On Degenerate Linear Stochastic Evolution Equations Driven by Jump Processes

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Abstract
We prove the existence and uniqueness of solutions of degenerate linear stochastic evolution equations driven by jump processes in a Hilbert scale using the variational framework of stochastic evolution equations and the method of vanishing viscosity. As an application of our result, we derive the existence and uniqueness of solutions of degenerate linear stochastic integro-differential equations in Sobolev spaces.

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1 Introduction
Let \((\Omega, F, P)\) be a complete probability space with a filtration \(F = (F_t)_{t \geq 0}\) satisfying the usual conditions of right-continuity and completeness. Let \((V_t)_{t \geq 0}\) be a real-valued \(F\)-adapted continuous non-decreasing process with \(V_0 = 0\). Let \((E, \| \cdot \|_E)\) be a Hilbert space and \((M_t)_{t \geq 0}\) be a \(E\)-valued \(F\)-adapted strongly càdlàg quasi-left continuous square integrable martingale issued from zero such that \(\langle M \rangle_t = V_t\) for all \(t \geq 0\). Set \(Q_t = d\langle M \rangle_t/dN_t, t \geq 0\). Let \((Z, \mathcal{Z})\) be a sigma-finite measure space and \(q(dt, dz)\) be a \(F\)-adapted quasi-left continuous stochastic martingale measure on \(R^+ \times Z\) with compensator \(\pi_t(dz) dV_t\) such that \(q(dt, dz)\) is uncorrelated with \((M_t)_{t \leq T}\). Let \(T > 0\) be a real number and \((H_\alpha)_{\alpha \in \mathbb{R}}\) be a Hilbert scale. For each real number \(\lambda \geq 1\), we consider the linear stochastic evolution equation

\[
du_t = (L u_t + f_t) dV_t + (M u_{t-} + g_t) dM_t + \int_Z (I_{t-z} u_{t-} + h_t(z)) q(dt, dz), \quad t \in [0, T],
\]
with an initial condition $u_0 = \varphi$ in $H_1$, where for each $(t, \omega) \in [0, T] \times \Omega$, $\mathcal{L}_{t, \omega}$ is a linear operator from $H_{1, t} \to H_{1, t-1}$, and $\mathcal{M}_{t, \omega} \circ Q_{t, \omega}^{1/2}$ is a linear operator from $H_{1, t}$ to the space of $H_1$-valued Hilbert-Schmidt operators on $E$, and $I_{t, \omega}$ is a linear operator from $H_{1, t} \to L^2(\mathbb{Z}, \mathbb{R}; \mathcal{P}_t \otimes d\mathbb{P}; H_{1, t})$. By virtue of Theorems 2.9 and 2.10 in [4], under some suitable conditions on the data $\varphi, f, g$, and $h$, if $\mathcal{L}$ satisfies a growth assumption and $\mathcal{L}, \mathcal{M}$, and $I$ satisfy a coercivity assumption in the normal triple $(H_{1, t}, H_1, H_{1, t-1})$, then there exists a unique solution $(u_t)_{t \leq T}$ of (1.1) that is strongly càdlàg in $H_1$ such that the $dV_t \otimes d\mathbb{P}$ equivalence class of $u$ belongs to $L^2([0, T] \times \Omega; \mathcal{O}_t, dV_t \otimes d\mathbb{P}; H_{1, t})$. In the present paper, under a weaker assumption than coercivity (see Assumption (2.1)(i)) and using the method of vanishing viscosity, we prove that there exists a unique solution $(u_t)_{t \leq T}$ of (1.1) that is strongly càdlàg in $H_{1, t-1}$ such that the $dV_t \otimes d\mathbb{P}$ equivalence class of $u$ belongs to $L^2([0, T] \times \Omega; \mathcal{O}_t, dV_t \otimes d\mathbb{P}; H_{1, t})$. Furthermore, under some additional assumptions on the operators $\mathcal{L}, \mathcal{M}$, and $I$ (see Assumption (2.3)(iii)) and the martingale measure $q(dt, dz)$, we can show that $u$ is weakly càdlàg in $H_1$.

The variational theory of deterministic degenerate linear elliptic and parabolic PDEs was established by OA Oleinik and EV Radkevich in [15] and [16]. In [17], E. Pardoux developed the variational theory of monotone stochastic evolution equations, which was extended in [11], [6], and [4] by N.V. Krylov, B.L. Rozovskiĭ, and I. Gyöngy. Degenerate parabolic stochastic partial differential equations (SPDEs) driven by continuous noise were first investigated by N.V. Krylov and B.L. Rozovskiĭ in [12]. These type of equations arise in the theory of non-linear filtering of continuous diffusion processes as the Zakai equation and as equations governing the inverse flow of continuous diffusions. In [2], the solvability of systems of linear SPDEs in Sobolev spaces was proved by M. Gerencsér, I. Gyöngy, and N.V. Krylov, and a small gap in the proof of the main result of [12] was fixed. In Chapters 2, 3, and 4 of [20], B.L. Rozovskiĭ offers a unified presentation and extension of earlier results on the variational framework of linear stochastic evolution systems and SPDEs driven by continuous martingales (e.g. [17], [10], [11], and [12]). Our existence and uniqueness result on degenerate linear stochastic evolution equations driven by jump processes (Theorem 3.1 below) extends Theorem 2 in Ch. 3 of [20] to include the important case of equations driven by jump processes.

Let $d_1, d_2 \geq 1$ be integers. Let $\tilde{N}(dt, dz) = N(dt, dz) - \pi_t(dz)dV_t$ be a $\mathbb{F}$-adapted compensated quasi-left continuous integer-valued random measure on $\mathbb{R}_+ \times Z$. As a special case of (1.1), we consider the system of stochastic integro-differential equation on $[0, T] \times \mathbb{R}^{d_1}$ for $u_t = u_t(x) = (u_t^k(x))_{1 \leq k \leq d_2}$ given by

$$du_t(x) = (\mathcal{L}_tu_t(x) + f_t(x))dV_t + \int_{Z} (I_{t, z}u_t(x) + h_t(x, z))\tilde{N}(dt, dz), \quad t \in [0, T],$$

$$u^k(0, x) = \psi^k(x),$$

where $\psi \in C^\infty_c(\mathbb{R}^{d_1}; \mathbb{R}^{d_2})$.

$$\mathcal{L}_k^t\psi(x) := \int_{Z} \left((\delta_{kl} + \rho_{kl}^t(x, z))\psi^l(x + H_t(x, z)) - \psi^l(x) - H_t^l(x, z)\partial_l\psi(x)\right)\pi_t(dz)$$

$$+ b_t^l(x)\partial_l\psi^l(x) + c_t^l(x)\psi^l(x)$$

$$I_{t, z}^k\psi(x) := (\delta_{kl} + \rho_{kl}^t(x, z))\psi^l(x + H_t(x, z)) - \psi^l(x).$$

The summation convention with respect to repeated indices is used here and below and $\delta_{kl}$ denotes the Kronecker delta. The summation over $i$ is performed over the set $\{1, \ldots, d_1\}$ and the summation over $j$ is performed over the set $\{1, \ldots, d_2\}$. The solution is weakly càdlàg in $[0, T] \times \mathbb{R}^{d_1}$ and strongly càdlàg in $[0, T] \times \mathbb{R}^{d_1}$. The solution is weakly càdlàg in $[0, T] \times \mathbb{R}^{d_1}$ and strongly càdlàg in $[0, T] \times \mathbb{R}^{d_1}$.
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over \( l \) is performed over the set \( \{1, \ldots, d_2 \} \). Let \( (H^{\alpha}(\mathbb{R}^{d_1}, \mathbb{R}^{d_2}))_{\alpha \in \mathbb{R}} \) be the \( L^2 \)-Sobolev-scale. For each integer \( m \geq 1 \), using our theorem on degenerate stochastic evolution equations discussed above, under suitable regularity conditions on \( h, c, \rho \) and \( H \) and the data \( \varphi, f, \) and \( h \), we obtain the existence of a unique solution \((u_t)_{t \in \mathbb{T}}\) of (1.2) that is weakly càdlàg in \( H^{m}(\mathbb{R}^{d_1}, \mathbb{R}^{d_2}) \) and strongly càdlàg in \( H^{m-1}(\mathbb{R}^{d_1}, \mathbb{R}^{d_2}) \) such that the \( dV_t \otimes dp \) equivalence class of \( u \) belongs to \( L^2([0, T] \times \Omega, O_T, dV_t \otimes dp; H^{m}(\mathbb{R}^{d_1}, \mathbb{R}^{d_2})) \).

Degenerate stochastic integro-differential equations of type (1.2) arise in the theory of non-linear filtering of semimartingales as the Zakai equation and as the equations governing the inverse flow of jump diffusion processes. In [1], I. Gyöngy and K. Dareiotis, proved the existence, uniqueness, and the positivity of solutions of non-linear non-degenerate stochastic integro-differential equations using a comparison principle. Applying the method of stochastic characteristics, in [13], we proved the existence and uniqueness of classical solutions of (1.2) in Hölder spaces. It is worth mentioning that the main estimate used in the proof of uniqueness in [13] (which is done in weighted \( L^2 \) spaces) is essentially the same as the main estimate used in the proof of the degenerate coercivity property of the operators \( L \) and \( I \) in (1.2).

This paper is organized as follows. We derive our existence and uniqueness result for (1.1) in Section 2 and for (1.2) in Section 3.

2 Degenerate Linear Stochastic Evolution Equations

2.1 Notation and Formulation of Result

Let \( \mathbb{N} \) be the set of natural numbers, \( \mathbb{R} \) be the set of real numbers, and \( \mathbb{R}_+ \) be the set of non-negative real numbers. For an integer \( d \geq 1 \), we denote by \( \mathbb{R}^d \) the \( d \)-dimensional Euclidean space and by \( | \cdot | \) the standard Euclidean norm. All Hilbert spaces considered in this paper are assumed to be separable with base field \( \mathbb{R} \). For any Hilbert space \( H \), we denote by \( H^* \) the dual of \( H \) and \( \mathcal{B}(H) \) the Borel sigma-algebra of \( H \). Whenever we say that a map \( \mathcal{H}^{\alpha}(\mathbb{R}^{d_1}, \mathbb{R}^{d_2}) \) the space of all \( S \)-measurable functions \( f : S \rightarrow H \) such that

\[
\int_S |f(s)|^2_H \mu(ds) < \infty,
\]

where we identify functions \( f, g : S \rightarrow H \) that are equal \( \mu \)-almost-everywhere. The linear space \( L^2(S, \mathcal{S}, \mu; H) \) is a Hilbert space when endowed with the inner product

\[
(f, g)_{L^2(S, \mathcal{S}, \mu; H)} := \int_S (f(s), g(s))_H \mu(ds).
\]

We now recall some definitions from [20]. Let \( X \) be a Hilbert space and \( \Lambda \) be a continuous injective linear map from \( X \) into \( H \) such that \( \Lambda(X) \) is dense in \( H \). Let \( X' \) be a Hilbert space and \( \Lambda' \) be a continuous injective linear map from \( H \) into \( X' \) such that \( \Lambda' H \) is dense in \( X' \). Assume that there exists a constant \( N > 0 \) such that for all \( x \in X \) and \( h \in H \),

\[
|(h, \Lambda x)_H| \leq N|h|_H |x|_{X'}.
\]  

(2.3)
2.1 Notation and Formulation of Result

For all \( x' \in X' \), by the denseness of the images of the operators \( \Lambda \) and \( \Lambda' \), there exists a sequence \((x_n)_{n \in \mathbb{N}} \subset X\) such that \( |\Lambda' \circ \Lambda x_n - x'|_{X'} \to 0 \) as \( n \to \infty \); define the pairing \([\cdot, \cdot] : X \times X' \to \mathbb{R}\) by

\[
[x, x'] := \lim_{n \to \infty} (\Lambda x, \Lambda x_n)_H
\]

for each \( x \in X \). Indeed, the right-hand side exists and does not depend on the choice of the sequence \((x_n)_{n \in \mathbb{N}} \). The bi-linear form \([\cdot, \cdot]\) is called the canonical bi-linear functional (CBF) of the normal triple of spaces \((X, H, X')\). It follows that

(i) for \( x \in X \) and \( x' \in X' \), \([x, x']\) \( \leq N|x|x'|_{X'}\);

(ii) \([x, x'] = (x, x')_H\) for every \( x' \in H\);

(iii) if for some sigma-finite measure space \((S, S, \mu)\), \( f : S \to X'\) is a strongly \( \mu\)-measurable integrable function (in the Bochner sense), then for all \( x \in X\),

\[
\left[ x, \int_S f_s \mu(ds) \right] = \int_S \{x, f_s\} \mu(ds).
\]

Recall that the Borel sigma-algebra of the separable Hilbert space \( X \) generated by the strong norm is the same as the sigma-algebra generated by the weak norm, since the Borel \( \sigma\)-algebra is generated by the continuous linear forms, which are the same for the strong and weak topology on \( X \). Moreover, it follows by Kuratowski’s theorem (see, e.g., [18] Ch. 1.3) that \( X \cap \mathcal{B}(H) = \mathcal{B}(X)\) and \( H \cap \mathcal{B}(X') = \mathcal{B}(H)\).

