC (32) yields
\[ Y_{\alpha ij} \left[ 6g^2 \left[ 2a \left( 4 - m \right) - 1 \right] C_F^i - 6g^2 C_F^j + 2 \frac{4 - m + n}{m n} \sum_{k=1}^{n} y_B^k - 12g^2a \frac{4 - m + n}{n} \sum_{k=1}^{n} C_F^k \right] = 0 \text{,} \tag{36} \]
which, depending on the particular value of the constant $a$ in Eq. (34), allows for exactly three types of solutions. For $a \neq 0$, the commutator in Prop. 3 entails
\[ Y_{\alpha ij} \left( C_F^i - C_F^j \right) = 0 \quad \forall \, i, j \in \{1, \ldots, n\} \text{,} \tag{37} \]
whereas, in the case $a = 0$, no such statement can be made. We summarize the solutions in form of

**Proposition 3:** In finite quantum field theories with Yukawa couplings satisfying the tensor structure $x = u \otimes v$ of Def. 3 and the ansatz $x^{\alpha a} = 6g^2a C_F^a + b$ of Eq. (34), all $y_F^i, i \in \{1, \ldots, n\}$, necessarily assume one of the following values:

**A:** For $a = 0$, there is only one common value for $y_F^i$, which involves the average $C_m := (C_F^i + C_F^j)/2$ of the Casimir eigenvalues:
\[ y_F^i \equiv y = 6g^2 \frac{m}{4 - m + n} C_m \quad \forall \, i \in \{1, \ldots, n\} \text{.} \]

**B:** For $a \neq (4 - m)^{-1}$, only one fermionic Casimir eigenvalue $C$ is allowed, that is, $(C_F)^i_j = \delta^i_j C$, and only one common value for $y_F^i$ is possible:
\[ y_F^i \equiv y = 6g^2 \frac{m}{4 - m + n} C \quad \forall \, i \in \{1, \ldots, n\} \text{.} \]

**C:** For $a = (4 - m)^{-1}$, different values for $y_F^i$ are allowed:
\[ y_F^i = 6g^2 \frac{m}{4 - m} \left( C_F^i - \frac{1}{4 - m + n} \sum_{k=1}^{n} C_F^k \right) \text{.} \]

**Remark 2:**

\footnote{This ansatz will prove to be consistent with the general solution of the YFC for tensorial $x = u \otimes v$}

\footnote{For a sketch of the proof, see Appendix C.}
1. We note explicitly that Prop. 3 is necessary and sufficient for finding solutions of the YFC which satisfy both $x = u \otimes v$ and the ansatz (44). However, it does not suffice to determine potentially finite theories since the two gauge-coupling finiteness conditions (9) and (10) overdetermine the YFC by restricting the particle content of such a theory. Formally, this fact becomes manifest by comparison of the value of $E(Y)$ with the group-theoretic quantity equivalent to $36 \, g^4 \, d_y \, Q_y \, F^2$:

$$E(Y) = 6 \, g^2 \sum_{i=1}^{n} C^i_y \, y^i_y = 36 \, g^4 \, d_y \, Q_y \, F^2 ? \quad (38)$$

2. For the purpose of solving the YFC (11), at least, it is neither necessary to demand $y^i_y \equiv y$ for all $i \in \{ 1, \ldots, n \}$ nor necessary to restrict the spectrum of solutions to $F = 1$. This observation rather stresses the importance of incorporating into an eventual proof of the necessity of conditions (9) and (10).

We call a quantum field theory “potentially finite” if its particle content fulfills both the finiteness condition (1) and the inequalities $0 < F^2 \leq 1$ for that quantity $F$ defined by Eq. (24), if the anomaly index of its fermionic representation, $R_y$, vanishes, if its bosonic representation, $R_B$, is real, $R_B \simeq R_B^*$, and if, at least, one fundamental invariant tensor, required for the decomposition (28) of $Y_{\alpha ij}$, exists.

In view of the structure of the quasi-linear YFC (24), the ansatz (44) for $x^{i\alpha}$ is independent of $\alpha$:

$$x^{i\alpha \beta} = \delta^{i\alpha} \delta^{i\beta} x^{ij} =: \delta^{i\alpha} \delta^{i\beta} x^{i} \quad (39)$$

Moreover, in our analysis of the system (26) only nonvanishing $y^i_y$, i.e., $y^i_y \neq 0$, enter. Hence, we may divide Eq. (13) by $x^j$ in order to get

$$M^{i\alpha} M_{\beta} + M^{1\beta} M_{\alpha} = 2 \, \delta^{i\alpha} \, 1_n \quad \forall \, \alpha, \beta \in \{ 1, \ldots, m \} \quad (40)$$

