A generalized Blakers–Massey theorem

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Abstract

We prove a generalization of the classical connectivity theorem of Blakers–Massey, valid in an arbitrary higher topos and with respect to an arbitrary modality, that is, a factorization system $(\mathcal{L}, \mathcal{R})$ in which the left class is stable by base change. We explain how to rederive the classical result, as well as the recent generalization of Chachólski, Scherer and Werndli (Ann. Inst. Fourier 66 (2016) 2641–2665). Our proof is inspired by the one given in homotopy-type theory in Favonia et al. (2016).

1. Introduction

The classical Blakers–Massey theorem, sometimes known as the homotopy excision theorem, is one of the most fundamental facts in homotopy theory. Given a homotopy pushout diagram of spaces

$$
\begin{array}{ccc}
A & \xrightarrow{g} & C \\
\downarrow{f} & & \downarrow{\iota} \\
B & \longrightarrow & D
\end{array}
$$

such that the map $f$ is $m$-connected and the map $g$ is $n$-connected, the theorem tells us that the canonical map $A \rightarrow B \times_D C$ to the homotopy pullback is in fact $(m + n)$-connected. (We direct readers, who find themselves surprised by the statement, to Remark 3.3.5 for an explanation of our indexing conventions for connected maps.) Among other things, the theorem gives rise to the Freudenthal suspension theorem and, thus, to stable homotopy theory.

Recently, a new proof of this theorem was found in the context of homotopy-type theory, a formal system originating in constructive mathematics and computer science which has been shown to provide an elementary axiomatization of homotopy theoretic reasoning [16]. One pleasant feature of this proof is that it is entirely homotopy invariant, neither relying on a particular model of homotopy types such as topological spaces or simplicial sets, nor requiring more sophisticated mathematical machinery such as transversality arguments or homology calculations. A second and perhaps more surprising consequence is that, written as it is in a formal language, it becomes subject to automatic verification by a computer. The interested reader may consult [8], where just such formalization is described in detail.

The reasoning formalized by homotopy-type theory is generally thought to serve as an ‘internal language’ for a particular class of higher categories, namely the $\infty$-topoi as developed

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by Rezk [14] and Lurie [11]. This is to say that each operation of the logic has a corresponding interpretation as a higher categorical construction. As a consequence, the original proof of [8] may be translated into the language of higher category theory, an undertaking which is carried out in unpublished work of Rezk [15], and which we revisit in this article. Our result is a much generalized theorem, applying not only to spaces, but to an arbitrary ∞-topos. As we will see in the companion article [1], the generalized theorem can be applied to an appropriate presheaf topos and yields an analogue of the Blakers–Massey theorem in the context of Goodwillie’s calculus of functors.

In order to pursue these sorts of applications, however, we will need to further generalize the theorem, beyond simply placing it in an abstract context. While on the face of it, the theorem speaks about the connectivity of certain maps, it turns out that we may in fact replace the property of ‘connectedness’ with any other property of morphisms which behaves sufficiently like it. The central observation here is that the $n$-connected maps form the left class of a factorization system $(\mathcal{L}, \mathcal{R})$ on the category of spaces with the additional property that the left class $\mathcal{L}$ is stable under base change. We refer to a factorization system satisfying this condition as a modality, a term originating in the literature on type theory [16, Section 7.7].

Concretely, then, our main theorem is the following.

**Theorem 4.1.1.** Let $\mathcal{E}$ be an ∞-topos and $(\mathcal{L}, \mathcal{R})$ a modality on $\mathcal{E}$. Write $\Delta h : A \to A \times_B A$ for the diagonal of a map $h : A \to B \in \mathcal{E}$ and $- \circ_Z -$ for the pushout product in the slice category $\mathcal{E}/Z$. Given a pushout square

\[
\begin{array}{ccc}
Z & \xrightarrow{g} & Y \\
\downarrow f & & \downarrow \downarrow \\
X & \to & W \\
\end{array}
\]

in $\mathcal{E}$, suppose that $\Delta f \circ_Z \Delta g \in \mathcal{L}$. Then the canonical map $(f, g) : Z \to X \times_W Y$ is also in $\mathcal{L}$.

In fact, a similar generalization of the Blakers–Massey theorem was recently obtained by Chachólski, Scherer and Werndli in [5], and their work provided inspiration for the statement of our main result. Our techniques, however, are quite different: their method involves the manipulation of weak cellular inequalities of spaces, as introduced in [7], whereas we focus on ∞-topos theoretic tools such as descent. (It is possible though to interpret our results as proving weak cellular inequalities for morphisms.) The present work can be seen as a synthesis and generalization of two new approaches to a classical result: via weak cellular inequalities and via higher topos theory/homotopy-type theory. Overall, the necessary input from classical homotopy theory has become almost invisible.

We also would like to draw the attention of the reader to the results in Subsection 3.7 where descent properties associated to the left class of a modality are proved. These results are important for our proof of the generalized Blakers–Massey theorem but might be of independent interest.

Let us now turn to an outline of the paper. Section 2 fixes our higher categorical conventions and recalls some elementary facts which will be used throughout the paper. We briefly review the definition of an ∞-topos, including the axiom of descent. Section 3 begins by introducing the notion of a factorization system, as well as the pushout product and pullback hom, two constructions which prove convenient for manipulating orthogonality relations between maps in a category. In a cartesian closed category such as an ∞-topos, orthogonality can be strengthened to an internal version, and we explore some of the properties of factorization systems compatible with this internalization, of which modalities will prove to be examples. We then give a short treatment of the $n$-connected/$n$-truncated factorization system in an ∞-topos. This archetypal
example of a modality will be important for extracting the classical Blakers–Massey theorem from our generalized version. Next, we introduce the notion of a modality itself, providing a number of examples and deriving some elementary properties, including the Dual Blakers–Massey theorem. We conclude the section by deriving what turns out to be the most crucial property of modalities for our purposes: the descent theorem for \( L \)-cartesian squares. Section 4 then turns to the proof of the generalized Blakers–Massey theorem itself, finishing with the derivation of the classical theorem, as well as that of Chachólski–Scherer–Wernli.

2. Higher topoi

2.1. Higher categorical conventions

Throughout this paper, we employ the language of higher category theory, considering only homotopy invariant constructions. Moreover, we use terminology which reflects this convention: by category we will always mean an \((x, 1)\)-category, saying 1-category explicitly to refer to an ordinary category if the occasion so arises. In particular, we will from now on refer to an \( \infty \)-topos simply as a topos. Similarly, we say simply limit and colimit for the higher categorical version, what would ordinarily be called the homotopy limit or homotopy colimit. All mapping spaces are ‘derived’, and composition of morphisms is associative up to coherent higher homotopy.

For readers unfamiliar with the literature on higher category theory, we have tried hard to make the paper nonetheless accessible. Indeed, our arguments involve only the elementary manipulation of homotopy limits and colimits, they are, in a sense, model independent. A reader more familiar with the theory of model categories should have no trouble interpreting our results in, for example, a simplicial model category. Of course, for a more precise discussion of the relationship between the higher categorical approach and the model category theoretic one, we refer the reader to [11].

We will use the word space to refer to an abstract homotopy type, what is often called an \( \infty \)-groupoid in the higher categorical literature. The reader is free to keep in mind any preferred model for these objects, such as topological spaces (compactly generated Hausdorff) or simplicial sets up to weak homotopy equivalence, but none of our arguments will depend on such a choice. We write \( \mathcal{S} \) for the category of spaces.

For two objects \( X \) and \( Y \) in a category \( \mathcal{C} \), we write \([X, Y]\) for the space of maps between \( X \) and \( Y \). The words map, morphism and arrow will be used interchangeably, as is common. For a category \( \mathcal{C} \), we let \( \mathcal{C}^{-} \) denote its category of arrows. By an isomorphism in \( \mathcal{C} \) will refer to a morphism which is invertible in \( \mathcal{C} \) in the higher categorical sense: for example, in \( \mathcal{S} \) the isomorphisms correspond to the weak homotopy equivalences when homotopy types are modeled as topological spaces. We will often write ‘\( X = Y \)’ to mean that two objects \( X \) and \( Y \) of \( \mathcal{C} \) are isomorphic, when the isomorphism is clear. Similarly, we will write \( f = g \) to mean that two maps \( f, g : X \to Y \) are homotopic, when the homotopy is clear. As, for example, in the statement of Proposition 3.2.3, we may also write \( f = g \) to mean that two maps are naturally isomorphic in the arrow category \( \mathcal{C}^{-} \) when the isomorphisms are clear. As the former ‘\( = \)’ is a special case of the latter we hope this does not cause confusion.

We encourage the interested reader to consult [16] for the homotopy-type theory perspective on the equality relation.

We will write 0 for initial and 1 for terminal objects.

Given a finite family of maps \( f_i : X \to Y_i \), where \( 1 \leq i \leq n \) in a category \( \mathcal{C} \), we will write

\[(f_1, \ldots, f_n) : X \to Y_1 \times \cdots \times Y_n\]

for the canonical map from \( X \) to the product of the \( Y_i \). Dually, for a finite family \( f^i : X_i \to Y \), where \( 1 \leq i \leq m \), we write

\[[f^1, \ldots, f^m] : X_1 \sqcup \cdots \sqcup X_m \to Y\]
for the canonical map from the coproduct of the $X_i$ to $Y$. More generally, for any doubly
indexed family $f_{ij} : X_i \to Y_j$, where $1 \leq i \leq m$ and $1 \leq j \leq n$, we have an induced ‘total’ map

$$T(f_{ij}) : X_1 \sqcup \cdots \sqcup X_m \to Y_1 \times \cdots \times Y_n$$

and we leave it to the reader to check that this map obeys the commutation relation

$$T(f_{ij}) = [(f_1^1, \ldots, f_1^n), \ldots, (f_m^1, \ldots, f_m^n)]$$

where

$$
\begin{bmatrix}
  f_1^1 & f_2^1 & \cdots & f_1^n \\
  f_2^1 & f_2^2 & \cdots & f_2^n \\
  \vdots & \vdots & \ddots & \vdots \\
  f_m^1 & f_m^2 & \cdots & f_m^n \\
\end{bmatrix}
\cdot
\begin{bmatrix}
  \frac{1}{f_1^1} & \frac{1}{f_1^2} & \cdots & \frac{1}{f_1^n} \\
  \frac{1}{f_2^1} & \frac{1}{f_2^2} & \cdots & \frac{1}{f_2^n} \\
  \vdots & \vdots & \ddots & \vdots \\
  \frac{1}{f_m^1} & \frac{1}{f_m^2} & \cdots & \frac{1}{f_m^n} \\
\end{bmatrix}
= (|f_1^1, \ldots, f_m^1|, \ldots, |f_1^n, \ldots, f_m^n|).
$$

The following special cases of the above notation will occur frequently enough that they will
merit some special terminology. Suppose we are given a commutative square

$$
\begin{array}{ccc}
Z & \xrightarrow{g} & Y \\
\downarrow{f} & & \downarrow{k} \\
X & \xrightarrow{h} & W
\end{array}
$$

in a category $\mathcal{C}$. Taking the pushout of the diagram $X \leftarrow Z \rightarrow Y$ or the pullback of the
diagram $X \rightarrow W \leftarrow Y$, we obtain two canonical maps which, using the previous notation, will
be denoted by

$$(f, g) : Z \to X \times_W Y$$

$$(h, k) : X \sqcup_Z Y \to W.$$  

We will refer to the first of these two maps as the cartesian gap map, or merely the gap map. The second will be referred to as the cocartesian gap map, or more briefly, the cogap map. This
notation is in fact mildly abusive since the maps in question depend on the data of the entire
commutative square. That is to say, the first map is a special case of our general notation when
regarded in the slice category $\mathcal{C}_{/W}$ and the second in the coslice category $\mathcal{C}_{Z/}$. In practice,
however, the remaining maps will be clear from the context.

2.2. Topoi and descent

There are many equivalent characterizations of the notion of a topos, but for the purposes of
this article we will adopt the position that a topos is simply a category satisfying a certain
collection of exactness conditions, that is, compatibilities between limits and colimits. While
this is perhaps not the most profound point of view on the subject, it nonetheless has the
benefit of practicality, making explicit the constructions which can be performed, and hence
will be adequate for our purposes here.

