WARPED PRODUCT APPROACH TO UNIVERSE WITH NON-SMOOTH SCALE FACTOR

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Abstract. In the framework of Lorentzian warped products, we study the Friedmann-Robertson-Walker cosmological model to investigate non-smooth curvatures associated with multiple discontinuities involved in the evolution of the universe. In particular we analyze non-smooth features of the spatially flat Friedmann-Robertson-Walker universe by introducing double discontinuities occurred at the radiation-matter and matter-lambda phase transitions in astrophysical phenomenology.

I. Introduction

Since the cosmic microwave background was discovered, there have been many ideas and proposals to figure out how the universe has evolved. The standard big bang cosmological model based on the Friedmann-Robertson-Walker (FRW) spacetimes has led to the inflationary cosmology [1] and nowadays to the M-theory cosmology with bouncing universes [2]. These spacetimes are foliated by a special set of spacelike hypersurfaces such that each hypersurface corresponds to an instant of time. From a physical point of view, these warped product spacetimes are interesting since they include classical examples of spacetime such as the FRW manifold and the intermediate zone of Reissner-Nordström (RN) manifold [3, 4].

The Lorentzian manifolds with non-smooth metric tensors have been extensively discussed from various viewpoints [5, 6, 7, 8, 9]. In a spacetime where the metric tensor is continuous but has a jump in its first and second derivatives across a submanifold in an admissible coordinate system, one can have a curvature tensor containing a Dirac delta function [10]. The support of this distribution may be of three, two, or one dimensional or may even consist of a single event. Moreover, Lichnerowicz’s formalism [7] for dealing with such tensors is modified so that one can obtain the Riemannian curvature tensor and Ricci curvature tensor defined in the sense of distributions.

A general theory for matching two solutions of the Einstein field equations has been proposed [8, 9] at arbitrary shock-wave interface across which the metric tensor is $C^0$-Lorentzian, namely at smooth surface across which the first derivatives of the metric suffer at worst a jump discontinuity, so that the simplest solution of Einstein equations can incorporate a shock-wave into a standard FRW metric whose equation of state accounts for the Hubble constant and the microwave background radiation temperature. There have been later presented the evolution of the one point probability distribution function of the cosmological density field based on an exact statistical treatment [11].

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On the other hand, the concept of a warped product manifold was introduced by Bishop and O’Neill long ago [12], and it was later connected to general relativity [13] and semi-Riemannian geometry [14] by elevating warped products to a central role. Warped product spaces has been also extended to a richer class of spaces involving multiply product spaces [5, 6]. One of us has investigated the curvature of a multiply warped product possessing $C^0$-warping functions with a discontinuity at a single point [5], and in this paper we will generalize this result to a warped product spacetime with multiple discontinuities associated with cosmological phenomenology. Of particular interest are spacetimes with metric tensors which fail to be $C^1$ across multiple points on the hypersurface, and is $C^\infty$ off the hypersurface. We will also study the Lorentzian metric which fails to be $C^0$ across multiple points on the hypersurfaces and is $C^\infty$ off the hypersurfaces.

In this paper, as a cosmological model we will exploit the FRW spacetimes $M_0 \times_f H$, which can be treated as a warped product manifold possessing warping function (or scale factor) $f$ with time dependence, to investigate the non-smooth curvature associated with the multiple discontinuities involved in the evolution of the universe. We will also analyze non-smooth features of the spatially flat FRW universe by introducing double discontinuities occurred at the radiation-matter and matter-lambda phase transitions in the astrophysical phenomenology.

In section 2 we will introduce the warped product spacetime with multiple warping functions and extend the warped product scheme to the case with multiple discontinuities in the FRW cosmological model in section 3. We will study the realistic cosmological phenomenology in the spatially flat FRW universe associated with the radiation-matter and matter-lambda phase transitions in section 4.

II. Warped product spacetime with multiple warping functions

In this section we briefly recapitulate the curvature of the warped product approach to spacetime with multiple $C^0$-warping functions at a single point.

