A $p$-adic exponential map for the Picard group
and its application to the Albanese map

Wataru Kai *
May 22, 2014

Abstract

We define an exponential map from the first cohomology group of the
structure sheaf to the Picard group of a proper flat scheme over a complete
DVR of characteristic $(0, p)$. To be precise, it is an isomorphism between
subgroups of each member. It is an analogue of the classical one defined
in complex geometry. This exponential map is then applied to prove a
surjectivity property concerning the Albanese map of a smooth projective
variety over a complete DVF.

Introduction

Let $\mathcal{O}_k$ be a complete discrete valuation ring of characteristic $(0, p)$ $(p \geq 0)$,
$f : \mathcal{X} \to Spec \mathcal{O}_k$ be a proper and flat scheme over $\mathcal{O}_k$ and $X$ its generic fiber.

The first aim of this paper is to construct an exponential map from the
cohomology group $H^1(X, \mathcal{O}_X)$ to the Picard group $Pic(X)$ which is a local
isomorphism.

Such an exponential map is known in complex geometry: if $X$ is a compact
complex manifold, we have an exact sequence

$$0 \to 2\pi \mathbb{Z} \hookrightarrow \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 1$$

and the induced exponential map on $H^1$ is a local isomorphism:

$$H^1(X, \mathcal{O}_X) \xrightarrow{\exp} Pic(X).$$

Of course, in our algebraic and non-Archimedean set-up the morphism $\exp$ does not make sense in the level of sheaves on $X$. However, we can overcome
this problem with the help of formal geometry and la propriété d’échange to
obtain the first main result:

---

*Graduate School of Mathematical Sciences, The University of Tokyo
**Theorem 0.1** (Theorem 1.6). Let $f : \mathcal{X} \to \text{Spec} \mathcal{O}_k$ be a proper flat scheme over a complete discrete valuation ring with characteristic $(0, p)$ ($p \geq 0$). Define

$H^1(\mathcal{X}, \mathcal{O}_\mathcal{X})^{(\nu)} := \text{Ker}[ H^1(\mathcal{X}, \mathcal{O}_\mathcal{X}) \rightarrow H^1(\mathcal{X}/(\pi^\nu), \mathcal{O}_\mathcal{X}/(\pi^\nu)) ]$,

$\text{Pic}(\mathcal{X})^{(\nu)} := \text{Ker}[ \text{Pic}(\mathcal{X}) \rightarrow \text{Pic}(\mathcal{X}/(\pi^\nu)) ]$

for $\nu \geq 1$.

Then, if $n$ is sufficiently large (a bound explicitly written), we have an isomorphism of filtered abelian groups

$\exp : H^1(\mathcal{X}, \mathcal{O}_\mathcal{X})^{(n)} \cong \text{Pic}(\mathcal{X})^{(n)}$

where each member is filtered decreasingly by $(- - -)^{(\nu)}$ ($\nu \geq n$).

The domain and the target of our exponential map turns out to be subgroups of $H^1(X, \mathcal{O}_X)$ and $\text{Pic}(X)$ respectively. If we take another $\mathcal{X}$ with the generic fiber $X$ unchanged, the two exponential maps

$H^1(X, \mathcal{O}_X) \rightarrow \text{Pic}(X)$,

defined on two open submodules, turn out to coincide on a smaller open submodule. So our exponential map is something that is considered as “the exponential map from $H^1(X, \mathcal{O}_X)$ and $\text{Pic}(X)$ defined within its convergence radius”.

Let us mention here the following classical theorem due to Mattuck [Mat, Theorem 6] on Jacobian varieties (from which he deduces the same statement on general Abelian varieties):

**Theorem 0.2** (Mattuck). Let $X$ be a proper smooth geometrically connected curve with genus $g$ over a complete (rank one) valuation field $k$. Suppose $X$ has a rational point. Then there is an open subgroup of $J_X(k)$ which is analytically isomorphic to $\mathcal{O}^{\oplus g}_k$.

The proof goes as follows. He begins with constructing a family of zero-cycles (some choice is involved)

$x^i_t - x^0_t$ ($i, t \in \{1, \ldots, g\} \times \pi^N \mathcal{O}_k$, $x^i_t \in X(k)$, $N >> 0$)

such that the map it defines

$\pi^N \mathcal{O}^{\oplus g}_k \rightarrow J_X(k)$

is an open immersion of topological spaces. On the other hand, he defines an analytic map mimicking the Abel-Jacobi map in complex geometry (here is another choice)

$\pi^N \mathcal{O}_k \rightarrow \mathcal{O}^{\oplus g}_k$

which is a local homeomorphism. Let $U \subset \mathcal{O}^{\oplus g}_k$ an open set on which the inverse map is defined. Then he proves that the resulting map

$U \rightarrow J_X(k)$

2
is a homomorphism on a neighborhood of the origin. This is done by reducing to the classical Jacobian theory over $\mathbb{C}$.

