Quasi Differential Quotients

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Abstract—We explore basic properties and some applications of Quasi Differential Quotients (QDQs) and the related QDQ-approximating multi-cones. A QDQ, which is a special kind of H. Sussmann’s Approximate Generalized Differential Quotient (AGDQ), consists in a notion of generalized differentiation for set-valued maps. QDQs have the advantage over AGDQs of allowing a genuine, non-punctured, Open Mapping result, so in optimal control to deduce Maximum Principles that involve set-valued maps.

The “Generalized Differential Quotients (GDQ)” and the “Approximate Generalized Differential Quotient” (AGDQ). This was part of a program designed to identifying generalized differentiation theories capable of approximating multi-cones. A QDQ is a quasi-differential tool in the program of extending Goh-like conditions for optimal control problems with nonsmooth dynamics. Such conditions involve set-valued Lie brackets, which happen to be QDQs of suitable compositions of flows.

In the present paper we wish to present the notion of QDQ, by proving some basic properties as well as by recalling some known facts. We also provide some significant examples and give sufficient or necessary conditions for a set to be a QDQ. This is intended as the beginning of a more articulate investigation addressed on one hand to the relation between QDQs with other notions of generalized differentiation (see e.g. [9], [10], and, on the other hand, to concrete applications.

Here is an outline of the article. In Section 2, we present the definition of QDQs in Euclidean spaces and on manifolds, and prove basic properties, comprising locality, linearity, product rules, and a Chain Rule. Section 3 is devoted to QDQ-approximating multi-cones and to the above-mentioned open mapping results. Section 4 provides several examples, as well as some results for QDQs of curves. Moreover a class of QDQs for set-valued maps taking values on “F-abundant” (see Def. 4.5) subsets is illustrated. In Section 5 we show that a set-valued Lie bracket \( [f, g]_{\text{QDQ}} \) of Lipschitz continuous vector fields \( f, g \) (see [9], [10], [8] and [3]) is a QDQ for the commutator-like multi-flow of \( f \) and \( g \) at \( q \).

II. DEFINITIONS AND BASIC PROPERTIES

A. Quasi-Differential Quotients

Definition 2.1: Assume that \( F: \mathbb{R}^n \to \mathbb{R}^m \) is a set-valued map, \( (\bar{x}, \bar{y}) \in \mathbb{R}^n \times \mathbb{R}^m \), \( \Lambda \subset \text{Lin} (\mathbb{R}^n, \mathbb{R}^m) \) is a compact set, and \( \Gamma \subset \mathbb{R}^n \) is any subset. We say that \( \Lambda \) is a Quasi Differential Quotient (QDQ) of \( F \) at \( (\bar{x}, \bar{y}) \) in the direction of \( \Gamma \) if there exists a \( \delta^* \in (0, +\infty] \) and modulus \( \rho: [0, +\infty) \to [0, +\infty] \) having the property that (*) for every \( \delta \in (0, \delta^*) \) there is a continuous map \( (L_\delta, h_\delta): (\bar{x} + B_{\delta}) \cap \Gamma \to \text{Lin}(\mathbb{R}^n, \mathbb{R}^m) \times \mathbb{R}^m \) such that, whenever \( x \in (\bar{x} + B_{\delta}) \cap \Gamma \),

\[
d(\Lambda(x), \Lambda) \leq \rho(\delta), \quad |h_\delta(x)| \leq \delta \rho(\delta)
\]

\( (1) \)

The definition of QDQ can be extended to (set-valued) functions from a manifold into another manifold.

Definition 2.2: Let \( \mathcal{N}, \mathcal{M} \) be differential manifolds of class \( C^1 \). Assume that \( \tilde{G}: \mathcal{N} \to \mathcal{M} \) is a set-valued map, \( (\bar{x}, \bar{y}) \in \mathcal{N} \times \mathcal{M}, \tilde{\Lambda} \subset \text{Lin}(T_{\bar{x}}\mathcal{N}, T_{\bar{y}}\mathcal{M}) \) is a compact set, \( \rho:[0, +\infty) \to [0, +\infty] \) such that \( \lim_{\delta \to 0} \rho(\delta) = 0 \)

1 We call modulus a non-decreasing non-negative function \( \rho: [0, +\infty) \to [0, +\infty] \).
and $\tilde{\Gamma} \subseteq \mathcal{N}$ is any subset. Moreover, let $\phi: U \rightarrow \mathbb{R}^n$ and $\psi: V \rightarrow \mathbb{R}^m$ be charts defined on neighborhoods $U$ and $V$ of $\tilde{x}$ and $\tilde{y}$, respectively, and assume that $\phi(\tilde{x}) = 0$, $\psi(\tilde{y}) = 0$. Consider the map $G := \psi \circ G \circ \phi^{-1}: U \rightarrow \mathbb{R}^m$ and extend it arbitrarily to a map $G: \mathbb{R}^n \rightarrow \mathbb{R}^m$ (we do not relabel). We say that $G$ is a Quasi Differential Quotient (QDQ) of $G$ at $(\tilde{x}, \tilde{y})$ in the direction of $\tilde{L}$ if $G := \phi((\Gamma \cap U))$. B. Basic properties

We now present some properties which make QDQs fit for proving Maximum Principles via set-separation techniques.

