Proton Recoil Energy and Angular Distribution of Neutron Radiative $\beta^-$–Decay

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We analyse the proton recoil energy and angular distribution of the radiative $\beta^-$–decay of the neutron to leading order in the large baryon mass expansion by taking into account the contributions of the proton–photon correlations. We show that the account for the proton–photon correlations does not contradict the description of the radiative corrections to the lifetime of the neutron and the proton recoil energy spectrum of the neutron $\beta^-$–decay in terms of the functions $(\alpha/\pi) g_\nu(E_\nu)$ and $(\alpha/\pi) f_\nu(E_\nu)$, where $E_\nu$ is the electron energy. In addition we find that the contributions of the proton–photon correlations in the radiative $\beta^-$–decay of the neutron to the proton recoil asymmetry $C$ are of order $10^{-4}$. They make the contributions of the radiative corrections to the proton recoil asymmetry $C$ symmetric with respect to a change $A_0 \leftrightarrow B_0$, where $A_0$ and $B_0$ are the correlation coefficients of the neutron $\beta^-$–decay.

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I. INTRODUCTION

Recently contributions of order $10^{-4}$ of interactions beyond the Standard model (SM) to the neutron $\beta^-$–decay have been investigated in [1]. For the analysis of such contributions to the proton recoil energy spectrum $\alpha(T_p)$, where $T_p$ is a kinetic energy of the decay proton, and the proton recoil asymmetry, defined by a correlation coefficient $C$, the calculation of the proton recoil energy and angular distribution has been performed by taking into account a complete set of corrections, caused by the “weak magnetism” and the proton recoil, calculated to next–to–leading order in the large baryon mass expansion, and the radiative corrections of order $\alpha/\pi \sim 10^{-3}$, calculated to leading order in the large baryon mass expansion [1]. For the cancellation of the infrared divergences, caused by one–virtual photon exchanges, the standard procedure [2, 3] has been used and the contribution of the radiative $\beta^-$–decay of the neutron has been added [1]. The calculation of the contribution of the radiative $\beta^-$–decay has been performed, first, by integrating over the proton 3–momentum and, second, by integrating over the antineutrino energy. This has led to the radiative corrections, defined by two functions $(\alpha/\pi) g_\nu(E_\nu)$ and $(\alpha/\pi) f_\nu(E_\nu)$, which were calculated for the first time by Sirlin [2] and Shann [4], respectively. However, as has been pointed out by Glück [5] (see also [6]), the calculation of the contribution of the radiative $\beta^-$–decay of the neutron to the proton recoil energy spectrum and the proton recoil asymmetry is much more complicated. For the calculation of such a contribution one has to integrate, first, over the antineutrino 3–momentum and then over other dynamical variables. In spite of the calculation of the radiative corrections to leading order in the large baryon mass expansion such an integration leads to correlations between a recoil proton and an emitted photon, i.e. the proton–photon correlations. The region of the integration over the photon energy spectrum should be divided into two parts [6], corresponding to an emission of i) the soft–photons, the contribution of which is responsible for a cancellation of the infrared divergences of one–virtual photon exchanges in the neutron $\beta^-$–decay, and of ii) the hard–photons, which define a part of the observable radiative corrections to the neutron $\beta^-$–decay. As has been pointed out by Glück [5], the calculation of the contributions of the hard–photons to $\beta^-$–decays may be carried out only numerically. For the aim of the calculation of the hard–photon corrections Glück has used the Monte Carlo simulation method [6].

In this paper we follow the paper by Glück [6] and revise the contribution of the radiative $\beta^-$–decay of the neutron, obtained in [1]. We perform the integration over the antineutrino 3–momentum and analyse the proton–photon correlations in the soft- and hard-photon energy regions.

The paper is organised as follows. In section [11] we write down the correction to the electron–proton energy and angular distribution of the radiative $\beta^-$–decay of the neutron, obtained in [1], which includes the proton–photon correlations. In section [111] we calculate the contribution of the soft–photon energy region. In section [1111] we numerically calculate the contribution of the hard–photon energy region. We show that the contributions of the
proton–photon correlations to the lifetime of the neutron $\tau_n$ and the proton recoil energy spectrum $a(T_p)$, integrated over the proton recoil energy, are of order $10^{-5}$, and can be neglected at the level of accuracy $10^{-4}$, accepted in [2]. This confirms the use of the radiative corrections to the lifetime of the neutron and the proton recoil energy spectrum, described by the functions $(\alpha/\pi)g_n(E_e)$ and $(\alpha/\pi)f_n(E_e)$ [2]. In turn the contributions of the proton–photon corrections to the proton recoil asymmetry $C$ are of order $10^{-4}$ and should be taken into account at the level of accuracy $10^{-5}$ [2]. We show that the account for the proton–photon correlations to the proton recoil asymmetry $C$ makes the contribution of the radiative corrections symmetric with respect to a change $A_0 \leftrightarrow B_0$ as well as the main term $C_0 = -xc(A_0 + B_0)$, calculated for the first time by Treiman [3], where $A_0$ and $B_0$ are the correlation coefficients of the neutron $\beta^-$-decay, calculated to leading order in the large baryon mass expansion $S$ [3] (see also [1]). In section VI we summarise the obtained results. In section VII we compare our results, obtained by Glück [10]. We show that i) the radiative corrections coefficient $a_0$ in the electron–antineutrino $(E_e, \cos \theta_{ee})$ distribution may be described by the function $f_n(E_e)$ and ii) the radiative corrections to the correlation coefficient $a_0$ in the proton–energy spectrum $a(T_p)$, described by the functions $g_n(E_e)$ and $f_n(E_e)$, are of order of magnitude larger compared with the radiative corrections from the proton recoil spectrum of the radiative $\beta^-$-decay of the neutron, caused by the proton–photon correlations. Since the proton–energy spectrum $a(T_p)$ is obtained by the integration of the electron–proton energy distribution $a(E_e, T_p)$ [2] over the electron energies, the results, concerning the radiative corrections to the proton–energy spectrum $a(T_p)$, are fully valid for the electron–proton energy distribution $a(E_e, T_p)$.

II. CORRECTION TO ELECTRON–PROTON ENERGY AND ANGULAR DISTRIBUTION OF NEUTRON $\beta^-$–DECAY ACCOUNTING FOR PROTON-PHOTON CORRELATIONS

Using the results, obtained in Appendices B and I of Ref.[1], we may write down the correction of the radiative $\beta^-$–decay of the neutron to the electron–proton energy and angular distribution of the neutron $\beta^-$–decay [1], taking into account the proton-photon correlations. We get

$$\frac{d^3\Delta\lambda_{\beta^-}(E_e, k_p, \theta_p, P)}{dE_e dk_p d\cos \theta_p} = \sum_{j=1,2,3} \frac{d^3\Delta\lambda^{(j)}_{\beta^-}(E_e, k_p, \theta_p, P)}{dE_e dk_p d\cos \theta_p},$$

where $E_e$ and $k_p$ are a total electron energy and an absolute value of the proton 3–momentum, $\theta_p$ is a polar angle between the proton 3–momentum and the neutron spin and $P$ is the neutron polarisation [1]. Then, we have denoted

$$\frac{d^3\Delta\lambda^{(1)}_{\beta^-}(E_e, k_p, \theta_p, P)}{dE_e dk_p d\cos \theta_p} = \frac{(1 + 3\lambda^2)}{\pi} \frac{G_F^2 |V_{ud}|^2}{4\pi^3} \int \frac{d\Omega_{ep}}{4\pi} \int \frac{d\Omega_{em}}{4\pi} \int \frac{d\Omega_{ep}}{4\pi} \int \frac{d\Omega_{em}}{4\pi} \left\{ (1 - B_0 \frac{\xi_n \cdot (\vec{k}_p + \vec{k}_e)}{|\vec{k}_p + \vec{k}_e|}) g^{(1)}_{\beta^- \gamma}(E_e, \mu) \right.\left. - \left( - a_0 \frac{\vec{k}_e \cdot (\vec{k}_p + \vec{k}_e)}{E_e |\vec{k}_p + \vec{k}_e|} + A_0 \frac{\xi_n \cdot \vec{k}_e}{E_e} \right) g^{(2)}_{\beta^- \gamma}(E_e, \mu) \right\} \delta(E_0 - E_e - |\vec{k}_p + \vec{k}_e|) k_e E_e k_p^2 F(E_e, Z = 1),$$

$$\frac{d^3\Delta\lambda^{(2)}_{\beta^-}(E_e, k_p, \theta_p, P)}{dE_e dk_p d\cos \theta_p} \times \left\{ \frac{k^2_e}{(E_e - \vec{n} \cdot \vec{k}_e)^2} + \frac{E_e + \omega}{E_e - \vec{n} \cdot \vec{k}_e} \right\} \left( \frac{\omega}{E_e} \right) \delta(E_0 - E_e - |\vec{k}_p + \vec{k}_e + \omega \vec{n}| - \omega) k_e E_e k_p^2 F(E_e, Z = 1),$$

and

$$\frac{d^3\Delta\lambda^{(3)}_{\beta^-}(E_e, k_p, \theta_p, P)}{dE_e dk_p d\cos \theta_p} = \frac{(1 + 3\lambda^2)}{\pi} \frac{G_F^2 |V_{ud}|^2}{4\pi^3} \int \frac{d\Omega_{ep}}{2\pi} \int \frac{d\Omega_{em}}{4\pi} \int \frac{d\Omega_{ep}}{4\pi} \int \frac{d\Omega_{em}}{4\pi} \left\{ B_0 \frac{\xi_n \cdot (\vec{k}_p + \vec{k}_e)}{|\vec{k}_p + \vec{k}_e|} \right.\left. - \frac{\xi_n \cdot (\vec{k}_p + \vec{k}_e + \omega \vec{n})}{|\vec{k}_p + \vec{k}_e + \omega \vec{n}|} \right\} \left( \frac{k^2_e}{(E_e - \vec{n} \cdot \vec{k}_e)^2} + \frac{\omega}{E_e - \vec{n} \cdot \vec{k}_e} \right) \delta(E_0 - E_e - |\vec{k}_p + \vec{k}_e + \omega \vec{n}| - \omega) k_e E_e k_p^2 F(E_e, Z = 1),$$

