UNIFORM BOUNDEDNESS FOR ALGEBRAIC GROUPS AND LIE GROUPS

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Abstract. Let $G$ be a semisimple linear algebraic group over a field $k$ and let $G^+(k)$ be the subgroup generated by the subgroups $R_u(Q)(k)$, where $Q$ ranges over all the minimal $k$-parabolic subgroups $Q$ of $G$. We prove that if $G^+(k)$ is bounded then it is uniformly bounded. Under extra assumptions we get explicit bounds for $\Delta(G^+(k))$: we prove that if $k$ is algebraically closed then $\Delta(G^+(k)) \leq 4\text{rank}(G)$, and if $G$ is split over $k$ then $\Delta(G^+(k)) \leq 28\text{rank}(G)$. We deduce some analogous results for real and complex semisimple Lie groups.

1. Introduction

In this paper we investigate the boundedness behaviour of a semisimple linear algebraic group $G$ over an infinite field $k$. (For definitions of boundedness and related notions, see Section 2.) If $k = \mathbb{R}$ then $G$ is a semisimple Lie group, and it is well known that $G$ is compact in the real topology if and only if it is anisotropic. The authors showed in [5, Thm. 1.2] that if $G$ is compact then $G$ is bounded but is not uniformly bounded; on the other hand, if $G$ has no simple compact factors then $G$ is uniformly bounded. Motivated by this, we make the following conjecture.

Conjecture 1.1. Let $G$ be a semisimple linear algebraic group over an infinite field $k$. Then $G^+(k)$ is uniformly bounded.

Here $G^+(k)$ denotes the subgroup of $G(k)$ generated by the subgroups $R_u(Q)(k)$, where $Q$ ranges over the minimal $k$-parabolic subgroups of $G$. If $k = \overline{k}$ then $G^+(k) = G(k)$, while if $G$ is anisotropic over $k$ then $G^+(k) = 1$. If $G$ has no anisotropic $k$-simple factors then $G^+(k)$ is dense in $G$. Note that a finite group is clearly uniformly bounded.
so Conjecture 1.1 and the other results below all hold trivially for a semisimple linear algebraic group over a finite field $k$.

We make some steps towards proving the conjecture.

**Theorem 1.2.** Let $G$ be a semisimple linear algebraic group over an infinite field $k$, and suppose $G(k) = G^+(k)$. Then $G(k)$ is finitely normally generated. Moreover, if $G(k)$ is bounded then $G(k)$ is uniformly bounded.

We want to give explicit bounds for $\Delta(G)$ in terms of Lie-theoretic quantities such as $\text{rank } G$ and $\text{dim } G$. We can do this in some special cases. The first improves the bound $4 \dim G$ from [5, Thm. 4.3].

**Theorem 1.3.** Let $G$ be a semisimple linear algebraic group over an algebraically closed field $k$. Then $\Delta(G(k)) \leq 4 \text{rank } G$.

**Theorem 1.4.** Let $G$ be a split semisimple linear algebraic group over an infinite field $k$. Then $\Delta(G^+(k)) \leq 28 \text{rank } G$.

When $k = \mathbb{R}$, we get the following result.

**Theorem 1.5.** Let $H$ be a real semisimple linear algebraic group with no compact simple factors. Then $H$ is uniformly bounded. Moreover, if $H$ is split then $\Delta(H) \leq 28 \text{rank } G$.

When $k = \mathbb{C}$, we get the following result.

**Theorem 1.6.** Let $H$ be a complex semisimple linear algebraic group. Then $H$ is uniformly bounded and $\Delta(H) \leq 4 \text{rank } G$.

The idea of the proofs is as follows. First we prove Theorem 1.3 (Section 4); the new ingredient is that we work in the quotient variety $G/\text{Inn}(G)$ rather than in $G$, which allows us to improve on the bound in [5, Thm. 4.3]. A key result underpinning our theorems for non-algebraically closed $k$ is Proposition 5.5. We prove this in Section 5 and deduce Theorem 1.2. When $G$ is split we obtain Theorem 1.4 from Proposition 5.5 and the Bruhat decomposition; see Section 6. In Section 7 we prove Theorems 1.5 and 1.6.

**Acknowledgements.** This work was funded by Leverhulme Trust Research Project Grant RPG-2017-159.

2. Boundedness and uniform boundedness

A conjugation-invariant norm on a group $H$ is a non-negative function $\| \| : H \to \mathbb{R}$ such that $\| \|$ is constant on conjugacy classes, $\| g \| = 0$ if and only if $g = 1$ and $\| gh \| \leq \| g \| + \| h \|$ for all $g, h \in H$. The diameter of $H$, denoted $\| H \|$, is $\sup_{g \in H} \| g \|$. A group $H$ is called bounded
if every conjugation-invariant norm has finite diameter. In [5] we introduced two stronger notions of boundedness. We briefly recall them now.

