HÖLDER COCYCLES AND ERGODIC INTEGRALS FOR TRANSLATION FLOWS ON FLAT SURFACES

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Abstract. The main results announced in this note are an asymptotic expansion for ergodic integrals of translation flows on flat surfaces of higher genus (Theorem 1) and a limit theorem for such flows (Theorem 2). Given an abelian differential on a compact oriented surface, consider the space $\mathcal{B}^+$ of Hölder cocycles over the corresponding vertical flow that are invariant under holonomy by the horizontal flow. Cocycles in $\mathcal{B}^+$ are closely related to G. Forni’s invariant distributions for translation flows [10]. Theorem 1 states that ergodic integrals of Lipschitz functions are approximated by cocycles in $\mathcal{B}^+$ up to an error that grows more slowly than any power of time. Theorem 2 is obtained using the renormalizing action of the Teichmüller flow on the space $\mathcal{B}^+$. A symbolic representation of translation flows as suspension flows over Vershik’s automorphisms allows one to construct cocycles in $\mathcal{B}^+$ explicitly. Proofs of Theorems 1, 2 are given in [5].

1. HÖLDER COCYCLES OVER TRANSLATION FLOWS.

Let $\rho \geq 2$ be an integer, let $M$ be a compact oriented surface of genus $\rho$, and let $\omega$ be a holomorphic one-form on $M$. Denote by $m = i(\omega \wedge \omega)/2$ the area form induced by $\omega$ and assume that $m(M) = 1$.

Let $h^+_t$ be the vertical flow on $M$ (i.e., the flow corresponding to $\Re(\omega)$); let $h^-_t$ be the horizontal flow on $M$ (i.e., the flow corresponding to $\Im(\omega)$). The flows $h^+_t$, $h^-_t$ preserve the area $m$. Take $x \in M$, $t_1, t_2 \in \mathbb{R}_+$ and assume that the closure of the set

$$\{h^+_{\tau_1}h^-_{\tau_2}x, 0 \leq \tau_1 < t_1, 0 \leq \tau_2 < t_2\}$$

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does not contain zeros of the form $\omega$. The set (1) is then called an admissible rectangle and denoted $\Pi(x, t_1, t_2)$. Let $\mathcal{T}$ be the semi-ring of admissible rectangles. Consider the linear space $\mathfrak{B}^+$ of Hölder cocycles $\Phi^+(x, t)$ over the vertical flow $h_t^+$ that are invariant under horizontal holonomy. More precisely, a function $\Phi^+(x, t) : M \times \mathbb{R} \to \mathbb{R}$ belongs to the space $\mathfrak{B}^+$ if it satisfies:

**Assumption 1.**

1. $\Phi^+(x, t + s) = \Phi^+(x, t) + \Phi^+(h_t^+s, x);$
2. There exists $t_0 > 0$, $\theta > 0$ such that $|\Phi^+(x, t)| \leq t^\theta$ for all $x \in M$ and all $t \in \mathbb{R}$ satisfying $|t| < t_0;$
3. If $\Pi(x, t_1, t_2)$ is an admissible rectangle, then $\Phi^+(x, t_1) = \Phi^+(h_{t_2}^+, x, t_1)$.

For example, a cocycle $\Phi_1^+$ defined by $\Phi_1^+(x, t) = t$ belongs to $\mathfrak{B}^+$.

In the same way define the space of $\mathfrak{B}^-$ of Hölder cocycles $\Phi^-(x, t)$ over the horizontal flow $h_t^-$ which are invariant under vertical holonomy, and set $\Phi_1^-(x, t) = t$.

Given $\Phi^+ \in \mathfrak{B}^+$, $\Phi^- \in \mathfrak{B}^-$, a finitely additive measure $\Phi^+ \times \Phi^-$ on the semi-ring $\mathcal{T}$ of admissible rectangles is introduced by the formula

$$\Phi^+ \times \Phi^-(\Pi(x, t_1, t_2)) = \Phi^+(x, t_1) \cdot \Phi^-(x, t_2).$$

In particular, for $\Phi^- \in \mathfrak{B}^-$, set $m_{\Phi^-} = \Phi_1^+ \times \Phi^-;

$$m_{\Phi^-}(\Pi(x, t_1, t_2)) = t_1 \Phi^-(x, t_2).$$

For any $\Phi^- \in \mathfrak{B}^-$ the measure $m_{\Phi^-}$ satisfies $(h_t^-)^* m_{\Phi^-} = m_{\Phi^-}$ and is an invariant distribution in the sense of G. Forni [9], [10]. For instance, $m_{\Phi^-} = m$.

