On mass polarization effect in three-body systems

I. Filikhin · R. Ya. Kezerashvili · V. M. Suslov · B. Vlahovic

Abstract We evaluate the mass polarization term of the kinetic-energy operator for different three-body nuclear AAB systems by employing the method of Faddeev equations in configuration space. For a three-boson system this term is determined by the difference of the doubled binding energy of the AB subsystem $2E_2$ and the three-body binding energy $E_3(V_{AA} = 0)$ when the interaction between the identical particles is omitted. In this case: $|E_3(V_{AA} = 0)| > 2 |E_2|$. In the case of a system complicated by isospins(spins), such as the kaonic clusters $K^-K^-p$ and $ppK^-$, the similar evaluation is impossible. For these systems it is found that $|E_3(V_{AA} = 0)| < 2 |E_2|$. A model with an $AB$ potential averaged over spin(isospin) variables transforms the later case to the first one. The mass polarization effect calculated within this model is essential for the kaonic clusters. Besides we have obtained the relation $|E_3| \leq |2E_2|$ for the binding energy of the kaonic clusters.

Keywords Mesic nuclei · Mass polarization · Faddeev equation · Nucleon-kaon interactions

1 Introduction

The mass polarization effect of the kinetic-energy operator is well known in atomic physics [1,2]. The kinetic energy operator in the Schrödinger equation for an $N$-electron atomic system with a finite nuclear mass $M$ in the centre-of-mass coordinate system is comprised of two parts: the kinetic energy term related to the introduction of the reduced mass and the mass polarization term (MPT) $-\frac{\hbar^2}{2M} \sum_{i<j} \nabla_i \cdot \nabla_j$,

where the indices $i$ and $j$ denote the $i$th and $j$th electrons, respectively. This term

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leads to the shift of atomic spectra and Hughes and Eckart [3] in 1930 were the ones who studied this effect. For decades, the mass polarization term has been treated differently in calculations: in the approximation of an infinitely heavy nucleus and using perturbation theory. In the approximation of an infinitely heavy nucleus the contribution of this term is zero, while the evaluation of this term within perturbation theory proved to be unreliable, as is pointed out in [4]. However, the contribution of the mass polarization term in atomic physics is always considered as a small correction due to large mass of the core nucleus [5,6]. For description of charged excitons and biexcitons in condensed matter physics the contribution of the mass polarization term cannot be ignored due to the comparable masses of electrons and holes and requires its careful consideration [7,8,9]. In nuclear few-body physics such correction can be essential. In particular, the mass polarization term of the three-body kinetic-energy operator can play an important role in the study of nuclear interactions in double hypernuclei [10,11] like the $^6\Lambda\Lambda\alpha$ considered within the three-body cluster model $\Lambda\Lambda\alpha$. If we write the Schrödinger equation for a three-body $AAB$ system using the non Jacobian coordinate set and neglect the MPT interaction between two identical particles, we obtain the trivial solution that binding energy is $2E_2$, where $E_2$ is the two-body $AB$ energy. The consideration of the MPT shifts the energy by adding the mass polarization energy. For a three-boson $AAB$ system, this contribution can be evaluated as [11]

$$\delta B = 2E_2 - E_3(V_{AA} = 0),$$

where $E_3(V_{AA} = 0)$ is the three-body energy of the $AAB$ system when interaction between two identical particles is omitted. Note that the contribution (1) is small for the $\Lambda\Lambda\alpha$ system due to the $B$-particle mass factor dependence which is expressed as $m_A/m_B$, where $m_A$ and $m_B$ are masses of non identical particles, and $m_B > m_A$. This mass ratio is approximately equal to 1/4 for the $\Lambda\Lambda\alpha$ system. When $m_B >> m_A$, the contribution of the term can be neglected [11].

Consideration of the mass polarization term is very important for the three-body $AAB$ system when mass ratio for non identical particles is not small, for example, for the kaonic cluster $K^-K^-p$ and $ppK^-$, the mass ratio of the kaon and nucleon is about 1/2. However, there are examples [12,13] in the literature when this term is ignored within a theoretical analysis of the kaonic clusters by proposing that $E_2 = E_3(V_{AA} = 0)/2$. In the presented work we focus on different nuclear $AAB$ systems involving two identical and one distinguishable particle to evaluate the mass polarization term of the kinetic-energy operator. We distinguish bosonic-like systems from systems having isospins(spins) dependent interactions. In the case of a system complicated by isospins(spins), such as the kaonic clusters $K^-K^-p$ and $ppK^-$, the evaluation (1) is impossible. For these systems it is found that $|E_3(V_{AA} = 0)| < 2|E_2|$, which gives a handy lower bound of $2E_2$ to the $E_3$ (see also [14] in this regard). For this case, the approach with average $AB$ potential may be applied to reduce it to a bosonic-like system and the mass polarization can be roughly evaluated by using Eq. (1). Our treatment is based on the Faddeev equations in configuration space. These equations allow us to separate components of the total wave function corresponding to the different particle rearrangements and to show the effects related to the exchange of identical particles and the difference of particle masses. The latter facts are hidden in each Faddeev component that corresponds to the interaction of any two particles in the presence of the third.
The paper is organized in the following way. In Sec 2 we present the formalism of the Faddeev equations in configuration space for a three-body system with two identical particles. We consider two cases, when the identical particles are fermions or bosons. The Faddeev equations are written for the cases of two identical bosons and two identical fermions in the s-wave approach and we consider the corresponding spin-isospin configurations, as well as an average potential approach. The analysis of the mass polarization energy for a three-boson system within the s-wave Faddeev approach is given in Sec. 3. The explanation for the mass polarization term and mass polarization effect is presented in Sec. 4 based on the Schrödinger equation for bosonic-like systems. The results of numerical evaluations for the double $\Lambda$-hypernucleus $^6\Lambda\Lambda$He, the kaonic clusters $K^-K^-p$ and $ppK^-$, and nucleus $^3H$ are presented and discussed in Sec. 5. The conclusions follow in Sec. 6.

