Canonical Quantization of Geometrized Mechanics

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**Abstract.** A particle in 3D space with certain potential will move in a curved trajectory like a comet in gravitational potential caused by the star. On the other hand, a free particle in curved space also moves according to geometry of that space. In this paper, the connection between potential energy and space metric will be discussed. So the formulation of classical mechanics in geometric terms can be found and the canonical quantization of it can be carried out. At the end of this paper, as an example, we will consider a particle under isotropic harmonic oscillator potential in two-dimensional sphere, carry out the canonical quantization, and then calculate the energies and their states.

1. **Introduction**

There is no unique ‘recipe’ for quantization of a classical theory. Several methods had been proposed to construct the Schrodinger equation from the classical theory. For example N T De Oliveira and R Lobo [1] found the Schrodinger equation of the classical theory of a free particle in curved space by using propagator and then expanding the corresponding action in series, taking the first two term. De Witt [2] found the Schrodinger equation of classical theory of a free particle in curved space with certain potential energy in a quite complicated way and expressed it in kinetic term containing space metric, quantum potential containing metric and Christoffel symbol, and the potential energy term. The other approach is using Wigner function and Weyl symbol as [3]. There is also another way to quantize classical theory of a particle in a curvature dependent space with harmonic oscillator potential influence done by Jose F Carinena, Manuel F Ranada, and Mariano Santander [4]. They construct the quantum Hamiltonian in two terms, i.e. the kinetic term containing curvature dependent space metric and the potential energy term. In this paper, we will formulate the Schrodinger equation of the classical theory of a free particle in curved space with certain potential energy by geometrizing it first [5]. Geometrizing is fusing the potential energy term with space metric to form new space metric. In this way, quantization of classical system can be done systematically and in more simple way.

2. **Hamiltonian Mechanics**

A mechanical system can be described by a set of dynamical variables which are functions of time. For example, in 3D Cartesian coordinate, a point of mass is expressed by \( \mathbf{r}(t) = (x(t), y(t), z(t)) \). This formulation can be generalized by using generalized coordinates \( q^\alpha(t) \) with \( \alpha = (1, 2, 3, \ldots, N) \) so \( \mathbf{r}(t) = \mathbf{r}(q^\alpha(t)) \).

Because \( \mathbf{r} \) is not an explicit function of \( t \),
\[ \dot{v} = \dot{r} = \frac{\partial \dot{r}}{\partial q^a} \dot{q}^a \rightarrow \frac{\partial \dot{r}}{\partial q^a} = \frac{\partial \dot{r}}{\partial q^a}. \]  

(1)

The above equation is named 'dot cancelation equation'. At fixed time, an infinitesimal shift in position is related to an infinitesimal shift in generalized coordinate,

\[ \delta \dot{r} = \frac{\partial \dot{r}}{\partial q^a} \delta q^a. \]  

(2)

The work in differential form is equal to,

\[ \delta W = \mathbf{F} \cdot \delta \dot{r} = \mathbf{F} \cdot \frac{\partial \dot{r}}{\partial q^a} \delta q^a = \mathcal{F}_a \delta q^a. \]  

(3)

\( \mathcal{F}_a \) is the \( a^{th} \) component of generalized force. The kinetic energy of a particle with mass \( m \) is \( T = \frac{1}{2} m \dot{r} \cdot \dot{r} \) and the relation with \( \mathcal{F}_a \) can be found as follows. The general equation of motion can be derived as follows,

\[ \frac{\partial T}{\partial q^a} = m \dot{r} \cdot \frac{\partial \dot{r}}{\partial q^a}, \quad \frac{\partial T}{\partial \dot{q}^a} = m \ddot{r} \cdot \frac{\partial \dot{r}}{\partial q^a} = m \ddot{r} \cdot \frac{\partial \dot{r}}{\partial q^a}, \]

\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}^a} \right) = m \dot{r} \cdot \frac{\partial \dot{r}}{\partial q^a} + m \ddot{r} \cdot \frac{\partial \dot{r}}{\partial q^a} = \mathcal{F}_a + \frac{\partial T}{\partial q^a}, \]

\[ \frac{d}{dt} \left( \frac{\partial T}{\partial q^a} \right) - \frac{\partial T}{\partial q^a} = \mathcal{F}_a. \]  

(4)

