Trivial-source endotrivial modules for sporadic groups

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Abstract
We determine the group of endotrivial modules (as an abstract group) for \( G \) a (quasi)simple group of sporadic type, extending previous results in the literature. In many sporadic cases we directly construct the subgroup of trivial-source endotrivial modules. We also resolve the question of whether certain simple modules for sporadic groups are endotrivial, posed by Lassueur, Malle and Schulte, in the majority of open cases. The results rely heavily on a recent description of the group of trivial-source endotrivial modules due to Grodal.

Keywords  Sporadic groups · Representation theory · Endotrivial module · Trivial-source module

Mathematics Subject Classification  Primary 20C34; Secondary 20C20

1 Introduction

Dade once wrote “There are just too many modules over \( p \)-groups”, and introduced endopermutation – modules \( M \) such that \( M \otimes M^* \) is a permutation module – and endotrivial – where it is the sum of a projective and the trivial module – modules as a consequence (Dade 1978). These modules have found their way into many parts of representation theory, from the structure of nilpotent blocks to equivalences of categories. It is fair to say that the endotrivial modules are one of the most important classes of modules in the representation theory of finite groups.

The set of isomorphism classes of indecomposable endotrivial modules for a finite group \( G \) possesses a natural abelian group structure induced by the tensor product, called the group of endotrivial modules, and denoted \( T(G) \). If \( G \) is a finite \( p \)-group then the structure and generators for \( T(G) \) have been completely determined (Carlson and Thévenaz 2004). Over the last decade, the focus has shifted to understanding \( T(G) \)

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for \( G \) an arbitrary finite group. Any attempt to apply the classification of finite simple groups and Clifford theory will necessarily require a detailed understanding of \( T(G) \) for \( G \) a finite simple group with Sylow \( p \)-subgroup \( S \). The restriction of an endotrivial module is always endotrivial, and so there is a group homomorphism \( T(G) \rightarrow T(S) \), which by Carlson et al. (2013), Mazza and Thévenaz (2007) is a split map. Write \( T(G, S) \) for the kernel, the subgroup of \( T(G) \) consisting of trivial-source endotrivial modules. The determination of \( T(G, S) \) is the main problem in endotrivial modules for arbitrary finite groups.

Lassueur and Mazza (2015b) made some headway on the problem of determining \( T(G, S) \) for \( G \) a quasisimple sporadic group by computing it for small sporadic groups (the Monster was considered in Grodal (2016)), and Carlson et al. (2009) and Carlson et al. (2010) determine \( T(G, S) \) for an alternating or symmetric group (their covering groups were studied in Lassueur and Mazza (2015a)). Both results contain minor errors \((T(G, S) \text{ for } G = J_3 \text{ and } p = 2 \text{ should be } 1, \text{ not } 3, \text{ and for } G = A_{2p}.A_{2p+1} \text{ and } p > 3 \text{ it should be } 4, \text{ not } 2 \times 2, \text{ the former corrected here and the latter in Grodal (2016). We complete the determination of } T(G, S) \text{ for all sporadic groups } G, \text{ and determine } T(G, S) \text{ for } G \text{ 'almost quasisimple’ of sporadic type, e.g., a double cover of a sporadic. Together with work recently announced by Carlson, Grodal, Mazza and Nakano (and still in preparation) that completes the determination of } T(G, S) \text{ for Lie type groups in non-defining characteristic, this completes the determination of } T(G, S) \text{ for all finite quasisimple groups } G.\)

We show that \( T(G, S) \neq 1 \) if \( G = J_3 \) and \( p = 3 \). This group has 3-rank 3, and is the only example here of a quasisimple group \( G \) having \( p \)-rank greater than 2 and with \( T(G, S) \neq 1 \). This compares with Lassueur and Malle (2015), where it is shown that if a quasisimple group \( G \) possesses a simple endotrivial module then it has \( p \)-rank at most 2. (The groups \( G = \text{SL}_2(p^a) \) have \( T(G, S) \cong C_{p^a-1} \), because they have Sylow \( p \)-subgroups that intersect trivially (see Carlson et al. 2006, and of course have arbitrarily large \( p \)-rank.)

**Theorem 1.1** Let \( G \) be a quasisimple sporadic group or the Tits group \( T = 2F_4(2)' \). The group \( T(G, S) \) is the abelianization (quotient by the derived subgroup) of the group in Table 1. If \( G \) is a quasisimple sporadic group such that the \( p \)-rank of \( G \) is at least 3 and \( T(G, S) \neq 1 \), then \( G = J_3 \) and \( p = 3 \) (which has \( p \)-rank exactly 3).

(The result for \( 3 \cdot J_3 \) in characteristic 2 differs from that in Lassueur and Mazza (2015b). There is a torsion endotrivial module for \( M_{11} \) and \( p = 2 \), but it is not trivial source (so does not occur in Table 1), and comes from the torsion element in \( T(S) \).) We must also determine the actions of outer automorphisms on the groups in Table 1, to understand the group \( T(G, S) \) when \( G \) is an almost simple sporadic group. This information is given in Sect. 4.3.

We also attempt to construct all modules in \( T(G, S) \) for \( G \) a quasisimple sporadic group as well. For \( G \) ‘small’ we can do so, but for ‘large’ groups, namely \( M_{11}, C_{O_3}, C_{O_2}, J_4, Th \) and \( BM \), we cannot explicitly construct all of the (non-trivial) elements of \( T(G, S) \).

We also consider the case of simple endotrivial modules for quasisimple sporadic groups \( G \). Lassueur et al. 2016, Table 6 give a partial classification of simple endotrivial
Table 1 The groups $N_G(S)/K_G$ (see Sect. 2) for $G$ a sporadic simple group and $S$ non-cyclic

| Group       | $p = 2$ | $p = 3$ | $p = 5$ | $p = 7$ | $p = 11$ | $p = 13$ |
|-------------|---------|---------|---------|---------|---------|---------|
| $M_{11}$    | 1 (SN)  | $SD_{16}$ (TI) |
| $M_{12}$    | 1 (SN)  | 1 ($K^\circ$) |
| $M_{22}$    | 1 (SN)  | $Q_8$ (r2) |
| $2 \cdot M_{22}$ | —       | SmallGroup(16,4) |
| $M_{23}$    | 1 (SN)  | 2 (r2) |
| $M_{24}$    | 1 (SN)  | 1 ($K^\circ$) |
| $J_1$       | 1 ($K^\circ$) |
| $J_2$       | 1 ($K^\circ$) | 2 (nnc) | 2 (r2) |
| $J_3$       | 1 ($K^\circ$) | 2 (nnc)* |
| $J_4$       | 1 (SN)  | 1 ($K^\circ$) | 5 $\times$ $2S_3$ (TI) |
| $HS$        | 1 (SN)  | 2 $\times$ 2 (r2) | 4 (nnc) |
| $2 \cdot HS$ | —       | $D_8$       | —       |
| $McL$       | 1 (SN)  | 1 ($K^\circ$) | 3 $\times$ 8 (TI) |
| $3 \cdot McL$ | —       | —           | 3 $\times$ (3 $\times$ 8) |
| $Suz$       | 1 ($K^\circ$) | 1 ($K^\circ$) | 2 (r2) |
| $He$        | 1 (SN)  | 1 ($K^\circ$) | 3 (r2) | 1 ($K^\circ$)* |
| $Ru$        | 1 ($K^\circ$) | 2 (const) | 1 ($K^\circ$)* |
| $HN$        | 1 ($K^\circ$) | 1 ($K^\circ$) | 1 ($K^\circ$) |
| $ON$        | 1 (SN)  | 1 ($K^\circ$)* | 1 ($K^\circ$) |
| $Ly$        | 1 (SN)  | 1 ($K^\circ$) | 1 ($K^\circ$) |
| $Co_3$      | 1 (SN)  | 1 ($K^\circ$) | 4 (nnc)* |
| $Co_2$      | 1 (SN)  | 1 ($K^\circ$) | $S_3$ (nnc)* |
| $Co_1$      | 1 (SN)  | 1 ($K^\circ$) | 1 ($K^\circ$)* | 1 ($K^\circ$)* |
| $Fi_{22}$   | 1 (SN)  | 1 ($K^\circ$) | $S_3$ (r2) |
| $i \cdot Fi_{22}$ | —       | —           | $i \times S_3$ |
| $Fi_{23}$   | 1 (SN)  | 1 ($K^\circ$)* | 2 (r2) |
| $Fi_{24}'$  | 1 (SN)  | 1 ($K^\circ$)* | 1 ($K^\circ$) | 1 ($K^\circ$)* |
| $Th$        | 1 (SN)  | 1 ($K^\circ$)* | 1 ($K^\circ$)* | 2 (r2)* |
| $BM$        | 1 (SN)  | 1 ($K^\circ$)* | 1 ($K^\circ$)* | 2 (r2)* |
| $M$         | 1 (SN)  | 1 ($K^\circ$) | 1 ($K^\circ$) | 1 ($K^\circ$) | 1 ($K^\circ$) |
| $T$         | 1 (SN)  | 1 ($K^\circ$) | 4$A_4$ (TI) |

For $T(G, S)$, take the abelianization of this group. The labels (SN) and so on are indicators of the proof, and are explained in Sect. 4. For $\hat{G}$ a central extension of $G$, we have $N_{\hat{G}}(S)/K_{\hat{G}} \cong N_G(S)/K_G$ unless this table states otherwise. For these lines of the table, a $-$ indicates that $N_{\hat{G}}(S)/K_{\hat{G}} \cong N_G(S)/K_G$. * indicates that the result does not appear elsewhere in the literature.
modules for quasisimple sporadic groups, with (up to duality) fourteen potential simple endotrivial modules undecided. Here we prove or disprove endotriviality for eleven of the fourteen modules left over in Lassueur et al. (2016), four of which were already determined in Lassueur and Malle (2015) (although we provide alternate proofs for those four modules here).

The proof of Theorem 1.1 uses a local-group-theoretic characterization of $T(G, S)$ found by Grodal (2016). Because that paper is long and is mainly homotopy theory, in Sect. 3 we give a short proof that shows that Grodal’s criterion is equivalent to the characterization of $T(G, S)$ as the group of ‘weak homomorphisms’ given in Balmer (2013). That result uses representation theory, and so together this removes the homotopy theory from the proof.

(The proof given here can be pieced together from various places in Grodal (2016) as well, and of course the homotopy theory is how the characterization was originally found. But it does mean that algebraists have a complete proof of all results in the field.)

Grodal’s characterization, given in the next section, turns the computation of $T(G, S)$ into a question about normalizers, which means finite group theory can be used to attack it. This enables us to obtain much better results much more easily than using previous methods; some of the work from several papers (Carlson et al. 2010, 2009; Lassueur and Mazza 2015a, b) can been condensed into just a few pages.

2 Preliminary results

Throughout this work $G$ is a finite group, $p$ is a prime and $k$ is an algebraically closed field of characteristic $p$. Let $S$ denote a Sylow $p$-subgroup of $G$, and write $N$ as shorthand for $N_G(S)$. Write $G'$ for the derived subgroup of $G$, and $G^{ab}$ for $G/G'$, the abelianization of $G$. Our notation for simple groups follows (Conway et al. 1985), and the notation for conjugacy classes as well. The notation is a number followed by a letter, e.g., 2B, where the number denotes the order of an element and the letter denotes the position of the class among all classes of elements of order 2, when ordered according to increasing class size. The structure of $N_G(S)$ for $G$ sporadic is given in Wilson (1998).