Definition 2.1 [5] A multiply warped products spacetime with base $(M_0, -dt^2)$, fibers $(F_i, g_i) i = 1, ..., n$ and warping functions $f_i > 0$ is the product manifold $(M_0 \times F_1 \times \cdots \times F_n, g)$ endowed with the Lorentzian metric:

\[
g = -\pi_{M_0}^* dt^2 + \sum_{i=1}^{n} (f_i \circ \pi_{M_0})^2 \pi_i^* g_i
\]

where $\pi_{M_0}, \pi_i (i = 1, ..., n)$ are the natural projections of $M_0 \times F_1 \times \cdots \times F_n$ onto $M_0$ and $F_1, ..., F_n$, respectively. For a specific case of $M_0 = R$ and $g_{M_0} = -dt^2$, the Lorentzian metric is given by

\[
g = -dt^2 + \sum_{i=1}^{n} f_i^2 g_i.
\]

Proposition 2.2 [5] Let $M = M_0 \times_f F_1 \times \cdots \times_f F_n$ be a multiply warped products with Riemannian curvature tensor $R$. If $X, Y \in \mathcal{L}(M_0), U_i, V_i, W_i \in \mathcal{L}(F_i) (n = 1, 2, ..., n), f_i \in C^0(S)$ at a single point $p \in M_0$, and $S = \{p\} \times_f F_1 \times \cdots \times_f F_n$. Then the covariant derivative of $X$ along $Y$ is given by

\[
\nabla_X Y = \left( \sum_{i=1}^{n} f_i^2 \nabla_{f_i X} f_i Y \right) + f_i \nabla_{f_i X} Y - \sum_{i=1}^{n} f_i \nabla_{Y f_i} X.
\]
\[ R_{XY}U_j = R_{U_iU_j}X = R_{U_jXU_i} = 0 \text{ for } i \neq j, \]

\[ R_{U_iU_j}X = U_iX^1Y^1 f''(t) + \delta(t - p)(f^{i+}_i - f^{i-}_i), \]

\[ R_{XU_i}U_j = R_{U_iXU_j} = R_{U_jXU_i} = 0, \text{ for } i \neq j, \]

\[ R_{XYU_i} = 0, \text{ for } i = 1, \ldots, n, \]

\[ R_{U_iU_j}U_j = 0, \text{ for } i \neq j, \]

\[ R_{U_iU_jV_j} = U_i(U_j, V_j)(f^{i+}_i + f^{i-}_i)(f^{j+}_j + f^{j-}_j)/f_i f_j, \text{ for } i \neq j, \]

\[ R_{U_iV_i}W_i = F^i R_{U_iV_i}W_i + ((U_i, W_i)V_i - (V_i, W_i)U_i) f^{i+}_i \mu(t - p) + f^{i-}_i \mu(p - t)/f_i^2, \]

where \( X = X^1 \partial/\partial t \) and \( Y = Y^1 \partial/\partial t \), and \( \mu(t - p) \) and \( \delta(t - p) \) are the unit step function and the delta function, respectively.

**Proposition 2.3** Let \( M = M_0 \times f_1 \times \ldots \times f_n \), \( F \), be a multiply warped products with Riemannian curvature tensor \( R \). If \( X, Y \in \mathcal{L}(M_0) \), \( U_i, V_i \in \mathcal{L}(F_i) \) \((n = 1, 2, \ldots, n), d_i = \dim F_i, f_i \in C^0(S) \) at a single point \( p \in M_0 \), and \( S = \{p\} \times f_1 \times \ldots \times f_n \), then

\[ \text{(i)} \quad \text{Ric}(X, Y) = -\sum_{i=1}^n d_i X^1 \partial/\partial t \frac{f''_i(t) + \delta(t - p)(f^{i+}_i - f^{i-}_i)}{f_i}, \]

\[ \text{(ii)} \quad \text{Ric}(X, U_i) = 0, \]

\[ \text{(iii)} \quad \text{Ric}(U_i, V_i) = F^i \text{Ric}(U_i, V_i) + (U_i, V_i) \frac{f''_i(t) + \delta(t - p)(f^{i+}_i - f^{i-}_i)}{f_i} \]

\[ + (U_i, V_i) \left( (d_i - 1) \frac{f^{i+}_i - f^{i-}_i}{f_i^2} + \sum_{j \neq i} d_j \frac{f^{j+}_j - f^{j-}_j}{f_i f_j} \right), \]

\[ \text{(iv)} \quad \text{Ric}(U_i, U_j) = 0, \text{ for } i \neq j, \]

where \( X = X^1 \partial/\partial t \) and \( Y = Y^1 \partial/\partial t \), and \( \delta(t - p) \) is the delta function.

