ON SEQUENCES WITH PRESCRIBED METRIC DISCREPANCY BEHAVIOR

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Abstract. An important result of H. Weyl states that for every sequence \((a_n)_{n \geq 1}\) of distinct positive integers the sequence of fractional parts of \((a_n \alpha)_{n \geq 1}\) is uniformly distributed modulo one for almost all \(\alpha\). However, in general it is a very hard problem to calculate the precise order of convergence of the discrepancy \(D_N\) of \((\{a_n \alpha\})_{n \geq 1}\) for almost all \(\alpha\).

By a result of R. C. Baker this discrepancy always satisfies \(ND_N = O\left(N^{1/2+\varepsilon}\right)\) for almost all \(\alpha\) and all \(\varepsilon > 0\). In the present note for arbitrary \(\gamma \in (0, \frac{1}{2}]\) we construct a sequence \((a_n)_{n \geq 1}\) such that for almost all \(\alpha\) we have \(ND_N = O\left(N^{\gamma}\right)\) and \(ND_N = \Omega\left(N^{\gamma-\varepsilon}\right)\) for all \(\varepsilon > 0\), thereby proving that any prescribed metric discrepancy behavior within the admissible range can actually be realized.

1. Introduction

H. Weyl [12] proved that for every sequence \((a_n)_{n \geq 1}\) of distinct positive integers the sequence \((\{a_n \alpha\})_{n \geq 1}\) is uniformly distributed modulo one for almost all reals \(\alpha\). Here, and in the sequel, \(\{\cdot\}\) denotes the fractional part function. The speed of convergence towards the uniform distribution is measured in terms of the discrepancy, which – for an arbitrary sequence \((x_n)_{n \geq 1}\) of points in \([0, 1)\) – is defined by

\[
D_N = D_N(x_1, \ldots, x_N) = \sup_{0 \leq a < b \leq 1} \left| \frac{A_N([a, b])}{N} - (b - a) \right|
\]

where \(A_N([a, b]) := \# \{1 \leq n \leq N \mid x_n \in [a, b)\}\). For a given sequence \((a_n)_{n \geq 1}\) it is usually a very hard and challenging problem to give sharp estimates for the discrepancy \(D_N\) of \((\{a_n \alpha\})_{n \geq 1}\) valid for almost all \(\alpha\). For general background on uniform distribution theory and discrepancy theory see for example the monographs [6, 9].

A famous result of R. C. Baker [3] states that for any sequence \((a_n)_{n \geq 1}\) of distinct positive integers for the discrepancy \(D_N\) of \((\{a_n \alpha\})_{n \geq 1}\) we have

\[
(1) \quad ND_N = O\left(N^{1/2} (\log N)^{3/2+\varepsilon}\right)
\]

as \(N \to \infty\) for almost all \(\alpha\) and for all \(\varepsilon > 0\).

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Note that (1) is a general upper bound which holds for all sequences \((a_n)_{n \geq 1}\); however, for some specific sequences the precise typical order of decay of the discrepancy of \(\{a_n \alpha\}_{n \geq 1}\) can differ significantly from the upper bound in (1). The fact that (1) is essentially optimal (apart from logarithmic factors) as a general result covering all possible sequences can for example be seen by considering so-called lacunary sequences \((a_n)_{n \geq 1}\), i.e., sequences for which \(\frac{a_{n+1}}{a_n} \geq 1 + \delta\) for a fixed \(\delta > 0\) and all \(n\) large enough. In this case for \(D_N\) we have

\[
\frac{1}{4\sqrt{2}} \leq \limsup_{N \to \infty} \frac{ND_N}{\sqrt{2N \log \log N}} \leq c_\delta
\]

for almost all \(\alpha\) (see [10]), which shows that the exponent \(1/2\) of \(N\) on the right-hand side of (1) cannot be reduced for this type of sequence. For more information concerning possible improvements of the logarithmic factor in (1), see [5].

