COMBINATORIAL YANG-BAXTER MAPS ARISING FROM TETRAHEDRON EQUATION

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ABSTRACT. We survey the matrix product solutions of the Yang-Baxter equation obtained recently from the tetrahedron equation. They form a family of quantum $R$ matrices of generalized quantum groups interpolating the symmetric tensor representations of $U_q(A_{n-1}^{(1)})$ and the anti-symmetric tensor representations of $U_{-q^{-1}}(A_{n-1}^{(1)})$. We show that at $q = 0$ they all reduce to the Yang-Baxter maps called combinatorial $R$, and describe the latter by explicit algorithm.

1. Introduction

Tetrahedron equation \cite{sergeev} is a generalization of the Yang-Baxter equation \cite{kuniba} and serves as a key to the integrability in three dimension (3D). Typically it has the form called $RRRR$ type or $RLLL$ type:

\[
R_{1,2,4}R_{1,3,5}R_{2,3,6}R_{4,5,6} = R_{4,5,6}R_{2,3,6}R_{1,3,5}R_{1,2,4},
\]

\[
L_{1,2,4}L_{1,3,5}L_{2,3,6}L_{4,5,6} = R_{4,5,6}L_{2,3,6}L_{1,3,5}L_{1,2,4}.
\]

Here $R \in \text{End}(F^\otimes 3)$ and $L \in \text{End}(V \otimes V \otimes F)$ for some vector spaces $F$ and $V$. The above equalities hold in $\text{End}(F^\otimes 6)$ and $\text{End}(V^\otimes 3 \otimes F^\otimes 3)$ respectively, and the indices specify the components on which $R$ and $L$ act nontrivially. We call the solutions $R$ and $L$ 3D $R$ and 3D $L$, respectively. The tetrahedron equations are reducible to the Yang-Baxter equation

\[
S_{1,2}S_{1,3}S_{2,3} = S_{2,3}S_{1,3}S_{1,2}
\]

if the spaces 4, 5, 6 are evaluated away appropriately \cite{sergeev}. Such reductions and the relevant quantum group aspects have been studied systematically in the recent work \cite{okado, sergeev, kuniba} by Okado, Sergeev and the author for the distinguished example of 3D $R$ and 3D $L$ originating in the quantized algebra of functions \cite{kuniba}. They correspond to the choice $V = \mathbb{C}^2$ and the $q$-oscillator Fock space $F = \bigoplus_{m \geq 0} \mathbb{C}|m\rangle$.

In this paper we first review the latest development in \cite{kuniba} concerning the reduction by trace. It generates $2^n$ solutions $S(z)$ to the Yang-Baxter equation from the $n$ product of $R$ and $L$. Any $S(z)$ is rational in the parameter $q$ and the (multiplicative) spectral parameter $z$. Symmetry of $S(z)$ is described by generalized quantum groups \cite{sergeev, kuniba} which include quantum affine \cite{kuniba, sergeev} and super algebras of type $A$.

In the last Section \cite{kuniba} we supplement a new result, Theorem \cite{kuniba}. It shows that at $q = 0$ each $S(z)$ yields a combinatorial $R$, a certain bijection between finite sets satisfying the Yang-Baxter equation. We describe it by explicit combinatorial algorithm generalizing \cite{kuniba, sergeev}.
The notion of combinatorial $R$ originates in the crystal base theory, a theory of quantum groups at $q = 0$ [10]. The motivation for $q = 0$ further goes back to Baxter’s corner transfer matrix method [11], where it corresponds to the low temperature limit manifesting fascinating combinatorial features of Yang-Baxter integrable lattice models. It has numerous applications including generalized Kostka-Foulkes polynomials, Fermionic formulas of affine Lie algebra characters, integrable cellular automata in one dimension and so forth. See for example [14, 25, 4, 24, 11, 17] and reference therein. Combinatorial $R$’s form most systematic examples of set-theoretical solutions to the Yang-Baxter equation (Yang-Baxter maps) [6, 26] arising from the representation theory of quantum groups.

In this paper the 3D $R$ and the 3D $L$ will mainly serve as the constituent of the $S(z)$ which tends to the combinatorial $R$ at $q = 0$. However they possess a decent combinatorial aspect by themselves as pointed out in [19, eq.(2.41)] for the 3D $R$. In fact their limits (39) define the maps

$$\lim_{q \to 0} R : \begin{pmatrix} i \\ j \\ k \end{pmatrix} \mapsto \begin{pmatrix} j + \max(i - k, 0) \\ \min(i, k) \\ j + \max(k - i, 0) \end{pmatrix}, \quad \lim_{q \to 0} L : \begin{pmatrix} i \\ j \\ k \end{pmatrix} \mapsto \begin{pmatrix} j + \max(i - j - k, 0) \\ \min(i, k + j) \\ \max(k + j - i, 0) \end{pmatrix}$$

on $(\mathbb{Z}_{\geq 0})^3$ and on $\{0, 1\} \times \{0, 1\} \times \mathbb{Z}_{\geq 0}$, respectively. The tetrahedron equations survive the limit nontrivially as the combinatorial tetrahedron equations, e.g.,

They constitute the local relations responsible for the Yang-Baxter equation of the combinatorial $R$ in Corollary [9].

The layout of the paper is as follows. In Section 2 we recall the definition of the 3D $R$ and 3D $L$. In Section 3 tetrahedron equations of type $RRRR$ and $RLLL$ are given with their generalization to $n$-layer case. In Section 4 the $2^n$ family of solutions $S(z)$ to the Yang-Baxter equation are constructed by applying the trace reduction. In Section 5 generalized quantum group symmetry of $S(z)$ is explained. Section 6 contains the main Theorem 6, which describes the combinatorial $R$ arising from $S(z)$ at $q = 0$ in terms of explicit combinatorial algorithm.

