The Complexity of the Numerical Semigroup Gap Counting Problem

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Abstract

In this paper, we prove that the numerical-semigroup-gap counting problem is \#NP-complete as a main theorem. A numerical semigroup is an additive semigroup over the set of all nonnegative integers. A gap of a numerical semigroup is defined as a positive integer that does not belong to the numerical semigroup. The computation of gaps of numerical semigroups has been actively studied from the 19th century. However, little has been known on the computational complexity. In 2005, Ramírez-Alfonsín proposed a question whether or not the numerical-semigroup-gap counting problem is \#P-complete. This work is an answer for his question. For proving the main theorem, we show the \#NP-completenesses of other two variants of the numerical-semigroup-gap counting problem.

1 Introduction

The numerical-semigroup-gap counting problem [25, 26, 32] has been actively researched since 19th century from the mathematical point of view. We denote this problem by \#NS GAPS. This problem is defined as follows. Given a set \( A \) of coprime positive integers \( a_1, \ldots, a_n \), the task is counting the number of all integers that cannot be represented as nonnegative integer combinations of \( A \). The set of those integers consists of an additive semigroup called a numerical semigroup [26] and denoted \( S(A) \). An integer not in \( S(A) \) is called a gap of \( S(A) \) and the set of all gaps of \( S(A) \) is denoted \( N(A) \). An integer in \( S(A) \) is called a nongap of \( S(A) \). For example, if \( A = \{6, 11, 15\} \) are given, then \( N(A) \) consists of 10 integers 1, 2, 3, 4, 7, 8, 9, 13, 14, 19. For a given set \( A \) of coprime positive integers, it is known that \( N(A) \) is finite [26, 25]. The maximum integer of \( N(A) \) is called the Frobenius number and denoted \( g(A) \).

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1.1 Computation of the numerical-semigroup-gap counting problem

In this paper, we assume that the reader has some acquaintance with the fundamentals of computational complexity theory. The reader is referred to textbooks; e.g., [2, 9, 23]. We denote the class of all polynomial-time computable function (search) problems by $\text{FP}$. In this paper, given an input, we assume its bit length to be the parameter. For example, given a finite set $A$ of positive integers as an input of a problem, the number $\sum_{a \in A} (\lfloor \log a \rfloor + 1)$ is its input size.

Little research has been known from the complexity theoretical point of view for the problem $\#\text{NS Gaps}$ while many results have been known from others. Ramírez-Alfonsín proposed the following question ([25], Problem A.4.2). “Is computing $N(a_1, \cdots, a_n)$ $\#\text{P}$-complete?” (In [25], Ramírez-Alfonsín uses the notation $N(A)$ for describing two different concepts; the set of all gaps of $S(A)$ and its cardinality, where $A$ is a finite set of coprime positive integers. From the context, it is clear to denote the latter in the above question.) This problem has remained open until the present work. This paper gives an answer, which is negative unless $\#\text{P} = \#\text{NP}$. $\#\text{P}$ is one of the best-known classes of counting problems [36] and the counting counterpart of $\text{NP}$. We formally review the class $\#\text{P}$ formally in Section 2.2.

On computation of $\#\text{NS Gaps}$, more results are known. Sylvester [32] showed $|N(a_1, a_2)|$ to be $(a_1 - 1)(a_2 - 1)/2$ for two coprime positive integers $a_1$ and $a_2$. ([25], Section 5) surveys results on equalities and inequalities for $\#\text{NS Gaps}$. Barvinok [3] found a polynomial-time algorithm for counting the number of elements in $N(A)$ under the assumption that the cardinality of $A$ is fixed. In more detail, Barvinok [3] showed a short rational function for a generating function that counts all the elements of $N(A)$. Unfortunately, this algorithm is doubly exponential in the cardinality $n$ of $A$.

1.2 Variants of the numerical-semigroup-gap counting problem

This paper concerns counting all elements of $N(A)$ for a given set $A$ of coprime positive integers. This problem is closely related to the problem finding the maximum element of $N(A)$, the Frobenius number $g(A)$, for a given set $A$ of coprime positive integers. This problem has been studied as actively as on $\#\text{NS Gaps}$. In this paper, we call this decision problem the Frobenius problem and denote it by $\text{FROBENIUS}$ although in general, the problem is often called by several different names; the linear Diophantine problem of Frobenius, the coin exchange problem of Frobenius, the money changing problem, and so on. Ramírez-Alfonsín [24] proved $\text{FROBENIUS}$ to be $\text{NP}$-hard under Cook reductions. More recently, Matsubara [20] improved this lower bound. In more detail, he proved the $\Sigma^P_2$-hardness of the Frobe-
nious problem under Karp reductions. We formally review the class \( \Sigma^P_2 \) in Section 2.2. These results suggest that the problem \(#\text{NS Gaps}\) is at least as hard as \(#\text{P}\).

However, if we assume the number of input integers to be fixed, then the Frobenius problem is known to be polynomial-time computable. Kannan [17] found a polynomial-time algorithm for solving the Frobenius problem under the assumption that the number of input integers is fixed. Barvinok [3] also found a polynomial-time algorithm under the same assumption.

Many experimentally fast algorithms were developed for solving Frobenius; e.g., [5, 11, 27], Section 1 in [25].

### 1.3 Complexity classes and polynomial-time reductions

In this paper, we concern computational complexity of counting. In particular, we prove \(#\text{NS Gaps}\) and its variants to be complete for \(#\text{NP}\) under two types of polynomial-time reductions, called parsimonious and relaxed subtractive reductions. The class \(#\text{NP}\) can be considered to be the counting counterpart of \( \Sigma^P_2 \). A relaxed subtractive reduction is a new type of reduction introduced in this paper.

Meyer and Stockmeyer [21] found a well-known hierarchy of decision problems, called the polynomial-time hierarchy and many results are known on this hierarchy [29, 30, 31, 39]. This hierarchy is contained in \( \text{PSPACE} \). We review this hierarchy formally in Section 2.

Valiant [36, 37] introduced \(#\text{P}\) and developed the theory of the complexity of counting. Many complete problems are found for \(#\text{P}\) [37]. In any problem \(#L \in \#\text{P}\), we count the number of witnesses for a given instance while in any problem \(L \in \text{NP}\), we check the existence of a witness for a given instance. The relationship between counting complexity classes \( \text{FP} \) and \(#\text{P}\) does not entirely correspond to the one of the decision complexity classes \( \text{P} \) and \( \text{NP} \), respectively. In more detail, there is a decision problem \(L \in \text{P}\) such that its counting counterpart is \(#\text{P}\)-complete. For example, the permanent problem [36] is \(#\text{P}\)-complete while its decision counterpart is the bipartite perfect matching problem, which is in \( \text{P} \). This phenomenon is often called “easy to decide but hard to count.”

The descriptive power of \(#\text{P}\) is known to be so strong. Toda proved the polynomial hierarchy to be Cook reducible to \(#\text{P}\) [34]. However, some natural counting problems appear to be not in \(#\text{P}\) [10, 16, 15]. Under such a background, several types of counting complexity classes were introduced. In this paper, we use classes in the following two types of counting classes \(#\Sigma^P_k\) [36] and \(#\cdot \Sigma^P_k\) [33], which are classes in counting variants of the polynomial hierarchy. As the complement of the latter hierarchy, we also use \(#\cdot \Pi^P_k\). We formally define these classes in Section 2.2.

A counting problem \(#\Sigma_k\text{SAT}\) is a representative of counting problems in a class wider than \(#\text{P}\), which is a counting variant of the \((k+1)\)-alternating
quantified Boolean satisfiability problem. Moreover, \( \#\cdot \Sigma_k \text{SAT} \) is shown to be \( \#\cdot \Sigma_k \text{P} \)-complete [10]. Reductions of problems in \( \#\cdot \Sigma_k \text{SAT} \) and \( \#\cdot \Pi_k \text{SAT} \) are sensitive for which type of reduction we adopt. Parsimonious reductions are weaker reductions for counting problems. \( \#\cdot \Sigma_k \text{SAT} \) and \( \#\cdot \Pi_k \text{SAT} \) are closed under parsimonious reductions. If two counting problems have similar structures, then we can find a parsimonious reduction. Unfortunately, it is not easy to find a parsimonious reduction for given problems in many cases. We review parsimonious reductions in Section 2.3. On the other hand, Cook reductions are too strong to reduce counting problems in the following sense. Toda and Watanabe [35] showed that the hierarchy \( \#\cdot \Sigma_k \text{P} \) is not closed under Cook reductions even if we restrict the number of its oracle calls to at most once. Durand et al. [10] introduced subtractive reductions. Subtractive reductions are intermediates between parsimonious and Cook reductions. They found some complete counting problems for \( \#\cdot \Pi_k \text{P} \) under subtractive reductions.

Schaefer [28] found a dichotomy theorem on the general satisfiability problems. By this theorem, we can classify all general satisfiability problems into one of P and NP-complete problems by some properties. Creignou and Hermann [8] found the counting version of the dichotomy theorem for the general satisfiability problems. By this theorem, we can classify all the general satisfiability counting problems into one of FP and \#P-complete problems. Moreover, Bauland et al. [4] found a trichotomy theorem for the class of counting problems. This theorem is a generalization of the dichotomy theorem in [8] to the classes \( \#\cdot \Sigma_k^P \) and \( \#\cdot \Pi_k^P \). They introduced another type of reduction, called a complementive reduction, and showed the trichotomy that classifies any \( k \)-alternating quantified satisfiability counting problem into a FP problem, a \#P-complete problem under Cook reductions, and a \#\cdot \Sigma_k^P\)-complete problem under complementive reductions for some \( k \geq 1 \). Note that, as described by [4], the “\#\cdot \Sigma_k^P\)-completeness under complementive reductions” does not have a more desirable property in structural complexity theory. In more detail, \#\cdot \Sigma_k^P\) is not closed under complementive reductions, although \#\cdot \Pi_k^P\) is closed under complementive reductions.

Hermann and Pichler [16] proved some counting problems on propositional abduction to be complete for some classes that are at least as hard as \#P. In that work, some of the completenesses are proved under subtractive reductions. Hermann and Pichler [15] also investigated some problems that count optimal solutions for given instances by using the counting complexity class \#\cdot \text{Opt}_kP\). The class \#\cdot \text{Opt}_kP\) is an extension of the class introduced by Krentel [18]. Hemaspaandra and Vollmer [14] surveyed results on several counting classes wider than \#P.
1.4 Contributions of this work

The main contribution of this work is the proof for the \#NP-completeness of \#NS Gaps. This result is also an answer for the open problem proposed by Ramírez-Alfonsín, which is a negative answer unless \#NP = \#P. That completeness will be proved under a new type of reduction, which we call a relaxed subtractive reduction. For the proof of the main theorem, we will prove the \#NP-completenesses of two variants of \#NS Gaps. One is a problem \#NonRep. The task of \#NonRep is counting the number of all positive integers that are in a given interval and cannot be represented as nonnegative integer combinations of a given set of positive integers. The other is \#Bounded NS Gaps. The task of \#Bounded NS Gaps is counting the number of all positive integers that are greater than or equal to a given bound and cannot be represented as nonnegative integer combinations of a given set of positive integers. We will show the \#NP-completenesses of those two variants under parsimonious reductions. Our proofs are self-contained for the readability. We prove almost all statements by using combinatorial methods and elementary properties on numerical semigroup. We do not use the existing notions and results in numerical semigroup if not necessary.

1.5 Organization of the paper

In Section 2, we define notions and notations for the proofs of the later sections. In Sections 2.1 and 2.2, we define the basic notions and notations on Boolean satisfiability and computational complexity, respectively. In Section 2.3, we introduce relaxed subtractive reductions. In Section 2.4, we list related computational problems. Section 2.5 describes the \#NP-completeness of \#·Π₁\text{IN3SAT}, a counting problem on Boolean satisfiability, under parsimonious reductions. Then, the remaining sections, we prove the \#NP-completenesses of three variants of the numerical-semigroup-gap counting problem. In Section 4, we prove the \#NP-completeness of \#NonRep under parsimonious reductions. In Section 5, we prove the \#NP-completeness of \#Bounded NS Gaps under parsimonious reductions. In Section 6, we prove the \#NP-completeness of \#NS Gaps under relaxed subtractive reductions. Section 7 concludes this paper with future work.

2 Preliminaries

We denote by \(\mathbb{N}\) and \(\mathbb{N}_+\) the sets of all nonnegative integers and all positive integers, respectively. For any two nonnegative integers \(a\) and \(b\) with \(a \leq b\), we denote the interval \(\{c \in \mathbb{N} : a \leq c \leq b\}\) by \([a, b]\). For any integer \(k\), we denote the set \(\{c \in \mathbb{N} : c \geq k\}\) by \([k, \infty]\). In this paper, for any integers \(a_1, \ldots, a_n\), we consider a nonnegative integer combination \(\sum_{i=1}^n c_ia_i\) to be
the sum of integers \( a_1, \ldots, a_n \), where \( c_i \in \mathbb{N} \) for every \( 1 \leq i \leq n \) and \( 1 \leq i_1 \leq \cdots \leq i_l \leq n \). For any \( n \in \mathbb{N}_+ \), we denote the number of 1s in the binary representation of \( n \) by \#_1(n).

Let \( A \) denote a set of positive integers \( a_1, \ldots, a_n \). In Section 1, when \( a_1, \ldots, a_n \) are coprime, we defined \( S(A) \) as the numerical semigroup, and defined \( N(A) \) as the set of all gaps of \( S(A) \). We extend these notations to any set of positive integers as follow. For any \( A \), we denote the set \( \{ \sum_{a \in A} c_a a : c_a \in \mathbb{N} \} \) by \( S(A) \), and the set \( \mathbb{N} \setminus S(A) \) by \( N(A) \). Needless to say, if \( a_1, \ldots, a_n \) are not coprime integers, then \( S(A) \) is not a numerical semigroup.

We identify any integer as its binary representation if no confusion arises.

We denote a characteristic function on a predicate \( p \) by \([p]\); i.e., \([p]\) is 1 if \( p = 1 \), and 0 otherwise. The notation \([\cdot]\) was introduced by Kenneth E. Iverson, and often called “Iverson bracket” ([13], Section 2.1). This notations is concise and thus often convenient.

### 2.1 Notions, notations, and assumptions on Boolean formulae

In this subsection, we describe notations and assumptions on Boolean formulae and satisfiability. We assume the reader to be familiar with them. The reader is referred to Chapter 4 in [23] if necessary. Let \( X \) be a set of Boolean variables. Let \( \psi \) be a Boolean formula over \( X \). We denote by \( \text{Var}(\psi) \) the set of all variables occurring in \( \psi \). We denote the set \( X \cap \text{Var}(\psi) \) by \( X_{\psi} \). Let \( \varphi \) be a CNF formula over \( X \). We denote by \( C_{\varphi} \) the set of all clauses in \( \varphi \). For any variable set \( Z \subseteq X \), we denote the set \( \{ z : z \in Z \} \) by \( Z \). For any literal \( \hat{x} \), we denote its complement by \( \hat{x}^c \).

