The evaluation of the 4-point Green functions in the 1+1 Schwinger model is presented both in momentum and coordinate space representations. The crucial role in our calculations play two Ward identities: i) the standard one, and ii) the chiral one. We demonstrate how the infinite set of Dyson-Schwinger equations is simplified, and is so reduced, that a given n-point Green function is expressed only through itself and lower ones. For the 4-point Green function, with two bosonic and two fermionic external ‘legs’, a compact solution is given both in momentum and coordinate space representations. For the 4-fermion Green function a selfconsistent equation is written down in the momentum representation and a concrete solution is given in the coordinate space. This exact solution is further analyzed and we show that it contains a pole corresponding to the Schwinger boson. All detailed considerations given for various 4-point Green functions are easily generizable to higher functions.

I. INTRODUCTION

Massless quantum electrodynamics in 1+1 space-time dimensions, known as the Schwinger Model (SM), proved to be a very fruitful example of quantum field theory. Thanks to its symmetries it is a completely solvable model, and therefore it is particularly well suited for studying nonperturbative effects.

One of the most important and well known observations is that, the initially massless boson, called hereafter a ‘photon’ (if one may say of ‘photons’ in two dimensions), acquires a mass — the so called Schwinger mass. As the consequence of this, the electromagnetic potential becomes a function exponentially decreasing in space and proportional to $e^{-\mu|x|}$, where $\mu = e/\sqrt{\pi}$ represents the Schwinger mass of the dressed photon. Responsible for this effect is the vacuum polarization which totally shields the charge [2]. The effect of charge screening is also known from perturbative calculations in the ordinary, 4-dimensional QED, giving rise to a weak deviation from the Coulomb law, particularly for small distances [3], while in SM the change is dramatic. The interpretation of this massive state (composite vs. elementary) depends on the particular field variables chosen to describe the model.

The photon mass generation mechanism appears already on the diagramatical level, because the exact (nonperturbative) vacuum polarization scalar $\Pi(k^2)$ possesses a first order pole at $k^2 = 0$, with the residuum equal to $\mu^2$. This is commonly known as the Schwinger mechanism. On the other hand, from the mathematical point of view, the nonzero photon mass results in SM from the noninvariance of the path integral fermion measure with respect to the local chiral gauge transformations — the $U_A(1)$ group — which in turn is a reflection of the presence of anomaly in the local chiral gauge transformations. On the other hand, from the mathematical point of view, the nonzero photon mass results in SM from the noninvariance of the path integral fermion measure with respect to the local chiral gauge transformations — the $U_A(1)$ group — which in turn is a reflection of the presence of anomaly in the model.

This vector meson mass generation through screening effects, is also of interest in electroweak theory, where the additional, and still unobserved, Higgs field has to be introduced “by hand”, to ensure the simultaneous renormalizability of theory and the nonzero masses of the intermediate bosons $W^\pm$ and $Z^0$.

Another important property of SM is the absence of the asymptotic fermionic states [2]. This in turn is interesting from the point of view of hadron structure investigations [8], where the permanent quark confinement and asymptotic freedom of QCD, giving rise to the nonperturbative mass scale $\Lambda_{QCD}$, as the necessary mathematical ingredient of the logarithmic fall off, also precludes the appearence of the asymptotic quark states. Yet the other similarity between SM and QCD is the existence of a fermion condensate [6,11], though this requires considering a nontrivial instanton sector. The above features of SM are also preserved in a generalization of SM, by allowing fermions to have a nonzero mass [12,13].

Thanks to its full solvability, the SM, on an equal footing with other models, as for instance the Thirring Model [14], may also be used to test various assumptions in the nonperturbative calculations in quantum field theories. As ex-
amples can serve here: i) the postulated infrared form of the vertex function in the 4-dimensional massive electrodynamics [15], applied later in the so called Gauge Technique [16] and other works in the context of nonperturbative solutions of QED (for which the transverse corrections may be found in SM explicitly [17]), ii) the renormalisation group methods [18], or iii) even the very formulation of the quantum field theory [19,20]. One should also mention in this context the generalized versions of SM, formulated on the compact manifolds as two-sphere [21], or torus [22] instead of the flat space as well as the light-cone formulation [23].

Although a number of papers have already been devoted to the investigation of propagators in SM, a relatively small interest, up to our knowledge, has been paid to higher order Green functions [24]. In this paper we plan to fill up the gap with the particular interest paid to the 4-point functions.

In the following sections we show how they can systematically be found. In Section II we consider the Ward identities in the momentum space and show how the infinite set of Dyson-Schwinger equations can be reduced to only one, fully solvable equation. Particularly simple solution is given in Section II B for the function corresponding to “Compton scattering”. For the 4-fermion Green function we derive in Section II C the integral equation which has a closed form (it does not contain any higher Green functions). In Section III we consider the same question in the coordinate space. Following Schwinger in the quoted work [1], we find explicit solutions for both 4-point Green functions: the 4-fermion and the 2-photon-2-fermion one. Both of them are expressible through the known scalar factors of the fermion propagator. For the most interesting case of four fermions we use the derived formula to show that the function contains a pole at $p^2 = \mu^2$, i.e. corresponding to the Schwinger boson. We also give a formula for the formfactor of the appropriate residue. In the appendix we give the definitions of all the Green functions considered in the present work.

II. MOMENTUM SPACE 4-POINT GREEN FUNCTIONS

In this section we are concentrating on the momentum space equations for the 4-point Green functions. Firstly, we deal with 2-fermion-2-boson function. We recapitulate Ward identities which allow us to represent it as the appropriate combination of the 3-point functions. These, however, are already known and expressible, once again due to Ward identities, through the full fermion propagator [25].

Secondly, we consider 4-fermion Green function. In this case the situation is much more difficult since we do not have at our disposal any identity which would permit to reduce the problem to lower functions. Therefore, we consider the Dyson-Schwinger equation which couples the 4-point function to 5-point one (with one boson and four fermion ‘legs’). Next, we apply both Ward identities to the latter, and in consequence obtain a selfconsistent integral equation which contains only the 4-fermion function (and lower ones).

