YANG-MILLS THEORY AND JUMPING CURVES

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Abstract. We study a smooth analogue of jumping curves of a holomorphic vector bundle, and use Yang-Mills theory over $S^2$ to show that any non-trivial, smooth Hermitian vector bundle $E$ over a smooth simply connected manifold, must have such curves. This is used to give new examples complex manifolds for which a non-trivial holomorphic vector bundle must have jumping curves in the classical sense, (when $c_1(E)$ is zero). We also use this to give new proof a theorem of Gromov on the norm of curvature of unitary connections, and make the theorem slightly sharper. Lastly we define a sequence of new non-trivial integer invariants of smooth manifolds, connected to this theory of smooth jumping curves, make some computations of these invariants and indicate their connection to symplectic geometry.

1. Introduction

To study a holomorphic vector bundle $E$ over a complex manifold $X$, one strategy is to extract information about $E$ from information obtained by restrictions of $E$ to (families) of complex genus 0 curves in $X$. By Birkhoff-Grothendieck theorem, a complex rank $r$ holomorphic vector bundles over $\mathbb{CP}^1$ is holomorphically isomorphic to

$$\sum_{1 \leq i \leq r} \mathcal{O}(d_i),$$

with weights $d_i$ uniquely determined, (up to order) which makes their theory particularly transparent. We shall say that a holomorphic curve $u : \mathbb{CP}^1 \to X$ is a jumping curve if the pullback of $E$ to $\mathbb{CP}^1$ by $u$ is holomorphically non-trivial. When $c_1(E) \neq 0$ the classical definition is different but for our purpose we stick to the above for convenience. It is not actually very hard to extend our arguments to the case of classical jumping curves (when $c_1(E) \neq 0$), but at the cost of much added complexity. In this case the strategy then consists of studying splitting types of jumping curves. For $X = \mathbb{CP}^n$, one may take this approach very far, see for example [13], in particular the authors prove that a holomorphic vector bundle over $\mathbb{CP}^n$ is non trivial if and only if it has jumping lines.

Jumping curves can be defined for a smooth Hermitian vector bundle with a connection, as a direct generalization of the holomorphic case. Let $X$ be a smooth manifold, $E$ a Hermitian vector bundle over $X$, and $A$ a Hermitian, i.e. unitary connection on $E$. Given a smooth map $u : S^2 \to X$, the pull-back bundle $u^*E$, with its pull back connection $u^*A$, has a natural induced holomorphic structure. It is the holomorphic structure whose Dolbeault operator is induced by the the anti-complex linear part of the covariant derivative operator corresponding to $A$, this operator will be denoted by $\bar{\partial}_A$. This Dolbeault operator is necessarily integrable, see for instance [1]. We shall say that $u$ is a smooth jumping curve if the $\bar{\partial}_{u^*A}$
holomorphic structure on $u^* E$ is non-trivial. We will also just say jumping curves where there is no possibility of confusion with holomorphic case.

**Theorem 1.1.** Let $X$ be a simply connected smooth manifold, $E$ a complex, non-trivial, smooth Hermitian vector bundle over $X$, then for any Hermitian connection $A$ on $E$, there are smooth jumping curves in any class $[u] \in \pi_2(X)$.

This is proved by studying the map $\Omega^2 X \to \Omega^2 BU(r)$ induced by the classifying map for $E$, via Yang-Mills theory, where $\Omega^2 X$ denotes the spherical mapping space.

Let us explain the relationship of the above to the holomorphic setting. Suppose that $E \to X$ is a holomorphic vector bundle. Pick a Hermitian structure $h$ on $E$. Then there is a unique compatible Hermitian connection $A_h$ on $E$, inducing the given holomorphic structure, [6]. From this it readily follows that a holomorphic curve $u: \mathbb{CP}^1 \to X$, is a jumping curve if and only if it is a smooth jumping curve with respect to $A_h$. Let $\text{Hol}[u](\mathbb{CP}^1, X)$ denote the space of based holomorphic maps in class $[u]$. Then an argument for theorem above, together with the discussion above give the following.

**Theorem 1.2.** Suppose that $E$ is a non-trivial (as a complex vector bundle) holomorphic vector bundle over $X$, with $\pi_1(X) = 0$ and suppose that for every $k > 0$ there is a class $[u_k]$ such that the inclusion $\text{Hol}[u_k](\mathbb{CP}^1, X) \to \Omega^2([u_k]X)$ induces an isomorphism of homotopy groups, in range $[0, k]$. Then $E$ has (holomorphic) jumping curves. In particular this holds for $X = \mathbb{CP}^n$, a generalized flag manifold, a toric manifold, and $X = \Omega G$, with $G$ compact Lie group.