Given a Boolean formula \( \psi \), we assume that there is a bijection from \( \text{Var}(\psi) \) to \([1, |\text{Var}(\psi)|]\). That is, all variables occurring in \( \psi \) have successive indices from 1. If \( \text{Var}(\varphi) = \{ x_1, \ldots, x_n \} \) for a given Boolean formula \( \psi \), then we denote a truth assignment \( \sigma \) by a binary representation \( b_1 \cdots b_n \) such that \( b_i = \sigma(x_i) \) for every \( 1 \leq i \leq n \). For any assignment \( \sigma \), we denote by \( T(\varphi, \sigma) \) the set of all literals \( l \) in \( \varphi \) such that \( \sigma(l) = 1 \). For any assignment \( \sigma \), we say that a literal \( l \) is true if \( \sigma(l) = 1 \); and false otherwise. We define the size of \( \varphi \) as the number of occurrences of literals in \( \varphi \).

Let \( \psi \) be a Boolean formula over the union of pairwise disjoint sets \( X_1, \ldots, X_n \), where \( n \geq 1 \). Following the above assumption, we assume every variable in \( \text{Var}(\varphi) \cap X_l \) to have an index in a suitable method such that all variables in \( \text{Var}(\varphi) \) have successive indices from 1. For example, for every \( 1 \leq l \leq n \), we may consider every variable in \( \text{Var}(\varphi) \cap X_l \) to have a positive integer in \( \sum_{l=1}^{l-1} |\text{Var}(\varphi) \cap X_i| + 1, \sum_{l=1}^{l} |\text{Var}(\varphi) \cap X_i| \) as its index. Thus, we can denote an assignment for all variables in \( \psi \) by \( \sigma_1 \cdots \sigma_n \), where \( \sigma_i \) is a partial assignment for \( \text{Var}(\psi) \cap X_i \) for every \( 1 \leq i \leq n \).
Let \( \varphi \) be a CNF formula. Throughout the paper, we assume \(|X_\varphi| \geq 2\) and \(|C_\varphi| \geq 2\). Furthermore, for any variable \( z \), every clause \( C \in C_\varphi \) includes at most one literal of \( z \). By these assumptions, we do not lose the generality.

### 2.2 Complexity classes and hierarchies

In this paper, we assume that the reader is familiar with basic notions and results on the computational complexity theory. The reader is referred to [2, 9, 23] if necessary. We assume the alphabet of a given problem or relation to be \( \{0, 1\} \) unless stated otherwise. Let \( \mathcal{C} \) be a class of decision problems. Let \( L \) be a decision problem in \( \mathcal{C} \). We denote by \( R_L \) a binary relation such that a pair \((u, v)\) is in \( R_L \) if and only if \( v \) is a witness of \( u \) in \( L \). We often call \( R_L \) the underlying binary relation of \( L \). We assume \( R_L \) to be polynomially balanced; i.e., for every pair \((u, v)\) in \( R_L \), \(|v| \leq p(|u|)\), where \( p \) is a polynomial. For every instance \( u \) of \( L \), we denote the set of all witnesses of \( u \) by \( W_L(u) \). Let \( f \) be a function from \( \{0, 1\}^* \) to \( \mathbb{N} \) such that \( f(u) = |W_L(u)| \) for a given \( u \in \{0, 1\}^* \). We call \( f \) the counting problem for \( L \). We often denote \( f \) by \( \#L \). For any binary relation \( R \), we assume that every pair of \( R \) is encoded over \( \{0, 1\}^* \). We denote by \( \# \cdot \mathcal{C} \) the class of all counting problems \( \#S \) such that \( R_S \in \mathcal{C} \).

We denote by \( M^A \) an oracle Turing machine with an oracle \( A \). We denote the problem that an oracle machine \( M^A \) accepts by \( L(M^A) \). We consider \( M^\emptyset \) to be \( M \). Let \( L^A \) be a problem \( L \) such that \( L(M^A) = L \) for some oracle machine \( M^A \). If \( \mathcal{B} \) is a class of decision problems and \( A \) is a decision problem, then we denote by \( \mathcal{B}^A \) the class of all problems \( B^A \) such that \( B \in \mathcal{B} \). If \( \mathcal{A} \) is a class of decision problems, then we denote by \( \# \cdot \mathcal{A} \) the class of all counting problems \( \#L \) such that \( L \in \mathcal{NP}^A \) for some \( A \in \mathcal{A} \). Note that for a given class \( \mathcal{C} \) of decision problems, if \( \co \mathcal{C} \) denotes the complement of \( \mathcal{C} \), then \( \# \co \mathcal{C} = \# \mathcal{C} \) by definition. If \( \mathcal{A} \) and \( \mathcal{B} \) are classes of decision problems, then we denote the class \( \{B^A : A \in \mathcal{A}\} \) by \( \mathcal{B}^\mathcal{A} \).

We define the classes \( \Sigma_k^p \) and \( \Pi_k^p \), where \( k \geq 0 \), introduced by [21] inductively as follows. \( \Sigma_0^p \) is the class \( \mathcal{P} \). For any \( k \geq 1 \), \( \Sigma_k^p \) is the class \( \mathcal{NP}^{\Sigma_{k-1}^p} \). For every \( k \geq 0 \), \( \Pi_k^p \) is the class of the complements of all problems in \( \Sigma_k^p \). We call the sequence \( \Sigma_0^p, \ldots, \Sigma_k^p, \ldots \) of the classes the polynomial hierarchy. We denote the polynomial hierarchy by \( \mathcal{PH} \). By definition, the classes \( \Sigma_1^p \) and \( \Pi_1^p \) coincide with \( \mathcal{NP} \) and \( \mathcal{coNP} \), respectively. The polynomial hierarchy has other characterizations. We can characterize \( \mathcal{PH} \) by polynomially balanced binary relations [39]. Let \( k \geq 1 \). A decision problem \( L \) is in \( \Sigma_k^p \) if and only if its underlying binary relation \( R_L \) is in \( \Pi_{k-1}^p \). The other characterization of \( \mathcal{PH} \) is by \( k \)-alternating Turing machines [6]. In this paper, we use those three characterizations. Let \( k \geq 0 \). We adopt a notation for decision problems in \( \Sigma_k^p \) and \( \Pi_k^p \). Let \( L \) be a decision problem. We define decision problems \( \Sigma_k L \) and \( \Pi_k L \) inductively as follow. \( \Sigma_1 L \) is \( L \). For every \( k \geq 2 \), \( \Sigma_k L \) is a decision problem \( S \) such that its underlying binary relation \( R_S \) is
in $\Pi_{k-1}L$. For every $k \geq 1$, $\Pi_k L$ is the complement of $\Sigma_k L$.

Given a decision problem in $\Sigma_k^P$ or $\Pi_k^P$, where $k \geq 0$, another type of counting problem were studied by Bauland et al. [4, 10]. Let $L$ be a decision problem. We denote by #-$\Sigma_k L$ and #-$\Pi_k L$ the following counting problems $f_1$ and $f_2$, respectively. Given $w \in \{0, 1\}^*$, $f_1(w) = |W_K(w)|$ and $f_2(w) = |W_J(w)|$, where $K$ and $J$ are decision problems such that $R_K \in \Sigma_k L$ and $R_J \in \Pi_k L$, respectively. Note that the decision counterpart of #-$\Pi_{k-1} L$ is $\Sigma_k L$ while the decision counterpart of #-$\Sigma_k L$ is $\Sigma_k L$. This asymmetry is inherent in counting problems and is from the task that counts the existing solutions for a given input.

By the definitions of the polynomial hierarchy and #-$C$ and #-$C$ for some decision problem class $C$, we immediately obtain the following hierarchies of counting complexity classes. We denote by #-$PH$ and #-$PH$ the sequences of #-$\Sigma_0^P$, #-$\Sigma_1^P$ and #-$\Sigma_0^P$, #-$\Sigma_1^P$ of the counting complexity classes, respectively.

2.3 Relaxed subtractive reductions and its special cases

Durand et al. [10] introduced subtractive reductions and found some complete problems on counting under this reductions. We introduce a new type of reduction by generalizing subtractive reductions.

**Definition 1.** Let $A$ and $B$ be decision problems, respectively. We define a strong relaxed subtractive reduction from $#A$ to $#B$ as a pair $(t_0, t_1)$, where $t_0$ and $t_1$ are polynomial-time computable functions from $\{0, 1\}^*$ to $\{0, 1\}^*$, which satisfy the following. There is a polynomially balanced binary relation $R_F \subset \{0, 1\}^* \times \{0, 1\}^*$ such that $#F \in FP$ and for any $w \in \{0, 1\}^*$,

1. $W_F(w) \subseteq W_B(t_0(w))$,
2. $W_B(t_1(w)) \subseteq W_B(t_0(w))$,
3. $W_B(t_1(w)) \cap W_F(w) = \emptyset$,
4. $|W_A(w)| = |W_B(t_0(w))| - |W_B(t_1(w))| - |W_F(w)|$.

Figure 1 illustrates an Euler diagram for three related sets in a strong relaxed subtractive reduction. Let $K$ and $L$ be problems. Let $\xi$ be a type of polynomial-time reduction. Then, we say that $K$ is $\xi$ reducible to $L$ if there is a $\xi$ reduction from $K$ to $L$. We say that a relation $R$ is the $\xi$ reducibility if $R$ consists of all pairs $(K, L)$ such that $K$ is $\xi$ reducible to $L$.

**Definition 2.** We call the reflexive and transitive relation of strong relaxed subtractive reducibility the relaxed subtractive reductibility; i.e., we call a sequence $(t_{01}, t_{11}), \ldots, (t_{0n}, t_{1n})$ a relaxed subtractive reduction if there is a sequence $A_0, \ldots, A_n$ of problems such that $(t_{0i}, t_{1i})$ is a strong relaxed subtractive reduction from $A_{i-1}$ to $A_i$ for every $1 \leq i \leq n$. 

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We define some special cases of (strong) relaxed subtractive reductions as follows. Let $A$, $B$, $(t_0, t_1)$, and $F$ be defined as in 1. We call $(t_0, t_1)$ a strong subtractive reduction if $WF(w) = \emptyset$ for every $w \in \{0, 1\}^*$. Similarly, we define subtractive reductions. We call $(t_0, t_1)$ a parsimonious reduction if $t_1(w) = WF(w) = \emptyset$ for every $w \in \{0, 1\}^*$. We usually denote a parsimonious reduction $(t_0, t_1)$ by simply $t_0$. Parsimonious reductions have the following significant property for discussing the completenesses for classes $\# \cdot \Sigma_k^P$ and $\# \cdot \Pi_k^P$.

**Theorem 3** ([4, 10]). The classes $\# \cdot \Sigma_k^P$ and $\# \cdot \Pi_k^P$ are closed under parsimonious reductions.

### 2.4 Related computational problems

For convenience, we summarize the computational problems that we describe in this paper.

#### 2.4.1 Decision problems

**Problem 4** (NS Gaps).

**Input.** A set $A$ of coprime positive integers $a_1, \cdots, a_n$ such that $2 \leq a_1 < \cdots < a_n$ and $n \geq 2$.

**Question.** Is there a gap of $S(A)$?

**Comment.** This problem is trivial since the answer is always “Yes”.

**Problem 5** (Bounded NS Gaps).

**Input.** A pair $(A, \kappa)$, where $A$ is a set of coprime positive integers $a_1, \cdots, a_n$ such that $2 \leq a_1 < \cdots < a_n$ and $n \geq 2$, and $\kappa \in \mathbb{N}$.

**Question.** Is there an integer in $N(A) \cap [\kappa, \infty]$?
Problem 6 (NonRep).

Input. A pair \((A, [κ_0, κ_1])\), where \(A\) is a set of positive integers \(a_1, \ldots, a_n\) such that \(2 \leq a_1 < \cdots < a_n\) and \(n \geq 2\), and \(κ_0, κ_1 \in \mathbb{N}\).

Question. Is there an integer in \(N(A) \cap [κ_0, κ_1]\)?

Comment. “NonRep” is an abbreviation for “nonrepresentable”.

Note. \(a_1, \ldots, a_n\) may not necessarily be coprime.

As a comprehensive survey of the decision problems in the 2nd and 3rd levels of the polynomial hierarchy and the results on them, we can refer to Marcus Schäfer and Umans [29, 30].

2.4.2 Counting problems

Problem 7 (\#NS Gaps).

Input. A set \(A\) of coprime positive integers \(a_1, \ldots, a_n\) such that \(2 \leq a_1 < \cdots < a_n\) and \(n \geq 2\).

Output. \(|N(A)|\); i.e., the number of gaps of \(S(A)\).

Problem 8 (\#Bounded NS Gaps).

Input. A pair \((A, κ)\), where \(A\) is a set of coprime positive integers \(a_1, \ldots, a_n\) such that \(2 \leq a_1 < \cdots < a_n\) and \(n \geq 2\), and \(κ \in \mathbb{N}\).

Output. \(|N(A) \cap [κ, \infty)|\); i.e., the number of gaps of \(S(A)\), which is greater than or equal to \(κ\).

Problem 9 (\#NonRep).

Input. A pair \((A, [κ_0, κ_1])\), where \(A\) is a set of positive integers \(a_1, \ldots, a_n\) such that \(2 \leq a_1 < \cdots < a_n\) and \(n \geq 2\), and \(κ_0, κ_1 \in \mathbb{N}\).

Output. \(|N(A) \cap [κ_0, κ_1]|\); i.e., the number of gaps of \(S(A)\) in \([κ_0, κ_1]\).

Comment. “NonRep” is an abbreviation for “nonrepresentable”.

Note. \(a_1, \ldots, a_n\) may not necessarily be coprime.

To formulate related counting problems on Boolean satisfiability, we define two sets \(Φ\) and \(Φ_{1/3}\) as follows.

Definition 10. \(Φ\) is the set of all triples \((k, ϕ, σ)\), where \(k\) is a positive integer, \(ϕ\) is a CNF-formula \(C_1 \land \cdots \land C_m\), and \(σ\) is a truth assignment, such that \((k, ϕ, σ) \in Φ\) if and only if \(k\) is even and \(|C_α \cap T(ϕ, σ)| \neq 0\) for every \(1 \leq α \leq m\) or \(k\) is odd and \(|C_α \cap T(ϕ, σ)| = 0\) for some \(1 \leq α \leq m\).