A subset \( S \subseteq H \) is said to \textit{normally generate} \( H \) if the union of the conjugacy classes of its elements generates \( H \). Thus, every element of \( H \) can be written as a word in the conjugates of the elements of \( S \) and their inverses. Given \( g \in H \), the length of the shortest such word that is needed to express \( g \) is the \textit{word norm} of \( g \) denoted \( \|g\|_S \). It is a conjugation-invariant norm on \( H \). The \textit{diameter} of \( H \) with respect to this word norm is denoted \( \|H\|_S \). For every \( n \geq 0 \) we define

\[
B^H_S(n) = \{ g \in H \mid \|g\|_S \leq n \},
\]

the ball of radius \( n \) (of all elements that can be written as a product of \( n \) or fewer conjugates of the elements of \( S \) and their inverses). When there is no danger of confusion we simply write \( B_S(n) \) (cf. Notation 3.1).

We will use the following result [5, Lem. 2.3] repeatedly: if \( X, Y \subseteq H \) and \( Y \subseteq B_X(m) \) then \( B_Y(n) \subseteq B_X(mn) \).

We say that \( H \) is \textit{finitely normally generated} if it admits a finite normally generating set. In this case we define

\[
\Delta^k(H) = \sup\{ \|H\|_S : S \text{ normally generates } H \text{ and } |S| \leq k \}
\]

\[
\Delta(H) = \sup\{ \|H\|_S : S \text{ normally generates } H \text{ and } |S| < \infty \}.
\]

A finitely normally generated group \( H \) is called \textit{strongly bounded} if \( \Delta^k(H) < \infty \) for all \( k \). It is called \textit{uniformly bounded} if \( \Delta(H) < \infty \). Notice that \( \Delta^k(H) \leq \Delta(H) \) for all \( k \in \mathbb{N} \), so uniform boundedness implies strong boundedness. It follows from [5, Corollary 2.9] that strong boundedness implies boundedness.

3. Linear algebraic groups

We recall some material on linear algebraic groups; see [2] and [9] for further details. Below \( k \) denotes an infinite field and \( G \) denotes a semisimple linear algebraic \( k \)-group; we write \( r \) for rank \( G \). We adopt the notation of [2]: we regard \( G \) as a linear algebraic group over the algebraic closure \( \overline{k} \) together with a choice of \( k \)-structure. We identify \( G \) with its group of \( k \)-points \( G(k) \). If \( H \) is any \( k \)-subgroup of \( G \) then we denote by \( H(k) \) the group of \( k \)-points of \( H \). More generally, if \( C \) is any subset of \( G \)—not necessarily closed or \( k \)-defined—then we set \( C(k) = C \cap G(k) \). By [2, V.18.3 Cor.], \( G(k) \) is dense in \( G \).

Fix a maximal split \( k \)-torus \( S \) of \( G \). Let \( L = C_G(S) \) and fix a \( k \)-parabolic subgroup \( P \) such that \( L \) is a Levi subgroup of \( P \). Set \( U = R_u(P) \). Then \( P \) is a minimal \( k \)-parabolic subgroup of \( G \), \( L \) and \( S \) are \( k \)-defined and \( P, S \) are unique up to \( G^+(k) \)-conjugacy [9, 15.4.7].
Fix a maximal $k$-torus $T$ of $G$ such that $S \subseteq T$ and a \( (\text{not necessarily } k\text{-defined}) \) Borel subgroup $B$ of $G$ such that $T \subseteq B \subseteq P$.

**Notation 3.1.** If $X \subseteq G^+(k)$ then we write $B_X(n)$ for $B^{G^+(k)}_X(n)$.

**Lemma 3.2.** Let $O, O'$ be nonempty open subsets of $G$. For any $g \in G(k)$, there exist $h \in O(k)$ and $h' \in O'(k)$ such that $g = hh'$.

**Proof.** Since $G$ is irreducible as a variety, $O^{-1}g \cap O'$ is an open dense subset of $G$. Since $G(k)$ is dense in $G$, we can choose $h' \in (O^{-1}g)(k) \cap O'(k)$. We can write $h' = h^{-1}h$ for some $h \in O(k)$. This yields $g = hh'$, as required. \qed

