Identities about double Eisenstein series

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Abstract  In this paper we consider certain classes of generalized double Eisenstein series by simple differential calculations of trigonometric functions. In particular, we give four new transformation formula for some double Eisenstein series. We can find that these double Eisenstein series are reducible to infinite series involving hyperbolic functions. Moreover, some interesting new examples are given.

Keywords  Eisenstein series; trigonometric function; hyperbolic function; Gamma function.

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1 Introduction

Let \( \mathbb{N} \) be the set of natural numbers, \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \), \( \mathbb{Z} \) the ring of integers, \( \mathbb{Q} \) the field of rational numbers, \( \mathbb{R} \) the field of real numbers, and \( \mathbb{C} \) the field of complex numbers. Let \( i = \sqrt{-1} \).

The subject of this paper are Eisenstein series and hyperbolic functions. Let \( \tau \) be a complex number with strictly positive imaginary part, the holomorphic Eisenstein series \( G_{2k}(\tau) \) of weight \( 2k \), where \( k \geq 2 \) is an integer, is defined by the following series:

\[
G_{2k}(\tau) := \sum_{m,n \in \mathbb{Z} \setminus (0,0)} \frac{1}{(m+n\tau)^{2k}}.
\]

This series absolutely converges to a holomorphic function of \( \tau \) in the upper half-plane. It is well known that the value of \( G_{4k}(i) \) can be expressed as

\[
G_{4k}(i) = \frac{\Gamma^2(1/4)}{2^{2k} \pi^{4k}} H_{4k} \quad (k \in \mathbb{N}),
\]

where \( H_{4m} \) are called the Hurwitz numbers (see [1,7,11]). When working with the \( q \)-expansion of the Eisenstein series, this alternate notation is frequently introduced:

\[
E_{2k}(\tau) := \frac{G_{2k}(\tau)}{2\zeta(2k)} = 1 + \frac{2}{\zeta(1-2k)} \sum_{n=1}^{\infty} \frac{n^{2k-1}q^n}{1-q^n},
\]

where \( q = \exp(2\pi \tau) \), and \( \zeta(s) \) denotes the Riemann zeta function. In Ramanujan’s notation, the three relevant Eisenstein series are defined for \( |q| < 1 \) by

\[
P(q) := 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n},
\]

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\[ Q(q) := 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1 - q^n}, \tag{1.5} \]
\[ R(q) := 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1 - q^n}. \tag{1.6} \]

Thus, for \( q = \exp(2\pi i \tau) \), \( E_4(\tau) = Q(q) \) and \( E_6(\tau) = R(q) \), which have weights 4 and 6, respectively. Since \((1.3)\) does not converge for \( j = 1 \), the Eisenstein series \( E_2(\tau) \) must be defined differently, which is defined by

\[ E_2(\tau) = P(q) - \frac{3}{\pi \text{Im} \tau}. \tag{1.7} \]

The functions \( P, Q \) and \( R \) were thoroughly studied in a famous paper \([6]\) by Ramanujan. Berndt \([2, 3]\) found a lot of identities about infinite series involving hyperbolic functions using certain modular transformation formula that originally stems from the generalized Eisenstein series. Further results of infinite series involving hyperbolic functions see Berndt’s books \([4, 5]\) and the references therein.

Recently, surprisingly little work has been done on double Eisenstein series involving hyperbolic functions. Tsumura \([11–14]\) and \([8]\) with Komori and Matsumoto studied many double Eisenstein series involving hyperbolic functions. For example, in 2008, Tsumura \([11]\) considered the following two double Eisenstein series of hyperbolic functions

\[ G_k(i) := \sum_{m,n \in \mathbb{Z}, m \neq 0} (-1)^n \frac{\sinh(m \pi) (m + ni)^k}{\text{Im} \tau}, \quad H_k(i) := \sum_{m,n \in \mathbb{Z}, m \neq 0} (-1)^n \frac{\cosh(m \pi) (m + ni)^k}{\text{Im} \tau} \quad (k \in \mathbb{N}). \]

He proved that \( G_{2k-1}(i) \) and \( H_{2k}(i) \) can be expressed in terms of \( \Gamma \) function and \( \pi \). Further, in 2009, Tsumura \([12]\) studied the closed form representations of sums

\[ C_k^v := \sum_{m,n \in \mathbb{Z}, m \neq 0} \coth^v(m \pi) (m + ni)^k \]

for \( k \in \mathbb{N} \) with \( k \geq 3 \) and \( v \in \mathbb{Z} \). Specially, he showed that

\[ C_k^v \in \mathbb{Q} \left[ \frac{1}{\pi}, \pi, \frac{\Gamma^8(1/4)}{\pi^2} \right] \]

for \( k \geq 3 \) and \( v \in \mathbb{N}_0 \) with \( k \equiv v \pmod{2} \).