(5)

If \( \mathbf{F} \) is a conservative force \( \mathbf{F} = -\nabla V(\mathbf{r}) \), then,

\[ W = \int \mathbf{F} \cdot d\mathbf{r} = - \int \nabla V(\mathbf{r}) \cdot d\mathbf{r} = - \int \nabla V(q) \cdot \frac{\partial \dot{r}}{\partial q^a} dq^a = - \int \frac{\partial V(q)}{\partial q^a} dq^a. \]  

(6)

Compare equation (6) with equation (3) then it follows,

\[ \mathcal{F}_a = \mathbf{F} \cdot \frac{\partial \dot{r}}{\partial q^a} = - \frac{\partial V(q)}{\partial q^a}. \]  

(7)

Substitute equation (7) to equation (4),

\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}^a} \right) - \frac{\partial T}{\partial q^a} = - \frac{\partial V}{\partial q^a}, \]

\[ \frac{d}{dt} \left( \frac{\partial (T - V)}{\partial q^a} \right) - \frac{\partial (T - V)}{\partial q^a} = 0. \]  

(8)

Define Lagrangian as \( L \equiv T - V \), then we get the famous Euler-Lagrange equation,
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^a} \right) - \frac{\partial L}{\partial q^a} = 0, \quad (9)
\]

\[
p_a = \frac{\partial L}{\partial \dot{q}^a}. \quad (10)
\]

Equation (9) is also solution of variational calculus problem, that extremize the following Lagrangian action,

\[
S = \int_{t_1}^{t_2} L(q(t), \dot{q}(t), t) dt. \quad (11)
\]

Then the Hamiltonian as function of phase space can be determined by Legendre transformation as follows,

\[
H(p, q, t) = p_a \dot{q}^a - L(q, q, t), \quad (12)
\]

\[
\frac{\partial H}{\partial p_c} dp_c + \frac{\partial H}{\partial q^a} dq^a + \frac{\partial H}{\partial t} dt = q^a dp_a - p_a dq^a - \frac{\partial L}{\partial t} dt, \quad (13)
\]

\[
\frac{\partial H}{\partial p_a} = q^a, \quad \frac{\partial H}{\partial q^a} = -p_a, \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}. \quad (14)
\]

3. Geometrization of Classical Mechanics

Let \((q^1(t), q^2(t), \ldots, q^N(t))\) be solution of Euler-Lagrange equations and one of them, i.e. \(q^1(t)\), can be solved algebraically to find \(t = t(q^1)\). Then substitute \(t = t(q^1)\) to \([N'] = [N] - [1]\) (IA expresses number) other equations to get \(q^{N'} = q^{N'}(q^1)\) with \(N' = 2, 3, 4, \ldots, N\).

In phase space, \(q^{N'}(q^1)\) which are parametrized by \(q^1\) are called the equation of particle’s trajectory. The time-free \(q^{N'}(q^1)\) are called kinematic variables and \(q^1(t)\) is called dynamical variable. The idea is how to modify \(S = \int L dt\) to \(S = \int L dq^1\). This mechanics is defined in geometric setting.

Euler-Lagrange equation for Lagrangian \(L(q, \dot{q})\) in matrix form is as follows,

\[
M_{ab} \ddot{q}^b = K_a, \quad (15)
\]

\[
M_{ab} = \frac{\partial^2 L}{\partial q^a \partial q^b}, \quad K_a = \frac{\partial L}{\partial \dot{q}^a} - \frac{\partial^2 L}{\partial q^b \partial q^a} \dot{q}^b. \quad (16)
\]

Using these notations \(\dot{q}^{N'} = \frac{dq^{N'}}{dt}\) and \(q^{N'*} = \frac{dq^{N'}}{dq^1}\), it follows,

\[
\dot{q}^{N'}(q^1) = \frac{d q^{N'}(t)}{dt} = q^{N'*} \dot{q}^1, \quad (17)
\]

\[
\ddot{q}^{N'}(q^1) = q^{N'*'}(q^1)^2 + q^{N'*} \ddot{q}^1. \quad (18)
\]

Then \(q^{1} = M^{1a} K_a = R^1\) and \(q^b = M^{ba} K_a = R^b\). So,
\[ q_{N^*}^* = \frac{R_{N^*} - q_{N^*}^* R^1}{(q^1)^2}. \] (19)

