GODBILLON-VEY SEQUENCE AND FRANÇOISE ALGORITHM

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Abstract. We consider foliations given by deformations $dF + \epsilon \omega$ of exact forms $dF$ in $\mathbb{C}^2$ in a neighborhood of a family of cycles $\gamma(t) \subset F^{-1}(t)$.

In 1996 Françoise gave an algorithm for calculating the first nonzero term of the displacement function $\Delta$ along $\gamma$ of such deformations. This algorithm recalls the well-known Godbillon-Vey sequences discovered in 1971 for investigation integrability of a form $\omega$. In this paper, we establish the correspondence between the two approaches and translate some results by Casale relating types of integrability for finite Godbillon-Vey sequences to the Françoise algorithm settings.

1. Introduction

Let $\gamma_0 \subset \mathbb{C}^2$ be a regular curve, $\Sigma$ a trasversal to $\gamma_0$, $F$ a holomorphic function defined on a tubular neighborhood $U \subset \mathbb{C}^2$ of $\gamma_0$, formed by regular curves $\gamma(t) \subset F^{-1}(t)$, $t \in F(\Sigma)$, with $\gamma_0 = \gamma(t_0)$.

Consider the integrable foliation $dF = 0$ and its holomorphic deformation $dF + \epsilon \omega = 0$ (1.1) in $U$. We are interested in the displacement function $\Delta$ (holonomy along $\gamma$ minus identity) of (1.1). Here $\Delta(t)$ denotes the holonomy of (1.1) along $\gamma(t)$. It can be developed as

$$\Delta(t) = \sum_{i \geq 1} \epsilon^i M_i(t).$$ (1.2)

The functions $M_i(t)$ are called Melnikov functions. If $\Delta \equiv 0$, this means that (1.1) has a first integral in a neighborhood of $\gamma(t)$. If not, then there exists a first non-zero Melnikov function $M_\mu$.

1.1. Françoise algorithm. Françoise algorithm allows to compute the first nonzero Melnikov function $M_\mu$. Let us first recall the following classical Lemma.

Lemma 1.1. Given a holomorphic one-form $\omega$ and a family of cycles $\gamma(t) \subset \{F^{-1}(t)\}$, the following conditions are equivalent:

(i) The form $\omega$ verifies

$$\int_{\gamma} \omega \equiv 0.$$ (1.3)

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(ii) There exists a function \( r \) holomorphic in a neighborhood of \( \gamma \) such that

\[
dF \wedge (\omega - dr) \equiv 0. \tag{1.4}
\]

(iii) There exist functions \( g \) and \( r \) holomorphic in a neighborhood of \( \gamma \) such that

\[
\omega = gdF + dr. \tag{1.5}
\]

Note that the functions \( g \) and \( r \) are univalued in \( U \) but in general do not extend to polynomial, nor even univalued functions in \( \mathbb{C}^2 \).

Recall, the classical result of Poincaré and Pontryagin:

\[
M_1(t) = - \int_{\gamma(t)} \omega.
\]

If \( M_1 \equiv 0 \), then, by Lemma 1.1,

\[
\omega = g_1dF + dr_1,
\]

and in that case, Françoise [3] proves the following theorem (see also [13], [7], [8], [4], [5], [11], [12], [10]):

**Theorem 1.2.** Let \( (1.2) \) be the displacement function of \( (1.1) \). Assume that \( M_i(t) \equiv 0 \), for \( i = 1, \ldots, k \). Then \( M_{k+1}(t) = (-1)^{k+1} \int_{\gamma(t)} g_k\omega \), where \( g_0 = 1 \) and \( g_i, r_i \) verify

\[
g_{i-1}\omega = g_idF + dr_i, \quad i = 1, \ldots, k. \tag{1.6}
\]

The existence of the decomposition \( (1.6) \), follows by induction from Lemma 1.1.

**Definition 1.3.** We call any pair \( (g_i, r_i) \), verifying \( (1.6) \) an \( i \)-th Françoise pair associated to the deformation \( (1.1) \) and call the sequence \( (g_i, r_i), i = 0, 1, \ldots \) a Françoise sequence. We say that the length of a Françoise sequence is \( \ell \), if \( \ell \) is the smallest index such that \( g_{\ell+1} = 0 \). If there does not exist such an index, we say that the sequence if of infinite length.