A family of Hilbert spaces \((H_\alpha)_{\alpha \in \mathbb{R}}\) with norms \(|\cdot|_\alpha\) \( \alpha \in \mathbb{R}\) is called a Hilbert scale if:

(i) for \( \beta, \alpha \in \mathbb{R}\) with \( \beta > \alpha\), the space \( H_\beta\) is continuously and densely embedded into \( H_\alpha\);

(ii) for every \( \alpha, \beta, \gamma \in \mathbb{R}\) with \( \alpha < \beta < \gamma\) and \( x \in H_\gamma\),

\[
|x|_\beta \leq |x|_\alpha^{(\gamma-\beta)/(\gamma-\alpha)} \cdot |x|_\gamma^{(\beta-\alpha)/(\gamma-\alpha)}.
\]

We recall the following facts from Ch. 4 Sec. 10 of [8] (see also Ch. 3 Sec. 2 of [20]). Given a Hilbert space \( H_1 \) that is continuously and densely embedded into a Hilbert space \( H_0\), there exists a unique positive self-adjoint linear operator \( \Lambda \) on \( H_0\) with domain \( D(\Lambda) = H_1\) such that for all \( x \in H_1\),

\[
|\Lambda x|_{H_0} = |x|_{H_1}.
\]

It is also clear that on \( H_1\), \((\Lambda^2 x, x)_{H_0} = |\Lambda x|_{H_0}^2 = |x|_{H_1}^2 \geq |x|_{H_0}^2\), and hence the spectrum of \( \Lambda \) is contained in \([1, \infty)\). By virtue of the spectral theorem of unbounded self-adjoint operators on a Hilbert space, there exists a unique projection-valued measure \( E_\lambda \) on \((\mathbb{R}, \mathcal{B}(\mathbb{R}))\), called the spectral-measure or resolution of the identity, with support in \([1, \infty)\) such that

\[
\Lambda = \int_1^{\infty} \lambda dE_\lambda.
\]

For each \( \alpha \in \mathbb{R}\), we define the linear operator \( \Lambda^\alpha\) by

\[
\Lambda^\alpha x = \lim_{n \to \infty} \int_1^n \Lambda^\lambda dE_\lambda x,
\]
Whenever the limit exists. For \( \alpha > 0 \), we denote by \( H_\alpha \) the set of all \( x \in H_0 \) such that the above limit exists. The space \( H_\alpha \) is a closed and dense subspace of \( H_0 \) and is a Hilbert space with the inner product

\[
(\cdot, \cdot)_\alpha := (\Lambda^\alpha, \Lambda^\alpha)
\]

and the corresponding norm denoted by \( | \cdot |_\alpha \). For \( \alpha < 0 \), let \( H_\alpha \) the completion of \( H_0 \) with respect to the norm

\[
| \cdot |_\alpha := |\Lambda^\alpha |_0.
\]

It follows that \((H_\alpha)_{\alpha \in \mathbb{R}} \) forms a unique Hilbert scale and \( H_\infty := \cap_{\alpha \in \mathbb{R}} H_\alpha \) is dense in \( H_\alpha \) (in the strong norm) for every \( \alpha \in \mathbb{R} \). Moreover, for every \( \alpha, \beta \in \mathbb{R} \) with \( \alpha > \beta \) the spaces \((H_{\alpha}, H_{\beta}, H_{2\beta - \alpha}) \) constitute a normal triple with CBF denoted by \( [\cdot, \cdot]_{\alpha, \beta} \), and the mapping from \( H_{2\beta - \alpha} \) to \( H_{\alpha} \) given by

\[
x \mapsto [\cdot, x]_{\alpha, \beta}
\]

is an isometric isomorphism of the spaces \( H_{\alpha} \) and \( H_{2\beta - \alpha} \). In particular, the mapping \( x \mapsto [\cdot, x]_{\alpha+1, \alpha-1} \) is an isometric isomorphism of the spaces \( H_{\alpha+1} \) and \( H_{\alpha-1} \).

Let \( \mathcal{P}, \mathcal{R}, \) and \( \mathcal{O} \) be the predictable, progressive, and optional \( \sigma \)-algebras, respectively, on \( \mathbb{R}_+ \times \Omega \). Let \( \mathcal{P}_T, \mathcal{R}_T, \) and \( \mathcal{O}_T \) be the predictable, progressive, and optional \( \sigma \)-algebras, respectively, on \([0, T] \times \Omega \). For the c\'adl\'ag process \( X \), we write \( \Delta X_t := X_t - X_{t-} \).

We denote the separable Banach space of nuclear operators on \( E \) by \( L_1(E) \) and the corresponding norm by \( | \cdot |_{L_1(E)} \). Recall that for all operators \( Q \in L_1(E) \), the trace of \( Q \), denoted \( \text{tr}(Q) \), is well-defined and finite. Denote by \( L_1^+(E) \) the subspace of \( L_1(E) \) consisting of all self-adjoint non-negative nuclear operators. For a Hilbert space \( H \), we denote the Hilbert space of Hilbert-Schmidt operators by \( L_2(E, H) \) and the corresponding norm by \( | \cdot |_{L_2(E, H)} \) and inner product by \( (\cdot, \cdot)_{L_2(E, H)} \). Recall that for every element \( Q \in L_1^+(E) \), there exists a unique operator \( Q^{1/2} \in L_2(E, E) \) such that \( (Q^{1/2})^2 = Q \) and \( |Q^{1/2}|_{L_2(E, E)} = |Q|_{L_1(E)} \). For an operator \( Q \in L_1^+(E, E) \) and a Hilbert space \( H \), we denote by \( L_Q(E, H) \) the set of all linear bounded operators \( A : Q^{1/2} \in L_2(E, E) \rightarrow H \) such that \( AQ^{1/2} \in L_2(E, H) \). For \( A, B \in L_Q(E, H) \) set \( |A|_{L_Q(E, H)} = |A|Q^{1/2}|_{L_2(E, H)} \) and \( (A, B)_{L_Q(E, H)} = (AQ^{1/2}, BQ^{1/2})_{L_2(E, H)} \) and note that

\[
|A|_{L_Q(E, H)} \leq |\text{tr}(Q)|^{1/2}.
\]

Denote the inner product of \( E \) by \( \langle \cdot, \cdot \rangle_E \) and the norm by \( | \cdot |_E \). Given \( x, y, z \in E \), recall that \( x \otimes y(z) = \langle y, z \rangle_E x \), \( x \otimes y \in L_1(E) \), and \( |x \otimes y|_{L_1(E)} = |x|_E |y|_E \). Since \( M \) is a quasi-left continuous square integrable \( E \)-valued martingale issued from zero, there exists a uniquely defined (up to the Doolean’s measure of \( |M|_E^2 \)) real-valued continuous \( \mathcal{P} \)-measurable non-decreasing process \( (\langle M \rangle_t)_{t \geq 0} \) with \( \langle M \rangle_0 = 0 \) such that \( |M|_E^2 - \langle M \rangle_t \) is a local martingale. We have assumed that \( \langle M \rangle_t = V_t \) for all \( t \geq 0 \). Furthermore, there exists a \( L_1^+(E) \)-valued strongly continuous \( \mathcal{P} \)-measurable non-decreasing process \( (\langle (M)_t \rangle_t)_{t \geq 0} \) with \( \langle M \rangle_0 = 0 \) such that \( M \otimes M_t - \langle (M) \rangle_t \) is \( L_1^+(E) \)-valued local martingale. It follows that \( \text{tr}(\langle (M)_t \rangle_t) = \langle (M) \rangle_t \), for all \( t \geq 0 \), and that there exists a unique \( \mathcal{P} \)-measurable process \( (Q_t)_{t \geq 0} \) taking values in \( L_1^+(E) \) such that

\[
\langle (M) \rangle_t = \int_0^t Q_s dV_s, \quad t \geq 0.
\]

For more information about the existence and uniqueness of the processes \((\langle M \rangle_t)_{t \geq 0}, (\langle (M) \rangle_t)_{t \geq 0}, \) and \((Q_t)_{t \geq 0} \), see sections 21 in [14].
We now state what is means for \( q(dt, dz) \) to be a quasi-left continuous stochastic martingale measure on \( \mathbb{R}_+ \times Z \) with compensator \( \pi_s(dz) dV_t \). This definition is taken from section 4 in [5] (see, also, section 8 in [9]). Let \( (Z_n)_{n \in \mathbb{N}} \) be an increasing sequence from \( Z \) such that \( Z = \bigcup_{n \in \mathbb{N}} Z_n \). For each \( n \in \mathbb{N} \), set \( Z_n = \{ \Gamma \in Z : \Gamma \subset Z_n \} \). For each \( n \in \mathbb{N} \) and every \( \Gamma \in Z_n \), \( q(t, \Gamma) = (q(t, \Gamma))_{t \geq 0} \) is a real-valued càdlàg quasi-left continuous square integrable martingale and for pairwise disjoint \( \Gamma_1, \ldots, \Gamma_n \in Z_n \), \( \mathbb{P} \)-a.s. for all \( t \in \mathbb{R}_+ \), we have

\[
q(t, \Gamma_1 \cup \ldots \cup \Gamma_n) = q(t, \Gamma_1) + \cdots + q(t, \Gamma_n).
\]

For each \( t \geq 0 \), the measure \( \pi_s(dz) \) on \( (Z, \Sigma) \) is finite on \( Z_n \) for each \( n \in \mathbb{N} \) and for each \( n \in \mathbb{N} \) and every \( \Gamma \in Z_n \), the process \( (\pi_t(\Gamma))_{t \geq 0} \) is \( \mathbb{P} \)-measurable. Moreover, for every \( (t, \omega) \in \mathbb{R}_+ \times \Omega \), \( n \in \mathbb{N} \), and \( \Gamma_1, \Gamma_2 \in Z_n \),

\[
\langle q(\Gamma), q(\Gamma) \rangle_t = \int_{[0,t]} \pi_s(\Gamma) dV_s < \infty,
\]

and

\[
\langle q(\Gamma_1), q(\Gamma_2) \rangle_t = \int_{[0,t]} \pi_s(\Gamma_1 \cap \Gamma_2) dV_s.
\]

We assume that for every \( n \in \mathbb{N} \) and \( \Gamma \in Z_n \), and \( e \in E \), \( \mathbb{P} \)-a.s. for all \( t \in \mathbb{R}_+ \),

\[
\langle Me, q(\Gamma) \rangle_t = 0.
\]

Let \( H \) be a Hilbert space. Let \( \Psi \) be a process such that for each \( (t, \omega) \in [0, T] \times \Omega \), \( \Psi_t, \omega \in L_{Q_t, \omega}(E, H) \). Moreover, assume that \( \Psi^1/2 e \) is a \( H \)-valued \( \mathcal{P}_T \)-measurable process satisfying

\[
\int_{[0,T]} |\Psi_t|^2_{L_{Q_t, \omega}(E, H)} dV_s < \infty, \quad \mathbb{P} \text{-a.s.}.
\]

Then the stochastic integral \( (\int_{[0,t]} \Psi_s dM_s)_{t \leq T} \) is well-defined and is a \( H \)-valued càdlàg locally square integrable martingale satisfying \( \mathbb{P} \)-a.s. for all \( t \leq T 

\[
\left\{ \int_{[0,t]} \Psi_s dM_s \right\}_t = \int_{[0,t]} |\Psi_s|_{L_{Q_t, \omega}(E, H)}^2 dV_s.
\]

For more information on stochastic integration in Hilbert spaces, see section 22 in [14]. Let \( \Phi : [0, T] \times \Omega \times Z \to H \) be \( \mathcal{P}_T \otimes \mathcal{Z} \)-measurable and

\[
\int_{[0,T]} |\Phi_s(z)|_{H}^2 \pi_s(dz) dV_s < \infty, \quad \mathbb{P} \text{-a.s.}.
\]

Then the stochastic integral \( (\int_{[0,t]} \Phi_s(z) q(ds, dz))_{t \leq T} \) is well-defined and is a \( H \)-valued càdlàg locally square integrable martingale satisfying \( \mathbb{P} \)-a.s. for all \( t \in [0, T] 

\[
\left\{ \int_{[0,t]} \Phi_s(z) q(ds, dz) \right\}_t = \int_{[0,t]} \int_Z |\Phi_s(z)|_{H}^2 \pi_s(dz) dV_s.
\]

Let \( (H_\alpha)_{\alpha \in \mathbb{R}} \) be a Hilbert scale connecting Hilbert spaces \( H_1 \) and \( H_0 \) with generator \( \Lambda \). For each \( \alpha \in \mathbb{R} \), denote by \( || \cdot ||_\alpha \) and the norm of \( H_\alpha \) and let \( [\cdot, \cdot]_\alpha \) the CBF of the triple \( (H_{\alpha+1}, H_\alpha, H_{\alpha-1}) \).
2.1 Notation and Formulation of Result

Denote by $\mathcal{R}_\alpha : H_\alpha \rightarrow H_\alpha^*$ the Riesz isomorphism. For operators $A, B \in L_{Q_t}(E,H_\alpha)$, we set $\|A\|_{\alpha; Q_t} = |A|_{L_{Q_t}(E,H_\alpha)}$ and $(A, B)_{\alpha; Q_t} = \langle A, B \rangle_{L_{Q_t}(E,H_\alpha)}$ and drop the dependence on $\omega$ when convenient.

For each real number $\lambda \geq 1$, we consider the linear stochastic evolution equation

$$
    du_t = (L_t u_t + f_t) dV_t + (M_t u_t - g_t) dM_t + \int_{\Omega} (\tilde{I}_{t,\omega} u_t + h_t(\omega)) q(\omega, d\omega), \quad t \in [0, T],
$$

(2.4)

with an initial condition $u_0 = \varphi$ in $H_\lambda$ that is $\mathcal{F}_0$-measurable. Here the operator $L$ is a mapping from $[0, T] \times \Omega \times H_{\lambda+1}$ into $H_{\lambda-1}$ that is linear in $H_{\lambda+1}$ and $f : [0, T] \times \Omega \rightarrow H_{\lambda-1}$. For each $t$ and $\omega$, $M_t$ is a linear operator from $H_{\lambda+1}$ into the space of Hilbert-Schmidt operators $L_{Q_\omega}(E,H_\lambda)$ and $g_t(\omega) \in L_{Q_t}(E,H_\lambda)$. For each $t$ and $\omega$, $\tilde{I}_{t,\omega}$ is a linear operator from $L^2(\Omega)$ into $L^2(\mathbb{Z} \times \Omega, \pi_{\omega}(d\omega); H_\lambda)$ and $h_t(\omega) \in L^2(\mathbb{Z}, \pi_{\omega}(d\omega); H_\lambda)$. We also assume that for each $v \in H_{\lambda-1}$, $L_v$ is $\mathcal{F}_T$-measurable, $M_v Q^{1/2} e$ is $\mathcal{P}_T$-measurable for each $e \in E$, and $T$ is $\mathcal{P}_T \otimes \mathcal{Z}$-measurable. Furthermore, we assume that $f$ is $\mathcal{F}_T$-measurable, $g^2 Q^{1/2} e$ is $\mathcal{P}_T$-measurable for each $e \in E$, and $h$ is $\mathcal{P}_T \otimes \mathcal{Z}$-measurable.

