Universal Bundle, Generalized Russian Formula and Non-Abelian Anomaly in Topological Yang-Mills Theory

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Abstract

We re-examine the geometry and algebraic structure of BRST’s of Topological Yang-Mills theory based on the universal bundle formalism of Atiyah and Singer. This enables us to find a natural generalization of the Russian formula and descent equations, which can be used as algebraic method to find the non-Abelian anomalies counterparts in Topological Yang-Mills theory. We suggest that the presence of the non-Abelian anomaly obstructs the proper definition of Donaldson’s invariants.

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1. Introduction

The topological origin of the non-Abelian anomaly have been investigated by Atiyah and Singer in terms of universal bundle formalism. Let $M$ denote a $2n - 2$ dimensional oriented compact Riemann manifold. Consider a principal $G$-bundle $P$ over base space $M$. Let $U$ denote the affine space of all connections on $P$ and $G$ denote bundle automorphism, which acts as the gauge symmetry group. Now consider a principal $G$-bundle over base space $(P \times U)/G$ where $G$ acts freely; $(P \times U, G, (P \times U)/G)$. The base space of the above bundle itself can be regarded as a principal $G$-bundle over $M \times U/G$, which is called the *universal bundle* $\left( (P \times U)/G, G, M \times U/G \right)$.

Let $\hat{F}$ denote curvature form over $M \times U/G$, then, one can define a characteristic class given by the rank $n$ invariant polynomial $\text{Tr} (\hat{F}^n)$, which yields a two-form on $U/G$ after integrating over $M$. The resulting two-form is related to the $2n - 2$ dimensional non-Abelian anomaly\[^{2}\]. A systematic algebraic method for obtaining $2n - 2$ dimensional non-Abelian anomaly from $2n$ dimensional Abelian anomaly, which is given by the rank $n$ invariant polynomial $\text{Tr} (F^n)$ on $M$, was developed using the *Russian formula* and *descent equation*\[^{2}\][\(^{3}\)]. And, of course, both methods are closely related.

Recently, the universal bundle formalism has been used for an another important application to Donaldson-Witten theory of the smooth four dimensional invariants\[^{4}\][\(^{5}\)]. Let $M$ be a four dimensional compact oriented Riemann manifold. Now one restricts the orbit space $U/G$ to the moduli space $\mathcal{M}$ of anti-self dual connections (the universal instanton bundle). One can define the invariant second rank polynomial $\text{Tr} (\hat{F}^2)$ which is an element of the cohomology class $H^4(M \times \mathcal{M}, R)$. If we restrict the base space to $Y \times \mathcal{M}$, where $Y$ is an $r$-dimensional submanifold of $M$, and by integrating $\text{Tr} (\hat{F}^2)$ along the fibers of the projection $Y \times \mathcal{M} \to \mathcal{M}$, we can get an element in $H^{4-r}(\mathcal{M}, R)$ which depends only on the homology class of $Y$, i.e. $H_r(M, R)$\[^{3}\]. Witten has used the elements $\hat{W}^{4-r_i} \in H^{4-r_i}(\mathcal{M}, R)$ as the basic observables of his Topological Yang-Mills Theory (TYMT in short), which is designed such that the physical configurations of theory are precisely the instanton moduli space, and interpreted Donaldson’s invariant as the expectation value of observables

$$\left< \hat{W}^{4-r_1} \ldots \hat{W}^{4-r_k} \right>, \quad (1.2)$$
\[ \sum_{i=1}^{k} (4 - r_i) = d(M), \quad (1.3) \]

where \( d(M) \) denote the dimension of moduli space. In particular, Witten has introduced fermionic variables \((\Psi, \Phi, \chi)\), whose zero modes precisely correspond to the cohomology classes of instanton complex\(^3\), with \(U\)-charges (the ghost numbers of Witten’s BRST-like operator \(\delta_W\)) \((1, -1, -1)\) such that

\[ d(M)_f = n_\Psi - n_\Phi - n_\chi, \quad (1.4) \]

where \( d(M)_f \) denote the formal dimension\(^3\) of moduli space and \( n_x \) denotes the number of zero-modes of fermionic fields. Witten’s action has the total ghost number zero and an observable \( \tilde{W}^{4-r_i} \) has the ghost number \((4 - r_i)\). Thus the non-zero dimension of moduli space implies the ghost number anomaly (an Abelian anomaly) and suitable set of observables should be inserted according to the superselection rule\(^{13}\) to compensate the ghost number anomaly\(^3\)\(^7\). That is, Donaldson’s invariants are manifestations of the Abelian anomaly.

