Morawetz Estimates Method for Scattering of Radial Energy Sub-critical Wave Equation∗

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Abstract

In this short paper we consider a semi-linear, energy sub-critical, defocusing wave equation

\[ \partial_t^2 u - \Delta u = -|u|^{p-1}u \quad (x,t) \in \mathbb{R}^3 \times \mathbb{R}; \]

\[ u(\cdot,0) = u_0; \]

\[ u_t(\cdot,0) = u_1. \]

(CP1)

This Cauchy problem is locally well-posed for any initial data \((u_0,u_1)\) in the critical Sobolev space \(\dot{H}^s \times \dot{H}^{s-1}(\mathbb{R}^3)\) with \(s_p = 3/2 - 2/(p-1)\), as shown in Lindblad and Sogge’s work [9]. There is also an energy conservation law for suitable initial data:

\[ E(u,u_t) = \int_{\mathbb{R}^3} \left( \frac{1}{2} |\nabla u(\cdot,t)|^2 + \frac{1}{2} |u_t(\cdot,t)|^2 + \frac{1}{p+1} |u(\cdot,t)|^{p+1} \right) \, dx = \text{Const}. \]

We then need to consider the global existence and asymptotic behaviour of solutions. The only fully understood case is the energy critical one with \(p = 5\). More than twenty years ago, M. Grillakis [4] proved that any solution with initial data in the energy space \(\dot{H}^1 \times L^2(\mathbb{R}^3)\) must scatter in both two time directions, i.e. the solution looks like a free wave as \(t\) goes to infinity.

A similar result is expected to hold for other \(p\) as well.

**Conjecture 1.1.** Any solution to (CP1) with initial data \((u_0,u_1)\) \(\in \dot{H}^s \times \dot{H}^{s-1}\) must exist for all time \(t \in \mathbb{R}\) and scatter in both two time directions

This is still an open problem in the field of analysis of PDEs, in spite of some progress. Roughly speaking, known results fall into two categories:

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A priori estimate The first type of results assume that a solution \( u \) satisfies an a priori estimate
\[
\sup_{t \in I} \| (u(\cdot, t), u_t(\cdot, t)) \|_{\dot{H}^{p} \times \dot{H}^{p-1}(\mathbb{R}^3)} < +\infty
\]
in the whole lifespan \( I \), then prove that \( u \) is a global solution in time and scatters. Please see table 1 for a list of these results. They are usually proved via a compactness-rigidity argument. Please note that our assumption (1) is automatically true in the energy critical case \( p = 5 \), thanks to the energy conservation law.

Table 1: Results of scattering with a priori estimates in critical space

| Dodson-Lawrie [1] | Shen [11] | Kenig-Merle [7] | Killip-Visan [8] |
|-------------------|-----------|-----------------|-----------------|
| \( 1 + \sqrt{2} < p \leq 3 \), radial | \( 3 < p < 5 \), radial | \( p > 5 \), radial | \( p > 5 \), non-radial |

Stronger assumptions on initial data The second type of results make additional assumptions on the initial data in order to prove the scattering of solutions.

- Conformal conservation laws (see \([3, 5]\)) can be used to prove the scattering of solutions for \( p \in [3, 5) \) if initial data satisfy an additional regularity-decay condition
\[
\int_{\mathbb{R}^3} \left[ (|x|^2 + 1)(|\nabla u_0(x)|^2 + |u_1(x)|^2) + |u_0(x)|^2 \right] \, dx < \infty.
\]
The key ingredient of the proof is the following conformal conservation law
\[
\frac{d}{dt} Q(t, u, u_t) = \frac{4(3 - p)t}{p + 1} \int_{\mathbb{R}^3} |u(x, t)|^{p+1} \, dx.
\]
Here \( Q(t, \varphi, \psi) = Q_0(t, \varphi, \psi) + Q_1(t, \varphi) \) is called the conformal charge with
\[
Q_0(t, \varphi, \psi) = \left\| \frac{\psi + t \nabla \varphi}{|x|} \right\|_{L^2(\mathbb{R}^3)}^2 + \left\| \frac{\psi + 2 \varphi}{|x|} \right\|_{L^2(\mathbb{R}^3)}^2 + |x| \nabla \varphi \right\|_{L^2(\mathbb{R}^3)}^2
\]

