Entanglement bounds for squeezed non-symmetric thermal state

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Abstract
I study the three parameters bipartite quantum Gaussian state called squeezed asymmetric thermal state, calculate Gaussian entanglement of formation analytically and the up bound of relative entropy of entanglement, compare them with coherent information of the state. Based on the result obtained, one can determine the relative entropy of entanglement of the state with infinitive squeezing.

1 Introduction
Quantum entanglement is one of the most important phenomenon in quantum theory. It exhibit the nature of nonlocal correlation between quantum systems, and plays an essential role in various fields of quantum information processing, such as quantum computation, quantum communication, quantum cryptography, quantum teleportation[1],and closely related to quantum channel[2]. After the first experiments [3] on quantum teleportation using two-mode squeezed states [4][5], a significant amount of work has been devoted to develop a quantum information theory of continuous variable systems. So far, most of the theoretical work has focused on the entanglement properties of the quantum states involved in all these experiments, the so-called Gaussian states. The first problem arisen is that if a given quantum Gaussian state is entangled, the problem of qualifying entanglement has been solved in the general bipartite setting [6][7]. But the efficiency of entanglement manipulation protocols used in practical quantum information processing critically depends on the quality of the entanglement that one can generate. It is therefore essential to be able to quantify the amount of entanglement in systems with continuous variables especially Gaussian states. Several measures have been proposed to quantify the amount of entanglement [8]. Among which are the entanglement of formation, relative entropy of entanglement and distillation entanglement. Entanglement of formation is one of the most important entanglement measures. For symmetric Gaussian state, this entanglement measure can be carried out analytically [9]. For general bipartite Gaussian state, another entanglement measure called
Gaussian entanglement of formation was introduced \cite{10}. Clearly it is the upper bound of the entanglement of formation. Relative entropy of entanglement \cite{11} \cite{12} on the other hand measures the distance between the state under consideration to the closest separable state. It has the advantage that separable states obviously correspond to zero distance. For pure states of bipartite systems it reduces to the von Neumann entropy of the reduced state of either subsystem. For mixed bipartite states it is usually difficult to be calculated, except for some specific states. The distance of the Gaussian state to the set of separable Gaussian states measured by the relative entropy was considered \cite{13}. Although one has yet no proof that there does not exist a non-Gaussian separable state which is closer to the Gaussian state under consideration than the closest separable Gaussian state, one has good reason to think that it is a good bound of entanglement measure. However, the expression derived \cite{13} is still not easy for a direct numeric calculation. Recently the bound on the relative entropy of entanglement was calculated for two modes squeezed symmetric thermal state \cite{14}. In this paper I will consider the more practical used state of two mode squeezed non-symmetric thermal state, which is to be distributed between two parties by means of a lossy optical fiber in asymmetric settings for an initial two mode squeezed vacuum state \cite{15}. I will concentrate on its Gaussian entanglement of formation and its upper bound of relative entropy of entanglement. According to the hypothesis of hashing inequality \cite{16}, all this entanglement measures should be lower bounded by coherent information. I will at last compare the results with coherent information of the state.

2 Squeezed non-symmetric thermal state

Gaussian state is completely specified by its mean and its covariance matrix, where the mean can be dropped by local unitary operation so that is irrelevant for entanglement problems. Let us consider a quantum Gaussian state $\rho$ of two system A and B acting on a Hilbert space $L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$. (In quantum optical term, it is called two modes.) It is convenient to describe Gaussian state density operator $\rho$ by its characteristic function. (e.g. \cite{17}) $\chi(z) = Tr(\rho V(z))$. Here $z = (q_A, q_B, p_A, p_B) \in \mathbb{R}^4$ is a real vector and $V(z) = \exp[-i \sum_{k=A,B} (q_k X_k + p_k P_k)]$ is Weyl operator (displacement operator) with $X_k$ and $P_k$ are operators of system A and B respectively, satisfying canonical commutation relations. A characteristic function $\chi$ uniquely defines a state $\rho_\chi$. Gaussian state is state whose $\chi$ is Gaussian function of $z$: $\chi(z) = \exp[i \eta^T z - \frac{1}{2} z^T \gamma z]$, where $\gamma$ is a real symmetric matrix, called correlation matrix.

