The Lattice of Closure Relations on a Poset

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Abstract

In this paper we show that the set of closure relations on a finite poset $P$ forms a supersolvable lattice, as suggested by Rota. Furthermore this lattice is dually isomorphic to the lattice of closed sets in a convex geometry (in the sense of Edelman and Jamison [EJ]). We also characterize the modular elements of this lattice (when $P$ has a greatest element) and compute its characteristic polynomial.

1 The lattice of closure relations

Let $P$ be a poset. A closure relation on $P$ is a map $H : P \to P$ such that for all $x, y \in P$:

1. $x \leq H(x)$
2. $x \leq y \Rightarrow H(x) \leq H(y)$ (monotone)
3. $H(H(x)) = H(x)$ (idempotent)

If $P = 2^S$, the set of all subsets of a set $S$, then we say $H$ is a closure on the set $S$. The structure of all closures on a set has been extensively studied by Ore [Or1,Or2].

We may partially order the set of all closure relations on a poset $P$ by setting $H \leq K$ if $H(x) \leq K(x)$ for all $x \in P$. A closure relation $H$ on $P$ may be regarded as a partition of the elements of $P$, by considering $x, y \in P$ to be in the same block of $H$ if $H(x) = H(y)$. If $H(x) \leq K(x)$ for all $x \in P$, then since $x \leq H(x) \leq K(x)$ for all $x \in P$, $H(x)$ must be in the same block as $x$ of $K$ (regarding $K$ as a partition of $P$.) Hence $H$ is a refinement of $K$ as partitions. The converse is evident, and so we have the useful observation that $H \leq K$ if and only if $H$ is a refinement of $K$ as partitions.

For the sake of simplifying the proofs, from now on, we will assume that the poset $P$ is finite. However, many of our results are valid under other finiteness conditions on $P$ (such as when $P$ has no infinite chains or no infinite ascending chains).

Theorem 1. The partial order of all closure relations on any poset $P$ forms a lattice, denoted $LC(P)$. It is a join-sublattice of the lattice $\Pi(P)$ of partitions of the elements of $P$, i.e the join in
$LC(P)$ is the same as the join in $\Pi(P)$.

Proof. Let $\{H_a\}$ be a set of closure relations on $P$. Then each $H_a$ represents a partition of the elements of $P$. Let $\bigvee H_a$ denote the join of the $\{H_a\}$ in the partition lattice $\Pi(P)$. We will first show that each block of $\bigvee H_a$ has a unique greatest element. Then we can define a map $J : P \to P$ by sending every $x \in P$ to the greatest element in its block of $\bigvee H_a$. Finally we show that $J$ obeys axiom 2 and so is a closure relation, and hence $J$ must be the join of the set $\{H_a\}$ in $LC(P)$.

To show each block of $\bigvee H_a$ has a greatest element, suppose we have two elements $a, b$ that are maximal in some block. By definition of join in $\Pi(P)$, this means that there exists a sequence $B_1, B_2, \ldots, B_n$ of blocks from the various $H_a$’s, such that $a \in B_1, b \in B_n$, and $B_k \cap B_{k+1} \neq \emptyset$. Let $z_k$ be the maximum element of $B_k$. We have $z_1 = a$ and $z_n = b$ by maximality, so if we prove that $z_k \leq a$ for all $k$, then we will have the contradiction $b \leq a$. Assume by induction that $z_k \leq a$ (true for $k = 1$), let $w \in B_k \cap B_{k+1}$, and assume $B_{k+1}$ is a block of the closure relation $H_i$. Then

$$z_{k+1} = H_i(w) \leq H_i(z_k) \leq H_i(a) = a.$$  

since if $H_i(a) > a$ then $a$ would not be maximal.

Now we show $J$ (as defined above) is a closure relation on $P$. Since $J$ sends each element to the maximum element in its block in $\bigvee H_a, x \leq J(x)$ and $J(J(x)) = J(x)$. Suppose $x \leq y$ in $P$. Define a sequence $x_1, x_2, \ldots \in P$ as follows: Let $x_0 = x$, and define $x_{k+1}$ by choosing a closure $H_k$ among $\{H_a\}$ satisfying $H_k(x_k) \neq x_k$, and then setting $x_{k+1} = H_k(x_k)$. By finiteness, this process must stop at some $x_n$, and $x_n = J(x)$. Define the sequence $y_1, y_2, \ldots \in P$ by applying the same sequence of $H_k$’s to $y$, and note that $x_k \leq y_k$ for all $k$ since each $H_k$ is a closure. Thus we have $J(x) = x_n \leq y_n \leq J(y).$ Thus $J$ is monotone, and hence defines a closure relation.

