Quantum Optical Construction of Generalized Pauli and Walsh–Hadamard Matrices in Three Level Systems

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Abstract

A set of generators of generalized Pauli matrices play a crucial role in quantum computation based on n level systems of an atom. In this paper we show how to construct them by making use of Rabi oscillations. We also construct the generalized Walsh–Hadamard matrix in the case of three level systems and present some related problems.
1 Introduction

Quantum Computation (or Computer) is a challenging task in this century for not only physicists but also mathematicians. See for example [1] as a general introduction to it.

Quantum Computation is in a usual understanding based on qubits which are based on two level systems (two energy levels or fundamental spins) of atoms, See [2], [3], [4], [5] as for general theory of two level systems.

In a realistic image of Quantum Computer we need at least one hundred atoms. However then we meet a very severe problem called Decoherence which will destroy a superposition of quantum states in the process of unitary evolution of our system. At the present it is not easy to control Decoherence. See for example [17] or recent [9] as an introduction.

By the way, an atom has in general infinitely many energy levels, while in a qubit method we use only two energy ones. We should use this possibility to reduce a number of atoms. We use \( n \) energy levels from the ground state (it is not realistic to take all energy levels into consideration at the same time). We call this \( n \) level systems a qudit theory, see for example [12], [14], [15], [16].

In quantum computation based on a qudit theory the generalized Pauli matrices \( \{ \Sigma_1, \Sigma_3 \} \) and the generalized Walsh–Hadamard matrix \( W \) play a central role, see [18], [19], [20], [13]. Therefore we must first of all construct them.

In a qubit case we need Rabi oscillations to construct quantum logic gates. See [10] as a simple introduction to quantum logic gates. Similarly we also need Rabi oscillations to construct them in a qudit space. However a general theory of Rabi oscillations in \( n \) level systems has not been developed enough as far as we know. For three level systems see [3], [4], [7] and [15].

Therefore we develop such a theory in this paper and construct the generalized Pauli matrices and Walsh–Hadamard matrix by making use of Rabi oscillations in three level systems.

In this paper we assume the rotating wave approximation (RWA) in our model from the beginning, otherwise we cannot solve the model. However there is no problem on the
approximation in the weak coupling regime. We note that some problems will appear when using this approximation in the strong coupling regime.

2 Two Level System

In this section we make a review of Rabi oscillations (or coherent oscillations) in two level systems of an atom and apply them to constructing quantum logic gates in quantum computation. For a quantum version of the Rabi oscillations see for example [5], [8], [11].

2.1 General Theory

Let \( \{\sigma_1, \sigma_2, \sigma_3\} \) be Pauli matrices:

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

and we set

\[
\sigma_+ = \frac{1}{2}(\sigma_1 + i\sigma_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \frac{1}{2}(\sigma_1 - i\sigma_2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\]

Let us consider an atom with 2 energy levels \( E_0 \) and \( E_1 \) (\( E_1 > E_0 \)). Its Hamiltonian is in the diagonal form given as

\[
H_0 = \begin{pmatrix} E_0 & 0 \\ 0 & E_1 \end{pmatrix}.
\]

This is rewritten as

\[
H_0 = E_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & E_1 - E_0 \end{pmatrix} = E_0 I_2 + \begin{pmatrix} 0 & 0 \\ 0 & \Delta \end{pmatrix}
\]

where \( \Delta = E_1 - E_0 \) is an energy difference.

We consider an atom with two energy levels which interacts with external (periodic) field with \( g \cos(\omega t + \phi) \). In the following we set \( \hbar = 1 \) for simplicity. The Hamiltonian in the dipole approximation is given by

\[
H = H_0 + g \cos(\omega t + \phi) \sigma_1,
\]
where \( \omega \) is the frequency of the external field, \( g \) the coupling constant between the external field and the atom. This model is complicated enough, see [2], [6].

In the following we assume the rotating wave approximation (which neglects the fast oscillating terms), namely

\[
\cos(\omega t + \phi) = \frac{1}{2}(e^{i(\omega t + \phi)} + e^{-i(\omega t + \phi)}) = \frac{1}{2}e^{i(\omega t + \phi)}(1 + e^{-2i(\omega t + \phi)}) \approx \frac{1}{2}e^{i(\omega t + \phi)},
\]

and

\[
\cos(\omega t + \phi)\sigma_1 = \begin{pmatrix} 0 & \cos(\omega t + \phi) \\ \cos(\omega t + \phi) & 0 \end{pmatrix} \approx \frac{1}{2} \begin{pmatrix} 0 & e^{i(\omega t + \phi)} \\ e^{-i(\omega t + \phi)} & 0 \end{pmatrix},
\]

therefore the Hamiltonian is given by

\[
H = E_0 \mathbf{1}_2 + \frac{\Delta}{2} (\mathbf{1}_2 - \sigma_3) + \frac{g}{2} \left( e^{i(\omega t + \phi)} \sigma_+ + e^{-i(\omega t + \phi)} \sigma_- \right)
\equiv E_0 \mathbf{1}_2 + \frac{\Delta}{2} (\mathbf{1}_2 - \sigma_3) + g \left( e^{i(\omega t + \phi)} \sigma_+ + e^{-i(\omega t + \phi)} \sigma_- \right)
\]

by the redefinition of \( g \) (\( g/2 \rightarrow g \)). It is explicitly

\[
H = E_0 \mathbf{1}_2 + \begin{pmatrix} 0 & ge^{i(\omega t + \phi)} \\ ge^{-i(\omega t + \phi)} & \Delta \end{pmatrix}.
\] (4)

We would like to solve the Schrödinger equation

\[
i \frac{d}{dt} \Psi = H \Psi.
\] (5)

For that let us decompose \( H \) in (4) into

\[
\begin{pmatrix} 0 & ge^{i(\omega t + \phi)} \\ ge^{-i(\omega t + \phi)} & \Delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ g^{-i(\omega t + \phi)} & \Delta \end{pmatrix} \begin{pmatrix} 0 & g \\ g & \Delta \end{pmatrix} \begin{pmatrix} 1 & e^{i(\omega t + \phi)} \\ e^{-i(\omega t + \phi)} & 1 \end{pmatrix},
\] (6)

so if we set

\[
\Phi = e^{itE_0} \begin{pmatrix} 1 \\ e^{i(\omega t + \phi)} \end{pmatrix} \Psi \iff \Psi = e^{-itE_0} \begin{pmatrix} 1 \\ e^{-i(\omega t + \phi)} \end{pmatrix} \Phi
\] (7)

then it is not difficult to see

\[
i \frac{d}{dt} \Phi = \begin{pmatrix} 0 & g \\ g & \Delta - \omega \end{pmatrix} \Phi,
\] (8)
which is easily solved. For simplicity we set the resonance condition

\[ \Delta = \omega, \quad (9) \]

then the solution of (8) is

\[ \Phi(t) = \exp\left\{ -igt \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\} \Phi(0) = \begin{pmatrix} \cos(gt) & -i\sin(gt) \\ -i\sin(gt) & \cos(gt) \end{pmatrix} \Phi(0). \]

As a result, the solution of the equation (5) is given as

\[ \Psi(t) = e^{-itE_0} \begin{pmatrix} 1 \\ e^{-i(\omega t + \phi)} \end{pmatrix} \Phi(t) \\
= e^{-itE_0} \begin{pmatrix} 1 \\ e^{-i(\omega t + \phi)} \end{pmatrix} \begin{pmatrix} \cos(gt) & -i\sin(gt) \\ -i\sin(gt) & \cos(gt) \end{pmatrix} \Phi(0) \tag{10} \]

by (7). If we choose \( \Phi(0) = (1, 0)^T \) as an initial condition, then

\[ \Psi(t) = e^{-itE_0} \begin{pmatrix} \cos(gt) \\ -ie^{-i(\omega t + \phi)}\sin(gt) \end{pmatrix}. \tag{11} \]

This is a well-known model of the Rabi oscillation (or coherent oscillation).

For the latter use we set

\[ U(t, 0) = e^{-itE_0} \begin{pmatrix} 1 \\ e^{-i(\omega t + \phi)} \end{pmatrix} \begin{pmatrix} \cos(gt) & -i\sin(gt) \\ -i\sin(gt) & \cos(gt) \end{pmatrix} \tag{12} \]

and

\[ U(t_f, t_i) = U(t_f - t_i, 0) \tag{13} \]

because \( \Delta = \omega \) due to the resonance condition (9). Moreover, for \( g = 0 \) (no interaction with external field), then we have

\[ V(t_f, t_i) = e^{-i(t_f - t_i)E_0} \begin{pmatrix} 1 \\ e^{-i\Delta(t_f - t_i)} \end{pmatrix}. \tag{14} \]

We construct several quantum logic gates useful by combining \( U(t_f, t_i) \) and \( V(t_f, t_i) \) for appropriate \( t_i < t_f \) in the following sections. Then we will change a time \( t \) and a phase \( \phi \) as free parameters (we don’t change the coupling constant \( g \)).
2.2 Quantum Logic Gates

As an exercise we would like to construct useful unitary matrices

\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_\theta = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}, \quad W = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \]

for any \( \theta \) explicitly.

First we choose \( t_1 \) as \( t_1 = \frac{3\pi}{2A} \) satisfying

\[ V(t_1, 0) = e^{-iE_0 t_1} \begin{pmatrix} 1 \\ e^{-i\Delta t_1} \end{pmatrix} = e^{-iE_0 t_1} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \]

and next choose \( t_2 - t_1 \) as \( t_2 - t_1 = \frac{\pi}{2g} \) satisfying

\[ U(t_2, t_1) = e^{-iE_0(t_2-t_1)} \begin{pmatrix} 1 \\ e^{-i(\Delta(t_2-t_1)+\phi)} \end{pmatrix} \begin{pmatrix} \cos(g(t_2-t_1)) & -i\sin(g(t_2-t_1)) \\ -i\sin(g(t_2-t_1)) & \cos(g(t_2-t_1)) \end{pmatrix} = e^{-iE_0(t_2-t_1)} \begin{pmatrix} 1 \\ e^{-i(\Delta(t_2-t_1)+\phi)} \end{pmatrix} \begin{pmatrix} -i \\ -i \end{pmatrix}. \]

Then

\[ U(t_2, t_1)V(t_1, 0) = e^{-iE_0 t_2} \begin{pmatrix} 1 \\ e^{-i(\Delta(t_2-t_1)+\phi)} \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix}. \]

Moreover multiplying \( V(t_3, t_2) \) from the left

\[ V(t_3, t_2)U(t_2, t_1)V(t_1, 0) = e^{-iE_0 t_3} \begin{pmatrix} 1 \\ e^{-i(\Delta(t_3-t_1)+\phi)} \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix}. \]

Here we choose \( t_3 = 2k\pi/E_0 \) \((> t_2 > t_1)\) as \( e^{-iE_0 t_3} = 1 \) and the phase \( \phi \) as \( e^{-i(\Delta(t_3-t_1)+\phi)} = i \), so that

\[ V(t_3, t_2)U(t_2, t_1)V(t_1, 0) = \begin{pmatrix} 1 \\ i \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \sigma_1. \]

Next we construct \( \sigma_\theta \). We choose \( t_1 \) as \( t_1 = 2\pi/g \)

\[ U(t_1, 0) = e^{-iE_0 t_1} \begin{pmatrix} 1 \\ e^{-i(\Delta t_1+\phi)} \end{pmatrix} \begin{pmatrix} \cos(g t_1) & -i\sin(g t_1) \\ -i\sin(g t_1) & \cos(g t_1) \end{pmatrix} = e^{-iE_0 t_1} \begin{pmatrix} 1 \\ e^{-i(\Delta t_1+\phi)} \end{pmatrix}. \]
Then multiplying $V(t_2, t_1)$ from the left

$$V(t_2, t_1)U(t_1, 0) = e^{-iE_0 t_2} \begin{pmatrix} 1 \\ e^{-i(\Delta t_2 + \phi)} \end{pmatrix}.$$  

Here we choose $t_2 = 2k\pi/E_0 (> t_1)$ for some $k \in \mathbb{N}$ and the phase $\phi$ as $e^{-i(\Delta t_2 + \phi)} = e^{i\theta}$ for any $\theta$, so that

$$V(t_2, t_1)U(t_1, 0) = \begin{pmatrix} 1 \\ e^{i\theta} \end{pmatrix} = \sigma_\theta.$$  

In particular we obtain

$$\begin{pmatrix} 1 \\ i \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

for $\theta = \pi/2, \pi, 3\pi/2$ respectively.

