Energy dependence of the entanglement entropy of composite boson (quasiboson) systems

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Abstract

Bipartite composite boson (quasiboson) systems, which admit realization in terms of deformed oscillators, were considered in our previous paper from the viewpoint of entanglement characteristics. These characteristics, including entanglement entropy and purity, were expressed through the relevant deformation parameter for different quasibosonic states. On the other hand, it is of interest to present the entanglement entropy and likewise the purity as function of energy for those states. In this work, the corresponding dependences are found for different states of composite bosons realized by deformed oscillators and, for comparison, also for the hydrogen atom viewed as a composite boson. The obtained results are expressed graphically and their implications discussed.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Composite bosons or quasibosons or cobosons, as non-elementary systems or (quasi-) particles built from two or more constituent particles, are widely encountered [1–7] in diverse branches of modern theoretical (quantum) physics. Among quasibosons there are mesons, diquarks/tetraquarks, odd–odd or even–even nuclei, positronium, excitons, cooperons, atoms, etc. In this work, we focus on the case of bipartite (two-component) composite bosons. Their creation and annihilation operators can be given through the typical ansatz,

\[ A_+^\alpha = \sum_{\mu \nu} \Phi_{1 \mu \nu}^\alpha a^\dagger_\mu b^\dagger_\nu, \quad A_-^\alpha = \sum_{\mu \nu} \Phi_{1 \mu \nu}^\alpha b_\nu a_\mu, \]

where \( a^\dagger_\mu \) and \( b^\dagger_\nu \) are the creation operators for the (distinguishable) constituents, which can be either both fermionic or both bosonic. In [8, 9], it was shown that the composite bosons of particular form (i.e. those that involve appropriate matrices \( \Phi_{1 \mu \nu}^\alpha \)) can be realized, in algebraic sense, by suitable deformed bosons (deformed oscillators).
As is known, among the measures characterizing degree of entanglement or correlation between the entangled constituents in a quasiboson there are Schmidt rank, Schmidt number, concurrence, purity and the entanglement entropy. The latter two are especially important in the context of (theoretical and experimental) quantum information research, quantum communication and teleportation [10, 1].

It is very important to know how the change of a system’s energy influences the quantum correlation and/or quantum statistics properties of the system under study. As is known, the characteristics of the entanglement between constituents of a quasiboson, which measure bosonic quality of the quasiboson [11–14], and their energy dependence are of importance in quantum information research: quantum communication, entanglement production [15], quantum dissociation processes [16], particle addition or subtraction in general and in the teleportation problem [17, 18], etc. Knowledge of the energy dependence of witnesses of quantum correlations, e.g. entanglement entropy or purity, allows one to relate these to the energy level of excitation that can be measured in experiments, see e.g. [19, 20, 16].

In other words, the energy of a quasiboson differs from the energy of the respective ideal boson by a term which depends on the quasiboson’s entanglement—the measure of deviation from the bosonic behavior. All this motivates to study the energy dependence of the entanglement entropy and other witnesses of entanglement. In this work, we analyze the interconnection between the energy of system (state) and the main two entanglement characteristics such as entanglement entropy or purity. Note that the relationship between entanglement and energy for composite bosons was discussed in [21, 22], for qubits in [23, 24] and for spin systems in [25].

For those composite bosons realizable by deformed oscillators, it is possible, as shown in [26], to directly link the relevant parameter of deformation with the entanglement characteristics of the composite bosons. Namely, the characteristics (or measures) of bipartite entanglement with respect to $a$- and $b$-subsystems, see the above ansatz, were found explicitly [26] for a single composite boson, for multi-quasiboson states and for a coherent state, corresponding to the quasibosons system under study.

Among the above-mentioned entanglement characteristics, the entanglement entropy $S_{\text{ent}}$ certainly is of primary interest. Therefore, in this work our attention is devoted to finding the explicit dependence of entanglement entropy $S_{\text{ent}}$ on the energy $E$ of the quasibosons system, i.e. of the corresponding state. This paper further develops the findings of [26]: we take the composite bosons system as being realized in terms of independent-modes deformed oscillators with the quadratic structure function $\phi(n) = (1 + \epsilon f^2)n - \epsilon f^2n^2$, where $\epsilon = +1/ -1$ for fermionic/bosonic constituents, respectively. Our analysis performed here is for the states considered as the examples in [26], and also for the hydrogen atom as an independent example. The obtained dependences $S_{\text{ent}}(E)$ of the entanglement entropy on energy are shown graphically for a few values of the deformation parameter $f$; one of the cases is compared with the situation emerging for the hydrogen atom.

Analogous treatment, although in a briefer fashion, is also performed for the purity–energy dependence. The paper is structured as indicated: in section 2 we sketch some facts necessary for what follows; main results on the energy dependence of the entanglement entropy of (multi-) quasiboson states are presented in sections 3 and 4, whereas similar treatment for the purity witness of bipartite entanglement is given in section 5. The paper ends with the discussion of the obtained results and of the most interesting physical implications from our viewpoint (section 6).

1 As proven in [9, 8] this is the only possibility in the case when both constituents are pure fermions (or pure bosons).

