Integral representations for the eigenfunctions of quantum open and periodic Toda chains from QISM formalism

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Abstract

The integral representations for the eigenfunctions of \( N \) particle quantum open and periodic Toda chains are constructed in the framework of Quantum Inverse Scattering Method (QISM). Both periodic and open \( N \)-particle solutions have essentially the same structure being written as a generalized Fourier transform over the eigenfunctions of the \( N - 1 \) particle open Toda chain with the kernels satisfying to the Baxter equations of the second and first order respectively. In the latter case this leads to recurrent relations which result to representation of the Mellin-Barnes type for solutions of an open chain. As byproduct, we obtain the Gindikin-Karepelevich formula for the Harish-Chandra function in the case of \( GL(N, \mathbb{R}) \) group.

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1 Introduction

This report is devoted to well-known quantum mechanical problem to find the simultaneous eigenfunction of commuting set of Hamiltonians for the periodic Toda chain. The first important step in this direction has been done by Gutzwiller [1] who solved the problem for the particular cases \( N = 2, 3 \) and 4 particles and found such important phenomena as quantization of spectrum and separation of multidimensional Baxter equation into the product of one dimension ones. In fact, he performed the quantization of periodic Toda chain in terms of separated variables introduced by Flaschka and McLaughlin [2]. Next important step was taken by Sklyanin [3] who constructed \( R \)-matrix formalism for both classical and quantum cases Toda chains and introduced the algebraic method of separation variables for an arbitrary number of particles. His approach drastically simplifies the derivation of the Baxter equation and works for wide spectrum of integrable models [4].

Our method to solve the spectral problem consists of analytical re-interpretation of Sklyanin’s algebraic ideas which allows to find the integral representation for the eigenfunctions of the periodic Toda chain as a kind of generalized Fourier transform with the eigenfunctions for the open Toda chain [5]. In turn, this method can be treated as a natural generalization of original Gutzwiller’s approach. The explicit solution for the eigenfunctions of the open Toda chain plays a key role in this construction.

It has been discovered by Kostant [6] that the commuting set of Hamiltonians of an open Toda chain coincides with the Whittaker model of the center of universal enveloping algebra. Hence, the Whittaker functions are in fact the eigenfunctions for the open Toda chain. In the usual group-theoretical way the Whittaker function is defined as a matrix element between compact and Whittaker vectors [7]-[9] in the principal series representation. There are many obstacles to generalize this approach to other quantum models or to loop groups.

The present approach to construct the eigenfunctions for both periodic and open chains is rather different [5, 10]: it bases on the Quantum Inverse Scattering Method for the periodic Toda chain [3]. One of the interesting results of analytical calculations in the \( R \)-matrix framework is the revealing of recurrent relation between \( N \) and \( N-1 \) particles eigenfunctions for the open Toda chain (in fact the idea to use a recurrent relation was pointed out by Sklyanin in [11]; our recurrent relations is an explicit realization of such an idea). This naturally leads to new integral representation for the Weyl invariant Whittaker functions to compare with classical results [7]-[4]. This representation is quite explicit and very useful to investigate the different asymptotics. In particular, the Gindikin-Karpelevich formula [12] for the Harish-Chandra function [13] can be obtained in a very simple way for particular case of \( GL(N, \mathbb{R}) \) group. The eigenfunction for the periodic Toda chain are constructed in a rather explicit form and have essentially the same form as the recurrent relation mentioned above. The integral formula for eigenfunctions can be considered as a representation of the Whittaker functions for \( \hat{GL}(N) \) group at the critical level.

The present approach can be generalized to other quantum integrable models. For example, the eigenfunctions for the relativistic Toda chain are calculated in [14] using the same QISM ideology.
2 Quantum Toda chain: description of the model

2.1 Periodic spectral problem

The quantum $N$-periodic Toda chain is a multi-dimensional eigenvalue problem with $N$ mutually commuting Hamiltonians $H_k(x_1,p_1;\ldots;x_N,p_N)$, $(k = 1,\ldots,N)$, where the simplest Hamiltonians have the form

$$H_1 = \sum_{k=1}^{N} p_k$$

$$H_2 = \sum_{k<m} p_k p_m - \sum_{k=1}^{N} e^{x_k-x_{k+1}}$$

$$H_3 = \sum_{k<m<n} p_k p_m p_n + \ldots,$$

(2.1)

and the phase variables $x_k, p_k$ satisfy the standard commutation relations $[x_k, p_m] = i\hbar \delta_{km}$. The main goal is to find the solution to the eigenvalue problem

$$H_k \Psi_E = E_k \Psi_E \quad k = 1,\ldots,N$$

(2.2)

with fast decreasing wave function $\Psi_E$. To be more precise, let us note that, due to translation invariance, the solution to (2.2) has the following structure:

$$\Psi_E(x_1,\ldots,x_N) = \tilde{\Psi}_E(x_1-x_2,\ldots,x_{N-1}-x_N) \exp \left\{ \frac{i}{\hbar} E_1 \sum_{k=1}^{N} x_k \right\}$$

(2.3)

One needs to find the solution to (2.3) such that $\tilde{\Psi}_E \in L^2(\mathbb{R}^{N-1})$. In equivalent terms, we impose the requirement

$$\int f(E_1) \Psi_E(x_1,\ldots,x_N) dE_1 \in L^2(\mathbb{R}^N)$$

(2.4)

for any smooth function $f(y)$, $(y \in \mathbb{R})$ with finite support.

