Influence of long-range interaction on degeneracy of eigenvalues of connection matrix of $d$-dimensional Ising system

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Abstract

We examine connection matrices of Ising systems with long-range interaction on $d$-dimensional hypercube lattices of linear dimensions $L$. We express the eigenvectors of these matrices as the Kronecker products of the eigenvectors for the one-dimensional Ising system. The eigenvalues of the connection matrices are polynomials of the $d$th degree of the eigenvalues for the one-dimensional system. We show that including of the long-range interaction does not remove the degeneracy of the eigenvalues of the connection matrix. We analyze the eigenvalue spectral density in the limit $L \to \infty$. In the case of the continuous spectrum, for $d \leq 2$ we obtain analytical formulas that describe the influence of the long-range interaction on the spectral density and the crucial changes of the spectrum.

Keywords: Ising model, eigenvalues, long-range interaction, Kronecker product

1. Introduction

The Ising model is widely used in various science areas. Commonly it describes a system of interacting particles in the nodes of hypercube lattices. In the book [1], one can find a classical review of various approaches to the analysis of the Ising systems and the obtained results. Applications of the Ising model to the studies of phase transitions in solids can be found in the book [2]. The monograph [3] describes applications of this model in spin glasses and neural
networks. Following the paper [4], there was a series of publications where the authors used the Ising model for training deep neural networks. A collective monograph [5] describes the relations between the Ising model and the problems of binary optimization. Useful references can also be found in [6].

In the present paper, we obtain exact expressions for the eigenvalues and eigenvectors of the Ising connection matrices on hypercube lattices taking into account interactions with an arbitrary number of neighbors. The exact eigenvalues obtained here can be used when calculating the free energy of a spin system [7], in the analysis of the role of long-range hopping in many-body localization for lattice systems of various dimensions (see [8–13] and references therein), and in many other applications.

For natural spin systems, the interaction constants are typically determined by the distances between the spins. Then truncating the number of interactions by accounting only for a finite number of neighbors is an approximation, which holds the better the stronger the interaction decays as a function of distance. However, for artificial spin systems with couplers, such as the ones used for quantum annealing (see for example [14–20]), the obtained expressions are exact.

In section 2, we obtain exact results for the eigenvalues and eigenvectors of the Ising connection matrices with discrete spectra. In section 3, we present the results for a continuous spectrum of the eigenvalues in the limit $L \to \infty$, where $L$ is a linear size of the system. Section 4 contains discussion and conclusions.

2. Eigenvalues spectrum

In this section, we obtain expressions for the eigenvalues and eigenvectors of the connection matrices of the Ising systems on hypercube lattices with an arbitrary long-range interaction and periodic boundary conditions. We, first, examine one-, two- and three-dimensional lattices and then generalize the results to the case of a hypercube.

2.1. 1D Ising model

Let us consider a chain of the length $L$ and set the distance between its nodes to be equal to one. For certainty, we suppose that $L$ is an odd number: $L = 2l + 1$. Let $J(k)$ be an $L \times L$ symmetric matrix that describes the interactions only between spins spaced by the distance $k$ ($k = 1, 2, \ldots, l$). The structure of the matrix $J(k)$ is as follows. The ones occupy the $k$th and $(L - k)$th its diagonals which are parallel to the main diagonal and the other matrix elements are equal to zero. The central row of the matrix $J(k)$ has the form

$$(0 \ldots 1 \ldots 0 \ldots 1 \ldots 0),$$

where the ones are at the distance $k$ from the center. We obtain all other rows by consequent cyclic shifts of the $(l + 1)$th row: shifting it to the left we obtain the $l$th row, the right shift gives the $(l + 2)$-row, and so on. For example, when $L = 7$, the matrices $J(1), J(2)$, and $J(3)$ are

$$J(1) = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \quad J(2) = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}.$$
All the matrices $J(k)$ commute and consequently they all have the same set of the eigenvectors $\{f_\alpha\}_{\alpha = 1}^L$. The components of the vectors $f_\alpha$ are well-known [21]:

$$f_\alpha(i) = \cos \varphi_\alpha i + \sin \varphi_\alpha i, \quad \varphi_\alpha = \frac{2\pi(\alpha - 1)(i - 1)}{L}, \quad i = 1, 2, \ldots, L. \quad (1)$$

Each matrix $J(k)$ has its own set of the eigenvalues $\{\lambda_\alpha(k)\}_{\alpha = 1}^L$:

$$J(k)f_\alpha = \lambda_\alpha(k)f_\alpha, \quad \lambda_\alpha(k) = 2 \cos \left( \frac{2\pi k(\alpha - 1)}{L} \right), \quad \alpha = 1, 2, \ldots, L, \quad k = 1, 2, \ldots, l. \quad (2)$$

Let $w(k)$ be the constant of interaction between spins that are at the distant $k$ from each other, where $k = 1, 2, \ldots, l$. Then for the one-dimensional lattice, the interaction matrix taking account for an arbitrary long-range interaction has the form:

$$A_0 = \sum_{k=1}^l w(k) J(k).$$

The equation (1) define the eigenvectors of this matrix and its eigenvalues, $A_0f_\alpha = \mu_\alpha f_\alpha$, are

$$\mu_\alpha = \sum_{k=1}^l w(k)\lambda_\alpha(k), \quad \alpha = 1, 2, \ldots, L. \quad (2)$$

For simplicity and universality, we introduce the notations,

$$J(0) = I, \quad \lambda_\alpha(0) = 1, \quad \alpha = 1, 2, \ldots, L,$$

where $I$ is an $L \times L$ unit matrix. In addition, we would like to recall the product rules for matrices and vectors that are the Kronecker products. Suppose we have a matrix $M = M_1 \otimes M_2$ and a vector $F = F_1 \otimes F_2$. Then

$$F^\dagger MF = F_1^\dagger \otimes F_2^\dagger M_1 \otimes M_2 F_1 \otimes F_2 = F_1^\dagger M_1 F_1 \cdot F_2^\dagger M_2 F_2. \quad (3)$$

For the following calculations, the rule (3) is very useful.

