Hardy–Littlewood and Ulyanov inequalities

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Abstract. We give the full solution of the following problem: obtain sharp inequalities between the moduli of smoothness \( \omega_\alpha(f,t)_q \) and \( \omega_\beta(f,t)_p \) for \( 0 < p < q \leq \infty \). A similar problem for the generalized \( K \)-functionals and their realizations between the couples \( (L_p,W^\psi_p) \) and \( (L_q,W^\varphi_q) \) is also solved.

The main tool is the new Hardy–Littlewood–Nikol’skii inequalities. More precisely, we obtained the asymptotic behavior of the quantity
\[
\sup_{T_n} \frac{\|D(\psi)(T_n)\|_q}{\|D(\varphi)(T_n)\|_p}, \quad 0 < p < q \leq \infty,
\]
where the supremum is taken over all nontrivial trigonometric polynomials \( T_n \) of degree at most \( n \) and \( D(\psi), D(\varphi) \) are the Weyl-type differentiation operators.

We also prove the Ulyanov and Kolyada-type inequalities in the Hardy spaces. Finally, we apply the obtained estimates to derive new embedding theorems for the Lipschitz and Besov spaces.

2010 Mathematics Subject Classification. Primary 41A63, 42B15, 26D10; Secondary 46E35, 26A33, 41A17, 26C05, 41A10.

Key words and phrases. Moduli of smoothness, \( K \)-functionals, Ulyanov’s inequalities, Hardy–Littlewood–Nikol’skii’s inequalities, embedding theorems, fractional derivatives.

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Contents

1. Introduction 5
2. Auxiliary results 15
3. Polynomial inequalities of Nikol’skii–Stechkin–Boas–types 22
4. Basic properties of fractional moduli of smoothness 28
5. Hardy–Littlewood–Nikol’skii inequalities for trigonometric polynomials 33
6. General form of the Ulyanov inequality for moduli of smoothness, $K$-functionals, and their realizations 54
7. Sharp Ulyanov inequalities for $K$-functionals and realizations 58
8. Sharp Ulyanov inequalities for moduli of smoothness 70
9. Sharp Ulyanov inequalities for realizations of $K$-functionals related to Riesz derivatives and corresponding moduli of smoothness 77
10. Sharp Ulyanov inequality via Marchaud inequality 80
11. Sharp Ulyanov and Kolyada inequalities in Hardy spaces 87
12. $(L_p, L_q)$ inequalities of Ulyanov-type involving derivatives 95
13. Embedding theorems for function spaces 100
References 110
Basic notation

Let $\mathbb{R}^d$ be the $d$-dimensional Euclidean space with elements $x = (x_1, \ldots, x_d)$, and $(x, y) = x_1 y_1 + \cdots + x_d y_d$, $|x| = (x, x)^{1/2}$, $|x|_1 = |x_1| + \cdots + |x_d|$, $|x|_\infty = \max_{1 \leq j \leq d} |x_j|$, and $\mathbb{T}^d = \mathbb{R}^d / 2\pi \mathbb{Z}^d$.

In what follows, $X = \mathbb{R}^d$ or $X = \mathbb{T}^d$. As usual, the space $L_p(X)$ consists of all measurable functions $f$ such that for $0 < p < \infty$

$$\|f\|_{L_p(X)} = \left( \int_X |f(x)|^p dx \right)^{1/p} < \infty$$

and for $p = \infty$

$$\|f\|_{L_\infty(X)} = \text{ess sup}_{x \in X} |f(x)| < \infty.$$

Note that $\|f\|_{L_p(X)}$ for $0 < p < 1$ is a quasi-norm satisfying $\|f + g\|_{L_p(X)}^p \leq \|f\|_{L_p(X)}^p + \|g\|_{L_p(X)}^p$. In this paper, we mostly deal with the $L_p(\mathbb{T}^d)$-setting. In this case, for simplicity, we write $\|f\|_p = \|f\|_{L_p(\mathbb{T}^d)}$.

We denote by $T_n$ the space of all trigonometric polynomials of order at most $n$, i.e.,

$$T_n = \text{span} \left\{ e^{i(k, x)}, \ k \in \mathbb{Z}^d, \ |k|_\infty \leq n \right\}.$$

Also, set

$$\mathcal{T} = \bigcup_{n \geq 0} T_n$$

and

$$\mathcal{T}' = \left\{ T \in T_n : T(x) \neq 0 \text{ and } \int_{\mathbb{T}^d} T(x) dx = 0 \right\}.$$

The Fourier transform of $f \in L_1(\mathbb{R}^d)$ is defined by

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(x) e^{-i(x, \xi)} dx.$$

Throughout the paper, we use the notation

$$A \lesssim B,$$

with $A, B \geq 0$, for the estimate $A \leq C B$, where $C$ is a positive constant independent of the essential variables in $A$ and $B$ (usually, $f$, $\delta$, and $n$). If $A \lesssim B$ and $B \lesssim A$ simultaneously, we write $A \asymp B$ and say that $A$ is equivalent to $B$. For two function spaces $X$ and $Y$, we will use the notation

$$Y \hookrightarrow X$$

if $Y \subset X$ and $\|f\|_X \lesssim \|f\|_Y$ for all $f \in Y$.

Let $1 \leq p \leq \infty$ and $0 < q \leq \infty$. In what follows, $p'$ and $q_1$ are given by

$$\frac{1}{p} + \frac{1}{p'} = 1 \quad \text{and} \quad q_1 := \begin{cases} q, & q < \infty; \\ 1, & q = \infty. \end{cases}$$
Finally, for any real number $a$, $[a]$ be the floor function, that is, $[a]$ is the largest integer not greater than $a$ and let

$$a_+ = \begin{cases} a, & a \geq 0; \\ 0, & \text{otherwise.} \end{cases}$$
1. Introduction

1.1. Ulyanov inequalities for moduli of smoothness. The problem of estimating the moduli of smoothness of a function in $L_q$ in terms of its moduli of smoothness in $L_p$ has a long history. The starting point was a study of embeddings of the Lipschitz spaces

\[ \text{Lip}(\alpha, p) = \left\{ f \in L_p(T) : \| f(x + h) - f(x) \|_{L_p(T)} = O(h^\alpha) \right\} \]

(E. Titchmarsh [103], G. H. Hardy and J. E. Littlewood [35], and S. M. Nikol'skii [69]; see also the references in [7, 49, 52, 109]). In particular, the classical Hardy–Littlewood embedding (see [35])

\[ \text{Lip}(\alpha, p) \hookrightarrow \text{Lip}(\alpha - \theta, q), \]

where

\[ 1 \leq p < q < \infty, \quad \theta = \frac{1}{p} - \frac{1}{q}, \quad \theta < \alpha \leq 1, \]

can easily be obtained from the $(L_p, L_q)$ inequality for moduli of smoothness proved by Ulyanov [109]

\[ \omega_k(f, \delta)_{L_q(T)} \lesssim \left( \int_0^\delta \left( t^{-\theta} \omega_k(f, t)_{L_p(T)} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}, \]

where

\[ 1 \leq p < q \leq \infty, \quad \theta = \frac{1}{p} - \frac{1}{q}. \]

Nowadays, inequality (1.1) is known as Ulyanov’s inequality.

Here and in what follows, the $k$-th order modulus of smoothness \( \omega_k(f, \delta)_{L_p(T^d)} \) is defined in the standard way by

\[ \omega_k(f, \delta)_{L_p(T^d)} := \sup_{|h| \leq \delta} \| \Delta_h^k f \|_{L_p(T^d)}, \]

where

\[ \Delta_h f(x) = f(x + h) - f(x), \quad \Delta_h^k = \Delta_h \Delta_h^{k-1}, \quad h \in \mathbb{R}^d, \quad d \geq 1. \]

It was known long ago that the straightforward application of the Ulyanov inequality (1.1) for $C^\infty$-functions gives only the estimate \( \omega_k(f, \delta)_{L_q(T^d)} = O(\delta^{-\theta}) \), which differs substantially from the best possible estimate \( \omega_k(f, \delta)_{L_q(T^d)} = O(\delta^{\theta}) \).

To overcome this shortcoming, recently (see [87, 104]) the sharp Ulyanov inequality was established using the fractional moduli of smoothness. We recall the definition of the modulus of smoothness \( \omega_\alpha(f, \delta)_{L_p(T^d)} \) of fractional order \( \alpha > 0 \) of a function \( f \in L_p(T^d) \), \( 0 < p \leq \infty \), which appeared for the first time in 1970’s (see [12, p. 788], [97]):

\[ \omega_\alpha(f, \delta)_{L_p(T^d)} := \sup_{|h| \leq \theta} \| \Delta_h^\alpha f \|_{L_p(T^d)}, \]

where

\[ \Delta_h^\alpha f(x) = \sum_{\nu=0}^\infty (-1)^\nu \binom{\alpha}{\nu} f \left( x + (\alpha - \nu)h \right), \quad h \in \mathbb{R}^d, \]
and \((\alpha) = \alpha(\alpha-1)\ldots(\alpha-\nu+1)\), \((\alpha) = 1\). It is clear that for integer \(\alpha\) we deal with the classical definition \((1.2)\). Also, in the case of \(0 < p < 1\), since
\[
\sum_{\nu=0}^{\infty} \left| \left( \frac{\alpha}{\nu} \right) \right|^{p} < \infty \quad \text{for} \quad \alpha \in \mathbb{N} \cup ((1/p - 1), \infty),
\]
it is natural to assume that \(\alpha \in \mathbb{N} \cup ((1/p - 1), \infty)\) while defining the fractional modulus of smoothness in \(L_{p}\). Note that the moduli of smoothness of fractional order satisfy all usual properties of the classical moduli of smoothness (see Section 4).

The sharp Ulyanov inequality for the fractional moduli of smoothness for \(1 < p < q < \infty\) reads as follows:
\[
\omega_{\alpha}(f, \delta)_{L_{q}(\mathbb{T})} \lesssim \left( \int_{0}^{\delta} \left( t^{-\theta} \omega_{\alpha+\theta}(f, t)_{L_{p}(\mathbb{T})} \right)^{\frac{q}{p}} \frac{dt}{t} \right)^{\frac{1}{q}}, \quad \theta = d \left( \frac{1}{p} - \frac{1}{q} \right).
\]
Moreover, this inequality is sharp over the class
\[
\text{Lip} (\omega(\cdot), \alpha + \theta, p) = \left\{ f \in L_{p}(\mathbb{T}) : \omega_{k+\theta}(f, \delta)_{p} = O (\omega(\delta)) \right\}
\]
(see [87]). More precisely, for any function \(\omega \in \Omega_{\alpha+\theta}\) (i.e., such that \(\omega(\delta)\) is non-decreasing and \(\delta^{-\alpha-\theta} \omega(\delta)\) is non-increasing), there exists a function
\[
f_{0}(x) = f_{0}(x, p, \omega) \in \text{Lip} (\omega(\cdot), \alpha + \theta, p)
\]
such that for any \(q \in (p, \infty)\) and for any \(\delta > 0\)
\[
\omega_{\alpha}(f_{0}, \delta)_{L_{q}(\mathbb{T})} \geq C \left( \int_{0}^{\delta} \left( t^{-\theta} \omega(t) \right)^{\frac{q}{p}} \frac{dt}{t} \right)^{\frac{1}{q}},
\]
where a constant \(C\) is independent of \(\delta\) and \(\omega\).

Nowadays the Ulyanov inequalities are used in approximation theory, function spaces, and interpolation theory (see, e.g., [7, 15, 19, 31, 32, 37, 48, 49, 51, 52, 53, 66, 76, 87, 99, 104]). Very recently, sharp Ulyanov inequalities were shown between the Lorentz–Zygmund spaces \(L_{p,r}(\log L)^{\alpha-\gamma}\), \(\alpha, \gamma > 0\), and \(L_{q,s}(\log L)^{\alpha}\) over \(\mathbb{T}^{d}\) in the case \(1 < p \leq q < \infty\), and the corresponding embedding results were established. In the case of quasi-normed Lebesgue spaces \(L_{p}, 0 < p < 1\), an Ulyanov inequality was established only in its not-sharp form given by \((1.1)\) (see [24, 93] and [15, Ch. 3.7]).

It is known (see [99]) that in the limiting cases \(p = 1, q < \infty\) or \(1 < p, q = \infty\), inequality \((1.5)\) does not hold in general. The best possible inequality is given by
\[
\omega_{\alpha}(f, \delta)_{L_{q}(\mathbb{T})} \lesssim \left( \int_{0}^{\delta} \left( t^{-\theta} \ln 2/t \right)^{\max(1/p', 1/q)} \omega_{\alpha+\theta}(f, t)_{L_{p}(\mathbb{T})}^{\frac{q}{p}} \frac{dt}{t} \right)^{\frac{1}{q}},
\]
where \((\ln 2/t)^{\max(1/p', 1/q)}\) cannot be replaced by \(o(\ln 2/t)^{\max(1/p', 1/q)}\).

Interestingly, in the case \(p = 1\) and \(q = \infty\), inequality \((1.5)\) is valid in the one-dimensional case (see [99]). Note that in this case \(\theta = 1\) and one can use the classical (non fractional) moduli of smoothness.
Another improvement of the Ulyanov inequality (1.1) was suggested by Kolyada [48]:

\[
\delta^{\alpha-\theta} \left( \int_{\delta}^{1} \left( t^{\theta-\alpha} \omega_{\alpha}(f, t)_{L_{q}(\mathbb{T})} \right)^{p} \frac{dt}{t} \right)^{\frac{1}{p}} \lesssim \left( \int_{0}^{\delta} \left( t^{\theta-\alpha} \omega_{\alpha}(f, t)_{L_{p}(\mathbb{T})} \right)^{q} \frac{dt}{t} \right)^{\frac{1}{q}},
\]

where \( 1 < p < q < \infty \) and \( \theta = 1/p - 1/q \). Unlike the one-dimensional case, in the multidimensional case \( (d \geq 2, \theta = d(1/p - 1/q)) \), the corresponding inequality also holds for \( p = 1 < q < \infty \).

Roughly speaking, the sharp Ulyanov inequality refines inequality (1.1) with respect to the right-hand side while the Kolyada inequality refines (1.1) with respect to the left-hand side. Note that the Kolyada inequality is not comparable to the sharp Ulyanov inequality.

In this paper, one of our main goals is to study sharp \((L_{p}, L_{q})\) inequalities of Ulyanov-type with variable parameters for moduli of smoothness, generalized \(K\)-functionals, and their realizations. More precisely, we solve the long-standing problem (see [49, 109]) of finding the sharp inequalities between the moduli of smoothness

\[
\omega_{\alpha}(f, t)_{L_{q}(\mathbb{T}^{d})} \quad \text{and} \quad \omega_{\beta}(f, t)_{L_{p}(\mathbb{T}^{d})}
\]

in the general case: \( 0 < p < q \leq \infty \) and \( d \geq 1 \).

One of our main contributions is the full description of sharp Ulyanov inequalities given by the following result.

**Theorem 1.1.** Let \( f \in L_{p}(\mathbb{T}^{d}) \), \( 0 < p < q \leq \infty \), \( \alpha \in \mathbb{N} \cup ((1 - 1/q)_{+}, \infty) \), and \( \alpha + \gamma \in \mathbb{N} \cup ((1/p - 1)_{+}, \infty) \). Then, for any \( \delta \in (0, 1) \), we have

\[
\omega_{\alpha}(f, \delta)_{L_{q}(\mathbb{T}^{d})} \lesssim \frac{\omega_{\alpha+\gamma}(f, \delta)_{L_{p}(\mathbb{T}^{d})}}{\delta^{\gamma}} \left( \frac{1}{\delta} \right) + \left( \int_{0}^{\delta} \left( \frac{\omega_{\alpha+\gamma}(f, t)_{L_{p}(\mathbb{T}^{d})}}{t^{d(\frac{1}{p} - \frac{1}{q})}} \right)^{q_{1}} \frac{dt}{t} \right)^{\frac{1}{q_{1}}},
\]

where

(1) if \( 0 < p \leq 1 \) and \( p < q \leq \infty \), then

\[
\sigma(t) := \begin{cases} 
  t^{d(\frac{1}{p} - 1)}, & \gamma > d \left( 1 - \frac{1}{q} \right)_{+}; \\
  t^{d(\frac{1}{p} - 1)}, & \gamma = d \left( 1 - \frac{1}{q} \right)_{+} \geq 1, d \geq 2, \text{ and } \alpha + \gamma \in \mathbb{N}; \\
  t^{d(\frac{1}{p} - 1) \ln \frac{1}{q} (t + 1)} \ln^{\frac{1}{q}}(t + 1), & \gamma = d \left( 1 - \frac{1}{q} \right)_{+} \geq 1, d \geq 2, \text{ and } \alpha + \gamma \notin \mathbb{N}; \\
  t^{d(\frac{1}{p} - 1) \ln \frac{1}{q} (t + 1)}, & 0 < \gamma = d \left( 1 - \frac{1}{q} \right)_{+} = 1 \text{ and } d = 1; \\
  t^{d(\frac{1}{p} - 1) \ln \frac{1}{q} (t + 1)}, & 0 < \gamma = d \left( 1 - \frac{1}{q} \right)_{+} < 1; \\
  t^{d(\frac{1}{p} - \frac{1}{q}) \gamma}, & 0 < \gamma < d \left( 1 - \frac{1}{q} \right)_{+}; \\
  t^{d(\frac{1}{p} - \frac{1}{q})}, & \gamma = 0,
\end{cases}
\]
(2) if $1 < p \leq q \leq \infty$, then

$$\sigma(t) := \begin{cases} 
1, & \gamma \geq d \left(\frac{1}{p} - \frac{1}{q}\right), \quad q < \infty; \\
1, & \gamma > \frac{d}{p}, \quad q = \infty; \\
\ln \frac{1}{\sigma^\prime}(t + 1), & \gamma = \frac{d}{p}, \quad q = \infty; \\
t^{d \left(\frac{1}{p} - \frac{1}{q}\right) - \gamma}, & 0 \leq \gamma < d \left(\frac{1}{p} - \frac{1}{q}\right).
\end{cases}$$

We make several comments on this result. The result is sharp in the sense that, for given $p$ and $q$ such that $p < q$, there exists a non-trivial function $f_0 \in L_q(\mathbb{T}^d)$ for which the left-hand side of (1.7) is equivalent to the right-hand side (see Section 8).

We see that the critical values of the parameter $\gamma$ in the definition of $\sigma$ in the cases $p > 1$ and $p \leq 1$ are different: $\gamma = d(1/p - 1/q)$ and $\gamma = d(1 - 1/q)$, correspondingly.

It is important to mention that the sharp form of the Ulyanov inequalities, that is, an optimal choice of the function $\sigma$, essentially depends on the method we choose to measure the smoothness or, in other words, how we define a modulus of smoothness. For simplicity, we explain this in the one dimensional case. On the one hand, it is well known that the classical modulus of smoothness is equivalent to the $K$-functional (for $1 \leq p \leq \infty$) and the realization functional (for $0 < p \leq \infty$) defined by means of the classical Weyl derivative. In this case, the sharp Ulyanov inequality is given by Theorem 1.1. On the other hand, if instead of the Weyl derivative in the definition of the realization concept, we consider the Riesz derivative, then the corresponding modulus of smoothness of order $\alpha > 0$ is given by (see (82))

$$\omega_\alpha(f, \delta)_{L_p(\mathbb{T})} = \sup_{0 < h \leq \delta} \left\| \sum_{\nu \in \mathbb{Z} \setminus \{0\}} \left( -\frac{\beta_{\nu}}{\beta_\alpha} (\alpha) \right) f(t + \nu h) - f(t) \right\|_{L_p(\mathbb{T})},$$

where

$$\beta_m(\alpha) = \sum_{j=m}^{\infty} (-1)^{j+1} 2^{-2j} \binom{\alpha/2}{j} \binom{2j}{j-m}, \quad m \in \mathbb{Z}_+.$$ 

For the modulus (1.8), a sharp form of Ulyanov inequality in critical cases, which is of most interest, is different that the one given in Theorem 1.1. In particular, for the limiting parameters $0 < p \leq 1$ and $q = \infty$ (in this case $\gamma = 1$), the corresponding inequality reads as follows (see Section 9)

$$\omega_\alpha(f, \delta)_{L_q(\mathbb{T})} \lesssim \omega_\alpha^{(\alpha+1)}(f, \delta)_{L_p(\mathbb{T})} \ln \left( \frac{1}{\delta} + 1 \right) + \int_0^\delta \left( \frac{\omega_\alpha^{(\alpha+1)}(f, t)_{L_p(\mathbb{T})}}{t^{\frac{1}{p} - \frac{1}{q}}} \right) dt,$$

where $\alpha + 1 \in (2\mathbb{N}) \cup (1/p - 1, \infty)$. Note that for $\alpha \in 2\mathbb{N}$, the modulus (1.8) coincides with the classical modulus of smoothness $\omega_\alpha(f, \delta)_{L_p(\mathbb{T})}$.

In the multidimensional case, new interesting effects occur not only when $q = \infty$. In particular, the sharp Ulyanov inequality (1.5) holds not only for $1 < p < q < \infty$ but also for $p = 1, q \leq \infty$, provided that $\alpha + d(1 - 1/q) \in \mathbb{N}$ and $d \geq 2$ (see Corollary 8.3).

Finally, we would like to mention two remarks. First, likewise to $(L_p, L_q)$ inequalities for the moduli of smoothness, several authors have studied a similar problem for the
1.2. Specifics of the case $0 < p < 1$. One of the main contributions of this paper is a thorough study of the case $0 < p < 1$. We recall that dealing with smoothness properties in $L_p$, $p < 1$, differs dramatically from the case $p \geq 1$.

First, let us illustrate this by discussing how the classical Bernstein inequality for the fractional derivatives of trigonometric polynomials (see [59])

(1.9) $\|T_n^{(\alpha)}\|_{L_p(\mathbb{T})} \lesssim n^\alpha \|T_n\|_{L_p(\mathbb{T})}$, $\alpha > 0$, $1 \leq p < \infty$,

changes in $L_p(\mathbb{T})$, $0 < p < 1$.

Recall that the fractional derivative of a polynomial $T_n(x) = \sum_{|k| \leq n} c_k e^{ikx} \in T_n$ in the sense of Weyl is given by

$$T_n^{(\alpha)}(x) = \sum_{|k| \leq n} (ik)^\alpha c_k e^{ikx}, \quad (ik)^\alpha = |k|^\alpha e^{\frac{\pi\alpha}{2} \text{sign} k}.$$

The Bernstein inequality in $L_p(\mathbb{T})$, $0 < p < 1$, reads as follows (see [3] and [47, 80] for the multidimensional case).

**Proposition 1.1.** Let $0 < p < 1$. Then

$$\sup_{\|T_n\|_{L_p(\mathbb{T})} \leq 1} \|T_n^{(\alpha)}\|_{L_p(\mathbb{T})} \asymp \begin{cases} n^\alpha, & \alpha \in \mathbb{Z}_+ \text{ or } \alpha \notin \mathbb{Z}_+ \text{ and } \alpha > \frac{1}{p} - 1; \\ n^{\frac{\alpha}{2} - 1}, & \alpha \notin \mathbb{Z}_+ \text{ and } \alpha < \frac{1}{p} - 1; \\ n^{\frac{1}{2} - \frac{1}{p}} \log^\frac{1}{p} n, & \alpha = \frac{1}{p} - 1 \notin \mathbb{Z}_+. \end{cases}$$

It is worth mentioning that if the polynomial $T_n$ is such that $\text{spec}(T_n) \subset [0, n]$, that is, $c_k = 0$ for $k < 0$, then the Bernstein inequality changes drastically: for any $0 < p \leq \infty$, one has

$$\|T_n^{(\alpha)}\|_{L_p(\mathbb{T})} \lesssim n^\alpha \|T_n\|_{L_p(\mathbb{T})}, \quad \alpha > 0;$$

see Belinskii’s paper [2].

Second, let us discuss the smoothness of functions in the sense of behaviour of their moduli of smoothness in $L_p$. As an example, we consider the splines of maximum smoothness. Denote by $h_m$ the following function sequence: $h_1(x) = \frac{x}{\pi} \text{sgn}(\cos x)$, $x \in \mathbb{T}$, and

$$h_m(x) = \int_0^x \left(h_{m-1}(t) - \frac{1}{2\pi} \int_{\mathbb{T}} h_{m-1}(z) dz \right) dt, \quad m = 2, 3, \ldots.$$

Note that up to a constant

$$h_m(x) \sim \sum_{k \in \mathbb{Z}} \frac{e^{i(2k+1)x}}{(2k+1)^m}.$$
It is well known (see [18, p. 359]) that, for $0 < p \leq \infty$ and $l \in \mathbb{N}$, we have

$$\omega_l(h_m, \delta)_{L^p(\mathbb{T})} \leq \begin{cases} \delta^{m-l+\frac{1}{p}}, & l \geq m; \\ \delta^l, & l < m. \end{cases}$$

(1.10)

Let us compare the classical and sharp Ulyanov inequalities for such functions in the case $0 < p < 1$. First, we have $\omega_k(h_{r+k}, \delta)_{L^p(\mathbb{T})} \approx \delta^k$, and the standard (not-sharp) Ulyanov inequality (1.1) implies $\omega_k(h_{r+k}, \delta)_{L^q(\mathbb{T})} \lesssim \delta^{k-(1/p-1/q)}$. At the same time, the sharp Ulyanov inequality (1.7) with $\gamma = r \in \mathbb{N}$ yields

$$\omega_k(h_{r+k}, \delta)_{L^q(\mathbb{T})} \lesssim \frac{\omega_{k+r}(h_{r+k}, \delta)_{L^p(\mathbb{T})}}{\delta^r} \delta^{1-\frac{1}{p}}$$

$$+ \left( \int_0^\delta \left( \frac{\omega_{r+k}(h_{r+k}, t)_{L^p(\mathbb{T})}}{t^{1-\frac{1}{q}}} \right)^{\frac{1}{q}} \frac{dt}{t} \right)^{\frac{q}{q-1}} \approx \delta^k,$$

which is the best possible estimate since $\omega_k(h_{r+k}, \delta)_{L^q(\mathbb{T})} \approx \delta^k$. Therefore, the sharp Ulyanov inequality implies better estimates for $L_p$-smooth functions when $0 < p < 1$.

It is interesting to note that, for any $f \in C^\infty(\mathbb{T})$ and for all $0 < p \leq \infty$, one has $\omega_\gamma(f, \delta)_{L^p(\mathbb{T})} \approx \delta^\gamma$. In other words, taking into account (1.10), we see that spline functions in the case $0 < p < 1$ are smoother than $C^\infty$-functions in the sense of the behavior of their moduli of smoothness. This phenomena disappears if we deal with absolutely continuously functions. More precisely, if $f^{(\alpha-1)} \in AC(\mathbb{T}), \alpha \in \mathbb{N}$, and $\omega_\alpha(f, \delta) = o(\delta^\alpha)$ as $\delta \to 0$, then $f \equiv \text{const}$ (see [96] and Proposition 12.1).

1.3. Moduli of smoothness, $K$-functionals, and their realizations. The key approach to obtain the sharp Ulyanov inequalities is to use the realizations of the $K$-functionals. For simplicity, we start with the one-dimensional case. It is known (see [5, p. 341]) that if $1 \leq p \leq \infty$, then the classical modulus of smoothness is equivalent to the $K$-functional given by

$$K_\alpha(f, \delta)_{L^p(\mathbb{T})} := \inf_{g^{(\alpha)} \in L^p(\mathbb{T})} \left( \| f - g \|_{L^p(\mathbb{T})} + \delta^\alpha \| g^{(\alpha)} \|_{L^p(\mathbb{T})} \right),$$

that is,

$$\omega_\alpha(f, \delta)_{L^p(\mathbb{T})} \approx K_\alpha(f, \delta)_{L^p(\mathbb{T})}, \quad 1 \leq p \leq \infty, \quad \alpha > 0.$$ (1.11)

This equivalence fails for $0 < p < 1$ since in this case $K_\alpha(f, \delta)_{L^p(\mathbb{T})} \equiv 0$ (see [22]). A suitable substitute for the $K$-functional for $p < 1$ is the realization concept given by

$$\mathcal{R}_\alpha(f, \delta)_{L^p(\mathbb{T})} = \inf_{T \in \mathcal{T}_{[1/\delta]}} \left( \| f - T \|_{L^p(\mathbb{T})} + \delta^\alpha \| T^{(\alpha)} \|_{L^p(\mathbb{T})} \right).$$ (1.12)

The next result was proved in [22] for $\alpha \in \mathbb{N}$. For any positive $\alpha$, the proof follows from Theorem 3.1 below.

**Proposition 1.2. (Realization result)** Let $f \in L^p(\mathbb{T}), 0 < p \leq \infty$, and $\alpha \in \mathbb{N} \cup ((1/p - 1)_+, \infty)$. Then, for any $\delta \in (0, 1)$, we have

$$\omega_\alpha(f, \delta)_{L^p(\mathbb{T})} \approx \mathcal{R}_\alpha(f, \delta)_{L^p(\mathbb{T})}.$$ (1.13)
In the multidimensional case, an analogue of equivalence (1.11) holds for the classical moduli of smoothness of integer order (see [5, p. 341]). More precisely, if $1 \leq p \leq \infty$ and $\alpha \in \mathbb{N}$, we have for $f \in L_p(T^d)$
\[
\omega_\alpha(f, \delta)L_p(T^d) \simeq \inf_{g \in W^\alpha_{\alpha p}(T^d)} (\|f - g\|_{L_p(T^d)} + \delta^\alpha \|g\|_{\dot{W}^\alpha_{\alpha p}(T^d)}),
\]
where
\[
\|f\|_{\dot{W}^\alpha_{\alpha p}(T^d)} = \sum_{|\nu| = \alpha} \|D^\nu f\|_{L_p(T^d)}, \quad D^\nu = \frac{\partial^{\nu_1 + \cdots + \nu_d}}{\partial x_1^{\nu_1} \cdots \partial x_d^{\nu_d}}.
\]

For the fractional moduli of smoothness ($\alpha > 0$), Wilmes [111] proved that, for $1 < p < \infty$,
\[
\omega_\alpha(f, \delta)L_p(T^d) \simeq \inf_{(-\Delta)^{\alpha/2}g \in L_p} (\|f - g\|_{L_p(T^d)} + \delta^\alpha \|(-\Delta)^{\alpha/2}g\|_{L_p(T^d)}).
\]
The corresponding realization result (see, e.g., [31]) is given by
\[
\omega_\alpha(f, \delta)L_p(T^d) \simeq \inf_{T \in T_{[1/\delta]}} (\|f - T\|_{L_p(T^d)} + \delta^\alpha \|(-\Delta)^{\alpha/2}T\|_{L_p(T^d)}),
\]
where
\[
(-\Delta)^{\alpha/2}T(x) = \sum_{|k| \leq \lfloor 1/\delta \rfloor} |k|^{\alpha} c_k e^{i(k,x)}.
\]

We will show (see Theorem 4.1 below) that for any $0 < p \leq \infty$ and for any $\alpha \in \mathbb{N} \cup (1/p - 1)_+, \infty$ the following realization result holds
\[
\omega_\alpha(f, \delta)L_p(T^d) \simeq \inf_{T \in T_{[1/\delta]}} \left(\|f - T\|_{L_p(T^d)} + \delta^\alpha \sup_{\xi \in \mathbb{R}^d, |\xi| = 1} \left\| \left( \frac{\partial}{\partial \xi} \right)^{\alpha} T \right\|_{L_p(T^d)} \right),
\]
where $\left( \frac{\partial}{\partial \xi} \right)^{\alpha} T$ is the directional derivative of order $\alpha$, that is,
\[
\left( \frac{\partial}{\partial \xi} \right)^{\alpha} T(x) = \sum_{|k| \leq \lfloor 1/\delta \rfloor} (i(k, \xi))^\alpha c_k e^{i(k,x)}.
\]
Equivalence (1.16) is a crucial relation to obtain sharp Ulyanov-type inequalities in the multidimensional case for all $0 < p \leq q \leq \infty$.

1.4. Main tool: Hardy–Littlewood–Nikol’skii polynomial inequalities.
The Hardy–Littlewood inequality for fractional integrals states that for any $f \in L_p(T^d)$ such that $\int_{T^d} f(x)dx = 0$, we have
\[
\|(\Delta)^{-\theta/2}f\|_{L_{q}(T^d)} \lesssim \|f\|_{L_{p}(T^d)}, \quad 1 < p < q < \infty, \quad \theta = d \left( \frac{1}{p} - \frac{1}{q} \right).
\]
Remark that (1.17) does not hold outside the range $1 < p < q < \infty$. On the other hand, the following Nikol’skii inequality (sometimes called the reverse Hölder inequality)
holds for any trigonometric polynomial $T_n$ of degree at most $n$:

$$\|T_n\|_{L_q(T^d)} \lesssim n^\theta \|T_n\|_{L_p(T^d)}, \quad 0 < p < q \leq \infty, \quad \theta = d \left( \frac{1}{p} - \frac{1}{q} \right).$$

Both inequalities are, in a way, the limiting cases of the following general estimate:

$$\|(-\Delta)^{-\gamma/2} T_n\|_{L_q(T^d)} \lesssim \sigma(n) \|T_n\|_{L_p(T^d)}, \quad 0 < p < q \leq \infty.$$  

The key step in our proof of Ulyanov-type inequalities, which is of its own interest, is to obtain a sharp asymptotic behavior of $\sigma(n) = \sigma(n, p, q, \gamma, d)$. Our result reads as follows (see Corollary 5.2).

1. If $0 < p \leq 1$ and $p < q \leq \infty$, then

$$\sup_{T_n \in T_n} \frac{\|(-\Delta)^{-\gamma/2} T_n\|_{L_q(T^d)}}{\|T_n\|_{L_p(T^d)}} \asymp \begin{cases} 
    n^{d\left(\frac{1}{p} - \frac{1}{q}\right)}, & \gamma > d \left(1 - \frac{1}{q}\right); \\
    n^{d\left(\frac{1}{p} - \frac{1}{q}\right) \ln (t + 1)}, & 0 < \gamma = d \left(1 - \frac{1}{q}\right); \\
    n^{d\left(\frac{1}{p} - \frac{1}{q}\right) - \gamma}, & 0 < \gamma < d \left(1 - \frac{1}{q}\right); \\
    n^{d\left(\frac{1}{p} - \frac{1}{q}\right)}, & \gamma = 0.
\end{cases}$$

2. If $1 < p < q \leq \infty$, then

$$\sup_{T_n \in T_n} \frac{\|(-\Delta)^{-\gamma/2} T_n\|_{L_q(T^d)}}{\|T_n\|_{L_p(T^d)}} \asymp \begin{cases} 
    1, & \gamma \geq d\left(\frac{1}{p} - \frac{1}{q}\right), \quad q < \infty; \\
    1, & \gamma > d\left(\frac{1}{p}\right), \quad q = \infty; \\
    \ln \left(\frac{n}{\gamma}\right) + 1, & \gamma = d\left(\frac{1}{p}\right), \quad q = \infty; \\
    n^{d\left(\frac{1}{p} - \frac{1}{q}\right) - \gamma}, & 0 \leq \gamma < d\left(\frac{1}{p} - \frac{1}{q}\right).
\end{cases}$$

In particular, this implies the following analogue of the Hardy–Littlewood inequality (1.17) for the limiting cases: If $1 \leq p < q \leq \infty$, $f \in L_p(T^d)$, and $\int_{T^d} f(x)dx = 0$, then

$$\|(-\Delta)^{-\gamma/2} f\|_{L_q(T^d)} \lesssim \|f\|_{L_p(T^d)}$$

holds provided $\gamma > d(1/p - 1/q)$, $p = 1$ or/and $q = \infty$.

In fact, we will study a more general problem than (1.19) by considering the Weyl derivatives defined by homogeneous functions in place of powers of Laplacians.

**1.5. Main goals and work organization.** In this paper, our main goals are the following:

1. To prove Hardy–Littlewood–Nikol’skii polynomial inequalities for the generalized Weyl derivatives with the whole range of parameters $0 < p < q \leq \infty$.
2. To obtain sharp ($L_p, L_q$) inequalities of Ulyanov-type with variable parameters for moduli of smoothness, generalized $K$-functionals, and their realizations.
3. To apply Ulyanov inequalities to obtain new optimal embedding theorems of Lipschitz-type and Besov spaces.

The paper is organized as follows. In Section 2, we collect auxiliary results on embedding theorems for Besov and Triebel–Lizorkin spaces, Fourier multiplier theorems, and Hardy–Littlewood theorems on Fourier series with monotone coefficients. Moreover, we obtain a result which connects the $L_p$-norms of the Fourier transform of a continuous function and $L_p$-norm of polynomial generated by this function.
Section 3 deals with Nikol’skii–Stechkin–Boas–type inequalities for the relationship between norms of derivatives and differences of trigonometric polynomials written as follows

\[
\sup_{\xi \in \mathbb{R}^d, |\xi| = 1} \left\| \left( \frac{\partial}{\partial \xi} \right)^\alpha T_n \right\|_{L_p(\mathbb{T}^d)} \asymp \delta^{-\alpha} \omega_\alpha (T_n, \delta)_{L_p(\mathbb{T}^d)}, \quad T_n \in \mathcal{T}_n,
\]

where \(0 < \delta \leq \pi/n\), \(0 < p \leq \infty\), and \(\alpha > 0\). In particular, this relation plays a crucial role in obtaining the equivalence between moduli of smoothness and realization concepts, see (1.16).

Basic properties of fractional multidimensional moduli of smoothness are given in Section 4.

In Section 5, we obtain sharp Hardy–Littlewood–Nikol’skii inequalities for the generalized Weyl derivatives of trigonometric polynomials in the multidimensional case. In more detail, we find the asymptotic behavior of

\[
\sup_{T_n \in \mathcal{T}_n} \frac{\| \mathcal{D}(\psi) T_n \|_{L_q(\mathbb{T}^d)}}{\| \mathcal{D}(\varphi) T_n \|_{L_p(\mathbb{T}^d)}}, \quad 0 < p < q \leq \infty,
\]

where \(\mathcal{D}(\eta)\) is the generalized Weyl-type differentiation operators, i.e., \(\mathcal{D}(\eta) : \sum_{\nu} c_\nu e^{i(\nu,x)} \mapsto \sum_{\nu \neq 0} \eta(\nu) c_\nu e^{i(\nu,x)}\), and \(\eta\) is a homogeneous function of a certain degree.

In Section 6, we study properties of the generalized \(K\)-functionals and their realizations and prove a general form of sharp Ulyanov inequalities.

Section 7 provides an explicit formula of the sharp Ulyanov inequality for the realization concepts.

Section 8 deals with the sharp Ulyanov inequality for the moduli of smoothness in one-dimensional and multi-dimensional cases, where we use results of Sections 6 and 7.

Section 9 is devoted to the proof of sharp Ulyanov inequalities for moduli of smoothness and \(K\)-functional related to the Riesz derivatives.

Section 10 deals with the sharp Ulyanov inequalities, which are obtained using Marchaud-type inequalities.

In Section 11, we treat the Ulyanov and Kolyada-type inequalities in the real and analytic Hardy spaces.

In Section 12, we prove the sharp Ulyanov inequalities in involving derivatives. In particular, we improve the known estimate (see [25])

\[
\omega_\alpha (f(r), \delta)_{L_q(\mathbb{T})} \lesssim \left( \int_0^\delta \left( t^{-r - (\frac{1}{p} - \frac{1}{q})} \omega_{r+\alpha} (f, t)_{L_p(\mathbb{T})} \right)^{\frac{q}{q_1}} \frac{dt}{t} \right)^{\frac{1}{q_1}}, \quad 0 < p < q \leq \infty,
\]

where \(1/p - 1/q < \alpha\).

Finally, in Section 13, we discuss new embedding theorems for smooth function spaces, which follow from inequalities for moduli of smoothness.

Acknowledgements. The first author was supported by the project AFFMA that has received funding from the European Union’s Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie grant agreement No 704030. The second
author was partially supported by MTM 2014-59174-P, 2014 SGR 289, and by the CERCA Programme of the Generalitat de Catalunya.
2. Auxiliary results

2.1. Besov and Triebel-Lizorkin spaces and their embeddings. Let us recall the definition of the Besov space $B_{p,q}^s(\mathbb{R}^d)$ (see, e.g., [105]). We consider the Schwartz function $\varphi \in \mathcal{S}(\mathbb{R}^d)$ such that $\text{supp} \varphi \subset \{\xi \in \mathbb{R}^d : 1/2 \leq |\xi| \leq 2\}$, $\varphi(\xi) > 0$ for $1/2 < |\xi| < 2$ and

$$\sum_{k=-\infty}^{\infty} \varphi(2^{-k} \xi) = 1 \quad \text{if} \quad \xi \neq 0.$$ 

We also introduce the functions $\varphi_k$ and $\psi$ by means of the relations

$$\mathcal{F} \varphi_k(\xi) = \varphi(2^{-k} \xi)$$

and

$$\mathcal{F} \psi(\xi) = 1 - \sum_{k=1}^{\infty} \varphi(2^{-k} \xi).$$

We say that $f \in \mathcal{S}'(\mathbb{R}^d)$ belongs to the (non-homogeneous) Besov space $B_{p,q}^s(\mathbb{R}^d)$, $s \in \mathbb{R}$, $0 < p, q \leq \infty$, if

$$\|f\|_{B_{p,q}^s(\mathbb{R}^d)} = \|\psi * f\|_{L^p(\mathbb{R}^d)} + \left( \sum_{k=1}^{\infty} 2^{sqk} \|\varphi_k * f\|_{L^q(\mathbb{R}^d)}^q \right)^{\frac{1}{q}} < \infty$$

(with the usual modification in the case $q = \infty$).

We will also deal with the Triebel-Lizorkin spaces. We say that $f \in \mathcal{S}'(\mathbb{R}^d)$, belongs to the (non-homogeneous) Triebel-Lizorkin space $F_{p,q}^s(\mathbb{R}^d)$, $s \in \mathbb{R}$, $0 < p, q \leq \infty$, if

$$\|f\|_{F_{p,q}^s(\mathbb{R}^d)} = \|\psi * f\|_{L^p(\mathbb{R}^d)} + \left\| \left( \sum_{k=1}^{\infty} 2^{sqk} |\varphi_k * f(\cdot)|^q \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^d)} < \infty$$

(with the usual modification in the case $q = \infty$).

In order to define the homogeneous Besov spaces and Triebel-Lizorkin spaces, recall that

$$\dot{\mathcal{S}}(\mathbb{R}^d) = \{ \varphi \in \mathcal{S}'(\mathbb{R}^d) : (D^\nu \varphi)(0) = 0 \text{ for all } \nu \in \mathbb{N}^d \cup \{0\} \},$$

where $\dot{\mathcal{S}}(\mathbb{R}^d)$ is the space of all continuous functionals on $\mathcal{S}(\mathbb{R}^d)$. We say that $f \in \dot{\mathcal{S}}'(\mathbb{R}^d)$ belongs to the homogeneous Besov space $\dot{B}_{p,q}^s(\mathbb{R}^d)$ if

$$\|f\|_{\dot{B}_{p,q}^s(\mathbb{R}^d)} = \left( \sum_{k=-\infty}^{\infty} 2^{sqk} \|\varphi_k * f\|_{L^p(\mathbb{R}^d)}^q \right)^{\frac{1}{q}} < \infty.$$ 

Similarly, $f \in \dot{\mathcal{S}}'(\mathbb{R}^d)$ belongs to the homogeneous Triebel-Lizorkin space $\dot{F}_{p,q}^s(\mathbb{R}^d)$ if

$$\|f\|_{\dot{F}_{p,q}^s(\mathbb{R}^d)} = \left\| \left( \sum_{k=-\infty}^{\infty} 2^{sqk} |\varphi_k * f(\cdot)|^q \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^d)} < \infty.$$
Recall also the following relations between the real Hardy spaces and the homogeneous Triebel-Lizorkin spaces. We have $H_p^s(\mathbb{R}^d) = F_p^0(\mathbb{R}^d)$ and $\|(-\Delta)^s/2 f\|_{H_p^s(\mathbb{R}^d)} = \|\hat{f}\|_{\dot{F}_p^{s,2}(\mathbb{R}^d)}$ for $0 < p < \infty$ (see [105, Ch. 5]).

We will use the following embeddings.

**Lemma 2.1.** For $0 < p < q < \infty$ and $\theta = d(1/p - 1/q) < \alpha$, we have

\begin{equation}
B_{p,q}^\theta(\mathbb{R}^d) \hookrightarrow F_{q,2}^0(\mathbb{R}^d) \quad \text{and} \quad \|f\|_{H_q^s(\mathbb{R}^d)} \lesssim \|f\|_{\dot{B}_{p,q}^s(\mathbb{R}^d)},
\end{equation}

and

\begin{equation}
F_{p,2}^\alpha(\mathbb{R}^d) \hookrightarrow B_{q,p}^{\alpha-\theta}(\mathbb{R}^d) \quad \text{and} \quad \|f\|_{\dot{B}_{q,p}^{\alpha-\theta}(\mathbb{R}^d)} \lesssim \|f\|_{F_{p,2}^\alpha(\mathbb{R}^d)}.
\end{equation}

**Proof.** The proofs of these embeddings one can find in [39] and [40] (see also, e.g., [105]). Let us only note that (2.1) was proved in [39] only for non-homogeneous spaces. The proof for homogeneous spaces can be easily obtained by using triangle inequality, the Nikol’skii-type inequality, and the fact that $\|\varphi_k * f\|_{H_q^s(\mathbb{R}^d)} \sim \|\varphi_k * f\|_{L_q^s(\mathbb{R}^d)}$ with the constants in $\sim$ independent of $f$ and $k$ (see, e.g., [73, p. 239]).

The following result was proved in [46] for the case $p = 1$. The case $0 < p < 2$ can be treated similarly (see [44]).

**Lemma 2.2.** Let $0 < p < 2$, $1 < q, r < \infty$, $s > d(1/p - 1 + 1/r)$, and $f \in \mathcal{S}^\prime(\mathbb{R}^d)$. Let us suppose that $q = r = 2$ or

$$
\frac{1 - \theta}{q} + \frac{\theta}{r} > \frac{1}{2}, \quad \theta = \frac{d}{s} \left(\frac{1}{p} - \frac{1}{2}\right).
$$

If, in addition, $f \in L_q^s(\mathbb{R}^d)$ and $(-\Delta)^{s/2} f \in L_r^s(\mathbb{R}^d)$, then $f \in \dot{B}_{2,p}^{s,\theta}(\mathbb{R}^d)$.

The next result easily follows from the definition of the Besov spaces (see, e.g., [105, 3.4.1, p. 206]; mind the typo in [105, 5.3.1, Remark 4, p. 239]).

