Finite Element Model for Free Vibration Analyses of FG-CNT Reinforced Composite Beams using Refined Shear Deformation Theories

Surojit Biswas¹ and Priyankar Datta¹,²

¹ Mechanical Engineering Department, Jadavpur University, Kolkata-700032, India
² Corresponding author: priyankardatta.mech@jadavpuruniversity.in; priyomech@gmail.com

Abstract. The present article deals with the free vibration of functionally graded carbon nanotube reinforced composite (FG-CNTRC) beams employing various refined deformation theories and validates the accuracy and feasibility of these proposed theories. The theories involved are the first order shear deformation theory (FSDT) and other refined theories involving additional higher order terms. Carbon nanotubes (CNTs) are assumed to be oriented along the axis of the beam. Uniform and three types of different functionally graded (FG) distributions of CNTs through the thickness of the beam are considered. The rule of mixture is used to describe the effective material properties of the beams. The governing equations are derived using Hamilton’s principle and solved using the finite element method (FEM). A FEM code is compiled in MATLAB considering a C⁰ finite element. The influences of different key parameters such as CNT volume fraction, distribution type of CNTs, boundary conditions and slenderness ratio on the natural frequencies of FG-CNTRC beams are investigated. It can be concluded that the above-mentioned parameters have significant influence on the free vibration of the beam and the accuracy of the proposed refined theories is good.

Keywords - carbon nanotube, finite element method, functionally graded carbon nanotube reinforced composite, shear deformation theories

1. Introduction
The use of composite materials for weight-dependent applications in spacecrafts, marine and automotive has escalated exponentially over the last few decades. Shortly after CNTs were observed by Iijima [1], researchers have been continuously investigating its properties and also trying to improve its method of synthesis. CNTs have gained popularity as reinforcement in composites due to its extraordinarily high strength, stiffness and aspect ratio with lower density. The superior electrical [2], mechanical [3–4], optical [5] and thermal [6–7] properties of CNTs have drawn the attention of researchers over past three decades. Yu et al. [3] conducted tensile test on a set of 19 multiwalled carbon nanotubes (MWCNTs). They loaded each MWCNTs individually and measured the tensile strength which varied from 11 to 63 GPa to identify the elastic modulus (E) of the outermost layer from the individual stress-strain curves and the value ranged from 270 to 950 GPa. Salvetat et al. [4] experimentally determined that CNTs have extremely high strength, flexibility and resilience and the result is consistent with the theoretical predictions. Wong et al. [8] used a cantilever beam model in which silicon carbide (SiC) nanorods (NRs) and MWCNTs were bent using an atomic force microscope (AFM) tip. The MWCNTs found to be two times stiffer than SiC NRs and the toughness is found to be 5 to 10 times compared to SiC NRs. CNTs
have thermal conductivity as high as diamond crystals [6]. Experimental observations found thermal conductance and thermal conductivity of a single-walled carbon nanotube (SWCNT) of length 2.6 µm and diameter 1.7 nm to be around 2.4 nW/K and 3500 Wm⁻¹K⁻¹ at room temperature [7]. An isolated CNT at room temperature is taken by Berber et al. [9] to determine the thermal conductivity of the same from molecular dynamics (MD) simulation. The simulation results indicated thermal conductivity of 6600 Wm⁻¹K⁻¹ which is exceptionally high like that of a high-purity diamond. Overney et al. [10] used a cantilever beam model to determine the elastic modulus of SWCNT and reported it to be 1,500 GPa which is 10 times greater than that of a similar beam made up of Iridium. Lu [11] used an empirical model to calculate moduli of SWCNTs and MWCNTs. Young’s modulus, shear modulus and bulk modulus are found to be approximately 1 TPa, 0.45 TPa and 0.74 TPa, respectively. It is also observed that elastic properties are independent of size and shape of CNTs. The results confirmed that CNTs are even stronger than diamond. Hata et al. [12] successfully synthesized SWCNTs of 2.5 mm height using chemical vapor deposition (CVD). Using MD simulations, Arash et al. [13] demonstrated how Young’s modulus of Poly methyl methacrylate (PMMA) increased 16 times when reinforced with CNTs compared to pure PMMA. Qian et al. [14] conducted a tensile test on polystyrene matrices to measure the influence of CNT reinforcement. Addition of 1 wt% MWCNTs improves elastic modulus by 36%–42% whereas the break stress increases around ~25%. Wuite and Adali [15] examined stresses developed in nanocomposite reinforced beams for multi scale model. Their results indicated that a small amount of CNT addition can lead to noteworthy increase in beam stiffness. CNTs can be reinforced along with other reinforcements like clay particles in hybrid composites to enhance its mechanical properties [16]. Due to overwhelming properties of CNTRC, many scholars have investigated its static and dynamic response. Kanu et al. [17] compared various applications, fabrication methods, cost and time of manufacturing of CNTRC. Jain et al. [18] presented various case studies to capture the widespread applications of CNTs in multiple areas like electronics, electrical, biomedical, agricultural, pollution control, etc.

Scientific community has shown huge interest to explore functionally graded materials (FGMs) since it was first proposed by Japanese scientists in the mid-1980s [19,20]. FGMs are a class of advanced materials, consisting of two (or more) different materials, fabricated in a specific way to have gradation of composition with dimensions [19]. This helps to achieve continuous variation of mechanical properties in a preferred direction. Kanu et al. [21] elaborately discussed about the various manufacturing processes of FGMs and its response during fracture. Recent developments in numerical and analytical methods to compute stress, vibration and buckling of FGMs have been reviewed. Drawing inspiration from the idea of FGMs, CNTs are embedded in an FG pattern which has resulted in a new class of composites i.e., FG-CNTRC. Beams are indispensable structural elements due to its wide application both in large and micro structures. They are commonly used in mechanical, civil, aerospace and marine structures. Various deformation theories are available to analyse dynamic behaviour of such structural elements. Understanding the feasibility and accuracy of various theories is of interest to both researchers and engineers. Nuttawit and Variddhi [22] analytically found solutions for vibration of CNTRC beams with elastic foundation and simply supported boundary. Different patterns of reinforcements are considered and the results are obtained employing multiple deformation theories. The result showed that the frequencies are inversely proportional to thickness ratios (L/h). Also, natural frequency is highest for the FG-X beam, eventually decreases in the order of UD, FG-V and O beams. The natural frequencies don’t vary significantly with various deformation theories. Rokni et al. [23] gave an exponential CNT distribution along the length of the FG-CNTRC micro-beam which maximize the natural frequency for a pre-set weight percent (wt.%) of MWCNTs. Penna et al [24] considered free vibrations of FG-CNTRC nano-beams with geometric imperfection and nonlinearity under initial pretension force. Equation of motions is derived with the help of stress-driven nonlocal integral model (SDM) for different boundary conditions. Yas and Samadi [25] investigated the effects of CNT volume fraction and pattern, foundation stiffness, aspect ratios and edge conditions on natural frequency of FG-CNTRC Timoshenko beam. For laminated FG-CNTRC beams, Vo-Duy [26] studied free vibration using FSDT. The results revealed that 0° orientation of CNT for all layers produces the largest frequency. With addition of layers, the frequency decreases for FG-X distribution but increases for FG-O and FG-V. However, the lamination sequence has negligible effect on the frequency. Thermal
environment is considered by Shen and Xiang [27] to carry out numerical studies on vibration of FG-CNTRC beams on Pasternak foundations with large deformations. Mohseni and Shakouri [28] employed Timoshenko beam theory to evaluate natural frequency of FG-CNTRC beams with variable thickness. Rokni et al. [29-30] developed FEM of a micro-beam to identify optimized CNT distribution for each vibration mode under different boundary conditions. A comparative study of nonlinear free vibration of FG-CNTRC beam employing FSDT and third-order shear deformation theory (TOSDT) was performed by Lin and Xiang [31]. Mohammadimehr et al. [32] scrutinized viscoelastically damped FG-CNTRC beams using MCST. The results revealed that micro composite beams reinforced with SWCNTs have higher natural frequency than that of MWCNTs. Kitipornchai et al. [33] presented free vibration of FG porous beams with graphene platelets (GPLs) as reinforcements. Formulations are derived using Timoshenko beam theory and the fundamental frequency is evaluated with the help of Ritz method. It is found that free vibrations can be most effectively improved by the symmetric and non-uniform distribution of both internal pores and GPLs. Momeni and Dehkordi [34] carried out frequency analysis on sandwich beam with flexible core and FG-CNT face sheets. Parametric studies of CNT volume fraction, types of CNT distribution, boundary conditions and geometry have been performed. Dabbagh et al. [35] introduced a new refined higher-order beam theory to study vibration of laminated hybrid nanocomposite beams. The beam is composed of polymer matrix reinforced with carbon fibres and CNTs. They presented both analytical and numerical solutions to the problem. Also, several recent research work can be found reporting the vibration control of both FG [36,37] structures and CNT reinforced [38-41] structures.