**Assumption 2.1 ($\lambda$).** There exist constants $L$ and $K$ such that following conditions hold for all $(t, \omega) \in [0, T] \times \Omega$:

(i) for $\alpha \in [0, \lambda)$ and all $v \in H_{\alpha+1}$,

$$
    2[v, L_t v]_\alpha + \|M_t v\|_{\alpha; Q_t}^2 + \int_Z |\tilde{I}_{t,\omega} v|^2_{\alpha} \pi_t(\omega) d\omega \leq L v^2_{\alpha};
$$

(2.5)

(ii) for $\alpha \in [0, \lambda - 1, \lambda]$ and all $v \in H_{\alpha+1}$,

$$
    |L_t v|^2_{\alpha-1} \leq K v^2_{\alpha-1}, \quad |M_t v|^2_{\alpha-1} \leq K v^2_{\alpha-1}, \quad \int_Z |\tilde{I}_{t,\omega} v|^2_{\alpha} \pi_t(\omega) d\omega \leq K v^2_{\alpha+1}.
$$

**Assumption 2.2 ($\lambda$).** We assume that

$$
    \mathbb{E} v^2_{\lambda} + \mathbb{E} \kappa^2_{\lambda}(T) < \infty,
$$

where

$$
    \kappa^2_{\lambda}(t) := \int_{[0,t]} \left( |f_t|^2_{\lambda+1} + |g_t|^2_{\lambda+1; Q_t} + \int_Z |h_t|^2_{\lambda+1} \pi_t(\omega) d\omega \right) dV_t, \quad t \in [0, T].
$$

**Definition 2.1.** An $H_0$-valued $\mathbb{F}$-adapted strongly càdlàg processes $(u_t)_{t \leq T}$ is said to be a solution of the stochastic evolution equation (2.4) on $[0, T]$ if its $dV_t \otimes d\mathbb{P}$ equivalence class belongs to $L^2([0, T] \times \Omega, \mathcal{P}_T, dV_t \otimes d\mathbb{P}; H_1)$, and if there exists a set $\hat{\Omega} \subseteq \Omega$ with $\mathbb{P}(\hat{\Omega}) = 1$ such that for all $(t, \omega) \in [0, T] \times \hat{\Omega}$ and $v \in H_1$,

$$
    (v, u_t) = (v, \varphi) + \int_{[0,t]} v_{\omega} (L_s \tilde{u}_s + f_s) dV_s + \int_{[0,t]} v_{\omega} (M_s \tilde{u}_s + g_s) dM_s + \int_{[0,t]} v_{\omega} \int_Z (\tilde{I}_{s,\omega} \tilde{u}_s + h_s(\omega)) q(\omega, d\omega),
$$

(2.6)

where $\tilde{u}$ is any $H_1$-valued $\mathcal{P}_T$-measurable $dV_t \otimes d\mathbb{P}$ version of $u$. We say a solution of (2.4) on $[0, T]$ is unique if for any two solutions $(u_t)_{t \leq T}$ and $(v_t)_{t \leq T}$ of (2.4) on $[0, T]$ we have

$$
    \mathbb{P} \left( \sup_{t \leq T} |u_t - v_t| > 0 \right) = 0.
$$
2.1 Notation and Formulation of Result

Remark 2.1. Since \((u_t)_{t \leq T}\) in Definition 2.1 is \(O_T\)-measurable and \(\hat{u} \in L^2([0, T] \times \Omega, O_T, dV_t \otimes dP; H_0)\), there always exists a \(H_1\)-valued \(O_T\)-measurable \(dV_t \otimes dP\) version of \(\hat{u}\). See Exercise 4.2.3 in [19] for details.

Remark 2.2. It is implicitly assumed that the integrals in (2.6) are well-defined. It is easy to check that if Assumptions 2.1(1)(ii) and 2.2(1) hold, then the integrals in (2.6) are well-defined. For all \(\nu \in H_1, P\text{-a.s. for all } t \in [0, T],\)

\[
\left(\nu, \int_{[0,t]} M_s \bar{u}_{s-} dM_s \right)_0 = \int_{[0,t]} \mathcal{R}_0 \nu M_s \bar{u}_{s-} dM_s.
\]

For each \((t, \omega) \in [0, T] \times \Omega\) and \(\nu \in H_1\), the operator \(\mathcal{R}_0 \nu M_t \bar{Q}^{1/2}_t e = (M_t \nu Q^{1/2}_t, \nu)_0\) for each \(u \in H_1\) and \(e \in E\). Furthermore, if \((e_i)_{i \in N}\) is an orthonormal basis of \(E\), then for each \((t, \omega) \in [0, T] \times \Omega\) and \(u, \nu \in H_1,\)

\[
\|\mathcal{R}_0 \nu M_t \bar{Q}^{1/2}_t e_i\|^2_{L^2(E)} = \sum_{i=1}^{\infty} \|\nu, M_t \bar{Q}^{1/2}_t e_i\|_0^2 \leq |\nu|_0^2 \sum_{i=1}^{\infty} |M_t \nu Q^{1/2}_t e_i|_0^2 \leq |\nu|_0^2 |M_t \nu Q^{1/2}_t e_i|_0^2.
\]

Thus, the stochastic integral against \((M_t)_{t \leq T}\) is well-defined in (2.6) if Assumption 2.1 (ii) holds.

The notation \(C = C(\cdot, \cdots, \cdot)\) is used to denote a positive constant depending only on the quantities appearing in the parentheses. In a given context, the same letter is often used to denote different constants depending on the same parameter.

Theorem 2.1. Assume that there is a positive constant \(N_T\) such that \(V_t \leq N_T\) for all \((t, \omega) \in [0, T] \times \Omega\) and that Assumptions 2.1(\(\lambda\)) and 2.2(\(\lambda\)) hold for some \(\lambda \geq 1\). Then there exist a unique solution \((u_t)_{t \leq T}\) of (2.4) that is a strongly càdlàg \(H_{\lambda-1}\)-valued process whose \(dV_t \otimes dP\) equivalence class belongs to \(L^2([0, T] \times \Omega, O_T, dV_t \otimes dP; H_{\lambda})\) and there is a constant \(C = C(L, K, N_T)\) such that

\[
E\sup_{t \leq T} |u_t|^2_{1, \lambda} + \sup_{t \leq T} E|u_t|^2_{1, \lambda} + E \int_{[0, T]} |u_t|^2 dV_t \leq C \left( E|\tilde{\nu}|_{1, \lambda} + E \bar{\lambda}^2(T) \right).
\]  

(2.7)

Under some additional assumptions, we can obtain an estimate of \(E\sup_{t \leq T} |u_t|^2_{1, \lambda}\). Assume that \(q(dt, dz) = \tilde{N}(dt, dz) = N(dt, dz) - \pi_t(dz)\) and that \(M\) is strongly continuous. Moreover, assume that for all \(\nu \in H_{\lambda+1}\) and \((t, \omega) \in [0, T] \times \Omega,\)

\[
\mathcal{L} \nu = \mathcal{A} \nu + \int_Z J_{t, \omega} \nu \pi_t(dz),
\]

where \(\mathcal{A}\) is a mapping from \([0, T] \times \Omega \times H_{\lambda+1}\) into \(H_{\lambda+1}\) that is linear in \(H_{\lambda+1}\) and \(J_{t, \omega}\) is a linear operator from \(H_{\lambda+1}\) into \(L^2(Z, \mathcal{F}, \pi_t(dz); H_{\lambda-1})\). We also assume that for each \(\nu \in H_{\lambda+1}, \mathcal{A} \nu\) is \(\mathcal{R}_T\)-measurable and \(J\) is \(\mathcal{P}_T \otimes \mathcal{Z}\)-measurable.

Assumption 2.3 (\(\lambda\)). There exist constants \(L\) and \(K\) such that the following conditions hold for all \((t, \omega) \in [0, T] \times \Omega:\)
2.2 Proof of Theorems 2.1 and 2.2

(i) For $\alpha \in [0, \lambda)$ and all $v \in H_\alpha$, 
\[ 2 \langle v, A_v \rangle + \| M_v \|_{L_2, Q}^2 \leq \frac{L}{2} \| v \|^2_{\alpha}, \quad \int_Z \left( 2 \langle v, J_{t,z} v \rangle + \| I_{t,z} v \|_{L_2}^2 \right) \rho_t(dz) \leq \frac{L}{2} \| v \|^2_{\alpha}; \]

(ii) For $\alpha \in [0, \lambda - 1, \lambda]$ and all $v \in H_\alpha$, 
\[ \| L_{\alpha-1} v \|_{\alpha+1}^2 \leq K \| v \|^2_{\alpha+1}, \quad \| M_v \|_{\alpha+1}^2 \leq K \| v \|^2_{\alpha+1}, \quad \int_Z | I_{t,z} v |^2 \pi_t(dz) \leq K \| v \|^2_{\alpha+1}; \]

(iii) 
\[ \| R_{\alpha} v M_v u \|_{L_{2, \rho}(E, R)}^2 \leq K \| v \|^4, \quad \int_Z \langle v, (I_{t,z} - J_{t,z}) v \rangle \pi_t(dz) \leq K \| v \|^4. \]

**Theorem 2.2.** Assume that $q(dt, dz) = \hat{N}(dt, dz)$, $M$ is strongly continuous, and that there is a positive constant $N_T$ such that $V_t \leq N_T$ for all $(t, \omega) \in [0, T] \times \Omega$. Moreover, assume that for all $v \in H_{\lambda+1}$ and $(t, \omega) \in [0, T] \times \Omega$, 
\[ L_v = A_v v + \int_Z J_{t,z} v \pi_t(dz) \]
and that Assumption 2.3(\lambda) and 2.2(\lambda) hold for some $\lambda \geq 1$. Then there exist a unique solution $(u_t)_{t \leq T}$ of (2.4) that is a weakly càdlàg $H_\lambda$-valued process and a strongly càdlàg $H_{\lambda-1}$-valued process. Moreover, the $dV_t \otimes dP$ equivalence class of $u$ belongs to $L^2([0, T] \times \Omega, \mathcal{F}_t, dV_t \otimes dP; H_\lambda)$ and there is a constant $C = C(L, K, N_T)$ such that 
\[ \mathbf{E} \sup_{t \leq T} | u_t |^2 + \mathbf{E} \int_{[0, T]} | u_t |^2 dV_t \leq C \left( \mathbf{E} | \sigma |^4 + \mathbf{E} \kappa^2(T) \right). \]  

(2.8)

2.2 Proof of Theorems 2.1 and 2.2

**Proof of Theorem 2.1.** Fix $\lambda \geq 1$. For all $\alpha \in \mathbb{N}$ and $(t, \omega) \in [0, T] \times \Omega$, set $L_{t, \omega}^{(n)} = L_{t, \omega} - \frac{1}{n} \Lambda^2$. A simple approximation argument shows that for all $v \in H^{\lambda+1}$, 
\[ | v |^2_{\lambda+1} = [ v, \Lambda^2 v ]_{\lambda}. \]  

(2.9)

Thus, by Assumption 2.1(i), for all $v \in H_{\lambda+1}$ and $(t, \omega) \in [0, T] \times \Omega$, 
\[ 2 \langle v, L_{t, \omega}^{(n)} v \rangle + \| M_v \|^2_{L_2, Q} + \int_Z | I_{t,z} v |^2 \leq - \frac{2}{n} | v |^2_{\lambda+1} + L | v |^2_{\lambda}. \]  

(2.10)

and by Assumption 2.1(ii), for all $v \in H_{\lambda+1}$ and $(t, \omega) \in [0, T] \times \Omega$, 
\[ | L_{t, \omega}^{(n)} v |_{\lambda-1} \leq | L_{t} v |_{\lambda-1} + \frac{1}{n} | \Lambda^2 v |_{\lambda-1} \leq \left( K + \frac{1}{n} \right) | v |_{\lambda+1}. \]  

(2.11)

Therefore, owing to (2.10), (2.11), and Assumption 2.2, by Theorems 2.9, 2.10, and 4.1 in [4] and a simple extension of the result in Section 4 of [5] to Hilbert space valued integrands (see Sec. 3 in [7] as well), for every $\alpha \in \mathbb{N}$, there exist a strongly càdlàg $H_\lambda$-valued processes $(u_t^{(n)})_{t \leq T}$ such that its
$dV_t \otimes d\mathbb{P}$ equivalence class belongs to $L^2([0, T] \times \Omega, O_T, dV_t \otimes d\mathbb{P}; H_{t+1})$, and there exists a set $\Omega \subseteq \Omega$ with $\mathbb{P}(\Omega) = 1$ such that for all $(t, \omega) \in [0, T] \times \Omega$ and $v \in H_{t+1}$,

\[
(v, u^{(n)}_t)_{\lambda} = (v, \varphi)_{\lambda} + \int_{[0,t]} \left[ v, (\mathcal{L}_s \bar{u}^{(n)}_s + f_s) \right] dV_s + \left( v, \int_{[0,t]} (\mathcal{M}_s \bar{u}^{(n)}_s + g_s) dM_s \right)_{\lambda} + \left( v, \int_{[0,t]} \int_{Z} (I_{s, z} \bar{u}^{(n)}_s + h_s(z)) q(ds, dz) \right)_{\lambda},
\]

(2.12)

where $\bar{u}^{(n)}$ is any $H_{t+1}$-valued $O_T$-measurable $dV_t \otimes d\mathbb{P}$ version of $u^{(n)}$. Moreover, if $(v^{(n)}_t)_{\leq T}$ is any other process satisfying these conditions, then

\[
\mathbb{P} \left( \sup_{t \leq T} |v^{(n)}_t - v^{(n)}_t|_{\lambda} > 0 \right) = 1.
\]

Furthermore, by virtue of Theorem 4.1 in [4], there is a constant $C(n) = C(n, L, K, N_T)$ such that

\[
\mathbb{E} \sup_{t \leq T} |u^{(n)}_t|_{\lambda}^2 + \mathbb{E} \int_{[0, T]} |u^{(n)}_t|_{\lambda+1}^2 dV_t \leq C(n) \left( \mathbb{E} |\varphi|_{\lambda}^2 + \mathbb{E} \kappa_2^2(T) \right),
\]

(2.13)

In the following lemma, we show that a similar estimate is valid with a constant independent of $n$.

