On Generalized $I$–Algebras and 4–valued Modal Algebras

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Abstract

In this paper we establish a new characterization of 4–valued modal algebras considered by A. Monteiro. In order to obtain this characterization we introduce a new class of algebras named generalized $I$–algebras. This class contains strictly the class of $C$–algebras defined by Y. Komori as an algebraic counterpart of the infinite–valued implicative Łukasiewicz propositional calculus. On the other hand, the relationship between $I$–algebras and commutative BCK–algebras, defined by S. Tanaka in 1975, allows us to say that in a certain sense G–algebras are also a generalization of these latter algebras.

1 Introduction

Y. Arai, K. Iseki and S. Tanaka [1] (see also [8, 9, 10, 11]) defined the class of BCK–algebras as algebras $\langle A, *, 0 \rangle$ of type $(2,0)$ which satisfies:

(A1) $((x * y) * (x * z)) * (z * y) = 0$,

(A2) $(x * (x * y)) * y = 0$,

(A3) $x * x = 0$,

(A4) $0 * x = 0$,

(A5) $x * y = 0, y * x = 0$ imply $x = y$.

From A1, . . . , A5 it follows

(A6) The relation $x \leq y$ if and only if $x * y = 0$ is an order on $A$. 
S. Tanaka [22], considered the subclass of conmutative BCK–algebras (or CBCK–algebras) and H. Yutani [24] proved that these are an equational class of algebras characterized by the following identities:

(B1) \((x * y) * z = (x * z) * y\),

(B2) \(x * (x * y) = y * (y * x)\),

(B3) \(x * x = 0\),

(B4) \(x * 0 = 0\).

S. Tanaka [22] also proved that every CBCK–algebra is a meet semilattice for the order defined by A6 where the infimum \(\land\) satisfies:

(B5) \(x \land y = x * (x * y)\).

K. Iseki and S. Tanaka [10] proved that every CBCK–algebra with last element 1 is a lattice where the supremum \(\lor\) verifies:

(B6) \(x \lor y = 1 * ((1 * x) \land (1 * y))\).

T. Traczyk [23] showed that every bounded CBCK–algebra with last element 1 is a distributive lattice.

Y. Komori [13] considered the equational classes of CN–algebras (named Wasjberg algebras by A. J. Rodriguez [21]) and C–algebras (which we name I–algebras). The CN–algebras are the algebraic counterpart of the infinite–valued Lukasiewicz propositional calculus with implication (\(\rightarrow\)) and negation (\(\sim\)). The C–algebras are the algebraic version of the implicational part of this calculus. These algebras are defined as follows.

A Wajsberg algebra (or W–algebra) is an algebra \(\langle A, \rightarrow, \sim, 1 \rangle\) of type \((2, 1, 0)\) which satisfies:

(W1) \(1 \rightarrow x = x\),

(W2) \((x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x\),

(W3) \((x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = 1\),

(W4) \((\sim x \rightarrow \sim y) \rightarrow (y \rightarrow x) = 1\). (see [6, 13, 21])

An I–algebra is an algebra \(\langle A, \rightarrow, 1 \rangle\) of type \((2, 0)\) which verifies:

(I1) \(1 \rightarrow x = x\),

(I2) \((x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x\),
(I3) \((x \to y) \to ((y \to z) \to (x \to z)) = 1\),

(I4) \(((x \to y) \to (y \to x)) \to (y \to x) = 1\). (see \[13, 4\])

An \(I^0\)-algebra is an algebra \(\langle A, \to, 1, 0 \rangle\) of type \((2, 0, 0)\) which satisfies:

(I5) \(0 \to x = 1\).

We are going to denote by \(\text{CBCK}, \text{W}, \text{I}\) and \(I^0\) the varieties of algebras described above respectively.

The following results show the relationship between these varieties

(CI) Let \(A \in \text{CBCK}\) be such that it verifies the additional identity:

\[(x \ast y) \ast ((x \ast y) \ast (y \ast x)) = 0.\]

If we define \(x \to y = y \ast x\) for all \(x, y \in A\), then \(\langle A, \to, 0 \rangle \in I\) and 0 is the last element of \(A\) for the dual order of \(A_6\).

(IC) Let \(A \in I\). If we define \(x \ast y = y \to x\) for all \(x, y \in A\) then \(\langle A, \ast, 1 \rangle \in \text{CBCK}\) and 1 is the first element of \(A\).

