THE LEAST NUMBER WITH PRESCRIBED LEGENDRE SYMBOLS

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Abstract. In this article we estimate the number of integers up to $X$ which can be properly represented by a positive-definite, binary, integral quadratic form of small discriminant. This estimate follows from understanding the vector of signs that arises from computing the Legendre symbol of small integers $n$ at multiple primes.

1. Introduction

A well-known and outstanding problem in number theory is the estimation of the least quadratic non-residue modulo a prime $p$. Recall, the least quadratic non-residue modulo $p$ is the integer

$$n_p = \min \left\{ n > 1 : \left( \frac{n}{p} \right) = -1 \right\},$$

where $\left( \frac{x}{p} \right)$ is the Legendre symbol of $x$ modulo $p$. Vinogradov conjectured that for any $\varepsilon > 0$, $n_p$ should be at most $p^\varepsilon$ provided $p$ is sufficiently large relative to $\varepsilon$. This conjecture is still open, however Linnik proved that any exceptions to it are sparse. Specifically, he proved the conjecture holds for all but $O(\log \log N)$ of the primes $p \leq N$, with the implied constant depending only on $\varepsilon$.

Now suppose $p_1$ and $p_2$ are distinct, odd primes and $n$ is an integer not divisible by either. There are four possibilities for the vector $\left( \left( \frac{n}{p_1} \right), \left( \frac{n}{p_2} \right) \right)$. How large must $N$ be so that all four vectors are realized by integers bounded by $N$? In general, one might ask the following.

**Problem 1.** Given distinct odd primes $p_1, \ldots, p_k$, how large must $N$ be before one has seen each of the $2^k$ different vectors of signs in $\{1, -1\}^k$ realized by a vector of the form

$$\left( \left( \frac{n}{p_1} \right), \ldots, \left( \frac{n}{p_k} \right) \right)$$
with \( 1 \leq n \leq N \)?

We believe that the above problem is intrinsically interesting, but there is further motivation studying it. Recall that the quadratic form \( F(x, y) = Ax^2 + Bxy + Cy^2 \) has discriminant \( d = B^2 - 4AC \), is said to represent \( q \) if \( F(x, y) = q \) has a solution \((x, y) \in \mathbb{Z}^2\) with \( x \) and \( y \), and is said to properly represent \( q \) if furthermore \( x \) and \( y \) can be taken to be relatively prime. We say the form is definite if \( d < 0 \). One could then ask,

**Problem 2.** *What is the least positive integer \( d \) such that \( q \) is represented by a quadratic form of discriminant \(-d\)?*

If we allow for indefinite forms, which is to say forms with positive discriminant, then this problem is less interesting. A number is a difference of squares if and only if it is not congruent to 2 mod 4. Thus, either \( q \) is not congruent to 2 mod 4 and \( q = x^2 - y^2 \) has a solution, or else \( q/2 \) is not congruent to 2 mod 4 in which case \( q = 2x^2 - 2y^2 \) has a solution.

As is outlined below, answering Problem 2 essentially amounts to the solution of Problem 1 because of the following theorem (see for instance [B, Proposition 4.1]).

**Theorem 1.** *The number \( q \) is properly represented by a binary quadratic form of discriminant \( d \) if and only if \( d \) is a square modulo \( 4q \).*

When \( q \) is an odd prime and \( q \equiv 3 \pmod{4} \), Problem 2 boils down to that of finding the least quadratic non-residue modulo \( q \). But when handling Problem 2 for general \( q \), rather than have \( \left( \frac{-d}{q} \right) = -1 \), we require that \( \left( \frac{d}{p} \right) = (-1)^{\frac{p-1}{2}} \) for each odd prime \( p \) dividing \( q \). Thus we are interested in prescribing the Legendre symbol of \( d \) at several primes, which returns us to Problem 1.

One goal of this article is to extend Linnik’s result on the least non-residue and show that one can usually prescribe the sign of the Legendre symbol simultaneously at many primes with a small integer, barring some “local” obstructions as described in the next section.
2. Notation, preliminary observations, and statement of results

To begin, let \( k \geq 1 \) be an integer and \( p_1, \ldots, p_k \leq N \) be distinct odd primes. Let \( p = (p_1, \ldots, p_k) \), \( q = p_1 \cdots p_k \) and write
\[
\left( \frac{n}{p} \right) = \left( \left( \frac{n}{p_1} \right), \ldots, \left( \frac{n}{p_k} \right) \right).
\]
Finally, denote by \( G = \{ \pm 1 \}^k \) the (multiplicative) group of all \( k \)-tuples with entries \( \pm 1 \). As described in the introduction, we are interested in the number
\[
q = \max_{e \in G} \min \left\{ n : n \geq 1, \left( \frac{n}{p} \right) = e, n \equiv 1 \mod 8 \right\}.
\]
Thus \( q \) is the least positive integer such that we observe all possible sign choices for \( \left( \frac{n}{p} \right) \) with integers less than \( q \) and congruent to 1 mod 8. The congruence condition allows us to further insist that we deal only with squares modulo a power of two, which is needed in applications.

