Categorical models of Linear Logic with fixed points of formulas
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Abstract—We develop a categorical semantics of $\mu LL$, a version of propositional Linear Logic with least and greatest fixed points extending David Baelde’s propositional $\mu MALL$ with exponentials. Our general categorical setting is based on Seely categories and on strong functors acting on them. We exhibit two simple instances of this setting. In the first one, which is based on the category of sets and relations, least and greatest fixed points are interpreted in the same way. In the second one, based on a category of sets equipped with a notion of totality (non-uniform totality spaces) and relations preserving it, least and greatest fixed points have distinct interpretations. This latter model shows that $\mu LL$ enjoys a denotational form of normalization of proofs.

I. INTRODUCTION

Propositional Linear Logic is a well-established logical system introduced by Girard in [1]. It provides a fine-grain analysis of proofs in intuitionistic and classical logic, and more specifically of their cut-elimination. LL features a logical account of the structural rules (weakening, contraction) which are handled implicitly in intuitionistic and classical logic. For this reason, LL has many useful outcomes in the Curry-Howard based approach to the theory of programming: logical understanding of evaluation strategies, new syntax of proofs/programs (proof-nets), connections with other branches of mathematics (linear algebra, functional analysis, differential calculus), new operational semantics (geometry of interaction) etc.

However propositional LL is not a reasonable programming language, by lack of data-types and iteration or recursion principles. This is usually remedied by extending propositional LL to the 2nd order, thus defining a logical system in which Girard’s System $F$ [2] can be embedded. Another option to turn propositional LL into a programming language – closer to usual programming – is to extend it with least and greatest fixed points of formulas. Such an extension was early suggested by Girard in an unpublished note [3], though the first comprehensive proof-theoretic investigation of such an extension of LL is recent: in [4] Baelde considers an extension $\mu MALL$ of Multiplicative Additive LL sequent calculus with least and greatest fixed points. His motivations arose from a proof-search and system verification perspective and therefore his $\mu MALL$ logical system is a predicate calculus. Our purpose is to develop a more Curry-Howard oriented point of view on LL with fixed points and therefore we stick to the proposition calculus setting of [2]. But, unlike [4] we include the exponentials in our system from the beginning, so we call it $\mu LL$ rather than propositional $\mu MALL$ and we consider it as an alternative to the “system $F$” approach to representing programs in LL. Our system $\mu LL$ could also have applications to session types, in the line of [5]. The $\nu$-introduction rule of $\mu LL$ (Park’s rule, that is rule ($\nu$-rec) of Section II-F1) leads to subtle cut-elimination rewrite rules for which Baelde could prove cut-elimination in $\mu MALL$, showing for instance that a proof of the type of integers $\mu \zeta (1 \oplus \zeta)$ necessarily reduces to an integer (in contrast with LL, $\mu MALL$ enjoys only a restricted form of sub-formula property). There are alternative proof-systems for the same logic, involving infinite or cyclic proofs, see [6], whose connections with the aforementioned finitary proof-system are not completely clear yet.

Since the proof-theory (and hence the “operational semantics”) of $\mu LL$ is still under development, it is important to investigate its categorical semantics, whose definition does not rely on the precise choice of inference and rewrite rules we equip $\mu LL$ with, see the Outcome § below. We develop here a categorical semantics of $\mu LL$ extending the standard notions of Seely category of classical LL, see [7]. Such a model of $\mu LL$ consists of a Seely category $\mathcal{L}$ and of a class of functors $\mathcal{L}^n \to \mathcal{L}$ for all possible arities $n$ which will be used for interpreting $\mu LL$ formulas with free variables. These functors have to equipped with a strength to deal properly with contexts in the rule ($\nu$-rec), see Section II-F2 for a discussion on these contexts in particular.

Then we develop a simple instance of this setting which consists in taking for $\mathcal{L}$ the category of sets and relations, a well-known Seely model of LL. The variable sets are the strong functors we consider on this category. They are the pairs $F = (F, F')$ where $F$ is the strength and $F' : \text{Rel}^n \to \text{Rel}$ is a functor which is Scott-continuous in the sense that it commutes with directed unions of morphisms. This property implies that $F'$ maps injections to injections and is cocontinuous on the category of sets and injections. There is no special requirement about the strength $F$ beyond naturality, monoidality and compatibility with the comultiplication of the comonad $\Delta$. Variable sets form a Seely model of $\mu LL$.

1Exponentials are not considered in $\mu MALL$ because some form of exponential can be encoded using inductive/coinductive types, however these exponentials are not fully satisfactory from our point of view because their denotational interpretation does not satisfy all required isomorphisms; specifically, the Seely iso is lacking.

2Sometimes called new-Seely category: it is a cartesian symmetric monoidal closed category with a $*$-autonomous structure and a comonad $\Delta$ with a strong symmetric monoidal structure from the cartesian product to the tensor product.
where linear negation is the identity on objects. The formulas $\mu \zeta F$ and $\nu \zeta F$ are interpreted as the same variable set, exactly as $\otimes$ and $\exists Y$ are interpreted in the same way (and similarly for additives and exponentials). This denotational “degeneracy” at the level of types is a well known feature of Rel which does not mean at all that the model is trivial. For instance normal multiplicative exponential LL proofs which have distinct relational interpretations have distinct associated proof-nets [8], [9].

Last we enrich this model Rel by considering sets equipped with an additional structure of totality: a non-uniform totality space (NUTS) is a pair $X = (|X|, T(X))$ where $|X|$ is a set and $T(X)$ is a set of subsets of $|X|$ which intuitively represent the total, that is, terminating computations of type $X$. This set $T(X)$ is required to coincide with its bidual for a duality expressed in terms of non-empty intersections. This kind of definition by duality is ubiquitous in $X$ the total, that is, terminating computations of type $X$ is required to coincide with its bidual for a duality expressed in terms of non-empty intersections. This kind of definition by duality is ubiquitous in $X$ the total, that is, terminating computations of type $X$.

Rather than considering them directly as functors, we define a strong functor $\mu \mathsf{LL}$ that it gives a value to all proofs of $\mu \mathsf{LL}$ which seems too weak in our Curry-Howard perspective. And indeed $\mu$-bicomplete categories do not provide the monoidal and exponential structures required for interpreting $\mu \mathsf{LL}$.

In [22], that we became aware of only recently (and seems related to the earlier report [23]), Loader extends the simply typed $\lambda$-calculus with inductive types and develops its denotational semantics. His models are cartesian closed categories $C$ equipped with a class of strong functors and seem very close to ours (Section II-F): one might think that any of our models yields a Loader model as its Kleisli category. This is not the case because in a Loader model the category $C$ is cocartesian whereas the Kleisli category of a Seely category is not cocartesian in general: this would require to have an iso between $(X \oplus Y)$ and $1X \oplus !Y$ which is usually absent. Loader studies two concrete instances of his models: one is based on recursion theory (partial equivalence relations) and the other on a notion of domains with totality described as a model of LL. This model might give rise to one of our Seely models, this point requires further studies. Our NUTS are quite different from Loader totality domains which feature a notion of “consistency” enforcing some kind of determinism and, combined with totality, allow the Kleisli category to be cocartesian as well. Our model is based on Rel and therefore is compatible with non-determinism [24] and PCF recursion. This is important for us because we would like to consider rules beyond Park’s rule for inductive and coinductive types, based on PCF fixed points – with further guardedness conditions for guaranteeing termination – in the spirit of [25], [26], [27] or even on infinite terms in the spirit of [6].

We mention also the work of Clairambault [28], [29] who investigates the game with totality semantics of an extension of intuitionistic logic with least and greatest fixed points (independently of [22], [23]). A Kleisli-like connection with his work should be sought too.

a) Outcome: One major benefit of this construction is that it gives a value to all proofs of $\mu \mathsf{LL}$, invariant by cut-elimination. Moreover, the fact that this value is total shows in a syntax independent way that when $\pi$ is for instance a $\mu \mathsf{LL}$ proof of $1 \otimes 1$ (the type of booleans), the value associated with $\pi$ is non-empty, that is, $\pi$ has a defined boolean value true or false. We could also obtain this by a normalization theorem: $\pi$ reduces to one of the two normal proofs of $1 \otimes 1$ (and if we prove for instance a Church-Rosser theorem we will know that this proof is unique). Such proofs would depend of course on the actual presentation of the syntax whereas our denotational argument does not.

b) Related work: There is a vast literature on extending logic with fixed point that we cannot reasonably

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3This new model is a major simplification wrt. notions of totality on coherence spaces [11] or Loader’s totality spaces [12] where biduality is much harder to deal with because it combines totality with a form of determinism.

4Or both because our NUTS model accepts non-determinism. By adding a non-uniform coherence relation as defined in [13], [14] to the model one can show that this value is actually a uniquely defined boolean. See also Section II-F.

5To account for the disjunction of his logical system which is crucial for defining interesting data-types such as the integers.
from $A$ to $B$ in $A$ (all the categories we consider are locally small). If $\mathcal{F} : A \times B \to C$ is a functor and $A \in \text{Obj}(A)$ then $\mathcal{F}_A : B \to C$ is the functor defined by $\mathcal{F}_A(B) = \mathcal{F}(A, B)$ and $\mathcal{F}_A(f) = \mathcal{F}(A, f)$ (we often write $A$ instead of $\text{Id}_A$).

Most proofs can be found in an Appendix.

II. Categorical models of LL

A. Seely categories.

We recall the basic notion of categorical model of LL. Our main reference is the notion of a Seely category as presented in [2]. We refer to that survey for all the technical material that we do not recall here.

A Seely category is a symmetric monoidal closed category (SMCC) $\mathcal{L} = (\otimes, 1, \lambda, \rho, \alpha, \gamma)$ where $\lambda_X \in \mathcal{L}(1 \otimes X, X)$, $\rho_X \in \mathcal{L}(X \otimes 1, X)$, $\alpha_{X,Y,Z} \in \mathcal{L}((X \otimes Y) \otimes Z, X \otimes (Y \otimes Z))$ and $\gamma_{X,Y} \in \mathcal{L}(X \otimes Y, Y \otimes X)$ are natural isomorphisms satisfying coherence diagrams that we do not record here. We use $X \rightarrow \otimes Y$ for the object of linear morphisms from $X$ to $Y$, $ev \in \mathcal{L}((X \rightarrow \otimes Y) \otimes Y, Y)$ for the evaluation morphism and $cur$ for the linear currying map $\mathcal{L}(\otimes X, Y) \rightarrow \mathcal{L}(X, \otimes Y)$. We assume $\mathcal{L}$ to be $*$-autonomous with dualizing object $D$ (this object is part of the structure of a Seely category).

We use $X^\bot$ for the object $X \rightarrow \otimes \bot$. The definition of these objects is explained above, and in particular the Seely isos. We use $\text{tr}([\cdot])$ for the “De Morgan dual” of $\mathcal{L}$. A linear map $\mathcal{L}(X,Y)$ has the form $\gamma^\otimes \circ \lambda^\otimes \circ \rho^\bot$ and similarly for morphisms.

B. Oplax monoidal comonads

Let $\mathcal{M}$ be a symmetric monoidal category (with the same notations as above for the tensor product) and $(\mathcal{T}, \varepsilon, \mu) : \mathcal{M} \to \mathcal{M}$ be a comonad ($\varepsilon$ is the unit and $\mu$ the multiplication). An oplax monoidal structure on $\mathcal{T}$ consists of a morphism $\theta^0 \in \mathcal{M}(\mathcal{T}1, 1)$ and a natural transformation $\theta_{X_1,X_2} : \mathcal{T}(\mathcal{T}X_1 \otimes \mathcal{T}X_2) \Rightarrow \mathcal{T}(\mathcal{T}(X_1 \otimes X_2), \mathcal{T}(X_1 \otimes \mathcal{T}X_2), \mathcal{T}(X_1) \otimes (\mathcal{T}X_2))$ subject to standard symmetric monoidality and compatibility with $\varepsilon$ and $\mu$, this latter reading $\varepsilon(X_1 \otimes X_2) = \varepsilon(X_1) \otimes \varepsilon(X_2)$ and:

$$T(X_1 \otimes X_2) \xrightarrow{\theta_{X_1,X_2}} TX_1 \otimes TX_2 \xrightarrow{\mu_{X_1} \otimes \mu_{X_2}} T^2X_1 \otimes T^2X_2$$

$$\xrightarrow{\mu_{X_1} \otimes \mu_{X_2}} T^2X_1 \otimes T^2X_2 \xrightarrow{\theta_{X_1,X_2}} TX_1 \otimes TX_2$$

$$T^2(X_1 \otimes X_2) \xrightarrow{T(\theta_{X_1,X_2})} T(TX_1 \otimes TX_2)$$

Then the Kleisli category $\mathcal{M}_T$ has a canonical symmetric monoidal structure, with unit 1 and tensor product $X_1 \otimes X_2$ defined as in $\mathcal{M}$ for objects. Given $f_i \in \mathcal{M}_T(X_i, Y_i)$, $f_1 \otimes_T f_2 \in \mathcal{M}_T(X_1 \otimes X_2, Y_1 \otimes Y_2)$ is defined as

$$f_1 \otimes_T f_2 \in \mathcal{M}_T(X_1 \otimes X_2, Y_1 \otimes Y_2)$$

Let $\mathcal{F}_T : \mathcal{M} \to \mathcal{M}_T$ be the functor which acts as the identity on objects and maps $f \in \mathcal{M}(X,Y)$ to $f \varepsilon_X \in \mathcal{M}_T(X,Y)$.