Lastly we construct the Walsh–Hadamard matrix which plays a central role in quantum computation based on qubits. For $V(t_1, 0), V(t_3, t_2)$ with $t_1 = 3\pi/2\Delta$ and $t_3 - t_2 = 3\pi/2\Delta$

$$V(t_1, 0) = e^{-iE_0 t_1} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \quad V(t_3, t_2) = e^{-iE_0 (t_3 - t_2)} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

and $U(t_2, t_1)$ with $t_2 - t_1 = \pi/4g$

$$U(t_2, t_1) = e^{-iE_0 (t_2 - t_1)} \begin{pmatrix} 1 \\ e^{-i(\Delta (t_2 - t_1) + \phi)} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -i/\sqrt{2} \\ -i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix},$$

we have

$$V(t_3, t_2)U(t_2, t_1)V(t_1, 0) = e^{-iE_0 t_3} \begin{pmatrix} 1 \\ e^{-i(\Delta (t_2 - t_1) + \phi)} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}. $$

We cannot remove the phase $e^{-iE_0 t_3}$, so we multiply the above by $V(t_4, t_3)$ to obtain

$$V(t_4, t_3)V(t_3, t_2)U(t_2, t_1)V(t_1, 0) = e^{-iE_0 t_4} \begin{pmatrix} 1 \\ e^{-i(\Delta (t_4 - t_3 + t_2 - t_1) + \phi)} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}. $$
By choosing \( t_4 = 2k\pi/E_0 \) for some \( k \in \mathbb{N} \) and the phase \( \phi \) as \( e^{-i(\Delta(t_4-t_3+t_2-t_1)+\phi)} = 1 \) we finally obtain

\[
V(t_4,t_3)V(t_3,t_2)U(t_2,t_1)V(t_1,0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = W.
\]

### 3 Three Level System

In this section we consider an atom with three energy levels \( \{|0\rangle, E_0\}, \{|1\rangle, E_1\}, \{|2\rangle, E_2\} \) which interacts with external fields. As for the external fields we use laser fields with frequencies equal to energy differences of the atom. Now we set

\[
\Delta_1 = E_1 - E_0, \quad \Delta_2 = E_2 - E_0 \iff E_1 - E_0 = \Delta_1, \quad E_2 - E_1 = \Delta_2 - \Delta_1
\]

for the latter convenience and assume \( E_1 - E_0 > E_2 - E_1 \). See the following picture:

\[
\begin{align*}
E_2 & \quad |2\rangle \\
E_1 & \quad |1\rangle \\
E_0 & \quad |0\rangle
\end{align*}
\]

Under the following conditions depending on external fields we solve the Schrödinger equations and obtain unitary transformations. These transformations will play an central role in constructing the generalized Pauli matrices and generalized Walsh–Hadamard matrix in the next section.
3.1 Unitary Transformation of type 0

First we consider an atom with no interaction with external fields. The Hamiltonian that we are treating is

\[
H_0 = \begin{pmatrix}
E_0 & E_1 \\
E_1 & E_2
\end{pmatrix} = E_0 \mathbf{1}_3 + \begin{pmatrix}
0 & \Delta_1 \\
\Delta_1 & \Delta_2
\end{pmatrix}.
\]  
(16)

We would like to solve the Schrödinger equation

\[
i \frac{d}{dt}\Psi = H_0 \Psi.
\]  
(17)

The solution is easily obtained to be

\[
\Psi(t) = e^{-itE_0} \begin{pmatrix}
1 \\
e^{-it\Delta_1} \\
e^{-it\Delta_2}
\end{pmatrix} \Psi(0).
\]  
(18)

For the latter use we set

\[
U_0(t, 0) = e^{-itE_0} \begin{pmatrix}
1 \\
e^{-it\Delta_1} \\
e^{-it\Delta_2}
\end{pmatrix}.
\]  
(19)

3.2 Unitary Transformation of type I

We consider the following case. That is, we shoot the atom with a laser field having the frequency equal to the energy difference \(E_1 - E_0\). See the following picture.
The Hamiltonian that we are treating is

\[
H_I = \begin{pmatrix}
E_0 & g e^{i(\phi_1 + \omega_1 t)} \\
g e^{-i(\phi_1 + \omega_1 t)} & E_1 \\
& E_2
\end{pmatrix} = E_0 1_3 + \begin{pmatrix}
0 & g e^{i(\phi_1 + \omega_1 t)} \\
g e^{-i(\phi_1 + \omega_1 t)} & \Delta_1 \\
& \Delta_2
\end{pmatrix}.
\] (20)

We would like to solve the Schrödinger equation

\[
i \frac{d}{dt} \Psi = H_I \Psi.
\] (21)

If we note the following decomposition

\[
\begin{pmatrix}
0 & g e^{i(\phi_1 + \omega_1 t)} \\
g e^{-i(\phi_1 + \omega_1 t)} & \Delta_1 \\
& \Delta_2
\end{pmatrix} = \begin{pmatrix}
1 & e^{-i(\phi_1 + \omega_1 t)} \\
e^{-i(\phi_1 + \omega_1 t)} & e^{-i\Delta_2}
\end{pmatrix} \begin{pmatrix}
0 & g \\
g & \Delta_1
\end{pmatrix} \begin{pmatrix}
1 & e^{i(\phi_1 + \omega_1 t)} \\
e^{i\Delta_2}
\end{pmatrix}
\] (22)

and define

\[
\Phi = e^{itE_0} \begin{pmatrix}
1 & e^{i(\phi_1 + \omega_1 t)} \\
e^{i\Delta_2}
\end{pmatrix} \Psi
\] (23)

, then it is not difficult to see

\[
i \frac{d}{dt} \Phi = \begin{pmatrix}
0 & g \\
g & \Delta_1 - \omega_1
\end{pmatrix} \Phi \equiv \tilde{H}_I \Phi.
\] (24)
Here we set the resonance condition

$$
\Delta_1 = \omega_1,
$$

(25)

so the solution is easily obtained to be

$$
\Phi(t) = \exp(-it\tilde{H})\Phi(0) = \begin{pmatrix}
\cos(gt) & -i\sin(gt) \\
-i\sin(gt) & \cos(gt)
\end{pmatrix} \Phi(0).
$$

(26)

As a result the solution we are looking for is

$$
\Psi(t) = e^{-itE_0} \begin{pmatrix} 1 \\ e^{-i(\phi_1+\omega_1t)} \end{pmatrix} \begin{pmatrix}
\cos(gt) & -i\sin(gt) \\
-i\sin(gt) & \cos(gt)
\end{pmatrix} \Phi(0).
$$

(27)

For the latter use we set

$$
U_1(t,0) = e^{-itE_0} \begin{pmatrix} 1 \\ e^{-i(\phi_1+\omega_1t)} \end{pmatrix} \begin{pmatrix}
\cos(gt) & -i\sin(gt) \\
-i\sin(gt) & \cos(gt)
\end{pmatrix}.
$$

(28)

### 3.3 Unitary Transformation of type II

We consider the following case. That is, we shoot the atom with a laser field having the frequency equal to the energy difference $E_2 - E_1$. See the following picture.
The Hamiltonian that we are treating is
\[
H_{11} = \begin{pmatrix} E_0 & E_1 & g e^{i(\phi_2 + \omega_2 t)} \\ g e^{-i(\phi_2 + \omega_2 t)} & E_2 & \end{pmatrix} = E_1 1_3 + \begin{pmatrix} -\Delta_1 & 0 & g e^{i(\phi_2 + \omega_2 t)} \\ 0 & g e^{-i(\phi_2 + \omega_2 t)} & \Delta_2 - \Delta_1 \end{pmatrix}.
\] (29)

We would like to solve the Schrödinger equation
\[
i \frac{d}{dt} \Psi = H_{11} \Psi. \tag{30}
\]

If we note the following decomposition
\[
\begin{pmatrix} -\Delta_1 \\ 0 & g e^{i(\phi_2 + \omega_2 t)} \\ g e^{-i(\phi_2 + \omega_2 t)} & \Delta_2 - \Delta_1 \end{pmatrix} = \begin{pmatrix} e^{it \Delta_1} & -\Delta_1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{-it \Delta_1} \\ 0 & g \Delta_2 - \Delta_1 \end{pmatrix} \begin{pmatrix} 1 \\ e^{i(\phi_2 + \omega_2 t)} \end{pmatrix},
\]
and define
\[
\Phi = e^{it E_1} \begin{pmatrix} e^{-it \Delta_1} \\ 1 \\ e^{i(\phi_2 + \omega_2 t)} \end{pmatrix} \Psi \tag{32}
\]
, then it is not difficult to see
\[
i \frac{d}{dt} \Phi = \begin{pmatrix} 0 \\ 0 & g \\ g & \Delta_2 - \Delta_1 - \omega_2 \end{pmatrix} \Phi \equiv \tilde{H}_{11} \Phi. \tag{33}
\]

Here we set the resonance condition
\[
\Delta_2 - \Delta_1 = \omega_2, \tag{34}
\]
so the solution is easily obtained to be
\[
\Phi(t) = \exp(-it \tilde{H}_{11}) \Phi(0) = \begin{pmatrix} 1 & \cos(gt) & -\sin(gt) \\ \cos(gt) & -i \sin(gt) & \cos(gt) \\ -i \sin(gt) & \cos(gt) \end{pmatrix} \Phi(0). \tag{35}
\]
As a result the solution we are looking for is
\[
Ψ(t) = e^{-itE_1} \begin{pmatrix} e^{it\Delta_1} & 1 & 1 \\ 1 & \cos(gt) & -\sin(gt) \\ e^{-i(\phi_2 + \omega_2 t)} & -\sin(gt) & \cos(gt) \end{pmatrix} \Phi(0).
\]
However let us rewrite this. It is easy to see
\[
e^{-itE_1} \begin{pmatrix} e^{it\Delta_1} & 1 \\ 1 & e^{-it\Delta_1} \\ e^{-i(\phi_2 + \omega_2 t)} \end{pmatrix} = e^{-itE_0} \begin{pmatrix} 1 & e^{-it\Delta_1} \\ e^{-i(\phi_2 + \omega_2 t)} & e^{-i\{\phi_2 + (\omega_2 + \Delta_1)t\}} \end{pmatrix}
\]
by the relation \(\Delta_1 = E_1 - E_0\), so
\[
Ψ(t) = e^{-itE_0} \begin{pmatrix} 1 & e^{-it\Delta_1} \\ e^{-i(\phi_2 + \omega_2 t)} & e^{-i\{\phi_2 + (\omega_2 + \Delta_1)t\}} \end{pmatrix} \begin{pmatrix} 1 & \cos(gt) & -\sin(gt) \\ -\sin(gt) & \cos(gt) \end{pmatrix} \Phi(0). \tag{36}
\]
For the latter use we set
\[
U_2(t, 0) = e^{-itE_1} \begin{pmatrix} 1 & e^{-it\Delta_1} \\ e^{-i(\phi_2 + \omega_2 t)} & e^{-i\{\phi_2 + (\omega_2 + \Delta_1)t\}} \end{pmatrix} \begin{pmatrix} 1 & \cos(gt) & -\sin(gt) \\ -\sin(gt) & \cos(gt) \end{pmatrix} \tag{37}
\]

### 3.4 Unitary Transformation of type III

We consider the following case. That is, we shoot the atom with a laser field having the frequency equal to the energy difference \(E_2 - E_0\). See the following picture.
The Hamiltonian that we are treating is

\[
H_{III} = \begin{pmatrix} E_0 & g e^{i(\phi_3 + \omega_3 t)} \\ e^{-i(\phi_3 + \omega_3 t)} & E_1 \\ g e^{i(\phi_3 + \omega_3 t)} & E_2 \end{pmatrix} = E_0 1_3 + \begin{pmatrix} 0 & g e^{i(\phi_3 + \omega_3 t)} \\ e^{-i(\phi_3 + \omega_3 t)} & \Delta_1 \\ g e^{-i(\phi_3 + \omega_3 t)} & \Delta_2 \end{pmatrix}.
\]  

(38)

We would like to solve the Schrödinger equation

\[
i \frac{d}{dt} \Psi = H_{III} \Psi.
\]  

(39)

If we note the following decomposition

\[
\begin{pmatrix} 0 & g e^{i(\phi_3 + \omega_3 t)} \\ e^{-i(\phi_3 + \omega_3 t)} & \Delta_1 \\ g e^{i(\phi_3 + \omega_3 t)} & \Delta_2 \end{pmatrix} =
\begin{pmatrix} 1 & \Delta_1 \\ e^{-i\Delta_1} & \Delta_2 \end{pmatrix} \begin{pmatrix} 0 & g \\ g & \Delta_2 \end{pmatrix} \begin{pmatrix} 1 \\ e^{i\Delta_1} \end{pmatrix},
\]

(40)

and define

\[
\Phi = e^{itE_0} \begin{pmatrix} 1 \\ e^{i\Delta_1} \\ e^{i(\phi_3 + \omega_3 t)} \end{pmatrix} \Psi
\]  

(41)

, then it is not difficult to see

\[
i \frac{d}{dt} \Phi = \begin{pmatrix} 0 & g \\ 0 & 0 \\ g & \Delta_2 - \omega_3 \end{pmatrix} \Phi \equiv \tilde{H}_{III} \Phi.
\]  