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2. Preliminaries

As already mentioned, we deal with composite bosons, which are realized by a mode-independent system of deformed bosons (deformed oscillators) given for one mode by the structure function $\varphi(n)$. This means that algebraically the quasiboson operators $A_{\alpha}, A_{\alpha}^\dagger$ and the number operator $N_{\alpha}$ satisfy on the states the same relations as the corresponding deformed oscillator creation, annihilation and occupation number operators:

$$A_{\alpha}^\dagger A_{\alpha} = \varphi(N_{\alpha}),$$  \hspace{1cm} (1)

$$[A_{\alpha}, A_{\beta}^\dagger] = \delta_{\alpha\beta} (\varphi(N_{\alpha} + 1) - \varphi(N_{\alpha})), \hspace{1cm} (2)$$

$$[N_{\alpha}, A_{\beta}^\dagger] = \delta_{\alpha\beta} A_{\beta}, \hspace{1cm} [N_{\alpha}, A_{\beta}] = -\delta_{\alpha\beta} A_{\beta}, \hspace{1cm} (3)$$

where the Kronecker deltas reflect mode independence. Such a realization is possible, see [9, 8], only when the structure function $\varphi(n)$ involving the discrete deformation parameter $f$ is quadratic in $n$, namely (recall that $\epsilon = \pm 1$)

$$\varphi(n) = \left(1 + \epsilon \frac{f}{2}\right)n - \epsilon \frac{f}{2}n^2, \hspace{1cm} f = \frac{2}{m}, \hspace{1cm} m \in \mathbb{N},$$  \hspace{1cm} (4)

whereas the matrices $\Phi_{\alpha}$ are of the form

$$\Phi_{\alpha} = U_1(d_{a}) \text{ diag}[0..0, \sqrt{f/2} U_{a}(m), 0..0]U_2^\dagger(d_b).$$  \hspace{1cm} (5)

Note that the state of the one composite boson

$$|\Psi_{\alpha}\rangle = \sum_{\mu \nu} \Phi_{\alpha}^{\mu\nu} |a_{\mu}\rangle \otimes |b_{\nu}\rangle, \hspace{1cm} |a_{\mu}\rangle \equiv a_{\mu}^\dagger |0\rangle, \hspace{1cm} |b_{\nu}\rangle \equiv b_{\nu}^\dagger |0\rangle,$$  \hspace{1cm} (6)

see the ansatz, is in general bipartite entangled with respect to the states of two constituent fermions (or two bosons); likewise, the state describing many composite bosons,

$$|\Psi\rangle = \sum_{\{n_{\gamma}\}} \Psi(\{n_{\gamma}\})(A_{\gamma_1}^\dagger)^{n_{\gamma_1}} \cdot \cdot \cdot (A_{\gamma_D}^\dagger)^{n_{\gamma_D}} |0\rangle$$  \hspace{1cm} (7)

is viewed as bipartite entangled with respect to $a$- and $b$-subsystems. The degree of entanglement can be measured by well-known characteristics such as the Schmidt rank, Schmidt number, purity, entanglement entropy and concurrence; see e.g. [1, 10] for their definition.

For the entanglement entropy in the case of the one composite boson, we obtain [26]

$$S_{\text{ent}} = \ln(m) = \ln \frac{2}{f},$$  \hspace{1cm} (8)

whereas for the multi-quasibosonic states (7), see [26],

$$S_{\text{ent}} = - \sum_{\{n_{\gamma}\}} |\Psi(\{n_{\gamma}\})|^2 \left(\frac{1}{m}\right)^{\sum n_{\gamma}} \prod_{j=1}^{D}(n_{\gamma_j})!^2 N_{\gamma}^{n_{\gamma_j}} \cdot \ln \left[|\Psi(\{n_{\gamma}\})|^2 \left(\frac{1}{m}\right)^{\sum n_{\gamma}} \prod_{j=1}^{D}(n_{\gamma_j})!^2\right].$$  \hspace{1cm} (9)
3. Energy dependence of the entanglement entropy

In order to find the energy dependence of the entanglement entropy, we need the expression for the Hamiltonian of the composite boson system. Different choices are possible here, but since quasibosons in our approach are realized by means of deformed oscillators, we adopt the corresponding Hamiltonian of the same form as e.g. in [27, 28]. That is, we take the following Hamiltonian of deformed oscillators (deformed bosons) which provide a realization of the composite bosons:

\[
H = \sum_a \frac{1}{2} \hbar \omega_a \left( \psi(N_a) + \psi(N_a + 1) \right).
\]  \(10\)

(a) Single composite boson (quasiboson) case. As our first example, consider the system which consists of a single composite boson. For the entanglement entropy in this case, we have [26]

\[
S_{\text{ent}} = \ln \frac{\epsilon}{2}.
\]  \(11\)

The expression for the energy of the one composite boson as follows from (10) along with (4) is

\[
E = \frac{\hbar \omega}{2} (\psi(1) + \psi(2)) = \hbar \omega \left( \frac{3}{2} - \epsilon \right) = \hbar \omega \left( \frac{3}{2} - \frac{\epsilon}{m} \right).
\]  \(12\)

Then for the entanglement entropy characterizing a single composite boson, we find

\[
S_{\text{ent}} = \ln \left( \frac{\epsilon}{2} - \frac{E}{\hbar \omega} \right) = \begin{cases} 
- \ln \left( \frac{3}{2} - \frac{E}{\hbar \omega} \right), & \epsilon = 1, \quad \frac{1}{2} \leq \frac{E}{\hbar \omega} \leq \frac{3}{2}, \\
- \ln \left( \frac{E}{\hbar \omega} - \frac{3}{2} \right), & \epsilon = -1, \quad \frac{3}{2} \leq \frac{E}{\hbar \omega} \leq \frac{5}{2}.
\end{cases}
\]  \(13\)

The plots corresponding to equation (13) are presented in figures 1 and 2. Note the important feature of the opposite behavior (increasing versus decreasing) of the energy dependence in the case of fermionic constituents with respect to the case of bosonic constituents. In both \(\epsilon = \pm 1\) cases, the entropy \(S_{\text{ent}}\) goes to infinity for the energy \(E = \frac{3}{2} \hbar \omega\), which implies maximal entanglement between constituents. In this case the constituents (fermionic or bosonic) become most tightly bound within a quasiboson, and the quasiboson is most close to the pure boson. In contrast, for \(E = \frac{1}{2} \hbar \omega\) and \(E = \frac{5}{2} \hbar \omega\), the entanglement entropy \(S_{\text{ent}} = 0\), i.e. the constituents are unentangled. From the physical viewpoint, in this case the constituents are in fact unbound.