C. Eilenberg-Moore category and free comodules

Let $\mathcal{L}$ be a Seely category. Since $\mathcal{L}$ is a comonad we can define the category $\mathcal{L}^!$ of $!*$-coalgebras (Eilenberg-Moore category of $\mathcal{L}$). An object of this category is a pair $P = (P, h_P)$ where $P \in \text{Obj}(\mathcal{L})$ and $h_P \in \mathcal{L}(P, P^!)$ is such that $\text{der}h_p = \text{Id}g \text{dig}_X \text{dig}_Y f = f \text{h}_p$. The functor $\mathcal{L}$ can be seen as a functor from $\mathcal{L}$ to $\mathcal{L}^!$ mapping $X$ to $(X, \text{d}g_X)$ and $f \in \mathcal{L}(X, Y)$ to $f^!$. It is right adjoint to the forgetful functor $\mathcal{L}^! \rightarrow \mathcal{L}$. Given $f \in \mathcal{L}^!(P, X)$, we use $f^! \in \mathcal{L}^!(P, X)$ the morphism associated with $f$ by this adjunction, one has $f^! = f \text{h}_p$. If $g \in \mathcal{L}^!(Q, P)$, we have $f^! g = (f g)^!$.

Then $\mathcal{L}^!$ is cartesian with final object $(1, h_1 = 0)$ still denoted as 1 and product $P \otimes P = (P \otimes P, h_{P \otimes P})$ with $h_{P \otimes P} = P \otimes h_P = \mu^\otimes = \mu^\otimes \mu^\bot$. This category is also cocartesian with initial object $(0, h_0)$ still denoted as 0 and coproduct $P \oplus P = (P \oplus P, h_{P \oplus P})$ with $h_{P \oplus Q} = (P \oplus Q, h_{P \oplus Q})$.

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We set $F_{P} = F_{rP} : L \to L[P]$. Girard showed in [30] that $L[P]$ is a Seely model of $LL$ with operations on objects defined in the same way as in $L$, and using the coalgebra structure of $P$ for the operations on morphisms. Intuitively, $P$ should be considered as a given context and $L[P]$ as a model in this context. This idea appears at various places in the literature, see for instance [31], [32]. Let us summarize this construction. If $f_{i} \in L[P](X_{i}, Y_{i})$ for $i = 1, 2$ then $f_{1} \otimes_{P} f_{2} = f_{1} \otimes_{rP} f_{2} \in L[P](X_{1} \otimes_{X_{2}}, Y_{1} \otimes_{X_{2}})$ is given by

$$
P \otimes X_{1} \otimes X_{2} \xrightarrow{\text{contr}_{P} \otimes \text{id}} P \otimes P \otimes X_{1} \otimes X_{2}
Y_{1} \otimes Y_{2} \xleftarrow{f_{1} \otimes_{P} f_{2}} P \otimes X_{1} \otimes P \otimes X_{2}
$$

The object of linear morphisms from $X$ to $Y$ in $L[P]$ is $X \otimes_{P} Y$, and the evaluation morphism $\text{ev}_{P} \in L[P]((X \otimes_{P} Y) \otimes_{P} X, Y)$ is simply $P_{r}(\text{ev})$. Then it is easy to check that if $f \in L[P](Z \otimes X, Y)$, then $f^{\perp} \in L[P](Z \otimes X, Y)$ satisfies the required monoidal closedness equations. With these definitions, the category $L[P]$ is $*$-autonomous, with $\perp$ as dualizing object. Specifically, given $f \in L[P](X, Y)$, then $f^{\perp}[P]$ is the following composition of morphisms:

$$
P \otimes Y^{\perp} \xrightarrow{\text{ev}^{\perp}} P \otimes (P \to X^{\perp}) \xrightarrow{\text{ev}} X^{\perp}
$$

using implicitly the iso between $(Z \otimes X)^{\perp}$ and $Z \to X^{\perp}$, and the $*$-autonomy of $L$ allows to prove that indeed $f^{\perp}[P][P] = f$. The category $L[P]$ is easily seen to be cartesian with $\perp$ as final object, $X_{1} \otimes_{P} X_{2}$ as cartesian product (and projections defined in the obvious way, applying $F_{rP}$ to the projections of $L$). Last we define a functor $!_{P} : L[P] \to L[P]$ by $!_{P}X = \{X, X^{\perp}\}$ and, given $f \in L[P](X, Y)$, we define $!_{P}f \in L[P](!X, !Y)$ as $P \otimes !X \xrightarrow{\text{h}_{P} \otimes \text{id}} !P \otimes !X \xrightarrow{\mu^{2}} !P \otimes !X \xrightarrow{!f} !Y$ and this functor has a comonad structure $(\text{der}[P], \text{dig}[P])$ defined by $\text{der}[P] = F_{rP}(\text{der})$ and $\text{dig}[P] = F_{rP}(\text{dig})$.

**Remark 1.** Any $p \in L'(P, Q)$ induces an arbitrary identity strong functor $\mathbb{T}$ and for each object $Y$ of $L$ there is an $n$-ary $Y$-valued constant strong functor $\mathbb{K}^{Y}$; in the first case the strength natural transformation is the identity morphism and in the second case, it is defined using $\mathbb{w}_{X}$. Let $\mathbb{F}$ be an $n$-ary strong functor and $\mathbb{G}_{1}, \ldots, \mathbb{G}_{n}$ be $k$-ary strong functors. Then one defines a $k$-ary strong functor $\mathbb{H} = \mathbb{F} \circ (\mathbb{G}_{1}, \ldots, \mathbb{G}_{n})$: the functorial component $\mathbb{H}$ is defined in the obvious compositional way. The strength is

$$
!X \otimes \mathbb{F}(!Y) \xrightarrow{\text{ev}} \mathbb{F}(!X \otimes \mathbb{F}(!Y)) \xrightarrow{!} \mathbb{F}(!X \otimes \mathbb{F}(!Y) \otimes \mathbb{F}(!X \otimes \mathbb{F}(!Y)))
$$

and is easily seen to satisfy the commutations of Figure [1].

**Remark 2.** Since the seminal work of Moggi [34] strong functors play a central role in semantics for representing effects. Our adaptation of this notion to the present $LL$ setting follows the definition of an $L$-tensorial strength in [35].

1) **Operations on strong functors:*** There is an obvious unary identity strong functor $\mathbb{T}$ and for each object $Y$ of $L$ there is an $n$-ary $Y$-valued constant strong functor $\mathbb{K}^{Y}$; in the first case the strength natural transformation is the identity morphism and in the second case, it is defined using $\mathbb{w}_{X}$. Let $\mathbb{F}$ be an $n$-ary strong functor and $\mathbb{G}_{1}, \ldots, \mathbb{G}_{n}$ be $k$-ary strong functors. Then one defines a $k$-ary strong functor $\mathbb{H} = \mathbb{F} \circ (\mathbb{G}_{1}, \ldots, \mathbb{G}_{n})$: the functorial component $\mathbb{H}$ is defined in the obvious compositional way. The strength is

$$
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$$

and is easily seen to satisfy the commutations of Figure [1].

Given an $n$-ary strong functor, we can define its De Morgan dual $\mathbb{F}^{\perp}$ which is also an $n$-ary strong functor. On objects, we set $\mathbb{F}^{\perp}(Y) = \mathbb{F}(Y^{\perp})^{\perp}$ and similarly for morphisms. The strength of $\mathbb{F}^{\perp}$ is defined as the Curry transpose of the following morphism (remember that $!X \to Y^{\perp} = (!X \otimes Y)^{\perp}$ up to canonical iso):

$$
!X \otimes \mathbb{F}(Y)^{\perp} \otimes \mathbb{F}(!X \to Y^{\perp}) \xrightarrow{!} !X \otimes \mathbb{F}(!X \to Y^{\perp}) \otimes \mathbb{F}(Y)^{\perp} \xrightarrow{!} !X \otimes \mathbb{F}(!X \to Y^{\perp}) \otimes \mathbb{F}(Y)^{\perp}
$$

Then it is possible to prove, using the $*$-autonomy of $L$, that $\mathbb{F}^{\perp}$ and $\mathbb{F}$ are canonically isomorphic (as strong functors). As a direct consequence of the definition of $\mathbb{F}^{\perp}$ and of the canonical iso between $\mathbb{F}^{\perp}$ and $\mathbb{F}$ we get:

**Lemma 1.** $(\mathbb{F} \circ (\mathbb{G}_{1}, \ldots, \mathbb{G}_{n}))^{\perp} = \mathbb{F}^{\perp} \circ (\mathbb{G}_{1}^{\perp}, \ldots, \mathbb{G}_{n}^{\perp})$ up to canonical iso.

The definition of $!_{P}f$ requires $P$ to be a $1$-coalgebra and not simply a commutative $\otimes$-comonoid. Of course if $!$ is the free exponential as in [30] the latter condition implies the former.

8The definition of $!_{P}f$ requires $P$ to be a $1$-coalgebra and not simply a commutative $\otimes$-comonoid. Of course if $!$ is the free exponential as in [30] the latter condition implies the former.
The bifunctor ⊠ can be turned into a strong functor: one defines the strength as \[ !X \otimes !Y \overset{\text{constr}_{A,X} \otimes \text{Id}}{\longrightarrow} !X \otimes !Y \otimes !Y \cong !X \otimes !Y \otimes !Y \]

By De Morgan duality, this endows \( X \) with a strength as well. The bifunctor \( \otimes \) is endowed with a strength, simply using the distributivity of \( \otimes \) over \( \oplus \) (which results from the monoidal closedness of \( \mathcal{L} \)). By duality again, \( \otimes \) inherits a strength. The functor \( !\) is equipped with the strength \[ !X \otimes !Y \overset{\text{constr}_{A,X} \otimes \text{Id}}{\longrightarrow} !X \otimes !Y \otimes !Y \otimes !Y \rightarrow (!!X \otimes Y) \] .

**E. Fixed Points of strong functors**

The following facts are standard in the literature on fixed points of functors.

**Definition 2.** Let \( A \) be a category and \( F : A \to A \) be a functor. A coalgebra of \( F \) is a pair \((A, f)\) where \( A \) is an object of \( A \) and \( f \in \text{Mor}(A, F(A)) \). Given two coalgebras \((A, f)\) and \((A', f')\) of \( F \), a coalgebra morphism from \((A, f)\) to \((A', f')\) is an \( h \in \text{Mor}(A, A') \) such that \( f' \circ h = F(h) \cdot f \). The category of coalgebras of the functor \( F \) will be denoted as \( \text{Coalg}_A(F) \). The notion of algebra of an endofunctor is defined dually (reverse the directions of the arrows \( f \) and \( f' \)) and the corresponding category is denoted as \( \text{Alg}_A(F) \).

By Lambek’s Lemma, if \((A, f)\) with \( f \in \text{Mor}(A, F(A)) \) is a final object in \( \text{Coalg}_A(F) \) then \( f \) is an isomorphism. We assume that this is iso in all the identifiers as this holds in our concrete models so that this final object \((\nu F, \text{Id})\) satisfies \( F(\nu F) = \nu F \). We focus on coalgebras rather than algebras for reasons which will become clear when we deal with fixed points of strong functors. This universal property of \( \nu F \) gives us a powerful tool for proving equalities of morphisms.

**Lemma 3.** Let \( A \in \text{Obj}(A) \) and let \( f_1, f_2 \in \text{Mor}(A, \nu F) \). If there exists \( l \in \text{Mor}(A, F(A)) \) such that \( F(l) \cdot f = f_1 \), for \( i = 1, 2 \), then \( f_1 = f_2 \).

**Lemma 4.** Let \( F : B \times A \to A \) be a functor such that, for all \( B \in \text{Obj}(B) \), the category \( \text{Coalg}_A(F_B) \) has a final object.

Then there is a functor \( \nu F : B \to A \) such that \((\nu F(B), \text{Id})\) is the final object of \( \text{Coalg}_A(F_B) \) (so that \( F(B, \nu F(B)) = \nu F(B) \)) for each \( B \in \text{Obj}(B) \), and, for each \( g \in \text{Mor}(B, B') \), \( \nu F(g) \) is uniquely characterized by \( F(g, \nu F(g)) = \nu F(g) \).