(4)
where $\lambda$, $\alpha$, $G_F$ and $V_{ud}$ are the axial, fine-structure and Fermi coupling constants and the Cabibbo–Kobayashi–Maskawa (CKM) matrix element, respectively. Then, $\phi_p$ is an azimuthal angle between the neutron spin and the proton 3-momentum, $d\Omega_{ep} = \sin \theta_{ep} d\theta_{ep} d\varphi_{ep}$ and $d\Omega_{pr} = \sin \theta_{pr} d\theta_{pr} d\varphi_{pr}$ are the solid angle elements of the 3-momenta of the electron and photon relative to a 3-momentum of the proton, $\omega$ and $\omega n$ are a photon energy and a 3-momentum, $\xi_0$ is a neutron polarisation vector such as $\xi_0 \cdot k_p = P k_p \cos \theta_p$ and $\xi_0 \cdot k_e = P k_p (\cos \theta_p \cos \theta_{ep} + \sin \theta_p \sin \theta_{ep} \cos (\phi_p - \phi_{ep}))$ with $P = |\xi_0|$. Then $a_0$, $A_0$ and $B_0$ are the correlation coefficients of the neutron $\beta^+$/decay, calculated to leading order in the large baryon mass expansion [8,9] (see also [1]). The functions $g^{(1)}_{\beta^-\gamma}(E_e, \mu)$ and $g^{(2)}_{\beta^-\gamma}(E_e, \mu)$, calculated in Appendix B of Ref.11 (see Eq.(B-28)), are equal to

$$g^{(1)}_{\beta^-\gamma}(E_e, \mu) = \left[ \ln \left( \frac{2(E_0 - E_e)}{\mu} \right) - \frac{3}{2} + \frac{1}{2} \frac{E_0 - E_e}{E_e} \left( 1 + \frac{1}{8} \frac{E_0 - E_e}{E_e} \right) \right] \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 + 1$$

$$+ \frac{1}{12} \frac{(E_0 - E_e)^2}{E_e^2} + \frac{1}{2} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - \frac{1}{4} \frac{\ln^2 \left( \frac{1 + \beta}{1 - \beta} \right)}{1 + \beta} + \frac{1}{\beta} L \left( \frac{2\beta}{1 + \beta} \right),$$

$$g^{(2)}_{\beta^-\gamma}(E_e, \mu) = \left[ \ln \left( \frac{2(E_0 - E_e)}{\mu} \right) - \frac{3}{2} + \frac{1}{2} \frac{E_0 - E_e}{E_e} \left( 1 + \frac{1}{8} \frac{E_0 - E_e}{E_e} \right) \right] \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 + 1$$

$$+ \frac{1}{2} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - \frac{1}{4} \ln^2 \left( \frac{1 + \beta}{1 - \beta} \right) + \frac{1}{\beta} L \left( \frac{2\beta}{1 + \beta} \right),$$

(5)

where $\mu$ is a finite photon mass or a Lorentz invariant infrared regularisation scale and $L(x)$ is the Spence function [11–14]. The Fermi function $F(E_e, Z = 1)$ takes into account the electron–proton final–state Coulomb interaction [1]. Having integrated in Eq.24 over the solid angle $\Omega_{ep}$, we arrive at the expression

$$\frac{d^4 \Delta \lambda^{(1)}_{\beta^-}(E_e, k_p, \theta_p, P)}{dE_e dk_p d\cos \theta_p} = \left( 1 + 3\lambda^2 \right) \frac{\alpha^2 G^2_F |V_{ud}|^2}{8\pi^3} \int_0^{2\pi} d\varphi_p \int \frac{d\Omega_{ep}}{4\pi} \int \frac{d\Omega_{pr}}{4\pi} \int \frac{d\Omega_{\omega}}{4\pi} \left\{ (E_0 - E_e - \omega) - B_0 \xi_0 \cdot (k_p + \bar{k}_p) \right\}$$

$$- a_0 \frac{(k_p + \bar{k}_p) \cdot \bar{k}_e}{E_e} + A_0 (E_0 - E_e - \omega) \frac{\xi_0 \cdot \bar{k}_e}{E_e} \ \left( E_e - \bar{\omega} \cdot \bar{k}_e \right)^2 \left[ \left( E_0 - E_e \right) \left( E_0 - E_e - 2\omega \right) - k_p^2 - k_e^2 \right] \frac{1}{2k_p k_e}$$

$$\times E_e k_p \ F(E_e, Z = 1) + \left( 1 + 3\lambda^2 \right) \frac{\alpha^2 G^2_F |V_{ud}|^2}{8\pi^3} \int_0^{2\pi} d\varphi_p \int \frac{d\Omega_{ep}}{4\pi} \int \frac{d\Omega_{pr}}{4\pi} \int \frac{d\Omega_{\omega}}{4\pi} \left\{ (E_0 - E_e - \omega) - B_0 \xi_0 \cdot (k_p + \bar{k}_p) \right\}$$

$$\times \left( k_p^2 - (\bar{n} \cdot \bar{k}_p)^2 \right) \frac{1}{(k_p - \bar{n})^2} - \frac{\omega}{(k_p - \bar{n})^2} + \left( - a_0 \left( \bar{k}_p + \bar{k}_e \right) + A_0 (E_0 - E_e - \omega) \frac{\xi_0}{E_e - \bar{n} \cdot \bar{k}_e} \right) \cdot$$

$$\left( \bar{k}_e + (E_e + \omega) \frac{\bar{n}}{E_e - \bar{n} \cdot \bar{k}_e} - m_p^2 \frac{\bar{n}}{(E_e - \bar{n} \cdot \bar{k}_e)^2} \right)$$

(6)

III. SOFT-PHOTON CONTRIBUTION TO PROTON RECOIL ENERGY AND ANGULAR DISTRIBUTION OF NEUTRON RADIATIVE $\beta^-$/DECAY

In this section we calculate the contributions of the soft–photons, which are responsible for the cancellation of the infrared divergences in the neutron $\beta^-$/decay. For the calculation of Eq.31 and Eq.32 in the soft–photon energy region we neglect i) correlations between a photon momentum $q = \omega n$ and a momentum $\bar{k}_p + \bar{k}_e$ and use ii) for regularisation of infrared divergent contributions a finite-photon mass regularisation (FPM) [2,4] (see also [1]). For the application of the FPM to the problem under consideration we transcribe the right–hand–side (r.h.s.) of Eq.33 as follows

$$d^4 \Delta \lambda^{(2)}_{\beta^-}(E_e, k_p, \theta_p, P) = \left( 1 + 3\lambda^2 \right) \frac{\alpha^2 G^2_F |V_{ud}|^2}{4\pi^3} \int_0^{2\pi} d\varphi_p \int \frac{d\Omega_{ep}}{4\pi} \int \frac{d\Omega_{pr}}{4\pi} \int \frac{d\Omega_{\omega}}{4\pi} \left\{ (E_0 - E_e - \omega) - B_0 \xi_0 \cdot (k_p + \bar{k}_p) \right\}$$

$$- a_0 \frac{(k_p + \bar{k}_e) \cdot \bar{k}_e}{E_e} + A_0 (E_0 - E_e - \omega) \frac{\xi_0 \cdot \bar{k}_e}{E_e} \ \left( E_e - \bar{n} \cdot \bar{k}_e \right)^2 \left[ \left( E_0 - E_e \right) \left( E_0 - E_e - 2\omega \right) - k_p^2 - k_e^2 \right] \frac{1}{2k_p k_e}$$

$$\times E_e k_p \ F(E_e, Z = 1) + \left( 1 + 3\lambda^2 \right) \frac{\alpha^2 G^2_F |V_{ud}|^2}{4\pi^3} \int_0^{2\pi} d\varphi_p \int \frac{d\Omega_{ep}}{4\pi} \int \frac{d\Omega_{pr}}{4\pi} \int \frac{d\Omega_{\omega}}{4\pi} \left\{ (E_0 - E_e - \omega) - B_0 \xi_0 \cdot (k_p + \bar{k}_p) \right\}$$

$$\times \left( k_p^2 - (\bar{n} \cdot \bar{k}_p)^2 \right) \frac{1}{(k_p - \bar{n})^2} - \frac{\omega}{(k_p - \bar{n})^2} + \left( - a_0 \left( \bar{k}_p + \bar{k}_e \right) + A_0 (E_0 - E_e - \omega) \frac{\xi_0}{E_e - \bar{n} \cdot \bar{k}_e} \right) \cdot$$

$$\left( \bar{k}_e + (E_e + \omega) \frac{\bar{n}}{E_e - \bar{n} \cdot \bar{k}_e} - m_p^2 \frac{\bar{n}}{(E_e - \bar{n} \cdot \bar{k}_e)^2} \right)$$

(6)
Integrating over directions of a photon velocity $\vec{v}$ or an infrared regularisation scale $q$, if $q = \sqrt{\omega^2 - p^2}$ and $v = q/\omega$ are a photon momentum and velocity, respectively, (see Appendix B of Ref. [1]). After the integration over $q_p$ and $\cos \theta_{ep}$, the r.h.s. of Eq. (7) takes the form

$$
\int \frac{d^3 \Delta \lambda^{(2)}_\beta(E_c, k_p, \theta_p, P)}{dE_c dk_p d\cos \theta_p} = (1 + 3\lambda^2) \frac{\alpha}{\pi} \frac{G_F^2 |V_{ud}|^2}{8\pi^3} \int \frac{q^2 dq}{\omega^3} \int \frac{d\Omega_\gamma}{4\pi} \left( \frac{(E_0 - E_c - \omega) - k_p^2 - k_\gamma^2}{2k_p E_c} \right) + P A_0 \cos \theta_p (E_0 - E_c - \omega)
$$

Integrating over directions of a photon velocity $\vec{v}$ and a momentum $q$ we obtain

$$
\int \frac{d^3 \Delta \lambda^{(2)}_{\beta\gamma}(E_c, k_p, \theta_p, P)}{dE_c dk_p d\cos \theta_p} = (1 + 3\lambda^2) \frac{\alpha}{\pi} \frac{G_F^2 |V_{ud}|^2}{8\pi^3} \left\{ \left( \frac{(E_0 - E_c - \omega) - k_p^2 - k_\gamma^2}{2k_p E_c} \right) \left( \frac{(E_0 - E_c - \omega) - k_p^2 - k_\gamma^2}{2k_p E_c} \right) \right\} + P A_0 \cos \theta_p (E_0 - E_c - \omega)
$$

Summing up the contributions of Eqs. (8) and (9) we obtain the expression, which does not depend on a photon mass or an infrared regularisation scale $\mu$, that is,

$$
\sum_{j=1,2} \int \frac{d^3 \Delta \lambda^{(2)}_{\beta\gamma}(E_c, k_p, \theta_p, P)}{dE_c dk_p d\cos \theta_p} = (1 + 3\lambda^2) \frac{\alpha}{\pi} \frac{G_F^2 |V_{ud}|^2}{8\pi^3} \left\{ \left( \frac{(E_0 - E_c) - P B_0 \cos \theta_p (E_0 - E_c) - \omega) - k_p^2 - k_\gamma^2}{2k_p E_c} \right) \left( \frac{(E_0 - E_c - \omega) - k_p^2 - k_\gamma^2}{2k_p E_c} \right) \right\} + P A_0 \cos \theta_p (E_0 - E_c - \omega)
$$

$$
\times E_c k_p F(E_c, Z = 1).
$$
Having integrated Eq. (10) over $\omega$ for the correction to the electron–proton energy and angular distribution of the neutron, we get:

\[
+(1 + 3\lambda^2) \frac{\alpha}{\pi} \frac{G^2_P |V_{ud}|^2}{8\pi^3} \int \frac{d\Omega}{4\pi} \int d\omega \left\{ -1 - PB_0 \cos \theta_p \frac{-2(E_0 - E_e)}{2k_p} - a_0 \frac{-2(E_0 - E_e)}{2E_e} \right. \\
+ PA_0 \cos \theta_p \left( -\frac{(E_0 - E_e)(E_0 - E_e - \omega) - k_p^2 - k_e^2}{2k_p E_e} + PA_0 \cos \theta_p (E_0 - E_e) \right) \\
\times E_e k_p F(E_e, Z = 1)
\]

\[
+(1 + 3\lambda^2) \frac{\alpha}{\pi} \frac{G^2_P |V_{ud}|^2}{8\pi^3} \int \frac{d\omega}{E_e} \int \frac{d\Omega}{4\pi} \left\{ \left( (E_0 - E_e - \omega) - PB_0 \cos \theta_p \frac{(E_0 - E_e)(E_0 - E_e - 2\omega) + k_p^2 - k_e^2}{2k_p} \\
+ PA_0 \cos \theta_p (E_0 - E_e - \omega) \right) \\
\times \frac{k_e^2 - (\vec{n} \cdot \vec{k}_e)^2}{E_e - \vec{n} \cdot \vec{k}_e} \right\} E_e k_p F(E_e, Z = 1).
\]