An $\mathbb{R}$-linear pairing between $\mathfrak{B}^+$ and $\mathfrak{B}^-$ is given, for $\Phi^+ \in \mathfrak{B}^+$, $\Phi^- \in \mathfrak{B}^-$, by the formula

$$\langle \Phi^+, \Phi^- \rangle = \Phi^+ \times \Phi^-(M).$$

Take an abelian differential $X = (M, \omega)$. The space $\mathfrak{B}^+_X = \mathfrak{B}^+(M, \omega)$ can be mapped to $H^1(M, \mathbb{R})$ in the following way. A continuous closed curve $\gamma$ on $M$ is called rectangular if

$$\gamma = \gamma_1^+ \cup \cdots \cup \gamma_k^+ \cup \gamma_1^- \cup \cdots \cup \gamma_k^-,$$

where $\gamma_i^+$ are arcs of the flow $h_t^+$, $\gamma_i^-$ are arcs of the flow $h_t^-$. For $\Phi^+ \in \mathfrak{B}^+$ define $\Phi^+(\gamma) = \sum_{i=1}^k \Phi^+(\gamma_i^+)$. It is easy to show that if $\gamma$ is homologous to $\gamma'$, then $\Phi^+(\gamma) = \Phi^+(\gamma')$. Using a similar construction for $\mathfrak{B}^-_X = \mathfrak{B}^-(M, \omega)$, we obtain maps

$$\hat{I}^+_X : \mathfrak{B}^+_X \to H^1(M, \mathbb{R}), \hat{I}^-_X : \mathfrak{B}^-_X \to H^1(M, \mathbb{R}).$$

For a generic abelian differential, the image of $\mathfrak{B}^-_X$ under $\hat{I}^-_X$ is the strictly unstable space of the Kontsevich-Zorich cocycle over the Teichmüller flow.

More precisely, let $\kappa = (\kappa_1, \ldots, \kappa_\sigma)$ be a nonnegative integer vector such that $\kappa_1 + \cdots + \kappa_\sigma = 2p - 2$. Denote by $M_\kappa$ the moduli space of pairs $(M, \omega)$, where $M$ is a Riemann surface of genus $p$ and $\omega$ is a holomorphic differential of area 1 with singularities of orders $\kappa_1, \ldots, \kappa_\sigma$. The space $M_\kappa$ is often called the stratum in the moduli space of abelian differentials.

The Teichmüller flow $g_s$ on $M_\kappa$ sends the modulus of a pair $(M, \omega)$ to the modulus of the pair $(M, \omega')$, where $\omega' = e^{s\Re(\omega)} + ie^{-s\Im(\omega)}$: the new complex structure on $M$ is uniquely determined by the requirement that the form $\omega'$ be
holomorphic. As shown by Veech, the space $\mathcal{M}_\kappa$ need not be connected; let $\mathcal{H}$ be a connected component of $\mathcal{M}_\kappa$.

Let $\mathbb{H}(\mathcal{H})$ be the fibre bundle over $\mathcal{H}$ whose fibre at a point $(M, \omega)$ is the cohomology group $H^1(M, \mathbb{R})$. The bundle $\mathbb{H}(\mathcal{H})$ carries the Gauss-Manin connection which declares continuous integer-valued sections of our bundle to be flat and is uniquely defined by that requirement. Parallel transport with respect to the Gauss-Manin connection along the orbits of the Teichmüller flow yields a cocycle over the Teichmüller flow, called the Kontsevich-Zorich cocycle and denoted $A = A_{KZ}$.

Let $\mathbb{P}$ be a $g_*$-invariant ergodic probability measure on $\mathcal{H}$. By definition, the Kontsevich-Zorich cocycle $A_{KZ}$ satisfies the assumptions of the Oseledets Theorem with respect to $\mathbb{P}$. For $X \in \mathcal{H}$, $X = (M, \omega)$, denote by $E^u_X \subset H^1(M, \mathbb{R})$ the space spanned by vectors corresponding to the positive Lyapunov exponents of $A_{KZ}$, and by $E^s_X \subset H^1(M, \mathbb{R})$ the space spanned by vectors corresponding to the negative Lyapunov exponents of $A_{KZ}$. As before, let $\mathcal{B}^+_X$, $\mathcal{B}^-_X$ be the spaces of Hölder cocycles corresponding to the vertical and the horizontal flows of $X$.

