Holographic two-point functions for 4d log-gravity

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ABSTRACT: We compute holographic one- and two-point functions of critical higher-curvature gravity in four dimensions. The two most important operators are the stress tensor and its logarithmic partner, sourced by ordinary massless and by logarithmic non-normalisable gravitons, respectively. In addition, the logarithmic gravitons source two ordinary operators, one with spin-one and one with spin-zero. The one-point function of the stress tensor vanishes for all Einstein solutions, but has a non-zero contribution from logarithmic gravitons. The two-point functions of all operators match the expectations from a three-dimensional logarithmic conformal field theory.

KEYWORDS: AdS/LCFT, log-gravity, logarithmic CFT, two-point functions

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1 Introduction

During the last couple of years, a lot of effort has been put into the understanding of higher-curvature gravity theories in three space-time dimensions. In these studies, one centre of attention has been so-called “critical tunings” in the space of coupling constants. The corresponding theories exhibit several features making them potentially interesting as candidates for models of quantum gravity.

The first instance of this criticality phenomenon was unravelled in cosmological topologically massive gravity (TMG) [1, 2]. The action of TMG consists of an Einstein–Hilbert term, a cosmological constant and a gravitational Chern–Simons term. In a seminal paper, Li, Song and Strominger [3] noted that there is a certain tuning of the Chern–Simons
coupling $\mu$ where several interesting phenomena simultaneously occur. First, one of the central charges of the boundary CFT vanishes and second, the linearised equations of motion degenerate.

The fact that the central charge is zero means that, if the theory is unitary, it has to be chiral. Therefore, the partition function would be trivially holomorphic, fulfilling the hope [4] apparently not realised [5] for pure AdS gravity. This was taken as a strong indication that TMG at the critical point may have a consistent quantum mechanical formulation.

In the same vein, the degeneration of the linearised equations of motion was taken as evidence that the massive graviton — a negative energy state for general tunings — is absent in the critical case. However, soon after the chiral gravity conjecture, it was realised that even at the critical point there is a physical bulk mode $\psi_{\log}$ with negative bulk energy [6]. As in logarithmic CFT [7], the Hamiltonian was shown to have a Jordan block structure on the Hilbert space including $\psi_{\log}$ and based upon this it was conjectured [6] that TMG at the critical point is holographically dual to a logarithmic CFT.

The logarithmic mode has a different asymptotic behaviour than other modes, it diverges more quickly. This is not a problem in itself — the variational principle is well-defined, and the required boundary conditions are consistent [8]. However, it opens up for the possibility to eliminate the mode by imposing certain boundary conditions. It was subsequently shown [10], that the asymptotic behaviour cannot be avoided by constructing wave-packets with compact support because of a linearisation instability.

There are therefore dual reasons to focus some attention on critical points in the study of higher-curvature gravity. On the one hand, they offer a potential mechanism eliminating problematic negative energy states. On the other hand — keeping the logarithmic modes — they offer holographic descriptions of logarithmic CFTs: an AdS/LCFT correspondence.

As its discovery, the first technical pieces of evidence for this correspondence were provided in the context of TMG. Two-point functions were computed according to (suitably generalised) standard AdS/CFT methods [11], resulting in perfect agreement with LCFT correlators, and these results were generalised to three-point functions in a technical tour de force [12]. The one-loop partition function was also demonstrated to match LCFT expectations [13]. These developments have since been generalised to various other versions of higher-curvature gravity in three dimensions exhibiting critical points. (For an extensive list of references see, e.g., [14–16].)

In particular, a certain combination of higher-curvature terms results in a theory that propagates unitary gravitons [17, 18]. This theory, which has acquired the increasingly misleading name “new massive gravity” (NMG), also allows for critical tunings leading to logarithmic behaviour [13, 19, 20].

Naturally, it is of interest to lift this entire discussion to higher dimensions. Such a venture was initiated by Lu and Pope [21] who formulated a higher-curvature gravity theory, being a four-dimensional analogue of critical NMG. This model was later generalised to arbitrary dimensions [22]. Subsequent works [23–25] have demonstrated and categorised

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1We note in passing that a similar idea was advocated not long ago to eliminate the intrinsic ghosts of conformal gravity in four dimensions [9].
the logarithmic excitations in the theory. However, the logarithmic structure is so far explored to a far lesser extent than in the aforementioned lower-dimensional literature. For instance, there are no results on any linearisation instability (potentially allowing for elimination of the logarithmic modes), there are no results on partition functions, and no correlators have been computed.

In this work we take a first step toward a more complete higher-dimensional AdS/LCFT correspondence: we compute the one- and two-point functions of several operators in four-dimensional critical gravity. These operators are the stress tensor, its logarithmic partner and two other operators corresponding to the transverse vector and scalar pieces of the logarithmic graviton. All results are completely consistent with an LCFT$_3$ dual. Our computations follow to a large extent the lead of [11] which also contains clear and comprehensible explanations of the associated logic and technicalities.

This paper is organised as follows. In section 2 we briefly review critical gravity in four dimensions and derive the first variation of the action. In section 3 we obtain the renormalised second variation of the action. This requires the construction and evaluation of several holographic counterterms. In section 4 we explicitly calculate the linearised modes and use them to derive the two-point correlators for critical gravity, and in section 5 we conclude. Three appendices are devoted to technicalities that are not presented in the main text.

## 2 First variation of the action

To compute one- and two-point functions according to the AdS/CFT dictionary [26–28], one functionally differentiates the on-shell action with respect to the relevant sources. In this section we perform the first variation of the action and compute the corresponding boundary term. This object will be differentiated further in later sections. We also evaluate the first variation with respect to the boundary metric allowing to determine the one-point function of the stress tensor. The computation of one-point functions in general requires holographic renormalisation, and the method we follow was developed in [29, 30]. (For reviews, see [31, 32].)

Our model is the one presented in [21] with bulk action

$$I_{\text{bulk}} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left( R - 2\Lambda + \alpha R^{\mu\nu} R_{\mu\nu} + \beta R^2 \right).$$  \hfill (2.1)

Such higher-curvature actions generally propagate a massless spin-two graviton, a massive spin-two field and a massive scalar [33, 34]. The two latter excitations are ghosts. If the parameters $\alpha$ and $\beta$ are tuned as $\alpha = -3\beta$, the spin-zero excitation is absent. If furthermore $\beta = -1/2\Lambda$ the theory becomes critical: the black holes have zero mass and entropy and the massive graviton becomes logarithmic [21, 23, 25].

Varying (2.1) with respect to the metric $g_{\mu\nu}$ produces

$$\delta I_{\text{bulk}} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left( \text{EOM}^{\mu\nu} \delta g_{\mu\nu} + \nabla_\sigma J^\sigma \right),$$ \hfill (2.2)
where
\[
\text{EOM}^{\mu\nu} = -G^{\mu\nu} - E^{\mu\nu},
\]
\[
J^\sigma = A^{\mu\nu} \delta \Gamma^\sigma_{\mu \nu} - A^{\mu\sigma} \delta \Gamma^\lambda_{\lambda \mu} + \left( \frac{1}{2} \nabla_\lambda A^{\sigma\lambda} g^{\mu\nu} - \nabla^\mu A^{\sigma\nu} + \frac{1}{2} \nabla^\sigma A^{\mu\nu} \right) g_{\mu\nu},
\]
where the tensor \( A^{\mu\nu} \) is defined as
\[
A^{\mu\nu} = (1 + 2 \beta R) g^{\mu\nu} + 2 \alpha R^{\mu\nu}.
\]
The tensors appearing in the equations of motion are furthermore
\[
G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} \nabla R^{\mu\nu} + \Lambda g^{\mu\nu},
\]
\[
E^{\mu\nu} = 2\alpha (R^{\mu\lambda} R^{\lambda\nu} - \frac{1}{4} R^{\lambda\sigma} R_{\lambda\sigma} g^{\mu\nu}) + 2\beta R(R^{\mu\nu} - \frac{1}{4} R g^{\mu\nu})
+ \alpha(\Box R^{\mu\nu} + \frac{1}{2} \Box R g^{\mu\nu} - 2\nabla \nabla (\mu R^{\lambda}) + 2\beta(g^{\mu\nu} \Box R - \nabla \nabla R).
\]
We shall from here on only consider the case \( \alpha = -3 \beta \). To reduce clutter we also fix \( \Lambda = -3 \).

Using the (twice contracted) Bianchi identity it is easy to show that, for \( \alpha = -3 \beta \), the tensor \( E^{\mu\nu} \) is traceless. Thus, taking the trace of the equations of motion then establishes
\[
R = 4\Lambda = -12.
\]
In particular the Ricci scalar is constant on-shell, eliminating several of the terms in Eq. (2.7). Using again the Bianchi identities the equations of motion simplify to
\[
G^{\mu\nu} + \frac{3\beta}{2} R^{\lambda\sigma} R_{\lambda\sigma} g^{\mu\nu} - 24\beta (R^{\mu\nu} + 3g^{\mu\nu}) - 3\beta(\Box R^{\mu\nu} + 2R^{\alpha\mu\lambda\nu} R^{\alpha\lambda}) = 0.
\]
Note that these are the correct equations of motion only in vacuum. Coupling the theory to matter would of course require keeping the full result (2.6)–(2.7).

Let us now fix Gaussian normal coordinates,
\[
ds^2 = dp^2 + \gamma_{ij}(\rho) \, dx^i \, dx^j.
\]
From (2.2) it is then clear that the on-shell variation reads
\[
\delta I_{\text{bulk}|\text{EOM}} = \frac{1}{2\kappa^2} \int_{\partial M} d^3 x \sqrt{-\gamma} \, J^\rho,
\]
with \( J^\mu \) defined in Eq. (2.4). Partially integrating the Christoffel symbols in \( J^\rho \) allows us to write the variation as
\[
\delta I_{\text{bulk}|\text{EOM}} = \frac{1}{2\kappa^2} \int_{\partial M} d^3 x \sqrt{-\gamma} \, J^\rho = \int_{\partial M} d^3 x \sqrt{-\gamma} \left( - (A^{ij} + A^{\rho\rho}) \delta K_{ij} + \left[ \frac{1}{2} \nabla_\rho (A^{ij} + A^{\rho\rho}) + \nabla_k A^{\rho k} \gamma_{ij} - \nabla_i A^{\rho j} + A^{\rho j} K_{ij} \right] \delta \gamma_{ij} \right).
\]
This is the expression that we shall differentiate further to obtain the two-point correlators in later sections. In doing so, we shall use Poincaré patch AdS_4 as a background, to obtain
the correlators corresponding to a CFT$_3$ on a flat background. This computation will also give us the one-point functions on a flat background.

Let us first, however, compute the one-point function of the stress tensor in global AdS. The advantage of this is that we can then obtain the conserved Poincaré charges of, e.g., a black hole. Our background metric is thus global AdS$_4$:

$$ds^2 = -\cosh^2 \rho \, dt^2 + d\rho^2 + \sinh^2 \rho \, d\Omega_2^2,$$

and the matrix $\gamma_{ij}$ in Eq. (2.10) is assumed to have the expansion

$$\gamma_{ij} = \gamma_{ij}^{(0)} e^{2\rho} + \gamma_{ij}^{(2)} \rho e^{-\rho} + \gamma_{ij}^{(3)} e^{-\rho} + \ldots$$

with the leading contributions fixed

$$\gamma_{ij}^{(0)} = \begin{pmatrix} -1/4 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/4 \sin^2 \theta \end{pmatrix}, \quad \gamma_{ij}^{(2)} = \begin{pmatrix} -1/2 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & -1/2 \sin^2 \theta \end{pmatrix}.$$ (2.15)

The term $\gamma_{ij}^{(3)}$ corresponds to the massless graviton and the term $\beta_{ij}^{(3)}$ is forced to vanish by the equations of motion except at the critical point $\beta = 1/6$, where it captures the logarithmic mode. For other tunings, the massive graviton has a different power law fall-off. We ignore such terms for brevity, but still keep $\beta$ arbitrary since it illuminates some of the results. It is important to keep in mind though, that our result for the stress tensor below is incomplete unless $\beta = 1/6$.