Of course, we can deduce this theorem (in the discrete valuation case) from Theorem 0.1 by taking a proper flat model of $X$ over $\mathcal{O}_k$. Compared to Mattuck’s method, our method has an advantage of being free of choices and reduction to $\mathbb{C}$.

Furthermore, following his idea and an argument of Saito-Sujatha [SS], our exponential map allows us to prove the second main result concerning the Albanese variety:

**Theorem 0.3** (Theorem 2.8). Let $X$ be a projective smooth geometrically connected scheme over $k$, $g = \dim H^1(X, \mathcal{O}_X)$, Fix a finite field extension $k'/k$ such that there exists a closed point $x$ of $X$ with $k(x) \cong k'$ over $k$. Then there exists a family of zero-cycles on $X$ of the form

$$x_t^i - x_0^i \ (i, t) \in I \times \pi^N \mathcal{O}_k, \ N >> 0, \ \# I = g, \ k(x_t^i) \cong k'$$

such that the map (given by the family and the Albanese map)

$$\pi^N \mathcal{O}_k^{(I)} \to \text{Alb}_k(X)(k)$$

$$(t_i)_{i \in I} \mapsto \sum_i \text{alb}_X(x_t^i - x_0^i)$$

is an open immersion of topological spaces.

Here, $\text{Alb}_k(X)(k)$ is endowed with the strong topology which comes from the topology of $k$. In the special case where $k$ is a finite extension of $\mathbb{Q}_p$, the compactness gives the following

**Corollary 0.4.** In the notation of previous theorem, assume further that $k$ is a finite extension of $\mathbb{Q}_p$. Concerning the image of the homomorphism

$$\text{alb}_X : CH_0(X)_{\deg = 0} \to \text{Alb}_k(X)(k),$$

we have

(i) If $k'/k$ is a finite extension with $X(k') \neq \emptyset$, the image of $k'$-valued points by $\text{alb}_X$ generates a finite-index subgroup of $\text{Alb}_k(X)(k)$.

(ii) There exists a finite extension $k''/k$ such that the image of $\text{alb}_X$ is generated by the image of $k''$-valued points.

An analogous exponential map to ours is defined in the context of formal groups of the Picard scheme. We will see in §3 that, if the formal completion of the Picard group functor is a smooth formal group, then our exponential map and the formal exponential map coincide. Note that it is a delicate problem whether a nice theory on the formal group of the Picard group of a given scheme holds or not, while our exponential map is defined whenever we have a proper flat scheme over $\mathcal{O}_k$. Moreover, the new exponential map is more suitable for the explicit computation as in §2.
Notation
Throughout this article (with few exceptions), $k$ denotes a complete discrete valuation field of characteristic $(0, p)$ with normalized valuation $v_k$ and a uniformizer $\pi$, $\mathcal{O}_k$ its ring of integers, $f: \mathcal{X} \to \text{Spec}(\mathcal{O}_k)$ a proper and flat morphism of schemes,

$$\mathcal{X}_\nu = \mathcal{X} \otimes \mathcal{O}_k \mathcal{O}_k/(\pi^n),$$

$$Y = \mathcal{X}_1 = \mathcal{X} \otimes \mathcal{O}_k \mathcal{O}_k/(\pi)$$

(beware that the numbering is contrary to the customs).

For a group functor $G$ and an integer $\nu \geq 0$, we set

$$G(\mathcal{X})^{(\nu)} = \text{Ker}[G(\mathcal{X}) \to G(\mathcal{X}_\nu)].$$

$(G(\mathcal{X})^{(\nu)})_{\nu \geq 0}$ is a decreasing filtration on $G(\mathcal{X})$. We will use this only for $G = H^1(-, \mathcal{O}_-)$ and $\text{Pic}$.

1 Exponential Map.
1.1 Review on the exponential power series.
1.1.1 Recall:

$$\exp(T) = 1 + T + \frac{1}{2!}T^2 + \cdots \in \mathbb{Q}[[T]],$$

$$\log(1 + T) = T - \frac{1}{2}T^2 + \frac{1}{3}T^3 - \cdots \in \mathbb{Q}[[T]].$$

We will use

$$\exp(\pi^nT) = 1 + \pi^nT + \frac{1}{2!}(\pi^nT)^2 + \cdots \in \mathbb{k}[[T]],$$

$$\log(1 + \pi^nT) = \pi^nT - \frac{1}{2}(\pi^nT)^2 + \frac{1}{3}(\pi^nT)^3 - \cdots \in \mathbb{k}[[T]].$$

Their $i$-th coefficients are $\pi^{ni}/i!$ and $(-\pi^n)^i/i$.