**Proposition 1.** For $\mathbb{P}$-almost all $X \in \mathcal{H}$ the map $\tilde{T}_X^+$ takes $\mathcal{B}^+_X$ isomorphically onto $E^u_X$, and the map $\tilde{T}_X^-$ takes $\mathcal{B}^-_X$ isomorphically onto $E^s_X$.

The pairing $\langle \cdot, \cdot \rangle$ is nondegenerate and is taken by the isomorphisms $\tilde{T}_X^+$, $\tilde{T}_X^-$ to the cup-product in the cohomology group $H^1(M, \mathbb{R})$.

**Remark.** In particular, if $\mathbb{P}$ is the Masur-Veech “smooth” measure $[17], [18]$, then almost surely, with respect to $\mathbb{P}$, we have

$$\dim \mathcal{B}^+_X = \dim \mathcal{B}^-_X = \rho.$$  

**Remark.** The isomorphisms $\tilde{T}_X^+$, $\tilde{T}_X^-$ are analogues of G. Forni’s isomorphism between his space of invariant distributions and the unstable space of the Kontsevich-Zorich cocycle.

**Remark.** Cocycles in $\mathcal{B}^+$ can be interpreted, in the spirit of Bonahon [4], as finitely-additive holonomy-invariant Hölder transverse measures on oriented measured foliations and also as finitely-additive invariant measures for interval exchange transformations. See the preprint [5] for details.

Consider the inverse isomorphisms $\tilde{T}_X^+ = (\tilde{T}_X^-)^{-1}$, $\tilde{T}_X^- = (\tilde{T}_X^+)^{-1}$. Let

$$1 = \theta_1 > \theta_2 > \cdots > \theta_l > 0$$

be the distinct positive Lyapunov exponents of the Kontsevich-Zorich cocycle $A_{KZ}$, and let

$$E^u_X = \bigoplus_{i=1}^{l} E^u_{X, \theta_i}$$

be the corresponding Oseledets decomposition at $X$.

**Proposition 2.** Let $v \in E^u_{X, \theta_i}$, $v \neq 0$, and denote $\Phi^+ = \tilde{T}_X^+(v)$. Then for any $\varepsilon > 0$ the cocycle $\Phi^+$ satisfies the Hölder condition with exponent $\theta_i - \varepsilon$ and for any $x \in M(X)$ we have

$$\limsup_{T \to \infty} \frac{\log |\Phi^+(x, T)|}{\log T} = \theta_i.$$
2. Approximation of weakly Lipschitz functions.

The space of Lipschitz functions is not invariant under \( h_t^\pm \), and a larger function space \( \text{Lip}_w^+(M, \omega) \) of weakly Lipschitz functions is introduced as follows. A bounded measurable function \( f \) belongs to \( \text{Lip}_w^+(M, \omega) \) if there exists a constant \( C \), depending only on \( f \), such that for any admissible rectangle \( \Pi(x, t_1, t_2) \) we have

\[
\left| \int_0^{t_1} f(h_t^+ x) dt - \int_0^{t_1} f(h_t^+ (h_t^- x)) dt \right| \leq C.
\]

Let \( C_f \) be the infimum of all \( C \) satisfying \( (6) \). We norm \( \text{Lip}_w^+(M, \omega) \) by setting

\[
\|f\|_{\text{Lip}_w^+} = \sup_M |f| + C_f.
\]

By definition, the space \( \text{Lip}_w^+(M, \omega) \) contains all Lipschitz functions on \( M \) and is invariant under \( h_t^\pm \). We denote by \( \text{Lip}_{w,0}^+(M, \omega) \) the subspace of \( \text{Lip}_w^+(M, \omega) \) of functions whose integral with respect to \( \mu \) is 0.

For any \( f \in \text{Lip}_{w,0}^+(M, \omega) \) and any \( \Phi^- \in \mathfrak{B}^- \), the integral \( \int_M f \mu_\Phi^- \) can be defined as the limit of Riemann sums.

