Circular Backbone Colorings: on matching and tree backbones of planar graphs

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Abstract. Given a graph $G$, and a spanning subgraph $H$ of $G$, a circular $q$-backbone $k$-coloring of $(G, H)$ is a proper $k$-coloring $c$ of $G$ such that $q \leq |c(u) - c(v)| \leq k - q$, for every edge $uv \in E(H)$. The circular $q$-backbone chromatic number of $(G, H)$, denoted by $CBC_q(G, H)$, is the minimum integer $k$ for which there exists a circular $q$-backbone $k$-coloring of $(G, H)$. The Four Color Theorem implies that whenever $G$ is planar, we have $CBC_2(G, H) \leq 8$. It is conjectured that this upper bound can be improved to 7 when $H$ is a tree, and to 6 when $H$ is a matching. In this work, we show that: 1) if $G$ is planar and has no $C_4$ as subgraph, and $H$ is a linear spanning forest of $G$, then $CBC_2(G, H) \leq 7$; 2) if $G$ is a plane graph having no two 3-faces sharing an edge, and $H$ is a matching of $G$, then $CBC_2(G, H) \leq 6$; and 3) if $G$ is planar and has no $C_4$ nor $C_5$ as subgraph, and $H$ is a matching of $G$, then $CBC_2(G, H) \leq 5$. These results partially answers questions posed by Broersma, Fujisawa and Yoshimoto (2003), and by Broersma, Fomin and Golovach (2007). It also points towards a positive answer for the Steinberg’s Conjecture.

Keywords: graph coloring, circular backbone coloring, matching, planar graph, Steinberg’s conjecture.

1 Introduction

For basic notions and terminology on Graph Theory, the reader is referred to [5]. In this text, we only consider simple graphs.

Let $G = (V, E)$ be a graph. A (proper) $k$-coloring of $G$ is a function $c : V(G) \to \{1, \ldots, k\}$ such that $c(u) \neq c(v)$, for every edge $uv \in E(G)$. $G$ is $k$-colorable if there exists a $k$-coloring of $G$. The chromatic number of $G$, denoted by $\chi(G)$, is the smallest $k$ for which $G$ has a $k$-coloring. $G$ is $k$-chromatic if $\chi(G) = k$. The VERTEX COLORING PROBLEM consists in determining $\chi(G)$, for a given graph $G$.

Among many practical problems that can be modeled using graph coloring, Frequency Assignment problems are perhaps the most famous ones [4]. There are several variations of the VERTEX COLORING PROBLEM that were defined in
In order to model the specific constraints of the practical applications related to frequency assignment in networks, the Backbone Coloring Problem was defined by Broersma et al. [6,7] in the context of Frequency Assignment Problems where certain channels of communication were more demanding than others.

Formally, given a graph $G$, a spanning subgraph $H$ of $G$, and two positive integers $q$ and $k$, a $q$-backbone $k$-coloring of $(G, H)$ is a $k$-coloring $c$ of $G$ for which $|c(u) - c(v)| \geq q$, for every $uv \in E(H)$. The $q$-backbone chromatic number of $(G, H)$, denoted by $BBC_q(G, H)$, is the minimum $k$ for which there exists a $q$-backbone $k$-coloring of $(G, H)$. The Backbone Coloring Problem consists in determining $BBC_q(G, H)$. In this work, we focus on the case $q = 2$ and thus we usually omit $q$ from the notation.

In their seminal article, Broersma et al. observe that $BBC(G, H) \leq 2 \cdot \chi(G) - 1$. (1)

This can be easily seen by considering an optimal coloring of $G$ that uses only odd colors. Note that, thanks to the Four Color Theorem [2,3], whenever $G$ is a planar graph and $H$ is any spanning subgraph of $G$, we get an upper bound of 7 to the backbone chromatic number of $(G, H)$. However, when $H$ is a spanning tree of $G$, Broersma et al. conjecture that this upper bound is in fact 6, and they show that this would be best possible [8].

**Conjecture 1** ([8]). If $G$ is a planar graph and $T$ is a spanning tree of $G$, then $BBC(G, T) \leq 6$.

In the literature, the only result approaching directly this conjecture shows that it holds whenever $T$ has diameter at most 4 [9].

The authors in [10,11,12] consider special backbone $k$-colorings where the color space is “circular”, i.e., it behaves as $\mathbb{Z}/k$. More formally, given a graph $G$, a spanning subgraph $H$ of $G$, and a positive integer $q$, a circular $q$-backbone $k$-coloring of $(G, H)$ is a function $c : V(G) \rightarrow \{1, \ldots, k\}$ such that $q \leq |c(u) - c(v)| \leq k - q$, for every $uv \in E(H)$. The circular $q$-backbone chromatic number of $(G, H)$, denoted by $CBC_q(G, H)$, is the smallest $k$ for which there exists a circular $q$-backbone $k$-coloring of $(G, H)$. Once more, we quite often omit the index $q$ whenever $q = 2$. In order to simplify the notation, we often write CBC-$k$-coloring instead of circular 2-backbone $k$-coloring.

Note that any CBC-$k$-coloring of $(G, H)$ is also a backbone $k$-coloring of $(G, H)$, and, conversely, if $c$ is a backbone $k$-coloring of $(G, H)$, then it can also be seen as a CBC-$(k + 1)$-coloring of $(G, H)$. Therefore we get:

$BBC(G, H) \leq CBC(G, H) \leq BBC(G, H) + 1$. (2)

Consequently, as far as Conjecture [4] is not proved to be true, then the following circular version of it is also opened:
Conjecture 2. If $G$ is a planar graph and $T$ is a spanning tree of $G$, then

$$\text{CBC}(G,T) \leq 7.$$ 

One may observe that a graph $G$ whose chromatic number is $\chi(G) = k$, satisfies $\text{CBC}_2(G,H) \leq 2k$, by combining Inequalities 1 and 2. Steinberg conjectures that every planar graph $G$ having no $C_4$ or $C_5$ as subgraph satisfies $\chi(G) \leq 3$ [13]. Consequently, one may wonder whether:

Conjecture 3. If $G$ is a planar graph having no $C_4$ or $C_5$ as subgraph, then $\text{CBC}_2(G,H) \leq 6$, for every backbone $H \subseteq G$.

Notice that Conjecture 3 is in fact equivalent to Steinberg’s Conjecture when $H = G$.

In this paper, we prove particular cases of Conjectures 2 and 3.

1.1 Matching Backbones

It is known that if $G$ is a 3-colorable graph and $M$ is a matching of $G$, then $\text{BBC}(G,M) \leq 4$ [7]. Combining this result with Inequality 2, we observe that if Steinberg’s Conjecture is true, then $\text{CBC}(G,M) \leq 5$, whenever $G$ is a planar graph without cycles of length 4 or 5, and $M$ is a matching of $G$. We first prove that this bound holds, giving yet more evidence to the validity of Steinberg’s Conjecture:

**Theorem 1.** If $G$ is a planar graph without cycles of length 4 or 5 as subgraph, and $M$ is a matching of $G$, then $\text{CBC}(G,M) \leq 5$.

In [7] the authors prove that $\text{BBC}(G,M) \leq 6$, whenever $G$ is a planar graph and $M$ is a matching. They also ask whether $\text{BBC}(G,M) \leq 5$ holds, and whether $\text{BBC}(G,M) \leq 6$ can be proved without using the Four Color Theorem. We partially answer both questions by showing that:

**Theorem 2.** If $G$ is a plane graph with no two faces of degree 3 that share an edge, and $M$ is a matching in $G$, then $\text{CBC}(G,M) \leq 6$.

