On Weighted Low-Rank Approximation
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Abstract

Our main interest is the low-rank approximation of a matrix in $\mathbb{R}^{m \times n}$ under a weighted Frobenius norm. This norm associates a weight to each of the $(m \times n)$ matrix entries. We conjecture that the number of approximations is at most $\min(m, n)$. We also investigate how the approximations depend on the weight-values.

Keywords: Weight, Low-rank, Factorization, Missing.

This is a short investigation concerning a best approximation of an arbitrary real matrix $X$ by a “weighted” low-rank approximation WLRA according to

$$X \approx A B', \quad \text{WLRA}$$

with a condition on the ranks

$$\text{rank}(A) = \text{rank}(B) = p \quad \text{and} \quad p < \text{rank}(X).$$

All matrices are real with $X \in \mathbb{R}^{m \times n}$, $A \in \mathbb{R}^{m \times p}$ and $B \in \mathbb{R}^{n \times p}$.

The attention is more focused on the approximations WLRA than on the factors $A$ and $B$. The approximations under consideration minimize the weighted Frobenius norm

$$\|X - A B'\|_{w^2}^2 = \sum_{i=1}^{m} \sum_{j=1}^{n} w_{i,j}^2 \left[X_{i,j} - (A B')_{i,j}\right]^2 = \text{Min}_{A,B},$$

as also defined by (1.3) of Higham (2002).

In some sense, the minimization we face goes much further than the indefinite least squares problem of Bojanczyk, Higham and Patel (2003). The fact that both factors $A$ and $B$ are of concern and the allocation of a separate weight to each of the $X$-entries seriously complicate the matter. Nevertheless, this structure naturally arises in some contexts such as the smoothing of a data matrix of measurements where each of the measured entity can have its own limited precision. The scientists working in this domain know that minimization (1) can have several solutions, a badly understood feature that can loosely be attributed to the lack of convexity of this norm in terms of all the entries of $A$ and $B$, although the norm is convex in the entries of $(A | \text{given } B)$ and those of $(B | \text{given } A)$.

What is the possible number of best approximations $\text{WLRA} = A B'$, under the weighted Frobenius norm (1)?

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This minimization is a NP-hard problem, as shown by Gillis and Glineur (2010). What exactly takes place is so little known that it may explain why many authors of application papers ignore the fact. Since Bradu and Gabriel (1978) introduced the biplot, many variations on approximating by a low-rank matrix have been investigated (Greenacre, 2012).

The paper leads to a conjecture on the number of solutions; they can be quite many. Having started with generalities, Section 2 addresses some aspects of algorithmic convergence. Then, we turn our attention to a dual problem where we consider that the weights can vary, while the matrix entries are seen as arbitrary constants. At Section 4, we even imagine that the squared weights can become negative and wonder what can be expected from such an algebraic trick. This is a domain where the experience is fairly limited, although the trick gives insight on our problem. Sections 5 and 6 report numerical experiments that lead to the conjecture. Eventually, some algorithmic notes are the object of Section 8.

1. Generalities.

There exist several algorithms for the evaluation of WLRA and the one with the lowest computational complexity is based on two weighted linear regression steps. Starting with an arbitrary estimate of $A$, $A^{(k)}$, it is improved by first evaluating the corresponding optimal $B^{(k)}$,

$$B^{(k)} : \| X - A^{(k)} B' \|^2_{w^2} = \text{Min}_B,$$

and then estimating an improved $A^{(k+1)}$,

$$A^{(k+1)} : \| X - A^{(k+1)} B' \|^2_{w^2} = \text{Min}_A,$$

until the limit

$$\text{WLRA}^{(\infty)} \approx A^{(k)} B^{(k)}'$$

is sufficiently well attained. This algorithm is said to be alternating and has been thoroughly investigated in statistics since Gabriel and Zamir (1979); their method does not necessarily converge to an absolute minimum point (see the discussion in Section 6, page 491).