FG-CNTRC is considered to be a promising material because it can sustain at high pressures and in harsh conditions, resistant to wear and tear and relieves stress concentrations. Knowledge about the natural frequency of any structural element is a pre-requisite for design. Over the years FEM has been established as a robust numerical modelling tool for structural analysis. Also, for composite and FG structures several shear deformation theories have been developed to accurately predict the natural frequency and deformation of structures. Further, researchers have refined the available theories to improve the accuracy of computation, especially for thick beams, plates and shells. However, the literature review disclosed that most of the refined theories have the mathematical form which requires $C^0$ finite element modelling and eventually this increases the computation cost when implemented on large and complicated structures. In this article, the authors have presented some deformation theories by refining the FSDT which is suitable for structures with high aspect ratio and also requires less computational cost as the FE modelling can be done using $C^0$ finite elements. Various articles on free vibrations of different FG-CNTRC structures have been reviewed due to completeness of the topic. However, the present work is restricted to study the free vibration of FG-CNTRC beams only.

2. Mechanical Properties of FG-CNTRC Beam

The UD and FG distributions (pattern ‘X’, ‘V’ and ‘O’) of CNTs along the thickness of the beam (z-direction) are shown in figure 1. We have considered a prismatic beam of length $L$ and cross-section $b \times h$ defined along the x-, y- and z-axes, respectively as depicted in the figure 2. Density of CNTs remains unchanged while the volume fraction changes along the thickness of the beam. Irrespective of the type of distributions, the total amount of CNT volume fraction ($V_{cnt}$) is same and distributed across the thickness based on the following equations [25].

\[
\text{UD: } V_{cnt} = V_{tcnt} \\
\text{FG-X: } V_{cnt} = 4 \left( \frac{|z|}{h} \right) V_{tcnt} \\
\text{FG-V: } V_{cnt} = \left( 1 + 2 \left( \frac{z}{h} \right) \right) V_{tcnt} \\
\text{FG-O: } V_{cnt} = \left( 2 - 4 \left( \frac{|z|}{h} \right) \right) V_{tcnt}
\]
Polymer matrix is reinforced with SWCNTs along the length direction such that there is no discontinuity at the fiber-matrix interface. In the present discussion, the isotropic matrix considered is PMMA having the following material properties: $\nu_m = 0.3$, $\rho_m = 1190 \, \text{kg/m}^3$, $E_m = 2.5 \, \text{GPa}$. It is reinforced with armchair (10,10) SWCNTs having material properties $\nu_{12}^{cnt} = 0.19$, $G_{12}^{cnt} = 17.2 \, \text{GPa}$, $E_{11}^{cnt} = 600 \, \text{GPa}$, $E_{22}^{cnt} = 10 \, \text{GPa}$, $\rho^{cnt} = 1400 \, \text{kg/m}^3$. Effective material properties are obtained by using the rule of mixture. Effective Young’s modulus ($E_{11}$ and $E_{22}$), shear modulus ($G_{12}$), Poisson’s ratios ($\nu_{12}$ and $\nu_{21}$) and density ($\rho$) can be obtained as given below [25]:

$$E_{11} = \eta_1 V_{cnt} E_{11}^{cnt} + V_m E_m$$  \hspace{1cm} (5)

$$\frac{\eta_2}{E_{22}} = \frac{V_{cnt}}{E_{22}^{cnt}} + \frac{V_m}{E_m}$$  \hspace{1cm} (6)

$$\frac{\eta_3}{G_{12}} = \frac{V_{cnt}}{G_{12}^{cnt}} + \frac{V_m}{G_m}$$  \hspace{1cm} (7)

$$V_m = 1 - V_{cnt}$$  \hspace{1cm} (8)

$$\nu_{12} = V_{cnt} \nu_{12}^{cnt} + V_m \nu_m$$  \hspace{1cm} (9)

$$\nu_{21} = \nu_{12} \frac{E_{22}}{E_{11}}$$  \hspace{1cm} (10)

$$\rho = V_{cnt} \rho^{cnt} + V_m \rho_m$$  \hspace{1cm} (11)

where $E_{11}^{cnt}$, $E_{22}^{cnt}$ represent the Young’s modulus of CNTs along the axial and the transverse direction, respectively; $G_{12}^{cnt}$ is the shear modulus of CNTs. $E_m$ and $G_m$ denote Young’s modulus and shear modulus of the matrix, respectively. $\nu_{12}^{cnt}$ and $\nu_m$ represent Poisson’s ratio of CNT and matrix, respectively, and $\rho^{cnt}$ and $\rho_m$ are the respective mass densities. $V_m$ denotes volume fraction of the matrix. CNT efficiency parameters $\eta_j$ ($j=1,2,3$) mentioned in equations (5) to (7) incorporates the size effect on the material properties and are calculated from the results of MD simulations. Value for the CNT efficiency parameters for some particular CNT volume fractions are known from the available literature. $\eta_1 = 1.2833$, $\eta_2 = 1.0556$ for $V_{cnt} = 0.12$; $\eta_1 = 1.3414$, $\eta_2 = 1.7101$ for $V_{cnt} = 0.17$; $\eta_1 = 1.3238$, $\eta_2 = 1.7380$ for $V_{cnt} = 0.28$. Also, it has been assumed that $\eta_3 = \eta_2$ [25].