We consider now the same \( \nu F \) operation applied to strong functors on a model \( L \) of \( \mathcal{L} \). Let \( F \) be an \( n + 1 \)-ary strong functor on \( L \) (so that \( F \) is a functor \( L^n \times L \to L \)). Assume that for each \( X \in \text{Obj}(L^n) \) the category \( \text{Coalg}_{L^n}(F_X) \) has a final object. We have defined \( \nu F : L^n \to L \) characterized by \( F(\nu F(X)) = \nu F(X) \) and \( F(f, \nu F(f)) = \nu F(f) \) for all \( f \in L^n(X, X') \) (Lemma 6). For each \( X, Y \in \text{Obj}(L) \), we define \( \nu F_{X,Y} : \text{obj}(L^n \times L^n) \to \text{obj}(L \times L) \) as the following diagram commutes

\[
\begin{array}{ccc}
!Y \otimes !Y & \longrightarrow & !Y \otimes !X \\
\text{ev}_{X,Y} & \downarrow & \text{ev}_{Y,X} \\
F(Y \otimes X, Y \otimes X) & \longrightarrow & F(X \otimes Y, X \otimes Y)
\end{array}
\]

**Lemma 5.** Let \( F \) be an \( n + 1 \)-ary strong functor on \( L \) such that for each \( X \in \text{Obj}(L^n) \), the category \( \text{Coalg}_{L^n}(F_X) \) has a final object \( \nu F_X \). Then there is a unique \( n + 1 \)-ary strong functor \( \nu F \) such that \( F(\nu F(X)) = \nu F_X \) (and hence \( F(f, \nu F(f)) = \nu F(f) \) for all \( f \in L^n(X, X') \), \( F(\text{ev}_{X,Y}) = \nu F_{X,Y} \) for all \( X, Y \))

**Lemma 6.** Let \( F \) be an \( n + 1 \)-ary strong functor on \( L \) such that for each \( X \in \text{Obj}(L^n) \), the category \( \text{Alg}_{L}(F_X) \) has an initial object \( \mu F_X \). Then there is a unique \( n + 1 \)-ary strong functor \( \mu F \) such that \( F(\mu F(X)) = \mu F_X \) (and hence \( F(f, \mu F(f)) = \mu F(f) \) for all \( f \in L^n(X, X') \) and \( F(\text{ev}_{X,Y}) = \mu F_{X,Y} \) for all \( X, Y \)). Moreover \( \mu F_X = \mu (\nu F X) \).

**Proof:** Apply Lemma 5 to the strong functor \( F \).

**F. A categorical axiomatization of \( \mu \mathcal{L} \)**

Our general definition of Seely categorical model of \( \mu \mathcal{L} \) is based on the notions and results above. We refer in particular
to Section II-D for the basic definitions of operations on strong functors in our LL categorical setting.

Definition 7. A categorical model or Seely model of $\mu LL$ is a pair $(L, \tilde{L})$ where

1. $L$ is a Seely category
2. $\tilde{L} = (L_n)_{n \in \mathbb{N}}$ where $L_n$ is a class of strong functors $L^n \rightarrow L$, and $L_0 = \text{Obj}(L)$
3. If $X \in L_n$ and $X_i \in L_k$ (for $i = 1, \ldots , n$) then $X \circ X_i \in L_k$ and all $k$ projection strong functors $L^k \rightarrow L$ belong to $L_k$
4. The strong functors $\otimes$ and $\&$ belong to $L_2$, the strong functor $\exists \otimes \subseteq L$ belongs to $L_1$ and, if $X \in L_n$, then $X \exists \otimes \in L_n$

5. and last, for all $X \in L_1$ the category $\text{Coalg}(X)$ (see Section II-D) has a final object. So for any $X \in L_{k+1}$ there is a strong functor $\nu X : L^k \rightarrow L$ (see Lemma 5). It is required that $\nu X \in L_k$.

Remark 3. By Conditions 2 and 5 (applied with $n = 0$), all constant strong functors are in $L_n$, for all $n$. Therefore given $X \in L_{k+1}$ and $X \in \text{Obj}(L)^k$, the strong functor $X(\ldots , X)$ is in $L_1$ by Condition 5. This explains why we can apply Lemma 5 in Condition 5.

Our goal is now to outline the interpretation of $\mu LL$ formulas and proofs in such a model. This requires first to describe the syntax of formulas and proofs.

Remark 4. One can certainly also define a notion of categorical model of $\mu LL$ in a linear-non-linear adjunction setting as presented in [7]. This is postponed to further work.

1) Syntax of $\mu LL$: We assume to be given an infinite set of propositional variables $V$ (ranged over by Greek letters $\zeta, \xi, \ldots$). We introduce a language of propositional LL formulas with least and greatest fixed points.

$$A, B, \cdots := 1 \mid \bot \mid A \otimes B \mid A \& B \mid A \triangledown B$$

The notion of closed formula is defined as usual, the two last constructions being the only binders.

Remark 5. In contrast with second-order LL or dependent type systems where open formulas play a crucial role, in the case of fixed points, all formulas appearing in sequents and other syntactical devices allowing to give types to programs will be closed. In our setting, open types/formulas appear only locally, for allowing the expression of (least and greatest) fixed points.

We can define two basic operations on formulas:

- **Substitution**: $A[B/\zeta]$, taking care of not binding free variables (uses $\alpha$-conversion).
- **Negation or dualization**: defined by induction on formulas $
abla = \bot$, $\nabla = \bot$, $(A \triangledown B) = A \triangledown B$, $(A \& B) = A \& B$, $0 = \bot$, $B \triangledown 0 = 0$, $(A \& B) = A \& B$, $(A \otimes B) = A \otimes B$, $(!A) = A$, $(\neg A) = A$, $\nabla = \nabla$, $\nabla = \nabla$, $(\mu A) = \nu A$ and $(\nu A) = \mu A$. Obviously $A^{\nabla \nabla} = A$ for any formula $A$.

Remark 6. The only subtle point of this definition is negation of propositional variables: $\nabla = \nabla$. This entails $(B[A/\zeta])^{\nabla} = B^{\nabla}[A^{\nabla/\zeta}]$ by an easy induction on $B$. If we consider $B$ as a compound logical connective with placeholders labeled by variables then $B^{\nabla}$ is its De Morgan dual. This definition of $\nabla$ is also a natural way of preventing the introduction of fixed points wrt. variables with negative occurrences. As an illustration, if we define as usual $A \rightarrow B$ as $A^{\nabla} \& B$ then we can define $E = \mu \zeta (1 \& (\nabla \rightarrow \zeta))$ which looks like the definition of a model of the pure $\lambda$-calculus as a recursive type. But this is only an illusion since we actually have $E = \mu \zeta (1 \& (\nabla \& \neg \zeta))$ so that $E \rightarrow 0$ is not a retract of $E$. And indeed if it were possible to define a type $D$ such that $!D \rightarrow D$ is isomorphic to (or is a retract of) $D$ then we would be able to type all pure $\lambda$-terms in our system and this would contradict the fact that $\mu LL$ enjoys strong normalization and has a denotational semantics based on totality as shown below.

Our logical system extends the usual unilateral sequent calculus of classical propositional LL [1], see also [7] Section 3.1 and 3.13. In this setting we deal with sequents $\vdash A_1, \ldots , A_n$ where the $A_i$’s are formulas. It is important to notice that the order of formulas in this list is not relevant, which means that we keep the exchange rule implicit as it is usual in sequent calculus. To the standard rule[13] of [7] Fig. 1, we add the two next introduction rules for fixed point formulas which are essentially borrowed to [3] (see Section III-F).

$$\vdash \Gamma, \mu \zeta F \quad (\mu\text{-fold})$$

By taking, in the last rule, $\Delta = A^{\nabla}$ and proving the left premise by an axiom, we obtain the following derived rule

$$\vdash \Delta, A^{\nabla}, F[A/\zeta]$$

The corresponding cut-elimination rule is described in Section III-F. For the other connectives (which are the standard connectives of LL), the cut-elimination rules are the usual ones as described in [1], [7].

2) Comments: Let us summarize and comment the differences between our system and Baelde’s $\mu MALL$.

- Baelde’s logical system is a predicate calculus whereas our system is a propositional calculus. Indeed, Baelde is mainly interested in applying $\mu MALL$ to program verification where the predicate calculus is essential for expressing properties of programs. We have a Curry-Howard perspective where formulas are seen at types and proofs as programs and where a propositional logical system is sufficient.

- Our system has exponentials whereas Baelde’s system has not because they can be encoded in $\mu MALL$ to some extent. However the exponentials encoded in that way do not satisfy all required iso (in particular the “Seely morphisms” are not iso with Baelde’s exponentials) and this is a serious issue if we want to encode some form of $\lambda$-calculus in the system and consider it as a programming language.

[13] Notice that the promotion rule of LL has a condition on contexts similar to that of the rule (r-rec) below: to deduce $\vdash \Delta, !A$ from $\vdash \Delta, A$ it is required that all formulas in the context $\Delta$ are of shape $B$, that is $\Delta = ?B$.\]
• Our (ν-rec) rule differs from Baelde’s by the fact that we admit a context in the right premise. Notice that all formulas of this context must bear a ?* modality: this restriction is absolutely crucial for allowing to express the cut-elimination rule in Section [0x1.0] which uses an operation of substitution of proofs in formulas and this operation uses structural rules on the context. The semantic counterpart of this operation is described in Section [0x1.0.1] where it appears clearly that it uses the fact that $P$ is an object of $L'$. Such a version of (ν-rec) with a context would be problematic in Baelde’s system by lack of built-in exponentials.

3) Syntactic functoriality of formulas: The reduction rule for the (μ-fold)/(ν-rec) cut requires the possibility of substituting a proof for a propositional variable in a formula. More precisely, let $(ζ, ξ_1, …, ξ_k)$ be a list of pairwise distinct propositional variables containing all the free variables of a formula $F$ and let $C = (C_1, …, C_k)$ be a sequence of closed formulas. Let $π$ be a proof of $Γ ⊢ ?Γ, A^⊥, B$, then one defines a proof $F \pi/ζ, C_1/ξ_1$ of

$Γ ⊢ ?Γ, (F[π/ζ, C_1/ξ_1, C_2/ξ_2])^⊥, F[B/ζ, C_1/ξ_1]$  

by induction on $F$, adapting the corresponding definition in [0x1.1]. We illustrate this definition by two inductive steps. Observe, in these examples, how the exchange rule is used implicitly.

Assume first that $F = μξ G$ (so that $(ζ, ξ, ξ_1, …, ξ_k)$ is a list of pairwise distinct variables containing all free variables of $G$). Let $G' = G[\bar{C}/ξ]$ whose only possible free variables are $ζ$ and $ξ$. The proof $F[\pi/ζ, C_1/ξ]$ is defined by

$Γ, (G'[\pi/ζ, C_1/ξ, C_2/ξ_2])^⊥, G[B/ζ, C_1/ξ]$  

(μ-fold)

$Γ, (G'[\pi/ζ, C_1/ξ, C_2/ξ_2])^⊥, G[B/ζ, C_1/ξ]$  

(ν-rec')

Notice that this case uses the additional parameters $C_i$ in the definition of this substitution with $k + 1$ parameters in the inductive hypothesis. To see that the last inference in this deduction is an instance of (ν-rec'), set $H = G'[\pi/ζ, C_1/ξ]$ and notice that $(G'[\pi/ζ, C_1/ξ, C_2/ξ_2])^⊥ = H[(μξ G')[B/ζ, C_1/ξ]]$ and $(μξ G'[B/ζ, C_1/ξ])^⊥ = νξ H$. Another example is $F = G_1 \otimes G_2$: $F[\pi/ζ, C_1/ξ]$ is defined as

$Γ, (G_1[\pi/ζ, C_1/ξ, C_2/ξ_2])^⊥, G_2[\pi/ζ, C_1/ξ]$  

(μ-fold)

$Γ, (G_1[\pi/ζ, C_1/ξ, C_2/ξ_2])^⊥, G_2[\pi/ζ, C_1/ξ]$  

(ν-rec')

Again the fact that the formulas of the context bear a ?* is absolutely necessary to make this definition possible.

Observe that we use in an essential way the fact that all formulas of the context are of shape $?H$ (even if $F$ is exponential-free) when we apply contraction rules on this context.

4) Cut elimination: The only reduction that we will mention here is (μ-fold)/(ν-rec). Let $θ$ be

\[ \vdash Λ, F[μξ G/ξ] \]

(μ-fold)

\[ \vdash Λ, μξ F \]

(μ-rec)

\[ \vdash Λ, Δ, A^⊥ \]

(μ-rec)

\[ \vdash Λ, Δ, T, (μξ F)^⊥ \]

(cut)

and let $ρ'$ be the proof

\[ \vdash Λ, Δ, ?Γ, ?θ \]

(μ-rec')

Then $θ$ reduces to the proof shown in Figure[0x1.2]. This reduction rule uses the functoriality of formulas as well as the ?-contexts in the (ν-rec) rule.

Remark 7. In [0x1.1.0] it is shown that μMALL enjoys cut-elimination. We will show in a further paper how this method based on reducibility can be adapted to our μLL. Notice that a cut-free proof has not the sub-formula property in general because of rule (ν-rec). Though, the normalization theorem makes sure that a proof of a sequent which does not contain any ν-formula reduces to a cut-free proof enjoying the subformula property.

