The meaning of ”anomalous weak values” in quantum and classical theories

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The readings of a highly inaccurate ”weak” quantum meter, employed to determine the value of a dichotomous variable \( S \) without destroying the interference between the alternatives, may take arbitrary values. We show that the expected values of its readings may take any real value, depending on the choice of the states in which the system is pre- and post-selected. Some of these values must fall outside the range of eigenvalues of \( A \), in which case they may be expressed as ”anomalous” averages obtained with negative probability weights, constructed from available probability amplitudes. This behaviour is a natural consequence of the Uncertainty Principle. The phenomenon of ”anomalous weak values” has no non-trivial analogue in classical statistics.

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I. INTRODUCTION

The interest in the so-called 'weak measurements' began with the publication of Ref. [1] entitled "How the result of a measurement of a component of the spin of a spin-1/2 particle can turn out to be 100". Recently the argument was extended to purely classical domain in [2], where the authors set to explain "How can a result of a single coin toss turn out to be 100 heads". The questions of this type invite two possible answers: either the measurement is not particularly good, or an error has been made in the analysis. Below we will show that the former is true in the case of [1], and the latter - in the case of Ref. [2]. A hint of what happens in the quantum case can be taken from D. Bohm’s warning [3] that "if the interference were not destroyed", "the quantum theory could be shown to lead to absurd results". The claim that a simple classical model may exhibit non-trivial anomalous weak values [2] is, on the other hand, based on a simple misunderstanding.

Other critique of Ref. [2] can be found in [4], and we refer the interested reader to the Refs. in the Dressel’s Comment [4] for some further developments in the field of quantum weak measurements.

II. 'NORMAL' AND 'ANOMALOUS' AVERAGES

Consider an average of the form

\[ \bar{s} = \sum_{n=1}^{N} s_n P_n, \quad \sum_{n} P_n = 1 \]  

where \( s_1 > s_2 \ldots > s_N \). We will call \( \bar{s} \) normal if it lies between \( s_1 \) and \( s_N \), \( s_1 \geq \bar{s} \geq s_N \), and anomalous otherwise. It is readily seen that \( \bar{s} \) is always normal if \( P_n \geq 0 \), \( n = 1, 2, \ldots N \), and to be anomalous it requires that at least one of the \( P_n \) is negative. To see how an anomalous average may be produced, consider \( N = 2 \), \( s_{1,2} = \pm 1 \), \( P_1 = 1001 \), \( P_2 = -1000 \), \( P_1 + P_2 = 1 \), and find that \( \bar{s} = 2001 \). Thus, a large anomalous value would occur where the moduli of \( P_n \) are large, but the sum of all \( P_n \) is unity due to a very precise cancellation. Multiplying the \( P_n \)'s by \( s_n \) destroys the cancellation, so that the resulting \( \bar{s} \) is unduly large. A more detailed example is given in [3], where a similar effect is found responsible for what appears to be super-luminal transmission of a wave packet across a potential barrier. Next we discuss in detail the appearance of such alternating distributions in quantum measurement theory.
FIG. 1. Two virtual routes connecting initial and final states $|\psi\rangle$ and $|\phi\rangle$. A meter acts to destroy the interference between them.

III. QUANTUM MEASUREMENTS WITH POST-SELECTION

Consider a two level quantum system (a spin $1/2$) with a hamiltonian $H$, pre- and post-selected (observed) in some states $\psi$ and $\phi$ at $t = 0$ and $t = T$, respectively. Choose an operator $S$ with eigenstates $|s_{1,2}\rangle$ and eigenvalues $s_{1,2}\pm 1$, e.g., the $z$-component of the spin, $S = \sigma_z$. Inserting, at some $0 < t' < T$ the unity $I = |s_1\rangle\langle s_1| + |s_2\rangle\langle s_2|$ into the transition amplitude $\langle \phi|\exp(-iHT)|\psi\rangle$, shows that the spin can reach the final state via two virtual routes, $\psi \rightarrow s_1 \rightarrow \phi$ and $\psi \rightarrow s_2 \rightarrow \phi$, as shown in Fig.1. Putting for simplicity $H = 0$, for the two corresponding amplitudes we have