**Proposition 2.3:** Let $\mathcal{M}, \mathcal{N}$ be $C^1$ real differential manifolds of dimension $m$ and $n$, respectively, and let $F, G: \mathcal{N} \rightarrow \mathcal{M}$ be set-valued maps. Assume that $\tilde{x} \in \mathcal{N}$, $\tilde{y}, \tilde{y}_F, \tilde{y}_G \in \mathcal{M}$, $\Gamma, \Gamma_F, \Gamma_G \subseteq \mathcal{M}$, and $\alpha, \beta \in \mathbb{R}$. Then:

1) *Locality* If $U$ is a neighborhood of $\tilde{x}$ and $F(x) = G(x)$ for $x \in U \cap \Gamma$, then $\Lambda$ is a QDQ for $F$ at $(\tilde{x}, \tilde{y})$ in the direction of $\Gamma$ if and only if it is a QDQ for $G$ at $(\tilde{x}, \tilde{y})$ in the direction of $\Gamma$.

2) *Linearity* If $\mathcal{M} = \mathbb{R}^m$ and $\Lambda_F$ and $\Lambda_G$ are QDQs for $F$ and $G$ at points $(\tilde{x}, \tilde{y}_F)$ and $(\tilde{x}, \tilde{y}_G)$ and in the direction of $\Gamma_F$ and $\Gamma_G$, respectively, then $\alpha \Lambda_F + \beta \Lambda_G$ is a QDQ for $\alpha F + \beta G$ at $(\tilde{x}, \tilde{y}_F)$ in the direction of $\Gamma_F \cap \Gamma_G$.

3) *Set product property* Under the same assumptions as in 2), except for $\mathcal{M}$ which here is allowed to be any manifold, $\Lambda_F \times \Lambda_G$ is a QDQ at $(\tilde{x}, (\tilde{y}_F, \tilde{y}_G))$, in the direction of $\Gamma_F \times \Gamma_G$ for the set-valued map $F \times G: x \mapsto F(x) \times G(x)$.

4) *Product Rule* If $\mathcal{M} = \mathbb{R}$, and still using the same notation as in 2), we have $F(\tilde{x})\Lambda_G + G(\tilde{x})\Lambda_F$ is a QDQ for $FG: x \mapsto F(x)G(x)$.

5) If $F$ is single-valued and $L \in \text{Lin}(T_{\mathcal{N}}' \mathcal{N}, T_{\mathcal{M}}' \mathcal{M})$, $\{L\}$ is a QDQ for $F$ at $(\tilde{x}, \tilde{y})$ in the direction of $\mathcal{M}$ if and only if $F$ is differentiable at $\tilde{x}$ and $L = DF(\tilde{x})$.

Proof: The first property, namely locality, is straightforward from the definition. To prove the linearity stated in 2) let $\rho^F$, $L^F$, $h^F$, $\rho^G$, $L^G$, $h^G$ be the maps verifying $(*)$ in Definition 2.1 for $\Lambda_F$ and $\Lambda_G$, respectively. Hence the thesis follows upon defining $h^G := \alpha h^F + \beta h^G$, $L^G := \alpha L^F + \beta L^G$ and $\rho := |\alpha|\rho^F + |\beta|\rho^G$.

Property 3) is straightforward. We omit the proof of 4) as well, for it can be trivially obtained by using the relation coming from $(*)$ in Definition 2.1.

Let us prove 5) in the case when $\mathcal{N} = \mathbb{R}^m$, $\mathcal{M} = \mathbb{R}^n$, the extension of the property to manifolds being trivial. The sufficiency of the differentiability in order $\{L\}$ be a QDQ at $(\tilde{x}, F(\tilde{x}))$ in the direction of $\mathcal{M}$ is trivial. To prove that it is also necessary, observe that from $L_F(x)(x - \tilde{x}) = F(x) - F(\tilde{x}) - h_F(x)$ and $|L_F(x) - L| < \rho(\delta)$, it follows that $|F(x) - F(\tilde{x}) - L(x - \tilde{x})| = |L_F(x)(x - \tilde{x}) + h_F(x)| \leq 2\delta \rho(\delta)$ holds for all $\delta > 0$, provided $|x - \tilde{x}| \leq \delta$. Hence $F$ is differentiable at $\tilde{x}$ and $L = DF(\tilde{x})$.

**Proposition 2.4 (Chain rule):** Let $\mathcal{N}, \mathcal{M}, \mathcal{L}$ be $C^1$ manifolds of dimensions $m, n, l$, respectively, let $F: \mathcal{N} \rightarrow \mathcal{M}$ and $G: \mathcal{M} \rightarrow \mathcal{L}$ be set-valued maps, and consider the composition $G \circ F: x \in \mathcal{N} \mapsto \bigcup G(y) \in \mathcal{L}$. Assume $\Lambda_F$ is a QDQ for $F$ at point $(\tilde{x}, \tilde{y})$ in the direction of $\Gamma_F$ and $\Lambda_G$ is a QDQ for $G$ at point $(\tilde{y}, \tilde{z})$ in any direction $\Gamma_G$ containing $F(\Gamma_F)$. Then the set $\Lambda := \Lambda_G \circ \Lambda_F$ of all compositions of elements of $\Lambda_G$ with elements of $\Lambda_F$ is a QDQ for $G \circ F$ at $(\tilde{x}, \tilde{z})$ in the direction of $\Gamma_F$.

Proof: Let us prove Proposition 2.4 when $\mathcal{M} = \mathbb{R}^m$, $\mathcal{N} = \mathbb{R}^n$, and $\mathcal{L} = \mathbb{R}^l$, the locality and chart independence of the notion of QDQ guaranteeing that the proof is still valid in the general case.

