Abstract. It has been shown recently that spectral flow admits a natural integer-valued extension to essential spectrum. This extension admits four different interpretations; two of them are singular spectral shift function and total resonance index. In this work we study resonance index outside essential spectrum.

Among results of this paper are the following.
1. Total resonance index satisfies Robbin-Salamon axioms for spectral flow.
2. Direct proof of equality “total resonance index = intersection number”.
3. Direct proof of equality “total resonance index = total Fredholm index”.
4. (a) Criteria for a perturbation $V$ to be tangent to the resonance set at a point $H$, where the resonance set is the infinite-dimensional variety of self-adjoint perturbations of the initial self-adjoint operator $H_0$ which have $\lambda$ as an eigenvalue. (b) Criteria for the order of tangency of a perturbation $V$ to the resonance set.
5. Investigation of the root space of the compact operator $(H_0 + sV - \lambda)^{-1}V$ corresponding to an eigenvalue $(s - r\lambda)^{-1}$, where $H_0 + r\lambda V$ is a point of the resonance set.

This analysis gives a finer information about behaviour of discrete spectrum compared to spectral flow.

Finally, many results of this paper are non-trivial even in finite dimensions, in which case they can be and were tested in numerical experiments.

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1. Introduction

1.1. Introduction. The spectral flow of a continuous path of self-adjoint operators \( \{H_r : r \in [0, 1] \} \) through a point \( \lambda \) which does not belong to the common essential spectrum \( \sigma_{ess} \) of operators \( H_r \) is naively understood as the number of eigenvalues of \( H_r \) which cross \( \lambda \) from left to right minus the number of eigenvalues of \( H_r \) which cross \( \lambda \) from right to left as the variable \( r \) moves from 0 to 1 \[APS\]. This naive definition was given a rigorous basis in \[RoSa\]. The spectral flow can also be defined as total Fredholm index of the path \( \{H_r\} \), \[Ph\] \[Ph2\], \[BCPRSW\] Section 4]. Further, the total Fredholm index can be interpreted as an integral of a one-form defined on some real affine space of self-adjoint operators \[Ga\] \[CP\] \[CP2\] \[ACS\] \[BCPRSW\], the possibility of such interpretation of spectral flow was first suggested by I. M. Singer in 1974. The spectral flow can also be interpreted as Maslov index \[RoSa\]. Finally, outside the essential spectrum the spectral flow is equal to the spectral shift function \[L\] \[Kr\], see e.g. \[ACDS\] \[ACS\] \[Pu\].

Any of these definitions of spectral flow is applicable only for numbers \( \lambda \) outside the common essential spectrum of a path of self-adjoint operators \( H_r \), with the exception of the spectral shift function. The spectral shift function is not integer-valued inside the essential spectrum and therefore it cannot be considered as a proper analogue of spectral flow for essential spectrum. In \[Az2\] \[Az3\] \[Az4\] an analogue of Lebesgue decomposition \( m = m^{(a)} + m^{(s)} \) of a measure \( m \) into its absolutely continuous and singular parts was suggested for the spectral shift function \( \xi \). It was shown that the singular part \( \xi^{(s)} \) of the spectral shift function \( \xi \), which can be correctly
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defined by formula

\[
\xi^{(s)}(\lambda) = \frac{d}{d\lambda} \int_0^1 \text{Tr}(VE_{\lambda}^{H^{(s)}_r}) \, dr, \text{ a.e. } \lambda,
\]

where \( H^{(s)}_r \) is the singular part of \( H_r = H_0 + rV \), is a function which takes integer values for a.e. value of the spectral parameter \( \lambda \), including those in \( \sigma_{\text{ess}} \), and which coincides with the spectral shift function \( \xi \), and thus with spectral flow, outside the essential spectrum. Apparently, it is difficult to work directly with the definition (1.1) of the singular spectral shift function \( \xi^{(s)} \).

In [Az5] (see also [Az6], Section 6) it was found that for trace class perturbations the singular spectral shift function can be interpreted as total resonance index (TRI),

\[
\xi^{(s)}(\lambda) = \sum_{r_\lambda \in [0,1]} \text{ind}_{\text{res}}(\lambda; H_{r_\lambda}, V),
\]

where \( \text{ind}_{\text{res}}(\lambda; H_{r_\lambda}, V) \) is the so-called resonance index of the triple \( (\lambda; H_{r_\lambda}, V) \), and where the sum is taken over real resonance points \( r_\lambda \) of the triple \( (\lambda; H_0, V) \), which belong to \([0,1]\). One of the ways to define the resonance points and the resonance index is as follows. Let

\[
\sigma_1^1(s), \sigma_2^1(s), \sigma_3^1(s), \ldots
\]

be the list of eigenvalues of the compact operator

\[
(H_0 + sV - z)^{-1}V, \quad s \in \mathbb{R}.
\]

It is not difficult to show that for each of these eigenvalues \( \sigma_j^1(s) \) there exists a number \( r_\lambda^j \) such that

\[
\sigma_j^1(s) = (s - r_\lambda^j)^{-1}.
\]

The numbers \( r_\lambda^j \) are resonance points of the triple \( (z, H_0, V) \). A real number \( r_\lambda \) is a real resonance point of the triple \( (\lambda, H_0, V) \), if at least one of the resonance points \( r_\lambda^j \) approaches \( r_\lambda \) as \( z = \lambda + iy \to \lambda + i0 \). The resonance index of the triple \( (\lambda, H_{r_\lambda}, V) \) is the integer

\[
N_+ - N_-, \quad \text{where } N_+ (\text{respectively, } N_-), \text{ is the number of resonance points which approach } r_\lambda \text{ in the upper complex half-plane (respectively, lower complex half-plane).}
\]

The resonance points \( r_\lambda^j \) in this definition should be counted according to their algebraic multiplicities, which are transferred from the corresponding eigenvalues \( \sigma_j^1(s) \).

If \( \lambda \) does not belong to the essential spectrum, then \( \xi^{(s)}(\lambda) = \xi(\lambda) \), as the definition (1.1) of the singular SSF turns into a well-known Birman-Solomyak formula for SSF. Thus, outside the essential spectrum the formula (1.2) turns into

\[
\text{spectral flow through } \lambda = \sum_{r_\lambda \in [0,1]} \text{ind}_{\text{res}}(\lambda; H_{r_\lambda}, V).
\]

Once this formula is obtained, one may choose to forget its origin and consider the right hand side as a new definition of spectral flow. Apart from the fact that, unlike other definitions of spectral flow, this definition makes perfect sense inside the essential spectrum, it has two other advantages. Firstly, it requires minimum assumptions in order to be defined, in particular, it includes as special cases the spectral flow for operators with compact resolvent and spectral shift function for relatively trace class perturbations. Secondly, it is defined in the language of complex analysis and it can be investigated using tools of complex analysis. It is well known that proofs based on complex analysis are as a rule considerably simpler (and also more beautiful, but this depends on one’s taste) and therefore more natural. As a historical example, Franz
Rellich’s perturbation theory of isolated eigenvalues was essentially simplified by introduction of Riesz idempotents.

In this paper we study spectral flow from the point of view of resonance index.

1.2. Basic assumption. In the following Assumption we collect basic assumptions, notation and terminology which will be assumed and used throughout this paper.

Assumption 1.1.

(1) $H_0$ is a self-adjoint operator on a separable complex Hilbert space $\mathcal{H}$ with dense domain $D$.

(2) $A_0$ is a real vector space of self-adjoint operators $V$ which are relatively compact with respect to $H_0$. The last means by definition that domain of $V$ contains $D$ and the product

$$R_z(H_0)V := (H_0 - z)^{-1}V$$

is compact for some and thus for any complex number $z$ which does not belong to the spectrum of $H_0$. Elements of the real vector space $A_0$ will also be called directions.

(3) $A$ is the real affine space $H_0 + A_0$ of self-adjoint operators. Elements of the real affine space $A$ will also be called points.

(4) It follows from (2) and the second resolvent identity, that all directions from $A_0$ are relatively compact with respect to any point from $A$. Therefore, by Weyl’s theorem (see e.g. [RS]) all points from $A$ have the same essential spectrum. This common essential spectrum we shall denote by $\sigma_{\text{ess}}$ and refer to it as the essential spectrum of the affine space $A$.

(5) There exists at least one real number $\lambda$ which does not belong to the essential spectrum $\sigma_{\text{ess}}$. Most of the time, the real number $\lambda$ will be fixed.

These are the only assumptions which we shall make in this paper. These assumptions are quite generic in Hilbert space perturbation theory. As a special case they include the case of self-adjoint operators $H_0$ with compact resolvent and a vector space of bounded perturbations $V$, which is the main setting of spectral flow theory in differential geometry and global analysis.

We shall consistently use the words “point” and “direction” instead of self-adjoint operator from $A$ and a self-adjoint perturbation operator from $A_0$. Elements of a vector space associated with an affine space are usually called vectors, but we shall not use this word for this purpose to avoid confusion.

The set of all points $H$ from $A$ for which $\lambda$ is an eigenvalue we call resonance set and denote it by $\mathcal{R}(\lambda)$. Elements of the set $\mathcal{R}(\lambda)$ will be called $\lambda$-resonant operators or $\lambda$-resonant points. Since the real number $\lambda$ will be fixed for most of the time, $\lambda$-resonant points will often be called resonant points. We say that a resonance point $H$ is simple, if $\lambda$ is an eigenvalue of $H$ of geometric multiplicity 1. An analytic path $H(s)$ will be said to be a resonant path if $H(s) \in \mathcal{R}(\lambda)$ for all $s$. Otherwise we say that $H(s)$ is a regular path. A regular path $H(s)$ may have resonant points on it, but the set of such points is discrete. A direction $V$ at a resonant point $H$ will be said to be regular, if the straight line $H + sV$ is a regular path. In my previous papers, whether published or not, regular directions were called regularising.

The terminology “$\lambda$-resonant operator” was used in [Az6] in a study of spectral flow inside essential spectrum $\sigma_{\text{ess}}$, but for a real number $\lambda$ outside the essential spectrum the definition of $\lambda$-resonant operator reduces to the one given above.

A function defined on a finite-dimensional real affine space is called analytic if it is given by a real analytic function in some and thus in any affine system of coordinates. A subset $R$ of a finite-dimensional real affine space is called an analytic set if $R$ is the set of zeros of one or
several real analytic functions. A subset $R$ of any real affine space is called an analytic set if the intersection of $R$ with any finite-dimensional affine subspace is an analytic set. By an affine space from now on we will always mean a real affine space. Finite subsets of an affine space and an affine space itself are analytic sets. Intersections and finite unions of analytic sets are also analytic. The resonance set $R(\lambda)$ is an analytic set, proof of this assertion follows verbatim that of [Az4] Theorem 4.2.5).

1.3. Preliminaries. The second resolvent identity

$$R_z(H_0 + V) - R_z(H_0) = -R_z(H_0 + V)V R_z(H_0) = -R_z(H_0)V R_z(H_0 + V).$$

holds for any pair of closed operators $H_0$ and $V$ provided the sum $H_0 + V$ is well-defined and $z$ belongs to the resolvent sets of both operators $H_0$ and $H_0 + V$. This identity can be rewritten as follows:

$$R_z(H_0 + V) = R_z(H_0) (1 + VR_z(H_0))^{-1} = (1 + R_z(H_0)V)^{-1} R_z(H_0).$$

Let

$$A_z(s) = R_z(H_s)V \quad \text{and} \quad B_z(s) = VR_z(H_s),$$

where

$$H_s = H_0 + sV.$$  

By Assumption 1.1 the operators $A_z(s)$ and $B_z(s)$ are compact. The second resolvent identity (1.3) implies that for $s, r \in \mathbb{C}$

$$A_z(s) = (1 + (s-r)A_z(r))^{-1} A_z(r),$$

and a similar equality holds for $B_z(s)$. This equality shows that $A_z(s)$ is a meromorphic function of $s$.

We start with a recap of some material of [Az4] Section 3].

Let $H_0 \in \mathcal{A}$, $V \in \mathcal{A}_0$ and let $H_s = H_0 + sV$, where $s \in \mathbb{C}$. Let $z = \lambda + iy \in \mathbb{C} \setminus \sigma_{ess}$. A point $z$ is a resonance point of the triple $(z; H_0, V)$, if any one of the following equivalent conditions hold:

1. $z$ is a pole of the meromorphic function $\mathbb{C} \ni s \mapsto A_z(s)$.
2. $z$ is a pole of the meromorphic function $\mathbb{C} \ni s \mapsto B_z(s)$.
3. The operator $1 + (r_z - s)A_z(s)$ has a non-zero kernel for some $s \in \mathbb{C}$. The kernel $\Upsilon_z^1(r_z; H_0, V)$ of this operator does not depend on $s$.
4. The operator $1 + (r_z - s)B_z(s)$ has a non-zero kernel. The kernel $\Psi_z^1(r_z; H_0, V)$ of this operator also does not depend on $s$.
5. The number $z$ is an eigenvalue of the operator $H_0 + r_z V =: H_{r_z}$.

With every resonance point $r_z$ of the triple $(z; H_0, V)$, one can associate an idempotent operator $P_z(r_z)$, the vector space $\Upsilon_z(r_z) := \text{im} P_z(r_z)$ and a nilpotent operator $A_z(r_z)$, which is reduced by the vector space $\Upsilon_z(r_z)$. They can be defined by formulas

$$P_z(r_z) = \frac{1}{2\pi i} \oint_{C_{r_z}} A_z(s) ds$$

and

$$A_z(r_z) = \frac{1}{2\pi i} \oint_{C_{r_z}} (s - r_z)A_z(s) ds,$$
where $C_{r_z}$ is a contour encircling $r_z$ and no other resonance points. Similarly one defines an idempotent operator $Q_z(r_z)$ and a nilpotent operator $B_z(r_z)$, by replacing $A_z(s)$ in (1.6) and (1.7) by $B_z(s)$. For these operators we have

$$B_z(r_z)V = VA_z(r_z) \quad \text{and} \quad (A_z(r_z))^* = B_z(\bar{r}_z).$$

The Laurent expansion of the meromorphic function $A_z(s)$ at a resonance point $r_z$ has the following form [Az6] (3.3.16):

$$A_z(s) = \tilde{A}_z(s) + (s - r_z)^{-1}P_z(r_z) + (s - r_z)^{-2}A_z(r_z) + \ldots + (s - r_z)^{-d}A_z^{d-1}(r_z),$$

where $d$ is the order of the resonance point $r_z$ (see below) and $\tilde{A}_z(s)$ is the holomorphic part of the Laurent series. The idempotent operators $P_z(r_z)$ and $Q_z(r_z)$ have the following properties [Az6].

If $r_z^{(1)}$ and $r_z^{(2)}$ are two different resonance points corresponding to $z$, then

$$P_z(r_z^{(1)})P_z(r_z^{(2)}) = 0.$$  

Further,

$$P_z^*(r_z) = Q_z(\bar{r}_z),$$  

and

$$VP_z(r_z) = Q_z(r_z)V.$$  

The operator $Q_z(\bar{r}_z)VP_z(r_z)$ is a finite-rank self-adjoint operator which in [Az6] is called resonance matrix. If $\lambda$ and $r_{\lambda}$ are real and $\lambda \notin \sigma_{ess}$, then

$$Q_{\lambda}(r_{\lambda})VP_{\lambda}(r_{\lambda}) = VP_{\lambda}(r_{\lambda}).$$

The vector space $\mathcal{Y}_z(r_z)$ consists of all vectors $\chi$ such that for some positive integer $k$

$$(1 + (r_z - s)A_z(s))^k\chi = 0,$$

where $s$ is any number which is not a pole of $A_z(s)$. This definition does not depend on the choice of $s$. A vector from $\mathcal{Y}_z(r_z)$ will be called a resonance vector. The equality (1.12) will be called resonance equation of order $k$. The smallest integer $k$ such that the equality (1.12) holds for some (and thus for any) $s$ will be called the order of $\chi$. The vector space of resonance vectors of order $k$ we denote $\mathcal{Y}_k(z)$. The operator $A_z(r_z)$ maps $\mathcal{Y}_k(z)$ onto $\mathcal{Y}_k^{k-1}(r_z)$, that is, $A_z(r_z)$ lowers the order of a resonance vector by 1. A resonance vector $\chi$ has depth at least $k$, if $\chi \in \text{im } A_z^k(r_z)$.

Dimensions of the vector spaces $\mathcal{Y}_z(r_z)$ and $\mathcal{Y}_z^1(r_z)$ we denote by $N$ and $m$ respectively. The vector space $\mathcal{Y}_z^1(r_z)$ is the eigenspace of $H_{r_z} = H_0 + r_zV$ corresponding to eigenvalue $z$, and we also denote it by $\mathcal{V}_z(r_z)$ or $\mathcal{V}_z$ if there is no danger of confusion.

The largest of positive integers $d$ such that $\mathcal{Y}_d(r_z)$ will be called the order of the perturbation $V$ at $H_{r_z}$ and the order of the resonance point $r_z$. A regular direction $V$ is said to be simple at a $\lambda$-resonance point $H$, if $V$ has order 1.

Jordan decomposition of the nilpotent operator $A_z(r_z)$ consists of $m$ Jordan blocks; we use lower case Greek letters $\nu$ and $\mu$ to enumerate them. The size of $\nu$th block we denote $d_\nu$ and assume that $d_1 \geq d_2 \geq \ldots \geq d_m$. A basis

$$\chi^{(j)}_\nu, \quad \nu = 1, \ldots, m, \quad j = 0, 1, \ldots, d_\nu - 1$$

of $\mathcal{Y}_z(r_z)$ is a Jordan basis if

$$A_z(r_z)\chi^{(j)}_\nu = \chi^{(j-1)}_\nu,$$

where it is assumed that $\chi^{(-1)}_\nu = 0$. In particular, $\ker(A_z(r_z)) \cap \mathcal{Y}_z(r_z) = \mathcal{Y}_z^1(r_z) = \mathcal{V}_z$. A Jordan basis can be depicted by either of the two Young diagrams shown next:
In such diagram each square represents a resonance vector from a Jordan basis, and the height of the square is the order of the vector. The number of squares is \( N \), the width is \( m \), and the height is \( d \). Each Jordan basis defines a direct sum decomposition of the resonance vector space \( \Upsilon_z(r_z) \), the \( \nu \)th summand of which we denote by \( \Upsilon_z^{[\nu]}(r_z) \). Thus,

\[
\Upsilon_z(r_z) = \Upsilon_z^{[1]}(r_z) + \ldots + \Upsilon_z^{[m]}(r_z).
\]

The height of \( \nu \)th column in the Young diagram is \( d_\nu = \dim \Upsilon_z^{[\nu]}(r_z) \).

The vector space \( \Upsilon_z^1(z; H_0, V) \) depends only on the operator \( \hat{H}_{r_z} = H_0 + r_z V \) and does not depend on \( V \), but the vector spaces \( \Upsilon_z^k(z; H_0, V) \), \( k \geq 2 \), depend on both \( H_{r_z} \) and \( V \).

A complex number \( r_z \) is a resonance point iff

\[
\sigma_z(s) = (s - r_z)^{-1}
\]

is an eigenvalue of \( A_z(s) \). The eigenvalue \( \sigma_z(s) \) has algebraic multiplicity \( N \) and geometric multiplicity \( m \).

Though \( z \) can be any complex number outside the essential spectrum, we are mainly interested in the case where \( z \) and the corresponding resonance point \( r_z \) are real numbers. In this case, if a real number \( \lambda \) is shifted to \( \lambda + iy \) with small \( y \) > 0, then the real eigenvalue \( \sigma_z(s) \) of \( A_z(s) \) splits into \( N_+ \) and \( N_- \) (where \( N_+ \geq 0 \), \( N_+ + N_- \geq 1 \)) eigenvalues in \( \mathbb{C}_+ \) and \( \mathbb{C}_- \) respectively, and all shifted eigenvalues are non-real. Resonance index of the triple \( (\lambda; H_{r_\lambda}, V) \) is by definition the difference

\[
\text{ind}_{\text{res}}(\lambda; H_{r_\lambda}, V) = N_+ - N_-.
\]

The objects such as \( P_z(r_z) \), \( \Upsilon_z(r_z) \), etc, depend on \( H_0 \) and \( V \) too, but since for the most part the operators \( H_0 \) and \( V \) are fixed, usually we do not indicate this dependence. If necessary we write \( P_z(r_z; H_0, V) \), etc, or \( P_z(H_{r_\lambda}, V) \), etc, where \( H_{r_\lambda} = H_0 + r_\lambda V \). The notation, such as \( P_z(H_{r_\lambda}, V) \) is not ambiguous, since \( P_z(H_{r_\lambda}, V) \) depends on operators \( H_{r_\lambda} \) and \( V \) but not on \( H_0 \).

### 1.4. Description of results.

#### 1.4.1. Section 2

Let \( H_{r_\lambda} \) be a resonance point, let \( V \) be a regular direction and let \( H_\lambda = H_{r_\lambda} + (s - r_\lambda)V \). We use the following notation: \( \mathcal{V}_\lambda \) is the eigenspace of \( H_{r_\lambda} \) corresponding to eigenvalue \( \lambda \), \( \mathcal{H} = \mathcal{V}_\lambda^\perp \), \( \hat{P} \) is the orthogonal projection onto \( \mathcal{H} \), \( \hat{H}_\lambda = \hat{P}H_{r_\lambda}\hat{P} \), \( \tilde{H} = \hat{P}V\hat{P} \), \( v = \hat{P}V\hat{P}^{-1} \), \( R_\lambda(\hat{H}_\lambda) \) is the resolvent of \( \hat{H}_\lambda \) on \( \mathcal{V}_\lambda^\perp \) and zero on \( \mathcal{V}_\lambda \), \( S_\lambda \) is the operator \( R_\lambda(\hat{H}_\lambda) \), \( \hat{A}_\lambda(s) = R_\lambda(\hat{H}_\lambda)V \).

**Theorem 1.2.** (Theorem 2.4) Let \( k \geq 2 \). If \( \phi \in \mathcal{H} \) is a resonance vector of order \( k \) then \( \phi \) belongs to the linear subspace

\[
\im R_\lambda(\hat{H}_\lambda)v + \im \hat{A}_\lambda(r_\lambda)R_\lambda(\hat{H}_\lambda)v + \ldots + \im \hat{A}_\lambda^{k-2}(r_\lambda)R_\lambda(\hat{H}_\lambda)v.
\]

The following theorem provides a criterion for a resonance vector to have depth at least 1. This criterion is used several times in the remaining sections.

**Theorem 1.3.** (Theorem 2.7) For a resonance vector \( \chi \) the following three assertions are equivalent: (i) \( V_\chi \perp \mathcal{V}_\lambda \), (ii) \( \chi \) has depth at least 1 and (iii) \( A_\lambda(r_\lambda)S_\lambda\chi = -\chi \). In particular, if \( V_\chi \perp \mathcal{V}_\lambda \), then \( S_\lambda\chi \) is a resonance vector.
The operator $S_\lambda$ satisfies the following equality.

**Theorem 1.4.** (Theorem 2.11) \[ -S_\lambda A_\lambda^j(r_\lambda) = \hat{P} A_\lambda^{j-1}(r_\lambda). \]

The last two theorems show that the operator $-S_\lambda$ behaves to a certain extent as the inverse of the nilpotent operator $A_\lambda(r_\lambda)$, in particular, the operator $S_\lambda$ increases the order of a resonance vector by 1, if there is a room for that. Since the operator $A_\lambda(r_\lambda)$ decreases order by 1 and increases depth by 1, this raises a natural question of whether $S_\lambda$ decreases depth by 1. This assertion is not proved, but Theorem 2.14 provides a criterion for this property.

**Theorem 1.5.** (Theorem 2.14) The following assertions are equivalent:

1. For all $j = 1, 2, \ldots, d - 1$, $\text{im}(S_\lambda A_\lambda^{j-1}) \subset \text{im}(A_\lambda^{j-1}).$
2. For all $j = 1, 2, \ldots, d - 1$, $\text{im}(\hat{P} A_\lambda^{j-1}) \subset \text{im}(A_\lambda^{j-1}).$

That is, the operator $S_\lambda$ decreases depth of resonance vectors by 1 iff the orthogonal projection onto the eigenspace $V_\lambda$ preserves depth.

1.4.2. **Section [3]** The eigenvalue $\lambda$ of multiplicity $m$ of the self-adjoint operator $H_\lambda$ splits into $m$ analytic eigenvalue functions $\lambda_\nu(s)$, $\nu = 1, \ldots, m$, of the operator $H_s = H_\lambda + (s - r_\lambda)V$, so that $\lambda_\nu(r_\lambda) = \lambda$ for all $\nu = 1, \ldots, m$. The corresponding eigenvector functions we denote by $\varphi_\nu(s)$. The eigenvalue functions $\lambda_\nu(s)$ are not necessarily distinct, in which case we list them according to their multiplicities, but in any case the analytic eigenvalue functions $\varphi_\nu(s)$ can be chosen to be pairwise orthogonal: for any $s$ and any $\nu \neq \mu$, $\langle \varphi_\nu(s), \varphi_\mu(s) \rangle = 0$. We assume such a choice throughout this paper.

**Theorem 1.6.** (Theorem 3.7) Let $k \geq 2$, let $\varphi(s)$ be an analytic path of eigenvectors of the path $H_s = H_0 + sV$. The following assertions are equivalent:

1. The vectors $V \varphi(r_\lambda)$, $V \varphi'(r_\lambda)$, $\ldots$, $V \varphi^{(k-2)}(r_\lambda)$ are orthogonal to the eigenspace $V_\lambda$.
2. The vectors $V \varphi(r_\lambda)$, $V \varphi'(r_\lambda)$, $\ldots$, $V \varphi^{(k-2)}(r_\lambda)$ are orthogonal to the eigenspace $V_\lambda$.
3. The equalities $\lambda'(r_\lambda) = 0$, $\ldots$, $\lambda^{(k-1)}(r_\lambda) = 0$, hold, where $\lambda(s)$ is an analytic path of eigenvalues of $H_s$ which corresponds to $\varphi(s)$.
4. For all $j = 1, 2, \ldots, k - 1$, $(H_\lambda - \lambda) \varphi^{(j)}(r_\lambda) = -jV \varphi^{(j-1)}(r_\lambda)$.
5. For all $j = 1, 2, \ldots, k - 1$, $A_\lambda(r_\lambda) \varphi^{(j)}(r_\lambda) = j \varphi^{(j-1)}(r_\lambda)$.
6. $\varphi(r_\lambda)$ is an eigenvector of depth at least $k - 1$.

An eigenpath $\varphi(s)$ will be said to have order at least $k$, if it has any of these properties. This definition is correct in the sense that an eigenpath $\varphi(s)$ has order $k$ if and only if an eigenpath $a(s) \varphi(s)$ has order $k$ for any non-zero analytic function $a(s)$, such that $a(r_\lambda) \neq 0$.

**Theorem 1.7.** (Lemma 3.9) If an eigenpath $\varphi(s)$ has order at least $k$, then the vectors $\varphi(r_\lambda)$, $\varphi'(r_\lambda)$, $\ldots$, $\varphi^{(k-1)}(r_\lambda)$ are resonance vectors of orders respectively $1, 2, \ldots, k$.

1.4.3. **Section[4]** It is interesting to find out conditions under which a straight line of operators $H_s = H_\lambda + (s - r_\lambda)V$ is tangent to the resonance set at a resonance point $H_\lambda$. We say that a direction $V$ is tangent to the resonance set $\mathbb{R}(\lambda)$ at $H_\lambda$ to order at least $k$, if there exists a resonant path $\{H(s)\} \subset \mathbb{R}(\lambda)$ such that for some (necessarily real) numbers $c_2, c_3, \ldots, c_{k-1}$

$$ H(s) = H_\lambda + (s - r_\lambda)V + \sum_{j=2}^{k-1} c_j(s - r_\lambda)^j V + O((s - r_\lambda)^k), \ s \to r_\lambda. $$

(1.13)
In this case we also say that the path $H(s)$ is tangent to $V$ at $H(r_\lambda) = H_{r_\lambda}$ to order at least $k$. The order of tangency of a direction $V$ to the resonance set $\mathcal{R}(\lambda)$ is the largest of positive integers $k$ such that for some resonance path $H(s)$ holds. We say that a direction $V$ is tangent at $H_{r_\lambda}$ if $V$ is tangent to order at least 2. If a direction $V$ is tangent only to order 1 at $H_{r_\lambda} \in \mathcal{R}(\lambda)$, then we say that $V$ is transversal at $H_{r_\lambda}$.

**Theorem 1.8.** (Theorem 4.2) Assume Assumption 1.1. Let $k \geq 1$, let $H_{r_\lambda}$ be a resonance point and $V$ be a regular direction at $H_{r_\lambda}$. If $H(s)$ is a resonant path tangent to $V$ at $H_{r_\lambda}$ to order at least $k$ and if $\chi(s)$ is a corresponding analytic resonant eigenpath, then

(i) vectors $\chi(r_\lambda), \chi'(r_\lambda), \ldots, \chi^{(k-1)}(r_\lambda)$ have orders respectively $1, 2, \ldots, k$,

(ii) the direction $V$ has order at least $k$,

(iii) for any $j = 1, 2, \ldots, k$

$$A_\lambda(r_\lambda)\chi^{(k-1)}(r_\lambda) = (k-1)\chi^{(k-2)}(r_\lambda) + \sum_{j=2}^{k-1} j! \binom{k-1}{j} c_j \chi^{(k-1-j)}(r_\lambda),$$

where the numbers $c_2, \ldots, c_k$ are as in (1.13), and

(iv) the eigenvector $\chi(r_\lambda)$ has depth at least $k - 1$.

If a resonant path $H(s)$ is tangent to $V$ to order $k$ at a resonant point $H_{r_\lambda}$, then changing the parameter $s$ if necessary we can always make the operators $H''(r_\lambda), \ldots, H^{(k-1)}(r_\lambda)$ equal to zero, so that the path $H(s)$ takes the form

$$H(s) = H_{r_\lambda} + (s - r_\lambda)V + O((s - r_\lambda)^k).$$

A path of this form will be called standard.

According to Theorem 4.2 with a resonant path tangent to order $k$ we can associate a set of resonance vectors $\chi_0, \ldots, \chi_{k-1}$ of respective orders $1, \ldots, k$; namely, the first $k$ coefficients

$$\chi_j = \frac{1}{j!} \chi^{(j)}(r_\lambda), \quad j = 0, 1, 2, \ldots$$

of the Taylor expansion of a resonant eigenpath $\chi(s)$.

**Proposition 1.9.** (Proposition 4.4) Assume Assumption 1.1. Let $V$ be a regular direction at $H_{r_\lambda}$ and let $H(s)$ be a resonant path tangent to $V$ at $H_{r_\lambda}$ to order $k$. The path $H(s)$ is standard if and only if for all $j = 1, 2, \ldots, k - 1$ there holds the equality

$$A_\lambda(r_\lambda)\chi_j(r_\lambda) = \chi_{j-1}(r_\lambda),$$

where $\chi(s)$ is a corresponding analytic path of eigenvectors.

Given a direction $V$ of order $d$ it is possible to present a resonant path $H(s)$ which is tangent to $V$ to order $d$. Namely, let $\chi$ be an eigenvector of $H(r_\lambda)$ corresponding to the eigenvalue $\lambda$, and let $W = \langle \chi, \cdot \rangle \chi$. We consider the intersection of the two-dimensional real affine plane

$$\alpha = H_{r_\lambda} + \mathbb{R}V + \mathbb{R}W$$

with the resonance set $\mathcal{R}(\lambda)$. A sufficiently small neighbourhood of $H_{r_\lambda}$ in the intersection $\alpha \cap \mathcal{R}(\lambda)$ consists of one and only one simple curve (Theorem 1.5). We denote this analytic curve by $\gamma_\lambda$. The curve $\gamma_\lambda$ can be normalised so that $\gamma_\lambda(r_\lambda) = \chi$ (Theorem 1.7).

**Theorem 1.10.** (Theorem 4.8) If $\chi$ has depth at least $k - 1$ then

1. the analytic curve $\gamma_\lambda$ is tangent to $V$ to order $k$,

2. for any parametrisation $\gamma(s)$ of $\gamma_\lambda$, the vectors $\gamma(r_\lambda), \gamma'(r_\lambda), \ldots, \gamma^{(k-1)}(r_\lambda)$ have orders respectively $1, 2, \ldots, k$. 

(3) if the parametrisation \( \gamma(s) \) of \( \gamma_\lambda \) is standard then for all \( j = 1, 2, \ldots, k - 1 \)
\[
A_\lambda(r_\lambda)\gamma^{(j)}(r_\lambda) = j\gamma^{(j-1)}(r_\lambda).
\]

Theorems 4.2 and 4.8 provide the following geometric interpretation of order of a direction \( V \).

**Theorem 1.11.** (Theorem 4.9) The order of tangency of a regular direction \( V \) at a resonance point \( H_{r_\lambda} \) to the resonance set is equal to the order of the direction \( V \).

1.4.4. Section 5. Assume that \( r_\lambda \) is a resonance point of algebraic multiplicity \( N \) and geometric multiplicity \( m \). A resonance point \( r_\lambda \) depends analytically on \( z \) outside essential spectrum, and as \( \lambda \) varies the resonance point \( r_\lambda \) splits into up to \( N \) resonance points \( r_\lambda^j \). Theorem 5.1 asserts that all these resonance points \( r_\lambda^j \) have geometric multiplicity 1 in some deleted neighbourhood of \( \lambda \). When \( z \) makes one round around \( \lambda \), these \( N \) resonance points undergo a permutation. Theorem 5.7 asserts that this permutation is a product of \( m \) disjoint cycles of lengths \( d_1, \ldots, d_m \).

We denote these cycles by \( r_\nu^{(j)}(z), \nu = 1, \ldots, m, \ j = 0, \ldots, d_\nu - 1 \).

Functions of each cycle \( r_\nu^{(j)}(z) \) represent branches of a multivalued holomorphic function. The idempotents \( P_z(r_\nu^{(j)}(z)) \) which correspond to these resonance points are also multivalued, but the sum
\[
P_z^{[\nu]} = \sum_{j=0}^{d_\nu-1} P_z(r_\nu^{(j)}(z))
\]
is single-valued in a neighbourhood of \( \lambda \). Proposition 5.3 asserts that this function admits analytic continuation to the point \( \lambda \). Thus, the limit operator \( P_\lambda^{[\nu]}(r_\lambda) \) is defined. Similarly, one can define operators \( Q_\lambda^{[\nu]} \), or they also can be defined by formula \( Q_z^{[\nu]} = (P_z^{[\nu]})^* \). The operators \( P_z^{[\nu]} \), including the case of \( z = \lambda \), have the following properties:
\[
P_z^{[\nu]}P_z^{[\mu]} = \delta_{\nu\mu}P_z^{[\nu]}, \quad A_z(r_\lambda)P_z^{[\nu]} = P_z^{[\nu]}A_z(r_\lambda), \quad P_z(r_\lambda) = \sum_{\nu=1}^{m} P_z^{[\nu]}, \quad VP_\lambda^{[\nu]} = P_\lambda^{[\nu]}V.
\]

Here \( P_z(r_\lambda) \) (respectively, \( A_z(r_\lambda) \)) is the sum of idempotents \( P_z(r_\nu^{(j)}(z)) \) (respectively, nilpotent operators \( A_z(r_\nu^{(j)}(z)) \)) over all resonance points \( r_\nu^{(j)}(z) \) of the group of \( r_\lambda \). In particular, the image \( \Upsilon_\lambda^{[\nu]} \) of the operator \( P_\lambda^{[\nu]} \) reduces the operator \( A_\lambda(r_\lambda) \). This reduction we denote by \( A_\lambda^{[\nu]} \).

Restriction of this operator to \( \Upsilon_\lambda^{[\nu]} \) is cyclic.

Proposition 5.16 asserts that in the Puiseux series (5.16)
\[
r_\nu^{(j)}(z) = \sum_{k=0}^{\infty} r_{k/d_\nu}^j e^{k_\nu j} (z - \lambda)^{k/d_\nu}, \ j = 0, \ldots, d_\nu - 1,
\]
of the function \( r_\nu^{(j)}(z) \) the coefficients \( r_{k/d_\nu} \) are real and \( r_{1/d_\nu} \neq 0 \).

