Modelling and Analysis of Fractional Order Systems using Ultradistributions

C.M.Grunfeld and M.C.Rocca

Departamento de Física, Fac. de Ciencias Exactas,
Universidad Nacional de La Plata.
C.C. 67 (1900) La Plata. Argentina.

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Abstract

In this paper we introduce a new mathematical tool to solve fractional equations representing models of fractional systems: The Ul-

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Ultradistributions permit us to unify the notion of integral and derivative in one only operation. Several examples of application of the results obtained are given.

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1 Introduction

The use of fractional calculus for modelling physical systems has been considered in many works. See for example [1, 2, 3]. We can find also works dealing with the application of this mathematical tool in control theory [4, 5, 6, 7].

Moreover, there are many physical systems that can be described by means of a fractional calculus. Some examples are: chaos [8], long electric lines [9], electrochemical process [10] and dielectric polarization [11].

In this paper we want to introduce a new mathematical framework to solve fractional equations representing models of fractional systems which was not treated in none of the previous works: The Ultradistributions.

The paper is organized as follow: in section 2 we introduce definition of fractional derivation and integration. In section 3 we give some examples of application of the formulae of section 2 using the Fourier Transform and the one-side Laplace Transform. In section 3 we present a circuital application. Finally in section 4 we discuss the results obtained in sections 1, 2 and 3.
2 Fractional Calculus

The purpose of this sections is to introduce definition of fractional derivation
and integration given in ref. [12]. This definition unifies the notion of integral
and derivative in one only operation. Let \( \hat{f}(x) \) a distribution of exponential
type and \( F(\Omega) \) the complex Fourier transformed Tempered Ultradistribution.

Then:
\[
F(\Omega) = \mathcal{U}[\Im(\Omega)] \int_0^\infty \hat{f}(x)e^{j\Omega x} \, dx - \mathcal{U}[-\Im(\Omega)] \int_{-\infty}^0 \hat{f}(x)e^{j\Omega x} \, dx \quad (2.1)
\]

(\( \mathcal{U}(x) \) is the Heaviside step function) and

\[
\hat{f}(x) = \frac{1}{2\pi} \oint_{\Gamma} F(\Omega)e^{-j\Omega x} \, d\Omega \quad (2.2)
\]

where the contour \( \Gamma \) surround all singularities of \( F(\Omega) \) and runs parallel to
real axis from \(-\infty\) to \(\infty\) above the real axis and from \(\infty\) to \(-\infty\) below the
real axis. According to [12] the fractional derivative of \( \hat{f}(x) \) is given by

\[
\frac{d^\lambda \hat{f}(x)}{dx^\lambda} = \frac{1}{2\pi} \oint_{\Gamma} (-j\Omega)^\lambda F(\Omega)e^{-j\Omega x} \, d\Omega + \oint_{\Gamma} (-j\Omega)^\lambda a(\Omega)e^{-j\Omega x} \, d\Omega \quad (2.3)
\]

Where \( a(\Omega) \) is entire analytic and rapidly decreasing. If \( \lambda = -1 \), \( d^\lambda /dx^\lambda \) is
the inverse of the derivative (an integration). In this case the second term of
the right side of (2.3) gives a primitive of \( \hat{f}(x) \). Using Cauchy’s theorem the
additional term is
\[
\oint \frac{a(\Omega)}{\Omega} e^{-j\Omega x} d\Omega = 2\pi a(0) \tag{2.4}
\]

Of course, an integration should give a primitive plus an arbitrary constant.

