On the generalized $\text{SO}(2n, \mathbb{C})$-opers

Indranil Biswas$^1$ · Laura P. Schaposnik$^2$ · Mengxue Yang$^2$

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Abstract
Since their introduction by Beilinson–Drinfeld (Opers, 1993. arXiv math/0501398; Quantization of Hitchin’s integrable system and Hecke eigensheaves, 1991), opers have seen several generalizations. In Biswas et al. (SIGMA Symmetry Integr Geom Methods Appl 16:041, 2020), a higher rank analog was studied, named generalized $B$-opers, where the successive quotients of the oper filtration are allowed to have higher rank and the underlying holomorphic vector bundle is endowed with a bilinear form which is compatible with both the filtration and the oper connection. Since the definition did not encompass the even orthogonal groups, we dedicate this paper to study generalized $B$-opers whose structure group is $\text{SO}(2n, \mathbb{C})$ and show their close relationship with geometric structures on a Riemann surface.

Keywords Oper · Orthogonal group · Differential operator · Connection

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1 Introduction
Motivated by the works of Drinfeld and Sokolov [8, 9], Beilinson and Drinfeld introduced opers, in [2, 3], for a semisimple complex Lie group $G$. A $G$-oper on a compact Riemann surface $X$ is

- a holomorphic principal $G$-bundle $P$ on $X$ equipped with a holomorphic connection $\nabla$, and
- a holomorphic reduction in structure group of $P$ to a Borel subgroup of $G$,
such that the reduction satisfies the Griffiths transversality condition with respect to the connection $\nabla$ and the second fundamental form of $\nabla$ for the reduction satisfies certain non-degeneracy conditions.

In recent years, different extensions of the above objects have been introduced and studied—examples are $g$-opers (a $g$-oper is a $\text{Aut}(g)$-oper [3]) and Miura opers [10], as well as $(G, P)$-opers [7]. Very recently, the authors introduced the notion of a generalized $B$-oper in [6]. The definition was inspired by [5], where a particular class of opers was studied for $G$, $P$-opers (a $G$-oper to be compatible with it. However, the work done in [6] did not apply to opers with structure group $SO(2n, \mathbb{C})$. The case of $SO(2n, \mathbb{C})$ is subtler than $SO(2n + 1, \mathbb{C})$ and $\text{Sp}(2n, \mathbb{C})$. We dedicate the present paper to the study the $SO(2n, \mathbb{C})$ case. The reader can find in Yang further descriptions of the relationships between the present work, the $(G, P)$-opers of [7] and the author’s work in [6].

We begin our work by considering filtered $SO(2n, \mathbb{C})$-bundles with connections in Sect. 2, leading to the introduction and study of a generalized $SO(2n, \mathbb{C})$-quasioper. A quadruple $(E, B_0, \mathcal{F}_\ast, D)$, where $(E, B_0, \mathcal{F}_\ast)$ is a filtered $SO(2n, \mathbb{C})$-bundle over a compact Riemann surface $X$, and $D$ is a holomorphic connection on $(E, B_0, \mathcal{F}_\ast)$ (see Definition 2.5). Let $2m + 1$ be the length of the filtration. Then, $r = n/(m + 1)$ is an integer.

The quasiopers have naturally induced isomorphic dual quasiopers, as shown in Proposition 2.6. The properties of generalized $SO(2n, \mathbb{C})$-quasiopers are studied in Sect. 3, in the spirit of [6] and in relation to the jet bundles.

The main goal of the paper is to introduce $SO(2n, \mathbb{C})$-opers and to show that generalized $SO(2n, \mathbb{C})$-opers are closely related to projective structures on the base Riemann surface $X$, and this is done in Sect. 4. After constructing and studying $SO(2n, \mathbb{C})$-opers through $SO(2n, \mathbb{C})$-quasiopers, we consider their relation to geometric structures on $X$.

Let $X$ be a compact connected Riemann surfaces of genus at least two. Fix positive integers $n$ and $m$ such that $r := n/(m + 1)$ is an integer. Let

$$\mathcal{O}_X(n, m)$$

denote the space of all isomorphism classes of generalized $SO(2n, \mathbb{C})$-opers on $X$ of filtration length $2m + 1$ (see Definitions 2.2 and 4.1). Let $\mathcal{C}_X$ be the space of all isomorphism classes of holomorphic principal $SO(r, \mathbb{C})$-bundles on $X$ equipped with a holomorphic connection, and let $\mathfrak{P}(X)$ be the space of all projective structures on the Riemann surface $X$.

We prove the following (see Theorem 4.3):

**Theorem 1.1** If the integer $r$ is odd, then there is a canonical bijection between $\mathcal{O}_X(n, m)$ and the Cartesian product

$$\mathcal{C}_X \times \mathfrak{P}(X) \times \left( H^0(X, K_X^{\otimes (m+1)}) \oplus \left( \bigoplus_{i=2}^m H^0(X, K_X^{\otimes 2i}) \right) \right) \times J(X)_2,$$

for $J(X)_2$, the group of holomorphic line bundles on $X$ of order two, and $K_X$ the holomorphic cotangent bundle of $X$.

If $r$ is even, then there is a canonical bijection between $\mathcal{O}_X(n, m)$ and the Cartesian product

$$\mathcal{C}_X \times \mathfrak{P}(X) \times \left( H^0(X, K_X^{\otimes (m+1)}) \oplus \left( \bigoplus_{i=2}^{m/2} H^0(X, K_X^{\otimes 2i}) \right) \right) \times J(X)_2,$$
\[ C_X \times \Psi(X) \times \left( H^0(X, K_X^{(m+1)}) \oplus \bigoplus_{i=2}^m H^0(X, K_X^{2i}) \right). \]

We note that in the cases of SO(2n + 1, C) and Sp(2n, C), the decomposition is the same for even and odd \( r \), unlike in Theorem 1.1.

2 Filtered SO(2n, C)-bundles with connections

Let \( X \) be a compact connected Riemann surface of genus \( g \), with \( g \geq 2 \). The holomorphic cotangent bundle and the holomorphic tangent bundle of \( X \) will be denoted by \( K_X \) and \( TX \), respectively. Let \( E \) be a holomorphic vector bundle on \( X \) of rank \( 2n \), where \( n \geq 2 \), such that

\[ \det E = \bigwedge^{2n} E = \mathcal{O}_X \]

An SO(2n, C) structure on \( E \) is a holomorphic symmetric bilinear form

\[ B_0 \in H^0(X, \text{Sym}^2(E^*)) \quad (2.1) \]
on \( E \) which is fiberwise nondegenerate. In other words, \( B_0(x) \) is a nondegenerate symmetric bilinear form on \( E_x \) for every \( x \in X \). A pair of the form \( (E, B_0) \), where \( B_0 \) is an SO(2n, C) structure on a holomorphic vector bundle on \( X \), will be denoted by \( X \).

We note that for an SO(2n, C)-bundle \( (E, B_0) \), the determinant line bundle \( \bigwedge^{2n} E \) is holomorphically identified with \( \mathcal{O}_X \) uniquely up to a sign. More precisely, for any \( x \in X \), consider all isomorphisms of \( (E_x, B_0(x)) \) with \( C^{2n} \) equipped with the standard symmetric bilinear form. Then, the space of corresponding isomorphisms of \( \bigwedge^{2n} E_x \) with \( \bigwedge^{2n} C^{2n} \) has exactly two elements, and these two elements just differ by a sign.

