Dedicated to A. Joseph on the occasion of his 60th birthday

INSTANTON COUNTING VIA AFFINE LIE ALGEBRAS II: FROM WHITTAKER VECTORS TO THE SEIBERG-WITTEN PREPOTENTIAL

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Abstract. Let $G$ be a simple simply connected algebraic group over $\mathbb{C}$ with Lie algebra $\mathfrak{g}$. Given a parabolic subgroup $P \subset G$, in [1] the first author introduced a certain generating function $Z_{aff}^{G,P}$. Roughly speaking, these functions count (in a certain sense) framed $G$-bundles on $\mathbb{P}^2$ together with a $P$-structure on a fixed (horizontal) line in $\mathbb{P}^2$. When $P = B$ is a Borel subgroup, the function $Z_{aff}^{G,B}$ was identified in [1] with the Whittaker matrix coefficient in the universal Verma module over the affine Lie algebra $\tilde{\mathfrak{g}}$ (here we denote by $\mathfrak{g}_{aff}$ the affinization of $\mathfrak{g}$ and by $\tilde{\mathfrak{g}}_{aff}$ the Lie algebra whose root system is dual to that of $\mathfrak{g}_{aff}$).

For $P = G$ (in this case we shall write $Z_{aff}^{G}$ instead of $Z_{aff}^{G,P}$) and $G = SL(n)$ the above generating function was introduced by Nekrasov (cf. [7]) and studied thoroughly in [5] and [8]. In particular, it is shown in loc. cit. that the leading term of certain asymptotic of $Z_{aff}^{G}$ is given by the (instanton part of the) Seiberg-Witten prepotential (for $G = SL(n)$). The prepotential is defined using the geometry of the (classical) periodic Toda integrable system. This result was conjectured in [7].

The purpose of this paper is to extend these results to arbitrary $G$. Namely, we use the above description of the function $Z_{aff}^{G,B}$ to show that the leading term of its asymptotic (similar to the one studied in [7] for $P = G$) is given by the instanton part of the prepotential constructed via the Toda system attached to the Lie algebra $\tilde{\mathfrak{g}}_{aff}$. This part is completely algebraic and does not use the original algebro-geometric definition of $Z_{aff}^{G,B}$. We then show that for fixed $G$ these asymptotic are the same for all functions $Z_{aff}^{G,P}$.

1. Introduction

1.1. The partition function. This paper has grown out of a (still unsuccessful) attempt to understand the following object. Let $K$ be a simple $\dagger$ simply connected compact Lie group and let $d$ be a non-negative integer. Denote by $\mathcal{M}^d_K$ the moduli space of (framed) $K$-instantons on $\mathbb{R}^4$ of second Chern class $-d$. This space can be naturally embedded into a larger Uhlenbeck space $\mathcal{U}^d_K$. Both spaces admit a natural action of the group $K$ (by changing the framing at $\infty$) and the torus $(S^1)^2$ acting on $\mathbb{R}^4$ after choosing an identification $\mathbb{R}^4 \cong \mathbb{C}^2$. Moreover, the maximal torus of $K \times (S^1)^2$ has unique fixed point on $\mathcal{U}^d_K$. Thus we may consider (cf. [1], [5] or [7] for precise definitions) the equivariant integral

$$\int_{\mathcal{U}^d_K} 1^d$$

$\dagger$In this paper by a simple Lie (or algebraic) group we mean a group whose Lie algebra is simple.
of the unit $K \times (S^1)^2$-equivariant cohomology class (which we denote by $1^d$) over $U^d_K$; the integral takes values in the field $K$ of fractions of the algebra $\mathcal{A} = H^*_K((S^1)^2(pt))^2$. Note that $\mathcal{A}$ is canonically isomorphic to the algebra of polynomial functions on $k \times \mathbb{R}^2$ (here $k$ denotes the Lie algebra of $K$) which are invariant with respect to the adjoint action of $K$ on $k$. Thus each $\int 1^d$ may be naturally regarded as a rational function of $a \in k$ and $(\varepsilon_1, \varepsilon_2) \in \mathbb{R}^2$.

Consider now the generating function $Z = \sum_{d=0}^{\infty} Q^d \int 1^d$. It can (and should) be thought of as a function of the variable $s$ as before.

In [7] it was conjectured that the first term of the asymptotic in the limit $\lim_{\varepsilon_1, \varepsilon_2 \to 0} \ln Z$ is closely related to the Seiberg-Witten prepotential of $K$. For $K = SU(n)$ this conjecture has been proved in [8] and [5]. Also in [7] an explicit combinatorial expression for $Z$ has been found.

1.2. Algebraic version. In [1] the first author has defined some more general partition functions containing the function $Z_K$ as a special case. Let us recall that definition. First of all, it will be convenient for us to make the whole situation completely algebraic.

Namely, let $G$ be a complex simple algebraic group whose maximal compact subgroup is isomorphic to $K$. We shall denote by $\mathfrak{g}$ its Lie algebra. Let also $S = \mathbb{P}^2$ and denote by $D_\infty \subset S$ the "straight line at $\infty$"; thus $S \setminus D_\infty = \mathbb{A}^2$. It is well-known that $M_K^d$ is isomorphic to the moduli space $\operatorname{Bun}_G^d(S, D_\infty)$ of principal $G$-bundles on $S$ of second Chern class $-d$ endowed with a trivialization on $D_\infty$. When it does not lead to a confusion we shall write $\operatorname{Bun}_G$ instead of $\operatorname{Bun}_G(S, D_\infty)$. The algebraic analog of $U^d_K$ has been constructed in [2]; we denote this algebraic variety by $U^d_G$. This variety is endowed with a natural action on $G \times (\mathbb{C}^*)^2$.

1.3. Parabolic generalization of the partition function. Let $C \subset S$ denote the standard horizontal line. Choose a parabolic subgroup $P \subset G$. Let $\operatorname{Bun}_{G,P}$ denote the moduli space of the following objects:

1) A principal $G$-bundle $\mathcal{F}_G$ on $S$;
2) A trivialization of $\mathcal{F}_G$ on $D_\infty \subset S$;
3) A reduction of $\mathcal{F}_G$ to $P$ on $C \cap D_\infty$ compatible with the trivialization of $\mathcal{F}_G$ on $C \cap D_\infty$.

