GAPS BETWEEN TOTIENTS

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Abstract. We study the set $D$ of positive integers $d$ for which the equation $\phi(a) - \phi(b) = d$ has infinitely many solution pairs $(a, b)$. We show that $\min D \leq 154$, exhibit a specific $A$ so that every multiple of $A$ is in $D$, and show that any progression $a \mod d$ with $4|a$ and $4|d$, contains infinitely many elements of $D$. We also show that the Generalized Elliott-Halberstam Conjecture, as defined in [6], implies that $D$ contains all positive, even integers.

1. Introduction

Let $\mathcal{V} = \{v_1, v_2, \ldots\}$ be the set of totients, that is, $\mathcal{V}$ is the image of Euler’s totient function $\phi(n)$. In this paper we study the set $D$ of positive integers which are infinitely often a difference of two elements of $\mathcal{V}$. A classical conjecture asserts that every even positive integer is infinitely often the difference of two primes, and this implies immediately that $D$ is the set of all positive, even integers. We are interested in what can be accomplished unconditionally, by leveraging the recent breakthroughs on gaps between consecutive primes by Zhang [8], Maynard [4], Tao (unpublished) and the PolyMath8b project [6]. We let $\mathcal{E}$ be the set of positive even numbers that are infinitely often the difference of two primes. Clearly $\mathcal{E} \subseteq D$. In this note we prove some results about $D$ which are not known for $\mathcal{E}$.

The behavior of the smallest elements of $D$ arose in recent work of Fouvry and Waldschmidt [2] concerning representation of integers by cyclotomic forms, and the problem of studying the differences of totients was also posted in a list of open problems by Shparlinski [7], Problem 56. Our paper is a companion of the recent work of the first author [1] concerning the equation $\phi(n + k) = \phi(n)$ for fixed $k$.

It is known [6] that $\min \mathcal{E} \leq 246$ and thus $\min D \leq 246$. We can do somewhat better.

Theorem 1. We have $\min D \leq 154$.

Although there is no specific even integer which is known to be infinitely often the difference of two primes, we give an infinite family of specific numbers that are in $D$.

Theorem 2. Let $a_0 = \prod_{p \leq 47} p$ and $b = \text{lcm}[1, 2, \ldots, 49]$. Then every multiple of $\phi(a_0 b_0) a_0$ lies in $D$.

Granville, Kane, Koukoulopoulos and Lemke-Oliver [3] showed that $\mathcal{E}$ has lower asymptotic density at least $\frac{1}{354}$ and thus so does $D$. We do not know how to prove a better lower bound for the density and leave this as an open problem.

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Central to the works \cite{4, 5, 6, 8} is the concept of an admissible set of linear forms. For positive integers \(a_i\) and integers \(b_i\), the set of affine-linear forms \(a_1x + b_1, \ldots, a_kx + b_k\) is admissible if, for every prime \(p\), there is an \(x \in \mathbb{Z}\) such that \(p \nmid (a_1x + b_1) \cdots (a_kx + b_k)\).

**Definition.** Hypothesis \(\text{DHL}[k, m]\) is the statement that for any admissible \(k\)-tuple of linear forms \(a_i n + b_i, 1 \leq i \leq k\), for infinitely many \(n\), at least \(m\) of them are simultaneously prime.

In this paper we are concerned with the statements \(\text{DHL}[k, 2]\). The Polymath8b project \cite{6}, plus subsequent work of Maynard \cite{5}, established \(\text{DHL}[50, 2]\) unconditionally.

The Elliott-Halberstam Conjecture implies \(\text{DHL}[5, 2]\), see \cite{4}. The Generalized Elliott-Halberstam Conjecture implies \(\text{DHL}[3, 2]\) (see \cite{6} for details).

**Theorem 3.** We have

(i) \(\text{DHL}[3, 2]\) implies that \(\mathcal{D} = \{2, 4, 6, 8, 10, \ldots\}\), the set of all positive even integers.

(ii) \(\text{DHL}[4, 2]\) implies that \(\mathcal{D}\) contains every positive multiple of 4.

(iii) \(\text{DHL}[5, 2]\) implies that \(\min \mathcal{D} \leq 6\).

(iv) \(\text{DHL}[6, 2]\) implies \(8 \in \mathcal{D}\).