5) Interpreting formulas and proofs (outline): We assume to be given a μLL Seely model $(L, L)$, see Section [0x1.0.2]. With any formula $A$ and any repetition-free sequence $ζ = (ζ_1, …, ζ_k)$ of type variables containing all the free variables of $A$, we associate $[A]_{L_ζ} ∈ L_k$ in the obvious way, for instance $[A ⊗ B]_{L_ζ} := [A]_{L_ζ} ⊗ [B]_{L_ζ}$. Though, the normalization theorem makes sure that a proof of a sequent which does not contain any ν-formula reduces to a cut-free proof enjoying the subformula property. These two facts allow to define a term $Λ_ζ$ for each type $ζ$ and $[A]_{L_ζ}$, considered as $[A]_{L_ζ}$ to a natural isomorphism. In this outline, we keep symmetric monoidality isomorphisms of $L$ and of $L'$ implicit (see for instance [0x1.0.2] and [0x1.1.0] how monoidal trees allow to take them into account). With any $Γ = (A_1, …, A_n)$ we associate an object $[Γ]_L$ of $L$ and with any proof $π$ of $Γ$ we associate a morphism $[π] : L(Γ, [Γ])$ using the categorial constructs of $L$ in a straightforward way, see [0x1.1.0]. Then one proves that if $π$ and $π'$ are proofs of $Γ$ and $π$ reduces to $π'$ by the cut-elimination rules, then $[π] = [π']$. This is done by an inspection of the various cut-elimination rules. In the case of (μ-fold)/(ν-rec) cut-elimination, we need the following lemma.

Lemma 8. Let $Γ = (D_1, …, D_n)$ be a sequence of closed formulas and let $P = [D_1] \otimes … \otimes [D_n]$, considered as an object of $L'$. Let $F$ be a formula and $ζ, ξ_1, …, ξ_k$ be a repetition-free list of variables containing all the free variables of $F$ so that $F[ζ, ξ_1/ξ, A^⊥]$ is a strong functor $L_k^{k+1} → L$ which belongs to $L_k$. As shown in Section [0x1.0.2] this strong functor lifts to a functor $F[P] : L[P]^{k+1} → L[P]$. Let $π$ be
a proof of \( \vdash ?T, A^\perp, B \), so that \([\pi] \in \mathcal{L}[P](\{[A], [B]\})\). Let 
\( \bar{C} = (C_1, \ldots, C_k) \) be a list of closed formulas. Then

\[
\mathcal{L}[P](\bar{F}(\{[A], [C_1], \ldots, [C_k]\}))\
\mathcal{L}[P](\bar{F}(\{[B], [C_1], \ldots, [C_k]\}))
\]

The proof of the lemma is a simple verification. Notice that we use the fact that the objects of \( \mathcal{L}[P] \) are the same as those of \( \mathcal{L} \).

III. SETS AND RELATIONS

The category \( \text{Rel} \) has sets as objects, and given sets \( E \) and \( F \), \( \text{Rel}(E, F) = \mathcal{P}(E \times F) \). Identity is the diagonal relation and composition is the usual composition of relations, denoted by simple juxtaposition. If \( t \in \text{Rel}(E, F) \) and \( a \subseteq E \) then \( t \cdot a = \{ b \in F \mid \exists a \in a \cup (a, b) \in t \} \).

A. \( \text{Rel} \) as a model of LL

This category is a well-known model of LL in which \( 1 = \perp = \{ \ast \} \), \( E \otimes F = (E \rightarrow F) = (E \forall F = E \times F \) so that \( E^\perp = E \). As to the additives, \( 0 = ? = \emptyset \) and \( \mathbf{E}_i \in \text{Rel}(E, F) \). The exponentials are given by \( \forall E = ?E = \mathcal{M}_{\text{fin}}(E) \) (finite multisets of elements of \( E \)) which are functions \( m : E \rightarrow \mathbb{N} \) such that \( m(a) \neq 0 \) for finitely many \( a \)'s; addition of multisets is defined in the obvious pointwise way, and \( [a_1, \ldots, a_k] \) is the multiset which maps \( a \) to the number of \( \ast \)'s such that \( a_1 = a \). For the additives and multiplicative, the operations on morphisms are defined in the obvious way. Let us be more specific about the exponentials. Given \( s \in \text{Rel}(E, F) \), \( s \in \text{Rel}(E, F) \) is \( \ast \) \( s \in \{ (a_1, \ldots, a_n), [b_1, \ldots, b_n] \mid n \in \mathbb{N} \) and \( \forall i \in \{ a_i, b_i \} \) \( b_i \in \text{Rel}(E, F) \).

B. Locally continuous functors on \( \text{Rel} \)

The following considerations on continuity of functors are standard, see [37]. A functor \( \mathbb{F} : \text{Rel} \rightarrow \text{Rel} \) is locally continuous if, for all \( \bar{E}, \bar{F} \in \text{Rel}^n \) and all directed set \( D \subseteq \text{Rel}^n(\bar{E}, \bar{F}) \), one has \( \mathbb{F}(\bigcup D) = \bigcup \{ \mathbb{F}((\bar{S}) \mid \bar{s} \in D) \} \) (we use \( \bigcup D \) for the component-wise union). This implies in particular that if \( \bar{s} \subseteq \bar{r} \), one has \( \mathbb{F}(\bar{s}) \subseteq \mathbb{F}(\bar{r}) \) (taking \( D = \{ \bar{s}, \bar{r} \} \)). To simplify notations assume that \( n = 1 \) (but what follows holds for all values of \( n \)).

Lemma 9. Let \( E \) and \( F \) be sets and let \( s \in \text{Rel}(E, F) \) and \( t \in \text{Rel}(F, E) \). Assume that \( t s = \text{Id}_E \) and that \( t s \subseteq s F \). Then \( s \) is the graph of an injective function and \( t = \{ (b, a) \in F \times E \mid (a, b) \in s \} \).

Lemma 10. Let \( \mathbb{F} : \text{Rel} \rightarrow \text{Rel} \) be a locally continuous functor. Assume that \( E \subseteq F \) and let \( \eta_{E,F}^+ = \{ (a, a) \mid a \in E \} \in \text{Rel}(E, F) \) and \( \eta_{E,F}^- = \{ (a, a) \mid a \in E \} \in \text{Rel}(F, E) \). Then \( \mathbb{F}(\eta_{E,F}^+) \in \mathbb{F}(\text{Id}_E) \) is an injective function.

\[ \mathbb{F} : \text{Rel} \rightarrow \text{Rel} \] is a strong functor when \( \mathbb{F} \) maps inclusions to injections, we shall call \( \mathbb{F} \) as the identity on objects. Obviously, \( \text{Rel} \) is complete.

Proposition 11. If \( \mathbb{F} : \text{Rel} \rightarrow \text{Rel} \) is locally continuous then \( \mathbb{F} \eta^+ : \text{Rel} \rightarrow \text{Rel} \) is directed-cocontinuous (that is, preserves the colimits of directed sets of objects).

The proof can be found in [37]. We know that a locally continuous functor \( \mathbb{F} \) maps inclusions to injections, we shall say that \( \mathbb{F} \) is strict if it maps inclusions to inclusions, that is, if \( E \subseteq F \) then \( \mathbb{F}(E) \subseteq \mathbb{F}(F) \) and \( \mathbb{F}(\eta_{E,F}^+) = \eta_{\mathbb{F}(E),\mathbb{F}(F)}^+ \) (which implies \( \mathbb{F}(\eta_{E,F}^-) = \eta_{\mathbb{F}(E),\mathbb{F}(F)}^- \)). As a direct consequence of Proposition 11 we get:

Lemma 12. If \( \mathbb{F} \) is strict locally continuous then, for any directed set of sets \( D \), one has \( \mathbb{F}(\bigcup D) = \bigcup_{E \in D} \mathbb{F}(E) \).

C. Variable sets and basic constructions on them

Definition 13. An \( n \)-ary variable set is a strong functor \( \mathbb{V} : \text{Rel}^n \rightarrow \text{Rel} \) such that \( \mathbb{V} \) is locally continuous and strict.

\[ \text{rel} \] Notice that it is not complete, for instance is has no final object.
By the general considerations of Section II-D there is a constant strong functor $Rel^n \to Rel$ with value $E$ for each set $E$. There are projection strong functors $Rel^n \to Rel$, $\times$ (that is $\otimes$) and $+$ (that is $\oplus$) define strong functors $Rel^2 \to Rel$, $\mathcal{M}_{\otimes}(\cdot)$ (that is $\amalg$) define a strong functor $Rel \to Rel$. Strong functors on $Rel$ are stable under composition, and if $V$ is a strong functor $Rel^n \to Rel$ then there is a “dual” strong functor $V^\perp$ (which is actually identical to $V$ in this very simple model). To check that these strong functors $V$ are variable sets we have only to check that the underlying functors $\nabla$ are strictly locally continuous.

We deal with $\nabla$ and composition, the other cases are similar. We already defined the functor $\nu$ in Section III-A. If $s \subseteq t \in Rel(E,F)$, it follows from the definition that $!s \subseteq !t$. Let $D \subseteq Rel(E,F)$ be directed, we prove $!((\bigcup D) \subseteq \bigcup_{s \in D}!s)$: an element of $!(\bigcup D)$ is a pair $((a_1, \ldots, a_k), (b_1, \ldots, b_k))$ with $(a_i, b_i) \in D$ for $i = 1, \ldots, k$. Since $D$ is directed, there is an $s \in D$ such that $(a_i, b_i) \in s$ for $i = 1, \ldots, k$ and the inclusion follows. Strictness is obvious.

Let $V_i : Rel_i \to Rel$ be variable sets for $i = 1, \ldots, k$ and let $\mathcal{W} : Rel^k \to Rel$ be a variable set. Then the functor $\mathcal{W} \circ V : Rel^n \to Rel$ is clearly strictly locally continuous (since these conditions are preservation properties) from which it follows that the strong functor $U = \mathcal{W} \circ V$ is a variable type.

1) Fixed point of a variable set: Let $F : Rel \to Rel$ be a strictly local continuous functor. Since $0 \subseteq \emptyset(0)$ we have $F^n(\emptyset) \subseteq F^{n+1}(\emptyset)$ for all $n \in \mathbb{N}$, by induction on $n$ and hence $F(\bigcup_{n=0}^\infty F^n(\emptyset)) = \bigcup_{n=0}^\infty F^n(\emptyset)$ by Lemma [12] since $\{F^n(\emptyset) | n \in \mathbb{N}\}$ is directed. Let $sF = \bigcup_{n=0}^\infty F^n(\emptyset)$, so that $(sF, \text{id}_{sF})$ is an $F$-coalgebra.

**Lemma 14.** The coalgebra $(sF, \text{id})$ is final in $\text{Coalg}_{\text{Rel}}(F)$.

Notice that $(sF, \text{id})$ is also an initial object in $\text{Alg}_{\text{Rel}}(F)$. When we insist on considering $sF$ as a final coalgebra, we denote it as $\nabla F$.

**Lemma 15.** Let $F : Rel^{n+1} \to Rel$ be a strictly locally continuous functor. The functor $\nabla F : Rel^n \to Rel$ is strictly locally continuous.

Let $V : Rel^{n+1} \to Rel$ be a variable set, by Lemma [5] there is a unique strong functor $\nabla V : Rel^n \to Rel$ which is characterized by $\nabla V(E) = \nabla V(F)$, for each $s \in \text{Rel}^n(E,F)$, $\nabla V(s) = \nabla V(\hat{s})$ and last $\nabla V(E \otimes F) = \nabla V(E,F)$.

**Lemma 16.** The functor $\nabla$ is a variable set.

**Proof:** By the conditions above satisfied by $\nabla$ we have that $\nabla \nabla = \nabla$ and hence $\nabla$ is strictly locally continuous by Lemma [15].

2) A model of $\mu\text{LL}$ based on variable sets: Let $\text{Rel}_n$ be the class of all $n$-ary variable sets, so that $\text{Rel}_0 = \text{Obj}(\text{Rel})$. The fact that $(\text{Rel}_n, (\text{Rel}_n)_{n \in \mathbb{N}})$ is a Seely model of $\mu\text{LL}$ in the sense of Definition [2] results mainly from the fact that we take all variable sets in the $\text{Rel}_n$’s so that there is essentially nothing to check. More explicitly: [1] holds by Section II-A, [2] holds by construction, [3] holds by the fact that variable sets compose as explained in Section II-C (notice that this condition refers to the general composition of strong functors defined at the beginning of Section II-D1), [4] holds by Section II-C and by the fact that the De Morgan dual of a strong functor is strong, see Section II-D1 and [5] holds by Lemma [16].

IV. NON-UNIFORM TOTALITY SPACES

We enrich the model of Section III with a notion of totality, we use notations from that section for operations on sets and relations.

A. Basic definitions.

Let $E$ be a set and let $T \subseteq \mathcal{P}(E)$. We define $T^\perp = \{u' \subseteq E | \forall u \in T \ u \cap u' \neq \emptyset\}$. If $S \subseteq T \subseteq \mathcal{P}(E)$ then $T^\perp \subseteq S^\perp$.

