NON-ARCHIMEDEAN QUANTUM MECHANICS VIA QUANTUM GROUPS

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Abstract. We present a new non-Archimedean realization of the Fock representation of the $q$-oscillator algebras where the creation and annihilation operators act on complex-valued functions, which are defined on a non-Archimedean local field of arbitrary characteristic, for instance, the field of $p$-adic numbers. This new realization implies that many quantum models constructed using $q$-oscillator algebras are non-Archimedean models, in particular, $p$-adic quantum models. In this framework, we select a $q$-deformation of the Heisenberg uncertainty relation and construct the corresponding $q$-deformed Schrödinger equations. In this way we construct a $p$-adic quantum mechanics which is a $p$-deformed quantum mechanics. We also solve the time-independent Schrödinger equations for the free particle, and a particle in a non-Archimedean box. In the last case, we show the existence of a discrete sequence of energy levels. We determine the eigenvalues of Schrödinger operator for a general radial potential. By choosing the potential in a suitable form we recover the energy levels of the $q$-hydrogen atom.

1. Introduction

In the last thirty-five years the connection between $q$-oscillator algebras and quantum physics have been studied intensively, see, e.g., [3], [5], [6], [15], [19], [20]-[21], [30], [38]-[39], [43], [44], [53], [59]-[62], and the references therein. From the seminal work of Biedenharn [6] and Macfarlane [43], it was clear that the $q$-analysis, [18], [28], [30], plays a central role in the representation of $q$-oscillator algebras which in turn has a deep physical meaning. In particular, the $q$-deformation of the Heisenberg algebra drives naturally to several types of $q$-deformed Schrödinger equations, see, e.g., [3], [39], [41], [59]-[62].

Also, in the last thirty years the $p$-adic quantum mechanics and the $p$-adic Schrödinger equations have been studied extensively, see, e.g., [11], [9]-[17], [31], [34]-[36], [46], [49], [54], [57]-[52], [63], among many available references. Here, we present a new perspective: a non-Archimedean quantum mechanics, which includes $p$-adic quantum mechanics as a particular case, is a $q$-deformation of the classical quantum mechanics. The construction is based on a new non-Archimedean realization of the Fock representation of the $q$-oscillator algebras, where the creation and annihilation operators act on complex-valued functions defined on a non-Archimedean local field $\mathbb{K}$ of arbitrary characteristic.

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On the other hand, the emergence of ultrametricity in physics, which is the occurrence of ultrametric spaces in physical models, has driven to the development of deep connections between $p$-adic analysis and physics, see, e.g., \cite{17, 56, 52, 32, 33, 51, 64} and the references therein. The existence of a Planck length implies that the spacetime considered as a topological space is completely disconnected, the points (which are the connected components) play the role of spacetime quanta. This is precisely the Volovich conjecture on the non-Archimedean nature of the spacetime below the Planck scale, \cite{55, 56, 51, Chapter 6}.

In the last forty years, the above mentioned ideas have motivated many developments in quantum field theory and string theory, see, e.g., \cite{13, 17, 23, 22, 27, 52, 55-56}, and more recently, \cite{4, 10, 11, 10, 25-26}, among others.

In \cite{3} Aref’eva and Volovich pointed out the existence of deep analogies between $p$-adic and $q$-analysis, and between $q$-deformed quantum mechanics and $p$-adic quantum mechanics. In this work we start the investigation of these matters. We present a new non-Archimedean realization of the Fock representation of $q$-oscillator algebras, where the creation and annihilation operators act on functions $f : \mathbb{K} \rightarrow \mathbb{C}$, where $\mathbb{K}$ is a non-Archimedean local field, for instance, the field of formal Laurent series:

$$F_q((T)) = \left\{ \sum_{k=k_0}^{\infty} a_k T^k; a_k \in F_q, a_{k_0} \neq 0 \text{ with } k_0 \in \mathbb{Z} \right\},$$

where $F_q$ is the finite field with $q$ elements. Our results imply, for instance, that the results on the $q$-deformed harmonic oscillator of Biedenharn \cite{6} and Macfarlane \cite{43} are valid in $F_q((T))$.

We also study some analogues of the Schrödinger equation coming from a $q$-deformation of the Heisenberg algebra. There are several different ways of choosing a $q$-deformation of the classical uncertainty relation which in turn produces several different $q$-deformations of the Schrödinger equation. In the case of the free particle, these $q$-deformed equations admit plane waves which are constructed using the classical exponential functions from $q$-analysis. There are two basic exponential functions, one admitting a meromorphic continuation to the complex plane, and the other admitting an entire analytic continuation to the complex plane. We pick a $q$-deformation of the uncertainty relation so that the corresponding $q$-deformed Schrödinger equation admits plane waves which are entire functions.

In this framework, we study, in a rigorous mathematical way, the time-independent $q$-deformed Schrödinger equations for a free particle, a particle confined in a non-Archimedean box, and a particle subject to an arbitrary radial potential.

We now discuss some applications of our results and do some comparisons. To fix ideas we discuss all the results using the field $F_q((T))$. The standard norm on $F_q((T))$ is defined as

$$|x| = \begin{cases} 0 & \text{if } x = 0 \\ q^{-k_0} & \text{if } x = \sum_{k=k_0}^{\infty} a_k T^k, a_{k_0} \neq 0. \end{cases}$$
Notice that $|T| = q^{-1}$. We denote by $S = \{ x \in \mathbb{F}_q ((T)) ; |x| = 1 \}$ the unit sphere. Then
\[ \mathbb{F}_q ((T)) \setminus \{ 0 \} = \bigcup_{k=-\infty}^{\infty} T^k S. \]
Which implies that $\mathbb{F}_q ((T)) \setminus \{ 0 \}$ is a self-similar set, and that $(\mathbb{Z}, +)$ is a scale group acting on $\mathbb{F}_q ((T)) \setminus \{ 0 \}$ as
\[ (k, x) \mapsto T^k x. \]
In the classical applications of the $q$-analysis to mathematical physics the background space is $\mathbb{R}$ or $\mathbb{C}$, these fields do not admit $(\mathbb{Z}, +)$ as a scale group. On these fields, by using the Jackson derivative, it is possible to construct certain fractals, see [19]. In the non-Archimedean setting, the background space is a non-Archimedean local field, which has a fractal nature.

We introduce a non-Archimedean version of the Jackson derivative. Let $n, m$ be non-negative integers, and $f : \mathbb{F}_q ((T)) \to \mathbb{C}$, we set
\[ (\partial(n, m)f) (x) := \frac{f(T^{-n}x) - f(T^mx)}{|T^{-n}x| - |T^mx|} = \frac{f(T^{-n}x) - f(T^mx)}{(q^n - q^m) |x|} \text{ for } x \neq 0. \]
This derivative measures the speed of deformation of function $f$ under the scale group of $\mathbb{F}_q ((T))$. Notice that this derivative is not defined at the origin. In this article we use mainly the case $\partial(1, 1) =: \partial$. We also use the operators $(q^{nN}f) (x) := f(T^x1x)$. The $q$-oscillator algebras $\mathcal{A}_q$, $\mathcal{A}_q^C$ are generated by the symbols $a, a^\dagger, q^{\pm N}$. By interpreting $a$ as $\partial$, $a^\dagger$ as the multiplication $|x|$ and using $q^{\pm N}$, we show the existence of a non-Archimedean realization of the Fock representation of the algebras $\mathcal{A}_q$, $\mathcal{A}_q^C$. The underlying Hilbert space of the representation is isometric to $L^2(\mathbb{K}, dx)$, where $dx$ is the normalized Haar measure of $(\mathbb{K}, +)$. This new realization implies that many models constructed using $q$-oscillator algebras are indeed non-Archimedean models, in particular, $p$-adic models.

