Internal shocks in relativistic jets with time–dependent sources

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ABSTRACT
We present a ballistic description of the formation and propagation of the working surface of a relativistic jet. Using simple laws of conservation of mass and linear momentum at the working surface, we obtain a full description of the working surface flow parametrised by the initial velocity and mass injection rate. This allows us to compute analytically the energy release at any time in the working surface. We compare this model with the results obtained numerically through a new hydrodynamical code applied to the propagation of a relativistic fluid in one dimension in order to test the limits of our study. Finally, we compare our analytical results with observed light curves of five long gamma ray bursts and show that our model is in very good agreement with observations using simple periodic variations of the injected velocity profiles. This simple method allows us to recover initial mass discharge and energy output ejected during the burst.

Key words: hydrodynamics – relativity – galaxies:jets – quasars: general – gamma rays: bursts.

1 INTRODUCTION

Apparent superluminal knots observed along relativistic jets of quasars and micro–quasars are generally interpreted as shock waves moving through the jet. It was first [Rees (1966)] who mathematically predicted the apparent superluminal motions observed in extragalactic jets, due to geometrical effects and relativistic velocities in the proper motion of knots inside them. The same author proposed that these observed knots were produced by a varying velocity of the flow that moves along the jet [Rees (1978)]. Additionally, observations of blazars have been carried through many years showing that the variability of the intensity and polarisation are most likely generated by a transverse shock propagating along the jet (see, e.g. Hagen-Thorn et al. 2007; Spada et al. 2001; Sahayanathan & Misra 2001). Recently, Jamil et al. (2008) have proposed an internal shock model for micro-quasar jets in order to investigate particle acceleration and radiation production in these astrophysical objects. Another situation where internal shock waves appear is the fireball model for long Gamma Ray Bursts (GRBs). In such model, an effective mechanism for the generation of the observed gamma rays is the existence of internal shock waves along the associated jet which are caused by velocity variations of the outflow (Rees & Meszaros 1992; Piran 2005). Despite the alternative explanations to the origin of internal shocks (e.g. Narayan & Kumar 2003) the fireball model has been widely accepted and we show in this article how observations match quite well with such picture.

Detailed numerical models explaining the origin and the characteristic radiation associated to internal shocks have been developed (Blandford & Konigl 1979; Hughes et al. 1985; Marscher 1996), with special attention to the polarisation of the observed radiation as a probe to the internal shocks accelerating the material. In this article we explore the formation of internal working surfaces propagating along a relativistic jet originated from the periodic variation of the source velocity and/or mass injection. Looking at the one–dimensional propagation of mass particles, we present an extension of the non–relativistic model formulated by Cantó et al. (2000). In that work, the formation and evolution of internal working surfaces is modelled for the ejecta corresponding to stellar jets (Raga & Kofman 1992; Raga et al. 2004). This model has the advantage of providing analytic expressions for the kinetic power radiated by the mass particles colliding inside the working surface. The extension of such analysis to the extreme relativistic regime is presented here. Assuming that the pressure gradients between the fluid particles are negligible, and that radiation timescales are much shorter than the time it takes to form a particular working surface, we are able to recover analytic expressions for the speed of the particles at both shock fronts and for the luminosity of the shocked gas. Our analytic model is compared to our new numerical aztekas code.
which completely solves the equations of hydrodynamics in a single dimension for the relativistic regime. In the light of our simulation, the limits for the validity of our model are discussed. A final and most important test for our model is the comparison with gamma ray observations from long GRBs. We present five cases in which the luminosity can be reproduced with our analysis and report on the inferred physical conditions of the ejecta that give way to the observed blast waves.

The article is organised as follows. In section 2 the dynamics of the setup are discussed and the analytical description of the problem is presented. In section 3 we provide an example for the case of a constant mass discharge with an oscillating initial velocity. In section 4 we present the numerical method used to solve the equations in one dimension with planar symmetry. The comparison with the analytical model is presented through the example of section 5. As a test for the limits of our analytic model we also run simulations where the pressure of the fluid is non–vanishing. In section 3 we compare the results of the numerical and analytic methods with each other and discuss the applications of our model. Finally, in section 5 we use our simple analytical model to fit the light curves of five long GRBs.