Let us describe the connected components of $\operatorname{Bun}_{G,P}$. Let $M$ be the Levi group of $P$. Denote by $\check{M}$ the Langlands dual group of $M$ and let $Z(\check{M})$ be its center. We denote by $\Lambda_{G,P}$ the lattice of characters of $Z(\check{M})$. Let also $\Lambda_{G,P}^{\text{aff}} = \Lambda_{G,P} \times \mathbb{Z}$ be the lattice of characters of $Z(\check{M}) \times \mathbb{C}^*$. Note that $\Lambda_{G,G}^{\text{aff}} = \mathbb{Z}$.

The lattice $\Lambda_{G,P}^{\text{aff}}$ contains a canonical semi-group $\Lambda_{G,P}^{\text{aff, pos}}$ of positive elements (cf. [2] and [1]). It is not difficult to see that the connected components of $\operatorname{Bun}_{G,P}$ are
parameterized by the elements of $\Lambda_{G,P}^{aff,\text{pos}}$:

$$\text{Bun}_{G,P} = \bigcup_{\theta_{aff} \in \Lambda_{G,P}^{aff,\text{pos}}} \text{Bun}_{G,P}^{\theta_{aff}}.$$ 

Typically, for $\theta_{aff} \in \Lambda_{G,P}^{aff}$ we shall write $\theta_{aff} = (d, \theta)$ where $\theta \in \Lambda_{G,P}$ and $d \in \mathbb{Z}$.

Each $\text{Bun}_{G,P}^{\theta_{aff}}$ is naturally acted on by $\mathbb{P} \times (\mathbb{C}^*)^2$; by embedding $M$ into $P$ we get an action of $M \times (\mathbb{C}^*)^2$ on $\text{Bun}_{G,P}^{\theta_{aff}}$. In [2] we define for each $\theta_{aff} \in \Lambda_{G,P}^{aff,\text{pos}}$ a certain Uhlenbeck scheme $U_{G,P}^{\theta_{aff}}$, which contains $\text{Bun}_{G,P}^{\theta_{aff}}$ as a dense open subset. The scheme $U_{G,P}^{\theta_{aff}}$ still admits an action of $M \times (\mathbb{C}^*)^2$.

We want to do some equivariant intersection theory on the spaces $U_{G,P}^{\theta_{aff}}$. For this let us denote by $\mathcal{A}_{M \times (\mathbb{C}^*)^2}$ the algebra $H_{M \times (\mathbb{C}^*)^2}(pt, \mathbb{C})$. Of course this is just the algebra of $M$-invariant polynomials on $\mathfrak{m} \times \mathbb{C}^2$. Let also $\mathcal{K}_{M \times (\mathbb{C}^*)^2}$ be its field of fractions. We can think about elements of $\mathcal{K}_{M \times (\mathbb{C}^*)^2}$ as rational functions on $\mathfrak{m} \times \mathbb{C}^2$ which are invariant with respect to the adjoint action.

Let $T \subset M$ be a maximal torus. Then one can show that $(U_{G,P}^{\theta_{aff}})^{T \times (\mathbb{C}^*)^2}$ consists of one point. This guarantees that we may consider the integral $\int_{U_{G,P}^{\theta_{aff}}} 1_{G,P}^{\theta_{aff}}$ where $1_{G,P}^{\theta_{aff}}$ denotes the unit class in $H_{M \times (\mathbb{C}^*)^2}^{\theta_{aff}}(U_{G,P}^{\theta_{aff}}, \mathbb{C})$. The result can be thought of as a rational function on $\mathfrak{m} \times \mathbb{C}^2$ which is invariant with respect to the adjoint action of $M$. Define

$$Z_{G,P}^{aff} = \sum_{\theta_{aff} \in \Lambda_{G,P}^{aff,\text{pos}}} q_{aff}^{\theta_{aff}} \int_{U_{G,P}^{\theta_{aff}}} 1_{G,P}^{\theta_{aff}}$$  \hspace{1cm} (1.1)$$

(we refer the reader to Section 2 of [1] for a detailed discussion of equivariant integration). One should think of $Z_{G,P}^{aff}$ as a formal power series in $q_{aff} \in Z(M) \times \mathbb{C}^*$ with values in the space of ad-invariant rational functions on $\mathfrak{m} \times \mathbb{C}^2$. Typically, we shall write $q_{aff} = (q, Q)$ where $q \in Z(M)$ and $Q \in \mathbb{C}^*$. Also we shall denote an element of $\mathfrak{m} \times \mathbb{C}^2$ by $(a, \varepsilon_1, \varepsilon_2)$ (note that for general $P$ (unlike in the case $P = G$) the function $Z_{G,P}^{aff}$ is not symmetric with respect to switching $\varepsilon_1$ and $\varepsilon_2$). Here is the main result of this paper:

**Theorem 1.4.** Let $P \subset G$ be a parabolic subgroup as a above.

1. There exists a function $\mathcal{F}^{\text{inst}} \in \mathbb{C}(a)[[Q]]$ such that

$$\lim_{\varepsilon_1 \rightarrow 0} \lim_{\varepsilon_2 \rightarrow 0} \varepsilon_1 \varepsilon_2 \ln Z_{G,P}^{aff} = \mathcal{F}^{\text{inst}}(a, Q).$$  \hspace{1cm} (1.2)$$

In particular, the above limit does not depend on $q$ and it is the same for all $P$.