**Lemma 2.3.** There is a constant $C = C(L, K, N_T)$ such that for all $n \in \mathbb{N}$,

\[
\mathbb{E} \sup_{t \leq T} |u^{(n)}_t|_{\lambda}^2 + \mathbb{E} \int_{[0, T]} |u^{(n)}_t|_{\lambda+1}^2 dV_t \leq C \left( \mathbb{E} |\varphi|_{\lambda}^2 + \mathbb{E} \kappa_2^2(T) \right).
\]

(2.14)

**Proof of Lemma 2.3.** For each $(t, \omega) \in [0, T] \times \Omega$, set $\phi_t = e^{-\frac{L + 2}{2} N_T}$ and note that $e^{-\frac{L + 2}{2} N_T} \leq \phi_t \leq 1$,

for all $(t, \omega) \in [0, T] \times \Omega$, since $V$ bounded uniformly in $t$ and $\omega$. By Itô’s product rule and (2.23), $\mathbb{P}$-a.s. for all $t \in [0, T]$ and all $v \in H_{t+1}$, we have

\[
(v, \phi_t u^{(n)}_t)_{\lambda} = (v, \varphi)_{\lambda} + \int_{[0,t]} \left[ v, \phi_s \left( \mathcal{L}_s \bar{u}^{(n)}_s - \frac{L + 2}{2} u^{(n)}_s + f_s \right) \right] dV_s + \left( v, \int_{[0,t]} \int_{Z} \phi_s (I_{s, z} \bar{u}^{(n)}_s + h_s(z)) q(ds, dz) \right)_{\lambda},
\]

(2.15)

where $\bar{u}^{(n)}$ is any $H_{t+1}$-valued $O_T$-measurable $dV_t \otimes d\mathbb{P}$ version of $u^{(n)}$. Applying Theorem 2 in [6] to (2.15), $\mathbb{P}$-a.s. for all $t \in [0, T]$, we obtain

\[
|\phi_t u^{(n)}_t|_{\lambda}^2 = |\varphi|_{\lambda}^2 + I_1(t) + I_2(t) + I_3(t) + m_t
\]

where

\[
I_1(t) := \int_{[0,t]} \phi_s^2 \left( 2 \left[ \bar{u}^{(n)}_s, \mathcal{L}_s \bar{u}^{(n)}_s \right]_{\lambda} + \|\mathcal{M}_s \bar{u}^{(n)}_s\|^2_{L^2(Q_s)} + \int_Z |I_{s, z} \bar{u}^{(n)}_s|^2 \pi_s(dz) - (L + 1) |u^{(n)}_s|_{\lambda}^2 \right) dV_s,
\]
2.2 Proof of Theorems 2.1 and 2.2

\[ I_2(t) := 2 \int_{[0,t]} \phi_s^2 \left( (u_s^{(n)}, f_s) + (M_s \bar{u}_s^{(n)}, g_s) \right) \, dV_s + \int_{\Gamma} (I_{s,\gamma} \bar{u}_s^{(n)}, h_s(z)) \, \pi_s(dz) - |u_s^{(n)}|^2 \, dV_s, \]

\[ I_3(t) := 2 \int_{[0,t]} \phi_s^2 \left( |g_s|^2 \right) + \int_{\Gamma} |h_s(z)|^2 \, \pi_s(dz) \, dV_s, \]

and \((m_t)_{t \leq T}\) is the local martingale given by

\[ m_t := 2 \int_{[0,t]} \phi_s^2 \mathcal{R}_s u_s^{(n)} (M_s \bar{u}_s^{(n)} + g_s) \, dM_s + 2 \int_{[0,t]} \phi_s^2 (\mathcal{R}_s u_s^{(n)} - M_s \bar{u}_s^{(n)} + h_s(z)) q(ds, dz) \]

\[ + \left( \int_{[0,t]} \phi_s (M_s \bar{u}_s^{(n)} + g_s) \, dM_s \right) - \left( \int_{[0,t]} \phi_s (M_s \bar{u}_s^{(n)} + g_s) \, dM_s \right) \]

\[ + \left( \int_{[0,t]} \phi_s (I_{s,\gamma} \bar{u}_s^{(n)} + h_s(z)) q(ds, dz) \right) - \left( \int_{[0,t]} \phi_s (I_{s,\gamma} \bar{u}_s^{(n)} + h_s(z)) q(ds, dz) \right). \]

Let \((\tau_k)_{k \in \mathbb{N}}\) be the sequence of bounded \(\mathbb{P}\)-stopping times defined by

\[ \tau_k = \inf \{ t \in [0, T] : |u_t^{(n)}|_1 \geq k \} \wedge T. \]

Since \((u_t^{(n)})_{t \leq T}\) is a strongly càdlàg \(H_1\)-valued processes, \(\tau_k \uparrow T\) as \(k\) tends to infinity. It follows that for all \(t \in [0, T]\) that

\[ 1_{[0,\tau_k]}(t) |u_t^{(n)}|_1 \leq k, \]

and hence by Assumption 2.1(ii), (2.13), and Young’s inequality, we have

\[ \mathbb{E} \int_{[0,T]} \phi_t^{(n)} \mathcal{R}_t u_t^{(n)} (M_t \bar{u}_t^{(n)} + g_t) \, dM_t \leq \mathbb{E} \int_{[0,T]} \left( \|M_t \bar{u}_t^{(n)}\|_{L_Q}^2 + \|g_t\|_{L_Q}^2 \right) \, dV_t \]

\[ \leq k \mathbb{E} \int_{[0,T]} \left( K |\bar{u}_t^{(n)}|_{L+1}^2 + \|g_t\|_{L_Q}^2 \right) \, dV_t < \infty. \]

Similarly, using Assumption 2.1(ii), (2.13), and Young’s inequality, we obtain

\[ \mathbb{E} \int_{[0,T]} \int_{\Gamma} \phi_t^{(n)} (I_{t,\gamma} \bar{u}_t^{(n)} + h_t(z)) \, dV_t \leq k \mathbb{E} \int_{[0,T]} \left( K |\bar{u}_t^{(n)}|_{L+1}^2 + \int_{\Gamma} |h_t(z)|_{\pi}^2 \, dz \right) \, dV_t < \infty. \]

Once again appealing to Assumption 2.1(ii), (2.13), and Young’s inequality, we get

\[ \mathbb{E} \int_{[0,T]} \phi_t^{(n)} \|M_t \bar{u}_t^{(n)} + g_t\|_{L,Q}^2 \, dV_t \leq 2 \mathbb{E} \int_{[0,T]} \left( K |\bar{u}_t^{(n)}|_{L+1}^2 + \|g_t\|_{L_Q}^2 \right) \, dV_t < \infty, \]

\[ \mathbb{E} \int_{[0,T]} \int_{\Gamma} \phi_t^{(n)} |I_{t,\gamma} \bar{u}_t^{(n)} + h_t(z)|_{\pi}^2 \, dV_t \leq 2 \mathbb{E} \int_{[0,T]} \left( K |\bar{u}_t^{(n)}|_{L+1}^2 + \int_{\Gamma} |h_t(z)|_{\pi}^2 \, dz \right) \, dV_t < \infty. \]

The above estimates imply \((\tau_k)_{k \in \mathbb{N}}\) is a localizing sequence for \((m_t)_{t \leq T}\). By virtue of (2.10), for all \((t, \omega) \in [0, T] \times \Omega\), we have

\[ I_1(t) \leq - \int_{[0,t]} |\phi_s \bar{u}_s^{(n)}|_{L}^2 \, dV_s. \]

Owing to Young’s inequality, for all \((t, \omega) \in [0, T] \times \Omega\),

\[ 2(u_t^{(n)}, f_t) \leq \frac{1}{3} |u_t^{(n)}|_1^2 + 3 |f_t|^2. \]
2.2 Proof of Theorems 2.1 and 2.2

Let \((e_i)_{i \in \mathbb{N}}\) be an orthonormal basis of \(E\). By Assumption 2.1(ii), for each \((t, \omega) \in [0, T] \times \Omega\), we have

\[
2 \left( \langle M_t \tilde{u}^{(n)}_t, g_t \rangle \right)_{L^2} \leq 2 \sum_{i=1}^{\infty} \left( M_t u^{(n)}_i Q^{1/2} e_i, g_t Q^{1/2} e_i \right) \leq 2 \|M_t \tilde{u}^{(n)}_t\|_{L^2} \|g_t\|_{L^2},
\]

with

\[
\|M_t \tilde{u}^{(n)}_t\|_{L^2} \leq \frac{1}{3} |\tilde{u}^{(n)}_t|_1 + 3K^2 \|g_t\|_{L^2}.
\]

and

\[
\int_Z 2 \left( \int_{t,z} \tilde{u}^{(n)}(z) h_t(z) \right) \pi_t(dz) \leq \frac{1}{3} |\tilde{u}^{(n)}_t|_1 + 3K^2 \int_Z |h_t(z)|_{L^2} \pi_t(dz).
\]

Hence, there is a constant \(C = C(K)\) such that for all \((t, \omega) \in [0, T] \times \Omega\),

\[
I_2(t) + I_3(t) \leq C_{\kappa_2}(t).
\]

Combining the above estimates, \(\mathbf{P}\)-a.s. for all \(t \in [0, T]\), we have

\[
|\phi_t u^{(n)}_t|_1^2 + \int_{[0,t]} |\phi_t \tilde{u}^{(n)}_s|_1^2 dV_s \leq |\varphi|_1^2 + C_{\kappa_2}(t) + m_t,
\]

and hence there is a constant \(C = C(L, K, N_T)\) such that \(\mathbf{P}\)-a.s. for all \(t \in [0, T]\),

\[
|u^{(n)}_t|_1^2 + \int_{[0,t]} |\tilde{u}^{(n)}_s|_1^2 dV_s \leq C \left( |\varphi|_1^2 + \kappa_2(t) + m_t \right)
\]

(2.18)

Stopping (2.18) along the localizing sequence \((\tau_k)_{k \in \mathbb{N}}\) and taking the expectation, we obtain,

\[
\sup_{t \leq T} \mathbf{E} \left( |u^{(n)}_{\tau_k} ... \right) + \mathbf{E} \int_{[0,T]} |\tilde{u}^{(n)}_s|_1^2 dV_s \leq C \left( \mathbf{E}|\varphi|_1^2 + \mathbf{E}\kappa_2(T) \right).
\]

Therefore, passing to the limit as \(k \to \infty\) in (2.19) and applying the monotone convergence theorem and Fatou’s lemma, and the fact that \((u^{(n)}_t)_{t \leq T}\) is a strongly càdlàg \(H_1\)-valued processes, we obtain (2.14).

Let \(\bar{V}_t = V_t + t\) for all \((t, \omega) \in [0, T] \times \Omega\). Let \(\overline{O_T}\) be the completion of \(O_T\) with respect to the measure \(d\bar{V}_t \otimes d\mathbf{P}\). Let \(S_T = (\Omega \times [0, T], \overline{O_T}, d\bar{V}_t \otimes d\mathbf{P})\). It follows from Lemma 2.3 that there is a sequence \((n_k)_{k \in \mathbb{N}}\) such that \(n_k \to \infty\) as \(k\) tends to infinity and \(u^{(n_k)} \to \bar{u}\) converges weakly in \(L^2(S_T; H_1)\) to some \(\bar{u} \in L^2(S_T; H_1)\) that satisfies

\[
\mathbf{E} \int_{[0,T]} |ar{u}|_1^2 d\bar{V}_t \leq C \left( \mathbf{E}|\varphi|_1^2 + \mathbf{E}\kappa_2(T) \right),
\]

where \(C\) is the constant from Lemma 2.3. We can always find a version of \((\bar{u}_t)_{t \leq T}\) that is \(O_T\)-measurable in \(H_1\).

Fix \(w \in H_{1-1}\) and a \(O_T\)-measurable process \((\eta_t)_{t \leq T}\) bounded by \(K\). Define the linear functional \(\Phi^L : L^2(S_T; H_1) \to \mathbb{R}\) by

\[
\Phi^L(v) = \mathbf{E} \int_{[0,T]} \eta_t \int_{[0,T]} [w, \mathcal{L}_s v_s]_{H_1} dV_s d\bar{V}_t, \quad \forall v \in L^2(S_T; H_1).
\]
Owing to Assumption (2.1)(ii) and the fact that \((\bar{V}_t)_{t \leq T}\) is uniformly bounded, there is a constant 

\[ C = C(K, N_T) \] 

such that for all \( v \in L^2(S_T; H_1) \),

\[ |\Phi^L(v)| \leq C |w|_{L^2} E \int_0^T |v_t|^2 dV_t, \]

which implies that \( \phi^L \) is a continuous linear functional on \( L^2(S_T; H_1) \), and hence that

\[ \lim_{k \to \infty} \Phi^L(u^{(n)}) = \Phi^L(\bar{u}). \] (2.20)

By (2.9), the uniform boundedness of \((\bar{V}_t)_{t \leq T}\), and Claim ??, there is a constant \( C = C(K, N_T) \) such that

\[ \lim_{k \to \infty} E \int_0^T \eta_t \int_{[0,t]} \frac{1}{n_k} [w, \Lambda^2 u_s^{(n)}]_L^1 dV_s d\bar{V}_t \leq |w|_{L^2} \lim_{k \to \infty} \frac{C}{n_k} \int_0^T |u_s^{(n)}|^2 dV_t \]

\[ \leq |w|_{L^2} \left( E|\varphi|^2 + E x_n^2(T) \right) \lim_{k \to \infty} \frac{C}{n_k} = 0. \] (2.21)

Define the linear functionals \( \Phi^M, \Phi^T : L^2(S_T, H_1) \to \mathbb{R} \) by

\[ \Phi^M(v) = E \int_{[0,T]} \eta_t \int_{[0,t]} R_{t-1} w M_s v_s dM_s d\bar{V}_t, \quad \forall v \in L^2(S_T; H_1) \]

and

\[ \Phi^T(v) = E \int_{[0,T]} \eta_t \int_{[0,t]} \int_{Z} (w, I_s z v_s)_L^1 q(ds, dz) d\bar{V}_t, \quad \forall v \in L^2(S_T; H_1). \]

Using Assumption 2.1(ii), the uniform boundedness of \((\bar{V}_t)_{t \leq T}\), and the Burkholder-Davis-Gundy inequality, we get that there is a constant \( C = C(K, N_T) \) such that for all \( v \in L^2(S_T; H_1) \),

\[ \Phi^M(v) \leq E \int_{[0,T]} \eta_t \sup_{t \leq T} \int_{[0,t]} M_s v_s dM_s^2 \leq CE \int_{[0,T]} |v_s|_L^2 dV_s \]

and

\[ \Phi^T(v) \leq CE \int_{[0,T]} |v_s|_L^2 dV_s \]

which implies that \( \phi^M \) and \( \phi^T \) are a continuous linear functionals on \( L^2(S_T; H_1) \). Thus, we have

\[ \lim_{k \to \infty} \Phi^M(u^{(n)}) = \Phi^M(\bar{u}) \quad \text{and} \quad \lim_{k \to \infty} \Phi^T(u^{(n)}) = \Phi^T(\bar{u}). \] (2.22)

By (2.12) and the fact that \( \Lambda^2 H_{16} \) is dense in \( H_{16}, P \)-a.s. for all \( t \in [0, T] \) and \( w \in H_1 \),

\[ E \int_{[0,T]} \eta_t (w, u_t^{(n)}) dV_t = E \int_{[0,T]} \eta_t \left( (w, \varphi)_{L^1} + \int_{[0,t]} (w, I_s u_s^{(n)} + f_s)_{L^1} dV_s \right) d\bar{V}_t \]

\[ + E \int_0^T \eta_t \int_{[0,t]} R_{t-1} w (M_s u_s^{(n)} + g_s) dM_s d\bar{V}_t \]

\[ + E \int_0^T \eta_t \int_{[0,t]} \int_{Z} (w, I_s z u_s^{(n)} + h_s(z))_{L^1} q(ds, dz) d\bar{V}_t. \] (2.23)
2.2 Proof of Theorems 2.1 and 2.2

