A distance measure of non-trace-preserving channels

Yu Shi$^1,\ast$ and Edo Waks$^{1,2,3,\dagger}$

$^1$Department of Electrical and Computer Engineering and Institute for Research in Electronics and Applied Physics, University of Maryland, College Park, Maryland 20742, USA

$^2$Joint Quantum Institute, University of Maryland, College Park, Maryland 20742, USA

$^3$Department of Physics, University of Maryland, College Park, Maryland 20742, USA

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The diamond distance can measure the error between real and ideal quantum processes. We can calculate the distance for trace-preserving channels using convex optimization. However, convex optimization cannot solve the distance for non-trace-preserving channels because they are nonlinear. We generalize the diamond distance for non-trace-preserving channels. We first compute the distance for a simple measurement and post-selection channel, then generalize the method to any non-trace-preserving channels. As an example, we analyze the Knill-Laflamme-Milburn (KLM) protocol in linear optical quantum computing with physical imperfections. Our method provides a well-behaved measure for general quantum channels, which can bridge the physical imperfections with the logical errors in quantum circuits. Hence, it has broad applications in analyzing real systems in quantum information processing, such as quantum computers, quantum teleportation, measurement-based quantum computation, etc.

The diamond distance [1], also called completely bounded trace distance [2] or stabilized process distance [3], measures the distinction between two quantum channels. Specifically, we use it to calibrate the worst-case error of a real quantum process from an ideal one. We can calculate the diamond distance between two trace-preserving channels by convex optimization [2]. However, non-trace-preserving channels are ubiquitous in physics. A typical one is the measurement and post-selection process. We cannot compute their distance using convex optimization because these channels are non-linear.

Another measure of quantum channels is by calculating the fidelity of their process matrices using channel-state duality [3]. This measure estimates the average-case error of a quantum process. Kiesel et al. [4] calculated the fidelity for a non-trace-preserving channel by renormalizing its process matrix, but the renormalized matrix is nonphysical because it can have negative eigenvalues. Additionally, the fault-tolerant analysis requires a threshold of the worst-case error rate. Therefore, we still lack a well-behaved error measure for real quantum processes [5].

We generalize the diamond distance for non-trace-preserving channels, which share the same properties and physical interpretation as the conventional diamond distance. We decompose the generalized distance to the addition of one conventional diamond distance and another distance referred as the renormalizing distance, which we can compute separately. As a demonstration of its application, we analyze the KLM protocol [6] in linear optical quantum computing with physical imperfections. Our method provides a well-behaved measure of imperfections in quantum systems, which has broad applications in analyzing real systems in quantum information processing.

We first consider a simple process of measurement followed by selecting a single outcome. The channel is described by $\mathcal{M}(\rho) = \frac{M\rho M^\dagger}{Tr[M\rho M^\dagger]}$, where $\rho$ is a density operator, and $M$ is a diagonal measurement operator. We consider a distance measure of $\mathcal{M}$ from an identity channel $\mathcal{I}$ as

$$d(\mathcal{I},\mathcal{M}) = \max_{\rho} \left\| \rho - \frac{(M \otimes I) \rho (M^\dagger \otimes I)}{Tr[(M \otimes I) \rho (M^\dagger \otimes I)]} \right\|_1,$$  \hspace{1cm} (1)$$

where $\| \cdot \|_1$ represents the matrix trace norm [7]. This definition adheres to the conventional diamond distance definition of considering a distance measure in the extended state space, so that it satisfies the stability property, i.e., $d(\mathcal{E} \otimes \mathcal{I},\mathcal{F} \otimes \mathcal{I}) = d(\mathcal{E},\mathcal{F})$ [3]. We will abbreviate $M \otimes I$ as $M$ for concision in the following manuscript. The chaining property for composed processes, $d(\mathcal{E} \otimes \mathcal{I},\mathcal{F} \circ \mathcal{M}) \leq d(\mathcal{E},\mathcal{F}) + d(\mathcal{I},\mathcal{M})$, follows straightforwardly from the triangle inequality of trace norm.