Definition 11. \(Φ_{1/3}\) is the set of all triples \((k, ϕ, σ)\), where \(k\) is a positive integer, \(ϕ\) is a CNF-formula \(C_1 \land \cdots \land C_m\), and \(σ\) is a truth assignment, such that \((k, ϕ, σ) \in Φ_{1/3}\) if and only if \(k\) is even and \(|C_α \cap T(ϕ, σ)| = 1\) for every \(1 \leq α \leq m\) or \(k\) is odd and \(|C_α \cap T(ϕ, σ)| \neq 1\) for some \(1 \leq α \leq m\).

Problem 12 (\#-Σ_k SAT, \(k \geq 0\)).

Input. A CNF formula \(ϕ\) over \(X \cup \bigcup_{1 \leq i \leq k} X_i\), where \(X, X_1, \ldots, X_k\) are pairwise disjoint.
Example 16. Let \( \sigma \in \{0, 1\}^{|\varphi|} \), \( \exists \sigma_1 \forall \sigma_2 \cdots \exists \sigma_k [(k, \varphi, \sigma_1 \cdots \sigma_k) \in \Phi] \), where \( \sigma_i \) is a partial assignment for \( X_{i,\varphi} \) for every \( i \) with \( 1 \leq i \leq k \).

Comment. It is known to be \( \# \cdot \Sigma_k^P \)-complete under parsimonious reductions [10].

Note. We consider the problem \( \# \cdot \Sigma_0 \text{SAT} \) to be \( \#\text{SAT} \).

Problem 13 (\( \# \cdot \Pi_k \text{SAT} \), \( k \geq 1 \)).

Input. A CNF formula \( \varphi \) over \( X \cup \bigcup_{1 \leq i \leq k} X_i \), where \( X, X_1, \ldots, X_k \) are pairwise disjoint.

Output. \( \{\sigma \in \{0, 1\}^{|\varphi|} : \forall \sigma_1 \exists \sigma_2 \cdots \exists \sigma_k [(k, \varphi, \sigma_1 \cdots \sigma_k) \in \Phi] \} \), where \( \sigma_i \) is a partial assignment for \( X_{i,\varphi} \) for every \( i \) with \( 1 \leq i \leq k \).

Comment. It is known to be \( \# \cdot \Pi_k^P \)-complete under parsimonious reductions [10].

Problem 14 (\( \# \cdot \Sigma_k \text{1IN3SAT} \), \( k \geq 0 \)).

Input. A 3-CNF formula \( \varphi \) over \( X \cup \bigcup_{1 \leq i \leq k} X_i \), where \( X, X_1, \ldots, X_k \) are pairwise disjoint.

Output. \( \{\sigma \in \{0, 1\}^{|\varphi|} : \exists \sigma_1 \forall \sigma_2 \cdots \exists \sigma_k [(k, \varphi, \sigma_1 \cdots \sigma_k) \in \Phi_{1/3}] \} \), where \( \sigma_i \) is a partial assignment for \( X_{i,\varphi} \) for every \( i \) with \( 1 \leq i \leq k \).

Note. We consider the problem \( \# \cdot \Sigma_0 \text{1IN3SAT} \) to be \( \#\text{1IN3SAT} \).

Problem 15 (\( \# \cdot \Pi_k \text{1IN3SAT} \), \( k \geq 1 \)).

Input. A 3-CNF formula \( \varphi \) over \( X \cup \bigcup_{1 \leq i \leq k} X_i \), where \( X, X_1, \ldots, X_k \) are pairwise disjoint.

Output. \( \{\sigma \in \{0, 1\}^{|\varphi|} : \forall \sigma_1 \exists \sigma_2 \cdots \exists \sigma_k [(k, \varphi, \sigma_1 \cdots \sigma_k) \in \Phi_{1/3}] \} \), where \( \sigma_i \) is a partial assignment for \( X_{i,\varphi} \) for every \( i \) with \( 1 \leq i \leq k \).

Example 16. Let \( \varphi_1 \) be a 3-CNF formula \( C_1 \land C_2 \land C_3 \land C_4 \) over \( X \cup Y \), where \( X \) and \( Y \) are pairwise disjoint, \( C_1 = \neg x_1 \lor x_2 \lor y_1, C_2 = x_1 \lor x_3 \lor \neg y_2, C_3 = \neg x_2 \lor x_4 \lor \neg y_3, C_4 = x_1 \lor y_2 \lor y_3 \), \( X_{\varphi_1} = \{x_1, x_2, x_3, x_4\} \) and \( Y_{\varphi_1} = \{y_1, y_2, y_3\} \). We can check the values of every assignment in Table 1. In this table, every cell corresponds to an assignment \( \sigma \) for \( \varphi_1 \) and contains the indices of all clauses \( C \) such that \( |T(C, \sigma)| = 1 \). In the case where \( \sigma_x = 000 \), if we let \( \sigma_y = 001 \), then exactly one literal is 1 in every clause. Similarly, in the case where \( \sigma_x = 1100 \), if we let \( \sigma_y = 010 \), then exactly one literal is 1 in every clause. However, if \( \sigma_x \) is one of the other \( X \)-assignments, no \( Y \)-assignment exists such that exactly one literal is 1 in every clause. Thus, by Table 1, we can observe that \( \# \cdot \Pi_1 \text{1IN3SAT}(\varphi_1) \) is 14; and moreover, \( \varphi_1 \) is an yes-instance of \( \Sigma_2 \text{1IN3SAT} \).

We summarize the relations between decision problems and their counting problems and their complexity classes in Table 2.

2.5 The complexities of \( \# \Sigma_k \text{1IN3SAT} \) and \( \# \Pi_k \text{1IN3SAT} \)

In this section, we describe the \( \# \cdot \Pi_k^P \)-completeness of \( \# \Pi_k \text{1IN3SAT} \). This statement is equivalent to Lemma 5.5 in [4] and they proved it by using a
Table 1: Clauses in \( \varphi_1 \), each of which contains exactly one true literal, where each cell corresponds to an assignment and contains indices of clauses.

| \( \sigma_x \) | 000 | 100 | 010 | 001 | 110 | 101 | 011 | 111 |
|----------------|-----|-----|-----|-----|-----|-----|-----|-----|
| 0000           | 1, 2| 1, 4| 1, 2, 3, 4| 4 | 2, 3, 4| 1, 3 | 3 |
| 1000           | — 1 | 2, 4 | 3, 4 | 1, 2, 4 | 1, 3, 4 | 2, 3 | 1, 2, 3 |
| 0100           | 2, 3 | 2, 3 | 3, 4 | 2, 4 | 3, 4 | 2, 4 | — | — |
| 0010           | 1 — | 1, 2, 4 | 1, 3, 4 | 2, 4 | 3, 4 | 1, 2, 3 | 2, 3 |
| 0001           | 1, 2, 4 | 2, 4 | 1 | 1, 2 | — | 2 | 1 | — |
| 1100           | 1, 3 | 3 | 1, 2, 3, 4 | 1, 4 | 2, 3, 4 | 4 | 1, 2 | 2 |
| 1010           | — 1 | 4 | 3, 4 | 1, 4 | 1, 3, 4 | 3 | 1, 3 |
| 1001           | 4 | 1, 4 | 2 | — | 1, 2 | 1 | 2 | 1, 2 |
| 0110           | 3 | 3 | 2, 3, 4 | 4 | 2, 3, 4 | 4 | 2 | 2 |
| 0101           | 2, 4 | 2, 4 | — | 2, 3 | — | 2, 3 | 3 | 3 |
| 0011           | 1, 4 | 4 | 1, 2 | 1 | 2 | — | 1, 2 | 2 |
| 1110           | 1, 3 | 3 | 1, 3, 4 | 1, 4 | 3, 4 | 4 | 1 | — |
| 1101           | 1, 4 | 4 | 1, 2 | 1, 3 | 2 | 3 | 1, 2, 3 | 2, 3 |
| 1011           | 4 | 1, 4 | — | — | 1 | 1 | — | 1 |
| 0111           | 4 | 4 | 2 | 3 | 2 | 3 | 2, 3 | 2, 3 |
| 1111           | 1, 4 | 4 | 1 | 1, 3 | — | 3 | 1, 3 | 3 |

theory for constraints satisfaction problems, which is a more general tool for investigating satisfiability problems than the one in this paper. For readability, we give a slightly weaker but more intuitive and elementary proof. The reduction in the following proof is a folklore, although any explicit literature has not been published, to the best of the author’s knowledge.

**Theorem 17.** \( \# \cdot \Sigma_k \text{1IN3SAT} \) is \( \# \cdot \Sigma_k^P \)-complete under parsimonious reductions for every \( k \geq 0 \).

**Proof.** By Theorem 2.1 in [10], \( \# \cdot \Sigma_k \text{SAT} \) is \( \# \cdot \Sigma_k^P \)-complete under parsimonious reductions. Since \( \# \cdot \Sigma_k \text{1IN3SAT} \) is a special case of \( \# \cdot \Sigma_k \text{SAT} \), the problem \( \# \cdot \Sigma_k \text{1IN3SAT} \) is in \( \# \cdot \Sigma_k^P \). Thus, it suffices to show that \( \# \cdot \Sigma_k \text{SAT} \) is parsimonious reducible to \( \# \cdot \Sigma_k \text{1IN3SAT} \). Let \( X, Y_1, \ldots, Y_{k-1} \) be pairwise disjoint sets of variables. Let \( Z \) be the set \( X \cup \bigcup_{i=1}^{k-1} Y_i \). Let \( \varphi \) be a 3-CNF formula over \( Z \). We will define a set \( V \) of new variables and then construct a 3-CNF formula \( \varphi' \) such that

\[
|\{ \sigma' \in \{0,1\}^{X \cup V} : \exists \sigma'_1 \lor \sigma'_2 \ldots Q \sigma'_{k-1}[(k, \varphi, \sigma'_1 \ldots \sigma'_{k-1}) \in \Phi_{k/3}]\}| = |\{ \sigma \in \{0,1\}^{X \cup X} : \exists \sigma_1 \lor \sigma_2 \ldots Q \sigma_{k-1}[(k, \varphi, \sigma_1 \ldots \sigma_{k-1}) \in \Phi]\}|
\]

where \( \sigma_i \) and \( \sigma'_i \) denote partial assignments for \( X_i \) for every \( i \) with \( 1 \leq i \leq k-1 \).
Table 2: Complexities of related computational problems.

| Counting problems | Upper bounds | Decision problems |
|-------------------|--------------|------------------|
|                    | [reduction type] |                  |
| #NonRep           | #NP-complete | NonRep           |
|                   | [parsimonious] | (Section 4)      |
| Bounded NS Gaps   | #NP-complete | Bounded NS Gaps  |
|                   | [parsimonious] | (Section 5)      |
| #NS Gaps          | #NP-complete | NS Gaps          |
|                   | [relaxed subtractive] | (Section 6) |
| #SAT              | #P-complete | SAT              |
|                   | [parsimonious] | [36]             |
| #· Σ₁SAT          | #NP-complete | SAT              |
|                   | [parsimonious] | [10]             |
| #· Πₖ₋₁SAT        | #· Πₖ₋₁-complete |               |
|                   | [parsimonious] | [10]             |
| #· ΣₖSAT          | #· Σₖ-complete | ΣₖSAT            |
|                   | [parsimonious] |                  |
| #· 1IN₃SAT        | #P-complete | 1IN₃SAT          |
|                   | [parsimonious] |                  |
| #· Σ₁1IN₃SAT      | #NP-complete | 1IN₃SAT          |
|                   | [parsimonious] |                  |
| #· Πₖ₋₁1IN₃SAT    | #· Πₖ₋₁-complete |               |
|                   | [parsimonious] |                  |
| #· Σₖ1IN₃SAT      | #· Σₖ-complete | Σₖ1IN₃SAT        |
|                   | [parsimonious] |                  |
Table 3: All satisfying assignments for the constructed formula $\varphi'_C$ from a clause in the proof for Theorem 17.

| $\tilde{z}_1$ | $\tilde{z}_2$ | $\tilde{z}_3$ | $\langle C \rangle_1$ | $\langle C \rangle_2$ | $\langle C \rangle_3$ | $\langle C \rangle_4$ | $\langle C \rangle_5$ | $\langle C \rangle_6$ | $\langle C \rangle_7$ | $\langle C \rangle_8$ | $\langle C \rangle_9$ |
|------|------|------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 0    | 0    | 1    | 0               | 1               | 0               | 0               | 1               | 0               | 1               | 0               | 0               |
| 0    | 1    | 0    | 1               | 0               | 0               | 0               | 1               | 0               | 1               | 0               | 0               |
| 0    | 1    | 1    | 0               | 0               | 0               | 0               | 0               | 1               | 1               | 0               | 0               |
| 1    | 0    | 0    | 0               | 0               | 1               | 0               | 1               | 0               | 0               | 1               | 0               |
| 1    | 0    | 1    | 0               | 0               | 1               | 0               | 0               | 0               | 1               | 1               | 0               |
| 1    | 1    | 0    | 0               | 0               | 1               | 0               | 1               | 0               | 0               | 1               | 0               |
| 1    | 1    | 1    | 0               | 0               | 0               | 1               | 0               | 0               | 0               | 1               | 1               |

Let $C$ be a clause $\tilde{z}_1 \lor \tilde{z}_2 \lor \tilde{z}_3$ in $\varphi$, where $\tilde{z}_1$, $\tilde{z}_2$, and $\tilde{z}_3$ are literals over $V \cup Z$. Let $\langle C \rangle_1, \ldots, \langle C \rangle_9$ be new variables. Then we define a 3-CNF formula $\varphi'_C$ as the conjunction of clauses

$\tilde{z}_1 \lor \langle C \rangle_1 \lor \langle C \rangle_2$, $\tilde{z}_2 \lor \langle C \rangle_2 \lor \langle C \rangle_3$, $\tilde{z}_3 \lor \langle C \rangle_5 \lor \langle C \rangle_6$, 
$\tilde{z}_3 \lor \langle C \rangle_6 \lor \langle C \rangle_7$, $\langle C \rangle_1 \lor \langle C \rangle_3 \lor \langle C \rangle_4$, $\langle C \rangle_5 \lor \langle C \rangle_7 \lor \langle C \rangle_8$, 
$\langle C \rangle_2 \lor \langle C \rangle_6 \lor \langle C \rangle_9$.

Table 3 illustrates all satisfying assignments for $\varphi'_C$. We define $V$ as the set

$\{ \langle C \rangle_i : C \text{ is a clause in } \varphi, 1 \leq i \leq 9 \}$.

Finally, we define $\varphi'$ as the conjunction of $\varphi'_C$ for all clauses $C$ in $\varphi$.