We also have $T \subseteq T^\perp$ and therefore $T^{\perp \perp} = T^\perp$. The biorthogonal closure has a nice and simple characterization.

**Lemma 17.** Let $T \subseteq \mathcal{P}(E)$, then $T^{\perp \perp} = \uparrow T = \{v \subseteq E | \exists u \in T \ u \subseteq v\}$.

**Proof:** The $\supseteq$ direction is obvious, the converse is not difficult either: let $u \in T^\perp$. Then $E \setminus u \notin T^\perp$, so there is $v \in T$ such that $v \cap (E \setminus u) = \emptyset$, that is $v \subseteq u$. Hence $u \in \uparrow T$.

A non-uniform totality space (NUTS) is a pair $X = ([X], T(X))$ where $|X|$ is a set and $T(X) \subseteq \mathcal{P}(|X|)$ satisfies $T(X) = T(X)^{\perp \perp}$, that is $T(X) = \uparrow T(X)$. Of course we set $X^\perp = ([X], T(X)^\perp)$.

**Example 18.** Let $X = (\mathbb{N}, T(X))$ where $T(X)$ is the set of all infinite subsets of $\mathbb{N}$. It is a NUTS because a superset of an infinite set is infinite. Then $|X^\perp| = \mathbb{N}$ and $T(X)^\perp$ is the set of all cofinite subsets of $\mathbb{N}$ (the subsets $u$ of $\mathbb{N}$ such that $\mathbb{N} \setminus u$ is finite). If, with the same web $\mathbb{N}$, we take $T(X) = \{u \subseteq \mathbb{N} | u \neq \emptyset\}$ (again $T(X) = \uparrow T(X)$ obviously), then $T(X)^\perp = \{\emptyset\}$.

We define four basic NUTS: $0 = (\emptyset, \emptyset)$, $\top = (\emptyset, \{\emptyset\})$. Given NUTS $X_1$ and $X_2$ we define a NUTS $X_1 \otimes X_2$ by $|X_1 \otimes X_2| = |X_1| \times |X_2|$ and $T(X_1 \otimes X_2) = \uparrow \{u_1 \otimes u_2 | u_1 \in T(X_1) \text{ for } i = 1, 2\}$ where $u_1 \otimes u_2 = u_1 \times u_2$. And then we define $X \rightarrow Y = (X \otimes Y)^\perp$.

**Lemma 19.** $t \in T(X \rightarrow Y) \iff \forall u \in T(X) \ t \cdot u \in T(Y)$.

We define the category $\text{Nuts}$ whose objects are the NUTS and $\text{Nuts}(X, Y) = T(X \rightarrow Y)$, composition being defined as the usual composition in $\text{Rel}$ (relational composition) and identities as the diagonal relations. Lemma [19] shows that we have indeed defined a category.

1) Multiplicative structure:

**Lemma 20.** Let $X$ and $Y$ be NUTS and $t \in \text{Nuts}(X, Y)$. Then $t$ is an iso in $\text{Nuts}$ iff $t$ is (the graph of) a bijection $|X| \rightarrow |Y|$ such that $\forall u \subseteq |X| \ u \in T(X) \leftrightarrow t(u) \in T(Y)$.
Lemma 21. Let \( t \subseteq |X| \times |Y| \). One has \( t \in \text{Nuts}(X, Y) \) iff \( t^\perp = \{(a, b) \mid (a, b) \in t\} \in \text{Nuts}(Y^\perp, X^\perp) \).

Proof: This is an obvious consequence of Lemma 19 and of the fact that \((X \to Y) = (X \otimes Y^\perp)^\perp\) and \((Y^\perp \to X^\perp) = (Y \otimes X)^\perp\).

Lemma 22. Let \( t \subseteq |X_1 \otimes X_2| \to Y \). One has \( t \in \text{Nuts}(X_1 \otimes X_2, Y) \) iff for all \( u_1 \in T(X_1) \) and \( u_2 \in T(X_2) \) one has \( t \cdot (u_1 \otimes u_2) \in T(Y) \).

Lemma 23. The bijection \( \alpha_{|X_1|,|X_2|,|Y|} \) is an isomorphism from \((X_1 \otimes X_2) \to Y \) to \( X_1 \to (X_2 \to Y) \).

We turn now \( \otimes \) into a functor, its action on morphisms being defined as in \( \text{Rel} \). Let \( t \in \text{Nuts}(X_1, Y) \) for \( i = 1, 2 \), we have \( t_1 \otimes t_2 \in \text{Nuts}(X_1 \otimes X_2, Y_1 \otimes Y_2) \) by Lemma 22 and by the equation \((t_1 \otimes t_2) \cdot (u_1 \otimes u_2) = (t_1 \cdot u_1) \otimes (t_2 \cdot u_2) \). This functor is monoidal, with unit \( 1 \) and symmetric monoidality isomorphisms \( \lambda, \rho, \gamma \) and \( \alpha \) defined as in \( \text{Rel} \). The only non-trivial thing to check is that \( \alpha \) is indeed a morphism, namely
\[
\alpha_{|X_1|,|X_2|,|Y|} \in \text{Nuts}(X_1 \otimes X_2 \otimes X_3, X_1 \otimes (X_2 \otimes X_3)).
\]
This results from Lemma 23 and from the observation that
\[
(\otimes (X_1 \otimes X_2) \otimes X_3) = (X_1 \otimes (X_2 \otimes X_3)),
\]
and hence it is \( \alpha \)-autonomous with dualizing object \( 1 = 1 \).

2) Additive structure: Let \((X_i)_{i \in I}\) be an at most countable family of objects of \( \text{Nuts} \). We define \( X = \bigoplus_{i \in I} X_i \) by:
\[|X| = \bigcup_{i \in I} \{i\} \times |X_i| \text{ and } T(X) = \{u \subseteq |X| \mid \forall i \in I \; \pi_i \cdot u \in T(X_i)\} \text{ where } \pi_i = \{((a, i), a) \mid a \in |X_i|\}.\]
It is clear that \( T(X) = \uparrow T(X) \) and hence \( X \) is an object of \( \text{Nuts} \). By definition of \( X \) and by Lemma 19 we have \( \forall i \in I \; \pi_i \in \text{Nuts}(X, X_i) \). Given \( \bar{t} = (t_i)_{i \in I} \in \text{Nuts}(Y, X) \), we have \( \bar{t} \in \text{Nuts}(Y, X) \) as easily checked (using Lemma 19 again). It follows that \((\bigoplus_{i \in I} X_i, (\pi_i)_{i \in I})\) is the cartesian product of the \( X_i \)’s in \( \text{Nuts} \). This shows that the category \( \text{Nuts} \) has all countable products and hence is cartesian. Since \( \otimes \) is autonomous, the category \( \text{Nuts} \) is also cocartesian, coproduct being given by \( \bigoplus_{i \in I} X_i = (\bigoplus_{i \in I} X_i^\perp)^\perp \). It follows that \( X = \bigoplus_{i \in I} X_i = (\bigoplus_{i \in I} X_i^\perp)^\perp \) satisfies \(|X| = \bigcup_{i \in I} \{i\} \times |X_i| \) and
\[T(X) = \{v \subseteq \bigcup_{i \in I} \{i\} \times |X_i| \mid \exists i \in I \; \exists u \in T(X_i) \{i\} \times u \subseteq v\} \]
as easily checked. Notice that the final object is \( T(\emptyset, \emptyset) \) and that \( 0 = \perp = (\emptyset, \emptyset) \).

3) Exponential: We extend the exponential of \( \text{Rel} \) with totality. If \( u \in \mathcal{P}(|X|) \) we set \( u^{(0)} = \mathcal{M}_{\text{fin}}(u) \in \mathcal{P}(|X|) \). Then we set \(|X|^* = \mathcal{M}_{\text{fin}}(|X|) \) and \( T(!X) = \{u^{(0)} \mid u \in T(X)\} \).
$\text{Nuts} \rightarrow \text{Rel}$ which maps $X$ to $|X|$ and acts as the identity on morphisms (p is right adjoint to u). Let $X$ be a unary VNUTS and let $\mathcal{X} : \text{Nuts} \rightarrow \text{Nuts}$ be the associated strong functor. Then we have $|X| = u \circ \mathcal{X} \circ p$ and $|X|_{E,F} = \mathcal{X}(p(E),p(F))$ for any sets $E$ and $F$. Last, given a NUTS $X$, we have that $\mathcal{T}(X)(X)$ is just the totality component of the NUTS $\mathcal{X}(X)$. This shows that the VNUTS which induces $\mathcal{X}$ can be retrieved from $\mathcal{X}$.

For this reason we use $X$ to denote the functor $\mathcal{X}$.

**Remark 8.** Another possibility would be to define a VNUTS at the first place as a strong functor $\mathcal{X} : \text{Nuts}^n \rightarrow \text{Nuts}$ satisfying additional properties whose purpose would be to make the definition of the underlying $X$ possible. This option, suggested by the reviewers, will be explored further. It is crucial to notice that these additional properties (that is, the existence of $\mathcal{X}$) are crucial in the proof of Theorem 30. In particular, it is essential that $|\mathcal{X}(X)|$ depends only on $|X|$.

Given $n \in \mathbb{N}$ let $\text{VNuts}_n$ be the class of strong $n$-ary VNUTS. We identify $\text{VNuts}_0$ with the class of objects of the Seely category $\text{Nuts}$. The following refers to Definition 7.

**Theorem 30.** $(\text{Nuts}(\text{VNuts}_n),n \in \mathbb{N})$ is a Seely model of $\mu_{\text{LL}}$.

**Partial proof:** We condition on $\mathcal{X}$. Let first $X = (|X|,\mathcal{T}(X))$ be a unary VNUTS. Let $E = \sigma|X|$ which is the least set such that $|X|(E) = E$, that is $E = \bigcup_{n=0}^{\infty} |X|^n(\emptyset)$. Let $\Phi : \text{Tot}(E) \rightarrow \text{Tot}(E)$ be defined as follows: given $S \in \text{Tot}(E)$, then $(E, S)$ is a NUTS, and we set $\Phi(S) = \mathcal{T}(X)(E, S) \in \text{Tot}(|X|(E)) = \text{Tot}(E)$. This function is monotone. Therefore $S_1, S_2 \in \text{Tot}(E)$ with $S_1 \subseteq S_2$. Then we have $\text{Id} \in \text{Nuts}((E, S_1), (E, S_2))$ and therefore, by Condition 1 satisfied by $X$, we have $\text{Id} = |X|(\text{Id}) \in \text{Nuts}(|X|(E, S_1), |X|(E, S_2)) = \text{Nuts}((E, \Phi(S_1)), (E, \Phi(S_2)))$ which means that $\Phi(S_1) \subseteq \Phi(S_2)$. By the Knaster Tarski Theorem (remember that $\text{Tot}(E)$ is a complete lattice), $\Phi$ has a greatest fixpoint $T$ that we can describe as follows. Let $\mathcal{T}_\lambda = \mathcal{T}(E)$, where $\mathcal{O}$ is the class of ordinals, be defined by: if $\mathcal{T}_0 = \mathcal{P}(E)$ (the largest possible notion of totality on $E$), then $\mathcal{T}_{\alpha+1} = \mathcal{T}(\mathcal{T}_\alpha)$ and $\mathcal{T}_\lambda = \bigcap_{\beta < \lambda} \mathcal{T}_\beta$ when $\mathcal{T}$ is a limit ordinal. This sequence is decreasing (easy induction on ordinals using the monotonicity of $\mathcal{X}$) and there is an ordinal $\theta$ such that $\mathcal{T}_{\theta+1} = \mathcal{T}_\theta$ (by a cardinality argument; we can assume that $\theta$ is the least such ordinal). The greatest fixpoint of $\Phi$ is then $\mathcal{T}_\theta$ as easily checked.

By construction $((E, \mathcal{T}_\theta), \text{Id})$ is an object of $\text{Coalg}_{\text{Nuts}}(\mathcal{X})$, we prove that it is the final object. So let $(Y, t)$ be another object of the same category. Since $(|Y|, t)$ is an object of $\text{Coalg}_{\text{Rel}}(\mathcal{X})$ and since $(E, \text{Id})$ is the final object in that category, we know by Lemma 11 that there is exactly one $e \in \text{Rel}(|Y|, E)$ such that $|X|(e) \cdot t = e$. We prove that actually $e \in \text{Nuts}(Y, (E, \mathcal{T}_\theta))$ so let $v \in \mathcal{T}(Y)$. We prove by induction on the ordinal $\alpha$ that $e \cdot v \in \mathcal{T}_\alpha$. For $\alpha = 0$ it is obvious since $\mathcal{T}_0 = \mathcal{P}(E)$. Assume that the property holds for $\alpha$ and let us prove it for $\alpha + 1$. We have $t \cdot v \in \mathcal{T}(X)(Y) = \mathcal{T}(\mathcal{T}(Y))$ and since $\mathcal{T}(e) \in \text{Nuts}(|X|, (E, \mathcal{T}_\theta))$ and $\mathcal{T}(E, \mathcal{T}_\theta) = (E, \mathcal{T}_{\alpha+1})$ we have $|X|(e) \cdot t \cdot v \in \mathcal{T}_{\alpha+1}$. Since $\mathcal{T}$ is a limit ordinal and if we assume $\forall \alpha < \lambda, e \cdot v \in \mathcal{T}_\alpha$ we have $e \cdot v \in \bigcap_{\alpha < \lambda} \mathcal{T}_\alpha = \mathcal{T}_\alpha$. Therefore $e \cdot v \in \mathcal{T}_\theta$. We use $\mathcal{T}(X)(e)$ to denote this final coalgebra $(E, \mathcal{T}_\theta)$ (its definition depends only on $X$ and does not involve the strength $\mathcal{X}$).