Although our result restricts the class of graphs when compared to the result presented in [7], it is stronger on this restricted class since we deal with circular backbone colorings instead. We mention that our result points to a positive answer to the question about whether $\text{BBC}(G,M) \leq 5$, and that our proof does not use the Four Color Theorem.
1.2 Linear Forest Backbones

Finally, we also study more general backbones. A forest is called linear if its components are paths.

In [4], the authors investigate $CBC(G, F)$ in the light of Steinberg’s Conjecture [13]. Araujo et al. prove that if $G$ is a planar graph with no cycles of length 4 or 5, then $CBC(G, F) \leq 7$ whenever $F$ is a spanning forest of $G$, and that $CBC(G, F) \leq 6$, whenever $F$ is a spanning linear forest of $G$ [4]. Observe that their results partially solve Conjectures 2 and 3.

The last result we present in this work is similar to theirs by considering planar graphs with no cycles of length 4 and linear forests as backbones.

**Theorem 3.** If $G$ is a planar graph without cycles of length 4 as subgraph, and $F$ is a linear spanning forest of $G$, then $CBC(G, F) \leq 7$.

Although in our proof we can consider graphs that have $C_5$ as subgraph, we need an extra color than in the previous result in the literature. However, this was expected since our efforts were done towards an answer to Conjecture 2.

The remainder of this text is organized as follows: in Section 2, we introduce basic notation and results. Then, we prove Theorems 1, 2 and 3 in Sections 3, 4, and 5 respectively.

2 Preliminaries

For the basic definitions about simple graphs and planar graphs, we refer the reader once again to [5].

Given a statement $P$, and a partially ordered set $(\mathcal{S}, \preceq)$, we denote by $P(\mathcal{S})$ the set $\{ S \in \mathcal{S} \mid P \text{ holds for } S \}$. And we say that $S \in \mathcal{S}$ is a minimal counterexample for $P$ if $S \not\in P(\mathcal{S})$, and $S' \in P(\mathcal{S})$ for every $S' \in \mathcal{S}$ such that $S' \prec S$. In our proofs, we consider minimal counterexamples to our theorems. For this, we consider a pair $(G', H')$ to be smaller than a pair $(G, H)$ if $G' \subset G$ and $H' \subseteq H$; in this case we say that $(G', H')$ is a subpair of $(G, H)$.

In what follows, given a minimal counterexample $(G, H)$ to one of our theorems, we get a contradiction by being successful in extending a partial CBC-$k$-coloring of $(G', H')$ to $(G, H)$, where $(G', H')$ is a subpair of $(G, H)$. The following lemma presented in [4] will be useful. It can be easily proved by considering a CBC-$k$-coloring of $(G - u, H - u)$ and observing that it can be extended to a CBC-$k$-coloring of $(G, H)$.

**Lemma 1 (4).** If $(G, H)$ is minimal such that $CBC(G, H) > k$, then, for every $u \in V(G)$, we have that $d_G(u) + 2d_H(u) \geq k$. 
The general technique used to prove the above lemma is also extensively applied in the remainder of the text. Because of this, we introduce the following definitions and notation.

Given a positive integer $k$, we denote the set $\{1, \ldots, k\}$ by $[k]$, and given $c \in [k]$, we denote by $\langle c \rangle$ the set $\{d \in [k] \mid |c - d| \leq 1 \text{ or } |c - d| \geq k - 1\}$ (the colors adjacent to $c$ in the circular space $[k]$). Also, we denote the power set of $[k]$ by $2^k$. Given a pair $(G,H)$, a subgraph $G' \subset G$, and a CBC-$k$-coloring $\psi$ of $(G',H[V(G')]$, we define, for each $u \in V(G) \setminus V(G')$, the set of available colors for $u$ in $\psi$:

$$A_\psi(u) = [k] \setminus (\psi(N_G'(u)) \cup \{\langle \psi(v) \rangle \mid v \in N_{H'}(u)\}).$$

Also, we denote $|A_\psi(u)|$ by $a_\psi(u)$.

## 3 Proof of Theorem

In order to prove Theorem we need the following lemma, proved in [4].

**Lemma 2 ([4])**. Let $G$ be a plane graph without cycles of length 4 or 5, $G \neq K_3$, and let $n$ and $f_3$ denote the number of vertices of $G$ and number of faces of degree 3 in $G$, respectively. Then,

$$\sum_{v \in V(G)} d(v) \leq 3n + \frac{3f_3}{2} - 6.$$

We use the discharging method to prove that if $(G,M)$ is a minimal counterexample to Theorem then Lemma does not hold for $G$. This means that no counterexample can exist and that the theorem holds. The following lemma will be useful.

**Lemma 3.** Let $(G,M)$ be a minimal counterexample to Theorem. Then, we have $\delta(G) \geq 3$. Furthermore, if $u \in V(G)$ has degree 3, then $u$ is incident to some edge in $M$, say $uw$, and $w$ is such that $d(w) \geq 5$.

**Proof.** Let $u \in V(G)$, and denote by $T$ the subgraph $(V(G),M)$. By Lemma and because $d_T(u) \leq 1$, we get that $d_G(u) \geq 3$. Similarly, if $d_G(u) \leq 4$, we must have $d_T(u) = 1$. So, suppose that $u \in V(G)$ has degree 3 and let $w \in V(G)$ be such that $uw \in M$. By contradiction, suppose that $d(w) \leq 4$, and let $\psi$ be a CBC-5-coloring of $(G - u - w, M - uw)$. Note that $a_\psi(u) \geq 3$ and $a_\psi(w) \geq 2$. Therefore, there exists a color $c \in A_\psi(w)$ such that $A_\psi(u) \setminus \langle c \rangle \neq \emptyset$. This implies that $\psi$ can be extended to $(G,M)$, a contradiction.

Denote by $F_3$ the set of faces of degree 3 of $G$. We start by giving charge $d(v) - 3$ for every $v \in V(G)$, and $-\frac{1}{2}$ for every $t \in F_3$. We want to distribute
the charge between the vertices of \( G \) and the faces in \( F_3 \) in such a way as to ensure that at the end, each vertex and each face in \( F_3 \) has nonnegative charge. Because the total amount of charge does not change, we get (below, \( f_3 \) and \( n \) represent \( |F_3| \) and \( |V(G)| \), respectively):

\[
\sum_{v \in V(G)} (d(v) - 3) - \frac{3f_3}{2} \geq 0 \iff \sum_{v \in V(G)} d(v) \geq 3n + \frac{3f_3}{2}.
\]

This contradicts Lemma \( \text{2} \). To prove this can be done, we apply the following discharging rules. Below, given \( u \in V(G) \), we denote by \( F_3(u) \) the set of faces of degree 3 containing \( u \).

**Rule 1** For each \( uw \in M \) such that \( d(u) = 3 \), send \( \frac{1}{2} \) charge from \( w \) to \( u \).

**Rule 2** For each \( u \in V(G) \) and each \( t \in F_3(u) \), send charge \( \frac{1}{2} \) from \( u \) to \( t \).

**Proof (of Theorem \( \text{1} \)).** For each \( x \in V(G) \cup F_3 \), denote by \( \mu_0(x), \mu_1(x), \mu_2(x) \) the charge of \( x \) before Rule \( \text{1} \) has been applied, before Rule \( \text{2} \) has been applied and after Rule \( \text{2} \) has been applied, respectively. Because \( M \) is a matching, no vertex is incident to more than one edge in \( M \). Thus, by Lemma \( \text{4} \) we get the following:

- If \( d(u) = 3 \), then \( \mu_1(u) = \frac{1}{2} \);
- If \( d(u) = 4 \), then \( \mu_1(u) = \mu_0(u) = 1 \); and
- If \( d(u) \geq 5 \), then \( \mu_1(u) \geq \mu_0(u) - \frac{1}{2} = \frac{2d(u) - 7}{2} \).