In this paper, continuing Tsumura et al’s work, we study the four double Eisenstein series

\[ \sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{f(m)(-1)^n}{(m + ani)^k}, \quad \sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{f(m)(-1)^n}{(m + a(2n+1)i)^k}, \quad \sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{g(m)}{(m + ani)^p}, \quad \sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{g(m)}{(m + a(2n+1)i)^p}, \]

where \( k \in \mathbb{N}, \ p \in \mathbb{N} \setminus \{1\}, \ a \in \mathbb{R} \setminus \{0\} \) and

\[ f(m) = o(1), \ g(m) = o(1/m), \ m \to \infty. \]

We prove that these double series can be expressed by single infinite series involving hyperbolic functions. Moreover, we consider some special cases. We can find that many double Eisenstein series involving hyperbolic functions can be expressed in terms of \( \Gamma \) function and \( \pi \).
2 Differential formulas of trigonometric functions

Let

\[ |r|_i := r_0 + r_1 + \cdots + r_l \quad (r_j \in \mathbb{N}_0). \]

**Lemma 2.1** Let \( I_{k,m} \) be a sequence and \( k \) and \( m \) are positive integers (including zero). If \( I_{k,m} \) satisfy a recurrence relation in the form

\[
I_{m,k} = a_m I_{k-1,m+1} - b_m I_{k-1,m} \tag{2.1}
\]

then

\[
I_{k,m} = \sum_{l=0}^{k-1} \left( \prod_{j=0}^{l} a_{m+j} \right) \left( \sum_{|r|_{l+1}=k-l-1} \prod_{h=0}^{l+1} b_{m_{h+1}} \right) I_{0,m+l+1}(-1)^{k-l+1} + (-1)^{k+1} b_m^k I_{0,m}, \tag{2.2}
\]

where \( a_m \) and \( b_m \) are constants.

**Proof.** The result (2.2) can be proved by mathematical induction. \( \square \)

**Theorem 2.2** For integers \( k \geq 0, \) \( m \geq 1 \) and complex number \( s \in \mathbb{C} \setminus \mathbb{N}_0, \) then

\[
\frac{d^{2k}}{ds^{2k}} \left( \frac{1}{\sin^{2m-1}(\pi s)} \right) = \pi^{2k} \sum_{l=0}^{k} \left( \frac{(2m+2l-2)!}{(2m-2)!} \right) \sum_{|r|_{l+1}=k-l-1} \prod_{h=0}^{l} (2m+2h-1)^{2r_h} \left( \frac{(-1)^{l-k-1}}{\sin^{2m+2l-1}(\pi s)} \right), \tag{2.3}
\]

\[
\frac{d^{2k+1}}{ds^{2k+1}} \left( \frac{1}{\sin^{2m-1}(\pi s)} \right) = \pi^{2k+1} \sum_{l=0}^{k} \left( \frac{(2m+2l-1)!}{(2m-2)!} \right) \sum_{|r|_{l+1}=k-l-1} \prod_{h=0}^{l} (2m+2h-1)^{2r_h} \left( \frac{(-1)^{l-k}}{\sin^{2m+2l-1}(\pi s)} \right) \left( \frac{\cos(\pi s)}{\sin^{2m+2l-1}(\pi s)} \right). \tag{2.4}
\]

**Proof.** A elementary calculation gives

\[
\frac{d^{2k}}{ds^{2k}} \left( \frac{1}{\sin^{2m-1}(\pi s)} \right) = 2m(2m-1)\pi^2 \frac{d^{2k-2}}{ds^{2k-2}} \left( \frac{1}{\sin^{2m+1}(\pi s)} \right)
- (2m-1)^2 \pi^2 \frac{d^{2k-2}}{ds^{2k-2}} \left( \frac{1}{\sin^{2m-1}(\pi s)} \right). \tag{2.5}
\]

So, setting \( I_{k,m} = \frac{d^{2k}}{ds^{2k}}(1/\sin^{2m-1}(\pi s))/ds^{2k} \), \( a_m = 2m(2m-1)\pi^2 \) and \( b_m = (2m-1)^2\pi^2 \) in (2.2) and combining (2.5) yields the desired result (2.3). Then, differentiating (2.3) with respect to \( s \), we may deduce the evaluation (2.4). \( \square \)