We can check the following statement. For every assignments $\sigma \in \{0,1\}^{X|\varphi|}$ and $\delta \in \{0,1\}^{Z|\varphi|}$, if $(k, \varphi, \sigma \delta) \in \Phi$ then there is exactly one $\sigma' \in \{0,1\}^{X|\varphi|+|V|}$ such that $(k, \varphi, \sigma' \delta) \in \Phi_{1/3}$. Moreover, we can check the statement of the converse direction. We can construct $\varphi'$ in polynomial time since we construct 9 variables and a formula of 7 clauses for every clause in a given $\varphi$. $\square$

Durand et al. [10] describes the $\# \cdot \Sigma^P_k$-completeness under parsimonious reductions explicitly without proofs. As they described in the paper [10], the proof for the statement is due to [39]. Although the proof in [39] is for the $\Sigma^P_k$-completenesses of $\Sigma_k\text{SAT}$, the reduction is parsimonious. Theorem 3.13 in [9] also give a more concise proof for the same statement.

By the proof for Theorem 17, we immediately obtain the following corollary.

**Corollary 18.** $\# \cdot \Pi_k\text{IN3SAT}$ is $\# \cdot \Pi^P_k$-complete under parsimonious reductions for every $k \geq 1$. 

14
Overview of three versions of the numerical-semigroup-gap counting problem

By definition, \#NonRep is a generalization of \#Bounded NS Gaps and \#Bounded NS Gaps is a generalization of \#NS Gaps. We can observe these relationships as follows. Given a set $A$ of coprime positive integers $a_1, \ldots, a_n$ with $2 \leq a_1 < \cdots < a_n$ and $n \geq 2$, the Frobenius number $g(A)$ is known to be smaller than $a_n^2$ \cite{38}. Since $g(A)$ is the largest element of $N(A)$ and $N(A)$ consists of only positive integers, the cardinality of $N(A)$ is less than $a_n^2$. For example, if $a_1 = 12, a_2 = 19, a_3 = 51, a_4 = 53$ and $A = \{a_1, a_2, a_3, a_4\}$, then $|N(A)| = 60, g(A) = 109$, and $a_n^2 = 2809$. Thus, we consider an instance $(A, \lambda)$ of \#Bounded NS Gaps to be an instance $(A, [\lambda, a_n^2])$ of \#NonRep; and an instance $A$ of \#NS Gaps to be an instance $(A, 1)$ of \#Bounded NS Gaps. Thus, the hardnesses of three problems are \#NS Gaps, \#Bounded NS Gaps, \#NonRep in the order from the weakest one.

On the other hand, the difficulties of reductions from \#·Π₁₁IN₃SAT appear to be in the reverse order for the ones of the problems in themselves. \#NonRep has a structure that is the most similar to \#·Π₁₁IN₃SAT among the three problems. \#NS Gaps has a structure that is the farthest to \#·Π₁₁IN₃SAT. We remark the following elementary fact. We use 19 in Sections 5 and 6, implicitly.

**Remark 19:** If $A_1$ and $A_2$ are sets of coprime positive integers such that $A_1 \subseteq A_2$, then $N(A_2) \subseteq N(A_1)$.

To prove the \#NP-completensess of three problems \#NonRep, \#Bounded NS Gaps, and \#NS Gaps, we use a standard approach for problems on numbers; e.g., the proofs for Theorem 3.5 in \cite{12} and Theorem 9.10 in \cite{23}. In Section 4, we will describe almost all parts of the reduction. In Sections 5 and 6, we will extend the reductions of the previous sections.

### 4 Complexity of \#NonRep

In this section, we investigate the complexity of \#NonRep. In particular, we prove the following theorem.

**Theorem 20.** \#NonRep is \#NP-complete under parsimonious reductions.

#### 4.1 Ideas of the proof

In this subsection, we describe the idea of the proof for Theorem 20. We can easily check the membership of \#NonRep to \#NP. Thus, the main part
of the proof is for the hardness of \( \#\text{NonRep} \). The proof is by constructing a relaxed subtractive reduction from \( \#\cdot \Pi_1^1 1\text{IN3SAT} \) to \( \#\text{NonRep} \) in Section 4.2. We first describe the reduction intuitively and illustrate an example.

Let \( \varphi \) be a 3-CNF formula over \( X \cup Y \), where \( X \) and \( Y \) are pairwise disjoint. To simulate the behavior of \( \varphi \), we define a positive integer for each pair of a truth assignment and a Boolean variable. Let us describe the binary representations for integers that we construct. Let \( \beta \) be one of such binary representations. We consider \( \beta \) to be partitioned into 8 zones. Each of 8 zones is classified into one of 5 types with its role. We explain each of the 5 types in the order from the smaller bits. The 1st type is for simulating an assignment for a variable in \( X \). The 2nd and 3rd types are for simulating which variable in \( X \) and \( Y \) we assign a Boolean value to, respectively. The 4th type is for simulating which clause we assign 1 to a literal in. The 5th type is introduced for a purpose not directly for simulating anything in an assignment for \( \varphi \) which we will describe below in more detail.

Each type of subrepresentation is allocated to a fixed range of positions, which depends on \( \varphi \). Figure 2 illustrates the binary representation of a constructed integer. In Figure 2, each of the 5 types of subrepresentations is distinguished by a kind of shading of gray. For a technical reason, the 2nd to 4th types of subrepresentations have copies of them. Such copies are not for the proof for Theorem 20 but for the ones for Theorems 25 and 30. We give each of 8 zones a name as in Figure 2. That is, the names of zones are \( 1S, C2, C1, Y2, Y1, X2, X1, \) and \( XA \) in the descending order from the highest bits.

We describe more details of integers that we construct. Let \( x_1, \ldots, x_{k_1} \) and \( y_1, \ldots, y_{k_2} \) be all variables in \( X \) and \( Y \), occurring in \( \varphi \), respectively. Let \( C_1, \ldots, C_{k_3} \) be all clauses in \( \varphi \). Every constructed integer is represented with \( 3k_1 + 2k_2 + 2k_3 + \lceil \log(3k_1 + 2k_2 + 2k_3) \rceil + 1 \) bits, where \( k_1 = |X\varphi| \), \( k_2 = |Y\varphi| \), and \( k_3 = |C\varphi| \). Each of the \( X2, X1, \) and \( XA \) zones is of length \( k_1 \). Each of the \( Y2 \) and \( Y1 \) zones is of length \( k_2 \). Each of the \( C2 \) and \( C1 \) zones is of length \( k_3 \).
of length $k_3$. For example, we construct integers for $\varphi_1$ in 16 as illustrated in Table 4. In Table 4, we denote the $IS$ zone with its decimal representation and the other zones with their binary representations for reflecting the role of each zone.

Table 4: Integers constructed from $\varphi_1 = C_1 \land C_2 \land C_3 \land C_4$, where $C_1 = \neg x_1 \lor x_2 \lor y_1$, $C_2 = x_1 \lor x_3 \lor \neg y_2$, $C_3 = \neg x_2 \lor x_4 \lor \neg y_3$, $C_4 = x_4 \lor y_2 \lor y_3$.

| Variables | Values | $IS$ | $C2$ | $C1$ | $Y2$ | $Y1$ | $X2$ | $X1$ | $XA$ |
|-----------|--------|------|------|------|------|------|------|------|------|
| $x_1$     | 1      | 5    | 0010 | 0010 | 000  | 000  | 0001 | 0001 | 0001 |
| $x_1$     | 0      | 5    | 0001 | 0001 | 000  | 000  | 0001 | 0001 | 0000 |
| $x_2$     | 1      | 5    | 0001 | 0001 | 000  | 000  | 0010 | 0010 | 0010 |
| $x_2$     | 0      | 5    | 0100 | 0100 | 000  | 000  | 0010 | 0010 | 0000 |
| $x_3$     | 1      | 5    | 0010 | 0010 | 000  | 000  | 0100 | 0100 | 0100 |
| $x_3$     | 0      | 3    | 0000 | 0000 | 000  | 000  | 0100 | 0100 | 0000 |
| $x_4$     | 1      | 7    | 1100 | 1100 | 000  | 000  | 1000 | 1000 | 1000 |
| $x_4$     | 0      | 3    | 1100 | 1100 | 000  | 000  | 1000 | 1000 | 1000 |
| $y_1$     | 1      | 4    | 0001 | 0001 | 001  | 001  | 0000 | 0000 | 0000 |
| $y_1$     | 0      | 2    | 0000 | 0000 | 001  | 001  | 0000 | 0000 | 0000 |
| $y_2$     | 1      | 4    | 1000 | 1000 | 010  | 010  | 0000 | 0000 | 0000 |
| $y_2$     | 0      | 4    | 0010 | 0010 | 010  | 010  | 0000 | 0000 | 0000 |
| $y_3$     | 1      | 4    | 1000 | 1000 | 100  | 100  | 0000 | 0000 | 0000 |
| $y_3$     | 0      | 4    | 0100 | 0100 | 100  | 100  | 0000 | 0000 | 0000 |

We describe the detail of the role of each zone of a binary representation that we construct. A constructed integer corresponds to a pair of a Boolean variable and value. Let $\beta$ be a binary representation that we construct.

Let us first consider $\beta$ to be one for simulating $x_i \in X$ assigned 1, where $1 \leq i \leq k_1$. In the $XA$ zone, we set the $i$-th bit to 1 and the other bits to 0s. This means that $x_i$ is assigned 1 and the other variables are assigned 0. In each of the $XI$ and $X2$ zones, we set the $i$-th bit to 1 and the other bits to 0s. This means that $x_i$ is assigned a Boolean value but the other variables are not assigned. In each of the $Y1$ and $Y2$ zones, we set every bit to 0. This means that $b$ is not for simulating the behavior of a variable in $Y$. Moreover, in each of the $C1$ and $C2$ zones, for every $1 \leq j \leq k_3$, we set the $j$-th bit to 1 if the clause $C_j$ includes the literal $x_i$; and the other bits to 0s otherwise. Let $v$ be the binary representation of the integer $3 + 2l$ such that $l$ is the numbers of 1s, occurring in the $C1$ zone. We allocate $v$ to the $IS$ zone of $\beta$.

Let us next consider $\beta$ to be one for simulating $x_i \in X$ assigned 0. We set every bit of the $XA$ zone to 0. This means that at least $x_i$ is assigned 1, but does not mean that each of the remaining variables is assigned 0.
The $X_1$, $X_2$, $Y_1$, and $Y_2$ zones are constructed in the same way as for the case where $x_i$ is assigned 1. In each of the $C1$ and $C2$ zones, for every $1 \leq j \leq k_3$, we set the $j$-th bit to 1 if the clause $C_j$ includes the literal $\neg x_i$; and the other bits to 0s otherwise. Let $v$ be the binary representation of the integer $2 + 2l$ such that $l$ is the numbers of 1s, occurring in the $C1$ zone. We allocate $v$ to the $1S$ zone of $\beta$.

Then, let us consider $\beta$ to be one for simulating $y_i \in Y$ assigned 1, where $1 \leq i \leq k_2$. In each of the $XA$, $X1$, and $X2$ zones, we set every bit to 0. This means that $b$ is not for simulating the behavior of a variable in $X$. In each of the $Y1$ and $Y2$ zones, we set the $i$-th bit to 1 and the other bits to 0s. This means that $y_i$ is assigned a Boolean value but the other variables are not assigned. Note that, in more general, we do not construct the “YA” part since our concern is on the counting of partial assignments for variables in $X_{\varphi}$. In each of the $C1$ and $C2$ zones, for every $1 \leq j \leq k_3$, we set the $j$-th bit to 1 if the clause $C_j$ includes the literal $y_i$; and the other bits to 0s otherwise. Let $v$ be the binary representation of the integer $2 + 2l$ such that $l$ is the numbers of 1s, occurring in the $C1$ zone. We allocate $v$ to the $1S$ zone of $\beta$. Similarly, we can consider the construction in the case where $\beta$ is one for simulating $y_i$ assigned 0.

As described above, we simulate an assignment to a variable with an integer. Moreover, for every variable, we construct two integers that correspond to assigned values. Let $A$ be the set of $(k_1 + k_2)$ integers such that for every variable in $X_{\varphi} \cup Y_{\varphi}$, exactly one of the two constructed integers is in $A$. Let $\alpha$ be the binary representation of $\sum_{a \in A} a$. Then, we can simulate an assignment for $\varphi$ with $\alpha$. Every bit in the $X1$, $X2$, $Y1$, and $Y2$ zones of $\alpha$ is 1. The $1S$ zone represent the integer $3\#_1(w_1) + 2\#_1(w_2) + 2\#_1(w_3)$ such that $w_1$, $w_2$, and $w_3$ are the $X1$, $Y1$, and $C1$ zones, respectively. The $XA$ zone corresponds a partial assignment for $X_{\varphi}$. Moreover, if every clause contains exactly one true literal, then every bit in the $C1$ and $C2$ zones is 1. That is, for every $X$-assignment $\sigma_x$, if there is a $Y$-assignment $\sigma_y$ such that exactly one true literal exists in every clause, then the assignment $\sigma_x \sigma_y$ is simulated as an integer in the interval $[\lambda, \mu]$, as illustrated in Figure 3.

The above description suggests that we can simulate all assignments for $\varphi$ if we correctly select exactly one of the corresponding two integers for every variable in $X$ and $Y$. However, the reverse direction is more complicated. It is more difficult to verify whether for every variable, exactly one of the corresponding two integers is selected, since, in #NONREP, we can repeatedly select an input integer. Moreover, for a variable $z$, the two integers constructed from the pair of $z$ and 0 and the pair of $z$ and 1 may be simultaneously selected. Another awkward phenomenon is carrying over in the additions of the integers that we select. Overcoming these difficulties motivates the introduction of the $1S$ zone. Let $a$ be the sum of some integers constructed from $\varphi$ and $\alpha$ be the binary representation of $a$. If $a$ is in $[\lambda, \mu]$ and the $1S$ zone of $\alpha$ represents the integer $3\#_1(w_1) + 2\#_1(w_2) + 2\#_1(w_3)$
Figure 3: Interval $[\lambda, \mu]$ that simulates assignments in $\varphi$, where $\lambda = (r + 1)^23k_1+2k_2+2k_3 - 2^k$ and $\mu = (r + 1)^23k_1+2k_2+2k_3 - 1$.

such that $w_1$, $w_2$, and $w_3$ are the $X_1$, $Y_1$, and $CI$ zones, respectively, then we can consider that no carry occurs in the additions for obtaining $a$.

In Table 4, we can check that for every pair $(\sigma_x, \sigma_y)$ of an $X$-assignment $\sigma_x \in \{0, 1\}^4$ and a $Y$-assignment $\sigma_y \in \{0, 1\}^3$, every clause of $\varphi_1$ includes exactly one true literal if and only if the sum of selected integers is in the interval $[\lambda, \mu]$ in Figure 3.

