Flow Decomposition for Multi-User Channels
- Part I

Jonathan Ponniah
Department of Electrical Engineering
San Jose State University

Liang-Liang Xie
Department of Electrical and Computer Engineering
University of Waterloo

Abstract—A framework based on the idea of flow decomposition is proposed to characterize the decode-forward region for general multi-source, multi-relay, all-cast channels with independent input distributions. The region is difficult to characterize directly when deadlocks occur between two relay nodes, in which both nodes benefit by decoding after each other. Rate-vectors in the decode-forward region depend ambiguously on the outcomes of all deadlocks in the channel. The region is characterized indirectly in two phases. The first phase assumes relays can operate non-causally. It is shown that every rate-vector in the decode-forward region corresponds to a set of flow decompositions, which describe the messages decoded at each node with respect to the messages forwarded by all the other nodes. The second phase imposes causal restrictions on the relays. Given an arbitrary set of (possibly non-causal) flow decompositions, necessary and sufficient conditions are derived for the existence of an equivalent set of causal flow decompositions that achieves the same rate-vector region.

I. INTRODUCTION

Multi-node channels in network information theory are not fully understood. Despite decades of research, only the capacity of the one-way point-to-point channel is known. The two-way channel is still unsolved as are all other channels with three or more nodes. We examine multi-node all-cast channels in which the source nodes send common messages to all of the destination nodes with the help of the relay nodes. The general discrete-memoryless channel models the input-output dynamics. Our analysis is confined to decode-forward schemes. Each relay decodes source messages and forwards some of these messages to destinations and other relays. Our objective is to characterize the region of rate vectors achievable by decode-forward schemes, otherwise known as the decode-forward region.

The original decode-forward scheme, first proposed for the one-source one-relay channel, also extends to other channels with particular numbers of source and relay nodes. In more general multi-node channels, deadlocks between pairs of relay nodes occur when both relays in a pair have an incentive to wait for the other to forward messages before decoding their own. Each deadlock yields two different outcomes depending on which of the two relays decode first, and each outcome generates different rate-vector regions.

To send a message, each source transmits a sequence of symbols through the channel. The mapping from messages to sequences is defined by a source codebook. Each transmitted symbol requires one use of the channel by the sources. The symbols received at any given node are correlated with the symbols simultaneously transmitted by the other nodes. Relays and destinations decode source messages if there are codebook sequences correlated (or “typical”) with the sequences of symbols they receive. A message “hop” occurs when a relay decodes and forwards a message. Source messages hop from node to node until they arrive at each destination.

Every message hop between two nodes has an encoding delay, defined as the number of channel uses from the start of the first node sending the message until the start of the second. Without loss of generality, the encoding delays can be positive integer-valued multiples of some fixed block of channel uses. In each block, the sources send unique messages and the relays forward unique combinations of past messages to other nodes.

To decode any set of messages, a node first identifies all of the prior blocks in which other nodes transmit some of the messages in the set, then identifies which codebook sequences are jointly typical with the sequences received during these blocks. This general procedure is called joint decoding. Since the knowledge of one message in any simultaneously transmitted set helps remove some uncertainty about the other messages (even if the messages are independent), it is better to decode messages together (or “jointly”) rather than separately.

The messages decoded by each node in a given block are unique and determined by the desired rate vector and the messages forwarded by other nodes in previous blocks. The encoding delays, which determine the messages forwarded by each relay, are constrained by the assumption of causality; relays can only forward messages they have already decoded. Any relay that jointly decodes a set of source messages to achieve a desired rate-vector must be the last node to jointly decode this set out of all of the preceding nodes. Encoding delays at this relay must be larger than the encoding delays of all the preceding hops. A deadlock occurs if the encoding delays at one relay can only satisfy these causality constraints at the expense of another relay.

An outer-bound on the decode-forward region can be derived from the capacity of the point-to-point channel. This outer-bound depends on the sequences of nodes traversed by the message hops from each source. We call each sequence a flow. For some arbitrarily assigned flows and some relay in the channel, the corresponding “super-source” consists of all the nodes preceding the relay. The “super-channel” is the the channel seen by the relay free of any interference from nodes.
not in the super-source. The sum rate of the source nodes in the super-source cannot exceed the capacity of the point-to-point super-channel. Invoking this argument for any set of flows creates an outer-bound on the decode-forward region. It turns out that this outer-bound is tight when deadlocks are absent from the channel. When deadlocks are present, the decode-forward region is too complicated to express explicitly; the outcomes of each deadlock introduce unique constraints that have no simple interpretation.

The decode-forward region is difficult to characterize directly, so we propose an indirect approach in two phases. The first phase allows relays to operate non-causally. For arbitrarily assigned flows and encoding delays, we show each node can achieve any rate-vector in the outer-bound by decoding a corresponding set of source messages. We introduce the concept of flow decompositions to describe the source messages decoded at a particular node relative to the messages encoded by all the other nodes. Every rate-vector in the decode-forward region corresponds to a set of flow decompositions, where each flow decomposition in the set belongs to a unique node in the channel.

The second phase reinforces causal restrictions on the relays. Flow decompositions are causal by definition if the corresponding relays only forward messages they have already decoded. The previous result and the assumption of causality imply that every rate-vector in the decode-forward region corresponds to a set of causal flow decompositions. An arbitrarily chosen set (which may include non-causal flow decompositions) is “feasible” by definition if there exists another set of causal flow decompositions that achieves the same rate-vector region. We derive necessary and sufficient conditions that determine whether or not an arbitrary set of flow decompositions is feasible. These conditions emerge naturally from the two-way two-relay channel, the simplest channel in which deadlocks occur, and restrict the flow decompositions that can be simultaneously assigned to the affected nodes.

No single decode-forward scheme is universally better than the others. Each scheme achieves a different region of rate-vectors. Collectively, these overlapping and interlocking regions recover the outer-bound defined by the point-to-point super-channel. Our approach exploits this underlying structure. However, we do not express the decode-forward region in the conventional way for channels with deadlocks. We also do not address which rate-vectors outside the decode-forward region are achievable. This fundamental open problem is not fully solved, even for the one-relay channel.

Concerning the organization of the paper, Section II defines the concept of flow, Section III defines the concept of flow decomposition and states the theorem that gives this concept significance, Section IV works through an important example that demonstrates some of the key ideas in the proof, Section V provides this proof, and Section VI concludes the paper and sets the stage for the one to follow.

II. Flows

Let \( \mathcal{N} \) denote the set of all nodes, \( \mathcal{I} \subseteq \mathcal{N} \) the set of nodes with inputs into the channel, \( \mathcal{S} \subseteq \mathcal{I} \) the set of source nodes, \( \mathcal{D} \subseteq \mathcal{N} \) the set of destination nodes, and \( \mathcal{Z} \subseteq \mathcal{I} \) the set of relay nodes, where \( \mathcal{Z} \subseteq \mathcal{D} \) and \( \mathcal{Z} \) and \( \mathcal{S} \) are not necessarily disjoint. Every destination decodes all of the source messages. Let \( y_d := \{(d, y_d) : d \in \mathcal{D}\} \) and \( x_I := \{(i, x_i) : i \in \mathcal{I}\} \). The input-output dynamics are modeled by the discrete memoryless channel:

\[
(\prod_{i \in \mathcal{Z}} X_i, p(y_d|x_I), \prod_{d \in \mathcal{D}} Y_d).
\]

Decode-forward schemes describe both the order in which source messages “hop” from one node to another until they reach the destination nodes as well as the encoding delays induced by each hop. In each “block” of \( n \) channel uses, the source nodes generate new messages and the relay nodes forward messages from the past. Encoding delays refer to the difference between the blocks in which the first and second nodes of a hop transmit the same message. Hops also occur between disjoint sets of nodes if the nodes in each set transmit the same message simultaneously, but both sets transmit the same message in different blocks. The encoding delays are non-negative integer multiples of \( n \) and the entire transmission period occurs over \( B \) blocks.

A flow \( f(s,d) \) is a sequence of hops that starts at source \( s \in \mathcal{S} \) and finishes at destination \( d \in \mathcal{D} \). Formally, \( f(s,d) = Z_1 \xrightarrow{k_1} Z_2 k_2 \xrightarrow{\cdots} Z_{q} k_q \xrightarrow{b_d} d \) where \( Z_1 := \{s\}, Z_l \subseteq \mathcal{Z} \setminus \{s,d\} \) for each \( l = 2, \ldots, q \), \( Z_l \cap Z_{l'} = \{\} \) for all \( l, l' = 1, \ldots, q \) when \( l \neq l' \), and \( k_l \in \mathbb{N} \) is the one-hop encoding delay of a message from \( s \) leaving any node in \( Z_l \). By convention \( ||f(s,d)|| := q \). For any \( Z \subseteq \mathcal{Z} \), \( Z \in f(s,d) \) if \( Z \subseteq Z_l \) for some \( 1 \leq l \leq q \). Note that \( d \notin f(s,d) \). For any \( i \in Z_l, k_{s,i} = \sum_{e=1}^{l} k_e \) is the encoding delay between source \( s \) and node \( i \). This definition extends to nodes that do not forward messages from \( s \). If \( i \notin f(s,d) \) for any \( d \in \mathcal{D} \) then \( k_{s,i} := \infty \).

A flow set \( F := \{f(s,d) : s \in \mathcal{S}, d \in \mathcal{D} \} \) specifies a flow for every source-destination pair \( (s,d) \). The space of flow sets is denoted by \( \mathcal{F} \). Each flow set induces a multi-edge directed graph on \( \mathcal{N} \). This induced graph may have cycles. Source \( s \in \mathcal{S} \) generates the message \( m_s(b) \in \{1, \ldots, 2^{nR_s}\} \) in block \( b \) and node \( i \in \mathcal{Z} \) sends the index \( w_i(b) \in \{1, \ldots, 2^{n(\sum_{e \in \mathcal{Z}} R_e)}\} \) assigned to the message vector \( w_i(b) := \{s, m(b - k_{s,i}) \} : s \in \mathcal{S} \) in the same block, where \( k_{s,i} \) is finite for each \( s \in \mathcal{S} \), and \( R_s \) is the rate of source \( s \).

Example 1. Let \( f(1,3) = 1 \xrightarrow{1} 2 \xrightarrow{\infty} 3 \) as depicted in Figure II(i). Then \( k_{1,2} = 1, k_{2,3} = \infty \), and \( k_{1,3} = k_{1,2} + k_{2,3} = \infty \). In block \( b \), node 1 sends \( m_1(b) \in \{1, \ldots, 2^{nR_1}\} \) and node 2 forwards \( m_1(b-1) \). Node 3 does not forward messages.