**Theorem 2.3** For integer \( k \in \mathbb{N}_0 \) and complex number \( s \in \mathbb{C} \setminus \mathbb{N}_0, \) we have

\[
\frac{d^{2k}}{ds^{2k}} \left( \cot(\pi s) \right) = \pi^{2k} \sum_{0 \leq l \leq j \leq k} \left\{ \binom{2k}{2j} (2l)! - \binom{2k}{2j+1} (2l+1)! \right\}
\times \left\{ \sum_{|r|_{l+1}=j-l} \prod_{h=0}^{l} (2h+1)^{2r_h} \right\} \frac{(-1)^{k-l} \cos(\pi s)}{\sin^{2l+1}(\pi s)}. \tag{2.6}
\]
\[
\frac{d^{2k+1}}{ds^{2k+1}}(\cot(\pi s)) = \pi^{2k+1} \sum_{1 \leq l \leq j \leq k+1} \left\{ \binom{2k+2}{2j} (2l-1)! - \binom{2k+2}{2j+1} (2l-1)! (2l+1) \right\} \\
\times \left\{ \sum_{|r| = j-l, h = 0}^{l} (2h+1)^{2r_h} \right\} \frac{(-1)^{k-l}}{\sin^{2l}(\pi s)}, \quad (2.7)
\]

where
\[
\binom{n}{k} := \frac{n!}{k!(n-k)!},
\]

if \( k > n \), then \( \binom{n}{k} = 0 \).

**Proof.** By a direct calculation we find that
\[
\frac{d^{2k}}{ds^{2k}} (\cot(\pi s)) = \frac{d^{2k}}{ds^{2k}} \left( \frac{\sin(\pi s)}{\cos(\pi s)} \right)
\]
\[
= \sum_{j=0}^{k} \binom{2k}{2j} \frac{d^{2j}}{ds^{2j}} \left( \frac{1}{\sin(\pi s)} \right) \frac{d^{2k-2j-1}}{ds^{2k-2j-1}} (\cos(\pi s))
\]
\[
- \sum_{j=0}^{k} \binom{2k}{2j+1} \frac{d^{2j+1}}{ds^{2j+1}} \left( \frac{1}{\sin(\pi s)} \right) \frac{d^{2k-2j-1}}{ds^{2k-2j-1}} (\cos(\pi s)). \quad (2.8)
\]

Hence, letting \( k = j \) and \( m = 1 \) in (2.3) and (2.4), then substituting it into (2.8) we obtain (2.6). Integrating (2.6) over the interval \((1/2, s)\) with respect to \( s \), a simple calculation gives the formula (2.7).

Further, changing \( s \) to \( 1/2 - s \) in Theorems 2.2 and 2.3, we can get the following corollaries.

**Corollary 2.4** For integer \( k \) and complex number \( s \) with \( s \neq \pm 1/2, \pm 3/2, \cdots \), then
\[
\frac{d^{2k}}{ds^{2k}} \left( \frac{1}{\cos(\pi s)} \right) = \pi^{2k} \sum_{l=0}^{k} \left( (2l)! \sum_{|r| = k-l, h = 0}^{l} (2h+1)^{2r_h} \right) \frac{(-1)^{k-l}}{\cos^{2l+1}(\pi s)}, \quad (2.9)
\]
\[
\frac{d^{2k+1}}{ds^{2k+1}} (\tan(\pi s)) = \pi^{2k+1} \sum_{l=0}^{k} \left( (2l+1)! \sum_{|r| = k-l, h = 0}^{l} (2h+1)^{2r_h} \right) \frac{(-1)^{k-l} \sin(\pi s)}{\cos^{2l+1}(\pi s)}. \quad (2.10)
\]

**Corollary 2.5** For integer \( k \) and complex number \( s \) with \( s \neq \pm 1/2, \pm 3/2, \cdots \), then
\[
\frac{d^{2k}}{ds^{2k}} (\tan(\pi s)) = \pi^{2k} \sum_{0 \leq l \leq j \leq k} \left\{ \binom{2k}{2j} (2l)! - \binom{2k}{2j+1} (2l+1)! \right\}
\]
\[
\times \left\{ \sum_{|r| = j-l, h = 0}^{l} (2h+1)^{2r_h} \right\} \frac{(-1)^{k-l} \sin(\pi s)}{\cos^{2l+1}(\pi s)}, \quad (2.11)
\]
\[
\frac{d^{2k+1}}{ds^{2k+1}} (\tan(\pi s)) = \pi^{2k+1} \sum_{1 \leq l \leq j \leq k+1} \left\{ \binom{2k+2}{2j} (2l-1)! - \binom{2k+2}{2j+1} (2l-1)! (2l+1) \right\}
\]
\[
\times \left\{ \sum_{|r| = j-l, h = 0}^{l} (2h+1)^{2r_h} \right\} \frac{(-1)^{k-l+1}}{\cos^{2l}(\pi s)}. \quad (2.12)
\]
3 Main Theorems and Corollaries

In this section we consider the following double Eisenstein series

\[ \sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{f(m)}{m + ani} \pi (-1)^n, \quad \sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{f(m)}{m + a(2n + 1)i} \pi (-1)^n \]

and

\[ \sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{g(m)}{(m + ani)^p}, \quad \sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{g(m)}{(m + a(2n + 1)i)^p}, \]

where \( k \in \mathbb{N}, \ p \in \mathbb{N} \setminus \{1\}, \ a \in \mathbb{R} \setminus \{0\} \) and

\[ f(m) = o(1), \ g(m) = o(1/m), \ m \to \infty. \]