Note that we can no longer control the class of jumping curves without stronger assumptions. The hypothesis are known to be satisfied for $X = \mathbb{CP}^n$ [17], flag manifolds for example [8], [11] toric manifolds [9], loop groups $\Omega G$, with $G$ compact Lie group, for example [4]. Thus 1.2 also reproduces a slightly weakened version of the result in [13] mentioned above, weaker because in principle we obtain only jumping curves rather than lines, and we must ask for topological non-triviality rather than holomorphic. For the other cases the above existence result on jumping curves seems to be new.

1.1. Some new integer invariants of smooth vector bundles and smooth manifolds. For a smooth complex rank $r$ vector bundle $E$ over $X$, and a unitary connection $A$, if $u: S^2 \to X$ is a smooth curve define

(1.1) $|u|_A = \sum_i |d_i|^2$,

(1.2) $|u|_{A, \infty} = \max_i |d_i|$,

with $d_i$ the weights in the Birkhoff-Grothendieck splitting of $(u^* E, \overline{\partial}_{u^* A})$. Note that $|u|_A$ is the energy of the homomorphism $S^1 \to U(n)$, with weights $d_i$, after appropriate choice of normalization of the bi-invariant metric on $U(n)$. This will come into the proof of 1.5.

Define:

$$\zeta_{YM}(E, n) = \inf_{A, [f], f' \in [f]} \sup_{s \in S^n} |f'(s)|_A \in \mathbb{N},$$

where the infimum is over all $[f] \in \pi_n(\Omega^2 X)$, s.t. the map $\Omega^2 X \to \Omega^2 BU(r)$ induced by the classifying map of $E$ is non-vanishing on $[f]$, if there is no such $[f]$.\]
set $\zeta_{YM}(E, n) = 0$. Here $\Omega^2 X$ denotes the component of the spherical mapping space in the constant class, (which is why the invariant is $\mathbb{N}$ valued). Also define:

$$
\zeta_{YM}(E) = \inf_n \zeta_{YM}(E, n) \in \mathbb{N},
$$
$$
K_{YM}(X, r) = \inf_{E} \zeta_{YM}(E) \in \mathbb{N},
$$
$$
K_{YM}(X) = \inf_{r > 0} K_{YM}(X, r) \in \mathbb{N}.
$$

where the second infimum is over all rank $r > 0$ complex vector bundles $E$ with some non-vanishing Chern number.

**Theorem 1.3.** If $E$ is a non-trivial complex vector bundle over a simply connected $X$, then the sequence of positive integers $\{\zeta_{YM}(E, n)\}$ cannot be identically 0 and hence $\zeta_{YM}(E) \neq 0$, $K_{YM}(X, r) \neq 0$, and $K_{YM}(X) \neq 0$.

**Conjecture 1.4.** For $f : X \to Y$ a smooth homotopy equivalence, and $E$ a complex vector bundle over $Y$:

$$
\zeta_{YM}(E) = \zeta_{YM}(f^* E),
$$

in particular the invariants $K_{YM}(X, r)$, $K_{YM}(X)$ of a smooth manifold $X$ are homotopy invariants.

This conjecture is extremely likely. It would follow immediately if the perturbation property needed in the proof of Theorem 1.5, held for every $X$ as opposed to just $X = S^m$. With the perturbation property, it also becomes much easier to compute (at least in principle) the above invariants.

We note that unlike Pontryagin and Chern numbers, the integers $\zeta_{YM}(E, n)$ are not stable. That is, for a direct sum $E' = E \oplus \epsilon^k$ of vector bundles with $\epsilon^k$ the trivial $\mathbb{C}^k$ vector bundle, we do not in general have $\zeta_{YM}(E, n) = \zeta_{YM}(E', n)$. In fact we have:

**Theorem 1.5.** For any rank $r$ complex vector bundle $E \to S^m$, and any $k > 0$, so that $n \leq 2(r + k) - 2$ we have

$$
\zeta_{YM}(E \oplus \epsilon^k, m - 2) \leq r + k.
$$

On the other hand for $l = 2$, or any $l$ of the form $l = \sum_{1 \leq i \leq r} |d_i|^2$, with $d_i \in \mathbb{Z}$, $\sum_i d_i = 0$, there is a complex rank $r$ vector bundle $E$ over $S^m$, with $\zeta_{YM}(E, m_l) = l$. In particular $K_{YM}(S^m) = 2$.