2 DYNAMICS OF RELATIVISTIC WORKING SURFACES

The formation of shocks along the structure of a relativistic jet has been explained by several phenomena such as the presence of inhomogeneities in the surrounding media, the deviations and changes in the geometry of jets, and the temporal fluctuations in the parameters of the ejection (see e.g. Rees & Meszaros 1994, Mendoza 2000, Mendoza & Longair 2001, 2002, Jamil et al. 2008, and references therein). Here we are concerned with the last situation. When the speed of the emitted mass particles varies with time, a faster but later fluid parcel eventually hits an earlier but slower ejection producing an initial discontinuity which gives rise to a working surface, i.e. a contact discontinuity surface, two shock waves, and two regions of shocked flow that must be at rest with respect to the contact, as the shocks recede from the contact surface in its frame of reference. The working surface travels along the jet with an average speed \( v_{\text{ws}} \), as measured in the frame of the jet source. This picture for the formation of radiation shock surfaces is known as the internal shock model (Rees & Meszaros 1994, Daigne & Mochkovitch 1998). Although several extensions and particular aspects of the model have been presented in the literature (see e.g. Panaitescu et al. 1999, Spada et al. 2001, Sahayanathan & Misra 2003), there are no simple analytical descriptions of this phenomenon. Here we present an analytical approximation to the formation of an internal working surface along a relativistic jet. We assume that the radiation timescales are small compared to the characteristic dynamical times of the problem (Spada et al. 2001). In consequence the pressure of the fluid is negligible and the collision is described ballistically. This assumption is valid if the flow within the jet is nearly adiabatic and non-turbulent (Sahayanathan & Misra 2003).

To follow the evolution of the working surfaces, we consider a source ejecting material in a preferred direction \( x \) with a velocity \( v(\tau) \) and a mass ejection rate \( \dot{m}(\tau) \), both dependent on time \( \tau \) as measured from the jet’s source.

Once the material has been ejected from the source, we assume it will flow in a free-streaming regime (see e.g. Raga et al. 1998). This approximation is valid since the Mach number of the flow is large and, as mentioned before, we emphasise that the radiation processes which cool down the fluid occur in times scales shorter than other dynamical times associated to the problem studied in this article (cf. Spada et al. 2001). The formation of a working surface is studied as the intersection of two different parcels of material ejected at times \( \tau_1 \) = 0 and \( \tau_2 \) with flow velocities \( v_1 = v(\tau_1) \) and \( v_2 = v(\tau_2) = v_1 + \alpha \tau_2 \) respectively (see Figure 1), with \( \alpha := (\text{d}v/\text{d}\tau)_{\text{in}} \). If \( \alpha > 0 \), the second parcel will eventually reach the first one. At time \( \tau_2 \), the distance between both parcels is given by \( v_1 (\tau_2 - \tau_1) \) and thus the time \( t_m \), measured in the reference frame of the central engine (i.e. the observer’s frame), when both of them merge is given by

\[
t_m = \frac{1}{\alpha} \left[ v_1^2 \left( 1 - v_1^2 - \alpha \tau_2 \right) \right],
\]

where \( \gamma(u) := 1/\sqrt{1 - u^2} \) represents the Lorentz factor of the flow with velocity \( u \), and we have assumed that the speed of light \( c = 1 \). The working surface is then formed at a distance

\[
d_f = (t_m + \Delta t)v_1,
\]

from the central engine, where \( \Delta t = \tau_2 - \tau_1 \).

Following the non-relativistic formalism first proposed by Cantó et al. (2000), we assume that the working surface is thin and that there are no mass losses within it (e.g. by sideways ejection of material (see Falcke & Raga 1994, 1992)). Since the flow is approximated as a free–streaming one, its velocity \( v(x, t) \) as a function of the \( x \) coordinate and time \( t \) is simply

\[
v(x, t) = v_0(\tau) = \frac{x}{t - \tau}.
\]

This relation implies that the position \( x_{\text{ws}} \) of the working surface from the downstream flow is given by

![Figure 1](image-url)
where the weighted mass $M_\tau$ of the working surface as a function of the times
is also given by (3) and (6) establish a relation between times $\tau_1$ and $\tau_2$ is

$$x_{ws} = v_1(t - \tau_1),$$  \hspace{1cm} (2)$$
and the one corresponding to the upstream flow takes the form

$$x_{ws} = v_2(t - \tau_2).$$  \hspace{1cm} (3)$$

Consistent with the assumption that the flow is free–streaming, the velocity of the working surface is given by the velocity $\gamma_{ws}$ of its centre of mass, which is determined by (Landau & Lifshitz 1994)

$$v_{ws} = \frac{1}{M_\gamma} \int_{\tau_1}^{\tau_2} \gamma(v(s)) \dot{m}(s)v(s) ds,$$  \hspace{1cm} (4)$$
where the weighted mass $M_\gamma$ ejected between times $\tau_1$ and $\tau_2$ is

$$M_\gamma = \int_{\tau_1}^{\tau_2} \gamma(v(s)) \dot{m}(s) ds.$$  \hspace{1cm} (5)$$