The assumption (2) is essential to guarantee the finiteness of the conformal charge \( Q(t, u, u_t) \) as defined above. The conformal conservation law then gives a global space-time integral
\[
\int_{|t| > 1} \int_{\mathbb{R}^3} |u(x, t)|^{p+1} \, dx \, dt \lesssim_p \sup_{t \in \mathbb{R}} Q_1(t, u) \leq \sup_{t \in \mathbb{R}} Q(t, u, u_t) = Q(0, u_0, u_1) < +\infty,
\]
which implies the scattering. One advantage of this argument is that the radial assumption is not necessary.

- In the author’s previous work \([12]\) we proved the scattering of solutions if the radial initial data \( (u_0, u_1) \in \dot{H}^1 \times L^2 \) satisfy
\[
\int_{\mathbb{R}^3} (1 + |x|)^{1+2\varepsilon} (|\nabla u_0|^2 + |u_1|^2) \, dx < \infty
\]
for a constant \( \varepsilon > 0 \). The assumptions on the decay of initial data are weaker than the conformal conservation law method above, reducing the exponent of \( |x| \) from 2 to slightly greater than 1. The proof depends on a conformal transformation
\[
v(y, \tau) = \frac{\sinh |y|}{|y|} e^{\varepsilon t} u \left( e^{\varepsilon t} \frac{\sinh |y|}{|y|}, y, t_0 + e^{\varepsilon} \cosh |y| \right), \quad (y, \tau) \in \mathbb{R}^3 \times \mathbb{R},
\]
which converts a solution $u$ as above to a finite-energy solution $v$ of another non-linear wave equation

$$v_{rr} - \Delta_p v = -\left(\frac{|y|}{\sinh |y|}\right)^{p-1}e^{-(p-3)r}|v|^{p-1}v.$$ 

This second equation turns out to be easier to deal with since its non-linear term has a good decay rate as $x$ or $t$ goes to infinity.

**Main Result** In this paper we prove the scattering result with even weaker assumptions on the decay rate of the initial data.

**Theorem 1.2.** Let $\kappa > \kappa(p) = \frac{2(3-p)}{p+3}$ be a constant. If initial data $(u_0, u_1)$ are radial and satisfy

$$\int_{\mathbb{R}^3} (|x| + 1)^\kappa \left(\frac{1}{2} |
abla u_0|^2 + \frac{1}{2} |u_1|^2 + \frac{1}{p+1} |u_0|^{p+1}\right) < +\infty.$$ 

Then the corresponding solution $u$ to (CP1) must scatter in both two time directions. More precisely, there exists $(u_0^+, u_1^+) \in (H^{1} \cap H^{s_1}(\mathbb{R}^3)) \times (L^2 \cap H^{s_2-1}(\mathbb{R}^3))$, so that for any $s' \in [s_p, 1]$

$$\lim_{t \to \pm \infty} \|u(\cdot, t)\|_{H^s} = S_L(t) = 0.$$ 

Here $S_L(t)$ is the linear wave propagation operator.

**Remark 1.3.** Given any initial data as in the theorem above, we have

$$\int_{\mathbb{R}^3} (|\nabla u_0|^q + |u_1|^q) \, dx \leq 2 \left[ \int_{\mathbb{R}^3} (|\nabla u_0|^q + |u_1|^q) (1 + |x|)^\kappa \, dx \right]^{q/2} \left[ \int_{\mathbb{R}^3} (1 + |x|)^{-\kappa q/(2-q)} \, dx \right]^{(2-q)/2}$$

as long as $\frac{6}{3+\kappa} < q < 2$. By the Sobolev embedding $W^{1,q} \times L^q \hookrightarrow \dot{H}^s \times \dot{H}^{s-1}$ with $\frac{1+q}{q} = \frac{1}{q} - \frac{1}{2} > 0$, we have

$$(u_0, u_1) \in \dot{H}^s \times \dot{H}^{s-1}(\mathbb{R}^3), \text{ for any } s \in \left[1 - \frac{\kappa}{2}, 1\right].$$

Since we have $s_p > \frac{5p-9}{2(p+3)} = 1 - \frac{\kappa(p)}{2} > 1 - \frac{\kappa}{2}$, our initial data is always contained in the critical Sobolev space.

