Global solutions for the relativistic Boltzmann equation in the homogeneous case on the Minkowski space-time

Norbert Noutchegueme ; Mesmin Erick Tetsadjio T.
Department of Mathematics, Faculty of Science, University of Yaounde I, PO Box 812, Yaounde, Cameroon
e-mail : nnoutch@uycdn.uninet.cm, nnoutch@justice.com

Abstract

We prove, for the relativistic Boltzmann equation in the homogeneous case, on the Minkowski space-time, a global in time existence and uniqueness theorem. The method we develop extends to the cases of some curved space-times such as the flat Robertson-Walker space-time and some Bianchi type I space-times.

1 Introduction

The relativistic Boltzmann equation is one of the basic equations of the relativistic kinetic theory. An essential tool to describe the dynamics of a kind of fast moving particles subject to mutual collisions is their distribution function, denoted by $f$, and that is a non-negative real-valued function, depending on the position and the 4-momentum of the particles; $f$ is physically interpreted as the "probability of the presence density" of the particles in a given volume, during their collisional evolution. In the case of binary and elastic collisions, the distribution function $f$ is determined by the Boltzmann equation, through a non-linear operator called the "collision operator", that describes, at each point where two particles collide with each other, the effects of the behavior imposed by the collision on the distribution function, taking into account the fact that the momentum of each particle is not the same, before and after the collision, only the sum of their two momenta being preserved.

Several authors proved global existence theorems for the non-relativistic Boltzmann equation and the original result is due to Carleman, in [1]. R. Illner and M. Shinbrot proved that result in [5], in the case of small initial data and without assuming symmetry. An analogous result in the relativistic case is not known. Several authors proved local existence theorems for the
relativistic Boltzmann equation, considering this equation alone, as Bichteler in [2], or coupling it to other fields equations, as Bancel and Choquet-Bruhat in [3] and [4]. More interesting would certainly be to look for a global existence theorem for the relativistic Boltzmann equation. R. T. Glassey and W. Strauss obtained such result in [6] for data near to that of an equilibrium solution with non-zero density. It would be interesting to extend to the relativistic case the global existence theorem in the case of small initial data. That was one of the objectives of Mucha, in [8] and [9], who studies the relativistic Boltzmann equation coupled to Einstein’s equations, for a flat Robertson-Walker space-time. Unfortunately, several points in that work are far from clear, namely:

- the formulation of the relativistic Boltzmann equation using formulae that are valid only in the non-relativistic case;
- the use of the fixed point theorem without specifying which complete metric space is mapped into itself by a contracting map;
- the use of an approximating operator $Q_n$ with unspecified domain, to approximate the collision operator;
- the use of an important property of the collision operator, that requires the symmetry of its kernel, without specifying this assumption used by Bancel, in [3].

In this work, we consider the relativistic Boltzmann equation in the homogeneous case, which means that the distribution function depends only on the time $t$ and the 4-momentum $p$ of the particles. Such cases are very useful for instance in cosmology. In order to simplify the proofs, we take as background space-time, the Minkowski space-time, but the techniques we develop easily extend to the case where the background space-time is the flat Robertson-Walker space-time or, some Bianchi type I space-time. We simplify and correct the method followed by Mucha, in [8], and we prove a global existence theorem for the relativistic Boltzmann equation whose right hand side is defined by the non-linear collision operator, whereas the left hand side is defined by a linear operator. Our method consists of:

1) constructing an approximation operator $Q_n$, with suitable properties, that will approximate the collision operator, in a suitable function space;
2) solving by usual methods, the approximating equation obtained by replacing the collision operator in the Boltzmann equation by $Q_n$, obtain a global solution $f_n$ and prove that $f_n$ converges, in a suitable function space, to a global solution of the relativistic Boltzmann equation.

We begin, first of all, by giving, following Glassey in [7], the correct formulation of the relativistic Boltzmann equation. Next, we construct a suitable framework and an approximating operator $Q_n$, that, in contrast to that defined in [3], has a clear domain, and for which we establish properties, under the symmetry assumption on the kernel of the collision operator. We then construct a complete metric space and a contracting map of this space onto itself, that gives, using the Banach fixed point theorem, the
solution \( f_n \) of the Cauchy problem for the approximating equation, provided that the initial data is sufficiently small, and we show that the solution \( f_n \) is global. Moreover, we obtain a very simple estimation of the global solution \( f_n \) by the initial data that leads to a considerable simplification of the proofs given in [8] by showing that there is no need to use, as the author of that paper does, neither the infinite products, nor the divergence of the numerical series \( \sum \frac{1}{n} \), to prove the convergence of \( f_n \) to a global solution of the relativistic Boltzmann equation. Finally, we prove that, if the Cauchy data are sufficiently small, then the Cauchy problem for the relativistic Boltzmann equation in the homogeneous case, on the Minkowski space-time, has a unique global solution \( f \) in a suitable function space, that admits a very simple estimation by the initial data.

The paper is organized as follows: in section 2, we specify the notations, we define the function spaces, and we introduce the relativistic Boltzmann equation and the collision operator. We end this section by a sketch of the strategy adopted to solve the equation. In section 3, we give the properties of the collision operator, some preliminary results, and we construct the sequence of the approximating operators. In Section 4 devoted to the global existence theorem, we solve the approximating equations, and we prove the global in time existence theorem.

2 Notations, function spaces and the relativistic Boltzmann equation

2.1 Notations and function spaces

We denote the Minkowski space-time by \((\mathbb{R}^4, \eta)\) with \(\eta = Diag(1, -1, -1, -1)\). A Greek index varies from 0 to 3 and a Latin index from 1 to 3. We adopt the Einstein summation convention:

\[
A_\alpha B^\alpha = \sum_{\alpha=0}^{3} A_\alpha B^\alpha
\]

For \(x = (x^\alpha) = (x^0, x^i) \in \mathbb{R}^4\), we set:

\[
x^0 = t \quad ; \quad \bar{x} = (x^i) \quad ; \quad |\bar{x}| = \left( \sum_{i=1}^{3} (x^i)^2 \right)^{1/2}.
\]

The framework we will refer to is \(L^1(\mathbb{R}^3)\) whose norm is denoted \(\| . \|\).