2 Formalism

2.1 Faddeev equations in configuration space

The wave function of the three-body system can be obtained by solving the Schrödinger equation. Alternatively, in the Faddeev method the total wave function is decomposed into three components:

$$\Psi = \Phi_1 + \Phi_2 + \Phi_3.$$

The Faddeev components $\Phi_i$ correspond to the separation of particles into configurations $i + (kl)$, $i \neq k \neq l = 1, 2, 3$. Each Faddeev component $\Phi_i(x_i, y_i)$ depends on its own set of the Jacobi coordinates $x_i$ and $y_i$. The components satisfy the Faddeev equations in the coordinate representation written in the form:

$$\left( H^i_0 + v_i(x_i) - E \right) \Phi_i(x_i, y_i) = -v_i(x_i) (\Phi_k(x_k, y_k) + \Phi_l(x_l, y_l)), \quad i \neq k \neq l = 1, 2, 3, \quad (2)$$

where $H^i_0 = -(\Delta x_i + \Delta y_i)$ is the kinetic energy operator and $v_i$ is the potential acting between the particles $(kl)$, $i \neq k \neq l$. We refer to (2) as the differential Faddeev equations (DFE). The mass scaled Jacobi coordinates $x_i$ and $y_i$ are expressed in terms of the particle coordinates $r_i$ and masses $m_i$ as:

$$x_i = \sqrt{\frac{2m_i m_k}{m_i + m_k}} (r_k - r_i), \quad y_i = \sqrt{\frac{2m_i (m_k + m_l)}{M}} \left( \frac{m_k r_k + m_l r_l}{m_k + m_l} - r_i \right), \quad M = m_i + m_k + m_l. \quad (3)$$

The orthogonal transformation between three different sets of the Jacobi coordinates has the form:

$$\begin{pmatrix} x_i \\ y_i \end{pmatrix} = \begin{pmatrix} C_{ik} & S_{ik} \\ -S_{ik} & C_{ik} \end{pmatrix} \begin{pmatrix} x_k \\ y_k \end{pmatrix}, \quad C^2_{ik} + S^2_{ik} = 1, \quad (4)$$

where

$$C_{ik} = -\frac{m_i m_k}{\sqrt{(M - m_i)(M - m_k)}}, \quad S_{ik} = (-1)^{k-i} \text{sign}(k-i) \sqrt{1 - C^2_{ik}}.$$
2.2 Faddeev equations for $AAB$ system

The objective of this work is a consideration of a three-body $AAB$ system with two identical particles. Particularly, we focus on the kaonic clusters $ppK^-$ and $K^-K^-p$, $^3\text{He}$ nucleus and the double $\Lambda$-hypernucleus $\Lambda\Lambda^\Lambda\text{He}$ in the framework of the $\Lambda\Lambda\Lambda$ cluster model. Therefore, let us rewrite the system (2) for a case of two identical particles. In this case the total wave function of the system is decomposed into the sum of the Faddeev components $U$ and $W$ corresponding to the $(AA)B$ and $A(AB)$ types of rearrangements: $\Psi = U + W \pm PW$, where $P$ is the permutation operator for two identical particles. These types of the particle rearrangements and corresponding Jacobi coordinates are graphically presented in Fig. 1. In the latter expression for $\Psi$, the sign "+" corresponds to two identical bosons, while the sign "−" corresponds to two identical fermions, respectively. For a three–body system with two identical particles the set of the Faddeev equations (2) is reduced to the system of two equations for the components $U$ and $W$ [18,19]:

\begin{align}
(H_0^U + V_{AA} - E)U &= -V_{AA}(W \pm PW), \\
(H_0^W + V_{AB} - E)W &= -V_{AB}(U \pm PW),
\end{align}

(5)

where the signs "+" and "−" correspond to two identical bosons and fermions, respectively. The wave function of the system $AAB$ is symmetrized with respect to two identical bosons, while it is antisymmetrized with respect to two identical fermions. The partial wave analysis of the DFE (5) can be performed by the $LS$ coupling scheme given in [17,18,20]. The $LS$ basis allows us to restrict the model space to the states with the total angular momentum $L = 0$.

2.3 Separation of spin(isospin) variables

The description of the aforementioned $AAB$ systems differs for the type of $AA$ and $AB$ interactions. Without losing any generality and for the simplicity of the presentation we employ the $s$-wave spin-isospin dependent $V_{AA}$ and $V_{AB}$ potentials. This requires to write the DFE in the $s$-wave approach and consider the corresponding spin-isospin configurations. The separation of spin(isospin) variables leads to the Faddeev equations for the three-body system $AAB$ in the following form:

\begin{align}
(H_0^U + V_{AA} - E)U &= -V_{AA}D(1 + p)W, \\
(H_0^W + V_{AB} - E)W &= -V_{AB}(D^T U + GpW),
\end{align}

(6)
where matrices $D$ and $G$ are defined by the nuclear system under consideration, the $W$ is a column matrix with the singlet and triplet parts of the $W$ component of the wave function of a nuclear system, and the exchange operator $p$ acts on the particles’ coordinates only.