**Remark 1.4.** The author believes that the lower bound of decay rate $\kappa(p) = \frac{3(5-p)}{p+3}$ given in the main theorem is by no means optimal. However, this decay rate is still lower than previously known results.

**Notations** In this work we use the following notations.

- If $u(x)$ is a radial function defined in $\mathbb{R}^3$, then by convention we define $u(r) = u(x)$ where $|x| = r$.
- The notation $A \lesssim B$ means that there exists a constant $c$ so that the inequality $A \leq cB$ holds. We can also add one or more parameter(s) as the subscript of $\lesssim$. This implies that the constant $c$ depends on the parameter(s) mentioned but nothing else.
2 Motivation

Because the initial data come with a finite energy, Energy-super criticality leads to the global existence of the corresponding solution \( u \). In order to obtain the scattering result, we need to use the following result:

**Proposition 2.1** (Scattering with a finite \( L^2(p-1)L^{2(p-1)} \) norm, see Proposition 3.8 of [12]). Let \( u \) be a solution to (CP1) with initial data \((u_0, u_1) \in (H^1 \cap \dot{H}^{s'}) \times (L^2 \cap \dot{H}^{s'-1})\). If \( \| u \|_{L^2(p-1)L^{2(p-1)}(\mathbb{R}^3)} < \infty \), then \( u \) scatters in both two time directions. More precisely, there exist two pairs \((u_0^{\pm}, u_1^{\pm}) \in (\dot{H}^1 \cap \dot{H}^{s'}) \times (L^2 \cap \dot{H}^{s'-1})\), so that the following limit holds for each \( s' \in [s, 1] \)

\[
\lim_{t \to \pm \infty} \| (u(\cdot, t), u_t(\cdot, t)) - S_L(t)(u_0^{\pm}, u_1^{\pm}) \|_{\dot{H}^{s'} \times \dot{H}^{-1}} = 0.
\]

As a result, it suffices to prove the global space-time integral estimate

\[
\int_{-\infty}^{\infty} \int_{\mathbb{R}^3} |u(x, t)|^{2(p-1)} dx dt < +\infty. \tag{3}
\]

The first known global space-time integral that comes into our mind is the Morawetz estimate

\[
\int_{-\infty}^{\infty} \int_{\mathbb{R}^3} \frac{|u(x, t)|^{p+1}}{|x|} dx dt \lesssim E.
\]

In the energy critical case, i.e. \( p = 5 \), we can apply inequality \(|x|^{1/2}|u(x, t)| \lesssim \| u(\cdot, t) \|_{\dot{H}^1} \lesssim E^{1/2} \) for radial \( \dot{H}^1 \) functions and the Morawetz estimate immediately gives us (3). In the energy sub-critical case, however, if we applied the best estimate for radial solutions the author knows (See Lemma 4.1 below)

\[
|u(x, t)| \lesssim_p E^{\frac{2}{p+3}} |x|^{-\frac{2}{p+3}}
\]

we would obtain

\[
\int_{-\infty}^{\infty} \int_{\mathbb{R}^3} \frac{|u(x, t)|^{2(p-1)}}{|x|^{\frac{2(p-2)}{p+3}}} dx dt = \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} \frac{|u(x, t)|^{p+1}}{|x|} \cdot \left(|x|^{\frac{2}{p+3}} |u(x, t)| \right)^{p-3} dx dt < \infty.
\]

This is still weaker than the desired inequality (3) as \(|x|\) is large. In this work we will solve this problem by a suitable power-like decay

\[
\int_{-\infty}^{\infty} \int_{|x| > R} \frac{|u(x, t)|^{p+1}}{|x|} dx dt \lesssim R^{-\infty}.
\]

3 Review of Morawetz Estimates

We are able to take a more careful look at this well-known global space-time integral estimate if we recall the original theorem given in Perthame and Vega’s work [10].