Theorem 5.2 and Proposition 5.3 assert that there is a natural one-to-one correspondence between cycles \( r_\nu^{(j)}(z) \) and eigenvalue functions \( \lambda_\nu(\cdot) \), and therefore, with eigenpaths \( \varphi_\nu(\cdot) \). Namely, restriction of one of the functions \( r_\nu^{(0)}(z) \) of a cycle \( r_\nu^{(j)}(z) \) to at least one of the half-intervals \([\lambda, \lambda + \varepsilon]\) or \((\lambda - \varepsilon, \lambda]\) takes real values and it is the inverse of \( \lambda_\nu(s) \) in some left or right neighbourhood of \( r_\lambda \). Such a function \( r_\nu^{(0)}(z) \) is unique in the sense that there is no other branch of \( r_\nu^{(j)}(z) \) which takes real values when restricted to the same half-interval as the function \( r_\nu^{(0)}(z) \).

Theorem 5.2 provides more properties of this one-to-one correspondence.
Theorem 5.7 provides further information about this correspondence. It asserts that for each $\nu = 1, \ldots, m$ the following numbers are equal:

1. order of the eigenpath $\varphi_{\nu}(s)$ (see Theorem 1.6),
2. the size of the cycle $r_{\nu}^{(j)}(s)$,
3. the size of the $\nu$th Jordan cell of $R_{\lambda}(H_{s})V$ corresponding to eigenvalue $(s - r_{\lambda})^{-1}$.

The number $d_{\nu}$ are their common value.

The Puiseux series of the idempotent $P_{z}(r_{\nu}^{(j)}(z))$ has the form (Proposition 5.20)

$$P_{z}(r_{\nu}^{(j)}(z)) = \tilde{P}_{\nu}^{(j)}(z) + \sum_{l=0}^{d_{\nu}-1} \frac{e^{-2\pi i l j}}{d_{\nu}^{l}} (z - \lambda)^{-l/d_{\nu}} P_{-l/d_{\nu}}$$

where $\tilde{P}_{\nu}^{(j)}(z)$ is continuous at $z = \lambda$. The informative part of this formula is the upper summation limit $d_{\nu} - 1$. Further, for each $\nu = 1, \ldots, m$ and for all $k \geq 0$ (Proposition 5.21)

$$\lim_{z \to \lambda} \sum_{j=0}^{d_{\nu}-1} (r_{\nu}^{(j)}(z) - r_{\lambda})^{k} P_{z}(r_{\nu}^{(j)}(z)) = P_{\lambda}^{(\nu)}(r_{\lambda}) A_{\lambda}^{k}(r_{\lambda})$$

Theorem 5.26 provides finer information about the connection of the eigenpath $\varphi_{\nu}$ to the vector space $\Upsilon_{\lambda}^{[\nu]}(r_{\lambda})$.

**Theorem 1.12.** (Theorem 5.25 and Corollary 5.26) Let $\varphi_{\nu}(s)$ be an eigenpath which corresponds to a cycle $\nu$ of length $d_{\nu}$. The set of vectors

$$\varphi_{\nu}(r_{\lambda}), \varphi_{\nu}^{(d_{\nu}-1)}(r_{\lambda})$$

is a basis of the vector space $\Upsilon_{\lambda}^{[\nu]}(r_{\lambda})$ and for any $j = 1, 2, \ldots, d_{\nu} - 1$

$$A_{\lambda}(r_{\lambda}) \varphi_{\lambda}^{(j)}(r_{\lambda}) = j \varphi_{\lambda}^{(j-1)}(r_{\lambda})$$

This theorem implies that the set of vectors (1.14) is a Jordan basis of $\Upsilon_{\lambda}(r_{\lambda})$.

The following theorem provides a relationship between different eigenpaths $\varphi_{\nu}(s)$.

**Theorem 1.13.** (Theorem 5.12, Corollary 5.27) Let $\varphi_{\nu}(s)$ and $\varphi_{\mu}(s)$ be two different eigenpaths of $H_{s}$, $s \in \mathbb{R}$. For all $j = 0, 1, \ldots, d_{\nu} - 1$ and all $k = 0, 1, \ldots, d_{\nu} - 1$,

$$\langle \varphi_{\nu}^{(j)}(r_{\lambda}), V \varphi_{\mu}^{(k)}(r_{\lambda}) \rangle = 0.$$

This theorem is a stronger version of Theorem 3.12 in that it shows that the numbers $d_{\nu}$ from its statement add up to $N$.

Proposition 5.22 gives an expression for the restriction of a power of the operator $A_{\lambda}(r_{\lambda})$ to the vector space $\Upsilon_{\lambda}^{[\nu]}(r_{\lambda})$ via Puiseux coefficients of $r_{\nu}^{(j)}(z)$ and $P_{z}(r_{\nu}^{(j)}(z))$: for all $\nu = 1, \ldots, m$ and for all $k = 1, 2, \ldots, d_{\nu} - 1$

$$P_{\lambda}^{(\nu)}(r_{\lambda}) A_{\lambda}^{k}(r_{\lambda}) = d_{\nu} \sum_{l=k}^{d_{\nu}-1} \left( \sum_{m_{1} + \ldots + m_{k} = l} r_{m_{1}/d_{\nu}} \ldots r_{m_{k}/d_{\nu}} \right) P_{-l/d_{\nu}}$$

where in the sum $m_{1}, \ldots, m_{k} \geq 1$.

Inside the essential spectrum geometric meaning of the total resonance index is obscure, but outside the essential spectrum it has a clear geometric interpretation, which is allowed by the fact that $r_{\lambda}$ depends on $\lambda$ analytically. Assume that $r_{0}$ is a resonance point corresponding to $\lambda_{0}$, that is, $\lambda_{0}$ is an eigenvalue of $H_{0} + r_{0}V$. If $r_{\lambda}$ is an increasing function of $\lambda$ in a neighbourhood $I$
of \( \lambda_0 \), and therefore, if \( \lambda(r) \) is an increasing function of \( r \), then intuitively contribution of the function \( \lambda(r) \) to the spectral flow through \( \lambda_0 \) is \( +1 \). Since \( r_\lambda \) is an increasing function, the derivative \( \frac{dr_\lambda}{d\lambda} \) is positive on \( I \). Therefore, according to the geometric interpretation of the derivative of a holomorphic function, when \( z = \lambda_0 \) is perturbed to \( z = \lambda_0 + iy \) with small \( y > 0 \), the real value \( r_0 \) of the function \( r_z \) rotates towards the half-plane \( C_+ \). This gives a contribution to the total resonance index of \( +1 \). Similarly, decreasing function \( r_\lambda \) contributes \( -1 \) to the TRI.

A more interesting situation occurs if at some value \( r_0 \) of the coupling constant the derivative \( \lambda'(r_0) \) vanishes. Geometrically, this means that the eigenvalue \( \lambda(r) \) of \( H_r \) stops at \( \lambda_0 \) when the coupling constant attains the value \( r_0 \) and may either turn back or go through \( \lambda_0 \), thus contributing one of the three numbers \(-1, 0, \) or \(+1\) to the spectral flow through \( \lambda_0 \). Since the derivative \( \lambda'(r_0) \) vanishes, the inverse function \( r(z) \) is not single-valued in a neighbourhood of \( z = \lambda_0 \). Accordingly, when \( \lambda_0 \) is shifted to \( \lambda_0 + iy \) with \( y > 0 \), the resonance point \( r_0 \) splits into two or more resonance points. About one half \( d_+ \) of those resonance points goes towards \( C_+ \) and another half \( d_- \) goes towards \( C_- \), resulting in the appropriate contribution to the total resonance index \( d_+ - d_- \). The overall number \( d = d_+ + d_- \) of the resonance points equals the smallest of positive numbers \( d \) such that \( \frac{d_\lambda}{d\lambda} \bigg|_{r=r_0} \neq 0 \). Moreover, those \( d \) resonance points undergo a cyclic permutation when \( z \) makes one round around \( \lambda_0 \). Combining this with the fact that \( r_z \) cannot be real for non-real \( z \), one can already infer that \( d_+ - d_- \) is to be equal to the contribution of \( \lambda(r) \) to the spectral flow. If \( \lambda \) has geometric multiplicity \( m > 1 \), then this argument applies to each of the \( m \) eigenvalue functions \( \lambda_\nu(s) \). This formal observation is made precise in Theorems 5.28 and 5.29.

**Theorem 1.14.** (Theorem 5.28) For each \( \nu = 1, \ldots, m \) and for all small enough \( \epsilon > 0 \) and \( y > 0 \), signs of the following real numbers coincide:

1. \( \lambda_\nu(r_\lambda + \epsilon) - \lambda_\nu(r_\lambda) \),
2. \( \left( \varphi_\nu(r_\lambda), V_\nu^{(d_\nu-1)}(r_\lambda) \right) \),
3. \( \text{Im} r_\nu(0)(z + iy) \), for all \( 0 < y << 1 \), and for all \( z \in I \), where \( I \) is one of the two intervals \( (\lambda, \lambda + \epsilon) \) or \( (\lambda - \epsilon, \lambda) \), on which the branch \( r_\nu(0)(z + iy) \) takes positive values (such an interval and such a branch exist and are unique).

This sign will be called the sign of a cycle \( \nu \) and denoted \( \text{sign}(\nu) \). Let

\[
\begin{align*}
\nu \quad b_\nu = \begin{cases} 
0, & \text{if } d_\nu \text{ is even,} \\
1, & \text{if } d_\nu \text{ is odd.}
\end{cases}
\end{align*}
\]

The intersection number through \( \lambda \) for a resonance point \( r_\lambda \) can be defined by equality

\[
(1.15) \quad \sum_{\nu=1}^{m} b_\nu \text{sign}(\nu).
\]

**Theorem 1.15.** (Theorem 5.29) The sum of the intersection numbers \( (1.15) \) of resonance points of a path \( H_r, r \in [0,1] \), through \( \lambda \) is equal to the total resonance index.

Theorem 1.14 allows to prove the following two theorems. The first of these theorems is an explicit formula for the idempotent \( P_\lambda(r_\lambda) \), the second theorem asserts that the resonance index equals the signature of the so-called resonance matrix. The second theorem holds for \( \lambda \) inside essential spectrum too [Az0, Section 9], but here we provide a new and simpler proof in the case \( \lambda \notin \sigma_{\text{ess}} \).
Theorem 1.16. (Theorem 5.30) The idempotent operator $P_\lambda(r_\lambda)$ can be written in the form

$$P_\lambda(r_\lambda) = \sum_{\mu=1}^{m} \sum_{\nu=1}^{m} \sum_{k=0}^{d_\nu-1} \sum_{j=0}^{d_\nu-1} \frac{1}{k!j!} \alpha_{\mu\nu}^{kj} \langle V\varphi_{\mu}^{(k)}(r_\lambda), \varphi_{\nu}^{(j)}(r_\lambda) \rangle,$$

where the $N \times N$ matrix $\alpha$ is a direct sum of self-adjoint skew-upper triangular Hankel matrices of sizes $d_1, \ldots, d_m$.

Theorem 1.17. (Theorem 5.33) The equality

$$\text{ind}_{\text{res}}(\lambda; H_{r_\lambda}, V) = \text{sign} \, V \, P_\lambda(r_\lambda)$$

holds.

1.4.5. Section 6. In section 6 we study dependence of resonance index $\text{ind}_{\text{res}}(\lambda; H_{r_\lambda}, V)$ on direction $V$. This requires some topology in $A$. We postulate that the topology of $A$ has the following properties: for some non-real complex number $z$ and for some $H_0 \in A$ (1) the product $VR_z(H_0)$ continuously depends on $V$ and (2) the product $V_1R_z(H_0)V_2$ is compact and jointly continuously depends on $V_1$ and $V_2$. These properties hold if $A_0$ consist of bounded operators and the topology of $A_0$ is the uniform topology.

The sets of regular and simple directions are open in norm of $A_0$ (Lemma 6.4). Restrictions of the mappings

$$A_0 \ni V \mapsto P_\lambda(H_{r_\lambda}, V) \quad \text{and} \quad A_0 \ni V \mapsto VP_\lambda(H_{r_\lambda}, V)$$

to the open set of simple directions are continuous in the norm of $A_0$ (Lemma 6.5). Further, the resonance index

$$A_0 \ni V \mapsto \text{ind}_{\text{res}}(\lambda; H_{r_\lambda}, V)$$

is a locally constant function on the set of simple directions (Theorem 6.7).

For regular directions these assertions are not true. But the following theorem holds. In this theorem for simplicity we assume that $r_\lambda = 0$.

Theorem 1.18. (Theorems 6.9 and 6.10) Let $V$ be a regular direction at a resonance point $H_0$. Let $W$ be a small (in the norm of $A_0$) perturbation of $V$ and let $H_0'$ be a small perturbation of $H_0$. Let $r_\lambda^1(H_0', W)$, $r_\lambda^2(H_0', W)$, $\ldots$ be resonance points of the triple $(\lambda; H_0', W)$ which belong to the group of the resonance point $s = 0$ of the triple $(\lambda; H_0, V)$, where $H_r = H_0 + rV$. Then the resonance index $\text{ind}_{\text{res}}(\lambda; H_0, V)$ is equal to the sum of resonance indices

$$\sum_j \text{ind}_{\text{res}}(\lambda; H_{r_\lambda}^j, W),$$

where the sum is taken over real resonance points of the group of $s = 0$.

This theorem allows to prove homotopy stability of the total resonance index.

Theorem 1.19. (Theorem 6.11) Let $H_0, H_1$ be two operators from $A$ such that $H_0$ and $H_1$ are not resonant at $\lambda \notin \sigma_{\text{ess}}$. Then there exist neighbourhoods $U_0$ and $U_1$ of $H_0$ and $H_1$ respectively such that for all $H_0' \in U_0$ and all $H_1' \in U_1$

$$\sum_{r \in [0,1]} \text{ind}_{\text{res}}(\lambda; H_r, V) = \sum_{r \in [0,1]} \text{ind}_{\text{res}}(\lambda; H_r', V'),$$

where $V' = H_1' - H_0'$ and $H_r' = H_0' + rV'$.
Theorem 6.12 asserts that the total resonance index satisfies Robbin-Salamon axioms for spectral flow. Since Robbin-Salamon axioms uniquely identify spectral flow (Theorem 6.13), this proves the equality of the TRI and the spectral flow.

In the last subsection of section 6 we give proof of some well-known properties of the resonance set $\mathcal{R}(\lambda) : \text{codim} \mathcal{R}(\lambda) = 1$, the set $\mathcal{R}(\lambda)$ has no cusps and that any plane section of $\mathcal{R}(\lambda)$ consists of no more than $m$ simple curves.

1.4.6. Section 7. In this section we also assume that the resonance point $r_\lambda$ is 0. In subsection 7.1 we observe that the finite-rank self-adjoint operator $VP_\lambda(H_0, V)$, which is called resonance matrix of the triple $(\lambda; H_0, V)$, preserves many properties of the initial direction $V$ : if $V$ is regular then so is $VP_\lambda$ (Theorem 7.1), further, the operators $P_\lambda$ and $A_\lambda$ are the same for the triples $(\lambda; H_0, V)$ and $(\lambda; H_0, VP_\lambda)$ (Theorem 7.3), and the resonance matrices of the directions $V$ and $VP_\lambda$ are equal (Theorem 7.4). In particular, resonance indices of the triples $(\lambda; H_0, V)$ and $(\lambda; H_0, VP_\lambda)$ coincide (Theorem 7.5):

$$\text{ind}_{\text{res}}(\lambda; H_0, V) = \text{ind}_{\text{res}}(\lambda; H_0, VP_\lambda).$$

All these assertions in essence follow from the following observation (Theorem 7.2):

$$R_\lambda(H_0 + sVP_\lambda)VP_\lambda = R_\lambda(H_0 + sV)VP_\lambda.$$

Further, the operators $V$ and $VP_\lambda$ are plane homotopic, that is, a direction $V$ can be deformed to a direction $VP_\lambda$ in the affine plane generated by these directions without crossing the resonance set (Theorem 7.7).

In subsection 7.2 we give a direct proof of equality of TRI and total Fredholm index, Theorem 7.10.

1.4.7. Section 8. In section 8 we give a direct proof of equality of the total resonance index and spectral shift function. This assertion is a special case of equality of TRI and singular spectral shift function [Az6, Section 6], [Az5]. It also follows from the fact that the spectral shift function satisfies Robbin-Salamon axioms. The proof presented in this subsection highlights a key idea of the proof of the more general result given in [Az6, Section 6], [Az5].

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2. 2 × 2 MATRIX REPRESENTATIONS

2.1. 2×2 representation of $A_2(s)$. Recall that we are working in the setting of Assumption 1.1. Let $H_{r_\lambda}$ be a resonance point of multiplicity $m$. Accordingly, let

$$\mathcal{H} = \hat{\mathcal{H}} \oplus \mathcal{V}_\lambda$$

be the orthogonal decomposition of the Hilbert space $\mathcal{H}$, on which $H_{r_\lambda}$ acts, into the sum of the $m$-dimensional eigenspace

$$\mathcal{V}_\lambda = \{ \chi \in \mathcal{H} : H_{r_\lambda} \chi = \lambda \chi \}$$

and its orthogonal complement which we hereby denote $\hat{\mathcal{H}}$. Operators acting on the Hilbert space $\mathcal{H} = \mathcal{H} \oplus \mathcal{V}_\lambda$ can be written as $2 \times 2$ matrices. The operator $H_{r_\lambda}$ has the matrix representation

(2.1)

$$H_{r_\lambda} = \begin{pmatrix} \hat{H}_{r_\lambda} & 0 \\ 0 & \lambda I_m \end{pmatrix}.$$
The operator $V$ has the form

$$V = \begin{pmatrix} \hat{V} & v \\ v^* & a \end{pmatrix},$$

where $\hat{V}$ is a self-adjoint operator in $\hat{H}$, $v$ is an operator from $V_\lambda$ to $\hat{H}$ and $a$ is a self-adjoint operator on $V_\lambda$. In [RoSa, p. 14] the operator $a$ is called the crossing operator.

We agree to identify an element $\chi$ of the Hilbert space $\hat{H}$ with an element $\begin{pmatrix} \chi \\ 0 \end{pmatrix}$ of $H$. Analogously, an operator $v: V_\lambda \to \hat{H}$ will also be considered as an operator from $H \to H$. This remark applies to other operators such as $\hat{H}_s$, $v^*$ and $a$.

By $\hat{P}$ we denote the operator of orthogonal projection from $H$ onto $\hat{H}$. Sometimes we write $\begin{pmatrix} \chi_0 \\ 0 \end{pmatrix}$ for $\hat{P}\chi$ and $\begin{pmatrix} 0 \\ \chi \end{pmatrix}$ for $\hat{P}^\perp\chi$, and thus $\chi$ and $\begin{pmatrix} \chi \\ 0 \end{pmatrix}$ are two ways to write the same vector.

Similarly, the number 1 is treated as the identity operator on $H$, or $\hat{H}$ or $\hat{H}^\perp$, depending on the context in which it appears. The components of the operator $V$ in its $2 \times 2$ representation (2.2) can be defined by formulas

$$\hat{V} = \hat{P}V\hat{P}, \quad v = \hat{P}V\hat{P}^\perp \quad \text{and} \quad a = \hat{P}^\perp V\hat{P}^\perp.$$

The $2 \times 2$ representation of the operator $H_s = H_{r_\lambda} + (s - r_\lambda)V$

is given by

$$H_s = \begin{pmatrix} \hat{H}_s & (s - r_\lambda)v \\ (s - r_\lambda)v^* & \lambda + (s - r_\lambda)a \end{pmatrix},$$

where $\hat{H}_s = \hat{H}_{r_\lambda} + (s - r_\lambda)\hat{V}$. According to our agreements made above, we can rewrite this formula as

$$H_s = \hat{H}_s + \lambda\hat{P}^\perp + (s - r_\lambda)(v + v^*) + (s - r_\lambda)a.$$

We shall often use both ways of writing these kind of formulas, but mostly we prefer the matrix formulas.

The following lemma is a well-known fact of linear algebra.

**Lemma 2.1.** Assume that $A$ is an invertible operator with bounded inverse. A block operator

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

has bounded inverse if and only if the operator $D - CA^{-1}B$ has bounded inverse. In this case the inverse of the block operator is given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} + A^{-1}BDCA^{-1} & -A^{-1}BD \\ -DCA^{-1} & D \end{pmatrix},$$

where

$$D = (D - CA^{-1}B)^{-1}.$$

It follows from this lemma that the inverse of the operator $H_s - z$ is given by

$$(H_s - z)^{-1} = \begin{pmatrix} \hat{H}_s - z & (s - r_\lambda)v \\ (s - r_\lambda)v^* & \lambda - z + (s - r_\lambda)a \end{pmatrix}^{-1}
= \begin{pmatrix} R_z(\hat{H}_s) + (s - r_\lambda)^2R_z(\hat{H}_s)v^*D_z(s)v^*R_z(\hat{H}_s) & (r_\lambda - s)R_z(\hat{H}_s)v^*D_z(s) \\ (r_\lambda - s)D_z(s)v^*R_z(\hat{H}_s) & D_z(s) \end{pmatrix}.$$
where
\begin{equation}
\mathcal{D}_v(s) = (\lambda - z + (s - r\lambda)a - (s - r\lambda)^2v^*R_z(\hat{H}_s)v)^{-1}.
\end{equation}
We use notation
\[ \hat{A}_z(s) = R_z(\hat{H}_s)\hat{V}. \]
We have,
\begin{equation}
\hat{A}_z(s) := (H_s - z)^{-1}V
\end{equation}
\begin{equation}
= \left( R_z(\hat{H}_s) + (s - r\lambda)^2R_z(\hat{H}_s)v\mathcal{D}_z(s)v^*R_z(\hat{H}_s) \ (r\lambda - s)R_z(\hat{H}_s)v\mathcal{D}_z(s) \mathcal{D}_z(s) \right) \left( \hat{V} \ v^* \ a \right) \\
= \left( \hat{A}_z(s) + (r\lambda - s)R_z(\hat{H}_s)v\mathcal{D}_z(s)v^*\mathcal{F}_z(s) \ R_z(\hat{H}_s)v(1 + (r\lambda - s)[\ldots]) \right) \mathcal{D}_z(s)v^*\mathcal{F}_z(s) \ [\ldots],
\end{equation}
where
\[ [\ldots] = \mathcal{D}_z(s) \left[ a + (r\lambda - s)v^*R_z(\hat{H}_s)v \right], \]
and
\begin{equation}
\mathcal{F}_z(s) = 1 + (r\lambda - s)\hat{A}_z(s),
\end{equation}
where 1 is the identity operator on $\hat{H}$. The operator $\mathcal{F}_z(s)$ is invertible and
\begin{equation}
\mathcal{F}_z^{-1}(s) = 1 + (s - r\lambda)\hat{A}_z(r\lambda).
\end{equation}
The second resolvent identity implies that
\begin{equation}
\mathcal{F}_z(s)\hat{A}_z(r\lambda) = \hat{A}_z(s).
\end{equation}
From now on we consider the case of $z = \lambda$.
For a regular direction $V$ we have from (2.4)
\begin{equation}
\mathcal{D}_\lambda(s) = ((s - r\lambda)a - (s - r\lambda)^2v^*R_\lambda(\hat{H}_s)v)^{-1}.
\end{equation}
\textbf{Lemma 2.2.} A direction $V$ given by (2.2) is regular at a resonance point $H_{r\lambda}$ given by (2.1) if and only if the matrix
\[ a + (r\lambda - s)v^*R_\lambda(\hat{H}_s)v \]
is defined and is invertible for some real value of $s$.
\textbf{Proof.} This follows from Lemma 2.1, 2 × 2 representation (2.3) of the resolvent $(H_s - \lambda)^{-1}$, and from the definition (2.9) of $\mathcal{D}_\lambda(s)$.

\textbf{Corollary 2.3.} If a direction $V$ given by (2.2) is regular at $H_{r\lambda}$ then for any non-zero eigenvector $\chi$ of $H_{r\lambda}$ at least one of the two vectors $a\chi$ or $v\chi$ is non-zero. That is, if $V$ is regular then for any non-zero eigenvector $\chi$ the vector $V\chi$ is also non-zero.

This necessary condition of regularity of a direction $V$ is not sufficient. A simple three-dimensional example, which demonstrates this, can be found in [Az6] §14.6.1.

Using 2 × 2 representation (2.5) of $A_\lambda(s)$ and the equality (2.9), a direct calculation gives the formulas
\begin{equation}
A_\lambda(s) = \begin{pmatrix}
\hat{A}_\lambda(s) + (r\lambda - s)R_\lambda(\hat{H}_s)v\mathcal{D}_\lambda(s)v^*\mathcal{F}_\lambda(s) & 0 \\
\mathcal{D}_\lambda(s)v^*\mathcal{F}_\lambda(s) & (s - r\lambda)^{-1}
\end{pmatrix}
\end{equation}
and

\begin{equation}
1 + (r_\lambda - s)A_\lambda(s) = \begin{bmatrix}
1 + (s - r_\lambda)^2 R_\lambda(\hat{H}_s)vD_\lambda(s)v^* & \mathcal{F}_\lambda(s) & 0 \\
(s - r_\lambda)D_\lambda(s)v^*\mathcal{F}_\lambda(s) & 0
\end{bmatrix}.
\end{equation}

The function \( A_\lambda(s) \) is not holomorphic at \( s = r_\lambda \). As can be seen from (2.10), apart from the \((2,2)\)-entry of \( A_\lambda(s) \), the only factor which violates holomorphicity of this function is \( D_\lambda(s) \), the other terms are holomorphic at \( s = r_\lambda \).

One can note that a vector \( \hat{\varphi} \in \hat{\mathcal{H}} \) has order 2 if and only if

\[
\begin{bmatrix}
1 + (r_\lambda - s)A_\lambda(s)
\end{bmatrix}
\begin{pmatrix}
\hat{\varphi} \\
0 \\
\cdots
\end{pmatrix}
= \begin{pmatrix}
0 \\
\cdots
\end{pmatrix},
\]

where the dots denote a non-zero vector. By (2.11), this equality is equivalent to

\[
\begin{bmatrix}
1 + (s - r_\lambda)^2 R_\lambda(\hat{H}_s)vD_\lambda(s)v^* \\
(r_\lambda - s)D_\lambda(s)v^*\mathcal{F}_\lambda(s)
\end{bmatrix}\mathcal{F}_\lambda(s)\hat{\varphi} = 0.
\]

It follows that \( \hat{\varphi} \) is a vector of order two if and only if

\[
\hat{\varphi} = -(s - r_\lambda)^2 \mathcal{F}_\lambda^{-1}(s)R_\lambda(\hat{H}_s)vD_\lambda(s)v^*\mathcal{F}_\lambda(s)\hat{\varphi}
= -(s - r_\lambda)^2 R_\lambda(\hat{H}_{r_\lambda})vD_\lambda(s)v^*\mathcal{F}_\lambda(s)\hat{\varphi}.
\]

Thus, for a vector \( \hat{\varphi} \) from \( \hat{\mathcal{H}} \) the following equivalence holds:

\begin{equation}
\hat{\varphi} \in \mathcal{Y}_\lambda^2 \iff \hat{\varphi} = -(s - r_\lambda)^2 R_\lambda(\hat{H}_{r_\lambda})vD_\lambda(s)v^*\mathcal{F}_\lambda(s)\hat{\varphi}.
\end{equation}

In particular, any resonance vector \( \hat{\varphi} \) of order two from \( \hat{\mathcal{H}} \) belongs to the image of the operator \( R_\lambda(\hat{H}_{r_\lambda})v \). The following theorem generalises this statement for vectors of arbitrary order.

**Theorem 2.4.** Let \( k \geq 2 \). A resonance vector \( \hat{\varphi} \) of order \( k \) from \( \hat{\mathcal{H}} \) belongs to the linear span of the images of operators

\[
R_\lambda(\hat{H}_{r_\lambda})v, \quad \hat{\Lambda}_\lambda(r_\lambda)R_\lambda(\hat{H}_{r_\lambda})v, \quad \ldots, \quad \hat{\Lambda}_\lambda^{k-2}(r_\lambda)R_\lambda(\hat{H}_{r_\lambda})v.
\]

**Proof.** The induction base with \( k = 2 \) follows from (2.12). Assume that the claim holds for a vector of order less than \( k \) and let \( \hat{\varphi}_k \in \mathcal{H} \) be a vector of order \( k \). Since the operator \( 1 + (r_\lambda - s)A_\lambda(s) \) decreases order of a resonance vector by one, the vector \( \hat{\varphi}_{k-1}(s) = (1 + (r_\lambda - s)A_\lambda(s))\hat{\varphi}_k \) has order \( k - 1 \). Using 2 \times 2 representation (2.11) of the operator \( 1 + (r_\lambda - s)A_\lambda(s) \), we have

\[
\begin{bmatrix}
1 + (s - r_\lambda)^2 R_\lambda(\hat{H}_s)vD_\lambda(s)v^*
\end{bmatrix}\mathcal{F}_\lambda(s)\hat{\varphi}_k = \hat{\varphi}_{k-1}(s)
\]

This equality can be rewritten as

\[
\hat{\varphi}_k + (s - r_\lambda)^2 \mathcal{F}_\lambda^{-1}(s)R_\lambda(\hat{H}_s)vD_\lambda(s)v^*\mathcal{F}_\lambda(s)\hat{\varphi}_k = \mathcal{F}_\lambda^{-1}(s)\hat{\varphi}_{k-1}(s),
\]

and by (2.7) this equality is equivalent to

\[
\hat{\varphi}_k = -(s - r_\lambda)^2 R_\lambda(\hat{H}_{r_\lambda})vD_\lambda(s)v^*\mathcal{F}_\lambda(s)\hat{\varphi}_k + (1 + (s - r_\lambda)\hat{\Lambda}_\lambda(r_\lambda))\hat{\varphi}_{k-1}(s).
\]

Using induction assumption, this equality proves the claim. \( \square \)
2.2. The operator $S_\lambda$. In what follows the product of operators $R_\lambda(\hat{H}_r)$ and $V$ will be encountered very often; for this reason, we shall introduce notation
\begin{equation}
S_\lambda = R_\lambda(\hat{H}_r)V
= \begin{pmatrix}
\hat{A}_\lambda(r_\lambda) & R_\lambda(\hat{H}_r)v \\
0 & 0
\end{pmatrix}.
\end{equation}

Restriction of the power $S_\lambda^k$ to $\hat{H}$ coincides with $\hat{A}_\lambda^k(r_\lambda)$, restriction of this power to $V_\lambda$ coincides with $\hat{A}_\lambda^{k-1}(r_\lambda)R_\lambda(\hat{H}_r)v$.

**Lemma 2.5.** For any $\hat{f} \in \hat{H}$
\begin{equation}
(1 + (r_\lambda - s)A_\lambda(s))S_\lambda \hat{f} = A_\lambda(s)\hat{f} + \begin{pmatrix}
(s - r_\lambda)R_\lambda(\hat{H}_s)vD_\lambda(s)v^*\hat{f} \\
-D_\lambda(s)v^*\hat{f}
\end{pmatrix}
\end{equation}

*Proof.* Since $\hat{f} \in \hat{H}$, we have $S_\lambda \hat{f} = \hat{A}_\lambda(r_\lambda)\hat{f}$. Hence,
\begin{equation}
(1 + (r_\lambda - s)A_\lambda(s))S_\lambda \hat{f} = (1 + (r_\lambda - s)A_\lambda(s))\hat{A}_\lambda(r_\lambda)\hat{f}.
\end{equation}

From (2.7) we have
\begin{equation}
\hat{A}_\lambda(r_\lambda) = (s - r_\lambda)^{-1}(F_\lambda^{-1}(r) - 1).
\end{equation}

Hence, combining the last two equalities and using $2 \times 2$ representation (2.11) of the operator $1 + (r_\lambda - s)A_\lambda(s)$, we get
\begin{align*}
(1 + (r_\lambda - s)A_\lambda(s))S_\lambda \hat{f} &= (1 + (r_\lambda - s)A_\lambda(s))\hat{A}_\lambda(r_\lambda)\hat{f} \\
&= (s - r_\lambda)^{-1}(1 + (r_\lambda - s)A_\lambda(s))(F_\lambda^{-1}(r) - 1)\hat{f} \\
&= -(s - r_\lambda)^{-1}(1 + (r_\lambda - s)A_\lambda(s))\hat{f} \\
&\quad + (s - r_\lambda)^{-1} \begin{pmatrix}
1 + (s - r_\lambda)2R_\lambda(\hat{H}_s)vD_\lambda(s)v^*\hat{f} \\
(r_\lambda - s)D_\lambda(s)v^*\hat{f}
\end{pmatrix} \\
&= A_\lambda(s)\hat{f} + \begin{pmatrix}
(s - r_\lambda)R_\lambda(\hat{H}_s)vD_\lambda(s)v^*\hat{f} \\
-D_\lambda(s)v^*\hat{f}
\end{pmatrix}.
\end{align*}

\[\square\]

**Theorem 2.6.** If a vector $\chi \in \mathcal{H}$ is such that the vector $V\chi$ is orthogonal to the eigenspace $\mathcal{V}_\lambda$, then
\begin{equation}
(1 + (r_\lambda - s)A_\lambda(s))S_\lambda \chi = A_\lambda(s)\chi.
\end{equation}

*Proof.* (A) The premise $V\chi \perp \mathcal{V}_\lambda$ means that the second component of $V\chi$ in the direct sum $\mathcal{H} = \hat{H} \oplus \mathcal{V}_\lambda$ is zero, that is,
\begin{equation}
v^*\chi + a\chi = 0.
\end{equation}

(B) If $V\chi \perp \mathcal{V}_\lambda$, then
\begin{equation}
(1 + (r_\lambda - s)A_\lambda(s))R_\lambda(\hat{H}_r)\chi = \begin{pmatrix}
(r_\lambda - s)R_\lambda(\hat{H}_s)vD_\lambda(s)v^*\chi \\
(s - r_\lambda)^{-1}\chi + D_\lambda(s)v^*\chi
\end{pmatrix}.
\end{equation}

Proof of (2.16).
Using (2.11), we have

\begin{equation}
\left( E \right) := (1 + (r_\lambda - s)A_\lambda(s))R_\lambda(\hat{H}_{r_\lambda})v_\chi = \left[ 1 + (s - r_\lambda)^2R_\lambda(\hat{H}_s)v^*D_\lambda(s)v^* \right] F_\lambda(s)R_\lambda(\hat{H}_{r_\lambda})v_\chi \nonumber \\
= \left[ 1 + (s - r_\lambda)^2R_\lambda(\hat{H}_s)v^*D_\lambda(s)v^* \right] R_\lambda(\hat{H}_s)v_\chi,
\end{equation}

where in the second equality the second resolvent identity (2.5) is used. The second component of this vector can be transformed as follows:

\[ (r_\lambda - s)D_\lambda(s)v^*R_\lambda(\hat{H}_s)v_\chi = \left[ a_\chi + (r_\lambda - s)v^*R_\lambda(\hat{H}_s)v_\chi \right] = (s - r_\lambda)^{-1}\hat{P}_\perp\chi - D_\lambda(s)a_\chi, \]

where the second equality follows from the definition (2.9) of \( D_\lambda(s) \). Combining this with (2.15) gives

\[ (r_\lambda - s)D_\lambda(s)v^*R_\lambda(\hat{H}_s)v_\chi = (s - r_\lambda)^{-1}\hat{P}_\perp\chi + D_\lambda(s)v^*\chi. \]

Substituting the right hand side into (2.17) yields

\begin{equation}
\left( E \right) = \left( R_\lambda(\hat{H}_s)v \left[ \chi + (s - r_\lambda)^2D_\lambda(s)v^*R_\lambda(\hat{H}_s)v_\chi \right] \right) \nonumber \\
= \left( R_\lambda(\hat{H}_s)v \left[ \chi + (s - r_\lambda)^2\hat{P}_\perp\chi + D_\lambda(s)v^*\chi \right] \right) \nonumber \\
= \left( (r_\lambda - s)R_\lambda(\hat{H}_s)vD_\lambda(s)v^*\chi \right),
\end{equation}

where in the last equality we used \( v^*\hat{P}_\perp = v \).