**Lemma 2.3.** We have

$$
\|f(\lambda \cdot)\|_{\dot{B}_{p,q}^s(\mathbb{R}^d)} \lesssim \lambda^{s \frac{d}{p}} \|f(\cdot)\|_{\dot{B}_{p,q}^s(\mathbb{R}^d)}.
$$

### 2.2. Fourier multipliers.

In this paper, we will use Fourier multipliers only for periodic functions. Recall that a function $m : \mathbb{R}^d \to \mathbb{C}$ is called a Fourier multiplier (we will write $m \in M_p$) if there exists a constant $C = C(m, p, d)$ such that for any $\varepsilon > 0$ and $f \in L_p^p(\mathbb{T}^d)$, $1 \leq p \leq \infty$, one has

$$
\left\| \sum_{k \in \mathbb{Z}^d} m(\varepsilon k) \hat{f}_k e^{i(k,x)} \right\|_p \leq C \left\| \sum_{k \in \mathbb{Z}^d} \hat{f}_k e^{i(k,x)} \right\|_p,
$$

where

$$
\hat{f}_k = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x) e^{-i(k,x)} dx, \quad k \in \mathbb{Z}^d,
$$

are the Fourier coefficients of the function $f$.

We need the following properties of Fourier multipliers, which can be found, for example, in [64], [73, p. 119] or [90, Ch. IV, 3.2].
Lemma 2.4. (i) (Mikhlin–Hörmander’s theorem) Let $1 < p < \infty$. If
\[
\sup_{\xi \in \mathbb{R}^d} |\xi|^{|\nu|_1} |D^\nu m(\xi)| < \infty \quad \text{for all} \quad 0 \leq |\nu|_1 \leq [n/2] + 1, \quad \nu \in \mathbb{Z}^d,
\]
then $m \in M_p$.

(ii) (Peetre’s theorem) If $m \in \dot{B}^{d/2}_{2,1}(\mathbb{R}^d)$, then $m \in M_1$ or, equivalently, $m \in M_\infty$.

2.3. Estimates of $L_p$-norm of an integrable function in terms of its Fourier coefficients. The following result allows us to estimate a (quasi-)norm of an integrable function via its Fourier coefficients. In the case $p = 1$, this lemma was proved in [107].

Lemma 2.5. Let $0 < p \leq 1$, $\Phi \in L_1(\mathbb{T}^d)$, and
\[
\Phi(x) \sim \sum_{k \in \mathbb{Z}^d} \varphi(k)e^{i(k,x)}.
\]
Let $\varphi_c$ be such that $\varphi_c(k) = \varphi(k)$ for any $k \in \mathbb{Z}^d$ and $\varphi_c \in \dot{B}^{d/p - 1/2}_{2,p}(\mathbb{R}^d)$ and $\varphi_c \in \dot{B}^{d/p - 1/2}_{2,p}(\mathbb{T}^d)$. Then
\[
\|\Phi\|_{L_p(\mathbb{T}^d)} \leq C \|\varphi_c\|_{\dot{B}^{d/p - 1/2}_{2,p}(\mathbb{T}^d)},
\]
where a constant $C$ is independent of $\Phi$.

Proof. For any $\varphi_c \in \dot{B}^{d}_{2,1}(\mathbb{R}^d)$, there exists $g \in L_1(\mathbb{R}^d)$ such that
\[
\varphi_c(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} g(t)e^{i(t,x)} dt
\]
(see, for example, [73, p. 119]). For $k \in \mathbb{Z}^d$, we get
\[
\varphi(k) = \varphi_c(k) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} g(t)e^{-i(k,t)} dt
\]
\[
= (2\pi)^{-d/2} \sum_{\mu \in \mathbb{Z}^d} \int_{\mathbb{T}^d + 2\pi \mu} g(t)e^{-i(k,t)} dt
\]
\[
= (2\pi)^{-d/2} \sum_{\mu \in \mathbb{Z}^d} \int_{\mathbb{T}^d} g(t + 2\pi \mu)e^{-i(k,t)} dt
\]
\[
= (2\pi)^{-d/2} \int_{\mathbb{T}^d} g_T(t)e^{-i(k,t)} dt,
\]
where
\[
g_T(t) \sim \sum_{\mu \in \mathbb{Z}^d} g(t + 2\pi \mu), \quad t \in \mathbb{T}^d.
\]
Using Beppo Levi’s theorem, we have $g_T \in L_1(\mathbb{T}^d)$. Since
\[
g_T(t) \sim \sum_{k \in \mathbb{Z}^d} \varphi(k)e^{i(k,t)}
\]
then
\[
\|\Phi\|_{L_p(\mathbb{T}^d)} = \|g_T\|_{L_p(\mathbb{T}^d)} \leq \|g\|_{L_p(\mathbb{R}^d)} \leq C \|\varphi_c\|_{\dot{B}^{d/p - 1/2}_{2,p}(\mathbb{R}^d)},
\]
where in the last estimate we have used the Bernstein-type inequality

\[ \|f\|_{L_p(\mathbb{R}^d)} \leq C\|\hat{f}\|_{\dot{B}^{d(\frac{1}{p} - \frac{1}{2})}_{2,p}(\mathbb{R}^d)} \]

(see, e.g., [73, p. 119]).

\[ \Box \]

## 2.4. Estimates of \( L_p \)-norms of polynomials related to Lebesgue constants.

We will widely use the following anisotropic result which connects the norms of the Fourier transform of a function and the polynomial generated by this function (see [108, p. 106]). Note that for \( p = 1 \), this question is closely related to a study of Lebesgue constants of approximation methods (see [1], [58], and [108, Ch. 4 and Ch. 9]).

**Theorem 2.1.** Let \( 0 < p \leq 1 \). Let \( \varphi \in C(\mathbb{R}^d) \) have a compact support and let \( \hat{\varphi} \in L_p(\mathbb{R}^d) \). Then

\[
\sup_{\varepsilon_{i} \neq 0, j = 1, \ldots, d} \left( \prod_{j=1}^{d} |\varepsilon_j| \right)^{-\frac{1}{p}} \| \Phi_{\varepsilon} \|_{L_p(\mathbb{T}^d)} = (2\pi)^{d/2} \| \hat{\varphi} \|_{L_p(\mathbb{R}^d)},
\]

where

\[
\Phi_{\varepsilon}(x) = \sum_{k \in \mathbb{Z}^d} \varphi(\varepsilon_1 k_1, \ldots, \varepsilon_d k_d) e^{i(k,x)}, \quad \varepsilon = (\varepsilon_1, \ldots, \varepsilon_d).
\]

Note that this theorem was proved in the case \( \varepsilon_1 = \cdots = \varepsilon_d \) in [1] for \( p = 1 \) and in [108, p. 106] for \( 0 < p \leq 1 \). In applications below, the anisotropic case \( \varepsilon_j \neq \varepsilon_i \) plays an essential role.

**Proof.** We follow the proof of Theorem 4.4.1 from [108]. First, let us verify the estimate from above. By the Poisson summation formula (see [90, Ch. VII]), we get

\[
\sum_{k \in \mathbb{Z}^d} \hat{\varphi} \left( \frac{x_1 + 2\pi k_1}{\varepsilon_1}, \ldots, \frac{x_d + 2\pi k_d}{\varepsilon_d} \right) = (2\pi)^{-d/2} \prod_{j=1}^{d} |\varepsilon_j| \sum_{k \in \mathbb{Z}^d} \varphi (-\varepsilon_1 k_1, \ldots, -\varepsilon_d k_d) e^{i(k,x)}
\]

\[
= (2\pi)^{-d/2} \prod_{j=1}^{d} |\varepsilon_j| \sum_{k \in \mathbb{Z}^d} \varphi (\varepsilon_1 k_1, \ldots, \varepsilon_d k_d) e^{-i(k,x)}.\]
Further, simple calculations imply the following relations:

\[
\left( \prod_{j=1}^{d} |\varepsilon_j| \right)^{p-1} \| \Phi_\varepsilon \|_{L_p(\mathbb{T}^d)}^p
\]

\[
= (2\pi)^{pd/2} \prod_{j=1}^{d} |\varepsilon_j|^{-1} \int_{\mathbb{T}^d} \left| \sum_{k \in \mathbb{Z}^d} \hat{\varphi} \left( \frac{x_1 + 2\pi k_1}{\varepsilon_1}, \ldots, \frac{x_d + 2\pi k_d}{\varepsilon_d} \right) \right|^p dx
\]

\[
\leq (2\pi)^{pd/2} \prod_{j=1}^{d} |\varepsilon_j|^{-1} \int_{\mathbb{T}^d} \left| \sum_{k \in \mathbb{Z}^d} \hat{\varphi} \left( \frac{x_1}{\varepsilon_1}, \ldots, \frac{x_d}{\varepsilon_d} \right) \right|^p dx
\]

\[
= (2\pi)^{pd/2} \prod_{j=1}^{d} |\varepsilon_j|^{-1} \int_{\mathbb{T}^d} \left| \hat{\varphi} \left( \frac{x_1}{\varepsilon_1}, \ldots, \frac{x_d}{\varepsilon_d} \right) \right|^p dx
\]

\[
= (2\pi)^{pd/2} \| \hat{\varphi} \|_{L_p(\mathbb{R}^d)}^p.
\]

To obtain the estimate from below, we use the substitution \( x_j \to \varepsilon_j x_j, j = 1, \ldots, d, \) which gives

\[
\left( \prod_{j=1}^{d} |\varepsilon_j| \right)^{p-1} \| \Phi_\varepsilon \|_{L_p(\mathbb{T}^d)}^p
\]

\[
= \left( \prod_{j=1}^{d} |\varepsilon_j| \right)^p \int_0^{2\pi} \cdots \int_0^{2\pi} \left| \sum_{k \in \mathbb{Z}^d} \varphi (\varepsilon_1 k_1, \ldots, \varepsilon_d k_d) e^{-i(\sum_{j=1}^{d} \varepsilon_j k_j x_j)} \right|^p dx.
\]

Assuming that the limit inferior of this value as \( |\varepsilon| = (\varepsilon_1^2 + \cdots + \varepsilon_d^2)^{1/2} \to 0 \) is finite and equals \( M, \) we will show that

\[
(2\pi)^{pd/2} \| \hat{\varphi} \|_{L_p(\mathbb{R}^d)}^p \leq M.
\]

Given arbitrary \( N > 0 \) and \( \delta > 0, \) we obtain for sufficiently small \( |\varepsilon| \)

\[
(2.2) \quad \int_{NT^d} \left| \sum_{k \in \mathbb{Z}^d} \varphi (\varepsilon_1 k_1, \ldots, \varepsilon_d k_d) e^{-i(\sum_{j=1}^{d} \varepsilon_j k_j x_j)} \prod_{j=1}^{d} \varepsilon_j \right|^p dx
\]

\[
\leq \left( \prod_{j=1}^{d} |\varepsilon_j| \right)^p \int_0^{2\pi} \cdots \int_0^{2\pi} \left| \sum_{k \in \mathbb{Z}^d} \varphi (\varepsilon_1 k_1, \ldots, \varepsilon_d k_d) e^{-i(\sum_{j=1}^{d} \varepsilon_j k_j x_j)} \right|^p dx \leq M + \delta.
\]

Noting that we deal with the Riemann integral sum in (2.2), we pass to the limit as \( |\varepsilon| \to 0 \) to obtain

\[
(2\pi)^{pd/2} \int_{NT^d} |\hat{\varphi}(x)|^p dx = \int_{NT^d} \left| \int_{\mathbb{R}^d} \varphi(y) e^{-i(x,y)} dy \right|^p dx \leq M + \delta.
\]

Letting \( N \to \infty \) and, consequently, \( \delta \to 0, \) we get the desired estimate from below. □
Corollary 2.1. Let \( 0 < p \leq 1 \) and \( \varphi \in C^\infty(\mathbb{R}^d) \) have a compact support. Then
\[
\sup_{\varepsilon_j \neq 0, j = 1, \ldots, d} \left( \prod_{j=1}^d |\varepsilon_j| \right)^{1 - \frac{1}{p}} \left\| \sum_{k \in \mathbb{Z}^d} \varphi(\varepsilon_1 k_1, \ldots, \varepsilon_d k_d) e^{i(k, x)} \right\|_{L_p(\mathbb{T}^d)} < \infty.
\]

2.5. Hardy–Littlewood type theorems for Fourier series with monotone coefficients. It is well known that trigonometric series with monotone coefficients have several important properties. In particular, Hardy and Littlewood [114] studied the series
\[
f(x) = \sum_{n \in \mathbb{Z}} a_n \cos n x, \quad a_n \geq a_{n+1},
\]
and proved that a necessary and sufficient condition for \( f \in L_p(\mathbb{T}), 1 < p < \infty \), is
\[
\sum_{n \in \mathbb{Z}} a_n^p n^{p-2} < \infty.
\]
Moreover,
\[
\|f\|_p \lesssim \left( \sum_{n \in \mathbb{Z}} a_n^p n^{p-2} \right)^{1/p}. \tag{2.3}
\]
It turns out that in applications one needs a more general condition on Fourier coefficients than just monotonicity. We will give a higher-dimensional generalization of the Hardy-Littlewood theorem for trigonometric series with general monotone coefficients.

A non-negative sequence \( a = \{a_m\}, m = (m_1, \ldots, m_d) \in \mathbb{N}^d, d \geq 1 \), satisfies the \( GM^d(\beta)\)-condition, written \( a \in GM^d(\beta) \), \( \beta = \{\beta_k\}, k = (k_1, \ldots, k_d) \in \mathbb{N}^d \) ([26]), if
\[
a_m \to 0 \quad \text{as} \quad |m|_1 \to \infty
\]
and
\[
\sum_{m_1 = k_1}^\infty \cdots \sum_{m_d = k_d}^\infty |\triangle^{(d)} a_m| \leq C \beta_k, \quad k = (k_1, \ldots, k_d) \in \mathbb{N}^d,
\]
where
\[
\triangle^{(d)} \equiv \prod_{j=1}^d \triangle^j \quad \text{and} \quad \triangle^j a_m = a_m - a_{m_1, \ldots, m_{j-1}, m_j+1, m_{j+1}, \ldots, m_d}.
\]

Lemma 2.6. (See [26, 27].) Let \( 1 < p < \infty, d \geq 1 \), and let the Fourier series of \( f \) be of the following type
\[
f(x) \sim \sum_{m_1 = 1}^\infty \cdots \sum_{m_d = 1}^\infty a_m \prod_{j \in B} \cos m_j x_j \prod_{j \in N \setminus B} \sin m_j x_j,
\]
where \( N = \{1, 2, \ldots, d\} \) and \( B \subseteq N \).

(i) Let \( a \in GM^d(\beta) \). Then
\[
\|f\|_{L_p(\mathbb{T}^d)} \lesssim \left( \sum_{m_1 = 1}^\infty \cdots \sum_{m_d = 1}^\infty \beta_m^p \left( \prod_{j=1}^d m_j \right)^{p-2} \right)^{1/p}.
\]


(ii) Let $a \in GM^d(\beta)$ with
\[
\beta_k = \sum_{m_1=k_1/c}^{\infty} \cdots \sum_{m_d=k_d/c}^{\infty} \frac{a_{m_1,\ldots,m_d}}{m_1 \cdots m_d}, \quad c > 1, \quad a_{m_1,\ldots,m_d} \geq 0.
\]

Then
\[
\|f\|_{L^p(T^d)} \asymp \left( \sum_{m_1=1}^{\infty} \cdots \sum_{m_d=1}^{\infty} a_m^p \left( \prod_{j=1}^{d} m_j \right)^{p-2} \right)^{1/p}.
\]

(2.5)

In particular (see also [65]), equivalence (2.5) holds for the series (2.4) satisfying the condition $\Delta^{(d)} a_m \geq 0$ for any $m \in \mathbb{N}^d$.

2.6. Weighted Hardy’s inequality for averages. We will frequently use the following weighted Hardy inequality (see, e.g., [56]).

**Lemma 2.7.** Let $1 < \lambda < \infty$. Suppose $u(x), v(x) \geq 0$ on $(0, \delta)$. Then
\[
\int_0^\delta u(x) \left[ \int_0^x \psi(t) \, dt \right]^{\lambda} \, dx \lesssim \int_0^\delta \psi^{\lambda}(x) v(x) \, dx
\]
holds for each $\psi(x) \geq 0$ if and only if
\[
\sup_{0<s<\delta} \left( \int_s^\delta u(x) \, dx \right)^{1/\lambda} \left( \int_0^s v(x)^{1-\lambda'} \, dx \right)^{1/\lambda'} < \infty.
\]

(2.6)
3. Polynomial inequalities of Nikol’skii–Stechkin–Boas–types

In its simplest form, the Nikol’skii–Stechkin–type inequality for a polynomial $T_n(x) = \sum_{|k| \leq n} c_k e^{ikx}$, $x \in \mathbb{T}$, states that, for $r \in \mathbb{N}$ and $1 \leq p \leq \infty$,

$$\| T_n^{(r)} \|_{L^p(\mathbb{T})} \leq \left( \frac{n}{2 \sin \frac{n\delta}{2}} \right)^r \| \Delta_\delta^n T_n \|_{L^p(\mathbb{T})}, \quad 0 < \delta \leq \frac{\pi}{n},$$

extending the classical Bernstein inequality (1.9). Boas derived that

$$\left( \frac{n}{2 \sin \frac{nh}{2}} \right)^r \| \Delta_h^n T_n \|_{L^p(\mathbb{T})} \leq \left( \frac{n}{2 \sin \frac{n\delta}{2}} \right)^r \| \Delta_\delta^n T_n \|_{L^p(\mathbb{T})}, \quad 0 < h \leq \delta \leq \frac{\pi}{n}.$$ 

In this section, we prove analogues of these results in the multidimensional case for all $0 < p \leq \infty$. We will use the following notion of the directional derivative. Let $f(x) \sim \sum_{k \in \mathbb{Z}^d} \hat{f}_k e^{i(k,x)}$, we denote the directional derivative of $f$ of order $\alpha > 0$ along a vector $\xi \in \mathbb{R}^d$ by

$$\left( \frac{\partial}{\partial \xi} \right)^\alpha f(x) = \sum_{k \in \mathbb{Z}^d} (i(k, \xi))^\alpha \hat{f}_k e^{i(k,x)}$$

(see, e.g., [111]).

**Theorem 3.1.** Let $0 < p \leq \infty$, $\alpha > 0$, $n \in \mathbb{N}$, and $\xi \in \mathbb{R}^d$, $0 < |\xi| < \pi/n$. Then, for any $T_n(x) = \sum_{|k| \leq n} c_k e^{i(k,x)} \in \mathcal{T}_n$,

we have

$$\left\| \left( \frac{\partial}{\partial \xi} \right)^\alpha T_n \right\|_p \lesssim \| \Delta_\xi^\alpha T_n \|_p,$$

where the constants in this equivalence depend only on $p$, $\alpha$, and $d$.

As we have already mentioned, inequalities of this type have a long and rich history starting from the 1940s. When $d = 1$ and $1 \leq p \leq \infty$, this is the result of Nikol’skii [67], Stechkin [88], and Boas [8] (for integer $\alpha$, see also [101, p. 214 and p. 251]), and Taberski [97] and Trigub [106] (for any positive $\alpha$). When $d = 1$ and $0 < p < 1$, (3.1) follows from [22] for $\alpha \in \mathbb{N}$ and from [43] for $\alpha > 0$. When $d \in \mathbb{N}$ and $1 \leq p \leq \infty$, the part $\lesssim$ was proved in [111, Theorem 3] for any $\alpha > 0$.

**Proof.** First, let us prove the estimate from above in (3.1). Let the function $v \in C^\infty(\mathbb{R})$, $v(s) = 1$ for $|s| \leq \pi$ and $v(s) = 0$ for $|s| \geq 3\pi/2$. We define

$$K_\xi(x) = \sum_{k \in \mathbb{Z}^d} \left( \frac{i(k, \xi)}{2i \sin \frac{|k, \xi|}{2}} \right)^\alpha v \left( \sqrt{(k_1 \xi_1)^2 + \cdots + (k_d \xi_d)^2} \right) e^{i(k,x)}$$
Integrating the above inequality over $x$. Note that $K_\xi$ is a trigonometric polynomial of degree at most $C/|\xi_j|$ in variable $x_j$. We can assume that $\xi_j \neq 0$ for any $j = 1, \ldots, d$, otherwise we consider the same problem in the space $\mathbb{R}^{d-k}$ with some $k > 0$.

Taking into account that

$$\Delta^\alpha_n(x) = \sum_{|k| \leq n} \left( 2i \sin \left( \frac{k \cdot \xi}{2} \right) \right)^\alpha \cos e^{i(k \cdot x + \xi_k)}$$

and $\left( (k_1^2) + \cdots + (k_d^2) \right)^{\frac{1}{2}} \leq |k| |\xi| \leq \pi$, one has that

$$(3.2) \quad \left( \frac{\partial}{\partial \xi} \right)^\alpha T_n(x) = (K_\xi * \Delta^\alpha_n) \left( x - \frac{\xi \cdot \alpha}{2} \right),$$

where $*$ stands for the convolution of periodic functions.

Let first $0 < p < 1$. We will show that

$$\left\| \left( \frac{\partial}{\partial \xi} \right)^\alpha T_n \right\|_p \lesssim \left( \prod_{j=1}^d |\xi_j| \right)^{1-\frac{1}{p}} \|K_\xi\| \|\Delta^\alpha_n T_n\|_p.$$  

For this we need the following Nikol’skii’s inequality ([68], see also [70])

$$\|U_{n_1, \ldots, n_d}\|_1 \lesssim \left( \prod_{j=1}^d n_j \right)^{\frac{1}{p}-1} \|U_{n_1, \ldots, n_d}\|_p,$$

where

$$U_{n_1, \ldots, n_d}(x) = \sum_{|k| \leq n_1} \cdots \sum_{|k| \leq n_d} \sum_{k \in \mathbb{Z}^d} a_k e^{i(k \cdot x)}, \quad k \in \mathbb{Z}^d.$$  

By using (3.2) and the above Nikol’skii inequality for the polynomial $K_\xi(x) \Delta^\alpha_n T_n(x - \xi \cdot \alpha/2 - t)$ of degree at most $C/|\xi_j|$ in $x_j$, we obtain

$$\left| \left( \frac{\partial}{\partial \xi} \right)^\alpha T_n(x) \right|^p \leq \frac{1}{(2\pi)^d} \left( \int_{\mathbb{T}^d} |K_\xi(t) \Delta^\alpha_n T_n(x - \xi \cdot \alpha/2 - t)| dt \right)^p \leq \left( \prod_{j=1}^d |\xi_j| \right)^{p-1} \int_{\mathbb{T}^d} |K_\xi(t) \Delta^\alpha_n T_n(x - \xi \cdot \alpha/2 - t)|^p dt.$$  

Integrating the above inequality over $x$, we get

$$\int_{\mathbb{T}^d} \left( \frac{\partial}{\partial \xi} \right)^\alpha T_n(x)^p dx \lesssim \left( \prod_{j=1}^d |\xi_j| \right)^{p-1} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} |K_\xi(t) \Delta^\alpha_n T_n(x - \xi \cdot \alpha/2 - t)|^p dt dx.$$

Thus, applying Fubini’s theorem derives (3.3).

We now show that $\left( \prod_{j=1}^d |\xi_j| \right)^{1-1/p} \|K_\xi\|_p$ is bounded from above. Set

$$g_\xi(t) = \left( \frac{(t, \xi)}{2 \sin \left( \frac{(t, \xi)}{2} \right)} \right)^\alpha \sqrt{(t_1^2 + \cdots + (t_d^2)^2)}.$$
Since \( g_1 \in C^\infty(\mathbb{R}) \), \( 1 = (1, \ldots, 1) \), and \( g_1 \) has a compact support, we obtain that, for any \( r \geq 1 \), the following holds
\[
\widehat{g}_1(y) = O\left(\frac{1}{|y|^r}\right) \quad \text{as} \quad |y| \to \infty.
\]
The latter together with \( |\widehat{g}_1(y)| \leq C \) for \( |y| \leq 1 \) gives that \( \widehat{g}_1 \in L_p(\mathbb{R}^d) \).

Now, by using Theorem 2.1, we obtain
\[
\left(\prod_{j=1}^{d} |\xi_j|^{1-\frac{1}{p}}\right) \|K_\xi\|_p = \left(\prod_{j=1}^{d} |\xi_j|^{1-\frac{1}{p}}\right) \left(\int_{\mathbb{T}^d} \left|\sum_{k \in \mathbb{Z}^d} g_\xi(k) e^{i(k,x)}\right|^p dx\right)^{\frac{1}{p}}
\]
\[
= \left(\prod_{j=1}^{d} |\xi_j|^{1-\frac{1}{p}}\right) \left(\int_{\mathbb{T}^d} \left|\sum_{k \in \mathbb{Z}^d} g_1(\xi_1 k_1, \ldots, \xi_d k_d) e^{i(k,x)}\right|^p dx\right)^{\frac{1}{p}}
\]
\[
\leq (2\pi)^{d/2} \|\widehat{g}_1\|_{L_p(\mathbb{R}^d)}.
\]
Thus, from the last inequality and (3.3) we obtain the part "\( \lesssim \)" in (3.1).

To prove the estimate "\( \gtrsim \)" we consider the polynomial
\[
\tilde{K}_\xi(x) = \sum_{k \in \mathbb{Z}^d} \left(2i \sin \left(\frac{k \cdot \xi}{2}\right)\right)^\alpha \left(\sqrt{(k_1 \xi_1)^2 + \cdots + (k_d \xi_d)^2}\right) e^{i(k,x)},
\]
the equality
\[
\Delta_\alpha^\xi T_n(x) = \left(\tilde{K}_\xi * \left(\partial_\xi^\alpha\right) T_n\right)\left(x + \frac{\xi\alpha}{2}\right)
\]
and repeat the above proof.

Let us consider the case \( 1 \leq p \leq \infty \). By (3.2) and Young’s inequality, we have
\[
\left\|\left(\partial_\xi^\alpha\right) T_n\right\|_p \lesssim \|K_\xi\|_1 \|\Delta_\alpha^\xi T_n\|_p.
\]
As above, we derive that \( \|K_\xi\|_1 \lesssim 1 \), which implies the inequality "\( \lesssim \)" in (3.1). The converse inequality can be proved similarly.

Using now the fact that
\[
\omega_\alpha(T_n, \delta)_p = \sup_{\xi \in \mathbb{R}^d, |\xi| = 1} \|\Delta_\alpha^\xi T_n\|_p,
\]
we obtain the following result.

**Corollary 3.1.** Let \( 0 < p \leq \infty \), \( \alpha > 0 \), \( n \in \mathbb{N} \), and \( 0 < \delta \leq \pi/n \). Then, for any \( T_n \in \mathcal{T}_n \), we have
\[
\sup_{\xi \in \mathbb{R}^d, |\xi| = 1} \left\|\left(\partial_\xi^\alpha\right) T_n\right\|_p \asymp \delta^{-\alpha} \omega_\alpha(T_n, \delta)_p.
\]

Further, using (3.4) we prove the following Bernstein type inequality for fractional directional derivative of trigonometric polynomials.
Corollary 3.2. Let $0 < p \leq \infty$, $\alpha \in \mathbb{N} \cup ((1/p - 1)_+, \infty)$, and $n \in \mathbb{N}$. Then, for any $T_n \in T_n$, we have

$$\sup_{\xi \in \mathbb{R}^d, |\xi|=1} \left\| \left( \frac{\partial}{\partial \xi} \right)^\alpha T_n \right\|_p \lesssim n^\alpha \| T_n \|_p. \quad (3.5)$$

Inequality (3.5) is the classical Bernstein inequality for $d = 1$ (see, e.g., [19, Ch. 4]). For the fractional $\alpha$ and $d = 1$ see [59] and [3], cf. [80]. For $1 \leq p \leq \infty$, $\alpha > 0$, and $d \in \mathbb{N}$, inequality (3.5) can be obtained from [111, Theorem 3].

Note also that (3.5) does not hold for $\alpha \not\in \mathbb{N} \cup ((1/p - 1)_+, \infty)$ (see Proposition 1.1).

Proof. To prove (3.5), one should take $\delta = 1/n$ in (3.4) and use the simple inequality

$$\| \Delta^\alpha f \|_p \lesssim \left( \sum_{\nu=0}^{\infty} \left\| \left( \frac{\alpha}{\nu} \right)^{\min(1, p)} \right\|_p \right)^{\frac{1}{\min(1, p)}} \| f \|_p \leq C(\alpha, p) \| f \|_p$$

(cf. property (c) in Section 4). \(\square\)

For moduli of smoothness of integer order, we have the following Nikol’skii–Stechkin–Boas result, which is new in the case $0 < p < 1$, $d \geq 2$.

Theorem 3.2. Let $0 < p \leq \infty$, $r \in \mathbb{N}$, and $0 < \delta \leq \pi/n$. Then, for any $T \in T_n$, $n \in \mathbb{N}$, one has

$$\| T_n \|_{\dot{W}^r_p} \asymp \delta^{-r} \omega_r(T_n, \delta)_p, \quad (3.6)$$

where the constants in this equivalence are independent of $T_n$ and $\delta$.

Proof. Let us first show that

$$\| T_n \|_{\dot{W}^r_p} \lesssim n^r \omega_r \left( T_n, \frac{\pi}{n} \right)_p. \quad (3.7)$$

Since

$$\| T_n \|_{\dot{W}^r_p} = \sum_{|k|_1=r} \left\| \frac{\partial^r T_n}{\partial x^{k_1}_1 \ldots \partial x^{k_d}_d} \right\|_p,$$

we have to verify that for any $k \in \mathbb{Z}^d_+ \setminus \{|k|_1 = r\}$, one has

$$\left\| \frac{\partial^r T_n}{\partial x^{k_1}_1 \ldots \partial x^{k_d}_d} \right\|_p \lesssim n^r \omega_r \left( T_n, \frac{\pi}{n} \right)_p.$$

Indeed, denoting $U_n(x) = \frac{\partial^{r-k_1}}{\partial x^{k_2}_2 \ldots \partial x^{k_d}_d} T_n(x)$ and applying Theorem 3.1 to $U_n$ in the one-dimensional case, we obtain

$$\left\| \frac{\partial^r T_n}{\partial x^{k_1}_1 \ldots \partial x^{k_d}_d} \|_p = \left\| \frac{\partial^{k_1}}{\partial x^{k_1}_1} U_n \right\|_p \lesssim n^{k_1} \| \Delta^{k_1}_{x;1} U_n \|_p$$

$$= n^{k_1} \left\| \frac{\partial^{r-k_1}}{\partial x^{k_2}_2 \ldots \partial x^{k_d}_d} \Delta^{k_1}_{x;1} T_n \right\|_p,$$
where $\Delta^{k_1}_{x_1} U_n$ is the $k_1$-th difference with respect to $x_1$. Repeating this procedure $d-1$ times, we get

$$\left\| \frac{\partial^r T_n}{\partial x_1^{k_1} \ldots \partial x_d^{k_d}} \right\|_p \lesssim n^r \Delta^{k_1}_{x_1} \ldots \Delta^{k_d}_{x_d} T_n \|_p \lesssim n^r \omega_{k_1, \ldots, k_d} \left( T_n, \frac{\pi}{n}, \ldots, \frac{\pi}{n} \right)_p,$$

where $\omega_{k_1, \ldots, k_d} \left( T_n, \frac{\pi}{n}, \ldots, \frac{\pi}{n} \right)_p$ is the mixed modulus of smoothness (see, e.g., [102]).

It remains only to apply the following equivalence result

$$\sum_{|k|_1=r} \omega_{k_1, \ldots, k_d} \left( T_n, \frac{\pi}{n}, \ldots, \frac{\pi}{n} \right)_p \asymp \omega_r \left( T_n, \frac{\pi}{n} \right)_p.$$ 

In the case $1 \leq p \leq \infty$, the proof of this equivalence is given in [102] and is based on the representation of the total difference of a function via the sum of the mixed differences. Hence, repeating the proof of Theorem 7 from [102], one can easily verify that the above equivalence holds in the case $0 < p < 1$ too. Thus, we have proved (3.7). Noting that Corollary 3.1 implies the equivalence $n^r \omega_r \left( T_n, \frac{\pi}{n}, \ldots, \frac{\pi}{n} \right)_p \asymp \delta^{-r} \omega_r \left( T_n, \delta \right)_p$, $0 < \delta < \pi/n$, we conclude the proof of the part ”$\lesssim$” in (3.6).

The reverse part follows from Corollary 3.1. □

We will also need the following result on equivalence between the Riesz derivatives (see (1.15)) and the directional derivatives.

**Corollary 3.3.** Let $1 < p < \infty$ and $\alpha > 0$. Then, for any $T \in \mathcal{T}$, one has

$$\sup_{\xi \in \mathbb{R}^d, |\xi| = 1} \left\| \left( \frac{\partial}{\partial \xi} \right)^\alpha T \right\|_p \asymp \|(-\Delta)^{\alpha/2} T\|_p.$$ 

Moreover, if $0 < p \leq \infty$ and $r \in \mathbb{N}$, then

$$\sup_{\xi \in \mathbb{R}^d, |\xi| = 1} \left\| \left( \frac{\partial}{\partial \xi} \right)^r T \right\|_p \asymp \| T \|_{W^r_p}.$$ 

**Proof.** The first part follows from (3.4) and the equivalence $\delta^{-\alpha} \omega_\alpha (T, \delta)_p \asymp \|(-\Delta)^{\alpha/2} T\|_p$, where $0 < \delta \leq 1/\deg T$ (see [111]). Equivalence (3.9) follows from Theorem 3.2 and Corollary 3.1. □

We conclude this section by the following new description of the classical Sobolev seminorm in terms of the norm defined by the directional derivatives. In the case $1 \leq p \leq \infty$, this extends (3.9) for any function $f \in W^r_p$.

**Theorem 3.3.** Let $1 \leq p \leq \infty$ and $r \in \mathbb{N}$. Then, for any $f \in W^r_p$, one has

$$\| f \|_{W^r_p} \asymp \sup_{\xi \in \mathbb{R}^d, |\xi| = 1} \left\| \left( \frac{\partial}{\partial \xi} \right)^r f \right\|_p.$$ 

**Proof.** The estimate

$$\sup_{\xi \in \mathbb{R}^d, |\xi| = 1} \left\| \left( \frac{\partial}{\partial \xi} \right)^r f \right\|_p \lesssim \| f \|_{W^r_p}$$

(3.10)
easily follows from the multinomial theorem given by
\[(x_1 + x_2 + \cdots + x_d)^r = \sum_{|k|=r} \frac{r!}{k_1!k_2!\cdots k_d!} x_1^{k_1} x_2^{k_2} \cdots x_d^{k_d}\]
and the triangle inequality.

Let us prove the upper estimate for \(\|f\|_{W^r_p}\). Let \(T_n \in \mathcal{T}_n\) be such that
\[\|f - T_n\|_{W^r_p} \to 0\quad\text{as}\quad n \to \infty.\]
Then, by (3.9) and (3.10), we obtain
\[\|f\|_{W^r_p} \leq \|T_n\|_{W^r_p} + \|f - T_n\|_{W^r_p} \lesssim \sup_{\xi \in \mathbb{R}^d, |\xi| = 1} \left\| \left( \frac{\partial}{\partial \xi} \right)^r T_n \right\|_p + \|f - T_n\|_{W^r_p},\]
\[\lesssim \sup_{\xi \in \mathbb{R}^d, |\xi| = 1} \left\| \left( \frac{\partial}{\partial \xi} \right)^r f \right\|_p + \sup_{\xi \in \mathbb{R}^d, |\xi| = 1} \left\| \left( \frac{\partial}{\partial \xi} \right)^r (f - T_n) \right\|_p + \|f - T_n\|_{W^r_p},\]
\[\lesssim \sup_{\xi \in \mathbb{R}^d, |\xi| = 1} \left\| \left( \frac{\partial}{\partial \xi} \right)^r f \right\|_p + \|f - T_n\|_{W^r_p}.\]
It remains only to pass to the limit as \(n \to \infty\). \(\square\)
4. Basic properties of fractional moduli of smoothness

Let us recall several basic properties of moduli of smoothness ([12, 82, 85, 97]). For \( f, f_1, f_2 \in L^p(\mathbb{T}^d) \), \( 0 < p \leq \infty \), and \( \alpha, \beta \in \mathbb{N} \cup ((1/p - 1)_+, \infty) \), we have

(a) \( \omega_\alpha(f, \delta)_p \) is a non-negative non-decreasing function of \( \delta \) such that \( \lim_{\delta \to 0^+} \omega_\alpha(f, \delta)_p = 0 \);

(b) \( \omega_\alpha(f_1 + f_2, \delta)_p \leq 2^{(1/p - 1)_+} \left( \omega_\alpha(f_1, \delta)_p + \omega_\alpha(f_2, \delta)_p \right) \);

(c) \( \omega_{\alpha + \beta}(f, \delta)_p \leq \left( \sum_{\nu = 0}^{\infty} \left| \binom{\beta}{\nu} \right|^{\min(1, p)} \right)^{1/(\min(1, p))} \omega_\alpha(f, \delta)_p \);

(d) for \( \lambda \geq 1 \),

\[
\omega_\alpha(f, \lambda \delta)_p \leq C(\alpha, p, d) \lambda^{\alpha + d(1/p - 1)_+} \omega_\alpha(f, \delta)_p .
\]

(e) for \( 0 < t \leq \delta \),

\[
\frac{\omega_\alpha(f, \delta)_p}{\delta^{\alpha + d(1/p - 1)_+}} \leq C(\alpha, p, d) \frac{\omega_\alpha(f, t)_p}{t^{\alpha + d(1/p - 1)_+}} .
\]

Statement (a) follows from \( |\omega_\alpha(f_1, \delta)_p - \omega_\alpha(f_2, \delta)_p| \lesssim \|f_1 - f_2\|_p \), inequalities in (b) and (c) are clear, and (d) and (e) will be proved later in this section.

Now, we prove several basic results in approximation theory. In what follows, \( E_n(f)_p \) is the best approximation of \( f \in L^p(\mathbb{T}^d) \) by trigonometric polynomials \( T \in \mathcal{T}_n \), i.e.,

\[
E_n(f)_p = \inf_{T \in \mathcal{T}_n} \|f - T\|_p .
\]

We start with the direct inequality.

**Proposition 4.1.** Let \( 0 < p \leq \infty \), \( \alpha \in \mathbb{N} \cup ((1/p - 1)_+, \infty) \), and \( n \in \mathbb{N} \). Then

\[
E_n(f)_p \lesssim \omega_\alpha \left( f, \frac{1}{n} \right)_p .
\]

Inequality (4.1) follows immediately from the Jackson-type inequality for the moduli of smoothness of integer order [95]. Indeed, for \( \alpha + r \in \mathbb{N} \) and \( r > (1/p - 1)_+ \), property (c) implies

\[
E_n(f)_p \lesssim \omega_{\alpha + r} \left( f, \frac{1}{n} \right)_p \lesssim \omega_\alpha \left( f, \frac{1}{n} \right)_p .
\]

Note that for \( 1 < p < \infty \) the technique from [17] allows us to obtain a sharper version of (4.1) for the fractional moduli of smoothness:

\[
\frac{1}{n^\alpha} \left( \sum_{k=0}^{n} (k + 1)^{\alpha p - 1} E_k(f)_p \right)^{\frac{1}{p}} \lesssim \omega_\alpha \left( f, \frac{1}{n} \right)_p ,
\]

where

\[
\rho = \rho(p) = \max(p, 2) .
\]

The inverse inequality is given as follows.
Proposition 4.2. Let $0 < p \leq \infty$, $\alpha \in \mathbb{N} \cup ((1/p - 1)_+, \infty)$, and $n \in \mathbb{N}$. Then we have

$$\omega_\alpha \left( f, \frac{1}{n} \right)_p \lesssim \frac{1}{n^\alpha} \left( \sum_{k=0}^{n} (k + 1)^{\alpha \tau - 1} E_k(f)_p^\tau \right)^{\frac{1}{\tau}},$$

where

$$\tau = \tau(p) = \begin{cases} \min(p, 2), & p < \infty; \\ 1, & p = \infty. \end{cases}$$

In the case $1 \leq p \leq \infty$, $\alpha \in \mathbb{N}$, this result is well known (see, e.g., [18, Ch. 7], [16], [101] and the reference therein). To the best of our knowledge, the case $p = 1$, $\infty$ and $\alpha > 0$ does not seem to have been considered before for this moduli of smoothness (cf. [21]). In the case $0 < p < 1$, $d > 1$ this result is new, while the case $0 < p < 1$ and $d = 1$ was recently considered in [82]. Moreover, the corresponding results in terms of different realizations of the $K$-functional (in particular, using the Laplace operator) were obtained in the papers [79, 82].

Proof. The proof follows the standard argument. We give the proof only for the case $0 < p \leq 1$ or $p = \infty$. The case $1 < p < \infty$ can be handled using [16].

Let $T_n \in \mathcal{T}_n$ be such that $E_n(f)_p = \|f - T_n\|_p$. Then for any $m \in \mathbb{N}$ we have with $p_1 = \min(p, 1)$

$$\omega_\alpha \left( f, \frac{1}{n} \right)_{p_1} \leq \omega_\alpha \left( f - T_{2^m+1}, \frac{1}{n} \right)_{p_1} + \omega_\alpha \left( T_{2^m+1}, \frac{1}{n} \right)_{p_1} \lesssim E_{2^m+1}(f)_{p_1} + \omega_\alpha \left( T_{2^m+1}, \frac{1}{n} \right)_{p_1}.$$

Next, by (3.4) and (3.5), choosing $m$ such that $2^m < n \leq 2^{m+1}$, we obtain

$$\omega_\alpha \left( T_{2^m+1}, \frac{1}{n} \right)_{p_1} \leq \omega_\alpha \left( T_1 - T_0, \frac{1}{n} \right)_{p_1} + \sum_{\nu=0}^{m} \omega_\alpha \left( T_{2^{\nu+1}} - T_{2^\nu}, \frac{1}{n} \right)_{p_1} \lesssim \frac{1}{n^\alpha p_1} \left( E_0(f)_{p_1} + \sum_{\nu=0}^{m} 2^{(\nu+1)\alpha p_1} E_{2^\nu}(f)_{p_1} \right) \lesssim \frac{1}{n^\alpha p_1} \sum_{k=0}^{2^m} (k + 1)^{\alpha p_1 - 1} E_k(f)_{p_1}.$$

Combining (4.4) and (4.5), we proved the desired result. \hfill \Box

Now, we will prove a realization result, which will be crucial in our further study.

Theorem 4.1. Let $f \in L_p(\mathbb{T}^d)$, $0 < p \leq \infty$, and $\alpha \in \mathbb{N} \cup ((1/p - 1)_+, \infty)$. Then, for any $\delta \in (0, 1)$, we have

$$\omega_\alpha(f, \delta)_p \asymp R_\alpha(f, \delta)_p,$$
where $\mathcal{R}_\alpha(f, \delta)_p$ is the realization of the K-functional, i.e.,

$$
\mathcal{R}_\alpha(f, \delta)_p := \inf_{T \in \mathcal{T}_{1/\delta}} \left\{ \| f - T \|_p + \delta^\alpha \sup_{\xi \in \mathbb{R}^d, |\xi| = 1} \left\| \left( \frac{\partial}{\partial \xi} \right)^\alpha T \right\|_p \right\}.
$$

Moreover, if $T \in \mathcal{T}_n$, $n = [1/\delta]$, is such that $\| f - T \|_p \lesssim E_n(f)_p$, then

$$
(4.7) \quad \omega_\alpha(f, \delta)_p \lesssim \| f - T \|_p + \delta^\alpha \sup_{\xi \in \mathbb{R}^d, |\xi| = 1} \left\| \left( \frac{\partial}{\partial \xi} \right)^\alpha T \right\|_p,
$$

This theorem is known for $d = 1$, $0 < p \leq \infty$, and $\alpha \in \mathbb{N}$ (see [22]); for $d = 1$, $0 < p < 1$, and $\alpha > 0$ see [45]; for $d = 1$, $1 \leq p \leq \infty$, and $\alpha > 0$ see [86]. In the case $d \geq 1$, $0 < p \leq \infty$, and $\alpha \in \mathbb{N}$, (4.7) was stated in [24] without the proof.

**Proof.** First, let us prove the estimate from above in (4.6). Let $n = [1/\delta]$ and $T_n \in \mathcal{T}_n$. By Corollary 3.1, we estimate

$$
\omega_\alpha(f, \delta)_p \lesssim \| f - T_n \|_p + \omega_\alpha(T_n, \delta)_p
$$

$$
\lesssim \| f - T_n \|_p + \delta^\alpha \sup_{\xi \in \mathbb{R}^d, |\xi| = 1} \left\| \left( \frac{\partial}{\partial \xi} \right)^\alpha T_n \right\|_p.
$$

It remains to take infimum over all $T_n \in \mathcal{T}_n$.

Let us prove the estimate from below. Let $T \in \mathcal{T}_n$ be such that $\| f - T \|_p \lesssim E_n(f)_p$. Then by (4.1) and (3.4), we obtain

$$
\mathcal{R}_\alpha \left( f, \frac{1}{n} \right)_p \lesssim \| f - T \|_p + n^{-\alpha} \sup_{\xi \in \mathbb{R}^d, |\xi| = 1} \left\| \left( \frac{\partial}{\partial \xi} \right)^\alpha T \right\|_p
$$

$$
\lesssim \omega_\alpha \left( f, \frac{1}{n} \right)_p + \omega_\alpha \left( T, \frac{1}{n} \right)_p \lesssim \omega_\alpha \left( f, \frac{1}{n} \right)_p,
$$

that is, (4.6) follows.

In fact, we have proved that

$$
\omega_\alpha \left( f, \frac{1}{n} \right)_p \lesssim \mathcal{R}_\alpha \left( f, \frac{1}{n} \right)_p
$$

$$
\lesssim \| f - T \|_p + n^{-\alpha} \sup_{\xi \in \mathbb{R}^d, |\xi| = 1} \left\| \left( \frac{\partial}{\partial \xi} \right)^\alpha T \right\|_p \lesssim \omega_\alpha \left( f, \frac{1}{n} \right)_p,
$$

which implies (4.7).

Similarly, taking into account Theorem 3.2, we can prove the following result.

**Theorem 4.2.** Let $f \in L_p(\mathbb{T}^d)$, $0 < p \leq \infty$, and $r \in \mathbb{N}$. Then, for any $\delta \in (0, 1)$, we have

$$
\omega_r(f, \delta)_p \asymp \mathcal{R}_r^2(f, \delta)_p,
$$

where

$$
\mathcal{R}_r^2(f, \delta)_p := \inf_{T \in \mathcal{T}_{1/\delta}} \left\{ \| f - T \|_p + \delta^r \| T \|_{W^r_p} \right\}.
$$
Moreover, if $T \in \mathcal{T}_n$, $n = [1/\delta]$, is such that $\|f - T\|_p \lesssim E_n(f)_p$, then
\[ \omega_r(f, \delta)_p \lesssim \|f - T\|_p + \delta
\]

In the case $d = 1$ and $0 < p \leq \infty$, this was proved in [22]; for $1 < p < \infty$ (see [21]). The result is new in the case $d > 1$, $0 < p \leq 1$, and $p = \infty$.

Next we prove the following important property of the moduli of smoothness.