We will need a couple of elementary facts about presentable categories, for whose complete
theory we refer the reader to [11, Chapter 5]. A presentable category has all limits and colimits,
and a functor $F : \mathcal{C} \to \mathcal{D}$ between presentable categories preserves all colimits if and only if it
has a right adjoint. We say that the colimits in a presentable category $\mathcal{C}$ are universal if the
base change functor

$$f^* : \mathcal{C}_{/Y} \to \mathcal{C}_{/X}.$$
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preserves colimits for any map $f : X \to Y$ in $\mathcal{C}$. In this case, the functor $f^*$ admits a right adjoint $f_*$ by the previous remarks. In particular, the base change functor $A \times (-) : \mathcal{C} \to \mathcal{C}/_A$ has a right adjoint

$$\Pi_A : \mathcal{C}/_A \to \mathcal{C}$$

for every object $A \in \mathcal{C}$. It follows that the category $\mathcal{C}$ is cartesian closed with internal hom

$$[A, B] = \Pi_A(A \times B, p_A)$$

for every $A, B \in \mathcal{C}$, where $p_A$ is the projection onto $A$.

We will say that a morphism $\alpha : f \to g$ in the arrow category $\mathcal{C}^- \to \mathcal{C}$ is cartesian if the corresponding square in $\mathcal{C}$ is cartesian. The composite of two cartesian morphisms is cartesian, since the composite of two cartesian squares is cartesian. We will denote by $\text{Cart}(\mathcal{C}^-)$ the (non-full) subcategory of cartesian morphisms of $\mathcal{C}^-.$

**Definition 2.2.1.** We say that a cocomplete category $\mathcal{C}$ satisfies the descent principle if the subcategory $\text{Cart}(\mathcal{C}^-)$ is closed under colimits.

The closure condition in the definition means two things: colimits exist in the subcategory $\text{Cart}(\mathcal{C}^-)$ and they are preserved by the inclusion functor $\text{Cart}(\mathcal{C}^-) \to \mathcal{C}^-.$ More precisely, a diagram $D : I \to \mathcal{C}^-$ is the same thing as a natural transformation $\alpha : D_0 \to D_1$ between two diagrams $D_0, D_1 : I \to \mathcal{C}$. The diagram $D$ belongs to the subcategory $\text{Cart}(\mathcal{C}^-)$ if and only if the natural transformation $\alpha : D_0 \to D_1$ is cartesian: that is, if the naturality square

$$\begin{array}{ccc}
D_0(i) & \xrightarrow{D_0(f)} & D_0(j) \\
\alpha(i) \downarrow & \Rightarrow & \alpha(j) \\
D_1(i) & \xrightarrow{D_1(f)} & D_1(j)
\end{array}$$

is cartesian for every arrow $f : i \to j$ in the category $I$. The colimit of $D : I \to \mathcal{C}^-$ is the map $\text{colim}(\alpha) : \text{colim} D_0 \to \text{colim} D_1$. The descent principle implies that the square

$$\begin{array}{ccc}
D_0(i) & \xrightarrow{\iota_0(i)} & \text{colim} D_0 \\
\alpha(i) \downarrow & \Rightarrow & \text{colim}(\alpha) \downarrow \\
D_1(i) & \xrightarrow{\iota_1(i)} & \text{colim} D_1
\end{array}$$

is cartesian for every $i \in I$, where $\iota_0(i)$ and $\iota_1(i)$ are the canonical maps. The principle also implies that a square

$$\begin{array}{ccc}
\text{colim} D_0 & \xrightarrow{u_0} & A \\
\text{colim}(\alpha) \downarrow & & \downarrow f \\
\text{colim} D_1 & \xrightarrow{u_1} & B
\end{array}$$

is cartesian if and only if the square

$$\begin{array}{ccc}
D_0(i) & \xrightarrow{\iota_0(i)} & A \\
\alpha(i) \downarrow & & \downarrow f \\
D_1(i) & \xrightarrow{\iota_1(i)} & B
\end{array}$$

is cartesian for every $i \in I$. 
The applications of the descent principle in the present work are all consequences of the following.

**Lemma 2.2.2.** If a category $\mathcal{C}$ satisfies the descent principle, then for every pushout

\[
\begin{array}{ccc}
  f & \xrightarrow{\alpha} & g \\
  \downarrow{\beta} & & \downarrow{\gamma} \\
  h & \xleftarrow{\delta} & k
\end{array}
\]

in $\mathcal{C}^\rightarrow$, such that $\alpha$ and $\beta$ are cartesian, $\gamma$ and $\delta$ are also cartesian.

It may be worth spelling out what the lemma says in the category $\mathcal{C}$ itself. A square of arrows in $\mathcal{C}^\rightarrow$ corresponds to a cubical diagram in $\mathcal{C}$ as follows:

The hypothesis of Lemma 2.2.2 then requires that the top and bottom horizontal squares are pushouts and that the back and left squares ($\alpha$ and $\beta$ in the definition above) are pullbacks. The conclusion then asserts that the front and right squares ($\gamma$ and $\delta$) are pullbacks as well.

For the category of spaces, this fact is well known and often referred to as Mather's cube lemma [12].

**Definition 2.2.3.** We say that a category $\mathcal{E}$ is a **topos** if:

(i) $\mathcal{E}$ is presentable;
(ii) colimits in $\mathcal{E}$ are universal;
(iii) $\mathcal{E}$ satisfies the descent principle.

**Example 2.2.4.** The category of spaces $\mathcal{S}$ is a topos, as is the category of presheaves $[\mathcal{C}^{\text{op}}, \mathcal{S}]$ for any small category $\mathcal{C}$. More generally, any left-exact localization of a presheaf category is a topos, and this in fact completely characterizes the class of topoi. See [11, Proposition 6.1.3.10].

Let us give a simple application of descent.

**Definition 2.2.5.** We will say that a map $f : A \to B$ in a topos $\mathcal{E}$ is a **monomorphism** if the square

\[
\begin{array}{ccc}
  A & \xrightarrow{\alpha} & A \\
  \downarrow{f} & & \downarrow{f} \\
  A & \xrightarrow{f} & B
\end{array}
\]

is cartesian.
This concept will be more thoroughly treated in Section 3.3. For example, in the category of spaces \( \mathcal{S} \), a map \( f : A \to B \) is a monomorphism if and only if it is (weakly equivalent to) an inclusion of a union of path components of \( B \) into \( B \).

**Proposition 2.2.6.** Consider a pushout square

\[
\begin{array}{ccc}
A & \xrightarrow{h} & C \\
\downarrow{f} & & \downarrow{g} \\
B & \xrightarrow{k} & D
\end{array}
\]

in a topos in which the arrow \( f \) is a monomorphism. Then \( g \) is a monomorphism and the square is cartesian.

**Proof.** Consider the cube:

The top square is trivially cocartesian and the back square is trivially cartesian. Note also that the left side is cartesian, since \( f \) is a monomorphism. Hence, the front and right squares are cartesian by descent. But the front face is just our original square, and the fact that the right square is cartesian says that \( g \) is a monomorphism.

Finally, we recall also what is sometimes called the fundamental theorem of topos theory \[11\, Proposition 6.3.5.1\].

**Proposition 2.2.7.** For any object \( X \) in a topos \( \mathcal{E} \), the slice category \( \mathcal{E}_{/X} \) is a topos.

### 3. Modalities

In this section, we introduce the prerequisite material on factorization systems and modalities which will allow us to state our generalized form of the Blakers–Massey theorem. Homotopy-unique factorization systems of the sort we consider here appear in a number of places in the literature. For example, from a model category theoretic perspective in \[3\], from a higher categorical perspective in \[10, 11\], and from a type theoretic one in \[16, Chapter 7\]. We recall some basic tools and ideas here in order to fix notation and conventions.

#### 3.1. Factorization systems

**Definition 3.1.1.** Let \( f : A \to B \) and \( g : X \to Y \) be two maps in a category \( \mathcal{E} \). We say that \( f \) and \( g \) are orthogonal if the following square is cartesian in \( \mathcal{E} \):

\[
\begin{array}{ccc}
A & \xrightarrow{h} & C \\
\downarrow{f} & & \downarrow{g} \\
B & \xrightarrow{k} & D
\end{array}
\]
We denote this relation by \( f \perp g \) and say that \( f \) is left orthogonal to \( g \) and that \( g \) is right orthogonal to \( f \).

If \( f \perp g \), then every commutative square

\[
\begin{array}{ccc}
A & \rightarrow & X \\
\downarrow f & & \downarrow g \\
B & \rightarrow & Y
\end{array}
\]

has a unique diagonal filler \( d : B \rightarrow X \); indeed, the cartesian gap map

\[
[B, X] \rightarrow [A, X] \times_{[A, Y]} [B, Y]
\]

of the square in Definition 3.1.1 is an isomorphism. Of course, ‘uniqueness’ means that the space of diagonal fillers of the square is contractible.

If \( M \) and \( N \) are classes of maps in a category \( C \), we will write \( M \perp N \) if we have \( u \perp f \) for every \( u \in M \) and \( f \in N \). Let us put

\[
\begin{align*}
\perp L & := \{ f \in C \mid u \perp f \text{ for every } u \in M \}, \\
\perp N & := \{ u \in C \mid u \perp f \text{ for every } f \in N \}.
\end{align*}
\]

Then the relations \( M \perp N \), \( M \subset \perp N \) and \( N \subset \perp M \) are equivalent.

**Definition 3.1.2.** A factorization system on a category \( C \) is the data of a pair \((L, R)\) of classes of maps in \( C \) such that:

(i) every map \( f \) in \( C \) admits a factorization \( f = R(f) \circ L(f) \), where \( L(f) \in L \) and \( R(f) \in R \);

(ii) \( L \perp R \) and \( L \subset \perp R \).

Here, \( L \) is called the left class and \( R \) is called the right class.

A well-known example of a factorization system in ordinary category theory is formed by the surjective and injective functions in the 1-category (in fact, 1-topos) \( \text{Set} \). There is a similar factorization system in any topos \( E \).

**Definition 3.1.3.** We say that a map in a topos \( E \) is a cover if it is left orthogonal to every monomorphism (defined in 2.2.5). We say that a family of maps \( \{ f_i : X_i \rightarrow X \}_{i \in I} \) is a coverage of the object \( X \) if the resulting map \( \bigsqcup_{i \in I} X_i \rightarrow X \) is a cover.

**Remark 3.1.4.** Covers are referred to as effective epimorphisms in [11]. We list some elementary facts about covers below.

(i) Every map in a topos can be factored as a cover followed by a monomorphism.

(ii) A map \( f : X \rightarrow Y \) in the category of spaces \( \mathcal{E} \) is a cover if and only if the induced map \( \pi_0(f) : \pi_0 X \rightarrow \pi_0 Y \) is surjective. Let 1 be the terminal object in \( \mathcal{E} \). A pointed space \((X, x)\) is connected if and only if the map \( x : 1 \rightarrow X \) is a cover.

(iii) A map \( f : X \rightarrow Y \) in a topos \( \mathcal{E} \) is a cover if and only if the base change functor \( f^* : \mathcal{E}/Y \rightarrow \mathcal{E}/X \) is conservative.
A family of maps \( \{f_i : X_i \to X\}_{i \in I} \) is a coverage if and only if the family of functors \( f^*_i : \mathcal{E}/X \to \mathcal{E}/X_i \) is collectively conservative, that is, if the functor

\[
(f^*_i)_{i \in I} : \mathcal{E}/X \to \prod_{i \in I} \mathcal{E}/X_i
\]

is conservative.

Let \( D : K \to \mathcal{E} \) be a diagram in a topos. Then the family of canonical maps \( i_k : D(k) \to \text{colim}(D) \) for \( k \in K \) is a coverage.

Note that, in light of the orthogonality requirement \( \mathcal{L} \perp \mathcal{R} \), factorizations are unique up to unique isomorphism: indeed, if \( f = rl = r'l' \) are two \( (\mathcal{L}, \mathcal{R}) \)-factorizations of a map \( f : X \to Y \), then the following squares have a unique diagonal filler \( d : Z \to Z' \) and \( d' : Z' \to Z \), respectively, since \( l \perp r' \) and \( l' \perp r \).

\[
\begin{array}{ccc}
X & \xrightarrow{l'} & Z' \\
\downarrow l & & \downarrow r' \\
Z & \xrightarrow{r} & Y \\
\end{array} \quad \begin{array}{ccc}
X & \xrightarrow{l} & Z \\
\downarrow l & & \downarrow r \\
Z & \xrightarrow{r} & Y \\
\end{array}
\]

We then have \( d'd = 1_Z \) and \( dd' = 1_{Z'} \), since the following squares have a unique diagonal filler.

\[
\begin{array}{ccc}
X & \xrightarrow{l} & Z \\
\downarrow l & & \downarrow r \\
Z & \xrightarrow{r} & Y \\
\end{array} \quad \begin{array}{ccc}
X & \xrightarrow{r'} & Z' \\
\downarrow l' & & \downarrow r' \\
Z' & \xrightarrow{r'} & Y \\
\end{array}
\]

Thus, \( d \) is an isomorphism and it is unique.

