BIHARMONIC HYPERSURFACES IN A RIEMANNIAN MANIFOLD WITH NON-POSITIVE RICCI CURVATURE

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Abstract. In this paper, we show that, for a biharmonic hypersurface \((M, g)\) of a Riemannian manifold \((N, h)\) of non-positive Ricci curvature, if \(\int_M |H|^2 v_g < \infty\), where \(H\) is the mean curvature of \((M, g)\) in \((N, h)\), then \((M, g)\) is minimal in \((N, h)\). Thus, for a counter example \((M, g)\) in the case of hypersurfaces to the generalized Chen’s conjecture (cf. Sect. 1), it holds that \(\int_M |H|^2 v_g = \infty\).

1. Introduction and statement of results.

In this paper, we consider an isometric immersion \(\varphi : (M, h) \to (N, h)\), of a Riemannian manifold \((M, g)\) of dimension \(m\), into another Riemannian manifold \((N, h)\) of dimension \(n = m + 1\). We have

\[
\nabla^N_{\varphi_*X}\varphi_*Y = \varphi_*(\nabla_X Y) + k(X, Y)\xi,
\]

for vector fields \(X\) and \(Y\) on \(M\), where \(\nabla, \nabla^N\) are the Levi-Civita connections of \((M, g)\) and \((N, h)\), respectively, \(\xi\) is the unit normal vector field along \(\varphi\), and \(k\) is the second fundamental form. Let \(A : T_x M \to T_x M\) \((x \in M)\) be the shape operator defined by \(g(AX, Y) = k(X, Y)\), \((X, Y) \in T_x M\), and \(H\), the mean curvature defined by \(H := \frac{1}{m} \text{Tr}_g(A)\). Then, let us recall the following B.Y. Chen’s conjecture (cf. [3], [4]):

Let \(\varphi : (M, g) \to (\mathbb{R}^n, g_0)\) be an isometric immersion into the standard Euclidean space. If \(\varphi\) is biharmonic (see Sect. 2), then, it is minimal.

This conjecture is still open up to now, and let us recall also the following generalized B.Y. Chen’s conjecture (cf. [3], [2]):

\[
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\]

\[
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\]

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Let \( \varphi : (M, g) \to (N, h) \) be an isometric immersion, and the sectional curvature of \((N, h)\) is non-positive. If \( \varphi \) is biharmonic, then, it is minimal.

Oniciuc ([9]) and Ou ([11]) showed this is true if \( H \) is constant.

In this paper, we show

**Theorem 1.1.** Assume that \((M, g)\) is complete and the Ricci tensor \( \text{Ric}^N \) of \((N, h)\) satisfies that

\[
\text{Ric}^N(\xi, \xi) \leq |A|^2.
\]

If \( \varphi : (M, g) \to (N, h) \) is biharmonic (cf. Sect. 2) and satisfies that

\[
\int_M H^2 v_g < \infty,
\]

then, \( \varphi \) has constant mean curvature, i.e., \( H \) is constant.

As a direct corollary, we have

**Corollary 1.2.** Assume that \((M, g)\) is a complete Riemannian manifold of dimension \( m \) and \((N, h)\) is a Riemannian manifold of dimension \( m+1 \) whose Ricci curvature is non-positive. If an isometric immersion \( \varphi : (M, g) \to (N, h) \) is biharmonic and satisfies that \( \int_M H^2 v_g < \infty \), then, \( \varphi \) is minimal.

By our Corollary 1.2, if there would exist a counter example (cf. [11]) in the case \( \text{dim } N = \text{dim } M + 1 \), then it must hold that

\[
\int_M H^2 v_g = \infty,
\]

which imposes the strong condition on the behaviour of the boundary of \( M \) at infinity. Indeed, (1.3) implies that either \( H \) is unbounded on \( M \), or it holds that \( H^2 \geq C \) on an open subset \( \Omega \) of \( M \) with infinite volume, for some constant \( C > 0 \).

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2. Preliminaries.

In this section, we prepare general materials about harmonic maps and biharmonic maps of a complete Riemannian manifold into another Riemannian manifold (cf. [5]).

