Abstract

In this paper, we consider a new approach for solving fuzzy integral equation. In this work, we find an approximate solution of fuzzy integral equation by using fuzzy Gauss quadrature formula. We first express the necessary materials and definitions, and then we introduce the concept of fuzzy Gauss quadrature formula. In the last section, we present an important application of fuzzy Gauss quadrature rule to solve a boundary value problem. In this problem, the boundary conditions appear as the form of fuzzy integral equations. However, the integrals in the boundary conditions are approximated by fuzzy Gauss quadrature method. Numerical examples given in the last part confirm the accuracy and validity of our new technique. In this example, we obtain the Hausdorff distance between exact solution and approximate solution.

Keywords: Boundary Value Problems, Fuzzy Integral Equation, Fuzzy Gauss Quadrature Formula, Hausdorff Distance

1. Introduction

The present paper is concerned with the numerical solution of the fuzzy integral equation. The concept of fuzzy integral equations has been considered by various authors and has been developed rapidly. Ralescu and Adams in presented the fuzzy integral of a positive, measurable function, with respect to a fuzzy measure. They showed that the monotone convergence theorem and Fatou’s lemma are true in new setting. In 1982, Dubois and Prade published a paper in which they investigated the concept of integration of fuzzy functions. In this paper, they define the integral of such fuzzy mappings over a crisp interval and provide a special analytical representation of the fuzzy mapping. In, the auto-continuity of a set function and Sugeno’s fuzzy measure with some conditions are introduced by Wang. He proved Egoroff’s theorem on a fuzzy measure space and some convergence theorems of sequence of fuzzy integrals. Goetschel and Voxman defined differentiation and integration of fuzzy-valued functions by using the usual vector space operations and a spatial metric in (4). Nanda in (5) generalized the integral introduced by Matloka (6) and also extends Riemann-Stieltjes integral over a closed interval to fuzzy mappings. However, several attempts have been made to apply numerical methods for fuzzy integral equations, and these techniques have been rapidly developed in recent years. The interested reader can see (7) (8) (9) (10) and (11) for some more provided methods to solve the fuzzy integral equations. In the present paper, we apply the fuzzy Gaussian quadrature method for solving fuzzy Fredholm integral equations. The fuzzy Gauss quadrature rule is one of the best and the most powerful techniques which gives fast and accurate solutions. In this study, we first introduce some properties and definitions of fuzzy numbers, and then we state the fuzzy Gaussian quadrature formula and its properties. However, we apply this method for solving a boundary value conditions with the boundary conditions as the form of fuzzy Fredholm integral equation. Finally, the efficiency and accuracy of this method is shown by an example.
2. Fuzzy Set Properties

A nonempty subset $B$ of $\mathbb{R}^n$ is called fuzzy convex if and only if $u(\lambda x + (1 - \lambda)y) \geq \min\{u(x), u(y)\}$ for every $x, y \in B$ and $\lambda \in [0,1]$. Let $H(\mathbb{R}^n)$ denote the space of non-empty compact convex subsets of $\mathbb{R}^n$. Recall that for any $A \subseteq H(\mathbb{R}^n)$ we defined

$$d(x, A) = \inf_{a \in A} d(x, a)$$

to be the distance of any $x \in \mathbb{R}^n$ from $A$, and for any positive number $\varepsilon$, we define the $\varepsilon$-neighborhood of $A$ as the set

$$N(A, \varepsilon) = \{x \in \mathbb{R}^n : |d(x, A)| < \varepsilon\}$$

Notice that the infimum in the definition of $d(x, A)$ is actually achieved, that is, there is a point $a \in A$ such that $d(x, A) = d(x, a)$, because $A$ is compact. For $A$ and $B \in H(\mathbb{R}^n)$, we define the Hausdorff separation $\rho(A, B)$ as below

$$\rho(A, B) = \inf_{\varepsilon > 0} |A \subseteq N(B, \varepsilon)|$$

We introduce the Hausdorff metric on $H(\mathbb{R}^n)$ as below

$$d_H(A, B) = \max\{\rho(A, B), \rho(B, A)\}$$

$$= \inf_{\varepsilon > 0} |A \subseteq N(B, \varepsilon) and B \subseteq N(A, \varepsilon)|$$

It is obvious that $(H(\mathbb{R}^n), d_H)$ is a complete and separable metric space. There are some definitions for the concept of fuzzy numbers. Here we define this concept as bellow.

