Parity Considerations in Rogers–Ramanujan–Gordon Type Overpartitions

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Abstract. In 2010, Andrews considers a variety of parity questions connected to classical partition identities of Euler, Rogers, Ramanujan and Gordon. As a large part in his paper, Andrews considered the partitions by restricting the parity of occurrences of even numbers or odd numbers in the Rogers-Ramanujan-Gordon type. The Rogers–Ramanujan–Gordon type partition was defined by Gordon in 1961 as a combinatorial generalization of the Rogers–Ramanujan identities with odd moduli. In 1974, Andrews derived an identity which can be considered as the generating function counterpart of the Rogers–Ramanujan–Gordon theorem, and since then it has been called the Andrews–Gordon identity. By revisiting the Andrews–Gordon identity Andrews extended his results by considering some additional restrictions involving parities to obtain some Rogers–Ramanujan–Gordon type theorems and Andrews–Gordon type identities. In the end of Andrews’ paper, he posed 15 open problems. Most of Andrews’ 15 open problems have been settled, but the 11th that “extend the parity indices to overpartitions in a manner” has not. In 2013, Chen, Sang and Shi, derived the overpartition analogues of the Rogers–Ramanujan–Gordon theorem and the Andrews–Gordon identity. In this paper, we post some parity restrictions on these overpartitions analogues to get some Rogers–Ramanujan–Gordon type overpartition theorems.

Keywords: Rogers-Ramanujan-Gordon theorem, Andrews-Gordon identity, parity, overpartitions

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1 Introduction

The celebrated combinatorial generalization of the Rogers–Ramanujan identities was given by Gordon [8] in 1961, which can be stated as follows:
Theorem 1.1 (Rogers-Ramanujan-Gordon) For \( k \geq a \geq 1 \), let \( B_{k,a}(n) \) denote the number of partitions of \( n \) of the form \( \lambda_1 + \lambda_2 + \cdots + \lambda_s \), where \( \lambda_j \geq \lambda_{j+1} \), \( \lambda_j - \lambda_{j+k-1} \geq 2 \) and part 1 appears at most \( a-1 \) times. Let \( A_{k,a}(n) \) denote the number of partitions of \( n \) into parts \( \neq 0, \pm a \pmod{2k+1} \). Then for any \( n \geq 0 \), we have

\[
A_{k,a}(n) = B_{k,a}(n).
\]

In 1967, Andrews [2] established his analytic generalization of the Rogers–Ramanujan identities with the moduli from 5 to all odd positive integers:

Theorem 1.2 For \( k \geq a \geq 1 \), we have

\[
\sum_{N_1 \geq N_2 \geq \cdots \geq N_{k-1} \geq 0} \frac{q^{N_1^2 + N_2^2 + \cdots + N_{k-1}^2 + N_{a+1} + \cdots + N_{k-1}}}{(q)_{N_1-N_2} \cdots (q)_{N_k-2-N_{k-1}} (q)_{N_{k-1}}} = (q^a, q^{2k+1-a}, q^{2k+1}; q^{2k+1})_\infty. \tag{1.1}
\]

Here and in the rest of this paper, we adopt the common notation as used in Andrews [3]. Let

\[
(a)_\infty = (a; q)_\infty = \prod_{i=0}^{\infty} (1 - aq^i),
\]

and

\[
(a)_n = (a; q)_n = \frac{(a)_\infty}{(aq^n)_\infty}.
\]

We also write

\[
(a_1, \ldots, a_k; q)_\infty = (a_1; q)_\infty \cdots (a_k; q)_\infty.
\]

Let us give an overview of some definitions. A partition \( \lambda \) of a positive integer \( n \) is a non-increasing sequence of positive integers \( \lambda_1 \geq \cdots \geq \lambda_s > 0 \) such that \( n = \lambda_1 + \cdots + \lambda_s \). The partition of zero is the partition with no parts. An overpartition \( \lambda \) of a positive integer \( n \) is also a non-increasing sequence of positive integers \( \lambda_1 \geq \cdots \geq \lambda_s > 0 \) such that \( n = \lambda_1 + \cdots + \lambda_s \) and the first occurrence of each integer may be overlined. For example, \((7, 7, 6, 5, 2, 1)\) is an overpartition of 28. For a partition or an overpartition \( \lambda \) and for any integer \( l \), let \( f_l(\lambda)(f_l(\lambda)) \) denote the number of occurrences of \( l \) non-overlined (overlined) in \( \lambda \).

In 2010, Andrews [4] investigated a variety of parity questions in classical partition identities. In particular, he revisited his generating function (1.1) and extended his results by considering some additional restrictions involving parities. The first theorem considered that in the case \( k \equiv a \pmod{2} \), the conditions are subject to that the occurrences of even numbers must be even.

Theorem 1.3 Suppose \( k \geq a \geq 1 \) are integers with \( k \equiv a \pmod{2} \). Let \( W_{k,a}(n) \) denote the number of those partitions enumerated by \( B_{k,a}(n) \) with the added restriction that even parts appear an even number of times. If \( k \) and \( a \) are both even, let \( G_{k,a}(n) \) denote the number of partitions of \( n \) in which no odd part is repeated and no even part \( \equiv 0 \pm a \pmod{2k+2} \). If \( k \) and \( a \) are both odd, let \( G_{k,a}(n) \) denote the number of partitions of \( n \) into parts that are neither \( \equiv 2 \pmod{4} \) nor \( \equiv 0 \pm a \pmod{2k+2} \). Then for all \( n \geq 0 \)

\[
W_{k,a}(n) = G_{k,a}(n).
\]
The generating function form of this theorem can be stated as follows:

**Theorem 1.4** For $k \geq a \geq 1$ and $k \equiv a \pmod{2}$, we have

$$
\sum_{n \geq 0} W_{k,a}(n) q^n = \frac{\sum_{N_1 \geq N_2 \geq \cdots \geq N_{k-1} \geq 0} q^{N_1^2+N_2^2+\cdots+N_{k-1}^2+2N_0+2N_{a+2}+\cdots+2N_{k-2}}}{(q^2; q^2)_{N_1-N_2-\cdots-N_{k-1}-N_{k-1}}(q^2; q^2)_{N_{k-1}}} = \frac{(-q; q^2)_\infty (q^a, q^{2k+2-a}, q^{2k+2}, q^{2k+2})_\infty}{(q^2; q^2)_\infty},
$$

(1.2)

When $k \not\equiv a \pmod{2}$, Andrews considered the partitions provided that the occurrences of odd parts must even.

**Theorem 1.5** Suppose $k \geq a \geq 1$ with $k$ odd and $a$ even. Let $\overline{W}_{k,a}(n)$ denote the number of those partitions enumerated by $B_{k,a}(n)$ with added restriction that odd parts appear an even number of times. Then

$$
\sum_{n \geq 0} \overline{W}_{k,a}(n) q^n = \frac{(q^a, q^{2k+2-a}, q^{2k+2}, q^{2k+2})_\infty}{(-q; q^2)_\infty (q; q)_\infty}.
$$

The generating function form of this theorem is also given by Andrews [3] as follows:

**Theorem 1.6** For $k \geq a \geq 1$ with $k$ odd and $a$ even, we have

$$
\sum_{N_1 \geq N_2 \geq \cdots \geq N_{k-1} \geq 0} \frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2+2N_0+2N_{a+2}+\cdots+2N_{k-2}}}{(q^2; q^2)_{N_1-N_2-\cdots-N_{k-1}-N_{k-1}}(q^2; q^2)_{N_{k-1}}} = \frac{(-q^2; q^2)_\infty (q^a, q^{2k+2-a}, q^{2k+2}, q^{2k+2})_\infty}{(q^2; q^2)_\infty} = \frac{(q^a, q^{2k+2-a}, q^{2k+2}, q^{2k+2})_\infty}{(-q; q^2)_\infty (q; q)_\infty}.
$$

(1.3)

Those generalizations are obtained by using double recurrences and the Defining $q$-Difference Equations Principle.

