GENERICALLY NEF VECTOR BUNDLES ON RULED SURFACES

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ABSTRACT. The present paper concerns the invariants of generically nef vector bundles on ruled surfaces. By Mehta - Ramanathan Restriction Theorem and by Miyaoka characterization of semistable vector bundles on a curve, the generic nefness can be considered as a weak form of semistability. We establish a Bogomolov type inequality for generically nef vector bundles with nef general fiber restriction on ruled surfaces with no negative section, see Theorem [3.1]. This gives an affirmative answer in this case to a problem posed by Th. Peternell in [P].

Concerning ruled surfaces with a negative section, we prove a similar result for generically nef vector bundles, with nef and balanced general fiber restriction and with a numerical condition on first Chern class, which is satisfied, for instance, if in its class there is a reduced divisor, see Theorem [3.5].

Finally, we use such results to bound the invariants of curve fibrations, which factorize through finite covers of ruled surfaces.

CONTENTS

1. Introduction 2
2. Notation and preliminaries 3
3. Bogomolov type inequalities 4
4. The normalized relative canonical divisor 9
5. Positivity results on \( L_\pi \) 14
5.1. Rational fibrations with uniform Tschirnhausen sheaf 15
5.2. Upper bounds on \( L_\pi^2 \) and primitive cyclic covers 16
5.3. A Beniamino Segre’s construction 17
6. Sharpness results 19
6.1. Existence of fibrations with the required properties: rational basis 19
6.2. Existence of fibrations with the required properties: other cases 20

References 20

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1. Introduction

The present paper concerns the invariants of generically nef vector bundles on ruled surfaces, see Definition 2.1. By Mehta - Ramanathan Restriction Theorem 2.2 and by Miyaoka characterization of semistable vector bundles on a curve, Theorem 2.3, the generic nefness can be considered as a weak form of semistability, see Corollary 2.4.

A question about a possible relation between generic nefness and Bogomolov type inequality has been posed by Th. Peternell in the paper [P, remarks after Theorem 3.8]. By considering the Harder - Narasimhan filtration of a vector bundle \( E \) on a projective surface, Miyaoka proved the inequality \( c_2(E) \geq 0 \), provided that \( E \) is generically nef and \( c_1(E) \) is nef.

In the present paper, under the hypotheses of generic nefness and the nefness of the generic fiber restriction, we give an affirmative answer to Peternell’s question for ruled surfaces with invariant \( e = -C_0^2 \leq 0 \) (see Theorem 3.1):

**Theorem 1.1.** Let \( Y \) be a ruled surface on a smooth curve \( B \) with invariant \( e = -C_0^2 \leq 0 \). Let \( E \) be a generically nef vector bundle of rank \( r \) on \( Y \) with nef generic fiber restriction. Then

\[
c_2(E) \geq \sum_{i=1}^{r-1} \frac{a_i}{2a} c_1(E)^2,
\]

where \( c_1(E) \equiv aC_0 + \delta L \) and \((a_1, \ldots, a_r)\) is the generic splitting type of \( E \).

In the \( e > 0 \) case, we prove a similar bound under two additional assumptions, namely that the general fiber restriction is balanced, and \( c_1(E) \cdot C_0 \geq -\frac{e}{2} \), see Theorem 3.5. The last condition is satisfied, for instance, if \( c_1(E) \) is nef or if \( c_1(E) \) is effective and \( C_0 \) is not contained in the base locus of \( |2c_1(E) - C_0| \); this assumption is typically satisfied by the Tschirnhausen sheaf associated with a finite cover of smooth surfaces with reduced branch divisor. Our results allow us to obtain some bounds on the invariants of fibred surfaces factoring through finite covers. In these specific cases the bounds found are better than the recent bounds found by X. Lu and K. Zuo in [LZ]. Moreover, in the case of primitive cyclic covers \( \pi : S \to Y \), we obtain the same bound \( \lambda_{g,0,n} \) given in [E, Remark 4.4], see Theorem 5.7.

The techniques involved concern vector bundles and algebraic surfaces techniques. We believe that our approach can be the starting point for further research in the theory of vector bundles on fibred surfaces in general.

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The examples contained in Subsection 6.1 give an answer to a question posed by Ciro Ciliberto at the conference New Trends in Algebraic Geometry, Università della Calabria, June 12-14 (2013), about the sharpness of some slope estimates.

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2. Notation and preliminaries

Let us introduce the definitions involved in the notion of generic nefness, see [P, Definition 3.1]).

**Definition 2.1.** A vector bundle $E$ on a smooth curve is nef if the tautological divisor of $\mathbb{P}(E)$ is nef.

A vector bundle $E$ on a projective variety $X$ of dimension $n \geq 2$ is called generically nef with respect to an ample divisor $H$ if the restriction $E|_C$ is nef for a general curve $C = D_1 \cap \cdots \cap D_{n-1}$, where $D_i \in |m_i H|$ are general and $m_i \gg 0$; such a curve $C$ is said to be MR-general, which means general in the sense of Mehta-Ramanathan, with respect to $H$ (w.r.t. $H$).

A vector bundle $E$ is generically nef if for every ample divisor $H$ on $X$, the restriction $E|_C$ is nef for a MR-general curve w.r.t. $H$.

We now recall Mehta-Ramanathan Restriction Theorem and Miyaoka characterization of semistable vector bundles on a curve, which imply that the generic nefness can be considered as a weak form of semistability.

**Theorem 2.2.** (Mehta-Ramanathan Restriction Theorem)

A locally free sheaf $E$ on a projective variety $X$ is semi-stable w.r.t. an ample divisor $H$ if and only $E|_C$ is semi-stable for $C$ MR-general w.r.t. $H$.

**Theorem 2.3.** (Miyaoka) Let $C$ be a smooth curve and $E$ a vector bundle on $C$. Then $E$ is semi-stable if and only if the $\mathbb{Q}$-bundle $E \otimes \frac{\det E^\vee}{\det E}$ is nef, where $E^\vee = \text{Hom}(E, \mathcal{O}_C)$ is the dual vector bundle.

**Corollary 2.4.** If $E$ is semi-stable w.r.t. $H$ and if $c_1(E) \cdot H \geq 0$, then $E$ is generically nef w.r.t. $H$.

Finally, since we shall consider vector bundles on ruled surfaces, we can talk of the general splitting type.

**Definition 2.5.** Let $p : Y \to B$ be a ruled surface over a smooth curve $B$, and let $E$ be a rank $r$ vector bundle on $Y$. We say that $(a_1, \ldots, a_r)$, with $a_1 \leq \cdots \leq a_r$, is the generic splitting type of $E$ if for a general fiber $L$ of $p$ we have

$$E \otimes \mathcal{O}_L \cong \bigoplus_{i=1}^r \mathcal{O}_\mathbb{P}^1(a_i).$$

We say that a fiber $L$ is a jumping line if

$$E \otimes \mathcal{O}_L \not\cong \bigoplus_{i=1}^r \mathcal{O}_\mathbb{P}^1(a_i).$$

Finally, we say that $E$ is uniform if it has no jumping lines.

Finally, let us recall that for a tensor product $V \otimes L$, where $V$ is a rank $r$ vector bundle and $L$ is a line bundle, we have:

$$c_1(V \otimes L) = c_1(V) + r \cdot c_1(L), \quad c_2(V \otimes L) = c_2(V) + (r-1)c_1(V) \cdot c_1(L) + \frac{r(r-1)}{2}c_1(L)^2.$$
3. Bogomolov Type Inequalities

**Theorem 3.1.** Let \( Y \) be a ruled surface on a smooth curve \( B \) with invariant \( e = -C_0^2 \leq 0 \). Let \( E \) be a generically nef vector bundle of rank \( r \) on \( Y \) with nef generic fiber restriction.

Then

\[
c_2(E) \geq \left( \frac{\sum_{i=1}^{r-1} a_i}{2a} \right) c_1(E)^2,
\]

where \( c_1(E) \equiv aC_0 + \delta L \) and \((a_1, \ldots, a_r)\) is the generic splitting type of \( E \).

**Proof.** We prove the statement by induction on \( r \geq 2 \). If \( r = 2 \), the claimed inequality has been proved in [BeZ, Theorem 2.8], in the slightly different setting of blown up ruled surfaces, and under the assumption on \( E \) to be weakly positive and nef general fiber restriction.

We observe, however, that the weak positivity of a vector bundle implies its generic nefness. Moreover, since the weak positivity is preserved for any quotient bundle, the same properties hold for such quotients. Then it is simple to check that the proof of the statement in the rank 2 case can be done with some little adaptations.

So let us assume \( r \geq 3 \) and suppose that the claim holds for any vector bundle of rank \( q \leq r - 1 \) satisfying the assumptions of the statement.