2.2 $GL(N-1, \mathbb{R})$ spectral problem

It turns out that solution to (2.3), (2.4) can be effectively written in terms of the wave functions corresponding to open $N-1$-particle Toda chain (quantum $GL(N-1, \mathbb{R})$ chain). The Hamiltonians of the latter system can be (formally) derived from (2.1) by cancelling out all the operators containing $p_N$ and $x_N$ thus obtaining exactly $N-1$ commuting Hamiltonians $h_k(x_1,p_1;\ldots;x_{N-1},p_{N-1})$ $(k = 1,\ldots,N-1)$. Let $\gamma = (\gamma_1,\ldots,\gamma_{N-1}) \in \mathbb{R}^{N-1}$, $x = (x_1,\ldots,x_{N-1}) \in \mathbb{R}^{N-1}$. We consider $GL(N-1, \mathbb{R})$ spectral problem

$$h_k \psi_\gamma(x) = \sigma_k(\gamma) \psi_\gamma(x) \quad k = 1,\ldots,N-1$$

(2.5)
where \( \sigma_k(\gamma) \) are elementary symmetric functions.

Obviously, in the asymptotic region \( x_{k+1} \gg x_k \) (\( k = 1, \ldots, N-2 \)) all potentials vanish and the solution to (2.5) is a superposition of plane waves. The problem is to find a solution to (2.5) satisfying the following properties:

(i) The solution vanishes very rapidly

\[
\psi_\gamma(x) \sim \exp \left\{ -\frac{2}{\hbar} e^{(x_k-x_{k+1})/2} \right\} \quad x_k - x_{k+1} \to \infty \tag{2.6}
\]

(ii) The function \( \psi_\gamma \) is Weyl-invariant, i.e. it is symmetric under any permutation

\[
\psi_{\gamma_j \ldots \gamma_k \ldots} = \psi_{\gamma_k \ldots \gamma_j \ldots} \tag{2.7}
\]

(iii) \( \psi_\gamma \) can be analytically continued to an entire function of \( \gamma \in \mathbb{C}^{N-1} \) and the following asymptotics hold:

\[
\psi_\gamma \sim |\gamma_j|^\frac{2N}{2} \exp \left\{ -\frac{\pi}{2\hbar} (N-2)|\gamma_j| \right\} \tag{2.8}
\]

as \( |\text{Re}\gamma_j| \to \infty \) in the finite strip of complex plane.

The properties (i)-(iii) define a unique solution to the spectral problem (2.2).

3 Main results

\textbf{Theorem 3.1} The following statements hold [3, 10]:

(i) Let a set \( ||\gamma_{jk}|| \) be the lower triangular \( (N-1) \times (N-1) \) matrix. The solution to the spectral problem (2.5)-(2.8) can be written in the form of multiple Mellin-Barnes integrals:\(^1\)

\[
\psi_{1,1,\ldots,1} \cdots \gamma_{N-1,1} \left( x_1, \ldots, x_{N-1} \right) = \\
= \left( 2\pi \hbar \right)^{(N-1)(N-2)/2} \prod_{k=1}^{N-2} k! \int \prod_{n=1}^{N-2} \prod_{j=1}^{n} \prod_{k=1}^{j+1} \Gamma \left( \frac{\gamma_{nj} - \gamma_{n+1,k}}{i\hbar} \right) \Gamma \left( \frac{\gamma_{nk} - \gamma_{nj}}{i\hbar} \right) \times \\
\exp \left\{ \frac{i}{\hbar} \sum_{n,k=1}^{N-1} x_n \left( \gamma_{nk} - \gamma_{n-1,k} \right) \right\} \prod_{j,k=1}^{N-2} d\gamma_{jk}
\]

where the integral should be understand as follows: first we integrate on \( \gamma_{11} \) over the line \( \Im \gamma_{11} > \max \{ \Im \gamma_{21}, \Im \gamma_{22} \} \); then we integrate on the set \( (\gamma_{21}, \gamma_{22}) \) over the lines \( \Im \gamma_{21} > \max \{ \Im \gamma_{31}, \Im \gamma_{32} \} \) and so on. The last integrations should be performed on the set of variables \( (\gamma_{N-2,1}, \ldots, \gamma_{N-2,N-2}) \) over the lines \( \Im \gamma_{N-2,k} > \max \{ \Im \gamma_{N-1,m} \} \).

\(^1\)We identify the set \( \gamma \) with the last row \( (\gamma_{N-1,1}, \ldots, \gamma_{N-1,N-1}) \).
(ii) In the region \(x_k \ll x_{k+1} (k = 1, \ldots, N - 1)\) the solution has the following asymptotics:

\[
\psi_{\gamma}(x) = \sum_{s \in W} \phi(s \gamma) e^{i \frac{\ell}{h}(s \gamma, x)} + O\left(\max\left\{e^{x_k - x_{k+1}}\right\}_{k=1}^{N-1}\right)
\]

(3.2)

where \((\cdot, \cdot)\) is a scalar product in \(\mathbb{R}^{N-1}\) and the summation is performed over the permutation group; \(\phi(\gamma)\) is (renormalized) Harish-Chandra function

\[
\phi(\gamma) = \frac{1}{\mathcal{H}} \prod_{j<k} \left| \Gamma\left(\frac{\gamma_j - \gamma_k}{i \hbar}\right) \right|^{-2}
\]

(3.3)

where \((\gamma, \rho) \equiv \frac{1}{2} \sum_{m=1}^{N-1} (N - 2m) \gamma_k\).