Seeing that

$$A B' = (A M) \times (M^{-1} B') \quad \text{where} \quad M \text{ is full rank},$$

the factors $A$ and $B$ are defined up to the matrix scaling factor $M$. Hence, clearly, we estimate too many parameters when we work on all entries of $A$ and $B$; the feature dramatically complicates the algorithmic test of convergence. To obviate the difficulty, several other algorithms have been investigated ranging from brute force minimization on all entries of $A$ (or $B$) to limiting the space search to a Grassmann product manifold, Manton et al. (2003). Various comparisons of the possible strategies have been reported and Srebro and Jaakkola (2003) deserves a special mention. Simonsson and Eldén (2010), Yan (2010), Markovsky (2010) as well as Markovsky and Van
Huffel (2007) cover more specific aspects. Note that Van Huffel’s school investigates a norm that is slightly generalized compared to (1); their weight structure permits to introduce correlations between the \( X \)-entries. The recent work of Usevich and Markovsky (2012) about structured matrices is worthy of attention. Maronna and Yohai (2008) pay particular attention to algorithm initialisation. Okatani, Yoshida and Deguchi (2011) compare several methods where a damping factor is helping to find the global minimum solution; their study is restricted to the field of computer vision.

2. **On convergence and conditioning.**

The two steps (2) and (3) lead to convergence seeing that each of them reduces the norm \( \| \cdot \|_{u_2}^2 \). This is a guarantee of convergence, although not of unicity of the approximation.

The iterations nicely converge inasmuch as the implied weighted regressions are sufficiently well conditioned. Focusing on (2), the \( j \)-th column of \( B' \) is evaluated by solving the equation

\[
\sum_{i=1}^{m} w_{i,j}^2 \left[ X_{i,j} - \left( \sum_{k=1}^{p} a_{ik} b_{jk} \right) \right]^2 = \text{Min}_{b_{jk}}, \quad \text{for } j = 1, 2, \ldots, n
\]

or

\[
(X - A b_j)' W_j (X - A b_j) = \text{Min}_{b_j}, \quad W_j = \text{diag} \left( w_{1,j}^2, \ldots, w_{m,j}^2 \right).
\]

Hence, the next \( m + n \) conditions arise

\[
\text{Det} \left( B' W^i B \right) \neq 0 \quad \text{and} \quad \text{Det} \left( A' W_j A \right) \neq 0, \quad (5)
\]

where

\[
W^i = \text{diag} \left( w_{1,i}^2, \ldots, w_{m,i}^2 \right).
\]

Their dependence on \( A \) and \( B \) is weak, seeing (4).

3. **Generalisation to weights seen as variables.**

The minimization problem (1) has for solution \( \text{WLRA} = (A B') \) that is such that the derivatives in its vicinity cancel. Hence, it can also be viewed as a ‘saddle point’ as well as a ‘fixed point’ of the mapping \( A \rightarrow A^{(\infty)} \) in

\[
A^{(\infty)} = A = \text{SaddlePoint} = \text{FixedPoint} = \text{Min}_{A,B}
\]

\[
(A B') : \begin{cases}
A^{(\infty)} = A = \text{SaddlePoint} = \text{FixedPoint} = \text{Min}_{A,B} \\
B = B(A) : \sum_{i=1}^{m} \sum_{j=1}^{n} w_{i,j}^2 [X_{i,j} - (A B')_{i,j}]^2 = \text{Min}_B, \\
A^{(\infty)} = A^{(\infty)}(B) : \sum_{i=1}^{m} \sum_{j=1}^{n} w_{i,j}^2 [X_{i,j} - (A^{(\infty)} B')_{i,j}]^2 = \text{Min}_{A^{(\infty)}}.
\end{cases}
\]

This equivalence between \( \text{Min}_{A,B} \), FixedPoint and SaddlePoint results from the convexity of \( \| X - A B' \|_{u_2}^2 \) in the vicinity of the solution. Again, observe that the convexity of (1) is in term of product \( A B' \) but not jointly in both \( A \) and \( B \), which practically means that the algorithms can only, if at all, guarantee convergence to a local minimum. We will shortly loose
this convexity but, for the time being, let us be a little more precise on the sort of derivatives we invoke when we speak of a saddle point.

