Abstract: In this paper, the following problem in the hyperbolic space $B_N$ will be considered

$$-\Delta_{B^N} u = f(x, u), \text{ in } B^N. \tag{1}$$

where, $\Delta_{B^N}$ denotes the Laplace Beltrami operator on $B^N$. And this problem can be converted into the following Euclidean problem

$$\begin{cases} -\text{div}(K(x) \nabla u) = 4K(x)^{N-2} f(x, u), & \text{in } B^N, \\ u(0) = 0, & \text{on } \partial B^N, \end{cases} \tag{2}$$

where, $K(x) := 1/(1 - |x|^2)^{N-2}$. Then, the existence of solution of problem (1) can be obtained by studying the existence of solution of problem (2). We will equip problem (2) with a weighted Sobolev space and prove the compact embedding theorem and the concentration compactness principle for the weighted Sobolev space. And we will prove that the maximum principle holds for the operator $-\text{div}(K(x)\nabla u)$.

When $f(x, u) = |u|^{q-2}u + \lambda u^{p-2}u$, $\lambda > 0$, $1 < q < 2^*$, using the variational method, the compact embedding theorem, the concentration compactness principle and the maximum principle, the existence of nonradial solutions of problem (2) will be obtained.

Keywords: Weighted Sobolev Space, Concentration compactness principle, Infinitely many solutions, Brezis-Nirenberg problem.

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1 Introduction

Let $\mathbb{B}^N := \{x \in \mathbb{R}^N : |x| < 1\}$ denote the unit disc in $\mathbb{R}^N$. The space $\mathbb{B}^N$ endowed with the Riemannian metric $g$ given by $g_{ij} = \left(\frac{2}{1-|x|^2}\right)^2 \delta_{ij}$ is called the ball model of the Hyperbolic space. We will denote the associated hyperbolic volume by $dV_{\mathbb{B}^N}$ and is given by $dV_{\mathbb{B}^N} = \left(\frac{2}{1-|x|^2}\right)^N dx$. The hyperbolic gradient $\nabla_{\mathbb{B}^N}$ and the hyperbolic Laplacian $\Delta_{\mathbb{B}^N}$ are given by

$$\nabla_{\mathbb{B}^N} = \left(1 - |x|^2\right)^2 \nabla, \quad \Delta_{\mathbb{B}^N} = \left(1 - |x|^2\right)^2 \Delta + \frac{1}{2}(N - 2) \frac{1}{2} \langle x, \nabla \rangle. \quad (1.1)$$

Let $H^1(\mathbb{B}^N)$ denote the Sobolev space on $\mathbb{B}^N$ with the above metric $g$, then we have the embeddings $H^1(\mathbb{B}^N) \rightarrow L^p(\mathbb{B}^N)$ for $2 \leq p \leq \frac{2N}{N-2}$ when $N \geq 3$ and $p \geq 2$ when $N = 2$. For more details on hyperbolic geometry we refer to [23].

The following Brezis-Nirenberg type problem was considered in [3, 16],

$$-\Delta_{\mathbb{B}^N} u - \lambda u = |u|^{2^*-2} u, u \in H^1(\mathbb{B}^N), \quad (1.2)$$

where $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent and $N \geq 3$. Problem (1.2) is a natural generalization of the famous Brezis-Nirenberg problem which was proposed by Brezis and Nirenberg in 1983 (see [4]). In recent forty years, great progress has been made on Brezis-Nirenberg problem. Readers can refer to [8, 9, 10, 14, 24] and its references for more progress on Brezis-Nirenberg problem.

Since the embedding $H^1(\mathbb{B}^N) \rightarrow L^p(\mathbb{B}^N)$ for $2 \leq p \leq \frac{2N}{N-2}$ is not compact, it is not easy to prove the existence of solutions of problem (1.2). In [3], Bhakta and Kunnath have studied problem (1.2) in the radial case, obtained the hyperbolic sobolev compact embedding theorem of this case, and used the result to prove the existence of radial solutions. The existence of infinite many radial solutions were obtained for (1.2) by Ganguly and Karmakar in [17]. For the case $N = 2$, Ganguly and Karmakar [17] considered the more general nonlinear equation

$$-\Delta_{\mathbb{B}^2} u = f(x, t), \quad u \in H^1(\mathbb{B}^2), \quad (1.3)$$

and used the Moser-Trudinger inequality of the hyperbolic space (see [2, 20]) to prove the existence of solutions for problem (1.3).

The common feature of the results in [3, 16, 17] is that the existence of solutions are obtained in the radial case in $H^1(\mathbb{B}^N)$. The reader can be referred to [21, 22, 25] for more results on Hyperbolic Laplacian equations.

In this paper, we consider the following problem

$$-\Delta_{\mathbb{B}^N} u = f(x, u), \text{in} \; \mathbb{B}^N, \quad (1.4)$$

where, $N \geq 3$, $f(x, u) = |u|^{2^*-2} u + \lambda u^{q-2}u, \lambda > 0, 1 < q < 2^*$. By (1.1), it is not difficult to deduce that problem (1.4) can be converted into the following Euclidean problem

$$\begin{cases}
-\text{div}(K(x)\nabla u) = 4K(x)^{\frac{2}{N-2}} f(x, u), & \text{in} \; \mathbb{B}^N, \\
u(0) = 0, & \text{on} \; \partial\mathbb{B}^N,
\end{cases} \quad (1.5)$$

where,

$$K := K(x) := \frac{1}{(1 - |x|^2)^{N-2}} = e^{-(N-2)\ln(1-|x|^{2})}. \quad (1.6)$$
Then, the existence of solution of problem (1.4) can be obtained by studying the existence of solution of problem (1.5). The main purpose of this paper is to equip problem (1.5) with a weighted Sobolev space, prove the compact embedding theorem, the maximum principle, and the concentration-compactness principle, and use these results to obtain the existence of nonradial solutions of problem (1.5).

We define the weighted Lebesgue spaces and the weighted smooth function spaces as follows:

\[ L^p_{K^{N/2}}(\mathbb{B}^N) := \left\{ u : \int_{\mathbb{B}^N} K^{N/2} |u|^p \, dx < \infty \right\}, \quad 1 \leq p < \infty, \]

\[ L^\infty_{K^{N/2}}(\mathbb{B}^N) := \left\{ u : K^{N/2} u \in L^\infty(\mathbb{B}^N) \right\}, \]

\[ C^\infty_{K^{N/2}}(\mathbb{B}^N) := \left\{ u : K^{N/2} u \in C^\infty(\mathbb{B}^N) \right\}, \]

and

\[ C^\infty_{0,K^{N/2}}(\mathbb{B}^N) := \left\{ u : K^{N/2} u \in C^\infty_0(\mathbb{B}^N) \right\}. \]

Similarly, we can also define \( L^p_K(\mathbb{B}^N), L^p_{K^{N/2}}(\mathbb{B}^N), C^\infty_{0,K}(\mathbb{B}^N), C^\infty_K(\mathbb{B}^N), \) etc. Readers are referred to [1, 5, 6, 11, 13, 26, 27] for several types of weighted Lebesgue spaces.

Similar to the proofs of page 32 and page 49 in [1], it is easy to see that the space \( L^p_{K^{N/2}}(\mathbb{B}^N) \) \((1 < p < \infty)\) is separable and reflexive Banach space endowed with the norm

\[ |u|_{K^{N/2},p} := \left( \int_{\mathbb{B}^N} K^{N/2} |u|^p \, dx \right)^{1/p} < \infty. \]

Now we define the weighted Sobolev spaces

\[ H^1_K(\mathbb{B}^N) := \left\{ u : \int_{\mathbb{B}^N} K |\nabla u|^2 \, dx < \infty \right\}, \]

and

\[ H^1_{0,K}(\mathbb{B}^N) := \left\{ u : \int_{\mathbb{B}^N} K |\nabla u|^2 \, dx < \infty, \text{ and } u = 0 \text{ on } \partial \mathbb{B}^N \right\}. \]

We easily know that \( H^1_K(\mathbb{B}^N) \) and \( H^1_{0,K}(\mathbb{B}^N) \) are Hilbert spaces equipped with the norm

\[ ||u||_{K,2} := \left( \int_{\mathbb{B}^N} K |\nabla u|^2 \, dx \right)^{1/2}. \] (1.6)

Similar to the proofs of page 71 and page 72 in [1], we can show that \( H^1_{0,K}(\mathbb{B}^N) \) can also be treated as the completion of \( C^\infty_K(\mathbb{B}^N) \) with respect to the norm (1.6).

This paper is organized as follows. In Section 2, we will prove that the embedding theorem and the concentration-compactness principle hold for the weighted Sobolev Space. Sections 3 will be devoted to the proof of the maximum principle for the operator \( -\text{div}(K(x) \nabla u) \). In Section 4, the Brezis-Nirenberg type problem of (1.5) i.e. \( f(x,u) = |u|^{q-2} u + \lambda u, \lambda > 0 \) will considered and the existence of nonradial solutions will be proved. In Section 5, we will consider the case \( f(x,u) = |u|^{q-2} u + \lambda |u|^{q-2} u, \lambda > 0, \) and obtain the existence of nonradial solutions. Our main results are Theorem 2.1, Theorem 2.2, Theorem 3.1, Theorem 3.2, Theorem 4.1, Theorem 5.1 and Theorem 5.2.
2 Embedding theorem and concentration-compactness principle

In this section, we will prove that the embedding theorem and the concentration-compactness principle hold for the weighted Sobolev Space.

**Theorem 2.1.** (Embedding theorem) For all $p \in [1, 2^*)$, the embedding $H^1_K(\mathbb{R}^N) \hookrightarrow L^p_{K^{N/2}}(\mathbb{R}^N)$ is continuous, and when $p \in [1, 2^*), the embedding is compact, i.e. $H^1_K(\mathbb{R}^N) \hookrightarrow L^p_{K^{N/2}}(\mathbb{R}^N)$.

**Theorem 2.2.** (Concentration-compactness principle) Let $1 < p < N, 2^* = \frac{2N}{N-2}$, $\{u_n\}$ be a bounded sequence in $H^1_{0,K}(\mathbb{R}^N)$, $u_n = 0$ on the outside of $\mathbb{R}^N$, and $\{u_n\}$ satisfy the following conditions

$$
\left\{\begin{array}{l}
u_n \rightharpoonup \nu, \\
\mu_n \rightharpoonup \mu, \\
\frac{\mu_n}{\nu_n} \rightharpoonup \frac{\mu}{\nu},
\end{array}\right.
$$

where $\mu, \nu$ are bounded positive measures on $\mathbb{R}^N$. Then

(i) There exist an at most countable set $J$, a family $\{x_j\}_{j \in J}$ of distinct points in $\mathbb{R}^N$ and a family $\{\nu_j\}_{j \in J} \subset (0, \infty)$, such that

$$
\nu = K^{N/2} u^2 + \sum_{j \in J} \nu_j \delta_{x_j}.
$$

(ii) There exist $\mu_j \geq S_K \nu_j^{\frac{2}{N-2}}$ such that

$$
\mu \geq K |\nabla u|^2 + \sum_{j \in J} \mu_j \delta_{x_j},
$$

where $S_K$ is the best constant of the embedding: $H^1_{0,K}(\mathbb{R}^N) \rightarrow L^2_{K^{N/2}}(\mathbb{R}^N)$, that is,

$$
S_K := \inf_{0 \neq u \in H^1_{0,K}(\mathbb{R}^N)} \frac{\|\nabla u\|_{L^2}}{|u|^2_{L^2_{K^{N/2},2^*}}}.
$$

**Proof of Theorem 2.1.** Below we will take three steps to prove Theorem 2.1.

**Step 1:** When $p = 2$ or $2^*$, the embedding $H^1_K(\mathbb{R}^N) \rightarrow L^p_{K^{N/2}}(\mathbb{R}^N)$ is continuous.