Andrews just considered $W_{k,a}(n)$ in the case that $k \equiv a \pmod{2}$ and $\overline{W}_{k,a}(n)$ in the case that $k$ is odd and $a$ is even. In 2013, Kim and Yee [?] derived the generating function of $W_{k,a}(n)$ and $\overline{W}_{k,a}(n)$ with $k$ and $a$ in other parities. For $k$ and $a$ have different parities, Kim and Yee derived the following result about $W_{k,a}(n)$.

**Theorem 1.7** For $k \geq a \geq 1$, $k \not\equiv a \pmod{2}$,

$$
\sum_{n \geq 0} W_{k,a}(n) q^n = \frac{(-q^3; q^2)_\infty (q^{a+1}, q^{2k+1-a}, q^{2k+2}, q^{2k+2})_\infty}{(q^2; q^2)_\infty} + \frac{q(-q^3; q^2)_\infty (q^{a-1}, q^{2k+3-a}, q^{2k+2}, q^{2k+2})_\infty}{(q^2; q^2)_\infty}.
$$

(1.4)
For $\overline{W}_{k,a}(n)$, Kim and Yee have the following result which is a “missing” case of Andrews.

**Theorem 1.8** Suppose $k \geq a \geq 1$ with $k$ even and $a$ odd. Then

$$\sum_{n \geq 0} \overline{W}_{k,a}(n)q^n = \frac{(-q^2;q^2)_{\infty}(q^{a+1}, q^{2k+1-a}, q^{2k+1};q^{2k+1})_{\infty}}{(q^2;q^2)_{\infty}}. \tag{1.5}$$

They also note the following relation for $\overline{W}_{k,a}(n)$.

**Theorem 1.9** For $k \geq a \geq 1$ with $a$ even and $n \geq 1$.

$$\overline{W}_{k,a}(n) = \overline{W}_{k,a-1}(n). \tag{1.6}$$

Then the generating function of $W_{k,a}(n)$ and $\overline{W}_{k,a}(n)$ in all parities have been derived. In this paper, we will give parity restrictions on the Rogers–Ramanujan–Gordon type overpartitions. We first revisit the Rogers–Ramanujan–Gordon type overpartition theorems.

In 2012, Chen, Sang and Shi [5] established the overpartition analogue of the Rogers–Ramanujan–Gordon theorem:

**Theorem 1.10 (The Rogers–Ramanujan–Gordon type overpartition theorem)** For $k \geq 2$ and $k \geq i \geq 1$, let $B_{k,i}(n)$ denote the number of overpartitions of $n$ of the form $\lambda_1 + \lambda_2 + \cdots + \lambda_s$ such that part 1 occurs as a non-overlined part at most $i - 1$ times, and $\lambda_j - \lambda_{j+k-1} \geq 1$ if $\lambda_j$ is overlined and $\lambda_j - \lambda_{j+k-1} \geq 2$ otherwise. For $k > i \geq 1$, let $A_{k,i}(n)$ denote the number of overpartitions of $n$ whose non-overlined parts are not congruent to 0, ±i modulo 2k and let $A_{k,k}(n)$ denote the number of overpartitions of $n$ with parts not divisible by $k$. Then we have

$$A_{k,i}(n) = B_{k,i}(n). \tag{1.7}$$

We shall focus on the parity of the frequencies of a number $l$ by consider $f_l + f_{\overline{l}}$. We give the following two definitions.

**Definition 1.11** For $k \geq a \geq 1$, let $U_{k,a}(n)$ denote the number of overpartitions of $n$ of the form $(f_1, f_2, f_3, \ldots)$ such that

(i) $f_1(\lambda) \leq a - 1 + f_1(\lambda)$;

(ii) $f_2l-1(\lambda) \geq f_2l-1(\lambda)$;

(iii) $f_2l(\lambda) + f_2l(\lambda) \equiv 0 \pmod{2}$;

(iv) $f_1(\lambda) + f_1(\lambda) + f_{l+1}(\lambda) \leq k - 1 + f_{l+1}(\lambda)$.

**Definition 1.12** For $k \geq a \geq 1$, let $\overline{U}_{k,a}(n)$ denote the number of overpartitions of $n$ of the form $(f_1, f_2, f_3, \ldots)$ such that

1. $f_1(\lambda) \leq a - 1 + f_1(\lambda)$;
2. $f_{2l}(\lambda) \geq f_{2l}(\lambda)$;

3. $f_{2l-1}(\lambda) + f_{2l+1}(\lambda) \equiv 0 \pmod{2}$;

4. $f_{i}(\lambda) + f_{i+1}(\lambda) \leq k - 1 + f_{i+1}(\lambda)$.

Here are our main results.

**Theorem 1.13** For $k \geq a \geq 1$ and $k \equiv a \pmod{2}$, we have

$$\sum_{n \geq 0} U_{k,a}(n)q^n = \frac{(-q;q)_\infty(q^a, q^{2k-a}, q^{2k}; q^2)_\infty}{(q^2; q^2)_\infty}. \quad (1.8)$$

**Theorem 1.14** For $k \geq a \geq 1$ and $k \not\equiv a \pmod{2}$, we have

$$\sum_{n \geq 0} U_{k,a}(n)q^n = \frac{(-q^2;q)_\infty(q^{a+1}, q^{2k-a-1}, q^{2k}; q^2)_\infty}{(q^2; q^2)_\infty} + xq(-q^2;q)_\infty(q^{a-1}, q^{2k-a+1}, q^{2k}; q^2)_\infty. \quad (1.9)$$

**Theorem 1.15** For $k \geq a \geq 2$ with $a$ even, we have

$$\sum_{n \geq 0} \overline{U}_{k,a}(n)q^n = \sum_{n \geq 0} \overline{U}_{k,a-1}(n)q^n = \frac{(-q^2;q)_\infty^2(q^a, q^{2k-a}, q^{2k}; q^2)_\infty}{(q^2; q^2)_\infty}. \quad (1.10)$$

By Theorem 1.13—Theorem 1.15 we get the generating functions of $U_{k,a}(n)$ and $\overline{U}_{k,a}(n)$ in all parities of $k$ and $a$.

This paper is organized as follows. In Section 2, we shall recall $\overline{Q}_{k,a}(x; q)$ which has been used in the proof of Theorem 1.10. And recall the defining $q$-difference equations principle. In Section 3, we prove Theorem 1.13 in the case that $k \equiv a \equiv 0 \pmod{2}$. In Section 4, we derive the generating function of $U_{k,a}$ in the case that $k \not\equiv a \pmod{2}$. In Section 5, we give the generating function of $\overline{U}_{2k,2a}(n)$. In Section 6, we derive the generating function of $U_{2k+1,2a+1}(n)$ and $\overline{U}_{2k+1,2a+1}(n)$ by two $q$-differential relations. In Section 7, we give the combinatorial interpretations of two identities, which reveal the relations among $U_{k,a}(n)$, $\overline{U}_{k,a}(n)$ and $\overline{B}_{k,a}(n)$.