Mimicking a proof given in Ref. 14, we show, in Appendix 2, that any set of matrices $M_{\alpha}$ satisfying these relations is equivalent to the union of the $n \times n$ unit matrix $1_n$ and the subset

$$\mathcal{B}_m = \{ N_{\alpha} \mid \{ N_{\alpha}, N_{\beta} \} = 2 \, \delta_{\alpha\beta} \, 1_n, \ N_{\alpha ij} = N_{\alpha ji} \in \mathbb{R}, \ \alpha = 1, \ldots, m - 1 \} \quad (41)$$

of real, symmetric, and anticommuting elements $N_{\alpha}$ of a representation of some Clifford algebra $C$:

$$\{ Y_{\alpha}, \ \alpha = 1, \ldots, m \} \sim \{ 1_n \} \cup \mathcal{B}_m \quad (42)$$

**Remark 3:** According to Remark 22, $F = 1$ is not necessary to allow for solutions of the YFC (11), which are equivalent to representations of Clifford algebras. Moreover, considering Case C of Prop. 3, even solutions for different $y^i_y$ are possible.

At this point, the restriction to an irreducible YFC becomes important. As a consequence of this irreducibility assumption, the fermionic dimension $n$ of the YFC has to coincide with the dimension of the Clifford algebra representation. We may even use (reducible) representations of different Clifford algebras $C_p_i$ with rank $C_{p_i} = p_i$ if the number $q_i$ of elements in $C_{p_i}$ belonging to $\mathcal{B}_m$ is large enough:

$$m - 1 \leq \min_i q_i \quad (40)$$

The rank $p_i$ of a Clifford algebra is either even, $p_i = 2 \nu_i$, or odd, $p_i = 2 \nu_i + 1$, with $\nu_i \in \mathbb{N}$. If $p_i = 2 \nu_i$, then $C_{p_i}$ is simple and its representations are isomorphic to the direct sums of $2^{\nu_i} \times 2^{\nu_i}$ matrices [24]. These matrices may be constructed by Kronecker products of Pauli matrices [21]. Exactly one half of them is totally symmetric, as required for $\mathcal{B}_m$. However, for $p_i = 2 \nu_i$, an additional symmetric basis element of the Clifford algebra, the product of all generators, exists, yielding $q_i = \nu_i + 1$ symmetric anticommuting elements. If $p_i = 2 \nu_i + 1$, then $C_{p_i}$ is the direct sum of two two-sided ideals and

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12 The matrices $M_{\alpha}$ satisfying Eq. (40) transform like bi-vectors under a change of basis. Therefore, the matrices $N_{\alpha}$ are also bi-vectors. This behaviour under basis transformations guarantees that just the symmetric and anticommuting elements of a Clifford algebra representation are relevant for $\mathcal{B}_m$. [ref]
there exist again $q_i = \nu_i + 1$ symmetric anticommuting elements \[21\]. Let $k_i$ be the multiplicity of $2^{\nu_i}$-blocks in some representation covering $\mathfrak{B}_m$. Then, with $\nu := \min_i \nu_i$, $n$ must satisfy the inequality

$$n = \sum_i k_i 2^{\nu_i} \geq 2^{\nu} \sum_i k_i \geq 2^{\nu} \geq 2^{m-2}.$$  \hspace{1cm} (43)

Of course, this is only a necessary condition for a set of matrices to be equivalent to a Clifford algebra representation. For our purposes, however, it suffices. The actual restrictivity of this inequality may be demonstrated by applying it directly to the class of $F = 1$ theories (cf. Remark 4), which entails

**Proposition 4:** There exist no potentially finite $F = 1$ solutions of the quasi-linear YFC \[32\] which simultaneously obey the inequality \[43\].

This means that Clifford solutions of the kind conjectured in Ref. \[15\] do not exist for an irreducible YFC.

**Remark 4:**

1. Regarding the conjecture \[15\] that there might be a connection between solutions of the YFC being isomorphic to Clifford algebra representations (in our sense) and $N = 4$ supersymmetry, Prop. 4 excludes any such connection for the case of an irreducible YFC.

2. Very crucial for the non-existence of $F = 1$ Clifford solutions of an irreducible YFC is the drastic restriction on the fermionic dimension imposed by the inequality \[43\]: $d_F = 2$ or $d_F = 4$.

### 7 Numerics

Having formulated the problem in a way accessible to systematic investigation, we are now going to apply Props. 2 and 3 and the inequality \[43\] to gauge theories with simple gauge group $G$. Because of the gauge invariance (12) of the Yukawa couplings $Y$, we have to make sure that a decomposition (28) of $Y$ into invariant tensors indeed exists. In order to list all interesting theories, we have developed a C package \[25\] which provides us with all potentially finite theories for a given simple Lie algebra $\mathfrak{A}$. For every potentially finite theory, this C package involves (optionally) a function constraint to be specified by the user, which we adopt to filter all theories obeying Props. 2 and 3 as well as Eq. (43).