**Theorem 4.3.** Let $f \in L_p(\mathbb{T}^d)$, $0 < p \leq \infty$, $\alpha \in \mathbb{N} \cup ((1/p - 1)_, \infty)$, $\lambda > 0$, and $t > 0$. Then
\[ \omega_\alpha(f, \lambda t)_p \lesssim (1 + \lambda)^{\alpha + d(1/p - 1)} + \omega_\alpha(f, t)_p. \]

In particular, for any $0 < h < t$, one has
\[ \frac{\omega_\alpha(f, t)_p}{t^{\alpha + d(1/p - 1)_+}} \lesssim \frac{\omega_\alpha(f, h)_p}{h^{\alpha + d(1/p - 1)_+}}. \]

When $d = 1$ and $0 < p \leq 1$, this was obtained in [71, 77] (for integer $\alpha$) and [82] (for positive $\alpha$). For the case $d \in \mathbb{N}$, $1 < p < \infty$, and $\alpha > 0$ see [111, Theorem 7].

**Proof.** We follow the idea from [79, p. 194]. We will consider only the case $0 < p \leq 1$. The case $p = \infty$ can be proved by the same way. The case $1 < p < \infty$ follows from (4.6) and (1.14).

Denote $\delta = (\delta_1, \ldots, \delta_d) \in \mathbb{R}^d$, $\delta_j > 0$, $j = 1, \ldots, d$, and
\[ K_\delta(x) = \sum_{k \in \mathbb{Z}^d} \varphi_\delta(k)e^{i(k,x)}, \]
where $\varphi_\delta(t) = v((\delta, t))$, $v \in C^\infty(\mathbb{R})$, $v(s) = 1$ for $|s| \leq 1/2$ and $v(s) = 0$ for $|s| > 1$.

We can assume that $\lambda > 1$. Suppose also that $|\delta| \leq h$ and $T \in \mathcal{T}_{1/h}$.

Using Theorem 3.1, we obtain
\begin{align*}
\|\Delta^\alpha_\delta f\|_p^p & \lesssim \|f - K_{\lambda \delta} * T\|_p^p + \|\Delta^\alpha_\delta(K_{\lambda \delta} * T)\|_p^p \\
& \lesssim \|f - T\|_p^p + \|T - K_{\lambda \delta} * T\|_p^p + \left\| \frac{\partial}{\partial(\lambda \delta)} \right\|^\alpha (K_{\lambda \delta} * T\|_p^p \\
& = \|f - T\|_p^p + \|T - K_{\lambda \delta} * T\|_p^p + \lambda^{\alpha p} \left\| K_{\lambda \delta} * \left\| \frac{\partial}{\partial(\delta)} \right\|\right\|^\alpha T\|_p^p.
\end{align*}

Now, we will show that
\[ \|T - K_{\lambda \delta} * T\|_p \lesssim \lambda^{\alpha + d(1/p - 1)} \left\| \frac{\partial}{\partial(\delta)} \right\|^\alpha T\|_p^p \]
and
\[ \left\| K_{\lambda \delta} * \left\| \frac{\partial}{\partial(\delta)} \right\|\right\|^\alpha T\|_p^p \lesssim \lambda^{d(1/p - 1)} \left\| \frac{\partial}{\partial(\delta)} \right\|^\alpha T\|_p^p.
\]

We will prove only the first inequality, the second one can be obtained similarly. It is easy to see that (4.11) is equivalent to the following inequality
\[ \|A_{\lambda \delta} * T\|_p \lesssim \lambda^{\alpha + d(1/p - 1)} \|T\|_p, \]
where
\[ A_\delta(x) = \sum_{k \in \mathbb{Z}^d} \psi_\delta(k)e^{i(k,x)} \]
and
\[ \psi_\delta(t) = \frac{(1 - \varphi_\delta(t))}{(it, \delta)^\alpha} \left( \sqrt{(t_1\delta_1)^2 + \cdots + (t_d\delta_d)^2} \right). \]

The same argument as in the proof of (3.3) implies
\[
\| A_{\lambda\delta} \ast T \|_p \lesssim \sum_{j=1}^d \delta_j \| A_{\lambda\delta} \|_p \| T \|_p \lesssim \| \hat{\psi}_{\lambda 1} \|_{L_p(\mathbb{R}^d)} \| T \|_p
\]
\[
\lesssim \lambda^{d\left(\frac{1}{p} - 1\right)} \| \hat{\psi}_{1} \|_{L_p(\mathbb{R}^d)} \| T \|_p \lesssim \lambda^{d\left(\frac{1}{p} - 1\right)} \| T \|_p,
\]
that is, (4.11) is verified.

Now, combining (4.10)–(4.12), we obtain
\[
\| \Delta_{\lambda\delta} f \|_p \lesssim \| f - T \|_p + \lambda^{\alpha + \gamma} \| \left( \frac{\partial}{\partial \delta}\right)^\alpha T \|_p
\]
\[
\lesssim \lambda^{\alpha + \gamma} \| f - T \|_p + \delta^\alpha \sup_{\xi} \left( \left( \frac{\partial}{\partial \xi}\right)^\alpha T \right)_p
\]
Finally, taking infimum over all \( T \in \mathcal{T}_{1/h} \) and using (4.6), we get
\[
\| \Delta_{\lambda\delta} f \|_p \lesssim \lambda^{\alpha + \gamma} \mathcal{R}_\alpha (f, h)_p \lesssim \lambda^{\alpha + \gamma} \omega_\alpha (f, h)_p
\]
which implies (4.8).

We finish this section by the Marchaud inequality for the fractional moduli of smoothness.

**Theorem 4.4.** Let \( f \in L_p(\mathbb{T}^d), 0 < p \leq \infty, \alpha \in \mathbb{N} \cup \{1/p - 1, \infty\}, \) and \( \gamma > 0 \) be such that \( \alpha + \gamma \in \mathbb{N} \cup \{1/p - 1, \infty\}. \) Then, for any \( \delta \in (0, 1), \) one has
\[
(4.13) \quad \omega_\alpha (f, \delta)_p \lesssim \delta^\alpha \left( \int_\delta^1 \left( \frac{\omega_{\gamma + \alpha} (f, t)}{t^\alpha} \right)^\tau dt \right)^{\frac{1}{\tau}},
\]
where \( \tau \) is given by (4.3).

The Marchaud inequality for the moduli of smoothness of integer order can be found, e.g., in [18, p. 48] and [20]. The case \( 1 < p < \infty \) for fractional moduli was handled in [104, Theorem 2.1].

**Proof.** The proof is a combination of the direct and inverse estimates (4.1) and (4.2) as well as the monotonicity property of modulus of smoothness \( \omega_\alpha (f, \delta)_p \preceq \omega_\alpha (f, 2\delta)_p \) (see (4.8)).
5. Hardy–Littlewood–Nikol’skii inequalities for trigonometric polynomials

We say that a continuous on \( \mathbb{R}^d \setminus \{0\} \) function \( \psi(\xi) \) belongs to the class \( H_\alpha, \alpha \in \mathbb{R} \), if \( \psi \) is a homogeneous function of degree \( \alpha \), i.e.,

\[
\psi(\tau \xi) = \tau^{\alpha} \psi(\xi), \quad \tau > 0, \quad \xi \in \mathbb{R}^d.
\]

In addition, if \( \alpha \leq 0 \), we assume that \( \psi \in C^\infty(\mathbb{R}^d \setminus \{0\}) \).

Any function \( \psi \in H_\alpha \) generates the Weyl-type differentiation operator as follows:

\[
\mathcal{D}(\psi) : \sum_{\nu \in \mathbb{Z}^d} c_\nu e^{i(\nu,x)} \rightarrow \sum_{\nu \in \mathbb{Z}^d} \psi(\nu)c_\nu e^{i(\nu,x)},
\]

where \( \sum_{\nu} a_\nu \) means \( \sum_{|\nu|>0} a_\nu \). Let us give several important examples of the Weyl-type operators:

1. the linear differential operator

\[
P_m(D)f = \sum_{k_1+\cdots+k_d=m, k \in \mathbb{Z}^d_+} a_k D^k f, \quad D^k = D^{k_1} \cdots D^{k_d} = \frac{\partial^{k_1+\cdots+k_d}}{\partial x_1^{k_1} \cdots \partial x_d^{k_d}},
\]

with

\[
\psi(\xi) = \sum_{k_1+\cdots+k_d=m, k \in \mathbb{Z}^d_+} a_k (i\xi_1)^{k_1} \cdots (i\xi_d)^{k_d};
\]

2. the fractional Laplacian \( (-\Delta)^{\alpha/2}f \) (here \( \psi(\xi) = |\xi|^{\alpha}, \xi \in \mathbb{R}^d \));

3. the classical Weyl derivative \( f^{(\alpha)} \) (here \( \psi(\xi) = (i\xi)^{\alpha}, \xi \in \mathbb{R} \)).

In this section, we study the sharp Hardy–Littlewood–Nikol’skii inequality given by

\[
\|\mathcal{D}(\psi)T\|_q \lesssim \eta(n)\|\mathcal{D}(\varphi)T\|_p, \quad T \in \mathcal{T}_n',
\]

where \( 0 < p \leq q \leq \infty \), \( \psi \in H_\alpha \), and \( \varphi \in H_{\alpha+\gamma} \).

Note that if \( \psi(\xi) = |\xi|^{\alpha}, \varphi \equiv 1, 1 < p < q < \infty \), and \( \alpha = d(1/p - 1/q) \), then \( \eta(n) = 1 \) by the Hardy–Littlewood theorem on fractional integration. On the other hand, if \( \psi = \varphi \equiv 1 \), then \( \eta(n) = n^{d(1/p - 1/q)} \), which corresponds to the Nikol’skii inequality for trigonometric polynomials (see Subsection 1.4).

The main result of this section is the following theorem.

**Theorem 5.1.** Let \( 0 < p < q \leq \infty \), \( \alpha > 0 \), \( \gamma \geq 0 \), \( \psi \in H_\alpha \), and \( \varphi \in H_{\alpha+\gamma} \). Let also \( \frac{\psi}{\varphi} \in C^\infty(\mathbb{R}^d \setminus \{0\}) \) and \( \frac{\varphi}{\psi} \in C^\infty(\mathbb{R}^d \setminus \{0\}) \). We have

\[
\sup_{T \in \mathcal{T}_n'} \|\mathcal{D}(\psi)T\|_q \lesssim \sigma(n),
\]

where \( \sigma(\cdot) \) is given as follows:
If $0 < p \leq 1$ and $p < q < \infty$, then
\[
\sigma(t) := \begin{cases} 
    t^d(\frac{1}{p} - 1), & \gamma > d(\frac{1}{p} - \frac{1}{q}); \\
    t^d(\frac{1}{p} - 1) \ln(\frac{t}{q}), & 0 < \gamma = d \left( 1 - \frac{1}{q} \right); \\
    t^d(\frac{1}{p} - 1 - \frac{1}{q}), & 0 < \gamma < d \left( 1 - \frac{1}{q} \right); \\
    t^d(\frac{1}{p} - 1), & \gamma = 0, 0 < p \leq 1 < q < \infty; \\
    t^d(\frac{1}{p} - \frac{1}{q}), & \gamma = 0, 0 < q \leq 1, \text{ and } \frac{\psi(\xi)}{\varphi(\xi)} \equiv \text{const}; \\
    t^d(\frac{1}{p} - 1) \ln(t + 1), & \gamma = 0, d = 1, 0 < q < 1, \text{ and } \frac{\psi(\xi)}{\varphi(\xi)} \not\equiv \text{const}; \\
    t^d(\frac{1}{p} - 1) \ln(t + 1), & \gamma = 0, d = 1, q = 1, \text{ and } \frac{\psi(\xi)}{\varphi(\xi)} \not\equiv \text{const},
\end{cases}
\]

If $0 < p \leq 1$ and $q = \infty$, then
\[
\sigma(t) := \begin{cases} 
    t^d(\frac{1}{p} - 1), & \gamma > d; \\
    t^d(\frac{1}{p} - 1), & \gamma = d = 1 \text{ and } \frac{\psi(\xi)}{\varphi(\xi)} = A|\xi|^{-\gamma} \text{ sign } \xi \text{ for some } A \in \mathbb{C} \setminus \{0\}; \\
    t^d(\frac{1}{p} - 1) \ln(t + 1), & \gamma = d = 1 \text{ and } \frac{\psi(\xi)}{\varphi(\xi)} \not\equiv A|\xi|^{-\gamma} \text{ sign } \xi \text{ for any } A \in \mathbb{C} \setminus \{0\}; \\
    t^d(\frac{1}{p} - 1) \ln(t + 1), & \gamma = d \geq 2; \\
    t^d(\frac{1}{p} - \frac{1}{q}), & 0 \leq \gamma < d(\frac{1}{p} - \frac{1}{q}).
\end{cases}
\]

If $1 < p < q \leq \infty$, then
\[
\sigma(t) := \begin{cases} 
    1, & \gamma \geq d(\frac{1}{p} - \frac{1}{q}), \quad q < \infty; \\
    1, & \gamma > \frac{d}{p}, \quad q = \infty; \\
    \ln^\frac{1}{p}(t + 1), & \gamma = \frac{d}{p}, \quad q = \infty; \\
    t^d(\frac{1}{p} - \frac{1}{q} - \frac{1}{q}), & 0 \leq \gamma < d(\frac{1}{p} - \frac{1}{q}).
\end{cases}
\]

**Proof.** The proof of Theorem 5.1 can be obtained by combining Lemmas 5.1–5.11 from the following two Subsections 5.1 and 5.2. In more detail, in the case (1) the proof follows from Lemmas 5.9 and 5.10 (the case $\gamma = 0$) and from Lemmas 5.1, 5.2, 5.6, and 5.7 (the case $\gamma > 0$). In the case (2), the proof follows from Lemmas 5.8 and 5.9. Concerning the case (3), see Lemma 5.11. \qed

In what follows, the de la Vallée Poussin type kernel is defined by
\[
V_n(x) := \sum_{k \in \mathbb{Z}^d} \left( \prod_{j=1}^d v \left( \frac{k_j}{n} \right) \right) e^{i(k,x)},
\]
where $v \in C^\infty(\mathbb{R})$ is monotonic for $t \geq 0$, $v(t) = v(-t)$, $v(t) = 1$ for $|t| \leq 1$ and $v(t) = 0$ for $|t| \geq 2$.

**Remark 5.1.** We do not know the sharp growth behavior of $\sup_{T \in \mathcal{T}_n} \|D(\psi/\varphi)\|, \sup_{T \in \mathcal{T}_n} \|D(\psi/\varphi)\|$, in the case $0 < p < q \leq 1$, $\gamma = 0$, $d \geq 2$, and $\frac{\psi(\xi)}{\varphi(\xi)} \not\equiv \text{const}$. However, we can show that
\[
\sup_{T \in \mathcal{T}_n} \|D(\psi/\varphi)\| \lesssim \|D(\psi/\varphi)\| \lesssim \|D(\psi/\varphi)\|.
\]
where \( V_n(x) = e^{ix} V_{n/4}(2x) \) (see Lemma 5.1 and the proof of Lemma 5.10).

**Remark 5.2.** Let again \( 0 < p < q \leq 1, \gamma = 0, d \geq 2, \) and \( \frac{\psi(\xi)}{\varphi(\xi)} \neq \text{const.} \) Let either \( \psi \) be a polynomial or \( d/(d+\alpha) < q \leq 1 \) and let either \( \varphi \) be a polynomial or \( d/(d+\alpha) < p < 1 \). Then, we can show the following estimate from above

\[
(5.3) \sup_{T \in T_n} \frac{\|D(\psi)T\|_q}{\|D(\varphi)T\|_p} \lesssim n^{d(\frac{1}{p} - 1)} \begin{cases} 1, & 0 < q < 1; \\ \ln(n+1), & q = 1. \end{cases}
\]

Since the proof of (5.3) is based on Theorem 10.1′, we will give it in Section 10.

Regarding the estimate from below, by Theorem 2.1 and Lemma 5.5 (ii), we have that \( \sup_{T \in T_n} \|D(\psi/\varphi)V_n\|_q \neq \infty \). Thus, by (5.3), there exists a sequence \( \{\varepsilon_n\} \) such that \( \varepsilon_n \to \infty \) and

\[
n^{d(\frac{1}{p} - 1)} \varepsilon_n \lesssim \sup_{T \in T_n} \frac{\|D(\psi)T\|_q}{\|D(\varphi)T\|_p},
\]

which provides optimality of (5.3) in some sense.

**Remark 5.3.** Without the assumption \( \hat{\varphi}/\hat{\psi} \in C^\infty(\mathbb{R}^d \setminus \{0\}) \) in Theorem 5.1, the result remains true if (5.1) is replaced by

\[
\sup_{T \in T_n} \|D(\psi/\varphi)V_n\|_q \lesssim \sigma(n).
\]

**Remark 5.4.** To illustrate a different behavior of \( \sigma(\cdot) \) in (2) for \( \gamma = d = 1 \), we consider the following example. Taking \( \varphi(\xi) = (i\xi) \) and \( \psi(\xi) = |\xi|^2 \), we get

\[
\sigma(n) \asymp n^{\frac{1}{p} - 1} \left\| D \left( \frac{i\xi}{|\xi|^2} \right) V_n \right\|_\infty \asymp n^{\frac{1}{p} - 1} \left\| \sum_{0 < |k| < n} \frac{e^{ikx}}{ik} \right\|_\infty \asymp n^{\frac{1}{p} - 1} \left\| \sum_{0 < k < n} \sin kx \right\|_\infty \asymp n^{\frac{1}{p} - 1}.
\]

On the other hand, considering \( \varphi(\xi) = |\xi| \) and \( \psi(\xi) = |\xi|^2 \) gives

\[
\sigma(n) \asymp n^{\frac{1}{p} - 1} \left\| D \left( \frac{1}{|\xi|} \right) V_n \right\|_\infty \asymp n^{\frac{1}{p} - 1} \left\| \sum_{0 < |k| < n} \frac{e^{ikx}}{|k|} \right\|_\infty \asymp n^{\frac{1}{p} - 1} \left\| \sum_{0 < k < n} \cos kx \right\|_\infty \asymp n^{\frac{1}{p} - 1} \ln n.
\]

Moreover, similar examples can be constructed to illustrate different behavior with respect to \( \psi \) and \( \varphi \) in (1) when \( 0 < q \leq 1 \).

Now, let us present two important cases of Theorem 5.1.
Corollary 5.1. Let $d = 1$, $0 < p < q \leq \infty$, $\alpha > 0$, and $\gamma \geq 0$. We have

$$\sup_{T \in T_n} \frac{\|T(\alpha)\|_q}{\|T(\alpha+\gamma)\|_p} \asymp \sigma(n),$$

where $\sigma(\cdot)$ is given as follows:

(1) if $0 < p \leq 1$ and $p < q \leq \infty$, then

$$\sigma(t) := \begin{cases} t^{\frac{1}{p}-1}, & \gamma \geq \left(1 - \frac{1}{q}\right) +; \\ t^{\frac{1}{p}-1} \ln \frac{1}{q} (t + 1), & 0 < \gamma = \left(1 - \frac{1}{q}\right) +; \\ t^{\frac{1}{p}-\frac{1}{q}} - \gamma, & 0 < \gamma < \left(1 - \frac{1}{q}\right) +; \\ t^{\frac{1}{p}-\frac{1}{q}}, & \gamma = 0, \end{cases}$$

(2) if $1 < p \leq q \leq \infty$, then

$$\sigma(t) := \begin{cases} 1, & \gamma \geq \frac{1}{p} - \frac{1}{q}, \quad q < \infty; \\ 1, & \gamma > \frac{1}{p}, \quad q = \infty; \\ \ln \frac{1}{q} (t + 1), & \gamma = \frac{1}{p}, \quad q = \infty; \\ t^{\frac{1}{p}-\frac{1}{q} - \gamma}, & 0 \leq \gamma < \frac{1}{p} - \frac{1}{q}. \end{cases}$$

Now, let us consider the corresponding result in the case of Riesz derivatives.

Corollary 5.2. Let $0 < p < q \leq \infty$, $\alpha > 0$, and $\gamma \geq 0$. We have

$$\sup_{T \in T_n} \frac{\|(-\Delta)^{\alpha/2} T\|_q}{\|(-\Delta)^{(\alpha+\gamma)/2} T\|_p} \asymp \sigma_\Delta(n),$$

where $\sigma_\Delta(\cdot)$ is given as follows:

(1) if $0 < p \leq 1$ and $p < q \leq \infty$, then

$$\sigma_\Delta(t) := \begin{cases} t^{d\left(\frac{1}{p}-1\right)}, & \gamma > d\left(1 - \frac{1}{q}\right) +; \\ t^{d\left(\frac{1}{p}-1\right)} \ln \frac{1}{q} (t + 1), & 0 < \gamma = d\left(1 - \frac{1}{q}\right) +; \\ t^{d\left(\frac{1}{p}-\frac{1}{q}\right)} - \gamma, & 0 < \gamma < d\left(1 - \frac{1}{q}\right) +; \\ t^{d\left(\frac{1}{p}-\frac{1}{q}\right)}, & \gamma = 0, \end{cases}$$

(2) if $1 < p < q \leq \infty$, then

$$\sigma_\Delta(t) := \begin{cases} 1, & \gamma \geq d\left(\frac{1}{p} - \frac{1}{q}\right), \quad q < \infty; \\ 1, & \gamma > \frac{d}{p}, \quad q = \infty; \\ \ln \frac{1}{q} (t + 1), & \gamma = \frac{d}{p}, \quad q = \infty; \\ t^{d\left(\frac{1}{p}-\frac{1}{q}\right) - \gamma}, & 0 \leq \gamma < d\left(\frac{1}{p} - \frac{1}{q}\right). \end{cases}$$

In particular, this and Theorem 5.1 imply the following two analogues of the Hardy–Littlewood fractional integration theorem.
Corollary 5.3. Let $1 \leq p < q \leq \infty$, $f \in L_p(\mathbb{T}^d)$, and $\int_{\mathbb{T}^d} f(x)dx = 0$. Then

$$\|(-\Delta)^{-\gamma/2} f\|_q \lesssim \|f\|_p$$

holds provided $\gamma > d(1/p - 1/q)$, $p = 1$ or/and $q = \infty$.

Corollary 5.4. Let $d = 1, 0 < p \leq 1$, $f \in AC(\mathbb{T})$, and $\int_{\mathbb{T}} f(x)dx = 0$. Then

$$\|f\|_\infty \lesssim \|f'\|_p.$$ 

5.1. Hardy–Littlewood–Nikol’skii $(L_p, L_q)$ inequalities for $0 < p \leq 1$. Recall that $V_n$ is given by (5.2).

Lemma 5.1. Let $0 < p \leq 1$, $p < q \leq \infty$, $\alpha > 0$, $\gamma \geq 0$, $\psi \in \mathcal{H}_\alpha$, and $\varphi \in \mathcal{H}_{\alpha+\gamma}$.

(i) We have

$$\sup_{T \in T^*_n} \frac{\|\mathcal{D}(\psi)T\|_q}{\|\mathcal{D}(\varphi)\|_p} \lesssim n^{d(\frac{1}{p} - 1)} \|\mathcal{D}(\psi/\varphi)V_n\|_q. \tag{5.4}$$

(ii) Assuming that $\gamma > 0$ and $\varphi(x) = 0$ if and only if $x = 0$, we have

$$n^{d(\frac{1}{p} - 1)} \|\mathcal{D}(\psi/\varphi)V_n\|_q \lesssim \sup_{T \in T^*_n} \frac{\|\mathcal{D}(\psi)T\|_q}{\|\mathcal{D}(\varphi)\|_p}. \tag{5.5}$$

(iii) Assuming that $\gamma > 0$ and $\frac{\psi}{\varphi}, \frac{\psi}{\varphi} \in C^\infty(\mathbb{R}^d \setminus \{0\})$, we have

$$\sup_{T \in T^*_n} \frac{\|\mathcal{D}(\psi)T\|_q}{\|\mathcal{D}(\varphi)\|_p} \lesssim n^{d(\frac{1}{p} - 1)} \|\mathcal{D}(\psi/\varphi)V_n\|_q.$$ 

Proof. (i) First, we consider the case $0 < q \leq 1$. We have

$$\mathcal{D}(\psi)T_n(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} (\mathcal{D}(\psi/\varphi)V_n(t)) (\mathcal{D}(\varphi)T_n(x - t)) dt.$$ 

Considering the product $(\mathcal{D}(\psi/\varphi)V_n(t)) (\mathcal{D}(\varphi)T_n(x - t))$ as a trigonometric polynomial of degree $3n$ in each variable $t_j, j = 1, \ldots, d$, and applying $(L_1, L_q)$ Nikol’skii’s inequality, we obtain

$$|\mathcal{D}(\psi)T_n(x)|^q \lesssim n^{d(1 - q)} \int_{\mathbb{T}_d} \|\mathcal{D}(\psi/\varphi)V_n(t)\| (\mathcal{D}(\varphi)T_n(x - t))|^q dt.$$ 

Integrating this inequality with respect to $x$ and applying the Fubini theorem and the $(L_q, L_p)$ Nikol’skii inequality, we get

$$\|\mathcal{D}(\psi)T_n\|_q \lesssim n^{d(\frac{1}{p} - 1)} \|\mathcal{D}(\psi/\varphi)V_n\|_q \|\mathcal{D}(\varphi)T_n\|_q \lesssim n^{d(\frac{1}{p} - 1)} \|\mathcal{D}(\psi/\varphi)V_n\|_q \|\mathcal{D}(\varphi)T_n\|_p.$$ 

Now, let $1 < q \leq \infty$. Then the Young convolution inequality and the Nikol’skii inequality between $(L_p, L_1)$ imply

$$\|\mathcal{D}(\psi)T_n\|_q \lesssim \|\mathcal{D}(\psi/\varphi)V_n\|_q \|\mathcal{D}(\varphi)T_n\|_1 \lesssim n^{d(1/p - 1)} \|\mathcal{D}(\psi/\varphi)V_n\|_q \|\mathcal{D}(\varphi)T_n\|_p.$$ 

Thus, the proof of (5.4) is complete.
(ii) Denote
\[ V_n^\varphi(x) := \mathcal{D}(1/\varphi)(V_n(x) - V_n(2x)). \]
Note that \( V_n^\varphi \in \mathcal{T}_4n \). Then
\[ \|\mathcal{D}(\varphi)V_n^\varphi\|_p = \|V_n(\cdot) - V_n(2\cdot)\|_p \leq 2^{1/p}\|V_n\|_p. \]
Set
\[ B(\xi) := \prod_{j=1}^d v(\xi_j). \]
Thus, taking into account that \( B \in C^\infty(\mathbb{R}^d) \) and it has a compact support, by Corollary 2.1 and (5.6), we obtain
\[ \|\mathcal{D}(\phi)V_n^\varphi\|_{L^p(T_4^n)} \lesssim \|B\|_{L^p(\mathbb{R}^d)} \lesssim n^{-d(1/p - 1)}. \]
Note also that
\[ \mathcal{D}(\psi/\varphi)V_n^\varphi(x) = \sum_{|k| \neq 0} \frac{\psi(k)}{\varphi(k)} B \left( \frac{k}{n} \right) e^{i(k,x)} - \sum_{|k| \neq 0} \frac{\psi(2k)}{\varphi(2k)} B \left( \frac{k}{n} \right) e^{i(2k,x)} = \mathcal{D}(\psi/\varphi)V_n(x) - \frac{1}{2^\gamma}\mathcal{D}(\psi/\varphi)V_n(2x) \]
and, therefore,
\[ \|\mathcal{D}(\psi)V_n^\varphi\|_q \geq C \|\mathcal{D}(\psi/\varphi)V_n\|_q, \]
where \( C = (1 - 2^{-\gamma})^{1/q}, \tilde{q} = \min(1, q) \).
Thus, combining inequalities (5.7) and (5.8), we derive
\[ \sup_{T \in \mathcal{T}_4^n} \frac{\|\mathcal{D}(\psi)T\|_q}{\|\mathcal{D}(\varphi)T\|_p} \geq \frac{\|\mathcal{D}(\psi)V_n^\varphi\|_q}{\|\mathcal{D}(\varphi)V_n^\varphi\|_p} \gtrsim n^{d(\tilde{q} - 1)}\|\mathcal{D}(\psi/\varphi)V_n\|_q. \]

(iii) Equivalence (5.5) follows from Lemma 5.2 in the case \( 0 < q \leq 1 \), Lemmas 5.6 and 5.7 in the case \( 1 < q < \infty \), and Lemma 5.8 in the case \( q = \infty \).
Then
\[
\|D(\psi/\varphi)V_n\|_q \lesssim \|D(\psi/\varphi)V_n\|_1 = \|V_n \ast f_{\psi,\varphi}\|_1 \lesssim \|V_n\|_1 \|f_{\psi,\varphi}\|_1.
\]
(5.10)

We have by Remark 5.6 below that
\[
\|f_{\psi,\varphi}\|_1 \lesssim \|f_{\psi,\varphi}\|_r \lesssim \|f_{\gamma}\|_r
\]
for some \(r > 1\), where
\[
f_{\gamma}(x) = \sum_{|k| \neq 0} \frac{e^{i(k,x)}}{|k|^\gamma}.
\]
The fact that \(f_{\gamma} \in L_r(\mathbb{T}^d)\) follows from
\[
f_{\gamma}(x) \asymp |x|^\gamma - d \quad \text{as} \quad x \to 0, \quad 0 < \gamma < d
\]
(see [110]). Therefore, by (5.10),
\[
\|D(\psi/\varphi)V_n\|_q \lesssim \|V_n\|_1 \lesssim 1,
\]
where the last estimate follows from Corollary 2.1.

To show
\[
\|D(\psi/\varphi)V_n\|_q \gtrsim 1,
\]
we again use the function \(f_{\psi,\varphi}\):
\[
\|f_{\psi,\varphi} - f_{\psi,\varphi} \ast V_n\|_q \gtrsim \|f_{\psi,\varphi} - f_{\psi,\varphi} \ast V_n\|_1 \to 0 \quad \text{as} \quad n \to \infty,
\]
since \(V_n\) is an approximate identity and \(f_{\psi,\varphi} \in L_1(\mathbb{T}^d)\). This implies that
\[
\|D(\psi/\varphi)V_n\|_q \gtrsim \|f_{\psi,\varphi}\|_q - \|f_{\psi,\varphi} - f_{\psi,\varphi} \ast V_n\|_q \gtrsim C(\psi, \varphi),
\]
since \(f_{\psi,\varphi} \in L_1(\mathbb{T}^d)\). \(\square\)

Now, we study the case \(0 < q \leq 1\) and \(\gamma = 0\).

**Lemma 5.3.** Let \(0 < q \leq 1\), \(\alpha > 0\), and \(\psi, \varphi \in \mathcal{H}_\alpha\). Let also \(\frac{\psi}{\varphi} \in C^\infty(\mathbb{R}^d \setminus \{0\})\) and \(\frac{\varphi}{\psi} \in C^\infty(\mathbb{R}^d \setminus \{0\})\). Then
\[
\|D(\psi/\varphi)V_n\|_q \asymp \left\{
\begin{array}{ll}
1, & d = 1, 0 < q < 1, \text{ and } \frac{\psi(\xi)}{\varphi(\xi)} \not\equiv \text{const}; \\
\ln(n+1), & d = 1, q = 1, \text{ and } \frac{\psi(\xi)}{\varphi(\xi)} \not\equiv \text{const}; \\
n^{d(1-\frac{1}{q})}, & d \geq 1 \text{ and } \frac{\psi(\xi)}{\varphi(\xi)} \equiv \text{const}.
\end{array}\right.
\]

**Proof.** The simplest case is when \(\frac{\psi(\xi)}{\varphi(\xi)} \equiv \text{const}\). In this case, the estimate from above follows from Corollary 2.1. At the same time, by Nikol’skii’s inequality, we have
\[
\|D(\psi/\varphi)V_n\|_q \gtrsim n^{d(1-\frac{1}{q})}\|D(\psi/\varphi)V_n\|_1 \gtrsim n^{d(1-\frac{1}{q})},
\]
completing the proof.

To prove the lemma in the case \(d = 1\) and \(\frac{\psi(\xi)}{\varphi(\xi)} \not\equiv \text{const}\), we need the following two auxiliary results.
Lemma 5.4. (See [108, p. 119].) Let \( f : \mathbb{R} \to \mathbb{C} \) be a function of bounded variation and let \( \lim_{|\xi| \to \infty} f(\xi) = 0 \). Then, for any \( \varepsilon > 0 \), we have
\[
\sup_{0 < |x| \leq \pi} \left| \sum_{k=-\infty}^{\infty} f(\varepsilon k)e^{ikx} - \int_{-\infty}^\infty f(u)e^{iux}du \right| \leq 2\varepsilon V_{-\infty}(f),
\]
where \( V_{-\infty}(f) \) is the total variation of \( f \) on \( \mathbb{R} \).

Lemma 5.5. ([80]) Let \( f_\beta \in C^\infty(\mathbb{R}^d \setminus \{0\}) \) be a homogeneous function of order \( \beta \geq 0 \) and not a polynomial and let \( \eta \in C^\infty(\mathbb{R}^d) \) have a compact support.

(i) We have
\[
|\widehat{f_\beta \eta}(\xi)| \leq C_1 (1 + |\xi|)^{-\beta - d}, \quad \xi \in \mathbb{R}^d.
\]

(ii) There exist \( \rho > 1, \theta > 0, \) and \( u_0 \in S^{d-1} \) such that
\[
|\widehat{f_\beta \eta}(\xi)| \geq C_2 |\xi|^{-\beta - d}, \quad \xi \in \Omega,
\]

where \( \Omega \equiv \Omega(\rho, \theta, u_0) = \{ \xi = ru : r \geq \rho, u \in S^{d-1}, \cos \theta \leq (u, u_0) \leq 1 \} \), \( S^{d-1} \) is the unit sphere in \( \mathbb{R}^d \) and \( C_1 \) and \( C_2 \) are some positive constants.

Denote \( f(y) = \frac{\psi(y)}{\varphi(y)}v(y) \). By Lemmas 5.4 and 5.5, we obtain
\[
\|D(f)V_n\|_q \lesssim n \left( \int_{\mathbb{T}^d} \left| \int_{\mathbb{R}} f(y)e^{iy\cdot x}dy \right|^q dx \right)^{1/q} + V_{-\infty}(f)
\]
\[
\lesssim n \left( \int_{\mathbb{T}^d} \left( \frac{dx}{(1 + |nx|)^q} \right)^{1/q} + V_{-\infty}(f) \right)
\]
\[
\lesssim \begin{cases} 
1, & 0 < q < 1; \\
\ln(n+1), & q = 1. 
\end{cases}
\]

Similarly, one can prove the estimate from below. The proof of Lemma 5.3 is now complete.

The next two lemmas deal with the case \( 1 < q < \infty \).

Lemma 5.6. Let \( 1 < q < \infty \) and \( \gamma \in \mathbb{R} \). Let also \( \mu \in \mathcal{H}_{-\gamma} \) be such that \( \mu, 1/\mu \in C^\infty(\mathbb{R}^d \setminus \{0\}) \). Then for any \( f \in L_q(\mathbb{T}^d) \) such that \( (-\Delta)^{-\gamma/2} f \in L_q(\mathbb{T}^d) \) the following two-sided inequalities hold
\[
\|D(\mu)f\|_q \approx \|(-\Delta)^{-\gamma/2} f\|_q, \quad \gamma \neq 0,
\]
and
\[
\|D(\mu)f\|_q \approx \|f - \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x)dx\|_q, \quad \gamma = 0.
\]

Remark 5.6. If we do not assume \( \mu \in C^\infty(\mathbb{R}^d \setminus \{0\}) \) in Lemma 5.6, then we only have
\[
\|D(\mu)f\|_q \lesssim \|(-\Delta)^{-\gamma/2} f\|_q, \quad \gamma \neq 0.
\]
PROOF. To obtain (5.13), we consider the function

\[ u(\xi) = |\xi|^\gamma \mu(\xi). \]

By properties of homogeneous functions, we have that the function \( D^\nu u \) is a homogeneous function of order \(-|\nu|_1\) and it belongs to \( C^\infty(\mathbb{R}^d \setminus \{0\})\) for any multi-index \( \nu \in \mathbb{Z}_+^d \). Hence,

\[ \sup_{\xi \in \mathbb{R}^d} |\xi|^{\nu_1} |D^\nu u(\xi)| < \infty. \]

Thus, by Lemma 2.4, we get that for any function \( f \in L_q(\mathbb{T}^d) \), \( 1 < q < \infty \),

\[ \|D(u)f\|_q \lesssim \|f\|_q. \]

The proof of the reverse inequality is similar using \( 1/\mu \in C^\infty(\mathbb{R}^d \setminus \{0\}) \).

Inequality (5.14) can be obtained similarly. \( \square \)

**Lemma 5.7.** Let \( 1 < q < \infty \) and \( \gamma \geq 0 \). We have, for any \( n \in \mathbb{N} \),

\[ \|(-\Delta)^{-\gamma/2}V_n\|_q \lesssim \begin{cases} 1, & \gamma > d \left(1 - \frac{1}{q}\right); \\ \ln\frac{1}{\gamma}(n + 1), & \gamma = d \left(1 - \frac{1}{q}\right); \\ n^{d\left(1 - \frac{1}{q}\right) - \gamma}, & 0 \leq \gamma < d \left(1 - \frac{1}{q}\right). \end{cases} \]

**Proof.** Let us prove (5.15) only for \( d = 2 \). For \( d = 1 \) or \( d > 2 \) the proof is similar. In what follows, \( \xi, \eta \in \mathbb{R} \),

\[ g_n(\xi, \eta) := \begin{cases} \frac{1}{(\xi^2 + \eta^2)^{\gamma/2}}, & |\xi| \leq 2n, \ |\eta| \leq 2n, \ |\xi| + |\eta| \neq 0; \\ 0, & \text{otherwise}, \end{cases} \]

and

\[ a_{k,l} = a_{k,l}^{(n)} = g_n(k,l) v\left(\frac{|k|}{n}\right) v\left(\frac{|l|}{n}\right), \quad a_{0,0} = 0. \]

We have

\[ (-\Delta)^{-\gamma/2}V_n(x, y) = \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} a_{k,l} e^{i(kx+ly)} = 2 \sum_{k=1}^{2n} a_{k,0} \cos kx + 2 \sum_{l=1}^{2n} a_{0,l} \cos ly + 4 \sum_{k=1}^{2n} \sum_{l=1}^{2n} a_{k,l} \cos kx \cos ly. \]

First, we estimate \( \|(-\Delta)^{-\gamma/2}V_n\|_q \) from below as follows

\[ \|(-\Delta)^{-\gamma/2}V_n\|_{L_q(\mathbb{T}^2)} \geq 4 \left\| \sum_{k=1}^{2n} \sum_{l=1}^{2n} a_{k,l} \cos kx \cos ly \right\|_{L_q(\mathbb{T}^2)} \left\| -4(2\pi)^{1/q} \sum_{k=1}^{2n} a_{k,0} \cos kx \right\|_{L_q(\mathbb{T}^1)}. \]

Using the Hardy–Littlewood theorem for series with monotone coefficients (see (2.3)), we get

\[ \left\| \sum_{k=1}^{2n} a_{k,0} e^{ikx} \right\|_{L_q(\mathbb{T}^1)} \lesssim \left( \sum_{k=1}^{2n} k^{q-2} v\left(\frac{k}{n}\right) \right)^{1/q} \lesssim \sigma_1(n). \]
Here we set

$$\sigma_d(n) := \begin{cases} 1, & \gamma > d \left( 1 - \frac{1}{q} \right); \\ \ln^{\frac{1}{q}}(n + 1), & \gamma = d \left( 1 - \frac{1}{q} \right); \\ n^{d(1 - \frac{1}{q}) - \gamma}, & \gamma < d \left( 1 - \frac{1}{q} \right), \end{cases}$$

where $d$ stands for the dimension.

To estimate the norm of the double sum, we use Lemma 2.6, which is a multidimensional analog of the Hardy-Littlewood theorem for series with monotone coefficients $a_{n,m}$ in the sense of Hardy, i.e., $\Delta^{(2)} a_{n,m} \geq 0$. Recall that for a sequence $\{a_{n,m}\}_{n,m} \in \mathbb{N}$ the differences $\Delta^{(2)} a_{n,m}$ is defined as follows:

$$\Delta^{1,0} a_{n,m} = a_{n,m} - a_{n+1,m}, \quad \Delta^{0,1} a_{n,m} = a_{n,m} - a_{n,m+1},$$

$$\Delta^{(2)} a_{n,m} = \Delta^{0,1}(\Delta^{1,0} a_{n,m}).$$

Since for $1 \leq k \leq 2n - 1$ and $1 \leq l \leq 2n - 1$, we have

$$\Delta^{(2)} g_n(k, l) = \int_k^{k+1} \int_l^{l+1} \left( \frac{\partial^2}{\partial \xi \partial \eta} \left( g_n(\xi, \eta) v \left( \frac{\xi}{n} \right) v \left( \frac{\eta}{n} \right) \right) \right) d\xi d\eta \geq 0,$$

then Lemma 2.6 implies

$$\left\| \sum_{k=1}^{2n} \sum_{l=1}^{2n} a_{k,l}^{(n)} \cos kx \cos ly \right\|_{L_q(\mathbb{T}^2)} \leq \left( \sum_{k=1}^{2n} \sum_{l=1}^{2n} \frac{k^{q-2} q^{-2}}{(k^2 + l^2)^{q \gamma / 2}} \left( v \left( \frac{k}{n} \right) v \left( \frac{l}{n} \right) \right)^q \right)^{1/q} \times \left( \sum_{k=1}^{2n} \sum_{l=1}^{2n} \frac{k^{q-2} q^{-2}}{(k^2 + l^2)^{q \gamma / 2}} \right)^{1/q} \times \left( \int_1^n \int_1^n \frac{(\xi \eta)^{q-2}}{(\xi^2 + \eta^2)^{q \gamma / 2}} d\xi d\eta \right)^{1/q} \approx \sigma_2(n).$$

Assume first that $\gamma \leq d(1 - 1/q)$. To complete the proof in this case, it only remains to use the fact that for sufficiently large $n$ and $\gamma \leq d(1 - 1/q)$ one has

$$\sigma_2(n) - (2\pi)^{1/q} \sigma_1(n) \geq \sigma_2(n),$$

which follows from (5.16) and (5.18).

Let now $\gamma > d(1 - 1/q)$. To estimate $\|(-\Delta)^{-\gamma/2} V_n\|_q$ from above, we use the inequality

$$\|(-\Delta)^{-\gamma/2} V_n\|_{L_q(\mathbb{T}^2)} \lesssim \left\| \sum_{k=1}^{2n} \sum_{l=1}^{2n} a_{k,l} \cos kx \cos ly \right\|_{L_q(\mathbb{T}^2)} + \left\| \sum_{k=1}^{2n} a_{k,0} \cos kx \right\|_{L_q(\mathbb{T}^1)} \lesssim 1,$$
where we again used (5.16) and (5.18). The estimate from below is given as follows
\[
\|(-\Delta)^{-\gamma/2}V_n\|_q \gtrsim \|(-\Delta)^{-\gamma/2}V_n\|_1 \\
\gtrsim \int_{\mathbb{R}^2} \left( \sum_{k,l \in \mathbb{Z}} a_{k,l} e^{i(k-1)x+i(l-1)y} \right) dx dy \\
\gtrsim a_{1,1} > 0.
\]

□

For the case \( q = \infty \), the quantity \( \|D(\psi/\varphi)V_n\|_q \) may have different growth properties depending on given \( \psi \) and \( \varphi \) (see also Remark 5.4). However, let us show that under some natural conditions it may happen only in the one-dimensional case.

**Lemma 5.8.** Let \( \alpha > 0, \gamma \geq 0, \psi \in H_\alpha, \varphi \in H_{\alpha+\gamma}, \) and \( \frac{\psi}{\varphi} \in C^\infty(\mathbb{R}^d \setminus \{0\}) \).

(i) If \( d = 1 \), then

\[
\|D(\psi/\varphi)V_n\|_\infty \asymp \left\{ \begin{array}{ll}
1, & \gamma > 1; \\
1, & \gamma = 1 \text{ and } \frac{\psi(\xi)}{\varphi(\xi)} = A|\xi|^{-\gamma} \text{ sign } \xi \text{ for some } A \in \mathbb{C} \setminus \{0\}; \\
\ln(n+1), & \gamma = 1 \text{ and } \frac{\psi(\xi)}{\varphi(\xi)} \neq A|\xi|^{-\gamma} \text{ sign } \xi \text{ for any } A \in \mathbb{C} \setminus \{0\}; \\
n^{1-\gamma}, & 0 \leq \gamma < 1,
\end{array} \right.
\]

(ii) if \( d \geq 2 \) and \( \frac{\xi}{\varphi} \in C^\infty(\mathbb{R}^d \setminus \{0\}) \), then

\[
\|D(\psi/\varphi)V_n\|_\infty \asymp \left\{ \begin{array}{ll}
1, & \gamma > d; \\
\ln n, & \gamma = d; \\
n^{d-\gamma}, & 0 \leq \gamma < d.
\end{array} \right.
\]

**Remark 5.7.** Note that without the condition \( \frac{\xi}{\varphi} \in C^\infty(\mathbb{R}^d \setminus \{0\}) \), the right-hand side estimate in (5.20), i.e. " \( \lesssim \) ", still holds.

**Proof.** (i) For \( d = 1 \), we write

\[
\frac{\psi(\xi)}{\varphi(\xi)} = \frac{1}{|\xi|^{\gamma}} \Phi(u),
\]

where \( u = \frac{\xi}{|\xi|} \in \{-1; 1\} \) and the function \( \Phi(u) \neq 0 \) is such that one of the following conditions hold:

(1) \( \Phi(1) = \Phi(-1) = A; \)
(2) \( \Phi(1) = A, \Phi(-1) = -A; \)
(3) \( \Phi(1) = A, \Phi(-1) = B, \) and \( |A| \neq |B|, \)

where \( A, B \in \mathbb{C} \).

In the first case, we have

\[
\|D(\psi/\varphi)V_n\|_\infty = 2|A| \left\| \sum_{k=1}^{2n} v(k) \frac{1}{k^{\gamma}} \cos kx \right\|_\infty \asymp \left\{ \begin{array}{ll}
1, & \gamma > 1; \\
\ln n, & \gamma = 1; \\
n^{1-\gamma}, & 0 \leq \gamma < 1.
\end{array} \right.
\]
In the second case, if \( \gamma \leq 1 \), by Bernstein’s inequality, we get
\[
\| D(\psi/\varphi) V_n \|_\infty = 2|A| \left\| \sum_{k=1}^{2n} v \left( \frac{k}{n} \right) \frac{1}{k^{\gamma}} \sin kx \right\|_\infty \\
\geq 2|A| \frac{n}{n} \left\| \sum_{k=1}^{2n} v \left( \frac{k}{n} \right) \frac{1}{k^{\gamma-1}} \cos kx \right\|_\infty \\
\gtrsim n^{1-\gamma}.
\]
If \( \gamma > 1 \), then
\[
\| D(\psi/\varphi) V_n \|_\infty \geq |A| \left\| \sum_{k=1}^{2n} v \left( \frac{k}{n} \right) \frac{1}{k^{\gamma}} \sin kx \right\|_1 \\
\geq |A| \int_0^{2\pi} \left( \sum_{k=1}^{2n} v \left( \frac{k}{n} \right) \frac{1}{k^{\gamma}} \sin kx \right) e^{-ix} \, dx \gtrsim 1.
\]
At the same time, by using the boundedness of \( \| V_n \|_1 \), we obtain
\[
\| D(\psi/\varphi) V_n \|_\infty = 2 \| V_n \ast \sum_{\nu=1}^{2n} \frac{|\nu x|}{\nu^{\gamma}} \|_\infty \leq 2 \| V_n \|_1 \left\| \sum_{\nu=1}^{2n} \frac{|\nu x|}{\nu^{\gamma}} \right\|_\infty \\
\lesssim \begin{cases} 
1, & \gamma \geq 1; \\
\ln(n+1), & 0 \leq \gamma < 1.
\end{cases}
\]
(see, e.g., [114, Ch. 5, §2]).