Let \( L(q, \dot{q}) \) has the following form,

\[ L = \frac{1}{2} m \delta_{cd} \ddot{q}^c q^d - U, \] (20)

and the Euler-Lagrange equation becomes,

\[
\frac{\partial^2 L}{\partial q^a \partial q^b} q^b = \frac{\partial L}{\partial q^a} - \frac{\partial^2 L}{\partial q^a \partial \dot{q}^b} \dot{q}^b, \\
m \dot{q}^a = -\frac{\partial U}{\partial q^a}, \\
q_{N^*}^* = \frac{\partial U}{\partial q^a} + q_{N^*}^* \frac{\partial U}{\partial \dot{q}^a}. \] (23)

To determine \( \dot{q}^1 \), determine the Hamiltonian associated with the above Lagrangian first,

\[ h = H = \frac{1}{2m} \delta^{cd} p_c p_d + U ; p_b = m \delta_{ba} q^a, \] (24)

\[ h = \frac{m}{2} (\delta_{cd} \ddot{q}^c q^d) + U = \frac{m}{2} (\delta_{cd} q^c q^d) (\dot{q}^1)^2 + U, \\
q^1 = \sqrt{\frac{2}{m \delta_{cd} q^c q^d} (h - U)} \] (25)

Because \( \sqrt{2 m \delta_{cd} q^c q^d} \) is not a function of \( t \), so it can be integrated to get \( q^1 = \sqrt{2 \frac{h - U}{m \delta_{cd} q^c q^d}} t + c \). The solution of equations of motion can be separated into two parts, the kinematic (19) dan dynamicals (25). The Hamiltonian equations of motion parametrized by \( q^1 \) are expressed as follows,

\[
q_{N^*}^* = \frac{\partial H}{\partial p_{N^*}}, \\
p_{N^*} = -\frac{\partial H}{\partial q_{N^*}}, \\
q_{N^*}^* = \frac{\partial H}{\partial \dot{p}_{N^*}}, \\
p_{N^*} = \frac{\partial H}{\partial \dot{q}_{N^*}}. \] (26, 27)

Assume \( p_1 \) can be solved algebraically from Hamiltonian function \( H(q^1, p_1, q_{N^*}, p_{N^*}) = h \) with \( h \) a constant,

\[ p_1 = -\bar{H}(q^1, q_{N^*}, p_{N^*}, h). \] (28)

Substitute equation (31) to \( H(q^1, p_1, q_{N^*}, p_{N^*}) = h \),

\[ h = H(q^1, -\bar{H}(q^1, q_{N^*}, p_{N^*}, h), q_{N^*}, p_{N^*}). \] (29)
\[
\frac{\partial H}{\partial q^{N'}} = \frac{\partial H}{\partial p_1} \frac{\partial \bar{H}}{\partial q^{N'}},
\]
\[
\frac{\partial H}{\partial p} = \frac{\partial H}{\partial \bar{H}} \frac{\partial \bar{H}}{\partial p_N},
\]

Modify the Hamiltonian equation of motion with help of equations (30) and (31),

\[
\dot{q}^{N'} = \frac{\partial H}{\partial p_N} = q^{N'} \frac{\partial H}{\partial p_1},
\]
\[
\frac{\partial H}{\partial p_1} \frac{\partial \bar{H}}{\partial p_N} = q^{N'} \frac{\partial H}{\partial p_1} \rightarrow q^{N'} = \frac{\partial \bar{H}}{\partial p_N},
\]
\[
p^{N'} = -\frac{\partial H}{\partial q^{N'}} = p_{N'} \frac{\partial H}{\partial p_1},
\]
\[
-\frac{\partial H}{\partial p_1} \frac{\partial \bar{H}}{\partial q^{N'}} = p_{N'} \frac{\partial H}{\partial p_1} \rightarrow p^{N'} = -\frac{\partial \bar{H}}{\partial q^{N'}}.
\]

Equations (35) and (36) are called Whittaker equation. So, the kinematic part consists of three following equations,

\[
p_1 = -\bar{H}(q^1, q^{N'}, p_{N'}, h),
\]
\[
q^{N*} = \frac{\partial \bar{H}}{\partial p_N},
\]
\[
p^{N*} = -\frac{\partial \bar{H}}{\partial p_N},
\]