If $H$ is a subgroup of $G$ write $(-)\downarrow_H$ for restriction to $H$. Write $V^*$ for the dual of a $kG$-module $V$, and $V^\pm$ to mean $V$ and/or $V^*$. When writing socle structures of a module, we use ‘/’ to delineate the layers, so that

$$A/B, C/D, E$$

is a module with socle $D \oplus E$, second socle $B \oplus C$ and third socle $A$.

A $kG$-module $V$ is endotrivial if $V \otimes V^*$ is the sum of the trivial module $k$ and a projective module. Every endotrivial module is the sum of a projective module and an indecomposable module. The collection of endotrivial modules forms a group under tensor product if one ignores projective factors, and this group is denoted $T(G)$. Write $T(G, S)$ for the subgroup of $T(G)$ of all trivial-source modules. Thus $V \in T(G)$ lies in $T(G, S)$ if and only if $V\downarrow_S$ is the sum of a single trivial module and a projective
module. Note that if $T(S)$ is torsion-free, and this is always the case unless $S$ is cyclic, quaternion or semidihedral (Carlson and Thévenaz 2004), then $V$ lies in $T(G, S)$ whenever $V$ is self-dual.

There is already a large literature on endotrivial modules for finite groups. We just need a few results from this, mainly for confirming that a given module actually is endotrivial.

**Lemma 2.1** Let $Q_1, \ldots, Q_n$ be representatives from the conjugacy classes of maximal elementary abelian $p$-subgroups of $G$ (i.e., maximal amongst all elementary abelian subgroups). Suppose that the $p$-rank of $G$ is at least 2. A $kG$-module $V$ is endotrivial if and only if $V \downarrow_{Q_i}$ is endotrivial for all $i$. Furthermore, if $G$ is a $p$-group of $p$-rank 2 then there is a homomorphism

$$T(G) \rightarrow \prod_i T(Q_i) \cong \mathbb{Z}^n.$$  

Both the kernel and cokernel of this map are finite. If the $p$-group $G$ is not a semidihedral 2-group then the kernel is trivial.

The first statement is (Carlson et al. 2006, Proposition 2.3(b)). That the kernel of the map is finite is (Carlson et al. 2006, Proposition 2.3(c)), and that the cokernel is finite follows from the fact that the kernel is finite and the rank of $T(G)$ is equal to $n$ by (Alperin 2001, Theorem 4). The last statement follows from (Carlson and Thévenaz 2005, Theorem 1.2).

We now define a particular subgroup for any finite group $G$ and any prime $p$.

**Definition 2.2** Let $G$ be a finite group with Sylow $p$-subgroup $S$. The subgroup $K_G$ of $G$ is defined as consisting of all elements $g \in N_G(S)$ such that there exist $x_1, \ldots, x_n, y \in G$ and subgroups $Q_1, \ldots, Q_n$ of $S$, with the following properties:

1. $g = x_1 \ldots x_n$;
2. $x_i \in O_p'(N_G(Q_i))$ for all $1 \leq i \leq n$;
3. $y^{x_1 \ldots x_i}$ lies in both $Q_i$ and $Q_{i+1}$, for all $1 \leq i < n$.

Let $K_G^o$ be the normal subgroup of $N_G(S)$ generated by $N_G(S) \cap O_p'(N_G(Q))$ for all $N_G(S)$-conjugacy classes of subgroups $1 < Q \leq S$.

It is clear that $K_G^o \leq K_G \leq N_G(S)$. Often $K_G = K_G^o$, but not always. The main result of Grodal’s is the following.

**Theorem 2.3** (Grodal 2016, Theorem 4.27) There is an isomorphism

$$(N_G(S)/K_G)^{ab} \cong T(G, S).$$

We therefore must compute $K_G$ for $G$ a (quasi)simple group. It is clear that for a specific group, if one can compute $O_p'(N_G(Q))$, say on a computer, then one can easily compute $K_G^o$. The computation of $K_G$ looks more difficult, but luckily in all but one case that we consider here $K_G^o = K_G$. (In fact, the author knows of only one
other case where $K_G \neq K_G^c$, that of $G_2(5)$ and $p = 3$, to which he was alerted by Grodal. It seems very rare for quasisimple groups that $K_G$ differs from $K_G^c$.

If a Sylow $p$-subgroup $S$ of $G$ is cyclic then we see that $K_G = S$ since $O^p(N_G(S)) = S$, so we obtain that $T(G, S)$ is isomorphic to the abelianization of $N_G(S)/S$, agreeing with Mazza and Thévenaz (2007). We therefore concern ourselves with groups with non-cyclic Sylow $p$-subgroups.

Another consequence of Grodal’s result is the following previously known statement (Carlson et al. 2006, Proposition 2.8).

**Lemma 2.4** If $G$ has trivial intersection Sylow $p$-subgroups then $K_G = S$, so $T(G, S) \cong (N_G(S)/S)^{ab}$.

If $K_G^c = N_G(S)$ then obviously $K_G = N_G(S)$ and we are done. However, if $K_G^c \neq N_G(S)$, then we need a criterion to guarantee $K_G^c = K_G$; the next lemma furnishes us with such a criterion.

**Lemma 2.5** If $O^p(N_G(Q)) \leq S$ for all $1 < Q < S$ not of order $p$, then $K_G = K_G^c$. In particular, if $|S| = p^2$ then $K_G = K_G^c$.

**Proof** From the assumption, the subgroups $Q_i$ in the definition of $K_G$ must all have order $p$, or else the $x_i$ all lie in $S$. However, if $Q_i$ has order $p$ then $Q_i = \langle y \rangle$ for all $i$, and so $g \in O^p(N_G(Q)) \leq K_G^c$. Thus $K_G \leq K_G^c$, as claimed. \(\square\)

We illustrate this with an example, solving some of the open cases from Lassueur and Mazza (2015b).

**Example 2.6** Let $G = J_3$ and $p = 3$. The order of $S$ is $3^5$, but the only subgroup $Q$ of $S$ for which $O^p(N_G(Q))$ is not contained in $S$ is of order 3. Thus $K_G = K_G^c$ by Lemma 2.5, and in this case $N_G(S)/K_G^c$ has order 2 by a direct computation on Magma (Bosma et al. 1997). We may also apply Lemma 2.5 to $Co_2$ and $Co_3$ for $p = 5$ (in both cases the size of $S$ is $5^5$), and in these situations $N_G(S)/K_G$ is isomorphic to $C_4$ and $S_3$ respectively.

We will also want a simple criterion to guarantee that a given element of $N_G(S)$ lies in $K_G^c$. Note that, trivially, if $Q$ has order $p$, $O^p(N_G(Q)) \leq C_G(Q)$. Often centralizers of $p$-elements in big sporadic groups satisfy $O^p(C_G(z)) = C_G(z)$, so we can just test if $g \in N_G(S)$ lies in $C_G(z)$ for some $z \in S$ with this property.

**Lemma 2.7** Let $z$ be an element of order $p$ in $G$ such that $O^p(C_G(z)) = C_G(z)$, and let $g \in N_G(S)$. If $C_S(g)$ contains a $G$-conjugate of $z$ then $g \in K_G^c$.

We give an example of how to use this lemma now, dealing with another open case from (Lassueur and Mazza 2015b).

**Example 2.8** Let $G = Th$, $p = 3$. The centralizer of a 3A element is $3 \times G_2(3)$, so satisfies the condition of Lemma 2.7. The group $N_G(S)/S$ is isomorphic to the Klein four group. Let $E$ be a Klein four complement to $S$ in $N_G(S)$.

To prove that every element of $E$ centralizes a 3A element, we check on a computer. We start with a 248-dimensional representation of $G$ from the online Atlas (Wilson...
et al. 2019), and a straight-line program that gives us one of the two 3-local subgroups that contain $N_G(S)$. Let $H$ be this group, with Sylow 3-subgroup $S$. To more easily compute inside $H$, it is better to produce a copy $H_2$ of $H$ as a power-commutator (or equivalently power-conjugate) group. We construct the subgroup $E$ inside $H_2$, then compute the centralizer in $S$ of each non-trivial element of $E$ (mapped back from $H_2$ into $H$) and check that they contain a 3A element.

Inside $H$ we can detect the conjugacy classes of elements of order 3 by the size of their fixed-point space on the representation of $G$. We provide Magma code for the whole procedure, starting from the point where we assume we have $H$:

```magma
> H2, phi := PCGroup(H); S := Sylow(H2,3);
> E := Sylow(Normalizer(H2,S),2);
> Order(E);
> for x in [E.1, E.2, E.1*E.2] do
> C := Centralizer(S, x) @@ phi;
> {Dimension(Eigenspace(i,1)) : i in C | Order(i) eq 3};
> end for;
{ 80^6, 86^26, 92^12 } { 80^6, 86^26, 92^12 } { 80^6, 86^26, 92^12 }
```

This shows that all three of the classes of elements of order 3 lie in the fixed-point space of each involution in $E$. In particular, $N_G(S) = K_G^0$.

In fact, we see from (Wilson 1988) that $O^3(C_G(z)) = C_G(z)$ for all elements $z$ of order 3, so we would just need to check that every element of $E$ has a fixed point on $S$, but for most groups not all centralizers of elements of order $p$ have this property, so this extra check needs to be implemented.

With these two tricks, the information on local subgroups from the Atlas (Conway et al. 1985) and some of their constructions on the online Atlas (Wilson et al. 2019), and the structures of $N_G(S)/S$ given in Wilson (1998), we can fairly quickly despatch all groups in Sect. 4.

### 3 Equivalence of Grodal’s and Balmer’s characterizations

Balmer (2013) gives a group-theoretic criterion for the determination of $T(G, S)$, with an algebraic proof. He defines a group of ‘weak $H$-homomorphisms’ $A(G, H)$, and for $H = S$ proves that $A(G, S) \cong T(G, S)$. Here we assume throughout that $H = S$, as we have no need for the more general definition. However, we want to generalize weak $S$-homomorphisms in a different way.

**Definition 3.1** A function $\rho : G \to \text{GL}_n(k)$ is a weak $n$-homomorphism if $\rho(x) = 1$ (i.e., the identity matrix) whenever $S \cap S^x = 1$ or $x$ is a $p$-element normalizing some non-trivial subgroup of $S$, and if $S \cap S^y \cap S^{xy} \neq 1$ then $\rho(x)\rho(y) = \rho(xy)$.

Let $A_n(G, S)$ denote the set of weak $n$-homomorphisms $G \to \text{GL}_n(k)$, and $A(G, S) = A_1(G, S)$. We will prove the following.
Theorem 3.2  The restriction map from $G$ to $N_G(S)$ induces a bijection

$$A_n(G, S) \rightarrow \text{Hom}(N_G(S)/K_G, \text{GL}_n(k)).$$

The case $n = 1$ shows that Balmer’s and Grodal’s characterizations of $T(G, S)$ are the same. For $n \geq 1$, Grodal showed (Grodal 2016, Theorem 3.10) that every indecomposable element of $\text{Hom}(N_G(S)/K_G, \text{GL}_n(k))$ is the Green correspondent of an indecomposable $kG$-module, whose restriction to $S$ is the sum of a free and a trivial module (of some dimension). For $n \geq 2$ the content of Theorem 3.2 is to give the correct extension of the definition of weak $S$-homomorphisms from Balmer (2013) to arbitrary dimensions.