(WI) If \(\langle A, \to, \sim, 1 \rangle \in \text{W}\) then the reduct \(\langle A, \to, 1 \rangle \in I\).

(IW) If \(\langle A, \to, 1, 0 \rangle \in I^0\) then defining \(\sim x = x \to 0\) for all \(x \in A\) we have that \(\langle A, \to, \sim, 1 \rangle \in W\).

In 1978 A. Monteiro introduced the 4–valued modal algebras (or \(M_4\)-algebras) as algebras \(\langle A, \land, \lor, \sim, \nabla, 1 \rangle\) of type \((2, 2, 1, 1, 0)\) which verify:

(M1) \(x \land (x \lor y) = x\),

(M2) \(x \land (y \lor z) = (z \land x) \lor (z \land y)\),

(M3) \(\sim \sim x = x\),

(M4) \(\sim (x \lor y) = \sim x \land \sim y\),

(M5) \(\nabla x \lor \sim x = 1\),

(M6) \(\nabla x \land \sim x = \sim x \land x\)\[14\] (see also \[5, 15\])

It is easy to see that every \(M_4\)-algebra satisfies:

(M7) \(1 \lor x = 1\).
From M1, M2, M7, M3, M4 it follows that \( \langle A, \land, \lor, \neg, 1 \rangle \) is a De Morgan algebra with last element 1. Taking into account [16,17] we have that three–valued Lukasiewicz algebras (or L₃–algebras) are M₄–algebras which satisfy:

\[(M^6) \quad \nabla(x \land y) = \nabla x \land \nabla y.\]

I. Loureiro [14], has proved:

\[(M^8) \quad \text{If } A \in M_4 \text{ is non trivial, then there exists a non empty set } X \text{ such that } A \text{ is isomorphic to a subalgebra of } T_4^X, \text{ where } T_4 = \langle T_4, \lor, \land, \neg, \nabla, 1 \rangle, T_4 = \{0, a, b, 1\} \text{ has the diagram of the figure 1 and } \neg, \nabla \text{ are defined by means of the following tables:}\]

\[
\begin{array}{ccc}
\hline
x & \neg x & \nabla x \\
0 & 1 & 0 \\
0 & a & 1 \\
0 & b & 1 \\
1 & 0 & 1 \\
\hline
\end{array}
\]

\[
\text{fig. 1}
\]

If \( A \in M_4 \), the operator \( \triangle \) is defined by the formula:

\[(M^9) \quad \triangle x = \neg \nabla \neg x.\]

Now we are going to indicate different operators of implication defined in an M₄–algebra A:

\[(M^{10}) \quad x \circ y = \neg x \lor y \text{ (this operation has been defined in the De Morgan algebras [7]),}\]

\[(M^{11}) \quad x \rightarrow y = \nabla \neg x \lor y, \text{ (see [5])}\]

\[(M^{12}) \quad x \mapsto y = (x \rightarrow y) \land (\nabla y \lor \neg x),\]

\[(M^{13}) \quad x \triangleright y = (x \mapsto y) \land ((x \circ y) \rightarrow (\triangle \neg x \lor y)).\]

Remark that if \( A \in M_4 \) verifies the Kleene condition \( x \land \neg x \leq y \lor \neg y \), or equivalently if \( A \in L_3 \) then the operators \( \mapsto \) and \( \triangleright \) defined by M12 and M13 respectively coincide with the Lukasiewicz implication.
In $\mathcal{T}_4$, the operations $\Delta$, $\succ$, $\to$, $\mapsto$ and $\succ$ have the following tables:

| $x$ | $\Delta x$ | $\succ$ | $\to$ | $\mapsto$ |
|-----|-------------|--------|------|--------|
| 0   | 0           | 0 1 1 1 1 1 | 0 1 1 1 1 1 | 0 1 1 1 1 1 |
| $a$ | $a$         | $a$ 1 1 1 1 | $a$ 1 1 1 1 | $a$ 1 1 1 1 |
| $b$ | $b$         | $b$ 1 1 1 1 | $b$ 1 1 1 1 | $b$ 1 1 1 1 |
| 1   | 1           | 0 1 1 1 1 1 | 1 0 1 1 1 1 | 1 0 1 1 1 1 |

Furthermore in $\mathcal{T}_4$ it holds

(L1) $x \lor y = (x \succ y) \succ y$,

(L2) $\sim x = x \succ 0$,

(L3) $\nabla x = \sim x \succ x$,

(L4) $x \land y = \sim (\sim x \lor \sim y)$.