The dynamical equation has following form.

\[
q^1 = \int \frac{\partial H}{\partial p_1} dt = \frac{\partial H}{\partial p_1} t + c.
\]

The form of Lagrangian parametrized by \(q^1\) is still not determined. The idea comes from Hamiltonization of \(L\), i.e. \(H = p_N q^N - L\) leads to Hamiltonian equation of motions \(q^{N} = \frac{\partial H}{\partial p_N}\) and \(p_{N'} = -\frac{\partial H}{\partial q^{N'}}\). Moreover, Lagrangian parametrized by \(q^1\) has Whittaker equation as the equation of motion \(q^{N*} = \frac{\partial \bar{H}}{\partial p_N}\) dan \(p_{N*} = -\frac{\partial \bar{H}}{\partial p_N}\). So, after comparing the two above equations, it’s sufficient to state that \(\bar{H} = p_N q^{N*} - L\). Then, do these following tricks to carry out the Lagrangian parametrized by \(q^1\).

\[
H = h = p_N q^N - L,
\]
\[
h = p_1 q^1 + p_{N'} q^{N'} - L,
\]
\[
h = \left( p_1 q^1 + p_{N'} q^{N*} \right) q^1 - L.
\]
\[
\mathcal{S} = \int L dq^1 = \int \left( p_{N'} q^{N*} - \bar{H} \right) dq^1
\]
Equation (39) can be rearranged to show $p_N q_N^* = \frac{h+L}{q^1} - p_1 q_1^*$ but $-\overline{H} = p_1$. 

$$\overline{S} = \int \overline{L} dq^1 = \int \frac{h+L}{q^1} dq^1 \rightarrow \overline{L} = \frac{h+L}{q^1}.$$  (40)

A particle with mass $m$ in curved space with metric $g_{ij}$ has Lagrangian and equation of motion as follows,

$$L = \frac{1}{2} mg_{ij} \dot{q}^i \dot{q}^j,$$  (41)

$$\ddot{q}^i = \frac{g^{ce}}{2} \left( \partial_\gamma g_{ab} - \partial_a g_{\gamma b} - \partial_b g_{\gamma a} \right) \dot{q}^a \dot{q}^b,$$  (42)

$$\ddot{q}^i = -\Gamma^{ce}_{ab} q^a \dot{q}^b \Leftrightarrow \ddot{q}^i + \Gamma^{ce}_{ab} q^a \dot{q}^b = 0.$$  (43)

A particle with mass $m$ moving in space with metric $\zeta_{ab}$ and have potential energy $U$ has Lagrangian and equation of motion as follows,

$$L = \frac{1}{2} m \zeta_{ij} \dot{q}^i \dot{q}^j - U,$$  (44)

$$L = L_0 - U,$$  (44)

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^c} \right) - \frac{\partial L}{\partial q^c} = 0,$$  (44)

$$\frac{d}{dt} \left( \frac{\partial L_0}{\partial \dot{q}^c} \right) - \frac{\partial L_0}{\partial q^c} + \frac{\partial U}{\partial q^c} = 0,$$  (45)

$$q^\alpha + \Gamma^{ce}_{ab} (\zeta) q^a \dot{q}^b + \frac{\zeta^{ce}}{m} \frac{\partial U}{\partial q^c} = 0.$$  (46)

Now, make the following substitution,

$$\zeta^{ce} = \chi(U) g^{ce} \Leftrightarrow \zeta_{ce} = \frac{q}{\chi(U)} g_{ce},$$  (47)

$$\Gamma^{ed}_{ab} (\zeta) = -\frac{\zeta^{ed}}{2} \left( \partial_a \zeta_{ab} - \partial_a \zeta_{db} - \partial_b \zeta_{da} \right),$$

$$\Gamma^{e}_{ab} (\zeta) = -\frac{\chi g^{ed}}{2} \left( \partial_a \left( \frac{1}{\chi} g_{ab} \right) - \partial_a \left( \frac{1}{\chi} g_{db} \right) - \partial_b \left( \frac{1}{\chi} g_{ad} \right) \right),$$

$$\chi g^{ed}_{ab} \frac{\partial_a \partial_b \partial_a \chi - g_{ab} \partial_a \partial_b \chi}{2 \chi^2},$$

$$\Gamma^{e}_{ab} (\zeta) = \Gamma^{e}_{ab} + \frac{g^{ed}}{2\chi} (g_{ab} \partial_a \chi - g_{ab} \partial_a \chi - g_{ab} \partial_b \chi).$$  (48)

