RANDOM GRAPHS FROM RANDOM MATRICES

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Abstract. In the paper [GPCI15], the authors introduced the order complex corresponding to a symmetric matrix. In this note, we use it to define a class of models of random graphs, and show some surprising experimental results, showing sharp phase transitions.

1. Introduction

In the paper [GPCI15] the authors introduce the ”order complex” associated to a (symmetric) matrix. Briefly, we view the symmetric $n \times n$ matrix $M$ (with its diagonal set to zero) as the adjacency matrix of the complete graph $K_n$, and now we produce an increasing family of graphs, starting with the completely disconnected graph on $n$ vertices, and then adding edges in order of increasing size of the corresponding entry of the matrix $M$, until $p*n(n-1)/2$ edges have been added (in other words, the edge density in the graph is $p$). It is now natural to look at different models of random matrices, use them to generate random graphs, and see what the properties of the random graphs are.

Example 1.1. Suppose $M$ is drawn from the ensemble of symmetric matrices with i.i.d Gaussian entries (note: for this model it is irrelevant what the mean of the Gaussian is). Then the random graphs are nothing but the much studied Erdős-Rényi random graphs.

Example 1.2. In the upcoming paper [CR19] we generate a random vector $v$ and look at the rank one matrix $M(v) = v^t v$ - in the case where the entries of $v$ are iid $N(0,1)$, this is a Wishart ensemble. However, if we pick the entries of $v$ to be uniform in $[0,1]$, we get a model with other properties $^1$.

Example 1.3. This is, in a way, the motivating example: consider a point cloud approximating some shape in $\mathbb{R}^n$ (usually for $n = 2, 3$), and let $M$ be the distance matrix of this cloud (that is, the $M_{ij}$ equals the distance between the $i$th and the $j$th points in the cloud.

In this paper we look at the Laplacian eigenvalues of the graphs we construct. There are (at least) two ways to define the Laplacian matrix of a graph. The first, and simplest is

$$L = D - A,$$

$^1$In the paper [CR19] we look at the associated clique complexes, not the graphs per se.

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where $D$ is the diagonal matrix of degrees of vertices and $A$ is the adjacency matrix of the graph $G$. The second is the normalized symmetric Laplacian (see [CG97]):

$$L = D^{-\frac{1}{2}}LD^{-\frac{1}{2}},$$

with $L$ as above.

It turns out that the normalized Laplacian is much better behaved. Note that the normalized Laplacian spectrum is contained between 0 and 2, and the mean is at 1, since the trace of $L$ is always equal to $n$.

2. Spectral Gap

2.1. Erdős-Rényi model. We see that the “raw” spectral gap - Figure 1a increases linearly from 0 to the value of the complete graph ($K_{2000}$ in this case), while the normalized gap - Figure 1b - is asymptotic to 1. The latter case has been studied - $\lambda_2 \asymp C1 - Cn^{-\frac{1}{2}}$, see [HKP19], but the former seems to be a new observation.

2.2. Positive rank one model. We notice that the raw spectral gap - Figure 2a- seems to increase like $\sqrt{p}$, while the normalized gap - Figure 2b - is increasing linearly.
To confirm the first observation, let us plot the square root of the gap: Note that Figure 3 is consistent with quadratic growth of the spectral gap. It is also interesting that the two ends (near the completely disconnected and complete graphs) seem symmetric.

2.3. Rank 1 Wishart model. The evolution of the spectral gap (see Figure 4) looks starkly different in the Wishart model. Part of the explanation is that (as noted in [CR19]), the graph stays bipartite for low density, until at (roughly) density
\[ p = \frac{1}{2} \] it becomes complete bipartite (recall that the Laplace eigenvalues of \( K_{m,n} \) are \( m+n, n,m, 0 \), with multiplicities \( 1, m-1, n-1, 1 \)). However, this explains only some of the features of the evolution (in particular, the sharp phase transition just before the graph becomes complete bipartite and the non-monotonicity of the function).

2.4. **Point clouds.** We now look at the "motivating examples" - point clouds in low-dimensional spaces. The point clouds we look at are the noisy circle and the noisy torus, both found in the Eirene ([HG16]) distribution - see Figures 5a and 5b. We convert these point clouds into distance matrices, and see the following spectral behavior: It is quite obvious to the naked eye that the spectral gap curves in Figures 6 and 7 are very similar to those in the positive rank one case (Figure 2).

### 3. Spectral densities

3.1. **Erdős-Rényi.** The spectral density of the Erdős-Rényi random graph has been extensively studied (see, for example [EKY+13]) - the "raw" spectrum seems to have...
been more extensively studied, and found to satisfy the semicircle law (as the reader might be convinced by looking at the figures 8 and 9).
We see that the shapes (whatever that means) of the curves stabilize fairly quickly, and only the width is shrinking with increasing $p$. It is thus natural to look at the width as a function of $p$. Instead of the width (which is a little hard to define, we just look at the standard deviation of the empirical distribution of eigenvalues. Let us do it for the normalized spectrum: We see in Figure 10 that the standard deviation rises sharply until $p = 1/n$, and then declines.

3.2. **Positive rank 1.** First let us look at the spectral distribution: The raw distribution is interesting (there is a large spike at $n$), but what is more interesting is that the normalized Laplacian has extreme concentration of eigenvalues at 1, completely unlike the Erdős-Rényi model. The standard deviation of the spectral distribution is (not surprisingly) much smaller, and it is also much less regular, the peak is also achieved for a far larger $p$.

3.3. **Wishart rank 1.** The Wishart rank one graphs show essentially the same behavior as the positive rank one case, with a very tight concentration around 1, and rapid decay, but also a massive concentration at 0 (indicating many connected components) for $p < 0.5$ See Figure 14. The standard deviation is quite different from
3.4. **Point Clouds.** Here we look at the spectral distribution of the point clouds (noisy circle and noisy torus). It is evident that these are very close to the positive rank one matrices - the reader can judge for his or her own self. The bulk density at the positive case — see Figure 15 — showing the usual phase transition at $p = 0.5$. 

![Figure 12](image1.png)

**Figure 12.** Positive rank 1 model, $p = 0.2$

![Figure 13](image2.png)

**Figure 13.** Standard deviation of spectral density for positive rank 1 model

![Figure 14](image3.png)

**Figure 14.** Spectral density of Wishart rank 1 model
Figure 15. Standard deviation of spectral density for Wishart rank 1 model

Figure 16. Spectral density at $p = 0.2$

$p = 0.2$ is in Figure 16. The evolution of standard deviation for the circle is given in Figure 17, for the torus in Figure 18.

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Figure 17. Standard deviation of spectral density for noisy circle

Figure 18. Standard deviation of spectral density for noisy torus

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