Tsallis distribution with complex nonextensivity parameter $q$

G. Wilk$^a$, Z.Włodarczyk$^b$

$^a$National Centre for Nuclear Research, Department of Fundamental Research, Hoża 69, 00-681 Warsaw, Poland; e-mail: wilk@fuw.edu.pl
$^b$Institute of Physics, Jan Kochanowski University, Świętokrzyska 15; 25-406 Kielce, Poland; e-mail: zbigniew.wlodarczyk@ujk.kielce.pl

19 March 2014

Abstract: We discuss a Tsallis distribution with complex nonextensivity parameter $q$. In this case the usual distribution is decorated with a log-periodic oscillating factor. Complex $q$ also means complex heat capacity which shall also be briefly discussed.

PACS: 05.70.Ln 05.90.+m 12.40.Ee

Keywords: Scale invariance, Log-periodic oscillation, Complex heat capacity

The two parameter Tsallis distribution,

$$f(X) = C \cdot \left[1 + \frac{X}{mT}\right]^{-m} \quad (1)$$

with scale parameter $T$ (identified in thermodynamical applications with the usual temperature) and with real power index $m = 1/(q - 1)$ ($q$ being known as the parameter of nonextensivity in statistical mechanical approaches) are nowadays very well known and applied in vast variety of situations [1]. For $m \to \infty$ (or $q \to 1$) this power-like distribution coincides with the usual exponential distribution $f(X) = C \exp(-X/T)$. Actually, Tsallis distribution can be regarded as generalizations to real power $m$ (or $q$) of such well known distributions as the Gosset-Student distribution ($X = t^2$, $m = (\nu + 1)/2$ with integer $\nu$, which for $\nu \to \infty$ becomes a Gaussian distribution and for $\nu = 1$ a Cauchy distribution).

In this note we investigate the case when $m$ (or $q$) in Eq. (1) is complex. It turns out that in such a case the Tsallis distribution retains its main quasi-power like form, but this form is now decorated with some specific log-periodic oscillations.
In fact, such behavior has been found in many places, such as earthquakes [2], escape probabilities in chaotic maps close to crisis [3], biased diffusion of tracers on random systems [4], kinetic and dynamic processes on random quenched and fractal media [5], diffusion limited aggregates [6], growth models [7], or stock markets near financial crashes [8], to name only a few examples. However, in all these cases the main distributions were scale free power law ones without any scale parameter (here \( T \)) and without a constant term governing their \( X < mT \) behavior. In the context of nonextensive statistical mechanics log-periodic oscillations have first been observed while analyzing the convergence dynamics of \( z \)-logistic maps [9].

We illustrate our point by an example of recent results obtained for the highest presently available energies of 7 TeV in two experiments performed at the Large Hadron Collider at CERN, namely CMS [10] and ATLAS [11]. In Fig. 1a we show the observed transverse momentum (\( p_T \)) distributions for secondaries produced in pp collisions in these experiments \( \star \). Albeit both fits look pretty good, closer inspection shows that the ratio of data/fit is not flat. It shows some kind of visible oscillations, cf. Fig. 1b. It turns out that these oscillations cannot be compensated or erased by any reasonable change of fitting parameters. Instead, to account for them distributions \( f(p_T) \) from Eq. (1) have to be multiplied by some log-periodic oscillating factor \( \star \star \):

\[
R(E) = a + b \cos [c \ln(E + d) + f].
\]

To explain the origin of such a dressing factor (and tacitly assuming that it is not an experimental artifact, as it was observed in both experiments), start from the known observation that, whereas Boltzmann-Gibbs (BG) distribution,

\[
f(E) = \frac{1}{T} \exp \left( -\frac{E}{T} \right),
\]

\( \star \)

These secondaries were produced at midrapidity, i.e., for \( y = \frac{1}{2} \ln \frac{E + p_L}{E - p_L} \approx 0 \) for which, for large transverse momentum, \( p_T > M \) (where \( M \) is mass of the particle), one has that, approximately, the energy of particle, \( E = \sqrt{M^2 + p_T^2} \cosh(y) \approx p_T \), i.e., it practically coincides with \( p_T \) (\( p_L = \sqrt{M^2 + p_L^2} \sinh(y) \) is longitudinal momentum of observed particle.)