Substitute equation (48) into equation (46),

$$q^\alpha + \left\{ \Gamma^{e}_{ab} + \frac{g^{ed}}{2\chi} (g_{ab} \partial_a \chi - g_{ab} \partial_a \chi - g_{ab} \partial_b \chi) \right\} q^a \dot{q}^b + \frac{\chi g^{ce}}{m} \partial_c U = 0,$$
\[ q^a + \Gamma^a_{ab} q^a q^b + g_{ab} \left( \frac{\partial_a X}{2X} g_{bc} q^c q^b + \frac{X}{m} \partial_q U \right) - \frac{\partial_a X}{X} q^a q^b = 0. \] (49)

Do Hamiltonization to Lagrangian \( L = \frac{1}{2} m \xi_{ij} q^i q^j - U, \)

\[ H = \frac{1}{2} m \xi_{ij} \dot{q}^i \dot{q}^j + U = \frac{m}{2X(U)} g_{ij} \dot{q}^i \dot{q}^j + U, \] (50)

\[ g_{ij} \dot{q}^i \dot{q}^j = \frac{h - U}{2X}, \] (51)

\[ \dot{q}^a + \Gamma^a_{ab} q^a q^b + g_{ab} \left( \frac{h - U}{m} \partial_a X + \frac{X}{m} \partial_a U \right) - \frac{X}{\dot{X}} \dot{q}^b = 0, \] (52)

\[ \ddot{q}^a + \Gamma^a_{ab} q^a \dot{q}^b \frac{\alpha}{\alpha} \dot{q}^d = 0. \] (53)

Equation (53) is the general geodesic equation of a free particle moving in a space with metric \( g_{ij}. \)
If \( \alpha = 1 \) the geodesic equation is called canonically parametrized and has the same form with equation (42). In order to make equation (53) as general geodesic equation the third term must vanish,

\[ \frac{h - U}{m} \partial_a X + \frac{X}{m} \partial_a U = 0, \] (54)

\[ \frac{\partial_a X}{X} = \frac{\partial_a (U - h)}{(U - h)} \rightarrow \chi = c(U - h), \] (55)

\[ \chi = \alpha. \] (56)

After finding the solution it can be concluded that a particle with mass \( m \) under influence of a potential and moving in a curved space– with metric \( \zeta_{ab} \)–is equivalent to a free particle with mass \( m \) moving in a curved space with metric \( g_{ab} = \chi(U) \zeta_{ab} = c(U - h) \zeta_{ab}. \) Conventionally the constant is taken as \( c = -\frac{1}{h}. \)

4. Quantization
The quantization of Hamiltonian is achieved by replacing the observables with operator,

\[ \hat{H} \psi = \frac{1}{2m} \partial^2 \psi, \] (57)

\[ x \rightarrow \xi = x, \quad p \rightarrow \hat{p} = i\hbar \hat{\nabla}, \quad H \rightarrow \hat{H} = i\hbar \frac{\partial}{\partial t}, \] (58)

\[ \hat{H} \psi = -\frac{\hbar^2}{2m} \nabla^2 \psi. \] (59)

The general form of Laplacian operator \( \nabla^2 \) operating in space with metric \( g_{ij} \) is named Laplace-Beltrami operator which has following form,

\[ \nabla^2 \psi = \frac{1}{\sqrt{g}} \partial_i \left( \sqrt{g} g^{ij} \partial_j \psi \right), \quad g_{ab} = \frac{(U - h)}{h} \zeta_{ab}, \] (60)

\[ \hat{H} \psi = -\frac{h}{h - U} \frac{\hbar^2}{2m} \frac{1}{\sqrt{\zeta}} \partial_i \left( \sqrt{\zeta} \zeta^{ij} \partial_j \psi \right). \] (61)
Suppose there is a particle with mass $m$ living in two-dimensional sphere with metric $\zeta_{ab} = \text{diag}(1, r^2, r^2 \sin^2 \theta)$ and influenced by isotropic harmonic oscillator potential $U = \frac{1}{2} m \omega^2 r^2$. This case is equivalent to a free particle in a space with following metric $g_{ab}$.