2. The function $\mathcal{F}^{\text{inst}}(a, Q)$ is equal to the instanton part of the Seiberg-Witten prepotential of the affine Toda system associated with the Langlands dual Lie algebra $\mathfrak{g}_{aff}$ (cf. Section 3 for the explanation of these words).
Since the function $Z^\text{aff}_G$ is symmetric in $\varepsilon_1$ and $\varepsilon_2$, Theorem 1.4 implies the following result:

**Corollary 1.5.** The function $\varepsilon_1\varepsilon_2 \ln Z^\text{aff}_G$ is regular when both $\varepsilon_1$ and $\varepsilon_2$ are set to 0. Moreover, one has

$$((\varepsilon_1\varepsilon_2 \ln Z^\text{aff}_G)|_{\varepsilon_1=\varepsilon_2=0} = \mathcal{F}^{\text{inst}}.$$  

Corollary 1.5 was conjectured by N. Nekrasov in [7] (in fact [7] contains only the formulation for $G = SL(n)$ but the generalization to other groups is straightforward). For $G = SL(n)$ Nekrasov’s conjecture was proved in [5] and [8]. Also, more recently, this conjecture was proved in [9] for all classical groups. These papers, however, utilize methods which are totally different from ours. In particular, in our approach the existence of the partition functions $Z^\text{aff}_{G,P}$ for $P \neq G$ (in particular, for $P$ being the Borel subgroup) plays a crucial role.

In fact, we are going to prove the following slightly stronger version of Theorem 1.4:

**Theorem 1.6.**

1. Theorem 1.4 holds for $P = B$.
2. For every parabolic subgroup $P \subset G$ one has

$$\lim_{\varepsilon_2 \to 0} \varepsilon_2 (\ln Z^\text{aff}_{G,P} - \ln Z^\text{aff}_G) = 0.$$  

1.7. **Plan of the proof.** Let us explain the idea of the proof of Theorem 1.6. The second part is in fact rather routine so let us explain the idea of the proof of the first part.

The "Borel" partition function $Z^\text{aff}_{G,B}$ was realized in [1] as the Whittaker matrix coefficient in the universal Verma module over the Lie algebra $\hat{\mathfrak{g}}^\text{aff}$. As a corollary one gets that the function $Z^\text{aff}_{G,B}$ is an eigenfunction of the non-stationary Toda hamiltonian associated with the affine Lie algebra $\hat{\mathfrak{g}}^\text{aff}$ (cf. Corollary 3.7 from [1] for the precise statement; we use [3] as our main reference about Toda hamiltonians).

It turns out that this is all that we have to use in order to prove Theorem 1.6(1). Namely, in this paper (cf. Section 2 and Section 3) we introduce the notion of the Seiberg-Witten prepotential (more precisely, its instanton part) for a very general class of non-stationary Schrödinger operators in such a way that by the definition it is equal to some asymptotic of the (in some sense) universal eigenfunction of this operator (we were unable to find such a definition in the literature). Usually the prepotential is attached to a classical completely integrable system (our main references on the definition of the Seiberg-Witten prepotential are [6] and [7]). We show that in the integrable case our definition of the prepotential coincides with the one from loc. cit.

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2. **Schrödinger operators and the prepotential: the one-dimensional case**
2.1. Schrödinger operators. Let \( x \in \mathbb{C} \) and let \( U(x) \) be a trigonometric polynomial in \( x \) - i.e. a polynomial in \( e^x \) and \( e^{-x} \). Let also \( h \) and \( Q \) be formal variables. We want to study the eigenvalues of the Schrödinger operator

\[
T = h^2 \frac{d^2}{dx^2} + QU(x).
\]

More precisely, for each \( a \in \mathbb{C} \) let \( W_a \) denote the space \( e^{ax} \mathbb{C}[e^x,e^{-x}][[Q]] \) with the natural action of the algebra of linear differential operators in \( x \). Then we would like to look for eigenfunctions of \( T \) in \( W_a \). After conjugating \( T \) with \( e^{ax} \) the operator \( T \) turns into the operator

\[
h^2 \frac{d^2}{dx^2} + 2ha \frac{d}{dx} + QU(x) + a^2.
\]

Let

\[
T^a = h^2 \frac{d^2}{dx^2} + 2ha \frac{d}{dx} + QU(x).
\]

We now want to look for eigenfunctions of \( T^a \) in \( W_0 \) (this problem is obviously equivalent to finding eigenfunctions of \( T \) in \( W_a \)). In fact, we want them to depend nicely on \( a \), so we set \( W = \mathbb{C}(a,h)[[e^x,e^{-x}]][[Q]] \) and we want to look for eigenfunctions of \( T^a \) (considered now as a differential operator with coefficients in \( \mathbb{C}(a,h)[[Q]] \)) in \( W \).

**Proposition 2.2.**

1. There exist \( \psi \in W \) and \( b \in Q\mathbb{C}(a,h)[[Q]] \) such that

\[
T^a \psi = b \psi \tag{2.1}
\]

and such that \( \psi = 1 + O(Q) \). Moreover, under such conditions \( b \) is unique and \( \psi \) is unique up to multiplication by an element of \( 1 + Q\mathbb{C}(a,h)[[Q]] \).

2. Let \( \phi = h \ln \psi \) (note that \( \phi \) is defined uniquely up to adding an element of \( Q\mathbb{C}(a,h)[[Q]] \)). Then \( \phi \) is regular at \( h = 0 \) provided this is true for its constant term.

3. The limit \( v(a,Q) := \lim_{h \to 0} b(a,h,Q) \) exists in \( \mathbb{C}(a)[[Q]] \).

**Proof.** Let us prove the first assertion. Let us write

\[
\psi = \sum_{n=0}^{\infty} \psi_n Q^n \quad \text{and} \quad b = \sum_{n=0}^{\infty} b_n Q^n.
\]

Note that \( \psi_0 = 1 \) and thus automatically \( b_0 = 0 \). Thus the equation (2.1) becomes

\[
h^2 \psi_n'' + 2ha \psi_n' + U(x) \psi_n - 1 = \sum_{i=0}^{n-1} b_{n-i} \psi_i. \tag{2.2}
\]

which should be valid for each \( n > 0 \) (here and in what follows the prime denotes the derivative of a function with respect to \( x \)). It is enough for us to prove that the system of equations (2.2) has a unique solution if we require that for all \( n > 0 \) the constant term of the function \( \psi_n \) is equal to 0.