### 3. Theory and Formulation

The refined theories considered here includes ESDT, TSDT and HSDT. These theories have been developed by extrapolating the displacement field of FSDT. Unlike FSDT, these theories exclude shear correction factor ($k$) for computing the shear stresses. Under the assumption of these refined theories, displacement field for the FG-CNTRC beams can be expressed as
\[ u(x, z, t) = u_0(x, t) + z \theta_z(x, t) + f(z) \phi_z(x, t) \]  
\[ v(x, z, t) = z \theta_y(x, t) + f(z) \phi_y(x, t) \]  
\[ w(x, t) = w_0(x, t) \]

where \( u_0 \) and \( w_0 \) denote change in position of any point on the mid-plane \((z=0)\) along the x and z axes, respectively. \( \theta_z \) and \( \phi_z \) indicates the rotations of a transverse normal about the y axis, whereas, \( \theta_y \) and \( \phi_y \) are the rotations of the same about the x axis. The expressions for \( f(z) \) in the equations (12-14) are assumed to be as follows:

\[
f(z) = \begin{cases} 
0, & \text{FSDT} \\
 z \exp\left[-2\left(\frac{z}{h}\right)^2\right], & \text{ESDT} \\
 \frac{h}{\pi} \sin \frac{xz}{h}, & \text{TSDT} \\
 z \cos h \left(\frac{1}{h}\right) - h \sin h \left(\frac{z}{h}\right), & \text{HSDT} 
\end{cases}
\]

We have assumed \( k = 5/6 \) for FSDT and \( k = 1 \) for the refined theories. The reference coordinate system is located on the mid-plane \((z=0)\) in such a way that \( L \) and \( b \) units along x and y axes, respectively define the extent of the beam. Top and the bottom surfaces have \( z \) coordinates \( \frac{h}{2} \) and \( -\frac{h}{2} \) respectively. For the ease of computation, translational and rotational component of the generalized displacement variables are arranged into two vectors \( \{d_t\} \) and \( \{d_r\} \), respectively.

\[
\{d_t\} = \begin{bmatrix} u_0 & w_0 \end{bmatrix}^T \quad \text{and} \quad \{d_r\} = \begin{bmatrix} \theta_x & \theta_y & \phi_x & \phi_y \end{bmatrix}^T
\]

Generalized displacement vectors \( \{d_t\} \) and \( \{d_r\} \) can be interpolated from the generalized nodal displacement vectors \( \{d_t^e\} \) and \( \{d_r^e\} \) with the help of shape functions \( \{N_t\} \) and \( \{N_r\} \) as follows:

\[
\{d_t\} = [N_t]\{d_t^e\} \quad \text{and} \quad \{d_r\} = [N_r]\{d_r^e\}
\]

in which

\[
\{d_t\} = \begin{bmatrix} u_{01} & w_{01} & u_{02} & w_{02} & u_{03} & w_{03} \end{bmatrix}^T \\
\{d_r\} = \begin{bmatrix} \theta_{x1} & \theta_{y1} & \phi_{x1} & \phi_{y1} & \theta_{x2} & \phi_{x2} & \phi_{y2} & \theta_{x3} & \theta_{y3} & \phi_{x3} & \phi_{y3} \end{bmatrix}^T
\]

\( \{N_t\} \) and \( \{N_r\} \) have been explicitly expressed in Appendix A. The state of strain at any point within the beam have been grouped into two vectors \( \{\varepsilon_b\} \) and \( \{\varepsilon_s\} \), which denote strains due to bending and strains due to shear, respectively. \( \varepsilon_{xx} \) denotes the normal strains along the x-direction, \( \gamma_{xy} \) is the in-plane shear strain and \( \gamma_{xz} \) and \( \gamma_{yz} \) are the transverse shear strains.

\[
\{\varepsilon_b\} = \begin{bmatrix} \varepsilon_{xx} & \gamma_{xy} \end{bmatrix}^T \quad \text{and} \quad \{\varepsilon_s\} = \begin{bmatrix} \gamma_{xz} & \gamma_{yz} \end{bmatrix}^T
\]

Using the displacement field from equations (12-14) and assuming linear strain–displacement relations, strains can be expressed as

\[
\{\varepsilon_b\} = [B_{tb}]\{d_t^e\} + [Z_1]\{B_{rb}\}\{d_r^e\} \\
\{\varepsilon_s\} = [B_{ts}]\{d_t^e\} + [Z_2]\{B_{rs}\}\{d_r^e\}
\]

Elaborated form of \( [B_{tb}], [B_{rb}], [B_{ts}], [B_{rs}], [Z_1] \) and \( [Z_2] \) mentioned in equation (20) have been shown in the Appendix A. The state of stress at any point can be expressed with the help of following two stress vectors:

\[
\{\sigma_b\} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \end{bmatrix}^T \quad \text{and} \quad \{\sigma_s\} = \begin{bmatrix} \sigma_{xz} & \sigma_{yz} \end{bmatrix}^T
\]

where \( \sigma_{xx} \) is the normal stress along the x-direction, \( \sigma_{xy} \) is the in-plane while \( \sigma_{xz} \) and \( \sigma_{yz} \) are the out of plane shear stresses. Let \( [C_b] \) and \( [C_s] \) be bending and transverse shear elastic coefficients, respectively. The constitutive relations are given by

\[
\{\sigma_b\} = [C_b]\{\varepsilon_b\} \quad \text{and} \quad \{\sigma_s\} = [C_s]\{\varepsilon_s\}
\]
where \([C_b]\) and \([C_s]\) have their usual form.

The governing equation is derived using the dynamic version of the principle of virtual work, which is given by

\[
\int_0^T (\delta U + \delta V - \delta K) = 0
\]  

(23)

where \(U\), \(V\) and \(K\) denote the strain energy, work done by the applied force and the kinetic energy, respectively. Using Virtual work principle on strain energy, we get

\[
\delta U = b \int_{-h/2}^{h/2} \int_0 \delta [\{\varepsilon_b\}^T \{\sigma_b\} + \delta [\{\varepsilon_s\}^T \{\sigma_s\}] \, dx \, dz
\]

(24)

Substituting the value from equation (20) and (22) into equation (24), we get

\[
= b \int_{-h/2}^{h/2} \int_0 \left[ \delta \{d_x\}^T (K_{\varepsilon}^e) \{d_x\} + \delta \{d_x\}^T (K_{\tau}^e) \{d_x\} + \delta \{d_x\}^T (K_{\sigma}^e) \{d_x\} \right] \, dx \, dz
\]

(25)

The elemental stiffness matrix appearing in equation (25) are given by

\[
\begin{align*}
[K_{\varepsilon}^e] &= [B_{\varepsilon}]^T [C_b] [B_{\varepsilon}]; \\
[K_{\tau}^e] &= [B_{\tau}]^T [C_s] [B_{\tau}]; \\
[K_{\sigma}^e] &= [B_{\sigma}]^T [C_s] [B_{\sigma}]; \\
\end{align*}
\]

(26)

For loading along z-direction, the elemental load vector \([\{F^e\}]\) is computed using virtual work done by the applied force and is given by

\[
[F^e] = \int_{-h/2}^{h/2} \int_0 \left[ N_t \right]^T q_0 [0 \ 1]^T \, dx
\]