Let $\delta^F$, $\delta^G$, $L^F$, $L^G$, $h^F$, $h^G$, $\rho^F$, $\rho^G$, with $0 < \delta < \delta^F := \min\{\delta^F, \delta^G\}$, the maps involved in the definition of the QDQs $\Gamma_F$ and $\Lambda_G$. In particular, one has, for all $\delta (0, \delta^F)$,

$$\tilde{y} + L^F_G((x - \tilde{x}) + h^G_F(x)) \in F(x), \forall x \in (\tilde{x} + B_\delta) \cap \Gamma_F, \quad (2)$$

$$\tilde{z} + L^G_G((y - \tilde{y}) + h^G_F(y)) \in G(y), \forall y \in (\tilde{y} + B_\delta) \cap F(\Gamma_F). \quad (3)$$

Let us set

$$M := \max \left\{ \frac{\max_{L \in \Lambda_F} |L'|}{\max_{L \in \Lambda_G} |L''|}, 1 \right\}, \quad \eta(\delta) := \frac{\delta}{3M}, \quad \forall \delta \geq 0.$$  

If needed, let us redefine $\delta^*$ in order that $\delta^* < 1$ and it small enough to guarantee $\rho^F(\eta(\delta)) \leq M$ for all $\delta < \delta^*$.

Now, let us consider any $x \in (\tilde{x} + B_\delta(\eta(\delta))) \cap \Gamma_F$ and let us set $y := \tilde{y} + L^F_G((x - \tilde{x}) + h^G_F(x))$. Clearly, we have $y \in F(\Gamma_F)$, and $|y - \tilde{y}| \leq \eta(\delta) |M + 2\rho^F(\eta(\delta))| \leq \delta^*$, so that $y \in (\tilde{y} + B_\delta) \cap F(\Gamma_F)$.

If we set $\xi(x) := \tilde{y} + L^F_G((x - \tilde{x}) + h^G_F(x))$, we get

$$\tilde{z} + L^G_G((\xi(x)) \cap L^F_G((x - \tilde{x}) + h^G_F(x)), \forall x \in (\tilde{x} + B_\delta(\eta(\delta))) \cap \Gamma_F. \quad (4)$$

where the continuous function $h^G_F(\eta(\delta))$, defined as

$$h^G_F(\eta(\delta)) := L^G_G((\xi(x)) \cap h^G_F((x - \tilde{x}) + h^G_F(x))$$

verifies

$$|h^G_F(\eta(\delta))| \leq M\eta(\delta)\rho^F(\eta(\delta)) + 3M\eta(\delta)\rho^F(3M\eta(\delta)). \quad (5)$$

The function between curly brackets in (4), i.e.,

$$\delta \mapsto L^G_G((\xi(x)) \cap L^F_G((x - \tilde{x}) + h^G_F(x)),$$

is also a continuous function as it is composition of continuous functions. Moreover, every image $L^G_G((\xi(x)) \cap L^F_G((x - \tilde{x}) + h^G_F(x))$ is at most $\rho^F(3M\eta(\delta))$ and whose norm is at most $M$ with a linear functional whose distance from $\Lambda_F$ is at most $\rho^F(3M\eta(\delta))$.

Therefore

$$d(L^G_G((\xi(x)) \cap L^F_G((x - \tilde{x}) + h^G_F(x)), \Lambda) \leq M(\rho^F(\eta(\delta)) + \rho^G(3M\eta(\delta))). \quad (6)$$

Finally, if one sets
\[ \rho(\eta(\delta)) := M \rho^F(\eta(\delta)) + 3p^G(3M\eta(\delta)) \]
then formulas (4)–(6) tell us that \( \Lambda \) is a QDQ for \( G \circ F \) at \((\bar{x}, \bar{z})\) in the direction of \( \Gamma_F \).

III. OPEN MAPPING AND SET-SEPARATION

A. Open mapping

Open mapping results are essential for any reasonable generalized differentiation theory. We recall here a result which, in particular, marks the difference between QDQs and AGDQs.

Theorem 3.1 (Open Mapping for QDQs): Let \( F : \mathbb{R}^n \to \mathbb{R}^m \) be a set-valued map and let \( \Gamma \) be a convex cone in \( \mathbb{R}^n \). Let \( \Lambda \) be a QDQ of \( F \) at \((\bar{x}, \bar{y})\) in the direction of \( \Gamma \) and assume that \( L \cdot \Gamma = \mathbb{R}^m \) for all \( L \) in \( \Lambda \). Then:

(i) There exist constants \( \alpha, \beta > 0 \) such that \( \forall \alpha \in (0, \alpha] \)
\[ \gamma + (B_\alpha \setminus \{0\}) \subseteq F(\bar{x} + B_{\alpha \beta} \cap \Gamma). \]

(ii) There exist \( \delta^* > 0 \) such that for all \( \delta < \delta^* \) and every \( (L_0, h_0) \) as in Definition 3.1 there exists \( x_0 \in (\bar{x} + B_{\delta}) \cap \Gamma \) such that:
\[ \gamma = L_0(x_0)(x_0 - \bar{x}) + h_0(x_0) \ (\in F(x_0)). \]

In particular, by possibly reducing \( \alpha \), the following inclusions hold for all \( \alpha \in (0, \alpha] \)
\[ \gamma + B_\alpha \subseteq F(\bar{x} + B_{\alpha \beta} \cap \Gamma). \]  \( (7) \)