2.1 Filtered SO(2n, C)-bundles

An SO(2n, C) structure \( B_0 \) on \( E \) produces a holomorphic isomorphism

\[ B : E \rightarrow E^* \quad (2.2) \]

that sends any \( v \in E_x, x \in X \), to the element of \( E^*_x \) defined by \( w \mapsto B_0(x)(w, v) \). The annihilator of a holomorphic subbundle \( F \subset E \), for the bilinear form \( B_0 \), will be denoted by \( F^\perp \). So, for any \( x \in X \), the subspace \( F^\perp_x \subset E_x \) consists of all \( v \in E_x \) such that \( B_0(x)(w, v) = 0 \) for all \( w \in F_x \). The bilinear form \( B_0 \) produces \( C^\infty \) homomorphisms

\[ E \otimes (E \otimes \mathcal{O}_X, 1_X) \rightarrow \mathcal{O}_X, 1_X \quad \text{and} \quad (E \otimes \mathcal{O}_X, 1_X) \otimes E \rightarrow \mathcal{O}_X, 1_X \]

simply by tensoring with the identity map of \( \mathcal{O}_X, 1_X \). Since the bilinear form \( B_0 \) is holomorphic, we have

\[ \overline{\partial} B_0(s, t) = B_0(\overline{\partial} E s, t) + B_0(s, \overline{\partial} t), \quad (2.3) \]

where \( s \) and \( t \) are locally defined \( C^\infty \) sections of \( E \) and \( \overline{\partial} E : C^\infty(X, E) \rightarrow C^\infty(X, E \otimes \mathcal{O}_X, 1_X) \) is the Dolbeault operator defining the
holomorphic structure on $E$. If $t$ is a locally defined holomorphic section of $F$ and $s$ is a locally defined $C^\infty$ section of $F^\perp$, then from (2.3), we have

$$B_0(\overline{\partial}_E s, t) = 0,$$

because $\overline{\partial}_E t = 0 B_0(s, t)$. This implies that $F^\perp$ is actually a holomorphic subbundle of $E$; its rank is $2n - \text{rank}(F)$.

**Definition 2.1** A filtration of an $\text{SO}(2n, \mathbb{C})$-bundle $(E, B_0)$ is a filtration of holomorphic subbundles of $E$ satisfying the following two conditions:

1. $\text{rank}(F_{i+1}/F_i) = 2 \cdot \text{rank}(F_1)$, and $\text{rank}(F_i/F_{i-1}) = \text{rank}(F_1)$, for all $i \in \{1, \cdots, 2m + 1\}\setminus\{m + 1\}$, and
2. $F_i^\perp = F_{2m+1-i}$ for all $0 \leq i \leq m$.

Note that, the first condition implies that $$(m + 1) \cdot \text{rank}(F_1) = n.$$ The second condition implies that the restriction $B_0|_{F_{m+1}}$ of the form $B_0$ to the subbundle $F_{m+1} \subset E$ has the following two properties:

- the subbundle $F_m \subset F_{m+1}$ is annihilated by $B_0|_{F_{m+1}}$, meaning $F_m \subset F_m^\perp$, and
- the restriction $B_0|_{F_{m+1}}$ descends to the quotient bundle $F_{m+1}/F_m$ as a fiberwise nondegenerate symmetric bilinear form.

For notational convenience, the filtration $\{F_i\}_{i=0}^{2m+1}$ in (2.4) will henceforth be denoted by $\mathcal{F}$.

**Definition 2.2** An $\text{SO}(2n, \mathbb{C})$-bundle equipped with a filtration will be called a filtered $\text{SO}(2n, \mathbb{C})$-bundle. The odd integer $2m + 1$ in (2.4) will be called the length of the filtration.

We shall always assume that $m \geq 2$. This is because

$$\text{SO}(4, \mathbb{C}) = (\text{SL}(2, \mathbb{C}) \times \text{SL}(2, \mathbb{C}))/\langle \mathbb{Z}/2\mathbb{Z} \rangle.$$ 

### 2.2 $\text{SO}(2n, \mathbb{C})$-quasiopers: filtered $\text{SO}(2n, \mathbb{C})$-bundles with connections

Recall that a holomorphic connection on a holomorphic vector bundle $E$ on $X$ is a first-order holomorphic differential operator

$$D : E \longrightarrow E \otimes K_X$$

satisfying the Leibniz identity, this is,

$$D(fs) = fD(s) + s \otimes df$$

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for any locally defined holomorphic function $f$ on $X$ and any locally defined holomorphic section $s$ of $E$ [1]. In particular, a holomorphic connection is automatically flat because $\Omega_{X}^{2,0} = 0$. The bilinear form $B_0$ in (2.1) produces holomorphic homomorphisms

$$E \otimes (E \otimes K_X) \longrightarrow K_X \quad \text{and} \quad (E \otimes K_X) \otimes E \longrightarrow K_X$$

simply by tensoring with the identity map of $K_X$. A holomorphic connection on an $\text{SO}(2n, \mathbb{C})$-bundle $(E, B_0)$ is a holomorphic connection $D$ on the holomorphic vector bundle $E$ satisfying the identity

$$\partial B_0(s, t) = B_0(D(s), t) + B_0(s, D(t))$$

for all locally defined holomorphic sections $s$ and $t$ of $E$.

We note that for a holomorphic connection $D$ on an $\text{SO}(2n, \mathbb{C})$-bundle $(E, B_0)$, the connection on the determinant line bundle $\bigwedge^{2n}E = \mathcal{O}_X$ induced by $D$ coincides with the trivial connection on the trivial holomorphic line bundle given by the de Rham differential $d$ (it is the unique rank one holomorphic connection on $X$ with trivial monodromy). Indeed, this follows immediately from the fact that the isomorphism $B$ in (2.2) takes the connection $D$ on $E$ to the dual connection on $E^*$ induced by $D$.

Let $D$ a holomorphic connection on $E$, and let $F_1 \subset F_2 \subset E$ and $F_3 \subset F_4 \subset E$ be holomorphic subbundles such that

$$D(F_1) \subset F_3 \otimes K_X \quad \text{and} \quad D(F_2) \subset F_4 \otimes K_X.$$

**Definition 2.3** The second fundamental form of $(F_1, F_2, F_3, F_4)$ for the connection $D$ is the map

$$S(D; F_1, F_2, F_3, F_4) : \frac{F_2}{F_1} \longrightarrow (\frac{F_4}{F_3}) \otimes K_X, \quad s \longmapsto D(\tilde{s})$$

that sends any locally defined holomorphic section $s$ of $F_2/F_1$ to the image of $D(\tilde{s})$ in $(\frac{F_4}{F_3}) \otimes K_X$, where $\tilde{s}$ is any locally defined holomorphic section of the subbundle $F_2$ that projects to $s$ under the quotient map $F_2 \longrightarrow F_2/F_1$.

It is straightforward to check that the image of $D(\tilde{s})$ in $(\frac{F_4}{F_3}) \otimes K_X$ does not depend on the choice of the above lift $\tilde{s}$ of $s$ (see [6, Lemma 2.10]).

**Definition 2.4** Let $(E, B_0, F_i)$ be a filtered $\text{SO}(2n, \mathbb{C})$-bundle. A holomorphic connection on $(E, B_0, F_i)$ is a holomorphic connection $D$ on $(E, B_0)$ satisfying the following three conditions:

1. $D(F_i) \subset F_{i+1} \otimes K_X$ for all $1 \leq i \leq 2m$ [see (2.4)],
2. the second fundamental form

$$S(D, i) : \frac{F_i}{F_{i-1}} \longrightarrow (\frac{F_{i+1}}{F_i}) \otimes K_X$$

is an isomorphism for all $i \in \{1, \ldots, 2m+1\}\setminus\{m, m+1\}$, and
3. the composition of homomorphisms
(S(D, m + 1) \otimes \text{Id}_{K_X}) \circ S(D, m) : F_m/F_{m-1} \rightarrow (F_{m+2}/F_{m+1}) \otimes K_X^{\otimes 2}

is an isomorphism.