In $q$-analysis, the parameter $q$ is a complex number, meanwhile in $\mathbb{K}$-analysis, $q = p^n$, where $p$ is a prime number and $n$ is a positive integer. By specializing the $q$ parameter to a power of $p$, we pass from $q$-analysis to $\mathbb{K}$-analysis.

The $q$-deformed harmonic oscillators have been studied intensively, see, e.g., [6], [21], [38]-[42], [43], [53], among others. These models can be formulated on $\mathbb{K} = \mathbb{F}_q ((T))$. The energy levels of these harmonic oscillators have the form
\[ E_n = \frac{1}{2} \hbar \omega \frac{\sinh \left( \frac{2n+1}{2} \ln q \right)}{\sinh \left( \frac{1}{2} \ln q \right)} \sim \frac{1}{2} \hbar \omega \frac{\exp \left( \frac{2n+1}{2} \ln q \right)}{\exp \left( \frac{1}{2} \ln q \right)} \text{ as } n \to \infty. \]
These energy levels are no longer uniformly spaced since $q$ is a power of a prime number. The interpretation of the non-uniform distribution of the energy levels of the $q$-harmonic oscillator is a challenging problem. In the non-Archimedean framework, they obey a scale law. Consider another background space $\mathbb{K}_m = \mathbb{F}_{q^m} ((T))$, which is a $\mathbb{K}$-vector space of
dimension $m \geq 2$. In this new background space, we have a copy of the $q$-deformed harmonic oscillator, with energy levels:

$$E_n^{(m)} = \frac{1}{2} \hbar \omega \frac{\sinh \left( \frac{2(n+1) \ln q^m}{2} \right)}{\sinh \left( \frac{\ln q^m}{2} \right)} \sim \frac{1}{2} \hbar \omega \left( \frac{\exp \left( \frac{2(n+1) \ln q}{2} \right)}{\exp \left( \frac{\ln q}{2} \right)} \right)^m.$$ 

Then

$$E_n^{(m)} \sim \left( \frac{E_n}{\frac{1}{2} \hbar \omega} \right)^m \text{ as } n \to \infty.$$ 

This scale law is a reinterpretation of a well-known number-theoretic result, which is available only in the non-Archimedean framework.

Let $V(|x|) : \mathbb{F}_q ([T]) \to \mathbb{R}$ be an arbitrary radial potential with a unique singularity at the origin. In this article we propose the following time-independent Schrödinger equation:

$$\left\{ \begin{array}{l}
\Psi_n : \mathbb{F}_q [\lbrack T \rbrack] \to \mathbb{R} \\
\Psi_n |_S = 0 \\
-\frac{\hbar^2}{2m} \left\{ (q^{-N} \partial)^2 + V(|x|) \right\} \Psi_n (x) = E_n \Psi_n (x),
\end{array} \right.$$ 

here $\mathbb{F}_q [\lbrack T \rbrack]$ is the unit ball centered at the origin. We show that the energy levels have the form

$$E_n = -\frac{\hbar^2}{2m} \left( 1 - q^{-2} \right) q^{4n-4} + V(q^{-n}), \text{ for } n = 1, 2, \ldots.$$ 

Here we determine only the point spectrum of $-\frac{\hbar^2}{2m} \left\{ (q^{-N} \partial)^2 + V(|x|) \right\}$. The determination of the whole spectrum is an open problem.

By a suitable selection of the potential $V(|x|)$, the energy levels of several $q$-models can be obtained from (1.1). For instance by taking,

$$V_{HA} (|x|) = \frac{\hbar^2}{2mq^2 |x|^4} - \frac{1}{2} mc^2 \left( \frac{e^2}{\hbar c} \right)^2 \frac{(q - q^{-1})^2}{(|x| - |x|^{-1})^2}, x \in \mathbb{F}_q [\lbrack T \rbrack],$$

formula (1.1) gives the energy levels of the Finkelstein $q$-hydrogen atom [21]:

$$E_n (\mu) = -\frac{1}{2} mc^2 \left( \frac{e^2}{\hbar c} \right)^2 \frac{q^{4n}}{[2n + 1]^2},$$

where $\mu$ is a real parameter, and $[j] = q^{j-\frac{1}{2}}$. In the limit $q$ tends to one, (1.1) gives the Balmer energy formula [21].

The limits $\text{lim } T \to 1, \text{ lim } q \to 1$ are completely different. To the best of our knowledge, the understanding of the first one requires motivic integration, while the second requires ordinary calculus, of course, after extending the parameter $q$ as a real variable. Notice that this difference is not very clear in the $p$-adic case, where $T$ is replaced by $p$ and $q = p^{-1}$, for this reason, we prefer $\mathbb{F}_q ((T))$ over $\mathbb{Q}_p$. It is interesting to mention that in the limit $p$ tends to one, the $p$-adic strings relate to ordinary strings see, e.g., [12] and the references therein.
The non-Archimedean difference equations introduced here are new mathematical objects. There are several open problems and intriguing connections between these $\pi$-difference equations with several mathematical theories.

2. The $q$-oscillator algebras and Fock representations

In this section we review some basic aspects of the $q$-oscillator algebras (also called $q$-boson algebras) and their Fock representations. For further details the reader may consult [30, Chapter 5].

2.1. The $q$-oscillator algebras. Let $q$ be a fixed complex number such that $q \neq \pm 1$. We recall that the one-dimensional harmonic oscillator algebra is generated by two elements $a, a^\dagger$ satisfying the commutation relation

\begin{equation}
[a, a^\dagger] = aa^\dagger - a^\dagger a = 1.
\end{equation}

In the Fock representation the generators $a, a^\dagger$ correspond to the annihilation and creation operators, respectively, and $N = a^\dagger a$ corresponds to the particle number operator. Furthermore,

\begin{equation}
[N, a^\dagger] = a^\dagger, \ [N, a] = -a.
\end{equation}

Definition 1. The centrally extended $q$-oscillator algebra $A_q^c$ is the associative, unital, $\mathbb{C}$-algebra generated by four elements $a, a^\dagger, q^N, q^{-N}$ subject to the relations

1A. $q^{-N}q^N = q^Nq^{-N} = 1,$
2A. $q^N a^\dagger = q a^\dagger q^N,$
3A. $q^N a = q^{-1} a q^N,$
4A. $[a, a^\dagger]_q := aa^\dagger - qa^\dagger a = q^{-N}.$

It is relevant to note that in the above definition $q^N, q^{-N}$, are symbols for two elements and that $N$ is not an element of the algebra $A_q^c$.

For $\alpha \in \mathbb{C}, k \in \mathbb{Z}$, we use the notation

$q^{kN+\alpha} := q^\alpha (q^N)^k$ and $[N + \alpha] := \frac{q^{N+\alpha} - q^{-N-\alpha}}{q - q^{-1}}$.

Definition 2. The symmetric $q$-oscillator algebra $A_q$ is the associative, unital, $\mathbb{C}$-algebra generated by four elements $a, a^\dagger, q^N, q^{-N}$ subject to the relations

1B. $q^{-N}q^N = q^Nq^{-N} = 1,$
2B. $q^N a^\dagger = q a^\dagger q^N,$
3B. $q^N a = q^{-1} a q^N,$
4B. $[a, a^\dagger]_q := aa^\dagger - qa^\dagger a = q^{-N},$
5B. $[a, a^\dagger]_{q^{-1}} := aa^\dagger - q^{-1} a^\dagger a = q^N.$

Note that relations (4B)-(5B) imply that

$a^\dagger a = [N]_q, \quad aa^\dagger = [N + 1]_q,$
which in turn imply that
\[ [N]_q a^\dagger = a^\dagger a a^\dagger = a^\dagger [N + 1]_q a, \text{ and } a [N]_q = a a^\dagger = [N + 1]_q a. \]

In the limit \( q \to 1 \) the relations (1B)-(5B) of the algebra \( A_q \) reduce to (2.1) and \( N = a^\dagger a \).