Now, for each \( u \in V(G) \), denote by \( f_3(u) \) the value \( |F_3(u)| \). Note that, since \( G \) has no cycles of length 4, no two faces in \( F_3 \) can share an edge. This implies that \( f_3(u) \leq \lfloor \frac{d(u)}{4} \rfloor \). One can verify by what is said above that \( \mu_1(u) \geq \frac{d(u)}{4} \geq f_3(u) \). This means that after distributing charge 1/2 to each \( t \in F_3(u) \), we get that \( u \) still has non-negative charge, i.e., \( \mu_2(u) \geq 0 \) for every \( u \in V(G) \). Finally, because each \( t \in F_3 \) receives charge 1/2 from each vertex in \( t \), we get \( \mu_2(t) = \mu_0(t) + 3/2 = 0 \).

4 Proof of Theorem \( \text{2} \)

Consider a plane graph \( G \) and its dual \( G^* \), and let \( F_3 \) be the set of faces of degree 3 in \( G \) (alternatively, the set of vertices of degree 3 in \( G^* \)). We denote the graph \( G^* - F_3 \) by \( G_4^* \), and say that a component of \( G_4^* \) is an island of \( G \). Also, if \( H \) is an acyclic component of \( G_4^* \) such that \( d_{G^*}(f) = 4 \), for every \( f \in V(H) \), then we say that \( H \) is a bad island of \( G \). We denote the set of bad islands of \( G \) by \( \Gamma \) and we let \( \gamma \) denote \( |\Gamma| \). Let \( f \in F_3 \) and \( H \) be an island of \( G \); we say that \( f \) share an edge with \( H \) if \( N_H(f) \neq \emptyset \) (i.e., if \( f \) and \( f' \) share an edge in \( G \) for some \( f' \in V(H) \)). Also, we denote by \( \Gamma(f) \) the set of bad islands that share an edge with \( f \).
Lemma 4. Let $G$ be a plane graph with no two faces of degree 3 sharing an edge, and let $f_3$ denote the number of faces of degree 3 in $G$. Then,

$$\sum_{v \in V(G)} d(v) \leq 5|V(G)| + \gamma - f_3 - 10.$$ 

Proof. Let $f_4$ denote the number of faces of degree 4 in $G$, $|E(G)|$ be denoted by $m$, $F$ denote the set of faces of $G$ and, given $f \in F$, let $|f|$ denote the degree of $f$. We claim that:

$$3f_3 + f_4 \leq m + \gamma \quad (3)$$

This implies that $\sum_{f \in F}(|f| - 5) \geq -2f_3 - f_4 \geq -m - \gamma + f_3$. On the other hand $\sum_{f \in F}(|f| - 5|F| = 2m - 5|F|$. Combining these and applying Euler’s Formula we get (below, $n$ denotes $|V(G)|$):

$$2m - 5(2 - n + m) \geq -m - \gamma + f_3 \iff 2m \leq 5n + \gamma - f_3 - 10$$

It remains to prove Inequality 3. For this, we partition $E(G)$ in $E_3$, $E_4$, where $E_3$ is described below and $E_4 = E(G) \setminus E_3$.

$$E_3 = \{e \in E(G) \mid e \text{ is in the boundary of some face of degree 3}\}.$$ 

Because $G$ has no two faces of degree 3 sharing an edge, we get $|E_3| = 3f_3$. We prove that $|E_3| \geq f_4 - \gamma$, thus finishing the proof. For this, note that if $e \in E_3$, then there is an edge $e^*$ in $G_4^*$ related to $e$. On the other hand, if $e^* \in E(G_4^*)$, then $e^*$ is related to an edge $e \in E(G)$ that separates faces of degree at least 4; hence, $e \in E_3$. Therefore, $|E_3| = |E(G_4^*)|$. Finally, because the number of edges in any graph is at least the number of vertices minus the number of acyclic components of the graph, we get:

$$|E_3| \geq |V(G_4^*)| - \gamma \geq f_4 - \gamma.$$ 

□

Now, by supposing that there exists a counterexample $(G, M)$ to Theorem 2, we use the discharging method to get a contradiction to Lemma 4. For this, start by giving charge $d(v) - 5$ to each $v \in V(G)$, charge 1 to each $f \in F_3$, and charge -1 to each $b \in B$. Then, we apply discharging rules and ensure that this initial charge can be redistributed in the graph in such a way that every vertex, every face of degree 3 and every bad island have non-negative charge. We get a contradiction since:

$$\sum_{v \in V(G)} (d(v) - 5) + f_3 - \gamma \geq 0 \iff \sum_{v \in V(G)} d(v) \geq 5n + \gamma - f_3.$$ 

We need the following lemma.
Lemma 5. Let \((G, M)\) be a minimal counterexample to Theorem 2. Then, we have \(\delta(G) \geq 4\). Furthermore, if \(u \in V(G)\) has degree 4, then \(u\) is incident to an edge of \(M\), say \(uw\), and \(d(w) \geq 6\).

Proof. Let \(T\) denote the subgraph \((V(G), M)\). By Lemma 1 we get \(\delta(G) \geq 4\), and that \(d_T(u) = 1\) whenever \(d_G(u) \leq 5\). So, consider \(u \in V(G)\) with degree 4, and suppose that \(d(w) \leq 5\), where \(w\) is such that \(uw \in M\). Let \(\psi\) be a CBC-6-coloring of \((G - \{u, w\}, M - \{u, w\})\). Then, \(a_\psi(w) \geq 3\) and \(a_\psi(u) \geq 2\). Therefore, there exists a color \(c \in A_\psi(u)\) such that \(A_\psi(w) \setminus \langle c \rangle \neq \emptyset\), which implies that \(\psi\) can be extended to \((G, M)\), a contradiction.

Let \(V_4\) be the set of vertices with degree 4 in \(G\), and for each \(u \in V_4\), denote by \(u^*\) the vertex such that \(uu^* \in M\). The discharging rules are the following:

Rule 1 For each \(f \in F_3\), send charge \(\frac{1}{3}\) from \(f\) to each \(b \in \Gamma(f)\).

Rule 2 For each \(u \in V_4\), send charge 1 from \(u^*\) to \(u\).

Proof (of Theorem 2). For each \(x \in V(G) \cup F_3 \cup \Gamma\), let \(\mu_0(x), \mu_1(x), \mu_2(x)\) denote the charge of \(x\) before Rule 1 after Rule 1 and after Rule 2 has been applied, respectively. Recall that \(\mu_0(v) = d(v) - 5\), for every \(v \in V(G)\); \(\mu_0(f) = 1\), for every \(f \in F_3\); and \(\mu_0(b) = -1\), for every \(b \in \Gamma\).

Because \(M\) is a matching and by Lemma 5 we get that \(\mu_2(v) \geq 0\), for every \(v \in V(G)\). Also, for each \(f \in F_3\), we have \(|\Gamma(f)| \leq 3\); hence \(\mu_2(f) = \mu_1(f) = \mu_0(f) - |\Gamma(f)|/3 \geq 0\). It remains to prove that each bad island also ends up with non-negative charge. So, consider a bad island of \(G\), i.e., an acyclic component \(H\) of \(G^*_4\) such that each \(f \in V(H)\) has degree 4 in \(G^*\). If \(|V(H)| \geq 2\), then \(H\) has at least one leaf, say \(f\); as before, we get that \(f\) is adjacent to at least 3 distinct vertices of \(F_3\). In any case, we get that \(y = |\{f \in F_3 \mid H \in \Gamma(f)\}| \geq 3\), which implies that \(\mu_2(H) = \mu_1(H) = \mu_0(H) + y/3 \geq 0\).