For \(r \in \mathbb{R}, r > 0\), we set:

\[
X_r = \{ f \in L^1(\mathbb{R}^3), f \geq 0 \text{ a.e.}, \| f \| \leq r \}.
\]

Endowed with the distance induced by \(L^1(\mathbb{R}^3)\), \(X_r\) is a complete and connected metric space.
Let $I \subset \mathbb{R}$ be a real interval; we denote by $C[I; L^1(\mathbb{R}^3)]$ the Banach space

$$C[I; L^1(\mathbb{R}^3)] = \{ f : I \rightarrow L^1(\mathbb{R}^3), f \text{ continuous and bounded} \}$$

endowed with the norm:

$$|| f || = \sup_{t \in I} || f(t) || .$$

We set:

$$C[I; X_r] = \{ f \in C[I, L^1(\mathbb{R}^3)], f(t) \in X_r, \forall t \in I \} .$$

Endowed with the distance induced by the distance $d(f, g) = || f - g ||$ defined by the norm $|| \cdot ||$, $C[I; X_r]$ is a complete metric space. We look for the distribution function $f$ of one kind of particles with rest mass $m = 1$, in a collisional evolution, in the space-time $(\mathbb{R}^4, \eta)$; $f$ is a non-negative real-valued function depending on the position $x = (t, \bar{x})$ and the momentum $p = (p^0, \bar{p}) \in \mathbb{R}^4$ of the particles. $f$ is then defined on the tangent bundle $T(\mathbb{R}^4)$ that is a 8-dimensional manifold with local coordinates $(x^\alpha, p^a)$, i.e.

$$f : T(\mathbb{R}^4) \rightarrow \mathbb{R}_+ ; (x^\alpha, p^a) \mapsto f(x^\alpha, p^a).$$

Now the particles are required to move only on the future sheet of the mass shell, whose equation is $\eta(p, p) = 1$, i.e.

$$(p^0)^2 - \sum_{i=1}^{3} (p^i)^2 = 1, \quad p^0 \geq 0$$

or equivalently:

$$p^0 = \sqrt{1 + |\bar{p}|^2}. \quad (2.1)$$

Hence, $f$ is in fact defined on the 7-dimensional subbundle of $T(\mathbb{R}^4)$ defined by (2.1), and whose local coordinates are $x^\alpha, p^i$. Notice that, in the case of the flat Minkowski space we consider, $\bar{p} = (p^i)$ also stands for the spatial velocity $v = (v^i)$ of the particles.

2.2 The relativistic Boltzmann equation

The relativistic Boltzmann equation in $f$, on the flat Minkowski space-time can be written

$$p^\alpha \frac{\partial f}{\partial x^\alpha} = Q(f, f) \quad (2.2)$$

where $Q$ is the collision operator we now introduce. In the case of binary and elastic collisions, $Q$ is defined as follows, $p$ and $q$ standing for the momenta of two particles before their collision, $p'$ and $q'$ for their momenta after the collision, and where $f$ and $g$ are two functions on $\mathbb{R}^3$: 4
1) \[ Q(f, g) = Q^+(f, g) - Q^-(f, g) \] (2.3)

with

2) \[ Q^+(f, g) = \int_{\mathbb{R}^3} \frac{d\bar{q}}{q_0} \int_{S^2} f(\bar{p}')g(\bar{q}')S(\bar{p}, \bar{q}, \bar{p}', \bar{q}')d\omega \] (2.4)

3) \[ Q^-(f, g) = \int_{\mathbb{R}^3} \frac{d\bar{q}}{q_0} \int_{S^2} f(\bar{p})g(\bar{q})S(\bar{p}, \bar{q}, \bar{p}', \bar{q}')d\omega \] (2.5)

in which:

4) \( S^2 \) is the unit sphere of \( \mathbb{R}^3 \) whose area element is denoted \( d\omega \), and

5) \( S \) is a non-negative real-valued function of the indicated arguments, called the kernel of the collision operator \( Q \), or the cross section of the collisions. We suppose that \( S \) is bounded i.e.

\[ 0 \leq S \leq C_1 \] (2.6)

where \( C_1 \) is a positive constant, and we require for \( S \) the symmetry assumption:

\[ S(\bar{p}, \bar{q}, \bar{p}', \bar{q}') = S(\bar{p}', \bar{q}', \bar{p}, \bar{q}) \] (2.7)

6) As consequences of the conservation law \( p + q = p' + q' \), that splits into

\[ p^0 + q^0 = p'^0 + q'^0 \] (2.8)

and

\[ \bar{p} + \bar{q} = \bar{p}' + \bar{q}' \] (2.9)

i) we have, using (2.1) and (2.8):

\[ \sqrt{1 + |\bar{p}|^2} + \sqrt{1 + |\bar{q}|^2} = \sqrt{1 + |\bar{p}'|^2} + \sqrt{1 + |\bar{q}'|^2} \]

which expresses the conservation of the quantity

\[ e = \sqrt{1 + |\bar{p}|^2} + \sqrt{1 + |\bar{q}|^2} \] (2.10)

called the energy of the unit rest mass particles,

ii) (2.9) can be expressed, following Glassey in [7], by setting

\[ \begin{cases} 
\bar{p}' = \bar{p} + a(\bar{p}, \bar{q}, \omega) \\
\bar{q}' = \bar{q} - a(\bar{p}, \bar{q}, \omega) \end{cases}, \omega \in S^2 \] (2.11)
where \( a \) is given by

\[
a(\bar{p}, \bar{q}, \omega) = \frac{2e(\hat{q} - \hat{\bar{p}})\sqrt{1 + |\bar{p}|^2}\sqrt{1 + |\bar{q}|^2}}{e^2 - [\omega.(\bar{p} + \bar{q})]^2} \tag{2.12}
\]

where

\[
\hat{\bar{p}} = \frac{\bar{p}}{\sqrt{1 + |\bar{p}|^2}} = \frac{\bar{p}}{p^0}
\]

with \( e \) defined by \( (2.10) \), the dot denoting the usual scalar product in \( \mathbb{R}^3 \), whereas the jacobian of the change of variables \((\bar{p}, \bar{q}) \rightarrow (\bar{p}', \bar{q}')\) defined by \( (2.11) \) is

\[
\frac{\partial(\bar{p}', \bar{q}')}{\partial(\bar{p}, \bar{q})} = -\frac{\sqrt{1 + |p'|^2}\sqrt{1 + |q'|^2}}{\sqrt{1 + |\bar{p}|^2}\sqrt{1 + |\bar{q}|^2}}. \tag{2.13}
\]