These identities give the relationships between the variety $W_3$ of 3–valued Wajsberg algebras and $L_3$.

On the other hand by L1,...,L4 and M8 it results that $M_4$–algebras may be characterized by means of the operations $\succ$ and 0, or $\succ$ and $\sim$.

This fact leads us to pose the following problems:

**Problem 1.** Axiomatize the matrix $\langle T_4, \succ, \sim, D = \{1\} \rangle$.

**Problem 2.** Characterize the $M_4$–algebras by means of the operations $\{\succ, 1, 0\}$ or $\{\succ, \sim, 1\}$.

In this paper we solve the second problem. It is to this end that we introduce a new class of algebras which we name generalized $I$–algebras because it strictly contains the class of $I$–algebras.
2 Generalized $I$–Algebras

Definition 2.1 An algebra $\langle A, \succ, 1 \rangle$ of type $(2, 0)$ is a generalized $I$–algebra (or $G$–algebra) if it satisfies:

(G1) $1 \succ x = x$,
(G2) $x \succ 1 = 1$,
(G3) $(x \succ y) \succ y = (y \succ x) \succ x$,
(G4) $x \succ (y \succ z) = 1$ implies $y \succ (x \succ z) = 1$.

Examples 2.1

(1) The algebra $\langle T_4, \succ, 1 \rangle$ where $T_4$ and $\succ$ are defined before is a G–algebra but it is not an I–algebra.

(2) Let $\langle A, *, 0 \rangle \in \text{CBCK}$. If we define $x \succ y = y * x$, for all $x, y \in A$, then $\langle A, \succ, 0 \rangle \in G$.

Lemma 2.1 If $A \in G$, then it holds:

(G5) $x \succ x = 1$,
(G6) $x \succ y = 1, y \succ x = 1$, imply $x = y$,
(G7) $x \succ (y \succ x) = 1$,
(G8) $x \succ ((x \succ y) \succ y) = 1$,
(G9) $x \succ (z \succ ((x \succ y) \succ y)) = 1$,
(G10) $x \succ y = 1, y \succ z = 1$, imply $x \succ z = 1$,
(G11) $(A, \leq)$ is a partially ordered set, where $\leq$ is given by $x \leq y$ if and only if $x \succ y = 1$,
(G12) $x \leq (x \succ y) \succ y$,
(G13) $y \leq (x \succ y) \succ y$,
(G14) $x \leq y$ implies $y \succ z \leq x \succ z$,
(G15) $x \leq z, y \leq z$ imply $(x \succ y) \succ y \leq z$,
(G16) $(A, \leq)$ is a join semilattice where the supremum, for all $x, y \in A$ are defined by $x \lor y = (x \succ y) \succ y$. 
Proof.

(G5) \( x \succ x = (1 \succ x) \succ x \), \[G1\]
    \[= (x \succ 1) \succ 1, \] \[G3\]
    \[= 1. \] \[G2\]

(G6) If

(1) \( x \succ y = 1 \),
(2) \( y \succ x = 1 \),
then

\[
x = 1 \succ x, \quad [G1]
= (y \succ x) \succ x, \quad [(2)]
= (x \succ y) \succ y, \quad [G3]
= 1 \succ y, \quad [(1)]
= y. \quad [G1]
\]

(G7) (1) \( y \succ (x \succ x) = 1 \), \[G5,G2\]
(2) \( x \succ (y \succ x) = 1 \). \([(1),G4]\)

(G8) It follows from G5 and G8.

(G9) (1) \( x \succ ((x \succ y) \succ y) = 1 \), \[G8\]
(2) \( z \succ (x \succ ((x \succ y) \succ y)) = z \succ 1 = 1 \), \([(1),G2]\]
(3) \( x \succ (z \succ ((x \succ y) \succ y)) = 1 \). \([(2),G3]\)

(G10) If

(1) \( x \succ y = 1 \),
(2) \( y \succ z = 1 \),
then

\[
(x \succ z) = x \succ (1 \succ z), \quad [G1]
= x \succ ((y \succ z) \succ z), \quad [(2)]
= x \succ ((z \succ y) \succ y), \quad [G3]
\]
\[ = x \triangleright ((z \triangleright y) \triangleright (1 \triangleright y)), \quad \text{[G1]} \]
\[ = x \triangleright ((z \triangleright y) \triangleright ((x \triangleright y) \triangleright y)), \quad \text{[(1)]} \]
\[ = 1. \quad \text{[G9]} \]

(G11) It follows from G5, G6 and G10.