Throughout the paper we assume that $q$ is not a root of unity and use the notations:

$$(z; q)_m = \prod_{k=1}^{m}(1 - zq^{k-1}), \quad (q)_m = (q; q)_m, \quad \binom{m}{k}_q = \frac{(q)_m}{(q)_k(q)_{m-k}}.$$
Let $F$ be a Fock space $F = \bigoplus_{m \geq 0} \mathbb{C}|m\rangle$ and $a^\pm, k \in \text{End}(F)$ be the operators on it called $q$-oscillators:

$$a^+|m\rangle = |m + 1\rangle, \quad a^-|m\rangle = (1 - q^{2m})|m - 1\rangle, \quad k|m\rangle = q^m|m\rangle. \tag{1}$$

They satisfy the relations

$$ka^\pm = q^{\pm 1}a^\pm k, \quad a^+a^- = 1 - k^2, \quad a^-a^+ = 1 - q^2k^2. \tag{2}$$

We define a three dimensional $R$ operator, 3D $R$ for short, $\mathcal{R} \in \text{End}(F^\otimes 3)$ by

$$\mathcal{R}(|i\rangle \otimes |j\rangle \otimes |k\rangle) = \sum_{a,b,c \geq 0} \mathcal{R}^{a,b,c}_{l,j,k}(a) \otimes |b\rangle \otimes |c\rangle, \tag{3}$$

where several formulas are known for the matrix element:

$$\mathcal{R}^{a,b,c}_{l,j,k} = \delta_{i+j+k} \sum_{\lambda + \mu = b} (1)^{\lambda} q^{i(c-j)+(k+1)\lambda + \mu(\mu-k)} \left(\frac{q^2 + \mu}{q^2}\right)^{c+\mu} \left(\frac{i}{\mu}\right)^{j} \left(\frac{j}{\mu}\right)^{\lambda - k}, \tag{4}$$

$$= \delta_{i+j+k} \sum_{\lambda + \mu = b} (1)^{\lambda} q^{ik+b+\lambda(c-a)+\mu(\mu-i-k-1)} \left(\frac{i}{\mu}\right)^{\lambda + a} \left(\frac{\lambda + a}{\mu}\right)^{j}, \tag{5}$$

$$= \delta_{i+j+k} \sum_{\lambda + \mu = b} (1)^{\lambda} q^{ik+b} \left(\frac{\lambda + a}{\mu}\right)^{j} \left(\frac{i}{\mu}\right)^{\lambda - k}, \tag{6}$$

where $\delta_i^j = \delta_{i,j}$ just to save the space. The sum (4) is over $\lambda, \mu \in \mathbb{Z}_{\geq 0}$ satisfying $\lambda + \mu = b$, $\mu \leq i$ and $\lambda \leq j$. The sum (5) is over $\lambda, \mu \in \mathbb{Z}_{\geq 0}$ satisfying $\lambda + \mu = b$ and $\mu \leq i$. The integral (6) encircles $u = 0$ anti-clockwise so as to pick the coefficient of $u^b$. Derivation of these formulas can be found in [19, Th.2] for (4), [15, Sec.4] for (5) and [20] for (6). The 3D $R$ can also be expressed as a collection of operators on the third component. For example (4) yields

$$\mathcal{R}(|i\rangle \otimes |j\rangle \otimes |k\rangle) = \sum_{a,b \geq 0} |a\rangle \otimes |b\rangle \otimes \mathcal{R}^{a,b}_{l,j,k}(c), \quad \mathcal{R}^{a,b}_{l,j,k} \in \text{End}(F), \tag{7}$$

$$\mathcal{R}^{a,b}_{l,j,k} = \delta_{i+j+k} \sum_{\lambda + \mu = b} (1)^{\lambda} q^{i\lambda + \mu - \lambda i} (\lambda + a)^{j} (\lambda + b)^{\mu - \lambda - \mu}, \tag{8}$$

where the sum is taken under the same condition as in (4), which guarantees that the powers of $q$-oscillators are nonnegative.

The 3D $R$ was first obtained as the intertwiner of the quantized coordinate ring $A_q(sl_3)$ by [15]. It was found later also from a quantum geometry consideration in a different gauge [2]. They were shown to be the same object in [19, eq.(2.29)]. See also [20, App. A] and [17, Sec. 4] for the recursion relations characterizing $\mathcal{R}$ and useful corollaries. Here we note the properties [15]

$$\mathcal{R} = \mathcal{R}^{-1}, \quad \mathcal{R}^{a,b,c}_{l,j,k} = \mathcal{R}^{c,b,a}_{k,j,i} \in q^\xi \mathbb{Z}(q^2), \quad \mathcal{R}^{a,b,c}_{l,j,k} = (q^2)^i (q^2)^j (q^2)^k \mathcal{R}^{a,b,c}_{l,j,k}, \tag{9}$$

where $\xi = 0, 1$ is specified by $\xi \equiv (a - j)(c - j) \mod 2$.\footnote{The formula for it on p194 in [17] contains a misprint unfortunately. The formula (4) here is a correction of it.}
Example 1. The following is the list of all the nonzero \( R_{a,b,c} \).
\[
R_{3,1,2}^{1,3,0} = -q^2(1 - q^4)(1 - q^6), \quad R_{3,1,2}^{2,2,1} = (1 + q^2)(1 - q^6)(1 - q^2 - q^6), \\
R_{3,1,2}^{3,3,0} = q^6, \quad R_{3,1,2}^{3,1,1} = -q^2(-1 - q^2 + q^6 + q^8 + q^{10}).
\]
We see \( \lim_{q \to 0} R_{3,1,2}^{a,b,c} = 0 \) in agreement with (39) with \( \epsilon = 0 \).

