Composition operators on reproducing kernel Hilbert spaces with analytic positive definite functions

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Abstract. In this paper, we specify what functions induce the bounded composition operators on a reproducing kernel Hilbert space (RKHS) associated with an analytic positive definite function defined on $\mathbb{R}^d$. We prove that only affine transforms can do so in a certain large class of RKHS. Our result covers not only the Paley-Wiener space on the real line, studied in previous works, but also much more general RKHSs corresponding to analytic positive definite functions, where existing methods do not work. Our method only relies on intrinsic properties of the RKHSs, and we establish a connection between the behavior of composition operators and asymptotic properties of the greatest zeros of orthogonal polynomials on a weighted $L^2$-space on the real line. We also investigate the compactness of the composition operators and show that any bounded composition operators cannot be compact in our situation.

1 Introduction

In this paper, we establish that the composition operator generated by a map in the Euclidean space $\mathbb{R}^d$ enjoys the boundedness property in a reproducing kernel Hilbert space (RKHS for short) only if the map is affine when the reproducing kernel is an analytic positive definite function with some conditions. In addition, we will characterize affine maps which induce bounded composition operators. We summarize several basic notation at the end of this section.

Recall the definition of composition operators. Let $f : E \to E'$ be a map from a set $E$ to a set $E'$, and let $V$ and $W$ be function spaces on $E$ and $E'$, respectively. The composition operator $C_f : f \mapsto h \circ f$ is the linear operator $(C_f, \mathcal{D}(C_f))$ from $W$ to $V$ whose domain is $\mathcal{D}(C_f) = \{h \in W : h \circ f \in V\}$. Since a composition operator is always a closed operator, it is worth remarking that the preserving property, that is, $\mathcal{D}(C_f) = W$ implies the boundedness of $C_f$

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if $V$ and $W$ are sufficiently good topological linear spaces, for example, Banach spaces.

We also recall the notion of RKHSs. A function $k$ defined on the cross product of $E \times E$, where $E$ is a set, is said to be positive definite if for any arbitrary function $X : E \to \mathbb{C}$ and for any finite subset $F$ of $E$, $\sum_{p,q \in F} X(p)X(q)k(p,q) \geq 0$.

A fundamental theorem in the theory of RKHSs is that such a function $k$ generates a unique reproducing kernel Hilbert space $H_k$. See [20], for example.

Now, let us state our main result. We will adopt the following definition of the Fourier transform:

$$\widehat{h}(\xi) = \mathcal{F}[h](\xi) := \int_{\mathbb{R}^d} h(x)e^{-2\pi i x \cdot \xi} dx.$$  

Let $0 \neq w \in L^1 \cap L^\infty(\subset L^2)$ and assume $w \geq 0$ almost everywhere. Then, $\widehat{w}$ is a positive definite function, namely,

$$k(x,y) := \widehat{w}(x-y)$$

becomes a positive definite kernel, and thus $k$ determine a RKHS $H_k$ (see Section 2 for more details). The RKHS $H_k$ is realized in the space of continuous and square integrable functions on $\mathbb{R}^d$ whose Fourier transform vanishes almost everywhere on $\{w = 0\}$, and its norm is given by

$$\|f\|_{H_k} = \sqrt{\int_{\mathbb{R}^d} |\widehat{f}(\xi)|^2 w(\xi)^{-1} d\xi} = \|\widehat{f}\|_{L^2(w^{-1})} \quad (f \in H_k).$$

It is worth considering the case of $d = 1$ and $w = 1_{[-1/2,1/2]}$. In this case, the RKHS constructed by $w$ is called the Paley-Wiener space. This space is composed of functions whose fourier transforms are supported on $[-1/2,1/2]$. As warping operators, many researchers have been studying the condition for which the $\mathcal{D}(C_f)$ is equal to $H_k$, which is equivalent to the boundedness of $C_f$, in the field of signal processing, a branch of engineering [1, 3, 5–7, 23]. In [17], a general dimensional case is treated although their setting slightly differs from ours. We also note that composition operators have recently attracted researchers in a field of data science and machine learning [13, 14, 16], and mathematical properties of composition operators are quite important to provide their theoretical guarantee.

For $z \in \mathbb{C}^d$, we denote by $m_z$ the pointwise multiplication operator on $L^2(w)$:

$$m_z[h](\xi) := e^{z^\top \xi} h(\xi),$$
where $\top$ stands for the transpose of vectors or matrices. For each $n \in \mathbb{N}$, the space $P_n \subset \mathbb{C}[\xi_1, \xi_2, \ldots, \xi_d]$ stands for the linear space of all polynomials of (total) degree at most $n$. In the case of $|\xi|^{2n} w(\xi) \in L^1$, we regard $P_n \subset L^2(w)$, and for any $y \in \mathbb{R}^d$, we define

$$E_n^+(y; w) := n \sqrt{\sup_{P \in P_n \setminus \{0\}} \frac{\|m_y P\|_{L^2(w)}}{\|P\|_{L^2(w)}}} = \|m_y| P_n \|_{L^1(w)}^{1/n},$$

$$E_n^-(y; w) := n \sqrt{\sup_{P \in P_n \setminus \{0\}} \frac{\|P\|_{L^2(w)}}{\|m_y P\|_{L^2(w)}}} = \|m_y|^{-1} P_n \|_{L^1(w)}^{1/n}.$$

We define

$$\mathcal{G}(w) := \left\{ A \in \text{GL}_d(\mathbb{R}) : w(A^\top \xi) \geq \lambda w(\xi) \text{ a.e. } \xi \text{ for some } \lambda > 0 \right\}.$$

We will check that $\mathcal{G}(w)$ coincides with the linear maps inducing a bounded composition operator on $H_k$ (see Proposition 9).

Under certain assumptions on $w$, the boundedness of composition operators force the original maps to be affine as our main result below shows:

**Theorem 1.** Let $w \in L^1 \cap L^\infty \setminus \{0\}$ be non-negative almost everywhere, and let $k(x, y) := \hat{w}(x - y)$.

We impose the following three assumptions on $w$:

(A) for any $a > 0$, there exists $c_a > 0$ such that $w(\xi) \leq c_a e^{-a|\xi|}$ for almost all $\xi \in \mathbb{R}^n$,

(B) there exists $B > 0$ such that for any $y \in \mathbb{R}^d$,

$$\limsup_{n \to \infty} E_n^-(y; w), \limsup_{n \to \infty} E_n^+(y; w) < B,$$

(C) $\mathcal{G}(w)$ spans $M_d(\mathbb{R})$, that is, $\langle \mathcal{G}(w) \rangle \mathbb{R} = M_d(\mathbb{R})$.

Then, for any open set $U \subset \mathbb{R}^d$ and any map $f : U \to \mathbb{R}^d$, the map $f$ is a restriction of an affine map of the form,

$$f(x) = Ax + b$$

with $A \in \mathcal{G}(w)$ and $b \in \mathbb{R}^d$ if and only if $D(C_f) = H_k$, and thus the composition operator $C_f : H_k \to H_{k|U^2}$ is bounded.
We will prove Theorem 1 as a corollary of Theorems 3 and 4. Our method is based on an intrinsic structure of $L^2(w)$, and thus can treat a quite general class of $w$ on higher dimensional Euclidean space $\mathbb{R}^d (d \geq 1)$. For some special cases, we have a more concrete result as follows:

**Theorem 2.** Let $w \in L^1 \cap L^\infty \setminus \{0\}$ be a nonnegative spherical function. Assume that there exists a locally $L^1$-function $Q : [0, \infty) \to \mathbb{R}$ such that $w(\xi) = e^{-Q(|\xi|)}$. We further assume that there exists $c \geq 0$ such that $Q(t) + ct$ is non-decreasing for sufficiently large $t \geq 0$ and that $Q(t + R) - Q(t) \to \infty$ as $t \to \infty$ for some $R > 0$. Then, the function $w$ satisfies Assumptions (A), (B), and (C). Thus, if a map induces bounded composition operators on the RKHS $H_k$ associated to $k(x, y) = \hat{w}(x - y)$, then the original map is affine.

For example, $w(\xi) = |\xi|^{1-|\xi|}, \Gamma(|\xi|)^{-1}, |\xi|^\alpha e^{-|\xi|\beta(|\xi|)}$, where $\alpha \geq 0$ and $\beta : [0, \infty) \to \mathbb{R}$ is a non-decreasing function going to $\infty$, satisfy the conditions in Theorem 2. We actually obtain a more general result (Theorem 6), and Theorem 2 above is a corollary (see Corollary 9) of Theorem 6. We also obtain a similar result in the case where $w$ is a tensor product of even functions on $\mathbb{R}$ (see Section 4.4 for details).

Our results cover the previous works [6, 7], namely, we see that Assumptions (A)–(C) in Theorem 1 hold if $d = 1$ and $w$ is compactly supported, since $m_z$ is a bounded linear isomorphism on $L^2(w)$ and $1 \in \mathcal{G}(w)$. Needless to say, $1_{[-\pi, \pi]}$ satisfies Assumptions (A)–(C) in Theorem 1. Thus, only affine maps can induce bounded linear operators on the Paley–Wiener space in this case.

Furthermore, our result also provides a non-trivial improvement even in a one-dimensional case. In fact, in [6, 7], their proof is based on finiteness of the order of the entire function $\hat{w}$. In their situation, the RKHS is composed of entire functions of order at most 1, and they directly use Pólya’s theorem [19] with some careful analysis, in other words, finiteness of the order of $\hat{w}$ is crucial in their proof. In contrast, our method does not need finiteness of the order, and by means of Theorem 2, we actually find an example of RKHS containing entire functions of infinite order, but only affine maps can induce bounded composition operators (see Section 4.5 for details) as in the following corollary:

**Corollary 1.** Assume $d = 1$. Let

$$w(\xi) = \sum_{n = -\infty}^{\infty} \frac{1_{[-1/2+n,1/2+n]}(\xi)}{|n|!}.$$ 

Then, if a map induces bounded composition operators on the RKHS $H_k$ associated to $k(x, y) = \hat{w}(x - y)$, then the original map is affine, but the RKHS $H_k$ contains entire functions of infinite order.
In addition, in previous works, they always impose some good properties, such as entireness, on the original map. On the other hand, we do not need to assume the map $f$ is entire in advance as we prove the boundedness of $C_f$ automatically induces the holomorphy and continuity of $f$ (Theorem 4).

The compactness of composition operators is an important problem as well. However, unfortunately, compactness fails as our next theorem shows.

**Corollary 2.** *Using the same notation as Theorem 1, let $w$ satisfy Assumptions (A)–(C). Then, no composition operators $C_f$ can be compact.*

We can rephrase Corollary 2 in Proposition 10 on the basis of Theorem 1, namely, thanks to Theorem 1, it suffices to prove that affine maps cannot induce compact composition operators. See Section 3.1 for the details.

Under Assumption (A), $H_k$ is composed of entire functions on $\mathbb{C}^d$, thus our study is also located as a study of composition operators on entire functions. This topic has been extensively studied, for example, as in [2, 8, 9, 21] and so on. However, the function space $H_k$ in our study is quite different from those in these previous literature. As a result, the behavior of composition operators dramatically changes compared with them, for example, composition operators therein can become compact linear operators, but does not in our situation as in the above corollary.

Let us explain the outline of the proof of Theorem 1 and 2. Regarding Theorem 1, the “if” part is not so hard, and we prove it in Section 3.1. The harder part of the proof is the “only if” part, and is obtained as a corollary of Theorems 3 and 4 in Section 3. Theorem 3 states that the “only if” part of Theorem 1 holds under Assumptions (A), (B), and (C), and assuming the existence of the holomorphic map $F : \mathbb{C}^d \to \mathbb{C}^d$ with $F|_U = f$. Since there exists a natural isomorphism $\Psi_w : H_k \cong L^2(w)$ (see Corollary 4), the study of composition operators is reduces to that of the corresponding operators on $L^2(w)$. Thus, by considering the action of the corresponding operators on spaces of polynomials of various degrees in $L^2(w)$, the boundedness of the composition operators enables us to control the derivative of the holomorphic map $F$. Theorem 4 deduces that under Assumption (A) and $\mathcal{D}(C_f) = H_k$, there exists a holomorphic map $F : \mathbb{C}^d \to \mathbb{C}^d$ with $F|_U = f$. The proof involves an explicit construction of the analytic continuation of $f$ in terms of the boundedness of $C_f$. Here, we emphasize that $f$ is originally a mere map defined on an open subset $U \subset \mathbb{R}^d$, not the whole space, but we prove that the boundedness of $C_f$ shows the map $f$ is a restriction of holomorphic map $F$ defined on $\mathbb{C}^d$. As for Theorem 2, the hardest part in the proof is to verify Assumption (B). We show that the left hand side in Assumption (B) is closely related to an asymptotic behavior of greatest zeros of orthogonal polynomials in a weighted $L^2$-space on $\mathbb{R}$. We improve...
Freud’s methods in his works [11, 12] on asymptotic behaviors of orthogonal polynomials, and check Assumption (B). We include the details in Section 4.