A. Notation and Definitions

SM may be defined through the two-dimensional lagrangian density

$$\mathcal{L}(x) = \bar{\Psi}(x) \left( i \gamma^\mu \partial_\mu - e A^\mu(x) \gamma_\mu \right) \Psi(x) - \frac{1}{4} F^\mu\nu(x) F_{\mu\nu}(x) - \frac{\lambda}{2} \left( \partial_\mu A^\mu(x) \right)^2, \quad (1)$$

where $\lambda$ is the gauge fixing parameter. For our calculations it will be convenient to choose later the Landau gauge by setting $\lambda \to \infty$. For the Dirac gamma matrices the following convention will be used

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^5 = \gamma^0 \gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and for the metric tensor

$$g^{00} = -g^{11} = 1.$$

The totally antisymmetric symbol $\varepsilon^{\mu\nu}$ is defined by

$$\varepsilon^{01} = -\varepsilon^{10} = 1, \quad \varepsilon^{00} = \varepsilon^{11} = 0.$$

The definitions of all the Green function that appear in the formulae below are collected together in the appendix.
B. Calculation of the 2-boson, and 2-fermion functions

We start this section with deriving Ward identities satisfied by the relevant Green function. The standard procedure in this derivation is to perform, under the functional integral \( \Lambda_1 \), the following infinitesimal local gauge transformation

\[
A^{\mu}(x) \rightarrow A^{\mu}(x) + \partial^{\mu}\omega(x), \\
\Psi(x) \rightarrow \Psi(x) - ie\omega(x)\Psi(x), \\
\overline{\Psi}(x) \rightarrow \overline{\Psi}(x) + ie\omega(x)\overline{\Psi}(x),
\]

and consider the resulting variational equation. Doing that way we get the relation satisfied by the generating functional \( W(\eta, \overline{\eta}, J) \)

\[
- \lambda \Box_x \partial_x^{\mu} \frac{\partial W}{\partial J^{\mu}(x)} - \partial_x^{\mu} J^{\mu}(x) - i\nu \overline{\eta}_a(x) \frac{\partial W}{\partial \overline{\eta}_a(x)} + i\nu \eta_a(x) \frac{\partial W}{\partial \eta_a(x)} = 0
\]

Now we have to functionally differentiate both sides of this equation over \( J^{\nu}(y), \overline{\eta}_b(z) \), and \( \eta_a(u) \). After having put all the external currents at zero value we obtain the following equation for the 4-point Green function \( \Gamma^{\mu\nu} \), defined in the appendix,

\[
i\lambda \Box_x \partial_x^{\mu} D^{\mu\nu}(x - y) = \partial_x^{\nu}\delta^{(2)}(x - y),
\]

which stresses that only the transverse part of \( D^{\mu\nu} \) is influenced by the interaction, and rewrite the expression in momentum space using the definitions of Figure 1, we obtain, after removing the common factors on both sides,

\[
i k^{\mu} S(p + q - k) \Gamma^{\mu\nu}(k, q, p) S(p) = e^2 \left[ S(p + q) \Gamma^{\nu}(q, p) S(p) - S(p + q - k) \Gamma^{\nu}(q, p - k) S(p - k) \right].
\]

Obviously, this equation does not define \( \Gamma^{\mu\nu} \) entirely, but only its longitudinal part (in index \( \mu \)). Fortunately, due to the vanishing electron mass the Lagrangian \( \mathcal{L} \) is invariant also with respect to the local chiral gauge transformation. In the infinitesimal version they read

\[
A^{\mu}(x) \rightarrow A^{\mu}(x) + \varepsilon^{\mu\nu}\partial_{\nu}\omega(x), \\
\Psi(x) \rightarrow \Psi(x) - i\varepsilon^{\mu\nu}\gamma^5\Psi(x), \\
\overline{\Psi}(x) \rightarrow \overline{\Psi}(x) - i\varepsilon^{\mu\nu}\gamma^5\overline{\Psi}(x),
\]

Similarly as it was done to obtain \( \Lambda_2 \) we can derive the following equation for the generating functional \( W \)

\[
\left( \Box_x + \frac{\epsilon^2}{\pi} \right) \varepsilon^{\mu\nu} \partial_x^{\nu} \frac{\delta W}{\delta J^{\mu}(x)} - \varepsilon^{\mu\nu} \partial_x^{\nu} J^{\mu}(x) - i\nu \overline{\eta}_a(x) \gamma^b \frac{\delta W}{\delta \overline{\eta}_b(x)} - i\nu \eta_a(x) \gamma^b \frac{\delta W}{\delta \eta_b(x)} = 0.
\]

One important difference in comparison with eq. \( \Lambda_2 \), which should be noted here, is the presence of the mass equal to \( \frac{\epsilon^2}{\pi} \) in the first term. This term results, as mentioned in the Introduction, from the noninvariance of the path integral measure with respect to the group of transformations \( \Lambda_2 \) and constitutes the well known chiral anomaly \( \Lambda_2 \). Following the same way as above, and using the chiral version of equation \( \Lambda_2 \)

\[
\left( \Box_x + \frac{\epsilon^2}{\pi} \right) \varepsilon^{\mu\nu} \partial_x^{\nu} D_{\nu\alpha}(x - y) = -\varepsilon_{\alpha\beta} \partial_x^{\nu} \delta^{(2)}(x - y),
\]
we get in momentum space

\[ i\varepsilon_{\mu\alpha}k^{\alpha}S(p + q - k)\Gamma^{\mu\nu}(k, q, p)S(p) = e^2 \left[ \gamma^5 S(p + q)\Gamma^{\nu}(q, p)S(p) + S(p + q - k)\Gamma^{\nu}(q, p - k)S(p - k)\gamma^5 \right]. \] (10)