**Definition 2.5** A *generalized SO*(2\(n\), \(\mathbb{C}\))-quasioper on \(X\) is a quadruple \((E, B_0, \mathcal{F}_*, D)\), where \((E, B_0, \mathcal{F}_*)\) is a filtered \(SO(2n, \mathbb{C})\)-bundle, and \(D\) is a holomorphic connection on the filtered \(SO(2n, \mathbb{C})\)-bundle \((E, B_0, \mathcal{F}_*)\).

Two generalized \(SO(2n, \mathbb{C})\)-quasiopers \((E, B_0, \mathcal{F}_*, D)\) and \((E', B'_0, \mathcal{F}'_*, D')\) are called *isomorphic* if there is a holomorphic isomorphism of vector bundles such that

\[
\Phi : E \rightarrow E'
\]
such that

- \(\Phi\) takes the bilinear form \(B_0\) on \(E\) to the bilinear form \(B'_0\) on \(E'\),
- \(\Phi\) takes the filtration \(\mathcal{F}_*\) of \(E\) to the filtration \(\mathcal{F}'_*\) of \(E'\), and
- \(\Phi\) takes the connection \(\nabla\) on \(E\) to the connection \(\nabla'\) on \(E'\).

**Proposition 2.6** Given a generalized \(SO(2n, \mathbb{C})\)-quasioper \((E, B_0, \mathcal{F}_*, D)\), there is a naturally associated isomorphic dual quasioper.

**Proof** Let \((E, B_0, \mathcal{F}_*, D)\) be a generalized \(SO(2n, \mathbb{C})\)-quasioper on \(X\), where \(\mathcal{F}_*\), as in (2.4), is a filtration of \(E\). Consider the dual vector bundle \(E^*\). It is equipped with a holomorphic connection \(D^*\) induced by the connection \(D\) on \(E\).

Since the symmetric bilinear form \(B_0\) on \(E\) is nondegenerate, it produces a holomorphic symmetric nondegenerate bilinear form \(B_0^*\) on \(E^*\). For any \(F_i\) in (2.4), define

\[
G_{2m+1-i} \subset E^*
\]

(2.6)
to be the kernel of the natural projection \(E^* \rightarrow (F_i)^*\). Then,

\[
(E^*, B_0^*, \{G_j\}_{j=0}^{2m+1}, D^*)
\]

(2.7)
is also a generalized \(SO(2n, \mathbb{C})\)-quasioper.

It is straightforward to check that the holomorphic isomorphism \(B\) in (2.2) takes the generalized \(SO(2n, \mathbb{C})\)-quasioper \((E, B_0, \mathcal{F}_*, D)\) to the generalized \(SO(2n, \mathbb{C})\)-quasioper \((E^*, B_0^*, \{G_j\}_{j=0}^{2m+1}, D^*)\) constructed in (2.7).

\[\square\]

### 3 Properties of a generalized \(SO(2n, \mathbb{C})\)-quasioper

Let \(W\) be holomorphic vector bundle over \(X\) equipped with a holomorphic connection \(D_W\), and let \(V \subset W\) be any holomorphic subbundle.
Lemma 3.1 There is a unique minimal holomorphic subbundle \( \hat{D}_W(V) \) of \( W \) containing \( V \) such that the connection \( D_W \) takes \( V \) into \( \hat{D}_W(V) \otimes K_X \).

Proof From Definition 2.3, consider the second fundamental form of \( V \) for the connection \( D_W \) by letting \( F_1 = 0, F_2 = F_3 = V, \) and \( F_4 = W \) in Eq. (2.5). Let

\[
T \subset ((W/V) \otimes K_X)/(S(D_W;V)(V))
\]

be the torsion part of the coherent analytic sheaf \(((W/V) \otimes K_X)/(S(D_W;V)(V))\). The inverse image of \( T \) under the quotient map

\[
(W/V) \otimes K_X \rightarrow ((W/V) \otimes K_X)/(S(D_W;V)(V))
\]

will be denoted by \( \mathcal{F} \). So \( \mathcal{F} \otimes TX \) is a holomorphic subbundle of

\[
(W/V) \otimes K_X \otimes TX = W/V.
\]

The inverse image of the subbundle \( \mathcal{F} \otimes TX \subset W/V \) under the quotient map \( W \rightarrow W/V \) will be denoted by \( \hat{D}_W(V) \).

Note that, \( \hat{D}_W(V) \) is a holomorphic subbundle of \( W \), because \( \mathcal{F} \otimes TX \) is a holomorphic subbundle of \( W/V \). Also, \( V \) is a holomorphic subbundle of \( \hat{D}_W(V) \). From the construction of \( \hat{D}_W(V) \), it is evident that we have

\[
D_W(V) \subset \hat{D}_W(V) \otimes K_X.
\]

Also, it is clear that \( \hat{D}_W(V) \) is the smallest among all subbundles \( U \) of \( W \) such that \( D_W(V) \subset U \otimes K_X \).

\( \square \)

Note that, \( V \) is preserved by the connection \( D_W \) if and only if \( \hat{D}_W(V) = V \), where \( \hat{D}_W(V) \) is constructed in Lemma 3.1.

The holomorphic subbundle \( \hat{D}_W(\hat{D}_W(V)) \subset W \) will be denoted by \( \hat{D}^2_W(V) \). Moreover, for ease of notation, inductively define the subbundle

\[
\hat{D}^{k+1}_W(V) := \hat{D}_W(\hat{D}^k_W(V)) \subset W,
\]

for \( k \geq 2 \). So \( \{\hat{D}^j_W(V)\}_{j \geq 1} \) is an increasing sequence of holomorphic subbundles of \( W \).

Through Lemma 3.1, we can construct a holomorphic subbundle of a generalized \( \text{SO}(2n, \mathbb{C}) \)-quasioper. Indeed, let

\[
(E, B_0, \mathcal{F}_*, D)
\]

be a generalized \( \text{SO}(2n, \mathbb{C}) \)-quasioper on \( X \), where \( \mathcal{F}_* \), as in (2.4), is a filtration

\[
0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_i \subset \cdots \subset F_{2m} \subset F_{2m+1} = E
\]

of length \( 2m + 1 \). For the holomorphic subbundle \( F_1 \subset E \) in (3.2), define the holomorphic subbundle

\[
\mathcal{F} := \hat{D}^{2m}_E(F_1) \subset E
\]
[see (3.1)]. We note that the subbundle \( i \) in general is not preserved by the connection \( D \) on \( E \).

Now consider the generalized \( \text{SO}(2n, \mathbb{C}) \)-quasioper \((E^*, B^*_0, \{G_j\}_{j=0}^{2m+1}, D^*) \) in (2.7) associated to \((E, B_0, F, D)\) via Proposition 2.6. As in (3.3), define the holomorphic subbundle

\[
\mathbb{G} := (D^*)_{E^*}^2m(G_1) \subset E^*,
\]

(3.4)

where \( G_1 \) is constructed in (2.6). The dual of the natural quotient map \( E^* \to E^*/\mathbb{G} \) is a fiberwise injective holomorphic homomorphism \((E^*/\mathbb{G})^* \to E^{**}\). Therefore, we have the holomorphic subbundle

\[
S := (E^*/\mathbb{G})^* \subset E^{**} = E
\]

(3.5)
given by the image of the above fiberwise injective homomorphism.