For the rest of the section we assume that $G$ is split over $k$; then $S = T$ and $P = B$. Let $\Psi_T$ denote the set of roots of $G$ with respect to $T$. For $\alpha \in \Psi_T$, we denote by $U_\alpha$ the corresponding root group. Let $\alpha_1, \ldots, \alpha_r$ be the base for the set of positive roots associated to $B$. Note that $U_{\alpha_i}$ commutes with $U_{-\alpha_i}$ if $i \neq j$ because $\alpha_i - \alpha_j$ is not a root. Let $U^-$ be the opposite unipotent subgroup to $U$ with respect to $T$. Let $G_{\alpha} = \langle U_\alpha \cup U_{-\alpha} \rangle$ for $\alpha \in \Psi_T$; then $G_{\alpha}$ is $k$-isomorphic to either $\text{SL}_2$ or $\text{PGL}_2$. Let $\alpha^\vee : \mathbb{G}_m \to G_{\alpha}$ be the coroot associated to $\alpha$. The image $T_\alpha$ of $\alpha^\vee$ is $G_{\alpha} \cap T$, and this is a maximal torus of $G_{\alpha}$.

We use the Bruhat decomposition for $G(k)$. We recall the necessary facts [2, Sec. V.14, Sec. V.21]. Fix a set $\tilde{W} \subseteq N_G(T)(k)$ of representatives for the Weyl group; we denote by $n_0 \in \tilde{W}$ the representative corresponding to the longest element of $W$ (note that $n_0^2 \in T(k)$ and $n_0U_0n_0^{-1} = U^-$). The Bruhat decomposition $G = \bigsqcup_{w \in \tilde{W}} BnB$ for $G$ yields a decomposition $G(k) = \bigsqcup_{n \in \tilde{W}} BnB(k)$ for $G(k)$ [2, Thm. V.21.15]. The double coset $Bn_0B$ is open and $k$-defined. The map $U \times B \to Bn_0B$, $(u, b) \mapsto un_0b$ is an isomorphism of varieties. Hence if $g \in Bn_0B(k)$ then $g = un_0b$ for unique $u \in U$ and $b \in B$, and it follows that $u \in U(k)$ and $b \in B(k)$. Likewise, multiplication gives $k$-isomorphisms of varieties

\[
U^- \times T \times U \to U^- \times B \to U^- B = n_0(Bn_0B),
\]

so $U^- B$ is open and $(U^- B)(k) = U^-(k)B(k) = U^-(k)T(k)U(k)$.

4. The algebraically closed case

Throughout this section $k$ is algebraically closed. We need to recall some results from geometric invariant theory [7, Ch. 3]. Let $H$ be a reductive group acting on an affine variety $X$ over $\overline{k}$. We denote the orbit of $x \in X$ by $H \cdot x$ and the stabiliser of $x$ by $H_x$. One may form the affine quotient variety $X/H$. The points of $X/H$ correspond to the
closed \( H \)-orbits. We have a canonical projection \( \pi_X : X \to X/H \). The closure \( \overline{H \cdot x} \) of any orbit \( H \cdot x \) contains a unique closed orbit \( H \cdot y \), and we have \( \pi_X(x) = \pi_X(y) \). If \( C \subseteq X \) is closed and \( H \)-stable then \( \pi_X(C) \) is closed.

In particular, \( H \) acts on itself by inner automorphisms—that is, by conjugation—and the orbit \( H \cdot h \) is the conjugacy class of \( h \). We denote the quotient variety by \( H/\text{Inn}(H) \) and the canonical projection by \( \pi_H : H \to H/\text{Inn}(H) \). If \( h = h_s h_u \) is the Jordan decomposition of \( h \) then \( H \cdot h_s \) is the unique closed orbit contained in \( \overline{H \cdot h} \); so \( H \cdot h \) is closed if and only if \( h \) is semisimple, and \( \pi_H(h) = \pi_H(1) \) if and only if \( h \) is unipotent. Fix a maximal torus \( T \) of \( H \). The Weyl group \( W \) acts on \( T \) by conjugation. The inclusion of \( T \) in \( G \) gives rise to a map \( \psi_T : T/W \to H/\text{Inn}(H) \); it is well known that \( \psi_T \) is an isomorphism of varieties.

Now assume \( G \) is simply connected. We can write \( G \cong G_1 \times \cdots \times G_m \), where the \( G_i \) are simple. Let \( \nu_i : G \to G_i \) be the canonical projection. Set \( r_i = \text{rank}(G_i) \) for \( 1 \leq i \leq m \).

**Lemma 4.1.** Let \( C \) be a closed \( G \)-stable subset of \( G \) such that \( C \nsubseteq Z(G) \). Then there exist \( g \in C \) and \( x \in G \) such that \([g, x]\) is not unipotent.

