Quark model relations for b-baryon decay

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Abstract
Properties of b-baryon decay matrix elements (amplitudes) are derived in a nonsymmetric quark model without any use of SU(6), SU(3), or SU(2) (isotopic spin) groups. Equalities between pairs of matrix elements are derived, and $\Lambda^-$-$\Sigma^0$ mixing is used to calculate the branching ratio for the transition $\Lambda_b \to \Sigma^0$.

1 INTRODUCTION
High energy accelerators are producing large numbers of heavy baryons whose decay processes are now being studied experimentally. Recent papers studying heavy baryons have generally been based on the group SU(3) [1,2]. However, there is no good evidence for SU(3) symmetry in the quark model, and all of the experimentally satisfied predictions that use SU(3) can be derived in the quark model without introducing any internal group.

Sum rules for baryon properties can be derived in the quark model by assuming the dominance of one and two-body interactions that are the same for each baryon in a particular sum rule. This two-body, baryon independent, approach was introduced some time ago using SU(6) quark wave functions [3,4]. It was later shown to give the same results without introducing any form of SU(6), SU(3), or SU(2) symmetry for the quark wave functions, in a ‘nonsymmetric quark model’ [5]. Any predictions for baryons can be produced by using three-quark spin wave functions, with no need for internal groups.

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In section 2 of this paper, we describe the nonsymmetric quark model. In section 3, we use the model to derive relations between matrix elements of b-baryons. Section 4 is a review of Λ − Σ mixing, which we then apply to the matrix element for a Λ_b to Σ^0 transition, which would vanish in the absence of the mixing. Our results are summarized in Section 5.

2 THE NONSYMMETRIC QUARK MODEL

The nonsymmetric quark model is described in sections III-VI of reference [5]. Here, we give a brief review of how it is used for the properties of spin one-half baryons.

A quark model wave function is given by

\[ q_1 q_2 q_3 \chi_1 \quad \text{or} \quad q_1 q_2 q_3 \chi_0, \]  

where \( \chi_1 \) and \( \chi_0 \) are the three-quark spin states

\[ \chi_1 = \frac{1}{\sqrt{6}} [2 \uparrow \uparrow \downarrow - \uparrow \downarrow \uparrow - \downarrow \uparrow \uparrow] \]  

and

\[ \chi_0 = \frac{1}{\sqrt{2}} [\uparrow \downarrow \uparrow - \downarrow \uparrow \uparrow]. \]  

The spin states correspond to a three-quark spin state of total spin 1/2 with the first two quarks having spin 1 (for \( \chi_1 \)) or spin 0 (for \( \chi_0 \)).

The quark order is not arbitrary, but must be coordinated with the spin wave function. For the spin function \( \chi_1 \) shown above, any two identical quarks must be chosen as the first two quarks in the wave function. This is to implement the Pauli principle, requiring identical quarks to be symmetrical with respect to interchange.

The Pauli principle only applies to two identical quarks like \( uu \) or \( dd \), but not to different quarks like \( ud \). There is no ‘extended Pauli principle’ because the \( u \) and \( d \) quarks are unrelated in the nonsymmetric model. They are not taken as two states of a single particle, as is presumed to implement the isotopic spin formalism.

For ‘flavor-degenerate’ baryons, composed of three different quark flavors, we choose the quark order to be in the order of their masses. This quark order will minimize any mixing between flavor-degenerate baryons. [6] The quark order can be changed, but then the quark spin vectors must be changed.

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along with the quark ordering to preserve the proper quark-spin coordination. (This process will be seen in the examples below.)

No symmetrization of these wave functions is necessary or helpful. The Pauli principle and the quark-spin structure shown gives the same results as would any internal group symmetrization.

3 b-BARYON DECAY MATRIX ELEMENTS

In this section, we study decays of the form \( B \rightarrow Y \) (or \( N \)) + \( S \), where \( B \) is a b-baryon decaying to a hyperon or a nucleon, plus a neutral object \( S \) with spin-parity \((1^-)\) (for instance, a photon, or the \( J/\psi \) meson). The transitions are from a spin \( \frac{1}{2} \) baryon to a spin \( \frac{1}{2} \) baryon and a spin \( 1^- \) meson or photon. Parity is not conserved in the transitions \( b \rightarrow d \) or \( b \rightarrow s \), so there will be a scalar and a pseudoscalar \((\sigma \cdot \mathbf{p})\) decay amplitude for b-baryon decay.