Using these expansions it is straightforward to obtain an expansion for the variation in Eq. (2.11). This computation is detailed in appendix A and the result is

$$\delta I_{\text{bulk}|EOM} = \frac{1}{2\kappa^2} \int_{\partial M} d^3 x \sqrt{-\gamma} \left[ 1 - 6\beta \right] (K_{ij} \delta \gamma_{ij} - 2 \gamma_{ij} \delta K_{ij}) - \frac{27}{2} \beta e^{-5\rho} \beta^{ij}_{(3)} \delta \gamma_{ij}. $$

(2.16)

Here $K_{ij}$ is the extrinsic curvature

$$K_{ij} = \frac{1}{2} \partial_{\rho} \gamma_{ij}. \quad (2.17)$$

The first term in this variation is divergent and contains variations of $\gamma_{ij}^{(3)}$ and $\beta_{ij}^{(3)}$, destroying a well-defined variational principle. These terms must in general be cancelled by holographic counterterms. Note however, that at the critical point $\beta = 1/6$ the first term vanishes. Thus, there is no need to add holographic counterterms for the critical case. This is in complete analogy with what happens in three dimensions for new massive gravity [20, 35].

Off the critical locus, counterterms are needed. It is shown in appendix A that the required terms are exactly the ones for pure Einstein gravity [36, 37]$^2$, multiplied with the appropriate prefactor:

$$I_{\partial M} = -\frac{1 - 6\beta}{2\kappa^2} \int_{\partial M} d^3 x \sqrt{-\gamma} \left( 4 - 2K + R[\gamma] \right).$$

(2.18)

$^2$For a less directly comparable, but earlier, computation of the counterterms for Einstein gravity, see [29, 30].
The boundary stress tensor $T^{ij}$ is defined as

$$\delta I_{\text{ren}}|_{EOM} = \frac{1}{2} \int_{\partial M} d^3 x \sqrt{-\gamma^{(0)}} \, T^{ij} \delta \gamma^{(0)}_{ij}, \quad (2.19)$$

or, shorter,

$$T_{ij} = -\frac{2}{\sqrt{-\gamma^{(0)}}} \frac{\delta I_{\text{ren}}|_{EOM}}{\delta \gamma^{(0)}_{ij}}, \quad (2.20)$$

where $I_{\text{ren}} = I_{\text{bulk}} + I_{\partial M}$. At the critical point the stress tensor takes the simple form

$$T_{ij}^{\text{crit}} = -\frac{9}{4\kappa^2} \beta^{(3)}_{ij}, \quad (2.21)$$

while for generic values of $\beta$ the result is

$$T_{ij} = \left(1 - 6\beta\right) \frac{3}{2\kappa^2} \gamma^{(3)}_{ij}. \quad (2.22)$$

The latter result of course contains contributions from the boundary action in Eq. (2.18), and, for $\beta = 0$, is consistent with [37]. We note again that the result (2.22) only contains the contribution from solutions captured by $\gamma^{(3)}_{ij}$, e.g., massless gravitons and black holes.

As shown in appendix A, the asymptotic equations of motion imply

$$\text{Tr} \beta^{(3)} \equiv \gamma^{ij}_{(0)} \beta^{(3)}_{ij} = \text{Tr} \gamma^{(3)} \equiv \gamma^{ij}_{(0)} \gamma^{(3)}_{ij} = 0, \quad (2.23)$$

and

$$\nabla^{(0)}_{(0)} \beta^{(3)}_{ki} = 0, \quad \left(1 - 6\beta\right) \nabla^{(0)}_{(0)} \gamma^{(3)}_{ki} = 0, \quad (2.24)$$

where the covariant derivative $\nabla^{(0)}$ is taken with respect to the boundary metric $\gamma^{(0)}_{ij}$. These equations imply that the boundary stress tensor — for general $\beta \neq 1/6$ at least the part we computed — is traceless and conserved. Note that $\gamma^{(3)}_{ij}$ is not necessarily transverse for the critical case. The non-transverse components make up part of the logarithmic graviton.

Our result (2.21) shows that any solution having a vanishing $\beta^{(3)}$ also has a vanishing stress tensor. In particular this means that any solution to Einstein gravity has vanishing mass and angular momentum, confirming the result of [21] for the mass. The situation is completely parallel to that in NMG [38].

### 3 Second variation of the action

To be able to compute the two-point correlators we first need the second variation of the action, or, equivalently, the first variation of the one-point functions. This entails computing the first variation of the coefficients multiplying $\delta \gamma_{ij}$ and $\delta K_{ij}$ in (2.12). To technically simplify this computation we shall perform it perturbatively around Poincaré patch AdS as opposed to the global metric. This means that the boundary is a plane, and that we obtain planar CFT correlation functions.

Thus, we consider a metric of the form

$$ds^2 = \frac{dy^2}{y^2} + \gamma_{ij} \, dx^i \, dx^j = \frac{dy^2 + \eta_{ij} \, dx^i \, dx^j}{y^2} + h_{ij} \, dx^i \, dx^j, \quad (3.1)$$
where $\eta_{ij}$ is the flat Minkowski metric on the boundary $\mathbb{R}^{2,1}$. The perturbation $h_{ij}$ is assumed to have a Fefferman–Graham expansion of the form

$$h_{ij} = b_{ij}^{(0)} \log \frac{y}{y^2} + h_{ij}^{(2)} \log y + g_{ij}^{(2)} y \log y + g_{ij}^{(3)} y + \ldots$$ \hspace{1cm} (3.2)

To compute the correlator of the logarithmic mode we must include the leading logarithmic term $b_{ij}^{(0)}$ that breaks the asymptotic AdS property of the metric. By the equations of motion, this also requires the inclusion of the term $b_{ij}^{(2)} \log y$. Furthermore, the equations of motion require $b_{ij}^{(0)}$ be traceless.

In AdS/CFT language, the expansion coefficients $h_{ij}^{(0)}$ and $b_{ij}^{(0)}$ represent sources for different operators. The former is a source for the stress-energy tensor ($T_{ij}$), whereas the latter contains sources for its logarithmic partner, but also for other operators. Collectively, we shall denote these operators by $t_{ij}$. The tracelessness of $b_{ij}^{(0)}$ carries over to the operator(s) $t_{ij}$ and imposes the constraint that $t_{ij}$ be traceless too. Correlators of these operators are then given by functionally differentiating the on-shell action with respect to $h_{ij}^{(0)}$ and $b_{ij}^{(0)}$.

Our normalisation of these operators will be defined by

$$\langle T_{ij} \rangle = 2 \delta \frac{\mathcal{I}_{\text{ren}}|_{\text{EOM}}}{\delta h_{ij}^{(0)}}, \hspace{1cm} \langle T_{ij} \ldots \rangle = -2i \delta \frac{\mathcal{I}_{\text{ren}}|_{\text{EOM}}}{\delta h_{ij}^{(0)}} \langle \ldots \rangle, \hspace{1cm} (3.3a)$$

$$\langle t_{ij} \rangle = 2 \delta \frac{\mathcal{I}_{\text{ren}}|_{\text{EOM}}}{\delta b_{ij}^{(0)}}, \hspace{1cm} \langle t_{ij} \ldots \rangle = -2i \delta \frac{\mathcal{I}_{\text{ren}}|_{\text{EOM}}}{\delta b_{ij}^{(0)}} \langle \ldots \rangle, \hspace{1cm} (3.3b)$$

where the ellipsis denotes any operator. The factor of $-i$ in the two rightmost expressions comes about because the generating function is actually $\sim i \mathcal{I}_{\text{ren}}$ (for Lorentzian signature), as explained in appendix B of [11].

### 3.1 Second variation of the bulk action

The computation of the second variation of the bulk action is lengthy but straightforward. The details and the conventions are presented in appendix B. The final result is

$$\delta^{(2)} \mathcal{I}_{\text{bulk}|_{\text{EOM}}} = \frac{1}{2\kappa^2} \int_{\partial M} d^3x \sqrt{-\gamma} \left\{ -\frac{3}{4} \delta b_{ij}^{(0)} \delta b_{ij}^{(0)} \frac{1}{y} + \left[ 2 \delta \frac{\mathcal{I}_{\text{ren}}|_{\text{EOM}}}{\delta h_{ij}^{(0)}} \delta b_{ij}^{(2)} - \frac{1}{2} \delta b_{ij}^{(0)} \delta b_{ij}^{(2)} - \frac{1}{2} \delta h_{ij}^{(0)} \delta g_{ij}^{(3)} + \frac{3}{4} \delta b_{ij}^{(0)} \delta g_{ij}^{(3)} - \frac{3}{4} \delta h_{ij}^{(0)} \delta b_{ij}^{(3)} \right] \right\}, \hspace{1cm} (3.4)$$

where all indices are raised by $\eta^{ij}$. We also have already symmetrised\(^3\) in the two variations, since only the symmetrised version is needed in computing the correlators.

We see that setting $b_{ij}^{(0)}$ and $b_{ij}^{(2)}$ to zero recovers our old result (2.21) for the stress tensor at the critical point. Furthermore, because of the many divergent terms it is clear that the computation of two-point correlators requires holographic renormalisation, a topic to which we now turn.

\(^3\)This means, e.g., setting $\delta_1 b_{ij}^{(0)} \delta_2 b_{ij}^{(2)} + \delta_2 b_{ij}^{(0)} \delta_1 b_{ij}^{(2)} = 2 \delta b_{ij}^{(2)} \delta b_{ij}^{(2)}$ and setting $\delta_1 b_{ij}^{(0)} \delta_2 b_{ij}^{(3)} - \delta_2 b_{ij}^{(0)} \delta_1 b_{ij}^{(3)} = 0$ and so on.
3.1.1 Auxiliary field formalism

To determine the correct counterterms it is useful to consider first the formulation of a well-defined boundary value problem with the cut-off radius not taken to infinity. Since we are dealing with a higher-curvature theory, it is not enough to fix the metric at the boundary, but also some derivatives must be held fixed. A convenient and enlightening way to set up the problem, and to choose a combination of derivatives to hold fixed, is through the auxiliary field formalism developed in \[17, 18, 35\]. We shall use a minutely tweaked version, the tweaking being the non-linear version of the field redefinition used, e.g., in \[19\]. The field redefinition is such that the auxiliary field vanishes for the AdS vacuum.

Running the risk of some redundancy, we present it in some small detail for general $\Lambda$ and $\beta$. We consider the bulk action

$$S = \int_M d^4x \sqrt{-g} \left[ (2\Lambda \beta + 1) R - (2\Lambda \beta + 1) 2\Lambda + \frac{3}{\Lambda} F^\mu{}_{\nu} G_{\mu\nu} + \frac{3}{4\beta \Lambda^2} (F^\mu{}_{\nu} F_{\mu\nu} - F^2) + 3F \right],$$

(3.5)

where $F^\mu{}_{\nu}$ is the auxiliary field, $F = g^\mu{}_{\nu} F_{\mu\nu}$ and $G_{\mu\nu} = R_{\mu\nu} - (R/2) g_{\mu\nu}$. Varying with respect to $F^\mu{}_{\nu}$, and using the trace of the resulting equation of motion, gives $F^\mu{}_{\nu}$ in terms of the metric:

$$F^\mu{}_{\nu} = \frac{\beta \Lambda}{3} \left( -6 R^\mu{}_{\nu} + R g^\mu{}_{\nu} + 2\Lambda g^\mu{}_{\nu} \right).$$

(3.6)

From this equation it is easy to see that $F^\mu{}_{\nu} = 0$ if and only if $G_{\mu\nu} + \Lambda g_{\mu\nu} = 0$, i.e., the auxiliary field vanishes exactly if the cosmological Einstein tensor vanishes.