To calculate $d(\mathcal{I},\mathcal{M})$, we first prove the distance optimized over a pure state $\psi \in X \otimes X$, where $X \otimes X$ denotes the extended state space. For a density operator $\rho \in X \otimes X$, we consider its purification $\psi \in X \otimes X \otimes R$ such that $\rho = Tr_R [\psi]$. We formulate Eq. 1 as

$$d(\mathcal{I},\mathcal{M}) = \max_{\psi \in X \otimes X \otimes R} \left\| Tr_R \left\{ \psi - \frac{M\psi M^\dagger}{Tr[M\psi M^\dagger]} \right\} \right\|_1,$$

$$\leq \max_{\psi \in X \otimes X \otimes R} \left\| \psi - \frac{M\psi M^\dagger}{\langle \psi |M^\dagger M |\psi \rangle} \right\|_1 \hspace{1cm} (2a)$$

$$= \max_{\psi \in X \otimes X} \left\| \psi - \frac{M\psi M^\dagger}{\langle \psi |M^\dagger M |\psi \rangle} \right\|_1, \hspace{1cm} (2b)$$

where Eq. 2a is from the contractility of trace distance [8], and Eq. 2b is from lemma 3.45 in [7]. Eq. 2b should
be trivial given that $M$ is a diagonal operator, but the proof also applies to general channels, which we will discuss later. Therefore, we calculate $d(I, M) = \max_\psi \|\psi \psi^\dagger - \phi \phi^\dagger\|_1$, where $|\phi\rangle = \frac{M|\psi\rangle}{\|M|\psi\rangle\|}$ is the renormalized state after measurement. The trace distance of two pure states is related to their inner product as $\|\psi \psi^\dagger - \phi \phi^\dagger\|_1 = 2\sqrt{1 - |\langle \psi | \phi \rangle|^2}$ [8]. Thus, to maximize $\|\psi \psi^\dagger - \phi \phi^\dagger\|_1$ is equivalent to minimizing $|\langle \psi | \phi \rangle|$.

We consider a diagonal matrix $M_n$ in an $n$-dimensional state space, where $M_n = diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$ and $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_n \geq 0$. We denote a unit vector $|u\rangle = \sqrt{x}|n\rangle + \sqrt{1-x}|v\rangle$, where $x \in [0, 1]$, $|n\rangle$ is the basis corresponding to the eigenvalue $\lambda_n$, and $|v\rangle$ is a unit vector in the $(n-1)$-dimensional subspace orthogonal to $|n\rangle$. The assumption of positive coefficients loses no generality because we can always redefine the basis with a phase factor. Applying $M_n$ to $|u\rangle$ gives

$$M_n |u\rangle = \lambda_n \sqrt{x} |n\rangle + \sqrt{1-x} M_{n-1} |v\rangle,$$

where $M_{n-1}$ is a diagonal matrix in the subspace. Applying $M_{n-1}$ to $|u\rangle$ will contract and rotate it as

$$M_{n-1} |v\rangle = r_{n-1} \cos \theta_{n-1} |v\rangle + r_{n-1} \sin \theta_{n-1} |v_\perp\rangle,$$

where $r_{n-1}$ and $\theta_{n-1}$ are the contracting factor and rotation angle corresponding to $|v\rangle$, and $|v_\perp\rangle$ is a unit vector orthogonal to $|v\rangle$. A specific rotating axis does not affect the following calculation because the axis is always orthogonal to $|n\rangle$. We calculate the contracting factor $r_n = |M_n |u\rangle|$ and the rotating angle $\cos \theta_n = \langle u |M_n |u\rangle / r_n$ for applying $M_n$ to $|u\rangle$ and get

$$\begin{cases} r_n = \sqrt{x \lambda_n^2 + (1-x) r_{n-1}^2} \\ \cos \theta_n = \frac{x \lambda_n + (1-x) r_{n-1} \cos \theta_{n-1}}{r_n} \end{cases} \quad (5),$$