(42)

Here we set the resonance condition

\[
\Delta_2 = \omega_3,
\]  

(43)

so the solution is easily obtained to be

\[
\Phi(t) = \exp(-it\tilde{H}_{III})\Phi(0) = \begin{pmatrix} \cos(gt) & -i\sin(gt) \\ -i\sin(gt) & \cos(gt) \end{pmatrix} \Phi(0).
\]  

(44)
As a result the solution we are looking for is
\[
\Psi(t) = e^{-itE_0} \begin{pmatrix} 1 & \cos(gt) & -i\sin(gt) \\ e^{-it} & 1 & \end{pmatrix} \Phi(0). \quad (45)
\]

For the latter use we set
\[
U_3(t, 0) = e^{-itE_0} \begin{pmatrix} 1 & \cos(gt) & -i\sin(gt) \\ e^{-i(\phi_3+\omega_3 t)} & 1 & \end{pmatrix}. \quad (46)
\]

### 3.5 Unitary Transformation of type IV

We consider the following case. That is, we shoot the atom with two laser fields having the frequencies equal to the energy differences \(E_1 - E_0\) and \(E_2 - E_1\) respectively. See the following picture.

\[\begin{array}{c}
\omega_2 \\
\hline \\
\omega_1 \\
\hline
\end{array}\]

The Hamiltonian that we are treating is
\[
H_{IV} = \begin{pmatrix} E_0 & ge^{i(\phi_1+\omega_1 t)} & 0 \\ ge^{-i(\phi_1+\omega_1 t)} & E_1 & ge^{i(\phi_2+\omega_2 t)} \\ 0 & ge^{-i(\phi_2+\omega_2 t)} & E_2 \end{pmatrix} = E_0 \mathbf{1}_3 + \begin{pmatrix} 0 & ge^{i(\phi_1+\omega_1 t)} & 0 \\ ge^{-i(\phi_1+\omega_1 t)} & \Delta_1 & ge^{i(\phi_2+\omega_2 t)} \\ 0 & ge^{-i(\phi_2+\omega_2 t)} & \Delta_2 \end{pmatrix}. \quad (47)
\]
We would like to solve the Schrödinger equation
\[ i \frac{d}{dt} \Psi = H_{IV} \Psi. \] (48)

If we note the following decomposition
\[
\begin{pmatrix}
0 & g e^{i(\phi_1 + \omega_1 t)} & 0 \\
ge^{-i(\phi_1 + \omega_1 t)} & \Delta_1 & g e^{i(\phi_2 + \omega_2 t)} \\
0 & g e^{-i(\phi_2 + \omega_2 t)} & \Delta_2
\end{pmatrix} =
\begin{pmatrix}
1 \\
e^{-i(\phi_1 + \omega_1 t)} \\
e^{-i(\phi_1 + \omega_1 t + \phi_2 + \omega_2 t)}
\end{pmatrix}
\begin{pmatrix}
0 & g & 0 \\
g & \Delta_1 & g \\
0 & g & \Delta_2
\end{pmatrix}
\begin{pmatrix}
1 \\
e^{i(\phi_1 + \omega_1 t)} \\
e^{i(\phi_1 + \omega_1 t + \phi_2 + \omega_2 t)}
\end{pmatrix}
\] (49)

and define
\[
\Phi = e^{itE_0} \begin{pmatrix}
1 \\
e^{i(\phi_1 + \omega_1 t)} \\
e^{i(\phi_1 + \omega_1 t + \phi_2 + \omega_2 t)}
\end{pmatrix} \Psi (50)
\]

, then it is not difficult to see
\[
\frac{i}{dt} \Phi = \begin{pmatrix}
0 & g & 0 \\
g & \Delta_1 - \omega_1 & g \\
0 & g & \Delta_2 - \omega_1 - \omega_2
\end{pmatrix} \Phi \equiv \tilde{H}_{IV} \Phi. \] (51)

Here we set the resonance condition
\[ \Delta_1 = \omega_1 \text{ and } \Delta_2 = \omega_1 + \omega_2 \iff \Delta_1 = \omega_1 \text{ and } \Delta_2 - \Delta_1 = \omega_2, \] (52)

so the solution is obtained to be
\[
\Phi(t) = \exp(-it\tilde{H}_{IV})\Phi(0)
= \frac{1}{2} \begin{pmatrix}
1 + \cos(\sqrt{2}gt) & -i\sqrt{2}\sin(\sqrt{2}gt) & -1 + \cos(\sqrt{2}gt) \\
-i\sqrt{2}\sin(\sqrt{2}gt) & 2\cos(\sqrt{2}gt) & -i\sqrt{2}\sin(\sqrt{2}gt) \\
-1 + \cos(\sqrt{2}gt) & -i\sqrt{2}\sin(\sqrt{2}gt) & 1 + \cos(\sqrt{2}gt)
\end{pmatrix} \Phi(0). \] (53)
As a result the solution we are looking for is

$$\Psi(t) = e^{-itE_0} \begin{pmatrix}
1 & e^{-i(\phi_1 + \omega_1 t)} & e^{-i(\phi_1 + \omega_1 t + \phi_2 + \omega_2 t)} \\
& & \\
& & \\
& & \\
& & \\
& & \\
\end{pmatrix} \times
\begin{pmatrix}
\frac{1 + \cos(\sqrt{2}gt)}{2} & -i \frac{\sin(\sqrt{2}gt)}{\sqrt{2}} & \frac{1 + \cos(\sqrt{2}gt)}{2} \\
-i \frac{\sin(\sqrt{2}gt)}{\sqrt{2}} & \cos(\sqrt{2}gt) & -i \frac{\sin(\sqrt{2}gt)}{\sqrt{2}} \\
\frac{1 + \cos(\sqrt{2}gt)}{2} & -i \frac{\sin(\sqrt{2}gt)}{\sqrt{2}} & \frac{1 + \cos(\sqrt{2}gt)}{2} \\
\end{pmatrix} \Phi(0).$$

(54)

For the latter use we set

$$U_4(t, 0) = e^{-itE_0} \begin{pmatrix}
1 & e^{-i(\phi_1 + \omega_1 t)} & e^{-i(\phi_1 + \omega_1 t + \phi_2 + \omega_2 t)} \\
& & \\
& & \\
& & \\
& & \\
& & \\
\end{pmatrix} \times
\begin{pmatrix}
\frac{1 + \cos(\sqrt{2}gt)}{2} & -i \frac{\sin(\sqrt{2}gt)}{\sqrt{2}} & \frac{1 + \cos(\sqrt{2}gt)}{2} \\
-i \frac{\sin(\sqrt{2}gt)}{\sqrt{2}} & \cos(\sqrt{2}gt) & -i \frac{\sin(\sqrt{2}gt)}{\sqrt{2}} \\
\frac{1 + \cos(\sqrt{2}gt)}{2} & -i \frac{\sin(\sqrt{2}gt)}{\sqrt{2}} & \frac{1 + \cos(\sqrt{2}gt)}{2} \\
\end{pmatrix}. \quad (55)

3.6 Unitary Transformation of type V

We consider the following case. That is, we shoot the atom with two laser fields having the frequencies equal to the energy differences $E_1 - E_0$ and $E_2 - E_0$ respectively. See the following picture.
The Hamiltonian that we are treating is

\[
H_V = \begin{pmatrix}
E_0 & ge^{i(\phi_1 + \omega_1 t)} & ge^{i(\phi_3 + \omega_3 t)} \\
ge^{-i(\phi_1 + \omega_1 t)} & E_1 & 0 \\
ge^{-i(\phi_3 + \omega_3 t)} & 0 & E_2
\end{pmatrix}
\]

\[= E_0 \mathbf{1}_3 + \begin{pmatrix}
0 & ge^{i(\phi_1 + \omega_1 t)} & ge^{i(\phi_3 + \omega_3 t)} \\
ge^{-i(\phi_1 + \omega_1 t)} & \Delta_1 & 0 \\
ge^{-i(\phi_3 + \omega_3 t)} & 0 & \Delta_2
\end{pmatrix}.
\]

We would like to solve the Schrödinger equation

\[i \frac{d}{dt} \Psi = H_V \Psi.\]

If we note the following decomposition

\[
\begin{pmatrix}
0 & ge^{i(\phi_1 + \omega_1 t)} & ge^{i(\phi_3 + \omega_3 t)} \\
ge^{-i(\phi_1 + \omega_1 t)} & \Delta_1 & 0 \\
ge^{-i(\phi_3 + \omega_3 t)} & 0 & \Delta_2
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 \\
eg^{-i(\phi_1 + \omega_1 t)} & 0 & 0 \\
eg^{-i(\phi_3 + \omega_3 t)} & 0 & 0
\end{pmatrix}\begin{pmatrix} 1 \\ g \\ g \end{pmatrix}
\begin{pmatrix} g \\ g \Delta_1 & 0 \\ g \Delta_2 & \Delta_2
\end{pmatrix}\begin{pmatrix} 1 \\ g \Delta_1 & 0 \\ g \Delta_2 & \Delta_2
\end{pmatrix}
\begin{pmatrix}
1 \\
eg^{i(\phi_1 + \omega_1 t)} \\
eg^{i(\phi_3 + \omega_3 t)}
\end{pmatrix}
\]

and define

\[
\Phi = e^{itE_0} \begin{pmatrix} 1 \\ \neg^{i(\phi_1 + \omega_1 t)} \\ \neg^{i(\phi_3 + \omega_3 t)}
\end{pmatrix} \Psi
\]

, then it is not difficult to see

\[i \frac{d}{dt} \Phi = \begin{pmatrix}
0 & g & g \\
g & \Delta_1 - \omega_1 & 0 \\
g & 0 & \Delta_2 - \omega_3
\end{pmatrix} \Phi \equiv \tilde{H}_V \Phi.
\]

Here we set the resonance condition

\[\Delta_1 = \omega_1 \quad \text{and} \quad \Delta_2 = \omega_3,
\]
so the solution is obtained to be

\[ \Phi(t) = \exp(-it\tilde{H}_V)\Phi(0) \]

\[ = \begin{pmatrix}
\cos(\sqrt{2}gt) & -i\frac{\sin(\sqrt{2}gt)}{\sqrt{2}} & -i\frac{\sin(\sqrt{2}gt)}{\sqrt{2}} \\
-i\frac{\sin(\sqrt{2}gt)}{\sqrt{2}} & 1+\cos(\sqrt{2}gt) & -1+\cos(\sqrt{2}gt) \\
-i\frac{\sin(\sqrt{2}gt)}{\sqrt{2}} & -1+\cos(\sqrt{2}gt) & 1+\cos(\sqrt{2}gt)
\end{pmatrix}\Phi(0). \] (62)

As a result the solution we are looking for is

\[ \Psi(t) = e^{-itE_0} \begin{pmatrix}
1 \\
e^{-i(\phi_1+\omega_1t)} \\
e^{-i(\phi_3+\omega_3t)}
\end{pmatrix} \times \begin{pmatrix}
\cos(\sqrt{2}gt) & -i\frac{\sin(\sqrt{2}gt)}{\sqrt{2}} & -i\frac{\sin(\sqrt{2}gt)}{\sqrt{2}} \\
-i\frac{\sin(\sqrt{2}gt)}{\sqrt{2}} & 1+\cos(\sqrt{2}gt) & -1+\cos(\sqrt{2}gt) \\
-i\frac{\sin(\sqrt{2}gt)}{\sqrt{2}} & -1+\cos(\sqrt{2}gt) & 1+\cos(\sqrt{2}gt)
\end{pmatrix}\Phi(0). \] (63)

For the latter use we set

\[ U_5(t, 0) = e^{-itE_0} \begin{pmatrix}
1 \\
e^{-i(\phi_1+\omega_1t)} \\
e^{-i(\phi_3+\omega_3t)}
\end{pmatrix} \times \begin{pmatrix}
\cos(\sqrt{2}gt) & -i\frac{\sin(\sqrt{2}gt)}{\sqrt{2}} & -i\frac{\sin(\sqrt{2}gt)}{\sqrt{2}} \\
-i\frac{\sin(\sqrt{2}gt)}{\sqrt{2}} & 1+\cos(\sqrt{2}gt) & -1+\cos(\sqrt{2}gt) \\
-i\frac{\sin(\sqrt{2}gt)}{\sqrt{2}} & -1+\cos(\sqrt{2}gt) & 1+\cos(\sqrt{2}gt)
\end{pmatrix}. \] (64)