1.1.1. More invariants of smooth manifolds or “quantum” Pontryagin numbers.

We may also use above construction to give invariants of a smooth manifold $X$ in terms of invariants $\zeta_{YM}(TX \otimes \mathbb{C}, n)$ of the complexified tangent bundle. These are possibly sensitive to the smooth structure, unlike (conjecturally) the invariants $K_{YM}(X), K_{YM}(X, r)$. Note however that Pontryagin numbers are topologically invariant by a theorem of Novikov. The “hope” is that instability of $\zeta_{YM}$ invariants introduces something new to the picture. The word “quantum” above is due to a rather direct but lengthy to describe connection with quantum characteristic classes, see more in Remark below.

There appear to be a couple of natural ways of proceeding, and at the moment it is far from clear if they give the same answers, we shall stick to one which appears to be the most interesting. For $X$ a smooth manifold, let $TM \otimes \mathbb{C}$ denote the complexification of the smooth tangent bundle of $X$. We may define

$$
\zeta_{YM}(X, n)
$$
as $\zeta_{YM}(TM \otimes \mathbb{C}, n)$ but taking infimum only over complexified Levi-Civita connections (for some Riemannian metrics on $X$) as opposed to all unitary connections. As a Corollary of 1.3 we get:

**Corollary 1.6.** For $X$ a smooth manifold, whose complexified tangent bundle is non-trivial the sequence of integers $\{\zeta_{YM}(X, n)\}$ cannot be identically zero.

**Remark 1.7.** There is a connection of the above invariants with symplectic geometry, and in fact the author arrived at 1.1 in that context first, (not written). Projectivizing the complex vector bundle $E$, we get a Hamiltonian (in fact $PU(n)$), $\mathbb{CP}^{r-1}$ bundle $P$ over $X$. And we may get obvious lower bounds for $\zeta_{YM}(E, n)$, from (unstable) quantum characteristic classes of this fibration, (see for example [15] for introduction to the subject). One may expect that they are in fact sharp. On the other hand these cohomology classes have a locality property: they are reconstructable from algebraic data that can be associated to trivializations of $P$. The framework for this is the global Fukaya category of $P$, [16]. This leads us to believe that $\zeta_{YM}(E, n)$ are themselves at least partially local and it would be very interesting to understand this intrinsically.

1.2. **Gromov norm.** Given a unitary connection $A$ on a rank $r$ complex vector bundle $E \to X$, a metric $g$ on $X$, and a fixed bi-invariant metric on $u(r)$, we can define the norm of it’s curvature $R_A$ by

$$|R_A|_g \equiv \sup_{|v \wedge w|_g = 1} |R_A(v, w)|,$$

where $R_A$ is considered as a $u(r)$ valued 2-form. Then 1.1 gives another very different in nature proof of the following theorem of Gromov.

**Theorem 1.8 ([7]).** For a simply connected Riemannian manifold $X, g$, and $E$ a non-trivial Hermitian vector bundle over $X$, we have

$$\inf_A |R_A|_g > 0.$$

We can sharpen it as follows.

**Theorem 1.9.** For $X, E, g$ as above:

$$\inf_A |R_A|_g \geq \inf_{n, [f], \{f'\}, s \in S^n} \sup_{\text{area}_g(f'(s))} \frac{|f'(s)|_{A, \infty}}{\text{area}_g(f'(s))} > 0,$$

with the supremum over all $[f] \in \pi_n(\Omega^2 X)$, s.t. the map $\Omega^2 X \to \Omega^2 BU(r)$ is non-vanishing on $[f]$, and with $\text{area}_g$ denoting the area functional with respect to $g$.

This is proved using Chern-Weil theory, in connection with our approach to jumping curves via Yang-Mills theory.

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3. Preliminaries

We briefly collect here some principal points of Yang-Mills theory, and set up our conventions and notation. Our notation will mostly follow [3] and [1].

The Yang-Mills functional denoted by $YM$ is defined on the space $A$ of $G$ connections on a principal $G$ bundle $P$ over $X$ by:

$$YM(A) = \int_X |F_A|^2;$$

for $A$ a $G$ connection, and $F_A$ the curvature 2-form on $X$, with values in $\text{Ad}P = P \times_{\text{Ad}G} \mathfrak{g}$. The norm $||$ above is the natural norm induced by a metric $g$ on $X$ and a fixed bi-invariant inner product on $\mathfrak{g},$ inducing a metric on $P \times_{\text{Ad}G} \mathfrak{g}$.