Reformulating Eq. 5 gives

$$\begin{cases} r_n^2 = x \lambda_n^2 + (1-x) r_{n-1}^2 \\ r_n \cos \theta_n = x \lambda_n + (1-x) r_{n-1} \cos \theta_{n-1} \end{cases} \quad (6),$$

i.e., the feasible point $(r_n^2, r_n \cos \theta_n)$ is a convex combination of $(\lambda_n^2, \lambda_n)$ and $(r_{n-1}^2, r_{n-1} \cos \theta_{n-1})$. By induction, we can compute the feasible points of $(r^2, \cos \theta)$ for applying $M$ to a vector, as

$$\begin{cases} r^2 = \sum_i p_i \lambda_i^2 \\ r \cos \theta = \sum_i p_i \lambda_i \end{cases} \quad (7),$$

where $\sum_i p_i = 1$. The feasible set of $(r^2, \cos \theta)$ is a polygon with vertices $(\lambda_i^2, \lambda_i)$. Fig. 1 shows the feasible set of $(r, \cos \theta)$. The red curve presents its boundary, and the blue points show the statistics from the Monte Carlo simulation by random input vectors. The minimum of $\cos \theta$ is on the boundary of the domain and equals $\frac{2\sqrt{\lambda_1 \lambda_n}}{\lambda_1 + \lambda_n}$.

Next, we compute the distance of two general quantum channels $\mathcal{E}$ and $\mathcal{F}$, defined as $\mathcal{E}(\rho) = \frac{\sum a_k \rho a_k^\dagger}{Tr[\sum a_k \rho a_k^\dagger]}$ and $\mathcal{F}(\rho) = \frac{\sum b_k \rho b_k^\dagger}{Tr[\sum b_k \rho b_k^\dagger]}$. $a_k$ and $b_k$ are Kraus operators, which map a state $\rho$ from input space $X$ to output space $Y$. These operators satisfy $\sum_k a_k^\dagger a_k \leq I$ and $\sum_k b_k^\dagger b_k \leq I$, i.e., the channels can be either trace-preserving or non-trace-preserving. We then represent the two channels in Stinespring form as $\mathcal{E}(\rho) = Tr_Z \{ A \rho A^\dagger \}$ and $\mathcal{F}(\rho) = Tr_Z \{ B \rho B^\dagger \}$, where $A = \sum_k a_k \otimes e_k$, $B = \sum_k b_k \otimes e_k$, and $e_i$ represents a basis in the auxiliary space $Z$ [7]. The distance between $\mathcal{E}$ and $\mathcal{F}$ is defined by

$$d(\mathcal{E}, \mathcal{F}) = \max_u \left\| Tr_Z \{ \frac{A uu^\dagger A^\dagger - B uu^\dagger B^\dagger}{(uu^\dagger A^\dagger - B uu^\dagger B^\dagger)} \} \right\|,$$

where $u$ is a pure state in state space $X \otimes X$, and we abbreviate $A \otimes I$ as $A$ and $B \otimes I$ as $B$.

We define a new state $\psi = \frac{A^\dagger}{\|A^\dagger\|}$, such that

$$d(\mathcal{E}, \mathcal{F}) = \max_\psi \left\| Tr_Z \{ \psi \psi^\dagger - \frac{(BA^{-1}) \psi \psi^\dagger (BA^{-1})}{(\psi (BA^{-1}) \psi)^\dagger} \} \right\|_1,$$