The two types of transition matrix elements are

- **Scalar**:
  \[
  A_s(B \rightarrow Y + S) = \langle nsq, \eta | K_{bq} T_{bq} | nsb, \xi \rangle,
  \]

- **Pseudoscalar**:
  \[
  A_{ps}(B \rightarrow Y + S) = \langle nsq, \eta | K_{bq}' \sigma_z(3) T_{bq} | nsb, \xi \rangle,
  \]

where the quark \( q \) is either a \( d \) or an \( s \) quark, and \( \eta \) and \( \xi \) are spin states. \( T_{bq} \) is a transition operator that converts a \( b \) quark to a \( d \) quark or an \( s \) quark. The constants \( K_{bq} \) and \( K_{bq}' \) are assumed to be the same (separately) for each baryon connected by \( T_{bq} \).

We first study decays where the quark flavor transition is \( b \rightarrow d \). For the transition \( \Xi_b \rightarrow \Sigma \), the scalar matrix elements are

\[
A_s(\Xi_b^0 \rightarrow \Sigma^0 + S) = \langle usd, \bar{\chi}_1 | K_{bd} T_{bd} | usb, \chi_0 \rangle
\]

\[
= \frac{K_{bd}}{2\sqrt{3}} \left( [2 \uparrow\downarrow - \uparrow\uparrow - \downarrow\uparrow] [\uparrow\uparrow - \downarrow\uparrow] \right)
\]

\[
= \frac{(2 + 1) K_{bd}}{2\sqrt{3}} = \frac{\sqrt{3}}{2} K_{bd}.
\]

(6)

\[
A_s(\Xi_b^- \rightarrow \Sigma^- + S) = \langle dsd, \bar{\chi}_1 | K_{bd} T_{bd} | dsb, \chi_0 \rangle
\]

\[
= \frac{K_{bd}}{2\sqrt{3}} \left( [2 \uparrow\downarrow - \uparrow\uparrow - \downarrow\uparrow] [\uparrow\uparrow - \downarrow\uparrow] \right)
\]

\[
= \frac{(2 + 1) K_{bd}}{2\sqrt{3}} = \frac{\sqrt{3}}{2} K_{bd}.
\]

(7)
Note that in these steps, the quark spin vectors for quark 2 and quark 3 had to be switched from spin state \( \chi_1 \) to a transposed state \( \tilde{\chi}_1 \) to coordinate with the switch from \( uds \) to \( usd \) (and \( dds \) to \( dsd \)) in the quark order.

The corresponding pseudoscalar matrix elements are

\[
A_{ps}(\Xi_b^0 \to \Sigma^0 + S) = \langle usd, \tilde{\chi}_1 | K'_{bd} T_{bd} \sigma_z(3) |usb, \chi_0 \rangle \\
= \frac{K'_{bd}}{2\sqrt{3}} \langle [2 \uparrow\downarrow\uparrow - \uparrow\uparrow\downarrow - \downarrow\downarrow\uparrow] \sigma_z(3) [\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow] \rangle \\
= \frac{(2 + 1)K'_{bd}}{2\sqrt{3}} = \frac{\sqrt{3}}{2} K_{bd}. \tag{8}
\]

\[
A_{ps}(\Xi_b^- \to \Sigma^- + S) = \langle dsd, \tilde{\chi}_1 | K'_{bd} T_{bd} \sigma_z(3) |dsb, \chi_0 \rangle \\
= \frac{K'_{bd}}{6} \langle [2 \uparrow\downarrow\uparrow - \uparrow\uparrow\downarrow - \downarrow\downarrow\uparrow] \sigma_z(3) [2 \uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow] \rangle \\
= \frac{(2 + 1)K_{bd}}{2\sqrt{3}} = \frac{\sqrt{3}}{2} K_{bd}. \tag{9}
\]

Comparing these four equations leads to the relation

\[
A(\Xi_b^0 \to \Sigma^0 + S) = A(\Xi_b^- \to \Sigma^- + S) \tag{10}
\]

for either the scalar or pseudoscalar matrix element. In DGGS, this relation is given as

\[
A(\Xi_b^0 \to \Sigma^0 + S) = \frac{1}{\sqrt{2}} A(\Xi_b^- \to \Sigma^- + S), \quad \text{(DGGS).} \tag{11}
\]

The difference in the two results is that DGGS uses the isotopic spin formalism where the \( ud \) spin one quark state is given as \( \frac{1}{\sqrt{2}}(ud + du) \), with \( ud \) and \( du \) considered as two different states. This results in the factor \( \frac{1}{\sqrt{2}} \) in Eq. (11).