This, together with (3), determines uniquely \( \Gamma^{\mu\nu} \) because in two dimensions there are only two independent space-time 2-vectors. Therefore we can write

\[ g^{\mu\nu} = \frac{1}{k^2} \left( k^{\mu}k^{\nu} - \varepsilon^{\mu\alpha}k_\alpha\varepsilon^{\nu\beta}k_\beta \right), \] (11)

and in consequence, any tensor \( A^{\mu\nu} \) may be written as

\[ A^{\mu\nu} = g^{\mu\lambda}A^{\nu}_\lambda = \frac{k^\mu}{k^2} \left( k_\lambda A^{\lambda\nu} \right) - \frac{\varepsilon^{\mu\alpha}k_\alpha}{k^2} \left( \varepsilon_{\lambda\beta}k^\beta A^{\lambda\nu} \right). \] (12)

Applying this to the 4-point function \( \Gamma^{\mu\nu} \), we find

\[ S(p + q - k)\Gamma^{\mu\nu}(k, q, p)S(p) = -ie^2 \left[ k\gamma^\mu S(p + q)\Gamma^{\nu}(q, p)S(p) - S(p + q - k)\Gamma^{\nu}(q, p - k)k\gamma^\mu S(p - k) \right]. \] (13)

In deriving this equation we made use of the fact, that \( k^{\mu} - \varepsilon^{\mu\alpha}k_\alpha\gamma^5 = k\gamma^\mu \), as well as that the propagator \( S \) is linear in gamma matrices \([\text{in} \{ S, \gamma^5 \}] = 0 \) (at least in the 0-instanton sector to which we restrict ourselves in the present paper). In that way the 4-point function is given in terms of the vertex function and the propagator. The vertex function, however, thanks to the analogous Ward identities, can be further reduced \([12, 22]\) to the form

\[ \Gamma^{\nu}(q, p) = \frac{1}{q^2} \left[ S^{-1}(p + q) - S^{-1}(p) \right] \not{g}\gamma^\nu. \] (14)

Applying this, we obtain our final equation for the 4-point Green function

\[ S(p + q - k)\Gamma^{\mu\nu}(k, q, p)S(p) = -ie^2 \left[ k\gamma^\mu \left[ S(p) - S(p + q) - S(p - k) + S(p + q - k) \right] \gamma^\nu \not{g}. \right. \] (15)

From Eq. (15) we see that the 4-point function is entirely expressible through 2-point functions which are already known. In the same way one can reduce to fermion propagators, by successively applying both Ward identities, any Green function with two fermion and \( n \) boson ‘legs’.

A different approach one has to make use of while dealing with functions with more than two fermions, since only the photon ‘legs’ may be removed the above way. We consider this question in the following paragraph.

The simple structure of \( \Gamma^{\mu\nu}(k, q, p) \) reflects the fact that in the Schwinger Model the external photons (i.e. nonperturbative, massive photons) are always coupled directly to the electron line, without an intermediate fermion loop, since any loop with more than two external photons turns out to be zero, if we consider all possible permutations of vertices.

### C. Selfconsistent equation for the 4-fermion function

In the case of the 4-fermion function the exploiting of the ordinary, and the chiral Ward identity does not solve the problem completely (although it is still very useful), since we cannot reduce fermion legs, in any way. However, we are able to obtain a selfconsistent equation for this function. To do so we start from the Dyson-Schwinger equation for the 4-fermion function which can be derived in a standard way. We consider the functional derivative over field \( \Psi(x) \) \([3]\), and write

\[ \int D\Psi D\overline{\Psi} DA \frac{\delta}{\delta\Psi(x)} e^I d^2x \left[ \mathcal{L}(x) + \overline{\Psi}(x)\Psi(x) + \overline{\Psi}(x)\eta(x) + J^{\mu}(x)A_\mu(x) \right] = 0. \] (16)
This gives the following relation for the generating functional
\[
\eta^{\mu}_{ab}(x) \frac{\delta W}{\delta \eta_{b}(x)} - e^{\delta W} \frac{\delta W}{\delta J_{\mu}(x)} \eta^{\mu}_{ab}(x) \frac{\delta W}{\delta \eta_{b}(x)} + e^{\delta W} \frac{\delta^{2} W}{\delta J_{\mu}(x) \delta \eta_{b}(x)} + \eta_{a}(x) = 0 .
\] (17)

Now, one has to differentiate the above equation over the fermionic currents \(\eta_{d}(w), \eta_{e}(z),\) and \(\eta_{c}(y),\) and at the end set all the currents \(\eta, \eta_{b}, J\) to be equal zero. The result is

\[
-i\eta^{\mu}_{ab}(x) \frac{\delta W}{\delta \eta_{b}(w)} = e^{2} \int \frac{d^{2}w_{1} d^{2}w_{2} d^{2}w_{3} d^{2}w_{4} S_{bg}(w_{1} - w_{2}) S_{ef}(w_{2} - w_{3}) S_{cg}(w_{3} - w_{4}) S_{eg}(w_{4} - w_{1}) S_{ab}(w_{1} - w_{2}) S_{ab}(w_{2} - w_{3}) S_{ab}(w_{3} - w_{4}) S_{ab}(w_{4} - w_{1})).
\]

A schematical representation of this equation is shown on Figure 2. There the thick lines correspond to full propagators and thin lines to the free ones. This equation indicates that the 4-point function depends on the 5-point one which is, of course, the typical behaviour of the set of Dyson-Schwinger equations, since the interacting Lagrangian contains always terms of at least third order in fields, resulting in an infinite interlacement of Green functions. We can, however, use here the method of previous paragraph and get rid of the 5-point function from the right hand side. Equation (15), transformed into the momentum space according to the definitions of Figure 4, reads

\[
i p_{ab} S_{bf}(p) S_{ef}(q + l - p) \Gamma_{fgrs}(p, q, l) S_{rc}(q) S_{sd}(l) = e^{2} S_{ef}(q + l - p) \Gamma_{f}(l - p, q) \times
\]

\[S_{bg}(q) D_{\mu \nu} \Gamma_{f}(l - p, q) \times
\]

\[e^{2} S_{ef}(q + l - p) \Gamma_{f}(l - p, q) S_{sd}(l) D_{\mu \nu} (p - q) \Gamma_{f}(p - k, q) S_{bc}(q) \times
\]

\[+ e^{2} \eta^{\mu}_{ab} \left( \frac{d^{2}k}{2\pi)^{2}} D_{\mu \nu} \Gamma_{f}(p - k, q) S_{bc}(q) \times
\]

\[S_{bf}(p - k) S_{eg}(q + l - p) \Gamma_{fgrs}(p - k, q, l) S_{rc}(q) S_{sd}(l) .
\] (19)