Any Gaussian state of two modes can be transformed into what we called the standard form, using local unitary operation only \cite{6} \cite{7}. The corresponding characteristic function has displacement $\eta = 0$ and the correlation matrix has the simple form of four parameters. It was proved that (see \cite{17} and reference therein, also see \cite{7} and reference therein): For arbitrary real symmetric matrix $\gamma$ there exists linear transformation (symplectic transformation) $S_p : z \rightarrow S_p z$, where
preserving canonical commutation relations, then
\[ \gamma \rightarrow \gamma_0 = S_p \gamma S_p^T = \text{diag} (\gamma_A, \gamma_B, \gamma_A, \gamma_B). \] (1)

This is to say that by proper symplectic transformation, Gaussian state can be transformed into a direct product of thermal states \( \rho \rightarrow \rho_0 = \rho_A^0 \otimes \rho_B^0 \). The one mode thermal state density operators are of the forms
\[ \rho_\mu^0 = (1 - v_\mu) \sum_{m=0}^{\infty} v_\mu^m |m\rangle_\mu \langle m|, \] (2)

where \( v_\mu = \frac{N_\mu}{N_\mu + 1}, \) \( N_\mu = \frac{1}{2} (\gamma_\mu - 1), (\mu = A, B) \). The symplectic transformation \( S_p \) induces a unitary operator \( U(S_p) \) so that \( \rho \rightarrow U(S_p) \rho U(S_p)^+ (S_p) = \rho_0 \) [18], and \( \rho = U^+ (S_p) \rho_0 U(S_p) \). For \( \rho_0 \) is a direct product of thermal states. It represents the randomness side of state \( \rho \). While the quantum correlation side of \( \rho \) should be caused by the two mode unitary operators \( U(S_p) \).

In this paper I will consider one of the special kind of Gaussian state which can be generated from the thermal states \( \rho_0 = \rho_A^0 \otimes \rho_B^0 \) with a simple form of \( S_p \).

\[ S_p = \begin{bmatrix} \cosh r & -\sinh r \\ -\sinh r & \cosh r \end{bmatrix} \oplus \begin{bmatrix} \cosh r & \sinh r \\ \sinh r & \cosh r \end{bmatrix}, \] (3)

Then \( \gamma = S_p^{-1} \gamma_0 (S_p^T)^{-1} \). The \( S_p^{-1} \) induced unitary operator \( U(S_p^{-1}) \) is just the two-mode squeezed operator \( S_2 (r) \)

\[ S_2 (r) = \exp (a_A^+ a_B^+ \tanh r) \exp \left[ - (a_A^+ a_A + a_B^+ a_B + 1) \ln \cosh r \right] \cdot \exp \left[ -a_A a_B \tanh r \right]. \] (4)

This kind of Gaussian states will be called squeezed non-symmetric thermal states. And in the rest of this paper, I will use \( \rho \) to specify the states.

\[ \rho = S_2 (r) \rho_0 (v_A, v_B) S_2^+ (r), \] (5)

Denote \( \lambda = \tanh r \), then the inseparability criterion reads

\[ \lambda > \sqrt{v_A v_B}. \] (6)