**Notation:** In this paper, we always work in the $d$-dimensional Euclidean space. For $r \geq 0$, we denote by $C^r$ the space of $C^r$ functions on $\mathbb{R}^d$. For a non-negative measurable function $w$ on $\mathbb{R}^d$, and $p \in [1, \infty)$, we denote by $L^p(w)$ the space composed of the equivalent classes of measurable functions $h$ vanishing on $\{w = \infty\}$ such that $\int_{\mathbb{R}^d} |h(x)|^p w(x) dx < \infty$. In the case of $w \equiv 1$, we abbreviate $L^p(1)$ to $L^p$. We also denote by $L^\infty$ the space of essentially bounded measurable functions on $\mathbb{R}^d$. For any measurable set $A$ in an Euclidean space, we denote by $1_A$ the characteristic function supported on $A$, and by $|A|$ the volume of $A$ with respect to the Lebesgue measure. We denote by $\mathbb{M}_d(\mathbb{R})$ (resp. $\text{GL}_d(\mathbb{R})$) the set of square real matrices of size $d$ (resp. regular real matrices of size $d$).

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## 2 Preliminaries

In this section, we review the notion of reproducing kernel Hilbert spaces associated with positive definite functions and composition operators and then show some of their basic properties.

### 2.1 Reproducing kernel Hilbert spaces with positive definite functions

Let $E$ be an arbitrary abstract (non-void) set and $k : E \times E \rightarrow \mathbb{C}$ be a map. Denote by $\mathbb{C}^E$ the linear space of all maps from $E$ to $\mathbb{C}$. A *reproducing kernel Hilbert space* (RKHS for short) with respect to $k$ is a Hilbert space $H_k \subset \mathbb{C}^E$ satisfying the following two conditions:

1. For any $x \in E$, the map $k_x := k(x, \cdot)$ is an element of $H_k$.
2. For any $x \in E$ and $h \in H_k$, we have $\langle h, k_x \rangle_{H_k} = h(x)$.

If such $H_k$ exists, $H_k$ is unique as a set and we call $k$ a *positive definite kernel*. We note that if $k$ is a positive definite function in the sense described in Introduction, there exists a unique Hilbert space $H_k \subset \mathbb{C}^E$ satisfying the above two
conditions (see, [20] for more detail). The second condition is known as the reproducing property of the RKHS $H_k$. Here, we define the feature map by

$$\phi_k : E \rightarrow H_k; \ x \mapsto k_x.$$ 

For any subset $F \subset E$, we define a closed subspace of $H_k$ by

$$H_{k,F} = \overline{\langle \phi_k(F) \rangle}_C,$$

which is the closure of the linear subspace generated by the set $\phi_k(F)$. Accordingly, we see that $H_{k,F}$ is isomorphic to $H_{k\vert_F^2}$:

**Proposition 1.** Let $k$ be a positive definite kernel on $E$, and let $F \subset E$. Then, the restriction map $\mathbb{C}^E \rightarrow \mathbb{C}^F$ induces an isomorphism $r_{k,F} : H_{k,F} \rightarrow H_{k\vert_F^2}$.

**Proof.** For any $x \in F$, the restriction map allocates $\phi_{k\vert_F^2}(x)$ to $\phi_k(x)$; thus, it induces the isomorphism $r_{k,F}$ between the Hilbert spaces.

**Definition 1.** A map $u : \mathbb{R}^d \rightarrow \mathbb{C}$ is a positive definite function if \( k(x,y) := u(x − y) \) is a positive definite kernel. We call $H_k$ the RKHS associated with $u$.

Thanks to Bochner’s theorem [15, p. 148], a positive definite function on $\mathbb{R}^d$ can be realized as a Fourier transform of a finite Borel measure; namely, we have the following proposition.

**Proposition 2.** Let $u$ be a non-zero $\mathbb{C}$-valued continuous function on $\mathbb{R}^d$. Then, $u$ is a positive definite function if and only if there exists a finite Borel measure $\mu$ on $\mathbb{R}^d$ such that

$$u(x) = \hat{\mu}(x) := \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \xi} \ d\mu(\xi).$$

We shall now prove a proposition for the feature map for an RKHS associated with a positive definite function $u$:

**Proposition 3.** Let $u$ be a non-zero positive definite function. Assume that $u$ is continuous and $u(x) \rightarrow 0$ as $|x| \rightarrow \infty$. Write $k(x,y) := u(x − y)$ as above. Then the feature map $\phi_k : \mathbb{R}^d \rightarrow H_k$ is injective and continuous. Moreover, the inverse $\phi_k^{-1} : \phi_k(\mathbb{R}^d) \rightarrow \mathbb{R}^d$ is also continuous.

**Proof.** Since $\|\phi_k(x) − \phi_k(y)\|_{H_k}^2 = 2u(0) − 2\text{Re}(u(x − y))$, the continuity of $u$ implies that of $\phi_k$. We will prove the injectivity of $\phi_k$. Suppose to the contrary that there exists $a, b \in \mathbb{R}^d$ with $a \neq b$ such that $\phi_k(a) = \phi_k(b)$. Then, for any $x \in \mathbb{R}^d$,

$$u(x) = \langle \phi_k(a+x), \phi_k(a) \rangle_{H_k} = \langle \phi_k(a+x), \phi_k(b) \rangle_{H_k} = u(x + (a − b)).$$
which contradicts the assumption that \( u \) vanishes at infinity. Thus, the feature map is injective. Now, we prove the continuity of \( \phi_k^{-1} : \phi_k(\mathbb{R}^d) \to \mathbb{R}^d \). Suppose that there exists a sequence \( \{ \phi_k(x_n) \}_{n \geq 1} \) such that \( \phi_k(x_n) \to \phi_k(a) \) for some \( a \in \mathbb{R}^d \) but that \( \{ x_n \}_{n \geq 0} \) does not converge to \( a \). Since \( \phi_k \) is injective and continuous, any convergent subsequence of \( \{ x_n \}_{n \geq 0} \) converges to \( a \); thus, we may assume \( |x_n| \to \infty \) as \( n \to \infty \). For any \( x \in \mathbb{R}^d \),

\[
\begin{align*}
  u(x) &= \langle \phi_k(x+a), \lim_{n \to \infty} \phi_k(x_n) \rangle_{H_k} \\
       &= \lim_{n \to \infty} \langle \phi_k(x+a), \phi_k(x_n) \rangle_{H_k} \\
       &= \lim_{n \to \infty} u(x+a-x_n) \\
       &= 0.
\end{align*}
\]

Since \( u \) is not a constant function, this is a contradiction. Thus, \( \phi_k^{-1} \) is continuous.

By the Riemann-Lebesgue theorem, we have the following corollary:

**Corollary 3.** Assume \( H_k \) is an RKHS associated with a positive definite function in the form of \( \hat{\omega} \) for some non-negative \( w \in L^1 \cap L^\infty \setminus \{0\} \). Then the feature map \( \phi_k \) is a homeomorphism from \( \mathbb{R}^d \) onto \( \phi_k(\mathbb{R}^d) \).

In this paper, we will only consider the RKHSs associated with a positive definite function in the form of \( \hat{\omega} \) for some \( w \in L^1 \cap L^\infty \setminus \{0\} \).

We have an explicit description of the RKHS \( H_k \) above:

**Proposition 4.** Let \( w \in L^1 \cap L^\infty \setminus \{0\} \), and let \( k(x,y) = \hat{\omega}(x-y) \). Then,

\[
H_k = \left\{ h \in C^0 \cap L^2 : \hat{h} \in L^2(w^{-1}) \right\}
\]

and its inner product is given by \( \langle g, h \rangle_{H_k} = \int_{\mathbb{R}^d} \hat{g}(\xi) \hat{h}(\xi) w(\xi)^{-1} d\xi \). In particular, the Fourier transform of any element in \( H_k \) is in \( L^1 \).

**Proof.** Since \( L^2 \supset L^2(w^{-1}) \), we define \( V := \mathcal{F}^{-1} \left( L^2(w^{-1}) \right) \) be a Hilbert space with inner product \( \langle g, h \rangle_V := \langle \hat{g}, \hat{h} \rangle_{L^2(w^{-1})} \). By direct computation, we see that \( k_x \in V \) and for any \( h \in V \), \( \langle h, k_x \rangle_V = h(x) \) almost all \( x \in \mathbb{R}^d \). We will show \( V \) is regarded as a subspace of \( C^0_\ast \), namely, any element \( h \in V \) has a continuous representative. For \( x \in \mathbb{R}^d \), let \( \hat{h}(x) := \langle h, k_x \rangle_V \). Since \( \hat{h}(x) = h(x) \) for almost all \( x \in \mathbb{R}^d \), it suffices to show that the map \( \hat{h} \) is continuous. In fact, it is a consequence of the following inequality:

\[
|\hat{h}(x) - \hat{h}(y)| \leq ||h||_V \cdot (k(x,x) + k(y,y) - k(x,y) - k(y,x))^{1/2} \quad (x, y \in \mathbb{R}^d).
\]
This inequality can be deduced from Schwartz’s inequality. Thus, we may re-
gard \( V \subset C^0 \subset \mathbb{C}^d \). By the uniqueness of the RKHS, we have \( H_k = V \).

We define \( \Psi_w : H_k \to L^2(w) \) by

\[
\Psi_w(h)(\xi) := w(\xi)^{-1} \hat{h}(\xi).
\]

Then, we immediately obtain the following corollary:

**Corollary 4.** Let \( w \in L^1 \cap L^\infty \setminus \{0\} \) be a non-negative measurable function. Write \( k(x, y) = \hat{w}(x - y) \). Then, \( \Psi_w \) is an isomorphism from \( H_K \) to \( L^2(w) \).

### 2.2 Properties of RKHSs for positive definite functions with a certain decay condition

Let \( w \in L^1 \cap L^\infty \setminus \{0\} \) be a non-negative measurable function. We define the following decay condition: For a positive integer \( n > 0 \) and positive real number \( a \geq 0 \), we define a function \( u \) that satisfies

\[
(\text{DC})_{n,a} \quad w(\xi) \leq L_\varepsilon (1 + |\xi|^{2n+d+\varepsilon})^{-1}e^{-4\pi a |\xi|} \quad \text{a.e. } \xi.
\]

For \( a > 0 \), we define

\[
X^d_a := \left\{ z = (z_i)_{i=1}^d \in \mathbb{C}^d : |\text{Im}(z)| < a \right\}.
\]

We also define

\[
X^d_0 := \left\{ z = (z_i)_{i=1}^d \in \mathbb{C}^d : \text{Im}(z) = 0 \right\} = \mathbb{R}^d.
\]

By virtue of Proposition 4, if \( u \) satisfies \( (\text{DC})_{n,a} \) for some \( a > 0 \) (resp. \( a = 0 \)), then any element of \( H_k \) is holomorphic on \( X^d_a \) (resp. \( C^n \) on \( X^d_0 \)). As a result, we have the following proposition:

**Proposition 5.** Let \( w \in L^1 \cap L^\infty \setminus \{0\} \) be a non-negative function satisfying the condition \( (\text{DC})_{n,a} \) for some integer \( n > 0 \) and \( a > 0 \), and let \( k(x, y) = \hat{w}(x - y) \). Then, we have

\[
H_{k,U} = H_k
\]

for any open subset \( U \subset \mathbb{R}^d \).