## 2 Background

We shall use a series $H_{k,i}(a; x; q)$ introduced by Andrews [11 2], which is defined by

$$H_{k,i}(a; x; q) = \sum_{n=0}^{\infty} x^{knq^{n^2+n-i}+n-a}a^n(1-x^iq^{2ni})(axq^{n+1})\infty(1/a)_n. \quad (2.11)$$

In his algebraic proof of the Rogers–Ramanujan–Gordon theorem, Andrews used the function $J_{k,i}(a; x; q)$ constructed based on $H_{k,i}(a; x; q)$,

$$J_{k,i}(a; x; q) = H_{k,i}(a; xq; q) - axqH_{k,i-1}(a; xq; q).$$
Chen, Sang and Shi [5] proved Theorem 1.10 by using a specialization of $H_{k,i}(a;x;q)$ which we denote it here by $Q_{k,i}(x;q)$, that

$$Q_{k,i}(x;q) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{kn} q^{kn^2+kn-m}(1-x^i q^{(2n+1)i})(-xq^{n+1})}{(q)_n (xq^{n+1})} (-q)_n,$$

(2.12)

to get the following result

$$Q_{k,i}(x;q) = \sum_{m,n \geq 0} B_{k,i}(n) x^m q^n.$$

In this paper, we also employ $Q_{k,i}(x;q)$ to prove Theorem 1.13–Theorem 1.15. We stated some results on $Q_{k,i}(x;q)$ which is derived from Andrews’ relations about $H_{k,i}(a;x;q)$ and have been stated by Chen, Sang and Shi in [5].

**Lemma 2.1** Let $Q_{k,i}(x;q) = \sum_{m,n \geq 0} Q_{k,i}(x;q) x^m q^n$, then we have the following initial values

$$Q_{k,i}(0,0) = 1, \quad \text{for } k \geq i \geq 1,$$

(2.13)

$$Q_{k,0}(m,n) = 0, \quad \text{for } k \geq 1, \quad m,n \geq 0,$$

(2.14)

$$Q_{k,i}(m,n) = 0, \quad \text{if } m \text{ or } n \text{ is zero but not both},$$

(2.15)

$$Q_{k,i}(m,n) = 0, \quad \text{if } m \text{ or } n < 0.$$

(2.16)

By (2.12), one can derive the following relation.

**Lemma 2.2** For $k \geq 1$, we have

$$Q_{k,-\frac{1}{2}}(x;q) = -(xq)^{-\frac{1}{2}} Q_{k,\frac{1}{2}}(x;q)$$

(2.17)

**Lemma 2.3** [3, P106, Lemma 7.1] We have

$$Q_{k,i}(x;q) - Q_{k,i-1}(x;q) = (xq)^i Q_{k,k-i}(x;q) + (xq)^{i-1} Q_{k,k-i+1}(x;q).$$

(2.18)

By relation (2.18), we shall give the following relation satisfied by $Q_{k,i}(m,n)$.

**Theorem 2.4** For $k \geq a \geq 1$, we have

$$Q_{k,a}(x;q) = \sum_{i=1}^{a} (xq)^i \left[ \sum_{h=1}^{k-i} (xq^2)^h Q_{k,k-h}(x;q^2) + \sum_{h=0}^{k-i-1} (xq^2)^h Q_{k,k-h}(x;q^2) \right]$$

$$+ \sum_{i=0}^{a-1} (xq)^i \left[ \sum_{h=1}^{k-i} (xq^2)^h Q_{k,k-h}(x;q^2) + \sum_{h=0}^{k-i-1} (xq^2)^h Q_{k,k-h}(x;q^2) \right].$$

(2.19)

**Proof.** By relation (2.18), we have that

$$Q_{k,a}(x;q) = Q_{k,a-1}(x;q) + (xq)^a Q_{k,k-a}(x;q) + (xq)^{a-1} Q_{k,k-a+1}(x;q).$$

(2.19)
Then by successively using (2.4) to \( \overline{Q}_{k,a}(x; q) \) in above identity, we have that

\[
\overline{Q}_{k,a}(x; q) = \overline{Q}_{k,a-2}(x; q) + (xq)^{a-1}\overline{Q}_{k,k-a+1}(xq; q) + (xq)^a\overline{Q}_{k,k-a+2}(xq; q)
\]

\[+ (xq)^a\overline{Q}_{k,k-a}(xq; q) + (xq)^{a-1}\overline{Q}_{k,k-a+1}(xq; q)\]

\[= \overline{Q}_{k,0}(x; q) + \sum_{i=1}^{a}(xq)^i\overline{Q}_{k,k-i}(xq; q) + \sum_{i=0}^{a-1}(xq)^i\overline{Q}_{k,k-i}(xq; q).\]

By (2.14), we have

\[
\overline{Q}_{k,a}(x; q) = \sum_{i=1}^{a}(xq)^i\overline{Q}_{k,k-i}(xq; q) + \sum_{i=0}^{a-1}(xq)^i\overline{Q}_{k,k-i}(xq; q).
\]

Now we successively employ (2) to \( \overline{Q}_{k,k-i}(x; q) \), to get that

\[
\overline{Q}_{k,a}(x; q) = \sum_{i=1}^{a}(xq)^i\left[\sum_{j=1}^{k-i}(xq)^j\overline{Q}_{k,k-j}(xq^2; q) + \sum_{h=0}^{k-i-1}(xq)^h\overline{Q}_{k,k-h}(xq^2; q)\right]
\]

\[+ \sum_{i=0}^{a-1}(xq)^i\left[\sum_{j=1}^{k-i}(xq)^j\overline{Q}_{k,k-j}(xq^2; q) + \sum_{h=0}^{k-i-1}\overline{Q}_{k,k-h}(xq^2; q)\right].\]

Recall the Defining \( q \)-Difference Equations Principle as follows [4].

\[\text{Theorem 2.5} \quad \text{Suppose that we have two sets of functions, for } 1 \leq i \leq r, \ f_i(x, q) \text{ and } g_i(x, q), \]

which are analytic in \( x \) and \( q \) for \( |q| < 1 \) and \( |x| < |q|^{-1} \). Further, suppose that for each \( i, f_i(0, q) = g_i(0, q), \)

\[f_i(x, q) = \sum_{j=1}^{r}h_{i,j}(x, q)f_j(xq^{e(i,j)}, q), \quad (2.20)\]

and

\[g_i(x, q) = \sum_{j=1}^{r}h_{i,j}(x, q)g_j(xq^{e(i,j)}, q), \quad (2.21)\]

where the \( e(i, j) \) are all positive integers and the \( h_{i,j}(x, q) \) are polynomials in \( x \) and \( q \). Then it follows by a double induction on the double power series coefficients that

\[f_i(x, q) = g_i(x, q), \quad 1 \leq i \leq r.\]

We can see that by the defining \( q \)-difference equations principle, Lemma 2.1 and Theorem 2.3, the functions \( \overline{Q}_{k,a}(x; q) \) are uniquely determined.