Since the vertices \(\Gamma_{ab}^{\mu}\) are perfectly known, what we need is only the relation which would allow us to express the 5-point function \(\Gamma_{abcd}^{\mu}\) through the 4-point one. The method is analogous to that shown in detail in the previous paragraph, and we do not repeat it here. The result is

\[
S(p) \otimes S(q + l - p - k) \cdot \Gamma^{\nu}(k, p, q, l) \cdot S(p) \otimes S(l) =
\]

\[- \frac{i e}{k^{2}} \left[ (\gamma_{\nu} S(p + k)) \otimes S(q + l - p - k) \cdot \Gamma(p + k, q, l) \cdot S(q) \otimes S(l)
\]

\[+ S(p) \otimes (\gamma_{\nu} S(q + l - p)) \cdot \Gamma(p, q, l) \cdot S(q) \otimes S(l) - S(p) \otimes S(q + l - p - k) \cdot \Gamma(p, q, l - k) \cdot S(q) \otimes (\gamma_{\nu} S(l - k))
\]

\[- S(p) \otimes S(q + l - p - k) \cdot \Gamma(p - k, q, l) \cdot (\gamma_{\nu} S(q - k)) \otimes S(l) \right] ,
\] (20)

where, for abbreviation, we have used the obvious notation \(S^{(1)} \otimes S^{(2)} \cdot \Gamma \cdot S^{(3)} \otimes S^{(4)}\) for defining an object: \(S_{abc}^{(1)} S_{def}^{(2)} \Gamma_{efgh} S_{gcd}^{(3)} S_{ab}^{(4)} .\) If we substitute (21), together with (14), into (11) we obtain the final equation

\[
i p_{ab} S_{bf}(p) S_{eg}(q + l - p) \Gamma_{fgrs}(p, q, l) S_{rc}(q) S_{sd}(l) =
\]

\[- \frac{e^{2}}{(l - p)^{2}} \left[ (\gamma_{\nu} S(p + k)) \otimes S(q + l - p)) \right]_{bc} D_{\mu \nu} (p - k) \gamma_{ab} S_{bc}(q)
\]

\[- e^{2} \eta^{\mu}_{ab} \left( \frac{d^{2}k}{2\pi)^{2}} D_{\mu \nu} \Gamma_{f}(p - k, q) S_{bc}(q)
\]

\[+ S(p - k) S_{bf}(p - k) S_{eg}(q + l - p + k) \Gamma_{fgrs}(p - k, q, l) S_{rc}(q) S_{sd}(l) - S(p - k) S_{bf}(p + l - p) \Gamma_{fgrs}(p - k, q, l) S_{rc}(q) (\gamma_{\nu} S(q - k))_{sd}
\]

\[- S(p - k) S_{bf}(p + l - p) \Gamma_{fgrs}(p - k, q, l) (\gamma_{\nu} S(q - k))_{sd} S_{ab}^{(1)} S_{def}^{(2)} \Gamma_{efgh} S_{gcd}^{(3)} S_{ab}^{(4)} \right] .
\] (21)
in which only the 4- and 2-point functions are involved.

This derivation shows how the infinite series of coupled Dyson-Schwinger equations may be reduced to only one integral equation, which in principle might be solved. Due to the complicated tensor structure of $\Gamma_{abcd}$ (it requires introducing several scalar coefficient function), and perplexed mathematical form of $\frac{21}$ we do not try to solve it here and will rather concentrate on finding the explicit form of the 4-fermion Green function in coordinate space.

It is a common feature of the Schwinger Model that coordinate space solutions are much simpler than the momentum space ones. In this case it will be even possible to express $\Gamma_{abcd}$ through the electron propagator, similarly as it was done for $\Gamma_{\mu\nu}$ in $\frac{20}$. This problem will constitute the subject of the next section.

The method of the present paragraph allows also to find a selfconsistent equation for any higher Green function. In particular, a function with $2n_f$ fermionic legs and $n_b$ bosonic legs should first be reduced, thanks to the consecutive $n_b$ applications of both Ward identities, to purely fermionic $2n_f$-point function, and than the selfconsistent equation for the latter can be obtained.

### III. 4-POINT GREEN FUNCTIONS IN COORDINATE SPACE

In this section we find the explicit formulae for the 4-point Green functions in coordinate space. In this case also the 4-fermion function may be given a compact form, instead of having it as a solution of an integral equation like $\frac{21}$, but extend it also for higher functions.

#### A. 2-boson and 2-fermion function

As it is known the generating functional $Z(\eta, \bar{\eta}, J)$ may be given the following form

$$Z(\eta, \bar{\eta}, J) = \exp \left[ -i \int d^2x d^2y \bar{\eta}(x) S(x, y; \delta / i \delta J \eta(y) \right] \times$$

$$\left[ \exp \left[ - \frac{i}{2} \int d^2x d^2y J_\mu(x) \Delta^{\mu\nu}(x - y; \frac{e^2}{\pi}) J_\nu(y) \right] \right] ,$$

where $S(x, y, A)$ is the classical electron propagator in the external electromagnetic field $A^\mu$, and is given by the formula

$$S(x, y, A) = S_0(x - y) \exp \left[ -i \left( \tilde{\phi}(x, A) - (\tilde{\phi}(y, A)) \right) \right] ,$$

with

$$\tilde{\phi}(x, A) = e \int d^2y \Delta(x - y) \gamma^\nu \gamma^\mu \partial_\mu A_\nu(y) ;$$