(iii) The functions (3.1) have the scalar product

\[
\int_{\mathbb{R}^{N-1}} \overline{\psi}_{\gamma}(x) \psi_{\gamma}(x) dx = \frac{\mu(\gamma)}{(N-1)!} \sum_{s \in W} \delta(s \gamma - \gamma') \quad (\gamma, \gamma' \in \mathbb{R}^{N-1})
\]

(3.4)

and obey the completeness condition

\[
\int_{\mathbb{R}^{N-1}} \mu(\gamma) \overline{\psi}_{\gamma}(x) \psi_{\gamma}(y) d\gamma = \delta(x - y)
\]

(3.5)

where

\[
\mu(\gamma) = \frac{1}{(N-1)!} \prod_{j<k} \left| \Gamma\left(\frac{\gamma_j - \gamma_k}{i \hbar}\right) \right|^{-2}
\]

(3.6)

is the Sklyanin measure [3].

The eigenfunctions for the periodic chain are constructed as a kind of Fourier transform with the function (3.1). Let

\[
t_N(\lambda; E) = \sum_{k=0}^{N} (-1)^k \lambda^{N-k} E_k
\]

(3.7)

and \(e_j\) denotes \(j\)-th basis vector in \(\mathbb{R}^{N-1}\).

**Theorem 3.2** The solution to the spectral problem (2.2), (2.4) can be represented as the integral over real variables \(\gamma = (\gamma_1, \ldots, \gamma_{N-1})\) in the following form:

\[
\Psi_{E}(x, x_N) = \frac{1}{2\pi} \int_{\mathbb{R}^{N-1}} \mu(\gamma) C(\gamma; E) \Psi_{\gamma, E_1}(x, x_N) d\gamma
\]

(3.8)

where
The function $\Psi_{\gamma,E_1}(x,x_N)$ is defined in terms of solution (3.1) to the $GL(N-1,\mathbb{R})$ spectral problem:

$$\Psi_{\gamma,E_1}(x,x_N) = \psi_\gamma(x) \exp\left\{ \frac{i}{\hbar} \left( E_1 - \sum_{m=1}^{N-1} \gamma_m \right) x_N \right\} \tag{3.9}$$

The function $C(\gamma;E)$ is the solution of multi-dimensional Baxter equations

$$t_N(\gamma_j;E)C(\gamma;E) = i^N C(\gamma + ihe_j;E) + i^{-N} C(\gamma - ihe_j;E) \tag{3.10}$$

which is symmetric entire function in $\gamma$-variables with the asymptotics

$$C(\gamma;E) \sim |\gamma_k|^{-N/2} \exp\left\{ -\frac{\pi N |\gamma_k|}{2\hbar} \right\} \tag{3.11}$$

as $\text{Re}\gamma_k \to \pm\infty$ in the strip $|\text{Im}\gamma_k| \leq \hbar$

The above restrictions imposed on solution to (3.10) are reformulation of the quantization condition (2.4) on the level of $\gamma$-representation. To obtain the explicit integral form for the eigenfunctions, we use the solution to (3.10), (3.11) in the Pasquier-Gaudin form [15] (see sect.7 below)

$$C(\gamma;E) = \prod_{j=1}^{N-1} \frac{c_+(\gamma_j;E) - \xi(E)c_-(\gamma_j;E)}{\prod_{k=1}^{N} \sinh \frac{\pi}{\hbar}(\gamma_j - \delta_k(E))} \tag{3.12}$$

where the entire functions $c_{\pm}(\gamma)$ are two Gutzwiller’s solutions [1] of the one-dimensional Baxter equation

$$t(\gamma;E)c(\gamma;E) = i^{-N} c(\gamma + i\hbar;E) + i^{N} c(\gamma - i\hbar;E) \tag{3.13}$$

and the parameters $\xi(E)$, $\delta = (\delta_1(E), \ldots, \delta_N(E))$ satisfy the Gutzwiller conditions (the energy quantization) [1] [15] (see sect.7 below). Then the multiple integral (3.8) can be explicitly evaluated. Let $y = (y_1, \ldots, y_N) \in \mathbb{R}^N$ be an arbitrary vector. We denote $y^{(s)} = (y_1, \ldots, y_{s-1}, y_{s+1}, \ldots, y_N)$ the corresponding vector in $\mathbb{R}^{N-1}$.