$$A_{1,2} = \langle \phi|s_{1,2}\rangle\langle s_{1,2}|\psi\rangle. \quad (2)$$

To see what actually happens at $t'$ we may employ a von Neumann pointer with the position $f$, initially decoupled from the spin. We set the pointer at some $f'$ by preparing it in a state $|M_{f'}\rangle = \int df G(f-f')|f\rangle$, where $G(f)$ is a function peaked around 0 with a width $\Delta f$, such that $\int |G(f)|^2 = 1$. For $t' \leq t'' \leq t' + \tau < T$ the pointer briefly interacts with the spin via $H_{\text{int}} = -i\tau^{-1}\partial_f S$, and then its final position is measured (read) exactly.

We can do the measurement in three steps [6]. First the meter acts, after which the entangled state of the system becomes

$$|\Phi\rangle = \int \left[G(f-s_1)\langle s_1|\psi\rangle|s_1\rangle + G(f-s_2)\langle s_2|\psi\rangle|s_2\rangle\right]|f\rangle df. \quad (3)$$

Then the pointer’s reading is found to be $f$, which leaves the spin polarised in along some axis in an (unnormalised) pure state $\langle f|\Phi\rangle$. Finally, in this state we measure the projector on $|\phi\rangle$, and keep the results only if the projection is successful, which happens with a probability $|\langle \phi|\langle f|\Phi\rangle|^2$. For the (unnormalised) probability of a reading $f$ in our pre- and post-selected
setup we have

\[ P(f|\phi \leftarrow \psi, \Delta f) = |B_1(f) + B_2(f)|^2 \]

\[ |B_1(f)|^2 + |B_2(f)|^2 + 2\text{Re}[B_1(f)B_2^*(f)] , \]

where \( B_{1,2} = G(f - s_{1,2})A_{1,2} \). Choosing, with no loss of generality, a Gaussian pointer,

\[ G(f) = (2/\pi\Delta f^2)^{1/4} \exp(-f^2/\Delta f^2), \]

for the expected value of the pointer’s reading, \( \bar{f} \) we find

\[ \bar{f} \equiv \int fP(f|\phi \leftarrow \psi, \Delta f)df = \{s_1|A_1|^2 + s_2|A_2|^2 \]

\[ + \text{Re}[A_1A_2^*](s_1 + s_2) \exp[-(s_1 - s_2)^2/2\Delta f^2]\}/N \]

\[ N \equiv |A_1|^2 + |A_2|^2 + 2\text{Re}[A_1A_2^*] \exp[-(s_1 - s_2)^2/2\Delta f^2]. \]

Equation (6), expresses \( \bar{f} \) in terms of the parameters which describe the measured variable and the transition, \( A_{1,2} \) and \( s_{1,2} \). It does not have the form (1), except it two special cases. We consider these cases next.

IV. ACCURATE "STRONG" MEASUREMENTS

Now \( \Delta f \) determines what is known about the initial position of the pointer and, therefore, the accuracy of the measurement. By sending \( \Delta f \to 0 \) we make the pointer position correlate exactly with the eigenvalues of \( S \), \( s_{1,2} \), so that

\[ P(f|\phi \leftarrow \psi, \Delta f \to 0) = \]

\[ |A_1|^2\delta(f - s_1) + |A_2|^2\delta(f - s_2). \]