**Definition 2.1.** A fuzzy number is a function such as $u: \mathbb{R}^n \rightarrow [0,1]$ satisfying the following properties:

1. $u$ is normal, i.e. $\exists x_0 \in \mathbb{R}^n$ such that $u(x_0) = 1$,
2. $u$ is fuzzy convex,
3. $u$ is upper semicontinuous,
4. $[u]^0 = \{x \in \mathbb{R}^n : u(x) > 0\}$ is compact.

The set of all fuzzy numbers is denoted by $E^n$. If $u$ is a fuzzy number in $E^n$, we define $[u]^\alpha = \{x \in \mathbb{R}^n : u(x) > \alpha\}$ the $\alpha$-level of $u$ with $0 < \alpha \leq 1$. For $\alpha = 0$ the support of $u$ is defined as $[u]^0 = \text{supp}(u) = \{x \in \mathbb{R}^n : u(x) > 0\}$. Clearly, for any $\alpha \in [0,1], [u]^\alpha$ is a bounded closed interval. For all $u, v \in E^n$ and for any real number $\lambda$ we obtain

$$[u + v]^\alpha = [u]^\alpha + [v]^\alpha, \quad [\lambda u]^\alpha = \lambda [u]^\alpha \forall \alpha \in [0,1]$$

In the what follows, we present another definition for a fuzzy number which will be used in the rest of this paper.

**Definition 2.2.** An arbitrary fuzzy number is showed by an ordered pair of functions $(\underline{u}(r), \overline{u}(r))$, $0 \leq r \leq 1$, which satisfies the following requirements:

1. $\underline{u}(r)$ is a bounded left semicontinuous non-decreasing function over $[0,1]$.
2. $\overline{u}(r)$ is a bounded left semicontinuous non-increasing function over $[0,1]$.
3. $\underline{u}(r) \leq \overline{u}(r), 0 \leq r \leq 1$.

In particular, if $\underline{u}, \overline{u}$ are linear functions we have triangular fuzzy number. A crisp number $u$ is simply represented by $(\underline{u}(r), \overline{u}(r)) = (\underline{u}(r), \overline{u}(r))$. For arbitrary fuzzy numbers $u = (\underline{u}(r), \overline{u}(r)), v = (\underline{v}(r), \overline{v}(r))$ we have algebraic operations bellow:

1. $ku = (k\underline{u}(r), k\overline{u}(r))$ and $kv = (k\underline{v}(r), k\overline{v}(r))$.
2. $u + v = (\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r))$.
3. $u - v = (\underline{u}(r) - \underline{v}(r), \overline{u}(r) - \overline{v}(r))$.

Now, we introduce the concepts of fuzzy Differentiability and integrability. The two following definitions are due to Ralesco in (1) and R. Goetschel and W. Voxman in (4).

**Definition 2.3.** A fuzzy function $F: (a, b) \rightarrow \mathbb{R}^n$ is called differentiable at $\xi_0 \in (a, b)$ if there exists $F'(\xi_0) \in \mathbb{R}^n$, such that the limits

$$\lim_{h \rightarrow 0^+} \frac{F(\xi_0 + h) - F(\xi_0)}{h} \text{ and } \lim_{h \rightarrow 0^-} \frac{F(\xi_0 + h) - F(\xi_0)}{h}$$

both exist and are equal to $F'(\xi_0)$.