(b) Hydrogen atom as a quasiboson. It is of interest to consider the hydrogen atom which constitutes a composite boson (entangled with respect to the proton and electron). In this case, however, the relevant matrices \(\Phi_{1}^{\mu\nu}\) are not of the form (5); therefore if it (H-atom) was realized by a deformed boson (this is an open problem), the latter should be different from the type mentioned above. So the creation operator for the hydrogen atom with zero total momentum and quantum number \(n\) can be written in the second quantization formalism (with discrete momenta) as

\[
A_{n}^{\dagger} = \frac{(2\pi \hbar)^{3/2}}{\sqrt{V}} \sum_{p} \phi_{p} \phi_{p}^{\dagger} b_{-p}^{\dagger}.
\]  \(14\)

\[\text{Note that a similar ansatz is used for the excitonic creation operators, see e.g. [6, 5].}\]
where \( a^\dagger_p \) and \( b^\dagger_p \) are the creation operators for the electron and proton, respectively, taken with opposite momenta; \( V \) is a large enough confining volume for the hydrogen atom.

The momentum–space wavefunction \( \phi_{p n} \) is determined by the Schrödinger equation

\[
\phi_{p n}(r) = \frac{1}{(2\pi \hbar)^{3/2}} \sqrt{V} \int \frac{2}{(2\pi)^{3/2}} e^{ipr} \phi_n(r) d^3r, \quad -\frac{\hbar^2 \nabla^2}{2m} \phi_n(r) + U(r) \phi_n(r) = E_n \phi_n(r).
\]

The expression for the hydrogen wavefunction in the momentum representation is given as [7]

\[
\phi_{pnlm} = e^{\pm im\theta_p} \left( \frac{(2l+1)(l-m)!}{2(l+m)!} \right)^{1/2} P^m_l(\cos \theta_p) \frac{\pi 2^{2l+4} l!}{(\gamma \hbar)^{3/2}} \left( \frac{\xi}{\xi^2 + 1} \right)^{l+1} C_{n-l-1}^{l+1} \left( \frac{\xi^2 - 1}{\xi^2 + 1} \right), \tag{15}
\]

where \( P^m_l \) is the associated Legendre polynomial, \( C_{n-l-1}^{l+1}(\ldots) \) is the Gegenbauer polynomial, \( \xi = (2\pi /\gamma \hbar)p, \gamma = Z/(na_0) \).

The expansion (14) can be viewed directly as the Schmidt decomposition for the state \( A_{\gamma 0}^\dagger |0\rangle \) with Schmidt coefficients \( \lambda_p = (2\pi \hbar)^{3/2} \left| \phi_{p n} \right|^2 \). Then the entanglement entropy for the hydrogen atom is given by the relation

\[
S_{\text{ent}} = -\sum_p |\lambda_p|^2 \ln |\lambda_p|^2 = -\sum_p \frac{(2\pi \hbar)^3}{V} |\phi_{p n}|^2 \ln \left( \frac{(2\pi \hbar)^3}{V} |\phi_{p n}|^2 \right), \tag{16}
\]

where the first equality is nothing but the definition of the entanglement entropy [1].

Figure 1. Dependence of the entanglement entropy \( S_{\text{ent}} \) on the energy \( E_\alpha \) for a single composite boson in the case of fermionic components, i.e. at \( \epsilon = +1 \).
Dependence of the entanglement entropy $S_{\text{ent}}$ on the energy $E$ for a single composite boson in the case of bosonic components, i.e. at $\epsilon = -1$.

Calculation of expression (16) is moved to the appendix. Performing a derivation, we obtain the following result:

$$S_{\text{ent}} = -\ln \left[ \frac{(2l + 1)(l - m)!}{(l + m)!} \frac{4\pi 2^{2l} (l!)^2 n(n - l - 1)!}{V(na_0)^3 (n + l)!} \right]$$

$$= \frac{(2l + 1)(l - m)!}{2(l + m)!} \int_{-1}^{1} \left| P^m_l(t) \right|^2 \ln \left| P^m_l(t) \right|^2 dt$$

$$= \frac{4^l (l!)^2 n(n - l - 1)!}{\pi / 2} \int_{-1}^{1} \frac{\sqrt{1 - x^2}}{n(n + l)!} \cdot \ln G_{nl}(x) dx$$

where $G_{nl}(x) = (1 - x^2)^{l/2} (1 - x)^{4(l+1)} C^{l+1}_{n-l-1}(x)^2$. Let us consider the simplest case when the quantum numbers $l = 0$ and $m = 0$. For these values,

$$S_{\text{ent}} = \ln \left[ \frac{V}{4\pi n^3 a_0^4} \right] - \frac{2}{\pi} \int_{-1}^{1} dx (1 - x^2)^{1/2} (1 - x) \left( C^{l}_{n-l-1}(x) \right)^2 \cdot \ln \left[ (1 - x)^4 \left( C^{l}_{n-l-1}(x) \right)^2 \right]$$

$$= S_{\text{ent}}^{(0)} - \ln \left[ 4\pi n^3 \right] - \frac{2}{\pi} \int_{-1}^{1} dx (1 - x^2)^{1/2} (1 - x) \left( C^{l}_{n-l-1}(x) \right)^2 \cdot \ln \left[ (1 - x)^4 \left( C^{l}_{n-l-1}(x) \right)^2 \right],$$

$$S_{\text{ent}}^{(0)} = \ln \frac{V}{a_0^4}.$$
Figure 3. Dependence of the entanglement entropy $\Delta S = S_{\text{ent}} - S_{\text{ent}}^{(0)}$ from (20) on the energy $E$ for the hydrogen atom.