\( \star \star \)

Detailed analysis of this phenomenon in the available high energy experimental data will be presented elsewhere.
Fig. 1. (Color online) (a) Fit to $p_T$ data for pp collisions at 7 TeV from CMS [10] and ATLAS [11] experiments using distribution (1) with parameters used $T = 0.145$ GeV and $m = 6.7$. Data points for the CMS experiment are scaled by a factor of 10 for better readability. (b) Fit to $p_T$ dependence of data/fit ratio for results presented in the left panel (a) using the function $R$ from Eq. (2) with: $a = 0.909$, $b = 0.166$, $c = 1.86$, $d = 0.948$ and $f = -1.462$.

comes from the simple equation,

$$\frac{df(E)}{dE} = -\frac{1}{T}f(E),$$

(4)

with the scale parameter $T$ being constant, the same equation, but with variable scale parameter in the form (known as preferential attachment in networks [12]),

$$T = T(E) = T_0 + \frac{E}{n},$$

(5)

is now,

$$\frac{df(E)}{dE} = -\frac{1}{T(E)}f(E) = -\frac{1}{T_0 + E/n}f(E),$$

(6)

and results in the Tsallis distribution

$$f(E) = \frac{n-1}{nT_0} \left(1 + \frac{E}{nT_0}\right)^{-n}.$$  

(7)

Now write Eq. (6) in the finite difference form, namely as

$$f(E + dE) = \frac{-ndE + nT + E}{nT + E} f(E),$$

(8)

and consider a situation in which $dE$ is not going to zero but always remains finite (albeit, depending on the value of the new scale parameter $\alpha$, it can be
very small) and equal to
\[ dE = \alpha nT(E) = \alpha(nT + E) \] (9)
(because one expects that changes \(dE\) are of the order of the temperature \(T\),
the scale parameter must be limited by \(1/n, \alpha < 1/n\)). In this case
\[ f[E + \alpha(nT + E)] = (1 - \alpha n)f(E). \] (10)
It can be further shown that Eq. (10), when expressed in a new variable \(x\),
\[ x = \left(1 + \frac{E}{nT}\right), \] (11)
formally corresponds to the following scale invariant relation:
\[ g[(1 + \alpha)x] = (1 - \alpha n)g(x). \] (12)
Now, it is known [13] that, if for some function \(O(x)\), one finds that \(O(x) = \mu O(\lambda x)\) then it is scale invariant and its form follows a simple power law,
\(O(x) = Cx^{-m}\) with \(m = \ln \mu/\ln \lambda\). This relation can be written as \(\mu \lambda^{-m} = 1 = e^{i2\pi k}\), where \(k\) is an arbitrary integer. It means therefore that, in general,
\(m = -\ln \mu/\ln \lambda + i2\pi k/\ln \lambda\), i.e., is a complex number, the imaginary part of which signals a hierarchy of scales leading to log-periodic oscillations. Coming now back to Eq. (12), in general,
\[ g(x) = x^{-m_k}, \quad m_k = -\frac{\ln(1 - \alpha n)}{\ln(1 + \alpha)} + ik\frac{2\pi}{\ln(1 + \alpha)}. \] (13)
The special case of \(k = 0\), i.e., the usual real power law solution with \(m_0\)
corresponding to fully continuous scale invariance[*][*][*][*], recovers in the limit \(\alpha \to 0\) the power \(n\) in the usual Tsallis distribution. In general one has
\[ g(x) = \sum_{k=0} w_k \cdot \text{Re} \left(x^{-m_k}\right) = x^{-\text{Re}(m_k)} \sum_{k=0} w_k \cdot \cos \left[\text{Im} (m_k) \ln(x)\right]. \] (14)
One therefore obtains a Tsallis distribution decorated by a weighted sum of log-oscillating factors (where \(x\) is given by Eq. (11)). Because usually in practice we do not a priori know the details of the dynamics of processes under consideration