We address a function \( F \) of a matrix \( X \); the function \( F(X) \) is in a metric space and varies continuously with variations of \( X \). We expect \( F(X) \) to be Gâteaux differentiable and, in a broadly generalized sense, we defined a ‘saddle point’ \( X_s \) as being a point where all the directional derivatives vanish, namely

\[
\frac{\partial}{\partial X} F(X) \bigg|_{X = X_s} = \lim_{h \to 0} \frac{F(X_s + h \Delta X) - F(X_s)}{h} = 0, \quad || \Delta X || > 0
\]

whatever the bounded perturbation \( \Delta X \) is. The present extension of the ‘saddle point’ concept places us at some distance of the remarkable work of Benzi, Golub and Liesen (2005).

At the risk of losing the convexity, we substitute pseudo-weights \( z_{i,j} \) to the non-negative \( w_{i,j}^2 \). Then, our problem takes the form

\[
(A B') : \begin{cases}
A^{(\infty)} = A = \text{FixedPoint} \\
B = B(A) : \sum_{i=1}^{m} \sum_{j=1}^{n} z_{i,j} [X_{i,j} - (A B')_{i,j}]^2 = \text{SaddlePoint}_B, \\
A^{(\infty)} = A^{(\infty)}(B) : \sum_{i=1}^{m} \sum_{j=1}^{n} z_{i,j} [X_{i,j} - (A^{(\infty)} B')_{i,j}]^2 = \text{SaddlePoint}_A^{(\infty)}
\end{cases}
\]

or rather

\[
WLRA = \begin{pmatrix} A^{(\infty)} \\ B^{(\infty)'} \end{pmatrix} = A B' = \text{FixedPoint}
\]

where

\[
(A B') : \begin{cases}
B^{(\infty)} : \frac{\partial}{\partial B^{(\infty)}} \sum_{i=1}^{m} \sum_{j=1}^{n} z_{i,j} \left[ X_{i,j} - \begin{pmatrix} A^{(\infty)} B^{(\infty)'} \\ A^{(\infty)} B^{(\infty)'} \end{pmatrix}_{i,j} \right]_{i,j}^2 = 0, \\
A^{(\infty)} : \frac{\partial}{\partial A^{(\infty)}} \sum_{i=1}^{m} \sum_{j=1}^{n} z_{i,j} \left[ X_{i,j} - \begin{pmatrix} A^{(\infty)} B^{(\infty)'} \\ A^{(\infty)} B^{(\infty)'} \end{pmatrix}_{i,j} \right]_{i,j}^2 = 0.
\end{cases}
\]

The difference between the last two formulations is computational.

4. Path following and anticipations.

What is the possible number of best approximations \( WLRA = A B' \), given a set of weights?

With this question as an investigation direction, we first converted the problem to the saddle point set-up (6). This induces us to consider the solutions as being functions of the pseudo-weights \( z_{i,j} \) and we wonder how these approximations vary when the pseudo-weights vary.

Clearly, independently varying all the pseudo-weights greatly increases the number of concerned variables and the complexity of the problem. In order to keep it simple and to be able to ‘see’ how the solutions behave, we limited ourselves to following a “path”. This is now supported with the help of a numerical example to clarify the details.

Starting with solutions on pseudo-weights, the path is in the space of pseudo-weights \( z_{i,j} \) and is parametrized by a parameter \( \tau \).
• For \( \tau = 0 \), the given set of pseudo-weights \( \{..., z_{i,j}, ...\} \) is of concern,

\[
\tau = 0 \quad \Rightarrow \quad Z_0 = \{..., z_{i,j}, ...\} = \{..., w^2_{i,j}, ...\}.
\]

• For \( \tau = 1 \), the unweighted case (with a unique minimum) is met with,

\[
\tau = 1 \quad \Rightarrow \quad Z_1 = \{\bar{z}, ..., \bar{z}\} \quad \text{where} \quad \bar{z} = \frac{\sum_{i,j} w^4_{i,j}}{\sum_{i,j} w^2_{i,j}}.
\]

• For \( |\tau - 1| \gg 0 \), the pseudo-weight set \( Z_\tau \) consists of positive and negative entries. Several saddle point solutions to (6) can be expected.