In the third case, assuming for definiteness that \( |A| > |B| \), we obtain
\[
\| D(\psi/\varphi) V_n \|_\infty = \left\| A \sum_{k=-2n}^{2n} v \left( \frac{k}{n} \right) \frac{1}{k^{\gamma}} e^{ikx} + (B - A) \sum_{k=-2n}^{-1} v \left( \frac{k}{n} \right) \frac{1}{k^{\gamma}} e^{ikx} \right\|_\infty \\
\gtrsim 2|A| \left\| \sum_{k=1}^{2n} \frac{1}{|k|^{\gamma}} v \left( \frac{k}{n} \right) - |B - A| \sum_{k=-2n}^{-1} \frac{1}{|k|^{\gamma}} v \left( \frac{k}{n} \right) \right\|_\infty \\
\gtrsim \begin{cases} 
1, & \gamma \geq 1; \\
\ln(n+1), & \gamma = 1; \\
n^{1-\gamma}, & 0 \leq \gamma < 1.
\end{cases}
\]
Estimate from above is similar to (5.21):
\[
\| D(\psi/\varphi) V_n \|_\infty \lesssim \begin{cases} 
1, & \gamma \geq 1; \\
\ln(n+1), & \gamma = 1; \\
n^{1-\gamma}, & 0 \leq \gamma < 1.
\end{cases}
\]
Thus, summarizing the above estimates and noting that the case (2) corresponds to the case \( \frac{\psi(\xi)}{\varphi(\xi)} = |A|^{1-\gamma} \text{sign} \xi \), we get (5.19) for \( d = 1 \).

(ii) Let now \( d \geq 2 \). As above, we have \( \frac{\psi(\xi)}{\varphi(\xi)} = \frac{1}{|\xi|^d} \Phi \left( \frac{\xi}{|\xi|} \right) \), where \( \Phi \in C^\infty(S^{d-1}) \).
Since \( \frac{\psi}{\varphi} \in C^\infty(\mathbb{R}^d \setminus \{0\}) \) and \( \frac{\psi}{\varphi} \in C^\infty(\mathbb{R}^d \setminus \{0\}) \), we obtain that \( \Phi(u) > 0 \) or \( \Phi(u) < 0 \).
for all $u \in S^{d-1}$. Moreover, one can find a constant $C > 0$ such that $|\Phi(u)| > C$ for all $u \in S^{d-1}$.

Thus,
\[
\|D(\psi/\varphi)V_n\|_\infty \geq |D(\psi/\varphi)V_n(0)|
\geq C \sum_{k_1=1}^{n'} \cdots \sum_{k_d=1}^{n'} \frac{1}{|k|^{\gamma}} \geq \begin{cases} 1, & \gamma > d; \\ \ln(n+1), & \gamma = d; \\ n^{d-\gamma}, & 0 \leq \gamma < d. \end{cases}
\]

On the other hand,
\[
\|D(\psi/\varphi)V_n\|_\infty \leq C \sum_{k_1=2n}^{2n'} \cdots \sum_{k_d=2n}^{2n'} \frac{1}{|k|^{\gamma}} \leq \begin{cases} 1, & \gamma > d; \\ \ln(n+1), & \gamma = d; \\ n^{d-\gamma}, & 0 \leq \gamma < d, \end{cases}
\]
which implies (5.20).

We are now in a position to obtain an explicit form of the Hardy-Littlewood-Nikol’skii inequality in the case $\gamma = 0$: for $p \leq 1 < q$, see Lemma 5.9 and for $p < q \leq 1$, Lemma 5.10.

**Lemma 5.9.** Let $0 < p \leq 1 < q \leq \infty$, $\alpha > 0$, and $\psi, \varphi \in H_\alpha$. Let also $\overline{\psi \varphi} \in C^\infty(\mathbb{R}^d \setminus \{0\})$ and $\overline{\varphi \varphi} \in C^\infty(\mathbb{R}^d \setminus \{0\})$. Then
\[
\sup_{T \in \mathcal{T}_n} \frac{\|D(\psi)T\|_q}{\|D(\varphi)T\|_p} \asymp n^{d(\frac{1}{p} - \frac{1}{q})}.
\]

**Proof.** The estimate from above in (5.22) follows from Lemmas 5.1 and 5.6 and Lemma 5.7 (the case $1 < q < \infty$) and Lemma 5.8 (the case $q = \infty$).

Let us prove the estimate from below. For this, we denote
\[
V_n^\varphi(x) = D \left( \frac{1}{\varphi} \right) (V_{Nn}(x) - V_n(x)),
\]
where $N \in \mathbb{N}$ will be chosen later.

By (5.7), we have
\[
\|D(\varphi)V_n^\varphi\|_p \lesssim n^{-d(\frac{1}{p} - 1)}.
\]

At the same time, using Lemma 5.6 and Lemma 5.7 (the case $1 < q < \infty$) and Lemma 5.8 (the case $q = \infty$), we derive that
\[
\|D(\psi)V_n^\varphi\|_q \geq \|D(\psi/\varphi)V_{Nn}^\varphi\|_q - \|D(\psi/\varphi)V_n^\varphi\|_q 
\geq C_1 (Nn)^{d(1 - \frac{1}{q})} - C_2 n^{d(1 - \frac{1}{q})} \geq n^{d(1 - \frac{1}{q})},
\]
where $N^{d(1-1/q)} \geq (C_2 + 1)/C_1$. Thus, combining the inequality
\[
\sup_{T \in \mathcal{T}_{2Nn}} \frac{\|D(\psi)T\|_q}{\|D(\varphi)T\|_p} \geq \frac{\|D(\psi)V_n^\varphi\|_q}{\|D(\varphi)V_n^\varphi\|_p}
\]
with (5.23) and (5.24), we obtain the estimate from below in (5.22). \qed
Lemma 5.10. Let $0 < p < q \leq 1$, $\alpha > 0$, and $\psi, \varphi \in \mathcal{H}_\alpha$. Let also $\frac{\psi}{\varphi} \in C^\infty(\mathbb{R}^d \setminus \{0\})$. Then

\begin{equation}
\sup_{T \in T_\alpha} \| D(\psi)T \|_q \lesssim \begin{cases} 
n^{d(\frac{1}{p}-1)}, & d = 1, 0 < q < 1, \text{ and } \frac{\psi(\xi)}{\varphi(\xi)} \not\equiv \text{const}; \\
n^{d(\frac{1}{p}-1)} \ln(n+1), & d = 1, q = 1, \text{ and } \frac{\psi(\xi)}{\varphi(\xi)} \not\equiv \text{const}; \\
n^{d(\frac{1}{p}-\frac{1}{q})}, & d \geq 1 \text{ and } \frac{\psi(\xi)}{\varphi(\xi)} \equiv \text{const}.
\end{cases}
\end{equation}

Proof. The estimate from above in all cases follows from Lemmas 5.1 and 5.3.

The estimate from below in (5.25) in the case $\frac{\psi(\xi)}{\varphi(\xi)} \equiv \text{const}$ can be proved the same way as in the proof of the corresponding estimate in Lemma 5.9. We only note that we use Lemma 5.3 instead of Lemmas 5.6, 5.7, and 5.8.

To prove the estimate from below in the case $d = 1$ and $\frac{\psi(\xi)}{\varphi(\xi)} \not\equiv \text{const}$, we put

\begin{equation}
V_n^\varphi(x) = D\left(\frac{1}{\varphi}\right) \left(\sum_{k \in \mathbb{Z}} v\left(\frac{k}{n}\right) e^{i(2k+1)x}\right).
\end{equation}

Then, by Corollary 2.1, we have

\begin{equation}
\| D(\psi)V_n^\varphi \|_p = \| V_n(2) \|_p \lesssim n^{\frac{1}{p} - 1}.
\end{equation}

Next, using the fact that $\psi/\varphi$ is a homogeneous function of order zero, we obtain

\begin{equation}
\| D(\psi)V_n^\varphi \|_q = \left\| \sum_{k \in \mathbb{Z}} \frac{\psi(2k+1)}{\varphi(2k+1)} v\left(\frac{k}{n}\right) e^{i(2k+1)x}\right\|_q = \| \tilde{V}_n \|_q,
\end{equation}

where

\begin{equation}
\tilde{V}_n(x) = \sum_{k \in \mathbb{Z}} f_n\left(\frac{k}{n}\right) v\left(\frac{k}{n}\right) e^{ikx}, \quad f_n(y) = \frac{\psi(2y + \frac{1}{n})}{\varphi(2y + \frac{1}{n})}.
\end{equation}

By properties of the Fourier transform,

\begin{equation}
\hat{f}_n v(\xi) = e^{i\frac{\xi}{2n}} \hat{f}_\infty v_n(\xi), \quad v_n(\xi) = v\left(y - \frac{1}{2n}\right).
\end{equation}

Thus, from (5.12) (see also the proof of (4.7) in [80]), it is easy to see that for sufficiently large $n$,

\begin{equation}
|\hat{f}_n v(\xi)| \geq C \frac{1}{|\xi|}, \quad |\xi| > \rho,
\end{equation}

where $C$ and $\rho$ do not depend on $n$.

Now, using Lemma 5.4 and (5.28), by analogy with the proof of Lemma 5.3, we derive

\begin{equation}
\| \tilde{V}_n \|_q \gtrsim \begin{cases} 
1, & 0 < q < 1; \\
\ln n, & q = 1.
\end{cases}
\end{equation}

It remains to combine inequalities (5.26), (5.27), and (5.29) with the inequality

\begin{equation}
\sup_{T \in T_{\alpha}} \| D(\psi)T \|_q \gtrsim \| D(\psi)V_n^\varphi \|_q).
\end{equation}
5.2. Hardy–Littlewood–Nikol’skii \((L_p, L_q)\) inequalities for \(1 < p < q \leq \infty\).

**Lemma 5.11.** Let \(1 < p < q \leq \infty\), \(\alpha > 0\), \(\gamma \geq 0\), \(\psi \in \mathcal{H}_\alpha\), and \(\varphi \in \mathcal{H}_{\alpha + \gamma}\). Let also \(\frac{\psi}{\varphi} \in C^\infty(\mathbb{R}^d \setminus \{0\})\) and \(\frac{\xi}{\psi} \in C^\infty(\mathbb{R}^d \setminus \{0\})\). We have

\[
\eta(n) := \sup_{T \in T_n^*} \frac{\|\mathcal{D}(\psi)T\|_q}{\|\mathcal{D}(\varphi)T\|_p} \leq \begin{cases} \\
1, & \gamma \geq d\left(\frac{1}{p} - \frac{1}{q}\right), \ q < \infty; \\
1, & \gamma > \frac{d}{p}, \ q = \infty; \\
\ln^n(n + 1), & \gamma = \frac{d}{p}, \ q = \infty; \\
n^{-\gamma d\left(\frac{1}{p} - \frac{1}{q}\right)} & 0 \leq \gamma < d\left(\frac{1}{p} - \frac{1}{q}\right).
\end{cases}
\]

**Proof.** To estimate \(\eta(n)\) from above in the case \(q < \infty\), we use the fact that

\[
\eta(n) \leq \sup_{T \in T_n^*} \frac{\|(-\Delta)^{-\gamma/2}T\|_q}{\|T\|_p}
\]

(see Lemma 5.6).

Let \(\gamma \geq d(1/p - 1/q)\) and \(q < \infty\). Using the Hardy-Littlewood inequality for fractional integrals, see (1.17), and taking into account the fact that by Lemma 2.4 the function

\[
u(\xi) = \frac{1 - v(2|\xi|)}{|\xi|^{\gamma - d\left(\frac{1}{p} - \frac{1}{q}\right)}}
\]
is a Fourier multiplier in \(L_p(\mathbb{T}^d)\), we arrive at

\[
\|(-\Delta)^{-\gamma/2}T\|_q \lesssim \|\mathcal{D}(u)T\|_p \lesssim \|T\|_p.
\]

If \(0 < \gamma < d(1/p - 1/q)\) and \(q < \infty\), the required estimate follows from the Hardy–Littlewood inequality for fractional integrals and the Bernstein inequality (see, e.g., [80]):

\[
\|(-\Delta)^{-\gamma/2}T\|_q = \|(-\Delta)^{-d\left(\frac{1}{p} - \frac{1}{q}\right)/2}(-\Delta)^{d\left(\frac{1}{p} - \frac{1}{q}\right)/2}T\|_q \
\lesssim \|(-\Delta)^{d\left(\frac{1}{p} - \frac{1}{q}\right)/2}T\|_p \
\lesssim n^{d\left(\frac{1}{p} - \frac{1}{q}\right)/2}\|T\|_p.
\]

If \(\gamma = 0\) and \(q < \infty\), the part ” \(\lesssim\) ” in \((5.30)\) follows from \(\|\mathcal{D}(\psi)T\|_p \lesssim \|\mathcal{D}(\varphi)T\|_p\) (see Lemma 5.6) and the classical Nikol’skii inequality:

\[
\eta(n) \lesssim \sup_{T \in T_n^*} \frac{\|\mathcal{D}(\psi)\|_q}{\|\mathcal{D}(\varphi)\|_p} \lesssim n^{d\left(\frac{1}{p} - \frac{1}{q}\right)/2}.
\]

Let now \(\gamma \geq 0\) and \(q = \infty\). In this case, the proof follows from the inequality

\[
\|\mathcal{D}(\psi/\varphi)T\|_\infty = \left\|\int_{\mathbb{T}^d} \left(\mathcal{D}(\psi/\varphi)V_n(t)\right)T_n(x-t)dt\right\|_\infty \
\lesssim \|\mathcal{D}(\psi/\varphi)\|_{p^*}\|T\|_p
\]

and Lemmas 5.6 and 5.7.

Now, we study the estimate of \(\eta(n)\) from below, i.e., the part ” \(\gtrsim\) ” in equivalence \((5.30)\).
If $\gamma = 0$ and $q < \infty$, then $\eta(n) \asymp \sup_{T \in T_n'} \|T\|_p$ and, therefore, (5.30) is the classical Nikol’skii’s inequality. The sharpness follows from the Jackson-type kernel example, see [101, § 4.9]:

$$T(x) = \left( \frac{\sin \frac{n^2 t}{2}}{n \sin \frac{t}{2}} \right)^2 r, \quad r \in \mathbb{N}.$$  

If $0 < \gamma < d(1/p - 1/q)$ and $q < \infty$, using Lemmas 5.6 and 5.7, we estimate

$$\eta(n) \geq \frac{\|(-\Delta)^{-\gamma/2} V_{n/2} - 1\|_q}{\|V_{n/2} - 1\|_p} \geq \frac{\|(-\Delta)^{-\gamma/2} V_{n/2}\|_q}{\|V_{n/2}\|_p + (2\pi)^d/p} \gtrsim n^{d(\frac{1}{p} - \frac{1}{q}) - \gamma}, \quad n \geq 2.$$  

If $\gamma \geq d(1/p - 1/q)$ and $q < \infty$, we have

$$\eta(n) \geq \frac{\|D(\psi) T^*\|_q}{\|D(\varphi) T^*\|_p} \asymp 1, \quad T^*(x) = \cos x_1,$$

and the desired result follows.

Assume now that $q = \infty$. Set

$$T_{n,\lambda}(x) = \sum_{|k| \leq n} \frac{e^{i(k,x)}}{|k|^\lambda}, \quad \lambda > 0.$$  

Note that Lemma 5.6 implies that

$$\eta(n) = \sup_{T \in T_n'} \frac{\|T\|_\infty}{\|D(\varphi/\psi)(T)\|_p} \gtrsim \sup_{T \in T_n'} \frac{\|T\|_\infty}{\|(-\Delta)^{\gamma/2} T\|_p}.$$  

We divide the rest of the proof into three cases.

1) Let first $\gamma = d/p$. Since

$$\|T_{n,d}\|_\infty \gtrsim \sum_{|k| \leq n} \frac{1}{|k|^d} \gtrsim \ln(n + 1),$$

then, by (5.31) and Lemma 5.7, we have

$$\eta(n) \gtrsim \frac{\|T_{n,d}\|_\infty}{\|(-\Delta)^{\gamma/2} T_{n,d}\|_p} \gtrsim \frac{\ln(n + 1)}{\left\| \sum_{|k| \leq n} \frac{e^{i(k,x)}}{|k|^{d(1/p - 1/q)}} \right\|_p} \gtrsim \frac{\ln(n + 1)}{\|(-\Delta)^{d(1/p - 1/q)}/2 V_n\|_p} \gtrsim \ln^\frac{1}{d} (n + 1).$$

2) Let now $\gamma > d/p$. Considering $T_{n,d+\gamma}$ and using again (5.31) and Lemma 5.7, we get

$$\eta(n) \gtrsim \frac{\|T_{n,d+\gamma}\|_\infty}{\|(-\Delta)^{\gamma/2} T_{n,d+\gamma}\|_p} \gtrsim \frac{\|T_{n,d+\gamma}\|_\infty}{\|(-\Delta)^{-d/2} V_n\|_p} \gtrsim 1.$$
3) Finally, if $0 \leq \gamma < d/p$, considering $T_{n,\lambda}$ with $\lambda = \gamma + (d - d/p)/2$ and using the same procedure, we arrive at

$$\eta(n) \gtrsim \frac{\|T_{n,\lambda}\|_\infty}{\|(-\Delta)^{\gamma/2}T_{n,\lambda}\|_p} \gtrsim \frac{\|T_{n,\lambda}\|_\infty}{\|(-\Delta)^{-(\lambda-\gamma)/2}V_n\|_p} \lesssim \frac{1}{n^{d-\lambda}} = n^{\frac{d}{d} - \frac{\gamma}{d}}.$$  

Here, we take into account that $0 < \lambda - \gamma < d(1 - 1/p)$, $0 < \lambda < d$, and

$$\|T_{n,\lambda}\|_\infty \gtrsim \sum_{|k| \leq n} 1\bigg|\sum_{|k| \leq n} 1\bigg|^{\lambda} \gtrsim n^{d-\lambda}. \tag{5.3}$$

\[\square\]

5.3. Hardy–Littlewood–Nikol’skii $(L_p, L_q)$ inequalities for directional derivatives. For applications, in particular, to obtain the sharp Ulyanov inequality for moduli of smoothness, it is important to use the Hardy-Littlewood-Nikol’skii inequalities for directional derivatives. Here the interesting case is $p \leq 1$, otherwise see equivalence (3.8) in Corollary 3.3.

**Lemma 5.12.** Let $0 < p \leq 1$, $1 < q < \infty$, $\alpha > 0$, $\gamma > 0$, and $\alpha + \gamma \neq 2k + 1$, $k \in \mathbb{Z}_+$. Then

$$\sup_{T_n \in T_d} \left\| \left( \frac{\partial}{\partial \xi} \right)^{\alpha+\gamma} T_n \right\|_p \lesssim \sigma(n),$$  

where $\sigma(\cdot)$ is given as follows:

1. if $0 < p \leq 1$ and $1 < q < \infty$, then

   $$\sigma(t) := \begin{cases} t^{d\left(\frac{1}{p} - 1\right)}, & \gamma > d\left(1 - \frac{1}{q}\right); \\ t^{d\left(\frac{1}{p} - 1\right)} \ln t + 1, & 0 < \gamma = d\left(1 - \frac{1}{q}\right); \\ t^{d\left(\frac{1}{p} - \frac{1}{q}\right) - \gamma}, & 0 \leq \gamma < d\left(1 - \frac{1}{q}\right), \end{cases}$$

2. if $1 < p \leq q < \infty$, then

   $$\sigma(t) := \begin{cases} 1, & \gamma \geq d\left(\frac{1}{p} - \frac{1}{q}\right), \ q < \infty; \\ t^{d\left(\frac{1}{p} - \frac{1}{q}\right) - \gamma}, & 0 \leq \gamma < d\left(\frac{1}{p} - \frac{1}{q}\right). \end{cases}$$

**Proof.** First, we note that

$$\|D^{\alpha+\gamma} T_n\|_p \lesssim \sum_{j=1}^{d} \left\| \left( \frac{\partial}{\partial x_j} \right)^{\alpha+\gamma} T_n \right\|_p \lesssim \sup_{|\xi| = 1, \xi \in \mathbb{R}^d} \left\| \left( \frac{\partial}{\partial \xi} \right)^{\alpha+\gamma} T_n \right\|_p,$$
where
\[ D^{\alpha+\gamma}T_n(x) := \sum_{j=1}^d \left( \frac{\partial}{\partial x_j} \right)^{\alpha+\gamma} T_n(x) = \sum_{|k|\leq n} \varphi(k)(\hat{T}_n)_k e^{i(k,x)} , \]

\((\hat{T}_n)_k\) is the \(k\)-th Fourier coefficient of \(T_n\), and \(\varphi(y) := (iy_1)^{\alpha+\gamma} + \cdots + (iy_d)^{\alpha+\gamma}\). Thus, using Corollary 3.3, we obtain

\[ \sup_{T_n \in \mathcal{T}_n^q} \left\| \left( \frac{\partial}{\partial \xi} \right)^{\alpha+\gamma} T_n \right\|_q \lesssim \sup_{T_n \in \mathcal{T}_n^p} \left\| (-\Delta)^{\alpha/2} T_n \right\|_q. \]  

(5.33)

Note that

\[ \varphi(y) = \cos \left( \frac{(\alpha + \gamma)\pi}{2} \left( |y_1|^{\alpha+\gamma} + \cdots + |y_d|^{\alpha+\gamma} \right) \right. \]

\[ + i \left( \sin \left( \frac{(\alpha + \gamma)\pi}{2} \right. \left. |y_1|^{\alpha+\gamma} + \cdots + \sin \left( \frac{(\alpha + \gamma)\pi}{2} \right. \left. |y_d|^{\alpha+\gamma} \right) \right). \]

Hence, it is easy to see that \(\text{Re}\ \varphi(y) \neq 0\) for \(y \neq 0\) and \(\alpha + \gamma \neq 2k+1, \ k \in \mathbb{Z}_+\).

Next, by using Lemma 5.1, we obtain for \(0 < p \leq 1 \) and \(1 < q < \infty\)

\[ \sup_{T_n \in \mathcal{T}_n^p} \left\| (-\Delta)^{\alpha/2} T_n \right\|_q \lesssim n^{d(1/p - 1)} \left\| D(\psi/\varphi)V_n \right\|_q, \]  

(5.35)

where

\[ \psi(y) = \frac{|y|^\alpha}{(iy_1)^{\alpha+\gamma} + \cdots + (iy_d)^{\alpha+\gamma}}. \]

Let us show that

\[ \left\| D(\psi/\varphi)V_n \right\|_q \lesssim \left\| (-\Delta)^{-\gamma/2} V_n \right\|_q. \]  

(5.36)

For this, it is sufficient to verify that the function

\[ h(y) = \frac{|y|^{\alpha+\gamma}}{(iy_1)^{\alpha+\gamma} + \cdots + (iy_d)^{\alpha+\gamma}} \]

is a Fourier multiplier in \(L_q(\mathbb{T}^d)\). This easily follows from (5.34) and Lemma 2.4. Thus, combining (5.33), (5.35), and (5.36) and using the bounds for \(\left\| (-\Delta)^{-\gamma/2} V_n \right\|_q\) from Lemma 5.7, we complete the proof. \(\square\)

Recall that the homogeneous Sobolev norm is given by \(\|f\|_{\dot{W}^\alpha_p} = \sum_{|\nu|=\alpha} \|D^\nu f\|_p\).

**Lemma 5.13.** Let \(d \geq 1\).

(i) We have, for \(1/q^* = (d - 1)/d\),

\[ \|T\|_{q^*} \leq \frac{1}{d} \sum_{j=1}^d \left\| \frac{\partial T}{\partial x_j} \right\|_1, \quad T \in \mathcal{T}'. \]  

(5.37)

(ii) We have

\[ \|T\|_\infty \lesssim \|T\|_{\dot{W}^\alpha_p}, \quad T \in \mathcal{T}'. \]  

(5.38)
Proof. (i) To prove (5.37) see [91, pp. 129–130].
(ii) First, we use the limiting case of the Sobolev embedding theorem (see, e.g., [7, Ch. 10], [74, Theorem 33]) given by

\[ \|T\|_\infty \lesssim \|T\|_{W^d_1} + \|T\|_1. \]

Then by (i), we estimate

\[ \|T\|_1 \lesssim \|T\|_{q^*} \lesssim \sum_{j=1}^d \left\| \frac{\partial T}{\partial x_j} \right\|_1. \]

Thus, (5.38) follows from

\[ (5.39) \quad \left\| \frac{\partial T}{\partial x_j} \right\|_1 \lesssim \left\| \frac{\partial^2 T}{\partial x_j^2} \right\|_1 \lesssim \cdots \leq \left\| \frac{\partial^d T}{\partial x_j^d} \right\|_1, \quad T \in \mathcal{T}', \quad j = 1, \ldots, d. \]

It remains to show (5.39). We write

\[ \frac{\partial T}{\partial x_j}(x) = \int_{x_j}^{x_j} \frac{\partial^2 T}{\partial x_j^2} T(x_1, \ldots, x_{j-1}, t_j, x_{j+1}, \ldots, x_d) dt_j + \frac{\partial T}{\partial x_j}(x_1, \ldots, x_{j-1}, z_j, x_{j+1}, \ldots, x_d). \]

Then integrating over \((0, 2\pi)\) by \(z_j\) and using the fact that \(\frac{\partial T}{\partial x_j}\) as a polynomial of \(x_j\) belongs to \(\mathcal{T}'\), we obtain

\[ 2\pi \frac{\partial T}{\partial x_j} = \int_0^{2\pi} \int_{x_j}^{x_j} \frac{\partial^2 T}{\partial x_j^2} T(x_1, \ldots, x_{j-1}, t_j, x_{j+1}, \ldots, x_d) dt_j dz_j, \]

which gives

\[ \left\| \frac{\partial T}{\partial x_j} \right\|_1 \lesssim \left\| \frac{\partial^2 T}{\partial x_j^2} \right\|_1. \]

Repeating this procedure \(d - 1\) times, we complete the proof of (5.39). \(\square\)

Now, we proceed with the case \(\alpha + \gamma \in \mathbb{N}\) and \(\gamma \geq 1\). It turns out that, in this case, an interesting effect in the Hardy-Littlewood inequality occurs. More precisely, in the limiting case \(\gamma = d(1 - 1/q)\), we can obtain an improved version of Lemma 5.12.

Lemma 5.14. Let \(0 < p \leq 1, 1 < q \leq \infty, d \geq 2, \alpha > 0, \gamma \geq 1, \) and \(\alpha + \gamma \in \mathbb{N}\). Suppose also that \(\alpha \in \mathbb{N}\) if \(q = \infty\). Then

\[ (5.40) \quad \sup_{T_n \in \mathcal{T}^{\alpha}_q} \sup_{|\xi| = 1, \xi \in \mathbb{R}^d} \left\| \left( \frac{\partial}{\partial \xi} \right)^\alpha T_n \right\|_q^{\alpha+\gamma} \lesssim n^{d(1-1/p)+(d(1-1/q)-(\alpha+\gamma)_+).} \]
Remark 5.8. It is worth mentioning that inequality (5.40) in the case of $\gamma = d(1 - 1/q)$ gives sharper bound than both inequality (5.32) and the following relation

$$\sup_{T_n \in T^\star} \frac{\|D(\psi)T_n\|_q}{\|D(\varphi)T_n\|_q} \asymp n^{d(\frac{1}{p} - 1)} \ln^{\frac{1}{q}}(n + 1) \quad \text{if} \quad 0 < p \leq 1 < q < \infty,$$

which follows from Theorem 5.1. We assume in (5.41) that $\psi \in \mathcal{H}_\alpha$, $\varphi \in \mathcal{H}_{\alpha + \gamma}$, and $\widetilde{\psi}, \widetilde{\varphi} \in C\infty(\mathbb{R}^d \setminus \{0\})$. Note that the left-hand sides of (5.40) and (5.41) are not equivalent.

Proof. By Corollary 3.3 and Nikol’skii’s inequality, we have

$$\|T_n\|_{W^{\alpha+\gamma}_1} \lesssim \sup_{|\xi|=1, \xi \in \mathbb{R}^d} \left\| \left( \frac{\partial}{\partial \xi} \right)^{\alpha+\gamma} T_n \right\|_1 \lesssim n^{d(\frac{1}{p} - 1)} \sup_{|\xi|=1, \xi \in \mathbb{R}^d} \left\| \left( \frac{\partial}{\partial \xi} \right)^{\alpha+\gamma} T_n \right\|_p,$$

and

$$\sup_{|\xi|=1, \xi \in \mathbb{R}^d} \left\| \left( \frac{\partial}{\partial \xi} \right)^{\alpha+\gamma} T_n \right\|_q \lesssim n^{d(\frac{1}{p} - 1)} I(T_n),$$

where

$$I(T_n) := \begin{cases} \frac{\|(-\Delta)^{\alpha/2} T_n\|_q}{\|T_n\|_{W^{\alpha+\gamma}_1}}, & 1 < q < \infty; \\ \frac{\|T_n\|_{W^{\alpha+\gamma}_1}}{\|T_n\|_{\dot{W}^\alpha}}, & q = \infty. \end{cases}$$

Let us first consider the case $1 < q < \infty$. The proof of (5.40) is in five steps.

1) Suppose that $\gamma = d(1 - 1/q) \geq 1$. Choose $q^* \in (1, q)$ such that

$$\gamma - 1 = d \left( \frac{1}{q^*} - \frac{1}{q} \right).$$

Then, using the Hardy-Littlewood inequality for fractional integrals and Lemma 5.13 (i), we have

$$\|(-\Delta)^{\alpha/2} T_n\|_q \lesssim \|(-\Delta)^{(\alpha+\gamma-1)/2} T_n\|_{q^*} \lesssim \|T_n\|_{W^{\alpha+\gamma-1}_{q^*}} \lesssim \|T_n\|_{\dot{W}^\alpha_{q^*}},$$

(5.42)

that is, $I(T_n) \lesssim 1$.

2) If $1 \leq \gamma < d(1 - 1/q)$, then we can choose $\widetilde{q} \in (1, q)$ such that

$$\gamma = d \left( 1 - \frac{1}{\widetilde{q}} \right).$$

Then, using Nikol’skii’s inequality and (5.42), we get

$$\|(-\Delta)^{\alpha/2} T_n\|_q \lesssim n^{d(\frac{1}{\widetilde{q}} - \frac{1}{q})} \|(-\Delta)^{\alpha/2} T_n\|_{\widetilde{q}} \lesssim n^{d(1 - \frac{1}{\widetilde{q}}) - \gamma} \|T_n\|_{W^{\alpha+\gamma}_1},$$

i.e., $I(T_n) \lesssim n^{d(1 - 1/q) - \gamma}.$
3) If \( d(1 - 1/q) < \gamma < d \), then we can choose \( q_\ast > q \) such that

\[
\gamma = d \left( 1 - \frac{1}{q_\ast} \right) > d \left( 1 - \frac{1}{q} \right).
\]

Then, using Hölder’s inequality and (5.42), we have

\[
\|(-\Delta)^{\alpha/2} T_n\|_q \lesssim \|(-\Delta)^{\alpha/2} T_n\|_{q_\ast} \lesssim \|T_n\|_{\dot{W}^{\alpha+\gamma}_1},
\]

i.e., \( I(T_n) \lesssim 1 \).

4) If \( \gamma > d \), then we have

\[
(-\Delta)^{\alpha/2} T_n(x) = (-\Delta)^{(\alpha+\gamma-d)/2} T_n * f_{\gamma-d}(x),
\]

where

\[
f_{\gamma-d}(x) = \sum_{k \neq 0} e^{i(k,x)} |k|^{-\gamma-d}.
\]

Note that \( f_{\gamma-d} \in L_q(\mathbb{T}^d) \) (see (5.11)). Thus, from the above and inequality (5.38) we obtain

\[
\|(-\Delta)^{\alpha/2} T_n\|_q \lesssim \|(-\Delta)^{(\alpha+\gamma-d)/2} T_n\|_q \|f_{\gamma-d}\|_1 \lesssim \|T_n\|_{\dot{W}^{\alpha+\gamma}_q} \lesssim \|T_n\|_{\dot{W}^{\alpha+\gamma}_1},
\]

which gives \( I(T_n) \lesssim 1 \).

5) Assume that \( \gamma = d \). Noting that \( \alpha \in \mathbb{N} \), Lemma 5.13 (ii) gives

\[
\|(-\Delta)^{\alpha/2} T_n\|_q \lesssim \|T_n\|_{\dot{W}^{\alpha}_q} \lesssim \|T_n\|_{\dot{W}^{\alpha}_1},
\]

i.e., \( I(T_n) \lesssim 1 \).

Finally, let us consider the case \( q = \infty \). If \( \gamma \geq d \), then applying Lemma 5.13 (ii), and using (5.39) in the case \( \gamma > d \), we obtain

\[
\|T_n\|_{\dot{W}^{\alpha}_\infty} \lesssim \|T_n\|_{\dot{W}^{\alpha+d}_1} \lesssim \|T_n\|_{\dot{W}^{\alpha+\gamma}_1}.
\]

If \( 1 \leq \gamma < d \), then we choose \( \tilde{q} > 1 \) such that

\[
\gamma = d \left( 1 - \frac{1}{\tilde{q}} \right).
\]

Now, applying the Nikol’skii inequality and (5.42), we get

\[
\|T_n\|_{\dot{W}^{\alpha}_\infty} \lesssim n^{\frac{d}{\tilde{q}}} \|T_n\|_{\dot{W}^{\alpha}_{\tilde{q}}} = n^{d-\gamma} \|T_n\|_{\dot{W}^{\alpha}_{\tilde{q}}} \lesssim n^{d-\gamma} \|(-\Delta)^{\alpha/2} T_n\|_{\tilde{q}} \lesssim n^{d-\gamma} \|T_n\|_{\dot{W}^{\alpha+\gamma}_1}.
\]

Thus, we have proved (5.40) for all cases. \( \square \)
6. General form of the Ulyanov inequality for moduli of smoothness, K-functionals, and their realizations

Let \( \psi \in \mathcal{H}_\alpha, \alpha > 0 \), and let \( W_p(\psi) \) be the space of \( \psi \)-smooth functions in \( L_p \), that is,

\[
W_p(\psi) = \{ g \in L_p(\mathbb{T}^d) : D(\psi)g \in L_p(\mathbb{T}^d) \}.
\]

We define the generalized K-functional by

\[
K_\psi(f, \delta)_p = \inf_{g \in W_p(\psi)} \{ \| f - g \|_p + \delta^\alpha \| D(\psi)g \|_p \}.
\]

It is known that \( K_\psi(f, \delta)_p = 0 \) if \( 0 < p < 1 \). This was shown for \( d = 1, \psi(\xi) = (i\xi)^r, r \in \mathbb{N} \), in [22] and in the general case in [79]. The suitable substitution of the K-functional is given by the realization concept

\[
R_\psi(f, \delta)_p = \inf_{T \in T_{[1/\delta]}} \{ \| f - T \|_p + \delta^\alpha \| D(\psi)T \|_p \}.
\]

We list below several important properties of the K-functionals and their realizations.

**Lemma 6.1.** ([79, Theorem 4.21]) Let \( \psi \in \mathcal{H}_\alpha, \alpha > 0 \), and \( 1 \leq p \leq \infty \). We have

\[
R_\psi(f, \delta)_p \asymp K_\psi(f, \delta)_p.
\]

The next lemma is a simple corollary of Lemma 6.1.

**Lemma 6.2.** ([79, Lemma 4.10]) Let \( \psi \in \mathcal{H}_\alpha, \alpha > 0 \), and \( 1 < p < \infty \). We have

\[
R_\psi(f, \delta)_p \asymp \inf_{(-\Delta)^{\alpha/2}g \in L_p} \{ \| f - g \|_p + \delta^\alpha \| (-\Delta)^{\alpha/2}g \|_p \}.
\]

The following result was proved in [79, Theorem 4.24] (with the remark that if \( \psi \) is a polynomial one can consider any \( p > 0 \)).

**Lemma 6.3.** Let \( \psi \in \mathcal{H}_\alpha, \alpha > 0 \), and either \( \psi \) be a polynomial and \( 0 < p \leq \infty \) or \( d/(d + \alpha) < p \leq \infty \). Then

\[
R_\psi(f, \delta)_p \asymp \| f - T \|_p + \delta^\alpha \| D(\psi)T \|_p,
\]

where \( T \in T_n, n = [1/\delta] \), is such that

\[
\| f - T \|_p \lesssim E_n(f)_p.
\]

The proof is standard using the Bernstein inequality

\[
\| D(\psi)T \|_p \lesssim n^\alpha \| T \|_p
\]
(see [80]).

The next lemma easily follows from the definition of the generalized K-functional.

**Lemma 6.4.** Let \( \psi \in \mathcal{H}_\alpha, \alpha > 0 \), and \( 1 \leq p \leq \infty \). We have

\[
K_\psi(f, \delta)_p \lesssim \delta^\alpha \| D(\psi)f \|_p.
\]

**Lemma 6.5.** ([79, Theorem 4.22]) Let \( \psi \in \mathcal{H}_\alpha, \alpha > 0 \), and \( 0 < p \leq \infty \). We have

\[
R_\psi(f, \delta)_p \leq R_\psi(f, n\delta)_p \lesssim n^{\alpha + d\left(\frac{1}{p} - 1\right)}R_\psi(f, \delta)_p, \quad n \in \mathbb{N}.
\]
From Lemma 6.3 and inequality (6.1), it is easy to obtain the following result.

**Lemma 6.6.** Let \( \alpha_1 \geq \alpha_2 > 0, \psi_1 \in \mathcal{H}_{\alpha_1}, \psi_2 \in \mathcal{H}_{\alpha_2}, \) and \( 0 < p \leq \infty. \) If \( \psi_1/\psi_2 \in C^\infty(\mathbb{R}^d \setminus \{0\}) \) and either \( \psi_1/\psi_2 \) is a polynomial or \( d/(d + \alpha_1 - \alpha_2) < p \leq \infty. \) Then

\[
\mathcal{R}_{\psi_1}(f, \delta)_p \lesssim \mathcal{R}_{\psi_2}(f, \delta)_p.
\]

The next result provides the general form of the sharp Ulyanov inequality for realizations. It will play the key role in our further study.

**Theorem 6.1.** Let \( f \in L_p(\mathbb{T}^d), \) \( 0 < p < q \leq \infty, \) \( \alpha > 0, \gamma \geq 0, \psi \in \mathcal{H}_\alpha, \) and \( \varphi \in \mathcal{H}_{\alpha + \gamma}. \)

**(A)** For any \( \delta \in (0, 1), \) we have

\[
(6.2) \quad \mathcal{R}_\psi(f, \delta)_q \lesssim \frac{\mathcal{R}_\varphi(f, \delta)_p}{\delta^{\gamma}} \eta \left( \frac{1}{\delta} \right) + \left( \int_0^\delta \left( \frac{\mathcal{R}_\varphi(f, t)_p}{t^{d(\frac{1}{p} - \frac{1}{q})}} \right)^{\frac{q}{n}} \frac{dt}{t} \right)^{\frac{1}{\eta}},
\]

where

\[
\eta(t) := \eta(t; \psi, \varphi, p, q, d) := \sup_{T \in \mathcal{T}_t} \frac{\|D(\psi)T\|_q}{\|D(\varphi)T\|_p}.
\]

**(B)** Let either \( \psi \) be a polynomial or \( d/(d + \alpha) < q \leq \infty. \) Let also \( \alpha + \gamma > d(1/p - 1/q), \) and

\[
n^{d(1/p - 1/q)} - \gamma \lesssim \eta(n) \quad \text{as} \quad n \to \infty.
\]

Then there exists a sequence of nontrivial polynomials \( T_n \in \mathcal{T}_n, n \in \mathbb{N}, \) such that

\[
(6.3) \quad \mathcal{R}_\psi(T_n, \delta)_q \lesssim \frac{\mathcal{R}_\varphi(T_n, \delta)_p}{\delta^{\gamma}} \eta \left( \frac{1}{\delta} \right) + \left( \int_0^\delta \left( \frac{\mathcal{R}_\varphi(T_n, t)_p}{t^{d(\frac{1}{p} - \frac{1}{q})}} \right)^{\frac{q}{n}} \frac{dt}{t} \right)^{\frac{1}{\eta}},
\]

where \( n = [1/\delta]. \)

**Remark 6.1.**

(i) The condition \( \alpha + \gamma > d(1/p - 1/q) \) is natural since by Lemma 6.5 it is clear that \( t^{\alpha + \gamma} \mathcal{R}_\varphi(f, 1)_p \lesssim \mathcal{R}_\varphi(f, t)_p, \quad t \in (0, 1), \quad 1 \leq p \leq \infty. \) Therefore, if

\[
\int_0^1 \left( \frac{\mathcal{R}_\varphi(f, t)_p}{t^{d(\frac{1}{p} - \frac{1}{q})}} \right)^{\frac{q}{n}} dt < \infty,
\]

then \( \alpha + \gamma > d(1/p - 1/q). \)

(ii) The condition \( n^{d(1/p - 1/q)} - \gamma \lesssim \eta(n) \) as \( n \to \infty \) is also natural since it always holds if \( \psi \in C^\infty(\mathbb{R}^d \setminus \{0\}) \) (see Theorem 7.1).

**Proof.** (A) Let \( T_n \in \mathcal{T}_n, \) \( n \in \mathbb{N}, \) be such that

\[
(6.4) \quad \|f - T_n\|_p + n^{-(\alpha + \gamma)} \|D(\varphi)T_n\|_p \leq 2 \mathcal{R}_\varphi(f, n^{-1})_p.
\]

From the Nikol’skii inequality (1.18), it follows that (see [24, Lemma 4.2])

\[
\|f - T_{2^n}\|_q \lesssim \left( \sum_{\nu=0}^{\infty} 2^{\nu q_1 d(\frac{1}{p} - \frac{1}{q})} \|f - T_{2^n}\|_p^{q_1} \right)^{\frac{1}{q_1}}
\]

\[
(6.5) \quad \lesssim \left( \sum_{\nu=0}^{\infty} 2^{\nu q_1 d(\frac{1}{p} - \frac{1}{q})} \mathcal{R}_\varphi(f, 2^{-\nu})_p^{q_1} \right)^{\frac{1}{q_1}}.
\]
By the definition of $\eta(t)$ and (6.4), we get

$$2^{-\alpha n}\|D(\psi)T_{2^n}\|_q \leq \eta(2^n)2^{-\alpha n}\|D(\varphi)T_{2^n}\|_p.$$  

(6.6)

Thus, taking into account (6.4), (6.5), and (6.6), we obtain

$$R_\psi(f, 2^{-n})_q \leq \|f - T_{2^n}\|_q + 2^{-\alpha n}\|D(\psi)T_{2^n}\|_q$$

$$\lesssim 2^{\gamma n}R_\varphi(f, 2^{-n})_p \eta(2^n) + \left(\sum_{\nu=n}^{\infty} \left(2^{\nu d\left(\frac{1}{p} - \frac{1}{q}\right)}R_\varphi(f, 2^{-\nu})_p\right)^{q_1}\right)^{\frac{1}{q_1}}.$$  

From the last inequality, (6.2) follows immediately.

(B) Let us prove the second part of the theorem. We choose a sequence of polynomials $T_n, n \in \mathbb{N}$, such that

$$\eta(n) \lesssim \frac{\|D(\psi)T_n\|_q}{\|D(\varphi)T_n\|_p}.$$  

(6.7)

Further, by Lemma 6.3, we have

$$R_\psi(T_n, 1/n) \gtrsim n^{-\alpha}\|D(\psi)T_n\|_q.$$  

(6.8)

Using the definition of the realization of $K$-functional, we get

$$B_n := R_\varphi(T_n, 1/n)_p n^{\gamma} \eta(n) + \left(\int_0^{1/n} \left(\frac{R_\varphi(T_n, t)_p}{t^{d\left(\frac{1}{p} - \frac{1}{q}\right)}}\right)^{q_1} dt\right)^{\frac{1}{q_1}}$$

(6.9)

$$\lesssim \|D(\varphi)T_n\|_p n^{-\alpha} \eta(n) + \|D(\varphi)T_n\|_p \left(\int_0^{1/n} \left(\frac{t^{\alpha + \gamma}}{t^{d\left(\frac{1}{p} - \frac{1}{q}\right)}}\right)^{q_1} dt\right)^{\frac{1}{q_1}}$$

$$\lesssim \|D(\varphi)T_n\|_p n^{-\alpha} \eta(n).$$

Thus, from (6.7), (6.8), and (6.9) we derive

$$R_\psi(T_n, 1/n)_q \gtrsim n^{-\alpha}\|D(\psi)T_n\|_q \gtrsim n^{-\alpha} \eta(n)\|D(\varphi)T_n\|_p \gtrsim B_n.$$  

This yields (6.3) by the monotonicity property given in Lemma 6.5.

□

Taking into account Lemma 6.1, we obtain the following result.

**Corollary 6.1.** Under all conditions of Theorem 6.1 if $p \geq 1$, we have

$$K_\psi(f, \delta)_q \lesssim \frac{K_\varphi(f, \delta)_p}{\delta^n} \eta\left(\frac{1}{\delta}\right) + \left(\int_0^{\delta} \left(\frac{K_\varphi(f, t)_p}{t^{d\left(\frac{1}{p} - \frac{1}{q}\right)}}\right)^{q_1} dt\right)^{\frac{1}{q_1}}.$$  

The next corollary easily follows from (6.2) and Nikol’skii’s inequality (1.18).

**Corollary 6.2.** Under all conditions of Theorem 6.1, if $\psi(x) = \varphi(x)$, we have

$$R_\varphi(f, \delta)_q \lesssim \left(\int_0^{\delta} \left(\frac{R_\varphi(f, t)_p}{t^{d\left(\frac{1}{p} - \frac{1}{q}\right)}}\right)^{q_1} dt\right)^{\frac{1}{q_1}}.$$  

(6.10)
Following the proof of Theorem 6.1 and taking into account Lemmas 6.3 and 6.6, it is easy to prove a slightly more general form of Theorem 6.1.