5 Linear Forest Backbone

We prove Theorem 2 in this section using the same general strategy, except that the structural properties needed are more complex. In the previous sections, a simple lemma concerning at most two vertices, say \(u\) and \(v\), was enough to say that a CBC-\(k\)-coloring \(\psi\) of \((G - u - v, H - u - v)\) could be extended to \((G, H)\). Here, the backbone is a linear tree and therefore we sometimes need to remove entire subpaths from a minimal counterexample \((G, H)\). For this, we work with the lists \(A_\psi\) in a more clever way. This is done in the next subsection.
5.1 Forbidden Structures

Let \((H, P)\) be such that \(P \subseteq H\), \(k\) be a positive integer, and \(\mathcal{L} : V(H) \to 2^{[k]}\). If there exists a CBC-\(k\)-coloring \(\psi\) of \((H, P)\) such that \(\psi(v) \in \mathcal{L}(v)\), for all \(v \in V(H)\), then we say that \((H, P)\) is \(\mathcal{L}\)-CBC-\(k\)-colorable. Throughout the proof, we sometimes consider \(\mathcal{L}\) to be smallest possible in the context. This is not a problem since whenever \((H, P)\) is \(\mathcal{L}\)-CBC-\(k\)-colorable and \(\mathcal{L}'\) is such that \(\mathcal{L}(v) \subseteq \mathcal{L}'(v)\), for every \(v \in V(H)\), we also have that \((H, P)\) is \(\mathcal{L}'\)-CBC-\(k\)-colorable.

Consider a pair \((H, P)\) such that \(P\) is a Hamiltonian path of \(H\), and write \(P\) as \((v_1, \ldots, v_n)\). Also, let \(\mathcal{L} : V(H) \to 2^{[7]}\) be a list assignment for \(H\), and \(\mathcal{L}' : V(H') \to 2^{[7]}\) be a list assignment for \(H' \subseteq H\). We use the reduction rule below to prove the non-existence of certain structures in a minimal counterexample to Theorem[5] We denote the values \(|\mathcal{L}(x)|\) and \(|\mathcal{L}'(x)|\) by \(\ell(x)\) and \(\ell'(x)\), respectively.

**Reduction Rule:** \(((H', P'), \mathcal{L}')\) is a reduction of \(((H, P), \mathcal{L})\) on \(v_1\) if:

- \(H' = H - v_1\);
- \(P' = P - v_1\);
- \(\ell'(v_2) \geq \ell(v_2) - 2\);
- \(\ell'(x) \geq \ell(x) - 1\), for every \(x \in N(v_1) \setminus \{v_2\}\);
- \(\ell'(x) = \ell(x)\), for every \(x \in V(H) \setminus N[v_1]\) and
- \(\mathcal{L}'(v_2) \setminus \mathcal{L}'(v_2) = \{c, d\}\), then \(|\{c\} \cup \{d\}| \leq 5\).

We say that a reduction \(((H', P'), \mathcal{L}')\) of \(((H, P), \mathcal{L})\) on \(v_1\) is extendable if every \(\mathcal{L}'\)-CBC-\(7\)-coloring of \((H', P')\) can be extended to an \(\mathcal{L}\)-CBC-\(7\)-coloring of \((H, P)\). The following lemma gives sufficient conditions for \(((H, P), \mathcal{L})\) to have an extendable reduction.

**Lemma 6.** Let \(H\) be any graph, \(P = (v_1, \ldots, v_n)\) be a Hamiltonian path of \(H\), and consider \(\mathcal{L} : V(H) \to 2^{[7]}\). If the conditions below hold, then \(((H, P), \mathcal{L})\) has an extendable reduction on \(v_1\).

1. \(d(v_1) \leq 4\);
2. \(\ell(v_1) \geq 1 + d(v_1)\); and
3. If \(d(v_1) = 4\), and \(c\) and \(d\) are the colors not in \(\mathcal{L}(v_1)\), then \(|\{c\} \cup \{d\}| \leq 5\).

**Proof.** Without loss of generality, suppose that \(\ell(v_1) = 1 + d(v_1)\). First, suppose that \(d(v_1) = 1\) If \(\mathcal{L}(v_1)\) has two consecutive colors, then remove both from \(\mathcal{L}(v_2)\); if \(\mathcal{L}(v_1) = \{c - 1, c + 1\}\) for some \(c \in [7]\), then remove \(c\) from \(\mathcal{L}(v_2)\); otherwise, do not change \(\mathcal{L}(v_2)\). Let \(\mathcal{L}'\) be the obtained function. One can see that \(((H - v_1, P - v_1), \mathcal{L}')\) is a reduction of \(((H, P), \mathcal{L})\) on \(v_1\). Let \(\psi\) be an \(\mathcal{L}'\)-CBC-\(7\)-coloring of \((H - v_1, P - v_1)\); if no such coloring exists, then the lemma holds by vacuity. By the choice of the removed colors, note that \(\mathcal{L}(v_1) \setminus \langle \psi(v_2) \rangle \neq \emptyset\), which means that \(\psi\) can be extended to \(v_1\).

Now, consider \(d(v_1) > 1\). First, suppose that there exists \(c \in \mathcal{L}(v_1)\) such that \(\{c - 1, c + 1\} \cap \mathcal{L}(v_1) = \emptyset\). Let \(\mathcal{L}'\) be obtained by removing \(c - 1\) and \(c + 1\) from
Lemma 8. Let \( (G, F) \) be a minimal counterexample to Theorem \( \text{X} \) and \( P \) be a heavy subpath of a component of \( F \). The following hold.

Lemma 7. Let \( (G, F) \) be a minimal counterexample to Theorem \( \text{X} \) then, we have \( \delta(G) \geq 3 \), and if \( v \in V(G) \) is such that \( d_{G}(v) \leq 4 \), then \( d_{F}(v) = 2 \).
(a) If $P$ has one vertex $v$ of degree 3, then $d(u) = 5$, $\forall u \in V(P) \setminus \{v\}$;

(b) If $P$ contains a leaf of $F$, then $d(u) = 5$, $\forall u \in V(P)$; and

(c) $P$ has at most two vertices of degree 4.

**Proof.** Below, we consider a subpath $P'$ of $P$, and denote by $H$ the subgraph $G[V(P')]$. We prove that whenever $P$ does not satisfy one of the assertions, then, letting $\psi$ be a CBC-7-coloring of $(G - H, F - H)$, we get that $(H, P')$ is $A_\psi$-CBC-7-colorable, contradicting the fact that $(G, F)$ is a minimal counterexample to Theorem 3 We recall that, by Lemma 4 we have $\delta(G) \geq 3$ and $d_G(u) \geq 5$ whenever $d_F(u) \leq 1$.