We can consider the push-pull map

\[
p^* p_* \mathcal{E}(-a_r C_0) \to \mathcal{E}(-a_r C_0);
\]

such a map is generically injective, hence it is an injective map of locally free sheaves. Moreover, the quotient sheaf is locally free outside a subscheme \( Z \) of codimension 2.

It follows that we have a Brosius type sequence (see [Br]):

\[
0 \to p^* p_* \mathcal{E}(-a_r C_0) \to \mathcal{E}(-a_r C_0) \to G \otimes I_Z \to 0.
\]

By setting \( A := p_*(\mathcal{E}(-a_r C_0)) \), \( A := c_1(A) = \) and \( \alpha := \deg(A) \), and by tensoring by \( O_Y(a_r C_0) \) we get:

\[
0 \to (p^* A)(a_r C_0) \to \mathcal{E} \to G(a_r C_0) \otimes I_Z \to 0,
\]

where \( G \) is a vector bundle of rank \( q \leq r \) and \( A \) is a vector bundle of rank \( r - q \geq 1 \). We observe that \( r - q \) is equal to the number of integers in the general splitting type \((a_1, \ldots, a_r)\), which are equal to \( a_r \), so that we have

\[
a = \sum_{i=1}^{r} a_i = \sum_{i=1}^{q} a_i + (r - q)a_r.
\]

This sequence gives

\[
c_2(E) = c_1(p^* A (a_r C_0)) \cdot c_1(G(a_r C_0)) + c_2(p^* A (a_r C_0)) + c_2(G(a_r C_0)) + Z.
\]

Set \( M := G(a_r C_0) \). Let us compute each term appearing in (3.4).

\[
c_1(M) \equiv \left( c_1(E) - p^* A - (r - q)a_r C_0 \right) \equiv a_M C_0 + (\delta - \alpha)L
\]

where we have set

\[
a_M := \sum_{i=1}^{q} a_i,
\]
and

\[(3.6) \quad c_1(p^*A (a_r C_0)) \cdot c_1(M) = \left( p^*A + (r-q)a_r C_0 \right) \cdot \left( a_M C_0 + (\delta - \alpha)L \right) = \]

\[= (r-q)a_r a_M C_0^2 + (\alpha a_M + (r-q)a_r(\delta - \alpha)). \]

By taking into account the relations \[(2.1)\], we get

\[(3.7) \quad c_2(p^*A (a_r C_0)) = (r-q-1)p^*A \cdot (a_r C_0) + \frac{(r-q)(r-q-1)}{2} a_r^2 C_0^2 = \]

\[= (r-q-1) a_r + \frac{(r-q)(r-q-1)}{2} a_r^2 C_0^2. \]

Moreover, since \(M\) is a quotient of \(E\) away from the zero-dimensional scheme \(Z\), \(M\) is a generically nef vector bundle of rank \(q < r\). Now we analyze the general fiber restriction of \(M\). Since \(E\) has nef general fiber restriction, we have that the general splitting type \((a_1, \ldots, a_r)\) satisfies

\[(3.8) \quad 0 \leq a_1 \leq \cdots \leq a_r. \]

Moreover, by construction the generic splitting type of \(M\) is

\[(a_1, \ldots, a_q) \]

where the integers \(a_i\) are the first \(q\) integers appearing in \[(3.8)\], hence also \(M\) is nef on the generic fiber restriction.

It follows that \(M\) satisfies the assumptions of the Theorem and we can apply the induction hypothesis, which gives the inequality

\[(3.9) \quad c_2(M) \geq \left( \frac{\sum_{i=1}^{q-1} a_i}{2 a_M} \right) c_1(M)^2. \]

By \[(3.5)\] we get

\[(3.10) \quad c_1(M)^2 = a_M^2 C_0^2 + 2 a_M (\delta - \alpha), \]

and by observing that

\[\sum_{i=1}^{q-1} a_i = a_M - a_q, \]

the inequality \[(3.9)\] becomes

\[(3.11) \quad c_2(M) \geq \left( \frac{a_M - a_q}{2} \right) \left( a_M C_0^2 + 2 (\delta - \alpha) \right). \]

By taking into account \[(3.6), (3.7), (3.11)\] and the fact that \(Z\) is effective, from \[(3.4)\] we get

\[(3.12) \quad c_2(E) \geq (r-q)a_r a_M C_0^2 + \alpha a_M + (r-q)a_r(\delta - \alpha) + \]

\[+ (r-q-1) a_r \alpha + \frac{(r-q)(r-q-1)}{2} a_r^2 C_0^2 + \left( \frac{\alpha a_M - a_q}{2} \right) a_M C_0^2 + (a_M - a_q)(\delta - \alpha). \]

We can rewrite the inequality above in the form

\[(3.13) \quad c_2(E) \geq d C_0^2 + (a_q - a_r) \alpha + c \delta, \]

where

\[d := (r-q)a_r a_M + \frac{1}{2} (r-q)(r-q-1)a_r^2 + \frac{1}{2} (a_M - a_q) a_M, \]
c := (r - q)a_r + a_M - a_q.

In particular, the coefficient of $\alpha$ in the expression (3.13) is

$$a_q - a_r < 0.$$  

Now we shall bound the integer $\alpha$. Recall that a generically nef vector bundle has a generically nef first Chern class, since exterior products of nef vector bundles are nef (see, for instance, [La2, Theorem 6.2.12, (iv)]). Hence $c_1(M)$ is generically nef. So for any nef divisor $xC_0 + yL$ with $x \geq 1$ and $y \geq -\frac{1}{2}x C_0^2$, we have

$$c_1(M) \cdot (xC_0 + yL) \geq 0,$$

that is

$$(a_M C_0 + (\delta - \alpha)L) \cdot (xC_0 + yL) = a_M(xC_0^2 + y) + x(\delta - \alpha) \geq 0,$$

which gives

$$\alpha \leq \delta + \frac{a_M(xC_0^2 + y)}{x},$$

and since this holds for any $x \geq 1$ and any $y \geq -\frac{1}{2}x C_0^2$, we get

$$\alpha \leq \delta + \frac{1}{2}a_M C_0^2.$$  

By substituting the righthand expression in (3.13), we get

$$c_2(E) \geq (d + \frac{1}{2}(a_q - a_r) a_M) C_0^2 + (c + a_q - a_r) \delta.$$

Next we shall suitably express the integer $\delta$. As $c_1(E)^2 = a^2C_0^2 + 2a\delta$, we can write

$$\delta = -\frac{a}{2}C_0^2 + \frac{c_1(E)^2}{2a}.$$

Then the inequality (3.14) becomes

$$c_2(E) \geq (d + \frac{1}{2}(a_q - a_r) a_M - \frac{a}{2}(c + a_q - a_r))C_0^2 + \frac{(c + a_q - a_r)}{2a}c_1(E)^2.$$  

By computing the coefficients in (3.15) we get that the coefficient of $C_0^2$ is zero an we get

$$c_2(E) \geq \left(\frac{a_M + (r - q - 1)a_r}{2a}\right) c_1(E)^2 = \left(\frac{a - a_r}{2a}\right) c_1(E)^2,$$

which is the bound in the statement. \hfill \Box

As a particular case, we can consider generically nef vector bundles $E$ with nef and balanced general fiber restriction, that is the restriction of $E$ to a general fiber of $Y$ is a balanced vector bundle with splitting type $(m, \ldots, m, m+1, \ldots, m+1)$.

**Corollary 3.2.** Let $Y$ be a ruled surface on a smooth curve $B$ with invariant $e = -C_0^2 \leq 0$. Let $E$ be a generically nef vector bundle of rank $r$ on $Y$, such that the restriction of $E$ to a general fiber of $Y$ is a nef and balanced vector bundle with splitting type $(m, \ldots, m, m+1, \ldots, m+1)$, and set $c_1(E) = aC_0 + \delta L$, $a = mr + k$ and $1 \leq k \leq r - 1$. 
Then
\[ c_2(\mathcal{E}) \geq \left( \frac{a-(m+1)}{2a} \right) c_1(\mathcal{E})^2 = \left( \frac{r-1}{2r} - \frac{r-k}{2ar} \right) c_1(\mathcal{E})^2. \]

Moreover, the equality holds if and only if \( \mathcal{E} \) is uniform and \( \mathcal{E} \) is an extension sitting in an exact sequence of the type
\[ 0 \to p^*\mathcal{B} \otimes \mathcal{O}_Y((m+1)C_0) \to \mathcal{E} \to p^*\mathcal{V} \otimes \mathcal{O}_Y(mC_0) \to 0, \]
where \( \mathcal{B} \) is a rank \( k \) vector bundle on \( B \) satisfying \( \deg c_1(\mathcal{B}) = \delta \), and \( \mathcal{V} \) is a rank \( r-k \) vector bundle on \( B \) satisfying \( c_1(\mathcal{V}) \equiv 0 \).