Having integrated Eq. (10) over $\omega$ in the limits $0 \leq \omega \leq (E_0 - E_e)$ and directions of a photon momentum we find the expression

\[
\sum_{j=1,2} \frac{d^4\Delta \lambda_{\beta,j}(E_e, k_p, \theta_p, P)}{dE_\ell d\vec{k}_{\rho} d\cos \theta_p} = (1 + 3\lambda^2) \frac{\alpha}{\pi} \frac{G^2_P |V_{ud}|^2}{8\pi^3} \left\{ \left( (E_0 - E_e) - PB_0 \cos \theta_p \frac{(E_0 - E_e)^2 + k_p^2 - k_e^2}{2k_p} \right) \\
\times \left\{ \left( \frac{3}{2} - \frac{1}{3} \frac{(E_0 - E_e)}{E_e} - \frac{1}{24} \frac{(E_0 - E_e)^2}{E_e^2} \right) \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \right. \\
- \frac{1}{12} \frac{(E_0 - E_e)^2}{k_p E_e} \right\} + \left. \right. \\
\times E_e k_p F(E_e, Z = 1)
\]

\[
+(1 + 3\lambda^2) \frac{\alpha}{\pi} \frac{G^2_P |V_{ud}|^2}{8\pi^3} \left\{ \left( \frac{1}{2} \frac{(E_0 - E_e)^2}{E_e} - PB_0 \cos \theta_p \frac{1}{2} \frac{(k_p^2 - k_e^2)}{k_p E_e} \right) \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \right. \\
\times E_e k_p F(E_e, Z = 1)
\]

\[
+ \frac{1}{12} \frac{(E_0 - E_e)^3}{E_e} + PB_0 \cos \theta_p \frac{1}{24} \frac{(E_0 - E_e)^3 - 3(k_p^2 + k_e^2)}{k_p E_e^2} \right\} \left( \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \right.
\]

\[
+ \left. \frac{1}{2} \frac{(k_p^2 - k_e^2)}{k_p E_e} \right) + \left( \frac{1}{12} \frac{(E_0 - E_e)^3}{E_e} \right) + PA_0 \cos \theta_p \frac{1}{24} \frac{(E_0 - E_e)^3 - 3(k_p^2 + k_e^2)}{k_p E_e^2} \right\} \left( \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \right.
\]

\[
+ \left. \right. \\
\times E_e k_p F(E_e, Z = 1).
\]

For the integration over directions of a photon momentum $\vec{q} = \omega \vec{n}$ we have used the formulas

\[
\int \frac{d\Omega}{4\pi} \frac{k_e^2 - (\vec{n} \cdot \vec{k}_e)^2}{(E_e - \vec{n} \cdot \vec{k}_e)^2} = \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2,
\]

\[
\int \frac{d\Omega}{4\pi} \frac{1}{E_e - \vec{n} \cdot \vec{k}_e} = \frac{1}{2E_e} \left( \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \right).
\]

Following the approximation, neglecting correlations between a photon momentum $\vec{q} = \omega \vec{n}$ and a momentum $\vec{k}_p + \vec{k}_e$, for the correction to the electron–proton energy and angular distribution of the neutron $\beta^-$ decay, given by Eq. (11).
we obtain the expression
\[
\frac{d^3 \Delta \lambda_{\beta e}^{(3)}(E_c, k_p, \theta_p, P)}{dE_c dk_p d \cos \theta_p} = (1 + 3 \lambda^2) \alpha \frac{G_F^2 |V_{ud}|^2}{8 \pi^3} \int \frac{d \Omega_{\gamma}}{4\pi} P \frac{B_0 \cos \theta_p}{2k_p} \left( -\frac{(2k_p^2 + k_n^2)(E_0 - E_e)}{k_n^2} \frac{k_n^2 - (\vec{n} \cdot \vec{k}_e)^2}{(E_n - \vec{n} \cdot \vec{k}_e)^2} - \frac{1}{12} \frac{(E_0 - E_e)^5 + 2(k_p^2 + k_n^2)(E_0 - E_e)^3}{k_p^2} \right) \\
\times \frac{1}{k_p E_e} \left( \frac{1}{E_n - \vec{n} \cdot \vec{k}_e} \frac{1}{E_e - \vec{n} \cdot \vec{k}_e} \right) - \frac{1}{12} \frac{(E_0 - E_e)^5 + 2(k_p^2 + k_n^2)(E_0 - E_e)^3}{k_p^2} \left( \frac{1}{E_n - \vec{n} \cdot \vec{k}_e} \frac{1}{E_e - \vec{n} \cdot \vec{k}_e} \right)
\]
\[
\Delta X^{(s)} = \frac{\alpha}{\pi} \alpha D X^{(s)} + a_0 \frac{\alpha}{\pi} \Delta X^{(s)} + P \cos \theta_p \left( A_0 \frac{\alpha}{\pi} \Delta X^{(s)} - B_0 \frac{\alpha}{\pi} \Delta X^{(s)} \right),
\]
where \(\Delta X^{(s)}, \Delta X^{(s)}_0, \Delta X^{(s)}_0\) and \(\Delta X^{(s)}_1\) are given by
\[
\Delta X^{(s)} = \int_{k_1}^{k_2} dk_p \int_{E_{\text{min}}}^{(E_{\text{max}})} dE_e (E_0 - E_e) \left\{ 1 + \frac{1}{3} (E_0 - E_e) + \frac{1}{12} \frac{(E_0 - E_e)^2}{E_e^2} \right\} \frac{1}{12} \frac{\ln \left( \frac{1 + \beta}{1 - \beta} \right)}{E_e} - 2 \\
+ \frac{1}{6} \frac{(E_0 - E_e)^2}{E_e^2} \right\} E_e k_p F(E_e, Z = 1) \\
+ \frac{1}{12} \frac{(E_0 - E_e)^2}{E_e^2} \left\{ \frac{1}{12} \frac{\ln \left( \frac{1 + \beta}{1 - \beta} \right)}{E_e} - 2 \right\} + \frac{1}{6} \frac{(E_0 - E_e)^2}{E_e^2} \right\} E_e k_p F(E_e, Z = 1) = 0.016051, \text{ MeV}^5
\]
\[
\Delta X^{(s)} = \int_{k_1}^{k_2} dk_p \int_{E_{\text{min}}}^{(E_{\text{max}})} dE_e \left\{ (E_0 - E_e)^2 - (E_0 - E_e)^2 - k_p^2 + k_n^2 \right\} \right\} E_e k_p F(E_e, Z = 1) = 0.016051, \text{ MeV}^5
\]
\[
\Delta X^{(s)}_0 = \int_{k_1}^{k_2} dk_p \int_{E_{\text{min}}}^{(E_{\text{max}})} dE_e \left\{ (E_0 - E_e)^2 - (E_0 - E_e)^2 - k_p^2 + k_n^2 \right\} \right\} E_e k_p F(E_e, Z = 1) = 0.016051, \text{ MeV}^5
\]
\[
\times \left[ \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \right] - \frac{1}{3} \left( \frac{E_0 - E_c}{E_c} \right)^3 \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - E_c (E_0 - E_c) \left[ \frac{3 - \beta^2}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 6 \right] \right) \\
\times k_p F(E_c, Z = 1) + \int_{E_2}^{(k_p)_{\text{max}}} \frac{dE_p}{E_p} \int_{(E_c)_{\text{min}}}^{(E_c)_{\text{max}}} dE_c \left\{ \left( (E_0 - E_c)^2 - \left( (E_0 - E_c)^2 - k_p^2 + k_c^2 \right) \right) \right. \\
\times \left( \frac{3}{2} \frac{1}{3} \frac{(E_0 - E_c)^2}{\beta^2 E_c} \frac{1}{24} \frac{(E_0 - E_c)^2}{\beta^2 E_c} \right) + \frac{1}{12} \left( \frac{E_0 - E_c}{E_c} \right)^4 + 3 \left( k_p^2 + k_c^2 \right) (E_0 - E_c)^2 \right) \\
\times \left[ \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \right] - \frac{1}{3} \left( \frac{E_0 - E_c}{E_c} \right)^3 \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - E_c (E_0 - E_c) \left[ \frac{3 - \beta^2}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 6 \right] \right) \\
\times k_p F(E_c, Z = 1) = -0.010201 \text{ MeV}^5, \quad (18) \\
\Delta X_{10}^{(s)} = \int_{k_1}^{k_2} \frac{dE_p}{E_p} \int_{(E_c)_{\text{max}}}^{(E_c)_{\text{max}}} dE_c \left\{ \left( (E_0 - E_c)^3 - (k_p^2 + k_c^2) (E_0 - E_c) \right) \right. \\
\times \left( \frac{3}{2} \frac{1}{3} \frac{(E_0 - E_c)^2}{\beta^2 E_c} \frac{1}{24} \frac{(E_0 - E_c)^2}{\beta^2 E_c} \right) + \frac{1}{6} \left( \frac{E_0 - E_c}{E_c} \right)^4 + 3 \left( k_p^2 + k_c^2 \right) (E_0 - E_c)^2 \right) \\
\times \left[ \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \right] - \frac{1}{3} \left( \frac{E_0 - E_c}{E_c} \right)^3 \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - E_c (E_0 - E_c) \left[ \frac{3 - \beta^2}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 6 \right] \right) \right. \\
\times F(E_c, Z = 1) = -0.010201 \text{ MeV}^5. \quad (19)
\]

and
\[
\Delta X_{11}^{(s)} = \int_{k_1}^{k_2} \frac{dE_p}{E_p} \int_{(E_c)_{\text{max}}}^{(E_c)_{\text{max}}} dE_c \left\{ E_c \left( (E_0 - E_c)^2 + k_p^2 + k_c^2 \right) \right. \\
\times \left( \frac{3}{2} \frac{1}{3} \frac{(E_0 - E_c)^2}{E_c} \frac{1}{24} \frac{(E_0 - E_c)^2}{E_c} \right) - \frac{1}{6} \left( E_0 - E_c \right)^4 + 3 \left( k_p^2 + k_c^2 \right) (E_0 - E_c)^2 \beta^2 \right) \\
\times \left[ \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \right] - \frac{1}{3} \left( \frac{E_0 - E_c}{E_c} \right)^3 \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - E_c (E_0 - E_c) \left[ \frac{3 - \beta^2}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 6 \right] \right) \right. \\
\times F(E_c, Z = 1) = -0.010201 \text{ MeV}^5. \quad (20)
\]