(C) We have

\begin{equation}
(1 + (r_\lambda - s)A_\lambda(s))S_\lambda\chi = (1 + (r_\lambda - s)A_\lambda(s))R_\lambda(\hat{H}_{r_\lambda})V\chi = (1 + (r_\lambda - s)A_\lambda(s))R_\lambda(\hat{H}_{r_\lambda})V\chi + (1 + (r_\lambda - s)A_\lambda(s))R_\lambda(\hat{H}_{r_\lambda})v_\chi.
\end{equation}

Applying Lemma 2.5 to the vector \( \hat{P}_\chi = \hat{\chi} \), we transform the first summand as follows

\begin{equation}
(1 + (r_\lambda - s)A_\lambda(s))R_\lambda(\hat{H}_{r_\lambda})V\chi = A_\lambda(s)\hat{\chi} + \left( \left( s - r_\lambda \right)R_\lambda(\hat{H}_s)vD_\lambda(s)v^*\hat{\chi} \right).\nonumber
\end{equation}

Since \( v^*\chi = v^*\hat{\chi} \), combining (2.19), (2.20) and (2.16) yields the equality

\[ (1 + (r_\lambda - s)A_\lambda(s))S_\lambda\chi = A_\lambda(s)\hat{\chi} + (s - r_\lambda)^{-1}\hat{P}_\perp\chi. \]

Since \( \hat{P}_\perp\chi \) is an eigenvector of \( H_{r_\lambda} \), we have \( A_\lambda(s)\hat{P}_\perp\chi = (s - r_\lambda)^{-1}\hat{P}_\perp\chi \). Hence,

\[ (1 + (r_\lambda - s)A_\lambda(s))S_\lambda\chi = A_\lambda(s)\hat{\chi} + A_\lambda(s)\hat{P}_\perp\chi = A_\lambda(s)\chi. \]

\[ \square \]
The argument of this proof shows that in general (without the assumption $V \chi \perp V_\lambda$) we have

\[(2.21) \quad (1 + (r_\lambda - s)A_\lambda(s))S_\lambda \chi = A_\lambda(s)\chi + \left((s - r_\lambda)R_\lambda(H_s)vD_\lambda(s)(v^* + a)\chi \right).\]

2.3. Depth of resonance vectors and depth criteria. Recall that a resonance vector $\chi$ has depth at least $k$ if there exists a resonance vector $\chi_1$ such that

$$\chi = (1 + (r_\lambda - s)R_\lambda(H_s)V_\lambda)^k \chi_1.$$Equivalently, a resonance vector $\chi$ has depth at least $k$ if there exists a resonance vector $\chi_1$ such that

$$\chi = A^k_\lambda \chi_1.$$A resonance vector has depth $k$ if it has depth at least $k$ but not at least $k + 1$.

The following theorem provides a criterion for a resonance vector to have depth $\geq 1$.

**Theorem 2.7.** Let $\chi$ be a resonance vector. The following assertions are equivalent.

(i) The vector $V \chi$ is orthogonal to $V_\lambda$.
(ii) The depth of the vector $\chi$ is at least 1.
(iii) The equality $A_\lambda(s)\chi = -(s-r_\lambda)A_\lambda S_\lambda \chi + \ldots$ holds.

Proof. (i) $\Rightarrow$ (iii). By (1.8), the meromorphic operator-function $A_\lambda(s)$ has the following Laurent expansion at $s = r_\lambda$:

\[(2.22) \quad A_\lambda(s) = \tilde{A}_\lambda(s) + \sum_{j=0}^{d-1} (s - r_\lambda)^{-j-1} \tilde{A}_\lambda^j,\]

where $\tilde{A}_\lambda^0 = P_\lambda$. From this we obtain

$$1 + (r_\lambda - s)A_\lambda(s) = (r_\lambda - s)\tilde{A}_\lambda(s) + (1 - P_\lambda) - \sum_{j=1}^{d-1} (s - r_\lambda)^{-j} \tilde{A}_\lambda^j.$$Applying the latter series to the vector $S_\lambda \chi$ and retaining only the term $(s - r_\lambda)^{-1}$ gives

$$(1 + (r_\lambda - s)A_\lambda(s))S_\lambda \chi = \ldots - (s - r_\lambda)^{-1} A_\lambda S_\lambda \chi + \ldots.$$Since $\chi$ is a resonance vector, the former series applied to the vector $\chi$ also gives

$$A_\lambda(s)\chi = \ldots + (s - r_\lambda)^{-1} \chi + \ldots.$$Since $V \chi \perp V_\lambda$, according to Theorem 2.6 the last two Laurent expansions are equal and therefore $A_\lambda S_\lambda \chi = -\chi$.

(iii) $\Rightarrow$ (ii). This is obvious.

(ii) $\Rightarrow$ (i). Let $\chi$ be a resonance vector of depth at least one. By definition of depth, there exists a resonance vector $\chi'$ such that $A_\lambda \chi' = \chi$. Hence, for any eigenvector $\varphi \in V_\lambda$ we have

$$\langle V \chi, \varphi \rangle = \langle VA_\lambda \chi', \varphi \rangle = \langle V \chi', A_\lambda \varphi \rangle = 0,$$where the last equality holds since $A_\lambda(r_\lambda)$ eliminates any eigenvector.

Question. Are the items of this theorem also equivalent to this one: (iv) $S_\lambda \chi$ is a resonance vector? The implication (iii) $\Rightarrow$ (iv) is obvious, so the question is whether (iv) $\Rightarrow$ (iii)?
2.4. $2 \times 2$ representations of $P_\lambda$ and $A_\lambda(r_\lambda)$. The meromorphic operator-valued function $\mathcal{D}_\lambda(s)$ defined in (2.9) acts on the finite-dimensional Hilbert space $\mathcal{V}_\lambda$. Since the function

$$\mathcal{D}_\lambda^{-1}(s) = (s - r_\lambda)a - (s - r_\lambda)^2v^*R_\lambda(\hat{H}_s)v$$

is holomorphic at $s = r_\lambda$, by the analytic Fredholm alternative, at the pole $r_\lambda$ the Laurent series of the function $\mathcal{D}_\lambda(s)$ has finitely many terms with negative powers.

**Lemma 2.8.** The meromorphic function $\mathcal{D}_\lambda(s)$ has a pole of order $d$ at $s = r_\lambda$.

**Proof.** Since by (2.22) the operator $A_\lambda(s)$ has a pole of order $d$ at $s = r_\lambda$, it follows from $2 \times 2$ representation (2.10) of this operator that the function $\mathcal{D}_\lambda(s)$ has a pole of order at least $d$ at $s = r_\lambda$, since other factors in this $2 \times 2$ representation are holomorphic at $s = r_\lambda$. That the order of this pole is not greater than $d$ can be observed from the second resolvent identity

$$R_\lambda(H_s) = R_\lambda(H_{s_0}) - (s - s_0)R_\lambda(H_s)V R_\lambda(H_{s_0})$$

and the fact that $\mathcal{D}_\lambda(s)$ is the $(2,2)$-entry of $R_\lambda(H_s)$, since the right hand side of the resolvent identity above has a pole of order at most $d$.

This lemma shows that the Laurent expansion of the function $\mathcal{D}_\lambda(s)$ can be written in the following form:

$$\mathcal{D}_\lambda(s) = \sum_{j = -d+1}^{\infty} D_{-j}(s - r_\lambda)^{j-1}. \quad (2.23)$$

The additional factor $(s - r_\lambda)^{-1}$ is introduced here for convenience; also, since we shall be working mainly with coefficients of negative powers of $(s - r_\lambda)$, we choose to denote the coefficient of $(s - r_\lambda)^{-1}$ by $D_{-j}$. Since the meromorphic function $\mathcal{D}_\lambda(s)$ depends only on the triple $(\lambda; H_{r_\lambda}, V)$, the operators $D_1, \ldots, D_{d-1}$ are also invariants of this triple. Further, since $\mathcal{D}_\lambda(s)$ is self-adjoint for real values of $s$, the operators $D_j$ are self-adjoint.

**Lemma 2.9.** We have

$$D_{d-1}a = 0,$$

$$D_{d-2}a = D_{d-1}v^*R_\lambda(\hat{H}_{r_\lambda})v,$$

$$D_{d-3}a = D_{d-2}v^*R_\lambda(\hat{H}_{r_\lambda})v - D_{d-1}v^*\hat{A}_\lambda(r_\lambda)R_\lambda(\hat{H}_{r_\lambda})v,$$

$$\ldots$$

$$D_1a = D_2v^*R_\lambda(\hat{H}_{r_\lambda})v - D_3v^*\hat{A}_\lambda(r_\lambda)R_\lambda(\hat{H}_{r_\lambda})v + \ldots + (-1)^{d-3}D_{d-1}v^*\hat{A}_{\lambda}^{d-3}(r_\lambda)R_\lambda(\hat{H}_{r_\lambda})v,$$

$$D_0a = D_1v^*R_\lambda(\hat{H}_{r_\lambda})v - D_2v^*\hat{A}_\lambda(r_\lambda)R_\lambda(\hat{H}_{r_\lambda})v + \ldots + (-1)^{d-2}D_{d-1}v^*\hat{A}_{\lambda}^{d-2}(r_\lambda)R_\lambda(\hat{H}_{r_\lambda})v + 1,$$

$$D_{-1}a = D_0v^*R_\lambda(\hat{H}_{r_\lambda})v - D_1v^*\hat{A}_\lambda(r_\lambda)R_\lambda(\hat{H}_{r_\lambda})v + \ldots + (-1)^{d-1}D_{d-1}v^*\hat{A}_{\lambda}^{d-1}(r_\lambda)R_\lambda(\hat{H}_{r_\lambda})v.$$

**Proof.** By definition of $\mathcal{D}_\lambda(s)$, we have $1 = (s - r_\lambda)\mathcal{D}_\lambda(s)(a + (r_\lambda - s)v^*R_\lambda(\hat{H}_s)v)$, that is,

$$1 = \sum_{j = -d+1}^{\infty} D_{-j}(s - r_\lambda)^j \left( a + \sum_{j = 0}^{\infty}(-1)^{j+1}(s - r_\lambda)^j v^*\hat{A}_{\lambda}^j(r_\lambda)R_\lambda(\hat{H}_{r_\lambda})v \right).$$

Comparing powers of $s - r_\lambda$ on both sides gives the required equalities.
The operators $P_\lambda$ and $A^j_\lambda$ are coefficients of the Laurent series of the meromorphic function $A_\lambda(s)$. Hence, using (2.10), (2.23) and the Neumann series

$$F_\lambda(s) = \sum_{k=0}^{\infty} (-1)^k (s - r_\lambda)^k A^k_\lambda(r_\lambda),$$

one can calculate $2 \times 2$ representations of these operators. We omit the straightforward calculations and present only the resulting formulas; to simplify them, we use notations

$$Y_j := R_\lambda(\hat{H}_{r_\lambda}) v D_j v^*$$

and

$$\{ \hat{A}_{\lambda}(r_\lambda), Y_j \} = \hat{A}_{\lambda}(r_\lambda) Y_j + \hat{A}_{\lambda}^{j-1}(r_\lambda) Y_j \hat{A}_{\lambda}(r_\lambda) + \hat{A}_{\lambda}^{j-2}(r_\lambda) Y_j \hat{A}_{\lambda}^{2}(r_\lambda) + \ldots + Y_j \hat{A}_{\lambda}^j(r_\lambda).$$

**Theorem 2.10.** $2 \times 2$ representations of operators $A^j_\lambda(r_\lambda)$, $j = 1, \ldots, d - 1$, $P_\lambda(r_\lambda)$ and $\hat{A}_\lambda(r_\lambda)$ are given by formulas

(2.24)

$$A^j_\lambda = \begin{pmatrix} -Y_{j+1} + \{ \hat{A}_\lambda(r_\lambda), Y_{j+2} \} - \{ \hat{A}_\lambda^2(r_\lambda), Y_{j+3} \} + \ldots + (-1)^{d-j-1} \{ \hat{A}_\lambda^{d-j-2}(r_\lambda), Y_{d-1} \} & 0 \\ D_j v^* - D_{j+1} v^* \hat{A}_\lambda(r_\lambda) + D_{j+2} v^* \hat{A}_\lambda^2(r_\lambda) + \ldots + (-1)^{d-j-1} D_{d-1} v^* \hat{A}_\lambda^{d-j-1}(r_\lambda) & 0 \end{pmatrix},$$

(2.25)

$$P_\lambda = \begin{pmatrix} -Y_1 + \{ \hat{A}_\lambda(r_\lambda), Y_2 \} - \{ \hat{A}_\lambda^2(r_\lambda), Y_3 \} + \ldots + (-1)^{d-1} \{ \hat{A}_\lambda^{d-2}(r_\lambda), Y_{d-1} \} & 0 \\ D_0 v^* - D_1 v^* \hat{A}_\lambda(r_\lambda) + D_2 v^* \hat{A}_\lambda^2(r_\lambda) + \ldots + (-1)^{d-1} D_{d-1} v^* \hat{A}_\lambda^{d-1}(r_\lambda) & 1 \end{pmatrix},$$

and

(2.26)

$$\hat{A}_\lambda(r_\lambda) = \begin{pmatrix} \hat{A}_\lambda(r_\lambda) - Y_0 + \{ \hat{A}_\lambda(r_\lambda), Y_1 \} - \{ \hat{A}_\lambda^2(r_\lambda), Y_2 \} + \ldots + (-1)^{d-1} \{ \hat{A}_\lambda^{d-1}(r_\lambda), Y_{d-1} \} & 0 \\ D_{-1} v^* - D_0 v^* \hat{A}_\lambda(r_\lambda) + D_1 v^* \hat{A}_\lambda^2(r_\lambda) + \ldots + (-1)^{d} D_{d-1} v^* \hat{A}_\lambda^{d}(r_\lambda) & 0 \end{pmatrix}.$$

In particular,

$$A^{d-1}_\lambda = \begin{pmatrix} 0 & 0 \\ D_{d-1} v^* & 0 \end{pmatrix}, \quad A^{d-2}_\lambda = \begin{pmatrix} -Y_{d-1} & 0 \\ D_{d-2} v^* - D_{d-1} v^* \hat{A}_\lambda(r_\lambda) & 0 \end{pmatrix}.$$

These formulas generalise those given in [Az0] §14.4 considering that in this setting the eigenvalue $\lambda$ is degenerate. Note that the $(2,2)$-entry of $P_\lambda$ is 1.

**Theorem 2.11.** In the $2 \times 2$ representation, for all $j = 1, 2, \ldots, d$, the only non-zero entry of the operator $-S_\lambda A^j_\lambda$ is the $(1,1)$-entry which is equal to the $(1,1)$-entry of $A^{j-1}_\lambda$. In other words,

(2.27)

$$-S_\lambda A^j_\lambda = \hat{P} A^{j-1}_\lambda.$$

**Proof.** Proof is a direct calculation based on $2 \times 2$ representation (2.24) of the operator $A^j_\lambda$. For $j \geq 1$, we have

$$S_\lambda A^j_\lambda(r_\lambda) = \begin{pmatrix} \hat{A}_\lambda(r_\lambda) & R_\lambda(\hat{H}_{r_\lambda}) v \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \sum_{k=0}^{d-j-2} (-1)^{k+1} \{ \hat{A}_\lambda^k(r_\lambda), Y_{k+j+1} \} & 0 \\ \sum_{k=0}^{d-j-1} (-1)^k D_{j+k} v^* \hat{A}_\lambda^k(r_\lambda) & 0 \end{pmatrix}.$$
The only non-zero entry of this product is \((1,1)\)-entry, which is equal to the following expression

\[
\begin{align*}
- \hat{A}_\lambda(r_\lambda)Y_{j+1} + \hat{A}_\lambda(r_\lambda) \left\{ \hat{A}_\lambda(r_\lambda), Y_{j+2} \right\} - \ldots + (-1)^{d-j-1} \hat{A}_\lambda(r_\lambda) \left\{ \hat{A}_\lambda^{d-j-2}(r_\lambda), Y_{d-1} \right\} \\
+ R_\lambda(\hat{H}_{r_\lambda})vD_jv^*-R_\lambda(\hat{H}_{r_\lambda})vD_{j+1}v^*\hat{A}_\lambda(r_\lambda) + \ldots + (-1)^{d-j-1} R_\lambda(\hat{H}_{r_\lambda})vD_{d-1}v^*\hat{A}_\lambda^{d-j-1}(r_\lambda)
\end{align*}
\]

\[
= -\hat{A}_\lambda(r_\lambda)Y_{j+1} + \hat{A}_\lambda(r_\lambda) \left\{ \hat{A}_\lambda(r_\lambda), Y_{j+2} \right\} - \ldots + (-1)^{d-j-1} \hat{A}_\lambda(r_\lambda) \left\{ \hat{A}_\lambda^{d-j-2}(r_\lambda), Y_{d-1} \right\}
\]

+ \(Y_j - Y_{j+1}\hat{A}_\lambda(r_\lambda) + \ldots + (-1)^{d-j-1}Y_{d-1}\hat{A}_\lambda^{d-j-1}(r_\lambda)\).

Re-arranging the summands in this expression we see that it is equal to

\[
Y_j - \left\{ \hat{A}_\lambda(r_\lambda), Y_{j+1} \right\} + \left\{ \hat{A}_\lambda^2(r_\lambda), Y_{j+2} \right\} - \ldots + (-1)^{d-j-1} \left\{ \hat{A}_\lambda^{d-j-1}(r_\lambda), Y_{d-1} \right\}
\]

Since this is the negative of the \((1,1)\)-entry of the operator \(A_\lambda^{j-1}\), we are done. \(\square\)

Subtracting \(A_\lambda^{j-1}\) from both sides of \eqref{2.27} and replacing \(j - 1\) by \(j\) in the resulting formula gives for each \(j = 0,1,2,\ldots,d-1\)

\[
(1 + S_\lambda A_\lambda)A_\lambda^j = \hat{P}^\perp A_\lambda^j.
\]

Since \(\hat{P}^\perp\) is the orthogonal projection onto the eigenspace \(V_\lambda\), combining this equality with \eqref{2.24} gives for \(j = 1,2,\ldots\)

\[
(2.28) \quad (1 + S_\lambda A_\lambda)A_\lambda^j = \sum_{k=0}^{d-1-j} (-1)^k D_jv^* \hat{A}_\lambda^k(r_\lambda).
\]

If \(j = 0\), then we have to use the formula \eqref{2.25} which has non-zero \((2,2)\)-entry equal to 1. This entry results in the additional summand \(\hat{P}^\perp\). Hence, we have

\[
(2.29) \quad (1 + S_\lambda A_\lambda)P_\lambda = \sum_{k=0}^{d-1} (-1)^k D_k v^* \hat{A}_\lambda^k(r_\lambda) + \hat{P}^\perp.
\]

**Theorem 2.12.** For any \(j = 1,2,\ldots,d-1\)

\[
D_jv^* = (1 + S_\lambda A_\lambda)A_\lambda^j (1 + A_\lambda \hat{A}_\lambda(r_\lambda)).
\]

Also,

\[
D_0v^* = -\hat{P}^\perp + (1 + S_\lambda A_\lambda)P_\lambda (1 + A_\lambda \hat{A}_\lambda(r_\lambda)).
\]

**Proof.** Using the equalities \eqref{2.28}, we obtain formulas

\[
D_{d-1}v^* = A_\lambda^{d-1},
\]

\[
D_{d-2}v^* = A_\lambda^{d-2} + S_\lambda A_\lambda^{d-1} + D_{d-1}v^* \hat{A}_\lambda(r_\lambda)
\]

\[
= A_\lambda^{d-2} + S_\lambda A_\lambda^{d-1} + A_\lambda^{d-1} \hat{A}_\lambda(r_\lambda)
\]

\[
= (1 + S_\lambda A_\lambda)A_\lambda^{d-2} (1 + A_\lambda \hat{A}_\lambda(r_\lambda)),
\]

\[
D_{d-3}v^* = A_\lambda^{d-3} + S_\lambda A_\lambda^{d-2} + D_{d-2}v^* \hat{A}_\lambda(r_\lambda) - D_{d-1}v^* \hat{A}_\lambda^2(r_\lambda)
\]

\[
= A_\lambda^{d-3} + S_\lambda A_\lambda^{d-2} + \left( A_\lambda^{d-2} + S_\lambda A_\lambda^{d-1} + A_\lambda^{d-1} \hat{A}_\lambda(r_\lambda) \right) \hat{A}_\lambda(r_\lambda) - A_\lambda^{d-1} \hat{A}_\lambda^2(r_\lambda)
\]

\[
= A_\lambda^{d-3} + S_\lambda A_\lambda^{d-2} + A_\lambda^{d-2} \hat{A}_\lambda(r_\lambda) + S_\lambda A_\lambda^{d-1} \hat{A}_\lambda(r_\lambda)
\]

\[
= (1 + S_\lambda A_\lambda)A_\lambda^{d-3} (1 + A_\lambda \hat{A}_\lambda(r_\lambda)).
\]
We proceed by induction. Moving all summands of the right side of (2.28), except the first one, to the left side and then using the induction assumption we obtain

\[ D_j v^* = (1 + S_\lambda A_\lambda)A^j_\lambda + \sum_{k=1}^{\infty} (-1)^{k+1}D_{j+k}v^*\hat{A}^k_\lambda(r_\lambda) \]

\[ = (1 + S_\lambda A_\lambda)A^j_\lambda + \sum_{k=1}^{\infty} (-1)^{k+1}(1 + S_\lambda A_\lambda)A^{j+k}_\lambda(1 + A_\lambda\hat{A}_\lambda(r_\lambda))\hat{A}^k_\lambda(r_\lambda) \]

\[ = (1 + S_\lambda A_\lambda)A^j_\lambda \left( 1 + \sum_{k=1}^{\infty} (-1)^{k+1}A^{k}_\lambda(1 + A_\lambda\hat{A}_\lambda(r_\lambda))\hat{A}^k_\lambda(r_\lambda) \right) \]

\[ = (1 + S_\lambda A_\lambda)A^j_\lambda \left( 1 + \sum_{k=1}^{\infty} (-1)^{k+1}A^k_\lambda\hat{A}^k_\lambda(r_\lambda) + \sum_{k=1}^{\infty} (-1)^{k+1}A_{\lambda}^{k+1}\hat{A}^{k+1}(r_\lambda) \right) \]

\[ = (1 + S_\lambda A_\lambda)A^j_\lambda \left( 1 + A_\lambda\hat{A}_\lambda(r_\lambda) \right). \]

The appearance of the additional summand $-\hat{P}^\bot$ in case of $j = 0$ was explained before the statement of this theorem. We also remark that the infinite sums above are in fact finite, but for simplicity the upper summation indexes are replaced by infinity.

Since $aS_\lambda = 0$, this theorem implies the equality

\[ (2.30) \quad aD_j v^* = aA^j_\lambda(1 + A_\lambda\hat{A}_\lambda(r_\lambda)), \quad j = 1, 2, \ldots, d - 1. \]

Using $2 \times 2$ representations of operators $P_\lambda, A_\lambda, \hat{A}_\lambda(r_\lambda)$ and Lemma (2.9) straightforward but somewhat lengthy calculations prove the following relations:

\[ P_\lambda + A_\lambda S_\lambda = \sum_{l=0}^{d-1} (-1)^l S^l_\lambda D_l(v^* + a), \]

and

\[ \hat{A}_\lambda(r_\lambda) + P_\lambda S_\lambda = S_\lambda + \sum_{l=0}^{d-1} (-1)^{l+1} S^{l+1}_\lambda D_l(v^* + a), \]

where $S^0_\lambda$ is the identity operator on the whole Hilbert space $\mathcal{H}$. Since these formulas are not used further, their proofs are omitted. We note that, since a vector $\chi$ has the property $V\chi \perp V_\lambda$ if and only if $(v^* + a)\chi = 0$, Theorem (2.7) immediately follows from the first of these formulas.

2.5. **Resonance points with property $B$.**

**Theorem 2.13.** The following assertions are equivalent:

1. $aD_j = 0$ for all $j = 1, 2, \ldots, d - 1$.
2. $aD_j v^* = 0$ for all $j = 1, 2, \ldots, d - 1$.
3. $aA_\lambda = 0$.
4. $v^*A_\lambda = 0$.
5. The function $(s - r_\lambda)D_\lambda(s)a$ is holomorphic at $s = r_\lambda$.
6. The function $(s - r_\lambda)vD_\lambda(s)a$ is holomorphic at $s = r_\lambda$.
7. $\text{im} \left( \hat{P}^\bot A_\lambda \right) \subset \text{im} (A_\lambda)$.
Proof. We prove the following equivalences: (1) $\Leftrightarrow$ (2), (2) $\Leftrightarrow$ (3), (3) $\Leftrightarrow$ (1), (1) $\Leftrightarrow$ (5), (2) $\Leftrightarrow$ (6), (3) $\Leftrightarrow$ (7).

(1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1). We shall prove that $vD_j a = 0$ implies $D_j a = 0$.

The equality $D_{d-1} a = 0$ follows from Lemma 2.9. The second equality in Lemma 2.9 also implies that $aD_{d-2} a = 0$. So, if it were that $D_{d-2} a \neq 0$ then by Corollary 2.13 we would have $vD_{d-2} a \neq 0$ which contradicts the premise. Hence, $D_{d-2} a = 0$.

The equalities $D_{d-1} a = 0$, $D_{d-2} a = 0$ and the third equality in Lemma 2.9 imply that $aD_{d-3} a = 0$. So, if it were that $D_{d-3} a \neq 0$ then by Corollary 2.3 we would have $vD_{d-3} a \neq 0$ which contradicts the premise. Hence, $D_{d-3} a = 0$. And so on.

(3) $\Rightarrow$ (2) follows immediately from formula (2.30).

(2) $\Rightarrow$ (3). Using (2.30) with $j = d - 1$ we infer that $0 = aD_{d-1} v^* = a\mathbf{A}_{\lambda}^{d-1}$. Hence, using (2.30) with $j = d - 2$ we infer that $0 = aD_{d-2} v^* = a\mathbf{A}_{\lambda}^{d-2}$. And so on: $0 = aD_1 v^* = a\mathbf{A}_{\lambda}^1$.

(3) $\Leftrightarrow$ (1). This equivalence follows from formula $(v^* + a)\mathbf{A}_{\lambda} = 0$ which holds for any resonance point (outside essential spectrum), by Theorem 2.7.

The equivalences (1) $\Leftrightarrow$ (5), (2) $\Leftrightarrow$ (6) obviously follow from the Laurent expansion (2.23) of $\mathcal{D}_{\lambda}(s)$.

(7) $\Rightarrow$ (3). Since $a = a\hat{P}^\perp$, we have $a \text{im} (\mathbf{A}_{\lambda}) = a \text{im} (\hat{P}^\perp \mathbf{A}_{\lambda})$. By the premise, we have $\text{im} (\hat{P}^\perp \mathbf{A}_{\lambda}) \subset \text{im} (\mathbf{A}_{\lambda})$. Combining this with the obvious inclusion $\text{im} (\hat{P}^\perp \mathbf{A}_{\lambda}) \subset \text{im} (\mathbf{A}_{\lambda}) \cap \mathcal{V}_{\lambda}$ gives $a \text{im} (\mathbf{A}_{\lambda}) = a \text{im} (\hat{P}^\perp \mathbf{A}_{\lambda}) \subset a (\text{im} (\mathbf{A}_{\lambda}) \cap \mathcal{V}_{\lambda}) = \{0\}$, where the last equality follows from the fact that by Theorem 2.7 eigenvectors $\chi$ of depth at least one are $V$-orthogonal to $\mathcal{V}_{\lambda}$, and therefore for such eigenvectors $a\chi = 0$.

(3) $\Rightarrow$ (7). Since $a = a\hat{P}^\perp$, we have $a\hat{P}^\perp \mathbf{A}_{\lambda} = 0$. Since the kernel of the crossing operator $a$ consists of vectors of depth at least one, we are done.

We say that a resonance point $r_{\lambda}$ has property $B$ if one of the equivalent conditions of Theorem 2.13 hold.

2.6. Resonance points with property $A$. Theorem 2.7 and formula (2.27) indicate that the operators $\mathbf{A}_{\lambda}$ and $-S_{\lambda}$ restricted to the vector space $\mathcal{Y}_{\lambda}(r_{\lambda})$ behave to a certain extent as inverses of each other. In particular, while the operator $\mathbf{A}_{\lambda}$ decreases order of a resonance vector $\chi$ by 1, the operator $S_{\lambda}$ increases order of $\chi$ by 1, provided there is some room for increasing the order. Another property of $\mathbf{A}_{\lambda}$ is that it increases depth of a resonance vector by 1 (this is, in fact, definition of the depth). It is therefore reasonable to ask whether the operator $S_{\lambda}$ decreases depth of a resonance vector by 1. We say that a resonance point has property $A$, if it possesses this property. We conjecture that all resonance points have the property $A$. In this subsection we give two conditions which are equivalent to property $A$.

Theorem 2.14. For a resonance point $r_{\lambda} = 0$ which does not belong to the essential spectrum, the following assertions are equivalent:

(1) For all $j = 1, 2, \ldots, d - 1$ $\text{im} (D_j v^*) \subset \text{im} \mathbf{A}_{\lambda}^j$.

(2) For all $j = 1, 2, \ldots, d - 1$ $\text{im} \left( S_{\lambda} \mathbf{A}_{\lambda}^j \right) \subset \text{im} \left( \mathbf{A}_{\lambda}^{j-1} \right)$.

(3) For all $j = 1, 2, \ldots, d - 1$ $\text{im} \left( \hat{P} \mathbf{A}_{\lambda}^{j-1} \right) \subset \text{im} \left( \mathbf{A}_{\lambda}^{j-1} \right)$.

(4) For all $j = 1, 2, \ldots, d - 1$ $\text{im} \left( \hat{P} \mathbf{A}_{\lambda}^{j-1} \right) \subset \text{im} \left( \mathbf{A}_{\lambda}^{j-1} \right)$. 
The definition of the order of an eigenpath is correct in the sense that it does not
Lemona 3.3. shall see later, the largest of orders of eigenpaths is equal to the order of the direction

We say that the eigenpath \( \phi \) is orthogonal to the eigenspace \( \mathcal{V}_\lambda \). Let \( r_\lambda \) be a \( \lambda \)-resonant operator. Eigenvectors of \( H_{r_\lambda} \) corresponding to the eigenvalue \( \lambda \) form a vector space \( \mathcal{V}_\lambda \), the dimension of which we denote by \( m \).

We denote by \( \lambda_\nu(s), \nu = 1, \ldots, m \), the eigenvalue functions of \( H_s \), and by \( \varphi_\nu(s) \) the corresponding eigenvector functions of \( H_s \).

**Proposition 3.1.** If \( \varphi_\nu(s) \) is an eigenpath of \( H_s \) then the vector \( \varphi_\nu(r_\lambda) \) is also an eigenvector of the crossing operator \( a = \hat{P}^{-1}V\hat{P}^{-1} \). Moreover, the corresponding eigenvalue is \( \lambda_\nu(r_\lambda) \).

**Proof.** Differentiating \( H_s \varphi_\nu(s) = \lambda_\nu(s)\varphi_\nu(s) \) and letting \( s = r_\lambda \) we obtain the equality

\[
V\varphi_\nu(r_\lambda) + H_{r_\lambda}\varphi_\nu'(r_\lambda) = \lambda_\nu(r_\lambda)\varphi_\nu(r_\lambda) + \lambda_\nu(r_\lambda)\varphi_\nu'(r_\lambda).
\]

Applying to both sides of this equality the operator \( \hat{P}^{-1} \) gives \( \hat{P}^{-1}V\varphi_\nu(r_\lambda) = \lambda_\nu(r_\lambda)\varphi_\nu(r_\lambda) \), which is what is required.

**Definition 3.2.** Let \( \varphi(s) \) be an analytic path of eigenvectors of \( H_s \) and let \( k \) be a positive integer. We say that the eigenpath \( \varphi(s) \) has order at least \( k \), if the vectors

\[
V\varphi(r_\lambda), V\varphi'(r_\lambda), \ldots, V\varphi^{(k-2)}(r_\lambda)
\]

are orthogonal to the eigenspace \( \mathcal{V}_\lambda \).

We also say that a path \( \varphi(s) \) has order \( k \), if in addition \( V\varphi^{(k-1)}(r_\lambda) \) is not orthogonal to \( \mathcal{V}_\lambda \).

Since for \( k = 1 \) the set of vectors (3.1) is empty, order of every eigenpath is at least 1. As we shall see later, the largest of orders of eigenpaths is equal to the order of the direction \( V \).

**Lemma 3.3.** The definition of the order of an eigenpath is correct in the sense that it does not depend on normalisation of the eigenpath \( \varphi(s) \). That is, if \( a(s) \) is an analytic function such that \( a(r_\lambda) \neq 0 \), then for the eigenpath \( \psi(s) = a(s)\varphi(s) \) the vectors

\[
V\psi(r_\lambda), V\psi'(r_\lambda), \ldots, V\psi^{(k-2)}(r_\lambda)
\]

are orthogonal to \( \mathcal{V}_\lambda \) if and only if so are the vectors (3.1).

**Proof.** This immediately follows from the Leibniz rule

\[
\psi^{(k)}(s) = \sum_{j=0}^{k} \binom{k}{j} a^{(j)}(s)\varphi^{(k-j)}(s).
\]
If \((H_0 + sV)\varphi(s) = \lambda(s)\varphi(s)\), then \(\lambda'(r_\lambda) = \langle \varphi(r_\lambda), V\varphi(r_\lambda) \rangle\), which is a well-known fact in perturbation theory (see e.g. [LL, §38]). In particular, if \(\lambda'(r_\lambda) = 0\), then the vector \(V\varphi(r_\lambda)\) is orthogonal to the vector \(\varphi(r_\lambda)\). The following lemma is a generalisation of this statement.

**Lemma 3.4.** Let \(k \geq 2\) and let \(\varphi(s)\) be an analytic path of eigenvectors of the path \(H_s = H_{r_\lambda} + sV\). The following assertions are equivalent:

1. the path \(\varphi(s)\) has order at least \(k\),
2. the vectors \(V\varphi(r_\lambda), V\varphi'(r_\lambda), \ldots, V\varphi^{(k-2)}(r_\lambda)\) are orthogonal to the vector \(\varphi(r_\lambda)\),
3. the equalities \(\lambda'(r_\lambda) = 0, \ldots, \lambda^{(k-1)}(r_\lambda) = 0\) hold, where \(\lambda(s)\) is an analytic path of eigenvalues of \(H_s\) which corresponds to \(\varphi(s)\),
4. for all \(j = 1, 2, \ldots, k - 1\) \((H_{r_\lambda} - \lambda)\varphi^{(j)}(r_\lambda) = -jV\varphi^{(j-1)}(r_\lambda)\).

**Proof.** Since \(\varphi(r_\lambda)\) is an eigenvector, (i) plainly implies (ii). The implication (iv) \(\Rightarrow\) (i) is also obvious. We shall prove that (ii) implies (iii) and that (iii) implies (iv).

(ii) \(\Rightarrow\) (iii). Differentiating \(k - 1\) times the eigenvalue equation \(H_s\varphi(s) = \lambda(s)\varphi(s)\) gives the equality

\[
(3.3) \quad (k - 1)V\varphi^{(k-2)}(s) + H_s\varphi^{(k-1)}(s) = \sum_{j=0}^{k-1} \binom{k-1}{j} \lambda^{(j)}(s)\varphi^{(k-1-j)}(s).
\]

Here we let \(s = r_\lambda\) and take the scalar product of both sides with the vector \(\varphi(r_\lambda)\). This leads to cancellation of the second summand of the left hand side with the first summand of the right hand side. Hence, we obtain the equality

\[
(3.4) \quad (k - 1)\langle \varphi(r_\lambda), V\varphi^{(k-2)}(r_\lambda) \rangle = \sum_{j=1}^{k-1} \binom{k-1}{j} \lambda^{(j)}(r_\lambda)\langle \varphi(r_\lambda), \varphi^{(k-1-j)}(r_\lambda) \rangle.
\]

If \(k = 2\) then \(\langle \varphi(r_\lambda), V\varphi(r_\lambda) \rangle = \lambda'(r_\lambda)\langle \varphi(r_\lambda), \varphi(r_\lambda) \rangle\). This equality implies the assertion for \(k = 2\). Assume that the claim holds for \(k < n\). Then from (3.4) with \(k = n\), using the induction assumption, we get

\[
(n - 1)\langle \varphi(r_\lambda), V\varphi^{(n-2)}(r_\lambda) \rangle = \lambda^{(n-1)}(r_\lambda)\langle \varphi(r_\lambda), \varphi(r_\lambda) \rangle.
\]

Since by the premise \(\langle \varphi(r_\lambda), V\varphi^{(n-2)}(r_\lambda) \rangle = 0\), this gives \(\lambda^{(n-1)}(r_\lambda) = 0\).