Although the DGGS result differs from ours for the matrix element \( A(\Xi_b^0 \to \Sigma^0 + S) \), it gives the same result for the decay rate. The decay rate to the \( \Sigma^0 \) depends on \( |A|^2 \), for which the DGGS value is one half of ours. But, in the isotopic spin formalism, the quark states \( ud \) and \( du \) are two different states, each of which leads to a separate final decay state. This restores the isotopic spin decay rate to the same value as given by the nonsymmetric quark model.

Using the same procedure as above, scalar matrix elements for other decay modes of b-baryons for the transition \( b \to d \) are given by
\[ \begin{align*}
A_s(\Xi^0_b \to \Lambda + S) &= \langle usd, \bar{\chi}_o | K_{bd} T_{bd} |usb, \chi_o \rangle \\
&= \frac{K_{bd}}{2} \langle \uparrow \uparrow \downarrow - \downarrow \uparrow \uparrow \downarrow \uparrow \downarrow - \downarrow \uparrow \uparrow \rangle \\
&= \frac{0 + 1}{2} K_{bd} = \frac{K_{bd}}{2}, \\
A_s(\Lambda_b \to n + S) &= \langle udd, \bar{\chi}'_1 | K_{bd} T_{bd} | udb, \chi_o \rangle \\
&= \frac{K_{bd}}{2 \sqrt{3}} \langle \left[ 2 \downarrow \uparrow \uparrow - \uparrow \downarrow \uparrow - \uparrow \uparrow \downarrow \right] \left[ \uparrow \downarrow \uparrow - \downarrow \uparrow \uparrow \right] \rangle \\
&= \frac{(-2 - 1) K_{bd}}{2 \sqrt{3}} = -\frac{\sqrt{3}}{2} K_{bd}. 
\end{align*} \]

Combining Eqs. (6), (7), (12), and (13) gives the relations

\[
\text{scalar : } \quad A_s(\Xi^0_b \to \Sigma^0 + S) = A_s(\Xi^-_b \to \Sigma^- + S) = \sqrt{3} A_s(\Xi^0_b \to \Lambda + S) = -A_s(\Lambda_b \to n + S). 
\]

Our equality (14) agrees with the corresponding equality (2.18) in DGGS when the isotopic spin factor $\frac{1}{\sqrt{2}}$ is taken into account. But our equalities (15) and (16) do not agree with the corresponding equalities (2.20), (2.21), and (2.22) in DGGS. This difference is probably due to our treatment of the spin functions in our derivations. The difference here between the DGGS result and ours would lead to a difference in the branching ratios.

The pseudoscalar matrix elements for the transitions in Eqs. (12) and (13) are

\[
\begin{align*}
A_{ps}(\Xi^0_b \to \Lambda + S) &= \langle usd, \bar{\chi}_o | K'_{bd} T_{bd} \sigma_z(3) |usb, \chi_o \rangle \\
&= \frac{K'_{bd}}{2} \langle \uparrow \uparrow \downarrow - \downarrow \uparrow \uparrow \downarrow \uparrow \downarrow - \downarrow \uparrow \uparrow \rangle \sigma_z(3) [\uparrow \downarrow \uparrow - \downarrow \uparrow \uparrow \rangle \\
&= \frac{0 + 1}{2} K'_{bd} = \frac{K'_{bd}}{2}, \\
A_{ps}(\Lambda_b \to n + S) &= \langle udd, \bar{\chi}'_1 | K'_{bd} T_{bd} \sigma_z(3) | udb, \chi_o \rangle \\
&= \frac{K'_{bd}}{2 \sqrt{3}} \langle \left[ 2 \downarrow \uparrow \uparrow - \uparrow \downarrow \uparrow - \uparrow \uparrow \downarrow \right] \sigma_z(3) [\uparrow \downarrow \uparrow - \downarrow \uparrow \uparrow \rangle \rangle \\
&= \frac{(-2 - 1) K'_{bd}}{2 \sqrt{3}} = -\sqrt{3} K'_{bd} / 2. 
\end{align*} \]
Combining Eqs. (8), (9), (17), and (18) gives the relations

\[ A_{ps}(\Xi^0_b \to \Sigma^0 + S) = A_{ps}(\Xi^-_b \to \Sigma^- + S) \]
\[ = \sqrt{3} A_{ps}(\Xi^0_b \to \Lambda + S) \]
\[ = -A_{ps}(\Lambda_b \to n + S). \]

DGGS did not consider the pseudoscalar matrix elements.