**Proof.** The bound is a direct consequence of the general bound. The characterization of the vector bundles attaining the equality can be directly obtained by imposing the equalities in all the bounds considered in the proof of Theorem 3.1.

We remark that vector bundles with balanced general fiber restriction are the natural generalization of vector bundles with semistable general fiber restriction, in which case the Bogomolov discriminant is non negative by Moriwaki’s Theorem [Mo, Theorem 2.2.1], which we recall.

**Theorem 3.3.** Let \( \varphi: Z \to C \) be a fibration from a smooth surface \( Z \) to a smooth curve \( C \). Let \( \mathcal{E} \) be a torsion free sheaf on \( Z \) such that the restriction of \( \mathcal{E} \) to a general fibre \( F \subset Z \) is a \( \mu \)-semistable locally free sheaf. Then the Bogomolov discriminant \( \Delta(\mathcal{E}) \) satisfies
\[ \Delta(\mathcal{E}) = c_2(\mathcal{E}) - \frac{rk(\mathcal{E}) - 1}{2rk(\mathcal{E})} c_1(\mathcal{E})^2 \geq 0. \]

Since in Moriwaki Theorem the only assumption is the semistability of the general fiber restriction, one can wonder, if the generic balancedness condition could be sufficient in order to have a Bogomolov type inequality. A negative answer is given by the following example.

**Example 3.4.** On \( Y = \mathbb{P}^1 \times \mathbb{P}^1 \) consider the rank two split vector bundle
\[ \mathcal{E} = \mathcal{O}_Y(m,b_1) \oplus \mathcal{O}_Y(m+1,b_2). \]
We have
\[ c_1(\mathcal{E})^2 = 2(2m+1)(b_1 + b_2), \quad c_2(\mathcal{E}) = m(b_1 + b_2) + b_1, \]
so we see that \( c_2(\mathcal{E}) \) can be arbitrarily lowered by adjusting \( b_1 \), even with \( c_1(\mathcal{E})^2 \) fixed.

Let us conclude this section with a result in the \( e > 0 \) case. We will consider only the balanced case, and we will need to assume \( c_1(\mathcal{E}) \cdot C_0 \geq -\frac{e}{2} \), which is satisfied, for instance, if \( c_1(\mathcal{E}) \) is nef, or if \( c_1(\mathcal{E}) \) is effective and \( C_0 \) is not contained in the base locus of \( |2c_1(\mathcal{E}) - C_0| \). The last condition is typically satisfied by Tschirnhausen sheaves associated with surface covers with reduced branch divisor.

**Theorem 3.5.** Let \( Y \) be a ruled surface on a smooth curve \( B \) with invariant \( e = -C_0^2 > 0 \). Let \( \mathcal{E} \) be a generically nef vector bundle of rank \( r \) on \( Y \), such that the restriction of \( \mathcal{E} \) to a general fiber of \( Y \) is a nef and balanced vector bundle with splitting type \( (m,\ldots,m,m+1,\ldots,m+1) \), \( c_1(\mathcal{E}) \equiv aC_0 + \delta L \), \( a = mr + k \) and \( 1 \leq k \leq r - 1 \).

Assume, moreover, that \( c_1(\mathcal{E}) \cdot C_0 \geq -\frac{e}{2} \).
the relation (3.12) becomes

\[
(3.16) \quad c_2(\mathcal{E}) \geq \left( \frac{a - (m + 1)}{2a} - \frac{a - k(m + 1)}{2a(a - 1)} \right) c_1(\mathcal{E})^2.
\]

**Proof.** The proof is similar to the one of Theorem 3.1. By assumption the restriction of \( \mathcal{E} \) to a general fiber \( L \) of \( p: Y \to \mathbb{P}^1 \) is balanced. Since by hypothesis \( c_1(\mathcal{E}) \cdot L = a = mr + k \), the general fiber restriction of \( \mathcal{E} \) is of the type

\[
\mathcal{E}|_L \cong \bigoplus_{k} \mathcal{O}_{\mathbb{P}^1}(m + 1) \oplus \bigoplus_{r-k} \mathcal{O}_{\mathbb{P}^1}(m).
\]

Then we have again a Brosius type exact sequence:

\[
(3.17) \quad 0 \to p^*p_*\mathcal{E}(-(m + 1)C_0) \to \mathcal{E}(-(m + 1)C_0) \to \mathcal{G} \otimes \mathcal{I}_Z \to 0,
\]

where \( p^*p_*\mathcal{E}(-(m + 1)C_0) \) has rank \( k \) and \( \mathcal{G} \) has rank \( (r - k) \).

Set \( \mathcal{A} := p_*\mathcal{E}(-(m + 1)C_0), \mathcal{B} := c_1(\mathcal{A}) =, \alpha := \deg(A) \) and \( \mathcal{M} := \mathcal{G}((m + 1)C_0) \), so that (3.17) becomes

\[
(3.18) \quad 0 \to p^*\mathcal{A}((m + 1)C_0) \to \mathcal{E} \to \mathcal{M} \otimes \mathcal{I}_Z \to 0,
\]

and

\[ c_1(\mathcal{M}) = c_1(\mathcal{E}) - p^*A - k(m + 1)C_0 \equiv (r - k)m C_0 + (\delta - \alpha)L. \]

The main difference in the present proof is the bound on \( c_2(\mathcal{M}) \). Since the restriction of the Brosius exact sequence (3.17) to the general fiber \( L \cong \mathbb{P}^1 \) gives

\[
0 \to \bigoplus_{k} \mathcal{O}_{\mathbb{P}^1} \to \bigoplus_{k} \mathcal{O}_{\mathbb{P}^1} \oplus \bigoplus_{r-k} \mathcal{O}_{\mathbb{P}^1}(-1) \to \bigoplus_{r-k} \mathcal{O}_{\mathbb{P}^1}(-1) \to 0,
\]

the restriction of \( \mathcal{G} \) to the general fiber of \( Y \) is \( \mu \)-semistable. Since \( \mathcal{M} \) is a twist of \( \mathcal{G} \) the same holds for the general fiber of \( \mathcal{M} \). Hence \( \mathcal{M} \) is Bogomolov semistable by Moriwaki Theorem 3.3 and we have

\[
c_2(\mathcal{M}) \geq \frac{(r - k - 1)}{2(r - k)} c_1(\mathcal{M})^2 = \frac{(r - k)(r - k - 1)}{2} m^2 C_0^2 + (r - k)m(\delta - \alpha).
\]

By observing that with the notations of the proof of Theorem 3.1 we have

\[ a_r = m + 1, \quad q = r - k, \quad a_M = (r - k)m, \]

the relation (3.12) becomes

\[
c_2(\mathcal{E}) \geq k(r - k)m(m + 1)C_0^2 + (\delta - \alpha)k(m + 1) + \alpha(r - k)m + (k - 1)(m + 1)\alpha +
\]

\[
+ \frac{k(k - 1)}{2} (m + 1)^2 C_0^2 + \frac{(r - k)(r - k - 1)}{2} m^2 C_0^2 + (r - k)m(\delta - \alpha),
\]

which simplifies as

\[
c_2(\mathcal{E}) \geq \left( k(r - k)m(m + 1) + \frac{k(k - 1)}{2} (m + 1)^2 + \frac{(r - k)(r - k - 1)}{2} m^2 \right) C_0^2 - \\
- \alpha + (k + (r - 1)m)\delta.
\]
Next we use the generic nefness of $c_1(E)$ to bound $\alpha = \deg(A)$. Let $H$ be a very ample divisor of $Y$ which avoids the points of $Z$ arising in the Brosius sequence (3.17). Since $E$ is generically nef, and since $M \otimes I_Z$ is a quotient of $E$, $G((m+1)C_0) \otimes O_{m_0H}$ is nef for $m_0 >> 0$, hence $c_1(M) \otimes O_{m_0H} \geq 0$.