Let \((M, g)\) be an \(m\)-dimensional complete Riemannian manifold, and the target space \((N, h)\) is an \(n\)-dimensional Riemannian manifold. For every \(C^\infty\) map \(\varphi\) of \(M\) into \(N\), and relatively compact domain \(\Omega\) in \(M\), the energy functional on the space \(C^\infty(M, N)\) of all \(C^\infty\) maps of \(M\) into \(N\) is defined by

\[
E_{\Omega}(\varphi) = \frac{1}{2} \int_{\Omega} |d\varphi|^2 v_g,
\]

and for a \(C^\infty\) one parameter deformation \(\varphi_t \in C^\infty(M, N)\) \((-\epsilon < t < \epsilon)\) of \(\varphi\) with \(\varphi_0 = \varphi\), the variation vector field \(V\) along \(\varphi\) is defined by \(V = \frac{d}{dt}\bigg|_{t=0} \varphi_t\). Let \(\Gamma_{\Omega}(\varphi^{-1}TN)\) be the space of \(C^\infty\) sections of the induced bundle \(\varphi^{-1}TN\) of the tangent bundle \(TN\) by \(\varphi\) whose supports are contained in \(\Omega\). For \(V \in \Gamma_{\Omega}(\varphi^{-1}TN)\) and its one-parameter deformation \(\varphi_t\), the first variation formula is given by

\[
\frac{d}{dt}\bigg|_{t=0} E_{\Omega}(\varphi_t) = -\int_{\Omega} \langle \tau(\varphi), V \rangle v_g.
\]

The tension field \(\tau(\varphi)\) is defined globally on \(M\) by

\[
\tau(\varphi) = \sum_{i=1}^{m} B(\varphi)(e_i, e_i), \quad (2.1)
\]

where

\[
B(\varphi)(X, Y) = \nabla_{\varphi_*X} \varphi_*Y - \varphi_*(\nabla_X Y)
\]

for \(X, Y \in \mathfrak{X}(M)\). Then, a \(C^\infty\) map \(\varphi : (M, g) \rightarrow (N, h)\) is harmonic if \(\tau(\varphi) = 0\). For a harmonic map \(\varphi : (M, g) \rightarrow (N, h)\), the second variation formula of the energy functional \(E_{\Omega}(\varphi)\) is

\[
\frac{d^2}{dt^2}\bigg|_{t=0} E_{\Omega}(\varphi_t) = \int_{\Omega} \langle J(V), V \rangle v_g
\]

where

\[
J(V) := \overline{\Delta} V - R(V),
\]

\[
\overline{\Delta} V := \nabla^* \nabla V = -\sum_{i=1}^{m} \{ \nabla_{e_i} (\nabla_{e_i} V) - \nabla_{\nabla_{e_i} e_i} V \},
\]

\[
R(V) := \sum_{i=1}^{m} R^N(V, \varphi_*(e_i)) \varphi_*(e_i).
\]
Here, $\nabla$ is the induced connection on the induced bundle $\varphi^{-1}TN$, and $R^N$ is the curvature tensor of $(N, h)$ given by $R^N(U, V)W = [\nabla^N_U, \nabla^N_V]W - \nabla^N_{[U, V]}W$ $(U, V, W \in \mathfrak{X}(N))$. 

The bienergy functional is defined by

$$E_{2, \Omega}(\varphi) = \frac{1}{2} \int_{\Omega} |\tau(\varphi)|^2 v_g,$$

and the first variation formula of the bienergy is given (cf. [7]) by

$$\frac{d}{dt} \bigg|_{t=0} E_{2, \Omega}(\varphi_t) = - \int_{\Omega} \langle \tau_2(\varphi), V \rangle v_g$$

where the bitension field $\tau_2(\varphi)$ is defined globally on $M$ by

$$\tau_2(\varphi) = J(\tau(\varphi)) = \Delta \tau(\varphi) - R(\tau(\varphi)),$$  \hspace{0.5cm} (2.2)

and a $C^\infty$ map $\varphi : (M, g) \to (N, h)$ is called to be biharmonic if

$$\tau_2(\varphi) = 0.$$  \hspace{0.5cm} (2.3)

3. Some Lemma for the Schrödinger type equation

In this section, we prepare some simple lemma of the Schrödinger type equation of the Laplacian $\Delta_g$ on an $m$-dimensional non-compact complete Riemannian manifold $(M, g)$ defined by

$$\Delta_g f := \sum_{i=1}^m e_i(e_i f) - \nabla_{e_i} e_i f \hspace{0.5cm} (f \in C^\infty(M)),$$  \hspace{0.5cm} (3.1)

where $\{e_i\}_{i=1}^m$ is a locally defined orthonormal frame field on $(M, g)$.

