Stabilizability and Bipartite Containment Control of Multi-Agent Systems Over Signed Directed Graphs

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This work was supported in part by the National Natural Science Foundation of China under Grant 11662002 and Grant 61973118, in part by the Project of Science and Technology Department of Jiangxi Province under Grant 20182BCB22009 and Grant 20192BAB217009, and in part by the Science and Technology Research Project of Jiangxi Provincial Educational Department under Grant GJJ180349.

ABSTRACT This paper investigates the stabilizability and bipartite containment control problem of general linear multi-agent systems over signed directed graphs, where the negative edges indicate the antagonistic interactions between agents. By employing the proposed bipartite containment control protocols over signed directed graphs, and combining the linear system stabilization theory and the signed Laplacian matrix, the necessary condition for the stabilization of multi-agent systems with multiple leaders is discussed. According to this necessary condition, a leader-follower matching method is presented to establish leader-follower multi-agent system networks. Based on the leader-follower topology established above and the designed state feedback gain, the stability of multi-agent systems is proved by using the system error and the algebraic Riccati equation. It is shown that the followers can gradually converge into the convex hull spanned by the states and the sign-inverted states of leaders, and the bipartite containment control of multi-agent systems can be realized. The simulation results verify the feasibility of the theoretical analysis.

INDEX TERMS Multi-agent systems, controlled followers selection, stabilization, bipartite containment control.

I. INTRODUCTION
During the past decades, the researches on distributed cooperative control of multi-agent systems had received extensive attention, such as consensus, flocking, formation, formation-containment, multi-target tracking and so on [1]–[6]. Besides, considering that some factors may affect the stability of whole systems, the consensus of multi-agent systems under the switching topologies [7], time delays [8], [9], heterogeneous interdependent group systems [10] and external disturbances [11] was studied. For the consensus control strategies, the methods based on event triggering [12], impulsive consensus algorithm [13] and aperiodic intermittent control [14] were investigated to achieve consensus of multi-agent systems.

As a special kind of consensus problem, the containment control of multi-agent systems also had been studied in many researches. Realizing containment control means that the followers can obtain the control information of multiple leaders in distributed way and eventually converge into the convex hull spanned by leaders. For example, in a multi-robot cooperative combat network, only a few robots need to receive control instructions or be equipped with sensors to sense the dangerous obstacles. These robots are regarded as leaders, and the remaining robots are followers. After the leaders formed a secure convex hull, the followers can autonomously move into the convex hull and be transported from one location to another. The related problems of containment control were first studied by Ji in [15]. After that, the necessary and sufficient conditions for multi-agent systems to achieve containment were given in [16]. On this basis, Cao [17] studied the containment control problem with stationary leaders and dynamic leaders under fixed and switching topologies. For agent dynamics, the containment control of heterogeneous [18] and higher-order [19] multi-agent systems had been investigated, and the robust containment control protocols were proposed to cope with external disturbances in [20]. For complex communication situations, the containment control of multi-agent systems with unbounded communication delays under fixed topologies and communication delays under switching topologies were respectively studied in [21] and [22].
In the researches of multi-agent systems, the controllability of the system is a basic concept. Tanner [23] first studied the influence of the controllability of multi-agent network topology on system controllability. Subsequently, the maximum matching algorithm was used to select the least driving nodes to ensure that the multi-agent networks are fully controlled in [24], [25]. In addition, the leaders selection problems were solved by input addition and in terms of downer branch and subgraphs in [26] and [27], respectively. For group systems, the controllability of the heterogeneous interdependent group systems [28] and the two-time-scale multi-agent systems [29] were studied. Based on previous researches, the important results and recent advances in the studies of controllability networked linear dynamical systems were reviewed in [30], and main methods of analyzing controllability were evaluated. Although the controllability of the networks has been studied widely, the researches on the stabilizability of the networks are still in the initial stage. In this direction, the stabilizations of first-order [31] and general linear [32] multi-agent systems under nonnegative networks were investigated.