It is convenient to identify the group \( G \) with \( \mathbb{F}_2^k = \left( \mathbb{Z}/2\mathbb{Z} \right)^k \), the vector space of dimension \( k \) over the field \( \mathbb{F}_2 \), in the natural way. To be concrete, set
\[
U_q(y) = \{ n : 1 \leq n \leq y, \ (n, q) = 1, \ n \equiv 1 \mod 8 \}.
\]
Consider the map \( \theta_q : U_q(y) \to \mathbb{F}_2^k \) given by
\[
\theta_q(n) = \left( \frac{1}{2} \left( 1 - \frac{n}{p_1} \right), \ldots, \frac{1}{2} \left( 1 - \frac{n}{p_k} \right) \right) \mod 2.
\]
The \( i \)th entry of \( \theta_q(n) \) is 1 mod 2 if \( n \) is a quadratic non-residue modulo \( p_i \) and 0 mod 2 if \( n \) is a quadratic residue modulo \( p_i \). The map \( \theta_q \) is also an additive function in the sense that \( \theta_q(mn) = \theta_q(m) + \theta_q(n) \), with the addition operation belonging to \( \mathbb{F}_2^k \). Moreover, for \( y \) sufficiently large the map \( \theta_q \) is surjective by Chinese Remainder Theorem and the fact that the primes \( p_i \) are distinct and odd.

Suppose we know that the integers in \( U_q(y) \) span \( \mathbb{F}_2^k \), in the sense that \( \{ \theta_q(n) : n \in U_q(y) \} \) contains a basis of \( \mathbb{F}_2^k \). Then \( q \leq y^k \). Indeed, any vector \( e \in \mathbb{F}_2^k \) is the sum of at most \( k \) basis vectors, each of which is of the form \( \theta_q(m) \) with \( m \leq y \). The product of these integers \( m \) gives
an integer \( n \) with \( \theta_q(n) = e \). It is therefore sensible to consider the number
\[
g_q = \min \{ y : \{ \theta_q(n) : n \in U_q(y) \} \text{ spans } \mathbb{F}_2^k \}.
\]
We record the above observation as a lemma.

**Lemma 1.** Let \( q = p_1 \cdots p_k \) be an odd, square-free integer. Then
\[
n_q \leq g_q^k.
\]

We have so far reduced the problem of bounding \( n_q \) to that of bounding \( g_q \). In the spirit of Linnik, we would like to estimate the number of \( q \) for which \( g_q \) is large. However, we need to be mindful of the following obstruction. If \( g_d > y \) for some divisor \( d \) of \( q \) then \( g_q > y \) as well. Thus, if \( d \) is small and \( g_d > y \), then \( g_q > y \) for at least \( \lfloor Q/d \rfloor \) numbers up to \( Q \) (the multiples of \( d \)), which is substantial. So, we need to restrict our attention to what we will call *eligible* \( q \), namely those which are not divisible by some \( d \) for which \( g_d \) is large. In fact, for technical reasons, we have need to consider the number \( g_{q,r} \) defined as
\[
g_{q,r} = \min \{ y : \{ \theta_q(n) : n \in U_{qr}(y) \} \text{ spans } \mathbb{F}_2^k \}.
\]
Notice that in this latter definition, we want to generate the full group of signs with numbers not just coprime to \( q \) but also to \( r \).

**Definition.** Let \( y \geq 1 \) be a parameter. We say \( q \) is \( y \)-eligible if for each divisor \( d \) of \( q \) with \( 1 < d < q \), we have \( g_{d,q} \leq y \). Otherwise, we say \( q \) is \( y \)-ineligible. If \( g_q > y \) and \( q \) is \( y \)-eligible, we say \( q \) is \( y \)-exceptional.

So, an odd prime \( p \) is always \( y \)-eligible, and is \( y \)-exceptional if \( n_p > y \). It also becomes clear why we need to introduce the the notion of \( g_{q,r} \): it may be that \( g_d \leq y \) for each proper divisor \( d \) of \( q \), but in order to get this full set of generators, we must use numbers which are coprime to \( d \) but not coprime to \( q \).

