Fiber Floer cohomology and conormal stops

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Introduction

Let $S$ be a closed, orientable and spin $n$-manifold. Consider the Liouville manifold $(T^*S, \lambda)$, where $\lambda := pdq$ and $\omega := d\lambda$ in coordinates $(q, p) \in T^*S$. The Liouville vector field is defined as $\omega(X, -) = \lambda$.

In this case $X = p\partial_p$.

- $\omega$ is symplectic ($\omega$ closed and non-degenerate)
- $T^*S$ can be exhausted by domains with convex boundary (with respect to $X$)
Introduction

The \((2n - 1)\)-dimensional boundary \(ST^*S\) is a contact manifold with contact form \(\alpha := \lambda|_{ST^*S}\).

\[\alpha \wedge d\alpha \wedge (n-1) \neq 0.\]

The Reeb vector field \(R\) is defined by

- \(d\alpha(R, -) = 0\)
- \(\alpha(R) = 1\)

In this case \(R = p\partial_q\).
Introduction

Let $F = T_\xi^* S$ be a cotangent fiber. The wrapped Floer cochain complex $CW^*(F, F)$ is generated by

- Reeb chords of $\partial F \subset ST^* S$
- One generator corresponding to a minimum of Morse function on $F$

The differential $\mu^1 : CW^*(F, F) \rightarrow CW^*(F, F)$ is defined by counting holomorphic disks with 2 boundary punctures.

Maps $\mu^k : CW^*(F, F)^\otimes k \rightarrow CW^*(F, F)$ counting holomorphic disks with $k + 1$ boundary punctures turn $CW^*(F, F)$ into an $A_\infty$-algebra.
Abbondandolo–Schwarz (2008) and Abouzaid (2012)

There is an isomorphism $HW^*(F, F) \cong H_{-*}(\Omega S)$ which intertwines the triangle product $\mu^2$ on $HW^*(F, F)$ with the Pontryagin product on $H_{-*}(\Omega S)$. 
Introduction

Let $K \subset S$ be a submanifold and consider its unit conormal bundle

$$
\Lambda_K := \{(x, p) \mid x \in K, \ |p| = 1, \ \langle p, T_x K \rangle = 0 \} \subset ST^* S.
$$

The smooth topology of $K$ can be studied via $\Lambda_K$:

- **Legendrian contact homology (LCH)** of $\Lambda_K$ is related to the Alexander polynomial of $K \subset \mathbb{R}^3$ and furthermore detects the unknot and torus knots (Ng 2008, Ekholm–Ng–Shende 2016, Cieliebak–Ekholm–Latschev–Ng 2016)

- Refined version of LCH of $\Lambda_K$ is a complete knot invariant of $K \subset \mathbb{R}^3$ (ENS 2016)

- Legendrian isotopy class of $\Lambda_K$ is a complete knot invariant of $K \subset \mathbb{R}^3$ (Shende 2016, ENS 2016)
Main results

Questions

1. How is $HW^*(F, F)$ affected by adding a stop at $\Lambda_K$?
2. Does the Legendrian isotopy class of $\Lambda_K$ know about the underlying smooth codimension 2 knot $K \subset S^n$ for $n \geq 4$?

The based loop space $\Omega S$ is homotopy equivalent to the space $BS$ of smooth piecewise geodesic loops.

Theorem A

Let $M_K := S \setminus K$. There is an isomorphism of $A_\infty$-algebras

$$\Psi : CW_{\Lambda_K}^*(F, F) \to C_{\text{cell}}^*(BM_K).$$

In particular we have $HW_{\Lambda_K}^*(F, F) \cong H_{-*}(\Omega M_K)$. 
Main results

**Theorem B**
\[ \Psi \text{ induces an isomorphism of } \mathbb{Z}[\pi_1(M_K)]\text{-modules} \]
\[ HW^*_K(F, F) \longrightarrow H_*(-(\Omega_\xi M_K)). \]

**Application**
For certain codimension 2 knots \( K \subset S^n \), \( HW^*_K(F, F) \) is related to the Alexander invariant of \( K \).