**Lemma 3.2** For the holomorphic subbundles \( \mathbb{F} \) and \( S \) of \( E \), in (3.3) and (3.5), respectively, the natural homomorphism

\[
\mathbb{F} \oplus S \to E
\]

is an isomorphism. Moreover, the resulting holomorphic decomposition

\[
E = \mathbb{F} \oplus S
\]
of \( E \) is orthogonal with respect to the bilinear form \( B_0 \) on \( E \).

**Proof** From the properties of the filtration \( F \), and the connection \( D \) it follows that the natural homomorphism

\[
\mathbb{F} \oplus S \to E
\]

is surjective. Note that, from the properties of the filtration \( F \), and \( D \), we also have \( \text{rank}(\mathbb{F}) = 2n - \frac{n}{m+1} = \text{rank}(E) - \text{rank}(F_1) \) and \( \text{rank}(S) = \frac{n}{m+1} = \text{rank}(F_1) \). Furthermore, we have \( B_0(\mathbb{F}, S) = 0 \). These together imply that \( \mathbb{F} \cap S = 0 \) and \( \mathbb{F}^\perp = S \).

In what follows, we shall describe an alternative construction of the subbundle \( S \) in (3.5) by considering the jet bundle approach given in [6]. Let

\[
Q := E/F_{2m}
\]

(3.6)

be the quotient in (3.2), and let

\[
q : E \to E/F_{2m} = Q
\]

(3.7)

be the quotient map.

For any nonnegative integer \( i \), let \( J^i(Q) \) be the \( i \)-th order jet bundle of \( Q \) in (3.6) (see [6, Section 3.1], [4, 5] for jet bundles). As shown in [6, Eq. (3.3)], [6, Eq. (3.5)], the connection \( D \) on \( E \) produces an \( \mathcal{O}_X \)-linear homomorphism

\[
f_i : E \to J^i(Q).
\]

(3.8)

We briefly recall the construction of \( f_i \) as this homomorphism plays a crucial role.

Take any point \( x \in X \), and let \( x \in U_x \subset X \) be a simply connected analytic open neighborhood of the point \( x \). For any \( v \in E_x \), let \( \tilde{v} \) be the unique flat section of \( E|_{U_x} \), for
the flat connection \( D \) on \( E \), such that \( \tilde{\nu}(x) = \nu \). Consider the holomorphic section \( q(\tilde{\nu}) \) of the vector bundle \( Q|_{U_i} \), in (3.6), where \( q \) is the projection in (3.7). Restricting this section \( q(\tilde{\nu}) \) to the \( i \)th order infinitesimal neighborhood of \( x \), we get an element of \( J^i(Q)_x \); this element of \( J^i(Q)_x \) given by \( q(\tilde{\nu}) \) will be denoted by \( q(\tilde{\nu})_i \). The map \( f_i \) in (3.8) sends any \( \nu \in E_x, x \in X \), to \( q(\tilde{\nu})_i \in J^i(Q)_x \) constructed above using \( \nu \) and the connection \( D \).

From the three conditions in Definition 2.4, it follows that the homomorphism

\[
f_{2m} : E \rightarrow J^{2m}(Q)
\]

in (3.8) is surjective. Moreover, the subbundle \( S \) in (3.5) coincides with the kernel of the above homomorphism \( f_{2m} \). Consequently, we have a short exact sequence of holomorphic vector bundles

\[
0 \rightarrow S = \text{kernel}(f_{2m}) \rightarrow E \xrightarrow{f_{2m}} J^{2m}(Q) \rightarrow 0
\]

(3.9)
on \( X \). Therefore, Lemma 3.2 has the following corollary.

**Corollary 3.3** The composition of homomorphisms

\[
\mathbb{F} \leftarrow E \xrightarrow{f_{2m}} J^{2m}(Q),
\]

where \( f_{2m} \) is the homomorphism in (3.8), and \( \mathbb{F} \) is the subbundle of \( E \) in Lemma 3.2, which is an isomorphism.

Let

\[
f'_{2m} : \mathbb{F} \xrightarrow{\sim} J^{2m}(Q)
\]

(3.10)
be the composition of homomorphisms in Corollary 3.3; recall from Corollary 3.3 that \( f'_{2m} \) is an isomorphism.

Using the decomposition \( \mathbb{F} \oplus S = E \) in Lemma 3.2, consider the composition of homomorphisms

\[
\mathbb{F} \leftarrow E \xrightarrow{D} E \otimes K_X \xrightarrow{q_F \otimes \text{Id}} \mathbb{F} \otimes K_X ,
\]

(3.11)
where \( q_F : E = \mathbb{F} \oplus S \rightarrow \mathbb{F} \) is the natural projection to factor \( \mathbb{F} \). This composition of homomorphisms is a holomorphic connection on \( \mathbb{F} \), because it satisfies the Leibniz identity. The holomorphic connection on \( \mathbb{F} \) constructed in (3.11) will be denoted by \( D^F \).

Similarly, the composition of homomorphisms

\[
S \leftarrow E \xrightarrow{D} E \otimes K_X \xrightarrow{q_S \otimes \text{Id}} S \otimes K_X ,
\]

(3.12)
where \( q_S : E = \mathbb{F} \oplus S \rightarrow S \) is the natural projection to factor \( S \) in Lemma 3.2, is a holomorphic connection on the holomorphic vector bundle \( S \). The holomorphic connection on \( S \) constructed in (3.12) will be denoted by \( D^S \).

The holomorphic connections \( D^F \) and \( D^S \), on \( \mathbb{F} \) and \( S \), respectively, together define a holomorphic connection \( D^F \oplus D^S \) on \( \mathbb{F} \oplus S \). It should be emphasized that the isomorphism \( \mathbb{F} \oplus S = E \) in Lemma 3.2 does not, in general, take the holomorphic connection \( D^F \oplus D^S \) on \( \mathbb{F} \oplus S \) to the connection \( D \) on \( E \). Indeed, for the connection \( D \) on \( E \), the subbundles \( \mathbb{F} \) and \( S \) of \( E \) may have nontrivial second fundamental form. On the other
hand, for the direct sum of connections $D^F \oplus D^S$, the second fundamental form of both $\mathbb{F}$ and $S$ vanish identically.

From Lemma 3.2, it follows immediately that the restrictions of the bilinear form $B_0$ on $E$ to both $\mathbb{F}$ and $S$ are nondegenerate. The holomorphic symmetric nondegenerate bilinear form on $\mathbb{F}$ (respectively, $S$) obtained by restricting $B_0$ to $\mathbb{F}$ (respectively, $S$) will be denoted by $B_0^\mathbb{F}$ (respectively, $B_0^S$); in particular, we have

$$
B_0^\mathbb{F} \in H^0(X, \text{Sym}^2(\mathbb{F}^*)) \quad \text{and} \quad B_0^S \in H^0(X, \text{Sym}^2(S^*)).
$$

As the decomposition $\mathbb{F} \oplus S = E$ in Lemma 3.2 is orthogonal, we have

$$
B_0 = B_0^\mathbb{F} \oplus B_0^S. \quad (3.13)
$$

Since the connection $D$ preserves the bilinear form $B_0$ on $E$, and (3.13) holds, from the constructions of the connections $D^\mathbb{F}$ in (3.11) and the connection $D^S$ in (3.12), we have the following:

**Corollary 3.4** The connection $D^\mathbb{F}$ on $\mathbb{F}$ in (3.11) preserves the bilinear form $B_0^\mathbb{F}$ on $\mathbb{F}$.

The connection $D^S$ on $S$ in (3.12) preserves the bilinear form $B_0^S$ on $S$.

**Proposition 3.5** The connection $D^\mathbb{F}$ on $\mathbb{F}$ produces a holomorphic connection $D_Q$ on $J^{2m}(Q)$.