Remark 3.2: Part (ii) is not true if \( \Lambda \) is just a AGDQ. In that case, one can only prove a “punctured” inclusion, namely in general \( \gamma \not\in F(\bar{x} + B_{\alpha \beta} \cap \Gamma). \)

B. Transversal cones

Let us recall the notions of transversality and strong transversality of cones. Let \( E \) be a finite-dimensional, real linear space, and let \( E^* \) be its dual space. A subset \( \mathcal{K} \subseteq E \) is a cone if \( \alpha k \in \mathcal{K} \) for all \( (\alpha, k) \in [0, +\infty) \times \mathcal{K} \). If \( D \subseteq E \) is any subset, let us set
\[ \text{Span}^+ D := \left\{ \sum_{i=1}^l \alpha_i v_i : \alpha_i \geq 0, v_i \in D, \forall i \leq l \right\}, \]
\[ D^\perp := \left\{ p \in E^* : p \cdot w \leq 0 \ \forall w \in D \right\} \subseteq E^*. \]

The convex cones \( \text{Span}^+ D \) and \( D^\perp \) are called the conic hull of \( D \) and the polar cone of \( D \), respectively. Let \( \mathcal{K}_1, \mathcal{K}_2 \subseteq E \) be convex cones. We say that \( \mathcal{K}_1 \) and \( \mathcal{K}_2 \subseteq E \) are transversal, if \( \mathcal{K}_1 - \mathcal{K}_2 := \{k_1 - k_2 : (k_1, k_2) \in \mathcal{K}_1 \times \mathcal{K}_2 \} = E. \mathcal{K}_1 \) and \( \mathcal{K}_2 \) are strongly transversal, if they are transversal and \( \mathcal{K}_1 \cap \mathcal{K}_2 \subseteq \{0\} \).

Proposition 3.3: Two convex cones \( \mathcal{K}_1, \mathcal{K}_2 \subseteq E \) are transversal if and only if they are either strongly transversal or complementary linear subspaces, namely \( \mathcal{K}_1 \oplus \mathcal{K}_2 = E \) (i.e., \( \mathcal{K}_1 + \mathcal{K}_2 = E \) and \( \mathcal{K}_1 \cap \mathcal{K}_2 = \{0\} \)).

Saying that two cones \( \mathcal{K}_1 \) and \( \mathcal{K}_2 \) are not strongly transversal is equivalent to saying that they are linearly separable:

**Proposition 3.4:** Two convex cones \( \mathcal{K}_1, \mathcal{K}_2 \subseteq E \) are not transversal if and only if \( \mathcal{K}_1 \) and \( \mathcal{K}_2 \) are linearly separable, by which we mean that \( \{\mathcal{K}_1 - \mathcal{K}_2 \} \setminus \{0\} \neq 0 \), namely, there exists a linear form \( \lambda \in E^\perp \setminus \{0\} \) such that, \( \forall (k_1, k_2) \in \mathcal{K}_1 \times \mathcal{K}_2, \lambda \cdot k_1 \geq 0 \), and \( \lambda \cdot k_2 \leq 0 \).

In a linear space \( E \), let us call convex multi-cone any family of convex cones of \( E \).

**Definition 3.5:** [7 Def. 2.5] Let \( \mathcal{M} \) be a \( \mathcal{C}^1 \) differentiable manifold, \( \mathcal{M} \subseteq \mathcal{N} \) a set, and \( z \in \mathcal{M} \). A QDQ approximating multi-cone to \( \mathcal{M} \) at \( z \) is a convex multi-cone \( \mathcal{K} \subseteq T_z \mathcal{N} \) such that there exist an integer \( n \geq 0 \), a set-valued map \( G : \mathbb{R}^n \to \mathcal{M} \), a convex cone \( \Gamma \subseteq \mathbb{R}^n \), and a Quasi Differential Quotient \( \Lambda \) of \( G \) at \((0, z)\) in the direction of \( \Gamma \) such that \( G(\Gamma) \subseteq \mathcal{M} \) and \( \mathcal{K} = \{L \cdot \Gamma : L \in \Lambda \} \). We say that such a triple \((G, \Gamma, \Lambda)\) generates the multi-cone \( \mathcal{K} \).

If the triple \((G, \Gamma, \Lambda)\) defining a QDQ approximating multi-cone \( \mathcal{K} \) can be chosen so that \( G(\Gamma) \subseteq \mathcal{M} \setminus \{z\} \), then we say that \( \mathcal{K} \) is z-ignoring.

**Remark 3.6:** Because of local character of the notion of QDQ for a set-valued map, one can equivalently say that a QDQ approximating multi-cone \( \mathcal{K} \) is z-ignoring if there is some \( \delta < 0 \) such that \( G(\mathcal{K} \cap \Gamma) \subseteq \mathcal{M} \setminus \{z\} \) for some \( \delta > 0 \).

**Remark 3.7:** For single-valued maps, QDQ approximating multi-cones consisting of a single cone are precisely Boltyanski’s approximating cones.

**Definition 3.8:** Let \( \mathcal{X} \) be a topological space, and let \( \mathcal{D}_1, \mathcal{D}_2 \subseteq \mathcal{X} \), \( y \in \mathcal{D}_1 \cap \mathcal{D}_2 \). We say that \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \) are locally separated at \( y \) provided there exists a neighborhood \( V \) of \( y \) such that \( \mathcal{D}_1 \cap \mathcal{D}_2 \cap V = \emptyset \).

The following open-mapping-based result, obtained in [7], characterizes set separation in terms of linear separation of QDQ approximating cones. It includes a crucial approximation property (see (ii) below) whenever one of the cones is z-ignoring.