$$g_{ab} = \left( \frac{h - \frac{1}{2} m \omega^2 r^2}{h} \right) \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{pmatrix}, \quad \sqrt{\zeta} = r^2 \sin \theta. \quad (62)$$

The associated Schrodinger equation is as follows,

$$\hbar \psi = -\frac{h}{h - \frac{1}{2} m \omega^2 r^2} \frac{\hbar^2}{2 m} \left[ \frac{1}{r^2} \partial_r (r^2 \partial_r) + \frac{1}{r^2 \sin \theta} \partial_\theta (\sin \theta \partial_\theta) + \frac{1}{r^2 \sin^2 \theta} \partial_\varphi \right] \psi \quad \text{(63)}$$

$$0 = \frac{\hbar^2}{2 m} \left[ \frac{1}{r^2} \partial_r (r^2 \partial_r) + \frac{1}{r^2 \sin \theta} \partial_\theta (\sin \theta \partial_\theta) + \frac{1}{r^2 \sin^2 \theta} \partial_\varphi \right] \psi + \left( E - \frac{1}{2} m \omega^2 r^2 \right) \psi. \quad (64)$$

To solve the Schrodinger equation to determine the energies and wave functions, let us assume $\psi = R(r)T(\theta)F(\varphi)$. The polar and azimuthal solutions can be obtained easily,

$$T(\theta)F(\varphi) = C_\alpha P_l^m(\cos \theta) e^{im\varphi}, \quad (65)$$

$$P_l^m(\cos \theta) = \frac{1}{2^l l!} (1 - \cos^2 \theta)^l \left( \frac{d}{d \cos \theta} \right)^{m+l} (\cos^2 \theta - 1)^l. \quad (66)$$

The radial equation has the following expression,

$$0 = \frac{\hbar^2}{2 m} \left[ \frac{1}{r^2} d_r (r^2 d_r R) - \frac{l(l + 1)}{r^2} R \right] + \left( E - \frac{1}{2} m \omega^2 r^2 \right) R. \quad (67)$$

First, make these substitutions,

$$R(r) = \frac{\varpi(r)}{r}, \quad d_r R = \frac{1}{r} d_r \varpi - \frac{\varpi}{r^2}, \quad d_r (r^2 d_r R) = r d_r^2 \varpi. \quad (68)$$

Then, multiply the radial equation with $\frac{1}{r}$, so we get the following equation,

$$0 = \frac{\hbar^2}{2 m} \left[ d_r^2 \varpi - \frac{l(l + 1)}{r^2} \varpi \right] + \left( E - \frac{1}{2} m \omega^2 r^2 \right) \varpi. \quad (69)$$

To simplify equation (69), make the following substitutions,

$$\varpi(r) = \sigma(\rho), \rho = \alpha r, \quad \alpha = \sqrt{\frac{m \omega}{h}}, \quad \varepsilon = \frac{E}{h \omega}, \quad (70)$$

$$d_r = \alpha d_\rho, \quad d_r^2 = \alpha^2 d_\rho^2. \quad (71)$$

So, we get the simplified radial equation,
\[0 = \left[ d_{\rho}^2 - \frac{l(l+1)}{\rho^2} + \rho^2 - 2\epsilon \right] \sigma. \] (72)

To find the solution of the simplified radial equation, one must consider the asymptotic behaviour for case \(\rho \to 0\),

\[0 = \left[ d_{\rho}^2 - \frac{l(l+1)}{\rho^2} \right] \sigma \to \sigma = \sigma_0 \rho^{l+1}, \] (73)

and for case \(\rho \to \infty\),

\[0 = \left[ d_{\rho}^2 + \rho^2 \right] \sigma \to \sigma = \sigma_0 e^{-\frac{\rho^2}{2}}. \] (74)

Hence the solution must have form \(\sigma(\rho) = \sigma_0 \rho^{l+1} e^{-\frac{\rho^2}{2}} f(\rho)\). Resubstitute to find \(f(\rho)\),

\[d_{\rho} \rho \sigma = \rho' e^{-\frac{\rho^2}{2}} \left\{ \rho d_{\rho} + (l+1) - \rho^2 \right\} f, \] (75)
\[d_{\rho}^2 \sigma = \rho' e^{-\frac{\rho^2}{2}} \left\{ \rho d_{\rho}^2 + 2(l+1)d_{\rho} + 2\rho^2 d_{\rho} + \rho^3 - (2l+3)\rho + (l+1)\rho^{-1} \right\} f. \] (76)