Given a map \( f : X \to Y \), we will occasionally write \( \|f\| \) for the object produced by factoring \( f \) with respect to a factorization system \( (\mathcal{L}, \mathcal{R}) \), so that we have a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\mathcal{L}(f)} & \|f\| \\
\downarrow f & & \downarrow \mathcal{R}(f) \\
& & Y
\end{array}
\]

In such a situation, the intended factorization system will always be clear from the context. The factorization \( f = \mathcal{R}(f)\mathcal{L}(f) \) of a map in \( \mathcal{E} \) is functorial in \( f \in \mathcal{E} \). More precisely, from a commutative square \( \alpha : f \to g \)

\[
\begin{array}{ccc}
A & \xrightarrow{h} & C \\
\downarrow f & \downarrow \alpha & \downarrow g \\
B & \xrightarrow{k} & D,
\end{array}
\]

we obtain two commutative squares

\[
\begin{array}{ccc}
A & \xrightarrow{h} & C \\
\downarrow \mathcal{L}(f) & \downarrow \mathcal{L}(\alpha) & \downarrow \mathcal{L}(g) \\
\|f\| & \xrightarrow{\|f\|\|\|} & \|g\| \\
\downarrow \mathcal{R}(f) & \downarrow \mathcal{R}(\alpha) & \downarrow \mathcal{R}(g) \\
B & \xrightarrow{k} & D,
\end{array}
\]

\[
\begin{array}{ccc}
A & \xrightarrow{h} & C \\
\downarrow \mathcal{L}(f) & \downarrow \mathcal{L}(\alpha) & \downarrow \mathcal{L}(g) \\
\|f\| & \xrightarrow{\|f\|\|\|} & \|g\| \\
\downarrow \mathcal{R}(f) & \downarrow \mathcal{R}(\alpha) & \downarrow \mathcal{R}(g) \\
B & \xrightarrow{k} & D.
\end{array}
\]
This defines two functors $L, R : C^\to \to C^\to$. Let us denote by $L^\to$ (respectively, $R^\to$) the full subcategory of $C^\to$ whose objects are the maps in $L$ (respectively, in $R$). Then, $L : C^\to \to L^\to$ and $R : C^\to \to R^\to$.

**Lemma 3.1.5.** The functor $L : C^\to \to L^\to$ is right adjoint to the inclusion $L^\to \subset C^\to$ and the functor $R : C^\to \to R^\to$ is left adjoint to the inclusion $R^\to \subset C^\to$.

**Proof.** The unit $\eta(f) : f \to R(f)$ and the counit $\epsilon(f) : L(f) \to f$ of the adjunctions are the squares of the following diagram:

$$
\begin{array}{ccc}
A & \xrightarrow{L(f)} & A \\
\|f\| & \downarrow & \|\|f\|\| \\
L(f) & \xrightarrow{\epsilon(f)} & B \\
\|f\| & \downarrow & \|\|f\|\| \\
\end{array}
$$

The following elementary closure properties of the right and left classes of a factorization system are standard.

**Lemma 3.1.6.** Given a factorization system $(L, R)$, then:

(i) $L$ and $R$ contain all isomorphisms;
(ii) $L$ and $R$ are closed under composition;
(iii) if $fg \in L$ and $g \in L$, then $f \in L$; dually, if $fg \in R$ and $f \in R$, then $g \in R$;
(iv) $L$ is stable by cobase change and $R$ is stable by base change (when they are well defined);
(v) the subcategory $L^\to$ of $C^\to$ is closed under colimits and the subcategory $R^\to$ of $C^\to$ is closed under limits.

Note that the closure property (5) follows from Lemma 3.1.5.

If $M$ is a class of maps in a category $C$ and $T \in C$ an object we let $M_T$ denote the class of maps in the slice category $C/T$ formed by all those maps whose image under the forgetful functor from $C/T$ to $C$ land in $M$. Then one can easily show that factorization systems are compatible with slicing.

**Lemma 3.1.7.** If $(L, R)$ is a factorization system in a category $C$, then the pair $(L_T, R_T)$ is a factorization system in the category $C/T$ for every object $T \in C$.

A fundamental fact about presentable categories is the following.

**Proposition 3.1.8.** Let $S$ be a set of maps in a presentable category $C$. Then the class $R = S^\perp$ is the right class of a factorization system with $L = \perp R$.

**Proof.** [11, Proposition 5.5.5.7].

### 3.2. Operations on maps

We fix a topos $\mathcal{E}$ throughout. Given two maps $u : A \to B$ and $v : S \to T$ in $\mathcal{E}$, we have a commutative diagram

$$
\begin{array}{ccc}
A \times S & \xrightarrow{A \times v} & A \times T \\
\downarrow_{u \times 2} & & \downarrow_{u \times T} \\
B \times S & \xrightarrow{B \times v} & B \times T
\end{array}
$$
where, for example, $A \times v$ stands for the map
\[ \text{id}_A \times v : A \times S \to A \times T. \]

We define the pushout product of $u$ and $v$, denoted $u \circ v$, as the cocartesian gap map of the square above. Explicitly, then
\[ u \circ v = (B \times S) \sqcup_{(A \times S)} (A \times T) \to B \times T. \]

**Example 3.2.1.** The pushout product has a number of important special cases.

(i) Recall that 1 is the terminal object of $\mathcal{E}$. A pointed object of $\mathcal{E}$ is a pair $(A, a)$, where $a$ is a map $1 \to A$. For $(A, a)$ and $(B, b)$ two pointed objects, one sees that the pushout product $a \circ b = (1 \to A) \circ (1 \to B)$ is the canonical inclusion of the wedge into the product
\[ A \vee B \to A \times B. \]

(ii) The join of two objects $A$ and $B$ in $\mathcal{E}$, denoted $A \ast B$, is the pushout of the diagram
\[\begin{array}{ccc}
A \times B & \longrightarrow & B \\
\downarrow & & \\
A
\end{array}\]

One finds immediately from the definition that
\[ (A \to 1) \circ (B \to 1) = (A \ast B) \to 1. \]

Let $S^0 = 1 \sqcup 1$ be the sphere of dimension 0 in $\mathcal{E}$. Then
\[ S^0 \ast A = \Sigma A \]
is the unreduced suspension of $A$. In this way one can define spheres in any topos: set $S^{-1} = 0$ and $S^{n+1} = S^0 \ast S^n$ for every $n \geq -1$. If $s_n : S^n \to 1$, then $s_0 \circ s_n = s_{n+1}$ for every $n \geq -1$.

(iii) For any map $u : A \to B$, $s_0 \circ u = \nabla u : B \sqcup_A B \to B$

is the codiagonal of $u$, that is, the map defined by the following diagram with a pushout square,

\[\begin{array}{ccc}
A & \xrightarrow{u} & B \\
\downarrow & & \downarrow \nabla u \\
B & \xleftarrow{i_1} & B \sqcup_A B
\end{array}\]

In particular, $\nabla(A \to 1) = s_0 \circ (A \to 1) = (\Sigma A \to 1)$.

(iv) The pushout product $f \circ g$ of two maps $f : X \to A$ and $g : Y \to B$ in a topos $\mathcal{E}$ can be thought as the ‘external’ join product of the fibers of $f$ and $g$. Indeed, letting $a : 1 \to A$ and $b : 1 \to B$ be points of $A$ and $B$, we define $f^{-1}(a)$ and $g^{-1}(b)$ to be the fibers of $f : X \to A$ and $g : Y \to A$ at $a$ and $b$, respectively:
\[\begin{array}{ccc}
f^{-1}(a) & \xrightarrow{\sim} & X \\
\downarrow & & \downarrow f \\
1 & \xrightarrow{a} & A
\end{array}\]
\[\begin{array}{ccc}
g^{-1}(b) & \xrightarrow{\sim} & Y \\
\downarrow & & \downarrow g \\
1 & \xrightarrow{b} & B.
\end{array}\]
The fiber of $f \circ g$ at $(a, b) : 1 \to A \times B$ is similarly defined by the pullback
\[
(f \circ g)^{-1}(a, b) \xrightarrow{\sim} (A \times Y) \sqcup (X \times Y) \to X \times B.
\]

By the universality of colimits in $\mathcal{E}$, we have
\[
(f \circ g)^{-1}(a, b) = f^{-1}(a) \cdot g^{-1}(b).
\]

(v) For an object $Z$ of $\mathcal{E}$, we will denote by $\cdot_{Z}$ the pushout product in the category $\mathcal{E}/Z$.

If $f : W \to Z$ is a map in $\mathcal{E}$, then the base change functor $f^* : \mathcal{E}/Z \to \mathcal{E}/W$ preserves pushout products: we have a canonical isomorphism
\[
f^*(u \cdot_{Z} v) = (f^* u) \cdot_{W} (f^* v)
\]
for any two maps $u$ and $v$ in the topos $\mathcal{E}/Z$. In fact, the map $u \cdot_{Z} v$ in the topos $\mathcal{E}/Z$ is a base change of the map $u \circ v$, which can be seen as living over $Z \times Z$, along the diagonal $Z \to Z \times Z$.

Dually, the pullback hom $\langle u, f \rangle$ of two maps $u : A \to B$ and $f : X \to Y$ in $\mathcal{E}$ is defined to be the cartesian gap map of the following commutative square in $\mathcal{E}$
\[
\begin{array}{ccc}
[B, X] & \xrightarrow{[B, f]} & [B, Y] \\
\downarrow \quad \quad \quad \quad \quad \quad \downarrow \quad \quad \quad \quad \quad \quad \downarrow \\
[A, X] & \xrightarrow{[A, f]} & [A, Y].
\end{array}
\]

REMARK 3.2.2. Note that the map $u$ is left orthogonal to the map $f$ if and only if the map $\langle u, f \rangle$ is invertible. The codomain of $\langle u, f \rangle$ is properly denoted by $[u, f]$, as it is exactly the space of maps $u \to f$ in the arrow category $\mathcal{E}^{\to}$.

As defined, the pullback hom $\langle u, f \rangle$ is a map of spaces. However, as a topos $\mathcal{E}$ is cartesian closed, it admits an internal hom $[-, -] : \mathcal{E}^{\to} \times \mathcal{E} \to \mathcal{E}$. We may thus define an internal pullback hom, denoted $\langle \langle u, f \rangle \rangle$ as the cartesian gap map of the diagram
\[
\begin{array}{ccc}
[B, X] & \xrightarrow{[B, f]} & [B, Y] \\
\downarrow \quad \quad \quad \quad \quad \quad \downarrow \quad \quad \quad \quad \quad \quad \downarrow \\
[A, X] & \xrightarrow{[A, f]} & [A, Y].
\end{array}
\]

The pushout product $- \circ -$ and the internal pullback hom $\langle \langle -, - \rangle \rangle$ are part of a symmetric monoidal structure on the category $\mathcal{E}^{\to}$. It follows that we have a natural isomorphism
\[
[u \circ v, f] = [u, \langle v, f \rangle]
\]
for all $u, v, f \in \mathcal{E}^{\to}$. Furthermore, this isomorphism can itself be internalized, giving

PROPOSITION 3.2.3. We have natural isomorphisms
\[
[u \circ v, f] = [u, \langle v, f \rangle] \quad \text{and} \quad \langle \langle u \circ v, f \rangle \rangle = \langle \langle u, \langle v, f \rangle \rangle \rangle.
\]

A useful property of the pushout product is the following: if $u$ is invertible, then so is the map $u \circ v$ for any map $v$. The pullback hom enjoys a similar absorption property: the map $\langle \langle v, f \rangle \rangle$ is invertible as soon as either $v$ or $f$ is.
Example 3.2.4. We note some useful special cases of the pullback hom.

(i) For any map \( f : X \to Y \),

\[ \langle s_0, f \rangle = \Delta f : X \to X \times_Y X \]

is the diagonal of \( f \), that is, the map defined by the following diagram with a pullback square:

(ii) For a pair of objects \( A \) and \( X \) in a cartesian closed category, the \( A \)-diagonal of \( X \) is defined to be the map

\[ \Delta_A(X) = \langle A \to 1, X \to 1 \rangle : X = [1, X] \to [A, X]. \]

Intuitively speaking, the map \( \Delta_A(X) \) associates to each element of \( X \) the map from \( A \) to \( X \) which is constant at that element. Of course, to make this precise in full generality we must speak of generalized elements, but we will not dwell on that issue here.

