Entanglement distribution in fermion model with long-range interaction

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(Dated: March 22, 2022)

How two-party entanglement (TPE) is distributed in the many-body systems? This is a fundamental issue because the total TPE between one party with all the other parties, $C^N$, is upper bounded by the Coffman, Kundu and Wootters (CKW) monogamy inequality, from which $C^N \leq \sqrt{N-1}$ can be proved by the geometric inequality. Here we explore the total entanglement $C^\infty$ and the associated total tangle $\tau^\infty$ in a $p$-wave free fermion model with long-range interaction, showing that $C^\infty \sim O(1)$ and $\tau^\infty$ may become vanishing small with the increasing of long-range interaction. However, we always find $C^\infty \sim 2\xi\tau^\infty$, where $\xi$ is the truncation length of entanglement, beyond which the TPE is quickly vanished, hence $\tau^\infty \sim 1/\xi$. This relation is a direct consequence of the exponential decay of the TPE induced by the long-range interaction. These results unify the results in the Lipkin-Meshkov-Glick (LMG) model and Dicke model and generalize the Koashi, Buzek and Imomo bound to the quantum many-body models, with much broader applicability.

Entanglement is essential for quantum information [1–4] and quantum computation [5–7]. In the many-body systems, entanglement can be distributed between two arbitrary parties, which hereafter will be named as two-party entanglement (TPE) [8–11]. In general, the long-range interaction is required for long-distance TPE [12–15]. However, the monogamy of entanglement from the Coffman, Kundu, Wootters (CKW) conjecture restricts the total entanglement shared between one party with all the other parties with [10, 11, 16–18]

$$\tau^N := \sum_{i \neq 1}^{N} \tau(\rho_{A_iA_i}) \leq \tau(\rho_{A_1|A_2A_3\cdots A_N}),$$

where the $N$ qubits are denoted by $A_i$, with the density matrix is denoted by $\rho_{A_1A_2\cdots A_N}$. This inequality also holds in all multimode Gaussian states [19, 20]. Here the tangle is given by $\tau(\rho) = C^2(\rho)$, where $C$ is the two-party concurrence [10]. The right-hand side of the inequality is bounded from above by $\tau(\rho_{A_1|A_2A_3\cdots A_N}) \leq 1$, which is the maximum entropy of a single particle density matrix measured for $A_1$. This inequality is tight and can be satisfied by some pure states (such as the W states); see [17, 21–24]. By the geometric inequality

$$C^N := \sum_{i \neq 1}^{N} C(\rho_{A_iA_i}) \leq \sqrt{N-1} < \sqrt{N},$$

in which the tightness can be achieved using some asymmetric $W$ states [21, 25]. However, in the realistic many-body models, the total entanglement $C^N$ is much smaller than the above upper bound. In the Lipkin-Meshkov-Glick (LMG) model with identical interaction between two arbitrary parties [26–31], it has been found that $\tau(\rho_{A_1A_1}) \sim 1/N^2$ and $C(\rho_{A_1A_1}) \sim 1/N$, thus while the total entanglement $C^N \rightarrow O(1) \ll \sqrt{N}$, the total tangle $\tau^N \sim 1/N$ becomes vanishing small as $N \rightarrow \infty$ [29, 32, 33]. The large $N$ limit is nothing but just the classical approximation of the quantum spin operator, based on which its phase transition can be solved. These results are also found in the Dicke state, spin coherent state and spin squeezed states [27, 28]. For the symmetric case, the upper bound $C(\rho_{A_1}) \leq 2/N$ is proven by Koashi, Buzek and Imomo (KBI) [34]. These results raise an important yet intriguing question that how the total entanglement is distributed in the general many-body models in the thermodynamic limit? Obviously, when the LMG and Dicke models approach the classical limit without TPE at $N \rightarrow \infty$, it will be totally different in the other quantum systems with nonzero short-distance TPE. The answer to this question maybe useful for us to understand the role of entanglement in the many-body models and their phase transitions.

We aim to explore the distribution of TPE restricted by the CKW monogamy inequality in the many-body system with long-range interaction, which may induce long-distance TPE. We will focus on an artificial $p$-wave superconducting model with long-range interaction in the thermodynamic limit. We find that: (I) Both the long-range interaction and large chemical potential are important for long-distance TPE; when the chemical potential is small, the TPE can not be distributed between distance qubits. (II) In the presence of long-distance TPE, the scaling laws of these long-distance TPE at the critical point can be used to diagnose the phase transition; however, the central charge from the entanglement entropy is still $c = 1$ for free bosons; (III) In the condition with long-distance TPE, we always have $C^\infty \sim 2\xi\tau^\infty$ and $C^\infty \sim O(1)$, which have the same feature as the LMG model by replacing the total number of qubits $N$ using the truncation length $\xi$. These results may represents some general features of TPE in the many-body systems.
We consider the following $p$-wave superconducting model with long-range interaction (see Fig. 1 (a))

