Integral mean value theorem for discontinuous function

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Abstract. This study examines the modification of the integral mean value theorem for discontinuous functions. Modification is studied by proving that a function that is not continuous at a certain and bounded interval can be integrated (finite integral) both in Riemann’s integral and Newton’s integral (integral as antiderivative). The discontinuity of the intended function, namely; \( f \) is defined on \([a, b]\) but the value of a function and its limit are not equal at some points or infinite points and countable on \((a, b)\), \( f \) is undefined on \([a, b]\) at some points or infinite points and countable on \((a, b)\) but its limit exists there. The results of this study provide a modification of the integral mean value theorem by replacing the value of \( f \) in the implication of the theorem with its limit value so that the integral mean value theorem is obtained for the non-continuous function.

1. Introduction
The mean value theorem for integral states that if a function \( f \) continues at \([a, b]\), there must be a number \( c \) in \((a, b)\) in which the result of integral of \( f \) on \([a, b]\) equals the multiplication of the value of \( f \) at \( c \) with the length of the interval \([a, b]\) [1,2]. This theorem requires that \( f \) must be continuous for the implications of the theorem to be valid, if \( f \) is not continuous then the implication does not necessarily apply. The purpose of this research is to modify the mean value theorem for integral by converting \( f \) into a function that does not have to be continuous to produce implications that are similar to or the same as the implications of the theorem.

The problem that arises if this is done is, it is not certain that functions that are not continuous can be integrated. But this can be solved by using research results from Sari [3] which have provided a modification of the second calculus fundamental theorem.

The results of this research are also supported by the explanation of Smithee [4] related to the expansion of the second calculus fundamental theorem which allows discontinuous functions to be integrated. The integral of a discontinuous function is also talked about by Klippert [5], where the main discussion is about the integral function derivative at a discontinuity point of its integrand. And similar to that, in [6], the integral of a discontinuous function is discussed when the discontinuity points of the function create a monotone sequence. So, it is possible to make modifications to the mean value theorem for integral for discontinuous function.

Besides that, the extension and process to find the integral mean value theorem are discussed in [7-11]. Some other results about this mean value theorem for integral can also be found at [12-15].
2. Methods
Modification of integral mean value theorem for discontinuous function is done by first to ensure that the discontinuous function can be integrated than find the form of which is similar or same as the implication of integral mean value theorem.

3. Results and Discussion

3.1. First Modification of Mean Value Theorem for Integral
At this segment, Theorem 3.1.2 is introduced as the integral mean value theorem first modification. The integral mean value theorem first modification is about the change of the implication of the theorem if the function \( f \) on that theorem is discontinuous in some points of the interval \( (a, b) \). Before the first modification is given, firstly Theorem 3.1.1 is given as the main base of the first modification.

**Theorem 3.1.1** If the function \( f \) is defined in \( [a, b] \) except at points \( x_1, x_2, \ldots, x_n \in (a, b) \) and \( F \) continues in \( [a, b] \) where \( F \) is an indefinite integral of \( f \) in \( [a, b] \) except at points \( x_1, x_2, \ldots, x_n \in (a, b) \) then

\[
\int_{a}^{b} f(x) \, dx = F(b) - F(a)
\]

Example.
Assume \( f(x) = \frac{1}{\sqrt{|x|}} \), where \( x \neq 0, x \in [-1, 1] \).
Graph \( f(x) = \frac{1}{\sqrt{|x|}} \) is given below.

![Graph](image)

**Figure 1.** Graph \( f(x) = \frac{1}{\sqrt{|x|}} \) for \( x \in [-1, 1], x \neq 0 \).

It is clear from the graph above that \( f \) is undefined at \( x = 0 \), where \( 0 \in [-1, 1] \) and the value of \( f \) tends to \( \infty \) when \( x \) approaches 0. The indefinite integral of \( f \) is

\[
F(x) = \begin{cases} 
-2\sqrt{-x}, & \text{for } x \in [-1, 0) \\
2\sqrt{x}, & \text{for } x \in [0, 1]
\end{cases}
\]

That satisfies \( F'(x) = f(x) \) for every \( x \) in \((-1,1)\) except at \( x = 0 \), and \( F \) continues in \([-1,1]\).