**Theorem 3.3** Assuming that $\delta_j(E) \neq \delta_k(E)$, the solution (3.8) can be written (up to an inessential numerical factor) in the equivalent form

$$\Psi_E(x,x_N) = \sum_{s=1}^{N} (-1)^{N-s} \sum_{n^{(s)} \in \mathbb{Z}^{N-1}} \Delta(\delta^{(s)} + i\hbar n^{(s)}) C_+^{(s)}(\delta^{(s)} + i\hbar n^{(s)}) \Psi_{\delta^{(s)} + i\hbar n^{(s)},E_1}(x,x_N) \tag{3.14}$$

where

$$C_+(\gamma) \equiv \prod_{j=1}^{N-1} c_+(\gamma_j;E) \tag{3.15}$$

and $\Delta(\gamma) = \prod_{j>k}(\gamma_j - \gamma_k)$ is the Vandermonde determinant.

**Remark 3.1** For $N = 2, 3$ and $N = 4$ formula (3.14) reproduces the results obtained by Gutzwiller [1].
4 \textit{R}-matrix approach

The Toda chain can be nicely described using the \textit{R}-matrix approach \cite{3}. It is well known that the Lax operator

\[ L_n(\lambda) = \begin{pmatrix} \lambda - p_n & e^{-x_n} \\ -e^{x_n} & 0 \end{pmatrix} \] (4.1)

satisfies the commutation relations

\[ R(\lambda - \mu)(L_n(\lambda)) \otimes I)(I \otimes L_n(\mu)) = (I \otimes L_n(\mu))(L_n(\lambda) \otimes I)R(\lambda - \mu) \] (4.2)

where

\[ R(\lambda) = I \otimes I + \frac{i\hbar}{\lambda} P \] (4.3)

is a rational \textit{R}-matrix. The monodromy matrix

\[ T_N(\lambda) \overset{\text{def}}{=} L_N(\lambda) \ldots L_1(\lambda) \equiv \begin{pmatrix} A_N(\lambda) & B_N(\lambda) \\ C_N(\lambda) & D_N(\lambda) \end{pmatrix} \] (4.4)

satisfies the analogous equation

\[ R(\lambda - \mu)(T(\lambda) \otimes I)(I \otimes T(\mu)) = (I \otimes T(\mu))(T(\lambda) \otimes I)R(\lambda - \mu) \] (4.5)

In particular, the following commutation relations hold:

\[ [A_N(\lambda), A_N(\mu)] = [C_N(\lambda), C_N(\mu)] = 0 \] (4.6)

\[ (\lambda - \mu + i\hbar)A_N(\mu)C_N(\lambda) = (\lambda - \mu)C_N(\lambda)A_N(\mu) + i\hbar A_N(\lambda)C_N(\mu) \] (4.7)

\[ (\lambda - \mu + i\hbar)D_N(\lambda)C_N(\mu) = (\lambda - \mu)C_N(\mu)D_N(\lambda) + i\hbar D_N(\mu)C_N(\lambda) \] (4.8)

From (4.5) it can be easily shown that the trace of the monodromy matrix

\[ \tilde{t}_N(\lambda) = A_N(\lambda) + D_N(\lambda) \] (4.9)

satisfies the commutation relations \([\tilde{t}(\lambda), \tilde{t}(\mu)] = 0\) and is a generating function for the Hamiltonians of the periodic Toda chain:

\[ \tilde{t}_N(\lambda) = \sum_{k=0}^{N} (-1)^k \lambda^{N-k} H_k \] (4.10)

We reformulate the spectral equations (2.2) as follows:

\[ \tilde{t}_N(\lambda)\Psi_{E} = t_N(\lambda; E)\Psi_{E} \] (4.11)

where

\[ t_N(\lambda; E) = \sum_{k=0}^{N} (-1)^k \lambda^{N-k} E_k \] (4.12)
On the other hand, it can be easily shown that the operator

\[ A_{N-1}(\lambda) \equiv \sum_{k=0}^{N-1} (-1)^k \lambda^{N-k-1} h_k(x_1, p_1; \ldots; x_{N-1}, p_{N-1}) \quad (4.13) \]

is nothing but the generating function for the Hamiltonians \( h_k \) of \( GL(N-1) \) Toda chain. Therefore, the \( GL(N-1, \mathbb{R}) \) spectral equations can be written in the form

\[ A_{N-1}(\lambda) \psi_\gamma(x) = \prod_{m=1}^{N-1} (\lambda - \gamma_m) \psi_\gamma(x) \quad (4.14) \]

Using the obvious relation

\[ C_N(\lambda) = -e^{xN} A_{N-1}(\lambda) \quad (4.15) \]

one obtains, as a trivial corollary of (4.14),

\[ C_N(\gamma_j) \psi_\gamma(x) = 0 \quad \forall \gamma_j \in \gamma \quad (4.16) \]

Remark 4.1 Equations (4.14) are an analytical analog of the notion of "operator zeros" introduced by Sklyanin [3].

5 Eigenfunctions for the open Toda chain

Suppose that solution to (4.14) satisfying (2.6)-(2.8) is given. Using the commutation relations (4.7), (4.8) together with (4.16), it is easy to show that the following relations hold

\[ A_N(\gamma_j) \psi_\gamma = i^{-N} e^{-xN} \psi_{\gamma - i\epsilon e_j} \quad (5.1a) \]

\[ D_N(\gamma_j) \psi_\gamma = i^N e^{xN} \psi_{\gamma + i\epsilon e_j} \quad (5.1b) \]

\((j = 1, \ldots, N - 1)\) where \( e_j \) is \( j \)-th basis vector in \( \mathbb{R}^{N-1} \). Note that (5.1b) is a corollary of (5.1a) since the quantum determinant of the monodromy matrix (4.4) is unity.

