Lyapunov Control on Quantum Open System in Decoherence-free Subspaces

W. Wang, L. C. Wang, X. X. Yi
School of Physics and Optoelectronic Technology, Dalian University of Technology, Dalian 116024, China

A scheme to drive and manipulate a finite-dimensional quantum system in the decoherence-free subspaces (DFS) by Lyapunov control is proposed. Control fields are established by Lyapunov function. This proposal can drive the open quantum system into the DFS and manipulate it to any desired eigenstate of the free Hamiltonian. An example which consists of a four-level system with three long-lived states driven by two lasers is presented to exemplify the scheme. We have performed numerical simulations for the dynamics of the four-level system, which show that the scheme works good.

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I. INTRODUCTION

Manipulating the time evolution of a quantum system is a major task required for quantum information processing. Several strategies for the control of a quantum system have been proposed in the past decade[1], which can be divided into coherent and incoherent control, according to how the controls enter the dynamics. Among the quantum control strategies, Lyapunov control plays an important role in quantum control theory. Several papers have been published recently to discuss the application of Lyapunov control to quantum systems[2–5]. Although the basic mathematical formulism for Lyapunov control is well established, many questions remain when one considers its applications in quantum information processing, for instance, the Lyapunov control on open system and the state manipulation in its decoherence-free subspace.

As a collection of states that undergo unitary evolution in the presence of decoherence, the decoherence-free subspaces (DFS) [6] and noiseless subsystem (NS) [7] are promising concept in quantum information processing. Experimental realizations of DFS have been achieved with photons [8] and in nuclear spin systems [9]. A decoherence-free quantum memory for one qubit has been realized experimentally with two trapped ions [10, 11]. An in-depth study of quantum stabilization problems for DFS and NS of Markovian quantum dynamics was presented in [12].

Most recently, we have proposed a scheme to drive an open quantum system into the decoherence-free subspaces [13]. This scheme works also for closed quantum system, by replacing the DFS with a desired subspace. The result suggests that it is possible to drive a quantum system to a set of states (for example, the DFS in the paper), however it is difficult to manipulate the system into a definite quantum state in the DFS. The aim of this paper is to design a Lyapunov control to drive an open system to a definite state in the DFS. The Lyapunov control has been proven to be a sufficient simple control to be analyzed rigorously, in particular, the control can be shown to be highly effective for systems that satisfy certain sufficient conditions, which roughly speaking are equivalent to the controllability of the linearized system. In Lyapunov control, Lyapunov functions which were originally used in feedback control to analyze the stability of the control system, have formed the basis for new control design. By properly choosing the Lyapunov function, our analysis and numerical simulations show that the control scheme works good.

This paper is organized as follows. In Sec. II we present a general analysis of Lyapunov control for open quantum systems, Lyapunov functions and control fields are given and discussed. To illustrate the general formulism, we exemplify a four-level system with 2-dimensional DFS in Sec. III showing that the system can be controlled to a desired state in the DFS by Lyapunov control. Finally, we conclude our results in Sec. IV

II. GENERAL FORMULISM

We can model a controlled quantum system either by a closed system, or by an open system governed by a master equation. In this paper, we restrict our discussion to a N-dimensional open quantum system, and consider its dynamics as Markovian and therefore the dynamics obeys the Markovian master equation ($\hbar = 1$, throughout this paper),

$$\dot{\rho} = -i[H, \rho] + L(\rho),$$

where $L(\rho) = \frac{1}{2} \sum_{m=1}^{M} \lambda_m (L_m \rho L_m^\dagger + L_m^\dagger L_m) + \sum_{n=1}^{F} f_n(t) H_n$. $\lambda_m (m = 1, 2, ..., M)$ are positive and time-independent parameters, which characterize the decoherence. $L_m (m = 1, 2, ..., M)$ are jump operators. $H_0$ is a free Hamiltonian and $H_n (n = 1, 2, ..., F)$ are control Hamiltonian, while $f_n(t) (n = 1, 2, ..., F)$ are control fields. Equation (1) is of Lindblad form, this means that the solution to Eq. (1) has all the required properties of a physical density matrix at all times.