2.2. The Fock representation of \( A_q \). In this section we assume that \( q > 0, q \neq 1 \). We do not use the bra and ket notation, we directly identify the generators of \( A_q \) with operators acting on a certain Hilbert space. This direct approach is more convenient for our purposes, see [53], [30, Section 5.3]

The Fock space \( F_q \) is an \( A_q \)-module with basis vectors

\[ v_n, \ n \in \mathbb{N} := \{0, 1, \ldots , l, l + 1, \ldots \}, \]

and the action of the generators of \( A_q \) is given by

\[ q^{\pm N} v_n = q^{\pm n} v_n, \quad a^\dagger v_n = \sqrt{|n + 1|} v_{n+1}, \quad av_n = \sqrt{|n|} v_{n-1}, \]

where

\[ [r] := (q^r - q^{-r})/(q - q^{-1}) = \frac{\sinh(r \ln q)}{\sinh(\ln q)}. \]

Notice that

\[ a^\dagger av_n = [n] v_n, \quad aa^\dagger v_n = [n + 1] v_n. \]

The Fock space becomes a Hilbert space with respect to the inner product \( \langle v_m, v_n \rangle := \delta_{m,n} \). Furthermore, \( a \) and \( a^\dagger \) are adjoint to each other, whereas those of \( q^N \) and \( q^{-N} \) are self-adjoint operators.

3. Non-Archimedean local fields

We recall that the field of rational numbers \( \mathbb{Q} \) admits two types of norms: the Archimedean norm (the usual absolute value), and the non-Archimedean norms (the \( p \)-adic norms) which are parameterized by the prime numbers. The field of real numbers \( \mathbb{R} \) arises as the completion of \( \mathbb{Q} \) with respect to the Archimedean norm. Fix a prime number \( p \), the \( p \)-adic norm is defined as

\[ |x|_p = \begin{cases} 0 & \text{if } x = 0 \\ p^{-\gamma} & \text{if } x = p^{\gamma} \frac{a}{b}, \end{cases} \]

where \( a \) and \( b \) are integers coprime with \( p \). The integer \( \text{ord}(x) := \gamma \), with \( \text{ord}(0) := \infty \), is called the \( p \)-adic order of \( x \). The field of \( p \)-adic numbers \( \mathbb{Q}_p \) is defined as the completion of the field of rational numbers \( \mathbb{Q} \) with respect to the \( p \)-adic norm \(| \cdot |_p\).

A non-Archimedean local field \( \mathbb{K} \) is a locally compact topological field with respect to a non-discrete topology, which comes from a norm \(| \cdot |_{\mathbb{K}}\) satisfying

\[ |x + y|_{\mathbb{K}} \leq \max \{|x|_{\mathbb{K}}, |y|_{\mathbb{K}}\}, \]

for \( x, y \in \mathbb{K} \). Such a norm is called an ultranorm or non-Archimedean. Any non-Archimedean local field \( \mathbb{K} \) of characteristic zero is isomorphic (as a topological field) to a finite extension of \( \mathbb{Q}_p \). The field \( \mathbb{Q}_p \) is the basic example of non-Archimedean local field of characteristic
zero. In the case of positive characteristic, \( K \) is isomorphic to the field of formal Laurent series \( \mathbb{F}_q ((T)) \) over a finite field \( \mathbb{F}_q \), where \( q \) is a power of a prime number \( p \).

**Notation 1.** From now on, we fix the parameter \( q \) to be a power of \( p \). In addition, we use \( q \) to denote only the cardinality of \( \mathbb{F}_q \).

The ring of integers of \( K \) is defined as

\[
R_K = \{ x \in K ; |x|_K \leq 1 \}.
\]

Geometrically \( R_K \) is the unit ball of the normed space \( (K, |\cdot|_K) \). This ring is a domain of principal ideals having a unique maximal ideal, which is given by

\[
P_K = \{ x \in K ; |x|_K < 1 \}.
\]

We fix a generator \( \pi \) of \( P_K \), i.e., \( P_K = \pi R_K \). Such a generator is also called a local uniformizing parameter of \( K \), and it plays the same role as \( p \) in \( \mathbb{Q}_p \).

The group of units of \( R_K \) is defined as

\[
R^*_K = \{ x \in R_K ; |x|_K = 1 \}.
\]

The natural map \( R_K \to R_K/P_K \cong \mathbb{F}_q \) is called the reduction mod \( P_K \). The quotient \( \overline{K} := R_K/P_K \cong \mathbb{F}_q \), \( q = p^l \), is called the residue field of \( K \). Every non-zero element \( x \) of \( K \) can be written uniquely as \( x = \pi^{\text{ord}(x)} u \), \( u \in R^*_K \). We call \( u := \text{ac}(x) \) the angular component of \( x \).

We set \( \text{ord}(0) = \infty \). The normalized valuation of \( K \) is the mapping

\[
K \to \mathbb{Z} \cup \{ \infty \}
\]

\[
x \to \text{ord}(x).
\]

Then \( |x|_K = q^{-\text{ord}(x)} \) and \( |\pi|_K = q^{-1} \).

We fix \( \mathcal{S} \subset R_K \) a set of representatives of \( \mathbb{F}_q \) in \( R_K \), i.e., the reduction mod \( P_K \) is a bijection from \( \mathcal{S} \) onto \( \mathbb{F}_q \). We assume that \( 0 \in \mathcal{S} \). Any non-zero element \( x \) of \( K \) can be written as

\[
(3.1) \quad x = \pi^{\text{ord}(x)} \sum_{i=0}^{\infty} x_i \pi^i,
\]

where \( x_i \in \mathcal{S} \) and \( x_0 \neq 0 \). This series converges in the norm \( |\cdot|_K \). Notice that \( \text{ac}(x) = \sum_{i=0}^{\infty} x_i \pi^i \).

A multiplicative character (or quasi-character) of the group \( (\mathbb{K}^\times, \cdot) \) is a continuous homomorphism \( \omega : \mathbb{K}^\times \to \mathbb{C}^\times \) satisfying \( \omega (xy) = \omega(x) \omega(y) \). Every multiplicative character has the form

\[
\omega(x) = |x|_K^s \omega_0 (\text{ac}(x)), \quad \text{for some } s \in \mathbb{C},
\]

where \( \omega_0 \) is the restriction of \( \omega \) to \( R^*_K \); \( \omega_0 \) is a continuous multiplicative character of \( (R^*_K, \cdot) \) into the complex unit circle.

For an in-depth exposition of non-Archimedean local fields, the reader may consult [58, 50], see also [2, 52].
4. NON-ARCHIMEDEAN ANALOGUES OF THE JACKSON DERIVATIVE

In this article we introduce several non-Archimedean analogues of the Jackson derivative. For an in-depth presentation of the classical $q$-analysis the reader may consult [18], [28].

Given $f : \mathbb{K} \to \mathbb{C}$ we define

$$\partial f (x) = \frac{f(\pi^{-1}x) - f(\pi x)}{(q - q^{-1}) |x|_\mathbb{K}}, \text{for } x \neq 0.$$ 

The existence of $\partial f (0)$ depends on $f$. In the classical case, the value at the origin of the Jackson derivative is given by the standard derivative, this approach cannot be used here.

If $g : \mathbb{K} \to \mathbb{C}$, then the following Leibniz-type rule holds true:

$$\partial (f (x) g(x)) = g(\pi x) \partial f (x) + f(\pi^{-1}x) \partial g(x)$$

(4.1)

Notice that for any function $f (ac(x))$, it holds that $\partial f (ac(x)) = 0$, for $x \neq 0$. In particular, $\partial \omega_0 (ac(x)) = 0$, $x \neq 0$, for any multiplicative character $\omega_0$ of $(R^K_\mathbb{K}, \cdot)$. We also have

$$\partial |x|^m = [m] |x|^{m-1} \text{ for } m \in \mathbb{N} \setminus \{0\}.$$ 

(4.2)

We set $[m] := \prod_{i=1}^{m} [i]$, with $[0]! = 1$.