Quite recently in [2] it was shown that also for a large class of sequences with polynomial growth behavior Baker’s result is essentially best possible. For example, the following result was shown there: Let \(f \in \mathbb{Z}[x]\) be a polynomial of degree larger or equal to 2. Then for the discrepancy \(D_N\) of \(\{f(n)\alpha\}_{n \geq 1}\) for almost all \(\alpha\) and for all \(\varepsilon > 0\) we have

\[
ND_N = \Omega \left( N^{\frac{1}{2} - \varepsilon} \right).
\]

On the other hand there is the classical example of the Kronecker sequence, i.e., \(a_n = n\), which shows that the actual metric discrepancy behavior of \(\{a_n \alpha\}_{n \geq 1}\) can differ vastly from the general upper bound in (1). Namely, for the discrepancy of the sequence \(\{n \alpha\}_{n \geq 1}\) for almost all \(\alpha\) and for all \(\varepsilon > 0\) we have

\[
ND_N = \mathcal{O} \left( \log N \left( \log \log N \right)^{1 + \varepsilon} \right),
\]

which follows from classical results of Khintchine in the metric theory of continued fractions (for even more precise results, see [II]). The estimate (2) of course also holds for \(a_n = f(n)\) with \(f \in \mathbb{Z}[x]\) of degree 1. In [2] further examples for \((a_n)_{n \geq 1}\) were given, where \((a_n)_{n \geq 1}\) has polynomial growth behavior of arbitrary degree, such that for the discrepancy of \(\{a_n \alpha\}_{n \geq 1}\) we have

\[
ND_N = \mathcal{O} \left( \left( \log N \right)^{2 + \varepsilon} \right)
\]

for almost all \(\alpha\) and for all \(\varepsilon > 0\); see there for more details.

These results may seduce to the hypothesis that for all choices of \((a_n)_{n \geq 1}\) for the discrepancy of \(\{a_n \alpha\}_{n \geq 1}\) for almost all \(\alpha\) we either have

\[
ND_N = \mathcal{O} \left( N^\varepsilon \right)
\]

or

\[
ND_N = \Omega \left( N^{\frac{1}{2} - \varepsilon} \right).
\]

This hypothesis, however, is wrong as was shown in [I]: Let \((a_n)_{n \geq 1}\) be the sequence of those positive integers with an even sum of digits in base 2, sorted in increasing order; that
is \((a_n)_{n \geq 1} = (3, 5, 6, 9, 10, \ldots)\). Then for the discrepancy of \(\{a_n\alpha\}_{n \geq 1}\) for almost all \(\alpha\) we have
\[
ND_N = O\left(N^{\kappa + \varepsilon}\right)
\]
and
\[
ND_N = \Omega\left(N^{\kappa - \varepsilon}\right)
\]
for all \(\varepsilon > 0\), where \(\kappa\) is a constant with \(\kappa \approx 0.404\). Interestingly, the precise value of \(\kappa\) is unknown; see [8] for the background.

The aim of the present paper is to show that the example above is not a singular counter-example, but that indeed “everything” between (3) and (4) is possible. More precisely, we will show the following theorem.

**Theorem 1.** Let \(0 < \gamma \leq \frac{1}{2}\). Then there exists a strictly increasing sequence \((a_n)_{n \geq 1}\) of positive integers such that for the discrepancy of the sequence \(\{a_n\alpha\}_{n \geq 1}\) for almost all \(\alpha\) we have
\[
ND_N = O\left(N^{\gamma}\right)
\]
and
\[
ND_N = \Omega\left(N^{\gamma - \varepsilon}\right)
\]
for all \(\varepsilon > 0\).

2. Proof of the Theorem

For the proof we need an auxiliary result which easily follows from classical work of H. Behnke [4].