(iii) \(\Rightarrow\) (iv). Letting \(s = r_\lambda\) in (3.3) gives the equality

\[
(3.5) \quad (k - 1)V\varphi^{(k-2)}(r_\lambda) + H_{r_\lambda}\varphi^{(k-1)}(r_\lambda) = \sum_{j=0}^{k-1} \binom{k-1}{j} \lambda^{(j)}(r_\lambda)\varphi^{(k-1-j)}(r_\lambda).
\]

By the premise, the right hand side simplifies to \(\lambda(r_\lambda)\varphi^{(k-1)}(r_\lambda) = \lambda\varphi^{(k-1)}(r_\lambda)\). Hence,

\[
(H_{r_\lambda} - \lambda)\varphi^{(k-1)}(r_\lambda) = -(k - 1)V\varphi^{(k-2)}(r_\lambda).
\]

This proof also shows that if an eigenpath \(\varphi(s)\) has order \(k\), then

\[
(3.5) \quad \langle \varphi(r_\lambda), V\varphi^{(k-1)}(r_\lambda) \rangle = \frac{1}{k} \lambda^{(k)}(r_\lambda)\langle \varphi(r_\lambda), \varphi(r_\lambda) \rangle.
\]

In particular, in this case the number \(\langle \varphi(r_\lambda), V\varphi^{(k-1)}(r_\lambda) \rangle\) is non-zero and real.

**Lemma 3.5.** Let \(k \geq 2\) and let \(\varphi(s)\) be an analytic path of eigenvectors of the path \(H_s = H_{0} + sV\). If \(\varphi(s)\) has order at least \(k\), then
Applying the sliced resolvent $R$ order is a resonance vector of order two. Theorem 2.7 also implies the equality (3.6) for the vector $\varphi$ where the last equality follows from the induction assumption, according to which and Lemma 3.5

Combining this with (3.4), we obtain

$\varphi$ Combining this with the premise $V\varphi$ can be rewritten as ($H$ by the induction assumption and item (iii) of Lemma 3.4 we have

Since $\varphi$ has depth at least $\lambda$ and, since $\varphi(s)$ has order $\lambda$, the relation $V\varphi$ holds. Hence, Theorem 2.7 implies that $\varphi$ is a resonance vector of order two. Theorem 2.7 also implies the equality (3.6) for $j = 2$.

(B) Assume that the claim holds for values of $k$ less than $n$ and let $\varphi(s)$ be an eigenpath of order $\geq n$. By item (iv) of Lemma 3.3 we have

$(H_{r_{\lambda}} - \lambda)\varphi^{(n-1)}(r_{\lambda}) = -(n - 1)V\varphi^{(n-2)}(r_{\lambda}).$

Applying the sliced resolvent $R_{\lambda}(H_{r_{\lambda}})$ to both sides of this equality gives

$$\frac{1}{n-1}\varphi^{(n-1)}(r_{\lambda}) = -S_{r_{\lambda}}\varphi^{(n-2)}(r_{\lambda}) + \text{order 1 vector}.$$ 

By the induction assumption the order of the vector $\varphi^{(n-2)}$ is $n - 1$ and, since $\varphi(s)$ has order $\lambda \geq n$, the relation $V\varphi^{(n-2)}$ holds. Hence, Theorem 2.7 implies that $\varphi^{(n-1)}$ is a vector of order $n$ and that (3.6) holds. □

Lemma 3.6. Let $k \geq 2$. If $\varphi(s)$ is an analytic path of eigenvectors of the path $H_s = H_0 + sV$ such that $\varphi(r_{\lambda})$ has depth at least $k - 1$, then the eigenpath $\varphi(s)$ has order at least $k$.

Proof. If $k = 2$ then the assertion follows from the equivalence for $\varphi(r_{\lambda})$ to have depth at least one and to be $V$-orthogonal to $V_{\lambda}$, see Theorem 2.7. Assume that the claim holds for $k < n$ and let $\varphi(r_{\lambda})$ be of depth at least $n - 1$. For the eigenvalue function $\lambda(s)$ which corresponds to $\varphi(s)$, by the induction assumption and item (iii) of Lemma 3.4 we have $\lambda'(r_{\lambda}) = \cdots = \lambda^{(n-2)}(r_{\lambda}) = 0$. Combining this with (3.4), we obtain

$$(n - 1) \langle \varphi(r_{\lambda}), V\varphi^{(n-2)}(r_{\lambda}) \rangle = \lambda^{(n-1)}(r_{\lambda}) \langle \varphi(r_{\lambda}), \varphi(r_{\lambda}) \rangle.$$ 

Since $\varphi(r_{\lambda})$ has depth at least $n - 1$, there exists a vector $f$ such that $A_{\lambda}^{n-1}f = \varphi(r_{\lambda})$. Hence,

$$\lambda^{(n-1)}(r_{\lambda}) \langle \varphi(r_{\lambda}), \varphi(r_{\lambda}) \rangle = (n - 1) \langle A_{\lambda}^{n-1}f, V\varphi^{(n-2)}(r_{\lambda}) \rangle$$

$$= (n - 1) \langle f, VA_{\lambda}^{n-1}\varphi^{(n-2)}(r_{\lambda}) \rangle$$

$$= 0,$$

where the last equality follows from the induction assumption, according to which and Lemma 3.5 the vector $\varphi^{(n-2)}(r_{\lambda})$ has order $n - 1$ and therefore $A_{\lambda}^{n-1}\varphi^{(n-2)}(r_{\lambda}) = 0$. This gives $\lambda^{(n-1)}(r_{\lambda}) = 0$. Hence, by Lemma 3.4 the proof is complete. □
Since the eigenvalue \( \lambda \) has geometric multiplicity \( m \), there are \( m \) eigenpaths \( \varphi_{\nu}(s) \), \( \nu = 1, 2, \ldots, m \), and their orders we denote by \( d_{\nu} \).

We summarise Lemmas 3.4, 3.5 and 3.6 in the following theorem.

**Theorem 3.7.** For each \( \nu = 1, \ldots, m \) the following assertions are equivalent:

(i) The eigenpath \( \varphi_{\nu}(s) \) has order \( \tilde{d}_{\nu} \).

(ii) The vectors \( V \varphi_{\nu}(r_{\lambda}) \), \( V \varphi'_{\nu}(r_{\lambda}) \), \ldots, \( V \varphi^{(\tilde{d}_{\nu} - 2)}_{\nu}(r_{\lambda}) \) are orthogonal to the vector \( \varphi_{\nu}(r_{\lambda}) \), and the vector \( V \varphi^{(\tilde{d}_{\nu} - 1)}_{\nu}(r_{\lambda}) \) is not.

(iii) The equalities \( \lambda^{(d_{\nu} - 1)}_{\nu}(r_{\lambda}) = 0 \), \ldots, \( \lambda^{(d_{\nu} - 1)}_{\nu}(r_{\lambda}) = 0 \), and the inequality \( \lambda^{(d_{\nu})}_{\nu}(r_{\lambda}) \neq 0 \) hold, where \( \lambda^{(d)}_{\nu}(s) \) is an analytic path of eigenvalues of \( H_{s} \) which corresponds to \( \varphi_{\nu}(s) \).

(iv) For all \( j = 1, 2, \ldots, \tilde{d}_{\nu} - 1 \) the equalities \( (H_{r_{\lambda}} - \lambda) \varphi^{(j)}_{\nu}(r_{\lambda}) = -jV \varphi^{(j - 1)}_{\nu}(r_{\lambda}) \) hold, but it fails for \( j = \tilde{d}_{\nu} \).

(v) For all \( j = 1, 2, \ldots, \tilde{d}_{\nu} - 1 \) the equalities \( A_{\lambda} \varphi^{(j)}_{\nu}(r_{\lambda}) = j \varphi^{(j - 1)}_{\nu}(r_{\lambda}) \) hold, but it fails for \( j = \tilde{d}_{\nu} \).

(vi) \( \varphi_{\nu}(r_{\lambda}) \) is an eigenvector of depth \( \tilde{d}_{\nu} - 1 \).

We shall refer to the positive integer \( \tilde{d}_{\nu} \) as the order of the eigenvalue function \( \lambda^{(d)}_{\nu}(s) \) too. Thus, the order of \( \lambda^{(d)}_{\nu}(s) \) is the smallest of positive integers \( \tilde{d}_{\nu} \), such that \( \lambda^{(d)}_{\nu}(s) \neq 0 \), or

\[
\lambda^{(d)}_{\nu}(s) = \lambda + \varepsilon_{\nu}(s - r_{\lambda})^{\tilde{d}_{\nu}} + O((s - r_{\lambda})^{\tilde{d}_{\nu} + 1}), \quad s \to r_{\lambda},
\]

where \( \varepsilon_{\nu} \neq 0 \).

**Remark 3.8.** The formula (3.6) is independent from a choice of normalisation of the eigenpath \( \varphi(s) \). That is, if \( \psi(s) = a(s)\varphi(s) \) is another eigenpath, where \( a(s) \) is an analytic function with \( a(r_{\lambda}) \neq 0 \), then \( A_{\lambda} \psi^{(j)}(r_{\lambda}) = j \psi^{(j - 1)}(r_{\lambda}) \). This equality also follows from (3.6) and (3.2).

**Remark 3.9.** In this paper we use notation \( \varphi^{(k-1)} \) to denote a vector of order \( k \). This is consistent with the usage of the bracketed superscript index \( (k) \) for the \( k \)-th order derivative. In [Az], the notation \( \varphi^{(k)} \) was used to denote a vector of order \( k \).

Assume that \( H_{r_{\lambda}} \) is a \( \lambda \)-resonance point of multiplicity \( m \). Let \( V \) be a regular direction and let \( H_{s} = H_{r_{\lambda}} + (s - r_{\lambda})V \). Let \( \lambda^{(d)}_{\nu}(s), \nu = 1, \ldots, m \), be eigenvalue functions of \( H_{s} \), which are listed counting multiplicities, and let \( \varphi_{\nu}(s) \), \( 1, \ldots, m \), be corresponding eigenvector functions. If the eigenvalue \( \lambda \) of \( H_{r_{\lambda}} \) is splitting, that is, if the functions \( \lambda_{1}(s), \ldots, \lambda_{m}(s) \) are distinct, then for \( \mu \neq \nu \) the vectors \( \varphi_{\nu}(s) \) and \( \varphi_{\mu}(s) \) are orthogonal as eigenvectors of a self-adjoint operator \( H_{s} \) corresponding to different eigenvalues \( \lambda^{(d)}_{\nu}(s) \) and \( \lambda^{(d)}_{\mu}(s) \). Even if the eigenvalue \( \lambda \) of \( H_{r_{\lambda}} \) is not splitting, it is always possible to choose the eigenvector functions \( \varphi_{\nu}(s) \) to be pairwise orthogonal. Thus, we can and do assume that for any \( s \)

\[
\langle \varphi_{\nu}(s), \varphi_{\mu}(s) \rangle = 0.
\]

In particular, \( \langle \varphi_{\nu}(r_{\lambda}), \varphi_{\mu}(r_{\lambda}) \rangle = 0 \). Hence, a regular direction \( V \) induces a natural orthogonal decomposition of the eigenspace. This orthogonal decomposition of the eigenspace has an additional property given by the following lemma.

**Lemma 3.10.** In the setting given above, if \( \mu \neq \nu \), then \( \langle V \varphi_{\mu}(r_{\lambda}), \varphi_{\nu}(r_{\lambda}) \rangle = 0 \).

**Proof.** Taking derivative of (3.5) we get

\[
\langle \varphi'_{\nu}(s), \varphi_{\mu}(s) \rangle + \langle \varphi_{\nu}(s), \varphi'_{\mu}(s) \rangle = 0.
\]
Further, for any $s$ from some neighbourhood of $r_{\lambda}$ we have

$$0 = \langle \lambda_{\nu}(s) \varphi_{\nu}(s), \varphi_{\mu}(s) \rangle = \langle H_{s} \varphi_{\nu}(s), \varphi_{\mu}(s) \rangle.$$  

Hence,

$$0 = \frac{d}{ds} \langle H_{s} \varphi_{\nu}(s), \varphi_{\mu}(s) \rangle$$

$$= \langle H'_{s} \varphi_{\nu}(s), \varphi_{\mu}(s) \rangle + \langle H_{s} \varphi'_{\nu}(s), \varphi_{\mu}(s) \rangle + \langle H_{s} \varphi_{\nu}(s), \varphi'_{\mu}(s) \rangle.$$  

With $s = r_{\lambda}$ this gives

$$0 = \langle V \varphi_{\nu}(r_{\lambda}), \varphi_{\mu}(r_{\lambda}) \rangle + \langle H_{r_{\lambda}} \varphi'_{\nu}(r_{\lambda}), \varphi_{\mu}(r_{\lambda}) \rangle + \langle H_{r_{\lambda}} \varphi_{\nu}(r_{\lambda}), \varphi'_{\mu}(r_{\lambda}) \rangle$$

$$= \langle V \varphi_{\nu}(r_{\lambda}), \varphi_{\mu}(r_{\lambda}) \rangle + \lambda \langle \varphi'_{\nu}(r_{\lambda}), \varphi_{\mu}(r_{\lambda}) \rangle + \lambda \langle \varphi_{\nu}(r_{\lambda}), \varphi'_{\mu}(r_{\lambda}) \rangle$$

$$= \langle V \varphi_{\nu}(r_{\lambda}), \varphi_{\mu}(r_{\lambda}) \rangle,$$

where the last equality follows from (3.9). □

**Lemma 3.11.** Let $\varphi_{\mu}(s)$ and $\varphi_{\nu}(s)$ be two (orthogonal) eigenpaths of $H_{0} + sV$ with corresponding eigenvalue functions $\lambda_{\mu}(s)$ and $\lambda_{\nu}(s)$. If the eigenpath $\varphi_{\mu}(s)$ has order at least $k$, then for all $j = 0, 1, \ldots, k - 1$, $\langle V \varphi_{\mu}^{(j)}(r_{\lambda}), \varphi_{\nu}(r_{\lambda}) \rangle = 0$.

**Proof.** It is sufficient to prove that $\langle V \varphi_{\mu}^{(k-1)}(r_{\lambda}), \varphi_{\nu}(r_{\lambda}) \rangle = 0$. Since by (3.8) the function $s \mapsto \langle H_{s} \varphi_{\mu}(s), \varphi_{\nu}(s) \rangle$ is zero, for any non-negative integer $k$ the Leibniz rule implies

$$0 = \frac{d^{k}}{ds^{k}} \langle H_{s} \varphi_{\mu}, \varphi_{\nu} \rangle$$

$$= \langle H_{s} \varphi_{\mu}^{(k)}, \varphi_{\nu} \rangle + \sum_{j=0}^{k-1} \binom{k}{j} \langle H_{s} \varphi_{\mu}^{(j)}, \varphi_{\nu}^{(k-j)} \rangle + \sum_{j=0}^{k-1} k \binom{k-1}{j} \langle V \varphi_{\mu}^{(j)}, \varphi_{\nu}^{(k-1-j)} \rangle.$$  

Here we have explicitly separated the first summand $\langle H_{s} \varphi_{\mu}^{(k)}, \varphi_{\nu} \rangle$ since it will be treated differently. After this special treatment, this summand will be returned into the sum.

In this equality we substitute $s$ by $r_{\lambda}$, but nevertheless, we write $\varphi_{\mu}$ instead of $\varphi_{\mu}(r_{\lambda})$, etc. We have $H_{r_{\lambda}} \varphi_{\mu} = \lambda \varphi_{\mu}$. By Lemma 3.4 the premise asserts that the path $\varphi_{\mu}(s)$ has order at least $k$. Hence, the item (iv) of Lemma 3.3 implies that for all $j = 0, 1, \ldots, k - 1$

$$H_{r_{\lambda}} \varphi_{\mu}^{(j)} = \lambda \varphi_{\mu}^{(j)} - jV \varphi_{\mu}^{(j-1)}.$$
Using these equalities, we get from (3.10) with $s = r_\lambda$

\[0 = \langle H_{r_\lambda} \varphi_{\mu}^{(k)}, \varphi_{\nu} \rangle + \sum_{j=0}^{k-1} \binom{k}{j} \langle H_{r_\lambda} \varphi_{\mu}^{(j)}, \varphi_{\nu}^{(k-j)} \rangle + \sum_{j=0}^{k-1} k \binom{k-1}{j} \langle V \varphi_{\mu}^{(j)}, \varphi_{\nu}^{(k-j)} \rangle \]

\[= \lambda \langle \varphi_{\mu}^{(k)}, \varphi_{\nu} \rangle + \lambda \sum_{j=0}^{k-1} \binom{k}{j} \langle \varphi_{\mu}^{(j)}, \varphi_{\nu}^{(k-j)} \rangle - \sum_{j=0}^{k-1} j \binom{k}{j} \langle V \varphi_{\mu}^{(j-1)}, \varphi_{\nu}^{(k-j)} \rangle + \sum_{j=0}^{k-1} k \binom{k-1}{j} \langle V \varphi_{\mu}^{(j)}, \varphi_{\nu}^{(k-j)} \rangle \]

\[= \lambda \sum_{j=0}^{k} \binom{k}{j} \langle \varphi_{\mu}^{(j)}, \varphi_{\nu}^{(k-j)} \rangle - \sum_{j=0}^{k-1} \binom{k}{j} \frac{k!}{(j-1)!(k-j)!} \langle V \varphi_{\mu}^{(j-1)}, \varphi_{\nu}^{(k-j)} \rangle \]

\[+ \sum_{j=0}^{k-1} \frac{k!}{j!(k-1-j)!} \langle V \varphi_{\mu}^{(j)}, \varphi_{\nu}^{(k-j)} \rangle. \]

The first summand of the last expression is equal to $\lambda \frac{d^k}{ds^k} \langle \varphi_{\mu}, \varphi_{\nu} \rangle \big|_{s=r_\lambda}$, which by (3.8) is zero. Hence,

\[0 = -\sum_{j=0}^{k-2} \frac{k!}{j!(k-1-j)!} \langle V \varphi_{\mu}^{(j)}, \varphi_{\nu}^{(k-j)} \rangle + \sum_{j=0}^{k-1} \frac{k!}{j!(k-1-j)!} \langle V \varphi_{\mu}^{(j)}, \varphi_{\nu}^{(k-j)} \rangle \]

\[= k \langle V \varphi_{\mu}^{(k-1)}, \varphi_{\nu} \rangle. \]

\[\square\]

**Theorem 3.12.** If $\varphi_{\mu}(s)$ and $\varphi_{\nu}(s)$ are two eigenpaths with orders $\tilde{d}_{\mu}$ and $\tilde{d}_{\nu}$ respectively, then for any $j = 0, 1, 2, \ldots, \tilde{d}_{\mu} - 1$ and $k = 0, 1, 2, \ldots, \tilde{d}_{\nu} - 1$

\[\langle V \varphi_{\mu}^{(j)}(r_\lambda), \varphi_{\nu}^{(k)}(r_\lambda) \rangle = 0.\]

**Proof.** As in Lemma 3.11, it is sufficient to prove $\langle V \varphi_{\mu}^{(\tilde{d}_{\mu}-1)}, \varphi_{\nu}^{(\tilde{d}_{\nu}-1)} \rangle = 0.$

Let $l = \tilde{d}_{\mu} + \tilde{d}_{\nu} - 2$. Since eigenpaths $\varphi_{\mu}(s)$ and $\varphi_{\nu}(s)$ are distinct we have

\[\langle H_s \varphi_{\mu}(s), \varphi_{\nu}(s) \rangle = \lambda_{\mu}(s) \langle \varphi_{\mu}(s), \varphi_{\nu}(s) \rangle = 0,\]

where $\lambda_{\mu}(s)$ is the eigenvalue function for $\varphi_{\mu}(s)$. Leibniz rule gives the equality

\[0 = \frac{d^{l+1}}{ds^{l+1}} \langle H_s \varphi_{\mu}, \varphi_{\nu} \rangle \]

\[= \sum_{j=0}^{l+1} \binom{l+1}{j} \langle H_s \varphi_{\mu}^{(j)}, \varphi_{\nu}^{(l+1-j)} \rangle + \sum_{j=0}^{l} (l+1) \binom{l}{j} \langle V \varphi_{\mu}^{(j)}, \varphi_{\nu}^{(l-j)} \rangle. \]
We transform the first summand as follows:
\[
\sum_{j=0}^{\tilde{d}_\nu} \binom{l+1}{j} \left( H_{s \nu}^{(j)} , \varphi_{\nu}^{(l+1-j)} \right) = \sum_{j=\tilde{d}_\nu}^{l+1} \binom{l+1}{j} \left( \varphi_{\nu}^{(j)} , H_{s \nu}^{(l+1-j)} \right) + \sum_{j=0}^{\tilde{d}_\nu - 1} \binom{l+1}{j} \left( H_{s \nu}^{(j)} , \varphi_{\nu}^{(l+1-j)} \right).
\]

Here we let \( s = r_\lambda \) and apply the equalities
\[
H_{r_\lambda} \varphi_{\nu}^{(l+1-j)} = \lambda \varphi_{\nu}^{(l+1-j)} - (l + 1 - j) V \varphi_{\nu}^{(l-j)}, \quad H_{r_\lambda} \varphi_{\mu}^{(j)} = \lambda \varphi_{\mu}^{(j)} - j V \varphi_{\mu}^{(j-1)},
\]
which follow from the premise that the eigenpaths \( \varphi_{\nu}(s) \) and \( \varphi_{\mu}(s) \) have orders \( \tilde{d}_\nu \) and \( \tilde{d}_\mu \) respectively, according to Theorem 3.7. This gives
\[
(E) := \sum_{j=0}^{l+1} \binom{l+1}{j} \left( H_{r_\lambda} \varphi_{\mu}^{(j)} , \varphi_{\nu}^{(l+1-j)} \right)
= \sum_{j=\tilde{d}_\nu}^{l+1} \binom{l+1}{j} \lambda \left( \varphi_{\mu}^{(j)} , \varphi_{\nu}^{(l+1-j)} \right) + \sum_{j=0}^{\tilde{d}_\nu - 1} \binom{l+1}{j} \lambda \left( \varphi_{\mu}^{(j)} , \varphi_{\nu}^{(l+1-j)} \right)
- \sum_{j=\tilde{d}_\nu}^{l+1} (l + 1 - j) \binom{l+1}{j} \left( \varphi_{\mu}^{(j)} , V \varphi_{\nu}^{(l-j)} \right) - \sum_{j=0}^{\tilde{d}_\nu - 1} j \binom{l+1}{j} \left( V \varphi_{\mu}^{(j-1)} , \varphi_{\nu}^{(l+1-j)} \right).
\]

This expression we can rewrite as follows:
\[
(E) = \lambda \sum_{j=0}^{l+1} \binom{l+1}{j} \left( \varphi_{\mu}^{(j)} , \varphi_{\nu}^{(l+1-j)} \right)
- \sum_{j=\tilde{d}_\nu}^{l+1} \frac{(l+1)!}{j!(l-j)!} \left( \varphi_{\mu}^{(j)} , V \varphi_{\nu}^{(l-j)} \right) - \sum_{j=1}^{\tilde{d}_\mu - 1} \frac{(l+1)!}{(j-1)!(l+1-j)!} \left( V \varphi_{\mu}^{(j-1)} , \varphi_{\nu}^{(l+1-j)} \right)
= 0 - \sum_{j=\tilde{d}_\nu}^{l+1} \frac{(l+1)!}{j!(l-j)!} \left( \varphi_{\mu}^{(j)} , V \varphi_{\nu}^{(l-j)} \right) - \sum_{j=0}^{\tilde{d}_\mu - 2} \frac{(l+1)!}{j!(l-j)!} \left( V \varphi_{\mu}^{(j)} , \varphi_{\nu}^{(l-j)} \right).
\]

Here the first summand is zero for the same reason as in Lemma 3.11. Combining this equality with (3.11) gives \( \left( V \varphi_{\mu}^{(\tilde{d}_\nu - 1)} , \varphi_{\nu}^{(\tilde{d}_\nu - 1)} \right) = 0. \]

In subsection 3.7 we give another proof of Theorem 3.12.

3.2. On ground eigenvalue. If \( H_{r_\lambda} \) is bounded below then its smallest eigenvalue, if it exists, is called the ground eigenvalue.

**Proposition 3.13.** Let \( \lambda \) be a simple eigenvalue of \( H_{r_\lambda} \). The order of the triple \( (\lambda; H_{r_\lambda}, V) \) is equal to \( d \) if and only if
\[
a = 0, \quad \left( R_\lambda (\tilde{H}_{r_\lambda}) v, v \right) = 0, \quad \ldots, \quad \left( \left( R_\lambda (\tilde{H}_{r_\lambda}) V \right)^{d-3} R_\lambda (\tilde{H}_{r_\lambda}) v, v \right) = 0
\]
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and

$$\left\langle \left( R_\lambda(\hat{H}_{r_\lambda})V \right)^{d-2} R_\lambda(\hat{H}_{r_\lambda})v, v \right\rangle \neq 0.$$

**Proof.** Since $\mathcal{V}_\lambda$ is one-dimensional, the matrix $v$ is equal to a vector $V\chi$, where $\chi$ is an eigenvector. Further, since $\lambda$ is a simple eigenvalue, by Theorem 2.7 the order is equal to $d$ if and only if

$$\chi \perp V\chi, \ S_\lambda\chi \perp V\chi, \ldots, S_{d-2}\chi \perp V\chi,$$

and $S_{d-1}\chi$ is not orthogonal to $V\chi$. It is left to note that

$$\left\langle \left( R_\lambda(\hat{H}_{r_\lambda})V \right)^j R_\lambda(\hat{H}_{r_\lambda})v, v \right\rangle = \left\langle S_{d+1}^j\chi, V\chi \right\rangle.$$

\[\square\]

**Theorem 3.14.** If $\lambda$ is a non-degenerate ground eigenvalue, then order of the corresponding resonance point is at most two.

**Proof.** Since $\lambda$ is the smallest eigenvalue, the sliced resolvent $R_\lambda(\hat{H}_{r_\lambda})$ is a strictly positive operator. Hence, if the order $d$ is greater than two, then Proposition 3.13 gives

$$\left\langle R_\lambda(\hat{H}_{r_\lambda})v, v \right\rangle = 0$$

and combining this with $R_\lambda(\hat{H}_{r_\lambda}) > 0$ gives $R_\lambda(\hat{H}_{r_\lambda})v = 0$, which is a contradiction. \[\square\]

If the order of a resonance point is even then the corresponding eigenvalue makes a U-turn. One may ask whether this is a left or a right U-turn. At the ground eigenvalue, the following theorem answers this question.

**Theorem 3.15.** If $\lambda$ is a non-degenerate ground eigenvalue and the order of a direction $V$ is equal to 2, then the corresponding U-turn is a left U-turn.

**Proof.** Since $d = 2$, using Theorem 3.7 one infers that

$$0 \neq \left\langle \varphi(r_\lambda), V\varphi'(r_\lambda) \right\rangle = -\left\langle \varphi(r_\lambda), VS_\lambda\varphi(r_\lambda) \right\rangle = -\left\langle \chi, VR_\lambda(\hat{H}_{r_\lambda})V\chi \right\rangle.$$

Since $\lambda$ is a ground eigenvalue, the operator $R_\lambda(\hat{H}_{r_\lambda})$ is positive. Combining this with the previous equality and inequality gives $\left\langle \varphi(r_\lambda), V\varphi'(r_\lambda) \right\rangle < 0$. It follows from this and formula (3.5) that $\lambda''(r_\lambda) < 0$. Hence, the U-turn is the left one. \[\square\]

4. Characterisation of order $k$ directions

Recall that a resonant path $H(s)$ is an analytic path of self-adjoint operators in an affine space $\mathcal{A}$ such that $\lambda$ is an eigenvalue of all $H(s)$. For any resonant path $H(s)$ there exists (see [Ka]) an analytic path of eigenvectors $\chi(s)$, that is,

$$H(s)\chi(s) = \lambda\chi(s).$$

Such an analytic path of eigenvectors $\chi(s)$ we shall call a resonant eigenpath. We say that a resonant path $H(s)$ is simple, if at least one point on that path is simple. If a resonant path is simple, then all its points except a discrete set are simple. Further, a simple resonant path has a unique resonant eigenpath up to a multiplicative constant, see e.g. [Ka].
4.1. **Tangent directions have high orders.** We say that a direction $V$ is **tangent** to the resonance set $\mathcal{R}(\lambda)$ at $H_{r_{\lambda}}$ to order at least $k$, if there exists a resonant path $H(s) \subset \mathcal{R}(\lambda)$ such that for some (necessarily real) numbers $c_2, c_3, \ldots, c_{k-1}$

$$H(s) = H_{r_{\lambda}} + (s - r_{\lambda})V + \sum_{j=2}^{k-1} c_j(s - r_{\lambda})^j V + O((s - r_{\lambda})^k), \quad s \to r_{\lambda}.$$  

In this case we also say that the path $H(s)$ is tangent to $V$ at $H(r_{\lambda}) = H_{r_{\lambda}}$ to order at least $k$. If $k$ is the greatest positive integer with this property, then we say that $V$ is tangent at $H_{r_{\lambda}}$ to order $k$.

We say that a direction $V$ is **tangent** at $H_{r_{\lambda}}$ if $V$ is tangent to order at least 2. If a direction $V$ is tangent only to order 1 at $H_{r_{\lambda}} \in \mathcal{R}(\lambda)$, then we say that $V$ is **transversal** at $H_{r_{\lambda}}$.

**Lemma 4.1.** Assume Assumption~\ref{Assumption 1.1} Let $H_{r_{\lambda}}$ be a resonant point and let $V$ be a regular direction. If the direction $V$ is tangent to the resonance set $\mathcal{R}(\lambda)$ at $H_{r_{\lambda}}$ then order of $V$ is at least 2.

**Proof.** As usual, we let $H_s = H_0 + sV$. Since $V$ is tangent to $\mathcal{R}(\lambda)$ at $H_{r_{\lambda}}$, there exists a resonant path $\{H(s)\}$ such that $H(r_{\lambda}) = H_{r_{\lambda}}$ and $H'(r_{\lambda}) = V$. Let $\chi(s)$ be a corresponding resonant eigenpath. Differentiating the eigenvalue equation \eqref{4.1} we get

$$H'(s)\chi(s) + H(s)\chi'(s) = \lambda\chi'(s).$$

Letting here $s = r_{\lambda}$ gives $V\chi(r_{\lambda}) + H_{r_{\lambda}}\chi'(r_{\lambda}) = \lambda\chi'(r_{\lambda})$. This equality can be rewritten as follows:

$$(1 + (r_{\lambda} - s)R_{\lambda}(H_s)V)\chi'(r_{\lambda}) = -R_{\lambda}(H_s)V\chi(r_{\lambda}),$$

where $s$ is any real number such that the operator $H_s - \lambda$ has bounded inverse (such real numbers $s$ exist since by the premise $V$ is a regular direction). The eigenvalue equation $H_{r_{\lambda}}\chi(r_{\lambda}) = \lambda\chi(r_{\lambda})$ is equivalent to $(1 + (r_{\lambda} - s)R_{\lambda}(H_s)V)\chi(r_{\lambda}) = 0$. Combining this equality with previous one gives

$$(1 + (r_{\lambda} - s)R_{\lambda}(H_s)V)\chi'(r_{\lambda}) = (r_{\lambda} - s)^{-1}\chi(r_{\lambda}).$$

Therefore, $(1 + (r_{\lambda} - s)R_{\lambda}(H_s)V)^2\chi'(r_{\lambda}) = 0$. Hence, if $V$ is tangent to $\mathcal{R}(\lambda)$, then $\chi'(r_{\lambda})$ is a vector of order 2 and therefore the direction $V$ has order at least 2. \hfill $\square$

**Theorem 4.2.** Assume Assumption~\ref{Assumption 1.1} Let $k \geq 1$, let $H_{r_{\lambda}}$ be a resonance point and $V$ be a regular direction at $H_{r_{\lambda}}$. If $H(s)$ is a resonant path tangent to $V$ at $H_{r_{\lambda}}$ to order at least $k$ and if $\chi(s)$ is a corresponding analytic resonant eigenpath, then

(i) vectors $\chi(r_{\lambda}), \chi'(r_{\lambda}), \ldots, \chi^{(k-1)}(r_{\lambda})$ have orders respectively 1, 2, $\ldots$, $k$,

(ii) the direction $V$ has order at least $k$,

(iii) for any $j = 1, 2, \ldots, k$

$$A_{\lambda}(r_{\lambda})\chi^{(k-1)}(r_{\lambda}) = (k - 1)\chi^{(k-2)}(r_{\lambda}) + \sum_{j=2}^{k-1} j!(k - 1) c_j \chi^{(k-1-j)}(r_{\lambda}),$$

where the numbers $c_2, \ldots, c_k$ are as in \eqref{4.2}, and

(iv) the eigenvector $\chi(r_{\lambda})$ has depth at least $k - 1$.

**Proof.** (i) We prove this item using induction on $k$.

That the vector $\chi(r_{\lambda})$ has order 1 is trivial. That in case of $k \geq 2$ the vector $\chi'(r_{\lambda})$ has order 2 was proved in Lemma~\ref{Lemma 4.1}. Now, assuming that the assertion holds for $k = n - 1$, we prove it for $k = n$; still, we write $k$ instead of $n$. 


Since a path \( H(s) \) which is tangent to \( V \) at \( H_{r_\lambda} \) to order at least \( k \) is also tangent to \( V \) to order at least \( k - 1 \), it follows from the induction assumption that the vectors

\[
\chi(r_\lambda), \ \chi'(r_\lambda), \ \chi''(r_\lambda), \ldots, \ \chi^{(k-2)}(r_\lambda)
\]

have orders 1, 2, \ldots, \( k - 1 \) respectively. Differentiating \( k - 1 \) times the eigenvalue equation (4.1) gives

\[
(\sum_{j=0}^{k-1} \binom{k-1}{j} H^{(j)}(s) \chi^{(k-j-1)}(s)) = \lambda \chi^{(k-1)}(s).
\]

Since \( H(s) \) is tangent to \( V \) at \( H_{r_\lambda} \) to order \( k \), the operators \( H'(r_\lambda), \ldots, H^{(k-1)}(r_\lambda) \) are co-linear to \( V \). Therefore, it follows from the previous equality, taken with \( s = r_\lambda \), that

\[
H_{r_\lambda} \chi^{(k-1)}(r_\lambda) + (k-1)V \chi^{(k-2)}(r_\lambda) + \tilde{c}_2 \chi^{(k-3)}(r_\lambda) + \ldots + \tilde{c}_{k-1} V \chi(r_\lambda) = \lambda \chi^{(k-1)}(r_\lambda),
\]

where

\[
\tilde{c}_j = \binom{k-1}{j} j! c_j
\]

and the numbers \( c_j \) are from (1.2). Adding to both sides of (4.4) the vector \( (s - r_\lambda) V \chi^{(k-1)}(r_\lambda) \) gives

\[
(H_s - \lambda) \chi^{(k-1)}(r_\lambda) + (k-1)V \chi^{(k-2)}(r_\lambda) + \tilde{c}_2 \chi^{(k-3)}(r_\lambda) + \ldots + \tilde{c}_{k-1} V \chi(r_\lambda) = (s - r_\lambda) V \chi^{(k-1)}(r_\lambda).
\]

Since \( V \) is regular, there exists \( s \in \mathbb{R} \) such that the inverse \( R_{\lambda}(H_s) = (H_s - \lambda)^{-1} \) exists. Multiplying both sides of the last equality by \( R_{\lambda}(H_s) \), we obtain the equality

\[
\chi^{(k-1)}(r_\lambda) + R_{\lambda}(H_s)V \left( (k-1) \chi^{(k-2)}(r_\lambda) + \tilde{c}_2 \chi^{(k-3)}(r_\lambda) + \ldots + \tilde{c}_{k-1} \chi(r_\lambda) \right)
\]

\[
= (s - r_\lambda) R_{\lambda}(H_s)V \chi^{(k-1)}(r_\lambda),
\]

which can be rewritten as

\[
(1 + (r_\lambda - s) R_{\lambda}(H_s)V) \chi^{(k-1)}(r_\lambda) = -R_{\lambda}(H_s) V \left( (k-1) \chi^{(k-2)}(r_\lambda) + \tilde{c}_2 \chi^{(k-3)}(r_\lambda) + \ldots + \tilde{c}_{k-1} \chi(r_\lambda) \right).
\]

By the induction assumption, the vector in the last pair of brackets has order \( k - 1 \). Since by the definition (1.12) of vectors of order \( k \) the operator \( R_{\lambda}(H_s)V \) preserves order of vectors, the vector in the right hand side has order \( k - 1 \) too. Since by the same definition (1.12) the operator \( 1 - s R_{\lambda}(H_s)V \) decreases order of vectors by 1, it follows that the vector \( \chi^{(k-1)}(r_\lambda) \) has order \( k \).