By contrast, for any \(k \geq 2\), \(\text{DHL}[k, 2]\) implies that \(\liminf p_{n+1} - p_n \leq a_k\), where \(a_k\) is the minimum of \(h_k - h_1\) over all admissible \(k\)-tuples \(n + h_1, \ldots, n + h_k\). We have \(a_3 = 6, a_4 = 8, a_5 = 12\) and \(a_6 = 16\).

We show parts of Theorem 3 (ii) and (iii) using a more general result.

**Theorem 4.** Assume \(\text{DHL}[k, 2]\), with \(k \geq 3\). Also assume that there are integers \(1 < m_1 < \ldots < m_k\) and \(\ell_{i,j}\) for \(1 \leq i < j \leq k\) and such that

\[
\frac{\ell_{i,j}m_i}{m_j - m_i} \in \mathcal{V}, \quad \frac{\ell_{i,j}m_j}{m_j - m_i} \in \mathcal{V} \quad (1 \leq i < j \leq k).
\]

Then \(L \leq 2 \max \ell_{i,j}\). Moreover, if \(\ell_{i,j} = \ell\) for all \(i, j\) then \(2\ell \in \mathcal{D}\).

In Section 3, we give a heuristic argument that there exist numbers \(m_1, \ldots, m_{50}\) satisfying the hypothesis of Theorem 4 with \(\ell_{i,j} = 2\) for all \(i < j\). In this case we achieve an unconditional proof that \(4 \in \mathcal{D}\). Actually finding such \(m_i\) seems computationally difficult, however.

Does every arithmetic progression \(a \mod d\) containing even numbers have infinitely many elements of \(\mathcal{D}\)? We answer in the affirmative if \(4 | d\) and \(4 | a\). The case \(a \equiv 2 \mod 4\) is more difficult; see our Remarks following the proof in Section 4.

**Theorem 5.** Let \(a, d\) be positive integers with \(4 | a, 4 | d\). Then the progressions \(a \mod d\) contains infinitely many elements of \(\mathcal{D}\).

Observe that, even assuming \(\text{DHL}[3, 2]\), there is no specific progression \(a \mod d\), not containing 0, which is known to contain a number that is infinitely often the difference of two primes.

When \(4 | d\) and \(a \equiv 2 \mod 4\) we can sometimes show that \(\mathcal{D}\) contains infinitely many elements that are \(\equiv a \mod d\); see the Remarks following the proof of Theorem 5 in Section 4.
2. Proof of Theorems 1–4

Proof of Theorem 1. Define

\[ S_1 = \{41, 43, 47, 53, 67, 71\}, \]
\[ S_2 = \{59, 61, 67, 71, 73, 83, 89, 101, 103, 107, 109, 113, 127, 131, 137, 139\}, \]
\[ S_4 = \{p \text{ prime : } 127 \leq p \leq 271\}. \]

and consider the collection of 50 linear forms

\[ n + a \ (a \in S_1), \quad 2n + a \ (a \in S_2), \quad 4n + a \ (n \in S_4). \]

This collection is admissible; indeed if \( p < 41 \) and \( n = 0 \) then all of them are coprime to \( p \). For \( p > 50 \) it is clear that there is an \( n \) for which all of them are coprime to \( p \). When \( p \in \{41, 43, 47\} \) we take \( n = 1, 3, 8 \) respectively, and then all of the forms are coprime to \( p \).

By DHL[50, 2], there are two of these forms that are simultaneously prime for infinitely many \( n \). If both forms are of the type \( n + a \ for \ a \in S_1 \), then this shows that \( \min D \leq 71 - 41 = 30 \).

Likewise, if both forms are of the type \( 2n + a \ for \ a \in S_2 \) then \( \min D \leq 139 - 59 = 80 \) and if both forms are of the type \( 4n + a \ where \ a \in S_4 \), then \( \min D \leq 271 - 127 = 144 \). Now suppose for infinitely many \( n \), \( n + a \) and \( 2n + b \) are both prime, where \( a \in S_1 \), \( b \in S_2 \). Then

\[ \phi(4(n + a)) = 2n + 2a - 2, \quad \phi(2n + b) = 2n + b - 1, \]

which shows that \( |b - 2a + 1| \in D \). We have \( b - 2a + 1 \neq 0 \) for all choices, and the maximum of \( b - 2a + 1 \) is 82, and hence \( \min D \leq 82 \). Similarly, if for infinitely many \( n \), \( 2n + a \) and \( 4n + b \) are both prime, where \( a \in S_2 \), \( b \in S_4 \), then \( |b - 2a + 1| \in D \). Hence \( \min D \leq 154 \).