We confine ourselves to theories where all irreducible representations able to evolve invariant tensors for Yukawa couplings (together with their respective partners, if necessary) indeed contribute. The C package \[25\] yields bosonic multiplicities $b_0^A$ and fermionic multiplicities $f_0^I$, each of them describing the multiplicity of a certain type of pairwise inequivalent irreducible representations. $R_F$ and $R_B$ are then completely determined by $f_0^I$ and $b_0^A$:

$$R_F = \bigoplus_I f_0^I R_I^I,$$

$$R_B = \bigoplus_A b_0^A R_A^A.$$

Now, with respect to that constant $a$ in Ansatz (34), Prop. 3 suggests to analyze the cases $a \neq 0$ and $a = 0$ separately:

- **Case $a \neq 0$:** For every $R_I \subset R_F$, we have to find those $R_J \subset R_F$ and $R_A \subset R_B$ which, according to Eqs. (12) and (28), satisfy $R_I \otimes R_J \otimes R_A \supset 1$, and, according to Prop. 3, have $C_F^I = C_F^J$.

Precisely the same procedure has to be applied to every $R_A \subset R_B$. An (admissible) irreducible non-orthogonal representation $R_A \subset R_B$ enforces a non-vanishing contribution of the complete real block $R^\mu \simeq R_A \oplus (R_A)^c$ (cf. Appendix B):

$$R_I \otimes R_J \otimes R_A \supset 1 \quad \text{if and only if} \quad R_I \otimes R_J \otimes R_A \supset 1 \quad \text{or} \quad R_I \otimes R_J \otimes (R_A)^c \supset 1.$$  \hspace{1cm} (45)

\[13\] For the proof, see Appendix B.

\[14\] This means, we do not delete the contribution of irreducible representations to the YFC by hand.
Every $R^I$ and $R^\mu$ which does not satisfy both requirements (44) and (45) has to be deleted from $R_F$ and $R_B$, respectively. This procedure yields new multiplicities $f_I$ and $b_\mu$. The corresponding irreducible representations then fulfill Eqs. (44) and (45). Furthermore, they define the subsets

$$R_F^{YFC} = \bigoplus_I f_I \, R^I \subset R_F,$$
$$R_B^{YFC} = \bigoplus_\mu b_\mu \, R^\mu \subset R_B,$$  \hspace{1cm} (46)

with the dimensions

$$n = \dim R_F^{YFC} = \sum_I f_I \, d_I \leq d_F,$$
$$m = \dim R_B^{YFC} = \sum_\mu b_\mu \, d_\mu \leq d_B.$$  

The remaining $R^I$ with non-vanishing multiplicities $f_I$ have to be searched for different Casimir eigenvalues. The number of different Casimir eigenvalues specifies whether Case B or Case C of Prop. 3 is relevant for that particular theory. Having decided which case is actually realized, we compute $F^2_{YFC}$, the value of $F^2$ resulting from the YFC. With $m$ and $n$ as given above and the abbreviations

$$Q_F = \sum_{R^I \subset R_F} f_I^0 \, S_I \, C_I,$$
$$Q_{YFC} = \sum_{R^I \subset R_F^{YFC}} f_I \, S_I \, C_I,$$
$$S_{YFC} = \sum_{R^I \subset R_F^{YFC}} f_I \, S_I,$$
$$C^0 = \frac{1}{4 - m + n} \sum_{R^I \subset R_F^{YFC}} f_I \, d_I \, C_I = \frac{S_{YFC} \, d_g}{4 - m + n},$$

we get, if $C^I_F = C$ for all $f_I \neq 0$,

$$F^2_{YFC} = \frac{m}{4 - m + n} \, \frac{Q_{YFC}}{Q_F},$$

else

$$F^2_{YFC} = \frac{m}{4 - m} \, \frac{Q_{YFC} - C^0 \, S_{YFC}}{Q_F}.$$  

The subroutine constraint also yields the value of $F^2$ which results from the particle content of the theory and which may be compared with the above $F^2_{YFC}$:

$$F^2_{YFC} = \frac{Q_F + Q_B + c_g \, (S_F - 2 \, c_g)}{3 \, Q_F}?.$$  \hspace{1cm} (47)

All theories giving equality may be regarded as good candidates for finite quantum field theories in the sense of Prop. 3. As final check, we apply Eq. (13) to theories passing the criterion (47).