Substituting equation (3.6) into the action (3.5) yields after a little algebra the action (2.1) for $\alpha = -3\beta$:

$$S = \int_M d^4x \sqrt{-g} \left[ R - 2\Lambda - 3\beta R^\mu{}_{\nu} R_{\mu\nu} + \beta R^2 \right].$$

(3.7)

Note that the action (3.5) is particularly simple for the critical tuning $2\Lambda \beta = -1$. In fact, with $\Lambda = -3$ and $\beta = 1/6$ we have

$$S^\text{aux.}_{\text{crit.}} = \int_M d^4x \sqrt{-g} \left[ -F^\mu{}_{\nu} G_{\mu\nu} + \frac{1}{2} (F^\mu{}_{\nu} F_{\mu\nu} - F^2) + 3F \right].$$

(3.8)

Apart from the term linear in $F$, this action looks precisely (up to a rescaling of $F^\mu{}_{\nu}$) like the auxiliary field formulation \[18\] of the pure higher-curvature theory studied by Deser \[39\].

The boundary value problem is now defined by requiring that the variation of $g_{\mu\nu}$ and $F^\mu{}_{\nu}$ both vanish at the boundary. Thus, the first variation of the on-shell action is allowed to contain boundary terms multiplying $\delta g_{\mu\nu}$ and $\delta F^\mu{}_{\nu}$, but no variation of the extrinsic curvature. Eliminating such terms requires adding a generalised Gibbons–Hawking term \[35\] which for the critical case (and in Gaussian normal coordinates) reads

$$I_{\text{GHH}} = \frac{1}{2\kappa^2} \int_{\partial M} d^3x \sqrt{-\gamma} F^{ij} (K_{ij} - K_{ij}).$$

(3.9)
We have now set up a natural-looking boundary value problem for this theory: we keep the metric fixed at the boundary, and, since non-zero $F_{\mu\nu}$ corresponds to non-Einstein modes, we keep the massive graviton fixed at the boundary. Note that this is nevertheless a choice — there are of course other possible boundary conditions.

Fixing this choice now limits the number of additional allowed boundary terms. Most importantly, a boundary term may not change the boundary value problem. This limits us to use combinations of boundary intrinsic metric quantities and the auxiliary field $F_{ij}$. In the next subsection we present a set of boundary terms that makes the second variation of the action finite.

3.1.2 Renormalised second variation

To regularise the action (3.4), one needs to find admissible counterterms that cancel all divergences. Since the computations themselves are not very illuminating, we defer them to appendix B. Let us here make some general comments.

First, the generalised Gibbons–Hawking term (3.9) on its own is very complicated when expanded to second order. In particular, there is a non-zero contribution at the order $\sim \log y/y^3$. This is even more divergent than the terms present in the variation of the bulk action. Furthermore, the first variation of $I_{\text{GGH}}$ does not vanish. There is, however, a term that remedies these deficiencies. The term

$$I_{\hat{F}} = \frac{1}{2\kappa^2} \int_{\partial M} d^3x \sqrt{-\hat{\gamma}} \hat{F}, \quad (3.10)$$

where $\hat{F} = \gamma^{ij} F_{ij}$ can be used for this purpose. In fact, the combination

$$I_{\text{GGH}} - 2 I_{\hat{F}} \quad (3.11)$$

has vanishing first variation, and the second variation starts only at order $\sim 1/y^3$. The full result for this term is presented in (B.24). This term alone, however, far from does the job. It only cancels one of the problematic terms in (3.4), and only at the price of adding another divergent term (at order $\sim \log y/y$).

To obtain a finite second variation one must add a number of terms involving the field $F_{\mu\nu}$, as well as the Ricci curvature $R^{(3)}_{ij}$ of the boundary metric $\gamma_{ij}$. We have not been able to construct a finite action using only such terms however. Instead, to reach our goal requires adding a term of the form

$$I_{\hat{h}\hat{F}} = \frac{1}{2\kappa^2} \int_{\partial M} d^3x \sqrt{-\hat{\gamma}} \hat{h} \hat{F} = \frac{1}{2\kappa^2} \int_{\partial M} d^3x \sqrt{-\gamma} \gamma^{ij} (\gamma_{ij} - \eta_{ij}/y^2) \hat{F}. \quad (3.12)$$

This term is Lorentz invariant at the boundary, but uses not only $\gamma_{ij}$ and $F_{ij}$ for its definition, but also the perturbation $h_{ij}$ explicitly. (Or — equivalently of course — it uses the background metric $\eta_{ij}$.). Although the counterterm (3.12) does not appear to be generally covariant we find that the boundary stress tensor is conserved. Thus the theory does not suffer from a diffeomorphism anomaly [40]. This suggests that the term (3.12) is actually equivalent to some covariant counterterm to second order in the perturbation.
The fully regularised action having a finite second variation reads

\[ I_{\text{ren}} = I_{\text{bulk}} + I_{\text{GGH}} - 2I_{\hat{F}} - \frac{1}{2\kappa^2} \int_{\partial M} d^3x \sqrt{-\gamma} \left[ \frac{1}{6} F^{ij} F_{ij} - F_{ij} R_{ij}^{(3)} + \frac{1}{18} F^{ij} \nabla^2 F_{ij} + \frac{5}{18} F^{ij} (D^2 F)_{ij} + \hat{h} \hat{F} \right]. \tag{3.13} \]

Here, the differential operator \( D^2 \) is defined in (B.7) and computes, up to a factor, the linearised Ricci tensor around three-dimensional Minkowski space. Expanded in the Fefferman–Graham expansion (3.2) the second variation of \( I_{\text{ren}} \) is

\[ \delta^{(2)} I_{\text{ren}}|_{\text{EOM}} = \frac{1}{2\kappa^2} \int_{\partial M} d^3x \sqrt{-\gamma} \left[ \frac{3}{2} \delta b^{ij}_{(0)} \delta b^{(3)}_{ij} + \frac{9}{4} \left( \delta b^{ij}_{(0)} g^{ij}_{(3)} - \delta h^{ij}_{(0)} \delta b^{(3)}_{ij} \right) \right]. \tag{3.14} \]

Already from this expression it is clear that all two-point correlators involving only Einstein modes vanish: if all the \( b^{(n)}_{ij} \) are zero, so is the second variation.

### 3.2 One-point functions

The result in Eq. (3.14) also contains the result for one-point functions around a flat background. For our operator defined in (3.3a) we get

\[ \langle T_{ij} \rangle = -\frac{9}{4\kappa^2} b^{(3)}_{ij}, \tag{3.15} \]

whereas for the operator \( t_{ij} \) defined in (3.3b) we have

\[ \langle t_{ij} \rangle = \frac{3}{2\kappa^2} \left( b^{(3)}_{ij} + \frac{3}{2} g^{(3)}_{ij} \right). \tag{3.16} \]

### 4 Two-point correlators

The two-point correlators are given by the second variation of the action, or alternatively, by the functional derivative of the one-point functions \( \langle T_{ij} \rangle \) and \( \langle t_{ij} \rangle \) with respect to the sources \( h^{(0)}_{ij} \) and \( b^{(0)}_{ij} \). Differentiating the expressions in Eqs. (3.15) and (3.16) for the one-point functions according to Eqs. (3.3a) and (3.3b) we obtain

\[ \langle T_{ij}(x) T_{kl}(x') \rangle = 0, \tag{4.1} \]

\[ \langle T_{ij}(x) t_{kl}(x') \rangle = \frac{9i}{2\kappa^2} \frac{\delta b^{(3)}_{ij}(x)}{\delta b^{(0)}_{kl}(x')}, \tag{4.2} \]

\[ \langle t_{ij}(x) t_{kl}(x') \rangle = -\frac{3i}{\kappa^2} \left( \frac{\delta b^{(3)}_{ij}(x)}{\delta h^{(0)}_{kl}(x')} + \frac{3}{2} \frac{\delta g^{(3)}_{ij}(x)}{\delta b^{(0)}_{kl}(x')} \right). \tag{4.3} \]

Note that there are two ways to compute the \( \langle T_{ij} t_{kl} \rangle \) correlator — either by differentiating \( \langle T_{ij} \rangle \) with respect to \( b^{(0)}_{ij} \), or by differentiating \( \langle t_{kl} \rangle \) with respect to \( h^{(0)}_{ij} \). In obtaining the expressions for the correlators we used the fact that

\[ \frac{\delta b^{(3)}_{ij}}{\delta h^{(0)}_{kl}} = 0, \tag{4.4} \]
meaning that \( h_{ij}^{(0)} \) does not source any modes with logarithmic behaviour. We shall see explicitly that this is true when studying the modes below.

As remarked earlier, correlators including only Einstein modes are identically zero. We immediately note that any different — log or otherwise — behaviour of the log-log correlator with respect to the log-Einstein correlator stems from the \( g_{ij}^{(3)} \) mode, as can be seen from Eq. (4.3). A logarithmic mode is only defined up to an arbitrary shift of the mode by Einstein modes, and this freedom can be used to eliminate the first term in Eq. (4.3). Thus, the non-trivial information in Eq. (4.3) comes from the last term:

\[
\langle t_{ij} t_{kl} \rangle = -\frac{2}{3} \langle T_{ij} t_{kl} \rangle - \frac{9i}{2\kappa^2} \frac{\delta g_{ij}^{(3)}}{\delta b_{kl}^{(0)}}.
\]  

(4.5)

Eliminating the first term would amount to redefining

\[
t_{ij} \rightarrow t_{ij} + \frac{1}{3} T_{ij}.
\]  

(4.6)

Thus, in order to compute the correlators, all we need to do is find the functional relations between the matrices \( b_{ij}^{(0)}, h_{ij}^{(0)}, b_{ij}^{(3)} \) and \( g_{ij}^{(3)} \). To achieve this we shall find all linearised modes in momentum space. This is the task of the next subsection.

### 4.1 Modes

We now aim to determine how the subleading Fefferman–Graham coefficients \( b_{ij}^{(3)} \) and \( g_{ij}^{(3)} \) functionally depend on \( b_{ij}^{(0)} \) and \( h_{ij}^{(0)} \). This dependence comes about as a combination of the equations of motion and the boundary conditions in the interior of AdS. In the global case, the latter consists of requiring regularity, and in the present case of Poincaré patch AdS, we require infalling boundary conditions at the Poincaré horizon.

In transverse gauge, the linearised equations of motion are rather simple. For the critical case they read

\[
(\Box + 2)(\Box + 2)\psi_{\mu\nu} = 0.
\]  

(4.7)

Here \( \Box \) is the wave operator on AdS\(_4\). The solution space consists of Einstein modes \( \psi_E^{\mu\nu} \) and logarithmic modes \( \psi_{\mu\nu}^{\log} \) satisfying

\[
(\Box + 2)\psi_E^{\mu\nu} = 0,
\]  

(4.8)

and

\[
(\Box + 2)^2\psi_{\mu\nu}^{\log} = 0, \quad \text{but} \quad (\Box + 2)\psi_{\mu\nu}^{\log} \neq 0,
\]  

(4.9)

respectively. Now, we need the modes expressed in Gaussian normal coordinates, which is not compatible with transverse traceless gauge in general. We shall anyhow proceed by solving (4.7) and then transform the solutions to Gaussian normal coordinates.

To find the logarithmic, as well as the Einstein solutions, we solve the equation

\[
(\Box + m^2)\psi_{\mu\nu}(m) = 0.
\]  

(4.10)
Then the Einstein and logarithmic modes are obtained as

\[ \psi^E_{\mu\nu} = \psi_{\mu\nu}(\sqrt{2}), \quad \text{and} \quad \psi^{\text{log}}_{\mu\nu} = \left. \frac{\partial \psi_{\mu\nu}}{\partial m} \right|_{m=\sqrt{2}}. \]  

(4.11)

We work in three-dimensional momentum space, making the separation ansatz

\[ \psi_{\mu\nu} = e^{-ip \cdot x} \tilde{\psi}_{\mu\nu}(p_i; y). \]  

(4.12)

Using Lorentz invariance of the background, we can fix a certain Lorentzian covector \( p_i \). If \( p_i \) is timelike we choose \( p_i = E \delta^t_i \) and if it is lightlike we choose \( p_i = E \delta^t_i + E \delta^x_i \).