3 \ The generating function of \( U_{2k,2a}(n) \)

In this section and in the sequel, we let \( U_{k,a}(m, n) \) (resp. \( \overline{U}_{k,a}(m, n) \)) denote the number of overpartitions enumerated by \( U_{k,a}(n) \) (resp. \( \overline{U}_{k,a}(n) \)) that have exactly \( m \) parts. The related
Proof. By the definition of the overpartitions enumerated by $U_{k,a}(n)$ and $U_{k,2a-2}(n)$ we know that $U_{k,2a}(n) - U_{k,2a-2}(n)$ is the number of overpartitions that enumerated by $U_{k,2a}(n)$ with the restrictions that
1. if there is an $\overline{T}$, then the number of 1 is $2a$ or $2a - 1$;

2. if there is no $\overline{T}$, then the number of 1 is $2a - 1$ or $2a - 2$.

Then we consider the set in the following four cases.

(i) The number of overpartitions enumerated by $U_{2k,2a}(n)$ with the restriction that is an $\overline{T}$ and the number of 1 is $2a$. If there is an $\overline{T}$, then the number of non-overlined 2 is at most $2k - 2a - 1$ with the restriction that $2m + 2f_{2l}$ is even. Suppose that the number of $2m + 2f_{2l}$ is $2h$, then the number of non-overlined 3 is at most $2k - 2h$ with one $\overline{T}$, and $2k - 2h - 1$ with no $\overline{T}$. We delete all the 1’s and 2’s, and then subtract 2 from each other parts. We get the overpartitions enumerated by $U_{2k,2k-2h}(m - 2a - 1 - 2h, n - 2m + a + 1)$. Then the generating function is

$$
(x q)^{2a + 1} \sum_{h=1}^{k-a} (x q^2)^{2h} U_{2k,2k-2h}(x q^2; q).
$$

(ii) Now, consider the overpartitions that enumerated by $U_{2k,2a}(n)$ with the restriction that there is an $\overline{T}$ and the number of 1 is $2a - 1$. By similar analysis, we can get the generating function as

$$
(x q)^{2a} \sum_{h=1}^{k-a} (x q^2)^{2h} U_{2k,2k-2h}(x q^2; q) + (x q)^{2a} \sum_{h=0}^{k-a-1} (x q^2)^{2h} U_{2k,2k-2h}(x q^2; q).
$$

(iii) The number of overpartitions that enumerated by $U_{2k,2a}(n)$ with the restriction that there is no $\overline{T}$ and the number of 1 is $2a - 1$. The generating function is

$$
(x q)^{2a - 1} \sum_{h=1}^{k-a+1} (x q^2)^{2h} U_{2k,2k-2h}(x q^2; q) + (x q)^{2a - 1} \sum_{h=0}^{k-a} (x q^2)^{2h} U_{2k,2k-2h}(x q^2; q).
$$

(iv) The number of overpartitions that enumerated by $U_{2k,2a}(n)$ with the restriction that there is an $\overline{T}$ and the number of 1 is $2a - 2$. The generating function is

$$
(x q)^{2a-2} \sum_{h=1}^{k-a+1} (x q^2)^{2h} U_{2k,2k-2h}(x q^2; q) + (x q)^{2a-2} \sum_{h=0}^{k-a} (x q^2)^{2h} U_{2k,2k-2h}(x q^2; q).
$$

Compute the summation of (3.25)–(3.29) we complete the proof.
By successively employing the above theorem we get the following result.

**Theorem 3.3** For \( k \geq a \geq 1 \), we have

\[
U_{2k,2a}(x; q) = (1 + xq) \left[ \sum_{i=1}^{a} (xq)^{2i} \left( \sum_{h=1}^{k-i} (xq^{2})^{2h} U_{2k,2k-2h}(xq^{2}; q) + \sum_{h=0}^{k-i-1} (xq^{2})^{2h} U_{2k,2k-2h}(xq^{2}; q) \right) \right] + \sum_{i=1}^{a} (xq)^{2i-2} \left( \sum_{h=1}^{k-i+1} (xq^{2})^{2h} U_{2k,2k-2h}(xq^{2}; q) + \sum_{h=0}^{k-i} (xq^{2})^{2h} U_{2k,2k-2h}(xq^{2}; q) \right].
\]

(3.30)

The proof of Theorem 3.3 Now we check the initial values. One can easily get that \( U_{k,0}(m, n) = 0 \), \( U_{k,a}(0, n) = U_{k,a}(m, 0) = 0 \) and \( U_{k,a}(0, 0) = 1 \). Then \( T_{2k,2a}(x; q) \) and \( U_{2k,2a}(x; q) \) have the same initial values. By Theorem 3.3 and the \( q \)-Difference Equations Principle (3.23), we complete the proof.

**Theorem 3.4** For \( k \geq a \geq 1 \), we have

\[
\sum_{n \geq 0} U_{2k,2a}(n)q^{n} = \frac{(-q; q)_{\infty}(q^{2a}, q^{4k-2a}, q^{4k}; q^{4})_{\infty}}{(q^{2}; q^{2})_{\infty}}.
\]

(3.31)

**Proof.** Let \( x = 1 \) in (3.22), we have

\[
U_{2k,2a}(1; q) = \sum_{m,n \geq 0} U_{2k,2a}(m,n)q^{n} = \sum_{n \geq 0} U_{2k,2a}(n)q^{n} = \frac{(-q; q)_{\infty}(q^{2a}, q^{4k-2a}, q^{4k}; q^{4})_{\infty}}{(q^{2}; q^{2})_{\infty}}.
\]

(3.32)

(3.33)

4 The generating function of \( U_{2k,2a-1}(n) \) and \( U_{2k+1,2a}(n) \)

In this section, we will give the generating function of \( U_{2k,2a-1}(n) \) and \( U_{2k+1,2a}(n) \). For \( 1 \leq i \leq a \), let \( U_{k,a}^{i}(m, n) \) denote the number of overpartitions enumerated by \( U_{k,a}(m, n) \) where if there is an \( n \), then the number of non-overlined 1 is exactly \( i \) otherwise the number of non-overlined 1 is exactly \( i - 1 \), and the generating function denoted as follows.

\[
U_{k,a}^{i}(x; q) := \sum_{m,n \geq 0} U_{k,a}^{i}(m, n)x^{m}q^{n}.
\]

**Theorem 4.1** For \( k \geq a \geq 1 \), we have

\[
U_{2k,2a-1}(x; q) = U_{2k,2a}(x; q) - U_{2k,2a}^{2a}(x; q),
\]

(4.32)

and

\[
U_{2k,2a}^{2a}(x; q) = xq(-xq^{3}; q^{2})_{\infty}[\overline{U}_{k,a}(x^{2}; q^{2}) - \overline{U}_{k,a-1}(x^{2}; q^{2})].
\]

(4.33)
Proof. The relation (4.32) is a directly result by considering the number of 1 and T.

To derive (4.33), we decompose the overpartitions enumerated by \( U_{2k,2a}^2(m,n) \) into the following four cases.

1. if there is an \( 1 \), then the number of nonoverlined 1 is 2a. Suppose there is an \( 2 \) then the number of nonoverlined 2 is odd and at most \( 2k - 2a - 1 \). Further, suppose there are exactly \( 2h - 1 \) non-overlined 2, then by deleting all the \( 1 \), 1's, 2's, and subtracting 2 from each of other parts, we get all overpartitions enumerated by \( U_{2k,2k-2h}(m-2a-2h-1,n-2a-4h-1) \).