**Lemma 1.** Let \((e_k)_{k \geq 1}\) be a strictly increasing sequence of positive integers. Let \(\varepsilon > 0\). Then for almost all \(\alpha\) there is a constant \(K(\alpha, \varepsilon) > 0\) such that for all \(r \in \mathbb{N}\) there exist \(M_r \leq e_r\) such that for the discrepancy of the sequence \(\{n^2\alpha\}_{n \geq 1}\) we have
\[
M_r D_{M_r} \geq K(\alpha, \varepsilon) \sqrt{\frac{e_r}{(\log e_r)^{1+\varepsilon}}}.
\]

**Proof.** For \(\alpha \in \mathbb{R}\) let \(a_k(\alpha)\) denote the \(k\)-th continued fraction coefficient in the continued fraction expansion of \(\alpha\). Then it is well-known that for almost all \(\alpha\) we have \(a_k(\alpha) = O(k^{1+\varepsilon})\) for all \(\varepsilon > 0\). Let \(\varepsilon > 0\) be given and let \(\alpha\) and \(c(\alpha, \varepsilon)\) be such that
\[
a_k(\alpha) \leq c(\alpha, \varepsilon) k^{1+\varepsilon}
\]
for all \(k \geq 1\).

Let \(q_l\) the \(l\)-th best approximation denominator of \(\alpha\). Then
\[
q_{l+1} \leq (c(\alpha, \varepsilon) l^{1+\varepsilon} + 1) q_l.
\]
Since \(q_l \geq 2^{\frac{l}{2}}\) in any case, we have \(l \leq \frac{2 \log q_l}{\log 2}\), and we obtain
\[
q_{l+1} \leq c_1(\alpha, \varepsilon) q_l (\log q_l)^{1+\varepsilon},
\]
for an appropriate constant \( c_1(\alpha, \varepsilon) \). In [4] it was shown in Satz XVII that for every real \( \alpha \) we have
\[
\left| \sum_{n=1}^{N} e^{2\pi i n^2 \alpha} \right| = \Omega \left( \frac{1}{N^{1/2}} \right).
\]
Indeed, if we follow the proof of this theorem we find that even the following was shown:
For every \( \alpha \) and for every best approximation denominator \( q_l \) of \( \alpha \) there exists an \( Y_l \) such that
\[
\left| \sum_{n=1}^{Y_l} e^{2\pi i n^2 \alpha} \right| \geq c_{abs} \sqrt{q_l}.
\]
Here \( c_{abs} \) is a positive absolute constant (not depending on \( \alpha \)).

Let now \( r \in \mathbb{N} \) be given and let \( l \) be such that \( q_l \leq e_r < q_{l+1} \), and let \( M_r := Y_l \) from above. Then by (6) and (7) we obtain, for an appropriate constant \( c_2(\alpha, \varepsilon) \),
\[
\left| \sum_{n=1}^{M_r} e^{2\pi i n^2 \alpha} \right| \geq c_{abs} \sqrt{q_l} \geq c_2(\alpha, \varepsilon) \sqrt{e_r} \geq c_2(\alpha, \varepsilon) \sqrt{e_l} \geq c_2(\alpha, \varepsilon) \sqrt{e_l} \geq c_2(\alpha, \varepsilon) \sqrt{e_l}.
\]
By the fact that (see Chapter 2, Corollary 5.1 of [9])
\[
M_r D_{M_r} \geq \frac{1}{4} \left| \sum_{n=1}^{M_r} e^{2\pi i n^2 \alpha} \right|,
\]
which is a special case of Koksma’s inequality, the result follows.

Now we are ready to prove the main theorem.