Analogously when \( \lambda = -2 \) (a double iterated integration) we have
\[
\oint \frac{a(\Omega)}{\Omega^2} e^{-j\Omega x} d\Omega = \gamma + \delta x \tag{2.5}
\]
where \( \gamma \) and \( \delta \) are arbitrary constants. With the change of variables \( s = -j\Omega \)
formulae (2.1) and (2.2) can be written as:
\[
G(s) = U[\Re(s)] \int_0^\infty \hat{f}(x) e^{-sx} \, dx - U[-\Re(s)] \int_{-\infty}^0 \hat{f}(x) e^{-sx} \, dx \tag{2.6}
\]
and
\[
\hat{f}(x) = \frac{1}{2\pi i} \oint_\Gamma G(s) e^{sx} \, ds \tag{2.7}
\]
where the contour \( \Gamma \) surround all singularities of \( G(S) \) and runs parallel to imaginary axis from \(-j\infty\) to \(j\infty\) to the right of the imaginary axis and from \(j\infty\) to \(-j\infty\) to the left of the imaginary axis. Formula (2.6) represents the two-sided Laplace Transform. The fractional derivative is now:
\[
\frac{d^\lambda \hat{f}(x)}{dx^\lambda} = \frac{1}{2\pi i} \oint_\Gamma s^\lambda G(s) e^{sx} \, ds + \oint_\Gamma s^\lambda a(s) e^{sx} \, ds \tag{2.8}
\]
For the one-side Laplace Transform we have
\[
G(s) = U[\Re(s)] \int_0^\infty \hat{f}(x) e^{-sx} \, dx \tag{2.9}
\]
\[ \hat{f}(x) = \frac{1}{2\pi j} \int_{a-j\infty}^{a+j\infty} G(s)e^{sx} \, ds \quad (2.10) \]

and for the fractional derivative:

\[ \frac{d^\lambda \hat{f}(x)}{dx^\lambda} = \frac{1}{2\pi j} \int_{a-j\infty}^{a+j\infty} s^\lambda G(s)e^{sx} \, ds \quad (2.11) \]

3 Examples

In this section we give some examples of the application of formulae of the precedent section. At first using the Fourier Transform and at second place using the one-side Laplace Transform.

The Fourier Transform

Let \( U(x) \) be the Heaviside step function.

\[ \hat{f}(x) = U(x) ; \quad \mathcal{F}(\Omega) = U[\mathcal{I}(\Omega)] \int_0^\infty e^{-j\Omega x} \, dx = \frac{jU[\mathcal{I}(\Omega)]}{\Omega} \quad (3.1) \]

The fractional derivative is:

\[ \frac{d^\lambda U(x)}{dx^\lambda} = \frac{je^{-j\pi\lambda/2}}{2\pi} \int \mathcal{I}(\Omega)|\Omega|^{\lambda-1} e^{-j\Omega x} \, d\Omega + \int \Omega^\lambda a(\Omega)e^{-j\Omega x} \, d\Omega = \]

\[ \frac{je^{-j\pi\lambda/2}}{2\pi} \int_{-\infty}^{\infty} (\omega + j0)^{\lambda-1} e^{-j\omega x} \, d\omega + \int \Omega^\lambda a(\Omega)e^{-j\Omega x} \, d\Omega \quad (3.2) \]
With the use of the result (see ref. [13])

\[
\int_{-\infty}^{\infty} (\omega + j0)^{\lambda-1} e^{-j\omega x} d\omega = -2\pi j \frac{e^{i\pi\lambda}}{\Gamma(1-\lambda)} x^{\lambda-1}
\]  

(3.3)

we obtain:

\[
\frac{d^\lambda U(x)}{dx^\lambda} = \frac{x^{\lambda}}{\Gamma(1-\lambda)} + \oint \Omega^\lambda a(\Omega) e^{-j\Omega x} d\Omega
\]  

(3.4)

When \( \lambda = n \)

\[
\left. \frac{x^{\lambda}}{\Gamma(1-\lambda)} \right|_{\lambda=n} = \delta^{(n-1)}(x)
\]  

(3.5)

\[
\oint \Omega^n a(\Omega) e^{-j\Omega x} d\Omega = 0
\]  

(3.6)

and we have the ordinary derivative:

\[
\frac{d^n U(x)}{dx^n} = \delta^{(n-1)}(x)
\]  

(3.7)

When \( \lambda = -n \)

\[
\frac{d^{-n} U(x)}{dx^{-n}} = \frac{x^n}{n!} + a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}
\]  

(3.8)

which is a \( n \)-times iterated integral.