Now it is natural to ask what is the non-Abelian anomaly counterpart in Donaldson-Witten theory. In this letter we suggest that the universal bundle formalism of Atiyah and Singer can provide an answer to the above question. In particular, we propose an extended Russian formula and descent equation which can be used to find the candidates for non-Abelian anomaly counterpart in TYMT. We also find that the universal bundle formalism can provide an unified picture of the various BRST algebras; the conventional BRS algebra\(^8\), Witten’s \(\delta_W\) algebra, the \(\delta_T\) algebra proposed by the authors of ref.\(^9\) and Horne’s BRS algebra\(^10\). Similar formalism had already proposed by Kanno\(^11\) and reformulated by Birmingham et. al.\(^12\). However, these authors had failed to uncover the complete structures the various BRST algebras. And, as the results, the extended Russian formula and descent equation were absent in their formalism.

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\(^1\) The formal dimension becomes actual dimension of moduli space when there are no \(\Phi\) and \(\chi\) zero-modes. The \(\Phi\) and \(\chi\) zero-modes are related to the singularities of moduli space, and Donaldson’s invariants are not well-defined in this case. In particular \(\Phi\) zero-modes arise when there are reducible connections. We shall discuss this issue later.
2. Universal Bundle and BRST Algebras

In this section we discuss the origin of the various BRST algebras - the conventional BRS ($\delta_{\text{BRS}}$) algebra\cite[10], Witten’s BRST ($\delta_{\text{W}}$) algebra\cite[5] and $\delta_{\text{T}}$ algebra\cite[9], of TYMT based on the universal bundle formalism\cite[1]. The presentations of this section are based on the ref.\cite[4], and we generally follow the conventions of ref.\cite[3].

It is convenient to start from a pull backed bundle $Q$ over $M \times U$ from the universal bundle (1.1). One can locally parametrize $U$ by

$$A = g^{-1} A g + g^{-1} dg,$$

(2.1)

where $A$ denotes a fixed connection one-form tangent to $M$, and $g$ is an element of gauge group $\mathcal{G}$ which depends on both the space-time coordinates $x^\mu$ and some group parameter $\lambda^i$. An arbitrary variation on $A$ is

$$\delta A = g^{-1} \delta A g - d_A (g^{-1} \delta g),$$

(2.2)

where we will interpret the operator $\delta$ as the exterior derivative tangent to $U$. If we decompose the operator $\delta$ as

$$\delta = \delta_T + \delta_{\text{BRS}},$$

(2.3)

such that $\delta_{\text{BRS}}$ is the variation (exterior derivation) along gauge group $\mathcal{G}$

$$\delta_{\text{BRS}} = d\lambda^i \frac{\partial}{\partial \lambda^i},$$

(2.4)

and $\delta_T$ is the exterior derivative over the orbit space $U/\mathcal{G}$

$$\delta^2 = \delta^{2}_T = \delta^2_{\text{BRS}} = \delta_T \delta_{\text{BRS}} + \delta_{\text{BRS}} \delta_T = 0.$$  

(2.5)

Then (2.2) becomes

$$\delta A = g^{-1} \delta_T A g - d_A (g^{-1} \delta_T g) - d_A (g^{-1} \delta_{\text{BRS}} g).$$

(2.6)

such that

$$\delta_T A = g^{-1} \delta_T A g - d_A (g^{-1} \delta_T g),$$

$$\delta_{\text{BRS}} A = -d_A (g^{-1} \delta_{\text{BRS}} g),$$

(2.7)

\footnote{We also naturally extend the action of $g$ to the Lie algebra valued-forms tangent to the orbit space $U/\mathcal{G}$.}
where we have used $\delta_{\text{BRS}} A = 0$.