We first notice that if $\rho$ is a weak $n$-homomorphism and $1 < Q \leq S$, then the restriction of $\rho$ to $N_G(Q)$ is a homomorphism to $\text{GL}_n(k)$. In particular, the restriction of any weak $n$-homomorphism to $N_G(S)$ is a map in $\text{Hom}(N_G(S), \text{GL}_n(k))$. Moreover, since $x \in N_G(Q)$ is a $p$-element then $\rho(x) = 1$, so therefore $O^p\rho(N_G(Q))$ lies in the kernel of this homomorphism.

(Comparing our definition to the original definition of weak 1-homomorphism from (Balmer 2013), this extra condition on $p$-elements is not present. However, since $k^\times$ has no $p$th roots of unity, if $x \in N_G(Q)$ is a $p$-element then $\rho(x) = 1$ for any $\rho \in A_1(G, S)$—since $\rho(x^\ell) = \rho(x)^\ell$ by the weak $n$-homomorphism property—so the condition always holds for weak $S$-homomorphisms in the sense of Balmer (2013).)

Let $g \in G$ and $A$ be a non-trivial subset of $S \cap S^{s^{-1}}$. An Alperin decomposition of $g$ with respect to $A$ (from Alperin’s fusion theorem (Alperin 1967)) is an expression $g = x_1 \cdots x_r z$, where $x_i \in N_G(Q_i)$ for some subgroup $1 < Q_i \leq S$, $z \in N_G(S)$, each $x_i$ is a $p$-element, and $A^{x_1 \cdots x_i} \leq Q_i$ for all $1 \leq i < r$.

Lemma 3.3  Let $\rho : G \rightarrow \text{GL}_n(k)$ be a weak $n$-homomorphism.

1. If $g = x_1 \cdots x_r z$ is an Alperin decomposition of $g$ with respect to any subset non-trivial $A$, then $\rho(g) = \rho(z)$.
2. If $\rho(g) = 1$ for all $g \in N_G(S)$ then $\rho(g) = 1$ for all $g \in G$.
3. The restriction map from $A_n(G, S)$ to $\text{Hom}(N_G(S), \text{GL}_n(k))$ is injective.

Proof  Suppose we have an Alperin decomposition $g = x_1 \cdots x_r z$. Since $x_i$ is a $p$-element of $N_G(Q_i)$ for $1 < Q_i \leq S$, we have that $\rho(x) = 1$. We claim that the weak $n$-homomorphism property implies that $\rho(x_1 \cdots x_i) = 1$ for all $1 \leq i \leq r$, and then that $\rho(g) = \rho(z)$.

We proceed by induction on $i$, the result obviously holding that $\rho(x_1) = 1$. Thus write $x = x_1 \cdots x_{i-1}, x' = x_i$ and suppose that $\rho(x) = 1$ (and $\rho(x') = 1$). We merely need to show that

$$S \cap S^{x'} \cap S^{xx'} \neq 1,$$

but this is equivalent to

$$S^{xx'} \cap S \cap S^x \neq 1;$$
the subgroup $R^x$ lies in $S^{x^{-1}}$, $S$, and $S^x$, so this triple intersection is indeed non-empty. Thus $\rho(x x') = \rho(x) \rho(x')$, and thus $\rho(x_1 \ldots x_r) = 1$. Finally, since $z \in N_G(S)$, we clearly have that the triple intersection property holds for $x_1 \ldots x_r$ and $z$, and therefore $\rho(g) = 1 \cdot \rho(z)$, proving (i).

We now prove (ii), so let $g \in G$. If $S \cap S^g = 1$ then $\rho(g) = 1$ by definition. If $S \cap S^g \neq 1$ then we obtain an Alperin decomposition of $g$ with respect to $S \cap S^{g^{-1}}$, and so $\rho(g) = \rho(z)$ for some $z \in N_G(S)$. However, $\rho(z) = 1$ by assumption, so $\rho(g) = 1$, as claimed.

We now prove (iii). Since the product of two weak $n$-homomorphisms is a weak $n$-homomorphism, if $\rho^{-1}$ is trivial on $N_G(S)$ then $\rho^{-1} = n \cdot 1_G$, but then $\rho = \sigma$, as claimed.

\[ \rho \in \text{Hom}(N_G(S)/K_G, GL_n(k)). \]

\textbf{Proof} By definition, $K_G$ consists of those elements $g$ of $N_G(S)$ with an Alperin decomposition $g = x_1 \ldots x_r z$ with $z = 1$. By Lemma 3.3(i), $\rho(g) = \rho(z) = 1$, and so $K_G \leq \ker(\rho \downarrow_{N_G(S)})$. This completes the proof.

We now must construct a map going the other direction. Thus, given any homomorphism $\rho : N_G(S) \rightarrow GL_n(k)$ with $K_G$ in the kernel, we need to extend this to a weak $n$-homomorphism $G \rightarrow GL_n(k)$ in $A_n(G, S)$. We do this as follows.

1. If $g \in G$ satisfies $S \cap S^g = 1$, then set $\rho(g) = 1$.
2. Thus $S \cap S^g \neq 1$, so we have an Alperin decomposition $g = x_1 \ldots x_r z$. Set $\rho(g) = \rho(z)$.

Call this the \textit{weak $n$-extension} of the map $\rho : N_G(S) \rightarrow GL_n(k)$. We claim it is well defined: let $x_1 \ldots x_r z$ be an Alperin decomposition of $g$ with respect to $S \cap S^{g^{-1}}$, and let $x'_1 \ldots x'_s z'$ be any other Alperin decomposition of $g$, with respect to some (non-trivial) subset $A'$ of $S$. We have that

$$x_1 \ldots x_r z = x'_1 \ldots x'_s z',$$

so

$$zz'^{-1} = x_r^{-1} \ldots x_1^{-1} x'_1 \ldots x'_s.$$

We claim that the right-hand side is an Alperin decomposition of $zz'^{-1}$, and therefore that $zz'^{-1}$ lies in $K_G$. Thus $\rho(z) = \rho(z')$, as needed.

We now check that it is indeed an Alperin decomposition. To see this, we set $A = A'^S \leq S$. It is easy to see that this expression satisfies the condition of being an Alperin decomposition with respect to $A$, so we see that $\rho$ is well defined.

\textbf{Proposition 3.5} If $\rho : N_G(S) \rightarrow GL_n(k)$ is a homomorphism with $K_G$ in the kernel, then the weak $n$-extension (also denoted $\rho$) to $G$ is a weak $n$-homomorphism.
Consequently, the restriction map induces an isomorphism

\[ A_n(G, S) \rightarrow \text{Hom}(N_G(S)/K_G, GL_n(k)), \]

and so in particular \( T(G, S) \) consists of the Green correspondents in \( G \) of elements of \( \text{Hom}(N_G(S)/K_G, k^\times) \).

**Proof** Let \( g \in G \). If \( g \in S \) or \( S \cap S^g = 1 \) then \( \rho(g) = 1 \) by construction. Thus we need to check that \( \rho(gh) = \rho(g)\rho(h) \) whenever \( S \cap S^h \cap S^{gh} \neq 1 \). Note that, in particular each of \( S \cap S^h, S \cap S^{gh}, \) and \( S \cap S^g \) is non-trivial. Let \( g = x_1 \ldots x_r z, \ h = x'_1 \ldots x'_r z' \), be Alperin decompositions of \( g \) and \( h \), with respect to the subsets \( S \cap S^{g^{-1}} \) and \( S \cap S^{h^{-1}} \) respectively. We claim that

\[ gh = \left[ x_1 \ldots x_r (x'_1 \ldots x'_s)z^{-1} \right] [zz'] \]  

is an Alperin decomposition of \( gh \), and therefore

\[ \rho(gh) = \rho(zz') = \rho(z)\rho(z') = \rho(g)\rho(h), \]

since \( z, z' \in N_G(S) \).

Certainly \( gh \) is equal to the expression, so we need to find a subset \( A \) that is transported by each \( x_i \) and \( (x'_i)^{-1} \). Set \( A = S^{h^{-1}g^{-1}} \cap S^{g^{-1}} \cap S \neq 1 \). Let \( x_i \in N_G(Q_i) \) and \( x'_i \in N_G(Q'_i) \) for each \( i \); we have that \( A \leq S \cap S^{g^{-1}} \), so that \( A^{x_1 \ldots x_i^{-1}} \leq Q_i \) by the definition of an Alperin decomposition.

Set \( B = A^{x_1 \ldots x_r} = A^{g^{-1}} \). Notice that since \( A^{g} \leq Q'_1 \), \( B \in (Q'_1)^{z^{-1}} \), and indeed we see that \( B^{x'_1 \ldots x'_{i-1}} \leq (Q'_i)^{z^{-1}} \) for all \( 1 \leq i < s \). Thus the expression in (1) is an Alperin decomposition, as needed. \( \square \)

**4 Determination of \( K^o_G \) and \( K_G \)**

If \( G \) is, say, \( M_{11} \), one may load the group directly into Magma and in a few seconds compute \( K^o_G \). For groups where there is a permutation representation available, and Magma can determine \( K^o_G \) in less than a couple of minutes, we omit a proof that \( K^o_G \) is as claimed. In all cases except for \( G = Ru \) for \( p = 3 \) and \( G = 3 \cdot J_3 \) for \( p = 2 \), \( K_G = K^o_G \) either by an application of Lemma 2.5 or because \( K^o_G = N_G(S) \). For \( Ru \) we construct an endotrivial module to demonstrate that \( T(G, S) \) has order at least 2, and for \( 3 \cdot J_3 \) we express an element of \( K_G \setminus K^o_G \) as the product of two elements \( x_1x_2 \), as in Definition 2.2.

The explanations in Table 1 in brackets are shorthand, to alert the reader as to which technique is used to conclude that \( K_G \) is as stated, rather than just \( K^o_G \). (SN) means that \( G \) has a self-normalizing Sylow 2-subgroup, so \( N_G(S) = S \). (K°) means \( K^o_G \) is...
already all of $N_G(S)$. (r2) means $S \cong C_p \times C_p$, so we may apply Lemma 2.5 above to see that $K_G = K_G^0$. (nnc) means that for all $Q \leq S$ with $Q$ not cyclic of order $p$, $O^p(N_G(Q)) \leq S$, hence $K_G = K_G^0$ by Lemma 2.5 above. (TI) means that $G$ has a trivial intersection Sylow $p$-subgroup, so that $K_G = K_G^0$ by Lemma 2.4. Finally, (const) means that $|N_G(S) : K_G^0|$ was found to be equal to 2, and a trivial-source endotrivial module other than $k$ was constructed directly.

This section proves the following result as a consequence of a case-by-case analysis.