**III. FRW metric with multiple discontinuities**

The FRW spacetime is one of the warped product manifold where the base is an open interval \( M_0 \) of \( R \) with usual metric reversed \((M_0, -dt^2)\), the fiber is a 3-dimensional Riemannian manifold \((F, g_F)\) and the warping function \( f \) is any positive function \( f \) on \( M_0 \). The Robertson-Walker spacetime is then the product manifold \( M = M_0 \times f H \) endowed with the Lorentzian metric \( g = -dt^2 + f^2(t)g_H \) with \( f \) being the scale factor of the FRW universe associated with universal expansion. This warping function \( f \) is a function of time alone and it measures how physical separations change with time. The dynamics of the expanding universe only appears implicitly in the time dependence of the warping function (or scalar factor) \( f \).
Consider the spacetime $(M, g)$ with metric $g = -dt^2 + f^2d\sigma^2$ in the form of warped products. Let $M = M_0 \times_f H$ be a warped product with $g_{M_0} = -dt^2$. Let $f > 0$ be smooth functions on $M_0 = (t_0, t_\infty)$. Assume $f \in C^\infty$ for $t \neq t_i$ and $f \in C^1$ at $t = t_i$ ($i = 1, 2, ..., n$). When $f \in C^1$ at points $t_i \in (t_0, t_\infty)$ and $S = \{t_i\} \times_f H$, we define $f \in C^1(S)$ as a collection of functions $\{f^{(i)}\}$ with $f^{(i)}$ piecewisely defined on the intervals $t_i \leq t \leq t_{i+1}$ ($i = 0, 1, 2, ..., n$) with $t_{n+1} = t_\infty$. Since $f \in C^1(S)$, we have $f^{(i-1)} = f^{(i)}$, $f^{(i-1)'} = f^{(i)'}$ but $f^{(i-1)''} \neq f^{(i)''}$. We shall use the unit step function $\mu$ for discontinuity of $f^{(i)''}$ at $t = t_i$.

Consider the FRW metric of the form

$$g = -dt^2 + f^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right)$$

where $k$ is a parameter denoting the spatially flat ($k = 0$), 3-sphere ($k = 1$) and hyperbolid ($k = -1$) universes.

**Proposition 3.1** Let $M = M_0 \times_f H$ be the FRW spacetime with Riemannian curvature $R$ and flow vector field $U = \partial_t$. If $f \in C^1(S)$, vector fields $X, Y, Z \in \mathfrak{L}(H)$ satisfy

\begin{align*}
(i) & \quad R_{XY}Z = \frac{f'^2 + k}{f^2} (\langle X, Z \rangle Y - \langle Y, Z \rangle X) \\
(ii) & \quad R_{XU}U = \frac{f''}{f} X \\
(iii) & \quad R_{XY}U = 0 \\
(iv) & \quad R_{XU}Y = \frac{f''}{f} \langle X, Y \rangle U
\end{align*}

where $f''$ is given by

\begin{equation}
\begin{aligned}
f'' &= \left( f^{(n)''} - f^{(n-1)''} + \frac{1}{n} \sum_{k=0}^{n-1} f^{(k)''} \right) \mu(t - t_n) \\
&\quad + \sum_{l=1}^{n-1} \left( -f^{(l-1)''} + \frac{1}{n} \sum_{k=0}^{n-1} f^{(k)''} \right) \mu(t - t_l) \\
&\quad + \frac{1}{n} \sum_{k=0}^{n-1} f^{(k)''} \mu(t_n - t) + \sum_{l=1}^{n-1} \left( -f^{(l)''} + \frac{1}{n} \sum_{k=0}^{n-1} f^{(k)''} \right) \mu(t_l - t),
\end{aligned}
\end{equation}

with $\mu(t - t_i)$ being the unit step function which becomes unity for $t > t_i$ and vanishes otherwise.