So we have proven the first part of Condition 5 in the definition of a Seely model of $\mu_{\text{LL}}$ (see Section 1). As to the second part, let $X$ be an $n + 1$-ary VNUTS. We know by the general Lemma 5 how to build a strong functor $\nu : \text{Nuts}^n \rightarrow \text{Nuts}$ with suitable properties. To end the proof, it suffices to exhibit an $n$-ary VNUTS $\nu = (|\nu|, \mathcal{T}(\nu))$ whose associated strong functor coincides with $\nu X$. The construction of $\nu X$ is essentially straightforward, using the constructions available in $\text{Rel}$.

**Remark 9.** For any closed formula $A$, the web of its interpretation $[A]_{\text{Nuts}}$ in $\text{Nuts}$ coincides with its interpretation $[A]_{\text{Rel}}$ in $\text{Rel}$. It is also easy to check that for any proof $\pi$ of $\vdash A$, one has $[\pi]_{\text{Nuts}} = [\pi]_{\text{Rel}}$ (this can be formalized using the functor $u : \text{Nuts} \rightarrow \text{Rel}$ introduced in the proof of Lemma 29 which acts trivially on morphisms).

**Remark 10.** The same method can be applied in many contexts. For instance, we can replace $\text{Rel}$ with the category of coherence spaces – where least and greatest fixpoints are interpreted in the same way – and $\text{Nuts}$ with the category of coherence spaces with totality where the interpretations will be different. One of the reviewers suggested that this situation might be generalized using the concept of topological functors, this option will be explored further work.

**C. Examples of data-types**

1) **Integers:** The type of “flat integers” is defined by $\iota = \mu X(1 \oplus \zeta)$. In $\text{Rel}$, $1 \oplus \zeta$ is interpreted as the unary variable set $[1 \oplus \zeta]_{\text{Rel}} : \text{Rel} \rightarrow \text{Rel}$ which maps a set $E$ to $1 \oplus E = \{1, E\} \cup \{(2, E) \times E\}$. Hence $[\iota]_{\text{Rel}}$ is the least set such that $[\iota] = \{1\} \cup \{(2 \times x, x)\}$ that is, the set of all tuples $\pi = (2, (2, \cdots (1, \cdots \cdots)) \cdots))$ where $n$ is the number of occurrence of 2 so that we can consider the elements of $[\iota]$ as integers. We have $|[\iota]_\text{Nuts}| = [\iota]_{\text{Rel}}$ and we compute $\mathcal{T}([\iota]_{\text{Nuts}})$ dually wrt. the proof of Theorem 30 it is the least fixed point of the operator $\Phi : \text{Tot}([\iota]_{\text{Rel}}) \rightarrow \text{Tot}([\iota]_{\text{Rel}})$ such that, if $T \in \text{Tot}([\iota]_{\text{Rel}})$ then $\Phi(T) = \{ u \subseteq [\iota]_{\text{Rel}} | [u]_{\text{Rel}} \subseteq [\pi + 1]_{\text{Rel}} \}$. Therefore $\text{Tot}([\iota]_{\text{Nuts}}) = \{ u \subseteq [\iota]_{\text{Rel}} | u \neq \emptyset\}$.

**Theorem 31.** If $\pi$ is a proof of $\vdash \iota$ then $[\pi]_{\text{Nuts}} = [\pi]_{\text{Rel}}$ is a non-empty subset of $[\iota]_{\text{Rel}}$.

Indeed we know that $|[\iota]_{\text{Rel}} = [\pi]_{\text{Nuts}} \in \mathcal{T}([\iota]_{\text{Nuts}})$. Using an additional notion of coherence (which can be fully compatible with $\text{Rel}$ as in the non-uniform coherence space models of [13, 14]) we can also prove that $[\iota]_{\text{Rel}}$ has at most one element, and hence is a singleton \{u\}. This is a denotational version of normalization expressing that indeed $\pi$ “has a value” (and actually exactly one, which expresses a weak form of confluence).
2) Binary trees with integer leaves: This type can be defined as $τ = \mu Nτ(ι \otimes (ζ \otimes ζ))$. Then an element of $[τ]^{Rel}$ is an element of the set described by the following syntax: $α, β, \cdots \vdash (n) | (α, β)$. A computation similar to the previous one shows that $\text{Tot}([τ]^{Nuts}) = \{u \subseteq [τ]^{Rel} | u \neq \emptyset\}$.

3) An empty type of streams of integers: After reading [6], one could be tempted to define the type of streams of integers as $σ_0 = νζ(ι \otimes ζ)$. The variable set $[ι \otimes ζ]^{Rel} : \text{Rel} → \text{Rel}$ maps a set $E$ to $N × E$. The least fixed point of this operation on sets is $0$ and hence $[σ_0]^{Nuts} = 0$ and notice that $\text{Tot}(0) = \{\emptyset, \{\emptyset\}\}$. In that case, the operation $Φ : \text{Tot}(0) → \text{Tot}(0)$ maps $T$ to $\{u \cup v | u ∈ T$ and $u ∈ P(N) \setminus \{\emptyset\}\}$ and hence $\{\emptyset\}$ to itself. It follows that $T(\{σ_0\}^{Nuts}) = \emptyset$, that is $[σ_0]^{Nuts} = T$, the final object of $\text{Nuts}$. What is the meaning of this trivial interpretation? It simply reflects that, though $σ_0$ has a lot of non-trivial proofs in $\mu LL$, it is impossible to extract any finite information from these proofs within $\mu LL$, and accordingly all these proofs are interpreted as $\emptyset$.

**Theorem 32. In $\mu LL$, there is no proof of $σ_0^⊥, τ$.**

In other words there is no proof of $σ_0^⊥ → τ$ in $\mu LL$: typically a function extracting the first element of a stream would be a proof of this type… if it would exist! Here is the argument: if $π$ were a proof of $σ_0^⊥ → τ$, we would have $[π] ∈ \text{Nuts}(\{σ_0\}^{Nuts}, \{\emptyset\}^{Nuts})$ and hence $[π] : 0 ∈ T(\{σ_0\}^{Nuts})$ which is not the case since $[π] : 0 = \emptyset$. If types like $σ_0$ are meaningful in a proof-search perspective, their relevance as data-types in a Curry-Howard approach to $\mu LL$ is dubious.

4) A non-empty type of streams of integers: We now set $σ = νζ(1 & (ι \otimes ζ))$. This type looks like the previous one, but the type 1 leaves space for partial empty streams. Warning: it is not a type of finite or infinite streams; the & means that this empty stream will not be a total element: it will have to be complemented by some total element from the right argument of the &. More precisely $[1 & (N \otimes ζ)]^{Rel} : \text{Rel} → \text{Rel}$ is the variable set which maps a set $E$ to $\{(1, *)\} \cup \{2\} × N × E$ so that up to renaming $[σ]^{Nuts} = N^{ωω}$ (all finite sequences of integers). In this case, the operator $Φ : \text{Tot}(N^{ωω}) → \text{Tot}(N^{ωω})$ maps $T$ to 

$$\{v \subseteq N^{ωω} | (\emptyset) ∈ v \text{ and } 3n ∈ N, u ∈ T \{n\} × u ≤ v\}$$

where we use () for the empty sequence. So for instance $Φ^0(P(N^{ωω})) = P(N^{ωω})$ $Φ^1(P(N^{ωω})) = \{u ∈ P(N^{ωω}) | (\emptyset) \in u\}$ $Φ^2(P(N^{ωω})) = \{u ∈ P(N^{ωω}) | 3n, τ, n_1, n_2, (n_1, n_1, n_2) ∈ u\}.

The greatest fixed point is reached in $ω$ steps:

$$\text{Tot}([σ]^{Nuts}) = \bigcap_{n < ω} Φ^n(P(N^{ωω}))$$

$$= \{u \subseteq N^{ωω} | 3f ∈ N^{ωω} ∀k < ω (f(1), \ldots, f(k)) ∈ u\}.$$ 

So a total subset of $[σ]^{Nuts}$ must contain (at least) an infinite stream of integer. For this type of streams $σ$ it is easy to build a proof of $σ^⊥, τ$ extracting the first element of a stream, interpreted as $\{(n), n | n ∈ N\}$.

**V. Conclusion and further work**

We will study next the semantics of infinite proofs of $\mu LL$ (whose definition extends that of [6] for $µ\text{MALL}$). A crucial step is to prove that these infinite proofs can be interpreted as total sets in $\text{Nuts}$, this will be presented in a further paper. This interpretation of proofs is based on the interpretation of their finite approximations in $\text{Rel}$ (remember that the interpretations of a $\mu LL$ proof in $\text{Nuts}$ and in $\text{Rel}$ are exactly the same set).

Our models will also serve as guidelines for the design of a functional language based on $\mu LL$, generalizing Gödel’s System $T$ in the spirit of [22] though, as explained in the Introduction, Loader’s syntax is not fully compatible with LL as it is based on cocartesian cartesian closed categories. Our system will primarily implement Park’s rule, but we will also consider other options based on polymorphism in the spirit of [38], [19] or [39], or general recursion with guardedness restrictions as in [25], [26], [27].

Its syntax will be based on the idea of representing data-types as positive formulas of $\mu LL$ interpreted in $\ell'$ and therefore equipped with morphisms implementing weakening, contraction and promotion: as noticed in [4], $\mu_\text{LL}$ is a positive operarion whereas $νζ$ is negative. In $\ell'$, the $∅$ of LL is a coproduct and the $\otimes$ is a cartesian product as expected (and $\otimes$ distributes over $+$). The targeted calculus will feature a notion of values (positive terms) accounting for the morphisms of $\ell'$, substitution in terms being allowed only for values because only them can safely be discarded and duplicated. Thanks to this choice of design, weakening and contraction will remain implicit operations as in the usual $\lambda$-calculus. Our calculus will have explicit promotion and dereliction operations, allowing to implement both CBN and CBV in the same setting, just as in Levy’s Call-by-push-value [40], [41].

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VI. APPENDIX

A. Proof of Lemma 4

Proof: We have $F(g, \nu F(B)) \in \mathcal{A}(\nu F(B), F(B', \nu F(B)))$ thus defining an $F$-coalgebra structure on $\nu F(B)$ and hence there exists a unique morphism $\nu F(g)$ such that

$$F(B', \nu F(g)) \circ F(g, \nu F(B)) = \nu F(g),$$

that is $F(g, \nu F(g)) = \nu F(g)$.

Functionality follows: consider also $g' \in B(B', B'')$, then we know that $h = \nu F(g')$ satisfies $F(g', h)$ by the definition above. Now $h' = \nu F(g') \nu F(g)$ satisfies the same equation by functionality of $F$ and because $F(g, \nu F(g)) = \nu F(g)$ and $F(g', \nu F(g')) = \nu F(g')$, and hence $h' = h$ by Lemma 3 taking $l = F(g', \nu F(B))$. In the same way one proves that $\nu F(Id) = Id$.

B. Proof of Lemma 5

Proof: The part of the statement which concerns the functor $\nu F$ is a direct application of Lemma 4 so we only have to deal with the strength. Let us prove naturality so let $f \in L''(X, X')$ and $g \in L(Y, Y')$, we must prove that the following diagram commutes

$$
\begin{array}{ccc}
Y \otimes \nu F(X) & \xrightarrow{\nu F(Y \otimes X)} & \nu F(Y \otimes X') \\
!(g \otimes \nu F(f)) & & \downarrow \nu F(Y \otimes X') \\
!Y' \otimes \nu F(X') & \xrightarrow{\nu F(Y' \otimes X')} & \nu F(Y' \otimes X')
\end{array}
$$

Let $h_1 = \nu F(Y \otimes X') \circ (g \otimes \nu F(f)) \nu F(Y, X)$ and $h_2 = (g \otimes \nu F(f)) \nu F(Y, X)$ be the two morphisms we must prove equal. We use Lemma 4 taking the following morphism $l$.

$$!(Y \otimes \nu F(X)) = !Y \otimes F(X, \nu F(X))$$

With these notations we have

$$\nu F(Y' \otimes X', h_1) = \nu F(Y' \otimes X', h_2)$$

so that $\nu F(Y' \otimes X', h_1) l = h_1$ as required. On the other hand we have

$$\nu F(Y' \otimes X', h_2) l = \nu F(Y' \otimes X', \nu F(Y \otimes X', \nu F(Y, X)))$$

C. Proof of Lemma 9

Proof: Let $a, b \in E$, since $(a, a) \in Id_E = t s$, there must exist $b \in F$ such that $(a, b) \in s$ and $(b, a) \in t$. If $(a, b') \in s$ then $(b, b') \in s \subseteq Id_F$ and hence $b' = b$. It follows that $s$ is a total function $E \rightarrow F$. Let $(a, b) \in s$ (that is $a \in E$ and $b = s(a)$). Since $t s = Id_E$, we must have $(b, a) \in t$. Conversely let $(b, a) \in t$, we have $(b, s(a)) \in s$ and hence $b = s(a)$. We have proven that $t = \{(s(a), a) \mid a \in E\}$. If $(a, a') \in s$ satisfies $s(a) = s(a')$ we have therefore $(a, a') \in t s = Id_E$ and hence $a = a'$; this shows that $s$ is injective.