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3By the "constant term" we shall always mean the constant term of a trigonometric polynomial. The reader should not confuse this with the notion of "free term" by which we always mean the coefficient of the 0-th power of the variable in a formal power series.
Equation (2.2) is equivalent to
\[ \hbar^2 \psi''_n + 2h a \psi'_n = -U(x) \psi_{n-1} + \sum_{i=0}^{n-1} b_{n-i} \psi_i. \] (2.3)

where the left hand side is just the differential operator \( D = \hbar \frac{d^2}{dx^2} + 2ha \frac{d}{dx} \) applied to \( \psi_n \), and the right hand side only depends on the \( \psi_i \)'s with \( i < n \).

Let us now argue by induction on \( n \). By the induction hypothesis we assume that \( \psi_i \) and \( b_i \) have already been uniquely determined for all \( i < n \). Note that the operator \( D \) has the following properties (whose verification is left to the reader):

1) \( \ker D \) consists of constant (i.e. independent of \( x \)) functions.
2) \( \text{im } D \) consists of all trigonometric polynomials whose constant term is equal to 0.

Observe now that the coefficient of \( b_n \) in the RHS of (2.3) is \( \psi_0 = 1 \). Thus property 2) above determines \( b_n \) uniquely – it has to be chosen so that the constant term of the RHS is equal to 0. If \( b_n \) is chosen in this way then there exists some \( \psi_n \) satisfying (2.3). 

A priori such \( \psi_n \) is defined uniquely up to adding a constant trigonometric polynomial, but the requirement that the constant term of \( \psi_n \) is equal to 0 determines \( \psi_n \) uniquely.

Let us prove the second and third assertions (this is a standard WKB argument which we include for the sake of completeness). Let us write
\[ \phi = h \ln \psi \]
(the logarithm is taken in the sense of formal power series in \( Q \); this makes sense because \( \psi_0 = 1 \)).

Let us rewrite (2.1) in terms of \( \phi \). We get
\[ (\phi')^2 + h \phi'' + 2a \phi' + QU(x) = b. \] (2.4)

Let us now look for a solution \( \phi \) of the form
\[ \phi = \sum_{n=1}^{\infty} \phi_n Q^n \]

Then (2.4) is equivalent to the following system of equations:
\[ h \phi''_1 + 2a \phi'_1 = b_1 - U(x) \] (2.5)

and
\[ h \phi''_n + 2a \phi'_n = b_n - \sum_{i=1}^{n-1} \phi_i' \phi'_{n-i} \] (2.6)

for all \( n > 1 \).

Without loss of generality we may assume that the constant term of all \( \phi_n \) is equal to zero. We need to show that under such conditions all \( \phi_n \) and \( b_n \) are regular when \( h = 0 \). Let us prove by induction in \( k \) that the statement is valid for \( n \leq k \). If \( k = 0 \), the statement is clear, so let \( k > 0 \); we need to prove the statement for \( n = k \). By the induction assumption we may assume that \( \sum_{i=1}^{n-1} \phi_i' \phi'_{n-i} \) is regular at \( h = 0 \). Arguing as before we see that if (2.6) has a solution then \( b_n \) has to be equal to the constant term
of \( \sum_{i=1}^{n-1} \phi_i \phi'_{n-i} \) for \( n > 1 \) and of \( U(x) \) for \( n = 1 \), and thus it is also regular at \( h = 0 \). Thus the right hand side of (2.6) is regular at \( h = 0 \). This immediately implies that the same is true for \( \phi_n \).

2.3. Explicit calculation of \( \lim_{h \to 0} b(a, h, Q) \) via periods. We now want to explain how to evaluate the function \( \lim_{h \to 0} b(a, h, Q) = v(a, Q) \) using period integrals on a certain algebraic curve. More precisely, we are going to express \( a \) as a (multi-valued) function of \( v \) and \( Q \) which will be written in terms of such periods. Let \( \varphi \) denote the limit of \( \phi \) as \( h \to 0 \). Then we have the equation

\[
(\varphi')^2 + 2a\varphi' = v - QU(x).
\]

In other words, \( \varphi' \) satisfies a quadratic equation. Thus we may write

\[
\varphi' = -a + \sqrt{a^2 + v - QU(x)}.
\]

This is an equality of formal power series in \( Q \). The square root is chosen in such a way that the right hand side is equal to 0 when \( Q = 0 \) (note we automatically have \( v = 0 \) when \( Q = 0 \)).

Recall, however that \( \varphi \) was a trigonometric polynomial. This implies that

\[
\int_0^{2\pi i} \varphi' dx = 0.
\]

This is equivalent to the equation

\[
2\pi ia = \int_0^{2\pi i} \sqrt{a^2 + v - QU(x)} dx \tag{2.8}
\]

Set \( w = e^x \) and recall that \( U(x) = P(w) \) for some polynomial \( P \) in \( w \) and \( w^{-1} \). Set also \( u = a^2 + v \) and consider the algebraic curve \( C = C_u \) which is the projectivization of the affine curve given by the equation

\[
z^2 + QP(w) = u.
\]

We claim that we may write \( a \) locally as a function \( a(u, Q) \) of \( u \) and \( Q \). Namely, first of all \( a_0 := a(u, 0) \) must satisfy \( a_0^2 = u \). Let us locally choose one of the square roots. Then the function \( a \) is found as a series \( a_0 + a_1 Q + \ldots + a_n Q^n + \ldots \), where \( a_i \) with \( i > 0 \) are found recursively.

Note that when \( Q = 0 \) the above curve breaks into two components corresponding to \( z = \pm a_0 \). Let \( A = A_{u, Q} \) denote the one-dimensional cycle in \( C \) satisfying the following conditions:

1) The projection of \( A \) to the \( w \)-plane is an isomorphism between \( A \) and the unit circle.

2) \( A \) depends continuously on \( Q \) and when \( Q = 0 \) it lies in the component of \( C \) corresponding to \( z = a_0 \).
Such a cycle is unique at least for small values of $Q$. Thus the equation (2.8) becomes equivalent to

$$a = \frac{1}{2\pi i} \oint_A z \frac{dw}{w}.$$  (2.9)

Note that $z \frac{dw}{w}$ is a well-defined meromorphic differential on $C$. Note also that $C$ and $A$ depend only on $Q$ and $u$; thus we may think of (2.8) as expressing $a$ as a function of $u$ and $Q$.