### 3.7 Unitary Transformation of type VI

We consider the following case. That is, we shoot the atom with two laser fields having the frequencies equal to the energy difference \( E_2 - E_0 \) and \( E_2 - E_1 \) respectively. See the following picture.
The Hamiltonian that we are treating is

\[ H_{VI} = \begin{pmatrix}
E_0 & 0 & g e^{i(\phi_3 + \omega_3 t)} \\
0 & E_1 & g e^{i(\phi_2 + \omega_2 t)} \\
ge^{-i(\phi_3 + \omega_3 t)} & ge^{-i(\phi_2 + \omega_2 t)} & E_2
\end{pmatrix} \]

\[ = E_0 \mathbf{1}_3 + \begin{pmatrix}
0 & 0 & g e^{i(\phi_3 + \omega_3 t)} \\
0 & \Delta_1 & g e^{i(\phi_2 + \omega_2 t)} \\
ge^{-i(\phi_3 + \omega_3 t)} & ge^{-i(\phi_2 + \omega_2 t)} & \Delta_2
\end{pmatrix}. \]  

(65)

We would like to solve the Schrödinger equation

\[ i \frac{d}{dt} \Psi = H_{VI} \Psi. \]  

(66)

If we note the following decomposition

\[ \begin{pmatrix}
0 & 0 & g e^{i(\phi_3 + \omega_3 t)} \\
0 & \Delta_1 & g e^{i(\phi_2 + \omega_2 t)} \\
ge^{-i(\phi_3 + \omega_3 t)} & ge^{-i(\phi_2 + \omega_2 t)} & \Delta_2
\end{pmatrix} = \begin{pmatrix}
1 \\
0 & e^{-i(\phi_3 + \omega_3 t - \phi_2 - \omega_2 t)} & 0 \\
0 & e^{-i(\phi_3 + \omega_3 t)} & e^{-i(\phi_2 + \omega_2 t)}
\end{pmatrix} \begin{pmatrix}
0 & 0 & g \\
0 & \Delta_1 & g \\
g & g & \Delta_2
\end{pmatrix} \begin{pmatrix}
1 \\
0 & e^{i(\phi_3 + \omega_3 t - \phi_2 - \omega_2 t)} & 0 \\
0 & e^{i(\phi_3 + \omega_3 t)} & e^{i(\phi_2 + \omega_2 t)}
\end{pmatrix} \]  

(67)

and define

\[ \Phi = e^{itE_0} \begin{pmatrix}
1 \\
e^{i(\phi_3 + \omega_3 t - \phi_2 - \omega_2 t)} & 0 \\
e^{i(\phi_3 + \omega_3 t)} & 0
\end{pmatrix} \Psi \]  

(68)
then it is not difficult to see
\[
i \frac{d}{dt} \Phi = \begin{pmatrix} 0 & 0 & g \\ 0 & \Delta_1 - \omega_3 + \omega_2 & g \\ g & g & \Delta_2 - \omega_3 \end{pmatrix} \Phi \equiv \tilde{H}_V \Phi. \tag{69}
\]

Here we set the resonance condition
\[
\Delta_1 = \omega_3 - \omega_2 \quad \text{and} \quad \Delta_2 = \omega_3 \iff \Delta_2 - \Delta_1 = \omega_2 \quad \text{and} \quad \Delta_2 = \omega_3, \tag{70}
\]
so the solution is obtained to be
\[
\Phi(t) = \exp(-it\tilde{H}_V) \Phi(0) = \begin{pmatrix} 1 & -1+\cos(\sqrt{2}gt) & -i\sin(\sqrt{2}gt) \\ -1+\cos(\sqrt{2}gt) & 1 & -i\sin(\sqrt{2}gt) \\ -i\sin(\sqrt{2}gt) & -i\sin(\sqrt{2}gt) & \cos(\sqrt{2}gt) \end{pmatrix} \Phi(0). \tag{71}
\]

As a result the solution we are looking for is
\[
\Psi(t) = e^{-itE_0} \begin{pmatrix} 1 \\ e^{i(\phi_3 + \omega_3 t - \phi_2 - \omega_2 t)} \\ e^{i(\phi_3 + \omega_3 t)} \end{pmatrix} \begin{pmatrix} 1 & -1+\cos(\sqrt{2}gt) & -i\sin(\sqrt{2}gt) \\ -1+\cos(\sqrt{2}gt) & 1 & -i\sin(\sqrt{2}gt) \\ -i\sin(\sqrt{2}gt) & -i\sin(\sqrt{2}gt) & \cos(\sqrt{2}gt) \end{pmatrix} \Phi(0). \tag{72}
\]

For the latter use we set
\[
U_6(t, 0) = e^{-itE_0} \begin{pmatrix} 1 \\ e^{i(\phi_3 + \omega_3 t - \phi_2 - \omega_2 t)} \\ e^{i(\phi_3 + \omega_3 t)} \end{pmatrix} \begin{pmatrix} 1 & -1+\cos(\sqrt{2}gt) & -i\sin(\sqrt{2}gt) \\ -1+\cos(\sqrt{2}gt) & 1 & -i\sin(\sqrt{2}gt) \\ -i\sin(\sqrt{2}gt) & -i\sin(\sqrt{2}gt) & \cos(\sqrt{2}gt) \end{pmatrix}. \tag{73}
\]
3.8 Unitary Transformation of type VII

We consider the following case. That is, we shoot the atom with three laser fields having the frequencies equal to the energy difference $E_1 - E_0$, $E_2 - E_1$ and $E_2 - E_0$ respectively. See the following picture.

\[ H_{VII} = \begin{pmatrix} E_0 & g e^{i(\phi_1 + \omega_1 t)} & g e^{i(\phi_3 + \omega_3 t)} \\ g e^{-i(\phi_1 + \omega_1 t)} & E_1 & g e^{i(\phi_2 + \omega_2 t)} \\ g e^{-i(\phi_3 + \omega_3 t)} & g e^{-i(\phi_2 + \omega_2 t)} & E_2 \end{pmatrix} \]

\[ = E_0 1_3 + \begin{pmatrix} 0 & g e^{i(\phi_1 + \omega_1 t)} & g e^{i(\phi_3 + \omega_3 t)} \\ g e^{-i(\phi_1 + \omega_1 t)} & \Delta_1 & g e^{i(\phi_2 + \omega_2 t)} \\ g e^{-i(\phi_3 + \omega_3 t)} & g e^{-i(\phi_2 + \omega_2 t)} & \Delta_2 \end{pmatrix}. \quad (74) \]

Here we assume the following condition

\[ \omega_3 = \omega_1 + \omega_2, \]

which is a kind of consistency condition. We note that without this assumption it is very difficult (almost impossible) to solve the following equation, see [15].

We would like to solve the Schrödinger equation

\[ i \frac{d}{dt} \Psi = H_{VII} \Psi. \quad (75) \]
If we note the following decomposition
\[
\begin{pmatrix}
0 & ge^{i(\phi_1 + \omega_1 t)} & ge^{i(\phi_3 + \omega_3 t)} \\
ge^{-i(\phi_1 + \omega_1 t)} & \Delta_1 & ge^{i(\phi_2 + \omega_2 t)} \\
ge^{-i(\phi_3 + \omega_3 t)} & ge^{-i(\phi_2 + \omega_2 t)} & \Delta_2
\end{pmatrix} =
\begin{pmatrix}
1 \\
e^{-i(\phi_1 + \omega_1 t)} \\
e^{i(\phi_1 + \omega_1 t + \phi_2 + \omega_2 t)}
\end{pmatrix}
\begin{pmatrix}
0 & g & ge^{i(\phi_3 - \phi_1 - \phi_2)} \\
g & \Delta_1 & g \\
ge^{-i(\phi_3 - \phi_1 - \phi_2)} & g & \Delta_2
\end{pmatrix}
\times
\begin{pmatrix}
ed^{i(\phi_1 + \omega_1 t)} \\
ed^{i(\phi_1 + \omega_1 t + \phi_2 + \omega_2 t)}
\end{pmatrix}
\]
and define
\[
\Phi = e^{itE_0} \begin{pmatrix}
1 \\
ed^{i(\phi_1 + \omega_1 t)} \\
ed^{i(\phi_1 + \omega_1 t + \phi_2 + \omega_2 t)}
\end{pmatrix} \Psi
\]
, then it is not difficult to see
\[
i \frac{d}{dt} \Phi = \begin{pmatrix}
0 & g & ge^{i(\phi_3 - \phi_1 - \phi_2)} \\
g & \Delta_1 - \omega_1 & g \\
ge^{-i(\phi_3 - \phi_1 - \phi_2)} & g & \Delta_2 - \omega_1 - \omega_2
\end{pmatrix} \Phi \equiv \tilde{H}_{VII} \Phi.
\]
Here we set the resonance condition
\[
\Delta_1 = \omega_1 \quad \text{and} \quad \Delta_2 = \omega_1 + \omega_2 = \omega_3,
\]
and moreover the phase condition
\[
\phi_3 = \phi_1 + \phi_2,
\]
so the solution is obtained to be
\[
\Phi(t) = \exp(-it\tilde{H}_{VII})\Phi(0)
\]
\[
= e^{igt} \begin{pmatrix}
2 + \exp(-i3gt) & -1 + \exp(-i3gt) & -1 + \exp(-i3gt) \\
-1 + \exp(-i3gt) & 2 + \exp(-i3gt) & -1 + \exp(-i3gt) \\
-1 + \exp(-i3gt) & -1 + \exp(-i3gt) & 2 + \exp(-i3gt)
\end{pmatrix} \Phi(0).
\]
As a result the solution we are looking for is

$$\Psi(t) = e^{-itE_0} \begin{pmatrix} 1 & e^{-i(\phi_1 + \omega_1 t)} & e^{-i(\phi_1 + \omega_1 t + \phi_2 + \omega_2 t)} \\ e^{-i(\phi_1 + \omega_1 t + \phi_2 + \omega_2 t)} \\ e^{igt} & \frac{2 + \exp(-i3gt)}{3} & -1 + \frac{\exp(-i3gt)}{3} & -1 + \frac{\exp(-i3gt)}{3} \\ -1 + \frac{\exp(-i3gt)}{3} & 2 + \frac{\exp(-i3gt)}{3} & -1 + \frac{\exp(-i3gt)}{3} \\ -1 + \frac{\exp(-i3gt)}{3} & -1 + \frac{\exp(-i3gt)}{3} & 2 + \frac{\exp(-i3gt)}{3} \end{pmatrix} \Phi(0).$$ (82)

For the latter use we set

$$U_7(t, 0) = e^{-itE_0} \begin{pmatrix} 1 & e^{-i(\phi_1 + \omega_1 t)} & e^{-i(\phi_1 + \omega_1 t + \phi_2 + \omega_2 t)} \\ e^{-i(\phi_1 + \omega_1 t + \phi_2 + \omega_2 t)} \\ e^{igt} & \frac{2 + \exp(-i3gt)}{3} & -1 + \frac{\exp(-i3gt)}{3} & -1 + \frac{\exp(-i3gt)}{3} \\ -1 + \frac{\exp(-i3gt)}{3} & 2 + \frac{\exp(-i3gt)}{3} & -1 + \frac{\exp(-i3gt)}{3} \\ -1 + \frac{\exp(-i3gt)}{3} & -1 + \frac{\exp(-i3gt)}{3} & 2 + \frac{\exp(-i3gt)}{3} \end{pmatrix} \Phi(0).$$ (83)

A comment is in order. We have prepared unitary transformations of eight types constructed from Rabi oscillations. In the following section by making use of these ones we will construct several unitary matrices indispensable in Quantum Computation based on three level systems.

4 N Level Systems · · · Basic Theory

In this section we introduce a generalization of Pauli matrices in section 2 which has been used in several situations in both Quantum Field Theory and Quantum Computation, and also introduce a generalized Walsh–Hadamard matrix that plays a crucial role in Quantum Computation based on \(n\) level systems. See for example [10], Appendix B.

We construct them in terms of Rabi oscillations made in the preceding section in the three level systems.
4.1 Basic Theory

First of all we summarize the properties of Pauli matrices. By \( \sigma_2 = i\sigma_1\sigma_3 \), so that the essential elements of Pauli matrices are \( \{\sigma_1, \sigma_3\} \) and they satisfy

\[
\sigma_1^2 = \sigma_3^2 = 1; \quad \sigma_1^\dagger = \sigma_1, \quad \sigma_3^\dagger = \sigma_3; \quad \sigma_3\sigma_1 = -\sigma_1\sigma_3 = e^{i\pi}\sigma_1\sigma_3. \quad (84)
\]

The Walsh–Hadamard matrix is defined by

\[
W = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \in O(2) \subset U(2). \quad (85)
\]

This matrix (or transformation) is unitary and it plays a very important role in Quantum Computation. Moreover it is easy to realize it in Quantum Optics as shown in section 2.

Let us list some important properties of \( W \):

\[
W^2 = 1, \quad W^\dagger = W = W^{-1}, \quad (86)
\]

\[
\sigma_1 = W\sigma_3W^{-1}, \quad (87)
\]

The check is very easy.