**Proof.** It suffices to prove that \( H_{k,U}^\perp = \{0\} \). Take an arbitrary \( g \in H_{k,U}^\perp \). Then, we see that for any \( x \in U \),

\[
g(x) = \langle \phi_k(x), g \rangle_{H_k} = 0.
\]

Since \( g \) is an analytic function on \( \mathbb{R}^d \), we have \( g = 0 \). Therefore, \( H_{k,U}^\perp = \{0\} \).
Under the condition $(DC)_{n,a}^d$, for each \( z \in \mathcal{X}_a^d \), we define \( e_z \in L^2(w) \) by
\[
e_z(\xi) := e^{-2\pi i \xi^T \xi},
\]
and we define the map,
\[
\varphi : \mathcal{X}_a^d \longrightarrow L^2(w); z \mapsto e_z.
\]

We should remark that in the case of \( a > 0 \), for any \( z \in \mathcal{X}_a^d \), \( \varphi(z) = \Psi_w(\phi_k(z)) \), where \( \phi_k(z)(x) \) is defined as the evaluation of the analytic continuation of \( u \) at \( x - z \) (note that \( u \) is originally defined on \( \mathbb{R}^d \)). Accordingly, we have the following proposition:

**Proposition 6.** Let \( w \in L^1 \cap L^\infty \setminus \{0\} \) be a non-negative function satisfying the condition $(DC)_{n,a}^d$ for some integer \( n > 0 \) and \( a > 0 \) (resp. \( a = 0 \)). Then the map \( \varphi \) is holomorphic (resp. differentiable) in \( \mathcal{X}_a^d \) in the sense that for any \( z = (z_j)_{j=1}^d \in \mathcal{X}_a^d \), the limit
\[
\partial_{z_j} \varphi(z) := \lim_{\varepsilon \to 0} \varepsilon^{-1} (\varphi(z + \varepsilon e_j) - \varphi(z))
\]
exists. Here, \( e_j := (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{C}^d \) denotes the \( j \)-th elementary vector. Moreover, for any \( d \)-variable polynomial (resp. \( d \)-variable polynomial of degree smaller than or equal to \( n \)) \( q \in \mathbb{C}[\xi_1, \ldots, \xi_d] \), we have
\[
\left[ q \left( \frac{i\partial_{z_1}}{2\pi}, \ldots, \frac{i\partial_{z_d}}{2\pi} \right) \right] (z) = q e_z.
\]

**Proof.** For any positive number \( \varepsilon > 0 \), any non-negative integer \( n \geq 0 \), \( j = 1, \ldots, d \), and any function \( \psi : \mathcal{X}_a^d \rightarrow L^2(w) \), we define
\[
\left( \Delta_{j,\varepsilon}^{(n)} \psi \right) (z) := (\varepsilon n!)^{-1} \sum_{r=0}^n (-1)^{n-r} \binom{n}{r} \psi(z + r\varepsilon e_j).
\]

Moreover, for \( n = (n_1, \ldots, n_d) \), we define
\[
\Delta_{\varepsilon}^{(n)} = \Delta_{1,\varepsilon}^{(n_1)} \cdots \Delta_{d,\varepsilon}^{(n_d)}.
\]

It suffices to show that
\[
\lim_{\varepsilon \to 0} \left( \frac{i}{2\pi} \right)^{|n|} \left( \Delta_{\varepsilon}^{(n)} \varphi \right) (z) = \xi_1^{n_1} \cdots \xi_d^{n_d} e_z
\]
for any \( n \) if \( a > 0 \), or \( |n| \leq n \) if \( a = 0 \). Here, we denote \( |n| := \sum_j n_j \). By direct computation, we have
\[
\| (\text{left hand side of (2)}) - (\text{right hand side of (2)}) \|_{L^2(w)}^2
= \int_{\mathbb{R}^d} \left| \prod_{j=1}^d \left( \frac{e^{-2\pi i \xi_j} - 1}{-2\pi i \xi_j} \right) - \prod_{j=1}^d \xi_j^{n_j} e^{2\pi \text{Im}(z)^\top \xi} w(\xi) d\xi \right|^2.
\]
Thus, from the definition of \((DC)_{n,a}^d\), we see that the last integral converges to 0.

### 2.3 Composition operators on RKHS

We give a definition of composition operators for our setting:

**Definition 2.** Let \( k \) and \( \ell \) be positive definite kernels on sets \( E \) and \( F \), respectively. For any map \( f : E \rightarrow F \), the composition operator \( C_f : H_\ell \rightarrow H_k \) is a linear operator defined by \( C_f(h) := h \circ f \) of domain

\[
\mathcal{D}(C_f) = \{ h \in H_\ell : h \circ f \in H_k \}.
\]

Since we see that a composition operators is closed, by the closed graph theorem, we have the following proposition:

**Proposition 7.** If \( \mathcal{D}(C_f) = H_\ell \), the composition operator \( C_f \) is a bounded operator.

Accordingly, the adjoint of \( C_f \) has the following property:

**Proposition 8.** Let \( k \) and \( \ell \) be a positive definite kernel on \( E \) and \( F \), respectively, and let \( f : E \rightarrow F \) be a map such that \( \mathcal{D}(C_f) = H_\ell \). Then, we have

\[
C_f^*(\phi_k(x)) = \phi_\ell(f(x)) \quad (x \in E).
\]

**Proof.** The proof entails a straightforward computation.

Consequently, we have the following corollary:

**Corollary 5.** Let \( E \subset \mathbb{R}^d \), and let \( f : E \rightarrow \mathbb{R}^d \) be a map. Suppose that we have a non-zero positive definite function \( u \) on \( \mathbb{R}^d \). Write \( k(x,y) := u(x-y) \) as above. Assume \( u \) is continuous and \( u(x) \rightarrow 0 \) as \( |x| \rightarrow \infty \). Then, if \( \mathcal{D}(C_f) = H_k \), the original map \( f \) is continuous.
Proof. From Propositions 1, 7 and 8, we see that
\[ f = \phi_k^{-1} \circ C_f^* \circ r_{k,E} \circ \phi_k. \]
Since the right-hand side is continuous, so is \( f \).

At the end of this section, we define another linear operator \( K_f \) under the condition \((\text{DC})^d_{n,a}\) for some \( a > 0 \), keeping in mind that \( H_k = H_{k,U} \) according to Proposition 5.

**Definition 3.** Let \( U \subset \mathbb{R}^d \) be an open subset. For any map \( f : U \to \mathbb{R}^d \) such that \( \mathcal{D}(C_f) = H_k \), define a linear operator \( K_f : L^2(w) \to L^2(w) \) by
\[
K_f : L^2(w) \xrightarrow{\psi_{w}^{-1}} H_k = H_{k,U} \xrightarrow{r_{k,U}} H_k \xrightarrow{C_f} H_k \xrightarrow{\psi_{w}} L^2(w). \tag{3}
\]
Here, \( r_{k,U} \) is the restriction map defined in Proposition 1.

## 3 Main results

Here, we prove the main results. We establish the criterion of the boundedness of composition operators in the case that the map \( f \) is affine.

### 3.1 Boundedness and compactness of composition operators for affine maps

Recall that we defined
\[
\mathcal{G}(w) = \bigcup_{\lambda > 0} \left\{ A \in \text{GL}_d(\mathbb{R}) : w(A^\top \xi) \geq \lambda w(\xi) \text{ for almost all } \xi \in \mathbb{R}^d \right\}.
\]
As the following proposition shows, \( \mathcal{G}(u) \) is a natural class.

**Proposition 9.** Let \( w \in L^1 \cap L^\infty \setminus \{0\} \) be a non-negative function. Write \( k(x, y) = \hat{w}(x-y) \) for \( x, y \in \mathbb{R}^d \). Let \( f : \mathbb{R}^d \to \mathbb{R}^d \) be an affine map, namely, \( f(x) = Ax + b \) such that \( A \in M_d(\mathbb{R}^d) \) and \( b \in \mathbb{R}^d \). Then, \( A \in \mathcal{G}(w) \) if and only if \( \mathcal{D}(C_f) = H_k \) (and \( C_f \) is bounded on \( H_k \)).

**Proof.** First, we prove the “if” part. Since any element of \( H_k \) vanishes at \( \infty \) and \( C_f \) preserves \( H_k \), the matrix \( A \) has to be regular. Let \( h \) be an arbitrary non-negative smooth function with compact support and vanishing in an open set including \( \{w = 0\} \). We define \( g := \Psi_w^{-1}(h^{1/2}w^{-1}) = \mathcal{F}^{-1}(h^{1/2}) \). Since \( C_f \) is bounded, there exists \( L > 0 \) such that
\[
L\|g\|_{H_k}^2 - \|C_fg\|_{H_k}^2 \geq 0.
\]
Since
\[ F(Cf)g(\xi) = \int_{\mathbb{R}^d} g(Ax + b) e^{-2\pi i x^\top \xi} \, dx \]
\[ = |\det A|^{-1} e^{2\pi i b^\top \xi} \int_{\mathbb{R}^d} g(x) e^{-2\pi i x^\top (A^{-\top} \xi)} \, dx \]
\[ = |\det A|^{-1} e^{2\pi i b^\top \xi} \mathcal{F}(g)(A^{-\top} \xi), \]
by Proposition 4, we see that
\[ \int_{\mathbb{R}^d} h(\xi) \left( |\det A| \cdot Lw(\xi) - 1 - w(A^\top \xi) \right) \, d\xi \geq 0. \]
Since \( h \) is arbitrary, we have
\[ w(A^\top \xi) \geq |\det A| L^{-1} w(\xi), \]
namely, \( A \in \mathcal{G}(w) \).

Now we prove the “only if” part. Let \( A \in \mathcal{G}(w) \), and let \( \lambda > 0 \) such that \( w(A^\top \xi) \geq \lambda w(\xi) \) for almost all \( \xi \). As in the same computation as above, for any \( g \in H_k \), we have
\[ \lambda^{-1} \|g\|_{H_k}^2 - \|C_f g\|_{H_k}^2 \geq 0. \]
Thus, we conclude that \( \mathcal{D}(C_f) = H_k \) and that \( C_f \) is bounded on \( H_k \).

We also observe that the composition operators induced by affine maps cannot be compact:

**Proposition 10.** Let \( w \in L^1 \cap L^\infty \setminus \{0\} \) be a non-negative function. Write \( k(x,y) = \hat{w}(x-y) \) for \( x,y \in \mathbb{R}^d \) as before. Let \( f : \mathbb{R}^d \to \mathbb{R}^d \) be such that \( A \in \mathcal{G}(w) \). Then the composition operator \( C_f \) cannot be a compact operator on \( H_k \).

**Proof.** Put \( u = \hat{w} \). It suffices to show that the operator \( K_f \) (Definition 3) cannot be compact. Assume to the contrary that \( K_f \) is compact. Fix a sequence \( \{x_n\} \_{n \geq 0} \) such that \( \inf_{m,n \geq R} |x_m - x_n| \to \infty \) as \( R \to \infty \) (for example, \( x_n = (n^2,0,\ldots,0) \)). Since \( K_f \) is compact, we may assume \( \{K_f(\phi(x_n))\} \) converges to an element \( h \in L^2(w) \). Meanwhile, \( \|K_f \phi(x_m) - K_f \phi(x_n)\|_{L^2(w)} = 2u(0) - 2\text{Re}(u)(A(x_m - x_n)) \). Since \( u(x) \) converges to 0 as \( |x| \to \infty \) by the Riemann–Lebesgue theorem, by taking the limit \( m,n \to \infty \) with \( m \neq n \), we find that \( u(0) = 0 \). Since \( u \) is a positive definite function, \( |u(x)| = |\langle k_x, k_0 \rangle_{H_k}| \leq u(0) \) by the Cauchy-Schwarz inequality. Thus, we have \( u = 0 \). This is a contradiction.
3.2 Affineness of holomorphic maps with bounded composition operators

In this section, we prove that maps are affine if they admit an analytic continuation and the domains of their composition operators are the whole space $H_k$, namely, the following theorem:

**Theorem 3.** Let $w \in L^1 \cap L^\infty \setminus \{0\}$ be a non-negative function, and let $k(x,y) = \hat{w}(x - y)$. We impose the following three assumptions on $w$:

(A) the function $w$ satisfies $(DC)^d_{a,n}$ for all $a > 0$,
(B) there exists a constant $B > 0$ such that for any $y \in \mathbb{R}^d$,

$$\limsup_{n \to \infty} \delta_n^-(y;w), \limsup_{n \to \infty} \delta_n^+(y;w) < B,$$

(C) $\langle \mathcal{G}(w) \rangle_{\mathbb{R}} = M_d(\mathbb{R})$.

Then, for any open set $U \subset \mathbb{R}^d$ and any map $f : U \to \mathbb{R}^d$ such that $F|_U = f$ for some holomorphic map $F : \mathbb{C}^d \to \mathbb{C}^d$, the map $f$ is affine in the form,

$$f(x) = Ax + b$$

with $A \in \mathcal{G}(w)$ and $b \in \mathbb{R}^d$ if and only if the composition operator $C_f : H_k \to H_k|_{L^2}$ is defined on the whole space $H_k$ and is bounded. Here, we recall that $P_n$ is the space of $d$-variable polynomials of degree smaller than or equal to $n$.