For all $u \in C^1_0(\mathbb{R}^N)$, let $v = K(x)^{1/2} u$, we have

$$
K^{1/2} \nabla u = \nabla v - \frac{v}{2} \nabla \omega
$$

and

$$
\int_{\mathbb{R}^N} |\nabla u|^2 K \ dx = \int_{\mathbb{R}^N} |\nabla v|^2 \ dx - \frac{1}{2} \int_{\mathbb{R}^N} 2v \nabla v \cdot \nabla \theta \ dx + \frac{1}{4} \int_{\mathbb{R}^N} v^2 |\nabla \omega|^2 \ dx,
$$

where,

$$
\omega = \ln K(x) = -(N-2) \ln (1 - |x|^2).
$$
We use integration by parts for the second term on the right-hand side of (2.6), and obtain
\[-\frac{1}{2} \int_{B^N} 2 v \nabla v \cdot \nabla \omega dx = \frac{1}{2} \int_{B^N} v^2 \Delta \theta dx. \tag{2.7}\]
Substituting (2.7) into (2.6), we deduce
\[
\int_{B^N} |\nabla u|^2 K dx = \int_{B^N} |\nabla v|^2 dx + \frac{1}{2} \int_{B^N} v^2 \Delta \omega dx + \frac{1}{4} \int_{B^N} v^2 |\nabla \omega|^2 dx
= \int_{B^N} |\nabla (K^{1/2} u)|^2 dx + \frac{1}{2} \int_{B^N} K u^2 \left( \Delta \omega + \frac{1}{2} |\nabla \omega|^2 \right) dx. \tag{2.8}\]
Using (2.8), it is easy to prove that
\[
\int_{B^N} |\nabla (K^{1/2} u)|^2 dx = \int_{B^N} K |\nabla u|^2 dx - \frac{1}{2} \int_{B^N} K u^2 \left( \Delta \omega + \frac{1}{2} |\nabla \omega|^2 \right) dx \tag{2.9}\]
and
\[
\int_{B^N} K |\nabla u|^2 dx \geq \frac{1}{2} \int_{B^N} K u^2 \left( \Delta \omega + \frac{1}{2} |\nabla \omega|^2 \right) dx. \tag{2.10}\]
From Sobolev’s inequality, we get
\[
\int_{B^N} |\nabla (K^{1/2} u)|^2 dx \geq S_0 \left( \int_{B^N} (K^{1/2} |u|)^2 dx \right)^{2/2'} = S_0 \left( \int_{B^N} K^{N/N-2} |u|^2 dx \right)^{2/2'}, \tag{2.11}\]
where \(S_0\) is the best Sobolev constant. Therefore, combining (2.9) and (2.11), we can get
\[
\int_{B^N} K |\nabla u|^2 dx \geq \frac{1}{2} \int_{B^N} K u^2 \left( \Delta \omega + \frac{1}{2} |\nabla \omega|^2 \right) dx + S_0 \left( \int_{B^N} K^{N/N-2} |u|^2 dx \right)^{2/2'} \tag{2.12}\]
\[
\geq S_0 \left( \int_{B^N} (K^{N/N-2} |u|)^2 dx \right)^{2/2'}. \]
Then the embedding \(H^1_k(B^N) \to L^2_{K^{N/N-2}}(B^N)\) is continuous, this is due to the fact that
\[
\Delta \omega + \frac{1}{2} |\nabla \omega|^2 = \frac{(N - 2)}{(1 - |x|^2)^2} \left[ N (1 - |x|^2) + 2(N - 1) |x|^2 + \frac{(N - 2) |x|^2}{(1 - |x|^2)^{N-2}} \right]
\geq \frac{(N - 2)}{(1 - |x|^2)^N} = (N - 2) K^{N/(N-2)}. \tag{2.13}\]
Combining inequalities (2.10) and (2.13), we can know
\[
\int_{B^N} K |\nabla u|^2 dx \geq (N - 2) \int_{B^N} K K^{N/N-2} u^2 dx \geq (N - 2) \int_{B^N} K^{N/N-2} u^2 dx, \tag{2.14}\]
which implies that the embedding \(H^1_k(B^N) \to L^2_{K^{N/N-2}}(B^N)\) is continuous.

**Step 2:** The embedding \(H^1_k(B^N) \to L^2_{K^{N/N-2}}(B^N)\) is compact.

We assume that \(\{u_n\} \subset H^1_k(B^N)\), and
\[
u_n \to 0 \text{ in } H^1_k(B^N), \quad \|u_n\|_{K, 2} \leq 1.
\]
When \( |x| \to 1 \), it is easy to show that
\[
\Delta \omega + \frac{1}{2} |\nabla \omega|^2 - \frac{1}{K^{N/2-N}} \to +\infty.
\] (2.15)

Therefore, for all \( \varepsilon > 0 \), there is a \( r > 0 \), for every \( x \in \{ x : 1 - r < |x| < 1 \} \), such that
\[
\Delta \omega + \frac{1}{2} |\nabla \omega|^2 = \frac{(N - 2)}{(1 - |x|^2)^N} \left[ N \left(1 - |x|^2\right) + 2(N - 1)|x|^2 + \frac{(N - 2)|x|^2}{(1 - |x|^2)^{N-2}} \right] \\
\geq \frac{2}{\varepsilon} K^{N/2-N}.
\] (2.16)

Then (2.10) and (2.16) imply that
\[
\int_{B^{N} \setminus B(0)} K |\nabla u_n|^2 \, dx \geq \frac{1}{\varepsilon} \int_{B^{N} \setminus B(0)} K u_n^2 K^{N/2-N} \, dx \geq \frac{1}{\varepsilon} \int_{B^{N} \setminus B(0)} K^{N/2-N} u_n^2 \, dx,
\]
that is
\[
\int_{B^{N} \setminus B(0)} K^{N/2-N} u_n^2 \, dx \leq \varepsilon \int_{B^{N} \setminus B(0)} K |\nabla u_n|^2 \, dx \leq \varepsilon \int_{B^{N}} K |\nabla u_n|^2 \, dx \leq \varepsilon. \tag{2.17}
\]

On the other hand, according to Sobolev’s compact embedding theorem, we know that
\[ u_n \to 0 \text{ in } L^2(B_r(0)). \]

Since \( K^{N/2-N} \in L^\infty(B_r(0)) \), the following inequality holds
\[
\int_{B_r(0)} K^{N/2-N} u_n^2 \, dx \leq \varepsilon.
\]

From (2.16) and (2.17), we have
\[
\int_{B^{N}} K^{N/2-N} u_n^2 \, dx \leq 2\varepsilon,
\]
which implies that
\[ u_n \to 0 \text{ in } L^2_{K^{N/2-N}}(B^{N}). \]

**Step 3:** \( H^1_K(\mathbb{R}^N) \hookrightarrow L^p_{K^{N/2-N}}(\mathbb{R}^N), \ p \in [1, 2^*]. \)

In step 1 and step 2, we have proved that the embedding \( H^1_K(\mathbb{R}^N) \hookrightarrow L^p_{K^{N/2-N}}(\mathbb{R}^N) \) is continuous for \( p = 2 \) and \( p = 2^* \). Hence, we only need to prove that the embedding is continuous and compact when \( p \in (1, 2^*). \)

Let \( p \in (2^*, 2^*) \) and
\[
t = \frac{2(2^* - p)}{p(2^* - 2)}, \quad a = \frac{N(2^* - p)}{(N - 2)(2^* - 2)}, \quad b = \frac{N(p - 2)}{(N - 2)(2^* - 2)}, \quad q_1 = \frac{2^* - 2}{2^* - p}, \quad q_2 = \frac{2^* - 2}{p - 2}.
\]

Obviously, \( q_1 + q_2 = 1 \). From (2.12) and (2.14), by Hölder inequality, we have
\[
\int_{\mathbb{R}^N} K^{N/2-N} |u|^p \, dx = \int_{\mathbb{R}^N} K^a |u|^{qt} K^b |u|^{p(1-t)} \, dx
\]
\[ \left( \int_{\mathbb{R}^N} K^{\alpha q_1} |u|^{p q_1} \, dx \right)^{1/q_1} \left( \int_{\mathbb{R}^N} K^{\alpha q_2} |u|^{p (1-\alpha) q_2} \, dx \right)^{1/q_2} \leq \left( \int_{\mathbb{B}^N} K^{\alpha N/N-2} |u|^2 \, dx \right)^{1/q_1} \left( \int_{\mathbb{B}^N} K^{\alpha N/N-2} |u|^2 \, dx \right)^{1/q_2} \leq \left( \frac{1}{N-2} \int_{\mathbb{B}^N} K|\nabla u|^2 \, dx \right)^{1/q_1} \left( \frac{1}{S_K} \int_{\mathbb{B}^N} K|\nabla u|^2 \, dx \right)^{2/(2q_2)} = C_p \left( \int_{\mathbb{B}^N} K|\nabla u|^2 \, dx \right)^{1/q_1 + 2/(2q_2)} \leq C_p \left( \int_{\mathbb{B}^N} K|\nabla u|^2 \, dx \right)^{p/2} , \]  

where, \( C_p = (N-2)^{-1/q_1} S_K^{-2q_2/2q_1} \), \( S_K \) is defined in (4.4). As a result, the embedding \( H^1_k(\mathbb{B}^N) \to L^p_{K^{N/N-2}}(\mathbb{B}^N) \) is continuous for \( p \in (1,2^*) \).

Now we assume that \( p \in (2,2^*) \), \( \alpha \in (0,1) \) and \( \frac{\alpha}{2} + \frac{\alpha}{2} = \frac{1}{p} \). Then by the interpolation inequality, we have

\[ |u|_{K^{N/N-2},p} \leq |u|_{K^{N/N-2},2}^{1-\alpha} |u|_{K^{N/N-2},2^*}^{\alpha} \text{ for all } u \in L^2_{K^{N/N-2}}(\mathbb{B}^N) \cap L^{2^*}_{K^{N/N-2}}(\mathbb{B}^N). \]

Since the embedding \( H^1_k(\mathbb{B}^N) \hookrightarrow L^2_{K^{N/N-2}}(\mathbb{B}^N) \) is compact, it follows from the above inequality that

\[ H^1_k(\mathbb{B}^N) \hookrightarrow L^p_{K^{N/N-2}}(\mathbb{B}^N), \ p \in [1,2^*). \]

This finishes the proof. \( \Box \)

**Proof of Theorem 2.2.** For all \( \phi \in C^1_0(\mathbb{B}^N) \), by Theorem 2.1 and the definition of \( S_K \), we obtain

\[ \left( \int_{\mathbb{B}^N} K^{\frac{N}{N-2}} |\phi u_n|^2 \, dx \right)^{\frac{1}{2}} S_K^{\frac{1}{2}} \leq \left( \int_{\mathbb{B}^N} K|\nabla (\phi u_n)|^2 \, dx \right)^{\frac{1}{2}} = \left( \int_{\mathbb{B}^N} K|\phi u_n + u_n \nabla \phi|^2 \, dx \right)^{\frac{1}{2}} \leq \left( \int_{\mathbb{B}^N} K|\phi|^2 |\nabla u_n|^2 \, dx \right)^{\frac{1}{2}} + \left( \int_{\mathbb{B}^N} K|\nabla \phi|^2 |u_n|^2 \, dx \right)^{\frac{1}{2}} . \]  

Since \( u_n \to u \) in \( L^2_{K^{N/N-2}}(\mathbb{B}^N) \), we have

\[ \int_{\mathbb{B}^N} K^{\frac{N}{N-2}} |u_n \nabla \phi - u \nabla \phi|^2 \, dx \leq \sup_{\mathbb{B}^N} |\nabla \phi|^2 \int_{\mathbb{B}^N} K^{\frac{N}{N-2}} |u_n - u|^2 \, dx \to 0, \text{ for all } \phi \in C^1_0(\mathbb{B}^N). \]  

Now Brezis-Lieb lemma implies that

\[ \int_{\mathbb{B}^N} K^{\frac{N}{N-2}} |u_n \nabla \phi|^2 \, dx - \int_{\mathbb{B}^N} K^{\frac{N}{N-2}} |u_n \nabla \phi - u \nabla \phi|^2 \, dx \to \int_{\mathbb{B}^N} K^{\frac{N}{N-2}} |u \nabla \phi|^2 \, dx. \]

Substituting (2.21) into (2.20), we deduce

\[ \int_{\mathbb{B}^N} K^{\frac{N}{N-2}} |u_n \nabla \phi|^2 \, dx \to \int_{\mathbb{B}^N} K^{\frac{N}{N-2}} |u \nabla \phi|^2 \, dx, \text{ for all } \phi \in C^1_0(\mathbb{B}^N). \]
Letting $n \to +\infty$ in (2.19), by (2.2) and (2.22), it follows that
\[
\left( \int_{B_N} |\phi|^2 \, d\nu \right)^{2^*} S_K^{1/2} \leq \left( \int_{B_N} |\phi|^2 \, d\mu \right)^{1/2} + \left( \int_{B_N} |u \nabla \phi|^2 \, dx \right)^{1/2}, \quad \text{for all } \phi \in C^1_0(\mathbb{B}^N). \tag{2.23}
\]

(1) Firstly, we consider the case: $u = 0$. Then (2.23) implies that
\[
\left( \int_{B_N} |\phi|^2 \, d\nu \right)^{2^*} S_K^{1/2} \leq \left( \int_{B_N} |\phi|^2 \, d\mu \right)^{1/2}, \quad \text{for all } \phi \in C^1_0(\mathbb{B}^N). \tag{2.24}
\]

Next we will show that the inequality in (2.24) also holds for any $\phi \in L^\infty(\mathbb{B}^N)$.