(27)

whereas the elemental mass matrix \([M^e]\) is computed using virtual kinetic energy and is expressed by

\[
[M^e] = b \int_{-h/2}^{h/2} \int_0 \rho \left[ N_s \right]^T \left[ N_s \right] dx \, dz
\]

(28)

Substituting the value from equations (25), (27) and (28) into equation (23), we get the elemental governing equation of motion as follows:

\[
[M^e] \{d_x^e\} + [K_{\varepsilon}^e] \{d_x^e\} + [K_{\tau}^e] \{d_x^e\} = \{F^e\}
\]

(29)

where, \([K_{\varepsilon}^e]\) = \([K_{\varepsilon}^e]\) + \([K_{\tau}^e]\); \([K_{\tau}^e]\) = \([K_{\varepsilon}^e]\) + \([K_{\tau}^e]\); \([K_{\sigma}^e]\) = \([K_{\tau}^e]\) + \([K_{\sigma}^e]\).

After assembling all elemental matrices, we get the global equation of motion

\[
[M] \{\ddot{x}_t\} + [K_{tt}] \{x_t\} + [K_{tt}] \{x_t\} = \{F\}
\]

(30)
where \([M]\) indicates the global mass matrix; \([K_x]\), \([K_y]\) and \([K_{rr}]\) are the global stiffness matrices; \([X_x]\) and \([X_y]\) are the global generalized nodal displacement vectors corresponding to \((d_x^2)\) and \((d_y^2)\), respectively and \((F)\) is the global nodal load vector. The stiffness matrix can be obtained by setting the time derivative terms to zero in the above equation. For free vibration, the eigenvalues of \([M]\)^{-1}[K] depicts square of circular natural frequency, \(\omega^2\).

4. Numerical Results and Discussion

A MATLAB code is developed for the present numerical analyses. Four different deformation theories i.e., FSDT, TSDT, ESDT and HSDT have been used to deal with the free vibration of FG-CNTRC beams. Several numerical results have been generated to validate the accuracy and feasibility of these proposed refined theories. There is not much variation in the results obtained using different theories. Effects of key parameters like CNT volume fraction, its distribution, \(L/h\) ratio and boundary conditions have been presented elaborately. Results obtained from the FEM indicated that these parameters strongly influence the free vibrations. The results are in well agreement with the published results. Results are obtained in terms of non-dimensional frequency parameters \(\bar{\Omega} = \omega L\sqrt{\rho m}[1 - (\nu m)^2]/E^m\).

4.1. Validation of Present Theories

Before studying the various trends in the vibration, we must establish the validity of the present FE model so that the results obtained can be presented accurately. First three non-dimensional natural frequency parameters \((\bar{\Omega})\) obtained using the present theories are compared for various boundary conditions with the available results for \(L/h=15\). The results are tabulated in table 1 and table 2. It can be inferred that the results obtained using these theories are in well agreement with the results in the published articles. Shi et al. [42] gave an exact solution whereas Yas and Samadi [25] assumed Timoshenko beam theory. Deformation theories considered in the present article and the reference articles are different.

Table 1. First three non-dimensional natural frequencies of (S-S) FG-CNTRC beam \((L/h=15)\).

| \(V_{CNT}\) | CNT Distribution Pattern | Present FSDT | Present TSDT | Present ESDT | Present HSDT | Shi [41] |
|---|---|---|---|---|---|---|
| UD | 0.28 | 1.4602 | 4.0559 | 6.8192 | 1.4014 | 4.0825 | 6.9280 | 1.4018 | 4.0836 | 6.9378 | 1.4012 | 4.0797 | 6.9202 | 1.3981 | 4.0956 | 6.8086 |
| FG-X | 1.0104 | 4.3969 | 7.1993 | 1.5819 | 4.2909 | 7.0863 | 1.5821 | 4.2937 | 7.0959 | 1.5823 | 4.2900 | 7.0789 | 1.6086 | 4.3926 | 7.1913 |
| FG-V | 1.2165 | 3.7799 | 6.6057 | 1.2136 | 3.7745 | 6.6431 | 1.2138 | 3.7769 | 6.6511 | 1.2136 | 3.7730 | 6.6363 | 1.3639 | 3.7703 | 6.6307 |

Table 2. First three non-dimensional natural frequencies of (C-C) FG-CNTRC beam \((L/h=15)\).

| \(V_{CNT}\) | CNT Distribution Pattern | Present FSDT | Present TSDT | Present ESDT | Present HSDT | Yas [25] |
|---|---|---|---|---|---|---|
| UD | 0.12 | 1.4828 | 3.0982 | 4.9512 | 1.5015 | 3.1675 | 5.0943 | 1.5027 | 3.1713 | 5.1013 | 1.5004 | 3.1641 | 5.0876 | 1.5085 | 3.1353 | 4.9979 |
| FG-X | 1.5779 | 3.2293 | 5.1085 | 1.6048 | 3.3164 | 5.2788 | 1.6065 | 3.3208 | 5.2863 | 1.6033 | 3.3123 | 5.2712 | 1.6000 | 3.2629 | 5.1514 |
| FG-V | 1.3792 | 2.9597 | 4.7863 | 1.3928 | 3.0183 | 4.9155 | 1.3938 | 3.0216 | 4.9219 | 1.3920 | 3.0153 | 4.9092 | 1.4068 | 2.9997 | 4.8363 |
| FG-O | 1.2884 | 2.8328 | 4.6309 | 1.2607 | 2.7712 | 4.5450 | 1.2604 | 2.7720 | 4.5485 | 1.2615 | 2.7720 | 4.5438 | 1.3180 | 2.8762 | 4.6840 |
| UD | 0.17 | 1.8783 | 3.9641 | 6.3631 | 1.8989 | 4.0440 | 6.5318 | 1.9003 | 4.0485 | 6.5404 | 1.8977 | 4.0400 | 6.5236 | 1.9144 | 4.0817 | 6.4348 |
| FG-X | 2.0175 | 4.1600 | 6.6066 | 2.0331 | 4.2306 | 6.7595 | 2.0350 | 4.2358 | 6.7690 | 2.0316 | 4.2259 | 6.7505 | 2.0498 | 4.2111 | 6.6753 |
| FG-V | 1.7344 | 3.7732 | 6.1877 | 1.7474 | 3.8348 | 6.2815 | 1.7485 | 3.8387 | 6.2893 | 1.7466 | 3.8314 | 6.2741 | 1.7721 | 3.8312 | 6.2139 |
| FG-O | 1.6104 | 3.5946 | 5.9187 | 1.5855 | 3.5379 | 5.8418 | 1.5854 | 3.5391 | 5.8462 | 1.5862 | 3.5384 | 5.8401 | 1.6050 | 3.6655 | 5.9970 |
| UD | 0.28 | 2.1211 | 4.3866 | 6.9766 | 2.1520 | 4.4965 | 7.1988 | 2.1540 | 4.5022 | 7.2090 | 2.1504 | 4.4912 | 7.1886 | 2.1618 | 4.4556 | 7.0745 |
| FG-X | 2.2772 | 4.6338 | 7.3057 | 2.2472 | 4.6238 | 7.3439 | 2.2490 | 4.6299 | 7.3555 | 2.2461 | 4.6188 | 7.3333 | 2.2369 | 4.7051 | 7.4093 |
| FG-V | 2.0064 | 4.2662 | 6.8715 | 2.0174 | 4.3319 | 7.0282 | 2.0189 | 4.3371 | 7.0384 | 2.0162 | 4.3272 | 7.0184 | 2.0504 | 4.3414 | 6.9783 |
| FG-O | 1.8823 | 4.0959 | 6.6638 | 1.8809 | 4.1155 | 6.7299 | 1.8820 | 4.1200 | 6.7397 | 1.8803 | 4.1122 | 6.7216 | 1.9284 | 4.1740 | 6.7728 |
4.2. Effect of CNT Distribution