Finally, if for infinitely many \( n \), \( n + a \) and \( 4n + b \) are both prime, where \( a \in S_1 \), \( b \in S_4 \), then \( |b - 4a + 3| \in D \) since

\[ \phi(8(n + a)) = 4(n + a - 1), \quad \phi(4n + b) = 4n + b - 1. \]

In all cases \( 0 < |b - 4a + 3| \leq 154. \)

\[ \square \]

Proof of Theorem 2. Let

\[ a_0 = \prod_{p \leq 47} p, \quad b_0 = \text{lcm}[1, 2, \ldots, 49]. \]

Let \( k \in \mathbb{N} \) and consider the admissible set of linear forms \( n + ka_0, n + 2ka_0, \ldots, n + 50ka_0 \).

Since DHL[50, 2] holds, for any \( k \in \mathbb{N} \) there exists \( j_k \in \{1, \ldots, 49\} \) such that the equation

\[ \phi(u) - \phi(v) = u - v = k_j a_0 \]

has infinitely many solutions in primes \( u, v \). Since \( a_0 \mid a_0b_0/j_k \), \( \phi(a_0b_0l/j_k) = \phi(a_0b_0l)/j_k \) for any \( l \in \mathbb{N} \). Therefore, we have

\[ \phi(a_0b_0u/j_k) - \phi(a_0b_0v/j_k) = \phi(a_0b_0)(\phi(u) - \phi(v))/j_k = \phi(a_0b_0)a_0k, \]

as required. \[ \square \]
Proof of Theorem 4. The set of forms \( m_i n - 1, \ldots, m_k n - 1 \) is clearly admissible. By DHL[\( k, 2 \)], for some pair \( i < j \) and for infinitely many \( n \), \( m_i n - 1 \) and \( m_j n - 1 \) are prime. Let \( \ell = \ell_{i,j} \) and suppose that \( x, y \) satisfy

\[
\phi(x) = \frac{\ell m_i}{m_j - m_i}, \quad \phi(y) = \frac{\ell m_j}{m_j - m_i}.
\]

Then for sufficiently large \( n \)

\[
\phi(x(m_j n - 1)) - \phi(y(m_i n - 1)) = \frac{-2\ell m_i + 2\ell m_j}{m_j - m_i} = 2\ell. \tag*{\Box}
\]

Proof of Theorem 4. (i) Let \( h \in \mathbb{N} \) and consider the triple \( \{n+1, n+2h+1, 2n+2h+1\} \). This is admissible, since when \( n = 0 \), all of the forms are odd, and similarly none are divisible by 3 for some \( n \in \{0, 1\} \). By DHL[3, 2], either (i) \( n+1 \) and \( n+2h+1 \) are infinitely often both prime, (ii) \( n+1 \) and \( 2n+2h+1 \) are infinitely often both prime or (iii) \( n+2h+1 \) and \( 2n+2h+1 \) are infinitely often both prime. In case (i) we have \( \phi(n+2h+1) - \phi(n+1) = 2h \), in case (ii) we have \( \phi(2n+2h+1) - \phi(4(n+1)) = 2h \), and in case (iii) we have \( \phi(4(n+2h+1)) - \phi(2n+2h+1) = 2h \).

(ii) The deduction \( 4 \in \mathcal{D} \) follows from Theorem 4 using \( \ell_{i,j} = 2 \) for all \( i, j \) and

\[
\{m_1, \ldots, m_4\} = \{6, 8, 9, 12\}.
\]

Now suppose that \( d \) is congruent to 0 or 4 modulo 12, and define \( a \) by \( d = 2(a+1) \). In particular, \( (a, 6) = 1 \). Thus, the set of forms \( m_i n - a, 1 \leq i \leq 4 \), are admissible. By DHL[4, 2], for some pair \( i < j \) and for infinitely many \( n \), \( m_i n - a \) and \( m_j n - a \) are prime. Suppose that \( x, y \) satisfy

\[
\phi(x) = \frac{2m_i}{m_j - m_i}, \quad \phi(y) = \frac{2m_j}{m_j - m_i}.
\]

Then for sufficiently large \( n \)

\[
\phi(x(m_j n - a)) - \phi(y(m_i n - a)) = 2(a+1) = d. \tag*{\Box}
\]

Hence, \( d \in \mathcal{D} \).