Let us mentioned that the consideration of the spin and isospin dependence is relevant for the $AB$ potentials in the neutron-proton and proton-kaon cases for the $^3\text{H}$ nucleus and kaonic clusters, respectively. In the first case the potential and components of $W$ are labeled according to the pair spin. In the latter case the potential is isospin dependent, but both channels have total spin 1/2. Furthermore, spin/isospin dependence is irrelevant to the $AA$ potential which is assumed as $s$-wave interaction that is a spin-singlet $nn$ or $pp$ potentials or an isospin-triplet $\bar{K}\bar{K}$ potential.

For the $^3\text{H}$ nucleus, considered as the $nnp$ system, the inputs into (6) are the following: the spin singlet $nn$ potential $V_{AA} = v_{nn}^s$ and $V_{AB} = \text{diag}\{v_{np}^s, v_{np}^t\}$ that is a diagonal $2 \times 2$ matrix with the spin singlet $v_{np}^s$ and triplet $v_{np}^t$ potentials, respectively, and

$$D = (-\frac{\sqrt{3}}{2}, \frac{1}{2}), \quad G = \begin{pmatrix} \frac{1}{2} - \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}, \quad W = \begin{pmatrix} W^s \\ W^t \end{pmatrix}, \quad \text{(7)}$$

where $W^s$ and $W^t$ are the spin singlet and spin triplet components of the $W$. For a neutron-proton interaction, we use the semi-realistic Malfliet and Tjon MT I-III potential with the correction [22]. It has to be noted that we do not use isospin formalism for the $nnp$ system. Thus, the protons and neutrons are not identical. The details of such treatment are presented in [23].

For $K^-K^-p$ and $ppK^-$, despite of the fact that there are two identical bosons and two identical fermions, respectively, due to symmetry of the spin-isospin configurations in the kaonic clusters, the $D$ and $G$ matrices in (6) for these clusters are the same and have the following form [24]:

$$D = (-\frac{\sqrt{3}}{2}, \frac{1}{2}), \quad G = \begin{pmatrix} \frac{1}{2} - \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}, \quad W = \begin{pmatrix} W^s \\ W^t \end{pmatrix}. \quad \text{(8)}$$

Unlike to (7) the corresponding superscripts $s$ and $t$ in (8) denote the isospin singlet $W^s$ and isospin triplet $W^t$ components of the $W$. For the $K^-K^-p$ kaonic cluster, $V_{AA} = v_{K\bar{K}}^t$ is the $\bar{K}\bar{K}$ potential in the triplet isospin state. For the $ppK^-$ cluster, $V_{AA} = v_{NN}^s$ is the $NN$ potential in the singlet spin state. For both systems, one has to take $V_{AB} = \text{diag}(v_{K\bar{K}}^t, v_{NN}^s)$. In the presented work, we used the $s$-wave Akaishi-Yamazaki (AY) [25] and Hyodo-Weise (HW) [26,27] effective potentials for $\bar{K}\bar{K}$ and $\bar{KN}$ interactions which include the coupled-channel dynamics into a single channel $\bar{K}N$ interaction. The graphical representation of the isospin configurations in $\bar{K}\bar{K}N$ and $NNK$ systems is given in Fig. 2.

The double-$\Lambda$ hypernucleus $^6\Lambda\Lambda \text{He}$ is treated within the potential three-body cluster $\alpha \Lambda \Lambda$ model using the frozen core approximation and thus, effects of core excitations are lacking. For this case, one has

$$D = G = 1, \quad W = W \quad \text{(9)}$$

in (6) and latter could be reduced to a scalar form [19]. For the $\alpha \Lambda \Lambda$ calculations, we use modified Tang-Henrdon (TH(M)) potential from [29] for the $\Lambda\alpha$ interaction.
The isospin configurations in $\bar{K}K$ system. The isospin configurations for $NN\bar{K}$ system can be obtained by replacing $N \leftrightarrow \bar{K}$. The pair potential $\bar{K}N$ has singlet and triplet isospin component ($I = 0$ or $I = 1$). The $\bar{K}\bar{K}$ and $NN$ potentials are presented by triplet components. The exchange of identical particles shown by arrows. The $\bar{K}N$ pair has a deep bound isospin singlet state.

2.4 Average potentials

In this section, we consider the case when the $AB$ potentials are spin(isospin) dependent. For example, in the $NN\bar{K}$ system, the $\bar{K}$ meson combines two nucleons into the bound state for two $\bar{K}N$ isospin configurations which are energetically favorable. The effective $\bar{K}N$ interactions have a strong attraction in the singlet $I = 0$ channel and a weak attraction in the triplet $I = 1$ channel. The $\bar{K}N$ pair is bound in the singlet state with the energy corresponding to one of the $\Lambda(1405)$ resonance. There is no a bound state in the triplet isospin state.

Below following [20,28], we consider the effective potential obtained by averaging of the initial potential over the spin variables, in the case of the $^3\text{H}$ nucleus, and the isospin variables, in case of kaonic clusters. The isospin averaged potential $V_{\bar{K}N}^{av}$ is defined as:

$$V_{\bar{K}N}^{av} = \frac{3}{4}v_{I=0}^{\bar{K}N} + \frac{1}{4}v_{I=1}^{\bar{K}N}.$$  \hspace{1cm} (10)

This potential has a moderate attraction in comparison with the strong attraction in the $I = 0$ channel. Note that this simplification changes the two-body threshold, which is not related to the $K^-p$ bound state as $\Lambda(1405)$.