Taking \( k \to \infty \) in (2.23) and using (2.20), (2.21), (2.22), we obtain for each \( w \in H_1, \)

\[
(w, \bar{u}_t)_{\lambda-1} = (w, \varphi)_{\lambda-1} + \int_{[0,t]} [w, (L_s \bar{u}_s + f_s)]_{\lambda-1} dV_s + \left( w, \int_{[0,t]} (M_s \bar{u}_s + g_s) dM_s \right)_{\lambda-1}
+ \left( w, \int_{[0,t]} \int_Z (I_{s,z} \bar{u}_s + h_s(z)) q(ds, dz) \right)_{\lambda-1}
\]  

(2.24)

for \( d\bar{V}_t \otimes d\mathbf{P} \)-almost-all \( (t, \omega) \in [0, T] \times \Omega \). Defining \( (u_t)_{t \in T} \) by

\[
\begin{align*}
    u_t &= \varphi + \int_{[0,t]} (L_s \bar{u}_s + f_s) dV_s + \int_{[0,t]} (M_s \bar{u}_s + g_s) dM_s + \int_{[0,t]} \int_Z (I_{s,z} \bar{u}_s + h_s(z)) q(ds, dz), \quad t \in [0, T],
\end{align*}
\]

we have \( u_t = \bar{u}_t \) for \( d\bar{V}_t \times d\mathbf{P} \)-almost-all \( (t, \omega) \in [0, T] \times \Omega \). It follows from Theorem 2 in [6] that \( (u_t)_{t \in T} \) is a strongly càdlàg \( H_{\lambda-1} \)-valued process and \( \mathbf{P} \)-a.s. for all \( t \in [0, T] \) and \( w \in H_1, \)

\[
(w, u_t)_{\lambda-1} = (w, \varphi)_{\lambda-1} + \int_{[0,t]} [w, (L_s \bar{u}_s + f_s)]_{\lambda-1} dV_s + \left( w, \int_{[0,t]} (M_s \bar{u}_s + g_s) dM_s \right)_{\lambda-1}
+ \left( w, \int_{[0,t]} \int_Z (I_{s,z} \bar{u}_s + h_s(z)) q(ds, dz) \right)_{\lambda-1}.
\]  

(2.25)

From (2.25) and the density of \( \Lambda_{2,\lambda-1}^H \omega \) in \( H_1 \), we see that \( (u_t)_{t \in T} \) is a solution of (2.4) (as defined in Definition 2.1) that is a strongly càdlàg \( H_{\lambda-1} \)-valued process whose \( dV_t \otimes d\mathbf{P} \) equivalence class belongs to \( L^2([0, T] \times \Omega, \mathbf{O}_T, dV_t \otimes d\mathbf{P}; H_1) \). Let \((v_t)_{t \in T}\) be another solution of (2.4). For each \( (t, \omega) \in [0, T] \times \Omega \), let \( \phi_t = e^{-LW_t} \). Proceeding as in Lemma 2.3, by Theorem 2 in [6] and Assumptions 2.1(i), \( \mathbf{P} \)-a.s. for all \( t \in [0, T] \), we have

\[
0 \leq |\phi_t (u_t - v_t)|_0^2 \leq m_t,
\]

where \( (m_t)_{t \in T} \) is a local martingale with \( m_0 = 0 \). Since \( (m_t)_{t \in T} \) is a positive local martingale, \( \mathbf{E} m_t = 0 \) for all \( t \in [0, T] \), which implies that

\[
\mathbf{P} \left( \sup_{t \leq T} |u_t - v_t|_0 > 0 \right) = 0.
\]

To complete the proof of this theorem, we only need to show that the estimate (2.7) holds. By the Banach-Saks-Mazur theorem, there is a sequence \((\nu^{(k)})_{k \in \mathbb{N}}\) such that for each \( k \in \mathbb{N} \), \( \nu^k \) is a finite convex combination of \((u^{(m_k)})_{k \in \mathbb{N}}\), and \( \nu^{(k)} \to \bar{u} \) strongly in \( L^2(\Omega \times [0, T], d\bar{V}_t \otimes d\mathbf{P}; H_1) \). From the sequence \((\nu^{(k)})_{k \in \mathbb{N}}\), we can extract a further subsequence \((\nu^{(k_h)})_{h \in \mathbb{N}}\) that converges strongly in \( L^2(\Omega, \mathcal{F}, d\mathbf{P}; H_1) \) to \( u \) for \( dt \)-almost-all \( t \in [0, T] \). Owing to Lemma 2.3, there is a constant \( C \) independent of \( n \) such that

\[
\sup_{t \leq T} \left( \mathbf{E} [\nu^{(k_h)}_t]_{\lambda}^2 \right)^{1/2} \leq C \left( \mathbf{E} [\varphi^k_1]^2 + \mathbf{E} \kappa^2_1(T) \right)^{1/2}.
\]  

(2.26)

Obviously, there is a countable dense subset \( Q \subseteq [0, T] \) such that

\[
\sup_{t \in Q} \mathbf{E} [\nu^{(k_h)}_t]_{\lambda}^2 \leq C \left( \mathbf{E} [\varphi^k_1]^2 + \mathbf{E} \kappa^2_1(T) \right).
\]  

(2.27)
2.2 Proof of Theorems 2.1 and 2.2

Let $t_0 \in [0, T]$ and $(t_n)_{n \in \mathbb{N}} \subseteq Q$ be such that $t_n > t_0$ for all $n \in \mathbb{N}$ and $t_n \to t_0$ as $n \to \infty$. The estimate (2.27) implies that there is a subsequence $(t_{n_k})_{k \in \mathbb{N}}$ and $w \in L^2(\Omega, \mathcal{F}, d\mathbb{P}; H_\lambda)$ such that $u_{n_k} \rightharpoonup w$ weakly in $L^2(\Omega, \mathcal{F}, d\mathbb{P}; H_\lambda)$. Since $(u_t)_{t \in T}$ is strongly càdlàg in $H_{\lambda-1}$ and agrees with $\bar{u}$ a.s. for all $t \in Q$, for any $F \in L^2(\Omega, \mathcal{F}, d\mathbb{P}; H_{\lambda-1})$ we have

$$
\mathbb{E}(\Lambda^2 F, u_{t_0})_{|_{\lambda-1}} = \lim_{k \to \infty} \mathbb{E}(\Lambda^2 F, u_{n_k})_{|_{\lambda-1}} = \lim_{k \to \infty} \mathbb{E}(F, u_{n_k})_{|_{\lambda}} = \mathbb{E}(F, w)_{|_{\lambda}} = \mathbb{E}(\Lambda^2 F, w)_{|_{\lambda-1}}. \tag{2.28}
$$

Since $\Lambda^2 H_{\lambda+1}$ is dense in $H_{\lambda-1}$ and $F$ was arbitrarily chosen in (2.28), it follows that $u_{n_k}$ converges to $u_0$ weakly in $L^2(\Omega, \mathcal{F}, d\mathbb{P}; H_\lambda)$. Therefore, by (2.27),

$$
\sup_{t \leq T} \mathbb{E}[|u_t|_{|_{\lambda-1}}^2] \leq C \left( \mathbb{E}[\phi_t^2] + \mathbb{E}\kappa_t^2(T) \right),
$$

and hence we have proved that there is a constant $C = C(L, K, N_T)$ such that

$$
\sup_{t \leq T} \mathbb{E}[|u_t|_{|_{\lambda-1}}^2] + \mathbb{E} \int_{[0, T]} |\bar{u}_t|_{|_{\lambda-1}}^2 dV_t \leq C \left( \mathbb{E}[\phi_T^2] + \mathbb{E}\kappa_T^2(T) \right). \tag{2.29}
$$

In the following lemma, we derive an estimate of $\mathbb{E} \sup_{t \leq T} |u_t|_{|_{\lambda-1}}^2$.

**Lemma 2.4.** There is a constant $C = C(L, K, N_T)$ such that

$$
\mathbb{E} \sup_{t \leq T} |u_t|_{|_{\lambda-1}}^2 \leq C \kappa_T^2(T). \tag{2.30}
$$

**Proof.** Applying Theorem 2 in [6], $\mathbb{P}$-a.s. for all $t \in [0, T]$, we obtain

$$
|u_t|_{|_{\lambda-1}}^2 = |\phi_t|_{|_{\lambda-1}}^2 + \sum_{i=1}^5 I_i(t),
$$

where

$$
I_1(t) := 2 \int_{[0, t]} [u_s, (L_s \bar{u}_s + f_s)]_{|_{\lambda-1}} dV_s,
$$

$$
I_2(t) := 2 \int_{[0, t]} \mathcal{R}_{t-} \bar{u}_t (M_t \bar{u}_s + g_s) dM_s, \quad I_3(t) := \left[ \int_{[0, t]} (M_t \bar{u}_s + g_s) dM_s \right]_t
$$

$$
I_4(t) := 2 \int_{[0, t]} \int_Z (u_{s-}, I_{sZ} \bar{u}_{s-} + h_s(z)) q(ds, dz), \quad I_5(t) := \left[ \int_{[0, t]} \int_Z (I_{sZ} \bar{u}_{s-} + h_s(z)) q(ds, dz) \right]_t.
$$

Let $(\tau_k)_{k \in \mathbb{N}}$ be the sequence of bounded $\mathbb{P}$-stopping times defined by

$$
\tau_k = \inf \{ t \in [0, T] : |u_t|_{|_{\lambda}} \geq k \} \wedge T.
$$

By Assumption 2.1(ii) and the fact that $f \in H_{\lambda-1}$, we have

$$
\mathbb{E} \sup_{t \leq T \wedge \tau_k} I_1(t) \leq \mathbb{E} \int_{[0, T]} \left( 2 |u_t|_{|_{\lambda-1}}^2 + K |\bar{u}_t|_{|_{\lambda-1}}^2 + |f_t|_{|_{\lambda-1}}^2 \right) dV_s.
$$
Using the Burkholder-Davis-Gundy inequality, Assumption 2.1(ii), and Young’s inequality, we get
\[
\mathbb{E} \sup_{t \leq T \land T_k} I_2(t) \leq 6\mathbb{E} \left( \int_{[0,T]} |\mathcal{R}_{t-1} \tilde{u}_s(M_s \tilde{u}_s + g_s)|_L^2 dV_s \right)^{1/2} \\
\leq \frac{1}{4} \mathbb{E} \left[ \sup_{t \leq T \land T_k} |u_t|_{\lambda-1}^2 \right] + 36 \mathbb{E} \int_{[0,T]} (K|\tilde{u}_s|_{\lambda-1}^2 + \|g_s\|_{L^2}^2) dV_s.
\]
Since the quadratic variation of a locally square integrable martingale is increasing, by Assumption 2.1(ii) and Young’s inequality, we obtain
\[
\mathbb{E} \sup_{t \leq T} I_5(t) = \mathbb{E} \left[ \int_{[0,T]} (M_s \tilde{u}_s + g_s) dM_s \right] = \mathbb{E} \left( \int_{[0,T]} (M_s \tilde{u}_s + g_s) dM_s \right)_T \\
= \mathbb{E} \int_{[0,T]} \|M_s \tilde{u}_s + g_s\|_{L^2}^2 dV_s \leq 2\mathbb{E} \int_{[0,T]} (K|\tilde{u}_s|_{\lambda-1}^2 + \|g_s\|_{L^2}^2) dV_s.
\]
In a similar way, we derive
\[
\mathbb{E} \sup_{t \leq T} I_4(t) \leq \frac{1}{4} \mathbb{E} \left[ \sup_{t \leq T \land T_k} |u_t|_{\lambda-1}^2 \right] + 36 \mathbb{E} \int_{[0,T]} (K|\tilde{u}_s|_{\lambda-1}^2 + \int_{Z} |h_s(z)|_{\lambda-1}^2 \pi_s(dz)) dV_s
\]
and
\[
\mathbb{E} \sup_{t \leq T} I_5(t) \leq 2\mathbb{E} \int_{[0,T]} (K|\tilde{u}_s|_{\lambda}^2 + \int_{Z} |h_s(z)|_{\lambda-1} \pi_s(dz)) dV_s.
\]
Combining the above estimates and (2.29), we see that there is a constant \( C = C(L, K, N_T) \) such that
\[
\mathbb{E} \left[ \sup_{t \leq T \land T_k} |u_t|_{\lambda-1}^2 \right] \leq C \left( \mathbb{E} |\varphi|_{\lambda-1}^2 + \kappa_{\lambda} (T) \right).
\] (2.31)
Therefore, passing to the limit as \( k \to \infty \) in (2.31) and applying the monotone convergence theorem and Fatou’s lemma, and the fact that \( (u_t)_{t \leq T} \) is a strongly càdlàg \( H_{\lambda-1} \)-valued processes, we obtain (2.30). This completes the proof the lemma and the proof of Theorem 2.1. □

**Proof of Theorem 2.2.** Fix \( \lambda \geq 1 \). It is clear that if Assumption 2.3 holds, then Assumption 2.1 holds. Thus, Theorem 2.1 applies and all statements in the proof of Theorem 2.2 obviously hold. We adopt the same notation and refer to processes introduced in the proof of Theorem 2.1 when convenient. Therefore, all that remains to be shown is that (3.35) holds and that \( u \) is a weakly càdlàg \( H_{\lambda} \)-valued process. Under Assumption (2.3), we can obtain a stronger estimate than the one obtained in Lemma 2.3.