### 2.2. 2D Ising model

In this subsection we discuss the 2D Ising model that is a system of spins in the nods of a square lattice. By $w(m, k)$ we denote the constant of interaction between spins that are shifted from each other at a distance $m$ along one of the lattice axis and at a distance $k$ along the other axis.
The connection matrix of such a system is an $L^2 \times L^2$ matrix $B_0$. It is convenient to present this matrix as an $L \times L$ block matrix

$$
B_0 = \begin{pmatrix}
A_0 & A_1 & A_2 & \ldots & A_l & A_{l-1} & \ldots & A_2 & A_1 \\
A_1 & A_0 & A_1 & \ldots & A_{l-1} & A_l & \ldots & A_2 & A_1 \\
A_2 & A_1 & A_0 & \ldots & A_{l-2} & A_{l-1} & \ldots & A_2 & A_1 \\
& \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
A_1 & A_2 & A_3 & \ldots & A_{l-1} & A_l & \ldots & A_2 & A_1 \\
A_1 & A_2 & A_3 & \ldots & A_l & A_{l-1} & \ldots & A_2 & A_1
\end{pmatrix},
$$

(4)

where $L \times L$ matrices $A_m$ have the form:

$$
A_m = \sum_{k=0}^{l} w(m, k) J(k), \quad J(0) \equiv I, \quad m = 0, 1, \ldots, l.
$$

(5)

In equation (5) we set $w(0, 0) = 0$ and, consequently, the self-interaction is equal to zero.

Note, the central block row of the matrix $B_0$ is

$$
(A_l A_{l-1} \ldots A_1 A_0 A_1 \ldots A_{l-1} A_l),
$$

all other block rows we obtain by evident cyclic shifts.

We can treat a two-dimensional spin system as a set of interacting one-dimensional chains (for example, the horizontal ones.) Then the matrix $A_0$ describes the interactions between the spins of the one horizontal chain and the matrices $A_m$ ($m \neq 0$) define interactions between the spins from different chains shifted vertically by $m$ nodes.

The matrix $B_0$ is a block Toeplitz matrix with the matrices $A_0$ on the main diagonal and the matrices $A_m$ on its $m$th and $(L - m)$th diagonals ($m \neq 0$). It is easy to show that the matrix $B_0$ is

$$
B_0 = \sum_{m=0}^{l} J(m) \otimes A_m = \sum_{m=0}^{l} \sum_{k=0}^{l} w(m, k) \cdot J(m) \otimes J(k).
$$

(6)

Since the matrices $A_m$ commute, we can write the eigenvectors of the matrix $B_0$ as the Kronecker products of the eigenvectors (1):

$$
F_{\alpha \beta} = f_\alpha \otimes f_\beta, \quad \alpha, \beta = 1, 2, \ldots, L.
$$

(7)

This means that we reduce the eigenvalue problem $B_0 F_{\alpha \beta} = \mu_{\alpha \beta} F_{\alpha \beta}$ to calculation of the value $\mu_{\alpha \beta} = F_{\alpha \beta}^+ B_0 F_{\alpha \beta}$. Then substituting the matrix $B_0$ in the form (6) and the vector $F_{\alpha \beta}$ in the form (7) with the aid of the identity (3) we obtain

$$
\mu_{\alpha \beta} = \sum_{m=0}^{l} \sum_{k=0}^{l} w(m, k) \cdot \lambda_\alpha(m) \cdot \lambda_\beta(k),
$$

(8)

where $w(0, 0) = 0$ and $\lambda_\alpha(0) = \lambda_\beta(0) = 1$. We see that the eigenvalues $\mu_{\alpha \beta}$ are polynomials of the second degree of the eigenvalues $\lambda_\alpha(k)$ calculated for the one-dimensional system.
As example, let us examine a special case of an isotropic interaction only with the nearest neighbors (the interaction constant is \( w(0, 1) = w(1, 0) = 1 \) and the next nearest neighbors (the interaction constant is \( w(1, 1) \)). Then the equation (8) takes the form

\[
\mu_{\alpha \beta} = \lambda_{\alpha}(1) + \lambda_{\beta}(1) + w(1, 1) \cdot \lambda_{\alpha}(1) \lambda_{\beta}(1). \tag{9}
\]

The equation (9) repeats the result obtained previously in [22, 23] where we discussed this special case.

2.3. 3D Ising model

Let us discuss the three-dimensional Ising system of interacting spins that are in the nodes of a cubic lattice. By \( w(n, m, k) \) we denote the constant of interaction between spins shifted relative to each other by a distance \( n \) along one axis, by a distance \( m \) along the other axis, and by a distance \( k \) along the third axis. In such a system, it is convenient to write the connection \( L^3 \times L^3 \) matrix \( C_0 \) in the block form:

\[
C_0 = \begin{pmatrix}
B_0 & B_1 & B_2 & \ldots & B_{L} & B_{L-1} & B_{L-2} & \ldots & B_2 & B_1 \\
B_1 & B_0 & B_1 & \ldots & B_{L-1} & B_{L-2} & B_{L-3} & \ldots & B_2 & B_1 \\
B_2 & B_1 & B_0 & \ldots & B_{L-2} & B_{L-3} & B_{L-4} & \ldots & B_2 & B_1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
B_{L} & B_{L-1} & B_{L-2} & \ldots & B_0 & B_1 & B_2 & \ldots & B_{L-1} & B_{L-2} \\
B_{L-1} & B_{L-2} & B_{L-3} & \ldots & B_0 & B_1 & B_2 & \ldots & B_{L-2} & B_{L-3} \\
B_{L-2} & B_{L-3} & B_{L-4} & \ldots & B_0 & B_1 & B_2 & \ldots & B_{L-3} & B_{L-4} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
B_2 & B_1 & B_0 & \ldots & B_{L-2} & B_{L-3} & B_{L-4} & \ldots & B_2 & B_1 \\
B_1 & B_2 & B_3 & \ldots & B_1 & B_2 & B_3 & \ldots & B_1 & B_2 \\
B_0 & B_1 & B_2 & \ldots & B_0 & B_1 & B_2 & \ldots & B_0 & B_1 \\
\end{pmatrix}, \tag{10}
\]

where \( B_n \) are \( L^2 \times L^2 \) matrices \( (n = 0, 1, 2, \ldots, l) \). To obtain the matrices \( B_n \) we have to generate a set of \( L \times L \) matrices \( A_n^{(m)} \),

\[
A_n^{(m)} = \sum_{k=0}^{l} w(n, m, k) J(k), \quad J(0) = I, \quad n, m = 0, 1, 2, \ldots, l. \tag{11}
\]