An ample divisor on a ruled surface admitting a negative section $C_0$ is of the type $H \in |xC_0 + yL|$, with $x > 0$ and $y > x e$. The condition $c_1(M) \cdot m_0 H \geq 0$ gives

\begin{equation}
\alpha \leq \delta + m(r - k) \frac{y - xe}{x};
\end{equation}

in particular $\alpha \leq \delta + m(r - k) \frac{1}{x}$ for any $x > 0$, so

\begin{equation}
\alpha \leq \delta.
\end{equation}

Moreover, using again the trick

\begin{equation}
\delta = -\frac{a}{2} C_0^2 + \frac{c_1(E)^2}{2a},
\end{equation}

we get

\[ c_2(E) \geq \left( k(r-k)m(m+1) + \frac{k(k-1)}{2} (m+1)^2 + \frac{(r-k)(r-k-1)}{2} m^2 - \frac{a}{2} (k + (r-1)m - 1) \right) C_0^2 + \frac{(k + (r-1)m - 1)}{2a} c_1(E)^2, \]

that is

\begin{equation}
\frac{c_2(E)}{C_0^2} \geq \frac{(r-k)m}{2} - \frac{a - (m+1)}{2a} c_1(E)^2.
\end{equation}

The last bound is not satisfactory, since $C_0^2 < 0$, so we finally use the assumption that $c_1(E) \cdot C_0 \geq \frac{C_0^2}{2}$, which gives $\delta \geq (\frac{1}{2} - a) C_0^2$. The expression (3.22) yields $C_0^2 \geq -\frac{c_1(E)^2}{a(a-1)}$, and by (3.23) we get

\[ c_2(E) \geq \left( \frac{a - (m+1)}{2a} - \frac{a - k(m+1)}{2a(a-1)} \right) c_1(E)^2. \]

\[ \square \]

4. The normalized relative canonical divisor

In this section we shall apply the Bogomolov type inequalities to the Tschirnhausen sheaf of a finite cover of a Hirzebruch surface. Indeed, by the Viehweg Weak Positivity Theorem [V], the Tschirnhausen sheaf is weakly positive away from the branch locus, and hence nef on the complement of the branch locus (see also [La]), so it is in particular generically nef. This will allow us to bound the relative Euler characteristic $\chi_f$ of a fibration factoring through a finite cover.

Moreover, we shall introduce the normalized relative canonical divisor of a finite morphism $\pi$ and we shall show that its selfintersection is related with the slope.

We first recall how to determine the invariants and the slope of a fibration factoring through a finite cover.
**Definition 4.1.** Let \( \pi : S \to Y \) be a finite cover of degree \( n \) between smooth surfaces. Then the sheaf \( \pi_* \omega_{S/Y} \) is locally free of rank \( n \) and we can consider the trace map, which is surjective:

\[
tr : \pi_* \omega_{S/Y} \to O_Y.
\]

The kernel \( \mathcal{E} \) is a locally free sheaf of rank \( n - 1 \) on \( Y \) the exact sequence

\[
0 \to \mathcal{E} \to \pi_* \omega_{S/Y} \to O_Y \to 0
\]

splits.

Following [Mi], it is customary to call \( \mathcal{E} \) the Tschirnhausen sheaf of the finite morphism \( \pi \); in fact, Miranda calls Tschirnhausen module the sheaf \( \mathcal{E}^\vee \).

**Lemma 4.2.** Let \( S, Y \) be smooth surfaces, and let \( \pi : S \to Y \) be a finite cover of degree \( n \) with relative canonical divisor \( K_{S/Y} \). Then in the rational Chow ring \( A(Y) \otimes \mathbb{Q} \) we have

\[
(1) \quad \pi_* K_{S/Y} \equiv 2c_1(\mathcal{E}), \\
(2) \quad \chi(O_S) = n \chi(O_Y) + \frac{1}{2}c_1(\mathcal{E}) \cdot K_Y + \frac{1}{2}c_1(\mathcal{E})^2 - c_2(\mathcal{E});
\]

**Proof.** The first relation is well known in the case of flat finite morphisms. The result follows from Grothendieck-Riemann-Roch Theorem applied to the morphism \( \pi : S \to Y \) and the sheaf \( \omega_{S/Y} \).

The Grothendieck-Riemann-Roch Theorem asserts that for a proper morphism \( \pi \) of smooth varieties we have

\[
ch(\pi! \omega_{S/Y}) \cdot td_TY = \pi! (ch \omega_{S/Y} \cdot td_TS).
\]

As \( R^1 \pi_* \omega_{S/Y} = 0 \) since \( \pi \) is finite, we have

\[
\pi_* \omega_{S/Y} = \pi_* \omega_{S/Y} = \mathcal{E}.
\]

This yields

\[
(n + c_1(\mathcal{E}) + \frac{1}{2}(c_1^2(\mathcal{E}) - c_2(\mathcal{E}))) \cdot (1 - \frac{1}{2}K_Y + \chi(O_Y)) = \pi_* ((1 + K_{S/Y} + \frac{1}{2}K_{S/Y}^2) \cdot (1 - \frac{1}{2}K_S + \chi(O_S))).
\]

The divisorial part satisfies

\[
c_1(\mathcal{E}) - \frac{n}{2}K_Y = \pi_* (K_{S/Y} - \frac{1}{2}K_S).
\]

As \( K_S \sim \pi^* K_Y + K_{S/Y} \), we have \( \pi_* K_S \equiv nK_Y + \pi_* K_{S/Y} \) and the first claim follows.

The equality between the codimension two cycles gives formula (2). \[ \square \]

**Definition 4.3.** A fibration \( f : S \to B \) is a flat surjective morphisms between a smooth surface \( S \) and a smooth curve \( B \) with connected fibers, such that if \( x \in B \) is general then \( F_x := f^{-1}(x) \) is a smooth curve.

Following Xiao [X], we can associate with \( f : S \to B \) a rational number \( s(f) \), called the slope of \( f \), defined as:

\[
s(f) := \frac{K_f^2}{\chi_f}
\]

where \( K_f = K_S - f^* K_B \) is the relative canonical divisor, \( \chi_f := \deg f_* \omega_f \), and \( \omega_f := O_S(K_f) \).

**Remark 4.4.** We recall the well known relations:

\[
K_f^2 = K_S^2 - 8(g - 1)(g(B) - 1), \quad \chi_f = \chi(O_S) - (g - 1)(g(B) - 1).
\]
Corollary 4.5. Let $f : S \to B$ be a fibration, which factorises through a finite cover $\pi : S \to Y$ of a ruled surface $Y$. Then

\begin{equation}
K_f^2 = K_{S/Y}^2 - \frac{4}{(g+n-1)}c_1(\mathcal{E})^2, \quad \chi_f = \frac{(g+n-2)}{2(g+n-1)}c_1^2(\mathcal{E}) - c_2(\mathcal{E}).
\end{equation}

\begin{equation}
s(f) = \frac{K_f^2}{\chi_f} = \frac{K_{S/Y}^2 - \frac{4}{(g+n-1)}c_1(\mathcal{E})^2}{\frac{(g+n-2)}{2(g+n-1)}c_1^2(\mathcal{E}) - c_2(\mathcal{E})}.
\end{equation}

Proof. We can write $K_S^2 = K_{S/Y}^2 + 2K_{S/Y} \cdot \pi^*K_Y + nK_Y^2$, then by projection formula and Lemma 4.2 $K_S^2 = K_{S/Y}^2 + nc_1(\mathcal{E}) \cdot K_Y + nK_Y^2$. As $K_f^2 = K_S^2 - 8(g-1)(b-1)$, where $b = g(B)$, we have

\[ K_f^2 = K_{S/Y}^2 + nc_1(\mathcal{E}) \cdot K_Y + nK_Y^2 - 8(g-1)(b-1). \]

Finally, by choosing the generators of the Neron-Severi group of $Y$ to be the classes $[C_0]$ and $[L]$ where $L \in \text{NS}(Y)$ is the class of a ruling and $C_0 \in \text{NS}(Y)$ is the class of a section of minimal selfintersection, we may write

\begin{equation}
c_1(\mathcal{E}) \equiv (g+n-1)C_0 + \left( \frac{c_1(\mathcal{E})^2}{2(g+n-1)} + (g+n-1)C_0^2 \right) L,
\end{equation}

and the first formula follows.

Taking into account that $\chi_f = \chi(\mathcal{O}_S) - (g-1)(b-1)$, the formula for $\chi_f$ follows from Lemma 4.2 (2). \qed

Now we introduce the normalised relative canonical divisor of a finite cover, and we shall see that it is closely related to the slope of the induced fibration. Such a connection is not surprising, as a similar argument has already been used in such a context.