**Proof.** Let \( g \in C \) such that \( g \notin Z(G) \). Note that \( g_s \in C \) as \( C \) is closed and conjugation-invariant. If \( g_s \) is not central in \( G \) then we can choose a maximal torus \( T' \) of \( G \) such that \( g_s \in T' \); then \([g_s, x]\) is a nontrivial element of \( T \) for some \( x \in N_G(T) \), and we are done. So we can assume \( g_s \) is central in \( G \). Then \( g_u \) is a nontrivial unipotent element of \( G \). By [3, Lem. 3.2], \( G \cdot g \) contains an element of the form \( g_u u \), where \( 1 \neq u \) belongs to some root group \( U_\alpha \). Let \( n \in N_{G_\alpha}(T_\alpha) \) represent the nontrivial element of the Weyl group \( N_{G_\alpha}(T_\alpha)/T_\alpha \). Recall that \( G_\alpha \) is isomorphic to \( \text{SL}_2 \) or \( \text{PGL}_2 \). Explicit calculations with \( 2 \times 2 \) matrices (cf. the proof of Lemma 6.1 below) show that \([u, n] = [g_u u, n]\) is not unipotent. This completes the proof. \( \square \)

Suppose we are given \( G \)-conjugacy classes \( C_1, \ldots, C_m \) of \( G \) such that for each \( i \), \( \nu_i(C_i) \) is noncentral in \( G_i \) (we do not insist that the \( C_i \) are all distinct). Set \( D_i = [C_i, G_i] \) and \( E_i = D_i = [C_i, G_i] \). Note that for each \( i \), \( D_i \) is conjugation-invariant and constructible, and \( D_i^{-1} = D_i \); likewise, \( E_i \) is conjugation-invariant and irreducible, and \( E_i^{-1} = E_i \).

**Proposition 4.2.** Let \( G \), etc., be as above, and set \( X = D_1 \cup \cdots \cup D_m \). Then \( B_X(r) \) contains a constructible dense subset of \( G \).
Proof. It suffices to prove that the constructible set $D_{i_1} \cdots D_{i_r}$ is dense in $G$ for some $i_1, \ldots, i_r$. It is enough to show that the constructible set $E_{i_1} \cdots E_{i_r}$ is a dense subset of $G$ for some $i_1, \ldots, i_r$.

Fix a maximal torus $T$ of $G$ and set $T_i = T \cap G_i$ for each $i$. Clearly it is enough to prove that $(E_i)^{\ast r}$ is a dense subset of $G_i$ for each $i$. For notational convenience, we assume therefore that $m = 1$ and $G = G_1$ is simple; then $T = T_1$. Set $C = C_1 = \nu_1(C_1)$ and $E = E_1$; we prove that $E^r$ is a dense subset of $G$. By hypothesis, $E = [C, G]$ is an irreducible positive-dimensional subvariety of $G$. Set $A = E \cap T$. We claim that $A$ has an irreducible component $A'$ such that $\dim(A') > 0$.

Set $F = \pi_G(E)$; note that $F$ is closed and irreducible because $E$ is closed, conjugation-invariant and irreducible. Suppose $\dim(F) = 0$. Since $1 \in E$, we have $F = \{\pi_G(1)\}$, which forces $E$ to consist of unipotent elements. But this is impossible by Lemma 4.1. We deduce that $\dim(F) > 0$. Clearly $\pi_G(A) \subseteq F$. Conversely, given $g \in E$, write $g = g_s g_u$ (Jordan decomposition). Since $E$ is conjugation-invariant, we can, by conjugating $g$, assume without loss that $g_s \in T$. We have $g_s \in [g_s g \cap T \subseteq A$ and $\pi_G(g_s) = \pi_G(g)$. This shows that $F \subseteq \pi_G(A)$. Hence $F = \pi_G(A)$. Let $\pi_W : T \to T/W$ be the canonical projection. Now $F' := \psi_T^{-1}(F')$ is an irreducible closed positive-dimensional subset of $T/W$, with $A = \pi_W^{-1}(F')$. Since $W$ is finite, $\pi_W$ is a finite map and the fibres of $\pi_W$ are precisely the $W$-orbits. Hence the irreducible components of $A$ are permuted transitively by $W$, and each surjects onto $F'$. Thus any irreducible component $A'$ of $A$ has the desired properties.

Let $A_1, \ldots, A_t$ be the $W$-conjugates of $A'$. The $A_i$ generate a non-trivial $W$-stable subtorus $S$ of $T$. Hence the subset $V$ of $X(T) \otimes \mathbb{R}$ spanned by $\{\chi \in X(T) \mid \chi(S) = 1\}$ is proper and $W$-stable. But $W$ acts absolutely irreducibly on $X(T) \otimes \mathbb{R}$, so $V = 0$. This forces $S$ to be the whole of $T$. So the $A_i$ generate $T$. By the argument of [5, Sec. 5] or [4, 7.5 Prop.], there exist $i_1, \ldots, i_r \in \{1, \ldots, t\}$ and $\epsilon_1, \ldots, \epsilon_r \in \{\pm 1\}$ such that $A_{i_1}^{\epsilon_1} \cdots A_{i_r}^{\epsilon_r}$ is a constructible dense subset of $T$. Hence $E^r$ contains a constructible dense subset of $T$, and we deduce that $E^r$ is a constructible dense subset of $G$. This completes the proof.