For the calculation of the contributions of the soft–photons we have followed the paper by Glück and restricted the photon–energy spectrum from above at \( \omega_m \). For numerical calculations we have set \( \omega_m = (E_0 - m_e)/3 \). The limits of the integration, plotted in Fig. 1, are equal to
\[
k_1 = (E_m - E_0) + \sqrt{E_m^2 - m_e^2}, \quad k_2 = (E_0 - E_m) + \sqrt{E_m^2 - m_e^2}, \quad (k_p)_{\text{max}} = \sqrt{E_0^2 - m_e^2}, \\
(\text{E_c})_{\text{min}} = \frac{(E_0 - k_p)^2 + m_e^2}{2(E_0 - k_p)}, \quad (\text{E_c})_{\text{max}} = \frac{(E_0 + k_p)^2 + m_e^2}{2(E_0 + k_p)}, \quad E_0 = \frac{m_e^2 - m_p^2 + m_e^2}{2m_e}, \quad E_m = E_0 - \omega_m. \quad (21)
\]
FIG. 1: The energy regions of the integration over the electron energy $E_e$ and the proton momentum $k_p$. The regions above and below $E_m = E_0 - \omega_m$ correspond to the soft- and hard-photon energy region, respectively.

FIG. 2: The maximal electron energy $(E_e)_{\text{max}}$ (green line) and its derivative (blue line) as functions of $k_p$ for $0 \leq k_p \leq (k_p)_{\text{max}}$.

The shaded regions above and below $E_m = E_0 - \omega_m$ correspond to the soft- and hard-photon energy region, respectively. The function $(E_e)_{\text{max}}$ is practically a straight line. It is not a surprise, since the function $(E_e)_{\text{max}}$ can be written as

$$(E_e)_{\text{max}} = \frac{(E_0 + k_p)^2 + m_e^2}{2(E_0 + k_p)} = \frac{1}{2} \left( E_0 + \frac{m_e^2}{E_0 + k_p} \right) + \frac{1}{2} k_p,$$

where $E_0 \gg m_e^2/(E_0 + k_p)$ for $0 \leq k_p \leq (k_p)_{\text{max}} = \sqrt{E_0^2 - m_e^2}$. In Fig. 2 we plot $(E_e)_{\text{max}}$ and the derivative $d(E_e)_{\text{max}}/dk_p$. One may see that the derivative of $(E_e)_{\text{max}}$ is practically constant. This confirms a behaviour of $(E_e)_{\text{max}}$ as a straight line.
IV. HARD-PHOTON CONTRIBUTION TO PROTON RECOIL ENERGY AND ANGULAR DISTRIBUTION OF NEUTRON RADIATIVE $\beta^+-$DECAY

In this section we calculate the contributions of the hard-photons with energies $\omega \geq \omega_m$. In this photon-energy region the functions $g^{(1)}_{\beta^- \gamma}(E_c, \mu)$ and $g^{(2)}_{\beta^- \gamma}(E_c, \mu)$ should be replaced by the functions $g^{(1)}_{\beta^- \gamma}(E_c, \omega_m)$ and $g^{(2)}_{\beta^- \gamma}(E_c, \omega_m)$, respectively, defined in Appendix B of Ref. [1] (see Eq.(B.15)). They are given by

$$
\begin{align*}
  g^{(1)}_{\beta^- \gamma}(E_c, \omega_m) &= \left\{ \ln\left( \frac{E_0 - E_c}{\omega_m} \right) - \frac{3}{2} + \frac{2 \omega_m}{E_0 - E_c} - \frac{1}{2} \frac{\omega_m^2}{(E_0 - E_c)^2} + \frac{1}{3} \frac{(E_0 - E_c - \omega_m)^3}{E_c(E_0 - E_c)^2} \\
  &+ \frac{1}{24} \frac{(E_0 - E_c - \omega_m)^3}{E_0 - E_c} \left( \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \right) + \frac{1}{12} \frac{(E_0 - E_c - \omega_m)^3}{E_0 - E_c} \left( \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \right) \right\} \\
  &+ \frac{1}{24} \frac{(E_0 - E_c - \omega_m)^3}{E_0 - E_c} \left( \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \right). 
\end{align*}
$$

(22)

In terms of the functions $g^{(1)}_{\beta^- \gamma}(E_c, \omega_m)$ and $g^{(2)}_{\beta^- \gamma}(E_c, \omega_m)$ we transcribe Eq. (21) into the form

$$
\begin{align*}
  &\frac{d^4 \Delta^{(1)}_{\beta^- \gamma}(E_c, k_p, \theta_p, P)}{dE_c dk_pd \cos \theta_p} = (1 + 3 \lambda^2) \frac{\alpha}{\pi} \frac{G_F^2 |V_{ud}|^2}{8 \pi^3} \left\{ - \frac{(E_0 - E_c) - PB_0 \cos \theta_p}{2k_p} \left( \frac{(E_0 - E_c)^2 + k_p^2 - k_e^2}{2k_p E_c} \right) g^{(1)}_{\beta^- \gamma}(E_c, \omega_m) \\
  &- \left( - a_0 \frac{(E_0 - E_c)^2 - k_p^2 + k_e^2}{2E_c} + PA_0 \cos \theta_p \frac{(E_0 - E_c)^2 - k_p^2 + k_e^2}{2k_p E_c} \right) g^{(2)}_{\beta^- \gamma}(E_c, \omega_m) \right\} E_c k_p F(E_c, Z = 1).
\end{align*}
$$

(23)

In the hard–photon energy region we analyse the sum of Eqs. (21) and (22). The result is

$$
\begin{align*}
  &\sum_{j=1,2,3} \frac{d^4 \Delta^{(j)}_{\beta^- \gamma}(E_c, k_p, \theta_p, P)}{dE_c dk_pd \cos \theta_p} = (1 + 3 \lambda^2) \frac{\alpha}{\pi} \frac{G_F^2 |V_{ud}|^2}{8 \pi^3} \int_0^{2 \pi} \frac{d \phi_p}{2 \pi} \int \frac{d \Omega_{exp}}{4 \pi} \int d \omega \int \frac{d \Omega_{\gamma}}{4 \pi} \left\{ \left( \frac{1 - B_0}{k_p + \frac{k_e}{E_c}} \ln \frac{k_p + \frac{k_e}{E_c} + \omega \tilde{n}}{k_p + \frac{k_e}{E_c} + \omega \tilde{n}} \right) \\
  &\times \left[ \frac{k_e^2 - (\tilde{n} \cdot \frac{k_e}{E_c})^2}{(E_c - \tilde{n} \cdot \frac{k_e}{E_c})^2} \left( 1 + \frac{\omega}{E_c} \right) + \frac{1}{E_c - \tilde{n} \cdot \frac{k_e}{E_c}} \right] + \left( - a_0 \frac{k_p + \frac{k_e}{E_c} + \omega \tilde{n}}{k_p + \frac{k_e}{E_c} + \omega \tilde{n}} + A_0 \tilde{n} \right) \cdot \left[ \frac{(k_e^2 - (\tilde{n} \cdot \frac{k_e}{E_c})^2)}{(E_c - \tilde{n} \cdot \frac{k_e}{E_c})^2} + \frac{\omega}{E_c - \tilde{n} \cdot \frac{k_e}{E_c}} \right] \\
  &\times \frac{k_p}{E_c} + \left( - \frac{m_e^2}{(E_c - \tilde{n} \cdot \frac{k_e}{E_c})^2} + \frac{\frac{E_c + \omega}{E_c - \tilde{n} \cdot \frac{k_e}{E_c}}}{E_c - \tilde{n} \cdot \frac{k_e}{E_c}} \right) \frac{\omega}{E_c - \tilde{n} \cdot \frac{k_e}{E_c}} \right\} \delta(E_0 - E_c - |k_p + \frac{k_e}{E_c} + \omega \tilde{n}| - \omega) k_e k_p^2 F(E_c, Z = 1).
\end{align*}
$$

(24)

Using energy conservation we obtain

$$
\begin{align*}
  &\sum_{j=1,2,3} \frac{d^4 \Delta^{(j)}_{\beta^- \gamma}(E_c, k_p, \theta_p, P)}{dE_c dk_pd \cos \theta_p} = (1 + 3 \lambda^2) \frac{\alpha}{\pi} \frac{G_F^2 |V_{ud}|^2}{8 \pi^3} \int_0^{2 \pi} \frac{d \phi_p}{2 \pi} \int \frac{d \Omega_{exp}}{4 \pi} \int d \omega \int \frac{d \Omega_{\gamma}}{4 \pi} \left\{ \left( (E_0 - E_c - \omega) \\
  &- B_0 \frac{\tilde{n}}{k_p + \frac{k_e}{E_c} + \omega \tilde{n}} \right) \left[ \frac{k_e^2 - (\tilde{n} \cdot \frac{k_e}{E_c})^2}{(E_c - \tilde{n} \cdot \frac{k_e}{E_c})^2} \left( 1 + \frac{\omega}{E_c} \right) + \frac{1}{E_c - \tilde{n} \cdot \frac{k_e}{E_c}} \right] + \left( - a_0 \frac{k_p + \frac{k_e}{E_c} + \omega \tilde{n}}{k_p + \frac{k_e}{E_c} + \omega \tilde{n}} + A_0 \tilde{n} \right) \cdot \left[ \frac{(k_e^2 - (\tilde{n} \cdot \frac{k_e}{E_c})^2)}{(E_c - \tilde{n} \cdot \frac{k_e}{E_c})^2} + \frac{\omega}{E_c - \tilde{n} \cdot \frac{k_e}{E_c}} \right] \\
  &\times \delta \left( (E_0 - E_c - \omega)^2 - (\tilde{k}_p + \frac{k_e}{E_c} + \omega \tilde{n})^2 \right) k_e k_p^2 F(E_c, Z = 1). \right\}
\end{align*}
$$

(25)

Since an analytical calculation of the integrals in Eq. (25) is not practically possible, we proceed to a numerical calculation. For this aim we define the scalar products in terms of the angular variables. We set

$$
\begin{align*}
  \tilde{n} \cdot k_p &= P k_p \cos \theta_p, \\
  \tilde{n} \cdot \tilde{k}_e &= P k_e \cos \theta_p \cos \theta_{\beta^+} + \sin \theta_p \sin \theta_{\beta^+} \cos (\phi_p - \phi_{\beta^+}), \\
  \tilde{n} \cdot \tilde{n} &= P \cos \theta_p \cos \theta_{\beta^+} + \sin \theta_p \sin \theta_{\beta^+} \phi_p - \phi_{\beta^+}).
\end{align*}
$$
After the integration of Eq.(25) over the energy difference, having integrated Eq.(23) and Eq.(28) over the electron energy, we get:

\[
\begin{aligned}
\bar{p}_e \cdot \bar{k}_e &= k_p k_e \cos \theta_{ep}, \\
\bar{k}_p \cdot \bar{n} &= k_p \cos \theta_{kp}, \\
\bar{k}_e \cdot \bar{n} &= k_e (\cos \theta_{ep} \cos \theta_{pr} + \sin \theta_{ep} \sin \theta_{pr} \cos (\phi_{ep} - \phi_{pr})).
\end{aligned}
\]