where $A^{-1}$ is the pseudoinverse of $A$ satisfying $A^{-1} A = I$. Applying singular value decomposition to $BA^{-1}$ gives $BA^{-1} = U MV^\dagger \otimes I_n$, where we formulate the identity operator $I_n$ in the extending space explicitly. $U$ is an isometry with a dimension of $mk' \times n$, where $n, m$ are
the dimensions of input state space $X$ and output state space $Y$, and $k'$ is the number of Kraus operators $b_k$. $V$ is an isometry with a dimension of $mk \times n$, where $k$ is the number of Kraus operators $a_k$. $M$ is a non-negative diagonal operator with a dimension of $n \times n$. The distance derives

$$d(\mathcal{E}, \mathcal{F}) = \max_{\psi} \left\| \rho_{10} - \frac{UMV^\dagger \psi_1 \psi_2 V M^\dagger}{\rho_1 \rho_2} \right\|_1,$$

By denoting $\psi = V \phi$ and using triangular inequality and contractility of trace distance, we can get

$$d(\mathcal{E}, \mathcal{F}) \leq \max_{\phi} \left\| \rho_{10} - \frac{M \phi \phi^\dagger M}{\phi \phi^\dagger} \right\|_1 + \max_{\phi} \left\| \phi \phi^\dagger - \frac{M \phi \phi^\dagger M}{\phi \phi^\dagger} \right\|_1. \quad (9)$$

Therefore, we can compute the diamond distance between $\mathcal{E}$ and $\mathcal{F}$ by

$$d(\mathcal{E}, \mathcal{F}) \leq d(\mathcal{U}, \mathcal{V}) + d(\mathcal{I}, \mathcal{M}), \quad (10)$$

where $\mathcal{U}(\rho) = Tr_2 \{U \rho U^\dagger\}$ and $\mathcal{V}(\rho) = Tr_2 \{V \rho V^\dagger\}$ are the renormalized channels which are trace-preserving, and $\mathcal{M} = \frac{M_{\rho} M_{\rho}^\dagger}{Tr(M_{\rho} M_{\rho}^\dagger)}$ is a simple measurement channel. We can calculate $d(\mathcal{U}, \mathcal{V})$ by convex optimization and $d(\mathcal{I}, \mathcal{M})$ by Eq. 8. In the following manuscript, we refer $M$ as the renormalizing matrix and $d(\mathcal{I}, \mathcal{M})$ as the renormalizing distance.

To demonstrate the application of the generalized diamond distance, we analyze the KLM protocol in linear optical quantum computing [6]. We consider two modules, the beam splitter and the nondeterministic conditional sign flip (NS) gate, with physical imperfections. Readers can analyze other imperfections similarly. The operator associated with a beam splitter $B_{\theta, \phi}$ is given by

$$U(B_{\theta, \phi}) = \begin{pmatrix} t \cos \theta & -r \sin \theta e^{i \phi} \\ r \sin \theta e^{-i \phi} & t \cos \theta \end{pmatrix},$$

where $r$ and $t$ are the photon loss rates for reflection and transmission, respectively. Given $r, t = 1$, the operator corresponds to an ideal beam splitter which we denote as $U_{id}$. We calculate the distance between real and ideal beam splitters of $B_{\pi/4, 0}$ as $d(U_{id}, U_r) = \max_\rho \| (U_{id} \otimes I) \rho - (U_r \otimes I) \rho \|_1$, where $U_{id}(\rho) = U_{id} \rho U_{id}^\dagger$ and $U_r(\rho) = \frac{U_r \rho U_r^\dagger}{Tr(U_r \rho U_r^\dagger)}$. Fig. 2(a) shows the distance as a function of the ratio of photon loss rate between reflection and transmission $\alpha = \frac{r}{t}$. The violin plot shows the statistics from the Monte Carlo simulation of the distance between the output states by random input states. The red curve shows the theoretical calculation, which gives a tight upper bound. It is because the renormalizing matrix $M = \sqrt{\frac{t^2 + r^2}{2}} I$, so that the renormalizing distance equals 0.