**Proof** This can be deduced from the fact that the homomorphism $f'_{2m}$ in (3.10) is an isomorphism. So $D_Q$ is the holomorphic connection on $J^{2m}(Q)$ that corresponds to the connection $D^\mathbb{F}$ on $\mathbb{F}$ by this isomorphism $f'_{2m}$.

We shall give a direct construction of this connection $D_Q$ on $J^{2m}(Q)$. Let

$$
0 \to Q \otimes K_X^{(2m+1)} \overset{i}{\to} J^{2m+1}(Q) \overset{q}{\to} J^{2m}(Q) \to 0
$$

be the natural short exact sequence of jet bundles. It fits in the following commutative diagram of homomorphisms:

$$
egin{array}{ccc}
0 & \to & Q \otimes K_X^{(2m+1)} \\
\downarrow & & \downarrow i \\
0 & \to & J^{2m+1}(Q) \\
\downarrow \lambda & & \downarrow q \\
J^{2m}(Q) \otimes K_X & \overset{\lambda'}{\to} & J^1(J^{2m}(Q)) \\
\downarrow & & \downarrow q' \\
J^{2m-1}(Q) \otimes K_X & \overset{=} {\to} & J^{2m-1}(Q) \otimes K_X \\
\downarrow & & \downarrow 0 \\
0 & & 0
\end{array}
$$

(see [5, p. 4, (2.4), p. 10, (3.4)]).

Consider the homomorphism

$$
(f_{2m+1}^\mathbb{F}) \circ (f'_{2m})^{-1} : J^{2m}(Q) \to J^{2m+1}(Q),
$$
where $f_{2m+1}|_\mathcal{F}$ is the restriction of the homomorphism in (3.8), and $f'_{2m}$ is the isomorphism in (3.10). It is straightforward to check that

$$q \circ (f_{2m+1}|_\mathcal{F}) \circ (f'_{2m})^{-1} = \text{Id}_{J^{2m}(\mathcal{Q})},$$

where $q$ is the projection in (3.14). Therefore, from the commutativity of the diagram in (3.14), we conclude that

$$q' \circ \lambda \circ (f_{2m+1}|_\mathcal{F}) \circ (f'_{2m})^{-1} = \text{Id}_{J^{2m}(\mathcal{Q})},$$

where $q'$ and $\lambda$ are the homomorphisms in (3.14). Consequently, the homomorphism

$$\lambda \circ (f_{2m+1}|_\mathcal{F}) \circ (f'_{2m})^{-1} : J^{2m}(\mathcal{Q}) \to J^1(J^{2m}(\mathcal{Q}))$$

produces a holomorphic splitting of the short exact sequence

$$0 \to J^{2m}(\mathcal{Q}) \otimes K_X \xrightarrow{i'} J^1(J^{2m}(\mathcal{Q})) \xrightarrow{q'} J^{2m}(\mathcal{Q}) \to 0$$

in (3.14). But (3.15) is the twisted dual of the Atiyah exact sequence for $J^{2m}(\mathcal{Q})$. More precisely, let

$$0 \to J^{2m}(\mathcal{Q})^* \otimes J^{2m}(\mathcal{Q}) \to J^1(J^{2m}(\mathcal{Q}))^* \otimes J^{2m}(\mathcal{Q})$$

be the dual of (3.15) tensored with $\text{Id}_{J^{2m}(\mathcal{Q})}$. Then, $\eta^{-1}(\text{Id}_{J^{2m}(\mathcal{Q})} \otimes TX) \subset J^1(J^{2m}(\mathcal{Q}))^* \otimes J^{2m}(\mathcal{Q})$ is the Atiyah bundle $\text{At}(J^{2m}(\mathcal{Q}))$ of $J^{2m}(\mathcal{Q})$; furthermore, the short exact sequence

$$0 \to \text{End}(J^{2m}(\mathcal{Q})) = J^{2m}(\mathcal{Q})^* \otimes J^{2m}(\mathcal{Q}) \to \text{At}(J^{2m}(\mathcal{Q})) \xrightarrow{\eta} TX \to 0$$

obtained from the above short exact sequence is in fact the Atiyah exact sequence for $J^{2m}(\mathcal{Q})$. Consequently, a holomorphic splitting of (3.15) is a holomorphic connection on the holomorphic vector bundle $J^{2m}(\mathcal{Q})$ [1]. Therefore, the above homomorphism $\lambda \circ (f_{2m+1}|_\mathcal{F}) \circ (f'_{2m})^{-1}$ is a holomorphic connection on $J^{2m}(\mathcal{Q})$.

The holomorphic connection on $J^{2m}(\mathcal{Q})$ defined by $\lambda \circ (f_{2m+1}|_\mathcal{F}) \circ (f'_{2m})^{-1}$ coincides with the holomorphic connection $\mathcal{D}_\mathcal{Q}$ on $J^{2m}(\mathcal{Q})$ produced by the connection $\mathcal{D}_\mathcal{F}$ on $\mathcal{F}$ [see (3.11)] using the isomorphism $f'_{2m}$ in (3.10). $\square$

Let $\mathcal{L}$ be the holomorphic line bundle on $X$ of order two. So the holomorphic line bundle $\mathcal{L} \otimes \mathcal{L}$ is holomorphically isomorphic to $\mathcal{O}_X$. Fix a holomorphic isomorphism

$$\rho : \mathcal{L} \otimes \mathcal{L} \to \mathcal{O}_X. \quad (3.16)$$

There is a unique holomorphic connection

$$\mathcal{D}^\mathcal{L} \quad (3.17)$$

on $\mathcal{L}$ such that the isomorphism $\rho$ in (3.16) takes the holomorphic connection $\mathcal{D}^\mathcal{L} \otimes \text{Id} + \text{Id} \otimes \mathcal{D}^\mathcal{L}$ on $\mathcal{L} \otimes \mathcal{L}$ to the trivial connection on $\mathcal{O}_X$ given by the de Rham differential $d$. It should be clarified that this connection $\mathcal{D}^\mathcal{L}$ does not depend on the choice of the isomorphism $\rho$. $\square$
Let \((\mathcal{E}, B_0, \mathcal{F}_0, D)\) be a generalized SO\((2n, \mathbb{C})\)-quasioper on \(X\). Consider the holomorphic vector bundle \(\mathcal{E}^1 := \mathcal{E} \otimes \mathcal{L}\). Note that, \(E^1 \otimes E^1 = (E \otimes E) \otimes (\mathcal{L} \otimes \mathcal{L})\), we conclude that \(E^1 \otimes E^1\) is a fiberwise nondegenerate symmetric holomorphic bilinear form on \(E^1\), where \(\mathcal{L}\) is the isomorphism in (3.16). The filtration \(\mathcal{F}_0\) of holomorphic subbundles of \(E\) produces a filtration \(\mathcal{F}_1\) of holomorphic subbundles of \(E^1\). The \(i\)th term \(F^1_i\) of \(\mathcal{F}_1\) is simply \(F_i \otimes \mathcal{L}\) [see (3.2)]. Let \(D^1 := D \otimes \text{Id}_\mathcal{L} + \text{Id}_E \otimes D\) (3.18) be the holomorphic connection on \(E \otimes \mathcal{L} = E^1\), where \(D\) is the holomorphic connection in (3.17).

The following lemma is straightforward to prove.

**Lemma 3.6** The quadruple 
\[(E^1, B^1_0, \mathcal{F}^1_0, D^1)\]
constructed above is a generalized SO\((2n, \mathbb{C})\)-quasioper on \(X\).