We now state our main results.

**Main Theorem.** Let \( a \geq 3 \) be fixed. Suppose \( Q(Q,a) \) is the set of all integers \( q \leq Q \) which are odd, square-free, and \((\log q)^a\)-exceptional. Then, for \( \delta > 0 \), we have
\[
Q(Q,a) \ll_{\delta,a} Q^{2/a+\delta}.
\]
Recall that the square-free radical of $q$ is $r = \prod_{p|q} p$.

**Corollary 1.** Let $\varepsilon \in (0, 1)$. There are positive numbers $a$ and $c$ which depend only on $\varepsilon$ and such that the following holds. For $Q$ sufficiently large in terms of $\varepsilon$, there are at least $cQ^\varepsilon (\log \log Q)^{-1}$ numbers $q \in [Q, Q + Q^\varepsilon]$ such that $g_r \leq (\log r)^a$, where $r$ is the square-free radical of $q$.

As applications of the Main Theorem, we have the following corollaries.

**Corollary 2.** Let $\varepsilon > 0$ and let $Q$ be sufficiently large in terms of $\varepsilon$. There is an integer in the interval $[Q, Q + Q^\varepsilon]$ which is properly represented by a definite, binary quadratic form of discriminant $-d$ with $d \leq Q^\varepsilon$.

**Corollary 3.** Let $\varepsilon > 0$ and let $Q$ be sufficiently large in terms of $\varepsilon$. There is an integer $q$ in the interval $[Q, Q + Q^\varepsilon]$ which can be written as

$$q = \frac{1}{u}x^2 + \frac{v}{u}y^2$$

with $0 \leq u, v \leq Q^\varepsilon$ and $u \neq 0$.

This final corollary can be compared with the problem of bounding the gaps between consecutive sums of two squares. Being that there are about $x(\log x)^{-1/2}$ integers up to $x$ which are a sum of two squares, one might expect that the gaps between such integers are at most $(\log x)^c$ for some positive constant $c$. However the best known bound, due to Bambah and Chowla [BC], is that there is a integer between $x$ and $x + O(x^{1/4})$ which is a sum of two squares. Corollary 3 says that one has much smaller gaps if we weaken squares to numbers which are in a sense “almost-squares”.

### 3. Facts from analytic number theory

Here we recall some required background results from analytic number theory. Let

$$S(x, y) = \{n \leq x : p|n \implies p \leq y \text{ and } p = 1 \text{ mod } 8\}$$
and for a positive integer \( q \), let

\[
S_q(x, y) = \{ n \in S(x, y) : (n, q) = 1 \}.
\]

We need to estimate these sets. To begin, we have the following which is a modification of Corollary 7.9 from [MV].

**Theorem 2.** Suppose \( a \) is in the range \( 2 \leq a < \left( \log x \right)^{1/2}/(2 \log \log x) \) and let \( \delta > 0 \). Then for \( x \) sufficiently large and \( q \leq x \),

\[
x^{1-1/a-\delta} \ll \delta, a \mid S_q(x, (\log x)^a) \mid \leq |S(x, (\log x)^a)| \ll \delta, a \cdot x^{1-1/a+\delta}.
\]

**Proof.** Let \( y \geq (\log x)^2 \) and let \( P = \{p_1, \ldots, p_T\} \) denote the set of primes \( p \leq y \) with \( p = 1 \mod 8 \) and which do not divide \( q \). Any product of primes in \( P \) which does not exceed \( x \) belongs to \( S_q(x, y) \).

By the Prime Number Theorem in arithmetic progressions (Corollary 11.21 in [MV]), we have

\[
\pi(y; 8, 1) = \frac{y}{\log y} \left( \frac{1}{4} + o(1) \right).
\]

Since \( \omega(n) \) is maximized when \( n \) is a primorial number

\[
\omega(q) \ll \frac{\log q}{\log \log q} \leq \frac{\log x}{\log \log x}.
\]

Because \( y \geq (\log x)^2 \), for \( x \) sufficiently large, \( T \geq \frac{y}{\log y} \). Now consider any product of primes in \( P \). Its logarithm is of the form

\[
\sum_{j=1}^{T} v_j \log p_j \leq \log y \sum_{j=1}^{T} v_j.
\]

Thus a lower bound for \( |S_q(x, y)| \) is the number of vectors

\[
N = \left| \left\{ (v_1, \ldots, v_T) \in \mathbb{Z}^T : v_j \geq 0, \sum_{j=1}^{T} v_j \leq \left[ \frac{\log x}{\log y} \right] \right\} \right|.
\]