**Theorem 6.1′.** Suppose that under conditions of Theorem 6.1, the function \( \varphi \) is either a polynomial or \( d/(d + \alpha + \gamma) < p \leq \infty \). Let also \( m > 0 \) and the function \( \phi \in \mathcal{H}_{\alpha+m} \) be either a polynomial or \( d/(d + \alpha + m) < p \leq \infty \). Then

\[
R_{\psi}(f, \delta)_q \lesssim \frac{R_{\varphi}(f, \delta)_p}{\delta^\gamma} \eta \left( \frac{1}{\delta} \right) + \left( \int_0^\delta \left( \frac{R_{\phi}(f, t)_p}{t^{d(\frac{1}{p} - \frac{1}{q})}} \right)^{\frac{1}{q_1}} \frac{1}{t^{\frac{1}{q_1}}} \right).
\]

An analogue of Theorem 6.1′ for the moduli of smoothness reads as follows.

**Theorem 6.2.** Let \( f \in L_p(\mathbb{T}^d), \ d \geq 1, \ 0 < p < q \leq \infty, \ \alpha > 0, \ \gamma, m \geq 0, \ \alpha \in \mathbb{N} \cup ((1/q - 1)_+, \infty), \ \text{and} \ \alpha + \gamma, \alpha + m \in \mathbb{N} \cup ((1/p - 1)_+, \infty). \) Then, for any \( \delta \in (0, 1) \), we have

\[
\omega_{\alpha}(f, \delta)_p \lesssim \omega_{\alpha+\gamma}(f, \delta)_p \eta \left( \frac{1}{\delta} \right) + \left( \int_0^\delta \left( \frac{\omega_{\alpha+m}(f, t)_p}{t^{d(\frac{1}{p} - \frac{1}{q})}} \right)^{\frac{1}{q_1}} \frac{1}{t^{\frac{1}{q_1}}} \right),
\]

where

\[
\eta(t) := \sup_{T \in \mathbb{T}^d} \frac{\omega_{\alpha}(T, 1/t)_q}{\omega_{\alpha+\gamma}(T, 1/t)_p}.
\]

The proof goes along the same line as in Theorem 6.1 for the realization of the \( K \)-functionals.
7. Sharp Ulyanov inequalities for $K$-functionals and realizations

The goal of this section is to prove the following theorem, which provides an explicit form of the sharp Ulyanov inequality. Here, we assume that $\psi/\varphi$ is a smooth function (cf. Theorem 6.1).

**Theorem 7.1.** Let $0 < p < q \leq \infty$, $\alpha > 0$, $\gamma \geq 0$, $\psi \in H_\alpha$, $\varphi \in H_{\alpha+\gamma}$, and $\frac{\psi}{\varphi} \in C^\infty(\mathbb{R}^d \setminus \{0\})$.

(A) Let $f \in L_p(\mathbb{T}^d)$. Then, for any $\delta \in (0, 1)$, we have

\[
R_\psi(f, \delta) q \lesssim \frac{R_{\frac{\psi}{\varphi}}(f, \delta)}{\delta^\gamma} \sigma \left( \frac{1}{\delta} \right) + \left( \int_0^\delta \left( \frac{R_{\frac{\psi}{\varphi}}(f, t) q}{t} \right)^{\frac{1}{q}} \right)^{\frac{1}{q}},
\]

where

(A1) If $0 < p \leq 1$ and $p < q < \infty$, then

\[
\sigma(t) := \begin{cases}
    t^{d \left( \frac{1}{p} - 1 \right)}, & \gamma > d \left( 1 - \frac{1}{q} \right) + ; \\
    t^{d \left( \frac{1}{p} - 1 \right) \ln \frac{1}{\delta}(t + 1)}, & 0 < \gamma = d \left( 1 - \frac{1}{q} \right) + ; \\
    t^{\left( \frac{1}{p} - \frac{1}{q} \right) - \gamma}, & 0 < \gamma < d \left( 1 - \frac{1}{q} \right) + ; \\
    t^{d \left( \frac{1}{p} - \frac{1}{q} \right)}, & \gamma = 0, 0 < p \leq 1 < q < \infty; \\
    t^{d \left( \frac{1}{p} - \frac{1}{q} \right)}, & \gamma = 0, 0 < q \leq 1, \text{ and } \frac{\psi(\xi)}{\varphi(\xi)} \equiv \text{const}; \\
    t^{d \left( \frac{1}{p} - 1 \right)}, & \gamma = 0, 0 < q < 1, \text{ and } \frac{\psi(\xi)}{\varphi(\xi)} \not\equiv \text{const}; \\
    t^{d \left( \frac{1}{p} - 1 \right) \ln(t + 1)}, & \gamma = 0, q = 1, \text{ and } \frac{\psi(\xi)}{\varphi(\xi)} \not\equiv \text{const},
\end{cases}
\]

(A2) If $0 < p \leq 1$ and $q = \infty$, then

\[
\sigma(t) := \begin{cases}
    t^{d \left( \frac{1}{p} - 1 \right)}, & \gamma > d; \\
    t^{d \left( \frac{1}{p} - 1 \right)}, & \gamma = d = 1 \text{ and } \frac{\psi(\xi)}{\varphi(\xi)} = A|\xi|^{-\gamma} \text{ sign } \xi \text{ for some } A \in \mathbb{C} \setminus \{0\}; \\
    t^{d \left( \frac{1}{p} - 1 \right) \ln(t + 1)}, & \gamma = d = 1 \text{ and } \frac{\psi(\xi)}{\varphi(\xi)} \not= A|\xi|^{-\gamma} \text{ sign } \xi \text{ for any } A \in \mathbb{C} \setminus \{0\}; \\
    t^{d \left( \frac{1}{p} - 1 \right) \ln(t + 1)}, & \gamma = d \geq 2; \\
    t^{d \left( \frac{1}{p} - \gamma \right)}, & 0 \leq \gamma < d,
\end{cases}
\]

(A3) If $1 < p < q \leq \infty$, then

\[
\sigma(t) := \begin{cases}
    1, & \gamma \geq d \left( \frac{1}{p} - \frac{1}{q} \right), q < \infty; \\
    1, & \gamma > \frac{d}{p}, q = \infty; \\
    \ln \frac{1}{\delta}(t + 1), & \gamma = \frac{d}{p}, q = \infty; \\
    t^{d \left( \frac{1}{p} - \frac{1}{q} \right) - \gamma}, & 0 \leq \gamma < d \left( \frac{1}{p} - \frac{1}{q} \right).
\end{cases}
\]

(B) Inequality (7.1) is sharp in the following sense. Let $\psi$ be either a polynomial or $d/(d+\alpha) < q \leq \infty$. Let $\frac{\psi}{\varphi} \in C^\infty(\mathbb{R}^d \setminus \{0\})$ and $\alpha + \gamma > d(1 - 1/q) +$. Let also $\psi(x) = C\varphi(x)$ for some constant $C$ in the case $\gamma = 0$ and $0 < p < q \leq 1$ or $\gamma = 0$ and...
0 < p ≤ 1, q = ∞. Then there exists a function \( f_0 \in L_q(\mathbb{T}^d) \), \( f_0 \not\equiv \text{const} \), such that

\[
R_\psi(f_0, \delta)_q \lesssim \frac{R_\varphi(f_0, \delta)_p}{\delta^\gamma} \sigma \left( \frac{1}{\delta} \right) + \left( \int_0^\delta \left( \frac{R_\varphi(f_0, t)_p}{t^{d(\frac{1}{p} - \frac{1}{q})}} \right)^{\frac{q_1}{q}} \frac{dt}{t} \right)^{\frac{1}{q_1}}
\]
as \( \delta \to 0 \).

In light of Lemma 6.1, Theorem 7.1 implies the following result.

**Corollary 7.1.** Under all conditions of Theorem 7.1 if \( p \geq 1 \), we have

\[
K_\psi(f, \delta)_q \lesssim K_\varphi(f, \delta)_p \delta^\gamma \sigma \left( \frac{1}{\delta} \right) + \left( \int_0^\delta \left( \frac{K_\varphi(f, t)_p}{t^{d(\frac{1}{p} - \frac{1}{q})}} \right)^{\frac{q_1}{q}} \frac{dt}{t} \right)^{\frac{1}{q_1}}.
\]

**Remark 7.1.** (i) Note that in the proof of Theorem 7.1, in (A1) with \( \gamma = 0 \), \( 0 < p < q \leq 1 \) and in (A2) we may not assume that \( \frac{\psi}{\varphi} \in C^\infty(\mathbb{R}^d \setminus \{0\}) \).

(ii) It is important to remark that in part (B) of Theorem 6.1, we show sharpness of the corresponding Ulyanov type inequality by constructing a sequence of functions which depends on \( \delta \), while in Theorem 7.1 we construct a function \( f_0 \) which is independent of \( \delta \).

Similarly to Theorem 6.1', one can obtain the following more general analogue of Theorem 7.1.

**Theorem 7.1'.** Suppose that under conditions of Theorem 7.1, the function \( \varphi \) is either a polynomial or \( d/(d + \alpha + \gamma) < p \leq \infty \). Let also \( m > 0 \) and the function \( \phi \in H_{\alpha + m} \) be either a polynomial or \( d/(d + \alpha + m) < p \leq \infty \). Then

\[
R_\psi(f, \delta)_q \lesssim \frac{R_\varphi(f, \delta)_p}{\delta^\gamma} \sigma \left( \frac{1}{\delta} \right) + \left( \int_0^\delta \left( \frac{R_\varphi(f, t)_p}{t^{d(\frac{1}{p} - \frac{1}{q})}} \right)^{\frac{q_1}{q}} \frac{dt}{t} \right)^{\frac{1}{q_1}}.
\]

**Proof of Theorem 7.1.** (A) Taking into account Theorem 6.1, it is enough to estimate

\[
\eta(t) = \sup_{T \in T_{[t]}} \|D(\psi)T\|_q / \|D(\varphi)T\|_p.
\]

Since \( \frac{\psi}{\varphi} \in C^\infty(\mathbb{R}^d \setminus \{0\}) \), the inequality

\[
\eta(t) \lesssim \sigma(t)
\]

follows from Theorem 5.1 and Remark 5.3 in all cases except the case \( 0 < p < q \leq 1 \), \( \gamma = 0 \), \( d \geq 2 \), and \( \frac{\psi}{\varphi} \not\equiv \text{const} \). In the latter case, the proof of this inequality follows from Corollary 10.1 stated below.

To prove (B), we will construct a nontrivial function \( f_0 \in L_q(\mathbb{T}^d) \) such that equivalence (7.2) holds as \( \delta \to 0 \).

**Part (B1): 0 < p ≤ 1 and p < q ≤ \infty.** We set

\[
f_0(x) = F_\varphi(x) - \frac{1}{N_{\alpha + \gamma}} F_\varphi(Nx),
\]
where
\begin{equation}
F_\varphi(x) \sim \sum_{k \neq 0} \frac{e^{i(k,x)}}{\varphi(k)}
\end{equation}
and \( N \) is a fixed sufficiently large integer that will be chosen later.

Let us show that \( F_\varphi \in L_{q^*}(\mathbb{T}^d) \), where \( q^* = \max(q, 1) \). For this, we consider the function
\[ \mathcal{F}_{\alpha+\gamma}(x) \sim \sum_{k \neq 0} \frac{e^{i(k,x)}}{|k|^\alpha+\gamma}. \]
It is known that \( \mathcal{F}_{\alpha+\gamma} \in L_{q^*}(\mathbb{T}^d) \) for \( \alpha + \gamma > d(1 - 1/q^*) \) (see (5.11)). At the same time, the function
\[ u(y) = \frac{|y|^{\alpha+\gamma-\varepsilon}}{\varphi(y)} (1 - v(|y|)) \quad 0 < \varepsilon < \alpha + \gamma, \]
is a Fourier multiplier in \( L_r(\mathbb{T}^d) \) for any \( 1 \leq r \leq \infty \). This follows from Lemma 2.4 (see also Lemma 2.2) and the fact that \( \varphi(y) = 0 \) if and only if \( y = 0 \). Take \( \varepsilon \) such that \( \alpha + \gamma - \varepsilon > d(1 - 1/q^*) \). We have that
\[ \sum_{k \neq 0} \frac{e^{i(k,x)}}{|k|^\alpha+\gamma-\varepsilon} \in L_{q^*}(\mathbb{T}^d). \]
Thus, taking into account that
\[ F_\varphi(x) \sim \sum_{k \neq 0} \frac{u(k)}{|k|^\alpha+\gamma-\varepsilon} e^{i(k,x)}, \]
we obtain that \( F_\varphi \in L_{q^*}(\mathbb{T}^d) \) and, therefore, \( f_0 \in L_{q^*}(\mathbb{T}^d) \).

Next,
\begin{equation}
\mathcal{R}_\varphi(f_0, n^{-1})_p \lesssim \|f_0 - f_0 \ast V_n\|_p + n^{-\alpha-\gamma}\|\mathcal{D}(\varphi)(f_0 \ast V_n)\|_p.
\end{equation}
Recall that \( V_n \) is given by (5.2).

Let us consider the second summand in (7.6). In light of
\[ (F_\varphi(N \cdot) \ast V_n)(x) = \sum_{k \neq 0} \prod_{j=1}^d v\left(\frac{|k_j|}{n}\right) \frac{e^{i(Nk,x)}}{\varphi(k)} \]
and since \( \varphi \) is homogeneous of order \( \alpha + \gamma \), we obtain
\[ \mathcal{D}(\varphi)(f_0 \ast V_n)(x) = \sum_{k \neq 0} \prod_{j=1}^d v\left(\frac{|k_j|}{n}\right) e^{i(k,x)} - \sum_{k \neq 0} \prod_{j=1}^d v\left(\frac{|k_j|}{n}\right) e^{i(Nk,x)} \]
\[ = V_n(x) - V_{n/N}(N x). \]
Using the last equalities and Corollary 2.1, we derive that
\begin{equation}
\|\mathcal{D}(\varphi)(f_0 \ast V_n)\|_p \lesssim \|V_n\|_p + \|V_{n/N}\|_p \lesssim n^{d(1 - \frac{1}{p})}.
\end{equation}

Now, let us consider the first summand in (7.6). We estimate
\begin{equation}
\|f_0 - f_0 \ast V_n\|_p \lesssim \|F_\varphi - F_\varphi \ast V_n\|_p + N^{-p(\alpha+\gamma)}\|F_\varphi(N \cdot) - F_\varphi(N \cdot) \ast V_n\|_p.
\end{equation}
It is enough to estimate the first term in (7.8). We have
\[
\|F_\varphi - F_\varphi * V_n\|_p = \left\| \sum_{k \in \mathbb{Z}^d} \left( 1 - \prod_{j=1}^d v\left( \frac{|k_j|}{n} \right) \right) e^{i(k,x)} \varphi(k) \right\|_p
\]
\[
= \frac{1}{n^{\alpha+\gamma}} \left\| \sum_{k \in \mathbb{Z}^d} \xi \left( \frac{k}{n} \right) e^{i(k,x)} \right\|_p,
\]
where
\[
\xi(y) = \frac{1 - \prod_{j=1}^d v(|y_j|)}{\varphi(y)}.
\]

By Lemma 2.5, using the fact that \( \xi \in \dot{B}_{2,p}^{d(\frac{1}{p} - \frac{1}{2})} \cap \dot{B}_{2,1}^{d} (\mathbb{R}^d) \) (see, e.g., Lemma 2.2), and the homogeneity property of the Besov (quasi-)norm (see Lemma 2.3), we get
\[
\|F_\varphi - F_\varphi * V_n\|_p \lesssim n^{-\alpha-\gamma} \left\| \xi \left( \frac{y}{n} \right) \right\|_{\dot{B}_{2,p}^{d(\frac{1}{p} - \frac{1}{2})} (\mathbb{R}^d)}
\]
\[
\lesssim n^{d(1 - \frac{1}{p}) - \alpha - \gamma} \left\| \xi \right\|_{\dot{B}_{2,p}^{d(\frac{1}{p} - \frac{1}{2})} (\mathbb{R}^d)} \lesssim n^{d(1 - \frac{1}{p}) - \alpha - \gamma}.
\]
Hence, from (7.8) we have
\[
\|f_0 - f_0 * V_n\|_p \lesssim n^{-\alpha-\gamma}.
\]

Thus, combining (7.6), (7.7), and (7.10), we arrive at
\[
\mathcal{R}_{\varphi} (f_0, n^{-1})_p \lesssim n^{d(1 - \frac{1}{p}) - \alpha - \gamma}.
\]

Let us assume that
\[
\mathcal{R}_{\psi} (f_0, 1/n)_q \gtrsim n^{-\alpha} \|\mathcal{D}(\psi/\varphi)V_n\|_q, \quad n \in \mathbb{N},
\]
and
\[
t^{d(1/p - 1/q) - \gamma} = \mathcal{O}(\eta(t)) \quad \text{as} \quad t \to \infty,
\]
where \( \eta \) is given by
\[
\eta(t) = \sup_{T \in T_{[t]}} \frac{\|\mathcal{D}(\psi) T\|_q}{\|\mathcal{D}(\varphi) T\|_p}.
\]
We will prove (7.12) and (7.13) later.
Now, comparing estimates (7.11) and (7.12), we derive the following inequalities for $\delta \to 0$

\[
\mathcal{R}_\psi(f_0, \delta)_q \gtrsim \delta^\alpha \|D(\psi/\varphi)V_{1/\delta}\|_q \quad \text{(by (7.12))}
\]

\[
= \delta^{\alpha + d(\frac{1}{p} - 1)} \frac{1}{\delta^{d(\frac{1}{p} - 1)}} \|D(\psi/\varphi)V_{1/\delta}\|_q
\]

\[
\gtrsim \delta^{\alpha + d(\frac{1}{p} - 1)} \eta(1/\delta) \quad \text{(by Lemma 5.1)}
\]

\[
\gtrsim \delta^{\alpha + d(\frac{1}{p} - 1)} \sigma(1/\delta) + \delta^{\alpha + d(\frac{1}{p} - 1)} \eta(1/\delta) \quad \text{(by Theorem 5.1)}
\]

\[
\gtrsim \mathcal{R}_\psi(f_0, \delta)_q^{\frac{\gamma}{\delta}} \sigma(1/\delta) + \delta^{\alpha + \gamma + d(\frac{1}{p} - 1)} \quad \text{(by (7.11))}
\]

\[
\gtrsim \mathcal{R}_\psi(f_0, \delta)_q^{\frac{\gamma}{\delta}} \sigma(1/\delta) + \left( \int_0^{\delta} \left( \mathcal{R}_\psi(f_0, t)_p \right)^{\frac{q_1}{q}} \frac{dt}{t} \right)^{\frac{1}{q}} \quad \text{(by (7.13))}
\]

(7.14)

where in the last inequality we used the fact that $\alpha + \gamma > d(1/p - 1/q) \geq d(1 - 1/q)$.

Therefore, (7.2) follows. To complete the proof of (B1), we need to verify (7.12) and (7.13).

**Proof of estimate (7.12).** We divide the proof of this fact into three steps.

**Step 1.** Assume that $0 < q \leq 1$ and $\gamma > 0$. Denoting

\[H_{\varphi,n}(x) := F_\varphi * V_n(x) - \frac{1}{N^\alpha + \gamma} F_\varphi(N \cdot) * V_n(x),\]

we estimate

(7.15)

\[\mathcal{R}_\psi(f_0, 1/n)_q \geq C(q) \left( \mathcal{R}_\psi(H_{\varphi,n}, 1/n)_q - \mathcal{R}_\psi(f_0 - H_{\varphi,n}, 1/n)_q \right).\]

Since $H_{\varphi,n}$ is a trigonometric polynomial of degree $2Nn$, using Lemma 6.3, we get

\[\mathcal{R}_\psi(H_{\varphi,n}, 1/n)_q \gtrsim n^{-\alpha} \|D(\psi)H_{\varphi,n}\|_q.\]

Using the representation

\[D(\psi)H_{\varphi,n}(x) = \sum_k \prod_{j=1}^d v \left( \frac{|k_j|}{n} \right) \frac{\psi(k)}{\varphi(k)} e^{i(k,x)} - N^{-\gamma} \sum_k \prod_{j=1}^d v \left( \frac{|k_j|}{n} \right) \frac{\psi(k)}{\varphi(k)} e^{i(Nk,x)},\]

we obtain

(7.16)

\[\mathcal{R}_\psi(H_{\varphi,n}, 1/n)_q \gtrsim \left( 1 - \frac{1}{N^{\gamma q}} \right)^{1/q} \frac{1}{n^{\alpha}} \|D(\psi/\varphi)V_n\|_q.\]

Further, applying Holder’s inequality and Lemma 6.4 yields

\[\mathcal{R}_\psi(f_0 - H_{\varphi,n}, 1/n)_q \lesssim \mathcal{R}_\psi(f_0 - H_{\varphi,n}, 1/n)_1 \lesssim K_\psi(f_0 - H_{\varphi,n}, 1/n)_1 \]

\[\lesssim n^{-\alpha} \|D(\psi)(f_0 - H_{\varphi,n})\|_1 \]

\[\lesssim n^{-\alpha} \left( \|F_{\varphi/\psi} - F_{\varphi/\psi} * V_n\|_1 + \|F_{\varphi/\psi}(N \cdot) - F_{\varphi/\psi} * V_{nN}(N \cdot)\|_1 \right),\]

where $F_{\varphi/\psi}(x) = \sum_{k \neq 0} \frac{\psi(k)}{\varphi(k)} e^{i(k,x)}$ (see (7.5)).
Since \( \|F_{\varphi/\psi}\|_1 \lesssim \|F_{\varphi/\psi}\|_{1+\varepsilon} \lesssim \|\mathfrak{F}\|_{1+\varepsilon} \lesssim 1 \) (see (5.11)), we have that \( F_{\varphi/\psi} \in L_1(\mathbb{T}^d) \) and, therefore,

\[
(7.17) \quad \mathcal{R}_\psi(f_0 - H_{\varphi,n}, n^{-1})_q = o(n^{-\alpha}) \quad \text{as} \quad n \to \infty.
\]

Thus, combining (7.15)–(7.17), we immediately obtain

\[
\mathcal{R}_\psi(f_0, 1/n)_q \gtrsim n^{-\alpha}\|D(\psi/\varphi)V_n\|_q - o(n^{-\alpha}).
\]

Note that by Lemma 5.2, there exists a constant \( C \) independent of \( n \) such that

\[
\|D(\psi/\varphi)V_n\|_q \geq C.
\]

Hence,

\[
\mathcal{R}_\psi(f_0, 1/n)_q \gtrsim n^{-\alpha}\|D(\psi/\varphi)V_n\|_q,
\]

which implies (7.12) for \( 0 < q \leq 1 \) and \( \gamma > 0 \).

Step 2. Let \( 0 < q \leq 1 \) and \( \gamma = 0 \). In this case, we assume that \( \varphi(y) = C\psi(y) \) with some constant \( C \). By Lemma 6.5, one has that

\[
\mathcal{R}_\psi(f_0, \delta)_q \gtrsim \mathcal{R}_\psi(f_0, 1)_q \delta^{\alpha + d(1/q - 1)}.
\]

On the other hand, (5.9) in Remark 5.5 implies that

\[
\|D(\psi/\varphi)V_1/\delta\|_q \gtrsim \delta^{d(1 - 1/q)}
\]

and hence

\[
\mathcal{R}_\psi(f_0, \delta)_q \gtrsim \delta^{\alpha}\|D(\psi/\varphi)V_1/\delta\|_q,
\]

which is (7.12).

Step 3. Assume that \( 1 < q \leq \infty \). Taking into account that \( f_0 * V_n \) is the near best approximant of \( f_0 \) in \( L_q \), that is, \( \|f_0 * V_n - f_0\|_q \leq CE_n(f_0)_q \), we have that Lemma 6.3 implies

\[
(7.18) \quad \mathcal{R}_\psi(f_0, 1/n)_q \gtrsim n^{-\alpha}\|D(\psi/\varphi)(f_0 * V_n)\|_q
\]

\[
\gtrsim n^{-\alpha}(\|D(\psi/\varphi)V_n\|_q - \frac{1}{N\gamma}\|D(\psi/\varphi)V_{n/N}\|_q).
\]

If \( 1 < q < \infty \) and \( \gamma \geq 0 \), Theorem 5.1 gives the exact growth order of \( \|D(\psi/\varphi)V_n\|_q \) as \( n \to \infty \), which yields

\[
(7.19) \quad \|D(\psi/\varphi)V_n\|_q - \frac{1}{N\gamma}\|D(\psi/\varphi)V_{n/N}\|_q \gtrsim \|D(\psi/\varphi)V_n\|_q
\]

for sufficiently large \( N \). Therefore, (7.12) follows.

If \( q = \infty \) and \( \gamma > 0 \), we take take into account that

\[
D(\psi/\varphi)V_{n/N}(x) = \sum_{k \neq 0} \psi(k) \prod_{j=1}^d \varphi(j) \left( \frac{N|k_j|}{n} \right) e^{i(k,x)}
\]

\[
= \sum_{k \neq 0} \psi(k) \prod_{j=1}^d \varphi(j) \left( \frac{N|k_j|}{n} \right) \left( \frac{|k_j|}{n} \right) e^{i(k,x)}
\]
and that the function $B(y) = \prod_{j=1}^{d} v(y_j)$ is a Fourier multiplier in $L_{\infty}(\mathbb{T}^d)$ (see Lemmas 2.4 and 2.2). Hence, we have

$$\|D(\psi/\varphi)V_{n/N}\|_{\infty} \lesssim \|B(Ny)\|_{L_{\infty}\rightarrow L_{\infty}} \|D(\psi/\varphi)V_{n}\|_{\infty} \lesssim \|D(\psi/\varphi)V_{n}\|_{\infty},$$

which yields (7.19) for sufficiently large $N$.

Finally, suppose that $q = \infty$ and $\gamma = 0$. In this case, we assume that $\varphi(y) = C \psi(y)$ with some constant $C$. Therefore,

$$\|D(\psi/\varphi)V_{n}\|_{\infty} \approx \|V_{n}\|_{\infty} \approx n^d.$$ 

Then, using (7.18) and (7.19) for sufficiently large $N$, we arrive at (7.12).

Thus, the proof of (7.12) is complete.

**Proof of estimate (7.13).** First, assume that $\gamma > 0$. By Lemma 5.1 (iii), $\eta(n) \approx n^{d(1/p-1)}\|D(\psi/\varphi)V_{n}\|_{q}$.

If $1 < q < \infty$, then Lemmas 5.6 and 5.7 describe the sharp growth order of $\|D(\psi/\varphi)V_{n}\|_{q}$ and, in particular,

$$\|D(\psi/\varphi)V_{n}\|_{q} \gtrsim n^{d(1-\frac{1}{q})-\gamma}.$$ 

This gives (7.13).

If $p < q \leq 1$, Lemma 5.2 yields

$$\eta(n) \gtrsim n^{d(\frac{1}{p}-1)} \gtrsim n^{d(\frac{1}{p}-\frac{1}{q})-\gamma}.$$ 

Finally, let $q = \infty$. In this case, by Lemma 5.8, we have

$$\|D(\psi/\varphi)V_{n}\|_{\infty} \gtrsim n^{d-\gamma},$$

which implies the desired result.

Second, assume that $\gamma = 0$ and $\varphi(\xi) = C \psi(\xi)$. Then (7.13) is equivalent to

$$n^{d(\frac{1}{p}-\frac{1}{q})} \lesssim \sup_{T_{n} \in T_{n}} \|T\|_{q}. $$

It is enough to consider extremizers for the classical Nikol’skii inequality, for example, the Jackson-type kernel (see [101, § 4.9]).

Hence, the proof of inequality (7.13) and the part (B1) are complete.

Part (B2): $1 < p < q < \infty$. In this case, if $\gamma \geq d(1/p-1/q)$, then we can take as $f_0$ any non-trivial function from $C_{\infty}(\mathbb{T}^d)$. Indeed, by Lemma 6.2 and (1.14), for any $1 < r < \infty$, we have

$$R_{\psi}(f_0, \delta) \approx \inf_{\chi} \|f_0 - \chi\|_{r} + \delta^\alpha \|(-\Delta)^{\alpha/2}\chi\|_{r} \approx \omega_{\alpha}(f_0, \delta) \approx \delta^\alpha.$$ 

Then,

$$R_{\psi}(f_0, \delta)_{q} \gtrsim \delta^\alpha \gtrsim \frac{\delta^{\alpha+\gamma}}{\delta^\gamma}$$

$$\gtrsim \frac{R_{\psi}(f_0, \delta)}{\delta^\gamma} + \left( \frac{\int_{0}^{\delta} \left( \frac{R_{\psi}(f_0, t)}{t^{d(\frac{1}{p}-\frac{1}{q})}} \right)^{\frac{q}{q-1}} dt}{\delta} \right)^{\frac{1}{q-1}}.$$
If $0 \leq \gamma < d(1/p - 1/q)$, then we take

$$f_0(x) = F_\varphi(x) \sim \sum_{k \neq 0} e^{i(k,x)} \varphi(k).$$

Let us first estimate $\mathcal{R}_\varphi(f_0, \delta)_p$ from above. We have

$$\mathcal{R}_\varphi(f_0, 2^{-n})_p \lesssim \|f_0 - f_0 \ast V_{2^n}\|_p + 2^{-n(\alpha + \gamma)}\|\mathcal{D}(\varphi)(f_0 \ast V_{2^n})\|_p,$$

where $V_{2^n}$ is given by (5.2).

Note that by (7.9), $\|f_0 - f_0 \ast V_{2^n}\|_1 \lesssim 2^{-(\alpha + \gamma)n}$. Then, using Lemma 4.2 from [24] and taking into account that $\alpha + \gamma > d(1 - 1/q) > d(1 - 1/p)$, we obtain

$$\|f_0 - f_0 \ast V_{2^n}\|_p \lesssim \left( \sum_{\nu=n}^{\infty} 2^{\nu pd(1 - 1/q)}\|f_0 - f_0 \ast V_{2^n}\|_1^{1/p} \right) \lesssim 2^{-(\alpha + \gamma + d(\frac{1}{p} - 1))n}.$$  

Further, using the multidimensional Hardy-Littlewood theorem for series with monotone coefficients (see Lemma 2.6), we derive that

$$\|\mathcal{D}(\varphi)(f_0 \ast V_{2^n})\|_p \preceq \|V_{2^n} - 1\|_p \prec \left\| \sum_{|k| \leq 2^n} e^{i(k,x)} \right\|_p \prec \left( \sum_{\nu=1}^{2^n} (\nu^{p-2})^{d/p} \right)^{1/p} \prec 2^{d(n(1 - 1/p)).}$$

Therefore,

$$\mathcal{R}_\varphi(f_0, \delta)_p \lesssim \delta^{\alpha + \gamma + d(\frac{1}{p} - 1)}.$$  

Let us estimate $\mathcal{R}_\psi(f_0, 1/n)_q$ from below. Taking into account that $f_0 \ast V_n$ is the near best approximant of $f_0$ in $L_q$ and using Lemmas 5.7 and 6.3 with $0 \leq \gamma < d(1/p - 1/q) < d(1 - 1/q)$, we get

$$\mathcal{R}_\psi(f_0, 1/n)_q \gtrsim n^{-\alpha}\|\mathcal{D}(\psi)(f_0 \ast V_n)\|_q \gtrsim n^{-\alpha}\|(-\Delta)^{-\gamma/2}V_n\|_q \gtrsim n^{d(1 - \frac{1}{q}) - (\alpha + \gamma)}. $$

Therefore, combining (7.24) and (7.25), we finally obtain

$$\frac{\mathcal{R}_\varphi(f_0, \delta)_p}{\delta^\gamma} \sigma(1/\delta) + \left( \int_0^\delta \left( \frac{\mathcal{R}_\varphi(f_0, t)_p}{t^{d(\frac{1}{p} - 1)}} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \lesssim \delta^{\alpha + d(\frac{1}{p} - 1)} \sigma(1/\delta) + \left( \int_0^\delta t^{q(\alpha + \gamma + d(\frac{1}{p} - 1))} \frac{dt}{t} \right)^{\frac{1}{q}} \lesssim \delta^{\alpha + d(\frac{1}{p} - 1)} \sigma(1/\delta) + \delta^{\alpha + \gamma + d(\frac{1}{p} - 1)} \lesssim \mathcal{R}_\psi(f_0, \delta)_q (\text{by (7.24)})$$

$$\lesssim \delta^{\alpha + d(\frac{1}{p} - 1)} \sigma(1/\delta) + \delta^{\alpha + \gamma + d(\frac{1}{p} - 1)} \lesssim \mathcal{R}_\psi(f_0, \delta)_q \lesssim \mathcal{R}_\psi(f_0, \delta)_q (\text{by the definition of } \sigma)$$
which proves (7.2).

Part (B3): \(1 < p < \infty\) and \(q = \infty\). First, let \(\gamma > d/p\). We take a non-trivial function \(f_0 \in C^\infty(T^d)\). Then, by Lemma 6.2, for any \(1 < r < \infty\), we have
\[
\mathcal{R}_\psi(f_0, \delta)_r \gtrsim \delta^\alpha \|(-\Delta)^{-\alpha/2} f_0\|_r \gtrsim \delta^\alpha.
\]
By (7.20), \(\mathcal{R}_\varphi(f_0, \delta)_p \approx \delta^{\alpha + \gamma}\) and (7.2) follows since in this case \(\sigma(t) = 1\) (see (7.21)).

Now, we assume that \(0 \leq \gamma < d/p\) and \(\sigma(t) = t^{d/p - \gamma}\). We consider the function
\[
f_0(x) = \sum_{k \in \mathbb{Z}^d} \frac{1}{|k|} \hat{\psi}(k) e^{i(k,x)}.
\]
Then, by Lemma 6.3, we have
\[
\mathcal{R}_\psi(f_0, 2^{-n})_\infty \gtrsim 2^{-\alpha n} \|D(\psi) (f_0 \ast V_2^n)\|_\infty = 2^{-\alpha n} \|(-\Delta)^{-\gamma/2} V_2^n\|_\infty \gtrsim 2^{n(d - \alpha - \gamma)}.
\]
(7.26)

On the other hand, as above, by (7.22) and (7.23), we obtain that
\[
\|f_0 - f_0 \ast V_2^n\|_p \lesssim 2^{-(\alpha + \gamma + d(1/p -1))n}
\]
and
\[
\|D(\varphi)(f_0 \ast V_2^n)\|_p \lesssim 2^{d n(1 - 1/p)}.
\]
Hence,
\[
\mathcal{R}_\varphi(f_0, 2^{-n})_p \lesssim 2^{-(\alpha + \gamma + d(1/p -1))n}.
\]
(7.27)

Thus, (7.26) and (7.27) give (7.2).

Finally, let \(\gamma = d/p\) and \(\sigma(t) = \ln^{1/p'}(t + 1)\). In this case the proof is more technical. Consider
\[
f_0(x) = \sum_{k \in \mathbb{Z}^d} \frac{1}{|k|} \hat{\psi}(k) e^{i(k,x)}.
\]
First, by Lemma 6.3,
\[
\mathcal{R}_\psi(f_0, 1/n)_\infty \gtrsim n^{-\alpha} \|D(\psi)(f_0 \ast V_n)\|_\infty \gtrsim n^{-\alpha} \sum_{|k| \leq n} \frac{1}{|k|} \gtrsim n^{-\alpha} \ln n.
\]
(7.28)

Note that
\[
\mathcal{R}_\varphi(f_0, \delta)_p \approx \mathcal{R}_\varphi(f_0^*, \delta)_p,
\]
where
\[
f_0^*(x) = \sum_{k \in \mathbb{Z}^d} \frac{1}{|k|} e^{i(k,x)}.
\]
Denoting by \(D_n\) the cubic Dirichlet kernel, i.e.,
\[
D_n(x) = \sum_{|k| \leq n} e^{i(k,x)},
\]
we have
\[ R_\varphi(f_0, 1/n)_p \lesssim R_\varphi(f_0^*, 1/n)_p \]
\[ \lesssim \|f_0^* - f_0^* * D_n\|_p + n^{-\alpha-\gamma}\|D(\varphi)(f_0^* * D_n)\|_p \]
(7.29)
\[ \lesssim \left\| \sum_{|k|\geq n} e^{i(k,x)} \right\|_p + n^{-\alpha-\gamma}\left\| \sum'_{|k|\leq n} e^{i(k,x)} \right\|_p. \]

By Lemma 5.7, we get
\[ n^{-\alpha-\gamma}\left\| \sum'_{|k|\leq n} e^{i(k,x)} \right\|_p \lesssim n^{-\alpha-\gamma}\ln^\frac{d}{p}(n + 1), \quad \gamma = \frac{d}{p}. \]

Let us prove that
(7.30)
\[ \left\| \sum'_{|k|\geq n} e^{i(k,x)} \right\|_p \lesssim n^{-\alpha-\gamma}. \]

For simplicity, we consider only the case \( d = 2 \). Taking into account the Hardy–Littlewood theorem for multiple series (Lemma 2.6), we get
\[ \left\| \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \cos k_1 x_1 \cos k_2 x_2 \right\|_p \]
\[ \lesssim \left( \sum_{k_1=1}^{\infty} k_1^{p-2} \sum_{k_2=1}^{\infty} k_2^{p-2} \left( \sum_{\nu_1=1}^{\infty} \sum_{\nu_2=1}^{\infty} |\Delta^{(2)} a_{\nu_1, \nu_2}| \right) \right)^{1/p} =: D, \]

where
\[ a_{\nu_1, \nu_2} = \left\{ \begin{array}{ll} |\nu|^{-(\alpha+d)}, & |\nu| \geq n; \\ 0, & |\nu| < n. \end{array} \right. \]

We divide this series into four parts \( D \lesssim D_1 + D_2 + D_3 + D_4 \) and estimate each of them separately. We have
\[ D_1 := \left( \sum_{k_1=1}^{n-2} \sum_{k_2=1}^{n-2} k_1^{p-2} k_2^{p-2} \left( \sum_{\nu_1=n-1}^{\infty} \sum_{\nu_2=n-1}^{\infty} |\Delta^{(2)} a_{\nu_1, \nu_2}| \right) \right)^{1/p} \]
\[ \lesssim n^{2(p-1)/p} \sum_{\nu_1=n-1}^{\infty} \sum_{\nu_2=n-1}^{\infty} |\Delta^{(2)} a_{\nu_1, \nu_2}| \]
\[ \lesssim n^{2(p-1)/p} a_{n-1, n-1} \lesssim n^{2(1-\frac{1}{p})} n^{\alpha+d}, \]

since \( p > 1 \) and \( \Delta^{(2)} a_{\nu_1, \nu_2} \geq 0 \) (see (5.17)).
Further,
\[ D_2 := \left( \sum_{k_1=1}^{n-2} \sum_{k_2=n-1}^{\infty} k_1^{p-2} k_2^{p-2} \left( \sum_{\nu_1=n-1}^{\infty} \sum_{\nu_2=k_2}^{\infty} \left| \Delta^{(2)} a_{\nu_1,\nu_2} \right| \right)^p \right)^{1/p} \]
\[ \lesssim \left( \sum_{k_1=1}^{n-2} \sum_{k_2=n-1}^{\infty} k_1^{p-2} k_2^{p-2} a_{n-1,k_2}^p \right)^{1/p} \]
\[ \lesssim n^{1-\frac{1}{p}} \left( \sum_{k_2=n-1}^{\infty} \frac{k_2^{p-2}}{(n^2 + k_2^{(\alpha+d)p})}\right)^{1/p} \]
\[ \lesssim n^{1-\frac{1}{p}} \left( \sum_{k_2=n-1}^{\infty} \frac{1}{k_2^{(\alpha+d)p-2}} \right)^{1/p}. \]

In light of \( \alpha + \gamma = \alpha + d/p > d \), we have \( (\alpha + d)p - p + 1 > 0 \) and, therefore,
\[ D_2 \lesssim n^{2(1-\frac{1}{p})/(n\alpha+d)}. \]

Similarly, we derive that
\[ D_3 := \left( \sum_{k_1=n-1}^{\infty} \sum_{k_2=1}^{n-2} k_1^{p-2} k_2^{p-2} \left( \sum_{\nu_1=k_1}^{\infty} \sum_{\nu_2=k_2}^{\infty} \left| \Delta^{(2)} a_{\nu_1,\nu_2} \right| \right)^p \right)^{1/p} \]
\[ \lesssim \left( \sum_{k_1=n-1}^{\infty} \sum_{k_2=1}^{n-2} k_1^{p-2} k_2^{p-2} a_{k_1,k_2}^p \right)^{1/p} \]
\[ \lesssim n^{1-\frac{1}{p}} \left( \sum_{k_2=1}^{n-2} \frac{1}{k_2^{(\alpha+d)p-1}} \right)^{1/p}. \]

Finally,
\[ D_4 := \left( \sum_{k_1=n-1}^{\infty} \sum_{k_2=n-1}^{\infty} k_1^{p-2} k_2^{p-2} \left( \sum_{\nu_1=k_1}^{\infty} \sum_{\nu_2=k_2}^{\infty} \left| \Delta^{(2)} a_{\nu_1,\nu_2} \right| \right)^p \right)^{1/p} \]
\[ \lesssim \left( \sum_{k_1=n-1}^{\infty} \sum_{k_2=n-1}^{\infty} \frac{k_2^{p-2}}{(k_1^2 + k_2^{(\alpha+d)p})}\right)^{1/p} \]
\[ \lesssim \left( \sum_{k_1=n-1}^{\infty} \sum_{k_2=n-1}^{k_1} \frac{k_2^{p-2}}{(k_1^2 + k_2^{(\alpha+d)p})}\right)^{1/p} + \left( \sum_{k_1=n-1}^{\infty} \sum_{k_2=k_1}^{\infty} \frac{k_2^{p-2}}{(k_1^2 + k_2^{(\alpha+d)p})}\right)^{1/p} \]
\[ =: D_4^* + D_4^{**}. \]

We estimate each term separately. First,
\[ D_4^* \lesssim \left( \sum_{k_1=n-1}^{\infty} \sum_{k_2=n-1}^{k_1} \frac{k_1^{p-2}}{k_1^{(\alpha+d)p}} \right)^{1/p} \]
\[ \lesssim \left( \sum_{k_1=n-1}^{\infty} \frac{k_1^{2p-3}}{k_1^{(\alpha+d)p}} \right)^{1/p}. \]

Using again the assumption \( \alpha + d/p > d \), it is easy to see that
\[ (\alpha + d)p - 2p + 2 > 0, \]
which implies that
\[ D_4^* \lesssim \frac{n^{2(1 - \frac{1}{p})}}{n^{\alpha + d}}. \]

Further,
\[
D_{4^*}^* \lesssim \left( \sum_{k_1 = n - 1}^{\infty} \frac{k_1^{p-2}}{k_1} \sum_{k_2 = k_1}^{\infty} \frac{k_2^{p-2}}{k_2^{(\alpha + d)p}} \right)^{1/p}
\]
\[
\lesssim \left( \sum_{k_1 = n - 1}^{\infty} \frac{k_1^{2p-3}}{k_1^{(\alpha + d)p}} \right)^{1/p}
\]
\[
\lesssim \frac{n^{2(1 - \frac{1}{p})}}{n^{\alpha + d}}.
\]

Combining these estimates, we arrive at inequality (7.30). Thus, by (7.29) and (7.30), we obtain that
\[
(7.31) \quad R_\psi(f_0, 1/n)_p \lesssim n^{-\gamma} \ln \frac{1}{p} (n + 1) + \frac{n^{2(1 - \frac{1}{p})}}{n^{\alpha + d}} \lesssim n^{-\alpha - \frac{d}{p}} \ln \frac{1}{p} (n + 1).
\]

Hence,
\[
\frac{R_\varphi(f_0, \delta)_p}{\delta^\gamma} \sigma(1/\delta) + \int_0^\delta \frac{R_\varphi(f_0, t)_p}{t^\frac{d}{(\alpha + d)p}} dt \lesssim \frac{R_\varphi(f_0, \delta)_p}{\delta^d/\alpha + d} \ln^{1/\alpha'} (1/\delta) + \int_0^\delta \frac{R_\varphi(f_0, t)_p}{t^\frac{d}{(\alpha + d)p}} dt
\]
\[
\lesssim \delta^\alpha \ln(1/\delta) + \int_0^\delta t^\alpha \ln \frac{1}{t} \frac{1}{t^\frac{d}{p}} \ln \frac{1}{p} (n + 1) + \frac{n^{2(1 - \frac{1}{p})}}{n^{\alpha + d}} \lesssim \delta^\alpha \ln(1/\delta) \lesssim R_\psi(f_0, \delta)_\infty
\]

and (7.2) follows.
8. Sharp Ulyanov inequalities for moduli of smoothness

8.1. One-dimensional results. For $d = 1$, by using Theorem 7.1 with $\psi(\xi) = (i\xi)^\alpha$ and $\varphi(\xi) = (i\xi)^{\alpha+\gamma}$, which correspond to the fractional Weyl derivatives, we obtain the following sharp inequality.

**Theorem 8.1.** Let $0 < p < q \leq \infty$, $\alpha \in \mathbb{N} \cup ((1/q - 1)_+, \infty)$ and $\gamma, m \geq 0$ be such that $\alpha + \gamma, \alpha + m \in \mathbb{N} \cup ((1/p - 1)_+, \infty)$.

(A) Let $f \in L_p(\mathbb{T})$. Then, for any $\delta \in (0, 1)$, we have

$$
\omega_\alpha(f, \delta)_q \leq \frac{\omega_{\alpha+\gamma}(f, \delta)_p}{\delta^\gamma} \sigma \left( \frac{1}{\delta} \right) + \left( \int_0^\delta \frac{(\omega_{\alpha+m}(f, t)_p)^q}{t^{p-\frac{1}{q}}} \, dt \right)^{\frac{1}{q_1}},
$$

where

(A1) if $0 < p \leq 1$ and $p < q \leq \infty$, then

$$
\sigma(t) := \begin{cases}
  \frac{t^{\frac{1}{p}-1}}{p}, & \gamma > (1 - \frac{1}{q})_+; \\
  \frac{t^{\frac{1}{p}-1}}{p} \ln(t + 1), & 0 < \gamma = (1 - \frac{1}{q})_+; \\
  \frac{1}{t^{\frac{1}{p}-\frac{1}{q}}} - \gamma, & 0 < \gamma < (1 - \frac{1}{q})_+; \\
  \frac{1}{t^{\frac{1}{p}}}, & \gamma = 0,
\end{cases}
$$

(A2) if $1 < p \leq q \leq \infty$, then

$$
\sigma(t) := \begin{cases}
  1, & \gamma \geq \frac{1}{p} - \frac{1}{q}, \quad q < \infty; \\
  1, & \gamma > \frac{1}{p}, \quad q = \infty; \\
  \ln(t + 1), & \gamma = \frac{1}{p}, \quad q = \infty; \\
  \frac{t^{\frac{1}{p}}}{(t + 1)^{\frac{1}{p} - \frac{1}{q}} - \gamma}, & 0 \leq \gamma < \frac{1}{p} - \frac{1}{q}.
\end{cases}
$$

(B) Inequality (8.1) is sharp in the following sense. Let $\alpha + \gamma > (1 - 1/q)_+$ and $m - \gamma \in \mathbb{Z}_+ \cup ((1/p - 1)_+, \infty)$. There exists a function $f_0 \in L_q(\mathbb{T})$, $f_0 \neq \text{const}$, such that the following equivalence holds

$$
\omega_\alpha(f_0, \delta)_q \asymp \frac{\omega_{\alpha+\gamma}(f_0, \delta)_p}{\delta^\gamma} \sigma \left( \frac{1}{\delta} \right) + \left( \int_0^\delta \frac{(\omega_{\alpha+m}(f_0, t)_p)^q}{t^{p-\frac{1}{q}}} \, dt \right)^{\frac{1}{q_1}}
$$

as $\delta \to 0$.