**Timelike modes**

Let us start with the timelike case. Solving (4.10) with \( \psi_{\mu\nu}(m) = e^{-iEt} \tilde{\psi}_{\mu\nu}(p_i; y) \), using the gauge condition \( \nabla^\mu \psi_{\mu\nu}(m) = 0 \), is fairly straightforward. The most general solution has ten undetermined coefficients.

An example of a solution where eight of the coefficients have been put to zero is

\[
\psi_{\mu\nu} = e^{-iEt} \begin{bmatrix}
C_1 j_\nu(Ey) + C_2 y_\nu(Ey) \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}_{\mu\nu},
\]  

(4.13)

where \( \nu = 1/2(-1 + i\sqrt{-17 + 4m^2}) \), \( j_\nu \) and \( y_\nu \) are spherical Bessel functions and \( C_{1,2} \) are constants. (We shall present all solutions only in their final form.)

The next step is to require infalling boundary conditions at the Poincaré horizon at \( y = \infty \). To achieve this we note that

\[
[C_1 j_\nu(Ey) + C_2 y_\nu(Ey)] \sim \left[ C_1 \cos(Ey - \frac{\pi}{2} - \frac{\nu\pi}{2}) + C_2 \sin(Ey - \frac{\pi}{2} - \frac{\nu\pi}{2}) \right] \frac{1}{y},
\]  

(4.14)

from which it is clear that infalling boundary conditions correspond to \( C_2 = iC_1 \). For the other solutions the reasoning is identical. In this way the number of undetermined components reduces to five.

The next step is to construct six Einstein and five logarithmic modes by using (4.11). Four of the Einstein modes are pure gauge, whereas all the logarithmic modes are physical. The last step is to go to Gaussian normal coordinates. The mode (4.13) is already in this gauge. For those that are not, it is simple to construct the corresponding gauge transformation.

Below we present the modes that are the result of these computations.

**Logarithmic modes**
The full set of logarithmic modes is:

\[
\begin{align*}
\psi_{ij}^{\text{log1}} &= e^{-iEt} \frac{F_1(Ey)}{y^2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij}, \\
\psi_{ij}^{\text{log2}} &= e^{-iEt} \frac{F_1(Ey)}{y^2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}_{ij}, \\
\psi_{ij}^{\text{log3}} &= e^{-iEt} \frac{F_4(Ey)}{y^2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}_{ij}, \\
\psi_{ij}^{\text{log4}} &= e^{-iEt} \frac{F_4(Ey)}{y^2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}_{ij}, \\
\psi_{ij}^{\text{log5}} &= e^{-iEt} \left( F_2(Ey) \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{ij} + F_3(Ey) \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} \right).
\end{align*}
\] (4.15)

\[
\begin{align*}
F_1(Ey) &= \text{Ei}(iEy) - e^{iEy}, \\
F_2(Ey) &= \text{Ei}(iEy) - e^{iEy} \left( \frac{3}{2} - \frac{iEy}{2} \right), \\
F_3(Ey) &= \left( \frac{E^2y^2}{2} - 3 \right) \text{Ei}(iEy) + 3e^{iEy}, \\
F_4(Ey) &= e^{-iEy} \left( -e^{2iEy} \left[ 2 + \frac{\pi(i + Ey)}{2} \right] + (Ey - i)(\pi + i \text{Ei}(2iEy)) \right).
\end{align*}
\] (4.18-4.20)

It is clear that the first two modes correspond to a vector representation of the little group SO(2), that \(\psi_{ij}^{\text{log3}}\) and \(\psi_{ij}^{\text{log4}}\) correspond to a traceless tensor representation and that \(\psi_{ij}^{\text{log5}}\) corresponds to the scalar representation. It is therefore possible to immediately write down the modes for general timelike \(p_i\).

To this end we denote by \(\epsilon_1^{1,2}\) an orthonormal basis in the orthogonal complement of \(p_i\), and construct from them two traceless tensors \(M_{ij}^1 = \epsilon_1^i \epsilon_1^j - \epsilon_2^i \epsilon_2^j\) and \(M_{ij}^2 = \epsilon_1^i \epsilon_2^j + \epsilon_2^i \epsilon_1^j\) in the same space. Then, if \(|p| \equiv \sqrt{-p^2}\), we have

\[
\begin{align*}
\psi_{ij}^{\text{log1,2}} &= e^{-ip \cdot x} F_1(|p|y) \frac{1}{y^2} \left( p_i \epsilon_1^i + p_j \epsilon_1^j \right), \\
\psi_{ij}^{\text{log3,4}} &= e^{-ip \cdot x} \frac{F_4(|p|y)}{y^2} M_{ij}^{1,2}, \\
\psi_{ij}^{\text{log5}} &= e^{-ip \cdot x} y \left( F_2(|p|y) \eta_{ij} - F_3(|p|y) \frac{p_ip_j}{|p|^2} \right).
\end{align*}
\] (4.21-4.23)

**Einstein modes**

Regarding the Einstein modes, only the traceless tensor modes are not pure gauge. Thus, log-modes corresponding to scalar and vector Einstein excitations should be identified as Proca modes \([25]\).

To find the pure gauge modes one simply makes an ansatz \(\psi_{\mu \nu} = \nabla_{(\mu} \xi_{\nu)}\) with \(\xi_{\nu} = \)
To derive how $b_{ij}^{(3)}$ and $q_{ij}^{(3)}$ depend on $b_{ij}^{(0)}$ and $h_{ij}^{(0)}$ we must now construct modes that have either of these two expansion coefficients vanishing. The Einstein modes already have $b_{ij}^{(0)} = 0$, so they correspond to varying only $h_{ij}^{(0)}$. To get a mode that has vanishing $h_{ij}^{(0)}$ a linear combination of the Einstein and logarithmic modes must be taken. To this end, an expansion around $y = 0$ for the functions $F_i$ is useful:

$$F_1(|p|y) = \log y + \left[ -1 + \gamma + \frac{i\pi}{2} + \log |p| \right] + \frac{|p|^2 y^2}{4} + \frac{i|p|^3 y^3}{9} + \ldots$$  \hspace{1cm} (4.25)$$

$$F_2(|p|y) = \log y + \left[ - \frac{3}{2} + \gamma + \frac{i\pi}{2} + \log |p| \right] - \frac{i|p|^3 y^3}{18} + \ldots$$  \hspace{1cm} (4.26)$$

$$F_3(|p|y) = -3 \log y + \left[ 3 - 3\gamma - \frac{3i\pi}{2} - 3 \log |p| \right] + \frac{|p|^2 y^2}{2} \log y$$
$$+ \frac{|p|^2 y^2}{4} \left[ -3 + 2\gamma + i\pi + 2 \log |p| \right] + \frac{i|p|^3 y^3}{6} + \ldots$$  \hspace{1cm} (4.27)$$

$$F_4(|p|y) = \log y + \left[ -2 + \gamma - i\pi + \log 2 + \log |p| \right] + \frac{|p|^2 y^2}{2} \log y$$
$$+ \frac{|p|^2 y^2}{2} \left[ \gamma - i\pi + \log 2 + \log |p| \right] - \frac{i|p|^3 y^3}{3} \log y$$
$$- \frac{i|p|^3 y^3}{9} \left[ -8 + 3\gamma + \log 8 + 3 \log |p| \right] + \ldots$$  \hspace{1cm} (4.28)$$

Consequently we find the following combinations of logarithmic modes and Einstein modes with vanishing $h_{ij}^{(0)}$:

$$\psi_{ij}^{L1,2} = \psi_{ij}^{log1,2} + (1 - \gamma - \frac{i\pi}{2} - \log |p|) \psi_{ij}^{E1,2},$$  \hspace{1cm} (4.29)$$

$$\psi_{ij}^{L3,4} = \psi_{ij}^{log3,4} + (2 - \gamma + i\pi - \log 2 - \log |p|) \psi_{ij}^{E3,4},$$  \hspace{1cm} (4.30)$$

$$\psi_{ij}^{L5} = \psi_{ij}^{log5} - \frac{1}{4} (3 - 2\gamma - i\pi - 2 \log |p|) \psi_{ij}^{E5}$$
$$+ \frac{3}{|p|^2} \left( 1 - \gamma - \frac{i\pi}{2} - \log |p| \right) \psi_{ij}^{E6}.$$

(4.31)
Let us start with the functional derivative
\[
\frac{\delta g_{ij}^{(3)}}{\delta h_{kl}^{(0)}}.
\]
This is the quantity of relevance for computing the two-point function of the energy-momentum tensor in Einstein gravity. Variation of \(g_{ij}^{(3)}\) with respect to \(h_{ij}^{(0)}\), keeping \(b_{ij}^{(0)}\) fixed, corresponds to analysing the relation between \(h_{ij}^{(0)}\) and \(g_{ij}^{(3)}\) for the Einstein modes.

We let the index \(I\) run over \(1, \ldots 6\) and define the basis \(\{e^I\} = \{h_{(0)}^E\}\), where the subscript \(h_{(0)}^E\) denotes the leading Fefferman–Graham coefficient of the corresponding Einstein mode \(\tilde{\psi}^E\) from\(^4\) Eq. (4.24). Similarly, by \(g_{ij}^{E(3)}\) below, we shall mean the \(O(y)\)-term in the Fefferman–Graham expansion of the corresponding Einstein mode. Explicitly we have
\[
\begin{align*}
    e_1^{ij} &= p_i e_j^1 + p_j e_i^1, &
    e_2^{ij} &= p_i e_j^2 + p_j e_i^2, \\
    e_3^{ij} &= e_1^1 e_1^j - e_1^i e_1^j, &
    e_4^{ij} &= e_1^i e_2^j + e_2^i e_1^j, \\
    e_5^{ij} &= -2\eta_{ij} , &
    e_6^{ij} &= p_i p_j .
\end{align*}
\]
The only modes having a non-zero \(g_{ij}^{(3)}\) are the traceless tensor modes (4.24b). Explicitly:
\[
\begin{align*}
    g_{(3)i}^{E3,4} &= \frac{i|p|}{3} e_{3,4}^{i} , &
    g_{(3)i}^{E\neq3,4} &= 0 .
\end{align*}
\]
For a general Einstein mode \(h_{ij} = \sum_I A_I \tilde{\psi}_{ij}^E\) we therefore have
\[
\begin{align*}
    h_{ij}^{(0)} &= \sum_I A_I e_{ij}^I , &
    g_{ij}^{(3)} &= \frac{i|p|}{3} \left( A_3 e_{ij}^3 + A_4 e_{ij}^4 \right) .
\end{align*}
\]
To compute the desired functional relation, we only have left to compute how \(A_3\) and \(A_4\) depend on \(h_{ij}^{(0)}\), and to achieve this we need to invert the first relation in (4.38).

Fortunately this is simply done. Noting that
\[
\begin{align*}
    e_{kj}^I e_{ij}^K &= 2\delta_{K}^{I}, &
    K &= 3, 4 &
    \text{and} &
    I &= 1, \ldots 6 ,
\end{align*}
\]
we have
\[
A_K = \frac{1}{2} e_{kj}^I h_{ij}^{(0)} , &
    K &= 3, 4 .
\]
Inserting this into the second relation in (4.38) we have
\[
\begin{align*}
    g_{ij}^{(3)} &= \frac{i|p|}{6} \left( e_{kl}^3 e_{ij}^3 + e_{kl}^4 e_{ij}^4 \right) h_{ij}^{(0)} ,
\end{align*}
\]
\(^4\)We actually take the Fefferman–Graham coefficient of the Fourier transform, i.e., we leave the exponential factor out. To avoid clutter, we do not put a tilde on \(h_{ij}^{(0)}\) and \(g_{ij}^{(3)}\). Whenever there is risk of confusion, we shall write out the argument — \(x\) or \(p\) — explicitly.
and therefore
\[
\frac{\delta g_{ij}^{(3)}}{\delta h_{kl}^{(0)}} = \frac{i|p|^3}{6} (e^3_{kl}e^3_{ij} + e^4_{kl}e^4_{ij}) .
\] (4.42)

From the present form it is not clear that this expression is independent of the explicit choice of the polarization vectors $\epsilon^{1,2}$, but it is simple to show that this is the case. In fact, defining the matrix
\[
\Theta_{ij}(p) = \eta_{ij}p^2 - p_ip_j ,
\] (4.43)
and using that
\[
\eta_{ij} = \frac{p_ip_j}{p^2} + \epsilon^1_i \epsilon^1_j + \epsilon^2_i \epsilon^2_j ,
\] (4.44)
it is straightforward to show that
\[
\frac{\delta g_{ij}^{(3)}}{\delta h_{kl}^{(0)}} = i \frac{|p|^3}{6|p|} (\Theta_{ik}\Theta_{jl} + \Theta_{il}\Theta_{jk} - \Theta_{ij}\Theta_{kl}) .
\] (4.45)

This expression is identical to the Fourier transform of the two-point function of the stress tensor in a three-dimensional conformal field theory [41]. (See, e.g., Eqs. (105) and (108) in Ref. [42].)