Letting \( U = \left[ \frac{\log x}{\log y} \right] \), it is a simple combinatorial argument (see Lemma 7.7 of [MV]) that the number of such vectors is

\[
N = \binom{T + U}{T}.
\]

By Stirling’s Formula,

\[
n! \asymp n^{n+1/2} e^{-n}
\]
so that
\[ N \gg (U + T)^{U + T + 1/2} e^{-U - T} \]
\[ \geq \left( \frac{U + T}{U} \right)^{U} \frac{1}{\sqrt{U}}. \]
The right hand side above is increasing in \( T \), thus
\[ N \gg \left( 1 + \frac{y}{5U \log y} \right)^{U} \frac{1}{\sqrt{U}}. \]
If \( u = \frac{\log x}{\log y} \) then \( u - 1 \leq U \leq \frac{y}{\log y} \). Thus
\[ N \gg \left( \frac{y}{5u \log y} \right)^{u-1} \frac{1}{\sqrt{u}}. \]
For the exponent \(-1\) and the factor \( 1/\sqrt{u} \) we note that
\[ \frac{1}{\sqrt{u}} \frac{5u \log y}{y} \geq \frac{1}{y} \]
and thus
\[ N \gg \frac{1}{y} \left( \frac{y}{5 \log x} \right)^{\log x / \log y} = \frac{x}{y} \exp \left( -\frac{\log x}{\log y} \log(5 \log x) \right) \]
Now we take \( y = (\log x)^a \) where \( 2 \leq a < (\log x)^{1/2}/(2 \log \log x) \). Then we get
\[ N \gg x^{1-\frac{1}{a}} \exp \left( -a \log \log x - \frac{\log 5 \log x}{a \log \log x} \right) \gg_{\delta,a} x^{1-1/a-\delta}. \]
For the upper bound, trivially \( |S_q(x, (\log x)^a)| \) is less than \( |S(x, (\log x)^a)| \), which is in turn at most the number of \((\log x)^a\)-smooth numbers up to \( x \). There are at most \( O_{\delta,a}(x^{1-1/a+\delta}) \) such numbers by Corollary 7.1 of [MV]. \( \square \)

The main ingredient we need for the proof of our main theorems is a Large Sieve inequality. This one can be found in [IK] Theorem 7.13.

**Theorem 3** (Large Sieve). Let \( Q \geq 1 \) and let \((a_n)_{n \leq x}\) be a sequence of complex numbers. Then
\[ \sum_{q \leq Q} \varphi(q) \sum' \left| \sum_{\chi \mod q} a_n \chi(n) \right|^2 \ll (Q^2 + x) \sum_{n \leq x} |a_n|^2. \]
In the above sum over \( \chi \) we mean that the summation occurs over all primitive characters \( \chi \) of modulus \( q \).

In order to make Lemma 1 useful, we need to have integers with a reasonable number of prime factors. The next lemma tells us such integers are ubiquitous.

**Lemma 2.** Let \( \varepsilon \in (0, 1) \) and \( K \geq 1 \) be fixed. For all \( Q \) sufficiently large in terms of \( \varepsilon \), there are at most \( O\left(\frac{Q^\varepsilon (\log \log Q)^{-K}}{\log \log Q} \right) \) integers \( q \in [Q, Q + Q^\varepsilon] \) with at least \( (\log \log Q)^{K+1} \) prime factors.

**Proof.** Let \( u \in [1, Q] \) be a number to be determined later. Write

\[
\omega_u(q) = \sum_{\substack{p|q \leq u}} 1.
\]

Then

\[
\omega(q) - \omega_u(q) = \sum_{\substack{p|q \leq u \quad \text{and} \quad p > u}} 1 \leq \frac{1}{\log u} \sum_{p|q \leq u} \log p \leq \frac{\log q}{\log u}.
\]

The number of integers \( q \) in question is at most

\[
(\log \log Q)^{-K-1} \sum_{\omega(q)} \omega(q) \leq (\log \log Q)^{-K-1} \sum_{Q \leq q \leq Q + Q^\varepsilon} \omega_u(q) + 2Q^\varepsilon (\log \log Q)^{-K-1} \frac{Q}{\log u}.
\]

To estimate the sum,

\[
\sum_{Q \leq q \leq Q + Q^\varepsilon} \omega_u(q) \leq \sum_{p \leq u} \frac{Q^\varepsilon}{p} + O(u) \ll Q^\varepsilon \log \log Q + O(u).
\]