### 1.2. Godbillon-Vey sequence

On the other hand, the classical Godbillon-Vey sequence is associated to a foliation defined by a single one form

\[
\omega = 0. \tag{1.7}
\]

It is a sequence of one-forms \( \omega_0 = \omega, \omega_i, i = 1, \ldots \) such that the formal one-form

\[
\Omega = d\epsilon + \omega_0 + \sum_{i=1}^{\infty} \frac{\epsilon^i}{i!} \omega_i \tag{1.8}
\]

in \( \mathbb{C}^2 \times \mathbb{C} \) verifies the formal integrability condition

\[
\Omega \wedge \tilde{d}\Omega = 0. \tag{1.9}
\]

Here \( \tilde{d} = d_x + d_y \) denotes the total differential with respect to all variables \( x, y, \epsilon \).

Condition \( (1.9) \) is equivalent to

\[
\begin{align*}
\ d\omega_0 &= \omega_0 \wedge \omega_1, \\
\ d\omega_1 &= \omega_0 \wedge \omega_2, \\
\ddots & \ \ \ \ddots \\
\ d\omega_n &= \omega_0 \wedge \omega_{n+1} + \sum_{k=1}^{n} \binom{n}{k} \omega_k \wedge \omega_{n-k+1}.
\end{align*}
\]

We say that the Godbillon-Vey sequence is of length \( n \) if the forms \( \omega_k \) vanish for \( k \geq n \).
Definition 1.4. Let $K$ be a differential field, $G$ a function and $K_G$ the extension of $K$ by $G$. We say that the extension $K_G$ is: Darboux, Liouville or Riccati, respectively, if it belongs to a finite sequence of field extensions starting from the field $K$. The extensions in each step are either algebraic or given respectively by solutions of the equations $dG = \eta_0$, $dG = G\eta_1 + \eta_0$ or $dG = G^2\eta_2 + G\eta_1 + \eta_0$, with $\eta_i$ one-forms with coefficients in the corresponding field extensions.

In that case, we call the function $G$ Darboux, Liouville or Riccati with respect to $K$.

In [1], Casale relates the length $n$ of the Godbillon-Vey sequence to the type of first integral of the foliation given by (1.7):

Theorem 1.5.

(i) There exists a Godbillon-Vey sequence of length 1 if and only if (1.7) has a Darboux first integral.

(ii) There exists a Godbillon-Vey sequence of length 2 if and only if (1.7) has a Liouville first integral.

(iii) There exists a Godbillon-Vey sequence of length 3 if and only if (1.7) has a Riccati first integral.

Here we develop a version of Godbillon-Vey sequences well-adapted to studying a deformation of an integrable foliation given by (1.1). Recall that on the level $\epsilon = 0$ it is integrable (with first integral $F$). The Godbillon-Vey sequence gives a condition for verifying if this integrability extends to $\epsilon \neq 0$.

We define the form

$$\Omega = Rde + (dF + \epsilon\omega)G,$$

(1.10)

with

$$G = \sum_{i=0}^{\ell} \epsilon^i G_i, \quad R = \sum_{i=0}^{\ell} \epsilon^i R_{i+1}$$

(1.11)

unknown functions and $G_0 \equiv 1$. The form (1.10) of $\Omega$ comes from the requirement to define the same foliation as (1.10) on each level $\epsilon = \text{const}$.

We give a relative version of the definition of different types of first integral for the deformation (1.1).

Definition 1.6. We denote by $K_{F,\omega}$ the field associated to the deformation (1.1). That is, the smallest differential field in a tubular neighborhood $U$ of a cycle $\gamma_0$ containing the functions given by coefficients of $dF$ and $\omega$.

Let $F_\ell = \sum_{i=0}^{\ell} \epsilon^i F_i, \quad \ell < \infty$, be a first integral of (1.1). We say that it is Darboux, Liouville or Riccati, respectively, if all $F_i$ are in the corresponding extension of the field $K_{F,\omega}$.

Theorem 1.7. [6] There exists a solution $(G, R)$ of the equation

$$\Omega \wedge d\Omega = 0,$$

(1.12)

if and only if the deformation preserves formal integrability along $\gamma$ i.e. $\Delta \equiv 0$.

