Mean Width of a Regular Simplex

STEVEN R. FINCH

November 21, 2011

Abstract. The mean width is a measure on \( n \)-dimensional convex bodies. An integral formula for the mean width of a regular \( n \)-simplex appeared in the electrical engineering literature in 1997. As a consequence, expressions for the expected range of a sample of \( n + 1 \) normally distributed variables, for \( n \leq 6 \), carry over to widths of regular \( n \)-simplices. As another consequence, precise asymptotics for the mean width become available as \( n \to \infty \).

Let \( C \) be a convex body in \( \mathbb{R}^n \). A width is the distance between a pair of parallel \( C \)-supporting planes (linear varieties of dimension \( n - 1 \)). Every unit vector \( u \in \mathbb{R}^n \) determines a unique such pair of planes orthogonal to \( u \) and hence a width \( w(u) \). Let \( u \) be uniformly distributed on the unit sphere \( S^{n-1} \subset \mathbb{R}^n \). Then \( w \) is a random variable and

\[
\mathbb{E}(w_3) = \frac{3}{2\pi} \arccos \left( -\frac{1}{3} \right)
\]

for \( C = \) the regular 3-simplex (tetrahedron) in \( \mathbb{R}^3 \) with edges of unit length and

\[
\mathbb{E}(w_4) = \frac{10}{3\pi^2} \left[ 3 \arccos \left( -\frac{1}{3} \right) - \pi \right]
\]

for \( C = \) the regular 4-simplex in \( \mathbb{R}^4 \) with edges of unit length. Our contribution is to extend the preceding mean width results to regular \( n \)-simplices in \( \mathbb{R}^n \) for \( n \leq 6 \). We similarly extend the following mean square width result:

\[
\mathbb{E}(w_3^2) = \frac{1}{3} \left( 1 + \frac{3 + \sqrt{3}}{\pi} \right)
\]

which, as far as is known, first appeared in [3].

The key observation underlying our work is due to Sun [4], which in turn draws upon material in [5, 6]. It does not seem to have been acknowledged in the mathematics literature. After most of this paper was written, we found [7], which assigns priority to to Hadwiger [8] and to Ruben [9] for closely related ideas.

\[\text{Copyright © 2011 by Steven R. Finch. All rights reserved.}\]
1. Order Statistics

Let \( X_1, X_2, \ldots, X_n \) denote a random sample from a Normal \((0, 1)\) distribution, that is, with density function \( f \) and cumulative distribution \( F \):

\[
f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), \quad F(x) = \int_{-\infty}^{x} f(\xi) d\xi = \frac{1}{2} \text{erf}\left(\frac{x}{\sqrt{2}}\right) + \frac{1}{2}.
\]

The first two moments of the range

\[
r_n = \max\{X_1, X_2, \ldots, X_n\} - \min\{X_1, X_2, \ldots, X_n\}
\]

are given by [10, 11]

\[
\mu_n = \mathbb{E}(r_n) = \int_{-\infty}^{\infty} \{1 - F(x)^n - [1 - F(x)]^n\} \, dx,
\]

\[
\nu_n = \mathbb{E}(r_n^2) = 2 \int_{-\infty}^{\infty} \int_{-\infty}^{y} \{1 - F(y)^n - [1 - F(x)]^n + [F(y) - F(x)]^n\} \, dx \, dy.
\]

For small \( n \), exact expressions are possible [12, 13, 14]:

\[
\mu_2 = \frac{2}{\sqrt{\pi}} = 1.128..., \quad \mu_3 = \frac{3}{\sqrt{\pi}} = 1.692..., \quad \mu_4 = \frac{6}{\sqrt{\pi}} (1 - 2S_2) = 2.058..., \quad \mu_5 = \frac{10}{\sqrt{\pi}} (1 - 3S_2) = 2.325..., \quad \mu_6 = \frac{15}{\sqrt{\pi}} (1 - 4S_2 + 2T_2) = 2.534..., \quad \mu_7 = \frac{21}{\sqrt{\pi}} (1 - 5S_2 + 5T_2) = 2.704..., \quad \mu_8 = \frac{28}{\sqrt{\pi}} (1 - 6S_2 + 7T_2) = 2.871..., \quad \mu_9 = \frac{36}{\sqrt{\pi}} (1 - 7S_2 + 9T_2) = 3.050..., \quad \mu_{10} = \frac{45}{\sqrt{\pi}} (1 - 8S_2 + 10T_2) = 3.238..., \quad \mu_{11} = \frac{55}{\sqrt{\pi}} (1 - 9S_2 + 12T_2) = 3.436...
\]