In this note $G$ will be $U(r)$, sometimes $SU(r)$, and $M = S^2$. The value $YM(A)$, will be called Yang-Mills energy of a connection and critical points will be called Yang-Mills connections. For $X = S^2$ or more generally a Riemann surface, the negative gradient flow for $YM$ exists for all time on the space of smooth connections, and moreover gradient lines converge to Yang-Mills connections. We shall call the negative gradient flow for $YM$ on $A$, with respect to the natural $L^2$ metric, the Yang-Mills flow. The functional $YM$ and the metric are equivariant with respect to the action of the gauge group of $P$, i.e. the group of it’s bundle automorphisms, and the flow is defined on the space of gauge equivalence classes. A crucial point Yang-Mills flow is that preserves the orbit of $A$ under the complexified gauge group.

In our case $P$ is associated to some complex vector bundle $E$, and this means that Yang-Mills flow preserves the holomorphic isomorphism type of $(E, \mathcal{F}_A)$.

For the proof of 1.5 we also need some kind of Morse homology for $YM$. Much additional work is required to set up Morse-Smale-Witten complex on the space of gauge equivalence classes connections. For $X = S^2$ this is started in [3]. Note however that unlike [3] it is critical for us to work with the reduced gauge group which we call $\mathcal{G}_0(P)$ and which consists of bundle automorphisms fixing the fiber over the base point $0 \in S^2$, as this group acts freely. When we say gauge equivalence class, we mean the equivalence class under the action of $\mathcal{G}_0$, as opposed to the full gauge group. In this case $YM$ on $A/\mathcal{G}_0$ only gives rise to a Morse-Bott complex, full details of this appear in [18].

4. Proofs

We need a few preliminary steps. Fix a rank $r$ complex vector bundle $E$ over $\mathbb{C}P^1$. Let $\mathcal{G}_0(E)$ denote the group of bundle automorphisms of $E$, fixing the fiber over 0. Denote by $A(E)$ the space of unitary connection on $E$. It is easy to check that the group $\mathcal{G}_0(E)$ acts freely on $A(E)$. Elements of $A(E)/\mathcal{G}_0(E)$ will be called gauge equivalence classes of connections. Let also $\mathcal{E}$ denote the universal $\mathbb{C}^r$ bundle over $BU(r)$, and $\Omega^{2,[v]}BU(r)$ denote the smooth spherical mapping space in the component of a map $u : S^2 \to BU(r)$. To make sense of smooth here as well as of smooth connections on the universal bundle to be used later, we only need to observe that $BU(r)$ has a homotopy model is a direct limit of smooth manifolds, $BU(r) \simeq \lim_n GrC(r, \mathbb{C}^n)$, with maps in the directed system also smooth. This will be sufficient to make sense of what follows. For $v \in \Omega^{2,[v]}BU(r)$ set $v^*\mathcal{E} = \mathcal{E}_v$.

**Lemma 4.1.** Given a smooth unitary connection $A$ on the universal $\mathbb{C}^r$-bundle $\mathcal{E}$ over $BU(r)$, there is a natural induced map $f_{A,u} : \Omega^{2,[v]}BU(r) \to A(\mathcal{E}_u)/\mathcal{G}_0(\mathcal{E}_u)$. 
Proof. For \( v \in \Omega^2[\cdot]BU(r) \), take a smooth path \( p \) from \( u \) to \( v \), (in \( \Omega^2[\cdot]BU(r) \)). Using the connection \( A \) on \( \mathcal{E}^r \), this determines a bundle map \( m_p : u^*\mathcal{E} \to v^*\mathcal{E} \). It is defined via parallel transport along the paths \( \{ t \mapsto p(t)(z) \} \), \( t \in [0, 1] \), \( z \in \mathbb{CP}^1 \).