Taking \( u = Q^\varepsilon \) gives the bound

\[
O\left(\frac{Q^\varepsilon}{(\log \log Q)^K} + \frac{Q^\varepsilon}{\varepsilon (\log \log Q)^{K+1}}\right) = O\left(\frac{Q^\varepsilon}{(\log \log Q)^K}\right)
\]

once \( Q \) is sufficiently large in terms of \( \varepsilon \). \( \square \)

Finally, in order to guarantee that we can find \( y \)-eligible numbers in short intervals, we will need a basic consequence of Brun’s Pure Sieve. This result can be read from Corollary 6.2 in [FI].
Theorem 4. Let \( a \geq 1 \) and \( \varepsilon > 0 \) be fixed. Then the number of integers in the interval \([Q, Q + Q^\varepsilon]\) with no prime divisor less than \( z \) is asymptotic to \( Q^\varepsilon V(z) \), where \( V(z) = \prod_{p < z} (1 - p^{-1}) \), provided \( z \) is in the range \( 1 \leq z \leq (\log Q)^a \).

4. PROOFS OF MAIN THEOREMS AND COROLLARIES

The following lemma provides the key reduction to a problem which is approachable by the Large Sieve.

Lemma 3. Let \( q \) be an odd, square-free, and \( y \)-exceptional number. Then for any \( n \in S_q(x, y) \), we have

\[
\left( \frac{n}{q} \right) = \prod_{p \nmid q} \left( \frac{n}{p} \right) = 1.
\]

Proof. For convenience, write \( k = \omega(q) \), write \( q = p_1 \cdots p_k \), and denote by

\[
\Theta_q(y) = \theta_q(U_q(y))
\]

the image of \( U_q(y) \) in \( \mathbb{F}_2^k \). Since \( q \) is \( y \)-exceptional we have \( g_q > y \), and so \( \Theta_q(y) \) spans a proper subspace of \( \mathbb{F}_2^k \), say \( H \). A number \( n \in S_q(x, y) \) is a product of primes in \( U_q(y) \). Thus, writing \( n = \prod_{p \nmid n} p^v_p \), we have

\[
\theta_q(n) = \sum_p v_p \theta_q(p) \in H
\]

since \( \theta_q(p) \in H \) for each \( p \) occurring in the sum. Since \( H \) is a proper subspace, there is a non-zero vector in \( H^\perp \). In other words, for some non-empty subset \( I \subseteq \{1, \ldots, k\} \), each vector \( (x_1, \ldots, x_k) \in H \) satisfies

\[
\sum_{i \in I} x_i = 0.
\]

In fact, we must have \( I = \{1, \ldots, k\} \). To see this, suppose that \( I \) were a proper subset with \( |I| = l < k \). Let \( d_I = \prod_{i \in I} p_i \). Then the projection \( \pi_I : H \rightarrow \mathbb{F}_2^l \) given by

\[
\pi_I(x_1, \ldots, x_k) = (x_i)_{i \in I}
\]

is not surjective. Indeed, any vector in the image has co-ordinates which sum to 0 mod 2. But this projection contains the image \( \theta_{d_I}(U_q(n)) \).
So the proper divisor \( d_I \) satisfies \( gd_{I,q} > y \), which violates the \( y \)-eligibility of \( q \). Thus the each element of \( H \) satisfies \( \sum_{i=1}^k x_i = 0 \) which means that
\[
\prod_{p|q} \left( \frac{n}{p} \right) = 1
\]
for any \( n \in S_q(x,y) \).

We now prove our main theorem. The proof boils down to Linnik’s result on the least number \( n_\chi \) for which a primitive character \( \chi \) satisfies \( \chi(n_\chi) \neq 1 \). The result is stated but not proved, in [DK]. A proof is given in [Pol1] (Lemma 5.3), and we will will follow in much the same manner.

**Proof of Main Theorem.** Fix \( \delta > 0 \). Let \( C_{a,\delta} \) be a constant depending only on \( a \) and \( \delta \) which is at our disposal, and let \( x > C_a \). We will work dyadically, and apply Theorem 3 with
\[
a_n = \begin{cases} 
1 & \text{if } n \in S(x^2, (\log x)^a) \\
0 & \text{otherwise}.
\end{cases}
\]

Suppose \( q \in Q(Q,a) \) is in the range \( x < q \leq 2x \). Then
\[S(x^2, (\log x)^a) \subseteq S(x^2, (\log q)^a).\]
So if \( n \in S(x^2, (\log x)^a) \), then by Lemma 3
\[
\prod_{p|q} \left( \frac{n}{p} \right) = \begin{cases} 
1 & \text{if } (n, q) = 1 \\
0 & \text{otherwise}.
\end{cases}
\]
It follows that
\[
\sum_{n \leq x^2} a_n \left( \frac{n}{q} \right) \geq |S_q(x^2, (\log x)^a)|
\]