Each node \( i \in \mathcal{I} \) has a codebook consisting of \( 2^n \sum_{e \in \mathcal{Z}} R_e \) \( n \)-length codewords generated by an i.i.d distribution on \( \mathcal{X} \). Every index \( w \in \{1, \ldots, 2^{n(\sum_{e \in \mathcal{Z}} R_e)}\} \) corresponds to a unique codeword \( \bar{x}_i(w) \). In block \( b \), node \( i \) transmits the codeword \( \bar{x}_i(w_i(b)) \).
Let \( \bar{R} := \{ (s, R_s) : s \in S \} \) denote the vector of rates allocated to the source nodes. For any \( S \subseteq S \backslash \{ d \} \), let \( F_d(S) := \{ i \in f(s, d) : s \in S \} \) denote the set of nodes covered by the flows terminating at node \( d \), let \( F_d(S) := \emptyset \) for \( d \in D \), and let \( R_S := \sum_{s \in S} R_s \). Since \( R_S \) is the rate of the “super-source” \( F_d(S) \), it follows that

\[
R_S < I(X_{F_d(S)}; Y_d | X_{\bar{F_d}(S)}),
\]

by invoking the capacity of the point-to-point channel on the “super-channel” between \( F_d(S) \) and node \( d \). Let \( R_d(F) \) denote the region of rate vectors \( \bar{R} := \{ (s, R_s) : s \in S \} \) that satisfy (2) for every subset \( S \subseteq S \backslash \{ d \} \). We have the following outer-bound on the decode-forward region:

\[
\bigcup_{F \in F} \bigcap_{d \in D} R_d(F).
\]

**Example 2.** Let \( f(1, 3) = 1 \rightarrow 2 \rightarrow 3 \) and \( F = \{ f(1, 3) \} \) as shown in Figure 2. Then \( R_3(F) = \{ R_1 : R_1 < I(X_1; X_2; Y_3) \} \).

The outer-bound in (3) is achievable in two important channel classes. The first consists of channels for which \(|N| \leq 3\) or \(|Z| = 1\). Examples of channels in this class previously appear in the literature, include the relay channel \( [1] \), the multiple-access channel \( [2] \), the multiple-access relay channel \( [3] \), the two-way channel \( [4] \), the cooperative multiple-access channel \( [5] \), the three-way all-cast channel \( [6] \), the two-way relay channel \( [7] \), and the one-way multi-relay channel \( [8] \). The rate regions in the cited work, though not expressed in terms of flows, reduce to (3) for independent input distributions, making their shared structure explicit. The second class of channels consists of block transition matrices \( p(y_d|x_T) \) in which each non-zero block is a transition matrix \( p(y_d|x_i) \) for some \( d \in D \) and \( i \in T \).

In general, (3) is not achievable. Multi-node channels with at least two sources and two relays have deadlocks that require two relays to decode the same messages before each other [9]. The decode-forward region depends ambiguously on the outcomes of all such deadlocks and does not have a convenient expression. Both classes of channels in which (3) is tight, artifically eliminate deadlocks by restricting the composition of nodes in the first case, or by simplifying the channel dynamics in the second. We will develop a way of indirectly characterizing the decode-forward region without these restrictions.

### III. Flow Decomposition

Decoding schemes specify the message vectors decoded in each block at each node. For any \( d \in D \), a layered partition \( L_d := \{ L_{d,0}, \ldots, L_{d,|L_d|-1} \} \) of \( F_d(S) \), is a vector of sets, some possibly empty, that satisfies the following conditions:

\[
L_{d,i} \subseteq F_d(S) \quad \text{for} \quad i = 0, \ldots, |L_d|; \quad L_{d,i} \cap L_{d,m} = \emptyset, \quad \text{for} \quad i, m = 0, \ldots, |L_d|; \quad i \neq m; \quad L_{d,0} l \neq \emptyset \quad \text{and} \quad F_d(S) = \cup_{i=0}^{|L_d|-1} L_{d,i}.
\]

If \( i \in L_{d,i} \), then \( \text{LAYER}(i) = l \). Each “layer” corresponds to some past block of channel uses. Deeper layers go deeper into the past. The layered partition \( L_d \) decomposes \( F \) so that each node in \( F_d(S) \) helps node \( d \) decode some subset of a message vector in some previous block. The pair \( (F, L_d) \) is a flow decomposition and describes the message vectors encoded and decoded at node \( d \) in each block.

**Example 3.** Let \( f(1, 3) = 1 \rightarrow 2 \rightarrow 3 \). Figure 2(iii) depicts \( L_3 = \{(2), \{1\}\} \). Figure 2(iv) depicts \( L_3 = \{(2), \{1\}\} \). Figure 2(v) depicts \( L_3 = \{(1), \{2\}\} \).

Let \( \bar{m}_d(b) := \{ (s, m_s(b_s)) : s \in S \} \) denote the message vector node \( d \) decodes in block \( b \) where \( 1 \leq b_s \leq b \) for each \( s \in S \). This message vector determines the side information at node \( d \), namely the set of message vectors \( \cup_{b=1}^{b_d} \bar{m}_d(b) \) node \( d \) decodes prior to block \( b \). Node \( i \) carries useful information to node \( d \) in block \( b \) if \( \bar{m}_d(b) \) splits \( \bar{w}_i(b-l) \) into a set of messages that \( d \) decodes in block \( b \) and a set of messages that \( d \) knows from its side information. Let \( \bar{A}_d := \{ i : \bar{w}_i(b-l) \cap \bar{m}_d(b) \neq \emptyset \} \) and \( \bar{A}_d := \{ i : \bar{w}_i(b-l) \subseteq \cup_{b=1}^{b_d} \bar{m}_d(b) \} \). Equivalently, \( \bar{A}_d := \{ i : \bar{w}_i(b-l) \subseteq \cup_{b=1}^{b_d} \bar{m}_d(b) \} \). If \( L_{d,0} \subseteq \bar{A}_d \), then \( \bar{m}_d(b) \) is a splitting vector.

Define the mapping \( u : S \times F_d(S) \rightarrow N \) as follows: for \( s \in S \) and \( i \in f(s, d) \), let \( u(s,i) := \text{LAYER}(i)+k_{s,i} \). Then \( M(s) := \{ j : j = \arg \min_{i \in f(s,d)} u(s,i) \} \). For each source \( s \in S \), there is a virtual source \( v(s) := \{ j : j = \arg \min_{i \in M(s)} |f(s,i)| \} \). The virtual source \( v(s) \) is a set of nodes \( Z \in f(s, d) \) that appear to node \( d \) as the original source \( s \). The encoding delay \( k_{s,v(s)} \) is well defined, since the nodes in \( v(s) \) experience the same encoding delay from source \( s \) by construction. Set:

\[
\bar{m}_d(b) := \{ (s, m_s(b-k_{s,v(s)}-\text{LAYER}(v(s))) : s \in S \}
\]

**Lemma 1.** \( \bar{m}_d(b) \) is a splitting vector for \( (F, L_d) \).
Proof. Given \( f(s, d) := Z_1 \xrightarrow{k_1} Z_2 \xrightarrow{k_2} \cdots \rightarrow Z_q \xrightarrow{k_q} d \), let \( f(s, d) := Z'_1 \xrightarrow{k'_1} Z'_2 \xrightarrow{k'_2} \cdots \rightarrow Z'_p \xrightarrow{k'_p} d \) be the flow that satisfies \( Z'_1 = v(s) \), \( Z'_2 \subseteq Z_{q-(p-1)} \), and \( Z'_n = Z_{q-(p-n)} \) for all \( 2 \leq m \leq p \leq q \). For \( i \in f(s, d) \) let \( k_{i,v,s} := k_{i-1} - k_{s,v,s} \). Let \( f(s, v, d) := Z'_1 \xrightarrow{k'_1} Z'_2 \xrightarrow{k'_2} \cdots \rightarrow Z'_p \xrightarrow{k'_p} d \) be the flow that satisfies \( Z'_1 = v(s) \), \( Z'_2 \subseteq Z_p \), and \( Z'_n = Z_m \) for all \( 1 \leq m \leq p - 1 \leq q - 1 \). For \( i \in f(s, v, d) \) let \( k_{s,v,i} := k_{s,v,s} - k_{i,v,s} \). We first derive the following inequalities:

\[
\text{LAYER}(v(s)) - \text{LAYER}(i) \leq k_{s,v,i}, \quad \forall i \in f(s, d)
\]

(5)

\[
\text{LAYER}(v(s)) > k_{i,v,s} \quad \forall i \in f(s, v)
\]

(6)

By construction, \( \text{LAYER}(v(s)) + k_{s,v,s} \leq \text{LAYER}(i) + k_{s,i} \) for all \( i \in f(s, v, d) \) which implies (5) and \( \text{LAYER}(v(s)) + k_{s,v,s} \leq \text{LAYER}(i) + k_{i,v,s} \) for all \( i \in f(s, v) \) which implies (6). Fix \( l \in \{1, \ldots, |L_d|-1\} \) and \( i \in L_d \). For every \( s \in S \) there are two cases to consider. If \( i \in f(s, d) \) then (5) implies \( (s, m(b - k_{s,i} - l)) \in \bigcup_{i=1}^{d} m_{d}(q) \). If \( i \in f(s, v, d) \) then (6) implies \( (s, m(b - k_{s,i} - l)) \in \bigcup_{i=1}^{d} m_{d}(q) \) which concludes the proof.

Example 5. Let \( f(1,3) = 1 \xrightarrow{1} 2 \xrightarrow{\infty} 3 \) and \( L_3 = (\{2\}, \{1\}) \), as depicted in Figure 7(ii). Since \( \text{LAYER}(2) - \text{LAYER}(1) = 1 \leq k_{1,2} \) and \( g(1,3) = 1 \xrightarrow{\infty} 3 \). Then \( m_{d}(b) = \{(1, m_1(b-1))\} \).

Example 6. Let \( f(1,3) = 1 \xrightarrow{1} 2 \xrightarrow{\infty} 3 \) and \( L_3 = (\{2\}, \{1\}) \) as depicted in Figure 7(iii). Since \( \text{LAYER}(1) - \text{LAYER}(2) = 2 > k_{1,2} \) and \( g(1,3) = 2 \xrightarrow{\infty} 3 \). Then \( m_{d}(b) = \{(1, m_1(b-1))\} \).

Example 7. Let \( f(1,3) = 1 \xrightarrow{1} 2 \xrightarrow{\infty} 3 \) and \( L_3 = (\{1\}, \{2\}) \) as depicted in Figure 7(iv). Since \( \text{LAYER}(1) - \text{LAYER}(2) = -1 < k_{1,2} \) and \( g(1,3) = 1 \xrightarrow{\infty} 3 \). Then \( m_{d}(b) = \{(1, m_1(b-1))\} \).

Example 8. Let \( f(1,3) = 1 \xrightarrow{2} 2 \xrightarrow{\infty} 3 \) and \( L_3 = (\{2\}, \{1\}) \) as depicted in Figure 7(v). Since \( \text{LAYER}(1) - \text{LAYER}(2) = 2 = k_{1,2} \) and \( g(1,3) = 1 \xrightarrow{\infty} 3 \). Then \( m_{d}(b) = \{(1, m_1(b-2))\} \).

For any subset \( S \subseteq S \) and \( 0 \leq l \leq |L_d| - 1 \), let

\[
A_l(S) := \{ i \in g(s, d) : s \in S \} \cap L_{d,l}
\]

(7)

\[
\hat{A}_l(S) := (\bigcup_{i=0}^{d} L_{d,l}) \setminus A_l(S)
\]

(8)

It follows from Lemma 1 that \( A_l(S) = A_l \) and \( \hat{A}_l(S) \subseteq \hat{A}_l \). We rely on the context to convey that \( A_l(S) \) and \( \hat{A}_l(S) \) depend on a particular \((F, L_d)\).