**Corollary 8.1.** Under the conditions of Theorem 8.1, inequality (8.1) with $m = \gamma$ implies the following inequality

$$
\omega_\alpha(f, \delta)_q \lesssim \left( \int_0^\delta \frac{(\omega_{\alpha+\gamma}(f, t)_p)^q}{t^{p-\frac{1}{q}}} \, dt \right)^{\frac{1}{q_1}}.
$$

Moreover, if $\alpha + \gamma > (1 - 1/q)_+$, then there exists a nontrivial function $f_0 \in L_q(\mathbb{T})$ such that

$$
\omega_\alpha(f_0, \delta)_q \asymp \left( \int_0^\delta \frac{(\omega_{\alpha+\gamma}(f_0, t)_p)^q}{t^{p-\frac{1}{q}}} \, dt \right)^{\frac{1}{q_1}}.
$$
Proof of Theorem 8.1. Part (A) follows from the realization result (1.13) and Theorem 7.1 with \( \psi(\xi) = (i\xi)^\alpha \) and \( \varphi(\xi) = (i\xi)^{\alpha + \gamma} \).

The appearance of \( m \) in the integral in the right-hand side of (8.1) follows from the fact that inequality (6.5) can be written as

\[
\|f - T_{2^n}\|_q \lesssim \left( \sum_{\nu = n}^{\infty} 2^{\nu q(n - \frac{1}{q})}\|f - T_{2^{\nu}}\|_p^{q_1} \right)^{\frac{1}{q_1}}
\]

(8.3)

\[
\lesssim \left( \sum_{\nu = n}^{\infty} \left( 2^{\nu q(n - \frac{1}{q})} \omega_{\alpha + m}(f_0, 2^{-\nu})_p \right)^{q_1} \right)^{\frac{1}{q_1}}
\]

\[
\lesssim \left( \int_0^{2^{-n}} \left( \frac{\omega_{\alpha + m}(f, t)_p}{t^{\frac{1}{p} - \frac{1}{q}}} \right)^{q_1} \frac{dt}{t} \right)^{\frac{1}{q_1}},
\]

and, therefore, (8.1) holds.

Part (B) follows from Theorem 7.1 (B) noting that in our case (7.14) is given by

\[
\omega_{\alpha + \gamma}(f_0, \delta)_q \gtrsim \omega_{\alpha + \gamma}(f_0, \delta)_p \sigma(1/\delta) + \left( \int_0^{\delta} \left( \frac{\omega_{\alpha + \gamma}(f_0, t)_p}{t^{\frac{1}{p} - \frac{1}{q}}} \right)^{q_1} \frac{dt}{t} \right)^{\frac{1}{q_1}}.
\]

Using the inequality \( \omega_{\alpha + \gamma}(f_0, t)_p \gtrsim \omega_{\alpha + m}(f_0, \delta)_p \) for \( m \geq \gamma \), we arrive at the statement of part (B). \qed

8.2. Comparison between the sharp and classical Ulyanov inequalities.

Let us compare the obtained inequality (8.1) and the classical Ulyanov inequality given by

\[
\omega_{\alpha}(f, \delta)_q \lesssim \left( \int_0^\delta \left( \frac{\omega_{\alpha}(f, t)_p}{t^{\frac{1}{p} - \frac{1}{q}}} \right)^{q_1} \frac{dt}{t} \right)^{\frac{1}{q_1}}, \quad 0 < p < q \leq \infty.
\]

(8.4)

First, if \( 1 < p < q < \infty \), the sharp version of this inequality, i.e.,

\[
\omega_{\alpha}(f, \delta)_q \lesssim \left( \int_0^\delta \left( \frac{\omega_{\alpha}(f, t)_p}{t^{\frac{1}{p} - \frac{1}{q}}} \right)^{q_1} \frac{dt}{t} \right)^{\frac{1}{q_1}}, \quad 0 < p < q \leq \infty.
\]

(8.5)

which coincides with (8.1), clearly gives a better estimate than (8.4). Moreover, (8.5) is sharp over the class

\[
\text{Lip} \left( \omega(\cdot), \alpha + \theta, p \right) = \left\{ f \in L_p(\mathbb{T}) : \omega_{\alpha + \theta}(f, \delta)_p = O(\omega(\delta)) \right\}
\]

(see [87]). In more detail, for any function \( \omega \in \Omega_{\alpha + \theta} \), there exists a function \( f_0(x) = f_0(x, p, \omega) \in \text{Lip} \left( \omega(\cdot), \alpha + \theta, p \right) \) such that for any \( q \in (p, \infty) \) and for any \( \delta > 0 \)

\[
\omega_{\alpha}(f_0, \delta)_q \geq C \left( \int_0^\delta \left( t^{-\theta} \omega(t) \right)^{q_1} \frac{dt}{t} \right)^{\frac{1}{q_1}},
\]

where a constant \( C \) is independent of \( \delta \) and \( \omega \).
Let us give several examples showing essential differences between (8.1) and (8.4).

**Example 8.1.** Let $1 < p < q < \infty$ and $\alpha > \theta$. Define

$$f_0(x) \sim \sum_{m=1}^{\infty} a_m \cos mx, \quad a_m = \frac{1}{m^{\alpha+1-\frac{1}{q}}}, \quad \varepsilon > 0.$$  

The Hardy–Littlewood theorem (see (2.3)) implies that $f_0 \in L_q(\mathbb{T})$. Moreover, realization (1.13) and Theorem 6.1 from [34] give

$$\omega_{\xi}(f_0,1/n)^{\nu} \preceq n^{-\xi} \left( \sum_{k=1}^{n} a_k^{\nu} k^{\xi+\nu-2} \right)^{1/\nu} + \left( \sum_{k=n+1}^{\infty} a_k^{\nu} k^{\nu-2} \right)^{1/\nu}, \quad \xi > 0, \quad p \leq \nu \leq q.$$  

Then it is easy to see that

$$\omega_{\alpha}(f_0, \delta)^{\nu} \preceq \delta^{\alpha+\gamma},$$

$$(\int_{0}^{\delta} \left( \frac{\omega_{\alpha+\theta}(f_0, t)^{\nu}}{t^{\frac{1}{\nu} - \frac{1}{q}}} \right)^{\frac{q}{t}} dt)^{\frac{1}{\nu}} \preceq \begin{cases} \delta^{\alpha} \ln^{\frac{1}{\nu}} \frac{1}{\delta}, & \varepsilon = 0; \\ \delta^{\alpha}, & \varepsilon > 0, \end{cases}$$

and

$$(\int_{0}^{\delta} \left( \frac{\omega_{\alpha}(f_0, t)^{\nu}}{t^{\frac{1}{\nu} - \frac{1}{q}}} \right)^{\frac{q}{t}} dt)^{\frac{1}{\nu}} \preceq \delta^{\alpha-\theta},$$

that is, (8.5) is essentially sharper than (8.4).

Second, if $0 < p < 1$, then an $L_p$-function may have certain pathological behavior in the sense of its smoothness properties. This phenomenon was observed earlier (see, e.g., [22], [55], and [72]). Let us now consider functions, which are smooth in $L_p$, $0 < p < 1$ (in the sense of behaviour of their moduli of smoothness), and show that (8.1), unlike (8.4), provides the best possible estimate for such functions.

**Example 8.2.** Let $0 < p < 1$, $p < q \leq \infty$, $\alpha \in \mathbb{N} \cup ((1/q - 1)_{\cup}^+, \infty)$, and let $\gamma > 0$ be such that $\alpha + \gamma \in \mathbb{N} \cup ((1/p - 1)_{\cup}^+, \infty)$ and $\alpha + \gamma > (1 - 1/q)_{\cup}^+$. Let, for example, $f = f_0$, where $f_0$ is given by (7.4) with $\varphi(\xi) = (i\xi)^{\alpha+\gamma}$. Then, by (1.13), (7.11), and Lemma 6.5, we have

$$\omega_{\alpha+\gamma}(f, \delta)^{\nu}_p \preceq \delta^{\alpha+\gamma+\frac{1}{p}-1}.$$  

At the same time, (1.13) and (7.12) with $\psi(x) = (ix)^{\alpha}$ imply

$$\omega_{\alpha}(f, \delta)^{\nu}_q \succeq \delta^{\alpha} \| V^{(\nu)}_{1/\delta} \|_q \succeq \delta^{\alpha} \sigma \left( \frac{1}{\delta} \right),$$

where $V_{1/\delta}$ is given by (5.2) and $\sigma$ is defined in Theorem 8.1. By using the above formulas for the moduli of smoothness, it is easy to calculate that

$$\omega_{\alpha}(f, \delta)^{\nu}_q \succeq \frac{\omega_{\alpha+\gamma}(f, \delta)^{\nu}_p}{\delta^\gamma} \sigma \left( \frac{1}{\delta} \right) + \left( \int_{0}^{\delta} \left( \frac{\omega_{\alpha+\gamma}(f, t)^{\nu}_p}{t^{\frac{1}{\nu} - \frac{1}{q}}} \right)^{\frac{q}{t}} dt \right)^{\frac{1}{\nu}} \succeq \delta^{\alpha} \sigma \left( \frac{1}{\delta} \right),$$
i.e., (8.1) is sharp for the function $f_0$. On the other hand, inequality (8.4) for $f_0$ implies only the following (non-sharp) estimate:

$$
\delta^\sigma \left( \frac{1}{\delta} \right) \asymp \omega_\alpha(f, \delta)_q \leq \left( \int_0^\delta \left( \frac{\omega_\alpha(f, t)_p}{t^{p-\frac{q}{q}}} \right)^q \frac{dt}{t^{\frac{1}{q}}} \right)^{\frac{1}{q}} \\
\asymp \delta^{\alpha-(\frac{1}{p}-\frac{1}{q})}.
$$

We conclude this subsection by the following example dealing with the case $p = 1$.

**Example 8.3.** Let $1 = p < q < \infty$. We define $f_0$ such that

$$
f_0(x) = \sum_{\nu \in \mathbb{Z}} a_{|\nu|} \frac{1}{i\nu} e^{i\nu x},
$$

where $\{a_\nu\}$ is a convex sequence of positive numbers, which can tend to zero very slowly. Then, by [114, Ch. V, (1.5), p. 183], one gets $f_0^{(\alpha+1-1/q)} \in L_1(\mathbb{T})$. By the realization result, we then have that $\omega_{\alpha+1-1/q}(f_0, t)_1 \lesssim t^{\alpha+1-1/q}$. Hence, using the Marchaud inequality, we obtain that $\omega_\alpha(f_0, t)_1 \lesssim t^\alpha$. Then the classical inequality (8.4) yields

$$
\omega_\alpha(f_0, t)_q \lesssim t^{-(1-1/q)}.
$$

On the other hand, sharp Ulyanov inequality (8.5) gives a much better estimate:

$$
\omega_\alpha(f_0, t)_q \lesssim t^\alpha \ln \frac{1}{t}.
$$

Moreover, this estimate is sharp in the sense that $\omega_\alpha(f_0, t)_q \gtrsim t^\alpha \ln \frac{1}{t} a[1/t]$.

**8.3. Multidimensional results.** Since for $1 < p < \infty$ one has

$$
(8.6) \quad \omega_\alpha(f, \delta)_p \asymp R_\psi(f, \delta)_p, \quad \psi(\xi) = |\xi|^\alpha
$$

(see Lemma 6.2), the sharp Ulyanov inequalities for moduli of smoothness in the case $1 < p < q < \infty$ immediately follows from Theorem 7.1. Note that inequality (8.6) is not true anymore if $p = 1$ or $p = \infty$. Therefore, since Theorems 6.1 and 7.1 cannot be applied, we will provide a direct proof of the sharp Ulyanov inequalities for moduli of smoothness. Our main result of this section reads as follows.

**Theorem 8.2.** Let $f \in L_p(\mathbb{T}^d)$, $d \geq 2$, $0 < p < q < \infty$, $\alpha \in \mathbb{N} \cup ((1-1/q)_+, \infty)$, and $\gamma, m \geq 0$ be such that $\alpha+\gamma, \alpha+m \in \mathbb{N} \cup ((1/p-1)_+, \infty)$. Then, for any $\delta \in (0, 1)$, we have

$$
(8.7) \quad \omega_\alpha(f, \delta)_q \lesssim \frac{\omega_{\alpha+\gamma}(f, \delta)_p}{\delta^\gamma} \sigma \left( \frac{1}{\delta} \right) + \left( \int_0^\delta \left( \frac{\omega_{\alpha+m}(f, t)_p}{t^{1-\frac{1}{q}}t^\frac{1}{q}} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}},
$$

where
(1) if $0 < p \leq 1$ and $p < q \leq \infty$, then

$$
\sigma(t) := \begin{cases} 
  t^{d\left(\frac{1}{p} - 1\right)}; & \gamma > d\left(1 - \frac{1}{q}\right)_+; \\
  t^{d\left(\frac{1}{p} - \frac{1}{q}\right)}; & \gamma = d\left(1 - \frac{1}{q}\right)_+ \geq 1 \text{ and } \alpha + \gamma \in \mathbb{N}; \\
  t^{d\left(\frac{1}{p} - \frac{1}{q}\right) \ln t}; & \gamma = d\left(1 - \frac{1}{q}\right)_+ \geq 1 \text{ and } \alpha + \gamma \notin \mathbb{N}; \\
  t^{d\left(\frac{1}{p} - \frac{1}{q}\right) \ln t}; & 0 < \gamma = d\left(1 - \frac{1}{q}\right)_+ < 1; \\
  t^{d\left(\frac{1}{p} - \frac{1}{q}\right) - \gamma}; & 0 < \gamma < d\left(1 - \frac{1}{q}\right)_+; \\
  t^{d\left(\frac{1}{p} - \frac{1}{q}\right) - \gamma}; & \gamma = 0; \\
\end{cases}
$$

(2) if $1 < p \leq q \leq \infty$, then

$$
\sigma(t) := \begin{cases} 
  1; & \gamma \geq d\left(\frac{1}{p} - \frac{1}{q}\right), \quad q < \infty; \\
  1; & \gamma > \frac{d}{p}, \quad q = \infty; \\
  \ln t; & \gamma = \frac{d}{p}, \quad q = \infty; \\
  t^{d\left(\frac{1}{p} - \frac{1}{q}\right) - \gamma}; & 0 \leq \gamma < d\left(\frac{1}{p} - \frac{1}{q}\right). \\
\end{cases}
$$

**Remark 8.1.** Equivalence (8.6) and Theorem 7.1 (B) give the sharpness of inequality (8.7) in the case $1 < p \leq \infty$.

**Corollary 8.2.** Under all conditions of Theorem 8.2, we have

$$
\omega_\alpha(f, \delta)_q \lesssim \left( \int_0^\delta \left( \frac{\omega_{\alpha + \gamma} (f, t)}{t^{\gamma}} \frac{\sigma \left( \frac{1}{t} \right)}{t} \right) \frac{dt}{t} \right)^{\frac{1}{\alpha}}.
$$

**Proof of Theorem 8.2.** By Theorem 6.2 and Corollary 3.1, we have that

$$
\eta(n) \asymp n^\gamma \sup_{T_n \in \mathcal{T}_n} \sup_{\xi = 1, \xi \in \mathbb{R}^d} \left\| \left( \frac{\partial}{\partial \xi} \right)^\alpha T_n \right\|_q.
$$

Hence, it is enough to prove that

$$
\sup_{T_n \in \mathcal{T}_n} \sup_{\xi = 1, \xi \in \mathbb{R}^d} \left\| \left( \frac{\partial}{\partial \xi} \right)^\alpha T_n \right\|_q \lesssim \sigma(n).
$$

(8.8)

Note that for $\gamma = 0$ and $0 < p < q \leq \infty$, this follows immediately from the classical Nikol’skii inequality (1.18).

The rest of the proof for $\gamma > 0$ is divided into three cases.

**Case 1.** Let $1 < p < q < \infty$. In this case, Theorem 5.1 yields (8.8) since, by Corollary 3.1, we have that

$$
\sup_{\xi = 1, \xi \in \mathbb{R}^d} \left\| \left( \frac{\partial}{\partial \xi} \right)^\alpha T_n \right\|_q \asymp \|(-\Delta)^{\alpha/2} T_n\|_q.
$$
Remark 5.3 with \( \varphi \) where
\[
| (8.9) \]
type inequality (see (10.16) below):
\[
\text{Hence, (8.9) and (8.10) imply that (8.7) holds in the following four cases:}
\[
\text{which is the desired estimate.}
\]

**Case 2.** Let \( 1 < p < q = \infty \). To show (8.8), we note that by Theorem 5.1 and Remark 5.3 with \( \varphi(y) \equiv 1 \) and \( \psi(y) = |y|^{-\gamma} \), we have, for any \( T_n \in \mathcal{T}_n \) and \( \xi \in \mathbb{R}^d, \) \( |\xi| = 1, \)
\[
\left\| \left( \frac{\partial}{\partial \xi} \right)^\alpha T_n \right\|_\infty \lesssim \sigma(n) \left\| (-\Delta)^{\gamma/2} \left( \frac{\partial}{\partial \xi} \right)^\alpha T_n \right\|_p.
\]
Applying twice Corollary 3.3, we continue estimating as follows
\[
\lesssim \sigma(n) \left\| (-\Delta)^{(\alpha+\gamma)/2} T_n \right\|_p \lesssim \sigma(n) \sup_{\xi \in \mathbb{R}^d, |\xi| = 1} \left\| \left( \frac{\partial}{\partial \xi} \right)^{\alpha+\gamma} T_n \right\|_p,
\]
which is the desired estimate.

**Case 3.** Let \( 0 < p \leq 1, p < q \leq \infty, \) and \( \gamma > 0 \). In this case, we use the Marchaud-type inequality (see (10.16) below):
\[
\omega_\alpha(f, \delta)_q \lesssim \delta^\alpha \left( \int_{\delta}^{1} \left( \frac{\omega_{\alpha+\gamma}(f, t)p}{t^{\alpha+d(\frac{\alpha}{\gamma} - \frac{1}{q})}} \right)^\tau dt \right)^{\frac{1}{\tau}} + \left( \int_{0}^{\delta} \left( \frac{\omega_{\alpha+m}(f, t)p}{t^{\alpha+d(\frac{\alpha}{\gamma} - \frac{1}{q})}} \right)^{\eta} dt \right)^{\frac{1}{\eta}},
\]
where
\[
\tau = \begin{cases} 
\min(q, 2), & q < \infty; \\
1, & q = \infty.
\end{cases}
\]
By (4.9), it is easy to see that (8.10)
\[
\delta^\alpha \left( \int_{\delta}^{1} \left( \frac{\omega_{\alpha+\gamma}(f, t)p}{t^{\alpha+d(\frac{\alpha}{\gamma} - \frac{1}{q})}} \right)^\tau dt \right)^{\frac{1}{\tau}} \lesssim \frac{\omega_{\alpha+\gamma}(f, \delta)p}{\delta^{\gamma+d(\frac{\alpha}{\gamma} - 1)}} \begin{cases} 
1, & \gamma > d \left( 1 - \frac{1}{q} \right); \\
\ln^{1/\tau} \left( \frac{1}{\delta} + 1 \right), & \gamma = d \left( 1 - \frac{1}{q} \right); \\
\left( \frac{1}{\delta} \right)^{d(1-\frac{1}{q})^{-\gamma}}, & 0 < \gamma < d \left( 1 - \frac{1}{q} \right).
\end{cases}
\]
Hence, (8.9) and (8.10) imply that (8.7) holds in the following four cases:
\[
\text{(i) } \gamma > 0 \text{ and } 0 < q \leq 1; \\
\text{(ii) } \gamma = d(1 - 1/q) \text{ and } 1 < q \leq 2 \text{ with } \alpha + \gamma \not\in \mathbb{N}; \\
\text{(iii) } \gamma = d(1 - 1/q) \text{ and } q = \infty \text{ with } \alpha + \gamma \not\in \mathbb{N}; \\
\text{(iv) } 0 < \gamma < d(1 - 1/q).+. \\
\]
Thus, it remains to consider:
\[
\text{(v) } \gamma = d(1 - 1/q) \geq 1, \alpha + \gamma \not\in \mathbb{N}, \text{ and } 2 < q < \infty \text{ (note that for such } q \text{ and } d \geq 2 \text{ we always have } d(1 - 1/q) \geq 1); \\
\text{(vi) } \gamma = d(1 - 1/q) \geq 1, 1 < q \leq \infty, \text{ and } \alpha + \gamma \in \mathbb{N}.
\]
We have that inequality (8.8) follows from Lemma 5.12 in the case (v) and from Lemma 5.14 in the case (vi).
We finish this section with an important corollary.

**Corollary 8.3.** Let $0 < p \leq 1 < q \leq \infty$ and $d \geq 2$. We have

\[
\omega_{\alpha}(f, \delta)_q \lesssim \left( \int_0^\delta \left( \frac{\omega_{\alpha+d(1-1/q)}(f, t)_p}{t^{d(1-1/q)}} \right)^{\frac{1}{q}} \frac{dt}{t} \right)^{\frac{1}{q}}
\]

provided that $\alpha + d(1 - 1/q) \in \mathbb{N}$. In particular, for any $d, \alpha \in \mathbb{N}$, we have

\[
\omega_{\alpha}(f, \delta)_\infty \lesssim \int_0^\delta \frac{\omega_{\alpha+d}(f, t)_1 dt}{t^d}.
\]

Note that (8.11) does not hold in the one-dimensional case. Comparing inequality (8.11) for $p = 1$ with (1.6), we observe new effects in the multivariate case. For $d = 1$, inequality (8.12) is known (see [99]).
9. Sharp Ulyanov inequalities for realizations of \(K\)-functionals related to Riesz derivatives and corresponding moduli of smoothness

Let \(d \geq 1\), \(\psi(\xi) = |\xi|^\alpha\), and \(\varphi(\xi) = |\xi|^\alpha+\gamma\). By analogy with the one-dimensional case, we define

\[
R_{(\alpha)}(f, \delta)_p = \inf_{T \in \mathcal{T}_{[0]}(\delta)} \{ \| f - T \|_p + \delta^\alpha \| (-\Delta)^{\alpha/2} T \|_p \},
\]

where

\[
(-\Delta)^{\alpha/2} T(x) = \sum_{|k|_\infty \leq n} |k|_\omega c_k e^{i(k,x)}, \quad x \in \mathbb{T}^d.
\]

Theorem 7.1 for such \(\psi\) and \(\varphi\) immediately implies the following sharp Ulyanov inequality for \(R_{(\alpha)}(f, \delta)_p\).

**Theorem 9.1.** Let \(f \in L_p(\mathbb{T}^d)\), \(d \geq 1\), \(0 < p < q \leq \infty\), \(\alpha > 0\), and \(\gamma \geq 0\).

(A) For any \(\delta \in (0, 1)\), we have

\[
R_{(\alpha)}(f, \delta)_q \leq \frac{R_{(\alpha+\gamma)}(f, \delta)_p}{\delta^\gamma} \sigma_\Delta \left( \frac{1}{\delta} \right) + \left( \int_0^\delta \left( \frac{R_{(\alpha+\gamma)}(f, t)_p}{t^{(\frac{1}{p} - \frac{1}{q})}} \right)^{\frac{1}{q}} \ dt \right)^{\frac{1}{n}},
\]

where

(1) if \(0 < p \leq 1\) and \(p < q \leq \infty\), then

\[
\sigma_\Delta(t) := \begin{cases}
  t^{d\left(\frac{1}{p} - 1\right)}, & \gamma > d \left(1 - \frac{1}{q}\right)_+ \\
  t^{d\left(\frac{1}{p} - 1\right)} \ln^{\frac{1}{p}}(t + 1), & 0 < \gamma = d \left(1 - \frac{1}{q}\right)_+ \\
  t^{d\left(\frac{1}{p} - \frac{1}{q}\right) - \gamma}, & 0 < \gamma < d \left(1 - \frac{1}{q}\right)_+ \\
  t^{d\left(\frac{1}{p} - \frac{1}{q}\right)}, & \gamma = 0
\end{cases}
\]

(2) if \(1 < p \leq q \leq \infty\), then

\[
\sigma_\Delta(t) := \begin{cases}
  1, & \gamma \geq d\left(\frac{1}{p} - \frac{1}{q}\right), \quad q < \infty \\
  1, & \gamma > \frac{d}{p}, \quad q = \infty \\
  \ln^\frac{1}{p}(t + 1), & \gamma = \frac{d}{p}, \quad q = \infty \\
  t^{d\left(\frac{1}{p} - \frac{1}{q}\right) - \gamma}, & 0 \leq \gamma < d\left(\frac{1}{p} - \frac{1}{q}\right)_+. \quad q \leq \infty
\end{cases}
\]

(B) Inequality (9.1) is sharp in the following sense. Let \(\alpha \in (2\mathbb{N}) \cup (d(1/q - 1)_+, \infty)\) and \(\alpha + \gamma > d(1 - 1/q)_+\). Then there exists a function \(f_0 \in L_q(\mathbb{T}^d)\), \(f_0 \not\equiv \text{const}\), such that

\[
R_{(\alpha)}(f_0, \delta)_q \approx \frac{R_{(\alpha+\gamma)}(f_0, \delta)_p}{\delta^\gamma} \sigma_\Delta \left( \frac{1}{\delta} \right) + \left( \int_0^\delta \left( \frac{R_{(\alpha+\gamma)}(f_0, t)_p}{t^{d\left(\frac{1}{p} - \frac{1}{q}\right)}} \right)^{\frac{1}{q}} \ dt \right)^{\frac{1}{n}}
\]

as \(\delta \to 0\).

**Remark 9.1.** We would like to emphasize that the main difference in the definitions of \(\sigma\) in Theorems 8.1, 8.2, and 9.1 is when \(\gamma\) is the critical parameter, i.e., \(\gamma = d\left(1 - 1/q\right)_+\).
Under certain additional restrictions, similarly to Theorem 7.1', one can prove the following more accurate inequality.

**Remark 9.2.** Let \( f \in L_p(\mathbb{T}^d) \), \( d \geq 1 \), \( 0 < p < q < \infty \), \( \alpha > 0 \), and \( \gamma, m \geq 0 \) be such that \( \alpha + \gamma, \alpha + m \in (2\mathbb{N}) \cup (d(1/p - 1)_+ + \infty) \). Then, for any \( \delta \in (0, 1) \), we have

\[
\mathcal{R}_{\alpha}(f, \delta)_q \lesssim \mathcal{R}_{\alpha + \gamma}(f, \delta)_p \| \sigma_\Delta \left( \frac{1}{\delta} \right) + \left( \int_0^\delta \left( \frac{\mathcal{R}_{\alpha + m}(f, t)_p}{t^{d(1/p - 1) + \frac{1}{q}}} \right)^\frac{1}{q} dt \right)^\frac{1}{q},
\]

where \( \sigma_\Delta(\cdot) \) is defined in Theorem 9.1.

Note that in some cases the realization \( \mathcal{R}_{\alpha} \) can be replaced by special moduli of smoothness (see, for example, [57] for the case \( q \geq 1 \) and [23] for the case \( 0 < q < 1 \); see also [82, 83, 84]).

**Example 9.1.** Let \( \alpha > 0 \), \( r \in \mathbb{N} \), and \( 1 \leq q \leq \infty \). Denote

\[
w_\alpha(f, \delta)_q := \left\| \int_{|u| \geq 1} \frac{\hat{\Delta}^{2r}_u f(\cdot)}{|u|^{d+\alpha}} du \right\|_q,
\]

where

\[
\hat{\Delta}^r_h = \hat{\Delta}_h(\hat{\Delta}^{r-1}_h), \quad \hat{\Delta}^r_h f(x) = f(x + h) - f(x - h), \quad h \in \mathbb{R}^d.
\]

It follows from [57], [81], and Lemma 6.1 that, for \( 1 \leq q \leq \infty \), one has

\[
w_\alpha(f, \delta)_q \asymp \mathcal{R}_{\alpha}(f, \delta)_q = \inf_{T \in [1, \varepsilon]} \left( \| f - T\|_q + \delta^\alpha (\| (\Delta)^{\alpha/2}T\|_q) \right)
\]

\[
\asymp K_{\alpha}(f, \delta)_q = \inf_{(\Delta)^{\alpha/2}g \in L_q(\mathbb{T}^d)} \left( \| f - g\|_q + \delta^\alpha (\| (\Delta)^{\alpha/2}g\|_q) \right).
\]

**Example 9.2.** Now, we consider the following moduli of smoothness, which were studied in [83] for any \( 0 < q \leq \infty \) (see also [23] for the case \( m = 1 \)):

\[
\tilde{w}^{(m)}_2(f, \delta)_q := \sup_{|h| \leq \delta} \left\| \frac{\xi_m}{d} \sum_{j=1}^d \sum_{\nu=-m}^m \frac{(-1)^\nu}{\nu^2} \left( \frac{2m}{m - \nu} \right) \left( f(\cdot + \nu \varepsilon_j) - f(\cdot) \right) \right\|_q,
\]

where \( m, d \in \mathbb{N} \) and

\[
\xi_m = \left( 2 \sum_{\nu=1}^m \frac{(-1)^\nu}{\nu^2} \left( \frac{2m}{m - \nu} \right) \right)^{-1}.
\]

Set

\[
q_{m,d} = \begin{cases} 0, & d = 1; \\
\frac{d}{d+2(m+1)}, & d = 2, m = 2k, k \in \mathbb{N}; \\
\frac{d}{d+2m}, & \text{otherwise}.
\end{cases}
\]

It is known [83] (see also [23, Theorem 3.2]) that

\[
\tilde{w}^{(m)}_2(f, \delta)_q \asymp \mathcal{R}_{(2)}(f, \delta)_q, \quad q_{m,d} < q \leq \infty.
\]

Thus, for all \( 1 \leq q \leq \infty \) and \( m \in \mathbb{N} \), we have

\[
w_2(f, \delta)_q \asymp \tilde{w}^{(m)}_2(f, \delta)_q \asymp \mathcal{R}_{(2)}(f, \delta)_q.
\]

Using the above equivalences and Remark 9.2, we obtain the following result.
Corollary 9.1. Let $f \in L_p(\mathbb{T}^d)$, $d \geq 1$. For any $\delta \in (0,1)$, one has the following three inequalities:

(i) if $1 \leq p < q \leq \infty$ and $\alpha, \gamma > 0$, then

$$w_\alpha(f, \delta)_q \lesssim \frac{w_{\alpha+\gamma}(f, \delta)_p}{\delta^\gamma} \sigma_\Delta \left( \frac{1}{\delta} \right) + \left( \int_0^\delta \left( \frac{w_{\alpha+\gamma}(f, t)_p}{t^{\frac{d}{p} - \frac{1}{q}}} \right) \frac{q_1}{q} \frac{dt}{t} \right)^{\frac{1}{q_1}},$$

(ii) if $p_{m,d} < p < q$, $1 \leq q \leq \infty$, and $0 < \gamma < 2$, then

$$w_{2-\gamma}(f, \delta)_q \lesssim \frac{\tilde{w}_{2}(m)(f, \delta)_p}{\delta^\gamma} \sigma_\Delta \left( \frac{1}{\delta} \right) + \left( \int_0^\delta \left( \frac{\tilde{w}_{2}(m)(f, t)_p}{t^{\frac{d}{p} - \frac{1}{q}}} \right) \frac{q_1}{q} \frac{dt}{t} \right)^{\frac{1}{q_1}},$$

(iii) if $p_{m,d} < p < q \leq \infty$, then

$$\tilde{w}_{2}(m)(f, \delta)_q \lesssim \left( \int_0^\delta \left( \frac{\tilde{w}_{2}(m)(f, t)_p}{t^{\frac{d}{p} - \frac{1}{q}}} \right) \frac{q_1}{q} \frac{dt}{t} \right)^{\frac{1}{q_1}},$$

where $\sigma_\Delta(\cdot)$ is defined in Theorem 9.1.

Example 9.3. A more complete picture can be obtained when $d = 1$. We will deal with the following special modulus of smoothness, which has been recently introduced by Runovski and Schmeisser [82] (see also [84]). For $\alpha > 0$ and $f \in L_p(\mathbb{T})$, we define

$$\omega(\alpha)(f, \delta)_p := \sup_{0 < h \leq \delta} \left\| \sum_{\nu \in \mathbb{Z} \setminus \{0\}} \left( \frac{-\beta h(\alpha)}{\beta_0(\alpha)} \right) f(\cdot + \nu h) - f(\cdot) \right\|_p,$$

where

$$\beta_m(\alpha) = \sum_{j=m}^{\infty} (-1)^{j+1} 2^{-2j} \left( \frac{\alpha/2}{j} \right) \left( \frac{2j}{j-m} \right), \quad m \in \mathbb{Z}_+. $$

It has been proved in [82] that, for $\alpha > 0$ and $1/(\alpha + 1) < p \leq \infty$, one has

(9.2) $$\omega(\alpha)(f, \delta)_p \asymp R(\alpha)(f, \delta)_p, \quad \delta > 0.$$  

Note also that, for all $\alpha \in 2\mathbb{N}$, the modulus $\omega(\alpha)(f, \delta)_p$ coincides with the classical modulus of smoothness $\omega_\alpha(f, \delta)_p$.

From equivalence (9.2) and Theorem 9.1, we have

Corollary 9.2. Let $f \in L_p(\mathbb{T})$, $0 < p < q \leq \infty$, $\gamma \geq 0$, $\alpha \in (2\mathbb{N}) \cup ((1/q -1)_+, \infty)$, and $\alpha + \gamma \in (2\mathbb{N}) \cup ((1/p -1)_+, \infty)$. Then, for any $\delta \in (0,1)$, we have

$$\omega(\alpha)(f, \delta)_q \lesssim \frac{\omega(\alpha+\gamma)(f, \delta)_p}{\delta^\gamma} \sigma_\Delta \left( \frac{1}{\delta} \right) + \left( \int_0^\delta \left( \frac{\omega(\alpha+\gamma)(f, t)_p}{t^{\frac{d}{p} - \frac{1}{q}}} \right) \frac{q_1}{q} \frac{dt}{t} \right)^{\frac{1}{q_1}},$$

where $\sigma(\cdot)$ is defined in Theorem 9.1 in the case $d = 1$. 

10. Sharp Ulyanov inequality via Marchaud inequality

This section is devoted to the study of new types of Ulyanov inequalities with the use of the Marchaud inequalities. Such inequalities are of great importance when investigating the embedding theorems in Section 13.

Throughout this section, we assume that

$$\tau = \tau(q) = \begin{cases} \min(q, 2), & q < \infty; \\ 1, & q = \infty. \end{cases}$$

10.1. Inequalities for realizations of $K$-functionals. First, we obtain an analogue of the Marchaud inequality (cf. Theorem 4.4) for the realizations of the $K$-functionals.

Lemma 10.1. Let $f \in L_q(T^d)$, $0 < q \leq \infty$, $\alpha > 0$, $\gamma \geq 0$, $\psi \in \mathcal{H}_\alpha$, and $\varphi \in \mathcal{H}_{\alpha+\gamma}$. Let also either $\psi$ be a polynomial or $d/(d+\alpha) < q \leq \infty$. Then, for any $\delta \in (0, 1)$, one has

\begin{equation}
\mathcal{R}_\psi(f, \delta)_q \lesssim \delta^\alpha \left( \int_0^1 \left( \frac{\mathcal{R}_\varphi(f, t)_q}{t^\alpha} \right)^{\tau} \frac{dt}{t} \right)^{\frac{1}{\tau}}.
\end{equation}

Proof. We start with the inverse estimate given by

\begin{equation}
\mathcal{R}_\psi(f, \delta)_q \lesssim \delta^\alpha \left( \int_0^1 \left( \frac{E_{1/\delta}(f, t)_q}{t^\alpha} \right)^{\tau} \frac{dt}{t} \right)^{\frac{1}{\tau}}.
\end{equation}

In the case $0 < q \leq 1$ and $q = \infty$, the proof of (10.2) can be found in [79, Theorem 4.26, p. 203]. In the case $1 < q < \infty$, it follows from Theorem 4.4 (see also [104, Theorem 2.1]) and (8.6).

To finish the proof of (10.1), we apply Jackson’s inequality, which follows from Lemma 6.3.

One of the main results in this section is the following sharp Ulyanov inequality via Marchaud inequality.

Theorem 10.1. Let $0 < p < q \leq \infty$, $\alpha > 0$, $\gamma \geq 0$, $\psi \in \mathcal{H}_\alpha$, and $\varphi \in \mathcal{H}_{\alpha+\gamma}$. Let also either $\psi$ be a polynomial or $d/(d+\alpha) < q \leq \infty$.

(A) Let $f \in L_p(T^d)$. Then, for any $\delta \in (0, 1)$, we have

\begin{equation}
\mathcal{R}_\psi(f, \delta)_q \lesssim \delta^\alpha \left( \int_0^1 \left( \frac{\mathcal{R}_\varphi(f, t)_p}{t^{\alpha+d/(d+\alpha)}} \right)^{\tau} \frac{dt}{t} \right)^{\frac{1}{\tau}} + \left( \int_0^\delta \left( \frac{\mathcal{R}_\varphi(f, t)_p}{t^{d/(d+\alpha)}} \right)^{q_1} \frac{dt}{t} \right)^{\frac{1}{q_1}}.
\end{equation}

(B) Inequality (10.3) is sharp in the following sense. Let $\frac{\psi}{\varphi}$ and $\frac{\psi}{\varphi}$ are in $C_\infty(\mathbb{R}^d \setminus \{0\})$, $\alpha + \gamma > d/(1-1/q)_+$, $\gamma > 0$, and

1. $\gamma \neq d/(1-1/q)_+$ if $2 < q < \infty$ and $0 < p \leq 1$;
2. $\gamma \neq 1$ if $d = 1$, $q = \infty$, $0 < p \leq 1$, and $\frac{\psi(\xi)}{\varphi(\xi)} = A|\xi|^{-\gamma}\text{sign}\,\xi$ for any $A \in \mathbb{C} \setminus \{0\}$;
3. $\gamma \neq d/(1/p-1/q)$ if $1 < p < q \leq \infty$. 
Then there exists a function $f_0 \in L_q(\mathbb{T}^d)$, $f_0 \not\equiv \text{const}$, such that

\begin{equation}
R_\varphi(f_0, \delta)_q \approx \delta^\alpha \left( \int_\delta^1 \left( \frac{R_\varphi(f_0, t)_p}{t^{d+\frac{\alpha}{p}-\frac{1}{q}}} \right)^{\frac{\tau}{q}} \frac{dt}{t} \right)^{\frac{1}{q}} + \left( \int_0^\delta \left( \frac{R_\varphi(f_0, t)_p}{t^{d+\frac{\alpha}{p}-\frac{1}{q}}} \right)^{q_1} \frac{dt}{t} \right)^{\frac{1}{q_1}}
\end{equation}

as $\delta \to 0$.

Similarly to Theorem 6.1', one can obtain the following analogue of Theorem 10.1.

**Theorem 10.1'** Under all conditions of Theorem 10.1, suppose that the function $\varphi$ is either a polynomial or $d/(d+\alpha+\gamma) < p \leq \infty$. Let also $m > 0$ and the function $\phi \in \mathcal{H}_{\alpha+m}$ be either a polynomial or $d/(d+\alpha+m) < p \leq \infty$. Then, for any $\delta \in (0, 1)$, we have

\begin{equation}
R_\psi(f, \delta)_q \lesssim \delta^\alpha \left( \int_\delta^1 \left( \frac{R_\varphi(f, t)_p}{t^{d+\frac{\alpha}{p}-\frac{1}{q}}} \right)^{\frac{\tau}{q}} \frac{dt}{t} \right)^{\frac{1}{q}} + \left( \int_0^\delta \left( \frac{R_\varphi(f, t)_p}{t^{d+\frac{\alpha}{p}-\frac{1}{q}}} \right)^{q_1} \frac{dt}{t} \right)^{\frac{1}{q_1}}.
\end{equation}

It is worth mentioning the following important corollary for the case $\gamma = 0$.

**Corollary 10.1.** Let $f \in L_p(\mathbb{T}^d)$, $0 < p < q \leq 1$, $\psi, \varphi \in \mathcal{H}_\alpha$, $\alpha > 0$. Let either $\psi$ be a polynomial or $d/(d+\alpha) < q \leq 1$ and let either $\varphi$ be a polynomial or $d/(d+\alpha) < p < 1$. Let also $m > 0$ and the function $\phi \in \mathcal{H}_{\gamma+m}$ be either a polynomial or $d/(d+\alpha+m) < p < 1$. Then, for any $\delta \in (0, 1)$, we have

\begin{equation}
R_\psi(f, \delta)_q \lesssim \left( \int_0^\delta \left( \frac{R_\varphi(f, t)_p}{t^{d+\frac{\alpha}{p}-\frac{1}{q}}} \right)^{q_1} \frac{dt}{t} \right)^{\frac{1}{q_1}} + \frac{R_\varphi(f, \delta)_p}{\delta^{d+\frac{\alpha}{p}-1}} \begin{cases} 1, & 0 < q < 1; \\ \ln \left( \frac{1}{\delta} + 1 \right), & q = 1. \end{cases}
\end{equation}

**Proof.** The proof of (10.6) follows from (10.5) and Lemma 6.5. \qed

**Remark 10.1.** (i) We note that all three cases (1)-(3) in part (B) of Theorem 10.1, where the optimality of inequality (10.3) cannot be proved, are those cases where the Ulyanov inequality, given by Theorem 7.1, provides a sharper estimate than inequality (10.3). Moreover, one can construct examples showing that equivalence (7.2) holds in these cases. Indeed, in cases (1) and (2), taking $f = f_0$ such that $R_\varphi(f_0, t)_p \sim t^{d(1/p - 1) + \alpha + \gamma}$, we see that the equivalence in (7.2) is true, while in (10.4) the equivalence for $f_0$ is not valid. In the case (3), we obtain the same conclusion by taking any $f_0 \in C^\infty(\mathbb{T}^d)$.

(ii) At the same time, taking $f_0 \in C^\infty(\mathbb{R}^d)$ and letting $0 < p < 1$ and $\gamma = d(1 - 1/q)_+$, we have that the sharp Ulyanov inequality given by (10.5) becomes the equivalence

\begin{equation}
R_\psi(f_0, \delta)_q \approx \delta^\alpha \left( \int_\delta^1 \left( \frac{R_\varphi(f_0, t)_p}{t^{d+\frac{\alpha}{p}-\frac{1}{q}}} \right)^{\frac{\tau}{q}} \frac{dt}{t} \right)^{\frac{1}{q}} + \left( \int_0^\delta \left( \frac{R_\varphi(f_0, t)_p}{t^{d+\frac{\alpha}{p}-\frac{1}{q}}} \right)^{q_1} \frac{dt}{t} \right)^{\frac{1}{q_1}}.
\end{equation}

On the other hand, the sharp Ulyanov inequality given by (7.3) implies only

\begin{equation}
R_\psi(f_0, \delta)_q \approx \delta^\alpha \ln^\alpha \frac{1}{\delta} \approx \frac{R_\varphi(f_0, \delta)_p}{\delta^\gamma} \left( \frac{1}{\delta} \right) + \left( \int_0^\delta \left( \frac{R_\varphi(f_0, t)_p}{t^{d+\frac{\alpha}{p}-\frac{1}{q}}} \right)^{q_1} \frac{dt}{t} \right)^{\frac{1}{q_1}}.
\end{equation}
Proof. (A) Using the Ulyanov type inequality given by (6.10) and then the Marchaud inequality given by (10.1), we get

$$R_{\psi}(f, \delta)_q \lesssim \delta^\alpha \left( \int_0^1 u^{-\alpha \tau} \left( \int_0^u \left( \frac{R_{\varphi}(f, t)_p}{t^{\frac{1}{p} - \frac{1}{q}}} \right)^{q_1} \frac{dt}{t} \right)^{\frac{\tau}{q_1}} \frac{du}{u} \right)^{\frac{1}{\tau}}$$

(10.7)

$$\lesssim \delta^\alpha \left( \int_0^1 u^{-\alpha \tau} \left( \int_0^\delta \left( \frac{R_{\varphi}(f, t)_p}{t^{\frac{1}{p} - \frac{1}{q}}} \right)^{q_1} \frac{dt}{t} \right)^{\frac{\tau}{q_1}} \frac{du}{u} \right)^{\frac{1}{\tau}}$$

+ $$\delta^\alpha \left( \int_\delta^1 u^{-\alpha \tau} \left( \int_\delta^u \left( \frac{R_{\varphi}(f, t)_p}{t^{\frac{1}{p} - \frac{1}{q}}} \right)^{q_1} \frac{dt}{t} \right)^{\frac{\tau}{q_1}} \frac{du}{u} \right)^{\frac{1}{\tau}} \equiv I_1 + I_2.$$  

We start with the estimate of the first integral:

$$I_1 = \delta^\alpha \left( \int_\delta^1 \frac{du}{u^{1+\alpha \tau}} \right)^{\frac{1}{\tau}} \left( \int_0^\delta \left( \frac{R_{\varphi}(f, t)_q}{t^{\frac{1}{p} - \frac{1}{q}}} \right)^{q_1} \frac{dt}{t} \right)^{\frac{\tau}{q_1}}$$

(10.8)

$$\lesssim \left( \int_0^\delta \left( \frac{R_{\varphi}(f, t)_q}{t^{\frac{1}{p} - \frac{1}{q}}} \right)^{q_1} \frac{dt}{t} \right)^{\frac{1}{q_1}}.$$  

To estimate $I_2$, we consider two cases: $\tau > q_1$ and $\tau \leq q_1$. In the first case, Hardy's inequality (see Lemma 2.7) yields

$$I_2 \lesssim \delta^\alpha \left( \int_\delta^1 \left( \frac{R_{\varphi}(f, t)_p}{t^{\alpha + \frac{\tau}{q}}} \right)^\tau \frac{dt}{t} \right)^{\frac{1}{\tau}}.$$  

(10.9)

In the second case, we use the discretization of integrals. Let $n, N \in \mathbb{Z}_+$ be such that $2^{-N} < \delta \leq u \leq 2^{-n}$. Since by Lemma 6.5 we have

$$R_{\varphi}(f, t)_p \asymp R_{\varphi}(f, 2t)_p,$$

then

$$\left( \int_\delta^u \left( \frac{R_{\varphi}(f, t)_p}{t^{\frac{1}{p} - \frac{1}{q}}} \right)^{q_1} \frac{dt}{t} \right)^{\frac{\tau}{q_1}} \lesssim \left( \int_{2^{-N}}^{2^{-n}} \left( \frac{R_{\varphi}(f, t)_p}{t^{\frac{1}{p} - \frac{1}{q}}} \right)^{q_1} \frac{dt}{t} \right)^{\frac{\tau}{q_1}}$$

$$\lesssim \left( \sum_{k=n}^N \left( 2^{kd} \frac{1}{p} - \frac{1}{q} \right) R_{\varphi}(f, 2^{-k})_p^{q_1} \right)^{\frac{\tau}{q_1}}$$

$$\lesssim \sum_{k=n}^N \left( 2^{kd} \frac{1}{p} - \frac{1}{q} \right) R_{\varphi}(f, 2^{-k})_p^\tau$$

$$\lesssim \int_{2^{-N}}^{2^{-n}} \left( \frac{R_{\varphi}(f, t)_p}{t^{\frac{1}{p} - \frac{1}{q}}} \right)^\tau \frac{dt}{t} \lesssim \int_\delta^u \left( \frac{R_{\varphi}(f, t)_p}{t^{\frac{1}{p} - \frac{1}{q}}} \right)^\tau \frac{dt}{t}.$$
Thus,

\[
I_2 \lesssim \delta^\alpha \left( \int_\delta^1 u^{-\alpha\tau} \int_\delta^u \left( \frac{R\varphi(f,t)_p}{u^{(p-1)/q}} \right)^\tau dt \frac{du}{u} \right)^{\frac{1}{\tau}} \\
\]

\[
= \delta^\alpha \left( \int_\delta^1 \left( \frac{R\varphi(f,t)_p}{t^{\alpha+d(p-1)/q}} \right)^\tau \frac{dt}{t} \right)^{\frac{1}{\tau}} \\
\lesssim \delta^\alpha \left( \int_\delta^1 \left( \frac{R\varphi(f,t)_p}{t^{\alpha+d(p-1)/q}} \right)^\tau \frac{dt}{t} \right)^{\frac{1}{\tau}}.
\]

(10.10)

Finally, combining (10.7)–(10.10), we get (10.3).