The holomorphic vector bundle \(\mathcal{F} = \tilde{D}_E(\mathcal{F}_1)\) in (3.3) has the following filtration of holomorphic subbundles
\[0 \subset F_1 \subset \tilde{D}_E(F_1) \subset \tilde{D}_E^2(F_1) \subset \cdots \subset \tilde{D}_E^{2m-1}(F_1) \subset \tilde{D}_E^{2m}(F_1) = \mathcal{F}. \quad (3.19)\]
From Definition 2.4, it follows that the filtration of \(\mathcal{F}\) in (3.19) coincides with the filtration of \(\hat{\mathcal{F}}\) obtained by intersecting the filtration of \(E\) in (3.2) with the subbundle \(\hat{\mathcal{F}}\) of \(E\). Moreover, the isomorphism \(f^j_{2m}\) in (3.10) takes the filtration of \(\hat{\mathcal{F}}\) in (3.19) to the filtration of \(J^{2m}(Q)\) given by the short exact sequence of jet bundles
\[0 \longrightarrow Q \otimes K_X^{2i} \longrightarrow J^i(Q) \longrightarrow J^{i-1}(Q) \longrightarrow 0 \quad (3.20)\]
for \(i \geq 1\). More precisely, for any \(1 \leq j \leq 2m - 1\), the subbundle \(\tilde{D}_E^j(F_1)\) in (3.19) corresponds to the kernel of the projection \(J^{2m}(Q) \longrightarrow J^{2m-j-1}(Q)\) by the isomorphism \(f^j_{2m}\) in (3.10).

Let
\[0 \longrightarrow Q \otimes K_X^{2m} \longrightarrow \mathcal{F} = J^{2m}(Q) \longrightarrow J^{2m-1}(Q) \longrightarrow 0 \quad (3.21)\]
be the short exact sequence of jet bundles, where \(\mathcal{F}\) is identified with \(J^{2m}(Q)\) using the isomorphism \(f^j_{2m}\) in (3.10). As explained before, the connection \(D\) on \(E\) need not preserve the subbundle \(S\) in (3.5). Consider the decomposition \(E = \mathcal{F} \oplus S\) in Lemma 3.2. Assume that
\[\tilde{D}_E(S) = S \oplus (Q \otimes K_X^{2m}) \subset S \oplus \mathcal{F} = E, \quad (3.22)\]
where \( Q \otimes K_X^{2m} \) is the subbundle of \( \mathbb{I} \) in (3.21), and \( \hat{D}_E(S) \subset E \) is the holomorphic subbundle given by Lemma 3.1. Then, the second fundamental form \( S(D;S) \) of \( S \) for the connection \( D \) is a holomorphic section
\[
S(D;S) \in H^0(X, \text{Hom}(S, Q \otimes K_X^{2m+1}))
\]
\[
\subset H^0(X, \text{Hom}(S, \mathbb{F})) = H^0(X, \text{Hom}(S, E/S));
\]

note that, Lemma 3.2 identifies \( E/S \) with \( \mathbb{F} \).

\section{Generalized SO\((2n, \mathbb{C})\)-opers and projective structures}

Through the construction of generalized SO\((2n, \mathbb{C})\)-quasiopers in Definition 2.5 and that of generalized \( B \)-opers in [6, Definition 2.11], we define a generalized SO\((2n, \mathbb{C})\)-oper.

\begin{definition}
A generalized SO\((2n, \mathbb{C})\)-oper on \( X \) is a generalized SO\((2n, \mathbb{C})\)-quasioper \((E, B_0, \mathcal{F}, D)\) on \( X \) (see Definition 2.5) satisfying the following three conditions:
\end{definition}

(1) \( S = Q \otimes K_X^m \), where \( S \) and \( Q \) are defined in (3.5) and (3.6), respectively,

(2) \( \hat{D}(S) = S \oplus (Q \otimes K_X^{2m}) \) (see (3.22) for this condition), and

(3) there is a holomorphic section
\[
\phi \in H^0(X, K_X^{m+1})
\]
such that the second fundamental form \( S(D;S) \) in (3.23) is:
\[
S(D;S) = \text{Id}_Q \otimes \phi.
\]

Note that, using the isomorphism \( S = Q \otimes K_X^m \) in (1), the second fundamental form \( S(D;S) \) in (3.23) is a holomorphic section of
\[
\text{Hom}(Q \otimes K_X^m, Q \otimes K_X^{2m+1}) = \text{End}(Q) \otimes K_X^{m+1};
\]

the condition says that this section \( S(D;S) \) coincides with \( \text{Id}_Q \otimes \phi \).

Two generalized SO\((2n, \mathbb{C})\)-opers are called isomorphic if the underlying generalized SO\((2n, \mathbb{C})\)-quasiopers are isomorphic (see Definition 2.5).

The following lemma is straightforward to prove.

\begin{lemma}
Take a holomorphic line bundle \( \mathcal{L} \) on \( X \) of order two, and fix a holomorphic isomorphism \( \rho \) as in (3.16). Let \((E, B_0, \mathcal{F}, D)\) be a generalized SO\((2n, \mathbb{C})\)-oper on \( X \). Then, the generalized SO\((2n, \mathbb{C})\)-quasioper \((E^1, B_0^1, \mathcal{F}^1, D^1)\) in Lemma 3.6 is also a generalized SO\((2n, \mathbb{C})\)-oper.
\end{lemma}

Fix integers \( n \) and \( m \) as in Definition 2.2; note that, \( r := n/(m+1) \) is an integer, in fact it is the rank of \( F_1 \) in (2.4). Let
\[
\cup_X(n,m)
\]
denote the space of all isomorphism classes of generalized $\text{SO}(2n, \mathbb{C})$-opers on $X$ of filtration length $2m + 1$ (see Definitions 2.2 and 4.1).

Let

$$J(X)_2 \subset \text{Pic}^0(X)$$

be the group of holomorphic line bundles on $X$ of order two; it is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{\otimes 2g}$, where $g = \text{genus}(X)$.

Let

$$\mathcal{C}_X$$

be the space of all isomorphism classes of holomorphic $\text{SO}(r, \mathbb{C})$-bundles on $X$ equipped with a holomorphic connection. So $\mathcal{C}_X$ in (4.1) parametrizes isomorphism classes of pairs $(V, B_V)$, where $V$ is a holomorphic vector bundle on $X$ of rank $r$ with $\bigwedge^r V = \mathcal{O}_X$, and $B_V \in H^0(X, \text{Sym}^2(V^*))$ is a fiberwise nondegenerate symmetric bilinear form on $V$. We recall that a holomorphic connection on $(V, B_V)$ is a holomorphic connection $D_V$ on $V$ such that

$$\partial B_V(s, t) = B_V(D_V(s), t) + B_V(s, D_V(t))$$

for all locally defined holomorphic sections $s$ and $t$ of $V$. Let

$$\mathfrak{P}(X)$$

be the space of all projective structures on $X$; see [4, 11] for projective structures on $X$. Then, one has the following correspondence between generalized $\text{SO}(2n, \mathbb{C})$-opers and geometric structures.