We always regard the space of $d$-variable polynomials $\mathbb{C}[\xi_1, \ldots, \xi_d]$ as a subspace of $L^2(\mathbb{R})$ as functions of $(\xi_1, \ldots, \xi_d)$. We also fix an open set $U \subset \mathbb{R}^d$ and a map $f : U \to \mathbb{R}^d$ such that $F|_U = f$ for some $F = (F_1, \ldots, F_d) : \mathbb{C}^d \to \mathbb{C}^d$.

The following simple proposition shows that the information of $F$ is included in $K_f$ although $K_f$ is defined by the map $f$ initially defined only on $U$:

**Proposition 11.** Assume Assumption (A) in Theorem 3. For any $z \in \mathbb{C}^d$, we have

$$K_f(\varphi(z)) = \varphi(F(z)).$$

**Proof.** By Proposition 6, both $K_f \circ \varphi$ and $\varphi \circ F$ are $L^2(w)$-valued holomorphic functions on $\mathbb{C}^d$. Since both holomorphic maps are identical on the open set $U \subset \mathbb{R}^d$ by definition, their values are the same on $\mathbb{C}^d$.

The following lemma is crucial for controlling the Jacobian matrix of $F$: 
Lemma 1. Assume Assumption (A) in Theorem 3 and $\mathcal{D}(C_f) = H_k$. Then for any $d$-variable homogeneous polynomial $q \in \mathbb{C}[\xi_1, \ldots, \xi_d]$, we have

$$K_f(qe_z) = \left( \mathcal{J} \left[ \frac{i}{2\pi} J_F(z) \right] q + r \right) e_{F(z)},$$

where $r \in \mathbb{C}[\xi_1, \ldots, \xi_d]$ is a polynomial of degree smaller than $\text{deg}(q)$ without a constant term, and $\mathcal{J}[A] : \mathbb{C}[\xi_1, \ldots, \xi_d] \to \mathbb{C}[\xi_1, \ldots, \xi_d]$ is the symmetric product of a matrix $A = (a_{ij})$ of size $d$, namely, the linear map defined on $\mathbb{C}[\xi_1, \ldots, \xi_d]$ via the correspondence $\xi_k \mapsto \sum_{m=1}^{d} a_{mk} \xi_m$.

Proof. It suffices to show that in the case of $q(\xi_1, \ldots, \xi_d) = \xi_{i_1} \cdots \xi_{i_k}$ ($i_j \in \{1, \ldots, d\}$),

$$K_f(qe_z) - \prod_{m=1}^{k} \left[ \sum_{j=1}^{d} \partial_{\xi_j} F_{im}(z) \cdot \frac{i\xi_j}{2\pi} \right] \cdot e_{F(z)} = rze_{F(z)},$$

where $r_z$ is a polynomial of degree smaller than $k$ and its coefficients are entire functions with respect to the variable $z$. We prove (5) by induction on $k$. In the case of $k = 1$, since $K_f$ is continuous, we see that

$$K_f[\partial_{\xi_1} \varphi(z)] = \partial_{\xi_1} (K_f[\varphi(z)]).$$

The left hand side is equal to $K_f[-2\pi i \xi_{i_1} e_z]$, and the right hand side is equal to $\left( \sum_{j=1}^{d} \partial_{\xi_j} F_{i_1}(z) \cdot \xi_j \right) \cdot e_{F(z)}$. Thus, we have (5). Let $k > 1$ and for $\varepsilon > 0$, put

$$\psi_\varepsilon := i \frac{\varphi(z + \varepsilon e_{i_k}) - \varphi(z)}{2\pi \varepsilon},$$

where $e_{i_k}$ is the $i_k$-th elementary vector in $\mathbb{C}^d$. Then, by induction hypothesis, we have

$$K_f[\xi_{i_1} \cdots \xi_{i_{k-1}} \psi_\varepsilon]$$

$$= \prod_{m=1}^{k-1} \left[ \sum_{j=1}^{d} \partial_{\xi_j} F_{im}(z + \varepsilon e_{i_k}) \cdot \frac{i\xi_j}{2\pi} \right] \cdot \psi_\varepsilon$$

$$+ \frac{ie_{F(z)}}{2\pi \varepsilon} \left\{ \prod_{m=1}^{k-1} \left[ \sum_{j=1}^{d} \partial_{\xi_j} F_{im}(z + \varepsilon e_{i_k}) \cdot \frac{i\xi_j}{2\pi} \right] - \prod_{m=1}^{k-1} \left[ \sum_{j=1}^{d} \partial_{\xi_j} F_{im}(z) \cdot \frac{i\xi_j}{2\pi} \right] \right\}$$

$$+ i \frac{s_{z+\varepsilon e_{i_k}} e_{F(z+\varepsilon e_{i_k})} - s_z e_{F(z)}}{2\pi \varepsilon}.$$
where \( s_z \in \mathbb{C}[\xi_1, \ldots, \xi_d] \) is a polynomial of degree smaller than \( k - 1 \) and their coefficients are entire with respect to \( z \). Then, if we take \( \varepsilon \) to 0, since \( K_f \) is continuous, we see that there exists \( r_z \in \mathbb{C}[\xi_1, \ldots, \xi_d] \) of degree smaller than \( k \) whose coefficients are entire with respect to \( z \) such that

\[
K_f[\varepsilon z] = \prod_{m=1}^{k} \left[ \sum_{j=1}^{d} \partial_{\xi_j} F_m(z) \cdot \frac{i\xi_j}{2\pi} \right] + r_z e_F(z).
\]

Thus, by induction, we prove (5).

For each \( n > 0 \), we denote the space of homogeneous polynomials of degree \( n \) by \( \mathbb{C}[\xi_1, \ldots, \xi_d]_n \). Then, we have the following corollary:

**Corollary 6.** The situation is the same as that in Lemma 1. Write

\[
Q_{n,z} = \{ q \varphi_z : q \in P_n \} \subset L^2(w).
\]

Then, the following diagram is commutative:

\[
\begin{array}{ccc}
Q_{n,z} & \overset{K_f}{\longrightarrow} & Q_{n,F(z)} \\
\downarrow \text{proj.} & & \downarrow \text{proj.} \\
Q_{n,z}/Q_{n-1,z} & \overset{[K_f]}{\longrightarrow} & Q_{n,F(z)}/Q_{n-1,F(z)} \\
\cong \left[ m_{-2\pi i z} \right] & & \cong \left[ m_{-2\pi i F(z)} \right] \\
P_n/P_{n-1} & & P_n/P_{n-1} \\
\cong \mathbb{C}[\xi_1, \ldots, \xi_d]_n & \overset{\mathcal{S}_n[iF(z)/2\pi]}{\longrightarrow} & \mathbb{C}[\xi_1, \ldots, \xi_d]_n.
\end{array}
\]

Here, proj. is the natural surjection to the quotient, \([\cdot]\) means the natural morphism induced by \((\cdot)\), and we define \( \mathcal{S}_n[iF(z)/2\pi] \) to be the restriction of \( \mathcal{S}_n[iF(z)/2\pi] \) to \( \mathbb{C}[\xi_1, \ldots, \xi_d]_n \).

**Proof.** This is clear from Lemma 1.

Regarding Corollary 6, we have the following lemma:

**Lemma 2.** The situation is the same as in Corollary 6. Then, we have the following inequality on the norm of the operators:

\[
\| [K_f] \| \leq \| K_f \|, \\
\| [m_{-2\pi i z}] \| \leq \| m_{2\pi \text{Im}(z)} \|_{P_n}, \\
\| [m_{-2\pi i F(z)}]^{-1} \| \leq \| m_{2\pi \text{Im}(F(z))} \|_{P_n}^{-1}.
\]
Here, the topologies of the quotients space the above operators act on are induced from $L^2(w)$.

**Proof.** This lemma immediately follows the fact that $\|[*]\|$ is the same as the norms of the compositions of an inclusion and a projection for subspaces of $L^2(w)$ with $\ast$.

Next, we have the following key lemma:

**Lemma 3.** Assume that Assumptions (A) and (B) in Theorem 3 hold. Then, the map $z \mapsto \text{tr} J_F(z)$ is a constant function.

**Proof.** For each $n > 0$, we denote by $\|\cdot\|_n$ the norm on $\mathbb{C}[\xi_1, \ldots, \xi_d]_n$ induced from $P_n/P_{n-1}$ via the isomorphism (see Corollary 6). Here, the norm of $P_n$ is the restriction of that of $H_k$, and the norm of $P_n/P_{n-1}$ is the quotient norm. Let $\alpha_z$ be an arbitrary eigenvalue of $J_F(z)$ that acts on $\mathbb{C}[\xi_1, \ldots, \xi_d]_1$. Also, let $v \in \mathbb{C}[\xi_1, \ldots, \xi_d]_1$ be its eigenvector. Then, we have

$$\|\alpha_z^n v^n\|_n = \|\mathcal{N}_n[J_F(z)]v^n\|_n (2\pi)^{-n}.$$  

By Corollary 6 and Lemma 2 (we use the notation in this corollary,) we have

$$|\alpha_z|^n \leq (2\pi)^{-n} \|K_F\| \cdot \|m_{-2\pi i} \| \cdot \|m_{-2\pi i F(z)}\|^{-1} \| \leq (2\pi)^{-n} \|K_F\| \cdot \|m_{2\pi i \text{Im}(z)} P_n\| \cdot \|m_{2\pi i \text{Im}(F(z))} P_n\|^{-1} \|$$

If we take the $n$-th root and then “$\lim \sup_{n \to \infty}$”, by combining this limit and Assumption (B), there exists $B > 0$ independent of $z$ such that

$$\sup_z |\alpha_z| \leq \frac{B^2}{2\pi}.$$  

Thus, any eigenvalue of $J_F(z)$ is bounded by a constant independent of $z$. In particular, the holomorphic function $\text{tr} J_F$ is bounded on $\mathbb{C}^d$, and hence, $\text{tr} J_F$ is constant by Liouville’s theorem.

Now we prove Theorem 3.

**Proof.** The “only if” part immediately follows from Proposition 9. The “if” part is proved as follows: If $f$ is such a map, then, by Lemma 3, $\text{tr} J_{A \circ f}(z) = \text{tr} (A J_F(z))$ is constant for any $A \in \mathcal{G}(w)$ as $C_A \circ C_f = C_{A \circ f}$ is bounded. By Assumption (C) in Theorem 3: $\langle \mathcal{G}(w) \rangle_{\mathbb{R}} = M_d(\mathbb{R})$, it follows that $J_F$ itself is independent of $z$. Thus, $f$ is an affine map.
3.3 Analytic continuation

In this section, we prove that any map inducing bounded composition operators in $H_k$ has an analytic continuation:

**Theorem 4.** Let $w \in L^1 \cap L^\infty \setminus \{0\}$ be a non-negative function, and let $k(x, y) = \hat{w}(x - y)$. We require that the function $u$ satisfies $(DC)_{n,a}^d$ for some $a > 0$. Then, for any open set $U \subset \mathbb{R}^d$ and any map $f : U \to \mathbb{R}^d$, there exists a holomorphic map $F : \mathbb{X}_a^d \to \mathbb{C}^d$ such that $F|_U = f$ as long as $\mathcal{D}(C_f) = H_k$ and $C_f$ is bounded.

First, we give a simple lemma to prove analyticity of $f$ on $U$, as in Lemma 5 below:

**Lemma 4.** Assume $u \in C^1$ and $|u(x)| \to 0$ as $|x| \to \infty$. Then, we have

$$\langle \{\nabla u(a)\}_{a \in \mathbb{R}^d} \rangle_C = \mathbb{C}^d.$$

**Proof.** Assume that $\text{Span}(\{\nabla u(a)\}_{a \in \mathbb{R}^d})$ is a proper subset of $\mathbb{R}^d$. Then by a change of coordinates with a linear transformation, we may assume that

$$\text{Span}(\{\nabla u(a)\}_{a \in \mathbb{R}^d}) \subset \{x_d = 0\}$$

but it contradicts the fact that $u(x) \to 0$ as $|x| \to \infty$.

**Lemma 5.** Let $w \in L^1 \cap L^\infty \setminus \{0\}$ be a non-negative function satisfying the condition $(DC)_{n,a}^d$ for some $n > 0$ and $a \geq 0$. Let $k(x, y) := \hat{w}(x - y)$. If $\mathcal{D}(C_f) = H_k$ and $C_f$ is bounded, then $f$ is a $C^1$-function on $U$.