$$H = -\mu \sum_{i=0}^{\infty} c_{i}^\dagger c_{i} + \sum_{i,j} [-t(d) c_{i}^\dagger c_{j} + \Delta(d) c_{i} c_{j} + \text{h.c.}], \quad (3)$$

where $\mu$ is the chemical potential, and $t(d) = t e^{-\alpha|d|+\alpha}$, $\Delta(d) = \Delta e^{-\beta|d|+\beta}$ are the hopping and pairing strengths between the sites $i$ and $j$ with separation $d = i - j$. This model is a direct generalization of the Kitaev chain for Majorana edge modes [35], which has been a toy model for searching of Majorana modes in experiments [36–38]. After a non-local Jordan-Wigner transformation, it is changed to the transverse XY model, which is the limiting case of a lot of spin models [39–42]. Thus the above model may have features that can be found in a lot of many-body models.

In the limit of short-range interaction $t(d) = t \delta_{d=\pm1}$, $\Delta(d) = \Delta \delta_{d=\pm1}$, this model has been studied in [35, 43, 44], showing of only singularity in the first derivative of the nearest-neighbor TPE with some kind of universal scaling laws. This is a general feature of short-ranged interacting models [12, 13]. For the realization of long-distance TPE, one may need the long-range interaction, as evidenced by the LMG and Dicke models [26–31]. When $\alpha \to 0$ and $\beta \to 0$, it can be regarded as an infinitely coordinated system [32], such as that in the LMG model. We emphasize that all the features demonstrated below do not depend on the particular forms of the long-range interaction. In the momentum space

$$H = \sum_k \left( c_k^\dagger \begin{pmatrix} \epsilon_1 & i \epsilon_2 \\ -i \epsilon_2 & -\epsilon_1 \end{pmatrix} c_{-k} \right), \quad (4)$$

where

$$\epsilon_1(k) = \mu + t \exp(\alpha) \frac{\sinh(\alpha) - \cosh(\alpha) + \cos(k)}{\cosh(\alpha) - \cos(k)}, \quad (5)$$

$$\epsilon_2(k) = \Delta \exp(\beta) \frac{\sin(k)}{\cosh(\beta) - \cos(k)}. \quad (6)$$

The eigenvalues are

$$E(k) = \pm \sqrt{\epsilon_1^2 + \epsilon_2^2}, \quad E_g = \min_{k \in [-\pi, \pi]} (2E(k)). \quad (7)$$

The second term defines the global energy gap used in Fig. 1 (b). The closing and reopening of the energy gap yields the phase boundaries at (independent of $\beta$ and $\Delta$)

$$\alpha^* = \ln(\mu/(2t \pm 0)). \quad (8)$$

For simplicity, hereafter, we let $t = 1$ as the basic energy scale of the above Hamiltonian. The phase diagram of $H$ is given in Fig. 1 (b), with typical band structure shown in Fig. 1 (c) for the two phase transitions at $k = 0$ and $k = \pi$, respectively. At these boundaries, the spectra are linearized, indicating of criticality with nonzero central charge from conformal field theory.

The TPE derived from the formation of entanglement [11, 45] is well defined, termed as concurrence. We find that the two-party density matrix with separation $d$ can be obtained as [46–50]

$$\rho_d = \rho = \frac{1}{4} \begin{pmatrix} a & 0 & 0 & f \\ 0 & b & e & 0 \\ 0 & e^* & b & 0 \\ f^* & 0 & 0 & d \end{pmatrix}, \quad (9)$$

where $a = P_{00} + 2P_{z0} + P_{zz}$, $d = P_{00} - 2P_{z0} + P_{zz}$, $b = P_{00} - P_{zz}$, $e = P_{xx} + P_{yy}$, $f = P_{xx} - P_{yy}$. These coefficients are determined by [44, 47, 48]

$$P_{00} = 1, \quad P_{zz} = G_{00}' - G_{0} G_{-d}, \quad P_{z0} = P_{0z} = -G_{0}, \quad P_{xx} = \det(T(G_{i-j-1})), \quad P_{yy} = \det(T(G_{i-j+1})), \quad (10)$$

where $T$ is a $d \times d$ Toplitz matrix with entries $T_{i,j}$, with

$$G_{x} = \int_{-\pi}^{\pi} \frac{\epsilon_2 \cos(kx) - \epsilon_1 \sin(kx)}{2 \pi \sqrt{\epsilon_1^2 + \epsilon_2^2}} dk. \quad (11)$$
The concurrence of this density matrix reads as [10, 11]

\[ C_d(\rho_d) = \max\{0, \frac{1}{2}(|f| - |b|), \frac{1}{2}(|e| - \sqrt{ad})\}. \]  