Decay matrix elements for the quark flavor transition \( b \to s \) are

scalar :

\[ A_s(\Xi^0_b \to \Xi^0 + S) = \langle uss, \bar{\chi}'_1 | K_{bs} T_{bs} | usb, \chi_0 \rangle \]
\[ = \frac{K_{bs}}{2 \sqrt{3}} \langle [2 \downarrow \uparrow \uparrow - \uparrow \downarrow \uparrow - \uparrow \uparrow \downarrow - \downarrow \uparrow \uparrow] [\uparrow \downarrow \uparrow - \downarrow \uparrow \uparrow] \rangle \]
\[ = \frac{(-2 - 1)K_{bs}}{2 \sqrt{3}} = -\frac{\sqrt{3}}{2} K_{bs}, \] (22)

\[ A_s(\Xi^-_b \to \Xi^- + S) = \langle dss, \bar{\chi}'_1 | K_{bs} T_{bs} | dsb, \chi_0 \rangle \]
\[ = \frac{K_{bs}}{2 \sqrt{3}} \langle [2 \uparrow \downarrow \downarrow - \uparrow \uparrow \downarrow - \downarrow \uparrow \uparrow ] [\uparrow \downarrow \uparrow - \downarrow \uparrow \uparrow] \rangle \]
\[ = \frac{(-2 - 1)K_{bs}}{2 \sqrt{3}} = -\frac{\sqrt{3}}{2} K_{bs}, \] (23)

\[ A_s(\Lambda_b \to \Lambda + S) = \langle uds, \chi_0 | K_{bs} T_{bs} | udb, \chi_0 \rangle \]
\[ = \frac{K_{bs}}{2} \langle [\uparrow \downarrow \uparrow - \downarrow \uparrow \uparrow] [\uparrow \downarrow \uparrow - \downarrow \uparrow \uparrow] \rangle \]
\[ = (1 + 1)K_{bs}/2 = K_{bs}, \] (24)

Combining Eqs. (22), (23), and (24) gives the relations

\[ A_s(\Xi^0_b \to \Xi^0 + S) = A_s(\Xi^-_b \to \Xi^- + S) \]
\[ = -\frac{\sqrt{3}}{2} A_s(\Lambda_b \to \Lambda + S). \] (25)

Equation (25) agrees with Eq. (2.15) in DGGS, but Eq. (26) disagrees with their Eq. (2.16).

The pseudoscalar matrix elements for the transition \( b \to s \) are

pseudoscalar :

\[ A_{ps}(\Xi^0_b \to \Xi^0 + S) = \langle uss, \bar{\chi}'_1 | K_{bs} T_{bs} \sigma_2(3) | usb, \chi_0 \rangle \]
$$= \frac{K'_{bs}}{2\sqrt{3}} \langle [2 \downarrow \uparrow \uparrow - \uparrow \downarrow \uparrow - \uparrow \uparrow \downarrow] \sigma_z(3) [\uparrow \downarrow \uparrow - \downarrow \uparrow \uparrow] \rangle$$
$$= \frac{(-2 - 1)K'_{bs}}{2\sqrt{3}} = -\frac{\sqrt{3}}{2} K'_{bs}, \quad (27)$$

$$A_{ps}(\Xi_b^- \to \Xi^- + S) = \langle dss, \chi'_{1} | K'_{bs} T_{bs} \sigma_z(3) | dsb, \chi_0 \rangle$$
$$= \frac{K'_{bs}}{2\sqrt{3}} \langle [2 \downarrow \uparrow \uparrow - \uparrow \downarrow \uparrow - \uparrow \uparrow \downarrow] \sigma_z(3) [\uparrow \downarrow \uparrow - \downarrow \uparrow \uparrow] \rangle$$
$$= \frac{(-2 - 1)K'_{bs}}{2\sqrt{3}} = -\frac{\sqrt{3}}{2} K'_{bs}, \quad (28)$$

$$A_{ps}(\Lambda_b \to \Lambda + S) = \langle uds, \chi_0 | K'_{bs} T_{bs} \sigma_z(3) | udb, \chi_0 \rangle$$
$$= \frac{K'_{bs}}{2} \langle [\uparrow \downarrow \uparrow - \downarrow \uparrow \uparrow] \sigma_z(3) [\uparrow \downarrow \uparrow - \downarrow \uparrow \uparrow] \rangle$$
$$= \frac{1 + 1)K'_{bs}/2 = K'_{bs}. \quad (29)$$