**Theorem 3.1.** Let \( u \) be a solution to (CP1) defined in a time interval \([0, T]\) with a finite energy \( E \). Then given any \( R > 0 \), we have the inequality

\[
\frac{1}{2R} \int_{0}^{T} \int_{|x| < R} (|\nabla u|^2 + |u_t|^2) dx dt + \frac{1}{2R^2} \int_{0}^{T} \int_{|x| = R} |u|^2 d\sigma_R dt + \frac{p-2}{(p+1)R} \int_{0}^{T} \int_{|x| < R} |u|^{p+1} dx dt \\
+ \frac{p-1}{p+1} \int_{0}^{T} \int_{|x| > R} \frac{|u|^{p+1}}{|x|} dx dt + \frac{1}{R^2} \int_{|x| < R} |u(x, T)|^2 dx \leq 2E. \tag{4}
\]
Remark 3.2. We focus on the 3D case with \( d = 3 \). Please note that the notations \( E \) and \( p \) were defined in a slightly different way in Perthame-Vega’s original paper. Here we rewrite the inequality in the setting of the current work. The author also believes that there is a minor typing mistake in the original inequality. The last term \( \frac{d^2-1}{4R^2}\int_{B(0,R)} |u(T)|^2 \, dx \) in the left hand side should have been \( \frac{d^2-1}{4R^2}\int_{B(0,R)} |u(T)|^2 \, dx \) instead, although the change of this coefficient plays no role in the argument of this work.

Careful look at Morawetz Estimate First of all, let us ignore the final term in the left hand and substitute \( T \) by \(+\infty\). Thanks to the energy conversation law, we are also able to substitute the lower limit of the integrals by \(-\infty\). Finally we can combine part of the third term above with the first term, then divide both sides by 2 and write

\[
\frac{1}{2R} \int_{-\infty}^{+\infty} \int_{|x|<R} \left( \frac{1}{2} |\nabla u|^2 + \frac{1}{2} |u_t|^2 + \frac{1}{p+1} |u|^{p+1} \right) \, dx \, dt + \frac{1}{4R^2} \int_{-\infty}^{+\infty} \int_{|x|=R} |u|^2 \, d\sigma_R \, dt
\]

\[
+ \frac{p-3}{2(p+1)R} \int_{-\infty}^{+\infty} \int_{|x|<R} |u|^{p+1} \, dx \, dt + \frac{p-1}{2(p+1)} \int_{-\infty}^{+\infty} \int_{|x|>R} |u|^{p+1} \, dx \, dt \leq E. \tag{5}
\]

Now we have an important observation that the first term in (5) is almost \( E \) when \( R \) is sufficiently large. In fact, the finite speed of propagation implies that for almost all \( t \in (-R, R) \), as long as \( |t| \) is not too close to \( R \), almost all energy concentrates in the region \( B(0, R) = \{ x \in \mathbb{R}^3 : |x| < R \} \). This means that the values of other terms have to be very small. More precisely, we can calculate

\[
\frac{p-1}{2(p+1)} \int_{-\infty}^{+\infty} \int_{|x|>R} |u|^{p+1} \, dx \, dt \leq E - \frac{1}{2R} \int_{-R}^{+R} \int_{|x|<R} \left( \frac{1}{2} |\nabla u|^2 + \frac{1}{2} |u_t|^2 + \frac{1}{p+1} |u|^{p+1} \right) \, dx \, dt
\]

\[
= \frac{1}{2R} \int_{-R}^{+R} \int_{|x|>R} \left( \frac{1}{2} |\nabla u|^2 + \frac{1}{2} |u_t|^2 + \frac{1}{p+1} |u|^{p+1} \right) \, dx \, dt. \tag{6}
\]

The right hand side is exactly the average amount of energy which escapes outside the ball \( B(0, R) \) for \( t \in [-R, +R] \). Now we calculate carefully the energy outside the ball under additional decay assumption of the initial data.