Using Lusin’s theorem, for all $\phi \in L^\infty(\mathbb{B}^N)$, for all $\epsilon > 0$, there exists a $\phi_0 \in C(\mathbb{B}^N)$ such that
\[
\sup_{x \in \mathbb{B}^N} |\phi_0(x)| \leq \sup_{x \in \mathbb{B}^N} |\phi(x)| \leq |\phi|_\infty, \tag{2.25}
\]
and
\[
\mu(A) < \left( \frac{\epsilon}{4 |\phi|_\infty} \right)^2, \quad v(A) < \left( \frac{\epsilon}{4 |\phi|_\infty} \right)^2. \tag{2.26}
\]

where
\[
A = \{ x \in \mathbb{B}^N \mid \phi_0(x) \neq \phi(x) \}.
\]

As a result, we obtain
\[
\left( \int_{B_N} |\phi_0 - \phi|^2 \, d\mu \right)^{1/2} \leq \left( \int_A |\phi_0 - \phi|^2 \, d\mu \right)^{1/2} \leq 2 |\phi|_\infty \mu(A)^{1/2} \leq \frac{\epsilon}{2}. \tag{2.27}
\]

In the same way, we get
\[
\left( \int_{B_N} |\phi_0 - \phi|^2 \, d\nu \right)^{1/2} \leq \frac{\epsilon}{2}. \tag{2.28}
\]

Let the function $\eta(x) \in C^\infty_0(\mathbb{R}^N)$ satisfy
\[
\eta(x) = 1, \quad x \in \mathbb{B}^N.
\]

Let $\psi = \phi_0 \eta \in C_0(\mathbb{R}^N)$, then $(\psi - \phi_0) \in L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N)$, and
\[
\int_{\mathbb{R}^N} K |\psi - \phi_0|^2 |\nabla u_n|^2 \, dx = \int_{\mathbb{R}^N \setminus \mathbb{B}^N} K |\psi - \phi_0|^2 |\nabla u_n|^2 \, dx = 0.
\]

According to the definition of $K |\nabla u_n|^2 \overset{w}{\to} \mu$, it follows from that
\[
\int_{\mathbb{R}^N} K |\psi - \phi_0|^2 |\nabla u_n|^2 \, dx \to \int_{\mathbb{R}^N} |\psi - \phi_0|^2 \, d\mu.
\]

Consequently,
\[
\int_{\mathbb{R}^N} |\psi - \phi_0|^2 \, d\mu = 0. \tag{2.29}
\]
In the same way, we have
\[ \int_{\mathbb{R}^N} |\psi(x) - \phi_0|^2 \, d\mu = 0. \quad (2.30) \]
Since \( \psi \in C_0(\mathbb{R}^N) \), by the uniform continuity of \( \psi \), there exists a \( h_0 = h_0(\varepsilon) > 0 \), such that
\[ |\psi(x) - \psi(y)| < \varepsilon \left( 2 \max \left\{ \mu\left( \mathbb{R}^N \right)^{\frac{1}{2}}, \nu\left( \mathbb{R}^N \right)^{\frac{1}{2}} \right\} \right)^{-1}, \quad x, y \in \mathbb{R}^N, \ |x - y| < h_0. \quad (2.31) \]
Using the theory of the mollifier in [1], we make the following mollification of \( \psi \),
\[ \psi_h(x) = \int_{\mathbb{R}^N} \rho_h(x-y) \psi(y) \, dy \in C_0^\infty(\mathbb{R}^N). \quad (2.32) \]
If \( h \leq h_0 \), (2.31) and (2.32) imply that
\[ |\psi_h(x) - \psi(x)| = \left| \int_{\mathbb{R}^N} \rho_h(x-y)[\psi(x) - \psi(y)] \, dy \right| \leq \int_{|y-x|<h} \rho_h(x-y)|\psi(x) - \psi(y)| \, dy < \frac{\varepsilon}{2\mu(\mathbb{R}^N)^{\frac{1}{2}}} \int_{|y-x|<h} \rho_h(x-y) \, dy. \]
Therefore, we have
\[ \left( \int_{\mathbb{R}^N} |\psi_h(x) - \psi(x)|^2 \, d\mu \right)^{\frac{1}{2}} < \frac{\varepsilon}{2}, \quad h \leq h_0. \quad (2.33) \]
In the same way, we have
\[ \left( \int_{\mathbb{R}^N} |\psi_h(x) - \psi(x)|^{2^*} \, d\nu \right)^{\frac{1}{2}} < \frac{\varepsilon}{2^*}, \quad h \leq h_0. \quad (2.34) \]
Now using (2.27) (2.29) and (2.33), if follows that
\[ \left( \int_{\mathbb{R}^N} |\psi_h(x) - \phi(x)|^2 \, d\mu \right)^{\frac{1}{2}} \leq \left( \int_{\mathbb{R}^N} |\psi_h(x) - \psi(x)|^2 \, d\mu \right)^{\frac{1}{2}} + \left( \int_{\mathbb{R}^N} |\psi_h(x) - \phi_0(x)|^2 \, d\mu \right)^{\frac{1}{2}} + \left( \int_{\mathbb{R}^N} \phi_0(x) - \phi(x) |^2 \, d\mu \right)^{\frac{1}{2}} < \frac{\varepsilon}{2} + 0 + \frac{\varepsilon}{2} = \varepsilon, \quad h \leq h_0. \]
For the same reason, from (2.28), (2.30) and (2.34), we obtain

$$\left( \int_{\mathbb{R}^N} |\psi_h(x) - \phi(x)|^{2^*} \, dv \right)^{\frac{1}{2^*}} < \varepsilon, \, h \leq h_0. \tag{2.35}$$

It proved that there exists \( \{ \phi_i \} \equiv \{ \psi_{h_i} \} \subset C_0^\infty (\mathbb{R}^N) \) such that

$$\int_{\mathbb{R}^N} |\phi_i - \phi|^2 \, d\mu = \int_{\mathbb{R}^N} |\psi_{h_i} - \phi|^2 \, d\mu \to 0, \text{ as } i \to \infty \,(h_i \to 0), \tag{2.36}$$

$$\int_{\mathbb{R}^N} |\phi_i - \phi|^{2^*} \, dv = \int_{\mathbb{R}^N} |\psi_{h_i} - \phi|^{2^*} \, dv \to 0, \text{ as } i \to \infty \,(h_i \to 0).$$

Obviously, we have

$$\int_{\mathbb{R}^N} |\phi_i|^2 \, d\mu = \int_{\mathbb{R}^N} |\phi|^2 \, d\mu, \tag{2.37}$$

$$\int_{\mathbb{R}^N} |\phi_i|^{2^*} \, dv = \int_{\mathbb{R}^N} |\phi|^{2^*} \, dv.$$

Since \( \{ \phi_i \} \subset C_0^\infty (\mathbb{R}^N) \), then (2.24) implies that

$$\left( \int_{\mathbb{R}^N} |\phi_i|^{2^*} \, dv \right)^{2^*} S_{\mathcal{K}} \leq \left( \int_{\mathbb{R}^N} |\phi_i|^2 \, d\mu \right)^{\frac{1}{2}} \leq \left( \int_{\mathbb{R}^N} |\phi|^2 \, d\mu \right)^{\frac{1}{2}}. \tag{2.38}$$

Let \( i \to \infty \) to find in (2.38)

$$\left( \int_{\mathbb{R}^N} |\phi|^{2^*} \, dv \right)^{2^*} S_{\mathcal{K}} \leq \left( \int_{\mathbb{R}^N} |\phi|^2 \, d\mu \right)^{\frac{1}{2}} \leq \left( \int_{\mathbb{R}^N} |\phi|^2 \, d\mu \right)^{\frac{1}{2}}, \tag{2.39}$$

that is, the inequality in (2.24) also holds for any \( \phi \in L^\infty (\mathbb{B}^N) \). Hence, by Lemma 2.3, we have that there exist an at most countable set \( J \), a family \( \{ x_j \}_{j \in J} \) of distinct points in \( \mathbb{R}^N \), and \( \{ v_j \}_{j \in J} \subset (0, \infty) \), such that

$$v = \sum_{j \in J} v_j \delta_{x_j}, \quad \mu \geq S_{\mathcal{K}} \sum_{j \in J} v_j^{2/2^*} \delta_{x_j}.$$

Set

$$\mu_j = S_{\mathcal{K}} v_j^{2/2^*},$$

then

$$\mu \geq \sum_{j \in J} \mu_j \delta_{x_j}.$$

Therefore, Theorem 2.2 holds when \( u = 0 \).

(2) Now we consider the case: \( u \neq 0 \). Let \( v_n = u_n - u \). Then (2.1) implies that

$$\begin{cases}
  v_n \to 0, \text{ in } H_{0,K}^1 (\mathbb{B}^N), \\
  v_n \to 0, \text{ in } L_{K,N-2}^2 (\mathbb{B}^N), \\
  v_n \to 0, \text{ in } L_{K,N-2}^2 (\mathbb{B}^N), \\
  v_n \to 0, \text{ a.e. in } \mathbb{B}^N.
\end{cases} \tag{2.40}$$
By the first and second convergence of (2.40), we know that \( \{K|\nabla v_k|^2\} \) and \( \{K^{\frac{N}{N-2}}|v_k|^2\} \) are bounded sequences of nonnegative functions in \( L^1(\mathbb{R}^N) \). According to the characterizations of weak convergence in \( L^1(\mathbb{R}^N) \), there exists a subsequence of \( \{v_k\} \) which we still denote by \( \{v_k\} \) and bounded measures \( \bar{\mu} \) and \( \bar{\nu} \), such that

\[
\begin{align*}
\left\{ \begin{array}{l}
K|\nabla v_n|^2 \rightharpoonup \bar{\mu}, \\
K^{\frac{N}{N-2}}|v_n|^2 \rightharpoonup \bar{\nu}.
\end{array} \right.
\end{align*}
\tag{2.41}
\]

Then (2.40) and (2.41) indicate that the sequence \( \{v_k\} \) conform to the case (1). Therefore, there exist an at most countable set \( J \), a family \( \{x_j\}_{j \in J} \) of distinct points in \( \mathbb{R}^N \), and \( \{v_j\}_{j \in J} \subset (0, \infty) \), such that

\[
\bar{\nu} = \sum_{j \in J} v_j \delta_{x_j}, \bar{\mu} \geq \sum_{j \in J} \mu_j \delta_{x_j}, v_j \geq S_K v_j. \tag{2.42}
\]

Using Brezis-Lieb lemma, for any \( \phi \in L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N) \), we get

\[
\int_{\mathbb{R}^N} K^{\frac{N}{N-2}} |\phi u_n|^2 \, dx - \int_{\mathbb{R}^N} K^{\frac{N}{N-2}} |\phi v_n|^2 \, dx \to \int_{\mathbb{R}^N} K^{\frac{N}{N-2}} |\phi u|^2 \, dx \tag{2.43}
\]

Now (2.2) and (2.41) yield that

\[
\int_{\mathbb{R}^N} K^{\frac{N}{N-2}} |\phi u_n|^p \, dx \to \int_{\mathbb{R}^N} |\phi|^p \, d\nu, \tag{2.44}
\]

\[
\int_{\mathbb{R}^N} K^{\frac{N}{N-2}} |\phi v_n|^p \, dx \to \int_{\mathbb{R}^N} |\phi|^p \, d\bar{\nu}.
\]

Then, by (2.43) - (2.46), if follows that

\[
\int_{\mathbb{R}^N} |\phi|^2 \, d\nu - \int_{\mathbb{R}^N} |\phi|^2 \, d\bar{\nu} = \int_{\mathbb{R}^N} K^{\frac{N}{N-2}} |\phi|^2 \, dx. \tag{2.45}
\]

Since \( \phi \in L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N) \), one has

\[
\nu - \bar{\nu} = K^{\frac{N}{N-2}}|u|^2. \tag{2.46}
\]

Using (2.44) and (2.42), we get (2.3).