All the above researches are based on the traditional agent communication network including only positive edges, i.e., there exist only cooperative relationships between agents. However, if networks contain both positive and negative edges, which are called signed graphs, there will exist not only cooperative relationship but also competitive relationship between agents. Altafini [33] first found that all agents will converge to a same value except for the sign on signed graphs under the consensus protocols, and obtained the corresponding necessary and sufficient conditions. On this basis, the modulus consensus model was extended to the case where the network topology can be structure unbalanced or arbitrary time-varying in [34] and [35]. Then the Laplacian matrix of signed directed graph (known as digraph) was studied by introducing rooted cycles, meanwhile, the relationship between it and the spanning tree condition was addressed in [36], and it was shown that all agents reach interval bipartite consensus if their associated signed digraph has a spanning tree. Further, by establishing an equivalence between bipartite consensus and conventional one, state feedback and output feedback control approaches are proposed to achieve bipartite consensus in [37]. In terms of stabilization, Liu [38] coped with the decentralized stabilizability and the formation control problems of first-order multi-agent systems over signed graphs. For the multi-leader first-order multi-agent systems, based on the researches of [36], the bipartite containment control was studied in [39]. For general linear dynamics, the bipartite output containment of continuous-time heterogeneous multi-agent systems and discrete-time multi-agent systems on signed graphs were investigated in [40] and [41], respectively, and the bipartite containment of high-order multi-agent systems over time-varying cooperation-competition networks was solved in [42].

As described in [39], for the first-order multi-agent systems over signed directed graphs, the followers can converge into the convex hull spanned by the leaders with the same modules but different signs. That is the “bipartite containment” is realized for signed networks. This paper copes with the situation that the linear multi-leader systems over signed networks can realize bipartite containment by selecting the least controlled followers. The main advantages of the proposed method to realize bipartite containment in this paper are as follows: 1. The characteristics of the exact and least controlled followers to achieve containment control of general linear multi-agent systems over arbitrary signed directed graphs are discussed, which cannot be dealt with well through the bipartite graph maximum matching algorithm mentioned in [30]. The obtained results can help to better study the issues related to multi-agent systems over signed graphs. 2. Compared with the researches for the first order multi-agent systems over signed directed graphs studied in [39], we investigate the containment control of general linear multi-agent systems over signed directed graphs, which is a more general case and brings some new features and difficulties.

The remainder of this paper is organized as follows: In Section II, some definitions and preliminaries are introduced. In Section III, the necessary condition of the multi-agent systems to achieve stabilization is discussed, the leader-follower matching method and some related proofs are also presented. In Section IV, some simulation examples are provided. Finally, some conclusions are given in Section V.

**Notations:** Given two sets $X, Y$, the set represented by $X \setminus Y$ belongs to $X$, but does not belong to $Y$. $\otimes$ represents the Kronecker product. $\lambda(A)$ represents the eigenvalue set of the matrix $A$, and $\text{Re}(\lambda_i)$ represents the real part of the eigenvector of $\Lambda$ in $\mathbb{R}$. $\bar{v}_m$ denotes the all-one column vector with $m$ dimensions. $\text{diag} \{b_1, \ldots, b_m\}$ denotes the diagonal matrix composed by diagonal elements $b_1, \ldots, b_m$. $|a|$ denotes the absolute value of $a$.

## II. PRELIMINARIES

### A. GRAPH THEORY

Agents and their interconnections can be represented by graphs with nodes and edges. For a multi-agent system with $n$ agents, $F = \{v_1, \ldots, v_m\}$ is follower set and $L = \{v_{m+1}, \ldots, v_n\}$ is leader set. If the directions of information transmissions between agents are directed, and the leaders cannot receive information from other agents, the corresponding digraph can be expressed as $G = (V, E, A)$, $V = \{v_1, \ldots, v_n\}$ denotes the set of nodes, $E \subseteq V \times V$ denotes the set of edges, $A = \{a_{ij}\} \in \mathbb{R}^{n \times n}$ denotes the adjacency matrix of $G$, $a_{ij}$ is the weight of edge $e_{ij} = (v_i, v_j)$, the positive and negative edges correspond to $a_{ij} = 1$ and $a_{ij} = -1$, respectively. If $v_i$ cannot receive information from $v_j$, then $a_{ij} = 0$, $a_{ij} \geq 0$ is assumed in this paper, which means the positive and negative edges cannot coexist between two agents. $D = \{d_{ij}\} \in \mathbb{R}^{n \times n}$ is the in-degree matrix of the graph $G$, where $d_{ii} = \sum_{j \in V} |a_{ij}|$, and $d_{ij} = 0, i \neq j$. $L_G$ is the Laplacian matrix of $G = (V, E, A)$, where $L_G = D - A$.