Another non-Archimedean Jackson-type derivative is defined as

$$\widetilde{\partial} f (x) = \frac{f(\pi^{-1}x) - f(x)}{(q - 1) |x|_\mathbb{K}}, \text{ for } x \neq 0,$$

where $f : \mathbb{K} \to \mathbb{C}$. Now, for $g : \mathbb{K} \to \mathbb{C}$, we have

$$\widetilde{\partial} (f (x) g(x)) = g(\pi^{-1}x) \widetilde{\partial} f (x) + f(x) \widetilde{\partial} g(x).$$

Notice that

$$\widetilde{\partial} |x|^m = [[m]] |x|^{m-1}, m \in \mathbb{N} \setminus \{0\},$$

(4.3)

where $[[m]] := \frac{a^{m-1}}{q-1}$. We also set $[[m]]! = \prod_{i=1}^{m} [i]$, with $[[0]]! := 1$. In case of functions depending on several variables, say $f(x, t)$, we use the notation $\partial_x f (x, t), \widetilde{\partial}_x f (x, t)$ to mean a derivative with respect to $x$.

5. A NON-ARCHIMEDEAN FOCK REPRESENTATION OF $\mathcal{A}_q$

5.1. Some operators. We introduce the operators:

$$a^\dagger f(x) = |x|_\mathbb{K} f(x), \quad a f(x) = \partial f (x), \quad q^N f(x) = f(\pi^{-1}x), \quad q^{-N} f(x) = f(\pi x),$$

which act on functions $f : \mathbb{K} \to \mathbb{C}$. We now fix a multiplicative character $\omega_{\text{vac}}$ of $(R^K_\mathbb{K}, \cdot)$. By simplicity we take $\omega_{\text{vac}} = 1$. We call such a function the vacuum eigenstate. We define

$$u_n (x) = \frac{|x|^{n}_\mathbb{K}}{\sqrt{|n|!}}, \text{ for } n \in \mathbb{N}.$$

Then

$$a^\dagger u_n (x) = \sqrt{|n+1|} u_{n+1} (x) \text{ for } n \in \mathbb{N},$$

(5.1)
\[
au_n(x) = \sqrt{|n|} u_{n-1}(x) \text{ for } n \in \mathbb{N} \setminus \{0\},
\]

(5.3) \[au_0(x) = 0,\]

(5.4) \[q^{\pm n} u_n(x) = q^{\pm n} u_n(x) \text{ for } n \in \mathbb{N}.\]

5.2. A non-Archimedean Bargmann-Fock type realization. Let \(F_q\) be the \(\mathbb{C}\)-vector space of formal series of the form

\[
f(x) = \sum_{n=0}^{\infty} c_n |x|^n_K, \quad x \in \mathbb{K}, \quad c_n \in \mathbb{C} \text{ for every } n.
\]

We introduce a sesquilinear form on \(F_q\) by taking

\[
(f, g) := \frac{\mathcal{F}(\partial) g(x)}{x=0},
\]

where \(f(\partial) := \sum_{n=0}^{\infty} c_n \partial^n\). If \(g(x) = \sum_{n=0}^{\infty} d_n |x|^n_K\), by using (4.2), we have

\[
(f, g) = \sum_{n=0}^{\infty} c_n d_n [n]!.
\]

We set \(\|f\|^2 := \sum_{n=0}^{\infty} |c_n|^2 [n]!\). Notice that

\[
(u_n(x), u_m(x)) = \delta_{n,m},
\]

i.e., \(\{u_n(x)\}_{n \in \mathbb{N}}\) is an orthonormal basis. We now set

\[
F_q := \left\{ f = \sum_{n=0}^{\infty} c_n u_n(x) \in F_q^\bullet; \|f\|^2 = \sum_{n=0}^{\infty} |c_n|^2 < \infty \right\}.
\]

Then, the space \(F_q\) endowed with the inner product \((\cdot, \cdot)\) becomes a Hilbert space having \(\{u_n(x)\}_{n \in \mathbb{N}}\) as an orthonormal basis. Also, \(F_q\) is isomorphic to the classical \(l^2(\mathbb{C})\) Hilbert space consisting of the square-summable complex sequences.

By formulae (5.1)-(5.4), the operators \(a^\dagger, a, q^N, q^{-N}\) are well-defined in the \(\mathbb{C}\)-vector space

\[
F_{q}^\text{fin} := \left\{ f(x) = \sum_{n=0}^{M} c_n u_n(x) \in F_q; \text{ for some } M = M(f) \right\},
\]

which is a dense subspace of \(F_q\). These operators extend to linear unbounded operators on \(F_q\). One easily verifies that \(a^\dagger\) is the adjoint of \(a\) and that \(q^N, q^{-N}\) are self-adjoint operators on \(F_q^\text{fin}\).

Finally, by using formulae (5.1)-(5.4), one verifies that the operators \(a^\dagger, a, q^N, q^{-N}\) satisfy the relations (1)-(5) given in Definition 2. A similar realization for the algebra \(\mathcal{A}_q\) exists.
5.3. \( L^2 \) is isometric to \( F_q \). We fix a Haar measure \( dx \) on the additive group \( (\mathbb{K}, +) \) satisfying that \( \int_{\mathbb{K}} dx = 1 \). The space \( L^2 (\mathbb{K}) \) consists of all the functions \( f : \mathbb{K} \rightarrow \mathbb{C} \) satisfying

\[
\|f\|_2^2 = \int_{\mathbb{K}} |f(x)|^2 \, dx < \infty.
\]

In the case \( \mathbb{K} = \mathbb{Q}_p \) it is well-known that \( L^2 (\mathbb{Q}_p) \) admits a countable orthonormal basis, see, e.g., [33], [36], [52]. Consequently \( L^2 (\mathbb{Q}_p) \) is isometric to \( l^2 (\mathbb{C}) \). This fact is indeed valid for any non-Archimedean local field. We set

\[
\omega_{rbk} (x) := q^{\frac{r}{2}} \chi_k (\pi^{-1} k (\pi^r x - b)) \Omega (|\pi^r x - b|_{\mathbb{Q}_p}),
\]

where \( r \in \mathbb{Z}, k \in \mathfrak{S} \setminus \{0\}, b \in \mathbb{K}/R_{\mathbb{K}}, b = \sum_{i=\beta}^{-1} n_i \pi^i \), with \( n_i \in \mathfrak{S}, \beta \in \mathbb{Z}, \beta < 0 \) (\( \mathfrak{S} \) is a set of representatives of \( \mathbb{F}_q \) in \( R_{\mathbb{K}} \), \( \chi_k \) is the standard additive character of the additive group \( (\mathbb{K}, +) \), i.e., \( \chi_k : \mathbb{K} \rightarrow \mathbb{C} \) is a continuous mapping satisfying: \( |\chi_k (x)| = 1, \chi_k (x + y) = \chi_k (x) \chi_k (y), \chi_k |_{\mathbb{K}/R_{\mathbb{K}}} = 1 \) but \( \chi_k |_{\mathbb{K} \setminus R_{\mathbb{K}}} \neq 1 \). Finally, \( \Omega (|\pi^r x - b|_{\mathbb{Q}_p}) \) denotes the characteristic function of the ball \( \pi^{-r} b + \pi^{-r} R_{\mathbb{K}} \). The family \( \{\omega_{rbk} (x)\}_{rbk} \) forms a complete orthonormal basis of \( L^2 (\mathbb{K}) \). The proof of this result follows using the argument of the case \( \mathbb{K} = \mathbb{Q}_p \), see [37] Theorem 2. Therefore \( L^2 (\mathbb{K}) \) is isometric to \( l^2 (\mathbb{C}) \), the Fock space.