**Definition 2.4.** Let $f: [a, b] \rightarrow \mathbb{R}^n$ be a fuzzy function. For each partition $P = \{t_0, t_1, \ldots, t_n\}$ of $[a, b]$ with $\Delta = \max_{i = 1, \ldots, n} |t_i - t_{i-1}|$ and for arbitrary $\xi_i \in [t_{i-1}, t_i]$, $i = 1, \ldots, n$, suppose

$$R_P = \sum_{i=1}^{N} f(\xi_i)(t_i - t_{i-1})$$

The definite integral of $f$ over $[a, b]$ is defined as bellow

$$\int_{a}^{b} f(t)dt = \lim_{\Delta \rightarrow 0} R_P$$
If the fuzzy function $f$ is continuous in, then this integral exists. Furthermore
$$\int_{a}^{b} f(t, r) dt = \int_{a}^{b} f(t, r) dt$$

Note that here, we stated the definition of fuzzy Riemann integral. The interested reader can refer to (11)\(^{11}\) for definitions and some properties of fuzzy Lebesgue integral, you can also see the fuzzy integral of a positive, measurable function with respect to a fuzzy measure, the monotone convergence theorem and Fatous lemma in the space of fuzzy number.

### 3. Fuzzy Gaussian Quadrature Rule

In this section, we introduce the fuzzy Gaussian quadrature formula for fuzzy integral equations. A quadrature rule is an approximation of the definite integral of a function with values at specified points.

**Definition 3.1.** Function $w: (a, b) \to \mathbb{R}$ is called a weight function if $w$ is continuous on $(a, b)$, $w(x) \geq 0$ for all $x \in (a, b)$, $w \neq 0$ and $\int_{a}^{b} w(x) dx$ exists.

**Definition 3.2.** The set of functions \( \{\varphi_0, \ldots, \varphi_n\} \) is called orthogonal on integral $(a, b)$ with respect to the weight function $w(x)$ if
$$\int_{a}^{b} w(x) \varphi_i(x) \varphi_j(x) dx = \begin{cases} 0 & \text{if } i \neq j \\ a_i & \text{if } i = j \end{cases}$$

**Definition 3.3.** Let $\Pi_n$ be the set of all polynomials in $\mathcal{C}$, that is:
$$\Pi_n = \left\{ \sum_{i=0}^{n} a_i x^i : a_i \in \mathcal{C} \right\}$$

Now we consider the general integrals of the form
$$I(f) = \int_{a}^{b} w(x) f(x) dx \quad (1)$$

Where $w$ is a given crisp weight function, and $f$ is a fuzzy function with parametric form
$$\tilde{f} = \left( f(x, r), \tilde{f}(x, r) \right) \quad 0 \leq r \leq 1, \quad a \leq x \leq b.$$

In this work, our main purpose is finding the approximate value for eq. (1) by using Gaussian quadrature rule. In the next definition, we state the concept of fuzzy Gaussian quadrature rule.

**Definition 3.4.** The fuzzy Gaussian quadrature formula has the form
$$\int_{a}^{b} w(x) \tilde{f}(x) dx \approx \sum_{k=0}^{n} a_k \tilde{f}(x_k) \quad (2)$$

If for any $p \in \Pi_{2n+1}$, we have
$$\int_{a}^{b} w(x) p(x) dx = \sum_{k=0}^{n} a_k p(x_k)$$

Where $x_0 < x_1 < \cdots < x_n$ and $a_0, a_1, \ldots, a_n$ are called quadrature nodes and quadrature weights, respectively.

**Remark 3.5.** The polynomial $p(x)$ of the maximum degree $n$ interpolates a given function $f(x)$ at points $x_0, x_1, \ldots, x_n$ if $p(x_i) = f(x_i)$ for $i = 0, 1, \ldots, n$.

In the next theorem, we show that the evaluation quadrature points $x_i (i = 0, 1, \ldots, n)$ are just the roots of a polynomial belonging to a class of orthogonal polynomials.