Proof of Theorem 1.3. We have $\Delta(\tilde{G}) \leq \Delta(G)$ by [5, Lem. 2.16], where $\tilde{G}$ is the simply connected cover of $G$. Hence there is no harm in assuming $G$ is simply connected. Let $X$ be a finite normal generating set for $G$. We can choose $x_1, \ldots, x_m \in X$ such that $\nu_i(x_i)$ is noncentral in $G_i$ for $1 \leq i \leq m$. Let $C_i = G \cdot x_i \subseteq X$, let $D_i = [C_i, G]$ and let $X' = D_1 \cup \cdots \cup D_m$. By Proposition 4.2, $B_{X'}(r)$ contains a dense constructible subset of $G$. Since $D_i \subseteq B_{C_i}(2)$ for each $i$, $B_X(2r)$ contains a nonempty
open subset \( U \) of \( G \). Now \( U^2 = G \) by \([2, \text{I.1.3 Prop.}]\), so \( B_X(4r) \supseteq B_X(2r)B_X(2r) \supseteq U^2 = G \). It follows that \( \Delta(G) \leq 4r \), as required. \( \square \)

5. The isotropic case

Now we consider the case of arbitrary semisimple \( G \). There is no harm in replacing \( G \) with the Zariski closure of \( G^+(k) \), which is the product of the isotropic \( k \)-simple factors of \( G \). Hence we assume in this section that \( G^+(k) \) is dense in \( G \).

We start by noting a corollary of Proposition 4.2. Let \( X \subseteq G^+(k) \) such that \( X \) is a finite normal generating set for \( G \). By Proposition 4.2, there exist \( i_1, \ldots, i_r \in \{1, \ldots, m\} \) such that the image of the map \( f : G^{2r} \to G \) defined by

\[
\begin{align*}
    f(h_1, \ldots, h_r, g_1, \ldots, g_r) &= (h_1 x_1 h_1^{-1} g_1 x_1^{-1} g_1^{-1}) \cdots (h_r x_r h_r^{-1} g_r x_r^{-1} g_r^{-1})
\end{align*}
\]

contains a nonempty open subset \( G' \) of \( G \). Now let \( O \) be a nonempty open subset of \( G \). Then \( f^{-1}(G' \cap O) \) is a nonempty open subset of \( G^{2r} \). But \( G^+(k) \) is dense in \( G \), so \( G^+(k)^{2r} \) is dense in \( G^{2r} \). It follows that \( f(h_1, \ldots, h_r, g_1, \ldots, g_r) \in O \) for some \( h_1, \ldots, h_r, g_1, \ldots, g_r \in G^+(k)^{2r} \). We deduce that for any nonempty open subset \( O \) of \( G \),

\[(5.1) \quad B_X(2r) \cap O \neq \emptyset.\]

**Remark 5.1.** Let \( C = \text{im}(f) \), where \( f \) is as above. It follows from Eqn. (5.1) and Lemma 3.2 that \( C(k)^2 = G(k) \). We cannot, however, conclude directly from this that \( B_X(2r)^2 = G(k) \): the problem is that although the map \( f : G^{2r} \to C \) is surjective on \( \overline{k} \)-points, it need not be surjective on \( k \)-points.

**Lemma 5.2.** There exists \( t \in P(k) \) such that \( t \) is regular semisimple.

**Proof.** Define \( f : G \times P \to G \) by \( f(g, h) = ghg^{-1} \). Then \( f \) is surjective since every element of \( G \) belongs to a Borel subgroup of \( G \). Let \( O \) be the set of regular semisimple elements of \( G \), a nonempty open subset of \( G \). By \([2, \text{Thm. 21.20(ii)}]\), \( P(k) \) is dense in \( P \), and we know that \( G(k) \) is dense in \( G \), so \( G(k) \times P(k) \) is dense in \( G \times P \). It follows that there is a point \( (g, t) \in (G(k) \times P(k)) \cap f^{-1}(O) \). Then \( gtg^{-1} \) is regular semisimple, so \( t \in P(k) \) is regular semisimple also. \( \square \)