where

\[
S_k = \frac{\sqrt{k}}{\pi} \int_{0}^{\pi/4} \frac{dx}{\sqrt{k + \sec(x)^2}} = \frac{1}{2\pi} \text{arcsec} \left( k + 1 \right),
\]

\[
T_k = \frac{\sqrt{k}}{\pi^2} \int_{0}^{\pi/4} \int_{0}^{\pi/4} \frac{dx \, dy}{\sqrt{k + \sec(x)^2 + \sec(y)^2}} = \frac{1}{2\pi^2} \int_{0}^{\pi} \text{arcsec} \left( 1 + \frac{k(k + 1)}{k - \tan(z)^2} \right) \, dz,
\]

\[

\]
Mean Width of a Regular Simplex

\[ U = \frac{1}{\pi^2} \int_0^1 \frac{\text{arcsec} \left( \frac{2t^2 + 4}{2t^2 + 1} \right)}{\sqrt{2t^2 + 3}} dt, \quad V = \frac{1}{\pi^2} \int_0^1 \frac{\text{arcsec} \left( \frac{t^2 + 5}{t^2 + 2} \right)}{\sqrt{t^2 + 4}} dt. \]

The preceding table complements an analogous table in [15] for first and second moments of \( \max\{X_1, X_2, \ldots, X_n\} \). Similar expressions for \( \mu_8 = 2.847... \) and \( \nu_8 = 8.778... \) remain to be found.

2. Key Observation

Let us rescale length so that the circumradius of the \( n \)-simplex is 1. Adjusted width will be denoted by \( \tilde{w}_n \). Using optimality properties of the \( n \)-simplex, Sun [4] deduced a formula for mean half width:

\[
\frac{1}{2} \mathbb{E} (\tilde{w}_n) = \frac{n + 1}{2} \sqrt{\frac{(n + 1)n}{2\pi}} \frac{\Gamma \left( \frac{n}{2} \right)}{\Gamma \left( \frac{n + 1}{2} \right)} \int_{-\infty}^{\infty} F \left( \frac{x}{\sqrt{2}} \right)^{n-1} f(x) \, dx
\]

\[
= \frac{(n + 1)^{3/2}}{\sqrt{2n}} \frac{\Gamma \left( \frac{n}{2} \right)}{\Gamma \left( \frac{n + 1}{2} \right)} \int_{-\infty}^{\infty} x F(x)^n f(x) \, dx
\]

(see Corollary 2 on p. 1581 and its proof on p. 1585; his \( M \) is the same as our \( n + 1 \)). We recognize the latter integral as \( \mu_{n+1}/(2(n + 1)) \); hence

\[
\mathbb{E} (\tilde{w}_n) = \sqrt{\frac{n + 1}{2n}} \frac{\Gamma \left( \frac{n}{2} \right)}{\Gamma \left( \frac{n + 1}{2} \right)} \mu_{n+1}
\]

and therefore

\[
\mathbb{E} (\tilde{w}_n) = \frac{1}{2} \frac{\Gamma \left( \frac{n}{2} \right)}{\Gamma \left( \frac{n + 1}{2} \right)} \mu_{n+1}
\]

because, in our original scaling, the circumradius is \( \sqrt{n/(2(n + 1))} \).

No similar integral expression for \( \mathbb{E} (\tilde{w}_n^2) \) appears in [4]. We circumvent this difficulty by noticing that the formula [16][17]

\[
\mathbb{E} \left( \sqrt{\sum_{k=1}^{n} X_k^2} \right) = \sqrt{2} \frac{\Gamma \left( \frac{n + 1}{2} \right)}{\Gamma \left( \frac{n}{2} \right)}
\]

bears some resemblance to the coefficient of \( \mu_{n+1} \) in our expression for \( \mathbb{E} (\tilde{w}_n) \). The square version

\[
\mathbb{E} \left( \sum_{k=1}^{n} X_k^2 \right) = n
\]
Mean Width of a Regular Simplex

is trivial and leads us to conjecture that

$$E\left(\frac{w^2}{n}\right) = \frac{1}{2n}\nu_{n+1}$$

by analogy. Numerical confirmation for \(n \leq 6\) is possible via the computer algebra technique described in [3].