[*][*][*][*] In this case power law exponent \(m_0\) still depends on \(\alpha\) and increases with it roughly as \(m_0 \simeq n + \frac{n}{2}(n + 1)\alpha + \frac{n}{12} \left(4n^2 + 3n - 1\right)\alpha^2 + \frac{n}{24} \left(6n^3 + 4n^2 - n + 1\right)\alpha^3 + \ldots\). Notice also that \(\alpha < 1/n\).
(i.e., we do not know the weights \( w_k \)), for fitting purposes one usually uses only \( k = 0 \) and \( k = 1 \). In this case one has, approximately,

\[
g(E) \simeq \left(1 + \frac{E}{nT}\right)^{-m_0} \left\{ w_0 + w_1 \cos \left[ \frac{2\pi}{\ln(1 + \alpha)} \ln \left(1 + \frac{E}{nT}\right) \right] \right\} \tag{15}
\]

and reproduces the general form of a dressing factor given by Eq. (2) and often used in the literature [13].

However, this is not the most general result possible. Notice that in our derivation presented by Eqs. (9)–(12) we only accounted for a single step evolution, whereas in reality we can have a whole hierarchy of evolutions. Then one has that

\[
E_i = E_{i-1} + \alpha_{i-1} (nT + E_{i-1}), \tag{16}
\]

each with its own scale parameter \( \alpha_i \). In the simplest situation, neglecting any fluctuations of consecutive scaling parameters, i.e., assuming that all \( \alpha_i = \alpha \), after \( \kappa \) steps one has that

\[
nT + E_\kappa = (1 + \alpha)^\kappa (nT + E_0). \tag{17}
\]

This means then that Eq. (12) should be replaced by a new scale invariant equation:

\[
g [(1 + \alpha)\kappa x] = (1 - \alpha n)^\kappa g(x). \tag{18}
\]

Whereas this equation does not change the slope parameter \( m_0 \), it significantly influences the frequency of oscillations which are now \( \kappa \) times smaller,

\[
c = \frac{2\pi}{\kappa \ln(1 + \alpha)}. \tag{19}
\]

For more complex behavior of intermediate scale parameters \( \alpha_i \) one gets more complicated expressions (we shall not discuss this here).

There are other consequences of allowing the parameter \( m \) to be complex. Namely, the complex power exponent in the Tsallis distribution, \( m_k = m' + i\cdot m'' \), also means a complex nonextensivity parameter \( q \).
Now, the complex nonextensive parameter $q$ has some profound consequences. This is because, as shown in [14] (and before in [15,16,17]), the nonextensivity parameter $q$ can be treated as a measure of the thermal bath heat capacity $C$ with

\[
C = \frac{1}{q - 1}.
\]  

It means therefore that, in general, the heat capacity becomes complex as well. As a matter of fact, such complex (frequency dependent) heat capacities (or generalized calorimetric susceptibilities) are known in the literature \[18\] under the form

\[
C = C' - iC'' = C_\infty + \frac{C_0 - C_\infty}{1 + (\omega \tau)^2} (1 - i\omega \tau).
\]

Here $C_\infty$ is the heat capacity related to the infinitely fast degrees of freedom of the system as compared to the frequency $\omega$, and $C_0$ is the total contribution at equilibrium (the frequency is set to zero) of the degrees of freedom, fast and slow, of the sample. The time constant $\tau$ is the kinetic relaxation time constant of a certain internal degree of freedom.