Definition 3.2.5. Let \( u : A \to B \) and \( f : X \to Y \) be two maps in a topos \( \mathcal{E} \). We say that \( u \) is internally left orthogonal to \( f \) (and \( f \) is internally right orthogonal to \( g \)) if the following square

\[ \begin{array}{ccc}
[B, X] & \xrightarrow{[B, f]} & [B, Y] \\
[u, X] \downarrow & & \downarrow [u, Y] \\
[A, X] & \xrightarrow{[A, f]} & [A, Y]
\end{array} \]

is cartesian. In this case, we will write \( u \perp f \).

Remark 3.2.6. Analogous to Remark 3.2.2 about the ordinary pullback hom and (external) orthogonality, internal orthogonality is detected by the internal pullback hom: \( u \) is internally left orthogonal to \( f \) if and only if the map \( \langle u, f \rangle \) is invertible.

Remark 3.2.7. For every object \( Z \) in \( \mathcal{E} \) one has \( [Z, \langle u, f \rangle] = \langle Z \times u, f \rangle \). Hence, it follows from the Yoneda lemma that the relation \( u \perp f \) is equivalent to the relation \( Z \times u \perp f \) for every object \( Z \) in \( \mathcal{E} \). In particular, \( u \perp f \) implies \( u \perp f \).

Example 3.2.8. Here are two special cases of internal orthogonality.

(i) The category of spaces \( \mathcal{S} \) is cartesian closed and we have \( \langle u, f \rangle = \langle u, f \rangle \) for any pair of maps \( u, f \in \mathcal{S} \). It follows that the relations \( u \perp f \) and \( u \perp f \) are the same in \( \mathcal{S} \).

(ii) Let \( A \) and \( X \) be objects in a cartesian closed category \( \mathcal{E} \). By Remark 3.2.6, \( (A \to 1) \perp (X \to 1) \) if and only if the \( A \)-diagonal of \( X \)

\[ \Delta_A(X) = \langle A \to 1, X \to 1 \rangle : X \to [A, X] \]

from Example 3.2.4 is invertible.
Proposition 3.2.9. Let \( S \) be a set of maps in a topos \( \mathcal{E} \). Then \( \mathcal{R} := S \downarrow \) is the right class of a factorization system \( (\mathcal{L}, \mathcal{R}) \) with \( \mathcal{L} := \downarrow \mathcal{R} = \uparrow \mathcal{R} \).

Proof. Let \( G \) be a set of generators of \( \mathcal{E} \) and put \( G \times S := \{ Z \times u \mid Z \in G, u \in S \} \). Let us show that \( (G \times S)\downarrow = S \downarrow \).

By Remark 3.2.7, the relation \( u \downarrow f \) is equivalent to the relation \( Z \times u \downarrow f \) for every \( Z \) in \( \mathcal{E} \). Thus, \( S \downarrow \subseteq (G \times S)\downarrow \).

Conversely, if \( f \in (G \times S)\downarrow \) and \( u \in S \), then we have \( Z \times u \downarrow f \) for every \( Z \in G \). This means that the map

\[
\langle Z \times u, f \rangle = [Z, \langle u, f \rangle]
\]

is invertible for every object \( Z \in G \). It follows that the map \( \langle u, f \rangle \) is invertible, since \( G \) is a set of generators. Thus, \( (G \times S)\downarrow \subseteq S \downarrow \).

From Proposition 3.1.8 now follows that \( \mathcal{R} := S \downarrow = (G \times S)\downarrow \) is the right class of a factorization system \( (\mathcal{L}, \mathcal{R}) \) with \( \mathcal{L} := \downarrow \mathcal{R} \). The relation \( \mathcal{L} \downarrow \mathcal{R} \) is left to the reader. \( \square \)

3.3.
Connectedness and truncation

The factorization system of covers and monomorphisms in a topos \( \mathcal{E} \) belongs to a whole family of factorization systems corresponding to \( n \)-connected and \( n \)-truncated maps, to which we now turn.

Definition 3.3.1. The notion of \( n \)-truncated map \( f : X \to Y \) in a topos \( \mathcal{E} \) is defined by induction on \( n \geq -2 \).

- \( f \) is said to be \((-2)\)-truncated if it is invertible.
- \( f \) is said to be \((n+1)\)-truncated if the diagonal map

\[
\Delta f : X \to X \times_Y X
\]

is \( n \)-truncated.

We write \( T_n(\mathcal{E}) \) for the class of \( n \)-truncated maps in a topos \( \mathcal{E} \). An object \( X \in \mathcal{E} \) is said to be \( n \)-truncated if the map \( X \to 1 \) is \( n \)-truncated.

By definition, a map is \((-1)\)-truncated if its diagonal is an isomorphism. Hence, a map is \((-1)\)-truncated if and only if it is a monomorphism as defined in Definition 2.2.5. An object \( X \in \mathcal{E} \) is \((-1)\)-truncated if and only if the map \( X \to 1 \) is a monomorphism.

A space \( X \) is \( n \)-truncated if and only if \( X \) is an \( n \)th Postnikov section, that is, if the homotopy group \( \pi_k(X) \) vanish for \( k > n \) and all basepoints. More generally, a map of spaces is \( n \)-truncated if and only if all of its fibers are \( n \)-truncated spaces. Thus, a space is \((-2)\)-truncated if it is contractible and a map is \((-2)\)-truncated if it is an equivalence. A space is \((-1)\)-truncated if it is either empty or contractible and a map is \((-1)\)-truncated if it is a monomorphism. Finally, a space is \( 0 \)-truncated if it is equivalent to a discrete space and a map is \( 0 \)-truncated if it is equivalent to a covering space map.

Remark 3.3.2. A map \( f : X \to Y \) in a topos \( \mathcal{E} \) is \( n \)-truncated if and only if the object \( (X, f) \) of \( \mathcal{E}/Y \) is \( n \)-truncated. If \( Z \) is an object of \( \mathcal{E} \), then a map \( f : (X, p) \to (Y, q) \) in \( \mathcal{E}/Z \) is \( n \)-truncated if and only if the map \( f : X \to Y \) in \( \mathcal{E} \) is \( n \)-truncated.

Recall that in Example 3.2.1(2), the \( n \)-sphere \( S^n \) for \( n \geq -1 \) is defined for an arbitrary topos \( \mathcal{E} \). Moreover, if \( s_n : S^n \to 1 \) is the canonical map, then \( s_{n+1} = s_0 \circ s_n \) for every \( n \geq -1 \).
Lemma 3.3.3. A map \( f : X \to Y \) is \( n \)-truncated if and only if \( s_{n+1} \trianglelefteq f \).

Proof. We wish to show that a map \( f : X \to Y \) is \( n \)-truncated if and only if the map \( \langle s_{n+1}, f \rangle \) is invertible. The proof proceeds by induction on \( n \geq -2 \). The result is clear if \( n = -2 \), since \( \langle s_{-1}, f \rangle = f \). Let us suppose \( n > -2 \). By definition, \( f \) is \( n \)-truncated if and only if the map \( \Delta f \) is \((n - 1)\)-truncated. By the induction hypothesis, the latter holds if and only if the map \( \langle s_n, \Delta f \rangle \) is invertible. But we have canonical isomorphisms:

\[
\langle s_n, \Delta f \rangle = \langle s_n, \langle s_0, f \rangle \rangle = \langle s_n \circ s_0, f \rangle = \langle s_{n+1}, f \rangle.
\]

Hence, the map \( \langle s_n, \Delta f \rangle \) is invertible if and only if the map \( \langle s_{n+1}, f \rangle \) is invertible. This shows that \( f \) is \( n \)-truncated if and only if the map \( \langle s_{n+1}, f \rangle \) is invertible. \( \square \)

Definition 3.3.4. A map \( f : X \to Y \) in a topos \( \mathcal{E} \) is said to be \( n \)-connected if it is left orthogonal to all \( n \)-truncated maps. An object \( X \) is said to be \( n \)-connected if the map \( X \to 1 \) is \( n \)-connected. We write \( C_n(\mathcal{E}) \) for the class of \( n \)-connected maps in a topos \( \mathcal{E} \).

A map is \((-1)\)-connected if and only if it is a cover. Every map is \((-2)\)-connected.

Remark 3.3.5. Note that this definition of \( n \)-connectedness, while consistent with the standard usage for objects (that is, say for topological spaces), differs from the convention for maps: an \( n \)-connected map in our sense is \((n + 1)\)-connected map in the traditional sense (see, for example, [9, p. 302]).

Proposition 3.3.6. The pair \( (C_n(\mathcal{E}), T_n(\mathcal{E})) \) is a factorization system in any topos \( \mathcal{E} \) and any \( n \geq -2 \).

Proof. If \( s_{n+1} = \{S^{n+1} \to 1\} \), then \( T_n(\mathcal{E}) = s_{n+1} \trianglelefteq \) by Lemma 3.3.3. The result then follows from Proposition 3.2.9. \( \square \)

In particular, \((C_{-2}(\mathcal{E}), T_{-2}(\mathcal{E}))\) is the factorization system of isomorphisms and all maps; \((C_{-1}(\mathcal{E}), T_{-1}(\mathcal{E}))\) is the factorization system of covers and monomorphisms.

The following corollary shows how the operations of Section 3.2 interact with connectedness and truncation.

Corollary 3.3.7. Suppose that \( u : A \to B \) is \( m \)-connected, \( v : C \to D \) is \( n \)-connected and \( f : X \to Y \) is \( p \)-truncated. Then:

(i) \( \langle s_k, f \rangle \) is \((p - k - 1)\)-truncated;
(ii) \( u \circ s_k \) is \((m + k + 1)\)-connected;
(iii) \( \langle u, f \rangle \) is \((p - m - 2)\)-truncated;
(iv) \( u \circ v \) is \((m + n + 2)\)-connected.

Proof. We use Proposition 3.3.6 and the adjunction formula in Proposition 3.2.3.

(i) By elementary properties of the join \( s_k \circ s_\ell = s_k \circ s_{\ell+1} \) for all \( k, \ell \geq -1 \). Now, by Lemma 3.3.3 \( f \) is \( p \)-truncated if and only if

\[
s_{p+1} \trianglelefteq f \iff (s_{p-k} \circ s_k) \trianglelefteq f \iff s_{p-k} \trianglelefteq \langle s_k, f \rangle.
\]

Again by Lemma 3.3.3, this is equivalent to the fact that \( \langle s_k, f \rangle \) is \((p - k - 1)\)-truncated.

(ii) Let \( h \) be any \((m + k + 1)\)-truncated map. Then

\[
(u \circ s_k) \trianglelefteq h \iff u \trianglelefteq \langle s_k, h \rangle.
\]
But \( \langle s_k, h \rangle \) is \((m + k + 1) - k - 1 = m\)-truncated by 1.

(iii) We have
\[
s_{p-m-1} \perp \langle u, f \rangle \iff (s_{p-m-1} \circ u) \perp f
\]
and \( s_{p-m-1} \circ u \) is \((p - m - 1) + m + 1 = p\)-connected by 2.

(iv) Let \( h \) be any \((m + n + 2)\)-truncated map, then
\[
(u \circ v) \perp h \iff u \perp \langle v, h \rangle.
\]
But the map \( \langle v, h \rangle \) is \((m + n + 2) - n - 2 = m\)-truncated by 3.

\[\square\]

**Proposition 3.3.8.** For \( n \geq -1 \), a map in a topos \( \mathcal{E} \) is \( n \)-connected if and only if it is a cover and its diagonal is \((n - 1)\)-connected.

**Proof.** See [11, Proposition 6.5.1.18].

\[\square\]

### 3.4. Modalities

We fix for this section a given topos \( \mathcal{E} \).

**Definition 3.4.1.** We say that a factorization system \((\mathcal{L}, \mathcal{R})\) in a topos \( \mathcal{E} \) is a modality if the class \( \mathcal{L} \) is stable under base change by any map in \( \mathcal{E} \).

The right class of a factorization system is always closed under base change by Lemma 3.1.6. Hence in a modality, both classes \( \mathcal{L} \) and \( \mathcal{R} \) are stable by base change.

**Example 3.4.2.** The are many examples of modalities.

(i) The factorization system of covers and monomorphisms in a topos \( \mathcal{E} \) is a modality.

(ii) More generally, the factorization system \((C_n(\mathcal{E}), T_n(\mathcal{E}))\) of \( n \)-connected maps and \( n \)-truncated maps in a topos \( \mathcal{E} \) is a modality. It is a factorization system by Proposition 3.3.6. It only remains to check that \( n \)-connected maps are stable under base change by any map which we will leave to the reader.

(iii) If \((\mathcal{L}, \mathcal{R})\) is a modality in a topos \( \mathcal{E} \), then for every object \( T \in \mathcal{E} \) the induced factorization system \((\mathcal{L}_T, \mathcal{R}_T)\) on \( \mathcal{E}/T \) described in Lemma 3.1.7 is also a modality.