**Theorem 4.1** If $G$ is a sporadic quasisimple group and $S$ is non-cyclic then the group $N_G(S)/K_G$ is given in Table 1. In addition, we have $K_G = K_G^0$, unless $G = 3 \cdot J_3$ and $p = 2$, in which case $|K_G : K_G^0| = 3$. Moreover, in this case $N_G(S) = K_G$.

We first compute $K_G$ in the case where $G$ is simple in Sect. 4.1. In Sect. 4.2 we determine for $G$ quasisimple but not simple, whether $K_G$ contains $Z(G)$. These two subsections furnish us with a proof of Theorem 4.1.

### 4.1 Simple groups

We begin by proving Theorem 4.1 for the sporadic simple groups $G$. We will give example Magma code in a few of the earlier cases to show how to compute the information we use.

#### 4.1.1 The groups $M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, J_1, J_2, HS, Mcl, He, Co_3, Co_2, T$

In all cases there is a small permutation representation of $G$ available via (Wilson et al. 2019) and one may easily determine $K_G^0$; enumerate all conjugacy classes of subgroups $Q$ of $S$, take all normalizers $N_G(Q)$ of them, and then take the subgroup generated by all normal closures of $S$ inside all $N_G(Q)$, to produce $K_G^0$. It is as in Table 1.

#### 4.1.2 The group $Ru$ ($p = 2, 3, 5$)

Unless $p = 3$, an easy computer calculation determines that $K_G^0 = N_G(S)$. For $p = 3$, we have that $|N_G(S) : K_G^0| = 2$, and we cannot apply Lemma 2.5. However, there is a simple endotrivial module with trivial source, namely the 406-dimensional simple module (Lassueur et al. 2016). One may explicitly check that it has trivial source, or note that it is self-dual and $T(S)$ is torsion-free. Thus $T(G, S) \cong C_2$, and in particular $K_G = K_G^0$.

#### 4.1.3 The group $Suz$ ($p = 2, 3, 5$)

If $p$ is odd then an easy computer calculation determines $K_G^0$. If $p = 2$ then $N_G(S) \cong S \times 3$ (see Wilson 1998). There are many 2-subgroups, so we give a quick theoretical proof that $K_G^0 = N_G(S)$. If $z$ is an involution in $G$ then $C_G(z)$ has no non-trivial odd quotients as the two centralizers have composition factors only simple groups and $C_2$ (see Gorenstein et al. 1998, Table 5.3o, p.276). Thus it suffices to check that
if $E$ denotes a Sylow 3-subgroup of $N_G(S)$, then $C_S(E) \neq 1$. This is the case since $|S| = 2^{13}$ and so $3 \nmid (|S| - 1)$. Thus $K^o_G = N_G(S)$.

### 4.1.4 The group $HN (p = 2, 3, 5)$

Here there is a permutation representation, but it is large, so we give more details to prove Theorem 4.1 quickly, again assisted by a computer to reduce errors. For $p = 2$, the same method of proof as for $Suz$ applies. This time the table is (Gorenstein et al. 1998, Table 5.3w, p.284), but as $|S| = 2^{14}$, in theory we could have $C_S(E) = 1$. In fact, $|C_S(E)| = 64$ by a Magma calculation:

```plaintext
> S:=Sylow(G,2); //G is a copy of HN
> NGS:=Normalizer(G,S); E:=Sylow(NGS,3);
> Order(Centralizer(S,E));
64
```

Let $p = 3$. Here $|S| = 3^6$, $|N_G(S)/S| = 8$. All elements $z$ of order 3 in $G$ satisfy $O^3(C_G(z)) = C_G(z)$ (again, see Gorenstein et al. 1998, Table 5.3w, p.284), so $K^o_G$ contains $N_G(S) \cap C_G(z) = C_N(z)$ (where $N = N_G(S)$) for all elements $z \in \Omega_1(S) \setminus \{1\}$. A quick computer calculation shows that every element of a Sylow 2-subgroup of $N$ lies in $C_N(z)$ for some $z$ of order 3, and therefore $K^o_G = N$.

```plaintext
> S:=Sylow(G,3); //G is a copy of HN
> NGS:=Normalizer(G,S); E:=Sylow(NGS,2);
> [Order(Centralizer(S,i)):i in E];
[ 729, 3, 9, 3, 9, 3, 9, 3 ]
```

Now let $p = 5$. Here $|S| = 5^6$, $|N_G(S)/S| = 8$. There are three classes of elements (classes 5B, 5C and 5D) that are fixed-point-free in the permutation representation on 1140000 points. For $z$ from each of these classes, $O^5(C_G(z)) = C_G(z)$, so $K^o_G$ contains the subgroup of $N = N_G(S)$ generated by $C_N(z)$ for all fixed-point-free $z \in S$. This is all of $N_G(S)$, as in the $p = 3$ case.

### 4.1.5 The group $Co_1 (p = 2, 3, 5, 7)$

If $p = 2$ then $N_G(S) = S$. If $p = 5, 7$, then $K^o_G = N_G(S)$ and the computation is easy on a computer. For $p = 3$ it is much more difficult because the Sylow 3-subgroup is large enough to make enumerating all subgroups challenging.

In this case, $|S| = 3^9$ and $|N_G(S)/S| = 8$. Let $z_1$ be an element of $S$ with centralizer $3 \cdot Suz$, and $z_2$ be an element of $S$ with centralizer $3 \times A_9$. The $G$-conjugates of $z_1$ in $S$ split into two $N_G(S)$-orbits, and the $G$-conjugates of $z_2$ in $S$ split into five $N_G(S)$-orbits. Then $N = N_G(S)$ is generated by $C_N(z_1)$ and $C_N(z_2)$, unless $z_2$ lies in the unique class with $N_G(S)$-centralizer of order 243. Thus if $z_2$ is chosen not to lie in this case, $N$ is generated by $C_N(z_1)$ and $C_N(z_2)$, both of which lie in $K^o_G$. Hence $K^o_G = N_G(S)$, as claimed.
4.1.6 The groups $Fi_{22}, Fi_{23}$ and $Fi'_{24}$ ($p = 2, 3, 5, 7$)

In each of these cases a permutation representation is available, and can easily be used to determine $K_G^\circ$ for $p = 5, 7$. (For $p = 7$ we have $G = Fi'_{24}$ and $K_G^\circ = N_G(S)$, and if $p = 5$ then $S \cong C_p \times C_p$, so we may apply Lemma 2.5.) If $p = 2$ then $N_G(S) = S$ so we are done, and therefore only $p = 3$ remains. As with $Co_1$, there are too many 3-subgroups to easily enumerate all of them, so we give more details.

If $G = Fi_{22}$, then $|S| = 3^9$ and $|N_G(S)/S| = 4$. Let $z$ be an element from class $3B$, whose centralizer has no $3'$-quotients. There are thirteen $N_G(S)$-classes of $3B$ elements $x_i$, and together the $C_N(x_i)$ generate $N_G(S)$.

If $G = Fi_{23}$, then $|S| = 3^{13}$ and $|N_G(S)/S| = 8$. Let $z$ be an element from class $3A$, whose centralizer is $3 \times O_7(3)$. There are seven $N_G(S)$-classes of $3A$ elements $x_i$, and together the $C_N(x_i)$ generate $N_G(S)$.

If $G = Fi'_{24}$, then $|S| = 3^{16}$ and $|N_G(S)/S| = 8$. All elements $z$ of order 3 in $G$ satisfy $O^3(C_G(z)) = C_G(z)$ (see Wilson 1987, p.82), so $K_G^\circ$ contains the subgroup of $N = N_G(S)$ generated by $C_N(z)$ for all $z \in S$. This is all of $N_G(S)$. One does not in fact need all elements. An element from class $3A$ has centralizer $3 \times O_7(3)$. There are seven $N_G(S)$-classes of $3A$ elements $x_i$, and together the $C_N(x_i)$ generate $N_G(S)$.

This is the end of the list of groups where there is a permutation representation available, and hence one can be very explicit. The rest only have matrix representations available, and so can require slightly more theoretical proofs, for the most part.

4.1.7 The group $J_4$ ($p = 2, 3, 11$)

If $p = 2$ then $N_G(S) = S$, and if $p = 11$ then $S$ is TI, so we may apply Lemma 2.4. Thus $p = 3$.

Here $|S| = 3^3$, $|N_G(S)/S| = 32$. Let $E$ denote a Sylow 2-subgroup of $N_G(S)$. The structure of $N_G(S)$ is given in (Wilson 1998, p.302); it is $(2 \times 3^{1+2} \times 8) \times 2$, and lies inside the maximal subgroup $2^{1+2} \cdot 3 \cdot M_{22} \times 2$, in fact inside the normalizer of an element of order 3, which is $6 \cdot M_{22}$. The element of order 3 therefore has centralizer $6 \cdot M_{22}$, so we may apply Lemma 2.7: if $E$ is generated by elements that are not fixed-point-free on $S$, then $K_G^\circ$ contains $E$. We can check this easily inside $6 \cdot M_{22}$, and find 24 such elements, obviously enough to generate a group of order 32. (Alternatively but similarly, we can also note that we already find in $K_G^\circ$ a subgroup of index 2 in $E$, coming from $C_G(Z(S)) \cong 6 \cdot M_{22}$, so we consider the sixteen elements in $E \setminus C_E(Z(S))$, and half of these centralize an element of order 3 on $S$.) Thus $K_G^\circ = N_G(S)$, as claimed.

4.1.8 The group $Th$ ($p = 2, 3, 5, 7$)

If $p = 2$ then $N_G(S) = S$, and if $p = 3$ then this is Example 2.8. Thus $p = 5, 7$.

If $p = 7$ then $|S| = 7^2$ (so we may apply Lemma 2.5 to obtain $K_G = K_G^\circ$) and $|N_G(S)/S| = 144$. There is a single class of elements $x$ of order 7 in $G$, with centralizer $C_G(x) \cong (x) \times PSL_2(7)$ (see Gorenstein et al. 1998, Table 5.3x, p.285). Thus $K_G^\circ$ is the subgroup generated by $C_N(x)$ for all $x \in S \setminus \{1\}$. This subgroup
has order $72 \cdot |S|$, and so $N_G(S)/K_G$ has order $2$. (To see this, note that all elements of order $7$ are conjugate in $G$, and therefore we obtain a subgroup of order $3$ acting non-trivially each of the eight subgroups of order $7$ and centralizing a complement. Inside $N_G(S)/S \cong 3 \times 2 \cdot S_4$, we find three classes of subgroups of order $3$, with one, four and eight subgroups in the class. Thus the subgroups of order $3$ must be the single class of eight, and they generate the subgroup $3 \times 2 \cdot A_4$, of index $2$ in $N_G(S)/S$.)

