An elementary optical gate for expanding entanglement web

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(Dated: March 13, 2008)

We introduce an elementary optical gate for expanding polarization-entangled W states, in which every pair of photons is entangled alike. The gate is composed of a pair of 50:50 beamsplitters and ancillary photons in the two-photon Fock state. By seeding one of the photons in an n-photon W state into this gate, we obtain an (n + 2)-photon W state after post-selection. This gate gives a better efficiency and a simpler implementation than previous proposals for W-state preparation.

PACS numbers: 03.67.Mn, 03.67.-a, 42.50.-p

I. INTRODUCTION

Quantum entanglement lies at the heart of most of the quantum information processing tasks, e.g., teleportation [1], key distribution (QKD) [2], and computation [3]. Entanglement between a pair of systems is fairly simple, in the sense that there is a maximally entangled state from which any entangled state can be generated under local operations and classical communication (LOCC). In contrast, multipartite entanglement among three or more systems exhibits more variety. For example, three qubits can be entangled in two inequivalent ways, namely, the GHZ state |GHZ⟩ = (|000⟩ + |111⟩)/√2 and the W state |W⟩ = (|001⟩ + |010⟩ + |100⟩)/√3 can never be converted to each other under LOCC, even probabilistically [4].

The distinction between these two types of entanglement becomes clearer if we consider their generalizations to the N-qubit case: |WN⟩ = |N − 1, 1⟩/√N and |GHZN⟩ = (|N, 0⟩ + |0, N⟩)/√2 where |N − k, k⟩ is the sum over all the terms with N − k modes in |0⟩ and k modes in |1⟩. In |WN⟩, every pair of qubits are entangled with each other directly, namely, the pairwise entanglement survives even after the rest of the qubits are discarded [5,6,7]. In fact, it was shown that the state |WN⟩ has the maximum amount of such pairwise entanglement shared by every pair [5]. It looks as if forming a web-like structure in which every qubit has a bond with every other qubit [see Fig. 1(a)]. On the other hand, the entanglement in |GHZN⟩ is sustained by all of the N qubits, and loss of only one particle destroys the entanglement completely. But if access to every qubit is allowed, it shows a maximal violation of local realism [8]. These distinct properties make the W and GHZ states interesting resources for multiparty tasks and fundamental studies of quantum mechanics. Thus, there have been many proposals and experimental implementations in photons [8,10,11,12,13,14,15,16,17,18,19,20,21,22], trapped ions [23,24], and NMR systems [25].

The distinction also shows up when we consider how one can increase the number of qubits forming W or GHZ states. In the case of GHZ states, there is a systematic way to extend its size without accessing all of the qubits: One can pick the N-th qubit of |GHZN⟩ and let it interact with a new qubit to produce |GHZN+1⟩. This is not surprising since (i) the marginal state of the remaining untouched N − 1 qubits is the same for |GHZN⟩ and |GHZN+1⟩, and (ii) the N-th qubit is pivotal such that if we remove and discard it, the rest of the qubits will be disentangled. On the other hand, it is not so trivial whether such a local extension of W states is possible or not. For one thing, the marginal states of N − 1 qubits are different for |WN⟩ and |WN+1⟩. Hence no unitary operation on the N-th qubit and a new qubit makes |WN+1⟩. In addition, newly added qubits must form the pairwise entanglement with each of the uninteracted N − 1 qubits [see Fig. 1(a)].

In this paper, we show that such a local extension of polarization-entangled photonic W states is possible using a surprisingly simple probabilistic gate composed of a two-photon Fock state, two 50:50 beam splitters (BS),
and a phase shifter (PS), based on post-selection. Interestingly, the same gate can be used for the expansion $|W_N\rangle \rightarrow |W_{N+2}\rangle$ of any size $N$ [see Fig. II(b)]. The gate also works for $N = 1, 2$ if we extrapolate the definition of W states naturally: seeding $|W_1\rangle = |1\rangle$ results in $|W_3\rangle$ [see Fig. II(c)], and seeding an EPR pair $|W_2\rangle = (|01\rangle + |10\rangle)/\sqrt{2}$ results in $|W_4\rangle$ [see Fig. II(d)]. In particular, the latter case is experimentally easier and more efficient than any other linear optical scheme of generating $|W_4\rangle$ proposed so far. Starting with (c) and applying (b) successively $N - 1$ times, we can prepare W-states with odd number of photons, $|W_{2N+1}\rangle$. In the same way, states with even number of photons $|W_{2N}\rangle$ can be prepared starting with (d). Thus, in principle it is possible to prepare any $|W_N\rangle$ using this gate.