Since there is no self-interaction, we set \( w(0, 0, 0) = 0 \). With the aid of the matrices (11) we generate the matrices \( B_n^{} \):

\[
B_n = \sum_{m=0}^{l} J(m) \otimes A_n^{(m)} = \sum_{m=0}^{l} \sum_{k=0}^{l} w(n, m, k) \cdot J(m) \otimes J(k).
\]

By analogy with the two-dimensional system, we can consider the three-dimensional lattice as a set of interacting planar lattices. Then the matrix \( B_0 \) describes the interactions of spins belonging to one (let us say, a horizontal) plane; the matrix \( B_n (n \neq 0) \) describes the interactions between the spins from two different planes shifted with respect to each other along the vertical axis by \( n \) nodes.

As we see, the matrix \( C_0 \) has a form of a block Toeplitz matrix with the matrices \( B_0 \) at its main diagonal and the matrices \( B_n (n \neq 0) \) at its \( n \)th and \( (L-n) \)th diagonals. Then, we can write the matrix \( C_0 \) as

\[
C_0 = \sum_{n=0}^{l} J(n) \otimes B_n = \sum_{n=0}^{l} \sum_{m=0}^{l} \sum_{k=0}^{l} w(n, m, k) \cdot J(n) \otimes J(m) \otimes J(k). \tag{12}
\]
Since the matrices $B_i$ commute, we can write the eigenvectors of the matrix $C_0$ as the Kronecker products of the eigenvectors (1):

$$F_{\alpha\beta\gamma} = f_\alpha \otimes f_\beta \otimes f_\gamma, \quad \alpha, \beta, \gamma = 1, 2, \ldots, L.$$ 

This means that we reduce the eigenvalues problem $C_0 F_{\alpha\beta\gamma} = \mu_{\alpha\beta\gamma} F_{\alpha\beta\gamma}$ to calculation of the values $\mu_{\alpha\beta\gamma} = F_{\alpha\beta\gamma} C_0 F_{\alpha\beta\gamma}$ and with account for equation (3), we obtain:

$$\mu_{\alpha\beta\gamma} = \sum_{n=0}^{l} \sum_{m=0}^{l} \sum_{k=0}^{l} w(n, m, k) \cdot \lambda_\alpha(n) \cdot \lambda_\beta(m) \cdot \lambda_\gamma(k), \quad \text{(12)}$$

where, as usually, $\lambda_\alpha(0) = \lambda_\beta(0) = \lambda_\gamma(0) = 1$. We see that in the three-dimensional case the eigenvalues are the polynomials of the third degree of the eigenvalues for the one-dimensional system.

As an example, let us discuss a special case of the three-dimensional isotropic Ising system that is $w(n, m, k) = w(k, n, m) = w(m, n, k)$. We suppose that only the interactions with the nearest neighbors, the next nearest and the third neighbors are nonzero. We set $w(0, 0, 1) = w(0, 1, 0) = w(1, 0, 0) = 1$, $w(0, 1, 1) = w(1, 0, 1) = w(1, 1, 0) = b_1$, and $w(1, 1, 1) = b_2$. Then from equation (12) we obtain

$$\mu_{\alpha\beta\gamma} = w_1 \left( \lambda_\alpha(1) + \lambda_\beta(1) + \lambda_\gamma(1) \right) + w_2 \left( \lambda_\alpha(1) \lambda_\beta(1) + \lambda_\alpha(1) \lambda_\gamma(1) + \lambda_\beta(1) \lambda_\gamma(1) \right) + w_3 \lambda_\alpha(1) \lambda_\beta(1) \lambda_\gamma(1).$$

The last expression coincides with result obtained previously in [22, 23], where we examined the same special case.

2.4. Ising system on hypercube

A generalization to the case of a $d$-dimensional lattice is evident. Let us introduce the constants of interaction between spins $w(k_1, k_2, \ldots, k_d)$, where $k_1$ is a relative distance between the spins along the axis $1$, $k_2$ is a relative distance between the spins along the axis $2$, and so on. We generate a set of $L \times L$ matrices

$$A_{k_1, k_2, \ldots, k_d}^{(k_1, \ldots, k_d)} = \sum_{k_1=0}^{l} w(k_1, k_2, \ldots, k_d) J(k_1),$$

where $k_j = 0, 1, \ldots, L$, and $i = 2, 3, \ldots, d$. We use these matrices to generate a set of $L^2 \times L^2$ matrices $B_{k_1, \ldots, k_d}^{(k_1, \ldots, k_d)} = \sum_{k_1=0}^{l} J(k_1) \otimes A_{k_1, k_2, \ldots, k_d}^{(k_1, \ldots, k_d)}$. In the same way the matrices $B_{k_1, \ldots, k_d}^{(k_1, \ldots, k_d)}$ allows us to construct the matrices $C_{k_1, \ldots, k_d}^{(k_1, \ldots, k_d)} = \sum_{k_1=0}^{l} J(k_1) \otimes B_{k_1, \ldots, k_d}^{(k_1, \ldots, k_d)}$ for the three-dimensional system, and so on until we obtain the matrix $U_0^{(d)}$ that defines the interactions in the $d$-dimensional system. This is a block matrix similar to the matrices (4) and (10), where the blocks are the $L^{d-1} \times L^{d-1}$ matrices that describe interactions in the $(d - 1)$-dimensional system. Omitting the intermediate calculations, we obtain

$$U_0^{(d)} = \sum_{k_1=0}^{l} \sum_{k_2=0}^{l} \ldots \sum_{k_d=0}^{l} w(k_1, k_2, \ldots, k_d) \cdot J(k_1) \otimes J(k_2) \otimes \ldots \otimes J(k_d), \quad \text{(13)}$$

where $w(0, 0, \ldots, 0) = 0$, and $J(0) \equiv I$. 