**Theorem 3.9:** [7 Thm. 2.3] Let \( \mathcal{D}_1, \mathcal{D}_2 \) be subsets of a \( \mathcal{C}^1 \) differentiable manifold \( \mathcal{M} \) and let \( z \in \mathcal{D}_1 \cap \mathcal{D}_2 \). Assume that \( \mathcal{K}_1, \mathcal{K}_2 \) are QDQ approximating cones of \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \), respectively, at \( z \).

(i) If \( \mathcal{K}_1 \) and \( \mathcal{K}_2 \) are strongly transversal, then the sets \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \) are not locally separated.

(ii) If \( \mathcal{K}_1 \) (or, equivalently, \( \mathcal{K}_2 \)) is z-ignoring and \( \mathcal{K}_1 \) and \( \mathcal{K}_2 \) are transversal then the sets \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \) are not locally separated.

IV. EXAMPLES

**Example 4.1** (The \( \delta \)-independent case): Let us begin observing that if, for a given set-valued map \( F : \mathbb{R}^n \to \mathbb{R}^m \), a point \((\bar{x}, \bar{y}) \in \mathbb{R}^n \times \mathbb{R}^m \), a subset \( \Gamma \subseteq \mathbb{R}^n \), \( \Lambda \subseteq \text{Lin}(\mathbb{R}^n, \mathbb{R}^m) \) is a compact set such that
\[ \lim_{\delta \to 0} \|d(L(x), \Lambda)L^{-1}(\bar{x} + B_{\delta} \Gamma)\| = 0, \quad \|h(x)\|L^{-1}(\bar{x} + B_{\delta} \Gamma) = o(\delta) \]

When a QDQ approximating multi-cone is a singleton, namely \( \mathcal{K} = \{K\} \), we say that \( K \) is a QDQ approximating cone to \( \delta \) at \( z \).
and \( y + L(x) \cdot (x - \bar{x}) + h(x) \in F(x) \), where 
\((L, h) : \Gamma \to \operatorname{Lin}(\mathbb{R}^m, \mathbb{R}^m) \times \mathbb{R}^m \) is a (\(\delta\)-independent) 
continuous function, then \(\Lambda\) is a \(\text{QDQ}\). Indeed, one can check that, if \((L_\delta, h_\delta) := (L, h)_{[\bar{x}+B_\delta \cap \Gamma]}\) and 
\[\rho(\delta) := \max \left\{ \|d(L(x),\Lambda)\|_{L^{\infty}(\bar{x}+B_\delta \cap \Gamma)}, \frac{\|h(x)\|_{L^{\infty}(\bar{x}+B_\delta \cap \Gamma)}}{\delta} \right\},\]
for every \(\delta > 0\), then the properties in Definition 2.1 are met.

However it is not, in general, possible to reduce \(\delta \mapsto (L_\delta, h_\delta)\) to a constant in the definition of \(\text{QDQ}\), as one can check already in the \(n = m = 1\) case below.

**Example 4.2 (Clarke’s generalized Jacobian):** Let \(\Omega \subseteq \mathbb{R}^n\) be an open set, let \(F : \Omega \to \mathbb{R}^m\) be a (single-valued) Liphsitz continuous map, and let \(x^* \in \Omega\). In \(\Pi\), we proved a sufficient condition for a set to be a \(\text{QDQ}\) via approximation of the function: it is possible to infer from that the Clarke’s generalized Jacobian \(\partial_K F(\bar{x})\) at \(\bar{x}\) is a \(\text{QDQ}\) for \(F\) at \((\bar{x}, F(\bar{x}))\) in the direction of \(\mathbb{R}^n\).

**A. The case \(n = 1\)**

**Example 4.3:** Since \([-1,1]\) is the Clarke’s generalized Jacobian of the function \(f(x) := |x|\) at \(x = 0\), in view of the previous example it is a \(\text{QDQ}\) for the function \(f\) at \((0,0)\) in the direction of \(\mathbb{R}\). However, we prefer here to give an explicit construction of the maps \(L_\delta, h_\delta\). For every \(\delta > 0\), let us set

\[g_\delta(x) := \begin{cases} \frac{1}{2} \left(\frac{x}{\delta}\right)^2 + \frac{\delta^2}{2} & \forall x \in [-\delta^2, \delta^2] \\ |x| & \forall x \in \mathbb{R} \setminus [-\delta^2, \delta^2]. \end{cases}\]

For every \(\delta > 0\), \(g_\delta\) is of class \(C^1\). Moreover, for every \(x \in [-\delta, \delta]\), one has \(f(x) = |x| = L_\delta(x) \cdot x + h_\delta(x)\), where

\[L_\delta(x) := \frac{1}{\delta^2} \mathbf{1}_{[-\delta^2, \delta^2]} + \operatorname{sgn}(x) \mathbf{1}_{[-\delta, -\delta^2] \cup \delta^2, \delta^2]}\]

and \(h_\delta(x) := \frac{\delta^2}{2} + (|x| - g_\delta(x)) \mathbf{1}_{[-\delta^2, \delta^2]}\). Notice that \(L_\delta([-\delta, \delta]) = [-1,1]\), and \(|h_\delta(x)| \leq \delta^2\), so \(\Lambda = [-1,1]\) meets Definition 2.1 with \(\rho(\delta) = \delta\).

