MANIFOLDS WITH SMALL TOPOLOGICAL COMPLEXITY

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Abstract. We study closed orientable manifolds whose topological complexity is at most 3 and determine their cohomology rings. For some of admissible cohomology rings we are also able to identify corresponding manifolds up to homeomorphism.

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1. Introduction

Topological complexity of a space $X$, denoted $TC(X)$ is a numerical homotopy invariant introduced by M. Farber [8] as a quantitative measure for the complexity of motion planning in a configuration space $X$ of some robot device. Although configuration spaces of robots can be quite general topological spaces (cf. Kapovich-Millson [16], Pavešić [20]), of particular importance are those that have the structure of a closed manifold (e.g., ordered configuration spaces of manifolds, see Cohen [3], configuration spaces of spidery linkages, see O’Hara [19] and of general parallel mechanisms, see Shvalb-Shoham-Blanc [24]). It is thus of interest to determine which closed manifolds $M$ have a given value of $TC(M)$. The case $TC(M) = 1$ is void, because a non-trivial closed manifold cannot be contractible. Grant, Lupton and Oprea [13, Corollary 1.2] showed that the only closed manifolds with topological complexity equal to 2 are the odd-dimensional spheres. In this paper we will make a further step and study closed oriented manifolds $M$ with $TC(M) = 3$. Some examples immediately spring to mind: even-dimensional spheres $S^{2n}$ by [8, Theorem 8] and products of two odd-dimensional spheres, by [8, Theorem 8 and Theorem 11]. The question is, are there any other examples? We will give an answer in the main result of the next section, Theorem 2.2 in which we exactly describe admissible cohomology rings of manifolds whose topological complexity is at most 3. In Section 3 we discuss explicit manifolds whose cohomology ring is described in the mentioned theorem.

For a topological space $X$ let $X^I$ denote the space of continuous paths $\alpha: I \to X$, and let $\pi: X^I \to X \times X$ be the evaluation map $\pi(\alpha) := (\alpha(0), \alpha(1))$. Topological complexity of a path-connected topological space $X$ is the least integer $TC(X) = n$ for which there exists a covering $U_1, \ldots, U_n$ of $X \times X$, where each $U_i$ is open and admits a continuous section to the map $\pi: X^I \to X \times X$ (cf. [8 Definition 2]). Note that if $X$ is a compact ANR space (which includes closed manifolds) then

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the requirement that the sets in the covering are open is superfluous, since by [21, Theorem 4.6], one can consider coverings of $X \times X$ by arbitrary subsets. Note also, that many authors (e.g., the above mentioned article [13]) use a normalized or reduced topological complexity, which is one less that in our definition, so that the topological complexity of a contractible space is 0 instead of 1.

The main properties of topological complexity are listed in the following proposition, where the value of $TC(X)$ is related to the Lusternik-Schnirelmann category $cat(X)$ (for which we refer to the classical monograph [5]), and to the nilpotency of certain ideal in the cohomology ring of $X \times X$.

**Proposition 1.1.**

(1) $TC(X) = 1$ if, and only if $X$ is contractible;

(2) Homotopy invariance:

$$X \simeq Y \Rightarrow TC(X) = TC(Y);$$

(3) Category estimate:

$$cat(X) \leq TC(X) \leq cat(X \times X);$$

(4) If $X$ is a topological group, then $TC(X) = cat(X)$;

(5) Cohomological estimate:

$$TC(X) \geq \text{nil}(\text{Ker } \Delta^*),$$

where $\Delta^*: H^*(X \times X; R) \to H^*(X; R)$ is the homomorphism induced by the diagonal map $\Delta: X \to X \times X$ on the cohomology with coefficients in a ring $R$, and $\text{nil}(\text{Ker } \Delta^*)$ is the minimal integer $k$ for which all $k$-fold products in $\text{Ker } \Delta^*$ are zero;