It then appears clearly that, since \( \bar{p}' \) and \( \bar{q}' \) can be expressed through \( (2.11) \) as functions of \( \bar{p}, \bar{q} \) and \( \omega \), and by \( (2.1) \), \( q^0 \) can be expressed as a function of \( \bar{q} \), the integrals \( (2.4) \) and \( (2.5) \) that are taken with respect to \( \bar{q} \) and \( \omega \), give two functions \( Q^+(f, g) \), \( Q^-(f, g) \) of the single variable \( \bar{p} \). In practice, we will consider functions \( f(t) \) defined on \( \mathbb{R}^3 \) with :

\[
f: I \times \mathbb{R}^3 \rightarrow \mathbb{R}; \quad (t, \bar{p}) \mapsto f(t, \bar{p})
\]

in which we set, for \( t \) fixed in the real interval \( I \) :

\[
f(t)(\bar{p}) = f(t, \bar{p}), \quad \bar{p} \in \mathbb{R}^3.
\]

Also notice the important fact that, since \( \bar{p} \) is not a variable in the integral \( (2.5) \) that defines \( Q^- \), the operator \( Q^- \) has the useful property that :

\[
Q^-(f, g) = fQ^-(1, g) \tag{2.14}
\]

**Remark 2.1** The expression \( a(\bar{p}, \bar{q}, \omega) = \omega.(\bar{p} - \bar{q}) \) used by the author in \( [8] \) and \( [9] \) is valid only in the non-relativistic case, for very low velocities; it should not have been used in the full relativistic case where fast moving particles, with arbitrary high velocities that could be not negligible compared to the speed of the light, are considered.

We consider the relativistic Boltzmann equation \( (2.2) \) in the homogeneous case, i.e \( f \) does not depend on \( \bar{x} = (x^i) \). In this case, \( (2.2) \) can be written :

\[
\frac{df}{dt} = \frac{1}{p^0}Q(f, f) \tag{2.15}
\]

**Remark 2.2** Equations \( (2.2) \) and \( (2.15) \) do not contain the derivatives of \( f \) with respect to \( p^i \). This implies that the dependence of \( f \) on \( \bar{p} \) will be understood only through the functional framework chosen. This is where the difference occurs between the flat case we consider, and the case where
the background space-time could be the flat Robertson-Walker space-time or, some Bianchi type I space-time, that are curved space-times. For example, in the flat Robertson-Walker space-time \((\mathbb{R}^4, g)\) where the metric \(g\) is defined by the line element:

\[
ds^2 = dt^2 - a^2(t)((dx^1)^2 + (dx^2)^2 + (dx^3)^2)
\]

where \(a > 0\) is a given regular function of \(t\), the homogeneous relativistic Boltzmann equation depends explicitly on the metric and can be written:

\[
\frac{\partial f}{\partial t} - 2\frac{\dot{a}}{a}p^i \frac{\partial f}{\partial p^i} = \frac{1}{p^0} Q'(f, f)
\]

(2.16)

where \(Q' = a^3Q\). Solving (2.16) is equivalent to solving the associated characteristic system that can be written, taking \(t\) as parameter:

\[
\frac{dp^i}{dt} = -2\frac{\dot{a}}{a} p^i; \quad \frac{df}{dt} = \frac{1}{p^0} Q'(f, f)
\]

(2.17)

The equations in \((p^i)\) being trivial, (2.17) shows that the real problem to solve is analogous to (2.16). We find similar results in the case where the background space-time is a Bianchi type I space-time whose line element can be written:

\[
ds^2 = dt^2 - a^2(t)((dx^1)^2 - b^2(t)((dx^2)^2 + (dx^3)^2)
\]

with \(a > 0, b > 0\) given regular functions of \(t\), and that reduces to the Robertson-Walker line element, when \(a = b\).

We end this section by indicating the strategy we adopt to solve (2.16). We consider the following Cauchy problem for (2.16):

\[
\begin{cases}
\frac{df}{dt} = \frac{1}{p^0} Q(f, f) \\
 f(0) = f_0
\end{cases}
\]

(2.18)

where \(f_0\) stands for the initial data. (2.16) is equivalent the following integral equation in \(f\):

\[
f(t, y) = f_0(y) + \int_0^t \frac{1}{p^0} Q(f, f)(s, y) ds
\]

(2.19)

where \(y \in \mathbb{R}^3\). Our strategy will then consist of:
1) constructing an approximating operator \(Q_n\), with suitable properties that will converge pointwise in \(L^1(\mathbb{R}^3)\), to the operator \(\frac{1}{p^0} Q\).
2) solving the following approximating integral equation:

\[
f(t, y) = f_0(y) + \int_0^t Q_n(f, f)(s, y) ds
\]

(2.20)
obtained by replacing in (2.19) \( \frac{1}{p^0} Q \) by \( Q_n \) to obtain a global solution \( f_n \) that will converge to a global solution \( f \) of (2.19), in a suitable function space, that will be, given the general framework adopted, a suitable metric subspace of \( C[0, +\infty; L^1(\mathbb{R}^3)] \), namely a space \( C[0, +\infty; X_r] \) for a convenient real number \( r > 0 \).

3 Preliminary results and approximating operators.

In this section, we establish some properties of the collision operator \( Q \), we construct a sequence \((Q_n)\) of approximating operators to \( \frac{1}{p^0} Q \) and we give some useful properties of \((Q_n)\).

**Proposition 3.1** If \( f, g \in L^1(\mathbb{R}^3) \), then \( \frac{1}{p^0} Q^+(f, g) \), \( \frac{1}{p^0} Q^-(f, g) \in L^1(\mathbb{R}^3) \) and :

\[
\| \frac{1}{p^0} Q^+(f, g) \| \leq C \| f \| \| g \| \tag{3.1}
\]

\[
\| \frac{1}{p^0} Q^-(f, g) \| \leq C \| f \| \| g \| \tag{3.2}
\]

\[
\| \frac{1}{p^0} Q^+(f, f) - \frac{1}{p^0} Q^+(g, g) \| \leq C (\| f \| + \| g \|) \| f - g \| \tag{3.3}
\]

\[
\| \frac{1}{p^0} Q^-(f, f) - \frac{1}{p^0} Q^-(g, g) \| \leq C (\| f \| + \| g \|) \| f - g \| \tag{3.4}
\]

\[
\| \frac{1}{p^0} Q(f, f) - \frac{1}{p^0} Q(f, g) \| \leq C (\| f \| + \| g \|) \| f - g \| \tag{3.5}
\]

where \( C = 8\pi C_1 \), with \( C_1 \) given by (2.6).