With no overlap between \( G(f - s_1) \) and \( G(f - s_2) \), finding a reading \( f = s_i \) leaves the spin in the state \( |s_i\rangle, i = 1, 2 \). The average meter reading (6) now has the form (1),

\[ \bar{f} = s_1P_{1}^{\text{strong}} + s_2P_{2}^{\text{strong}} \equiv \bar{f}_{\text{strong}}, \]

where

\[ P_i^{\text{strong}} = \frac{|A_i|^2}{|A_1|^2 + |A_2|^2}, \quad i = 1, 2 \]

are non-negative probability weight. Thus, expressed in terms of \( A_{1,2} \) and \( s_{1,2} \), \( \bar{f}_{\text{strong}} \) is always a "normal" average. This is just a way of saying that, for all \( \psi \) and \( \phi \), the average reading of an accurate meter always lies between \( s_1 \) and \( s_2 \).
V. A 'CLASSICALLY' INACCURATE METER

Suppose next that we still have an accurate 'strong' meter with $\Delta f \ll s_1 - s_2$, but for some reason are unable to set it precisely to zero. Rather, initially the pointer reads $f'$ with a probability $W(f') = W(-f')$, in a range of a width $\delta f'$ around zero. The initial state is now mixed,

$$\rho_M = \int |M_{f'}\rangle W(f')\langle M_{f'}| df', \quad \int W(f')df' = 1,$$

and finding a final reading $f$ leaves the spin in a mixed state

$$\rho_{\text{final spin}} = |s_1\rangle W(f - s_1)\langle s_1| + |s_2\rangle W(f - s_2)\langle s_2|.$$

For $\delta f' >> s_1 - s_2$, the pointer’s final readings are distributed with the probability $P(f|\phi \leftarrow \psi, \Delta f' = \int P(f - f'|\phi \leftarrow \psi, \Delta f)W(f')df'$ over a broad range $\sim \delta f'$. The meter appears to have lost correlation with the eigenvalues $s_{1,2}$. Yet some information about them may be recovered, provided one is only interested in average values. Thus, for $\delta f' >> s_1 - s_2$ and $\Delta f << s_1 - s_2$, the average pointer position is still given by Eq. (8)

$$\bar{f} = \int f P(f|\phi \leftarrow \psi, \Delta f \to 0)df = \bar{s}_{\text{strong}}.$$

No matter how large the classical uncertainty, the mean pointer reading would still lie between $s_1$ and $s_2$. A different result is achieved if the initial pointer position is made uncertain in the quantum sense.

VI. HOW 'NEGATIVE PROBABILITIES' ENTER QUANTUM MEASUREMENT THEORY

Next consider a pointer, prepared initially in a very broad pure state, $\Delta f \to \infty$. The accuracy of the measurement is very low, since the initial pointer position is highly uncertain, albeit in a different sense. There is only a probability amplitude, $G(f)$, and not the probability, for it to be set to a particular $f$. As in Sect. V, the probability distribution of pointer’s reading is very broad, so that $\bar{f}$ can, in principle, take any values as $\Delta f \to \infty$.

With both exponentials in Eq.(6) tending to unity, for $\bar{f}$ we have

$$\bar{f} = s_1 P_{1}^{\text{weak}} + s_2 P_{2}^{\text{weak}} \equiv \text{Re}\bar{s}_{\text{weak}}.$$

\[\text{(13)}\]
where
\[
P_{\text{weak}}^{1/2} = \text{Re} \frac{A_{1,2}}{A_1 + A_2},
\]
\[
\bar{s}_{\text{weak}} = \frac{(s_1 A_1 + s_2 A_2)}{(A_1 + A_2)}
\]
is quantum mechanical weak value of the operator $S$, which can take complex values (for a recent discussion see, for example \[8\]). For $P_{\text{weak}}^{1/2}$ having opposite opposite signs, $\text{Re}\bar{s}_{\text{weak}}$ in Eq.(13) has the appearance of an ”anomalous” average obtained with negative probability weights \[9\]. This is an elaborate way to say that the average reading of a meter, highly inaccurate in the quantum sense, is not apriori restricted in its magnitude.