4.2 Formal discussion for Theorem 20

In this subsection, we prove Theorem 20. As a preparation, we formally define the zones of the binary representation of a positive integer, as informally described in Section 4.1. Let $\beta$ be the binary representation of a positive integer. Then, we call the bit sequence from the 1st bit to the $k_1$-th bit the $XA$ zone of $\beta$; the sequence from the $(k_1+1)$-th bit to the $(2k_1)$-th bit the $XI$ zone of $\beta$; the sequence from the $(2k_1+1)$-th bit to the $(3k_1)$-th bit the $X2$ zone of $\beta$; the sequence from the $(3k_1+1)$-th bit to the $(3k_1+k_2)$-th bit the $Y1$ zone of $\beta$; the sequence from the $(3k_1+k_2+1)$-th bit to the $(3k_1+k_2+k_3)$-th bit the $Y2$ zone of $\beta$; the sequence from the $(3k_1+k_2+k_3+1)$-th bit to the $(3k_1+k_2+k_3+1)$-th bit the $C1$ zone of $\beta$; the sequence from the $(3k_1+k_2+k_3+1)$-th bit to the $(3k_1+k_2+k_3+1)$-th bit the $C2$ zone of $\beta$; and the sequence from the $(3k_1+k_2+k_3+1)$-th bit to the $(3k_1+k_2+k_3+1)$-th bit the $IS$ zone of $\beta$.

Proof of Theorem 20. Since $\#NP = \#coNP$ [33], we may prove that $\#NonRep$ is $\#coNP$-complete. We first show that $\#NonRep$ is in $\#coNP$. Let $(A, [\kappa_0, \kappa_1])$ be an instance of $\#NonRep$. Let $M^L$ be an oracle machine, where $L$ is the decision version of the integer knapsack problem ([22], Section 15.7). This problem was proved to be NP-complete ([22], Section 15.7). First $M^L$ guesses a gap $e$ of $S(A)$ in the interval $[\kappa_0, \kappa_1]$. Then, $M^L$ verifies $e$ to be in $N(A) \cap [\kappa_0, \kappa_1]$ by using oracle $L$. Consequently, $\#NonRep$ is in $\#coNP$.

In the rest of the proof, we show that $\#NonRep$ is $\#coNP$-hard under
Figure 4: Constructed integer that simulates a literal $\tilde{z}$ in a 3-CNF formula $\varphi$. 

$$s = \lceil \log(3k_1+2k_2+2k_3) \rceil + 1$$
parsimonious reductions by reducing $\# \cdot \Pi_1 1 \text{IN3SAT}$ to $\#\text{NonRep}$. In particular, we show a polynomial-time computable one-to-one mapping $s$ as a parsimonious reduction. We fix $\varphi$ to be a 3-CNF formula over $X \cup Y$ and define an instance $s(\varphi)$ of $\#\text{NonRep}$, where $X \cap Y = \emptyset$.

Let $k_1$ and $k_2$ be the integers $|X_{\varphi}|$ and $|Y_{\varphi}|$, respectively. Let $C_1, \cdots, C_{k_3}$ be all clauses of $\varphi$. By the assumption in Section 2.1, $k_3$ is larger than 1. We suppose that $X_{\varphi} = \{x_1, \cdots, x_{k_1}\}$ and $Y_{\varphi} = \{y_1, \cdots, y_{k_2}\}$. We will define $(H, [\lambda, \mu])$ as an instance of $\#\text{NonRep}$ below, where $H$ is a set of positive integers and $[\lambda, \mu]$ is an interval of positive integers. Every integer in $H$ is of the form in Figure 4.

We define $d_0$ as the integer $2^{3k_1 + 2k_2 + 2k_3}$. For every $\tilde{x}_1 \in X_{\varphi}$, we define an integer $h(\tilde{x}_1)$ as follows.

$$h(\tilde{x}_1) = (|\{C_j : \tilde{x}_1 \in C_j, 1 \leq j \leq k_3\}| + 3) d_0 + \sum_{\tilde{x}_1 \in C_j, 1 \leq j \leq k_3} (2^{3k_1 + 2k_2 + k_3 + j - 1} + 2^{k_1 + 2k_2 + j - 1}) + 2^{k_1 + i - 1} + 2^{k_1 + i - 1} + 2^{i - 1} [\tilde{x}_1 = x_i].$$

Similarly, for every $\tilde{y}_i \in Y_{\varphi}$, we define an integer $h(\tilde{y}_i)$ as follows.

$$h(\tilde{y}_i) = (|\{C_j : \tilde{y}_i \in C_j, 1 \leq j \leq k_3\}| + 2) d_0 + \sum_{\tilde{y}_i \in C_j, 1 \leq j \leq k_3} (2^{3k_1 + 2k_2 + k_3 + j - 1} + 2^{k_1 + 2k_2 + j - 1}) + 2^{k_1 + k_2 + i - 1} + 2^{3k_1 + i - 1}.$$

We define $H$ as $\{h(\tilde{z}) : \tilde{z} \in X_{\varphi} \cup Y_{\varphi}\}$. We define positive integers $\lambda$ and $\mu$ as follows.

$$\lambda = (3k_1 + 2k_2 + 2k_3) d_0 + \sum_{i=k_1}^{3k_1 + 2k_2 + 2k_3 - 1} 2^i,$$

$$\mu = (3k_1 + 2k_2 + 2k_3 + 1) d_0 - 1.$$

We define $s(\varphi)$ as $(H \cup \{d_0\}, [\lambda, \mu])$.

Then, we check the time complexity of the reduction. 1 shows the whole of the reduction. In this algorithm, $v$ denotes $h(\tilde{z})$ for every $\tilde{z} \in X_{\varphi} \cup Y_{\varphi}$ where $\tilde{z}$ denotes $x_1$ if $Z = X_{\varphi}$ and $p = 1$; $-x_1$ if $Z = X_{\varphi}$ and $p = 0$; $y_i$ if $Z = Y_{\varphi}$ and $p = 1$; and $-y_i$ otherwise. By 1, the reduction can compute $(H \cup \{d_0\}, [\lambda, \mu])$ in polynomial time.

**Claim 21.** $s(\varphi)$ can be constructed in polynomial time on the size of $\varphi$.

**Proof of Claim 21.** It suffices to check $\lambda$, $\mu$, and all integers in $H$ to be computable in time polynomial in $k_1$, $k_2$, and $k_3$. Let $\tilde{z}$ be a literal in $X_{\varphi} \cup Y_{\varphi}$.
Algorithm 1 Reduction from $\# \cdot \Pi_1 \text{1IN3SAT}$ to $\text{NonRep}$

**Input.** A 3-CNF formula $\varphi = (C_1 \land \cdots \land C_{k_3})$ over $X \cup Y$, where $|X_\varphi| = k_1$ and $|Y_\varphi| = k_2$.

**Output.** $(H \cup \{d_0\}, [\lambda, \mu])$.

1. $d_0 \leftarrow 2^{3k_1 + 2k_2 + 2k_3}$
2. $\lambda \leftarrow (3k_1 + 2k_2 + 2k_3)d_0 + \sum_{i=k_1}^{3k_1 + 2k_2 + 2k_3 - 1} 2^i$
3. $\mu \leftarrow (3k_1 + 2k_2 + 2k_3 + 1)d_0 - 1$
4. $H \leftarrow \emptyset$
5. for each $Z \in \{X_\varphi, Y_\varphi\}$ do
6.   for $i = 1$ to $|Z|$ do
7.     for $p \in \{0, 1\}$ do
8.       $c \leftarrow 2$
9.       if $Z = X_\varphi$ then
10.      $v \leftarrow 2^{2k_1 + i - 1} + 2^{k_1 + i - 1}$
11.      if $p = 1$ then
12.         $v \leftarrow v + 2^{i - 1}$
13.         $c \leftarrow c + 1$
14.         $\tilde{z} \leftarrow x_i$
15.       else
16.         $\tilde{z} \leftarrow \neg x_i$
17.       end if
18.     else
19.       $v \leftarrow 2^{3k_1 + k_2 + i - 1} + 2^{3k_1 + i - 1}$
20.       if $p = 1$ then
21.         $\tilde{z} \leftarrow y_i$
22.       else
23.         $\tilde{z} \leftarrow \neg y_i$
24.       end if
25.     end if
26.   end for
27.   for $j = 1$ to $k_3$ do
28.     if $\tilde{z} \in C_j$ then
29.       $v \leftarrow v + 2^{3k_1 + 2k_2 + k_3 + j - 1} + 2^{3k_1 + 2k_2 + j - 1}$
30.       $c \leftarrow c + 1$
31.     end if
32.   end for
33. end for
34. $H \leftarrow H \cup \{v\}$
35. return $(H, [\lambda, \mu])$
The integer $h(\tilde{z})$ consists of at most $3k_1 + 2k_2 + 2k_3 + \lceil \log(3k_1 + 2k_2 + 2k_3) \rceil + 1$ bits. We can construct each of the $X_A$, $X_1$, $X_2$, $Y_1$, and $Y_2$ zones of $h(\tilde{z})$ in time linear in $k_1$, $k_2$ and $k_3$. For every clause $C$ in $\varphi$, we can check whether $\tilde{z}$ is in $C$ in constant polynomial. Thus, we construct the $C1$ and $C2$ zones of $h(\tilde{z})$ in time polynomial in $k_1$, $k_2$, and $k_3$. After all the zones except the $1S$ zone are constructed, we can count 1s in all the zones except the $1S$ zone in time linear in $k_1$, $k_2$, and $k_3$. Thus, we can calculate the $1S$ zone in time polynomial in $k_1$, $k_2$, and $k_3$. It follows that $h(\tilde{z})$ is computed in time polynomial in $k_1$, $k_2$, and $k_3$. The integer $h(\tilde{z})$ is calculated in time polynomial in $k_1$, $k_2$, and $k_3$. The set $H$ has $2(k_1 + k_2)$ elements since an integer is defined as an element for every literal in $\tilde{\varphi}$.

Rest of the proof for Theorem 20, we fix $\alpha$ if $k$ computed in time polynomial in $\tilde{\varphi}$, since an integer is defined as an element for every literal in $X_{\varphi} \cup \tilde{Y}_\varphi$. By the similar method in the above discussion, the integers $\lambda$ and $\mu$ can be computed in time polynomial in $k_1$, $k_2$, and $k_3$. Consequently, the claim holds.

We will check the validity of the above reduction. We show that there is exactly one true literal in every clause in $\varphi$ for a given assignment $\sigma$ if and only if there is a set $S \subseteq H \cup \{d_0\}$ such that $\sum_{\tau \in S} \tau$ is in $[\lambda, \mu]$. The “only if” part is straightforward by definition. We show the “if” part here. In the rest of the proof for Theorem 20, we fix $\alpha_1, \cdots, \alpha_i$ to be positive integers in $H$, and $\alpha$ to be the sum of $\alpha_1, \cdots, \alpha_i$, where $i \geq 1$. For every $\tilde{z} \in X_{\varphi} \cup \tilde{Y}_\varphi$, we denote the number $|\{i \leq i \leq \iota\}|$ by $c(\tilde{z})$.

**Claim 22.** For every $z \in X_{\varphi} \cup Y_{\varphi}$, exactly one of the following conditions holds.

1. $c(z) = 0$ and $c(\neg z) = 1$.
2. $c(z) = 1$ and $c(\neg z) = 0$.

To prove Claim 22, it show a more general statement in Claim 24. As preparation, we define some notions and notations for any nonnegative integer $e$. For every $1 \leq i \leq 3$, let $I_{i,e}$ be a set of integers in $[1, k_i]$. Let $I_{0,e}$ be a subset of $I_{1,e}$. We say that $e$ is consistent with $\varphi$ if $e$ can be written as

\[
(3|I_{1,e}| + 2|I_{2,e}| + 2|I_{3,e}|)d_0 + \sum_{i \in I_{3,e}} (2^{k_3} + 1)2^{3k_2 + 2k_1 + i - 1} \\
+ \sum_{i \in I_{2,e}} (2^{k_2} + 1)2^{3k_1 + i - 1} + \sum_{i \in I_{1,e}} (2^{k_1} + 1)2^{k_1 + i - 1} + \sum_{i \in I_{0,e}} 2^{i - 1}.
\]

Note that if $e$ is in $[\lambda, \varphi]$, then $e$ is consistent with $\varphi$. For every $1 \leq i \leq \iota$, we denote the sum $\sum_{i=1}^{\iota} e_i$ by $\xi_i$. We may assume the addition of $e_1, \cdots, e_\iota$ to do in the left associative manner; i.e., $(e_1 + e_2) + \cdots + e_{\iota - 1} + e_\iota$, without loss of generality. Claim 23 is essential although it can be immediately obtained from definitions.
Claim 23. \( \xi_i \) is consistent with \( \varphi \) for every \( 1 \leq i \leq \ell \) if and only if no carry occurs at the \( XA, X1, X2, Y1, Y2, C1, \) and \( C2 \) zones in the addition of \( \xi_{i-1} \) and \( \alpha_i \) for every \( 2 \leq i \leq \ell \).

Proof of Claim 23. The “if ” part is trivial by the definitions of integers in \( H \). Thus, we prove the “only if” part below. The proof is by induction on \( \ell \geq 1 \). Suppose that \( \xi_i \) is consistent with \( \varphi \) for every \( 1 \leq i \leq \ell \). By induction hypothesis, no carry occurs at the \( XA, X1, X2, Y1, Y2, C1, \) and \( C2 \) zones in the addition \( \xi_i + \alpha_{i-1} \) for every \( 2 \leq i \leq \ell - 1 \). By definition, every element of \( H \) is consistent with \( \varphi \). Thus, \( \alpha_i \) is consistent with \( \varphi \). To show that no carry occurs at the \( XA, X1, X2, Y1, Y2, C1, \) and \( C2 \) zones in the addition \( \xi_{i-1} + \alpha_i \), we separately describe the cases depending on whether a carry occurs at one of the \( X1, X2, Y1, Y2, C1, \) and \( C2 \) zones in that addition. First, assume that a carry occurs at one of the the \( X1, X2, Y1, Y2, C1, \) and \( C2 \) zones in that addition. Then,

\[
\#_1(\xi_i \mod d_0) < \#_1(\xi_{i-1} \mod d_0) + \#_1(\alpha_i \mod d_0).
\]

Since \( \xi_i \) is consistent with \( \varphi \),

\[
\#_1(\xi_i \mod d_0) = \#_1(\xi_{i-1} \mod d_0) + \#_1(\alpha_i \mod d_0).
\]

However, in the addition \( \xi_{i-1} + \alpha_i \), if a carry occurs at the \( XA \) zone, then at least two carries occur at the \( X1 \) and \( X2 \) zones since \( I_{0,\xi_{i-1}} \subseteq I_{1,\xi_{i-1}} \) and \( I_{0,\alpha_i} \subseteq I_{1,\alpha_i} \) by their consistencies. It contradicts to the assumption. Hence, no carry occurs at the \( XA \) zone. Next, assume that no carry occurs at the \( X1, X2, Y1, Y2, C1, \) and \( C2 \) zones in the addition \( \xi_{i-1} + \alpha_i \). Then, by the definitions of the \( XA, X1, \) and \( X2 \) zones, no carry occurs at the \( XA \) zone. Consequently, no carry occurs at the \( XA, X1, X2, Y1, Y2, C1, \) and \( C2 \) zones in the addition \( \xi_{i-1} + \alpha_i \) for every \( 2 \leq i \leq \ell \). The proof is complete. \( \Box \) (Claim 23)

The following claim follows from Claim 23.