Finally, if \( d \equiv 8 \) (mod 12), write \( d = 2(b - 1) \), so that \( (b, 6) = 1 \). Similarly, the set of forms \( m_i n + b, 1 \leq i \leq 4 \), are admissible and we conclude that \( d \in \mathcal{D} \).

(iii) Consider the admissible set of forms \( \{f_1(n), \ldots, f_5(n)\} = \{n, n+2, 2n+1, 4n-1, 4n+3\} \). Indeed, if \( n \equiv 11 \) (mod 30) then all of the forms are coprime to 30. By DHL[5, 2], for some \( i < j \) and infinitely many \( n \), \( f_i(n) \) and \( f_j(n) \) are both prime. Say \( f_i(n) = an + b \) and \( f_j(n) = cn + d \) with \( c/a \in \{1, 2, 4\} \). Then

\[
\phi((2c/a)(an + b)) = (c/a)(an + b - 1) = cn + (c/a)(b - 1)
\]

and

\[
\phi(cn + d) = cn + d - 1.
\]

Thus, \( |(c/a)(b - 1) - (d - 1)| \in \mathcal{D} \). In all cases, \( |(c/a)(b - 1) - (d - 1)| \leq 6 \).

(iv) Use Theorem 4 with the set

\[
\{h - 72, h - 66, h - 64, h - 63, h - 60, h\}, \quad h = 120193920,
\]

and \( \ell_{i,j} = 4 \) for all \( i, j \). We used PARI/GP to verify that \( 4n_i/(n_j - n_i) \in \mathcal{V} \) and \( 4n_j/(n_j - n_i) \in \mathcal{V} \) for all \( i < j \).
3. A heuristic argument

In this section, we give an argument that there should exist \( m_1, \ldots, m_{50} \) satisfying the hypothesis of Theorem 4 with \( \ell_{i,j} = 2 \) for all \( i, j \). We first give a general construction of numbers with \( \frac{n_j}{n_j - n_i} \) all integers.

**Lemma 3.1.** For any positive integer \( b \) and and \( k \geq 2 \) there is a set \( \{ n_1, \ldots, n_k \} \) of positive integers with \( n_1 < n_2 < \cdots < n_k \) and with

\[
(3.1) \quad b \big| \frac{n_j}{n_j - n_i} \quad (1 \leq i < j \leq k).
\]

**Proof.** Induction on \( k \). When \( k = 2 \) take \( \{ 2b - 1, 2b \} \). Now assume (3.1) holds for some \( k \geq 2 \). Let \( M \) be the least common multiple of the \( \binom{k}{2} \) numbers

\[
n_j - n_i \quad (1 \leq i < j \leq k),
\]

and let \( K \) be the least common multiple of the numbers

\[
M, bM - n_1, \ldots, bM - n_k.
\]

We claim that the set

\[
\{ n'_1, \ldots, n'_{k+1} \} = \{ Kb - bM + n_1, \ldots, Kb - bM + n_k, Kb \}
\]

satisfies (3.1). Indeed, when \( 1 \leq i < j \leq k \) we have

\[
\frac{n'_j}{n'_j - n'_i} = \frac{Kb - bM + n_j}{n_j - n_i},
\]

which by hypothesis is divisible by \( b \). Finally, for any \( i \leq k \),

\[
\frac{n'_{k+1}}{n'_{k+1} - n'_i} = \frac{Kb}{bM - n_i},
\]

which is also divisible by \( b \). \( \square \)

Now let \( b = \prod_{p \leq 2450} p \), and apply Lemma 3.1 with \( k = 50 \). There is a set \( \{ n_1, \ldots, n_{50} \} \) such that for all \( i < j \),

\[
(3.2) \quad b \big| \frac{n_j}{n_j - n_i}.
\]

Let \( M \) be the least common multiple of the \( \binom{50}{2} \) numbers

\[
n_j - n_i \quad (1 \leq i < j \leq 50).
\]