Suppose \( x \) is large enough so that it satisfies \( 2^{-a} > \log(x^2)^{-\delta/16} \), which will be guaranteed by increasing \( C_{a,\delta} \) as necessary. By Theorem 2
\[
|S_q(x^2, (\log x)^a)| = |S_q(x^2, 2^{-a}(\log x^2)^a)|
\geq |S_q(x^2, (\log x^2)^{a-\delta/16})|
\gg_{\delta,a} x^{2-2/(a-\delta/16)-\delta/16}
\gg_{\delta,a} x^{2-2/a-\delta/8}
\]
by increasing $C_{a,\delta}$ if necessary. Now we sum over all $q \in \mathcal{Q}_x = \mathcal{Q}(Q, a) \cap (x, 2x]$ to get

$$\sum_{q \in \mathcal{Q}_x} x^{4-4/a-\delta/4} \ll_{\delta, a} \sum_{q \in \mathcal{Q}_x} \frac{\varphi(q)}{\chi_{\mod q}} \sum'_{n \leq x^2} a_n \chi(n) \left| \sum_{n \leq x^2} a_n \chi(n) \right|^2$$

$$\ll_{\delta, a} x^2 |S(x^2, (\log x^2)^a)|$$

$$\ll_{\delta, a} x^{4-2/a+\delta/4}.$$

Rearranging, we see that

$$|\mathcal{Q}_x| \ll_{\delta, a} x^{2/a+\delta/2} \leq Q^{2/a+\delta/2}$$

for $x \leq Q.$ Summing over all $x = 2^j$ in the range $\log_2(C_{a,\delta}) \leq j \leq \log_2 Q$ we get

$$|\mathcal{Q}(Q, a)| \ll_{\delta, a} \sum_{j \leq \log Q} |\mathcal{Q}_{2^j}| \ll_{\delta, a} Q^{2/a+\delta}.$$

Here we have used that there is a contribution of $O_{a,\delta}(1)$ from the terms with $2 \leq q \leq \log_2(C_{a,\delta})$ and the fact that $\log Q \ll_{\delta} Q^{\delta/2}$.  

Before proving the first corollary, we need some lemmas concerning eligibility.

**Lemma 4.** If $q$ is odd, square-free and $y$-ineligible, and if the least prime dividing $q$ is $p$, then $q$ has a proper divisor $d$ which is $\min\{y, p-1\}$-exceptional. In particular, if $q$ is odd, square-free and $y$-ineligible, and if all prime factors of $q$ exceed $y+1$, then $q$ has a proper divisor $d$ which is $y$-exceptional.

**Proof.** Since $q$ is $y$-ineligible it has a divisor $d$ such that $g_{d,q} > y.$ Let

$$d_0 = \min\{d : d|q, \ 1 < d < q, \ g_{d,q} > y\}.$$

Then $d_0$ is $y$-eligible. Indeed, if $d_0$ were not $y$-eligible, it would have a proper divisor $d'$ for which $g_{d',d_0} > y.$ But then, since $d_0|q,$ we have $g_{d',q} \geq g_{d',d_0} > y$ contradicting the minimality of $d_0.$ So $d_0$ is $y$-eligible but $g_{d_0,q} > y.$ Either $g_{d_0} > y$ as well and so $d_0$ is $y$-exceptional, or else $g_{d_0} \leq y.$ In the latter case, the vectors $\theta_{d_0}(n)$ with $n \in U_{d_0}(y)$ generate the full group of signs for $d_0,$ but those with $n \in U_q(y)$ do not. However, all of the elements of $U_{d_0}(y) \setminus U_q(y)$ are divisible by some
prime which is at least as big as $p$, and so $g_{d_0} > p - 1$. It may now be the case, since $p - 1 < y$, that $d_0$ is not $p - 1$-eligible. If so, let

$$d_1 = \min\{d : d|d_0, \ 1 < d < d_0, \ g_{d,d_0} > p - 1\}.$$

As before, $d_1$ is $p - 1$-eligible and $g_{d,d_0} > p - 1$. But in fact, since $d_0$ is only divisible by primes greater than $p$, $U_{d_0}(p - 1) = U_{d_1}(p - 1)$, and $g_{d_1} > p - 1$. Hence $d_1$ is $p - 1$-exceptional. The second statement of the lemma follows immediately. □

**Lemma 5.** If $q$ is an odd, square-free integer which is $(\log q)^a$-ineligible, and if all prime factors of $q$ are at least $2(\log q)^a$, then $q$ has a divisor $d$ which is $(\log d)^a$-exceptional.