First, suppose that either (a) or (b) does not hold, and let $P' = (v_1, v_2, \ldots, v_q)$ be a shortest subpath of $P$ such that $q \geq 2$, $d(v_1) \leq 4$, and either $d(v_q) = 3$ or $v_q$ is a leaf in $P$. Also, let $\psi$ be a CBC-7-coloring of $(G - H, F - H)$. We construct a sequence $R_1, \ldots, R_q$ such that $R_1 = ((H, P'), A_\psi)$; $R_i$ is an extendable reduction of $R_{i-1}$ on $v_{i-1}$, for each $i \in \{2, \ldots, q\}$; and the list available for $v_q$ in $R_q$, say $A_q$, is nonempty. Observe that this leads to a contradiction since a coloring of $v_q$ with any $c \in A_q$ can be extended to an $A_\psi$-CBC-7-coloring of $(H, P')$ by the definition of extendable reduction. For each $i \in \{1, \ldots, q\}$ we write $R_i$ as $((H_i, P_i), A_i)$. Observe that $P_i = (v_1, \ldots, v_q)$ and that $H_i = H[(v_i, \ldots, v_q)]$, and denote by $\ell_i(v)$ the value $|A_i(v)|$, for each $v \in \{v_i, \ldots, v_q\}$. In order to obtain the desired sequence of extendable reductions, we want to apply Lemma 6. For this, we need to ensure that, at the beginning and after each step $i$ of the procedure, the inequalities below hold.

\begin{align*}
\ell_i(v_j) &\geq d_{H_i}(v_j) + 2, \text{ for every } j \text{ such that } i < j < q. \quad (5) \\
\ell_i(v_i) &\geq d_{H_i}(v_i) + 1, \text{ if } i < q. \quad (6) \\
\ell_i(v_q) &\geq \begin{cases} 
\ell_i(v_q) + 2, & \text{if } i < q \\
1, & \text{otherwise}
\end{cases} \quad (7)
\end{align*}

**Claim.** If Inequalities (5), (6), and (7) hold for $R_i$, with $1 \leq i < q$, then $R_i$ has an extendable reduction on $v_i$.

**Proof:** Because $d(v_j) \leq 5$, for every $v_j \in V(P')$, and by Inequality (6), we get that Conditions (1) and (2) of Lemma 6 hold. Now, suppose that $d_{H_i}(v_i) = 4$. Recall that $d_G(v_1) \leq 4$; hence $1 < i < q$, which implies that $d_H(v_i) = 5$. But since $d_{H_i}(v_i) = 4$, this means that $N_G(v_i) = N_{H_i}(v_i) \cup \{v_{i-1}\}$, which implies that $A_{i-1}(v_i) = [7]$. Then, Condition (3) follows by the definition of reduction.

We first argue that these inequalities initially hold. Recall that $H_1 = H$, $P_1 = P'$, and $A_1 = A_\psi$. First, consider any $j \in \{2, \ldots, q - 1\}$. Since $F$ is a linear tree, we have that $N_F(v_j) \subseteq P'$, which means that $\ell_1(v_j) \geq 7$.
\[ d_{G-H}(v_j) = 7 - (d_G(v_j) - d_H(v_j)). \] By the choice of \( v_1 \) and \( v_q \), we know that \( d_G(v_j) = 5 \), which in turn implies Inequality \([5]\). Now, by Lemma \([6]\) we know that \( d_F(v_1) = 2 \); so let \( v \in N_F(v_1) \setminus \{v_2\} \). Note that \( v \) forbids 3 colors for \( v_1 \), while each other colored neighbor of \( v_1 \) forbids just one color. This gives us that

\[ \ell_1(v_1) \geq 7 - (d_G(v_1) + 2d_F(v_1)) = 5 - (d_G(v_1) - d_H(v_1)) \geq d_H(v_1) + 1. \]

Analogously, for \( v_q \) we get: if \( d(v_q) = 3 \), then \( d_F(v_q) = 2 \) and \( \ell_1(v_q) \geq d_H(v_q) + 2 \); and if \( v_q \) is a leaf in \( P \), then by Lemma \([7]\) we get \( d_G(v_q) = 5 \), and as before \( \ell_1(v_q) = 7 - d_{G-H}(v_q) \geq d_H(v_q) + 2 \).

Now, suppose that we are at step \( i \) of our construction, \( 1 \leq i < q \), and let \( R_{i+1} \) be an extendable reduction of \( R_i \). We want to prove that Inequalities \([5]\), \([6]\), and \([7]\) also hold for \( R_{i+1} \). First, note that if \( v_j \in N(v_i) \setminus \{v_{i+1}\} \), then both \( d_{H_{i+1}}(v_j) \) and \( \ell_{i+1}(v_j) \) decrease by exactly 1; hence, Inequality \([6]\) holds, as well as Inequality \([7]\) in the case where \( i < q - 1 \). Similarly, \( d_{H_{i+1}}(v_{i+1}) \) decreases by 1, while \( \ell_{i+1}(v_{i+1}) \) decreases by at most 2; hence, if \( i < q - 1 \), we have that \( \ell_{i+1}(v_{i+1}) \geq d_{H_1}(v_{i+1}) + 2 \), which means that Inequality \([6]\) also holds for \( R_{i+1} \). Finally, suppose that \( i = q - 1 \). Then \( \ell_{q-1}(v_q) \geq d_{H_{q-1}}(v_q) + 2 = 3 \), and by the definition of reduction we get that \( \ell(q(v_q)) \geq 1 \), i.e., Inequality \([7]\) holds also when \( i = q - 1 \), and we are done proving \([5]\) and \([6]\).

Finally, in order to prove \([5]\), suppose that \( d(v) \geq 4 \), for every \( v \in V(P) \), and let \( u, v, w \in V(P) \) be the closest three vertices of degree 4 in \( P \), where \( v \) is between \( u \) and \( w \). Write the subpath of \( P \) between \( u \) and \( w \) as \( P' = (v_1 = u, v_2, \ldots, v_q = w) \) and let \( v_p = v \). Denote \( G[V(P')] \) by \( H \), and let \( \psi \) be a \( CBC-7 \)-coloring of \( (G - H, F - H) \). Note that:

- For each \( z \in V(P') \setminus \{u, v, w\} \), we get \( a_\psi(z) \geq 7 - d_{G-H}(z) = 7 - (d_G(z) - d_H(z)) = 2 + d_H(z) \);  
- For \( z \in \{u, w\} \), we get \( a_\psi(z) \geq 4 - (d_G(z) - 1) = 1 + d_H(z) \); and  
- \( a_\psi(v) = 7 - d_{G-H}(v) = 3 + d_H(v) \).

By arguments similar to the ones made for the first two cases, one can verify that a series of extendable reductions can be made on \( P' \), from \( v_1 \) up to \( v_{p-1} \), and from \( v_q \) down to \( v_{p+1} \), until we end up with just \( v_p \) with non-empty list.

### 5.2 Discharging Method

In this section, we finish the proof of Theorem \([8]\) For this, we use a definition similar to the one used in the proof of Theorem \([2]\). We make an abuse of language and use the same nomenclature. Consider a plane graph \( G \) and its dual \( G^* \), and let \( F_3 \) be the set of faces of degree 3 in \( G \) (alternatively, the set of vertices of degree 3 in \( G^* \)). We denote the graph \( G^* - F_3 \) by \( G_3^* \), and say that a component of \( G_3^* \) is an island of \( G \). Also, if \( H \) is an acyclic component of \( G_3^* \) such that \( d_{G^*}(f) = 5 \), for every \( f \in V(H) \), then we say that \( H \) is a bad island of \( G \). We denote the set of bad islands of \( G \) by \( \Gamma \) and we let \( \gamma \) denote \( |\Gamma| \). Also, for \( v \in V(G) \), we denote by \( \Gamma(v) \) the set of bad islands containing \( v \), and by \( \gamma(v) \)
the value $|\Gamma(v)|$. If $X \subseteq V(G)$, then $\Gamma(X) = \bigcup_{x \in X} \Gamma(x)$, and $\gamma(X) = |\Gamma(X)|$.

In the remainder of the text, although we refer to $G$ as being planar, we are implicitly considering a planar embedding of $G$ and its islands.