**Proof.**

We derive $f''$ in terms of the collection of functions $\{f^{(i)}\}$ with $f^{(i)}$ piecewisely defined on the intervals $t_i \leq t \leq t_{i+1}$ ($i = 0, 1, 2, ..., n$) with $t_{n+1} = t_\infty$. For a single discontinuity $n = 1$ case, $f$ is trivially given by

$$f'' = f^{(1)''} \mu(t - t_1) + f^{(0)''} \mu(t_1 - t)$$
which fulfills (1). For double discontinuities $n = 2$ case, $f$ is similarly given by

$$ f'' = \left( f^{(2)\nu} - \frac{1}{2} f^{(1)\nu} + \frac{1}{2} f^{(0)\nu} \right) \mu(t - t_2) + \left( \frac{1}{2} f^{(1)\nu} - \frac{1}{2} f^{(0)\nu} \right) \mu(t - t_1) $$

(2)

$$ + \left( \frac{1}{2} f^{(1)\nu} + \frac{1}{2} f^{(0)\nu} \right) \mu(t_2 - t) + \left( -\frac{1}{2} f^{(1)\nu} + \frac{1}{2} f^{(0)\nu} \right) \mu(t_1 - t), $$

which also fulfills (1). By using iteration method, one can obtain (1) for an arbitrary $n$ case.

For the case of $f \in C^0(S)$ we use the derivative of the unit step function $\mu(t_i)$. For all $t \neq t_i$ this is well-defined, $\mu'(t) = 0$. However, at $t = t_i$ there exists a jump discontinuity so that we cannot define classical derivative and thus we use the $\delta$-function, $\mu'(t - t_i) = \delta(t - t_i)$ to obtain the follow results.

**Proposition 3.2** Let $M = M_0 \times fH$ be the FRW spacetime with Riemannian curvature $R$ and flow vector field $U = \partial_t$. If $f \in C^0(S)$, vector fields $X, Y, Z \in \mathcal{L}(H)$ then satisfy

(i) $R_{XY}Z = \frac{f'^2 + k}{f^2}((X, Z)Y - (Y, Z)X)$

(ii) $R_{XU}U = \frac{f''}{f}X$

(iii) $R_{XY}U = 0$

(iv) $R_{XU}Y = \frac{f''}{f}(X, Y)U$

where $f'$ and $f''$ are given by

$$ f' = \left( f^{(n)\nu} - f^{(n-1)\nu} + \frac{1}{n} \sum_{k=0}^{n-1} f^{(k)\nu} \right) \mu(t - t_n) $$

$$ + \sum_{l=1}^{n-1} \left( -f^{(l-1)\nu} + \frac{1}{n} \sum_{k=0}^{n-1} f^{(k)\nu} \right) \mu(t - t_l) $$

(3)

$$ + \frac{1}{n} \sum_{k=0}^{n-1} f^{(k)\nu} \mu(t_n - t) + \sum_{l=1}^{n-1} \left( -f^{(l)\nu} + \frac{1}{n} \sum_{k=0}^{n-1} f^{(k)\nu} \right) \mu(t_l - t) $$

$$ + \sum_{l=1}^{n-1} \left( -f^{(l)\nu} + \frac{1}{n} \sum_{k=0}^{n-1} f^{(k)\nu} \right) \mu(t_l - t) $$

$$ + \frac{1}{n} \sum_{k=0}^{n-1} f^{(k)\nu} \mu(t_n - t) + \sum_{l=1}^{n-1} \left( -f^{(l)\nu} + \frac{1}{n} \sum_{k=0}^{n-1} f^{(k)\nu} \right) \mu(t_l - t) $$

$$ + \left( f^{(n)\nu} - f^{(n-1)\nu} \right) \delta(t - t_n) + \sum_{l=1}^{n-1} \left( f^{(l)\nu} - f^{(l-1)\nu} \right) \delta(t - t_l), $$

(4)
with \( \mu(t - t_i) \) and \( \delta(t - t_i) \) being the unit step function and the delta function, respectively.

**Proof.**

Similar to (1) in Proposition 3.1, one can readily obtain \( f' \). Differentiating \( f' \) with respect to \( t \) and using the definition of the delta function \( \mu'(t - t_i) = \delta(t - t_i) \) at \( t = t_i \), one can also obtain \( f'' \).

**Proposition 3.3** Let \( M = M_0 \times fH \) be the FRW spacetime with Riemannian curvature \( R \) and flow vector field \( U = \partial_t \). If \( f \in C^0(S) \) and \( X, Y \in \mathfrak{L}(H) \), then Ricci curvature is given by

\[
(i) \quad \text{Ric}(U, U) = \frac{3f''}{f} \\
(ii) \quad \text{Ric}(U, X) = 0 \\
(iii) \quad \text{Ric}(X, Y) = \left(\frac{2(f'r^2 + k)}{f^2} + \frac{f''}{f}\right) \langle X, Y \rangle, \quad \text{if} \quad X, Y \perp U
\]

where \( f' \) and \( f'' \) are given by (3) and (4).