Let us now turn to the functional derivative $\frac{\delta b_{ij}^{(3)}}{\delta b_{kl}^{(0)}}$, which follows from a very similar computation. Since we are differentiating with respect to $b_{kl}^{(0)}$ we are interested in the modes $\tilde{\psi}^{L_I}$ having vanishing $h_{ij}^{(0)}$. We denote the corresponding Fefferman–Graham coefficients by $b_{L_I}^{(0)}ij$ and so on.

Note first that we have
\[
b_{L_I}^{(0)}ij = e^I_I , \quad I = 1, \ldots, 4 , \quad b_{L_5}^{(0)}ij = -\frac{1}{2} e^5 e^6 - \frac{3}{p^2} p_i p_j = \eta_{ij} - \frac{3}{p^2} p_i p_j .
\] (4.46)

For an arbitrary mode $h_{ij} = \sum_I A_I \psi_I^{L_I}$ we furthermore have\footnote{The source $b_{ij}^{(0)}$ has only five degrees of freedom, because it is traceless by the equations of motion (B.15). For similar reasons $g_{ij}^{(3)}$ and $b_{ij}^{(3)}$ have only two degrees of freedom — they are transverse and traceless.}
\[
b_{ij}^{(0)} = \sum_{I=1}^4 A_I e^I_{ij} + A_5 (\eta_{ij} - \frac{3}{p^2} p_i p_j) ,
\] (4.47)
and
\[
b_{ij}^{(3)} = -\frac{i|p|^3}{3} (A_3 e^3_{ij} + A_4 e^4_{ij}) .
\] (4.48)

Using exactly the same construction as before, we therefore find
\[
\frac{\delta b_{ij}^{(3)}}{\delta h_{kl}^{(0)}} = -\frac{\delta g_{ij}^{(3)}}{\delta h_{kl}^{(0)}} = - \frac{i}{6|p|} (\Theta_{ik}\Theta_{jl} + \Theta_{il}\Theta_{jk} - \Theta_{ij}\Theta_{kl}) .
\] (4.49)
We have now only left to compute $\delta g^{(3)}_{ij} / \delta b^{kl}$. As noted before, the $g^{(3)}_{ij}$ corresponding to non-Einstein modes is not necessarily transverse (but always traceless), and these excitations correspond to several distinct operators in the CFT. Only the transverse traceless tensor part gives the logarithmic partner of the stress tensor.

Using the York decomposition of a traceless tensor, we can split up the operator $t_{ij}$:

$$t_{ij} = \nabla_i V_j + \nabla_j V_i + t^{TT}_{ij} + \left(\nabla_i \nabla_j - \frac{1}{3} \eta_{ij} \nabla^2\right)S,$$

(4.50)

where $V$ is a transverse covector ($\nabla^i V_i = 0$), $t^{TT}_{ij}$ is transverse traceless and $S$ is a scalar operator. The logarithmic partner of the stress tensor is the operator $t^{TT}_{ij}$.

Correspondingly, we shall split up the (traceless) Fefferman–Graham components into three pieces:

$$h_{ij} = \nabla_i v_j + \nabla_j v_i + h^{TT}_{ij} + \left(\nabla_i \nabla_j - \frac{1}{3} \eta_{ij} \nabla^2\right)s.$$

(4.51)

In Fourier space this translates to

$$\tilde{h}_{ij} = -i (p_i \tilde{v}_j + p_j \tilde{v}_i) + \tilde{h}^{TT}_{ij} - \left(p_i p_j - \frac{1}{3} \eta_{ij} p^2\right)\tilde{s},$$

(4.52)

with

$$p_i \tilde{v}^i = 0 \quad \text{and} \quad p^i \tilde{h}^{TT}_{ij} = 0.$$

(4.53)

Thus, the transverse vector part of $g^{(3)}_{ij}$ corresponds to the $e^1_{ij}$ and $e^2_{ij}$ expansion coefficients, the transverse tensor part to $e^3_{ij}$ and $e^4_{ij}$, and the scalar part to the $e^5_{ij}$ coefficient.

Let us denote “transverse vector” by TV, “transverse tensor” by TT and “scalar” by S. Then, computing the derivatives

$$\left.\frac{\delta g^{(3)}_{ij}}{\delta b^{kl}(0)}\right|_{\text{TV}}, \quad \left.\frac{\delta g^{(3)}_{ij}}{\delta b^{kl}(0)}\right|_{\text{TT}} \quad \text{and} \quad \left.\frac{\delta g^{(3)}_{ij}}{\delta b^{kl}(0)}\right|_{\text{S}}$$

(4.54)

corresponds to letting only $A_{1,2}$, $A_{3,4}$ and $A_5$ be non-zero, respectively. Expanding an arbitrary mode $h_{ij} = \sum_I A_I \tilde{h}^{TV}_{ij}$ to order $y$ produces

$$g^{(3)}_{ij} = \frac{i|p|}{9} \left(\frac{A_1 e^1_{ij}}{1} + A_2 e^2_{ij}\right) + \frac{i|p|}{3} \left(C - 2 \log |p|\right) \left(\frac{A_3 e^3_{ij}}{1} + A_4 e^4_{ij}\right) + \frac{i|p|^3}{18} A_5 \left(\frac{1}{2} e^5 + \frac{3}{p^2} e^6\right),$$

(4.55)

where

$$C = 14/3 - 2\gamma + \pi i - 2 \log 2$$

(4.56)

is a numerical constant that might in the end be absorbed through redefining the logarithmic mode by adding suitable number times the corresponding Einstein mode. Note that the $\log |p|$ term cannot be cancelled in this way. This is the term responsible for the logarithmic behaviour of the correlators in an LCFT.
To compute the desired variations, we now need to invert Eq. (4.47) and insert the result in (4.55). Again this is straightforward, using that \( \{ e_{ij}^{1,2}, \ldots, e_{ij}^{3,4}, b_{ij}^{(0)L^5} \} \) is an orthogonal basis, and

\[
e_{1,2,3,4}^{ij} = -2|p|^2, \quad e_{3,4}^{ij} = 2, \quad b_{ij}^{(0)L^5} = 6.
\] (4.57)

The result is

\[
\delta g_{ij}^{(3)} \frac{\delta g_{kl}^{(3)}}{\delta b_{ij}^{(0)}} = \delta g_{ij}^{(3)} \frac{\delta g_{kl}^{(3)}}{\delta b_{ij}^{(0)}} |_{TV} + \delta g_{ij}^{(3)} \frac{\delta g_{kl}^{(3)}}{\delta b_{ij}^{(0)}} |_{S} + \delta g_{ij}^{(3)} \frac{\delta g_{kl}^{(3)}}{\delta b_{ij}^{(0)}} |_{S}.
\] (4.58)

with

\[
\delta g_{ij}^{(3)} \frac{\delta g_{kl}^{(3)}}{\delta b_{ij}^{(0)}} |_{TV} = \frac{2i}{9|p|} p_i (\Theta_j \Theta_k),
\] (4.59a)

\[
\delta g_{ij}^{(3)} \frac{\delta g_{kl}^{(3)}}{\delta b_{ij}^{(0)}} |_{TT} = (C - 2 \log |p|) \delta g_{ij}^{(3)} \frac{\delta g_{kl}^{(3)}}{\delta b_{ij}^{(0)}},
\] (4.59b)

\[
\delta g_{ij}^{(3)} \frac{\delta g_{kl}^{(3)}}{\delta b_{ij}^{(0)}} |_{S} = \left( p_i p_j - \frac{p^2}{3} \eta_{ij} \right) \left( -\frac{i}{12|p|} \right) \left( p_k p_l - \frac{p^2}{3} \eta_{kl} \right).
\] (4.59c)

In (4.59a) the symmetrisations are taken over \( ij \) and \( kl \) and are defined with the usual factor of \( 1/2 \).

**Spacelike modes**

The above construction of modes captures only the timelike case. There are no obstructions to a very similar analysis for spacelike modes — provided we make some obvious changes. For example we will have to choose different boundary conditions at the Poincaré horizon; we demand the modes be regular in the bulk. Finally we end up with results akin to (4.45) and (4.59b):

\[
\delta g_{ij}^{(3)} \frac{\delta g_{kl}^{(3)}}{\delta b_{ij}^{(0)} |_{TT}} = \frac{1}{6|p|} (\Theta_{ij} \Theta_{kl} + \Theta_{il} \Theta_{jk} - \Theta_{ij} \Theta_{kl}),
\] (4.60)

\[
\delta g_{ij}^{(3)} \frac{\delta g_{kl}^{(3)}}{\delta b_{ij}^{(0)} |_{TT}} = C - 2 \log |p|^2 \left( \frac{1}{6|p|} \right) \left( \Theta_{ij} \Theta_{kl} + \Theta_{il} \Theta_{jk} - \Theta_{ij} \Theta_{kl} \right).
\] (4.61)

**Lightlike modes**

Similarly we can solve (4.10) with the ansatz \( \psi_{\mu\nu}(m) = e^{iE+(+x)} \tilde{\psi}_{\mu\nu}(p_i; y) \) and the gauge condition \( \nabla^\mu \psi_{\mu\nu}(m) = 0 \). However, the modes that we find are all power series in \( y \). Therefore imposing boundary conditions, such as non-singularity of the modes in the bulk, kills one half of all solutions. Moreover, we kill all possible \( g_{ij}^{(3)} \) and \( b_{ij}^{(3)} \). Thus, the integral over the lightlike momenta does not contribute to the correlators. In fact, when actually performing the Fourier transform in the next subsection we will temporarily pass to Euclidean signature and the complication of having lightlike momenta will not play a role. For completeness, we present all lightlike modes in appendix C.
4.2 The correlators

We have already derived all results needed for computing the correlators in momentum space — in fact, up to numerical factors, the functional derivatives in the above subsection are the momentum space correlators. What remains is to translate these into a configuration space form. We shall perform this Fourier transform in Euclidean signature and then continue back to Lorentzian signature.