For the convenience of further application, one can express \( \rho \) in coherent state representation as

\[ \langle \alpha_A, \alpha_B | \rho | \beta_A, \beta_B \rangle = C_0 \exp \left[ -\frac{1}{2} \left( |\alpha_A|^2 + |\alpha_B|^2 + |\beta_A|^2 + |\beta_B|^2 \right) \right] \cdot \exp \left[ \tau (\alpha_A^* \alpha_B + \beta_A^* \beta_B) + \omega_A \alpha_A^* \beta_A + \omega_B \alpha_B^* \beta_B \right], \] (7)
where \( C_0 = \frac{(1-v_A)(1-v_B)(1-\lambda^2)}{1-v_A v_B \lambda^2} \), \( \tau = \frac{\lambda(1-v_A v_B)}{1-v_A v_B \lambda^2} \), \( \omega_A = \frac{v_A(1-\lambda^2)}{1-v_A v_B \lambda^2} \), \( \omega_B = \frac{v_B(1-\lambda^2)}{1-v_A v_B \lambda^2} \). With coherent state representation the reduced state can be easily obtained by integration. The reduced density matrix \( \rho_A = \text{Tr}_B \rho \) and \( \rho_B = \text{Tr}_A \rho \) turn out to be one mode thermal states

\[
\rho_{\mu} = \left( 1 - v_{rd \mu} \right) \sum_{m=0}^{\infty} (v_{rd \mu})^m |m\rangle_{\mu} \langle m|, \\
(\mu = A, B)
\]

with parameters \( v_{rd A} \) and \( v_{rd B} \) respectively,

\[
v_{rd A} = v_A (1 - v_B) + \lambda^2 (1 - v_A) \left/ \left( 1 - v_B + \lambda^2 (1 - v_B) \right) \right., \\
v_{rd B} = v_B (1 - v_A) + \lambda^2 (1 - v_B) \left/ \left( 1 - v_A + \lambda^2 (1 - v_A) \right) \right. \\
\]

(8)

3 Gaussian entanglement of formation

Entanglement of formation is defined as an infimum

\[
E_F (\rho) = \inf \left\{ \sum_k p_k E (\Psi_k) \left| \rho = \sum_k p_k |\Psi_k\rangle \langle \Psi_k| \right. \right\} \\
\]

(9)

over all (possibly continuous) convex decompositions of the state into pure states with respective entanglement being von Neumann entropy of the reduced state. By its definition calculating \( E_F \) is a highly non-trivial optimization problem, which becomes numerically intractable very rapidly if we increase the dimensions of the Hilbert spaces. Remarkably, there exist analytical expressions for two-qubit systems as well as for highly symmetric states. Recently, \( E_F \) was calculated for symmetric Gaussian states of two modes\[9\]. But for general two mode Gaussian states, it is still not easy if not impossible to carry out the \( E_F \). For these reasons the Gaussian entanglement of formation (GEoF) \( E_G \) is introduced \[10\] to quantify the entanglement of bipartite Gaussian states by taking the infimum in (9) only over decompositions into pure Gaussian states.

For any two-mode Gaussian state with correlation matrix \( \gamma = \gamma_q \oplus \gamma_p \), it is proved \[10\] that the Gaussian entanglement of formation is given by the entanglement of the least entangled pure state with \( \gamma_{pure} = X \oplus X^{-1} \) which is such that

\[
\det (X - \gamma_q) = \det \left( X^{-1} - \gamma_p \right) = 0. \\
\]

As a part of the correlation matrix of the entangled pure state, the symmetric \( 2 \times 2 \) matrix \( X \) can always be written in the form of

\[
X = \begin{bmatrix} y x \cosh r_g & y \sinh r_g \\ y \sinh r_g & y x^{-1} \cosh r_g \end{bmatrix}, \\
\]