**Proof**

1) The expression (2.4) for \( Q^+ \) gives, using (2.6):

\[
I = \| \frac{1}{p^0} Q^+(f, g) \| \leq C_1 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{d\tilde{p}d\tilde{q}}{p^0 q^0} \int_{S^2} |f(p')| |g(q')| d\omega
\]

Now we consider the change of variables \((p, q) \rightarrow (\tilde{p}', \tilde{q}')\) defined by (2.11). The Jacobian (2.13) of that transformation gives, using (2.1) that:

\[
d\tilde{p}d\tilde{q} = \frac{p^0 q^0}{(p')^2 (q')^2} dp'dq'.
\]

Hence, the above inequality gives, using \( \frac{1}{(p')^2 (q')^2} \leq 1 \),

\[
I \leq C_1 \int_{\mathbb{R}^3} |f(p')| dp' \int_{\mathbb{R}^3} |g(q')| dq' \int_{S^2} d\omega = 4\pi C_1 \| f \| \| g \|
\]
and (3.1) follows.

2) The expression (2.5) for $Q^-$ gives, using (2.6) and $\frac{1}{p_0^0} \leq 1$,

$$\| \frac{1}{p_0^0} Q^-(f, g) \| \leq C_1 \int_{\mathbb{R}^3} |f(\bar{p})| d\bar{p} \int_{\mathbb{R}^3} |g(\bar{q})| d\bar{q} \int_{S^2} d\omega = 4\pi C_1 \| f \| \| g \|$$

and (3.2) follows.

3) The inequalities (3.3), (3.4), (3.5) are direct consequences of (3.1), (3.2), $Q = Q^+ - Q^-$ and the fact that by (2.4), (2.5), the operators $Q^+, Q^-$ are bilinear and this allows us to write:

$$\frac{1}{p_0^0} Q^+(f, f) - \frac{1}{p_0^0} Q^+(g, g) = \frac{1}{p_0^0} Q^+(f, f - g) + \frac{1}{p_0^0} Q^+(f - g, g)$$

$$\frac{1}{p_0^0} Q^-(f, f) - \frac{1}{p_0^0} Q^-(g, g) = \frac{1}{p_0^0} Q^-(f, f - g) + \frac{1}{p_0^0} Q^+(f - g, g)$$

This completes the proof of Proposition 3.1. ■

**Remark 3.2** Inequality (3.5) shows that the operator $\frac{1}{p_0^0} Q$ is locally Lipschitzian in the $L^1(\mathbb{R}^3)$-norm, with respect to $f$. So, the standard theorem for ordinary differential equations in Banach spaces gives the local existence of the solution $f$ of the Cauchy problem (2.18). Moreover, given the conservation of the $L^1$-norm during the evolution, one could deduce the global existence of the solution $f$ of the Cauchy problem (2.18). But in our case, the physical background imposes to take non-negative initial data $f_0$ that evolves to give a non-negative solution $f$ of the Boltzmann equation. This is why, rather than using the above standard theorem for ordinary differential equation that does not give the positivity property, we develop a method that gives global non-negative solutions from non-negative initial data, provided that the initial data is sufficiently small.

We now state and prove the following result on which relies the construction of the approximating operators.

In what follows $C$ is the constant defined in Proposition 3.1.

**Proposition 3.3** Let $r_0 = \frac{1}{C_1}$, then, for every $r \in [0, r_0]$, for every $v \in X_r$ and for every integer $n \geq 2$, the equation

$$nu - \frac{n}{p_0^0} Q(u, u) = v$$

(3.6)

has a unique solution $u_n \in X_r$.

**Proof**: Consider a number $r > 0$ and take $v \in X_r$; $v$ being fixed. We look for a solution $u$ of (3.6) that can also be written, using $Q = Q^+ - Q^-$:

$$nu - \frac{n}{p_0^0} Q^+(u, u) + \frac{n}{p_0^0} Q^-(u, u) = v$$

(3.6)
from which we deduce, using \(Q^-(u,u) = uQ^-(1,u)\), for \(u \geq 0\) a.e.

\[
u = \frac{v + \frac{p}{p^0}Q^+(u,u)}{n + \frac{p}{p^0}Q^-(1,u)}
\]

So, \(u\) can be considered as a fixed point of the map \(F_n\) defined by:

\[
h \mapsto F_n(h) = \frac{v + \frac{p}{p^0}Q^+(h,h)}{n + \frac{p}{p^0}Q^-(1,h)} \quad (a)
\]

We deduce from (3.1) that \(F_n: X_r \to L^1(\mathbb{R}^3)\). Let us show that one can choose \(r > 0\) and \(n \in \mathbb{N}\) so that \(F_n\) is a contracting map from the complete metric space \(X_r\) into itself. We will then solve the problem by applying the fixed point theorem.

1) We deduce from (a), using (3.1) that, if \(h \in X_r\), we have since \(v \in X_r\):

\[
\| F_n(h) \| \leq \frac{\| v \|}{n} + C \| h \| \| h \| \leq \frac{r}{n} + Cr^2
\]

which shows that, we have:

\[
\| F_n(h) \| \leq r \quad \text{if} \quad n \geq 2 \quad \text{and} \quad 0 < r \leq \frac{1}{2C} \quad (b)
\]

So (b) gives conditions under which \(F_n\) maps \(X_r\) into itself.