**Theorem C**
Let \( n = 5 \) or \( n \geq 7 \). Then there exists a codimension 2 knot \( K \subset S^n \) with \( \pi_1(M_K) \cong \mathbb{Z} \) such that \( \Lambda_K \cup \Lambda_x \not\cong \Lambda_{\text{unknot}} \cup \Lambda_x \) where \( x \in M_K \).
Wrapped Floer cohomology without Hamiltonian

Let $CW^*(F, F)$ denote the wrapped Floer cochain complex of the fiber $F \subset T^*S$.

Generators

- Reeb chords of $\partial F \subset ST^*S$
- One generator corresponding to a minimum of Morse function on $F$

Grading

(Negative of the) Conley–Zehnder index
Wrapped Floer cohomology without Hamiltonian

Parallel copies of \( F \)

Choose family of positive Morse functions

\[
G_k : F \rightarrow \mathbb{R} \quad g_k : \partial F \rightarrow \mathbb{R}
\]

- \( G_k \) has one minimum on \( F \) and no other critical points
- \( F_k = \) flow of \( F \) by the Hamiltonian vector field associated to \( G_k \)
- \( \partial F_k \) is a small pushoff of \( \partial F \) in the positive Reeb direction
Wrapped Floer cohomology without Hamiltonian

\( \mathcal{A}_\infty \)-structure

\[ \mu^k : CW^*(F, F') \otimes k \to CW^*(F, F') \]
counts solutions to the equation

\[
\overline{\partial} J u = \frac{1}{2} \left( \frac{\partial u}{\partial s} + J(u) \frac{\partial u}{\partial t} \right) = 0
\]
with \( k \) inputs, 1 output and switching boundary conditions.

No covers can be multiply covered! Boundary bubbling is precluded for topological reasons.

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Wrapped Floer cohomology without Hamiltonian

Anchoring

Disks are allowed to have internal punctures asymptotic to Reeb orbits. Anchoring means that we cap off the Reeb orbits by holomorphic planes.

Transversality of holomorphic planes requires abstract perturbations.
Let $K \subset S$ be any submanifold and consider its unit conormal bundle $\Lambda_K \subset ST^*S$.

Attach a handle along $\Lambda_K$ modeled on $D_\varepsilon T^*([0, \infty) \times \Lambda_K)$. $CW^*_{\Lambda_K}(F, F)$ is defined as $CW^*(F, F)$ computed in $W_K$.
The complement Lagrangian

\[ \text{Skel}(W_K) = S \cup L_K, \text{ and } S \cap L_K = K \text{ is a clean intersection.} \]

Lagrange surgery along \( K \) gives an exact Lagrangian \( M_K \cong S \setminus K \)

\[ [0, \infty) \times A_K \]
The complement Lagrangian

Metric on $M_K$

Pick a generic metric $g$ in $S$ away from $N(K)$. Then define

$$h = \begin{cases} 
  dt^2 + f(t) \, g|_{\partial N(K)} & \text{in } [0, \infty) \times \partial N(K) \\
  g & \text{in } S \setminus N(K)
\end{cases}$$
Moduli space of half strips

Let

- $k \geq 1$
- $a = a_1, \ldots, a_k$ generators of $CW^*_{A_K}(F, F')$

Consider $\mathcal{M}(a)$ moduli space of holomorphic disks
Moduli space of half strips

The moduli space $\mathcal{M}(a)$ has the following properties:

- Transversely cut out.
- Compact after adding broken disks. Denote the compactification by $\overline{\mathcal{M}}(a)$.
- $\dim \overline{\mathcal{M}}(a) = -1 + k - \sum_{i=1}^{k} |a_i|$
- There exists a family of fundamental chains $[\overline{\mathcal{M}}(a)]$, which is compatible with orientations and the boundary stratification.
The evaluation map

\[ \text{ev}: \overline{\mathcal{M}}(a) \longrightarrow \Omega_{\xi}M_{K} \]

\[ u \longmapsto \gamma. \]