Eventually, we follow the path from \( \tau \ll 0 \) to \( \tau \gg 1 \), passing for \( \tau = 0 \) by our set of interest, \( Z_0 \). Such a path can be described by

\[
Z_\tau = Z_0 + \tau (Z_1 - Z_0).
\]

(7)

Note that more sophisticated forms could be of interest.

Bearing our attention on the solutions of the saddle point problem (6) while following this path,

• we know that

\[
\tau = 1 \quad \Rightarrow \quad \text{a unique solution}.
\]

• By continuity and remaining in the vicinity of the least squares approximation, we anticipate

\[
|\tau - 1| \ll 1 \quad \Rightarrow \quad \text{a unique solution}.
\]

• However, we have no clear indication yet on what occurs further away

\[
|\tau - 1| \gg \epsilon \quad \Rightarrow \quad \text{any number of solutions}.
\]

5. Some numerical observations.

A numerical example let see how the approximations WLRA vary as a function of \( \tau \).

We search for the (unique or multiple) rank-1 approximations to matrix \( X \) under weights \( (w_{i,j}) \) in minimization (1),

\[
X = \begin{pmatrix} 6 & 0 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad Z_0 = (w^2_{i,j}) = \begin{pmatrix} 0.04 & 0.68 \\ 0.84 & 0.40 \end{pmatrix}.
\]

(8)

and it turns out that the data set (8) has 2 rank-1 approximations under weighted Frobenius norm (1),

\[
\text{WLRA}_{b,\tau=0} = \begin{pmatrix} 5.871 & 0.202 \\ 1.032 & 0.036 \end{pmatrix}.
\]
and
\[
\text{WLRA}_{c, \tau=0} = \begin{pmatrix}
0.101 & 0.194 \\
1.030 & 1.968
\end{pmatrix}.
\]

Further on, the subscripts \(b\) and \(c\) will appear at Table 1 – These two approximations clearly are very different and they also are of different qualities. Defining a weighted root mean square error, \(\text{Rmse}\), by
\[
\text{Rmse}^2 = \frac{\sum_{i,j} w^2_{i,j} (\text{WLRA}_{i,j} - X_{i,j})^2}{\sum_{i,j} w^2_{i,j}},
\]
they respectively yield
\[\text{Rmse}_b = 0.8958\] and \[\text{Rmse}_c = 0.8507.\]

The set of weights (8) has the average
\[
\bar{z} = \frac{\sum_{i,j} w^4_{i,j}}{\sum_{i,j} w^2_{i,j}} = 0.67837
\]
and the path \(Z_\tau\) is defined by Rule (7), the set of pseudo-weights linearly passing through the two sets \(Z_0\) and \(Z_1 = \begin{pmatrix} 0.67837 & 0.67837 \\ 0.67837 & 0.67837 \end{pmatrix} \).

The solutions to saddle point problem (6) vary along this path. Each approximation can be seen as a point of a 1-dimension “Curve” parametrized by \(\tau\) in the \((m \times n)\)-dimension space of rank-\(p\) approximations. For instance, the curve corresponding to the SVD-approximation runs through the points
\[
\text{WLRA}_{\tau=0.9} = \begin{pmatrix} 5.978 & 0.335 \\ 1.099 & 0.062 \end{pmatrix}, \quad \text{with Rmse} = 0.9000,
\]
\[
\text{WLRA}_{\tau=1} = \begin{pmatrix} 5.978 & 0.351 \\ 1.110 & 0.065 \end{pmatrix} = \text{SVD}, \quad \text{with Rmse} = 0.9011,
\]
and
\[
\text{WLRA}_{\tau=1.1} = \begin{pmatrix} 5.978 & 0.373 \\ 1.125 & 0.070 \end{pmatrix} \quad \text{with Rmse} = 0.9028.
\]

This curve starts at \(\tau = -0.05227\) and terminates at \(\tau = 5.19696\). The right end point of this curve will be referred to as being a “Cut”; there is a \(w^2_{i,j}(\tau)\) that approximately cancels at this curve end, \((Z_{5.19697})_{2,1} = 0\). This curve also passes through the worst of the two solutions, \(\text{WLRA}_{b, \tau=0}\).