Introducing the connection one-form on $U$

$$-G_A d_A^* \delta A \equiv C, \quad (2.8)$$

where

$$G_A = (d_A^* d_A)^{-1}.$$  

The connection one-form $C$ can be also decomposed as

$$C = -G_A d_A^* \delta A = -G_A d_A^* \delta_T A - G_A d_A^* \delta_{\text{BRS}} A, \quad (2.9)$$

Then one can define the Faddev-Popov ghost $v$ as

$$-G_A d_A^* \delta_{\text{BRS}} A = g^{-1} \delta_{\text{BRS}} g \equiv v, \quad (2.10)$$

which is the connection one-form along $G$, and the BRS algebra naturally follows

$$\delta_{\text{BRS}} A = -dv - A v - v A \equiv -d_A v, \quad (2.11)$$

The total connection one-form over $M \times U$ is

$$A + C = A - G_A d_A^* \delta A, \quad (2.12)$$

and total curvature over $M \times U$ is

$$\hat{F} = (d + \delta) (A - G_A d_A^* \delta A) + (A - G_A d_A^* \delta A)^2 \quad = F + (1 - d_A G_A d_A^*) \delta A - \delta (G_A d_A^* \delta A) + (G_A d_A^* \delta A)^2, \quad (2.13)$$

which can be written in components

$$\hat{F}^{2,0} \equiv F = dA + A^2,$$
$$\hat{F}^{1,1} = (1 - d_A G_A d_A^*) \delta A, \quad (2.14)$$
$$\hat{F}^{0,2} = -\delta (G_A d_A^* \delta A) + (G_A d_A^* \delta A)^2.$$

Using the decomposition \([2.7]\) and \([2.7](2.9)(2.10)(2.11)\) we can get

$$\hat{F}^{1,1} = (1 - d_A G_A d_A^*) (\delta_T A + \delta_{\text{BRS}} A), \quad (2.15)$$
$$= (1 - d_A G_A d_A^*) \delta_T A, $$
\[ \hat{F}^{0,2} = - \delta_t (G_A d_A^* \delta_t A) + (G_A d_A^* \delta_t A)^2 + \delta_{\text{BRS}} v + v^2 \\
+ \delta_t v - \delta_{\text{BRS}} (G_A d_A^* \delta_t A) - \{G_A d_A^* \delta_t A, v\} \\
= - \delta_t (G_A d_A^* \delta_t A) + (G_A d_A^* \delta_t A)^2. \]

Thus, \( \hat{F} \) is also given by

\[ \hat{F} = F + (1 - d_A G_A d_A^*) \delta_t A - \delta_t (G_A d_A^* \delta_t A) + (G_A d_A^* \delta_t A)^2, \] (2.17)

which means \( \hat{F} \) is the total curvature over \( M \times \mathcal{U}/\mathcal{G} \). We can call the above relation an extended Russian formula. That is, if one restricts the variation of \( A \) in (2.12) to the gauge group direction, the above two equations (2.13)(2.17) lead to the well-known Russian formula.

Note that \((1 - d_A G_A d_A^*)\) is the horizontal projection. Then

\[ \hat{F}^{1,1} = (1 - d_A G_A d_A^*) \delta_t A \equiv \delta^H A, \] (2.18)

where \( \delta^H \) denotes the operator of horizontal tangent vector at \( \mathcal{U}/\mathcal{G} \). Furthermore, direct calculation shows that

\[ \delta^H (\delta^H A) = -d_A \hat{F}^{0,2}, \quad \delta^H \hat{F}^{0,2} = 0. \] (2.19)

Being the horizontal variation, \( \delta^H A \) should satisfy

\[ d_A^*(\delta^H A) = 0. \] (2.20)

Applying \( \delta^H \) to the above condition, we can get

\[ \delta^H (d_A^* \delta^H A) = [\delta^H \ast A, \delta^H A] - d_A^*(\delta^H (\delta^H A)) \\
= [\delta^H \ast A, \delta^H A] + d_A^* d_A \Phi \] (2.21)

which can be read as

\[ \Phi = \hat{F}^{0,2} = -G_A [\delta^H \ast A, \delta^H A]. \] (2.22)

Thus we have obtained Atiyah-Singer’s results.