**Proof.** Put $u = \hat{w}$. By Lemma 4, we can find vectors $a_1, a_2, \ldots, a_d$ such that $\{\nabla u(a_i)\}_{i=1}^d$ spans $\mathbb{C}^d$. Fix $b \in U$ arbitrarily. It suffices to show that $f$ is $C^1$ at a neighborhood of $b$ in $U$. Define

$$f_b(x) = (\phi_k(f(b) - a_1))(x), \ldots, (\phi_k(f(b) - a_d))(x)) = (u(a_1 + x - f(b)), u(a_2 + x - f(b)), \ldots, u(a_d + x - f(b))).$$

Then,

$$J_{f_b}(f(b)) = (\nabla u(a_1), \ldots, \nabla u(a_d)),$$

so that $f_b$ induces a bijective $C^1$-map on the open ball $U_b$ centered at $f(b)$ into $\mathbb{R}^d$, and $f_b^{-1}$ is also a $C^1$-map on $f_b(U_b)$. Since $\mathcal{D}(C_f) = H_k$,

$$[C_f f_b](x) := \{C_f[\phi_k(f(b) - a_i)](x)\}_{i=1}^d.$$
is also a $C^1$-function defined on $\mathbb{R}^d$. Furthermore,
\[
[C_f f_b](x) = \{C_f [\phi (f(b) - a_i)](x)\}_{i=1}^d \\
= \phi (f(b) - a_i)(f(x)) \\
= \{u(a_i + f(x) - f(b))\}_{i=1}^d \\
= f_b \circ f(x).
\]

Consequently, $[C_f f_b](b) = f_b(f(b)) \in f_b(U_b)$. Since $f$ is continuous by Corollary 5, we can find a neighborhood $V_b$ of $b$ such that $[C_f f_b](V_b) \subset f_b(U_b)$. Thus, on $x \in V_b$, we have $f(x) = f_b^{-1} \circ [C_f f_b](x)$. Therefore, $f$ is a $C^1$-function on a neighborhood of $b$.

Now let us prove Theorem 4.

**Proof.** First, we claim that we may replace $w$ with $\hat{w}(\xi) := (w(\xi) + w(-\xi))/2$. In particular, this allows us to assume that $w$ is an even function, and thus, $-1 \in \mathcal{D}(w)$. Let $v := \hat{w}$, and we define $\ell(x, y) := v(x - y)$. We show the claim as follows: in fact, it is obvious that $v$ satisfies $(DC)^d_{n,a}$. We prove that the composition operator from $H_\ell$ to $H_\ell|_{l^2}$ is defined everywhere and bounded. We define a densely defined linear map $\tilde{K}_f : L^2(\hat{w}) \to L^2(\hat{w})$ with domain $\mathcal{D}(\tilde{K}_f) = \langle \{e_x\}_{x \in U} \rangle_{C}$ by allocating $e_{f(x)}$ to $e_x$ (Here, we may define such a linear map since $e_x$’s are linearly independent). Let $u := \hat{w}$, and let $h = \sum_{j=1}^r a_j e_x \in \mathcal{D}(\tilde{K}_f)$ ($a_j \in \mathbb{C}$ and $x_j \in U$ for $j = 1, \ldots, r$). Then
\[
\left\| \tilde{K}_f h \right\|_{L^2(v)}^2 = \sum_{i,j=1}^r a_i a_j v(f(x_i) - f(x_j)) \\
= \sum_{i,j=1}^r \frac{a_i a_j + a_j a_i}{2} u(f(x_i) - f(x_j)) \\
\leq \|K_f\|^2 \sum_{i,j=1}^r \frac{a_i a_j + a_j a_i}{2} u(x_i - x_j) \\
= \|K_f\|^2 \cdot \|h\|^2.
\]

Thus, we see that $\tilde{K}_f$ is bounded and we can uniquely extend $\tilde{K}_f$ as a bounded linear operator on $L^2(w)$. We define
\[
\tilde{C}_f : H_\ell \overset{\psi_\ell}{\cong} L^2(\hat{w}) \overset{\psi_\ell^{-1}}{\cong} H_\ell = H_{\ell, U} \overset{r \psi_\ell^{-1}}{\cong} H_{\ell|_{l^2}}.
\]
For any \( h \in H_{\ell} \), we have 
\[
[\tilde{C}_f h](x) = \langle \tilde{C}_f h, \phi_{\ell}(x) \rangle_{H_k} = \langle h, \phi_{\ell}(f(x)) \rangle_{H_k} = h(f(x)).
\]
Therefore, \( \tilde{C}_f \) is simply the composition operator from \( H_{\ell} \) to \( H_{\ell|_{U^2}} \), which is defined everywhere and bounded.

By the above claim, we may assume that \( w \) is an even function and \( -1 \in G(u) \). Fix \( y = (y_1, \ldots, y_d) \in U \), and define the holomorphic map 
\( F : \mathcal{X}_a^d \rightarrow L^2(w) \otimes L^2(w) \) by
\[
F(z) = \sum_{j=1}^d \int_{y_j}^z \left[ \partial_{z_j} \varphi \otimes \varphi \right](z_1, \ldots, z_{j-1}, w, y_{j+1}, \ldots, y_d) \, dw,
\]
Let \( m : L^2(w) \otimes L^2(w) \rightarrow L^1(w) \) be the natural multiplication map, and let \( t : \mathbb{C}^d \rightarrow \sum_{i=1}^d \mathbb{C} \xi_i \subset L^1(w) \) be the linear isomorphism defined by allocating \( \xi_j \) to the vector \( e_j := (0, \ldots, 0, 1, 0, \ldots, 0) \). Then, we have the following proposition:

**Proposition 12.** Under the above notation, let \( U_0 \) be the connected component of \( U \) including \( y \). Then, \( m \circ (K_f \otimes K_{-f}) \circ F : \mathcal{X}_a^d \rightarrow L^1(w) \) is holomorphic and its image of \( U_0 \) is contained in the finite-dimensional vector space \( \sum_{i=1}^d \mathbb{C} \xi_i \subset L^1(w) \). Moreover, for any \( y \in U_0 \), we have
\[
(t^{-1} \circ m \circ (K_f \otimes K_{-f}) \circ F)(x) + f(x_0) = f(x).
\]

**Proof.** Since \( m \) and \( (K_f \otimes K_{-f}) \) are bounded linear operators and \( F \) is obviously holomorphic, the composition \( m \circ (K_f \otimes K_{-f}) \circ F \) is also holomorphic. Let \( x \in U_0 \). Thanks to Proposition 6 and Lemma 1, we see that
\[
(m \circ (K_f \otimes K_{-f}) \circ F)(x) = \sum_{j=1}^d (f_j(x) - f_j(x_0)) \xi_j,
\]
where \( f := (f_1, \ldots, f_d) \).

Now, we complete the proof. Since Lemma 6 below implies
\[
(m \circ F)(\mathcal{X}_a^d) \subset \sum_{i=1}^d \mathbb{C} \xi_i,
\]
\( F := f(x_0) + (t \circ m \circ (K_f \otimes K_{-f}) \circ F) : \mathcal{X}_a^d \rightarrow \mathbb{C}^d \) is well defined on \( \mathcal{X}_a^d \) and gives the analytic continuation of \( f \).
Lemma 6. Let $U \subset \mathbb{R}^d$ be an open set, and let $X \subset \mathbb{C}^d$ be a connected open set containing $U$. Also, let $V$ be a locally convex space over $\mathbb{C}$, and let $\gamma : X \to V$ be a weakly holomorphic map. If $\gamma(U)$ is contained in a finite-dimensional subspace $V_0 \subset V$ over $\mathbb{C}$, then we have $\gamma(X) \subset V_0$.

Proof. Suppose that $\gamma(z_0) \notin V_0$ for some point $z_0 \in X$. Then, the Hahn-Banach theorem guarantees that there is a continuous linear functional $\lambda : V \to \mathbb{C}$ such that $\lambda(\gamma(z_0)) = 1$ and $\lambda(V_0) = \{0\}$. Therefore, $\lambda \circ \gamma : X \to \mathbb{C}$ is a holomorphic function vanishing at $U$; thus, $\lambda \circ \gamma \equiv 0$ on $X$. This contradicts $\lambda \circ \gamma(z_0) = 1$.

4 Boundedness for special positive definite functions

In this section, we investigate the boundedness of composition operators on RKHSs associated to a spherical positive definite function and a convolution of real positive definite functions on $\mathbb{R}$. Unless $w$ is compactly supported, Assumption (B), that is the existence of $B > 0$ such that for any $y \in \mathbb{R}^d$,

$$\limsup_{n \to \infty} e_n^-(y;w), \limsup_{n \to \infty} e_n^+(y;w) < B,$$

is the hardest condition to verify in our main theorem (Theorem 1). In the case where $w$ is spherical or convolution of real positive definite functions, we may relate the left-hand-side to an asymptotic behavior of greatest zeros of certain orthogonal polynomials. On the other hand, the asymptotic properties of greatest zeros of orthogonal polynomials are extensively studied in previous works (see, for example, [18, Part 2]), and we can utilize various techniques developed there. Analysis illustrated in this section also provides a non-trivial consequence even in a one-dimensional case. Indeed, our result is still valid even if an RKHS contains an entire function of infinite order where previous frame work does not work.

4.1 Orthogonal polynomials

First, we briefly review basic properties of orthogonal polynomials. For more details, see [18, 22]. Let $\mu$ be a Borel measure on $\mathbb{R}$. Assume, for any integer $n \geq 0$,

$$\int_{\mathbb{R}} |t|^n d\mu(t) < \infty. \quad (6)$$

Then, we define (normalized) orthogonal polynomials by polynomials

$$p_0(t;\mu), p_1(t;\mu), \ldots \in \mathbb{R}[t]$$
such that $p_n(t; \mu)$ is of degree $n$ and

$$\int \mathbb{R} p_m(t; \mu)p_n(t; \mu)d\mu(t) = \delta_{m,n}$$

for any $m, n \geq 0$. We note that each $p_n(t; \mu)$ is uniquely determined up to sign. We denote by $X_n(\mu)$ the greatest zero of $p_n(t; \mu)$.

We denote by $\gamma_n(\mu)$ the leading coefficient of $p_n(t; \mu)$, and we define the Christoffel function $\lambda_n(s; \mu)$ by

$$\lambda_n(s; \mu) := \left( \sum_{k=0}^{n-1} p_k(s; \mu)^2 \right)^{-1}.$$  \hspace{1cm} (7)

We recall the following somewhat known properties:

**Proposition 13.** Let $\mu, \nu$ be finite Borel measures on $\mathbb{R}$ satisfying (6) such that $\nu - \mu \geq 0$.

Then

$$\gamma_n(\mu)^{-2} \leq \gamma_n(\nu)^{-2}$$  \hspace{1cm} (8)

$$\lambda(s; \mu) \leq \lambda(s; \nu).$$  \hspace{1cm} (9)

**Proof.** We invoke the following formulas (see, for example, [22, Theorem 3.1.2], and [18, (4.1.1)]): for any Borel measure satisfying (6), we have

$$\gamma_n(\mu)^{-2} = \min_{p \in P_{n-1}} \int \mathbb{R} (t^n + p(t))^2 d\mu(t),$$

$$\lambda_n(s; \mu) = \min_{p \in P_{n-1}, p(s=1)} \int \mathbb{R} p(t)^2 d\mu(t).$$

**4.2 Asymptotic estimation of greatest zeros of orthogonal polynomials**

We denote by $L^p(\mathbb{R})$ the usual $L^p$-space with respect to the Lebesgue measure on the real line $\mathbb{R}$ for $p \in (0, \infty]$. We denote by $\| \cdot \|_p$ the $L^p$-norm of $L^p(\mathbb{R})$. Let $W \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ be such that $W(t) > 0$ and $W(t) = W(-t)$ for almost all $t \geq 0$. Let

$$\mu := W(t)dt.$$

Before getting into the main body of this section, we introduce some notation:

For any $L \in \mathbb{R}$ and for any measurable function $Q : [0, \infty) \to \mathbb{R}$, we define

$$Q_L(t) := Q(t) - Lt.$$
We also define the “monotonic” part and “oscillated” part for \( Q \) as follows:

\[
Q^m(t) := \inf \{ T \in \mathbb{R} : |\{ s \geq t : T \geq Q(s)\}| > 0 \},
\]

\[
Q^o := Q - Q^m.
\]

We note that \( Q^o \) is non-negative, and \( Q^m \) is non-decreasing and upper semi-continuous (thus right continuous: \( \lim_{t \searrow s} Q^m(t) = Q(s) \)). Since \( Q^m_1 + Q^m_2 \leq (Q_1 + Q_2)^m \),

\[
Q^o_1 + Q^o_2 \geq (Q_1 + Q_2)^o.
\] (10)

For simplicity, we abbreviate \( (Q_L)^m \) and \( (Q_L)^o \) to \( Q^m_L \) and \( Q^o_L \).