**Example 3.4.3.** Let \( A \in \mathcal{S} \) be a space. A space \( X \) is \( A \)-null if the diagonal map
\[
\Delta_A(X) : X \to [A, X],
\]
defined in Example 3.2.4, is an equivalence, that is, if every function \( A \to X \) is uniquely homotopic to a constant map. Equivalently, a space \( X \) is \( A \)-null if and only if \((A \to 1) \perp (X \to 1)\). If \( \mathcal{A} \) is a set of spaces, a space \( X \) is \( \mathcal{A} \)-null if it is \( A \)-null for every object \( A \in \mathcal{A} \). If \( A_\mathcal{A} \) is the set of maps \( A \to 1 \) with \( A \in \mathcal{A} \), then \( S_\mathcal{A} = \mathcal{R}_\mathcal{A} \) is the right class of a factorization system \((\mathcal{L}_\mathcal{A}, \mathcal{R}_\mathcal{A})\) by Proposition 3.1.8. Moreover, it can be shown that this factorization system is in fact a modality. Indeed, the modality \((C_n(\mathcal{E}), T_n(\mathcal{E}))\) of the previous example is obtained from this construction by setting \( A = S^{n+1} \).

For a given space \( X \), factoring the terminal map \( X \to 1 \) produces an object \( P_\mathcal{A} X \) called the \( \mathcal{A} \)-nullification of \( X \) which is initial among all \( \mathcal{A} \)-null spaces admitting a map from \( X \). More generally, the stability of factorizations by pullback in a modality implies that the factorization of a map \( f : X \to Y \) may be seen as a fiberwise application of the \( \mathcal{A} \)-nullification functor. Classical accounts of this construction may be found in [6, 7, 13].
The nullification functor $P_A$ is the source of the weak cellular inequalities of [7] which are the main tool in the generalization of the Blakers–Massey theorem of [5]: a space $A$ kills $X$, written $X \triangleright A$, if $P_A X = 1$.

The next class of examples of modalities is important for applications to Goodwillie calculus in the companion paper [1].

**Example 3.4.4.** If $\mathcal{E}$ is a topos and $L : \mathcal{E} \to \mathcal{E}$ is a left exact localization with unit $\eta : \text{id}_\mathcal{E} \to L$, then we obtain a modality $(\mathcal{L}, \mathcal{R})$ on $\mathcal{E}$ by taking $\mathcal{L}$ to be the class of maps which are sent to isomorphisms by $L$. The factorization of a map $f : X \to Y$ may be obtained by considering the pullback diagram

$$
\begin{array}{ccc}
X & \xrightarrow{L f} & LX \\
\downarrow{f} & & \downarrow{L f} \\
Y & \xrightarrow{\eta Y} & LY
\end{array}
$$

The required properties of a modality follow easily from the hypothesis on $L$.

### 3.5. Dual Blakers–Massey theorem

As a first application of the concept of modality, we derive the following ‘dual’ Blakers–Massey theorem, which is in fact an elementary consequence of the definition.

**Theorem 3.5.1 (Dual Blakers–Massey).** Let $(\mathcal{L}, \mathcal{R})$ be a modality. Suppose we are given a pullback square

$$
\begin{array}{ccc}
X & \xrightarrow{h} & Z \\
\downarrow{f} & & \downarrow{g} \\
Y & \xrightarrow{k} & W
\end{array}
$$

and suppose that the map $k \circ g \in \mathcal{L}$. Then the cogap map $[k, g] : Y \sqcup_X Z \to W$ is in $\mathcal{L}$.

**Proof.** The pushout product $k \circ g$ is, by definition, the cogap map of the following square:

$$
\begin{array}{ccc}
Y \times Z & \longrightarrow & W \times Z \\
\downarrow & & \downarrow \\
Y \times W & \longrightarrow & W \times W
\end{array}
$$

and one can easily check that by pulling back this square along the diagonal map $W \to W \times W$, we obtain our original square. It follows, then, by universality of colimits that the pullback of the map $k \circ g$ is in fact the map $[k, g]$. Since we have $k \circ g \in \mathcal{L}$ by assumption, and $\mathcal{L}$ is stable by base change in light of the fact that it is the left class of a modality, we are done.

**Corollary 3.5.2 [9, Theorem 2.4].** Suppose we are given a pullback square

$$
\begin{array}{ccc}
X & \xrightarrow{h} & Z \\
\downarrow{f} & & \downarrow{g} \\
Y & \xrightarrow{k} & W
\end{array}
$$

in $\mathcal{S}$, where $g$ is $m$-connected and $k$ is $n$-connected, then the cogap map $[k, g] : Y \sqcup_X Z \to W$ is $(m + n + 2)$-connected.
Proof. Just apply Proposition 3.5.1 together with Corollary 3.3.7(4).

3.6. Modalities and local classes
Recall from Definition 3.1.3, that a family of maps \( \{ g_i : Y_i \to Y \} \) in a topos is called a coverage if the induced map \( \bigsqcup_{i \in I} Y_i \to Y \) is a cover.

**Definition 3.6.1** [11, Proposition 6.2.3.14]. Let \( \mathcal{M} \) be a class of maps in a topos \( \mathcal{E} \) stable by base change. We will say that \( \mathcal{M} \) is local if for every coverage \( \{ g_i : Y_i \to Y \} \) and every map \( f : X \to Y \),

\[
\begin{array}{ccc}
Y_i \times_Y X & \longrightarrow & X \\
g_i^* (f) \downarrow \ & \ & \downarrow f \\
Y_i & \longrightarrow & Y
\end{array}
\]

if \( g_i^* (f) \in \mathcal{M} \) for all \( i \in I \), then \( f \in \mathcal{M} \).

**Remark 3.6.2.** It is not hard to see that the above definition can be reformulated as follows: a class of maps \( \mathcal{M} \) is local if and only if it is closed under coproducts and for any pullback square

\[
\begin{array}{ccc}
X' & \longrightarrow & X \\
f' \downarrow \ & \ & \downarrow f \\
Y' & \longrightarrow & Y
\end{array}
\]

with \( g \) a cover, \( f' \in \mathcal{M} \) implies \( f \in \mathcal{M} \).

**Remark 3.6.3.** If \( \mathcal{M} \) is a local class in the category of spaces \( \mathcal{S} \), then a map \( f : X \to Y \) belongs to \( \mathcal{M} \) if and only if all its fibers \( f^{-1}(y) \) (the maps \( f^{-1}(y) \to 1 \)) belong to \( \mathcal{M} \). This is because the set of all maps \( 1 \to Y \) is a coverage of the space \( Y \).

**Remark 3.6.4.** Let \( \mathcal{A} \) be a class of objects in the category of spaces \( \mathcal{S} \) that is closed under isomorphisms (also known as weak homotopy equivalences). It is easy to verify that the class of maps having all their fibers in \( \mathcal{A} \) is a local class. Conversely, every local class \( \mathcal{M} \) in the topos \( \mathcal{S} \) is of this form for some class of objects \( \mathcal{A} \).

**Proposition 3.6.5.** The classes \( \mathcal{L} \) and \( \mathcal{R} \) of a modality in a topos \( \mathcal{E} \) are local.

Proof. Let us show that \( \mathcal{R} \) is local. Given \( f : X \to Y \) and a coverage \( \{ g_i : Y_i \to Y \} \) such that \( g_i^* (f) \in \mathcal{R} \) for all \( i \in I \), we need to show that \( f \in \mathcal{R} \).

For this, choose a factorization \( f = pu : X \to Z \to Y \) with \( p \in \mathcal{R} \) and \( u \in \mathcal{L} \). We will prove that \( u \) is invertible, and hence that \( f \in \mathcal{R} \). The base change functors \( g_i^* : \mathcal{S}_Y \to \mathcal{S}_{Y_i} \) are collectively conservative, since the family \( \{ g_i : i \in I \} \) is a coverage. Hence, it suffices to show that the map \( g_i^* (u) \) is invertible for every \( i \in I \). We have \( g_i^* (u) \in \mathcal{L} \) and \( g_i^* (p) \in \mathcal{R} \), since the classes \( \mathcal{R} \) and \( \mathcal{L} \) are closed under base changes. Thus, \( g_i^* (u) \) is invertible by uniqueness of a \( (\mathcal{L}, \mathcal{R}) \)-factorization, since \( g_i^* (f) = g_i^* (p) g_i^* (u) \) and \( g_i^* (f) \in \mathcal{R} \). This proves that \( u \) is invertible and hence that \( f \in \mathcal{R} \). We have proved that \( \mathcal{R} \) is a local class.

A similar argument shows that \( \mathcal{L} \) is a local class. \( \square \)
Corollary 3.6.6. Let \((\mathcal{L}, \mathcal{R})\) be a modality in the category of spaces \(\mathcal{S}\). Then a map \(f : X \to Y\) belongs to \(\mathcal{L}\) (respectively, \(\mathcal{R}\)) if and only if the map \(f^{-1}(y) \to 1\) belongs to \(\mathcal{L}\) (respectively, \(\mathcal{R}\)) for every \(y \in Y\).

Proof. This follows from Proposition 3.6.5 and Remark 3.6.3.

\[\Box\]

3.7. Descent for \(\mathcal{L}\)-cartesian squares

A key tool in the proof of the generalized Blakers–Massey theorem are descent properties of \(\mathcal{L}\)-cartesian squares. The reader may wish to compare the results of this section with those of \(\cite{4}\) where similar notions are considered.

Definition 3.7.1. Let \(\mathcal{L}\) be a local class of maps in a topos \(\mathcal{E}\). We say that a commutative square

\[
\begin{array}{ccc}
A' & \xrightarrow{f'} & B' \\
\downarrow{u} & & \downarrow{v} \\
A & \xrightarrow{f} & B
\end{array}
\]

is \(\mathcal{L}\)-cartesian if its cartesian gap map \((u, f') : A' \to A \times_B B'\) belongs to \(\mathcal{L}\).

We will say that a morphism \(\alpha : f \to g\) in \(\mathcal{E}^{-\text{cart}}\) is \(\mathcal{L}\)-cartesian if the corresponding square in \(\mathcal{E}\) is \(\mathcal{L}\)-cartesian. By Lemma 3.7.4, the composite of two \(\mathcal{L}\)-cartesian morphisms is \(\mathcal{L}\)-cartesian.

Proposition 3.7.2. Let \((\mathcal{L}, \mathcal{R})\) be a modality on a topos \(\mathcal{E}\) and let

\[
\begin{array}{ccc}
f & \xrightarrow{\alpha} & g \\
\downarrow{\beta} & & \downarrow{\gamma} \\
h & \xrightarrow{\delta} & k
\end{array}
\]

be a pushout square in \(\mathcal{E}^{-\text{cart}}\). If the squares \(\alpha\) and \(\beta\) are \(\mathcal{L}\)-cartesian, then so are the squares \(\delta\) and \(\gamma\).

The proof of Proposition 3.7.2 will be given after several preparatory lemmas which establish some basic properties of \(\mathcal{L}\)-cartesian squares. The following lemma connects \(\mathcal{L}\)-cartesian squares to cartesian squares. Recall from Lemma 3.1.5 that the functor \(\mathcal{R} : \mathcal{C} \to \mathcal{R}\) is left adjoint to the inclusion \(\mathcal{R}^{-1} \subset \mathcal{C}^{-\text{cart}}\).

Lemma 3.7.3. Let \((\mathcal{L}, \mathcal{R})\) be a modality in a topos \(\mathcal{E}\). If a square \(\alpha : f \to g\) is \(\mathcal{L}\)-cartesian,

\[
\begin{array}{ccc}
A & \xrightarrow{\alpha_1} & C \\
\downarrow{f} & & \downarrow{g} \\
B & \xrightarrow{\alpha_2} & D
\end{array}
\]

then the square \(\mathcal{R}(\alpha) : \mathcal{R}(f) \to \mathcal{R}(g)\) is cartesian.

Proof. Consider the diagram with two pullback squares.
By construction, \( f = ts(f, \alpha_1) \). We have \( s \in \mathcal{L} \), since the square (a) is cartesian, and we have \( t \in \mathcal{R} \), since the square (b) is cartesian. Moreover, we have \((f, \alpha_1) \in \mathcal{L} \), since the square \( \alpha \) is \( \mathcal{L} \)-cartesian by hypothesis. Thus, \( s(f, \alpha_1) = \mathcal{L}(f) \) and \( t = \mathcal{R}(f) \) by uniqueness of the factorization \( f = \mathcal{R}(f) \mathcal{L}(f) \). Thus, \( \mathcal{R}(\alpha) \) is the bottom square (b). This shows that \( \mathcal{R}(\alpha) \) is cartesian, since the square (b) is cartesian.