**Proposition 3.4** Let \( M = M_0 \times fH \) be the FRW spacetime with Riemannian curvature \( R \) and flow vector field \( U = \partial_t \). If \( f \in C^0(S) \), the Einstein scalar curvature is given by

\[
R = 6 \left(\frac{f'r^2}{f^2} + \frac{f''}{f} + \frac{k}{f^2}\right),
\]

where \( f' \) and \( f'' \) are given by (3) and (4).

**Proposition 3.5** For every plane containing a vector field of \( U = \partial_t \), if \( f \in C^0(S) \) and \( X, Y \in \mathfrak{L}(H) \), we have a sectional curvature \( K \) on the spacetime \( (M, g) \) for an arbitrary plane containing a vector field of \( U = \partial_t \) and \( W = \alpha U + \beta Y \)

\[
K(W, X) = \frac{-\alpha^2f'' + \beta^2(f' + k)}{(-\alpha^2 + \beta^2)f^2}
\]

where \( f' \) and \( f'' \) are given by (3) and (4).

**Proof.**

The result is follows from \( K(W, X) = \frac{g(Rw, X)}{g(W, W)g(X, X) - [g(W, X)]^2} \) of the nondegenerate 2-plane with basis \( (W, X) \).

**IV. Cosmology of spatially flat FRW metric with double discontinuities**

In the spatially flat FRW cosmology with \( k = 0 \), the early universe was radiation dominated, the adolescent universe was matter dominated, and the present universe is now entering into lambda-dominate phase in the absence of vacuum energy. If the universe underwent inflation, there was a very early period when the stress-energy was dominated by vacuum energy. The Friedmann equation may be integrated to give the age of the universe in terms of present cosmological parameters. We have the scale factor \( f \) as a function of time \( t \) which scales as \( f(t) \propto t^{1/2} \) for a radiation-dominated (RD) universe, and scales as \( f(t) \propto t^{2/3} \) for a matter-dominated (MD) universe, and scales as \( f(t) \propto e^{Kt} \) for a lambda-dominated (LD) universe. Note that the transition from the radiation-dominated phase to the matter-dominated is
For vector fields \( X \) of Lorentzian signature such that \( C_f \) note that in the spatially flat FRW model, (6) with the boundary conditions \( f \) to yield \( R \) to \( U \) and \( \sigma \), flow vector field \( \mathcal{K} \). Moreover \( K_{c_1} \) products. Let \( M = M_0 \times f H \) be a warped product with \( g_{M_0} = -dt^2 \).

**Definition 4.1** A \( C^0 \)-Lorentzian metric on \( M \) is a nondegenerate \((0,2)\) tensor of Lorentzian signature such that

\[
\begin{align*}
(i) & \quad g \in C^0 \text{ on } S \\
(ii) & \quad g \in C^\infty \text{ on } M \cap S^c \\
(iii) & \quad \text{for all } p \in S, \text{ and } U(p) \text{ partitioned by } S, \ g|_{U^+} \text{ and } g|_{U^-} \text{ have smooth extensions to } U. \text{ We call } S \text{ a } C^0 \text{-singular hypersurface of } (M,g).
\end{align*}
\]

Consider \( M_0 \) as a \( C^0 \)-singular hypersurface of \((M,g)\). In the spatially flat FRW spacetime, \( f > 0 \) is smooth functions on \( M_0 = (t_0, t_\infty) \) except at \( t \neq t_i \) \((i = 1, 2)\), that is \( f \in C^\infty(S) \) \((S = \{t_i\} \times f H)\) for \( t \neq t_i \) and \( f \in C^0(S) \) at \( t = t_i \in M_0 \) to yield

\[
f = \begin{cases} 
    f^{(0)} = c_0 t_1^{1/2}, & \text{for } t < t_1 \\
    f^{(1)} = c_1 t_2^{2/3}, & \text{for } t_1 \leq t \leq t_2 \\
    f^{(2)} = c_2 e^{Kt}, & \text{for } t > t_2 
\end{cases}
\]

with the boundary conditions \( c_0 t_1^{1/2} = c_1 t_2^{2/3} = c_2 e^{Kt} \).