Using the isospin (spin) averaging, Eqs. (6) can be reduced to the scalar form by an algebraic transformation. Taking into account that $W = DW, V^{av}_{AB} = DV_{AB}D^T$ and $DD^T = 1, DV_{AB}GD^T = V^{av}_{AB}$ we obtain

$$(H^U_0 + V_{AA} - E)U = -V_{AA}(1 + p)W,$$

$$(H^W_0 + V^{av}_{AB} - E)W = -V^{av}_{AB}(U + pW).$$ \hspace{1cm} (11)

In this case, one can evaluate the mass polarization term $[\bar{1}]$ as

$$\delta B^{av} = 2E_2^{av} - E_3^{av}(V_{AA} = 0).$$ \hspace{1cm} (12)

Here, $E_2^{av}$ is the two-body energy for the $AB$ pair with the averaged potential and $E_3^{av}(V_{AA} = 0)$ is the three-body energy with the averaged potential when the $AA$ interaction is omitted. The three-body $E_3^{av}(V_{AA} = 0)$ energies are calculated by using Eq. (11) with the averaged potential. It has to be noted that the two-body energies $E_2^{av}$ and $E_2$ are different due to the difference of the averaged and singlet $\bar{K}N$ potentials.
The averaged potentials for the \( np \) (averaged in the spin space) is constructed by the same way as in [20], while for the \( KKN \) systems (averaged in the isospin space) it is defined by [19].

3 When \( E_3 = 2E_2 \)

Let us consider the \( s \)-wave approach for the Faddeev equations [11] for the \( AAB \) system when particles \( A \) and \( B \) are interacting with the \( V_{AB} \) potential. We assume that the interaction between two identical particles is omitted, therefore the potential \( V_{AA} = 0 \). The \( s \)-wave DFE is reduced toa single equation for the Faddeev component \( W(x, y) \):

\[
(-\frac{\hbar^2}{2\mu} \partial_x^2 - \frac{\hbar^2}{2\nu} \partial_y^2 + V_{AB}(x) - E)W(x, y) = -V_{AB}(x) \int_{-1}^{1} \frac{1}{2} dy \frac{x'y}{x'y} W(x', y'), \tag{13}
\]

where \( x, y \) are the Jacobi coordinates and \( u=\cos(\hat{xy}) \), where \( \hat{xy} \) is the angle between \( x \) and \( y \). The identical particles in the system are labelled as 2 and 3 and \( m_2 = m_3 = m \), while the \( m_1 \) is the mass of the \( B \) particle. The appropriate transformation of coordinates and reduced masses are given by the following expressions:

\[
x' = \left(\left(\frac{m_1}{m_2} x\right)^2 + y^2 - 2 \frac{m_1}{m_2} xy\right)^{1/2},
\]

\[
y' = \frac{\nu}{m} \left(\left(\frac{m_2}{m_1} x\right)^2 + y^2 + 2 \frac{m_2}{m_1} xy\right)^{1/2},
\]

\[
\nu = \frac{m(m_1 + m)}{m_1 + 2m}, \quad \mu = \frac{m_1 m}{m_1 + m}.
\]

Two types of configurations for the particles in the \( AAB \) system and Jacobi coordinates were presented in Fig. 1. The configuration shown in Fig. 1B corresponds to one described by Eq. (13). The configuration shown in Fig. 1A corresponds to one described by Eq. (13).

The analysis of expressions (14) shows that when \( m_1 \gg m \), one can write \( x' \approx y, y' \approx x \) and \( \nu \approx \mu \). Within this approximation, after the integration by the variable \( u \), Eq. (13) can be rewritten in the following symmetric form:

\[
(-\frac{\hbar^2}{2\mu} \partial_y^2 - \frac{\hbar^2}{2\mu} \partial_x^2 + V_{AB}(x) + V_{AB}(y) - E)W(x, y) =
\]

\[
= V_{AB}(y)W(x, y) - V_{AB}(x)W(y, x). \tag{15}
\]

By averaging both sides of (15) one obtains on the right hand side the integral \( \int dx dy (W(x, y)V_{AB}(y)W(x, y) - W(x, y)V_{AB}(x)W(y, x)) \). If the function \( W(x, y) \) yields factorization and \( W(x, y) = W(y, x) \), then variables are separated and the values of this integral is equal to zero, due to the symmetry related to the replacement of variables \( x \rightarrow y, y \rightarrow x \). In this case, the function \( W(x, y) \) is factorized as \( W(x, y) = \phi(x)\phi(y) \). Taking into account that \((-\frac{\hbar^2}{2\mu} \partial_x^2 + V_{AB}(x))\phi(x) = E_2\phi(x) \), one obtains the relation \( E_3(V_{AA} = 0) = 2E_2 \) for the three-body ground state energy. Let us note that the relation \( E_3^{av}(V_{AA} = 0) = 2E_2^{av} \) is valid for the spin(isospin) averaged potential as well.

One can separate two effects related to the mass polarization term. The first is the exchange effect which is related to the right hand side of (15) and includes
the permutation operator of the identical particles (see (5)). In the limit of large
B particle mass the above mentioned integral is equal to zero. In an opposite case,
the latter integral violates the relation $E_3 = 2E_2$. At the same time, the second
effect is related to the difference of the reduced masses $\mu$ and $\nu$. This difference
violates $x$-$y$ symmetry on the left hand side of (15).