The following is the list of all the nonzero \( R_{3,1} \).
\[
R_{3,1}^{1,3} = (a^-)^3 \alpha^+ - q^{-4}(1 + q^2 + q^4)(a^-)^2 \kappa^2, \quad R_{3,1}^{4,0} = \alpha^+ \kappa^3, \\
R_{3,1}^{3,1} = q^{-2}(1 + q^2 + q^4)\alpha^- \alpha^+ \kappa^2 - q^{-2} \kappa^4, \quad R_{3,1}^{0,4} = -q^{-2}(a^-)^3 \kappa.
\]

A part of them will be used in Example 2.

Let us proceed to the 3D \( L \). Set \( V = \mathbb{C}v_0 \oplus \mathbb{C}v_1 \). We define a three dimensional \( L \) operator, 3D \( L \) for short, \( L \in \text{End}(V^\otimes 2 \otimes F) \) by a format parallel with (7):
\[
L(v_\alpha \otimes v_\beta \otimes |m\rangle) = \sum_{\gamma,\delta} v_\gamma \otimes v_\delta \otimes \mathcal{L}_{\alpha,\beta}^{\gamma,\delta} |m\rangle, \quad (10)
\]
where \( \mathcal{L}_{\alpha,\beta}^{\gamma,\delta} \in \text{End}(F) \) are zero except the following six cases:
\[
\mathcal{L}_{0,0}^{0,0} = L_{1,1}^{1,1} = 1, \quad \mathcal{L}_{0,0}^{0,1} = -q \kappa, \quad \mathcal{L}_{1,0}^{0,1} = \kappa, \quad \mathcal{L}_{1,0}^{1,0} = \alpha^-, \quad \mathcal{L}_{0,1}^{1,0} = \alpha^+. \quad (11)
\]
Thus \( L \) may be regarded as defining a six-vertex model \([1]\) whose Boltzmann weights take values in the \( q \)-oscillators. One may also write (10) like (3) as
\[
L(v_\alpha \otimes v_\beta \otimes |m\rangle) = \sum_{\gamma,\delta,j} \mathcal{L}_{\alpha,\beta}^{\gamma,\delta,j} v_\gamma \otimes v_\delta \otimes |j\rangle,
\]

\[
\mathcal{L}_{0,0,m}^{0,0,j} = L_{1,1,m}^{1,1,j} = \delta_m^j, \quad \mathcal{L}_{0,1,m}^{1,0,j} = -\delta_m^j q^{m+1}, \quad \mathcal{L}_{1,0,m}^{0,1,j} = \delta_m^j q^m, \quad \mathcal{L}_{1,0,m}^{1,0,j} = \delta_m^j q^{m+1}. \quad (12)
\]
The other \( \mathcal{L}_{\alpha,\beta}^{\gamma,\delta,j,m} \) are zero.

We assign a solid arrow to \( F \) and a dotted arrow to \( V \), and depict the matrix elements of 3D \( R \) and 3D \( L \) as
\[
R_{i,j,k}^{a,b,c}, \quad L_{i,j,k}^{a,b,c}.
\]
We will also depict \( R \) and \( L \) by the same diagrams with no indices.

3. Tetrahedron equation

The \( R \) satisfies the tetrahedron equation of \( RRRR \) type \([15]\)
\[
R_{1,2,4}R_{1,3,5}R_{2,3,6}R_{4,5,6} = R_{4,5,6}R_{2,3,6}R_{1,3,5}R_{1,2,4}, \quad (13)
\]
which is an equality in \( \text{End}(F^\otimes 6) \). Here \( R_{i,j,k} \) acts as \( R \) on the \( i, j, k \) th components from the left in the tensor product \( F^\otimes 6 \), and as identity elsewhere \([4]\). By denoting the \( F \) at the \( i \) th component by a solid arrow with \( i \), \( [15] \) is depicted as follows:

\[\text{Diagram for } R_{i,j,k}^{a,b,c}\]
The $L$ satisfies the tetrahedron equation of $RLLL$ type \[2\]
\[L_{1,2,4}L_{1,3,5}L_{2,3,6}R_{4,5,6} = R_{4,5,6}L_{2,3,6}L_{1,3,5}L_{1,2,4},\] (14)
which is an equality in $\text{End}(V \otimes F \otimes F \otimes F)$. The indices are assigned according to the same rule as in (13). By denoting the $V$ at the $i$th component by a dotted arrow with $i$, (14) is depicted as follows:

Viewed as an equation on $R$, (14) is equivalent to the intertwining relation for the irreducible representations of the quantized coordinate ring $A_q(sl_3)$ [15], [19, eq.(2.15)] in the sense that the both lead to the same solution given in (4)–(6) up to an overall normalization.

One can concatenate the tetrahedron equations to form the $n$-layer versions mixing the two types (13) and (14) arbitrarily. To describe them we introduce the notation unifying $F, V$ and $R, L$.

\[W^{(\epsilon)} = \begin{cases} F, & \epsilon = 0 \\ V, & \epsilon = 1 \end{cases}, \quad S^{(\epsilon)} = \begin{cases} \mathcal{R}, & \epsilon = 0 \\ \mathcal{L}, & \epsilon = 1 \end{cases}, \quad S^{(\epsilon)}_{a,b} = \begin{cases} a_{i,j}^{a,b}, & \epsilon = 0 \\ a_{i,j}^{a,b}, & \epsilon = 1 \end{cases}, \quad S^{(\epsilon)}_{a,b,c} = \begin{cases} a_{i,j,k}^{a,b,c}, & \epsilon = 0 \\ a_{i,j,k}^{a,b,c}, & \epsilon = 1 \end{cases}.\] (15)

Note that
\[S^{(\epsilon)}_{a,b,c} = 0 \text{ unless } (a+b+b+c) = (i+j+k).\] (16)

Now (13) and (14) are written as
\[S^{(\epsilon)}_{1,2,4}S^{(\epsilon)}_{1,3,5}S^{(\epsilon)}_{2,3,6}R^{(\epsilon)}_{4,5,6} = R^{(\epsilon)}_{4,5,6}S^{(\epsilon)}_{2,3,6}S^{(\epsilon)}_{1,3,5}S^{(\epsilon)}_{1,2,4} \quad (\epsilon = 0, 1)\] (17)
which is an equality in $\text{End}(W^{(\epsilon)} \otimes W^{(\epsilon)} \otimes W^{(\epsilon)} \otimes F \otimes F \otimes F)$.