Claim 24. If \( \alpha \) is consistent with \( \varphi \), then the following statements hold.

1. For every \( x_j \in X_\varphi \) with \( 1 \leq j \leq k_1 \),
   - if \( j \in I_{0,\alpha} \cap I_{1,\alpha} \), then \( c(x_j) = 1 \) and \( c(\neg x_j) = 0 \).
   - if \( j \in I_{1,\alpha} \setminus I_{0,\alpha} \), then \( c(x_j) = 0 \) and \( c(\neg x_j) = 1 \).
   - otherwise \( c(x_j) = c(\neg x_j) = 0 \).

2. For every \( y_j \in Y_\varphi \) with \( 1 \leq j \leq k_2 \),
   - if \( j \in I_{2,\alpha} \), then \( c(y_j) = 1 \) and \( c(\neg y_j) = 0 \).
   - otherwise \( c(y_j) = c(\neg y_j) = 0 \).

Consequently, the theorem holds. \( \Box \) (Theorem 25)
5 Complexity of \#Bounded NS Gaps

In Section 4, we proved \#\cdot\Pi_1 \Pi_1 \Pi_1 \Pi_1 3SAT to be parsimonious reducible to \#NonRep. In this section, we prove the following theorem by extending the reduction in the proof for Theorem 20 to one from \#\cdot\Pi_1 \Pi_1 \Pi_1 \Pi_1 3SAT to \#Bounded NS Gaps.

**Theorem 25.** \#Bounded NS Gaps is \#NP-complete under parsimonious reductions.

5.1 Ideas of the proof

In this subsection, let \( \varphi \) be fixed to a 3-CNF formula over \( X \cup Y \), as in the proof for Theorem 20, where \( X \) and \( Y \) are disjoint. Similarly, let \( \lambda, \mu, h, \) and \( d_0 \) be defined as in the proof for Theorem 20. For extending the reduction in the proof for Theorem 20 to one from \#\cdot\Pi_1 \Pi_1 \Pi_1 \Pi_1 3SAT to \#Bounded NS Gaps, we specify all gaps of \( S(H \cup \{d_0\}) \) in \( [\lambda, \mu] \) by using another set \( s_0(\varphi) \) that contains \( H \) as a subset. In more detail, we construct the mapping \( s_0 \) so that \( \mathcal{N}(H \cup \{d_0\}) \cap [\lambda, \mu] = \mathcal{N}(s_0(\varphi)) \cap [\lambda, \infty] \).

5.1.1 Observation

First, we describe how to specify every integer in \( \mathcal{N}(H \cup \{d_0\}) \cap [\lambda, \mu] \) as a gap of another numerical semigroup, which is larger than a bound. Let us first consider a toy example of \#NonRep instead of an input constructed from a 3-CNF formula. Let \( S \) be the set \{8, 12, 13\}. We observe an example illustrated in Figure 5. In Figure 5, all the circles on the three lines mean integers in \([0, 49]\). On every line, the grayed circles mean positive integers given as inputs. The black circles mean positive integers, each of which can be written as a nonnegative integer combination of the input positive integers but is not one of the input positive integers themselves. The white circles mean other integers in \([0, 49]\); i.e., integers that cannot be written as nonnegative integer combinations of the corresponding input integers. The circles on the 1st line illustrate a part of \( S(S) \). That is, we can observe an instance \((S, [28, 31])\) of \#Bounded NS Gaps on the 1st line.

Let us next specify two gaps 30 and 31 of \( S(S) \) in \([28, 31]\) with another numerical semigroup and only the lower bound 28 but without the upper bound 31. For this purpose, we may find a numerical semigroup \( B \) such that \((\mathbb{N}\setminus B) \cap [28, \infty] = \mathcal{N}(S) \cap [28, 31] \). A simple but naive approach to specify \( B \) is by using \( T = S \cup [32, 39] \) as its generators. The 2nd line of Figure 5 illustrates \( S(T) \). By definition, \( T \) contains successive integers from 32 to 39. The number of these successive integers is not smaller than the smallest element of \( T \). This fact guarantees that \( \mathcal{N}(T) \) does not include any integer greater than 31. Unfortunately, the size of \( T \) is of exponential order in the
Figure 5: Numerical semigroups generated by $S = \{8, 12, 13\}$, $T = \{8, 12, 13, 32, 33, 34, 35, 36, 37, 38, 39\}$, and $U = \{8, 12, 13, 27\}$. 
size of $S$ since we assume the size of an integer to be its bit length. This
defect is caused by the cardinality of $T$.

For making the number of generators of $B$ smaller, it suffices to specify
every integer between 32 and 39 as a nonnegative integer combination of a
smaller set of positive integers. More generally, it is not so difficult since we
can specify every integer in an interval $[\kappa, \kappa + m - 1]$ of positive integers as a
nonnegative integer combination of $l$ positive integers for some $l = O(\log m)$. However, we have to be careful to preserve all gaps of $S(S)$ between 28 and 31. If $U$ is the set $S \cup \{27\}$, then $U$ satisfies the above conditions. As
illustrated on the 3rd line in Figure 5, all integers in $N(S)$ greater than 31
are included by $N(U)$, which are 35 and 43, and $N(S)$ does not contain any
gap of $S(S)$ in $[28, 31]$, which are 30 and 31.

### 5.1.2 An example of our reduction

Next, we illustrate the reduction that we will formulate in Section 5.2 by
using an example. Let $\varphi_2$ be a 3-CNF formula $C_1 \land C_2$, where $C_1 = \neg x_1 \lor x_2 \lor y_1$ and $C_2 = x_1 \lor x_2 \lor y_2$. Table 5 shows clauses, each of which contains
exactly one true clause for every assignment for $\varphi_2$.

Table 5: Clauses in $\varphi_2$, each of which contains exactly one true literal, where
each cell contains indices of clauses for an assignment.

| $\sigma_x$ | 00 | 10 | 01 | 11 |
|------------|----|----|----|----|
| 00         | 1  | —  | 1,2| 2  |
| 10         | 2  | 1,2| —  | 1  |
| 01         | 2  | 2  | —  | —  |
| 11         | 1  | —  | 1  | —  |

Tables 6 and 7 show integers constructed from $\varphi_2$ by using the reduction
in the proof for Theorem 20. Tables 8 and 9 show integers that we will
construct. Moreover, we define $d_{0,\varphi_2}$ as the integer $2^{14}$. This is defined from
$\varphi_2$ by the same method for $d_0$ in the reduction in the proof for Theorem 20.

The numerical semigroup generated by the integers in Tables 8 or 9
satisfies the conditions that we described in 5.1.1. For every integer $e \geq \lambda_{\varphi_2}$,
if $e$ is greater than $\mu_{\varphi_2}$, then $e$ can be represented as a nonnegative integer
combination of $d_{0,\varphi_2}$ and integers in Tables 8 or 9; and otherwise, $e$ cannot
be represented by the same way. Intuitively speaking, every integer in Tables
8 or 9 is a kind of dummy. In Tables 8 and 9, every row is tagged with a
variable or a clause in the 1st column, although any integer does not have
a role for simulating a 3-CNF formula. If we select integers, one each for
every variable and clause in Tables 8 and 9, then we can represent an integer
greater than $\mu_{\varphi_2}$ as their sum, and vice versa.
Table 6: Integers constructed from $\varphi_2$ by the method in the proof for Theorem 20.

| Variables | Values | $I_S$ $C_2$ $C_1$ $Y_2$ $Y_1$ $X_2$ $X_1$ $X_A$ |
|-----------|--------|--------------------------------------------------|
| $x_1$     | 1      | 5 10 10 00 00 01 01 01 |
| $x_1$     | 0      | 5 01 01 00 00 01 01 00 |
| $x_2$     | 1      | 7 11 11 00 00 10 10 10 |
| $x_2$     | 0      | 3 00 00 00 00 10 10 00 |
| $y_1$     | 1      | 4 01 01 01 01 00 00 00 |
| $y_1$     | 0      | 2 00 00 01 01 00 00 00 |
| $y_2$     | 1      | 4 10 10 10 10 00 00 00 |
| $y_2$     | 0      | 2 00 00 10 10 00 00 00 |

Table 7: Endpoints of the interval constructed from $\varphi_2$ by the method in the proof for Theorem 20.

| Endpoints | Constructed integers |
|-----------|----------------------|
| $\lambda_{\varphi_2}$ | $I_S$ $C_2$ $C_1$ $Y_2$ $Y_1$ $X_2$ $X_1$ $X_A$ |
| $\mu_{\varphi_2}$   | 14 11 11 11 11 11 00 |
|                       | 14 11 11 11 11 11 11 11 |
We describe the details of integers in Tables 8 or 9. Let \( \beta \) be the binary representation of an integer between the 1st and 12th rows in Table 8. \( \beta \) is tagged with a variable in \( X_{\varphi^2} \). Let \( l \) be 1 if \( \beta \) is tagged with \( x_1 \); and 2 otherwise. In each of the \( XA, XI, \) and \( X2 \) zones of \( \beta \), 1 occurs at most once. Conversely, in at least one of the \( XA, XI, \) and \( X2 \) zones, 1 occurs. In each of the \( XA \) and \( X2 \) zones, if 1 occurs, then its position is the \( l \)-th bit of the zone. In the \( XI \) zone, if 1 occurs, then its position is the \(( (l + 1) \mod 2 )\)-th bit of the zone, where the modulus 2 is from the cardinality of \( X_{\varphi^2} \). As described above, every integer in Tables 8 or 9 is for representing all integers larger than \( \mu_{\varphi^2} \) but not for simulating a 3-CNF formula. Thus, for avoiding improperly simulating a 3-CNF formula, the position of the bit of 1 is shifted in the \( XI \) zone.

Similarly, in Tables 8 or 9, every integer between the 13th and 18th rows is tagged with a variable in \( Y_{\varphi^2} \). Moreover, every integer between the 19th and 24th rows is tagged with a clause in \( \varphi^2 \).

For every \( z \in X_{\varphi^2} \cup Y_{\varphi^2} \cup C_{\varphi^2} \), let \( \beta_z \) be a binary representation in Table 8, which is tagged with \( z \). In \( \beta_z \), the integer represented by its 1S zone is equal to \( c \), where \( c \) is the maximum number of occurrences of 1s in its remaining zones among all integers tagged with \( z \) in Table 8. On the other hand, in Table 9, the integer represented in its 1S zone is equal to \( c + 1 \), where \( c \) is the maximum number of occurrences of 1s in its remaining zones. The difference of the 1S zones in integers in Tables 8 and 9 is for representing every integer greater than \( \mu_{\varphi^2} \) as a sum of \( d_0 \) and integers that we select one each for every variable and clause in \( \varphi^2 \).

5.2 Formal discussion for Theorem 25

Proof of Theorem 25. First, we prove that \#Bounded NS Gaps is in \#\( \Pi^P_1 \), i.e., \#NP. By Theorem 20, it suffices to show that \#Bounded NS Gaps is a special case of \#NonRep as follows. Let \((A, \kappa) \) be an instance of \#Bounded NS Gaps. Let \( a \) be the maximum element of \( A \). By definition, all integers in \( A \) are coprime. Thus, there is the maximum element \( g(A) \) in \( N(A) \). By an upper bound \( a^2 \) for \( g(A) \) [38], \( N(A) \cap [\kappa, \infty] \) is equal to \( N(A) \cap [\kappa, a^2] \). Thus, \#Bounded NS Gaps is an special case of \#NonRep. Consequently, \#Bounded NS Gaps is in \#NP.