In the digraph $G$, For each pair of vertices $v_i, v_j$, if there is a
path from $v_i$ to $v_j$, and there also exists a path from $v_j$ to $v_i$, then we call $G$ a strongly connected graph.

**B. BASIC DEFINITIONS AND LEMMAS**

**Definition 1** [32]: An independent strongly connected component (iSCC) of a digraph $G = (V, E, A)$ is an induced subgraph $\bar{G} = (\bar{V}, \bar{E}, \bar{A})$ that is maximal, and satisfies $(v_i, v_j) \notin E$ for any $v_i \in V \backslash \bar{V}, v_j \in \bar{V}$.

**Definition 2** [39]: If all nodes in signed digraph $G$ can be divided into two disjoint nonempty node sets $V_\alpha, V_\beta$, such that $a_{ij} \geq 0, \forall v_i, v_j \in V_\alpha$ and $a_{ij} \leq 0, \forall v_i, v_j \in V_\beta$, then we call $G$ is structurally balanced. Otherwise, $G$ is structurally unbalanced.

**Definition 3** [38]: If an iSCC of a signed digraph $G$ is structurally balanced, then we call it structurally balanced independent strongly connected component (SBiSCC). Otherwise, it is known as structurally unbalanced independent strongly connected component (SU iSCC).

**Remark 1**: If one of two disjoint node sets of a SBiSCC is empty, then the SBiSCC is an iSCC in which all edges are positive. Obviously, if an iSCC contains only one node, it can also be called a SBiSCC [33].

**Definition 4** [40]: Let $X = \{x_{m+1}, -x_{m+1}, \ldots, x_n, -x_n\}$ be the states and the sign-inverted states of leaders. Then the convex hull containing all points in $X$ can be described as

$$Co (X) = \left\{ \sum_{i=m+1}^n (a_ix_i - b_ix_i)|a_i \geq 0, b_i \geq 0, \sum_{i=m+1}^n (a_i + b_i) = 1 \right\},$$

where $\sum_{i=m+1}^n (a_ix_i - b_ix_i)$ is the convex combination of points in $X$.

**Lemma 1** [1]: $L_G \in \mathbb{R}^{N \times N}$ is the Laplacian matrix of a nonnegative digraph $G$, if $G$ has a directed spanning tree, then 0 is a simple eigenvalue, the corresponding eigenvector is $1_N$, and all remaining eigenvalues have positive real parts.

**Lemma 2** [36]: In a signed digraph, for each follower, there exists at least one leader that has a directed path to the follower, then all the eigenvalues of $H$ have positive real parts, where $H = \Delta_F + \sum_{i=m+1}^{n} a_{ii} \bar{B}_{ii}, \bar{B}_{ii} = diag \{|b_{ii}|, \ldots, |b_{ml}|\}$. $\Delta_F$ is the Laplacian matrix of follower network, and $H^{-1}$ is nonnegative.

**Lemma 3** [37]: For a structurally balanced signed digraph $G$, an equivalent condition is $\exists D \in \mathbb{R}^+$ such that $DAD$ has no negative elements, where

$$D = \{diag(\sigma_1, \ldots, \sigma_n) | \sigma_i \in \{1, -1\}\}.$$

**III. MAIN RESULTS**

**A. STABILIZATION CONDITION FOR MULTI-AGENT SYSTEMS**

Considering the continuous-time agent linear dynamics system, the dynamic of each agent is described by

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t), \quad i = 1, \ldots, n,$$

where $x_i \in \mathbb{R}^N$ is the state of $i$th agent, $u_i \in \mathbb{R}^p$ is the control input, $A \in \mathbb{R}^{N \times N}$ is the system matrix, $B \in \mathbb{R}^{N \times p}$ is the input matrix.

**Assumption 1**: The pair $(A, B)$ is stabilizable.

**Assumption 2**: Weakly connected signed digraph $G = (V, E, A)$ contains $s$ SBiSCCs, described by $G_i = (V_i, E_i, A_i), i = 1, \ldots, s, 1 \leq s \leq m$, and no SU iSCCs.

Considering the agent control protocols:

$$u_l(t) = 0, \quad l \in L,$$

and

$$u_l(t) = -K \sum_{j=1}^m a_{lj} (sgn(a_{lj})x_l - x_j)$$

$$- K \sum_{l=m+1}^{n} b_{lj} (sgn(b_{lj})x_l - x_l), \quad i \in F,$$

where $K$ is the feedback gain matrix, if there is positive edge between $v_i$ and $v_j, l \in L$, then $b_{lj} = 1$, and if the edge between $v_i$ and $v_j$ is negative, then $b_{lj} = -1$, otherwise $b_{lj} = 0$.