So far we have shown that $LC(P)$ is a join-sub-semilattice of $\Pi(P)$. To show it is a lattice, it suffices to note that it has a minimum element, viz. the identity closure, $I(x) = x$ for all $x \in P$, which must be less than any closure on $P$. □

The following proposition characterizes when $LC(P)$ is a sublattice of $\Pi(P)$ for posets $P$ with a greatest element.

**Proposition 2.** Let $P$ have a greatest element $\hat{1}$. Then $LC(P)$ is a sublattice of $\Pi(P)$ if and only if $\hat{0} + P$ is a lattice, where $\hat{0} + P$ is the poset obtained by adjoining a new least element $\hat{0}$ to $P$.

Remark: A similar (but harder to state) proposition holds even if $P$ has no greatest element.

Proof. ($\Rightarrow$): If $\hat{0} + P$ is not a lattice then there must exist four elements $a, b, c, d \in P$ with $c, d$ both maximal lower bounds for $a$ and $b$. Define the closure $H_a$ by $H_a(x) = a$ if $x \leq a$ and $H_a(x) = \hat{1}$ otherwise. Define $H_b$ similarly. If we form $H_a \land H_b$ in $\Pi(P)$ then both $c$ and $d$ will be maximal in the same block. Hence this partition does not correspond to a closure relation. Thus $H_a$ and $H_b$ cannot have the same meet in both $\Pi(P)$ and $LC(P)$.

($\Leftarrow$) Suppose $\hat{0} + P$ is a lattice and let $\{H_a\}$ be a set of closure relations on $P$. Since the meet operation in $LC(P)$ is precisely intersection of the blocks of the individual closure relations, their meet in $\Pi(P)$ and $LC(P)$ coincide if and only if every block in their meet in $\Pi(P)$ has a
greatest element. So suppose this is not the case, i.e. let \( a, b \) be maximal in some block and \( H_\alpha(a) = H_\alpha(b) = c_\alpha, \forall \alpha \). Let \( d \) be the greatest lower bound of the \( c_\alpha \) in the lattice \( \emptyset + P \). Then \( c_\alpha \geq d \geq a, b \) for all \( \alpha \), which implies that \( H_\alpha(d) = c_\alpha \) for all \( \alpha \). Hence \( d \) is in the same block as \( a \) and \( b \), contradicting their maximality. \( \square \)

## 2 Convexity and mlb-closure

In this section we relate the lattice of closure relations to the notion of a convex geometry as studied by Edelman and Jamison [EJ]. The following proposition is the crucial observation necessary for what follows:

**Proposition 3.** Let \( H \) be a closure relation on \( P \), and \( A \) its set of closed elements, i.e.

\[
A = \{ x \in P : H(x) = x \}.
\]

Then for any subset \( B \subseteq A \), all maximal lower bounds of \( B \) are in \( A \) (we call such a set \( A \) mlb-closed). Conversely, any \( A \subseteq P \) which is mlb-closed defines the closed elements of a unique closure relation \( H \).

**Remark:** Note that taking \( B \) to be the empty set, we have that any mlb-closed set \( A \) must contain all of the maximal elements of \( P \).

**Proof.** Let \( H \) be a closure relation on \( P \), with closed elements \( A \). If \( B \subseteq A \) has some maximal lower bound \( x \), then

\[
x \leq H(x) \leq H(b) = b
\]

for all \( b \in B \). Hence \( H(x) \) is also a lower bound of \( B \), so by maximality, \( H(x) = x \) and \( x \in A \). Conversely, given a set \( A \) which is mlb-closed, we claim that for any \( x \) in \( P \) there exists a unique least element of \( A_{\geq x} \), where \( A_{\geq x} \) is defined to be \( \{ a \in A : a \geq x \} \). To prove this claim, note that since \( A \) contains the maximal elements of \( P \), \( A_{\geq x} \neq \emptyset \). If there were two such minimal elements \( y, z \in A_{\geq x} \), then they would have a maximal lower bound \( w \) above \( x \), contradicting their minimality. Thus, given any mlb-closed set \( A \), we define \( H(x) \) to be the minimum of \( A_{\geq x} \). It remains to show that \( H \) is the unique closure relation with closed elements \( A \). That \( H \) satisfies axioms 1 and 3 is evident. To show axiom 2: if \( x \leq y \), then we have \( A_{\geq x} \supseteq A_{\geq y} \) and hence

\[
H(x) = \min A_{\geq x} \leq \min A_{\geq y} = H(y).
\]