For $p = 5$, we have $|S| = 5^3$ and $|N_G(S)/S| = 96$. There is a single $N_G(S)$-class of subgroups $Q$ of order $25$, with normalizer (it is a maximal subgroup of $G$) $5^2 \times GL_2(5)$, so the subgroup we are interested in is $5^2 \times SL_2(5)$. The intersection of this with $N = N_G(S)$ is $S \times 4$, with the $4$ acting as a torus element on $Q$. This determines a unique subgroup $5^2 \times 4$ of $N_N(Q)$, and so we may work entirely inside $N$. Taking the normal closure in $N$ of the subgroup $S \times 4$ yields all of $N$, so $K_G = K_G^o = N_G(S)$. We give some example Magma code to ease the reader’s verification of this.

```magma
// Assume you are given a copy NGS of the // normalizer of a Sylow 5-subgroup of Th
S:=Sylow(NGS,5);
Q:=Subgroups(NGS:OrderEqual:=25) \ [1];
a:=Q.1; b:=Q.2; Q eq sub<Q|a,b>;
// Check that this is true
NNQ:=Normalizer(NGS,Q);
g:=[i:i in NNQ|a^i eq a^2 and b^i eq b^-2];#g;
// Check that this has order 25
g:=g[1]; A:=sub<NGS|S,g>;
Index(NGS,NormalClosure(NGS,A));
// Should equal 1
```

### 4.1.9 The Group $L_2(p = 2, 3, 5)$

If $p = 2$ then $N_G(S) = S$. If $p = 5$ then $N_G(S)$ is contained in a subgroup $5^3 \cdot PSL_3(5)$. This is already a $5$-local subgroup, so $K_G^o = N_G(S)$.

Thus $p = 3$. We ape the proof in Example 2.8, with a slight twist to speed things up. The order of $S$ is $3^7$, and $N_G(S)/S \cong 2 \times SD_{16}$. Let $E$ denote a Sylow $2$-subgroup of $N_G(S)$. We see that $N_G(S)$ is contained in a maximal subgroup $3^5 \times (2 \times M_{11})$, so there is a $3$-local subgroup $3^5 \times M_{11}$. This proves that the $SD_{16}$ subgroup $E_1$ of $E$ is contained in $K_G^o$, so we simply need to find one element $z$ of $E \setminus E_1$ in $K_G^o$ and we are done. Note that the centralizer of a $3A$ element is $3 \cdot McL$, so we may apply Lemma 2.7.

There are two obvious candidates for $z$: the two central involutions in $E \setminus E_1$. In both cases $C_S(z)$ has order $9$ and contains no element from $3A$. However, of the fourteen other elements, ten have $3A$ elements in $C_S(z)$. (One may tell $3A$ and $3B$ apart because they have different fixed points (dimensions $39$ and $21$ for $3A$ and $3B$ respectively) on the $111$-dimensional representation over $F_5$. Six elements from $E \setminus E_1$ have both classes in their centralizer.)
4.1.10 The group $BM$ ($p = 2, 3, 5, 7$)

If $p = 2$ then $N_G(S) = S$. If $p = 7$, then $|S| = 7^2$, and the structure of $N_G(S)$ is

$$(2^2 \times (S \rtimes (3 \times 2A_4))).2.$$

It lies in the maximal subgroup $(2^2 \times F_4(2)).2$ (Wilson 1998, Table II), which is the
centralizer of a 2C involution. (The outer 2 swaps the two 2A involutions in the centre
of the derived subgroup.) There is a single conjugacy class of elements of order 7,
with centralizer of structure $7 \times 2 \cdot PSL_3(4)$. This lies in the centralizer of a 2A involution, so the 2A involution in $C_G(S)$ lies in $K_G$. 

Also, for each subgroup $Q$ of order 7, $K_G^o$ contains an element of order 3 that
centralizes $Q$ and normalizes a complement in $S$. Each of these eight subgroups 3 must
be different, so $K_G^o$ contains all of $3 \times 2A_4$. Thus $K_G^o$ contains all of $2^2 \times 3 \times 2A_4$,
so a subgroup of index 2 in $N_G(S)$. (This can also be seen as $Th \leq BM$.)

We claim this is all of $K_G$, and hence $K_G$ by Lemma 2.5. To see this note that this
subgroup of index 2 has two conjugacy classes of subgroups of order 7, which are
interchanged by the top 2 in $N_G(S)$. Thus this top 2 does not centralize any subgroup
of order 7, and therefore cannot appear in $C_G(Q)$. Thus $K_G = K_G^o$ has index 2 in
$N_G(S)$, as claimed.

Now let $p = 5$. By (Wilson 1998, Table I), $N_G(S)/S \cong 4 \times 4$. Furthermore,
$5^3 \cdot PSL_3(5)$ is a 5-local subgroup (Wilson 1987), and clearly has no $5'$-quotients; it
also contains $N_G(S)$, so $K_G^o = N_G(S)$, as needed.

Finally, let $p = 3$. By (Wilson 1998, Table I), $N_G(S)/S \cong 2^3$. Note that $S$, and
indeed $N_G(S)$, is contained in the subgroup $H \cong Fi_{23}$, and since $N_G(S) = K_H^o$ as
we have seen above, $N_G(S) = K_G^o$ as well.

4.1.11 The Group $M$ ($p = 2, 3, 5, 7, 11, 13$)

If $p = 2$ then $N_G(S) = S$. Suppose that $p = 13$. By (Wilson 1998, Table I),
$N_G(S)/S \cong 3 \times 4S_4$, and $S \cong 13^{1+2}$. Since $N_G(S)$ is a maximal subgroup of $G$
(it is the normalizer of a 3B element), and we cannot work inside $G$ directly, we will
use other 13-local subgroups and decide which elements of $N_G(S)$ lie in them. The
centralizer of a 13A element is $13 \times PSL_3(3)$, and there is also maximal subgroup
$13^2 \rtimes SL_2(13).4$, with the $13^2$ being pure, and hence consisting of 13B elements.

There are two $N_G(S)$-classes of subgroups of order $13^2$, one with an involution
in $N_G(S)$ inverting all elements of it and the other without. Thus we have identified
a subgroup $Q$ of $S$ with normalizer $13^2 \rtimes SL_2(13).4$. Fix an element $z$ of order 2
inverting $Q$ (all such elements lie in $Q(z)$), and we aim to find a Borel subgroup of
the $SL_2(13)$. There are 104 elements of order 12 powering to $z$ in $N_S(Q)$, yielding
two $N_G(S)$-conjugacy classes of subgroups $(13^2) \rtimes (13 \times 12)$. However, the normal
closure in $N_G(S)$ of any of the 104 such subgroups is all of $N_G(S)$, so $K_G^o = N_G(S)$, 
as needed.

Next, we have $p = 11$. By (Wilson 1998, Table I), $N_G(S)/S \cong 5 \times 2A_5$, and
$S \cong 11^2$. The centralizer of an element of order 11 is $11 \times M_{12}$. Thus for each
subgroup $Q$ of order 11, $K^\circ_G$ contains an element of order 5 that centralizes $Q$ and normalizes a complement in $S$. Each of these twelve subgroups 5 must be different, so $K^\circ_G$ contains all of $5 \times 2A_5$. (Alternatively one may use that $K^\circ_G$ now contains $5 \times 5$ and is normal in $N_G(S)$.)

Next, we have $p = 7$, where $|N_G(S)/S| = 36$. From (Conway et al. 1985, p.234), the centralizers of 7A and 7B elements are $7 \times He$ and $7_{1+4}.2A_7$, so we may apply Lemma 2.7. If $N_G(S)$ is generated by $S$ and elements $x$ such that $|C_S(x)| > 1$, then $K^\circ_G = N_G(S)$.

Since $N_G(S)$ is contained in a maximal subgroup $7_{1+4} \rtimes (3 \times 2 \cdot S_7)$ (this is the normalizer of a 7B element (Conway et al. 1985, p.234), and contains $N_G(S)$ by (Wilson 1998, Table II)), we can easily calculate $C_S(x)$ for $x \in N_G(S)$. Of the 36 elements $x$ in a complement $E$ to $S$ in $N_G(S)$, 24 satisfy $|C_S(x)| > 1$, and hence lie in $K^\circ_G$. Thus $K^\circ_G = N_G(S)$, as claimed.

For $p = 5$, by (Wilson 1998, Table I), $N_G(S)/S \cong 4 \times 4 \times S_3$, and it is contained in the normalizer of a 5B element by (Wilson 1998, Table III). The centralizers of 5A and 5B elements are $5 \times HN$ and $5_{1+6}.2J_2$ by (Conway et al. 1985, p.234). In particular, we may apply Lemma 2.7, as in the case $p = 7$ above. This time there are again only twelve elements $x$ in a complement $E$ to $S$ in $N_G(S)$ such that $|C_S(x)| = 1$, so again $K^\circ_G = N_G(S)$.

Finally, we have $p = 3$. By (Wilson 1998, Table I), $N_G(S)/S \cong 2 \times 2 \times SD_{16}$. As with the previous two cases, we apply Lemma 2.7. The centralizers of 3A, 3B and 3C are $3Fi'_{24}$, $3_{1+12} \cdot S_4$ and $3 \times Th$ respectively, so again we need to check that a complement $E$ to $S$ in $N_G(S)$ is generated by elements that are not fixed-point-free. We use the group $3^{2+5+10}.(M_{11} \times 2S_4)$ for this, as in Wilson (1998), which contains $N_G(S)$. This time we find no fixed-point-free elements at all, and this completes the proof.

4.2 Central extensions

We must also prove Theorem 4.1 for central extensions of sporadic groups. In all cases but $2 \cdot BM$, the central extension $\hat{G}$ can be constructed in Magma and we can test whether $Z(\hat{G})$ is contained in some $O^p'(N_{\hat{G}}(Q))$, thus proving $Z(\hat{G}) \leq K^\circ_{\hat{G}}$ directly. If this is not the case, we may still have that $Z(\hat{G}) \leq K^\circ_{\hat{G}}$, and can easily test that on a computer as well. In the cases where $Z(\hat{G}) \not\leq K^\circ_{\hat{G}}$, we can also check whether $K^\circ_G = K^\circ_{\hat{G}}$ using the same reasoning as with the simple case. For the various $\hat{G}$ and $p$, we give a choice of $Q$ for which $Z(\hat{G}) \leq O^p'(N_{\hat{G}}(Q))$ (when this is the case).

4.2.1 All groups for the prime 2

We begin with $p = 2$, so $|Z(\hat{G})| = 3$. There are six sporadic groups with Schur multiplier a multiple of 3.

- If $\hat{G} = 3 \cdot M_{22}$, $Q$ may be taken to be a subgroup $2^2$ with normalizer of order $2^5 \cdot 3^3$. 

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If $\hat{G} = 3 \cdot J_3$ there is no such $Q$, so $Z(\hat{G}) \nless K^o_\hat{G}$. However, $Z(\hat{G}) \leq K_G$: this is because $z \in Z(\hat{G}) \setminus \{1\}$ may be expressed as a product $x_1x_2$ where $x_1 \in O^{2'}(N_G(Q_1))$ and $x_2 \in O^{2'}(N_G(Q_2))$. For $Q_1$ we choose a Klein four group with normalizer of order $2^7 \cdot 3^3$ (there are two classes, and the other has normalizer of order $2^5 \cdot 3^2$). For $Q_2$ we choose an elementary abelian group of order 16 (unique up to conjugacy in $G$). We may choose $Q_1$ and $Q_2$ so that $Q_1 \leq Q_2$, so any element of $N_G(S)$ that may be written as $x_1x_2$ lies in $K_G$. The central 3 is such an element. In the next section we confirm by direct computation that $T(\hat{G}, S) = 1$.