**Lemma 3.7.4.** Let \( (\mathcal{L}, \mathcal{R}) \) be a modality on a topos \( \mathcal{E} \). Then the composite of two \( \mathcal{L} \)-cartesian squares is \( \mathcal{L} \)-cartesian. Moreover, if the composite of the squares in the following diagram is \( \mathcal{L} \)-cartesian, the left-hand square (a) is \( \mathcal{L} \)-cartesian, and the map \( f \) is a cover, then the right-hand square (b) is \( \mathcal{L} \)-cartesian.

\[
\begin{array}{ccc}
A' & \overset{f}{\longrightarrow} & B' \\
\downarrow^{(u, f')} & & \downarrow^{g} \\
A & \overset{f}{\longrightarrow} & B \\
\end{array}
\]

\[
\begin{array}{ccc}
A' & \overset{f'}{\longrightarrow} & B' \\
\downarrow^{(u', f')} & & \downarrow^{g'} \\
A \times_B B' & \overset{(v, g')}{\longrightarrow} & C' \\
\end{array}
\]

\[
\begin{array}{ccc}
A \times_C C' & \longrightarrow & B \times_C C' \\
\downarrow & & \downarrow^{w} \\
A & \overset{f}{\longrightarrow} & B \\
\end{array}
\]

The cartesian gap map of the square (b) in the diagram 3.7.4 is \((v, g')\), the cartesian gap map of the square (a) is \((u, f')\) and the cartesian gap map of composite square \((a) + (b)\) is the composite \( f^*(v, g')(u, f') \). If the maps \((u, f')\) and \((v, g')\) belongs to \( \mathcal{L} \), then so is the map \( f^*(v, g')(u, f') \), since the class \( \mathcal{L} \) is closed under base changes and composition. Conversely, if the maps \( f^*(v, g')(u, f') \) and \((u, f')\) belongs to \( \mathcal{L} \), then \( f^*(v, g') \) belongs to \( \mathcal{L} \) by property (3) in Lemma 3.1.6. Thus, \((v, g') \in \mathcal{L} \) when \( f \) is cover, since the class \( \mathcal{L} \) is local by Proposition 3.6.5.

**Lemma 3.7.5.** Let \( \mathcal{M} \) be a local class of maps in a topos \( \mathcal{E} \). Consider a pushout diagram in \( \mathcal{E} \) and suppose that the squares \( \alpha \) and \( \beta \) are cartesian.
Then $f, g, h \in \mathcal{M}$ implies $k \in \mathcal{M}$.

Proof. The commutative square above corresponds to a cube in $\mathcal{E}$:

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
| & | & | \\
\downarrow f & \downarrow \delta_0 & \downarrow \gamma_0 \\
C & \xrightarrow{g} & D \\
| & | & | \\
\downarrow h & \downarrow \gamma_1 & \downarrow k \\
E & \xrightarrow{G} & F & \xrightarrow{\delta_1} & H
\end{array}
\]

(3.1)

By descent, the front and right faces of the cube are cartesian, since the back and left faces are cartesian by hypothesis. Hence, the base changes of $k$ along $\delta_1 : G \to H$ and $\gamma_1 : F \to H$ belong to $\mathcal{M}$. But the two maps $\delta_1$ and $\gamma_1$ form a coverage of $H$ by item (5) of Remark 3.1.4. Thus, $k \in \mathcal{M}$, since $\mathcal{M}$ is a local class.

**Lemma 3.7.6.** Let $(\mathcal{L}, \mathcal{R})$ be a modality in a topos $\mathcal{E}$ and let

\[
\begin{array}{ccc}
f & \xrightarrow{\alpha} & g \\
\beta & \downarrow & \gamma \\
h & \xrightarrow{\delta} & k
\end{array}
\]

be a pushout square in $\mathcal{E}^\wedge$. Suppose that the square $\mathcal{R}(\alpha)$ and $\mathcal{R}(\beta)$ are cartesian. Then so are the squares $\mathcal{R}(\delta)$ and $\mathcal{R}(\gamma)$.

Proof. As before, our square of arrows corresponds to a cube (3.1) in $\mathcal{E}$ such that the top and bottom squares are pushouts. Consider the following two-story building, obtained by factoring the vertical maps $f$, $h$ and $g$ with respect to the modality $(\mathcal{L}, \mathcal{R})$.
The middle floor of the building is a pushout by construction. It follows that the upper and lower cubes are pushout diagram in $\mathcal{E}^\to$, since the top and bottom floors of the building are pushout squares. The composite cube is also a pushout diagram in $\mathcal{E}^\to$ for the same reason. Thus, $k = ts$ since the cube (3.1) is a pushout diagram in $\mathcal{E}^\to$. We have $s \in \mathcal{L}$, since the class $\mathcal{L}$ is closed under all colimits in $\mathcal{E}^\to$ by Lemma 3.1.6, and the upper cube exhibits $s$ as a colimit of maps in $\mathcal{L}$. The back and left vertical faces of the lower cube are cartesian, since the squares $R(p\alpha q)$ and $R(p\beta q)$ are cartesian by hypothesis. Hence, the front and right faces of the lower cube are cartesian by descent. Moreover, we have $t \in \mathcal{R}$ by Lemma 3.7.5. Thus, $t = \mathcal{R}(k)$ and $s = \mathcal{L}(k)$, since $k = ts$. It follows that the front face of the lower cube is equal to $\mathcal{R}(\delta)$ and the right face is equal to $\mathcal{R}(\gamma)$. This shows that the squares $\mathcal{R}(\delta)$ and $\mathcal{R}(\gamma)$ are cartesian. 

**Proof of Proposition 3.7.2.** As in the previous lemmas, our square corresponds to a cube (3.1) in $\mathcal{E}$.

\[
\begin{array}{c}
A \\
\downarrow f \\
E \\
\downarrow \\
G
\end{array}
\quad
\begin{array}{c}
B \\
\downarrow g \\
D \\
\downarrow \\
H
\end{array}
\quad
\begin{array}{c}
C \\
\downarrow h \\
F \\
\downarrow \\
H
\end{array}
\quad
\begin{array}{c}
D
\end{array}
\]

By pulling back the bottom face of the cube along the map $k : D \to H$, we obtain the following decomposition

\[
\begin{array}{c}
A \\
\downarrow f' \\
E \times_H D \\
\downarrow h' \\
G \\
\downarrow \\
F \\
\downarrow k \\
H.
\end{array}
\quad
\begin{array}{c}
B \\
\downarrow g' \\
F \times_H D \\
\downarrow \\
D
\end{array}
\quad
\begin{array}{c}
C \\
\downarrow \beta' \\
G \times_H D \\
\downarrow \gamma' \\
D
\end{array}
\quad
\begin{array}{c}
D
\end{array}
\]

The top cube of this diagram is a square in the category $\mathcal{E}^\to$,

\[
\begin{array}{c}
f' \\
\downarrow \beta' \\
h' \\
\downarrow \gamma'
\end{array}
\quad
\begin{array}{c}
a' \\
\downarrow \gamma'
\end{array}
\quad
\begin{array}{c}
g' \\
\downarrow 1_D
\end{array}
\]

One finds easily from the composition of pullback squares that the gap map of the square $\beta'$ coincides with the gap map of the square $\beta$. Hence, the square $\beta'$ is $\mathcal{L}$-cartesian, since the square $\beta$ is $\mathcal{L}$-cartesian by hypothesis. By Lemma 3.7.3, it follows that $\mathcal{R}(\beta')$ is cartesian. Similarly, the face $\mathcal{R}(\alpha')$ is cartesian. So by Lemma 3.7.6, the faces $\mathcal{R}(\gamma')$ and $\mathcal{R}(\delta')$ are cartesian. We
have $\mathcal{R}(1_D) = 1_D$, since $1_D \in \mathcal{R}$. Hence, the maps $\mathcal{R}(g')$ and $\mathcal{R}(h')$ are invertible. It follows that $g' \in \mathcal{L}$ and $h' \in \mathcal{L}$. Hence, the square $\delta$ is $\mathcal{L}$-cartesian, since $h'$ is the gap map of $\delta$. Similarly, the square $\gamma$ is $\mathcal{L}$-cartesian, since $g'$ is the gap map of $\gamma$.

**Remark 3.7.7.** Proposition 3.7.2 is a special case of a more general results about $\mathcal{L}$-cartesian squares. Let us denote by $\text{Cart}_{\mathcal{L}}(\delta^{-})$ the (non-full) subcategory of $\delta^{-}$ whose morphisms are $\mathcal{L}$-cartesian. Then the subcategory $\text{Cart}_{\mathcal{L}}(\delta^{-})$ of $\delta^{-}$ is closed under colimits. As in the case of ordinary descent, the closure condition here means two things: both that colimits exists in the subcategory $\text{Cart}_{\mathcal{L}}(\delta^{-})$ and that they are preserved by the inclusion functor $\text{Cart}_{\mathcal{L}}(\delta^{-}) \to \delta^{-}$.

4. The Blakers–Massey theorem

In this section, we at last turn to the formulation and proof of our generalized Blakers–Massey theorem, as well as some immediate applications.

4.1. The statement

**Theorem 4.1.1 (Generalized Blakers–Massey).** Let $(\mathcal{L}, \mathcal{R})$ be a modality in a topos $\mathcal{E}$. Consider a pushout square:

$$
\begin{array}{ccc}
Z & \xrightarrow{g} & Y \\
\downarrow f & & \downarrow k \\
X & \xrightarrow{h} & W
\end{array}
$$

Suppose that $\Delta f \circ_Z \Delta g \in \mathcal{L}$. Then the square is $\mathcal{L}$-cartesian.

**Remark 4.1.2.** While the use of the relative pushout product $\circ_Z$, in the statement of the theorem is the most general result, in practice it is often simpler to check that $\Delta f \circ \Delta g \in \mathcal{L}$, as will be done in all the applications of Section 4.3. Indeed, according to Example 3.2.1(v), the map $\Delta f \circ_Z \Delta g$ is a base change of the map $\Delta f \circ \Delta g$ and hence contained in $\mathcal{L}$ as soon as the latter is.

Any set of maps in a topos generates a modality [2]. This leads to the following reformulation of the main theorem.

**Corollary 4.1.3.** The cartesian gap map of the square in Theorem 4.1.1 belongs to the left class of the modality generated by the map $\Delta f \circ_Z \Delta g$.

A special case of the generalized Blakers–Massey theorem 4.1.1 is obtained by considering the modality whose left class consists of the isomorphisms and whose right class is given by all maps. We refer to this as the ‘Little Blakers–Massey Theorem’, and the statement is the following:

**Corollary 4.1.4 (Little Blakers–Massey theorem).** Consider a pushout square in a topos $\mathcal{E}$:

$$
\begin{array}{ccc}
Z & \xrightarrow{g} & Y \\
\downarrow f & & \downarrow k \\
X & \xrightarrow{h} & W
\end{array}
$$
If the map $\Delta f \circ_Z \Delta g$ is invertible, then the square is cartesian.