2. if there is an \( 1 \), then the number of nonoverlined 1 is 2a. Further, suppose there is no \( 2 \), then the number of nonoverlined 2 is odd and at most \( 2k - 2a - 2 \). Suppose there are exactly \( 2h \) non-overlined 2, then by deleting all \( 1 \), 1's and 2's, and subtracting 2 from each of other parts, we get all overpartitions enumerated by \( U_{2k,2k-2h}(m-2a-2h-1,n-2a-4h-1) \).

3. if there is no \( 1 \), then the number of nonoverlined 1 is 2a - 1. Suppose there is an \( 2 \), then the number of nonoverlined 2 is odd and at most \( 2k - 2a - 1 \). Further, suppose there are exactly \( 2h - 1 \) non-overlined 2, then by deleting all \( 1 \), 1's, 2's and 2's, and subtracting 2 from each of other parts, we get all overpartitions enumerated by \( U_{2k,2k-2h}(m-2a-2h+1,n-2a-4h+1) \).

4. if there is no \( 1 \), then the number of nonoverlined 1 is 2a - 1. Suppose there is no \( 2 \) then the number of nonoverlined 2 is odd and at most \( 2k - 2a \). Further, suppose there are exactly \( 2h \) non-overlined 2, then by deleting all \( 1 \), 1's, 2's, and subtracting 2 from each of other parts, we get all overpartitions enumerated by \( U_{2k,2k-2h}(m-2a-2h+1,n-2a-4h+1) \).

Thus, we get the following relation

\[
U_{2k,2a}^2(x;q)
= (xq)^{2a+1} \sum_{h=1}^{k-a} (xq^2)^{2h} U_{2k,2k-2h}(xq^2;q) + (xq)^{2a+1} \sum_{h=0}^{k-a-1} (xq^2)^{2h} U_{2k,2k-2h}(xq^2;q)
+ (xq)^{2a-1} \sum_{h=1}^{k-a+1} (xq^2)^{2h} U_{2k,2k-2h}(xq^2;q) + (xq)^{2a-1} \sum_{h=0}^{k-a} (xq^2)^{2h} U_{2k,2k-2h}(xq^2;q).
\]

In Section 3, we have proved that \( U_{2k,2a}(x;q) = (-xq;q^2)_{\infty} \varphi_{k,a}(x^2;q^2) \), then we have

\[
U_{2k,2a}^2(x;q)
= (xq)^{2a+1} (-xq^3;q^2)_{\infty} \sum_{h=1}^{k-a} [(xq^2)^{2h} \varphi_{k-k-h}(x^2q^4;q) + (xq^2)^{2h-2} \varphi_{k-k-h-1}(x^2q^4;q)]
+ (xq)^{2a-1} (-xq^3;q^2)_{\infty} \sum_{h=1}^{k-a+1} [(xq^2)^{2h} \varphi_{k-k-h}(x^2q^4;q) + (xq^2)^{2h-2} \varphi_{k-k-h-1}(x^2q^4;q)].
\]

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By (2.18), we have

\[ U_{2k,2a}^2(x; q) = (xq)^{2a+1}(-xq^3; q^2) \sum_{h=1}^{k-a} [Q_{k,h}(x^2q^2; q^2) - \overline{Q}_{k,h-1}(x^2q^2; q^2)] \]
\[ + (xq)^{2a-1}(-xq^3; q^2) \sum_{h=1}^{k-a+1} [Q_{k,h}(x^2q^2; q^2) - \overline{Q}_{k,h-1}(x^2q^2; q^2)] \]
\[ = (xq)^{2a+1}(-xq^3; q^2) Q_{k,k-a}(x^2q^2; q^2) + (xq)^{2a-1}(-xq^3; q^2) \overline{Q}_{k,k-a+1}(x^2q^2; q^2) \]
\[ = xq(-xq^3; q^2)[Q_{k,a}(x^2q^2; q^2) - \overline{Q}_{k,a-1}(x^2q^2)]. \]

This completes the proof.

**Theorem 4.2** For \( k \geq a \geq 1 \), we have

\[ \sum_{n \geq 0} U_{2k,2a-1}(n)q^n = \frac{(-q^2; q^2)_\infty(q^{2a}, q^{4k-2a}, q^{4k}; q^{4k})_{\infty}}{(q^2; q^2)_\infty} + xq(-q^2; q^2)_\infty(q^{2a-2}, q^{4k-2a+2}, q^{4k}; q^{4k})_{\infty} \]  (4.34)

**Proof.** By Theorem 3.1 (4.32) and (4.38), we compute \( U_{2k,2a-1}(x; q) \) as

\[ U_{2k,2a-1}(x; q) = U_{2k,2a}(x; q) - U_{2k,2a}^2(x; q) \]
\[ = (-xq; q^2)_\infty Q_{k,a}(x^2; q^2) - xq(-xq^3; q^2)_\infty \overline{Q}_{k,a}(x^2; q^2) + xq(-xq^3; q^2)_\infty \overline{Q}_{k,a-1}(x^2; q^2) \]
\[ = (-xq^3; q^2)_\infty Q_{k,a}(x^2; q^2) + xq(-xq^3; q^2)_\infty \overline{Q}_{k,a-1}(x^2; q^2). \]  (4.35)

Let \( x = 1 \), we get the generating function of \( U_{2k,2a-1}(n) \).

\[ \sum_{n \geq 0} U_{2k,2a-1}(n)q^n = U_{2k,2a-1}(1; q) \]
\[ = \frac{(-q^2; q^2)_\infty(q^{2a}, q^{4k-2a}, q^{4k}; q^{4k})_{\infty}}{(q^2; q^2)_\infty} + xq(-q^2; q^2)_\infty(q^{2a-2}, q^{4k-2a+2}, q^{4k}; q^{4k})_{\infty}. \]  (4.36)

Now we begin to consider the generating function of \( U_{2k+1,2a}(n) \). The way is similar as deriving the generating function of \( U_{2k,2a-1}(n) \).

**Theorem 4.3** For \( k \geq a \geq 1 \), we have

\[ U_{2k+1,2a}(x; q) = U_{2k+1,2a+1}(x; q) - U_{2k+1,2a+1}^2(x; q) \]  (4.37)

and

\[ U_{2k,2a}^2(x; q) = xq(-xq^3; q^2)_\infty(\overline{Q}_{k,a}(x^2; q^2) - \overline{Q}_{k,a-1}(x^2; q^2)). \]  (4.38)

The proof is similar as that of Theorem 4.1 so we omit it here.
Theorem 4.4

\[
\sum_{n \geq 0} U_{2k+1,2a}(n)q^n = \frac{(-q^2; q)_\infty(q^{2a+1}, q^{4k-2a+1}, q^{4k+2}; q^{4k+2})_\infty}{(q^2; q^2)_\infty} + \frac{xq(-q^2; q)_\infty(q^{2a-1}, q^{4k-2a+3}, q^{4k+2}; q^{4k+2})_\infty}{(q^2; q^2)_\infty}. \tag{4.39}
\]