By definition, DFS is composed of states that undergo unitary evolution. Considering the fact that there are many ways for a quantum system to evolve unitarily,
we focus on the DFS here that the dissipative part $L(\rho)$ of the master equation is zero, leading to the following conditions for DFS\cite{3}. A space spanned by $H_{DFS} = \{ |\psi_1\rangle, |\psi_2\rangle, ..., |\psi_D\rangle \}$ is a decoherence-free subspace for all time $t$ if and only if (1) $H_{DFS}$ is invariant under $H_0$; (2) $L_m|\psi_n\rangle = c_m|\psi_n\rangle$ and (3) $\Gamma|\psi_n\rangle = g|\psi_n\rangle$ for all $n = 1, 2, ..., D$ and $m = 1, 2, ..., M$ with $g = \sum_{l=1}^{M} \lambda_l |c_l|^2$, and $\Gamma = \sum_{m=1}^{M} \lambda_m L_m$. With these notations, the goal of this paper can be formulated as follows. We wish to apply a specified set of control fields $\{ f_i(t), n = 1, 2, ..., F \}$ in Eq. (1) such that $\rho(t)$ evolves into a desired state in the DFS and stays there forever. In contrast to the conventional control problem\cite{3}, we here develop the control strategy to open system.

We use

$$V(\rho) = \text{Tr}(\rho \hat{A}) \quad (2)$$

as a Lyapunov function, where $\hat{A}$ is hermitian and time-independent. First, we analyze the structure of critical points for $V(\rho)$ with restriction $\text{Tr}(\rho) = 1$. To determine the structure of $V(\rho)$ around one of its critical points, for example $\rho_c = \sum_j p_j^c |A_j\rangle \langle A_j|$, we consider a finite variation $\delta \rho$ such that $\text{Tr}(\rho_c + \delta \rho) = 1$. Here we denote the normalized eigenvectors and eigenvalues of $\hat{A}$ by $|A_j\rangle$ and $A_j$ $(i = 1, 2, 3, ..., N)$, respectively. Express $(\rho_c + \delta \rho)$ in the basis of the eigenvectors of $\hat{A}$,

$$\rho_c + \delta \rho = \sum_j p_j^c |A_j + \delta A_j\rangle \langle A_j + \delta A_j|,$$

$$|A_j + \delta A_j\rangle = |A_j\rangle + \sum_{\alpha=1}^{N} \delta_{\alpha j} |A_\alpha\rangle. \quad (3)$$

The normalization condition $\text{Tr}(\rho_c + \delta \rho) = 1$ follows,

$$\sum_j p_j^c (\delta^*_{\alpha j} + \delta_{j \alpha}) + \sum_j p_j^c \sum_{\alpha \neq \alpha} \delta^*_{\alpha j} \delta_{\alpha j} = 0. \quad (4)$$

Then

$$V(\rho_c + \delta \rho) - V(\rho_c) = \sum_j p_j^c \sum_{\alpha \neq \alpha} (A_\alpha - A_j) \delta^*_{\alpha j} \delta_{\alpha j}. \quad (5)$$

Considering $\delta^*_{\alpha j}$ as variation parameters and noting $\delta^*_{\alpha j} \delta_{\alpha j} \geq 0$, we find that the structure of $V(\rho)$ around the critical point $\rho_c$ depends on the ordering of the eigenvalues: $\rho_c$ is a local maximum as a function of the variations $\delta_{\alpha j}$ if and only if $A_j$ is the largest eigenvalue, a local minimum if $A_j$ is the smallest eigenvalue and a saddle point otherwise. This observation leads us to suspect that the minimum of $V$ is asymptotically attractive, in other words, the control field based on this Lyapunov function would drive the open system to the eigenstate of $\hat{A}$ with the smallest eigenvalue. We will show through an example that this is exactly the case.

Now we establish the control fields $f_n(t)$. $V(\rho) = \text{Tr}(\rho \hat{A})$ yields,

$$\hat{V} = \text{Tr}(L(\rho) \hat{A}) - i \text{Tr}(\rho [\hat{A}, \sum_n f_n(t) H_n]),$$

where we choose $[\hat{A}, H_0] = 0$, because $(\hat{a}, \hat{b}$ any operators) $\text{Tr}[\hat{a}, \hat{b}] = 0$, i.e., the commutator can never be sign definite. The choice of $[\hat{A}, H_0] = 0$ implies that $H_0$ and $\hat{A}$ must have the same eigenvectors, then the control field would drive the open system into an eigenstate of the Hamiltonian $H_0$. To make $\hat{V} \leq 0$, we choose a $f_{j_0}(t)$ such that

$$f_{j_0}(t) = -i \frac{\text{Tr}(L(\rho) \hat{A})}{\text{Tr}([\hat{A}, H_0] \rho)} ,$$

$$f_j(t) = -i \kappa_j (\text{Tr}([\hat{A}, H_j] \rho))^*, \quad j \neq j_0. \quad (6)$$