(11)

then the pure bipartite state with correlation matrix \( \gamma_{pure} \) can be constructed by successively applying two local unitary operations to the two mode squeezed vacuum state, where the two local unitary operations are \( S_{p1} = \text{diag} \{ \sqrt{y}, \sqrt{y^{-1}}, \sqrt{y^{-1}}, \sqrt{y} \} \) and \( S_{p2} = \text{diag} \{ \sqrt{x}, \sqrt{x^{-1}}, \sqrt{x^{-1}}, \sqrt{x} \} \) respectively in simplectic form.
Denote \( n = (N_1 + N_2 + 1) \cosh 2r + N_1 - N_2 \), \( m = (N_1 + N_2 + 1) \cosh 2r - N_1 + N_2 \), \( k = (N_1 + N_2 + 1) \sinh 2r \). One has

\[
\gamma_q = \begin{bmatrix} n & k \\ k & m \end{bmatrix}, \quad \gamma_p = \begin{bmatrix} n & -k \\ -k & m \end{bmatrix}
\]

Eq. (10) then can be written as

\[
\begin{align*}
\left(\frac{n}{x} + mx\right) \cosh r_g - 2k \sinh r_g - \left[\frac{(nm - k^2)}{y} + y\right] &= 0, \\
\left(nx + m/x\right) \cosh r_g - 2k \sinh r_g - \left[\frac{(nm - k^2)y + 1}{y}\right] &= 0.
\end{align*}
\]

So that one has \( (n + m) \left(\frac{1}{x} + x\right) \cosh r_g - 4k \sinh r_g - (nm - k^2 + 1)(1/y + y) = 0 \). Clearly \( r_g \) is a monotonical increase function of \( (1/y + y) \), and when \( y = 1 \), the minimal value of \( r_g \) will be achieved. One can subsequently obtains \( x = 1 \) by substracting Eq. (12) from Eq. (13). The Gaussian entanglement of formation \( E_G \) will be

\[
E_G = g(\sinh^2 r_g).
\]

where \( g(x) = (x + 1) \log(x + 1) - x \log x \) is the bosonic entropy function, and

\[
r_g = r - \frac{1}{2} \ln \frac{1 + \sqrt{v_{AB}}}{1 - \sqrt{v_{AB}}}
\]

The second term at the right hand side can be written of as \( r_0 \), with \( \sqrt{v_{AB}} = \tanh r_0 \), representing the noise side of the bipartite state. While \( r \) represents quantum correlation side of the state. The difference of the two gives the GEoF squeezing parameter. The compaison of the GEoFs for different ratio of average photon numbers \( N_B/N_A \) is displayed in Fig. 1.

4 Up bound for relative entropy of entanglement

The relative entropy of entanglement for bipartite quantum state \( \sigma \) is defined by [11]:

\[
E_r(\sigma) \equiv \min_{\sigma \in D} \{ S(\sigma \| \tilde{\sigma}) \}
\]

where \( D \) is the set of all disentangled states, and \( S(\sigma \| \tilde{\sigma}) \equiv Tr \{ \sigma (\log \sigma - \log \tilde{\sigma}) \} \) is the relative entropy of \( \sigma \) with respect to \( \tilde{\sigma} \). Consider the relative entropy of entanglement of squeezed thermal state \( \rho \), if one chooses a subset of \( D \) which contain all Gaussian separable state, or more specifically all separable squeezed thermal state to substitute the district \( D \) to carry out the minimum of relative entropy, clearly such minimums are local minimums. They can be utilized
consider the relative entropy of $\rho$ where Neumann entropy of state $\rho$, generally speaking the identity can not be achieved. Even for relative entropy of pure Gaussian state, the minimum is achieved by non-gaussian separable state [12]. Clearly the above way of obtaining up bound by shrinking the district of minimization can be applied to other Gaussian state or any other state. In order to obtain the up bound of the relative entropy of entanglement of squeezed thermal state $\rho$, Let us first consider the relative entropy of $\rho$ with respect to seperable squeezed thermal state $\rho(\tilde{v}_A, \tilde{v}_B) = S_2(\tilde{r}) \rho_0(\tilde{v}_A, \tilde{v}_B) S_2^+(\tilde{r})$, with $\tilde{r} = \text{tanh} \tilde{r} \leq \sqrt{v_A v_B}$. The von Neumann entropy of state $\rho$ is [17]