(ii) This item follows from (i).

(iii) Taking contour integrals of both sides of (4.5) over a contour enclosing the resonance point \( s = r_\lambda \) and using the formulas (1.6) and (1.7) we obtain

\[
A_{\lambda}(r_\lambda) \chi^{(k-1)}(r_\lambda) = (k-1) \chi^{(k-2)}(r_\lambda) + \tilde{c}_2 \chi^{(k-3)}(r_\lambda) + \ldots + \tilde{c}_{k-1} \chi(r_\lambda).
\]

(iv) This item follows immediately from (iii).

If a resonant path \( H(s) \) is tangent to \( V \) to order \( k \) at a resonant point \( H_{r_\lambda} \), then changing the parameter \( s \) if necessary we can always make the operators \( H''(r_\lambda), \ldots, H^{(k-1)}(r_\lambda) \) equal to zero, so that the path \( H(s) \) takes the form

\[
H(s) = H_{r_\lambda} + (s - r_\lambda)V + O((s - r_\lambda)^k), \ s \to r_\lambda.
\]

For example, assuming that \( r_\lambda = 0 \), to eliminate the coefficient \( c_2 \) one can replace \( s \) by \( t - c_2 t^2 \). Once \( c_2 \) is eliminated, the change of variables \( s = t - c_3 t^3 \) eliminates \( c_3 \), and so on.

**Definition 4.3.** A path of the form (4.6) will be called standard.
According to Theorem 4.2, with a resonant path tangent to order \( k \) we can associate a set of resonance vectors \( \chi_0, \ldots, \chi_{k-1} \) of respective orders 1, \ldots, \( k \); namely, the first \( k \) coefficients
\[
\chi_j = \frac{1}{j!} \chi^{(j)}(r_\lambda), \quad j = 0, 1, 2, \ldots
\]
of the Taylor expansion of a resonant eigenpath \( \chi(s) \).

**Proposition 4.4.** Assume Assumption 1.1. Let \( V \) be a regular direction at \( H_{r_\lambda} \) and let \( H(s) \) be a resonant path tangent to \( V \) at \( H_{r_\lambda} \) to order \( k \). The path \( H(s) \) is standard if and only if for all \( j = 1, 2, \ldots, k - 1 \) there holds the equality
\[
A_\lambda(r_\lambda) \chi_j = \chi_{j-1},
\]
where \( \chi(s) \) is a corresponding analytic path of eigenvectors.

**Proof.** This follows from item (iii) of Theorem 4.2. \( \square \)

4.2. **A resonance curve associated with an eigenvector.** Assume that \( V \) is a regular direction at a resonance point \( H_{r_\lambda} \). We choose another direction \( W \) and consider the real affine plane \( \alpha = H_{r_\lambda} + \mathbb{R}V + \mathbb{R}W \) in the affine space \( \mathcal{A} \) determined by the point \( H_{r_\lambda} \) and the directions \( V \) and \( W \). It is possible that the intersection of \( \alpha \) and the resonance set \( \mathcal{R}(\lambda) \) in a neighbourhood of \( H_{r_\lambda} \) consists of only one point \( H_{r_\lambda} \). To avoid this one can choose \( W \) to be transversal to the resonance set. Since, as we shall see later, the resonance set has co-dimension 1, this will ensure that the intersection of the plane with the resonance set is a curve.

**Theorem 4.5.** Assume Assumption 1.1. Further, let \( H_{r_\lambda} \) be a resonance point and let \( V \) be a regular direction. Let \( \chi \) be an eigenvector of \( H_{r_\lambda} \) and let \( W = \langle \chi, \cdot \rangle \chi \). Then the intersection of the real affine plane
\[
\alpha := H_{r_\lambda} + \mathbb{R}V + \mathbb{R}W
\]
with a sufficiently small neighbourhood of the point \( H_{r_\lambda} \) in \( \mathcal{R}(\lambda) \setminus \{ H_{r_\lambda} + \mathbb{R}W \} \) consists of one and only one one-dimensional analytic curve \( \gamma \). Moreover, this curve is simple.

**Proof.** First we show that any neighbourhood of \( H_{r_\lambda} \) has a resonance point in the affine plane \( \alpha \) which is not in \( H_{r_\lambda} + \mathbb{R}W \). Assume the contrary. Then there exists a convex neighbourhood \( O \) of \( H_{r_\lambda} \) which does not have a resonance point outside the line \( H_{r_\lambda} + \mathbb{R}W \). Fix small enough \( s_0 \) so that \( H_{r_\lambda} \pm s_0W \) are in the neighbourhood \( O \). The operators \( H_{r_\lambda} + s_0W \) and \( H_{r_\lambda} - s_0W \) have the eigenvalue \( \lambda \) of multiplicity \( m - 1 \) and non-degenerate eigenvalues \( \lambda + s_0 \) and \( \lambda - s_0 \) respectively.
Since \(V\) is regular, for all small enough non-zero \(\varepsilon\) the operators \(H_{r\lambda} + \varepsilon V\) are non-resonant. We choose a small enough \(\varepsilon_0 > 0\) so that all operators \(H_{r\lambda} + sW + \varepsilon V, (s, \varepsilon) \in [-s_0, s_0] \times [-\varepsilon_0, \varepsilon_0]\), are in the neighbourhood \(O\). We also choose \(\varepsilon_0\) small enough so that the perturbed eigenvalues \(\lambda + s_0 + \ldots\) and \(\lambda - s_0 + \ldots\) of the perturbed operators \(H_{r\lambda} \pm s_0 W + \varepsilon_0 V\) are on the same side of \(\lambda\) as the original non-perturbed eigenvalues \(\lambda + s_0\) and \(\lambda - s_0\) respectively. The other \(m - 1\) eigenvalues of the operators

\[
H_{r\lambda} \pm s_0 W + \varepsilon_0 V|_{\varepsilon=0}
\]

will move away from \(\lambda\) as \(\varepsilon\) becomes non-zero, since by assumption there are no resonance points in the neighbourhood \(O\) except the line \(H_{r\lambda} + \mathbb{R}W\). Thus, when we deform \(H_{r\lambda} - s_0 W + \varepsilon_0 V\) to \(H_{r\lambda} + s_0 W + \varepsilon_0 V\) by changing \(-s_0\) to \(s_0\), the number of eigenvalues on each side of \(\lambda\) changes. Hence, for some \(s\) between \(-s_0\) and \(s_0\) the operator \(H_{r\lambda} + sW + \varepsilon_0 V\) must have \(\lambda\) as an eigenvalue. This operator is therefore resonant and belongs to the neighbourhood. This is a contradiction with our assumption.

The assertion that there can be only one simple analytic resonance curve in a neighbourhood of \(H_{r\lambda}\) in the plane \(\alpha\) follows from the fact that the operator \(W\) is non-negative and hence an eigenvalue of \(H_{r\lambda} + sW + \varepsilon_0 V\) can only move in the positive direction as \(s\) increases. □

**Definition 4.6.** The curve \(\gamma\) which exists and is uniquely determined by Theorem 4.5 we denote by \(\gamma_{\lambda}\), or by \(\gamma_{\lambda}(\lambda, H_{r\lambda}, V)\), if necessary.

Since the resonant curve \(\gamma_{\lambda}\) is simple, there exists only one (up to scaling) analytic resonant eigenpath \(\chi(s)\) corresponding to \(\gamma_{\lambda}\). Therefore, to an eigenvector \(\chi\) we can assign another eigenvector \(\chi(r_{\lambda})\).

**Theorem 4.7.** Assume Assumption 1.1 and let \(H_{r\lambda}, V\) and \(\chi\) be as in Theorem 4.5. If \(\chi(s)\) is a resonant eigenpath corresponding to the curve of operators \(\gamma_{\lambda}\) then the vectors \(\chi(r_{\lambda})\) and \(\chi\) are co-linear.

**Proof.** If for \(H_{r\lambda}\) the eigenvalue \(\lambda\) is simple, then the assertion is trivial. We shall reduce the general case to the case of simple eigenvalue \(\lambda\), by slightly perturbing \(H_{r\lambda}\).

Let \(G_t = H_{r\lambda} + t W\), where \(W\) as in Theorem 4.5. The operator \(G_t\) has an eigenvalue \(\lambda\) of multiplicity \(m - 1\) and, for all small enough \(t\), it has an eigenvalue \(\lambda + t\) of multiplicity 1 corresponding to the eigenvector \(\chi\). Since \(\lambda + t\) is a simple eigenvalue of \(G_t\), the curve

\[
\gamma_{\lambda}(\lambda + t, G_t, V)
\]

has a resonant eigenpath \(\chi_t(s)\) which starts at \(\chi\). When \(t\) is deformed to zero, the curve \(\gamma_{\lambda}(\lambda + t, G_t, V)\) is analytically deformed to \(\gamma_{\lambda}(\lambda, H_{r\lambda}, V)\). Thus, the resonant eigenpath \(\chi_t(s)\) gets deformed to \(\chi_0(s)\) in such a way that the base of this resonant eigenpath stays co-linear to \(\chi\) for all \(t\). Hence, in the final position \(\chi_0(s)\) of this deformation the base of \(\chi_0(s)\) will still be co-linear to \(\chi\). Finally, it is left to note that any two eigenpaths corresponding to \(\gamma_{\lambda}\) are co-linear, since \(\gamma_{\lambda}\) is a simple curve. Hence, \(\chi(s)\) starts at \(\chi\) too. □

### 4.3. High order directions are tangent.
Now we are going to prove the reverse to Theorem 4.2 if \(V\) is a direction of order \(k\) at \(H_{r\lambda}\) then \(V\) is tangent to \(\mathcal{R}(\lambda)\) at \(H_{r\lambda}\) to order \(k\).

**Theorem 4.8.** Assume Assumption 1.1 and let \(k\) be an integer greater than 1. If \(\chi_0\) is an eigenvector of depth at least \(k - 1\) for the triple \((\lambda; H_{r\lambda}, V)\), then

1. the direction \(V\) is tangent to the resonance curve \(\gamma_{\chi_0}\) to order at least \(k\),
2. in the Taylor expansion

\[
\chi(s) = \sum_{j=0}^{\infty} (s - r_{\lambda})^j \chi_j
\]
of a resonant eigenpath $\chi(s)$ corresponding to $\gamma_{\chi_0}$, the vectors $\chi_0, \chi_1, \ldots, \chi_{k-1}$ have orders respectively $1, 2, \ldots, k$.

(3) for any resonant eigenpath (4.9) corresponding to $\gamma_{\chi_0}$ and for all $j = 1, 2, \ldots, k - 1$ the vector $A_{\lambda}X_j$ is a linear combination of vectors $\chi_0, \chi_1, \ldots, \chi_{j-1}$. Moreover, if the parametrisation of the curve $\gamma_{\chi_0}$ is standard then $A_{\lambda}X_j = \chi_{j-1}$.

(4) the vectors $\chi_0, \chi_1, \ldots, \chi_{k-2}$, have depth at least one and are $V$-orthogonal to $\mathcal{V}_\lambda$.

Proof. Let $H(s)$ be a parametrisation of the resonance curve $\gamma_{\chi_0}$. Since the curve $\gamma_{\chi_0}$ is the intersection of the resonance set by the plane $H_{r_{\lambda}} + \mathbb{R}V + \mathbb{R}W$, where $W = \langle \chi_0, \cdot \rangle \chi_0$, the Taylor expansion of the path $H(s)$ has the form

\begin{equation}
H(s) = H_{r_{\lambda}} + \sum_{j=1}^{\infty} (s - r_{\lambda})^j (\alpha_j V + \beta_j W).
\end{equation}

By Theorem 4.7 a resonant eigenpath $\chi(s)$ corresponding to this path has Taylor series

\begin{equation}
\chi(s) = \chi_0 + (s - r_{\lambda})\chi_1 + (s - r_{\lambda})^2 \chi_2 + \ldots
\end{equation}

which starts at the vector $\chi_0$, that is, $\chi(r_{\lambda}) = \chi_0$. Comparing the coefficients of $s - r_{\lambda}$ on both sides of the eigenvalue equation (4.1) gives

\begin{equation}
(H_{r_{\lambda}} - \lambda)\chi_1 = - (\alpha_1 V + \beta_1 W)\chi_0.
\end{equation}

The vector $(H_{r_{\lambda}} - \lambda)\chi_1$ is orthogonal to the eigenspace $\mathcal{V}_\lambda$ and in particular it is orthogonal to $\chi_0$. Hence,

\begin{equation}
\langle \chi_0, (\alpha_1 V + \beta_1 W)\chi_0 \rangle = 0.
\end{equation}

Since $\chi_0$ has depth at least one, it follows from Theorem 2.7 that $\langle \chi_0, V\chi_0 \rangle = 0$. Combining this equality with previous one implies that $\beta_1 = 0$, and therefore, $H(s)$ is tangent to $V$ at $H_{r_{\lambda}}$ (to order at least 2). So, we have $(H_{r_{\lambda}} - \lambda)\chi_1 = - \alpha_1 V\chi_0$. This equality implies that

\begin{equation}
\chi_1 = - \alpha_1 S_{\lambda}\chi_0 + \text{order 1 vector}.
\end{equation}

Since $\chi_0$ has depth at least 1, the vector $V\chi_0$ is orthogonal to $\mathcal{V}_\lambda$. Hence, by Theorem 2.7 the previous equality implies $A_{\lambda}\chi_1 = \alpha_1 \chi_0$. In particular, $\chi_1$ is a vector of order 2. Further, if the parametrisation of $\gamma_{\chi_0}(s)$ is standard, then $\alpha_1 = 1$.

We have proved the theorem in case of $k = 2$. We proceed by induction on $k$. So, assume that the claim holds for values of $k$ not greater than $n$ and let $\chi_0$ be an eigenvector of depth $\geq n$.

Differentiating $n$ times the eigenvalue equation $H(s)\chi(s) = \lambda \chi(s)$ we obtain

\begin{equation}
\sum_{j=0}^{n} \binom{n}{j} H^{(j)}(s) \chi^{(n-j)}(s) = \lambda \chi^{(n)}(s).
\end{equation}

Letting $s = r_{\lambda}$ and replacing $\chi^{(j)}(r_{\lambda})/j!$ by $\chi_j$ gives

\begin{equation}
H_{r_{\lambda}} \chi_n + \sum_{j=1}^{n} (\alpha_j V + \beta_j W)\chi_{n-j} = \lambda \chi_n.
\end{equation}

By the induction assumption, we have

\begin{equation}
\beta_1 = \ldots = \beta_{n-1} = 0.
\end{equation}

Hence,

\begin{equation}
(H_{r_{\lambda}} - \lambda)\chi_n + \sum_{j=1}^{n-1} \alpha_j V\chi_{n-j} + (\alpha_n V + \beta_n W)\chi_0 = 0.
\end{equation}
Since $\chi_0$ has depth at least $n$, for some vector $g$ we have $\chi_0 = A^n g$. Since, by the induction assumption, $\chi_j$, $j = 0, 1, \ldots, n-1$, is a vector of order $j+1$, it follows that for all $j = 0, 1, \ldots, n-1$

$$\langle \chi_0, V\chi_j \rangle = \langle A^n g, V\chi_j \rangle = \langle g, V A^n \chi_j \rangle = 0.$$ 

Hence, it follows from (4.12) by taking the scalar product of the left hand side and $\chi_0$ that

\begin{equation}
\beta_n = 0
\end{equation}

and so

$$(H_{r\lambda} - \lambda)\chi_n + \sum_{j=1}^{n} \alpha_j V\chi_{n-j} = 0.$$ 

It follows from (4.10), (4.11) and (4.13) that $V$ is tangent to $\gamma_{\chi_0}$ to order at least $n+1$. Further, the vector $(H_{r\lambda} - \lambda)\chi_n$ is orthogonal to $V_{\lambda}$ and the vectors $V\chi_0$, $\ldots$, $V\chi_{n-2}$ are also orthogonal to $V_{\lambda}$ by the induction assumption (since the vectors $\chi_0, \ldots, \chi_{n-2}$ have depth at least one). Hence, according to the last equality, so is the vector $V\chi_{n-1}$. According to the last equality, we also have

$$\chi_n + \sum_{j=1}^{n} \alpha_j S_{\lambda} \chi_{n-j} = \text{order 1 vector}.$$ 

Hence, by Theorem 2.7 it follows that

$$A_{\lambda} \chi_n = \sum_{j=1}^{n} \alpha_j \chi_{n-j}.$$ 

Since, by the induction assumption, the vectors $\chi_0, \ldots, \chi_{n-2}$ have depth at least one, the last equality implies that the vector $\chi_{n-1}$ also has depth at least one.

Further, if the parametrisation of $\gamma_{\chi_0}(s)$ is standard, then $\alpha_2 = \ldots = \alpha_n = 0$ and $\alpha_1 = 1$ and therefore $A_{\lambda} \chi_n = \chi_{n-1}$. \hfill \Box

Theorem 4.8 combined with Theorem 4.2 proves the following theorem.

**Theorem 4.9.** Assume Assumption 1.1. A regular direction at any resonance point $H_{r\lambda}$ is tangent to order at least $k$ if and only if the order of the direction is at least $k$.

Theorem 4.9 has the following corollary.

**Theorem 4.10.** Assume Assumption 1.1. A regular direction at a resonance point is simple if and only if it is transversal.

The last two theorems give geometric interpretation of order of a regular direction in the case where $\lambda$ is outside the essential spectrum.

5. Resonance points as functions of the spectral parameter

In this section we study resonance points $r_z$ as functions of the spectral parameter $z$. To stress on this, we will often write $r(z)$ instead of $r_z$. 


5.1. **Order of a resonance function** $r_e = r(z)$. In this subsection we consider the following question. Every resonance point $r_e$ corresponding to a complex number $z$ from the complement of the essential spectrum has an order, which is a positive number. A natural question is how the order of $r_e$ depends on $z$. In the following theorem we show that the order of $r_e$ is equal to 1 for all values of $z$ except a discrete set, provided that $r_e$ admits analytic continuation to a gap in the essential spectrum in the real axis and that the analytic continuation has a real value at least at one point. We conjecture that this property holds in general without this assumption, but since we are interested in analytic continuation of $r_\lambda$, this hypothesis automatically holds in our case.

**Theorem 5.1.** Let $r_\lambda$ be a real resonance point of the triple $(\lambda; H_0, V)$. Analytic continuation of $r_\lambda$ as a function of the complex variable $\lambda$ has order 1 except a discrete set.

**Proof.** Assume the contrary: there exists a non-empty open subset $G$ of the domain of holomorphy of $r_e$ such that for all $z \in G$ the resonance point $r_e$ has order at least two. Then by [Az1] Corollary 3.4.7 there exist holomorphic vector-functions $\chi_1(z)$ and $\chi_2(z)$ such that

\begin{equation}
(H_0 + r(z) V - z) \chi_1(z) = 0
\end{equation}

and

\begin{equation}
(H_0 + r(z) V - z) \chi_2(z) = -V \chi_1(z).
\end{equation}

Differentiation of the equality (5.1) with respect to $z$ gives

\begin{equation}
(r'(z) V - 1) \chi_1(z) + (H_0 + r(z) V - z) \chi_1'(z) = 0.
\end{equation}

Let $\varphi(\bar{z})$ be an anti-resonance vector-function of order 1, that is,

\begin{equation}
(H_0 + \bar{r}(z) V - \bar{z}) \varphi(\bar{z}) = 0.
\end{equation}

Since

\[ \langle \varphi(\bar{z}), (H_0 + r(z) V - z) \chi_1'(z) \rangle = \langle (H_0 + \bar{r}(z) V - \bar{z}) \varphi(\bar{z}), \chi_1'(z) \rangle = 0, \]

taking the scalar product of $\varphi(\bar{z})$ with both sides of the equality (5.3) gives

\[ \langle \varphi(\bar{z}), (r'(z) V - 1) \chi_1(z) \rangle = 0. \]

Further, the equality (5.2) implies that

\[ -\langle \varphi(\bar{z}), V \chi_1(z) \rangle = \langle \varphi(\bar{z}), (H_0 + r(z) V - z) \chi_2(z) \rangle = \langle (H_0 + \bar{r}(z) V - \bar{z}) \varphi(\bar{z}), \chi_2(z) \rangle = 0. \]

Combining this with the previous equality implies

\[ \langle \varphi(\bar{z}), \chi_1(z) \rangle = 0. \]

Since $r(z)$ takes real values in some interval $I$ of the real axis, we can take $\chi_1(z)$ to be holomorphic extension of a first order vector-function in $I$, and we can take $\varphi(\bar{z})$ to be anti-holomorphic extension of the same function. Since the scalar product is anti-linear in the first argument, the scalar product of this pair of holomorphic and anti-holomorphic vector-functions will be holomorphic and it would vanish on the interval $I$ of the real axis. This implies that both these functions are zero in the gap $I$ and therefore everywhere. Since $\chi_1(z)$ is an eigenvector, this gives a contradiction. \qed
5.2. Cycles of resonance points. Let $r_\lambda$ be a resonance point of geometric multiplicity $m$ and algebraic multiplicity $N$. When $\lambda$ is shifted to $z = \lambda + iy$ with small $y > 0$, the resonance point $r_\lambda$ splits into $N = \dim Y_\lambda(r_\lambda)$ resonance points $r_\lambda^{(j)}$, counting algebraic multiplicities. The resonance points $r_\lambda^{(j)}$ are holomorphic functions of $z$. When $z$ makes one round around $r_\lambda$, these $N$ holomorphic functions undergo a permutation. We shall show in this subsection that this permutation consists of $m$ disjoint cycles of lengths $d_1, \ldots, d_m$, where $m$ is the number of Jordan cells of the compact operator $A_\lambda(s)$ corresponding to the eigenvalue $(s - r_\lambda)^{-1}$, and $d_\nu$ is the size of $\nu$-th cell.

For convenience, in this subsection we shall often indicate dependence of a resonance point $r_z$ on the spectral parameter $z$ in the usual way as $r(z)$ instead of using subindex notation. If there is no danger of confusion, we may choose to drop the variable $z$ from the notation altogether.

In section 3 we studied eigenvalues $\lambda_\nu(r)$ of the operator $H_0 = H_0 + rV$ as functions of the coupling constant $r$. The coupling constant $r$ was treated as a real variable. In this section we consider the coupling constant $r$ as a function of the spectral parameter $\lambda$, but unlike section 3, we shall treat both variables $r$ and $\lambda$ as complex variables. Since the spectral variable treated as a complex variable is denoted by $z$, the functions under study are $r_\nu(z)$, which are inverses of $\lambda_\nu(r)$. According to Theorem 5.7, eigenvalue functions $\lambda_\nu(r)$ of order $d_\nu > 1$ satisfy $\lambda_\nu'(r_\lambda) = 0$. Hence, in general the corresponding inverse function $r_\nu(z)$ is a branching multi-valued holomorphic function in a neighbourhood of $z = \lambda$.

**Theorem 5.2.** (a) For each cycle $r_\nu^{(j)}(z)$ of resonance points there exists $\varepsilon > 0$ and a resonance point $r_\nu^{(0)}(z)$ of this cycle which takes real values for all $z$ from an interval $I$, where $I$ is either $[\lambda, \lambda + \varepsilon)$ or $(\lambda - \varepsilon, \lambda]$.

(b) The number of real resonance points in a cycle for $z \in I$ is either one or two.

(c) In the case there are two real resonance points $r'$ and $r''$ in a cycle for $z \in I$, the numbers $r' - r_\lambda$ and $r'' - r_\lambda$ have different signs.

(d) The numbers of non-real resonance points in each cycle for $z \in I$ are the same in $\mathbb{C}_+$ and $\mathbb{C}_-$.

(e) In the case there are two real resonance points $r'$ and $r''$ in a cycle for $z \in I$, they shift to different half-planes $\mathbb{C}_+$ and $\mathbb{C}_-$ as $z \in I$ is shifted to $z + iy$ with small $y > 0$.

**Proof.** Part (a) follows from the fact that an isolated eigenvalue $\lambda$ of $H_0 + r_\lambda V$ is stable, that is, an eigenvalue $\lambda_\nu(s)$ of $H_s$ depends on $s$ continuously.

Parts (b) and (c) follow from the formula (see (3.7))

$$\lambda_\nu(s) = \lambda + \varepsilon_\nu(s - r_\lambda)^{\tilde{d}_\nu} + O((s - r_\lambda)^{\tilde{d}_\nu + 1}), \ s \to r_\lambda,$$

where $\tilde{d}_\nu$ is order of $\lambda_\nu(s)$. Namely, with $\varepsilon > 0$ sufficiently small, if the order $\tilde{d}_\nu$ is odd, then there is one and only one real resonance point for all $z \in I$, where $I = [\lambda, \lambda + \varepsilon)$ or $I = (\lambda - \varepsilon, \lambda]$. In this case, the interval $I$ can be chosen to be either of the intervals $[\lambda, \lambda + \varepsilon)$ or $(\lambda - \varepsilon, \lambda]$. If the order $\tilde{d}_\nu$ is even, then there are exactly two real resonance points for all $z \in I$ and they are located on different sides of $r_\lambda$ in the real axis of the coupling constant. In this case, if $\varepsilon_\nu > 0$, then the interval $I$ is $[\lambda, \lambda + \varepsilon)$ and if $\varepsilon_\nu < 0$, then the interval $I$ is $(\lambda - \varepsilon, \lambda]$.

Proof of part (d). By Lemma 6.3, if $r_z$ is a resonance point corresponding to $z$, then $\tilde{r}_z$ is a resonance point corresponding to $\tilde{z}$. Hence, the set of resonance points corresponding to a real value of $\lambda$ is symmetric with respect to the real axis. We still need to show that if a resonance
point \( r^{(j)}_\nu(z) \) belongs to a cycle \( \nu \), then its conjugate also belongs to the same cycle, but this readily follows from the Schwarz reflection principle.

Proof of part (e). By part (d), for real values of \( z \) from \( I \) the set of resonance points in a cycle is symmetric with respect to the real axis. Combining this with the fact that for non-real values of \( z \) there can be no real resonance points, one can infer the claim.

The following figure demonstrates this theorem. In this figure there are two cycles of lengths \( d_1 = 5 \) (black dots) and \( d_2 = 4 \) (white dots).

\[
\begin{align*}
z &= \lambda & z &= \lambda + \varepsilon & z &= \lambda + \varepsilon + iy, \ 0 < y \ll 1 \\
r_\lambda & & r_\lambda & & r_\lambda
\end{align*}
\]

Proposition 5.3. The number of cycles of the permutation of the \( N \) resonance points \( r^{(j)}_z \) is equal to geometric multiplicity \( m \) of the resonance point \( r_\lambda \). More precisely, there is a natural one-to-one correspondence between cycles of resonance points \( r^{(j)}_\nu(z) \) and the eigenvalue functions \( \lambda_\nu(s) \) of the operator \( H_s \), given by the following diagram:

(5.6) \[
\varphi_\nu(s) \leftrightarrow \lambda_\nu(s) \leftrightarrow r^{(j)}_\nu(z).
\]

That is, with an eigenpath \( \varphi_\nu(s) \) we associate an eigenvalue function \( \lambda_\nu(s) \) and the inverse of this eigenvalue function is the multi-valued holomorphic function \( r^{(j)}_\nu(z) \).

Proof. This immediately follows from parts (a), (b) and (c) of Theorem 5.2.

Corollary 5.4. Order \( \tilde{d}_\nu \) of eigenpath \( \varphi_\nu(s) \) is equal to the size \( \hat{d}_\nu \) of a cycle corresponding to the eigenpath \( \varphi_\nu(s) \).

Proof. By items (i) and (iii) of Theorem 5.7 the order \( \tilde{d}_\nu \) of \( \varphi_\nu(s) \) is a number determined by the equality (3.7). This implies that the inverse of the function \( \lambda_\nu(s) \) in a neighbourhood of \( s = r_\lambda \) is a multivalued function \( r^{(j)}_\nu(z) \) with \( \tilde{d}_\nu \) branches in a neighbourhood of \( z = \lambda \), and these branches form a single cycle. Hence, \( \tilde{d}_\nu = \hat{d}_\nu \).

Corollary 5.5. The sum of orders of eigenpaths \( \varphi_\nu(s) \) is equal to the algebraic multiplicity \( N \) of the resonance point \( r_\lambda \).

Corollary 5.6. The \( m \) sets of vectors 

\[
\frac{1}{j!} \varphi^{(j)}_\nu(r_\lambda), \ \nu = 1, \ldots, m, \ j = 0, 1, \ldots, \hat{d}_\nu - 1
\]

form a Jordan basis for the nilpotent operator \( A_\lambda(r_\lambda) \).

Proof. By item (v) of Theorem 5.7 for each \( \nu = 1, \ldots, m \) and for each \( j = 0, 1, \ldots, \hat{d}_\nu - 1 \) we have \( A_\lambda(r_\lambda) \varphi^{(j)}_\nu(r_\lambda) = \varphi^{(j-1)}_\nu(r_\lambda) \). Since by Corollary 5.5 the sum of numbers \( \hat{d}_\nu \) is equal to the algebraic multiplicity \( N \), we are done.

We collect these assertions in the following theorem.
Theorem 5.7. For each \( \nu = 1, \ldots, m \), the following numbers are equal (assuming that they are arranged in decreasing order):

1. the order of eigenpath \( \varphi_\nu(s) \),
2. the size of the cycle \( \nu \) of resonance points of the group of \( r_\lambda \),
3. the size of the \( \nu \)th Jordan block of the nilpotent operator \( A_\lambda(r_\lambda) \).

Moreover, the vectors \( \frac{1}{z} \varphi_\nu^{(j)}(r_\lambda) \), \( \nu = 1, \ldots, m \), \( j = 0, 1, \ldots, d_\nu - 1 \), form a Jordan basis of the nilpotent operator \( A_\lambda(r_\lambda) \).

5.3. Decomposition of \( P_\lambda(r_\lambda) \). Let \( r_\nu^{(0)}, \ldots, r_\nu^{(d_\nu - 1)} \) be the cycle \( \nu \). The function

\[
P_z^{[\nu]} := \sum_{j=0}^{d_\nu - 1} P_z(r_\nu^{(j)}(z))
\]

is single-valued in a neighbourhood of \( \lambda \).

Proposition 5.8. The function \( P_z^{[\nu]} \) of \( z \) is holomorphic in a neighbourhood of \( \lambda \).

Proof. By [Ka, Theorem II.1.8], the Laurent expansion of the function \( P_z^{[\nu]} \) can have only finitely many terms with negative powers of \( (z - \lambda) \). The sum \( \sum_{\nu=1}^{m} P_z^{[\nu]} \) converges to \( P_\lambda(r_\lambda) \) as \( z \to \lambda \) and therefore is bounded in a neighbourhood of \( \lambda \). Since \( P_z^{[\nu]} P_z^{[\mu]} = \delta_{\nu\mu} P_z^{[\nu]} \), each of the \( m \) functions \( P_z^{[\nu]} \) is also bounded in a neighbourhood of \( \lambda \). \( \square \)

Therefore, this operator has a limit as \( z \to \lambda \), which we denote by

\[
P_\lambda^{[\nu]}(r_\lambda) := \lim_{z \to \lambda} P_z^{[\nu]}.
\]

We have the equalities

\[
P_\lambda^{[\nu]} P_\lambda^{[\mu]} = \delta_{\nu\mu} P_\lambda^{[\nu]}
\]

and

\[
P_\lambda(r_\lambda) = \sum_{\nu=1}^{m} P_\lambda^{[\nu]}(r_\lambda).
\]

Since \( P_z(r_\nu^{(j)}(z)) P_z(r_\nu^{(k)}(z)) = 0 \) for different resonance points \( r_\nu^{(j)} \) and \( r_\nu^{(k)} \), the operator \( P_z^{[\nu]} \) is an idempotent.

Lemma 5.9. The operators \( P_\lambda^{[\nu]}(r_\lambda) \) and \( A_\lambda(r_\lambda) \) commute.

Proof. Since the limits of \( P_z^{[\nu]} \) and \( A_z(s) \) as \( z \to \lambda \) exist, it is enough to take limits of both sides of the equality \( P_z^{[\nu]} A_z(s) = A_z(s) P_z^{[\nu]} \), and then use Laurent expansion of \( A_\lambda(s) \). \( \square \)

In a similar way, we introduce operators \( Q_z^{[\nu]} \) by formula

\[
Q_z^{[\nu]} := \sum_{j=0}^{d_\nu - 1} Q_z(r_\nu^{(j)}(z)).
\]

Many properties of \( Q_z^{[\nu]} \) are analogues to those of \( P_z^{[\nu]} \).

It follows from (1.11) that \( V P_z^{[\nu]} = Q_z^{[\nu]} V \). Taking in this equality the limit \( z \to \lambda \), we obtain

\[
V P_\lambda^{[\nu]} = Q_\lambda^{[\nu]} V.
\]

Let \( \Upsilon_\lambda^{[\nu]} := \text{im } P_\lambda^{[\nu]} \).
Corollary 5.10. The nilpotent operator $A_\lambda(r_\lambda)$ is reduced by the vector space $\Upsilon_{\lambda}^{[\nu]}$.

We denote restriction of $A_\lambda(r_\lambda)$ to the vector space $\Upsilon_{\lambda}^{[\nu]}$ by $A_\lambda^{[\nu]}(r_\lambda)$. In previous sections we denoted by $d_\nu$ the sizes of the Jordan cells of the nilpotent operator $A_\lambda(r_\lambda)$. The last lemma shows that numbers $d_\nu$ and dimensions of the vectors spaces $\Upsilon_{\lambda}^{[\nu]}$, and therefore lengths of the cycles, are equal.

Lemma 5.11. Dimension of the vector space $\Upsilon_{\lambda}^{[\nu]} \cap \mathcal{V}_\lambda$ is equal to 1.

Proof. By Corollary 5.10 the dimension of $\Upsilon_{\lambda}^{[\nu]} \cap \mathcal{V}_\lambda$ is at least 1. Hence, this dimension is to be 1, since otherwise we get a contradiction with Proposition 5.3.

Lemma 5.12. The operator $A_\lambda^{[\nu]}$ is cyclic.

Proof. This follows immediately from Corollary 5.10 and 5.11.

5.4. Two lemmas. Since by Theorem 5.1 the functions $r_{\nu}^{(j)}(z)$ have order 1, the operator valued function $A_{\chi}(s)$ near the (real) point $r_\lambda$ has the Laurent expansion

\begin{equation}
A_{\chi}(s) = \tilde{A}_{\chi}(s) + \sum_{\nu=1}^{m} \sum_{j=0}^{d_\nu-1} \frac{P_\chi(r_{\nu}^{(j)}(z))}{s - r_{\nu}^{(j)}(z)},
\end{equation}

where $\tilde{A}_{\chi}(s)$ is a meromorphic function which has no poles in a neighbourhood of $r_\lambda$ which includes all $r_{\nu}^{(j)}(z)$. The functions $\frac{P_\chi(r_{\nu}^{(j)}(z))}{s - r_{\nu}^{(j)}(z)}$ of $z$ (with $s$ fixed), taken individually, are not single-valued in a neighbourhood of $\lambda$, unless $d_\nu = 1$, but each of the $m$ sums

\begin{equation}
\sum_{j=0}^{d_\nu-1} \frac{P_\chi(r_{\nu}^{(j)}(z))}{s - r_{\nu}^{(j)}(z)}
\end{equation}

is a single-valued function of $z$ in a neighbourhood of $\lambda$. We also have the equality

\begin{equation}
P_\chi^{[\nu]}A_{\chi}(s) = \sum_{j=0}^{d_\nu-1} \frac{P_\chi(r_{\nu}^{(j)}(z))}{s - r_{\nu}^{(j)}(z)}.
\end{equation}

Lemma 5.13. As $z \to \lambda$, the fractional part

$P_\chi(r_\lambda)A_{\chi}(s) = \sum_{\nu=1}^{m} \sum_{j=0}^{d_\nu-1} \frac{P_\chi(r_{\nu}^{(j)}(z))}{s - r_{\nu}^{(j)}(z)}$

of the Laurent expansion (5.10) of the meromorphic function $A_{\chi}(s)$ at $N$ poles $r_{\nu}^{(j)}(z)$, $\nu = 1, \ldots, m$, $j = 0, \ldots, d_\nu - 1$,

of the group of $r_\lambda$ converges in the norm topology to the fractional part of the Laurent expansion of $A_{\chi}(s)$ at the pole $r_\lambda$, which is

\begin{equation}
P_{\lambda}(r_\lambda)A_{\lambda}(s) = \frac{P_\lambda(r_\lambda)}{s - r_\lambda} + \frac{A_{\lambda}(r_\lambda)}{(s - r_\lambda)^2} + \ldots + \frac{A_{\lambda}^{d_\lambda-1}(r_\lambda)}{(s - r_\lambda)^{d_\lambda-1}}.
\end{equation}

The convergence is uniform with respect to $s$ on compact subsets of a deleted neighbourhood of $r_\lambda$.