Note that if we denote \( \hat{F}^{1,1} \equiv \Psi, \hat{F}^{0,2} = \Phi \) such that

\[ \hat{F} = F + \Psi + \Phi, \] (2.23)
and $\delta^H \equiv \delta_w$, we can get Witten’s BRST algebra
\begin{align}
\delta_w A &= \Psi, \\
\delta_w \Psi &= -d_A \Phi, \\
\delta_w \Phi &= 0.
\end{align}
(2.24)

Let
\begin{align}
C \equiv -G_A d_A^* \delta_r A,
\end{align}
(2.25)
such that (2.9) becomes
\begin{align}
C = C + v.
\end{align}
(2.26)

Then (2.15)(2.16) lead to the $\delta_r$ algebra \[9\]
\begin{align}
\delta_r A &= \Psi - d_A C, \\
\delta_r C &= \Phi - C^2, \\
\delta_r \Psi &= -[C, \Psi] - d_A \Phi, \\
\delta_r \Phi &= -[C, \Phi],
\end{align}
(2.27)
where the last two relation follow from $\delta_r^2 = 0$. Using
\begin{align}
\delta_r A &= \Psi - d_A C,
\end{align}
(2.28)
one can find that
\begin{align}
-G_A d_A^* \delta_r A = -G_A d_A^* \Psi + C.
\end{align}
(2.29)

Note that the following condition is crucial to the self-consistency of $\delta_r$ algebra
\begin{align}
d_A^* \Psi = 0,
\end{align}
(2.30)
which is identical to (2.20). We can also find $\delta_{\text{BRST}}$ algebra from eq. (2.11)(2.14)(2.15)(2.16)
\begin{align}
\delta_{\text{BRST}} A &= -d_A v, \\
\delta_{\text{BRST}} v &= -v^2, \\
\delta_{\text{BRST}} \mathcal{F} &= -[v, \mathcal{F}], \\
\delta_{\text{BRST}} \Psi &= -[v, \Psi], \\
\delta_{\text{BRST}} \Phi &= -[v, \Phi],
\end{align}
(2.31)
with
\begin{align}
\delta_{\text{BRST}} C = -\delta_r v - \{C, v\},
\end{align}
(2.32)
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which is identical to the additional BRS algebra of ref. [10]. We shall see that this additional BRS structure to \( \delta_T \) and \( \delta_W \) is crucial for non-Abelian anomalies in TYMT. Note that we have a natural bi-grading structure of ghost numbers such that the ghost numbers of the fields \((A, C, \Psi, \Phi, v)\) are \((0, 1, 1, 2, 0)\) for \( \delta_T \) algebra and \((0, 0, 0, 1)\) for \( \delta_{BRS} \)-algebra.

Let \( \text{Tr} (\hat{F}^n) \) denote an invariant polynomial of degree \( n \), such that

\[
\delta_{BRS} \text{Tr} (\hat{F}^n) = 0, \quad (d + \delta_T) \text{Tr} (\hat{F}^n) = 0,
\]

where the first relation follows from (2.31) and the second one from the Bianchi identity

\[
(d + \delta_T) \hat{F} + [A + C, \hat{F}] = 0.
\]

We can expanding the invariant polynomials \( \text{Tr} (\hat{F}^n) \) in terms of the \( \delta_T \) ghost number

\[
\text{Tr} (\hat{F}^n) = \tilde{W}_{2n-0} + \tilde{W}_{2n-1} + \cdots + \tilde{W}_0,
\]

where the superscripts indicate \( \delta_{BRS} \) and \( \delta_T \) ghost numbers, respectively, and the subscript indicate the space-time form degree. It is well-known that the integration of each terms \( \tilde{W}_{2n-k} \) in (2.35) over \( 2n - k \) cycle \( \gamma_{2n-k} \) are Witten’s observables for Donaldson’s polynomial invariant [3][11]:

\[
\int_{\gamma_{2n-k}} \tilde{W}_{2n-k} = \tilde{W}^{0,k}.
\]

Combining (2.33)(2.35) we get

\[
d\tilde{W}_{2n-0} = 0,
\]

\[
d\tilde{W}_{2n-1} + \delta_T \tilde{W}_{2n-0} = 0,
\]

\[
d\tilde{W}_{2n-2} + \delta_T \tilde{W}_{2n-1} = 0,
\]

\[
\vdots
\]

\[
d\tilde{W}_0 + \delta_T \tilde{W}_1 = 0,
\]

\[
\delta_T \tilde{W}_0 = 0,
\]

which is identical to the topological descent equation of ref. [3][11]. Integrating \( i \)th \((i = 2, \ldots, 2n)\) relation of (2.37) over \( 2n + 2 - i \) cycle, we can see that Witten’s observables \( \tilde{W}_{2n-k} \) are \( \delta_T \) closed;

\[
\delta_T \int_{\gamma_{2n-k}} \tilde{W}_{2n-k} = \delta_T \tilde{W}^{0,k} = 0.
\]
3. Extended Russian Formula and Descent Equation