**Theorem 4.3** First, assume that the integer $r = n/(m + 1)$ is odd. There is a canonical bijection between $\mathcal{O}_X(n, m)$ and the Cartesian product

$$\mathcal{C}_X \times \mathfrak{P}(X) \times \left( H^0(X, K_X^{\otimes(m+1)}) \oplus \left( \bigoplus_{i=2}^m \bigwedge^{2i} H^0(X, K_X^{\otimes i}) \right) \right) \times J(X)_2. \quad (4.1)$$

If $r$ is even, then there is a canonical bijection between $\mathcal{O}_X(n, m)$ and the Cartesian product

$$\mathcal{C}_X \times \mathfrak{P}(X) \times \left( H^0(X, K_X^{\otimes(m+1)}) \oplus \left( \bigoplus_{i=2}^m \bigwedge^{2i} H^0(X, K_X^{\otimes i}) \right) \right). \quad (4.2)$$

**Proof** Assume that $r = n/(m + 1)$ is an odd integer. Take an element

$$(\alpha, \beta, \gamma, \delta, \mathcal{L}) \quad (4.2)$$

in

$$\mathcal{C}_X \times \mathfrak{P}(X) \times \left( H^0(X, K_X^{\otimes(m+1)}) \oplus \left( \bigoplus_{i=2}^m \bigwedge^{2i} H^0(X, K_X^{\otimes i}) \right) \right) \times J(X)_2,$$

such that
\[ \alpha = (V, B_V, D_V) \in \mathcal{C}_X, \text{ where } (V, B_V) \text{ is a holomorphic } SO(r, C)\)-bundle on } X \\
\text{equipped with a holomorphic connection } D_V, \\
\beta \text{ is a projective structure on } X, \\
\gamma \text{ is a holomorphic section} \\
\gamma \in H^0(X, K^{\otimes (m+1)}_X), \quad (4.3) \\
\delta \in \bigoplus_{i=2}^m H^0(X, K^{\otimes 2i}_X), \text{ and} \\
\mathcal{L} \text{ is a holomorphic line bundle on } X \text{ of order two.} \\

Using [6, Theorem 4.6], the triple \((\alpha, \beta, \delta)\) produces the following:

- a nondegenerate holomorphic symmetric bilinear form \(B_1\) on \(J^{2m}(V \otimes K^{-\otimes m}_X)\), and
- a holomorphic connection \(D_1\) on \(J^{2m}(V \otimes K^{-\otimes m}_X)\) that preserves the bilinear form \(B_1\).

Furthermore, the triple \((J^{2m}(V \otimes K^{-\otimes m}_X), B_1, D_1)\), together with the filtration of \(J^{2m}(V \otimes K^{-\otimes m}_X)\) given by

\[ 0 = A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_{2m} \subset A_{2m+1} = J^{2m}(V \otimes K^{-\otimes m}_X), \quad (4.4) \]

where \(A_i\) is the kernel of the natural projection \(J^{2m}(V \otimes K^{-\otimes m}_X) \longrightarrow J^{2m-i}(V \otimes K^{-\otimes m}_X)\), define a generalized \(B\)-oper (see [6, Definition 2.11, Theorem 4.6]).

Now, consider the holomorphic vector bundle

\[ E := J^{2m}(V \otimes K^{-\otimes m}_X) \oplus V \]
on \(X\). Note that, it is equipped with nondegenerate holomorphic symmetric bilinear form \(B_1 \oplus B_V\). The holomorphic connection \(D_1\) on \(J^{2m}(V \otimes K^{-\otimes m}_X)\) and the holomorphic connection \(D_V\) on \(V\) together produce the holomorphic connection \(D_1 \oplus D_V\) on \(J^{2m}(V \otimes K^{-\otimes m}_X) \oplus V = E\). This connection \(D_1 \oplus D_V\) on \(E\) evidently preserves the bilinear form \(B_1 \oplus B_V\) on \(E\).

Using the holomorphic connection \(D_1 \oplus D_V\) on \(E\) and the section \(\gamma\) in (4.3), we shall construct another holomorphic connection on \(E\). Since \(E = J^{2m}(V \otimes K^{-\otimes m}_X) \oplus V\), using the filtration in (4.4), we have

\[ \text{Hom}(V, V \otimes K^{-\otimes m}_X) = \text{Hom}(V, A_1) \subset \text{Hom}(V, J^{2m}(V \otimes K^{-\otimes m}_X)); \quad (4.5) \]

in (4.4), note that \(A_1 = V \otimes K^{-\otimes m}_X\). Similarly, we have

\[ \text{Hom}(V \otimes K^{-\otimes m}_X, V) = \text{Hom}(A_{2m+1}/A_{2m}, V) \subset \text{Hom}(J^{2m}(V \otimes K^{-\otimes m}_X), V); \quad (4.6) \]
in (4.4), note that

\[ A_{2m+1}/A_{2m} = V \otimes K^{-\otimes m}_X, \]
so the quotient map \(A_{2m+1} \longrightarrow A_{2m+1}/A_{2m}\) produces the inclusion map

\[ \text{Hom}(A_{2m+1}/A_{2m}, V) \hookrightarrow \text{Hom}(J^{2m}(V \otimes K^{-\otimes m}_X), V). \]

On the other hand,
\[ \text{Hom}(V, J^{2m}(V \otimes K_X^{-\otimes m})) \oplus \text{Hom}(J^{2m}(V \otimes K_X^{-\otimes m}), V) \]
\[ \subset \text{End}(J^{2m}(V \otimes K_X^{-\otimes m}) \oplus V) = \text{End}(E). \]

Hence, from (4.5), (4.6), we conclude that
\[ (\text{End}(V) \otimes K_X^{-\otimes m}) \oplus (\text{End}(V) \otimes K_X^{-\otimes m}) = \text{Hom}(V, V \otimes K_X^{-\otimes m}) \oplus \text{Hom}(V \otimes K_X^{-\otimes m}, V) \]
\[ \subset \text{Hom}(V, J^{2m}(V \otimes K_X^{-\otimes m})) \oplus \text{Hom}(J^{2m}(V \otimes K_X^{-\otimes m}), V) \subset \text{End}(E). \]  
(4.7)

From (4.7), it follows immediately that
\[ (\text{Id}_V \otimes \gamma, -\text{Id}_V \otimes \gamma) \in H^0(X, \text{End}(E) \otimes K_X), \]
where \( \gamma \) is the section in (4.3).

Any two holomorphic connections on the holomorphic vector bundle \( E \) differ by a holomorphic section of \( \text{End}(E) \otimes K_X \). Since \( D_1 \oplus D_V \) is a holomorphic connection on \( E \), from (4.8), we conclude that
\[ D_E := (D_1 \oplus D_V) + (\text{Id}_V \otimes \gamma, -\text{Id}_V \otimes \gamma) \]
(4.9)
is a holomorphic connections on the holomorphic vector bundle \( E \). Since the connection \( D_1 \oplus D_V \) on \( E \) preserves the bilinear form \( B_1 \oplus B_V \) on \( E \), from the construction of \( (\text{Id}_V \otimes \gamma, -\text{Id}_V \otimes \gamma) \in H^0(X, \text{End}(E) \otimes K_X) \) in (4.8), it follows that the connection \( D_E \) on \( E \) in (4.9) also preserves the bilinear form \( B_1 \oplus B_V \) on \( E \).

Using the filtration of \( J^{2m}(V \otimes K_X^{-\otimes m}) \) in (4.4), we shall construct a filtration of holomorphic subbundles on \( E \). Let
\[ 0 = A_0' \subset A_1' \subset A_2' \subset \cdots \subset A_{2m}' \subset A_{2m+1}' = E = J^{2m}(V \otimes K_X^{-\otimes m}) \oplus V \]  
(4.10)
be the filtration, where \( A_i' = A_i \oplus 0 \) for all \( 0 \leq i \leq m \) and \( A_i' = A_i \oplus V \) for all \( m+1 \leq i \leq 2m+1 \).

From the above, we have that the holomorphic vector bundle \( E \), the bilinear form \( B_1 \oplus B_V \), the filtration \( \{A_i'\}_{i=0}^{2m+1} \) in (4.10), and the holomorphic connection \( D_E \) in (4.9) together define a generalized \( \text{SO}(2n, \mathbb{C}) \)-oper.