**Lemma 2.5.** There is a constant \( C = C(L, K, N_T) \) such that for all \( n \in \mathbb{N} \),
\[
\mathbb{E} \sup_{t \leq T} |u_t^{(n)}|_{\lambda}^2 \leq C \left( \mathbb{E} |\varphi|_{\lambda}^2 + \kappa_{\lambda}^2 (T) \right).
\] (2.32)
Proof of Lemma 2.5. Applying Theorem 2 in [6] to (2.12), P-a.s. for all \( t \in [0, T] \), we have
\[
|\mu_t^{(n)}|_1^2 = \mathcal{Q}_{\lambda} + I_2(t) + I_3(t) + I_4(t) + m_t
\]
where
\[
I_1(t) := \int_{[0, t]} \left( 2 \left[ \tilde{\mu}_s^{(n)}, \mathcal{M}_s \tilde{\mu}_s^{(n)} \right]_1 + \| \mathcal{M}_s \tilde{\mu}_s^{(n)} \|_{1, Q_t}^2 - \frac{2}{n} |\lambda_t^{(n)}|^2 \right) dV_s,
\]
\[
I_2(t) := \int_{[0, t]} \left( 2 [\tilde{\mu}_s^{(n)}, \tilde{\mu}_s^{(n)}]_1 + I_s, z \tilde{\mu}_s^{(n)}]_1 \right) N(ds, dz),
\]
\[
I_3(t) := \int_{[0, t]} \left( 2 [\tilde{\mu}_s^{(n)}, f_s]_1 + (M_s \tilde{\mu}_s^{(n)}, g_s) \right) dV_s + \int_{[0, t]} \int_Z (I_s, z \tilde{\mu}_s^{(n)} + h_s(z))_1 N(ds, dz),
\]
\[
I_4(t) := \int_{[0, t]} \| g_s \|_{1, Q_t}^2 dV_s + \int_{[0, t]} \int_Z |h_s(z)|_1^2 N(ds, dz),
\]
and \( m_t := 2 \int_{[0, t]} \mathcal{M}_s \tilde{\mu}_s^{(n)} + g_s) dM_s + 2 \int_{[0, t]} \int_Z [\tilde{\mu}_s^{(n)}, (I_s, z - J_s, z) \tilde{\mu}_s^{(n)} + h_s(z)]_1 N(ds, dz),
\]
and \( \tilde{\mu}^{(n)} \) is any \( H_{t+1} \)-valued \( \mathcal{F}_t \)-measurable \( dV_t \otimes d\mathcal{P} \) version of \( \mu^{(n)} \). Owing to Assumption (2.3)(i), for all \( (t, \omega) \in [0, T] \times \Omega \), we have
\[
I_1(t) + I_2(t) \leq L \int_{[0, t]} |\tilde{\mu}_s^{(n)}|^2 dV_s.
\]
By Assumption (2.3)(iii), for \( dV_t \otimes d\mathcal{P} \)-almost all \( (t, \omega) \in [0, T] \times \Omega \),
\[
|\mathcal{R}_3 \tilde{\mu}_t^{(n)} (\mathcal{M}_t \tilde{\mu}_t^{(n)} + \tilde{g}_t)|^2_{L_t(Q_t, \mathcal{R})} \leq 2K|\tilde{\mu}_t^{(n)}|_1^4 + 2|\tilde{\mu}_t^{(n)}|_1^2 \| g_t \|^2_{1, Q_t},
\]
and
\[
\int_Z [\tilde{\mu}_t^{(n)}, (I_t, z - J_t, z) \tilde{\mu}_t^{(n)} + h_t(z)]_1^2 \pi_t(dz) \leq 2K|\mu_t^{(n)}|_1^4 + 2|\tilde{\mu}_t^{(n)}|_1^2 \int_Z |h_t(z)|_1^2 \pi_t(dz).
\]
Thus, P-a.s. for all \( t \in [0, T] \) we have
\[
d\langle m \rangle_t \leq \left( 8K|\tilde{\mu}_t^{(n)}|_1^4 + 4|\tilde{\mu}_t^{(n)}|_1^2 \| g_t \|^2_{1, Q_t} + \int_Z |h_t(z)|_1^2 \pi_t(dz) \right) dV_t,
\]
and hence by the Burkholder-Davis-Gundy inequality and Young’s inequality, there is a constant \( C = C(K) \) such that for any stopping time \( \tau \leq T \),
\[
\mathbb{E} \sup_{t \leq \tau} m_s \leq 3 \mathbb{E} \langle m \rangle_t^{1/2} \leq \mathbb{E} \sup_{t \leq \tau} |\mu_t|_1 \left( \int_{[0, t]} \left( 8K|\mu_t|_1^4 + 4 \| g_t \|^2_{1, Q_t} + \int_Z |h_t(z)|_1^2 \pi_t(dz) \right) dV_t \right)^{1/2}
\]
\[
\leq \frac{1}{2} \mathbb{E} \sup_{t \leq \tau} |\mu_t|_1^2 + CE \int_{[0, t]} |\mu_t|_1^2 + \| g_t \|^2_{1, Q_t} + \int_Z |h_t(z)|_1^2 \pi_t(dz) dV_t.
\]
Let \( y_t = |\mu_t^{(n)}|_1^2, t \in [0, T] \). Then estimating as in Lemma 2.3 and combining the above estimates, we get that there is a constant \( C = C(L, K) \) such that for any stopping time \( \tau \leq T \),
\[
\mathbb{E} \sup_{t \leq \tau} y_t \leq 2 \mathbb{E} |\phi|_1^2 + CE \int_{[0, \tau]} y_t dV_t + CE^2(\tau).
\]
Therefore, by Lemma 2 in [3], we obtain (2.32), which completes the proof of this lemma. \( \square \)
We claim that
\[ \text{ESup}_{t \in T} \{ |u^k_t|^2 \} \leq C \left( \mathbb{E}[\varphi]_T + \mathbb{E}[\varphi^2](T) \right), \] (3.33)

From the sequence \((v^{(h_n)})_{n \in \mathbb{N}}\) (see Proof of Theorem 2.1), we can extract a further subsequence, denoted again as \((v^{(h_n)})_{n \in \mathbb{N}}\), that converges strongly in \(H^1\) to \(u\) for \(dt \otimes d\mathbb{P}\)-almost-all \((t, \omega) \in [0, T] \times \Omega\). Thus, there exists a countable dense subset \(Q \subseteq [0, T]\) such that \((v^{(h_n)})_{n \in \mathbb{N}}\) converges strongly in \(H^1\) to \(u\), \(\mathbb{P}\)-a.s. for all \(t \in Q\). Since \((u_t)_{t \in T}\) is strongly càdlàg in \(H^1\) and \(\Lambda^2 H_{1+1}\) is dense in \(H^1\), we have

\[
\begin{align*}
\sup_{t \in T} |u^1_t| &= \sup_{t \in T} \sup_{|\alpha| = 1} \left| \left( \Lambda^2 v, u^1_t \right)_{1-1} \right| = \sup_{t \in T} \sup_{|\alpha| = 1} \left| \left( \Lambda^2 v, u^1_t \right)_{1-1} \right|
\leq \sup_{t \in T} \sup_{|\alpha| = 1} \lim inf_{n \to \infty} \left| \left( v, u^{(h_n)}_t \right)_{1} \right| \\
&\leq \lim inf_{n \to \infty} \sup_{t \in T} |u^{(h_n)}_t| = \lim inf_{n \to \infty} \sup_{t \in T} |u^{(h_n)}_t| \
\end{align*}
\]

and hence (3.33) holds. It then follows immediately that \(u\) is a weakly càdlàg \(H^1\)-valued process. This completes the proof of the Theorem 2.2. □

3 Degenerate Linear Stochastic Integro-Differential Equations

3.1 Notation and Formulation of Result

For each \(i \in \{1, \ldots, d_1\}\), let \(\partial_i = \partial_i x_i\) and \(\partial_0\) be the identity operator. Moreover, for a multi-index \(\gamma = (\gamma_1, \ldots, \gamma_{d_1}) \in (\mathbb{N} \cup \{0\})^{d_1}\) of length \(|\gamma| = \gamma_1 + \cdots + \gamma_{d_1}\), let \(\partial^\gamma = \partial_1^{\gamma_1} \cdots \partial_{d_1}^{\gamma_{d_1}}\), where we use the convention \(\partial_i^0 = \partial_0\), for each \(i \in \{1, \ldots, d_1\}\). Let \(\Delta := \sum_{i=1}^{d_1} \partial_i^2\) be the Laplace operator on \(\mathbb{R}^{d_1}\).

It is well known that \(\Lambda^2 := I - \Delta\) is a positive self-adjoint unbounded operator on the Hilbert space \(H_0^0(\mathbb{R}^{d_1}; \mathbb{R}^{d_1}) := L^2(\mathbb{R}^{d_1}; \mathbb{R}^{d_1})\), for each integer \(d_1 \geq 1\). For each integer \(d_1 \geq 1\), we define the Hilbert space \((H_0^0(\mathbb{R}^{d_1}; \mathbb{R}^{d_1}))(\gamma)_{\alpha, \theta} \in \mathbb{R}\) associated with \(\Lambda := \sqrt{\Lambda^2}\) as in Section 2.1, and write \(|\cdot|_{\alpha, \theta}\) and \((\cdot, \cdot)_{\alpha, \theta}\) for the corresponding norms and inner products. It is well-known that \(C_c^{\infty}(\mathbb{R}^{d_1}; \mathbb{R}^{d_1})\) is dense in \(H^m\) for even integer \(m\), where \(C_c^{\infty}(\mathbb{R}^{d_1}; \mathbb{R}^{d_1})\) is the space of infinitely differentiable \(\mathbb{R}^{d_1}\)-valued functions defined on \(\mathbb{R}^{d_1}\) with compact support. We set \(H^m = H^m(\mathbb{R}^{d_1}; \mathbb{R}^{d_1})\), \(C_c^{\infty} = C_c^{\infty}(\mathbb{R}^{d_1}; \mathbb{R}^{d_1})\), \(|\cdot|_\theta = |\cdot|_{\alpha, \theta}\), \((\cdot, \cdot)_\theta = (\cdot, \cdot)_{\alpha, \theta}\), and denote by \([\cdot, \cdot]_\theta\) the CBF of the normal triple \((H^{m+1}, H^m, H^{m-1})\).

Consider the system of stochastic integro-differential equation on \([0, T] \times \mathbb{R}^{d_1}\) for \(u_t = u_t(x) = (u^k_t(x))_{1 \leq k \leq d_2}\) given by

\[
du_t(x) = (L_t u_t(x) + f_t(x)) \, \text{d}V_t + \int_\mathcal{Z} (I_{x,z} u_t(x) + h_t(x,z)) \, \text{d}\pi_t(z), \quad t \in [0, T],
\]

\[
u(0, x) = \varphi(x),
\]

where for \(\psi \in C_c^{\infty}\),

\[
L^k_x \psi(x) := \int_\mathcal{Z} \left( \left( \delta_{kl} + \rho_{lk}^h(x, z) \right) \left[ \psi^l(x + H_l(x,z)) - \psi^l(x) \right] - H_l^i(x,z) \partial_i \psi(x) \right) \pi_i(dz)
\]

\[+ b_{kl}^i(x) \partial_i \psi^l(x) + c_{kl}^i(x) \psi^l(x) \]

\[I^k_{x,z} \psi(x) := \left( \delta_{kl} + \rho_{lk}^h(x, z) \right) \psi^l(x + H_l(x,z)) - \psi^l(x), \quad k \in \{1, \ldots, d_2\}.
\]
3.1 Notation and Formulation of Result

We assume the random fields
\[ c_l(x) = \left( c_{l,i}^j(x) \right)_{1 \leq i, j \leq d_l}, \quad b_l(x) = \left( b_{l,i}^j(x) \right)_{1 \leq i, j \leq d_l}, \quad f_l(x) = \left( f_{l,i}^j(x) \right)_{1 \leq i, j \leq d_l}, \]
on \([0, T] \times \Omega \times \mathbb{R}^{d_1}\) are \( \mathcal{F}_T \otimes \mathcal{B}(\mathbb{R}^{d_1})\)-measurable and the random fields
\[ H_l(x, z) = \left( H_{l,i}^j(x, z) \right)_{1 \leq i, j \leq d_l}, \quad \rho_l(x, z) = \left( \rho_{l,i}^j(x, z) \right)_{1 \leq i, j \leq d_l}, \]
on \([0, T] \times \Omega \times \mathbb{R}^{d_1} \times \mathbb{Z}\) are \( \mathcal{F}_T \otimes \mathcal{B}(\mathbb{R}^{d_1}) \otimes \mathcal{Z}\)-measurable.

We will need the following assumption on the jump function \( H_l(x, z) \).

**Assumption 3.1.** For all \((t, \omega, x, y, z) \in [0, T] \times \Omega \times \mathbb{R}^{d_1} \times Z\),
\[ |H_l(x, z)| + |\nabla H_l(x, z)| \leq \alpha_l(z) \]
and
\[ |H_l(x, z) - H_l(x, y)| \leq \beta_l(z)|x - y|, \]
where \( \alpha, \beta : [0, T] \times \Omega \times Z \to \mathbb{R}_+ \) are \( \mathcal{F}_T \otimes \mathcal{Z}\)-measurable processes such that for all \((t, \omega) \in [0, T] \times \Omega\),
\[ \alpha_l(z) + \beta_l(z) + \int_Z \left( \alpha_l(z)^2 + \beta_l(z)^2 \right) \pi_l(dz) \leq C_0, \]
and \( C_0 \) is a positive constant. Further, we assume that there is a constant \( \eta < 1 \) such that for all \((t, \omega, z) \in (t, \omega, z) \in [0, T] \times \Omega \times Z : \beta_l(\omega, z) > \eta\), \( K_l(x, \theta, z) = (I_{d_l} + \theta \nabla H_l(x, z))^{-1} \) exists and
\[ |K_l(x, \theta, z)| \leq C_0. \]

**Remark 3.1.** It follows from Lemma 6.8 in [13] that for all \((t, \omega, \theta, z) \in [0, T] \times \Omega \times [0, 1] \times Z\), the mapping \( \tilde{H}_{t,\theta}(x, z) := x + \theta H_l(x, z) \) is a global diffeomorphism and for all \((t, \omega, x, \theta, \zeta) \in [0, T] \times \Omega \times \mathbb{R}^{d_1} \times [0, 1] \times Z\),
\[ \tilde{H}_{t,\theta}^{-1}(x, z) = x - \theta H_l(\tilde{H}_{t,\theta}^{-1}(x, z), z). \]
Moreover, there is a constant \( C = C(C_0) \) such that for all \((t, \omega, \theta, x, z) \in [0, T] \times \Omega \times [0, 1] \times \mathbb{R}^{d_1} \times Z\),
\[ |\det \nabla \tilde{H}_{t,\theta}^{-1}(x, z)| \leq C, \]
\[ |\det \nabla \tilde{H}_{t}^{-1}(x, z) - 1| \leq C|H_l(x, z)|, \]
and
\[ |\det \nabla \tilde{H}_{t}^{-1}(x, z) - 1 + \text{div} H_l(x, z)| \leq C|H_l(x, z)|^2. \]

Here, \( \det \cdot \) is the usual determinant operator on \( d_1 \times d_1\)-matrices.

We will also need the following assumptions for integers \( m \geq 1 \).