II. WORKING PRINCIPLE OF THE ELEMENTARY GATE $T_{+2}^W$

In Fig. II we show the schematic of the proposed gate. The gate receives one photon from mode 1 as the input, and mixes it by a 50:50 beamsplitter (BS1) with two ancilla photons in horizontal (H) polarization (we denote it by $|2H\rangle_2$, where the subscript number signifies the spatial mode). One of the output modes of BS1 is further divided into two modes by another 50:50 beamsplitter (BS2). The gate operation is successful when each of the output modes 4, 5, and 6 has a photon. The phase shifter (PS), which is a half-wave plate introducing a $\pi$-phase shift between H and V (vertical) polarizations, is in place just in order to keep the final W state in the standard symmetric form.

First we analyze how the gate works when the input photon in mode 1 is H polarized ($|1H\rangle_1$) or V polarized ($|1V\rangle_1$). The action of BS1 on H polarization is represented by the transformation $\hat{a}_{1H}^\dagger = (\hat{a}_{1H}^\dagger - \hat{a}_{2H}^\dagger)/\sqrt{2}$ and $\hat{a}_{2H}^\dagger = (\hat{a}_{1H}^\dagger + \hat{a}_{2H}^\dagger)/\sqrt{2}$, where $\hat{a}_{1H}^\dagger$ is the photon creation operator for mode $j$ in H polarization. We assume that the BS1 is polarization-independent, namely, the transformation for V polarization has the same form. Using these relations, we see that the initial states $|1H(V)\rangle_1 \otimes |2H\rangle_2 = 2^{-1/2}\hat{a}_{1H(V)}^\dagger(\hat{a}_{2H}^\dagger)^2|0\rangle$ evolve as

\[
|1H\rangle_1 |2H\rangle_2 \rightarrow \frac{\sqrt{3}}{2\sqrt{2}} |3H\rangle_3 |0\rangle_4 + \frac{1}{2\sqrt{2}} |2H\rangle_3 |1H\rangle_4 - \frac{1}{2\sqrt{2}} |1H\rangle_3 |2H\rangle_4 - \frac{\sqrt{3}}{2\sqrt{2}} |0\rangle_3 |3H\rangle_4, \\
|1V\rangle_1 |2H\rangle_2 \rightarrow \frac{1}{2\sqrt{2}} |1V2H\rangle_3 |0\rangle_4 + \frac{1}{2\sqrt{2}} |1H1V\rangle_3 |1H\rangle_4 + \frac{1}{2\sqrt{2}} |1V\rangle_3 |2H\rangle_4 - \frac{1}{2\sqrt{2}} |2H\rangle_3 |1V\rangle_4 - \frac{1}{2\sqrt{2}} |1H\rangle_3 |1H1V\rangle_4 - \frac{1}{2\sqrt{2}} |0\rangle_3 |1V2H\rangle_4. 
\]

For the gate operation to be successful, there must be two photons in mode 3 and one photon in mode 4. Hence we are interested only in the underlined terms. The states $|2H\rangle_3$ and $|1H1V\rangle_3$ appearing in the underlined terms are transformed at BS2 as

\[
|2H\rangle_3 \rightarrow \frac{1}{2} |2H\rangle_5 |0\rangle_6 + \frac{1}{\sqrt{2}} |1H\rangle_5 |1H\rangle_6 + \frac{1}{2} |0\rangle_5 |2H\rangle_6, \\
|1H1V\rangle_3 \rightarrow \frac{1}{2} |1H1V\rangle_5 |0\rangle_6 + \frac{1}{\sqrt{2}} |1H\rangle_5 |1V\rangle_6 + \frac{1}{2} |1V\rangle_5 |1H\rangle_6 + \frac{1}{\sqrt{2}} |0\rangle_5 |1H1V\rangle_6. 
\]