Making the replacement $x = \cos \alpha$ in the integral in (18), and using the formula $C_{n-1}^{1}(\cos \alpha) = \frac{\sin(n\alpha)}{\sin \alpha}$, we infer

$$S_{\text{ent}} = S_{\text{ent}}^{(0)} - \ln[4\pi n^3] - \frac{2}{\pi} \int_0^{\pi} d\alpha (1 - \cos \alpha) \sin^2(n\alpha) \cdot \ln \left[ (1 - \cos \alpha) \frac{\sin^2(n\alpha)}{\sin^2 \alpha} \right].$$  \hspace{1cm} (19)

From the well-known expression for the energy of the H-atom, $E = -\frac{\text{Ry}}{n^2}$, we have $n = \sqrt{-\text{Ry}/E}$. Substituting this into (19), we finally obtain

$$\Delta S(E) = S_{\text{ent}}(E) - S_{\text{ent}}^{(0)} = -\ln \left[ 4\pi \left( -\frac{\text{Ry}}{E} \right)^{3/2} \right] - \frac{2}{\pi} \int_0^{\pi} d\alpha (1 - \cos \alpha) \sin^2 \left( \sqrt{-\frac{\text{Ry}}{E}} \alpha \right)$$

$$\cdot \ln \left[ (1 - \cos \alpha)^2 \frac{\sin^2 \left( \sqrt{-\frac{\text{Ry}}{E}} \alpha \right)}{\sin^2 \alpha} \right].$$  \hspace{1cm} (20)

The derived energy dependence is shown graphically in figure 3. As can be seen, the character of the energy dependence here essentially differs from that of the single quasiboson (two-fermion composite) case above, see figure 1. The main reason for the distinction lies in that the matrices $\Phi_{\mu\nu}^{ij}$ of composite bosons realized by deformed oscillators are different from the corresponding matrices of the hydrogen atom given by its wavefunction $\phi_p$. From the physics viewpoint, this implies that the effective interaction between the constituents in the above quasiboson is different from the Coulomb interaction within the hydrogen atom.

Of course, it would be useful to perform an analysis of the H-atom system by taking into account the fact that the proton, in its turn, also has a composite structure (three-quark system), thus differing from the fundamental (elementary) fermionic entity.
Figure 4. Dependence of the entanglement entropy $S_{ent}$ see (22), on the number of quasibosons $n_\alpha$ for the one-mode multi-quasibosonic system: the case $\epsilon = +1$ of fermionic components.

Let us recall once more that the example of the H-atom, treated as a quasiboson, is included here for comparative purposes only. We suppose, however, that some realization (by deformed oscillators) for the hydrogen atom as a composite quasiboson does also exist, though it may be rather complicated and non-algebraic. In fact, this task is to be solved in two stages: first we have to construct a realization of the three-fermion (three-quark) state for the proton subsystem by some deformed fermionic oscillator, and second that of a quasiboson formed from the (protonic) quasifermion and one more fermion, i.e. an electron. We hope to solve this very involved problem in the near future.

4. Entanglement entropy versus energy for the multi-quasiboson system

Now we examine the case of multi-quasiboson states. Taking into account the Hamiltonian (10), the total energy of the system (at mode-independence) is expressed as

\[ E = \sum_{\alpha} \hbar \omega_{\alpha} \left[ n_\alpha + \frac{1}{2} - \frac{\epsilon f}{2 n_\alpha^2} \right]. \]  

(a) Quasiboson Fock state. Let us find the entanglement entropy as function of energy for the normalized Fock state of $n_\alpha$ quasibosons, $[\phi(n_\alpha)!^{-1/2}(A_\alpha^\dagger)^{n_\alpha}|0\rangle$, in a fixed mode $\alpha$. The entropy of entanglement between $a$- and $b$-subsystems for the two values of $\epsilon$ is equal, see [26], to, respectively,

\[ S_{ent|\epsilon=+1} = \ln C_{2/f}^{n_\alpha}, \quad S_{ent|\epsilon=-1} = \ln C_{2/f+n_\alpha-1}^{n_\alpha}. \]  

The latter dependences $S_{ent} = S_{ent}(n_\alpha)$ are shown in figures 4 and 5.
By inverting equation (21), we have the dependence of the occupation number $n_\alpha$ of quasibosons in the $\alpha$th mode on the corresponding energy $E_\alpha$ of quasibosons:

$$n_\alpha^\pm (E_\alpha) = \frac{1 \pm \sqrt{1 - 2\epsilon f (\frac{E_\alpha}{\hbar \omega_\alpha} - \frac{\epsilon}{2})}}{2\epsilon f}.$$  