The NS gate applies a nonlinear phase shift on one mode using two ancilla modes. It transforms the state

$$(\alpha_0 |0\rangle + \alpha_1 |1\rangle + \alpha_2 |2\rangle) \otimes |10\rangle_a \rightarrow (\alpha_0 |0\rangle + \alpha_1 |1\rangle - \alpha_2 |2\rangle) \otimes |10\rangle_a,$$

by linear optical devices and post-selection of the ancillas, where $|n\rangle$ represents the photon number state. We consider an imperfection from the dark count, which is an erroneously registered count without any incident photon. We denote the input state space $X = \{|0\rangle, |1\rangle, |2\rangle\}$ and the output state space $Y = \{|0\rangle, |1\rangle, |2\rangle, |3\rangle\}$. The ideal NS gate transforms the input state as $NS_{id}(\rho) = \frac{E_{00} \rho E_{00}^\dagger}{Tr[E_{00} \rho E_{00}^\dagger]}$, and the real NS gate applies $NS_r(\rho) = \frac{E_{00} \rho E_{00}^\dagger + \gamma E_{00} \rho E_{00}^\dagger}{Tr[E_{00} \rho E_{00}^\dagger + \gamma E_{00} \rho E_{00}^\dagger]}$, where $\gamma$ is the dark count rate. $E_{10}$ and $E_{00}$ are the operators conditioned on the measurement of the ancillas with outcomes $|10\rangle_a$ and $|00\rangle_a$, given

$$E_{10} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$E_{00} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2-\gamma}{2+\gamma} & 0 \\ 0 & \frac{-2+\gamma}{2+\gamma} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{6} \left( \frac{3}{2+\gamma} - \frac{2}{2+\gamma} \right) \end{pmatrix}.$$
We calculate the distance between real and ideal NS gates as \( d(\mathcal{N}S_{id}, \mathcal{N}S_r) = \max_\rho \| (\mathcal{N}S_{id} \otimes I) \rho - (\mathcal{N}S_r \otimes I) \rho \|_1 \). Fig. 2(b) shows the distance as a function of the dark count rate. The violin plot shows the statistics from the Monte Carlo simulation of the distance between the output states by random input states. The red curve shows the theoretical calculation, where the gap is from the inequality in Eq. 10.

Many previous studies considered stochastic errors (noise) in quantum systems, representing a completely positive trace-preserving channel using Pauli matrices. However, errors can also arise from systematic imperfections, such as manufacturing, which are usually non-trace-preserving. The generalized distance provides a simple and well-behaved measure for non-trace-preserving channels, which can bridge the physical imperfections with the logical errors in quantum circuits. Hence, it has broad applications in analyzing real systems in quantum information processing, such as quantum computers, quantum teleportation, measurement-based quantum computation, etc.

[1] D. Aharonov, A. Kitaev, and N. Nisan, Quantum Circuits with Mixed States, Conference Proceedings of the Annual ACM Symposium on Theory of Computing, 20 (1998), arXiv:9806029 [quant-ph].
[2] J. Watrous, Semidefinite programs for completely bounded norms, (2009), arXiv:0901.4709.
[3] A. Gilchrist, N. K. Langford, and M. A. Nielsen, Distance measures to compare real and ideal quantum processes, Physical Review A - Atomic, Molecular, and Optical Physics 71, 062310 (2005), arXiv:0408063 [quant-ph].
[4] N. Kiesel, C. Schmid, U. Weber, R. Ursin, and H. Weinfurter, Linear Optics Controlled-Phase Gate Made Simple, Physical Review Letters 95, 210505 (2005).
[5] T. Rudolph, Why I am optimistic about the silicon-photonic route to quantum computing, APL Photonics 2, 030901 (2017).
[6] E. Knill, R. Laflamme, and G. J. Milburn, A scheme for efficient quantum computation with linear optics, Nature 2001 409:6816 409, 46 (2001).
[7] J. Watrous, The Theory of Quantum Information (2018).
[8] A. N. Michael and L. C. Isaac, Quantum Computation and Quantum Information 10th Anniversary Edition (2010).