\vspace{1cm}

\textbf{VII. ANOMALOUS VALUES AND THE UNCERTAINTY PRINCIPLE}

One purpose of this paper is to establish why the meters in Sect. V and Sect. VI behave so differently. Suppose that one chooses a transition, and after series of weak measurements finds a mean value of 100 for a spin of $1/2$. We may suspect that this is a ’wrong’ result obtained with a malfunctioning meter. The problem is, we cannot produce the ’correct’ answer, and a brief look into quantum mechanical text book shows that it may not even exist.

Equation (4) suggests that we are dealing with a simple version of Young’s two-slit experiment. The pointer ”arrives” to a ’point on the screen” $f$ by passing through ”two slits”, corresponding to spin values of 1 and $-1$, with the probability amplitudes $B_1(f)$ and $B_2(f)$. The problem is well known in literature \[10\]. A strong measurement of Sects. IV and V destroys the interference between the paths, converting two virtual routes into two real ones, to each of which one can now ascribe a probability. A weak measurement of Sect.VI leaves the interference intact, and the probabilities in Eq.(4) contain an interference term, which involves both virtual routes at the same time (cf. Fig.2). One’s inability to refer the interference term to any one route is reflected in the Uncertainty Principle, which states that two interfering routes cannot be told apart and should be considered a single pathway \[10\].

In our example, different values of the spin’s components label different virtual paths in Fig. 1, and the mean value of $S$ may tell us something about how these paths are travelled. Obtaining in a series of strong measurements $\bar{s}_{\text{strong}} > 0$ allows to conclude that the lower route in Fig.1 is travelled more frequently, and vice versa. No such conclusion can be drawn
if the measurements are weak. If by asking "what was, on average, the value of $S$ if we hadn't destroyed coherence between the two routes?" one hoped to learn something about which path was actually travelled, he must be disappointed. The bizarre mean value of 100 stands as a reminder that indivisible cannot, after all, be divided [11].

Still, this result needs to be interpreted, and there is a choice: to follow Bohm [3] in discarding it as "absurd", or to follow the authors of [1] in trying to ascribe to it a degree of importance and "reality", simply because such a measurement can be made? In the next Section we will extend Bohm’s argument in favour of strong measurements, and look at all, rather than just one, possible transitions.

VIII. AN ANSWER TO THE AWKWARD QUESTION

Usually it is accurate 'strong' measurements, which provide one with useful information. To know whether two or three holes have been actually cut in the screen, we shine on it a light of wave-length short enough (a strong measurement) to produce just two, and not three, bright spots behind it. Shining light of a very large wave length (a weak measurement) would produce a very broad spot, whose centroid [cf. Eq.(12)] may lie far away from the two bright spots observed previously. Based on this, it would be hard to guess the number of holes actually made.

Similarly, to find out what sort of spin has an electron, one can devise a Stern-Gerlach
FIG. 3. Dependence of the mean meter reading $\bar{f}$ on the accuracy of the measurement, $\Delta f$ for $|\psi\rangle = |1\rangle + |2\rangle|/\sqrt{2}$. Final states $|\phi\rangle$ are chosen so that, as $\Delta f \to \infty$, $s_{\text{weak}} = -20, -21...0,...20$.}

experiment [3] with pre- and post-selection for the spin variable in the states $|\psi\rangle$ and $|\phi\rangle$,

$$
|\psi\rangle = \frac{|1\rangle + a|2\rangle}{\sqrt{1 + |a|^2}}, \quad |\phi\rangle = \frac{|1\rangle + b|2\rangle}{\sqrt{1 + |b|^2}}.
$$

For all choices of these states an accurate measurement of $\sigma_z$ would produce only the values of $\pm 1$. In all cases the average of $\sigma_z$ will lie between $-1$ and $1$. This is how one knows that spin of $1/2$ is an intrinsic property of the electron, and is later able to write its wave function as a Pauli spinor in situations much more general than the original Stern-Gerlach setup.