**Lemma 9.** Let $G$ be a planar graph without cycles of length 4 as subgraph. Then,

$$|E(G)| \leq 2|V(G)| - 4 + \frac{\gamma}{3}. $$

**Proof.** Let $f_3, f_5$ denote the number of faces of degree 3 and 5, respectively, and let $|E(G)|$ be denoted by $m$. Also, denote by $F$ the set of faces of $G$ and by $|f|$ the degree of a face $f \in F$. We claim that:

$$3f_3 + f_5 \leq m + \gamma \quad (8)$$

This implies that $t = \sum_{f \in F} (|f| - 6) \geq -3f_3 - f_5 \geq -m - \gamma$. On the other hand $t = \sum_{f \in F} (|f| - 6) = 2m - 6|F|$. Combining these and applying Euler's Formula we get (below, $n$ denotes $|V(G)|$):

$$2m - 6(2 - n + m) \geq -m - \gamma \iff m \leq 2n - 4 + \frac{\gamma}{3}$$

It remains to prove Inequality $8$. For this, we partition $E(G)$ in $E_3, \overline{E}_3$, where $E_3$ is described below and $\overline{E}_3 = E(G) \setminus E_3$.

$$E_3 = \{ e \in E(G) \mid e \text{ is contained in some face of degree 3} \}.$$ 

Because $G$ has no cycle of length 4, we trivially get that $|E_3| = 3f_3$. We prove that $|\overline{E}_3| \geq f_5 - \gamma$, thus finishing the proof. For this, note that if $e \in \overline{E}_3$, then there is an edge $e^*$ in $G_5$ related to $e$. On the other hand, if $e^* \in E(G_5)$, then $e^*$ is related to an edge $e \in E(G)$ that separates faces of degree at least 5; hence, $e \in \overline{E}_3$. Therefore, $|\overline{E}_3| = |E(G_5)|$. Finally, because the number of edges in any graph is at least the number of vertices minus the number of acyclic components of the graph, we get:

$$|\overline{E}_3| \geq |V(G_5^*)| - \gamma \geq f_5 - \gamma.$$ 

Supposing that $(G, F)$ is a minimal counterexample to Theorem 8 we apply the discharging method to prove that $\sum_{v \in V(G)} d(v) \geq 4|V(G)| + \frac{\gamma}{3}$, contradicting Lemma 9. For this, we start by giving charge $d(v) - 4$ to every $v \in V(G)$, and charge $-2/3$ to every bad island. The discharging rules ensure that every vertex and every bad island end up with a non-negative charge (i.e., Property 1 below holds), which clearly contradicts Lemma 9. The rules are applied in the order they are presented. Also, given $x \in V(G) \cup \Gamma$, the initial charge of $x$ is denoted by $\mu_0(x)$, and the charge of $x$ after Rule $i$ is applied is denoted by $\mu_i(x)$, for each $i \in \{1, \ldots, 5\}$.

**Property 1.** After Rule $i$ is applied, we have that $\mu_i(v) \geq 0$ and $\mu_i(b) \geq 0$, for every vertex $v$ iterated in Rule $i$ and every bad island $b$ containing $v$. 

The proof following each rule is a proof that Property 1 holds after the corresponding rule has been applied.

**Rule 1** For every $v \in V(G)$ with $d(v) \geq 6$, send $2/3$ from $v$ to each $b \in \Gamma(v)$.

**Proof.** Consider $v \in V(G)$ with $d(v) \geq 6$. Because every island containing $v$ receives $2/3$, we just need to prove that $\mu_1(v) \geq 0$. Because $G$ has no cycles of length 4, observe that $\gamma(v) \leq \frac{d(v)}{2}$. This gives us that:

$$\mu_1(v) \geq d(v) - 4 - \frac{2}{3} \gamma(v) \geq d(v) - 4 - \frac{2}{3} \cdot \frac{d(v)}{2} \geq \frac{2}{3} d(v) - 4 \geq 0. \quad (9)$$

The following proposition will be useful in the remainder of the text. Observe that it holds because at least one face containing $uv$ cannot be a face of degree 3, as otherwise we get a cycle of length 4.

**Proposition 1.** If $G$ is a graph without cycles of length 4, and $uv \in E(G)$, then there exists a face of degree greater than 3 containing $uv$.

**Rule 2** Let $P = (v_1, \ldots, v_q)$ be a heavy subpath containing no vertex with degree smaller than 5. We have the following cases:

R2.1 If $P$ is a component of $F$, send charge $2/3$ from $\mu_1(v_1) + \mu_1(v_2)$ to every $b \in \Gamma\{v_1, v_2\}$. After this, if $q \geq 3$, then for each $i \in \{3, \ldots, q\}$, send charge $2/3$ from $v_i$ to $b \in \Gamma(v_i) \setminus \Gamma(v_{i-1})$.

R2.2 Otherwise, let $v_0 \in N_F(v_1) \setminus \{v_2\}$. For every $i \in \{1, \ldots, q\}$, send charge $2/3$ from $v_i$ to $b \in \Gamma(v_i) \setminus \Gamma(v_{i-1})$.

**Proof.** First, note that $\mu_1(v_i) = 1$, for every $i \in \{1, \ldots, q\}$. Suppose that $P$ is a component of $F$. Note that Lemma 1 implies that $q \geq 2$. By Proposition 1, we get that $\gamma\{v_1, v_2\} \leq 3$, and that, when $q \geq 3$, then for every $i \in \{3, \ldots, q\}$ we get $|\Gamma(v_i) \setminus \Gamma(v_{i-1})| \leq 1$. Property 1 follows.

Now, suppose that $P$ is not a component of $G$, in which case we can suppose, without loss of generality, that $v_0$ exists. By the definition of heavy path, we know that $d(v_0) \geq 6$, which, by Rule 2, implies that the island in $\Gamma(v_0) \cap \Gamma(v_1)$ has non-negative charge. Now, applying Proposition 1, for each $v_i \in V(P)$ we get that $|\Gamma(v_i) \setminus \Gamma(v_{i-1})| \leq 1$. Hence, Property 1 follows.

**Rule 3** Let $P = (v_1, \ldots, v_q)$ be a heavy subpath containing exactly one vertex with degree smaller than 5, namely $v_p$, and let $v_0 \in N_F(v_1) \setminus P$ and $v_{q+1} \in N_F(v_q) \setminus P$. We have the following cases.

R3.1 If $q \geq 2$, we can suppose that $p < q$.

(i) Send charge $2/3$ from $v_i$ to $b \in \Gamma(v_i) \setminus \Gamma(v_{i-1})$, for each $i \in \{1, \ldots, p-1\}$;

(ii) Send charge $2/3$ from $v_i$ to $b \in \Gamma(v_i) \setminus \Gamma(v_{i+1})$, for each $i \in \{p+2, \ldots, q\}$;
(iii) If \( d(v_p) = 3 \), then \( v_{p+1} \) sends charge 1 to \( v_p \). Otherwise, \( v_{p+1} \) sends charge \( 2/3 \) to \( b \in \Gamma(v_p) \cap \Gamma(v_{p+1}) \).