Let us introduce the key object - the auxiliary function

\[ \Psi_{\gamma, \epsilon}(x_1, \ldots, x_N) \overset{\text{def}}{=} \psi_\gamma(x) \exp \left\{ \frac{i}{\hbar} \left( \epsilon - \sum_{m=1}^{N-1} \gamma_m \right) x_N \right\} \quad (5.2) \]

where \( \epsilon \) is an arbitrary parameter. From (4.14), (4.15) and (5.1) it is readily seen that this function satisfies to equations

\[ C_N(\lambda) \Psi_{\gamma, \epsilon} = -e^{xN} \prod_{j=1}^{N-1} (\lambda - \gamma_j) \Psi_{\gamma, \epsilon} \quad (5.3a) \]
\[ A_N(\lambda) \Psi_{\gamma,\epsilon} = (\lambda - \epsilon + \sum_{m=1}^{N-1} \gamma_m) \prod_{j=1}^{N-1} (\lambda - \gamma_j) \Psi_{\gamma,\epsilon} + i^{-N} \sum_{j=1}^{N-1} \Psi_{\gamma-i\hbar,\epsilon} \prod_{m \neq j} \frac{\lambda - \gamma_m}{\gamma_j - \gamma_m} \] (5.3b)

\[ D_N(\lambda) \Psi_{\gamma,\epsilon} = i^N \sum_{j=1}^{N-1} \Psi_{\gamma+i\hbar,\epsilon} \prod_{m \neq j} \frac{\lambda - \gamma_m}{\gamma_j - \gamma_m} \] (5.3c)

In particular,
\[ \tilde{t}_N(\gamma_j) \Psi_{\gamma,\epsilon} = i^N \Psi_{\gamma,\epsilon} + i^{-N} \Psi_{\gamma,\epsilon} \] (5.4)

The problem is to find the corresponding solution for \( GL(N, \mathbb{R}) \) Toda chain using the above information, i.e. in terms of the function \( \Psi_{\gamma,\epsilon}(x) \) construct the Weyl invariant function \( \psi_{\lambda_1,\ldots,\lambda_N}(x_1,\ldots,x_N) \) satisfying to equations

\[ A_N(\lambda) \psi_{\lambda_1,\ldots,\lambda_N} = \prod_{k=1}^{N} (\lambda - \lambda_k) \psi_{\lambda_1,\ldots,\lambda_N} \] (5.5a)

\[ A_{N+1}(\lambda_n) \psi_{\lambda_1,\ldots,\lambda_N} = i^{-N-1} e^{x_{N+1}} \psi_{\lambda_1,\ldots,\lambda_{n-1},-i\hbar,\ldots,\lambda_N} \quad (n = 1, \ldots, N) \] (5.5b)

and obeying the similar to (2.6)-(2.8) conditions

**Lemma 5.1** Let \( \Psi_{\gamma,\epsilon}(x, x_N) \) be the auxiliary function (5.2). Let \( \lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{C}^N \) be the set of indeterminates. Let \( \mu(\gamma) = (2\pi \hbar)^{-N} \prod_{j<k} \left\{ \Gamma\left(\frac{\gamma_j - \gamma_k}{i\hbar}\right) \Gamma\left(\frac{\gamma_k - \gamma_j}{i\hbar}\right) \right\}^{-1} \) (5.6)

\[ Q(\gamma_1, \ldots, \gamma_{N-1}|\lambda_1, \ldots, \lambda_N) = \prod_{j=1}^{N-1} \prod_{k=1}^{N} h \frac{\gamma_j - \lambda_k}{i\hbar} \Gamma\left(\frac{\gamma_j - \lambda_k}{i\hbar}\right) \] (5.7)

Then the Weyl invariant solution to the spectral problem (5.5a)-(5.5b) with the properties discussed above is given by recurrent formula

\[ \psi_{\lambda_1,\ldots,\lambda_N}(x_1,\ldots,x_N) = \int_{\mathcal{C}} \mu(\gamma) Q(\gamma; \lambda) \Psi_{\gamma;\lambda_1+\ldots+\lambda_N}(x_1,\ldots,x_N) d\gamma \] (5.8)

where the integration is performed along the horizontal lines with \( \text{Im} \gamma_j > \max_k \{ \text{Im} \lambda_k \} \).

**Proof.** One needs to calculate the action of the operators \( A_N(\lambda) \) and

\[ A_{N+1}(\lambda) = (\lambda - p_{N+1}) A_N(\lambda) + e^{-x_{N+1}} C_N(\lambda) \] (5.9)

on the function (5.8) using the formulae (5.3b) and (5.5a), (5.3a). The shifted contours can be deformed to original ones using the facts that integrand in (5.8) is an entire function fast decreasing in any finite horizontal strip of complex plane as \( |\text{Re} \gamma_j| \to \infty \). The last step is to use the difference equations for the parts of the integrand with respect to the shifts \( \pm i\hbar \).
Proof of Theorem 3.1. The proof of (3.1) is straightforward resolution of the recurrent relations (5.8) starting with trivial eigenfunction $\psi_{\gamma_{11}}(x_1) = \exp\{\frac{i}{\hbar}\gamma_{11} x_1\}$. Obviously, the function (3.1) is symmetric under the permutation of parameters $\gamma_k$. The asymptotics (2.6) can be proved using the steepest descent method. Using the Stirling formula for the $\Gamma$-functions as $\gamma_{N-1,k} \equiv \gamma_k \to \pm \infty$, it is easy to see that the asymptotics (2.8) hold. Hence, (3.1) is an appropriate solution to the spectral problem.