Proof. Indeed, if there exists a solution $(G, R)$ of (1.12), then by Frobenius theorem, $\Omega$ defines a foliation in a neighborhood of $(\gamma(t), 0)$ in $\mathbb{C}^3$ transversal to $\epsilon = 0$. It follows from the existence of this foliation that the integrability on the level $\epsilon = 0$ is preserved on nearby levels. \qed
We will also consider the Godbillon-Vey equation up to order \(k\) with \(\Omega, G, R\) given by (1.10) and (1.11):

\[
\Omega \wedge \tilde{d} = 0 \mod \varepsilon^{k+1}. 
\] (1.13)

**Definition 1.8.** We call any pair \((G_i, R_i)\), verifying (1.13) an \(i\)-th Godbillon-Vey pair associated to the deformation (1.1), \((G_i, R_i), i = 0, 1, \ldots\) is the Godbillon-Vey sequence associated to the deformation. We say that the length of a Godbillon-Vey sequence associated to the deformation is \(\ell\), if \(\ell\) is the smallest index such that \(G_{\ell+1} = 0\). If there does not exist such an index, we say that the sequence if of infinite length.

**Remark 1.9.** Note that the length is associated to any Françoise sequence or Godbillon-Vey sequence associated to the deformation (1.1). However, one deformation (1.1) can have Françoise sequences (or Godbillon-Vey sequence) of different lengths. The minimal length is well defined and one can choose a Françoise sequence so that all \(g_k = 0\), for \(k > \ell\). The same applies for the Godbillon-Vey sequences.

Françoise pairs \((g_i, r_i)\) and Godbillon-Vey pairs \((G_i, R_i)\) exist for all \(i = 1, 2, 3, \ldots\) if and only if the deformation preserves integrability along \(\gamma\).

2. **Main theorems**

In this section we state our two main results. The first establishes the relationship between the Françoise pairs and the Godbillon-Vey pairs associated to the deformation. In particular it shows that the minimal length of Françoise sequences and Godbillon-Vey sequences coincide:

**Theorem 2.1.**

(i) The Melnikov functions \(M_i, i = 1, \ldots, k\), are identically equal to zero if and only if one can solve the equation

\[
\Omega \wedge \tilde{d} = 0 \mod \varepsilon^{k+1}, 
\] (2.1)

(ii) For each choice of the Françoise sequence \((g_i, r_i), i = 1, \ldots, k\), the Godbillon-Vey sequence \((G_i, R_i), i = 1, \ldots, k\), can be chosen verifying the equations

\[
G_i = (-1)^i g_i, \quad R_i = (-1)^{i+1} ir_i. 
\] (2.2)

(iii) If \(\Omega\) verifies (2.1) then

a) there exists a function \(N = 1 + \sum_{i=1}^{k} \varepsilon^i n_i\) such that

\[
\Omega = \tilde{N}dF \mod \varepsilon^{k+1}.
\]

Then the function \(F\) is of the form

\[
F = F + \sum_{i=1}^{k} (-1)^{i+1} \varepsilon^i r_i. 
\] (2.3)

and

\[
\tilde{d}F = \tilde{R}d\varepsilon + \tilde{G}(dF + \varepsilon\omega). 
\] (2.4)

b) Let \(\tilde{G}\) and \(\tilde{R}\) be given in (2.4) and \((G_i, R_i), i = 1, \ldots, k\), be its coefficients as in (1.11). Then the functions \((g_i, r_i), i = 1, \ldots, k\), given by (2.2) are Françoise pairs.
Our second result gives the type of local first integral $F_\epsilon$ of the deformation (1.1) if the length of its Françoise sequence is finite. The first result is that the first integral is in a finite sequence of extensions of Darboux type. The second shows that it is in a single extension of Liouvillian type.

**Theorem 2.2.** Let $\eta_\epsilon = dF + \epsilon \omega$ as in (1.1) be such that there exists a Françoise sequence of finite length $\ell$.

(i) Then (1.1) admits a univalued first integral which is Darboux with respect to the field $K_{F,\omega}$ of the deformation (1.1).