- If $\hat{G} = 3 \cdot McL$, $Q$ may be taken to be a subgroup $2^2$ with normalizer of order $2^7 \cdot 3^3$.
- If $\hat{G} = 3 \cdot Suz$, $Q$ may be taken to be the subgroup generated by a 2B involution with normalizer $2^{1+6} \cdot PSU_4(2).2$ in the quotient $G$.
- If $\hat{G} = 3 \cdot ON$, $Q$ may be taken to be the subgroup generated by an involution (all are conjugate), which has normalizer $4_2 \cdot PSL_3(4).2_1$ in the quotient $G$.
- If $\hat{G} = 3 \cdot Fi'_24$, $Q$ may be taken to be the subgroup generated by a 2B involution (a 2-central involution), which has normalizer $2^{1+12} \cdot 3 \cdot PSU_4(3).2$ in the quotient $G$.

4.2.2 Odd primes and all groups except $2 \cdot BM$

If $p$ is odd then it is generally much easier to find a candidate subgroup $Q$ (as there are far fewer options for $Q$), and we simply list those groups $\hat{G}$ and primes $p$ such that $Z(\hat{G})$ is contained in $O^p(N_G(Q))$.

- For $p = 3$, if $\hat{G}$ is one of $2 \cdot M_{12}, 2 \cdot J_2, 2 \cdot Suz, 2 \cdot Co_1$ and $2 \cdot Fi_{22}$, then there exists a non-trivial $p$-subgroup $Q$ such that $Z(\hat{G})$ is contained in $O^p(N_G(Q))$.

If $\hat{G} = 2 \cdot HS$, then $Z(\hat{G})$ is not contained in $K^o_\hat{G}$, and not in $K_G$ by Lemma 2.5. However, in this case, the image of $Z(\hat{G})$ in $N_G(S)/K_G$ is contained in the derived subgroup, so $T(\hat{G}, S) \cong T(G, S)$.

If $\hat{G} = 4 \cdot M_{22}$, then the central involution lies in $K^o_\hat{G}$, but the whole of the centre does not (hence not in $K_G$ by Lemma 2.5). If $\hat{G} = 2 \cdot Ru$ then $Z(\hat{G})$ does not lie in $K^o_\hat{G}$, but the proofs that these central elements also do not lie in $K_G$ follow from the direct construction of elements of $T(\hat{G}, S)$ below.

- For $p = 5$, if $\hat{G}$ is one of $2 \cdot J_2, 2 \cdot HS, 2 \cdot Suz, 3 \cdot Suz$ (and therefore $6 \cdot Suz$), $2 \cdot Ru$ and $2 \cdot Co_1$, then there exists a non-trivial $p$-subgroup $Q$ such that $Z(\hat{G})$ is contained in $O^p(N_G(Q))$.

For $3 \cdot Fi'_24$, there is no subgroup $Q$ such that $Z(\hat{G}) \leq O^5(N_G(Q))$, but we still have that $Z(\hat{G}) \leq K^o_\hat{G}$.

For $2 \cdot Fi_{22}$ and $3 \cdot Fi_{22}$, and therefore $6 \cdot Fi_{22}$, $Z(\hat{G})$ does not lie in $K^o_\hat{G}$, which is equal to $K_G$ by Lemma 2.5. The quotient $N_G(S)/K_G$ is $S_3 \times Z(\hat{G})$. In particular, $T(\hat{G}, S) \cong 2 \times Z(\hat{G})$.

For $3 \cdot McL$, since $S$ is TI, we have that $K_G = S$, so $N_G(S)/K_G \cong N_G(S)/K_G \times Z(\hat{G})$. In particular, $T(\hat{G}, S) \cong C_{24}$. 

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• For \( p = 7 \), if \( \hat{G} \) is one of \( 2 \cdot C_{01} \) and \( 3 \cdot Fi_{24}^\prime \), then there exists a non-trivial \( p \)-subgroup \( Q \) such that \( Z(\hat{G}) \) is contained in \( O^p(\hat{N}_G(Q)) \).

For \( 3 \cdot ON \), there is no subgroup \( Q \) such that \( Z(\hat{G}) \leq O^7(\hat{N}_G(Q)) \), but we still have that \( Z(\hat{G}) \leq K^\circ_G \).

4.2.3 The group \( 2 \cdot BM \)

The only group with a central extension that isn’t easy to work in is \( G = BM \). The group \( BM \) has a 4370-dimensional representation over \( \mathbb{F}_2 \), and the dimensions of the fixed spaces of elements from \( 2A, 2B, 2C \) and \( 2D \) are 2510, 2322, 2212 and 2202 respectively. (This can be found by powering appropriate elements of large order in \( BM \), which can be obtained using the words in the online Atlas (Wilson et al. 2019.).)

From Conway et al. (1985), we see that (only) the \( 2C \) involution has preimage of order 4 in \( 2 \cdot G \), so if \( O^p(\hat{N}_G(Q)) \) possesses a \( 2C \) involution then the preimage in \( 2 \cdot G \) is a non-split central extension.

The subgroup \( 5 \times HS \) contains a \( 2C \) involution, so this deals with the case \( p = 5 \).

For \( p = 3 \), the centralizer of a \( 6A \) element in \( M \) is the intersection of the centralizer of a \( 2A \) element – which is \( 2 \cdot BM \) – and a \( 3A \) element – which is \( 3 \cdot Fi_{24}^\prime \). There are only two classes of involutions in \( 3 \cdot Fi_{24}^\prime \), and by checking orders we see that inside \( Fi_{24}^\prime \) the centralizer is \( 2 \cdot Fi_{22} \cdot 2 \). Thus inside \( BM \), the preimage of \( Fi_{22} \) in \( 2 \cdot BM \) is \( 2 \cdot Fi_{22} \), so there is a subgroup \( 3 \times 2 \cdot Fi_{22} \) in \( 2 \cdot BM \), proving that \( Z(\hat{G}) \) is in \( K^\circ_G \), as needed.

The centralizer of the \( 7 \) is \( 7 \times 2 \cdot PSL_3(4).2 \), which is contained in the centralizer of an involution \( 2 \cdot 2E_6(2).2 \). We claim that the preimage of this centralizer of an involution in \( 2 \cdot BM \) is \( 2^2 \cdot 2E_6(2).2 \). This, however, is easy to see: the subgroup \( 2^2 \cdot 2E_6(2).S_3 \) is the normalizer of a pure \( 2A \) subgroup of order 4 in the Monster \( M \), and therefore the centralizer of a point on the \( 2A \) subgroup in \( M \) is \( 2^2 \cdot 2E_6(2).2 \). As this centralizes a \( 2A \) element, this lies inside the centralizer of a \( 2A \) element of \( M \), namely \( 2 \cdot BM \).

As there is a copy of \( S_3 \) permuting the central involutions of the \( 2^2 \cdot 2E_6(2) \), all central extensions \( 2 \cdot 2E_6(2) \) are isomorphic. As the centralizer of an element of order \( 7 \) in \( 2 \cdot 2E_6(2).2 \) is \( 7 \times 2 \cdot PSL_3(4).2 \), the preimage in \( 2^2 \cdot 2E_6(2).2 \), hence in \( 2 \cdot BM \), of this subgroup must be \( 7 \times 2^2 \cdot PSL_3(4).2 \), as otherwise not all of the quotients by a central involution would be isomorphic. In particular, this means that \( Z(\hat{G}) \) lies inside \( O^7(\hat{N}_G(Q)) \), and therefore \( K^\circ_G \) contains \( Z(\hat{G}) \), as needed.

4.3 Outer automorphisms

This section details the actions of the outer automorphism of the simple group \( G \) (if it has one) on \( N_G(S)/K_G \), thereby computing the group \( N_{\text{Aut}(G)}(S)/K_G \). We of course need only do this when \( K_G \neq N_G(S) \), and when \( G \neq \text{Aut}(G) \). We include the cases where \( S \) is TI, even though then it is simply \( N_{\text{Aut}(G)}(S)/S \).

1. For \( M_{22} \) and \( p = 3 \), \( N_{\text{Aut}(G)}(S)/K_G \cong SD_{16} \).
2. For $J_2$, if $p = 3$ then $N_{\text{Aut}(G)}(S)/K_G \cong 2 \times 2$, and if $p = 5$ then $N_{\text{Aut}(G)}(S)/K \cong 4$.
3. For $J_3$, if $p = 3$ then $N_{\text{Aut}(G)}(S)/K_G \cong 2 \times 2$.
4. For $HS$, if $p = 3$, $5$ then $N_{\text{Aut}(G)}(S)/K_G \cong D_8$.
5. For $McL$, if $p = 5$ then $N_{\text{Aut}(G)}(S)/K_G \cong 3 \times E$ for a 2-group $E$, with $C_4 \times C_2$ as quotient. (The group $E$ has order 16 and a cyclic subgroup of index 2, but is not of maximal class. This determines $E$ uniquely.)
6. For $Suz$, if $p = 5$ then $N_{\text{Aut}(G)}(S)/K_G \cong 2 \times 2$.
7. For $He$, if $p = 5$ then $N_{\text{Aut}(G)}(S)/K_G \cong S_3$.
8. For $Fi_{22}$, if $p = 5$ then $N_{\text{Aut}(G)}(S)/K_G \cong D_{12}$.
9. For $T_2$, if $p = 5$ then $N_{\text{Aut}(G)}(S)/K_G \cong 4S_4$, which has $C_4$ as quotient.

5 Constructing endotrivial modules

We attempt to identify the trivial-source endotrivial modules where there are non-trivial ones, and the simple endotrivial modules that were not determined in Lassueur et al. (2016).

The characters of many of these have previously been identified in Lassueur and Mazza (2015b), but here we give the full socle series of the modules. The simple trivial-source, endotrivial modules that are given in Sect. 5.1 were all found already in Lassueur et al. (2016).

5.1 Trivial-source endotrivial modules

For the small sporadic groups, we simply induce up the modules for $N_G(S)/K_G$ to $G$, passing through a maximal subgroup between $N_G(S)$ and $G$ when one is available. We then remove all summands not from the appropriate block (the Brauer corresponding of the block containing the module for $N_G(S)$), and then remove any projective summands if we can find them. At this point, we just ask Magma to write the module as a direct sum.

For larger modules, we are sometimes able to write the induction (cut by the appropriate block) as a sum of a projective and another module: if Magma can construct the projective module $P$, the command $\text{AHom}(P, V)$ constructs all $kG$-module homomorphisms from $P$ to $V$, so we may take the cokernel of an injective map in this space. In such cases where we wish to do this, we give a direct sum decomposition of the induction to avoid the reader having the rediscover the projective part of the induction.

5.1.1 The groups $M_{11}$, $M_{22}$ and $M_{23}$

In all three cases we have $p = 3$. For $G = M_{11}$ we have $T(G, S) \cong C_2 \times C_2$. The four modules are $1$, $10_1$ (the self-dual 10-dimensional module), a module $10_2/5/1, 24/5^* /10_2^*$.
and a module
\[5^*/((5/1, 24/5^*) \oplus 10_1)/5.\]

For \(G = M_{22}\) and \(\hat{G} = 2 \cdot M_{22}\) we have \(T(G, S) \cong C_2 \times C_2\) and \(T(\hat{G}, S) \cong C_4 \times C_2\). The four modules in \(T(G, S)\) are 1, 55, 49/1, 55/49*, 49*/1, 55/49.