Example 9. Let \( f(1,3) = 1 \xrightarrow{1} 2 \xrightarrow{\infty} 3 \) and \( L_3 = (\{2\}, \{1\}) \) as depicted in Figure 7(ii). Then \( A_0(\{1\}) = \{2\} \) and \( A_1(\{1\}) = \{1\} \).

Example 10. Let \( f(1,3) = 1 \xrightarrow{1} 2 \xrightarrow{\infty} 3 \) and \( L_3 = (\{2\}, \{1\}) \) as depicted in Figure 7(iii). Then \( A_0(\{1\}) = \{2\}, A_1(\{1\}) = \{1\} \) and \( A_2(\{1\}) = \{\} \).

Example 11. Let \( f(1,3) = 1 \xrightarrow{1} 2 \xrightarrow{\infty} 3 \) and \( L_3 = (\{1\}, \{2\}) \) as depicted in Figure 7(iv). Then \( A_0(\{1\}) = \{1\}, A_1(\{1\}) = \{1\} \) and \( A_2(\{1\}) = \{\} \).

Example 12. Let \( f(1,3) = 1 \xrightarrow{2} 2 \xrightarrow{\infty} 3 \) and \( L_3 = (\{2\}, \{1\}) \) as depicted in Figure 7(v). Then \( A_0(\{1\}) = \{2\}, A_1(\{1\}) = \{1\} \) and \( A_2(\{1\}) = \{\} \).

In block \( b \), node \( d \) decodes \( m_{d}(b) \) as defined in (4) by finding the message vector \( \{s, m : s \in S\} \) that satisfies the following typicality checks for \( 0 \leq l \leq |L_d| - 1 \):

\[
(\{x_i(w_l(b - l)) : i \in A_l(S)\}, \{X_i(b - l) : i \in \hat{A}_l(S)\}, \bar{Y}_d(b - l)) \in T^{(\infty)}(X_{d,0:a(L_d,b),Y_d}).
\]

An error event occurs if some subset of source messages \( S \subseteq S \) is decoded incorrectly. The probability of such an event goes to zero if the following constraint is satisfied:

\[
R_S < \sum_{l=0}^{|L_d|-1} I(X_{A_l(S)}; Y_d|X_{\hat{A}_l(S)})
\]

(9)

Let \( R(F, L_d) \) denote the region of rate vectors \( \{s, R_a : s \in S\} \) that satisfy (9) for every \( S \subseteq \hat{S} \setminus \{d\} \).

Example 13. Let \( f(1,3) = 1 \xrightarrow{1} 2 \xrightarrow{\infty} 3 \), \( F = \{f(1,3), L_3 = (\{2\}, \{1\}) \) as depicted in Figure 7(ii). Then \( m_{d}(b) = \{(1, m_1(b-1))\} \) is the unique \( m \in \{1, \ldots, 2^{R_0}\} \) that satisfies \((x_2(m), Y_3(b)) \in T^{(\infty)}(X_2, Y_3)\) and \((x_1(m), X_2(b-1), Y_3(b-1)) \in T^{(\infty)}(X_1, X_2, Y_3)\). It follows that \( R(F, L_d) = \{R_1 : R_1 < I(X_2; Y_3) + I(X_1; Y_3|X_2) = I(X_1; Y_3)\} \).

Remark 1. The rate region \( R_3(F, L_d) \) achieved in Example 13 coincides with the decode-forward outerbound \( R_{d,F} \) in Example 2.

Example 14. Let \( f(1,3) = 1 \xrightarrow{1} 2 \xrightarrow{\infty} 3 \), \( F = \{f(1,3), L_3 = (\{2\}, \{1\}) \) as depicted in Figure 7(iii). Then \( m_{d}(b) = \{(1, m_1(b-1))\} \) is the unique \( m \in \{1, \ldots, 2^{R_0}\} \) that satisfies \((x_2(m), Y_3(b)) \in T^{(\infty)}(X_2, Y_3)\). It follows that \( R(F, L_d) = \{R_1 : R_1 < I(X_2; Y_3)\} \).

Remark 2. In Example 12 node 1 is excluded from the virtual flow and the typicality check because the message it sends in block 2 – 2 is already decoded by node 3 in block b. The corresponding achievable region \( R_3(F, L_d) \) is less than the decode-forward outerbound in Example 2.
Example 15. Let \( f(1,3) = 1 \xrightarrow{\infty} 2 \xrightarrow{\infty} 3 \), \( F = \{ f(1,3) \} \), and \( \bar{L}_3 = \{(1, \{2\})\} \) as depicted in Figure 7(iii). Then \( \bar{m}_3(b) = \{(1, m(b))\} \) is the unique \( m \in \{1, \ldots, 2^{nR_1}\} \) that satisfies \((\bar{x}_1(m), Y_5(b)) \in T_{w}^n(X_1, X_5)\). It follows that \( R(F, \bar{L}_3) = \{R_1: R_1 < I(X_1; Y_5)\} \).

Remark 3. Node 2 is excluded from the virtual flow and the typicity check because the message it sends in block \( b-1 \) is already decoded by node 3 in block \( b \). The less restrictive constraint \( R_1 < I(X_1; Y_5|X_2) \) is also achievable since node 3 has already decoded the message that node 2 sends in block \( b \). However both regions are less than the decode-forward outerbound in Example 2.

Example 16. Let \( f(1,3) = 1 \xrightarrow{\infty} 2 \xrightarrow{\infty} 3 \), \( F = \{ f(1,3) \} \) and \( \bar{L}_3 = \{(2), \{\}, \{1\}\} \) as depicted in Figure 7(iv). Then \( \bar{m}_3(b) = \{(1, m_1(b-2))\} \) is the unique \( m \in \{1, \ldots, 2^{nR_1}\} \) that satisfies \((\bar{x}_2(m), Y_5(b)) \in T_{w}^n(X_1, X_2, Y_5) \) and \((\bar{x}_1(m), X_2(b-2), Y_5(b-2)) \in T_{w}^n(X_1, X_2, Y_5)\). It follows that \( R(F, \bar{L}_3) = \{R_1: R_1 < I(X_2; Y_5) + I(X_1; X_2|X_3; Y_5)\} \).

Remark 4. The rate regions \( R_d(F, \bar{L}_3) \) in Examples 13 and 16 are the same even though the corresponding flow decompositions are different. Flow decompositions that recover the same region of rate-vectors are equivalent by definition.

Although different flow decompositions recover different rate-vector regions, these regions share a fundamental relationship.

Theorem 1. If \( \bar{R} \in R_d(F) \) then \( \bar{R} \in R(F, \bar{L}_d) \) for some \( \bar{L}_d \).

Proof. See Section 2.

Examples 6 and 7 achieve strictly lower rates (seen in Examples 13 and 15) than Example 5 which matches the outer-bound. The diamond relay channel is a more suitable example of the concepts introduced in this section.

IV. EXAMPLE: THE DIAMOND RELAY CHANNEL

The diamond relay channel is defined by the flows \( f(1, 5) = 1 \xrightarrow{\infty} 2 \xrightarrow{\infty} 3 \xrightarrow{\infty} 4 \xrightarrow{\infty} 5 \) and \( f(2, 5) = 2 \xrightarrow{\infty} 4 \xrightarrow{\infty} 5 \), both depicted in Figure 3(ii), where \( S = \{1, 2\} \), \( Z = \{2, 3, 4\} \) and \( D = \{5\} \). In block \( b \), node 1 encodes \( \bar{w}_1(b) = \{(1, m(b))\} \), node 2 encodes \( \bar{w}_2(b) = \{(1, m_1(b-1)), \{2, m_2(b)\}\} \), node 3 encodes \( \bar{w}_3(b) = \{(1, m_1(b-3))\} \), and node 4 encodes \( \bar{w}_4(b) = \{(2, m_2(b-1))\} \). Let \( F = \{f(1,5), f(2,5)\} \). By definition, \( R_5(F) \) is set of rate vectors \( \bar{R} = (R_1, R_2) \) that satisfies:

\[
\begin{align*}
R_1 &< I(X_1 X_2 X_3; Y_5|X_4) \quad (10) \\
R_2 &< I(X_2 X_3 Y_5; X_1 X_2) \quad (11) \\
R_1 + R_2 &< I(X_1 X_2 X_3 X_4; Y_5) \quad (12)
\end{align*}
\]

Four different decoding schemes collectively achieve all the rate vectors in \( R_5(F) \).

A. The first decoding scheme

Set \( L_5 = \{(3), \{4\}, \{2\}, \{1\}\} \). It follows that \( v(1) = 1 \) since \( \text{Layer}(1) - \text{Layer}(2) = 1 \leq k_{1,2} \) and \( \text{Layer}(1) - \text{Layer}(3) = 3 \leq k_{1,3} \) which satisfy (5). Similarly, \( v(2) = 4 \) since \( \text{Layer}(2) - \text{Layer}(4) = 2 > 1 \) which satisfies (6). Then \( g(1, 5) = 1 \xrightarrow{\infty} 2 \xrightarrow{\infty} 3 \xrightarrow{\infty} 5 \) and \( g(2, 5) = 2 \xrightarrow{\infty} 4 \xrightarrow{\infty} 5 \). The virtual flows are depicted in Figure 2(iii).

In block \( b \), node 5 decodes \( \bar{m}_5(b) = \{(1, m_1(b-3)), \{2, m_2(b-1)\}\} \) by finding the unique pair \( m_1 \in \{1, \ldots, 2^{nR_1}\} \) and \( m_2 \in \{1, \ldots, 2^{nR_2}\} \) that jointly satisfy the following typicality checks:

\[
\begin{align*}
(x_3(m_1), Y_5(b)) \in T_{w}^n(X_3, Y_5) \\
(x_4(m_2), X_3(b-1), Y_5(b-1)) \in T_{w}^n(X_3, X_4, Y_5) \\
(x_2(m_1, m_2), X_3(b-2), Y_5(b-2)) \in T_{w}^n(X_2, X_3, X_4, Y_5) \\
(x_1(m_1), X_2(b-3), X_3(b-3), X_4(b-3), Y_5(b-3)) \in T_{w}^n(X_2, X_3, X_4, Y_5)
\end{align*}
\]
The message from node 1 forwarded by node 2 in block 5 and \( f(2, 5) = 2 \overset{5}{\rightarrow} 4 \overset{5}{\rightarrow} 5 \) (ii) \( \bar{L}_5 = (\{3, 4\}, \{\}, \{2\}, \{1\}) \) (iii) \( L_5 = (\{3\}, \{\}, \{2\}, \{1\}) \) (iv) \( L_5 = (\{3\}, \{\}, \{1, 4\}, \{2\}) \) (v) \( L_5 = (\{3\}, \{\}, \{1, 4\}, \{2\}) \). Original flows are depicted with dotted lines and virtual flows with solid lines in (ii)-(v).

\[ \in T_c^{(n)}(X_2, X_3, X_4, Y_5) \]

The probability of error goes to zero if \( R \) satisfies the following conditions:

\[
\begin{align*}
R_1 &< I(X_3; Y_5) + I(X_2; Y_5 | X_3 X_4) \\
&+ I(X_1; Y_5 | X_2 X_3 X_4) \\
&= I(X_3; Y_5) + I(X_1 X_2; Y_5 | X_3 X_4) \\
R_2 &< I(X_2; Y_5 | X_3) + I(X_2; Y_5 | X_3 X_4) \\
R_1 + R_2 &< I(X_1; Y_5) + I(X_1 X_3 X_4; Y_5) \\
&= I(X_1 X_2 X_3 X_4; Y_5)
\end{align*}
\]

C. The third decoding scheme

Set \( \bar{L}_5 = (\{3\}, \{\}, \{1, 4\}, \{2\}) \). It follows that \( v(1) = 1 \) since \( \text{LAYER}(1) - \text{LAYER}(2) = 0 \leq k_{1,2} \) and \( \text{LAYER}(3) = 3 = k_{1,3} \) which satisfy (5). Similarly, \( v(2) = 2 \) since \( \text{LAYER}(2) - \text{LAYER}(4) = 1 = k_{2,4} \) which satisfies (5). Then \( g(1, 5) = 1 \overset{5}{\rightarrow} 3 \overset{5}{\rightarrow} 5 \) and \( g(2, 5) = 2 \overset{5}{\rightarrow} 4 \overset{5}{\rightarrow} 5 \). The virtual flows are depicted in Figure 2 (iv).