6
Since in equation (13) the matrices $J(k)$ commute, the eigenvectors of the matrix $U_0$ are the Kronecker products of the eigenvectors for the one-dimensional system:

$$F_{\alpha_1 \alpha_2 \cdots \alpha_d}^{(1)} \otimes F_{\alpha_2 \alpha_3 \cdots \alpha_d}^{(2)} \otimes \cdots \otimes F_{\alpha_d}^{(d)}, \quad \alpha_i = 1, 2, \ldots, L, i = 1, 2, \ldots, d.$$

Then again we reduce the eigenvalue problem $U_0^{(d)} F_{\alpha_1 \alpha_2 \cdots \alpha_d} = \mu_{\alpha_1 \alpha_2 \cdots \alpha_d} F_{\alpha_1 \alpha_2 \cdots \alpha_d}$ to calculation of the values $\mu_{\alpha_1 \alpha_2 \cdots \alpha_d}$. With account for equation (3) we obtain

$$\mu_{\alpha_1 \alpha_2 \cdots \alpha_d} = \sum_{k_1=0}^{l} \sum_{k_2=0}^{l} \cdots \sum_{k_d=0}^{l} w(k_1, k_2, \ldots, k_d) \prod_{i=1}^{d} \lambda_{\alpha_i}(k_i). \quad (14)$$

Concluding this section, let us make some remarks. The set of the eigenvectors $\{F_m\}$ and the eigenvalues $\{\mu_m\}$ we obtained above (for simplicity we use the indices $m = 1, 2, \ldots, N$, $N = L^d$ in place of the more complex indices $\alpha_1, \alpha_2, \ldots, \alpha_L = 1, 2, \ldots, L$) forms the eigenbasis of the Hamiltonian operator for the Ising model:

$$\hat{H} = \sum_{m=1}^{N} \mu_m \langle F_m | \hat{H} F_m \rangle = \mu_m F_m.$$

We can present each spin configuration $S^+ = (s_1, s_2, \ldots, s_N)$, $s_i = \pm 1$ as an expansion in these basis eigenvectors: $S = \sum_{m=1}^{N} a_m F_m$. Our analysis shows that account for long-range interactions does not change the basic eigenvectors $\{F_m\}$ as well as the expansion coefficients $a_m$. The situation differs when we examine the energy of the state, which we can write as

$$E(S) = S^+ \hat{H} S = \sum_{m=1}^{N} a_m^2 \mu_m.$$ 

A change of the long-range interaction does not influence the coefficients $a_m^2$, but the eigenvalues $\mu_m$ change. In other words, the long-range interaction does not change the expansion of the configuration in the basis eigenvectors; however, it leads to a shift of the energy of the state $E(S)$.

The authors of paper [7] developed an algorithm for calculation of the free energy of a finite spin system using eigenvalues and eigenvectors of its connection matrix. In this paper, they analyzed the one- and two-dimensional Ising systems accounting for interactions with the nearest neighbors only. In particular, they showed that their approach allowed us to approximate accurately the known exact expressions for the free energy even when the size of the spin system is not very large. We find it interesting to estimate the influence of the long-range interaction on the behavior of the free energy using the obtained expressions (2), (8), and (12).

### 3. Density of eigenvalue spectrum

In the previous sections, we obtained the expressions for the eigenvalues of the connection matrices in multidimensional Ising systems, which allow us to estimate the degeneracy of their spectra. However, in the limit $L \to \infty$ it is more efficient to pass from the discrete to continuous spectrum and analyze the spectral density $P(\mu)$ of the eigenvalue spectrum, where $P(\mu) d\mu$ is the number of the eigenvalues in the interval $[\mu, \mu + d\mu]$. In this limit we succeed in deriving analytical expressions only for one- and two-dimensional systems when we account for...
interactions with the nearest and the next nearest neighbors. For certainty, we suppose that the constant of interaction with the next nearest neighbor $b$ is positive: $b \geq 0$.

### 3.1. Spectrum density of 1D system

Let us examine the one-dimensional system. Let $w(1) = 1$ and $w(2) = b$ be the constants of interaction with the nearest and the next nearest spins, respectively. Then from equation (2) it follows that $\mu_\alpha = 2 \cos \varphi_\alpha + 2b \cos 2\varphi_\alpha$, where $\varphi_\alpha = 2\pi(\alpha - 1)/L$ and $\alpha = 1, 2, \ldots, L$. We obtain the density $P(\mu)$ by integrating the delta function \( \delta(\mu - 2 \cos \varphi_\alpha - 2b \cos 2\varphi_\alpha) \) with respect to the variable $\varphi_\alpha \in [0, 2\pi]$. In what follows, we show that the spectrum changes drastically when $b = 1/4$. This is the reason why we examine separately the spectrum to the left and to the right from this point.

We begin with the simplest case $b = 0$. After rather simple calculations we obtain

\[
P(\mu) = \frac{L}{2\pi \sqrt{4 - \mu^2}}, \quad -2 < \mu < 2.
\]

We see that the spectrum is limited to the interval $\mu \in (-2, 2)$ and there is a divergence at the spectrum ends (figure 1).