2.4. Eigenfunctions of non-stationary Schrödinger operators. Let us now change our problem a little. Introduce one more variable $\kappa$ and define new operators

$$\mathcal{L} = T - \kappa Q \frac{\partial}{\partial Q} \quad \text{and} \quad \mathcal{L}^a = T^a - \kappa Q \frac{\partial}{\partial Q}.$$  

Let us now look for solutions of the equation

$$\mathcal{L}^a \Psi = 0 \quad \text{(2.10)}$$

where $\Psi \in \mathcal{C}(a, \hbar, \kappa)[e^x, e^{-x}][[\kappa, Q]]$ (we shall denote this space by $W(\kappa)$). Of course this equation is equivalent to the equation

$$\mathcal{L}(e^{\frac{ax}{\hbar}} \Psi) = a^2 e^{\frac{ax}{\hbar}} \Psi.$$  

More precisely, we want to look for the asymptotic of these eigenfunctions when both $\hbar$ and $\kappa$ go to 0.

**Proposition 2.5.**

1. There exists unique solution $\Psi$ of (2.10) in $W(\kappa)$ such that $\Psi = 1 + O(Q)$.
2. This solution $\Psi$ takes the form

$$\Psi = e^{\frac{\Phi}{\hbar} + g} \quad \text{(2.11)}$$

where $g \in Q\mathcal{C}(a, \hbar)[e^x, e^{-x}][[\kappa, Q]]$ and $\Phi \in Q\mathcal{C}(a, \hbar)[[\kappa, Q]]$.
3. One has

$$\hbar Q \frac{\partial \Phi}{\partial Q} = b.$$  

4. The limit

$$\mathcal{F}^{\text{inst}} = \lim_{\hbar \to 0} \hbar \Phi(a, \hbar, Q)$$

exists and one has

$$Q \frac{\partial \mathcal{F}^{\text{inst}}}{\partial Q} = v \quad \text{(2.12)}$$

**Remark.** We will explain the origin of the notation a little later.

**Proof.** Let us first prove (1). Let us write

$$\Psi = \sum_{n=0}^{\infty} \Psi_n Q^n, \quad \Psi_0 = 1.$$  

Then (2.10) becomes equivalent to the sequence of equations:

$$\hbar^2 \Psi_n'' + 2ha \Psi_n' - \kappa n \Psi_n = U(x) \Psi_{n-1}. \quad \text{(2.13)}$$
Let $D_n$ denote the differential operator $\hbar^2 \frac{d^2}{dx^2} + 2\hbar a \frac{d}{dx} - \kappa n$. Then it is easy to see that $D_n$ is invertible when acting on $C(a, \hbar, \kappa)[e^x, e^{-x}]$ (it is diagonal in the basis given by the functions $\{e^{kx}\}_{k \in \mathbb{Z}}$ with non-zero eigenvalues). Thus by induction we get a unique solution for each $\Psi_n, n \geq 1$.

Let $F = \ln \Psi$. First of all, we claim that $\kappa F$ is regular when $\kappa = 0$. This is proved exactly in the same way as part (2) of Proposition 2.2 and we leave it to the reader.

Let us now write

$$F = \sum_{n=-1}^{\infty} F_n \kappa^n.$$  

We want to compute $F_{-1}$.

Equation (2.10) is equivalent to the equation

$$\hbar^2 ((F')^2 - F'' + 2aF' + QU(x) = \hbar \kappa Q \frac{\partial F}{\partial Q}. \quad (2.14)$$

Decomposing this in a power series in $\kappa$ and looking at the coefficient of $\kappa^{-2}$ we see that $\Phi = F_{-1}$ satisfies the equation $(\Phi')^2 = 0$; in other words $\Phi$ is indeed independent of $x$.

Let us now look at the free term (in $\kappa$) in the above identity (it is easy to see that the coefficient of $\kappa^{-1}$ is automatically 0 on both sides). We get the equation

$$\hbar^2 (F_0')^2 + \hbar^2 F_0'' + 2aF_0' + QU(x) = \hbar Q \frac{\partial F}{\partial Q}. \quad (2.15)$$

Note now that (2.15) is basically the same equation as (2.4) if we set $b = \hbar Q \frac{\partial \phi}{\partial Q}$ and $F_0 = \hbar^{-1} \phi$. Since obviously $F_0|_{Q=0} = 0$, the uniqueness statement from Proposition 2.2(1) implies (3). Now (4) is equivalent to Proposition 2.2(3).

\[\square\]

**Definition 2.6.** The function $F_{\text{inst}}(a, Q)$ is called the instanton part of the prepotential.

**Remark.** In the context of integrable systems one is usually interested in the full Seiberg-Witten prepotential $F$ which is defined as the sum of $F_{\text{inst}}$ and $F_{\text{pert}}$; here $F_{\text{pert}}$ is called the perturbative part of the prepotential and it is usually given by some simple formula. We don’t know if there is a canonical choice of $F_{\text{pert}}$ in our generality. However, we may observe that in all the known cases $F_{\text{pert}}$ satisfies the equation

$$Q \frac{\partial F_{\text{pert}}}{\partial Q} = a^2.$$  

This fixes $F_{\text{pert}}$ uniquely up to adding a function which is independent of $Q$. Note that if we now define $F = F_{\text{inst}} + F_{\text{pert}}$ (for any choice of $F_{\text{pert}}$ satisfying the above equation) then the equation (2.12) gets simplified: it is now equivalent to

$$Q \frac{\partial F}{\partial Q} = u. \quad (2.16)$$
3. Schrödinger Operators in Higher Dimensions and Integrable Systems

We now want to generalize the results of the previous section to higher dimensional situation.