• Case $a \equiv 0$: According to Prop. 3 let $(C^I_F, C^\mu_F)$ for $R^I, R^\mu \subset R_F$ be a pair of Casimir eigenvalues which may couple invariantly, and let $C_m = (C^I_F + C^\mu_F)/2$. $R^I$ and $R^\mu$, and every $R^\mu \subset R_B$ with $R^I \otimes R^I \otimes R^\mu \supset 1$, contribute to the YFC. If there exist further pairs $(C^K_F, C^L_F) \neq (C^I_F, C^\mu_F)$ for $R^K, R^L \subset R_F$, with the same $C_m$, which allow for invariant couplings, we collect all contributing irreducible representations in form of

$$R_F^{YFC}(C_m) = \bigoplus_I f_I \, R^I,$$
$$R_B^{YFC}(C_m) = \bigoplus_\mu b_\mu \, R^\mu,$$  \hspace{1cm} (48)

We are allowed to use $n$ and $m$ as in Def. 1 because we assume that all irreducible representations in Eq. (13) with non-vanishing multiplicity actually contribute to the YFC.
Remark 5: Our analysis is based on the standard form (26) of the YFC, which is naturally related to $x$ as in Def. 3 and the ansatz (34) for $y$ in $\mathbb{R}$ merely the consequence of the bi-linearity of the YFC and its invariance under gauge transformations. No invariant tensor exists which contributes to the system (26), on the other hand. The standard form (23) of the YFC turns out to be content of a theory under consideration, on the one hand, and the degrees of freedom which actually contribute to the system (26), on the other hand. The standard form (23) of the YFC on a more fundamental level, we worked out the importance of distinguishing carefully between the full particle quantum field theories. Apart from the re-derivation of the standard form (23) of the YFC on a more

$$F_{YFC}^2 = \frac{m}{4-m+n} \frac{C_m S_{YFC}}{Q_F} = \frac{Q_F + Q_B + c_g (S_F - 2c_g)}{3Q_F}$$

(49)

Finally, Eq. (13) has to be checked. This procedure has to be applied to all values of $C_m$ allowed by $R_F$.

8 Summary, Conclusions, and Outlook

Motivated by recent findings in the analysis of $F^2 = 1$ theories [15], we discussed particular properties and solutions of the one-loop finiteness condition for the Yukawa couplings in general renormalizable quantum field theories. Apart from the re-derivation of the standard form (23) of the YFC on a more fundamental level, we worked out the importance of distinguishing carefully between the full particle content of a theory under consideration, on the one hand, and the degrees of freedom which actually contribute to the system (23), on the other hand. The standard form (23) of the YFC turns out to be merely the consequence of the bi-linearity of the YFC and its invariance under gauge transformations. A comprehensive characterization of this standard form is provided by blockdiagonality of $R_F$ in each irreducible representation and of $R_B$ in each real block. Demanding $y_F^p = 6g^2 C_F^p$ for all $i \in \{1, \ldots, n\}$ suffices to reduce the (troublesome) cubic YFC (26) to a quadratic system of the “$F = 1$ form” (23).

The crucial observation leading to our notion of “reducibility” of the YFC in the sense of Def. 2 was that, in general, $R_F$ and $R_B$ may contain subsets of irreducible representations which completely decouple from each other. Our intention is to examine the existence of Clifford-like Yukawa couplings in finite theories, first, by considering an irreducible YFC. For $F = 1$, the situation is summarized in

**Theorem 1:** Let the YFC be irreducible, and assume $x = u \otimes v$. Then there does not exist any $F = 1$ solution of the YFC obeying the following criteria:

1. The fermionic representation $R_F$ has vanishing anomaly index.
2. The bosonic representation $R_B$ is real.
3. The beta function for the gauge coupling $g$ vanishes in one-loop approximation.

Hence, there cannot exist any connection between $N = 4$ supersymmetry and such Clifford solutions.

By means of the physically motivated ansatz (34), using our C package [25], we were able to prove

**Theorem 2:** Let us consider a simple Lie algebra

$$\mathfrak{a} \in \{ A_r, B_r, C_r, D_r, E_6, E_7, E_8, F_4, G_2 \mid r = \text{rank} \mathfrak{a} \leq 8 \} ,$$

\footnote{Apart from the fact that in Theorem 1 not all bosonic representations having appropriate partners in $R_F$ are required to contribute, Theorem 2 is a generalization of Theorem 1.}
let the YFC be irreducible, and assume \( x = u \otimes v \). Then there does not exist any solution of the YFC with
\[
x^{i\alpha} = 6 g^2 a C_F^i + b \quad , \quad a, b \in \mathbb{C}
\]

obeying the following criteria:

1. The fermionic representation \( R_F \) has vanishing anomaly index.
2. The bosonic representation \( R_B \) is real.
3. The beta function for the gauge coupling \( g \) vanishes in one- and two-loop approximation.
4. Irreducible blocks \( R^i_B \subset R_B \) and \( R^l_F \subset R_F \), with multiplicities \( b_i \), \( f_I \), and \( f_J \), respectively, which allow for invariant couplings, i.e., \( R^i_B \otimes R^l_F \otimes R^j_F \supset 1 \), contribute to the YFC such that
\[
y_F |_{f_I} \times R_f \neq 0
\]
and
\[
y_B |_{b_i} \times R_B \neq 0
\]

In order to complete the investigations started here, at least two directions have to be pursued: First, all possibilities for a reducible YFC must be analyzed in an identical manner; this ambitious goal will be approached in forthcoming papers [22, 23]. Secondly, by relaxing the last criterion in Theorem 4, a search for Yukawa solutions with arbitrary amount of contribution to the YFC should be performed.