(12)

Due to the translational symmetry in \(H\), it is sufficient to characterize the total entanglement between one arbitrary party with all the other parties using

\[ C^N = 2 \sum_{d=1}^{N} C_d(\rho_d), \quad \tau^N = 2 \sum_{d=1}^{N} \tau_d(\rho_d), \]  

(13)

where \(\tau_d = C^N_d\) and the factor 2 comes from the inversion symmetry of \(C_{-d} = C_d\). We find a truncation length \(\xi\), which means the farthest distance \(d\) that the TPE can be achieved. This distance is well-defined for the reason that when the distance \(|d| > \xi\), \(C_d\) will quickly approach zero (see Fig. 2 (a) - (b)). When \(d < \xi\), \(C_d\) will decay exponentially with the increase of separation. This is in stark contrast to the model with short-range interaction [43, 44], in which \(\xi \sim 1/3\), thus only the nearest-neighbor sites have significant TPE. Moreover, only the entanglement of \(C_{\pm 1}\) exhibits singularity at the critical boundaries, which has been used to characterize the phase transitions [44, 51]. Hence we can define the total entanglement shared between one party with all the other parties as \(C^\infty = 2 \sum_{d=1}^{\infty} C_d\) and \(\tau^\infty = 2 \sum_{d=1}^{\infty} \tau_d\).

The long-range interaction is necessary but not sufficient to induce long-distance TPE. The truncation length \(\xi\) as a function of \(\mu\) and \(\alpha\) (assuming \(\alpha = \beta\)) are shown in Fig. 2 (c) - (d), showing that when \(\mu = 0\), \(\xi\) will approach a small value \(\xi \sim 1\). On the other hand, for fixed \(\mu\), \(\xi\) will also approach a constant independent of \(\mu\) when \(\alpha \to 0\) (the long-range limit), for the reason of divergence of \(\epsilon_1\), \(\epsilon_2\) and the energy gap \(E_\gamma\) at \(\alpha \to 0\). However, we emphasize that while the long-range interaction can induce long-distance TPE, at the phase boundary the many-body entanglement measured using the Shannon entropy is not dramatically affected. By fitting the entanglement entropy using \(S_A = \frac{\xi}{2} \log_2 L + s_0\), we obtain \(c = 1\) [46, 49, 50]; however, the longer the interaction is, the longer the length \(L\) is required to obtain this perfect linear relation (see Fig. 2 (f)).

The long-range interaction can induce long-distance TPE, which exhibit some universal scaling laws in their first derivative. We want to show that (see Fig. 3) [43, 52]

\[ \frac{\partial C_d}{\partial \alpha} \approx k_d \ln(|\alpha - \alpha^*|), \quad k_d = q|d| + q', \]  

(14)

where \(\alpha^*\) is the critical values. This relation is also true as a function of the other parameters. The interesting point is that the long-range interaction can induce not only short-range TPE (by \(C_1\)), but also long-distance TPE (by \(C_d\) for \(|d| > 1\)), which is strictly forbidden in the transverse XY model [43]. The slope \(k_d\) depends strongly on the value of \(d\) in a linear manner (inset of Fig. 3 (b)). The same scaling form can be obtained as a function of chemical potential with critical \(\mu^*\). A detailed analysis shows that the above logarithmic divergence arises from the logarithmic divergence of \(\partial G_\lambda/\partial \alpha\), which is independent of the particular forms of long-range interaction.

The upper bounds of CKW and KBI inequalities yield some dilemma to the total entanglement in the thermodynamic limit. These relations in the few-party systems (using the entangled photon pairs) have been intensively verified [53–57]; yet in the many-body quantum systems they are rarely discussed. In 29, the concurrence in the LMG model has been studied using

\[ H = -\lambda \sum_{i<j} (\sigma^z_i \sigma^z_j + \gamma \sigma^y_i \sigma^y_j) - \sum_i \sigma^x_i, \]  

finding \((\lambda < 1)\)

\[ C_d = \frac{1 - \sqrt{1 - \gamma^2}}{N - 1}, \quad \tau^N \sim \frac{1}{N}. \quad C^N \sim N \tau^N. \]  

(15)

This model can be derived from the Dicke model [58–60]. In [27, 28], Wang et. al. have found the same scaling rela-
FIG. 3. Universal scaling of TPE induced by the long-range interaction $\alpha$. (a) The first derivation of $C_d$ as a function of $\alpha$, with critical point at $\alpha^* = 0.09$ (using Eq. 8). Here not only the nearest-neighbor TPE, but also the long-distance TPE, diverge at the same $\alpha^*$. (b) $\partial C_d/\partial \alpha$ as a function of $\ln|\alpha - \alpha^*|$ for various separation $d$. Inset shows $k_d$ as a function of $d$, with $q = 0.0012$ and $q' = -0.022$. In all figures $\mu = -23.23722$, $\Delta = 1.3$ and $\beta = 0.027$.