Substitution of this expression into (22) leads us to the two-branch form of the concerned dependence $S_{\text{ent}}^\pm (E_\alpha)$ for the case $\epsilon = +1$:

$$S_{\text{ent}}^\pm (E_\alpha)|_{\epsilon = +1} = \ln \left( \frac{C_{\frac{1+\sqrt{1+4\epsilon f^2 E_\alpha/\hbar \omega_\alpha}}{2f} -(1/2)f}}{2} \right),$$  

where $\frac{E_\alpha}{\hbar \omega_\alpha} \leq \frac{1+f}{2f}$ for both $S_{\text{ent}}^+$ and $S_{\text{ent}}^-$ branches, and $\frac{E_\alpha}{\hbar \omega_\alpha} \geq \frac{1}{2}$ for the $S_{\text{ent}}^-$-branch. For $\epsilon = -1$, we have the single monotonous branch

$$S_{\text{ent}}^\pm (E_\alpha)|_{\epsilon = -1} = \ln \left( \frac{C_{\frac{1-\sqrt{1+4\epsilon f^2 E_\alpha/\hbar \omega_\alpha}}{2f} +(1/2)f}}{2} \right),$$  

where $\frac{E_\alpha}{\hbar \omega_\alpha} \geq \frac{1}{2}$. The corresponding functions are presented graphically in figures 6 and 7.

(b) The state with one quasiboson per mode. Now let us turn to the example 2 from [26]. In this case, the quasibosons are all in different modes, i.e. the quasibosonic system is in the state

$$|\Psi\rangle = A_{\gamma_1}^\dagger \cdots A_{\gamma_n}^\dagger |0\rangle, \quad \gamma_i \neq \gamma_j, \ i \neq j, \ i, j = 1, \ldots, n.$$  

For the entanglement entropy, for $\epsilon = \pm 1$, we have

$$S_{\text{ent}} = n \ln(m) = n \ln \frac{2}{f}.$$  

(25)
Figure 6. Dependence of the entanglement entropy $S_{\text{ent}}$, see (23), on the energy $E_\alpha$ for the one-mode multi-quasibosonic system: the case $\epsilon = +1$ of fermionic components.

The energy of the system depends on the dispersion relation of $\omega_{\gamma_j}$ as a function of $\gamma_j$. Taking it in a linear (in $\gamma_j$) form, namely $\omega_{\gamma_j} = \omega_0 + (\gamma_j - \gamma_1) \frac{\delta \omega}{\delta \gamma_j}$, and also using (21) and $n_{\gamma_j} = 1$, we arrive at the following expression for the energy:

$$E = 3 - \epsilon f \left( \hbar \omega_0 n + \frac{1}{2} \hbar \delta \omega n(n-1) \right), \quad (26)$$

where $\delta \omega = \frac{\partial \omega}{\partial \gamma_j} \delta \gamma_j$. Solving the latter yields $n$ as a function of energy, namely

$$n(E) = -1 + \frac{1}{2} \frac{\delta \omega}{\omega_0} + \sqrt{\left(1 - \frac{1}{2} \frac{\delta \omega}{\omega_0} \right)^2 + 4 \frac{\delta \omega}{\omega_0} \frac{1}{3-\epsilon f} \frac{E}{\hbar \omega_0}}. \quad (27)$$

Then (25) yields

$$S_{\text{ent}}(E) = -1 + \frac{1}{2} \frac{\delta \omega}{\omega_0} + \sqrt{\left(1 - \frac{1}{2} \frac{\delta \omega}{\omega_0} \right)^2 + 4 \frac{\delta \omega}{\omega_0} \frac{1}{3-\epsilon f} \frac{E}{\hbar \omega_0}} \ln \frac{2}{f}. \quad (28)$$

As in the previous case we obtain the corresponding plots, which are shown in figures 8 and 9 ($S$ is given in units of $\omega/\delta \omega$ and $E$ in units of $\hbar \omega^2/\delta \omega$; besides, $\omega = |\omega_0 - \frac{1}{2} \delta \omega|$ is used).
(c) Coherent state of quasibosons. As our last example we consider the coherent state of the composite boson system in the \( \alpha \)th mode, see example 3 in [26]:

\[
|\Psi_\alpha\rangle = \tilde{C}(A; m) \sum_{n=0}^{\infty} \frac{A^n}{\phi(n)!} (A_\alpha^\dagger)^n |0\rangle,
\]

\[
\tilde{C}(A; m) = \left( \sum_{n=0}^{\infty} \frac{|A|^{2n}}{\phi(n)!} \right)^{-1/2} = \left[ \frac{(m - 1)I_{m-1}(z)}{(z/2)^{m-1}} \right]^{-1/2}
\]

\[
= e^{-|A|^2/2} \left[ 1 + \frac{1}{4} \frac{|A|^4}{m} + \cdots \right], \quad z = 2\sqrt{m}|A|,
\]

where \( I_{m-1}(z) \) is the modified Bessel function of order \( m - 1 \). For the mean energy of the system in this state, we have

\[
E_{\alpha} = \langle \Psi_\alpha | \frac{1}{2} \hbar \omega_{\alpha} [\phi(N_\alpha) + \phi(N_\alpha + 1)] | \Psi_\alpha \rangle
\]

\[
= \frac{1}{2} \hbar \omega_{\alpha} |\tilde{C}|^2 \cdot \sum_{n=0}^{\infty} \frac{|A|^{2n}}{\phi(n)!} \phi(n) + \frac{1}{2} \hbar \omega_{\alpha} |\tilde{C}|^2 \sum_{n=0}^{\infty} \frac{|A|^{2n}}{\phi(n)!} \phi(n + 1)
\]

\[
= \hbar \omega_{\alpha} |\tilde{C}|^2 |A|^2 \sum_{n=0}^{\infty} \frac{|A|^{2n}}{\phi(n)!} \phi(n) + \frac{1}{2} \hbar \omega_{\alpha} |\tilde{C}|^2 \sum_{n=0}^{\infty} \frac{|A|^{2n}}{\phi(n)!} \cdot [\phi(n + 1) - \phi(n)]
\]

\[
= \hbar \omega_{\alpha} |A|^2 + \frac{1}{2} \hbar \omega_{\alpha} |\tilde{C}|^2 \sum_{n=0}^{\infty} \frac{|A|^2}{\phi(n)!} \left[ 1 + \frac{2n}{m} \right]
\]
Figure 8. Dependence of the entanglement entropy $S_{\text{ent}}$ on energy $E$ for the multi-quasibosonic system with one quasiboson per mode: the case $\epsilon = +1$ of fermionic constituents.