Since azimuthal angles enter in the differences, the azimuthal angle \(\phi_{pr}\) may be excluded. This gives:

\[
\begin{aligned}
\overline{\xi}_n \cdot \bar{k}_p &= P k_p \cos \theta_{kp}, \\
\overline{\xi}_n \cdot \bar{k}_e &= P k_e (\cos \theta_p \cos \theta_{ep} + \sin \theta_p \sin \theta_{ep} \cos (\phi_{kp} - \phi_{ke})), \\
\overline{\xi}_n \cdot \bar{n} &= P (\cos \theta_p \cos \theta_{pr} + \sin \theta_p \sin \theta_{pr} \cos (\phi_{kp})), \\
\bar{k}_p \cdot \bar{n} &= k_p k_e \cos \theta_{kp}, \\
\bar{k}_p \cdot \bar{k}_e &= k_p k_e \cos \theta_{kp}, \\
\bar{k}_e \cdot \bar{n} &= k_e (\cos \theta_{ep} \cos \theta_{pr} + \sin \theta_{ep} \sin \theta_{pr} \cos (\phi_{ep} - \phi_{pr})).
\end{aligned}
\]

After the integration of Eq.(25) over \(\omega\) in the limits \(\omega_m \leq \omega \leq (E_0 - m_e)/2\) we arrive at the expression:

\[
\sum_{j=2,3} \frac{d^3 \Delta \lambda^{(j)}(E_e, k_p, \theta_{kp}, P)}{dE_e dp_k d \cos \theta_{kp}} = (1 + 3 \lambda^2) \frac{\alpha G_F^2 V_{ud}^2}{16 \pi^3} \int_0^{2 \pi} \frac{d \phi_{kp}}{2 \pi} \int_0^{2 \pi} \frac{d \phi_{ep}}{2 \pi} \int_{-1}^{+1} d \cos \Theta_{kp} \int_{-1}^{+1} d \cos \Theta_{pr} \\
\times \left\{ (E_0 - m_e) - \frac{(E_0 - E_e)^2 - k_p^2 - k_e^2 - 2 \bar{k}_e \cdot \bar{k}_p}{(E_0 - E_e) + \bar{k}_p \cdot \bar{n} + \bar{k}_e \cdot \bar{n}} \right\} - \Theta \left( \omega_m - \frac{1}{2} \frac{(E_0 - E_e)^2 - k_p^2 - k_e^2 - 2 \bar{k}_e \cdot \bar{k}_p}{(E_0 - E_e) + \bar{k}_p \cdot \bar{n} + \bar{k}_e \cdot \bar{n}} \right) \\
\times \frac{1}{E_0 - E_e + \bar{k}_p \cdot \bar{n} + \bar{k}_e \cdot \bar{n}} \left\{ \left( (E_0 - E_e - \omega) - B_0 \bar{\xi}_n \cdot (\bar{\xi}_n + \omega \bar{n}) \right) \left( k_e^2 - (\bar{n} \cdot \bar{k}_e)^2 \right) \left( E_0 - \bar{n} \cdot \bar{k}_e \right) \left( E_0 - \bar{n} \cdot \bar{k}_e \right) + \frac{1}{E_0 - E_e - \omega} \right\} \\
+ \frac{m_e^2}{E_0 - \bar{n} \cdot \bar{k}_e} \left( E_0 - \bar{n} \cdot \bar{k}_e \right) \left( E_0 - \bar{n} \cdot \bar{k}_e \right) \frac{1}{E_0 - \bar{n} \cdot \bar{k}_e} \right\} k_e k_p F(E_e, Z = 1),
\]

where \(\omega\) is the function, given by

\[
\omega = \frac{1}{2} \frac{(E_0 - E_e)^2 - k_p^2 - k_e^2 - 2 \bar{k}_e \cdot \bar{k}_p}{(E_0 - E_e) + \bar{k}_p \cdot \bar{n} + \bar{k}_e \cdot \bar{n}}.
\]

and \(\Theta(z)\) is the Heaviside step function.

Now we may define the contributions of the hard photons to the proton recoil angular distribution of the neutron \(\beta^-\)-decay. Having integrated Eq.(23) and Eq.(28) over the electron energy \(E_e\) and the proton momentum \(k_p\), we obtain \(\Delta X_2^{(h)}, \Delta X_0^{(h)}, \Delta X_{10}^{(h)}\) and \(\Delta X_{11}^{(h)}\) equal to the following:

i) \(\Delta X_2^{(h)}:\)

\[
\begin{aligned}
\Delta X_2^{(h)}(E_e, k_p) &= \int_0^{k_1} dk_p \int_{(E_e)_{max}}^{(E_e)_{min}} dE_e \left\{ -2 \left( \frac{E_0 - E_e}{E_0 - E_e} \right) g_\gamma^{(1)}(E_e, \omega_m) + \Delta g_n(E_e, k_p) k_e k_p \right\} E_e k_p F(E_e, Z = 1) \\
&= -0.005268 \text{ MeV}^5
\end{aligned}
\]

with

\[
\Delta g_n(E_e, k_p) = \int_0^{2 \pi} \frac{d \phi_{kp}}{2 \pi} \int_{-1}^{+1} d \cos \Theta_{kp} \int_{-1}^{+1} d \cos \Theta_{pr} \left\{ \Theta \left( \omega_m - \frac{1}{2} \frac{(E_0 - E_e)^2 - k_p^2 - k_e^2 - 2 \bar{k}_e \cdot \bar{k}_p}{(E_0 - E_e) + \bar{k}_p \cdot \bar{n} + \bar{k}_e \cdot \bar{n}} \right) \\
- \Theta \left( \omega_m - \frac{1}{2} \frac{(E_0 - E_e)^2 - k_p^2 - k_e^2 - 2 \bar{k}_e \cdot \bar{k}_p}{(E_0 - E_e) + \bar{k}_p \cdot \bar{n} + \bar{k}_e \cdot \bar{n}} \right) \right\} \frac{(E_0 - E_e - \omega)}{(E_0 - E_e + \bar{k}_p \cdot \bar{n} + \bar{k}_e \cdot \bar{n})} \left( \frac{k_e^2 - (\bar{n} \cdot \bar{k}_e)^2}{E_0 - \bar{n} \cdot \bar{k}_e} \right) \left( \frac{1}{E_0 - \bar{n} \cdot \bar{k}_e} \right)
\]
ii) $\Delta X_0^{(b)}$:

\[
\Delta X_0^{(b)} = \int_0^{k_1} dk_p \int_{E_{e}\text{ min}}^{(E_e)\text{ max}} dE_e \left\{ \frac{(E_0 - E_e)^2 - k_p^2 + k_e^2}{E_e} g_{\mu e}^{(2)}(E_e, \omega_m) + \Delta f_a(E_e, k_p) k_e k_p \right\} E_e k_p F(E_e, Z = 1)
\]

\[+ \int_{k_1}^{k_2} dk_p \int_{E_{e}\text{ min}}^{E_{e}\text{ max}} dE_e \left\{ \frac{(E_0 - E_e)^2 - k_p^2 + k_e^2}{E_e} g_{\mu e}^{(2)}(E_e, \omega_m) + \Delta f_a(E_e, k_p) k_e k_p \right\} E_e k_p F(E_e, Z = 1) = 0.000049 \text{MeV}^5 (32)
\]

with

\[
\Delta f_a(E_e, k_p) = \int_0^{2\pi} d\phi_E \frac{1}{2\pi} \int_{-1}^{+1} d\cos \theta_E \int_{-1}^{+1} d\cos \theta_p \left\{ \Theta \left( E_0 - m_e - \frac{(E_0 - E_e)^2 - k_p^2 + k_e^2 - 2 k_e \cdot \vec{k}_p}{E_0 - E_e + k_p \cdot \vec{n} + k_e \cdot \vec{n}} \right) \right\}
\]

\[\times \left\{ \frac{1}{\omega (E_e - \vec{n} \cdot \vec{k_e})^2 + \frac{1}{E_e - \vec{n} \cdot \vec{k_e}}} - \frac{k_p \cdot \vec{k_e} + k_e^2 + \omega \vec{n} \cdot \vec{k_e}}{E_e} \right\}, (33)
\]

iii) $\Delta X_0^{(b)}$:

\[
\Delta X_0^{(b)} = \int_0^{k_1} dk_p \int_{E_{e}\text{ min}}^{(E_e)\text{ max}} dE_e \left\{ \frac{(E_0 - E_e)^2 - k_p^2 + k_e^2}{k_p E_e} g_{\mu e}^{(2)}(E_e, \omega_m) + \Delta f_A(E_e, k_p) k_e k_p \right\} E_e k_p F(E_e, Z = 1)
\]

\[+ \int_{k_1}^{k_2} dk_p \int_{E_{e}\text{ min}}^{E_{e}\text{ max}} dE_e \left\{ \frac{(E_0 - E_e)^2 - k_p^2 + k_e^2}{k_p E_e} g_{\mu e}^{(2)}(E_e, \omega_m) + \Delta f_A(E_e, k_p) k_e k_p \right\} E_e k_p F(E_e, Z = 1) = -0.012227 \text{MeV}^5 (34)
\]

with

\[
\Delta f_A(E_e, k_p) = \int_0^{2\pi} d\phi_E \frac{1}{2\pi} \int_{-1}^{+1} d\cos \theta_E \int_{-1}^{+1} d\cos \theta_p \left\{ \Theta \left( E_0 - m_e - \frac{(E_0 - E_e)^2 - k_p^2 + k_e^2 - 2 k_e \cdot \vec{k}_p}{E_0 - E_e + k_p \cdot \vec{n} + k_e \cdot \vec{n}} \right) \right\}
\]

\[\times \left\{ \frac{\cos \theta_E k_e}{E_e} \frac{1}{\omega (E_e - \vec{n} \cdot \vec{k_e})^2 + \frac{1}{E_e - \vec{n} \cdot \vec{k_e}}} + \cos \theta_p \frac{1}{E_e} - \frac{m_e^2}{E_e - \vec{n} \cdot \vec{k_e}} + \frac{E_e + \omega}{E_e - \vec{n} \cdot \vec{k_e}} \right\}, (35)
\]

and iv) $\Delta X_{11}^{(b)}$:

\[
\Delta X_{11}^{(b)} = \int_0^{k_1} dk_p \int_{E_{e}\text{ min}}^{(E_e)\text{ max}} dE_e \left\{ -\frac{(E_0 - E_e)^2 + k_p^2 - k_e^2}{k_p} g_{\mu e}^{(1)}(E_e, \omega_m) + \Delta f_B(E_e, k_p) k_e k_p \right\} E_e k_p F(E_e, Z = 1)
\]

\[+ \int_{k_1}^{k_2} dk_p \int_{E_{e}\text{ min}}^{E_{e}\text{ max}} dE_e \left\{ -\frac{(E_0 - E_e)^2 + k_p^2 - k_e^2}{k_p} g_{\mu e}^{(1)}(E_e, \omega_m) + \Delta f_B(E_e, k_p) k_e k_p \right\} E_e k_p F(E_e, Z = 1) = -0.007939 \text{MeV}^5 (36)
\]

with

\[
\Delta f_B(E_e, k_p) = \int_0^{2\pi} d\phi_E \frac{1}{2\pi} \int_{-1}^{+1} d\cos \theta_E \int_{-1}^{+1} d\cos \theta_p \left\{ \Theta \left( E_0 - m_e - \frac{(E_0 - E_e)^2 - k_p^2 + k_e^2 - 2 k_e \cdot \vec{k}_p}{E_0 - E_e + k_p \cdot \vec{n} + k_e \cdot \vec{n}} \right) \right\}
\]

\[\times \left\{ k_p + k_e \cos \theta_E + \omega \cos \theta_p \right\} \left\{ \frac{k_p^2 - (\vec{n} \cdot \vec{k_e})^2}{(E_e - \vec{n} \cdot \vec{k_e})^2} \left[ \frac{1}{E_e} + \frac{1}{E_e - \vec{n} \cdot \vec{k_e}} \frac{\omega}{E_e} \right] \right\}, (37)
\]