Further, the formula (3.2) can be proved as follows. The integrand in (5.8) decreases exponentially as $|\gamma_j| \to \infty$, $(j = 1, \ldots, N-1)$ in the lower half-plane and, as consequence, the integrals over large semi-circles in the lower half-plane vanish. Using the Cauchy formula to calculate the integral (5.8) in the asymptotic region $x_{k+1} \gg x_k$, $(k = 1, \ldots, N-1)$, it is easy to see that the asymptotics of the function $\psi_\gamma$ are determined precisely in terms of the corresponding Harish-Chandra function (3.3).

The scalar product (3.4) is the consequence of the Plancherel formula proved in [16] for the $SL(N, \mathbb{R})$ case. The formula (3.5) can be proved by induction.

Remark 5.1 In [3] (eqs.(4.7),(4.18)) the eigenfunctions for an open Toda chain have been constructed in terms of Whittaker function which has a standard integral representation corresponding to the Iwasawa decomposition of semi-simple group (see for example [9]). Our expression (3.1), being obtained in the framework of Quantum Inverse Scattering Method, seems quite different. Nevertheless, both representations do define the same function (this can be shown by comparing the corresponding asymptotics and analytical properties). Hence, one can consider the representation (3.1) as a new one for Whittaker function [7]-[9].

6 Periodic chain: $\gamma$-representation, eigenfunctions, and Plancherel formula

Let $\Psi_E(x, x_N)$ be the fast decreasing solution of the problem (4.11). We define the function $C(\gamma; E)$ by the generalized Fourier transform:

$$\delta(E_1 - \epsilon) C(\gamma; E) = \int_{\mathbb{R}^{N-1}} \Psi_E(x, x_N) \overline{\Psi}_{\gamma,x}(x_0, x) dxdx_N$$  (6.1)

Lemma 6.1 The function $C(\gamma)$ possesses the following properties:

(i) It is a symmetric function with respect to $\gamma$-variables.

(ii) It is an entire function of $\gamma \in \mathbb{C}^{N-1}$.
(iii) The function \( C(\gamma) \) obeys the asymptotics

\[
C(\gamma; E) \sim |\gamma_k|^{-N/2} \exp \left\{-\frac{\pi N |\gamma_k|}{2\hbar}\right\}
\]

as \( \text{Re} \gamma_k \to \pm \infty \) in the strip \( |\text{Im} \gamma_k| \leq \hbar \).

(iv) The function \( C(\gamma) \) satisfies the multi-dimensional Baxter equation

\[
t(\gamma_j; E)C(\gamma; E) = i^N C(\gamma + i\hbar e_j; E) + i^{-N} C(\gamma - i\hbar e_j; E)
\]

where \( t(\gamma; E) \) is defined by (4.12).

**Proof.** The symmetry of the function \( C(\gamma) \) is obvious. We present here only a sketch of the proof of the statements (ii) and (iii).

The statement (ii) follows from the assertion that the auxiliary function \( \Psi_{\gamma, \epsilon} \) is an entire one while the solution of the periodic chain vanishes very rapidly as \( |x_k - x_{k+1}| \to \infty \).

(iii) The asymptotics (6.2) is a combination of two factors. The first one comes from the asymptotics (2.8) while the additional factor \( \sim |\gamma_k|^{-1} \exp\{-\pi |\gamma_k|/\hbar\} \) results from the stationary phase method while calculating the multiple integral including the function (3.1). The calculation is based heavily upon the exact asymptotics of the function \( \Psi_E(x, x_N) \) as \( |x_k - x_{k+1}| \to \infty \).

The proof of (iv) is simple. Using the definition (4.11) and integrating by parts (evidently, boundary terms vanish), one obtains

\[
\delta(E_1 - \epsilon) t(\gamma_j; E)C(\gamma) \equiv \int_{\mathbb{R}^{N-1}} \left\{ \hat{t}(\gamma_j) \Psi_E(x, x_N) \right\} \overline{\Psi_{\gamma, \epsilon}(x, x_N)} dxdx_N = \int_{\mathbb{R}^{N-1}} \Psi_E(x, x_N) \overline{\hat{t}(\gamma_j) \Psi_{\gamma, \epsilon}(x, x_N)} dxdx_N
\]

Taking into account the relation (5.4), the Baxter equation (6.3) follows from definition (6.1).