Finally, any closure relation is defined uniquely by specifying its closed elements so \( H \) is unique. \( \square \)

Denote by \( \overline{A} \) the mlb-closure of a set \( A \subseteq P \), i.e. \( \overline{A} \) is the intersection of all mlb-closed sets which contain \( A \). It can easily be shown that this defines a closure on the set \( P \) (i.e. a closure relation on the poset \( 2^P \)). Let \( L_{mlb} \) denote the lattice of mlb-closed subsets of \( P \) ordered under inclusion, and let \( M \subseteq P \) be the set of maximal elements of \( P \). We then have

**Theorem 4.** The order-dual of \( LC(P) \) is isomorphic to the interval between \( \overline{M} \) and \( P \) in \( L_{mlb}(P) \).

**Proof.** Define a map \( f : LC(P) \to 2^P \) by

\[
f(H) = \{ x \in P : H(x) = x \}.
\]
Figure 1: A six-element poset $P$ and its lattice $LC(P)$.

Proposition 3 shows that the image of $f$ is exactly the interval from $\overline{M}$ to $P$ in $L_{mib}(P)$. It remains then to show that $H \leq K$ if and only if

$$\{x \in P : K(x) = x\} \subseteq \{x \in P : H(x) = x\}.$$  

Clearly, if $H \leq K$ and $K(x) = x$ then

$$x = K(x) \geq H(x) \geq x$$

so $H(x) = x$. Conversely, if

$$\{x \in P : K(x) = x\} \subseteq \{x \in P : H(x) = x\},$$ 

then by the proof of Proposition 3, we have

$$H(y) = \min(\{x \in P : H(x) = x\}_{\geq y}) \leq \min(\{x \in P : K(x) = x\}_{\geq y}) = K(y).$$

**Example.** Consider the poset $P$ shown in figure 1. Here $M = \{a, b\}$ and $\overline{M} = \{a, b, c\}$. The lattice $LC(P)$ is shown with the closure relations pictured as partitions of the set $P$.

In [EJ], Edelman and Jamison define a closure $\overline{\cdot}$ on a set $S$ to be a *convex closure* if it satisfies the following *anti-exchange axiom*: Given distinct $x, y \in S$ and a closed set $A = \overline{A} \subseteq S$ with $x, y \notin A$, we have

$$x \in \overline{y \cup A} \Rightarrow y \notin \overline{x \cup A}.$$ 

**Proposition 5** Mlb-closure is a convex closure on the set $P$.

**Proof.** As before let $\overline{A}$ denote the mib-closure of $A$, and let $x$ and $y$ be distinct elements not in $A$, a closed set. If $x \in \overline{y \cup A}$ then $x$ is a maximal lower bound of some set of maximal lower bounds of some set of maximal lower bounds, etc., of some subsets of $y \cup A$. In this expression for $x$, if $y$ never appears then $x \in \overline{A} = A$, a contradiction. Since $y$ does appear, $x \leq y$. Thus we cannot have $y \leq x$, and so $y \notin \overline{x \cup A}$. $\square$
Edelman and Jamison [EJ] show that convex closures enjoy a number of interesting properties. Corollary 6 states a few properties of $LC(P)$ which are consequences of $mlb$ being a convex closure. The reader is referred to [EJ] for proofs.

**Corollary 6**

1. $LC(P)$ is a *join-distributive* lattice, i.e. every atomic interval in $LC(P)$ is a Boolean algebra.
2. $LC(P)$ is upper-semimodular, and consequently ranked.
3. If $K \in LC(P)$ is defined by the $mlb$-closed set $A \subseteq P$, then the rank function of $LC(P)$ is given by $r(K) = card(P) - card(A)$, where $card(A)$ is also the number of blocks in $K$ when regarded as a partition of the elements in $P$.