Using formula (4) for the velocity $v_{ws}$, it follows that the position of the working surface is given by

$$x_{ws} = (t - \tau_2)v_{ws} + \frac{1}{M_\gamma} \int_{\tau_1}^{\tau_2} \gamma(v(s)) \dot{m}(s) v(s) (\tau_2 - s) ds.$$  \hspace{1cm} (6)$$
From equations (2) and (3) it follows that the position of the working surface as a function of the times $\tau_1$ and $\tau_2$ is also given by

$$x_{ws} = \frac{v_1 v_2}{v_2 - v_1} (\tau_2 - \tau_1).$$  \hspace{1cm} (7)$$
In the same manner, from the same set of equations, we calculate the time $t$ as a function of $\tau_1$ and $\tau_2$, giving

$$t = \frac{\tau_2 v_2 - \tau_1 v_1}{v_2 - v_1}.$$  \hspace{1cm} (8)$$
For a given value of the position $x_{ws}$, expressions (2), (5) and (6) establish a relation between times $\tau_1$ and $\tau_2$. Taking $\tau_2$ as a parameter, we can construct the position and velocity of the working surface as a function of $\tau_2$ and calculate the values of relevant quantities to the problem, such as the energy available on the moving working surface. Such a relation is one to one as long as the ejection speed $v(\tau)$ increases monotonically.

In order to calculate the amount of kinetic energy radiated away as the working surface moves, we take into account the energy $E_0$ the material had when it was ejected, which is well approximated by

$$E_0 = \int_{\tau_1}^{\tau_2} \dot{m}(s) \gamma(v(s)) ds,$$  \hspace{1cm} (9)$$
and the energy $E_{ws}$ of the material inside the working surface, which is given by

$$E_{ws} = m_\gamma v_{ws},$$  \hspace{1cm} (10)$$
where the Lorentz factor $\gamma_{ws}$ of the working surface material is such that $\gamma_{ws}^2 = 1 - v_{ws}^2$.

If we assume now that the energy loss along the jet, $E_r = E_0 - E_{ws}$ is completely radiated away, then the luminosity $L := dE_i/dt$ of the working surface is given by

$$L = \int \dot{m}(\tau_2) \frac{d\tau_2}{d\tau} \left[ \gamma_{ws} + \frac{m_\gamma v_{ws}}{v_{ws}} (v_{ws} - 1) - \gamma_{ws} (v_{ws} - 1) \right],$$  \hspace{1cm} (11)$$
where the Lorentz factors $\gamma_{ws}^2 := 1 - v_{ws}^2$ and, as we did before, we keep $\tau_2$ as the free parameter in the expansion. In consequence, the luminosity $L$ in equation (11) is found by writing down the expressions for $\tau_1$, $v_1$, $v_2$, $v_{ws}$ and the derivatives $d\tau_1/d\tau_2$ as well as $d\tau_2/d\tau_1$ as functions of the free parameter $\tau_2$.

3 A CONSTANT DISCHARGE FLOW

As an example of our analytic description, let us consider the particular case of a constant discharge $\dot{m}$ and calculate the luminosity $L$ that is obtained through simple oscillations of the particle emission speed. We assume that the injected velocity $v$ has a periodic form given by

$$v(\tau) = \eta_0 + \eta \sin \tau,$$  \hspace{1cm} (12)$$
in which the constant $\eta \ll 1$. This type of oscillatory emission speeds have been widely used for the description of internal shocks in both the Newtonian and the relativistic cases (cf. Cantó et al. 2008, Panaitescu et al. 1999). For this example we choose a system of units in which $\dot{m} = 1$. In addition we set the time unit so that the oscillation frequency is $\omega = 1$. As a consequence of this assumption, the luminosity $L$ is such that its dimensions are the same as the dimensions of $\dot{m}$, i.e.

$$[L] = [\dot{m}],$$  \hspace{1cm} (13)$$
and is thus dimensionless.

If one assumes that the injected flow is highly relativistic, it is then possible to solve analytically equations (1)–(11) at $O(\gamma^{-1})$. However, the analytic expressions are long and cumbersome because, as opposed to the non–relativistic case, the Lorentz factor appears in this description ubiquitously. We have performed numerical integration of equations (4) through (11) with a velocity profile given by equation (12), using the values $v_0 = 0.9$ and $\eta^2 = 0.09$. The results are shown in Figure 2 and are very similar in shape to the ones obtained by Cantó et al. (2000).

The abrupt bump in the luminosity shows that the kinetic energy must be radiated very effectively giving rise to emissions of high energy photons at various wavelengths depending on the strength of the shock.