**Remark 1.** (i) Given \( f, g, \partial f, \partial g \in L^2 (\mathbb{K}) \), by using changes of variables, one verifies that

\[
\langle g, \partial f \rangle := \int_{\mathbb{K}} g(x) \partial f (x) \, dx = - \int_{\mathbb{K}} \partial g(x) f (x) \, dx.
\]

(ii) The operators \( |x|_{\mathbb{K}}, \partial \) are well-defined on the space of test functions which is dense in \( L^2 (\mathbb{K}) \). However these operators cannot be directly interpreted as creation and annihilation operators in \( L^2 (\mathbb{K}) \). Let \( i : L^2 (\mathbb{K}) \rightarrow l^2 (\mathbb{C}) \) be the above mentioned isometry. Then the creation operator, respectively annihilation operator, in \( L^2 (\mathbb{K}) \) are \( i^{-1} \circ |x|_{\mathbb{K}} \circ i \), respectively \( i^{-1} \circ \partial \circ i \).

6. **THE NON-ARCHIMEDEAN HARMONIC OSCILLATOR**

Motivated by Biedenharn’s work [4], see also [30] and the references therein, we introduce the \( \pi \)-momentum operator \( \Pi \) and the \( \pi \)-position operator \( Q \), in terms of \( a^\dagger = |x|_{\mathbb{K}}, a = \partial \), as

\[
\Pi := i \sqrt{\frac{\hbar \omega}{2}} (a^\dagger - a), \quad Q := \sqrt{\frac{\hbar}{2m\omega}} (a^\dagger + a).
\]

The Hamiltonian of the \( \pi \)-harmonic oscillator (or non-Archimedean oscillator) is defined as

\[
H = \frac{\Pi^2}{2m} + \frac{m\omega^2}{2} Q^2 = \frac{1}{2} \hbar \omega (aa^\dagger + a^\dagger a).
\]

Then \( H u_n (x) = E_n u_n (x) \), i.e., \( H \) is diagonal on the eigenstates \( u_n (x) \) with eigenvalues

\[
E_n := \frac{1}{2} \hbar \omega ([n + 1] + [n]) = \frac{1}{2} \hbar \omega \frac{\sinh (\frac{2n+1}{2} \ln q)}{\sinh (\frac{1}{2} \ln q)}.
\]
Since \( q \) is a power of a prime number, the energy levels are no longer uniformly spaced. In the limit \( q \to 1 \) these numbers give the eigenvalues of the usual quantum harmonic oscillator, see, e.g., [3]. The interpretation of the non-uniform distribution of the energy levels of the \( q \)-harmonic oscillator is a challenging problem.

In the non-Archimedean framework, the nonuniform spacing of the energy levels of the \( \pi \)-harmonic oscillator obey to a scale law. We set \( S_r = \{ x \in \mathbb{K} ; |x|_\mathbb{K} = q^r \} \) for the sphere with center at the origin and radius \( r \in \mathbb{Z} \). Notice that \( S_0 = \bigcup_j (j + \pi R_\mathbb{K}) \), where \( j \in \mathcal{S} \setminus \{0\} \), see (3.1), and each ball \( j + \pi R_\mathbb{K} \) can be identified with an infinite regular rooted tree. The regularity means that each vertex has exactly \( q \) children. The set \( \mathbb{K} \setminus \{0\} \) is the disjoint union of a countable number of copies of scaled trees. More precisely,

\[
\mathbb{K} \setminus \{0\} = \bigcup_{r=-\infty}^{\infty} S_r = \bigcup_{r=-\infty}^{\infty} \pi^{-r} S_0.
\]

The group \((\mathbb{Z}, +)\) is a scale group acting on \( \mathbb{K} \setminus \{0\} \) as

\[
\mathbb{Z} \times (\mathbb{K} \setminus \{0\}) \to \mathbb{K} \setminus \{0\}, \quad (r, x) \to \pi^{-r} x.
\]

Then \( \mathbb{K} \setminus \{0\} \) is a self-similar set obtained from \( S_0 \) by the action of the scale group \((\mathbb{Z}, +)\).

Let us fix \( \mathbb{K} \) a non-Archimedean local field, which plays the role of a one-dimensional background space. Now, let \( \mathbb{K}_m \) be an extension of \( \mathbb{K} \) of degree \( m \), this means that \( \mathbb{K}_m \) is a local field, containing \( \mathbb{K} \), which is a \( \mathbb{K} \)-vector space of dimension \( m \geq 2 \). The ring of integers \( R_\mathbb{K} \) of \( \mathbb{K} \) is a subring of the ring of integers \( R_{\mathbb{K}_m} \) of \( \mathbb{K}_m \). The local uniformizing parameter \( \pi \) of \( \mathbb{K} \) generates an ideal \( \pi R_{\mathbb{K}_m} \) in \( R_{\mathbb{K}_m} \). Since any ideal of \( R_{\mathbb{K}_m} \) has the form \( \pi_m^{\alpha} R_{\mathbb{K}_m} \), where \( \pi_m \) denotes a local uniformizing parameter of \( \mathbb{K}_m \), we have \( \pi R_{\mathbb{K}_m} = \pi_m^{\alpha} R_{\mathbb{K}_m} \), for some positive integer \( \alpha \) (called the ramification index of the extension \( \mathbb{K}/\mathbb{K}_m \)). By a well-known result, we have \( m = e \alpha \), where the positive integer \( \alpha \) (called the inertia index of the extension \( \mathbb{K}/\mathbb{K}_m \)) is the dimension of \( \mathbb{K}_m \) considered as a \( \mathbb{K} \)-vector space, i.e., \( \mathbb{K}_m = \mathbb{F}_{q^\alpha} \), see, e.g., [58].

Since any function \( f : \mathbb{K}_m \to \mathbb{C} \) has a restriction to \( \mathbb{K} \), the operators \( a_i^j = |x|_{\mathbb{K}_m} \), \( a_m = \partial \) have natural restrictions which act on functions defined on \( \mathbb{K} \). We denote these restrictions as \( a_i^j, a \). Then, we may assume the existence of ‘two identical copies’ of a non-Archimedean harmonic oscillator, one in \( \mathbb{K} \) and the other in \( \mathbb{K}_m \). The energy levels of these oscillators are

\[
E_n (\mathbb{K}) = \frac{1}{2} \hbar \omega \sinh \left( \frac{2n+1}{2} \ln \sqrt{q} \right) \sinh \left( \frac{1}{2} \ln q \right), \quad E_n (\mathbb{K}_m) = \frac{1}{2} \hbar \omega \sinh \left( \frac{2n+1}{2} \ln q^\alpha \right) \sinh \left( \frac{1}{2} \ln q^\alpha \right), \tag{6.1}
\]

respectively.

By using (6.1),

\[
E_n (\mathbb{K}) \sim \frac{1}{2} \hbar \omega \exp \left( \frac{2n+1}{2} \ln q \right) \exp \left( \frac{1}{2} \ln q \right), \quad E_n (\mathbb{K}_m) \sim \frac{1}{2} \hbar \omega \left[ \frac{\exp \left( \frac{2n+1}{2} \ln q \right)}{\exp \left( \frac{1}{2} \ln q \right)} \right]^\alpha
\]

for \( n \to \infty \), then

\[
\left( \frac{E_n (\mathbb{K}_m)}{\frac{1}{2} \hbar \omega} \right) \sim \left( \frac{E_n (\mathbb{K})}{\frac{1}{2} \hbar \omega} \right)^\alpha \quad \text{for} \quad n \to \infty.
\]
7. Non-Archimedean quantum mechanics

In this section we introduce a new class of $q$-deformed Schrödinger equations and study the cases of the free particle and a particle in a non-Archimedean box.