(a) Let examine the values of $b$ inside the interval $0 < b < 1/4$. In this case the spectral density is nonzero only inside the interval $\mu \in (-2 + 2b, 2 + 2b)$, where

\[
P(\mu) = \frac{L}{2\pi \sqrt{(2 + 2b - \mu)(\mu + 2 - 2b)}}, \quad -2 + 2b \leq \mu \leq 2 + 2b. \quad (15)
\]

In equation (15) $Q_0 = Q_0(\mu)$ is a slow function without singularities on the interval in question:

\[
Q_0 = \left( \frac{1 + 4b + \sqrt{1 + 4b(\mu + 2b)}}{1 + 4b(\mu + 2b)} \right)^{1/2}.
\]

We see that account for the interaction with the next nearest neighbors shifts the spectrum in the positive direction by a value equal to $2b$. In the same time, divergences at the ends of the spectrum ($\mu \to \pm 2 + 2b$) remain. Note, when $b \to 1/4$ the left end of the spectrum, where $P(\mu) \approx L/(2\pi \sqrt{(1 - 4b)(\mu + 2 - 2b)})$, is much higher its right end where $P(\mu) \approx L/(2\pi \sqrt{(1 + 4b)(2 + 2b - \mu)})$.

(b) When $b = 1/4$ the spectral density is nonzero on the interval $\mu \in [-3/2, 5/2]$. In this case equation (15) takes the form

\[
P(\mu) = \frac{L}{2\pi \sqrt{(3/2 + \mu)^{1/2} (5/2 - \mu)^{1/2}}}, \quad -3/2 \leq \mu \leq 5/2.
\]

As we see, in this case the divergence at the left end of the spectrum ($\mu \to -3/2$) is much stronger.

(c) Let us analyze the values of $b > 1/4$. We introduce the notations

\[
\mu_{\text{min}} = -2b - \frac{1}{4b}, \quad \mu_{av} = -2 + 2b, \quad \mu_{\text{max}} = 2 + 2b.
\]

Here $\mu_{\text{min}}$ and $\mu_{\text{max}}$ are the lower and the upper boundaries of the spectrum, respectively. Next, $\mu_{av}$ is a point between the spectrum boundaries: $\mu_{\text{min}} < \mu_{av} < \mu_{\text{max}}$. The spectrum
Figure 1. Spectral density $P(\mu)$ for 1D Ising system for different values of parameter $b$. (a) $0 \leq b \leq 0.25$; we show graphs for $b = 0, 0.125, 0.25$. (b) $b > 0.25$; we show graphs for $b = 0.5$ and 1.

is a composite: in the different regions of the interval it is described by two different functions that does not match

$$P(\mu) = \frac{L}{\pi \sqrt{\mu - \mu_{\text{min}}}} \left( \frac{Q_1}{\sqrt{\mu_{\text{av}} - \mu}} + Q_2 \right) \quad \text{when } \mu_{\text{min}} \leq \mu < \mu_{\text{av}},$$

$$P(\mu) = \frac{L}{\pi Q_1 \sqrt{(\mu - \mu_{\text{min}})(\mu_{\text{max}} - \mu)}} \quad \text{when } \mu_{\text{av}} \leq \mu \leq \mu_{\text{max}},$$

where $Q_1 = Q_1(\mu)$ and $Q_2 = Q_2(\mu)$ are slow functions without singularities at the above-mentioned intervals

$$Q_1 = \frac{4b - 1 + \sqrt{4b(\mu - \mu_{\text{min}})}}{4b + 1 + \sqrt{4b(\mu - \mu_{\text{min}})}}^{1/2}, \quad Q_2 = \frac{\sqrt{4b}}{Q_1(4b + 1)^2 - 4b (\mu - \mu_{\text{min}})}^{1/2}.$$

We see that at the point $\mu = \mu_{\text{av}}$ the function $P(\mu)$ is discontinues: when $\mu \to \mu_{\text{av}} - 0$, there is a singularity $P(\mu) \sim (\mu - \mu_{\text{min}})^{-1/2}$, when $\mu \to \mu_{\text{av}} + 0$ the spectrum density is finite: $P(\mu) = 2bL/\pi(4b - 1)^{3/2}$. In figure 1, we show how the spectrum changes when $b$ increases.

The following picture arises from our analysis. If $b \leq 1/4$, the spectrum as a whole shifts by the value $2b$; the lower and upper spectrum boundaries are $\mu_{\text{min}} = -2 + 2b$ and $\mu_{\text{max}} = 2 + 2b$, respectively; the width of the spectrum remains constant. When $b > 1/4$, an increase of $b$ leads to the spectrum broadening. Namely, the value of $\mu_{\text{min}}$ decreases: $\mu_{\text{min}} = -2b - 1/4b$ and the value of $\mu_{\text{max}}$ increases: $\mu_{\text{max}} = 2 + 2b$. The spectrum width also increases as $\mu_{\text{max}} - \mu_{\text{min}} = 2 + 4b + 1/4b$. For an arbitrary $b$ there are divergences both at the lower and upper spectrum boundaries. Moreover, if $b > 1/4$ an additional divergence at the point $\mu = \mu_{\text{av}} \equiv -2 + 2b$ appears. In figure 2(a), we show this picture schematically.

3.2. Spectral density of 2D system

We analyze the two-dimensional Ising system in the simplest case supposing an isotropic interaction and accounting for interactions with the nearest and next nearest neighbors only. Let the constant of interaction with the nearest neighbors be equal to one and by $b = w(1, 1)$ we denote the interaction with the next nearest neighbors. Then from equation (9), we obtain $\mu_{\alpha \beta} = 2 \cos \varphi_{\alpha} + 2 \cos \varphi_{\beta} + 4b \cos \varphi_{\alpha} \cos \varphi_{\beta}$, where $\varphi_{\alpha} = 2\pi(\alpha - 1)/L$ and $\varphi_{\beta} = 2\pi(\beta - 1)/L$. 

Figure 2. Dependences of values $\mu_{\text{max}}$, $\mu_{\text{min}}$, $\mu_{\text{av}}$ on interaction constant $b$. Solid lines correspond to $\mu_{\text{min}}$ and $\mu_{\text{max}}$, which appears when $b > 1/4$. (a) 1D Ising model. Dashed line describes movement of singular point $\mu_{\text{av}}$, which appears when $b > 1/4$. (b) 2D Ising model. Dashed line and dotted line show movement of singular point $\mu_{\text{av}}$ and jump point $\mu_0$, respectively.