A thorough space search led to the finding of four different curves. They are briefly described at Table 1.

This numerical example is small-size and bigger matrices are worthy of attention. We now search for the (unique or multiple) rank-2 approximations to the next matrix \(X\) under weights \((w_{i,j})\) in minimization (1),
\[
X = \begin{pmatrix} 6 & 4 & 6 \\ 2 & 2 & 9 \\ 9 & 0 & 7 \\ 1 & 3 & 1 \end{pmatrix} \quad \text{and} \quad Z_0 = (w^2_{i,j}) = \begin{pmatrix} 0.04 & 0.84 & 0.72 \\ 0.56 & 1 & 0.68 \\ 0.12 & 0.40 & 0.52 \\ 0.60 & 0.48 & 0.32 \end{pmatrix}. \quad (9)
\]
Table 1: Four curves and the cuts of example (8).

| Subscript | \(\tau\) (Left end) | \(\tau\) (Right end) | Special \(\tau\)-values |
|-----------|----------------------|-----------------------|------------------------|
| a         | -8.27280             | -0.06485              | -1.43695 = \text{Cut}_{(2,2)} |
| b         | -0.18799             | 0.09357               | -0.06266 = \text{Cut}_{(1,1)} |
| c         | -0.05227             | 5.19696               | 0 and 1                 |
| d         | 4.07575              | 5.19696               | 5.19697 = \text{Cut}_{(2,1)} \text{;} 416.500 = \text{Cut}_{(1,2)} |

Example (9) has 3 solutions, namely the approximations

\[
\begin{pmatrix} 9.372 & 3.431 & 6.079 \\ 2.152 & 1.704 & 9.052 \\ 6.448 & 2.668 & 6.754 \\ 1.550 & 0.618 & 1.427 \end{pmatrix}, \quad \begin{pmatrix} 2.114 & 3.974 & 6.095 \\ 3.424 & 2.112 & 8.486 \\ 3.355 & -0.239 & 7.572 \\ 0.290 & 2.875 & 1.584 \end{pmatrix}
\]

and

\[
\begin{pmatrix} -0.065 & 3.563 & 6.285 \\ 2.908 & 2.774 & 8.363 \\ 7.371 & -0.743 & 7.320 \\ 0.104 & 1.295 & 2.433 \end{pmatrix}.\]

As for example (8), we report the main observations at Table 2.

6. **Discussion of numerical findings**

The tables must be read keeping in mind that the reported values of curve end-points have a limited precision. They have been identified by following the curves until they terminate and this procedure is somewhat coarse.

The two cases reported for illustration are somewhat different. Example (8) is so small that that the sheer appearance of a zero weight yields to a rank reduction such that the ‘approximations’ realise exact fits (at \(p = 1\)). This is not the case for \(\min m, n > 1\) as in Example (9).

Let us now list our main observations. When referring to given curves of Tables 1 and 2, we use the subscripts specified in those two Tables.

(a) Cuts play a crucial role as end-points of curves (see a, c, d, k, l).

(b) Most end-points are not in the vicinity of cuts.

(c) Curves corresponding to paths (7) may pass through cuts (see g, i, j, m, o).

(d) For \(p > 1\), we observe that conditions (5) are usually satisfied at curve ends.