In summary, we have mean width results

$$E(w_2) = \frac{3}{\pi} = 0.954929658551372..., \quad E(w_3) = \frac{3}{2}(1 - 2S_2) = 0.912260171954089..., \quad E(w_4) = \frac{20}{3\pi}(1 - 3S_2) = 0.874843256085440..., \quad E(w_5) = \frac{45}{16}(1 - 4S_2 + 2T_2) = 0.842274297659162..., \quad E(w_6) = \frac{56}{5\pi}(1 - 5S_2 + 5T_2) = 0.813743951590337...,$$

and mean square width results

$$E\left(\frac{w^2}{2}\right) = \frac{1}{2}\left(1 + \frac{3\sqrt{3}}{2\pi}\right) = 0.913496671566344..., \quad E\left(\frac{w^2}{3}\right) = \frac{1}{3}\left(1 + \frac{3 + \sqrt{3}}{\pi}\right) = 0.835419517991054..., \quad E\left(\frac{w^2}{4}\right) = \frac{1}{4}\left(1 + \frac{5\sqrt{3}}{2\pi} + \frac{30}{\pi}S_{1/2} - \frac{5\sqrt{3}}{\pi}S_3\right) = 0.769572883591771..., \quad E\left(\frac{w^2}{5}\right) = \frac{1}{5}\left(1 + \frac{5(9 + 2\sqrt{3})}{2\pi} - \frac{90}{\pi}S_2 - \frac{15\sqrt{3}}{\pi}S_3\right) = 0.714241915072694..., \quad E\left(\frac{w^2}{6}\right) = \frac{1}{6}\left(1 + \frac{35\sqrt{3}}{4\pi} + \frac{210}{\pi}S_{1/2} - \frac{105}{\pi}S_2 - \frac{35\sqrt{3}}{\pi}S_3 + \frac{35\sqrt{3}}{2\pi}T_3 + \frac{210}{\pi}U - \frac{420}{\pi}V\right) = 0.667314714095430...,
3. Asymptotics

We turn now to the asymptotic distribution of \( r_n \) as \( n \to \infty \). Define \( a_n \) to be the positive solution of the equation \([12, 18]\)

\[
2\pi a_n^2 \exp \left( a_n^2 \right) = n^2,
\]

that is,

\[
a_n = \sqrt{W \left( \frac{n^2}{2\pi} \right)} \sim \sqrt{2 \ln(n) - \frac{1}{2} \ln(\ln(n)) - \ln(4\pi)} \sqrt{\frac{2 \ln(n)}{n}}
\]

in terms of the Lambert \( W \) function \([19]\). It can be proved that the required density is a convolution \([20, 21]\):

\[
\lim_{n \to \infty} \frac{d}{dy} \mathbb{P} \left( \sqrt{2 \ln(n)} (r_n - 2a_n) < y \right) = \int_{-\infty}^{\infty} \exp(-x - e^{-x}) \exp(-(y - x) - e^{-(y-x)}) dx
\]

\[
= 2 e^{-y} K_0 \left( 2 e^{-y/2} \right)
\]

where \( K_0 \) is the modified Bessel function of the second kind \([22]\). A random variable \( Y \), distributed as such, satisfies

\[
\mathbb{E}(Y) = 2\gamma, \quad \mathbb{E}(Y^2) = \frac{\pi^2}{3} + 4\gamma^2
\]

where \( \gamma \) is the Euler-Mascheroni constant \([23]\). This implies that

\[
\mu_n \sim 2 \left( a_n + \frac{\gamma}{\sqrt{2 \ln(n)}} \right) \sim 2\sqrt{2 \ln(n)} - \frac{\ln(\ln(n)) + \ln(4\pi) - 2\gamma}{\sqrt{2 \ln(n)}}
\]