(ii) Then there exists a meromorphic form $\tilde{\eta}_\epsilon$ verifying the Godbillon-Vey sequence of length 2:

\[
\begin{align*}
\tilde{d}\eta_\epsilon &= \eta_\epsilon \wedge \tilde{\theta}_\epsilon \\
\tilde{d}\tilde{\theta}_\epsilon &= 0,
\end{align*}
\]

such that there exists a (possibly multivalued) first integral $\tilde{F}_\epsilon$ of (1.1) verifying

\[
\tilde{d}\tilde{F}_\epsilon = f\eta_\epsilon,
\]

where

\[
df = f\tilde{\theta}_\epsilon
\]

is a (possibly multivalued) function in a tubular neighborhood $U$ of the cycle $\gamma_0$.

In particular, the function $f$ belongs to a Liouville extension of $K_{F,\omega}$ and $\tilde{F}_\epsilon$ belongs to a Darboux extension of this Liouville extension.

**Remark 2.3.** Note that we are restricting our study to a tubular neighborhood $U$ of a cycle $\gamma_0$. A first integral $F_\epsilon$ which is Darboux in $U$ can be more complicated (Liouville, Riccati,...) when studied globally.

**Remark 2.4.** In Theorem 2.2 (ii) we prove in particular that if the deformation (1.1) has a finite Françoise sequence, then it has a Liouvillian first integral. The converse is an interesting question.

**Remark 2.5.** In Theorem 2.2 we suppose that (1.1) has a Françoise sequence of finite order. What happens in the case of $\ell = \infty$? In particular, is it possible to give a condition assuring that a deformation (1.1) has a Liouville or a Riccati first integral in these terms?

3. **Proof of Theorem 2.1**

**Proof.** We first prove the direct implication of the statement (i), the converse will follow from (iii)(b). If the functions $M_i$ identically vanish for $i = 1, \ldots, k$, then one can build a first integral $F_\epsilon$ of $dF + \epsilon \omega \mod \epsilon^{k+1}$ in the following way: extend $F$ to transversal $\Sigma$ to $\gamma \times \{0\}$ in $\mathbb{C}^3 = \mathbb{C}^2_{x,y} \times \mathbb{C}_\epsilon$ as $F(x, y, \epsilon) = F(x, y)$, and extend it to a neighborhood $U$ of $\gamma \times \{0\}$ in $\mathbb{C}^3$ by the flow. The extension $F_\epsilon$ is a multivalued function, but different branches of $F_\epsilon$ agree $\mod \epsilon^{k+1}$ on $\Sigma$ by assumption, and therefore everywhere in $U$. In other words, in the decomposition $F_\epsilon = F + \sum_{i \geq 1} \epsilon F_i, F_i = F_i(x, y)$, the functions $F_i$ are univalued for $i = 1, \ldots, k$.

This implies that in the decomposition

\[
\tilde{d}F_\epsilon = (F_\epsilon)_\epsilon' \, d\epsilon + (dF + \epsilon \omega) G,
\]
the coefficients \((F_\epsilon)_\epsilon\), \(G\) are univalued modulo terms of order \(\geq k + 1\), and we take
\[
\Omega := j^k_\epsilon (F_\epsilon)_\epsilon dy + (dF + \omega) j^k_\epsilon G,
\]
where \(j^k_\epsilon\) denotes the \(k\)-th jet with respect to \(\epsilon\). Then \(\Omega\) verifies \((2.1)\).

Using \((1.6)\), the proof of (ii) follows from the computation:
\[
(dF + \omega) \left(1 + \sum_{i=1}^{k} (-1)^i \epsilon^i g_i\right) = dF + \sum_{i=1}^{k} \epsilon^i \left((-1)^i g_i dF + (-1)^{i+1} g_{i-1} \omega\right)
\]
\[
= dF + \sum_{i=1}^{k} \epsilon^i (-1)^{i+1} dr_i \mod \epsilon^{k+1},
\]
where \(g_0 \equiv 1\). Therefore, by \((2.2)\) and \((1.11)\),
\[
\Omega = R d\epsilon + G(dF + \omega) = \left(\sum_{i=1}^{k} (-1)^{i-1} i \epsilon^i r_i\right) d\epsilon + dF + \sum_{i=1}^{k} \epsilon^i (-1)^{i+1} dr_i
\]
\[
= d\left(F + \sum_{i=1}^{k} \epsilon^i (-1)^{i+1} r_i\right) \mod \epsilon^{k+1}
\]
is closed up to order \(\epsilon^{k+1}\) and therefore satisfies \((2.1)\).