Based on theorem 3.1.1, then
\[ \int_{-1}^{1} f(x) \, dx = F(1) - F(-1) = 2 + 2 = 4 \]

so

\[ \int_{-1}^{1} f(x) \, dx = 4 \]

**Theorem 3.1.2** If \( f \) is defined and continues in \([a,b]\) except at points \( x_1, x_2, \ldots, x_n \in (a,b) \), and limit of \( f \) at its discontinuous points is exists, and also if the anti derivative of \( f \) in \([a,b]\) is \( F \) such that \( F'(x) \) is exist and \( F'(x) = f(x) \) for each \( x \in (a,b) \) except at points \( x_1, x_2, \ldots, x_n \in (a,b) \), then there must be a number \( c \) in \((a,b)\) so that \( \int_{a}^{b} f(x) \, dx = \lim_{x \to c} f(x) (b - a) \).

**Proof:**

Because \( F \) continues at \([a,b]\) and \( F'(x) = f(x) \) in every \( x \in (a,b) \) except at points \( x_1, x_2, \ldots, x_n \in (a,b) \) and \( f \) continues in \([a,b]\) except at points \( x_1, x_2, \ldots, x_n \in (a,b) \), so based on theorem 3.1.1 then

\[ \int_{a}^{b} f(x) \, dx = F(b) - F(a) \]

Since \( f \) has a limit at points \( x_1, x_2, \ldots, x_n \in (a,b) \) and \( f \) is continuous other than that point, then \( F'(x) = \lim_{t \to x} f(t) \) in every \( x \in (a,b) \).

We know that \( \int_{a}^{b} f(x) \, dx = F(b) - F(a) = K \) is exist, it means \( K \neq \pm \infty \).

Because \( F'(x) = \lim_{t \to x} f(t) \) in every \( x \in (a,b) \), then

\[ F'(c) = \frac{F(b) - F(a)}{b - a} \]

\[ \lim_{x \to c} f(x) = \frac{F(b) - F(a)}{b - a} \]

\[ \lim_{x \to c} f(x) = \frac{\int_{a}^{b} f(x) \, dx}{b - a} \]

Because \( K \neq \pm \infty \), then \( \lim_{x \to c} f(x) = \frac{\int_{a}^{b} f(x) \, dx}{b - a} \neq \pm \infty \), where \( c \in (a,b) \).

Then, there must be \( c \) in \((a,b)\) so that

\[ \int_{a}^{b} f(x) \, dx = \lim_{x \to c} f(x) (b - a) \]

3.2. **Second Modification of Mean Value Theorem for Integral**

At this segment, **Theorem 3.2.2** is introduced as the integral mean value theorem second modification. The integral mean value theorem second modification is about the change of the implication of the theorem if the function \( f \) on that theorem is discontinuous at \( x_n \in (a,b) \), \( n \in N \) where \( x_n \) is a monotone sequence. Before the first modification is given, firstly **Theorem 3.2.1** is given as the main base of the second modification.

**Theorem 3.2.1** Given the monotone sequence \( x_n \) in \((a,b)\).

If at \([a,b]\) is \( f \) defined except at points \( x_n \in (a,b) \), \( n \in N \) and we can determine the indefinite integral function \( F \) of \( f \) in \([a,b]\) that continues in \([a,b]\) such that \( F'(x) = f(x) \) for every \( x \in (a,b) \) except at points \( x_n \in (a,b) \), \( n \in N \) then
\[
\int_a^b f(x)\,dx = F(b) - F(a)
\]

**Teorema 3.2.2** If \( f \) is defined and continues in \([a,b]\) except at points \( x_n \in (a,b), n \in \mathbb{N} \) with \( x_n \) is a monotone sequence, and limit of \( f \) at its discontinuous points is exist, and also if the anti derivative of \( f \) in \([a,b]\) is \( F \) so that \( F'(x) \) is exist and \( F'(x) = f(x) \) in every \( x \in (a,b) \) except at points \( x_n \in (a,b), n \in \mathbb{N} \), then there must be \( c \in (a,b) \) so that \( \int_a^b f(x)\,dx = \lim_{x\to c} f(x) (b - a) \).

4. **Conclusion**

Integral mean value theorem for discontinuous functions as the integral mean value theorem modification is obtained by replacing the value of \( f \) in the implication of the theorem with its limit value.

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