\[
S(\rho) = - Tr \rho \log \rho = g(N_A) + g(N_B)
\]  
with $N_\mu = \frac{v_\mu}{\sqrt{v_\mu}}$, ($\mu = A, B$). And one gets $\log \tilde{\rho} = S_2(\tilde{r}) (\log \rho_0(\tilde{v}_A, \tilde{v}_B)) S_2^+(\tilde{r})$ by the unitary of $S_2(\tilde{r})$. Put $\rho_0(\tilde{v})$ in explicit operator form,

\[
\rho_0(\tilde{v}) = (1 - \tilde{v}_A)(1 - \tilde{v}_B) a_A^\dagger a_A - a_B^\dagger a_B.
\]

Then the second part of relative entropy will be

\[
Tr \rho \log \tilde{\rho} = \log(1 - \tilde{v}_A) + \log(1 - \tilde{v}_B) + Tr \{S_2(\tilde{r}) \rho_0(\tilde{v}_A, \tilde{v}_B) S_2^+(\tilde{r})\}
\]

By utilizing $S_2(\tilde{r}) a_A S_2^+(\tilde{r}) = a_A \cosh r - a_B^\dagger \sinh r$ and with the property that operator can be cycled under the trace, after some algebra one obtains

\[
Tr \rho \log \tilde{\rho} = \log(1 - \tilde{v}_A) + \log(1 - \tilde{v}_B) + \left[\frac{v_A}{1 - v_A} \cosh^2(\tilde{r} - \tilde{r}) + \frac{1}{1 - v_B} \sinh^2(\tilde{r} - \tilde{r})\right] \log \tilde{v}_A
\]

\[
+ \left[\frac{v_B}{1 - v_B} \cosh^2(\tilde{r} - \tilde{r}) + \frac{1}{1 - v_A} \sinh^2(\tilde{r} - \tilde{r})\right] \log \tilde{v}_B
\]

where $Tr[(a_A^+ a_B^+ + a_A a_B) \rho_0(\tilde{v}_A, \tilde{v}_B)] = 0$ is applied. Let us first find out the maximum point of Eq. (20) by partial differentiation with respect to $\tilde{r}$ and $\tilde{v}_A, \tilde{v}_B$ regardless the fact that $\tilde{\lambda} = \text{tanh} \tilde{r} \leq \sqrt{v_A v_B}$, it will be at the point $(\tilde{r}, \tilde{v}_A, \tilde{v}_B) = (r, v_A, v_B)$. Meanwhile it is noticed that there is no other maximum point. Then let us add the condition of $\tilde{\lambda} \leq \sqrt{v_A v_B}$, the maximum should be achieved at the edge of $D_{ST}$, that is $\tilde{\lambda} = v_A v_B$. After one of the parameter is determined, the remain problem is to seek out the maximum with respect to $\tilde{v}_A, \tilde{v}_B$. If the maximum is achieved at $(\tilde{v}_A, \tilde{v}_B) = (\tilde{v}_A^*, \tilde{v}_B^*)$, and denote $\sqrt{v_A v_B} = \text{tanh} \tilde{r}^*$, The up bound of relative entropy of entanglement for squeezed thermal state will be
\[ E_{ur} (\rho) = -g(N_A) - g(N_B) - \log(1 - \bar{v}_A^*) + \log(1 - \bar{v}_B^*) \]
\[ + \left[ \frac{\bar{v}_A}{1 - \bar{v}_A} \cosh^2 (r - \bar{r}^*) + \frac{1}{1 - \bar{v}_A} \sinh^2 (r - \bar{r}^*) \right] \log \bar{v}_A^* \]
\[ + \left[ \frac{\bar{v}_B}{1 - \bar{v}_B} \cosh^2 (r - \bar{r}^*) + \frac{1}{1 - \bar{v}_B} \sinh^2 (r - \bar{r}^*) \right] \log \bar{v}_B^* \quad (21) \]