**Lemma 5.3.** Let \( t \in P(k) \) be regular semisimple. Then \( U(k) \subseteq B_t(2) \).

**Proof.** Define \( f : U \to U \) by \( f(u) = utu^{-1}t^{-1} \). The conjugacy class \( U \cdot t \) is closed because orbits of unipotent groups are closed, so \( \text{im}(f) \) is a closed subvariety of \( U \). Since \( t \) is regular, it is easily checked that \( f \) is injective and the derivative \( df_u \) is an isomorphism for each \( u \in U \). It follows from Zariski’s Main Theorem that \( f \) is an isomorphism of
varieties. As \( f \) is defined over \( k \), \( f \) gives a bijection from \( U(k) \) to \( U(k) \), and the result follows.

\[ \text{Lemma 5.4.} \quad \text{Let} \ X \ \text{be a finite normal generating subset for} \ G^+(k). \ \text{Then} \ X \ \text{normally generates} \ G. \]

\[ \text{Proof.} \ \text{There exists} \ d \in \mathbb{N} \ \text{such that} \ (G(k) \cdot X)^d = G^+(k). \ \text{So the} \ \text{constructible set} \ (G \cdot X)^d \ \text{contains} \ G^+(k) \ \text{and is therefore dense in} \ G. \ \text{This implies that} \ (G \cdot X)^d \ \text{contains a nonempty open subset of} \ G, \ \text{so} \ (G \cdot X)^d(G \cdot X)^d = G. \ \text{Hence} \ X \ \text{is a finite normal generating set for} \ G. \]

\[ \text{Proposition 5.5.} \quad \text{Let} \ X \ \text{be a finite subset of} \ G^+(k) \ \text{such that} \ X \ \text{normally generates} \ G. \ \text{Then} \ U(k) \subseteq B_X(8r). \]

\[ \text{Proof.} \ \text{The big cell} \ Pn_0P \ \text{is open, so by Eqn. (5.1), we can choose} \ g \in B_X(2r) \cap Pn_0P. \ \text{We can write} \ g = xn_0x' \ \text{for some} \ x, x' \in P(k). \ \text{Since} \ B_X(2r) \ \text{is conjugation-invariant, there is no harm in replacing} \ g \ \text{with} \ (x')^{-1}gx', \ \text{so we can assume that} \ x' = 1 \ \text{and} \ g = xn_0. \ \text{Let} \ C_1 = \{n_0x_1 \mid x_1 \in P, xn_0^2x_1 \ \text{is regular semisimple}\}. \ \text{Let} \ O_1 = P \cdot C_1 = U \cdot C_1; \ \text{then} \ O_1 \ \text{is a constructible dense subset of} \ G. \ \text{By Eqn. (5.1), there exists} \ g \in B_X(2r) \cap O_1. \ \text{We can write} \ g = un_0x_1u^{-1} \ \text{where} \ xn_0^2x_1 \ \text{is regular semisimple and} \ u \in U. \ \text{Since} \ g \in G(k), \ \text{both} \ u \ \text{and} \ x_1u^{-1} \ \text{belong to} \ G(k). \ \text{Hence} \ n_0x_1 \in B_X(2r) \cap C_1. \ \text{It follows that} \ t := xn_0^2x_1 \ \text{is regular semisimple and belongs to} \ B_X(4r). \ \text{We have} \ t \in B_X(4r) \cap P(k), \ \text{so} \ U(k) \subseteq B_t(2) \subseteq B_X(8r) \ \text{by Lemma 5.3}. \ \text{This completes the proof.} \]

\[ \text{Proof of Theorem 1.2.} \ \text{Suppose} \ G(k) = G^+(k). \ \text{By Lemma 5.2, there exists} \ t \in P(k) \ \text{such that} \ t \ \text{is regular semisimple. By Lemma 5.3,} \ B_t(2) \ \text{contains} \ U(k). \ \text{Since} \ G(k) \ \text{is generated by the} \ G(k)-\text{conjugates of} \ U(k), \ \text{we deduce that} \ \{t\} \ \text{normally generates} \ G(k). \ \text{Hence} \ G(k) \ \text{is finitely normally generated.}

\text{Now suppose further that} \ G(k) \ \text{is bounded. Fix a finite normal generating set} \ Y \ \text{for} \ G(k). \ \text{Then} \ G(k) = B_Y(s) \ \text{for some} \ s \in \mathbb{N} \ \text{and} \ Y \subseteq B_{U(k)}(d) \ \text{for some} \ d \in \mathbb{N}. \ \text{Let} \ X \ \text{be any finite normal generating set for} \ G(k). \ \text{Then} \ X \ \text{is normally generates} \ G \ \text{by Lemma 5.4}. \ \text{By Proposition 5.5,} \ U(k) \subseteq B_X(8r). \ \text{So}

\[ G(k) = B_Y(s) \subseteq B_{U(k)}(sd) \subseteq B_X(8r ds). \]