(e) There are vicinities in \(\tau\)-values where no curve seem to exist. They correspond to combinations of positive and negative pseudo-weights.
Table 2: Curves and cuts of example (9).

| Subscript | $\tau$ (Left end) | $\tau$ (Right end) | Special $\tau$-values |
|-----------|-------------------|---------------------|-----------------------|
| e         | -3.36121          | -3.08193            | -10.1509 = Cut$_{(4,1)}$ |
|           |                   |                     | -5.65039 = Cut$_{(2,1)}$ |
|           |                   |                     | -3.73810 = Cut$_{(3,3)}$ |
| f         | -2.15933          | -2.10259            | -2.67994 = Cut$_{(4,2)}$ |
| g         | -1.82206          | -1.01699            | -1.54376 = Cut$_{(3,2)}$ |
|           |                   |                     | -0.94365 = Cut$_{(4,3)}$ |
| h         | -0.94702          | -0.32216            | -0.22259 = Cut$_{(3,1)}$ |
| i         | -0.24797          | 0.01177             | 0                      |
| j         | -0.06533          | 0.27532             | 0                      |
|           |                   |                     | -0.06461 = Cut$_{(1,1)}$ |
| k         | -0.06461          | 2.93359             | 0 and 1                |
| l         | 2.55480           | 2.93348             | 2.93348 = Cut$_{(2,2)}$ |
| m         | 3.75504           | 8.40023             | 4.64366 = Cut$_{(1,2)}$ |
| n         | 5.07162           | 6.66613             | 11.8243 = Cut$_{(1,3)}$ |
| o         | 8.51944           | > 20.00             | 32.5488 = Cut$_{(2,3)}$ |

(f) The best solution at $\tau = 0$ cannot always be attained from the SVD-solution at $\tau = 1$ (no curve may join these two solutions, see $b$ and $c$ of Table 1).

(g) The number of solutions at $\tau = 0$ is at least 1 and at most the minimum dimension of the approximated matrix.

Table 3 reports the findings based on a large population of $(X, Z_0)$-pairs.

7. A conjecture

Given a $m \times n$-matrix $X$, it has up to $\text{Min}(m, n)$ weighted low-rank approximations. This is observed in the context of weighted Frobenius norm (1).

8. Algorithmic notes

Two main numerical difficulties are encountered and both are approached by the use of “closest bases”. We first introduce this concept and then describe the two difficulties, namely the path following procedure and the thorough space search.

- Closest basis.
Given a set of vectors $a_1, ..., a_p$, the Gram-Schmidt process is standard to build up a basis that is normed and orthogonal and that permits a perfect decomposition of these original vectors. This Gram-Schmidt basis is clearly not unique, seeing that different sets of such bases can be constructed by simply permuting the original vectors; unfortunately, this lack of uniqueness seriously complicates the numerical steps.

The classical Gram-Schmidt process is now reminded by (10), before being slightly modified into (11).

\[
\begin{align*}
\text{for } i := 1 \text{ to } p & \text{ do} \\
& e_i := a_i \quad \text{Base vector initialisation.} \\
& \delta_i := \sum_{j < i} (e_j' e_i) e_j \quad \text{Projection on previous } e_j. \\
& e_i := e_i - \delta_i \quad \text{Deflation.} \\
& e_i := e_i / (e_i' e_i)^{1/2} \quad \text{Normalisation.} \\
& a_i := e_i \quad \text{Substitution.}
\end{align*}
\]

(10)

The stability can be greatly improved by constructing a basis that is “close” in directions to the original set $a_1, ..., a_p$.

The “closest basis” is derived according to the next algorithm, an original method as far as we know.

\[
\begin{align*}
\text{Repeat} \quad \text{for } i := 1 \text{ to } p & \text{ do} \\
& e_i := a_i / (a_i' a_i)^{1/2} \quad \text{Initialisation and normalisation.} \\
\text{for } i := 1 \text{ to } p & \text{ do} \\
& \delta_i := \frac{1}{2} \sum_{j \neq i} (e_j' e_i) e_j \quad \text{Half projection on all others.} \\
\text{for } i := 1 \text{ to } p & \text{ do} \\
& e_i := e_i - \delta_i \quad \text{Deflation.} \\
& a_i := e_i \quad \text{Substitution.}
\end{align*}
\]

(11)

Algorithm (11) is iterative and therefore is slower than the Gram-Schmidt standard by (10). Its convergence is quadratic and the global loop is little run. The extreme stability of this “closest basis” justifies the expense.