Lemma 1.1. If $n > c/(p-1)$, $\lim_{i \to \infty} \pi^{ni}/i! = 0$ and $\lim_{i \to \infty} (-\pi^n)^i/i = 0$ in $\mathbb{k}$, and we have

$$\exp(\pi^nT) = 1 + \pi^nT + (\text{terms } \in \pi^{n+1}\mathcal{O}_k[[T]]) \in \mathcal{O}_k[[T]],$$

$$\log(1 + \pi^nT) = \pi^nT + (\text{terms } \in \pi^{n+1}\mathcal{O}_k[[T]]) \in \mathcal{O}_k[[T]].$$

Moreover, we have the following identities in $\mathcal{O}_k[[T]]$:

$$\exp(\pi^nT)|_{T = \log(1 + \pi^nT^\prime)/\pi^n} = 1 + \pi^nT^\prime,$$

$$\log(1 + \pi^nT)|_{T = \{\exp(\pi^nT^\prime) - 1\}/\pi^n} = \pi^nT^\prime.$$
Proof. If we let \( i = i_0 + i_1 p + \cdots \) \((0 \leq i_\nu \leq p - 1)\) be the \( p \)-adic expansion of \( i \), we have

\[
v_k(i!) = e \cdot (\left\lfloor \frac{i}{p} \right\rfloor + \left\lfloor \frac{i}{p^2} \right\rfloor + \cdots) = \frac{e - \sum \nu i_\nu}{p - 1}.
\]

The first and the second assertions thus follow immediately. The identities hold since they hold in \( k[[T]] \).

\[\square\]

1.1.2

Let \( n > e/(p - 1) \) and let \( A \) be an \( \mathcal{O}_k \)-algebra which is complete with respect to the filtration \((\pi_\nu A)_{\nu \geq 0}\) (it is the case, for example, when \( \pi \) is nilpotent in \( A \)). Note that \( \pi \cdot 1 \in \text{rad}(A) \). In this case, for any \( f \in A \), the elements

\[
\exp(\pi^n T)|_{T=f} \in 1 + \pi^n A \subset A^* \text{ and } \\
\log(1 + \pi^n T)|_{T=f} \in \pi^n A
\]

make sense by Lemma 1.1 (we can substitute \( T = f \) because the series \( \exp(\pi^n T) \) and \( \log(1 + \pi^n T) \) belong to \( \mathcal{O}_k[[T]] \)).

Lemma 1.2. We have well-defined homomorphisms of groups:

\[
\exp : \quad \pi^n A \to 1 + \pi^n A \\
\log : \quad 1 + \pi^n A \to \pi^n A
\]

which are inverse to each other. Moreover, these homomorphisms are independent of the choice of the uniformizer \( \pi \in \mathcal{O}_k \).

To see they are well-defined maps, we have to verify that, for any \( f \) and \( f' \in A \), \( \pi^n f = \pi^n f' \) implies \( \exp(\pi^n T)|_{T=f} = \exp(\pi^n T)|_{T=f'} \) and \( \log(1 + \pi^n T)|_{T=f} = \log(1 + \pi^n T)|_{T=f'} \). It suffices to show

\[
\left( \frac{1}{p^n} \pi^n T^i \right)|_{T=f} = \left( \frac{1}{p^n} \pi^n T^i \right)|_{T=f'}
\]

for \( i \geq 1 \). We have

\[
\left( \frac{1}{p^n} \pi^n T^i \right)|_{T=f} - \left( \frac{1}{p^n} \pi^n T^i \right)|_{T=f'} = \left( \frac{1}{p^n} \pi^n \right)(f - f')(f^{i-1} + \cdots + f'^{i-1}),
\]

so, by \( \pi^n(f - f') = 0 \), we are reduced to showing \( v_k(\pi^n(i-1)/i!) \geq 0 \). We have (letting \( i = i_0 + i_1 p + \cdots \) be the \( p \)-adic expansion)

\[
v_k(\pi^n(i-1)/i!) = n(i - 1) - \frac{i - \sum \nu i_\nu}{p - 1} e
\]

\[
\geq \frac{i - 1}{p - 1} e - \frac{i - \sum \nu i_\nu}{p - 1} e
\]

\[
\geq 0,
\]

\[5\]
thus the maps are well-defined. They are obviously homomorphisms and are inverse to each other because of the identities in Lemma 1.

Remark 1.3. Everything in this subsection is functorial in $A$.

1.2 La propriété d’échange.

By [EGA] III, 7.7.6, there is an $\mathcal{O}_k$-module $M$ of finite type which represents the covariant functor

$$ A \mapsto \Gamma(X_A, \mathcal{O}_{X_A}) $$

(i.e. $\Gamma(X_A, \mathcal{O}_{X_A}) = \text{Hom}_{\mathcal{O}_k-mod}(M, A)$) from the category of $\mathcal{O}_k$-algebras to the category of Abelian groups.