Then define \( f_{A,u}(v) = m_p^*A_u \in \mathcal{A}(\mathcal{E}_u)/\mathcal{G}_0(\mathcal{E}_u) \), where \( A_u \) is the connection \( v^*A \) on \( \mathcal{E}_u^r \). We need to check that this is well defined, i.e. independent of the choice of \( p \). This is immediate for if \( p' \) is another path from \( u \) to \( v \), the connections \( m_p^*A_u, m_{p'}^*A \) are gauge equivalent with respect to the bundle automorphism \( m_{p^{-1} \cdot p'} \), obtained via \( A \)-parallel transport as before with \( p^{-1} \cdot p' \) denoting concatenation.

\[ \square \]

**Proposition 4.2.** \( f_{A,u} \) is a homotopy equivalence for any \( A \).

Proof. The spaces \( \Omega^2[\cdot]BU(r) \) and, \( \mathcal{A}(\mathcal{E}_u)/\mathcal{G}_0(\mathcal{E}_u) \) are homotopy equivalent, [1, Proposition 2.4], it will then suffice to show that \( f_{A,u} \) is injective on homotopy groups. Thus, suppose that \( f'_1 = f_{A,u} \circ f_1, f'_2 = f_{A,u} \circ f_2 \), for \( f_i : S^k \to \Omega^2[\cdot]BU(r) \) are homotopic. As \( \mathcal{A}(\mathcal{E}_u)/\mathcal{G}_0(\mathcal{E}_u) \simeq \mathcal{B}(\mathcal{E}_u) \), to a given \( f : S^k \to \mathcal{A}(\mathcal{E}_u)/\mathcal{G}_0(\mathcal{E}_u) \) we have associated principal \( \mathcal{G}_0(\mathcal{E}_u) \)-bundles \( \mathcal{P}_f \). Let \( \mathcal{P}_t \) denote the principal \( \mathcal{G}_0(\mathcal{E}_u) \) bundles associated to \( f'_1 \), and let \( pr : E_t \to S^k \) denote the associated \( \mathcal{E}_u \) bundles, (a bundle with fiber modeled on the complex vector bundle \( E_u \) over \( \mathbb{CP}^1 \)). We may also think of \( \mathcal{P}_t \) as total spaces of complex \( \mathbb{C}^r \) vector bundles over \( S^k \times S^2 \). For distinction let us call them \( E_t \). It is elementary to verify that \( E_t \) are classified by the maps \( f_i : S^k \times S^2 \to BU(r) \), induced by \( f_i \). If \( H : S^k \times [0, 1] \to \mathcal{A}(\mathcal{E}_u)/\mathcal{G}_0(\mathcal{E}_u) \) is a homotopy between \( f'_1, f'_2 \), we get again an induced \( \mathbb{C}^r \) vector bundle \( E_H \) over \( S^k \times S^2 \times [0, 1] \). By observation above it’s classifying map to \( BU(r) \) induces a homotopy between \( \tilde{f}_i \), and consequently a homotopy between \( f_i \).

\[ \square \]

**Proof of 1.1.** Suppose \( E \) has complex rank \( r \). Clearly we may suppose \( c_1(E) = 0 \). As \( E \) is non-trivial, the classifying map \( c_E : X \to BU(r) \), must be non-vanishing on some \( \pi_k \), with \( k > 2 \). Consequently, since \( X \) is simply connected the induced map \( c_E : \Omega^2 X \to \Omega^2 BU(r) \), is non-vanishing on some \( [f] \in \pi_k(\Omega^2 X) \) with \( k > 0 \), with the spherical mapping spaces taken in the component of the constant map. Approximate the classifying map by a smooth embedding \( c_E \), and push-forward the connection \( A \) to a connection on the universal bundle \( \mathcal{E} \) over the image of \( f_E \), then extend to a connection \( \tilde{A} \) on \( \mathcal{E} \) over \( BU(r) \). (There are no obstructions to existence of such an extension, it can be constructed via partitions of unity for example). Now suppose that there are no jumping curves in \( X \) for \( E \), \( A \) in the trivial holonomy class, then we claim that for any representative \( f \) of \( [f] \), the map

\[ \tilde{f} = f_{\tilde{A},[\text{const}]} \circ c_E \circ f : S^k \to \mathcal{A}(\epsilon^r)/\mathcal{G}_0(\epsilon^r), \]

is null-homotopic where \( \text{const} \) denotes the constant based map \( \text{const} : S^2 \to BU(r) \), and \( \epsilon^r \) is shorthand for the trivial \( \mathbb{C}^r \) vector bundle \( \mathcal{E}_{\text{const}} \). This will be a contradiction and hence conclude the proof.