Several numerical results are presented in table 3 to table 6. First three non-dimensional frequencies are obtained for UD, FG-X, FG-V and FG-O beams for various CNT volume fractions and boundary conditions. It is observed that FG-X produces the maximum frequency, whereas minimum frequency is obtained with FG-O. This trend is obtained for all vibration modes, irrespective of CNT volume fraction and boundary conditions. This happens as the pattern of CNT along the thickness direction also distributes the stiffness in a graded manner. The FG-X distribution has higher CNT concentration near the top and bottom surfaces. Thus, the stiffness is low near mid plane and high near the top and bottom surfaces which increases resistance against bending and the natural frequency also goes up. Since the FG-O pattern is reverse of FG-X, it gives lowest natural frequency. It can be concluded that symmetric distributions of CNTs are more effective in increasing or decreasing the frequency compared to uniform and unsymmetric distributions.

Table 3. First three non-dimensional natural frequencies of FG-CNTRC beams (L/h= 15) using FSDT.

| V<sub>CNT</sub> | CNT Distribution Pattern | Clamped-Free | Clamped-Clamped | Simply-Supported |
|-----------|---------------------------|-------------|-----------------|------------------|
| 0.12      | UD                        | 0.3633      | 1.6677          | 3.6146           |
|           | FG-X                      | 0.4278      | 1.8194          | 3.8316           |
|           | FG-V                      | 0.3081      | 1.5142          | 3.3866           |
|           | FG-O                      | 0.2705      | 1.3926          | 3.1955           |
| 0.17      | UD                        | 0.4430      | 2.0927          | 4.9121           |
|           | FG-X                      | 0.5248      | 2.3028          | 4.9042           |
|           | FG-V                      | 0.3734      | 1.8858          | 4.2798           |
|           | FG-O                      | 0.3272      | 1.7249          | 4.0192           |
| 0.28      | UD                        | 0.5437      | 2.4123          | 5.1587           |
|           | FG-X                      | 0.6411      | 2.6506          | 5.5277           |
|           | FG-V                      | 0.4610      | 2.2189          | 4.9117           |
|           | FG-O                      | 0.4054      | 2.0485          | 4.6495           |

Table 4. First three non-dimensional natural frequencies of FG-CNTRC beams (L/h= 15) using TSDF.

| V<sub>CNT</sub> | CNT Distribution Pattern | Clamped-Free | Clamped-Clamped | Simply-Supported |
|-----------|---------------------------|-------------|-----------------|------------------|
| 0.12      | UD                        | 0.3637      | 1.6312          | 3.6722           |
|           | FG-X                      | 0.4288      | 1.8432          | 3.9078           |
|           | FG-V                      | 0.3083      | 1.5249          | 3.4433           |
|           | FG-O                      | 0.2692      | 1.3681          | 3.1255           |
| 0.17      | UD                        | 0.4435      | 2.1094          | 4.6653           |
|           | FG-X                      | 0.5247      | 2.3152          | 4.9364           |
|           | FG-V                      | 0.3735      | 1.8954          | 4.3270           |
|           | FG-O                      | 0.3262      | 1.7033          | 3.9549           |
| 0.28      | UD                        | 0.5445      | 2.4387          | 5.2517           |
|           | FG-X                      | 0.6357      | 2.6177          | 4.9576           |
|           | FG-V                      | 0.4608      | 2.2264          | 4.9580           |
|           | FG-O                      | 0.4050      | 2.0459          | 4.6555           |

Table 5. First three non-dimensional natural frequencies of FG-CNTRC beams (L/h= 15) using ESDF.

| V<sub>CNT</sub> | CNT Distribution Pattern | Clamped-Free | Clamped-Clamped | Simply-Supported |
|-----------|---------------------------|-------------|-----------------|------------------|
| 0.12      | UD                        | 0.3638      | 1.6843          | 3.6758           |
|           | FG-X                      | 0.4289      | 1.8449          | 3.9124           |
|           | FG-V                      | 0.3083      | 1.5257          | 3.4364           |
|           | FG-O                      | 0.2692      | 1.3678          | 3.1257           |
| 0.17      | UD                        | 0.4435      | 2.1106          | 4.6608           |
|           | FG-X                      | 0.5248      | 2.3169          | 4.9617           |
|           | FG-V                      | 0.3736      | 1.8963          | 4.3306           |
|           | FG-O                      | 0.3261      | 1.7031          | 3.9555           |
| 0.28      | UD                        | 0.5446      | 2.4405          | 5.2573           |
|           | FG-X                      | 0.6357      | 2.6194          | 5.4813           |
|           | FG-V                      | 0.4608      | 2.2277          | 4.9626           |
|           | FG-O                      | 0.4050      | 2.0468          | 4.6594           |
Table 6. First three non-dimensional natural frequencies of FG-CNTRC beams ($L/h = 15$) using HSDT.

| $V_{CNT}$ | CNT Distribution Pattern | $\Omega_1$ | $\Omega_2$ | $\Omega_3$ | $\Omega_1$ | $\Omega_2$ | $\Omega_3$ | $\Omega_1$ | $\Omega_2$ | $\Omega_3$ |
|-----------|--------------------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|
| 0.12      | UD                       | 0.3637     | 1.6823     | 3.6690     | 1.5004     | 3.1641     | 5.0876     | 0.9463     | 2.8339     | 4.8728     |
|           | FG-X                     | 0.4287     | 1.8418     | 3.9036     | 1.6033     | 3.3123     | 5.2712     | 1.0876     | 3.0588     | 5.1046     |
|           | FG-V                     | 0.3083     | 1.5242     | 3.4306     | 1.3920     | 3.0153     | 4.9092     | 0.8178     | 2.5962     | 4.6209     |
|           | FG-O                     | 0.2693     | 1.3689     | 3.1268     | 1.2615     | 2.7720     | 4.5438     | 0.7203     | 2.3423     | 4.2312     |
| 0.17      | UD                       | 0.4434     | 2.1084     | 4.6528     | 1.8977     | 4.0400     | 6.5236     | 1.1640     | 3.5698     | 6.2163     |
|           | FG-X                     | 0.5247     | 2.3138     | 4.9518     | 2.0316     | 4.2259     | 6.7505     | 1.3433     | 3.8605     | 6.5062     |
|           | FG-V                     | 0.3735     | 1.8947     | 4.3241     | 1.7466     | 3.8314     | 6.2741     | 0.9980     | 3.2416     | 5.8556     |
|           | FG-O                     | 0.3262     | 1.7039     | 3.9559     | 1.5862     | 3.5384     | 5.8401     | 0.8790     | 2.9319     | 5.3910     |
| 0.28      | UD                       | 0.5445     | 2.4372     | 5.2466     | 2.1504     | 4.4912     | 7.1886     | 1.4012     | 4.0797     | 6.9202     |
|           | FG-X                     | 0.6357     | 2.6168     | 5.4714     | 2.2461     | 4.6188     | 7.3333     | 1.5823     | 4.2900     | 7.0789     |
|           | FG-V                     | 0.4608     | 2.2254     | 4.9538     | 2.0162     | 4.3272     | 7.0184     | 1.2136     | 3.7730     | 6.6363     |
|           | FG-O                     | 0.4050     | 2.0455     | 4.6528     | 1.8803     | 4.1122     | 6.7216     | 1.0817     | 3.5002     | 6.2973     |