In fact, we know more precisely that [Il, 3.8] $Rf_*(\mathcal{O}_X)$ is isomorphic in the derived category of $\mathcal{O}_k$-modules to a complex in the form

$$ \begin{align*}
0 &\to \mathcal{O}_k^{\oplus a} \\
&\quad \quad \quad \to \mathcal{O}_k^{\oplus b} \\
&\quad \quad \quad \to \cdots \\
&\quad \quad \quad \to \mathcal{O}_k^{\oplus z} \\
&\quad \quad \quad \to 0,
\end{align*} $$

(1)

where the matrix $T$ is in the form

$$ T = \begin{pmatrix}
\pi^{m_1} \\
\vdots \\
\pi^{m_s} \\
0
\end{pmatrix}, \quad 0 \leq m_1 \leq \cdots \leq m_s < \infty
$$

and that $M$ is realized as the cokernel of the transposed map

$$ \mathcal{O}_k^{\oplus b} \xrightarrow{T^T} \mathcal{O}_k^{\oplus a}. $$

We define the integer $m$ to be the $m_s$ above. We can also define it intrinsically by

$$ \text{Ann}(M_{\text{tors}}) = \pi^m \mathcal{O}_k. $$

If we choose an isomorphism

$$ M \cong \mathcal{O}_k^{\oplus r} \bigoplus_{i=1}^s \mathcal{O}_k/\pi^{m_i} \mathcal{O}_k $$

where $1 \leq m_1 \leq \cdots \leq m_s = m$ and $r + s = a$, we have

$$ \Gamma(X_A, \mathcal{O}_{X_A}) = A^{\oplus r} \bigoplus_{i=1}^s A[\pi^{m_i}] $$

(2)

where $[\pi^{m_i}]$ means the submodule annihilated by $\pi^{m_i}$. 
1.3 Definition of the exponential map.

We denote by $\mathfrak{X}$ the formal completion of $\mathcal{X}$ along $Y$. First, we have an exact sequence if $n > e/(p-1)$:

\[
\begin{array}{cccccccc}
0 & \rightarrow & \pi^n\mathcal{O}_X & \rightarrow & \mathcal{O}_X & \rightarrow & \mathcal{O}_X / (\pi^n) & \rightarrow & 0 \\
\exp & \cong & \log & & & & & & \\
0 & \rightarrow & 1 + \pi^n\mathcal{O}_X & \rightarrow & \mathcal{O}_X^* & \rightarrow & \mathcal{O}_X^* / (\pi^n) & \rightarrow & 0
\end{array}
\]

where the vertical isomorphism is due to Lemma 1, and hence long exact sequences:

\[
\begin{array}{cccccccc}
H^0(\mathcal{O}_X) & \rightarrow & H^0(\mathcal{O}_X^n) & \rightarrow & H^1(\pi^n\mathcal{O}_X) & \rightarrow & H^1(\mathcal{O}_X) & \rightarrow & 0 \\
\exp \cong \log & & & & \cong & & & \\
H^0(\mathcal{O}_X) & \rightarrow & H^0(\mathcal{O}_X^n) & \rightarrow & H^1(1 + \pi^n\mathcal{O}_X) & \rightarrow & \text{Pic}(\mathcal{X}) & \rightarrow & 0
\end{array}
\]

Note that by [EGA III, (5.1.2), (5.1.4)], $H^i(\mathcal{X}, \mathcal{O}_X)$ and $\text{Pic}(\mathcal{X})$ are identified respectively with $H^i(\mathcal{X}, \mathcal{O}_X)$ and $\text{Pic}(\mathcal{X})$. We claim that this diagram induces an isomorphism $H^1(\mathcal{O}_X^n) \rightarrow \text{Pic}(\mathcal{X})^n$.

Proposition 1.4. Let $n > m + e/(p-1)$. Then, $\text{Im}(\partial_{\text{add.}})$ and $\text{Im}(\partial_{\text{mult.}})$ coincide under the identification $H^1(\pi^n\mathcal{O}_X) \cong H^1(1 + \pi^n\mathcal{O}_X)$.

First we prove a lemma.

Lemma 1.5. Suppose $n > m$. Then we have

\[
\begin{align*}
H^0(\mathcal{O}_X^n) &= \text{Im}[H^0(\mathcal{O}_X) \rightarrow H^0(\mathcal{O}_X^n)] + H^0(\pi^n\mathcal{O}_X^n), \\
H^0(\mathcal{O}_X^n) &= \text{Im}[H^0(\mathcal{O}_X) \rightarrow H^0(\mathcal{O}_X^n)] \cdot H^0(1 + \pi^n\mathcal{O}_X^n).
\end{align*}
\]

(If $n \leq m$, the assertion is trivial.)

Proof. By (2), we have a commutative diagram:

\[
\begin{array}{cccccccc}
H^0(\mathcal{O}_X^n) & \cong & (\mathcal{O}_k/(\pi^n)) \oplus \bigoplus_{j=1}^{s} \mathcal{O}_k/(\pi^{n-m}j) \\
\uparrow & & & & & & \uparrow \\
H^0(\mathcal{O}_X) & \cong & (\mathcal{O}_k)^r
\end{array}
\]

This shows that we have

\[
H^0(\mathcal{O}_X^n) = \text{Im}[H^0(\mathcal{O}_X) \rightarrow H^0(\mathcal{O}_X^n)] + H^0(\mathcal{O}_X^n)[\pi^m].
\]