If the pairing \( \langle , \rangle \) induces an isomorphism between \( \mathfrak{B}^+ \) and the dual \( (\mathfrak{B}^-)^\ast \), then one can assign to a function \( f \in \text{Lip}_{w}^+(M, \omega) \) the cocycle \( \Phi_f^+ \in \mathfrak{B}^+ \) by the formula

\[
\langle \Phi_f^+, \Phi^- \rangle = \int_M f \mu_\Phi^- , \Phi^- \in \mathfrak{B}^-.
\]

By definition, \( \Phi_{f h_t^+}^+ = \Phi_f^+ \).

**Theorem 1.** Let \( \mathbb{P} \) be a \( \text{g}_s \)-invariant ergodic probability measure on \( \mathcal{H} \). For any \( \varepsilon > 0 \) there exists a constant \( C_\varepsilon \) depending only on \( \mathbb{P} \) such that for \( \mathbb{P} \)-almost every \( X \in \mathcal{H} \), any \( f \in \text{Lip}_{w,0}^+(X) \), any \( x \in M \) and any \( T > 0 \) we have

\[
\left| \int_0^T f \circ h_t^+(x) dt - \Phi_f^+(x, T) \right| \leq C_\varepsilon \|f\|_{\text{Lip}_w^+}(1 + T^\varepsilon).
\]

Consider the case in which the Lyapunov spectrum of the Kontsevich-Zorich cocycle is simple in restriction to the space \( E^u \) (as, by the Avila-Viana theorem [2], is the case with the Masur-Veech smooth measure). Let \( l_0 = \dim E^u \) and let \( 1 = \theta_1 > \theta_2 > \cdots > \theta_{l_0} \) be the corresponding simple expanding Lyapunov exponents.

Let \( \Phi_1^+ \) be given by the formula \( \Phi_1^+(x, t) = t \). Introduce a basis \( \Phi_1^+, \Phi_2^+, \ldots, \Phi_{l_0}^+ \) in \( \mathfrak{B}_X^+ \) in such a way that \( \mathfrak{J}_X (\Phi_1^+) \) lies in the Lyapunov subspace with exponent \( \theta_1 \). By Proposition 2, for any \( \varepsilon > 0 \) the cocycle \( \Phi_1^+ \) satisfies the Hölder condition with exponent \( \theta_1 - \varepsilon \), and for any \( x \in M(X) \) we have

\[
\limsup_{T \to \infty} \frac{\log |\Phi_1^+(x, T)|}{\log T} = \theta_1.
\]

Let \( \Phi_1^-, \ldots, \Phi_{l_0}^- \) be the dual basis in \( \mathfrak{B}_X^- \). Clearly, \( \Phi_1^-(x, t) = t \).

By definition, we have

\[
\Phi_f^+ = \sum_{i=1}^{l_0} m_{\Phi_i^-}(f) \cdot \Phi_i^+.
\]
Noting that by definition we also have $m_{\Phi^-} = m$, we derive from Theorem 1 the following corollary.

**Corollary 1.** Assume that $\mathbb{P}$ is an invariant ergodic probability measure for the Teichmüller flow such that, with respect to $\mathbb{P}$, all positive Lyapunov exponents of the Kontsevich-Zorich cocycle are simple.

Then for any $\varepsilon > 0$ there exists a constant $C_\varepsilon$ depending only on $\mathbb{P}$ such that for $\mathbb{P}$-almost every $X \in \mathcal{H}$, any $f \in \text{Lip}_{w,0}^+(X)$, any $x \in X$ and any $T > 0$ we have

$$\left| \int_0^T f \circ h^+_t(x) dt - T \cdot \int_M f dm - \sum_{i=1}^l m_{\Phi^-}(f) \cdot \Phi^+_i(x, T) \right| \leq C_\varepsilon \|f\|_{\text{Lip}_{w,0}^+}(1 + T^{\varepsilon}).$$

**Remark.** If $\mathbb{P}$ is the Masur-Veech smooth measure on $\mathcal{H}$, then the work of G. Forni [9], [10], [11] and S. Marmi, P. Moussa, J.-C. Yoccoz [16] implies that the left-hand side is bounded for any $f \in C^{1+\varepsilon}(M)$ (in fact, for any $f$ in the Sobolev space $H^{1+\varepsilon}$). In particular, if $f \in C^{1+\varepsilon}(M)$ and $\Phi^+_f = 0$, then $f$ is a coboundary.