Next, we prove that \#Bounded NS Gaps is \#\( \Pi^P_1 \)-hard under parsimonious reductions. For this purpose, we reduce \#\( \Pi^P_1 \)1IN3SAT to \#Bounded NS Gaps. Let \( X, Y, \varphi, k_1, k_2, \) and \( k_3 \) be given as in the proof for Theorem 20. \( \varphi \) is a 3-CNF formula over \( X \cup Y \). \( k_1, k_2, \) and \( k_3 \) are \( |X_{\varphi}|, |Y_{\varphi}|, \) and \( |C_{\varphi}| \), respectively. Let \( h, H, [\lambda, \mu] \) and \( d_0 \) be defined as in the proof for Theorem 20. \( h \) is a polynomial-time computable function from 3-CNF-formulae to positive integers. \( H \) is a set of positive integers. \([\lambda, \mu]\) is an interval of positive integers. and \( d_0 \) is a positive integer. By the proof for Theorem 20, we can construct \( h, H, [\lambda, \mu] \) and \( d_0 \) in polynomial time.
Table 8: Integers constructed from $\varphi_2$, which are introduced for the reduction to Bounded NS Gaps.

| Variables or clauses | Values | Constructed integers |
|----------------------|--------|----------------------|
|                      |        | $I_1$ $S$ $C_1$ $C_2$ $Y_1$ $Y_2$ $X_1$ $X_2$ $X_A$ |
| $x_1$                |        |                     |
| 001                  | 3      | 00 00 00 00 00 00 00 00 01 |
| 010                  | 3      | 00 00 00 00 00 00 10 00 00 |
| 011                  | 3      | 00 00 00 00 00 01 00 01 01 |
| 100                  | 3      | 00 00 00 00 00 01 00 00 01 |
| 101                  | 3      | 00 00 00 00 00 01 00 10 01 |
| 110                  | 3      | 00 00 00 00 00 01 01 00 01 |
| 111                  | 3      | 00 00 00 00 00 01 10 01 10 |
| $x_2$                |        |                     |
| 001                  | 3      | 00 00 00 00 00 00 00 00 10 |
| 010                  | 3      | 00 00 00 00 00 00 00 10 00 |
| 011                  | 3      | 00 00 00 00 00 00 01 00 10 |
| 100                  | 3      | 00 00 00 00 00 00 10 00 00 |
| 101                  | 3      | 00 00 00 00 00 00 10 00 10 |
| $y_1$                |        |                     |
| 01                   | 2      | 00 00 00 00 10 00 00 00 00 |
| 10                   | 2      | 00 00 00 01 00 00 00 00 00 |
| 11                   | 2      | 00 00 00 01 10 00 00 00 00 |
| $y_2$                |        |                     |
| 01                   | 2      | 00 00 00 00 01 00 00 00 00 |
| 10                   | 2      | 00 00 00 10 00 00 00 00 00 |
| 11                   | 2      | 00 00 00 10 01 00 00 00 00 |
| $c_1$                |        |                     |
| 01                   | 2      | 00 10 00 00 00 00 00 00 00 |
| 10                   | 2      | 01 00 00 00 00 00 00 00 00 |
| 11                   | 2      | 01 10 00 00 00 00 00 00 00 |
| $c_2$                |        |                     |
| 01                   | 2      | 00 01 00 00 00 00 00 00 00 |
| 10                   | 2      | 10 00 00 00 00 00 00 00 00 |
| 11                   | 2      | 10 01 00 00 00 00 00 00 00 |

Table 9: Integers constructed from $\varphi_2$, which are introduced for the reduction to Bounded NS Gaps

| Variables or clauses | Values | Constructed integers |
|----------------------|--------|----------------------|
|                      |        | $I_1$ $S$ $C_1$ $C_2$ $Y_1$ $Y_2$ $X_1$ $X_2$ $X_A$ |
| $x_2$                |        |                     |
| 110                  | 4      | 00 00 00 00 00 10 01 00 |
| 111                  | 4      | 00 00 00 00 00 10 01 10 |
For \( \varphi \), we will define a function \( s_0 \) such that

\[
|\{(\sigma_x \in \{0,1\}^{k_1} : (\forall \sigma_y \in \{0,1\}^{k_2}) (2, \varphi, \sigma_x, \sigma_y) \in \Phi_{1/3}\}| \]

is equal to \(|N(s_0(\varphi)) \cap [\lambda, \infty]|\), where \( \Phi_{1/3} \) is the set defined in Section 2.4. Then, we will define dummy integers described in Section 5.1. For every \( x_i \in X_\varphi \), \( b_0, b_1, b_2 \in \{0, 1\} \), we define an integer \( d(x_i, b_2 b_1 b_0) \) as

\[
3d_0 + 2^{2k_1+i-1}b_2 + 2^{k_1+(i \mod k_1)}b_1 + 2^{i-1}b_0.
\]

Figure 6 illustrates the binary representation of \( d(x_i, b_2 b_1 b_0) \). Similarly, for every \( y_i \in Y_\varphi \), \( b_1, b_2 \in \{0, 1\} \), we define an integer \( d(y_i, b_2 b_1) \) as

\[
2d_0 + 2^{3k_1+k_2+i-1}b_2 + 2^{3k_1+(i \mod k_2)}b_1.
\]

Figure 7 illustrates the binary representation of \( d(y_i, b_2 b_1) \). For every clause \( C_i \) with \( 1 \leq i \leq k_3 \) and \( b_1, b_2 \in \{0, 1\} \), we define an integer \( d(C_i, b_2 b_1) \) as

\[
2d_0 + 2^{3k_1+2k_2+k_3+i-1}b_2 + 2^{3k_1+2k_2+(i \mod k_3)}b_1.
\]

Figure 8 illustrates the binary representation of \( d(C_i, b_2 b_1) \). For every \( x \in X_\varphi \), we denote the set \( \{d(x, \beta) : \beta \in \{0, 1\}^{3} \setminus \{000\}\} \) by \( D(x) \). For every \( z \in Y_\varphi \cup C_\varphi \), we denote the set \( \{d(z, \beta) : \beta \in \{0, 1\}^{2} \setminus \{00\}\} \) by \( D(z) \).
Figure 7: Integer $d(y_i, b_2b_1)$ constructed in the proof for Theorem 25.

Figure 8: The integer $d(C_i, b_2b_1)$ constructed in the proof for Theorem 25.
For every \( x \in X_\varphi \) and \( \beta \in \{0, 1\}^3 \), we define \( d_+(x, \beta) \) as \( d(x, \beta) + d_0 \); i.e., \( d_+(x, \beta) \) is the integer obtained from \( d(x, \beta) \) by incrementing its 1S zone. We denote the set \( \bigcup_{z \in \mathcal{X}_\varphi} D(z) \) by \( D \). Moreover, we denote the set
\[
(D \{ d(x_{k_1}, 110), d(x_{k_1}, 111) \}) \cup \{ d_+(x_{k_1}, 110), d_+(x_{k_1}, 111) \}
\]
by \( D_+ \). We define \( s_0(\varphi) \) as the set
\[
H \cup D_+ \cup \{d_0\}.
\]
By 2, the reduction can compute \( (H \cup D_+ \cup \{d_0\}, \lambda) \) in polynomial time.

Then, we show the validity of the reduction. In the rest of the proof for Theorem 25, let \( \alpha_1, \cdots, \alpha_t \) be integers in \( s_0(\varphi) \) and let \( \alpha \) be the sum of \( \alpha_1, \cdots, \alpha_t \). For every \( 1 \leq \kappa \leq t \), let \( \xi_\kappa \) be \( \sum_{i=1}^\kappa \alpha_i \). That is, \( \xi_1 = \alpha_1 \) and \( \xi_i = \alpha \). For any nonnegative integer \( e \), we say that \( e \) is consistent with \( \varphi \) if \( e \) satisfies the same condition as in the proof for Theorem 20. Note that this notion is well-defined for any nonnegative integer. Moreover, by definition, if \( \alpha \) is in \( [\lambda, \varphi] \), then \( \alpha \) is consistent with \( \varphi \). We can prove the following claim by the similar discussion as in the proof for Claim 23.

**Claim 26.** \( \xi_\kappa \) is consistent with \( \varphi \) for every \( 1 \leq \kappa \leq t \) if and only if no carry occurs at the \( XA, X_1, X_2, Y_1, Y_2, C_1, \) and \( C_2 \) zones in the addition \( \xi_{\kappa-1} + \alpha_i \) for every \( 2 \leq i \leq t \).

**Claim 27.** If \( \alpha \in [\lambda, \mu] \) and there is no \( S \subseteq H \) such that \( \sum_{\tau \in S} \tau = \alpha \), then \( \alpha \in N(s_0(\varphi)) \).

**Proof of Claim 27.** Let \( \alpha \) be in \( [\lambda, \mu] \). Let \( S_1 \) be a subset of \( H \) and \( S_2 \) be a subset of \( D_+ \) such that \( \sum_{\tau \in S_1 \cup S_2} \tau = \alpha \). By assumption, if \( S_2 = \emptyset \), then \( S_1 \subseteq H \) and \( \sum_{\tau \in S_1} \tau = \alpha \), which is a contradiction. Thus, \( S_2 \) is nonempty. Assume that \( S_1 \neq \emptyset \) and \( \alpha \) is in \( [\lambda, \mu] \). Then, by the definitions of integers in \( H \) or \( D_+ \), at least one carry occurs in one of the \( X_1, X_2, Y_1, Y_2, C_1, C_2 \) zones. This contradicts to Claim 26. \( \square \)(Claim 27)

**Claim 28.** If \( \alpha \geq \mu + 1 \), then we can find a set \( S \subseteq D_+ \) such that \( \sum_{\tau \in S} \tau = \alpha \).

**Proof of Claim 28.** By the definitions of integers in \( D_+ \), we can represent every integer in \( [\mu + 1, \mu + d_0] \) as a nonnegative integer combination of \( D_+ \). Thus, the claim holds. \( \square \)(Claim 28)

**Claim 29.** All integers in \( s_0(\varphi) \) are coprime.

**Proof of Claim 29.** By definition, \( D \) contains two successive integers; e.g., \( d(x_1, 010) \) and \( d(x_1, 011) \). By Euclidean algorithm ([13], Chapter 4), two successive integers are coprime. Thus, all integers in \( s_0(\varphi) \) are coprime. \( \square \)(Claim 29)

Moreover, we can check that Claim 24 holds even if \( \alpha \) is defined as an integer represented as a nonnegative integer combination of \( H \cup D_+ \cup \{d_0\} \). The proof for Theorem 25 is complete. \( \square \)(Theorem 25)

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Algorithm 2 Reduction from $\# \cdot \Pi_1 \cdot 1\text{IN3SAT}$ to Bounded NS Gaps

**Input.** A 3-CNF formula $\varphi = (C_1 \land \cdots \land C_k)$ over $X \cup Y$, where $|X_\varphi| = k_1$ and $|Y_\varphi| = k_2$.

**Output.** $(H \cup D_+ \cup \{d_0\}, \lambda)$.

Execute the same process in 1 (After executing this process, we obtain $d_0$, $\lambda$, and $H$)

for each $l \in \{k_1, k_2, k_3\}$ do
  if $l = k_1$ then
    $q \leftarrow k_1$
  else if $l = k_2$ then
    $q \leftarrow 3k_1$
  else
    $q \leftarrow 3k_1 + 2k_2$
  end if
  for each $i = 1$ to $l$ do
    for each $b_2, b_1 \in \{0, 1\}$ such that $b_2b_1 \neq 00$ if $l \neq k_1$ do
      $v \leftarrow 0$
      if $b_2 = 1$ then
        $v \leftarrow v + 2^{q+l+i-1}$
      end if
      if $b_1 = 1$ then
        $v \leftarrow v + 2^{q+(i \mod l)}$
      end if
      if $l = k_1$ then
        for each $b_0 \in \{0, 1\}$ do
          if $b_2b_1b_0 \neq 000$ then
            if $i = k_1$ and $b_2b_1 = 11$ then
              $D_+ \leftarrow D_+ \cup \{v + 4d_0 + 2^{i-1}b_0\}$
            else
              $D_+ \leftarrow D_+ \cup \{v + 3d_0 + 2^{i-1}b_0\}$
            end if
          end if
        end for
      else
        $D_+ \leftarrow D_+ \cup \{v + 2d_0\}$
      end if
    end for
  end for
return $(H \cup D_+ \cup \{d_0\}, \lambda)$
6 Complexity of #NS Gaps

In this section, we prove Theorem 30 by extending the reduction in the proof for Theorem 25 to a reduction from \(\# \cdot \Pi^P_k\) to \#NS Gaps. Theorem 30 is a main theorem in this paper.

**Theorem 30.** #NS Gaps is \#NP-complete under relaxed subtractive reductions.

6.1 Closure property of \# \cdot \Pi^P_k under relaxed subtractive reductions

In Section 2.3, we introduced a new type of polynomial-time reduction that we call a relaxed subtractive reduction. In this subsection, we prove that \# \cdot \Pi^P_k is closed under relaxed subtractive reductions. The proof takes the same approach as the proofs for the closure property of \# \cdot \Pi^P_k under subtractive reductions in [10] and complementive reductions in [4].

**Theorem 31.** For every \(k \geq 1\), the class \# \cdot \Pi^P_k is closed under relaxed subtractive reductions.

**Proof of Theorem 31.** Let \(\#A\) and \(\#B\) be counting problems. Suppose that \(\#B \in \# \cdot \Pi^P_k\). Let a pair \((t_0, t_1)\) of polynomial-time computable functions be a relaxed subtractive reduction from \#A to \#B and \(F \in \mathbb{P}\) be a decision problem such that \#F \(\in \mathbb{P}\Sigma^P_k\), which satisfy the following conditions. For every \(w \in \{0, 1\}^*\),

1. \(W_F(w) \subseteq W_B(t_0(w))\)
2. \(W_B(t_1(w)) \subseteq W_B(t_0(w))\)
3. \(W_F(w) \cap W_B(t_1(w)) = \emptyset\)
4. \(|W_A(w)| = |W_B(t_0(w))| - |W_B(t_1(w))| - |W_F(w)|\).

These 4 conditions are from the definition of a strong relaxed subtractive reduction. For proving the theorem, it suffices to show that

(S1) there is a decision problem \(A'\) such that \(R_{A'} \in \mathbb{P}\Sigma^P_k\),
(S2) \(|W_{A'}(w)| = |W_A(w)|\) for every \(w \in \{0, 1\}^*\).

By the equality \# \cdot \Pi^P_k = \# \cdot \mathbb{P}\Sigma^P_k\) [33], the statement (S1) implies that \#A' \(\in \# \cdot \Pi^P_k\). The statements (S1) and (S2) imply that \#A \(\in \# \cdot \Pi^P_k\).

Let \(N^B\) be a deterministic oracle machine that recognizes \(B\), where \(L_B\) is an oracle in \(\Sigma^P_k\). By the equality \# \cdot \Pi^P_k = \# \cdot \mathbb{P}\Sigma^P_k\), \(R_B\) is in \(\mathbb{P}\Sigma^P_k\). Let \(R_{A'}\) consist of all pairs \((w, w_0 \# w_1 \# v)\) such that \(w_0 = t_0(w), w_1 = t_1(w),\) and \(v \in (W_B(w_0) \backslash W_B(w_1)) \backslash W_F(w)\), where \# is a new symbol. We show
that \( w_0 \# w_1 \# v \) is a witness of \( w \) by using a deterministic oracle machine \( M^{L_B} \). \( M^{L_B} \) simulates \( N^{L_B} \) as a subroutine with overhead bounded by a polynomial in the size of \( w \).

Let \( \gamma \) be \( w_0 \# w_1 \# v \). We can extract \( w_0 \) from \( \gamma \) in polynomial time. Moreover, we can check whether \( w_0 \) is obtained from \( w \) in polynomial time, since \( t_0 \) is a polynomial-time computable function. We can do the same process for \( w_1 \) in \( \gamma \). After doing these processes, we extract \( (w_0, v) \) from \( \gamma \). We can do this process in polynomial time. Then, we check that \( v \) is in \( W_B(w_0) \). By using \( N^{L_B} \) as a subroutine, we can check that \( v \) is in \( W_B(w_0) \) in polynomial time. Similarly, we can execute the following steps in polynomial time. We extract \( (w_1, v) \) from \( \gamma \) and check that \( v \) are not in \( W_B(w_1) \). Finally, we check \( (w, v) \) to be not in \( R_F \). We can execute this step in polynomial time since we assume \#FP to be in FP. Thus, \#\( A' \) is in \#\( \cdot \Pi^p_k \).