The entanglement entropy for the coherent state (29) is given, see [26], by the formula

$$
S_{\text{ent}} = 2 \sum_{n=0}^{\infty} (|A|^2)^n \frac{1}{(n!)^2 C_{n+m-1}^2} \ln \left[ \frac{(n!)^2 (C_{n+m-1}^2)^2}{C^2 (|A|^2 m)^n} \right].
$$

Hence we have nothing but the dependence of $S_{\text{ent}}$ on $E_\alpha$ in the parametric form (unfortunately, we cannot solve (30) for $|A|$, here $|A|$ being the parameter, in order to insert the solution into (31); that is why we merely use the parametric presentation of the $S_{\text{ent}} = S_{\text{ent}}(E)$ dependence). The plot of this dependence is given in figure 10.

5. Energy dependence of other measures (witnesses) of entanglement

There exist some other widely used witnesses of entanglement: Schmidt rank, concurrence, Schmidt number $K$ or its inverse $P = 1/K$ termed purity [1, 10]. Energy dependences of these
entanglement witnesses, Schmidt rank, concurrence and purity have a somewhat simpler form and can be calculated in a similar way using explicit formulas from [26].

Since entanglement characteristics such as purity are exploited in connection with the issue of entanglement creation in scattering processes [15] and others [17, 23], let us pay some attention to $P$.

For the entangled system consisting of one quasiboson, the purity in [26] was found to be connected with the deformation parameter $m = \frac{2}{f}$ in a simple way:

$$P = \sum_k \lambda_k^4 \frac{1}{m}, \quad \text{or} \quad P = \text{Tr} \left( \rho^{(\alpha)} \right)^2 = \text{Tr} \left( \rho^{(0)} \right)^2 = \frac{1}{m}. \quad (32)$$

Then, the energy dependence for purity in the case of a single composite boson readily follows by combining (32) with (12), which gives

$$P = \frac{f}{2} = \frac{1}{\epsilon} \left( \frac{3}{2} - \frac{E}{\hbar \omega} \right), \quad (\epsilon = 1, \quad \frac{1}{2} \leq \frac{E}{\hbar \omega} \leq \frac{3}{2}),$$

$$= \left\{ \begin{array}{ll}
\frac{3}{2} - \frac{E}{\hbar \omega}, & \epsilon = 1, \quad \frac{1}{2} \leq \frac{E}{\hbar \omega} \leq \frac{3}{2}, \\
\frac{E}{\hbar \omega} - \frac{3}{2}, & \epsilon = -1, \quad \frac{3}{2} \leq \frac{E}{\hbar \omega} \leq \frac{5}{2}
\end{array} \right. \quad (33)$$

Thus, the dependence of purity on energy is linear for both $\epsilon = +1$ and $\epsilon = -1$. Observe however the two mutually opposite (i.e. falling versus raising) types of behavior of purity with increasing energy in these cases of fermionic versus bosonic constituents.
Figure 10. Dependence of the entanglement entropy $S_{\text{ent}}$ in (31) on the mean energy $E_\alpha$, see (30), for the quasibosonic coherent state.

In a similar way, for purity in the case of single-mode multi-quasibosonic Fock states on the basis of [26], we obtain

$$P_{\pm}(E_\alpha) |_{\epsilon=\pm 1} = \left( e^{\frac{1}{2} \sqrt{\frac{1}{f} + \frac{1}{2} E_\alpha / \hbar \omega}} \right)^{-1},$$

(34)

the functions (34) and (35) of energy are presented graphically in figures 11 and 12, respectively. Note the peculiar shape of the curves in figure 11 (non-monotonic behavior, with two pieces of monotonicity for each curve).

Likewise, for multi-quasibosonic states with one quasiboson per mode, using the expression for purity calculated in [26], we easily find

$$P(E) = \exp(-S_{\text{ent}}(E)).$$

(36)

The corresponding plots are shown in figures 13 and 14.

As figures 12–14 demonstrate, purity is falling from its maximal possible value $P = 1$ (zero entanglement), attained at $E = \frac{\hbar \omega}{2}$ implying absence of quanta, to $P = 0$ at very large energies of many-particle states. A peculiar behavior of purity as a function of energy is seen in figure 11: purity drops from $P = 1$ with energy growing to some maximum $E_{\text{max}}(f)$ (the latter is determined by the parameter $f$), and then further decreases from $P_f \equiv P(E_{\text{max}}(f))$ with the energy decreasing from $E_{\text{max}}(f)$ to the smallest values. It is tempting to interpret the existence of two regimes as follows: both addition and subtraction [17, 18] of quanta (of quasiboson) can result in lowering purity. The two regimes, linked with the existence of two branches, differ in the starting value of purity $P$ (whether it is zero or $P(E_{\text{max}})$).
6. Discussion

In conclusion, let us make some comments on the obtained dependences of entanglement entropy on energy, and their visualization with the corresponding plots. For the state of the one composite boson realized by the deformed oscillator, using the Hamiltonian (10) we find that the entanglement entropy monotonically grows with energy if the components are fermions, and decreases if the components are bosons (figures 1 and 2).