\footnote{The term $\int \gamma(s) \dot{m}(s) ds$ is subdominant in our problem as long as the variation of the velocity does not dramatically drop the speed to very small values with respect to the speed of light.}
4 1–D NUMERICAL SOLUTION

In order to compare our analytical calculations and the results obtained in section 3, we use planar symmetry in equations (13)-(16), with a null pressure model. At position \( x = 0 \) of our domain, we inject at any time \( t \) a constant injected particle number per unit time flow \( \dot{n} = 1 \), which in turn implies that the particle number density \( n \) at the point \( x = 0 \) is given by

\[
  n(t, x = 0) = \frac{\dot{n}}{\gamma v(t)} = \frac{\sqrt{1 - v^2(t)}}{v(t)}.
\]

In order to analyse a single flash of luminosity, such as the one described in Figure 2, the value of the velocity is assumed constant after the time \( t = 2\pi \), i.e. \( v(t > 2\pi, x = 0) = 0.9 \). The initial conditions for the flow are chosen such that \( v(t = 0, x) = 0.9 \), \( p(t = 0, x) = 0.001 \) and \( n(t = 0, x) = \sqrt{1 - v^2(t = 0, x)/v(t = 0, x)} \). The shock waves obtained by the varying velocity of the flow are captured numerically by introducing an artificial viscosity (Book et al. 1972). Once the position of both shock waves are known, then at each time, the energy \( E_{ws} \) of the flow between both shock waves (the working surface) is calculated as the sum \( \sum n\gamma \Delta x \), where the summation is done for each
numerical cell of width $\Delta x$ that lies between both shocks. The energy of the input flow is calculated as in equation \((19)\) and the derivative that appears in the calculation of the luminosity $L := \frac{d(E_0 - E_{\infty})}{dt}$ is performed numerically and softened using the flux-corrector method presented by Book et al. (1979). Figure 3 presents the numerical result as compared with the analytic prescription described before. This shows that the analytical approximation is a good way to describe the dynamics and energetics of working surfaces formed by a varying input velocity.

We have also performed calculations for two more cases in which the pressure $p$ in the hydrodynamical equations has been written as $\propto \frac{n}{\kappa}$, with $\zeta = 0.01, 0.02$. The pressure and the density are assumed to follow a polytropic relation of the form $p \propto n^{4/3}$, which is in agreement with relation \((17)\), and $\kappa = 4/3$, as described by Rooper (1968). The initial and boundary conditions were not modified. However, the input energy $E_0$, given by equation \((9)\) is now

$$E = \int_0^t \left[ 1 + (3 + v^2) \frac{dp}{d\sigma} \frac{t}{n^2} - (3 + v^2) \frac{p}{n^2} \right] \gamma dt,$$  \((19)\)

according to equation \((15)\). The energy $E_{\infty}$ within the working surface is calculated as the sum $E_{\infty} = \sum n\gamma \Delta x + \sum p\gamma (3 + v^2) \Delta x$, where the summation is taken all cells of width $\Delta x$ of the domain, which lie between both shock waves. As it can be seen from the results presented in Figure 3 the peak of the luminosity is formed at the same time. However, the intensity of the pulse increases with an increasing $\zeta$. The case $\zeta = 1$, which corresponds to a full ultrarelativistic flow has not been drawn in Figure 3 since its luminosity peak has a much greater value.

5 ASTROPHYSICAL APPLICATIONS

The model developed in the previous sections can be applied to shocks within jets emerging from AGNs, $\mu$-QSRs and long GRBs. In order to see how this simple prescription can account for some of the light curves of these objects, we select five long gamma-ray bursts and show that our luminosity function fits quite well their observed light curves.

Our sample consists of five GRBs: GRB051111, GRB060206, GRB060904B, GRB070318 and 0GRB080413B with known redshift, observed by the BAT instrument on board the Swift (Gehrels 2004) satellite (taken from the public database at \texttt{ftp://legacy.gsfc.nasa.gov/swift/}), in the energy range from 15 - 150 KeV. In order to obtain the Flux $F$, the spectra taken at different sections of the light curve for each individual event were adjusted as a power law, with a normalisation $N$, and a photon index $\alpha$ (for a more detailed explanation of spectral analysis see Firmani et al. 2008, and references therein).