For instance, the Cornalba - Harris theory for bounding the slope of any fibration $f$ relies on the study of the normalized relative canonical divisor of a fibration $f : S \to B$

\[ \mathbb{K}_f := K_f - \frac{1}{g}f^*c_1(f_*\mathcal{O}_f), \quad \mathbb{O}_f := \mathcal{O}_S(\mathbb{K}_f) \]

and on the normalized Hodge bundle

\[ \mathbb{E}_f := f_*\mathbb{O}_f. \]

Indeed, the pseudo-effectivity of $f_*(\mathbb{K}_f^2)$, proved by Cornalba and Harris in [CH, Theorem 1.1, Proposition 2.9 and Section 4], under the assumption that the Hilbert points of the general fibre are semistable, is a crucial step in their proof of the classical bound on the slope

\[ s(f) \geq 4 - \frac{4}{g}. \]

A similar task has been used by Fedorchuk and Jensen in [FJ], who obtained as a straight consequence of the positivity of $c_1(f_*\mathbb{O}_f^{\otimes 2})$ the result that if $S \to B$ is a flat family of Gorenstein curves with the generic fiber a canonically embedded curve whose 2nd Hilbert point is semistable (e.g. with the generic fiber a general trigonal curve), then the slope satisfies the inequality $s(f) \geq 5 - \frac{6}{g}$. 

Also in the context of projective vector bundles \( \pi : \mathbb{P}(\mathcal{G}) \to Y \) fibered in \( \mathbb{P}^{r-1} \) over a variety \( Y \) a similar divisor is studied, namely the so called \textit{normalized tautological divisor}

\[
\mathbb{T}_\mathcal{G} := T_{\mathbb{P}(\mathcal{G})} - \frac{1}{r} \pi^* c_1(\mathcal{G}) = -\frac{1}{r} K_{\mathbb{P}(\mathcal{G})/Y},
\]

and the nefness of such a divisor has been investigated by N. Nakayama \([N]\). More precisely, Nakayama proved the following result:

**Theorem 4.6.** Let \( \mathcal{G} \) be a rank \( r \) vector bundle on a smooth complex projective variety \( Y \) of dimension \( d \geq 2 \). Then the following conditions are equivalent:

- \( \mathbb{T}_\mathcal{G} \) is nef;
- \( \mathcal{G} \) is \( \mu \)-semistable and \( \left(c_2(\mathcal{G}) - \frac{(r-1)}{2r} c_1(\mathcal{G})^2 \right) \cdot A^{d-2} = 0 \) for an ample divisor \( A \).

It turns out that in our context, as we are dealing with a finite morphism and a rank \((n-1)\) torsion free sheaf \( \mathcal{E} \) with \( c_1(\mathcal{E}) = c_1(\pi_* \omega_{S/Y}) \), it is natural to give the following Definition:

**Definition 4.7.** Let \( \pi : S \to Y \) be a finite morphism of degree \( n \). The \( \mathbb{Q} \)-divisor

\[
\Lambda_\pi := K_{S/Y} - \frac{1}{(n-1)} \pi^*(c_1(\pi_* \omega_{S/Y})).
\]

is called the \textit{normalised relative canonical divisor} of \( \pi \).

To explain the reason which leads to the definition of \( \Lambda_\pi \), we need to recall the following well-known result (see \([CE]\)).

**Theorem 4.8.** Let \( Y \) be an integral surface and let \( \pi : S \to Y \) be a Gorenstein cover of degree \( n \geq 3 \). There exists a unique \( \mathbb{P}^{n-2} \)-bundle \( \pi_Y : \mathbb{P} \to Y \) and an embedding \( i : S \to \mathbb{P} \) such that \( \pi = \pi_Y \circ i \).

Moreover \( \mathbb{P} \cong \mathbb{P}(\mathcal{E}) \) and the ramification divisor \( R \) satisfies:

\[
\mathcal{O}_S(R) \cong i^* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1).
\]

**Remark 4.9.** Consider the embedding \( S \subseteq \mathbb{P}(\mathcal{E}) \). Since \( (T_{\mathcal{P}(\mathcal{E})})_S = K_{S/Y} \), we get that

\[
\Lambda_\pi = (T_{\mathcal{E}})_S.
\]

The connection between the slope and \( \Lambda_\pi \) is given by the following:

**Proposition 4.10.** Let \( f : S \to B \) be a fibration, with general fiber \( F \) a smooth curve of genus \( g \).

Assume that \( f \) factorises through a finite degree \( n \) cover \( \pi : S \to Y \), where \( Y \) is a ruled surface, and assume that the general fiber restriction of the Tschirnhausen sheaf \( \mathcal{E} \) is a twist of the trivial sheaf.

By setting

\[
\chi_0^{\text{max}} := \frac{g}{2(n-1)(g+n-1)} c_1(\mathcal{E})^2,
\]

we have

\[
\chi_0^{\text{max}} \geq \chi_f
\]

and the following equality holds:

\[
K_f^2 = F(n, g) \chi_0^{\text{max}} + \Lambda_\pi^2,
\]

where

\[
F(n, g) = 6 - \frac{2}{n-1} - \frac{2n}{g}.
\]
is the conjectural bound of Stankova (see [S] Conjecture 13.3). In particular

\begin{equation}
(4.9) \quad s(f) \geq F(n, g) + \frac{\Lambda^2_\pi}{\chi^{\text{max}}_0}.
\end{equation}

**Proof.** By Theorem 3.3 we have \( c_2(\mathcal{E}) \geq \frac{n-2}{2(n-1)}c_1(\mathcal{E})^2 \), and since by Corollary 4.5 it holds \( \chi_f = \frac{(g+n-2)}{2(g+n-1)}c_1^2(\mathcal{E}) - c_2(\mathcal{E}) \), it follows that \( \chi_f \leq \frac{(g+n-2)}{2(g+n-1)}c_1^2(\mathcal{E}) - \frac{n-2}{2(n-1)}c_1(\mathcal{E})^2 = \chi^{\text{max}}_0 \).

With a direct computation one can obtain the equality (4.8), and the inequality (4.9) follows immediately.

As a consequence, the problem of bounding the slope can be rephrased in a problem of bounding \( \Lambda^2_\pi \).

The Bogomolov-type inequalities given in Corollary 3.2 and Theorem 3.5 allow us to derive a similar result also in the non divisible case \((n-1) \nmid g\). Indeed, the Tschirnhausen sheaf is nef outside the branch locus by the Weak Positivity Theorem of Viehweg [V, 3.4].

If we assume that the general fiber restriction of \( \mathcal{E} \) is balanced, which can be rephrased with the assumption that the general fiber of the fibration corresponds to a point outside the Maroni locus of a suitable Hurwitz scheme or of the moduli space, and that the branch divisor is reduced, which corresponds to impose the open condition that \( \pi \) has generically simple ramifications, then the Corollary 3.2 and Theorem 3.5 apply.

Therefore, recalling that \( \chi_f = \frac{(g+n-2)}{2(g+n-1)}c_1^2(\mathcal{E}) - c_2(\mathcal{E}) \) and setting \( g + n - 1 = (n-1)m + k \), we have

\[
\chi_f \leq \begin{cases} 
\frac{(g+n-2)}{2(g+n-1)}c_1(\mathcal{E})^2 - \frac{n-1-k}{2(n-1)(g+n-1)}c_1(\mathcal{E})^2 & \text{if } C^2_0 \geq 0 \\
\frac{(g+n-2)}{2(g+n-1)}c_1(\mathcal{E})^2 + \frac{(n-1-k)(m+1)}{2(n-1)(g+n-1)}c_1(\mathcal{E})^2 & \text{if } C^2_0 < 0.
\end{cases}
\]

that is

\begin{equation}
(4.10) \quad \chi_f \leq \chi^{\text{max}}_k := \begin{cases} 
\frac{m}{2(g+n-1)}c_1(\mathcal{E})^2 & \text{if } C^2_0 \geq 0 \\
\frac{m}{2(g+n-1)} + \frac{(n-1-k)m}{2(g+n-1)(g+n-2)}c_1(\mathcal{E}) & \text{if } C^2_0 < 0.
\end{cases}
\end{equation}

Then we can write

\[
K^2_f = K^2_\mathcal{S}/Y - \frac{4}{g + n - 1}c_1(\mathcal{E})^2 = \Lambda^2_\pi + F(n, g, k)\chi^{\text{max}}_k
\]

where

\[
F(n, g, k) := \begin{cases} 
\frac{(g+1-k) - n-1}{(g+n-1)(g+n-2)}c_1(\mathcal{E})^2 & \text{if } C^2_0 \geq 0 \\
\frac{2k}{(g+1-k)(g+2n-3-k)} & \text{if } C^2_0 < 0.
\end{cases}
\]

This shows that the function \( F(n, g, k) \) can be used to replace the function \( F(n, g) \) in the non divisible case; more precisely, we have:

**Proposition 4.11.** Let \( f : S \rightarrow B \) a semistable fibration over a rational curve \( B \). Assume that \( f \) factorizes through a finite cover of degree \( n \) of ruled surface and that the Tschirnhausen sheaf is...
balanced on the general fiber. If the genus of the general fiber of $f$ is $g = (m - 1)(n - 1) + k$, where $1 \leq k \leq n - 2$, then

\[ s(f) \geq \mathcal{F}(n, g, k) + \frac{\Lambda_\pi^2}{\chi_{\max}}. \]

Remark 4.12. The bounds (4.9) and (4.11) given in Propositions 4.10 and 4.11 hold also for fibrations $f : S \to B$, which are the relatively minimal model of fibrations satisfying the given hypotheses. Indeed, it is enough to observe that the selfintersection of the relative canonical divisor of a relatively minimal model of a given fibration can not decrease, and the relative Euler characteristic is unchanged.