Note that if \( k = 1 \) in the first two sums, then

\[ \sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{f(m)(-1)^n}{m + ani} = \sum_{m \in \mathbb{Z}, m \neq 0} \lim_{N \to \infty} \sum_{-N \leq n \leq N} \frac{f(m)(-1)^n}{m + ani}, \]

\[ \sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{f(m)(-1)^n}{m + a(2n + 1)i} = \sum_{m \in \mathbb{Z}, m \neq 0} \lim_{N \to \infty} \sum_{-N \leq n \leq N} \frac{f(m)(-1)^n}{m + a(2n + 1)i}. \]

3.1 Four Theorems

According to the partial fraction expansion of trigonometric function

\[ \frac{\pi}{\sin(\pi s)} = \sum_{n \in \mathbb{Z}} (-1)^n \frac{\pi}{n + s}, \quad \frac{\pi}{\cos(\pi s)} = 2 \sum_{n \in \mathbb{Z}} (-1)^n \frac{1}{2n + 1 - 2s}, \]

\[ \pi \cot(\pi s) = \sum_{n \in \mathbb{Z}} \frac{1}{n + s}, \quad \pi \tan(\pi s) = 2 \sum_{n \in \mathbb{Z}} \frac{1}{2n + 1 - 2s}, \]

elementary calculations show that

\[ \sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{f(m)}{(m + ani)^k} \pi (-1)^n = \frac{(-1)^{k-1}i^k}{a^k(k-1)!} \sum_{m=1}^{\infty} \left( f(m) + (-1)^k f(-m) \right) \frac{d^{k-1}}{ds^{k-1}} \left( \frac{\pi}{\sin(\pi s)} \right)_{s=mi/a}, \]

\[ \sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{g(m)}{(m + ani)^p} = \frac{(-1)^{p-1}i^p}{a^p(p-1)!} \sum_{m=1}^{\infty} \left( g(m) + (-1)^p g(-m) \right) \frac{d^{p-1}}{ds^{p-1}} \left( \pi \cot(\pi s) \right)_{s=mi/a}, \]

\[ \sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{f(m)(-1)^n}{(m + a(2n + 1)i)^k} = \frac{(-1)^{k}i^k}{(2a)^k(k-1)!} \sum_{m=1}^{\infty} \left( f(m) + (-1)^k f(-m) \right) \frac{d^{k-1}}{ds^{k-1}} \left( \frac{\pi}{\cos(\pi s)} \right)_{s=mi/2a}, \]

\[ \sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{g(m)}{(m + a(2n + 1)i)^p} = \frac{(-1)^{p}i^p}{(2a)^p(p-1)!} \sum_{m=1}^{\infty} \left( g(m) + (-1)^p g(-m) \right) \frac{d^{p-1}}{ds^{p-1}} \left( \pi \tan(\pi s) \right)_{s=mi/2a}. \]
In general, we have

\[
\sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{f(m)(-1)^n}{(bm + c + ani)^k} = \frac{(-1)^{k-1}i^k}{a^k(k-1)!} \sum_{m=1}^{\infty} \left\{ f(m) \frac{d^{k-1}}{ds^{k-1}} \left( \frac{\pi}{\sin(\pi s)} \right)_{s = (bm+c)i/a} \right\},
\]

(3.1)

\[
\sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{g(m)}{(bm + c + ani)^p} = \frac{(-1)^{p-1}i^p}{a^p(p-1)!} \sum_{m=1}^{\infty} \left\{ g(m) \frac{d^{p-1}}{ds^{p-1}} \left( \pi \cot(\pi s) \right)_{s = (bm+c)i/a} \right\},
\]

(3.2)

\[
\sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{f(m)(-1)^n}{(bm + c + a(2n+1)i)^p} = \frac{(-1)^k i^k}{(2a)^k(k-1)!}
\times \sum_{m=1}^{\infty} \left\{ f(m) \frac{d^{p-1}}{ds^{p-1}} \left( \frac{\pi}{\cos(\pi s)} \right)_{s = (bm-c)i/2a} \right\},
\]

(3.3)

\[
\sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{g(m)}{(bm + c + a(2n+1)i)^p} = \frac{(-1)^{p}i^{p}}{(2a)^{p}(p-1)!}
\times \sum_{m=1}^{\infty} \left\{ g(m) \frac{d^{p-1}}{ds^{p-1}} \left( \pi \tan(\pi s) \right)_{s = (bm-c)i/2a} \right\},
\]

(3.4)

where \( a, b \in \mathbb{R} \setminus \{0\} \) and \( c \in \mathbb{R} \). Then with the help of Theorems 2.2, 2.3 and Corollaries 2.4, 2.5, we can get the following theorems.