In order to fit our model to the observations, we have made use of our null pressure analytical model with the same assumptions as the ones used in section 3 but with $v_0 = 0.99$ and $n^0 = 0.009$, and so the Lorentz factor of the injected flow varies from $\sim 50$ to $\sim 500$ in an oscillating sinusoidal way. To obtain the Flux $F$ from the analytical approximation, we divide the analytical Luminosity $L$ by $4\pi D^2$, where $D$ is the luminosity distance, with cosmological parameters given by

\begin{align*}
H_0 &= 71 \text{ km s}^{-1}\text{Mpc}^{-1},
\Omega_{\text{matter}} = 0.27 \text{ and } \Omega_{\text{vacuum}} = 0.73.
\end{align*}

Figure 3. The figure shows the dimensionless luminosity $L$ of the working surface as a function of the dimensionless time $t$ for a single period ($t = 2\pi$) of oscillation with an input velocity given by equation \((20)\) and a constant discharge flow. The solid line shows the analytical approximation under the assumption of zero pressure. From bottom to top, the dotted curves show the numerical computations made for the cases $\zeta = 0, 0.01, 0.02$. The reason as to why the luminosity doesn’t go to zero at sufficiently large times ($\gtrsim 35$) for the numerical solution is because the input velocity reaches a constant value $= 0.9$ after a period of oscillation, whereas the analytical approximation decays to a null value.

Note that in order to get dimensional quantities, the dimensionless luminosity $L$ has to be multiplied by $\dot{m}c^2$, with the unknown quantity $\dot{m}$. In other words, we will use the fact that $L_{\text{obs}} \propto L$, and so $F_{\text{obs}} \propto F$. In the same manner, the dimensionless time $t$ and the observer time $t_{\text{obs}}$ must be proportional to each other, i.e. $t_{\text{obs}} \propto t$. The proportionality factors are obtained by a linear regression analysis applied to both $F$’s and $t$’s separately. The results of these fits are shown in Figure 4. Note that the observed luminosity $L_{\text{obs}}$ represents an upper limit for the luminosity, since we have assumed that the efficiency factor $\varepsilon$ of converting injected kinetic energy to radiation has been taken as one. In reality $\varepsilon \lesssim 1$ and according to the results of Stern & Poutanen (2008) is close to one for Lorentz factors greater than 40.

Complicated GRB light curve profiles may have to be adjusted by a mixture of a sum of sinusoidal variations of not only the velocity as shown in equation \((19)\), but also as a sum of periodic sinusoidal variations of the mass discharge $\dot{m}$, something outside the scope of this article.

6 CONCLUSION

We have constructed a full relativistic solution applied to the problem of a jet with varying periodic injection velocities and/or mass discharge. This was done by assuming that the working surfaces formed along the jet are ballistic. Under these circumstances we were able to obtain a full analytic description of their behaviour and with this, the luminosity

\begin{align*}
F &= \frac{\dot{m}c^2}{4\pi D^2},
\end{align*}

\begin{align*}
L &= \frac{\dot{m}c^2}{4\pi D^2} \frac{\Delta x}{\Delta t},
\end{align*}

\begin{align*}
\frac{dE}{dt} &= \dot{m}c^2
\end{align*}
along the jet was calculated. We have also made numerical comparisons of our analytic approximations using a 1D RHD code, the current status of the aztekas code.

We proved that the analytic approximations are in very good agreement with a full numerical solution under the assumption of null pressure gradients along the flow. Using the numerical code, the initial hydrodynamical quantities such as the particle discharge $\dot{n}$, the particle number density $n$, the velocity $v$ and the pressure $p$ can be chosen to be intricately periodic functions of time.

We have shown how to fit astronomical observations to five long GRBs using a simple regression analysis technique, assuming a constant discharge flow, a simple sinusoidal variation of the velocity of the injected flow, and a ballistic flow approximation to the problem. It remains to test the analysis and match observations with more complicated shapes of luminosity curves. These could be produced by more complex flows like the sum of periodic sinusoidal functions for the injected velocity and mass discharge. Such possibilities are left for future research.

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Figure 4. The figure shows the burst light curves (represented by dots with error bars at a 1σ level) of five long GRBs, from top to bottom, left to right, GRB051111, GRB060206, GRB060904B, GRB070318, and GRB080413B observed with the BAT instrument onboard the SWIFT satellite. The continuous curves on each graph are the best adjustments using the analytical model built in this article using a single period of oscillation for the variation of the velocity having a sinusoidal form given by equation (12) (see section 5 for more details). The fits give values of the isotropic luminosity given by (12.73, 141.8, 2.042, 1.771, 367.6) x 10^{52} ergs s^{-1} for each burst respectively, which divided by the velocity of light squared imply mass injection discharges $\dot{m} \sim (10^{-1} - 10^{-2}) M_\odot s^{-1}$. 

\[ $\dot{m} \sim (10^{-1} - 10^{-2}) M_\odot s^{-1}$. \]