In view of Lemma 4.2, the above generalized \( \text{SO}(2n, \mathbb{C}) \)-oper
\[ (E, B_1 \oplus B_V, \{A_i'\}_{i=0}^{2m+1}, D_E) \]
and the line bundle \( \mathcal{L} \) in (4.2) together produce a generalized \( \text{SO}(2n, \mathbb{C}) \)-oper. It is evident that this generalized \( \text{SO}(2n, \mathbb{C}) \)-oper is an element of \( \mathcal{O}_X(n, m) \).

Now, assume that the integer \( r \) is even. Let \( V \) be a holomorphic vector bundle on \( X \) of rank \( r \), and let \( B_V \in H^0(X, \text{Sym}^2(V^*)) \) is a fiberwise nondegenerate symmetric bilinear form on \( V \). Then, we have \( \bigwedge^r V = \mathcal{O}_X \), because \( r \) is even. Therefore, if \( (V, B_V) \) is an holomorphic \( \text{SO}(r, \mathbb{C}) \)-bundle, then for any \( \mathcal{L} \in \mathcal{J}(X)_2 \), that pair \( (V \otimes \mathcal{L}, B_V \otimes \rho) \), where \( \rho : \mathcal{L}^{\otimes 2} \rightarrow \mathcal{O}_X \) is an isomorphism [as in (3.16)], is again a holomorphic \( \text{SO}(r, \mathbb{C}) \)-bundle. Hence, in the case of even \( r \), when we consider \( \mathcal{C}_X \), tensoring with line bundles of order two is already taken into account, so we no longer need to take line bundles of order two separately (which was needed in the previous case of \( r \) being odd).

Therefore, the above constructions identify \( \mathcal{O}_X(n, m) \) with
We shall now describe the reverse construction. Again, first assume that the integer \( r \) is odd.

Let \( \mathcal{O} \) be a generalized \( \text{SO}(2n, \mathbb{C}) \)-oper on \( X \). Consider the decomposition in Lemma 3.2. As noted in (3.13), we have that \( \mathcal{F} \oplus S = E \) in Lemma 3.2. As noted in (3.13), we have that \( B_0 = B_F \oplus B_S \).

Moreover, from Corollary 3.4, the connection \( D^\mathcal{F} \) (respectively, \( D^S \)) on \( \mathcal{F} \) (respectively, \( S \)) preserves the bilinear form \( B_\mathcal{F} \) (respectively, \( B_S \)). The vector bundle \( \mathcal{F} \) has a filtration of holomorphic subbundles [see (3.19)], which we shall denote by \( \mathcal{F} \). Recall that the isomorphism \( f'_{2m} \) in (3.10) takes the filtration \( \mathcal{F} \) to the filtration of \( J_{2m}(Q) \) given by the exact sequences in (3.20).

Note that, \( (\mathcal{F}, B_\mathcal{F}, \mathcal{F}, D^\mathcal{F}) \) satisfies all the conditions needed to define a generalized \( B \)-oper (see [6, Definition 2.11]) except possibly the only condition

\[
\det \mathcal{F} = \mathcal{O}_X.
\]

In any case,

\[
\mathcal{L} := \det \mathcal{F} \in J(X)_2. \tag{4.11}
\]

For the vector bundle \( \mathcal{F}' := \mathcal{F} \otimes \mathcal{L} \), we have \( \det \mathcal{F}' = \mathcal{O}_X \).

Let \( \rho : \mathcal{L} \otimes \mathcal{L} \rightarrow \mathcal{O}_X \) be an isomorphism [as in (3.16)]. Define the nondegenerate symmetric bilinear form

\[
B'_\mathcal{F} := B_\mathcal{F} \otimes \rho
\]
on \( \mathcal{F}' = \mathcal{F} \otimes \mathcal{L} \). Tensoring the above filtration \( \mathcal{F} \), of \( \mathcal{F} \) by \( \mathcal{L} \), we get a filtration of holomorphic subbundles of \( \mathcal{F} \); this filtration of \( \mathcal{F}' \) will be denoted by \( \mathcal{F}' \). The holomorphic connection \( D^\mathcal{F} \) on \( \mathcal{F} \) and the canonical connection \( D^\mathcal{L} \) on \( \mathcal{L} \) in (3.17) together define a holomorphic connection

\[
D'_\mathcal{F} := D^\mathcal{F} \otimes \text{Id}_\mathcal{L} + \text{Id}_\mathcal{F} \otimes D^\mathcal{L}
\]
on \( \mathcal{F}' \) [as done in (3.18)].

Now, \( (\mathcal{F}', B'_\mathcal{F}, \mathcal{F}', D'_\mathcal{F}) \) is a generalized \( B \)-oper [6]. Therefore, from [6, Theorem 4.6], we obtain a triple

\[
(\alpha, \beta, \delta) \in \mathcal{C}_X \times \mathcal{P}(X) \times \left( \bigoplus_{i=2}^m H^0(X, K_X^{\otimes 2i}) \right) \tag{4.12}
\]

associated to \( (\mathcal{F}', B'_\mathcal{F}, \mathcal{F}', D'_\mathcal{F}) \).
Next, consider the second fundamental form for the subbundle $\mathcal{S} \subset \mathcal{E} = \mathcal{F} \oplus \mathcal{S}$ for the connection $D$ on $E$. Let

$$S(D; \mathcal{S}) \in H^0(X, \text{Hom}(\mathcal{S}, \mathcal{F}) \otimes K_X)$$

be the second fundamental form for the subbundle $\mathcal{S}$ for the connection $D$ on $E$. From Definition 4.1 and (3.23), we know that

$$S(D; \mathcal{S}) \in H^0(X, \text{Hom}(\mathcal{S}, F_1) \otimes K_X) = H^0(X, \text{Hom}(\mathcal{S}, Q \otimes K_X^{\otimes (2m+1)}))$$

$$\subset H^0(X, \text{Hom}(\mathcal{S}, F) \otimes K_X);$$

we note that $F_1 = Q \otimes K_X^{\otimes 2m}$, this follows from the fact that isomorphism $f_{2m}$ in (3.10) takes the filtration $\mathcal{F}'_i$ to the filtration of $J^m(Q)$ given by the exact sequences in (3.20). Since $\mathcal{S} = Q \otimes K_X^m$ (see Definition 4.1), we have

$$S(D; \mathcal{S}) \in H^0(X, \text{End}(\mathcal{S}) \otimes K_X^{\otimes (m+1)}).$$

We recall from Definition 4.1 that $S(D; \mathcal{S}) = \text{Id}_\mathcal{S} \otimes \phi$, where $\phi \in H^0(X, K_X^{\otimes (m+1)}).$ Then, we have

$$(\alpha, \beta, \phi, \delta, \mathcal{L}) \in C_X \times \mathcal{P}(X) \times \left( H^0(X, K_X^{\otimes (m+1)}) \oplus \bigoplus_{i=2}^m H^0(X, K_X^{\otimes 2i}) \right) \times J(X)_2,$$

where $(\alpha, \beta, \phi, \delta)$ is constructed in (4.12) and $\mathcal{L}$ is the line bundle in (4.11). It is straightforward to check that the two constructions are inverses of each other.

When the integer $r$ is even, the above reverse construction is simpler because in that case, $(\mathcal{F}', \mathcal{B}_\mathcal{F}', \mathcal{F}_\mathcal{F}', \mathcal{D}_\mathcal{F}')$ is already a generalized $B$-oper, so the construction of $(\mathcal{F}', \mathcal{B}_\mathcal{F}', \mathcal{F}_\mathcal{F}', \mathcal{D}_\mathcal{F}')$ from it is not needed. $\square$

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