Restriction of \( u \) to the boundary arc between the punctures \( \zeta_{\pm} \)
The $A_\infty$-homomorphism

We define

$$
\Psi_k : \text{CW}_{A_K}^*(F, F)^\otimes k \longrightarrow C^\text{cell}_{-*}(\Omega_\xi M_K)
$$

$$
a_k \otimes \cdots \otimes a_1 \longmapsto \text{ev}_*([\overline{M}(a)])
$$

**Proposition**

$$
\Psi := \{\Psi_k\}_{k=1}^\infty \text{ is an } A_\infty\text{-homomorphism}.
$$
The $A_{\infty}$-homomorphism

Proof.
We look at the boundary of $\overline{M}(a)$ of dimension $d$. The codimension 1 boundary is covered by strata of the form:

(a)

Strata of the form (a) contributes to $\partial \Psi_k$ with terms of the form

$$\sum_{r+s+t=k} \Psi_{r+1+t}(\text{id} \otimes r \otimes \mu^s \otimes \text{id} \otimes t).$$
The $A_\infty$-homomorphism

Proof. Strata of the form (b) contributes to $\partial \Psi_k$ with terms of the form

$$\sum_{k_1 + k_1 = k} P(\Psi_{k_2} \otimes \Psi_{k_1}).$$

All the terms together make up the codimension 1 boundary of a compact oriented manifold, so

$$\partial \Psi_k = \sum_{r+s+t = k} \Psi_{r+1+t}(\id^\otimes r \otimes \mu^s \otimes \id^\otimes t) + \sum_{k_1 + k_1 = k} P(\Psi_{k_2} \otimes \Psi_{k_1}).$$
Correspondence between generators

Each geodesic in $M_K$ of index $\lambda$ corresponds to a generator of $C^\text{cell}_{-\ast}(BM_K)$ in degree $\lambda$.

Length filtrations

- For $c \in CW^*_\Lambda K (F, F')$ a Reeb chord, define $\alpha(c) := \int_c \lambda$
- For $\sigma \in C^\text{cell}_{-\ast}(BM_K)$, define $\alpha(\sigma) := \max_{x \in [0,1]} L(\sigma(x))$, where
  
  $$L(\gamma) = \int_0^1 |\gamma(t)| \, dt.$$  

Lemma

*There is a one-to-one correspondence*

$$\left\{ \text{Reeb chords of } \partial F \text{ of index } -\lambda \text{ and action } A \right\} \leftrightarrow \left\{ \text{Geodesic loops in } M_K \text{ of index } \lambda \text{ and length } A \right\}$$
Correspondence between generators

Proof.

We consider the “trivial holomorphic strip” over a Reeb chord $a$:

$T_{(q,p)} \text{im } u_0 = \text{span } \{ \text{Reeb, Liouville} \}$

Consider the 2-form $d\beta$ where

$$\beta := \text{cutoff}(|p|) \cdot \frac{pdq}{|p|} .$$

Integrating $d\beta$ over any holomorphic half strip $u$ asymptotic to $a$ yields $a(a) - L(\gamma) \geq 0$. In the case of $u_0$ we have equality. \qed

\[ a \]

\[ u_0 \]

\[ \xi \]

\[ \gamma \]

\[ \xi \]
Towards the proof of the main theorem

Lemma

\( u_0 \) is transversely cut out

Corollary

For any \( a \in CW^*_{\Lambda K}(F, F) \) we have \( a(a) = a(\Psi_1(a)) \).

Proof of lemma.

Let \( v \in \ker D_{u_0} \) and define for \( \varepsilon > 0 \)

\[
\begin{cases}
    u_\varepsilon := \exp_{u_0}(\varepsilon v) \\
    \gamma_\varepsilon := ev(u_\varepsilon)
\end{cases}
\]
Towards the proof of the main theorem

Proof of lemma.