Much less can be learned about the electron if only inaccurate weak measurements are made. It is sufficient to consider real valued $a$ and $b$ in equation (15) to show that

$$
\bar{f} \equiv Re s_{\text{weak}} = (A_1 - A_2)/(A_1 + A_2)
$$

(16)
can have any real value $Z$, $-\infty \leq Z \leq \infty$, provided

$$
ab = (1 - Z)/(1 + Z),
$$

(17)
[see Fig. 3]. In particular, the choice $b = a$ yields a ”normal” $s_{\text{weak}} = \langle \psi | A | \psi \rangle$ which coincides with the expectation value of $A$ in the state $|\psi\rangle$. The choice $b = 1/a$ leads to $A_1 = A_2$, so that $s_{\text{weak}}$ coincides with $s_{\text{strong}}$, and also is ”normal”. Finally, for $b = -99/101a$ $|\psi\rangle$ and $|\phi\rangle$ are nearly orthogonal, and $s_{\text{weak}} = 100$ is an ”anomalous” average first obtained in [1]. Such values are properties of particular transitions, and tell very little about the electron’s own properties [12]. The answer to the question which, as we argued above should not have a meaningful answer, is, in this case, ”anything at all, depending on the circumstances”. 
IX. A CAT AND ITS SMILE

Let us illustrate the above conclusion by a more recent example [13]. Consider a system consisting of two 2-level systems, pre- and post-selected, as before, in yet unknown states $\psi$ and $\phi$. There altogether four orthogonal projectors, two for each degree of freedom: $\Pi_L = |L\rangle\langle L|$, $\Pi_R = |R\rangle\langle R|$, $s_+ = |+\rangle\langle +|$, $s_- = |-\rangle\langle -|$. Following [13] we will say that the system is on the left (right) if it is in the eigenstate of $\Pi_L (\Pi_R)$, and its spin is up (down) along the $z$-axis, if it is in an eigenstate of $s_+ (-s_-)$. With four possible intermediate measurements, there are four routes connecting $|\psi\rangle$ with $|\phi\rangle$, with the amplitudes $A_{ij} = \langle \phi | \Pi_i s_j | \phi \rangle$, $i = L, R$, $j = \pm$ [14]. There are also two spin operators, $\sigma^L_z = \Pi_L \sigma_z$ and $\sigma^R_z = \Pi_R \sigma_z$, $\sigma_z = s_+ - s_-$, for a system found on the left, and on the right respectively. With both operators measured accurately, the four routes are travelled with the probabilities $P^\text{strong}_{ij} = |A_{ij}|^2$ and, obviously, the found value of $\Pi_L \sigma_z$ is zero, if the system has chosen the right route, and vice versa.

The authors of [13] have demonstrated that this is no longer the case, if the measurements are weak, so that the interference between the routes is not destroyed. In particular, with the special choice of $\psi$ and $\phi$ it is possible to have $\Pi^\text{Lweak} = (A_{L+} + A_{L-})/\sum_{ij} A_{ij} = 1$, $\Pi^\text{Rweak} = (A_{R+} + A_{R-})/\sum_{ij} A_{ij} = 0$, $\sigma^L_{z\text{ weak}} = (A_{L+} - A_{L-})/\sum_{ij} A_{ij} = 1$, and $\sigma^R_{z\text{ weak}} = (A_{R+} - A_{R-})/\sum_{ij} A_{ij} = 1$. So the authors of [13] conclude that the system (the 'cat') is in one place, while its spin (its 'smile') is elsewhere.