Let $n$ be a positive integer. Given an arbitrary sequence $(\epsilon_1, \ldots, \epsilon_n) \in \{0, 1\}^n$, we set
\[W = W^{(\epsilon_1)} \otimes \cdots \otimes W^{(\epsilon_n)}.\] (18)
Let \( W^{(e_1)} \), \( W^{(e_2)} \), \( W^{(e_3)} \) be copies of \( W^{(e_i)} \), where \( e_1, e_2, \) and \( e_3 \) are distinct labels. Replacing the spaces 1, 2, 3 by them in (17) we have
\[
\mathcal{S}^{(e_1)} \mathcal{S}^{(e_2)} \mathcal{S}^{(e_3)} \mathcal{S}^{(e_4)} \mathcal{S}^{(e_5)} \mathcal{S}^{(e_6)} = \mathcal{R}^{(4,5,6)} = \mathcal{R}^{(4,5,6)} \mathcal{S}^{(e_1)} \mathcal{S}^{(e_2)} \mathcal{S}^{(e_3)} \mathcal{S}^{(e_4)} \mathcal{S}^{(e_5)} \mathcal{S}^{(e_6)},
\]
for each \( i \). Thus for any \( i \) one can let \( \mathcal{R}^{(4,5,6)} \) penetrate \( \mathcal{S}^{(e_1)} \mathcal{S}^{(e_2)} \mathcal{S}^{(e_3)} \mathcal{S}^{(e_4)} \mathcal{S}^{(e_5)} \mathcal{S}^{(e_6)} \) to the left transforming it into the reverse order product \( \mathcal{S}^{(e_1)} \mathcal{S}^{(e_2)} \mathcal{S}^{(e_3)} \mathcal{S}^{(e_4)} \mathcal{S}^{(e_5)} \mathcal{S}^{(e_6)} \). Repeating this \( n \) times leads to
\[
(\mathcal{S}^{(e_1)} \mathcal{S}^{(e_2)} \mathcal{S}^{(e_3)} \mathcal{S}^{(e_4)} \mathcal{S}^{(e_5)} \mathcal{S}^{(e_6)}) \cdots (\mathcal{S}^{(e_1)} \mathcal{S}^{(e_2)} \mathcal{S}^{(e_3)} \mathcal{S}^{(e_4)} \mathcal{S}^{(e_5)} \mathcal{S}^{(e_6)}) = \mathcal{R}^{(4,5,6)} (\mathcal{S}^{(e_1)} \mathcal{S}^{(e_2)} \mathcal{S}^{(e_3)} \mathcal{S}^{(e_4)} \mathcal{S}^{(e_5)} \mathcal{S}^{(e_6)}) \cdots (\mathcal{S}^{(e_1)} \mathcal{S}^{(e_2)} \mathcal{S}^{(e_3)} \mathcal{S}^{(e_4)} \mathcal{S}^{(e_5)} \mathcal{S}^{(e_6)}).
\]
This is an equality in \( \text{End}(\mathcal{W} \otimes \mathcal{W} \otimes \mathcal{W} \otimes \mathcal{F} \otimes \mathcal{F} \otimes \mathcal{F}) \), where \( \mathcal{W} = W^{(e_1)} \cdots \otimes W^{(e_n)} \). The spaces \( \mathcal{W} \) and \( \mathcal{W} \) should be understood similarly. They are just copies of \( \mathcal{W} \) in (18). The relation (19) is depicted as follows:

Here the broken arrows represent either solid or dotted arrows depending on whether the corresponding \( e_i \) is 0 or 1. The vertices on the \( i \)th layer \( \mathcal{S}^{(e_i)} \) should also be understood as \( \mathcal{R} \) or \( \mathcal{L} \) accordingly.

4. Solution to the Yang-Baxter equation

One can reduce (19) to the Yang-Baxter equation involving spectral parameters. In this paper we shall only consider the reduction by trace. See [22, 21] for another reduction by using boundary vectors.

Define \( h \in \text{End}(F) \) by \( h(m) = m|m \). By (10), \( [x^{h_1 + h_2} y^{h_3 + h_4} \mathcal{R}^{(4,5,6)}] = 0 \) holds for parameters \( x \) and \( y \), where the indices specify the spaces on which the operators act nontrivially. Multiply \( \mathcal{R}^{(4,5,6)} x^{h_1 + h_2} y^{h_3 + h_4} = x^{h_1 + h_2} y^{h_3 + h_4} \mathcal{R}^{(4,5,6)} \) from the left to (19) and take the trace over the space \( F^\otimes 3 \) corresponding to \( 4, 5, 6 \). The result becomes the Yang-Baxter equation
\[
\mathcal{S}_{\alpha, \beta}(x) \mathcal{S}_{\alpha, \gamma}(xy) \mathcal{S}_{\gamma, \beta}(y) = \mathcal{S}_{\beta, \gamma}(y) \mathcal{S}_{\alpha, \gamma}(xy) \mathcal{S}_{\alpha, \beta}(x) \in \text{End}(\mathcal{W} \otimes \mathcal{W} \otimes \mathcal{W}) \quad (20)
\]
for the matrix \( \mathcal{S}_{\alpha, \beta}(z) \in \text{End}(\mathcal{W} \otimes \mathcal{W}) \) constructed as
\[
\mathcal{S}_{\alpha, \beta}(z) = \text{Tr}_3 \left( z^{h_3} \mathcal{S}^{(e_3)} \cdots \mathcal{S}^{(e_n)} \mathcal{S}_{\alpha, \beta, 3} \right),
\]
\[
S_{\alpha, \beta}(z) = \text{Tr}_3 \left( z^{h_3} \mathcal{S}_{\alpha, \beta, 3} \cdots \mathcal{S}^{(e_n)} \right),
\]
where \( \mathcal{S}_{\alpha, \beta, 3} \) is understood as \( \mathcal{R} \) or \( \mathcal{L} \).
where $3$ denotes a copy of $F$. To describe the matrix elements of $S_{\alpha, \beta}(z)$ we write the basis of (18) as