Since $\mathcal{O}_X$ is $\pi$-torsion free, we have $H^0(\mathcal{O}_X^n)[\pi^m] = H^0(\pi^{n-m}\mathcal{O}_X^n)$, showing the first equality. The second equality follows from the first. \qed
Proof of Proposition. If $n > m + e/(p-1)$, the following diagram commutes:

$$
\begin{align*}
H^0(\pi^{n-m}\mathcal{O}_X) & \xrightarrow{\text{Lem. 1.5}} \ker[H^0(\mathcal{O}_X) \to H^0(\mathcal{O}_X) \to H^1(\pi^n\mathcal{O}_X)] \\
\exp & \downarrow \exp \\
H^0(1 + \pi^{n-m}\mathcal{O}_X) & \xrightarrow{\text{Lem. 1.5}} \ker[H^0(\mathcal{O}_X) \to H^0(\mathcal{O}_X) \to H^1(1 + \pi^n\mathcal{O}_X)]
\end{align*}
$$

Indeed, take any $a \in H^0(\pi^{n-m}\mathcal{O}_X)$ and let $a_i \in \Gamma(U_i, \pi^{n-m}\mathcal{O}_X)$ ($Y = \bigcup U_i$) be its local liftings. Then, since $a$ and $a_i$ belong to the domain of definition of $\exp$, the maps are calculated as follows:

$$
\begin{align*}
& a \\
\xrightarrow{\exp(a)} & \text{1-cocycle } a_j - a_i \\
\xrightarrow{\exp(a_i)} & \exp(a_i) - \exp(a_j) = \exp(a_j - a_i)
\end{align*}
$$

Thus Proposition follows immediately from the commutative diagram. \qed

Therefore we have proved

\textbf{Theorem 1.6.} If $f : \mathcal{X} \to \text{Spec}(\mathcal{O}_k)$ is proper and flat, and $n > m + e/(p-1)$, then we have a canonical isomorphism of filtered abelian groups

$$
H^1(\mathcal{O}, \mathcal{O}_X)^{(n)} \cong \text{Pic}(\mathcal{X})^{(n)}. \quad (4)
$$

In terms of formal geometry, the map $\exp$ is described by Čech 1-cocycles as:

$$
((U_i), (\pi^n f_{ij})) \mapsto ((U_i), (\exp(\pi^n f_{ij}))).
$$

\textbf{Remark 1.7.} The restriction maps

$$
H^1(\mathcal{X}, \mathcal{O}_X)^{(n)} \to H^1(\mathcal{X}, \mathcal{O}_X) \\
\text{Pic}(\mathcal{X})^{(n)} \to \text{Pic}(\mathcal{X})
$$

are injective if $n > m$. This is a consequence of \cite{Ill} and \cite{Ray} 6.4.4.

\subsection*{1.4 Functoriality}

\textbf{Proposition 1.8.} Theorem 1.6 holds functorially for any Noetherian flat $\mathcal{O}_k$-algebra $A$ which is complete with respect to the filtration $(\pi^n A)_\nu$. In other words, we have functorial isomorphisms of filtered abelian groups

$$
H^1(\mathcal{X}_A, \mathcal{O}_{X_A})^{(n)} \cong \text{Pic}(\mathcal{X}_A)^{(n)}
$$

where $H^1(\mathcal{O}_{X_A})^{(n)} := \text{Ker}[H^1(\mathcal{O}_{X_A}) \to H^1(\mathcal{O}_{X_A}/(\pi^n))]$ etc. as before.
Proof. In fact, the whole proof of Theorem 1.6 still works since \( \pi \) is a non zero divisor in \( A \) (and hence in \( \mathcal{O}_{X_0} \)) and \( A \) is Noetherian and \( \pi \)-adically complete. Since every diagram in the proof is functorial in \( A \), \( \exp \) is functorial in \( A \).

Next, suppose we have a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & X' \\
\downarrow & & \downarrow \\
\text{Spec}(\mathcal{O}_k) & \to & \text{Spec}(\mathcal{O}_{k'})
\end{array}
\]

with the same assumptions on \( f' : X' \to \text{Spec}(\mathcal{O}_{k'}) \) as those on \( f : X \to \text{Spec}(\mathcal{O}_k) \), and \( \mathcal{O}_{k'} \) being faithfully flat over \( \mathcal{O}_k \). Quantities associated to \( f' \) are indicated by primes (\(^\prime\)). Set \( \epsilon = e'/e \).

In this notation, we have (whose proof is obvious)

**Proposition 1.9.** If \( n > e/(p-1) + \max(m, m') \), we have a commutative diagram

\[
\begin{array}{ccc}
H^1(X, \mathcal{O}_X)^{(n)} & \overset{\exp_{\nu}}{\longrightarrow} & \text{Pic}(X)^{(n)} \\
\downarrow & & \downarrow \\
H^1(X', \mathcal{O}_{X'})_{(\epsilon n)} & \overset{\exp_{\nu}}{\longrightarrow} & \text{Pic}(X')_{(\epsilon n)}
\end{array}
\]