Combining Eqs. (27), (28), and (29), gives the relations

$$A_{ps}(\Xi_b^0 \to \Xi^0 + S) = A_{ps}(\Xi_b^- \to \Xi^- + S) \quad (30)$$
$$= -\frac{\sqrt{3}}{2} A_{ps}(\Lambda_b \to \Lambda + S). \quad (31)$$

Equation (30) agrees with Eq. (2.15) in DGGS, but Eq. (31) disagrees with their Eq. (2.16).

Because of the orthogonality of the $\Lambda_b$ and $\Sigma^0$ spin functions, the $\Lambda_b \to \Sigma^0$ transition matrix elements vanish:

$$A_s(\Lambda_b \to \Sigma^0 + S) = \langle uds, \chi_1 | K'_{bs} T_{bs} | udb, \chi_0 \rangle$$
$$= \frac{K_{bs}}{2\sqrt{3}} \langle [2 \uparrow \uparrow \uparrow - \uparrow \downarrow \uparrow - \downarrow \uparrow \uparrow] [\uparrow \downarrow \uparrow - \downarrow \uparrow \uparrow] \rangle$$
$$= ( -1 + 1) K_{bs}/2\sqrt{3} = 0. \quad (32)$$

$$A_{ps}(\Lambda_b \to \Sigma^0 + S) = \langle uds, \chi_1 | K'_{bs} T_{bs} \sigma_z(3) | udb, \chi_0 \rangle$$
$$= \frac{K'_{bs}}{2\sqrt{3}} \langle [2 \uparrow \uparrow \uparrow - \uparrow \downarrow \uparrow - \downarrow \uparrow \uparrow] \sigma_z(3) [\uparrow \downarrow \uparrow - \downarrow \uparrow \uparrow] \rangle$$
$$= ( -1 + 1) K'_{bs}/2\sqrt{3} = 0. \quad (33)$$

We show below that including $\Lambda - \Sigma$ mixing leads to a small non-vanishing branching ratio for the decay mode $\Lambda_b \to \Sigma^0 + S$. 

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4 LAMBDA-SIGMA MIXING

There have been a number of calculations of $\Lambda - \Sigma$ mixing in the quark model \[6,12\], most of which have relied on SU(3) symmetry and other assumptions. Here, we summarize the derivation of the $\Lambda - \Sigma$ mixing angle in \[6\], using the nonsymmetric quark model. The derivative is model independent in that it does not depend on any feature of the quark interactions, and makes no use of any group property.

The $\Lambda - \Sigma$ transition matrix element is

$$
\epsilon = \langle uds, \chi_0 | H | uds, \chi_1 \rangle.
$$

Using the nonsymmetric quark wave functions, it is shown in \[6\] that the transition matrix element is given by

$$
\epsilon = \frac{\sqrt{3}}{4} (D^0_{us} - D^1_{us} + D^1_{ds} - D^0_{ds}),
$$

where $D^s_{ij}$ is a two-body interaction energy for the two labeled quarks, with the combined spin of the two quarks being either 0 or 1. The quark mixing angle $\theta_m$ is then given by

$$
\tan(2\theta_m) = \frac{-2\epsilon}{(M_\Sigma - M_\Lambda)}.
$$

The combination $(D^0_{us} - D^1_{us} + D^1_{ds} - D^0_{ds})$ is also related to two sums of baryon masses, so $\epsilon$ can be given by either \[5\]

$$
\epsilon_\Sigma = \frac{(M_{\Sigma^*_+} - M_{\Sigma^*_+} + M_{\Sigma^*_+} - M_{\Sigma^-})}{2\sqrt{3}} = -1.07 \pm 0.02\text{MeV}
$$

or\[4\]

$$
\epsilon_\chi = \frac{(M_{\chi^*_+} - M_{\chi^*_+} + M_{\chi^*_+} - M_{\chi^-})}{2\sqrt{3}} = -1.07 \pm 0.02\text{MeV}.
$$