**Theorem 3.1** For positive integer \( k \) and real \( a \in \mathbb{R} \setminus \{0\} \), we have

\[
\sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{f(m)(-1)^n}{m + ani} = \frac{\pi^{2k}}{a^{2k}(2k-1)!} \sum_{l=0}^{k-1} (2l+1)! \left\{ \sum_{|r|=k-l} \prod_{h=0}^{l} (2h + 1)^{2r_h} \right\}
\times \sum_{m=1}^{\infty} \frac{(f(m) + f(-m)) \cosh(m \pi / a)}{\sinh^{2l+2}(m \pi / a)},
\]

(3.5)

\[
\sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{f(m)(-1)^n}{m + ani} = \frac{\pi^{2k-1}}{a^{2k-1}(2k-2)!} \sum_{l=0}^{k-1} (2l)! \left\{ \sum_{|r|=k-l} \prod_{h=0}^{l} (2h + 1)^{2r_h} \right\}
\times \sum_{m=1}^{\infty} \frac{(f(m) - f(-m))}{\sinh^{2l+1}(m \pi / a)},
\]

(3.6)

**Theorem 3.2** For positive integer \( k \) and real \( a \in \mathbb{R} \setminus \{0\} \), we have

\[
\sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{g(m)}{m + ani} = \frac{\pi^{2k}}{a^{2k}(2k-1)!} \sum_{1\leq i \leq j \leq k} \left\{ \binom{2k}{2j}(2l-1)! - \binom{2k}{2j+1}(2l-1)!(2l+1) \right\}
\]

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Theorem 3.3 For positive integer $k$ and real $a \in \mathbb{R} \setminus \{0\}$, we have

\[
\sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{f(m)(-1)^n}{(m + a(2n + 1)i)^{2k}} = \frac{\pi^{2k_i}}{(2a)^{2k}(2k - 1)!} \sum_{l=0}^{k-1} (-1)^{l-1}(2l + 1)! \left\{ \sum_{|r| = k-1-l} \prod_{l=0}^{l} (2l + 1)^{2r_h} \right\} \times \sum_{m=1}^{\infty} \frac{(f(m) - f(-m)) \sinh(m\pi/2a)}{\sinh^{2l+1}(m\pi/2a)}. \tag{3.9}
\]

Theorem 3.4 For positive integer $k$ and real $a \in \mathbb{R} \setminus \{0\}$, we have

\[
\sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{g(m)}{(m + a(2n + 1)i)^{2k}} = \frac{\pi^{2k}}{(2a)^{2k}(2k - 1)!} \sum_{1 \leq l \leq k} \left\{ \left( \frac{2k}{2j} \right) (2l - 1)! - \left( \frac{2k}{2j + 1} \right) (2l - 1)! (2l + 1) \right\} \times (-1)^l \left\{ \sum_{|r| = j-l} \prod_{l=0}^{l} (2l + 1)^{2r_h} \right\} \sum_{m=1}^{\infty} \frac{g(m) + g(-m)}{\cosh^{2l+1}(m\pi/2a)}, \tag{3.11}
\]

\[
\sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{g(m)}{(m + a(2n + 1)i)^{2k+1}} = \frac{\pi^{2k+1}}{(2a)^{2k+1}(2k)!} \sum_{0 \leq l \leq k} \left\{ \left( \frac{2k}{2j} \right) (2l)! - \left( \frac{2k}{2j + 1} \right) (2l + 1)! \right\} (-1)^l \times \left\{ \sum_{|r| = j-l} \prod_{l=0}^{l} (2l + 1)^{2r_h} \right\} \sum_{m=1}^{\infty} \frac{(g(m) - g(-m)) \sinh(m\pi/a)}{\sinh^{2l+1}(m\pi/a)}. \tag{3.12}
\]
3.2 Corollaries

From Theorems 3.1-3.4 we give the following corollaries.

**Corollary 3.5** For real \( a \in \mathbb{R} \setminus \{0\} \), we have

\[
\sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{f(m)(-1)^n}{m+ani} = \frac{\pi}{a} \sum_{m=1}^{\infty} \frac{f(m) - f(-m)}{\sinh(m\pi/a)},
\]

\( (3.13) \)

\[
\sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{f(m)(-1)^n}{(m+ani)^2} = \frac{\pi^2}{a^2} \sum_{m=1}^{\infty} \frac{(f(m) + f(-m)) \cosh(m\pi/a)}{\sinh^2(m\pi/a)},
\]

\( (3.14) \)

\[
\sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{f(m)(-1)^n}{(m+ani)^3} = \frac{\pi^3}{2a^3} \sum_{m=1}^{\infty} \left( \frac{f(m) - f(-m)}{\sinh(m\pi/a)} + 2 \frac{f(m) - f(-m)}{\sinh^3(m\pi/a)} \right),
\]