Now consider the extended Russian formula of eqs.(2.13)--(2.17), which can be written
as
\[ \hat{F} = (d + \delta_\tau + \delta_{BRS})(A + C + v) + (A + C + v)^2, \]
(3.1)
which reduces to the familiar Russian formula\[2][3][13]
\[ F = (d + \delta_{BRS})(A + v) + (A + v)^2 = dA + A^2, \]
(3.2)
for \( C = 0 \), i.e. \( (\Psi = \Phi = 0) \). Note that \( \hat{F} \) satisfies Bianchi identities
\[ 0 = (d + \delta_\tau)\hat{F} + [A + C, \hat{F}] \]
\[ = (d + \delta_\tau + \delta_{BRS})\hat{F} + [A + C + v, \hat{F}] = 0. \]
(3.3)
Thus we can see that the invariant polynomial of degree \( n \), \( \text{Tr} (\hat{F}^n) \) satisfies
\[ (d + \delta_\tau)\text{Tr} (\hat{F}^n) = (d + \delta_\tau + \delta_{BRS})\text{Tr} (\hat{F}^n) = 0. \]
(3.4)
Then, by the Poincaré lemma, (3.4) implies
\[ \text{Tr} (\hat{F}^n) = (d + \delta_\tau)W_{2n-1}(A + C, \hat{F}) \]
\[ = (d + \delta_\tau + \delta_{BRS})W_{2n-1}(A + C + v, \hat{F}), \]
(3.5)
where \( W_{2n-1} \) denotes the generalized Chern-Simons form. Expanding \( W_{2n-1}(A+C+v, \hat{F}) \)
with powers of \( v \)
\[ W_{2n-1}(A + C + v, \hat{F}) = W_{2n-1}^0(A + C, \hat{F}) + W_{2n-2}^1 + \cdots + W_0^{2n-1}, \]
(3.6)
where the superscript indicate the power of \( v \) (\( \delta_{BRS} \) ghost number) and the subscript indicates the space-time form degree plus the \( \delta_\tau \) ghost number. Then (3.5) reduce to
\[ (d + \delta_\tau)W_{2n-2}^1 + \delta_{BRS}W_{2n-1}^0 = 0, \]
\[ (d + \delta_\tau)W_{2n-3}^2 + \delta_{BRS}W_{2n-2}^1 = 0, \]
\[ \vdots \]
\[ (d + \delta_\tau)W_0^{2n-1} + \delta_{BRS}W_1^{2n-2} = 0, \]
\[ \delta_{BRS}W_0^{2n-1} = 0. \]
(3.7)
We shall call the above relations an *extended descent equation*, which is an obvious extension of the usual descent equation\[^2\][^3][^13] of Yang-Mills theory (the original equation can be recovered for \(C = 0\), i.e. \(\Psi = \Phi = 0\)).

We can expand the relations of (3.7) in terms of \(\delta_r\) ghost number. In particular, consider the second relation of (3.7)

\[
(d + \delta_r)W_{2n-3}^2 + \delta_{BRS} W_{2n-2}^1 = 0, \tag{3.8}
\]

which leads to

\[
dW_{2n-3}^{2,0} + \delta_{BRS} W_{2n-2}^{1,0} = 0,
\]

\[
dW_{2n-4}^{2,1} + \delta_r W_{2n-3}^{2,0} + \delta_{BRS} W_{2n-3}^{1,1} = 0,
\]

\[
dW_{2n-5}^{2,2} + \delta_r W_{2n-4}^{2,1} + \delta_{BRS} W_{2n-4}^{1,2} = 0,
\]

\[
\vdots
\]

\[
dW_0^{2,2n-3} + \delta_r W_1^{2,2n-4} + \delta_{BRS} W_1^{1,2n-3} = 0,
\]

\[
\delta_r W_0^{2,2n-3} + \delta_{BRS} W_0^{1,2n-2} = 0,
\]

where we have used the expansions in terms of \(\delta_r\) ghost number

\[
W_{2n-2}^1 = W_{2n-2}^{1,0} + W_{2n-3}^{1,1} + \cdots + W_0^{1,2n-2},
\]

\[
W_{2n-3}^2 = W_{2n-3}^{2,0} + W_{2n-4}^{2,1} + \cdots + W_0^{2,2n-3},
\]

such that the superscripts indicate \(\delta_{BRS}\) and \(\delta_r\) ghost numbers, respectively, and the subscript indicate the space-time form degree.