R3.2 If \( q = 1 \) and \( d(v_1) = 3 \), let \( b \in \Gamma(v_1) \). Send charge 1 from \( \mu_2(v_0) + \mu_2(v_2) + \mu_2(b) \) to \( v_1 \).

\[ \text{Proof.} \] By Lemma \( \Box \) we know that \( v_0 \) and \( v_q+1 \) exist, and, by Rule \( \Box \) we know that the islands in \( \Gamma(v_0) \cap \Gamma(v_1) \) and \( \Gamma(v_q) \cap \Gamma(v_{q+1}) \) have non-negative charge. First, suppose that \( q \geq 2 \). By arguments similar to the ones in the previous demonstrations, one can see that the vertices in \( \{v_1, \ldots, v_{p-1}, v_{p+2}, \ldots, v_q\} \), as well as the islands containing them, have non-negative charge. Also, note that, by Proposition \( \Box \) either \( d(v_p) = 3 \) and the only island containing \( v_p \) also contains \( v_{p-1} \) and \( v_{p+1} \), or \( d(v_p) = 4 \) and the island in \( \Gamma(v_p) \cap \Gamma(v_{p+1}) \) is the only one that might not be satisfied yet. In either case, one can verify that the rule satisfies \( v_p \) or the referred island, depending on the case.

Now, suppose that \( q = p = 1 \). If \( d(v_1) = 4 \), then \( \Gamma(v_1) \subseteq \Gamma(v_0) \cup \Gamma(v_2) \) and nothing needs to be done; so suppose otherwise. First note that, because \( d(v_1) = 3 \), the island \( b \in \Gamma(v_1) \) also contains \( v_0 \) and \( v_2 \). This means that \( b \) has received charge from both \( v_0 \) and \( v_2 \) when Rule \( \Box \) is applied; hence \( \mu_2(b) = 2/3 \). We end the proof by showing that \( \mu_2(v_2) = \mu_1(v_2) \geq 2/3 \). Note that, since \( d(v_1) = 3 \) and because \( G \) has no cycle of length 4, we can suppose that \( v_1 \) has no common neighbor with \( v_2 \). Therefore, if \( d(v_2) = 6 \), then \( \gamma(v_2) = 2 \), and applying the first part of Inequality \( \Box \) we get that \( \mu_2(v_2) = 6 - 4 - 4/3 = 2/3 \). On the other hand, if \( d(v_2) \geq 7 \), we get \( \mu_2(v) \geq 2/3 \) by Inequality \( \Box \).

In the next discharging rule, given \( X \subseteq V(G) \), we denote \( \sum_{x \in X} \mu_3(x) \) by \( \mu_3(X) \).

**Rule 4** Let \( P = (v_1, \ldots, v_\ell) \) be a heavy subpath containing exactly two vertices with degree smaller than 5, namely \( v_p \) and \( v_q \), \( p < q \). Let \( v_0 \in N_F(v_1) \setminus P \) and \( v_{\ell+1} \in N_F(v_\ell) \setminus P \). Define

\[ \beta = \Gamma(V(P)) \setminus \Gamma(\{v_0, v_{\ell+1}\}) \],

and

\[ \mu = \mu_3(V(P)) + \frac{2}{3} |\Gamma(v_0) \cap \Gamma(v_{\ell+1})| \].

If \( \mu \geq \frac{2}{3} |\beta| \), then send \( 2/3 \) from \( V(P) \) and \( \Gamma(v_0) \cap \Gamma(v_{\ell+1}) \) to each \( b \in \beta \).

By the condition under which it is applied, Rule \( \Box \) clearly satisfies Property \( \Box \). However, we still need a final rule for the paths on which the condition \( \mu \geq \frac{2}{3} |\beta| \) does not hold. Before we present the rule, we give sufficient conditions for Rule \( \Box \) to be applied.

**Lemma 10.** If \( P \) is a heavy subpath containing exactly two vertices with degree smaller than 5, and either \( |V(P)| \geq 4 \), or \( \gamma(V(P)) \leq |V(P)| \), then \( \mu \geq \frac{2}{3} |\beta| \).
Proof. Consider $P, v_p, v_q, v_0, v_{\ell+1}, \beta, \mu$ be all defined as in Rule 8 (recall that $v_0, v_{\ell+1}$ exist by Lemma 8). First note that

$$|\beta| = \gamma(V(P)) - |\Gamma(V(P)) \cap \Gamma\{v_0, v_{\ell+1}\}|.$$

Also, by Proposition 11 we have

$$\gamma(V(P)) \leq 2\ell - (\ell - 1) = \ell + 1.$$

Finally, by Lemma 8 we get that $d(v_p) = d(v_q) = 4$, and $d(v_i) = 5$, for every $v_i \in V(P) \setminus \{v_p, v_q\}$. Hence

$$\mu_3(V(P)) = \ell - 2.$$

Now, denote by $t$ the value $|\Gamma(V(P)) \cap \Gamma\{v_0, v_{\ell+1}\}|$. By Proposition 11 we know that $t \geq 1$. We analyse the following cases:

- If $t = 1$, then the islands in $\Gamma(v_0) \cap \Gamma(v_1)$ and $\Gamma(v_2) \cap \Gamma(v_{\ell+1})$ must be the same, i.e., $\Gamma(v_0) \cap \Gamma(v_{\ell+1}) \neq \emptyset$, and $|\beta| = \gamma(V(P)) - 1$. Therefore,

  $$\mu \geq \mu_3(V(P)) + \frac{2}{3} = \ell - 2 + \frac{2}{3} = \ell - \frac{4}{3}.$$

  If $\ell \geq 4$, then $|\beta| \leq \ell$ and $\mu \geq \ell - \frac{4}{3} \geq 2\ell \geq \frac{2}{3}|\beta|$. And if $\gamma(V(P)) \leq \ell$, then $|\beta| \leq \ell - 1$, and, since $\ell \geq 2$, we get $\mu = \ell - 2 + \frac{2}{3} = \ell - 1 = \ell + 1.$

- Now, if $t \geq 2$ and $\ell \geq 4$, then $|\beta| \leq \ell - 1$, and $\mu \geq \ell - 2 \geq \frac{2}{3}(\ell - 1) \geq \frac{2}{3}|\beta|$. Finally, if $t \geq 2$ and $\gamma(V(P)) \leq \ell$, then $|\beta| \leq \ell - 2$ and clearly $\mu \geq \ell - 2 \geq |\beta| \geq \frac{2}{3}|\beta|$.

Now, consider $P$ as in Rule 8 and suppose that the rule is not applied, which means that there might still exist some bad island intersecting $V(P)$ with negative charge. If such an island exists, we call such a path defective. Before we present the last discharging rule, we need the lemmas below. We mention that by Lemma 11 if $P$ is defective then $\ell \leq 3$ and $\gamma(V(P)) \geq \ell + 1$, where $\ell = |V(P)|$.

**Lemma 11.** Let $P$ be a defective path of size $\ell$ with extremities $v_1$ and $v_{\ell}$, and denote by $v_0$ the neighbor of $v_1$ in $P$ (hence, it might happen that $\ell = 2$). Also, let $v_0 \in N_F(v_1) \setminus \{v_2\}$, and $v_{\ell+1} \in N_F(v_{\ell}) \setminus \{v_{\ell-1}\}$. Then, for each $i \in \{1, 2, \ell\}$, we have that $v_i$ is contained in exactly two bad islands (which means that $v_i$ is contained in two 3-faces that separate these bad islands), and $v_{i-1}v_{i+1} \notin E(G)$.

**Proof.** First, suppose that $i \in \{1, 2, \ell\}$ is such that $v_i$ is contained in at most one triangle, which means that $\gamma(v_i) \leq 1$. Note that if $\ell = 3$, then $|\Gamma(v_1) \cap \Gamma(v_2) \cap \Gamma(v_3)| - |\Gamma(v_1) \cap \Gamma(v_3)| \leq 0$. This justifies the second line in the equation below.

$$\gamma(V(P)) = \left| \bigcup_{v_j \in V(P)} \Gamma(v_j) \right|$$

$$\leq \sum_{j \in \{1, 2, \ell\}} \gamma(v_j) - \sum_{j \in \{1, \ell - 1\}} |\Gamma(v_j) \cap \Gamma(v_{j+1})|$$

$$\leq \sum_{j \in \{1, 2, \ell\} \setminus \{i\}} \gamma(v_j) + \gamma(v_i) - (\ell - 1)$$

$$\leq 2(\ell - 1) + 1 - \ell + 1 = \ell$$
This means that $P$ satisfies Lemma 10, a contradiction. Note also that this actually implies that each $v_i$ is contained in exactly two bad islands.