We prove statement (iii)(a) and (iii)(b) simultaneously by induction. We define weights of monomials by posing \(w(x) = w(y) = w(dx) = w(dy) = 0\) and \(w(\epsilon) = w(d\epsilon) = 1\), so \(d\) and \(\tilde{d}\) preserve weights. We will denote by \(o_k\) any collection of terms of weight \(\geq k\). In these notations, \((3.1)\) is equivalent to
\[
\Omega \wedge \tilde{d}\Omega = o_{k+1}. \tag{3.1}
\]
Let \(N_j = 1 + \sum_{i=1}^{j} \epsilon^i n_i\). We construct the function \(N = N_k\) by induction.

Consider first \(k = 0\). A simple computation shows that
\[
\Omega \wedge \tilde{d}\Omega = dF \wedge d\epsilon \wedge (\omega - dR_1) + o_1,
\]
so, simplifying by \(\wedge d\epsilon\), \((3.1)\) for \(k = 0\) is equivalent to
\[
dF \wedge (\omega - dR_1) = 0.
\]
By Lemma \((1.6)\) this equation can be solved if and only if \(\int_p \omega \equiv 0\), i.e. if and only if the first Françoise condition \(M_1 \equiv 0\) is satisfied. Therefore, the existence of \(\Omega\) satisfying \((2.1)\) for \(k = 0\) is equivalent to the first Françoise condition, and we can choose \(r_1\) in \((1.6)\) to be equal to \(R_1\),
\[
\omega = d\epsilon + g_1 dF.
\]
Hence,
\[
\Omega = r_1 d\epsilon + (dF + \omega)(1 + \epsilon \beta_1) + o_1 \tag{3.2}
\]
for some function \(\beta_1\). Therefore,
\[
\Omega = [r_1 d\epsilon + (dF + \omega)(1 - \epsilon g_1)] (1 + \epsilon (\beta_1 + g_1)) + o_1 = N_1 \tilde{d} F_{\epsilon,1} + o_1,
\]
where \(N_1 = 1 + \epsilon (\beta_1 + g_1)\) and \(F_{\epsilon,1} = F + \epsilon r_0\).

Now, let \(k > 0\) and assume \((3.1)\). In particular, it means that \(\Omega \wedge \tilde{d}\Omega = o_k\). By induction, we have
\[
\Omega = N_{k-1} \tilde{d} F_{\epsilon,k-1} + o_{k-1}, \quad \text{where} \quad F_{\epsilon,k-1} = F + \epsilon r_0 - \ldots + (-1)^{k-2} \epsilon^{k-1} r_{k-1}.
\]
Define
\[
\Theta = N_{k-1} \Omega = \tilde{d} F_{\epsilon,k-1} + \theta_k + o_k,
\]
where \( \theta_k \) is homogeneous of weight \( k \). We have \( \Theta \wedge \tilde{d}\Theta = \Omega \wedge \tilde{d}\Omega = o_k \).

But \( \tilde{d}\Theta = \tilde{d}\theta_k \) has weight \( k \). Therefore

\[
\Theta \wedge \tilde{d}\Theta = dF \wedge \tilde{d}\theta_k + o_k.
\]

Note that \( \Theta \) has form \((1.10)\), with \( G_i, R_i \) as in \((2.2)\) for \( i \leq k - 1 \). Separating terms of weight \( k \), we get

\[
\theta_k = \epsilon^{k-1}R_k \epsilon + \epsilon^kG_k dF + (-1)^{k-1} \epsilon^k g_{k-1} \omega.
\]

Therefore

\[
0 = dF \wedge \tilde{d}\theta_k = \epsilon^{k-1}dF \wedge d\epsilon \wedge ((-1)^{k-1}kg_{k-1}\omega - dR_k).
\]

As \( \Theta \) is a solution of \((2.1)\), this equation is solvable, which, by Lemma 1.1, means that \( \int g_{k-1} \omega \equiv 0 \), i.e. that the \( k \)-th Melnikov function vanishes identically. Moreover, \((3.3)\) implies

\[
kg_{k-1} \omega = (-1)^{k-1}dR_k + kg_k dF,
\]

i.e. Françoise decomposition \((1.6)\) of \( g_{k-1} \omega \) with \( k \)-th Françoise pair \((g_k, r_k)\), such that \( R_k = (-1)^{k-1}kr_k \).