Proof. By using Theorem 4.3, we get

\[
U_{2k+1,2a}(x; q) = U_{2k+1,2a+1}(x; q) - U_{2k+1,2a+1}(x; q)
\]

\[
= (-xq; q^2)_\infty Q_{\frac{k}{2}, \frac{a}{2} + \frac{1}{2}}(x; q^2) - xq(-xq^3; q^2)_\infty[Q_{\frac{k}{2}, \frac{a}{2} + \frac{1}{2}}(x; q^2) - Q_{\frac{k}{2}, \frac{a}{2} - \frac{1}{2}}(x; q^2)]
\]

\[
= (-xq^3; q^2)_\infty Q_{\frac{k}{2}, \frac{a}{2} + \frac{1}{2}}(x; q^2) + xq(-xq^3; q^2)_\infty Q_{\frac{k}{2}, \frac{a}{2} - \frac{1}{2}}(x; q^2). \tag{4.40}
\]

Let \(x = 1\), we reach that

\[
\sum_{n \geq 0} U_{2k+1,2a}(n)q^n = U_{2k+1,2a}(1; q)
\]

\[
= \frac{(-q^2; q)_\infty(q^{2a+1}, q^{4k-2a+1}, q^{4k+2}; q^{4k+2})_\infty}{(q^2; q^2)_\infty} + \frac{xq(-q^2; q)_\infty(q^{2a-1}, q^{4k-2a+3}, q^{4k+2}; q^{4k+2})_\infty}{(q^2; q^2)_\infty}. \tag{4.41}
\]

5 The generating function of \(\overline{U}_{2k,2a}(n)\)

Now we begin to derive the generating function of \(\overline{U}_{k,a}(n)\). We can derive the following relation by noticing the property that \(f_1 + f_2\) is even.

Theorem 5.1 For \(k \geq a \geq 2\) with \(a\) even, we have

\[
\overline{U}_{k,a}(n) = \overline{U}_{k,a-1}(n). \tag{5.42}
\]

In this section, we just derive the generating function of \(\overline{U}_{2k,2a}(n)\). By the combinatorial interpretation of \(\overline{U}_{2k,2a}(n)\), we can get the relation between \(\overline{U}_{2k,2a}(n)\) and \(U_{2k,2a}(n)\) as follows.

Theorem 5.2 For \(k \geq a \geq 1\), we have

\[
\overline{U}_{2k,2a}(x; q) - \overline{U}_{2k,2a-2}(x; q) = (xq)^{2a}U_{2k,2k-2a}(x; q) + (xq)^{2a-2}U_{2k,2k-2a+2}(x; q). \tag{5.43}
\]

Proof. By the definition of the overpartitions enumerated by \(\overline{U}_{2k,2a}(n)\) and \(\overline{U}_{2k,2a-2}(n)\), we know that \(\overline{U}_{2k,2a}(n) - \overline{U}_{2k,2a-2}(n)\) is the number of overpartitions that enumerated by \(\overline{U}_{2k,2a}(n)\) with the restriction that: if there is an \(1\), then the number of \(1\) is \(2a - 1\), otherwise the number of \(1\) is \(2a - 2\).
Then we consider the overpartition \( \lambda \) enumerated by \( \overline{U}_{2k,2a}(n) - \overline{U}_{2k,2a-2}(n) \) in the following cases.

(i) \( f_1(\lambda) = 1 \) and \( f_1(\lambda) = 2a - 1 \).

If \( f_1(\lambda) = 1 \), then \( f_2(\lambda) \leq 2k - 2a \). By deleting all the 1’s, and then subtracting 1 from each other parts, we get the overpartitions enumerated by \( U_{2k,2k-2a}(m - 2a, n - m) \) which have the \( \overline{T} \) as a part.

If \( f_1(\lambda) = 0 \), then \( f_2(\lambda) \leq 2k - 2a - 1 \). By deleting all the 1’s, and then subtracting 1 from each other parts, we get the overpartitions enumerated by \( U_{2k,2k-2a}(m - 2a, n - m) \) which do not have the \( \overline{T} \) as a part.

(ii) \( f_1(\lambda) = 0 \) and \( f_1(\lambda) = 2a - 2 \).

If \( f_1(\lambda) = 1 \), then \( f_2(\lambda) \leq 2k - 2a + 2 \). By deleting all the 1’s, and then subtracting 1 from each other parts, we get the overpartitions enumerated by \( U_{2k,2k-2a+2}(m - 2a + 2, n - m) \) which have the \( \overline{T} \) as a part.

If \( f_1(\lambda) = 0 \), then \( f_2(\lambda) \leq 2k - 2a + 1 \). By deleting all the 1’s, and then subtracting 1 from each other parts, we get the overpartitions enumerated by \( U_{2k,2k-2a-2}(m - 2a + 2, n - m) \) which do not have the \( \overline{T} \) as a part.

In the previous section, we have derived the formula of \( U_{2k,2a}(x;q) \). Here, we derive the formula of \( \overline{U}_{2k,2a}(x;q) \).

**Theorem 5.3** For \( k \geq a \geq 1 \), we have

\[
\sum_{n \geq 0} \overline{U}_{2k,2a}(x;q) = (-xq^2; q^2)^\infty \overline{Q}_{k,a}(x^2; q^2).
\]  
(5.44)

**Proof.** We employ Theorem \[5.33\] and \[3.1\] get the following result.

\[
\overline{U}_{2k,2a}(x;q) - \overline{U}_{2k,2a-2}(x;q) = (xq)^{2a} U_{2k,2k-2a}(x;q) + (xq)^{2a-2} U_{2k,2k-2a+2}(x;q)
\]

\[
= (xq)^{2a} (-xq^2; q^2)^\infty \overline{Q}_{k,a}(x^2; q^2) + (xq)^{2a-2} (-xq^2; q^2)^\infty \overline{Q}_{k,a+1}(x^2; q^2).
\]

By using \[2.18\], we have

\[
\overline{U}_{2k,2a}(x;q) - \overline{U}_{2k,2a-2}(x;q) = (-xq^2; q^2)^\infty (\overline{Q}_{k,a}(x^2; q^2) - \overline{Q}_{k,a-1}(x^2; q^2)).
\]  
(5.45)

We successively apply \[5.45\] to get the following relation

\[
\overline{U}_{2k,2a}(x;q) = (-xq^2; q^2)^\infty \overline{Q}_{k,a}(x^2; q^2).
\]  
(5.46)

Letting \( x = 1 \), we get the generating function of \( \overline{U}_{2k,2a}(n) \).

**Theorem 5.4** For \( k \geq a \geq 1 \), we have

\[
\sum_{n \geq 0} \overline{U}_{2k,2a}(n)q^n = \frac{(-q^2; q^2)^\infty (q^{2a}; q^{4k-2a}, q^{4k}; q^{4k})^\infty}{(q^2; q^2)^\infty}.
\]

(5.47)
6 The generating function of $U_{2k+1,2a+1}(n)$ and $\overline{U}_{2k+1,2a+1}(n)$

In this section we consider the generating function $U_{2k+1,2a+1}(x; q)$ and $\overline{U}_{2k+1,2a+1}(x; q)$. By using the combinatorial tool we give the $q$-differential relations between $U_{2k+1,2a+1}(x; q)$ and $\overline{U}_{2k+1,2a+1}(x; q)$. Then by the initial values, we can get the formulas of $U_{2k+1,2a+1}(x; q)$ and $\overline{U}_{2k+1,2a+1}(x; q)$. We first check the following relations.