5. Positivity results on $\Lambda_\pi$

From the results of the previous section it follows, that a bound on $\Lambda_\pi^2$ gives a bound also on the slope. Therefore we are going to establish some conditions, under which the normalized relative canonical divisor has non-negative selfintersection. Some very similar problems have been studied in [BS1] and [BS2], but their results don’t apply to our case.

We first analyse the restriction of the divisor $\Lambda_\pi$ to the general fiber.

Proposition 5.1. Let $F$ be a general fiber of the fibration $f : S \to B$. If $g = (n - 1)(m - 1)$ and $F$ is not contained in the Maroni locus, the restriction of $\Lambda_\pi$ to $F$ satisfies:

1. $(\Lambda_\pi)|_F \sim K_F - (m - 2)\Gamma_F$, where $\Gamma_F \in g_1^1$ is a gonal divisor;
2. $h^0(\mathcal{O}_F(K_F - (m - 2)\Gamma_F)) = n - 1$;
3. the linear system $|K_F - (m - 1)\Gamma_F|$ is base point free.

Proof. We have

\[(\Lambda_\pi)|_F = (K_{S/Y} - \pi^*(mC_0 + kL))|_F = (K_S - \pi^*K_Y - \pi^*(mC_0 + kL))|_F \sim \]

\[ \sim K_F - (m - 2)\Gamma_F, \]

where $\Gamma_F \in g_1^1$ is a gonal divisor, which proves (1).

Let us compute $h^0(\mathcal{O}_F((m - 2)\Gamma_F))$ using the Geometric Riemann Roch Theorem:

\[ h^0(\mathcal{O}_F((m - 2)\Gamma_F)) = (m - 2)n - \dim((m - 2)\Gamma_F), \]

where $((m - 2)\Gamma_F) \subset \mathbb{P}^{g-1}$ is the linear span on the canonical model of the curve $F$. Now recall that since $F$ is Maroni general, the canonical model of $F$ lies on $W \cong \mathbb{P}(\oplus^{g-1}\mathcal{O}_{\mathbb{P}^1}(m-2))$ embedded in $\mathbb{P}^{g-1}$ by the tautological linear system. It follows that

\[ \dim((m - 2)\Gamma_F) = (m - 2)(n - 2) + m - 3, \]

hence by the Geometric Riemann Roch Theorem we have

\[ h^0(\mathcal{O}_F((m - 2)\Gamma_F)) = (m - 2)n - (m - 2)(n - 2) - (m - 3) = m - 1, \]

hence by Riemann Roch

\[ h^0(\mathcal{O}_F(K_F - (m - 2)\Gamma_F)) = h^0(\mathcal{O}_F((m - 2)\Gamma_F)) - ((\deg(m - 2)\Gamma_F) - g + 1) = \]

\[ = m - 1 - (m - 2)n + g - 1 = n - 1, \]

which proves (2).
Finally, assume by contradiction that \( P \) is a base point of the linear system \( |K_F - (m-2)\Gamma_F| \). Then by the Geometric Riemann Roch Theorem we would have
\[
\dim \langle K_F - (m-2)\Gamma_F \rangle = \dim \langle K_F - (m-2)\Gamma_F - P \rangle + 1,
\]
and
\[
\dim \langle (m-2)\Gamma_F \rangle = \dim \langle (m-2)\Gamma_F + P \rangle.
\]
We claim that the last equality cannot be satisfied. Indeed, the subspace \( \langle (m-2)\Gamma_F \rangle \) cuts on \( W \) exactly \( m-2 \) fibers; indeed, since the minimum degree of a unisecant curve on \( W \) is \( m-2 \), the subspace \( \langle (m-2)\Gamma_F \rangle \) contains no horizontal component. It follows that the divisor cut out by \( \langle (m-2)\Gamma_F \rangle \) on the canonical model of \( F \) is exactly \( (m-2)\Gamma_F \).

\[\square\]

**Corollary 5.2.** Let \( f : S \to B \) be a fibration in irreducible curves, with general fiber \( F \) a smooth curve of genus \( g \). Assume that \( f \) factorises through a finite degree \( n \) cover \( \pi : S \to Y \), where \( Y \) is a ruled surface, with Tschirnhausen sheaf generically a twist of the trivial sheaf.

If the restriction map \( H^0(O_S(\Lambda_\pi)) \to H^0(O_F(\Lambda_\pi)) \) is surjective, then \( \Lambda_\pi^2 \geq 0 \).

**Proof.** We claim that the linear system \( |\Lambda_\pi| \) has no horizontal base locus.

Assume by contradiction that \( \Lambda_\pi \) has a horizontal component \( \kappa \) in its base locus. Then \( \kappa \cap F \) is contained in the base locus of \( |(\Lambda_\pi)|_F \). But the latter linear system is base point free by Proposition 5.1, (3). This proves that \( |\Lambda_\pi| \) has no horizontal base locus.

Finally, since all the fibers of \( f \) are irreducible, \( \Lambda_\pi \) has no vertical base locus.

Summing up, as \( \Lambda_\pi \) is effective and \( |\Lambda_\pi| \) has no positive dimensional base locus, we have \( \Lambda_\pi^2 \geq 0 \).

\[\square\]

5.1. **Rational fibrations with uniform Tschirnhausen sheaf.** In the following Proposition we shall prove that in the divisible case, the fibrations over a rational curve, with uniform and generically balanced Tschirnhausen sheaf and with semistable unisecant restriction, satisfy the assumption of Proposition 5.1. We remark that by a recent result given in [DP], a sufficiently general curve in the Hurwitz scheme of degree \( n \) covers of curves of given genus \( p \geq 0 \) has a \( \mu \)-semistable Tschirnhausen sheaf. Therefore the assumption of semistability on unisecant restrictions can be read as a generality assumption concerning the family of pull-back curves of a family of general unisecant curves.

**Proposition 5.3.** Let \( f : S \to \mathbb{P}^1 \) be a fibration with irreducible fibres, with general fiber \( F \) a smooth curve of genus \( g \) and gonality \( n \), where \( n \geq 5 \), such that \( (n-1)g \) and with balanced reduced gonal direct image sheaf. Assume that \( f \) factorises through a finite morphism \( \pi : S \to Y \), where \( p : Y \to \mathbb{P}^1 \) is a Hirzebruch surface.

If the restriction of \( \mathcal{E} \) to some unisecant ample divisor is semistable and if \( \mathcal{E} \) is uniform, then the restriction map \( H^0(O_S(\Lambda_\pi)) \to H^0(O_F(\Lambda_\pi)) \) is an isomorphism.

In particular, \( s(f) \geq \mathcal{F}(n, g) \).

**Proof.** We set \( c_1(\mathcal{E}) = (g + n - 1)C_0 + \delta L \), where \( C_0 \) is a section with \( C_0^2 \leq 0 \) and \( L \) a fiber of ruling on \( Y = \mathbb{F}_e \).