2 Application to the Albanese map.

Let \( X \) be an arbitrary proper \( k \)-scheme. Then by the compactification theorem of Nagata \cite{Con}, \( X \) admits a proper model (hence also a proper flat model by taking closure of \( X \)) over \( \mathcal{O}_k \). If \( X_1 \) and \( X_2 \) are two proper flat models of \( X \), there exists a third one \( X_3 \) which dominates both \( X_1 \) and \( X_2 \):

\[
\begin{array}{ccc}
X_3 & \to & X \\
\downarrow & \leftarrow & \downarrow \\
X_i & \leftarrow & i = 1, 2
\end{array}
\]

Namely, we can take \( X_3 \) to be the closure of the locally closed diagonal subscheme \( X \hookrightarrow X_1 \times_{\mathcal{O}_k} X_2 \). Then by Proposition 1.9 we have

**Corollary 2.1.** Two exponential maps defined on open \( \mathcal{O}_k \)-submodules

\[ H^1(X, \mathcal{O}_X) \to \text{Pic}(X) \]

with respect to given two proper flat models of \( X \) coincide on a smaller open submodule.

**Corollary 2.2.** The topology on \( \text{Pic}(X) \) defined by \( (\text{Pic}(X)^{(\nu)})_{\nu} \), where \( X \) is a proper flat model of \( X \), is independent of \( X \).
Theoreom 2.3. Let $X$ be a smooth projective geometrically connected curve over $k$ with genus $g$. Fix a finite field extension $k'/k$ such that there exists a closed point $x$ of $X$ with $k(x) \cong k'$ over $k$. Then there exists a family of zero-cycles on $X$ of the form

$$x^i_i - x^0_i \quad ((i, t) \in I \times \pi^N \mathcal{O}_k, \# I = g, \ N \gg 0)$$

with $k(x^i_i) \cong k'$ such that the map they define

$$\pi^N \mathcal{O}_k^{(I)} \to \text{Pic}^0(X)$$

$$(t_i)_{i \in I} \mapsto \sum_i x^i_i - x^0_i$$

is an open immersion of topological spaces.

Proof. In this proof we will call a closed point of $X$ whose residue field is isomorphic to $k'$ a $k'$-point (not a morphism $\text{Spec}(k') \to X$). Let $D = x_1 + \cdots + x_l$ be a divisor on $X$ consisting of distinct $k'$-points and satisfies $H^1(X, \mathcal{O}_X(D)) = 0$.

Such a divisor exists because there are infinitely many $k'$-points.

Let $\mathcal{X}$ be a regular proper flat model of $X$ over $\mathcal{O}_k$ which exists because of two-dimensional desingularization theorem of Abhyankar [Abh]. The divisor $D$ extends naturally to a Cartier divisor on $\mathcal{X}$ denoted by the same symbol: $\mathcal{D} = \sum_{i=1}^l \{x_i\}^-$. Since $\mathcal{O}_k$ is Henselian, the set $\{x_i\} \cap Y$ consists of just one point which we denote by $y_i$. By blowing up $\mathcal{X}$ at some $y_i$'s finitely many times, we may assume $y_i$ are distinct and the reduced schemes $\overline{\{x_i\}}$ are normal. This last assumption is possible by the following easy lemma.

Lemma 2.4. Let $A$ be a Noetherian one-dimensional Henselian local integral domain and $\tilde{A}$ its normalization. Let $X = \text{Spec}(A), \tilde{X} = \text{Spec}(\tilde{A})$ and $f : X' \to X$ be the blowing up of $X$ at the closed point. Then $f$ is a finite morphism and, for $f$ to be an isomorphism, it is necessary and sufficient that $A$ is normal.

Consequently, if we let $A_0 = A$ and $A_{i+1}$ be the blowing-up of $A_i$ at the closed point, the sequence $A_0 = A, A_1, \ldots$ terminates at some $A_n$ and this $A_n$ is equal to $\tilde{A}$.

We can write

$$(\overline{\{x_i\}}) = \text{Spec}\mathcal{O}_{X,y_i}/(t_i)$$

for some $t_i \in \mathcal{O}_{X,y_i}$. Since $\overline{\{x_i\}}$ are disjoint, we have

$$H^0(\mathcal{X}, \mathcal{O}_X(D)/\mathcal{O}_X) = \bigoplus_{i=1}^l \frac{1}{t_i} \mathcal{O}_{X,y_i}/\mathcal{O}_{X,y_i}.$$
and we choose \( g \) elements
\[
a_{ij}/t_i \in \frac{1}{t_i} \mathcal{O}_{X,y_i}, \quad (a_{ij} \in \mathcal{O}_{X,y_i}, \quad (i,j) \in I, \#I = g)
\]
which are linearly independent over \( \mathcal{O}_k \) in \( H^1(\mathcal{O}_X) \).

Since \( H^1(X, \mathcal{O}_X(D)) \) is a torsion \( \mathcal{O}_k \)-module of finite type, the cokernel of
\[
\bigoplus_{(i,j) \in I} \frac{a_{ij}}{t_i} \mathcal{O}_k \rightarrow H^1(\mathcal{O}_X)
\]
is a torsion \( \mathcal{O}_k \)-module of finite type. We denote by \( (\pi^c) \) its annihilator. Then for any \( i \) and \( b \in \mathcal{O}_{X,y_i} \), we have
\[
\partial\left(\pi^c b\right) = \sum_{(i',j') \in I} \partial\left(\frac{a_{i'j'}b_{i'j'}}{t_i'}\right)
\]
for some \( b_{i'j'} \in \mathcal{O}_k, \ (i',j') \in I \).