(6) Product formula:

$$TC(X \times Y) \leq TC(X) + TC(Y) - 1.$$

Recall that the value of $\Delta^*$ on the cross-product $u \times v \in H^*(X \times X; R)$ of elements $u, v \in H^*(X; R)$ can be given in terms of their cup-product as

$$\Delta^*(u \times v) = u \cdot v,$$

and the cup-product of elements $u \times v$ and $u' \times v'$ is given as

$$(u \times v) \cdot (u' \times v') = (-1)^{|v||u'|}(u \cdot u') \times (v \cdot v'),$$

where $|v|, |u'|$ are the dimensions of cohomology classes $v$ and $u'$ (see [14, pp. 215-216]). This explains why Farber [3, Definition 6] called $\text{Ker } \Delta^*$ the ideal of zero-divisors of $H^*(X; R)$. For every $u \in H^*(X; R)$ we have

$$\Delta^*(u \times 1 - 1 \times u) = u \cdot 1 - 1 \cdot u = 0,$$

therefore $(u \times 1 - 1 \times u) \in \text{Ker } \Delta^*$. Indeed, if $H^*(X; R)$ is a free $R$-module (which implies that $H^*(X \times X; R) \cong H^*(X; R) \otimes H^*(X; R)$ by Kunneth theorem), then one can easily show that $\text{Ker } \Delta^*$ is generated as an ideal by elements of the form $(u \times 1 - 1 \times u)$. 
2. Admissible cohomology rings

Computation of topological complexity of closed surfaces was completed in the orientable case by Farber [3] Theorem 9 and in the non-orientable case by Dranishnikov [7] and Cohen and Vandembroucq [4]. By mentioned results the only closed surfaces whose topological complexity is 3 are the sphere $S^2$ and the torus $S^1 \times S^1$.

To avoid making unnecessary exceptions, from this point on let $M$ denote a closed, orientable $m$-dimensional manifold with $m \geq 3$.

We will base our computations on the following corollary of a theorem proved by Dranishnikov, Katz and Rudyak [6].

**Proposition 2.1.** If $\text{TC}(M) \leq 3$, then $\pi_1(M)$ is free.

*Proof.* If $\text{TC}(M) \leq 3$, then $\text{cat}(M) \leq 3$ by Proposition [11] (3), and then the claim follows immediately from [6] Theorem 1.1. □

We will proceed by considering several cases and sub-cases. Let $g$ denote the generator of $H^m(M; R)$ and for every $u \in H^*(M; R)$ let

$$\hat{u} := u \times 1 - 1 \times u \in H^*(M \times M; R)$$

be the shorthand for the corresponding zero-divisor.

(1) Let us first assume that the rank of $\pi_1(M)$ is at least 2 and consider the cup-product pairing

$$H^1(M; \mathbb{Z}) \times H^{m-1}(M; \mathbb{Z}) \rightarrow H^m(M; \mathbb{Z})$$

which is non-singular by [14] Proposition 3.38. Since the rank of $H^1(M; \mathbb{Z})$ is at least 2, non-singularity of the pairing implies that there exist linearly independent elements $u, v \in H^1(M; \mathbb{Z})$ and $u', v' \in H^{m-1}(M; \mathbb{Z})$ such that $u \cdot u' = v \cdot v' = g$, and furthermore $u \cdot v' = v \cdot u' = 0$. Then we have the following non-trivial four-fold product

$$\hat{u} \cdot \hat{u}' \cdot \hat{v} \cdot \hat{v}' = 2(g \cdot g) \neq 0.$$ 

Therefore, if $\text{rank}(\pi_1(M)) \geq 2$, then $\text{TC}(M) \geq 5$ by Proposition [11] (6).

(2) If $\pi_1(M) \cong \mathbb{Z}$ let $u$ be a generator of $H^1(M; \mathbb{Z}) \cong \mathbb{Z}$ and let, as before, $v \in H^{m-1}(M; \mathbb{Z})$ be such that $u \cdot v = g$. If $m - 1$ is even, then we have

$$\hat{v}^2 \cdot \hat{u} = -2(v \times v) \cdot \hat{u} = -2(g \times u - u \times g) \neq 0,$$

(note that $v^2 = 0$ for dimensional reasons) and thus $\text{TC}(M) \geq 4$ by Proposition [11] (6). On the other hand, if $m - 1$ is odd, and if there exists a non-zero element $w \in H^i(M; \mathbb{Z})$ for some $2 \leq i \leq m - 2$, then

$$\hat{u} \cdot \hat{v} \cdot \hat{w} = w \times g - g \times w \pm uw \times v - v \times uw \neq 0,$$

so again $\text{TC}(M) \geq 4$.

We conclude that if $\pi_1(M) \cong \mathbb{Z}$ and $\text{TC}(M) = 3$, then $H^*(M; \mathbb{Z})$ is generated by two cohomology classes in dimensions 1 and $m - 1$, which are Poincaré duals to each other, and furthermore $m - 1$ must be odd. In other words, $H^*(M; \mathbb{Z}) \cong \wedge(x_1, x_k)$ for some odd integer $k > 1$. 