Since \( \mathcal{E} \) is uniform, the restriction to any fiber \( L \) of the ruling satisfies \( \mathcal{E}|_L \cong \bigoplus_{i=1}^{n-1} O_{\mathbb{P}^1}(m) \), where \( m = \frac{n}{n-1} + 1 \). Then the injective map of sheaves \( p^*p_*\mathcal{E}(-mC_0) \to \mathcal{E}(-mC_0) \) is an
isomorphism, so

\[ \mathcal{E}(-mC_0) \cong \bigoplus_{i=1}^{n-1} \mathcal{O}_Y(p^*A_i). \]

By the assumption that \( \mathcal{E} \) is semistable with respect to an ample divisor \( H \sim C_0 + (e + a)L \), we get \( \deg \Lambda_i = k_i \), for any \( i \), and \( k_i = \frac{e}{(n-1)} \geq 0 \), so that

(5.1) \[ c_1(\mathcal{E}) \sim (n - 1)(mC_0 + kL). \]

We finally get

\[ h^0(\mathcal{O}_S(\Lambda_\pi)) = h^0(\pi_* \mathcal{O}_S(\Lambda_\pi)) \cong h^0(\mathcal{O}_Y(-(mC_0 + kL)) \oplus \mathcal{E}(-(mC_0 + kL))) = h^0(\bigoplus_{i=1}^{n-1} \mathcal{O}_Y) = n-1. \]

On the other hand,

\[ h^0(\mathcal{O}_S(\Lambda_\pi - F)) = h^0(\pi_* \mathcal{O}_S(\Lambda_\pi - F)) \cong h^0(\mathcal{E}(-(mC_0 + (k + 1)L))) = h^0(\bigoplus_{i=1}^{n-1} \mathcal{O}_Y(-L)) = 0. \]

Hence the restriction exact sequence

\[ 0 \rightarrow \mathcal{O}_S(\Lambda_\pi - F) \rightarrow \mathcal{O}_S(\Lambda_\pi) \rightarrow \mathcal{O}_F(\Lambda_\pi) \rightarrow 0 \]

determines the isomorphism of the statement. \( \square \)

**Remark 5.4.** We observe that in the case when \( \mathcal{E} \) is a uniform vector bundle, that is \( \Delta(\mathcal{E}) = 0 \), and if \( \mathcal{E} \) is \( \mu \)-semistable with respect to some ample divisor, then the normalized tautological divisor \( T_\mathcal{E}(\mathcal{E}) - \frac{1}{(n-1)}p^*c_1(\mathcal{E}) \) where \( p : \mathbb{F}(\mathcal{E}) \rightarrow Y \) is nef by Nakayama’s Theorem [4.6]. It follows that \( \left( T_\mathcal{E}(\mathcal{E}) - \frac{1}{(n-1)}p^*c_1(\mathcal{E}) \right)_S = K_{S/Y} - \frac{1}{(n-1)}T^* \pi^*(\mathcal{E}) = \Lambda_\pi \) is also nef. Since the restriction map is surjective by Proposition [5.3], all this implies directly that in such a case \( \Lambda_\pi^2 \geq 0 \).

### 5.2 Upper bounds on \( \Lambda_\pi^2 \) and primitive cyclic covers.

We recall that for finite covers, we have the following upper bound on \( R^2 \) in terms of \( c_1(\mathcal{E})^2 \), which is a consequence of the Hodge Index Theorem applied to the \( \mathcal{O} \)-divisor \( R - \pi^* \frac{2}{n}c_1(\mathcal{E}) \):

**Lemma 5.5.** Let \( \pi : S \rightarrow Y \) be a Gorenstein cover degree \( n \), and let \( \mathcal{E} \) be the Tschirnhausen sheaf. Then

(5.2) \[ K_{S/Y}^2 \leq \frac{4}{n}c_1(\mathcal{E})^2. \]

**Proof.** See [BeZ, Lemma 3.12]. \( \square \)

**Corollary 5.6.** With the assumptions of Lemma [5.5], we have

\[ \Lambda_\pi^2 \leq \frac{(n-2)^2}{n(n-1)^2}c_1(\mathcal{E})^2. \]

**Proof.** Since \( \Lambda_\pi = K_{S/Y} - \frac{1}{n-1} \pi^*c_1(\mathcal{E}) \), we have

\[ \Lambda_\pi^2 = K_{S/Y}^2 - 2K_{S/Y} \cdot \frac{1}{n-1} \pi^*c_1(\mathcal{E}) + \frac{1}{(n-1)^2} (\pi^*c_1(\mathcal{E}))^2, \]

and by projection formula and by Lemma [4.2] (1), it follows

\[ \Lambda_\pi^2 = K_{S/Y}^2 - \frac{4}{n-1}c_1(\mathcal{E})^2 + \frac{n}{(n-1)^2}c_1(\mathcal{E})^2 = K_{S/Y}^2 - \frac{3n-4}{(n-1)^2}c_1(\mathcal{E})^2 \leq \frac{(n-2)^2}{n(n-1)^2}c_1(\mathcal{E})^2. \]

\( \square \)
We remark that the Hodge Index Theorem also implies, that the equality holds in (5.2) if and only if

\[ K_{S/Y} \sim \frac{1}{n} \pi^* 2c_1 E \equiv \frac{1}{n} \pi^* B. \]

Such a condition is satisfied, for instance, when all ramification points of \( \pi \) are total ramification points, that is of maximal ramification index \( n \).

A typical context, when this happens, is the one of \emph{primitive cyclic covers} \( \pi : S \to Y \), that is covers such that there exist an effective divisor \( A \subset Y \) and an effective divisor \( D \subset S \) such that

\[ S \cong \text{Spec} \oplus_{i=0}^{n-1} \mathcal{O}_Y(iA), \]

and such that \( \pi : S \to Y \) does not factorise through two covers of smaller degree. In this case the following hold:

1. \( B \pi \sim nA \) and \( \pi^* B = nD \);
2. \( K_{S/Y} \sim (n-1)D \);
3. \( \pi_* \mathcal{O}_S(K_{S/Y}) \cong \bigoplus_{i=0}^{n-1} \mathcal{O}_Y(iA) \);
4. \( c_1(E) = \frac{n(n-1)}{2} A, \quad c_2(E) = \frac{n(n-1)(n-2)(3n-1)}{12} A^2. \)

From this we obtain that

\[ K_{S/Y} \sim \frac{\pi^*}{n} 2c_1(E), \quad K_{S/Y}^2 = \frac{4}{n} c_1(E)^2, \quad \Lambda^2 = \frac{(n-2)^2}{n(n-1)^2} c_1(E)^2. \]

Since \( c_1(E) = \frac{n(n-1)}{2} A \), if \( A^2 \geq 0 \), we get \( \Lambda^2 \geq 0 \). This gives a bound, which is exactly the bound \( \lambda_{g,0,n} \) given in [E, Remark 4.4]:

\[ \text{Theorem 5.7.} \quad \text{Let} \quad f : S \to B \quad \text{be the relatively minimal model of a finite cyclic cover} \quad \pi : \tilde{S} \to Y \quad \text{of a ruled surface} \quad Y. \]

Then

\[ s(f) \geq \frac{24(g-1)(n-1)}{(n^2 + 4ng - 3n + 2 - 2g)} = 6 - \frac{6}{2n-1} - \frac{12n(n^2 - 1)}{2g(2n-1) + (n-1)(n-2)}. \]

\[ \text{Proof.} \quad \text{Recall that the selfintersection of the relative canonical divisor of a relatively minimal model of a given fibration can not decrease, and the relative Euler characteristic is unchanged. Then we can apply formula (4.8).} \]

\[ \square \]

\[ \text{Remark 5.8.} \quad \text{We observe that for cyclic covers the Tschirnhausen sheaf is uniform.} \]

5.3. \textbf{A Beniamino Segre’s construction.} \ Following closely [ACG, Chapter 21 section 12] we shall show that for a general \( [C] \in \mathcal{M}_{g,n}^1 \), where \( \mathcal{M}_{g,n}^1 \) is the closure of the \( n \)-gonal locus in the moduli space \( \mathcal{M}_g \) of curves of genus \( g \), if \( p : C \to \mathbb{P}^1 \) is the gonal covering, then the corresponding Tschirnhausen sheaf \( \mathcal{E}_C \) is balanced.

\[ \text{Theorem 5.9.} \quad \text{Let} \quad g \geq 3. \quad \text{For any integer} \quad n \quad \text{such that} \quad 3 \leq n \leq g + \frac{g}{2}, \quad \text{there exists a smooth curve} \quad C \quad \text{of genus} \quad g \quad \text{admitting a complete} \quad g^1_n \quad \text{without base points, and such that the corresponding Tschirnhausen sheaf} \quad \mathcal{E}_C \quad \text{is balanced.} \]

\[ \text{Proof.} \quad \text{Let} \quad \mathbb{F}_e \text{ be a Hirzebruch surface with invariant} \quad e = -C_0^2 \geq 0. \quad \text{Consider the complete linear system} \]

\[ \Sigma_{n,h} = |nC_0 + hL|. \]
Assume that $h > \frac{2e}{n}$. Then $\Sigma_{n,h}$ is very ample and the image of the morphism $\phi_{\Sigma_{n,h}} : \mathbb{P}_e \to \mathbb{P}^N$ associated with $\Sigma_{n,h}$ is a smooth surface; hence by Bertini’s theorem the general member of $\Sigma_{n,h}$ is smooth. Then, by adjunction, the genus of the general element $\Gamma \in \Sigma_{n,h}$ is

$$g_{n,h} = (n - 1)(h - 1) - \frac{ne}{2}(n - 1).$$

Moreover, by Riemann-Roch we have

$$\dim \Sigma_{n,h} = g_{n,h} + 2n + 2h - 1 - ne.$$

Following [ACG, Theorem 12.16 see page 870] we immediately see that, given any fixed integer $n \geq 3$ and any fixed integer $e \geq 0$, the intervals