Experimental values for \( t_1 \) and \( t_2 \) are given by \( t_1 = 4.7 \times 10^4 \) yr and \( t_2 = 9.8 \) Gyr \([15]\). Moreover \( c_1 \) and \( c_2 \) are given in terms of \( c_0 \) and \( t_1 \) and \( t_2 \) as follows

\[
c_1 = c_0 t_1^{-1/6}, \quad c_2 = c_0 t_1^{-1/6} t_2^{2/3} e^{-Kt_2}.
\]

Note that in the spatially flat FRW model, \( f \in C^0(S) \) since if we assume \( f \in C^1(S) \) one could have the boundary conditions \( \frac{1}{3} c_0 t_1^{-1/2} = \frac{2}{3} c_1 t_1^{-1/3} \) and \( \frac{1}{3} c_1 t_2^{-1/3} = K c_2 e^{Kt_2} \), which cannot satisfy the above boundary conditions \([15]\) simultaneously.

**Proposition 4.2** Let \( M = M_0 \times H \) be the spatially flat FRW spacetime with Riemannian curvature \( R \), flow vector field \( U = \partial_t \) and warping function \( f \in C^0(S) \). For vector fields \( X, Y, Z \in \mathfrak{L}(H) \) we have

\[
\begin{align*}
(i) & \quad R_{XY}Z = \frac{f'^2}{f^2}((X, Z)Y - (Y, Z)X) \\
(ii) & \quad R_{XU}U = \frac{f''}{f} X \\
(iii) & \quad R_{XY}U = 0 \\
(iv) & \quad R_{XU}Y = \frac{f''}{f} (X, Y)U
\end{align*}
\]
where $f$ is given by \[5\] and $f'$ and $f''$ are given by

\[
\begin{align*}
  f' &= \left(\frac{1}{4}c_0 t^{-1/2} - \frac{1}{3}c_1 t^{-1/3} + Kc_2 e^{Kt}\right) \mu(t - t_2) \\
  &\quad + \left(-\frac{1}{4}c_0 t^{-1/2} + \frac{1}{3}c_1 t^{-1/3}\right) \mu(t - t_1) \\
  &\quad + \left(\frac{1}{4}c_0 t^{-1/2} + \frac{1}{3}c_1 t^{-1/3}\right) \mu(t_2 - t) \\
  &\quad + \left(\frac{1}{4}c_0 t^{-1/2} - \frac{1}{3}c_1 t^{-1/3}\right) \mu(t_1 - t) \\
\end{align*}
\]

and

\[
\begin{align*}
  f'' &= \left(-\frac{1}{8}c_0 t^{-3/2} + \frac{1}{9}c_1 t^{-4/3} + K^2 c_2 e^{Kt}\right) \mu(t - t_2) \\
  &\quad + \left(\frac{1}{8}c_0 t^{-3/2} - \frac{1}{9}c_1 t^{-4/3}\right) \mu(t - t_1) \\
  &\quad + \left(-\frac{1}{8}c_0 t^{-3/2} - \frac{1}{9}c_1 t^{-4/3}\right) \mu(t_2 - t) \\
  &\quad + \left(-\frac{1}{8}c_0 t^{-3/2} + \frac{1}{9}c_1 t^{-4/3}\right) \mu(t_1 - t) \\
  &\quad + \left(-\frac{2}{3}c_1 t^{-1/3} + Kc_2 e^{Kt}\right) \delta(t - t_2) \\
  &\quad + \left(-\frac{2}{3}c_1 t^{-1/3}\right) \delta(t - t_1),
\end{align*}
\]

with $\mu(t - t_i)$ and $\delta(t - t_i)$ being the unit step function and the delta function, respectively.