(3) If $M$ is simply-connected, then we consider four sub-cases according to the structure of the group

$$\tilde{H}(M; R) := \bigoplus_{i=2}^{m-2} H_i(M; R).$$

(3a) If $\tilde{H}(M; \mathbb{Q}) \neq 0$ we argue similarly as in case (2). First of all we note that $H^*(M; \mathbb{Z})$ is not all torsion, so by [14, Corollary 3.39] we may find elements $u, v, w \in H^*(M; \mathbb{Z})$ of infinite order, such that $u \cdot v = g$. As in case (2), if either $u$ or $v$ is of even degree, then we can find a non-trivial product of three zero-divisors, and then $TC(M) \geq 4$. Therefore, if $TC(M) \leq 3$, then both $u$ and $v$ must be of odd degree, which as before implies that $H^*(M; \mathbb{Z})$ contains a subring of the form $\mathbb{Z}[x_k, x_l]$ where $k, l$ are odd integers and $1 < k \leq l < m - 1$. Furthermore, if there exists an element $w \in H^*(M; \mathbb{Z})$ which is not contained in the mentioned subring, then $\hat{u} \cdot \hat{v} \cdot \hat{w} \neq 0$ similarly as in the second part of case (2). Thus, $\hat{H}(M; \mathbb{Q}) \neq 0$ and $TC(M) = 3$ imply $H^*(M; \mathbb{Z}) \cong \bigwedge(x_k, x_l)$.

(3b) Let us now assume that $\hat{H}(M; \mathbb{Q}) = 0$ but $\hat{H}(M; \mathbb{F}_p) \neq 0$ for some odd prime $p$, and let $k$ be the minimal $k \geq 2$ for which $H_k(M; \mathbb{Z})$ has $p$-torsion. By the universal coefficient theorem for cohomology (see [14, Theorem 3.2]) $H^i(M; \mathbb{F}_p) \neq 0$ for $i = k, k + 1$. It then follows by Poincaré duality that $H^i(M; \mathbb{F}_p) \neq 0$ for $i = m - k - 1, m - k$. Therefore, $H^i(M; \mathbb{F}_p) \neq 0$ in three different dimensions, unless $m = 2k + 1$. In the first case, we may find (as in case (2)) three non-trivial cohomology classes $u, v, w$ of different dimension (with $u \cdot v = g$ by [14, Corollary 3.39]), for which $\hat{u} \cdot \hat{v} \cdot \hat{w} \neq 0$ and thus $TC(M) \geq 4$.

On the other hand, if $m = 2k + 1$, then let $u \in H^k(M; \mathbb{F}_p)$ and $v \in H^{k+1}(M; \mathbb{F}_p)$ be such that $u \cdot v = g$. If $k$ is even, then

$$\hat{u}^2 \cdot \hat{v} = 2(u \times g - g \times u) + v \times u^2 - u^2 \times v \neq 0.$$ 

Similarly, if $k$ is odd, then $\hat{u} \cdot \hat{v}^2 \neq 0$, so in both cases $TC(M) \geq 4$.

(3c) The next sub-case arises if $\hat{H}(M; \mathbb{Q}) = 0$ and $\hat{H}(M; \mathbb{F}_p) = 0$ for $p$ odd but $\hat{H}(M; \mathbb{F}_2) \neq 0$. The argument is similar as in (3b), except if $m = 2k + 1$, since in that case the proof that $\hat{u}^2 \cdot \hat{v} \neq 0$ for $k$ even (or that $\hat{u} \cdot \hat{v}^2 \neq 0$ for $k$ odd) breaks down because of 2-torsion. On the other hand, if $u \in H^k(M; \mathbb{F}_2)$ and $v \in H^{k+1}(M; \mathbb{F}_2)$ such that $u \cdot v = g$, and if additionally $u^2 \neq 0$, then

$$\hat{u}^2 \cdot \hat{v} = u^2 \times v + v \times u^2 \neq 0,$$

which implies $TC(M) \geq 4$. Therefore, under the assumptions of (3c), if $TC(M) = 3$ then $H^*(M; \mathbb{F}_2) \cong \bigwedge(x_k, x_{k+1}) \otimes \mathbb{F}_2$.

(3d) The final possibility is that $\hat{H}(M; R) = 0$ for all coefficient rings $R$, which clearly implies that $H^*(M; \mathbb{Z}) \cong \bigwedge(x_k)$.