$$I_{n,h} = [g_{n,h} - n - h + 1 + \frac{ne}{2}, g_{n,h}]$$

cover the half line $[0, \infty)$. Then there exist an integer $\delta$ and an integer $h$ such that $0 \leq \delta \leq h + n - 1 - \frac{ne}{2}$ and

$$(5.3) \quad g = g_{n,h} - \delta.$$

A simple computation shows that

$$\delta \leq g_{n,h} \leq \dim \Sigma_{n,h} - 2\delta - 1.$$

By Castelnuovo’s theorem applied to $\Sigma_{n,h}$, see for instance [ACG, Theorem 12.6 pag.865], it follows that given $\delta$ general points $a_1, \ldots, a_\delta \in \mathbb{P}_e$, there exists an irreducible curve $\Gamma \in \Sigma_{n,h} = |nC_0 + hl|$ having $\delta$ nodes at $a_1, \ldots, a_\delta$ and no other singularities. Let $\nu : Z \to \mathbb{P}_e$ be the blow-up at $a_1, \ldots, a_\delta$ and let $E_i := \nu^{-1}(a_i), i = 1, \ldots, \delta$. The normalisation $C$ of $\Gamma$ is contained in $Z$ and $\nu|_C : C \to \Gamma$ is the normalisation morphism. Let $H_0 := \nu^{-1}(C_0)$ and $\bar{L} := \pi^{-1}(L)$. The smooth curve $C$ has a $g^1_n$ induced by the ruling of $\mathbb{P}_e$. Let us denote by $D$ a divisor of the $g^1_n$. Then

$$D \sim L|_C, \quad C \in |nH_0 + \bar{L} - \sum_{i=1}^\delta E_i|.$$

By standard surface theory we get

$$K_Z \sim -2H_0 - (2 + e)L + \sum_{i=1}^\delta E_i.$$

Then $(n - 2)H_0 + (h - e - 2 - v)L - \sum_{i=1}^\delta E_i \sim K_Z + C - vL$ and by adjunction theory on surfaces we have the following exact sequence

$$(5.4) \quad 0 \to \mathcal{O}_Z(K_Z - vL) \to \mathcal{O}_Z(K_Z + C - vL) \to \omega_C(-vD) \to 0.$$

Notice now that by projection formula and by Serre duality we have

$$h^1(Z, \mathcal{O}_Z(K_Z - vL)) = h^1(Z, \mathcal{O}_Z(vL)) = h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(vP)) = h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2 - vP)) = 0,$$

since $v \geq 0$. As $h^0(Z, \mathcal{O}_Z(K_Z - vL)) = 0$, by considering the long cohomology sequence associated with the sequence (5.4), we obtain that the restriction morphism

$$H^0(Z, \mathcal{O}_Z(K_Z + C - vL)) \to H^0(C, \omega_C(-vD))$$

is an isomorphism. In particular $h^0(C, \omega_C(-vD)) = h^0(Z, \mathcal{O}_Z(K_Z + C - vL))$. 

Since \( a_1, \ldots, a_\delta \) are general points, we have
\[
\max\{-1, h^0(Z, \mathcal{O}_Z(K_Z + C - vL))\} = \max\{-1, \dim \Sigma_{n-2, h-(e+2+v)} - \delta\}.
\]
Hence if we assume that \( h - (e + 2 + v) \geq \frac{(n-2)e}{2} \), then \( \dim \Sigma_{n-2, h-(e+2+v)} - \delta = g - (n-1)v \).

We have shown that if \( h - (e + 2 + v) \geq \frac{(n-2)e}{2} \) and if \( k \) is the unique integer such that \( 1 \leq k \leq n - 2 \) and \( g = m(n - 1) + k \) then
\[
h^0(C, \omega_C(-vD)) = (m - v)(n - 1) + k
\]
if \( m \geq v \).

Finally, let \( p: C \to \mathbb{P}^1 \) be the gonal morphism; then \( p_*\omega_C = \omega_{\mathbb{P}^1} \oplus \mathcal{E}_C(-2) \), where
\[
\mathcal{E}_C = \mathcal{O}_{\mathbb{P}^1}(m_1) \oplus \mathcal{O}_{\mathbb{P}^1}(m_2) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(m_{n-1}),
\]
with \( m_1 \leq m_2 \leq \ldots \leq m_{n-1} \). By projection formula and by the equation (5.5) it follows that in fact \( m_1 = \ldots = m_{n-1-k} = m - 1 \) and \( m_{n-k} = \ldots = m_{n-1} = m \), that is \( \mathcal{E}_C \) is balanced. \( \square \)

6. Sharpness results

In this Section we construct some examples, that realize the bound on the slope given in Proposition 5.3.

6.1. Existence of fibrations with the required properties: rational basis. Let \( Y = \mathbb{P}^1 \times \mathbb{P}^1 \) and \( T = Y \times \mathbb{P}^1 \). Let \( \pi_i: T \to \mathbb{P}^1 \) be the projection with respect to the \( i \)-th factor and set \( \mathcal{L}_i := \pi_i^*\mathcal{O}_{\mathbb{P}^1}(1) \) where \( i = 1, 2, 3 \). Let
\[
S \in |nL_1 + nL_2 + nL_3|
\]
be a general element, where \( n \geq 3 \). If \( \langle x_0, x_1 \rangle = H^0(T, \mathcal{L}_3) \), \( \langle y_0, y_1 \rangle = H^0(T, \mathcal{L}_2) \) and \( \langle z_0, z_1 \rangle = H^0(T, \mathcal{L}_1) \), then
\[
S = V(F) \quad \text{where} \quad F = \sum_{i=0}^n a_i((y_0 : y_1), (z_0 : z_1))x_0^{n-i}x_1^i.
\]
Thus if \( a_i((y_0 : y_1), (z_0 : z_1)) \in H^0(Y, \mathcal{O}_Y(n, n)) \) for \( i = 0, \ldots, n \) are general, the morphism \( \pi: S \to Y \) is finite of degree \( n \). Consider now the composition \( f: S \to \mathbb{P}^1 \) of the inclusion \( j: S \hookrightarrow T \) with the natural morphism \( \rho: T \to Y \) followed by the projection on the first factor \( \pi'_1: Y \to \mathbb{P}^1 \). The fiber over \( z_0 = a, z_1 = b \) is the curve \( C_{[a:b]} = V(F_{[a:b]}) \) where \( F_{[a:b]} = \sum_{i=0}^n a_i((y_0 : y_1), (a : b))x_0^{n-i}x_1^i \) inside \( \mathbb{P}^1 \times \mathbb{P}^1 \). In particular \( f: S \to \mathbb{P}^1 \) is a fibration in curves of genus \( g = (n-1)^2 \) and gonality \( n \) such that
\[
s(f) = \mathcal{F}(n, g).
\]
Moreover \( \Lambda_\pi \) is effective and it is induced on \( S \) by the linear system \( (0, 0, n-2) \), hence \( \Lambda_\pi^2 = 0 \). Finally by performing the push-forward of the standard exact sequence
\[
0 \to \mathcal{O}_T(K_T|Y) \to \mathcal{O}_T(K_T|Y + S) \to \omega_{S|Y} \to 0,
\]
by projection formula and by relative duality it follows that the Tschirnhausen sheaf satisfies \( \mathcal{E} \cong \mathcal{O}_Y(n, n)^{\oplus n-1} \), so it is a uniform and balanced vector bundle which is also semistable on the \((0, 1)\)-sections of the projection \( \pi'_1: Y \to \mathbb{P}^1 \). This shows that the bound given in Proposition 5.3 is sharp. Note that instead of \( S \in |nL_1 + nL_2 + nL_3| \) we can take \( S \in |n_1L_1 + n_2L_2 + n_3L_3| \) where \( n_i \geq 1 \) to obtain similar results.
6.2. Existence of fibrations with the required properties: other cases. Let $T = C_1 \times \mathbb{P}^1 \times \mathbb{P}^1$, where $C_1$ is a smooth curve of genus $g_1 > 0$, and let $L := \pi^* \mathcal{O}_{C_1}(L_1) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^1}(n) \otimes \pi_3^* \mathcal{O}_{\mathbb{P}^1}(n)$. As above we obtain a semistable $n$-gonal fibration over $C_1$ of genus $g = (n - 1)^2$ such that:

$$s(f) = 6 - \frac{6}{(n - 1)^2} = \mathcal{F}(n, g) + 4\frac{n - 2}{(n - 1)^2}.$$ 

Also in this case $\Lambda_2^n$ is zero and the Tsirnhausen sheaf is uniform and with balanced fiber restriction.

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