**Proof of the Theorem.** Let \((m_j)_{j \geq 1}\) and \((e_j)_{j \geq 1}\) be two strictly increasing sequences of positive integers, which will be determined later. We will consider the following strictly increasing sequence of positive integers, which will be our sequence \((a_n)_{n \geq 1}\):
\[
1, 2, 3, \ldots, m_1,
\]
\[
m_1 + 1^2, m_1 + 2^2, m_1 + 3^2, m_1 + 4^2, \ldots, m_1 + e_1^2 =: B_1
\]
\[
B_1 + 1, B_1 + 2, B_1 + 3, \ldots, B_1 + m_2, =: A_2
\]
\[
A_2 + 1^2, A_2 + 2^2, A_2 + 3^2, A_2 + 4^2, \ldots, A_2 + e_2^2, =: B_2
\]
\[
B_2 + 1, B_2 + 2, B_2 + 3, \ldots, B_2 + m_3, =: A_3
\]
ON SEQUENCES WITH PRESCRIBED METRIC DISCREPANCY BEHAVIOR

A₃ + 1², A₃ + 2², A₃ + 3², A₃ + 4², ..., A₃ + e₃²,

:= B₃.

Furthermore, let

\[ F_s := \sum_{i=1}^{s} m_i + \sum_{i=1}^{s-1} e_i \quad \text{and} \quad E_s := \sum_{i=1}^{s} m_i + \sum_{i=1}^{s} e_i. \]

The sequence \((a_n)_{n \geq 1}\) is constructed in such a way that it contains sections where it grows like \((n)_{n \geq 1}\) as well as sections where it grows like \((n^2)_{n \geq 1}\). By this construction we exploit both the strong upper bounds for the discrepancy of \((\{n\alpha\})_{n \geq 1}\) and the strong lower bounds for the discrepancy of \((\{n^2\alpha\})_{n \geq 1}\), in an appropriately balanced way, in order to obtain the desired discrepancy behavior of the sequence \((a_n\alpha)_{n \geq 1}\). In our argument we will repeatedly make use of the fact that

\[ D_N(x_1, \ldots, x_N) = D_N(\{x_1 + \beta\}, \ldots, \{x_N + \beta\}) \]

for arbitrary \(x_1, \ldots, x_N \in [0,1]\) and \(\beta \in \mathbb{R}\), which allows us to transfer the discrepancy bounds for \((\{n\alpha\})_{n \geq 1}\) and \((\{n^2\alpha\})_{n \geq 1}\) directly to the shifted sequences \((\{(M + n)\alpha\})_{n \geq 1}\) and \((\{(M + n^2)\alpha\})_{n \geq 1}\) for some integer \(M\).

Let \(\alpha\) be such that it satisfies (5) with \(\varepsilon = \frac{1}{2}\). Then it is also well-known (see for example [9]) that for the discrepancy \(D_N\) of the sequence \((\{n\alpha\})_{n \geq 1}\) we have

\[ ND_N \leq \bar{c}_1(\alpha) (\log N)^{\frac{1}{2}} \]

for all \(N \geq 2\).

By the above mentioned general result of Baker, that is by (11), we know that for almost all \(\alpha\) for the discrepancy \(D_N\) of the sequence \((\{n^2\alpha\})_{n \geq 1}\) we have

\[ ND_N \leq c_3(\alpha, \varepsilon) N^{\frac{1}{2}} (\log N)^{\frac{3}{4} + \varepsilon} \]

for all \(\varepsilon > 0\) and for all \(N \geq 2\), for an appropriate constant \(c_3(\alpha, \varepsilon)\). Actually an even slightly sharper estimate was given for the special case of the sequence \((\{n^2\alpha\})_{n \geq 1}\) by Fiedler, Jurkat and Körner in [7], who proved that

\[ ND_N \leq c_4(\alpha, \varepsilon) N^{\frac{1}{2}} (\log N)^{\frac{1}{4} + \varepsilon} \]

for almost all \(\alpha\) and for all \(\varepsilon > 0\) and all \(N \geq 2\).