Let \( \{\Sigma_1, \Sigma_3\} \) be the following matrices in \( M(n, \mathbb{C}) \)

\[
\begin{align*}
\Sigma_1 &= \begin{pmatrix}
0 & 1 \\
1 & 0 \\
1 & 0 \\
\vdots & \\
1 & 0
\end{pmatrix}, \\
\Sigma_3 &= \begin{pmatrix}
1 & \sigma \\
\sigma^2 & \ddots \\
\vdots & \\
\sigma^{n-1} & 
\end{pmatrix}
\end{align*}
\quad (88)
\]

where \( \sigma \) is a primitive root of unity \( \sigma^n = 1 \) ( \( \sigma = e^{2\pi i/n} \)). We note that

\[
\bar{\sigma} = \sigma^{n-1}, \quad 1 + \sigma + \cdots + \sigma^{n-1} = 0.
\]

The two matrices \( \{\Sigma_1, \Sigma_3\} \) are generalizations of Pauli matrices \( \{\sigma_1, \sigma_3\} \), but they are not hermitian. Here we list some of their important properties:

\[
\Sigma_1^n = \Sigma_3^n = 1_n; \quad \Sigma_1^\dagger = \Sigma_3^{n-1}, \quad \Sigma_3^\dagger = \Sigma_1^{n-1}; \quad \Sigma_3\Sigma_1 = \sigma\Sigma_1\Sigma_3. \quad (89)
\]
If we define a Vandermonde matrix $W$ based on $\sigma$ as

$$ W = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & \sigma^{n-1} & \sigma^{2(n-1)} & \ldots & \sigma^{(n-1)^2} \\ 1 & \sigma^{n-2} & \sigma^{2(n-2)} & \ldots & \sigma^{(n-1)(n-2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \sigma^{2} & \sigma^{4} & \ldots & \sigma^{2(n-1)} \\ 1 & \sigma & \sigma^{2} & \ldots & \sigma^{n-1} \end{pmatrix}, \quad (90) $$

and

$$ W^\dagger = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & \sigma & \sigma^{2} & \ldots & \sigma^{n-1} \\ 1 & \sigma^{2} & \sigma^{4} & \ldots & \sigma^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \sigma^{n-2} & \sigma^{2(n-2)} & \ldots & \sigma^{(n-1)(n-2)} \\ 1 & \sigma^{n-1} & \sigma^{2(n-1)} & \ldots & \sigma^{(n-1)^2} \end{pmatrix}, \quad (91) $$

then it is not difficult to see

$$ \Sigma_1 = W\Sigma_3W^\dagger = W\Sigma_3W^{-1}. \quad (92) $$

For example, for $n = 3$

$$ \Sigma_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \Sigma_3 = \begin{pmatrix} 1 \\ \sigma \\ \sigma^2 \end{pmatrix}, \quad W = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \sigma^2 & \sigma \\ 1 & \sigma & \sigma^2 \end{pmatrix} \quad (93) $$

where $\sigma = \frac{e^{2\pi i}}{2} = \frac{1}{2}(-1 + i\sqrt{3})$ and

$$ W\Sigma_3W^\dagger = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \sigma^2 & \sigma \\ 1 & \sigma & \sigma^2 \end{pmatrix} \begin{pmatrix} 1 \\ \sigma \\ \sigma^2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \sigma^2 & \sigma \\ 1 & \sigma & \sigma^2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \sigma & \sigma^2 \\ 1 & \sigma^2 & \sigma \end{pmatrix} $$

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\[
= \frac{1}{3} \begin{pmatrix} 1 & \sigma & \sigma^2 \\ 1 & 1 & 1 \\ 1 & \sigma^2 & \sigma \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \sigma & \sigma^2 \\ 1 & \sigma^2 & \sigma \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 0 & 0 & 3 \\ 3 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix} = \Sigma_1,
\]

where we have used that \(\sigma^3 = 1\), \(\bar{\sigma} = \sigma^2\) and \(1 + \sigma + \sigma^2 = 0\).

That is, \(\Sigma_1\) can be diagonalized by making use of \(W\).

**A comment is in order.** Since \(W\) corresponds to the Walsh–Hadamard matrix (85), so it may be possible to call \(W\) the generalized Walsh–Hadamard matrix.

Now we define a matrix

\[
K = \begin{pmatrix} 1 \\ & 1 \\ & & 1 \\ & & & \ddots \\ & & & & 1 \\ & & & & & 1 \end{pmatrix} \in M(n, \mathbb{C})
\]  

(94)

This matrix plays an important role in constructing the exchange gate in qudit theory, see [13]. However when \(n = 2\) this becomes just the identity.

Let us list some important properties of \(W\) corresponding to (86), (87):

\[
W^2 = K, \quad W^\dagger = KW = W^{-1}, \quad \Sigma_1 = W\Sigma_3W^{-1},
\]  

(95) \hspace{1cm} (96)

The check is easy.

### 4.2 Quantum Logic Gates on Three Level Systems

From here we shall restrict to the case of \(n = 3\) and construct both \(\{\Sigma_1, \Sigma_3\}\) and \(W\).

Let us make a comment once more that we will change a time \(t\) and phases \(\phi_j\) as free parameters (we don’t change the coupling constant \(g\)).
Noting the fact

\[ \Sigma_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \]  \hspace{1cm} (97)

we have only to construct each component. First let us construct the matrix

\[ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \]

From (19) we choose \( t_a \) as \( \Delta_1 t_a = (3/2)\pi \) such as

\[ U_0(t_a, 0) = e^{-iE_0t_a} \begin{pmatrix} 1 \\ e^{-i\Delta_1 t_a} \\ e^{-i\Delta_2 t_a} \end{pmatrix} = e^{-iE_0t_a} \begin{pmatrix} 1 \\ i \\ e^{-i\Delta_2 t_a} \end{pmatrix}. \]

Next from (28) we choose \( t_b (> t_a) \) as \( g(t_b - t_a) = \pi/2 \) such as

\[ U_1(t_b, t_a) = e^{-iE_0(t_b-t_a)} \begin{pmatrix} 1 \\ e^{-i(\omega_1(t_b-t_a)+\phi_1)} \\ e^{-i\Delta_2(t_b-t_a)} \end{pmatrix} \begin{pmatrix} -i \\ -i \\ 1 \end{pmatrix}. \]

We choose \( t_c (> t_b) \) as \( \Delta_1(t_c-t_b) = (3/2)\pi \) again in (19) such as

\[ U_0(t_c, t_b) = e^{-iE_0(t_c-t_b)} \begin{pmatrix} 1 \\ i \\ e^{-i\Delta_2(t_c-t_b)} \end{pmatrix}. \]

From (46) we choose \( t_d (> t_c) \) as \( g(t_d - t_c) = 2\pi \) such as

\[ U_3(t_d, t_c) = e^{-iE_0(t_d-t_c)} \begin{pmatrix} 1 \\ e^{-i\Delta_1(t_d-t_c)} \\ e^{-i(\omega_3(t_d-t_c)+\phi_3)} \end{pmatrix}. \]
Therefore we have

\[
U_3(t_d, t_c)U_0(t_c, t_b)U_1(t_b, t_a)U_0(t_a, 0)
= e^{-iE_0t_d} \begin{pmatrix}
1 & e^{-i(\omega_1(t_b-t_a)+\Delta_1(t_d-t_c)+\phi_1)} & e^{-i(\omega_3(t_d-t_c)+\Delta_2 t_c+\phi_3)} \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

In this case we cannot remove the phase $e^{-iE_0t_d}$. Here we again multiply the above matrix by the matrix

\[
U_0(t_e, t_d) = e^{-iE_0(t_e-t_d)} \begin{pmatrix}
1 & e^{-i\Delta_1(t_e-t_d)} & e^{-i\Delta_2(t_e-t_d)} \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

and obtain

\[
U_0(t_e, t_d)U_3(t_d, t_c)U_0(t_c, t_b)U_1(t_b, t_a)U_0(t_a, 0)
= e^{-iE_0t_e} \begin{pmatrix}
1 & e^{-i(\omega_1(t_b-t_a)+\Delta_1(t_e-t_c)+\phi_1)} & e^{-i(\omega_3(t_d-t_c)+\Delta_2(t_e-t_d+t_c)+\phi_3)} \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

with

\[
\Delta_1 t_a = (3/2)\pi, \quad g(t_b-t_a) = \pi/2, \quad \Delta_1(t_e-t_b) = (3/2)\pi, \quad g(t_d-t_c) = 2\pi.
\]

If we choose $t_e (> t_d)$ as

\[
E_0t_e = 2\pi k \quad \text{for some} \ k \in \mathbb{N}
\]

and the phases $\phi_1, \phi_3$ as

\[
e^{-i(\omega_1(t_b-t_a)+\Delta_1(t_e-t_c)+\phi_1)} = 1, \quad e^{-i(\omega_3(t_d-t_c)+\Delta_2(t_e-t_d+t_c)+\phi_3)} = 1,
\]

then we finally obtain

\[
U_0(t_e, t_d)U_3(t_d, t_c)U_0(t_c, t_b)U_1(t_b, t_a)U_0(t_a, 0) = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]
Next we construct
\[
\begin{pmatrix}
  0 & 0 & 1 \\
  0 & 1 & 0 \\
  1 & 0 & 0 \\
\end{pmatrix}.
\]
Similarly in the preceding
\[
U_0(t_e, t_d)U_1(t_d, t_c)U_0(t_c, t_b)U_3(t_b, t_a)U_0(t_a, 0)
= e^{-iE_0 t_e}
\begin{pmatrix}
  1 \\
  e^{-i(\omega_1 (t_d - t_c) + \Delta_1 (t_e - t_d + t_c) + \phi_1)} \\
  e^{-i(\omega_3 (t_b - t_a) + \Delta_2 (t_e - t_c) + \phi_3)}
\end{pmatrix}
\begin{pmatrix}
  0 & 0 & 1 \\
  0 & 1 & 0 \\
  1 & 0 & 0 \\
\end{pmatrix}
\]
with
\[
\Delta_2 t_a = (3/2)\pi, \quad g(t_b - t_a) = \pi/2, \quad \Delta_2 (t_e - t_b) = (3/2)\pi, \quad g(t_d - t_c) = 2\pi.
\]
If we choose \( t_e (> t_d) \) as
\[
E_0 t_e = 2\pi k \quad \text{for some } k \in \mathbb{N}
\]
and the phases \( \phi_1, \phi_3 \) as
\[
e^{-i(\omega_1 (t_d - t_c) + \Delta_1 (t_e - t_d + t_c) + \phi_1)} = 1, \quad e^{-i(\omega_3 (t_b - t_a) + \Delta_2 (t_e - t_c) + \phi_3)} = 1,
\]
then we finally obtain
\[
U_0(t_e, t_d)U_1(t_d, t_c)U_0(t_c, t_b)U_3(t_b, t_a)U_0(t_a, 0) =
\begin{pmatrix}
  0 & 0 & 1 \\
  0 & 1 & 0 \\
  1 & 0 & 0 \\
\end{pmatrix}.
\]  
(99)

We note here that the matrix \( K \) in (94) can be constructed as
\[
\begin{pmatrix}
  0 & 0 & 1 \\
  0 & 1 & 0 \\
  1 & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
  0 & 1 & 0 \\
  0 & 0 & 1 \\
  1 & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
  0 & 0 & 1 \\
  0 & 1 & 0 \\
  1 & 0 & 0 \\
\end{pmatrix}
= K.
\]  
(100)
Next we construct
\[
\Sigma_3 = \begin{pmatrix}
1 \\
\sigma \\
\sigma^2
\end{pmatrix}
\] (101)
where \(\sigma = \exp(2\pi i/3)\).