4 Mass polarization term and mass polarization effect

To better understand the effect induced by the different masses of $A$ and $B$ particles
in a three-body $AAB$ system one can use the non Jacobian form of the Schrödinger
equation from [11] that is written in a self-explanatory notation to analyze the
contribution of the mass polarization term:

$$\left(-\frac{\hbar^2}{2m_A}\nabla^2 r_{A_1} - \frac{\hbar^2}{2m_B}\nabla^2 r_{A_2} - \frac{\hbar^2}{m_B}\nabla r_{A_1}\nabla r_{A_2} + V_{AB}(r_{A_1}) + V_{AB}(r_{A_2}) - \bar{E}\right)\Psi(r_{A_1},r_{A_2}) = 0.$$ (16)

In the latter equation the third term is the mass polarization term, $T_{MPT} =
-\frac{\hbar^2}{m_B}\nabla r_{A_1}\nabla r_{A_2}$, the interaction between two identical particles is omitted, $V_{AA} = 0$, and $E \equiv E_3(V_{AA} = 0)$ corresponds to the binding energy of the $AAB$ system
when the interaction between two identical particles is neglected. The mass of each
particle $m_A$, $m_B$ is always bigger than the reduced mass $\mu m_B > m_A > \mu$ and the
reduced mass is always more close to the mass of the lightest particle. In the case
$m_A > m_B$ the contribution of the mass polarization term can be the same order
as the contribution of the other two differential operators in Eq. (16) due to the
comparable mass factors of these operators which are approximately $1/m_B$. In the
case $m_B > m_A$ the contribution of this term has the factor $1/m_B$, while the mass
factors of the other differential operators are about $1/m_A$. When $m_B >> m_A$
the contribution of the mass polarization term can be neglected.

Within the first order of perturbation theory, when $<T_{MPT}> << |E_3(V_{AA} = 0)|$ (case $m_B > m_A$) the initial wave function is factorized as

$$\Psi(r_{A_1},r_{A_2}) = \phi(r_{A_1})\phi(r_{A_2}),$$

where $\phi(r_{A_1})$ ($\phi(r_{A_2})$) is a solution of two-body Schrödinger equation for the $AB$
subsystem. Averaging of Eq. (16) leads to the following relation between $E_2$ and
$E_3(V_{AA} = 0)$:

$$<T_{MPT}> = 2E_2 - E_3(V_{AA} = 0).$$ (17)

We have obtained the evaluation for the mass polarization term when $m_B > m_A$.
For the simplest case, when the MPT is ignored in Eq. (16), $<T_{MPT}> = 0$, we
have $E_3(V_{AA} = 0, without MPT) = 2E_2$. For the general case, the effect of the mass
polarization term can be estimated as

$$E_3(V_{AA} = 0, without MPT) - E_3(V_{AA} = 0).$$

This estimation is valid for any mass ratio $m_B/m_A$. The relation (17) is known in
nuclear physics as the mass polarization effect [11] expressed as the deference of
$2E_2$ and $E_3(V_{AA} = 0)$ according Eq. (1). Therefore, the $\delta B$ in Eq. (1) is a direct
estimation of the MPT for bosonic-like systems, $\delta B = <T_{MPT}>$, when $m_B > m_A$. 

Let us mention that the MPT is not an artefact of not using Jacobi coordinates. In the system of reference presented in Eq. (16) this is a kinematical effect related to the presence of the third particle $A$ when the other one interact with the particle $B$. The presence of the third particle gives the redistribution of kinetic energy and as a result $AB$ subsystem is off the energy shell. It is well know that a physical result does not depend on the system of references. If one considers the $AAB$ using Jacobi coordinates by employing the Faddeev equations the latter fact is hidden in each Faddeev component that corresponds to the interaction of any two particles in the presence of the third.

5 Numerical Results

5.1 Bosonic-like system

Let us consider a three-boson system to exemplify the formalism presented above. When $m_1 > m$, the value $\delta B$ is mainly determined by the right hand side of Eq. (15). To illustrate this statement, we consider the $\Lambda^{6}\Lambda$He nucleus within the cluster model as a three-body $\Lambda\Lambda\alpha$ system and show a correlation between a type of $AB$ potential and the mass polarization term $\delta B$. We assume the frozen core approximation (there is no dynamical change of the core-nuclear structure) and the $s$-wave approach is based on Eq. (6) with the definitions (9). We note this model as "bosonic-like" due to a similarity of these equations to ones for a system of three bosons. In Eq. (1) $E_2$ is the ground state energy of the $\Lambda^{5}\Lambda$He nucleus within the two-body cluster model. $E_3(V_{\Lambda\Lambda} = 0)$ is the ground state energy of $\Lambda^{6}\Lambda$He nucleus in the framework of the three-body cluster model $\Lambda\Lambda\alpha$ when $V_{\Lambda\Lambda} = 0$. In previous calculations [10,11] it was determined that $2E_2 - E_3(V_{\Lambda\Lambda} = 0) \neq 0$ and the latter is related to the effect of the mass polarization term of the kinetic-energy operator $\delta B_{\Lambda\Lambda}$. Results of our calculation and calculations [19] for the $\Lambda\Lambda\alpha$ system are presented in Table 1. In our previous calculations [19] we consider the several $\Lambda\alpha$ potentials. These potentials have different shapes, while reproduce closely the experimental value of the binding energy for the $\Lambda^{5}\Lambda$He hypernucleus. The difference of the $\Lambda\alpha$ potentials can be clarified by indicating the corresponding $\Lambda\alpha$ scattering lengths. In particular, the scattering length characterizes the behavior of the potential of pair interaction at large distances. As example, in Table 1 the Tang-Herndon (TH) potential [30] is an attractive potential with no repulsive core. The Isle potential [34] has a weak repulsive core and decreases slowly at large distances. In Table 1 the potentials are arranged by increasing the values of the scattering length from 3.63 fm for the TH potential to 4.24 fm for the potential Isle, consequently. As follows from Table 1 the similar pattern appears for the $\delta B - E_2$ that is calculated for the same set of potentials. Thus, the mass polarization energy has the essential dependence on the type of $\Lambda\alpha$ interaction. However, there is no correlation between $\delta B$ and $E_2$ within the $\Lambda\Lambda\alpha$ cluster model as this follows from Table 1.