7.1. A non-Archimedean Heisenberg uncertainty relations. In the algebra $\mathcal{A}_q$, it verifies that $\partial |x|_K - q^{-1}|x|_K \partial = q^N$, and by using $q^{-1}q^{-N}|x|_K = |x|_K q^{-N}$, we have

$$1 = q^{-N}\partial |x|_K - q^{-1}q^{-N}|x|_K \partial = q^{-N}\partial |x|_K - q^{-2}|x|_K q^{-N}\partial$$

(7.1)

We propose using operator $-i\hbar q^{-N}\partial$ as a non-Archimedean analogue of the momentum operator, and propose operator $|x|_K$ as an analogue of the position operator. By using (7.1), the Heisenberg uncertainty formula becomes

$$[-i\hbar q^{-N}\partial, |x|_K]_q^{-2} = -i\hbar.$$  

(7.2)

The relation (7.2) is a $q$-deformation of the classical Heisenberg uncertainty relation. In the limit $q \to 1$ the relation (7.2) becomes the standard Heisenberg uncertainty relation.

7.2. Some mathematical results. We review some results on $q$-analysis following [18], [30, Chapter 2], [28]. In this framework $q$ is a complex parameter, since here $q$ represents the cardinality of a finite field, we use a complex parameter $\rho$ in our review of the $\rho$-analysis, later on we specialize $\rho$ to $q$.

For any nonzero complex number $\rho$, the $\rho$-number $[a]_\rho, a \in \mathbb{C}$, is defined as

$$[a]_\rho = \frac{\rho^a - \rho^{-a}}{\rho - \rho^{-1}} = \frac{\sinh(a \ln \rho)}{\sinh(\ln \rho)}.$$  

We also define

$$[[a]]_\rho = \frac{\rho^a - 1}{\rho - 1} = \rho^{\frac{a(a-1)}{2}}[a]_\rho^{1}.$$  

In the case $\rho = q$, we use the simplified notation $[a]_q = [a]$, $[[a]]_q = [[a]]$. We use this convention for any function depending on $\rho$.

For $m \in \mathbb{N}$, we set $\rho$-factorial $[m]_{\rho}! := \prod_{j=1}^{m} [m]_{\rho}$ with $[0]_{\rho}! := 1$, and $[[m]]_{\rho}! := \prod_{j=1}^{m} [[m]]_{\rho}$ with $[[0]]_{\rho}! := 1$. By convention, $[m]_q! = [m]!, [[m]]_q! = [[m]]!$.

We also set for $m \in \mathbb{N}$,

$$(a; \rho)_m := \begin{cases} (1 - a)(1 - a\rho) \cdots (1 - a\rho^{m-1}) & \text{for } m \geq 1 \\ 1 & \text{for } m = 0. \end{cases}$$

Then

$$[m]_{\rho}! = \frac{\rho^{\frac{m(m-1)}{2}}}{(1 - \rho^2)^{m}} \left(\rho^2; \rho^2\right)_m.$$  

(7.3)

For $|\rho| < 1$, we set

$$(a; \rho)_\infty := \prod_{j=1}^{\infty} (1 - a\rho^{j-1}).$$
This product converges for all \( a \in \mathbb{C} \) and defines an analytic function.

7.2.1. The \( \rho \)-exponential functions. There are two \( \rho \)-analogues of the exponential function:

\[
e^{\rho} (z) := \sum_{n=0}^{\infty} \frac{z^n}{(\rho; \rho)_n} = \frac{1}{(z; \rho)_\infty} \quad \text{for } z, \rho \in \mathbb{C}, \text{ with } |\rho| < 1,
\]

\[
E^{\rho} (z) := \sum_{n=0}^{\infty} \rho^{\frac{n(n-1)}{2}} \frac{z^n}{(\rho; \rho)_n} = (-z; \rho)_\infty \quad \text{for } z, \rho \in \mathbb{C}, \text{ with } |\rho| < 1.
\]

Furthermore, \( e^{\rho} (z) E^{\rho} (z) = 1, \)

\[
e^{\rho} (z) := 1 + \sum_{n=1}^{\infty} \frac{z^n}{(1 - \rho)(1 - \rho^2) \cdots (1 - \rho^n)} = \sum_{n=1}^{\infty} \frac{z^n}{[[n]]_\rho!},
\]

\[
E^{\rho} (z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_\rho!} \prod_{j=1}^{\infty} \left( 1 - \frac{z}{1 - \rho^j} \right),
\]

and also

\[
E^{\rho^2} (z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_\rho!} \prod_{j=1}^{\infty} \left( 1 + \frac{z}{1 - \rho^j} \right).
\]

7.2.2. The \( \pi \)-exponential functions. For \( \lambda \in \mathbb{C} \) and \( x \in \mathbb{K} \), by using \( \rho = q^{-1} \) in (7.5) and \([n]_\rho = [n]_{q^{-1}}\), we set

\[
E (x, \lambda) := \sum_{l=0}^{\infty} \frac{\lambda^l |x|^l}{[l]!} = E^{q^{-2}} \left( \frac{\lambda |x|}{1 - q^{-2}} \right) = \prod_{j=1}^{\infty} \left( 1 + \frac{\lambda |x|}{1 - q^{-2} q^{-2j+2}} \right),
\]

and

\[
e (x, \lambda) := \sum_{l=0}^{\infty} \frac{\lambda^l |x|^l}{([l]_q^{-1})!} = e^{q^{-1} \lambda} \left( (1 - q^{-1}) \lambda |x| q^{-j+1} \right) = \frac{1}{\prod_{j=1}^{\infty} (1 - (1 - q^{-1}) \lambda |x| q^{-j+1})}.
\]

Notice that the series \( e (x, \lambda) \) converges for \( |\lambda| |x| < \frac{1}{1-q^{-1}} \), and that \( E (x, \lambda) \) and \( e (x, \lambda) \) are radial functions of \( x \). We call \( e (x, \lambda), E (x, \lambda) \) the \( \pi \)-exponential functions.

By using (4.3), we have

\[
\tilde{\partial}_x e (x; \lambda) = \lambda^m e (x, \lambda), \text{ for } m \in \mathbb{N} \setminus \{0\}.
\]

On the other hand, \( \partial_x E (x, \lambda) = \lambda E \left( \pi^{-1} x, \lambda \right) = \lambda q^{N} E (x, \lambda) \), i.e.,

\[
\left( q^{-N} \partial_x \right) E (x, \lambda) = \lambda E (x, \lambda).
\]
By induction on $m$,

\[(q^{-N}\partial_x)^m E(x;\lambda) = \lambda^m E(x,\lambda), \text{ for } m \in \mathbb{N} \setminus \{0\}.\]

### 7.2.3. $\rho$-Trigonometric functions

The $\rho$-trigonometric functions attached to $E_\rho(z)$ are defined as

\[
\sin_\rho(z) = \frac{1}{2i} (E_\rho(iz) - E_\rho(-iz)) = \sum_{n=0}^{\infty} \frac{(-1)^n \rho(2n+1)\mu^{2n+1}}{[2n+1]_\rho!} z^{2n+1},
\]

\[
\cos_\rho(z) = \frac{1}{2} (E_\rho(iz) + E_\rho(-iz)) = \sum_{n=0}^{\infty} \frac{(-1)^n \rho(2n)\mu^{2n}}{[2n]_\rho!} z^{2n},
\]

where $i = \sqrt{-1}$.