Now, suppose that $i \in \{1, 2, \ell\}$ is such that $v_{i-1}v_{i+1} \in E(G)$. Note that if $\ell = 3$ and $i = 2$, then $\gamma(V(P)) = \gamma(\{v_1, v_3\})$, and the island in $\Gamma(v_0) \cap \Gamma(v_1)$ also contains $v_3$. This implies that $\gamma(V(P)) = 3$, contradicting Lemma 10. So suppose, without loss of generality, that $i = 1$ and let $b$ be the island containing $v_0v_2$. Note that $\Gamma(v_1) \subseteq \Gamma(\{v_0, v_2\})$; therefore, $\beta = \Gamma(\{v_2, v_4\}) \backslash \Gamma(\{v_0, v_{\ell+1}\})$. First consider $\ell = 2$. If $b$ also contains $v_3$, then $|\beta| \leq |\Gamma(v_2) \backslash \{b\}| = 1$, and $\Gamma(v_0) \cap \Gamma(v_3) \neq \emptyset$, which implies $\mu \geq \frac{2}{3}|\beta|$. And if $b$ does not contain $v_3$, then $\Gamma(v_2) \subseteq \Gamma(\{v_0, v_3\})$, in which case $\beta = \emptyset$. Both cases are contradictions.

Therefore, suppose that $\ell = 3$, and let $B$ denote $\Gamma(\{v_0, v_4\})$. Note that:

$$|\beta| = |\Gamma(\{v_2, v_3\}) \backslash B| = |(\Gamma(v_2) \backslash B) \cup (\Gamma(v_3) \backslash B)| \leq |\Gamma(v_2) \backslash B| + |\Gamma(v_3) \backslash B| \leq 2.$$

The last part holds since $b \in \Gamma(v_2) \cap B$, and $\Gamma(v_3) \cap \Gamma(v_4) \neq \emptyset$ (Proposition 11). If $|\beta| \leq 1$ we are done since $\mu \geq 1$. Therefore, suppose $|\beta| = 2$, in which case we must have $(\Gamma(v_2) \backslash B) \cap (\Gamma(v_3) \backslash B) = \emptyset$. So, let $b_i \in (\Gamma(v_i) \backslash B)$, for $i = 2$ and $i = 3$, and let $b^* \in \Gamma(v_2) \cap \Gamma(v_3)$. Because $\Gamma(v_2) \cap \Gamma(v_3) \neq \emptyset$ and $b_2 \neq b_3$, we get $b = b^*$, i.e., $b \in \Gamma(v_0) \cap \Gamma(v_4)$. Therefore, we get $\mu \geq 1 + \frac{2}{3} > \frac{4}{3} = \frac{2}{3}|\beta|$, a contradiction.

The next lemma is the final step before we can present the last discharging rule. We denote by $\Theta$ the set of bad islands with negative charge, and by $D$ the set of vertices of degree 5 which are contained in some island in $\Theta$.

**Lemma 12.** Let $b \in \Theta$, and $f$ be a face of degree 5 in $b$. Then $f$ contains at least one vertex of $D$ and, if it contains exactly one such vertex, namely $u$, then $b$ is the only island in $\Theta$ that contains $u$.

**Proof.** Let $f = (v_1, \ldots, v_5)$ be such that $v_i$ is contained in some defective path, for each $i \in \{1, \ldots, 5\}$. Without loss of generality, suppose that $d(v_i) = 4$, for every $i \in \{1, \ldots, 4\}$. First, we want to prove that $(v_1, \ldots, v_5)$ is an induced cycle in $G$. So suppose that $v_1v_5 \in E(G)$. Since $f$ is a 5-face in $G$, we must have that the edge $v_1v_5$ is traced in the outer side of $f$. Because $\delta(G) \geq 3$, one can verify that this implies that $(v_1, v_2, v_3)$ is not a 3-face in $G$, which in turn implies that $v_1$ is contained in at most one bad island, contradicting Lemma 11. Observe that the same argument can be applied to conclude that $v_iv_j \notin E(G)$, for every $i \in \{1, \ldots, 4\}$ and every $j \in \{1, \ldots, 5\} \setminus \{i\}$. Now observe that, by Lemma 11, there must exist $u_1, \ldots, u_5$, where $u_5 \in N(v_1) \cap N(v_5)$, and $u_i \in N(v_i) \cap N(v_{i+1})$, for each $i \in \{1, \ldots, 4\}$. This means that every island in $\Theta$ is a face of degree 5. We claim that $d(v_5) = 5$. Supposing it holds, let $w \in N(v_5) \setminus \{v_1, v_4, u_5\}$; also let $f_1$ be the face containing $u_4v_5$ different from $(v_4, v_5, u_5)$, and $f_2$ be the face containing $u_5v_5$ different from $(u_5, v_5, v_1)$. Because $G$ has no cycles of length 4, we know that $f_1$ and $f_2$ have degree bigger than 3, and that they share the edge
v_5 w. This means that f_1 and f_2 are within the same island t, which implies that t \notin \Theta, and the lemma follows, i.e., b is the only island in \Theta containing u. It remains to prove our claim.

Suppose by contradiction that d(v_5) = 4, and let H denote the induced subgraph G[[v_1, \ldots, v_5, u_1, \ldots, u_5]]. Because d_F(v_i) = 2 and N(v_i) \subseteq V(H), for every i \in \{1, \ldots, 5\}, we know that H must contain every edge in F incident to \{v_1, \ldots, v_5\}. For each \ell, let E_\ell denote the set \{\ell v_i \in E(F)\}; we know that |E_\ell| = 2. Therefore, if E_i \cap E_j = \emptyset, for every i, j \in \{1, \ldots, 5\}, i \neq j, then |E(H) \cap E(F)| = |\bigcup_{i=1}^5 E_i| = \sum_{i=1}^5 |E_i| = 10 = |V(H)|, contradicting the fact that F is acyclic. We can then suppose, without loss of generality, that v_1v_2 \in E(F). By Lemmas S and T we get that \{u_5v_1, u_2v_2\} \subseteq E(F). Also, by Lemma T we get |\{v_3v_4, v_3v_5\} \cap E(F)| \leq 1 and |\{v_4v_5, u_4v_5\} \cap E(F)| \leq 1. This implies that \{u_5v_5, u_2v_3\} \subseteq E(F). It is easy to verify that no matter the choice of edges in E_4, we get a cycle in F, a contradiction.

The lemma above implies the correctness of our final discharging rule.

Rule 5 Let K = (D, E) be such that uv \in E if and only if u and v are within the same bad island b \in \Theta. For each component K’ of K, apply one of the following:

R5.1 If |V(K’)| \geq 2, let T be a spanning tree of K’ and let uv \in E(T). Send charge 2/3 from \{u, v\} to each island in \Gamma(\{u, v\}), and for every w \in V(T) \setminus \{u, v\}, send charge 2/3 from w to the island in \Gamma(w) \setminus \Gamma(w’), where w’ \in N_T(w) separates w from uv.

R5.2 If V(K’) = \{u\}, send 2/3 from u to the bad island in \Theta containing u.

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