Let \( \delta(x) \) the Dirac’s delta distribution. For it we have:

\[
\hat{f}(x) = \delta(x) \quad ; \quad F(\Omega) = \frac{\text{Sgn}[\mathcal{F}(\Omega)]}{2}
\]  

(3.9)

The fractional derivative is:

\[
\frac{d^\lambda \delta(x)}{dx^\lambda} = \frac{x^{\lambda-1}}{\Gamma(-\lambda)} + \oint \Omega^\lambda a(\Omega) e^{-j\Omega x} d\Omega
\]  

(3.10)
When \( \lambda = n \):

\[
\frac{d^n \delta(x)}{dx^n} = \delta^{(n)}(x)
\]  

(3.11)

and when \( \lambda = -n \):

\[
\frac{d^{-n} \delta(x)}{dx^{-n}} = \frac{x^{n-1}}{(n-1)!} + a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1}
\]

(3.12)

Let us consider now the fractional derivative of \( e^{ibx} \)

\[
\hat{f}(x) = e^{ibx} \quad \text{F}(\Omega) = \frac{j}{\Omega + b}
\]

(3.13)

We have:

\[
\frac{d^\lambda e^{ibx}}{dx^\lambda} = \frac{j}{2\pi} \int \frac{(-j\Omega)^\lambda e^{-j\Omega x}}{\Omega + b} d\Omega + \int \Omega^\lambda a(\Omega)e^{-j\Omega x} d\Omega =
\]

(3.14)

\[
\frac{ie^{-i\pi \lambda}}{2\pi} \int_{-\infty}^{\infty} \frac{(\omega + j0)^\lambda}{\omega + b + j0} e^{-j\omega x} d\omega - \frac{ie^{-i\pi \lambda}}{2\pi} \int_{-\infty}^{\infty} \frac{(\omega - j0)^\lambda}{\omega + b - j0} e^{-j\omega x} d\omega + \int \Omega^\lambda a(\Omega)e^{-j\Omega x} d\Omega
\]

(3.15)

From ref. [14] we obtain:

\[
\int_{-\infty}^{\infty} \frac{(x + \gamma)^\lambda}{x + \beta} e^{-ipx} dx =
\]

\[
2\pi \mathcal{U}(p)e^{\frac{ip\pi(1-\lambda)}{\Gamma(1-\lambda)}} p^{-\lambda} e^{i\beta p} \phi[-\lambda, 1 - \lambda, j(\gamma - \beta)p]
\]

(3.16)
where $\phi$ is the confluent hypergeometric function. Thus the fractional derivative is:

$$\frac{d^\lambda e^{jb x}}{dx^\lambda} = \frac{(x + j0)^{-\lambda}}{\Gamma(1 - \lambda)} \phi(1, 1 - \lambda, jbx) + \oint_{\Gamma} \Omega^{\lambda} a(\Omega) e^{-j\Omega x} d\Omega$$  \hfill (3.17)

With the use of equality:

$$\phi(1, 1 - \lambda, jbx) = (jbx)^{\lambda} e^{jb x} \left[ \Gamma(1 - \lambda) + \lambda \Gamma(-\lambda, jbx) \right]$$  \hfill (3.18)

where $\Gamma(z_1, z_2)$ is the incomplete gamma function, (3.17) takes the form:

$$\frac{d^\lambda e^{jb x}}{dx^\lambda} = (jb)^{\lambda} e^{jb x} \left[ 1 + \frac{\lambda}{\Gamma(1 - \lambda)} \Gamma(-\lambda, jbx) \right] + \oint_{\Gamma} \Omega^{\lambda} a(\Omega) e^{-j\Omega x} d\Omega$$  \hfill (3.19)