$$\mathcal{W} = \bigoplus_{m_1, \ldots, m_n} \mathbb{C}|m_1, \ldots, m_n\rangle, \quad |m_1, \ldots, m_n\rangle = |m_1\rangle^{(\epsilon_1)} \otimes \cdots \otimes |m_n\rangle^{(\epsilon_n)}, \quad (22)$$

$$|m\rangle^{(0)} = |m\rangle \in F \quad (m \in \mathbb{Z}_{\geq 0}), \quad |m\rangle^{(1)} = v_m \in V \quad (m \in \{0, 1\}). \quad (23)$$

The range of the indices $m_i$ are to be understood as $\mathbb{Z}_{\geq 0}$ or $\{0, 1\}$ according to $\epsilon_i = 0$ or $1$ as in (23). It will crudely be denoted by $0 \leq m_i \leq 1/\epsilon_i$. We use the shorthand $|\mathbf{m}\rangle = |m_1, \ldots, m_n\rangle$ for $\mathbf{m} = (m_1, \ldots, m_n)$ and write (22) as $\mathcal{W} = \bigoplus_{\mathbf{m}} \mathbb{C}|\mathbf{m}\rangle$. We set $|\mathbf{m}| = m_1 + \cdots + m_n$.

Let $S(z) \in \text{End}(\mathcal{W} \otimes \mathcal{W})$ denote the solution (21) of the Yang-Baxter equation, where the inessential labels $\alpha, \beta$ are now suppressed. Remember, however, that $S(z)$ depends on the choice $(\epsilon_1, \ldots, \epsilon_n) \in \{0, 1\}^n$. We write its action as

$$S(z)(|i\rangle \otimes |j\rangle) = \sum_{\mathbf{a}, \mathbf{b}} S(z)_{i,j}^{\mathbf{a}, \mathbf{b}} |\mathbf{a}\rangle \otimes |\mathbf{b}\rangle. \quad (24)$$

Then the matrix elements are given by

$$S(z)_{i,j}^{\mathbf{a}, \mathbf{b}} = \text{Tr}_F \left( z^{b} S^{(\epsilon_1)}_{a_1, b_1} \cdots S^{(\epsilon_n)}_{a_n, b_n} \right)_{i, j, \mathbf{a}, \mathbf{b}} \quad (25)$$

$$= \sum_{c_0, \ldots, c_{n-1}} z^{c_0} S^{(\epsilon_1)}_{a_1, b_1} \cdots S^{(\epsilon_n)}_{a_n, b_n, c_n} \quad (26)$$

The operators in (25) are defined by (15), (11) and (8). From (10) it follows that

$$S(z)_{i,j}^{\mathbf{a}, \mathbf{b}} = 0 \quad \text{unless} \quad |\mathbf{a}| = |i|, |\mathbf{b}| = |j|. \quad (27)$$

Given such $\mathbf{a}, \mathbf{b}, i$ and $j$, (10) further reduces the sums over $c_i \in \mathbb{Z}_{\geq 0}$ in (26) effectively into a single sum. The latter property in (27) implies the direct sum decomposition:

$$S(z) = \bigoplus_{l, m \geq 0} S_{l,m}(z), \quad S_{l,m}(z) \in \text{End}(\mathcal{W}_l \otimes \mathcal{W}_m), \quad \mathcal{W}_l = \bigoplus_{|\mathbf{m}| = l} \mathbb{C}|\mathbf{m}\rangle \subset \mathcal{W}, \quad (28)$$

where the former sum ranges over $0 \leq l, m \leq n$ if $\epsilon_1 \cdots \epsilon_n = 1$ and $l, m \in \mathbb{Z}_{\geq 0}$ otherwise. The formula (25) is depicted as

$$S(z)_{i,j}^{\mathbf{a}, \mathbf{b}} = \text{Tr}_F \left( z^{b} S^{(\epsilon_1)}_{a_1, b_1} \cdots S^{(\epsilon_n)}_{a_n, b_n} \right)_{i, j, \mathbf{a}, \mathbf{b}} \quad (29)$$

Here the broken arrows represent either solid or dotted arrows according to $\epsilon_i = 0$ or $1$ at the corresponding site. Thus (25) is a matrix product construction of $S(z)$ in terms of 3D $R$ and 3D $L$ with the auxiliary space $F$.

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3 The labels $\alpha, \beta \ldots$ introduced for the exposition of (20) will no longer be used in the rest of the paper, and should not be confused with the indices of $S_{l,m}(z)$ in (28).
Example 2. Take \( n = 3 \) and \((\epsilon_1, \epsilon_2, \epsilon_3) = (1, 0, 1)\). Then one has

\[
S(z)(|031\rangle \otimes |110\rangle) = S_{031,110}^{031,110}(z)|031\rangle \otimes |110\rangle + S_{031,110}^{040,101}(z)|040\rangle \otimes |101\rangle + S_{031,110}^{121,020}(z)|121\rangle \otimes |020\rangle + S_{031,110}^{130,011}(z)|130\rangle \otimes |011\rangle,
\]
where the matrix elements are expressed as

\[
S_{031,110}^{031,110}(z) = \text{Tr}(z^h \mathbf{L}_{0,1}^{0,1,3,1} \mathbf{L}_{1,1}^{1,0}), \quad S_{031,110}^{040,101}(z) = \text{Tr}(z^h \mathbf{L}_{0,1}^{0,1,3,1} \mathbf{L}_{1,1}^{1,0}),
\]

\[
S_{031,110}^{121,020}(z) = \text{Tr}(z^h \mathbf{L}_{0,1}^{0,1,2,2} \mathbf{L}_{1,1}^{1,0}), \quad S_{031,110}^{130,011}(z) = \text{Tr}(z^h \mathbf{L}_{0,1}^{0,1,3,1} \mathbf{L}_{1,1}^{1,0}).
\]