**Acknowledgements**

M. M. would like to thank H. Urbantke for a fruitful discussion concerning parts in the construction of invariants. We are also indebted to A. Prets and W. Spitzer for a critical reading of the manuscript. M. M. was supported by “Fonds zur Förderung der wissenschaftlichen Forschung in Österreich,” project 09872-PHY, by the Institute for High Energy Physics of the Austrian Academy of Sciences, and by a grant of the University of Vienna.
A Invariance of the Operator $x$ on $V_F \times V_B$

Let $Z: V_F \times V_B \to V_F \times V_B$ be defined by its components according to

$$Z^{i\alpha}_{j\beta} := \sum_k (\Upsilon \otimes Y)^{\alpha k}_{\beta kj} = \sum_k \Upsilon^{\alpha k}_{\beta kj},$$

where the dual tensor $\Upsilon$ of $Y$, which acts on the product space $V_F^* \times V_B^* \times V_F^*$, has been introduced. From this definition, we infer that $Z$ behaves under gauge transformations like an operator on $V_F \times V_B$:

$$Z^{i\alpha}_{j\beta} = \sum_k (R_F)^i_k (R_B)^{\alpha}_{\gamma} (R_F^c)^l_j (R_B^c)^{\delta}_{\beta} Z^{k\gamma}_{l\delta}. \quad (51)$$

The operator $x$, defined by Eq. (13), is equal to half the sum of $Z$ and its Hermitean conjugate, $Z^\dagger$:

$$2x = Z + Z^\dagger.$$

Consequently, in order to prove the invariance of $x$ under $R_F \otimes R_B$, it is sufficient to show this for $Z$.

Obviously, the gauge-transformed $Z$, $Z'$, must be related to $Y' = (R_F^c \otimes R_B^c \otimes R_F^c) Y$ according to

$$Z'^{i\alpha}_{j\beta} = \sum_k \Upsilon^{\alpha k}_{\beta kj}.$$

However, recalling the gauge invariance of $Y$ as expressed by Eq. (12), $Y' = Y$, we get invariance of the operator $Z$ too:

$$Z'^{i\alpha}_{j\beta} = Z^{i\alpha}_{j\beta}. \quad (51)$$

With Eq. (51), this observation may be rephrased in the form

$$\sum_{k,\gamma} (R_F)^i_k (R_B)^{\alpha}_{\gamma} Z^{k\gamma}_{j\beta} = \sum_{k,\gamma} Z^{i\alpha}_{k\gamma} (R_F)^k_j (R_B)^{\gamma}_{\beta}$$

or by the commutator

$$[R_F \otimes R_B, Z] = 0,$$

which makes the invariance of $Z$ with respect to $R_F \otimes R_B$ (on the product space $V_F \times V_B$) manifest.

B Decomposition of $Y$ into Fundamental Invariant Tensors

We owe to the reader a discussion of the precise relation of Eq. (28) to the bases $B_F$ and $B_B$ in which $R_F$ and $R_B$ are of the blockdiagonal form (17). Recall that Eq. (17) corresponds to a decomposition according to invariant subspaces $V_I$ and $V_\mu$ of $V_F$ and $V_B$, respectively, with multiplicities $f_I$ and $b_\mu$ (11). Performing the unitary transformation induced by

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i1 \\ i1 & 1 \end{pmatrix}$$

in each unitary reducible orthogonal block $R^\mu \subset R_B$ of Eq. (17),

$$U^\dagger R^\mu U = R^A \oplus (R^A)^c, \quad (52)$$

we may express $R_B$ as the direct sum over irreducible representations $R^A$ with multiplicities $b_A$. We introduce a fermionic basis $B_F$ and a bosonic basis $B_B$ by

$$B_F = \{ e_I \otimes e_{i_I} \mid I = 1, \ldots, \sum_I f_I; \ i_I = 1, \ldots, d_I \}, \quad B_B = \{ f_A \otimes f_{\alpha_A} \mid A = 1, \ldots, \sum_A b_A; \ \alpha_A = 1, \ldots, d_A \}.$$