Consequently, to obtain the spectral density $P(\mu)$ we have to integrate the delta function $\delta(\mu - \mu_{\alpha\beta})$ with respect to the variables $\phi_{\alpha} \in [0, 2\pi]$ and $\phi_{\beta} \in [0, 2\pi]$. We examine separately the behavior of the spectrum to the left and to the right from the point $b = 1/2$, since at this point it changes drastically.

(a) We begin with the case $0 \leq b < 1/2$, where the spectral density is nonzero only when $-4 + 4b \leq \mu \leq 4 + 4b$.

When $b = 0$, we obtain

$$P(\mu) = \frac{L^2}{2\pi^2 K(m)}, \quad m = \frac{1 - \mu^2/16}{\mu} \quad \text{when} \quad \mu \in [-4, 4].$$

(16)

In equation (16), $K(m)$ is a complete elliptic integral of the first kind. We see that the spectrum is symmetric with respect to the point $\mu = 0$ and since $K(m) \approx -\ln|\mu|$ at $m \to 1$, it has a logarithmic divergence when $\mu \to 0$ [see figure 3(a)].

When $b \neq 0 (b < 1/2)$, we obtain a more general expression

$$P(\mu) = \frac{L^2}{2\pi^2 \sqrt{1 + b\mu}} K(m), \quad m = \frac{1 - (b - \mu/4)^2}{1 + b\mu} \quad \text{when} \quad \mu \in [-4 + 4b, 4 + 4b].$$

(17)

As it follows from equation (17), when $b$ increases, the spectrum as a whole shifts to the right by the value of $4b$. In the same time, the maximum of the function $P(\mu)$ at $\mu = -4b$, where there is the logarithmic singularity at $m \to 1$, shifts to the left. In addition, at the right end of the spectrum ($\mu = 4 + 4b$) the value of the spectral density decreases as $P(\mu) = L^2/4\pi(1 + 2b)$ and at the left end ($\mu = -4 + 4b$) it increases as $P(\mu) = L^2/4\pi(1 + 2b)$. We show the changes of $P(\mu)$ in figure 3(a).

(b) When $b = 1/2$, equation (16) takes the form

$$P(\mu) = \frac{L^2}{2\pi^2 \sqrt{1 + \mu/2}} K(m), \quad m = \frac{6 - \mu}{8} \quad \text{when} \quad \mu \in [-2, 6].$$

As we see in figure 3(b), the entire spectrum is to the right from the point $\mu = -2$. Near this point when $\mu \to -2$ and, consequently, $m \to 1$, we have
At the left end of the second interval there is also the same logarithmic discontinuity where $P(\mu)$ decreases gradually and reaches its minimal value $P(\mu) = L^2/8\pi$ at the right end of the spectrum where $\mu = 6$.

(c) The case $b > 1/2$. Now the spectral density is nonzero at the interval $-4b \leq \mu \leq 4 + 4b$. There are three parts of this interval where the discontinuous function $P(\mu)$ is described by three different expressions:

$$P(\mu) = \frac{L^2}{2\pi^2} R K(m),$$

where

$$R = \frac{2}{\sqrt{(b - \mu/4)^2 - 1}}, \quad m = \frac{(b + \mu/4)^2}{(b - \mu/4)^2 - 1} \quad \text{when} -4b \leq \mu < -1/b;$$

$$R = \frac{2}{b + \mu/4}, \quad m = \frac{(b - \mu/4)^2 - 1}{(b + \mu/4)^2} \quad \text{when} -1/b < \mu \leq -4 + 4b;$$

$$R = \frac{1}{\sqrt{1 + b\mu}}, \quad m = \frac{1 - (b - \mu/4)^2}{1 + b\mu} \quad \text{when} -4 + 4b \leq \mu \leq 4 + 4b.$$

At the left end of the first interval $\mu \in [-4b, -1/b]$, the density $P(\mu)$ has a finite value $P(\mu) = L^2/2\pi\sqrt{4b^2 - 1}$; at the right end of this interval there is a logarithmic divergence since $m \to 1$ and $P(\mu) \sim -\ln |\mu + 1/b|/(4b^2 - 1)$ when $\mu \to -1/b - 0$.

At the left end of the second interval there is also the same logarithmic discontinuity when $\mu \to -1/b + 0$ and at the right end of this interval the function $P(\mu)$ has a finite value $P(\mu) = L^2/2\pi(2b - 1)$. The function $P(\mu)$ has no singularities on the third interval $\mu \in [-4 + 4b, 4 + 4b]$. When $\mu$ increases, the density decreases smoothly from the value $P(\mu) = L^2/4\pi(2b - 1)$ when $\mu = -4 + 4b$ up to the value $P(\mu) = L^2/4\pi(2b + 1)$ when $\mu = 4 + 4b$. In figure 3(b) we show all the transformations of the function $P(\mu)$ when $b$ increases.

Let us summarize the analysis of this subsection. When $b$ changes inside the interval $0 \leq b \leq 1/2$, the entire spectrum shifts to the right by the value of $4b$: the lower boundary of

**Figure 3.** Spectral density $P(\mu)$ for 2D Ising system and different values of constant of interaction with next nearest neighbors. (a) When $0 \leq b < 0.5$, $b = 0$, 0.2, and 0.4; (b) when $b \geq 0.5$, $b = 0.5$, 0.8, and 1.2. Dotted lines define jumps of spectral density.
the spectrum is \( \mu_{\text{min}} = -4 + 4b \) and its upper boundary is \( \mu_{\text{max}} = 4 + 4b \). Inside the interval there is a logarithmic discontinuity at the point \( \mu = \mu_{av} \) (\( \mu_{av} = -4b \) when \( 0 \leq b \leq 1/2 \)). When \( b > 1/2 \), the further increase in \( b \) leads to the broadening of the spectrum and change of its form. The lower spectrum boundary \( \mu_{\text{min}} = -4b \) decreases and its upper boundary \( \mu_{\text{max}} = 4 + 4b \) increases. The spectrum density has two singularities: the divergence at \( \mu = \mu_{av} \) and the finite jump at \( \mu = \mu_0 \), where \( \mu_{av} = -1/b \) and \( \mu_0 = -4 + 4b \). Figure 2(b) presents this picture schematically.