**Theorem 1**: Under Assumption 1 and 2, consider a multi-agent system which consists of leaders and followers and satisfies (1). Assuming that some eigenvalues of the system matrix $A$ contain positive real parts in (1), then the necessary condition to ensure stabilization of this multi-agent system is, for each SBiSCC in follower topology, there exists agent $v_h$ can receive information from any leader’s information directly, i.e., $b_{hl} \neq 0, l \in L$.

**Proof**: For a follower network with $s$ SBiSCCs, there exists a row permutation matrix $E \in \mathbb{R}^{m \times m}$, the Laplacian matrix corresponding to the follower network can be described by

$$\tilde{L} = E L_F E^T = \begin{bmatrix}
L_1 & 0 & \cdots & 0 & 0 \\
0 & L_2 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & L_s & 0 \\
L_01 & L_02 & \cdots & L_{0s} & L_0
\end{bmatrix},$$

where, $L_F$ is the Laplacian matrix of follower topology, $L_1, \ldots, L_s$ are the Laplacian matrix blocks corresponding to $s$ SBiSCCs, $L_01, \ldots, L_{0s}$ correspond to the interconnections between the SBiSCCs and other followers, $L_0$ corresponds to the followers which does not belong to any SBiSCC. According to Lemma 2, all eigenvalues of $L_0$ have positive real parts, i.e., $L_0$ is invertible.

Here we employ two counterexamples to prove the necessity.

**Case 1**: Suppose that, all edges of some SBiSCCs are positive, and there does not exist such agent $v_h$ in these SBiSCCs, i.e., these SBiSCCs cannot receive information from leaders. Without loss of generality, we assume that the SBiSCC corresponding to $L_q, 1 \leq q \leq s$, cannot receive any leader’s information. According to (1), (3) and $b_{lj} = 0$, we have

$$\dot{x} = (L_q \otimes A - L_q \otimes BK)x,$$

where $x \in \mathbb{R}^N$ is the state of $i$th agent, $u_i \in \mathbb{R}^p$ is the control input, $A \in \mathbb{R}^{N \times N}$ is the system matrix, $B \in \mathbb{R}^{N \times p}$ is the input matrix.
SBiSCC can be treated as the SBiSCC where all edges are removed. According to the properties of strongly connected components, each SBiSCC has at least one directed spanning tree. Then according to the Lemma 1, \( L_q \) have one zero eigenvalue, therefore, we have \( A - \zeta B K = A \) on the basis of (4), \( \zeta \) denotes the zero eigenvalue of \( L_q \). As we assumed before, some eigenvalues of system matrix \( A \) contain positive real parts, hence partial eigenvalues of \( I_m \otimes A - L_q \otimes B K \) contain positive real parts according to \( \zeta = 0 \). Furthermore, according to the eigenvalue stability criterion of linear systems, we can conclude that the SBiSCCs without receiving leaders’ information is not asymptotically stable which means that the multi-agent system is not stabilizable.

It is easy to see that if there is only one node in a SBiSCC, which cannot also receive leader’s information, it can be treated as a special case of case 1. The corresponding proof is omitted here.

Case 2: Suppose that, some SBiSCCs contain both positive and negative edges, and there also does not exist such agent \( v_l \) in these SBiSCCs. Without loss of generality, we assume that the SBiSCC corresponding to \( L_r \), \( 1 \leq r \leq s \), cannot receive any leader’s information, the corresponding adjacency matrix is \( A_r \). By Lemma 3, we have \( D \in \mathcal{D} \), such that \( \tilde{A}_r = DA_r D \) has no negative elements, meanwhile, this SBiSCC can be treated as the SBiSCC where all edges are nonnegative. Then the Laplacian matrix of \( A_r \) is \( L_r = DL_r D \). By lemma 1, \( L_r \) has 0 simple eigenvalue. Therefore, \( L_r \) also has one 0 eigenvalue. The following stability analysis process about the whole system is similar to that in case 1, which is omitted here. The multi-agent system is also not stabilizable because of the existence of 0 eigenvalue.