\( (3.15) \)

\[
\sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{f(m)(-1)^n}{(m+ani)^4} = \frac{\pi^4}{6a^4} \sum_{m=1}^{\infty} \left( f(m) + f(-m) \right) \left( \cosh(m\pi/a) \right)^3 + 6 \cosh^4(m\pi/a) .
\]

\( (3.16) \)

**Corollary 3.6** For real \( a \in \mathbb{R} \setminus \{0\} \), we have

\[
\sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{g(m)}{(m+ani)^2} = \frac{\pi^2}{a^2} \sum_{m=1}^{\infty} \frac{g(m) + g(-m)}{\sinh^2(m\pi/a)},
\]

\( (3.17) \)

\[
\sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{g(m)}{(m+ani)^3} = \frac{\pi^3}{a^3} \sum_{m=1}^{\infty} \frac{(g(m) - g(-m)) \cosh(m\pi/a)}{\sinh^3(m\pi/a)}.
\]

\( (3.18) \)

**Corollary 3.7** For real \( a \in \mathbb{R} \setminus \{0\} \), we have

\[
\sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{f(m)(-1)^n}{m + a(2n+1)i} = -\frac{\pi i}{2a} \sum_{m=1}^{\infty} \frac{f(m) + f(-m)}{\cosh(m\pi/2a)},
\]

\( (3.19) \)

\[
\sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{f(m)(-1)^n}{(m + a(2n+1)i)^2} = -\frac{\pi^2 i}{4a^2} \sum_{m=1}^{\infty} \frac{(f(m) - f(-m)) \sinh(m\pi/2a)}{\cosh^2(m\pi/2a)},
\]

\( (3.20) \)

\[
\sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{f(m)(-1)^n}{(m + a(2n+1)i)^3} = -\frac{\pi^3 i}{16a^3} \sum_{m=1}^{\infty} \left( \frac{f(m) + f(-m)}{\cosh(m\pi/2a)} - 2 \frac{f(m) + f(-m)}{\cosh^3(m\pi/2a)} \right),
\]

\( (3.21) \)

\[
\sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{f(m)(-1)^n}{(m + a(2n+1)i)^4} = -\frac{\pi^4 i}{96a^4} \sum_{m=1}^{\infty} \left( f(m) - f(-m) \right) \left( \frac{\sinh(m\pi/2a)}{\cosh^2(m\pi/2a)} - 6 \frac{\sinh(m\pi/2a)}{\cosh^4(m\pi/2a)} \right).
\]

\( (3.22) \)

**Corollary 3.8** For real \( a \in \mathbb{R} \setminus \{0\} \), we have

\[
\sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{g(m)}{(m + a(2n+1)i)^2} = -\frac{\pi^2}{4a^2} \sum_{m=1}^{\infty} \frac{g(m) + g(-m)}{\cosh^2(m\pi/2a)},
\]

\( (3.23) \)

\[
\sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{g(m)}{(m + a(2n+1)i)^3} = -\frac{\pi^3}{8a^3} \sum_{m=1}^{\infty} \frac{(g(m) - g(-m)) \sinh(m\pi/2a)}{\cosh^3(m\pi/2a)},
\]

\( (3.24) \)

\[
\sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{g(m)}{(m + a(2n+1)i)^4} = -\frac{\pi^4}{48a^4} \sum_{m=1}^{\infty} \left( 2 \frac{g(m) + g(-m)}{\cosh^2(m\pi/2a)} - 3 \frac{g(m) + g(-m)}{\cosh^4(m\pi/2a)} \right).
\]

\( (3.25) \)
3.3 Examples

Since the four infinite series involving hyperbolic functions

\[
\sum_{n=1}^{\infty} \frac{1}{\sinh^{2p}(n\pi)}, \quad \sum_{n=1}^{\infty} \frac{1}{\cosh^{2p}(n\pi)}, \quad \sum_{n=1}^{\infty} \frac{n^{2p}}{\sinh^{2}(n\pi)} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{n^{2p}}{\cosh^{2}(n\pi)} \quad (p \in \mathbb{N})
\]

can be evaluated by Gamma function and \( \pi \) (explicit evaluations see [9, 15]). Hence, from Corollaries 3.5-3.8, we give some well-known and new results of double Eisenstein series involving hyperbolic functions.