\text{This shows that} \ G(k) \ \text{is uniformly bounded, as required.} \]

\[ \text{Remark 5.6.} \ \text{The hypothesis that} \ G^+(k) = G(k) \ \text{holds in many cases if} \ G \ \text{is} \ k-\text{simple and simply connected—this is the content of the Kneser-Tits conjecture, which holds, for example, when} \ k \ \text{is a local field.} \]
Example 5.7. It is well known that the abelianisation of $\text{SO}_3(\mathbb{Q})$ is $\mathbb{Q}^*/(\mathbb{Q}^*)^2$, which is an infinitely generated abelian group. It follows that $\text{SO}_3(\mathbb{Q})$ is not finitely normally generated. Note that $\text{SO}_3^+(\mathbb{Q}) = 1$ since $\text{SO}_3$ is anisotropic over $\mathbb{Q}$.

6. The split case

In this section we assume $G$ is split over $k$. If $G$ is simply connected then the Kneser-Tits Conjecture holds for $G$, so $G^+(k) = G(k)$ in this case.

Lemma 6.1. Suppose $(*)$ each $G_\alpha$ is isomorphic to $\text{SL}_2$. Let $t_i \in T_{\alpha_i}(k)$ for $1 \leq i \leq r$ and set $t = t_1 \cdots t_r$. There exist $u_i, w_i \in U_{\alpha_i}(k)$ and $v_i, x_i \in U_{-\alpha_i}(k)$ for $1 \leq i \leq r$ such that $t = x_r \cdots x_1 u_r \cdots u_1 v_r \cdots v_1 w_r \cdots w_r$.

Proof. We use induction on $r$. The case $r = 0$ is vacuous. Now consider the case $r = 1$. Then $G \cong \text{SL}_2$. For any $a, b, c, d \in k$ we have
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & ab \\ b & 1 \end{pmatrix} = \begin{pmatrix} 1+ab+cd+abd & \ast \\ \ast & \ast \end{pmatrix}.
\]
Let $x \in k^*$. Set $a = -x, b = x^{-1} - 1, c = 1$ and $d = x - 1$; then the matrix above becomes \(
\begin{pmatrix} 0 & 0 \\ 0 & x^{-1} \end{pmatrix}.
\) Hence the result holds when $r = 1$.

Now suppose $r > 1$. Let $H$ be the semisimple group with root system spanned by $\pm \alpha_1, \ldots, \pm \alpha_{r-1}$. Clearly condition $(*)$ holds for $H$. Let $s = t_1 \cdots t_{r-1}$. By our induction hypothesis, there exist $u_i, w_i \in U_{\alpha_i}(k)$ and $v_i, x_i \in U_{-\alpha_i}(k)$ for $1 \leq i \leq r-1$ such that
\[
s = x_{r-1} \cdots x_1 u_{r-1} \cdots u_1 v_1 \cdots v_{r-1} w_1 \cdots w_{r-1}.
\]
By the $\text{SL}_2$ case considered above, $t_r = x_r' u_r' v_r' w_r'$ for some $u_r, w_r \in U_{\alpha_r}$ and some $v_r, x_r \in U_{-\alpha_r}$. Set $x_r = sx_r', s^{-1}, u_r = sw_r', s^{-1}, v_r = v_r'$ and $w_r = w_r'$. We have
\[
x_r x_{r-1} \cdots x_1 u_r u_{r-1} \cdots u_1 v_1 \cdots v_{r-1} w_1 \cdots w_{r-1} w_r
\]
\[
= x_r u_r x_{r-1} \cdots x_1 u_r \cdots u_1 v_1 \cdots v_{r-1} v_r w_1 \cdots w_{r-1} w_r
\]
\[
= x_r u_r s v_r w_r
\]
\[
= s x_r' u_r' v_r' w_r'
\]
\[
= st_r = t.
\]
The result follows by induction.

$\square$

Proposition 6.2. Suppose $G$ is simply connected. Let $X \subseteq G^+(k)$ such that $U(k) \subseteq X$. Then $B_X(7) = G^+(k)$.

Proof. Since $G$ is simply connected, $(*)$ holds for $G$ and the map $\psi: \mathbb{G}_m^r \to T$ given by $\psi(a_1, \ldots, a_r) = \alpha_1^r(a_1) \cdots \alpha_r^r(a_r)$ is a $k$-isomorphism. It follows that $T(k) = T_{\alpha_1}(k) \cdots T_{\alpha_r}(k)$, so $T(k) \subseteq B_X(4)$ by Lemma 6.1.
Hence $U^-(k)B(k) = U^-(k)T(k)U(k) \subseteq B_X(1)B_X(4)B_X(1) \subseteq B_X(6)$. Now $G(k) = (U^-B)^{-1}(k)(U^-B)(k)$ by Lemma 3.2. But