**Theorem 3.6.** Let $x_0, x_1, \ldots, x_n \in [a, b]$, such that for all $q \in \Pi_n$ we have
$$\int_{a}^{b} w(x) q_{n+1}(x) q(x) dx = 0, \quad q_{n+1}(x) = \prod_{j=0}^{n} (x - x_j)$$

Then
$$\int_{a}^{b} w(x) \tilde{f}(x) dx \approx \sum_{k=0}^{n} a_k \tilde{f}(x_k)$$

is a fuzzy Gaussian quadrature formula for given function $\tilde{f}$.

**Theorem 3.7.** The weights of Gaussian quadrature are all positive.

**Proof.** We put
$$f_k(x) = \left( \frac{q_{n+1}(x)}{x - x_k} \right)^2 = \left( \prod_{j=0}^{n} (x - x_j) \right)^2 \in \Pi_{2n}$$

That is, Gaussian quadrature formula is accurate for $f_k(x)$. Therefore:
$$\sum_{j=0}^{n} a_j f_k(x_j) = \int_{a}^{b} w(x) f_k(x) > 0 \quad (3)$$

On the one hand:
The Solve of Fuzzy Integral Equation by using Quadrature Formula

\[ 0 \neq f_k(x_k) = \left[ \prod_{j=0}^{n} (x_k - x_j) \right] > 0 \]

From (3) we obtain
\[ f_k(x_k) \neq 0 \]
\[ f_k(x_k) = 0 \quad \forall j \neq k \]
So, from (3) we have
\[ a_k f_k(x_k) > 0 \Rightarrow a_k > 0 \]

4. An Application of Fuzzy Quadrature Methods

In this section, we consider a fuzzy boundary value problem, in which boundary conditions appear in the form of fuzzy integral equations. We apply fuzzy Gaussian quadrature rule to solve these fuzzy integral equations. Consider the following problem:
\[ \frac{\partial \tilde{u}}{\partial t} - \frac{\partial^2 \tilde{u}}{\partial x^2} = \tilde{f}(x, t) \]
\[ \tilde{u}(0, t) = \int_0^b k_0(x)\tilde{u}(x, t)dx + \tilde{g}_0(t) \]
\[ \tilde{u}(x, 0) = \tilde{g}(x) \]

where \( \tilde{f}, \tilde{g}_0, \tilde{g}_1 \) and \( \tilde{g} \) are given fuzzy functions. \( k_0(x) \) and \( k_1(x) \) are continuous crisp functions, and we suppose that \( k_0(x) \) and \( k_1(x) \) are positive for all \( x \in [a, b] \). In this work, the partial differential is approximated by explicit difference scheme, and we apply Gauss quadrature formula for approximation of integrals. Let \( \tilde{u} \) is a fuzzy function of the independent crisp variables \( x \) and \( t \). We define
\[ I = \{(x, t)|0 \leq x \leq 1, 0 \leq t \leq T\} \]
A \( \alpha \)-cut of \( \tilde{u}(x, t) \) and its the parametric form will be \( \tilde{u}(x, t)[\alpha] = [u(x, t;r), \tilde{u}(x, t;r)] \)
Let \( \tilde{u}(x, t;r) \) and \( \tilde{u}(x, t;r) \) have continuous partial differential, therefore \( \frac{\partial \tilde{u}}{\partial t} - \frac{\partial^2 \tilde{u}}{\partial x^2} \) and \( \frac{\partial^2 \tilde{u}}{\partial x^2} \) are continuous for all \( (x, t) \in I \) and all \( r \in [0, 1] \). We divide the domain \([0.11 \times [0, T]\) in to \( M \times N \) with spatial step size \( h = \frac{1}{N} \) in \( x \)-direction and \( k = \frac{1}{M} \) in \( t \)-direction.