In block \( b \), node 5 decodes \( \bar{m}_5(b) = \{(1, m_1(b - 3)), (2, m_2(b - 3))\} \) by finding the unique pair \( m_1 \in \{1, \ldots, 2^n R_1\} \) and \( m_2 \in \{1, \ldots, 2^n R_2\} \) that jointly satisfy the following typicality checks:

\[
(\bar{x}_3(m_1), \bar{Y}_5(b)) \in T_c^{(n)}(X_3, Y_5) \\
(\bar{x}_4(m_2), \bar{X}_4(b - 2), \bar{Y}_5(b - 2)) \in T_c^{(n)}(X_3, X_4, Y_5) \\
(\bar{x}_1(m_1), \bar{x}_2(m_2), X_3(b - 3), X_4(b - 3), \bar{Y}_5(b - 3)) \in T_c^{(n)}(X_1, X_2, X_3, X_4, Y_5)
\]  

The message from node 1 forwarded by node 2 in block \( b - 3 \) has already been decoded by node 5 in block \( b \), so \( \bar{x}_2 \) in (20) effectively depends on \( m_2 \) alone. The probability of error goes to zero if \( R \) satisfies the following conditions:

\[
\begin{align*}
R_1 &< I(X_3; Y_5) + I(X_1; Y_5 | X_2 X_3 X_4) \\
R_2 &< I(X_4; Y_5 | X_3) + I(X_2; Y_5 | X_3 X_4 X_5) \\
R_1 + R_2 &< I(X_1; Y_5) + I(X_2 X_3 X_4; Y_5)
\end{align*}
\]

D. The fourth decoding scheme

Set \( \bar{L}_5 = (\{3\}, \{\}, \{1, 4\}, \{2\}) \). It follows that \( v(1) = 1 \) since \( \text{LAYER}(1) - \text{LAYER}(2) = 0 \leq k_{1,2} \) and \( \text{LAYER}(3) = 3 = k_{1,3} \) which satisfy (5). Similarly, \( v(2) = 2 \) since \( \text{LAYER}(2) - \text{LAYER}(4) = 1 = k_{2,4} \) which satisfies (5). Then \( g(1, 5) = 1 \overset{5}{\rightarrow} 3 \overset{5}{\rightarrow} 5 \) and \( g(2, 5) = 2 \overset{5}{\rightarrow} 4 \overset{5}{\rightarrow} 5 \). The virtual flows are depicted in Figure 2 (v).

In block \( b \), node 5 decodes \( m_5(b) = \{(1, m_1(b - 3)), (2, m_2(b - 4))\} \) by finding the unique pair \( m_1 \in \{1, \ldots, 2^n R_1\} \) and \( m_2 \in \{1, \ldots, 2^n R_2\} \) that jointly satisfy the following typicality checks:

\[
(\bar{x}_3(m_1), \bar{Y}_5(b)) \in T_c^{(n)}(X_3, Y_5) \\
(\bar{x}_1(m_1), \bar{x}_4(m_2), \bar{X}_4(b - 3), \bar{Y}_5(b - 3)) \in T_c^{(n)}(X_1, X_2, X_3, X_4, Y_5)
\]  

The message from node 1 forwarded by node 2 in block \( b - 4 \) has already been decoded by node 5 in block \( b \), so \( \bar{x}_2 \) in (20) effectively depends on \( m_2 \) alone. The probability of error goes
to zero if $\bar{R}$ satisfies the following conditions:
\begin{align*}
R_1 &< I(X_1; Y_2) + I(X_1; Y_3 X_4) \\
R_2 &< I(X_4; Y_3 | X_1) + I(X_2; Y_3 X_1 X_4) \\
R_1 + R_2 &< I(X_3; Y_2) + I(X_1 X_4; Y_3 | X_3) \\
&+ I(X_2; Y_3 | X_1 X_3 X_4) \\
&= I(X_1 X_2 X_3 X_4; Y_5) \\
\end{align*}
(24) (25) (26)

E. The achievability of $R_5(F)$

To prove that $R_5(F)$ is achievable, we show that any rate vector in the region defined by (10)-(12) is in the region defined by (14)-(16) or (17)-(19) or (21)-(23) or (24)-(26). Suppose $\bar{R}$ is not in (14)-(16). Since (14) and (16) define the boundaries of $R_5(F)$, $\bar{R}$ must violate (15). Then (15) implies that $\bar{R}$ satisfies (17). If $\bar{R}$ satisfies (18) then the proof is finished since (19) is a boundary of $R_5(F)$. If $\bar{R}$ violates (18) then (19) implies that $\bar{R}$ satisfies (21). If $\bar{R}$ satisfies (22) then the proof is finished since (23) is a boundary of $R_5(F)$. Otherwise (23) implies that $\bar{R}$ satisfies (24). Here the proof is finished since (25) and (26) are boundaries of $R_5(F)$.

Remark 5. A fundamentally distinctive feature of the flow decomposition framework is that the encoding scheme is iterative multi-access channel and the two-source multi-access that must violate (15). Then (15) implies that $\bar{R}$ satisfies (17). If $\bar{R}$ satisfies (18) then the proof is finished since (19) is a boundary of $R_5(F)$. If $\bar{R}$ violates (18) then (19) implies that $\bar{R}$ satisfies (21). If $\bar{R}$ satisfies (22) then the proof is finished since (23) is a boundary of $R_5(F)$. Otherwise (23) implies that $\bar{R}$ satisfies (24). Here the proof is finished since (25) and (26) are boundaries of $R_5(F)$.

Example 19. Let $f(1,5) = 1 \xrightarrow{1} 2 \xrightarrow{2} 3 \xrightarrow{\infty} 5$, $f(2,5) = 2 \xrightarrow{4} 4 \xrightarrow{\infty} 5$, and $F = \{f(1,5), f(2,5)\}$. If $L_5 = (\{3, 4, 1, 2\})$, then $(F, L_5)$ is complete on $S = \{1, 2\}$ and $S = \{2\}$ as depicted in Figure 2(iv).

Example 20. Let $f(1,5) = 1 \xrightarrow{1} 2 \xrightarrow{2} 3 \xrightarrow{\infty} 5$, $f(2,5) = 2 \xrightarrow{4} 4 \xrightarrow{\infty} 5$, and $F = \{f(1,5), f(2,5)\}$. If $L_5 = (\{3, 4, 1, 2\})$, then $(F, L_5)$ is complete on $S = \{1, 2\}$ and $S = \{2\}$ as depicted in Figure 2(v).

Lemma 2. The constraints (22) and (23) coincide in $S$ if $(F, L_d)$ is complete.

Proof. $F_d(S) = G_d(S)$. Hence, $F_d(S) = \bigcup_{i=0}^{L_d-1} A_d(S)$. □

Lemma 3. A complete $(F, L_d)$ exists.

Proof. In Section III, the set of virtual flows $\{g(s, d) : s \in S\}$ derives from a particular flow decomposition $(F, L_d)$. To prove Lemma 5, it is useful to derive some corresponding $L_d$ from a particular set of virtual flows $\{g(s, d) : s \in S\}$ and “initial conditions” $\{\text{LAYER}(v(s)) : s \in S\}$. Let $g(s, d) = Z_1 \xrightarrow{k_1} Z_2 \xrightarrow{k_2} \cdots \cdots \xrightarrow{k_q} d$ where $Z_1 := v(s)$. The corresponding $L_d$ by definition must for each $l = 1, \ldots, q$ satisfy $\text{LAYER}(Z_l) = \text{LAYER}(v(s)) - \sum_{e=1}^{l-1} k_e$. (27)

For any $\{g(s, d) : s \in S\}$ such that $G_d(S) = F_d(S)$, it follows from (27) that every corresponding $L_d$ is completely determined by the choice of $\{\text{LAYER}(v(s)) : s \in S\}$. Given any directed graph defined by the flows $\{g(s, d) : s \in S\}$ there exists a directed acyclic spanning subgraph defined by disjoint virtual flows $\{g(s, d) : s \in S\}$. Since $G_d(S) = F_d(S)$, setting $\text{LAYER}(v(s)) = \sum_{e=1}^{q} k_e$ for each $s \in S$ suffices to define a corresponding $L_d$. □

Example 21. Let $f(1,5) = 1 \xrightarrow{1} 2 \xrightarrow{2} 3 \xrightarrow{\infty} 5$, $f(2,5) = 2 \xrightarrow{4} 4 \xrightarrow{\infty} 5$ and $F = \{f(1,5), f(2,5)\}$. Let $g(1,5) = 1 \xrightarrow{1} 2 \xrightarrow{2} 3 \xrightarrow{\infty} 5$ and $g(2,5) = 4 \xrightarrow{\infty} 5$ where $v(1) = 1$ and $v(2) = 4$. Then $\{g(1,5), g(2,5)\}$ is a directed acyclic spanning subgraph of $F$ where $g(1,5)$ and $g(2,5)$ are disjoint. Setting $\text{LAYER}(v(1)) = 3$ and $\text{LAYER}(v(2)) = 0$ gives $L_5 = (\{3, 4\}, \{1\})$ as depicted in Figure 2(ii).

The flow decomposition $(F, L_d)$ bifurcates $F_d(S)$ into $I \subseteq \mathcal{D}$ by definition, if there exists an $l'$ such that $I \subseteq \bigcup_{l'=1}^{L_d-1} L_{d,l'}$ and $\{F_d(S) \setminus I\} \subseteq \bigcup_{l'=1}^{L_d-1} L_{d,l'}$. For any $S \subseteq S$, define the boolean variable $E(S) := \text{TRUE}$ if $(F, L_d)$ bifurcates $F_d(S)$ into $F_d(S)$, and $E(S) := \text{FALSE}$ if otherwise.

Example 22. Let $f(1,5) = 1 \xrightarrow{1} 2 \xrightarrow{2} 3 \xrightarrow{\infty} 5$, $f(2,5) = 2 \xrightarrow{4} 4 \xrightarrow{\infty} 5$, $F = \{f(1,5), f(2,5)\}$ and $L_5 = (\{3, 4\}, \{1\})$ as depicted in Figure 2(ii). Then $(F, L_5)$ bifurcates $F_d(\{1, 2\})$ into $F_d(\{1\})$ since $F_d(\{1\}) \subseteq L_{5.3} \cup L_{5.2} \cup L_{5.1} \cup L_{5.0}$ and $F_d(\{1, 2\}) \setminus F_d(\{1\}) \subseteq L_{5.0}$.