### 3. Limit Theorems for Translation Flows.

#### 3.1. Time integrals as random variables.

As before, $(M, \omega)$ is an abelian differential, and $h^+_t$, $h^-_t$ are, respectively, its vertical and horizontal flows. Take $\tau \in [0, 1]$, $s \in \mathbb{R}$, a real-valued $f \in \text{Lip}_{w,0}^+(M, \omega)$ and introduce the function

$$\mathcal{G}[f, s; \tau, x] = \int_0^{\tau \exp(s)} f \circ h^+_t(x) dt. \quad (9)$$

For fixed $f$, $s$ and $x$ the quantity $\mathcal{G}[f, s; \tau, x]$ is a continuous function of $\tau \in [0, 1]$; therefore, as $x$ varies in the probability space $(M, m)$, we obtain a random element of $C[0, 1]$. In other words, we have a random variable $\mathcal{G}[f, s] : (M, m) \to C[0, 1]$ defined by the formula $(9)$.

For any fixed $\tau \in [0, 1]$ the formula (9) yields a real-valued random variable $\mathcal{G}[f, s; \tau] : (M, m) \to \mathbb{R}$ whose expectation, by definition, is zero. Our first aim is to estimate the growth of its variance as $s \to \infty$. Without loss of generality, one may take $\tau = 1$.

#### 3.2. The growth rate of the variance.

Let $\mathbb{P}$ be an invariant ergodic probability measure for the Teichmüller flow such that, with respect to $\mathbb{P}$, the second Lyapunov exponent $\theta_2$ of the Kontsevich-Zorich cocycle is positive and simple (recall that, as Veech and Forni showed, the first one, $\theta_1 = 1$, is always simple [20, 10] and that, by the Avila-Viana theorem [2], the second one is simple for the Masur-Veech smooth measure).

For an abelian differential $X = (M, \omega)$, denote by $E^+_{2,X}$ the one-dimensional subspace in $H^1(M, \mathbb{R})$ corresponding to the second Lyapunov exponent $\theta_2$, and let $\mathcal{B}^+_{2,X} = I_X(E^+_{2,X})$. Similarly, denote by $E^-_{2,X}$ the one-dimensional subspace in $H^1(M, \mathbb{R})$ corresponding to the Lyapunov exponent $-\theta_2$, and let $\mathcal{B}^-_{2,X} = I_X(E^-_{2,X})$. Recall that the space $H^1(M, \mathbb{R})$ is endowed with the Hodge norm $| \cdot |_H$; the isomorphisms $I^+_X$ take the Hodge norm to a norm on $\mathcal{B}^+_X$; slightly abusing notation, we denote the latter norm by the same symbol.
Introduce a multiplicative cocycle $H_2(s, X)$ over the Teichmüller flow $g_s$ by taking $v \in E^+_2 X$, $v \neq 0$, and setting
\begin{equation}
H_2(s, X) = \frac{|A(s, X)v|_H}{|v|_H}.
\end{equation}
By definition, we have $\lim_{s \to -\infty} \frac{\log H_2(s, X)}{s} = \theta_2$.
Now take $\Phi_2^+ \in \mathcal{B}^+_2 X$, $\Phi_2^- \in \mathcal{B}^-_2 X$ in such a way that $\langle \Phi_2^+, \Phi_2^- \rangle = 1$.

**Proposition 3.** There exists a constant $\alpha > 0$ depending only on $\mathbb{P}$ and positive measurable functions

\[ C : \mathcal{H} \times \mathcal{H} \to \mathbb{R}_+, \quad V : \mathcal{H} \to \mathbb{R}_+, \quad s_0 : \mathcal{H} \to \mathbb{R}_+ \]

such that the following is true for $\mathbb{P}$-almost all $X \in \mathcal{H}$. If $f \in \text{Lip}_{v_0}(X)$ satisfies $m_{\Phi_2^-}(f) \neq 0$, then for all $s \geq s_0(X)$ we have
\begin{equation}
\left| \frac{\text{Var}_m \mathcal{G}[f, s; 1]}{V(g_s X)(m_{\Phi_2^-}(f) |\Phi_2^+(f)|^2 (H_2(s, X))^2)} - 1 \right| \leq C(X, g_s X) \exp(-\alpha s).
\end{equation}

**Remark.** Observe that the quantity $(m_{\Phi_2^-}(f)|\Phi_2^+|)^2$ does not depend on the specific choice of $\Phi_2^+ \in \mathcal{B}^+_2 X$, $\Phi_2^- \in \mathcal{B}^-_2 X$ such that $\langle \Phi_2^+, \Phi_2^- \rangle = 1$.