Let us continue by describing the map $\Delta f \circ_Z \Delta g$ in more detail. If $f : Z \to X$ is a map in $\mathcal{E}$, then the map $\Delta f : Z \to Z \times_X Z$ is a section of the projection $p_1 : Z \times_X Z \to Z$ onto the first factor. Consequently, we can view $Z \times_X Z$ as a pointed object of the slice topos $\mathcal{E}_{/Z}$ with structure map given by the first projection $p_1 : Z \times_X Z \to Z$ and basepoint given by the diagonal map $\Delta f : Z \to Z \times_X Z$. Similarly, if $g : Z \to Y$ is another map in $\mathcal{E}$, then we can view $Z \times_Y Z$ as a pointed object of the slice topos $\mathcal{E}_{/Z}$ with structure map given by the first projection $p_1 : Z \times_Y Z \to Z$ and basepoint given by the diagonal map $\Delta g : Z \to Z \times_Y Z$:

\[
\begin{array}{ccc}
Z \times_X Z & \xrightarrow{p_1} & Z \\
\uparrow{\Delta f} & & \uparrow{\Delta g} \\
Z & \xrightarrow{p_1} & Z
\end{array}
\]

Recall from Example 3.2.1(2) that for any two pointed objects $a : 1 \to A$ and $b : 1 \to B$ of a topos $\mathcal{E}$, the pushout product $a \circ b : A \vee B \to A \times B$ is the canonical inclusion of the wedge into the product. This applies to the map $\Delta f \circ_Z \Delta g$ viewed in $\mathcal{E}_{/Z}$:

\[
(Z \times_X Z) \cup_Z (Z \times_Y Z) \xrightarrow{\Delta f \circ_Z \Delta g} (Z \times_X Z) \times_Z (Z \times_Y Z)
\]

It is instructive to compute the fibers of the maps $p$ and $q$ at $z \in Z$. The fiber of the projection $p_1 : Z \times_X Z \to Z$ at $z \in Z$ can be identified with the fiber $f^{-1}(f(z))$ of $f$ at $f(z)$ since the following square is cartesian:

\[
\begin{array}{ccc}
Z \times_X Z & \xrightarrow{p_2} & Z \\
\uparrow{f} & & \uparrow{f} \\
Z & \xrightarrow{f} & X
\end{array}
\]

(4.1)

Similarly, the fiber of the projection $p_1 : Z \times_Y Z \to Z$ at $z \in Z$ can be identified with the fiber $g^{-1}(g(z))$ of $g$ at $g(z)$. It follows that the fiber of the map $p$ at $z \in Z$ can be identified with the wedge $f^{-1}(f(z)) \vee_Z g^{-1}(g(z))$, while the fiber of $q$ can be identified with the product $f^{-1}(f(z)) \times g^{-1}(g(z))$. The map $\Delta f \circ_Z \Delta g$ is given fiberwise by the canonical maps

\[
f^{-1}(f(z)) \vee_Z g^{-1}(g(z)) \to f^{-1}(f(z)) \times g^{-1}(g(z))
\]

as $z$ ranges over $Z$. To summarize, there is a commuting diagram

\[
\begin{array}{ccc}
(Z \times_X Z) \cup_Z (Z \times_Y Z) & \xrightarrow{\Delta f \circ_Z \Delta g} & (Z \times_X Z) \times_Z (Z \times_Y Z) \\
\downarrow{f} & & \downarrow{q} \\
Z & & Z
\end{array}
\]

whose vertical lines are fiber sequences over $z \in Z$. 
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For the upcoming proof, we will need yet another description of $\Delta f \circ_Z \Delta g$. By definition, it is the cogap map of the following square

$$
\begin{array}{ccc}
Z & \xrightarrow{\Delta_f} & \Delta \times_Y Z \\
\downarrow & & \downarrow \Delta \times_Z (Z \times_Y Z) \\
Z \times_X Z & \xrightarrow{\Delta \times_Z (Z \times_Y Z)} & (Z \times_X Z) \times_Z (Z \times_Y Z)
\end{array}
$$

(4.2)

where $(Z \times_X Z) \times_Z (Z \times_Y Z)$ is the fiber product over $Z$ of the projections

$$p_1 : Z \times_X Z \to Z, \ (z_1, z_2) \mapsto z_1$$

and

$$p_1 : Z \times_Y Z \to Z, \ (z_1, z_2) \mapsto z_1.$$

We will use a more explicit notation for the various maps in the square. For example, $\Delta f = (1_Z, 1_Z) : Z \to Z \times_X Z$ since $(\Delta f) (z) = (z, z)$ for every $z \in Z$. Similarly, $\Delta g = (1_Z, 1_Z) : Z \to Z \times_Y Z$. The notation is not ambiguous as long as the domain and the codomain of the maps are clear. For example, the vertical map on the right in (4.2) is

$$\Delta f \times_Z (Z \times_Y Z) = (p_1, p_1, p_1, p_2)$$

since

$$(\Delta f \times_Z (Z \times_Y Z)) (z_1, z_2) = (z_1, z_1, z_1, z_2).$$

Beware that the fiber product over $Z$ is using the projection $p_1 : Z \times_Y Z \to Z$ onto the first factor. Similarly,

$$(Z \times_X Z) \times_Z \Delta g = (p_1, p_2, p_1, p_1),$$

since

$$(Z \times_X Z) \times_Z \Delta g(z_1, z_2) = (z_1, z_2, z_1, z_1).$$

Beware again that the fiber product over $Z$ is using $p_1 : Z \times_X Z \to Z$, and not $p_2$, because $p_1$ is the structure map of $Z \times_X Z$ over $Z$.

With this notation, the square (4.2) becomes

$$
\begin{array}{ccc}
Z & \xrightarrow{(1_Z, 1_Z)} & Z \times_Y Z \\
\downarrow (1_Z, 1_Z) & & \downarrow (p_1, p_1, p_1, p_2) \\
Z \times_X Z & \xrightarrow{(p_1, p_2, p_1, p_1)} & (Z \times_X Z) \times_Z (Z \times_Y Z)
\end{array}
$$

(4.3)

In terms of the notation introduced in Section 2.1, the cogap map of the square has the following description:

$$\Delta f \circ_Z \Delta g = [(p_1, p_2, p_1, p_1), (p_1, p_1, p_1, p_2)]$$

$$= \begin{bmatrix} p_1 & p_2 & p_1 & p_1 \\ p_1 & p_1 & p_1 & p_2 \end{bmatrix}$$

$$= [(p_1, p_1), [p_2, p_1], [p_1, p_1], [p_1, p_2]],$$

which will prove useful for calculations.
4.2. The proof

We will use the fact that the map $\Delta f \circ_Z \Delta g$ is isomorphic to two closely related maps:

$$
\delta_{XY} : (Z \times_X Z) \cup_Z (Z \times_Y Z) \to Z \times_X Z \times_Y Z
$$

$$
\delta_{YX} : (Z \times_X Z) \cup_Z (Z \times_Y Z) \to Z \times_Y Z \times_X Z
$$

which are defined by

$$
\delta_{XY} := \{(p_2, p_1, p_1), (p_2, p_2, p_1)\}
$$

$$
= \begin{bmatrix}
  p_2 & p_1 & p_1 \\
  p_2 & p_2 & p_1
\end{bmatrix}
$$

$$
= [(p_2, p_1), [p_1, p_2], [p_1, p_1]]
$$

$$
\delta_{YX} := \{(p_2, p_2, p_1), (p_2, p_1, p_1)\}
$$

$$
= \begin{bmatrix}
  p_2 & p_2 & p_1 \\
  p_2 & p_1 & p_1
\end{bmatrix}
$$

$$
= [(p_2, p_2), [p_2, p_1], [p_1, p_1]].
$$

**Lemma 4.2.1.** The maps $\Delta f \circ_Z \Delta g$, $\delta_{XY}$, and $\delta_{YX}$ are isomorphic as objects of the category $\delta^{\rightarrow^\rightarrow}$.

**Proof.** The maps $\delta_{XY}$ and $\delta_{YX}$ are isomorphic since the following square commutes

$$
(Z \times_X Z) \cup_Z (Z \times_Y Z) \xrightarrow{\sigma_X \cup \sigma_Y} (Z \times_X Z) \cup_Z (Z \times_Y Z)
$$

$$
\downarrow \delta_{XY}
$$

$$
\begin{array}{ccc}
Z \times_X Z \times_Y Z & \to & Z \times_Y Z \times_X Z \\
(p_3, p_2, p_1) & \to & (p_2, p_1, p_1)
\end{array}
$$

and the maps $(p_1, p_2, p_1)$, $\sigma_X := (p_2, p_1)$ and $\sigma_Y := (p_2, p_1)$ are invertible. It is easy to see that the map

$$
\theta := (p_2, p_1, p_2, p_3) : Z \times_X Z \times_Y Z \to (Z \times_X Z) \times_Z (Z \times_Y Z)
$$

is invertible. We have a commutative diagram

$$
(Z \times_X Z) \cup_Z (Z \times_Y Z) \xrightarrow{id \cup_Z \sigma_Y} (Z \times_X Z) \cup_Z (Z \times_Y Z)
$$

$$
\downarrow \delta_{XY}
$$

$$
\begin{array}{ccc}
(Z \times_X Z) \times_Z (Z \times_Y Z) & \leftarrow & Z \times_X Z \times_Y Z \\
\theta & \leftarrow & \delta_{XY} \circ (id \cup_Z \sigma_Y)
\end{array}
$$

since

$$
\Delta f \circ_Z \Delta g = [(p_1, p_1), [p_2, p_1], [p_1, p_1], [p_1, p_2])
$$

$$
= \theta \circ [(p_2, p_1), [p_1, p_1], [p_1, p_2])
$$

$$
= \theta \circ [(p_2, p_2 \circ \sigma_Y), [p_1, p_2 \circ \sigma_Y], [p_1, p_1 \circ \sigma_Y]]
$$

$$
= \theta \circ [(p_2, p_2), [p_1, p_2], [p_1, p_1]) \circ (id \cup_Z \sigma_Y)
$$

$$
= \theta \circ \delta_{XY} \circ (id \cup_Z \sigma_Y) .
$$
This shows that the maps $\Delta f \circ_Z \Delta g$ and $\delta_{XY}$ are isomorphic, since the horizontal maps of the diagram are invertible.

Our next step in proving Theorem 4.1.1 will be to show that we may assume without loss of generality that the map $g$ is a cover. To see this, choose a factorization of $g$

\[
\begin{array}{ccc}
Z & \xrightarrow{g} & Y \\
\downarrow{s} & & \downarrow{m} \\
Y' & \xrightarrow{\delta_{XY}} & Z
\end{array}
\]

where $s$ is a cover and $m$ is a monomorphism. Now consider the diagram formed by taking pushouts:

\[
\begin{array}{ccc}
Z & \xrightarrow{s} & Y' & \xrightarrow{m} & Y \\
\downarrow{f} & \downarrow{(a)} & \downarrow{(b)} & \downarrow{} & \downarrow{} \\
X & \xrightarrow{=} & W' & \xrightarrow{=} & W.
\end{array}
\]

**Lemma 4.2.2.** The gap map $(f, g) : Z \to X \times_W Y$ as an object of the category $\mathcal{E}$ is isomorphic to the gap map $(f, s) : Z \to X \times_W Y'$. Similarly, the map $\Delta g : Z \to Z \times_Y Z$ is isomorphic to the map $\Delta s : Z \to Z \times_Y Z$.

**Proof.** Since in each case, the maps have the same domain, it suffices to exhibit an isomorphism between the codomains (making the appropriate triangle commute, a detail we leave to the reader). In the first case, consider the diagram:

\[
\begin{array}{ccc}
X \times_W Y' & \xrightarrow{=} & Y' & \xrightarrow{=} & Y \\
\downarrow{r} & \downarrow{r} & \downarrow{} & \downarrow{} & \downarrow{} \\
X & \xrightarrow{=} & W' & \xrightarrow{=} & W.
\end{array}
\]

The left square is a pullback by construction. The right square is a pullback by Proposition 2.2.6. It follows that the object $X \times_W Y$ is isomorphic to the object $X \times_W Y'$.

In the second case, we consider the diagram:

\[
\begin{array}{ccc}
Z \times_Y Z & \xrightarrow{s} & Z & \xrightarrow{s} & Z \\
\downarrow{} & \downarrow{s} & \downarrow{s} & \downarrow{} & \downarrow{} \\
Z & \xrightarrow{s} & Y' & \xrightarrow{m} & Y \\
\downarrow{} & \downarrow{m} & \downarrow{} & \downarrow{} & \downarrow{} \\
Z & \xrightarrow{s} & Y' & \xrightarrow{m} & Y.
\end{array}
\]

The upper left square is cartesian by construction. The bottom right square is cartesian since $m$ is a monomorphism. The remaining two squares are trivially cartesian and hence so is the outer square. We conclude that the object $Z \times_Y Z$ is isomorphic to the object $Z \times_{Y'} Z$ as claimed.