We may summarize the above discussion in the following theorem:

**Theorem 2.2.** Assume that $M$ is a closed, orientable manifold with $TC(M) \leq 3$. Then $\pi_1(M)$ is a free group and one of the following alternatives holds:

1. $H^*(M; \mathbb{Z}) \cong \bigwedge(x_k)$, $1 \leq k$, or
2. $H^*(M; \mathbb{Z}) \cong \bigwedge(x_k, x_l)$, $k, l$ odd, $1 \leq k \leq l$, or
3. $H_i(M; \mathbb{Z}) = 0$ for $i \neq 0, k, m$ and $H^*(M; \mathbb{F}_2) \cong \bigwedge(x_k, x_{k+1}) \otimes \mathbb{F}_2$. 


3. Some manifolds with small TC

Theorem 2.2 clearly shows that the condition $\mathrm{TC}(M) \leq 3$ is much more restrictive than the analogous condition $\text{cat}(M) \leq 3$. Indeed the class of manifolds whose Lusternik-Schnirelmann category is at most 3 includes all surfaces, two-fold products of sphere, all $(n-1)$-connected $2n$-manifolds and a variety of other examples. In this section we will try to collect some information about the actual manifolds $M$ satisfying $\mathrm{TC}(M) \leq 3$. For some cohomology rings we will be able to determine exactly the corresponding manifolds, while in other cases we will only present suitable candidates and compute their Lusternik-Schnirelmann category.

1. The simplest case to consider are manifolds whose cohomology ring is given by Theorem 2.2(1). In fact it is straightforward that $H^*(M;\mathbb{Z}) \cong \bigwedge (x_k)$ implies that $M$ is homotopy equivalent to $S^k$. The positive solution to the Poincaré conjecture then implies that $M$ is actually homeomorphic to $S^k$.

2. If $H^*(M;\mathbb{Z}) \cong \bigwedge (x_1, x_k)$ as in Theorem 2.2(2), then (since $S^1 \cong K(\mathbb{Z}, 1)$) there is a map $f_1: M \to S^1$ which represents

$$x_1 \in H^1(M;\mathbb{Z}) \cong [M, S^1].$$

Similarly, there exists a map $f_k: M \to K(\mathbb{Z}, k)$ representing the cohomology class $x_k \in H^k(M;\mathbb{Z}) \cong [M, K(\mathbb{Z}, k)]$.

It is well-known that $K(\mathbb{Z}, k)$ can be obtained by attaching cells of dimension bigger or equal to $k+2$ to the sphere $S^k$. Since the dimension of $M$ is $n = k+1$, we may assume by cellular approximation theorem that the image of $f_k$ is contained in $S^k$, and so we have a map $f: X \to S^k$. It is easy to check that the resulting map

$$(f_1, f_k): M \to S^1 \times S^k$$

is an isomorphism on the integral cohomology and thus a homotopy equivalence, since $\pi_1(M) \cong \mathbb{Z}$. Then a result of Kreck and Lück [17, Theorem 0.13(a)] implies that $M$ is actually homeomorphic to $S^1 \times S^k$.

3. If $H^*(M;\mathbb{Z}) \cong \bigwedge (x_k, x_k)$ with $k$ odd, then $M$ is a $(k-1)$-connected $2k$-dimensional manifold and one has C.T.C. Wall’s classification [27] by which $M \approx S^k \times S^k$ (cf. also [2, Theorem 3.1]).

4. The instances of Theorem 2.2(2) when $H^*(M;\mathbb{Z}) \cong \bigwedge (x_k, x_l)$ for $1 < k < l$ with $k, l$ odd are more complicated. The products of odd spheres of the form $S^k \times S^l$ have the abovementioned cohomology ring and $\mathrm{TC}(S^k \times S^l) = 3$. Note that by the above-mentioned result of Kreck and Lück, a manifold that is homotopy equivalent to a product of odd spheres is actually homeomorphic to that product. Another example is the special unitary group $SU(3)$ whose cohomology is $H^*(SU(3);\mathbb{Z}) \cong \bigwedge (x_3, x_5)$. Indeed, Singhof [23, Theorem 1(a)] proved that $\text{cat}(SU(3)) = 3$, therefore by Proposition 1.3(1), we have $\mathrm{TC}(SU(3)) = 3$, as well. Another potential candidate is the symplectic group $Sp(2)$ whose cohomology is $H^*(Sp(2);\mathbb{Z}) \cong \bigwedge (x_3, x_7)$. However, Schweitzer [23] used secondary cohomology operations to prove that $\text{cat}(Sp(2)) = 4$, which in turn implies that $\mathrm{TC}(Sp(2)) = 4$. Hilton and Roitberg [15] discovered three more examples of H-spaces whose cohomology is isomorphic to $\bigwedge (x_3, x_7)$, which are usually denoted $E_{3n}$, $E_{4n}$, $E_{5n}$. 