Next, multiply the radial equation with \(\rho^{-l} e^{-\frac{\rho^2}{2}}\) to get the following equation,

\[0 = \left[ d_{\rho}^2 + 2 \left\{ \frac{l+1}{\rho} - \rho \right\} d_{\rho} + (2\epsilon - (2l+3)) \right] f. \] (77)

Then after making these substitutions,

\[\sigma(\rho) = \xi(y), y = \rho^2, \] (78)
\[d_{\rho} = 2 \sqrt{y} d_{y}, \quad d_{\rho}^2 = 2 d_{y} + 4 y d_{y}^2, \] (79)

the equation becomes very elegant,

\[0 = \left[ y d_{y}^2 + \left\{ \left( l + \frac{3}{2} \right) - y \right\} d_{y} + \left\{ \epsilon - \frac{1}{2} \left( l + \frac{3}{2} \right) \right\} \right] \xi. \] (80)

Let \(l + \frac{3}{2} = b\) and \(\left\{ \epsilon - \frac{1}{2} \left( l + \frac{3}{2} \right) \right\} = -a\) so,

\[0 = \left[ y d_{y}^2 + (b - y) d_{y} - a \right] \xi. \] (81)

The second order differential equation above has solution called the hypergeometric series [6].
\[ \xi = H_0 H(a; b, y) = H_0 \sum_{r=0}^{\infty} \frac{\Gamma(a + r)}{\Gamma(a)} \frac{y^r}{r!}. \tag{82} \]

Therefore, the radial wave function is,

\[ R(r) = R_0 \left( \frac{m\omega}{\hbar} \right)^{l+\frac{1}{2}} r^l e^{-\frac{m\omega}{2\hbar} r^2} H \left( \frac{1}{2} \left( l + \frac{3}{2} \right) - \frac{\varepsilon}{2} \right); \left( l + \frac{3}{2}, \frac{m\omega}{\hbar} r^2 \right). \tag{83} \]

Now define \( n_r = \frac{\varepsilon}{2} - \frac{1}{2} \left( l + \frac{3}{2} \right) \rightarrow \varepsilon = 2n_r + l + \frac{3}{2} \). So,

\[ E = \hbar \omega \left( 2n_r + l + \frac{3}{2} \right) = \hbar \omega \left( n + \frac{3}{2} \right) = \hbar. \tag{84} \]

Unlike the radial solution of Hydrogen atom, in this case there is no \( l \leq n \) rule found when solving the radial equation. But there is \( -l \leq m \leq l \) rule arising from the angular part of the wave function. So, the states can be written as 1s, 1p, 1d, ..., 2s, 2p, 2d, ..., 3s, 3p, 3d, ... etc. Moreover after some analysis, we can find there are \( g(n) = \frac{1}{2} (n + 1)(n + 2) \) degeneracies in every energy level.

According to classical formulation, a particle with mass \( m \) in a sphere and influenced by isotropic harmonic oscillator is equivalent to a free particle within curved space with metric (62). Classical mechanics allows any value of energy but after quantization, the particle’s trajectory can only lie on certain energy surfaces, i.e. \( E = \hbar \omega \left( 2n_r + l + \frac{3}{2} \right) = \hbar \).

5. Conclusion

In summary, we have studied the Hamiltonian mechanics and have shown the equivalence of the force formulation and the geometric formulation of a single particle system. We showed the way to express the potential energy of a particle as the curvature of the space in which the particle exists and the metric contains potential energy term (60). Hence, the equation of motion is equivalent to the general geodesic equation.

The quantization of the geometrized classical mechanics is carried out by replacing the observables with operators which leads to general Schrodinger equation (61). As an application for this quantization process, we consider a particle under isotropic harmonic oscillator potential in two-dimensional sphere. After the geometrization process which produces the metric (62), we quantize the system by using the general Schrodinger equation. Solving this Schrodinger equation, we obtain the radial wave function (83) which is given in terms of the hypergeometric series and the quantized energies (84) which is described by the quantum number \( (2n_r + l) \).

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