4 An Energy Escaping Estimate

Our argument relies on

Proposition 4.1. Let \( u \) be a solution to (CP1) with a finite energy and satisfy

\[
I = \int_{\mathbb{R}^3} |x|^\kappa \left( \frac{1}{2} |\nabla u_0|^2 + \frac{1}{2} |u_t|^2 + \frac{1}{p+1} |u_0|^{p+1} \right) \, dx < \infty.
\]

Then we have the function

\[
I(t) = \int_{|x|>t} (|x|-t)^\kappa \left( \frac{1}{2} |\nabla u|^2 + \frac{1}{2} |u_t|^2 + \frac{1}{p+1} |u|^{p+1} \right) \, dx \leq I, \quad t > 0.
\]
Proof. It immediately follows a basic calculation of the derivative

\[ I'(t) = - \int_{|x| > t} \kappa(|x| - t)^{\kappa - 1} \left( \frac{1}{2} |\nabla u|^2 + \frac{1}{2} |u_t|^2 + \frac{1}{p + 1} |u|^{p+1} \right) dx \]

\[ + \int_{|x| > t} (|x| - t)^{\kappa} \left( \nabla u \cdot \nabla u_t + u_t u_{tt} + |u|^{p-1} uu_t \right) dx \]

\[ = - \int_{|x| > t} \kappa(|x| - t)^{\kappa - 1} \left( \frac{1}{2} |\nabla u|^2 + \frac{1}{2} |u_t|^2 + \frac{1}{p + 1} |u|^{p+1} \right) dx \]

\[ + \int_{|x| > t} \left\{ (|x| - t)^{\kappa} \left( u_t u_{tt} + |u|^{p-1} uu_t \right) - u_t \text{div} \left( (|x| - t)^{\kappa} \nabla u \right) \right\} dx \]

\[ = - \int_{|x| > t} \kappa(|x| - t)^{\kappa - 1} \left( \frac{1}{2} |\nabla u|^2 + \frac{1}{2} |u_t|^2 + \frac{1}{p + 1} |u|^{p+1} \right) dx \]

\[ \leq 0. \]

Here we have assumed that \( u \) is sufficiently smooth. Otherwise we can apply smooth approximation techniques.

Remark 4.2. We can also consider the negative time direction and conclude

\[ I(t) = \int_{|x| > |t|} (|x| - |t|)^{\kappa} \left( \frac{1}{2} |\nabla u|^2 + \frac{1}{2} |u_t|^2 + \frac{1}{p + 1} |u|^{p+1} \right) dx \leq I, \quad t \in \mathbb{R}. \]

Energy escaping the ball \( B(0, R) \) Now we have \( t \in (-R, R) \)

\[ \int_{|x| > R} \left( \frac{1}{2} |\nabla u|^2 + \frac{1}{2} |u_t|^2 + \frac{1}{p + 1} |u|^{p+1} \right) dx \]

\[ \leq (R - |t|)^{-\kappa} \int_{|x| > R} (|x| - |t|)^{\kappa} \left( \frac{1}{2} |\nabla u|^2 + \frac{1}{2} |u_t|^2 + \frac{1}{p + 1} |u|^{p+1} \right) dx \]

\[ \leq (R - |t|)^{-\kappa} I(t) \]

\[ \leq (R - |t|)^{-\kappa} I. \]

Combining this inequality with (6), we obtain the decay rate of space-time integral of \( |u|^{p+1}/|x| \)

\[ \int_{-\infty}^{+\infty} \int_{|x| > R} \frac{|u|^{p+1}}{|x|} dx dt \lesssim_{p, \kappa} IR^{-\kappa}. \quad (7) \]