More generally, let us consider fibre bundles \( p: M \to S^l \) with fibre \( S^k \) for odd integers \( 1 < k < l \). The cohomology of \( M \) is easily computed using Gysin sequence, so we obtain \( H^*(M; \mathbb{Z}) \cong \bigwedge(x_k, x_l) \) and the manifold itself admits a CW-decomposition of the form

\[
M = S^k \cup_\alpha e^l \cup_\beta e^{k+l},
\]

with attaching maps \( \alpha: S^{l-1} \to S^k \) and \( \beta: S^{k+l-1} \to S^k \cup_\alpha e^l \). If \( \alpha \) is a suspension (e.g., if \( l < 2k - 1 \) so that \( \pi_{l-1}(S^k) \) is in the stable range), then \( \text{cat}(S^k \cup_\alpha e^l) = 2 \) and therefore \( \text{cat}(M) = 3 \). This includes important special cases as the complex and quaternionic Stiefel manifolds, \( V_2(\mathbb{C}^n) = U(n)/U(n-2) \) whose cohomology ring is \( H^*(V_2(\mathbb{C}^n); \mathbb{Z}) \cong \bigwedge(x_{2n-1}, x_{2n-3}) \), and \( V_2(\mathbb{H}^n) = Sp(n)/Sp(n-2) \) with \( H^*(V_2(\mathbb{H}^n); \mathbb{Z}) \cong \bigwedge(x_{4n-1}, x_{2n-5}) \).

If the attaching map \( \alpha \) is not a suspension, then \( \text{cat}(S^k \cup_\alpha e^l) = 3 \). In that case \( \text{cat}(M) = 3 \) if, and only if certain set of Hopf invariants \( H(\beta) \) contains the zero class (see Chapter 6 of [3], in particular Theorem 6.19). As we see, there are many sphere bundles over spheres, whose category is 3. Unfortunately, we are currently lacking a general method to determine their topological complexity, so this remains an interesting open problem. One possible approach is to apply certain higher Hopf invariants, a method that was recently developed by Gonzalez, Grant and Vandembroucq [12]. They managed to compute topological complexity of many two-cell complexes, but the technical details are quite formidable, and the full analysis of three-cell complexes is probably very hard. Nevertheless, we were able to combine some of their computations with our results from [22] that relate topological complexity of a space with topological complexity of its skeleta, to show that some sphere bundles over spheres have topological complexity at least 4. We will work in the so-called meta-stable range and assume that \( 2k < l < 3k - 1 \).

Under this assumption one can associate to every map \( \alpha: S^{l-1} \to S^k \) a generalized Hopf invariant \( H_0(\alpha): S^{l-1} \to S^{2k-1} \) (see [12] Section 5) for relevant definitions and results), which allow to determine \( \text{TC}(S^k \cup_\alpha e^l) \geq 4 \).

Proposition 3.1. Let \( k \) be an odd integer and let \( 2k < l < 3k - 1 \). Assume that \( M \) has a CW-decomposition of the form \( M = S^k \cup_\alpha e^l \cup_\beta e^{k+l} \) with attaching maps \( \alpha: S^{l-1} \to S^k \) and \( \beta: S^{k+l-1} \to S^k \cup_\alpha e^l \) (this in particular applies if \( M \) is an \( S^l \)-bundle over \( S^k \)). If \( H_0(\alpha) \neq 0 \), then \( \text{TC}(M) \geq 4 \).

Proof. Note that the inclusion \( S^k \cup_\alpha e^l \hookrightarrow M \) is a \( (k + l - 1) \)-equivalence because \( S^k \cup_\alpha e^l \) is the \( (k + l - 1) \)-skeleton of \( M \). Topological complexity of \( S^k \cup_\alpha e^l \) was bounded from below in [12] Theorem 5.6]: \( \text{TC}(S^k \cup_\alpha e^l) \geq 4 \). On the other hand, [22] Theorem 3.6] implies that \( \text{cat}(M) \geq \text{cat}(S^k \cup_\alpha e^l) = 3 \), therefore \( \text{TC}(M) \geq 3 \). Then we may apply [22] Theorem 3.1], which states that if

\[
2 \dim(S^k \cup_\alpha e^l) < k(\text{TC}(M) - 1) + (k + l - 1)
\]