Using \( L_{i,j}^{a,b} \) and \( R_{i,j}^{a,b} \) in Example 1 one calculates them for instance as

\[
S_{031,110}^{040,101}(z) = \text{Tr}(z^h (-qk)a^+ k^3a^-) = -q^{-2}\text{Tr}(k^4zha^+a^-)
\]

\[
= -q^{-2} \sum_{m=0}^{\infty} (q^4z^m (1 - q^{2m})) = -q^2(1 - q^2)(1 - q^6z)
\]

Similar calculations lead to

\[
S_{031,110}^{031,110}(z) = \frac{q^3(q^2 - z)}{(1 - q^4z)(1 - q^6z)}, \quad S_{031,110}^{121,020}(z) = \frac{-q^2(1 - q^5)(q^2 - z)z}{(1 - q^2z)(1 - q^4z)(1 - q^6z)},
\]

\[
S_{031,110}^{130,011}(z) = \frac{-(1 - q^2)z(q^2 - z - q^2z + q^{10}z)}{(1 - q^2z)(1 - q^4z)(1 - q^6z)}.
\]

In general \( S(z)^a\) is a rational function of \( q \) and \( z \).

Example 3. For \( 0 \leq a, b, i, j \leq 1 \), \( R_{i,j}^{a,b} \) and \( L_{i,j}^{a,b} \) are the same except \( R_{1,1}^{1,1} = a^a a^+ - k^2 \) and \( L_{1,1}^{1,1} = 1 \). This implies that \( S(z)^a \) with \( (a, b, i, j, \epsilon_a) = (1, 1, 1, 1) \) depends on \( \epsilon_a = 0, 1 \). The following table shows such examples, in which the case \((\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) = (0, 0, 0, 0)\) is omitted since the expression is too bulky.

| \((\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)\) | \((0,1,0,1)\) | \((0,1,0,0)\) | \((0,0,0,1)\) |
|---|---|---|---|
| \(S_{011,0111}^{011,0111}(z)\) | \((1 - q^2z)/(1 - q^2z)(1 - q^4z)\) | \((1 - q^2z)(1 - q^2z + q^4z)\) | \((1 - q^2z)(1 - q^2z)(1 - q^4z)\) |
| \(S_{0121,11011}(z)\) | \((1 - q^2z)/(1 - q^2z)(1 - q^4z)\) | \((1 - q^2z)(1 - q^2z + q^4z)\) | \((1 - q^2z)(1 - q^2z)(1 - q^4z)\) |

5. Generalized quantum group symmetry

The \( S(z) \) constructed in the previous section possesses the generalized quantum group symmetry. Recall that \( (\epsilon_1, \ldots, \epsilon_n) \in \{0, 1\}^n \) is an arbitrary sequence. Set

\[
q_i = (-1)^{\epsilon_i} q_i^{1-2\epsilon_i}, \quad D_{i,j} = \prod_{k \in \{i+1, \ldots, j\}} (q_k)^{2\delta_{i,j}-1} \quad (i, j \in \mathbb{N}).
\]

We introduce the \( \mathbb{C}(q) \)-algebra \( \mathcal{U}_A = \mathcal{U}_A(\epsilon_1, \ldots, \epsilon_n) \) generated by \( e_i, f_i, k_i^{\pm 1} \) \((i \in \mathbb{N})\) obeying the relations

\[
k_i k_i^{-1} = k_i^{-1} k_i = 1, \quad [k_i, k_j] = 0,
\]

\[
k_i e_j = D_{i,j} e_j k_i, \quad k_i f_j = D_{i,j}^{-1} f_j k_i, \quad [e_i, f_j] = \delta_{i,j} k_i - k_i^{-1} \frac{q}{q - 1}.
\]

We endow it with the Hopf algebra structure with coproduct \( \Delta \), counit \( \varepsilon \) and antipode \( S \) as follows:

\[
\Delta k_i^{\pm 1} = k_i^{\pm 1} \otimes k_i^{\pm 1}, \quad \Delta e_i = 1 \otimes e_i + e_i \otimes k_i, \quad \Delta f_i = f_i \otimes 1 + k_i^{-1} \otimes f_i, \quad \varepsilon(k_i) = 1, \quad \varepsilon(e_i) = \varepsilon(f_i) = 0, \quad S(k_i^{\pm 1}) = k_i^{\mp 1}, \quad S(e_i) = -e_i k_i^{-1}, \quad S(f_i) = -k_i f_i.
\]
With a supplement of appropriate Serre relations, the homogeneous cases \( \epsilon_1 = \cdots = \epsilon_n \) are identified with the quantum affine algebras \([3, 12]\) as

\[
U_A(0, \ldots, 0) = U_q(A^{(1)}_{n-1}), \quad U_A(1, \ldots, 1) = U_{-q^{-1}}(A^{(1)}_{n-1}).
\]

In general \( U_A(\epsilon_1, \ldots, \epsilon_n) \) is an example of generalized quantum groups \([8, 9]\) including an affinization of quantum super algebra \( sl_q(k, n-k) \). See \([21]\) Sec.3.3 for more detail.