Let $R^A_B \subset R_B$, $R^I_F \subset R_F$, and $R^I_B \subset R_B$ be any three irreducible representations. For these irreducible representations, $N(A, I, J)$ invariant tensors $\Lambda^{(k)}$ exist if and only if $R^A_B \otimes R^I_F \otimes R^I_B \supset 1_{N(A, I, J)} \otimes 1.$
Then the expansion of the Yukawa couplings \( Y \) in the tensor basis \( \mathcal{B} = \{ f^A \otimes e^I \otimes e^J \otimes f^{\alpha A} \otimes e^{\iota I} \otimes e^{\jmath J} \} \), with coefficients \( p_{\alpha\iota\jmath\jmath}^{(k)} \in \mathbb{C} \), assumes the form \( \text{(28)} \):

\[
Y_{\alpha\iota\jmath\jmath} = Y_{(A,\alpha A)(I,\iota I)(J,jJ)} = \sum_{k=1}^{N(A,I,J)} p_{\alpha\iota\jmath\jmath}^{(k)} \left( A^{(k)} \right)_{\alpha\iota\jmath\jmath}.
\]

Applying to this expansion the transformation inverse to Eq. \( \text{(52)} \) gives

\[
Y_{\alpha\iota\jmath\jmath} = Y_{(\mu,\alpha \mu)(I,\iota I)(J,jJ)} = \sum_{k=1}^{N(\mu,I,J)} p_{\alpha\iota\jmath\jmath}^{(k)} \left( A^{(k)} \right)_{\alpha\iota\jmath\jmath},
\]

where \( N(\mu,I,J) = N(A,I,J) + N(A^C,I,J) \). Hence, invariant contributions to a fixed real block \( R^\mu \) arise from \( R^A \) and \( (R^A)^C \); the splitting of the YFC into subsystems is between complete real blocks.

### C  The Three Types of Solutions of the Yukawa Finiteness Condition

We solve the quasi-linear YFC \( \text{(36)} \) for the ansatz \( \text{(34)} \) by distinguishing between the following three cases:

- **Case A**: \( a = 0 \). From the ansatz \( \text{(34)} \), we immediately conclude that \( y_F^i = y \) for all \( i \in \{1,\ldots,n\} \). With this, Eq. \( \text{(36)} \) yields, for all \( i \in \{1,\ldots,n\} \) and for all \( \alpha \in \{1,\ldots,m\} \),

\[
x^\alpha = y_F^i = y = 6 g^2 \frac{m}{4 - m + n} C_m,
\]

with \( C_m := (C_F^i + C_F^j)/2 \).

- **Case B**: \( a \neq (4 - m)^{-1} \). We take advantage of the commutator \( \text{(37)} \) in order to simplify Eq. \( \text{(36)} \) to

\[
Y_{\alpha\iota\jmath\jmath} \left[ 6 g^2 \left[ (4 - m) a - 1 \right] C_F^i + \frac{4 - m + n}{m n} \sum_{k=1}^{n} y_F^k - 6 g^2 \frac{a}{n} \sum_{k=1}^{n} C_F^k \right] = 0. \tag{53}
\]

The invertibility of \( Y_{\alpha\iota\jmath\jmath} \) expressed by Eq. \( \text{(33)} \) allows to sum in Eq. \( \text{(53)} \) over all \( i = 1,\ldots,n \):

\[
\sum_{k=1}^{n} y_F^k = 6 g^2 \frac{m}{4 - m + n} \sum_{k=1}^{n} C_F^k. \tag{54}
\]

This intermediate result may be re-inserted into Eq. \( \text{(53)} \):

\[
6 g^2 \left[ (4 - m) a - 1 \right] C_F^i + \frac{6 g^2}{n} \sum_{k=1}^{n} C_F^k - 6 g^2 \frac{a}{n} \sum_{k=1}^{n} C_F^k = 0,
\]

which clearly implies \( C_F^i = C \) for all \( i \in \{1,\ldots,n\} \), that is, \( (C_F)^i_j = \delta^i_j C \). Consequently, from Eq. \( \text{(34)} \), we find, for all \( i \in \{1,\ldots,n\} \),

\[
y_F^i = y = 6 g^2 \frac{m}{4 - m + n} C.
\]

- **Case C**: \( a = (4 - m)^{-1} \). By a line of reasoning analogous to the one applied to Case B, we get

\[
y_F^i = 6 g^2 \frac{m}{4 - m} \left( C_F^i - \frac{1}{4 - m + n} \sum_{k=1}^{n} C_F^k \right).
\]
D  Equivalence to the Representation of a Clifford Algebra

Following Ref. [14], let us briefly demonstrate that any set of \( n \times n \) matrices \( Y_\alpha \) satisfying the relations
\[
(Y^{\dagger \alpha} Y_\beta + Y^{\dagger \beta} Y_\alpha)_{ij} = 2 \delta^{\alpha \beta} \delta_{ij} \quad \forall \alpha, \beta \in \{1, \ldots, m\}
\]
(55)
is equivalent to a representation of a Clifford algebra. First of all, the block structure of \( Y_\alpha \) enforced by Prop. 3 can be used to split the fermionic representation space \( V_F \) into subspaces \( V_F^\prime \) with \( y_F |_{V_F^\prime} = y_F^\prime \).