- Path following.

The path $Z_\tau$ by (7) is with respect to the pseudo-weights and is associated to varying approximations

\[Z_\tau = Z_0 + \tau (Z_1 - Z_0) \rightarrow \text{WLRA}_\tau = (A B')_\tau = A_\tau B'_\tau.\]

We remark that, even if $(A B')_\tau$ is defined, $A_\tau$ remains indeterminate seeing (4). However, if we knew $A_\tau$, $B_\tau$ would immediately result from (2),

\[B_\tau : \| X - A_\tau B'_\tau \|_{w_\tau^2}^2 = \text{Min}_B.\]

In order to follow the path (7), we impose a smooth variation on $A_\tau$. On the one hand, we restrict $A_\tau$ to be orthonormed and, on the other hand, to be slowly varying.
Given a $A_\tau$ solution of (6) and corresponding to a given $\tau$-value, its $p$ column vectors $a_1, \ldots, a_p$ have size $m$, $m > p$, and we orthonormalise by algorithm (11). This is our first solution.

Slightly modifying the first $\tau$-value and with the help of the first solution as initialisation, we derive a new solution $A_\tau$. This is our second solution and it lies in a tight vicinity of the first (due to the fact that we used a closest basis rather than the classical Gram-Schmidt process). We have obtained two points of a curve.

Further on, we apply a predictor-corrector algorithm with step length adaptation. Several strategies of Bates et al. (2008) are relevant.

• Thorough space search.

Counting the number of solutions existing for a given $\tau$-value is quite a problem. We resorted to applying a space search strategy. We start with an arbitrary initialisation and solve minimization (1). This yields a given solution WLRA$_\tau$ (at $\tau = 0$). Repeating with other initialisations, we obtain either new WLRA$_\tau$ or repeats of the already known solutions.

The main trouble with the above strategy is due to possibly small radii of convergence. Some solutions WLRA$_\tau$ can be discovered only when an initialisation is performed in their immediate vicinity; starting too far away, the minimizations converge toward some “dominant” solutions.

Hence, it is peremptory to apply initialisations which evenly span the search space.

Initialising $N$ times, we need $N$ good starting $A_\tau$, each with $m \times p$ entries. These entries will be the coordinates of $N$ points on a ball in $\mathbb{R}^{m \times p}$; eventually, each of the $A_\tau$-points is projected onto a subspace by orthonormalisation into a closest basis. Gross (2011) and Recht (2011) discuss what the sample size $N$ must be.

First, the $N$ points on the ball are assigned as being the summits of a nearly regular polyhedron; then, they are slightly shifted as if they were exerting a repulsive force on the other points. We construct a dispersed phase of points on the ball surface.

References

Bates D.J., Hauenstein J.D., Sommese A.J. and Wampler C.W (2008). Adaptive Multiprecision Path Tracking, *SIAM J. Numer. Anal.*, 46, 722-746.

Benzi M., Golub G.H. and Liesen J. (2005). Numerical solution of saddle point problems, *Acta Numerica*, 14, 1 - 137

Bojanczyk A., Higham N.J. and Patel H. (2003). Solving the indefinite least squares problem by hyperbolic QR factorization, *SIAM J. Matrix Anal. Appl.*, 24, 914 - 931.
Bradu, D. and Gabriel, K.R. (1978). The biplot as a diagnostic tool for model of two-way tables, *Technometrics*, 20, 47-68.

Gabriel K.R. and Zamir S. (1979). Lower Rank Approximation of Matrices by Least Squares with Any Choice of Weights, *Technometrics*, 21, 489-498.

Gillis N. and Glineur F. (2010). Low-rank matrix approximation with weights or missing data is NP-hard, [http://arxiv.org/abs/1012.0197](http://arxiv.org/abs/1012.0197).