and hence

\[
\mathbb{E}(w_n) = \frac{1}{2} \frac{\Gamma \left( \frac{n+1}{2} \right)}{\Gamma \left( \frac{n+1}{2} \right)} \mu_{n+1} = \frac{1}{2} \frac{\Gamma \left( \frac{n}{2} \right)}{\Gamma \left( \frac{n+1}{2} \right)} \cdot \frac{\mu_{n+1}}{\mu_n} \cdot \mu_n
\]

\[
\sim \frac{1}{\sqrt{2n}} \left( 1 + \frac{1}{4n} \right) \cdot \left( 1 + \frac{1}{2n \ln(n)} \right) \cdot 2 \left( a_n + \frac{\gamma}{\sqrt{2 \ln(n)}} \right)
\]

\[
\sim 2 \sqrt{\frac{\ln(n)}{n}} - \frac{\ln(\ln(n)) + \ln(4\pi) - 2\gamma}{2\sqrt{n \ln(n)}}.
\]

More terms in the asymptotic expansion are possible.
If we rescale length so that the inradius of the \(n\)-simplex is 1 and denote adjusted width by \(\tilde{w}_n\), then

\[
E(\tilde{w}_n) \sim \sqrt{2n \cdot 2\sqrt{\frac{\ln(n)}{n}}} \sim 2\sqrt{2n \ln(n)}
\]

because, in our original scaling, the inradius is \(\sqrt{1/(2n(n+1))}\). This first-order approximation is consistent with \([2]\).

### 4. Regular Octahedron

As an aside, we return to the setting of \(\mathbb{R}^3\) and review our computational methods for \(C = \) the regular octahedron with edges of unit length.

For simplicity, let \(\mathcal{O}\) be the octahedron with vertices

\[
\begin{align*}
    v_1 &= (1, 0, 0), & v_2 &= (-1, 0, 0), & v_3 &= (0, 1, 0), \\
    v_4 &= (0, -1, 0), & v_5 &= (0, 0, 1), & v_6 &= (0, 0, -1).
\end{align*}
\]

At the end, it will be necessary to normalize by \(\sqrt{2}\), the edge-length of \(\mathcal{O}\).

Also let \(\mathcal{O}'\) be the union of six overlapping balls of radius \(1/2\) centered at \(v_1/2, v_2/2, v_3/2, v_4/2, v_5/2, v_6/2\). Clearly \(\mathcal{O} \subset \mathcal{O}'\) and \(\mathcal{O}'\) has centroid \((0, 0, 0)\). A diameter of \(\mathcal{O}'\) is the length of the intersection between \(\mathcal{O}'\) and a line passing through the origin.

Computing all widths of \(\mathcal{O}\) is equivalent to computing all diameters of \(\mathcal{O}'\). The latter is achieved as follows. Fix a point \((a, b, c)\) on the unit sphere. The line \(L\) passing through \((0, 0, 0)\) and \((a, b, c)\) has parametric representation

\[
x = ta, \quad y = tb, \quad z = tc, \quad t \in \mathbb{R}
\]

and hence \(y = (b/a)x, z = (c/a)x\) assuming \(a \neq 0\). The nontrivial intersection between first sphere and \(L\) satisfies

\[
(x - \frac{1}{2})^2 + \left(\frac{b}{a}x\right)^2 + \left(\frac{c}{a}x\right)^2 = \frac{1}{4}
\]

thus \(x_1 = a^2\) since \(a^2 + b^2 + c^2 = 1\); the nontrivial intersection between second sphere and \(L\) satisfies

\[
(x + \frac{1}{2})^2 + \left(\frac{b}{a}x\right)^2 + \left(\frac{c}{a}x\right)^2 = \frac{1}{4}
\]

thus \(x_2 = -a^2\). The nontrivial intersection between third/fourth sphere and \(L\) satisfies

\[
x^2 + \left(\frac{b}{a}x \mp \frac{1}{2}\right)^2 + \left(\frac{c}{a}x\right)^2 = \frac{1}{4}
\]

thus \(x_3 = ab, x_4 = -ab\). The nontrivial intersection between fifth/sixth sphere and \(L\) satisfies

\[
x^2 + \left(\frac{b}{a}x\right)^2 + \left(\frac{c}{a}x \mp \frac{1}{2}\right)^2 = \frac{1}{4}
\]
thus \( x_5 = a \cdot c \), \( x_6 = -a \cdot c \).