The grade points are given by:
\[ t_i = ik, \quad i = 0, 1, ..., M \]
Denote the value of mentioned functions at the representative mesh point \( p(x_i, t_j) \) by:
\[ \tilde{u}_p = \tilde{u}(x_j, t_i) = \tilde{u}^i_j \]
\[ \tilde{f}_p = \tilde{f}(x_j, t_i) = \tilde{f}^i_j \]
\[ \tilde{g}_0 = (t_i) = (\tilde{g}_0)^i \]
\[ \tilde{g}_1 = (t_i) = (\tilde{g}_1)^i \]
\[ k_0 = (x_j) = (k_0)^j \]
\[ k_1 = (x_j) = (k_1)^j \]

The integrals in the boundary conditions are discretized by the Gaussian quadrature rule.
\[ \int_0^1 f(x)dx \approx \sum_{k=0}^n a_k p(x_k), \quad a_k > 0 \]
To obtain the quadrature weights, we put \( p(x) = x^m, \quad m = 0, 1, 2, ..., \), therefore
\[ \int_0^1 x^m dx \approx \sum_{k=0}^n a_k(x_k)^m \]
and
\[ \lim_{m \to \infty} \sum_{k=0}^n a_k(x_k)^m = \int_0^1 x^m dx m = 0, 1, 2, ... \]
From the first boundary condition, we have
\[ \tilde{u}(0, t) = \int_0^1 k_0(x)\tilde{u}(x, t)dx + \tilde{g}_0(t) \]
\[ \tilde{u}^{i+1}_0 = \tilde{u}(x_0, t_{i+1}) = \int_0^1 k_0(x)\tilde{u}(x, t_{i+1})dx + \tilde{g}_0(t_{i+1}) \]
\[ = \sum_{k=0}^n a_k(k_0)^k \tilde{u}^{i+1}_k + (\tilde{g}_0)^{i+1} \]
Hence:
\[ \tilde{u}^{i+1}_0 = \tilde{u}_0 + \sum_{k=0}^{n-1} a_k(k_0)^k \tilde{u}^{i+1}_k + \sum_{j=1}^{n} a_j(k_0)^j \tilde{u}_j^{i+1} + (\tilde{g}_0)^{i+1} \]
After simplifying, we obtain
\[ (1 - a_0(k_0)^0)\tilde{u}^{i+1}_0 - a_0(k_0)^0\tilde{u}^{i+1}_0 = \sum_{j=1}^{n} a_j(k_0)^j \tilde{u}_j^{i+1} + (\tilde{g}_0)^{i+1} \]
Similarly, we have
We put 

\[ \alpha = \sum_{j=1}^{n-1} a_j(k_0)_j \tilde{u}_j^{i+1} + (g_0_i)^{i+1} \]  

and 

\[ \beta = \sum_{j=1}^{n-1} a_j(k_1)_j \tilde{u}_j^{i+1} + (g_1_i)^{i+1} \]  

From (4) and (5) we have 

\[ a_0 (k_1)_0 (1-a_n (k_1)_n) \tilde{u}_0^{i+1} - a_0 (k_1)_0 a_n (k_0)_n \tilde{u}_n^{i+1} = a_0 (k_1)_0 \alpha \]  

and 

\[ -a_0 (k_1)_0 (1-a_n (k_1)_n) \tilde{u}_0^{(i+1)} + (1-a_n (k_1)_n) (1-a_n (k_1)_n) \tilde{u}_n^{(i+1)} = (1-a_n (k_1)_n) \beta. \]  

Let 

\[ d = (1 - a_0 (k_1)_0) (1 - a_n (k_1)_n) - a_0 a_n (k_0)_0 (k_0)_n \]  

Let 

\[ (g_0_i)^{i+1} = (1 - a_0 (k_1)_0) (g_1_i)^{i+1} + a_0 (k_1)_0 (g_0_i)^{i+1}, \]  

so from (9) we get 

\[ \tilde{u}_n^{i+1} = \sum_{j=1}^{n-1} B_j \tilde{u}_j^{i+1} + G_0_i^{i+1} \]  

where 

\[ B_j = d^{-1} [(1 - a_0 (k_1)_0) a_j (k_0)_j + a_0 (k_0)_n a_j (k_1)_j] \]  