We will prove the following theorem:

**Theorem 5.** Let \( W \) be as above. Suppose that there exists a function \( Q : [0, \infty) \to \mathbb{R} \) such that \( W(t) = e^{-Q(|t|)} \). Let \( L > 0 \) be a positive number and assume \( t^\mu e^{\mu W(t)} \in L^1(\mathbb{R}) \) for all \( n \geq 0 \). For \( \sigma \leq L \), we define

\[
W^\sigma(t) := e^{\sigma |t|} W(t),
\]

\[
\mu^\sigma := W^\sigma(t) dt.
\]

Assume in addition the following two conditions:

\[
\lim_{t \to \infty} \frac{Q^m_L(t)}{t} = +\infty, \quad (11)
\]

and there exists \( B > 0 \) such that for any sufficiently large \( t > 0 \),

\[
\int_0^{\pi/2} Q^o_L(t \cos \theta) d\theta < B + \log t. \quad (12)
\]

Then, \( X_n(\mu^\sigma) n^{-1} \) uniformly converges to 0 over \( \sigma \leq L \).

The proof of Theorem 5 is based on some auxiliary estimates. The following lemma is essential:

**Lemma 7.** Under the same notation as Theorem 5, for any \( \rho > 0 \), we define

\[
\Xi_{\rho, \sigma} := \| |t| \rho W^\sigma \|_\infty,
\]

\[
\xi_{\rho, \sigma} := \sup_{\varepsilon > 0} \left[ \text{ess inf}_{t \geq 0} \{ t \rho W^\sigma(t) > \Xi_{\rho, \sigma} - \varepsilon \} \right]
\]

Then, for any \( \sigma \leq L \), we have

\[
X_n(\mu^\sigma) \leq \left( 2 + \frac{2}{3\pi n} \exp \left( \frac{2}{\pi} \int_0^{\pi/2} Q^o_L(\xi_{4n, \sigma} \cos \theta) d\theta \right) \right) \xi_{4n, L}
\]
In order to prove Lemma 7, we improve results in [10]. Before proving Lemma 7, we provide several fundamental inequalities.

For \( \xi \geq 0 \) and a function \( W \) as in the beginning of this section, as Freud did in [10], we define

\[
W_\xi(t) := W(t) \cdot 1_{[-\xi, \xi]}(t),
\]

\[
G_\xi(W) := \exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} \log |W(\xi \cos \theta)| \, d\theta \right\},
\]

and we denote \( \mu_\xi := W_\xi(t) \, dt \). First, we prove an elementary inequality for holomorphic functions. Lemma 8 below will substitute for the technique used in [10], where Freud used the Hardy space \( H_2 \) over the unit disc.

**Lemma 8.** Let \( U \subset \mathbb{C} \) be an open subset containing \( \overline{D} := \{ z \in \mathbb{C} : |z| \leq 1 \} \). Let \( f : U \to \mathbb{C} \) be a holomorphic map such that \( f(0) \neq 0 \). Then,

\[
|f(0)| \leq \exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{i\theta})| \, d\theta \right\}.
\]

**Proof:** Let \( \alpha_1, \ldots, \alpha_n \in \overline{D} \) be the zeros of \( f \) contained in \( \overline{D} \). Then, we may assume that \( f \) takes the form:

\[
f(z) = e^{g(z)} \prod_{i=1}^n (z - \alpha_i)^{m_i},
\]

where \( g \) is a holomorphic function in \( \overline{D} \) and \( m_1, \ldots, m_n > 0 \). Thus, it suffices to prove the case of \( f = e^g \) or \( f = z - \alpha \) for some \( \alpha \in \overline{D} \setminus \{0\} \). In the case of \( f = e^g \), it immediately follows from the mean value theorem for harmonic function. In the case of \( f(z) = z - \alpha \), we employ the following well-known fact for \( |\alpha| \leq 1 \):

\[
\frac{1}{2\pi} \int_0^{2\pi} \log |e^{i\theta} - \alpha| \, d\theta = 0.
\]

In this case, the inequality (15) is equivalent to the validity of the inequality for \( |\alpha| \leq 1 \).

Then, we have an estimation of \( \gamma_n(\mu) \):

**Lemma 9.** For any \( \xi > 0 \) and \( n \in \mathbb{N} \), we have

\[
\gamma_n(\mu)^2 \leq \gamma_n(\mu_\xi)^2 \leq \frac{2}{\pi \xi} \left( \frac{2}{\xi} \right)^{2n} G_\xi(W).
\]
Proof. The first inequality is a direct consequence of Proposition 13. We prove the second inequality. We define an entire function $f_n$ by
\[
f_n(z) := z^n p_n\left(\xi \left(z + z^{-1}\right)/2; \mu_\xi\right)
\]
keeping in mind that
\[
z^n p_n\left(\xi \left(z + z^{-1}\right)/2; \mu_\xi\right) = \sum_{k=0}^{n} a_n z^n \left(\frac{\xi}{2} \left(z + z^{-1}\right)\right)^k,
\]
where we write $p_n(t; \mu_\xi) = \sum_{k=0}^{n} a_k t^k$. Since
\[
\gamma_n(\mu_\xi) = \left(\frac{\xi}{2}\right)^n a_n, \quad \frac{1}{2\pi} \int_{0}^{2\pi} \log |\sin \theta| d\theta = -\log 2,
\]
by using Lemma 8, and the Jensen inequality, we have
\[
\left(\frac{\xi}{2}\right)^{2n} \gamma_n(\mu_\xi)^2 G_\xi(W) = |f_n(0)|^2 |G_\xi(W)\]
\[
\leq 2 \exp \left\{ \frac{1}{2\pi} \int_{0}^{2\pi} \log \left[ |f_n(e^{i\theta})|^2 W(\xi \cos \theta) \sin \theta \right] d\theta \right\}
\]
\[
\leq \frac{1}{\pi} \int_{0}^{2\pi} |f_n(e^{i\theta})|^2 |W(\xi \cos \theta)| \sin \theta |d\theta|
\]
As for the last integral, since $|f_n(e^{i\theta})| = |p_n(\xi \cos \theta; \mu_\xi)|$ from the definition of $f_n$ and $\mu_\xi = W_\xi(t) dt$, we have
\[
\frac{1}{\pi} \int_{0}^{2\pi} |f_n(e^{i\theta})|^2 |W(\xi \cos \theta)| \sin \theta |d\theta
\]
\[
= \frac{1}{\pi} \int_{0}^{2\pi} p_n(\cos \theta; \mu_\xi)^2 W(\xi \cos \theta) \sin \theta |d\theta
\]
\[
= \frac{4}{\pi} \int_{0}^{\pi/2} p_n(\cos \theta; \mu_\xi)^2 W(\xi \cos \theta) \sin \theta |d\theta
\]
\[
= \frac{2}{\xi \pi} \int_{-\xi}^{\xi} p_n(t; \mu_\xi)^2 d\mu_\xi(t)
\]
\[
= \frac{2}{\xi \pi}.
\]
Here, we employed a formula $p_n(t; \mu_\xi) = (-1)^n p_n(-t; \mu_\xi)$ as $W_\xi$ is an even function for the second equality. Therefore, we have
\[
\left(\frac{\xi}{2}\right)^{2n} \gamma_n(\mu_\xi)^2 G_\xi(W) \leq \frac{2}{\xi \pi},
\]
which is equivalent to the second inequality of (16).

Then, we have an estimation of the Christoffel function as a corollary of this lemma:

**Corollary 7.** For any $\xi > 0$ and $s \in \mathbb{R}$ such that $|s| > \xi > 0$, we have

$$
\lambda_n(s; \mu)^{-1}(s) \leq \lambda_n(s; \mu_{\xi})^{-1}(s) \leq \frac{8}{3\pi \xi} G_{\xi}(W)^{-1} \left( \frac{2|s|}{\xi} \right)^{2n-2} \quad (17)
$$

**Proof.** The proof is completely the same as [10, Lemma 2], where Freud factorized $p_n(z; \mu_{\xi})$ and used the fact that any zeros of $p_n(t; \mu_{\xi})$ are contained in $[-\xi, \xi]$ and that $\xi^2 \geq \xi^2 - a^2$ for all $\xi, a \in \mathbb{R}$ with $|\xi| \geq |a|$.

According to Chebychev’s theorem (see [10, (17), (18)])

$$
X_n(\mu) = \sup_{P \in P_{n-1}} \left\{ \int_{-\infty}^{\infty} xP(x)^2 W(x)dx \div \int_{-\infty}^{\infty} P(x)^2 W(x)dx \right\}.
$$

As Freud did in [10], we split the integral

$$
\int_{-\infty}^{\infty} xP(x)^2 W(x)dx = \left( \int_{-A\xi}^{A\xi} + \int_{\mathbb{R}\setminus[-A\xi,A\xi]} \right) xP(x)^2 W(x)dx
$$

and we can prove the following estimate:

**Lemma 10.** For any $\xi > 0$ and $A \geq 1$, we have

$$
X_n(\mu) \leq A\xi + \frac{8}{3\pi} \left( \frac{2}{\xi} \right)^{2n-1} G_{\xi}(W)^{-1} \int_{A\xi}^{\infty} t^{2n-1} d\mu(t). \quad (18)
$$

**Proof.** The proof is completely the same as [10, Theorem 2].

As a corollary of this lemma, we have the following estimation:

**Corollary 8.** Under the same notation as Lemma 10. For any $\rho > 0$, we define

$$
\Xi_\rho := \| | \cdot |^\rho W\|_\infty,
$$

$$
\xi_\rho := \sup_{\varepsilon > 0} \left[ \text{ess inf} \left\{ t \geq 0 : t^\rho W(t) > \Xi_\rho - \varepsilon \right\} \right]
$$

Then, we have

$$
X_n(\mu) \leq \left( 2 + \frac{2\Xi_{4n}\xi_{4n}}{3\pi n G_{\xi_{4n}}(W)} \right) \xi_{4n}.
$$
Proof. Let \( A = 2 \) and \( \xi = \xi_{4n} \) in Lemma 10. Since
\[
\int_{2\xi_{4n}}^{\infty} t^{2n-1} W(t) \, dt \leq \Xi_{4n} \int_{2\xi_{4n}}^{\infty} t^{-1-2n} \, dt
\]
we have the desired estimation.

Before proving Lemma 7, we prove a several simple lemmas:

**Lemma 11.** Under the same notation as Lemma 7, \( \xi_{\rho, \sigma} \leq \xi_{\rho, L} \) for any \( \sigma \in [0, L] \).

**Proof.** Let \( \varepsilon > 0 \) be an arbitrary positive number. Let \( \Omega_{\varepsilon, \sigma} := \{ t \geq 0 : t^\rho W^\sigma(t) > \Xi_{\rho, \sigma} - \varepsilon \} \), and let \( \eta_{\varepsilon, \sigma} := \text{ess inf} \Omega_{\varepsilon, \sigma} \). Then, by definition, for almost all \( s \in [0, \eta_{\varepsilon, \sigma}] \) and almost all \( t \in \Omega_{\varepsilon, \sigma} \), we have \( t \geq \eta_{\varepsilon, \sigma} \) and
\[
s^\rho W^\sigma(s) \leq \Xi_{\rho, \sigma} - \varepsilon < t^\rho W^\sigma(t).
\]
We write
\[
\delta(t) := (e^{(L-\sigma)t} - e^{(L-\sigma)\eta_{\varepsilon, \sigma}}) t^\rho W^\sigma(t).
\]
By multiplying the both sides by \( e^{(L-\sigma)t} \), we have
\[
s^\rho W^L(s) < t^\rho (e^{(L-\sigma)t} - e^{(L-\sigma)\eta_{\varepsilon, \sigma}} + e^{(L-\sigma)t}) W^\sigma(t)
\]
\[
< t^\rho W^L(t) - \delta(t)
\]
\[
\leq \Xi_{\rho, L} - \delta(t).
\]
Then, by definition of \( \Omega_{\delta(t), L} \), we see that \( \Omega_{\delta(t), L} \cap [0, \eta_{\varepsilon, \sigma}] \) is a null set, thus, we have \( \eta_{\varepsilon, \sigma} \leq \eta_{\delta(t), L} \leq \xi_{\rho, L} \). Hence, we have \( \xi_{\rho, \sigma} \leq \xi_{\rho, L} \).

We refer back to the proof of Lemma 7.