2) Let \(g, h \in X_r\) and \(n \geq 2\) be given. We deduce from (a) that:

\[
F_n(h) - F_n(g) = \frac{v + \frac{p}{p^0}Q^+(h,h)}{n + \frac{p}{p^0}Q^-(1,h)} - \frac{v + \frac{p}{p^0}Q^+(g,g)}{n + \frac{p}{p^0}Q^-(1,g)} \quad (c)
\]

Notice that, since \(g \geq 0\), \(h \geq 0\) a.e.; we have:

\[
(n + \frac{n}{p^0}Q^-(1,h))(n + \frac{n}{p^0}Q^-(1,g)) \geq n^2 > 1.
\]

So we deduce from (c) using once more (2.14), i.e., the relation: \(Q^-(f,g) = fQ^-(1,g)\), that:

\[
\| F_n(h) - F_n(g) \| \leq \frac{1}{p^0}Q^+(h,h) - \frac{1}{p^0}Q^+(g,g) + \frac{1}{p^0}Q^-(v,g) - \frac{1}{p^0}Q^-(v,h) + \frac{1}{p^0}Q^-(1,p^0Q^+(h,h),g) - \frac{1}{p^0}Q^-(1,p^0Q^+(g,g),h) \quad (d)
\]

We can write, using the bilinearity of \(Q^-\):

\[
\frac{1}{p^0}Q^-(v,g) - \frac{1}{p^0}Q^-(v,h) = \frac{1}{p^0}Q^-(v,g - h).
\]

So, if in (d) we apply (3.3) to the first term and (3.2) to the second term, we obtain since \(\| g \| \leq r\), \(\| h \| \leq r\):

\[
\| \frac{1}{p^0}Q^+(h,h) - \frac{1}{p^0}Q^+(g,g) \| \leq 2Cr \| h - g \|
\]

\[
\| \frac{1}{p^0}Q^-(v,g) - \frac{1}{p^0}Q^-(v,h) \| \leq Cr \| h - g \| \quad (e)
\]
Now concerning the third term in (d), we can write, using once more the bilinearity of $Q^-$:

$$
\frac{1}{p^0} Q^-[\frac{1}{p^0} Q^+(h,h), g] - \frac{1}{p^0} Q^-[\frac{1}{p^0} Q^+(g,g), h] = \frac{1}{p^0} Q^-[\frac{1}{p^0} Q^+(h,h), g-h] + \frac{1}{p^0} Q^-[\frac{1}{p^0} Q^+(h,h) - \frac{1}{p^0} Q^+(g,g), h] \quad (f)
$$

Now we apply:

i) to the first term in the right hand side of (f), the property (3.2) of $Q^-$ followed by the property (3.1) of $Q^+$,

ii) to the second term in the right hand side of (f), the property (3.2) of $Q^-$ followed by the property (3.3) of $Q^+$.

And (f) gives, since $\|g\| \leq r$, $\|h\| \leq r$:

$$
\| \frac{1}{p^0} Q^-[\frac{1}{p^0} Q^+(h,h), g] - \frac{1}{p^0} Q^-[\frac{1}{p^0} Q^+(g,g), h] \| \leq 3C^2 r^2 \| g-h \| \quad (g)
$$

Thus (d) gives, using (e) and (g)

$$
\| F_n(h) - F_n(g) \| \leq 3Cr(1+Cr) \| h-g \| \quad (h)
$$

So, if we take $r_0 = \frac{1}{3Cr}$, then for every $r \in [0, r_0]$, we have $3Cr(1+Cr) \leq 1/2$ and (b) and (h) show that for every integer $n \geq 2$, $F_n$ is a contracting map from $X_r$ into itself. By the fixed point theorem, $F_n$ has a unique fixed point $u_n$ that is the unique solution of (3.6).

This ends the proof of Proposition 3.3. ■

If $r$ and $n$ are fixed as indicated in Proposition 3.3, to every $v \in X_r$ corresponds a unique $u \in X_r$ that satisfies (3.6). We are then led to the following definition:

**Definition 3.4** Let $r_0 = \frac{1}{3Cr}$. Let $r \in [0, r_0]$ and $n \in \mathbb{N}$, $n \geq 2$ be given.

1) Define the operator:

$$
R(n,Q) : X_r \to X_r \quad u \mapsto R(n,Q)u
$$

as follows: for $u \in X_r$, $R(n,Q)u$ is the unique element of $X_r$ such that:

$$
nR(n,Q)u - \frac{n}{p^0} Q[R(n,Q)u, R(n,Q)u] = u \quad (3.7)
$$

2) Define the operator $Q_n$ on $X_r$ by

$$
Q_n(u,u) = n^2 R(n,Q)u - nu. \quad (3.8)
$$

We give the properties of the operators $R(n,Q)$ and $Q_n$. 

---

11
Proposition 3.5 \textit{Let }r_0=\frac{1}{C}. \textit{Let }r \in]0, r_0[ \text{ and } n, m \in \mathbb{N}, n \geq 2, m \geq 2 \text{ be given : Then we have } \forall u, v \in X_r:\]

\| nR(n, Q)u \| = \| u \| \quad (3.9)

\| nR(n, Q)u - u \| \leq \frac{K}{n} \quad (3.10)

\| R(n, Q)u - R(n, Q)v \| \leq \frac{2}{n} \| u - v \| \quad (3.11)

Q_n(u, u) = \frac{1}{p^0} Q[nR(n, Q)u, nR(n, Q)u] \quad (3.12)

\| Q_n(u, u) - Q_n(v, v) \| \leq K \| u - v \| \quad (3.13)

\| Q_n(u, u) - \frac{1}{p^0} Q(v, v) \| \leq \frac{K}{n} + K \| u - v \| \quad (3.14)

\| Q_n(u, u) - Q_m(v, v) \| \leq K \| u - v \| + \frac{K}{n} + \frac{K}{m} \quad (3.15)

Here \( K = K(r) \) is a continuous function of \( r \).

\textbf{Proof :} 1) \( (3.9) \) will be a consequence of the relation :

\[ \int_{\mathbb{R}^3} \frac{1}{p^0} Q(f, g) d\bar{p} = 0 \quad \forall f, g \in L^1(\mathbb{R}^3) \quad (3.16) \]

we now establish.

This is where we need the symmetry assumption (2.7) on the collision kernel i.e. \( S(\bar{p}, \bar{q}, \bar{p}', \bar{q}') = S(\bar{p}', \bar{q}', \bar{p}, \bar{q}) \).