\[
\sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{m^2(-1)^n}{\sinh(m\pi)(m + ni)} = -\frac{1}{4\pi} + \frac{\Gamma^8(1/4)}{768\pi^5},
\]

\[
\sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{m^4(-1)^n}{\sinh(m\pi)(m + ni)} = \frac{\Gamma^8(1/4)}{640\pi^6},
\]

\[
\sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{m^6(-1)^n}{\sinh(m\pi)(m + ni)} = \frac{\Gamma^{16}(1/4)}{57344\pi^{11}},
\]

\[
\sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{m^8(-1)^n}{\sinh(m\pi)(m + ni)} = \frac{3\Gamma^{16}(1/4)}{40960\pi^{12}},
\]

\[
\sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{m^2(-1)^n}{\cosh(m\pi)(m + (2n + 1)i/2)} = i \left( \frac{\Gamma^4(1/4)}{32\pi^3} - \frac{\Gamma^8(1/4)}{768\pi^5} \right),
\]

\[
\sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{m^4(-1)^n}{\cosh(m\pi)(m + (2n + 1)i/2)} = i \left( \frac{3\Gamma^8(1/4)}{5120\pi^6} - \frac{\Gamma^{12}(1/4)}{8192\pi^8} \right),
\]

\[
\sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{m^6(-1)^n}{\cosh(m\pi)(m + (2n + 1)i/2)} = -i \left( \frac{9\Gamma^{12}(1/4)}{32768\pi^9} + \frac{3\Gamma^{16}(1/4)}{1835008\pi^{11}} \right),
\]

\[
\sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{m^8(-1)^n}{\cosh(m\pi)(m + (2n + 1)i/2)} = -i \left( \frac{21\Gamma^{16}(1/4)}{5242880\pi^{12}} + \frac{33\Gamma^{20}(1/4)}{8388608\pi^{14}} \right),
\]

\[
\sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{m^2}{\cosh^2(m\pi/2)(m + 2ni)^2} = -\frac{1}{4} + \frac{\Gamma^8(1/4)}{768\pi^4},
\]

\[
\sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{1}{\sinh^2(m\pi)(m + ni)} = -\frac{11}{45}\pi^2 + \frac{2}{3}\pi + \frac{\Gamma^8(1/4)}{960\pi^4},
\]

\[
\sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{1}{\sinh(2m\pi)(m + ni)^3} = -\frac{11}{90}\pi^2 + \frac{1}{3}\pi^2 + \frac{\Gamma^8(1/4)}{1920\pi^4},
\]

\[
\sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{(-1)^n}{\sinh(m\pi)(m + (2n + 1)i/2)^2} = i \left( \pi^2 - \pi - \frac{\Gamma^4(1/4)}{8\pi} \right),
\]

\[
\sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{(-1)^n}{\sinh(m\pi)(m + (2n + 1)i/2)^4} = i \left( \frac{\pi^3}{2} - \frac{5}{6}\pi^4 + \frac{\pi}{16}\Gamma^4(1/4) + \frac{\Gamma^8(1/4)}{96\pi^2} \right),
\]

\[
\sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{(-1)^n}{\cosh(m\pi)(m + (2n + 1)i/2)^2} = i \left( \pi - 1 - \frac{\Gamma^4(1/4)}{8\pi^2} \right),
\]
\[
\sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{(-1)^n}{\cosh(m\pi)(m + (2n+1)i/2)^3} = i \left( \frac{\pi^2}{6} - \frac{\pi^3}{2} + \frac{\Gamma^4(1/4)}{48} + \frac{\Gamma^8(1/4)}{96\pi^3} \right),
\]
\[
\sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{1}{\sinh^2(m\pi/2)(m + (2n+1)i)^2} = \pi - \frac{\pi^2}{3},
\]
\[
\sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{1}{\cosh^2(m\pi)(m + (2n+1)i/2)^4} = \frac{4}{45} \pi^3 - \frac{\pi^4}{3} + \frac{\pi^5}{90} \Gamma^4(1/4) + \frac{\Gamma^8(1/4)}{288\pi^2} + \frac{\Gamma^{12}(1/4)}{1280\pi^5},
\]
\[
\sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{1}{\sinh(2m\pi)(m + (2n+1)i/2)^3} = \frac{\pi^3}{2} - \frac{\pi^2}{3} - \frac{\Gamma^4(1/4)}{24} - \frac{\Gamma^8(1/4)}{192\pi^5}.
\]

The tenth equation appear as example of Example 3 in the [8]. It should be emphasized that the reference [8] also contains many other types of double Eisenstein series.

It is possible that many other evaluations of double Eisenstein series involving hyperbolic functions can be obtained by using the methods and techniques of the present paper. For example, by Theorems 3.1-3.4, it is clear that the twelve double Eisenstein series involving hyperbolic functions