**Assumption 3.2 (m).** For all \((t, \omega, x) \in [0, T] \times \Omega \times \mathbb{R}^{d_1}\) and multi-indices \( \gamma \) with \(|\gamma| \leq m\),
\[ |\partial^\gamma b_l(x)| + |\partial^\gamma c_l(x)| \leq C_0 \]
and for all \((t, \omega, x, z) \in [0, T] \times \Omega \times \mathbb{R}^{d_1} \times Z\) and multi-indices \( \gamma \) with \(|\gamma| \leq m + 1\),
\[ |\partial^\gamma H_l(x, z)| + |\partial^\gamma \rho_l(x, z)| \leq \alpha_l(z). \]
Moreover, we assume that
\[ \text{Assumption 3.3 (m). For all } (t, \omega) \in [0, T] \times \Omega, \text{ the free terms } f_t \text{ and } h_t \text{ take values in } H^m \text{ and } L^2(Z, \mathbb{Z}, \pi_t(dz), H^{m+1}), \text{ respectively, and the initial value } \varphi \text{ is a } \mathcal{F}_0\text{-valued } H^m\text{-valued random variable. Moreover, we assume that} \]

\[ \mathbb{E}|\varphi|^2_m + \mathbb{E}^2_m(T) < \infty, \]

where
\[ \kappa^2_m(T) := \int_{[0,T]} \left( |f_t|^2_m + \int_Z |h_t(z)|^2 \pi_t(dz) \right) dV_t. \]

For \( \psi \in C^\infty_c \), let
\[ \Omega^{1,k}_t \psi(x) = -\int_0^1 \int_Z \left[ \psi^k(x + \theta H_t(x,z)) - \psi^k(x) \right] \partial_t H^k_t(x,z) \pi(dz) d\theta \]
\[ -\int_0^1 \theta \int_Z \partial_x \psi^k(x + \theta H_t(x,z)) \partial_t H^k_t(x,z) H^k_t(x,z) \pi(dz) d\theta \]
\[ + \int_Z \rho^H_t(x,z)(\psi^k(x + H_t(x,z)) - \psi^k(x)) \pi(dz) + b^H_t(x) \partial_x \psi^k(x) + c^H_t(x) \psi^k(x) \]
\[ \Omega^{2,k,i}_t \psi(x) = -\int_0^1 \int_Z \left[ \psi^k(x + \theta H_t(x,z)) - \psi^k(x) \right] H^k_t(x,z) \pi(dz) d\theta, \quad i \in \{1, 2, \ldots, d_1\}. \]

**Definition 3.1.** An \( H^0 \)-valued \( \mathbb{F} \)-adapted strongly càdlàg processes \( (u_t)_{t \leq T} \) is said to be a solution of the stochastic integro-differential equation (3.34) on \([0, T]\) if its \( dV_t \otimes d\mathbb{P} \) equivalence class belongs to \( L^2([0, T] \times \Omega, \mathcal{O}_T, dV_t \otimes d\mathbb{P}; H^1) \), and if there exists a set \( \bar{\Omega} \subseteq \Omega \) with \( \mathbb{P}(\bar{\Omega}) = 1 \) such that for all \((t, \omega) \in [0, T] \times \bar{\Omega} \) and \( \psi \in C^\infty_c \),

\[ (v, u_t)_0 = (v, \varphi)_0 + \int_{[0,t]} \left( (v, \Omega^1_s \dot{u}_s + f_s)_0 + (\partial_t v, \Omega^{2,1}_s \dot{u}_s)_0 + (\partial_t v, \Omega^{2,1}_s \dot{u}_s)_0 \right) dV_s \]
\[ + \int_{[0,t]} \int_Z (v, I_{s,z} \tilde{u}_s - h_s(z))_0 \mathcal{N}(ds, dz), \]

where \( \tilde{u} \) is any \( H_1 \)-valued \( \mathcal{O}_T \)-measurable \( dV_t \otimes d\mathbb{P} \) version of \( u \). We say a solution of (2.4) on \([0, T]\) is unique if for any two solutions \((u_t)_{t \leq T}\) and \((v_t)_{t \leq T}\) of (3.34) on \([0, T]\), we have
\[ \mathbb{P} \left( \sup_{t \leq T} |u_t - v_t| > 0 \right) = 0. \]

We will prove the following existence and uniqueness result using Theorem 2.1.

**Theorem 3.1.** Assume that there is a positive constant \( N_T \) such that \( V_t \leq N_T \) for all \((t, \omega) \in [0, T] \times \Omega \) and that Assumptions 3.1, 3.2(m), and 3.3(m) hold for some integer \( m \geq 1 \). Then there exist a unique solution \((u_t)_{t \leq T}\) of (2.4) that is a weakly càdlàg \( H^m \)-valued process and a strongly càdlàg \( H^{m-1} \)-valued process. Moreover, the \( dV_t \otimes d\mathbb{P} \) equivalence class of \( u \) belongs to \( L^2([0, T] \times \Omega, \mathcal{O}_T, dV_t \otimes d\mathbb{P}; H^m) \) and there is a constant \( C = C(L, K, N_T) \) such that
\[ \mathbb{E} \sup_{t \leq T} |u_t|^2_m + \mathbb{E} \int_{[0,T]} |u_t|^2_m dV_t \leq C \left( \mathbb{E}|\varphi|^2_m + \mathbb{E}^2_m(T) \right). \]
3.2 Proof of Theorem 3.1

Proof of Theorem 3.1. We will begin the proof by formulating (3.34) as a stochastic evolution equation of type (2.4) so that Theorem 2.1 may be applicable. Fix \((t, \omega) \in [0, T] \times \Omega\) and let \(C = C(C_0, d_1, d_2)\) be a constant independent of \(t\) and \(\omega\). By the Cauchy-Schwartz inequality, Hölder’s inequality, and Assumption 3.2(1), for all \(\phi, \psi \in C^\infty_c\), we have

\[
| - (\phi, \tilde{L}_t^1 \psi) | \leq C |\phi|_0 \left( \int_{\mathbb{R}^d} \int_0^1 \int_Z |\psi(x + \theta H_t(x, z)) - \psi(x)|^2 \pi_t(dz) d\theta dx \right)^{1/2} \\
+ C |\phi|_0 \left( \int_{\mathbb{R}^d} \int_0^1 \int_Z |\nabla \psi(x + \theta H_t(x, z))|^2 \pi_t(dz) d\theta dx \right)^{1/2} \\
+ C |\phi|_0 \left( \int_{\mathbb{R}^d} \int_Z |\psi(x + H_t(x, z)) - \psi(x)|^2 \pi_t(dz) d\theta dx \right)^{1/2} + C |\phi|_0 |\phi|_1
\]

and

\[
- \sum_{i=1}^{d_1} (\partial_i \phi, \tilde{L}_t^1 \psi) | \leq C |\phi|_0 \left( \int_{\mathbb{R}^d} \int_Z |\psi(x + \theta H_t(x, z)) - \psi(x)|^2 \pi_t(dz) d\theta dx \right)^{1/2} .
\]

Applying the change of variable formula and Assumption 3.1, we see that for each \(\psi \in C^\infty_c\),

\[
\int_{\mathbb{R}^d} \int_0^1 \int_Z \nabla \psi(x + \theta H_t(x, z)) |\nabla \pi_t(dz) |^2 d\theta dx \\
\leq \int_0^1 \theta \int_\mathbb{R}^d \nabla \pi_t(z)^2 \int_{\mathbb{R}^d} |\nabla \psi(x)|^2 dx \det \nabla \pi_t^{-1}(x, z) dx \pi_t(dz) d\theta 
\]

By Taylor’s formula, the change of variable formula, and Assumption 3.1, for all \(\psi \in C^\infty_c\), we obtain

\[
\int_{\mathbb{R}^d} \int_0^1 \int_Z |\psi(x + \theta H_t(x, z)) - \psi(x)|^2 \pi_t(dz) d\theta dx \\
\leq \int_0^1 \theta^2 \int_\mathbb{R}^d \int_0^1 \int_Z |\nabla \psi(x + \theta H_t(x, z))|^2 dx |H_t(x, z)|^2 dx \pi_t(dz) d\theta dx \\
\leq \int_{\mathbb{R}^d} |\nabla \psi(x)|^2 \int_0^1 \theta^2 \int_\mathbb{R}^d \int_0^1 \nabla \pi_t(z)^2 dx \det \nabla \pi_t^{-1}(x, z) dx \pi_t(dz) d\theta dx 
\]

Similarly, we derive

\[
\int_{\mathbb{R}^d} \int_Z |\psi(x + H_t(x, z)) - \psi(x)|^2 \pi_t(dz) dx \leq C |\psi|_1^2. \tag{3.36}
\]

Combining the above estimates, we get that for all \(\phi, \psi \in C^\infty_c\),

\[
| - (\phi, \tilde{L}_t^1 \psi) - \sum_{i=1}^{d_1} (\partial_i \phi, \tilde{L}_t^1 \psi) | | \leq C |\psi|_1 |\phi|_1,
\]

which implies there exists a unique linear operator \(\tilde{L}_t : H^1 \rightarrow H^{-1}\) such that for each \(w, v \in H^1\),

\[
[w, \tilde{L}_t u]_0 = -(w, \tilde{L}_t^1 u)_0 - \sum_{i=1}^{d_1} (\partial_i w, \tilde{L}_t^2 v)_0 \tag{3.37}
\]
and
\[ |\bar{L}_t \nu|_{L_1}^2 \leq C|\nu|_{H_1}^2. \]  
(3.38)

Moreover, \( \bar{L}u \) is \( R_T \)-measurable for all \( u \in H^1 \). Owing to Young’s inequality, the change of variable formula, and Assumptions 3.1 and 3.2(1), for all \( v \in H^1 \), we have
\[
\int_{\mathbb{R}_t} |I_{t,z}^T v_0^T \pi_t (dz) | \leq 2 \int_{\mathbb{R}_t} \int_{\mathbb{R}^d} \left( |v(x + H_t(x,z)) - v(x)|^2 + |\rho_t(x,z)|^2 |v(x + H_t(x,z))|^2 \right) dx \pi_t (dz)
\]
\[
\leq C|\nabla \psi_0^2| + \int_{\mathbb{R}_t} \alpha_t(z)^2 \int_{\mathbb{R}^d} |v(x)|^2 \det \nabla H_t^{-1}(x,z) dx \pi_t (dz) \leq C|\nu|_{H_1}^2.
\]  
(3.39)

where the second to last inequality follows from (3.36). Thus, \( I_t : H^1 \rightarrow L^2(\mathbb{R}, Z, \pi_t (dz); H^0) \) and it is clear that \( I_t \) is \( \mathcal{P}_T \otimes \mathcal{Z} \)-measurable for all \( u \in H^1 \). We can now formulate (3.34) as a stochastic evolution equation of type (2.4):
\[
u_t = \varphi + \int_{[0,t]} (\bar{L}_t u_t + f_t) dt + \int_{[0,t]} (I_{t,z}^T u_t + h_t(z)) \mathcal{N}(dt, dz), \quad t \in [0, T].
\]  
(3.40)

By (3.37) and the fact that \( C_v^\infty \) is dense in \( H^1 \), if \( u \) is a solution as given in Definition 2.1, then \( u \) is a solution as given in Definition 3.1. Applying Minkowski’s integral inequality, the Cauchy-Schwartz inequality, the change of variable formula, and Assumptions 3.1 and 3.2(1), for all \( v \in H^2 \), we obtain
\[
\int_{\mathbb{R}^d} \left( \int_0^1 (1 - \theta) \int_{\mathbb{R}_t} (\delta_t + \rho_t^i(x,z)) \partial_i \partial_j \nu^i(x + \theta H_t(x,z)) H_t^i(x,z) H_t^j(x,z) \pi_t (dz) d\theta \right)^2 dx
\]
\[
\leq \left( \int_0^1 (1 - \theta) \int_{\mathbb{R}^d} (\delta_t + \rho_t^i(x,z)) \partial_i \partial_j \nu^i(x + \theta H_t(x,z)) H_t^i(x,z) H_t^j(x,z) |dx| \right)^{1/2} \pi_t (dz) d\theta \right)^2
\]
\[
\leq \left( \int_0^1 (1 - \theta) \int_{\mathbb{R}^d} (1 + \alpha_t(z)) \alpha_t(z) \left( \int_{\mathbb{R}^d} |\partial_i \partial_j \nu^i(x)|^2 \det \nabla H_t^{-1}(x,z) dx \right)^{1/2} \pi_t (dz) d\theta \right)^2 \leq C|\nu|_{H_2}^2.
\]

Thus, for all \( v \in H^2 \),
\[
|\bar{L}_t \nu|_{L_0}^2 \leq C|\nu|_{H_2}^2.
\]  
(3.41)

Then integrating by parts, we have that for all \( v \in H^2 \) and \( w \in H^1 \),
\[
[w, \bar{L}_t \nu]_0 = (w, \bar{L}_t \nu)_0,
\]  
(3.42)

and hence the restriction of \( \bar{L}_t \) to \( H^2 \) agrees with \( \bar{L}_t \).

For all \( \psi \in C_v^\infty \), we have
\[
\mathcal{L}_t \psi = \mathcal{A}_t \psi + \int_Z J_{t,z} \psi \pi_t (dz)
\]
where
\[
\mathcal{A}_t \psi(x) := b_t^i \partial_i \psi^k(x) + c_t^k(x) \psi^i(x)
\]
and
\[
J_{t,z} \psi(x) := \int_Z \left( (\delta_{kl} + \rho_{kl}^i(x,z)) [\psi^j(x + H_t(x,z)) - \psi^j(x)] - H_t^j(x,z) \partial_l \psi(x) \right) \pi_t (dz).
\]
3.2 Proof of Theorem 3.1

For all \( v \in H^1 \) and \( (t, \omega) \in [0, T] \times \Omega \), we have

\[
\tilde{L}_t v = \bar{A}_t v + \int_Z \tilde{J}_{t,z} v \pi_t(dz) = \bar{A}_t v + \int_Z \tilde{J}_{t,z} v \pi_t(dz),
\]

where \( \tilde{J}_{t,z} \) is defined as the difference of \( \tilde{L}_t \) and \( A_t \). It is also clear that the restriction of \( \tilde{J}_{t,z} \) to \( H^2 \) agrees with \( \tilde{J}_{t,z} \) and \( J_u \) is \( \mathcal{P}_T \otimes \mathcal{Z} \)-measurable for all \( u \in H^1 \).