The four faithful modules in \(T(\hat{G}, S)\) are all simple, and are 10, 10*, 154 and 154*.

For \(G = M_{23}\) we have \(T(G, S) \cong C_2\), and the two modules are 1 and 253.

### 5.1.2 The groups \(J_2, J_3\) and \(J_4\)

For \(G = J_2\) we have \(p = 3, 5\), and in both cases \(T(G, S) \cong C_2\). The non-trivial module in \(T(G, S)\) for \(p = 3\) and \(p = 5\) have structures

\[
21_1, 21_2/133/21_1, 21_2 \text{ and } 189/14, 21, 189/14, 21, 189/189
\]

respectively.

For \(G = J_3\) and \(\hat{G} = 3 \cdot J_3\), if \(p = 3\) then \(T(G, S) \cong C_2\). The two modules are 1 and a summand \(V\) of the 25840-dimensional induced module from \(N_{G}(S)\). This induced module is the sum of the projectives \(P(2754) = 2754/324/2754, 1215_1\) and \(1215_2\), the simple module 324 and \(V\). The module \(V\) has dimension 17254, 85 composition factors, 25 socle layers, and socle structure

\[
168/306, 934/36, 168/306/168/306/36, 168/1^2, 306/168/306, 934/36, 168/1^2, 306/36, 168/1^4, 306, 934/36/168/1^2, 306^2/168^2/1^4, 306^2/168^2/1^4, 306^2/168^2/1^4, 306^2/36, 168^3/1^4, 306^2/36^2, 934^2/36^2, 168^3/1^4, 306^2/36^2, 168^3/1^4, 306^2/36^2, 934^2/36^2, 168^3/1^4, 306^2/36^2, 934^2/36^2, 168^3/1^4, 306^2/36^2, 934^2/36^2, 168^3/1^4, 36, 168^2, 934^2/1^3, 36, 168^2, 934^2/1^3, 36, 168, 306/168, 934.
\]

(Here, \(934^2\) means two copies of 934, and so on.)

If \(p = 2\) then \(T(\hat{G}, S)\) is trivial. Since \(K_{\hat{G}} \neq K_{\hat{G}}^0\), and (Lassueur and Mazza 2015b) is in disagreement, we confirm directly that \(K_{\hat{G}} = N_{\hat{G}}(S)\). To do this we induce one of the 1-dimensional characters of the group \(3 \times (2^5 \cdot A_5)\) up to all of \(\hat{G}\), as suggested in (Lassueur and Mazza 2015b, 4.2). The summands have dimensions 12681, 4770 and 8712, so the one of dimension 12681 is the Green correspondent. However, its restriction to \(S\) is not the sum of a trivial and a free module, as would be needed. Indeed, upon removing the free summands, there is a module of dimension 1161 left over. The problem in Lassueur and Mazza (2015b) appears to have been a communication problem between Carlson and the authors of Lassueur and Mazza (2015b), and Carlson’s algorithm produces the correct answer.

For \(J_4\) and \(p = 11\), \(N_{G}(S)\) is maximal, and so it is very difficult to obtain any information about the elements of \(T(G, S)\), beyond the fact that it is isomorphic to \(C_{10}\). In Lassueur et al. (2016), candidate simple endotrivial modules are given as

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the two dual modules of dimension 88 7778, and the self-dual module of dimension 39 476 5284. In the next section we show that the latter cannot be endotrivial, and that the former is endotrivial, but does not have trivial source. We cannot construct any non-trivial element of \( T(G, S) \).

5.1.3 The group \( HS \)

Let \( G = HS \). If \( p = 3 \), we have \( T(G, S) \cong C_2 \times C_2 \). The four modules are 1, 154, 1542 and 1543.

If \( p = 5 \), we have \( T(G, S) \cong C_4 \). Choose one of the two dual generators \( V_1 \) for \( T(G, S) \), and let \( V_2 \) denote the element of order 2 from \( T(G, S) \). Let \( W_i \) denote the induction of the Green correspondent of \( V_i \) in \( NG(S) \) to \( G \). Thus \( \dim(W_i) = 22 176 \).

We may decompose \( W_1 \) as

\[
V_1 \oplus P(280^*) \oplus P(650) \oplus P(1925) \oplus 1925 \oplus 1750.
\]

The module \( V_1 \) has dimension 2876, and has socle series

\[
98/1, 133_1, 133_2/21, 98/55, 55, 210, 518/21, 21, 98, 98, 280/1, 55, 133_1, 133_2, 518/98.
\]

(The heart of this module is indecomposable, and a Hasse diagram for it is too complicated to draw nicely.) Its dual has the same structure except for 280 in place of 280.

The other module is a self-dual 12 376-dimensional module \( V_2 \) with 64 composition factors and seven socle layers. The decomposition of \( W_2 \) is

\[
V_2 \oplus P(1925) \oplus 175 \oplus 1750 \oplus 2750,
\]

and the precise socle structure of \( V_2 \) is

\[
210, 518^2/21^2, 98^2, 280_1, 280_2/55, 133_1, 133_2, 210^2, 518^2/21^4, 98^5, 280^2_1, 280^2_2/1^4, 55^2, 133^2_1, 133^2_2, 210^3, 518^5/21^4, 98^5, 280^2_1, 280^2_2/55, 210, 518^2.
\]

5.1.4 The group \( McL \)

Let \( G = McL \) and \( p = 5 \). Writing \( \hat{G} = 3 \cdot McL \), we have that \( T(G, S) \cong C_8 \) and \( T(\hat{G}, S) \cong C_{24} \). There are four simple endotrivial modules for \( \hat{G} \), all of dimension 126. However, their sources are \( \Omega^\pm(k) \), so they do not lie in \( T(\hat{G}, S) \). On the other hand, the tensor product of the two 126-dimensional modules in the same block is indecomposable and has trivial source, so does lie in \( T(\hat{G}, S) \). It has socle structure

\[
1035/126_2/1233/126_1/1035/126_2/639, 1233/126_1, 126_1/1035, 4752/126_2, 126_2, 153, 846/639, 1233/126_1/1035.
\]
By finding its Green correspondent in $N_{G}(S)$, we see that it has order 6 in $T(\hat{G}, S)$. It seems difficult to construct the other non-trivial elements of $T(\hat{G}, S)$.

5.1.5 The groups $Suz$, He, $Co_{3}$, $Co_{2}$ and $Fi_{23}$

In these cases $p = 5$. If $G = Suz$ then $T(G, S) \cong C_{2}$, and the two modules are 1 and 1001. If $G = He$ then $T(G, S) \cong C_{3}$, and the three modules are 1, 51 and 51*. If $G = Fi_{23}$, then $T(G, S) \cong C_{2}$, and the two modules are 1 and 111826.

If $G = Co_{3}$ then $T(G, S) \cong C_{4}$. We cannot construct the two elements of order 4, but we can construct the element of order 2, by inducing the non-trivial 1-dimensional module from the maximal subgroup $McL \rtimes 2$. (Note that the other trivial-source endotrivial modules can be induced from modules for $McL \rtimes 2$, but we cannot construct them either: see Sect. 5.1.4.) The induction has structure $2^{3}/2^{2}$. The contribution $V$ from the principal 5-block has composition factors 23, 23, 23, and the two modules are 1 and 1001.

If $G = Co_{2}$ then $T(G, S) \cong C_{2}$. Unfortunately we cannot construct the non-trivial element of $T(G, S)$ in this case, but it also comes from a non-trivial 1-dimensional module, this time for the maximal subgroup $HS \times 2$. This induced module $W$ has dimension 476928, and from an ordinary character calculation, and the known decomposition matrix for $Co_{2}$, we see that the contribution $V$ from the principal 5-block has composition factors 23, 23, 2254, 2254, 29624. By restricting 23 and 2254 down to $HS \times 2$ and using the Nakayama relation, we find that $\text{soc}(V) = 23@2254$. However, this is not enough to give us the complete structure of $V$, since $\text{Ext}^{1}(23, 2254) \neq 0$, so $V$ need not have structure 23, 2254/29624/23, 2254.

5.1.6 The group $Ru$

For $G = Ru$ and $\hat{G} = 2 \cdot Ru$, we have $p = 3$. We have $T(G, S) \cong C_{2}$ and $T(\hat{G}, S) \cong C_{4}$. The elements of order 4 in $T(\hat{G}, S)$ are the modules 28 and 28*; the symmetric square of $28^{\pm}$ is 406, which is the non-trivial module in $T(G, S)$.

5.1.7 The group $Fi_{22}$

For $Fi_{22}$, $p = 5$, we have $T(G, S) \cong C_{2}$. The two modules are 1 and 1001.

In general, $T(\hat{G}, S) \cong C_{2} \times Z(\hat{G})$. If $|Z(\hat{G})|$ has order 2 there are two faithful trivial-source endotrivial modules. Both have dimension 61776 = $|G : H|$ for $H$ the maximal subgroup $Z(\hat{G}) \rtimes \Omega_{8}^{+}(2).S_{3}$, and are the inductions of 1-dimensional modules from this maximal subgroup. The two modules have structure

$$13376_{i}/(352 \oplus (5824_{i}/23024_{i}/5824_{i}^{*})))/13376_{i}$$

for $i = 1, 2$, with an appropriate labelling of the simple modules. (Note that this contains $5824_{i}^{\pm}$, which we need in the next subsection.)

For $\hat{G} = 3 \cdot Fi_{22}$, the two 351-dimensional simple modules are endotrivial and trivial source, being the elements of order 3 in $T(\hat{G}, S)$. The two elements of order 6 are dual and also must be Galois conjugates, so are ‘Galois dual’. The composition factors, from an ordinary character calculation and the decomposition matrix on the online
Modular Atlas (determined from Hiss and White 1994), are 2079, 5643^2, 48411. By Galois duality, the module must be 5643/2079, 48411/5643.

In the case where \( \mathbb{Z}^\hat{\mathcal{G}} \) has order 6, the four 61776-dimensional modules are simple and endotrivial. This is easy to check because the inductions of the relevant 1-dimensional modules for \( H \) are, in fact, simple.

5.1.8 The groups \( Th \) and \( BM \)

For \( Th \) and \( BM \), \( p = 7 \), there are two elements in \( T(G, S) \), but it is impossible with our current technology to construct the non-trivial element of it. (However, it is easy to understand the corresponding linear character of \( N_G(S) \), i.e., the Green correspondent of that element of \( T(G, S) \).)

Note that if \( G = BM \) then there is a subgroup \( H \leq G \) isomorphic with \( Th \). Indeed, up to conjugacy \( N_G(S) \leq H \), and therefore the Green correspondents of the elements of \( T(H, S) \) are the elements of \( T(G, S) \).

Furthermore, since the maximal subgroup \( L \cong 2^2 \cdot F_4(2).2 \) contains \( N_G(S) \), we see that the Green correspondents of the elements of \( T(G, S) \) are the two linear characters of \( L \).