We now examine all pairwise distances, squared, between the six intersection points:

\[
(x_i - x_j)^2 + \left( \frac{b}{a} x_i - \frac{b}{a} x_j \right)^2 + \left( \frac{c}{a} x_i - \frac{c}{a} x_j \right)^2
\]

\[
= \begin{cases} 
4a^2 & \text{if } i = 1, j = 2 \\
1 - 2ab - c^2 & \text{if } i = 1, j = 3 \text{ or } i = 2, j = 4 \\
1 + 2ab - c^2 & \text{if } i = 1, j = 4 \text{ or } i = 2, j = 3 \\
1 - 2ac - b^2 & \text{if } i = 1, j = 5 \text{ or } i = 2, j = 6 \\
1 + 2ac - b^2 & \text{if } i = 1, j = 6 \text{ or } i = 2, j = 5 \\
4b^2 & \text{if } i = 3, j = 4 \\
(b - c)^2 & \text{if } i = 3, j = 5 \text{ or } i = 4, j = 6 \\
(b + c)^2 & \text{if } i = 3, j = 6 \text{ or } i = 4, j = 5 \\
4c^2 & \text{if } i = 5, j = 6 
\end{cases}
\]

and define

\[
g(a, b) = \max \{ 4a^2, 1 - 2ab - c^2, 1 + 2ab - c^2, 1 - 2ac - b^2, 1 + 2ac - b^2, 4b^2, (b - c)^2, (b + c)^2, 4c^2 \}.
\]

The mean width for \( C \) is

\[
\frac{1}{\sqrt{2}} \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \sqrt{g(\cos \theta \sin \varphi, \sin \theta \sin \varphi, \cos \varphi)} \sin \varphi \, d\varphi \, d\theta = \frac{3}{\pi} \arccos \left( \frac{1}{3} \right)
\]

and the mean square width is

\[
\frac{1}{2} \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} g(\cos \theta \sin \varphi, \sin \theta \sin \varphi, \cos \varphi) \sin \varphi \, d\varphi \, d\theta = \frac{2}{3} \left( 1 + \frac{2\sqrt{3}}{\pi} \right).
\]

Here are details on the final integral. A plot of the surface

\[
(\theta, \varphi) \mapsto \sqrt{g(\cos \theta \sin \varphi, \sin \theta \sin \varphi, \cos \varphi)}
\]

appears in Figure 1, where \( 0 \leq \theta \leq 2\pi \) and \( 0 \leq \varphi \leq \pi \). Figure 2 contains the same surface, but viewed from above. Our focus will be on the part of the surface to the right of the bottom center, specifically \( 0 \leq \theta \leq \pi/4 \) and \( \pi/2 \leq \varphi \leq 9/4 \). The volume under this part is \( 1/24 \text{th} \) of the volume under the full surface.
We need to find the precise upper bound on $\varphi$ as a function of $\theta$. Recall the formula for $g$ as a maximum over nine terms; let $g_{\ell}$ denote the $\ell^{th}$ term, where $1 \leq \ell \leq 9$. Then the upper bound on $\varphi$ is found by solving the equation

$$g_1(\cos \theta \sin \varphi, \sin \theta \sin \varphi, \cos \varphi) = g_9(\cos \theta \sin \varphi, \sin \theta \sin \varphi, \cos \varphi)$$

for $\varphi$. We obtain $\varphi(\theta) = 2 \arctan(h(\theta))$, where

$$h(\theta) = \cos \theta + \sqrt{\frac{3 + \cos(2\theta)}{2}}$$

and, in particular,

$$\varphi(0) = 2 \arctan \left( 1 + \sqrt{2} \right) \approx 2.3562,$$

$$\varphi(\pi/4) = 2 \arctan \left( \left( 1 + \sqrt{3} \right) / \sqrt{2} \right) \approx 2.1862.$$