An immediate consequence of the previous lemma is that we have $(f, g) \in \mathcal{L}$ if and only if $(f, s) \in \mathcal{L}$. Similarly, we have $\Delta f \circ_Z \Delta g \in \mathcal{L}$ if and only if $\Delta f \circ_Z \Delta s \in \mathcal{L}$. Hence, the Blakers–Massey theorem for the square $(a)$ of (4.4) implies the Blakers–Massey theorem for the square $(a) + (b)$. It therefore suffices to prove Theorem 4.1.1 in the case where $g$ (or, by symmetry, $f$) is a cover.
The proof of Theorem 4.1.1 will hinge on a careful analysis of the following cubical diagram

\[
\begin{array}{ccc}
(Z \times_Z Z) \sqcup_Z (Z \times_Y Z) & \xrightarrow{\rho_X} & Z \times_Z Z \\
\downarrow \rho_Y & & \downarrow \rho_1 \\
Z \times_Y Z & \xrightarrow{g_{[p_2,p_2]}} & Z \\
\downarrow f_{[p_2]} & & \downarrow p_1 \\
Z & \xrightarrow{g} & Y \\
\downarrow f & & \downarrow k \\
X & \xrightarrow{h} & W
\end{array}
\] (4.5)

in which we set \( d := hf = kg \),

\[
\rho_X := [1_{Z \times Z}, \Delta f \circ p_1] = [(p_1,p_2), (p_1,p_1)] = \begin{bmatrix} p_1 & p_2 \\ p_1 & p_1 \end{bmatrix} = ([p_1,p_1], [p_2,p_1])
\]

and

\[
\rho_Y := [\Delta g \circ p_1, 1_{Z \times_Y Z}] = [(p_1,p_1), (p_1,p_2)] = \begin{bmatrix} p_1 & p_1 \\ p_1 & p_2 \end{bmatrix} = ([p_1,p_1], [p_1,p_2]).
\]

Let us pause to verify that the cube is indeed commutative. First of all, the bottom face commutes by hypothesis. The top face commutes since

\[
p_1 \circ \rho_X = p_1 \circ ([p_1,p_1], [p_2,p_1]) = [p_1,p_1] = p_1 \circ ([p_1,p_1], [p_1,p_2]) = p_1 \circ \rho_Y.
\]

Next, the map \( f : Z \to X \) coequalizes the maps \( p_1,p_2 : Z \times_X Z \to Z \) since the square \((4.1)\) commutes. Hence, the front face commutes since \( dp_1 = hfp_1 = hfp_2 \). The right face commutes by a similar argument. Finally, the left face commutes since

\[
f_{[p_2]}\rho_Y = f_{[p_2]}([p_1,p_1], [p_1,p_2]) = f[p_1,p_2] = [f_{p_1}, f_{p_2}] = [f_{p_2}, f_{p_2}] = f[p_2,p_2]
\]

and similarly for the back face.
Lemma 4.2.3. The top face of the cube \((4.5)\)

\[
\begin{array}{c}
(Z \times_X Z) \cup_Z (Z \times_Y Z) \xrightarrow{\rho_X} Z \times_X Z \\
\downarrow \rho_Y & \downarrow p_1 \\
Z \times_Y Z & \rightarrow Z
\end{array}
\]

is cocartesian.

Proof. Let us first show that for any pair of pointed objects \((A, a)\) and \((B, b)\) in a topos, the following square is cocartesian,

\[
\begin{array}{cccc}
A \lor B & \xrightarrow{|0_A, 1_B|} & B \\
\downarrow |1_A, 0_B| & \downarrow \downarrow \\
A & \rightarrow & 1
\end{array}
\]

is cocartesian, where \(0_A : A \rightarrow 1 \xrightarrow{b} B\) and \(0_B : B \rightarrow 1 \xrightarrow{a} A\). For this, consider the following commutative diagram

\[
\begin{array}{cccc}
1 & \xrightarrow{a} & A \\
\downarrow & \downarrow (a) & \downarrow A \lor B \\
A \lor B & \xrightarrow{|0_A, 1_B|} & B \\
\downarrow & \downarrow (b) & \downarrow (c) \\
1 & \xrightarrow{a} & A & \rightarrow & 1.
\end{array}
\]

The square \((a)\) in this diagram is cocartesian by construction. The squares \((a) + (b)\) and \((b) + (c)\) are trivially cocartesian. Hence, the square \((b)\) is cocartesian, since the squares \((a)\) and \((a) + (b)\) are cocartesian. It follows that the square \((c)\) is cocartesian, since \((b)\) and \((b + c)\) are cocartesian. Regarding \(Z \times_X Z\) and \(Z \times_Y Z\) as pointed objects of the topos \(\delta_{/Z}\) with structure maps given by \(p_1\) and base points given by the diagonal maps \(\Delta f\) and \(\Delta g\), respectively, we see that the top face of the cube \((4.5)\) has the form above, since \(\rho_X = [1_{Z \times_X Z}, \Delta f \circ p_1]\) and \(\rho_Y = [\Delta g \circ p_1, 1_{Z \times_Y Z}]\) by definition.

We are now nearly in a position to apply Proposition 3.7.2 on descent of \(\mathscr{L}\)-cartesian squares to the cube \((4.5)\). For this we need to show that the back and left faces of the cube \((4.5)\) are \(\mathscr{L}\)-cartesian.

Lemma 4.2.4. Suppose \(\Delta f \circ_Z \Delta g \in \mathscr{L}\). Then the squares \((a)\) and \((b)\)

\[
\begin{array}{c}
Z \times_Y Z & \xleftarrow{\rho_Y} (Z \times_X Z) \cup_Z (Z \times_Y Z) \xrightarrow{\rho_X} Z \times_X Z \\
\downarrow f_{p_2} & \downarrow (p_2, p_2) & \downarrow g_{p_2} \\
X & \leftarrow Z & \rightarrow Y
\end{array}
\]

are \(\mathscr{L}\)-cartesian.
Proof. The square (a) admits the following decomposition:

\[
\begin{array}{cccc}
(Z \times X Z) \sqcup_Z (Z \times Y Z) & \xrightarrow{\delta_{XY}} & Z \times X Z \times Y Z & \xrightarrow{\delta_{p,p_1}} Z \times Y Z \xrightarrow{\sigma} Z \times Y Z\\
\downarrow_{[p_2,p_2]} & & \downarrow_{p_1} & \downarrow_{f} \\
Z & \xrightarrow{(c)} & Z & \xrightarrow{(d)} X
\end{array}
\]

since

\[
\sigma \circ (p_2,p_1) \circ \delta_{XY} = (p_3,p_2) \circ \delta_{XY} = (p_3,p_2) \circ ([p_2,p_2],[p_1,p_2],[p_1,p_1]) = ([p_1,p_1],[p_1,p_2]) = \rho_Y.
\]

On the other hand, the square (d) in the diagram above is cartesian by construction and hence so is the square (d) + (e) since \(\sigma\) is an isomorphism. Consequently, the map \(\delta_{XY}\) is isomorphic to the gap map of the square (e) + (d) + (e) = (a). By Lemma 4.2.1, \(\delta_{XY}\) is isomorphic to the map \(\Delta f \circ_Z \Delta g\) which is in \(\mathcal{L}\) by hypothesis. Hence, the square (a) is \(\mathcal{L}\)-cartesian.

The square (b) admits a similar decomposition which takes the form

\[
\begin{array}{cccc}
(Z \times X Z) \sqcup_Z (Z \times Y Z) & \xrightarrow{\delta_{XY}} & Z \times Y Z \times X Z & \xrightarrow{\delta_{p,p_1}} Z \times X Z \\
\downarrow_{[p_2,p_2]} & & \downarrow_{p_1} & \downarrow_{g_{p_1}} \\
Z & \xrightarrow{g} Z & \xrightarrow{Y} & \xrightarrow{Y}
\end{array}
\]

A similar application of Lemma 4.2.1 to the map \(\delta_{YX}\) completes the proof.

Proof of Theorem 4.1.1. The bottom face of the cube cube (4.5) is cocartesian by the hypothesis and the top face is cocartesian by Lemma 4.2.3. The previous lemma shows that the back and left faces of the cube are \(\mathcal{L}\)-cartesian, since \(\Delta f \circ_Z \Delta g \in \mathcal{L}\) by the hypothesis. It then follows from Proposition 3.7.2 that the front and right faces are also \(\mathcal{L}\)-cartesian. Now, the right face of the cube (4.5) is the composite of the following two squares since \(d = h f\).

\[
\begin{array}{cccc}
Z \times X Z & \xrightarrow{p_2} Z & \xrightarrow{g} Y \\
\downarrow_{p_1} & \downarrow_{f} & \downarrow_{k} \\
Z & \xrightarrow{f} X & \xrightarrow{h} W
\end{array}
\]

Recall from Lemma 4.2.2 that we may assume that the map \(f : Z \to X\) is a cover. The left-hand square of the above diagram is cartesian by construction and the composite square is \(\mathcal{L}\)-cartesian by the previous considerations. It follows from Lemma 3.7.4 that the right-hand square is \(\mathcal{L}\)-cartesian, since \(f : Z \to X\) is a cover.

4.3. Applications

Our main application is a Blakers–Massey theorem in the context of Goodwillie’s calculus of homotopy functors. This is developed in our second joint article [1]. It was indeed this application to Goodwillie calculus that motivated the whole project outlined in the two papers.
Let us now show how easily the classical Blakers–Massey theorem follows from our main theorem. The classical theorem in the category of spaces $\mathcal{S}$ is a special case of the following Blakers–Massey theorem in an arbitrary topos. Recall from Remark 3.3.5 that our topos theoretic definition of $n$-connected map differs from the classical convention in homotopy theory by a shift of one.

**Corollary 4.3.1 (Classical Blakers–Massey for topoi).** Given a pushout square

$$
\begin{array}{ccc}
A & \xrightarrow{g} & C \\
\downarrow f & & \downarrow \\
B & \xrightarrow{h} & D
\end{array}
$$

in a topos, such that $f$ is $m$-connected and $g$ is $n$-connected, then the cartesian gap map $(f, g) : A \to B \times_D C$ is $(m + n)$-connected.

**Proof.** By Example 3.2.4(1), the diagonal map $\Delta f = \langle s_0, f \rangle$ is $(m - 1)$-connected using Proposition 3.3.8. Similarly, $\Delta g$ is $(n - 1)$-connected. Since the $(m + n)$-connected maps form a modality, and $\Delta f \circ \Delta g$ is $(m + n)$-connected by Corollary 3.3.7(4), the result now follows from Theorem 4.1.1.

As a further application we explain how our generalized version yields the improvement of the Blakers–Massey theorem by Chachólski, Scherer and Werndli in [5]. We will need some preparation.

In the category $\mathcal{S}$, the modality generated by a map $u : A \to B$ coincides with the modality generated by the maps $u^{-1}(b) \to 1$ for all $b \in B$. This follows from Remark 3.6.3.

If $x$ and $y$ are two points of a space $X$, let us denote by $X_{x,y}$ the space of paths $x \to y$. By construction, $X_{x,y}$ is the fiber of the diagonal map $X \to X \times X$ at $(x, y) \in X \times X$. Note that $X_{x,y} = \Omega_x X$ when $x$ and $y$ are homotopic and that $X_{x,y} = \emptyset$ otherwise. If $(A, a)$ is a pointed space, then $A(x, a)$ is the fiber at $x \in A$ of the map $a : 1 \to A$.

**Lemma 4.3.2.** Let $(A, a)$ and $(B, b)$ be pointed spaces. Then the fiber of the canonical map $A \vee B \to A \times B$ at $(x, y) \in A \times B$ is the join $A(x, a) \ast B(y, b)$.

**Proof.** One has $A \vee B \to A \times B = (a : 1 \to A) \circ (b : 1 \to B)$. The fiber $(a \circ b)^{-1}(x, y)$ is the join of the fibers $a^{-1}(x)$ and $b^{-1}(y)$ by Example 3.2.1(4). The result follows, since $a^{-1}(x) = A(x, a)$ and $b^{-1}(y) = B(y, b)$.

Finally, we now rederive the weak cellular inequality of Chachólski, Scherer and Werndli [5] in the case of a pushout. For a pointed space $(A, a)$ we denote the set of spaces $A(x, a)$ for $x \in A$ by $\mathcal{P}(A, a)$.

**Theorem 4.3.3.** In the category of spaces, the cartesian gap map of a pushout

$$
\begin{array}{ccc}
Z & \xrightarrow{g} & X \\
\downarrow f & & \downarrow \\
Y & \xrightarrow{h} & W
\end{array}
$$

belongs to the modality generated by the set of maps $S \ast T \to 1$ for $S \in \mathcal{P}(f^{-1}f(z), z)$, $T \in \mathcal{P}(g^{-1}g(z), z)$ and $z \in Z$. 
Proof. We saw above that $\Delta f \circ Z \Delta g$, viewed fiberwise over $Z$, can be regarded as a sum over $z \in Z$ of the canonical map

$$i_z : f^{-1}(f(z)) \vee g^{-1}(g(z)) \to f^{-1}(f(z)) \times g^{-1}(g(z)).$$

Hence, the modality generated by $\Delta f \circ Z \Delta g$ is also generated by the maps $i_z$ for all $z \in Z$. By Lemma 4.3.2, the fibers of $i_z$ are of the form $S \times T$, where $S$ is a fiber of the map $z : 1 \to f^{-1}(f(z))$ and $T$ is a fiber of the map $z : 1 \to g^{-1}(g(z))$. Hence, the modality generated by the map $\Delta f \circ Z \Delta g$ is generated by the maps $S \times T \to 1$ for $S \in \mathcal{P}(f^{-1}(f(z), z), T \in \mathcal{P}(g^{-1}(g(z), z)$ and $z \in Z$.

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