For the space \( \mathcal{W}_l \) \([28]\) and a parameter \( x \), the following map \( \pi_x(l) : U_A(\epsilon_1, \ldots, \epsilon_n) \to \text{End}(\mathcal{W}_l) \) gives an irreducible finite dimensional representation \([4]\)

\[
e_i|m\rangle = x^{\delta_{i,0}}[m_i]|m - e_i + e_{i+1}\rangle,
\]

\[
f_i|m\rangle = x^{-\delta_{i,0}}[m_{i+1}]|m + e_i - e_{i+1}\rangle,
\]

\[
k_i|m\rangle = (q_i)^{-m_i}(q_{i+1})^{m_{i+1}}|m\rangle,
\]

where \( |m\rangle = q^m - q^{-m} |1\rangle \) and \( e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^n \). The vectors \( |m'\rangle = |m'_1, \ldots, m'_n\rangle \) on the rhs of \([28]\) are to be understood as zero unless \( 0 \leq m'_i \leq 1/\epsilon_i \) for all \( 1 \leq i \leq n \). In the homogeneous case, the representation \( \pi_x(l) \) is equivalent to

- degree \( l \) symmetric tensor rep. of \( U_q(A^{(1)}_{n-1}) \) for \( \epsilon_1 = \cdots = \epsilon_n = 0 \).
- degree \( l \) anti-symmetric tensor rep. of \( U_{-q^{-1}}(A^{(1)}_{n-1}) \) for \( \epsilon_1 = \cdots = \epsilon_n = 1 \).

Let \( \Delta' \) denote the opposite (i.e., the left and the right components interchanged) coproduct of \( \Delta \) in \([31]\).

**Theorem 4.** (\(21\) Th.5.1) For any \( l, m \in \mathbb{Z}_{\geq 0} \), the following commutativity holds:

\[
\Delta'(g) S_{l,m}(z) = S_{l,m}(z) \Delta(g) \quad \forall g \in U_A(\epsilon_1, \ldots, \epsilon_n),
\]

where \( \Delta(g) \) and \( \Delta'(g) \) stand for the tensor product representations \( (\pi_x(l) \otimes \pi_y(m)) \Delta(g) \) and \( (\pi_x(l) \otimes \pi_y(m)) \Delta'(g) \) of \([28]\) with \( z = x/y \).

If \( \mathcal{W}_l \otimes \mathcal{W}_m \) is irreducible, Theorem \([4]\) characterizes \( S_{l,m}(z) \) up to an overall scalar. Therefore \( S_{l,m}(z) \) is identified with the quantum \( R \) matrix in the sense of \([12]\) associated with \( U_A(\epsilon_1, \ldots, \epsilon_n) \)-module \( \mathcal{W}_l \otimes \mathcal{W}_m \). Although we expect that \( \mathcal{W}_l \otimes \mathcal{W}_m \) is irreducible for arbitrary \( (\epsilon_1, \ldots, \epsilon_n) \), it has hitherto been proved rigorously only for \( (\epsilon_1, \ldots, \epsilon_n) \) of the form \( (1^\kappa, 0^{n-k}) \) with \( 0 \leq \kappa \leq n \) \([21]\). Anyway the family \( S_{l,m}(z) \) \([25]\) interpolates the quantum \( R \) matrices for the symmetric tensor representations of \( U_q(A^{(1)}_{n-1}) \) and the anti-symmetric tensor representations of \( U_{-q^{-1}}(A^{(1)}_{n-1}) \) as the two extreme cases \( \kappa = 0 \) and \( n \). In \([21]\) Prop.2.1, it was also shown that \( S_{l,m}(z) \)'s associated with \( (\epsilon_1, \ldots, \epsilon_n) \) and \( (\epsilon'_1, \ldots, \epsilon'_n) \) are connected by a similarity transformation if the two sequences are permutations of each other. Thus one can claim that all the \( S_{l,m}(z) \) \([25]\) are equivalent to the quantum \( R \) matrices of some generalized quantum group.

6. **Combinatorial \( R \)**

In this section we study \( S_{l,m}(z) \) \([28]\) at \( q = 0 \). Let \( (\epsilon_1, \ldots, \epsilon_n) \in \{0, 1\}^n \) be an arbitrary sequence and introduce the crystal

\[
B_l = \{ a = (a_1, \ldots, a_n) \in (\mathbb{Z}_{\geq 0})^n \mid |a| = l, \ 0 \leq a_i \leq 1/\epsilon_i \ (1 \leq i \leq n) \},
\]

\footnote{Image \( \pi_x(l) \) is denoted by \( g \) for simplicity.}
which is a finite labeling set of the basis of $W_l$. We identify $\mathbf{a} = (a_1, \ldots, a_n) \in B_l$ with the depth $n$ column shape tableau containing $a_i$ dots in the $i$-th box from the top ($1 \leq i \leq n$). See the diagrams given below. Call the dots in the $i$-th box bosonic if $\epsilon_i = 0$ and fermionic if $\epsilon_i = 1$. Thus there are $l$ dots in the tableau in total among which $\epsilon_1 a_1 + \cdots + \epsilon_n a_n$ are fermionic and the rest are bosonic.

We are going to define a map $R = R_{l,m} : B_l \otimes B_m \to B_m \otimes B_l$ and a function $H = H_{l,m} : B_l \otimes B_m \to \mathbb{Z}_{\geq 0}$ by combinatorial algorithm, where $\otimes$ may just be understood as a product of sets. Thus for a given pair of tableaux $i \otimes j \in B_l \otimes B_m$, we are to specify the right hand sides of

$$R(i \otimes j) = b \otimes a \in B_m \otimes B_l, \quad H(i \otimes j) = w \in \mathbb{Z}_{\geq 0}. \quad (35)$$

For $l \geq m$, it is done by the algorithm (i)–(iii) given below:

(i) Choose a dot, say $d$, in $j$ and connect it to a dot $d'$ in $i$ to form a pair. If $d$ is bosonic (resp. fermionic), $d'$ should be the lowest one among those located strictly higher (resp. not strictly lower) than $d$. If there is no such dot, take $d'$ to be the lowest one in $i$. Such a pair is called winding. The lines pairing the dots are called $H$-lines.