D. Proof of Lemma 14

Proof: Let $(E, t)$ be an $F$-coalgebra. We define a sequence $e_n \in \text{Rel}(E, \sigma F)$ as follows: $e_0 = \emptyset$ and $e_{n+1} = F(e_n)$. Then $e_n \subseteq e_{n+1}$ for all $n$ by an easy induction, using the fact that $F$ is locally continuous. Let $e = \bigcup_{n=0}^{\infty} e_n \in \text{Rel}(E, \sigma F)$, by local continuity of $F$ we have $F(e) t = \bigcup_{n=0}^{\infty} F(e_n) t = \bigcup_{n=0}^{\infty} e_n = e$ which means that

$$e \in \text{Coalg}_{\text{Rel}}(F)((E, t), (\sigma F, Id)).$$

We end the proof by showing that $e$ is the unique such morphism, so let $e' \in \text{Coalg}_{\text{Rel}}(F)((E, t), (\sigma F, Id))$ which means that $e' \in \text{Rel}(E, \sigma F)$ and $F(e') t = e'$.

Let $i_0 \in \text{Rel}(\sigma F, \sigma F)$ be defined by induction by $i_0 = \emptyset$ and $i_{n+1} = F(i_n)$. Then $(i_n)_{n \in \mathbb{N}}$ is monotone and $\bigcup_{n=0}^{\infty} i_n = Id$ by definition of $\sigma F$. We prove by induction on $n$ that $\forall n \in \mathbb{N} i_n e' = i_n e$. Clearly $i_0 e' = i_0 e = \emptyset$. Next

$$i_{n+1} e' = F(i_n) F(e') t = F(i_n e') t$$

by inductive hypothesis

$$= i_{n+1} e.$$

Therefore $e' = (\bigcup_{n \in \mathbb{N}} i_n) e' = \bigcup_{n \in \mathbb{N}} (i_n e') = \bigcup_{n \in \mathbb{N}} (i_n e) = e$. 

\[\blacksquare\]
E. Proof of Proposition [7]

Proof: Let \( \mathcal{D} \) be a directed set of sets and let \( H \) be a set. For each \( E \in \mathcal{D} \) let \( s_E \in \Rel(F(E), H) \) so that \((s_E)_{E \in \mathcal{D}}\) defines a cocone, that is, for each \( E, F \in \mathcal{D} \) such that \( E \subseteq F \), one has \( s_E = s_F \circ (\eta_{E,F}) \). Let \( L = \bigcup \mathcal{D} \). Let \( s \in \Rel(F(L), H) \) be given by \( s = \bigcup_{E \in \mathcal{D}} s_E \circ (\eta_{E,L}) \). Let \( E \in \mathcal{D} \), we have \( s_E \circ (\eta_{E,L}) = \bigcup_{E \in \mathcal{D}} s_F \circ (\eta_{E,F}) \circ (\eta_{F,L}) \) so that \( s_E \subseteq s \circ (\eta_{E,L}) \) (since \( s_E \circ (\eta_{E,F}) \circ (\eta_{F,L}) = s_E \) when \( F = E \)).

We prove the converse inclusion. Let \( F \in \mathcal{D} \) and let \( G \in \mathcal{D} \) be such that \( E, F \subseteq G \) (remember that \( \mathcal{D} \) is directed). We have

\[
s_F \circ (\eta_{E,F}^+) = s_F \circ (\eta_{G,F}^+) \circ (\eta_{G,L}^+) \circ (\eta_{E,G}^+)
\]

we have where we used the fact that \( \eta_{E,G}^+ \circ \eta_{F,G}^+ \subseteq \Id_G \) and hence \( \eta_{E,G}^+ \circ \eta_{F,G}^+ \subseteq \Id_D \) by local continuity of \( F \).

So \( s_F \circ (\eta_{E,L}^+) \subseteq s_E \) for all \( F \in \mathcal{D} \) and hence \( s \circ (\eta_{E,L}^+) \subseteq s \circ (\eta_{E,F}^+) \) as contended.

Let now \( s \in \Rel(F(L), H) \) be such that \( s \circ (\eta_{E,F}^+) = s_E \) for each \( E \in \mathcal{D} \), we show that \( s \) is \( s \) in \( s \) so that uniqueness the particular case of the universal property. For \( E \in \mathcal{D} \), let \( \theta_E = \eta_{E,L}^+ \in \Rel(L, H) \). Then \((\theta_E)_{E \in \mathcal{D}}\) is a directed family (for \( \subseteq \)) and \( \bigcup_{E \in \mathcal{D}} \theta_E = \Id_L \). By local continuity of \( F \), we have

\[
s = \bigcup_{E \in \mathcal{D}} s \circ (\eta_{E,F}^+) \circ (\eta_{E,L}^+) = s \circ (\eta_{E,L}^+) \subseteq s \circ (\eta_{E,F}^+)
\]

by our assumption on \( s \) and by definition of \( s \). This shows that the cocone \((F(\eta_{E,L}^+))_{E \in \mathcal{D}}\) on \( F \circ \eta^+ \) is collimating, thus proving that \( F \circ \eta^+ \) is directed cocontinuous.

F. Proof of Lemma [17]

Proof: As usual we assume that \( n = 1 \) to increase readability. We need to prove first that \( \nu\mathcal{F} \) is monoton on morphisms, so let \( s, t \in \Rel(F, E) \) with \( s \leq t \). We have \( \nu\mathcal{F}(s) = \bigcup_{n \in \mathbb{N}} s_n \) and \( \nu\mathcal{F}(t) = \bigcup_{n \in \mathbb{N}} t_n \) with \( s_0 = t_0 = 0 \), \( s_{n+1} = \mathcal{F}(s_n) \) and \( t_{n+1} = \mathcal{F}(t_n) \) (we use the action of \( \mathcal{F} \) on morphisms resulting from Lemma [14] and from the characterization of the morphisms to the final object given in the proof of Lemma [14]). By induction and homomorphism of \( \mathcal{F} \) we have \( \forall n \in \mathbb{N} \) \( s_n \leq t_n \) and hence \( \nu\mathcal{F}(s) \leq \nu\mathcal{F}(t) \). Let us prove now local continuity so let \( D \subseteq \Rel(F, E) \) be directed and let \( t = \bigcup D \), we prove that \( \nu\mathcal{F}(t) = \bigcup_{s \in D} \nu\mathcal{F}(s) \circ \Rel(F(E), \nu\mathcal{F}(F)) \) using Lemma [5] (with the notations of that lemma, we take \( t = \mathcal{F}(t, \nu\mathcal{F}(E)) \)). We have

\[
\mathcal{F}(\nu\mathcal{F}(t)) \mathcal{F}(t, \nu\mathcal{F}(E)) = \nu\mathcal{F}(t)
\]

by definition of the functor \( \nu\mathcal{F} \) and

\[
\mathcal{F}(t, \nu\mathcal{F}(E)) \mathcal{F}(t, \nu\mathcal{F}(E)) = \nu\mathcal{F}(t)
\]

In the second equation, we used the facts that \( D \) is directed and the monotonicity of \( \mathcal{F} \) and \( \nu\mathcal{F} \) on morphisms.

Let \( E \subseteq F \), we prove that \( \nu\mathcal{F}(E) \subseteq \nu\mathcal{F}(F) \). This results from the observation that if \( E' \subseteq F', \) then \( \mathcal{F}(E') \subseteq \mathcal{F}(F') \) and hence \( \forall n \in \mathbb{N} \) \( \mathcal{F}(E')(n) \subseteq \mathcal{F}(E')(n) \). Let us check that \( \nu\mathcal{F}(\eta_{E,F}) = \eta_{\nu\mathcal{F}(E),\nu\mathcal{F}(F)} \in \Rel(\nu\mathcal{F}(E), \nu\mathcal{F}(F)) \). We have

\[
\mathcal{F}(F, \nu\mathcal{F}(E)) \mathcal{F}(\eta_{E,F}, \nu\mathcal{F}(E)) = \mathcal{F}(\eta_{E,F}, \nu\mathcal{F}(E))
\]

by definition of the functor \( \nu\mathcal{F} \) and

\[
\mathcal{F}(F, \nu\mathcal{F}(E)) \mathcal{F}(\eta_{E,F}, \nu\mathcal{F}(E)) = \mathcal{F}(\eta_{E,F}, \nu\mathcal{F}(E))
\]

by strictness of \( \mathcal{F} \). The equation follows by Lemma [3] so that the functor \( \nu\mathcal{F} \) is strict.

G. Proof of Lemma [18]

Proof: Let \( t \in \mathcal{T}(X \rightarrow Y) \) and let \( u \in \mathcal{T}(X) \). Let \( v' \in \mathcal{T}(Y^-) \), since \( u \times v' \in \mathcal{T}(X \times Y^-) \) we have \( t \times u \times v' \neq \emptyset \) and hence \( (t \times u) \cap (u \times v') \neq \emptyset \). Therefore \( t \times u \in \mathcal{T}(Y^-) \). Conversely assume that \( \forall u \in \mathcal{T}(X) \) \( t \times u \in \mathcal{T}(Y^-) \). Let \( u \in \mathcal{T}(X) \) and \( v \in \mathcal{T}(X) \) \( t \times u \cap (u \times v') \neq \emptyset \) and hence \( t \times (u \times v') \neq \emptyset \) and this shows that \( t \in \mathcal{T}(X \rightarrow Y) \).

H. Proof of Lemma [20]

Proof: Assume that \( t \) is an iso in \( \mathcal{N} \) so that there is \( t' \in \mathcal{N}(Y, X) \) such that \( t \times t' = \Id_{X \times X} \) and \( t' \times t = \Id_{Y \times Y} \) and since we know that the isos in \( \mathcal{N} \) are the bijections we know that \( t \) is a bijection. The fact that \( \forall u \in \mathcal{T}(X) \) \( u \times t \in \mathcal{T}(X) \) \( t \times u \in \mathcal{T}(X) \) \( t \times u \times v' \neq \emptyset \) and hence \( (t \times t') \cap (t \times v') \neq \emptyset \) and this shows that \( t \in \mathcal{T}(X \rightarrow Y) \).

I. Proof of Lemma [22]

Proof: The condition is obviously necessary, let us prove that it is sufficient so assume that \( t \) fulfills it and let us prove that \( t \in \mathcal{T}(X_1 \times X_2 \rightarrow Y) \). To this end it suffices to prove that \( t \in \mathcal{T}(Y \rightarrow T) \). So let \( v' \in \mathcal{T}(Y^-) \) and let us prove that \( t \times v' \in \mathcal{T}(X \times X \rightarrow Y \rightarrow Y) \). Let \( u_i \in \mathcal{T}(X) \) for \( i = 1, 2 \). We know that \( t \times (u_i \times u_j) \in \mathcal{T}(Y) \) and hence \( (t \times (u_i \times u_j)) \cap (t \times v') \neq \emptyset \), that is \( (u_i \times u_j) \cap (t \times v') \neq \emptyset \), proving our contention.
J. Proof of Lemma 24
Proof: Let \( t \in \mathcal{T}((X_1 \times X_2) \rightarrow Y) \) and let us prove that \( s = \alpha \cdot t \in \mathcal{T}(X_1 \rightarrow (X_2 \rightarrow Y)) \). Given \( u_1 \in \mathcal{T}(X_1) \) is suffices to prove that \((t' \cdot u_1) \cdot u_2 \in \mathcal{T}(Y)\) which results from the fact that \((s \cdot u_1) \cdot u_2 = t \cdot (u_1 \otimes u_2)\). Conversely let \( s \in \mathcal{T}(X_1 \rightarrow (X_2 \rightarrow Y)) \) and let us prove that \((t' \cdot u_1) \cdot u_2 \in \mathcal{T}(X_1 \times X_2 \rightarrow Y)\). This results from Lemma 22 and from the equation \((s \cdot u_1) \cdot u_2 = t \cdot (u_1 \otimes u_2)\).

K. Proof of Lemma 25
Proof: The condition is obviously necessary, so let us assert that it holds. By Lemma 24 it suffices to prove that \((t' \cdot u_1) \cdot u_2 \in \mathcal{T}(Y)\). Let \( v' \in \mathcal{T}(Y) \) and let \( u \in \mathcal{T}(X) \), since \( t \cdot u(t') \in \mathcal{T}(Y) \) and hence \((t \cdot u(t')) \cap v' \neq \emptyset\), that is \((t' \cdot v') \cap u(t) \neq \emptyset\).