From (28) and (46) we choose \(t_a\) as \(gt_a = 2\pi\) and \(t_b\) (> \(t_a\)) as \(g(t_b - t_a) = 2\pi\) such as
\[
U_1(t_a, 0) = e^{-iE_0t_a} \begin{pmatrix}
1 \\
e^{-i(\omega_1t_a + \phi_1)} \\
e^{-i\Delta_2t_a}
\end{pmatrix}
\]
and
\[
U_3(t_b, t_a) = e^{-iE_0(t_b - t_a)} \begin{pmatrix}
1 \\
e^{-i\Delta_1(t_b - t_a)} \\
e^{-i(\omega_3(t_b - t_a) + \phi_3)}
\end{pmatrix},
\]
we have
\[
U_3(t_b, t_a)U_1(t_a, 0) = e^{-iE_0t_b} \begin{pmatrix}
1 \\
e^{-i(\Delta_1(t_b - t_a) + \omega_1t_a + \phi_1)} \\
e^{-i(\omega_3(t_b - t_a) + \Delta_2t_a + \phi_3)}
\end{pmatrix}.
\]
As we cannot remove the phase \(e^{-iE_0t_b}\) we multiply the above matrix by \(U_0(t_c, t_b)\) in (19)
\[
U_0(t_c, t_b)U_3(t_b, t_a)U_1(t_a, 0) = e^{-iE_0t_c} \begin{pmatrix}
1 \\
e^{-i(\Delta_1(t_c - t_a) + \omega_1t_a + \phi_1)} \\
e^{-i(\omega_3(t_b - t_a) + \Delta_2(t_c - t_b + t_a) + \phi_3)}
\end{pmatrix}.
\]
Here if we choose \(t_c\) as
\[E_0t_c = 2\pi k\] for some \(k \in \mathbb{N}\)
and the phases \(\phi_1, \phi_3\) as
\[
e^{-i(\Delta_1(t_c - t_a) + \omega_1t_a + \phi_1)} = e^{\frac{2\pi i}{3}} = \sigma, \quad e^{-i(\omega_3(t_b - t_a) + \Delta_2(t_c - t_b + t_a) + \phi_3)} = e^{\frac{4\pi i}{3}} = \sigma^2,
\]
then we finally obtain

\[ U_0(t_c, t_b)U_3(t_b, t_a)U_1(t_a, 0) = \begin{pmatrix} 1 \\ \sigma \\ \sigma^2 \end{pmatrix} = \Sigma_3. \] (102)

Moreover if we choose the phases \( \phi_1, \phi_3 \) as

\[ e^{-i(\Delta_1(t_c-t_a)+\omega t_a+\phi_1)} = i, \quad e^{-i(\omega_3(t_b-t_a)+\Delta_2(t_c-t_b+t_a)+\phi_3)} = i, \]

then we obtain

\[ I \equiv U_0(t_c, t_b)U_3(t_b, t_a)U_1(t_a, 0) = \begin{pmatrix} 1 \\ i \\ i \end{pmatrix} \] (103),

which will become useful in the following.

Lastly we construct the Walsh–Hadamard matrix

\[ W = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \sigma^2 & \sigma \\ 1 & \sigma & \sigma^2 \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1-i\sqrt{3}/2 & -1+i\sqrt{3}/2 \\ 1 & -1+i\sqrt{3}/2 & -1-i\sqrt{3}/2 \end{pmatrix} \] (104)

because

\[ \sigma = e^{\frac{2\pi i}{3}} = -\frac{1+i\sqrt{3}}{2}, \quad \sigma^2 = e^{\frac{4\pi i}{3}} = -\frac{1-i\sqrt{3}}{2}. \]

This construction is not easy. As preliminaries let us construct the matrix

\[ F = \begin{pmatrix} 1 \\ e^{i\frac{\pi}{4}} \begin{pmatrix} \cos(\frac{\pi}{4}) & -i\sin(\frac{\pi}{4}) \\ -i\sin(\frac{\pi}{4}) & \cos(\frac{\pi}{4}) \end{pmatrix} \end{pmatrix}. \] (105)

From (37) we choose \( t_a \) as \( gt_a = \pi/4 \) such as

\[ U_2(t_a, 0) = e^{-iE_0t_a} \begin{pmatrix} 1 \\ e^{-i\Delta_1t_a} \\ e^{-i(\omega_2+\Delta_1)t_a+\phi_2} \end{pmatrix} \begin{pmatrix} 1 & \cos(\frac{\pi}{4}) & -i\sin(\frac{\pi}{4}) \\ & -i\sin(\frac{\pi}{4}) & \cos(\frac{\pi}{4}) \end{pmatrix}. \]
Multiplying the above matrix by the matrix in (28)

\[ U_1(t_b, t_a) = e^{-iE_0(t_b-t_a)} \begin{pmatrix} 1 \\ e^{-i(\omega_1(t_b-t_a)+\phi_1)} \\ e^{-i\Delta_2(t_b-t_a)} \end{pmatrix} \]

with \( g(t_b - t_a) = 2\pi \) and by the matrix \( U_0(t_c, t_b) \) in (19), we have

\[ U_0(t_c, t_b)U_1(t_b, t_a)U_2(t_a, 0) = e^{-iE_0t_c} \times \begin{pmatrix} 1 \\ e^{-i(\Delta_1(t_c-t_b)+\omega_1(t_b-t_a)+\phi_1)} \\ e^{-i(\Delta_2(t_c-t_a)+(\omega_2+\Delta_1)t_a+\phi_2)} \end{pmatrix} \begin{pmatrix} 1 \\ \cos(\frac{\pi}{4}) \\ -i\sin(\frac{\pi}{4}) \end{pmatrix}. \]

Here if we choose \( t_c \) as

\[ E_0t_c = 2\pi k \quad \text{for some } k \in \mathbb{N} \]

and the phases \( \phi_1, \phi_2 \) as

\[ e^{-i(\Delta_1(t_c-t_b)+\omega_1(t_b-t_a)+\phi_1)} = e^{i\frac{\pi}{4}}, \quad e^{-i((\omega_2+\Delta_1)t_a+\Delta_2(t_c-t_a)+\phi_2)} = e^{i\frac{\pi}{4}} \]

then we finally obtain the desired matrix \( F \).

From (64)

\[ U_5(t_a, 0) = e^{-iE_0t_a} \begin{pmatrix} 1 \\ e^{-i(\omega_1t_a+\phi_1)} \\ e^{-i(\omega_3t_a+\phi_3)} \end{pmatrix} \times \begin{pmatrix} \cos(\sqrt{2}gt_a) & -i\sin(\sqrt{2}gt_a) & -i\sin(\sqrt{2}gt_a) \\ -i\sin(\sqrt{2}gt_a) & 1+\cos(\sqrt{2}gt_a) & -1+\cos(\sqrt{2}gt_a) \\ -i\sin(\sqrt{2}gt_a) & -1+\cos(\sqrt{2}gt_a) & 1+\cos(\sqrt{2}gt_a) \end{pmatrix} \]

we choose \( t_a \) as

\[ \cos(\sqrt{2}gt_a) = \frac{1}{\sqrt{3}} \quad \text{and} \quad \sin(\sqrt{2}gt_a) = \frac{\sqrt{2}}{\sqrt{3}} \iff gt_a = \frac{\theta}{\sqrt{2}} \quad \text{for some } \theta; \]

then

\[ U_5(t_a, 0) = e^{-iE_0t_a} \begin{pmatrix} 1 \\ e^{-i(\omega_1t_a+\phi_1)} \\ e^{-i(\omega_3t_a+\phi_3)} \end{pmatrix} \begin{pmatrix} 1 & -i & -i \\ -i & 1+\frac{\sqrt{3}}{2} & 1-\frac{\sqrt{3}}{2} \\ -i & 1-\frac{\sqrt{3}}{2} & 1+\frac{\sqrt{3}}{2} \end{pmatrix}. \]
To remove the phase we multiply the above matrix by $U_0(t_b, t_a)$ in (19)

\[
U_0(t_b, t_a)U_5(t_a, 0) = e^{-i E_0 t_b} \begin{pmatrix} 1 & e^{-i(\Delta_1(t_b-t_a)+\omega_1 t_a+\phi_1)} & e^{-i(\Delta_2(t_b-t_a)+\omega_3 t_a+\phi_1)} \\
\frac{1}{\sqrt{3}} & 1 + \frac{\sqrt{3}}{2} & 1 - \frac{\sqrt{3}}{2} \\
\frac{1}{\sqrt{3}} & 1 - \frac{\sqrt{3}}{2} & 1 + \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} 1 & -i & -i \\
-i & \frac{1 + \sqrt{3}}{2} & \frac{1 - \sqrt{3}}{2} \\
-i & \frac{1 - \sqrt{3}}{2} & \frac{1 + \sqrt{3}}{2} \end{pmatrix}
\]

, and choose $t_b (\gt t_a)$ as $E_0 t_b = 2\pi k$ for some $k \in \mathbb{N}$

and the phases $\phi_1$, $\phi_3$ as

\[
e^{-i(\Delta_1(t_b-t_a)+\omega_1 t_a+\phi_1)} = 1, \quad e^{-i(\Delta_2(t_b-t_a)+\omega_3 t_a+\phi_3)} = 1
\]

then we have

\[
U_0(t_b, t_a)U_5(t_a, 0) = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & -i & -i \\
-i & \frac{1 + \sqrt{3}}{2} & \frac{1 - \sqrt{3}}{2} \\
-i & \frac{1 - \sqrt{3}}{2} & \frac{1 + \sqrt{3}}{2} \end{pmatrix}.
\]

Next we multiply the above matrix by the matrix $I$ in (103) to become

\[
IU_0(t_b, t_a)U_5(t_a, 0)I = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\
1 & \frac{1 - \sqrt{3}}{2} & \frac{1 + \sqrt{3}}{2} \\
1 & \frac{1 + \sqrt{3}}{2} & \frac{1 - \sqrt{3}}{2} \end{pmatrix}.
\]

For the unitary matrix

\[
e^{\pi i \begin{pmatrix} \cos(\frac{\pi}{4}) & -i \sin(\frac{\pi}{4}) \\
-i \sin(\frac{\pi}{4}) & \cos(\frac{\pi}{4}) \end{pmatrix}} = \frac{1 + i}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\
-\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1 + i}{2} \begin{pmatrix} 1 & -i \\
-i & 1 \end{pmatrix},
\]

it is easy to check

\[
\frac{1 + i}{2} \begin{pmatrix} 1 & -i \\
-i & 1 \end{pmatrix} \begin{pmatrix} \frac{1 - \sqrt{3}}{2} & \frac{1 + \sqrt{3}}{2} \\
\frac{1 + \sqrt{3}}{2} & \frac{1 - \sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} \frac{1 - i \sqrt{3}}{2} & \frac{1 + i \sqrt{3}}{2} \\
\frac{1 + i \sqrt{3}}{2} & \frac{1 - i \sqrt{3}}{2} \end{pmatrix},
\]

\[
\frac{1 + i}{2} \begin{pmatrix} 1 & -i \\
-i & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 \\
1 \end{pmatrix}.
\]
Therefore multiplying (106) by (105) we finally obtain

\[
\begin{pmatrix}
1 \\
e^{i\frac{\pi}{4}} \\

\end{pmatrix}
\begin{pmatrix}
\cos\left(\frac{\pi}{4}\right) & -i\sin\left(\frac{\pi}{4}\right) \\
-i\sin\left(\frac{\pi}{4}\right) & \cos\left(\frac{\pi}{4}\right)
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 1 \\
\frac{1}{\sqrt{3}} & -\frac{1-\sqrt{3}}{2} & -\frac{1+\sqrt{3}}{2} \\
1 & -\frac{1+\sqrt{3}}{2} & -\frac{1-\sqrt{3}}{2}
\end{pmatrix}
\]

\[
= \frac{1}{\sqrt{3}}
\begin{pmatrix}
1 & 1 & 1 \\
1 & -\frac{1-i\sqrt{3}}{2} & -\frac{1+i\sqrt{3}}{2} \\
1 & -\frac{1+i\sqrt{3}}{2} & -\frac{1-i\sqrt{3}}{2}
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 1 \\
1 & \sigma^2 & \sigma \\
1 & \sigma & \sigma^2
\end{pmatrix}
= W. \quad (107)
\]

As shown in the proof, to construct the Walsh–Hadamard matrix \( W \) in terms of Rabi oscillations is not easy.

### 4.3 Quantum Logic Gates on N Level Systems · · · Problems

As in the three level case it is easy to construct generalized Pauli matrices \( \Sigma_1, \Sigma_3 \) (88).

For example, in the four level one we can construct \( \Sigma_1, \Sigma_3 \) like

\[
\Sigma_1 = 
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
1 & 0 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
1 & 0 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
1 & 0 \\
1 & 0
\end{pmatrix}
\]

and

\[
\Sigma_3 = 
\begin{pmatrix}
1 & \sigma \\
1 & \sigma^2 \\
1 & \sigma^3
\end{pmatrix}
\begin{pmatrix}
1 & \sigma \\
1 & \sigma^2 \\
1 & \sigma^3
\end{pmatrix}
\begin{pmatrix}
1 & \sigma \\
1 & \sigma^2 \\
1 & \sigma^3
\end{pmatrix}
\]

However, it is not easy to construct the generalized Walsh–Hadamard matrix \( W \) (90).

In fact, we don’t know how to construct it, so we present

**Problem** Construct the generalized Walsh–Hadamard matrix by making use of Rabi oscillations (in a general level system).

or, as an easier version,
Construct the generalized Walsh–Hadamard matrix by making use of Rabi oscillations in four level systems.

That is,

\[
W = \frac{1}{2} \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & -i & -1 & i \\
1 & -1 & 1 & -1 \\
1 & i & -1 & -i
\end{pmatrix}
\]  

(108)

since \(\sigma = \exp(\frac{2\pi}{2}) = i\).

4.4 Formal Construction of U(3)

Here we give a formal construction to \(SU(3)\) by using a generalization of the Euler angle parametrization in \(SU(2)\) by [21], [22] and next give a construction to \(U(3)\) by adding phases.