To illustrate the procedure, a general \( h_{ij}^{(0)}(x) \) can be written as [c.f. (4.38)]

\[
 h_{ij}^{(0)}(x) = \frac{1}{(2\pi)^3} \int d^3p e^{-ipx} \sum_{I} A_I(p) e_{ij}^I,
\]

where the factor \( 1/(2\pi)^3 \) is purely conventional. An Einstein mode sourced by this \( h_{ij}^{(0)}(x) \) would have an \( O(y) \) contribution

\[
 g_{ij}^{(3)}(x) = \frac{1}{(2\pi)^3} \int d^3p e^{-ipx} \frac{ip^3}{3} (A_3(p)c_{ij}^3 + A_4(p)c_{ij}^4) .
\]

Inverting the Fourier transform and solving for \( A_{3,4} \) in (4.62) gives

\[
 A_{3,4}(p) = \frac{1}{2} c_{3,4}^{kl}(p) \int d^3x' e^{ip(x-x')} h_{kl}^{(0)}(x') .
\]

Thus, we have

\[
 g_{ij}^{(3)}(x) = \frac{1}{(2\pi)^3} \int d^3p \int d^3x' e^{ip(x-x')} \frac{ip^3}{6} (c_{3}^{kl}c_{ij}^{3} + c_{4}^{kl}c_{ij}^{4}) h_{kl}^{(0)}(x') .
\]

Comparing with Eq. (4.42), we recognise the Fourier space functional derivative in the integrand. The configuration space derivative reads

\[
 \frac{\delta g_{ij}^{(3)}(x)}{\delta h_{kl}^{(0)}(x')} = \frac{1}{(2\pi)^3} \int d^3p e^{ip(x-x')} \frac{\delta g_{ij}^{(3)}}{\delta h_{kl}^{(0)}}(p) .
\]

Therefore, for the correlators we obtain

\[
 \langle T_{ij}(x) t_{kl}(0) \rangle = -\frac{1}{(2\pi)^3} \frac{9i}{2\kappa^2} \int d^3p e^{ipx} \frac{\delta g_{ij}^{(3)}}{\delta h_{kl}^{(0)}}(p) ,
\]

\[
 \langle t_{ij}(x) t_{kl}(0) \rangle = -\frac{2}{3} \langle T_{ij}(x) t_{kl}(0) \rangle - \frac{1}{(2\pi)^3} \frac{9i}{2\kappa^2} \int d^3p e^{ipx} \frac{\delta g_{ij}^{(3)}}{\delta h_{kl}^{(0)}}(p) .
\]

To compute these integrals we use the formula

\[
 \frac{1}{|x|^{2\alpha}} = C(\alpha) \int d^3p \frac{e^{ipx}}{|p|^{2(3/2-\alpha)}} ,
\]

\[
 C(\alpha) = \frac{1}{4^{\alpha/2} \pi^{3/2}} \frac{\Gamma(\frac{3}{2} - \alpha)}{\Gamma(\alpha)} ,
\]

and its generalisation

\[
 -\frac{1}{C(\alpha)} \ln \frac{|x|^{2\alpha}}{C(\alpha)} = \int d^3p \frac{e^{ipx}}{|p|^{2(3/2-\alpha)}} ,
\]
obtained by differentiation with respect to $\alpha$. The formulas (4.69) and (4.70) suffice to
calculate all configuration space correlators.

As shown in [41] the two-point correlator of a spin-two operator $O_{ij}$ in three dimensions
is given by

$$\langle O_{ij}(x)O_{kl}(0) \rangle = 48A \frac{1}{|x|^6} \left( \frac{1}{2} (I_{ik}I_{jl} + I_{il}I_{jk}) - \frac{1}{3} \eta_{ij} \eta_{kl} \right),$$  \hspace{1cm} (4.71)

$$I_{ij} = \eta_{ij} + 2 \frac{x_i x_j}{|x|^2},$$  \hspace{1cm} (4.72)

where $A$ is a numerical constant. We shall, however, find it more convenient to use the
form advocated in [42]

$$\langle O_{ij}(x)O_{kl}(0) \rangle = A \hat{\Delta}_{ij,kl} \frac{1}{|x|^2},$$  \hspace{1cm} (4.73)

where

$$\hat{\Delta}_{ij,kl} = \frac{1}{2} (\hat{\Theta}_{ik}\hat{\Theta}_{jl} + \hat{\Theta}_{il}\hat{\Theta}_{jk}) - \frac{1}{2} \hat{\Theta}_{ij}\hat{\Theta}_{kl},$$  \hspace{1cm} (4.74)

$$\hat{\Theta}_{ij} = \partial_i \partial_j - \eta_{ij} \Box.$$  \hspace{1cm} (4.75)

Note that $\hat{\Theta}_{ij}$ defined in (4.75), is the Fourier transform of $\Theta_{ij}$ defined in (4.43), used to
express the correlators in momentum space. Therefore, performing the Fourier transform,
keeping $\hat{\Delta}_{ij,kl}$, is quite trivial and we only need (4.69) and (4.70) for $\alpha = 1$. For the
correlator (4.67) we get

$$\langle T_{ij}(x)t_{kl}(0) \rangle = A \hat{\Delta}_{ij,kl} \frac{1}{|x|^2},$$  \hspace{1cm} (4.76)

with

$$A = \frac{1}{(2\pi)^3} \frac{6\pi}{\kappa^2}.$$  \hspace{1cm} (4.77)

The $t_{ij}$ operator contains, as explained before, three different pieces. The transverse
traceless part corresponds to the logarithmic partner of the stress tensor. It is clear from
Eqs. (4.59) that the correlators of the other two operators — $V_i$ and $S$ from (4.50) — have
the usual form for a spin-one and a spin-zero operator:

$$\langle V_i(x)V_j(0) \rangle = A_V \hat{\Theta}_{ij} \frac{1}{|x|^2},$$  \hspace{1cm} (4.78)

$$\langle S(x)S(0) \rangle = A_S \frac{1}{|x|^2},$$  \hspace{1cm} (4.79)

where

$$A_V = \frac{1}{(2\pi)^3} \pi \frac{3}{\kappa^2} \quad \text{and} \quad A_S = -\frac{1}{(2\pi)^3} \frac{3\pi}{2\kappa^2}.$$  \hspace{1cm} (4.80)

Note that the two-point correlator of the spin-zero operator is negative. This is another
manifestation of the non-unitarity of the theory and means that the theory would be non-
unitary even if the logarithmic partner of the stress tensor would be absent.
Evaluation of the transverse traceless part of (4.68) yields, via Eq. (4.59b),
\[
\langle t_{ij}(x) t_{kl}(0) \rangle = A \Delta_{ij,kl} \frac{\log |x|^2 + C + 2\gamma - 2/3}{|x|^2},
\]
(4.81)
generalising (4.73).

The ambiguity of the log-mode with respect to addition of Einstein modes is evident by the linearity of the operator \( \Delta_{ij,kl} \). Thus, we can again use the freedom of redefining \( t_{ij} \) to get rid of the factor \( C + 2\gamma - 2/3 \) via, \( t_{ij} \rightarrow t_{ij} - (C/2 + \gamma - 1/3)T_{ij} \).

We have now found all two-point functions of four-dimensional critical gravity. The two correlators (4.76) and (4.81) together with the result (4.77) for the quantity \( A \) constitute the main quantitative results of this paper.

5 Conclusions

This work is part of an effort to increase our understanding of higher-curvature gravity in four dimensions. In particular, we study the critical tuning of the coupling constants in order to determine to what extent the AdS/LCFT duality, discovered in three dimensions, extends to the four-dimensional case. As a first step in this direction we computed the one- and two-point functions for the critical theory.

There are four operators in the boundary theory, which are categorised by their sources. The stress tensor \( T_{ij} \) is sourced by \( h_{ij}^{(0)} \) and its one-point function is transverse and traceless and given by
\[
\langle T_{ij} \rangle = -\frac{9}{4\kappa^2} b_{ij}^{(3)}.
\]
(5.1)
The operators \( t_{ij}^{TT}, V_i \) and \( S \) are sourced by the corresponding components of \( b_{ij}^{(0)} \), and the one-point functions are
\[
\langle t_{ij}^{TT} \rangle = \frac{3}{2\kappa^2} \left( b_{ij}^{(3)} + 3 g_{ij}^{(3),TT} \right),
\]
(5.2)
\[
\langle V_i \rangle = \frac{9}{4\kappa^2} V_i^{(3)},
\]
(5.3)
\[
\langle S \rangle = \frac{9}{4\kappa^2} S^{(3)},
\]
(5.4)
where \( g_{ij}^{(3),TT}, V_i^{(3)} \) and \( S^{(3)} \) are defined by the York decomposition of \( g_{ij}^{(3)} \):
\[
g_{ij}^{(3)} = \nabla_i V_j^{(3)} + \nabla_j V_i^{(3)} + g_{ij}^{(3),TT} + (\nabla_i \nabla_j - \frac{1}{3} \eta_{ij} \nabla^2) S^{(3)}.
\]
(5.5)
Note that the one-point function of the vector operator satisfies a Ward identity of the form \( \nabla^i \langle V_i \rangle = 0 \).

The fact that the stress tensor vanishes for all Einstein solutions is consistent with the result that the mass and entropy of black holes in the theory vanish [21].
The non-trivial two-point correlators all match the expectations for a logarithmic CFT, $T_{ij}$ and $t_{ij}^{TT}$ forming a rank-two logarithmic pair. Explicitly,

\begin{align}
\langle T_{ij}(x) T_{kl}(0) \rangle &= 0, \\
\langle T_{ij}(x) t_{kl}^{TT}(0) \rangle &= \frac{1}{(2\pi)^3} \frac{6\pi}{\kappa^2} \frac{1}{|x|^2}, \\
\langle t_{ij}^{TT}(x) t_{kl}^{TT}(0) \rangle &= \frac{1}{(2\pi)^3} \frac{6\pi}{\kappa^2} \frac{1}{|x|^2} \log(|x|^2 m^2),
\end{align}

where we parameterised the freedom to add a multiple of an Einstein mode to the log-mode by a fiducial mass scale $m$ in (5.6c). The two-point functions of $V_i$ and $S$ have the form expected for ordinary spin-one and spin-zero operators (4.78)–(4.80), and all “mixed” correlators vanish. The relatively simple form of the log-log correlator (5.6c) suggests a natural generalisation to arbitrary dimensions:

\begin{align}
\langle t_{ij}^{TT}(x) t_{kl}^{TT}(0) \rangle \propto \Delta_{ij,kl}^{(d)} \log(|x|^2),
\end{align}

where

\begin{align}
\Delta_{ij,kl}^{(d)} = \frac{1}{2} (\hat{\Theta}_{ik} \hat{\Theta}_{jl} + \hat{\Theta}_{il} \hat{\Theta}_{jk}) - \frac{1}{d-1} \hat{\Theta}_{ik} \hat{\Theta}_{jl}.
\end{align}

To explicitly derive this result, and to compute the constant of proportionality, could be a worthwhile exercise.

The result (5.6) demonstrates that a lot of the story from three dimensions is repeated in the present case. There are, however, also differences. In three dimensions, the logarithmic graviton corresponds to two degrees of freedom, each playing the role of one chiral component of the logarithmic partner of the stress tensor. In the present case, this part is played by the transverse and traceless tensor modes. But now there are three additional degrees of freedom, parameterised by $V_i$ and $S$ above. These were called Proca modes in [25], and their normalisable representatives do not have logarithmic fall-off at the conformal boundary ($b_{ij}^{(3)} = 0$). The spin-zero operator gives rise to a negative two-point function, see (4.79) together with (4.80). This is another indication of the non-unitarity of the theory.

The Proca modes have zero energy by (5.1). It would be very interesting to know whether these modes are subject to a linearisation instability akin to that found in [10]. If they do show a linearisation instability, then all logarithmic modes can possibly be truncated by imposing boundary conditions, leaving a theory propagating only massless (and energyless!) spin-two gravitons. Otherwise, such a truncation is impossible. We note that there are other consistency requirements for a truncation to work. The boundary conditions must be consistent and the higher-point functions between truncated and untruncated modes must vanish. See, e.g., the discussions in [12, 43].

Another interesting extension of the present work would be to extend the analysis to the higher-derivative gravity models recently presented in [44]. Degeneration of multiple massive modes with each other and/or the Einstein modes might lead to gravity duals for higher-rank LCFTs in arbitrary dimensions. Such an extension would then involve the calculation of two-point functions in higher-rank Jordan cells.
Furthermore, while critical gravities are non-unitary [45] (as are all duals of LCFTs),
critical tunings of higher-derivative models that lead to odd-rank LCFTs, might open up
the interesting possibility of unitary truncations [46]. First steps of such a venture in the
gravity context are taken in [47].

Last, but not least, it would of course be very rewarding to find a suitable condensed
matter application for higher-dimensional LCFT, and to construct a phenomenological
holographic model of it, using the model explored in this paper as canvas.

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A Global coordinates: Fefferman–Graham expansion and first variation

A.1 First variation of the action

In the Gaussian normal coordinates of Eq. (2.10) we have

\[ K_{ij} = \frac{1}{2} \partial_{\rho} \gamma_{ij}, \quad \Gamma^j_{i\rho} = K^j_{i\rho}, \quad \Gamma^\rho_{ij} = -K_{ij}. \tag{A.1} \]

The matrix \( \gamma_{ij} \) is assumed to have the expansion (2.14) with the leading contributions
fixed.