Proof. The meromorphic function $A_{\chi}(s)$ converges in norm to $A_{\lambda}(s)$ as $z \to \lambda$ uniformly on compact subsets of a deleted neighbourhood of the pole $r_\lambda$ in $\mathbb{C}$. $P_\chi(r_\lambda)$ converges to $P_{\lambda}(r_\lambda)$ as $z \to \lambda$ in norm (in fact, even in trace class norm, see $A_{\chi}$ Lemma 5.2.2). The claim follows.
Lemma 5.14. For each \( \nu = 1, \ldots, m \), the following equality holds:

\[
\lim_{z \to \lambda} \sum_{j=0}^{d_{\nu} - 1} \frac{P_z(r^{(j)}_\nu)(z)}{s - r^{(j)}_\nu(z)} = \frac{P^{[\nu]}_\lambda(r_\lambda)}{s - r_\lambda} + \frac{A^{[\nu]}_\lambda(r_\lambda)}{(s - r_\lambda)^2} + \ldots + \frac{(A^{[\nu]}_\lambda(r_\lambda))^{d_{\nu} - 1}}{(s - r_\lambda)^{d_{\nu} - 1}},
\]

where \( d_{\nu} \) is the size of the cycle \( \nu \).

Proof. By Proposition 5.8,

\[
\lim_{z \to \lambda} P^{[\nu]}_z A_z(s) = P^{[\nu]}_\lambda A_\lambda(s),
\]

where the convergence is in norm and is uniform on compact subsets of a deleted neighbourhood of \( r_\lambda \). This equality is the same as (5.14). \( \square \)

5.5. Puiseux series for \( r^{(j)}_\nu(z) \). For a positive integer \( d \) let

\[
\varepsilon_d = e^{2\pi i/d}.
\]

For an integer \( k \) and a positive integer \( d \) let

\[
[d|k] = \begin{cases} \ 0, & \text{if} \ d \ \text{does not divide} \ k, \\ 1, & \text{if} \ d \ \text{divides} \ k. \end{cases}
\]

For any integer \( k \) we have

\[
\sum_{j=0}^{d-1} \varepsilon_d^{jk} = [d|k] \cdot d.
\]

For each \( \nu = 1, \ldots, m \), the functions

\[
r^{(0)}_\nu(z), \ldots, r^{(d_{\nu} - 1)}_\nu(z)
\]

are different branches of a multi-valued holomorphic function with Puiseux series

\[
r^{(j)}_\nu(z) = \sum_{k=0}^{\infty} r_{k/d_{\nu}}(z - \lambda)^{k/d_{\nu}}, \quad j = 0, \ldots, d_{\nu} - 1.
\]

The Puiseux series for \( r^{(j)}_\nu(z) \) does not have the part with negative powers of \( (z - \lambda) \), since this function is continuous at \( z = \lambda \).

Proposition 5.15. The coefficients \( r_{k/d_{\nu}} \) of the Puiseux series (5.16) are real.

Proof. More precisely, these numbers can be chosen to be real. For \( j = 0 \) we have

\[
r^{(0)}_\nu(z) = \sum_{k=0}^{\infty} r_{k/d_{\nu}}(z - \lambda)^{k/d_{\nu}}.
\]

Recall that \( \lambda \) is an isolated eigenvalue of the self-adjoint operator \( H_{r_\lambda} \). Due to stability of isolated eigenvalues, one of the functions \( r^{(j)}_\nu(z) \) is to take real values for real values of \( z \) close to \( \lambda \) on the left or on the right. From this one can infer that the numbers \( r_{k/d_{\nu}} \) are to be real. \( \square \)

Proposition 5.16. The first coefficient \( r_{1/d_{\nu}} \) of the Puiseux series (5.16) for \( r^{(j)}_\nu(z) \) is non-zero.
Proof. Let $r_{f_\nu}/d_\nu$ be the first non-zero coefficient of the Puiseux series (5.16). Since $r_{\nu}^{(1)}(z)$ is the inverse of $\lambda_\nu(s)$ (see diagram (5.64)), it follows from item (iii) of Theorem 5.7 and (5.16) that

$$\lambda_\nu(s) = \lambda + \varepsilon_\nu(s - r_\lambda)^{d_\nu/f_\nu} + O((s - r_\lambda)^{d_\nu/f_\nu+1})$$

with non-zero $\varepsilon_\nu$. Since $\lambda_\nu(s)$ is analytic at $s = r_\lambda$, $f_\nu$ divides $d_\nu$. If $f_\nu > 1$, then this would imply that the set of $d_\nu$ numbers $\{r_{\nu}^{(j)}(z): j = 0, \ldots, d_\nu - 1\}$ is not a cycle. Hence, the only possibility is that $f_\nu = 1$ and we are done. \hfill \Box

5.6. Puiseux series for $P_z(r_{\nu}^{(j)}(z))$. For each $\nu = 1, \ldots, m$, the operator-functions

$$P_z(r_{\nu}^{(0)}(z)), \ldots, P_z(r_{\nu}^{(d_\nu-1)}(z))$$

are different branches of a multi-valued holomorphic function with Puiseux series

$$(5.17) \quad P_z(r_{\nu}^{(j)}(z)) = \tilde{P}_\nu(z) + \sum_{l=0}^{p} \varepsilon_{d_\nu}^{-l}(z - \lambda)^{-1/d_\nu} P_{-l/d_\nu}, \quad j = 0, \ldots, d_\nu - 1$$

where $\tilde{P}_\nu(z)$ is the part with positive powers of $z - \lambda$. Since by definition (1.6) the operator $P_z(r_{\nu}^{(j)}(z))$ is an eigenprojection of the compact operator $A_z(s)$ corresponding to a non-zero eigenvalue

$$(s - r_{\nu}^{(j)}(z))^{-1},$$

by [Ka] Theorem II.1.8, the part of the Puiseux expansion (5.17) with negative powers is finite.

Our first aim is to prove that the upper limit $p$ in the Puiseux series (5.17) is equal to $d_\nu - 1$. For this we need two auxiliary lemmas.

Lemma 5.17. For each cycle of resonance points $r_{\nu}^{(j)}(z)$,

$$(5.18) \quad \frac{d_\nu-1}{\sum_{j=0}^{d_\nu-1} \frac{P_z(r_{\nu}^{(j)}(z))}{s - r_{\nu}^{(j)}(z)}} = \frac{d_\nu-1}{\sum_{j=0}^{d_\nu-1} \frac{A_{\nu}^{(j)}(z)}{(s - r_{\nu}^{(0)}(z))(s - r_{\nu}^{(j)}(z))}},$$

where for $j = 0, 1, \ldots, d_\nu - 1$

$$(5.19) \quad A_{\nu}^{(j)}(z) = \sum_{b=j}^{d_\nu-1} (r_{\nu}^{(b)}(z) - r_{\nu}^{(0)}(z)) \cdots (r_{\nu}^{(b)}(z) - r_{\nu}^{(j-1)}(z)) P_z(r_{\nu}^{(b)}(z)).$$

Proof. This lemma has general character as it holds for any set of numbers $r_{\nu}^{(j)}(z)$ and any bounded operators $P_z(r_{\nu}^{(j)}(z))$. For this reason we write $r_j$ for $r_{\nu}^{(j)}(z)$, $P_j$ for $P_z(r_{\nu}^{(j)}(z))$, $d$ for $d_\nu$, $A_j$ for $A_{\nu}^{(j)}(z)$.

We prove the equality (5.18) by induction on $d$. For $d = 1$ the assertion is trivial. Assume that the assertion holds for smaller values of $d$. Since the operators $A_j$ depend on $d$, we will indicate this dependence by writing $A_j^{(d)}$.

By the induction assumption

$$\sum_{j=0}^{d-1} \frac{P_j}{s - r_j} = \sum_{j=0}^{d-2} \frac{A_j^{(d-1)}}{(s - r_0) \cdots (s - r_j)} + \frac{P_{d-1}}{s - r_{d-1}}.$$

We need to find operators $A_j^{(d)}$ such that

$$\sum_{j=0}^{d-1} \frac{A_j^{(d)}}{(s - r_0) \cdots (s - r_j)} = \sum_{j=0}^{d-2} \frac{A_j^{(d-1)}}{(s - r_0) \cdots (s - r_j)} + \frac{P_{d-1}}{s - r_{d-1}}.$$
Multiplying both sides of this equality by $\prod_{k=0}^{d-1} (s - r_k)$ gives

$$\sum_{j=0}^{d-1} A_j^{(d)} \prod_{k=j+1}^{d-1} (s - r_k) = \sum_{j=0}^{d-1} A_j^{(d-1)} \prod_{k=j+1}^{d-1} (s - r_k) + \prod_{k=0}^{d-2} (s - r_k) P_{d-1}.$$  

Replacing $s$ by $r_{d-1}$ gives

$$A_{d-1}^{(d)} = 0 + \prod_{k=0}^{d-2} (r_{d-1} - r_k) P_{d-1}.$$  

Replacing $s$ by $r_{d-2}$ gives

$$A_{d-1}^{(d)} + A_{d-2}^{(d)} (r_{d-2} - r_{d-1}) = A_{d-2}^{(d-1)} (r_{d-2} - r_{d-1}) + 0.$$  

This gives

$$A_{d-2}^{(d)} = \frac{A_{d-1}^{(d)}}{r_{d-1} - r_{d-2}} + A_{d-2}^{(d-1)}$$  

$$= \prod_{k=0}^{d-3} (r_{d-1} - r_k) P_{d-1} + \sum_{b=d-2}^{d-2} (r_b - r_0) \ldots (r_b - r_{j-1}) P_b$$  

$$= \prod_{k=0}^{d-3} (r_{d-1} - r_k) P_{d-1} + (r_{d-2} - r_0) \ldots (r_{d-2} - r_{d-3}) P_{d-2}$$  

$$= \sum_{b=d-2}^{d-1} (r_b - r_0) \ldots (r_b - r_{d-3}) P_b.$$  

where the second equality follows from the previous one and the induction assumption.  

And so on.  

**Lemma 5.18.** Let $r^* \in \mathbb{C}$ and let $r_0^n, \ldots, r_{d-1}^n$, $n = 1, 2, \ldots$, be $d$ sequences of complex numbers such that $r_j^n \to r^*$ as $n \to \infty$ for all $j = 0, 1, \ldots, d - 1$. Assume that $B_0, \ldots, B_{d-1}$ are bounded operators and that $A_0^n, \ldots, A_{d-1}^n$, $n = 1, 2, \ldots$, are $d$ sequences of bounded operators such that for all $s \in \mathbb{C} \ \setminus \ \{r^*\}$

$$\lim_{n \to \infty} \left( \frac{A_0^n}{s - r_0^n} + \frac{A_1^n}{(s - r_0^n)(s - r_1^n)} + \ldots + \frac{A_{d-1}^n}{(s - r_0^n)(s - r_1^n) \ldots (s - r_{d-1}^n)} \right) = \frac{B_0}{s - r^*} + \frac{B_1}{(s - r^*)^2} + \ldots + \frac{B_{d-1}}{(s - r^*)^d},$$

where the limit is in norm and is uniform on compact subsets of $\mathbb{C} \ \setminus \ \{r^*\}$. Then for all $j = 0, 1, \ldots, d - 1$ the sequence $A_j^n$ converges in norm to $B_j$ as $n \to \infty$.

**Proof.** Multiplying the equality in the lemma by

$$\lim_{n \to \infty} (s - r_0^n) \ldots (s - r_{d-1}^n) = (s - r^*)^d$$

gives for all $s \neq r^*$ the equality

$$\lim_{n \to \infty} (A_0^n(s - r_1^n) \ldots (s - r_{d-1}^n) + \ldots + A_{d-2}^n(s - r_{d-1}^n) + A_{d-1}^n) = B_0(s - r^*)^{d-1} + \ldots + B_{d-1}.$$  

Since the convergence is uniform, we can differentiate the left hand side under the limit sign (see e.g. [Di] 3IX.12). Hence, applying the operator $\frac{d^d}{ds^d}$ to both sides of the last equality, we infer that $A_0^n \to B_0$. Applying $\frac{d^{d-1}}{ds^{d-1}}$, we infer that $A_1^n \to B_1$. And so on.  

Lemma 5.19. For each cycle \( \nu \) and for each \( k = 0, 1, \ldots, d_\nu - 1 \) the limits \( \lim_{z \to \lambda} A^{(k)}_\nu (z) \) exist and
\[
\lim_{z \to \lambda} A^{(k)}_\nu (z) = A^{(k)}_\lambda (r_\lambda) P_\lambda^{[\nu]} (r_\lambda),
\]
where the convergence is in norm.

Proof. Combining Lemmas 5.14 and 5.17 we get
\[
\lim_{z \to \lambda} \sum_{j=0}^{d_\nu-1} \frac{A^{(j)}_\nu (z)}{(s - r^{(0)}_\nu (z)) \ldots (s - r^{(j)}_\nu (z))} = \frac{P^{[\nu]}_\lambda (r_\lambda)}{(s - r_\lambda)^2} + \ldots + \frac{\left( A^{[\nu]}_\lambda (r_\lambda) \right)^{d_\nu-1}}{(s - r_\lambda)^{d_\nu-1}},
\]
where the convergence is in norm uniformly in \( s \) on compact subsets of a deleted neighbourhood of \( r_\lambda \). Hence, Lemma 5.18 completes the proof. □

Proposition 5.20. The integer \( p \) from Puiseux series (5.17) for \( P_z (r^{(j)}_\nu (z)) \) is equal to \( d_\nu - 1 \) :
\[
p = d_\nu - 1.
\]

Proof. By Lemma 5.19 we have
\[
\lim_{z \to \lambda} A^{(d_\nu-1)}_\nu (z) = A^{(d_\nu-1)}_\lambda (r_\lambda) P_\lambda^{[\nu]} (r_\lambda) \neq 0.
\]
Here the last inequality (\( \neq 0 \)) follows from the fact that the operator \( A^{(d_\nu-1)}_\lambda (r_\lambda) \) is reduced by the image of the idempotent \( P^{[\nu]}_\lambda (r_\lambda) \) and that this reduction is cyclic, see Lemma 5.12. By (5.19), we have
\[
A^{(d_\nu-1)}_\nu (z) = (r^{(d_\nu-1)}_\nu (z) - r^{(0)}_\nu (z)) \ldots (r^{(d_\nu-1)}_\nu (z) - r^{(d_\nu-2)}_\nu (z)) P_z (r^{(d_\nu-1)}_\nu (z)).
\]
Since by Proposition 5.16 the first Puiseux series coefficient \( r^{(j)}_1/d_\nu \) for \( r^{(j)}_\nu (z) \) is non-zero, the first non-zero fractional term of the product
\[
(r^{(d_\nu-1)}_\nu (z) - r^{(0)}_\nu (z)) \ldots (r^{(d_\nu-1)}_\nu (z) - r^{(d_\nu-2)}_\nu (z))
\]
is \( (z - \lambda)^{d_\nu-1} \). The term with the smallest power in Laurent expansion of \( P_z (r^{(d_\nu-1)}_\nu (z)) \) is \( (z - \lambda)^{-p} \). Hence, if \( p < d_\nu - 1 \), then the limit in (5.22) would be zero, and if \( p > d_\nu - 1 \), then the limit in (5.22) would diverge. □

Proposition 5.21. Let \( \nu = 1, \ldots, m \). For any \( k \geq 0 \) the function
\[
\sum_{j=0}^{d_\nu-1} (r^{(j)}_\nu (z) - r_\lambda)^k P_z (r^{(j)}_\nu (z))
\]
is analytic at \( z = \lambda \). Moreover, the limit of this sum as \( z \to \lambda \) is equal to \( P^{[\nu]}_\lambda A^{(k)}_\lambda (r_\lambda) \).

Proof. The function is symmetric with respect to \( r^{(j)}_\nu (z), j = 1, \ldots, d_\nu \), and therefore it is single-valued in a neighbourhood of \( \lambda \). By Proposition 5.20, this function also cannot have whole negative powers of \( z - \lambda \) in its power series expansion at \( \lambda \). Hence, it is analytic at \( \lambda \). In particular, if \( k = 0 \), then its limit as \( z \to \lambda \) is equal to \( P^{[\nu]}_\lambda (r_\lambda) \).

Further, we have
\[
\sum_{j=0}^{d_\nu-1} (r^{(j)}_\nu (z) - r_\lambda) P_z (r^{(j)}_\nu (z)) = \sum_{j=0}^{d_\nu-1} (r^{(j)}_\nu (z) - r^{(0)}_\nu (z)) P_z (r^{(j)}_\nu (z)) + \sum_{j=0}^{d_\nu-1} (r^{(0)}_\nu (z) - r_\lambda) P_z (r^{(j)}_\nu (z)).
\]
The first summand converges to $P^{[\nu]}_{\lambda}(r_{\lambda})$ by Lemma 5.19 and (5.19) (taken with $j = 1$). The second summand converges to zero, since by Proposition 5.8

$$\sum_{j=0}^{d_{\nu}-1} P_z(r_{\nu}^{(j)}(z))$$

is analytic and $r_{\nu}^{(0)}$ is continuous at $z = \lambda$. Thus, the claim holds for $k = 1$. Using this, for any $k \geq 1$ we have

$$\lim_{z \to \lambda} \sum_{j=0}^{d_{\nu}-1} (r_{\nu}^{(j)}(z) - r_{\nu}) P_z(r_{\nu}^{(j)}(z)) = \lim_{z \to \lambda} \left( \sum_{j=0}^{d_{\nu}-1} (r_{\nu}^{(j)}(z) - r_{\nu}) P_z(r_{\nu}^{(j)}(z)) \right)^k = P^{[\nu]}_{\lambda} A^{k}(r_{\lambda}),$$

where the first equality follows from (1.9).

Since $p = d_{\nu} - 1$, we have

$$\lim_{z \to \lambda} (z - \lambda) P_z(r_{\nu}^{(j)}(z)) = 0.$$

**Proposition 5.22.** For all $\nu = 1, \ldots, m$ and for all $k = 1, 2, \ldots, d_{\nu} - 1$

$$(5.23) \quad P^{[\nu]}_{\lambda}(r_{\lambda}) A^{k}(r_{\lambda}) = d_{\nu} \cdot \sum_{l=k}^{d_{\nu}-1} \left( \sum_{m_1 + \ldots + m_k = l} \prod_{j=1}^{k} r_{m_j/d_{\nu}} \right) P_{-l/d_{\nu}},$$

where in the sum $m_1, \ldots, m_k \geq 1$.

**Proof.** We shall prove this only for $k = 2$. The general case is proved by the same calculation.

We have, by Proposition 5.21

$$P^{[\nu]}_{\lambda}(r_{\lambda}) A^{2}(r_{\lambda}) = \lim_{z \to \lambda} \sum_{j=0}^{d_{\nu}-1} (r_{\nu}^{(j)}(z) - r_{\nu})^2 P_z(r_{\nu}^{(j)}(z))$$

$$= \lim_{z \to \lambda} \sum_{j=0}^{d_{\nu}-1} \left( \sum_{k=1}^{\infty} r_{k/d_{\nu}} e_{k}^{kj} (z - \lambda)^k/d_{\nu} \right) \left( \sum_{m=1}^{\infty} r_{m/d_{\nu}} e_{m}^{mj} (z - \lambda)^m/d_{\nu} \right) P_{-l/d_{\nu}}$$

$$= \lim_{z \to \lambda} \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \sum_{l=1}^{d_{\nu}-1} \sum_{j=0}^{d_{\nu}-1} e_{d_{\nu}}^{(k+m-l)j} (z - \lambda)^{k+m-l}/d_{\nu} r_{k/d_{\nu}} r_{m/d_{\nu}} P_{-l/d_{\nu}}.$$

In this sum, there are finitely many terms with negative powers of $z - \lambda$ and the sum of positive powers of $z - \lambda$ converges absolutely in some neighbourhood of $\lambda$. Hence, all the interchanges of summations, which have been performed so far and which are about to follow, are justified.

Let $x = k + m$. Then

$$P^{[\nu]}_{\lambda}(r_{\lambda}) A^{2}(r_{\lambda}) = \lim_{z \to \lambda} \sum_{x=2}^{\infty} \sum_{m=1}^{x-1} \sum_{l=1}^{d_{\nu}-1} \sum_{j=0}^{d_{\nu}-1} e_{d_{\nu}}^{(x-l)j} (z - \lambda)^{(x-l)/d_{\nu}} r_{(x-m)/d_{\nu}} r_{m/d_{\nu}} P_{-l/d_{\nu}}$$

$$= d_{\nu} \cdot \lim_{z \to \lambda} \sum_{x=2}^{\infty} \sum_{m=1}^{x-1} \sum_{l=1}^{d_{\nu}-1} [d_{\nu} |x - l|] (z - \lambda)^{(x-l)/d_{\nu}} r_{(x-m)/d_{\nu}} r_{m/d_{\nu}} P_{-l/d_{\nu}},$$
where in the last equality we have used (5.15). The terms with \( x > l \) disappear after taking the limit \( z \to \lambda \). Since \( x \geq 2 \) and \( 1 \leq l \leq d_\nu - 1 \), there are no non-zero terms with \( x < l \) and \( d_\nu \mid x - l \). Hence, the factor \([d_\nu \mid x - l]\) can be replaced by Kronecker’s symbol \( \delta_{xl} \). This gives

\[
P^{[\nu]}_\lambda(r_\lambda) A^2_\lambda(r_\lambda) = d_\nu \cdot \lim_{z \to \lambda} \sum_{x=2}^{d_\nu-1} \sum_{l=1}^{d_\nu-2} \delta_{xl}(z - \lambda)^{(x-l)/d_\nu} \left( \sum_{m=1}^{x-1} r(m)/d_\nu r_m/d_\nu \right) P_{-l/d_\nu}.
\]

Two special cases of (5.23) are the formulas

\[
P^{[\nu]}_\lambda(r_\lambda) A_\lambda(r_\lambda) = d_\nu \sum_{l=1}^{d_\nu-1} r_l/d_\nu P_{-l/d_\nu},
\]

and

\[
P^{[\nu]}_\lambda(r_\lambda) A^{d_\nu-1}_\lambda(r_\lambda) = d_\nu \cdot r^{d_\nu-1}/d_\nu P_{-d_\nu/d_\nu}.
\]

**Corollary 5.23.** For any \( j = 1, \ldots, d_\nu - 1 \) the operator \( P^{[\nu]}_\lambda(r_\lambda) A^j_\lambda(r_\lambda) \) is a linear combination of operators

\[
P_{-d_\nu/d_\nu}, \ldots, P_{-d_\nu/d_\nu}
\]

from the Puiseux expansion of \( P_z(r^{(j)}_\nu(z)) \) at \( z = \lambda \).

We summarise results of this section in the following theorem.

**Theorem 5.24.** Let \( r_\lambda \) be a real resonance point of the line \( H_{r_\lambda} + (s - r_\lambda)V \), corresponding to a point \( \lambda \) outside essential spectrum. Let \( N \) and \( m \) be respectively algebraic and geometric multiplicities of \( r_\lambda \). Let \( d_1, \ldots, d_m \) be the sizes of Jordan cells of the compact operator \( A_\lambda(s) \) corresponding to the eigenvalue \((s - r_\lambda)^{-1}\). Let \( r^{(1)}_z, \ldots, r^{(N)}_z \) be resonance points of the group of \( r_\lambda \) corresponding to \( z \approx \lambda \in \mathbb{C} \). Then

1. As \( z \) makes one round about \( \lambda \), the set of \( N \) resonance points undergoes a permutation which is the product of \( m \) disjoint cyclic permutations of resonance points

\[
r^{(0)}_\nu(z), \ldots, r^{(d_\nu-1)}_\nu(z), \quad \nu = 1, \ldots, m,
\]

and, as the notation indicates, the sizes of these cyclic permutations are the same as the sizes of the Jordan cells \( d_1, \ldots, d_m \).

2. For real values of \( z \) close to \( \lambda \), there is either one or two real resonance points in each of these \( m \) cycles of resonance points. In case there are two real resonance points in a cycle, one of them is greater than \( r_\lambda \) and the other is smaller than \( r_\lambda \); further, as \( z \) is shifted off the real axis, these two real resonance points of a cycle shift off to different complex half-planes.

3. The Puiseux series of the function \( r^{(j)}_\nu(z) \) has the form

\[
r^{(j)}_\nu(z) = \sum_{k=0}^{\infty} r_k/d_\nu e^{2\pi i j/k}(z - \lambda)^{k/d_\nu}
\]

with \( r_1/d_\nu \neq 0 \) (where \( r_0 = r_\lambda \)) and all coefficients \( r_k/d_\nu \) are real.
(4) The Puiseux series of the idempotent $P_z(r^{(j)}_\nu(z))$ has the form

$$P_z(r^{(j)}_\nu(z)) = \tilde{P}^{(j)}_\nu(z) + \sum_{l=0}^{d_\nu-1} e^{\frac{2\pi i j l}{d_\nu}} (z - \lambda)^{-l/d_\nu} P_{-l/d_\nu},$$

where $\tilde{P}^{(j)}_\nu(z)$ is continuous at $z = \lambda$.

(5) For each $\nu = 1, \ldots, m$ and for all $k \geq 0$,

$$\lim_{z \to \lambda} \sum_{j=0}^{d_\nu-1} (r^{(j)}_\nu(z) - r_\lambda)^k P_z(r^{(j)}_\nu(z)) = P^{(j)}_\lambda(r_\lambda) A^k_\lambda(r_\lambda),$$

where $P^{(j)}_\lambda(r_\lambda)$ is an idempotent of rank $d_\nu$ which commutes with $A_\lambda(r_\lambda)$ and is given by

$$P^{(j)}_\lambda(r_\lambda) = \lim_{z \to \lambda} \sum_{j=0}^{d_\nu-1} P_z(r^{(j)}_\nu(z)).$$

5.7. Relation of $\Upsilon^{[\nu]}_\chi$ to $\varphi_\nu(s)$. The total algebraic multiplicity $N$ of a real resonance point $r_\lambda$ can be naturally split into the sum of $m$ integers $d_1, \ldots, d_m$, where $m$ is the geometric multiplicity of $r_\lambda$. Namely, $d_\nu$ is the size of the $\nu$th Jordan cell of the compact operator $A_\lambda(s)$ corresponding to the eigenvalue $(s - r_\lambda)^{-1}$. To a triple $(\lambda; H_\lambda, V)$ we can also associate $m$ eigenpaths $\varphi_\nu(s)$, $s \in \mathbb{R}$, of $H_s = H_0 + sV$. With each of these eigenpaths $\varphi_\nu(s)$ we can also associate an integer, its order. Finally, there is a third way to split $N$ into the sum $d_1 + \ldots + d_m$, where $d_\nu$ is the length of the $\nu$th cycle of resonance points. Theorem 5.7 shows that these three sets of integers are identical, so our usage of the same notation $d_\nu$ in all these instances is justified. We now show that $d_\nu$ is equal to $\dim \Upsilon^{[\nu]}_\chi$. One can expect that there should be a deeper connection between this vector space and the eigenpath $\varphi_\nu(s)$. The following theorem demonstrates this connection.

**Theorem 5.25.** If $\varphi_\nu(s)$ is an eigenpath of order $d_\nu$, then

(i) the vectors $\varphi_\nu(r_\lambda)$, $\varphi_\nu^{(d_\nu-1)}(r_\lambda)$, $\ldots$, $\varphi_\nu^{(d_\nu-1)}(r_\lambda)$ belong to the vector space $\Upsilon^{[\nu]}_\chi(r_\lambda)$, and

(ii) for each $j = 1, \ldots, d_\nu - 1$, $A_\lambda(r_\lambda) \varphi_\nu^{(j)}(r_\lambda) = j \varphi_\nu^{(j-1)}(r_\lambda)$.

**Proof.** This proof uses divided differences $\varphi_\nu^{[j]}$ of the function $\varphi_\nu$ taken at resonance points of cycle $\nu$.

(i) The function $\varphi_\nu(s)$ is a holomorphic function of $s$ in a neighbourhood of $s = r_\lambda$. For each $j = 1, 2, \ldots, d_\nu$ we consider the divided difference of order $j$ for this function evaluated at $j$ resonance points $r^{(0)}_\nu(z), \ldots, r^{(j-1)}_\nu(z)$:

$$\varphi_\nu^{[j]}(r^{(0)}_\nu(z), \ldots, r^{(j-1)}_\nu(z)).$$

As $z \to \lambda$, all resonance points $r^{(0)}_\nu(z), \ldots, r^{(j-1)}_\nu(z)$ approach $r_\lambda$, and therefore, by a well-known property of high order divided differences, this function approaches the $j$th derivative $\varphi_\nu^{(j)}(r_\lambda)$ of $\varphi_\nu(s)$ up to a constant factor:

$$\lim_{z \to \lambda} \varphi_\nu^{[j]}(r^{(0)}_\nu(z), \ldots, r^{(j-1)}_\nu(z)) = \frac{1}{j!} \varphi_\nu^{(j)}(r_\lambda).$$
Hence, using this equality, Proposition 5.8 and (5.7), we get
\[
\lim_{z \to \lambda} d_{\nu-1} \sum_{k=0}^{d_{\nu-1}} P_z(r^{(k)}_{\nu}(z)) \varphi[j]_{\nu}(r^{(0)}_{\nu}(z), \ldots, r^{(j-1)}_{\nu}(z))
\]
\[
= \left( \lim_{z \to \lambda} d_{\nu-1} \sum_{k=0}^{d_{\nu-1}} P_z(r^{(k)}_{\nu}(z)) \right) \left( \lim_{z \to \lambda} \varphi[j]_{\nu}(r^{(0)}_{\nu}(z), \ldots, r^{(j-1)}_{\nu}(z)) \right)
\]
\[
= \frac{1}{j!} P_{\lambda}(r_{\nu}) \varphi[j]_{\nu}(r_{\nu}).
\]
On the other hand, since
\[
(5.25)
\]
\[
P_z(r^{(k)}_{\nu}(z)) \varphi[j]_{\nu}(r^{(0)}_{\nu}(z), \ldots, r^{(j-1)}_{\nu}(z)) = \delta_{kl} \varphi[j]_{\nu}(r^{(l)}_{\nu}(z))
\]
and since \( P_z(r^{(k)}_{\nu}(z)) P_z(r^{(l)}_{\nu}(z)) = 0 \) for \( k \neq l \), we also have
\[
\lim_{z \to \lambda} d_{\nu-1} \sum_{k=0}^{d_{\nu-1}} P_z(r^{(k)}_{\nu}(z)) \varphi[j]_{\nu}(r^{(0)}_{\nu}(z), \ldots, r^{(j-1)}_{\nu}(z)) = \lim_{z \to \lambda} \varphi[j]_{\nu}(r^{(0)}_{\nu}(z), \ldots, r^{(j-1)}_{\nu}(z))
\]
\[
= \frac{1}{j!} \varphi[j]_{\nu}(r_{\nu}).
\]
Hence, \( P_{\lambda}(r_{\nu}) \varphi[j]_{\nu}(r_{\nu}) = \varphi[j]_{\nu}(r_{\nu}) \) and therefore \( \varphi[j]_{\nu}(r_{\nu}) \in Y_{\lambda}^{[\nu]}(r_{\nu}) \).

(ii) This proof uses the same argument. We denote by \( M \varphi \) the function
\[
(M \varphi)(s) = (s - r_{\nu}) \varphi(s).
\]
On one hand, using Proposition 5.21 (with \( k = 1 \)) and (5.24), we have
\[
(E) := \lim_{z \to \lambda} d_{\nu-1} \sum_{k=0}^{d_{\nu-1}} (r^{(k)}_{\nu}(z) - r_{\nu}) P_z(r^{(k)}_{\nu}(z)) \varphi[j]_{\nu}(r^{(0)}_{\nu}(z), \ldots, r^{(j-1)}_{\nu}(z)) = \frac{1}{j!} A_{\lambda}^{[\nu]}(r_{\nu}) \varphi[j]_{\nu}(r_{\nu}).
\]
On the other hand, using (5.25) one can infer that
\[
\sum_{k=0}^{d_{\nu-1}} (r^{(k)}_{\nu}(z) - r_{\nu}) P_z(r^{(k)}_{\nu}(z)) \varphi[j]_{\nu}(r^{(0)}_{\nu}(z), \ldots, r^{(j-1)}_{\nu}(z)) = (M \varphi)^{[j]}(r^{(0)}_{\nu}(z), \ldots, r^{(j-1)}_{\nu}(z)),
\]
so that
\[
(E) = \lim_{z \to \lambda} (M \varphi)^{[j]}(r^{(0)}_{\nu}(z), \ldots, r^{(j-1)}_{\nu}(z)),
\]
\[
= \frac{1}{j!} \frac{d^j}{ds^j} (M \varphi)(s) \bigg|_{s=r_{\nu}}
\]
\[
= \frac{1}{(j-1)!} \varphi^{(j-1)}(r_{\nu}),
\]
where the second equality follows from (5.24). Hence, \( A_{\lambda}^{[\nu]}(r_{\nu}) \varphi[j]_{\nu}(r_{\nu}) = j \varphi^{(j-1)}(r_{\nu}). \)

**Corollary 5.26.** Dimension of the vector space \( Y_{\lambda}^{[\nu]}(r_{\nu}) \) is equal to \( d_{\nu} \).

**Proof.** The sum of dimensions of the vector spaces \( Y_{\lambda}^{[\nu]}(r_{\nu}), \nu = 1, \ldots, m, \) is equal to the algebraic multiplicity \( N \) of the resonance point \( r_{\nu} \). The sum of numbers \( d_{\nu}, \nu = 1, \ldots, m \) is also equal to \( N \). Since in addition to this, by Theorem 5.25 we have \( \dim Y_{\lambda}^{[\nu]}(r_{\nu}) \geq d_{\nu} \) for all \( \nu \), it follows that \( \dim Y_{\lambda}^{[\nu]}(r_{\nu}) = d_{\nu} \). \qed
Now we are in position to give a second proof of Theorem 3.12.

**Corollary 5.27.** Let \( \varphi_\nu(s) \) and \( \varphi_\mu(s) \) be two different eigenpaths of \( H_s \), \( s \in \mathbb{R} \). For all \( j = 0, 1, \ldots, d_\nu - 1 \) and all \( k = 0, 1, \ldots, d_\mu - 1 \),

\[
\left\langle \varphi_\nu^{(j)}(r_\lambda), V \varphi_\mu^{(k)}(r_\lambda) \right\rangle = 0.
\]

**Proof.** By Theorem 5.25 and Corollary 5.26, the vectors \( \varphi_\nu(r_\lambda) \), \( \varphi'(r_\lambda) \), \ldots, \( \varphi^{(d_\nu-1)}(r_\lambda) \) span the vector space \( \mathcal{Y}^{[\nu]}_\lambda = \text{im} P_\lambda^{[\nu]} \). Hence, using formulas (5.9) and (5.8), we have

\[
\left\langle \varphi_\nu^{(j)}(r_\lambda), V \varphi_\mu^{(k)}(r_\lambda) \right\rangle = \left\langle \varphi_\nu^{(j)}(r_\lambda), V P_\lambda^{[\mu]} \varphi_\mu^{(k)}(r_\lambda) \right\rangle = \left\langle P_\lambda^{[\mu]} \varphi_\nu^{(j)}(r_\lambda), V \varphi_\mu^{(k)}(r_\lambda) \right\rangle = 0.
\]

\( \square \)

### 5.8. Sign of a cycle.

Previous results show that the decomposition

\[
d_1 + d_2 + \ldots + d_m
\]

of algebraic multiplicity \( N \) admits a number of interpretations: \( d_\nu \) is the length of the \( \nu \)-th cycle, and the size of the \( \nu \)-th Jordan cell, and the largest of numbers such that \( \lambda_\nu^{(d_\nu)}(r_\lambda) \neq 0 \). In this subsection we show that to each cycle one can assign a sign \( \pm 1 \). This can be done in several equivalent ways.