Example 23. Let $f(1,5) = 1 \xrightarrow{1} 2 \xrightarrow{2} 3 \xrightarrow{\infty} 5$, $f(2,5) = 2 \xrightarrow{4} 4 \xrightarrow{\infty} 5$, $F = \{f(1,5), f(2,5)\}$ and $L_5 = (\{3, 4\}, \{1\})$ as depicted in Figure 2(ii). Then $(F, L_5)$ bifurcates $F_d(\{1, 2\})$ into $F_d(\{1\})$ since $F_d(\{1\}) \subseteq L_{5.3} \cup L_{5.2} \cup L_{5.1} \cup L_{5.0}$ and $F_d(\{1, 2\}) \setminus F_d(\{1\}) \subseteq L_{5.0}$.
Remark 6. In Example 22 the constraint 2 evaluated at $S = \{1\}$ is given by (44) and coincides with the boundary condition 3 at $S = \{1\}$ given by (47).

Example 23. Let $f(1,5) = 1 \frac{1}{2} 2 3 5$, $f(2,5) = 2 \frac{1}{4} 3 \frac{1}{5}$, $F = \{f(1,5), f(2,5)\}$, and $L_0 = [3, \{1, 2, 4\}, \{2\}]$ as depicted in Figure 2(i). Then $(F, L_0)$ bifurcates $F_1 = \{(1,2)\}$ into $F_2 = \{\{2\}\}$ since $F_0 = \{(1,2)\} \subseteq L_5 \cup \{L_4, L_5\}$ and $F_3 = \{(1,2)\} \subseteq L_5 \cup L_4 \cup \{L_5\}$.

Remark 7. In Example 23 the constraint 2 evaluated at $S = \{2\}$ is given by (23) and coincides with the boundary condition 3 at $S = \{2\}$ given by (17).

Given $(F, L_0)$, let $\delta(S) := 1 - \mathbb{I}_{E(S)}$ and let $(F, L'_0) := \text{SHIFT}((F, L_0), S)$ for some $S \subseteq S$, if for every $i \in G_d(S)$,

$$
\text{LAYER}'(i) = \begin{cases} 
1 & i \in A_i(S) \setminus A_i(S) \\
1 + \delta(S) & i \in A_i(S) 
\end{cases}
$$

Example 24. Let $f(1,5) = 1 \frac{1}{2} 2 3 \frac{1}{5}$, $f(2,5) = 2 \frac{1}{4} 3 \frac{1}{5}$ and $F = \{f(1,5), f(2,5)\}$. Let $L_5 = (\{3, 4\}, \{1\}, \{2\}, \{1\})$ as depicted in Figure 2(ii) and $L_5 = (\{3\}, \{4\}, \{1\}, \{2\})$ as depicted in Figure 2(iii). Then $(F, L'_5) = \text{SHIFT}((F, L_5), \{2\})$.

Example 25. Let $f(1,5) = 1 \frac{1}{2} 2 3 \frac{1}{5}$, $f(2,5) = 2 \frac{1}{4} 3 \frac{1}{5}$ and $F = \{f(1,5), f(2,5)\}$. Let $L_5 = (\{3\}, \{4\}, \{1\}, \{2\})$ as depicted in Figure 2(iv). Then $(F, L'_5) = \text{SHIFT}((F, L_5), \{2\})$.

Example 26. Let $f(1,5) = 1 \frac{1}{2} 2 3 \frac{1}{5}$, $f(2,5) = 2 \frac{1}{4} 3 \frac{1}{5}$ and $F = \{f(1,5), f(2,5)\}$. Let $L_5 = (\{3\}, \{4\}, \{1\}, \{2\})$ as depicted in Figure 2(v). Then $(F, L'_5) = \text{SHIFT}((F, L_5), \{2\})$.

Example 27. Let $f(1,5) = 1 \frac{1}{2} 2 3 \frac{1}{5}$, $f(2,5) = 2 \frac{1}{4} 3 \frac{1}{5}$ and $F = \{f(1,5), f(2,5)\}$. Let $L_5 = (\{3\}, \{4\}, \{1\}, \{2\})$ as depicted in Figure 2(vi). Then $(F, L'_5) = \text{SHIFT}((F, L_5), \{2\})$.

Lemma 4. $(F, L'_d)$ is complete if $(F, L_d)$ is complete.

Proof. From the definition of shift in (28).

Given some $(F, L_d)$ and $R \in \mathcal{R}_d(F)$, let $V \subseteq S$ denote the set of source nodes that satisfy 2 for all $S \subseteq V$, let $U \subseteq S$ denote the largest subset of $S$ that violates 3, and let $A_i(U)$ and $A_i(U)$ be defined in (7) and (8). Let $(F, L'_d) := \text{SHIFT}((F, L_d), U)$. The corresponding $A'_i(V)$, $A_i(U)$, $V'$ and $U'$ are defined with respect to $(F, L'_d)$.

Lemma 5. $V' = (S \setminus U) \cup V$.

Proof. The proof is by contradiction. Suppose there is some $S \subseteq (S \setminus U) \cup V$ that violates 3 for $(F, L_d)$. By assumption,

$$
R_S > \sum_{l=0}^{\lfloor |L'_d| \rfloor} I(X_{A'_i(S)}; Y_d|X_{A'_i(S)})
$$

Case 1: $S \cap U = \{\}$. Then,

$$
R_{U \cup S} > \sum_{l=0}^{\lfloor |L'_d| \rfloor} I(X_{A'_i(U \cup S)}; Y_d|X_{A'_i(U \cup S)})
$$

which contradicts the assumption that $U$ is the largest subset that violates 3 for $(F, L_d)$. (34) follows from (29) and the definition of $U$. To justify (31) - (33), fix any $l \in \{0, \ldots, L_d\}$. Since $A_i(U \cup S \setminus A_i(U)) \subseteq A'_i(U \cup S \setminus A_i(U))$, (31) follows from (32). Since $A_i(S \setminus A_i(U)) \subseteq A'_i(U \cup S \setminus A_i(U))$, (33) follows from (32).

Case 2: $S \cap U \neq \{\}$. Since $(S \setminus U) \cap U = \{\}$, the first case implies:

$$
R_{S \cap U} < \sum_{l=0}^{\lfloor |L'_d| \rfloor} I(X_{A'_i(S \cap U)}; Y_d|X_{A'_i(S \cap U)})
$$

It follows that,

$$
R_{S \cup U} > \sum_{l=0}^{\lfloor |L'_d| \rfloor} I(X_{A'_i(S \cup U)}; Y_d|X_{A'_i(S \cup U)})
$$

which contradicts the assumption that all subsets of $V$ satisfy 3 for $(F, L_d)$. Note that $(S \setminus U) \subseteq V$ since $S \subseteq (S \setminus U) \cup V$.

To justify (35) - (37), fix any $l \in \{0, \ldots, L'_d\}$. Then,

$$
I(X_{A'_i(S \cup U)}; Y_d|X_{A'_i(S \cup U)})
$$

(38)

$$
I(X_{A'_i(S \cup U)}; Y_d|X_{A'_i(S \cup U)})
$$

(39)

$$
I(X_{A'_i(S)}; Y_d|X_{A'_i(S)})
$$

(40)
Since $\tilde{A}_1(S \cap U) \subseteq \tilde{A}_1(S \setminus U)$ and $A_1(S \cap U) \subseteq \tilde{A}_1(S \setminus U)$, (39) follows from (38). By inspection, (36) follows from (29), (35), and (40). Since $A_1(S \cap U) \subseteq \tilde{A}_1(S \setminus U)$ and $A_1(S \cap U) = \tilde{A}_1(S \setminus U)$, (39) follows from (36) which finishes the proof of Lemma 5.

**Lemma 6.** $U \not\subset U'$

**Proof.** The proof is by contradiction. Suppose $U \subset U'$. Then,

\[
R_{U'} > \sum_{i=0}^{\left\lfloor \frac{|L'_d|}{2} \right\rfloor} I(X_{A'_1(U')}; Y_d | X_{\tilde{A}_1(U')}) \quad (41)
\]

\[
\geq \sum_{i=0}^{\left\lfloor \frac{|L_d|}{2} \right\rfloor} I(X_{A'_1(U')}; Y_d | X_{\tilde{A}_1(U')}) \quad (42)
\]

which contradicts the assumption that $U$ is the largest subset that violates (9) for $(f, L_d)$. Since $\tilde{A}_1(U') = A_1(U')$ and $A_1(U) \subseteq A_1(U')$, (42) follows from (41) which finishes the proof of Lemma 6.

Lemmas 5 and 6 are difficult to demonstrate in channels with fewer than three sources. Examples of (29)-(22) are provided in Appendix C for the three-source multiple-access relay channel.

For any $R \in \mathcal{R}_d(F)$, define the sequence of flow decompositions $\{(F, L_d)_k\}$ and the set of sequences $\{U_k\}$ where $(F, L_d)_{k+1} = \text{SHIFT}((F, L_d)_k, U_k)$. $(F, L_d)_0$ is complete, and $U_k$ is the largest subset of $S$ that violates (9) with respect to $(F, L_d)_k$. The sequence $\{U_k\}$ converges, by definition, if $U_k = U$ all but finitely often (a.b.fo), for some possibly empty $\subset \mathcal{S}$. In addition, let $\{V_k\}$ be the sequence of subsets such that all subsets of $V_k$ satisfy (9) with respect to $(F, L_d)_k$. The sequence $\{V_k\}$ converges, by definition, if $V_k = \mathcal{S}$ a.b.fo. Note that $\{V_k\}$ converges iff $\{U_k\}$ converges to $\{(F, L_d)_k\}$.

**Lemma 7.** $\{V_k\}$ converges.

**Proof.** Since $(F, L_d)_0$ is complete (Lemma 3), each flow decomposition in the sequence $\{(F, L_d)_k\}$ is complete (Lemma 4). Then $V_k \subset V_{k+1}$ for any $k$ such that $U_k \neq U_{k-1}$, where the inclusion is strict (Lemma 5 and Lemma 6). Therefore $\{U_k\}$ must converge to some $U \subset \mathcal{S}$. It remains to show that $U$ is empty.

Suppose $U$ is non-empty. First we show that $E(U) = \text{TRUE}$ with respect to $(F, L_d)_k$ a.b.fo. By inspection of $\mathcal{R}_d(F)$, $\{(F, L_d)_k\}$ bifurcates $F_d(S)$ into some $G_d(U)$ a.b.fo. If $G_d(U) = F_d(U)$ then $E(U) = \text{TRUE}$. Suppose $G_d(U) \neq F_d(U)$. For some $s \in U$ and $(F, L_d) \in \{(F, L_d)_k\}$, consider the case $i \in f(v(s), d)$ and $i \notin G_d(U)$. The property of $v(s)$ expressed in (5) implies that $\text{LAYER}(v(s)) > \text{LAYER}(i) < k_{v(s),2}$. Let $(F, L'_d) = \text{SHIFT}((F, L_d), U)$ and let $v'(s)$ be the corresponding virtual source. By inspection of (39), \text{LAYER}(v'(s)) > \text{LAYER}(v(s))$ but $\text{LAYER}(i) = \text{LAYER}(i)$. Shifting by $U$ eventually subsumes $i$ into the virtual flow.