**Proof.**
Substituting $f$ in \[5\] into \[3\] and \[4\] in Proposition 3.2, one can readily obtain \[7\] and \[8\].

**Proposition 4.3** Let $M = M_0 \times H$ be the spatially flat FRW spacetime with Riemannian curvature $R$, flow vector field $U = \partial_t$ and warping function $f \in C^0(S)$. For vector fields $X, Y, Z \in \mathfrak{L}(H)$, the Ricci curvature is given by

\[
\begin{align*}
  (i) \quad &\text{Ric}(U, U) = -\frac{3f''}{f} \\
  (ii) \quad &\text{Ric}(U, X) = 0 \\
  (iii) \quad &\text{Ric}(X, Y) = \left(\frac{2f^2}{f^2} + \frac{f''}{f}\right) \langle X, Y \rangle, \text{ if } X, Y \perp U
\end{align*}
\]

where $f$, $f'$ and $f''$ are given by \[5\], \[7\] and \[8\], respectively.

**Proposition 4.4** Let $M = M_0 \times H$ be the spatially flat FRW spacetime with Riemannian curvature $R$, flow vector field $U = \partial_t$ and warping function $f \in C^0(S)$. The Einstein scalar curvature is then given by

\[
R = 6 \left(\frac{f'^2}{f^2} + \frac{f''}{f}\right),
\]

where $f$, $f'$ and $f''$ are given by \[5\], \[7\] and \[8\], respectively.
**Proposition 4.5** For every plane containing a vector field of \( U = \partial_t \) and \( f \in C^0(S) \), if \( X, Y \in \Sigma(H) \) we have a sectional curvature \( K \) on the FRW spacetime \((M, g)\) for an arbitrary plane containing a vector field of \( U = \partial_t \) and \( W = \alpha U + \beta Y \)

\[
K(W, X) = \frac{-\alpha^2 f'' + \beta^2 f'}{(-\alpha^2 + \beta^2) f^2}
\]

where \( f, f' \) and \( f'' \) are given by (5), (7) and (8), respectively.

**Proposition 4.6** Let \( M = M_0 \times H \) be the spatially flat FRW spacetime with Riemannian curvature \( R \), flow vector field \( U = \partial_t \) and warping function \( f \in C^0(S) \).

The evolution equations are then given by

\[
(i) \quad 3f'^2 = 8\pi \rho + \Lambda \\
(ii) \quad 3f'' = -4\pi(\rho + 3P) + \Lambda,
\]

where \( f, f' \) and \( f'' \) are given by (5), (7) and (8), respectively. Here \( \rho, P \) and \( \Lambda \) are the mass density and pressure of matter and the cosmological constant.

**Proof.**

Consider the Einstein equation

\[
G_{\mu\nu} + \Lambda g_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}
\]

where \( G_{\mu\nu} \) is the Einstein tensor, and \( T_{\mu\nu} \) is the stress-energy tensor for all the field present-matter, radiation and so on. To be consistent with the symmetries of the metric, the total stress-energy tensor \( T_{\mu\nu} \) must be diagonal, and by isotropy the spatial components must be equal. The simplest realization of such a stress-energy tensor is that of a perfect fluid characterized by a time-dependent energy density \( \rho(t) \) and pressure \( p(t) \),

\[
T_{\mu\rho} = \text{diag}(\rho, -p, -p, -p).
\]

Substituting (10) into (9), together with the Ricci and Einstein curvatures given in Proposition 4.3 and Proposition 4.4, one can readily obtain the above evolution equations.

**Remarks 4.7** The \( \mu = 0 \) component of the conservation of stress-energy tensor, \( T^\mu_{\nu}\delta^\nu_\rho = 0 \), gives the first law of thermodynamics of the familiar form \( d(\rho f^3) = -p df(3f^3) \) or equivalently, \( d[f^3(\rho + p)] = f^3 dp \). The change in energy in a co-moving volume element, \( d(\rho f^3) \), is equal to minus the pressure times the change in volume element, \( -pd(f^3) \). For the simple equation of state \( p = \omega \rho \), where \( \omega \) is independent of time, the energy density evolves as \( \rho \propto f^{-3(1+\omega)} \). Examples of interest include: radiation \( (p = \frac{1}{3} \rho, \rho \propto f^{-4}) \), matter \( (p = 0, \rho \propto f^{-3}) \) and vacuum energy \( (p = -\rho, \rho \propto \text{const.}) \) phases.

V. Conclusions
We have considered the FRW cosmological model in the warped product scheme to investigate the non-smooth curvature associated with the multiple discontinuities involved in the evolution of the universe. In particular we have analyzed the non-smooth features of the spatially flat FRW universe by introducing double discontinuities occurred at the radiation-matter and matter-lambda phase transitions in the astrophysical phenomenology.

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