Based on the counterexamples presented above, the whole multi-agent system is not stabilizable when there exist SBiSCCs without leaders’ control in follower network, which contradicts that the multi-agent system is stabilizable in Theorem 1. The proof of necessity is thus completed.

B. LEADER–FOLLOWER MATCHING METHOD

Combined with the above necessary condition for system stabilization and graph theory, we give a matching method between followers and leaders. The procedures are as follows:

(1) Employ Tarjan algorithm to select all strongly connected components in graph \( G_F \) composed of followers.

(2) Select all SBiSCCs that have no edges ending at them and contain zero eigenvalues in all strongly connected components.

(3) Select arbitrary follower \( v_l \) in one SBiSCC, and select arbitrary leader \( v_l \) in leader set to ensure \( b_{vl} \neq 0 \). Repeat this step, until there exists no SBiSCC in the follower network.

(4) If there exists one (or more) leader which is not matched to followers in the end, it is only necessary to choose the follower arbitrarily to match the leader.

Remark 2: Obviously, the time complexity of the leader-follower matching method mainly depends on finding strongly connected components. When employing the Tarjan algorithm, each node is traversed once, and each edge is traversed once, the corresponding time complexity is \( O (m + \varphi) \), where \( m \) and \( \varphi \) are the number of followers and edges, respectively.

C. STABILITY ANALYSIS OF BIPARTITE CONTAINMENT CONTROL

For the multi-agent system composed of \( m \) followers and \( n - m \) leaders, combining with the leader-follower matching method given above, the agent network \( G \) is formed.

Theorem 2: Consider the multi-agent system consisting of the dynamics (1). If the system satisfies the Assumption 1 and 2, using the leader-follower matching method to establish the leader-follower network, and employing the control protocols (2), (3) and feedback gain

\[
K = \max \left\{ 1, 1 / \min_{\lambda_i \in \lambda(H)} \Re(\lambda_i) \right\} B^T P,
\]

where \( P \) is the only positive definite solution of the algebraic Riccati equation

\[
PA + A^T P + I_N - PBB^T P = 0,
\]

the whole system will be asymptotically stable and all followers gradually converge into the convex hull spanned by the states and the sign-inverted states of leaders.

Proof: Combining (1) and (3), we have

\[
\dot{x}(t) = (I_m \otimes A - H \otimes B K) x(t) + \sum_{l=m+1}^{n} (B_{0l} \otimes B K) \tilde{x}_l(t),
\]

where \( B_{0l} = \text{diag} \{ b_{1l}, \ldots, b_{ml} \} \), \( H = L_F + \sum_{l=m+1}^{n} \tilde{B}_{0l} \), \( \tilde{B}_{0l} = \text{diag} \{ b_{1l}, \ldots, b_{ml} \} \), \( L_F \) is the Laplacian matrix corresponding to the follower network without external input, \( \tilde{x}_l = 1_m \otimes x_l \).

According to (3), the overall error can be given as

\[
e(t) = (H \otimes I_N) x - \left( \sum_{l=m+1}^{n} B_{0l} \otimes I_N \right) \tilde{x}_l = (H \otimes I_N) \left( x - \left( \sum_{l=m+1}^{n} H^{-1} B_{0l} \otimes I_N \right) \tilde{x}_l \right).
\]

Let \( \tilde{x}_l = x_l - \sum_{l=m+1}^{n} \xi_{il} x_l \), \( \xi_{il} \) is the \( i \)th row elements of \( H^{-1} B_{0l} 1_m \), then we have

\[
\dot{x} = x - \sum_{l=m+1}^{n} H^{-1} B_{0l} 1_m \otimes x_l = x - \sum_{l=m+1}^{n} \left( H^{-1} B_{0l} \otimes I_N \right) \tilde{x}_l.
\]

Take the derivative of (8), then we can obtain

\[
\dot{\tilde{x}} = \dot{x} - \sum_{l=m+1}^{n} \left( H^{-1} B_{0l} \otimes I_N \right) \dot{\tilde{x}}_l = (I_m \otimes A - H \otimes B K) x + \sum_{l=m+1}^{n} (B_{0l} \otimes B K) \tilde{x}_l
\]
A similarity transformation is performed on $H$, which is turned into Jordan form, there is $Q^{-1}HQ = J$, where $Q \in R^{m \times m}$. Applying a similarity transformation to $I_m \otimes A - H \otimes BK$, corresponding Jordan form is

$$
(I_m \otimes A - H \otimes BK) (Q \otimes I_m)
$$

where * denotes the matrix block. The matrices have the same eigenvalues after the similarity transformation, therefore, the asymptotically stable equivalence condition for (6) is that for any $\lambda_i \in \lambda(H), A - \lambda_i BK$ is Hurwitz. Further, all eigenvalues known by Lemma 2 have positive real parts, and the system satisfies Assumption 1, so that there exists a matrix $K$ to ensure $A - \lambda_i BK$ is Hurwitz.