Assume that \(\alpha\) satisfies (10) with \(\varepsilon = \frac{1}{8}\). Then

\[ ND_N \leq \bar{c}_2(\alpha) N^{\frac{1}{2}} (\log N)^{\frac{3}{8}} \]
for all $N \geq 2$. Now for such $\alpha$ and for arbitrary $N$ we consider the discrepancy $D_N$ of the sequence $\{a_n\alpha\}_{n \geq 1}$.

**Case 1.**
Let $N = F_l$ for some $l$. Then $ND_N \leq E_{l-1}D_{E_{l-1}} + (N - E_{l-1})D_{E_{l-1},F_l}$, where $D_{x,y}$ denotes the discrepancy of the point set $\{a_n\alpha\}_{n=x+1,x+2,...,y}$. Hence by (8), (9) and by the trivial estimate $D_{B_l-1} \leq 1$ we have

$$ND_N \leq E_{l-1} + \mathfrak{c}_1(\alpha) (\log m_l)^{\frac{3}{2}}$$

$$\leq 2 (\log m_l)^2$$

$$\leq 2 (\log N)^2$$

for all $l$ large enough, provided that (condition (i)) $m_l$ is chosen such that $(\log m_l)^2 \geq E_{l-1}$.

**Case 2.**
Let $F_l < N \leq E_l$ for some $l$. Then by Case 1 and by (8) and (11) we have for $l$ large enough that

$$ND_N \leq F_lD_{F_l} + (N - F_l)D_{F_l,N}$$

$$\leq 2 (\log F_l)^2 + \mathfrak{c}_2(\alpha) (N - F_l)^{\frac{1}{2}} (\log (N - F_l))^\frac{3}{2}.$$  

Note that $0 < N - F_l < \epsilon_l$.

We choose (condition (ii))

$$e_l := \left\lceil \frac{F_l^{2\gamma}}{\log (F_l^{2\gamma})} \right\rceil.$$  

Note that conditions (i) and (ii) do not depend on $\alpha$. Now assume that $l$ is so large that $2 (\log F_l)^2 < \frac{F_l^{2\gamma}}{2}$. Then

$$\frac{F_l^{2\gamma}}{2} \leq 2 (\log F_l)^2 + (e_l \log e_l)^{\frac{1}{2}} \leq 2F_l^{\gamma}$$

and (note that $\gamma \leq \frac{1}{2}$)

$$F_l < N \leq E_l = F_l + e_l \leq 2F_l.$$  

Hence

$$ND_N \leq \max (1, \mathfrak{c}_2(\alpha)) 2F_l^{\gamma}$$

$$\leq \max (1, \mathfrak{c}_2(\alpha)) 2^{1+\gamma}N^\gamma.$$  

**Case 3.**
Let $E_l < N < F_{l+1}$ for some $l$. Then by Case 2 and by (8) and (9) we have

$$ND_N \leq E_lD_{E_l} + (N - E_l)D_{E_l,N}$$

$$\leq E_l^{\gamma} + \mathfrak{c}_1(\alpha) (\log (N - E_l))^2$$
ON SEQUENCES WITH PRESCRIBED METRIC DISCREPANCY BEHAVIOR

\[
\leq 2N^\gamma
\]

for \( N \) large enough.

It remains to show that for every \( \varepsilon > 0 \) we have \( ND_N \geq N^{\gamma - \varepsilon} \) for infinitely many \( N \). Let \( l \) be given and let \( M_l \leq e_l \) with the properties given in Lemma \([1]\). Let \( N := F_l + M_l \). Then by Lemma \([1]\) Case 1, \([8]\), \([12]\) and \([13]\) for \( l \) large enough we have

\[
ND_N \geq M_l D_{F_l,N} - F_l D_{F_l} \\
\geq K(\alpha, \varepsilon) \sqrt{\frac{e_l}{(\log e_l)^{1+\varepsilon}}} - 2 (\log m_l)^2 \\
\geq \frac{F_l^\gamma}{(\log F_l)^3} \\
\geq N^{\gamma-\varepsilon}.
\]

This proves the theorem. \( \square \)

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