3.1. The setup. In this section we are going to work with the following general setup. Let \( h \) be a finite dimension vector space over \( \mathbb{C} \) and let \( \Lambda \subset h \) be a lattice. We denote by \( H \) the algebraic torus whose lattice of co-characters is \( \Lambda \) (analytically one may think of \( H \) as \( h/2\pi i\Lambda \) by means of the map \( x \mapsto e^x \)); we let \( \mathbb{C}[H] \) denote the algebra of polynomial functions on \( H \); we might think of elements of \( \mathbb{C}[H] \) as trigonometric polynomials on \( h \). We assume that \( h \) is endowed with a non-degenerate bilinear form \( \langle \cdot, \cdot \rangle \) which takes integral values on \( \Lambda \).

Let \( K \) denote the field of rational functions on \( h^* \times \mathbb{C}^2 \) (typically, we denote an element in \( h^* \times \mathbb{C}^2 \) by \((a, h, \kappa)\) with \( a \in h^* \); so, sometimes we shall write \( \mathbb{C}(a, h, \kappa) \) instead of \( K \)). Let \( Q \) be another indeterminate. We are going to be interested in the space \( W(\kappa) := K[H][[Q]] \); its elements are power series in \( Q \) whose coefficients lie in \( K \).

Let \( \Delta \) denote the Laplacian on \( h \) (or \( H \)) corresponding to the bilinear form fixed above. Fix now any \( P \in \mathbb{C}[H] \). We shall denote by \( U \) the corresponding function on \( h \) given by the formula
\[
U(x) = P(e^x).
\]

Now, following the previous section we define the operators
\[
T = h^2 \Delta + QU(x); \quad T^a = h^2 \Delta + 2h\langle \nabla, a \rangle + QU(x).
\]

Here for a function \( \psi \) we denote by \( \nabla \psi \) its differential in the \( h \)-direction. Similarly, we define
\[
\mathcal{L} = T - \kappa Q \frac{\partial}{\partial Q} \quad \text{and} \quad \mathcal{L}^a = T^a - \kappa Q \frac{\partial}{\partial Q}.
\]

Here \( a \in h^* \). Note that as before for a fixed \( a \) the operator \( T^a + \langle a, a \rangle \) is formally conjugate to \( T \), and the operator \( \mathcal{L}^a + \langle a, a \rangle \) is formally conjugate to \( \mathcal{L} \).

As before, we set \( W = \mathbb{C}(a, h)[H][[Q]] \). Then with such notations Proposition 2.2 and Proposition 2.3 hold as stated in the current situation as well. The proofs are just word-by-word repetitions of those from the one-dimensional situation.

However, generalizing the results of Section 2.3 turns out to be a little bit more tricky. In order to do this we need to make some integrability assumptions.

3.2. Integrability. Let us denote by \( \mathcal{D} \) the subalgebra of the algebra of differential operators on \( H \) with coefficients in \( \mathbb{C}[h, Q] \) consisting of all differential operators of the form \( \sum h^i D_i \) (where \( D_i \) is a differential operator on \( H \) with coefficients in \( \mathbb{C}[Q] \)) such that the order of \( D_i \) is \( \leq i \). It is clear that \( \mathcal{D}/h\mathcal{D} \) is canonically isomorphic to \( \mathcal{O}(T^*H) \otimes \mathbb{C}[Q] = \mathcal{O}(T^*H \times \mathbb{C}) \) (here \( T^*H \) denotes the cotangent bundle to \( H \) and \( \mathcal{O}(T^*H) \) is the algebra of regular functions on it). Note that \( T^*H = H \times h^* \). We shall denote the resulting map from \( \mathcal{D} \) to \( \mathcal{O}(T^*H) \otimes \mathbb{C}[Q] \) by \( \sigma \) and call it the symbol map.

Similarly we let \( \mathcal{D}^a = \mathcal{D} \otimes \mathcal{O}(h^*) \); we have \( \mathcal{D}^a/h\mathcal{D}^a \simeq \mathcal{O}(T^*H \times \mathbb{C} \times h^*) \). We let \( \sigma^a : \mathcal{D}^a \rightarrow \mathcal{O}(T^*H \times \mathbb{C} \times h^*) \) denote the corresponding symbol map.
From now on we want to change our point of view a little bit and think about $T^a$ as a differential operator on $H$ rather than on $\mathfrak{h}$. Note that if we do so then $T^a$ lies in $\mathcal{D}^a$.

We now assume that in addition to the above data we are given the following:

a) An affine algebraic variety $S$ such that $\dim S = \dim H$;

b) A finite morphism $\pi : \mathfrak{h}^* \to S$;

c) An injective homomorphism $\eta : \mathcal{O}(S) \to \mathcal{D}$.

These data must satisfy the following conditions:

1) $T$ lies in the image of $\eta$; we let $C \in \mathcal{O}(S)$ denote the (unique) function for which $\eta(C) = T$.

2) $\eta \mid_{Q=0}$ is equal to the composition of $\pi^* : \mathcal{O}(S) \to \mathcal{O}(\mathfrak{h}^*)$ with the natural embedding $\mathcal{O}(\mathfrak{h}^*) \to \mathcal{D}$ which sends every function $h \in \mathcal{O}(\mathfrak{h}^*)$ which is homogeneous of degree $d$ to $\hbar^d D_h$ where $D_h$ is the differential operator with constant coefficients corresponding to $h$.

In this case we shall say that $T$ is integrable. Note that if $\dim H = 1$ then $T$ is automatically integrable.

Let $\eta^a : \mathcal{O}(S) \to \mathcal{D}^a$ denote the composition of $\eta$ with the conjugation by $e^{(a,x)}$. Note that $T^a = \eta^a(C) - \langle a, a \rangle$.

Let also $p : T^* H \times \mathbb{C} \to S$ denote the morphism such that for every $f \in \mathcal{O}(S)$ we have

$$p^*(f) = \sigma \circ \eta(f).$$

This morphism represents the classical integrable system, which is the classical limit of the quantum integrable system defined by $\eta$.

3.3. **Computation of $\lim_{\hbar \to 0} b$ via periods in the integrable case.** We now want to explain how to generalize the results Section 2.3 to our multi-dimensional situation in the integrable case.