L. Proof of Lemma 26
Proof: We deal with the case \( k = 2 \). The condition is necessary since, if \( u_1 \in \mathcal{T}(X_1) \) and \( u_2 \in \mathcal{T}(X_2) \), then \( u_1 \otimes u_2 \in \mathcal{T}(X_1 \times X_2) \). So assume that it holds. Let \( t' = \text{cur}(t) \in \mathcal{R}el(\mathcal{V}_1 \rightarrow (\mathcal{V}_2 \rightarrow Y)) \). Let \( u \in \mathcal{T}(X_1) \), we have \( t' \cdot u(t') \in \mathcal{P}(\mathcal{V}_2 \rightarrow Y) \). Let \( u_2 \in \mathcal{T}(X_2) \), we have \((t' \cdot u(t')) \cdot u_2 \in \mathcal{T}(Y) \) by our assumption. It follows by Lemma 24 that \((t' \cdot u(t')) \in \mathcal{T}(\mathcal{V}_2 \rightarrow Y) \) and since this holds for any \( u \in \mathcal{T}(X_1) \) we actually have \( t' \in \mathcal{N}uts(X_1, X_2 \rightarrow Y) \). It follows that \( t = \text{cur}^{-1}(t') \in \mathcal{N}uts(X_1 \times X_2, Y) \) as contended.

M. Proof of Lemma 27
Proof: Given an object \( X \) of \( \mathcal{N}uts \), we set \( \mathcal{D}er_X = \mathcal{D}er_{\mathcal{V}_1} \in \mathcal{R}el(\mathcal{V}_1 \times X) \) and \( \mathcal{D}ig_X = \mathcal{D}ig_{\mathcal{V}_1} \in \mathcal{R}el(\mathcal{V}_1 \times X) \). Given \( u \in \mathcal{T}(X) \), we have \( \mathcal{D}er_X \cdot u(t') = u \in \mathcal{T}(X) \) and \( \mathcal{D}ig_X \cdot u(t') = u(t) \in \mathcal{T}(\mathcal{V}_1) \). It follows by Lemma 24 that \( \mathcal{D}er_X \in \mathcal{N}uts(X, X) \) and \( \mathcal{D}ig_X \in \mathcal{N}uts(X, X) \).

N. Full proof of Theorem 33
Proof: Concerning Condition 3, let \( (X_i)_{i=1}^n \) be elements of \( \mathcal{N}uts_n \) and let \( X \in \mathcal{N}uts_k \). Considering \( X \) and the \( X_i \)'s as strong functors, we know that \( \mathcal{X} \circ \mathcal{X} \) is a strong functor \( \mathcal{N}uts_n \rightarrow \mathcal{N}uts \). We simply have to exhibit a \( \mathcal{N}uts \) whose associated strong functor is \( \mathcal{X} \circ \mathcal{X} \). Let \( \mathcal{F} = \mathcal{F} \) (composition of variable sets, Section 3)

Let \( X \in \mathcal{N}uts_n \), each \( \mathcal{Y}(X_i) \) is an object of \( \mathcal{N}uts \) and hence \( \mathcal{F}(\mathcal{Y}(X_i), \mathcal{Y}(X_{i+1}), \ldots, \mathcal{Y}(X_n)) \) is a \( \mathcal{N}uts \). Moreover given \( f \in \mathcal{N}uts_n(X, Y) \), we know that for each \( i = 1, \ldots, k \), one has \( \mathcal{Y}(i) \in \mathcal{N}uts(\mathcal{Y}(X), \mathcal{Y}(Y)) \) since \( X_i \) is a \( \mathcal{N}uts \). Since \( X \) is a \( \mathcal{N}uts \) we have
\[
\mathcal{F}(\mathcal{Y}(i) \in \mathcal{N}uts(\mathcal{Y}(X_1), \ldots, \mathcal{Y}(X_k))(i)) \]

Let \( X \in \mathcal{O}bj(\mathcal{N}uts) \) and \( \mathcal{Y} \in \mathcal{O}bj(\mathcal{N}uts_k) \). For \( i = 1, \ldots, k \) we know that \( \mathcal{Y}(i) \in \mathcal{N}uts(\mathcal{X}(X_i)(Y), \mathcal{X}(X_i)(Y)) \).

Therefore
\[
\mathcal{Y}(i) \in \mathcal{N}uts(\mathcal{X}(X_i)(Y), \mathcal{X}(X_i)(Y))_{i=1}^k
\]
and hence
\[
\mathcal{Y}(i) \mathcal{X}(X_i)(Y)_{i=1}^k
\in \mathcal{N}uts(\mathcal{X}(X_i)(Y), \mathcal{X}(X_i)(Y))_{i=1}^k
\]

Moreover we have
\[
\mathcal{F}(\mathcal{Y}(i) = \mathcal{Y}(i) \mathcal{X}(X_i)(Y))_{i=1}^k
\]
and
\[
\mathcal{Y}(i) \mathcal{X}(X_i)(Y)_{i=1}^k
\]
using again the fact that \( X \) and the \( X_i \)'s are VNUTS. This shows that the pair \( Y = (\mathcal{Y}, \mathcal{Y}) \) given by \( \mathcal{Y} \) and \( \mathcal{Y} \) is a VNUTS whose associated strong functor is \( X \circ X \) thus proving our contention.

Concerning Condition 4, let us deal only with the case of \( \mathcal{L} \), the others being similar. We have to exhibit a unary VNUTS \( X \) whose associated strong functor \( \mathcal{N}uts \rightarrow \mathcal{N}uts \) coincides with \( \mathcal{L} \) (which is known to be a strong functor \( \mathcal{N}uts \rightarrow \mathcal{N}uts \) by Section 4 and by the general considerations of Section 8). For \( X \), which has to be a variable set \( \mathcal{R}el \rightarrow \mathcal{R}el \), we take the interpretation \( \mathcal{E} \) of \( \mathcal{L} \) in the model \( \mathcal{R}el \) (Section 3) which is an element of \( \mathcal{R}el \), that is, a unary variable set. Next, given \( X \in \mathcal{O}bj(\mathcal{N}uts) \), we take \( \mathcal{Y}(X) = \mathcal{Y}(X) \). Condition 1 in the definition of VNUTS holds by functoriality of \( \mathcal{L} \) on \( \mathcal{N}uts \). Condition 2 holds by the definition of \( \mathcal{F}(\mathcal{X}(X)) \) as described in Section 8 which coincides with \( \mathcal{L} \) as bit more challenging.
1) Fixed Points of VNUTS: Let first $X = \langle |X|, T(X) \rangle$ be a unary VNUTS. Let $E = \sigma|X|$ which is the least set such that $\overline{X}(E) = E$, that is $E = \bigcup_{n=0}^{\infty} \overline{X}^n(\emptyset)$. Let $\Phi : \text{Tot}(E) \to \text{Tot}(E)$ be defined as follows: given $S \in \text{Tot}(E)$, then $(E, S)$ is a NUTS, and we set $\Phi(S) = T(\overline{X})(E, S) \in \text{Tot}(\overline{X}(E)) = \text{Tot}(E)$. This function $\Phi$ is monotonic. Let indeed $S_1, S_2 \in \text{Tot}(E)$ with $S_1 \subseteq S_2$. Then we have $\text{Id} \in \text{Nuts}((E, S_1), (E, S_2))$ and therefore, by Condition (5) satisfied by $\overline{X}$, we have

$$\text{Id} = \overline{X}(\text{Id}) \in \text{Nuts}(\overline{X}(E, S_1), \overline{X}(E, S_2)) = \text{Nuts}((E, \Phi(S_1)), (E, \Phi(S_2)))$$

which means that $\Phi(S_1) \subseteq \Phi(S_2)$. By the Knaster Tarski Theorem (remember that $\text{Tot}(E)$ is a complete lattice), $\Phi$ has a greatest fixpoint $T$ that we can describe as follows. Let $(T_0)_{\alpha \in \Delta}$ where $\Delta$ is the class of ordinals, be defined by: $T_0 = \mathcal{P}(E)$ (the largest possible notion of totality on $E$), $T_{\alpha+1} = \Phi(T_\alpha)$ and $T_\lambda = \bigcap_{\alpha < \lambda} T_\alpha$ when $\lambda$ is a limit ordinal. This sequence is decreasing (easy induction on ordinals using the monotonicity of $\Phi$) and there is an ordinal $\theta$ such that $T_{\theta+1} = T_\theta$ (by a cardinality argument; we can assume that $\theta$ is the least such ordinal). The greatest fixpoint of $\Phi$ is then $T_\theta$ as easily checked.

By construction, $((E, T_\theta), \text{Id})$ is an object of $\text{Coalg}_{\text{Nuts}}(\overline{X})$, we prove that it is the final object. So let $(Y, t)$ be another object of the same category. Since $\langle |Y|, t \rangle$ is an object of $\text{Coalg}_{\text{Rel}}(\overline{|X|})$ and since $(E, \text{Id})$ is the final object in that category, we know by Lemma [14] that there is exactly one $e \in \text{Rel}(|Y|, E)$ such that $\overline{X}(e) \cdot t = e$. We prove that actually $e \in \text{Nuts}(Y, (E, T_\theta))$ so let $v \in T(Y)$. We prove by induction on the ordinal $\alpha$ that $e \cdot v \in T_\alpha$. For $\alpha = 0$ it is obvious since $T_0 = \mathcal{P}(E)$. Assume that the property holds for $\alpha$ and let us prove it for $\alpha + 1$. We have $t \cdot v \in T(Y)(Y) = T(\overline{X}(Y))$ since $t \in \text{Nuts}(Y, \overline{X}(Y))$. Since $\overline{X}(e) \in \text{Nuts}(\overline{X}(Y), \overline{X}(E, T_\theta))$ and since $\overline{X}(E, T_\theta) = (E, T_{\alpha+1})$ we have $\overline{X}(e) \cdot v \in T_{\alpha+1}$, that is $e \cdot v \in T_\alpha$. Last if $\lambda$ is a limit ordinal and if we assume $\forall \alpha < \lambda \ e \cdot v \in T_\alpha$ we have $e \cdot v \in \bigcap_{\alpha < \lambda} T_\alpha = T_\lambda$. Therefore $e \cdot v \in T_\theta$. We use $\overline{X}$ to denote this final coalgebra $(E, T_\theta)$ (its definition only depends on $\overline{X}$ and does not involve the strength $\check{X}$).

So we have proven the first part of Condition (5) in the definition of a Seely model of $\mu \text{LL}$ (see Section [7]). As to the second part, let $X$ be an $n+1$-ary VNUTS. We know by the general Lemma [5] that there is a uniquely defined strong functor $\nu \overline{X} : \text{Nuts}^n \to \text{Nuts}$ such that

- $\nu \overline{X}(X) = \nu(\overline{X}^n)$, so that $\overline{X}(\nu(X), \nu(X)) = \nu(\overline{X})$, for all $\overline{X} \in \text{Obj}(\text{Nuts}_n)$,
- $\overline{X}(\nu(X), \nu(X)) = \nu_\overline{X}(\overline{X}, \nu(X), \nu(X)) = \nu_\overline{X}(\overline{X}, \nu(X))$ for all $Y \in \text{Obj}($Nuts$)$ and $\overline{X} \in \text{Obj}(\text{Nuts}_n)$.

To end the proof, it will be enough to exhibit an $n$-ary VNUTS $Y = (|Y|, T(Y))$ whose associated strong functor coincides with $\nu \overline{X}$. We know that $|X|$ is a variable set $\text{Rel}^{n+1} \to \text{Rel}$ so let $\overline{F} = \nu(|X| = \sigma|X|$ which is a variable set $\text{Rel}^n \to \text{Rel}$ (see Section [11-C1]). Let $\overline{X} \in \text{Obj}(\text{Nuts}_n)$, we have $\overline{\nu}(\overline{X}) = \nu(\overline{X}) = \nu(\overline{X}_X) = \nu(\overline{X}_X, \nu(\overline{X}_X)) = \nu(\overline{X})$, for all $\overline{X} \in \text{Obj}(\text{Nuts}_n)$,}$

\[ T(Y)(\overline{X}) = T(\nu(\overline{X})) \]

In $\text{Nuts}((\nu(\overline{X})), (\nu(\overline{Y})), (\nu(\overline{Y})))$ which satisfies $\overline{X}(\nu(X), \nu(\overline{X})) = \nu(\overline{X})$ and similarly for $\overline{Y}$. Last since $\overline{F}(\nu(X), \nu(\overline{X})) = \nu(\overline{X}) \cdot \nu(\overline{X})$ we know that $Y = (|Y|, T(Y))$ is a VNUTS whose associated strong functor is $\nu \overline{X}$. This ends the proof that $(\text{Nuts}, (\text{Nuts}_n)_{n \in \mathbb{N}})$ is a Seely model of $\mu \text{LL}$.

$\blacksquare$