**Theorem 6.1** For $k \geq a \geq 1$, we have

\[
U_{2k+1,2a+1}(x; q) - U_{2k+1,2a-1}(x; q) \\
= (1 + xq)(xq)^{2a+1}U_{2k+1,2k-2a}(xq; q) + (1 + xq)(xq)^{2a-1}U_{2k+1,2k-2a+2}(xq; q),
\]

and

\[
U_{2k+1,1}(x) = x^2q^2U_{2k+1,2k}(xq) + U_{2k+1,2k+2}(xq; q).
\]

**Proof.** We prove the relation (6.48) by consider the overpartitions enumerated by $U_{2k+1,2a+1}(m, n) - U_{2k+1,2a-1}(m, n)$.

1. there is an $\overline{T}$ and there are $2a+1$ or $2a$ 1’s. Since $f_2 + f_\overline{T}$ is even, if there is an $\overline{T}$, then there are at most $2k-2a-1$ non-overflowed 2’s or if there is no $\overline{T}$, then there are at most $2k-2a-2$ nonoverlined 2’s. By deleting all 1’s and $\overline{T}$ and subtracting 1 from each of other parts, we get all overpartitions enumerated by $U_{2k+1,2k-2a}(m-2a-1, n-m)$ and $U_{2k+1,2k-2a}(m-2a-2, n-m)$. Then the generating function is $(1 + xq)(xq)^{2a+1}U_{2k+1,2k-2a}(xq; q)$.

2. There is no $\overline{T}$ and there are $2a + 1$ 1’s. since $f_2 + f_\overline{T}$ is even, if there is an $\overline{T}$, then there are at most $2k-2a+1$ nonoverlined 2’s or if there is no $\overline{T}$, then there are at most $2k-2a$ nonoverlined 2’s. By deleting all 1’s and subtracting 1 from each of other parts, we get all overpartitions enumerated by $U_{2k+1,2k-2a+1}(m-2a, n-m)$ and $U_{2k+1,2k-2a+1}(m-2a+2, n-m)$. Then the generating function is $(1 + xq)(xq)^{2a-1}U_{2k+1,2k-2a-2}(xq; q)$.

To prove (6.49), we consider the overpartitions enumerated by $U_{2k+1,1}(m, n)$. If there is an $\overline{T}$ then there is a nonoverflowed 1, otherwise there are no 1’s. Deleting all 1 and $\overline{T}$ and subtracting 1 from each other part, we get the overpartitions enumerated by $U_{2k+1,2k-1}(m-2, n-m)$ and the overpartitions enumerated by $U_{2k+1,2a+1}(m, n-m)$. The generating function is $x^2q^2U_{2k+1,2k-1}(xq; q) + U_{2k+1,2k+1}(xq; q)$. This completes the proof.

**Theorem 6.2** For $k \geq a \geq 1$, we have

\[
\overline{U}_{2k+1,2a}(n) - \overline{U}_{2k+1,2a-2}(n) \\
= (xq)^{2a}U_{2k+1,2k-2a+1}(xq; q) + (xq)^{2a-2}U_{2k+1,2k-2a+3}(xq; q),
\]

and

\[
\overline{U}_{2k+1,2}(x; q) = (xq)^2U_{2k+1,2k-1}(xq; q) + U_{2k+1,2k+1}(xq; q).
\]

The proof is similar as Theorem 6.1 so we omit it here.

Now we derive the following result, by which we can get the generating function of $U_{2k+1,2a+1}(n)$ and $\overline{U}_{2k+1,2a}(n)$.
\textbf{Theorem 6.3} For \( k \geq a \geq 0 \), we have
\[
U_{2k+1,2a+1}(x; q) = (-xq; q^2)_{\infty} \overline{Q}_{k+\frac{1}{2},a+\frac{1}{2}}(x^2; q^2). \tag{6.52}
\]
\[
\overline{U}_{2k+1,2a}(x; q) = \overline{U}_{2k+1,2a-1}(x; q) = (-xq^2; q^2)_{\infty} \overline{Q}_{k+\frac{1}{2},a}(x^2; q^2). \tag{6.53}
\]

\textit{Proof.} Let
\[
T_{2k+1,2a+1}(x; q) := (-xq; q^2)_{\infty} \overline{Q}_{k+\frac{1}{2},a+\frac{1}{2}}(x^2; q^2)
\]
and
\[
\overline{T}_{2k+1,2a}(x; q) := (-xq^2; q^2)_{\infty} \overline{Q}_{k+\frac{1}{2},a}(x^2; q^2). \tag{6.54}
\]
We shall prove that
\[
T_{2k+1,2a+1}(x; q) = U_{2k+1,2a+1}(x; q)
\]
and
\[
\overline{T}_{2k+1,2a}(x; q) = \overline{U}_{2k+1,2a}(x; q) = \overline{U}_{2k+1,2a-1}(x; q).
\]
We first check the initial values
\[
1 = T_{2k+1,2a+1}(0; q) = T_{2k+1,2a+1}(x; 0) = \overline{T}_{2k+1,2a}(0; q) = \overline{T}_{2k+1,2a}(x; 0). \tag{6.55}
\]
By (2.17) and Lemma 2.3, one can derive the following relation
\[
T_{2k+1,1}(x; q) = (-xq; q^2)_{\infty} \overline{Q}_{k+\frac{1}{2},\frac{1}{2}}(x^2; q^2)
\]
\[
= (-xq^3; q^2)_{\infty}(1+xq) \overline{Q}_{k+\frac{1}{2},\frac{1}{2}}(x^2; q^2)
\]
\[
= (-xq^3; q^2)_{\infty}xq[\overline{Q}_{k+\frac{1}{2},\frac{1}{2}}(x^2; q^2) - \overline{Q}_{k+\frac{1}{2},-\frac{1}{2}}(x^2; q^2)]
\]
\[
= (-xq^3; q^2)_{\infty}xq[(xq)\overline{Q}_{k+\frac{1}{2},k}(x^2q^2; q^2) - (xq)^{-1}\overline{Q}_{k+\frac{1}{2},k+1}(x^2q^2; q^2)]
\]
\[
= x^2q^2T_{2k+1,2k}(xq; q) + T_{2k+1,2k+2}(xq; q).
\]
One also can check that
\[
T_{2k+1,2a+1}(x; q) - T_{2k+1,2a-1}(x; q)
\]
\[
= (-xq; q^2)_{\infty}[\overline{Q}_{k+\frac{1}{2},a+\frac{1}{2}}(x^2; q^2) - \overline{Q}_{k+\frac{1}{2},a-\frac{1}{2}}(x^2; q^2)]
\]
\[
= (-xq; q^2)_{\infty}[(xq)^{2a+1}\overline{Q}_{k+\frac{1}{2},k-a}(x^2q^2; q^2) + (xq)^{2a-1}\overline{Q}_{k+\frac{1}{2},k-a+1}(x^2q^2; q^2)].
\]
By (6.54),
\[
T_{2k+1,2a}(xq; q) = (-xq^3; q^2)_{\infty} \overline{Q}_{k+\frac{1}{2},a}(x^2q^2; q^2),
\]
then we have
\[
T_{2k+1,2a+1}(x; q) - T_{2k+1,2a-1}(x; q)
\]
\[
= (1+xq)[(xq)^{2a+1}T_{2k+1,2k-2a}(xq; q) + (xq)^{2a-1}T_{2k+1,2k-2a+2}(xq; q)]. \tag{6.56}
\]
Similarly, we can get the following relation

\[
T_{2k+1,2a}(x; q) - T_{2k+1,2a-2}(x; q) \\
= (-xq^2; q^2)\psi_{k+\frac{1}{2},a}(x^2; q^2) - \psi_{k+\frac{1}{2},a-1}(x^2; q^2) \\
= (-xq^2; q^2)\psi_{k+\frac{1}{2},a-\frac{1}{2}}(x^2; q^2) + (x^2q^2)\psi_{k+\frac{1}{2},a+\frac{1}{2}}(x^2; q^2) \\
= (xq)^{2a}T_{2k+1,2k-2a+1}(xq; q) + (xq)^{2a-2}T_{2k+1,2k-2a+3}(xq; q).
\]