Therefore

\[
\Theta = (r_0 + \ldots + \epsilon^{k-1}(-1)^{k-1}kr_k) d\epsilon
+ (dF + \epsilon\omega) (1 + \ldots + (-1)^{k-1}k^{k-1}g_{k-1} + \epsilon^kG_k) + o_k =
(1 + \epsilon^k(G_k + (-1)^{k-1}g_k)) dF_{e,k} + o_k,
\]

where \( F_{e,k} = F + \epsilon r_o - \ldots + (-1)^{k-1}\epsilon^k r_k \), and

\[
\Omega = N_k \tilde{d}F_{e,k} + o_k, \quad N_k = N_{k-1} (1 + \epsilon^k(G_k + (-1)^{k-1}g_k)),
\]

as required. \(\square\)

4. Proof of Theorem 2.2

Proof: Proof of (i): Let \((g_i, r_i), i = 0, 1, 2, \ldots \) be a Françoise sequence and assume that \( g_i = 0 \), for \( i \geq \ell + 1 \) (see Remark 1.9). Let

\[
\eta_e = dF + \epsilon\omega, \quad G = \sum_{i=0}^\ell (-1)^i \epsilon^i g_i, \quad F_e = F + \sum_{i=1}^\ell (-1)^i \epsilon^{i+1} r_i.
\]

It follows from the definition of Françoise pairs \((1.6)\) that

\[
G\eta_e = dF_e.
\]

Differentiating \((1.6)\) and dividing by \( dF \) (that is, applying the Gelfand-Leray derivative), one obtains

\[
dg_i = \frac{dg_{i-1} \wedge \omega}{dF} + g_{i-1} \frac{d\omega}{dF} = \eta_{i-1}.
\]

By induction, from the Definition 1.4, \( g_i \) is Darboux, for \( i = 0, \ldots, \ell \). It now follows from \((1.6)\) that \( r_i, i = 1, \ldots, \ell \), is Darboux as well and by Definition 1.6 the first integral \( F_e \) is Darboux with respect to the field \( K_{F,\omega} \).

Proof of (ii): Let \( \eta_e \), \( G \), and \( F_e \) be as in \((1.6)\). Now from \((1.2)\) it follows that

\[
d\eta_e = \frac{dG^{-1}}{G^{-1}} \wedge G^{-1}dF_e = \theta_e \wedge \eta_e, \quad \text{for} \quad \theta_e = \frac{dG^{-1}}{G^{-1}}.
\]
Hence, \( d\theta = 0 \). The two equations together give a Godbillon-Vey sequence of length 2 in a Liouville extension of the space of forms with coefficients in \( K_{F,\omega} \) in \((x, y) \in U\) and holomorphic with respect to the parameter \( \epsilon \).

Now Singer’s theorem \cite{9} (see also \cite{2}) gives that there exists a form \( \tilde{\theta} \) with coefficients in \( K_{F,\omega} \) (holomorphic with respect to \( \epsilon \)) verifying the same Godbillon-Vey equations:

\[
\begin{align*}
d\eta & = \eta \wedge \tilde{\theta}, \\
d\tilde{\theta} & = 0.
\end{align*}
\]

That is \( \tilde{\theta} \) is closed. Hence, there exists a (possibly multivalued) function \( f \) defined in \( U \) such that

\[
df = f\tilde{\theta}.
\]

One verifies that the form \( f\tilde{\theta} \) is closed. This means that there exists a (possibly multivalued) function \( \tilde{F} \) verifying

\[
d\tilde{F} = f\eta.
\]

□

5. Classical Godbillon-Vey sequences and examples

Let \( \Omega \) be the form given by (1.10). We apply the classical Godbillon-Vey condition (1.9) to the form

\[
\Omega \equiv \frac{d\epsilon}{R} = d\epsilon + \eta_0 + \epsilon \eta_1 + \ldots + \frac{\epsilon^i}{i!} \eta_i + \ldots
\]