\[
\sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{\coth^2(p(m\pi))}{(m + ni)^{2k+2}},
\]
\[
\sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{\coth^{2p+1}(m\pi)}{(m + ni)^{2k+1}},
\]
\[
\sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{\tanh^{2p}(m\pi)}{(m + (2n+1)i/2)^{2k+2}},
\]
\[
\sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{\tanh^{2p+1}(m\pi)}{(m + (2n+1)i/2)^{2k+1}},
\]
\[
\sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{(-1)^n}{(m + ni)^{2k} \cosh(m\pi) \sinh^{2p}(m\pi)},
\]
\[
\sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{(-1)^n}{(m + ni)^{2k-1} \sinh^{2p+1}(m\pi)},
\]
\[
\sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{1}{(m + ni)^{2k+1} \cosh(m\pi) \sinh^{2p+1}(m\pi)},
\]
\[
\sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{1}{(m + ni)^{2k} \sinh^{2p+2}(m\pi)},
\]
\[
\sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{(-1)^n}{(m + (2n+1)i/2)^{2k} \sinh(m\pi) \cosh^{2p}(m\pi)},
\]
\[
\sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{(-1)^n}{(m + (2n+1)i/2)^{2k-1} \cosh^{2p+1}(m\pi)},
\]
\[
\sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{1}{(m + (2n+1)i/2)^{2k+1} \sinh(m\pi) \cosh^{2p+1}(m\pi)}.
\]
\[\sum_{m,n \in \mathbb{Z}, m \neq 0} \frac{1}{(m + (2n + 1)i/2)^{2k} \cosh^{2p+2}(m \pi)}.\]

can be represented by \[\sum_{n=1}^{\infty} \frac{1}{\sinh^{2l}(n \pi)}\] or \[\sum_{n=1}^{\infty} \frac{1}{\cosh^{2l}(n \pi)},\] which implies that these double Eisenstein series can be expressed in terms of \(\Gamma\) function and \(\pi\). Here \(k \in \mathbb{N}\) and \(p \in \mathbb{N}_0\). For example, the evaluations of the first and second sums can be obtained by Theorem 3.2. The evaluations of the third and forth sums can be deduced by Theorem 3.4.

Similarly, by Theorems 2.2, 2.3, Corollaries 2.4, 2.5, and formulas (3.1)-(3.4) we can give explicit evaluations of the sums

\[\sum_{m,n \in \mathbb{Z}} \frac{f(m)(1)^n}{(2bm + b + ani)^k}, \quad \sum_{m,n \in \mathbb{Z}} \frac{g(m)}{(2bm + b + ani)^p}.
\]

\[\sum_{m,n \in \mathbb{Z}} \frac{f(m)(1)^n}{(2bm + b + a(2n + 1)i)^k} \quad \sum_{m,n \in \mathbb{Z}} \frac{g(m)}{(2bm + b + a(2n + 1)i)^p},\]

where \(a, b \in \mathbb{R} \setminus \{0\}\). For instance, a simple calculation gives

\[\sum_{m,n \in \mathbb{Z}} \frac{f(m)(1)^n}{(2bm + b + ani)^2} = \frac{\pi^{2k}}{a^{2k}(2k - 1)!}\sum_{l=0}^{k-1} (2l + 1)! \left\{ \sum_{|r| = k - 1 - l} \prod_{h=0}^{l} (2h + 1)^{2r} \right\} \times \sum_{m=1}^{\infty} \frac{(f(m - 1) + f(-m)) \cosh((2m - 1)b\pi/a)}{\sinh^{2l+2}((2m - 1)b\pi/a)}. \quad (3.26)\]

Setting \(k = 1\) in equation above yields

\[\sum_{m,n \in \mathbb{Z}} \frac{f(m)(1)^n}{(2bm + b + ani)^2} = \frac{\pi^2}{a^2} \sum_{m=1}^{\infty} \frac{(f(m - 1) + f(-m)) \cosh((2m - 1)b\pi/a)}{\sinh^2((2m - 1)b\pi/a)}. \quad (3.27)\]

Hence, we can deduce the two evaluations

\[\sum_{m,n \in \mathbb{Z}} \frac{(-1)^n}{\cosh((2m + 1)\pi/2)(2m + 1 + 2ni)^2} = -\frac{\pi}{4} + \frac{\Gamma^4(1/4)}{32\pi},\]

\[\sum_{m,n \in \mathbb{Z}} \frac{(-1)^n}{\cosh^3((2m + 1)\pi/2)(2m + 1 + 2ni)^2} = -\frac{\pi}{2} + \frac{\Gamma^4(1/4)}{32\pi},\]

where we used the three well known results (see [9, 10])

\[\sum_{n=1}^{\infty} \frac{1}{\sinh^2(n \pi)} = \frac{1}{6} - \frac{1}{2\pi},\]

\[\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sinh^2(n \pi)} = -\frac{1}{6} + \frac{\Gamma^4(1/4)}{32\pi^3},\]

\[\sum_{n=1}^{\infty} \frac{1}{\sinh^2((2n - 1)\pi/2)} = -\frac{1}{2\pi} + \frac{\Gamma^4(1/4)}{16\pi^3}.\]

Note that in [9, 10], \(u = \frac{\Gamma^4(1/4)}{8\pi}\). It is obvious that the results of Tsumura [11–13] can be established by using the methods of the present paper.
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