By definition of \(\text{QDQ}\) any compact set \(\Lambda' \supseteq \Lambda\) is again a \(\text{QDQ}\) for the same \(f\) at the same point \((0,0)\) in the same direction. Hence, in this example \([-1,1]\) is the smallest possible \(\text{QDQ}\) for \(f\) at \((0,0)\) in the direction of \(\mathbb{R}\).

Actually, while the previous example is a particular case of Clarke’s generalized Jacobian (see below), it is also an instance of the following basic fact:

**Proposition 4.4:** Let \(\mathcal{M}\) be an \(m\)-dimensional manifold of class \(C^1\) and \(f : \mathcal{M} \to \mathbb{R}\) a continuous curve admitting left and right derivatives \(f'(\bar{t}^-)\) and \(f'(\bar{t}^+)\) at a point \(\bar{t} \in \mathbb{R}\). Any compact set \(\Lambda \subseteq \operatorname{Lin}(\mathbb{R}, T_{f(\bar{t})}M)\) containing an arc connecting \(f'(\bar{t}^-)\) and \(f'(\bar{t}^+)\) is a \(\text{QDQ}\) for \(f\) at \((\bar{t}, f(\bar{t}))\) in the direction of \(\mathbb{R}\). Conversely, any \(\text{QDQ}\) for \(f\) at \((\bar{t}, f(\bar{t}))\)

have vanishing distances from the compact set \(\Lambda\) as \(\delta \to 0\). At the same time, they converge to \(f'(\bar{t}^+)\), respectively. Hence \(\{f'(\bar{t}^-), f'(\bar{t}^+)\} \subseteq \Lambda\). On the other hand let us assume by contradiction that \(\Lambda\) can be split into two disjoint compact sets \(C_-\) and \(C_+\) containing \(f'(\bar{t}^-)\) and \(f'(\bar{t}^+)\) respectively. The distance function \(d(x,y) : C_- \times C_+ \to \mathbb{R}\) is a continuous function defined on a compact set, hence it has a minimum \(\epsilon\) and \(\epsilon > 0\) and the sets \(C_\pm = \{L \in \operatorname{Lin}(\mathbb{R}, \mathbb{R}^m) : d(L, C_\pm) \leq \frac{\epsilon}{2}\}\) are still disjoint. If \(\delta\) is small enough to guarantee \(\rho(\delta) \leq \frac{\epsilon}{2}\) and \(|h_\delta(\pm \delta) - f'(\bar{t}^\pm)| \leq \frac{\epsilon}{2}\), the connected set \(L_\delta([-\delta, \delta])\), consisting only of points whose distance from \(\Lambda\) is smaller than \(\rho(\delta)\), is necessarily wholly contained in just one of the \(C_\pm\), which is in contradiction with the fact that \(L_\delta(\pm \delta) \in C_\pm\).
Remark 4.5: As a consequence of the above Proposition, if \( \mathcal{M} \) is a 1-dimensional \( C^1 \) manifold and if \( f : \mathbb{R} \to \mathcal{M} \) is a continuous curve admitting left and right derivatives \( f'(t) \) and \( f'(\bar{t}) \) at a point \( t \in \mathbb{R} \), then the segment

\[
[\min\{f'(t^-), f'(t^+)\}, \max\{f'(t^-), f'(t^+)\}]
\]

is the smallest QDQ of \( f \) at \( t \) in the direction of \( \mathbb{R} \). However, as soon as \( m > 1 \), the QDQs of curves, even those which are minimal with respect to inclusion, may very well be non-convex subsets (but always containing a connected set).

B. Set-valued functions ranging in “F-abundant” subsets

Let us begin with the definition of F-abundant subset:

Definition 4.6: Let \( \mathcal{E} \subseteq \mathcal{E} \subseteq \mathbb{R}^m \) be subsets and let \( F : \mathbb{R}^n \to E \subseteq \mathbb{R}^m \) be a (single-valued) map. We say that \( \mathcal{E} \) is F-abundant if, for every \( \eta > 0 \), there exists a continuous map \( \theta_\eta : \mathcal{E} \to \mathcal{E} \) such that \( |F(x) - \theta_\eta  \circ F(x)| < \eta \), for all \( x \in \mathbb{R}^n \).

We illustrate here a situation where a known QDQ \( \Lambda \) of a map \( F : \mathbb{R}^n \to E \subseteq \mathbb{R}^m \) happens to be also a QDQ for a set-valued map \( \tilde{F} : \mathbb{R}^n \to \mathcal{E} \subseteq \mathbb{R}^m \), provided \( \mathcal{E} \) is F-abundant in \( E \). Such a case is found in applications, in particular for the infimum gap problem, e.g. when approximating the reachable set of a control system with (QDQ) approximating cones to the larger reachable set of the convexified system (see (7)).

Example 4.7: Let \( F : \mathbb{R}^n \to E \subseteq \mathbb{R}^m \) be a (single-valued) map, \( \bar{x} \in \mathbb{R}^n \), and let \( \Lambda \) be a QDQ of \( F \) at \( (\bar{x}, F(\bar{x})) \) in the direction of a given set \( \Gamma \subseteq \mathbb{R}^n \). If \( \mathcal{E} \) is F-abundant in \( E \) then \( \Lambda \) is a QDQ of the set-valued map

\[
\tilde{F} : \mathbb{R}^n \to \mathcal{E} \ni x \mapsto \tilde{F}(x) := \bigcup_{\eta > 0} \theta_\eta \circ F(x),
\]

(where the maps \( \theta_\eta \) are as in Definition 4.6) at \( (\bar{x}, F(\bar{x})) \) in the direction of \( \Gamma \subseteq \mathbb{R}^n \).