4.3. Effect of CNT Volume Fraction

It is evident from figure 3 and figure 4 that vibration frequency increase with increase in CNT volume fraction. This effect is more pronounced for higher modes of vibration. This trend can be observed irrespective of distribution and boundary conditions. The results indicate that addition of CNT fibres increases the stiffness of the beam significantly.

![Figure 3](image1.png)

Figure 3. First five non-dimensional natural frequencies of FG-CNTRC beams ($L/h = 15$) with various CNT volume fraction using TSDT. The CNT distribution is FG-X. Two different boundary conditions are considered: (a) C-C (b) S-S.

![Figure 4](image2.png)

Figure 4. First five non-dimensional natural frequencies of FG-CNTRC beams ($L/h = 15$) with various CNT volume fraction. The CNT distribution is FG-X and the boundary condition is C-F. Two deformation theories are considered: (a) ESDT (b) HSDT.
4.4. Effect of Boundary Conditions
Boundary conditions play a major role in free vibration. Figure 5 (a) illustrates the variation of frequency due to boundary conditions using ESDT. The data showed that maximum frequency is obtained for C-C boundary conditions and the minimum frequency is obtained for C-F boundary conditions. However, for higher modes of vibration, there is no significant difference in frequency for C-C and S-S boundary conditions. As similar trends are observed with other deformation theories, they are not shown here. It can be inferred from figure 5 (b) that boundary conditions strongly influence the first mode of vibration both for thin and thick beams.

![Figure 5. (a) First five non-dimensional natural frequencies of FG-X beams (L/h= 15) with various boundary conditions using ESDT. (b) First non-dimensional natural frequencies of FG-CNTRC beams (FG-X) with various length-to-thickness ratios using TSDT. $V_{tcnt} = 0.28$.](image)

4.5. Effect of Geometry
Figure 6 illustrates the influence of slenderness ratio on free vibration. It can be concluded that frequency is inversely proportional to L/h ratio. The stiffness of the beam reduces with increase in slenderness ratio. The trend observed is same irrespective of CNT distributions. However, for thin beams, variation of frequencies with CNT distributions is not very pronounced. To make it concise results using HSDT are presented only.

![Figure 6. First non-dimensional natural frequencies of FG-CNTRC beams (C-F) with various length-to-thickness ratios using HSDT. $V_{tcnt} = 0.17$.](image)

4.6. Accuracy of the Present Theories
From the above results, it is clear that these theories can be considered to be very accurate for solving engineering problems. The actual shear stresses profile along the thickness of the beam is better represented by a quadratic function or higher-order polynomial. Expanding the displacement fields of FSDT by adding few higher-order polynomial, variation of transverse shear strain and warpage of the cross section can be taken care of. Therefore, the accuracy of the results obtained using TSST, ESDT and HSDT are generally higher than FSDT. Table 7 and 8 presents the non-dimensional natural frequencies for thick and thin beams, respectively. The results obtained using different deformation theories are fairly consistent. For thick beams, the frequency predicted by FSDT is smaller because
FSDT does not take into account warpage of the cross section. The higher order theories like TSDT, ESDT and HSĐT better approximate the profile of the cross section of thick beams. However, the traction free boundary condition is not accounted for in the present refined theories. Hence, higher order theories predict higher results for thick beams compared to FSDT. Further study of this refined theories is necessary for deflection and distribution of transverse shear stress across the thickness of the beams to conclude their efficacy compared to FSDT. However, for thin beams, the difference in the frequency obtained using various theories is practically negligible. Hence, FSDT is the most suitable to analyse thin beams as the computational cost is minimum and the finite element formulation is comparatively easier.

Table 7. Comparison of natural frequencies obtained using different shear deformation theories for thick beam ($L/h=5$).

| VCNT | CNT Distribution Pattern | Clamped-Free | Clamped-Clamped | Simply-Supported |
|------|--------------------------|--------------|----------------|-----------------|
| 0.12 | UD                       | 0.7303 0.7424 0.7431 0.7419 | 1.7810 1.8734 1.8764 1.8703 | 1.6032 1.6245 1.6265 1.6228 |
|      | FG-X                     | 0.7805 0.7976 0.7985 0.7969 | 1.8019 1.9044 1.9071 1.9015 | 1.6721 1.7024 1.7049 1.7001 |
|      | FG-V                     | 0.6755 0.6843 0.6848 0.6839 | 1.7634 1.8509 1.8537 1.8478 | 1.5240 1.5407 1.5423 1.5392 |
|      | FG-O                     | 0.6285 0.6182 0.6183 0.6184 | 1.7408 1.7407 1.7444 1.7373 | 1.4533 1.4077 1.4077 1.4087 |
| 0.17 | UD                       | 0.9233 0.9366 0.9374 0.9360 | 2.3093 2.4214 2.4251 2.4173 | 2.0489 2.0721 2.0744 2.0701 |
|      | FG-X                     | 0.9963 1.0093 1.0103 1.0085 | 2.3476 2.4593 2.4631 2.4552 | 2.1542 2.1693 2.1721 2.1669 |
|      | FG-V                     | 0.8473 0.8560 0.8565 0.8555 | 2.2883 2.3912 2.3949 2.3872 | 1.9371 1.9522 1.9541 1.9506 |
|      | FG-O                     | 0.7835 0.7744 0.7746 0.7746 | 2.2543 2.2603 2.2647 2.2564 | 1.8378 1.7939 1.7940 1.7947 |
| 0.28 | UD                       | 1.0468 1.0670 1.0680 1.0661 | 2.4860 2.6247 2.6297 2.6203 | 2.2713 2.3075 2.3106 2.3047 |
|      | FG-X                     | 1.1278 1.1235 1.1246 1.1227 | 2.5613 2.6585 2.6634 2.6531 | 2.3981 2.3601 2.3633 2.3576 |
|      | FG-V                     | 0.9844 0.9949 0.9949 0.9935 | 2.5115 2.6307 2.6351 2.6259 | 2.2065 2.2127 2.2154 2.2104 |
|      | FG-O                     | 0.9200 0.9222 0.9228 0.9217 | 2.4827 2.5553 2.5610 2.5495 | 2.1060 2.0980 2.1000 2.0968 |