$S_0$ being the free propagator. In the Landau gauge which we now use, $\Delta^{\mu\nu}$ takes the form

$$\Delta^{\mu\nu}(x - y, m^2) = \int d^2z \left[ g^{\mu\nu} \delta^{(2)}(x - z) - \partial_\xi^\mu \partial_\xi^\nu \Delta(x - z) \right] \Delta(y - z; m^2) ,$$

where $\Delta(x, m^2)$ and $\Delta(x)$ are respectively Klein-Gordon and d’Alambert propagators. For the exponential factor in $\frac{23}$ one often uses the abbreviated and useful form

$$\tilde{\phi}(x, A) - \tilde{\phi}(y, A) = - \int d^2z A^\mu(z) J_\mu(z; x, y) ,$$

with the (nonconserved) current $J_\mu$ stisfying

$$\partial_\xi^\nu J_\mu(z; x, y) = e \left[ \delta^{(2)}(x - z) - \delta^{(2)}(y - z) \right] .$$

This current has sources at every point, where charged particles are created, or anihilated, and is closely related with the notion of th so called “compensating current” $\frac{28}$. Now the 4-point Green function, considered in Section $\frac{IIB}$ (for functions with no external ‘legs’ amputated we reserve symbol $G$ with appropriate indices), is given by
\( G^\mu_\nu(x_1, x_2; x_3, x_4) = \frac{\delta^4}{\delta J_\mu(x_1) \delta J_\nu(x_2) \delta \bar{\eta}_a(x_3) \delta \eta_b(x_4)} Z(\bar{\eta}, \eta, J) \) \quad (\text{connected})
\[
\text{curr.} = 0
\]

where we have explicitly written that only connected graphs are considered. This gives

\[ G^\mu_\nu(x_1, x_2; x_3, x_4) = i \frac{\delta^2}{\delta J_\mu(x_1) \delta J_\nu(x_2)} S_{ab}(x_3, x_4; \delta / i \delta J) \]

\[ \exp \left[ -i \frac{1}{2} \int d^2 x d^2 y J^\alpha(x) \triangle_{\alpha\beta} (x - y; e^2 / \pi) J^\beta(y) \right] \quad (\text{connected}) \]
\[
\text{J}_{\alpha}(x; x_3, x_4) \right) \bigg|_{J = 0}^{\text{connected}}
\]

If we make use of the explicit form of \( S \), given in equations (23,24), leading to the representation in the form of a series of derivatives, and note that \( \exp (\frac{z}{2} \frac{\partial}{\partial z}) f(x) = f(x + z) \), we can write

\[ G^\mu_\nu(x_1, x_2; x_3, x_4) = S_0^{ac}(x_3 - x_4) \Delta^\mu_\nu (x_1 - x_2; e^2 / \pi) \times \]

\[ \exp \left[ -i \frac{1}{2} \int d^2 x d^2 y J^\alpha(x; x_3, x_4) \Delta_{\alpha\beta} (x - y; e^2 / \pi) J^\beta(y; x_3, x_4) \right] \quad (\text{connected}) \]

\[ -i S_0^{ac}(x_3 - x_4) \int d^2 z d^2 w \Delta_{\mu\lambda} (x_1 - w; e^2 / \pi) J^\lambda_{cd}(w; x_3, x_4) \Delta_{\nu\rho} (x_2 - z; e^2 / \pi) \times \]

\[ J^\rho_{de}(z; x_3, x_4) \exp \left[ -i \frac{1}{2} \int d^2 x d^2 y J^\alpha(x; x_3, x_4) \Delta_{\alpha\beta} (x - y; e^2 / \pi) J^\beta(y; x_3, x_4) \right] \quad (\text{connected}) \]

Now we recall that the full propagator \( S \) has the form \([1]\)

\[ S_{ab}(u - w) = S_0^{ac}(u - w) \exp \left[ -i \frac{1}{2} \int d^2 x d^2 y J^\alpha(x; u, v) \Delta_{\alpha\beta} (x - u; e^2 / \pi) J^\beta(y; u, v) \right] \quad (\text{connected}) \]

This allows us to write equation (31) in the form

\[ G^\mu_\nu(x_1, x_2; x_3, x_4) = S_0^{ab}(x_3, x_4) \Delta^\mu_\nu (x_1 - x_2; e^2 / \pi) - i S_{ac}(x_3, x_4) \]

\[ \int d^2 z d^2 w \Delta_{\mu\lambda} (x_1 - w; e^2 / \pi) J^\lambda_{cd}(w; x_3, x_4) \Delta_{\nu\rho} (x_2 - z; e^2 / \pi) J^\rho_{de}(z; x_3, x_4) \quad (\text{connected}) \]

The first term, constituting the nonconnected contribution, should now be rejected. For the amputated Green function \( \Gamma \) considered in Section [13], with the use of the definition of \( \mathcal{J} \) and \( \phi \), as well as the fact that \( S^{\gamma}_\mu \gamma^\nu = \gamma^\nu \gamma^\mu S \), we obtain

\[ \int d^2 u d^2 w S(x_3 - u) \Gamma^\mu_\nu (x_1, x_2; u, w) S(w - x_4) \]

\[ = -ie^2 \Theta_{x_1} (\Delta(x_3 - x_1) - \Delta(x_4 - x_1)) \gamma^\mu S(x_3 - x_4) \gamma^\nu \Theta_{x_2} (\Delta(x_3 - x_2) - \Delta(x_4 - x_2)) \]

where we have omitted the spinor indices. One can easily verify that this is the coordinate space representation of \((15)\). It is a common feature of all two-fermion Green functions that they can be represented through fermion propagator in both coordinate and momentum spaces. As it was mentioned this is possible since the external photons are coupled directly to the incoming and outgoing electron lines, and no intermediate fermions loops are possible, apart from those producing the photon mass \( e^2 / \pi \). On the other hand, for the 4-fermion function, we deal with in the following section, a much more complicated structure appears and an explicit and compact expression is possible to be given only in coordinate space.
B. 4-fermion function