Here $\kappa_j > 0$ will be refereed as the strength of the control. Then the evolution of the open system with Lyapunov control can be described by the following nonlinear equations

$$\rho(t) = -i[H_0 + \sum_n f_n(t) H_n, \rho(t)] + L(\rho), \quad (6)$$

where $f_n(t)$ is determined by Eq. (6). It should be emphasized that $f_{j_0}$ always exists. To find $f_{j_0}$, $\text{Tr}([\hat{A}, H_{j_0}] \rho) \neq 0$ is required. This can be done by construction. Now we show that $f_{j_0}$ is real. By the definition of $L(\rho)$, $L(\rho)$ is hermite, then $\text{Tr}(L(\rho) \hat{A})$ can be treated as the time derivative of $\langle \hat{A} \rangle$ and thus is real. Identifying $\hat{A}$ with a hermitian operator for a system described by the Hamiltonian $H_{j_0}$, we have $i \frac{d}{dt} \langle \hat{A} \rangle = [\hat{A}, H_{j_0}]$, so $\text{Tr}([\hat{A}, H_{j_0}] \rho)$ is real. By the same virtue, we can show that all the control fields are real as long as the control Hamiltonian $H_j$ $(j = 1, 2, 3,..)$ are hermitian.

By the LaSalle’s invariant principle\cite{3}, the autonomous dynamical system Eq. (6) converges to an invariant set defined by $E = \{ \hat{V} = 0 \}$. This set is in general not empty and of finite dimension, indicating that it is easy to manipulate an open system to a set of states but difficult to control it from an arbitrary initial state to a given target state. Fortunately, by elaborately designing the control Hamiltonian and the operator $\hat{A}$, we can solve this problem as follows. The invariant set defined by $E = \{ \hat{V} = 0 \}$ is an intersection of all sets $E_j$ $(j = 1, 2, 3,..)$, each one satisfies,

$$\text{Tr}(\hat{A} H_j \rho - H_j \hat{A} \rho) = 0,$$

leading to $[\hat{A}, \rho] = 0$, $[\hat{A}, H_j] = 0$ or $[H_j, \rho] = 0$. By elaborately choosing $H_j$ $(j = 1, 2, 3,..)$, we can set the contribution of $[\hat{A}, H_j] = 0$ and $[H_j, \rho] = 0$ to the intersection (i.e., the invariant set $E_j$) to zero. In this case, the invariant set is a collection of state $\{ \rho_m \}$ that satisfies $[\hat{A}, \rho_m] = 0$. Considering that only the states in DFS are stable, we claim that we can manipulate the system from any initial state to the target state in DFS. In other words, we can design $\hat{A}$ such that $E \cap DFS$ contains only the target state. We emphasize that although the control field $f_{j_0}(t)$ was specified to cancel $\text{Tr}(L(\rho) \hat{A})$ in $\hat{V}$, it makes contribution to the dynamics of the open system.
for this purpose, we choose the control Hamiltonian \( H_c = \sum_{\sigma_j = 1} f_\sigma(t) H_j \) with

\[
H_0 = \sum_{j=0}^{2} \Delta_j |j\rangle \langle j| + \sum_{j=1}^{2} \Omega_j |j\rangle \langle j| + h.c.,
\]

where \( \Omega_j \) (\( j = 1, 2 \)) are coupling constants. Without loss of generality, in the following the coupling constants are parameterized as \( \Omega_1 = \Omega \cos \phi \) and \( \Omega_2 = \Omega \sin \phi \) with \( \Omega = \sqrt{\Omega_1^2 + \Omega_2^2} \). The excited state \(|0\rangle\) is not stable, it decays to the three stable states with rates \( \gamma_1, \gamma_2 \) and \( \gamma_3 \), respectively. We assume this process is Markovian and can be described by the Liouvillian,

\[
\mathcal{L}(\rho) = \sum_{j=1}^{3} \gamma_j (\sigma_j^- \rho \sigma_j^+ - \frac{1}{2} \sigma_j^+ \sigma_j^- \rho - \frac{1}{2} \rho \sigma_j^- \sigma_j^+) + \hbar \omega \rho
\]

with \( \sigma_j^- = |0\rangle \langle j| \) and \( \sigma_j^+ = (\sigma_j^-)^\dagger \). It is not difficult to find that the two degenerate dark states

\[
|D_1\rangle = \cos \phi |2\rangle - \sin \phi |1\rangle, \\
|D_2\rangle = |3\rangle,
\]