It follows that $g = g_1$ for $0 \leq \theta \leq \pi/4$ and $\pi/2 \leq \varphi \leq 2 \arctan(h)$. Now we have

$$\frac{1}{48\pi} \int \frac{g_1(\cos \theta \sin \varphi, \sin \theta \sin \varphi, \cos \varphi) \sin \varphi \, d\varphi}{(1 + \cos(2\theta))(\cos(3\varphi) - 9 \cos(\varphi))}$$

and

$$\cos(3\varphi)|_{\pi/2}^{2\arctan(h)} = \frac{(1 + 4h + h^2)(1 - 4h + h^2)(1 - h^2)}{(1 + h^2)^3},$$

$$\cos(\varphi)|_{\pi/2}^{2\arctan(h)} = \frac{1 - h^2}{1 + h^2},$$

therefore

$$\frac{1}{24\pi} \int_{\pi/2}^{2\arctan(h)} g_1(\cos \theta \sin \varphi, \sin \theta \sin \varphi, \cos \varphi) \sin \varphi \, d\varphi$$

$$= \frac{(h^4 + 4h^2 + 1)(h^4 - 1)(1 + \cos(2\theta))}{6\pi(1 + h^2)^3}.$$

Integrating this expression from 0 to $\pi/4$ gives the desired formula for $E(w_{\text{octa}}^2)$.

5. $n$-Cubes

After having written the preceding, we discovered [7], which gives the mean width for a regular $n$-simplex in $\mathbb{R}^n$ as

$$E(w_n) = \frac{n(n + 1)}{\sqrt{2\pi}} \frac{\Gamma \left( \frac{n}{2} \right)}{\Gamma \left( \frac{n + 1}{2} \right)} \int_{-\infty}^{\infty} e^{-2x^2} \left( 1 + \text{erf}(x) \right)^{n-1} \, dx.$$
Consistency is readily established; nothing is said in [7] about the connection between $\mathbb{E}(w_n)$ and order statistics from a normal distribution (more precisely, the expected range $\mu_{n+1}$).

By contrast, the mean width for an $n$-cube with edges of unit length is elementary:

$$\mathbb{E}(w_{n\text{-cube}}) = \frac{n}{\sqrt{\pi}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}$$

and we conjecture that

$$\mathbb{E}(w_{n\text{-cube}}^2) = 1 + \frac{2(n-1)}{\pi}.$$

6. $n$-CROSSPOLYTOPES

A regular $n$-crosspolytope with edges of unit length has mean width [7, 24]

$$\mathbb{E}(w_{n\text{-crosspolytope}}) = \frac{2\sqrt{2n(n-1)}}{\pi} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} \int_0^{\infty} e^{-2x^2} \text{erf}(x)^{n-2} dx;$$

the case $n = 3$ corresponds to the octahedron discussed earlier. It is not surprising that a connection exists with order statistics from a half-normal (folded) distribution. We will examine this later, as well as relevant expressions from [25]. An appropriate mean square conjecture also needs to be formulated in this scenario.

References

[1] K. Böröczky, About the mean width of simplices, *Period. Polytech. Mech. Engrg.* 36 (1992) 291–297; MR1269513 (95b:52024).

[2] K. Böröczky and R. Schneider, Circumscribed simplices of minimal mean width, *Beiträge Algebra Geom.* 48 (2007) 217–224; MR2326411 (2008j:52005).

[3] S. R. Finch, Width distributions for convex regular polyhedra, [arXiv:1110.0671](http://arxiv.org/abs/1110.0671).

[4] Y. Sun, Stochastic iterative algorithms for signal set design for Gaussian channels and optimality of the L2 signal set, *IEEE Trans. Inform. Theory* 43 (1997) 1574–1587; MR1476788 (99a:94064).

[5] C. L. Weber, *Elements of Detection and Signal Design*, Springer-Verlag, 1987, pp. 149–214.

[6] A. V. Balakrishnan, A contribution to the sphere-packing problem of communication theory, *J. Math. Anal. Appl.* 3 (1961) 485–506; MR0219340 (36 #2423).
[7] M. Henk, J. Richter-Gebert and G. M. Ziegler, Basic properties of convex polytopes, *Handbook of Discrete and Computational Geometry*, CRC Press, 1997, 243–270; MR1730169.

[8] H. Hadwiger, Gitterpunktanzahl im Simplex und Wills’sche Vermutung, *Math. Annalen* 239 (1979) 271–288; MR0522784 (80d:52015).