Our interest here is to show that if we go over all possible transitions we will be able to find systems with all possible distributions of the spin between where the system is, and where it is not. To show this, we write

$$|\psi\rangle = [\alpha_1 |L\rangle + \alpha_2 |L\rangle - \alpha_3 |R\rangle + \alpha_4 |R\rangle]/\sqrt{\sum_{i=1}^4 |\alpha_i|^2};$$

$$|\phi\rangle = [\beta_1 |L\rangle + \beta_2 |L\rangle - \beta_3 |R\rangle + \beta_4 |R\rangle]/\sqrt{\sum_{i=1}^4 |\beta_i|^2};$$

and require that while the cat is weakly on the left side, its smile is distributed between left and right,

$$\Pi^\text{Lweak} = 1, \quad \Pi^\text{Lweak} = 0, \quad \sigma^L_{z\text{ weak}} = X, \quad \sigma^R_{z\text{ weak}} = Y,$$

where $X$ and $Y$ are any real numbers. A simple algebra shows that for Eqs. (19) to hold, $\alpha$'s and $\beta$'s in Eq. (18) may be chosen, for example, as

$$\alpha_1 = \beta_1 = 1, \quad \alpha_2 = \beta_2^* = \sqrt{(1 - X)/(1 + X)},$$

(20)
\[ \alpha_3 = \beta_3^* = \sqrt{Y/(1 + X)}, \quad \alpha_4 = \alpha_3 = -\beta_4^*. \]

As in the previous Section, this choice is not unique, and there are enough parameters in the wave functions \( \psi \) and \( \phi \), to make any choice of the weak values (19) possible. Thus Alice, having chosen \( \alpha_2 = \beta_2 = \alpha_3 = \beta_3 = 0 \), would find the smile firmly on the cat’s face, \( X = 1, Y = 0 \). Bob, working with \( \alpha_2 = \beta_2 = \alpha_3 = \beta_3 = 1 \), finds, as did the authors of [13], the smile completely detached from the cat, \( X = 0 \), and \( Y = 1 \). Finally Carol, choosing, \( \alpha_2 = \beta_2 = 0 \) and \( \alpha_3 = \beta_3^* = 100i \), finds a smile where the cat is, \( X = 1 \), but also a substantial "frown", \( Y = -200 \), where the cat isn’t. Together, Alice, Bob, Carol and co-workers must conclude that the only thing they learnt about a cat’s relation with its smile by making weak measurements, is that this relation may take any form at all. Bob and Carol may be surprised by their results, and it is the prime objective of our discussion to demonstrate that they shouldn’t be.

**X. THE WAYS TO MAKE A MEASUREMENT "WEAK"**

A brief remark is in order. Consider again the von Neumann Hamiltonian, this time with a adjustable parameter \( \lambda \),

\[ H_{\text{int}} = -i\lambda \frac{d}{df}S. \]

There are three equivalent ways to ensure that the meter perturbs the system as little as possible.

(A) Reduce the interaction strength, \( \lambda \to 0 \) and leave the initial meter state \( G(f) \) as it was, as was done in [I]. Hence the adjective "weak" widely used in this context.

(B) Put the coupling strength to unity, and rescale the meter’s position \( f \to f/\lambda \). As \( \lambda \to 0 \) the width of the pointer’s state increases, \( \Delta f \to \Delta f/\lambda \). The mean pointer position is then proportional to the weak value of the operator \( S \), \( \bar{f} \to \langle \phi | S | \psi \rangle / \langle \phi | \psi \rangle \). The value is "anomalous" should it lie outside the interval \([s_1, s_2]\). This is the convention we followed throughout this paper.

(C) Incorporate \( \lambda \) into the new operator \( S' \), \( S' = \lambda S \) with the eigenvalues \( \lambda s_{1,2} \) and send \( \lambda \to 0 \). The mean pointer position is then proportional to the weak value of the measured operator \( S' \), \( \bar{f} \to \lambda \langle \phi | S' | \psi \rangle / \langle \phi | \psi \rangle \). The value is "anomalous" should it lie outside the narrow interval \([\lambda s_1, \lambda s_2]\).
In all cases we achieve the essential condition for the ”weakness”: the spectrum of the measured operator ”fits” under the Gaussian $G(f)$, so the interference between the two pathways is not destroyed. It is important, however, to follow the chosen convention consistently, as we illustrate in the next Section.