Greenacre M.J. (2012) Biplots: the joy of singular value decomposition, *Wiley Interdisciplinary Reviews: Computational Statistics*, 4, 399-406.

Gross D. (2011) Recovering low-rank matrices from few Coefficients in any basis *IEEE Transactions on Information Theory*, 57, 1548 - 1566.

Higham, N.J. (2002) Computing the nearest correlation matrix, a problem from finance, *IMA Journal of Numerical Analysis*, 22, 329-343.

Manton J.H., Mahony, R. and Hua, Y. (2003). The geometry of weighted low-rank approximations. *IEEE Transactions on Signal Processing*, 51, 500-514.

Markovsky I. (2010). Algorithms and literate programs for weighted low-rank approximation with missing data, Preprint.

Markovsky I. and Van Huffel S. (2007). Left vs right representations for solving weighted low-rank approximation problems. *Linear Algebra and its Applications*, 422, 540-552.

Maronna R.A. and Yohai V.J. (2008). Robust lower-rank approximation of data matrices with element-wise contamination. *Technometrics*, 50, 295-304.

Okatani T., Yoshida T. and Deguchi, K. (2011). Efficient algorithm for low-rank matrix factorization with missing components and performance comparison of latest algorithms. *2011 IEEE International Conference on Computer Vision (ICCV)*, 6-13 Nov. 2011, 842-849.

Recht B. (2011) A simpler approach to matrix completion, *Journal of Machine Learning Research*, 12, 3413-3430

Simonsson L. and Eldén L. (2010). Grassmann algorithms for low-rank approximation of matrices with missing values, *BIT Numerical Mathematics*, 50, 173-191.

Srebro N. and Jaakkola T. (2003). Weighted low-rank approximations. In *ICML, 20th International Conference on Machine Learning*, 720-727.

Usevich K. and Markovsky I. (2012) Variable projection for affinely structured low-
rank approximation in weighted 2-norm. [http://arxiv.org/pdf/1211.3938](http://arxiv.org/pdf/1211.3938).

Yan G. (2010). Structured low-rank Matrix Optimization Problems: A Penalty Approach, Thesis.
Table 3: Maximum number of solutions.

| Dimensions | Approximation rank | Maximum number of solutions |
|------------|--------------------|-----------------------------|
| $m$  $n$   |                    |                             |
| 2  2       | 1                  | 2                          |
| 2  3       | 1                  | 2                          |
| 2  4       | 1                  | 2                          |
| 2  5       | 1                  | 2                          |
| 2  6       | 1                  | 2                          |
| 3  2       | 1                  | 2                          |
| 4  2       | 1                  | 2                          |
| 5  2       | 1                  | 2                          |
| 6  2       | 1                  | 2                          |
| 3  3       | 1                  | 3                          |
| 3  4       | 1                  | 3                          |
| 3  4       | 2                  | 3                          |
| 3  5       | 1                  | 3                          |
| 3  5       | 2                  | 3                          |
| 3  6       | 1                  | 3                          |
| 4  3       | 1                  | 3                          |
| 4  3       | 2                  | 3                          |
| 5  3       | 1                  | 3                          |
| 5  3       | 2                  | 3                          |
| 6  3       | 1                  | 3                          |
| 4  4       | 2                  | 4                          |
| 4  5       | 2                  | 4                          |
| 4  5       | 3                  | 4                          |
| 4  6       | 2                  | 4                          |
| 4  6       | 3                  | 4                          |
| 5  4       | 2                  | 4                          |
| 5  4       | 3                  | 4                          |
| 6  4       | 2                  | 4                          |
| 6  4       | 3                  | 4                          |
| 5  6       | 2                  | 5                          |
| 5  6       | 3                  | 5                          |
| 5  6       | 4                  | 5                          |
| 6  5       | 2                  | 5                          |
| 6  5       | 3                  | 5                          |
| 6  5       | 4                  | 5                          |