Rewrite (5), then

$$
\left( A - BB^T P \right)^T P + P \left( A - BB^T P \right) = -I_N - \eta BB^T P.
$$

Let $\eta = \sigma + j\omega$, $\tau := \sigma - 1 \geq 0$, subtract $2\eta BB^T P$ from both sides of (11), then get a equation as follows:

$$
-I_N - (2\theta + 1) BB^T P
$$

$$
= \left( A - BB^T P \right)^T P + P \left( A - BB^T P \right) - 2\eta BB^T P
$$

$$
= \left( A - (1 + \theta) BB^T P \right)^T P + P \left( A - (1 + \theta) BB^T P \right)
$$

$$
= \left( A - (\sigma + j\omega) BB^T P \right)^H P + P \left( A - (\sigma + j\omega) BB^T P \right).
$$

Since (12) satisfies the Lyapunov equation, then for any $\sigma \geq 1$, $A - \eta BB^T P$ is Hurwitz. Since $\lambda_i \max \{1, 1/\min_{\lambda_i \in \lambda(H)} Re(\lambda_i) \} \geq 1$, when $K = \max \{1, 1/\min_{\lambda_i \in \lambda(H)} Re(\lambda_i) \} BB^T P$, $A - \lambda_i BK$ is Hurwitz, the system is asymptotically stable.

$A - \lambda_i BK$ is Hurwitz means that all the eigenvalues of $I_m \otimes A - H \otimes BK$ have negative real parts. According to the eigenvalue stability criterion, it can be obtained that the equilibrium state $\bar{x} = 0$ in (9) is asymptotically stable, i.e.,

$$
\lim_{t \to \infty} (x - \sum_{l=m+1}^{n} H^{-1}B_{0l} 1_m \otimes x_l) = 0 \text{ as } t \to \infty.
$$

Let $\bar{A} = [a_{ij}]$ denotes the nonnegative adjacency matrix corresponding to the follower network without external input, $L_F = D - \bar{A}$, $F_L = L_F/(n-m)$, since $(L_F \otimes I_N) \bar{x}_l = 0$, (7) can be written as

$$
e(t) = (H \otimes I_N) x - \left[ \sum_{l=m+1}^{n} \left( L_{Fl} + B_{0l} \right) \otimes I_N \right] \bar{x}_l
$$

$$
= (H \otimes I_N) \left\{ \left[ \sum_{l=m+1}^{n} H^{-1} \left( L_{Fl} + B_{0l} \right) \otimes I_N \right] \bar{x}_l \right\}
$$

$$
= (H \otimes I_N) \left\{ \left[ \sum_{l=m+1}^{n} H^{-1} \left( L_{Fl} + B_{0l} \right) \right] \otimes I_N \right\} \bar{x}_l.
$$

Let

$$
\Phi_l = L_{Fl} + \frac{1}{2(n-m)} (\bar{A} - A) + \frac{1}{2} (\bar{B}_{0l} + B_{0l})
$$

$$
\Psi_l = \frac{1}{2(n-m)} (\bar{A} - A) + \frac{1}{2} (\bar{B}_{0l} - B_{0l})
$$

Then

$$e(t) = (H \otimes I_N) \left( x - \left( \sum_{l=m+1}^{n} H^{-1} (\Phi_l + \Psi_l) 1_m \right) \otimes I_N \right).
$$

Note that

$$
\sum_{l=m+1}^{n} H^{-1} (\Phi_l + \Psi_l) 1_m
$$

$$
= \sum_{l=m+1}^{n} H^{-1} \left[ \frac{1}{n-m} L_F + \bar{B}_{0l} \right] 1_m
$$

$$
= H^{-1} \sum_{l=m+1}^{n} \left[ \frac{1}{n-m} L_F + \bar{B}_{0l} \right] 1_m = I_m.
$$