**Proof.** Write \( \xi = \xi_{4n, \sigma} \) and \( \Xi = \Xi_{4n, \sigma} \). By the definitions of \( \xi \) and \( \Xi \), there exists a non-negative sequence \( \{t_m\}_{m \geq 1} \) such that \( t_m \to \xi \) and \( W^\sigma(t_m) \to \Xi_{\xi^{-4n}} \) as \( m \to \infty \). Put \( W_1(t) = e^{(\sigma-L)|t|} \), \( W_2 = e^{-G_{\xi}(|t|)} \), and \( W_3 := e^{-G_{\xi}(|t|)} \). Observe that \( W^\sigma = W_1 W_2 W_3 \) and that
\[
\limsup_{m \to \infty} \frac{W_1(t_m)}{G_{\xi}(W_1)} = 1.
\]
So, it suffices to show that

\[
\limsup_{m \to \infty} \frac{W_2(t_m)}{G_\xi(W_2)} \leq 1, \tag{20}
\]

\[
\limsup_{m \to \infty} \frac{W_3(t_m)}{G_\xi(W_3)} \leq \exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} Q^m_L(\xi | \cos \theta|) d\theta \right\}. \tag{21}
\]

As for (20), since $Q^m_L$ is decreasing, we see that

\[
\frac{W_2(t_m)}{G_\xi(W_2)} = \exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} Q^m_L(\xi | \cos \theta|) - Q^m_L(t_m) d\theta \right\}
\]

\[
\leq \exp \left\{ \frac{2}{\pi} \int_0^\infty \mathbf{1}_{[\text{max}]}(t) (Q^m_L(t) - Q^m_L(t_m)) \frac{dt}{\sqrt{\xi^2 - t^2}} \right\}
\]

Thus, by the Lebesgue convergence theorem, we have (20). The inequality (21) is immediate as $W_3(t_m) \leq 1$ and the right hand side is just $G_\xi(W_3)^{-1}$.

Now, we prove Theorem 5.

**Proof.** Thanks to Lemma 7, it suffices to prove that

\[
\frac{\xi_{\rho,L}}{\rho} \to 0
\]
as $\rho \to \infty$. Put $\xi_{\rho} := \xi_{\rho,L}$, $\Xi_{\rho} := \Xi_{\rho,L}$ for simplicity. Let $V(t) := e^{-Q^m_L(t)}$.

\[
V(t) = \|W^L_1|_{t,\infty}||_{\infty}.
\]

First, we claim that

\[
\Xi_{\rho} = \sup_{t \geq 0} t^{\rho} V(t) = \lim_{t \searrow \xi_{\rho}} t^{\rho} V(t). \tag{22}
\]

In fact, we first prove $\Xi_{\rho} \geq \sup_{t \geq 0} t^{\rho} V(t)$. Since for any $t \geq 0$ and $\delta > 0$, by definition, the set $S := \{ s \geq t : W^L(s) > V(t) - \delta \}$ has positive measure, we see that $\Xi_{\rho} \geq s^{\rho} W^L(s) \geq t^{\rho} V(t) - t^{\rho} \delta$ for almost all $s \in S$. It implies the desired inequality. As $\sup_{t \geq 0} t^{\rho} V(t) \geq \lim_{t \searrow \xi_{\rho}} t^{\rho} V(t)$ is obvious, it suffices to show that

\[
\lim_{t \searrow \xi_{\rho}} t^{\rho} V(t) \geq \Xi_{\rho}. \quad \text{For} \ \delta > 0, \ \text{let}
\]

\[
\Omega_{\delta} := \{ t \geq 0 : t^{\rho} W^L(t) > \Xi_{\rho} - \delta \},
\]
and define $\eta_\delta := \text{ess inf } \Omega_\delta$. Take arbitrary $\eta_1 > \eta_\delta$. Then, we see that $(\eta_\delta, \eta_1) \cap \Omega_\delta$ has positive volume and is contained in

$$\{ t \geq \eta_\delta : W(t) > (\Xi_\rho - \delta) \eta_1^{-\rho} \}.$$ 

Thus, by definition of $V(\eta_\delta)$, we have $V(\eta_\delta) \geq (\Xi_\rho - \delta) / \eta_1^\rho$. Since $\eta_1$ is arbitrary and $V$ is right continuous, we see that $\Xi_\rho - \delta \leq \eta_\delta^\rho V(\eta_\delta)$ for sufficiently small $\delta > 0$. Since, $\eta_\delta \nearrow \xi_\rho$ as $\delta \to 0$, we have $\lim_{t \nearrow \xi_\rho} t^\rho V(t) \geq \Xi_\rho$.

Let us assume $\xi_\rho \geq 1$. By (22), we have

$$\lim_{s \nearrow \xi_\rho} s^\rho V(s) \geq t^\rho V(t)$$

for all $0 \leq t \leq \xi_\rho$, or equivalently,

$$\lim_{s \nearrow \xi_\rho} Q^m_L(s) - Q^m_L(t) \leq \rho \log(\xi_\rho / t).$$

Let $q > 0$ be an arbitrarily positive number. By multiplying the both sides by $qt^{q-1}$ and integrating them from 0 to $\xi_\rho$, we have

$$\xi_\rho^q \lim_{s \nearrow \xi_\rho} Q^m_L(s) - q \int_0^{\xi_\rho} t^{q-1} Q^m_L(t) dt \leq \rho \xi_\rho^q. \quad (23)$$

Since

$$\left| q \int_0^{\xi_\rho} t^{q-1} Q^m_L(t) dt - Q^m_L(0) \right|$$

$$\leq \int_0^1 \left| Q^m_L(t^{1/q}) - Q^m_L(0) \right| dt + q \int_1^{\xi_\rho} t^{q-1} |Q^m_L(t)| dt$$

$$\leq \int_0^1 \left| Q^m_L(t^{1/q}) - Q^m_L(t) \right| dt + q \int_1^{\xi_\rho} |Q^m_L(t)| dt,$$

we see that the second term of (23) goes to $Q^m_L(0)$ when $q \to 0$ as $Q^m_L$ is right continuous and bounded on $[0, 1]$. Thus, by taking limit $q \to 0$ on (23), we have

$$\lim_{s \nearrow \xi_\rho} Q^m_L(s) - Q^m_L(0) \leq \rho.$$

Therefore, we have

$$\lim_{\rho \to +\infty} \lim_{s \nearrow \xi_\rho } Q^m_L(s) / s \leq \lim_{\rho \to +\infty} \rho / \xi_\rho.$$

By (11), we see that the left hand side is $+\infty$, thus so is the right hand side.
4.3 Boundedness for spherical positive definite functions

Let \( w \) be a spherical function on \( \mathbb{R}^d \), and let \( L > 0 \). Take \( W : [0, \infty) \to \mathbb{R} \) be a function such that \( w(\xi) = W(|\xi|) \). Assume \( |\xi|^n e^{2L|\xi|} |w(\xi)| \in L^1 \) for all \( n \geq 0 \). For \( y \in \mathbb{R}^d \) with \( |y| \leq L \), we introduce

\[
\tilde{E}_n(y; w) := \sqrt{\sup_{P \in P_n \setminus \{0\}} \frac{\|e^{|y| \cdot P}\|_{L^2(w)}}{\|P\|_{L^2(w)}}}.
\]

First, we show a relation between the zeros of orthogonal polynomials and \( \delta^+_y(w) \).

Lemma 12. Let \( w, W \) and \( L \) be as above. For \( 0 \leq \sigma \leq L \), we define the measure \( \nu_\sigma \) on \( \mathbb{R} \) by

\[
\nu_\sigma = |t|^{d-1} e^{2\sigma|t|} W(|t|) dt.
\]

Then, for \( y \in \mathbb{R}^d \) with \( |y| \leq L \), we have

\[
\delta^+_n(y; w) \leq \tilde{E}_n(y; w) \leq \exp \left( \frac{1}{n} \int_0^{|y|} X_{n+2}(\nu_t) dt \right).
\]

Proof. The first inequality is obvious in terms of the Cauchy-Schwarz inequality. We prove the second inequality. For any real-coefficients one-variable polynomial \( p \in \mathbb{R}[t] \), let

\[
\phi_p(s) := \int_{\mathbb{R}} p(t)^2 \nu_s(t).
\]

By the Cauchy-Schwarz inequality, we have

\[
\begin{align*}
\frac{\phi'_p(s)}{\phi_p(s)} &= 2 \cdot \frac{\int_{\mathbb{R}} |t| p(t)^2 \nu_s(t)}{\int_{\mathbb{R}} p(t)^2 \nu_s(t)} \\
&\leq 2 \left( \frac{\int_{\mathbb{R}} t^2 p(t)^2 \nu_s(t)}{\int_{\mathbb{R}} p(t)^2 \nu_s(t)} \right)^{1/2}.
\end{align*}
\]

By Theorem 1 of [12], we have

\[
\frac{\phi'_p(s)}{\phi_p(s)} \leq 2 |X_{n+2}(\nu_s)|. \tag{24}
\]

Therefore, by integrating both sides of (24), we have

\[
\frac{\phi_p(s)}{\phi_p(0)} \leq \exp \left( 2 \int_0^s |X_{n+2}(t)| dt \right). \tag{25}
\]
On the other hand, by a direct computation, we have
\[ \| e^{y|\cdot|} P \|_{L^2(w)}^2 \leq \int_{\mathbb{R}^d} P(\xi)^2 e^{2y|\cdot|\xi} w(\xi) d\xi, \]
\[ = \int_0^\infty \left( \int_{S^{d-1}} P(ru)^2 du \right) r^{d-1} e^{2y|\cdot|r} W(r) dr, \]
where \( S^{d-1} \) is the unit sphere, and \( du \) is a suitable invariant measure on the sphere \( S^{d-1} \). Put \( h(r) := \int_{S^{d-1}} P(ru)^2 du \). Since \( h \) is an even polynomial, we have \( h(t) = h(|t|) \) for \( t \in \mathbb{R} \). Thus,
\[ \| m_y P \|_{L^2(w)}^2 = \frac{1}{2} \int_{\mathbb{R}} h(r) d\nu_{y|r}. \]
Since \( h \) is a positive polynomial of degree at most 2n, it is well known that there exists a finite collection of polynomials \( h_1, \ldots, h_K \) of degree at most \( n \) such that \( h = h_1^2 + \cdots + h_K^2 \) (for example, see [4]). Thus we have
\[ \frac{\| m_y P \|_{L^2(w)}^2}{\| P \|_{L^2(w)}^2} \leq \frac{\sum_i \phi_{h_i}(|y|)}{\sum_i \phi_{h_i}(0)} \leq \max_{i=1,\ldots,K} \phi_{h_i}(|y|) \phi_{h_i}(0). \]
Therefore, by (25), we have the desired inequality.

We prove the following theorem:

**Theorem 6.** Let \( w \in L^1 \cap L^\infty \setminus \{0\} \) satisfying Assumption (A). Assume there exists a measurable function \( Q : [0, \infty) \to \mathbb{R} \) such that \( w(\xi) = e^{-Q(|\xi|)} \) and for all \( L > 0 \), there exists \( B > 0 \) such that for any sufficiently large \( t > 0 \),
\[ \int_0^{\pi/2} Q_L^0(t \cos \theta) d\theta < B + \log t. \] (26)
Then, the function \( w \) further satisfies Assumptions (B) and (C).

**Proof.** Since \( w \) is a spherical function, the set \( \mathcal{G}(w) \) contains all the orthogonal matrices, thus, \( \mathcal{G}(u) \) generates the space of matrices, namely, (C) holds. Let us prove Assumption (B). First, we prove
\[ \limsup_{n \to \infty} \tilde{\delta}_n(y; w) \leq 1. \]
Thanks to Lemma 7, it suffices to prove that \( R_L(t) := Q(t) - (d - 1) \log t - Lt \) satisfies the conditions (11) and (26) for all \( L > 0 \). The condition (11) is immediate since \( w \) satisfies Assumption (A). Regarding condition (26), since

\[
R_L^0(t) \leq Q^{0}_{L+d-1}(t) + (d - 1)(t - \log t)1_{[0,1]}(t),
\]

we have

\[
\int_0^{2\pi} R_L^0(t \cos \theta) d\theta \\
\leq \int_0^{2\pi} Q_{L+d-1}^0(t \cos \theta) d\theta + (d - 1) \int_0^{\min(1,t)} (s - \log s) \frac{ds}{\sqrt{t^2 - s^2}}.
\]

Since the second term is constant for sufficiently large \( t \), we see that \( R_L \) satisfies condition (26) as \( Q_{L+d-1}^0 \) satisfies (26). Thus, \( \limsup_{n \to \infty} \tilde{\epsilon}_n^{-}(y; w) \leq 1 \) by the left inequality of Lemma 12. We prove \( \limsup_{n \to \infty} \tilde{\epsilon}_n^{-}(y; w) \leq 1 \). Let \( w_0(\xi) := e^{-2|\xi|} \). Let \( P \in P_n \). By the Cauchy–Schwarz inequality, we have

\[
\|m_y P\|_{L^2(w)} \geq \|Pe^{-|y|}1\|_{L^2(w)} = \|P\|_{L^2(w_0)}.
\]

Then, we have

\[
\frac{\|P\|_{L^2(w)}}{\|m_y P\|_{L^2(w)}} \leq \frac{\|e^{-|y|} P\|_{L^2(w_0)}}{\|P\|_{L^2(w_0)}} \leq \tilde{\epsilon}_n^{-}(y; w_0).
\]

Thus, we may use Lemma 12 and Theorem 5 as in the above argument, and we obtain

\[
\limsup_{n \to \infty} \tilde{\epsilon}_n^{-}(y; w) \leq \limsup_{n \to \infty} \tilde{\epsilon}_n^{-}(y; w_0) \leq 1.
\]

Therefore, Assumptions (A), (B), and (C) hold.