**Theorem 5.28.** For each \( \nu = 1, \ldots, m \) and for all small enough \( \varepsilon > 0 \) and \( y > 0 \), signs of the following real numbers coincide:

1. \( \lambda_\nu(r_\lambda + \varepsilon) - \lambda_\nu(r_\lambda) \),
2. \( \left\langle \varphi_\nu(r_\lambda), V \varphi_\nu^{(d_\nu-1)}(r_\lambda) \right\rangle \),
3. \( \text{Im} r_\nu^{(0)}(z + iy) \), for all \( 0 < y << 1 \), and for all \( z \in I \), where \( I \) is one of the two intervals \((\lambda, \lambda + \varepsilon)\) or \((\lambda - \varepsilon, \lambda)\), on which the branch \( r_\nu^{(0)}(z + iy) \) takes positive values (such an interval and such a branch exist and are unique).

**Proof.** That signs of the first and second numbers coincide follows from (3.5). Equality of the signs of the first and third numbers can be inferred by considering the four cases: \( \lambda_\nu^{(d_\nu)}(r_\lambda) > 0 \) and \( \lambda_\nu^{(d_\nu)}(r_\lambda) < 0 \) for even and odd \( d_\nu \). This comparison is straightforward and therefore is omitted. \( \square \)

This sign will be called the *sign of a cycle* \( \nu \).

If an eigenvalue \( \lambda_\nu(s) \) crosses the threshold value \( \lambda \) from one side to the other, as \( s \) crosses \( r_\lambda \) in positive direction, then the sign of the corresponding cycle \( \nu \) is the contribution of the eigenvalue \( \lambda_\nu(s) \) to the spectral flow through \( \lambda \). If an eigenvalue \( \lambda_\nu(s) \) makes a U-turn at the threshold value \( \lambda \), then there is a dichotomous ambiguity in the way of assigning a sign to the cycle \( \nu \). Theorem 5.28 provides one way of choosing the sign of an eigenvalue making a U-turn, though, since such an eigenvalue does not contribute to the spectral flow, it is not essential which way to choose.

### 5.9. Resonance index and intersection number.

Let \( \text{sign}(\nu) \) be the sign of cycle \( \nu \), and let

\[
b_\nu = \begin{cases} 
0, & \text{if } d_\nu \text{ is even,} \\
1, & \text{if } d_\nu \text{ is odd.} 
\end{cases}
\]

The intersection number through a resonance point \( r_\lambda \) is equal to

\[
(5.26) \quad \sum_{\nu=1}^{m} b_\nu \text{sign}(\nu).
\]
Indeed, the value of $b_{\nu}$ determines whether the corresponding eigenvalue function $\lambda_{\nu}(s)$ makes a U-turn or not at $s = r_{\lambda}$, and if it does not, the value of $\text{sign}(\nu)$ shows whether the eigenvalue $\lambda_{\nu}(s)$ crosses the threshold value $\lambda$ in positive or in negative direction. Whatever the sign of cycle $\nu$ is, it does not contribute to the intersection number, if $b_{\nu} = 0$. Further, Theorems 5.2, 5.7 and 5.28 imply that cycles with even $d_{\nu}$ do not contribute to the resonance index, while cycles with odd $d_{\nu}$ contribute the number $\text{sign}(\nu)$. Hence, we proved the following

**Theorem 5.29.** The intersection number (5.26) of eigenvalues of a path $H_{r}$, $r \in [0,1]$, through $\lambda$ is equal to the total resonance index.

Since each cycle $\nu$ contributes one of the three numbers $\pm 1$ or 0 to the TRI, it follows that

$$|\text{ind}_{\text{res}}(\lambda; H_{r}, V)| \leq m.$$  

This is the U-turn inequality which holds for a.e. $\lambda$ inside essential spectrum too, [Az6] Theorem 10.1.6].

5.10. **A representation of $P_{\lambda}(r_{\lambda})$.** Let $T$ be an operator of rank $N < \infty$. If $b_{1}, \ldots, b_{N}$ is a basis of $\text{im}(T)$ and if $\alpha$ is an invertible $N \times N$ matrix, then there exists a unique basis $a_{1}, \ldots, a_{N}$ of $\text{im}(T^{*})$, such that

$$T = \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_{ij} \langle a_{i}, \cdot \rangle b_{j}. \tag{5.28}$$

Also, if $(a_{i})$ and $(b_{i})$ are bases of $\text{im}(T^{*})$ and $\text{im}(T)$ respectively, then there exists a unique invertible matrix $\alpha$ such that the equality (5.28) holds. In the case of a finite-rank operator $P_{\lambda}(r_{\lambda})$, there exists one natural basis

$$\frac{1}{j!} \varphi_{\nu}^{(j)}(r_{\lambda}), \quad \nu = 1, \ldots, m, \quad j = 0, 1, \ldots, d_{\nu} - 1,$$

of the vector space $\text{im}(P_{\lambda}(r_{\lambda})) = \Upsilon_{\lambda}(r_{\lambda})$, provided by part (i) of Theorem 5.25 and Corollary 5.26. Since by (1.11) and (1.10) $\text{im}(P_{\lambda}^{*}(r_{\lambda})) = V \text{im}(P_{\lambda}(r_{\lambda}))$, we also have a natural basis

$$\frac{1}{j!} V \varphi_{\nu}^{(j)}(r_{\lambda}), \quad \nu = 1, \ldots, m, \quad j = 0, 1, \ldots, d_{\nu} - 1,$$

of the vector space $\text{im}(P_{\lambda}^{*}(r_{\lambda})) = \Psi_{\lambda}(r_{\lambda})$. Hence, there exists a unique invertible $N \times N$ matrix $\alpha$, such that

$$P_{\lambda}(r_{\lambda}) = \sum_{\mu=1}^{m} \sum_{\nu=1}^{m} \sum_{k=0}^{d_{\nu}-1} \sum_{j=0}^{d_{\nu}-1} \frac{1}{k! j!} \alpha_{\mu \nu}^{k j} \langle V \varphi_{\mu}^{(k)}(r_{\lambda}), \cdot \rangle \varphi_{\nu}^{(j)}(r_{\lambda}). \tag{5.29}$$

Since $P_{\lambda}(r_{\lambda})$ is an idempotent, we have $P_{\lambda}(r_{\lambda}) \varphi_{\nu}^{(j)}(r_{\lambda}) = \varphi_{\nu}^{(j)}(r_{\lambda})$. Therefore, the matrix $\alpha$ is the transpose-inverse of the $N \times N$ matrix

$$(b_{\mu \nu}^{k j}) := \left( \frac{1}{k! j!} \langle V \varphi_{\mu}^{(k)}(r_{\lambda}), \varphi_{\nu}^{(j)}(r_{\lambda}) \rangle \right).$$

Previous results imply that the numbers $b_{\mu \nu}^{k j}$ are real. By Corollary 5.27, this matrix has the following property:

$$b_{\mu \nu}^{k j} = \delta_{\mu \nu} b_{\mu \nu}^{k j}. \tag{5.30}$$

Hence, the matrix $b$ is a direct sum of $m$ matrices $b_{\nu}^{k j}$ of size $d_{\nu} \times d_{\nu}$. 
Further, by part (ii) of Theorem 5.25, the scalar product
\[ \frac{1}{k!j!} \langle V \varphi^{(k)}(r_\lambda), \varphi^{(j)}(r_\lambda) \rangle \]
depends only on \( k+j \), and if \( k+j \leq d_\nu - 2 \), then it is zero. Hence, the matrix \( (b_{k,j}^{\nu}) \) is a Hankel matrix with zeros above the skew-diagonal. It follows that the inverse of \( (b_{k,j}^{\nu}) \) is a Hankel matrix with zeros below the skew-diagonal. Thus, we have proved the following theorem.

**Theorem 5.30.** The idempotent operator \( P_\lambda(r_\lambda) \) can be written in the form (5.29), where the \( N \times N \) matrix \( \alpha \) is a direct sum of self-adjoint skew-upper triangular Hankel matrices of sizes \( d_1, \ldots, d_m \).

### 5.11. Signature of the resonance matrix \( VP_\lambda(r_\lambda) \)

In this subsection we prove the equality
\[(5.31) \quad \text{ind}_{\text{res}}(\lambda; H_{r_\lambda}, V) = \text{sign} VP_\lambda(r_\lambda).\]

This equality was proved in [Az6] in a more general setting of \( \lambda \) from the essential spectrum. Here we give a new proof, which easily follows from previous results. The proof is based on the following two well-known lemmas.

**Lemma 5.31.** The signature of a self-adjoint skew-upper triangular \( d \times d \) Hankel matrix is zero, if \( d \) is even, and is equal to the sign of the skew-diagonal entry, if \( d \) is odd.

**Lemma 5.32.** If \( b_1, \ldots, b_d \) are linearly independent vectors in a Hilbert space and \( \alpha \) is a \( d \times d \) self-adjoint matrix then the signature of the self-adjoint operator
\[ T = \sum_{i=1}^{d} \sum_{j=1}^{d} \alpha_{ij} \langle b_i, \cdot \rangle b_j \]
is equal to the signature of \( \alpha \).

**Theorem 5.33.** The equality (5.31) holds.

**Proof.** For each \( \nu = 1, \ldots, m \), signs of the main skew-diagonal entries of the \( d_\nu \times d_\nu \) Hankel matrices (5.30) and \( (\alpha_{k,j}^{\nu}) \) from (5.29) are equal. Hence, by Lemma 5.31 and Theorems 5.28 and 5.29 the total resonance index is equal to the signature of the matrix \( (\alpha_{k,j}^{\nu}) \).

We have from (5.29)
\[ VP_\lambda(r_\lambda) = \sum_{\mu,k} \sum_{\nu,j} \frac{1}{k!j!} \alpha_{k,j}^{\nu} \langle V \varphi^{(k)}(r_\lambda), \cdot \rangle V \varphi^{(j)}(r_\lambda). \]

By Lemma 5.32 signature of this operator is equal to the signature of the matrix \( \alpha \).

\[ \square \]

### 6. On stability of resonance index

In this section we study the behaviour of the resonance index \( \text{ind}_{\text{res}}(\lambda; H_{r_\lambda}, V) \) as a function of the perturbation \( V \).
Topology of the vector space of directions. In previous sections we worked with a fixed direction $V$ and for this reason there was no need in having some topology in the real vector space of directions $A_0$. Now we are going to consider stability of resonance index $\text{ind}_{res}(\lambda; H_{\lambda}, V)$ with respect to small perturbations of $V$ and therefore we need to discuss the topology of $A_0$.

The approach taken is to impose conditions on the topology of the affine space $A$ which allow to prove stability results. The conditions imposed on the topology of $A$ hold trivially if the vector space of directions $A_0$ consists of bounded self-adjoint operators and if the topology of $A_0$ is the norm topology or stronger.

**Assumption 6.1.** The real vector space $A_0$ is endowed with a norm $\| \cdot \|_{A_0}$ such that for some non-real $z$ and for some $H_0 \in A$ the following three conditions hold:

1. (VR) the function $A_0 \ni V \mapsto VR_z(H_0) \in B(H)$ is continuous.
2. (VRV1) the product $V_1 R_z(H_0)V_2$ is compact.
3. (VRV2) the product $V_1 R_z(H_0)V_2$ is a continuous function of $V_1$ and $V_2$.

Since Assumption 1.1 implies compactness of $V_1 \text{Im} R_z(H_0)V_2$, the condition (VRV1) is equivalent to compactness of $V_1 \text{Re} R_z(H_0)V_2$.

Topology in $A_0$ induces a topology in the affine space $A$. We assume that $A$ is endowed with this topology.

**Lemma 6.2.**

(i) For any $V \in A_0$ and any $H \in A$ \(\lim_{y \to \infty} \|VR_{\lambda + iy}(H)\| = 0\).

(ii) For any non-real $z$ and for any $H \in A$ the operator $VR_z(H)$ jointly continuously depends on $z \in \mathbb{C} \setminus \mathbb{R}$, $V \in A_0$ and $H \in A$, and

(iii) For any non-real $z$ the operator $R_z(H)$ jointly continuously depends on $z \in \mathbb{C} \setminus \mathbb{R}$ and on $H \in A$.

**Proof.** (i) We have $VR_{\lambda + iy}(H) = VR_{\lambda + i}(H) \frac{H - \lambda - i}{H - \lambda + iy}$. The operator $VR_{\lambda + i}(H)$ is compact, and the operator $\frac{H - \lambda - i}{H - \lambda + iy}$ converges to zero in $*$-strong topology as $y \to \infty$. Hence, [Y] Lemma 6.1.3 completes proof.

(ii) The first resolvent identity combined with (VR) implies that $VR_z(H_0)$ depends jointly continuously on $z$ and $V$. Further, by the second resolvent identity we have

\begin{equation}
VR_z(H) = VR_z(H_0)(1 + (H - H_0)R_z(H_0))^{-1}.
\end{equation}

For non-real $z$ the operator $1 + (H - H_0)R_z(H_0)$ is invertible. Hence, this equality shows that the operator $VR_z(H)$ depends continuously on $z$, $V$ and $H$. The item (iii) is proved by the same argument. \hfill \Box

**Lemma 6.3.** If $A_0$ is endowed with a norm which satisfies Assumption 6.1, then the operator $V_1 R_z(H_0)V_2$ is compact for any non-real $z$ and any $H \in A$, and it jointly continuously depends on $z \in \mathbb{C} \setminus \mathbb{R}$, $V_1, V_2 \in A_0$ and $H \in A$.

**Proof.** By the first resolvent identity we have

\begin{equation}
V_1 R_z(H_0)V_2 - V_1 R_w(H_0)V_2 = (z - w)V_1 R_z(H_0)R_w(H_0)V_2.
\end{equation}

Since by Assumption 1.1 the right hand side is compact, the operator $V_1 R_z(H_0)V_2$ is compact for any non-real value of $z$, provided it is compact for some value of $z$. \hfill \Box
For $H = H_0 + V_3$ we have
\[
V_1 R_z(H)V_2 = V_1 R_z(H_0)(1 + V_3 R_z(H_0))^{-1}V_2
= V_1 R_z(H_0) \sum_{k \geq 0} (-1)^k (V_3 R_z(H_0))^k V_2
(6.3)
= V_1 R_z(H_0)V_2 + \sum_{k \geq 1} (-1)^k (V_3 R_z(H_0))^{k-1} \cdot [V_3 R_z(H_0)V_2].
\]

For large $y = \text{Im} z$ convergence of the series and compactness of this operator follow from Lemma 6.2(i) and (VRV1). The first summand and the product in the pair of square brackets are continuous by the assumption (VRV2). Since for large enough $y$ the geometric series converges uniformly, for such $y$ the last series depends continuously on $V_3$ by assumption (VR). For other values of $y$ the claim can now be inferred from (6.2). \hfill \Box

Usually we denote a resonance point of a triple $(\lambda; H_0, V)$, where $H_0$ is a $\lambda$-regular operator, by $r_\lambda$. But in this section for convenience we assume that $r_\lambda = 0$, so that the operator $H_0$ itself is $\lambda$-resonant.

**Lemma 6.4.** If $\mathcal{A}_0$ is endowed with a norm which satisfies Assumption 6.1 then the sets of all (a) regular and (b) simple directions at a resonance point $H_0$ are open in the norm of $\mathcal{A}_0$.

**Proof.** (a) Let $V$ be a regular direction. By continuity of the mapping $V \mapsto R_z(H)V$ (Lemma 6.2(ii)), there exists a neighbourhood $O_V$ of $V$ in $\mathcal{A}_0$, such that for all $W \in O_V$
\[
\|R_\lambda(H_0 + V)(V - W)\| < 1.
(6.4)
\]
Hence, by the second resolvent identity, for all $W \in O_V$
\[
R_\lambda(H_0 + W) = (1 + R_\lambda(H_0 + V)(V - W))^{-1}R_\lambda(H_0 + V),
\]
where the inverse exists due to (6.4). It follows that all directions from $O_V$ are regular.

(b) In this proof we use this characterisation of simple directions: a regular direction $V$ at a resonance point $H_0$ is simple if and only if the algebraic multiplicity $N$ of the eigenvalue $s^{-1}$ of the compact operator $R_\lambda(H_0 + sV)V$ is equal to the geometric multiplicity $m$ of that eigenvalue.

Since a simple direction is regular, by part (a) there exists a neighbourhood $O_V$ of a simple direction $V$ such that all directions $W$ from $O_V$ are regular. By Lemma 6.2(ii), the operator $R_\lambda(H_0 + W)W$ depends continuously on $W \in \mathcal{A}_0$ in some neighbourhood of $V$. Further, since $V$ is simple, the operator $R_\lambda(H_0 + sV)V$ has $s^{-1}$ as an eigenvalue of algebraic and geometric multiplicity $m$. Perturbation of the direction $V$ may not change the geometric multiplicity of the eigenvalue $s^{-1}$, since this number is the dimension of the eigenspace $V_\lambda$ of $H_0$ and thus it depends only on $H_0$. Hence, a perturbation of $V$ may not decrease the algebraic multiplicity of $s^{-1}$, but it may increase it.

The operator $R_\lambda(H_0 + sW)W$ also has $s^{-1}$ as an eigenvalue of geometric multiplicity $m$. Since the operator $R_\lambda(H_0 + sW)W$ is close to $R_\lambda(H_0 + sV)V$ in the operator norm for all $W$ close enough to $V$ in the norm of $\mathcal{A}_0$, it follows that there exists a neighbourhood $\hat{O}_V$ of $V$ such that for all $W$ from $\hat{O}_V$ the algebraic multiplicity $m$ of $s^{-1}$ does not increase for $R_\lambda(H_0 + sW)W$.

Thus, all directions from $\hat{O}_V$ are simple. \hfill \Box

6.2. Continuous dependence of $P_\lambda$ and $VP_\lambda$ on simple directions.

**Lemma 6.5.** Let $H_0$ be a resonant point and let $V$ be a regular direction of order 1. Then the idempotent $P_\lambda(H_0, V)$ depends on the direction $V$ continuously, that is, for any $\varepsilon > 0$ there
exists \( \delta > 0 \) such that if \( W \) is a regular direction with \( \| V - W \|_{A_0} < \delta \) then
\[
\| P_\lambda(H_0, V) - P_\lambda(H_0, W) \| < \varepsilon.
\]  
(6.5)

Proof. Let \( m \) be the geometric multiplicity of the resonance point \( H_0 \). By definition of the idempotent \( P_\lambda \), we have
\[
P_\lambda(H_0, V) = \frac{1}{2\pi i} \oint_C R_\lambda(H_0 + sV)V\, ds,
\]
where the contour \( C \) encloses only the resonance point \( s = 0 \) of the path \( H_0 + sV \), and this resonance point has both geometric and algebraic multiplicity \( m \), since \( V \) is a simple direction. By upper semi-continuity of spectrum we can choose \( \delta > 0 \) small enough so that, for all \( W \) with \( \| V - W \|_{A_0} < \delta \), inside the contour \( C \) there will be only one resonance point \( s = 0 \) of the path \( H_0 + sW \), and this resonance point will have geometric and algebraic multiplicities both equal to \( m \) (indeed, the total algebraic multiplicity of all resonance points inside \( C \) is at least \( m \) since \( H_0 \) has geometric multiplicity \( m \) and the total algebraic multiplicity is at most \( m \) due to upper semi-continuity of spectrum). Now, compactness of the contour \( C \) and joint continuity of \( R_\lambda(H_0 + sV)W \) (Lemma 6.2(ii)) imply that there exists a possibly smaller \( \delta > 0 \), if necessary, such that (6.5) holds as long as \( \| V - W \|_{A_0} < \delta \). \( \square \)

For directions of order greater than 1 this proof does not work, since in this case the algebraic multiplicity of the resonance point \( s = 0 \) is greater than \( m \), and as a consequence of this, while for the perturbed path \( H_0 + sW \) the point \( s = 0 \) will still have geometric multiplicity \( m \), other resonance points can appear inside the contour \( C \) which could have split from the resonance point \( s = 0 \).

**Theorem 6.6.** Let \( H_0 \) be a resonance point and let \( V \) be a regular direction of order 1. Then the resonance matrix \( VP_\lambda(H_0, V) \) depends on the direction \( V \) continuously, that is, for any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that if \( W \) is a regular direction such that \( \| V - W \|_{A_0} < \delta \) then
\[
\| VP_\lambda(H_0, V) - WP_\lambda(H_0, W) \| < \varepsilon.
\]

Proof. The mapping \( V \mapsto VR_\lambda(H_0 + sV)V \) is continuous by Lemma 6.3. Since the direction \( V \) is simple, it has a neighbourhood consisting of simple directions. Hence, inside a small enough contour \( C \), enclosing the resonance point \( s = 0 \), for all \( W \) from the neighbourhood there will be no other resonance points of the triple \( (\lambda; H_0, W) \). Hence, the formula
\[
VP_\lambda(H_0, V) = \frac{1}{2\pi i} \oint_C VR_\lambda(H_0 + sV)V\, ds
\]
completes the proof. \( \square \)

### 6.3. Homotopy stability of total resonance index.

We recall some definitions from previous sections. A point (that is, a self-adjoint operator) \( H_0 \) of the affine space \( A \) is resonant if a fixed real number \( \lambda \) which does not belong to the common essential spectrum \( \sigma_{ess} \) of operators from \( A \) is an eigenvalue of \( H_0 \). A resonant point \( H_0 \) is simple, if the eigenvalue \( \lambda \) has multiplicity one.

A regular direction \( V \) is simple at a resonance point \( H_0 \) if \( V \) is not tangent to \( \mathcal{R}(\lambda) \) at \( H_0 \); by this we mean that \( V \) is not the tangent vector of any differentiable path in \( \mathcal{R}(\lambda) \) which passes through \( H_0 \).

**Theorem 6.7.** Let \( H_0 \) be a resonance point and let \( V \) be a simple direction. Resonance index \( \text{ind}_{res}(\lambda; H_0, V) \) is stable under small perturbations of \( V \) within any finite-dimensional subspace of \( A_0 \).
Proof. Since order of $V$ is equal to 1, the rank $N$ of the resonance matrix $VP_\lambda(H_0, V)$ is equal to $m$, and, by Theorem 4.10, the direction $V$ is transversal. Hence, there exists a small enough convex neighbourhood of $V$ in the finite-dimensional subspace such that all directions from that neighbourhood are also transversal, and therefore, by Theorem 4.10 have order 1. Hence, the ranks of the resonance matrices $WP_\lambda(H_0, W)$ for all directions $W$ from the neighbourhood are equal to $m$. Since by Theorem 6.6 the resonance matrix depends continuously on $V$ for simple directions $V$, it follows that signature of the resonance matrix $VP_\lambda(H_0, V)$ is stable under small perturbations of $V$. \hfill \Box

While the resonance index of a direction is stable if the direction is simple, in general this is not true. Geometrically, the reason is that a tangent direction may cross the resonance set however small a perturbation of that direction is. It leads to a sudden change of the intersection number of that direction. This connection of resonance index with intersection number of eigenvalues was discussed earlier. Analytically, the reason for the instability of the resonance index is that a tangent direction may cross the resonance set however small a perturbation of that direction is. It leads to a sudden change of the intersection number of that direction. This connection of resonance index with intersection number of eigenvalues was discussed earlier. Analytically, the reason for the instability of the resonance index is that the resonance point may split into two or more resonance points as a non-simple direction $V$ is perturbed. In other words, as a non-simple direction $V$ is perturbed to a close direction $W$, near a resonance point $s = 0$ there may appear other resonance points, which may “take away” part of the resonance index. Thus, while the resonance index is not stable, the total resonance index is. In this subsection we prove the corresponding theorems.

Lemma 6.8. If a non-real complex number $r_\lambda^j$ is a resonance point of the triple $(\lambda; H_0, V)$ then the conjugate of $r_\lambda^j$ is also a resonance point of the triple $(\lambda; H_0, V)$ and moreover it has the same algebraic multiplicity.

This lemma is [Az6, Corollary 3.1.5].

Let $H_0$ be a resonance point and let $V$ be a regular direction. With every pair $(H_0, V)$ we can associate the set of resonance points of the pair. If the direction $V$ is slightly perturbed and if a resonance point is degenerate then it can split. The resonance point $s = 0$ itself will not move when $V$ is changed, since $H_0$ is resonant, but some other resonance points may break away from $s = 0$, if the direction $V$ is not simple. The following figure shows one of the possible scenarios.

\[ \text{Resonance points of } (H_0, V) : \bullet \ \ \ \ \bullet \ \ \ \ \bullet \ \ \ \ \bullet \ \ \ \ \text{Resonance points of } (H_0, W), \ \ \text{where } W \approx V : \]

\[ \text{Theorem 6.9. Let } V \text{ be a regular direction at a resonance point } H_0. \text{ Let } W \text{ be a small perturbation of } V \text{ and let } r_\lambda^1(H_0, W), \ r_\lambda^2(H_0, W), \ldots \text{ be resonance points of the triple } (\lambda; H_0, W) \text{ which belong to the group of the resonance point } s = 0 \text{ of the triple } (\lambda; H_0, V), \text{ where } H_r = H_0 + rV. \text{ Then the resonance index } \text{ind}_{res}(\lambda; H_0, V) \text{ is equal to the sum of resonance indices} \]

\[ \sum_j \text{ind}_{res}(\lambda; H_{r_\lambda^j}, W), \]

\[ \text{where the sum is taken over real resonance points of the group of } s = 0. \]

Proof. The resonance index $\text{ind}_{res}(\lambda; H_0, V)$ is equal to the difference $N_+ - N_-$, where $N_\pm$ is the number of resonance points of the triple $(\lambda + iy; H_0, V)$ for small enough $y$ which belong to the group of $s = 0$ and lie in $\mathbb{C}_\pm$. If the direction $V$ is deformed to $W$, the resonance point $s = 0$ of
the pair \((H_0, V)\) will in general split to some number of resonance points including the original resonance point \(s = 0\); the algebraic multiplicity of this resonance point may decrease but the geometric multiplicity will stay the same. We shall also refer to these resonance points of the pair \((H_0, W)\) as resonance points of the group of \(s = 0\). Some of these resonance points can be real and some can be non-real.

If \(\lambda\) is perturbed slightly to \(\lambda + iy\) with small positive \(y\) the non-real resonance points of the pair \((H_0, W)\) which belong to the group of \(s = 0\) will stay in the same half-plane, and the real resonance points of the group of \(s = 0\)

\[
\begin{align*}
r_1^\lambda(H_0, W), & \quad r_2^\lambda(H_0, W), \ldots \\
\end{align*}
\]

will shift from the real axis, thus giving a sum of resonance indices for the pair \((H_0, W)\). We have to show that this sum is equal to \(N_+ - N_-\).

Let \(M_+\) (respectively, \(M_-\)) be the number of resonance points of the pair \((H_0, W)\) which belong to the group of \(s = 0\) and which appear in the upper (respectively, lower) half plane, as \(\lambda\) is shifted to \(\lambda + iy\), with small \(y > 0\). The difference \(M_+ - M_-\) is equal to the resonance index of the triple \((\lambda; H_0, V)\), since we can deform \(V\) to \(W\) with \(y > 0\) fixed, and as we do so resonance points of \(V\) will get deformed to resonance points of \(W\) without crossing \(\mathbb{R}\). Hence, equality of the total resonance index of the pair \((H_0, W)\) to the resonance index of \((H_0, V)\) follows from Lemma 6.8 according to which the numbers of non-real resonance points of the group of \(s = 0\) in both half-planes are the same. \(\square\)

**Theorem 6.10.** Let \(V\) be a regular direction at a resonance point \(H_0\). Let \(H_0'\) be a small perturbation of \(H_0\) and let \(r_1^\lambda(H_0', V), \ r_2^\lambda(H_0', V), \ldots\) be resonance points of the triple \((\lambda; H_0', V)\) which belong to the group of the resonance point \(s = 0\) of the triple \((\lambda; H_0', V)\), where \(H_r' = H_0' + rV\). Then the resonance index \(\text{ind}\_{\text{res}}(\lambda; H_r', V)\) is equal to the sum of resonance indices

\[
\sum_j \text{ind}\_{\text{res}}(\lambda; H_r', V),
\]

where the sum is over real resonance points of the group of \(s = 0\).

Proof of this theorem follows almost verbatim the argument used in the proof of Theorem 6.9 and therefore is omitted. The only difference is that the resonance point \(s = 0\) itself may not only split but also shift.

![Resonance points diagram](image)

**Theorem 6.11.** Let \(H_0, H_1\) be two operators from \(A\) such that \(H_0\) and \(H_1\) are not resonant at \(\lambda \notin \sigma_{\text{ess}}\). Then there exist neighbourhoods \(U_0\) and \(U_1\) of \(H_0\) and \(H_1\) respectively such that for all \(H_0' \in U_0\) and all \(H_1' \in U_1\)

\[
\sum_{r \in [0, 1]} \text{ind}\_{\text{res}}(\lambda; H_r, V) = \sum_{r \in [0, 1]} \text{ind}\_{\text{res}}(\lambda; H_r', V'),
\]

where \(V' = H_1' - H_0'\) and \(H_r' = H_0' + rV'\).

Proof. This theorem follows immediately from Theorems 6.9 and 6.10. \(\square\)
6.4. Robbin-Salamon axioms for spectral flow and resonance index. In [RoSa] it was shown that for operators with compact resolvent the spectral flow of a path of operators can be uniquely characterised as a mapping which satisfies five axioms: Homotopy, Constancy, Catenation, Direct Sum, and Normalisation. In this subsection we show that the total resonance index satisfies these Robbin-Salamon axioms, following closely their paper [RoSa].

Assume Assumptions 1.1 and 6.1.

For any real numbers $a$ and $b$, $a < b$, and any two operators $H_a$ and $H_b$ from $\mathcal{A}$ let $PC^1([a, b], H_a, H_b)$ be the set of all continuous piecewise-$C^1$ paths $\{H_s, s \in [a, b]\}$, of operators from $\mathcal{A}$ such that (1) $\lambda$ does not belong to the spectrum of $H_a$ and $H_b$, (2) $\lambda$ does not belong to the spectrum of the vertex points of the $C^1$-subpaths of $H_s$, and (3) all operators $H_s, s \in [a, b]$, belong to a finite dimensional subspace of $\mathcal{A}$. The conditions (2) and (3) are not necessary, but they will allow to avoid unnecessary technical complications.

By definition, the total resonance index of a path $\{H_s, s \in [a, b]\}$ from $PC^1[a, b]$ is the total resonance index of a continuous piecewise linear path $K_s$ from $PC^1[a, b]$ which has the same end-points $H_a$ and $H_b$ and such that $\sup_{s \in [a,b]} \|H_s - K_s\|_{A_0}$ is small enough. Using a standard compactness argument and the homotopy invariance of the total resonance index, one can show that this definition is correct.

Let $\mu$ be a mapping $PC^1([a, b], H_a, H_b) \to \mathbb{Z}$, which satisfies the following Robbin-Salamon axioms for spectral flow:

(i) (Homotopy) If paths $f$ and $g$ from $PC^1([a, b], H_a, H_b)$ are homotopic, then $\mu(f) = \mu(g)$.

(ii) (Constancy) If a path $f$ from $PC^1([a, b], H_a, H_b)$ is constant then $\mu(f) = 0$.

(iii) (Catenation) If $f \in PC^1([a, b], H_a, H_b)$ and $g \in PC^1([b, c], H_b, H_c)$ and $f(b) = g(b)$, then $\mu(f \circ g) = \mu(f) + \mu(g)$, where $\circ$ denotes concatenation of paths.

(iv) (Direct Sum) If $f \in PC^1([a, b], H_a, H_b)$ and if $g \in PC^1([a, b], \tilde{H}_a, \tilde{H}_b)$, where $\tilde{H}_a, \tilde{H}_b$ are operators from another affine space $\tilde{\mathcal{A}}$, satisfying the above conditions, then $\mu(f \oplus g) = \mu(f) + \mu(g)$.

(v) (Normalisation) Let $\mathcal{H} = \mathbb{C}$ be a one-dimensional Hilbert space, let $a < \lambda < b$ and let $f(t) = t, f \in PC^1([a, b], a, b)$. Then $\mu(f) = 1$.

**Theorem 6.12.** Total resonance index satisfies all five Robbin-Salamon axioms.

**Proof.** All axioms except Homotopy Axiom are trivially satisfied by the total resonance index. The Homotopy Axiom is satisfied by Theorem 6.11.

The Normalisation Axiom is also trivially satisfied, but nevertheless we shall check it. So, let $\mathcal{H} = \mathbb{C}$, $a < \lambda < b$, $H_s = s$. Then $V = 1$, and

$$A_{\lambda + iy}(s) = R_{\lambda + iy}(H_s)V = (s - \lambda - iy)^{-1}.$$

For $y > 0$ the only pole $s = \lambda + iy$ of this meromorphic function, which belongs to the group of the pole $s = \lambda$ of $A_{\lambda}(s)$, is situated in the upper half-plane, and therefore $\text{ind}_{res}(\lambda; H_{\lambda}, V) = 1$. □

**Theorem 6.13.** A mapping $\mu$ which satisfies Robbin-Salamon axioms exists and is unique.

**Proof.** Original proof of this theorem from [RoSa] was given for operators with compact resolvent, but an inspection of that proof shows that with minor obvious changes it applies verbatim for this case too. Nevertheless, for readers’ convenience here we outline this proof.

In [RoSa] proof of the existence part is based on checking that the intersection number satisfies Robbin-Salamon axioms. Here for the existence part we can also refer to Theorem 6.12.

For the uniqueness, in [RoSa] it is shown that any putative spectral flow mapping coincides with the intersection number. Again, here we show that any putative spectral flow mapping coincides with total resonance index.
(A) For every path $H(s)$ from $PC^1([a,b], H_a, H_b)$ there exist an integer $n$ and a path $B(s)$ of self-adjoint $n \times n$ matrices such that $H \oplus B$ is homotopic to a constant path.

Proof. Firstly, the homotopy axiom allows to replace the path $H(s)$ by a path with only simple crossings (that is, resonance points with algebraic multiplicity $N = 1$). Further, the catenation axiom allows to reduce the problem to the case where $H(s)$ has only one simple crossing.

Without loss of generality we assume that $a = -1$, $b = 1$, $\lambda = 0$, and that the crossing point is $s = 0$. Let $\chi$ be the eigenvector of $H_0$, that is, $H_0 \chi = 0$. Let $B(s) = -s$, $B \in PC^1([-1,1], \mathbb{C})$, and let $\varphi(s), s \in [-1,1]$, be an eigenpath of $H_s$, that is, $H_s \varphi(s) = 0$. Let

$$\tilde{H}_{s,t} = \begin{pmatrix} H_s & t\varphi(s) \\ t\varphi^*(s) & -s \end{pmatrix}$$

Then $\tilde{H}_{s,0} = H_s \oplus B(s)$ and for $t > 0$ the operator $\tilde{H}_{s,t}$ is invertible for all $s \in [-1,1]$. Indeed, assume the contrary. Then, since $\lambda = 0$ does not belong to the essential spectrum, there exists a non-zero vector $\tilde{f} = f \oplus x \in H \oplus \mathbb{C}$ such that $\tilde{H}_{s,t}\tilde{f} = 0$. It follows that $t \langle \varphi(s), f \rangle = sx$, and $H_s f = -tx\varphi(s)$. The latter equality implies that

$$-tx \langle \varphi(s), \varphi(s) \rangle = \langle H_s \varphi(s), f \rangle = 0,$$

and hence, $x = 0$. Combining this with the former equality gives $\varphi(s) \perp f$. Also, $H_s f = 0$, and therefore, since $\lambda$ is a simple eigenvalue, $f$ is co-linear with $\varphi(s)$. Hence, $f = 0$.

So, the path $H_s \oplus B(s)$ is homotopic to a path $K_s$ without resonance points. Such a path can be continuously deformed to a constant path.

(B) Let $\tilde{\mu}$ be a putative spectral flow mapping. A piecewise linear path of self-adjoint matrices $B(s), s \in [a,b]$, is homotopic to a path of diagonal matrices. Hence, by the homotopy, direct sum and normalisation axioms both $\tilde{\mu}$ and the total resonance index of the path $B(s)$ are equal to

$$\frac{1}{2} \text{sign } B(b) - \frac{1}{2} \text{sign } B(a).$$

Now let $H(s)$ be any curve from $PC^1([a,b], H_a, H_b)$ and choose $B(s)$ as in part (A). Then it follows from homotopy and constant axioms that $\tilde{\mu}(H \oplus B) = 0$. Hence, by the direct sum axiom,

$$\tilde{\mu}(H) = -\tilde{\mu}(B) = -\text{TRI}(B) = \text{TRI}(H).$$

The proof in [RoSa] does not use the catenation axiom, which therefore follows from the other four axioms. In the proof above we used the catenation axiom for simplicity, though it is not necessary.