Consider the first relation of (3.9)

\[
dW_{2n-3}^{2,0} + \delta_{BRS} W_{2n-2}^{1,0} = 0,
\]

which leads the well-known Wess-Zumino consistency condition\[^{14}\]

\[
\delta_{BRS} \int_{\gamma_{2n-2}} W_{2n-2}^{1,0} = 0, \tag{3.10}
\]

that is, \(W_{2n-2}^{1,0}\) gives the non-Abelian anomaly of local Yang-Mills theory in \(2n - 2\) dimensions. Integrating the i-th relation of (3.9) over \(2n - 1 - i\) cycle

\[
\delta_{BRS} W_{2n-2}^{1,0} = 0,
\]

\[
\delta_r W_{2n-2}^{2,\ell-1} + \delta_{BRS} W_{2n-2}^{1,\ell} = 0, \quad \text{for } \ell = 1, \ldots, 2n - 2 \tag{3.11}
\]
where

\[ W_{2n-2-\ell 1,\ell} = \int_{\gamma_{2n-2-\ell}} W_{2n-2-\ell 1,\ell}, \quad W_{2n-2-\ell 2,\ell-1} = \int_{\gamma_{2n-2-\ell}} W_{2n-2-\ell 2,\ell-1}. \]

We conjecture that eq. (3.11) is the extended consistency condition and \( W_{2n-2-\ell 1,\ell} \) are the non-Abelian anomaly counterparts in TYMT.

If we expand the first relation of (3.7) in terms of \( \delta_T \) ghost number

\[
\begin{align*}
  dW_{2n-2}^{1,0} + \delta_{\text{BRS}} W_{2n-1}^{0,0} &= 0, \\
  dW_{2n-3}^{1,1} + \delta_T W_{2n-2}^{1,0} + \delta_{\text{BRS}} W_{2n-2}^{0,1} &= 0, \\
  &\vdots \\
  dW_0^{1,2n-2} + \delta_T W_1^{1,2n-3} + \delta_{\text{BRS}} W_1^{0,2n-2} &= 0, \\
  \delta_T W_0^{1,2n-2} + \delta_{\text{BRS}} W_0^{0,2n-1} &= 0,
\end{align*}
\]

which leads

\[
\delta_T W_{2n-2-j}^{1,j} = -\delta_{\text{BRS}} W_{2n-2-j}^{0,j+1},
\]

where \( j = 0, \ldots, 2n-2 \). Thus \( W_{2n-2-j}^{1,j} \) is \( \delta_{\text{BRS}} (\delta_T) \) closed up to \( \delta_T (\delta_{\text{BRS}}) \) exact term. In addition \( W_{2n-2-j}^{1,j} \) depends only on the homology class of \( \gamma \) up to BRST exact term like Witten’s observables \( \tilde{W}^{0,k} \). That is, if \( \gamma_{2n-2-j} \) is a boundary, say \( \gamma_{2n-2-j} = \partial \beta_{2n-1-j} \), then

\[
\begin{align*}
  W_{2n-2-j}^{1,j} &= \int_{\gamma_{2n-2-j}} W_{2n-2-j}^{1,j} \\
  &= \int_{\beta_{2n-1-j}} dW_{2n-2-j}^{1,j} \\
  &= -\delta_T \int_{\beta_{2n-1-j}} W_{2n-1-j}^{1,j-1} - \delta_{\text{BRS}} \int_{\beta_{2n-1-j}} W_{2n-1-j}^{0,j}.
\end{align*}
\]

Note that the algebraic structures of both the usual descent equation and the extended one (3.7) are identical if we replace \( d \) with \( d + \delta_T \) and \( A \) with \( A + C \). Thus we can follow the same procedure discussed in [3]. By introducing the one-parameter family of connection 1-forms over \( M \times U/G \)

\[
\hat{A}_t = t(A + C),
\]
and associated field strengths
\[ \hat{F}_t = (d + \delta_r) \hat{A}_t + \hat{A}_t^2 \]
\[ = t(d + \delta_r)(A + C) + t^2(A + C)^2, \]
we can obtain the Chern-Weil formula\[3\]
\[ \text{Tr} (\hat{F}^n) = n(d + \delta_r) \int_0^1 dt \text{Tr} (\hat{A} \hat{F}_t^{n-1}), \]
\[ = (d + \delta_r) W_{2n-1} (A + C, \hat{F}). \]