When $\lambda = n$

$$\frac{d^n e^{jb x}}{dx^n} = (jb)^n e^{jb x}$$  \hfill (3.20)

and when $\lambda = -n$:

$$\frac{d^{-n} e^{jb x}}{dx^{-n}} = (jb)^{-n} e^{jb x} + a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}$$  \hfill (3.21)

### The Laplace Transform

If we use the one-side Laplace transform to evaluate the fractional derivative of $U(x)$, then:

$$\hat{f}(x) = U(x) \quad ; \quad G(s) = U[\mathfrak{H}(s)] \int_0^\infty e^{-sx} dx = \frac{U[\mathfrak{H}(s)]}{s}$$  \hfill (3.22)
and as a consequence:

\[
\frac{d^\lambda U(x)}{dx^\lambda} = \frac{1}{2\pi j} \int_{a-j\infty}^{a+j\infty} U[\Re(s)]s^{\lambda-1}e^{sx} \, ds = \tag{3.23}
\]

\[
\frac{e^{-ax}}{2\pi} \int_{-\infty}^{\infty} \frac{e^{jsx}}{(a + js)^{1-\lambda}} \, ds = \frac{x^{\lambda-1}_+}{\Gamma(1-\lambda)} \tag{3.24}
\]

\[
\frac{d^\lambda U(x)}{dx^\lambda} = \frac{x^{\lambda-1}_+}{\Gamma(1-\lambda)} \tag{3.25}
\]

When \( \lambda = n \) we obtain

\[
\frac{d^n U(x)}{dx^n} = \delta^{(n-1)}(x) \tag{3.26}
\]

which coincides with (3.7). When \( \lambda = -n \) the result is:

\[
\frac{d^{-n} U(x)}{dx^{-n}} = \frac{x^n_+}{n!} \tag{3.27}
\]

In a analog way we obtain for Dirac’s delta distribution:

\[
\frac{d^\lambda \delta(x)}{dx^\lambda} = \frac{x^{\lambda-1}_+}{\Gamma(-\lambda)} \tag{3.28}
\]

\[
\frac{d^n \delta(x)}{dx^n} = \delta^{(n)}(x) \tag{3.29}
\]

\[
\frac{d^{-n} \delta(x)}{dx^{-n}} = \frac{x^{n-1}_+}{(n-1)!} \tag{3.30}
\]

Finally we consider the fractional derivative of \( e^{ibx} \):

\[
\hat{f}(x) = U(x)e^{ibx} \quad ; \quad G(s) = \frac{U[\Re(s)]}{s - ib} \tag{3.31}
\]
According to (2.11):

\[
\frac{d^\lambda U(x)e^{jbx}}{dx^\lambda} = \frac{1}{2\pi j} \int_{a-j\infty}^{a+j\infty} \frac{U[\Re(s)]}{s-jb} s^\lambda e^{sx} ds = \tag{3.32}
\]

\[
= e^{-j\lambda x/2} \int_{-\infty}^{\infty} \frac{(s+j0)^\lambda}{s+b+j0} e^{-jsx} ds \tag{3.33}
\]

And thus:

\[
\frac{d^\lambda U(x)e^{jbx}}{dx^\lambda} = \frac{U(x)x^{-\lambda}}{\Gamma(1-\lambda)} \Phi(1, 1-\lambda, jbx) \tag{3.34}
\]

Using (3.18), (3.34) transforms into:

\[
\frac{d^\lambda U(x)e^{jbx}}{dx^\lambda} = (jb)^\lambda U(x)e^{jbx} \left[ 1 + \frac{\lambda}{\Gamma(1-\lambda)} \Gamma(-\lambda, jbx) \right] \tag{3.35}
\]