We have, using this relation, the definition (2.4) of \( Q^+ \), \( p^0 = \sqrt{1 + |\bar{p}|^2} \), the change of variables \((\bar{p}, \bar{q}) \rightarrow (\bar{p}', \bar{q}') \) given by (2.11) and whose Jacobian is given by (2.13) :

\[ I_0 = \int_{\mathbb{R}^3} \frac{1}{p^0} Q^+(f, g) d\bar{p} = \int_{\mathbb{R}^3} d\bar{p}' \int_{\mathbb{R}^3} d\bar{q}' \int_{S^2} f(\bar{p}') g(\bar{q}') S(\bar{p}', \bar{q}', \bar{p}, \bar{q}) d\omega \]

On the other hand, we have, using definition (2.5) of \( Q^- \) and again \( p^0 = \sqrt{1 + |\bar{p}|^2} \):

\[ J_0 = \int_{\mathbb{R}^3} \frac{1}{p^0} Q^-(f, g) d\bar{p} = \int_{\mathbb{R}^3} d\bar{p} \int_{\mathbb{R}^3} d\bar{q} \int_{S^2} f(\bar{p}) g(\bar{q}) S(\bar{p}, \bar{q}, \bar{p}', \bar{q}') d\omega. \]

It then appears that \( I_0 = J_0 \) and \( (3.10) \) follows from \( Q = Q^+ - Q^- \). Given (3.16), (3.9) is obtained by integrating (3.7) over \( \mathbb{R}^3 \), since \( u \geq 0 \) a.e.
2) The definition (3.7) of $R(n, Q)$ gives, using the property (3.8) of $Q$ with $f = R(n, Q)u$ and $g = 0$:

\[
\| \ nR(n, Q)u - u \| = \| \frac{n}{p^0} Q[R(n, Q)u, R(n, Q)u] \| \leq Cn(\| R(n, Q)u \|)^2
\]

(3.10) then follows from (3.9) and $\| u \| \leq r$.

3) Subtracting relation (3.7) written for $u$ and $v$ gives:

\[
R(n, Q)u - R(n, Q)v = \frac{u - v}{n} + \frac{1}{p^0} Q[R(n, Q)u, R(n, Q)u] - \frac{1}{p^0} Q[R(n, Q)v, R(n, Q)v]
\]

which gives, using property (3.10) of $Q$ and (3.9) that gives $\| R(n, Q)u \| \leq \frac{r}{n}$,

\[
\| R(n, Q)v \| \leq \frac{r}{n}
\]

(3.11) then follows from $n \geq 2$ and the definition of $r_0$ that gives: $Cr < 1/2$.

4) Property (3.12) of $Q_n$ is obtained by multiplying (3.7) by $n$, using the bilinearity of the operator $Q$, the definition (3.8) of $Q_n$.

5) Property (3.13) is obtained by using (3.12), property (3.9) of $Q$, followed by properties (3.9) and (3.11) of $R(n, Q)$.

6) Notice that, by (3.2), $nR(n, Q)u \in X_r$, if $u \in X_r$. Then use the property (3.12) of $Q_n$, the property (3.5) of $Q$ to obtain:

\[
\| Q_n(u, u) - \frac{1}{p^0} Q(v, v) \| \leq 2Cr \| nR(n, Q)u - v \| \leq 2Cr(\| nR(n, Q)u - u \| + \| u - v \|)
\]

Hence, (3.10) gives (3.14).

7) (3.12) gives:

\[
Q_n(u, u) - Q_m(v, v) = \frac{1}{p^0} Q[nR(n, Q)u, nR(n, Q)u] - \frac{1}{p^0} Q[mR(m, Q)v, mR(m, Q)v].
\]

then, (3.15) follows (3.5), (3.9) and (3.10), adding and subtracting $u$ and $v$.

This ends the proof of Proposition 3.5. ■

**Remark 3.6**

1) [3.14] shows, by taking $u = v$, that $Q_n$ converges pointwise in the $L^1(\mathbb{R}^3)$-norm to the operator $\frac{1}{p^0} Q$. From there, the terminology of “approximating operators” we give to the sequence $Q_n$.

2) In [3], the author defined the operator $R(n, Q)$ whose domain is $X_r$ using the equation $nu - \frac{1}{p^0} Q(u, u) = v$, instead of (3.4) and he associated to it, the approximating operator $Q_n(u, u) = nR(n, Q)nu - nu$. But it is clear that, even if $u \in X_r$, for $n$ sufficiently large, $nu$ is no longer in $X_r$; so the domain of this operator $Q_n$ is unspecified, in contrast to $Q_n$ defined by (3.3), for which such a problem does not exist.

3) The useful relation (3.16) was also used in [3], but without specifying the symmetry assumption (2.7) on the collision kernel $S$, on which this result is based.
We now have all the tools we need to prove the global existence theorem for the homogeneous relativistic Boltzmann equation.

4 The global existence theorem

With the operator \( Q_n \) defined above, we will first state and prove a global existence theorem for the equation (2.20) we call “approximating equation”. Next we establish that the solution \( f_n \) of (2.20) converges, in a sense to be specified, to a global solution of (2.19). We prove these results under the smallness assumption on the initial data.

**Proposition 4.1** Let \( r_0 = \frac{1}{7}C \). Let \( r \in [0, r_0] \), \( g_0 \in X_r \), \( n \in \mathbb{N} \), \( n \geq 2 \) be given. Then, for every \( t_0 \in [0, +\infty[ \), the integral equation:

\[
f(t_0 + t, y) = g_0(y) + \int_{t_0}^{t_0+t} Q_n(f,f)(s,y)ds, \quad t \geq 0 \quad (4.1)
\]

has a unique global solution \( f_n \in C[t_0, +\infty; X_r] \). Moreover, \( f_n \) satisfies the inequality:

\[
\| f_n \| \leq \| g_0 \| \quad (4.2)
\]

**Proof**: We give the proof in two steps.

**Step 1**: Local existence and estimation.

Let us consider equation (4.1) on an interval \([t_0, t_0 + \delta]\), \( \delta > 0 \), i.e.

\[
f(t_0 + t, y) = g_0(y) + \int_{t_0}^{t_0+t} Q_n(f,f)(s,y)ds, \quad t \in [0, \delta], \quad \delta > 0 \quad (4.3)
\]

The relations:

\[
\frac{d}{dt}[e^{nt}f(t_0 + t, y)] = e^{nt}\frac{df}{dt} + nf(t_0 + t, y); \quad Q_n(f,f) + nf = n^2R(n,Q)f \quad [Cf \quad (3.8)]
\]

show that (4.3) is equivalent to:

\[
f(t_0 + t, y) = e^{-nt}g_0(y) + \int_{t_0}^{t} n^2e^{-n(t-s)}R(n,Q)f(t_0 + s, y)ds \quad (4.4)
\]

we solve (4.4) by the fixed point theorem.