It is known that any element \(U\) in \(SU(3)\) can be written as

\[
U \equiv U(\alpha, \beta, \gamma, \theta, a, b, c, \phi) = e^{i\alpha \lambda_3} e^{i\beta \lambda_2} e^{i\gamma \lambda_3} e^{i\theta \lambda_5} e^{ia \lambda_1} e^{ib \lambda_2} e^{ic \lambda_3} e^{i\phi \lambda_8}
\]  

(109)

by using a generalization of Euler angles \((\alpha, \beta, \gamma, \theta, a, b, c, \phi)\), where \(\lambda_2, \lambda_3, \lambda_5, \lambda_8\) are well-known Gell-Mann matrices defined by

\[
\lambda_2 = \begin{pmatrix}
0 & -i & 0 \\
i & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad
\lambda_3 = \begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad
\lambda_5 = \begin{pmatrix}
0 & 0 & -i \\
0 & 0 & 0 \\
i & 0 & 0
\end{pmatrix}, \quad
\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -2
\end{pmatrix}
\]  

(110)

Explicitly written

\[
e^{i\alpha \lambda_3} = \begin{pmatrix}
e^{i\alpha} & 0 & 0 \\
0 & e^{-i\alpha} & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad
e^{i\beta \lambda_2} = \begin{pmatrix}
\cos(\beta) & \sin(\beta) & 0 \\
-\sin(\beta) & \cos(\beta) & 0 \\
0 & 0 & 1
\end{pmatrix},
\]
\[
\begin{pmatrix}
\cos(\theta) & 0 & \sin(\theta) \\
0 & 1 & 0 \\
-\sin(\theta) & 0 & \cos(\theta)
\end{pmatrix},
\begin{pmatrix}
e^{i\phi/\sqrt{3}} & 0 & 0 \\
0 & e^{i\phi/\sqrt{3}} & 0 \\
0 & 0 & e^{-2i\phi/\sqrt{3}}
\end{pmatrix}.
\]

It is not difficult to construct the above matrices by making use of unitary operations \(U_0(t, 0) \sim U_7(t, 0)\) in section 3. We leave it to the readers.

**Exercise**  Construct them.

Any element \(U\) in \(U(3)\) can be obtained by multiplying an element in \(SU(3)\) by a phase matrix, for example

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & e^{i\epsilon} & 0 \\
0 & 0 & e^{i\delta}
\end{pmatrix}.
\]

Then it is easy to construct it, so we finish a formal construction of \(U(3)\).

A comment is in order. This construction is only formal, so it is not useful in a realistic scene. For example, when we want to construct the Walsh–Hadamard matrix with this form, it is almost impossible to find the angles above.

### 5 Discussion

We have given the explicit constructions to the generalized Pauli matrices and generalized Walsh–Hadamard matrix in the three level systems by making use of Rabi oscillations. Therefore we can perform important quantum logic gates used in [18], [19], [20], [13] at least in the three level systems.

For a qudit theory it is enough for us to consider three and four level systems. However we have not succeeded in constructing the generalized Walsh–Hadamard matrix in four level systems. We leave it to the readers.

We conclude this paper by expecting that some experimentalists in quantum optics will check our method.
Appendix

In this appendix we give slightly generalized versions of the contents in subsections 3.5, 3.6, 3.7, 3.8. That is, we consider models with two coupling constants \(g_1, g_2\) and solve them exactly. Unfortunately in the case of VII (three coupling constants) we cannot give an explicit solution owing to some technical difficulty.

**Unitary Transformation of type IV**

The Hamiltonian that we are treating is

\[
H_{IV} = \begin{pmatrix}
E_0 & g_1 e^{i(\phi_1 + \omega_1 t)} & 0 \\
g_1 e^{-i(\phi_1 + \omega_1 t)} & E_1 & g_2 e^{i(\phi_2 + \omega_2 t)} \\
0 & g_2 e^{-i(\phi_2 + \omega_2 t)} & E_2
\end{pmatrix}
\]

\[
= E_0 1_3 + \begin{pmatrix}
0 & g_1 e^{i(\phi_1 + \omega_1 t)} & 0 \\
g_1 e^{-i(\phi_1 + \omega_1 t)} & \Delta_1 & g_2 e^{i(\phi_2 + \omega_2 t)} \\
0 & g_2 e^{-i(\phi_2 + \omega_2 t)} & \Delta_2
\end{pmatrix}.
\]

(112)

We would like to solve the Schrödinger equation

\[
i \frac{d}{dt} \Psi = H_{IV} \Psi.
\]

(113)

If we note the following decomposition

\[
\begin{pmatrix}
0 & g_1 e^{i(\phi_1 + \omega_1 t)} & 0 \\
g_1 e^{-i(\phi_1 + \omega_1 t)} & \Delta_1 & g_2 e^{i(\phi_2 + \omega_2 t)} \\
0 & g_2 e^{-i(\phi_2 + \omega_2 t)} & \Delta_2
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & g_1 \\
e^{-i(\phi_1 + \omega_1 t)} & 1 & \Delta_1 \\
e^{-i(\phi_1 + \omega_1 t + \phi_2 + \omega_2 t)} & 0 & 1
\end{pmatrix}
\begin{pmatrix}
0 & g_1 & 0 \\
g_1 & \Delta_1 & g_2 \\
0 & g_2 & \Delta_2
\end{pmatrix}
\begin{pmatrix}
1 & e^{i(\phi_1 + \omega_1 t)} \\
e^{i(\phi_1 + \omega_1 t + \phi_2 + \omega_2 t)} & 1
\end{pmatrix}
\]

(114)
and define
\[
\Phi = e^{itE_0} \begin{pmatrix}
1 \\
e^{i(\phi_1 + \omega_1 t)} \\
e^{i(\phi_1 + \omega_1 t + \omega_2 t)}
\end{pmatrix} \Psi
\] (115)

, then it is not difficult to see
\[
i \frac{d}{dt} \Phi = \begin{pmatrix}
0 & g_1 & 0 \\
g_1 & \Delta_1 - \omega_1 & g_2 \\
0 & g_2 & \Delta_2 - \omega_1 - \omega_2
\end{pmatrix} \Phi \equiv \tilde{H}_{IV} \Phi.
\] (116)

Here we set the resonance condition
\[
\Delta_1 = \omega_1, \quad \Delta_2 = \omega_1 + \omega_2,
\] (117)

so the solution is obtained to be
\[
\Phi(t) = \exp(-it \tilde{H}_{IV}) \Phi(0) =
\begin{pmatrix}
\frac{g_1^2 \cos(\sqrt{g_1^2 + g_2^2} t) + g_2^2}{g_1^2 + g_2^2} & -i g_1 \sin(\sqrt{g_1^2 + g_2^2} t) & \frac{g_1 g_2 \cos(\sqrt{g_1^2 + g_2^2} t) - g_1 g_2}{g_1^2 + g_2^2} \\
-i g_1 \sin(\sqrt{g_1^2 + g_2^2} t) & \cos(\sqrt{g_1^2 + g_2^2} t) & -i g_2 \sin(\sqrt{g_1^2 + g_2^2} t) \\
g_1 g_2 \cos(\sqrt{g_1^2 + g_2^2} t) - g_1 g_2 & -i g_2 \sin(\sqrt{g_1^2 + g_2^2} t) & \frac{g_2^2 \cos(\sqrt{g_1^2 + g_2^2} t) + g_1^2}{g_1^2 + g_2^2}
\end{pmatrix} \Phi(0).
\] (118)

As a result the solution we are looking for is
\[
\Psi(t) = e^{-itE_0} \begin{pmatrix}
1 \\
e^{-i(\phi_1 + \omega_1 t)} \\
e^{-i(\phi_1 + \omega_1 t + \phi_2 + \omega_2 t)}
\end{pmatrix} \times
\begin{pmatrix}
\frac{g_1^2 \cos(\sqrt{g_1^2 + g_2^2} t) + g_2^2}{g_1^2 + g_2^2} & -i g_1 \sin(\sqrt{g_1^2 + g_2^2} t) & \frac{g_1 g_2 \cos(\sqrt{g_1^2 + g_2^2} t) - g_1 g_2}{g_1^2 + g_2^2} \\
-i g_1 \sin(\sqrt{g_1^2 + g_2^2} t) & \cos(\sqrt{g_1^2 + g_2^2} t) & -i g_2 \sin(\sqrt{g_1^2 + g_2^2} t) \\
g_1 g_2 \cos(\sqrt{g_1^2 + g_2^2} t) - g_1 g_2 & -i g_2 \sin(\sqrt{g_1^2 + g_2^2} t) & \frac{g_2^2 \cos(\sqrt{g_1^2 + g_2^2} t) + g_1^2}{g_1^2 + g_2^2}
\end{pmatrix} \Phi(0).
\] (119)

For the latter use we set
\[
\tilde{U}_4(t, 0) = e^{-itE_0} \begin{pmatrix}
1 \\
e^{-i(\phi_1 + \omega_1 t)} \\
e^{-i(\phi_1 + \omega_1 t + \phi_2 + \omega_2 t)}
\end{pmatrix} \times
\begin{pmatrix}
\frac{g_1^2 \cos(\sqrt{g_1^2 + g_2^2} t) + g_2^2}{g_1^2 + g_2^2} & -i g_1 \sin(\sqrt{g_1^2 + g_2^2} t) & \frac{g_1 g_2 \cos(\sqrt{g_1^2 + g_2^2} t) - g_1 g_2}{g_1^2 + g_2^2} \\
-i g_1 \sin(\sqrt{g_1^2 + g_2^2} t) & \cos(\sqrt{g_1^2 + g_2^2} t) & -i g_2 \sin(\sqrt{g_1^2 + g_2^2} t) \\
g_1 g_2 \cos(\sqrt{g_1^2 + g_2^2} t) - g_1 g_2 & -i g_2 \sin(\sqrt{g_1^2 + g_2^2} t) & \frac{g_2^2 \cos(\sqrt{g_1^2 + g_2^2} t) + g_1^2}{g_1^2 + g_2^2}
\end{pmatrix} \Phi(0).
\]
Unitary Transformation of type V

The Hamiltonian is

$$H_V = \begin{pmatrix}
E_0 & g_1 e^{i(\phi_1 + \omega_1 t)} & g_3 e^{i(\phi_3 + \omega_3 t)} \\
g_1 e^{-i(\phi_1 + \omega_1 t)} & E_1 & 0 \\
g_3 e^{-i(\phi_3 + \omega_3 t)} & 0 & E_2
\end{pmatrix}$$

$$= E_0 1_3 + \begin{pmatrix}
0 & g_1 e^{i(\phi_1 + \omega_1 t)} & g_3 e^{i(\phi_3 + \omega_3 t)} \\
g_1 e^{-i(\phi_1 + \omega_1 t)} & \Delta_1 & 0 \\
g_3 e^{-i(\phi_3 + \omega_3 t)} & 0 & \Delta_2
\end{pmatrix}.$$  \hspace{1cm} (121)

We would like to solve the Schrödinger equation

$$i \frac{d}{dt} \Psi = H_V \Psi.$$  \hspace{1cm} (122)

If we note the following decomposition

$$\begin{pmatrix}
0 & g_1 e^{i(\phi_1 + \omega_1 t)} & g_3 e^{i(\phi_3 + \omega_3 t)} \\
g_1 e^{-i(\phi_1 + \omega_1 t)} & \Delta_1 & 0 \\
g_3 e^{-i(\phi_3 + \omega_3 t)} & 0 & \Delta_2
\end{pmatrix} = \begin{pmatrix}
1 & e^{-i(\phi_1 + \omega_1 t)} \\
e^{-i(\phi_1 + \omega_1 t)} & 0 & \Delta_2
\end{pmatrix} \begin{pmatrix}
0 & g_1 & g_3 \\
g_1 & \Delta_1 & 0 \\
g_3 & 0 & \Delta_2
\end{pmatrix} \begin{pmatrix}
1 & e^{i(\phi_1 + \omega_1 t)} \\
& & \\
& & e^{i(\phi_3 + \omega_3 t)}
\end{pmatrix}.$$  \hspace{1cm} (123)

and define

$$\Phi = e^{itE_0} \begin{pmatrix}
1 \\
e^{i(\phi_1 + \omega_1 t)} \\
e^{i(\phi_3 + \omega_3 t)}
\end{pmatrix} \Psi$$  \hspace{1cm} (124)
, then it is not difficult to see
\[ i \frac{d}{dt} \Phi = \begin{pmatrix} 0 & g_1 & g_3 \\ g_1 & \Delta_1 - \omega_1 & 0 \\ g_3 & 0 & \Delta_2 - \omega_3 \end{pmatrix} \Phi \equiv \hat{H}_V \Phi. \tag{125} \]