As a result, we obtain a simple sufficient condition for \( w \) so as to satisfy Assumption (B):

**Corollary 9.** Let \( w \in L^1 \cap L^\infty \setminus \{0\} \) be a nonnegative spherical function. Assume that there exists a locally \( L^1 \)-function \( Q : [0, \infty) \to \mathbb{R} \) such that \( w(\xi) = e^{-Q(|\xi|)} \). We further assume that there exists \( c \geq 0 \) such that \( Q(t) + ct \) is non-decreasing for sufficiently large \( t \geq 0 \) and that \( Q(t + R) - Q(t) \to \infty \) as \( t \to \infty \) for some \( R > 0 \). Then, the function \( w \) satisfies Assumptions (A), (B), and (C).

**Proof.** We deduce from the assumptions of the corollary that \( Q \) is bounded any interval \([a, b]\) as long as \( b > a \gg 1 \). Assumption (A) immediately follows from
the condition that \( Q(t + R) - Q(t) \to \infty \) as \( t \to \infty \). We will prove that for any 
\( L > 0 \), \( Q^L_t(t) \) is bounded for any sufficiently large \( t > 0 \). We easily see that 
the boundedness of \( Q^L_t \) implies the condition (26) in Theorem 5, and thus, \( w \) 
satisfies the condition (B) and (C) by Theorem 6.

First, we claim that we may assume \( Q \) is non-decreasing. In fact, Let \( R' > 0 \) 
be a positive number such that \( Q_-(t) = Q(t) + ct \) is non-decreasing for \( t \geq R' \).
We define a non-decreasing function \( \tilde{Q} \) by

\[
\tilde{Q}(t) := (Q_-(t) - Q_-(R')) 1_{[R', \infty)}.
\]

Note that we immediately see that \( \tilde{Q}(t + R) - \tilde{Q}(t) \to \infty \) as \( t \to \infty \). Then, we have

\[
\begin{align*}
Q_L &= Q_1_{[0, R')} + Q_1_{[R', \infty]} + Q_L + \tilde{Q}_L + cR', \\
(Q_1_{[0, R')} - \tilde{Q}_L + cR')^0 &\leq (|Q| + |Q^m(0)| + cR') 1_{[0, R')}.
\end{align*}
\]

Thus, we have

\[
Q^0_L \leq (|Q| + |Q^m(0)| + cR') 1_{[0, R')} + \tilde{Q}_L + cR'.
\]

Since the first term does not affect the condition (26), we may replace \( Q(t) \) 
with the non-decreasing function \( \tilde{Q} \) to prove the condition (26).

Now, we assume \( Q \) is non-decreasing. Fix an arbitrary sufficiently large 
number \( s \geq 0 \) satisfying \( Q(t + R) - Q(t) \geq LR \) for any \( t \geq s \). Since for any \( t \in [s, s + R] \), we have \( Q_L(t + nR) - Q_L(t) > 0 \) for all positive integer \( n \geq 1 \). Thus, 
for any arbitrary \( \varepsilon > 0 \) there exists \( t_\varepsilon \in [s, s + R] \) such that \( Q_L^0(t_\varepsilon) + \varepsilon > Q_L(t_\varepsilon) \).
Then, by definition of \( Q^0_L \) and by the fact that \( Q \) is non-decreasing, we see that

\[
Q^0_L(s) - \varepsilon \leq Q_L(s) - Q_L(t_\varepsilon) \leq Q(s) - Q(t_\varepsilon) + LR \leq LR.
\]

Since \( \varepsilon \) is arbitrary, we have \( Q^0_L(s) \leq LR \), which means that for any sufficiently 
large \( s > 0 \), \( Q^0_L(s) \) bounded.

### 4.4 Boundedness for positive definite functions of tensor products of 
even functions

In this subsection, we discuss the case where the non-negative function \( w \) is a 
tensor product of even functions on \( \mathbb{R} \). First, we prove the following lemma:

**Lemma 13.** Let \( w \in L^1 \cap L^\infty \setminus \{0\} \) be a nonnegative measurable function. Assume there exists \( w_1, \ldots, w_d : \mathbb{R} \to \mathbb{R} \) such that each \( w_i \) satisfies Assumption
(A) and \( w(\xi_1, \ldots, \xi_d) = w_1(\xi_1) \cdots w_d(\xi_d) \). If each \( w_i \) satisfies Assumption (B), namely, there exists \( B_i > 0 \) such that for all \( y \in \mathbb{R} \),

\[
\limsup_{n \to \infty} \mathcal{E}_n^\pm(y; w_i) < B_i,
\]

then \( w \) also satisfies Assumption (B).

**Proof.** Fix \( y = (y_i)_{i=1}^d \in \mathbb{R} \). Then, for any \( P \in P_n \), we see that

\[
\frac{\|m_y P\|_{L^2(w)}^2}{\|P\|_{L^2(w)}^2} = \prod_{i=1}^d \int_{\mathbb{R}} e^{y_i \xi} P_i(\xi) w_i(\xi) d\xi \int_{\mathbb{R}} P_i(\xi) w_i(\xi) d\xi,
\]

where \( P_i \) is a one-variable polynomial of degree at most \( 2n \).

\[
P_i(\xi) := \int_{\mathbb{R}^{d-1}} |P(\xi_1, \ldots, \xi_{i-1}, \xi, \xi_{i+1}, \ldots, \xi_d)|^2 w_1(\xi_1) \cdots w_{i-1}(\xi_{i-1}) w_i(\xi_{i+1}) \cdots w_d(\xi_d) d\xi_1 \cdots d\xi_{i-1} d\xi_{i+1} \cdots d\xi_d
\]

Since each \( P_i \) is a positive polynomial, by the Hilbert’s 17th problem (see, for example [4]), there exists a finite collection of one-variable polynomials \( Q_{i,1}, \ldots, Q_{i,K_i} \) of degree at most \( n \) such that \( P_i = Q_{i,1}^2 + \cdots + Q_{i,K_i}^2 \). Thus, we have

\[
\frac{\int_{\mathbb{R}} e^{y_i \xi} P_i(\xi) w_i(\xi) d\xi}{\int_{\mathbb{R}} P_i(\xi) w_i(\xi) d\xi} \leq \max_{j=1, \ldots, K_i} \frac{\int_{\mathbb{R}} e^{y_j \xi} Q_{i,j}(\xi) w_i(\xi) d\xi}{\int_{\mathbb{R}} Q_{i,j}(\xi) w_i(\xi) d\xi} \leq \mathcal{E}_n^+(y_i; w_i)^2.
\]

Therefore, we see that

\[
\limsup_{n} \mathcal{E}_n^+(y, w) \leq B_1 \cdots B_d.
\]

We also obtain

\[
\limsup_{n} \mathcal{E}_n^-(y, w) \leq B_1 \cdots B_d.
\]

in the same manner as above.

Then, we obtain a similar theorem to Theorem 6:

**Theorem 7.** Let \( w \in L^1 \cap L^\infty \setminus \{0\} \) satisfying Assumption (A). Assume there exists a measurable function \( Q_1, \ldots, Q_d : [0, \infty) \to \mathbb{R} \) such that

\[
w(\xi_1, \ldots, \xi_d) = \prod_{i=1}^d e^{-Q_i(|\xi_i|)},
\]
for all $i, j \in \{1, \ldots, d\}$, $|Q_i - Q_j|$ is bounded, and for each $i = 1, \ldots, d$ and any $L > 0$, there exists $B_i > 0$ such that for any sufficiently large $t > 0$,

$$
\int_0^{\pi/2} (Q_i)_{L}^i (t \cos \theta) d\theta < B_i + \log t.
$$

(27)

Then, the function $w$ further satisfies Assumptions (B) and (C).

**Proof.** By Lemma 13 with Theorem 6, it suffices to show that $w$ satisfies Assumption (C). The boundedness of $|Q_i - Q_j|$ implies $\mathcal{G}(w)$ contains all the symmetric group $\mathfrak{S}_d \subset \text{GL}_d(\mathbb{R})$. Since each $Q_i$ is an even function, $\mathcal{G}(w)$ also contains $C_d^2 \subset \text{GL}_d(\mathbb{R})$, the group of diagonal matrices $A$ satisfying $A^2 = I$. Since the group generated by $\mathfrak{S}_d$ and $C_d^2$ generates $\mathcal{M}_d(\mathbb{R})$ as a linear space over $\mathbb{R}$, we obtain (C).

We also have a similar corollary to Corollary 9:

**Corollary 10.** Let $w \in L^1 \cap L^\infty \setminus \{0\}$ satisfying Assumption (A). Assume there exists a measurable function $Q_1, \ldots, Q_d : [0, \infty) \to \mathbb{R}$ such that

$$
w(\xi_1, \ldots, \xi_d) = \prod_{i=1}^d e^{-Q_i(|\xi_i|)},
$$

for all $i, j \in \{1, \ldots, d\}$, $|Q_i - Q_j|$ is bounded. We further assume that for each $i = 1, \ldots, d$, there exists $c_i$ such that $Q_i(t) + c_i t$ is non-decreasing for sufficiently large $t \geq 0$ and $Q_i(t + R_i) - Q_i(t) \to \infty$ as $t \to \infty$ for some $R_i > 0$. Then, the function $w$ satisfies Assumption (A), (B), and (C).

**Proof.** By Lemma 13 with Corollary 9, it suffices to show that $w$ satisfies Assumption (C), but its proof is the same as in that of Theorem 7.

For example, $w(\xi) = e^{-|\xi|^p}$ ($p > 1$) satisfies the condition in Corollary 10, where $|\xi|^p := (|\xi_1|^p + \cdots + |\xi_d|^p)^{1/p}$ for $\xi = (\xi_i)_{i=1}^d \in \mathbb{R}^d$.

### 4.5 An example of entire functions of infinite order

Even in one dimensional case, our result contains an essentially new contribution. In the case of $d = 1$ and $w = 1_{[-1/2, 1/2]}$, there are several works treated the boundedness of composition operators in RKHS [6, 7]. Their method based on finiteness of the order of the entire function $\hat{w}$. As the RKHS associated to the positive definite function $\hat{w}$ is composed of entire functions of order 1, they [6, 7] directly apply Pólya’s theorem [19] with some careful analysis, and deduce affiness of original maps inducing bounded composition operators. We
may apply this method without using ours discribed in this paper if the RKHS is composed of entire functions of finite order. However, there exists an example of RKHSs containing entire functions of infinite order, but only affine maps can induce bounded composition operators on the RKHS based on our framework.

Let us explain the example. We define

\[ w(\xi) = \sum_{n=-\infty}^{\infty} \frac{1_{[-1/2+n,1/2+n]}(\xi)}{|n|!}. \]

Then, we easily see that

\[ \hat{w}(z) = \sin(\pi z) \sum_{n=-\infty}^{\infty} \frac{e^{2\pi inz}}{|n|!} = (e^{e^{2\pi iz}} + e^{-e^{2\pi iz}} - 1) \cdot \frac{\sin(\pi z)}{\pi z}. \]

The entire function \( \hat{w}(z) \) is of infinite order since \( |\hat{w}(iy)| = O(e^{2\pi y}) \). Let \( Q := -\log w \). Then, we immediately see that \( Q \) is non-decreasing and \( Q(t+1) - Q(t) \to \infty \) as \( t \to \infty \) since \( Q(t+1) - Q(t) = \log(n+1) \) for \( t \in [-1/2+n,1/2+n] \). Thus, thanks to Corollary 9 and Theorem 1, if a composition operator on the RKHS associated with the positive definite function \( \hat{w} \) is bounded, then the original map is an affine map.
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