Consider the operator \( A \) defined on \( C[t_0, t_0 + \delta; X_r] \) by the right hand side of (4.4), i.e.

\[
Af(t_0 + t, y) = e^{-nt}g_0(y) + \int_{t_0}^{t} n^2e^{-n(t-s)}R(n,Q)f(t_0 + s, y)ds \quad (4.5)
\]
Let us show that one can find $\delta > 0$ such that $A$ is a contracting map of the complete metric space $C[t_0, t_0 + \delta; X_r]$ into itself.

i) Let $f \in C[t_0, t_0 + \delta; X_r]$. Since $\| g_0 \| \leq r$ and using (3.9) that gives $\| nR(n, Q)f(t_0 + s) \| = \| f(t_0 + s) \| \leq r$, (1.5) gives for every $t \in [0, \delta]$:

$$\| Af(t_0 + t) \| \leq e^{-nt} r + nre^{-nt} \int_0^t e^{ns} ds \leq e^{-nt} r + re^{-nt}[e^{nt} - 1] = r$$

Then $\| Af \| \leq r$ and this shows that $A$ maps $C[t_0, t_0 + \delta; X_r]$ into itself.

ii) Let $f, g \in C[t_0, t_0 + \delta; X_r]$. (1.5) gives:

$$(Af - Ag)(t_0 + t, y) = \int_0^t n^2 e^{-n(t-s)}[R(n, Q)f - R(n, Q)g](t_0 + s, y) ds$$

which gives, using property (3.11) of $R(n, Q)$ and $e^{-n(t-s)} \leq 1$:

$$\| Af - Ag \| \leq 2n\delta \| f - g \|$$

It then appears that $A$ is a contracting map in any space $C[t_0, t_0 + \delta; X_r]$ where $2n\delta \leq 1/2$ i.e $\delta \in [0, \frac{1}{2n}]$. Taking $\delta = \frac{1}{4n}$, we conclude that $A$ has a unique fixed point $f_0^n \in C[t_0, t_0 + \frac{1}{4n} ; X_r]$, that is the unique solution of (4.3) and hence, the unique solution of (4.3).

b) Estimation

Notice that (4.4) gives, taking $t = 0, f(t_0, y) = g_0(y) ; f_0^n$ then satisfies:

$$f_0^n(t_0 + t, y) = e^{-nt} f_0^n(t_0, y) + \int_0^t n^2 e^{-n(t-s)} R(n, Q)f_0^n(t_0 + s, y) ds$$

which gives, multiplying by $e^{nt}$, using once more (3.9), and for every $t \in [0, \frac{1}{4n}]$:

$$\| e^{nt} f_0^n(t_0 + t) \| \leq \| f_0^n(t_0) \| + n \int_0^t \| e^{ns} f_0^n(t_0 + s) \| ds$$

this gives by Gronwall’s Lemma: $e^{nt} \| f_0^n(t_0 + t) \| \leq e^{nt} \| f_0^n(t_0) \| ;$ hence;

$$\| f_0^n \| \leq \| f_0^n(t_0) \|$$

(4.6)

Step 2: Global existence, Estimation and Uniqueness.

Let $k \in \mathbb{N}$. Taking in (4.3) $\delta = \frac{1}{4n}$ and replacing in that integral equation, $t_0$ by $t_0 + \frac{k}{4n} ; t_0 + \frac{2k}{4n} ; \cdots ; t_0 + \frac{k}{4n} ; \cdots$, step 1 tells us that on each interval $I_k = [t_0 + \frac{k}{4n}, t_0 + \frac{k}{4n} + \frac{1}{4n}]$ whose length is $\frac{1}{4n}$, the initial value problem for the corresponding integral equation has a unique solution $f_k^n \in C[I_k ; X_r]$ provided that the initial data we denote $f_k^n(t_0 + \frac{k}{4n})$, is a given element of $X_r$; $f_k^n$ then satisfies:

$$f_k^n(t_0 + \frac{k}{4n} + t, y) = f_k^n(t_0 + \frac{k}{4n}, y) + \int_{t_0 + \frac{k}{4n} + t}^{t_0 + \frac{k}{4n} + t} Q_n(f_k^n, f_k^n)(s, y) ds$$

$$f_k^n(t_0 + \frac{k}{4n}) \in X_r, k \in \mathbb{N}, t \in [0, \frac{1}{4n}]$$
We give the proof in two steps. Local existence and estimation.

**Proof**

Define the function $f$ satisfies the estimation:

$$\int_0^t (t_0 + s, y) ds, \quad t \geq 0$$

this implies, using the property (3.13) of $Q_n$ that:

$$\| (f - g)(t_0 + t) \| \leq K \int_0^t \| (f - g)(t_0 + s) \| ds \quad t \geq 0$$

which implies, by the Gronwall’s Lemma, that $f = g$ and the uniqueness is proved. This ends the proof of Proposition 4.1.

**Remark 4.2** Equation (2.20) is the particular case of equation (4.1) when $t_0 = 0$, $g_0 = f_0$.

We now state and prove the global existence theorem.

**Theorem 4.3** Let $r_0 = \frac{1}{rC}$. Let $r \in [0, r_0]$ and $f_0 \in X_r$ be given. Then the Cauchy problem for the homogeneous Boltzmann equation on the Minkowski space, with initial data $f_0$, has a unique global solution $f \in C[0, +\infty; X_r]$; $f$ satisfies the estimation:

$$\sup_{t \in [0, +\infty]} \| f(t) \| \leq \| f_0 \|. \quad (4.7)$$

**Proof**: We give the proof in two steps.

Step 1: Local existence and estimation

Let $t_0 \in \mathbb{R}$, $g_0 \in X_r$ be given, the proposition 4.1 gives for every $n \in \mathbb{N}$, $n \geq 2$, the existence of a solution $f_n \in C[t_0, +\infty; X_r]$ of (4.1) that satisfies (4.2). It is important to notice that this solution depends on $n$.