3.3. Spectrum density for \( d \geq 3 \)

In the case of Ising systems of higher dimensions, due to significant mathematical difficulties we were unable to obtain analytical expressions for the spectral density. However, it is possible to make some general conclusions. For example, when we account for the nearest neighbors only, we can present the spectral density \( P_d(\mu) \) for the \( d \)-dimensional system as a convolution of the spectral densities of the systems of lower dimensions:

\[
P_d(\mu) = \int_{-\infty}^{\infty} P_{d_1}(x)P_{d_2}(\mu - x)dx, \quad d_1 + d_2 = d.
\]

Due to the integration the divergence of \( P_d(\mu) \) is weaker the divergences of \( P_{d_1}(x) \) and \( P_{d_2}(x) \). However, there are many cases when the density \( P_d(\mu) \) has singularities. For example, suppose that \( P_{d_1}(x) \) and \( P_{d_2}(x) \) have rather strong divergences at the points \( x_1 \) and \( x_2 \), respectively: \( P_{d_1}(x \rightarrow x_1) \sim |x - x_1|^{-r_1} \) and \( P_{d_2}(x \rightarrow x_2) \sim |x - x_2|^{-r_2} \), where \( 0.5 \leq r_1, r_2 < 1 \). Then the density \( P_d(\mu) \) (\( d = d_1 + d_2 \)) also has a singularity at the point \( \mu = x_1 + x_2 \). This may be a power singularity \( P_d(\mu \rightarrow x_1 + x_2) \sim |\mu - x_1 - x_2|^{-r} \) when \( r > 0 \), or a logarithmic singularity \( P_d(\mu \rightarrow x_1 + x_2) \sim -\ln |\mu - x_1 - x_2| \) when \( r = 0 \). In both formulas \( r = r_1 + r_2 - 1 \). Moreover, the singularity may be a combination of the logarithmic and power singularities. To avoid misunderstanding we note that the functions \( P_{d_1}(x) \) and \( P_{d_2}(x) \) are normalized functions and according definition may have only integrable singularities.

4. Discussion and conclusions

The analysis of the discrete spectra in section 2 shows that when the dimension of the space \( d \) increases the degeneracy of the eigenvalues increases too. Let us discuss the influence of account for the long-range interaction on the spectrum degeneracy.

In the one-dimensional system, all the eigenvalues of the interaction matrix are two-fold degenerate excluding the non-degenerate eigenvalue \( \mu_1 = \sum_{k=1}^{L} w(k) \lambda_1(k) = 2 \sum_{k=1}^{l} w(k) \). Such degeneracy takes place when we account for the interaction with the nearest neighbors and, as it follows from equation (2), it does not change when we include long-range interactions.

The situation becomes more complicated when the dimension of the lattice is larger. Let us start with the isotropic case when the interaction constant \( w(k_1, k_2, \ldots, k_d) \) does not change when rearranging the numbers \( k_1, k_2, \ldots, k_d \). As a result, the eigenvalue \( \mu_{\alpha_1 \alpha_2 \cdots \alpha_d} \) defined by equation (14) does not depend on the order of the indices \( \alpha_1, \alpha_2, \ldots, \alpha_d \), where \( \alpha_i \in [0, L] \), \( i = 1, 2, \ldots, d \). For example, in the two-dimensional system \( w(n, m) = w(m, n) = \mu_{\alpha_1, \alpha_2} \), and \( \mu_{\alpha_1, \alpha_2} = \mu_{\alpha_2, \alpha_1} \). In the isotropic \( d \)-dimensional case, the degeneracy of the eigenvalue \( \mu_{\alpha_1 \alpha_2 \cdots \alpha_d} \) is equal to \( 2^d d! \), where \( r \) is the number of indices that are equal to one. The factor \( 2^d d! \) appears due to the double degeneracy of the eigenvalues in the one-dimensional model and the factor \( d! \) is equal
to the number of permutations of the indices $\alpha_1, \alpha_2, \ldots, \alpha_d$. The same is true if we account only for interaction with the nearest neighbors and the result does not change when we include long-range interactions.

In anisotropic systems, where the interaction constants along different axes are not the same the situation is somewhat different. In this case, the value of $w(k_1, k_2, \ldots, k_d)$ changes when we rearrange the arguments $k_1, k_2, \ldots, k_d$. This means that the eigenvalue $\mu_{\alpha_1\alpha_2\ldots\alpha_d}$ depends significantly on the order of its indices $\alpha_1, \alpha_2, \ldots, \alpha_d$. Then the degeneracy of the eigenvalue $\mu_{\alpha_1\alpha_2\ldots\alpha_d}$ is equal to $2^{d-r}$ where $r$ is the number of indices that are equal to one. As before, the same is true when we account for the nearest neighbors only.

Let us summarize. First, the degeneracy of the eigenvalues increases rapidly when the dimension of the lattice $d$ increases. Second, the long-range interaction does not remove the degeneracy that has place when we account for the interaction with the nearest neighbors only. Third, in the case of a multidimensional lattice an anisotropic interaction decreases the degeneracy by $d!$ times.

Concluding we would like to note a fact that is significant when $d \geq 2$. For simplicity, we suppose that the interaction is isotropic and we account for the nearest neighbors only. If $L$ was even, there would be a lot of zero-valued eigenvalues. For example, in the case of the two-dimensional system there are $L-2$ pairs of the indices $\alpha$ and $\beta$, for which $\lambda_0(1) = -\lambda_0(1)$ and $\mu_{\alpha\beta} = 0$. For the three-dimensional system there are about $L^2$ zero-valued eigenvalues. When $d$ increases the degeneracy of the zero-valued eigenvalue increases as $L^{d-1}$. Everywhere above we examined the odd values of $L$. In this case, the degeneracy of the zero-valued eigenvalue was not so large. However, when $L$ increases the number of the eigenvalues with close to zero values increases rapidly. This fact becomes evident when we turn to the case of the continuous spectrum.