Indeed let \( \phi_{\cdot,M}(t) \) denote the time \( t \) flow map for the negative gradient flow for the Yang-Mills flow on \( \mathcal{A}(\epsilon^r)/\mathcal{G}_0(\epsilon^r) \). We need a pair of facts, first the isomorphism class of the induced holomorphic structure on \( \mathcal{E}_0 \) by elements \( [A] \in \mathcal{A}(\epsilon^r)/\mathcal{G}_0(\epsilon^r) \) is constant along negative gradient flow lines of the Yang-Mills functional, which follows by [1, 8.12]. Next, we know that a critical \( |A| \) for the Yang-Mills functional (a.k.a a Yang-Mills connection) determines a non-trivial holomorphic structure on
\[ \epsilon^r \text{, so long as it is not the gauge equivalence class of a trivial connection. Consequently, as the holomorphic structure induced by each } \tilde{f}(s), s \in S^k \text{ is trivial by assumption, the negative gradient lines } t \mapsto \phi_{YM}(t)(\tilde{f}(s)) \text{ must converge to the gauge equivalence class of a trivial connection. Reparameterizing Yang-Mills flow, (by Yang-Mills energy), we obtain a null-homotopy of } \tilde{f}.

The same argument would show that for a general class \([u] \in \pi_2(X)\), either there are jumping curves in class \([u]\), or the map \(\Omega^2[u]X \to \Omega^2 BU(r)\) induced by the classifying map vanishes on all homotopy groups, but the latter would imply that \(\Omega^2 X \to \Omega^2 BU(r)\) vanishes on all homotopy groups, which contradicts our assumptions. Consequently only the former is possible.

**Proof of 1.3.** This follows readily by the proof of 1.1.

**Proof of 1.2.** As previously observed the map \(\widetilde{c}_E : \Omega^2 X \to \Omega^2 BU(r)\) induced by the classifying map for \(E\), must be non-vanishing on some \([f] \in \pi_k(\Omega^2(X))\), with \(k > 0\). Let \([u_k]\) be as in the hypothesis and suppose that there are no class \([u_k]\), holomorphic jumping curves. Let \(\tilde{f} : S^k \to \Omega^2[u_k](X)\), be a representative for \(iso \circ f\), for \(iso : \Omega^2 X \to \Omega^2[u_k]\) the canonical homotopy equivalence, so that \(\tilde{f}\) is in the image of the inclusion \(Hol([u_k])(\mathbb{CP}^1, X) \to \Omega^2[u_k]X\). Consequently there are no smooth jumping curves in the image of \(\tilde{f}\). Thus running the argument in 1.1 we get that \(\widetilde{c}_E \circ \tilde{f}\) is null-homotopic, but then \(\widetilde{c}_E\) must be vanishing on the class \([f]\) as well, a contradiction.

**Proof of 1.9.** This follows by the proof of 1.1 once we note:

**Lemma 4.3.** Let \(E, X, A, g\) be as in the hypothesis. Suppose \(u : \mathbb{CP}^1 \to X\) is a smooth jumping curve. Then the norm of the curvature of \(u^*E, u^*A\), \(|R(u^*A)|_g\) is at least \(|u|_{A, \infty}/area_g(u)\).

**Proof.** Indeed for the holomorphic structure on \(u^*E\), induced by \(u^*A, u^*A\) preserves the Grothendieck splitting of \(u^*E\). Consequently the norm of the curvature of \(u^*A\) on the maximal weight subspace \(O_{max}\) of \(u^*E\) cannot exceed the norm of curvature of \(u^*A\) on \(u^*E\), see [6, Page 79]. This follows from “positivity” of curvature on a holomorphic vector bundle.

As \(c_1(O_{max}) = |u|_{A, \infty}\), by assumption, Chern-Weil theory gives a ready estimate

\[ |R(u^*A|_{O_{max}})|_{u^*g} \cdot area_g(u) \geq |u|_{A, \infty}. \]

**Proof of 1.5.** As we are working in the component of \(\Omega^2 X\) in the constant class we may restrict our arguments to \(SU(r)\) connections on \(E\), and \(A(\epsilon^r)\) will denote the space of \(C^\infty SU(r)\) connections on \(\epsilon^r\).

Morse theory for the Yang-Mills functional \(YM\) on \(A^r \equiv A(\epsilon^r)/G_\theta(\epsilon^r)\) is known to be essentially equivalent to Morse theory on \(\Omega SU(r)\) for the energy functional, [5], [19]. Although we run the argument for Yang-Mills theory, it will be helpful to refer to the correspondence in a few instances. Moreover the energy flow picture, as developed in great detail in [14], is probably more accessible, and may help the reader to get intuition. The correspondence is given via radial trivialization. We briefly sketch this. Given an element \(|A| \in A^r\) we get a loop \(Rad(|A|) \in \Omega SU(r)\),...
by fixing an $S^1$-family of rays $\{r_0\}$ from 0 to $\infty \in \mathbb{CP}^1$, identifying the fibers over 0, $\infty$ via $A$-parallel transport along $r_0$, and then getting a loop in $SU(r)$ via $A$-parallel transport along rays $\{r_0\}$, for $A \in |A|$, which is well defined with respect to equivalence under action of $G_0(E^r)$. The functions and their flow are then shown to behave well with respect to this correspondence. In particular critical points and their indices correspond. Thus radial trivialization of a Yang-Mills connection is an $S^1$ subgroup of $SU(r)$.