Choose \( \eta(t) \in C^1([0, \infty)) \) such that

\[
\eta(t) = \begin{cases} 
1, & \text{for } 0 \leq t \leq \frac{1}{2}, \\
0, & \text{for } t \geq 1,
\end{cases} \tag{2.47}
\]

where, \( 0 \leq \eta(t) \leq 1, |\eta'(t)| \leq 3, \text{ for all } t \in [0, \infty). \)

For all \( \epsilon > 0, j \in J \), we choose \( \phi(x) = \eta \left( \frac{|x-x_j|}{\epsilon} \right) \) in (2.23), then the Hölder inequality implies that

\[
\phi_j^\frac{1}{2} S_j^\frac{1}{2} = \nu \left( \left\{ x_j \right\} \right)^\frac{1}{2} S_j^\frac{1}{2} \leq \left( \int_{\mathbb{R}^N} \eta \left( \frac{|x-x_j|}{\epsilon} \right)^{2^*} \, d\nu \right)^\frac{1}{2} S_j^\frac{1}{2}
\]

11
\[
\leq \left( \int_{\mathbb{R}^N} \left| \eta \left( \frac{x - x_j}{\varepsilon} \right) \right|^2 \mu(dx) \right)^{\frac{1}{2}} + \left( \int_{\mathbb{R}^N} \left| \eta \left( \frac{x - x_j}{\varepsilon} \right) \right|^N \left| u(x)^2 \right| dx \right)^{\frac{1}{2}} \\
\leq \mu(B_{\varepsilon}(x_j))^{\frac{1}{2}} + \left( \int_{\mathbb{R}^N} \left| \eta \left( \frac{x - x_j}{\varepsilon} \right) \right|^N \left| u(x)^2 \right| dx \right)^{\frac{1}{2}}. 
\]  
(2.48)

By the variable substitution \( x = x_j + \varepsilon \bar{y} \) and (2.47), we obtain
\[
\int_{\mathbb{R}^N} \left| \eta' \left( \frac{x - x_j}{\varepsilon} \right) \right|^N \varepsilon^N dx = \int_{\mathbb{R}^N} \left| \eta'(|y|) \right|^N \varepsilon^N dy \leq 3^N \omega_N. 
\]  
(2.49)

Substituting (2.49) into (2.48), we deduce
\[
v_j^\frac{1}{2} S_k^\frac{1}{2} \leq \mu(B_{\varepsilon}(x_j))^{\frac{1}{2}} + 3^N \omega_N \left( \int_{B_{\varepsilon}(x_j)} \left| u(x)^2 \right| dx \right)^{\frac{1}{2}}. 
\]  
(2.50)

Let \( \varepsilon \to 0 \) in (2.50) to find
\[
v_j^\frac{1}{2} S_k^\frac{1}{2} \leq \mu(B_{\varepsilon}(x_j))^{\frac{1}{2}}, \quad \text{for all } j \in J,
\mu(B_{\varepsilon}(x_j)) \geq S v_j^\frac{1}{2}, \quad \text{for all } j \in J. 
\]  
(2.51)

Since \( \nabla u_n \rightharpoonup \nabla u \) in \( L^2_K(\mathbb{B}) \), by Lemma 2.4, for any nonnegative function \( \varphi \in L^\infty(\mathbb{R}^N) \), we have
\[
\int_{\mathbb{R}^N} K \varphi \left| \nabla u_n \right|^2 dx \leq \lim_{n \to \infty} \int_{\mathbb{R}^N} K \varphi \left| \nabla u_n \right|^2 dx. 
\]  
(2.52)

Now, \( K \left| \nabla u_n \right|^2 \rightharpoonup w \mu \) implies that
\[
\int_{\mathbb{R}^N} K \varphi \left| \nabla u_n \right|^2 dx \to \int_{\mathbb{R}^N} \varphi d\mu, \quad \text{for all } \varphi \in L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N). 
\]  
(2.53)

Combining (2.52) and (2.53), for any nonnegative function \( \varphi \in L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N) \), one has
\[
\int_{\mathbb{R}^N} K \varphi \left| \nabla u_n \right|^2 dx \leq \int_{\mathbb{R}^N} \varphi d\mu. 
\]  
(2.54)

Next we will show that (2.54) holds for \( \varphi \in L^\infty(\mathbb{R}^N) \). Let \( \psi \in L^\infty(\mathbb{R}^N) \) and \( \psi \geq 0 \) a.e. in \( \mathbb{R}^N \). For all \( \varepsilon > 0 \), using the absolute continuity, there exists a \( \delta = \delta(\varepsilon) > 0 \), such that
\[
\int_{E} K \left| \nabla u \right|^2 dx < \frac{\varepsilon}{2\| \psi \|_{\infty}}, \quad \text{if } E \subset \mathbb{R}^N, m(E) < \delta,
\]  
where \( m \) is the Lebesgue measure. By Lusin Theorem, for the measurable function \( \psi \) on \( \mathbb{R}^N \),
there exists a \( \varphi \in C(\mathbb{R}^N) \) such that the set \( F = \{ x \in \mathbb{R}^N \mid \varphi(x) \neq \psi(x) \} \) satisfies

\[
m(F) < \delta, \mu(F) < \frac{\varepsilon}{2|\psi|_{\infty}} \sup_{x \in \mathbb{R}^N} |\varphi(x)| \leq \sup_{x \in \mathbb{R}^N} |\psi(x)| = |\psi|_{\infty}.
\]

As a result, \( |\varphi| \in L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N) \) and

\[
\left| \int_{\mathbb{R}^N} K|\varphi||\nabla u|^2 dx - \int_{\mathbb{R}^N} K\psi|\nabla u|^2 dx \right| \leq 2|\psi|_{\infty} \int_F |K\nabla u|^2 dx < \varepsilon,
\]

\[
\left| \int_{\mathbb{R}^N} |\varphi| d\mu - \int_{\mathbb{R}^N} \psi d\mu \right| \leq 2|\psi|_{\infty} \int_F d\mu < \varepsilon. \tag{2.55}
\]

This proves that for any nonnegative function \( \psi \in L^\infty(\mathbb{R}^N) \), there exists a sequence \( \{\varphi_n\} \in L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N) \) such that

\[
\int_{\mathbb{R}^N} K|\varphi_n||\nabla u|^2 dx \to \int_{\mathbb{R}^N} K\psi|\nabla u|^2 dx \quad \text{as} \quad n \to \infty, \tag{2.56}
\]

and

\[
\int_{\mathbb{R}^N} |\varphi_n| d\mu \to \int_{\mathbb{R}^N} \psi d\mu \quad \text{as} \quad n \to \infty. \tag{2.57}
\]

For \( |\varphi_n| \in L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N) \), let \( \varphi = |\varphi_n| \) in (2.54), we obtain

\[
\int_{\mathbb{R}^N} K|\varphi_n||\nabla u|^2 dx \leq \int_{\mathbb{R}^N} |\varphi_n| d\mu \tag{2.58}
\]

Letting \( n \to \infty \) and using (2.56) and (2.57), we have

\[
\int_{\mathbb{R}^N} K\psi|\nabla u|^2 dx \leq \int_{\mathbb{R}^N} \psi d\mu, \quad \text{for all} \quad \phi \in L^\infty(\mathbb{R}^N). \tag{2.59}
\]

For any measurable set \( A \), taking \( \psi \) as the characteristic function on \( A \) and substituting it into (2.59), it follows that

\[
\int_A K|\nabla u|^2 dx \leq \mu(A).
\]

For any measurable set \( A \subset \mathbb{R}^N \), it follows from the above inequality that

\[
\mu(A) = \mu(A \setminus \{x_j\}_{j \in J}) + \mu(A \cap \{x_j\}_{j \in J}) \geq \int_{A \setminus \{x_j\}_{j \in J}} K|\nabla u|^2 dx + \mu(A \cap \{x_j\}_{j \in J}) \tag{2.60}
\]

\[
= \int_A K|\nabla u|^2 dx + \mu(A \cap \{x_j\}_{j \in J}).
\]
which implies that
\[ \mu(\mathbb{R}^N) \geq \int_{\mathbb{R}^N} K|\nabla u|^2 \, dx + \sum_{j \in J} \mu(|x_j|). \]  \hspace{1cm} (2.61)

Then (2.51) and (2.61) yield
\[ \mu \geq K|\nabla u|^2 + \sum_{j \in J} \mu_j \delta_{x_j}, \]
where, \( \mu_j \geq S_{K}^{\frac{1}{Q_j}} \). This completes the proof.

\[ \square \]

**Lemma 2.3** ([18, 19]). Assume that \( 1 \leq p < q < \infty, \mu \) and \( v \) are two bounded measures on \( \mathbb{B}^N \).

If there exists \( C > 0 \) such that
\[ \left( \int_{\mathbb{R}^N} |\phi|^q \, dv \right)^{\frac{1}{q}} = C \left( \int_{\mathbb{R}^N} |\phi|^p \, d\mu \right)^{\frac{1}{p}}, \quad \text{for all } \phi \in L^\infty(\mathbb{R}^N), \]
then there exist an at most countable set \( J \), a family \( \{x_j\}_{j \in J} \) of distinct points in \( \mathbb{R}^N \), and \( \{v_j\}_{j \in J} \subset (0, \infty) \), such that
\[ v = \sum_{j \in J} v_j \delta_{x_j}, \quad \mu \geq C^{-p} \sum_{j \in J} v_j^{p/q} \delta_{x_j}. \]

In particular, there holds \( \sum_{j \in J} v_j^{p/q} < \infty \).

**Lemma 2.4** ([18, 19]). Let \( 1 \leq p < \infty, \{u_k\} \subset L^p_{\mathcal{K}}(\mathbb{B}^N), u \in L^p(\mathbb{B}^N) \). If \( u_n \rightharpoonup u \) in \( L^p(\mathbb{B}^N) \), then for all nonnegative functions \( \phi \in L^\infty(\mathbb{B}^N) \), we have
\[ \int_{\mathbb{B}^N} K\phi |u|^p \, dx \leq \lim_{k \to \infty} \int_{\mathbb{B}^N} K\phi |u_n|^p \, dx. \]  \hspace{1cm} (2.62)

In particular,
\[ |u|_{k,p} \leq \lim_{k \to \infty} |u_n|_{k,p}. \]  \hspace{1cm} (2.63)

**Proof.** Let \( \left(L^p_{\mathcal{K}}(\mathbb{B}^N)\right)^* \) denote the dual space of \( L^p_{\mathcal{K}}(\mathbb{B}^N) \). For all \( f \in \left(L^p_{\mathcal{K}}(\mathbb{B}^N)\right)^* \), \( f \) can be expressed as
\[ f(u) = \int_{\mathbb{B}^N} Kuv \, dx, \quad v \in L^{p'}_{\mathcal{K}}(\mathbb{B}^N). \]

If \( u_k \rightharpoonup u \) in \( L^p_{\mathcal{K}}(\mathbb{B}^N) \), that is
\[ \int_{\mathbb{B}^N} Ku_n \psi \, dx \rightharpoonup \int_{\mathbb{B}^N} Ku \psi \, dx, \quad \text{for all } \psi \in L^{p'}(\mathbb{B}^N), \quad p' = \frac{p}{p-1}. \]  \hspace{1cm} (2.64)

Then for all nonnegative functions \( \phi \in L^\infty(\mathbb{B}^N) \), taking \( \psi = \phi |u|^{p-2} u \in L^{p'}_{\mathcal{K}}(\mathbb{B}^N) \) in (2.64), we deduce
\[ \int_{\mathbb{B}^N} Ku_n |u|^{p-2} u \phi \, dx \rightharpoonup \int_{\mathbb{B}^N} Ku |u|^{p} \phi \, dx. \]  \hspace{1cm} (2.65)

Now by the Hölder inequality, we have
\[ \int_{\mathbb{B}^N} Ku_n |u|^{p-2} u \phi \, dx \leq \left( \int_{\mathbb{B}^N} K |u_n|^p \phi \, dx \right)^{\frac{1}{p}} \left( \int_{\mathbb{B}^N} K |u|^p \phi \, dx \right)^{\frac{1}{p}}. \]  \hspace{1cm} (2.66)
Taking the limit on both sides of (2.66), and using (2.65), we get

\[
\int_{\mathbb{B}^N} K|u|^p \phi dx \leq \lim_{n \to \infty} \left( \int_{\mathbb{B}^N} K|u_n|^p \phi dx \right)^{\frac{1}{p}} \left( \int_{\mathbb{B}^N} |u|^p \phi dx \right)^{\frac{1}{p}},
\]

which implies that (2.62). In (2.62), we choose \( \phi \equiv 1 \) and obtain (2.63). \( \square \)

3 Maximum principle

Let

\[ Lu := -\text{div}(K(x)\nabla u) = f(x). \]  \( (3.1) \)

Then we have

**Theorem 3.1.** (Weak maximum principle). Assume \( f(x) \leq 0 \). If \( u(x) \in C^2(\mathbb{B}^N) \cap C^1(\overline{\mathbb{B}^N}) \) satisfies (3.1), then

\[ \max_{\mathbb{B}^N} u(x) = \max_{\partial\mathbb{B}^N} u(x). \]

**Theorem 3.2.** Assume that \( B_R \) is a ball of radius \( R(0 < R \leq 1) \) in \( \mathbb{H}^N(N \geq 3) \). If \( u \in C^2(B_R) \cap C^1(\overline{B_R}) \) satisfies

1. \( Lu \leq 0, x \in B_R; \)
2. there exists \( x_0 \in \partial B_R \) such that \( u(x) \) attains the strict nonnegative maximum at \( x_0 \in \overline{B_R} \), i.e. \( u(x_0) = \max_{\overline{B_R}} u(x) \geq 0 \), and if \( x \in B_R, u(x) < u(x_0) \), we have

\[ \frac{\partial u}{\partial v} \bigg|_{x=x_0} > 0, \]

where, the angle between \( v \) and the outer unit normal \( n \) at \( x_0 \in \partial B_R \) is less than \( \frac{\pi}{2} \).