Comparing to the closed form (1.10), we conclude that the forms \( \eta_1, \ldots \), defined by

\[
\eta_i = \sum_{i=0}^{\infty} \frac{\epsilon^i}{i!} \eta_i = \frac{dF_{\epsilon}}{d\epsilon F_{\epsilon}} = \frac{dF + \sum_{i=1}^{\infty} \epsilon^i (-1)^{i-1} dr_i}{\sum_{i=1}^{\infty} (-1)^{i-1} i\epsilon^{i-1} r_i}
\]

from a Godbillon-Vey sequence of \( \frac{dF}{R_1} \).

\[
\eta_0 = R_1^{-1} dF, \quad \eta_1 = R_1^{-1} (2R_2 dF + dR_1), \ldots,
\]

and the forms

\[
\eta_1 = 2R_1^{-1} (R_2 dF + dR_1), \ldots, \eta_i = R_1^{i-1} \eta_i,
\]

form a Godbillon-Vey sequence of \( dF \). This sequence could be infinite.

In classical setting one starts from a given foliation \( \omega = 0 \) for \( \epsilon = 0 \), and looks for a simplest perturbation \( \omega_\epsilon = 0 \) such that the form \( d\epsilon + \omega_\epsilon \) is integrable. Results of \cite{2} say that if the foliation \( \omega = 0 \) is Darboux integrable, Liouville integrable or Riccati integrable, then one can find perturbations such that \( \omega_\epsilon \) either does not depend on \( \epsilon \) or is polynomial in \( \epsilon \) of degree 1 or 2, respectively, i.e., that the Godbillon-Vey sequence has finite length.

In this paper, given a perturbation (1.1), and we construct a one-form \( \Omega \) such that its restriction to the planes \( \{ \epsilon = \text{const} \} \) defines the same foliation as the initial one. In other words, unlike the classical settings, here the perturbation of the foliation is almost uniquely prescribed, the only freedom being the coefficients \( G, R \) in (1.10). Thus the length of the corresponding Godbillon-Vey sequence can be infinite even if \( \omega_\epsilon \) is Liouville integrable for all \( \epsilon \).
Example 5.1. Let \( F = x^2 + y^2 \) and \( \omega = y^2 dx \). For symmetry reasons, the perturbation (1.1) is integrable. Computation shows that
\[
g_n = \frac{(-1)^n}{n!} x^n, \quad r_n = \frac{(-1)^n}{n!} \left( \frac{2}{n+2} x^{n+2} - x^n y^2 \right). \tag{5.2}
\]
Then a first integral is given by
\[
F_\epsilon = x^2 + y^2 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \epsilon^n r_n = e^{\epsilon x} \left( y^2 + 2 \frac{x}{\epsilon} - 2 \epsilon^{-2} \right),
\]
\[
\tilde{d}F_\epsilon = e^{\epsilon x} (ey^2 + 2x) dx + 2e^{\epsilon x} y dy,
\]
and therefore the Godbillon-Vey forms \( \omega_i \) are Taylor coefficients in \( \epsilon \) of
\[
\left( \frac{\partial}{\partial \epsilon} F_\epsilon \right)^{-1} dF_\epsilon = dF + \sum_{n=1}^{\infty} \epsilon^n \omega_i. \tag{5.3}
\]
One can see that this series is not polynomial in \( \epsilon \), though the first integral \( F_\epsilon \) is of Liouville type. Some authors call this type of functions generalized Darboux.