These complex heat capacities are known as dynamic heat capacities and are intensively explored from both experimental and theoretical perspectives. It is expected that dynamic calorimetry can provide an insight into the energy landscape dynamics, cf., for example, \[19\|20\|21\|22\]. Usually one associates the imaginary part of linear susceptibility with the absorption of energy by the sample from the applied field.

In the case of temperature fluctuations $\delta T(t)$ the deviation of the energy from its equilibrium value $\delta U(t)$ is, for a certain linear operator $\hat{C}(t)$, some linear function of the corresponding variation of the temperature,

\[
\delta U(t) = \hat{C} \delta T(t).
\]

If the temperature of the reservoir changes infinitely slowly in time, then the system can keep up with any changes in the reservoir and its susceptibility is just the specific heat of the system $C_V$. However, in general, the behavior of the
system is described by a generalized susceptibility $C_V(\omega)$, which can be called the complex and $\omega$-dependent heat capacity of the system.

A complex $C_V(\omega)$ means that $\delta U$ and $\delta T$ are shifted in phase and that the entropy production in the system differs from zero [22]. The corresponding fluctuation-dissipation theorem for the frequency dependent heat capacity was established in [21]. According to this result, the frequency-dependent heat capacity may be expressed within the linear response approximation as a linear susceptibility describing the response of the system to arbitrarily small temperature perturbations away from equilibrium,

$$C_V(\omega) = \frac{1}{T_0^2} \left( \langle U^2 \rangle_0 - i\omega \int_0^\infty dt e^{-i\omega t} \langle U(0)U(t) \rangle \right)$$

(25)

(the $\omega$ denotes frequency with which temperature field is varying with time).

The above results for heat capacity can now be used to a new phenomenological interpretation of the complex $q$ parameter discussed before. Namely, one can argue that (we denote now $T_0 = \langle T \rangle$)

$$q - 1 = \frac{\text{Var}(T)}{\langle T \rangle^2} - i\frac{S(T)}{(\langle T \rangle^2)}$$

(26)

where

$$S(T) = \omega \int \langle \text{Cov}[T(0), T(t)] \rangle e^{-i\omega t} dt$$

(27)

is the spectral density of temperature fluctuations (i.e., the Fourier transform of the covariance function averaging over the nonequilibrium density matrix).

To summarize: Log-periodic structures in the data indicate that the system and/or the underlying physical mechanisms have characteristic scale invariance behavior. This is interesting as it provides important constraints on the underlying physics. The presence of log-periodic features signals the existence of important physical structures hidden in the fully scale invariant description. It is important to recognize that Eq. (6) represents an averaging over highly `non-

We would like to stress at this point that, in a sense, Eq. (26) can be regarded as a generalization of our old proposition for interpreting $q$ as a measure of nonstatistical intrinsic fluctuations in the system under consideration [23] (which corresponds to the real part of (26)) by adding the effect of spectral density of such fluctuations (via the imaginary part of (26)).
smooth’ processes and, in its present form, suggests rather smooth behavior. In reality, there is a discrete time evolution for the number of steps. To account for this fact, one replaces a differential Eq. (4) by a difference quotient and expresses $dt$ as a discrete step approximation given by Eq. (9) with parameter $\alpha$ being a characteristic scale ratio. It can also be shown that discrete scale invariance and its associated complex exponents can appear spontaneously, without a pre-existing hierarchical structure.

Acknowledgements

This research was supported in part by the National Science Center (NCN) under contract DEC-2013/09/B/ST2/02897. We would like to warmly thank Dr Eryk Infeld for reading this manuscript.

References

[1] C. Tsallis, J. Statist. Phys. 52 (1988) 479 (1988); Eur. Phys. J. A 40 (2009) 257 (2009) and Introduction to Nonextensive Statistical Mechanics (Springer, 2009). For an updated bibliography on this subject, see http://tsallis.cat.cbpf.br/biblio.htm.