Then for any \( h \in \mathbb{N} \), the set \( \{ n_1 + hbM, \ldots, n_{50} + hbM \} \) has the same property. The collection of 2450 linear forms (in \( h \))

\[
\frac{2(n_i + hbM)}{n_j - n_i} + 1 = \frac{2(n_j + hbM)}{n_j - n_i} - 1 \quad (1 \leq i < j \leq 50)
\]

and

\[
\frac{2(n_j + hbM)}{n_j - n_i} + 1 \quad (1 \leq i < j \leq 50)
\]
is admissible by \((3.2)\), and the Prime \(k\)-tuples conjecture implies that all of these are prime for some \(h\). We need only the existence of one \(h\), and then the hypotheses of Theorem \(4\) hold with \(\ell_{i,j} = 2\) for all \(i, j\), and consequently \(4 \in D\). Discovering such an \(h\), however, appears to be computationally infeasible.

4. Totient gaps in progressions: proof of Theorem \(5\)

Lemma 4.1. Suppose that \(D \in \mathbb{N}\) and \(4|a\). Then there exist \(v_1\) and \(v_2\) such that \((D, v_1) = (D, v_2) = 1\) and \((v_1 - 1)(v_2 - 1) \equiv a \pmod D\) or \((v_1 + 1)(v_2 - 1) \equiv a \pmod D\).

Proof. We use the Chinese Remainder Theorem. We will prove that if \(D = p^{\alpha}\) is a prime power, then for \(p \neq 3\) we can find a pair \(v_1\) and \(v_2\) such that \((D, v_1) = (D, v_2) = 1\) and \((v_1 - 1)(v_2 - 1) \equiv a \pmod D\) and another pair \(v'_1\) and \(v'_2\) such that \((D, v'_1) = (D, v'_2) = 1\) and \((v'_1 + 1)(v'_2 - 1) \equiv a \pmod D\). If \(p = 3\) then we will find appropriate \(v_1\) and \(v_2\) such that one of desired congruences hold. This will suffice for the proof of the lemma.

If \(p \neq 3\) and for any \(a, 4|a\), a pair \(v_1\) and \(v_2\) exists such that \((D, v_1) = (D, v_2) = 1\) and \((v_1 - 1)(v_2 - 1) \equiv a \pmod D\), then there also exists a pair \(v'_1\) and \(v'_2\) as well. Indeed, take any possible \(D\) and \(a\). Then, by our supposition, there are \(v_1\) and \(v_2\) such that \((D, v_1) = (D, v_2) = 1\) and \((v_1 - 1)(v_2 - 1) \equiv -a \pmod D\). Then for \(v'_1 = -v_1\) and \(v'_2 = v_2\) the desired congruence \((v'_1 + 1)(v'_2 - 1) \equiv a \pmod D\) holds.

Consider \(p = 2\). Take \(v_1 \equiv 3 \pmod{p^{\alpha}}\) and \(v_2 \equiv \frac{3}{2} + 1 \pmod{p^{\alpha}}\). Then \((v_1 - 1)(v_2 - 1) \equiv a \pmod{p^{\alpha}}\) and both \(v_1, v_2\) are odd.

Consider \(p = 3\). If \(a \equiv 0 \pmod{3}\) or \(a \equiv 1 \pmod{3}\), let \(v_1 \equiv 2 \pmod{3^{\alpha}}\) and \(v_2 \equiv a + 1 \pmod{3^{\alpha}}\). Then we have \((v_1 - 1)(v_2 - 1) \equiv a \pmod{3^{\alpha}}\). If \(a \equiv -1 \pmod{3}\) let \(v_1 \equiv -2 \pmod{3^{\alpha}}\) and \(v_2 \equiv -a + 1 \pmod{3^{\alpha}}\). Then we have \((v_1 + 1)(v_2 - 1) \equiv a \pmod{3^{\alpha}}\).

Consider \(p > 3\). Take \(v_2\) so that \(v_2 \notin \{0, 1, 1-a\} \pmod{p}\). Then there is some \(v_1 \neq 0 \pmod{p}\) such that \((v_2 - 1)(v_1 - 1) \equiv a \pmod{p^{\alpha}}\).

Proof of Theorem \(5\). Let \(D\) be any positive integer satisfying

\(a\) \(d|D;\)

\(b\) \(D, D^2, \ldots, D^{49}\) are all in \(\mathcal{V}\).