5.1.9 The group \( T \)

For the Tits group \( T \), \( p = 5 \), we have \( T(G, S) \cong C_6 \). A generator for this group is the simple module 26 (or 26^*) with its tensor square splitting as the sum of the simple module 325 and a 351-dimensional module in \( T(G, S) \), which has structure

\[
109_1/27/1, 78/27^*/109_2,
\]

for some suitable labelling of 109_1 and 27.

For the final module we may work over \( \mathbb{F}_5 \), so that \( 109_1 \oplus 109_2 \) becomes 218 and \( 460_1 \oplus 460_2 \) becomes 920. The induction \( W \) from \( N_G(S) \) to \( G \) has dimension 14976, and splits up as

\[
V \oplus P(920) \oplus 300^{\oplus 2} \oplus 1300_1 \oplus 1300_2,
\]

where \( V \) is the trivial-source endotrivial module, of dimension 2976. It has socle structure

\[
27/920/27^*/1, 351/27^*, 52/218, 351^*/27, 27/1, 920/27^*.
\]

(Here we have chosen 27 so that there is a module 1/27^* but not 1/27, and 351 so that \( \Lambda^2(27) = 351^* \).)
Table 2 Known simple endotrivial modules for quasisimple sporadic groups

| Group | Prime | Module | Source | Previously known? |
|-------|-------|--------|--------|------------------|
| $M_{11}$ | 3 | $101$ | $k$ | Lassueur et al. (2016) |
| $M_{11}$ | 3 | $10\frac{3}{2}$ | $\Omega^{\pm 2}(k)$ | Lassueur et al. (2016) |
| $M_{22}$ | 3 | 55 | $k$ | Lassueur et al. (2016) |
| $2 \cdot M_{22}$ | 3 | $10^{\frac{3}{2}}$ and $154^{\frac{3}{2}}$ | $k$ | Lassueur et al. (2016) |
| $M_{23}$ | 3 | 253 | $k$ | Lassueur et al. (2016) |
| $HS$ | 3 | $154_1$, $154_2$ and $154_3$ | $k$ | Lassueur et al. (2016) |
| $3 \cdot McL$ | 5 | $126^{\frac{3}{2}}$ and $126^{\frac{3}{2}}_2$ | $\Omega^{\pm 2}(k)$ | Lassueur et al. (2016) |
| $Suz$ | 5 | 1001 | $k$ | Lassueur et al. (2016) |
| $He$ | 5 | $51^{\pm}$ | $k$ | Lassueur et al. (2016) |
| $Ru$ | 3 | 406 | $k$ | Lassueur et al. (2016) |
| $2 \cdot Ru$ | 3 | $28^{\pm}$ | $k$ | Lassueur et al. (2016) |
| $3 \cdot ON$ | 7 | $342^{\frac{3}{2}}_1$ and $342^{\frac{3}{2}}_2$ | $U_1$ | Lassueur et al. (2016) |
| $Fi_{22}$ | 5 | 1001 | $k$ | Lassueur et al. (2016) |
| $2 \cdot Fi_{22}$ | 5 | $5824^{\frac{3}{2}}_1$ and $5824^{\frac{3}{2}}_2$ | $\Omega^{\pm 5}(k)$ | No |
| $3 \cdot Fi_{22}$ | 5 | $351^{\pm}$ | $k$ | Lassueur et al. (2016) |
| $3 \cdot Fi_{22}$ | 5 | $12474^{\frac{3}{2}}_1$ and $12474^{\frac{3}{2}}_2$ | $\Omega^{\pm 5}(k)$ | No |
| $6 \cdot Fi_{22}$ | 5 | $61776^{\frac{3}{2}}_1$ and $61776^{\frac{3}{2}}_2$ | $k$ | Lassueur and Malle (2015) |
| $Fi_{23}$ | 5 | 111826 | $k$ | Lassueur et al. (2016) |
| $J_4$ | 11 | 887778 | $\Omega^{\pm 8}(k)$ | No |
| $T$ | 3 | $26^{\pm}$ | $U_2$ | Lassueur et al. (2016) |
| $T$ | 5 | $26^{\pm}$ | $k$ | Lassueur et al. (2016) |
| $T$ | 5 | $351^{\pm}$ | $\Omega^{\pm 6}(k)$ | Lassueur et al. (2016) |

For definitions of $U_1$ and $U_2$, see the proof of Proposition 5.1

Table 3 Remaining candidate simple endotrivial modules for quasisimple sporadic groups

| Group | Prime | Module | Notes |
|-------|-------|--------|-------|
| $Th$ | 7 | $27000^{\pm}$ | Known to be simple |
| $BM$ | 7 | $9287037474$ | Simplicity unknown, trivial source if endotrivial |
| $BM$ | 7 | $775438738408125$ | Simplicity unknown, trivial source if endotrivial |

5.2 Simple endotrivial modules

This section tabulates the known simple endotrivial modules for sporadic groups, and gives their sources. It also gives the candidate simple endotrivial modules from (Lassueur et al. 2016) whose status we have not determined here.

At the time of writing, the known simple endotrivial modules are given in Table 2 (if the module was known before this paper, a reference is given in the table), and the remaining candidates for simple endotrivial modules are in Table 3.
Proposition 5.1 If $V$ appears in Table 2, then $V$ is a simple endotrivial module.

Proof If $V$ appears in (Lassueur et al. 2016, Table 7) without a ‘?’ then $V$ was shown to be endotrivial in that paper. The sources of simple modules of small dimension for small $G$ can easily be found by computer. If $V$ is self-dual then it must be trivial source as $p$ is odd, and thus $T(S)$ is torsion-free by Carlson and Thévenaz (2004), as we stated in Sect. 2.

Of those simple modules whose endotriviality was established in Lassueur et al. (2016), the only ones that are difficult to construct (because they do not have very small dimension and are unavailable in the online Atlas (Wilson et al. 2019)) are the 342-dimensional modules for $3 \cdot ON$. One may find these simple modules as composition factors of a tensor product $45 \otimes 495_i$, and so their source can be accessed relatively easily. It can be described as follows: there are three conjugacy classes of subgroups $Q_i$ of order 49 in $N_G(S)$, with (say) $Q_1$ and $Q_2$ being exchanged by the outer automorphism. The action of both $Q_1$ and $Q_3$ is (up to projectives) $\Omega^3(k)$, and on $Q_2$ it is $\Omega^{-11}(k)$. This determines an endotrivial module $U_1$ of $S$ uniquely up to outer automorphism of $G$, by Lemma 2.1.

For the Tits group, if $p = 5$ then the sources are as in Table 2, but for $p = 3$ the source is more difficult to describe, as it was for the O’Nan group. Let $Q_1$ and $Q_2$ be representatives of the two $N_G(Q)$-classes of subgroups $3^2$ in $S$. The module $U_2$ has the property that $\Omega^3(U_2)$ has restriction to $Q_1$ trivial plus projective, and $\Omega^{-3}(U_2)$ has restriction to $Q_2$ trivial plus projective. This determines $U_2$ uniquely up to outer automorphism of $G$, which swaps $Q_1$ and $Q_2$.

We are left with those modules that were not proved to be endotrivial in Lassueur et al. (2016): these are the modules for $2 \cdot Fi_{22}$ and $6 \cdot Fi_{22}$. The 61776-dimensional modules $V$ are, by an ordinary character calculation, induced from 1-dimensional modules for the maximal subgroup $H \cong 6 \times \Omega^3_{22} \cdot S_3$, which is of course of index 61776. Thus these are trivial-source modules. For endotriviality we either note that they arise from characters on $N_G(S)/K_G$, or we note that by the Mackey formula we need to understand $H^S \cap S$ for $g$ running over all 2472 $(H, S)$-double-coset representatives. All but one of these intersections is trivial, so $V \downarrow_S$ is the sum of a trivial and a free module.

For $V$ one of $5824_{i}^\pm$ for $\hat{G} = 2 \cdot Fi_{22}$, one may find these simple modules in the elements of $T(\hat{G}, S)$ that were given in the previous section. Once they are given in a computer it is easy to restrict to $S$ and remove free summands. One is left with $\Omega^\pm(k)$, so $V$ is endotrivial with $\Omega^\pm(k)$ as source.

For $V$ one of the $12474_i^\pm$ for $\hat{G} = 3 \cdot Fi_{22}$, it is much harder to construct this module, and I am grateful to Richard Parker for running this computation on Meataxe64. One finds a 12474-dimensional composition factor in the tensor product $5864 \otimes 78$ over $\mathbb{F}_{25}$. Restricting down to $S$ and removing free summands, we are again left with $\Omega^\pm(k)$.

For $G = J_4$, we can confirm directly that the exterior squares of the 1333-dimensional modules are endotrivial. This can be proved by restricting to the (unique up to conjugacy) maximal elementary abelian subgroup of order $11^2$. One cannot restrict the 887778-dimensional module directly, so we restrict the 1333-dimensional module, remove any free summands – which reduces the dimension to 244 – then take the exterior square. This is the direct sum of a free module and the module $\Omega^\pm(k)$, so
the exterior square of the 1333-dimensional module for $G$ is endotrivial by Lemma 2.1. I am grateful to Thomas Breuer who proved simplicity of the reduction modulo 11 of this ordinary character: using GAP (The GAP Group 2018), he showed that there is no consistent breaking up of the mod 11 reduction of the ordinary character to two maximal subgroups $2^{1+12} \times M_{22} \times 2$.

**Proposition 5.2** Let $\hat{G}$ be a quasisimple sporadic group and $V$ is a non-trivial, simple $k\hat{G}$-module. If $V$ does not appear in Tables 2 or 3 then $V$ is not endotrivial.

**Proof** We start with (Lassueur et al. 2016, Table 7), and therefore need to eliminate the modules that appear in that table but not in Tables 2 and 3.

First, notice that if $V$ is self-dual and endotrivial then $V$ lies in $T(\hat{G}, S)$. Thus if $T(\hat{G}, S) = 1$, or is non-trivial but $V$ does not appear in the constructions in the previous section then $V$ is not endotrivial. This eliminates the module of dimension 74887473024 for $Fi'_{24}$ and the module of dimension 722691036263122062500 for $M$, but does not necessarily eliminate the module of dimension 394765284 for $J_4$ for $p = 11$. (The modules for $M$ and $Fi'_{24}$ were also eliminated in Lassueur and Malle 2015.)

To do this, since in this case $V$ is self-dual, it must be the unique non-trivial self-dual element of $T(G) \cong \mathbb{Z} \times C_{10}$. Assuming $V$ is endotrivial, the Green correspondent of $V$ is the non-trivial real linear character of $NG(S)$, and this takes value $-1$ on one class of elements of order 22 in $G$. However, from (Conway et al. 1985), we see that $V$ takes value 1 on both classes of elements of order 22 in $G$. Since projective characters take value 0 on $p$-singular classes, this means that the Green correspondent of this linear character cannot be the reduction modulo 11 of this ordinary character, and so the module $V$ of dimension 394765284 cannot be endotrivial.

We cannot do anything with the modules for $Th$, since they are not self-dual, and therefore they could just be elements of infinite order in $T(G)$. For $BM$, at most one of the two modules listed is endotrivial, because it would have to lie in $T(G, S)$ for this group, which has order 2.

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