By the divergence theorem and Assumption (3.2)(1), for all \( v \in H^1 \) and \( (t, \omega) \in [0, T] \times \Omega \), we have

\[
2[\nu, \bar{A}_t v]_0 = 2(\nu, A_t v)_0 = -\langle \nabla \phi H \rangle_0 + 2(\nu, c v)_0 \leq 3N_0 |v|_0^2.
\]

We will now show that there is a constant \( C = C(C_0, d_1, d_2) \) such that for all \( v \in H^1 \) and \( (t, \omega) \in [0, T] \times \Omega \),

\[
\int_Z \left( 2[v, \tilde{J}_{t,z} v]_0 + |I_{t,z} v|^2_0 \right) \pi_t(dz) \leq C|v|_0^2.
\]

By (3.42), in order to show (3.44), it suffices to show that there is a constant \( C = C(C_0, d_1, d_2) \) such that for all \( \psi \in C_c^\infty \) and \( (t, \omega) \in [0, T] \times \Omega \),

\[
\int_Z \left( 2(\psi, \tilde{J}_{t,z} \psi)_0 + |I_{t,z} \psi|^2_0 \right) \pi_t(dz) \leq C|\psi|_0^2.
\]

Fix \( (t, \omega) \in [0, T] \times \Omega \) and an arbitrary \( \psi \in C_c^\infty \) and let \( C = C(C_0, d_1, d_2) \) be a constant independent of \( t \) and \( \omega \) and \( \psi \). A simple calculation to yields the following identities:

\[
|I_{t,z} \psi|^2_0 = |\psi(\tilde{H}_t(z)) - \psi|^2_0 + 2[\psi(\tilde{H}_t(z)) - \psi, \rho_t(z) \psi(\tilde{H}_t(z))_0 + |\rho_t(z) \psi(\tilde{H}_t(z))_0^2,
\]

\[
2(\psi, [\psi(\tilde{H}_t(z)) - \psi]_0 + |\psi(\tilde{H}_t(z)) - \psi|^2_0 = |\psi(\tilde{H}_t(z))|^2_0 - |\psi|^2_0
\]

and

\[
2(\psi, \rho_t(z) [\psi(\tilde{H}_t(z)) - \psi]_0 + 2(|\psi(\tilde{H}_t(z)) - \psi, \rho_t(z) \psi(\tilde{H}_t(z))_0
\]

\[
= 2(\psi(\tilde{H}_t(z)), \rho_t(z) \psi(\tilde{H}_t(z))_0 - 2(\psi, \rho_t(z) \psi(\tilde{H}_t(z))_0
\]

Making use of the identities (3.46), (3.47), and (3.48), the change of variable formula, the divergence theorem, and Assumptions 3.1 and 3.2(1), we obtain

\[
\int_Z \left( 2(\psi, \tilde{J}_{t,z} \psi)_0 + |I_{t,z} \psi|^2_0 \right) \pi_t(dz) = \left( \psi, \int_Z \left[ \det \nabla \tilde{H}_t^{-1}(z) - 1 + \text{div}H_t(z) \right] \pi_t(dz) \psi \right)_0
\]

\[
+ 2 \left( \psi, \int_Z \left[ \rho_t(\tilde{H}_t^{-1}(z), z) \det \nabla \tilde{H}_t^{-1}(z) - \rho_t(z) \pi_t(dz) \psi \right)_0
\]

\[
+ \int_{R^d} |\psi(x)|^2 \int_Z |\rho_t(\tilde{H}_t^{-1}(x, z), z)|^2 \det \nabla \tilde{H}_t^{-1}(x, z) dx \pi_t(dz).
\]

By Taylor’s formula and Assumption 3.1 and 3.2(1), we have

\[
\left| \int_Z \left[ \rho_t(\tilde{H}_t^{-1}(x, z), z) \det \nabla \tilde{H}_t^{-1}(x, z) - \rho_t(x, z) \right] \pi_t(dz) \right|
\]

\[
\leq \int_0^1 \int_Z \left| \nabla \rho_t(x - \theta H_t^{-1}(x, z), z) \right| H_t(\tilde{H}_t^{-1}(x, z), z) \pi_t(dz) d\theta
\]
3.2 Proof of Theorem 3.1

\[
+ \int_Z |\rho_t(\hat{H}_t^{-1}(x,z))| \det \hat{H}_t^{-1}(x,z) - 1|\pi_t(dz) \leq C. \tag{3.50}
\]

Combining the identities (3.46) and (3.49) and applying Hölder’s inequality, (3.50), and Assumptions 3.1 and 3.2(1), we obtain (3.45), and hence (3.44). Applying the divergence theorem, Young’s inequality, and Assumption (3.2(1)), for all \( v \in H^1 \) and \((t, \omega) \in [0, T) \times \Omega\), we get

\[
\int_Z [v,(I_{t,z} - \hat{a}_{t,z})v]_0^2 \pi_t(dz) = \int_Z \left( \frac{1}{2} (v, \partial_i H_i^t(z)v_0 + (v,\rho_t(z)v_0) \right) \pi_t(dz) \leq 3N_0|v|^4.
\]

Thus far, we have shown that under Assumptions 3.1 and 3.2(1), there are constants \( L = L(C_0, d_1, d_2) \) and \( K = K(C_0, d_1, d_2) \) such that for all \( v \in H^1 \) and \((t, \omega) \in [0, T) \times \Omega\),

\[
2[v, \mathcal{A} v]_0 \leq \frac{L}{2}|v|^2_0, \quad \int_Z [2[v, \mathcal{J}_{t,z}]_0 + |I_{t,z}v|^2_0] \pi_t(dz) \leq \frac{L}{2}|v|^2_0,
\]

\[
|\mathcal{L}_v|^2_{-1} \leq K|v|^2_1, \quad |\mathcal{L}_v|^2_{0} \leq K|v|^2_2, \quad \int_Z |I_{t,z}v|^2_0 \pi_t(dz) \leq K|v|^2_1,
\]

and

\[
\int_Z [v,(I_{t,z} - \hat{a}_{t,z})v]_0^2 \pi_t(dz) \leq K|v|^4.
\]

Furthermore, if we increase the dimension of the state space and modified the dimensions of \( c \) and \( \rho \) accordingly, then these estimates will still hold under the same assumptions— albeit with a different constants \( K \) and \( L \), since, in general, the constant depends on the dimension of the state space. We will prove the following claim for \( m \geq 0 \).

**Claim 3.2 (m).** Under Assumptions 3.1 and 3.2((m \lor 1)), there exist constants \( L = L(m, C_0, d_1, d_2) \) and \( K = K(m, C_0, d_1, d_2) \) such that the following conditions hold for all \((t, \omega) \in [0, T) \times \Omega\):

(i) for \( \alpha \in [0, m) \) and all \( v \in H^{\alpha+1} \),

\[
2[v, \mathcal{L}_v]_\alpha + \int_Z |I_{t,z}v|^2_0 \pi_t(dz) \leq L|v|^2_0;
\]

(ii) for \( \alpha \in (0, (m-1) \lor 0, m) \) all \( v \in H^{\alpha+1} \),

\[
|\mathcal{L}_v|^2_{m-1} \leq K|v|^2_{m+1}, \quad \text{and} \quad \int_Z |I_{t,z}v|^2_0 \pi_t(dz) \leq K|v|^2_{m+1};
\]

(iii)

\[
\int_Z [v,(I_{t,z} - \hat{a}_{t,z})v]_m^2 \pi_t(dz) \leq K|v|^4_m.
\]

**Proof.** Obviously, we only need to derive the estimates in the claim for \( \alpha = m \), since Assumption 3.2(m) implies Assumption 3.2(m \lor 1). We will prove this claim by induction. First, let us introduce some notation. For an integer \( m \geq 1 \) and \( v \in H^m \), let

\[
\tilde{v} = \begin{bmatrix} v \\ \nabla v \end{bmatrix} \in H^{m-1}((0, T) \times \Omega) 
\]
and denote $\tilde{v}^{(k,r)} = \partial_r v^k$, for $k \in \{1, \ldots, d_2\}$ and $r \in \{0, 1, \ldots, d_1\}$. Using the integration by parts formula, for each integer $m \geq 0$ and all $\psi, \phi \in C^\infty_c$, we have
\[
(\phi, \psi)_{m+1} = ((1 - \Delta)^{m+1}\phi, \psi)_0 = ((1 - \Delta)^{m}\partial_r \phi, \partial_r \psi)_0 = (\partial_r \phi, \partial_r \psi)_m = (\tilde{\phi}, \tilde{\psi})_{m, d_2 \times (d_1 + 1)} =: ((\tilde{\phi}, \tilde{\psi}))_m.
\]
Using a density argument, for all $w, \nu \in H^{m+1}$, we get
\[
(w, \nu)_{m+1} = ((\tilde{w}, \tilde{\nu}))_m.
\]
We set $\| \cdot \|_m = \| \cdot \|_{m, d_2 \times (d_1 + 1)}$ for each $m \geq 1$. Thus, proceeding formally, for all $\psi, \phi \in C^\infty_c$ and $(t, \omega) \in [0, T] \times \Omega$, we obtain
\[
(\phi, L_t \psi)_{m+1} = ((\tilde{\phi}, \tilde{L}_t \psi))_m, \quad |L_t \psi|_{m+1}^2 = \|L_t \psi\|_{m}^2, \quad \text{and} \quad \int_I |I_{t, z} \psi|^2_{m+1} \pi_t(dz) = \int_I \|I_{t, z} \psi\|_{m}^2 \pi_t(dz).
\]
A simple calculation shows that for all $\psi \in C^\infty_c(\mathbb{R}^d, \mathbb{R}^{d_2})$ and $(t, \omega, z) \in [0, T] \times \Omega \times Z$,
\[
\tilde{L}_t \psi = \hat{L}_t \psi \quad \text{and} \quad \tilde{I}_{t, z} \psi = \hat{I}_{t, z} \psi,
\]
where
\[
\hat{L}_t^{(k,r)} \tilde{\psi}(x) := \int_Z \left( \delta_{(k,r), (l,r')} \tilde{\psi}^{(l,r')(x,z)}(x + H_t(x,z)) - \tilde{\psi}^{(l,r')(x)}(x) \right) \pi_t(dz)
\]
\[
+ b_{i}^t(x) \partial_i \tilde{\psi}^{(k,r)}(x) + c_{i}^t(x) \tilde{\psi}^{(l,r')(x)},
\]
\[
\hat{I}_{t, z}^{(k,r)} \tilde{\psi}(x) := \delta_{(k,r), (l,r')} \tilde{\psi}^{(l,r')(x,z)}(x + H_t(x,z)) + \tilde{\psi}^{(k,r)}(x),
\]
\[
\hat{\rho}_{t}^{(k,r), (l,r')} \tilde{\psi}(x) := \delta_{r,r'} \rho_{t}^{(k,l)}(x,z) + 1_{r \geq 1} [\delta_{0,r'} \partial_r \rho_{t}^{(k,l)}(x,z) + 1_{r \geq 1} (\delta_{kl} + \rho_{t}^{(k,l)}(x,z)) \partial_r H_t^{(k,r)}(x,z)],
\]
\[
\hat{c}_{t}^{(k,r), (l,r')} \tilde{\psi}(x) := \delta_{r,r'} c_{t}^{(k,l)}(x) + 1_{r \geq 1} [\delta_{0,r'} \partial_r c_{t}^{(k,l)}(x) + 1_{r \geq 1} (\delta_{kl} \partial_r b_{t}^{(k,l)}(x) + \int_Z \rho_{t}^{(k,l)}(x,z) \partial_r H_t^{(k,r)}(x,z) \pi_t(dz))],
\]
for $k, l \in \{1, \ldots, d\}$ and $r, r' \in \{0, 1, \ldots, d_1\}$, and where $1_{r \geq 1}$ is the characteristic function defined by $1_{r \geq 1} = 1$ if $r \geq 1$ and $1_{r \geq 1} = 0$ if $r = 0$. We define $\hat{A}$ and $\hat{J}$ such that for all $\psi \in C^\infty_c$ and $(t, \omega) \in [0, t] \times \Omega$
\[
\hat{L}_t \psi = \hat{A}_t \psi + \int_Z \hat{J}_{t, z} \psi \pi_t(dz),
\]
as we did for $L_t$.
Assume that the claim is true for some integer $m \geq 0$. Then there exist constants $L = L(m, C_0, d_1, d_2 \times (d_1 + 1))$ and $K = K(m, C_0, d_1, d_2 \times (d_1 + 1))$ such that for all $\psi \in C^\infty_c$ and $(t, \omega) \in [0, T] \times \Omega$,
\[
(\psi, \hat{A}_t \psi)_{m+1} = ((\tilde{\psi}, \tilde{A}_t \tilde{\psi}))_m \leq \frac{L}{2} \|\tilde{\psi}\|_m^2 = \frac{L}{2} \|\psi\|_{m+1}^2,
\]
\[
\int_Z (2(\psi, J_{t, z} \psi))_{m+1} + |I_{t, z} \psi|^2_{m+1} \pi_t(dz) = \int_Z (2((\tilde{\psi}, \tilde{J}_{t, z} \tilde{\psi}))_m + \|\tilde{I}_{t, z} \tilde{\psi}\|_{m}^2) \pi_t(dz) \leq \frac{L}{2} \|\tilde{\psi}\|_m^2 = \frac{L}{2} \|\psi\|_{m+1}^2,
\]
\[
|L_t \psi|_{m}^2 = \|\tilde{L}_t \tilde{\psi}\|_{m-1}^2 \leq K \|\psi\|_{m+2}^2; \quad \int_Z |I_{t, z} \psi|^2_{m+1} \pi_t(dz) \leq K \|\psi\|_{m+2}^2,
\]
and
\[ \int_Z (\psi, (I_{t,z} - J_{t,z}) \psi)^2 m^2 \pi_1(dz) \leq K|\psi|_{m+2}^4 \]
provided that Assumptions 3.1 and 3.2 hold with $\tilde{c}$ and $\tilde{\rho}$ instead of $c$ and $\rho$. It is easy to see that Assumption 3.2(m) holds with $\tilde{c}$ and $\tilde{\rho}$ if Assumption (3.2)(m + 1) holds for $c$, $b$, $H$, and $\rho$. This completes the proof of the claim.

It follows directly from Claim 3.2(m) that if Assumption 3.2(m) holds, then Assumption (2.3)(m) holds for $L, A, J$, and $I$. Obviously, Assumption 3.3(m) implies Assumption 2.2(m). Moreover, it clear that for all $m \geq 1$, $\mathcal{A}v$ is $\mathcal{R}_T$-measurable for all $v \in H^{m+1}$, $\mathcal{J}v$ is $\mathcal{P}_T$-measurable for all $v \in H^{m+1}$, and $\mathcal{I}v$ is $\mathcal{P}_T \otimes \mathcal{Z}$-measurable for all $v \in H^{m+1}$. Thus, we have verified all the required assumptions of Theorem 2.1, and hence we have completed the proof of this theorem.

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