(G12) It follows from G8 and G11.

(G13) (1) \( y \triangleright ((x \triangleright y) \triangleright y) = 1, \quad \text{[G7]} \)
(2) \( y \leq (x \triangleright y) \triangleright y. \quad \text{[(1),G11]} \)

(G14) If
(1) \( x \leq y, \)
then
(2) \( y \leq (y \triangleright z) \triangleright z, \quad \text{[G12]} \)
(3) \( x \leq (y \triangleright z) \triangleright z, \quad \text{[(1),(2),G11]} \)
(4) \( 1 = x \triangleright ((y \triangleright z) \triangleright z), \quad \text{[(3),G11]} \)
\[ = (y \triangleright z) \triangleright (x \triangleright z), \quad \text{[G4]} \]
(5) \( y \triangleright z \leq x \triangleright z. \quad \text{[(4),G11]} \)

(G15) If
(1) \( x \leq z, \)
(2) \( y \leq z, \)
then
(3) \( z \triangleright y \leq x \triangleright y, \quad \text{[(1),G14]} \)
(4) \( (x \triangleright y) \triangleright y \leq (z \triangleright y) \triangleright y, \quad \text{[(3),G14]} \)
(5) \( (x \triangleright y) \triangleright y \leq (y \triangleright z) \triangleright z, \quad \text{[(4),G3]} \)
(6) \( (x \triangleright y) \triangleright y \leq z. \quad \text{[(5),(2),G11,G1]} \)

(G16) It follows from G12, G13 and G15.

\[ \square \]

**Definition 2.2** \( A \in G \) is bounded (or \( G^0 \)-algebra) if there exists \( 0 \in A \) such that

(G17) \( 0 \leq x \) for all \( x \in A. \)
If $A \in \mathbf{G}^0$, we can define the unary operation $\sim$ (called negation) by means of the

\begin{align*}
(G18) \quad \sim x &= x \succ 0.
\end{align*}

**Lemma 2.2** If $A \in \mathbf{G}^0$ then it holds:

\begin{align*}
(G19) \quad \sim \sim x &= x, \\
(G20) \quad x \leq y \text{ implies } \sim y &\leq \sim x, \\
(G21) \quad \sim (\sim x \lor y) &\leq x, \\
(G22) \quad \sim (\sim x \lor y) &\leq y, \\
(G23) \quad z \leq x, z \leq y \text{ imply } z \leq \sim (\sim x \lor y), \\
(G24) \quad A \text{ is a meet semilattice where for all } x, y \in A \text{ the infimum satisfies} \\
&\quad x \land y = \sim (\sim x \lor \sim y) = (((x \succ 0) \succ (y \succ 0)) \succ (y \succ 0)) \succ 0, \\
(G25) \quad \sim (x \land y) = \sim x \land \sim y, \\
(G26) \quad \sim (x \land y) = \sim x \lor \sim y, \\
(G27) \quad (x \succ y) \lor (y \succ z) &\leq (x \land y) \succ z.
\end{align*}

**Proof.** We only prove G27.

\begin{align*}
(G27) \quad (1) \quad x \land y &\leq x, & [G21,G24] \\
(2) \quad x \land y &\leq y, & [G22,G24] \\
(3) \quad x \succ z &\leq (x \land y) \succ z, & [(1),G14] \\
(4) \quad y \succ z &\leq (x \land y) \succ z, & [(2),G14] \\
(5) \quad (x \succ z) \lor (y \succ z) &\leq (x \land y) \succ z. & [(3),(4),G15,G16]
\end{align*}

\[ \square \]

**Definition 2.3** $A \in \mathbf{G}^0$ is distributive (or $\text{DG}^0$–algebra) if it satisfies:

\begin{align*}
(DG1) \quad (x \land y) \succ z &\leq (x \succ z) \lor (y \succ z).
\end{align*}

If $A \in \text{DG}^0$ then it holds

\begin{align*}
(DG2) \quad (x \land y) \succ z &\leq (x \succ z) \lor (y \succ z).
\end{align*}
Examples 2.2

(1) The $G^0$–algebra $\langle T_4, \succ, 1 \rangle$ is distributive.