(6.57)

Now we consider the initial value of \( T_{2k+1,2}(x; q) \).

\[
T_{2k+1,2}(x; q) = (-xq^2; q^2)\psi_{k+\frac{1}{2},1}(x^2; q^2) \\
= (-xq^2; q^2)\psi_{k+\frac{1}{2},0}(x^2; q^2) \\
= (xq)^{2a}T_{2k+1,2k-1}(xq; q) + T_{2k+1,2k+1}(xq; q) \\
= (xq)^{2a}U_{2k+1,2a+1}(xq; q) + U_{2k+1,2a+1}(xq; q).
\]

(6.58)

By the initial values and double inductions we can derive that

\[ T_{2k+1,2a+1}(x; q) = U_{2k+1,2a+1}(x; q) \]

and

\[ T_{2k+1,2a}(x; q) = U_{2k+1,2a}(x; q) = U_{2k+1,2a-1}(x; q). \]

Then we completes the proof.

Letting \( x = 1 \), we get the following theorem.

**Theorem 6.4**

\[
\sum_{n=0}^{\infty} U_{2k+1,2a+1}(n)q^n = \frac{(-q; q)_{\infty}(q^{2a+1}, q^{4k+1-2a}, q^{4k+2}, q^{4k+2})_{\infty}}{(q^2; q^2)_{\infty}}. \tag{6.59}
\]

\[
\sum_{n=0}^{\infty} U_{2k+1,2a}(n)q^n = \sum_{n=0}^{\infty} U_{2k+1,2a-1}(n)q^n = \frac{(-q^2; q^2)_{\infty}(q^{2a}, q^{4k-2a+2}, q^{4k+2}, q^{4k+2})_{\infty}}{(q^2; q^2)_{\infty}}. \tag{6.60}
\]

### 7 The combinatorial relations with \( \overline{B}_{k,a}(n) \)

By the generating function, we can get some relations among \( U_{k,a}(n), \overline{U}_{k,a}(n) \) and \( \overline{B}_{k,a}(n) \). In this section, we shall give the combinatorial interpretation of these relations.

**Theorem 7.1** For \( k \geq a \geq 1 \), we have

\[
\sum_{n=0}^{\infty} U_{2k,2a}(n)q^n = (-q^2; q^2)_{\infty} \sum_{n=0}^{\infty} \overline{B}_{k,a}(n)q^{2n}, \tag{7.61}
\]

and

\[
\sum_{n=0}^{\infty} \overline{U}_{2k,2a}(n)q^n = (-q^2; q^2)_{\infty} \sum_{n=0}^{\infty} \overline{B}_{k,a}(n)q^{2n}. \tag{7.62}
\]
Proof. Firstly, we prove (7.61). Let \( \lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_m \) be a partition enumerated by \( U_{2k,2a}(n) \), i.e., \( f_1(\lambda) + f_\ell(\lambda) + f_{\ell+1}(\lambda) \leq 2 - 1 + f_{2\ell+1}(\lambda) \) and \( f_2(\lambda) + f_{2\ell}(\lambda) \) are even for each \( \ell \).

If there is a number \( 2l - 1 \) such that \( f_{2\ell-1}(\lambda) + f_{2\ell-1}(\lambda) \) is odd, then we remove one nonoverlined \( 2l - 1 \). We let \( \gamma \) be the partition consisting of those removed odd parts and \( \beta \) be the overpartition of the remaining parts of \( \lambda \). Then it is obvious that \( \gamma \) has distinct odd parts, and \( \beta \) is an overpartition such that \( f_l(\beta) + f_\ell(\beta) \) are even, for each \( \ell \) with \( f_1(\beta) \leq 2a - 1 + f_{2\ell}(\beta) \).

Now, we add two repeated parts of \( \beta \) and then divide them by 2, to obtain an overpartition \( \beta' \). If there is an overlined \( l \) in \( \beta \), then there is an \( \ell \) in \( \beta' \). We will check that \( \beta' \) is an overpartition enumerated by \( \overline{B}_{k,a}((n - |\gamma|)/2) \).

If \( f_1(\lambda) + f_1(\lambda) + f_{\ell+1}(\lambda) + f_{\ell+1}(\lambda) \leq 2k - 2 \), then after the operation, \( f_1(\beta') + f_\ell(\beta') + f_{\ell+1}(\beta') \leq k - 1 \).

If \( f_1(\lambda) + f_1(\lambda) + f_{\ell+1}(\lambda) + f_{\ell+1}(\lambda) = 2k - 1 \), then the odd number in \{\( l, l + 1 \)\} appears odd number of times (including overlined and nonoverlined) so that it should be removed. Thus, after the operation, \( f_1(\beta') + f_\ell(\beta') + f_{\ell+1}(\beta') = k - 1 \).

If \( f_1(\lambda) + f_1(\lambda) + f_{\ell+1}(\lambda) + f_{\ell+1}(\lambda) = 2k \) then \( f_{\ell+1}(\lambda) = 1 \). Hence, after the operation, \( f_1(\beta') + f_\ell(\beta') + f_{\ell+1}(\beta') = k \) and \( f_{\ell+1}(\beta') = 1 \), so \( f_1(\beta') + f_\ell(\beta') + f_{\ell+1}(\beta') = k - 1 \).

Then we can see that \( \beta' \) is an overpartition that \( f_1(\beta') + f_\ell(\beta') + f_{\ell+1}(\beta') \leq k - 1 \).

Now we check that \( f_1(\beta') \leq a - 1 \). If \( f_1(\lambda) = 1 \) then together with that \( f_1(\lambda) + f_\ell(\lambda) \leq 2a \), we have \( f_1(\beta') + f_{\ell+1}(\beta') \leq a \) and \( f_1(\beta') = 1 \). If \( f_1(\lambda) = 0 \), then \( f_1(\beta') = 2a - 1 \). If \( f_1(\lambda) \neq 2a - 1 \), then \( f_1(\beta') \leq a - 1 \), otherwise \( f_1(\lambda) \) is odd, so we should remove one 1. Then after the operation \( f_1(\beta') = a - 1 \).

Now, we have proved that \( \beta' \) is an overpartition enumerated by \( \overline{B}_{k,a}((n - |\gamma|)/2) \), i.e. (7.61).

The combinatorial interpretation of (7.62) is similar, so we omit it here.

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