A transformation of \( Y_\alpha \) of the form
\[
M_\alpha = D Y_\alpha D^T,
\]
which preserves the symmetry of \( Y_\alpha \) in its fermionic indices, i.e., \( M_\alpha = M_\alpha^T \), with the diagonal matrix \( D \) defined by
\[
D_{ij} := \delta_{ij} \frac{1}{\sqrt{x_i}} ,
\]
leads to
\[
(M^{\dagger \alpha} M_\beta + M^{\dagger \beta} M_\alpha)_{ij} = 2 \delta^{\alpha \beta} \delta_{ij} \quad \forall \alpha, \beta \in \{1, \ldots, m\} .
\]
(56)

Now, each \( M_\alpha \) may be decomposed like \( M_\alpha = M_\alpha^I + i M_\alpha^J \) with real symmetric matrices \( M_\alpha^I \) and \( M_\alpha^J \). For \( \alpha = \beta \), Eq. (56) implies
\[
[M_\alpha^I, M_\alpha^J] = 0 .
\]

Hence, there exists an orthogonal transformation \( U_0 \) such that, for a fixed \( \alpha_0 \), \( M_{\alpha_0} \) becomes a diagonal matrix of pure phases \( \zeta_j \),
\[
M_{\alpha_0}^I = U_0 M_{\alpha_0}^I U_0^T = (\delta_{ij} \exp(i \zeta_j)) ,
\]
while, for \( \{M_\alpha^I\} \), the analogue of Eq. (54) still holds. Since the YFC is invariant under \( U(d_F) \otimes O(d_B) \) transformations, these matrices \( M_\alpha^I \) are the solutions of the YFC transformed by \( D \) and \( U_0 \). We may use the invariance of the YFC under phase transformations to rotate \( M_{\alpha_0}^I \) into the \( n \times n \) unit matrix:
\[
M_{\alpha_0}^I = 1_n .
\]
Combining orthogonal and phase transformations, Eq. (56) becomes
\[
M_\beta^I + \overline{M_\beta^I} = 0 \quad \forall \beta \neq \alpha_0 := m ,
\]
where \( \overline{M_\beta^I} \) is the matrix complex conjugated to \( M_\beta^I \). This, in turn, implies
\[
M_\beta^I = \frac{i}{2} N_{\beta \beta} \quad \forall \beta \in \{1, \ldots, m-1\} ,
\]
with real and symmetric matrices \( N_{\beta \beta} \), i.e., \( N_{\beta \beta} = N_{\beta \beta}^T \in \mathbb{R} \). In terms of the latter, Eq. (56) reads
\[
\{N_\alpha, N_\beta\} = 2 \delta_{\alpha \beta} 1_n \quad \forall \alpha, \beta \in \{1, \ldots, m-1\} .
\]
(57)

E  Clifford Solutions of the Irreducible Yukawa Finiteness

Condition for \( F = 1 \)

By a straightforward inspection, we are able to preclude the existence of Clifford-like \( F = 1 \) solutions of an irreducible YFC. For \( F = 1 \), Eqs. (22) and (23) entail
\[
4 + d_F = 2m \leq 2d_B ,
\]
(58)
which tells us that \( d_F \) must be even. This relation for \( m \) and the inequalities (23) conspire to restrict the possible values of \( d_F \):
\[
d_F \geq n \geq 2^{m-2} = 2^{d_F/2}
\]
can be fulfilled only by \( d_F = 2, 3, 4 \). Hence, we have to investigate two possibilities: \( d_F = 2 \) or \( d_F = 4 \). The complete list of \( F = 1 \) theories with a particle content satisfying the one-loop finiteness condition \( [3] \) for the gauge coupling, with an anomaly-free fermionic representation \( R_F \), and with a real bosonic representation \( R_B \), is provided by our C package [25], the subroutine constraint checking for \( F = 1 \).

However, irreducible representations \( R^I \) with dimensions \( d_I \leq d_F \leq 4 \) exist only for the Lie algebras \( \mathfrak{g} = A_1, A_2, A_3, C_2 \). Merely one theory in our list, for \( A_1 \), is consistent with the requirement \( d_F \leq 4 \).