**Example 28.** Let $f(1, 5) = 1 \nrightarrow 2 \nrightarrow 3 \nrightarrow 5$, $f(2, 5) = 2 \nrightarrow 4 \nrightarrow 5$, and $F = \{(1, 5), (f(2, 5))\}$. Let $L_5 = \{(3), \{1\}, \{4\}, \{1, 2\}\}$ as depicted in Figure 2(ii).

Fig. 3: The Shift Operation. $f(1, 5) = 1 \nrightarrow 2 \nrightarrow 3 \nrightarrow 5$, $f(2, 5) = 2 \nrightarrow 4 \nrightarrow 5$, and $F = \{(1, 5), (f(2, 5))\}$.

(i) $L'_5 = \{(3), \{1\}, \{4\}, \{1, 2\}\}$.
(ii) $L'_5 = \{(3), \{1\}, \{4\}, \{1\}\}$ as depicted in Figure 2(ii).

Now assume $i \in f(s, v(s))$ and $i \notin G_d(U)$. The property of $v(s)$ expressed in (5) implies that $\text{LAYER}(i) = \text{LAYER}(v(s)) > k_{v(s),2}$. The same arguments from the previous case apply. Therefore $\{(F, L_d)_k\}$ bifurcates $F_d(S)$ into $F_d(U)$ a.b.fo.

**Example 29.** Let $f(1, 5) = 1 \nrightarrow 2 \nrightarrow 3 \nrightarrow 5$, $f(2, 5) = 2 \nrightarrow 4 \nrightarrow 5$, and $F = \{(1, 5), (f(2, 5))\}$ as depicted in Figure 2(ii). It follows that $v(1) = 1$ and $g(1, 5) = 1 \nrightarrow 3$. Now let $(F, L'_d) = \text{SHIFT}((F, L_d), \{i\})$. Then $L'_5 = \{(3), \{1\}, \{4\}, \{1\}\}$ as depicted in Figure 2(ii).

Remark 8. In Example 29 prior to the shift, $2 \subset f(1, 5)$ but not $g(1, 5)$ since LAYER(1) − LAYER(2) = 0 < $k_{1,2} = 1$. Hence, $G_5(\{i\}) = \{1, 3\} \neq F_5(\{i\}) = \{1, 2, 3\}$. After the shift $2 \subset g(1, 5)$ since LAYER(1) − LAYER(2) = 1 = $k_{1,2}$. Hence $G_5(\{i\}) = \{1, 2, 3\} = F_5(\{i\}) = \{1, 2, 3\}$.

Now assume $i \in f(s, v(s))$ and $i \notin G_d(U)$. The property of $v(s)$ expressed in (5) implies that LAYER(i) − LAYER(v(s)) > $k_{v(s),2}$. The same arguments from the previous case apply. Therefore $\{(F, L_d)_k\}$ bifurcates $F_d(S)$ into $F_d(U)$ a.b.fo.

**Example 29.** Let $f(1, 5) = 1 \nrightarrow 2 \nrightarrow 3 \nrightarrow 5$, $f(2, 5) = 2 \nrightarrow 4 \nrightarrow 5$, and $F = \{(1, 5), (f(2, 5))\}$ as depicted in Figure 2(ii). It follows that $v(1) = 1$ and $g(1, 5) = 1 \nrightarrow 3$. Now let $(F, L'_d) = \text{SHIFT}((F, L_d), \{2\})$. Then $L'_5 = \{(3), \{4\}, \{2\}, \{1\}\}$ as depicted in Figure 2(ii). It follows that $v'(2) = 2$ and $g'(2, 5) = 2 \nrightarrow 4 \nrightarrow 5$.

**Remark 9.** In Example 29 prior to the shift, $2 \subset f(2, 5)$ but not $g(2, 5)$ since LAYER(2) − LAYER(4) = 2 > $k_{2,4} = 1$. Hence $G_5(\{2\}) = \{4\} \neq F_5(\{2\}) = \{2, 4\}$. After the shift, $2 \subset g'(2, 5)$ since LAYER(2) − LAYER(4) = 1 = $k_{2,4}$. Hence $G_5'(\{2\}) = \{2, 4\} = F_5(\{2\}) = \{2, 4\}$.

Now we show that $U$ must be empty. Suppose $U$ is not empty. Since each flow decomposition in $\{(F, L_d)_k\}$ is
complete and \( \{ (F, \mathcal{L})_k \} \) bifurcates \( F_2(\mathcal{S}) \) into \( F_2(\mathcal{U}) \) a,b.f.o., it follows that \( \{ (F, \mathcal{L})_k \} \) is complete on \( \mathcal{U} \) and that (2) and (3) agree on \( S = \mathcal{U} \) a,b.f.o., which is a contradiction. This finishes the proof of Lemma 7.

**Example 30.** Let \( f(1,5) = 1 \triangleright 2 \triangleright 2 \triangleright 3 \xrightarrow{\infty} 5, \)
\( f(2,5) = 2 \triangleright 4 \xrightarrow{\infty} 5, \) \( F = \{ f(1,5), f(2,5) \} \) and \( \mathcal{L}_5 = \{ \{ 4 \}, \{ 1 \} \} \) as depicted in Figure 2ii.

**Remark 10.** In Example 39 (\( F, \mathcal{L}_5 \)) is complete on \( \{ 1, 2 \} \) and bifurcates \( F_5(\{ 1, 2 \}) \) into \( F_5(\{ 1 \}) \). Hence, (\( F, \mathcal{L}_5 \)) is complete on \( \{ 1 \} \). The constraint (9) evaluated at \( S = \{ 1 \} \) is given by (2) and coincides with the boundary condition (2) evaluated at \( S = \{ 1 \} \) given by (17).

**Example 31.** Let \( f(1,5) = 1 \triangleright 2 \triangleright 2 \triangleright 3 \xrightarrow{\infty} 5, \)
\( f(2,5) = 2 \triangleright 4 \xrightarrow{\infty} 5, \) \( F = \{ f(1,5), f(2,5) \} \) and \( \mathcal{L}_5 = \{ \{ 3 \}, \{ 1 \}, \{ 2 \} \} \) as depicted in Figure 2v.

**Remark 11.** In Example 37 (\( F, \mathcal{L}_5 \)) is complete on \( \{ 1, 2 \} \) and bifurcates \( F_5(\{ 1, 2 \}) \) into \( F_5(\{ 2 \}) \). Hence, (\( F, \mathcal{L}_5 \)) is complete on \( \{ 2 \} \). The constraint (9) evaluated at \( S = \{ 2 \} \) is given by (25) and coincides with the boundary condition (2) evaluated at \( S = \{ 2 \} \), given by (17).

Lemma 7 finishes the proof of Theorem 1. The flow decompositions in Figure 2(iii) and Figure 3(ii) are “equivalent” in the sense that they share the same virtual flows, the same non-empty layers, and the same achievable region of rate vectors. Another example of equivalent flow decompositions is discussed in Remark 4. Any flow decomposition in Figure 3 can be “shifted” into a flow decomposition equivalent with any other in Figure 2. The concept of equivalent flow decompositions will play a fundamental role in characterizing the decode-forward region for multi-source multi-relay all-cast channels.

**VI. Conclusion**

It is conventional in network information theory to express rate-vector in terms of mutual information. This approach is inadequate for general multi-node channels. The interdependencies between the messages decoded by each node and the messages encoded by all the other nodes make a direct characterization of the decode-forward region intractable.

We provide a way of circumventing this problem by describing instead, any encoding and decoding scheme at a given node that recovers a particular rate-vector in the decode-forward region. Flow decompositions are mathematical abstractions of these schemes. Theorem 1 implies that every rate-vector in the decode-forward region corresponds to a set of causal flow decompositions, each assigned to a unique node in the channel. The companion paper in Part II characterizes the decode-forward region indirectly, by identifying necessary and sufficient conditions that determine whether an arbitrary set of (possibly non-causal) flow decompositions maps to an equivalent set of causal flow decompositions.

**APPENDIX A**

**THE COOPERATIVE MULTIPLE-ACCESS CHANNEL**

The cooperative multiple-access channel is defined by the flows \( f(1,3) = 1 \xrightarrow{\infty} 2 \xrightarrow{\infty} 3 \) and \( f(2,3) = 2 \xrightarrow{\infty} 3 \), both depicted in Figure 3i, where \( S = \{ 1, 2 \}, \)
\( Z = \{ 1, 2 \} \) and \( D = \{ 3 \} \). In block \( b \), node 1 encodes \( w_1(b) \{ \{ 1, m_1(b) \}, (2, m_2(b - 1)) \} \) and node 2 encodes \( w_2(b) \{ \{ 1, m_1(b) \}, (2, m_2(b - 1)) \} \).

Let \( F = \{ f(1,3), f(2,3) \} \). By definition, \( \mathcal{R}_3(F) \) is set of rate vectors \( \hat{R} = \{ R_1, R_2 \} \) that satisfies:

\[
R_1 < I(X_1X_2; Y_3) \quad (43)
\]

\[
R_2 < I(X_2X_1; Y_3) \quad (44)
\]

\[
R_1 + R_2 < I(X_1X_2; Y_3) \quad (45)
\]

Three different decoding schemes collectively achieve all the rate vectors in \( \mathcal{R}_3(F) \).

**A. The first decoding scheme**

Set \( \hat{L}_3 = \{ \{ 1 \}, \{ 2 \} \} \). It follows that \( v(1) = 1 \) since \( \text{LAYER}(1) - \text{LAYER}(2) = -1 \leq k_{1,2} \) which satisfies (5). Similarly, \( v(2) = 2 \) since \( \text{LAYER}(2) - \text{LAYER}(1) = 1 = k_{2,1} \) which satisfies (5). Then \( g(1,3) = 1 \xrightarrow{\infty} 3 \) and \( g(2,5) = 4 \xrightarrow{\infty} 5 \) as depicted in Figure 4ii.

In block \( b \), node 3 decodes \( m_3(b) = \{ \{ 1, m_1(b) \}, (2, m_2(b - 1)) \} \) by finding the unique pair \( m_1 \in \{ 1, \ldots, 2^{R_1} \} \) and \( m_2 \in \{ 1, \ldots, 2^{2R_2} \} \) that jointly satisfy the following typicality checks:

\[
(x_1(m_1, m_2), y_3(b) \in T^{(m)}_3(X_1, Y_3)
\]

\[
(x_2(m_2), x_1(b - 1), y_3(b - 1) \in T^{(m)}_3(X_1, X_2, Y_3)
\]

(46)

In each block, node 2 sends its own source message and a source message from node 1. The message \( m_1(b - 2) \), though encoded by node 2 in block \( b - 1 \) has already been decoded by node 3 in block \( b \), so \( \hat{x}_2 \) in (46) effectively depends on \( m_2 \) alone. The probability of error goes to zero if \( \hat{R} \) satisfies the following conditions:

\[
R_1 < I(X_1; Y_3) \quad (47)
\]

\[
R_2 < I(X_1; Y_3) + I(X_2; Y_3|X_1)
\]

\[
= I(X_1X_2; Y_3) \quad (48)
\]

\[
R_1 + R_2 < I(X_1; Y_3) + I(X_2; Y_3|X_1)
\]

\[
= I(X_1X_2; Y_3) \quad (49)
\]

Each of the three inequalities above addresses an error event. The probability that \( m_2 \) is decoded correctly and \( m_1 \) is not, goes to zero if (47) is satisfied. Similarly, the probability that \( m_1 \) is decoded correctly and \( m_2 \) is not, goes to zero if (48) is satisfied. Finally, the probability that both \( m_1 \) and \( m_2 \) are not decoded correctly goes to zero if (49) is satisfied.