Now, for \( t \in \pi^2 \mathcal{O}_k, \ t_i - ta_{ij} \in \mathcal{O}_{X,y_i} \) is a part of a system of regular parameters and hence the set
\[
X \cap Spec\mathcal{O}_{X,y_i}/(t_i - ta_{ij})
\]
consists of a single point which we denote by \( x_{(i,j)}^t \). We have \( k(x_{(i,j)}^t) \cong k' \) if \( t \) is sufficiently close to 0 by the implicit function theorem on \( X_{k'} \) (cf. [Mat, Theorem 3]). This is the definition of the family.

Let us introduce a symbol: if \( Z \subset X \) is a closed subset contained in \( Spec\mathcal{O}_{X,y_i} \) and \( s \) is a section on \( (Spec\mathcal{O}_{X,y_i} - Z) \) of a sheaf of groups \( F \) on \( X \), we denote by \( h^1(y_i, s; F) \) the cohomology class in \( H^1(X, F) \) represented by the open covering \( X = (X - Z) \cup U \) and the cocycle \( s' \in F((X - Z) \cap U) \), where \( U \) is an open neighborhood of \( Spec\mathcal{O}_{X,y_i} \) and \( s' \) is an extension of \( s \).

We have an easy lemma.

**Lemma 2.5.**  (i) The connecting homomorphism
\[
\bigoplus_i \frac{1}{t_i} \mathcal{O}_{X,y_i}/\mathcal{O}_{X,y_i} \rightarrow H^1(X, \mathcal{O}_X)
\]

sends \( \sum a_i/t_i \ (a_i \in \mathcal{O}_{X,y_i}) \) to \( h^1(y_i, \frac{a_i}{t_i}; \mathcal{O}_X) \).

(ii) The map
\[
Div(X) \rightarrow Pic(X) = H^1(X, \mathcal{O}_X^*)
\]
sends the Cartier divisor \( \{x_{(i,j)}^t\} - \{x_{(i,j)}^{t_0}\} \) to \( -h^1(y_i, 1 - \frac{a_i}{t_i}; \mathcal{O}_X^*) \).

By this lemma and the definition of the exponential map we see that for a sufficiently large \( N \) the composite map
\[
\bigoplus \pi^N \mathcal{O}_k \xrightarrow{\text{family}} Pic(X)^{(N)} \xrightarrow{\log} H^1(X, \mathcal{O}_X)^{(N)} \subset \bigoplus I \mathcal{O}_k \cdot h^1(y_i, \frac{a_{ij}}{t_i}; \mathcal{O}_X)
\]
is written in power series and its Jacobian matrix at the origin is invertible by
the formula (6).

This completes the proof of the theorem. □

Remark 2.6. In this proof we picked a regular proper flat model \( X \) and a family
\((x^i_{(i,j)})\) such that the reduced schemes \( \{x^i_{(i,j)}\} \subset X \) are all regular.

This theorem is generalized to the higher dimensional case. Before stating
that, let us suppose \( X \) is a smooth projective geometrically integral curve of
genus \( g \). Then, we have a canonical injection of Abelian groups

\[ \text{Pic}^0(X) \hookrightarrow J_X(k) \]

where \( J_X \) denotes the Jacobian variety of \( X \). The group \( J_X(k) \) inherits the
topology of the topological field \( k \).

**Proposition 2.7.** The above injection is an open immersion of analytic topological
groups.

**Proof.** We use the family \( \{x^i_1 - x^0_i\} \) in Theorem 2.3. We proved that the map
given by this family:

\[ \pi^N \mathcal{O}_k^{\oplus g} \to \text{Pic}^0(X) \quad N \gg 0 \]

is an open immersion of analytical topological spaces. On the other hand, the composite

\[ \pi^N \mathcal{O}_k^{\oplus g} \to \text{Pic}^0(X) \subset J_X(k) \]

is also an open immersion. Indeed, it is true when \( X \) has a rational point (and
\( x^i \) are taken to be rational) as is proved by using the symmetric product \( X^{(g)} \)
of \( X \) (cf. [Mat, Theorem 5]). When \( X \) does not have rational points, choose a
finite Galois extension \( k'/k \) such that \( X(k') \neq \emptyset \). Our assertion is deduced from
the assertion for \( X_{k'} \) and the fact that

\[ (H^1(X \otimes \mathcal{O}_{k'}, \mathcal{O})^{(n)})^{\text{Gal}(k'/k)} = H^1(X, \mathcal{O})^{(n)}, \quad J_X(k')^{\text{Gal}(k'/k)} = J_X(k) \]

as topological spaces, where \( X \) is any proper flat model of \( X \). This completes
the proof. □

Now we recall what the Albanese variety is. Let \( K \) be an arbitrary field. Let
\( X \) be a smooth geometrically integral \( K \)-scheme of finite type. The \( K \)-Albanese
variety \( \text{Alb}_K(X) \) is the universal Abelian variety over \( K \) among morphisms

\[ X \times X \xrightarrow{f} A(\text{Abelian variety}) \]

which vanish on the diagonal \( (\text{Gab}, \text{Section 2}) \). Note that if \( X \) has a rational
point, such an \( f \) is written as

\[ f(x, y) = f'(x) - f'(y) \]
where $f'$ is a morphism from $X$ to $A$ (cf. [Mil, Theorem 3.4]).