Now, instead of (28), we have

\[ G_{abc;cd}(x_1, x_2; x_3, x_4) = \frac{\delta^4}{\delta \theta_{abc}(x_1) \delta \theta_{b}(x_2) \delta \eta_{c}(x_3) \delta \eta_{d}(x_4)} Z(\eta, \eta, J) \bigg|_{\text{connected}} = \]  

\[ = [S_{ac}(x_1, x_3; \delta/i\delta J)S_{bd}(x_2, x_4; \delta/i\delta J) - S_{ad}(x_1, x_4; \delta/i\delta J)S_{bd}(x_2, x_3; \delta/i\delta J)] Z(J) \bigg|_{J=0}. \]

The differentiations over external current \( J \), hidden in propagators \( S \), can be performed similarly as it was done to obtain (21), although it must be done with greater care than before due to the tensor structure of \( J \). In particular, using the notation of paragraph II C we find

\[ S(x_1, x_3; \delta/i\delta J) \otimes S(x_2, x_4; \delta/i\delta J) \cdot Z(J) = \]  

\[ = S_0(x_1 - x_3) \otimes S_0(x_2 - x_4) \cdot Z(J \otimes 1 \cdot J + J(x_1, x_3) \otimes 1 + 1 \otimes J(x_2, x_4)), \]

where for abbreviating we have not explicitly written the first argument of the currents \( J \) over which the integration in the generating functional \( Z \) is taken (see equations (22,24,26)). Now, we concentrate only on the last factor of the above expression \( (Z) \), which, after setting \( J = 0 \), takes the form

\[ Z(J(x_1, x_3) \otimes 1 + 1 \otimes J(x_2, x_4)) = \exp \left[ -\frac{i}{2} \int d^d x d^d y \left( J^\mu(x, x_1, x_3) \otimes 1 + \right. \right. \]  

\[ \left. + 1 \otimes J^\mu(x, x_2, x_4) \right) \Delta_{\mu\nu}(x - y; e^2/\pi) \left( J^\nu(y, x_1, x_3) \otimes 1 + 1 \otimes J^\nu(y, x_2, x_4) \right) \].

The expressions for both \( J^\mu \) and \( \Delta_{\mu\nu} \) are known so it is only a matter of patience to get the formula for the above exponential. Thanks to the fact that in two dimensions \( \gamma^\mu \gamma^\nu \gamma_\mu = 0 \) the “diagonal” terms of the kind \( J^\mu \otimes 1 \cdot \Delta_{\mu\nu} \cdot J^\nu \otimes 1 \) produce only expressions of the tensor structure \( 1 \otimes 1 \), whereas mixed terms as \( J^\mu \otimes 1 \cdot \Delta_{\mu\nu} \cdot 1 \otimes J^\nu \) give both \( 1 \otimes 1 \) and \( \gamma^5 \otimes \gamma^5 \). We skip this calculation here to save the reader’s time, and give below only the final result

\[ Z(J(x_1, x_3) \otimes 1 + 1 \otimes J(x_2, x_4)) = \frac{1}{2} (1 \otimes 1 + \gamma^5 \otimes \gamma^5) \times \]  

\[ \exp \left[ i e^2 \beta(x_1 - x_2) + \beta(x_1 - x_3) - \beta(x_1 - x_4) - \beta(x_2 - x_3) - \beta(x_2 - x_4) + \beta(x_3 - x_4) \right] \]  

\[ + \frac{1}{2} \left( 1 \otimes 1 - \gamma^5 \otimes \gamma^5 \right) \exp \left[ i e^2 \left( -\beta(x_1 - x_2) + \beta(x_1 - x_3) + \beta(x_1 - x_4) + \beta(x_2 - x_3) - \beta(x_2 - x_4) - \beta(x_3 - x_4) \right) \right], \]

where the function \( \beta \) is defined by

\[ \beta(x) = \int \frac{d^d p}{(2\pi)^d} \left( 1 - e^{i p x} \right) \frac{1}{(p^2 - e^2/\pi + i e)(p^2 + i e)} = \]  

\[ = \begin{cases} \frac{i}{2e^2} \left[ -\frac{i}{2e^2} + \gamma_E + \ln \sqrt{e^2 x^2/4\pi} + \frac{i}{2p} H_0^{(1)}(\sqrt{e^2 x^2/\pi}) \right] & x \text{ timelike} \\ \frac{i}{2e^2} \left[ \gamma_E + \ln \sqrt{-e^2 x^2/4\pi} + K_0(\sqrt{-e^2 x^2/\pi}) \right] & x \text{ spacelike}, \end{cases} \]

and is in fact a function of \( x^2 \) only. \( \gamma_E \) is here the Euler constant and functions \( H_0^{(1)} \) and \( K_0 \) are Hankel function of the first kind, and Basset function respectively [29]. Since we have

\[ S(x) = S_0(x) \exp \left[ -i e^2 \beta(x) \right]. \]

we can write down the final formula for the 4-fermion Green function.
\[
G_{abcd}(x_1, x_2; x_3, x_4) = \frac{1}{2} \left[ S_{ac}(x_1 - x_3) S_{bd}(x_2 - x_4) + \left( S(x_1 - x_3) \gamma^5 \right)_{ac} \left( S(x_2 - x_4) \gamma^5 \right)_{bd} \right] \times \\
\exp \left[ i e^2 (\beta(x_1 - x_2) - \beta(x_1 - x_4) - \beta(x_2 - x_3) + \beta(x_3 - x_4)) \right] \\
+ \frac{1}{2} \left[ S_{ac}(x_1 - x_3) S_{bd}(x_2 - x_4) - \left( S(x_1 - x_3) \gamma^5 \right)_{ac} \left( S(x_2 - x_4) \gamma^5 \right)_{bd} \right] \\
\exp \left[ - i e^2 (\beta(x_1 - x_2) - \beta(x_1 - x_4) - \beta(x_2 - x_3) + \beta(x_3 - x_4)) \right] \\
- \left\{ x_3 \leftrightarrow x_4, \quad c \leftrightarrow d \right\} .
\]

(41)

We see that in the coordinate space both 4-point functions (4-fermion and 2-boson-2-fermion) may perfectly be found and are given by compact formulae. Since \( \beta \)'s are related with the full fermion propagator \( S \), one can say that knowing \( S \) one knows “everything”. The calculation of higher functions may be lead very much similarly to what was given in this paragraph, and one will always obtain a product of electron propagators and exponentials of \( \beta \) function.