In Gaussian normal coordinates and after partially integrating the variations of the
Christoffel symbols, the boundary term coming from the first variation of the bulk action
becomes

\[ \delta I_{\text{bulk}|\text{EOM}} = \frac{1}{2\kappa^2} \int_{\partial M} d^3 x \sqrt{-\gamma} J^\rho = \int_{\partial M} d^3 x \sqrt{-\gamma} \left( - (A_{ij} + A^{\rho\rho} \gamma_{ij}) \delta K_{ij} \right. \]

\[ \left. + \frac{1}{2} \nabla_\rho (A_{ij} + A^{\rho\rho} \gamma_{ij}) + \nabla_k A^{\rho\rho} \gamma_{ij} - \nabla_i A^{\rhoj} + A^{\rhoj} K^{ij} \right) \delta \gamma_{ij}. \tag{A.2} \]

For the case of interest \( \alpha = -3\beta \), the tensor \( E_{\mu\nu} \) of Eq. (2.3) is traceless, and the trace
of the equations of motion thus implies \( R = 4\Lambda = -12 \). Asymptotically this implies the
Putting these expressions together yields for the variation of the on-shell action
\[
\delta I_{\text{bulk}}|_{EOM} = \frac{1}{2K^2} \int_{\partial M} d^3x \sqrt{-\gamma} J^\rho = 
\]
\[
= \frac{1}{2K^2} \int_{\partial M} d^3x \sqrt{-\gamma} \left( [1 - 6\beta](K^{ij} \delta \gamma_{ij} - 2\gamma^{ij} \delta K_{ij}) - \frac{27\beta}{2} e^{-5p} \beta_i^j \delta \gamma_{ij} \right),
\]
(A.19)
which is the expression quoted in the main text.

For $\beta = 1/6$ we do not need any holographic counterterms to get a well-defined variational principle. This is similar to the three-dimensional case, where we did not need any counterterms for the logarithmic point of NMG [20, 35]. The stress tensor can be directly read off from the above equation and is proportional to $\beta_{ij}^{(3)}$.

If $\beta \neq 1/6$ (in which case $\beta_{ij}^{(3)}$ vanishes by virtue of the equations of motion) we need a holographic counterterm. It turns out that the same counterterms as for pure four-dimensional gravity [36, 37] multiplied with a numerical factor does the job.

After adding this counterterm, the action reads

$$I_{\text{ren}} = \frac{1}{2\kappa^2} \int_M d^4x \sqrt{-g} \left[ R - 2\Lambda + \beta(R^2 - 3R^\mu_\nu R_{\mu\nu}) \right]$$

$$- \frac{1 - 6\beta}{2\kappa^2} \int_{\partial M} d^3x \sqrt{-\gamma} \left( 4 - 2K + R[\gamma] \right),$$

(A.20)

where $R[\gamma]$ is the contracted Ricci tensor of the induced metric $\gamma_{ij}$ on the boundary. Expressed in the components of the full Ricci tensor this quantity is given by

$$R[\gamma]^{ij} = R^{ij} + \partial_\rho K^{ij} + KK^{ij} + 2K^{il}K^j_l.$$

(A.21)

Using the expansions (A.4)–(A.11) it is now straightforward to obtain

$$R[\gamma]^{ij} = -\gamma_{ij}^{(2)} e^{-4\rho} + 2\gamma_{ij}^{(0)} e^{-4\rho}.$$  

(A.22)

Armed with this equation we derive the variation of the boundary term:

$$\delta \left( \sqrt{-\gamma}(4 - 2K + R[\gamma]) \right) = \sqrt{-\gamma} \left( -2\gamma^{ij}\delta K_{ij} + [2K^{ij} - \gamma^{ij} + \gamma^{ij}_{(2)} e^{-4\rho}] \delta \gamma_{ij} \right)$$

$$+ O(\rho e^{-\rho}).$$

(A.23)

This allows us to determine the variation of the full action (A.20). Adding the contributions from Eqs. (A.19) and (A.23) produces ($\beta_{ij}^{(3)} = 0$)

$$\delta I_{\text{ren}}|_{\text{EOM}} = \frac{1}{2\kappa^2} \int_{\partial M} d^3x \sqrt{-\gamma^{(0)}} \frac{3}{2} (1 - 6\beta) \gamma^{ij}_{(3)} \delta \gamma_{ij}^{(0)}.$$  

(A.24)

This shows that the stress tensor in this case is proportional to $\gamma_{ij}^{(3)}$. As noted in the main text, because we ignore other fall-off behaviours than present in (2.14), this result does not contain contributions from massive gravitons, but only from massless gravitons and black holes.

A.2 Asymptotic equations of motion

Let us now turn to the asymptotic equations of motion. By taking the trace of the EOMs we already established tracelessness of $\gamma_{ij}^{(3)}$ and $\beta_{ij}^{(3)}$. To show that the stress tensors for the critical case ($\beta = 1/6$) and the non-critical case ($\beta \neq 1/6$) are conserved we need to take a look at the $i\rho$-components of the EOMs. For $\beta = 1/6$ we use the ansatz (2.14) and
plug it into the equations of motion. Using the simplification that follows from that $R$ is constant and throwing away terms proportional to the traces of $\gamma^{(3)}_{ij}$ and $\beta^{(3)}_{ij}$ we obtain

$$EOM_{i\rho} = \frac{9}{4} \nabla^k (0) \beta^{(3)}_{ki} e^{-3\rho} + O(\rho e^{-4\rho}).$$

(A.25)

Here $\nabla(0)$ is the covariant derivative with respect to $\gamma(0)_{ij}$. This proves the conservation law for the critical case.

In the non-critical case $\beta \neq 1/6$ we do not have logarithmic modes and therefore we have to omit the $\beta^{(3)}_{ij}$ term in the expansion (2.14). Again the $i\rho$-components of the EOMs yield

$$EOM_{i\rho} = \frac{3}{2} (1 - 6\beta) \nabla^k (0) \gamma^{(3)}_{ki} e^{-3\rho} + O(\rho e^{-4\rho}),$$

(A.26)

up to trace terms. Thus, the stress tensors (2.21) and (2.22) for the action (2.1) are traceless and conserved for any value of $\beta$.

B Poincaré coordinates: Fefferman–Graham expansion and second variation

We expand around Poincaré patch AdS:

$$ds^2 = \frac{dy^2}{y^2} + \gamma_{ji} dx^i dx^j = \frac{dy^2 + \eta_{ij} dx^i dx^j}{y^2} + h_{ij} dx^i dx^j,$$

(B.1)

where $\eta_{ij}$ is the flat 3d Minkowski metric, and $h_{ij}$ has the expansion

$$h_{ij} = \frac{b^{(0)}_{ij} \log y}{y^2} + \frac{h^{(0)}_{ij}}{y^3} + b^{(2)}_{ij} \log y + g^{(2)}_{ij} + \frac{b^{(3)}_{ij} y \log y + g^{(3)}_{ij} y}{y^2} + \ldots$$

(B.2)

near $y = 0$. Our goal is to compute the variation of the terms multiplying the variations $\delta \gamma_{ij}$ and $\delta K_{ij}$ in (A.2). To give them names, let us define

$$\delta I_{bulk}|_{EOM} = \frac{1}{2\kappa^2} \int_{\partial M} d^3x \sqrt{-\gamma} J^\rho = \int_{\partial M} d^3x \sqrt{-\gamma} \left( R^{ij}_1 \delta K_{ij} + R^{ij}_2 \delta \gamma_{ij} \right).$$

(B.3)

We have computed $R^{ij}_1$ and $R^{ij}_2$ only on-shell in (A.2), but this is all we need. In fact, to compute the correlators, we put in variations that satisfy the linearised equations of motion. Therefore, it is enough to vary the on-shell quantities, and we may, along the way, use on-shell relations between the Fefferman–Graham coefficients.

For the purpose of expanding $R^{ij}_1$ and $R^{ij}_2$, and to expand the equations of motion, we need the Ricci tensor to the first order in $h_{ij}$. It is most simply obtained using some
Note that the operator $D_g$ corresponding to a metric perturbation $\mu_{\nu}$ just ordinary derivatives. We also defined the differential operator $\eta$ taken with respect to the metric $g$.

Let us start by expanding the linearised version of EOM.

### B.1 Linearised equations of motion

**Evaluation of these quantities, which all must vanish, yields the linearised equations of motion.** From the constancy of the Ricci scalar we obtain the following relations ($R^{(1)}_0$, $R^{(1)}_1$, and $R^{(1)}_2$ are all manifestly zero)

\begin{align}
R^{(1)}_0 &= -4 \text{Tr} \ b^{(0)} = 0, \\
R^{(1)}_1 &= \nabla^2 \text{Tr} \ b^{(0)} - \nabla_i \nabla_j b^{ij} - 4 \text{Tr} \ b^{(2)} = 0, \\
R^{(1)}_2 &= \nabla^2 \text{Tr} \ h^{(0)} - \nabla_i \nabla_j h^{ij} - 4 \text{Tr} \ g^{(2)} = 0, \\
R^{(2)}_1 &= -3 \text{Tr} \ b^{(3)} = 0, \\
R^{(2)}_2 &= -3 \text{Tr} \ g^{(3)} - 2 \text{Tr} \ b^{(3)} = 0,
\end{align}

and the linearised Ricci scalar as

\begin{equation}
R^{(1)} = \sum_{n=0}^{\infty} \left[ R^{L}_{n} y^{n} \log y + R_{n} y^{n} \right].
\end{equation}

In these equations, and always when it comes to the quantities $g^{(n)}_{ij}$ and $b^{(n)}_{ij}$, the trace is taken with respect to the metric $\eta_{ij}$, i.e., $\text{Tr} \ b^{(n)} = \eta^{ij} b_{ij}^{(0)}$. Also, the indices of $g^{(n)}_{ij}$ and $b^{(n)}_{ij}$ are always raised by $\eta^{ij}$ and all covariant derivatives are with respect to $\eta_{ij}$, i.e., are just ordinary derivatives. We also defined the differential operator $D^2$ as

\begin{equation}
(D^2 g^{(n)})_{ij} = \nabla_i \nabla_j (\text{Tr} \ g^{(n)}) + \nabla^2 g^{(n)}_{ij} - (\nabla_i \nabla_k g^{(n)}_{kj}) + (\nabla_j \nabla_k g^{(n)}_{ki}).
\end{equation}

Note that the operator $D^2$ computes, up to a factor of $-1/2$, the linearised Ricci scalar corresponding to a metric perturbation $g^{(n)}_{ij}$ around a flat background.
from which we deduce
\[ \text{Tr } b^{(0)} = \text{Tr } g^{(3)} = \text{Tr } b^{(3)} = 0, \quad (B.15) \]
\[ \nabla_i \nabla_j b_{(0)}^{ij} + 4\text{Tr } b^{(2)} = 0, \quad (B.16) \]
\[ \nabla^2 \text{Tr } h^{(0)} - \nabla_i \nabla_j h_{(0)}^{ij} - 4\text{Tr } g^{(2)} = 0. \quad (B.17) \]

Using these equations to simplify the expressions coming from \((\text{EOM}_{b}^{L})_{ij}\) and \((\text{EOM}_{0})_{ij}\) yields
\[
(D^2 b^{(0)})_{ij} = 2b^{(2)}_{ij} + 2\text{Tr } b^{(2)} \eta_{ij}, \quad (B.18)
\]
\[
(D^2 h^{(0)})_{ij} = -\frac{1}{2} \nabla^2 b^{(0)}_{ij} - 3b^{(2)}_{ij} + 2g^{(2)}_{ij} + \left[ 2\text{Tr } g^{(2)} + \text{Tr } b^{(2)} \right] \eta_{ij}. \quad (B.19)
\]

### B.2 Second variation of the action

Using the equations of motion derived in the last subsection, we now want to compute the action to second order in the perturbation and expand it in \(y\). Also this is best done using computer algebra. When we present the result we shall not keep track of the two variations separately since, for computing correlators, all we need are the symmetrised variations. More explicitly, if, say
\[
\delta^2 S[\delta_1 \gamma_{ij}, \delta_2 \gamma_{ij}] = \frac{1}{2\kappa^2} \int_{\partial M} d^3 x \sqrt{-\gamma} \left( \delta_1 h_{(0)}^{ij} \delta_2 g_{(3)}^{ij} + \ldots \right), \quad (B.20)
\]
we shall only write
\[
\delta^2 S = \frac{1}{2\kappa^2} \int_{\partial M} d^3 x \sqrt{-\gamma} \left( \delta h_{(0)}^{ij} \delta g_{(3)}^{ij} + \ldots \right). \quad (B.21)
\]

In this appendix, we shall often even leave the integral sign, the factors of \(1/2\kappa^2\) and \(\sqrt{-\gamma}\), and the \(\delta\) out and write
\[
\delta^2 S = h_{(0)}^{ij} g_{(3)}^{ij} + \ldots \quad (B.22)
\]
when actually meaning \((B.20)\), hoping that this does not lead to confusion.