We note one further consequence not given in [EJ]. Recall the definition of the *characteristic polynomial* of a ranked finite poset $Q$ with $0, 1$ and rank function $r$:

$$
\chi(Q; \lambda) = \sum_{q \in Q} \mu(\hat{0}, q)\lambda^{r(\hat{1})-r(q)}
$$

where $\mu$ is the *Mobius function* of the poset $Q$ (see Rota [Ro]).

**Proposition 7** If $L$ is a join-distributive lattice with $a$ atoms, then

$$
\chi(L, \lambda) = (\lambda - 1)^a \lambda^{r(\hat{1})-a}.
$$

*Proof.* Since $\mu(\hat{0}, q) = 0$ unless $q$ is a join of atoms ([4], Prop. 2), the only $q \in L$ that contribute to the sum are those in the Boolean algebra $B$ generated by the atoms of $L$. Thus we have

$$
\chi(L, \lambda) = \sum_{q \in B} \mu(\hat{0}, q)\lambda^{a-r(q)}\lambda^{r(\hat{1})-a}
$$

$$
= \chi(B, \lambda)\lambda^{r(\hat{1})-a}
$$

$$
= (\lambda - 1)^a \lambda^{r(\hat{1})-a}
$$

using the well-known fact (see e.g. [Ro]) that $\chi(B, \lambda) = (\lambda - 1)^a$. □

Using proposition 7, we get the following form for $\chi(L, \lambda)$:

**Corollary 8** Let $s$ be the number of elements of $P$ which are covered by a unique element, and let $m$ be the cardinality of $\overline{M}$ (= the $mlb$-closure of the maximal elements of $P$). Then

$$
\chi(LC(P), \lambda) = (\lambda - 1)^s \lambda^{\text{card}(P)-m-s}.
$$

*Proof.* By Corollary 6 and Theorem 4 we have $r(\hat{1}) = \text{card}(P) - m$ and so to apply Proposition 7 we only need to show that the atoms of $LC(P)$ correspond to elements of $P$ covered by a unique element. Let $H$ be an atom of $LC(P)$ with $A$ the set of its $mlb$-closed elements. By Corollary 6, $\text{card}(P) - \text{card}(A) = r(H) = 1$ so there is exactly one non-closed element $x$ for $H$. Hence $H(x)$ covers $x$, and if any other element $y$ covers $x$, then $H(y) \geq H(x)$ implies $H(y) \neq y$, a contradiction.

Conversely, specifying $x$ to be the only unclosed element does define a closure relation. □

**Example.** Figure 1 shows the atoms of $LC(P)$, the values of $\mu(\hat{0}, x)$ and $\chi(LC(P), \lambda)$. 5
3 Modular elements and supersolvability

Using Theorem 4, it is easy to characterize which closure relations correspond to various distinguished classes of elements of $LC(P)$, such as the atoms, coatoms, join-irreducibles, and meet-irreducibles. One interesting class for which this is non-trivial are the modular elements of $LC(P)$. Recall (see Stanley [St]) that an element $H$ in a lattice $L$ is modular if and only if for all $K \leq K' \in L$ we have

$$H \lor K = H \lor K', \text{ and } H \land K = H \land K' \implies K = K'.$$

**Theorem 9.** Assume $P$ has a greatest element $\overline{1}$. Then a closure relation $H$ on $P$ is a modular element of $LC(P)$ if and only if $H$ satisfies the following cover property:

For all $x, y \in P$, if $y$ covers $x$ and $H(x) \neq x$, then $H(y) = H(x)$.

**Proof ($\Rightarrow$):** Let $H$ be a closure relation on $P$ not having the cover property, i.e. there is some $y$ covering $x$ such that $H(x) \neq x$, but $H(y) \neq H(x)$. Let $K'$ be the closure relation having closed elements $\{\overline{1}, y\} \cup P_{<x}$, and $K$ the closure relation having closed elements $\{1, x, y\} \cup P_{<x}$ (note that both of these sets are mlb-closed). Then $K < K'$, and it is easily seen that $H \lor K = H \lor K'$ and $H \land K = H \land K'$, violating the modularity of $H$.