When \( \lambda = n \):

\[
\frac{d^n e^{jbx}}{dx^n} = (jb)^n U(x)e^{jbx} \tag{3.36}
\]

and when \( \lambda = -n \):

\[
\frac{d^{-n} e^{jbx}}{dx^{-n}} = (jb)^{-n} U(x)e^{jbx} \tag{3.37}
\]

4 Circuital Application

As circuital application we consider a semi-infinite cable with a voltage \( V = V_0 e^{jot} \) applied at one end. We use first the Fourier transform and then the Laplace transform for see the differences between both treatments.
The Fourier Transform

We should solve the system:

\[
\begin{cases} 
\frac{\partial^2 f(x,t)}{\partial x^2} - RC \frac{\partial f(x,t)}{\partial t} = 0 \ ; \ x > 0 \\
f(0,t) = V_0 e^{i\omega t}
\end{cases}
\]  

(4.1)

where \( R \) is the resistance per unit length and \( C \) is the capacitance per unit length. Let \( V(x,t) \) the voltage along the semi-infinite cable. We use a formalism developed in ref. [15] to solve the system (4.1). It consist in to define:

\[
\begin{cases} 
V(x,t) = U(x)f(x,t) \\
g(t) = \frac{\partial f(x,t)}{\partial x} \bigg|_{x=0}
\end{cases}
\]  

(4.2)

The differential equation in (4.1) transforms into:

\[
\frac{\partial^2 V(x,t)}{\partial x^2} - RC \frac{\partial V(x,t)}{\partial t} = \delta'(x)V_0 e^{i\omega t} + \delta(x)g(t)
\]

(4.3)

Taking the Fourier transform of (4.3) we obtain:

\[
\hat{V}(\alpha_1, \alpha_2) = \mathcal{F}[V(x, t)]
\]

(4.4)

\[
\hat{V}(\alpha_1, \alpha_2) = \pi j V_0 \delta(\alpha_1 + \omega) \left[ \frac{1}{\alpha_2 - \frac{j}{\sqrt{2}} \sqrt{-\alpha_1 RC}} + \frac{1}{\alpha_2 + \frac{j}{\sqrt{2}} \sqrt{-\alpha_1 RC}} \right] - \frac{\hat{g}(\alpha_1)}{(1 - j) \sqrt{-2\alpha_1 RC}}
\]
\[
\frac{1}{\alpha_2 - \frac{1-j\sqrt{2}}{\sqrt{2}} \alpha_1 RC} - \frac{1}{\alpha_2 + \frac{1-j\sqrt{2}}{\sqrt{2}} \alpha_1 RC} \tag{4.5}
\]

Deprecating the exponential increasing in the solution we obtain:

\[
\hat{g}(\alpha_1) = -(1 + j)\pi \sqrt{-2\alpha_1 RC} \delta(\alpha_1 + \omega) \tag{4.6}
\]

and then we obtain:

\[
V(x, t) = V_0 U(x) e^{-\frac{\sqrt{\omega RC}}{x_0} x} e^{j(\omega t - \sqrt{\omega RC} x)} \tag{4.7}
\]

\[
g(t) = -(1 + j)\sqrt{\frac{\omega RC}{2}} V_0 e^{j\omega t} \tag{4.8}
\]

The current \(i(x, t)\) is:

\[
i(x, t) = \left(1 + j\right) \sqrt{\frac{\omega C}{2R}} V_0 e^{-\frac{\sqrt{\omega RC}}{x_0} x} e^{j(\omega t - \sqrt{\omega RC} x)} ; \quad x > 0 \tag{4.9}
\]

As:

\[
\frac{\partial V(x, t)}{\partial x} = \left(1 + j\right) \sqrt{\frac{\omega RC}{2}} V_0 e^{-\frac{\sqrt{\omega RC}}{x_0} x} e^{j(\omega t - \sqrt{\omega RC} x)} ; \quad x > 0 \tag{4.10}
\]

then:

\[
i(x, t) = \left(1 + j\right) \sqrt{\frac{\omega C}{2R}} V_0 e^{-\frac{\sqrt{\omega RC}}{x_0} x} e^{j(\omega t - \sqrt{\omega RC} x)} ; \quad x > 0 \tag{4.11}
\]