We infer that for larger energies the two-fermion quasiboson becomes more tightly bound, whereas the two-boson quasiboson becomes less bound. In both the cases, the energy $E = \frac{1}{2} \hbar \omega$ corresponds to the most entangled quasiboson which shows itself here as closest to the pure boson.

If we compare the case of a two-fermion quasiboson state with the case of the hydrogen atom viewed as a two-fermion composite almost boson, we observe that in the case of the H-atom the dependence of the entanglement entropy on energy decreases and thus strongly differs from the two-fermionic quasiboson case (compare figures 1 and 3). The reason may be rooted in the specified proton–electron interaction and/or in the non-elementary nature of the proton, one of the two constituent fermions of the H-atom (see also the last paragraphs in section 3).

In the case of the multi-quasiboson state for a single fixed mode and when there are two fermionic components, we observe two branches—one decreasing and the other increasing—see figure 6 and the last but one paragraph of this section. For the rest of the considered multi-quasibosonic states with fermionic components (the fixed one mode case, or with one quasiboson per mode, or coherent state) the entanglement entropy is monotonously growing with energy, see figures 7–10, while purity is falling monotonically as in figures 12–14, or
Figure 12. Dependence of the purity $P$, see (35), on the energy $E_\alpha$ for the one-mode multi-quasibosonic system: the case $\epsilon = -1$ of bosonic components.

Figure 13. Dependence of the purity $P$ on energy $E$ for the multi-quasibosonic system with one quasiboson per mode: the case $\epsilon = +1$ of fermionic constituents.
with some peculiarities (two regimes or branches of monotonicity, see figure 11 and the end of section 5).

What about the role of deformation parameter \( f \)? We have a quite natural feature: the entanglement entropy is increasing with decreasing values of \( f \), i.e. with approaching truly bosonic behavior, either for the Fock states at fixed mode or for the coherent states.

For varying energy, from figures 7–10, we find: the two-fermion quasiboson state, multi-quasibosonic states for two-fermion quasibosons within a fixed mode, multi-quasibosonic states with one quasiboson per mode, and the coherent ones are the more entangled the larger the energy. This suggests the possibility of enhancing the entanglement (i.e. its entropy) by increasing the energy of the (multi-)quasiboson state. The states of two-boson quasibosons show an opposite behavior as they are less entangled for larger energies. From the physics viewpoint, we thus have an unambiguous relation between the degree (strength) of entanglement and say the energy level of the considered multi-quasibosonic states.

In some cases the dependences obtained above, e.g. those in figure 1 and figures 7–10, can be viewed in the context of entanglement production or enhancement, (see [15, 29]), and this provides another possible physical implication of our results. As can be seen, the entanglement becomes greater with increasing energy (particle addition?) for the listed cases. Unlike those ones, in the case of two-boson quasibosons entanglement creation is observed when the energy is decreased. That is, when the energy of system is lowered (particle subtraction?), the entanglement entropy grows. Possibly, this could be checked for some physical examples of the treated systems, especially from the particle addition/subtraction viewpoint [17, 18].

Let us make few more remarks on possible experimental verification of the obtained results, which may concern the dependences shown e.g. in figures 1 and 2. As for the first case
(two-fermionic quasiboson), one may consider electron–electron or electron–hole composites (excitons). To test the properties for the two-boson composite we could take bi-photons or the H-molecule H₂ for the corresponding relevant experiments. Besides, the multi-quasibosonic dependences presented in figures 8 and 9 may also be of practical or physical interest.

Finally, let us note the intriguing appearance of ‘bifurcations’ (or existence of two branches) that are in figures 6 and 11. Which of the branches is physically realized could be an intriguing possible issue for verification. Say, 2-electron, 2k-electron, 2k-photon, k-exciton systems, etc could be used for dedicated experiments. Besides the quasibosonic states studied in this work, non-pure e.g. thermal states of quasibosons are also of interest for future analysis.

We hope to extend the above treatment, the obtained results and physical implications to more complex quasiboson (or quasifermion) systems in our forthcoming works, and also to compare with other real physical examples (like the H-atom considered above).

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Appendix. Derivation of the entanglement entropy for the hydrogen atom

Transforming the sum in (16) into an integral (which implies very large volume V) and substituting the hydrogen wavefunctions from (15), for $S_{\text{ent}}$ we obtain

$$S_{\text{ent}} = - \int Vd^3\mathbf{p} \left( \frac{2\pi \hbar}{2(2\pi \hbar)^3} \right) V|\phi_{\text{psint}}|^2 \ln \left( \frac{2\pi \hbar}{2(2\pi \hbar)^3} |\phi_{\text{psint}}|^2 \right)$$

$$= - \int \sin \theta d\theta d\phi d^2\mathbf{p} \frac{(2l + 1)(l - m)!}{2(l + m)!} \left| \left| P_{\ell m}^{\text{psint}}(\cos \theta) \right| \right|^2$$

$$\cdot \left( \frac{\pi^{2l+1}}{(2l)!} \right)^2 \frac{n(n - l - 1)!}{(n + l)!} \left( \frac{\xi^2}{\xi^2 + 1} \right)^2 \left( C_{n-l-1}^{l+1} \left( \frac{\xi^2 - 1}{\xi^2 + 1} \right) \right)^2$$