($\Leftarrow$): We show in general that any closure relation with the cover property is modular. Suppose $H$ satisfies the cover condition, and assume we have a pair of closure relations $K \leq K'$ such that $H \lor K = H \lor K'$ and $H \land K = H \land K'$. We must show that $K = K'$, so it would suffice to show $K \geq K'$. So assume $K(x) = x$ for a given $x$, and we will prove that $K'(x) = x$. From $H \lor K = H \lor K'$, we may assume that $H(x) \neq x$. Our strategy: We claim that any $y \geq x$ must be comparable to $H(x)$.

Next we use this claim to show $K'(x) < H(x)$ by contradiction. Finally, we show $K'(x) = x$.

To prove the claim, suppose $y \notin H(x)$. Then $H(y) \neq H(x)$, so considering a maximal chain from $x$ to $y$, there must exist $y'$ covering $y$ such that $H(y') = H(x)$ but $H(y') \neq H(x)$. Now by the cover property, we must have $H(x) = H(y') = y'$. Therefore $y \geq y' = H(x)$ and the claim is proved.

Our next goal is to show that $K'(x) < H(x)$. By the claim we know that these two elements must be comparable. So suppose for contradiction that $K'(x) \geq H(x)$. Let $K'_{H}, H_{c}$ denote the sets of closed elements of $K', H$ respectively. Since $min((H_{c})_{\geq x} = H(x)$ and $min((K'_{H})_{\geq x} = K'(x) > H(x)$, any maximal lower bound $z \geq x$ of a subset $A \subseteq H_{c} \cup K'_{H}$ must be comparable to $H(x)$ (by the claim), and hence $z \geq H(x)$ by maximality. Thus

$$(H \land K')(x) = min((K'_{H})_{\geq x} \geq H(x) > x,$$

contradicting the fact that $(H \land K')(x) = (H \land K')(x) = x$. Hence $K'(x) < H(x)$.

Lastly, we show $K(x) = x$. Let $k' \in K'$ and $h \in H_{c}$ have a maximal lower bound $z \geq x$. By our claim either $k' \geq H(x)$, in which case $z \geq H(x)$, or $k' < H(x)$, in which case $z = k'$. This shows that

$$(K'_{H} \cup H_{c})_{< H(x)} = (K'_{H} < H(x)).$$

But

$$x = (H \land K)(x) = (H \land K')(x) \in (K'_{H} \cup H_{c})_{< H(x)},$$

so $x \in K'$, i.e. $K'(x) = x$. □

As a corollary, we have
Theorem 10 $LC(P)$ is supersolvable, i.e. it contains a maximal chain of modular elements. (See Stanley [St] for alternate definitions and consequences of supersolvability).

**Proof.** We first prove the theorem with the assumption that $P$ has a greatest element $\hat{1}$, and then deduce the theorem in general.

Let $p_1, p_2, \ldots, p_n$ be any linear extension of the partial order on $P$. Let $H_0$ be the greatest element of $LC(P)$, having $\hat{1}$ as its only closed element, and for $i = 1, 2, \ldots, n$ let $H_i$ be the closure relation having closed elements $\{\hat{1}, p_1, p_2, \ldots, p_i\}$. One can check that this set is in fact mlb-closed, since any maximal lower bounds of subsets of $\{\hat{1}, p_1, p_2, \ldots, p_i\}$ must either be $\hat{1}$ or lie in the ideal $\{p_1, p_2, \ldots, p_i\}$. It is easy to see that each $H_i$ satisfies the cover condition of Theorem 9, and hence is a modular element of $LC(P)$: if $y$ covers $x$ and $H_i(x) \neq x$, then $x \notin \{\hat{1}, p_1, p_2, \ldots, p_i\}$, so either $y = \hat{1}$ or else $y \notin \{\hat{1}, p_1, p_2, \ldots, p_i\}$. In either case, $H_i(x) = 1 = H_i(y)$.

To prove the theorem in general, let $P$ be an arbitrary poset with maximal elements $M$. Adjoin a greatest element $\hat{1}$ to $P$ to obtain the poset $P + 1$. Let $H$ be the closure relation on $P + 1$ with closed elements $\{\hat{1}\} \cup M$ (an mlb-closed set). Then by Theorem 4, we have that $LC(P)$ is isomorphic to the interval between the identity closure and $H$ in $LC(P + \hat{1})$. Since $LC(P + \hat{1})$ is supersolvable, this interval is also supersolvable ([St], Proposition 3.29(i)), and hence $LC(P)$ is supersolvable.$\square$

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