of the free Hamiltonian \( H_0 \) form a DFS. Now we show how to control the system to a desired target state in the DFS. For this purpose, we choose the control Hamiltonian \( H_c = \sum_{\sigma_j = 1} f_\sigma(t) H_j \) with

\[
H_1 = \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
\end{pmatrix},
H_2 = |D_1\rangle \langle D_2| + |D_2\rangle \langle D_1|,
H_3 = |0\rangle \langle D_2| + |D_2\rangle \langle 0|.
\]
For example, when $\hat{A} = \hat{A}_1 = |D_2\rangle\langle D_2| - |D_1\rangle\langle D_1|$, the system can be controlled into $|D_1\rangle$ (see Fig.4), whereas $\hat{A} = -\hat{A}_1$ can drive the system into $|D_2\rangle$ (see Fig.4). Based on the formalism in Sec. II, $\hat{A} = \hat{A}_1$ together with the controls could drive the system to the eigenstate of $\hat{A}_1$ with smallest eigenvalue (namely, $|D_1\rangle$), and to $|D_2\rangle$ with $\hat{A} = -\hat{A}_1$. As figure 4 shows, however, the fidelity is not 1 for some initial states, for example $\beta_1 = 0$. The reason is as follows. Though the choice of $\hat{A} = -\hat{A}_1$ benefits the target state $|D_2\rangle$, since $|D_2\rangle$ is the eigenstate of $\hat{A}$ with smallest eigenvalue, the control $H_3 = (|0\rangle\langle 0| + h.c.$) does not favor the control target $|D_2\rangle$, because $H_3$ couples the states $|0\rangle$ and $|D_2\rangle$, and $|0\rangle$ decays to the three stable states equally. This observation suggests that $h_3 = (|0\rangle\langle 0| + h.c.)$ instead of $H_3$ could help the control when the target is $|D_2\rangle$. Indeed further numerical simulations confirm this prediction that the control $h_3$ can drive the system into $|D_2\rangle$ with almost perfect fidelity 99%. The fidelity of the open system in the target state depends on the strength $\kappa_3 = \kappa$ of the control $f_3(t)$, the dependence is plotted in Fig.5. With large $\kappa$, the system would asymptotically converge to the target state as Fig.5 shows. As expected, the control fields $f_2(t)$ and $f_3(t)$ tend to zero when the open system converges to the target state, see Fig.6.

FIG. 4: Fidelity of the control versus initial states. The target state is $|D_2\rangle$ (or $\hat{A} = -|D_2\rangle\langle D_2| + |D_1\rangle\langle D_1|$), $\kappa_3 = 15$, the other parameters chosen are the same as in Fig.2 are chosen for this plot.

FIG. 5: Fidelity as a function of the control strength $\kappa_3 = \kappa$. (a) $\hat{A} = |D_2\rangle\langle D_2| - |D_1\rangle\langle D_1|$, (b) $\hat{A} = -|D_2\rangle\langle D_2| + |D_1\rangle\langle D_1|$. $\phi = \frac{\pi}{4}$, $\beta_1 = \frac{\pi}{4}$, $\beta_2 = -\frac{\pi}{4}$, $\beta_3 = \frac{\pi}{4}$, and $\kappa_2 = 1$. The other parameters chosen are the same as Fig.2.

FIG. 6: Control field $f_2(t)$ and $f_3(t)$ as a function of time. $\Omega = 5$, $\phi = \frac{\pi}{4}$, $\beta_1 = \frac{\pi}{4}$, $\beta_2 = \frac{\pi}{4}$, $\beta_3 = \frac{\pi}{4}$, $\kappa_2 = 1$, and $\kappa_3 = 15$. $\hat{A} = |D_2\rangle\langle D_2| - |D_1\rangle\langle D_1|$, $f_3(t)$ is zero in this scheme.

IV. CONCLUSION

In summary, we have proposed a scheme to manipulate an open quantum system in the decoherence-free subspaces. This study was motivated by the fact that for Lyapunov control, it is usually difficult to optimally control the system from an arbitrary initial state to a given target state, this is due to the LaSalle’s invariant principle. Our present study suggests that it is possible to drive a quantum system to a desired state in DFS by elaborately designing the controls. The results do not break the LaSalle’s role, instead it reduces the invariant set $E$ to include the target state only. To demonstrate the proposal we exemplify a four-level system and numerically simulate the controlled dynamics. The dependence of the fidelity on initial states as well as the control fields are calculated and discussed. This scheme put the Lyapunov control on quantum open system one step forward, and shed light on the quantum control in DFS.

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