5.2 Kaonic systems

5.2.1 $\bar{K}\bar{K}N$ system
The ground state energy $E_3$ of the $\bar{K}KN$ system was calculated using the effective AY and HW potentials describing the $\bar{K}N$ and $\bar{K}\bar{K}$ interactions. These effective $\bar{K}N$ s-wave potentials implicitly include $\pi\Sigma$ coupling and are widely used for description of the few-body kaonic clusters. The numerical results are presented in Table 2. In Table 2 we compare our results with the results obtained in [28] within a variational approach with a Gaussian expansion method. The small discrepancies in calculations are related to the different $K$-meson mass used in the calculations. We used the value 493.677 MeV for the $K^{-}$ mass from [35], while in [28] the authors used the mass 495.7 MeV. The consideration of the same mass as in [28] changes $E_2$ energy for the AY potential from our value of -30.3 MeV to the value of -30.6 MeV [28]. However, $E_3$ does not become -32.3 MeV as in [28]. Results of our calculation are different from ones reported in [28] by 0.3 MeV for both versions of the $\bar{K}\bar{K}$ AY potentials: AY(104) and AY(70). To check the accuracy of our calculations, we compare our results for the $nnp$ system and one reported in [37]. For $E_3$ energy we obtained the value of -8.534 MeV that is very close to the value of -8.535 MeV [37], when the MT I-III nucleon-nucleon potential is used. Note that the computer codes for $E_3$ calculation are the same for the both $nnp$ and $\bar{K}KN$ systems with taking into consideration the exchange for potentials, masses, matrices $D$ and $G$ in (6).

One can see from Table 2 that the energy calculated under the condition $V_{\bar{K}K} = 0$ has the larger absolute value comparing with one obtained within the full potential model that includes all interactions between particles. This is possible due to the repulsive $K^-\bar{K}^-$ interaction for the both AY and HW potentials. The absolute value of the ground state energy $E_3$ is larger than one for the ground state energy $E_2$ of the $K^-p$ pair (singlet isospin state) for both $KN$ potentials and for the both cases $V_{\bar{K}K} = 0$ and $V_{\bar{K}K} \neq 0$ so that the relation $2E_2 - E_3 < 0$ is satisfied.

The model with averaged potential demonstrates the opposite relation between $E_3(V_{\bar{K}K} = 0)$ and $E_2$: $2E_2 - E_3(V_{\bar{K}K} = 0) > 0$. The isospin components of the wave function present in (6) with different coefficients due to the non-trivial matrix $G$ and $D$ ("isospin complication"). Within the averaged potential approach, both kaons interact with the proton by the same average potential. There is a similar case when the singlet potential is equal to the triplet $\bar{K}N$ potential. Eq. (6) is reduced also to one with the trivial $G$ and $D$ matrices.

5.2.2 $NN\bar{K}$ system:
On mass polarization effect in three-body systems

Table 2  The ground state energies: $E_2$ for the $K^{-}p$, and $E_3$ for the $KKN$ system with the AY and HW potentials for the $\bar{K}K$ and $\bar{K}N$ interactions. The two-body $E_2^{ad}$ and three-body $E_3^{ad}$ energies are presented for the averaged potential $\frac{1}{2}v_{KN} + \frac{1}{2}v_{KN}$. The mass polarization term of the three-body kinetic-energy operator is evaluated by applying (12). $m_K$ is the $K^{-}\pi$ meson mass used for the calculations. The averaged nucleon mass of 938.9 MeV is used as the input for the proton mass. All entries are given in MeV.

| $KK$, $KN$ | $E_2$ | $E_3$ | $E_2^{ad}$ | $E_3^{ad}$ | $\delta B^{av}$ |
|------------|-------|-------|------------|------------|--------------|
| $m_K=493.66$ | AY , AY | -30.3 | -31.66 | -9.63 | -16.1 |
| $V_{K\bar{K}} = 0$, AY | | -35.18 | -21.7 | 2.44 |
| HW , HW | -11.16 | -b | -1.11 | -1.62 |
| $V_{K\bar{K}} = 0$, HW | | -12.18 | -3.07 | 0.85 |
| $m_K=495.7$ | AY , AY | -30.6 | -32.0 | -35.6 |
| $V_{K\bar{K}} = 0$, AY | | -11.42 | -b | -12.5 |
| HW , HW | -11.40 | -36.0 |
| $V_{K\bar{K}} = 0$, HW | | -12.6 |

*R. Ya. Kezerashvili, S. M. Tsiklauri, I. N. Filikhin, V. M. Suslov, and B. Vlahovic, ppK$^-$ and K$^-\bar{K}$–p Clusters, The 21st International Conference on Few-Body Problems in Physics (FB21), Chicago, Illinois, USA, May 18-22, 2015.

Table 3  Ground state energies $E_3$ of the $NN\bar{K}$ system (in MeV) with the AY and HW potentials for the $\bar{K}N$ interaction and T (the Tamagaki potential)[25] and simulating AV18 potential (sAV18)[38] for the singlet $NN$ interaction.

| $NN$, $\bar{K}N$ | $E_3$ | Ref. [38] | Ref. [25] |
|------------------|-------|-----------|-----------|
| T                | AY    | -46.35    | -39.0     | -48       |
| sAV18            | AY    | -46.21    | -45.8$^a$ | -            |
| sAV18            | HW    | -20.57    | -20.3$^b$ | -            |

$^a$ Energy dependent $\bar{K}N$ potential defined by the ansatz $M_N + m_K - B_K/2$.

$^b$ Different chiral $\bar{K}N$ potentials.

In Table 3 we compare our results for the $NN\bar{K}$ system with the variational calculations from [38] and [25]. The results are in acceptable agreement taking into account the difference of the models and methods. The details of discrepancy for the ground state energies -39 MeV and -48 MeV presented in Table 3 are discussed in [38].