**B. The second decoding scheme**

Set \( \hat{L}_3 = \{ \{ 1, 2 \} \}. It follows that \( v(1) = 1 \) since \( \text{LAYER}(1) - \text{LAYER}(2) = 0 \leq k_{1,2} \) which satisfies (5). Similarly, \( v(2) = 2 \) since \( \text{LAYER}(2) - \text{LAYER}(1) = 0 \leq k_{2,1} \) which satisfies (5).
The message $m$ effectively depends on $v$ Similarly, $R$ satisfies the following typicality check:

$$\langle x_1(m_1), x_2(m_2), y_3(b) \rangle \in T_{\epsilon}^{(m)}(X_1, X_2, Y_3)$$

The probability of error goes to zero if $R$ satisfies the following conditions:

$$R_1 < I(X_1; Y_3|X_2)$$
$$R_2 < I(X_2; Y_3|X_1)$$
$$R_1 + R_2 < I(X_1X_2; Y_3)$$

C. The third decoding scheme

Set $L_3 = \{(2), (1)\}$. It follows that $v(1) = 1$ since $LAYER(1) - LAYER(2) = 1 \geq k_{1,2}$ which satisfies $\delta$. Similarly, $v(2) = 2$ since $LAYER(2) - LAYER(1) = -1 < k_{2,1}$ which satisfies $\delta$. Then $g(1, 3) = 1 \xrightarrow{\mathcal{I}} 2 \xrightarrow{\mathcal{I}} 3$ and $g(2, 3) = 2 \xrightarrow{\mathcal{I}} 1 \xrightarrow{\mathcal{I}} 3$ as depicted in Figure 4(iv).

In block $b$, node 3 decodes $m_3(b) = \{(1, m_1(b - 1)), (2, m_2(b))\}$ by finding the unique pair $m_1 \in \{1, \ldots, 2^{nR_1}\}$ and $m_2 \in \{1, \ldots, 2^{nR_2}\}$ that jointly satisfies the following typicality checks:

$$(x_1(m_1), x_2(b - 1), y_3(b - 1)) \in T_{\epsilon}^{(m)}(X_1, X_2, Y_3)$$

The message $m_2(b - 2)$ forwarded by node 1 in block $b - 1$ has already been decoded by node 3 in block $b$, so $x_1$ in $LAYER(3)$ effectively depends on $m_2$ alone. The probability of error goes to zero if $R$ satisfies the following conditions:

$$R_1 < I(X_2; Y_3) + I(X_1; Y_3|X_2)$$
$$= I(X_1X_2; Y_3)$$
$$R_2 < I(X_2; Y_3)$$
$$R_1 + R_2 < I(X_1X_2; Y_3)$$

D. The achievability of $\mathcal{R}_3(F)$

To show that $\mathcal{R}_3(F)$ is achievable, we show that any rate vector in the region defined by $\mathcal{R}_4(F)$ is in the region defined by $\mathcal{R}_3(F)$, $R$ must violate $\mathcal{R}_3(F)$. Then $\mathcal{R}_3(F)$ implies that $R$ satisfies $\mathcal{R}_3(F)$. Now the proof is finished since $\mathcal{R}_3(F)$ are boundaries of $\mathcal{R}_3(F)$.

APPENDIX B

THE TWO-SOURCE MULTIPLE-ACCESS RELAY CHANNEL

The multiple-access relay channel is defined by the flows $f(1, 3) = 1 \xrightarrow{\mathcal{I}} 3 \xrightarrow{\mathcal{I}} 4$ and $f(2, 3) = 2 \xrightarrow{\mathcal{I}} 3 \xrightarrow{\mathcal{I}} 4$, both depicted in Figure 5(i), where $S = \{1, 2\}$, $Z = \{3\}$ and $D = \{4\}$. In block $b$, node 1 encodes $w_1(b) = \{(1, m_1(b))\}$, node 2 encodes $w_2(b) = \{(2, m_2(b))\}$, and node 3 encodes $w_3(b) = \{(1, m_1(b - 1)), (2, m_2(b - 1))\}$.

Let $F = \{f(1, 4), f(2, 4)\}$. By definition, $\mathcal{R}_4(F)$ is set of rate vectors $R = (R_1, R_2)$ that satisfies:

$$R_1 < I(X_1X_3; Y_4|X_2)$$
$$R_2 < I(X_2X_3; Y_4|X_1)$$
$$R_1 + R_2 < I(X_1X_2X_3; Y_4)$$

Three different decoding schemes collectively achieve all the rate vectors in $\mathcal{R}_4(F)$.

A. The first decoding scheme

Set $L_4 = \{(2), \{1\}\}$. It follows that $v(1) = 1$ since $LAYER(1) - LAYER(3) = 1 \geq k_{1,3}$ which satisfies $\delta$. Similarly, $v(2) = 2$ since $LAYER(2) - LAYER(3) = 0 \leq k_{2,3}$ which satisfies $\delta$. Then $g(1, 3) = 1 \xrightarrow{\mathcal{I}} 3 \xrightarrow{\mathcal{I}} 4$ and $g(2, 4) = 2 \xrightarrow{\mathcal{I}} 3 \xrightarrow{\mathcal{I}} 4$ as depicted in Figure 5(ii).

In block $b$, node 3 decodes $m_3(b) = \{(1, m_1(b - 1)), (2, m_2(b))\}$ by finding the unique pair $m_1 \in \{1, 2\}$.

\[\]
The probability that \( m \) satisfies the following typicality check:
\[
(x_3(m_1), x_2(m_2), y_4(b)) \in T_{c}^{(n)}(X_2, X_3, Y_4)
\]
\[
(x_1(m_1), x_3(b-1), x_2(b-1), y_4(b-1)) \in T_{c}^{(n)}(X_1, X_2, X_3, Y_4)
\]
Similarly, \( s \) satisfies. Finally, the probability that both \( m \) jointly satisfies the following typicality check:
\[
(x_3(m_1), x_2(m_2), y_4(b)) \in T_{c}^{(n)}(X_2, X_3, Y_4)
\]
\[
(x_1(m_1), x_3(b-1), x_2(b-1), y_4(b-1)) \in T_{c}^{(n)}(X_1, X_2, X_3, Y_4)
\]

In each block, node 3 sends source messages from node 1 and node 2. The message \( m_2(b-1) \), though encoded by node 3 in block \( b \) has already been decoded by node 4 in block 4, so \( x_3 \) in (60) effectively depends on \( m_1 \) alone. The probability of error goes to zero if \( R \) satisfies the following conditions:
\[
R_1 < I(X_1; Y_4 | X_2 X_3) + I(X_5; Y_2 | X_1)
\]
\[
= I(X_1 X_3; Y_2 | X_2)
\]
\[
R_2 < I(X_1; Y_4 | X_3)
\]
\[
R_1 + R_2 < I(X_2 X_3; Y_4) + I(X_1; Y_4 | X_2 X_3)
\]
\[
= I(X_1 X_2 X_3; Y_4)
\]
Each of the three inequalities above addresses an error event. The probability that \( m_2 \) is decoded correctly and \( m_1 \) is not, goes to zero if (61) is satisfied. Similarly, the probability that \( m_1 \) is decoded correctly and \( m_2 \) is not, goes to zero if (62) is satisfied. Finally, the probability that both \( m_1 \) and \( m_2 \) are not decoded correctly goes to zero if (63) is satisfied.

B. The second decoding scheme

Set \( L_4 = (\{3\}, \{2\}) \). It follows that \( v(1) = 1 \) since \( LAYER(1) - LAYER(3) = 1 \leq k_{1,3} \) which satisfies (5). Similarly, \( v(2) = 2 \) since \( LAYER(2) - LAYER(3) = 1 \leq k_{2,3} \) which satisfies (5). Then \( g(1, 4) = 1 \rightarrow 3 \rightarrow 4 \) and \( g(2, 4) = 2 \rightarrow 3 \rightarrow 4 \) as depicted in Figure 5(ii).

In block 4, node 3 decodes \( m_3(b) = \{1, m_1(b-1), 2, m_2(b-1)\} \) by finding the unique pair \( m_1 \in \{1, \ldots, 2^n R_1\} \) and \( m_2 \in \{1, \ldots, 2^n R_2\} \) that jointly satisfies the following typicality check:
\[
(x_3(m_1), x_2(m_2), y_4(b)) \in T_{c}^{(n)}(X_2, X_3, Y_4)
\]
\[
(x_1(m_1), x_3(b-1), x_2(b-1), y_4(b-1)) \in T_{c}^{(n)}(X_1, X_2, X_3, Y_4)
\]

C. The third decoding scheme

Set \( L_4 = (\{1, 3\}, \{2\}) \). It follows that \( v(1) = 1 \) since \( LAYER(1) - LAYER(3) = 1 \leq k_{1,3} \) which satisfies (5). Similarly, \( v(2) = 2 \) since \( LAYER(2) - LAYER(3) = 1 \leq k_{2,3} \) which satisfies (5). Then \( g(1, 3) = 1 \rightarrow 4 \) and \( g(2, 4) = 2 \rightarrow 3 \rightarrow 4 \) as depicted in Figure 5(iv).

In block 4, node 5 decodes \( m_5(b) = \{1, m_1(b), 2, m_2(b-1)\} \) by finding the unique pair \( m_1 \in \{1, \ldots, 2^n R_1\} \) and \( m_2 \in \{1, \ldots, 2^n R_2\} \) that jointly satisfies the following typicality checks:
\[
(x_1(m_1), x_2(m_2), y_3(b)) \in T_{c}^{(n)}(X_1, X_3, Y_4)
\]
\[
(x_2(m_2), x_1(b-1), x_3(b-1), y_4(b-1)) \in T_{c}^{(n)}(X_1, X_2, X_3, Y_4)
\]

The message \( m_1(b-1) \) forwarded by node 3 in block 4 has already been decoded by node 4 in block b, so \( x_3 \) in (67) effectively depends on \( m_2 \) alone. The probability of error goes to zero if \( R \) satisfies the following conditions:
\[
R_1 < I(X_1; Y_4 | X_3)
\]
\[
R_2 < I(X_3; Y_4 | X_1)
\]
\[
R_1 + R_2 < I(X_3; Y_4) + I(X_2; Y_4 | X_1 X_3)
\]
\[
= I(X_1 X_2 X_3; Y_4)
\]
D. The achievability of $\mathcal{R}_4(F)$

To show that $\mathcal{R}_3(F)$ is achievable, we show that any rate vector in the region defined by (57)–(59) is in the region defined by (61)–(63) or (64)–(66) or (68)–(69). Suppose $R$ is not in (61)–(63). Since (61) and (63) define the boundaries of $\mathcal{R}_4(F)$, $R$ must violate (62). Then (63) implies that $R$ satisfies (64). If $R$ satisfies (65) the proof is finished since (66) is a boundary of $\mathcal{R}_4(F)$. If $R$ violates (65) then (66) implies that $R$ satisfies (68). Now the proof is finished since (69) and (70) are boundaries of $\mathcal{R}_4(F)$.