Table 8. Comparison of natural frequencies obtained using different shear deformation theories for thin beam ($L/h=50$).

| VCNT | CNT Distribution Pattern | Clamped-Free | Clamped-Clamped | Simply-Supported |
|------|--------------------------|--------------|----------------|-----------------|
| 0.12 | UD                       | 0.1171 0.1171 0.1171 0.1171 | 0.6966 0.6974 0.6975 0.6974 | 0.3261 0.3261 0.3261 0.3261 |
|      | FG-X                     | 0.1423 0.1423 0.1423 0.1423 | 0.8214 0.8232 0.8233 0.8230 | 0.3944 0.3945 0.3945 0.3945 |
|      | FG-V                     | 0.0972 0.0972 0.0972 0.0972 | 0.5905 0.5910 0.5910 0.5910 | 0.2714 0.2714 0.2714 0.2714 |
|      | FG-O                     | 0.0843 0.0843 0.0843 0.0843 | 0.5183 0.5161 0.5161 0.5162 | 0.2358 0.2355 0.2355 0.2355 |
| 0.17 | UD                       | 0.1415 0.1415 0.1415 0.1415 | 0.8493 0.8502 0.8503 0.8501 | 0.3944 0.3944 0.3944 0.3944 |
|      | FG-X                     | 0.1722 0.1722 0.1722 0.1722 | 1.0072 1.0069 1.0071 1.0068 | 0.4782 0.4781 0.4781 0.4781 |
|      | FG-V                     | 0.1170 0.1170 0.1170 0.1170 | 0.7156 0.7160 0.7160 0.7159 | 0.3268 0.3268 0.3268 0.3267 |
|      | FG-O                     | 0.1014 0.1014 0.1014 0.1014 | 0.6269 0.6252 0.6252 0.6253 | 0.2837 0.2835 0.2835 0.2835 |
| 0.28 | UD                       | 0.1776 0.1776 0.1776 0.1776 | 1.0431 1.0446 1.0447 1.0445 | 0.4935 0.4935 0.4935 0.4935 |
|      | FG-X                     | 0.2162 0.2159 0.2159 0.2159 | 1.2319 1.2212 1.2213 1.2214 | 0.5983 0.5967 0.5967 0.5967 |
|      | FG-O                     | 0.1464 0.1464 0.1464 0.1464 | 0.8837 0.8833 0.8834 0.8833 | 0.4082 0.4081 0.4081 0.4081 |
|      | FG-O                     | 0.1270 0.1270 0.1270 0.1270 | 0.7767 0.7761 0.7762 0.7762 | 0.3547 0.3546 0.3546 0.3546 |

5. Conclusions
In this article, three deformation theories have been developed from modifying the FSDT to numerically investigate free vibration of FG-CNTRC beams. An efficiency parameter is used to account for the multi scale effect while representing the effective material properties of the FG-CNTRC beams. The governing finite element equations are derived using virtual work principle. The effect of different parameters on free vibration of the beam are numerically computed. The results obtained are compared with the published articles in tabular form and the comparison of different refined shear deformation theories are graphically presented. The following conclusions can be drawn from the results presented here:
- Frequency parameters increases with increase in CNT volume fraction.
- Beams with FG-X CNT distributions exhibit maximum fundamental frequency, whereas FG-O distributions has lowest fundamental frequency.
- Boundary conditions significantly affect the vibration characteristics. C-C beams have the highest natural frequency which is followed by S-S and C-F, respectively.
• Frequency parameters decrease with increase in slenderness ratio.
• Refined theories i.e., TSDDT, ESDT and HSDT predict fundamental frequency higher than FSDT for thick to moderately-thick beams.
• For thin beams, there is negligible difference in the natural frequency obtained using different theories.

Appendix A

\[ \begin{bmatrix} N_t \end{bmatrix} \] and \[ \begin{bmatrix} N_r \end{bmatrix} \] of equation (17) have the following expressions:

\[ N_t = \begin{bmatrix} N_1 I_2 & N_2 I_2 & N_3 I_2 \end{bmatrix} ; \quad N_r = \begin{bmatrix} N_4 I_4 & N_2 I_4 & N_3 I_4 \end{bmatrix} \quad (A1) \]

Expressions \([B_{tb}], [B_{rb}], [B_{rs}], [Z_1] \) and \([Z_2] \) mentioned in equation (20) are as follows,

\[ B_{tb} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & 0 \end{bmatrix} ; \quad B_{rb} = \frac{\partial}{\partial x} I_4 \quad (A2) \]

\[ Z_1 = \begin{bmatrix} z I_2 \\ f(x) I_2 \end{bmatrix} \quad \text{and} \quad Z_2 = \frac{\partial}{\partial z} \quad (A2) \]

where \( I_j \) is a \( j \times j \) identity matrix.

REFERENCES

[1] Iijima S, Helical microtubules of graphitic carbon. Nature. 1991 Nov;354(6348):56-8
[2] Sandler J, Shaffer MS, Prasse T, Bauhofer W, Schulte K, Windle AH. Development of a dispersion process for carbon nanotubes in an epoxy matrix and the resulting electrical properties. Polymer. 1999 Oct 1;40(21):5967-71.
[3] Yu MF, Lourie O, Dyer MJ, Moloni K, Kelly TF, Ruoff RS. Strength and breaking mechanism of multiwalled carbon nanotubes under tensile load. Science. 2000 Jan 28;287(5453):637-40.
[4] Salvat JP, Bonard JM, Thomson NH, Kulik AJ, Forro L, Benoit W, Zuppiroli L. Mechanical properties of carbon nanotubes. Applied Physics A. 1999 Sep 1;69(3):255-60.
[5] Kataura H, Kumazawa Y, Maniwa Y, Umezu I, Suzuki S, Ohtsuka Y, Achiba Y. Optical properties of single-wall carbon nanotubes. Synthetic metals. 1999 Jun 1;103(1-3):2555-8.
[6] Che J, Cagin T, Goddard III WA. Thermal conductivity of carbon nanotubes. Nanotechnology. 2000 Jun;11(2):65.
[7] Pop E, Mann D, Wang Q, Goodson K, Dai H. Thermal conductance of an individual single-wall carbon nanotube above room temperature. Nano letters. 2006 Jan 11;6(1):96-100.
[8] Wong EW, Sheehan PE, Lieber CM. Nanobeam mechanics: elasticity, strength, and toughness of nanorods and nanotubes. Science. 1997 Sep 26;277(5334):1971-5.
[9] Berber S, Kwon YK, Tománek D. Unusually high thermal conductivity of carbon nanotubes. Physical review letters. 2000 May 15;84(20):4613.
[10] Overney G, Zhong W, Tomanek D. Structural rigidity and low frequency vibrational modes of long carbon tubules. Zeitschrift für Physik D Atoms, Molecules and Clusters. 1993 Mar 1;27(1):93-6.
[11] Lu JP, Elastic properties of single and multilayered nanotubes. Journal of physics and chemistry of solids. 1997 Nov 1;58(11):1649-52.
[12] Hata K, Futaba DN, Mizuno K, Namai T, Yumura M, Iijima S, Water-assisted highly efficient synthesis of impurity-free single-walled carbon nanotubes. Science. 2004 Nov 19;306(5700):1362-4.
[13] Arash B, Wang Q, Varadan VK, Mechanical properties of carbon nanotube/polymer composites. Scientific reports. 2014 Oct 1;4:6479.
[14] Qian D, Dickey EC, Andrews R, Rantell T, Load transfer and deformation mechanisms in carbon nanotube-polystyrene composites. Applied physics letters. 2000 May 15;76(20):2868-70.
[15] Wuite J, Adali S. Deflection and stress behaviour of nanocomposite reinforced beams using a multiscale analysis. Composite Structures. 2005 Dec 1; 71(3-4):388-96.