According to [38], each spin(isospin) configuration of the system is represented by the corresponding Faddeev component of the wave function. The relative contributions of the isospin Faddeev components to the total wave function of the $NN\bar{K}$ system can be seen from Fig. 3. The model [24] with the AY and MT I-III potentials is used for the calculations. The first configuration $N(N\bar{K})$ with the singlet isospin state of the pair $(\bar{K}N)$ dominates with the maximum value of the $W^s$ component about 1. The area of rearrangement for the wave function of the $(\bar{K}N)N$ singlet configuration, shown in Fig. 3a, is visibly restricted along the coordinate $x$ (the distance between $\bar{K}$ and $N$) due to the existence of the strong bound $\bar{K}N$ state. At the same time, the function is quite prolongate along the $y$
axis due to the relatively weak bound of the third particle (N) with the pair $KN$ having the energy $|E_3 - E_2| < |E_2|$. The numerical solution reported in [24] gives $E_3 = -46.01$ MeV with $E_2 = -30.26$ MeV.

![Fig. 3](image)

**Fig. 3** Contour plots of the s-wave Faddeev components in the kaonic $NN\bar{K}$ system for the configurations of subsystems: a) $(\bar{K}N)N$ with the singlet isospin state of the pair $(\bar{K}N)$; b) $(\bar{K}N)N$ with the triplet isospin state of the pair $(\bar{K}N)$; c) $(NN)\bar{K}$.

The second configuration $(\bar{K}N)N$, shown in Fig. 3b, with the triplet isospin state of the $\bar{K}N$ pair is suppressed by the factor of 10 and the maximum of this component does not exceed the value of 0.1. The maximum of the $W^t$ component of the $(NN)\bar{K}$ configuration presented in Fig. 3c is relatively larger than the maximum of the component presented in Fig. 3b. The similar picture was obtained for the $K^-K^-p$ wave function. The similarity is due to the use of the same Eq. (13) for the description of the both $K^-K^-p$ and $ppK^-$ clusters.

The small contribution of the triplet isospin $W^t$ component for the $(\bar{K}N)N$ $(\bar{K}(NN)\bar{K})$ configuration to the total wave function, as is illustrated in Fig. 3, motivated us to evaluate $E_3$ by neglecting the triplet component of $\bar{K}N$ potential along with the nucleon-nucleon potential and, thus, keeping the singlet $\bar{K}N$ potential only as non-zero one. In the case of the $K^-K^-p$ cluster, the potentials of the triplet $\bar{K}N$ state and $\bar{K}K$ pair are zero. As a result, the singlet $W^s$ component only is a non trivial in Eq. (13). The results of calculations are presented in Table 4. Analysis of the results shows that $K^-K^-p$ and $ppK^-$ clusters are still bound by the singlet component of the $\bar{K}N$ potential when the triplet component of the potential is equal to zero, and the $KK(I = 1)$ and $NN$ potentials are equal to zero, respectively for the corresponding cluster. This is due to the domination of the singlet component of the $\bar{K}N$ potential over the triplet one. The results shown in Table 4 support the relation $2E_2 - E_3 < 0$ for the both cases: $V_{NN} = 0$ and $V_{NN} \neq 0$.

For description of the $NN\bar{K}$ cluster we take into account the contribution of the s-wave of $NN$ potential. When the spin state of the two protons is restricted to $S = 0$, the orbital momentum of the $NN$ pair is $l = 0, 2, 4$. As is shown in Ref. [24], the contributions of the higher orbital are small enough and the s-wave consideration is reasonable.

5.2.3 The relation $|E_3| \leq |2E_2|$
Table 4  Ground state energies $E_3$ of the $\bar{K}K$ and $NN\bar{K}$ clusters (in MeV) with the $A\bar{Y}$ potential for the $KN$ interaction and the $MT$ I-III potential for the $NN$ interaction. Results for the HW potential are given in parentheses. The difference of the two-body $2E_2$ and three-body $E_3$ energies are presented.

| System | Potentials | $E_3$ | $2E_2 - E_3$ |
|--------|------------|-------|--------------|
| $K\bar{K}$ | $V_{\bar{K}K} = 0, v_{\bar{K}N} = 0$ | -31.62 | -28.90 |
| | $V_{\bar{K}K} = 0, A\bar{Y}$ | -35.18 | -25.34 |
| $NN\bar{K}$ | $v_{NN} = 0, v_{\bar{N}K} = 0$ | -36.15 (-12.46) | -24.37 (-9.86) |
| | $v_{NN} = 0, A\bar{Y}$ | -42.94 (-17.11) | -17.58 (-5.21) |
| | $MT$ I-III, $v_{\bar{N}K} = 0$ | -41.47 (-17.08) | -19.15 (-5.24) |
| | $MT$ I-III, $A\bar{Y}$ | -46.01 (-20.46) | -14.51 (-1.86) |

For the both kaonic clusters, we found that $2E_2 - E_3(V_{AA} = 0) < 0$ (see Table 4). This is due to the difference of the strengths of the $\bar{K}N$ potential components for the $I = 0$ and $I = 1$. When $v_{\bar{K}N}^e = v_{\bar{K}N}^t = v_{AB}$, Eq. (6) reduced to the "scalar" form [11] in the same way as using the $D$-matrix transformation and taking into account that $V_{AB} = v_{AB}I$, where $I$ is the identity matrix. In this case the definition of the averaged potential is not necessary and the relation $2E_2 - E_3(V_{AA} = 0) > 0$ (the same as for the $\Lambda\Lambda\alpha$ case) will be automatically satisfied.