Indeed, for every \( \delta > 0 \), let \( \rho(\delta) \), \( L_\delta \), and \( h_\delta \) be as in Definition 2.1. Defining, for every \( x \in \mathbb{R}^n \) and any \( \delta > 0 \),

\[
L_\delta(x) := L_\delta(x), \quad h_\delta(x) := (\theta_{\delta\rho(\delta)} \circ F(x) - F(x) + h_\delta(x) \quad \text{and} \quad \rho(\delta) := 2\rho(\delta)\]

we obtain

\[
\min_{L \in \Lambda} |L_\delta(x) - L| \leq \rho(\delta), \quad |h_\delta(x)| \leq \delta \rho(\delta),
\]

\[
y + L_\delta(x) \cdot (x - \bar{x}) + h_\delta(x) = \theta_\delta \circ F(x) \in \tilde{F}(x)
\]

whenever \( x \in (\bar{x} + B_\delta) \cap \Gamma \). Hence \( \Lambda \) is QDQ of \( \tilde{F} \) at \( (\bar{x}, F(\bar{x})) \) in the direction of \( \Gamma \subseteq \mathbb{R}^n \).

V. Lie Brackets of Lipschitz vector fields

Let \( f, g : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^n \) be two Lipschitz vector fields and \( q \in \Omega \). For \( \eta > 0 \) sufficiently small, and for \( t \in (-\eta, \eta) \), we shall use \( \Phi_t^\eta(q) \) to denote the value at \( t \) of the solution of

\[
\begin{cases}
  y'(t) = f(y(t)) \\
  y(0) = q
\end{cases}
\]

As soon as \( f, g \in C^1 \), it is well known that the multi-flow

\[
\Psi_t(q) := [\Phi_t^\eta \circ \Phi_t^{-\eta} \circ \Phi_t^\eta \circ \Phi_t^{-\eta}](q)
\]

verifies \( \Psi_t(q) = q + t[f,g](q) + o(t^2) \), where \([f,g] \) is the Lie bracket, namely \([f,g] = Dg \cdot f - Df \cdot g \). In (9), a set-valued generalization \( [f,g]_{\text{set}} \) of the Lie bracket has been introduced for Lipschitz vector fields. It is defined by setting, for every \( q \) in the common domain of \( f \) and \( g \),

\[
[f,g]_{\text{set}}(q) = \mathcal{E} \left( \lim_{n \to \infty} [f,g](q_n) \right);
\]

where limits are taken over sequences \( q_n \to q \), \( q_n \in \text{Diff}(f) \cap \text{Diff}(g) \), \( \text{Diff}(f) \) and \( \text{Diff}(g) \) denoting the sets of differentiability points of \( f \) and \( g \), respectively. Using this bracket, the authors of (9) proved a Chow’s type theorem to non-smooth fields. Furthermore, in (10) it was shown that these multi-valued Lie brackets can be used to test commutativity of non-smooth fields and to estimate \( \Psi_t(q) - q \) to the second order, while in (8) they were used to extend Frobenius’ Theorem. Lastly, in an oncoming article (11) we are going to establish Geh-like second order conditions for optima, by means of these set-valued Lie brackets. For this aim we need the following result:

Proposition 5.1: Let \( \Omega \) be a compact subset of \( \mathbb{R}^n \), \( f, g : \Omega \to \mathbb{R}^n \) be Lipschitz vector fields and \( q \in \Omega \). The set \([f,g]_{\text{set}}(q) \) is a QDQ for the function

\[
F : \mathbb{E} \to \Psi(\sqrt{\tau}(q))
\]

at point \((0,q)\) in the direction of \( \mathbb{R}^+ \).

Proof: We are going to make use of the following integral formula, established in (9),

\[
\Phi_{\sqrt{\tau}}^\times \circ \Phi_{\sqrt{\tau}}^\times \circ \Phi_{\sqrt{\tau}}^\times \circ \Phi_{\sqrt{\tau}}^\times(q) =
\]

\[
q + \int_0^{\sqrt{\tau}} \int_0^{\sqrt{\tau}} [X_1,X_2](\theta(q, \sigma, \sqrt{\tau}, \tau)) d\tau d\sigma + o(\varepsilon),
\]

where \( X_1, X_2 \) are \( C^1 \) vector fields, and \( \theta(x, \sigma, t, \tau) = \Phi_{\sqrt{\tau}}^\times \circ \Phi_{\sqrt{\tau}}^\times \circ \Phi_{\sqrt{\tau}}^\times(x) \). Let us consider a mollifier, i.e. a \( C^\infty \) function \( \phi(x) \geq 0 \) having support contained in \( B_1(0) \) and \( L^1 \) norm equal to 1. For every \( \eta > 0 \), we set

\[
F_\eta(q) := \Phi_{\sqrt{\tau}}^\times \circ \Phi_{\sqrt{\tau}}^\times \circ \Phi_{\sqrt{\tau}}^\times \circ \Phi_{\sqrt{\tau}}^\times(q),
\]

where \( f_\eta \) and \( g_\eta \) denote the convolutions \( f \ast \phi_\eta \) and \( g \ast \phi_\eta \), respectively, with \( \phi_\eta := \frac{1}{\eta} \phi \left( \frac{x}{\eta} \right) \). If we take \( X_1 := f_\eta, X_2 := g_\eta \) in (8), and moreover, we set

\[
\eta = \eta(\varepsilon) := \varepsilon^2,
\]

we get

\[
F_\eta(\varepsilon) - F_0(0) =
\]

\[
= \int_0^{\sqrt{\tau}} \int_0^{\sqrt{\tau}} [f(\varepsilon), g(\varepsilon)](\theta(q, \sigma, \sqrt{\tau}, \tau)) d\tau d\sigma + o(\varepsilon).
\]

5The notion of F-abundant set generalizes the control-theoretical concept of abundant set, which was introduced by J. Warga (13) and successively elaborated by B. Kaskosz (3).