In the asymptotic limit $L \to \infty$, it is more effective to examine not the discrete spectrum but the eigenvalue density $P(\mu)$. The analysis of section 3 shows that in the most of the discussed cases the function $P(\mu)$ has singularities (the logarithmic and power divergences). We have shown that the long-range interaction leads to a significant increase of the degeneracy in some regions of the continuous eigenvalue spectrum.

There is one other line of research where the obtained expressions for the eigenvalues may be useful. We can try to answer the following question: how does the account for the long-range interaction influences the density of the states of the one-dimensional Ising system. Let us explain what we have in mind.

In the case of the one-dimensional Ising model with the interaction of the nearest neighbors only, the energy distribution of states is well-known (see [24]): the frequency of the state with the energy $E_r = -2(N - 4r)$ is equal to $D(r) = 2 \cdot \left(\frac{N}{2r}\right)$, where $r = 0, 1, 2, \ldots, N/2$. We also know the joint energy and magnetization distribution of states: the frequency of the state with the energy $E_r$ and magnetization $M_n$ is equal to

$$D(n, r) = \frac{2Nr}{(N-n)n} \cdot \left(\frac{N-n}{r}\right) \left(\frac{n}{r}\right),$$

where $M_n = N - 2n$, $n = 1, 2, \ldots, N/2$; $E_r = -2(N - 4r)$, $r = 1, 2, \ldots, n$, and $D(N, 0) = 2$.

(see [25]). The frequencies $D(n)$ and $D(n, r)$ were obtained by means of combinatorial calculations. We hope that it will be possible to obtain them using only the expression for the energy of the state.
\begin{align}
E(s) \sim s^+ J(1) s = \sum_{k=1}^{N} \lambda_k(1)(s, f_k)^2. \tag{18}
\end{align}

We would like to note that in the expression (18) only the eigenvalues \{\lambda_k(1)\} depend on the matrix \( J(1) \); however, the scalar products \((s, f_k)^2\) of the configuration vectors and the eigenvectors does not depend on the number of the connection matrix \( J(k) \). Using this fact, we expect to obtain an analytical expression for the state density when the connection matrix is a sum \( J(1) + J(2) \). This will allow us to obtain an analytical expression for the free energy of the one-dimensional Ising system when we will account for interactions with the nearest and the next nearest neighbors. From the papers \cite{26--28}, it follows that this is a long-waited result in the theory of the one-dimensional Ising systems. We also hope that this direction is perspective when it is necessary to analyze two- and three-dimensional Ising models.

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**References**

[1] Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (London: Academic)
[2] Stanley H E 1987 *Introduction to Phase Transitions and Critical Phenomena* (Oxford: Oxford University Press)
[3] Dotsenko V S 1994 *Introduction to the Theory of Spin-Glasses and Neural Networks* (Singapore: World Scientific)
[4] Hinton G E, Osindero S and Teh Y-W 2006 *Neural Comput.* 18 1527
[5] Hartmann A K and Rieger H (ed) 2004 *New Optimization Algorithms in Physics* (New York: VCH)
[6] Karandashev I and Kryzhanovsky B 2013 *J. Glob. Optim.* 56 1167
[7] Dixon J M, Tuszyński J A and Nip M L A 2001 *Physica A* 289 137
[8] Nag S and Garg A 2019 *Phys. Rev. B* 99 224203
[9] Iyer S, Oganesyan V, Refael G and Huse D A 2013 *Phys. Rev. B* 87 134202
[10] Kjall J A, Bardarson J H and Pollmann F 2014 *Phys. Rev. Lett.* 113 107204
[11] Nandkishore R and Huse D A 2015 *Annu. Rev. Condens. Matter Phys.* 6 15
[12] Tikhonov K S and Mirlin A D 2018 *Phys. Rev. B* 97 214205
[13] Wu Y-L and Das Sarma S 2016 *Phys. Rev. A* 93 022332
[14] Kelly B, Paul B, Jack R and Aidan R 2019 *Next-Generation Topology of D-Wave Quantum Processors Technical Report 2019-02-25* (Barnaby, BC: D-Wave Systems Inc)
[15] Klymenko C, Sullivan B D and Humble T S 2014 *Quantum Inf. Process.* 13 709
[16] Puclen K L, Albash T and Lidar D A 2014 *Nat. Commun.* 5 3243
[17] Vinci W, Albash T and Lidar D A 2016 *Quantum Inf.* 2 16017
[18] Singh R R P and Young A P 2017 *Phys. Rev. E* 96 022139
[19] Katzgraber H G, Hamze F and Andrist R S 2014 *Phys. Rev. X* 4 021008
[20] Kirkpatrick S, Gelatt C D and Vecchi M P 1983 *Science* 220 671
[21] Grenander U and Szego G 1958 *Toeplitz Forms And Their Applications* (Berkeley, CA: University of California Press)
[22] Kryzhanovsky B V and Litinskii L B 2019 *Dokl. Phys.* 64 414
[23] Litinskii L B and Kryzhanovsky B V 2020 *Physica A* 558 124929
[24] Becker R and Doring W 1939 *Ferromagnetism* (Berlin: Springer)
[25] Kryzhanovsky B V and Litinskii L B 2018 Opt. Mem. Neural Netw. 27 235
[26] Campa A, Dauxois T and Ruffo S 2009 Phys. Rep. 480 57
[27] Hovhannisyan V V et al 2017 Phys. Rev. E 96 062103
[28] Campa A, Gori G, Hovhannisyan V, Ruffo S and Trombettoni A 2019 J. Phys. A: Math. Theor. 52 344002