Clearly, only the underlined terms in Eq. (2) contributes to the successful operation. Therefore, if we postselect

FIG. 2: The schematic diagram of the setup for $T_{+2}^W$ gate.
the successful events, the action of the gate is given by

\[
\begin{align*}
|1_H⟩_1|2_H⟩_2 & \rightarrow \frac{1}{4} |1_H⟩_4|1_H⟩_5|1_H⟩_6, \\
|1_V⟩_1|2_H⟩_2 & \rightarrow \frac{1}{4} |1_H⟩_4|1_H⟩_5|1_V⟩_6 + \frac{1}{4} |1_H⟩_4|1_V⟩_5|1_H⟩_6 + \frac{1}{4} |1_V⟩_4|1_H⟩_5|1_H⟩_6, \\
\end{align*}
\]

where we have included the effect of PS. There are two essential features in this gate operation: One is the symmetrization among the input photon and the ancilla photons, and the other is that the success probability (1/16) for the \(|1_H⟩_1\) input is one third of the probability (3/16) for the \(|1_V⟩_1\) input. In other words, all the four terms appearing in Eq. (3) and Eq. (4) have the same amplitude.

The above calculation may be physically understood as follows. When the input is a V-polarized photon, we can always determine the origin of an output photon, namely, distinguish whether it has come from the mode 1 or the mode 2 by looking at its polarization. Hence the result is the same as in the case of classical distinguishable particles, and the symmetrization in Eq. (4) can be understood classically. On the other hand, when the input is an H-polarized photon, quantum interference comes into play. In this case, there are three indistinguishable paths leading to the final state \(|1_H⟩_4|1_H⟩_5|1_H⟩_6\), depending on which one of the three photons originates from the input mode 1. One of the path (the input photon going to mode 4) bears the opposite sign from the other two, and hence the probability becomes one third due to destructive interference.

III. SEEDING AND EXPANDING POLARIZATION ENTANGL ED W-STATES

Since the right hand side of Eq. (4) is \(|W_3⟩\), we can prepare state \(|W_3⟩\) with a success probability of 3/16 by applying the gate \(T_{W_2}^V\) on the input photon in state \(|1_V⟩_1\). If we prepare an EPR pair \(|1_H⟩_0|1_V⟩_1 + |1_V⟩_0|1_H⟩_1\)/\(\sqrt{2}\) and feed the spatial mode 1 into the \(T_{W_2}^V\) gate, the two terms \(|1_V⟩_1\) and \(|1_H⟩_1\) should evolve coherently as in Eq. (3) and Eq. (4), leading to the \(|W_4⟩\) state:

\[
|W_4⟩ = \frac{1}{2}[|1_H⟩_0|1_H⟩_4|1_H⟩_5|1_V⟩_6 \\
+ |1_H⟩_0|1_V⟩_4|1_V⟩_5|1_H⟩_6 \\
+ |1_V⟩_0|1_H⟩_4|1_H⟩_5|1_H⟩_6 \\
+ |1_V⟩_0|1_V⟩_4|1_V⟩_5|1_H⟩_6]
\]

with success probability 1/8. These values of success probability are significant improvements over other linear optics-based schemes. For instance, the most efficient schemes so far are those in [12] and [13], respectively for \(|W_4⟩\) and \(|W_3⟩\) with the corresponding success probabilities of 2/27 and 1/9, which are lower than those of our proposal.