$$(U^-B)^{-1}(k)(U^-B)(k) = B(k)U^-(k)U^-(k)B(k) = U(k)T(k)U^-(k)T(k)U(k)$$

$$= U(k)U^-(k)T(k)U(k) = U(k)U^-(k)B(k) \subseteq B_X(1)B_X(6) \subseteq B_X(7),$$

so we are done. \qed

**Proof of Theorem 1.4.** Let $\tilde{G}$ be the split form of the simply connected cover of $G$ and let $\psi: \tilde{G} \to G$ be the canonical projection. Then $\psi$ is a $k$-defined central isogeny, so by [2, V.22.6 Thm.], the map $\tilde{B} \mapsto \psi(\tilde{B})$ gives a bijection between the set of $k$-Borel subgroups of $\tilde{G}$ and the set of $k$-Borel subgroups of $G$; moreover, for each $\tilde{B}$, $\psi$ gives rise to a $k$-isomorphism from $R_u(\tilde{B})$ to $R_u(B)$ [2, Prop. V.22.4]. It follows that $\psi(\tilde{G}^+(k)) = G^+(k)$. By [5, Lem. 2.16] we have $\Delta(G^+(k)) \leq \Delta(\tilde{G}^+(k))$, so we can assume without loss that $G$ is simply connected. In particular, $G^+(k) = G(k)$.

Let $X$ be a finite normal generating set for $G(k)$. Then $X$ is a finite normal generating set for $G$ (Lemma 5.4), so by Eqn. (5.1) there exists $t \in B_X(2r)$ such that $t$ is regular semisimple. We have $U(k) \subseteq B_t(2)$ by Lemma 5.3 and $G(k) \subseteq B_{U(k)}(7)$ by Proposition 6.2. So

$$G(k) \subseteq B_{U(k)}(7) \subseteq B_t(14) \subseteq B_X(28r).$$

This shows that $\Delta(G(k)) \leq 28r$, as required. \qed

**Example 6.3.** (a) Let $G = \text{SL}_n(k)$ where $n \geq 3$, let $g$ be the elementary matrix $E_{1n}(1)$ and let $X = G(k) \cdot g$. By [5, Prop. 6.23], $X$ generates $G(k)$. One sees easily by direct computation that the centraliser $C_G(g)$ has dimension $n^2 - 2n + 1$, so $\dim(G \cdot g) = 2n - 2$. A simple dimension-counting argument shows that if $t < \frac{1}{2} \text{rank}(G)$ then $X^t$ is a proper closed subvariety of $G$. Since $G(k)$ is dense in $G$, it follows that $X^t$ does not contain $G(k)$, so $X(k)^t$ does not contain $G(k)$. We deduce that $\Delta(G(k)) \geq \frac{1}{2} \text{rank}(G)$.

(b) The bounds in Theorems 1.3 and 1.4 are far from sharp. Aseeri has shown by direct calculation that $3 \leq \Delta(\text{SL}_3(\mathbb{C})) \leq 6$ and that $\Delta(\text{SL}_2(\mathbb{C})^m) = 3m$ and $\Delta(\text{PGL}_2(\mathbb{C})^m) = 2m$ for every $k \in \mathbb{N}$ [1, Thm. 8.0.2, Thm. 7.2.10, Thm. 7.2.6], whereas Theorem 1.3 yields the bounds $\Delta(\text{SL}_3(\mathbb{C})) \leq 8$ and $\Delta(\text{SL}_2(\mathbb{C})^m), \Delta(\text{PGL}_2(\mathbb{C})^m) \leq 4m$. Aseeri also showed that $3 \leq \Delta(\text{SL}_3(\mathbb{R})) \leq 4$, whereas Theorem 1.4 gives $\Delta(\text{SL}_3(\mathbb{R})) \leq 56$. 


7. Semisimple Lie groups

Proof of Theorems 1.5 and 1.6. Let $H$ be a linear semisimple Lie group such that $H$ has no compact simple factors. By [6, Thm. III.2.13], there is a complex semisimple algebraic group $G$ defined over $\mathbb{R}$ such that $G^+(\mathbb{R}) = H$. Now $Z(H)$ is finite, so $H$ is finitely normally generated and bounded by [5, Thm. 1.2]. It follows from Theorem 1.2 that $H$ is uniformly bounded. If $H$ is split then $G$ is split over $\mathbb{R}$, so $\Delta(H) \leq 28 \text{rank}(H)$ by Theorem 1.4.

The argument for the complex case is similar: if $H$ is a semisimple linear complex Lie group then there is a semisimple complex algebraic group $G$ such that the complex Lie group associated to $G$ is $H$ (cf. [8, Ch. 4, Sec. 2, Problem 12], and $G$ is isomorphic to $H$. The result now follows from Theorem 1.3. □

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