[2] Y. Huang, H. Saleur, C. Sammis and D. Sornette, Europhys. Lett. 41 (1998) 43; H. Saleur, C. G. Sammis and D. Sornette, J. Geophys. Res. 101 (1996) 17661.

[3] A. Krawiecki, K. Kacperski, S. Matyjaskiewicz and J. A. Holyst, Chaos, Solitons, Fractals 18 (2003) 89.

[4] J. Bernasconi and W. R. Schneider, J. Stat. Phys. 30 (1983) 355; D. Stauffer and D. Sornette, Physica A 252 (1998) 271; D. Stauffer, Physica A 266 (1999) 35.

[5] B. Kutnjak-Urbanc, S. Zapperi, S. Milosevic and H. E. Stanley, Phys. Rev. E 54 (1996) 272; R. F. S. Andrade, Phys. Rev. E, 61 (2000) 7196; M. A. Bab, G. Fabricius and E. V. Albano, Phys. Rev. E 71 (2005) 36139; H. Saleur and D. Sornette, J. Phys. I 6 (1996) 327.

[6] D. Sornette, A. Johansen, A. Arneodo, J.-F. Muzy and H. Saleur, Phys. Rev. Lett. 76 (1996) 251.

[7] Y. Huang, G. Ouillon, H. Saleur and D. Sornette, Phys. Rev. E 55 (1997) 6433.

[8] D. Sornette, A. Johansen and J.-P. Bouchaud, J. Phys. I 6 (1996) 167; N. Vanderwalle, Ph. Boveroux, A. Minguet and M. Ausloos, Physica A 255 (1998) 201; N. Vanderwalle and M. Ausloos, Eur. J. Phys. B 4 (1998) 139; N. Vanderwalle, M. Ausloos, Ph. Boveroux and A. Minguet, Eur. J. Phys. B 9 (1999) 355; J. H. Wosnitza and J. Leker, Physica A 401 228.

[9] F. A. B. F. de Moura, U. Tirnakli and M. L. Lyra, Phys. Rev. E 62 (2000) 6361.
[10] V. Khachatryan et al. (CMS Collaboration), JHEP 02 (2010) 041 and JHEP 08 (2011) 086; Phys. Rev. Lett. 105 (2010) 022002.

[11] G. Aad et al. (ATLAS Collaboration), New J. Phys. 3 (2011) 053033.

[12] G. Wilk and Z. Włodarczyk, Acta Phys. Polon. B 35 (2004) 871 and B 36 (2005) 2513; Eur. Phys. J. A 48 (2012) 162.

[13] D. Sornette, Phys. Rep. 297 (1998) 239.

[14] T. S. Biró, G. G. Barnaföldi and P. Ván, Eur. Phys. J. A 49 (2013) 110.

[15] M. Campisi, Phys. Lett. A 366 (2007) 335.

[16] A. R. Plastino and A. Plastino, Phys. Lett. A 193 (1994) 140; M. P. Almeida, Physica A 300 (2001) 424.

[17] G. Wilk and Z. Włodarczyk, Eur. Phys. J. A 40 (2009) 299; 48 (2012) 161; Cent. Eur. J. Phys. 10 (2012) 568.

[18] J. E. K. Schawe, Thermochim. Acta 260 (1995) 1; J.-L. Garden, Thermochimica Acta 460 (2007) 85.

[19] J. L. Garden, Thermochim. Acta 452 (2007) 85.

[20] J. L. Garden and J. Richard, Thermochim. Acta 461 (2007) 57.

[21] J. K. Nielsen and C. Dyre, Phys. Rev. B 54 (1996) 15754.

[22] G. I. Salistra, Sov. Phys. JETP 26 (1968) 173.

[23] G. Wilk and Z. Włodarczyk, Phys. Rev. Lett. 84 (2000) 2770; T. S. Biró and A. Jakovác, Phys. Rev. Lett. 94 (2005) 132302.