Here we set the resonance condition
\[ \Delta_1 = \omega_1, \quad \Delta_2 = \omega_3, \tag{126} \]
so the solution is obtained to be
\[
\Phi(t) = \exp(-i \hat{H}_V t) \Phi(0) = \\
= \begin{pmatrix} \cos(\sqrt{g_1^2 + g_3^2} t) & -i g_1 \sin(\sqrt{g_1^2 + g_3^2} t) & -i g_3 \sin(\sqrt{g_1^2 + g_3^2} t) \\ -i g_1 \sin(\sqrt{g_1^2 + g_3^2} t) & g_1^2 \cos(\sqrt{g_1^2 + g_3^2} t) + g_3^2 \cos(\sqrt{g_1^2 + g_3^2} t) & g_1 g_3 \cos(\sqrt{g_1^2 + g_3^2} t) - g_1 g_3 \\ -i g_3 \sin(\sqrt{g_1^2 + g_3^2} t) & g_1 g_3 \cos(\sqrt{g_1^2 + g_3^2} t) - g_1 g_3 & g_3^2 \cos(\sqrt{g_1^2 + g_3^2} t) + g_1^2 \cos(\sqrt{g_1^2 + g_3^2} t) \end{pmatrix} \Phi(0). \tag{127} \]

As a result the solution we are looking for is
\[
\Psi(t) = e^{-i E_0 t} \begin{pmatrix} 1 \\ e^{-i(\phi_1 + \omega_1 t)} \end{pmatrix} \times \\
= \begin{pmatrix} \cos(\sqrt{g_1^2 + g_3^2} t) & -i g_1 \sin(\sqrt{g_1^2 + g_3^2} t) & -i g_3 \sin(\sqrt{g_1^2 + g_3^2} t) \\ -i g_1 \sin(\sqrt{g_1^2 + g_3^2} t) & g_1^2 \cos(\sqrt{g_1^2 + g_3^2} t) + g_3^2 \cos(\sqrt{g_1^2 + g_3^2} t) & g_1 g_3 \cos(\sqrt{g_1^2 + g_3^2} t) - g_1 g_3 \\ -i g_3 \sin(\sqrt{g_1^2 + g_3^2} t) & g_1 g_3 \cos(\sqrt{g_1^2 + g_3^2} t) - g_1 g_3 & g_3^2 \cos(\sqrt{g_1^2 + g_3^2} t) + g_1^2 \cos(\sqrt{g_1^2 + g_3^2} t) \end{pmatrix} \Phi(0). \tag{128} \]

For the latter use we set
\[
\tilde{U}_5(t, 0) = e^{-i E_0 t} \begin{pmatrix} 1 \\ e^{-i(\phi_1 + \omega_1 t)} \end{pmatrix} \times \\
= \begin{pmatrix} \cos(\sqrt{g_1^2 + g_3^2} t) & -i g_1 \sin(\sqrt{g_1^2 + g_3^2} t) & -i g_3 \sin(\sqrt{g_1^2 + g_3^2} t) \\ -i g_1 \sin(\sqrt{g_1^2 + g_3^2} t) & g_1^2 \cos(\sqrt{g_1^2 + g_3^2} t) + g_3^2 \cos(\sqrt{g_1^2 + g_3^2} t) & g_1 g_3 \cos(\sqrt{g_1^2 + g_3^2} t) - g_1 g_3 \\ -i g_3 \sin(\sqrt{g_1^2 + g_3^2} t) & g_1 g_3 \cos(\sqrt{g_1^2 + g_3^2} t) - g_1 g_3 & g_3^2 \cos(\sqrt{g_1^2 + g_3^2} t) + g_1^2 \cos(\sqrt{g_1^2 + g_3^2} t) \end{pmatrix} \Phi(0). \tag{129} \]
Unitary Transformation of type VI

The Hamiltonian is

\[
H_{VI} = \begin{pmatrix}
E_0 & 0 & g_3 e^{i(\phi_3 + \omega_3 t)} \\
0 & E_1 & g_2 e^{i(\phi_2 + \omega_2 t)} \\
g_3 e^{-i(\phi_3 + \omega_3 t)} & g_2 e^{-i(\phi_2 + \omega_2 t)} & E_2
\end{pmatrix}
\]

\[
= E_0 1_3 + \begin{pmatrix}
0 & 0 & g_3 e^{i(\phi_3 + \omega_3 t)} \\
0 & \Delta_1 & g_2 e^{i(\phi_2 + \omega_2 t)} \\
g_3 e^{-i(\phi_3 + \omega_3 t)} & g_2 e^{-i(\phi_2 + \omega_2 t)} & \Delta_2
\end{pmatrix}.
\]  

(130)

We would like to solve the Schrödinger equation

\[
i \frac{d}{dt} \Psi = H_{VI} \Psi.
\]  

(131)

If we note the following decomposition

\[
\begin{pmatrix}
0 & 0 & g_3 e^{i(\phi_3 + \omega_3 t)} \\
0 & \Delta_1 & g_2 e^{i(\phi_2 + \omega_2 t)} \\
g_3 e^{-i(\phi_3 + \omega_3 t)} & g_2 e^{-i(\phi_2 + \omega_2 t)} & \Delta_2
\end{pmatrix}
\]

\[
= \begin{pmatrix}
1 & 0 & g_3 \\
0 & \Delta_1 & g_2 \\
g_3 & g_2 & \Delta_2
\end{pmatrix}
\begin{pmatrix}
0 & 0 & g_3 e^{i(\phi_3 + \omega_3 t - \phi_2 - \omega_2 t)} \\
0 & \Delta_1 & g_2 e^{i(\phi_3 + \omega_3 t - \phi_2 - \omega_2 t)} \\
g_3 & g_2 & \Delta_2 e^{i(\phi_3 + \omega_3 t)}
\end{pmatrix}
\]

(132)

and define

\[
\Phi = \begin{pmatrix}
1 & e^{i(\phi_3 + \omega_3 t - \phi_2 - \omega_2 t)} \\
e^{i(\phi_3 + \omega_3 t)}
\end{pmatrix} \Psi
\]

(133)

, then it is not difficult to see

\[
i \frac{d}{dt} \Phi = \begin{pmatrix}
0 & 0 & g_3 \\
0 & \Delta_1 - \omega_3 + \omega_2 & g_2 \\
g_3 & g_2 & \Delta_2 - \omega_3
\end{pmatrix} \Phi \equiv \tilde{H}_{VI} \Phi.
\]  

(134)
Here we set the resonance condition

\[ \Delta_1 = \omega_3 - \omega_2, \quad \Delta_2 = \omega_3, \]  

so the solution is obtained to be

\[
\Phi(t) = \exp(-it\hat{H}_{IV})\Phi(0) = \\
\begin{pmatrix}
g_3^2\cos(\sqrt{g_1^2 + g_2^2}) + g_2^2 \\
g_3g_2\cos(\sqrt{g_1^2 + g_2^2} - g_1g_2) \\
-i g_3\sin(\sqrt{g_1^2 + g_2^2})
\end{pmatrix} \\
\begin{pmatrix}
g_3^2\cos(\sqrt{g_1^2 + g_2^2}) - g_1g_2 \\
g_3g_2\cos(\sqrt{g_1^2 + g_2^2} + g_1g_2) \\
-i g_3\sin(\sqrt{g_1^2 + g_2^2})
\end{pmatrix} \Phi(0). \tag{136}
\]

As a result the solution we are looking for is

\[
\Psi(t) = e^{-itE_0} \begin{pmatrix} 1 \\
e^{-i(\phi_3 + \omega_3 t - \phi_2 - \omega_2 t)} \\
e^{-i(\phi_3 + \omega_3 t)} 
\end{pmatrix} \times \\
\begin{pmatrix}
g_3^2\cos(\sqrt{g_1^2 + g_2^2}) + g_2^2 \\
g_3g_2\cos(\sqrt{g_1^2 + g_2^2} - g_1g_2) \\
-i g_3\sin(\sqrt{g_1^2 + g_2^2})
\end{pmatrix} \\
\begin{pmatrix}
g_3^2\cos(\sqrt{g_1^2 + g_2^2}) - g_1g_2 \\
g_3g_2\cos(\sqrt{g_1^2 + g_2^2} + g_1g_2) \\
-i g_3\sin(\sqrt{g_1^2 + g_2^2})
\end{pmatrix} \Phi(0). \tag{137}
\]

For the latter use we set

\[
\tilde{U}_6(t, 0) = e^{-itE_0} \begin{pmatrix} 1 \\
e^{-i(\phi_3 + \omega_3 t - \phi_2 - \omega_2 t)} \\
e^{-i(\phi_3 + \omega_3 t)} 
\end{pmatrix} \times \\
\begin{pmatrix}
g_3^2\cos(\sqrt{g_1^2 + g_2^2}) + g_2^2 \\
g_3g_2\cos(\sqrt{g_1^2 + g_2^2} - g_1g_2) \\
-i g_3\sin(\sqrt{g_1^2 + g_2^2})
\end{pmatrix} \\
\begin{pmatrix}
g_3^2\cos(\sqrt{g_1^2 + g_2^2}) - g_1g_2 \\
g_3g_2\cos(\sqrt{g_1^2 + g_2^2} + g_1g_2) \\
-i g_3\sin(\sqrt{g_1^2 + g_2^2})
\end{pmatrix}. \tag{138}
\]

**Unitary Transformation of type VII**
The Hamiltonian is

\[
H_{VI} = \begin{pmatrix}
E_0 & g_1 e^{i(\phi_1 + \omega_1 t)} & g_3 e^{i(\phi_3 + \omega_3 t)} \\
g_1 e^{-i(\phi_1 + \omega_1 t)} & E_1 & g_2 e^{i(\phi_2 + \omega_2 t)} \\
g_3 e^{-i(\phi_3 + \omega_3 t)} & g_2 e^{-i(\phi_2 + \omega_2 t)} & E_2
\end{pmatrix}
\]

\[= E_0 1_3 + \begin{pmatrix}
0 & g_1 e^{i(\phi_1 + \omega_1 t)} & g_3 e^{i(\phi_3 + \omega_3 t)} \\
g_1 e^{-i(\phi_1 + \omega_1 t)} & \Delta_1 & g_2 e^{i(\phi_2 + \omega_2 t)} \\
g_3 e^{-i(\phi_3 + \omega_3 t)} & g_2 e^{-i(\phi_2 + \omega_2 t)} & \delta_2
\end{pmatrix}.
\]

(139)

Here we assume the following

\[
\omega_3 = \omega_1 + \omega_2,
\]

which is a kind of consistency condition.

We would like to solve the Schrödinger equation

\[
i \frac{d}{dt} \Psi = H_{VI} \Psi.
\]

(140)

If we note the following decomposition

\[
\begin{pmatrix}
1 & e^{-i(\phi_1 + \omega_1 t)} & e^{-i(\phi_1 + \omega_1 t + \phi_2 + \omega_2 t)} \\
e^{i(\phi_1 + \omega_1 t)} & \Delta_1 & e^{i(\phi_1 + \omega_1 t + \phi_2 + \omega_2 t)}
\end{pmatrix}
\begin{pmatrix}
0 & g_1 & g_3 e^{i(\phi_3 - \phi_1 - \phi_2)} \\
g_1 & \Delta_1 & g_2 \\
g_3 e^{-i(\phi_3 - \phi_1 - \phi_2)} & g_2 & \Delta_2
\end{pmatrix}
\]

\[
= \begin{pmatrix}
1 & e^{i(\phi_1 + \omega_1 t)} & e^{i(\phi_1 + \omega_1 t + \phi_2 + \omega_2 t)}
\end{pmatrix}
\]

and define

\[
\Phi = e^{iE_0 t} \begin{pmatrix}
1 \\
e^{i(\phi_1 + \omega_1 t)} \\
e^{i(\phi_1 + \omega_1 t + \phi_2 + \omega_2 t)}
\end{pmatrix} \Psi
\]

(142)
, then it is not difficult to see

\[
\frac{d}{dt} \Phi = \begin{pmatrix}
0 & g_1 & g_3 e^{i(\phi_3 - \phi_1 - \phi_2)} \\
g_1 & \Delta_1 - \omega_1 & g_2 \\
g_3 e^{-i(\phi_3 - \phi_1 - \phi_2)} & g_2 & \Delta_2 - \omega_1 - \omega_2
\end{pmatrix} \Phi \equiv \tilde{H}_{VII} \Phi. \tag{143}
\]

Here we set the resonance condition

\[
\Delta_1 = \omega_1 \quad \text{and} \quad \Delta_2 = \omega_1 + \omega_2 = \omega_3, \tag{144}
\]

and moreover the phase condition

\[
\phi_3 = \phi_1 + \phi_2. \tag{145}
\]

To obtain the solution we must calculate

\[
\exp(-it\tilde{H}_{VII}) = \exp \left\{ -it \begin{pmatrix}
0 & g_1 & g_3 \\
g_1 & 0 & g_2 \\
g_3 & g_2 & 0
\end{pmatrix} \right\}. \tag{146}
\]

However we cannot do it (this calculation seems to be very difficult), so we present it as

**Problem** Calculate it.
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