Now we prove Theorem 3.2. Using the completeness condition

\[
\int_{\mathbb{R}^N} \mu(\gamma) \overline{\Psi_{\gamma, \epsilon}(x, x_N)} \overline{\Psi_{\gamma, \epsilon}(y, y_N)} d\gamma d\epsilon = 2\pi h \delta(x - y) \delta(x_N - y_N)
\]

which is a corollary of (3.5), the inversion of the formula (6.1) results to expression

\[
\Psi_E(x, x_N) = \frac{1}{2\pi} \int_{\mathbb{R}^{N-1}} \mu(\gamma) C(\gamma; E) \overline{\Psi_{\gamma, E_1}(x, x_N)} d\gamma
\]

\(^2\) Actually, the boundary conditions have the same importance here as the requirement of compact support in the theory of the analytic continuation for the usual Fourier transform.
The integral (6.6) is correctly defined. Indeed, the measure (3.6) is an entire function. Therefore, there are no poles in the integrand. Moreover, 

$$\mu(\gamma) \sim |\gamma_k|^{N-2} \exp \left\{ \frac{\pi}{\hbar} (N-2)|\gamma_k| \right\}$$  \hspace{1cm} (6.7)$$
as $|\gamma_k| \to \infty$. Taking into account the asymptotics (2.8) and (6.2) one concludes that the integrand has the behavior $\sim |\gamma_k|^{-1} \exp \{-\pi|\gamma_k|/\hbar\}$ as $|\gamma_k| \to \infty$. Therefore, the integral (6.5) is convergent.

One can directly prove the spectral problem (4.11) calculating the action of the operator $\hat{t}_N(\lambda) = A_N(\lambda) + D_N(\lambda)$ on the right hand side of (6.6) with the help of the formulae (5.3b), (5.3c). The calculation is performed similarly to those of Lemma 5.1, using the analytical properties of the integrand and the Baxter equation (6.3) (see [5] for details).

The last step is to prove that the function (6.6) satisfies to integrability requirement (2.4). Using the scalar product

$$\int_{\mathbb{R}^N} \Psi'_{\epsilon',\epsilon}(x,x_N)\Psi_{\epsilon,\epsilon}(x,x_N)dx dx_N = (2\pi \hbar)^{N} \frac{\mu(\gamma)}{(N-1)!} \delta(\epsilon - \epsilon') \sum_{s \in W} \delta(s\gamma - \gamma')$$  \hspace{1cm} (6.8)$$
one can write the Plancherel formula

$$2\pi \hbar \int_{\mathbb{R}^N} \Psi_{\epsilon}(x,x_N)\Psi_{\epsilon}(x,x_N)dx dx_N = \delta(E_1 - E_1') \int_{\mathbb{R}^{N-1}} \mu(\gamma) C(\gamma; E') C(\gamma; E) d\gamma$$  \hspace{1cm} (6.9)$$
The integral in the r.h.s. of (6.9) is absolutely convergent due to asymptotics (6.2) and (6.7). Hence, the norm $||\Psi_E||$ is finite modulo $GL(1)$ $\delta$-function $\delta(E_1 - E_1')$ (see the corresponding factor in (2.3) which leads to this function) and requirement (2.4) is fulfilled. Hence, Theorem 3.2 is proved.

### 7 Solution of Baxter equation

It is well known [1, 15] (see also [3] for details) that the solution to the Baxter equation (3.10) with the asymptotics (3.11) can be written in the following separated form:

$$C(\gamma; E) = \prod_{j=1}^{N-1} \frac{c_+(\gamma_j; E) - \xi c_-(\gamma_j; E)}{\prod_{k=1}^{N} \sinh \frac{\pi}{\hbar} (\gamma_j - \delta_k)}$$  \hspace{1cm} (7.1)$$
where \( \xi \) and \( \delta_k \) are arbitrary constants and the entire functions \( c_{\pm}(\gamma) \) are defined in terms of \( N \times N \) determinants:

\[
c_{+}(\gamma) = \frac{1}{\prod_{k=1}^{N} \frac{1}{i} \Gamma(1 - \frac{1}{N}(\gamma - \lambda_k))} \left| \begin{array}{cccc}
1 & t(\gamma + ih) & 0 & \ldots & \ldots & \ldots \\
1 & t(\gamma + 2ih) & 1 & t(\gamma + 2ih) & 0 & \ldots \\
0 & 1 & t(\gamma + 3ih) & 1 & t(\gamma + 3ih) & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{array} \right| (7.2a)
\]

\[
c_{-}(\gamma) = \frac{1}{\prod_{k=1}^{N} \frac{1}{i} \Gamma(1 + \frac{1}{N}(\gamma - \lambda_k))} \left| \begin{array}{cccc}
1 & t(\gamma - ih) & 0 & \ldots & \ldots & \ldots \\
1 & t(\gamma - 2ih) & 1 & t(\gamma - 2ih) & 0 \\
0 & 1 & t(\gamma - 3ih) & 1 & t(\gamma - 3ih) & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{array} \right| (7.2b)
\]

and \( \lambda_k \equiv \lambda_k(E) \) are the roots of the polynomial \( t(\gamma) \equiv t_{\gamma}(\gamma; E) \).