For example, let \(P\) be the largest prime factor of \(d\), \(\gamma\) sufficiently large and

\[ D = d \prod_{p \leq P} p^{\gamma}. \]

Indeed, if

\[ D = \prod_{p \leq P} p^{\alpha(p)}, \quad \prod_{p \leq P} (p - 1) = \prod_{p \leq P} p^{\beta(p)}, \]

then, assuming \(\gamma \geq \max \beta(p)\), for all \(j \geq 1\) we have

\[ \phi\left( \prod_{p \leq P} p^{\alpha(p) - \beta(p) + 1} \right) = D^j. \]

Now take any \(D\) satisfying \((a)\) and \((b)\) above, and let \(v\) be coprime to \(D\). Then the set

\[ f_j(x) = D^j x - v, \quad j = 1, \ldots, 50, \]
of linear forms is admissible. Indeed, if \( p \nmid v \), then \( f_1(0) \cdots f_{50}(0) = v^{50} \neq 0 \) (mod \( p \)), and if \( p | v \) then \( p \nmid D \) and \( f_1(1) \cdots f_k(1) \equiv D_1^{1225} \neq 0 \) (mod \( p \)). Since DHL[50, 2] holds, there are \( j_1 < j_2 \) such that for infinitely many positive integers \( x \) both numbers \( p_1 = D^{j_1} x - v \), \( p_2 = D^{j_2} x - v \) are primes. Denote \( j = j_2 - j_1 \). There exists \( l \) such that \( \phi(l) = D^j \). If \( x \) is large enough, then \( (p_1, l) = (p_2, l) = 1 \). We have

\[
(4.1) \quad \phi(p_2) - \phi(p_1 l) = (p_2 - 1) - (p_1 - 1)D^j = (v + 1)(D^j - 1).
\]

Let \( v_1, v_2 \) be as in Lemma 4.1, and let \( v \) satisfy

\[
\begin{align*}
    v & \equiv -v_1 \pmod D, v > 0 & \text{if } (v_1 - 1)(v_2 - 1) & \equiv a \pmod D, \\
    v & \equiv v_1 \pmod D, v < -1 & \text{otherwise}.
\end{align*}
\]

Fix a prime \( q \equiv v_2 \pmod D \) with \((q, l) = 1\), and assume that \( p_1, p_2 > q \). Then

\[
(4.2) \quad \phi(p_2q) - \phi(p_1 l q) = (q - 1)(v + 1)(D^j - 1).
\]

Thus, \(|(q - 1)(v + 1)(D^j - 1)| \in D\). The right side of \((1.2)\) is \((q - 1)(v - 1) \equiv (v_2 - 1)(v - 1) \pmod D\) and has sign equal to the sign of \( v \). If \((v_1 - 1)(v_2 - 1) \equiv a \pmod D\), then \( v > 0 \) and thus the right side of \((1.2)\) is positive and congruent to \((v_2 - 1)(v_1 - 1) \equiv a \pmod D\). Otherwise, \( v < 0 \) and the right side of \((1.2)\) is negative and congruent to \((v_2 - 1)(v_1 - 1) \equiv -a \pmod D\). By varying \( q \), we find that there are infinitely many elements of \( D \) in the residue class \( a \pmod d \).

\[ \square \]

**Remarks.** Equation \((4.1)\) holds for any \( v \) coprime to \( D \). Thus, if \( a \equiv 2 \pmod 4 \) and either \((a + 1, D) = 1\) or \((a - 1, D) = 1\) then the residue class \( a \pmod d \) contains infinitely many elements of \( D \); take \( v \equiv -a - 1 \pmod d \), \( v > 0 \) if \((a + 1, D) = 1\) and \( v \equiv a - 1 \pmod d \), \( v < -1 \) if \((a - 1, D) = 1\). Thus, if \( d \) has at most two distinct prime factors, and (b) holds for some \( D \) composed only of the primes dividing \( d \), then every residue class \( a \pmod d \), with \( 2 | a \) contains infinitely many elements of \( D \). Note that in this case, for all \( a \) with \( 2 | a \), either \((a + 1, d) = 1\) or \((a - 1, d) = 1\). In particular this holds with \( d \) of the form \( 2^k, 2^k3^\ell \), or \( 2^k5^\ell \) with \( k \geq 2 \), since in each case (a) and (b) hold with \( D = d \). Item (b) also holds with \( d = D = 28 \) (verified with PARI/GP). We do not know how to derive the same conclusion if \( d \) has 3 or more prime factors, e.g. \( d = 60 \).

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