The agreement of these two measures of $\epsilon$ supports the assumption of baryon independence for the two-body interactions. The baryon masses for these equations have been taken from the experimental summary of PDG \[13\].

\[1\]The sum rule for $\epsilon_\chi$ was originally derived in reference \[9\] using SU(6) quark wave functions.
Combining these two values for $\epsilon$, the $\Lambda - \Sigma$ mixing angle is given by

$$\theta_m = \frac{-\epsilon}{(M_\Sigma - M_\Lambda)} = \frac{1.07}{17} = 0.014 = 0.80^\circ \pm 0.02^\circ. \quad (39)$$

We have made a small angle approximation for $\theta_m$. The value $\theta_m = 0.80 \pm 0.02^\circ$ is the mixing angle value that should be used for heavy baryon decays.

The mixing of the $\Lambda$ with the $\Sigma^0$ in $\Lambda_b \rightarrow \Sigma^0 + S$ decay allows this decay mode to occur, even though their spin wave functions are orthogonal. With $\Lambda - \Sigma$ mixing, the $\Lambda_b \rightarrow \Sigma^0$ matrix element becomes (for either $A_s$ or $A_{ps}$)

$$A(\Lambda_b \rightarrow \Sigma^0 + S) = 0 + \sin \theta_m \times A(\Lambda_b \rightarrow \Lambda + S). \quad (40)$$

Then the branching ratio for the rates of the two decays, including a phase space correction factor [14] $\Phi_{\Lambda_b}$, is

$$\mathcal{R} = \Phi_{\Lambda_b} \left| \frac{A(\Lambda_b \rightarrow \Sigma^0 + S)}{A(\Lambda_b \rightarrow \Lambda + S)} \right|^2 = \Phi_{\Lambda_b} \sin^2 \theta = 1.058(0.014)^2 = (2.1 \pm 0.3) \times 10^{-4}. \quad (41)$$

The LHCb collaboration has measured an upper limit of $\mathcal{R} < 21 \times 10^{-4}$ at 95% confidence level [14]. Our prediction is 10% of that experimental upper limit.

5 SUMMARY

We have derived a number of relations between $b$-baryon decay amplitudes in a nonsymmetric quark model with no use of the SU(6), SU(3), or SU(2) groups. All our predictions for branching ratios are well below current experimental limits, so tests of the predictions depend on improved measurement levels.

References

[1] C. Q. Geng, C.-W. Liu, and T.-H. Tsai, Phys. Rev. D 101, 054005 (2020), arXiv:2002.09583 [hep-ph].

[2] A. Dery, M. Ghosh, Y. Grossman, et al., J. High Energ. Phys. 165 (2020). https://doi.org/10.1007/JHEP03(2020)165

We refer to this paper as DGGS.
[3] P. Federmann, H. R. Rubinstein, and I. Talmi, Phys. Lett. **22**, 208 (1966).

[4] H. R. Rubinstein, F. Scheck, and R. H. Socolow, Phys. Rev. **154**, 1608 (1967).

[5] J. Franklin, Phys. Rev. **172**, 1807 (1968).
   The ‘hidden spin’ in the title of this paper was a precursor of ‘color’,
   but the model in the paper is the usual quark model.

[6] J. Franklin, D. B. Lichtenberg, W. Namgung, and D. Carydas, Phys. Rev. D **24**, 2910 (1981).

[7] A. J. Macfarlane and E. C. G. Sudarshan, Nuovo Cimento **31**, 1176 (1964).

[8] R. H. Dalitz and F. Von Hippel, Phys. Lett. **10**, 153 (1964).

[9] A. Gal and F. Scheck, Nucl. Phys. B2, 110 (1967).

[10] A. De Rujula, H. Georgi, and S. L. Glashow, Phys. Rev. D **12**, 147 (1975).

[11] N. Isgur, Phys. Rev. D **21**, 779 (1980), [Erratum: Phys. Rev. D **23**, 817 (1981)].

[12] Z. R. Kordov, R. Horsley, Y. Nakamura, H. Perlt, P. E. L. Rakow, G. Schierholz, H. Stben, R. D. Young, and J. M. Zanotti, (2019), [arXiv:1911.02186 [hep-lat]].

[13] M. Tanabashi et al. (Particle Data Group), Phys. Rev. D**98**, 030001 (2018) and 2019 update.

[14] R. Aaij et al. (LHCb), Phys. Rev. Lett. **124**, 111802 (2020), [arXiv:1912.02110 [hep-ex]].