By definition, \( |W_{A'}(w)| \) is equal to \(|(W_B(t_0(w)) \setminus W_B(t_1(w))) \setminus W_F(w)| \). By assumption, \( W_A(w) = W_B(t_0(w)) \setminus W_B(t_1(w)) \setminus W_F(w) \). Thus, \( |W_{A'}(w)| \) is equal to \( |W_A(w)| \). This implies that \#\( A' = \#A \). Consequently, \#\( A \) is in \#\( \cdot \Pi^p_k \). The proof is complete.

\section{Ideas and examples of the reduction}

In this subsection, we describe main ideas of our reduction from \#\( \cdot \Pi_1 \)1IN3SAT to \#NS GAPS. In Section 5, we proved that \#\( \cdot \Pi_1 \)1IN3SAT is parsimonious reducible to \#BOUNDED NS GAPS. This implies that \#\( \cdot \Pi_1 \)1IN3SAT is also relaxed subtractive reducible to \#BOUNDED NS GAPS. By extending the reduction in the proof for Theorem 25, we will construct a strong relaxed subtractive reduction from \#\( \cdot \Pi_1 \)1IN3SAT to \#NS GAPS.

Let \( \varphi, [\lambda, \mu], H, h \), and \( s_0 \) be as in the proof for Theorem 25. Our goal is obtaining a function \( s_1 \) and a problem \( F \in P \), which satisfy the following conditions.

\begin{enumerate}
\item[(C1)] Every integer in \( N(s_0(\varphi)) \cap [\lambda, \mu] \) is in \( N(s_0(\varphi)) \setminus N(s_1(\varphi)) \).
\item[(C2)] \( F \) coincides \( (N(s_0(\varphi)) \setminus N(s_1(\varphi))) \cap [1, \lambda - 1] \).
\end{enumerate}

As a result, we want to construct a (strong) relaxed subtractive reduction \((s_0, s_1)\) such that

\[ |N(s_0(\varphi)) \cap [\lambda, \infty] | = |N(s_0(\varphi))| - |N(s_1(\varphi))| - |W_F(\varphi)|. \]

Let us informally describe the relaxed subtractive reduction \((s_0, s_1)\) by using examples. For observing essential properties of numerical semigroups for our reduction, we first consider a simple example such that a pair of a set of positive integers and an interval are given but may not be constructed from a 3-CNF formula. Let \((S, [28, 31])\) be given, where \( S = \{8, 9, 14\} \). We simulate every integer in \( N(S) \cap [1, 27] \) as an integer in \( N(T) \cup U \) as in Figure 9, where \( T = \{8, 9, 10, 14\} \) and \( U = \{10, 19, 20\} \). \( U \) coincides with
In Figure 9, every circle means an integer. The 1st two lines correspond to the numerical semigroups generated by $S$ and $T$, respectively. On these two lines, every black circle means a nongap of $S(S)$ or $S(T)$. Every gray circle means an element of $S$ or $T$. Every white circle means a gap of $S(S)$ or $S(T)$. The 3rd line corresponds to $U$. On this line, every white circle means an element of $U$.

We can specify all integers of $N(S) \cap [1, 27]$ by its two partitions. This partitions are defined by depending on whether an integer is in $N(T)$. In Figure 9, $[1, 7] \cup [11, 13] \cup \{15, 21\}$ coincides with $N(S) \cap [1, 27] \cap N(T)$. $U$ coincides with $(N(S) \cap [1, 27]) \setminus N(T)$.

Let us next consider an instance of $\#NS Gaps$, which is defined for an instance of $\#\cdot \Pi_1^1 \text{IN}3\text{SAT}$. Let $\varphi_2$ be the 3-CNF formula as in 5.1.2. Let $s_0(\varphi_2)$ be a set consisting of $d_0$ and all integers in Tables 6, 8 or 10. Let $s_1(\varphi_2)$ be a set consisting of $d_0$ and all integers in Tables 6, 8, 9, or 10. Then, we specify all integers in $N(s_0(\varphi_2)) \cap [\lambda_{\varphi_2}, \infty]$ as follows. Since $s_0(\varphi_2)$ is a subset of $s_1(\varphi_2)$, $N(s_0(\varphi_2))$ contains $N(s_1(\varphi_2))$ as a subset; i.e., any integer that cannot be represented as a nonnegative integer combination of $s_0(\varphi_2)$ cannot be done as of $s_1(\varphi_2)$. Thus, the task that is left is a specification of $(N(s_0(\varphi_2)) \setminus N(s_1(\varphi_2))) \cap [1, \lambda_{\varphi_2} - 1]$. Let $F$ be $(N(s_0(\varphi_2)) \setminus N(s_1(\varphi_2))) \cap [1, \lambda_{\varphi_2} - 1]$. Moreover, we can observe that $N(s_1(\varphi_2))$ does not include any integer of $N(s_0(\varphi)) \cap [\lambda_{\varphi_2}, \infty]$ by checking each bit in the binary representation of each integer in $s_1(\varphi_2)$ and the integer $\lambda_{\varphi_2}$.

Table 10: Integers constructed from $\varphi_2$, which are introduced for the reduction to $\text{NS Gaps}$.

| Variables or clauses | Values | Constructed integers |
|----------------------|--------|----------------------|
|                      |        | $IS$  $C2$  $C1$  $Y2$  $Y1$  $X2$  $X1$  $XA$ |
| $x_2$                | 110    | 3 00 00 00 00 10 01 00 |
|                      | 111    | 3 00 00 00 00 10 01 10 |
6.3 Formal discussion for Theorem 30

Proof of Theorem 30. By definition \#NS Gaps is a special case of \#Bounded NS Gaps. Thus, by Theorem 25, \#NS Gaps is in \#NP. All that is left is to prove \#NS Gaps to be \#NP-hard under relaxed subtractive reductions.

Figure 10: Euler diagram for the strong relaxed subtractive reduction that we construct in the proof for Theorem 30.

Let \( \varphi \) be given as in the proof for Theorem 25. Let \( d, \lambda, \mu, H, h, \) and \( s_0 \) be defined as in the proof for Theorem 25. We will define a function \( s_1 \) and find a decision problem \( F \) such that

\[
\begin{align*}
(C1) & \quad N(s_1(\varphi)) \subseteq N(s_0(\varphi)), \\
(C2) & \quad W_F(\varphi) \subseteq N(s_0(\varphi)), \\
(C3) & \quad N(s_1(\varphi)) \cap W_F(\varphi) = \emptyset, \\
(C4) & \quad \text{There is a } \sigma_x \in \{0, 1\}^{X_\varphi} \text{ such that } (k, \varphi, \sigma_x \sigma_y) \notin \Phi_{1/3} \text{ for every } \sigma_y \in \{0, 1\}^{Y_\varphi} \text{ if and only if there is an integer in } (N(s_0(\varphi)) \setminus N(s_1(\varphi))) \setminus W_F(\varphi).
\end{align*}
\]

Figure 10 illustrates relationships of \( N(s_0(\varphi)), N(s_1(\varphi)), \) and \( W_F(\varphi) \). Let \( \tilde{D}_+ \) be the set \( D_+ \cup \{d(x, 110), d(x, 111)\} \). \( \tilde{D}_+ \) coincides with \( D \cup \{d_+(x, 110), d_+(x, 111)\} \). We define \( s_1(\varphi) \) as

\[
H \cup \tilde{D}_+ \cup \{d_0\}, \quad \text{i.e.,} \quad s_0(\varphi) \cup \{d(x_{k_1}, 110), d(x_{k_1}, 111)\}.
\]

We can check that \( s_1 \) can be constructed in polynomial time by the same discussion as for \( s_0 \) in the proof of Theorem 25.

Next, we define \( F \) as a decision problem. We define the polynomially balanced binary relation \( R_F \) as follows. A pair \((\varphi, v)\) is in \( R_F \) if and only if \( f \) is a 3-CNF formula and \( v \) is the sum of all positive integers in a set \( B_\varphi \). \( B_\varphi \) satisfies the following conditions.

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• Exactly one of \(d_{\varphi}(x_{k_1}, 110)\) and \(d_{\varphi}(x_{k_1}, 111)\) is in \(B_{\varphi}\).

• For every \(x \in X_{\varphi} \setminus \{x_{k_1}\}\), at most one element in \(\{d_{\varphi}(x, \beta) : \beta \in \{0, 1\}^3, \beta \neq 000\}\) is in \(B_{\varphi}\).

• For every \(z \in Y_{\varphi} \cup C_{\varphi}\), at most one element in \(\{d_{\varphi}(z, \beta) : \beta \in \{0, 1\}^2, \beta \neq 00\}\) is in \(B_{\varphi}\).

Then, an instance of \(F\) over \(X \cup Y\) is in \(P\). Moreover, \(\#F(\varphi)\) is \(2 \cdot 8^{k_1-1} \cdot 4^{k_2+k_3}\); i.e., \(2^{3k_1+2k_2+2k_3-2}\), which is computable in time polynomial in \(k_1, k_2, k_3\). Thus, the counting problem \(\#F\) is in \(\text{FP}\).

In the rest of the proof for Theorem 30, we will show the validity of the above construction. Let \(s_1, \ldots, s_{\iota}\) be integers in \(s_1(\varphi)\) and let \(\alpha\) be the sum of \(s_1, \ldots, s_{\iota}\). For every \(1 \leq \kappa \leq \iota\), let \(\xi = \sum_{i=1}^{\kappa} s_i\). For any nonnegative integer \(e\), we say that \(e\) is consistent with \(\varphi\) if \(e\) satisfies the same condition as in the proof for Theorem 25. We can prove the following claim by the similar discussion in the proof for Theorems Theorem 20 and Theorem 25.

**Claim 32.** \(\xi_{i_{\iota}}\) is consistent with \(\varphi\) for every \(1 \leq i \leq \iota\) if and only if no carry occurs at the \(x_A, x_1, x_2, y_1, y_2, c_1, c_2\) parts in the addition \(\xi_{i_{\iota}} - \alpha_i\) for every \(2 \leq i \leq \iota\).

By Claim 32, we can verify Claims 33, 34, and 35.

**Claim 33.** If \(\alpha \geq \lambda\), then we can find a set \(S \subseteq \tilde{D}_+ \cup \{d_0\}\) such that \(\sum_{\tau \in S} \tau = \alpha\).

**Claim 34.** If \(\alpha \in [\lambda, \mu]\) and there is no \(S \subseteq H\) such that \(\sum_{\tau \in S} \tau = \alpha\), then \(\alpha \in N(s_0(\varphi))\).

**Claim 35.** Let \(\alpha\) be less than \(\lambda\). \(\alpha \in W_{F}(\varphi)\) if and only if there is a \(K \subseteq s_1(\varphi)\) such that \(\sum_{\tau \in K} \tau = \alpha\) and there is no \(L \subseteq s_0(\varphi)\) such that \(\sum_{\tau \in L} \tau = \alpha\).

We can observe the validities of the statements of Claims 33, 34, and 35 by using examples of Figure 9. Let \(S, T, U\) be as in Section 6.2. \(S, T\) and \(U\) correspond to \(s_0(\varphi), s_1(\varphi)\), and \(W_{F}(\varphi)\), respectively. Claim 33 corresponds to the fact that every integer greater than or equal to the lower endpoint 28 can be represented as a nonnegative integer combination of \(T\). Claim 34 corresponds to the fact that 29 cannot be represented as a nonnegative integer combination of \(S\). Claim 35 corresponds to the fact that 10, 19, and 20 are in \(U\) and cannot be represented as nonnegative integer combinations of \(T\). The following claim follows from Claim 29.

**Claim 36.** All integers in \(s_1(\varphi)\) are coprime.

By definition, the pair \((s_0, s_1)\) satisfies the conditions of strong relaxed subtractive reductions. The proof for Theorem 30 is complete. \(\square\)
7 Future work

In this paper, we proved the \#NP-completeness of \#NS Gaps under relaxed subtractive reductions as a main result. A relaxed subtractive reduction is a new type of polynomial-time reduction. Moreover, we proved the \#NP-completeness of \#NonRep and \#Bounded NS Gaps, which are variants of \#NS Gaps, under parsimonious reductions. This section describes future work.

7.1 \#NP-hardness of \#NS Gaps under parsimonious reductions

We showed the \#NP-completeness of \#NS Gaps under relaxed subtractive reductions. It is natural to ask whether \#NS Gaps is \#NP-complete under parsimonious reductions. Unfortunately, it appears to be quite difficult to answer this question.

In this paper, we mainly use combinatorial and logical approaches for constructing reductions, but use little result in number theory. Thus, it may be possible to prove \#NS Gaps to be \#NP-complete under parsimonious reductions by using some tools developed in number theory. Cook [7] said that the primality testing problem requires a deep insight on number theory. Indeed, over thirty years later, [1] found a polynomial-time algorithm by an approach from number theory.

7.2 Proof methods for the \#NP-hardness of \#Bounded NS Gaps

To prove the \#NP hardness of \#Bounded NS Gaps, we reduced \#\cdot \Pi_1 \text{1IN3SAT} to \#Bounded NS Gaps. However, a reduction from \#NonRep to \#Bounded NS Gaps is another natural reduction. Since the latter reduction is a self-reduction, it appears to be more natural than our reduction. Nevertheless, we could not adopt this approach by the following obstacle.

A parsimonious reduction from \#NonRep to \#Bounded NS Gaps requires a deep insight into number theoretical aspects of numerical semigroups. Given an instance \((A', [\lambda, \mu])\) of \#NonRep, no polynomial-time algorithm is known for finding an instance \((A, \kappa)\) of \#Bounded NS Gaps such that

\((C1) \ |N(A) \cap [\kappa, \infty)| = |N(A') \cap [\lambda, \mu]|\) and

\((C2) \ the \ size \ of \ (A, \kappa) \ is \ of \ polynomial \ order \ in \ the \ size \ of \ (A', [\lambda, \mu]).\)

To the best of the author’s knowledge, little relationship is known for an instance of \#Bounded NS Gaps and an instance of \#NonRep. Our reduction in Section 5 makes use of properties originated from an instance of \#\cdot \Pi_1 \text{1IN3SAT}.\)
7.3 Completeness for $\# \cdot \Pi^p_k$ under more general reductions

We introduced relaxed subtractive reductions as a generalization of subtractive reductions. Moreover, relaxed subtractive reductions are also special case of polynomial-time bounded truth-table reductions [19]. As a property of this type of reduction, relaxed subtractive reductions have nonadaptivity; i.e., all oracle queries must be decided before starting the other processes. Moreover, in every relaxed subtractive reduction, its normal processes are also independent of its two oracle queries. We can consider that a $\# \cdot \Pi^p_k$-complete problem exists under relaxed subtractive reductions due to this property. However, we do not know whether there is a more general type of nonadaptive reduction under which $\# \cdot \Pi^p_k$ is closed. This is an interesting subject since $\# \cdot \Pi^p_k$ is not closed under polynomial-time 1 Turing reductions [35].

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