(2) Let $\langle A, \succ, 1 \rangle \in G^0$ where $A = \{0, a, b, c, 1\}$ and $\succ$ is given by the following table

| $\succ$ | 0   | a   | b   | c   | 1   |
|---------|-----|-----|-----|-----|-----|
| 0       | 1   | 1   | 1   | 1   | 1   |
| a       | a   | 1   | b   | c   | 1   |
| b       | b   | a   | 1   | c   | 1   |
| c       | c   | a   | 1   | b   | 1   |
| 1       | 0   | a   | b   | c   | 1   |

Then $\langle A, \leq \rangle$ has the diagram of figure 2

\[ \text{fig. 2} \]

A is not distributive, since $(a \land b) \succ c = 1$, $(a \succ b) \lor (b \succ c) = c$ and $1 \neq c$.

Theorem 2.1 If $A \in DG^0$ then $A$ is a De Morgan algebra for the operations $\lor, \land, \sim$ defined in L1, L4 and L2.

Proof. Taking into account 2.5, 2.8, G29 and G25 we only prove the distributive law B7 or equivalently the cancellation law

(CL) $x \land y = x \land z$, $x \lor y = x \lor z$ imply $y = z$:

If

(1) $x \land y = x \land z$,
(2) $x \lor y = x \lor z$,

then
(3) 1 = (x \land z) \triangleright z, \quad [G24,G11]
    = (x \land y) \triangleright z, \quad [1]
    = (x \triangleright z) \lor (y \triangleright z), \quad [DG2]

(4) 1 = y \triangleright (y \lor z), \quad [G11,G16]
    = y \triangleright (x \lor z), \quad [2]
    = y \triangleright ((x \triangleright z) \triangleright z), \quad [G16]
    = (x \triangleright z) \triangleright (y \triangleright z), \quad [G4]

(5) x \triangleright z \leq y \triangleright z, \quad [(4),G11]

(6) 1 = y \triangleright z, \quad [(5),(3),G16]

(7) y \leq z. \quad [(6),G11]

Similar we prove

(8) z \leq y.

CL It result from (6), (7) and G11.

\[\square\]

3 Modal \textit{G}^{0}_{4}–Algebras

It is easy to prove

**Lemma 3.1** Let \(A \in G\). The following identities are equivalent:

\((G'28)\) \(((x \triangleright (x \triangleright y)) \triangleright x) \triangleright x = 1,\)

\((G'29)\) \((x \triangleright (x \triangleright y)) \lor x = 1,\)

\((G'30)\) \((x \triangleright (x \triangleright y)) \triangleright x = x.\)

**Definition 3.1** \(A \in G\) or \(A \in G^{0}\) is a \(G_{4}\)–algebra or \(G_{4}^{0}\)–algebra respectively if it satisfies \(G'28\).

**Example 3.1** Let \(C_{n+1} = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\}\), where \(n \geq 3\). For \(x, y \in C_{n+1}\) we define \(x \triangleright y = \text{min} \{1, 1 - x + y\}\), then \(\langle C_{n+1}, \triangleright, 1 \rangle\) is a \(G^{0}\)–algebra which in not an \(G_{4}^{0}\)–algebra.

For each \(A \in G^{0}_{4}\) we define the operators \(\triangleright, \triangleright\), by means of the formulas:

\((G'31)\) \(x \triangleright y = x \triangleright (x \triangleright y),\)
Lemma 3.2 If $A \in G_4^0$, then it holds:

\((G'32)\) $\nabla x = \sim x \succ x$.

\((G'33)\) $\sim x \rightarrow x = \sim x \succ x$,
\[(G'34)\] $x \leq \nabla x$.

**Proof.** We only prove $G'33$

\[(G'33)\] (1) $1 = x \lor ((x \succ (x \succ 0)))$,
= $(x \succ (x \succ \sim)) \succ (x \succ \sim x)$,
= $(x \rightarrow \sim x) \succ (x \succ \sim x)$,
\[(2)\] $x \rightarrow \sim x \leq x \rightarrow \sim x$.
\[(3)\] $x \rightarrow \sim x \leq x \rightarrow \sim x$.