(which is clearly satisfied if \( l < 3k - 1 \), then \( \text{TC}(M) \geq \text{TC}(S^k \cup_\alpha e^l) \geq 4 \). \( \square \)

Observe that the first ‘undecided’ cases arise in dimension 10, which is of interest if one considers configuration spaces of specific mechanical systems.
From a different perspective, one may also consider the Morse decomposition of a manifold $M$ with $H^*(M;\mathbb{Z}) \cong \wedge(x_k, x_l)$ as above. Smale [26] Theorem G] showed that if dimension of $M$ is at least 6, then it has a Morse decomposition with the minimal number of handles compatible with its homology. Therefore, $M$ admits a decomposition with four handles whose indices are 0, $k$, $l$ and $k+l$ respectively. The union of the 0- and $k$-handles depends on the framing which is given by an element of $\pi_{k-1}(O(l))$. This group is known to be trivial for $k \not\equiv 1(\text{mod } 8)$, therefore the union of the first two handles is homeomorphic to $S^k \times D^l$. By the same argument the union of the $l$- and $(k+l)$-handles is also homeomorphic to $S^k \times D^l$. We may conclude that under these assumptions ($\text{TC}(M) = 3$, $H^*(M;\mathbb{Z}) \cong \wedge(x_k, x_l)$ with $k \not\equiv 1(\text{mod } 8)$) the manifold $M$ can be obtained by glueing together two copies of $S^k \times D^l$ along the common boundary $S^k \times S^l$.

5. Let us finally consider manifolds that satisfy condition (3) of Theorem 2.2. The lowest dimensional case is a simply-connected 5-dimensional manifold whose $\mathbb{F}_2$ cohomology is $H^*(M;\mathbb{F}_2) \cong \wedge(x_2, x_3) \otimes \mathbb{F}_2$. Barden [1] showed that every simply-connected 5-dimensional manifolds can be decomposed as a connected sum of certain basic 5-manifolds. We are not dwelling into details but one can easily check that the only 5-manifold that satisfies the above condition is the famous Wu manifold $SU(3)/SO(3)$. It admits a CW-decomposition $SU(3)/SO(3) = S^2 \cup e^3 \cup e^9$, where the 3-cell is attached by a degree 2 map, therefore the 3-skeleton of $SU(3)/SO(3)$ is the Moore space $M(\mathbb{Z}/2, 2)$. The category of a Moore space is 2, therefore the category of the Wu manifold is 3. However, we were not able to check whether its topological complexity is also 3. One can construct higher analogues of the Wu manifold using handle decompositions, for example by gluing together two copies of a (twisted of untwisted, depending on the dimension) $D^k \times D^l$ over $S^k$ along a suitable homeomorphism of the boundary. All of these spaces have a CW-decomposition with the top-cell attached to a suspension, so their category is equal to 3.

We should also mention an interesting result that was recently proved by S. Mescher [13, Proposition 6.2]. He used weighted cohomology classes to show that a closed oriented manifold $M$ with $\text{TC}(M) \leq 3$ is either a rational homology sphere or it admits a degree 1 map from a closed oriented manifold of the form $S^1 \times P$ (i.e., it is dominated by some product of a manifold of dimension $\dim(M) - 1$ with a circle).

Let us conclude with a brief discussion on two possible extensions of the presented results. Theorem [2.2] gives a precise description of cohomology rings of closed orientable manifolds whose topological complexity is at most 3, so it is natural to ask what can be said about non-orientable closed manifolds $M$ with $\text{TC}(M) \leq 3$. As in the orientable case, the fundamental group $\pi_1(M)$ must be free. That rank of $\pi_1(M)$ cannot exceed 1 can be seen similarly as in Section 2. On the other hand, $\pi_1(M)$ cannot be trivial, because $M$ is non-orientable. We thus conclude that $H^*(M;\mathbb{F}_2) \cong \wedge(x_1, x_{m-1}) \otimes \mathbb{F}_2$, and the corresponding manifolds are the generalized Klein-bottles (non-orientable $S^{m-1}$-bundles over $S^1$). Their category is 3 but we do not know whether their topological complexity can be, at least in some cases, also equal to 3.
Another extension that could be pursued is determination of manifolds whose topological complexity is at most 4. Although the general case seems to be beyond reach because we have very little information on manifolds whose category is 4, we believe that some reasonable progress could be achieved on closed manifolds $M$ satisfying $\text{TC}(M) \leq 4$ and $\text{cat}(M) \leq 3$.

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