**Proof of Theorem 3.1.** The proof is divided into two parts.

1. Assume that \( f(x) < 0 \). If \( u(x) \) attains the maximum at \( x_0 \in \mathbb{B}^N \), i.e.

\[ u(x_0) = \max_{x \in \mathbb{B}^N} u(x) \geq 0. \]

From the theorem of multivariable calculus, we have \( \nabla u(x_0) = 0 \) and the Hessian matrix is nonnegative definite. Now we compute the trace of the Hessian matrix, and obtain

\[ \Delta u(x_0) = \text{tr} \left( \nabla^2 u(x_0) \right) \leq 0. \]

Hence, we get

\[
Lu(x_0) = -\text{div}(K(x)\nabla u)\bigg|_{x_0} = -\frac{1}{(1 - |x|^2)^{N-2}} \Delta u\bigg|_{x_0} - \frac{(N-2)}{(1 - |x|^2)^{N-1}} (x \cdot \nabla u)\bigg|_{x_0} = -\frac{1}{(1 - |x|^2)^{N-2}} \Delta u(x_0) - \frac{(N-2)}{(1 - |x|^2)^{N-1}} (x_0 \cdot \nabla u(x_0)) \]

\[ = -\frac{1}{(1 - |x|^2)^{N-2}} \Delta u(x_0) = f(x_0) \geq 0. \]

\( \square \)
This contradicts the assumption that \( f(x) < 0 \). Therefore, \( u(x) \) cannot attain its maximum in \( B^N \).

(2) Now we assume that \( f(x) \leq 0 \). For all \( \varepsilon > 0 \), set
\[
\nu(x) = u(x) - \varepsilon \left( 1 - |x|^2 \right).
\]

Obviously, we have
\[
\nu(x) \leq u(x) \leq \nu(x) + \varepsilon.
\]

Hence, it follows that
\[
\nu = \nabla \left( u - \varepsilon \left( 1 - |x|^2 \right) \right) = - \nabla \left( K(x) \nabla u - \varepsilon \nabla \left( K(x) \nabla |x|^2 \right) \right) = \nabla \left( f(x) - \varepsilon \nabla |x|^2 \right) < 0.
\]

According to (1), \( \nu \) can not attain its maximum value on \( B^N \), i.e
\[
\max_{\partial B^N} \nu(x) = \max_{\partial B^N} \nu(x).
\]

Consequently, we get
\[
\max_{\partial B^N} u(x) = \max_{\partial B^N} \nu(x) + \varepsilon \leq \max_{\partial B^N} \nu(x) + \varepsilon \leq \max_{\partial B^N} u(x) + \varepsilon.
\]

Let \( \varepsilon \to 0 \) to find
\[
\max_{\partial B^N} u(x) = \max_{\partial B^N} u(x).
\]

This completes the proof. \( \Box \)

**Proof of Theorem 3.2.** According to the assumptions of Theorem 3.2, we can easily know that
\[
\frac{\partial u}{\partial v}|_{x=x_0} \geq 0.
\]

Hence, we need to prove that
\[
\frac{\partial u}{\partial v}|_{x=x_0} > 0.
\]

We consider the following auxiliary functions on the spherical shell \( B^*_R = \{ x \in B_R \mid \frac{R}{2} < |x| \leq R \} \),
\[
w(x) = u(x) - u(x_0) + \varepsilon \nu(x),
\]

where, \( a > 0 \) and \( \nu(x) \) is is selected later. We can choose \( \nu(x_0) = 0 \), then \( w(x_0) = 0 \). Consequently, \( w(x) \) has nonnegative maximum on \( B^*_R = \{ x \in \mathbb{R}^N \mid \frac{R}{2} \leq |x| \leq R \} \). If we can choose \( \varepsilon > 0 \) and \( \nu(x) \) such that \( w(x) \) attains its nonnegative maximum \( w(x_0) \) at \( x_0 \), then
\[
0 \leq \frac{\partial w}{\partial v}|_{x=x_0} = \left( \frac{\partial u}{\partial v}|_{x=x_0} + \varepsilon \frac{\partial v}{\partial v}|_{x=x_0} \right).
\]
that is,
\[
\frac{\partial u}{\partial v} \bigg|_{x=x_0} \geq -\varepsilon \frac{\partial v}{\partial v} \bigg|_{x=x_0}.
\]

In order to make \( w(x) \) attained its nonnegative maximum on the boundary of spherical shell \( B^*_R \), by Theorem 3.1, we only need to propose a condition on spherical shell \( B^*_R \) such that \( Lw \leq 0 \). Therefore, we only need to construct function \( v \) such that
\[
Lw \leq 0, \quad x \in B^*_R,
\]
and
\[
\frac{\partial v}{\partial v} \bigg|_{x=x_0} < 0.
\]

If I can do the above, I can complete the proof.

By reason of the symmetry of spherical shell \( B^*_R \), we consider the radial symmetric function,
\[
v(r) = e^{-ar^2} - e^{-aR^2},
\]
We easily see that
\[
v'(r) = -2\alpha re^{-ar^2},
\]
\[
v''(r) = e^{-ar^2} \left( 4\alpha^2 r^2 - 2\alpha \right).
\]
On the spherical shell \( B^*_R \), we deduce
\[
Lw = -\frac{1}{(1 - r^2)^{N-2}} v''(r) - \frac{1}{(1 - r^2)^{N-2}} \left( \frac{N - 1}{r} + \frac{2r(N - 2)}{1 - r^2} \right) v'(r)
\]
\[
= \frac{2\alpha - 4\alpha^2 r^2}{(1 - r^2)^{N-2}} e^{-ar^2} + \frac{2\alpha r}{(1 - r^2)^{N-2}} \left( \frac{N - 1}{r} + \frac{2r(N - 2)}{1 - r^2} \right) e^{-ar^2}
\]
\[
= \frac{e^{-ar^2}}{(1 - r^2)^{N-2}} \left( -4\alpha^2 r^2 + 2\alpha N + \frac{4\alpha r(N - 2)}{1 - r^2} \right)
\]
\[
= \frac{e^{-ar^2}}{(1 - r^2)^{N-2}} \left( -4\alpha^2 r^2 - 2\alpha N + 8\alpha + \frac{4\alpha (N - 2)}{1 - r^2} \right).
\]

(3.4)

Taking \( \alpha > 0 \) large enough, we have \( Lw \leq 0 \). Then we have \( Lw \leq 0 \) on the spherical shell \( B^*_R \). Therefore by Theorem 3.1, \( w(x) \) attains its maximum on \( \partial B^*_R \). On the inner sphere of the spherical shell \( \partial B^*_R \cap B_R = \partial B_\frac{R}{2} \), we have
\[
\beta := \max_{|x|=\frac{R}{2}} (u(x) - u(x_0)) < 0.
\]

Now we chose \( \varepsilon > 0 \) small enough such that
\[
w|_{\partial B_\frac{R}{2}} \leq \beta + \varepsilon e^{-aR^2} \left( e^{\frac{3\alpha R^2}{2}} - 1 \right) < 0.
\]

On the outer surface of the spherical shell \( \partial B^*_R \cap \partial B_R = \partial B_R \), \( v(x) = 0 \). Obviously, \( w(x) \leq
$w(x_0) = 0$. Therefore, $w(x)$ attains its maximum at $x_0$. Since $v(x) = 0$ on $\partial B_R$, we have

$$\frac{\partial u}{\partial v} \bigg|_{x=x_0} \geq -\varepsilon \frac{\partial v}{\partial n} \bigg|_{x=x_0} \cos(v, n)$$

$$= 2\varepsilon \alpha R e^{-\alpha R^2} \cos(v, n).$$

(3.5)

Taking into account that the angle between $v$ and the outer unit normal $n$ at $x_0 \in \partial B_R$ is less than $\frac{\pi}{2}$, it follows that

$$\frac{\partial u}{\partial v} \bigg|_{x=x_0} > 0.$$

This completes the proof. □

4 Brezis-Nirenberg type equation

In this section, we consider the following problem

$$\begin{aligned}
&-\text{div}(K \nabla u) = 4K_{N/2}^N |u|^{2^*-2} u + 4\lambda K_{N/2}^N u, \text{ in } B_N \\
u = 0, \text{ on } \partial B_N
\end{aligned}$$

(4.1)

Let

$$I_\lambda(u) = \frac{1}{2} \int_{B_N} K|\nabla u|^2 \, dx - \frac{1}{2} \int_{B_N} 4\lambda K_{N/2}^N u^2 \, dx - \frac{1}{2^*} \int_{B_N} 4K_{N/2}^N |u|^{2^*} \, dx, \ u \in H^1_{0,K}(B_N).$$

(4.2)

And we know that the critical points of $I_\lambda(u)$ are just the weak solutions of problem (4.1).

Now we consider the following eigenvalue problem

$$\begin{aligned}
&-\text{div}(K \nabla u) = 4\lambda K_{N/2}^N u, \text{ in } B_N \\
u = 0, \text{ on } \partial B_N
\end{aligned}$$

(4.3)

From Theorem 2.1 and compact operator theory (see[12, page 730]), taking the similar proof of page 359 in [12], we easily know that problem (4.3) has a list of discrete eigenvalues $0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$.

**Theorem 4.1.** If $N \geq 4$, then for any $\lambda \in (0, \lambda_1)$, there exists a (positive) solution of problem (4.1).

**Proposition 4.2.** (Mountain Pass Theorem) Assume that $X$ is a Banach space, $E \in C^1(X, \mathbb{R})$ satisfies the following geometrical conditions:

1. $E(0) = 0$, and there exists $\rho > 0$, such that $E|_{\partial B_\rho(0)} \geq \alpha > 0$;
2. There exists $e \in X \setminus B_\rho(0)$, such that $E(\bar{e}) \leq 0$. Let $\Gamma$ is the set of all paths which connect 0 and $e$, i.e.,
\[
\Gamma = \{ g \in C([0,1], X) \mid g(0) = 0, g(1) = \bar{e} \}
\]

and
\[
c = \inf_{g \in \Gamma} \max_{t \in [0,1]} E(g(t)).
\]
Then \( c \geq \alpha \) is finite and \( E \) possesses a \((PS)_c\) sequence at level \( c \); Furthermore, if \( E \) satisfies the \((PS)\) condition (or the \((PS)_c\) condition at level \( c \)), then \( c \) is a critical value of \( E \).

Set

\[
S_{\lambda,K}(u) = \frac{||u||_{K,2}^2 - 4\lambda||u||_{K,N/N-2,2}^2}{||u||_{K,N/N-2,2}^2},
\]

and

\[
S_{\lambda,K} = \inf \left\{ \frac{||u||_{K,2}^2 - 4\lambda||u||_{K,N/N-2,2}^2}{||u||_{K,N/N-2,2}^2} : u \in H^1_{0,K}(\mathbb{B}^N) \setminus \{0\} \right\}.
\]

Let

\[
S_K = \inf \left\{ \frac{||u||_{K,2}^2}{||u||_{K,N/N-2,2}^2} : u \in H^1_{0,K}(\mathbb{B}^N) \setminus \{0\} \right\}
\]

denote the best constant for the embedding \( H^1_{0,K}(\mathbb{B}^N) \rightarrow L^2_{K,N/N-2}(\mathbb{B}^N) \).