The exact expression for the 4-fermion function, we have obtained, allows the analysis of its analytical properties.

We concentrate below on the presence of the fermion-antifermion pole (\( t \)-channel) corresponding to the Schwinger boson. Let us denote the first two terms on the right hand side of (41) by \( G_{abcd}^1 \) and \( G_{abcd}^2 \) respectively. The remaining terms represented by the curly brackets can contribute to the eventual pole in the \( u \)-channel only, and therefore we omit them in the present discussion.

While looking for a pole we first identify the “in” and “out” coordinates (in the \( t \)-channel) of fermion and antifermion: \( u \equiv x_1 = x_3, \quad v \equiv x_2 = x_4 \) and next consider the expression Fourier transformed in the variable \( z \equiv v - u \). The identification has to be performed with care, for instance in the following way:

1. for the time coordinates we put
   \[
   x_1^0 = x_3^0 \rightarrow u^0 \quad \text{and} \quad x_2^0 = x_4^0 \rightarrow v^0 ,
   \]
2. for the spacial coordinates we assume
   \[
   x_1^1 \rightarrow u^1 , \quad x_3^1 \rightarrow u^1 + \varepsilon , \quad x_2^1 \rightarrow v^1 , \quad x_4^1 \rightarrow v^1 + \eta ,
   \]
3. for the function depending on \( \varepsilon \) and \( \eta \) we take the fully symmetric limit
   \[
   \lim_{\varepsilon \rightarrow 0, \eta \rightarrow 0} \frac{\text{sym.}}{f(\varepsilon, \eta)} = \frac{1}{4} \lim_{\varepsilon \rightarrow 0, \eta \rightarrow 0} \left[ f(\varepsilon, \eta) + f(-\varepsilon, \eta) + f(\varepsilon, -\eta) + f(-\varepsilon, -\eta) \right] .
   \]

In that limit \( G^1 \) and \( G^2 \) become only \( z \) dependent. For instance for \( G^1 \) we have

\[
G^1(z) = \frac{1}{8\pi^2} (\gamma^0 \otimes \gamma^0 + \gamma^1 \otimes \gamma^1) \lim_{\varepsilon \rightarrow 0, \eta \rightarrow 0} \frac{1}{\varepsilon \eta} \times \\
\exp \left[ i e^2 \left( \beta(z) - \beta(0, \varepsilon) - \beta(z^0, z^1 + \eta) - \beta(z^0, z^1 - \varepsilon) - \beta(0, \eta) + \beta(z^0, z^1 - \varepsilon + \eta) \right) \right] ,
\]

where, when it was necessary, we wrote explicitly both coefficients of the two-vector argument of the \( \beta \) function
\[
\beta(x) = \beta(-x) \equiv \beta(x^0, x^1) .
\]

The symmetric limit above may be performed in a straightforward way, since the \( \beta \) function is perfectly known, and we obtain

\[
G^1(z) = -\frac{i e^2}{8\pi^2} (\gamma^0 \otimes \gamma^0 + \gamma^1 \otimes \gamma^1) \frac{d^2}{dz^2} \beta(z) .
\]

(43)

The same limit for \( G^2 \) gives

\[
G^2(z) = -\frac{i e^2}{8\pi^2} (\gamma^0 \otimes \gamma^0 - \gamma^1 \otimes \gamma^1) \frac{d^2}{dz^2} \beta(z) .
\]

(44)
If we now apply explicitly the definition of $\beta$ given by (39), and perform the Fourier transform over $z$, we find the following expression for the “polar” part of $G$

$$G_{\text{polar}}(k) = -\frac{ie^2}{4\pi^2} \gamma^0 \otimes \gamma^0 \frac{(k_1)^2}{(k^2 - e^2/\pi + i\epsilon)(k^2 + i\epsilon)} \rightarrow -\frac{i}{4\pi} \gamma^0 \otimes \gamma^0 \frac{(k_1)^2}{(k^2 - e^2/\pi + i\epsilon)},$$

from which a pole corresponding to the Schwinger boson may clearly be seen.

It should be noted that the similar analysis, although much more complicated, may be done without indentifying the “in” and “out” coordinates. One can for example introduce the new c.m. variables

$$u = \frac{1}{2}(x_1 + x_3), \quad v = \frac{1}{2}(x_2 + x_4),$$

and the relative ones

$$x = x_1 - x_3, \quad y = x_2 - x_4.$$

The Fourier transform of $G$ performed over $z = v - u$ displays now much richer analytical structure (branch points at $k^2 = n^2e^2/\pi$, $n = 2, 3, ...$) and the residue in the Schwinger pole depends on the relative coordinates $x$ and $y$

$$G_{\text{polar}}(x, y; k) = -4i\pi (S(x) \gamma^5) \otimes (S(y) \gamma^5) \frac{\sin[kx/2] \sin[ky/2]}{(k^2 - e^2/\pi + i\epsilon)},$$

where $S$ is given by (40). For $x, y \rightarrow 0$ (in a symmetrical way) we reproduce the result given by (45). It may be noted that the formfactor $F(x) \sim S(x) \gamma^5 \sin kx/2$ is square normalizable in the sense $\int_{-\infty}^{\infty} dx^1 |F(0, x^1)|^2$.