Now $G'33$ follows from (2), (3) and G11.

\[\square\]

The following Lemma is an immediately consequence of the results and definitions given above

**Lemma 3.3** Let $A \in G_4^0$. The following identities are equivalent:

\[(G'35)\] $\nabla x \rightarrow 0 = \nabla x \succ 0$,
\[(G'36)\] $\sim \nabla x \succ \nabla x = \nabla x$,
\[(G'37)\] $\sim \nabla x \lor \nabla x = 1$,
\[(G'38)\] $\nabla x \lor \sim \nabla x = 0$.

**Remark 3.1** We do not know if $G'35$ holds in any $G_4^0$-algebra. We shall denote by $DG_4^0$ the class of distributive $G_4^0$-algebras.

**Definition 3.2** $A \in DG_4^0$ is a modal $G_4^0$-algebra (or $MDG_4^0$-algebra) if it verifies $G'35$.

**Theorem 3.1** Let $A \in MDG_4^0$ and $\lor, \sim, \nabla, \land$ be defined by the formulas L1, .. , L4. Then $B = (A, \lor, \land, \sim, \nabla, 1) \in M_4$ and it satisfies $x \rightarrow y = (x \leftrightarrow y) \land ((x \circ y) \rightarrow (\triangle \sim x \lor y))$, where $\triangle, \circ, \rightarrow, \leftrightarrow$ are defined by M9, .. , M12 respectively.

**Proof.** Taking into account the theorem 2.12, we only need to prove M5 and M6.

\[(M5)\] $x \lor \nabla x$,
\[ \sim x \lor (\sim x \succ x), \quad \text{[G'}33] \]
\[ = \sim x \lor (\sim x \rightarrow x), \quad \text{[G'}29] \]
\[ = 1. \]
(M6) \[ \sim x \land \nabla x = (\sim x \land \nabla x) \lor 0 = (\sim x \land \nabla x) \land \sim 1, \quad \text{[M5]} \]
\[ = (\sim x \land \nabla x) \lor (\sim x \lor \nabla x), \quad \text{[G25,G19]} \]
\[ = (\sim x \land \nabla x) \lor (x \land \sim \nabla x), \]
\[ = (\sim x \land \nabla x \land x) \lor (\sim x \land \nabla x \land \sim \nabla x), \quad \text{[G'}34,G'}38 \]
\[ = \sim x \land x. \]

\[ \blacksquare \]

The converse of this results follows immediately of the representation theorem M8 of I. Loureiro.

**Theorem 3.2** Let \( \langle A, \lor, \land, \sim, \nabla, 1 \rangle \in M_4 \). If we define \( \succ \) by means of M13, then \( \langle A, \succ , 1 \rangle \in MDG^0_4 \), where \( 0 = \sim 1 \), and it satisfies L1, \ldots ,L4.

**Final Conclusion**

From the above results it follows that an axiomatization for the 4–valued modal algebras defined by the means of the operations \{\( \succ, \sim, 1 \}\} is:

(C1) \( 1 \succ x = x, \)
(C2) \( x \succ 1 = 1, \)
(C3) \( (x \succ y) \succ y = (y \succ x) \succ x, \)
(C4) If \( x \succ (y \succ z) = 1 \), then \( y \succ (x \succ z) = 1, \)
(C5) \( ((x \succ (x \succ y)) \succ x) \succ x = 1, \)
(C6) \( \sim 1 \succ x = 1, \)
(C7) \( x \succ \sim 1 = \sim x, \)
(C8) \( ((x \lor \sim y) \succ z) \succ ((x \succ z) \land (y \succ z)) = 1, \)

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where \( a \lor b \) denotes \((a \triangleright b) \triangleright b\).

We have named modal algebras to the algebras which satisfy \( G'35 \) because it is in this variety where the operator defined by \( G'33 \) has the modal properties M5 and M6.

On the other hand, for all \( A \in G \) we can define the operators \( \Rightarrow_i, i = 0, 1, \ldots \) by means of the formulas \( x \Rightarrow_0 y = y, \ x \Rightarrow_{i+1} y = x \triangleright (x \Rightarrow_i y) \). Then we say that \( A \in G \) is \((n + 1)\)-valued (or \( G_{n+1} \)-algebra) if it satisfies the identity:

\[
(x \Rightarrow_n y) \lor x = 1.
\]

We believe that \( G_{n+1} \)-algebras are an interesting generalization of the class of \((n + 1)\)-valued \( I \)-algebras. This terminology is analogous to the \((n + 1)\)-valued Wajsberg algebras, and taking it into account we think that it is more appropriated to call 3-valued modal algebras to the 4-valued modal algebras.

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