**Lemma 4.3.** For any \( \lambda \in \mathbb{R} \) and \( m \in \mathbb{N} \) the functional \( I_\lambda(u) \) satisfies the \((PS)_c\) condition in \((-\infty, (1/N)S^{N/2})\) in the following sense:

If \( c < (1/N)S^{N/2} \) and \( \{u_n\} \) is a sequence in \( H^1_{0,K}(\mathbb{B}^N) \) such that as \( n \rightarrow \infty \), \( I_\lambda(u_n) \rightarrow c \) and \( I'_\lambda(u_n) \rightarrow 0 \) strongly in \( H^1_{K}(\mathbb{B}^N) \), then \( \{u_n\} \) contains a subsequence strongly converging in \( H^1_{K}(\mathbb{B}^N) \).

**Proof.** 1. The \((PS)_c\) sequence is bounded in \( H^1_{0,K}(\mathbb{B}^N) \).

Assume that \( \{u_n\} \) is a \((PS)_c\) sequence, i.e., \( I_\lambda(u_n) \rightarrow c \) and \( I'_\lambda(u_n) \rightarrow 0 \) as \( n \rightarrow \infty \). A direct computation shows that, as \( m \rightarrow \infty \), there holds

\[
o(1) \left( 1 + ||u_n||_{2,K} \right) + 2c \geq 2I_\lambda(u_n) - <DI_\lambda(u_n), u_n> = \left( 1 - \frac{2}{2^*} \right) \int_{\mathbb{B}^N} 4K_{N/2}^{N/2} |u_n|^{2^*} \, dx \\
\geq C \left( \int_{\mathbb{B}^N} 4K_{N/2}^{N/2} |u_n|^{2^*} \, dx \right)^{2^*/2},
\]

which means the boundedness of \( \{u_n\} \) in \( L^2(\mathbb{B}^N) \) and in \( L^2_{K,N/N-2}(\mathbb{B}^N) \). Hence there holds

\[
||u_n||_{2,K}^2 = 2I_\lambda(u_n) + \lambda \int_{\mathbb{B}^N} 4K_{N/2}^{N/2} |u_n|^2 \, dx + \frac{2}{2^*} \int_{\mathbb{B}^N} 4K_{N/2}^{N/2} |u_n|^{2^*} \, dx \\
\leq C + o(1) ||u_n||_{2,K},
\]

and it follows that \( \{u_n\} \) is bounded in \( H^1_{0,K}(\mathbb{B}^N) \).

2. Medium convergence.

From the boundedness of \( \{u_n\} \) in \( H^1_{0,K}(\mathbb{B}^N) \), up to a subsequence, we may assume that \( u_n \rightarrow u \) weakly in \( H^1_{0,K}(\mathbb{B}^N) \), and then from the boundedness of \( \mathbb{B}^N \) and Theorem 2.1, \( u_n \rightarrow u \) strongly in \( L^p_{N/(N-2)}(\mathbb{B}^N) \) for \( p < 2^* \). Then as \( n \rightarrow \infty \), there hold

\[
\int_{\mathbb{B}^N} K^\frac{N}{N-2} |\nabla u_n|^2 \, dx = \int_{\mathbb{B}^N} K^\frac{N}{N-2} |\nabla (u_n - u)|^2 \, dx + \int_{\mathbb{B}^N} K^\frac{N}{N-2} |\nabla u|^2 \, dx + o(1),
\]

\[
\int_{\mathbb{B}^N} K^\frac{N}{N-2} (|u_n|^{2^*} - |u_n - u|^{2^*}) \, dx
\]

19
\[
\int_{\mathbb{B}^N} K^{\frac{N}{2}} \int_0^1 \frac{d}{dt} [u_n + (t - 1)|u|^{2^*}] \, dx
\]
\[
= 2^* \int_0^1 \int_{\mathbb{B}^N} K^{\frac{N}{2}} (u_n + (t - 1)u)|u_n + (t - 1)|u|^{2^* - 2} u \, dx \, dt
\]
\[
\to 2^* \int_0^1 \int_{\mathbb{B}^N} K^{\frac{N}{2}} I_4(u) \, dx \, dt = \int_{\mathbb{B}^N} K^{\frac{N}{2}} |u|^{2^*} \, dx
\] (4.6)

and
\[
\int_{\mathbb{B}^N} K^{\frac{N}{2}} (u_n|u_n|^{2^* - 2} - u|u|^{2^* - 2}) (u_n - u) \, dx
\]
\[
= \int_{\mathbb{B}^N} K^{\frac{N}{2}} (|u_n|^2 - u_n|u_n|^{2^*}) \, dx + o(1)
\] (4.7)

Hence from (4.5)–(4.7), there hold
\[
I_4(u_n) = I_4(u) + I_0(u_n - u) + o(1),
\]

and
\[
o(1) = \langle u_n - u, I_4'(u_n) \rangle = \langle u_n - u, I_4'(u_n) - I_4'(u) \rangle
\]
\[
= \int_{\mathbb{B}^N} K |\nabla (u_n - u)|^2 \, dx - \int_{\mathbb{B}^N} 4K^{\frac{N}{2}} |u_n - u|^{2^*} \, dx + o(1).
\] (4.8)

In particular, from the last equation it follows that
\[
I_0(u_n - u) = \frac{1}{N} \int_{\mathbb{B}^N} K |\nabla (u_n - u)|^2 \, dx + o(1).
\] (4.9)

3. \( u_n \to u \) strongly in \( H^1_{0, K}(\mathbb{B}^N) \).

From step 2, as \( n \to \infty \), there holds
\[
\langle u, I_4'(u_n) \rangle = \int_{\mathbb{B}^N} K \nabla u_n \cdot \nabla u \, dx - \int_{\mathbb{B}^N} 4K^{\frac{N}{2}} \left( Ku_n u + u_n |u_n|^{2^* - 2} u \right) \, dx
\]
\[
\to \int_{\mathbb{B}^N} K |\nabla u|^2 \, dx - \int_{\mathbb{B}^N} 4K^{\frac{N}{2}} \left( \lambda |u|^2 + |u|^{2^*} \right) \, dx
\]
\[
= \langle u, I_4(u) \rangle = 0,
\]

and hence
\[
I_4(u) = \left( \frac{1}{2} - \frac{1}{2^*} \right) \int_{\mathbb{B}^N} 4K^{\frac{N}{2}} |u|^{2^*} \, dx = \frac{1}{N} \int_{\mathbb{B}^N} 4K^{\frac{N}{2}} |u|^{2^*} \, dx \geq 0.
\]

Then there holds
\[
I_0(u_n - u) = I_4(u_n) - I_4(u) + o(1)
\]
\[
\leq I_4(u_n) + o(1) \leq c < \frac{1}{n} S_0^{N/2}, \quad \text{as } n \to \infty.
\] (4.10)

Therefore
\[
||u_n - u||_{2, K}^2 \leq nc < S_0^{N/2}, \quad \text{as } n \to \infty.
\] (4.11)
From Theorem 2.1 and (4.8), there holds
\[ \|u_n - u\|_{2,K}^2 \left( 1 - S_0^{-2/2} \|u_n - u\|_{2,K}^{-2} \right) \leq \int_{\mathbb{R}^N} K|\nabla (u_n - u)|^2 \, dx - \int_{\mathbb{R}^N} 4K^{\frac{N}{2}} |u_n - u|^{2^*} \, dx = o(1), \quad (4.12) \]
which means that \( u_n \to u \) strongly in \( H_{0,K}^1(\mathbb{R}^N) \).

\begin{proof}
We make a normalization \( u_n \to u \) strongly in \( H_{0,K}^1(\mathbb{R}^N) \). From Lemma 4.3 and Mountain Pass Theorem, it suffices to show that for \( \lambda \in (0, \lambda_1) \) for \( N \geq 4 \), there holds
\[ c = \inf_{g \in \Gamma} \max_{t \in [0,1]} I_\lambda(g(t)) < c^* = \frac{1}{N} S_0^{N/2}, \quad (4.13) \]
where \( \Gamma \) is the set of all paths which connect 0 and \( e \), i.e.,
\[ \Gamma = \{ g \in C([0,1], H_{0,K}^1(\mathbb{R}^N)) | g(0) = 0, g(1) = e \}. \]

1. It is easy to verify that \( I_\lambda \) satisfies the geometric condition of Mountain Pass Theorem when \( \lambda < \lambda_1 \), that is, \( I_\lambda(0) = 0 \); there exists \( \rho > 0 \), such that \( I_\lambda|_{B_\rho(0)} \geq \alpha > 0 \); and there exists \( e \in X \setminus B\rho(0) \), such that \( I_\lambda(e) \leq 0 \).

From Lemma 4.3 and Mountain Pass Theorem, it suffices to show that for \( \lambda \in (0, \lambda_1) \) for \( N \geq 4 \), there holds
\[ c \leq \sup_{0 \leq t \leq \infty} I_\lambda(te_1) = \sup_{0 \leq t \leq \infty} \left( \frac{t^2}{2} S_{\lambda,K}(u) - \frac{t^{2^*}}{2^*} \right) = \frac{1}{N} S_{\lambda,K}^{N/2}(u). \quad (4.15) \]
Thus (4.15) implies (4.14).

On the other hand, for each \( g \in \Gamma \), since \( \lambda < \lambda_1 \), for \( u = g(t) \) with \( t \) close to 0, we have \( (I_\lambda(u), u) > 0 \). If \( u = g(1) = e \), we have
\[ (I_\lambda(e), e) < 2I_\lambda(e) \leq 0. \]
As a result, by the intermediate value theorem, there exists \( u \in g \) such that \( u \neq 0 \) and
\[ \langle u, I_\lambda'(u) \rangle = \int_{\mathbb{R}^N} K|\nabla u|^2 \, dx - \int_{\mathbb{R}^N} 4K^{\frac{N}{2}} \left( \lambda |u|^2 + |u|^{2^*} \right) \, dx = 0. \quad (4.16) \]
Thus by a direct computation, for such \( u \), we can show that
\[ S_{\lambda,K}(u) = \left( \int_{\mathbb{R}^N} \left( K|\nabla u|^2 - 4AK^{\frac{N}{2}} |u|^2 \right) \, dx \right)^{1-2/2^*} = (NI_\lambda(u))^{2/N} \leq \left( N \sup_{u \in g} I_\lambda(u) \right)^{2/N}. \quad (4.17) \]
That is,  \[ c = \inf_{g \in \Gamma} \max_{t \in [0,1]} I_{\lambda}(g(t)) = \frac{1}{N} S_{A,K}(u)^{2/N}. \]

Thus (4.14) implies (4.15).

3. It suffices to show that (4.14) holds in the case \( N \geq 4 \).

Note that the best Sobolev constant is achieved by any of the following functions:

\[
U_\varepsilon(x) = \frac{(N(N - 2)\varepsilon^2)^{(N-2)/4}}{\left(\varepsilon^2 + |x|^2\right)^{(N-2)/2}}, \quad \varepsilon > 0.
\]

Then from [27, Theorem 1.42] we have

\[
|\nabla U_\varepsilon|_2^2 = |U_\varepsilon|_2^2 = S_0^{N/2}.
\]

Consider

\[
u_\varepsilon(x) = \varphi(x)U_\varepsilon(x) \quad x \in \mathbb{B}^N,
\]

where \( \varphi \in C_0^\infty(\mathbb{B}^N) \) such that \( \varphi(x) = \frac{1}{\sqrt{\lambda(x)}} \) in a \( B_\rho(0), \rho = \frac{1-\varepsilon^2}{2} \). Then we compute

\[
\nabla \varphi(x)U_\varepsilon(x) = \frac{N - 2}{2} \frac{-2x}{1 - |x|^2}\varphi(x)U_\varepsilon(x),
\]

\[
\nabla U_\varepsilon(x)\varphi(x) = \frac{N - 2}{2} \frac{2x}{\varepsilon^2 + |x|^2}U_\varepsilon(x)\varphi(x),
\]

\[
\nabla u_\varepsilon(x) = \nabla \varphi(x)U_\varepsilon(x) + \varphi(x)\nabla U_\varepsilon(x)
\]

\[
= (N - 2) \left( \frac{1}{\varepsilon^2 + |x|^2} - \frac{1}{1 - |x|^2} \right) xU_\varepsilon(x)\varphi(x).
\]

Therefore, we have

\[
|\nabla u_\varepsilon(x)| < \varphi^2(x)|\nabla U_\varepsilon(x)|^2.
\]