**Proof.** Since $q$ is not $(\log q)^a$-eligible then it has a divisor $d_1$ which is $(\log q)^a$-exceptional, by Lemma 4. Either $d_1$ is also $(\log d_1)^a$-exceptional (and we are done) or else it is not $(\log d_1)^a$-eligible. In the latter case it has a proper divisor $d_2$ which is either $(\log d_2)^a$-exceptional, or else not $(\log d_2)^a$-eligible. Continuing in this fashion, we must arrive at a divisor $d$ of $q$ which is $(\log d)^a$-exceptional. Indeed, eventually we would either stop or arrive at a prime $p$, and this prime is $(\log p)^a$-exceptional since all primes are $y$-eligible for all $y > 1$. □

**Proof of Corollary 1.** Assume $\varepsilon < 1/2$ without any loss of generality and let $a$ be a positive number at our disposal, which will depend only on $\varepsilon$. Let $B$ be the set of all numbers in $[Q, Q + Q^\varepsilon]$ which have no prime factors smaller than $2(\log Q)^a$. By Theorem 4 we have

$$|B| \sim Q^\varepsilon \cdot V(2(\log Q)^a).$$

Now, by Mertens’ theorem,

$$- \log V(z) = \sum_{p < z} - \log (1 - p^{-1}) \leq \sum_{p < z} \frac{1}{p - 1} = \log \log z + O(1).$$

So

$$V(2(\log Q)^a) \gg \frac{1}{a \log \log Q}$$

and hence

$$|B| \gg \frac{Q^\varepsilon}{a \log \log Q}.$$
Let $B' \subseteq B$ be the subset of all elements of $q \in B$ which have a square-free radical $r = \prod_{p|q} p$ satisfying $g_r > (\log r)^a$. For such $q$, $r$ is either $(\log r)^a$-exceptional or $(\log r)^a$-ineligible. Now, all prime factors of $r$ exceed $2(\log r)^a$. So, if $r$ is $(\log r)^a$-ineligible, it follows from Lemma 5 that $r$ is divisible by a number $d$ which is $(\log d)^a$-exceptional. In either case, $q$ has a divisor $d$ which is square-free and $(\log d)^a$-exceptional. Moreover, this divisor $d$ satisfies $d \geq (\log Q)^a$ since it is a non-empty product of primes exceeding $(\log Q)^a$. Hence every integer in $B'$ is divisible by some $d \in Q(2Q, a)$ which is at least $(\log Q)^a$. It follows that the size of $B'$ is at most

$$\sum_{d \geq (\log Q)^a} \left( \frac{Q^\varepsilon}{d} + O(1) \right) \ll Q^\varepsilon \int_{(\log Q)^a}^{2Q} \frac{Q(u, a)}{u^2} du + O(Q(2Q, a))$$

by partial summation. We apply our Main Theorem with $\delta = \varepsilon/4$. The integral is at most

$$O(1) \cdot Q^\varepsilon \int_{(\log Q)^a}^{\infty} u^{2/a-2+\varepsilon/4} du \ll_{\varepsilon, a} \frac{Q^\varepsilon}{(\log Q)^{a(1-\varepsilon/4)-2}} \ll_{\varepsilon, a} \frac{Q^\varepsilon}{(\log Q)^{a/2}},$$

while

$$Q(2Q, a) \ll \varepsilon (2Q)^{2/a+\varepsilon/4} \ll \varepsilon Q^{\varepsilon/2}$$

for $a$ sufficiently large in terms of $\varepsilon$. Thus once $a$ is sufficiently large in terms of $\varepsilon$ and $Q$ is sufficiently large in terms of $a$, we have $|B \setminus B'| \gg \varepsilon^{\frac{Q^\varepsilon}{\log \log Q}}$. The integers in $B \setminus B'$ all have a square-free radical $r$ with $g_r \leq (\log r)^a$. □

**Proof of Corollary 2**. We can assume $\varepsilon$ is small without any loss of generality. By Theorem 1, it is enough to find some number $q \in [Q, Q + Q^\varepsilon]$ and a discriminant $-d$ which is a square modulo $4q$. If $(d, 4q) = 1$, then in order for $-d$ to be a square modulo $4q$, it suffices that

1. $d \equiv 7 \mod 8$,
2. $\left( \frac{d}{q} \right) = 1$ for $p|q$ and $p \equiv 1 \mod 4$,
3. $\left( \frac{d}{q} \right) = -1$ for $p|q$ and $p \equiv 3 \mod 4$.