In the same way 

\[ \tilde{u}_0^{i+1} = \sum_{j=1}^{n-1} A_j \tilde{u}_j^{i+1} + G_1_i^{i+1} \]  

where 

\[ A_j = d^{-1} [(1 - a_0 (k_1)_0) a_j (k_0)_j + a_0 (k_0)_n a_j (k_1)_j] \]  

and also 

\[ d = (1 - a_0 (k_1)_0) (1 - a_n (k_1)_n) - a_0 a_n (k_0)_0 (k_0)_n \]  

As we said at the beginning of this section, we intend to solve this problem by using an explicit difference scheme. In this method, we use the forward difference approximation for \( \frac{\partial \tilde{u}}{\partial t} \), and the central difference approximation for \( \frac{\partial^2 \tilde{u}}{\partial x^2} \). This scheme is as below 

\[ \frac{\partial \tilde{u}}{\partial t} - \frac{\partial^2 \tilde{u}}{\partial x^2} = f(x, t) \]

\[ \tilde{u}_j^{i+1} = r \tilde{u}_j^{i+1} + (1 - 2r) \tilde{u}_j^i + r \tilde{u}_{j-1}^{i+1} + kf_j^i \]  

It is straightforward to derive 

\[ \tilde{u}_0^{i+1} = \frac{1}{2} \tilde{u}_1^{i} + \frac{1}{2} \tilde{u}_0^{i} \]

5. Numerical Example

In this section we present a numerical example to illustrate our method, whose exact solution is known to us. Consider the fuzzy heat equation 

\[ \frac{\partial U}{\partial t} (x, t) - \frac{1}{\pi^2 \partial x^2} (x, t) = 0 \]

Subject to the nonlocal boundary conditions 

\[ \frac{\partial U}{\partial t} (0, t) = \int_0^1 x \tilde{U}(x, t) dx + \left(1 + \frac{2}{\pi^2}\right) \exp(-t) \]

and the initial condition 

\[ U(x, 0) = \tilde{K} \cos \pi x \]

and 

\[ \tilde{K}[\alpha] = \begin{bmatrix} k(\alpha), k(\alpha) \end{bmatrix} = \begin{bmatrix} \alpha - 1, 1 - \alpha \end{bmatrix}. \]

which is easily seen to have exact solution for 

\[ \frac{\partial U}{\partial t} (x, t; \alpha) - \frac{1}{\pi^2 \partial x^2} (x, t; \alpha) = 0 + \alpha \]

and 

\[ \frac{\partial U}{\partial t} (x, t; \alpha) - \frac{1}{\pi^2 \partial x^2} (x, t; \alpha) = 0 - \alpha \]

are 

\[ U(x, t; \alpha) = \begin{cases} k(\alpha) \exp(-t) \cos \pi x & 0 < x < \frac{1}{2} \\ k(\alpha) \exp(-t) \cos \pi x & \frac{1}{2} < x < 1 \end{cases} \]

and 

\[ U(x, t; \alpha) = \begin{cases} k(\alpha) \exp(-t) \cos \pi x & 0 < x < \frac{1}{2} \\ k(\alpha) \exp(-t) \cos \pi x & \frac{1}{2} < x < 1 \end{cases} \]

The exact and approximate solutions are shown in figure (1) at the point (0.2, 0.001) with \( h = 0.005 \), \( k = 0.00001 \).
The Hausdorff distance between solutions in this case is $7.598 \times 10^{-4}$.

6. Conclusion

The purpose of the current study is to apply the Gaussian quadrature method for solving fuzzy integral equation over a finite interval $[a, b]$. Since this integration yields fuzzy number in parametric form, we use the parametric form of the methods.

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