Note that Tr (\hat{F}^n) is also given by
\[ \text{Tr} (\hat{F}^n) = n(d + \delta_r + \delta_{\text{BRS}}) \int_0^1 dt \text{Tr} ((\hat{A} + v) \hat{F}_t^{n-1}), \]
\[ = (d + \delta_r + \delta_{\text{BRS}}) W_{2n-1} (A + C + v, \hat{F}). \]

Now we can explicitly expand \( W_{2n-1} (A + C + v, \hat{F}) \) in powers of \( v \) as (3.6) and obtain the first order in \( v \)
\[ W_{2n-2} = n(n - 1) \int_0^1 dt (1 - t) \text{Str} \left( v(d + \delta_r) \left( (A + C) \hat{F}_t^{n-2} \right) \right), \]
where \( \text{Str} \) denotes the symmetrized trace\[3\]
\[ \text{Str}(B_1, B_2, \ldots, B_n) \equiv \sum_{\text{Perm.}} \frac{1}{n!} \text{Tr} (B_{p(1)} \cdots B_{p(n)}). \]

The next step is to expand (3.13) in terms of \( \delta_r \) ghost number, and the resulting expressions reduce to \( W_{2n-2-j} \). To be definite, let us consider the explicit expression for the non-Abelian anomaly counterpart in TYMT of 4-dimension, i.e. \( n = 3; \)
\[ W_4^1 = \text{Tr} \left( v(d + \delta_r) \left( (A + C)(d + \delta_r)(A + C) + \frac{1}{2}(A + C)^3 \right) \right), \]
which can be expanded in terms of \( \delta_r \) ghost number
\[ W_4^{1,0} = \text{Tr} \left[ v d \left( A F - \frac{1}{2} A^3 \right) \right], \]
\[ W_3^{1,1} = \text{Tr} \left[ v \delta_r \left( A F - \frac{1}{2} A^3 \right) + v d \left( A(\Psi - \frac{1}{2}[A, C]) + C(F - \frac{1}{2} A^2) \right) \right], \]
\[ W_2^{1,2} = \text{Tr} \left[ v \delta_r \left( A(\Psi - \frac{1}{2}[A, C]) + C(F - \frac{1}{2} A^2) \right) \right. \]
\[ + v d \left( C(\Psi - \frac{1}{2}[A, C]) + A(\Phi - \frac{1}{2} C^2) \right) \right], \]
\[ W_1^{1,3} = \text{Tr} \left[ v \delta_r \left( C(\Psi - \frac{1}{2}[A, C]) + A(\Phi - \frac{1}{2} C^2) \right) + v d \left( C\Phi - \frac{1}{2} C^3 \right) \right], \]
\[ W_0^{1,4} = \text{Tr} \left[ v \delta_r \left( C\Phi - \frac{1}{2} C^3 \right) \right]. \]
The complete explicit solutions for the extended descent equation (3.7) will not be discussed here [13].

4. Discussion

We have seen that the various BRST structures of TYMT can be naturally unified in terms of the universal bundle formalism. In particular, the BRS ($\delta_{\text{BRST}}$) structure was crucial in extending Russian formula and descent equation, which enable us to find the non-Abelian anomaly counterpart in TYMT.

Note that $W_{2n-2-j}^{1,j}$ has the $\delta_{\text{BRST}}$ ghost number 1, while the observable for Donaldson’s invariant $\tilde{W}_{0,k}$ has the $\delta_{\text{BRST}}$ ghost number 0. Thus, it is closely related to the zero-modes of the Faddev-Popov ghost $\nu$, which exist either for reducible connections or the Gribov ambiguity [16][17]. This is due to the obstruction for existence of global cross section of $U \to U/G$ [18]. The origin of the non-Abelian anomaly ($W_{2n-2}^{1,0}$) of local Yang-Mills theory had already been interpreted in this way [19]. Topological Yang-Mills theory has further source of such obstruction, because it is defined on the space of anti-self dual connection modulo $G$. Note also that the reducible connection is a source of singularity in the instanton moduli space and Donaldson’s invariants are not well-defined in this case [3]. Thus, the generalized non-Abelian anomaly can be regarded as an obstruction to defining Donaldson’s invariants.

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