(ii) Repeat (i) for yet unpaired dots until all dots in $j$ are paired to some dots in $i$.

(iii) Move the $l-m$ unpaired dots in $i$ horizontally to $j$. The resulting tableaux define $b \otimes a$. $w$ is the winding number (number of winding pairs).

The above example is for $n = 5$, $(\epsilon_1, \ldots, \epsilon_5) = (0, 1, 0, 1, 0)$, $B_l \otimes B_m = B_8 \otimes B_4$ and shows

$$R(01313 \otimes 10210) = 01012 \otimes 10511, \quad H(01313 \otimes 10210) = 2.$$

Remark 5.

(1) In (i) and (ii), the $H$-lines depend on the order of choosing the dots from $j$. However, the final result of $b \otimes a$ and $w$ can be shown to be independent of it.

(2) The $H$-lines in the winding case are naturally interpreted as going up periodically along the tableaux.

(3) The condition of being bosonic or fermionic in (i) only refers to $d$ and does not concern $d'$.

(4) When $l = m$, $R$ is trivial in that $R(i \otimes j) = i \otimes j$, but $H(i \otimes j)$ remains nontrivial.
Theorem 6. Let

\[ \lambda R = R \text{ when } \rho \text{ introduced for } \mathfrak{r} \text{ of crystal base theory [16] of } \mathfrak{U} \text{, i.e. } \forall q \text{ interchange of } \rho \text{ and 'lower'. We suppose this is due to the right relation in (32) indicating the B of the form (0')1^n, r, n = 0, R \text{ holds when } R_{\emptyset, l}^{\emptyset, a} = b \otimes a \text{ or equivalently } R_{\emptyset, l}^{\emptyset, a} = i \otimes j. \]

The bijection map R and the function H are called (classical part of) combinatorial R and energy, respectively. For \( (\epsilon_1, \ldots, \epsilon_n) \) of the form \( (0^n1^{n-r}) \), it was first introduced for \( r = 0 \) and \( n = 1 \) as Rule 3.10 in [23] and Rule 3.11 in [34] in the framework of crystal base theory \( [16] \) of \( U_q(\widehat{sl}_n) \), and later for general \( r \) in [10] based on a realization of \( U_q(\mathfrak{g}(r, n-r)) \) crystals in [34]. Note that our algorithm for \( r = 0 \) case, i.e. \( \forall \epsilon_i = 1 \) coincides with [23] Rule 3.10 after reversing the conditions 'higher' and 'lower'. We suppose this is due to the right relation in (32) indicating the interchange of \( q = 0 \) and \( q = \infty \) in the two papers.

We define the matrix element of the combinatorial R as

\[ R_{i,j}^{a,b} = \begin{cases} 1 & \text{if } R(i \otimes j) = b \otimes a, \\ 0 & \text{otherwise.} \end{cases} \]  

Now we state the main result.

**Theorem 6.** Let \( S_{i,j}^{a,b}(z) \) be the element [23] of \( S(z) = S_{i,m}(z) \) [23]. Set \( R = R_{\emptyset, m} \) and \( H = H_{\emptyset, m} \). Then the following equality is valid:

\[ (1 - z)^{\delta_{i,j}} \lim_{q \to 0} q^{-(m-l)} \cdot S_{i,j}^{a,b}(z) = z^{H(i \otimes j)} R_{i,j}^{a,b}. \]

**Proof.** Setting \( \tilde{S}_{i,m}(z) = P S_{i,m}(z) \) with \( P(u \otimes v) = v \otimes u \), one can show the inversion relation \( \tilde{S}_{i,m}(z) \tilde{S}_{m,l}(z^{-1}) = \rho(z) \) \( \text{id}_{W_m \otimes W_l} \) with an explicit scalar function \( \rho(z) \) by
Taking the limit in agreement with (38).

Another check of (38), where the last line is due to Example 3.

Example 8. Another check of (58) where the last line is due to Example 4.

Example 7. Taking the limit $q \to 0$ in Example 2 one has

$$\lim_{q \to 0} S(z) = S_{4,2}(z).$$

This agrees with the combinatorial $R$ and the energy

$$R(031 \otimes 110) = 011 \otimes 130, \quad H(031 \otimes 110) = 2.$$
Let us describe the Yang-Baxter equation satisfied by the combinatorial $R$ as a corollary of (20). In order to properly treat the spectral parameter we introduce the affine crystal

$$\text{Aff}(B_l) = \{ a[d] \mid a \in B_l, d \in \mathbb{Z} \}. $$

It allows us to unify the classical part of the combinatorial $R$ and the energy $H$ in (35) in the (full) combinatorial $R$ $R_{l,m} : \text{Aff}(B_l) \otimes \text{Aff}(B_m) \to \text{Aff}(B_m) \otimes \text{Aff}(B_l)$ as

$$R(i[d] \otimes j[e]) = b[e - H(i \otimes j)] \otimes a[d + H(i \otimes j)],$$

where $b \otimes a$ is specified by $b \otimes a = R(i \otimes j)$.

**Corollary 9.** The combinatorial $R$ satisfies the Yang-Baxter equation

$$(R_{l,m} \otimes 1)(1 \otimes R_{k,l})(R_{k,m} \otimes 1) = (1 \otimes R_{k,m})(R_{k,l} \otimes 1)(1 \otimes R_{l,m})$$

as maps $\text{Aff}(B_k) \otimes \text{Aff}(B_l) \otimes \text{Aff}(B_m) \to \text{Aff}(B_m) \otimes \text{Aff}(B_l) \otimes \text{Aff}(B_k)$.

**Example 10.** Let $n = 4$ and $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) = (1, 0, 1, 0)$. We apply the two sides of Corollary 9 on the element from $\text{Aff}(B_4) \otimes \text{Aff}(B_2) \otimes \text{Aff}(B_1)$ in the bottom line.

At the top line the two sides coincide, confirming the Yang-Baxter equation.

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