**Lemma 3.1.** Assume that $(M, g)$ is a complete non-compact Riemannian manifold, and $L$ is a non-negative smooth function on $M$. Then, every smooth $L^2$ function $f$ on $M$ satisfying the Schrödinger type equation

$$\Delta_g f = L f \hspace{0.5cm} (\text{on } M)$$  \hspace{0.5cm} (3.2)

must be a constant.

**Proof.** Take any point $x_0$ in $M$, and for every $r > 0$, let us consider the following cut-off function $\eta$ on $M$:

$$\begin{align*}
0 &\leq \eta(x) \leq 1 \quad (x \in M), \\
\eta(x) &\equiv 1 \quad (x \in B_r(x_0)), \\
\eta(x) &\equiv 0 \quad (x \notin B_{2r}(x_0)), \\
|\nabla \eta| &\leq \frac{2}{r} \quad (\text{on } M),
\end{align*}$$  \hspace{0.5cm} (3.3)
where \( B_r(x_0) = \{ x \in M : d(x, x_0) < r \} \), and \( d \) is the distance of \((M, g)\). Multiply \( \eta^2 f \) on (3.2), and integrate it over \( M \), we have
\[
\int_M (\eta^2 f) \Delta_g f v_g = \int_M L \eta^2 f^2 v_g. \tag{3.4}
\]
By the integration by part for the left hand side, we have
\[
\int_M (\eta^2 f) \Delta_g f v_g = -\int_M g(\nabla (\eta^2 f), \nabla f) v_g. \tag{3.5}
\]
Here, we have
\[
g(\nabla (\eta^2 f), \nabla f) = 2 \eta f g(\nabla \eta, \nabla f) + \eta^2 g(\nabla f, \nabla f)
= 2 \eta f \langle \nabla \eta, \nabla f \rangle + \eta^2 | \nabla f |^2, \tag{3.6}
\]
where we use \( \langle \cdot, \cdot \rangle \) and \( | \cdot | \) instead of \( g(\cdot, \cdot) \) and \( g(u, u) = |u|^2 \) \((u \in T_x M)\), for simplicity. Substitute (3.6) into (3.5), the right hand side of (3.5) is equal to
\[
RHS \text{ of (3.5)} = -\int_M 2 \eta f \langle \nabla \eta, \nabla f \rangle v_g - \int_M \eta^2 | \nabla f |^2 v_g \\
= -2 \int_M \langle f \nabla \eta, \eta \nabla f \rangle v_g - \int_M \eta^2 | \nabla f |^2 v_g. \tag{3.7}
\]
Here, applying Young’s inequality: for every \( \epsilon > 0 \), and every vectors \( X \) and \( Y \) at each point of \( M \),
\[
\pm 2 \langle X, Y \rangle \leq \epsilon |X|^2 + \frac{1}{\epsilon} |Y|^2, \tag{3.8}
\]
to the first term of (3.7), we have
\[
RHS \text{ of (3.7)} \leq \epsilon \int_M |\eta \nabla f|^2 v_g + \frac{1}{\epsilon} \int_M |f \nabla \eta|^2 v_g - \int_M \eta^2 |\nabla f|^2 v_g \\
= -(1 - \epsilon) \int_M \eta^2 |\nabla f|^2 v_g + \frac{1}{\epsilon} \int_M f^2 |\nabla \eta|^2 v_g. \tag{3.9}
\]
Thus, by (3.5) and (3.9), we obtain
\[
\int_M L \eta^2 f^2 v_g + (1 - \epsilon) \int_M \eta^2 | \nabla f |^2 v_g \leq \frac{1}{\epsilon} \int_M f^2 | \nabla \eta |^2 v_g. \tag{3.10}
\]
Now, putting \( \epsilon = \frac{1}{2} \), (3.10) implies that
\[
\int_M L \eta^2 f^2 v_g + \frac{1}{2} \int_M \eta^2 | \nabla f |^2 v_g \leq 2 \int_M f^2 | \nabla \eta |^2 v_g. \tag{3.11}
\]
Since $\eta = 1$ on $B_r(x_0)$ and $|\nabla \eta| \leq \frac{2}{r}$, and $L \geq 0$ on $M$, we have
\[
0 \leq \int_{B_r(x_0)} L f^2 v_g + \frac{1}{2} \int_{B_r(x_0)} |\nabla f|^2 v_g \leq \frac{8}{r^2} \int_M f^2 v_g. \tag{3.12}
\]
Since $(M, g)$ is non-compact and complete, $r$ can tend to infinity, and $B_r(x_0)$ goes to $M$. Then we have
\[
0 \leq \int_M L f^2 v_g + \frac{1}{2} \int_M |\nabla f|^2 v_g \leq 0 \tag{3.13}
\]
since $\int_M f^2 v_g < \infty$. Thus, we have $L f^2 = 0$ and $|\nabla f| = 0$ (on $M$) which implies that $f$ is a constant. \qed