By Corollary 1 there is a positive number $a$ such that the number of integers in $q \in [Q, Q + Q^\varepsilon]$ with $g_q \leq (\log q)^a$ is at least

$$c_\varepsilon Q^\varepsilon (\log \log Q)^{-1}$$
for some constant \( c \) depending only on \( \varepsilon \). By Lemma 2, one of these numbers will have \( \omega(q) \leq (\log \log Q)^K \) for some \( K \) sufficiently large in terms of \( \varepsilon \). Let \( p_0 \) be the smallest positive prime which is congruent to 7 modulo 8 and which does not divide \( q \). Then, since \( q \) has \( k = \omega(q) \leq (\log \log Q)^K \) prime factors, \( p_0 \) is at most the \( k+1 \)'th prime congruent to 7 mod 8 which is at most \( O(k \log k) = O_{\varepsilon}(Q^{\varepsilon/2}) \) by the Prime Number Theorem. Since

\[
 n_q \leq q^{\omega(q)} \leq (\log q)^a(\log \log q)^K = \mathcal{O}(Q^{\varepsilon/2}),
\]

we can find an integer \( d_0 \ll_{\varepsilon} Q^{\varepsilon/2} \) which is relatively prime to \( q \), congruent to 1 modulo 8 so that \( d_0 p_0 = 7 \mod 8 \), and such that \( \left( \frac{d_0 p_0}{p} \right) \) is prescribed as needed for each \( p \) dividing \( q \). The number \( d = d_0 p_0 \) satisfies the desired properties and the corollary is proved. \( \square \)

Recall that a binary quadratic form \( q(x, y) = Ax^2 + Bxy + Cy^2 \) of discriminant \( -d \) is called reduced if \(|B| \leq A \leq C\). In this case \( d = 4AC - B^2 \geq 3AC \), so that all coefficients are bounded by \( d \). We say two quadratic forms \( q_1(x, y) \) and \( q_2(x, y) \) are equivalent if one can be obtained from the other by an invertible, integral change of variables. To be precise, we have \( q_2(\alpha x + \beta y, \gamma x + \delta y) = q_1(x, y) \) for some integers \( \alpha, \beta, \gamma, \delta \) with \( \alpha \delta - \beta \gamma = 1 \). It is clear that equivalent forms represent the same numbers. This, combined with the following theorem, \( \{B, \text{Theorem 2.3}\} \), shows that there is no loss of generality in working with reduced forms.

**Theorem 5.** Every binary quadratic form of discriminant \( -d \) is equivalent to a reduced form of the same discriminant.

We are now ready to prove our final corollary.

**Proof of Corollary.** By Corollary 2 we can represent an integer \( q \) in the range \( Q \leq q \leq Q + Q^{\varepsilon} \) by some positive definite binary quadratic form \( Q \) of discriminant \( -d \) with \( d \) at most \( Q^{\varepsilon} \). Without loss of generality we may assume this form is reduced, and thus write \( q = Ax^2 + Bxy + Cy^2 \) with \( B^2 - 4AC = -d \) for some \( A, B, C \) bounded in absolute value by \( Q^{\varepsilon} \). It follows that \( A, C > 0 \), and by completing
the square we see
\[ q = \frac{1}{4A} \left((2Ax + By)^2 + dy^2\right). \]

\[ □ \]

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**References**

[BC] R. P. Bambah and S. Chowla, *On numbers which can be expressed as a sum of two squares*, Proc. Nat. Inst. Sci. India 13, (1947). 101-103.

[B] D. A. Buell, *Binary Quadratic Forms, Classical theory and modern computations*, Springer-Verlag, New York, 1989. x+247 pp.

[DK] W. Duke and E. Kowalski, *A problem of Linnik for elliptic curves and mean-value estimates for automorphic representations*, Invent. Math. 139 (2000) 1-39.

[FI] J. Friedlander and H. Iwaniec, *Opera de Cribro*, American Mathematical Society Colloquium Publications, 57. American Mathematical Society, Providence, RI, 2010.

[IK] H. Iwaniec and E. Kowalski, *Analytic Number Theory*, American Mathematical Society Colloquium Publications, 53. American Mathematical Society, Providence, RI, 2004.

[MV] H. L. Montgomery and R. C. Vaughan, *Multiplicative Number Theory 1. Classical Theory*, Cambridge, 2007.

[Pol1] P. Pollack, *The average least quadratic nonresidue modulo m and other variations on a theme of Erdős*, J. Number Theory 132 (2012), 1185-1202.
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