6For any subset \( S \subseteq \mathbb{R}^n \), \( \overline{\text{conv}}(S) \) denotes the closure of the convex hull of \( S \).

7More precisely, the authors of (9) proved an exact integral formula, of which (8) is a straightforward consequence.
From this inequality, and observing that there exists a constant \( c > 0 \) (depending on the Lipschitz constant and \( L^n \) norm of \( f \) and \( g \)) such that
\[
|F_\eta(\varepsilon) - F(\varepsilon)| < c\eta(\varepsilon) \quad \forall \eta, \varepsilon > 0,
\]
we get
\[
F(\varepsilon) - F(0) = \varepsilon L(\varepsilon) + o(\varepsilon)
\]
where
\[
L(\varepsilon) := \frac{1}{\varepsilon} \int_0^{\sqrt{\varepsilon}} \int_0^{\sqrt{\varepsilon}} [f_\eta(\varepsilon), g_\eta(\varepsilon)](\theta(q, \sigma, \sqrt{\varepsilon}, \tau)) d\tau d\sigma.
\]
It is clear that \( L(\varepsilon) \) depends continuously on \( \varepsilon \) and that the same holds true for the map
\[
h(\varepsilon) := F(\varepsilon) - F(0) - \varepsilon L(\varepsilon) = o(\varepsilon).
\]
In view of Example 4.1, to prove that \([f, g]_{\text{set}}(q)\) is indeed a \( QDQ \) of \( F \) at \((0, q)\) in the direction of \( \mathbb{R}^+ \), it only remains to check that \( L(\varepsilon) \) has vanishing distance from it as \( \varepsilon \) approaches 0. To this aim, let us set \( D := \text{Diff}(f) \cap \text{Diff}(g) \), and, for every \( \varepsilon > 0 \), \( D^\varepsilon(\theta) := B_1(0) \cap \{ v : \theta + \eta(\varepsilon)v \in D \} \), and
\[
Dg_\eta(\varepsilon)(\theta) = \int_{D^\varepsilon(\theta)} \phi(v)Dg(\theta + \eta(\varepsilon)v) dv
\]
\[
Df_\eta(\varepsilon)(\theta) = \int_{D^\varepsilon(\theta)} \phi(v)Df(\theta + \eta(\varepsilon)v) dv.
\]
Therefore, by the definition of Lie bracket we get
\[
[f_\eta(\varepsilon), g_\eta(\varepsilon)](\theta) = \int_{D^\varepsilon(\theta)} \phi(v)[f, g](\theta) dv + E_{f,g}(\eta(\varepsilon))(\theta) \quad (9)
\]
where
\[
E_{f,g}(\eta(\varepsilon))(\theta) :=
\int_{D^\varepsilon(\theta)} \phi(v)Dg(\theta + \eta(\varepsilon)v)[f(\theta + \eta(\varepsilon)v) - f_\eta(\varepsilon)(\theta)] dv -
\int_{D^\varepsilon(\theta)} \phi(v)Df(\theta + \eta(\varepsilon)v)[g(\theta + \eta(\varepsilon)v) - g_\eta(\varepsilon)(\theta)] dv.
\]
If \( r > 0 \), it is easy to show that, if we choose a positive constant \( \varepsilon > 0 \) small enough, we have \( \theta(q, \sigma, \sqrt{\varepsilon}, \tau) \in B_r(q) \), for all \( \varepsilon < \varepsilon \). Using the Lipschitz continuity of \( f \) and \( g \) and the convergence of the mollified fields to the original fields, one can easily prove that \( |E_{f,g}(\eta)(\theta)| \leq C\eta \), for some constant \( C \) and for all \( \theta \in B_r(q) \).
Therefore, by (9) we get
\[
[f_\eta(\varepsilon), g_\eta(\varepsilon)](\theta(q, \sigma, \sqrt{\varepsilon}, \tau)) = \int_{D^\varepsilon(\theta(q, \sigma, \sqrt{\varepsilon}, \tau))} \phi(v)[f, g](\theta) dv \in \mathbb{C}(\{f, g\}(x), x \in D \cap B_r(q)). \quad (10)
\]
If we also take \( \varepsilon \) small enough so that \( \varepsilon < \varepsilon \) implies \( \eta = \eta(\varepsilon) < \frac{\varepsilon}{\varepsilon} \), we have proved that for any \( r > 0 \) there is \( \varepsilon \) small enough such that
\[
L(\varepsilon) =
\frac{1}{\varepsilon} \int_0^{\sqrt{\varepsilon}} \int_0^{\sqrt{\varepsilon}} \left( \int_{D^\varepsilon(\theta)} \phi(v)[f, g](\theta) dv + E_{f,g}(\eta)(\theta) \right) d\tau d\sigma,
\]
where we have written \( \theta \) instead of \( \theta(q, \sigma, \sqrt{\varepsilon}, \tau) \), has distance less than \( r \) from \( \mathbb{C}(\{f, g\}(x), x \in D \cap B_r(q)) \).