FIG. 4. (a) - (b) The total entanglement $C_N$ as a function of $N$, with fixed $\alpha = \beta = 0.015$. (c) - (d) The corresponding total tangle $\tau_N$ for the above two cases. (e) - (f) Relation between the total entanglement $C_\infty$, total tangle $\tau_\infty$ and the truncation length of entanglement $\xi$, showing of Eq. 16. In (e) $\alpha = \beta = 0.015$; and (f) $\mu = 100$. In all figures $\Delta = 1.3$.

This relation can be regarded as a generalized KBI bound [34], replacing $N$ by $\xi$. It is a direct consequence of exponential decaying of TPE with respect to $d$, assuming $C_d = C_1 e^{-|d|/\xi}$ (see Fig. 2 (a) - (b)). Using the upper bound of $\tau_\infty$ from the CKW inequality, we have $C_1 \leq 1/\sqrt{\xi}$. The actual condition is, we find $C_\infty \sim O(1)$, thus $C_1 < 1/\xi$, and $\tau_\infty \sim 1/\xi$. Hence the bigger $\xi$ is, the smaller the tangle $\tau_\infty$ to be. This relation unifies Eq. 15 in the LMG model and Dicke model by replacing $N$ with $\xi$ and generalizes its validity to much more quantum many-body models.

To conclude, we examine the distribution of TPE in a fermion model with long-range interaction, which is upper bounded by the CKW monogamy inequality. We show that the long-range interaction induces long distant TPE, all of which can exhibit universal scaling laws at the critical boundaries. A finite truncation length $\xi$ is required to describe the TPE. Based on this quantity, we find a general relation $C_\infty \sim 2\xi \tau_\infty \sim O(1)$, which holds in both the gapped and gapless phases. Our new relation is a generalized KBI bound, which has much broader applicability in the quantum many-body models.

Acknowledgments: This work is supported by the National Key Research and Development Program in China (Grants No. 2017YFA0304504 and No. 2017YFA0304103), the strategic Priority Research Program(B) of the Chinese Academy of Sciences (Grant No. XDB01030300 and No. XDB01030100) and the National Natural Science Foundation of China (NSFC) with No. 11774328, No. 11674306 and No. 92065113.

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[61] We summarize the major pairwise entanglement in several many-body states. In [27, 28], Wang et. al. find that: (i) For the Dicke state $|\psi\rangle = |N/2, M\rangle$ in the Dicke model, the entanglement $C = A(N,M)/N$, where $1 \leq A(N,M) \leq 2$ for all $|M| \leq N/2$. (ii) For the spin squeezed state in the Kitagawa-Ueda Hamiltonian $H = \chi \mathbf{S}_z^2$, with $|\psi\rangle = e^{-i\chi \mathbf{S}_z^2} |11 \cdots 11\rangle$, $C \leq 1/N$ for all $\chi t$. For the GHZ state without entanglement in its subset, $C = 0$. (iii) For the Einstein-Podolsky-Rosen state with $|\psi\rangle \propto \sum_{l=0}^{N} |l\rangle \otimes |l\rangle$, $C = 1/N$. (iv) For the isotropic and anisotropic Heisenberg models, $C = 0$ in their thermal states. Thus in all these states, we have, at most, $C \sim 1/N$. The saturation of the total entanglement yields $C^N \sim O(1)$ and $\tau^N \sim 1/N$. In [24], Geetha et. al. find that for the W state $|W⟩ = \cos(\frac{\theta}{N}) |000 \cdots 0\rangle + \sin(\frac{\theta}{N}) |010 \cdots 0\rangle + \cdots + |N00 \cdots 0\rangle) / \sqrt{N}$, $C_d = (1 - \cos(\theta))/N \leq 2/N$, hence $C^N = (1 - \cos(\theta)) \leq 2$ and $\tau^N \sim 1/N$. This upper bound is tight and can be reached by the W state. These results are comparable with the KBI upper bound [34]. In the Greenberger-Horne-Zeilinger (GHZ) state $|\psi\rangle = (|000 \cdots 0\rangle + |111 \cdots 1\rangle) / \sqrt{2}$, where the two-party density matrix is independent of $N$, we expect $C_d \leq 2/N$ for all $N$, implying of $C_d = 0$. This property is well-known that tracing out one party of the GHZ state will completely destroys the entanglement, yielding a fully mixed separable state.