Furthermore, using (4.19) and [27, Lemma 1.46], we deduce that

\[
\int_{\mathbb{B}^N} K|\nabla u_\varepsilon|^2 \, dx \leq \int_{\mathbb{B}^N} K|\nabla U_\varepsilon|^2 \varphi^2 \, dx + O(\varepsilon^{N-2})
\]

\[
= \int_{\mathbb{B}^N} |\nabla U_\varepsilon|^2 \, dx + O(\varepsilon^{N-2})
\]

\[
= S_0^{N/2} + O(\varepsilon^{N-2}),
\]

\[
\int_{\mathbb{B}^N} K \frac{N}{N-2} |u_\varepsilon|^2 \, dx = \int_{\mathbb{B}^N} K \frac{N}{N-2} |U_\varepsilon|^2 \varphi^2 \, dx + O(\varepsilon^N)
\]

\[
= \int_{\mathbb{B}^N} |U_\varepsilon|^2 \, dx + O(\varepsilon^N)
\]

\[
= S_0^{N/2} + O(\varepsilon^N)
\]

and

\[
\int_{\mathbb{B}^N} K \frac{N}{N-2} |u_\varepsilon|^2 \, dx \geq \int_{B_\varepsilon(0)} |U_\varepsilon|^2 \, dx
\]
\[ \geq \int_{B_{\varepsilon}(0)} \frac{(N(N-2)\varepsilon^2)^{(N-2)/2}}{(2\varepsilon^2)^{N-2}} \, dx \]
\[ + \int_{B_{\varepsilon}(0)} \frac{N(N-2)\varepsilon^2)^{(N-2)/2}}{(2|x|^2)^{N-2}} \, dx \]
\[ = c_1 \varepsilon^2 + c_2 \varepsilon^{n-2} \int_{\varepsilon}^{r_3-N} r^{3-N} \, dr \]
\[ = \begin{cases} c_1 \varepsilon^2 + O(\varepsilon^{N-2}), & \text{if } N > 4, \\ c_1 \varepsilon^2 |\ln \varepsilon| + O(\varepsilon^2), & \text{if } N = 4, \\ c_1 \varepsilon + O(\varepsilon^2), & \text{if } N = 3, \end{cases} \]

where, \( c, c_1, c_2 > 0 \). Thus for \( \varepsilon > 0 \) sufficiently small, we obtain
\[ S_{\lambda,K}(u) = \frac{\|u\|^2_{K,2} - 4\lambda \|u\|^2_{K,N-2,2}}{\|u\|^2_{K,N-2,2}} < S_0, \]

which implies (4.14).

\[ \blacksquare \]

5 General critical growth problem

In this section, we consider the following problem
\[ \begin{cases} - \text{div}(K(x) \nabla u) = 4K(x) \frac{N}{N-2} |u|^{N-2} u + 4\lambda K(x) \frac{N}{N-2} |u|^{q-2} u, & \text{in } \mathbb{R}^N, \\ u = 0, & \text{on } \partial \mathbb{B}^N. \end{cases} \tag{5.1} \]

Let
\[ I_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^N} K|\nabla u|^2 \, dx - \frac{1}{2} \int_{\mathbb{R}^N} 4K \frac{N}{N-2} |u|^{N-2} \, dx - \frac{1}{q} \int_{\mathbb{R}^N} 4\lambda K \frac{N}{N-2} |u|^q \, dx, \quad u \in H_{0,k}^1(\mathbb{R}^N). \tag{5.2} \]

And we know that the critical points of \( I_\lambda(u) \) are just the weak solutions of problem (5.1).

**Theorem 5.1.** If \( 2 < q < 2^* \), there exists \( \lambda_0 > 0 \) such that problem (5.1) has a nontrivial solution for all \( \lambda \geq \lambda_0 \).

**Theorem 5.2.** If \( 1 < q < 2 \), there exists \( \lambda_0' > 0 \), such that problem (5.1) has infinitely many solutions for \( 0 < \lambda < \lambda_1 \).

Next, we will use the methods in [15] to prove Theorem 5.1 and Theorem 5.2.

**Lemma 5.3.** Let \( \{u_n\} \subset H_{0,k}^1(\mathbb{R}^N) \) be a (PS)_c sequence for \( I_\lambda \), defined by (5.2), that is,
\[ I_\lambda(u_n) \to c, \quad I_\lambda'(u_n) \to 0 \quad \text{in } H_{-1}^1(\mathbb{R}^N). \]

Then, we have
\[ (a) \text{ If } 2 < q < 2^*, \text{ and } c < S_{N/2}^{N/2}/N, \text{ there exists a subsequence of } \{u_n\}, \text{ which we still denote by } \{u_n\}, \text{ strongly convergent in } H_{0,k}^1(\mathbb{R}^N). \]
(b) If $1 < q < 2$, and $c < S^N_k/|N - c_0|^{2\beta}$, where $\beta = 2^*/(2^* - q)$ and $c_0$ depends on $q, N$ and $\mathbb{B}^N$, then there exists a subsequence of $\{u_k\}$, which we still denote by $\{u_n\}$, strongly convergent in $H^1_{0,k}(\mathbb{B}^N)$.

**Proof.** In both cases, by Theorem 2.2, it is easy to prove that the sequence $\{u_n\}$ is bounded in $H^1_{0,k}(\mathbb{B}^N)$. Then, if we take the appropriate subsequence, we can assume in both cases

$$u_n \to u, \text{ in } H^1_{0,k}(\mathbb{B}^N),$$

$$u_n \to u, \text{ in } L^q_{K^{N/2-1}}(\mathbb{B}^N), 1 < q < 2^*, \text{ and a.e.,}$$

$$K^{N/2^*} |u_{n}|^{2^*} \to v = K^{N/2}|u|^{2^*} + \sum_{j \in J} v_j \delta_{x_j}, \quad (5.3)$$

$$K |\nabla u_n|^{2} \to \mu \geq K |\nabla u|^{2} + \sum_{j \in J} \mu_j \delta_{x_j}.$$ 

Take $x_k \in \mathbb{B}^N$ in the support of the singular part of $\mu, \nu$. We consider $\varphi \in C^\infty_0(\mathbb{R}^N)$, such that

$$\varphi \equiv 1 \text{ on } B(x_k, \varepsilon), \varphi \equiv 0 \text{ on } B(x_k, 2\varepsilon)^c, |\nabla \varphi| \leq 2/\varepsilon. \quad (5.4)$$

It is easy that the sequence $\{\varphi u_n\}$ is bounded in $H^1_{0,k}(\mathbb{B}^N)$; then, using hypothesis of Lemma 5.3, we obtain $\lim_{n \to \infty} \langle J'_k(u_n), \varphi u_n \rangle = 0$, and

$$\int_{\mathbb{B}^N} 4\varphi \, dv + \lambda \int_{\mathbb{B}^N} 4K^{N/2^*} |\varphi|^{q} \, dx - \int_{\mathbb{B}^N} \varphi \, d\mu = \lim_{n \to \infty} \int_{\mathbb{B}^N} K u_n \nabla u_n \cdot \nabla \varphi \, dx. \quad (5.5)$$

By (5.4), (5.5), and the Hölder inequality, we have

$$\lim_{n \to \infty} \left| \int_{\mathbb{B}^N} K u_n \nabla u_n \cdot \nabla \varphi \, dx \right| \leq \left( \int_{\mathbb{B}^N} K^{N/2^*} |u|^{2^*} \, dx \right)^{1/2^*} \to 0 \text{ as } \varepsilon \to 0.$$

Then

$$0 = \lim_{\varepsilon \to 0} \left\{ \int_{\mathbb{B}^N} 4\varphi \, dv + \lambda \int_{\mathbb{B}^N} 4K^{N/2^*} |\varphi|^{q} \, dx - \int_{\mathbb{B}^N} \varphi \, d\mu \right\} = \nu_k - \mu_k. \quad (5.6)$$

By Lemma 2.2, $\mu_k \geq S_k \nu_k^{2/2^*}$, i.e. $\nu_k \geq S_k \nu_k^{2/2^*}$. That is, $\nu_k = 0$, or

$$\nu_k \geq S_k^{N/2}. \quad (5.7)$$

Below we will show that (5.7) is not possible.

Let us assume that there exists a $k_0$ with $\nu_{k_0} \neq 0$ i.e. $\nu_{k_0} \geq S_k^{N/2}$. Using hypothesis of Lemma 5.3 and (5.3), we get

$$c = \lim_{n \to \infty} I_\lambda(u_n) \geq I_\lambda(u) + \left( \frac{1}{2} - \frac{1}{2^*} \right) \sum_k \nu_k \geq I_\lambda(u) + \frac{1}{N} S_k^{N/2}.$$

But, by hypothesis, $c < S_k^{N/2}/N$; then, $I_\lambda(u_n) < 0$. In particular, $u \neq 0$, and

$$0 < \frac{1}{2} \int_{\mathbb{B}^N} K |\nabla u|^{2} \, dx < \frac{1}{2^*} \int_{\mathbb{B}^N} 4K^{N/2^*} |u|^{2^*} \, dx + \frac{\lambda}{q} \int_{\mathbb{B}^N} 4K^{N/2} |u|^{q} \, dx.$$
That is, 
\[ c = \lim_{n \to \infty} I_{\lambda}(u_n) = \lim_{n \to \infty} \left\{ I_{\lambda}(u_n) - \frac{1}{2} (J_{\lambda}'(u_n)u_n, u_n) \right\} \]
\[ \geq \frac{1}{N} \int_{\mathbb{B}^N} 4K^{\frac{N}{2}} |u|^2 \ dx + \frac{1}{N} S_K^{N/2} + \lambda \left( 1 - \frac{1}{q} \right) \int_{\mathbb{B}^N} 4K^{\frac{N}{2}} |u|^q \ dx. \]  

(5.8)

Now we distinguish two cases:

(a) If \( 2 < q < 2^* \), then \( c > S_K^{N/2}/N \), and this inequality contradicts the hypothesis for this case. Then, for all \( k, \nu_k = 0 \) and

\[ \lim_{n \to \infty} \int_{\mathbb{B}^N} 4K^{\frac{N}{2}} |u_n|^2 \ dx = \int_{\mathbb{B}^N} 4K^{\frac{N}{2}} |u|^2 \ dx. \]

By using (5.3), we conclude that \( u_n \to u \) in \( L^{2^*}_{K^{N/2}}(\mathbb{B}^N) \) and \( u_n \to u \) in \( H^1_{0,K}(\mathbb{B}^N) \).

(b) If \( 1 < q < 2 \), applying the Hölder inequality at (5.8), we have

\[ c \geq \frac{1}{N} S_K^{N/2} + \frac{1}{N} \int_{\mathbb{B}^N} 4K^{\frac{N}{2}} |u|^2 \ dx - \lambda \left( 1 - \frac{1}{q} \right) \left| \mathbb{B}^N \right|^{(2^*-q)/2^*} \left( \int_{\mathbb{B}^N} 4K^{\frac{N}{2}} |u|^2 \ dx \right)^{q/2^*}. \]

Let \( h(t) = c_1 t^{2^*} - \lambda c_2 t^q \). We easily know that \( h(t) \) attains its absolute minimum (for \( t > 0 \)) at the point \( t_0 = (\lambda c_2 q/2^* c_1)^{1/(2^*-q)} \), that is,

\[ h(t) \geq h(t_0) = -\lambda c_2^{2^*/(2^*-q)}. \]

But this result contradicts the hypothesis, then \( \nu_k = 0 \) for all \( k \).

\[ \square \]

**Proof of Theorem 5.1.** Firstly, we easily see that \( I_{\lambda}(u) \) satisfies the conditions (1) and (2) of Proposition 4.2.

If we can prove that \( c < \frac{1}{N} S_K^{N/2} \), then Proposition 4.2 and Lemma 5.3 imply the existence of the critical point of \( I_{\lambda}(u) \).

In order to obtain that \( c < \frac{1}{N} S_K^{N/2} \), we choose \( v_0 \in H^1_{0,K}(\mathbb{B}^N) \), with

\[ |v_0|_{K^{N/2},2^*_2} = 1, \quad \lim_{t \to \infty} I_{\lambda}(tv_0) = -\infty; \]

then, \( \sup_{t \geq 0} I_{\lambda}(tv_0) = I_{\lambda}(t_1v_0) \), for some \( t_1 > 0 \). Thus \( t_1 \) verifies

\[ 0 = t_1 \int_{\mathbb{B}^N} K |\nabla v_0|^2 \ dx - t_1^{2^*-1} \int_{\mathbb{B}^N} 4K^{\frac{N}{2}} |v_0|^2 \ dx - \lambda t_1^{q-1} \int_{\mathbb{B}^N} 4K^{\frac{N}{2}} |v_0|^q \ dx \]

and we get

\[ 0 = t_1^{q-1} \left( \int_{\mathbb{B}^N} K |\nabla v_0|^2 \ dx \right) - t_1^{2^*-q} - \lambda \int_{\mathbb{B}^N} 4K^{\frac{N}{2}} |v_0|^q \ dx \].