(B) The proof of this part goes along the same lines as the proof of part (B) of Theorem 7.1. In what follows, we denote for the sake of simplicity

\[
UM(f,\delta)_p := \delta^\alpha \left( \int_\delta^1 \left( \frac{R\varphi(f,t)_p}{t^{\alpha+d(p-1)/q}} \right)^\tau \frac{dt}{t} \right)^{\frac{1}{\tau}} + \left( \int_0^\delta \left( \frac{R\varphi(f,t)_p}{t^{\alpha+d(p-1)/q}} \right)^{q_1} \frac{dt}{t} \right)^{\frac{1}{q_1}}.
\]

Part (B1): \(0 < p \leq 1\) and \(p < q \leq \infty\). Let \(f_0\) be given by (7.4), i.e.,

\[
f_0(x) = F\varphi(x) - \frac{1}{N^{a+\gamma}} F\varphi(Nx), \quad F\varphi(x) = \sum_{k \neq 0} e^{i(k,x)} F\varphi(k),
\]

and \(N\) is a fixed sufficiently large integer which will be chosen later.

Let \(\delta \asymp [1/n]\). By (7.11) and (7.12), we have, respectively,

(10.11)

\[
R\varphi(f_0,\delta)_p \lesssim \delta^{d(\frac{1}{p}-1)+\alpha+\gamma}
\]

and

(10.12)

\[
R\psi(f_0,1/n)_q \gtrsim n^{-\alpha} ||D(\psi/\varphi)V_n||_q.
\]

Note that we have already obtained the following two-sided bounds of \(||D(\psi/\varphi)V_n||_q\):

(a) if \(0 < q \leq 1\), Lemma 5.2 gives

\[
||D(\psi/\varphi)V_n||_q \asymp 1,
\]

(b) if \(1 < q < \infty\), Lemmas 5.6 and 5.7 imply

\[
||D(\psi/\varphi)V_n||_q \asymp \begin{cases} 
1, & \gamma > d \left(1 - \frac{1}{q}\right); \\
\ln^{\frac{1}{\gamma}}(n+1), & \gamma = d \left(1 - \frac{1}{q}\right); \\
n^{d(\frac{1}{q}-1)-\gamma}, & 0 < \gamma < d \left(1 - \frac{1}{q}\right),
\end{cases}
\]
(c) if \( q = \infty \), by Lemma 5.8,
\[
\|D(\psi/\varphi)V_n\|_q \leq \begin{cases} 
1, & \gamma > d; \\
1, & \gamma = d = 1 \text{ and } \frac{\psi(\xi)}{\varphi(\xi)} = A|\xi|^{-\gamma} \text{ sign } \xi \text{ for some } A \in \mathbb{C} \setminus \{0\}; \\
\ln(n+1), & \gamma = d = 1 \text{ and } \frac{\psi(\xi)}{\varphi(\xi)} \neq A|\xi|^{-\gamma} \text{ sign } \xi \text{ for any } A \in \mathbb{C} \setminus \{0\}; \\
\ln(n+1), & \gamma = d \geq 2; \\
n^{d-\gamma}, & 0 < \gamma < d.
\end{cases}
\]

Now, using (10.11) and taking into account that \( \alpha + \gamma > d(1-1/q)_+ \), we obtain
\[
UM(f_0, \delta)_p \lesssim \delta^{\alpha+\gamma-d(1-1/q)} + \delta^\alpha \left( \int_{\delta}^1 \left( t^{d(1-1/q)} \right)^\frac{1}{t} \right)^\frac{1}{q}
\]
(10.13)
\[
\lesssim \delta^\alpha \begin{cases} 
1, & \gamma > d \left( 1 - \frac{1}{q} \right) \text{ and } 0 < q \leq \infty; \\
\ln^+ \left( \frac{1}{q} + 1 \right), & \gamma = d \left( 1 - \frac{1}{q} \right) \text{ and } 1 < q \leq \infty; \\
\delta^{\gamma-d(1-1/q)}, & 0 < \gamma < d \left( 1 - \frac{1}{q} \right) \text{ and } 1 < q \leq \infty.
\end{cases}
\]

Thus, combining (10.12) and (10.13), we derive that for \( \gamma > d(1-1/q)_+ \) the required inequality
\[
UM(f_0, \delta)_p \lesssim \mathcal{R}_\psi(f_0, \delta)_q
\]
holds in all cases except
(i) \( 2 < q < \infty \) and \( \gamma = d(1-1/q) \),
(ii) \( \gamma = d = 1, q = \infty \), and \( \frac{\psi(\xi)}{\varphi(\xi)} = A|\xi|^{-\gamma} \text{ sign } \xi \) for some \( A \in \mathbb{C} \setminus \{0\} \).

Part (B2): \( 1 < p < q < \infty \) and \( \gamma \neq d(1/p - 1/q) \). In this case, if \( \gamma > d(1/p - 1/q) \), then we can take as \( f_0 \) any function from \( C^\infty(\mathbb{T}^d) \). Then (7.20) implies
\[
\mathcal{R}_\varphi(f_0, \delta)_q \gtrsim \delta^\alpha \quad \text{and} \quad \mathcal{R}_\psi(f_0, \delta)_q \gtrsim \delta^{\alpha+\gamma}.
\]
Hence,
\[
UM(f_0, \delta)_p \gtrsim \mathcal{R}_\varphi(f_0, \delta)_q \gtrsim \delta^\alpha.
\]

If \( 0 \leq \gamma < d(1/p - 1/q) \), then we take
\[
f_0(x) = F_\varphi(x) = \sum_{k \neq 0} \frac{\psi(k,x)}{\varphi(k)}.
\]

By (7.24), we have
\[
\mathcal{R}_\varphi(f_0, \delta)_p \lesssim \delta^{\alpha+\gamma+d(\frac{1}{q} - 1)}
\]
and by (7.25)
\[
\mathcal{R}_\psi(f_0, \delta)_q \gtrsim \delta^{\alpha+\gamma+d(\frac{1}{q} - 1)}.
\]
Therefore, since \( \gamma < d(1/p - 1/q) < d(1-1/q) \), we have
\[
UM(f_0, \delta)_p \lesssim \delta^{\alpha+\gamma+d(\frac{1}{q} - 1)} \lesssim \mathcal{R}_\psi(f_0, \delta)_q.
\]
Part (B3): $1 < p < \infty$, $q = \infty$, and $\gamma \neq d(1/p - 1/q)$. Let first $\gamma > d/p$. Then we consider $f_0 \in C^\infty(T^d)$. In this case, we have the same equivalences as in (10.14) and (10.15).

Finally, assume that $0 < \gamma < d/p$. Here we deal with the function

$$f_0(x) = \sum_{k \neq 0} e^{i(k,x)} |k|^{\gamma} \psi(k).$$

Then, (7.26) yields

$$\mathcal{R}_\psi(f_0, \delta)_\infty \gtrsim \delta^{\alpha+\gamma-d}.$$

On the other hand, by (7.27),

$$\mathcal{R}_\varphi(f_0, \delta)_p \lesssim \delta^{\alpha+\gamma+d(\frac{1}{p}-1)}.$$

Thus, applying the last two estimates, we have

$$UM(f_0, \delta)_p \lesssim \delta^{\alpha+\gamma-d} \lesssim \mathcal{R}_\psi(f_0, \delta)_q.$$  

\[\Box\]

10.2. Inequalities for moduli of smoothness. Using Theorem 4.4, the Ulyanov inequality (Theorems 8.1 and 8.2 for $\gamma = 0$), and repeating arguments that were used in the proof of Theorem 10.1 (A), we obtain the following result.

**Theorem 10.2.** Let $f \in L^p(T^d)$, $d \geq 1$, $0 < p < q \leq \infty$, $\alpha \in \mathbb{N} \cup ((1/p - 1) +, \infty)$, and $\gamma, m > 0$ be such that $\alpha + \gamma, \alpha + m \in \mathbb{N} \cup ((1/p - 1) +, \infty)$. We have

$$\omega_\alpha(f_0, \delta)_q \lesssim \delta^{\alpha+\gamma} \left( \int_{\delta}^{1} \left( \frac{\omega_{\alpha+\gamma}(f_0, t)_p}{t^{\alpha+\gamma\left(\frac{1}{p} - \frac{1}{q}\right)}} \right)^{\frac{q}{q-1}} \frac{dt}{t} \right)^{\frac{1}{q-1}} + \left( \int_{0}^{\delta} \left( \frac{\omega_{\alpha+m}(f_0, t)_p}{t^{\alpha+\gamma\left(\frac{1}{p} - \frac{1}{q}\right)}} \right)^{\frac{q}{q-1}} \frac{dt}{t} \right)^{\frac{1}{q-1}}.$$  

**Remark 10.2.** (i) Using Theorem 10.1 (B) and realization result (1.13), we obtain that inequality (10.16) is sharp in the following sense. Let $d = 1$ and $\alpha + \gamma > (1-1/q)_+$. Let also

1. $\gamma \neq 1 - 1/q > 0$ if $2 < q \leq \infty$ and $0 < p \leq 1$,
2. $\gamma \neq 1/p - 1/q$ if $1 < p < q \leq \infty$.

Then there exists a function $f_0 \in L^q(T)$, $f_0 \neq \text{const}$, such that we have

$$\omega_\alpha(f_0, \delta)_q \asymp \delta^{\alpha} \left( \int_{\delta}^{1} \left( \frac{\omega_{\alpha+\gamma}(f_0, t)_p}{t^{\alpha+\gamma\left(\frac{1}{p} - \frac{1}{q}\right)}} \right)^{\frac{q}{q-1}} \frac{dt}{t} \right)^{\frac{1}{q-1}} + \left( \int_{0}^{\delta} \left( \frac{\omega_{\alpha+\gamma}(f_0, t)_p}{t^{\alpha+\gamma\left(\frac{1}{p} - \frac{1}{q}\right)}} \right)^{\frac{q}{q-1}} \frac{dt}{t} \right)^{\frac{1}{q-1}}$$

as $\delta \to 0$.

(ii) Note that two cases (1) and (2) where the sharpness of inequality (10.3) cannot be proved are those cases where the Ulyanov inequality, given by Theorem 8.2, provides a sharper bound than the one given by inequality (10.16).

Let us formulate an important corollary which will be used to obtain embedding theorems in Section 13.
Corollary 10.2. Under the conditions of Theorem 10.2, we have

\[
\omega_\alpha(f, \delta)_q \lesssim \left( \int_0^\delta \left( \frac{\omega_{\alpha+m}(f, t)_p}{q^t} \right)^{\frac{1}{q^t}} dt \right)^{\frac{1}{q^t}} + \frac{\omega_{\alpha+\gamma}(f, \delta)_p}{\delta^{\gamma+d(\frac{1}{p}-\frac{1}{q})}} \begin{cases} 1, & \gamma > d \left(1 - \frac{1}{q}\right) \gamma; \\ \ln \frac{1}{\tau} \left(1 + \frac{1}{\delta}\right), & q = 1. \end{cases}
\]

(10.17)

The proof of the corollary follows from inequality (10.16) and property (e) of moduli of smoothness from Section 4.

We conclude this section with the proof of Remark 5.2.

Proof of Remark 5.2. Let \( \phi(\xi) = \varphi(\xi)|\xi|^m, m \in 2\mathbb{N}, \) and \( m > d(1/p - 1). \)

First, by Corollary 10.1, we obtain the following analogue of (10.17):

\[
\mathcal{R}_\psi(f, \delta)_q \lesssim \left( \int_0^\delta \left( \frac{\mathcal{R}_\phi(f, t)_p}{t^{d(\frac{1}{p}-\frac{1}{q})}} \right)^{\frac{1}{q^t}} dt \right)^{\frac{1}{q^t}} + \frac{\mathcal{R}_\phi(f, \delta)_p}{\delta^{d(\frac{1}{p}-1)}} \begin{cases} 1, & 0 < q < 1; \\ \ln \frac{1}{\delta} + 1, & q = 1. \end{cases}
\]

(10.18)

Next, let \( f = T_n \in T_n, n = \lfloor 1/\delta \rfloor. \) Then by the definition of the realization, we have

\[
\mathcal{R}_\psi(T_n, 1/n)_q \approx n^{-\alpha} \|D(\psi)T_n\|_q.
\]

(10.19)

and

\[
\mathcal{R}_\phi(T_n, 1/n)_p \approx n^{-\alpha} \|D(\varphi)T_n\|_p.
\]

(10.20)

Combining (10.18)–(10.21), we derive

\[
n^{-\alpha} \|D(\psi)T_n\|_q \lesssim n^{-\alpha-m+d(\frac{1}{p}-\frac{1}{q})} \|D(\phi)T_n\|_p
\]

(10.22)

Finally, taking into account that by the classical Bernstein inequality

\[
\|D(\phi)T_n\|_p \lesssim n^m \|D(\varphi)T_n\|_p,
\]

we get from (10.22) that

\[
\|D(\psi)T_n\|_q \lesssim \|D(\varphi)T_n\|_p \begin{cases} n^{d(\frac{1}{p}-1)}, & 0 < q < 1; \\ n^{d(\frac{1}{p}-1)} \ln n, & q = 1. \end{cases}
\]

which gives (5.3). \( \Box \)
11. Sharp Ulyanov and Kolyada inequalities in Hardy spaces

In this section, we obtain sharp Ulyanov and Kolyada type inequalities in Hardy spaces. We start with the real Hardy spaces on $\mathbb{R}^d$ and continue by Subsection 11.2 dealing with analytic Hardy spaces.

11.1. Sharp Ulyanov and Kolyada inequalities in $H^p(\mathbb{R}^d)$. Let us recall that the real Hardy spaces $H^p(\mathbb{R}^d)$, $0 < p < \infty$, is the class of tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$
\|f\|_{H^p} = \|f\|_{H^p(\mathbb{R}^d)} = \|\sup_{t > 0} |\varphi_t \ast f(x)|\|_{L^p(\mathbb{R}^d)} < \infty,
$$

where $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $\hat{\varphi}(0) \neq 0$, and $\varphi_t(x) = t^{-d}\varphi(x/t)$ (see [92, Ch. III]).

The $K$-functionals and moduli of smoothness in $H^p(\mathbb{R}^d)$ are defined, respectively, as follows (see [112]):

$$
K_\alpha(f, \delta)_{H^p} := \inf_g (\|f - g\|_{H^p} + \delta^\alpha \|(-\Delta)^{\alpha/2}g\|_{H^p})
$$

and

$$
\omega_\alpha(f, \delta)_{H^p} = \sup_{|h| < \delta} \|\Delta_h^\alpha f\|_{H^p},
$$

where the fractional difference $\Delta_h^\alpha f$ is given by (1.4).

Note that for any $f \in H^p(\mathbb{R}^d)$, $0 < p < \infty$, $\alpha \in \mathbb{N} \cup ((1/p - 1)_+, \infty)$, and $\delta > 0$ we have

(11.1)

$$
\omega_\alpha(f, \delta)_{H^p} \asymp K_\alpha(f, \delta)_{H^p}.
$$

In the case $\alpha \in \mathbb{N}$, this equivalence was proved in [60, p. 175]. The case $\alpha > (1/p - 1)_+$ can be obtained similarly by using the Fourier multiplier technique (see [111]).

In this section, we restrict ourselves to consideration of $K$-functionals defined by the Laplacian operators since the general case $K_\psi(f, \delta)_{H^p}$ with $\psi \in H_\alpha$ can be reduced to $K_{(-\Delta)^{\alpha/2}}(f, \delta)_{H^p}$. This follows from an analogue of Lemma 5.6 in $H^p(\mathbb{R}^d)$, $0 < p < \infty$ (see, e.g., [30, Ch. III, § 7]).

Using boundedness properties of the fractional integrals in the Hardy spaces (see [54]) and following the proof of Theorem 6.1, we obtain the sharp Ulyanov inequality in $H^p(\mathbb{R}^d)$.

**Theorem 11.1.** Let $f \in H^p(\mathbb{R}^d)$, $0 < p < q < \infty$, $\alpha > 0$, and $\theta = d(1/p - 1/q)$. Then, for any $\delta \in (0, 1)$, we have

$$
K_\alpha(f, \delta)_{H_q} \lesssim \left( \int_0^\delta \left( \frac{K_{\alpha+\theta}(f, t)_{H_p}}{t^\theta} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}.
$$

In addition, if $\alpha \in \mathbb{N} \cup ((1/q - 1)_+, \infty)$ and $\alpha + \theta \in \mathbb{N} \cup ((1/p - 1)_+, \infty)$, then

$$
\omega_\alpha(f, \delta)_{H_q} \lesssim \left( \int_0^\delta \left( \frac{\omega_{\alpha+\theta}(f, t)_{H_p}}{t^\theta} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}.
$$
Hence, unlike the case of Lebesgue spaces (cf. [99]), the sharp Ulyanov inequality in Hardy spaces has the same form both in quasi-Banach spaces \(0 < p < 1\) and Banach spaces \(p \geq 1\).

Now, we concern with the Kolyada-type inequality. We recall that in the Lebesgue spaces \(L_p(\mathbb{T}^d), 1 < p < q < \infty,\) inequality (11.1) was improved by Kolyada [48] as follows:

\[
\delta^{1-\theta} \left( \int_0^1 \frac{(t^{1-\omega} \omega_1(f,t)^p dt)}{t} \right)^{\frac{1}{p}} \lesssim \left( \int_0^\delta \frac{(t^{-\omega} \omega_1(f,t)^q dt)}{t} \right)^{\frac{1}{q}}, \quad \theta = d \left( \frac{1}{p} - \frac{1}{q} \right).
\]

This estimate is sharp over the classes

\[
\text{Lip} \,(\omega(\cdot), 1, p) = \{ f \in L_p(\mathbb{T}^d) : \omega_1(f, \delta)_p = \mathcal{O} (\omega(\delta)) \},
\]

that is, for any \( \omega \in \{ \omega(0) = 0, \omega \uparrow, \omega(\delta_1 + \delta_2) \leq \omega(\delta_1) + \omega(\delta_2) \} \), there exists a function \( f_0 \in \text{Lip} \,(\omega(\cdot), 1, p) \) such that, for any \( \delta > 0, \)

\[
\delta^{1-\theta} \left( \int_0^1 \frac{(t^{1-\omega} \omega_1(f_0,t)^p dt)}{t} \right)^{\frac{1}{p}} \gtrsim \left( \int_0^\delta \frac{(t^{-\omega} \omega(t)^q dt)}{t} \right)^{\frac{1}{q}}.
\]

It is worth mentioning that (11.2) is not valid for \( p = 1, d = 1 \) but is true for \( p = 1, d \geq 2 \). Note that recently, Trebels [104] (see also [33]) obtained an analogue of inequality (11.2) for moduli of smoothness of fractional order in \(1 \leq p < \infty\).

Kolyada [48] also proved (11.2) for functions belonging to the analytic Hardy spaces on the disc, that is for \( f \in H_p(D) \) and \(0 < p < q < \infty\). Below, we extended this results to the real Hardy spaces \(H_p(\mathbb{R}^d)\).

To study Kolyada’s inequalities in \(H_p(\mathbb{R}^d)\), we need the following straightforward modification of the Holmstedt formulas (see [5] and [38])

\[
K(f, t^{\theta}; X, (X,Y)_{\theta,q}) \asymp t^{\theta} \left( \int_0^\infty \left[ s^{-\theta} K(f, s) \right] \frac{ds}{s} \right)^{\frac{1}{\theta}}
\]

and

\[
\tilde{K}(f, t^{1-\theta}; (X,Y)_{\theta,q}, Y) := \inf_{g \in Y} \{ |f - g|_{\theta,q} + t^{1-\theta} |g|_Y \}
\]

\[
\asymp \left( \int_0^t \left[ s^{1-\theta} K(f, s) \right] \frac{ds}{s} \right)^{\frac{1}{\theta}}
\]

where \((X, \| \cdot \|_X)\) is a quasi-Banach space, \(Y \subset X\) is a complete subspace with seminorm \( \| \cdot \|_Y \) and \( \| \cdot \|_Y = \| \cdot \|_X + | \cdot |_Y \),

\[
K(f, t) \equiv K(f, t; X, Y) := \inf_{g \in Y} (\| f - g \|_X + t \| g \|_Y)
\]

is Peetre’s \(K\)-functional, and

\[
(X,Y)_{\theta,q} := \left\{ f \in X : |f|_{\theta,q} = \left( \int_0^\infty \left[ t^{-\theta} K(f, t) \right] \frac{dt}{t} \right)^{\frac{1}{\theta}} < \infty, \quad 0 < \theta < 1 \right\}
\]

is the interpolation spaces.
We will use the following facts:

\[(H_p(\mathbb{R}^d), F_{p,2}^\alpha(\mathbb{R}^d))_{\theta,q} = B_{p,q}^{\theta\alpha}(\mathbb{R}^d), \quad \alpha > 0, \quad 0 < \theta < 1,\]

\[|f|_{B_{p,q}^\alpha(\mathbb{R}^d)} \asymp \|f\|_{B_{p,q}^{\theta\alpha}(\mathbb{R}^d)},\]

\[|f|_{F_{p,q}^\alpha(\mathbb{R}^d)} \asymp \|f\|_{F_{p,q}^{\theta\alpha}(\mathbb{R}^d)},\]

and

\[(11.6) \quad |f|_{F_{p,2}^\alpha(\mathbb{R}^d)} \asymp \|(-\Delta)^{\alpha/2} f\|_{H_p(\mathbb{R}^d)}\]

(see, for example, [105]).

The following theorem can be proved with the help of the same ideas as the ones used in the proof of Theorem 2.6 in [104].

**Theorem 11.2.** Let \( f \in H_p(\mathbb{R}^d), \ 0 < p < q < \infty, \ \theta = d(1/p - 1/q), \) and \( \alpha > \theta. \) Then

\[(11.7) \quad \delta^{\alpha - \theta} \left( \int_\delta^\infty \left( \frac{K_\alpha(f,t)_{H_q}}{t^{\alpha - \theta}} \right)^p \frac{dt}{t} \right)^{\frac{1}{p}} \lesssim \left( \int_0^\delta \left( \frac{K_\alpha(f,t)_{H_p}}{t^{\theta}} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}.\]

In addition, if \( \alpha \in \mathbb{N} \cup ((1/p - 1)_+, \infty), \) then

\[(11.8) \quad \delta^{\alpha - \theta} \left( \int_\delta^\infty \left( \frac{\omega_\alpha(f,t)_{H_q}}{t^{\alpha - \theta}} \right)^p \frac{dt}{t} \right)^{\frac{1}{p}} \lesssim \left( \int_0^\delta \left( \frac{\omega_\alpha(f,t)_{H_p}}{t^{\theta}} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}.\]

**Proof.** By using (11.6) and (11.3), we obtain

\[I := \delta^{1 - \frac{\theta}{\alpha}} \left( \int_\delta^\infty \left( \frac{t^{\theta - \alpha} K_\alpha(f,t)_{H_q}}{t} \right)^p \frac{dt}{t} \right)^{\frac{1}{p}} \lesssim \delta^{1 - \frac{\theta}{\alpha}} \left( \int_\delta^\infty \left( \frac{s^{\theta - 1} K(f,s; H_q, F_{q,2}^\alpha)}{s} \right)^p \frac{ds}{s} \right)^{\frac{1}{p}} \lesssim K \left( f, \delta^{1 - \frac{\theta}{\alpha}} H_q, (H_q, F_{q,2}^\alpha)_{1 - \frac{\theta}{\alpha}, p} \right).\]

Using now \( \|f\|_{H_q(\mathbb{R}^d)} \lesssim \|f\|_{B_{p,q}^{\theta\alpha}(\mathbb{R}^d)} \) and

\[\|f\|_{B_{p,q}^{\theta\alpha}(\mathbb{R}^d)} \lesssim \|f\|_{F_{p,2}^\alpha(\mathbb{R}^d)} \asymp \|(-\Delta)^{\alpha/2} f\|_{H_p(\mathbb{R}^d)},\]

(see Lemma 2.1), we get

\[I \lesssim \|f - g\|_{H_q} + \delta^{1 - \frac{\theta}{\alpha}} \|g\|_{B_{p,q}^{\theta\alpha}} \lesssim \|f - g\|_{B_{p,q}^\theta} + \delta^{1 - \frac{\theta}{\alpha}} \|g\|_{F_{p,2}^\alpha} \lesssim |f - g|_{B_{p,q}^\theta} + \delta^{1 - \frac{\theta}{\alpha}} |g|_{F_{p,2}^\alpha} \]

for all \( g \in F_{p,2}^\alpha. \)
Next, taking infimum over all \( g \in F_{p,2}^\alpha \) and applying (11.4) and (11.5) as well as (11.6), we have

\[
I \lesssim \tilde{K} \left( f, \delta l \cdot \frac{q}{p}; (H_{p}, F_{p,2}^\alpha) \frac{q}{p}, (F_{p,2}^\alpha) \frac{q}{p} \right) \\
\leq \left( \int_0^\delta \left( s^{-\frac{q}{p}} K(f,s;H_{p},F_{p,2}^\alpha) \right)^\frac{q}{p} ds \right)^\frac{1}{q} \\
\leq \left( \int_0^\delta \left( t^{-\theta} H_{p}(f,t) \right)^q dt \right)^\frac{1}{q}.
\]

The last inequality implies (11.7). Finally, (11.1) yields (11.8). \( \square \)

**Remark 11.1.** Note that inequality (11.8) implies the sharp Ulyanov inequality for \( f \in H_p(\mathbb{R}^d), \) 0 < \( p < q \leq 2 \), given by

\[
\omega_{\alpha-\theta}(f,\delta)_{H_q} \lesssim \left( \int_0^\delta \left( \frac{\omega_{\alpha}(f,t)_{H_p}}{t^\theta} \right)^q \frac{dt}{t} \right)^\frac{1}{q}, \quad \alpha > \theta.
\]

This follows from the Marchaud inequality in \( H_q(\mathbb{R}^d) \):

\[
\omega_{\alpha-\theta}(f,\delta)_{H_q} \lesssim \delta^{\alpha-\theta} \left( \int_\delta^\infty \left( \frac{\omega_{\alpha}(f,t)_{H_p}}{t^{\alpha-\theta}} \right)^q \frac{dt}{t} \right)^\frac{1}{q}, \quad 0 < q \leq 2.
\]

The latter can be proved by using the standard technique with the help of Bernstein’s and Jackson’s inequalities (see, e.g., [112] and [105, Ch. 1]).

The above results in this section remain true in the periodic real Hardy classes \( H_p(\mathbb{T}^d) \). Recall that the real periodic Hardy spaces \( H_p(\mathbb{T}^d), \) 0 < \( p < \infty \), are defined as a class of the tempered distributions \( f \in S'(\mathbb{T}^d) \) such that

\[
\|f\|_{H_p(\mathbb{T}^d)} = \sup_{t > 0} \left\| \sum_{k \in \mathbb{Z}^d} \hat{f}(tk) a_k(f) e^{i(k,x)} \right\|_{L_p(\mathbb{T}^d)} < \infty,
\]

where \( a_k(f) \) are Fourier coefficients of the distribution \( f \) (see, e.g., [105, Ch. 9] and [28, p. 156]).

In particular, we have the following analogue of Theorem 11.2.

**Theorem 11.3.** Let \( f \in H_p(\mathbb{T}^d), \) 0 < \( p < q < \infty, \) \( \theta = d(1/p - 1/q), \) and \( \alpha > \theta. \) Then, for any \( \delta \in (0,1), \) we have

\[
\delta^{\alpha-\theta} \left( \int_\delta^1 \left( \frac{K_{\alpha}(f,t)_{H_p(\mathbb{T}^d)}}{t^{\alpha-\theta}} \right)^p \frac{dt}{t} \right)^\frac{1}{p} \lesssim \left( \int_0^\delta \left( \frac{K_{\alpha}(f,t)_{H_p(\mathbb{T}^d)}}{t^\theta} \right)^q \frac{dt}{t} \right)^\frac{1}{q}.
\]

In addition, if \( \alpha \in \mathbb{N} \cup ((1/p - 1)_+, \infty), \) then, for any \( \delta \in (0,1), \) we have

\[
\delta^{\alpha-\theta} \left( \int_\delta^1 \left( \frac{\omega_{\alpha}(f,t)_{H_p(\mathbb{T}^d)}}{t^{\alpha-\theta}} \right)^p \frac{dt}{t} \right)^\frac{1}{p} \lesssim \left( \int_0^\delta \left( \frac{\omega_{\alpha}(f,t)_{H_p(\mathbb{T}^d)}}{t^\theta} \right)^q \frac{dt}{t} \right)^\frac{1}{q}.
\]
Since $H_p(\mathbb{T}^d) = L_p(\mathbb{T}^d)$ for $1 < p < \infty$, both (11.9) and (11.10) remain true for the Lebesgue spaces. In fact, inequality (11.10) also holds for $L_p(\mathbb{T}^d)$, $p \geq 1$, when $d \geq 2$. This was proved by Kolyada [48] in the case $\alpha = 1$. We extend this result to moduli of smoothness of arbitrary integer order. For the case $p > 1$ see also [32, 66, 104].

**Theorem 11.4.** Let $f \in L_p(\mathbb{T}^d)$, $d \geq 2$, $1 \leq p < q < \infty$, $\theta = d(1/p - 1/q)$, and $r \in \mathbb{N}$, $r > \theta$. Then, for any $\delta \in (0, 1)$, we have

$$\delta^{r-\theta} \left( \int_{\delta}^{1} \left( \frac{\omega_r(f, t) L_q(\mathbb{T}^d)}{t^{r-\theta}} \right)^p \frac{dt}{t} \right)^\frac{1}{p} \lesssim \left( \int_{0}^{\delta} \left( \frac{\omega_r(f, t) L_q(\mathbb{T}^d)}{t^\theta} \right)^q \frac{dt}{t} \right)^\frac{1}{q}.$$

**Proof.** As it was mentioned above, the case $p > 1$ follows from Theorem 11.3. Let us consider the case $p = 1$. It follows from [50, Theorem 4] that for any $f \in W^r_q(\mathbb{T}^d)$, $r \in \mathbb{N}$, and $1 < q < \infty$ the following embedding holds

$$\|f\| \lesssim \|f\|_{\dot{B}^{r-\theta}_{q, 1}}.$$

Note also that

$$\|f\|_{L^q_q} \lesssim \|f\|_{\dot{B}^\theta_{1, q}}$$

for $f \neq \text{const}$ (see (2.1)). At the same time, we have

$$L_q, W^r_q \rightleftharpoons \dot{B}^{r-\theta}_{q, 1} \quad \text{and} \quad (L_1, W^r_1) \rightleftharpoons \dot{B}^\theta_{1, q}$$

(see, e.g., [105, Sec. 2.5]). Thus, using (11.11)–(11.13) and repeating the proof of Theorem 11.2, we obtain (11.9).

11.2. Sharp Ulyanov and Kolyada inequalities in analytic Hardy spaces. Let us recall that an analytic function $f$ on the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ belongs to the space $H_p = H_p(D)$, if

$$\|f\|_{H_p} = \sup_{0 < \rho < 1} \left( \int_0^{2\pi} |f(\rho e^{it})|^p dt \right)^{\frac{1}{p}} < \infty.$$

We define the modulus of smoothness, and realization of $K$-functional in $H_p(D)$ by analogy with (1.3) and (1.12), correspondingly.

By the Burkholder-Gundy-Silverstein theorem on interrelation between analytic and real Hardy spaces [11] (see also [29, p. 111]), using Theorems 11.1–11.3, we obtain both the sharp Ulyanov and Kolyada inequalities in $H_p(D)$.

**Theorem 11.5.** Let $f \in H_p(D)$, $0 < p < q < \infty$, $\alpha \in \mathbb{N} \cup ((1/q - 1)_+, \infty)$, $\theta = 1/p - 1/q$, and $\alpha + \theta \in \mathbb{N} \cup (1/p - 1, \infty)$. Then, for any $\delta \in (0, 1)$, we have

$$\omega_{\alpha}(f, \delta)_{H_q} \lesssim \left( \int_{0}^{\delta} \left( \frac{\omega_{\alpha+\theta}(f, t)_{H_p}}{t^\theta} \right)^q \frac{dt}{t} \right)^\frac{1}{q}. $$
Theorem 11.6. Let \( f \in H_p(D) \), \( 0 < p < q < \infty \), \( \theta = 1/p - 1/q \), and \( \alpha \in \mathbb{N} \cup (\theta, \infty) \). Then, for any \( \delta \in (0, 1) \), we have

\[
\delta^{\alpha - \theta} \left( \int_{\delta}^{1} \left( \frac{\omega_\alpha(f, t)}{t^{\alpha - \theta}} \right)^q dt \right)^{\frac{1}{q}} \lesssim \left( \int_0^{\delta} \left( \frac{\omega_\alpha(f, t)}{t^\theta} \right)^q dt \right)^{\frac{1}{q}}.
\]

Remark 11.2. Note that inequalities (11.14) and (11.15) do not hold in \( L_p, p \leq 1 \) (see [48]).

Let us give an alternative proof of Theorem 11.6 in the case \( q \geq 2 \). This proof does not rely on the interpolation properties of function spaces and uses only summability properties of Taylor coefficients.

To prove the theorem we will need the following refinement of the Hardy-Littlewood inequality (see [63]).

Lemma 11.1. Let \( f(z) = \sum_{k=1}^{\infty} a_k z^k \in H_p(D), \ 0 < p \leq 1, \ 0 < q < \infty \). Then

\[
\sum_{n=0}^{\infty} 2^{np(1-(1/q+1/p))} \left( \sum_{2^n \leq |k| < 2^{n+1}} |a_k|^q \right)^{\frac{p}{q}} \lesssim \|f\|_{H_p}.
\]

Remark 11.3. Note that since \( 1 - (1/q + 1/p) < 0 \), by Hardy’s inequality, estimate (11.16) can be equivalently written as

\[
\sum_{n=0}^{\infty} 2^{np(1-(1/q+1/p))} \left( \sum_{k=1}^{2^{n+1}} |a_k|^q \right)^{\frac{p}{q}} \lesssim \|f\|_{H_p}.
\]

Proof of Theorem 11.6. We follow the scheme proposed in [75]. Set

\[
V_n(f)(x) := \sum_{k=0}^{2^n} v\left( \frac{k}{n} \right) c_k e^{ikx},
\]

where \( v \) is defined by (5.2) (in addition, we suppose that \( v \) is a monotonic function on \((0, \infty)\)), \( c_k = c_k(f) \) are the Taylor coefficients of the analytic function \( f \).

First, let us note that the realization result can be formulated as follows:

\[
\|f - V_{2^n}(f)\|_{H_p} + 2^{-\alpha n} \|V_{2^n}^{(\alpha)}(f)\|_{H_p} \asymp \omega_\alpha(f, 2^{-n})_{H_p}.
\]

The estimate “\( \lesssim \)” in (11.18) is clear, see the definition of the \( K \)-functional. The proof of the estimate “\( \gtrsim \)” follows from the inequalities

\[
\|f - V_{2^n}(f)\|_{H_p} \lesssim E_{2^n}(f)_{H_p} \lesssim \omega_\alpha(f, 2^{-n})_{H_p}.
\]

Here, the first inequality can be obtained by using the fact that the function \( v \) is a Fourier multiplier in \( H_p \) (see [108, § 7.3]) and the second one is the Jackson inequality in \( H_p \) (see [94]).
Now, let \( n \in \mathbb{N} \) be such that \( 2^{-n} \leq \delta < 2^{-n+1} \). By standard calculations, using properties of moduli of smoothness and (11.18), we derive that

\[
\delta^{(\alpha-\theta)p} \int_{\delta}^{1} \left( \frac{\omega_{\alpha}(f, t)}{t^{\alpha-\theta}} \right)^{p} \frac{dt}{t} \lesssim 2^{-(\alpha-\theta)n} \sum_{m=0}^{n} \omega_{\alpha}(f, 2^{-m})^{p}_{H_{q}} 2^{(\alpha-\theta)m_{p}}
\]

(11.20)

\[
\lesssim 2^{-(\alpha-\theta)n} \left( \sum_{m=0}^{n} 2^{-m_{p}} \| V_{2^{m}}^{(\alpha)}(f) \|^{p}_{H_{q}} + \sum_{m=0}^{n} 2^{m(\alpha-\theta)p} \| f - V_{2^{m}}^{(\alpha)}(f) \|^{p}_{H_{q}} \right).
\]

Using the Hausdorff-Young inequality with \( q \geq 2 \) and (11.17), we obtain

\[
\sum_{m=0}^{n} 2^{-m_{p}} \| V_{2^{m}}^{(\alpha)}(f) \|^{p}_{H_{q}} \lesssim \sum_{m=0}^{n} 2^{-m_{p}} \left( \sum_{\nu=1}^{2^{m+1}} \left| v \left( \frac{\nu}{2^{m}} \right) c_{\nu} \nu^{q'(\alpha)'q'} \right) \right)^{\frac{p}{q'}}
\]

(11.21)

\[
= \sum_{m=0}^{n} 2^{mp(1/(1/q'+1/p))} \left( \sum_{\nu=1}^{2^{m+1}} |v \left( \frac{\nu}{2^{m}} \right) c_{\nu} \nu^{q'(\alpha)'q'} \right)^{\frac{p}{q'}}
\]

\[
\lesssim \| V_{2^{n}}^{(\alpha)}(f) \|^{p}_{H_{p}}.
\]

Next, applying the inequality

\[
\| f - V_{2^{m}}^{(\alpha)}(f) \|_{q} \lesssim \| f - V_{2^{n}}^{(\alpha)}(f) \|_{q} + \| V_{2^{m}}^{(\alpha)}(f) - V_{2^{n}}^{(\alpha)}(f) \|_{q},
\]

we get

\[
\sum_{m=0}^{n} 2^{m(\alpha-\theta)p} \| f - V_{2^{m}}^{(\alpha)}(f) \|^{p}_{H_{q}}
\]

(11.22)

\[
\lesssim \| f - V_{2^{n}}^{(\alpha)}(f) \|^{p}_{H_{q}} \sum_{m=0}^{n} 2^{m(\alpha-\theta)p} + \sum_{m=0}^{n-1} 2^{m(\alpha-\theta)p} \| V_{2^{m}}^{(\alpha)}(f) - V_{2^{n}}^{(\alpha)}(f) \|^{p}_{H_{q}}
\]

\[
\lesssim 2^{n(\alpha-\theta)p} \| f - V_{2^{n}}^{(\alpha)}(f) \|^{p}_{H_{q}} + \sum_{m=0}^{n-1} 2^{m(\alpha-\theta)p} \left( \sum_{s=m+1}^{n} \| V_{2^{s}}^{(\alpha)}(f) - V_{2^{s-1}}^{(\alpha)}(f) \|_{H_{q}} \right)^{p}.
\]

Let us consider the second summand in the right-hand side of (11.22). Applying the Hardy inequality for sums, the Hausdorff-Young inequality, and Lemma 11.1, we
Thus, the right-hand side of (11.15) is bounded and, therefore, derive

\[
\sum_{m=0}^{n-1} 2^{m(\alpha-\theta)p} \left( \sum_{s=m+1}^{n} \|V_{2s}^\alpha(f) - V_{2s-1}^\alpha(f)\|_{H_p} \right)^p \\
\lesssim \sum_{m=0}^{n-1} 2^{m(\alpha-\theta)p} \|V_{2m+1}^\alpha(f) - V_{2m}^\alpha(f)\|_{H_p}^p
\]

(11.23)

\[
\lesssim \sum_{m=0}^{n-1} 2^{m(\alpha-\theta)p} \left( \sum_{\nu=2^m}^{2^{m+2}} \left| v \left( \frac{\nu}{2^{m+1}} \right) c_\nu \right|^q \right)^{\frac{p}{q}}
\]

\[
\lesssim \sum_{m=0}^{n-1} 2^{m(\alpha-\theta)p} \left( \sum_{\nu=2^m}^{2^{m+1}} \left| v \left( \frac{\nu}{2^m} \right) c_\nu \right|^q \right)^{\frac{p}{q}}
\]

\[
\lesssim \sum_{m=0}^{n-1} 2^{-m\beta p} \left( \sum_{\nu=2^m}^{2^{m+1}} \left| v \left( \frac{\nu}{2^m} \right) \nu^\alpha c_\nu \right|^q \right)^{\frac{p}{q}} \lesssim \|V_n^\alpha(f)\|_{H_p}.
\]

Now, we estimate the first summand in the right-hand side of (11.22). Taking into account the first inequality in (11.19) and applying Lemma 4.2 from [24], we obtain

(11.24)

\[
\|f - V_{2n}^\alpha(f)\|_{H_p}^q \lesssim \sum_{\nu=1}^{\infty} 2^{\nu q(\frac{1}{p} - \frac{1}{q})} E_{2\nu}^\alpha(f)_{H_p}^q.
\]

Thus, combining inequalities (11.18) and (11.20) and (11.21)–(11.24), we finally arrive at

\[
\delta(\alpha-\theta) \int_{\delta}^{1} \left( \frac{\omega_\alpha(f, t)_{H_p}}{t^{\alpha-\theta}} \right)^p dt \lesssim 2^{-n(\alpha-\theta)p} \left( \|V_n^\alpha(f)\|_{H_p}^p + 2^n(\alpha-\theta)p \|f - V_{2n}^\alpha(f)\|_{H_q}^p \right)
\]

\[
\lesssim 2^{n\beta p} \omega_\alpha(f, 2^{-n})_{H_p}^p + \left( \sum_{\nu=1}^{\infty} 2^{\nu q(\frac{1}{p} - \frac{1}{q})} E_{2\nu}^\alpha(f)_{H_p}^q \right)^{\frac{p}{q}}
\]

\[
\lesssim \left( \int_{0}^{\delta} \left( \frac{\omega_\alpha(f, t)_{H_p}}{t^{\theta}} \right)^q dt \right)^{\frac{p}{q}}.
\]

\[\square\]

**Corollary 11.1.** Let \( f \in H_p(D) \), \( 0 < p < q < \infty \), \( \theta = 1/p - 1/q \), and \( \alpha \in \mathbb{N} \cup (\theta, \infty) \). Suppose also that the \( \alpha \)-th derivative of \( f \) in the sense of Weyl belongs to \( H_p(D) \). Then \( f(e^{it}) \in B_{q,p}^{\alpha-\theta}(\mathbb{T}) \).

**Proof.** By Theorem 2.4 in [45], one has

\[
\omega_\alpha(f, \delta)_{H_p} \lesssim \delta^\alpha \|f^\alpha\|_{H_p}.
\]

Thus, the right-hand side of (11.15) is bounded and, therefore, \( f(e^{it}) \in B_{q,p}^{\alpha-\theta}(\mathbb{T}) \). \[\square\]
12. \((L_p, L_q)\) inequalities of Ulyanov-type involving derivatives

We start this section with the following result due to Marcinkiewicz \[62\]:

\[
\|f'\|_p \lesssim \left( \int_0^1 \left( \frac{\omega_2(f, t)_p}{t} \right)^\tau \frac{dt}{t} \right)^{\frac{1}{\tau}},
\]

where \(f \in L_p(\mathbb{T})\), \(1 < p < \infty\), and \(\tau = \min(2, p)\). Related inequalities were also studied by Besov \[6\]. A generalization of the previous estimate to the case of the \(k\)-th derivative is given by

\[
\|f^{(k)}\|_p \lesssim \left( \int_0^1 \left( \frac{\omega_r(f, t)_p}{t^k} \right)^\tau \frac{dt}{t} \right)^{\frac{1}{\tau}}, \quad 0 < k < r,
\]

which, in particular, follows from the known embedding between Sobolev and Besov spaces.

Related inequalities for the moduli of smoothness are given by

**Theorem 12.1.** (See \[25, 86\].) Let \(f \in L_p(\mathbb{T})\), \(1 < p < \infty\), \(\varsigma = \max(2, p)\), \(\tau = \min(2, p)\), \(\alpha, r > 0\), and \(\delta \in (0, 1)\). Then

\[
\left( \int_0^\delta \left( t^{-r} \omega_{r+\alpha}(f, t)_p \right)^{\tau} \frac{dt}{t} \right)^{\frac{1}{\tau}} \lesssim \omega^\alpha(f^{(r)}, \delta)_p
\]

\[
\lesssim \left( \int_0^\delta \left( t^{-r} \omega_{r+\alpha}(f, t)_p \right)^{\tau} \frac{dt}{t} \right)^{\frac{1}{\tau}}.
\]

Note that for \(p = 1\) or \(p = \infty\) and \(\alpha > 0\) one has only weaker estimates

\[
t^{-r} \omega_{r+\alpha}(f, t)_p \lesssim \omega^\alpha(f^{(r)}, t)_p
\]

and

\[
(12.1) \quad \omega^\alpha(f^{(r)}, \delta)_p \lesssim \int_0^\delta t^{-r} \omega_{r+\alpha}(f, t)_p \frac{dt}{t},
\]

see \[18, p. 46, 178\] for \(\alpha \in \mathbb{N}\) and \[86\] for \(\alpha > 0\). Both inequalities clearly hold also for \(1 < p < \infty\).