Now let us summarise the obtained results.
V. CONCLUSION

We have analysed the contributions of the proton–photon correlations to the proton recoil energy and angular distribution of the neutron $\beta^-$–decay, calculated in $\Gamma$. As has been shown in $\Gamma$, the radiative corrections to the proton recoil energy and angular distribution can be described by the functions $(\alpha/\pi) g_n(E_e)$ and $(\alpha/\pi) f_n(E_e)$, which were calculated by neglecting the proton–photon correlations in the radiative $\beta^-$–decay of the neutron. As has been pointed out by Glück $\Gamma$, the problem of the proton–photon correlations in the proton recoil energy spectrum of the radiative $\beta^-$–decay of the neutron and the contributions of these correlations to the energy and angular distributions of the neutron $\beta^-$–decay should be thoroughly investigated by means of a numerical analysis of the hard-photon energy region. For the calculation of the contributions of the hard photons Glück used the Monte Carlo simulation method $\Gamma$. For the analysis of the contributions of the proton–photon correlations in the radiative $\beta^-$–decay of the neutron we have defined a correction to the proton recoil energy and angular distribution of the neutron $\beta^-$–decay from the proton–photon correlations in the radiative $\beta^-$–decay of the neutron. In our analysis of this correction we have followed the paper by Glück $\Gamma$. We have divided the photon–energy spectrum into two parts, corresponding to the soft and hard photons. The contribution of the soft photons we have calculated analytically, whereas the contribution of the hard photons has been calculated numerically. For this aim we have used MATHEMATICA by Wolfram $\Gamma$. The soft- and hard-photon energy regions we have divided by $\omega_m$. For numerical calculations we have set $\omega_m = (E_0 - m_e)/3 = 0.260$ MeV. Integrating over the electron energy $E_e$ and the proton momentum $k_p$ we have obtained the contributions, caused by the proton–photon correlations in the radiative $\beta^-$–decay of the neutron, to the proton angular distribution of the neutron $\beta^-$–decay

$$\frac{d\Delta\lambda_{\beta^-}(\theta_p, P)}{d\cos\theta_p} = \frac{(1 + 3\lambda^2) G_F^2 |V_{ud}|^2}{16\pi^3} \left\{ \frac{\alpha}{\pi} \Delta X_2^{(s+h)} + a_0 \frac{\alpha}{\pi} \Delta X_0^{(s+h)} + P \cos\theta_p \left( A_0 \frac{\alpha}{\pi} \Delta X_{10}^{(s+h)} - B_0 \frac{\alpha}{\pi} \Delta X_{11}^{(s+h)} \right) \right\},$$

(38)

where $\Delta X_2^{(s+h)}$, $\Delta X_0^{(s+h)}$, $\Delta X_{10}^{(s+h)}$ and $\Delta X_{11}^{(s+h)}$, defined by the contributions of the soft and hard photons, are equal to

$$\Delta X_2^{(s+h)} = +0.010783 \text{ MeV}^5,$$
$$\Delta X_0^{(s+h)} = -0.010152 \text{ MeV}^5,$$
$$\Delta X_{10}^{(s+h)} = -0.057472 \text{ MeV}^5,$$
$$\Delta X_{11}^{(s+h)} = -0.028563 \text{ MeV}^5.$$

(39)

The contributions of $\Delta X_2^{(s+h)}$ and $\Delta X_0^{(s+h)}$, multiplied by $\alpha/\pi$, to the proton recoil energy and angular distribution are of order $2.4 \times 10^{-5}$ and can be neglected at the level of accuracy $10^{-5}$ accepted in $\Gamma$. Such a neglect of the contributions of $\Delta X_2^{(s+h)}$ and $\Delta X_0^{(s+h)}$ confirms also the correctness of the use of the radiative corrections, described by the functions $(\alpha/\pi) g_n(E_e)$ and $(\alpha/\pi) f_n(E_e)$, for the analysis of the proton–energy spectrum $a(T_p)$, pointed out in $\Gamma$. The contributions of $\Delta X_{10}^{(s+h)}$ and $\Delta X_{11}^{(s+h)}$ to the proton recoil asymmetry $C$, multiplied by $\alpha/\pi$, are equal to $(\alpha/\pi) \Delta X_{10}^{(s+h)} = -1.335 \times 10^{-4}$ and $(\alpha/\pi) \Delta X_{11}^{(s+h)} = -0.664 \times 10^{-4}$. Since these corrections are of order $10^{-4}$ and larger than accuracy of about $10^{-5}$ $\Gamma$, they should be taken into account. This means that in the proton recoil angular distribution of the neutron $\beta^-$–decay, defined by Eq.(I-21) in Ref.$\Gamma$, the parameters $X_{10}$ and $X_{11}$ should be replaced by

$$X_{10} \rightarrow \bar{X}_{10} = X_{10} + \Delta X_{10}^{(s+h)} = -2.214586 \text{ MeV}^5,$$
$$X_{11} \rightarrow \bar{X}_{11} = X_{11} + \Delta X_{11}^{(s+h)} = +2.215636 \text{ MeV}^5,$$

(40)

where $X_{10} = -2.157114 \text{ MeV}^5$ and $X_{11} = 2.244291 \text{ MeV}^5$ $\Gamma$. Denoting the new parameters as $\bar{X}_{10} = -X_{\text{eff}}$ and $\bar{X}_{11} = X_{\text{eff}}$, where $X_{\text{eff}} = 2.215111 \text{ MeV}^5$ is valid with an accuracy of about $5.5 \times 10^{-5}$% in the proton recoil angular distribution, we may transcribe the proton recoil angular distribution of the neutron $\beta^-$–decay, calculated in $\Gamma$, into the form

$$\frac{d\lambda_{\beta^-}(\theta_p, P)}{d\cos\theta_p} = \frac{(1 + 3\lambda^2) G_F^2 |V_{ud}|^2}{16\pi^3} \left\{ X_1 + \frac{\alpha}{\pi} X_2 + \frac{1}{M} \left[ X_3 + (1 + 3\lambda^2) (X_4 + Y_1) - (1 - \lambda^2)^2 (X_5 + Y_2) \right.ight.$$
$$+ (\lambda^2 + 2(\kappa + 1)\lambda) X_6 - (\lambda^2 - 2(\kappa + 1)\lambda) X_7] + P \cos\theta_p \left[ - (A_0 + B_0) \left( X_8 + \frac{\alpha}{\pi} X_{\text{eff}} \right) + A_0 X_9 \right.$$
$$+ \frac{1}{M} \left[ X_{12} - (\kappa + 1)\lambda X_{13} - (2\kappa + 1)\lambda X_{14} - \lambda(1 + \lambda)(X_{15} + Y_3) + \lambda(1 - \lambda)(X_{16} + Y_4) \right]\right\}.$$  

(41)
This leads to the following change of the correlation coefficient $C$

$$C = -\left(x_C + \frac{\alpha}{\pi} x_{\text{eff}}\right) \left(A_0 + B_0\right) + \frac{1}{2} X_2 A_0 + \frac{1}{M} \frac{1}{1 + 3\lambda^2} \left(\frac{1}{2} X_{12} \right) - (\kappa + 1) \left(\frac{1}{2} X_{13} \right) - (2\kappa + 1) \left(\frac{1}{2} X_{14} \right)$$

\begin{align*}
-\lambda(1 + \lambda) \left\{\frac{1}{2} X_{15} Y_3 + \lambda(1 - \lambda) \left(\frac{1}{2} X_{16} + Y_3\right)\right\} + \left(A_0 + B_0\right) X_8 X_1 \left(\frac{1}{\pi} \frac{1}{2} X_1\right) + \frac{1}{M} \frac{1}{1 + 3\lambda^2} \left(\frac{1}{2} X_3 \right) + (1 + 3\lambda^2) \\
\times \frac{1}{2} X_4 + Y_1 - (1 - \lambda^2) \frac{1}{2} X_5 + Y_2 + (\lambda^2 + 2(\kappa + 1)\lambda) \frac{1}{2} X_6 - (\lambda^2 - 2(\kappa + 1)\lambda) \frac{1}{2} X_7, \quad (42)
\end{align*}

where the contribution of the radiative corrections is symmetric with respect to a change $A_0 \leftrightarrow B_0$ as well as the main term $-x_C(A_0 + B_0)$, which has been calculated for the first time by Treiman [1]. The factor $x_C = X_8/2X_1 = 0.27591$, calculated in [1], agrees well with the factor $x_C = 0.27594$, calculated by Glück [15], and $x_{\text{eff}} = X_{\text{eff}}/2X_1 = 4.71210$. We would like to remind the reader that the appearance of the term $A_0X_9/2X_1$, violating a symmetry with respect to a change $A_0 \leftrightarrow B_0$, is related to the deviation of the Fermi function $F(E_c, Z = 1)$, caused by the final-state electron–proton Coulomb interaction, from unity [1]. The parameters $X_i$ and $Y_j$ for $i = 1, 2, \ldots, 16$ and $j = 1, \ldots, 4$ are defined in Appendix I of Ref.[1].