Since spectral flow is deemed to be characterised by Robbin-Salamon axioms, Theorem 6.12 shows that total resonance index and spectral flow are identical notions. Nevertheless, in subsection 7.2 we give a direct proof of the equality

“total resonance index = total Fredholm index”.

6.5. Geometric properties of the resonance set. In this subsection we give proofs of some well-known geometric properties of the resonance set, with the aim to provide an intuitive interpretation of spectral flow in terms of resonance set.

Theorem 6.14. Assume Assumption 1.3. The resonance set $\mathcal{R}(\lambda)$ has co-dimension 1.

Proof. If codimension of $\mathcal{R}(\lambda)$ is $\geq 2$, then there exists a two-dimensional affine plane in $\mathcal{A}$ which intersects the resonance set transversally at a $\lambda$-resonant point $H_0$. A point $H_0 + V$ on a small circle neighbourhood of $H_0$ in this plane can be deformed to $H_0 - V$ along the circle. By
Theorem 6.7, all deformations are simple and have constant resonance index. This contradicts the equality \( \text{ind}_{\text{res}}(\lambda; H_0, -V) = -\text{ind}_{\text{res}}(\lambda; H_0, V) \).

At the same time, for \( \lambda \) inside essential spectrum the spectral flow is not path-independent, see [Az4] §8.3. This indicates that the resonance set may have co-dimensions greater than 1 for \( \lambda \) inside essential spectrum.

Theorem 6.14 implies that the intersection of the resonance set \( \mathcal{R}(\lambda) \) with any \( k \)-dimensional affine space which passes through a resonance point \( H_0 \) and which is parallel to a simple direction \( V \) has dimension \( k - 1 \).

According to Theorem 6.14, the resonance set \( \mathcal{R}(\lambda) \) divides a small enough neighbourhood of any simple resonance point \( H_0 \) into two parts. The operators in one of those parts have an eigenvalue slightly larger than \( \lambda \), the operators in the other part have an eigenvalue slightly smaller than \( \lambda \). We shall call these parts positive and negative. Resonance hyper-surfaces divide a small enough neighbourhood of a resonant operator \( H_0 \) into several parts, which will be called cells. If \( V \) is a regular direction at a resonance point \( H_0 \), then it belongs to one and only one of those cells, by which we mean \( \exists \varepsilon > 0 \forall s \in (0, \varepsilon) \) the operator \( H_0 + sV \) belongs to the cell.

Theorem 6.15. If a plane section of the resonance set consists of only simple curves then the number of curves in a neighbourhood of \( H_0 \) is not greater than the geometric multiplicity \( m \).

Proof. Each curve divides the plane into two parts: positive and negative. If there are \( M > m \) curves then some points of the plane section will be positive for all \( M \) curves and resonance negative for none, and some points will be negative for all \( M \) curves and positive for none. The total resonance index of a continuous piece-wise linear path starting from one of the latter points to one of the former points will be greater than \( m \). Those two points can also be connected by a continuous piece-wise linear path which has only one resonance point \( H_0 \). By homotopy stability of the total resonance index, this path will have total resonance index greater than \( m \) too. This contradicts to the U-turn inequality (5.27).

We say that a plane section of the resonance set is simple, if the section does not have non-simple resonance curves.

Corollary 6.16. In any simple plane section of the resonance set there are no more than \( 2m \) resonance cells in a neighbourhood of a resonance point \( H_0 \), where \( m \) is the geometric multiplicity of \( H_0 \).

Theorem 6.17. The resonance set does not have cusps.

Proof. If a cusp exists then at a vertex of a cusp there exists a simple direction \( V \) which can be continuously deformed to the direction \( -V \) in the set of simple directions. This implies that resonance indices of \( V \) and \( -V \) are equal, which is false.

7. Resonance index and Fredholm index

In this section we consider relationship of the total resonance index with a traditional definition of spectral flow, the total Fredholm index.

7.1. Resonance matrix as direction reduction. In this subsection for convenience we denote a resonance point by \( H_0 \), instead of usual \( H_{r\lambda} \).

Given a \( \lambda \)-resonant operator \( H_0 \) and a regular direction \( V \), to a triple \( \langle \lambda; H_0, V \rangle \) we can assign a finite-rank self-adjoint operator \( V P_\lambda \). Results of this subsection demonstrate that it is the operator \( V P_\lambda \) which is responsible for spectral flow generated by the direction \( V \).
Theorem 7.1. If $V$ is a regular direction at a resonance point $H_0$, then the direction $VP_\lambda$ is also regular.

Proof. Assume the contrary. Then for any $s \in \mathbb{R}$ there exists non-zero vector $f(s)$ such that

\begin{equation}
(H_0 + sVP_\lambda)f(s) = \lambda f(s).
\end{equation}

This equality can be rewritten as $(H_0 - \lambda)f(s) = -sVP_\lambda f(s)$. This equality implies that the vector $VP_\lambda f(s)$ is orthogonal to the eigenspace $\mathcal{V}_\lambda$. It also implies that

\begin{equation}
f(s) = -sS_\lambda P_\lambda f(s) + \text{order 1 vector}.
\end{equation}

Since $VP_\lambda f(s) \perp \mathcal{V}_\lambda$ and since $P_\lambda f(s)$ is a resonance vector, it follows from Theorem 2.4 that the vector $S_\lambda P_\lambda f(s)$ is a resonance vector. Hence, by the last equality, so is the vector $f(s)$. That is, $P_\lambda f(s) = f(s)$. Combined with (7.1), this gives $(H_0 + sV)f(s) = \lambda f(s)$. This equality contradicts the regularity of $V$. \hfill \Box

Theorem 7.2. For any regular direction $V$ at a resonance point $H_0$ and for any non-resonant $s \in \mathbb{C}$

\begin{equation}
R_\lambda(H_0 + sVP_\lambda)VP_\lambda = R_\lambda(H_0 + sV)VP_\lambda,
\end{equation}

where $P_\lambda = P_\lambda(H_0, V)$.

Proof. Applying the second resolvent identity (1.4) to the pair of self-adjoint operators $H_0 + sV$ and $H_0 + sVP_\lambda = H_0 + sV - sV(1 - P_\lambda)$ gives

\begin{equation}
(E) := R_\lambda(H_0 + sVP_\lambda)VP_\lambda = \left[1 - sR_\lambda(H_0 + sV)V(1 - P_\lambda)\right]^{-1}R_\lambda(H_0 + sV)VP_\lambda.
\end{equation}

Using the notation $A_\lambda(s) = R_\lambda(H_s)V$, we can rewrite this equality as follows:

\begin{equation}
(E) = \left[1 - sA_\lambda(s)(1 - P_\lambda)\right]^{-1}A_\lambda(s)P_\lambda.
\end{equation}

It follows from (1.8) that $A_\lambda(s)(1 - P_\lambda) = \tilde{A}_\lambda(s)$, where $\tilde{A}_\lambda(s)$ is the holomorphic (at $s = 0$) part of the meromorphic function $A_\lambda(s)$. Hence, for all small enough $s$ we can write

\begin{equation}
(E) = \left[1 - s\tilde{A}_\lambda(s)\right]^{-1}A_\lambda(s)P_\lambda = \left[1 + s\tilde{A}_\lambda(s) + s^2\tilde{A}_\lambda^2(s) + \ldots\right]A_\lambda(s)P_\lambda.
\end{equation}

Since the operators $A_\lambda(s)$ and $P_\lambda$ commute and since $\tilde{A}_\lambda(s)P_\lambda = 0$, it follows that $(E) = A_\lambda(s)P_\lambda$. Since both sides of this equality are holomorphic, this equality holds for all, not necessarily small, $s$. This is what is required. \hfill \Box

Theorem 7.3. For any regular direction $V$ at a resonance point $H_0$

\[ P_\lambda(H_0, VP_\lambda(H_0, V)) = P_\lambda(H_0, V) \]

and

\[ A_\lambda(H_0, VP_\lambda(H_0, V)) = A_\lambda(H_0, V). \]

Proof. Using definition (1.6) of the idempotent $P_\lambda$ and Theorem 7.2, we have

\[ P_\lambda(H_0, VP_\lambda(H_0, V)) = \frac{1}{2\pi i} \oint_C R_\lambda(H_0 + sVP_\lambda)VP_\lambda ds = \frac{1}{2\pi i} \oint_C R_\lambda(H_0 + sV)VP_\lambda ds = P_\lambda^2 = P_\lambda, \]
where $C$ is a contour enclosing the resonance point $s = 0$. Proof of the second equality is the same, but uses (1.7) instead of (1.6).

**Theorem 7.4.** Let $V$ be a regular direction at a resonance point $H_0$. Resonance matrices of directions $V$ and $VP_\lambda$ are equal.

*Proof.* The resonance matrix of the direction $V$ is equal to $VP_\lambda$ and the resonance matrix of the direction $VP_\lambda$ is equal to $VP_\lambda \cdot P_\lambda(H_0, VP_\lambda)$. By Theorem 7.3, these two operators are equal. □

Since the resonance index of a direction is equal to the signature of its resonance matrix (Theorem 5.3), we have the following corollary of Theorem 7.4.

**Theorem 7.5.** Let $V$ be a regular direction. Resonance indices of directions $V$ and $VP_\lambda$ are equal.

This theorem is important in that aspect that it often allows to replace a direction $V$ by a finite rank direction $VP_\lambda$.

**Definition 7.6.** We say that two directions $V_1$ and $V_2$ at a resonance point $H_0$ are plane homotopic, if $\exists \varepsilon > 0 \forall (s_1, s_2) \in [0, \varepsilon]^2 \setminus \{(0, 0)\}$ the operators $H_0 + s_1 V_1 + s_2 V_2$ and $H_0 - s_1 V_1 - s_2 V_2$ are regular at $\lambda$.

Geometrically, two directions $V_1$ and $V_2$ are plane homotopic, if one of them can be deformed to another within the affine plane they generate so that the half-interval being deformed stays outside the resonance set.

**Theorem 7.7.** If $V$ is a regular direction at a resonance point $H_0$, then the directions $V$ and $VP_\lambda$ are plane homotopic.

*Proof.* Firstly, by Theorem 7.1, the direction $VP_\lambda$ is regular. Let

$$H_{s,t} = H_0 + sV + tVP_\lambda.$$ 

Since $V$ is regular, for all small enough $s$ the operator $H_0 + sV - \lambda$ is invertible. We need to show that for all small enough $s$ and $t$ the operator $H_0 + sV + tVP_\lambda - \lambda$ is also invertible. By the second resolvent identity, we have

$$R_\lambda(H_{s,t}) = (1 + tR_\lambda(H_s)VP_\lambda)^{-1} R_\lambda(H_s)$$
$$= (1 + tA_\lambda(s)P_\lambda)^{-1} R_\lambda(H_s),$$

whenever the right hand side makes sense. From this equality we can see that to prove the claim it is enough to prove the following assertion: if $1 + tA_\lambda(s)$ is invertible (which is equivalent to existence of $R_\lambda(H_0 + (s + t)V)$), then $1 + tA_\lambda(s)P_\lambda$ is also invertible. Assume the contrary. Then, since $A_\lambda(s)P_\lambda$ is compact, there exists a non-zero vector $\chi$ such that

$$(1 + tA_\lambda(s)P_\lambda)\chi = 0.$$

Since $A_\lambda(s)$ and $P_\lambda$ commute, this equality implies $\chi = -tP_\lambda A_\lambda(s)\chi$. It follows that $P_\lambda \chi = \chi$ and therefore $(1 + tA_\lambda(s))\chi = 0$. Hence, $1 + tA_\lambda(s)$ is not invertible. □

**Theorem 7.7** provides another proof of the equality of the resonance index and of the signature of the resonance matrix for the case where $\lambda$ does not belong to the essential spectrum.

**Proposition 7.8.** Any two regular non-negative (or-non-positive) directions are plane homotopic.
Proof. Let $V_1$ and $V_2$ be two regular and non-negative directions at a resonance point $H_0$. Since $V_1$ is regular, for all small enough $s > 0$ the operator $H_0 + sV_1$ is non-resonant and near $\lambda$ there are only eigenvalues of $H_0 + sV_1$ which are larger than $\lambda$. Adding $tV_2$ can only increase these eigenvalues. Choosing $s$ and $t$ small enough we can also ensure that there are no other eigenvalues $H_0 + sV_1 + tV_2$ near $\lambda$. \qed

**Theorem 7.9.** Plane homotopic directions have equal resonance indices.

This is a special case of homotopy stability of resonance index, Theorem [6.11].

7.2. **Resonance index and Fredholm index.** In this section we consider relationship of the resonance index with the Fredholm index. For reader’s convenience we recall here some well-known definitions and theorems.

A bounded operator $T$ acting from a Hilbert space $\mathcal{H}$ to a Hilbert space $\mathcal{K}$ is Fredholm, if $T$ has closed range and if the kernel of $T$ and the kernel of $T^*$ are finite-dimensional. In this case the *index* of $T$ is the integer number

$$\text{ind}(T) = \dim \ker(T) - \dim \ker(T^*).$$

The word “bounded” in the definition of a Fredholm operator can be replaced by the word “closed”, and “Hilbert” can be replaced by “Banach”, but we do not need this. Since the index is sensitive to the choice of domain and range, one may also write $\text{ind}_{\mathcal{H},\mathcal{K}}(T)$.

We denote by $\mathcal{F}(\mathcal{H},\mathcal{K})$ the set of all (bounded) Fredholm operators from $\mathcal{H}$ to $\mathcal{K}$. If $\mathcal{H} = \mathcal{K}$ then one writes $\mathcal{F}(\mathcal{H})$ for $\mathcal{F}(\mathcal{H},\mathcal{H})$. The set $\mathcal{F}(\mathcal{H},\mathcal{K})$ has the following properties, proofs of which can be found in e.g. [Hoer, Chapter XIX].

1. For any compact operator $K$ on a Hilbert space $\mathcal{H}$ the operator $1 + K$ is Fredholm and $\text{ind}(1 + K) = 0$.

2. The set of Fredholm operators is open in the norm topology. The index $\text{ind}(T)$ is stable in the norm topology, that is, $\text{ind}$ is a locally constant function on $\mathcal{F}(\mathcal{H},\mathcal{K})$.

3. If $T \in \mathcal{F}(\mathcal{H},\mathcal{K})$ and $K: \mathcal{H} \to \mathcal{K}$ is compact, then $T + K \in \mathcal{F}(\mathcal{H},\mathcal{K})$ and $\text{ind}(T + K) = \text{ind}(T)$. (1) is a special case of this property.

4. If $T \in \mathcal{F}(\mathcal{H}_1,\mathcal{H}_2)$ and $S \in \mathcal{F}(\mathcal{H}_2,\mathcal{H}_3)$, then $ST \in \mathcal{F}(\mathcal{H}_1,\mathcal{H}_3)$ and $\text{ind}(ST) = \text{ind}(S) + \text{ind}(T)$.

5. $T \in \mathcal{F}(\mathcal{H},\mathcal{K})$ if and only if there exists a bounded operator $S: \mathcal{K} \to \mathcal{H}$ such that $ST - 1_{\mathcal{K}}$ and $TS - 1_{\mathcal{H}}$ are compact. Such an operator $S$ is called *parametrization* of $T$ and it is also Fredholm. Moreover, $\text{ind}(S) = - \text{ind}(T)$.

A pair of orthogonal projections $(P,Q)$ is called a Fredholm pair, if the operator $PQ: Q\mathcal{H} \to P\mathcal{H}$ is Fredholm. The *Essential co-dimension* of the pair $(P,Q)$ is the index of the operator $PQ$, it is denoted by $\text{ec}(P,Q)$.

A pair $(P,Q)$ is Fredholm iff $\|\pi(P - Q)\| < 1$, where $\pi: B(\mathcal{H}) \to Q(\mathcal{H})$ is the canonical epimorphism of the $C^*$-algebra of bounded operators $B(\mathcal{H})$ onto the Calkin $C^*$-algebra $Q(\mathcal{H}) = B(\mathcal{H})/K(\mathcal{H})$, where $K(\mathcal{H})$ is the norm-closed ideal of compact operators, for a proof see e.g. [BCPRSW, Lemma 4.1]. If $(P_1,P_2)$ and $(P_2,P_3)$ are Fredholm pairs, then so are the pairs $(P_1,P_3)$ and $(P_1,P_2)$ and

$$\text{ec}(P_1,P_2) = \text{ec}(P_1,P_2) + \text{ec}(P_2,P_3), \quad \text{ec}(P_2,P_1) = - \text{ec}(P_1,P_2).$$

Now we proceed to a discussion of J. Phillips’ definition [Ph1, Ph2] of spectral flow as total Fredholm index. The theory of spectral flow was developed for self-adjoint operators $H_0$ with compact resolvent and with some summability condition such as $p$-summability and $\theta$-summability (though $p$-summability implies $\theta$-summability which in its turn implies compactness of resolvent, we choose to mention both). This assumes that the spectrum of $H_0$ is discrete and so it has
no essential spectrum. The spectral flow theory originates in the analysis of elliptic differential operators $D$ acting on sections of vector bundles over compact manifolds, such as Dirac operators on spin manifolds, and these operators satisfy the condition of compact resolvent and some summability assumptions. Our aim here is to demonstrate directly that the spectral flow as total Fredholm index and the total resonance index are identical notions. But since the theory of the former was developed for operators with compact resolvent we assume here this condition.

An inspection of proofs of basic theorems of spectral flow theory shows that the summability conditions are used essentially. It is quite possible that double operator integral techniques may allow to adjust the theory so that it becomes applicable to operators with essential spectrum as long as zero (or more generally a point $\lambda$) does not belong to it, but carrying out this plan may or may not be straightforward. In any case, as it was demonstrated in [ACS], it is sufficient to assume only compactness of resolvent without summability conditions.

For the rest of this subsection, we assume that $H_0$ has compact resolvent and that $A_0$ is a subspace of the algebra of bounded operators with the operator norm. If $H_s$ is a continuous path in $A$ such that $H_0$ and $H_1$ are not $\lambda$-resonant, then the spectral flow of $\{H_s\}_{s \in [0,1]}$ through $\lambda$, by definition of J. Phillips, is the number

$$\text{sf}(\{H_s\}) = \sum_{j=1}^n \text{ec}(P_{s_{j-1}}, P_{s_j}),$$

where $P_s$ is the spectral projection $E_{[\lambda, \infty)}^{H_s}$, and $\{s_j\}_{j=0}^n$ is a partition of the interval $[0,1]$. The spectral flow is well-defined for all partitions with small enough diameter $\max_j |s_j - s_{j-1}|$, and it does not depend on the choice of such a partition, which easily follows from the additivity property of the essential co-dimension. Moreover, spectral flow is homotopy invariant. Since this is one of several definitions of spectral flow, we shall call it here total Fredholm index.

This preliminary material can be found in e.g. [Ph, Ph], [BCPRSW], [Az, §§1.5, 1.6] and [Hoer] Chapter XIX.

We give a direct proof of the equality of the total resonance index and total Fredholm index, which does not allude to the Robbin-Salamon uniqueness Theorem 6.13.

By Fredholm index of a regular direction $V$ at $H_0$ we mean the number

$$\text{ec} \left( E^{H_0 - \varepsilon V}_{[\lambda, \infty)}, E^{H_0 + \varepsilon V}_{[\lambda, \infty)} \right),$$

where $\varepsilon > 0$ is a small enough number. General theory [Ph, Ph2] shows that this essential co-dimension is independent from the choice of sufficiently small $\varepsilon > 0$.

**Theorem 7.10.** Total resonance index coincides with total Fredholm index.

**Proof.** Let $H_r$ be a continuous piecewise linear path which connects two $\lambda$-regular operators $H_0$ and $H_1$. Using a small perturbation of this path, we can modify it in such a way that all crossings of this path with the resonance set will occur at simple points and at simple directions. Since both the total resonance index and the total Fredholm index are homotopy stable (Theorem 6.11), the Catenation Axiom (which trivially holds for both the total resonance index and the total Fredholm index) reduces the matter to the case of a path $H_r$ which intersects the resonance set only once at a simple point and at a simple direction. Let $V$ be the direction. By Theorem 7.7 the direction $V$ is plane homotopic to $VP_\lambda$. By the Homotopy Axiom, it follows that directions $V$ and $VP_\lambda$ have the same resonance index and Fredholm index. Since $V$ is a simple direction at a simple point, the resonance matrix $VP_\lambda$ is a rank one self-adjoint operator. Depending on whether the resonance index of $V$ is $+1$ or $-1$, the operator $VP_\lambda$ is plane deformable to $\langle \chi, \cdot \rangle \chi$ or $-\langle \chi, \cdot \rangle \chi$, where $\chi$ is an eigenvector of the simple point where $H_r$ crosses the resonance set.
Finally, it is left to note that the Fredholm index of the regular direction $\langle \chi, \cdot \rangle \chi$ (respectively, $-\langle \chi, \cdot \rangle \chi$) is obviously equal to 1 (respectively, $-1$).

\section{Resonance index and spectral shift function}

The aim of this section is to demonstrate the equality

\text{spectral shift function} = \text{total resonance index}

outside the essential spectrum. Since this is a special case of an essentially stronger result which asserts that the total resonance index is equal to the singular spectral shift function for a.e. $\lambda$ \cite{Az1}, \cite{Az2}, we do not formulate any theorems here. It is also well-known that the spectral shift function outside essential spectrum satisfies Robbin-Salamon axioms, and therefore the total resonance index and spectral shift function coincide by uniqueness Theorem \cite{Az5, Az6}. Nevertheless, here we demonstrate the argument of the proof of the above-mentioned more general result in this special case, where it simplifies quite significantly while retaining one of the key points of the proof.

Given two self-adjoint operators $H_0$ and $H_1$ with trace class difference $V = H_1 - H_0$, the spectral shift function of the pair $(H_0, H_1)$ is a unique real-valued integrable function $\xi \in L^1(\mathbb{R})$, such that for all compactly supported functions $\varphi$ of class $C^2$ the Lifshitz-Krein trace formula \cite{Kr, L} holds:

$$\text{Tr} (\varphi(H_1) - \varphi(H_0)) = \int_{-\infty}^{\infty} \varphi'(\lambda) \xi(\lambda) \, d\lambda.$$  

The Birman-Solomyak formula \cite{BS} gives another remarkable representation for the spectral shift function, but this formula treats the SSF as a distribution:

$$\xi(\varphi) = \int_{0}^{1} \text{Tr} \left( V \varphi(H_r) \right) \, dr,$$

where $H_r = H_0 + rV$. This formula indicates that the SSF is an integral of a one-form $\text{Tr} \left( V \varphi(H_r) \right)$ on the affine space of trace class perturbations of $H_0$. This form is exact, so that the straight line $H_0 + rV$ connecting the operators $H_0$ and $H_1$ can be replaced by a piecewise smooth path \cite{AzS}. In the spectral flow theory there exist analytic integral formulas for spectral flow (due to Getzler and Carey-Phillips) which can be considered as analogues of the Birman-Solomyak formula with specifically chosen distribution $\varphi$, though the settings of the operator theoretic spectral shift function and the differential geometric spectral flow were different (the former with relatively trace class conditions imposed on the perturbation $V$ with more or less arbitrary self-adjoint $H_0$, while in the latter the perturbation $V$ is an arbitrary bounded self-adjoint operator with summability conditions imposed on $H_0$). Alan Carey indicated many times that the idea of possibility of expressing the spectral flow as an integral of one-form belongs to I. M. Singer (1974).

Starting with the Birman-Solomyak formula as the definition of the SSF, one can show that it satisfies Krein’s trace formula in the form

$$\text{Tr} (\varphi(H_1) - \varphi(H_0)) = \int_{-\infty}^{\infty} \varphi'(\lambda) \, dm_\xi(\lambda),$$

where $m_\xi$ is the spectral shift measure. Though proof of the Birman-Solomyak formula is somewhat simpler than that of the Lifshitz-Krein formula, it does not allow to prove that the SSF is absolutely continuous, so the original proof of Lifshitz-Krein formula is indispensable. Nevertheless, there are sufficient indications, some of which were mentioned above, that the Birman-Solomyak formula is more fundamental.
Outside the essential spectrum the spectral shift function coincides with the spectral flow. This connection was demonstrated in [ACS], though it seems unlikely that this connection was not known in some form before. Since the spectral flow is an inherently integer-valued function, the SSF is also integer-valued outside the essential spectrum, but inside the essential spectrum this is not the case. The celebrated Birman-Krein formula, which connects the SSF and the scattering matrix,

$$\det S(\lambda; H_1, H_0) = e^{-2\pi i \xi(\lambda)},$$

indicates that the reason for non-integrality of SSF is existence of a non-trivial scattering matrix. It turns out however [Az2, Az4] that the SSF admits a natural decomposition as a sum of two components, the absolutely continuous $\xi^{(a)}$ and singular spectral shift functions $\xi^{(s)}$, such that the second component $\xi^{(s)}$ is integer-valued almost everywhere inside the essential spectrum too. The definitions of the functions $\xi^{(a)}$ and $\xi^{(s)}$ are obtained by modified Birman-Solomyak formulas:

$$\xi^{(a)}(\varphi) = \int_0^1 \text{Tr}(V \varphi(H^{(a)}_r)) \, dr$$

and

$$\xi^{(s)}(\varphi) = \int_0^1 \text{Tr}(V \varphi(H^{(s)}_r)) \, dr,$$

where $H^{(a)}$ and $H^{(s)}$ stand for the absolutely continuous and singular parts of a self-adjoint operator $H$ respectively. It is shown in [Az4] that for trace class perturbations the measure $\xi^{(a)}$ is absolutely continuous and its density $\xi^{(a)}(\lambda)$ satisfies a modified Birman-Krein formula

$$\det S(\lambda; H_1, H_0) = e^{-2\pi i \xi^{(a)}(\lambda)}.$$

This formula combined with Birman-Krein formula implies integrality of $\xi^{(s)}$.

Definitions of functions $\xi^{(a)}$ and $\xi^{(s)}$ indicate that the generalised spectral flow one-form $\text{Tr}(V \delta(H))$ naturally splits into two components: absolutely continuous $\text{Tr}(V \delta(H^{(a)}))$ and singular $\text{Tr}(V \delta(H^{(s)}))$ spectral flow one-forms. But unlike the spectral flow one-form, these two components are not exact, see a counter-example in [Az4, §8.3]. Outside the essential spectrum the absolutely continuous spectral flow one-form drops out, and thus in this case the remaining singular part becomes exact. In terms of the resonance set $\mathcal{R}(\lambda)$ exactness of the singular spectral flow one-form is equivalent to the equality $\text{codim} \mathcal{R}(\lambda) = 1$, and while outside the essential spectrum the resonance set $\mathcal{R}(\lambda)$ always has co-dimension one, inside the essential spectrum this is not the case.

The definition of $\xi^{(s)}$ is hardly suitable for calculation of this function. In [Az3] (see also [Az6], Section 6) it was shown that the singular spectral shift function is equal to the total resonance index, which is something far easier to work with. Proof of this equality for values of $\lambda$ inside essential spectrum cannot be explained in a short space, but if $\lambda$ is outside the essential spectrum then the proof simplifies sufficiently to present it here.

It is probably worthwhile to stress here that the notion of resonance index was discovered in the course of work on the singular spectral shift function and scattering theory. Since the singular spectral shift function coincides with spectral shift function outside the essential spectrum, it was immediately clear that the total resonance index coincides with spectral flow, and this is how this paper was originated.

We assume that a self-adjoint perturbation $V$ of a self-adjoint operator $H_0$ is trace class. This assumption is not necessary for the equality in question:

$$\text{if } \lambda \notin \sigma_{ess} \text{ then } \xi(\lambda) = \sum_{r \in [0,1]} \text{ind}_{res}(\lambda; H_r, V),$$
but it will significantly simplify the proof. For relatively trace class perturbations the proof, which includes the main essential spectrum case too, will appear in [AzD]. We conjecture that this equality holds as long as $V$ is a relatively compact perturbation of $H_0$.

The Birman-Solomyak formula implies the equality

$$
\frac{1}{\pi} \int_{\mathbb{R}} \Im R_{\lambda+iy}(x) \xi(x) \, dx = \frac{1}{\pi} \int_{0}^{1} \Tr \left( V \Im R_{\lambda+iy}(H_r) \right) \, dr.
$$

The left hand side of this equality is the Poisson integral of the function $\xi$. Hence, a well-known property of the Poisson integral implies that as $y \to 0^+$, the left hand side converges to $\xi(\lambda)$ for a.e. $\lambda$. Hence, we will be done if we show that the right hand side converges to the total resonance index. Since $\lambda$ lies outside the common essential spectrum of operators $H_r$, the operator $\Im R_{\lambda+iy}(H_r)$ has zero limit for all values of $r$ from the interval $[0,1]$, except some special values of $r$ for which the resolvent $R_{\lambda+0i}(H_r) = R_\lambda(H_r)$ does not exist. Since $\lambda$ is outside the essential spectrum, these values of $r$ are those for which $\lambda$ is an eigenvalue of $H_r$, that is, they are resonance points. In other words, a hindrance for taking the limit $y \to 0^+$ in the right hand side is presence of resonance points. If there were no resonance points, the limit of the right hand side would be zero, which agrees with the fact that no eigenvalue of $H_r$ reached the point $\lambda$ and therefore spectral flow through $\lambda$ is zero.

So, the presence of resonance points in the domain $[0,1]$ of integration is thus a hindrance, but it is the presence of resonance points which makes the RHS non-zero and interesting. To overcome this hindrance we note that the integrand of the RHS

$$
\Tr \left( V \Im R_{\lambda+iy}(H_r) \right)
$$

is a meromorphic function of $r$. A small neighbourhood of the interval $[0,1]$ in the coupling constant complex plane for sufficiently small $y > 0$ contains poles of the functions $VR_{\lambda+iy}(H_s)$ and $VR_{\lambda-iy}(H_s)$, and these poles converge to the poles $r_\lambda^j$ of $VR_\lambda(H_s)$ from $[0,1]$ as $y \to 0$. We represent the path of integration $[0,1]$ as the sum of a path $L_1$, shown below, which circumvents the poles from above, and closed contours $C_1^j$, encircling poles of the group of $r_\lambda^j$ lying in $\mathbb{C}_+$. Clearly, as $y \to 0^+$ the integral over $L_1$ vanishes: there are no obstructions in the form of poles on $L_1$ in the limit and the integrand is zero when $y = 0$. (The following figures are taken from [Az7].)
We show that
\[
\frac{1}{\pi} \oint_{C^+} \Tr \left( V \Im R_{\lambda+iy}(H_s) \right) ds = \text{ind}_{\text{res}}(\lambda; H_{r_{\lambda}}, V)
\]
and this will complete the proof. It is enough to do this for one contour, so we omit the superscript index. We have
\[
\text{LHS} = \frac{1}{2\pi i} \oint_{C^+} \Tr \left( V R_{\lambda+iy}(H_s) - V R_{\lambda-iy}(H_s) \right) ds = \Tr \left( \frac{1}{2\pi i} \oint_{C^+} R_{\lambda+iy}(H_s)V ds \right) - \Tr \left( \frac{1}{2\pi i} \oint_{C^+} R_{\lambda-iy}(H_s)V ds \right).
\]
We note that these two integrals are equal to \( P_{\lambda+iy}^\uparrow (r_{\lambda}) \) and \( P_{\lambda-iy}^\uparrow (r_{\lambda}) \). It is left to note that traces of these idempotents are \( N_+ \) and \( N_- \) respectively.

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\( a \), restriction of \( V \) to the eigenspace \( V_\lambda \), p. 13

\( \mathcal{A} \), real affine space of self-adjoint operators \( H \), p. 4

\( \mathcal{A}_0 \), real vector space of self-adjoint operators \( V \), p. 4

\( A_z(s) \), the operator \( R_z(H)V \)

\( A_z(r_z) \), nilpotent operator, p. 5

\( A_\lambda^{[\nu]} \), restriction of \( A_\lambda(r_\lambda) \) to the vector space \( \mathcal{Y}_\lambda^{[\nu]} \), p. 10 14

\( B_z(s) \), the operator \( VR_z(H) \)

\( d \), order of a resonance point

\( d_\nu \), size of a \( \nu \)th Jordan cell, size of a \( \nu \)th resonance cycle

\( d_\nu \), order of eigenpath \( \varphi_\nu(s) \)

\( D_\lambda(s) \), the \((2,2)\) entry of resolvent \((H_r - \lambda)^{-1}\), (2.9), p. 16

\( D_j \), Laurent coefficients of \( D_\lambda(s) \), (2.23), p. 21

\( F_z(s) \), (2.6), p. 16

\( \mathcal{H} \), Hilbert space

\( \mathcal{H} \), Hilbert space, orthogonal complement of eigenspace \( V_\lambda \), p. 14

\( H \), self-adjoint operator from an affine space \( \mathcal{A} \)

\( \hat{H} \), restriction of \( H \) to \( \mathcal{H} \), p. 14

\( H_z \), a path of operators \( H_0 + sV \)

\( H(s) \), a path of operators for which \( \lambda \) is an eigenvalue, pp. 8 4 33

\( \text{ind}_{res}(\lambda; H, V) \), resonance index

\( m \), geometric multiplicity of eigenvalue \( \lambda \)

\( N \), algebraic multiplicity of eigenvalue \( \lambda \)

\( \mathcal{P} \), orthogonal projection onto \( \mathcal{H} \), p. 15

\( P_z(r_z) \), idempotent operator, p. 5

\( P_\lambda^{[\nu]} \), idempotent, (5.7), p. 43

\( P_z(r_\lambda) \), idempotent operator, p. 10

\( \mathcal{R}(\lambda) \), the resonance set, the set of operators \( H \) from \( \mathcal{A} \) for which \( \lambda \) is an eigenvalue

\( r \), coupling constant, usually a real number

\( r_\lambda \), a resonance point, p. 5

\( r_z \), a resonance point, p. 5

\( r_\nu^{(j)}(z) \), a resonance point from \( \nu \)th cycle, p. 41

\( r_\nu^{(j)}(z) \), a cycle of resonance points, p. 41

\( s \), coupling constant, a complex number

\( S_\lambda \), the operator \( R_\lambda(H_z)V \), (2.13), p. 13

\( V \), self-adjoint operator from the real vector space \( \mathcal{A}_0 \)

\( \hat{V} \), restriction of \( V \) to \( \mathcal{H} \), p. 15
$\nu$, the eigenspace of a resonant operator, p.14

$\hat{P} \hat{V} \hat{P}^\perp$, the $(1, 2)$ matrix element of $V$, p.15

$y$, the imaginary part of spectral parameter $z$

$Y$, p.22

$z$, spectral parameter, a complex number outside the common essential spectrum $\sigma_{\text{ess}}$

$\gamma$, a curve of resonant operators, p.37

$\lambda$, an eigenvalue, a real number outside the common essential spectrum $\sigma_{\text{ess}}$

$\lambda_\nu(s)$, a path of eigenvalues of $H_0 + sV$

$\nu$, eigenvalue function index, resonance cycle index, Jordan cell index

$\sigma_\lambda(s)$, equal to $(s - r_\lambda)^{-1}$, eigenvalue of $A_\lambda(s)$

$Y_z(r_z)$, the range of $P_z(r_z)$, p.5

$Y^{[\nu]}_\lambda$, the range of $P^{[\nu]}_\lambda$, p.43

$\varphi_\nu(s)$, a path of eigenvectors of $H_s = H_0 + sV$

$\chi(s)$, a path of operators for which $\lambda$ is an eigenvalue

Curve,

— regular, p.4
— resonant, p.4

Direction,

— of order $d$, p.6
— regular, p.4
— simple, p.6
— tangent, p.9
— tangent to order $k$, pp.34
— transversal, pp.9,34

Path,

— regular, p.4
— resonant, p.4
— standard, p.9

Point,

— of geometric multiplicity $m$
— resonance, p.4
— simple, p.4

Order of eigenpath p.26

— strict, p.26

Resonance index, p.8

— total, p.8

Resonance vector, p.6

— of depth $k$, p.6
— of order $k$, p.6

TRI, total resonance index, p.8