Denoting the \( d_I \)-dimensional irreducible representation \( R^I \) of \( A_1 \) by \([d_I]\), this theory is specified by
\[
R_F = [4] , \quad R_B = 20 \{2\} \oplus 7 \{3\} ,
\]
where direct sums are implicitly understood. For these representations, invariant tensors to construct gauge-invariant Yukawa couplings exist only for \([3] \otimes [4] \otimes [4] \). From the decomposition (28) of \( Y \) into invariant tensors, \( m \) may take values in \( \{3, 6, 9, \ldots, 21\} \), whereas Eq. (58) for \( d_F = 4 \) implies \( m = 4 \).
References

[1] D. R. T. Jones and L. Mezincescu, Phys. Lett. B 136 (1984) 242; 138 (1984) 293; P. West, Phys. Lett. B 137 (1984) 371; A. Parkes and P. West, Phys. Lett. B 138 (1984) 99.

[2] D. R. T. Jones, L. Mezincescu, and Y.-P. Yao, Phys. Lett. B 148 (1984) 317; J. León and L. Pérez-Mercader, Phys. Lett. B 164 (1985) 95.

[3] C. Lucchesi, O. Piguet, and K. Sibold, Phys. Lett. B 201 (1988) 241; X.-D. Jiang and X.-J. Zhou, Phys. Rev. D 42 (1990) 2109.

[4] P. S. Howe, K. S. Stelle, and P. C. West, Phys. Lett. B 124 (1983) 55.

[5] A. Parkes and P. West, Phys. Lett. B 127 (1983) 353; J.-M. Frère, L. Mezincescu, and Y.-P. Yao, Phys. Rev. D 29 (1984) 1196; 30 (1984) 2238.

[6] I. G. Koh and S. Rajpoot, Phys. Lett. B 135 (1984) 397; F.-X. Dong, T.-S. Tu, P.-Y. Xue, and X.-J. Zhou, Phys. Lett. B 140 (1984) 333; J.-P. Derendinger, S. Ferrara, and A. Masiero, Phys. Lett. B 143 (1984) 133; X.-D. Jiang and X.-J. Zhou, Phys. Lett. B 144 (1984) 370; S. Kalara, D. Chang, R. N. Mohapatra, and A. Gangopadhyaya, Phys. Lett. B 145 (1984) 323; P. Fayet, Phys. Lett. B 153 (1985) 397.

[7] P. S. Howe, K. S. Stelle, and P. K. Townsend, Nucl. Phys. B 214 (1983) 519; 236 (1984) 125.

[8] W. Lucha and H. Neufeld, Phys. Rev. D 34 (1986) 1089.

[9] W. Lucha and H. Neufeld, Phys. Lett. B 174 (1986) 186.

[10] M. Böhm and A. Denner, Nucl. Phys. B 282 (1987) 206.

[11] W. Lucha and H. Neufeld, Helvetica Physica Acta 60 (1987) 699.

[12] W. Lucha, Phys. Lett. B 191 (1987) 404.

[13] W. Lucha and M. Moser, Int. J. Mod. Phys. A9 (1994) 2773.

[14] G. Kranner, Universelle Bedingungen für endliche renormierbare Quantenfeldtheorien, doctorate thesis, Technical University of Vienna (1990).

[15] G. Kranner and W. Kummer, Phys. Lett. B 259 (1991) 84.

[16] H. Skarke, Int. J. Mod. Phys. A9 (1994) 711.

[17] P. Grandits, Mod. Phys. Lett. A9 (1994) 1093; A9 (1994) 2555; Int. J. Mod. Phys. A10 (1995) 1507.

[18] C. H. Llewellyn Smith, Phys. Lett. B 46 (1973) 233; J. M. Cornwall, D. N. Levin, and G. Tiktopoulos, Phys. Rev. D 10 (1974) 1145.

[19] T. P. Cheng, E. Eichten, and L.-F. Li, Phys. Rev. D 9 (1974) 2259.

[20] M. E. Machacek and M. T. Vaughn, Nucl. Phys. B 222 (1983) 83.

[21] H. Boerner, Darstellungen von Gruppen (Springer, Berlin – Heidelberg, 1967).

[22] W. Lucha and M. Moser, Clifford algebras in finite quantum field theories, II: Reducible Yukawa finiteness condition, HEPHY-PUB 654/96, UWTPh-1996-50 (in preparation).

[23] W. Lucha and M. Moser, Clifford algebras in finite quantum field theories, III: Origin of Clifford solutions, HEPHY-PUB 655/96, UWTPh-1996-51 (in preparation).

[24] M. Riesz, Clifford Numbers and Spinors, in Fundamental Theories of Physics, E. F. Bolinder and P. Louesto, eds., (Kluwer Academic Pub., Dordrecht, 1993).

[25] W. Lucha and M. Moser, FINBASE, a C package for potentially finite quantum field theories, HEPHY-PUB 656/96, UWTPh-1996-52 (in preparation).

[26] R. U. Sexl and H. K. Urbantke, Relativität, Gruppen, Teilchen (Springer, Wien, 1992).