In the case \(0 < p \leq 1\), it is shown in \[25\] that for \(\alpha \in \mathbb{N}\) one has

\[
(12.2) \quad \omega^\alpha(f^{(r)}, \delta)_p \lesssim \left( \int_0^\delta \left( t^{-r} \omega_{r+\alpha}(f, t)_p \right)^{\frac{1}{p}} \frac{dt}{t} \right)^{\frac{1}{\frac{1}{p}}},
\]

where \(f^{(r)}\) is the derivative in the sense of \(L_p\), which is defined as follows. A function \(f \in L_p(\mathbb{T})\), \(0 < p < \infty\), has the derivative \(f^{(r)}\) of order \(r \in \mathbb{N}\) in the sense \(L_p\) if

\[
\left\| \frac{\Delta_h^r f}{h^r} - f^{(r)} \right\|_p \to 0 \quad \text{as} \quad h \to 0.
\]
Below we study sharp Ulyanov \((L_p, L_q)\) inequalities for moduli of smoothness involving derivatives. Our goal is to improve the known estimate (\cite{25})

\[\omega_\alpha(f^{(r)})(\delta)_q \lesssim \left( \int_0^\delta \left( t^{-r-(\frac{1}{p}-\frac{1}{q})} \omega_{\alpha+r} (f,t)_p \right)^q dt \right)^{\frac{1}{qn}} , \quad 0 < p < q \leq \infty,\]

where \(1/p - 1/q < \alpha, d = 1\), and \(\alpha, r \in \mathbb{N}\).

**Theorem 12.2.** Let \(f \in L_p(\mathbb{T})\), \(0 < p < q \leq \infty\), \(r \in \mathbb{N}\), \(\sigma \in \mathbb{N} \cup ((1/q - 1)_+, \infty)\), and \(\gamma \geq 0\) be such that \(\alpha + \gamma \in \mathbb{N} \cup ((1/p - 1)_+, \infty)\). Then, for any \(\delta \in (0,1)\), we have

\[\omega_\alpha(f^{(r)})(\delta)_q \lesssim \left( \int_0^\delta \left( \frac{\omega_{\alpha+\gamma} (f,t)_p \sigma (\frac{1}{t})}{t^{\gamma + r}} \right)^q dt \right)^{\frac{1}{qn}},\]

where

(1) if \(0 < p \leq 1\) and \(p < q \leq \infty\), then

\[\sigma(t) := \begin{cases} t^{\frac{1}{p}-1}, & \gamma > \left(1 - \frac{1}{q}\right)_+; \\ t^{\frac{1}{p}-1} \ln^\frac{1}{q}(t+1), & 0 < \gamma = \left(1 - \frac{1}{q}\right)_+; \\ t^{\frac{1}{p}-\frac{1}{q}} - \gamma, & 0 < \gamma < \left(1 - \frac{1}{q}\right)_+; \\ t^{\frac{1}{p}-\frac{1}{q}}, & \gamma = 0. \end{cases}\]

(2) if \(1 < p \leq q \leq \infty\), then

\[\sigma(t) := \begin{cases} 1, & \gamma \geq \frac{1}{p} - \frac{1}{q}, \quad q < \infty; \\ 1, & \gamma > \frac{1}{p}, \quad q = \infty; \\ \ln^\frac{1}{p}(t+1), & \gamma = \frac{1}{p}, \quad q = \infty; \\ t^{(\frac{1}{p} - \frac{1}{q}) - \gamma}, & 0 \leq \gamma < \frac{1}{p} - \frac{1}{q}. \end{cases}\]

**Remark 12.1.** (i) For \(q = \infty\), the condition \(\alpha + \gamma \in \mathbb{N} \cup ((1/p - 1)_+, \infty)\) can be weaken as follows: \(\alpha + \gamma + r \in \mathbb{N} \cup ((1/p - 1)_+, \infty)\).

(ii) Inequality (12.4) means that if the integral on the right-hand side is finite, then \(f^{(r)}\) exists in the sense of \(L_q\) and estimate (12.4) holds (see also \cite{25}).

(iii) Inequality (12.3) follows from Theorem 12.2 when \(\gamma = 0\).

**Proof of Theorem 12.2.** Let first \(0 < p < q < \infty\). Theorem 8.1 yields

\[\omega_\alpha(f^{(r)})(\delta)_q \lesssim \left( \int_0^\delta \left( \frac{\omega_{\alpha+\gamma} (f^{(r)}), t)_p \sigma \left( \frac{1}{t} \right)^q dt \right)^{\frac{1}{q}}, \quad 0 < p < q \leq \infty,\]

By Theorem 12.1 and inequalities (12.1)–(12.2), we obtain that

\[\omega_{\alpha+\gamma}(f^{(r)})(\delta)_p \lesssim \left( \int_0^\delta \left( t^{-r} \omega_{\alpha+\gamma+r} (f,t)_p \right)^{\frac{1}{p}} dt \right)^{\frac{1}{p}}, \quad \bar{p} = \min(1,p).\]

Here we take into account that the proof of inequality (12.2) for fractional moduli of smoothness repeats the proof given in \cite{25} for moduli of smoothness of integer order using the realization result (1.13).
Applying Hardy’s inequality given by Lemma 2.7, we derive
\[
\omega_\alpha(f^{(r)}(\delta),q) \lesssim \left( \int_0^\delta u(t) \left\{ \int_0^t \left[ s^{-r} \omega_{\alpha+\gamma+r}(f,s)_p \right] \frac{\tilde{p} ds}{s} \right\}^{q/\tilde{p}} dt \right)^{\frac{1}{q}}
\]
\[
\lesssim \left( \int_0^\delta v(t) \left[ t^{-r-1/\tilde{p}} \omega_{\alpha+\gamma+r}(f,t)_p \right]^q dt \right)^{\frac{1}{q}}
\]
\[
= \left( \int_0^\delta \left[ \frac{1}{t^{\gamma+r}} \sigma\left(\frac{1}{t}\right) \omega_{\alpha+\gamma+r}(f,t)_p \right]^{q/\tilde{p}} \frac{dt}{t} \right)^{\frac{1}{q}},
\]
where
\[
u(t) = \left( \frac{1}{t^\gamma} \sigma\left(\frac{1}{t}\right) \right)^q \frac{1}{t}, \quad v(t) = \frac{t^{q/\tilde{p}}}{t} \left( \frac{1}{t^\gamma} \sigma\left(\frac{1}{t}\right) \right)^q.
\]
It is easy to verify that (2.6) holds for the couple \((u,v)\) with \(\lambda = q/\tilde{p} > 1\) and \(\lambda' = q/(q - \tilde{p})\), which gives the desired result.

Suppose that \(0 < p < q = \infty\). Applying first inequality (12.1) and then Theorem 8.1, Fubini’s theorem gives
\[
\omega_\alpha(f^{(r)}(\delta),q) \lesssim \int_0^\delta \frac{1}{t} \omega_{\alpha+\gamma+r}(f,t)_p \frac{dt}{t}
\]
\[
\lesssim \int_0^\delta t^{-r} \int_0^t \frac{\omega_{\alpha+r\gamma}(f,u)_p}{u^\gamma} \sigma\left(\frac{1}{u}\right) \frac{du}{u} \frac{dt}{t}
\]
\[
\lesssim \int_0^\delta \frac{\omega_{\alpha+r\gamma}(f,u)_p}{u^\gamma} \sigma\left(\frac{1}{u}\right) \int_u^\infty t^{-r} \frac{dt}{t} \frac{du}{u},
\]
completing the proof.

Now, we consider the multi-dimensional case. It is well known that for \(f \in L_q(\mathbb{T}^d)\), \(1 \leq q \leq \infty\), \(d \geq 2\), one has
\[
\omega_\alpha(D^\beta f,\delta)_q \lesssim \left( \int_0^\delta \left( t^{-|\beta|} \omega_{\alpha+|\beta|}(f,t)_q \right)^\varrho \frac{dt}{t} \right)^{1/\varrho},
\]
where \(\alpha \in \mathbb{N}\), \(\beta \in \mathbb{Z}_+^d\), and
\[
\varrho = \begin{cases} 
\min(2,q), & q < \infty, \\
1, & q = \infty
\end{cases}
\]
(see [5, Ch. 4], [41] and [104]).

**Remark 12.2.** If \(1 < q < \infty\), the proof of (12.5) given in [104, Theorem 2.3] can be extended to the case \(\alpha > 0\).

Note also that the reverse inequality to (12.5) looks as follows (see [41]):
\[
\omega_{\alpha+r}(f,\delta)_q \lesssim \delta^r \sup_{|\beta|_1 = r} \omega_\alpha(D^\beta f,\delta)_q, \quad \alpha, r \in \mathbb{N}.
\]

By using inequality (12.5) and Theorem 8.2, we can prove the following multidimensional analog of Theorem 12.2.
Theorem 12.3. Let \( f \in L_p(\mathbb{T}^d) \), \( d \geq 2 \), \( 0 < p < q \), \( 1 \leq q \leq \infty \), \( \alpha \in \mathbb{N} \), \( \gamma \geq 0 \), \( \beta \in \mathbb{Z}_+^d \), and \( \alpha + |\beta|_1 + \gamma \in \mathbb{N} \cup ((1/p - 1)_+, \infty) \). Then, for any \( \delta \in (0, 1) \), we have

\[
(12.6) \quad \omega_{\alpha}(D^\beta f, \delta)_q \lesssim \left( \int_0^\delta \left( \frac{\omega_{\alpha + |\beta|_1 + \gamma}(f, t)_p}{t^{|\beta|_1 + \gamma}} \sigma \left( \frac{1}{\delta} \right) \right)^q dt \right)^{1/q},
\]

where

1. if \( 0 < p \leq 1 \) and \( 1 \leq q \leq \infty \), then

\[
\sigma(t) := \begin{cases} 
 t^{d(\frac{1}{p} - 1)}, & \gamma > d \left( 1 - \frac{1}{q} \right); \\
 t^{d(\frac{1}{p} - 1)}, & \gamma = d \left( 1 - \frac{1}{q} \right) \geq 1 \text{ and } \alpha + \gamma \in \mathbb{N}; \\
 t^{d(\frac{1}{p} - 1)} \ln \frac{1}{\delta t}, & \gamma = d \left( 1 - \frac{1}{q} \right) \geq 1 \text{ and } \alpha + \gamma \notin \mathbb{N}; \\
 t^{d(\frac{1}{p} - \frac{1}{q}) - \gamma}, & 0 < \gamma = d \left( 1 - \frac{1}{q} \right) < 1; \\
 t^{d(\frac{1}{p} - \frac{1}{q})}, & 0 < \gamma < d \left( 1 - \frac{1}{q} \right);
\end{cases}
\]

2. if \( 1 < p \leq q \leq \infty \), then

\[
\sigma(t) := \begin{cases} 
 1, & \gamma \geq d(\frac{1}{p} - \frac{1}{q}), \quad q < \infty; \\
 1, & \gamma > \frac{d}{p}, \quad q = \infty; \\
 \ln \frac{1}{\delta t}, & \gamma = \frac{d}{p}, \quad q = \infty; \\
 t^{d(\frac{1}{p} - \frac{1}{q}) - \gamma}, & 0 \leq \gamma < d(\frac{1}{p} - \frac{1}{q}).
\end{cases}
\]

Proof. The proof is slightly different from the proof of Theorem 12.2. We use first (12.5) and then we apply Theorem 8.2 to get

\[
(12.6) \quad \omega_{\alpha}(D^\beta f, \delta)_q \lesssim \left( \int_0^\delta \left( \frac{\omega_{\alpha + |\beta|_1 + \gamma}(f, t)_p}{t^{|\beta|_1 + \gamma}} \sigma \left( \frac{1}{\delta} \right) \right)^q dt \right)^{1/q}.
\]

To complete the proof, we only need the Hardy inequality for monotone functions (see, e.g., [18, Theorem 3.5, p. 28]). Let \( \xi > 0 \), \( 0 < \lambda < \infty \), and \( \phi \) be a non-negative monotone function on \( \mathbb{R}_+ \). Then the following inequality holds

\[
\int_0^\infty \left( t^{-\xi} \int_0^t \sigma(s) ds \right)^{\lambda} dt \lesssim \int_0^\infty \left( t^{-\xi} \phi(t) \right)^{\lambda} dt.
\]

Noting that this inequality also holds for weakly decreasing functions \( \phi \), that is, satisfying \( \phi(x) \leq C\phi(y) \) for \( x \geq y \geq 2x \) (see, e.g., [100]), we arrive at (12.6).

Remark 12.3. (i) Note that in Theorem 12.3, \( q \geq 1 \) and we deal with the usual partial derivatives \( D^\beta f \).

(ii) Assuming in Theorem 12.3 that \( p \geq 1 \), inequality (12.6) holds if we replace \( q \) by \( q_1 \), which gives a stronger result. This can be shown as in the proof of Theorem 12.2. Indeed, first, we use Theorem 8.2 for the derivative \( D^\beta f \) and then we apply inequality (12.5).
(iii) If $1 < q < \infty$, then (12.6) holds for $\alpha > 0$ (see Remark 12.2).

We finish this section with the following result related to the discussion in Subsection 1.2. We show that while dealing with absolutely continuously functions the pathological behavior of smoothness properties in $L^p$, $0 < p < 1$, disappears.

**Proposition 12.1.** Let $0 < p < 1$, $\alpha \in \mathbb{N}$, $f^{(\alpha-1)} \in AC(\mathbb{T})$, and
\[
\omega_\alpha(f, \delta)_p = o(\delta^\alpha), \quad \delta \to 0,
\]
then $f \equiv \text{const}$.

**Proof.** Let $0 < p < 1$, $f \in AC(\mathbb{T})$ and
\[
\omega_1(f, \delta)_p = o(\delta), \quad \delta \to 0,
\]
then $f \equiv \text{const}$ (see [96]). Using this and the estimate
\[
\omega_1(f^{(\alpha-1)}, \delta)_p \lesssim \left( \int_0^\delta \left( \frac{\omega_\alpha(f, \delta)_p}{t^\alpha} \right)^p t^{p-1} dt \right)^{1/p} = o(\delta), \quad \delta \to 0,
\]
(see (12.2)), we arrive at $f^{(\alpha-1)} \equiv \text{const}$ and hence the desired result follows. \qed
13. Embedding theorems for function spaces

As mentioned earlier, both sharp Ulyanov’s and Kolyda’s inequalities are closely related to embedding theorems for smooth function spaces (Lipschitz, Nikol’skii–Besov, etc.).

In what follows, we restrict ourselves to the classical Lipschitz spaces

\[ \text{Lip}(\alpha, k, p) = \left\{ f \in L^p(T^d) : \omega_k(f, \delta)_p = O(\delta^{\alpha}) \right\} \]

and their generalizations given by

\[ \text{Lip}^\beta(\alpha, k, p) = \left\{ f \in L^p(T^d) : \omega_k(f, \delta)_p = O(\delta^{\alpha} \ln^{\beta} 1/\delta) \right\}. \]

The latter spaces are the well-known logarithmic Lipschitz spaces, widely used in functional analysis (see, e.g., [10, 78]) and differential equations (see, e.g., [14, 113]).

The theory of embedding theorems for function spaces has been studied for a long time, beginning with the work of Hardy and Littlewood (see [35, 36]). They proved that

\[ \text{Lip}(\alpha, 1, p) \hookrightarrow \text{Lip}(\alpha - \theta, 1, q), \]

where

\[ 1 \leq p < q < \infty, \quad \theta = 1/p - 1/q, \quad \theta < \alpha \leq 1, \quad d = 1. \]

Later, this result was extended by many authors mostly in the case \( 1 \leq p < q \leq \infty \) (see [7, 33, 37, 48, 49, 66, 87, 86, 93, 99, 104, 109]). In particular, Kolyada’s inequality (see (11.2)) implies that

\[ \text{Lip}(\alpha, \alpha, p) \hookrightarrow B^{\alpha - \theta}_{q, p}, \quad 1 < p < q < \infty. \]

Recall that for an important limit case \( \alpha = k \), we have \( \text{Lip}(\alpha, \alpha, p) \equiv W^\alpha_p(T) \) for \( 1 < p \leq \infty \) and

\[ f \in \text{Lip}(\alpha, \alpha, 1) \iff \begin{cases} \mathcal{D}^{\alpha - 1} f \in BV(T), & \alpha > 1; \\ f \in BV(T), & \alpha = 1; \\ I^{1-\alpha} f \in BV(T), & \alpha < 1, \end{cases} \]

where \( BV \) is the space of all functions which are of bounded variation on every finite interval and \( I^{1-\alpha} \) is the fractional integral (see [114, XII, § 8, (8.3), p. 134]). By \( \mathcal{D}^\alpha f \) we denote the Liouville–Grünwald–Letnikov derivative of order \( \alpha > 0 \) of a function \( f \) in the \( L_p \)-norm: if for \( f \in L_p(T) \) there exists \( g \in L_p(T) \) such that

\[ \lim_{h \to 0^+} \left\| h^{-\alpha} \Delta_h^\alpha f(\cdot) - g(\cdot) \right\|_p = 0, \]

then \( g = \mathcal{D}^\alpha f \) (see [12], [85, § 20, (20.7)]). Note that for any \( f \in L_p(T), p \geq 1 \), we have \( f^{(\alpha)}(x) = \mathcal{D}^\alpha f(x) \) a.e., where \( f^{(\alpha)} \) is the Weyl derivative of \( f \).

In the case \( 0 < p < 1 \), the space \( \text{Lip}(\alpha, k, p) \) with limiting smoothness (that is \( \alpha = 1/p + k - 1 \)) consists only of ”trivial” functions. Let us state it more precisely. We have

\[ f \in \text{Lip}(1/p + k - 1, k, p) \]
if and only if after correction of \( f \) on a set of measure zero its \((k-1)\)-st derivative is of the form

\[
f^{(k-1)}(x) = d_0 + \sum_{x_i < x} d_i,
\]

where \( \{x_i\} \) is a sequence of different points from \([0, 2\pi)\) and \( \sum_x |d_i|^p < \infty \) (see [42] and [108, 4.8.26], see also [55] for the case \( k = 1 \)).

### 13.1. New embedding theorems for Lipschitz spaces

Our main goal in this section is to apply Ulyanov inequalities to study optimal embeddings of the type

\[
\text{Lip}(\eta, \alpha, p) \leftrightarrow X_q, \quad 0 < p < q \leq \infty,
\]

where \( X_q \) is a Lipschitz-type or Besov space with respect to \( L_q \)-norm.

It is worth mentioning again that the space \( \text{Lip}(\eta, \alpha, p) \) for \( 0 < p \leq 1 \) and \( \eta > \alpha + d(1/p - 1) \) consists only of a.e. constant functions. This follows from Theorem 4.3. Note also that the embedding \( \text{Lip}(\eta, \alpha, p) \leftrightarrow L_q \) holds for \( 0 < p < q \leq \infty \) if and only if \( \eta > \theta = d(1/p - 1)/q \). Thus, it is natural to assume below that \( \eta > \theta \) and \( \eta \leq \alpha + d(1/p - 1) \) if \( 0 < p < 1 \).

**Theorem 13.1.** Let \( d \geq 1, 0 < p < q \leq \infty, \alpha \in \mathbb{N} \cup ((1/p - 1)_+, \infty), \theta = d(1/p - 1)/q. \)

(A) If \( 0 < p \leq 1 \) and \( p < q \leq \infty \), then, for any \( \theta < \eta \leq \alpha + d(1/p - 1) \), one has

\[
\text{Lip}(\eta, \alpha, p) \leftrightarrow \begin{cases} 
\text{Lip}(\eta - \theta, \alpha - d(1 - \frac{1}{q}), q), & \frac{d}{d-1} \leq q \leq \infty \text{ and } \alpha \in \mathbb{N}; \\
\text{Lip}(\eta - \theta, \alpha, q), & \frac{d}{d-1} \leq q \leq 2 \text{ and } \alpha \notin \mathbb{N}; \\
\text{Lip}(\eta - \theta, \alpha, q), & q < \frac{d}{d-1} \text{ and } d \geq 2; \\
\text{Lip}(\eta - \theta, \alpha, q), & q \leq 2 \text{ and } d = 1; \\
\text{Lip}(\eta - \theta, \alpha, q) \cap \text{Lip}^{1/2}(\eta - \theta, \alpha - d(1 - \frac{1}{q}), q), & 2 < q < \infty \text{ and } \alpha \notin \mathbb{N}; \\
\text{Lip}(\eta - \theta, \alpha, q) \cap \text{Lip}^{1/2}(\eta - \theta, \alpha - d(1 - \frac{1}{q}), q), & 2 < q < \infty \text{ and } d = 1; \\
\text{Lip}(\eta - \theta, \alpha, q), & q = \infty, d \geq 2, \text{ and } \alpha \notin \mathbb{N}; \\
\text{Lip}(\eta - \theta, \alpha, q), & q = \infty, d = 1, \text{ and } \alpha \notin \mathbb{N}.
\end{cases}
\]

(B) If \( 1 < p < q \leq \infty \), then, for \( \theta < \eta = \alpha \), one has

\[
\text{Lip}(\eta, \alpha, p) \leftrightarrow \begin{cases} 
B^\alpha_{q,p}, & 1 < p \leq \min(2, q); \\
\text{Lip}(\eta - \theta, \alpha - \theta, q) \cap B^\alpha_{q,p}, & \min(2, q) < p < q < \infty; \\
\text{Lip}(\eta - \theta, \alpha, q) \cap \text{Lip}^{1/p'}(\eta - \theta, \alpha - \theta, q), & 1 < p < q = \infty,
\end{cases}
\]

while, for any \( \theta < \eta < \alpha \), one has

\[
\text{Lip}(\eta, \alpha, p) \leftrightarrow \text{Lip}(\eta - \theta, \alpha, q).
\]
Remark 13.1. In the special case $1 = p < q < \infty$, $d \geq 2$, and $\eta = \alpha \in \mathbb{N}$, some embeddings from Theorem 13.1 (A) can be improved as follows (see Theorem 11.4)
\[ \text{Lip}(\eta, \alpha, p) \hookrightarrow B_{q,1}^{\alpha - \theta}, \quad \theta = d\left(1 - \frac{1}{q}\right). \]

Note that by the Marchaud inequality (4.13), we have
\[ B_{q,1}^{\alpha - \theta} \hookrightarrow \text{Lip}\left(\eta - \theta, \alpha - d\left(1 - \frac{1}{q}\right), q\right), \]
see also the proof of part (B) below.

Remark 13.2. In the scale of Lipschitz spaces, embeddings in Theorem 13.1 are sharp. This follows from Theorem 13.2.

Let us illustrate the embeddings, which follow from Theorem 13.1, for the most important case $\eta = \alpha$ and $0 < p < q \leq \infty$. In Fig. 1 and Fig. 2, we use the following notation:

- $X_1 = \text{Lip}\left(\alpha - \theta, \alpha - d\left(1 - \frac{1}{q}\right), q\right)$,
- $X_2 = \text{Lip}(\alpha - \theta, \alpha, q)$,
- $X_3 = B_{q,p}^{\alpha - \theta}$,
- $X_4 = \text{Lip}(\alpha - \theta, \alpha - \theta, q) \cap X_3$,
- $X_5 = \text{Lip}(\alpha - \theta, \alpha - \theta, q) \cap \text{Lip}_{p'}(\alpha - \theta, \alpha - \theta, q)$.

We use the solid line ——— (or the dashed line - - - -) to emphasize that the corresponding boundary is included (or excluded).

**Fig. 1:** The sets of pairs $(1/p, 1/q)$ for which the embeddings $\text{Lip}(\alpha, \alpha, p) \hookrightarrow X_i$, $i = 1, 5$, from Theorem 13.1 are fulfilled in the case $d \geq 2$ and $\eta = \alpha \in \mathbb{N}$. 
Fig. 2: The sets of pairs \((1/p, 1/q)\) for which the embeddings \(\text{Lip}(\alpha, \alpha, p) \hookrightarrow X_i, i = 1, 5,\) from Theorem 13.1 are fulfilled in the case \(d \geq 2\) and \(\eta = \alpha \notin \mathbb{N}\).

**Proof.** (A) We need to compare the embeddings, which three main inequalities (the classical Ulyanov inequality, the sharp Ulyanov inequality, and the Ulyanov–Marchaud inequality) provide.

First of all, let us compare the sharp Ulyanov inequality and the Ulyanov–Marchaud inequality. In the case \(0 < p \leq 1\), we have from the sharp Ulyanov inequality given by Theorems 8.1 and 8.2 that

\[
\omega_{\alpha-\gamma}(f, \delta)_q \lesssim \left( \int_0^{\delta} \left( \frac{\omega_{\alpha}(f, t)_p}{t^{\frac{1}{d(1 - \frac{1}{q})}} - \frac{1}{q}} \right)^{\frac{1}{q_1}} dt \right)^{\frac{1}{q_1}} \]

\[
+ \frac{\omega_{\alpha}(f, \delta)_p}{\delta^{\gamma + d(\frac{1}{p} - 1)}} \begin{cases} 
1, & \gamma > d \left(1 - \frac{1}{q}\right)_+; \\
1, & \gamma = d \left(1 - \frac{1}{q}\right)_+ \geq 1, d \geq 2, \text{ and } \alpha \in \mathbb{N}; \\
\ln^{1/q_1} \left(\frac{1}{\delta} + 1\right), & \gamma = d \left(1 - \frac{1}{q}\right)_+ \geq 1, d \geq 2, \text{ and } \alpha \notin \mathbb{N}; \\
1, & \gamma = d - 1 \text{ and } q = \infty; \\
\ln^{1/q} \left(\frac{1}{\delta} + 1\right), & 0 < \gamma = d \left(1 - \frac{1}{q}\right)_+ < 1; \\
\left(\frac{1}{\delta}\right)^{d(1 - \frac{1}{\gamma})}, & 0 < \gamma < d \left(1 - \frac{1}{q}\right)_+; \\
\left(\frac{1}{\delta}\right)^{d(1 - \frac{1}{\gamma})}, & \gamma = 0.
\end{cases}
\]
At the same time, from the Ulyanov–Marchaud inequality (10.16) and property (e) given in Section 4, we obtain

\[
\omega_{\alpha-\gamma}(f,\delta)_q \lesssim \left( \int_0^\delta \left( \frac{\omega_\alpha(f,t)_p}{t^{d\left(\frac{1}{q} - \frac{1}{p}\right)}} \right)^{q_1} \frac{dt}{t} \right)^{\frac{1}{q_1}}
\]

(13.4)

\[ + \frac{\omega_\alpha(f,\delta)_p}{\delta^{\gamma + d(\frac{1}{p} - 1)}} \begin{cases} 1, & \gamma > d \left(1 - \frac{1}{q}\right); \\ \ln^{1/\tau} \left(\frac{1}{\delta} + 1\right), & \gamma = d \left(1 - \frac{1}{q}\right); \\ \left(\frac{1}{\delta}\right)^{d\left(1 - \frac{1}{q}\right) - \gamma}, & \gamma < d \left(1 - \frac{1}{q}\right), \end{cases} \]

where

\[ \tau = \tau(q) = \begin{cases} \min(q,2), & q < \infty; \\ 1, & q = \infty. \end{cases} \]

It is easy to see that (13.4) gives the same estimates as inequality (13.3) in all cases except the following three cases:

1) \(2 < q < \infty\) and \(\gamma = d\left(1 - 1/q\right)\). In this case, by (13.3), for \(\alpha \notin \mathbb{N}\), we have that

\[ \text{Lip}(\eta,\alpha,p) \hookrightarrow \text{Lip}^{\frac{1}{q}} \left(\eta - \theta, \alpha - d \left(1 - \frac{1}{q}\right), q\right) \]

and, for \(\alpha \in \mathbb{N}\), we have

\[ \text{Lip}(\eta,\alpha,p) \hookrightarrow \begin{cases} \text{Lip} \left(\eta - \theta, \alpha - d \left(1 - \frac{1}{q}\right), q\right), & d \left(1 - \frac{1}{q}\right) \geq 1; \\ \text{Lip}^{\frac{1}{q}} \left(\eta - \theta, \alpha - d \left(1 - \frac{1}{q}\right), q\right), & d \left(1 - \frac{1}{q}\right) < 1, \end{cases} \]

2) \(0 < q \leq 2, \gamma = d\left(1 - 1/q\right)_+ \geq 1\), and \(\alpha \in \mathbb{N}\),

3) \(q = \infty, \gamma = d = 1\) or \(\gamma = d \geq 2, \alpha \in \mathbb{N}\).

In the last two cases, (13.3) implies that

\[ \text{Lip}(\eta,\alpha,p) \hookrightarrow \text{Lip} \left(\eta - \theta, \alpha - d \left(1 - \frac{1}{q}\right), q\right). \]

Let us note that the Ulyanov–Marchaud inequality is a direct consequence of the classical Ulyanov inequality and the Marchaud inequality (see the proof of Theorem 10.1). Thus, the classical Ulyanov inequality always gives more optimal embeddings than the Ulyanov–Marchaud inequality. Taking into account this remark, to prove (A), it is sufficient to compare the embedding

\[ \text{Lip}(\eta,\alpha,p) \hookrightarrow \text{Lip}(\eta - \theta,\alpha,q), \]

which follows from the classical Ulyanov (see (8.1) and (8.7) with \(\gamma = 0\)), and the corresponding embeddings, which follow from the sharp Ulyanov inequality in the above exceptional cases.

1) If \(2 < q < \infty, \gamma = d\left(1 - 1/q\right)\), by Theorem 13.2, \(\text{Lip}(\eta - \theta,\alpha,q)\) and \(\text{Lip}^{1/q}(\eta - \theta,\alpha - d\left(1 - 1/q\right),q)\) are not comparable. Therefore, for \(\alpha \notin \mathbb{N}\), we have

\[ \text{Lip}(\eta,\alpha,p) \hookrightarrow \text{Lip}(\eta - \theta,\alpha,q) \cap \text{Lip}^{\frac{1}{q}} \left(\eta - \theta, \alpha - d \left(1 - \frac{1}{q}\right), q\right). \]
Let us consider the case $\alpha \in \mathbb{N}$. First, let $d(1-1/q) \geq 1$. In this case, it is clear that
\[
\text{Lip} \left( \eta - \theta, \alpha - d \left( 1 - \frac{1}{q} \right), q \right) \hookrightarrow \text{Lip}(\eta - \theta, \alpha, q).
\]

Second, let $d(1-1/q) < 1$. Then, by Theorem 13.2, $\text{Lip}(\eta - \theta, \alpha, q)$ and $\text{Lip}^{1/q} (\eta, \alpha - d(1-1/q), q)$ are not comparable.

2) In the second case, $0 < q < 2$, $\gamma = d(1-1/q) \geq 1$, $\alpha \in \mathbb{N}$, that is, $d/(d-1) \leq q < 2$, we have that
\[
(13.5) \quad \text{Lip}(\eta, \alpha, q) \hookrightarrow \text{Lip} \left( \eta - \theta, \alpha - d \left( 1 - \frac{1}{q} \right), q \right) \hookrightarrow \text{Lip}(\eta - \theta, \alpha, q),
\]
which implies the desired embedding.

3) If $q = \infty$, $\gamma = d = 1$ or $\gamma = d \geq 2$, $\alpha \in \mathbb{N}$, then as in the case 2), embeddings (13.5) hold.

Combining the above embeddings, we complete the proof of (A).

(B) Let $1 < p < q \leq \infty$. As above, we see that from the Ulyanov–Marchaud inequality (10.16) and property (e) we obtain
\[
\omega_{\alpha-\gamma}(f, \delta) \lesssim \left( \int_0^\delta \left( \frac{\omega_\alpha(f, t)_p}{t^{d/(d-\frac{1}{q})}} \right)^{q_1} \frac{dt}{t} \right)^{\frac{1}{q_1}} + \frac{\omega_\alpha(f, \delta)_p}{\delta^\gamma} \begin{cases} 1, & \gamma > \theta; \\ \ln^{1/\gamma} \left( \frac{1}{\delta} + 1 \right), & \gamma = \theta; \\ \left( \frac{1}{\delta} \right)^{\theta - \gamma}, & \gamma < \theta. \end{cases}
\]
This implies that in all cases except the case $\gamma = \theta$ the classical Ulyanov inequality gives the same estimates as the sharp Ulyanov inequality (see (8.1) for $d = 1$ and (8.7) for $d \geq 2$). Thus, to prove (B), we compare the three following embeddings:

(1) \[ \text{Lip}(\eta, \alpha, p) \hookrightarrow \text{Lip}(\eta - \theta, \alpha, q), \]
which follows from the classical Ulyanov inequality (see (8.1) and (8.7) in the case $\gamma = 0$);

(2) \[ \text{Lip}(\eta, \alpha, p) \hookrightarrow \text{Lip}(\eta - \theta, \alpha - \theta, q), \quad p < q < \infty; \]
\[ \text{Lip}^{\frac{1}{q'}}(\eta - \theta, \alpha - \theta, q), \quad q = \infty, \]
which follows from the sharp Ulyanov inequalities (8.1) and (8.7) in the case $\gamma = \theta$;

(3) \[ \text{Lip}(\eta, \alpha, p) \hookrightarrow B_{q,p}^{\eta-\theta}, \]
which follows from Kolyada’s inequality in the case $1 < p < q < \infty$ and $\eta = \alpha$.

First, if $\eta < \alpha$, comparing embeddings (1) and (2) and taking into account the Marchaud inequality (4.13), we obtain
\[
\text{Lip}(\eta - \theta, \alpha - \theta, q) \equiv \text{Lip}(\eta - \theta, \alpha, q) \hookrightarrow \text{Lip}^{\frac{1}{q'}}(\eta - \theta, \alpha - \theta, q),
\]
which implies (13.2). Second, if $\eta = \alpha$ and $q = \infty$, the same argument and Theorem 13.2 yield (13.1). Finally, if $\eta = \alpha$ and $q < \infty$, we use the following embedding

$$W^\alpha_p \hookrightarrow B^{\alpha-\theta}_{q,p}, \quad 1 < p < q < \infty,$$

which easily follows from Kolyada’s inequality (see Theorem 11.3) and the fact that $W^\alpha_p$ coincides with the Lipschitz-type space $\{ f \in L_p : \omega_\alpha(f, \delta) = O(\delta^\alpha) \}$. Thus, combining the above embedding with the well-known interrelation between the fractional Riesz and Besov spaces $W^{\alpha-\theta}_p$ and $B^{\alpha-\theta}_{q,p}$ (see [91, p. 155]), we easily obtain (13.1) in the case $q < \infty$. □

We conclude this subsection with a remark on more general function spaces. Define

$$\text{Lip}(\omega(\cdot), l, X) := \left\{ f \in X(\mathbb{T}^d) : \omega_t(f, \delta)_X = O(\omega(\delta)), \quad \delta \to 0 \right\},$$

where $\omega(\cdot)$ is a non-decreasing function on $[0, 1]$ such that $\omega(\delta) \to 0$ as $\delta \to 0$ and $\delta^{1-l}\omega(\delta)$ is non-increasing (note that this class is natural class of majorant for fractional moduli of smoothness, see [98]) and $X(\mathbb{T}^d)$ is an appropriate function space.

For the Lebesgue spaces, Theorems 8.1 and 8.2 imply that

$$\text{Lip}(\omega(\cdot), \alpha + \gamma, L_p(\mathbb{T}^d)) \subset \text{Lip}(\tilde{\omega}(\cdot), \alpha, L_q(\mathbb{T}^d)), \quad 0 < p < q \leq \infty,$$

provided that

$$\left( \int_0^\delta \left( \frac{\omega(t)}{t^\gamma} \frac{1}{\sigma(t)} \right)^{\frac{1}{q_1}} \frac{dt}{t} \right)^{\frac{1}{q_1}} = O\left( \tilde{\omega}(t) \right),$$

where $\sigma(\cdot)$ is defined in Theorems 8.1 and 8.2.

This approach can also be used to obtain embedding theorems for general Calderón-type spaces

$$\Lambda^l(G, E) = \left\{ f \in G : \|f\|_G + \|\omega_l(f, \cdot)_G\|_E < \infty \right\},$$

introduced by Calderón [13]. Note that the classical Besov spaces $B^{\alpha}_{p,q}$ are a particular case of the Calderón spaces.

13.2. Embedding properties of Besov and Lipschitz-type spaces. Here, we study sharp embedding theorems between Lipschitz and logarithmic Lipschitz spaces (Theorem 13.2) and between Lipschitz and Besov spaces (Theorem 13.3). This problem is of interest in its own right.

**Theorem 13.2.** Let $\alpha, \beta > 0$, $1 \leq q \leq \infty$, $\tau = \left\{ \begin{array}{ll} \min(2, q), & 1 \leq q < \infty; \\ 1, & q = \infty, \end{array} \right.$ and $\varepsilon \in (0, 1/\tau)$. We have

1) $\text{Lip}(\alpha, \alpha + \beta, q)$ and $\text{Lip}(\varepsilon, \alpha, q)$ are subclasses of $\text{Lip}(1/\tau)(\alpha, \alpha, q)$;
2) $\text{Lip}(\alpha, \alpha + \beta, q)$ and $\text{Lip}(\varepsilon, \alpha, q)$ are not comparable.

Fig. 3 illustrates the embeddings in Theorem 13.2.
The proof of the first part follows from Marchaud inequality (4.13). The proof of the second part is based on the following three results.

**Proposition 13.1.** Let $1 \leq q, r \leq \infty$, $\alpha, \beta > 0$, and $\varepsilon > 0$. Then there exists a function $f_1 \in \text{Lip}^\varepsilon(\alpha, \alpha, q) \setminus \text{Lip}(\alpha, \alpha + \beta, r)$.

**Proof.** Let us define the lacunary series

$$f_1(x) = \sum_{n=1}^{\infty} c_n \cos \lambda_n x, \quad \lambda_{n+1}/\lambda_n \geq \lambda > 1.$$ 

Zygmund’s theorem (see [114, Ch. 8]) implies that $\|f_1\|_q \asymp \|f_1\|_2$ for $1 \leq q < \infty$ and $\|f_1\|_\infty \asymp \|\{c_n\}_{n=1}^{\infty}\|_{\ell_1}$ (see [89]). Taking

$$\lambda_n = 2^{2^n}, \quad c_n = 2^{n\varepsilon} \lambda_n^{-\alpha}$$

and choosing $s$ for any natural $N$ such that $\lambda_s \leq N < \lambda_{s+1}$, we get by the realization results (1.13) that

$$\omega_\alpha(f_1, 1/N)_q \lesssim N^{-\alpha} \left\| S_N^{(\alpha)}(f_1) \right\|_q + \|f_1 - S_N(f_1)\|_q$$

$$\lesssim N^{-\alpha} \left( \sum_{\lambda_\nu \leq N} c_\nu^2 \lambda_\nu^{2\alpha} \right)^{1/2} + \left( \sum_{\lambda_\nu \geq N} c_\nu^2 \right)^{1/2}$$

$$= N^{-\alpha} \left( \sum_{\nu=0}^{s} 2^{2\nu \varepsilon} \lambda_\nu^{2\alpha} \right)^{1/2} + \left( \sum_{\nu=s+1}^{\infty} \frac{2^{2\nu \varepsilon}}{\lambda_\nu^{2\alpha}} \right)^{1/2}$$

$$\lesssim N^{-\alpha} \ln^\varepsilon(N + 1),$$

i.e., $f_1 \in \text{Lip}^\varepsilon(\alpha, \alpha, q)$ for $q < \infty$. If $q = \infty$, similar calculation implies that

$$\omega_\alpha(f_1, 1/N)_q \lesssim N^{-\alpha} \left\| S_N^{(\alpha)}(f_1) \right\|_q + \|f_1 - S_N(f_1)\|_q$$

$$\lesssim N^{-\alpha} \sum_{\lambda_\nu \leq N} c_\nu \lambda_\nu^\alpha + \sum_{\lambda_\nu \geq N} c_\nu \lesssim N^{-\alpha} \ln^\varepsilon(N + 1).$$
On the other hand, Jackson’s inequality (4.1) for \( N = \lambda_s \) gives

\[
\omega_{\alpha+\beta}(f_1, 1/N) \gtrsim E_{N-1}(f_1) \gtrsim \|f_1 - S_{N-1}(f_1)\| \gtrsim \left( \sum_{\nu=s}^{\infty} 2^{2\nu} \right)^{1/2} \geq \frac{2^{\varepsilon}}{\lambda_s^\alpha}.
\]

Assuming that \( f_1 \in \text{Lip}(\alpha, \alpha + \beta, r) \), we arrive at the contradiction

\[
\frac{1}{\lambda_s^\alpha} \gtrsim \omega_{\alpha+\beta}(f_1, 1/N) \gtrsim \frac{2^{\varepsilon}}{\lambda_s^\alpha}.
\]

Thus, \( f_1 \notin \text{Lip}(\alpha, \alpha + \beta, r) \) for \( r < \infty \). If \( r = \infty \)

\[
\omega_{\alpha+\beta}(f_1, 1/\lambda_s) \gtrsim E_{\lambda_s-1}(f_1) \gtrsim c_{\lambda_s} = \frac{2^{\varepsilon}}{\lambda_s^\alpha}.
\]

\[\square\]

**Proposition 13.2.** Let \( 1 \leq q, r \leq \infty, \alpha, \beta > 0 \), and \( \varepsilon \in (0, 1/2) \) for \( r < \infty \) and \( \varepsilon \in (0, 1) \) for \( r = \infty \). Then there exists a function \( f_2 \in \text{Lip}(\alpha, \alpha + \beta, q) \setminus \text{Lip}^\varepsilon(\alpha, \alpha, r) \).

**Proof.** We consider

\[
f_2(x) = \sum_{n=1}^{\infty} c_n \cos 2^n x, \quad c_n = \frac{1}{2^{\alpha n}}.
\]

It is clear that \( f_2 \in L_q(\mathbb{T}), 1 \leq q \leq \infty \). Moreover, since \( \|f_2\|_q \asymp \|f_2\|_2 \) for \( 1 \leq q < \infty \), we get by (1.13) that for integer \( N \in [2^n, 2^{n+1}) \)

\[
\omega_{\alpha+\beta}(f_2, 1/N)_q \lesssim N^{-\alpha-\beta} \left\| S_N^{(\alpha+\beta)}(f_2) \right\|_q + \|f - S_N(f_2)\|_q
\]

\[
\lesssim 2^{-n(\alpha+\beta)} \left( \sum_{k=1}^{n} c_k^2 \right)^{1/2} + \left( \sum_{k=n+1}^{\infty} c_k^2 \right)^{1/2} \lesssim 2^{-\alpha n} \lesssim N^{-\alpha}.
\]

i.e., \( f_2 \in \text{Lip}(\alpha, \alpha + \beta, q) \) if \( q < \infty \). For \( q = \infty \) the same holds using \( \|f_2\|_\infty \asymp \|\{c_n\}_{n=1}^{\infty}\|_{\ell_1} \).

Moreover, for \( 1 \leq r < \infty \),

\[
\omega_{\alpha}(f_2, 1/N)_r \gtrsim N^{-\alpha} \left\| S_N^{(\alpha)}(f_2) \right\|_r \gtrsim 2^{-\alpha n} \left( \sum_{k=1}^{n} 1 \right)^{1/2} \gtrsim N^{-\alpha} \ln^{1/2}(N + 1)
\]

and

\[
\omega_{\alpha}(f_2, 1/N)_\infty \gtrsim N^{-\alpha} \left\| S_N^{(\alpha)}(f_2) \right\|_\infty \gtrsim N^{-\alpha} \ln(N + 1).
\]

Then \( f_2 \notin \text{Lip}^\varepsilon(\alpha, \alpha, r) \) for any \( \varepsilon \in (0, 1/2) \) when \( r < \infty \) and for any \( \varepsilon \in (0, 1) \) when \( r = \infty \). \[\square\]

**Proposition 13.3.** Let \( 1 < q \leq 2, \alpha, \beta > 0 \), and \( \varepsilon \in (0, 1/\tau) \). Then there exists a function \( f_3 \in \text{Lip}(\alpha, \alpha + \beta, q) \setminus \text{Lip}^\varepsilon(\alpha, \alpha, q) \).
Proof. Define

\[ f_3(x) = \sum_{n=1}^{\infty} a_n \cos nx, \quad a_n^q = \frac{1}{n^{\alpha q + q - 1}}. \]

Since

\[ \sum_{n=1}^{\infty} a_n^q n^{q-2} = \sum_{n=1}^{\infty} \frac{1}{n^{\alpha q + 1}} < \infty, \]

by the Hardy–Littlewood theorem (2.3) we get that \( f_3 \in L_q(\mathbb{T}) \). Moreover, realization (1.13) and Theorem 6.1 from [34] gives

\[ \omega_{\alpha + \beta}(f_3, 1/n) \asymp n^{-\alpha - \beta} \left( \sum_{k=1}^{n} a_{k}^{q} k^{(\alpha + \beta)q + q - 2} \right)^{1/q} + \left( \sum_{k=n+1}^{\infty} a_{k}^{q} k^{q - 2} \right)^{1/q} \lesssim n^{-\alpha}, \]

i.e., \( f_3 \in \text{Lip}(\alpha, \alpha + \beta, q) \).

On the other hand,

\[ \omega_{\alpha}(f_3, 1/n) \gtrsim n^{-\alpha} \left\| S_n(f_3)^{(\alpha)} \right\|_q \]

\[ \gtrsim n^{-\alpha} \left( \sum_{k=1}^{n} a_{k}^{q} k^{\alpha q + q - 2} \right)^{1/q} \gtrsim n^{-\alpha} \left( \sum_{k=1}^{n} \frac{1}{k} \right)^{1/q} \gtrsim \frac{\ln^{1/q}(n + 1)}{n^{\alpha}}, \]

which yields that \( f_3 \notin \text{Lip}^\varepsilon(\alpha, \alpha, q) \) for any \( \varepsilon \in (0, 1/\tau) \). \( \square \)

The next theorem shows the relationship for the spaces \( \text{Lip}(\alpha - \theta, \alpha - \theta, q) \) and \( B_{q,p}^{\alpha - \theta} \).

**Theorem 13.3.** Let \( 1 < p < q < \infty, \theta = d(1/p - 1/q), \) and \( \alpha > \theta \). We have

1) if \( p \leq \min(2, q), \) then \( B_{q,p}^{\alpha - \theta} \subseteq \text{Lip}(\alpha - \theta, \alpha - \theta, q); \)
2) if \( \min(2, q) < p, \) then the spaces \( \text{Lip}(\alpha - \theta, \alpha - \theta, q) \) and \( B_{q,p}^{\alpha - \theta} \) are not comparable.

The first part is a simple consequence of the Marchaud inequality (see Theorem 4.4). To prove the second part, we consider the following functions:

\[ f_1(x) = \sum_{\nu=1}^{\infty} a_{\nu} \cos \nu x, \quad a_{\nu} = \frac{1}{\nu^{1 - \frac{1}{p} + \alpha - \theta}(\ln(\nu + 1))^{\frac{1}{p}}}, \]

and

\[ f_2(x) = \sum_{\nu=1}^{\infty} a_{2^{\nu}} \cos 2^{\nu} x, \quad a_{2^{\nu}} = \frac{1}{2^{\nu(\alpha - \theta)}(\nu^{\frac{1}{2}})}, \]

for which we have,

\[ f_1 \in \text{Lip}(\alpha - \theta, \alpha - \theta, q) \setminus B_{q,p}^{\alpha - \theta} \text{ if } p \leq \min(2, q), \]

\[ f_1 \in \text{Lip}(\alpha - \theta, \alpha - \theta, q) \setminus B_{q,p}^{\alpha - \theta} \text{ and } f_2 \in B_{q,p}^{\alpha - \theta} \setminus \text{Lip}(\alpha - \theta, \alpha - \theta, q) \text{ if } \min(2, q) < p \]

(see [87]).
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