4. Biharmonic isometric immersions.

In this section, we consider a hypersurface $M$ of an $(m+1)$-dimensional Riemannian manifold $(N, h)$. Recently, Y-L. Ou showed (cf. [10])

**Theorem 4.1.** Let $\varphi : (M, g) \to (N, h)$ be an isometric immersion of an $m$-dimensional Riemannian manifold $(M, g)$ into another $(m + 1)$-dimensional Riemannian manifold $(N, h)$ with the mean curvature vector field $\eta = H \xi$, where $\xi$ is the unit normal vector field along $\varphi$. Then, $\varphi$ is biharmonic if and only if the following equations hold:
\[
\begin{cases}
\Delta_g H - H |A|^2 + H \text{Ric}^N(\xi, \xi) = 0, \\
2 A (\nabla H) + \frac{m}{2} \nabla (H^2) - 2 H (\text{Ric}^N(\xi))^T = 0,
\end{cases} \tag{4.1}
\]
where $\text{Ric}^N : T_x N \to T_x N$ is the Ricci transform which is defined by $h(\text{Ric}^N(Z), W) = \text{Ric}^N(Z, W)$ ($Z, W \in T_x N$), $(\cdot)^T$ is the tangential component corresponding to the decomposition of $T_{\varphi(x)} N = \varphi_*(T_x M) \oplus \mathbb{R} \xi_x$ ($x \in M$), and $\nabla f$ is the gradient vector field of $f \in C^\infty(M)$ on $(M, g)$, respectively.

Due to Theorem 4.1 and Lemma 3.1, we can show immediately our Theorem 1.1.

*(Proof of Theorem 1.1.)*

Let us denote by $L := |A|^2 - \text{Ric}^N(\xi, \xi)$ which is a smooth non-negative function on $M$ due to our assumption. Then, the first equation is reduced to the following Schrödinger type equation:
\[
\Delta_g f = L f, \tag{4.2}
\]
where $f := H$ is a smooth $L^2$ function on $M$ by the assumption (1.2).
Assume that $M$ is compact. In this case, by (4.2) and the integration by part, we have

$$0 \leq \int_M L f^2 v_g = \int_M f (\Delta_g f) v_g = -\int_M g(\nabla f, \nabla f) v_g \leq 0,$$

which implies that $\int_M g(\nabla f, \nabla f) v_g = 0$, that is, $f$ is constant.

Assume that $M$ is non-compact. In this case, we can apply Lemma 3.1 to (4.2). Then, we have that $f = H$ is a constant. □

(Proof of Corollary 1.2.)
Assume that $\text{Ric}^N$ is non-positive. Since $L = |A|^2 - \text{Ric}^N(\xi, \xi)$ is non-negative, $H$ is constant due to Theorem 1.1. Then, due to (4.1), we have that $HL = 0$ and $H (\text{Ric}^N(\xi))^T = 0$. If $H \neq 0$, then $L = 0$, i.e.,

$$\text{Ric}^N(\xi, \xi) = |A|^2.$$  (4.4)

By our assumption, $\text{Ric}^N(\xi, \xi) \leq 0$, and the right hand side of (4.4) is non-negative, so we have $|A|^2 = 0$, i.e., $A \equiv 0$. This contradicts $H \neq 0$. We have $H = 0$. □

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