XI. THERE ARE NO ANOMALOUS MEAN VALUES IN A CLASSICAL THEORY

For each measurement scheme, quantum mechanics does in the end produce a classical statistical ensemble with non-negative probabilities, so all quantum averages are of the normal type. Yet, in Sect. V it was shown that ’negative probabilities’ enter the theory if we try to rewrite the ”normal” mean position of a quantum pointer, which has lost correlation with the measured system, in terms of the variables describing the system and the transition. Quantum mechanics operates with probability amplitudes from which one can construct for Eq.(1) weights of either sign. The Uncertainty Principle requires some of the averages written in this way to be ”anomalous".

Classical statistics operates only with non-negative probabilities. There is no analogue of the Uncertainty Principle, and no quantities, similar to probability amplitudes, from which to construct negative weights in Eq.(1). Thus, no matter how elaborate a measurement scheme, no averages in a purely classical theory can be anomalous.

As an illustration, consider a strong measurement of an operator $S$ using the meter (21) with $\lambda = 1$. With the interference destroyed, the meter is classical, its outcomes are $\pm 1$, and the mean of its readings, $\bar{s}$, lie in the interval $[-1, 1]$. As was shown in Sect.V, making the initial position of such a meter uncertain, does not lead to anomalous values. There are, however, two trivial ways to obtain a mean outside this interval: (i) shift the origin for the pointer by $f_0$, $G(f) \to G(f - f_0)$; (ii) recalibrate the meter, so that a value $\alpha s$ is recorded instead of $s$, by choosing in (21) $\lambda = \alpha$. Obviously, the new mean values, $\bar{s}' = \bar{s} - f_0$ and $\bar{s}'' = \alpha \bar{s}$ should not be considered 'anomalous means' of the operator $S$, but rather the 'normal' means of $S + f_0$ and $\alpha S$, respectively.
FIG. 4. A coin, which may be flipped in transit, is passed from Alice to Bob and then returned to Alice, Bob keeps the statistics only if Alice receives it heads up.

XII. AN EXAMPLE

We conclude by briefly reviewing the original proposal in Ref. [2]. In the scheme illustrated in Fig. 4, Alice sends Bob a coin heads up which, on arrival, may change side with a probability $\alpha \geq 0$. Bob records the result (1 for heads and -1 for tails) on a piece of paper, and then sends the coin back to Alice. On its way the coin may again change the side, this time with a probability $1 - \delta \geq 0$, and Alice tells Bob to keep his note only if the coin arrives to her heads up. The authors of [2] claim that the mean recorded by Bob is $1/(1 - \delta)$, so that for $\delta = 0.99$ ”outcome of the toss coin is 100 heads”. This, however, is not possible, for the simple reason that adding the 1’s and -1’s, and dividing the sum by the number of notes he has kept, Bob would always get a result $a_w$ between -1 and 1. Equivalently, $a_w$ is a normal average [1] obtained with non-negative weights $P_1 = (1 - \alpha)\delta/[(1 - \alpha)\delta + (1 - \delta)\alpha]$ and $P_2 = \alpha(1 - \delta)/[(1 - \alpha)\delta + (1 - \delta)\alpha]$ and, therefore, must be contained between -1 and 1. The only way for Bob to obtain the average of 100 is to resort to one of the possibilities outlined in the previous Section. Indeed, the authors of [2] ”recalibrate” Bob, making him calculate the mean not of s, but of s divided by a small parameter $\lambda$, $s' = s/\lambda$ [cf. Eq. (19) of [2]]. It is easy to check that in this way they obtain a perfectly normal average of a larger variable $S/\lambda$. Thus, it may be concluded that a result of a single toss of a coin is 100 provided the values in excess of 100, say, ±200 are attributed to the two sides of the coin. The suggestion that anomalous weak values are ”a purely statistical feature of pre- and post-tselection with disturbance” is, however, wrong [15].
XIII. CONCLUSIONS