If we take \(\lambda = 1/2\) in (3.19) we obtain:

\[
\frac{d^{\frac{1}{2}} e^{j\omega t}}{dt^{\frac{1}{2}}} = (j\omega)^{\frac{1}{2}} e^{j\omega t} \left[ 1 + \frac{1}{2\sqrt{\pi}} \Gamma\left(-\frac{1}{2}, j\omega t\right) \right] + \oint_{\gamma} Z^{\frac{1}{2}} a(Z)e^{-jZt}dZ \tag{4.12}
\]
\[
\frac{\partial^{\frac{1}{2}} V(x,t)}{\partial t^{\frac{1}{2}}} = (j\omega)^{\frac{1}{2}} \left[ 1 + \frac{1}{2\sqrt{\pi}} \Gamma\left(-\frac{1}{2}, j\omega t\right) \right] e^{-\sqrt{\frac{\omega RC}{2}} x} e^{j(\omega t - \sqrt{\frac{\omega RC}{2}} x)} + \\
\oint_{\Gamma} Z^{\frac{1}{2}} a(Z,x) e^{-jZt} dZ
\]

(4.13)

Thus we have a relation between the current and the time derivative of the voltage:

\[
i(x,t) = \sqrt{\frac{C}{R}} \left\{ \left[ \frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}} - \frac{(j\omega)^{\frac{1}{2}} \Gamma\left(-\frac{1}{2}, j\omega t\right)}{2\sqrt{\pi}} \right] V(x,t) - \\
\oint_{\Gamma} Z^{\frac{1}{2}} a(Z,x) e^{-jZt} dZ \right\}
\]

(4.14)

If we consider only the first term in the right side of (4.14) we obtain the more habitual result:

\[
i(x,t) = \sqrt{\frac{C}{R}} \frac{\partial^{\frac{1}{2}} V(x,t)}{\partial t^{\frac{1}{2}}}
\]

(4.15)

**The Laplace Transform**

If we use the Laplace transform in place of the Fourier transform to evaluate the fractional derivatives, (4.12), (4.13) and (4.14) are replaced by:

\[
\frac{d^{\frac{1}{2}} e^{j\omega t}}{dt^{\frac{1}{2}}} = (j\omega)^{\frac{1}{2}} e^{j\omega t} \left[ 1 + \frac{1}{2\sqrt{\pi}} \Gamma\left(-\frac{1}{2}, j\omega t\right) \right]
\]

(4.16)

\[
\frac{\partial^{\frac{1}{2}} V(x,t)}{\partial t^{\frac{1}{2}}} = (j\omega)^{\frac{1}{2}} \left[ 1 + \frac{1}{2\sqrt{\pi}} \Gamma\left(-\frac{1}{2}, j\omega t\right) \right] e^{-\sqrt{\frac{\omega RC}{2}} x} e^{j(\omega t - \sqrt{\frac{\omega RC}{2}} x)}
\]

(4.17)

\[
i(x,t) = \sqrt{\frac{C}{R}} \left[ \frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}} - \frac{(j\omega)^{\frac{1}{2}} \Gamma\left(-\frac{1}{2}, j\omega t\right)}{2\sqrt{\pi}} \right] V(x,t)
\]

(4.18)
Difference between this results and the precedents is the term that contain a contour integral.

5 Discussion

In this paper we have shown that Ultradistribution Theory is an adequate framework to define a Fractional Caculus and its applications. This definition unifies the notion of integral and derivative in one only operation. Several examples of application of fractional derivative are given, including a circuital application: a semi-infinite cable with a voltage $V = V_0 e^{j\omega t}$ applied at one end.
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