By Lemma 2, $H^{-1}$ is nonnegative, meanwhile, $\bar{L}_F 1_m = 0$, $([A] - A)$, and $(\bar{B}_{0l} - B_{0l})$ are also nonnegative. Then we have, $\sum_{l=m+1}^{n} H^{-1} \Phi_l 1_m$ and $\sum_{l=m+1}^{n} H^{-1} \Psi_l 1_m$ are both nonnegative. By the definition of the convex hull spanned by the leaders in Definition 4, \[\sum_{l=m+1}^{n} H^{-1} (\Phi_l + \Psi_l) 1_m \otimes x_l\] is a column vector of the convex combinations of points in $X$. According to $\lim_{t \to \infty} (x - \sum_{l=m+1}^{n} H^{-1}B_{0l} 1_m \otimes x_l) = 0$, we have $\lim_{t \to \infty} e(t) = 0$. Therefore, the follower will eventually converge into the convex hull which is spanned by the states and the sign-inverted states of leaders. 

Remark 3: Consider a special case that there exist SUISCCs which do not receive information from leaders in multi-agent systems. According to Corollary 3 in [33], all eigenvalues of Laplacian matrix of these SUISCCs have positive real parts.
which means the state 0 is a stable state of these iSCCs. We may as well treat these SUiSCCs as leaders whose all states are ultimately zero, then by Theorem 2, the whole system is also asymptotically stable.

IV. SIMULATION RESULTS
Consider a multi-agent system consists of followers $F = \{1, 2, \ldots, 13\}$ and leaders $L = \{14, 15, 16\}$, the network consisting of followers is shown in Fig. 3.

The controlled followers are selected by applying the bipartite graph maximum matching method to the follower topology. In order to represent the influence of the loss of SBiSCCs on the stability of the whole system, a set of maximum matching edge sets is determined as $1^+ \rightarrow 3^-, 3^+ \rightarrow 5^-, 4^+ \rightarrow 7^-, 5^+ \rightarrow 1^-, 6^+ \rightarrow 12^-, 8^+ \rightarrow 2^-, 9^+ \rightarrow 10^-, 10^+ \rightarrow 13^-, 11^+ \rightarrow 4^+, 13^+ \rightarrow 9^-$, the non-matching nodes determined by this maximum matching edge set are 6, 8 and 11, which are selected controlled follower nodes. Then the network topology composed of leaders and followers is shown in Fig. 4. After the state information of the leaders 14, 15, 16 are obtained, the state trajectories of all agents are shown in Fig. 5.

As can be seen from Fig. 5, some followers cannot converge well into the convex hull spanned by the leaders. Since the SBiSCCs in the follower network topology contain multiple individuals, such components cannot be recognized well when the bipartite graph maximum matching method is used to determine the controlled nodes. The individuals 1, 3, 5 cannot receive the information of leaders, and they can only be affected by each other, therefore, these followers cannot converge to the inside of the convex hull. Meanwhile, because of the existence of SUiSCCs, the number of selected followers is not the least.

The SBiSCCs determined by employing the leader-follower matching method in Fig. 3 are $\{1, 3, 5\}$ and $\{11\}$, individuals 3, 8, and 11 are selected as controlled followers, and individual 8 is an arbitrarily selected individual. The network established by connecting the leaders and followers is shown in Fig. 6. Under the same state information of leaders 14, 15, 16, the state trajectories of all agents are shown in Fig. 7.
When the states of leaders are sign-symmetric as shown in Fig. 7, the followers eventually converge into the convex hull spanned by the states and the sign-inverted states of leaders. The bipartite containment control of multi-agent systems is achieved as shown in Fig. 8.

V. CONCLUSIONS
It has great practical significance for cooperative control of linear multi-agent systems to study how to select the least number of controlled nodes to realize the stability of signed digraphs. The necessary condition is discussed by using the tools from algebra and graph theory to achieve bipartite containment control in the case of multi-agent systems with multiple leaders. Based on the network topology composed of the followers, the leader-follower matching method to establish the leader-follower signed network is further given. The corresponding control protocol and state feedback gain are designed for the followers, which impels the followers to gradually converge into the convex hull spanned by the states and the sign-inverted states of leaders to effectively realize the bipartite containment control of the multi-agent systems. The simulation results show that when selecting the controlled followers to achieve bipartite containment control, the method proposed in this paper can avoid the influences of SBiSCCs and need fewer controlled followers when achieving the stability of whole system.

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