We define the $\pi$-trigonometric functions attached to $E(x,\lambda)$ as

\[
\cos(x,\mu) := \sum_{n=0}^{\infty} \frac{(-1)^n q^n\mu^{2n}}{[2n]!} x^{2n},
\]

\[
\sin(x,\mu) := \sum_{n=0}^{\infty} \frac{(-1)^n q^n\mu^{2n+1}}{[2n+1]!} x^{2n+1},
\]

for $x \in \mathbb{K}, \mu \in \mathbb{R}$. We call $\cos(x,\mu)$, resp. $\sin(x,\mu)$, the $\pi$–cosine function, resp., the $\pi$–sine function. Some useful properties of these trigonometric functions are the following:

\[
\cos(0,\mu) = 1, \quad \cos(x,-\mu) = \cos(x,\mu), \quad q^{-N}\partial \cos(x,\mu) = -\mu \sin(x,\mu),
\]

\[
\sin(0,\mu) = 0, \quad \sin(x,-\mu) = -\sin(x,\mu), \quad q^{-N}\partial \sin(x,\mu) = \mu \cos(x,\mu).
\]

### 7.3. A non-Archimedean analogue of the Schrödinger equation

Given a function $\Psi(t,x) : \mathbb{K} \times \mathbb{K} \to \mathbb{C}$, we set

\[
\partial_t u(t,x) := \frac{u(\pi^{-1}t,x) - u(\pi t,x)}{(q - q^{-1})|t|_\mathbb{K}}, \quad q^{-N}\partial_x u(t,x) := \frac{u(t,x) - u(t,\pi^2 x)}{(1 - q^{-2})|x|_\mathbb{K}}.
\]

We propose the following non-Archimedean analogue of the Schrödinger equation:

\[(7.10) \quad \frac{i\hbar\partial_t}{2m} \Psi(t,x) = \left\{ \frac{-\hbar^2}{2m} (q^{-N}\partial_x)^2 + V(t,x) \right\} \Psi(t,x), \]

where $\Psi(t,x)$ is a wave function, $m$ is the mass of the particle, and $V(t,x)$ is the potential. The time and spatial variables are elements of a non-Archimedean local field of arbitrary characteristic, but the wave functions are complex-valued.

### 7.4. Free particle

In the case of the free particle, the time-independent Schrödinger equation takes the form

\[(7.11) \quad \frac{-\hbar^2}{2m} (q^{-N}\partial_x)^2 \Psi(x) = E\Psi(x). \]

We study the solutions of (7.11) in spaces of type $\mathcal{F}_q^\bullet$. We first notice that if $\Psi(x)$ is a solution of (7.11), then $h(ac(x))\Psi(x)$ is also a solution of (7.11), where $h(ac(x))$ is an
arbitrary function. This means that a regularization at the origin for $\Psi(x)$ is required. By using that $(q_x^{-N}\partial_x)^2 E(x; \lambda) = \lambda^2 E(x, \lambda)$, we obtain that

$$
\Psi(x) = c_0 E\left(x, i\sqrt{\frac{2mE}{\hbar^2}}\right) + c_1 E\left(x, -i\sqrt{\frac{2mE}{\hbar^2}}\right)
$$

(7.12)

$$
= a_0 \cos\left(x, \sqrt{\frac{2mE}{\hbar^2}}\right) + a_1 \sin\left(x, \sqrt{\frac{2mE}{\hbar^2}}\right)
$$

where $a_0, a_1$ are functions depending only on the angular component of $x$. Since the angular component is not defined at the origin, function $\Psi(x)$ is not defined at the origin. Since $\cos\left(0, \sqrt{\frac{2mE}{\hbar^2}}\right) = 1$, $\sin\left(0, \sqrt{\frac{2mE}{\hbar^2}}\right) = 0$, the simplest way of regularizing $\Psi(x)$ at the origin is by choosing $a_0 = 0$ and $a_1$ as a nonzero constant, i.e., by choosing

$$
\Psi(x) = a_1 \sin\left(x, \sqrt{\frac{2mE}{\hbar^2}}\right).
$$

(7.13)

The functions $E\left(x, \pm i\sqrt{\frac{2mE}{\hbar^2}}\right)$ are the analogs of the classical plane waves. These functions are radial functions defined in $\mathbb{K}$ by a convergent, complex-valued series, see (7.6).

Another possible version of a non-Archimedean, $q$-deformed, time-independent Schrödinger equation is

$$
-\frac{\hbar^2}{2m} \partial_x^2 \Psi(x) = E \Psi(x).
$$

(7.14)

By (7.8), the planes waves have the form $e\left(x, \pm i\sqrt{\frac{2mE}{\hbar^2}}\right)$ are solutions of (7.14). These series converge when $|x|_\mathbb{K} \sqrt{\frac{2mE}{\hbar^2}} < \frac{1}{1-q^{-1}}$, see (7.7). For this reason we focus on equations of type (7.11).

7.5. **Particle in a box.** We now consider a potential of the form

$$
V(x) = \begin{cases} 
0 & \text{if } |x|_\mathbb{K} \leq q^{L-1} \\
\infty & \text{if } |x|_\mathbb{K} > q^L
\end{cases}
$$

where $L$ is a fixed integer. We look for a solution of the Schrödinger equation

$$
\left\{ -\frac{\hbar^2}{2m} (q_x^{-N}\partial_x)^2 + V(x) \right\} \Psi(x) = E \Psi(x)
$$

subjected to the conditions: $\Psi(0) = 0$, $\Psi(q^L) = 0$. The first is a regularization condition and the second condition guarantees that the particle is confined in the box $\{ x \in K; |x|_\mathbb{K} \leq q^{L-1} \}$. The first condition implies that the solution has the form (7.13), with $E = \frac{\hbar^2}{2m\mu^2}$, $\mu \in \mathbb{R}$. Notice that $\Psi(x) = \Psi(|x|_\mathbb{K})$. To satisfy the second condition, we need

$$
\Psi(q^L) = B_1 \sin\left(q^L, \sqrt{\frac{2mE}{\hbar^2}}\right) = 0.
$$
There is a sequence of positive real numbers $\omega_1 < \omega_2 < \cdots < \omega_k \cdots$, such that the zeros of $\sin (x, \mu) = 0$ are $|x|_K \mu = \omega_k$, $k \geq 1$, then $\mu = q^{-L} \omega_k$, $k \geq 1$. This fact follows from Theorem 5.1 in [14]. Indeed, the exponential function in [14, formula 2.2] is exactly $E((1 - \rho^2) z)$, with $q = \rho^2$. Therefore

$$E_k = \frac{\hbar^2 \omega_k^2}{2mq^{2L}}, \text{ for } k \geq 1,$$

are the energy levels for a particle confined in a non-Archimedean box.

8. The non-Archimedean Schrödinger equation with a radial potential

The ball $B_L$ with center at the origin and radius $L \in \mathbb{Z}$ is defined as

$$B_L = \{ x \in \mathbb{K}; |x|_K \leq q^L \}.$$

The sphere $S_L$ with center at the origin and radius $L \in \mathbb{Z}$ is defined as

$$S_L = \{ x \in \mathbb{K}; |x|_K = q^L \}.$$

We denote by $[-\infty, +\infty]$ the extended numeric line. We fix a function $V : [0, +\infty) \to [-\infty, +\infty]$. We denote by $\text{Sing}(V)$ the set of singularities of $V$. A point $x \in [0, +\infty)$ is a singular point of $V$, if $V$ is not continuous at $x$, or if $V(x) = \pm \infty$. The function $V(|x|_K)$ is a radial potential. We now consider the following eigenvalue problem:

(8.1)

$$\begin{cases}
\Psi : B_L \to \mathbb{C} \\
\left\{ -\frac{\hbar^2}{2m} q^{-N} \partial_x^2 + V(|x|_K) \right\} \Psi(x) = \mathcal{E} \Psi(x).
\end{cases}$$

By using (7.9) with $m = 2$, and taking $\Psi(x) = \mathcal{E}(|x|_K, \lambda)$, one gets

$$q^{-N} \partial_x^2 \mathcal{E}(|x|_K; \lambda) = \lambda^2 \mathcal{E}(|x|_K, \lambda).$$

Now, by replacing $\Psi(x)$ in (8.1), we obtain the condition

(8.2)

$$\left\{ -\frac{\hbar^2}{2m} \lambda^2 + V(|x|_K) - \mathcal{E} \right\} \mathcal{E}(|x|_K, \lambda) = 0.$$

Consider the points $|x|_K = q^l \leq q^L$ such that $x \notin \text{Sing}(V)$. Then (8.2) becomes

(8.3)

$$\left\{ -\frac{\hbar^2}{2m} \lambda^2 + V(q^l) - \mathcal{E} \right\} \mathcal{E}(q^l, \lambda) = 0.$$

By (7.6) the zeros of $\mathcal{E}(q^l, \lambda) = 0$ satisfy $1 + \frac{\lambda q^j}{1 - q^{-2}} q^{-2j+2} = 0$, for $j = 1, 2, \ldots$, therefore, the energy levels $E = E_{l, \lambda}$ have the form

$$E_{l, \lambda} = -\frac{\hbar^2}{2m} \lambda^2 + V(q^l),$$

where $l$ is an integer satisfying $l \leq L$ such that $S_l \notin \text{Sing}(V)$, and

$$\lambda \in \mathbb{R} \setminus \left\{ - (1 - q^{-2}) q^{2j-l-2}; \ j \geq 1 \right\}.$$
8.1. **Potentials supported in the unit ball.** To obtain a more precise description of the energy levels, it is necessary to impose boundary conditions and some additional restrictions to the function $V$.