Next we discuss how this gate can be used to expand a general W-state \(|W_N⟩ = |N−1,1⟩/\sqrt{N}\), where \(|N−1,1⟩\) is the sum over all the terms with \(N−1\) modes in \(|1_H⟩\) and one mode in \(|1_V⟩\). This state may be rewritten as \(|N−1,1⟩ ⊗ |1_H⟩_1 + |N−1,0⟩ ⊗ |1_V⟩_1⟩/\sqrt{N}. If we apply the gate \(T_{W_3}^V\) on mode 1, we obtain \(|N−2,1⟩ ⊗ |3,0⟩ + |N−1,0⟩ ⊗ |2,1⟩⟩/\sqrt{N} = |N + 1,1⟩/\sqrt{N}\), implying that the gate produces \(|W_{N+2}⟩\) with success probability \((N+2)/(16N)\), which approaches a constant 1/16 when \(N\) becomes large. Note that while this probability partly comes from the inefficiency associated with linear optics schemes, the probabilistic nature itself plays an essential role of updating the marginal state of each of the untouched photons from \(ρ_N = (N−1)|1_H⟩⟨1_H| + |1_V⟩⟨1_V|\) to that of \(ρ_{N+2}\). By cascading \(T_{W_2}^V\) gates, we can prepare W states over 5 or more photons. Starting with \(|1_V⟩\) as an input and cascading the gate \(k\) times, one can prepare \(2(k+1)\)-photon W state, \(|W_{2k+1}\rangle\), provided that coincidence detection is observed at \(2k+1\) output spatial modes. The success probability of such an event is given by \(p_{\text{success}} = (2k + 1)2^{−4k}\). Similarly, starting with an EPR pair and cascading \(k\) gates, one can prepare \(2(k+1)\)-photon W state, \(|W_{2(k+1)}⟩\), with a success probability of \(p_{\text{success}} = (k+1)2^{−4k}\).

IV. FEASIBILITY ANALYSIS FOR \(|W_4⟩\)

So far, several linear optical schemes for preparing \(|W_4⟩\) have been proposed [12, 13, 14], but no experiments have been done yet. It is thus interesting to consider the feasibility of our scheme with practical pho-
ton sources, namely, parametric down-conversion (PDC) and/or weak coherent pulses (WCP) obtained by attenuating laser pulses. First, suppose that an EPR pair and ancillary two photons are both generated from PDC with rates $\gamma$ and $g$, respectively, which are $\sim 10^{-4}$ in typical experiments. In this case, the errors are mainly caused by generation of three pairs of photons in total. Such events occur with rate $O(\gamma^2 g)$ or $O(\gamma g^2)$, which is small compared with the rate $O(\gamma g)$ of the desired events. Alternatively, we may also use WCP instead of PDC for the ancillary photons in mode 2. If the mean photon number of WCP is $\nu$, the desired events occur with rate $O(\gamma \nu^2)$. With the requirement $\nu \ll 1$ as usual, the main source of errors in this case is two-pair production at PDC, indeed, the same set of the two BSs can also be used to satisfy the output modes 4, 5, and 6 has a photon, the transformation is given by $|1\psi_1\rangle \rightarrow 2^{-3/2} |1\psi_1\rangle |1\psi_1\rangle |1\psi_1\rangle$. Similarly, an $H$-polarized input will be transformed as $|1H_1\rangle \rightarrow 2^{-3/2} |1H_1\rangle |1H_1\rangle |1H_1\rangle$. It is then obvious that this gate achieves $|\text{GHZ}_N\rangle \rightarrow |\text{GHZ}_{N+2}\rangle$ up to a local unitary, with a success probability of $1/8$.

VI. CONCLUSION

In summary, we have proposed a simple elementary optical gate based on post-selection for expanding the symmetrically shared entanglement in polarization entangled W states. With a proper seeding, the gate can also be used for preparation of W states, and it has a larger success probability than other preparation methods. We believe that the proposed gate is easy to implement and feasible with the current experimental technologies. In our gate, polarization-dependent components play no essential role, and the desired transformation is achieved by multi-photon interference between the input photon and the ancilla photons. Note that this does not require sub-wavelength adjustments. We have also shown that just by changing the state of the ancilla photons, we obtain a gate for extending GHZ states.

Acknowledgments

This work was supported by 21st Century COE Program by the JSPS and by a MEXT Grant-in-Aid for Young Scientists (B) 17740265.
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