We have to note here, that the value of $2E_2 - E_3(V_{AA} = 0)$ has exact physical interpretation for the spin/isospin averaged approach. The value estimates the mass polarization term of the kinetic operator by Eq. (12). For general case of the spin/isospin dependent systems, this value is negative. It is important to mention that for the binding energy of the kaonic cluster $NN\bar{K}$ ($K\bar{K}N$) $|E_3| \leq |2E_2|$ due to the weakly attractive (repulsive) $AA$ potential. In particularly, $|E_3|$ somewhat is increased relatively the value of $|E_3(V_{NN} = 0)|$ by the attracting $NN$ force and with account of other possible physical channels [39].

5.3 Comparison of mass polarization effect for several $AAB$ systems

To illustarate a correlation between the mass ratio $m_A/m_B$ and the contribution of the mass polarization term to the three-body ground energy, we have calculated the ground state energy for different systems under the condition $V_{AA} = 0$. The considered systems are $K^-K^-p$, $ppK^-$, $^3H$ and $^6\Lambda\Lambda\text{He}$. The models with the averaged potentials are used for the calculation. The mass ratios $m_A/m_B$ are rounded to fractions and the corresponding mass ratio $m_A/m_B$ varies in the range from $1/4$ to 2. In Table 5 we have presented the results of calculations for the two-body $E_2^u$ and three-body $E_3^u$ ground state energies. The mass polarization term of the three-body kinetic-energy operator is evaluated as $\delta B^{av}$ according to Eq. (12). When $m_A < m_B$ there is a strong correlation of the $-\delta B^{av}/E_3^u$ with $m_A/m_B$. The value of $-\delta B^{av}/E_3^u$ increases when the ratio $m_A/m_B$ increases from $\approx 1/4$ to $\approx 1$. The case when $m_A > m_B$ takes place for the $ppK^-$ cluster. The analysis of the results in Table 5 leads to the conclusion that the relative contribution of $\delta B^{av}$ into $E_3$ energy is larger than one for the $K^-K^-p$ cluster.
Table 5 The two-body $E_{2}^{av}$ and three-body $E_{3}^{av}$ ground state energies (in MeV) are presented for different systems with averaged potentials (excluding the $ΛΛα$ system, as a system described by "scalar" equation). The mass polarization term of the three-body kinetic-energy operator $δB^{av}$ is evaluated in MeV using (12). The mass ratio $m_A/m_B$ is rounded to a fraction.

| System                  | $m_A/m_B$ | $E_{2}^{av}$ | $E_{3}^{av}$ | $δB^{av}$ | $−δB^{av}/E_{3}^{av}$ |
|-------------------------|-----------|--------------|--------------|-----------|-----------------------|
| $nnp$ ($V_{NN} = 0$)    | $1$       | $−1.53$      | $−3.87$      | $0.81$    | $21\%$               |
| $K^−K^−p$ ($V_{KK} = 0$) | $1/2$    | $−9.63$      | $−21.7$      | $2.44$    | $11\%$               |
| $ΛΛα$ ($V_{ΛΛ} = 0$)   | $1/4$     | $−3.12$      | $−6.49$      | $0.25$    | $4.3\%$              |
| $ppK^−$ ($V_{NN} = 0$) | $2/1$     | $−9.63$      | $−24.7$      | $5.44$    | $22\%$               |

6 Conclusions

In this paper, we have considered several three-body $AAB$ systems with two identical particles. In the case of the systems described by the scalar form of Eq. (6), the mass ratio $m_A/m_B$ correlates clearly with the mass polarization term for the three-body energy. This contribution also weakly depends on the $AB$ potential and correlates with the $AB$ scattering length. We have shown that the additional energy related to the mass polarization term is exactly estimated using Eq. (1) for any mass ratio $m_A/m_B$.

The relation (1) cannot be satisfied for a three-body system $AAB$ with a spin(isospin) dependent $AB$ interaction, such as the kaonic clusters $KKN$ and $NNK$. The "isospin complication" leads to the following evaluation for three-body ground state energy of the kaonic clusters: $|E_{2}| < |E_{3}(V_{AA} = 0)| < |2E_{2}|$. The relation gives the upper value which can be reached by using isospin formalism. For the $ppK^−$ cluster, $|E_{3}|$ is slightly larger then $|E_{3}(V_{NN} = 0)|$ due to the weakly attracting $NN$ force and taking into account other possible physical channels. However, $|E_{3}|$ is less than $|2E_{2}|$ and is essentially less than the experimentally motivated value about 100 MeV [40,41].

For kaonic clusters, the configuration with the singlet isospin state of the pair $KN$ dominates. This makes possible that the $K^−K^−p$ and $ppK^−$ systems can be bound in the cases when the pair $KK(I = 1)$ and triplet $KN$ potentials are equal to zero ($K^−K^−p$ cluster) and when the $NN$ and triplet $KN$ potentials are equal to zero ($ppK^−$ cluster).

The mass polarization effect for the kaonic clusters evaluated using the averaged potential approach is essential. It has to be taken into account when one attempts to construct a two body $KN$ potential using a relation between two- and three-body binding energies within the "particle picture" approach [12].

Finally it should be mentioned that we present the calculations for the kaonic system in the framework of the approximation with the effective complex $KN$ potential. The complete treatment of the $KNN$ system should be done within a coupled channel approach that explicitly includes effects due to the $πΣ$ coupling. Such consideration can lead to a possible modification of the results of our calculations. However, this will not change the qualitative conclusion that follows from our approach with the effective $KN$ potential which implicitly includes the $πΣ$ coupling.
Acknowledgements This work is supported by the National Science Foundation grant Supplement to the NSF grant HRD-1345219 and NASA (NNX09AV07A). R.Ya. K. partially supported by MES RK, the grant 3106/GP4.

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