IV. SUMMARY

Below we would like to recapitulate the results we obtained in the present work. At first, in Section IV, we considered chiral symmetry in momentum space satisfied by the 4- and 5-point Green functions. Thanks to the local chiral symmetry of the Lagrangian, apart from the ordinary gauge invariance, we derived two identities. In the 2-dimensional world these two identities suffice to entirely describe the considered Green function, and express it through lower ones. Each application of these identities allows us to reduce the number of external photons by one. Following that way we were able to reduce the 2-boson-2-fermion function to well known electron propagator. In the case of 4-fermion function the situation turned out to be much more severe since we have no photon legs to reduce. The alternative approach was, therefore, introduced in Section II C. The starting point was here the Dyson-Schwinger equation which, on one hand, introduces 5-point function, but on the other permits to reduce it to the function we are looking for. This leads to a selfconsistent integral equation which contains apart from the unknown function only propagators which are perfectly known. We were, unfortunately, unable to solve this integral equation because of its complicated mathematical character, which is not unexpected since in the Schwinger Model even the fermion propagator cannot be given an explicit form in momentum space. The selfconsistent equation obtained in this section may, however, be a starting point for an analysis in momentum space constituting an alternative for taking the six-variable (two integrations may be separated out to give the Dirac delta function) Fourier transform.

In Section IV we considered the same functions in coordinate space. We used the generating functional which had already been found in the original Schwinger’s work [1]. The Green functions are, of course, given as the appropriate derivatives of this functional over the external currents. The problem which one only has to take care of is the tensor structure of the functions. The final compact expressions for the all 4-point functions were found and are shown to be expressible through the fermion propagator. All the methods of this, as well as of preceding section, may easily be generalized to any higher Green functions.

For the most interesting case — the 4-fermion function — we were able to show that (11) contains a pole, in the fermion-antifermion channel, corresponding to the Schwinger boson. It is interesting to note that the formfactor in the residue of the pole turns out to be normalisable in the 1-space direction if we set the relative time to zero. However, we do not treat this observation as any “proof” that the Schwinger boson is a “bound electron-positron state”, as is here and there suggested [3].

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APPENDIX A: DEFINITIONS OF THE GREEN FUNCTIONS

In this Appendix we give the definitions of various Green functions used in the formulae of sections [1] and [11]. If we introduce the generating functionals \( Z \) and \( W \) by the formula

\[
Z(\eta, \overline{\eta}, J) = \exp iW(\eta, \overline{\eta}, J) = \int D\Psi D\overline{\Psi} DA \exp \left[ i \int d^2x \left( \mathcal{L}(x) + \overline{\Psi}(x) \Psi(x) + \overline{\Psi}(x) \eta(x) + J^\mu(x) A_\mu(x) \right) \right],
\]

we can define the connected Green functions through derivatives of the functional over the external currents as follows

\[
\frac{\delta^2 W}{\delta \overline{\eta}_a(x) \delta \eta_b(y)} \bigg|_{\text{curr.}=0} = S_{ab}(x - y), \quad (A2)
\]

\[
\frac{\delta^2 W}{\delta J^\mu(x) \delta J^\nu(y)} \bigg|_{\text{curr.}=0} = -D_{\mu\nu}(x - y), \quad (A3)
\]

\[
\frac{\delta^3 W}{\delta J^\mu(x) \delta \overline{\eta}_a(y) \delta \eta_b(z)} \bigg|_{\text{curr.}=0} = -e \int d^2 w_1 d^2 w_2 d^2 w_3 D_{\mu\nu}(x - w_1) S_{ac}(y - w_2) \Gamma^\nu_{cd}(w_1; w_2, w_3) \times S_{db}(w_3 - z), \quad (A4)
\]

We also need the 4- and 5-point functions

\[
\frac{\delta^4 W}{\delta \overline{\eta}_a(x) \delta \overline{\eta}_b(y) \delta \eta_c(z) \delta \eta_d(w)} \bigg|_{\text{curr.}=0} = -i \int d^2 w_1 d^2 w_2 d^2 w_3 d^2 w_4 S_{ac}(x - w_1) S_{bf}(y - w_2) \times \Gamma_{efgh}(w_1, w_2; w_3, w_4) S_{gd}(w_4 - u) S_{hd}(w_4 - u), \quad (A5)
\]

\[
\frac{\delta^4 W}{\delta J^\mu(x) \delta J^\nu(y) \delta \overline{\eta}_a(z) \delta \eta_b(w)} \bigg|_{\text{curr.}=0} = -i \int d^2 w_1 d^2 w_2 d^2 w_3 d^2 w_4 D_{\mu\alpha}(x - w_1) S_{ac}(z - w_3) \times \Gamma^\nu_{cd}(w_1, w_2; w_3, w_4) S_{db}(w_4 - u) D_{\beta\nu}(w_2 - y), \quad (A6)
\]

\[
\frac{\delta^5 W}{\delta J^\mu(x) \delta \overline{\eta}_a(y) \delta \overline{\eta}_b(z) \delta \eta_c(u) \delta \eta_d(w)} \bigg|_{\text{curr.}=0} = \int d^2 w_1 d^2 w_2 d^2 w_3 d^2 w_4 d^2 w_5 D_{\mu\alpha}(x - w_1) S_{ac}(y - w_2) S_{bf}(z - w_3) \times S_{gd}(w_4 - u) S_{hd}(w_5 - u) \Gamma_{efgh}(w_1; w_2, w_3, w_4, w_5), \quad (A7)
\]

Thanks to the translational invariance of the theory these functions depend in fact only on the differences of arguments. The corresponding definitions in momentum space, after having pulled apart the Dirac delta function of the whole two-momentum, are given on Fig. [1].

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FIG. 1. Definitions of arguments in the vertex function and 4- and 5-point Green functions: a) $\Gamma_{ab}(k, p)$, b) $\Gamma_{ab,cd}(p, q, l)$, c) $\Gamma^{\mu\nu}_{ab}(k, q, p)$, d) $\Gamma^{\mu}_{ab,cd}(k, p, q, l)$.

FIG. 2. Graphic representation of the Dyson-Schwinger equation for the 4-point Green function: $\Gamma_{ab,cd}$. Thick lines correspond to full propagators and thin to free ones.
Fig. 1
Fig. 2