### 5 Hashing inequality and comparison of the bounds with coherent information

One of the most concerned problems in quantum information is the quantum capacities. It is shown that the rate of quantum information transformation is bounded by the maximal attainable rate of coherent information. If there exists hashing inequality, that is, for any bipartite state the one-way distillable entanglement is no less than coherent information, then one obtains Shannon-like formulas for the capacities \[16\]. The coherent information of bipartite state \( \sigma \) with reductions \( \sigma_A \) and \( \sigma_B \) is defined as \[19\]

\[ I^\mu (\sigma) = S(\sigma^\mu) - S(\sigma), \quad (22) \]

for \( S(\sigma^\mu) - S(\sigma) \geq 0 \) and \( I^\mu (\sigma) = 0 \) otherwise. The hypothetical hashing inequality mentioned above is

\[ D_\rightarrow (\sigma) \geq I^\mu (\sigma) \quad (23) \]

where \( D_\rightarrow (\sigma) \) is forward classical communication aided distillable entanglement of the state. We know that relative entropy of entanglement \( E_r (\sigma) \) is no less than two way distillation of entanglement \( D_{\leftrightarrow} (\sigma) \) \[20\], the later is no less than one way distillation of entanglement \( D_\rightarrow (\sigma) \). Combining with hashing inequality one has

\[ E_{ur} (\sigma) \geq E_r (\sigma) \geq D_{\rightarrow} (\sigma) \geq D_{\leftrightarrow} (\sigma) \geq I^B (\sigma). \quad (24) \]

One the other hand, it is proved that relative entropy is lower bounded by coherent information \[21\], so \( E_r (\sigma) \geq I^B (\sigma) \) is always true irrespective of the hypothesis of hashing inequality.

Consider the squeezed non-symmetric thermal state \( \rho \), the reduced states \( \rho^A = Tr_B \rho \) and \( \rho^B = Tr'_A \rho \) turn out to be not the same. The entropies of its reduced states are \( g(N^d_A) \) and \( g(N^d_B) \) respectively, where \( N^d_\mu = \bar{v}^d_\mu / (1 - \bar{v}^d_\mu) \) is the average particle number of the reduced state. This leads to two different coherent information \( I^A (\rho) \) and \( I^B (\rho) \). They are

\[ I^\mu (\rho) = \max \{ g(N^d_\mu) - g(N_A) - g(N_B), 0 \} \quad (25) \]

The numerical results are shown in the Fig. 2.
6 Conclusions

A subset of quantum Gaussian mixed state is given and scrutinizingly investigated. The so called squeezed non-symmetric thermal state has several merits: It can be described by three parameters, one for quantum correlation, the other two for randomness; The separability of the state readily follows from comparison of the first parameter and the geometric average of the last two parameters, if the randomness is stronger, the state is separable, otherwise it is entangled; All the three parameters have range from zero to one. It is the simplest Gaussian mixed state other than the two parameters squeezed thermal state. The $\lambda, v_A, v_B$ description of state presented in this paper enable the calculation more comparable. With the local unitary operation method, I obtain the GEoF analytically and write it in a simple style with clearly physical meaning. By the numeric results I find that: As the squeezing parameter tends towards infinitive, the up bound of relative entropy of entanglement tend to coincide with the coherent information, so that one can determine the relative entropy of entanglement, and if the hashing inequality is right, the distillation entanglement can be determined at infinitive squeezing.

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Figure 1: Comparison of GEs of. With $\lambda =0.99$, dash-dot, solid, dash, and dot line for $N_B/N_A =0.5,1.0,1.5,2.0$ respectively.

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Figure 2: Comparison of the GEoF, up bound relative entropy of entanglement of with coherent information. With $\lambda = 0.99$, and $N_B/N_A = 0.5$, dot-dash for $E_{ur}$, solid for $I^A$, and dash line for GEoF.