First of all, the operator $T^a$ has simple spectrum in $W$; therefore the function $\psi$ which is an eigenfunction of $T^a$, is automatically an eigenfunction of every operator of the form $\eta^a(f)$ ($f \in \mathcal{O}(S)$). More precisely, we get a homomorphism $b : \mathcal{O}(S) \to \mathcal{O}(\mathfrak{h}^*)[[Q]]$ such that for each $f \in \mathcal{O}(S)$ we have

$$\eta^a(f)(\psi)(t,a,Q,h) = f(b(a,Q,h))\psi(t,a,Q,h).$$

Note that $b = b^*(C) - \langle a, a \rangle$. It is easy to see that the limit $\lim_{\hbar \to 0} b^*(C)$ exists; we denote it by $u$. By the definition $u$ is a map from $\mathfrak{h}^* \times \Sigma$ to $S$ where $\Sigma$ denotes the formal disc with coordinate $Q$. It is clear that $u \mid_{Q=0} = \pi$.

Let us now look at the function $\varphi = \lim_{\hbar \to 0} (\hbar \ln \psi)$. Then we have

$$p(d\varphi(t,a,Q) + a, Q) = u(a,Q).$$

On the other hand, for any $\lambda \in \Lambda$ considered as a morphism $\lambda : \mathbb{C}^* \to H$ we must have

$$\oint \lambda^* d\varphi = 0$$

4Here $t \in H$ (i.e. we think about $\psi$ as a function on $H$ rather than on $\mathfrak{h}$).
where $\oint$ denotes the integral over the unit circle in $C^*$. Let us think of $d\varphi$ as a morphism $H \to T^*H$ (which depends on $a$ and $Q$). We denote by $\alpha$ the canonical one-form on $T^*H$. Let also $L_\lambda$ denote the image of the unit circle under $\lambda$. Then (3.2) is equivalent to

$$\oint_{L_\lambda} (d\varphi) \ast \alpha = 0.$$  

(3.3)

We can now again write $a$ locally as a function of $u$ and $Q$: $a = a(u,Q)$. To do this we must make a (local) choice of $a_0 := a_0(u,Q)$. Note that $a_0$ must satisfy

$$\pi(a_0) = u$$

and therefore choosing $a_0$ amounts to choosing a local branch of $\pi$. Let now $\lambda$ be as above. Then we denote by $A_{\lambda,u,Q}$ the unique 1-dimensional cycle in $T^*H$ such that:

1) The projection of $A_{\lambda,u,Q}$ to $H$ is equal to $L_\lambda$;
2) $A_{\lambda,u,Q} \subset p^{-1}(u)$;
3) $A_{\lambda,u,Q}$ depends continuously on $Q$ and for $Q = 0$ it lies in the above chosen branch of $\pi$.

Then (3.1) says that for every $\lambda \in \Lambda$ we have

$$\langle a, \lambda \rangle = \frac{1}{2\pi i} \oint_{A_{\lambda,u,Q}} \alpha.$$  

(3.4)

3.4. Some variants. Let us choose a closed cone $\mathfrak{h}_+^* \subset \mathfrak{h}^*$ which is integral with respect to $\Lambda$ (i.e. given by finitely many inequalities given by elements of $\Lambda$). We assume also that $a \in \mathfrak{h}_+^*$ implies that $-a \not\in \mathfrak{h}_+^*$ for $a \neq 0$ (i.e. 0 is an extremal point of $\mathfrak{h}_+$). Set $\Lambda^\vee_+ = \Lambda^\vee \cap \mathfrak{h}_+^*$. We denote by $\widehat{W}$ the corresponding completion of $W$; by the definition it consists of all formal sums

$$\sum c_\gamma e^{\langle \gamma, x \rangle}$$

where $\gamma \in \Lambda^\vee$ and such that for each $\tilde{\lambda} \in \Lambda^\vee$ the set

$$\{ \gamma \in \tilde{\lambda} - \Lambda_+ \mid \text{such that } c_\gamma \neq 0 \}$$

is finite.

It is easy to see that the results of this section generalize immediately to the situation when the initial Schrödinger operator $T$ takes the form

$$T = \hbar^2 \Delta + U(Q,x)$$

where $U \in \mathbb{C}[H][[Q]]$ subject to the following condition:

- The function $U(0,x)$ is a linear combination of $e^{\langle \tilde{\lambda}, x \rangle}$ with $\tilde{\lambda} \in \mathfrak{h}_+^*$, $\tilde{\lambda} \neq 0$.

In this case the eigenfunctions $\psi$ and $\Psi$ should be elements of respectively $\widehat{W}$ and $\widehat{W}(\kappa)$.

---

5More precisely, this means the following: for $Q = 0$ the map $p$ is equal to the composition of the natural projection $T^*H \to \mathfrak{h}^*$ and $\pi : \mathfrak{h}^* \to S$. Thus for every $u$ we have $p|_{Q=0}(u) = H \times \pi^{-1}(u)$. We require that $A_\lambda$ lie in the product of $H$ and the corresponding branch of $\pi$. 

The above condition guarantees in particular that 0 is an eigenvalue of $T^a$ on $\mathring{W}$.

Here is the basic example of the above situation. Let

$$\mathfrak{h} = \{(x_1, \ldots, x_n) \in \mathbb{C}^n\}/\mathbb{C}(1, \ldots, 1) \quad \Lambda = \{(x_1, \ldots, x_n) \in \mathbb{Z}^n\}/\mathbb{Z}(1, \ldots, 1).$$

Clearly,

$$\mathfrak{h}^* = \{(a_1, \ldots, a_n) | \sum a_i = 0\}$$

and we set

$$\mathfrak{h}_+^* = \{(a_1, \ldots, a_n) \in \mathfrak{h}^* | a_1 + a_2 + \ldots + a_k \geq 0 \text{ for each } 1 \leq k \leq n\}.$$

Let

$$U(Q, x) = 2(e^{x_1-x_2} + e^{x_2-x_3} + \ldots + e^{x_n-1-x_n} + Qe^{x_n-x_1})$$

be the periodic Toda potential. It is clear that the condition above is satisfied and therefore we may speak of the corresponding prepotential. In the next section we explain its connection with the standard physical definition of the prepotential.