We shall use the construction of Morse-Bott-Smale complex for $YM$, in [18]. The critical sets of $YM$ on $\mathcal{A}'$ split into disjoint unions of submanifolds $\Lambda_d$, composed of (Yang-Mills) connections, inducing the holomorphic structure of type $d$, with $d$ by Birkhoff-Grothendieck can be taken to mean an unordered collection of $r$ integers. The manifolds $\Lambda_d$, which are isomorphic to the manifolds of $S^1$ subgroups of $SU(r)$, conjugate to the diagonal $S^1$ subgroup with weights $d$, are certain complex flag manifolds and admit perfect Morse functions. Consequently, the homology of $A_r$ is computed by the cascade Morse-Smale-Witten complex, [2], which has vanishing differentials, as the Morse-Bott index in this case is always even, (see elaboration below). We will not mention the auxilliary Morse-Smale functions on the critical manifolds explicitly, and just write $C^*(YM)$, for the above complex.

Up to some dimension $\dim(r)$ all the generators of $C_r(YM)$, come from critical manifolds $\Lambda_d$, with $d$ having all weights 1, $-1$ or 0. This readily follows upon computing the Morse-Bott index along $\Lambda_d$, (i.e. the normal component of Morse index). As the Morse index for $YM$ of a critical $|A| \in \mathcal{A}'$, coincides with the Morse index of $rad(|A|) \in \Omega SU(r)$, we may use the classical calculation of Morse index of geodesics in $SU(r)$, see for example [12]. This tells us that for a homomorphism $\gamma : S^1 \rightarrow SU(r)$, conjugate to $\gamma = \begin{pmatrix} e^{2\pi i a_1} & & \\ & e^{2\pi i a_2} & \\ & & \vdots \end{pmatrix}$

with weights $a_1, \ldots, a_r$, organized so that $a_1 \geq a_2 \geq \ldots \geq a_r$ the Morse index is given by

$$\sum_{i<j} 2|a_i - a_j| - 2.$$  

It is then easy to verify that if $d = \{a_1, \ldots, a_r\}$ does not satisfy that all $a_i$ are either 1, $-1$ or 0, then the Morse index is strictly greater than $2r - 2$, see [12, page 132] for a similar calculation.

There is a dual stratification of $\mathcal{A}'$ by finite co-dimension strata, intersecting the cascade unstable manifolds transversally, corresponding to cascade stable manifolds for $YM$. It follows that given a map $f : S^k \rightarrow \mathcal{A}'$ with $k \leq 2r - 2$, after a small perturbation of $f$ the (reparametrized by Yang-Mills energy) Yang-Mills flow will give a homotopy of $f$ into the $2r - 2$ skeleton of $\mathcal{A}'$, by which we mean the critical sub-level set for $YM$, so that all the generators of $C_r(YM)$, in the sub-level set have total index at most $2r - 2$. That is, we just have to take a perturbation of $f$ so that we are transverse to the finite codimension strata. Let’s now apply this.

Following the proof of 1.1, we have a map

$$\Omega^2 S^m \rightarrow \mathcal{A}^{r+k}$$
induced by the classifying map for $E \otimes e^{kn}$. Take $k$ so that $n \leq 2(r + k) - 2$. It follows by discussion above that the composition

$$\tilde{f} = f_{A, [\text{const}]} \circ \tilde{c}_{E \otimes e^{kn}} \circ f : S^n \to \mathcal{A}^{r+k}$$

can be perturbed so that the energy flow (after reparametrization) takes $\tilde{f}$ into the $2(r + k) - 2$ skeleton of $\mathcal{A}^{r+k}$.

Moreover we have:

**Lemma 4.4.** Any abstract perturbation of $\tilde{f}$, can be obtained via perturbation of the connection $A$ on $E$.

If we accept this for the moment then the first part of the theorem follows, as the Birkhoff-Grothendieck splitting type of elements in the $2(r + k) - 2$ skeleton, by preliminary discussion above, is such that all weights are either 1, $-1$ or 0. Consequently, (after perturbation of $A$) $|f(s)|_A \leq r + k$ for each $s \in S^n$.