Using these conventions, and using the equations of motion, the second variation of the bulk action reads
\[
\delta^2 I_{\text{bulk}}|_{\text{EOM}} = -\frac{3}{4} b_{(0)}^{ij} b_{(0)}^{ij} \frac{1}{y^3} + \left[ \frac{1}{2} b_{(0)}^{ij} b_{(2)}^{ij} + 2h_{(0)}^{ij} b_{(2)}^{ij} - 2b_{(0)}^{ij} g_{(2)}^{ij} \right] \frac{1}{y}
\]
\[ - \frac{3}{2} b_{(0)}^{ij} b_{(2)}^{ij} \log y - \frac{3}{4} b_{(0)}^{ij} g_{(3)}^{ij} - \frac{3}{4} h_{(0)}^{ij} b_{(3)}^{ij}. \]

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B.3 Boundary terms

We now apply the same computational and notational framework to various counterterms. The terms in question are \( (F_{\mu \nu} \text{ is the auxiliary field and } \hat{F} = \gamma^{ij} F_{ij}) \)

\[
I_{GGH} = \frac{1}{2\kappa^2} \int d^3 x \sqrt{-\gamma} F^{ij} (K \gamma_{ij} - K_{ij}) , \quad I_{\hat{F}} = \frac{1}{2\kappa^2} \int d^3 x \sqrt{-\gamma} \hat{F} , \\
I_{FF} = \frac{1}{2\kappa^2} \int d^3 x \sqrt{-\gamma} F^{ij} F_{ij} , \quad I_{FR} = \frac{1}{2\kappa^2} \int d^3 x \sqrt{-\gamma} F^{ij} R_{ij}^{(3)} , \\
I_{F\nabla F} = \frac{1}{2\kappa^2} \int d^3 x \sqrt{-\gamma} F^{ij} \nabla^2 F_{ij} , \quad I_{FDF} = \frac{1}{2\kappa^2} \int d^3 x \sqrt{-\gamma} F^{ij} (D^2 F)_{ij} , \\
I_{h\hat{F}} = \frac{1}{2\kappa^2} \int d^3 x \sqrt{-\gamma} \gamma^{ij} (\gamma_{ij} - \eta_{ij}/y^2) \hat{F} . \tag{B.23}
\]

The second variations of these quantities are\(^6\)

\[
\delta^{(2)} (I_{GGH} - 2I_{\hat{F}}) = \frac{3}{2} b_{ij}^{(0)} b_{ij}^{(0)} \frac{1}{y^3} + 3b_{ij}^{(0)} b_{ij}^{(2)} \log y \frac{1}{y} + \left[ b_{ij}^{(0)} b_{ij}^{(2)} - h_{ij}^{(0)} h_{ij}^{(2)} + \text{Tr} b^{(2)} \text{Tr} h^{(0)} + 4b_{ij}^{(0)} g_{ij}^{(2)} \right] \frac{1}{y} , \tag{B.24}
\]

\[
\delta^{(2)} I_{FF} = \frac{9}{2} b_{ij}^{(0)} b_{ij}^{(0)} \frac{1}{y^3} + \left[ -3b_{ij}^{(0)} b_{ij}^{(2)} - 6b_{ij}^{(0)} b_{ij}^{(2)} + 6\text{Tr} b^{(2)} \text{Tr} h^{(0)} + 6b_{ij}^{(0)} g_{ij}^{(2)} \right] \frac{1}{y} - 9b_{ij}^{(0)} b_{ij}^{(3)} , \tag{B.25}
\]

\[
\delta^{(2)} I_{FR} = -3b_{ij}^{(0)} b_{ij}^{(2)} \log y + \left[ -3b_{ij}^{(0)} b_{ij}^{(2)} + 3\text{Tr} b^{(2)} \text{Tr} h^{(0)} \right] \frac{1}{y} , \tag{B.26}
\]

\[
\delta^{(2)} I_{F\nabla F} = \left[ -27b_{ij}^{(0)} b_{ij}^{(2)} - 18b_{ij}^{(0)} b_{ij}^{(2)} + 18\text{Tr} b^{(2)} \text{Tr} h^{(0)} + 18b_{ij}^{(0)} g_{ij}^{(2)} \right] \frac{1}{y} , \tag{B.27}
\]

\[
\delta^{(2)} I_{FDF} = 9b_{ij}^{(0)} b_{ij}^{(2)} \frac{1}{y} , \tag{B.28}
\]

\[
\delta^{(2)} I_{h\hat{F}} = 2\text{Tr} b^{(2)} \text{Tr} h^{(0)} \frac{1}{y} . \tag{B.29}
\]

Combining these results and defining

\[
I_{\text{tot}} = I_{\text{bulk}} + I_{GGH} - 2I_{\hat{F}} - \frac{1}{2} I_{FF} + I_{FR} - \frac{1}{18} I_{F\nabla F} - \frac{5}{18} I_{FDF} - I_{h\hat{F}} , \tag{B.30}
\]

it is straightforward to show that all divergent terms cancel. The final result is

\[
\delta^{(2)} I_{\text{tot}|\text{EOM}} = \frac{1}{2\kappa^2} \int d^3 x \sqrt{-\gamma} \left[ \frac{3}{2} b_{ij}^{(0)} b_{ij}^{(3)} + \frac{9}{4} (b_{ij}^{(0)} g_{ij}^{(3)} - h_{ij}^{(0)} b_{ij}^{(3)}) \right] . \tag{B.31}
\]

\(^6\)The generalised Gibbons–Hawking term \( I_{GGH} \) actually has a non-vanishing first variation, but a linear combination of \( I_{GGH} \) and \( I_{\hat{F}} \) has vanishing first variation. The variation of this combination is displayed in (B.24).
C Lightlike Modes

To obtain the lightlike modes we solve (4.10) with the ansatz
\[ \psi_{\mu\nu}(m) = e^{iE(t+x)}\tilde{\psi}_{\mu\nu}(p_i; y) \]
and the gauge condition \( \nabla^{\mu}\psi_{\mu\nu}(m) = 0 \). Going to Gaussian normal coordinates and choosing combinations of log and Einstein modes such that all \( h^{(0)}_{ij} \) are zero we find the following log-modes:

\[
\psi_{\mu\nu}^{\log 1} = e^{iE(t+x^1)} \frac{c_3 \log y + c_4 y^3 \log y}{y^2} \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 1 & 0
\end{pmatrix},
\]
\[
\psi_{\mu\nu}^{\log 2} = e^{iE(t+x^1)} \frac{c_9 \log y + c_{10} y^3 \log y}{y^2} \begin{pmatrix}
1 & 1 & 0 \\
1 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

These are vector modes orthogonal to our chosen \( p_i \). Three of the other log-modes turn out to be Proca modes, their respective Einstein modes are pure gauge modes. Note that there are no explicit logarithms, but they are identified as log-modes by the equations of motion:

\[
\psi_{\mu\nu}^{\text{Proca1}} = e^{iE(t+x^1)} c_2 y \left( 1 + \frac{y^2 E^2}{25} \right) \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 1 \\
1 & 1 & 0
\end{pmatrix},
\]
\[
\psi_{\mu\nu}^{\text{Proca2}} = e^{iE(t+x^1)} c_6 y \left[ 1, 0, 0 \right] + \frac{2y^2 E^2}{25} \begin{pmatrix}
2 & 0 & 0 \\
0 & -2 & 0 \\
0 & 0 & -1
\end{pmatrix} + \frac{4y^4 E^4}{1225} \begin{pmatrix}
1 & 1 & 0 \\
1 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix},
\]
\[
\psi_{\mu\nu}^{\text{Proca3}} = e^{iE(t+x^1)} c_8 y \left[ -1, 0, 0 \right] - \frac{2y^2 E^2}{25} \begin{pmatrix}
1 & 1 & 0 \\
1 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

Finally, we also have

\[
\psi_{\mu\nu}^{\log 3} = e^{iE(t+x^1)} \frac{c_1}{y^2} \log y \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{pmatrix} - \frac{y^2 E^2}{4} (3 + 2 \log y) \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 1 \\
1 & 1 & 0
\end{pmatrix},
\]
\[
\psi_{\mu\nu}^{\log 4} = e^{iE(t+x^1)} \frac{c_7}{y^2} \log y \begin{pmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -2
\end{pmatrix} - \frac{y^2 E^2}{2} (3 + 2 \log y) \begin{pmatrix}
1 & 1 & 0 \\
1 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix},
\]

and

\[
\psi_{\mu\nu}^{\log 5} = e^{iE(t+x^1)} \frac{c_5}{y^2} \log y \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix} + \frac{y^2 E^2}{4} (2 + \log y) \begin{pmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -2
\end{pmatrix}
+ \frac{y^2 E^2}{4} \log y \begin{pmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -2
\end{pmatrix} + \frac{y^4 E^4}{32} (5 - 4 \log y) \begin{pmatrix}
1 & 1 & 0 \\
1 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]
The corresponding 'simple' Einstein modes read

$$\psi_{\text{Einst}1}^{\mu\nu} = e^{iE(t+x^1)} \frac{c3 + y^2 c4}{y^2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad (C.10)$$

$$\psi_{\text{Einst}2}^{\mu\nu} = e^{iE(t+x^1)} \frac{c9 + y^3 c10}{y^2} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (C.11)$$

The Einstein modes corresponding to the log-modes $\psi^{\log 3}$, $\psi^{\log 4}$ and $\psi^{\log 5}$ are

$$\psi_{\text{Einst}3}^{\mu\nu} = e^{iE(t+x^1)} \frac{c1}{y^2} \left[ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} - \frac{y^2 E^2}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \right], \quad (C.12)$$

$$\psi_{\text{Einst}4}^{\mu\nu} = e^{iE(t+x^1)} \frac{c7}{y^2} \left[ \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} + \frac{y^2 E^2}{2} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right], \quad (C.13)$$

$$\psi_{\text{Einst}5}^{\mu\nu} = e^{iE(t+x^1)} \frac{c5}{y^2} \left[ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{y^2 E^2}{2} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix} - \frac{y^4 E^4}{8} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right]. \quad (C.14)$$

The gauge modes should be constructed from

$$\psi_{\text{Egauge}1}^{\mu\nu} = e^{iE(t+x^1)} \frac{E}{y^2} \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \psi_{\text{Egauge}2}^{\mu\nu} = e^{iE(t+x^1)} \frac{E}{y^2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (C.15)$$

$$\psi_{\text{Egauge}3}^{\mu\nu} = e^{iE(t+x^1)} \frac{E}{y^2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad (C.16)$$

$$\psi_{\text{Egauge}4}^{\mu\nu} = e^{iE(t+x^1)} \frac{1}{y^2} \left[ \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \frac{y^2 E^2}{2} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right]. \quad (C.17)$$

As mentioned in the main text all lightlike modes are power series in $y$ (some multiplied by logarithms). Requiring non-singularity of the modes at $y \to \infty$ kills all the $g^{(3)}_{ij}$ and $b^{(3)}_{ij}$ parts. Therefore, the lightlike modes do not contribute to the correlators.

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