Some of the confusion (see, e.g.,[2]) currently surrounding the issue of weak values, arises from the interpretation the authors of [1] have given to their original result. It is, therefore, worth looking at the issue from a different prospective. A value of "100" obtained for a spin of 1/2 appears "surprising" if one expects the observed values to be limited to the eigenvalue range of the measured operator \(A\). In the weak limit, this expectation is, however, unfounded. With its back action minimised, a meter is trying to distinguish between two routes combined into one by quantum interference. Faced with a question which should have no answer, one has two possibilities. One is to refuse to give an answer, a luxury not permitted to an experimental setup. The other is to give an answer containing no information about the studied object, since the relevant information does not exist. Thus, for various transitions, a weak quantum measurement produces results both 'normal' and 'absurd' in the sense of Bohm [3], in fact, any results at all. The anomalous expectation values are unique to quantum mechanics, and cannot arise in a purely classical theory, except in the trivial sense, as explained in Sect. X.

XIV. ACKNOWLEDGEMENTS:

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[5] D. Sokolovski and E. Akmatskaya, Ann. Phys. 339, 307 (2013).
[6] Often one chooses to first post-select the spin in its final state, and then read the meter. Since the two operations commute, their order is of no importance. Our choice serves to emphasise
the fact that an inaccurate meter leaves the spin in a rather complicated linear combination of the up- and down-states.

[7] D. Sokolovski, Phys. Rev. A, 76, 042125 (2007).

[8] A. Hosoya and Y. Shikano, J. Phys. A: Math. Theor. 43, 385307 (2010).

[9] It is worth noting that the 'improper' distribution (14) doesn’t fully replace $P^{\text{strong}}_{1,2}$ as $\Delta f \to 0$. While for a accurate strong measurement we have $\bar{f}^2 = s_1^2 P^{\text{strong}}_1 + s_2^2 P^{\text{strong}}_2$, $\bar{f}^2$ for an inaccurate weak meter does not have such simple form [7].

[10] R. P. Feynman, R. Leighton, and M. Sands, The Feynman Lectures on Physics III (Dover Publications, Inc., New York, 1989), Sect. 22.8, p.600

[11] One can easily imagine a "bad" question in a purely classical context. Suppose there are two water drops on a surface. One puts a grain of sand into the first drop, and then tilts the surface so the two drops coalesce, with the grain of sand still inside. In which of the two drops is it now? Under the new circumstances the question has no meaningful answer.

[12] The argument is easily extended to a Hilbert space with dimension $N > 2$, in which the measured quantity $S$ is represented by an $N \times N$ matrix.

[13] Y. Aharonov, S. Popescu, D. Rohrlich, and P. Skrzypczyk, New J. Phys., 15, 113015 (2013).

[14] For a similar analysis of the 3-box case and teh Hardy’s paradox, see D. Sokolovski, I. Puerto Gimnez and R. Sala Mayato, Phys. Lett. A, 372, 6578 (2008) and ibid, 372, 3784 (2008).

[15] In Phys. Rev. A, 85, 022123 (2012) J. Dressel and N.A. Jordan considered a scheme similar to the one shown in Fig. 4 in which an inaccurate classical detector, set to measure a variable $s = \pm 1$, records, with a certain probability, the value $-1(1)$, given the input 1 ($-1$). The authors show that to obtain the same mean reading with an ideal accurate detector, the measured variable must take 'contextual' values $\pm 25$. This observation is not directly relevant to our discussion of anomalous means, since for both the inaccurate and the imagined accurate detector the mean reading $\bar{s}$ is normal, lying within the range $[-1, 1]$. To put $\bar{s}$ outside the range of contextual values the detector must err with a negative probability, which, of course, is not possible.