**Proposition 4.** There exists a positive measurable function $V : \mathcal{H} \times \mathcal{H} \to \mathbb{R}_+$ such that for $\mathbb{P}$-almost all $X \in \mathcal{H}$, we have
\begin{equation}
\text{Var}_m \mathcal{G}(x, e^s) = V(g_s X)|\Phi_2^+(f)|^2 (H_2(s, X))^2.
\end{equation}
In particular, we have $\text{Var}_m \Phi_2^+(x, e^s) \neq 0$ for any $s \in \mathbb{R}$. The function $V(X)$ is given by $V(X) = \frac{\text{Var}_m \Phi_2^+(x, 1)}{|\Phi_2^+(x, 1)|^2}$ (observe that the right-hand side does not depend on a particular choice of $\Phi_2^+ \in \mathcal{B}^+_2 X$, $\Phi_2^- \neq 0$).

### 3.3. The limit theorem.
Go back to the $C[0, 1]$-valued random variable $\mathcal{G}[f, s]$ and denote by $m[f, s]$ the distribution of the normalized random variable
\begin{equation}
\frac{\mathcal{G}[f, s]}{\sqrt{\text{Var}_m \mathcal{G}[f, s; 1]}}
\end{equation}
By definition, $m[f, s]$ is a Borel probability measure on $C[0, 1]$; furthermore, if $\xi = \xi(t) \in C[0, 1]$, then we have $\xi(0) = 0$ almost surely with respect to $m[f, s]$; $E_{m[f, s]}(\xi(\tau)) = 0$ for any fixed $\tau \in [0, 1]$; $\text{Var}_{m[f, s]}(\xi(1)) = 1$.
We are interested in the weak accumulation points of $m[f, s]$ as $s \to \infty$. Consider the space $\mathcal{H}'$ given by the formula
\[ \mathcal{H}' = \{ X' = (M, \omega, v), v \in E^+_2 (M, \omega), |v|_H = 1 \}. \]
By definition, the space $\mathcal{H}'$ is a $\mathbb{P}$-almost surely defined two-to-one cover of the space $\mathcal{H}$. The skew-product flow of the Kontsevich-Zorich cocycle over the Teichmüller flow yields a flow $g'_s$ on $\mathcal{H}'$ given by the formula
\[ g'_s(X, v) = \left( g_s X, \frac{A(s, X)v}{|A(s, X)v|_H} \right). \]
Given $X' \in \mathcal{H}$, set $\Phi_{2,X'}^+ = \mathcal{I}^+(v)$. Take $\tilde{v} \in E_\mathcal{F}(M, \omega)$ such that $\langle v, \tilde{v} \rangle = 1$ and set $\Phi_{2,X'}^- = \mathcal{I}^-(v)$, $m_{\tilde{v},X'} = m_{\tilde{v},X'}$. By Proposition 4 for any $\tau > 0$ we have $\text{Var}_m \Phi_{2,X'}^+(x, \tau) \neq 0$.

Let $\mathfrak{M}$ be the space of all probability distributions on $C[0, 1]$. Introduce a $\mathbb{P}$-almost surely defined map $D_2^+: \mathcal{H} \to \mathfrak{M}$ by setting $D_2^+(X')$ to be the distribution of the $C[0, 1]$-valued normalized random variable

$$\frac{\Phi_{2,X'}^+(x, \tau)}{\sqrt{\text{Var}_m \Phi_{2,X'}^+(x, 1)}}, \tau \in [0, 1].$$

By definition, $D_2^+(X')$ is a Borel probability measure on the space $C[0, 1]$; it is a compactly supported measure as its support consists of equibounded Hölder functions with exponent $\theta_2/\theta_1 - \varepsilon$.