A dependence of our results on $\omega_m$ makes our analysis to some extent qualitative. Nevertheless, one may show that the orders of magnitudes of the obtained corrections are rather stable under variations of $\omega_m$. For example, for a cut–off $\omega_m = 0.10$ MeV, at which the contribution of the quantum corrections to the radiative $\beta^–$–decay of the neutron may be neglected with respect to the classical one, proportional to $1/\omega$, we get

$$\begin{align*}
\Delta X_2^{(s+h)} &= -0.015290 \text{ MeV}^5, \\
\Delta X_0^{(s+h)} &= -0.005509 \text{ MeV}^5, \\
\Delta X_{10}^{(s+h)} &= -0.085248 \text{ MeV}^5, \\
\Delta X_{11}^{(s+h)} &= -0.029908 \text{ MeV}^5. \quad (43)
\end{align*}$$

These values retain the orders of magnitudes of $(\alpha/\pi) \Delta X_2$ and $(\alpha/\pi) \Delta X_0$ and our assertion that the radiative corrections to the lifetime of the neutron $T_n$ and the proton–energy spectrum $a(T_p)$ can be described by the functions $(\alpha/\pi) g_n(E_c)$ and $(\alpha/\pi) f_n(E_c)$. A description of the radiative corrections to the lifetime of the neutron by the function $(\alpha/\pi) g_n(E_c)$ agrees well with the results, obtained by Glück [10]. For $\omega_m = 0.10$ MeV we may introduce $X_{\text{eff}} = 2.228282 \text{ MeV}^2$. At the level of the accuracy $10^{-5}$ this does not contradict to the results obtained above. An approximate coincidence of the results, obtained for $\omega_m = 0.26$ MeV and $\omega_m = 0.10$ MeV, may serve for a confirmation of an approximate independence of our calculations of the proton–photon correlations of the cut–off $\omega_m$.

VI. COMPARISON WITH RESULTS, OBTAINED BY GLÜCK [10]

Now let us discuss our results in comparison to the results, obtained by Glück [10]. We analyse the contributions of the radiative corrections to the electron–antineutrino $(E_e, \cos \theta_{e\bar{\nu}})$ distribution and the proton–energy spectrum $a(T_p)$.

A. Electron–antineutrino $(E_e, \cos \theta_{e\bar{\nu}})$ distribution

The electron–antineutrino $(E_e, \cos \theta_{e\bar{\nu}})$ distribution of the neutron $\beta^–$–decay with unpolarised neutron and decay electron and proton, where $\cos \theta_{e\bar{\nu}} = \vec{k} \cdot \vec{k}_{\bar{\nu}}/E_{e\bar{\nu}}$ and $\vec{k}$ and $\vec{k}_{\bar{\nu}}$ are the antineutrino and electron 3–momenta, can be obtained from Eq.(6) of Ref.[1]. One gets

$$\frac{d^2 \lambda_n(E_e, \vec{k}_{\bar{\nu}}, \vec{k})}{dE_{\text{cut}} d\cos \theta_{e\bar{\nu}}} = (1 + 3\lambda^2) \frac{G_F^2 |V_{ud}|^2}{4\pi^3} (E_0 - E_e)^2 \sqrt{E_e^2 - m_e^2} E_e F(E_e, Z = 1) \zeta(E_e) \left(1 + \frac{\alpha}{\pi} g_n(E_e)\right) \times \left\{1 + a^{(W)}(E_e) \left(1 + \frac{\alpha}{\pi} f_n(E_e)\right) \beta P_1(\cos \theta_{e\bar{\nu}}) - 2 a_0 E_e \frac{E_e}{M} P_2(\cos \theta_{e\bar{\nu}})\right\}, \quad (44)$$

where the function $\zeta(E_e)$, including the corrections of order $1/M$, caused by the “weak magnetism” and the proton recoil, is given in [1]. Then, $P_1(\cos \theta_{e\bar{\nu}}) = \cos \theta_{e\bar{\nu}}$ and $P_2(\cos \theta_{e\bar{\nu}}) = (3 \cos^2 \theta_{e\bar{\nu}} - 1)/2$ are the Legendre polynomials
The correlation coefficient \( a^{(W)}(E_e) \), containing the leading and next-to-leading terms in the large \( M \) expansion only, is given by

\[
a^{(W)}(E_e) = a_0 \left\{ 1 + \frac{1}{M} \left( \frac{1}{1 - \lambda^2 (1 + 3 \lambda^2)} \left( a_1 E_0 + a_2 E_e + a_3 \frac{m_e^2}{E_e} \right) \right) \right.,
\]

where

\[
a_1 = 4M(\lambda^2 + 1)(\lambda - (\kappa + 1)),
\]

\[
a_2 = -26\lambda^4 + 8(\kappa + 1)\lambda^3 - 20\lambda^2 + 8(\kappa + 1)\lambda - 2,
\]

\[
a_3 = -2\lambda(\lambda^2 - 1)(\lambda - (\kappa + 1)).
\]

We would like to accentuate that the \((E_e, \cos \theta_{e\nu})\) distribution Eq. (44) is calculated in Ref. [1] by integrating first over the proton 3-momentum in the final state of the continuum-state \( \beta^- \)-decay and the radiative \( \beta^- \)-decay of the neutron, respectively. As has been pointed out by Glück [3], there are no proton–photon correlations between decay protons and virtual photons in one–virtual photon exchanges in the continuum-state \( \beta^- \)-decay of the neutron. The integration over the 3-momentum of the decay proton in the radiative \( \beta^- \)-decay of the neutron leads to the absence of the proton–photon correlations to order \( \alpha/\pi \) or to leading order in the large \( M \) expansion. One may see that after the integration over the decay proton 3-momentum the proton–photon correlations in the radiative \( \beta^- \)-decay of the neutron appear to order \( \alpha/\pi \) \((E_0/M) \sim 10^{-6} \) only, which may be neglected in comparison to contributions of order \( \alpha/\pi \sim 10^{-4} \) (see also Ref. [12]). The radiative corrections, described by the functions \( (\alpha/\pi) g_n(E_e) \) and \( (\alpha/\pi) f_n(E_e) \), are defined by the contributions of the radiative \( \beta^- \)-decay of the neutron and one–virtual photon exchanges in the continuum-state \( \beta^- \)-decay of the neutron (see Eq. (D-58) of Ref. [1]).

\[
g_n(E_e) = \frac{3}{2} \frac{\beta}{\hat{\beta}} \left\{ \frac{3}{8} \left[ \frac{1}{2} \frac{\beta}{\hat{\beta}} \left( \frac{1 + \beta}{1 - \beta} \right)^2 - \frac{1}{2} \frac{\beta}{\hat{\beta}} \left( \frac{1 + \beta}{1 - \beta} \right) - 1 \right] \left[ 2 \frac{E_0 - E_e}{m_e} - \frac{3}{2} + \frac{1}{2} \frac{E_0 - E_e}{E_e} \right] + \frac{2}{\beta} \left( \frac{2}{1 + \beta} \right) \right.
\]

\[
+ \frac{1}{2\beta} \frac{\beta}{\hat{\beta}} \left( \frac{1 + \beta}{1 - \beta} \right) \left[ 1 + \beta + \frac{1}{12} \frac{(E_0 - E_e)^2}{E_e^2} - \frac{1}{12} \frac{(E_0 - E_e)^2}{E_e^2} - \frac{1}{2\beta} \left( \frac{1 + \beta}{1 - \beta} \right) \right] + CWZ,
\]

\[
f_n(E_e) = \frac{2}{3} \frac{E_0 - E_e}{E_e} \left( 1 + \frac{1}{8} \frac{E_0 - E_e}{E_e} - \frac{1}{2\beta} \left[ \frac{1}{2\beta} + \frac{1}{6} \frac{E_0 - E_e}{E_e} - \frac{1}{2\beta} \left( \frac{1 + \beta}{1 - \beta} \right) - 1 \right] - \frac{1}{12} \frac{(E_0 - E_e)^2}{E_e^2} + \frac{1 - \beta^2}{2\beta} \frac{\beta}{\hat{\beta}} \left( \frac{1 + \beta}{1 - \beta} \right).
\]

where the constant \( CWZ = 10.429 \) is caused by the electroweak boson exchanges and QCD corrections (see a discussion in Appendix D of Ref. [1]). The contribution of \( CWZ = 10.249 \) to the neutron \( \beta^- \)-decay can be described by the parameter \( \Delta_R = (\alpha/\pi) CWZ = 0.0238 \) (see, for example, Ref. [1]). This value agrees well with the value \( \Delta_R = 0.024 \) used in Ref. [11].

In Ref. [11] the radiative corrections to the \((E_e, \cos \theta_{e\nu})\) distribution are described by the function \( r_{e\nu}(x, \cos \theta_{e\nu}) \), where \( x = (E_e - m_e)/(E_0 - m_e) \) and \( E_0 - m_e = ((m_n - m_e)^2 - m_n^2)/2m_n = 0.7817 \) MeV is the \( Q \)-value of the neutron \( \beta^- \)-decay. The function \( r_{e\nu}(x, \cos \theta_{e\nu}) \), defined in terms of the function \( f_n(E_e) \), is

\[
r_{e\nu}(x, \cos \theta_{e\nu}) = 100 \frac{\alpha}{\pi} f_n(E_e).
\]

Unlike the results, obtained in Ref. [11], the function Eq. (47) does not depend on \( \cos \theta_{e\nu} \). Such an independence of \( \cos \theta_{e\nu} \) is exact to leading order in the large \( M \) expansion and caused by the integration over the 3-momentum of the decay proton that leads to the proton–photon decorrelation. The contribution of the radiative corrections in Ref. [11] has been compared with the contribution, calculated in Ref. [20] and given by the function \((\alpha/\pi) f_n(E_e)\). As has been pointed out by Glück [20], the contribution of the radiative corrections, calculated in Ref. [11], is of order of magnitude larger compared to the contribution of the radiative corrections, defined by the function \((\alpha/\pi) f_n(E_e)\). In Fig. 3 we plot the function \( r_{e\nu}(x, \cos \theta_{e\nu}) \), given by Eq. (47) and defined in terms of the function \( f_n(E_e) \). The numerical values of this function are added in Table I. Apart from the independence of \( \cos \theta_{e\nu} \) one may see that the values of the radiative corrections \( r_{e\nu}(x, \cos \theta_{e\nu}) \), defined by the function \( f_n(E_e) \), are commensurable with the values of the radiative corrections, calculated by Glück (see Table V of Ref. [10]). We would like to emphasise that the radiative corrections, given by the function \( f_n(E_e) \), are defined by both the contributions of the soft and hard photons [1].

| \( x \) | 0.0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| \( r_{e\nu}(x, \cos \theta_{e\nu}) \) | 0.28 | 0.23 | 0.19 | 0.16 | 0.14 | 0.12 | 0.10 | 0.09 | 0.08 | 0.07 | 0.06 |

 TABLE I: The numerical values of the radiative corrections \( r_{e\nu}(x, \cos \theta_{e\nu}) \) to the electron–antineutrino \((E_e, \cos \theta_{e\nu})\) distribution.
FIG. 3: The radiative corrections \( r_\beta(x, \cos \theta_\beta) \) to the electron–antineutrino \((E_\beta, \cos \theta_\beta)\) distribution, defined in terms of the function \( f_\eta(E_\beta) \) for \( 0 \leq x \leq 1 \) with \( x = (E_\beta - m_\beta)/(E_0 - m_\beta) \).