APPENDIX C

THE THREE-SOURCE MULTIPLE-ACCESS RELAY CHANNEL

The three-source multiple-access relay channel is defined by the flows $f(1,5) = 1 \xrightarrow{4} 4 \xrightarrow{5}$, $f(2,3) = 2 \xrightarrow{4} 4 \xrightarrow{5}$ and $f(3,5) = 3 \xrightarrow{4} 4 \xrightarrow{5}$ depicted in Figure 6(i), where $S = \{1, 2, 3\}$, $Z = \{4\}$ and $D = \{5\}$. In block b, node 1 encodes $w_1(b) = \{(1, m_1(b))\}$, node 2 encodes $w_2(b) = \{(2, m_2(b))\}$, node 3 encodes $w_3(b) = \{(3, m_3(b))\}$ and node 4 encodes $w_4(b) = \{(1, m_1(b) - 1), (2, m_2(b) - 1), (3, m_3(b) - 1)\}$.

Let $F = \{f(1, 5), f(2, 5), f(3, 5)\}$. By definition, $\mathcal{R}_3(F)$ is set of rate vectors $\vec{R} = (R_1, R_2, R_3)$ that satisfies:

$$R_1 < I(X_1; X_4; Y_5|X_2X_3)$$
$$R_2 < I(X_2; X_4; Y_5|X_1X_3)$$
$$R_3 < I(X_3; X_4; Y_5|X_1X_2)$$
$$R_1 + R_2 < I(X_1X_2; X_4; Y_5)$$
$$R_1 + R_3 < I(X_1X_3; X_4; Y_5)$$
$$R_2 + R_3 < I(X_2X_3; X_4; Y_5)$$
$$R_1 + R_2 + R_3 < I(X_1X_2X_3; X_4; Y_5)$$

The proofs in Appendices A and B rely on the two-dimensional structure of the rate regions in channels with two sources, and do not extend to higher-dimensional regions such as $\mathcal{R}_4(F)$. In general, the number of constraints needed to express the decode-forward region increases exponentially in the number of sources, which is evident when comparing (71)–(77) with (57)–(59). The number of decode-forward schemes required in the proof of achievability is also subject to exponential scaling.

Instead of explicitly characterizing these schemes and piecing their rate regions together as before, we will focus on four particular decode-forward schemes (out of the many other possibilities) in order to illustrate the arguments in Lemmas 5 and 6. These lemmas are the basis for the proof of Theorem 1. Using the notation of layered partitions, the decode-forward schemes of interest are: $L_5 = \{(2, 4), \{1, 3\}\}$, $L_5 = \{(4), \{1, 2, 3\}\}$, $L_5 = \{(2, 3, 4), \{1\}\}$, and $L_5 = \{(2), \{3, 4\}, \{2\}\}$, depicted in Figures 6(ii), (iii), (iv), and (v) respectively.

A. The first decoding scheme

Set $L_5 = \{(4, 2), \{1, 3\}\}$. It follows that $g(1,5) = 1 \xrightarrow{4} 4 \xrightarrow{5}$, $g(2,5) = 2 \xrightarrow{4} 5$, and $g(3,5) = 3 \xrightarrow{4} 4 \xrightarrow{5}$ as depicted in Figure 6(i). In block b, node 5 decodes $\tilde{m}_5(b) = \{(1, m_1(b - 1))\}$. $\tilde{m}_5(b)$ is found by finding the unique triple $m_1 \in \{1, \ldots, 2^nR_1\}$, $m_2 \in \{1, \ldots, 2^nR_2\}$ and $m_3 \in \{1, \ldots, 2^nR_3\}$ that jointly satisfies the following typicality checks:

$$\tilde{x}_5(m_1, m_3), \tilde{Y}_5(b) \in T_\infty^{(n)}(X_2, X_3, Y_5)$$
$$\tilde{x}_1(m_1), \tilde{x}_3(m_3), X_4(b - 1), X_2(b - 1)$$
$$Y_5(b - 1) \in T_\infty^{(n)}(X_1, X_2, X_3, X_4, Y_5)$$

The probability of error goes to zero if $R$ satisfies the following conditions:

$$R_1 < I(X_1; Y_5|X_2X_3X_4) + I(X_4; Y_5)$$
$$R_2 < I(X_2; Y_5|X_1X_3X_4) + I(X_4; Y_5)$$
$$R_3 < I(X_3; Y_5|X_1X_2X_4) + I(X_4; Y_5)$$
$$R_1 + R_2 < I(X_1X_2; Y_5|X_3X_4) + I(X_4; Y_5)$$
$$R_1 + R_3 < I(X_1X_3; Y_5|X_2X_4) + I(X_4; Y_5)$$
$$R_2 + R_3 < I(X_2X_3; Y_5|X_1X_4) + I(X_4; Y_5)$$
$$R_1 + R_2 + R_3 < I(X_1X_2X_3; Y_5|X_4) + I(X_4; Y_5)$$

C. The third decoding scheme

Set $L_5 = \{(2, 4, 3), \{1\}\}$. It follows that $g(1, 5) = 1 \xrightarrow{4} 4 \xrightarrow{5}$, $g(2, 5) = 2 \xrightarrow{4} 5$, and $g(3, 5) = 3 \xrightarrow{4} 5$ as depicted in Figure 6(iv). In block b, node 5 decodes
\( m_5(b) = \{(1, m_1(b-1)), (2, m_2(b)), (3, m_3(b))\} \) by finding the unique triple \( m_1 \in \{1, \ldots, 2^{nR_1}\}, m_2 \in \{1, \ldots, 2^{nR_2}\} \) and \( m_3 \in \{1, \ldots, 2^{nR_3}\} \) that jointly satisfies the following typicality checks:

\[
(\bar{x}_2(m_2), \bar{x}_4(m_1), \bar{x}_3(m_3), \bar{y}_5(b)) \in T_{e_{(n)}}(X_2, X_3, X_4, Y_5) \quad (83)
\]

\[
(\bar{x}_1(m_1), \bar{x}_2(b-1), \bar{x}_4(b-1), \bar{x}_3(b-1), \bar{y}_5(b-1)) \in T_{e_{(n)}}(X_1, X_2, X_3, X_4, Y_5)
\]

The messages \( m_2(b-1) \) and \( m_3(b-1) \) forwarded by node 4 in block 4 have already been decoded by node 5 in block 4, so \( x_4 \) in (83) effectively depends on \( m_1 \) alone. The probability of error goes to zero if \( R \) satisfies the following conditions:

\[
R_1 < I(X_1; Y_5|X_2X_3) + I(X_4; Y_5|X_2X_3)
\]

\[
R_2 < I(X_2; Y_5)
\]

\[
R_3 < I(X_3; Y_5|X_2X_4)
\]

\[
R_1 + R_2 < I(X_1; Y_5|X_2X_3) + I(X_2X_4; Y_5|X_3)
\]

\[
R_1 + R_3 < I(X_1; Y_5|X_2X_3) + I(X_3X_4; Y_5|X_2)
\]

\[
R_2 + R_3 < I(X_2X_3; Y_5|X_4)
\]

\[
R_1 + R_2 + R_3 < I(X_1; Y_5|X_2X_3) + I(X_2X_3X_4; Y_5)
\]

\[
= I(X_1X_2X_3X_4; Y_5)
\]

**D. The fourth decoding scheme**

In block 4, node 5 decodes \( m_5(b) = \{(1, m_1(b-2)), (2, m_2(b)), (3, m_3(b-1))\} \) by finding the unique triple \( m_1 \in \{1, \ldots, 2^{nR_1}\}, m_2 \in \{1, \ldots, 2^{nR_2}\} \) and \( m_3 \in \{1, \ldots, 2^{nR_3}\} \) that jointly satisfies the following typicality checks:

\[
(\bar{x}_2(m_2), \bar{y}_5(b)) \in T_{e_{(n)}}(X_2, Y_5)
\]

\[
(\bar{x}_4(m_1), \bar{x}_3(m_3), \bar{x}_2(b-1), \bar{y}_5(b-1)) \in T_{e_{(n)}}(X_1, X_2, X_3, X_4, Y_5)
\]

The messages \( m_2(b-2) \) and \( m_3(b-2) \) forwarded by node 4 in block 4 have already been decoded by node 5 in block 4, so \( x_4 \) in (87) effectively depends on \( m_1 \) alone. The probability of error goes to zero if \( R \) satisfies the following conditions:

\[
R_1 < I(X_1; Y_5|X_2X_3) + I(X_4; Y_5|X_2X_3)
\]

\[
= I(X_1X_2X_3X_4; Y_5)
\]

\[
R_1 + R_2 < I(X_1; Y_5|X_2X_3) + I(X_4; Y_5|X_2X_3)
\]

\[
= I(X_1X_2X_3X_4; Y_5)
\]

\[
R_1 + R_3 < I(X_1; Y_5|X_2X_3) + I(X_3X_4; Y_5|X_2)
\]

\[
= I(X_1X_2X_3X_4; Y_5)
\]

\[
R_2 + R_3 < I(X_2X_3; Y_5|X_4)
\]

\[
R_1 + R_2 + R_3 < I(X_1; Y_5|X_2X_3) + I(X_2X_3X_4; Y_5)
\]

\[
= I(X_1X_2X_3X_4; Y_5)
\]

Scenarios one and two described below will demonstrate the arguments in Lemma 5 and 8 respectively.
E. Scenario One

Let \( \hat{L}_5 = \{(2, 3, 4), \{1\}\) as depicted in Figure 6(iv). Pick any \( R \in \mathcal{R}_5(F) \) and let \( V = \{2\} \) and \( U = \{2, 3\} \). Recall that \( V \) is the largest subset of \( S \) such that all subsets of \( V \) satisfy \([9]\) and \( U \) is the largest subset of \( S \) that violates \([9]\). By definition of \( U \) and \( V \), \( R \) violates \([84]\) and \([85]\). Set \((F, L_5) = \text{SHIFT}((F, \hat{L}_5), U) \). Then \( L_5' = \{(4), \{1, 2, 3\}\) as depicted in Figure 6(iii). We will check that \( V' = \{1, 2\} \). Suppose, as in the hypothesis of Lemma 5 that there is some \( S \subseteq V' = \{1, 2\} \) that violates \([9]\). There are two cases to consider.

**Case 1**: \( S \cap U = \{\} \). Since \( U = \{2, 3\} \) and \( S \subseteq V' = \{1, 2\} \), this condition implies \( S = \{1\} \). From \([81]\) and the definition of \( S \), \( R_1 > I(X_1; Y_5|X_2X_3X_4) + I(X_4; Y_5) \)

\[ R_1 + R_2 + R_3 > I(X_1; Y_5|X_2X_3X_4) + I(X_4; Y_5) \]

which violates \([86]\) and contradicts the assumption that \( U = \{2, 3\} \) is the largest subset of \( S \) that violates \([9]\) for \((F, \hat{L}_5)\). In this example \( U = \{1, 2\} \). Note that \([87]\) and \([88]\) correspond with \([41]\) and \([42]\) respectively. This argument could be replayed for any \( U' \subseteq S \) such that \( U \subseteq U' \).

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