Then

1. From \( u_0: a(a) = L(\gamma_0) \)
2. Can show: \( a(a) - L(\gamma_\varepsilon) > 0 \)

This implies

\[
L(\gamma_\varepsilon) - L(\gamma_0) < 0 \implies E_{**}(\pi_*v, \pi_*v) < 0.
\]

By unique continuation, the restriction

\[
\pi_*: \ker D_{u_0} \longrightarrow \{ w \in T_{\gamma}B M_K \mid E_{**}(w, w) < 0 \},
\]

is injective, and finally \( \dim \ker D_{u_0} \leq \text{ind} \gamma_0 = \text{ind} D_{u_0} \).
Outline of proof of the main theorem

**Lemma**

*The evaluation*

\[ \text{ev}: \overline{M}(a) \rightarrow BM_K, \]

*is a submersion at* \( u_0 \in \overline{M}(a) \).

**Theorem**

*For any* \( A > 0 \), *there is an isomorphism of* \( A_\infty \)-*algebras*

\[ F_A CW^*_{A_K}(F, F) \cong \mathcal{F}_AC_{cell}^{cell}(BM_K). \]

*Finally take colimits as* \( A \rightarrow \infty \), *and we obtain the main theorem.*
\[ \mathbb{Z}[\pi_1(M_K)] \text{-module structure in cohomology} \]

In cohomology we can equip \( HW^*_{\Lambda_K}(F, F) \) and \( H_*(\Omega M_K) \) with \( \mathbb{Z}[\pi_1(M_K)] \)-module structures.

\[
\pi_1(M_K) \times H_*(\Omega M_K) \longrightarrow H_*(\Omega M_K)
([\gamma], \sigma) \longmapsto P(\sigma \otimes \sigma_{\gamma})
\]

\[
\pi_1(M_K) \times HW^*_{\Lambda_K}(F, F) \longrightarrow HW^*_{\Lambda_K}(F, F)
([\gamma], a) \longmapsto \mu^2(a \otimes a_{\gamma})
\]
\[ \mathbb{Z}[\pi_1(M_K)] \text{-module structure in cohomology} \]

**Theorem B**

\[ \Psi \text{ induces an isomorphism of } \mathbb{Z}[\pi_1(M_K)] \text{-modules} \]

\[ HW^*_A(F, F) \longrightarrow H_{-\ast}(\Omega M_K). \]
The Alexander invariant

We now let $n = 5$ or $n \geq 7$ and $K \subset S^n$ a codimension 2 knot.

- Alexander invariant of $K$ is $H_*(\widetilde{M}_K)$ viewed as a $\mathbb{Z}[t^{\pm 1}]$-module
- It is used to show non-triviality of certain knots $K$ with $\pi_1(M_K) \cong \mathbb{Z}$.

Plumbing of spheres

Let $n = p + q + 1$ where $p \geq 2$, $q \geq 4$ or $p = q = 2$. Consider $S^p, S^q \subset S^n$. 
The Alexander invariant

Plumbing of spheres

Define $\mathcal{A}$ to be the boundary of the plumbing above.
The Alexander invariant and partially wrapped Floer cohomology

Theorem C
Let \( n = p + q + 1 \) with \( p \) and \( q \) as above. Then there exists a codimension 2 knot \( K \subset S^n \) with \( \pi_1(M_K) \cong \mathbb{Z} \) such that \( \Lambda_K \cup \Lambda_x \not\cong \Lambda_{\text{unknot}} \cup \Lambda_x \) where \( x \in M_K \).

Sketch of proof.
By using the Leray–Serre spectral sequence, we can compute that in either case

\[
H_p(\tilde{M}_K) \cong \mathbb{Z}[t^{\pm 1}]/(t - 1) \otimes_{\mathbb{Z}[t^{\pm 1}]} HW_{\Lambda_K}^{1-p}(F, F).
\]

Implies \( HW_{\Lambda_K}^{1-p}(F, F) \not\cong HW_{\Lambda_{\text{unknot}}}^{1-p}(F, F) \) and hence \( \Lambda_K \cup \Lambda_x \not\cong \Lambda_{\text{unknot}} \cup \Lambda_x \) where \( x \in M_K \).
Thank you!