The periodic Toda potential is equal to the Toda potential defined by the affine Lie algebra $\mathfrak{sl}_n$ (cf. for example [3]). One can easily see that the Toda potential for any affine Lie algebra (cf. [3]) satisfies our conditions and thus the corresponding prepotential is well-defined.

Note also that the operator

$$\hbar^2 \Delta + U(Q, x)$$

turns into the operator

$$\hbar^2 \Delta + 2Q^{1/n}(e^{x_1-x_2} + \ldots + e^{x_{n-1}-x_n} + e^{x_n-x_1})$$

after the change of variables

$$x_j \mapsto x_j + \frac{j \ln Q}{n}.$$

Thus when computing the prepotential we may deal with the latter operator (a similar statement is true for all affine Lie algebras).

**Remark.** The variable $Q$ that we are using is connected with the variable $\Lambda$ (which is commonly used by physicists - cf. [6], [7] etc.) by the formula

$$Q = \Lambda^{2n}.$$
open pieces in the Jacobians of the Seiberg-Witten curves; thus periods of a regular one-form over these fibers are equal to the periods of a certain meromorphic one-form over the curves themselves and it is not difficult to check that we get exactly the same periods as we need. Some generalization of this fact will be considered in much more detail in a further publication.

4. Proof of Nekrasov’s conjecture

In this section we want to prove Theorem 1.6 (and thus also Theorem 1.4). The first part of Theorem 1.6 is an immediate corollary of Corollary 3.7 from II combined with the definition of $F_{\text{inst}}$ given by Definition 2.6. Thus it remains to prove the second part of Theorem 1.6. The proof is based on the following result.

**Theorem 4.1.** Let $P \subset G$ be a parabolic and let $(d, \theta) \in N_{G, P}^{\text{aff}, +}$. Then one of the following is true:

a) Both $\int_{U^d_{G, P}} 1$ and $\int_{U^d_G} 1$ are 0.

b) $\int_{U^d_G} 1 \neq 0$ and the ratio

$$\frac{\int_{U^d_{G, P}} 1}{\int_{U^d_G} 1}$$

is regular when $\varepsilon_2 \to 0$.

Let us first explain why Theorem 4.1 implies Theorem 1.6. First of all, we claim that for any $d \leq d'$ we have

$$\int_{U^{d'}_{G, P}} 1 = A_d \int_{U^d_G} 1$$

where $A_d$ is a regular function on $\mathfrak{h} \times \mathbb{C}^2$ (in particular, it is regular when $\varepsilon_2 \to 0$). Indeed, according to II there exists a closed $G \times (\mathbb{C}^*)^2$-equivariant embedding $U^d \rightarrow U^{d'}$. Since $U^{d'}$ is contractible we have $H^*_G((\mathbb{C}^*)^2)(U^{d'}_G) = A_{G \times (\mathbb{C}^*)^2}$. Thus it follows that the direct image $(i_d)_* 1$ of the equivariant unit cohomology class is equal to some $A_d \in A_{G \times (\mathbb{C}^*)^2}$.

Now it follows from Theorem 4.1 that the ratio

$$\frac{Z_{G, P}^{\text{aff}}(q, Q, a, \varepsilon_1, \varepsilon_2)}{Z_{G}^{\text{aff}}(Q, a, \varepsilon_1, \varepsilon_2)}$$

is regular when $\varepsilon_2 \to 0$. This means that

$$\lim_{\varepsilon_2 \to 0} (\ln Z_{G, P}^{\text{aff}} - \ln Z_{G}^{\text{aff}}) = 0.$$ 

This is the statement of Theorem 1.6(2).

Thus to complete the proof we need to prove Theorem 4.1. The proof is based on the following general lemma.
Lemma 4.2. Let $L_1$ and $L_2$ be two algebraic tori and let $L = L_1 \times L_2$. We let $l_1$ and $l_2$ denote the corresponding Lie algebras. We shall denote a typical element in $l$ by $(l_1, l_2)$, where $l_i \in l_i$.

Let $\pi : X \to Y$ be a morphism of $L$-varieties. Assume that:
1) Both $X^L$ and $Y^L$ are proper.
2) The natural map $X^{L_1} \to Y^{L_1}$ is proper.

Then if $\int_1$ is zero then $\int_1$ is also zero (here we consider both integrals in $L$-equivariant cohomology). If $\int_1 \neq 0$ then the ratio

$$\frac{\int_X 1}{\int_Y 1}$$

(where the integral is taken in $L$-equivariant cohomology) is regular when $l_2 \to 0$.

Lemma 4.2 is an easy corollary of the definition of the above integrals given in Section 2 of [1] and we leave the proof to the reader.

4.3. End of the proof. First of all, we need to show that

$$\int_{U^d} 1 \neq 0.$$

We now want to apply Lemma 4.2 to the case when $X = U^d_{G,P}$, $Y = U^d_C$, $L_1 = T \times \mathbb{C}^*$ where the $\mathbb{C}^*$ factor corresponds to $\varepsilon_1$ and $L_2 = \mathbb{C}^*$ corresponding to $\varepsilon_2$. To avoid confusion in the notation we shall denote the “first” (i.e. horizontal) copy of $\mathbb{C}^*$ by $\mathbb{C}^*_1$ and the other copy by $\mathbb{C}^*_2$. We need to show that the map $(U^d_{G,P})^{T \times \mathbb{C}^*_1} \to (U^d_C)^{T \times \mathbb{C}^*_1}$ is proper. In fact, we claim that the following stronger statement is true:

Lemma 4.4. The map $(U^d_{G,P})^{\mathbb{C}^*_1} \to (U^d_C)^{\mathbb{C}^*_1}$ is an isomorphism.

Proof. This is an easy corollary of Theorem 10.2 of [2]. In loc. cit a natural stratification of $U_{G,P}$ is described and it follows immediately that

$$(U^d_{G,P})^{\mathbb{C}^*_1} = (U^d_C)^{\mathbb{C}^*_1} = \text{Sym}^d(X \setminus \{\infty\})$$

(recall that $X$ denotes the “vertical” axis in $\mathbb{P}^2$). \qed

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