5 Completion of the Proof

Now we need the following point-wise estimate on solutions

Lemma 5.1. If a radial function \( u \) satisfies

\[ \int_{\mathbb{R}^3} (|\nabla u|^2 + |u|^{p+1}) dx \leq E, \]

then we have \( |u(x)| \lesssim_{p} E^{2/(p+3)} |x|^{-4/(p+3)}. \)
Proof. Let $|u(r_0)| = S$. For any $r \in (r_0, r_0 + r_0^2S^2/4E)$ we have

$$|u(r) - u(r_0)| \leq \int_{r_0}^{r} |u_r(s)|ds \leq \left( \int_{r_0}^{r} s^2|u_r(s)|^2ds \right)^{1/2} \left( \int_{r_0}^{r} s^{-2}ds \right)^{1/2},$$

$$\leq E^{1/2} \left( \frac{1}{r_0} - \frac{1}{r} \right)^{1/2} \leq \left[ E \cdot \frac{r_0^2S^2/4E}{r_0r} \right]^{1/2} \leq \frac{S}{2}.$$ 

Therefore $u$ satisfies $|u(r)| \geq S/2$ for all $r \in (r_0, r_0 + r_0^2S^2/4E)$. Now we use the $L^{p+1}$ bound

$$\left( \frac{S}{2} \right)^{p+1} r_0^p \cdot \frac{r_0^2S^2}{4E} \leq \int_{r_0}^{r_0 + r_0^2S^2/4E} |u(r)|^{p+1}r^2dr \leq \int_{R^3} |u(x)|^{p+1}dx \leq E.$$ 

This immediately gives the pointwise estimate.

**Global Integral Estimate** We start by applying Lemma 5.1 and obtain

$$|u(x,t)|^{2(p-1)} = |u(x,t)|^{p-3} \cdot |u(x,t)|^{p+1} \lesssim_p E^{2\frac{p-3}{p+3}} |x|^{-\frac{4(p-3)}{p+3}} \cdot |u(x,t)|^{p+1}. \quad \text{(8)}$$

We use the inequality above, recall the decay rate estimate (7) and deduce

$$\int_{-\infty}^{\infty} \int_{|x| > R} \frac{|u(x,t)|^{2(p-1)}}{|x|^{n(p-1)\alpha}} dxdt \lesssim_p \int_{-\infty}^{\infty} \int_{|x| > R} E^{2\frac{p-3}{p+3}} |x|^{-\frac{4(p-3)}{p+3}} \cdot |u(x,t)|^{p+1} dxdt$$

$$= E^{2\frac{p-3}{p+3}} \int_{-\infty}^{\infty} \int_{|x| > R} |u(x,t)|^{p+1} dxdt$$

$$\lesssim_p E^{2\frac{p-3}{p+3}} IR^{-\kappa}.$$ 

Since $\kappa > \frac{3(5-p)}{p+3}$, the inequality above implies

$$\int_{-\infty}^{\infty} \int_{|x| > R} |u(x,t)|^{2(p-1)} dxdt \lesssim_p E^{2\frac{p-3}{p+3}} IR^{-\left(\kappa - \frac{3(5-p)}{p+3}\right)}. \quad \text{(9)}$$

This gives a finite upper bound for the integral of $|u|^{2(p-1)}$ in the region with large $x$. In order to find an upper bound of the integral in the region with small $x$, we can use (8) again and obtain

$$\int_{-\infty}^{\infty} \int_{|x| < R} \frac{|u(x,t)|^{2(p-1)}}{|x|^{-n(p-1)\alpha}} dxdt \lesssim_p E^{2\frac{p-3}{p+3}} \int_{-\infty}^{\infty} \int_{|x| < R} |u(x,t)|^{p+1} dxdt \lesssim_p E^{2\frac{p-1}{p+3}}.$$ 

As a result we have

$$\int_{-\infty}^{\infty} \int_{|x| < R} |u(x,t)|^{2(p-1)} dxdt \lesssim_p E^{\frac{3(p-1)}{p+3}} R^{-\frac{3(5-p)}{p+3}}. \quad \text{(10)}$$

We choose an arbitrary $R > 0$, combine (9) and (10) and finally conclude

$$\int_{-\infty}^{\infty} \int_{R^3} |u(x,t)|^{2(p-1)} dxdt \leq C(p, \kappa, E, I) < +\infty.$$ 

This finishes the proof.
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