Consider the set $\mathfrak{M}_1$ of probability measures $m$ on $C[0, 1]$, $\xi = \xi(t)$, the conditions:

1. the equality $\xi(0) = 0$ holds $m$-almost surely;
2. for any $\tau \in [0, 1]$ we have $E_m \xi(\tau) = 0$;
3. we have $\text{Var}_m \xi(1) = 1$ and for any $\tau \neq 0$ we have $\text{Var}_m \xi(\tau) \neq 0$.

By Proposition 4 we have $D_2^+(\mathcal{H}') \subset \mathfrak{M}_1$.

Consider a semi-flow $J_s$ on the space $C[0, 1]$ defined by the formula

$$J_s \xi(t) = \xi(e^{-s}t), \ s \geq 0.$$

Introduce a semi-flow $G_s$ on $\mathfrak{M}_1$ by the formula

$$G_s m = \frac{(J_s)_s m}{\text{Var}_m (\xi(e^{-s}))}, m \in \mathfrak{M}_1.$$

By definition, the diagram

$$\begin{array}{ccc}
\mathcal{H}' & \xrightarrow{D_2^+} & \mathfrak{M}_1 \\
\downarrow G_s & & \uparrow G_s \\
\mathcal{H}' & \xrightarrow{D_2^+} & \mathfrak{M}_1
\end{array}$$

is commutative.

Let $d_{L, P}$ be the Lévy-Prohorov metric on $\mathfrak{M}$ (see, e.g., [3]).

**Theorem 2.** Let $\mathbb{P}$ be a $g_s$-invariant ergodic probability measure on $\mathcal{H}$ such that the second Lyapunov exponent of the Kontsevich-Zorich cocycle is positive and simple with respect to $\mathbb{P}$.

There exists a positive measurable function $C : \mathcal{H}' \times \mathcal{H} \to \mathbb{R}_+$ and a positive constant $\alpha$ depending only on $\mathbb{P}$ such that for $\mathbb{P}$-almost every $X' \in \mathcal{H}'$ and any $f \in \text{Lip}_{w_0}(X)$ satisfying $m_{2,X}(f) > 0$ we have

$$d_{L, P}(m[f, s], D_2^+(g_s X')) \leq C(X', g_s X') \exp(-\alpha s).$$

Fix $\tau \in \mathbb{R}$ and let $m_2(X', \tau)$ be the distribution of the $\mathbb{R}$-valued random variable

$$\frac{\Phi_{2,X'}^+(x, \tau)}{\sqrt{\text{Var}_m \Phi_{2,X'}^+(x, \tau)}},$$

For brevity, write $m_2(X', 1) = m_2(X')$. 
Proposition 5. For $\mathbb{P}$-almost any $X' \in \mathcal{H}'$, the measure $m_2(X', \tau)$ admits atoms for a dense set of $\tau \in \mathbb{R}$.

By definition, $m_2(X')$ is always compactly supported; the following Proposition shows, however, that the family $\{m_2(X'), X' \in \mathcal{H}'\}$ is in general not closed. Let $\delta_0$ stand for the delta-measure at zero.

Proposition 6. Let $\mathcal{H}$ be endowed with the Masur-Veech smooth measure. Then the measure $\delta_0$ is an accumulation point for the set $\{m_2(X'), X' \in \mathcal{H}'\}$ in the weak topology.

4. A symbolic coding for translation flows.

By Vershik’s Theorem [21], every ergodic automorphism of a Lebesgue probability space can be represented as a Vershik automorphism of a Markov compactum. For an interval exchange transformation, an explicit representation is obtained using Rohlin towers given by Rauzy-Veech induction (see [12]). Passing to Veech’s zipperred rectangles and their bi-infinite Rauzy-Veech expansions, one represents a minimal translation flow as a flow along the leaves of the asymptotic foliation of a bi-infinite Markov compactum. In this representation, the cocycles in $\mathcal{B}^+$ become finitely-invariant measures on the asymptotic foliation of a Markov compactum.

Thus, after passage to a finite cover (namely, the Veech space of zipperred rectangles), the moduli space of abelian differentials is represented as a space of Markov compacta. The Teichmüller flow and the Kontsevich-Zorich cocycle admit a simple description in terms of this symbolic representation, and the cocycles in $\mathcal{B}^+$ are constructed explicitly. Theorems 1, 2 are then derived from their symbolic counterparts. Detailed proofs are given in [5].

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