**Proof of Lemma 4.4.** Let us then verify the claim about the perturbation. This is where the assumption $X = S^n$ comes into the proof. Take a representative $f : S^n \to \Omega^2 S^m$, for $[f]$ so that the induced map $g : S^n \times S^2 \to S^m$ is an embedding outside the submanifolds $S^n \times \{0\}$, $\{0\} \times S^2$ which are collapsed to a point. Given an abstract perturbation $f'$ of $\tilde{f}$, with $\hat{F} : S^n \times I \to \mathcal{A}^{r+k}$ a homotopy between $\tilde{f}$, and $\hat{f}'$, we get an associated structure group $\mathcal{G}_0(\epsilon^{r+k})$ bundle

$$\pi_{\hat{F}} : P_{\hat{F}} \to S^n \times [0, 1]$$

with fiber $\mathcal{C}^{r+k} \times S^2$. We also get a bundle $G_{\hat{F}} \to S^n \times [0, 1]$, with fiber over $(s, t)$ the space of connections on $\pi_{\hat{F}}^{-1}(s, t)$, in the gauge equivalence class of $\hat{F}(s, t)$. Note that this is in general not isomorphic to the principal $\mathcal{G}_0(\epsilon^{r+k})$ bundle pulled back by $\hat{F}$. By construction $G_{\hat{F}}$ has a section over $S^n \times \{0\}$, use homotopy lifting property to obtain a section $S_{\hat{F}}$ over $S^n \times \{1\}$. $S_{\hat{F}}$ can be thought of as smooth family of connections $\{A_s\}$ over $\{(s) \times S^2 \subset S^n \times S^2\}$, for the $\mathcal{C}^{r+k}$ bundle $E_{\hat{F}} \to S^n \times S^2$ canonically associated to $P_{\hat{F}}$. Note that $E_{\hat{F}}$ is canonically trivial over $S^n \times \{0\}$, $\{0\} \times S^2$, consequently we may extend $\{A_s\}$ to a connection $A''$ of $E_{\hat{F}} \to S^n \times S^2$, trivial over $S^n \times \{0\}$, $\{0\} \times S^2$. (The extension may be obtained via classical partition of unity argument.) It follows that we may push forward $A''$ via $g$ to a connection $A'$ on $E$, with exactly the right property. \hfill $\square$

We now verify the second part of the theorem. We will say that a skeleton of $\mathcal{A}'$ (in the previous sense) has energy $l$, if all the generators of $\mathcal{C}_n(YM)$ in it have YM energy less then or equal to $l$. As $\mathcal{C}_n(YM)$ is perfect, skeletons with distinct energy have distinct homology. From this it follows that a skeleton $S$ with energy $l$ admits a non null-homotopic based map $f : S^{m_{l-2}} \to S$, which cannot be pushed below the energy level $l$. As otherwise $S$ could be compressed into a lower energy skeleton, which would contradict the discussion above. (The compression argument is just Whitehead’s theorem, see for example [10]). As the map $f_{A, \text{const}}$ is a homotopy equivalence for every $A$, we may find a $g : S^{m_{l-2}} \to \Omega^2 BU(r)$, s.t. $f_{A, \text{const}}[g] = [f]$. Following the proof of Lemma 4.4 we get a family $\{A'_{s}\}$ of connections on $\{E'_{f'(s)} \times S^2\}$ where $E'_{f'}$ is the $\mathcal{C}'$ vector bundle over $S^{m_{l-2}} \times S^2$ associated to $f$ as in the proof of 4.4. Extend $\{A'_{s}\}$ to a connection $A'_f$ on $E_f$, trivial over $S^{m_{l-2}} \times \{0\}$, where the bundle is canonically trivialized. By construction $A'_f$
is also a trivial connection over \( \{0\} \times S^2 \). It follows that \( E'_f \) induces a vector bundle over \( \sum^{m_l} = \sum^{m_l-2} \sqrt{S^2} \), and \( A'_f \) induces a connection \( A_f \) on \( E_f \). By construction the infinum in \( \zeta_{YM}(E_f, m_l) \) is attained on \( A_f \).

The proof is finished as the normalization for the bi-invariant metric on \( SU(r) \) that we take, is such that an \( S^1 \) geodesic with weights \( \{d_i\} \), has energy given by (1.1). \( \square \)

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