[9] H. Ruben, On the geometrical moments of skew-regular simplices in hyperspherical space, with some applications in geometry and mathematical statistics, *Acta Math.* 103 (1960) 1–23; MR0121713 (22 #12447).

[10] L. H. C. Tippett, On the extreme individuals and the range of samples taken from a normal population, *Biometrika* 17 (1925) 364–387.

[11] K. V. Mardia, Tippett’s formulas and other results on sample range and extremes, *Annals Inst. Statist. Math.* 17 (1965) 85–91; MR0178522 (31 #2779).

[12] H. A. David, *Order Statistics*, 2nd ed., Wiley, 1981, pp. 38–43, 53, 258–269; MR0099101 (20 #5545).

[13] H. Ruben, On the moments of the range and product moments of extreme order statistics in normal samples, *Biometrika* 43 (1956) 458-460; MR0082769 (18,607d).

[14] Y. Watanabe, M. Isida, S. Taga, Y. Ichijo, T. Kawase, G. Niside, Y. Takeda, A. Horisuzi, and I. Kuriyama, Some contributions to order statistics, *J. Gakugei, Tokushima Univ.* 8 (1957) 41-90; MR0099101 (20 #5545).

[15] S. R. Finch, Extreme value constants, *Mathematical Constants*, Cambridge Univ. Press, 2003, pp. 363–367; MR2003519 (2004i:00001).

[16] I. Ben Yaacov, Continuous and random Vapnik-Chervonenkis classes, *Israel J. Math.* 173 (2009) 309–333; MR2570671 (2011j:03072); [arXiv:0802.0068](https://arxiv.org/abs/0802.0068).

[17] H. O. Lancaster, Chi distribution, *Encyclopedia of Statistical Sciences*, v. 1, ed. S. Kotz, N. L. Johnson and C. B. Read, Wiley, 1982, p. 439; MR0646617 (83j:62001a).

[18] P. Hall, On the rate of convergence of normal extremes, *J. Appl. Probab.* 16 (1979) 433–439; MR0531778 (80d:60025).

[19] R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey and D. E. Knuth, On the Lambert W function, *Adv. Comput. Math.* 5 (1996) 329–359; MR1414285 (98j:33015).
[20] E. J. Gumbel, The distribution of the range, *Annals of Math. Statistics* 18 (1947) 384–412; MR0022331 (9,195a).

[21] D. R. Cox, A note on the asymptotic distribution of range, *Biometrika* 35 (1948) 310–315; MR0028562 (10,466b).

[22] F. W. J. Olver, Bessel functions of integer order, *Handbook of Mathematical Functions*, ed. M. Abramowitz and I. A. Stegun, Dover, 1992, pp. 374–377; MR1225604 (94b:00012).

[23] S. R. Finch, Euler-Mascheroni constant, *Mathematical Constants*, Cambridge Univ. Press, 2003, pp. 28–40; MR2003519 (2004i:00001).

[24] U. Betke and M. Henk, Intrinsic volumes and lattice points of crosspolytopes, *Monatsh. Math.* 115 (1993) 27–33; MR1223242 (94g:52010).

[25] Z. Govindarajulu, Exact lower moments of order statistics in samples from the chi-distribution (1 d.f.), *Annals Math. Statist.* 33 (1962) 1292–1305; MR0141179 (25 #4590).

[26] S. R. Finch, Simulations in R involving colliding dice and mean widths, [http://algo.inria.fr/csolve/rsimul.html](http://algo.inria.fr/csolve/rsimul.html)

Steven R. Finch
Dept. of Statistics
Harvard University
Cambridge, MA, USA
*Steven.Finch@inria.fr*
Figure 1: Surface plot of $\sqrt{g/2}$, where $0 \leq \theta \leq 2\pi$ and $0 \leq \varphi \leq \pi$.

Figure 2: Another view of $\sqrt{g/2}$, with contours of intersection.
This figure "Figure01.jpg" is available in "jpg" format from:

http://arxiv.org/ps/1111.4976v1
This figure "Figure02.jpg" is available in "jpg" format from:

http://arxiv.org/ps/1111.4976v1