[16] Lal A, Kanu N. The nonlinear deflection response of CNT/nanoclay reinforced polymer hybrid composite plate under different loading conditions. IOP Conference Series: Materials Science and Engineering. 2020; 814:012033.

[17] Kanu N, Bapat S, Deodhar H, Gupta E, Singh G, Vates U et al. An Insight into Processing and Properties of Smart Carbon Nanotubes Reinforced Nanocomposites. Smart Science. 2021:1-16.

[18] Jain N, Gupta E, Kanu N. Plethora of Carbon Nanotubes Applications in Various Fields – A State-of-the-Art-Review. Smart Science. 2021:1-24.

[19] Niino M, Maeda S. Recent development status of functionally gradient materials. ISIJ International. 1990 Sep 15; 30(9):699-703.

[20] Kawasaki A, Watanabe R. Concept and P/M fabrication of functionally gradient materials. Ceramics international. 1997 Jan 1; 23(1):73-83.

[21] Kanu N, Vates U, Singh G, Chavan S. Fracture problems, vibration, buckling, and bending analyses of functionally graded materials: A state-of-the-art review including smart FGMS. Particulate Science and Technology. 2018; 37(5):583-608.

[22] Wattanasakulpong N, Unghakorn V. Analytical solutions for bending, buckling and vibration responses of carbon nanotube-reinforced composite beams resting on elastic foundation. Computational Materials Science. 2013 Apr 1; 71:201-8.

[23] Rokni H, Milani AS, Seethaler RJ. Size-dependent vibration behavior of functionally graded CNT-reinforced polymer microcantilevers: modeling and optimization. European Journal of Mechanics-A/Solids. 2015 Jan 1; 49:26-34.

[24] Penna R, Feo L, Fortunato A, Luciano R. Nonlinear free vibrations analysis of geometrically imperfect FG nano-beams based on stress-driven nonlocal elasticity with initial pretension force. Composite Structures. 2020; 255:112856.

[25] Yas MH, Samadi N. Free vibrations and buckling analysis of carbon nanotube-reinforced composite Timoshenko beams on elastic foundation. International Journal of Pressure Vessels and Piping. 2012 Oct 1; 98:119-28.

[26] Vo-Duy T, Ho-Huu V, Nguyen-Thoi T. Free vibration analysis of laminated FG-CNT reinforced composite beams using finite element method. Frontiers of Structural and Civil Engineering. 2019 Apr 1; 13(2):324-36.

[27] Shen HS, Xiang Y. Nonlinear analysis of nanotube-reinforced composite beams resting on elastic foundations in thermal environments. Engineering Structures. 2013 Nov 1; 56:698-708.

[28] Mohseni A, Shakouri M. Vibration and stability analysis of functionally graded CNT-reinforced composite beams with variable thickness on elastic foundation. Proceedings of the Institution of Mechanical Engineers, Part L: Journal of Materials: Design and Applications. 2019 Dec; 233(12):2478-89.

[29] Rokni H, Milani AS, Seethaler RJ. Maximum natural frequencies of polymer composite micro-beams by optimum distribution of carbon nanotubes. Materials & Design. 2011 Jun 1; 32(6):3389-98.

[30] Rokni H, Milani AS, Seethaler RJ. 2D optimum distribution of carbon nanotubes to maximize fundamental natural frequency of polymer composite micro-beams. Composites Part B: Engineering. 2012 Mar 1; 43(2):779-85.

[31] Lin F, Xiang Y. Numerical analysis on nonlinear free vibration of carbon nanotube reinforced composite beams. International Journal of Structural Stability and Dynamics. 2014 Jan 12; 14(01):1350056.

[32] Mohammadimehr M, Monajemi AA, Afshari H. Free and forced vibration analysis of viscoelastic damped FG-CNT reinforced micro composite beams. Microsystem Technologies. 2017 Dec 27:1-5.

[33] Kitipornchai S, Chen D, Yang J. Free vibration and elastic buckling of functionally graded porous beams reinforced by graphene platelets. Materials & Design. 2017 Feb 15; 116:656-65.
[34] Momeni M, Dehkordi MB, Frequency analysis of sandwich beam with FG carbon nanotubes face sheets and flexible core using high-order element. *Mechanics of Advanced Materials and Structures*. 2019 May 3;26(9):805-15.

[35] Dabbagh A, Rastgoo A, Ebrahimi F, Finite element vibration analysis of multi-scale hybrid nanocomposite beams via a refined beam theory. *Thin-Walled Structures*. 2019 Jul 1;140:304-17.

[36] Bodaghi M, Damanpack AR, Aghdam MM, Shakeri M. Geometrically non-linear transient thermo-elastic response of FG beams integrated with a pair of FG piezoelectric sensors. *Composite Structures*. 2014 Jan 1;107:48-59.

[37] Datta P. Active vibration control of axially functionally graded cantilever beams by finite element method. *Materials Today: Proceedings*. 2021 Jan 1;44:2543-50.

[38] Nguyen-Quang K, Vo-Duy T, Dang-Trung H, Nguyen-Thoi T, An isogeometric approach for dynamic response of laminated FG-CNT reinforced composite plates integrated with piezoelectric layers. *Computer Methods in Applied Mechanics and Engineering*. 2018 Apr 15;332:25-46.

[39] Datta P, Ray MC. Effect of carbon nanotube waviness on smart damping of geometrically nonlinear vibrations of fuzzy-fiber reinforced composite plates. *Journal of Intelligent Material Systems and Structures*. 2019 Apr;30(7):977-97.

[40] Manoj A. Kumbhalkar; D.V. Bhope; A.V. Vanalkar, “Evaluation of Frequency Excitation of Helical Suspension Spring Using Finite Element Analysis”, International Journal of Computer Aided Engineering and Technology, Vol. 9, No. 4, pp 420-433, 2017.

[41] Song ZG, Zhang LW, Liew KM. Active vibration control of CNT-reinforced composite cylindrical shells via piezoelectric patches. *Composite Structures*. 2016 Dec 15;158:92-100.

[42] Shi Z, Yao X, Pang F, Wang Q, An exact solution for the free-vibration analysis of functionally graded carbon-nanotube-reinforced composite beams with arbitrary boundary conditions. *Scientific reports*. 2017 Oct 10;7(1):1-8.