Note that

\[ t_1^{2^*-q} + \lambda \int_{\mathbb{B}^N} 4K^{\frac{N}{2}} |v_0|^q \ dx \to \infty \text{ as } \lambda \to \infty; \]

therefore, (5.9) implies \( t_1 \to 0 \), as \( \lambda \to \infty \). From the continuity of \( I_{\lambda} \), we have

\[ \lim_{\lambda \to \infty} \left( \sup_{t \geq 0} I_{\lambda}(tv_0) \right) = 0. \]
then, there exists $\lambda_0$ such that for all $\lambda \geq \lambda_0$,
\[
\sup_{t \geq 0} I_\lambda (tv_0) < \frac{S_K^{N/2}}{N}.
\]

Let $X$ be a Banach space, and $\Sigma$ the class of the closed and symmetric with respect to the origin subsets of $X \setminus \{0\}$. For $A \in \Sigma$, we define the genus $\gamma(A)$ by
\[
\gamma(A) = \min \left\{ k \in \mathbb{N} : \exists \phi \in C\left( A; \mathbb{R}^k \setminus \{0\} \right), \phi(x) = -\phi(-x) \right\}.
\]
If such a minimum does not exist then we define $\gamma(A) = +\infty$. The main properties of the genus are the following (see [?] or [7] for the details):

**Lemma 5.4.** Let $A, B \in \Sigma$. Then

1. If there exists $f \in C(A, B)$, odd, then $\gamma(A) \leq \gamma(B)$.
2. If $A \subset B$, then $\gamma(A) \leq \gamma(B)$.
3. If there exists an odd homeomorphism between $A$ and $B$, then $\gamma(A) = \gamma(B)$.
4. If $S^{N-1}$ is the sphere in $\mathbb{R}^N$, then $\gamma\left( S^{N-1} \right) = N$.
5. $\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$.
6. If $\gamma(B) < +\infty$, then $\gamma(A - B) \geq \gamma(A) - \gamma(B)$.
7. If $A$ is compact, then $\gamma(A) < +\infty$, and there exists $\delta > 0$ such that $\gamma(A) = \gamma\left( N_\delta(A) \right)$ where $N_\delta(A) = \{ x \in X : d(x, A) \leq \delta \}$.
8. If $X_0$ is a subspace of $X$ with codimension $m$, and $\gamma(A) < m$, then $A \cap X_0 \neq \emptyset$.

From the hypothesis $q < 2$ and the definition of $S_K$, we obtain for the functional $I_\lambda(u)$
\[
I_\lambda(u) \geq \frac{1}{2} \int_{\mathbb{R}^N} K|\nabla u|^2 \, dx - \frac{4}{2^*S_K^{2/2}} \left( \int K|\nabla u|^2 \, dx \right)^{2^*/2} - \frac{4\lambda}{q} C_{2,q} \left( \int K|\nabla u|^2 \, dx \right)^{q/2}.
\]
If we define
\[
m(t) = \frac{1}{2} t^2 - \frac{1}{2^*S_K^{2/2}} t^{2^*} - \frac{\lambda}{q} C_{2,q} t^q
\]
then
\[
I_\lambda(u) \geq m\left( \|\nabla u\|_{K,2} \right)
\]
There exists $\lambda'_0 > 0$ such that, if $0 < \lambda \leq \lambda'_0$, $m(t)$ attains its positive maximum (see Figure).
Let us assume $0 < \lambda \leq \lambda'$. By choosing $t_0$ and $t_1$ as in Figure 5.1, we make the following truncation of the functional $I_\lambda(u)$: Take $\tau : \mathbb{R}^+ \to [0, 1]$, nonincreasing and $C^\infty$, such that
\[
\begin{cases}
\tau(t) = 1 & \text{if } t \leq t_0, \\
\tau(t) = 0 & \text{if } t \geq t_1.
\end{cases}
\]
Let $\varphi(u) = \tau(\|\nabla u\|_{2,K})$. We consider the truncated functional
\[
\overline{I}_\lambda(u) = \frac{1}{2} \int_{\mathbb{B}^N} K|\nabla u|^2 \, dx - \frac{1}{2} \int_{\mathbb{B}^N} 4K^{\frac{N}{2}}|u|^2 \varphi(u) \, dx - \frac{1}{q} \int_{\mathbb{B}^N} 4\lambda K^{\frac{N}{2}}|u|^q \, dx.
\] (5.11)
As in (5.10), $\overline{I}_\lambda(u) \geq \overline{m}(\|\nabla u\|_{2,K})$, with
\[
\overline{m}(t) = \frac{1}{2} t^2 - \frac{1}{2} \frac{S_K^{2/2}}{t^2} t^2 \tau(t) - \frac{\lambda}{q} C_{2,q} t^q.
\] (5.12)
(see Figure 5.2).

We easily know that $t \leq t_0, \overline{m}(t) = m(t)$, and for $t \geq t_1$
\[
\overline{m}(t) = \frac{1}{2} t^2 - \frac{\lambda}{q} C_{2,q} t^q.
\]

**Lemma 5.5.** (1) $\overline{I}_\lambda(u) \in C^1\left(H^1_{0,K}(\mathbb{B}^N), \mathbb{R}\right)$.
(2) If $\overline{I}_\lambda(u) \leq 0$, then $\|\nabla u\|_{2,K} < t_0$, and $I_\lambda(v) = \overline{I}_\lambda(u)$ for all $v$ in a small enough neighbourhood of $u$.
(3) There exists $\lambda'_0 > 0$, such that, if $0 < \lambda < \lambda'_0$, then $\overline{I}_\lambda(u)$ verifies a local Palais-Smale condition for $c \leq 0$.

**Proof.** (1) and (2) are obvious. To prove (3), note that all $(PS)_c$ sequences for $\overline{I}_\lambda(u)$ with $c \leq 0$ must be bounded; then, by Lemma 5.3, if $\lambda$ verifies $S_K^{N/2}/N - c_0\lambda^p \geq 0$ there exists a convergent subsequence.

**Lemma 5.6.** Given $m \in \mathbb{N}$, there is $\varepsilon = \varepsilon(m) > 0$, such that
\[
\gamma\left(\left\{u \in H^1_{0,K}(\mathbb{B}^N) : J(u) \leq -\varepsilon\right\}\right) \geq m.
\]

**Proof.** Fix $m$, let $E_m$ be an $m$-dimensional subspace of $H^1_{0,K}(\mathbb{B}^N)$. We take $u_m \in E_m$, with norm
\[ \| \nabla u_m \|_{2,K} = 1 \] For \(0 < \rho < t_0\), we have
\[
\overline{\mathcal{F}}(\rho u_n) = I_1(\rho u_n) = \frac{1}{2} \rho^2 - \frac{1}{2} \rho^2 \int_{\mathbb{B}^N} 4K_\infty |u|^2 \, dx - \frac{1}{\rho^2} \int_{\mathbb{B}^N} 4K_\infty |u|^q \, dx.
\]

Since \(E_m\) is a space of finite dimension, all the norms are equivalent. Then, if we define
\[
\alpha_n = \inf \left\{ \int_{\mathbb{B}^N} 4K_\infty |u|^2 \, dx : u \in E_m, \| \nabla u_m \|_{2,K} = 1 \right\} > 0,
\]
\[
\beta_n = \inf \left\{ \int_{\mathbb{B}^N} 4K_\infty |u|^q \, dx : u \in E_m, \| \nabla u_m \|_{2,K} = 1 \right\} > 0,
\]
we have
\[
\overline{\mathcal{F}}(\rho u_n) \leq \frac{1}{2} \rho^2 - \frac{\alpha_n}{2} \rho^2 - \frac{\beta_n}{q} \rho^q,
\]
and we can choose \(\epsilon\) (which depends on \(m\)), and \(\eta < t_0\), such that \(\overline{\mathcal{F}}(\eta u) \leq -\epsilon\) if \(u \in E_m\), and \(\| \nabla u \|_{2,K} = 1\).

Let \(S_\eta = \{ u \in H^1_{0,K} \left( \mathbb{B}_N \right) : \| \nabla u \|_{2,K} = \eta \}\), \(S_\eta \cap E_m \subset \{ u \in H^1_{0,K} \left( \mathbb{B}_N \right) : \overline{\mathcal{F}}(u) \leq -\epsilon \}\). Therefore, by Lemma 5.4,
\[
\gamma(\{ u \in H^1_{0,K} \left( \mathbb{B}_N \right) : \overline{\mathcal{F}}(u) \leq -\epsilon \}) \geq \gamma(S_\eta \cap E_m) = m.
\]

\[\square\]

**Lemma 5.7.** Let \(\Sigma_k = \{ C \subset H^1_{0,K} \left( \mathbb{B}_N \right) \setminus \{ 0 \} : C \text{ is closed, } C = -C, \gamma(C) \geq k \}. \) Let \( c_k = \inf_{C \in \Sigma_k} \sup_{u \in C} \overline{\mathcal{F}}(u), \ K_c = \{ u \in H^1_{0,K} \left( \mathbb{B}_N \right) : \overline{\mathcal{F}}(u) = 0, \overline{\mathcal{F}}(u) = c \}, \) and suppose \(0 < \lambda < \lambda_0\), where \(\lambda_0\) is the constant of Lemma 5.5. Then, if \(c = c_k = c_{k+1} = \cdots = c_{k+r}, \gamma(K_c) \geq r + 1\). (In particular, the \(c_k\)'s are critical values of \(\overline{\mathcal{F}}(u)\).)

**Proof.** Next we will use Lemma 5.6 and the classical deformation lemma ((see [7])) to prove this lemma.

For the sake of convenience, we let \(\overline{\mathcal{F}}(u) : \overline{\mathcal{F}}(u) \leq -\epsilon \}. \) From Lemma 5.6, for all \(k \in \mathbb{N}\), there exists \(\epsilon > 0\) such that \(\gamma(\overline{\mathcal{F}}(u)) \geq k\).

Since \(\overline{\mathcal{F}}(u)\) is continuous and even, \(\overline{\mathcal{F}}(u) \in \Sigma_k\). Then for all \(k \in \mathbb{N}\), \(c_k \leq -\epsilon(k) < 0\). But \(\overline{\mathcal{F}}(u)\) is bounded from below, hence, \(c_k \geq -\infty\).

Let us assume that \(c = c_k = \cdots = c_{k+r}\), and observe that \(c < 0\). Therefore, \(\overline{\mathcal{F}}(u)\) satisfies the (PS) condition in \(K_c\), and it is easy to show that \(K_c\) is a compact set.

If \(\gamma(K_c) \leq r\), there is a closed and symmetric set \(U, K_c \subset U\), such that \(\gamma(U) \leq r\). (Since \(c < 0\), we can choose \(U \subset \overline{T}_d^0\).) Now using the deformation lemma, there exists an odd homeomorphism
\[
\eta : H^1_{0,K} \left( \mathbb{B}_N \right) \rightarrow H^1_{0,K} \left( \mathbb{B}_N \right),
\]
such that \(\eta(\overline{T}_d^\epsilon \setminus U) \subset \overline{T}_d^\epsilon\), for some \(\delta > 0\). (Again, we must choose \(0 < \delta < -c\), because \(\overline{T}_d(u)\) satisfies the (PS) condition on \(\overline{T}_d^0\), and we need \(\overline{T}_d^\epsilon \subset \overline{T}_d^0\).) According to the definition, we have
\[
c = c_{k+r} = \inf_{C \in \Sigma_{k+r}} \sup_{u \in C} \overline{\mathcal{F}}(u)(u).\]
Consequently, there exists $A \in \Sigma_{k+r}$, such that $\sup_{u \in \overline{A}} \overline{I}_t(u) < c + \delta$; i.e., $A \subset \overline{T}_{c+\delta}$. Then, $\eta(A - U) \subset \eta(\overline{T}_{c+\delta} \setminus U) \subset \overline{T}_{c-\delta}$.

(5.13)

But $\gamma(A \setminus U) \geq \gamma(A) - \gamma(U) \geq k$, and $\gamma(\eta(A \setminus U)) \geq \gamma(\overline{A \setminus U}) \geq k$. Then, $\eta(A \setminus U) \in \Sigma_k$. And this contradicts (5.13), in fact,

$$\eta(A \setminus U) \in \Sigma_k \implies \sup_{u \in \eta(A \setminus U)} \overline{I}_t(u) \geq c_k = c$$

Finally, Lemma 5.7 implies that Theorem 5.2 holds.

Remark 5.8. If we assume that $f(x, u) > 0$ in (4.1) and (5.1), then by the maximum principle in section 3, we have $u > 0$.

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