On the other hand, the solution (7.1) is not an entire function in general since the denominator in (7.1) has an infinite number of poles at \( \gamma = \delta_k + i\hbar n_k, \ n_k \in \mathbb{Z}, \ k = 1, \ldots, N \). The poles are cancelled only if the following conditions hold:

\[
c_{+}(\delta_k + i\hbar n_k) = \xi c_{-}(\delta_k + i\hbar n_k) \tag{7.3}
\]

In turn, this means that the Wronskian

\[
W(\gamma) = c_{+}(\gamma)c_{-}(\gamma + ih) - c_{+}(\gamma + ih)c_{-}(\gamma) \tag{7.4}
\]

vanishes at \( \gamma = \delta_k + i\hbar n_k \). The Wronskian is \( i\hbar \)-periodic function and possesses exactly \( N \) real roots \( \delta_k(E) \). Therefore, the solution (7.1) has no poles if one takes \( \delta_k = \delta_k(E) \) provided that the constant \( \xi \) is chosen such a way that

\[
\xi = \frac{c_{+}(\gamma)}{c_{-}(\gamma)} \bigg|_{\gamma=\delta_k(E)} \quad k = 1, \ldots, N \tag{7.5}
\]

Hence, one arrives at the following

**Lemma 7.1** [6, 7] The function

\[
C(\gamma; E) = \prod_{j=1}^{N-1} \frac{c_{+}(\gamma_j; E) - \xi(E)c_{-}(\gamma_j; E)}{\prod_{k=1}^{N} \sinh \frac{\pi}{\hbar}(\gamma_j - \delta_k(E))} \tag{7.6}
\]

where \( \delta_k(E) \) are real zeros of the Wronskian (7.4) and the constant \( \xi \) is chosen according to (7.5), satisfies to conditions of Lemma 6.1.
The quantization conditions
\[
\frac{c_+(\delta_1)}{c_-(\delta_1)} = \cdots = \frac{c_+(\delta_N)}{c_-(\delta_N)}
\] (7.7)
determine the energy spectrum of the problem. They have been obtained for the first time by Gutzwiller [1] using quite different method.

To prove Theorem 3.3, one should substitute the solution (7.6) into the integral formula (3.8) and calculate the residues coming from individual terms
\[
\frac{c_\pm(\gamma_j; E)}{\prod_{k=1}^N \sinh \frac{\pi}{\hbar} (\gamma_j - \delta_k(E))}
\] (7.8)
The result is exactly the sum over all possible poles of expressions (7.8) and essentially coincides with (3.14) (see careful analysis in [5]).

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References

[1] M.Gutzwiller, *The quantum mechanical Toda lattice II*, Ann. of Phys., (1981), 133, 304-331.
[2] H.Flachka, D.McLaughlin, *Canonically conjugate variables for the Korteweg-de Vries equation and the Toda lattice with periodic boundary conditions*, Progr.Theor.Phys., (1976), 55, 438-456.
[3] E.Sklyanin, *The quantum Toda chain*, Lect.Notes in Phys., (1985), 226, 196-233.
[4] E.Sklyanin, *Separation of variables. New trends*, Progr.Theor.Phys.Suppl. (1995), 118, 35-60; solv-int/9504001.
[5] S.Kharchev, D.Lebedev, *Integral representation for the eigenfunctions of a quantum periodic Toda chain*, Lett.Math.Phys., (1999), 50, 53-77; hep-th/9910265.
[6] B. Kostant, Quantization and representation theory In: Representation theory of Lie Groups. Proc.SRC/LMS research Symp., Oxford 1977, London Math. Soc. Lecture Notes, (1979), 34, 287-316.

[7] H. Jacquet, Fonctions de Whittaker associées aux groupes de Chevalley, Bull. Soc. Math. France, (1967), 95, 243-309.

[8] G. Schiffmann, Intégrales d’entrelacement et fonctions de Whittaker, Bull. Soc. Math. France, (1971), 99, 3-72.

[9] M. Hashizume, Whittaker models for real reductive groups, J. Math. Soc. Japan, (1979), 5, 394-401.

[10] S. Kharchev, D. Lebedev, Eigenfunctions of $GL(n, \mathbb{R})$ Toda chain: The Mellin-Barnes representation, Pis’ma v ZhETF, (2000), 71, 338-343; hep-th/0004065.

[11] E. Sklyanin, Quantum Inverse Scattering Method. Selected Topics, in Quantum Group and Quantum Integrable Systems, (Nankai Lectures in Mathematical Physics), ed. Molin Ge, Singapore: World Scientific, (1992), 63-97; hep-th/9211111.

[12] S. Gindikin, F. Karpelevich, Plancherel measure of Riemann symmetric spaces of non-positive curvature, Soviet Math. Dokl., 3, (1962), 962-965.

[13] Harish-Chandra, Spherical functions on a semisimple Lie group. I, Amer. J. Math., (1958), 80, 241-310.

[14] S. Kharchev, D. Lebedev, M. Semenov-Tian-Shansky, Wave functions for the open relativistic Toda chain, in preparation.

[15] V. Pasquier, M. Gaudin, The periodic Toda chain and a matrix generalization of the Bessel function recursion relations, J. Phys., (1992), A25, 5243-5252.

[16] M. Semenov-Tian-Shansky, Quantum Toda lattices. Spectral theory and scattering. Preprint LOMIR-3-84. Leningrad, 1984. 64p.

M. Semenov-Tian-Shansky, Quantization of Open Toda Lattices. Encyclopaedia of Mathematical Sciences, vol. 16. Dynamical Systems VII. Ch. 3. Springer Verlag, 1994, pp. 226-259.