$$\cdot \frac{n(n - l - 1)!}{(n + l)!} \left( \frac{\xi^2}{\xi^2 + 1} \right)^2 \left( C_{n-l-1}^{l+1} \left( \frac{\xi^2 - 1}{\xi^2 + 1} \right) \right)^2$$

$$\cdot \frac{2^{2l+4}(l)!^2}{\pi} \frac{n(n - l - 1)!}{(n + l)!} \left( \frac{\xi^2}{\xi^2 + 1} \right)^2 \left( C_{n-l-1}^{l+1} \left( \frac{\xi^2 - 1}{\xi^2 + 1} \right) \right)^2$$

$$\cdot \frac{\xi^2}{\xi^2 + 1} \left( \frac{\xi^2}{\xi^2 + 1} \right)^2 \left( C_{n-l-1}^{l+1} \left( \frac{\xi^2 - 1}{\xi^2 + 1} \right) \right)^2 \left( \frac{2^{2l+6}(l)!^2}{\pi} \right) \frac{n(n - l - 1)!}{(n + l)!} \left( \frac{\xi^2}{\xi^2 + 1} \right)^2 \left( C_{n-l-1}^{l+1} \left( \frac{\xi^2 - 1}{\xi^2 + 1} \right) \right)^2 =: \text{A.1}$$

Recall that $C_{n-l-1}^{l+1}(\ldots)$ is the Gegenbauer polynomial. For convenience, introduce new variable $x = \frac{\xi^2 - 1}{\xi^2 + 1}$ and the function $G_{nl}(x) = (1 - x^2)z'(1 - x^2)(C_{n-l-1}^{l+1}(x))^2$. Then we arrive at the
expression

\[ S_{\text{ent}} = - \frac{(2l + 1)(l - m)! 2^{2l}(l!)^2}{(l + m)!} \frac{n(n - l - 1)!}{\pi} \int_{-1}^{1} dt |P_{l}^{nm}(t)|^2 \cdot \int_{-1}^{1} dx \frac{\sqrt{1 - x^2}}{(1 - x)^{3/2}} G_{nl}(x) \]

\[ \times \ln \left( \frac{|P_{l}^{nm}(t)|^2}{(l + m)!} \frac{(2l + 1)(l - m)! \pi 4^{l+1}(l!)^2 n(n - l - 1)!}{V(na_0)^3} \right) \]

\[ = - \frac{(2l + 1)(l - m)! 4^l(l!)^2}{(l + m)!} \frac{n(n - l - 1)!}{\pi} \cdot \int_{-1}^{1} dt |P_{l}^{nm}(t)|^2 \int_{-1}^{1} dx \frac{\sqrt{1 - x^2}}{(1 - x)^{3/2}} G_{nl}(x) \]

\[ \cdot \left\{ \ln \left( \frac{(2l + 1)(l - m)! \pi 4^{l+1}(l!)^2 n(n - l - 1)!}{(l + m)!} \right) + \ln |P_{l}^{nm}(t)|^2 + \ln G_{nl}(x) \right\}. \quad (A.2) \]

Using the normalization condition for \( P_{l}^{nm}(t) \), that is, \( \int_{-1}^{1} |P_{l}^{nm}(t)|^2 dt = \frac{2(l+m)!}{(2l+1)(l-m)!} \), for \( S_{\text{ent}} \) we derive

\[ S_{\text{ent}} = - \frac{2^{2l+1}(l!)^2 n(n - l - 1)!}{\pi} \ln \left[ \frac{(2l + 1)(l - m)! \pi 4^l(l!)^2 n(n - l - 1)!}{V(na_0)^3} \right] \]

\[ \times \int_{-1}^{1} dx \frac{\sqrt{1 - x^2}}{(1 - x)^{3/2}} G_{nl}(x) - \frac{(2l + 1)(l - m)! 4^l(l!)^2 n(n - l - 1)!}{(l + m)!} \frac{\pi}{\pi} \]

\[ \times \int_{-1}^{1} dx \frac{\sqrt{1 - x^2}}{(1 - x)^{3/2}} G_{nl}(x) \int_{-1}^{1} dt |P_{l}^{nm}(t)|^2 \ln |P_{l}^{nm}(t)|^2 - \frac{4^l(l!)^2 n(n - l - 1)!}{\pi/2} \]

\[ \times \int_{-1}^{1} dx \frac{\sqrt{1 - x^2}}{(1 - x)^{3/2}} G_{nl}(x) \ln G_{nl}(x). \quad \text{(A.3)} \]

Using the orthonormalization condition and recurrence relation for Gegenbauer polynomials, we have

\[ \int_{-1}^{1} dx \frac{\sqrt{1 - x^2}}{(1 - x)^{3/2}} G_{nl}(x) = \frac{\pi 2^{-1-2l} (n + l)!}{(n - l - 1)! (l!)^2}. \quad \text{(A.4)} \]

Then from (A.3), we finally obtain

\[ S_{\text{ent}} = - \ln \left[ \frac{(2l + 1)(l - m)! \pi 2^{2l}(l!)^2 n(n - l - 1)!}{V(na_0)^3} \right] \]

\[ - \frac{(2l + 1)(l - m)!}{2(l + m)!} \int_{-1}^{1} dt |P_{l}^{nm}(t)|^2 \ln |P_{l}^{nm}(t)|^2 \]

\[ - \frac{4^l(l!)^2 n(n - l - 1)!}{\pi/2} \int_{-1}^{1} dx \frac{\sqrt{1 - x^2}}{(1 - x)^{3/2}} G_{nl}(x) \ln G_{nl}(x). \quad \text{(A.5)} \]

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