Example 5.2. For a trivially integrable perturbation \( \omega = gdF \), the Françoise pairs are given by
\[
g_i = g^i, \quad r_i \equiv 1, \quad r_i = 0 \text{ for } i = 2, \ldots.
\]
Therefore the first integral is
\[
F_\epsilon = F + \epsilon, \quad R \equiv 1, \quad G = 1 + \sum_{i=1}^{\infty} (-1)^i \epsilon^i g^i = (1 - \epsilon g)^{-1},
\]
and
\[
\Omega = d\epsilon + (dF + \epsilon gdF)(1 - \epsilon g)^{-1} = dF_\epsilon.
\]

Example 5.3. For a Darboux integrable perturbation (1.1) with \( \omega = F \frac{dr}{r \log r} \), we have
\[
g_i = (-1)^i \frac{\log r}{i!}, \quad r_i = -F g_i,
\]
so
\[
F_\epsilon = F + \sum_{i=1}^{\infty} (-1)^{i-1} \epsilon^i r_i = F + \epsilon F \sum_{i=1}^{\infty} \frac{\epsilon^i}{i!} \log r = F e^{\epsilon \log r} = Fr^\epsilon.
\]
Then
\[
\tilde{d}F_\epsilon = Fr^\epsilon \log r d\epsilon + \left( r^\epsilon dF + \epsilon Fr^{\epsilon-1} dr \right) = Fr^\epsilon \log r \left[ d\epsilon + \left( \frac{dF}{F \log r} + \epsilon \frac{dr}{r \log r} \right) \right].
\]
Therefore the forms
\[
\omega_1 = \frac{dr}{r \log r}, \quad \omega_i = 0, \quad i = 2, \ldots
\]
form a Godbillon-Vey sequence for \( \frac{dF}{F \log r} \), and hence
\[
\tilde{\omega}_1 = \frac{dr}{r \log r} + \frac{dF}{F} + \frac{dr}{r}, \quad \tilde{\omega}_i = 0, \quad i = 2, \ldots
\]
form a Godbillon-Vey sequence for \( dF \), so this Godbillon-Vey sequence for \( dF + \epsilon \omega \) has length 1.

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References

[1] Casale, G. Suites de Godbillon-Vey et intégrales premières. (French. English, French summary) [Godbillon-Vey sequences and first integrals] C. R. Math. Acad. Sci. Paris 335 (2002), no. 12, 1003-1006.

[2] Casale, G. Sur le groupoïde de Galois d’un feuilletage, Thesis, Université Paul Sabatier, Toulouse, (2004), 1-92.

[3] Francoise, J.-P., Successive derivatives of a first return map, application to the study of quadratic vector fields. Ergodic Theory Dynam. Systems 16 (1996), no. 1, 87-96.

[4] Gavrilov, L. Higher order Poincaré-Pontryagin functions and iterated path integrals, Ann. Fac. Sci. Toulouse Math. (6) 14 (2005), no. 4, 663-682.

[5] Gavrilov, L., Iliev, I. D., The displacement map associated to polynomial unfoldings of planar Hamiltonian vector fields, Amer. J. Math. 127 (2005), no. 6, 1153-1190.

[6] Godbillon, C., Vey, J. Un invarant des feuilletages de codimension 1. (French) C. R. Acad. Sci. Paris Sér. A-B 273 1971 A92-A95.

[7] Jebrane, A., Mardešić, P., Pelletier, M., A generalization of Françoise’s algorithm for calculating higher order Melnikov functions, Bull. Sci. Math. 126 (2002), no. 9, 705-732.

[8] Jebrane, A., Mardešić, P., Pelletier, M., A note on a generalization of Françoise’s algorithm for calculating higher order Melnikov functions, Bull. Sci. Math. 128 (2004), no. 9, 749-760.

[9] Singer, M. Liouvillian first integrals of differential equations. Trans. Amer. Math. Soc. 333 (1992), no. 2, 673-688. Ann. Sci. École Norm. Sup. (4) 7 (1974), 405-430.

[10] Pontigo Herrera, J. The first nonzero Melnikov function for a family of good divides, to appear in RACSAM.

[11] Uribe, M. Principal Poincaré-Pontryagin function of polynomial perturbations of the Hamiltonian triangle, J. Dyn. Control Syst. 12 (2006), no. 1, 109-134.

[12] Uribe, M., Principal Poincaré-Pontryagin function associated to polynomial perturbations of a product of \((d+1)\) straight lines. J. Differential Equations 246 (2009), no. 4, 1313-1341.

[13] Yakovenko, S. A geometric proof of the Bautin theorem. Concerning the Hilbert 16th problem, 203-219, Amer. Math. Soc. Transl. Ser. 2, 165, Adv. Math. Sci., 23, Amer. Math. Soc., Providence, RI, 1995.

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