We take

$$V(|x|_K) : B_0 \to [-\infty, +\infty],$$

such that $\text{Sing}(V)$ is just the origin. We consider the following eigenvalue problem:

$$\begin{cases}
\Psi : B_0 \to \mathbb{C} \\
\Psi |_{S_0} = 0 \\
\left\{ \frac{\hbar^2}{2m} (q_x^{-N} \partial_x)^2 + V(|x|_K) \right\} \Psi(x) = E \Psi(x).
\end{cases}$$

(8.4)

We take $\Psi(x) = E (|x|_K, \lambda)$. To satisfy the condition $\Psi|_{S_0} = 0$, we require

$$1 + \frac{\lambda}{1 - q^{-2} q^{-2j+2}} = 0, \text{ for } j = 1, 2, \ldots,$$

i.e.,

(8.5)

$$\lambda = - (1 - q^{-2}) q^{2j-2}, \text{ for } j = 1, 2, \ldots.$$

Take $|x|_K = q^{-r} < 1$, notice that $x \notin \text{Sing}(V)$, then (8.3), with $-r = l$, is satisfied if $\lambda$ satisfies (8.5). We pick $j = r$, then the energy levels are given by

$$E_r = - \frac{\hbar^2}{2m} (1 - q^{-2}) q^{4r-4} + V(q^{-r}), \text{ for } r = 1, 2, \ldots,$$

(8.6)

and the functions

$$\Psi_r(x) = \mathcal{E} (|x|_K, -(1 - q^{-2}) q^{2r-2}) \text{ for } r = 1, 2, \ldots,$$

are eigenfunctions. The determination of all the possible eigenfunctions requires solving an equation of the form

$$- \frac{\hbar^2}{2m} \left( \frac{1 - q^{-2}}{q^2} \right)^2 q^4 + V(y^{-1}) = E,$$

for $y \in (0, 1)$, where $E$ is known, and then take $r = \frac{-\ln y}{\ln q} \in \mathbb{N}$.

8.2. **π-Hydrogen atom.** By a suitable selection of the potential $V(|x|_K)$, the energy levels of several $q$-models can be obtained from (8.6). For instance by taking,

$$V_{HO}(|x|_K) = \frac{\hbar^2}{2m q^2 |x|^2} \left( 1 - q^{-2} \right)^2 + \frac{1}{2} \hbar \frac{\sinh \left( \frac{1}{2} \ln q \right)}{\sinh \left( \frac{1}{2} \ln q \right)}$$

$$= \frac{\hbar^2}{2m q^2 |x|^2} \left( 1 - q^{-2} \right)^2 + \frac{1}{2} \hbar \frac{q^2 |x| - q^{-1}}{q^2 - q^{-2}},$$

formula (8.6) gives the energy levels of the $q$-harmonic oscillator, see [6].

Many versions of the $q$-hydrogen atom have been studied. In [21], Finkelstein studied a model of a $q$-hydrogen atom with energy levels of the form

$$E_n(\mu) = - \frac{1}{2} mc^2 \left( \frac{e^2}{\hbar c} \right)^2 \frac{q^{4\mu}}{|2n + 1|^2},$$

(8.7)
where $\mu$ is a real parameter. This result was established using classical $q$-analysis on $\mathbb{C}$. The potential

$$V_{HA}(|x|) = \frac{\hbar^2 (1 - q^{-2})^2}{2 m q^2 |x|^4} - \frac{1}{2} m c^2 \left(\frac{e^2}{\hbar c}\right)^2 q^{4 \mu} \frac{\sinh^2 (\ln q)}{\sinh^2 (\ln |x|)}$$

produces the energy levels (8.7). In the limit $q$ tends to one, (8.7) becomes

$$E = -\frac{1}{2} m c^2 \left(\frac{e^2}{\hbar c}\right)^2 \frac{1}{(2n + 1)^2},$$

which is the Balmer energy formula, where the principal quantum number is $2n + 1$, see [21].

9. Some open problems

The construction of non-Archimedean quantum mechanics as a $q$-deformation of the classical quantum mechanics gives rise to several new mathematical problems and intriguing connections.

9.1. Semigroups with non-Archimedean time. A central problem is to determine if there is a semigroup attached to Schrödinger equation (7.10), i.e., if there is a family of operators $\{S_t\}_{t \in \mathbb{K}}$ such that

$$\Psi(t, x) = S_t \Psi_0(x), \text{ with } \Psi(0, x) = \Psi_0(x) : \mathbb{K} \to \mathbb{K},$$

is the solution of the initial valued problem attached to (7.10).

9.2. A non-Archimedean version of the Frobenius method. Set $D = (q^{-N} \partial)$ and $A_i(x) = \sum_{l=0}^{\infty} \frac{c_{i,l}|x|^l}{l!}$ for $i = 0, 1, \ldots, M$. To determine if a $\pi$-difference equation of the form

$$\sum_{i=1}^{M} A_i(x) D^i \Phi(x) = 0$$

(9.1)

admits a solution of the from $\Phi(x) = \sum_{i=0}^{\infty} \frac{d_i|x|^i}{i!} : \mathbb{K} \to \mathbb{C}$. To the best of our knowledge, there is no a theory for equations of type (9.1). It is important to mention here, that nowadays there are at least three different types of theories of $p$--adic differential equations, see [2], [8], [29], [33], [36], [48], and [64].

9.3. Non-Archimedean representations of $q$-oscillatory algebras. Suppose that $g : \mathbb{K} \to \mathbb{K}$. We define the operators

$$\Delta g(x) = \frac{g(\pi^{-1}x) - g(\pi x)}{(\pi^{-1} - \pi) x}, \text{ for } x \neq 0,$$

and

$$\tilde{\Delta} g(x) = \frac{g(\pi^{-1}x) - g(x)}{(\pi^{-1} - 1) x}, \text{ for } x \neq 0.$$
Is it possible to construct a Fock-type representation of $A_q$, where $ag = df$ and $a^\dagger g = xg$? A solution of this problem will allow constructing non-Archimedean quantum mechanics with $\mathbb{K}$-valued wave functions via quantum groups.

Another relevant problem is to study $\pi$-difference equations of type

$$L \sum_{j=1}^L a_j (x) \Delta_j g(x) = 0,$$

where $a_j (x) = \sum_{k=0}^\infty d_{j,k} x^k$ with $d_{j,k} \in \mathbb{K}$, and $g : \mathbb{K} \to \mathbb{K}$. Notice that equations of type (9.1) are radically different to those of type (9.2).

9.4. Sato-Bernstein-type theorems. A very relevant problem consists in studying the existence of Sato-Bernstein theorems on algebras of type $\mathbb{C} [x|_{\mathbb{K}}, \partial, q^{-N}, q^N]$, see, e.g., [7].

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