**B. Proton–energy \( a(T_p) \) spectrum**

The proton–energy spectrum is defined by (see Appendix I of Ref. [1])

\[
\frac{d\lambda_n(T_p)}{dT_p} = M (1 + 3\lambda^2) \frac{G_F^2 |V_{ud}|^2}{4\pi^3} a(T_p) \left( 1 - b_F \frac{m_e}{E_e} \right)
\]

(48)

where \( b_F \) is the Fierz term, \( \langle m_e/E_s \rangle_{SM} = 0.6556 \) [1] and \( a(T_p) \) is defined by

\[
a(T_p) = g_p^{(1)}(T_p) + \frac{\alpha}{\pi} f_p^{(1)}(T_p) + a_0 \left( g_p^{(2)}(T_p) + \frac{\alpha}{\pi} \left( f_p^{(2)}(T_p) + f_p^{(3)}(T_p) \right) \right) + b_p f_p^{(4)}(T_p).
\]

(49)

The functions \( g_p^{(1)}(T_p) \) and \( g_p^{(2)}(T_p) \) are defined by the integrals

\[
g_p^{(1)}(T_p) = \int_{(E_e)_{min}}^{(E_e)_{max}} \zeta_1(E_e, T_p) F(E_e, Z = 1) E_e dE_e,
\]

\[
g_p^{(2)}(T_p) = \int_{(E_e)_{min}}^{(E_e)_{max}} \left( \zeta_2(E_e, T_p) + \frac{1}{1 - \lambda^2} \left( \frac{E_0}{M} \right)^2 \right) F(E_e, Z = 1) E_e dE_e,
\]

(50)

where the functions \( \zeta_1(E_e, T_p) \) and \( \zeta_2(E_e, T_p) \) are defined by Eq. (I-15) in Appendix I of Ref. [1]. They include the next-to-leading order corrections in the large \( M \) expansion. The functions \( f_p^{(1)}(T_p) \) and \( f_p^{(2)}(T_p) \) determine the radiative corrections to the proton–energy spectrum. They are given by the integrals over the electron–energy spectrum \((E_e)_{min} \leq E_e \leq (E_e)_{max}\) in terms of the functions \( g_n(E_\beta) \) and \( f_n(E_\beta) \) (see Appendix I of Ref. [1])

\[
f_p^{(1)}(T_p) = \int_{(E_e)_{min}}^{(E_e)_{max}} (E_0 - E_e) g_n(E_\beta) F(E_\beta, Z = 1) E_e dE_e,
\]

\[
f_p^{(2)}(T_p) = -\frac{1}{2} \int_{(E_e)_{min}}^{(E_e)_{max}} \left( (E_0 - E_e)^2 + E_e^2 - m_\beta^2 - 2MT_p \right) \left( g_n(E_\beta) + f_n(E_\beta) \right) F(E_\beta, Z = 1) dE_e.
\]

(51)

The radiative corrections, described by the function \( f_p^{(3)}(E_e) \), are given by the proton–photon correlations in the radiative \( \beta^- \) decay of the neutron. It reads

\[
f_p^{(3)}(T_p) = \left( \Theta(T_p - T_1) - \Theta(T_p - T_2) \right) \int_{(E_e)_{min}}^{(E_e)_{max}} \left( \frac{1}{(E_0 - E_e)^2} + \frac{1}{24} \frac{(E_0 - E_e)^4}{\beta^2 E_e^2} + \frac{1}{12} \frac{(E_0 - E_e)^4}{\beta^2 E_e^2} \right) dE_e
\]

\[
\times \left( \frac{3}{2} - \frac{1}{3} (2MT_p - E_e^2 - m_\beta^2) / \beta^2 E_e \right) \left( \frac{1}{2} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \right) - \frac{1}{3} \frac{(E_0 - E_e)^3}{E_e} \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - E_e (E_0 - E_e)
\]

\[
+ \frac{(2MT_p - E_e^2 + m_\beta^2)(E_0 - E_e)}{\beta^2 E_e} \right) \left[ \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \right] - \frac{1}{3} \frac{(E_0 - E_e)^3}{E_e} \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - E_e (E_0 - E_e)
\]

\[
+ \frac{(2MT_p - E_e^2 + m_\beta^2)(E_0 - E_e)}{\beta^2 E_e} \right) \left[ \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \right] - \frac{1}{3} \frac{(E_0 - E_e)^3}{E_e} \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - E_e (E_0 - E_e)
\]
where $T_j = k_j^2/2M$ and $j = 1, 2$ with $k_1$ and $k_2$ defined in Eq. (21). Then, the function $f_p^{(4)}(T_p)$ is given by

$$f_p^{(4)}(T_p) = m_e \int_{(E_e)_{min}}^{(E_e)_{max}} (E_0 - E_e) F(E_e, Z = 1) \, dE_e \quad (53)$$

Following Glück [10] we introduce the functions

$$r_1(y) = 100 \frac{\alpha}{\pi} f_p^{(1)}(T_p), \quad r_2(y) = 100 \frac{\alpha}{\pi} f_p^{(2)}(T_p),$$

$$r_3(y) = 100 \frac{\alpha}{\pi} f_p^{(3)}(T_p), \quad r_4(y) = 100 \frac{\alpha}{\pi} f_p^{(4)}(T_p),$$

where $y = T_p/(T_p)_{max}$.

TABLE II: The numerical values of the functions $r_1(y)$, $r_2(y)$ and $r_4(y)$, describing the radiative corrections, and the function $r_4(y)$, defining the contributions of the Fierz term, to the proton–energy spectrum $a(T_p)$ for $0 \leq y \leq 1$ with $y = T_p/(T_p)_{max}$. The functions $r_1(y)$, $r_2(y)$, $r_3(y)$ and $r_4(y)$ are obtained by the integration of the electron–proton energy $a(E_e, T_p)$ distribution over the energies of the decay electron $(E_e)_{min} \leq E_e \leq (E_e)_{max}$ and measured in MeV$^3$.
where \( y = (E_p - m_p)/(E_{p\text{max}} - m_p) = T_p/(T_{\text{pmax}}) \) with \((T_p)_{\max} = (E_0^2 - m_e^2)/2M\). The functions \( r_1(y) \), \( r_2(y) \), \( r_3(y) \) and \( r_4(y) \) are plotted in Fig. 4. The numerical values of these functions for \( 0 \leq y \leq 1 \) are added in Table II. The contribution of the function \( r_4(y) \) to \( a(T_p) \) is proportional to 0.01 \( bF \pi/\alpha = 4.31 bF \).

One may see that the contributions of the radiative corrections, described by the function \( r_3(y) \) and induced by the proton–photon correlations, are of order of magnitude smaller compared with the contributions of the radiative corrections, described by the function \( r_4(y) \) and determined by the functions \( g_n(E_e) \) and \( f_n(E_e) \). Since in addition the function \( r_3(y) \) changes a sign around \( y \simeq 0.45 \) or \( T_p \simeq 0.340\text{keV} \), the result of the integration of \( r_3(y) \) over the proton energy spectrum \( 0 \leq k_p \leq (k_{p\text{max}}) \) or \( 0 \leq y \leq 1 \) confirms our assertion, concerning a negligibility of the term, proportional to \( a_0 \), in the proton recoil angular distribution Eq. (35).

In Fig. 5 we plot the proton–energy spectrum \( a(T_p) \). A maximum of the proton–energy spectrum \( a(T_p) \) is located around \( T_p \sim 0.391\text{keV} \) or around \( y \simeq 0.52 \) (see also [21]). One may see that for the decay protons with energies \( 0.40 \leq y \leq 0.55 \) or \( 0.300\text{keV} \leq T_p \leq 0.413\text{keV} \) the contributions of the radiative corrections \( r_3(y) \), caused by the proton–photon correlations can be fairly neglected in comparison to the contributions of the radiative corrections \( r_2(y) \), defined by the functions \( g_n(E_e) \) and \( f_n(E_e) \).

Of course, for the decay protons, detected from the energy region \( 0.413\text{keV} \leq T_p \leq 0.680\text{keV} \) or \( 0.55 \leq y \leq 0.90 \) with the aim to measure the contributions of the Fierz term [24, 26], defined by scalar and tensor interactions beyond the SM [1], one should take into account the contributions of the proton–photon correlations, given by the function \( r_3(y) \). A recent estimate of the Fierz term \( bF = 3.2(2.3) \times 10^{-3} \) has been carried out in [27]. The Fierz term, multiplied by 4.31 \( r_4(y) \), gives the contributions of order \( 10^{-4} \) to the proton–energy spectrum \( a(T_p) \).

Comparing the function \( r_2(y) \) with the function \( r_3(y) \) (see Table IV of Ref. [10]), one may see that these functions differ in a sign and a behaviour at \( y \rightarrow 1 \). Indeed, the function \( r_2(y) \) is negative in the interval \( 0.00 \leq y \leq 0.45 \) and changes sign at \( y \approx 0.45 \). whereas the function \( r_3(y) \) is positive in the interval \( 0.00 \leq y \leq 0.62 \) and changes sign at \( y \approx 0.62 \). Then, the function \( r_2(y) \) is positive in the interval \( 0.45 \leq y \leq 1 \) and vanishes at \( y = 1 \), whereas the function \( r_3(y) \) is negative in the interval \( 0.62 \leq y \leq 1 \) and does not vanish at \( y = 1 \).

The vanishing of the function \( r_2(y) \) is obvious, since this function is defined by the integral over the electron energies \((E_e)_{\min} \leq E_e \leq (E_e)_{\max} \) and \((E_e)_{\min} = (E_0 - m_e^2)/2E_0 \) for \( y = 0 \) and \((E_e)_{\max} = (E_e)_{\max} = E_0 \) for \( y = 1 \) or for \( T_p = 0 \) and \( T_p = (T_{\text{pmax}}) \), respectively. The sign of the function \( r_2(y) \) is defined by the sign of the function \( f(E_e, T_p) = -((E_0 - E_e)^2 + E_e^2 - m_e^2 - 2MT_p)/2 \) in the integrand of the integral over the electron energies, which is changed at \( T_p = (E_0^2 - 2m_e^2 E_e)/4M \) and \( E_e = E_0/2 \).

C. Electron–proton energy \( a(E_e, T_p) \) distribution

We have analysed in detail the radiative corrections to the proton–energy spectrum \( a(T_p) \). Since the proton–energy spectrum \( a(T_p) \) is related to the electron–proton energy distribution \( a(E_e, T_p) \) as [1]

\[
a(T_p) = \int_{(E_e)_{\min}}^{(E_e)_{\max}} a(E_e, T_p) F(E_e, Z = 1) E_e dE_e,
\]

FIG. 5: The proton–energy spectrum \( a(T_p) \). The maximal value \( (T_p)_{\max} = 0.235\text{MeV}^2 \) is located around \( T_p = 0.391\text{keV} \) or \( y = 0.52 \).
a dominance of the radiative corrections, described by the functions $g_n(E_e)$ and $f_n(E_e)$, with respect to the radiative corrections, caused by the proton–photon correlations, is also valid for the electron–proton energy distribution.

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