Let $T > 0$ be given; we have $f_n \in C[t_0, t_0 + T; X_r]$ for $n \in \mathbb{N}$, $n \geq 2$. Let us prove that the sequence $(f_n)$ converges in $C[t_0, t_0 + T; X_r]$ to a solution of the integral equation:

$$f(t_0 + t, y) = g_0(y) + \int_{t_0}^{t_0 + t} \frac{1}{p^0} Q(f, f)(s, y) ds, \quad t \in [0, T] \quad (4.8)$$
Consider two integers \( n, m \geq 2 \). We deduce from (4.1) that \( \forall t \in [0, T] \):
\[
f_n(t_0 + t, y) - f_m(t_0 + t, y) = \int_{t_0}^{t_0 + t} [Q_n(f_n, f_n) - Q_m(f_m, f_m)](s, y) \, ds
\]
this gives using the property (3.15) of the operators \( Q_n, Q_m \):
\[
\| f_n(t_0 + t) - f_m(t_0 + t) \| \leq \left( \frac{K}{n} + \frac{K}{m} \right) T + K \int_0^t \| (f_n - f_m)(t_0 + s) \| \, ds
\]
which gives, using Gronwall’s Lemma :
\[
\| f_n - f_m \| \leq \left( \frac{1}{n} + \frac{1}{m} \right) KT e^{KT}
\]
this prove that \( f_n \) is a Cauchy sequence in the complete metric space \( C[t_0, t_0 + T; X_r] \). Then, there exists \( f \in C[t_0, t_0 + T; X_r] \) such that
\[
f_n \text{ converges to } f \text{ in } C[t_0, t_0 + T; X_r]
\]
Let us show that \( f \) satisfies (4.8), (4.9) implies that : \( f_n(t_0 + t) \) converges to \( f(t_0 + t) \) in \( X_r \) for every \( t \in [0, T] \); this implies in particular, by taking \( t = 0 \), and since \( f_n(t_0) = g_0 \) that \( f(t_0) = g_0 \). Next we have, using property (3.14) of the operators \( Q_n \) and \( Q \):
\[
\| \int_{t_0}^{t_0 + t} [Q_n(f_n, f_n) - \frac{1}{p_0} Q(f, f)](s, y) \, ds \| \leq \frac{KT}{n} + KT \| f_n - f \|
\]
which shows that :
\[
\int_{t_0}^{t_0 + t} Q_n(f_n, f_n)(s) \, ds \text{ converges to } \int_{t_0}^{t_0 + t} \frac{1}{p_0} Q(f, f)(s) \, ds \text{ in } L^1(\mathbb{R}^3)
\]
\[
\text{hence, since } f_n \text{ satisfies (4.1), } f \text{ satisfies (4.8) } \forall t \in [0, T] \text{ and a.e. with respect to } y \in \mathbb{R}^3.
\]
Finally (4.2) and (4.9) imply
\[
\| f \| \leq \| f(t_0) \|
\]
Step 2 : Global existence, Estimation and Uniqueness.
Since \( t_0 \in [0, +\infty[ \) is arbitrary and since \( \forall n \geq 2 \), the solution \( f_n \) of (4.1) is globally defined on \( [t_0, +\infty[ \); Step 1 tells us that, given \( T > 0 \), by taking in the integral equation (4.8) : \( t_0 = 0, t_0 = T, t_0 = 2T, \ldots, t_0 = (k - 1)T, t_0 = kT, \ldots, k \in \mathbb{N} \), then, on each interval \( J_k = [kT, (k + 1)T] \) whose length is \( T \), the initial value problem for the corresponding integral equation has a solution \( f^k \in C[J_k; X_r] \), provided that the initial data we denote \( f^k(kT) \) is a given element of \( X_r \). We then proceed exactly as in Step 2 of the proof of Proposition 4.1 by writing for \( T > 0 \) given, that :
\[
[0, +\infty[ = \bigcup_{k \in \mathbb{N}} [kT, (k + 1)T],
\]
to overlap the local solutions \( f^k \in C[kT; (k + 1)T; X_r], k \in \mathbb{N} \) with 
\( f^0(0) = f_0 \) and obtain a global solution \( f \in C[0, +\infty; X_r] \) of the equation :

\[
f(t, y) = f_0(y) + \int_0^t \frac{1}{p_0} Q(f, f)(s, y) ds
\]

that satisfies, using (4.10) that gives \( || f^k || \leq || f^k(kT) ||, \forall k \in \mathbb{N} \), the estimation :

\[
|| f || \leq || f_0 ||
\]  

(4.12)

Now if \( f, g \in C[0, +\infty; X_r] \) are two solutions of (4.11), property (5.5) of \( Q \) gives :

\[
|| (f - g)(t) || \leq 2Cr \int_0^t || (f - g)(s) || ds \quad t \geq 0
\]

which gives by Gronwall’s Lemma \( f = g \) and proves the uniqueness. This completes the proof of Theorem 4.3.

Acknowledgement: The authors thank A.D. Rendall for helpful comments and suggestions. This work was supported in part by the VolkswagenStiftung, Federal Republic of Germany.

References

[1] Carleman, T., Sur la théorie de l’équation intégro-différentielle de Boltzmann, Acta Math. 60, 91-146. (1933)

[2] Bichteler, K., On the Cauchy problem for relativistic Boltzmann equation, Comm. Math. Phys. 4, 352-364, (1967).

[3] Bancel, D., Problème de Cauchy pour l’équation de Boltzmann en Relativité Générale, Ann. Inst. Henri Poincaré, vol. XVIII, 3, p.263-284, (1973).

[4] Bancel, D.; Choquet-Bruhat, Y., Uniqueness and local stability for the Einstein-Maxwell-Boltzmann system, Comm. Math. Phys. 33, 83-96, (1973).

[5] Illner, R.; Shinbrot, M., The Boltzmann equation, global existence for a rare gas in an infinite vacuum, Comm. Math. Phys. 95, 217-226, (1984)

[6] Glassey, R.T.; Strauss, W., Asymptotic stability of the relativistic Maxwellian, Publ. Math. RIMS Kyoto, 29, 301-347, (1992).

[7] Glassey, R.T., The Cauchy problem in kinetic theory, SIAM, Indiana University, Bloomington Indiana (1996).
[8] Mucha, P.B., *Global existence for the Einstein-Boltzmann equation in the flat Robertson-Walker space-time*, Comm. Math. Phys. **203**, 107-118, (1999).

[9] Mucha, P.B., *Global existence of solutions of the Einstein-Boltzmann equation in the spatially homogeneous case*, in Evolution Equations Existence, Regularity and Singularities, Banach Center Publications, volume 52, Institute of Mathematics, Polish Academy of Sciences, Warszawa, (2000).