If $K$ is a perfect field, we have a canonical homomorphism of abelian groups

$$alb_X : CH_0(X)_{deg=0} \rightarrow Alb_K(X)(K),$$

which is defined by using $f'$ when one considers a cycle consisting of rational

points, and by Galois descent in general. The fact that it respects rational

equivalence relies on the theory of Jacobian varieties (but we do not need this

last fact).

By Proposition 2.7 we already know that the one-dimensional case of the

following theorem is true.

**Theorem 2.8.** Let $X$ be a projective smooth geometrically connected scheme

over $k$, $g = \dim H^1(X, \mathcal{O}_X)$, $K/k$ a field extension as above. Then, there is a

family of zero-cycles of the above form such that the following map give

$$\pi^N \mathcal{O}^{(I)}(k) \rightarrow CH_0(X)_{deg=0} \rightarrow Alb_k(X)(k)$$

is an open immersion of topological spaces.

By a Bertini theorem [Gab, 1.7, 2.4], there exists a hypersurface section $H$

of $X$ such that $H$ is smooth and geometrically connected, $H$ contains a $k'$-point

and $Alb_k(H) \rightarrow Alb_k(X)$ is smooth. By induction on dimension, there exists a

family which has desired properties for $H$

$$x_i^t - x_i^0 \ (i, t) \in I_H \times \pi^N \mathcal{O}_k$$

where $\# I_H = \dim H^1(H, \mathcal{O}_H)$. By the smoothness, we can choose a subset

$I \subset I_H$ with $\# I = g$ such that the family restricted to $I \times \pi^N \mathcal{O}_k$ satisfies the

desired property for $X$. This completes the proof.

**Remark 2.9.** With an appropriate Bertini theorem over discrete valuation rings,

we will be able to show that Remark 2.6 is also true in the higher dimensional

case (possibly with some “good” reduction condition). So far, by using [JS, Theorem 1.2] instead of [Gab, 1.7, 2.4], we can show the next corollary where

we can control the reduction of $x_i^t$’s but lose strict control of their residue fields.

**Corollary 2.10.** In Corollary 2.8 assume further that $X$ has strictly quasi-

semistable reduction. Then there exist a model $\mathcal{X}$ of $X$ and a family of zero-
cycles

$$x_i^t - x_i^0 \ ((t, i) \in \pi^N \mathcal{O}_k \times I, \ N >> 0, \ \# I = \dim H^1(X, \mathcal{O}_X))$$

such that the residue fields $k(x_i^t)$ are all isomorphic over $k$, the reduced closed

subschemes $\{x_i^t\} \subset \mathcal{X}$ are all regular and the map

$$\pi^N \mathcal{O}^{(I)}_k \rightarrow Alb_k(X)(k)$$

is an open immersion of topological spaces.

We take a strictly quasi-semistable model $\mathcal{X}$ of $X$ and apply the Bertini

theorem [JS, Theorem 1.2] (to the effect that we can choose a hypersurface

section of $\mathcal{X}$ which is strictly quasi-semistable, in particular regular) to reduce

the problem to the one-dimensional case. Then we repeat the analysis done in the

proof of Theorem 2.8.
3 Comparison with Formal Exponential Map.

Let \( f : X \to \text{Spec} \mathcal{O}_k \) be proper and flat. We are going to show that, when Picard group functor has good properties, the formal exponential map coincides with our exponential map.

We denote by \( \text{Pic}_{X/\mathcal{O}_k} \) the following functor
\[
(Sch/\mathcal{O}_k) \to \text{(Abelian groups)}
\]
\[
T \mapsto \text{Pic}(X_T)/f^*_T(\text{Pic}(T))
\]
and we assume, throughout this section,
\[(*) \text{ Let } (\text{Art}/\mathcal{O}_k) \text{ be the category of local Artinian } \mathcal{O}_k\text{-algebras with the same residue field as that of } \mathcal{O}_k, \text{ and } \text{Pic}_{X/\mathcal{O}_k}\hat{\ } \text{ denote the functor}
\]
\[
(\text{Art}/\mathcal{O}_k) \to \text{(Abelian groups)}
\]
\[
A \mapsto \text{Ker}[ \text{Pic}(X_A) \to \text{Pic}(Y) ].
\]

Then the functor \( \text{Pic}_{X/\mathcal{O}_k}\hat{\ } \) is pro-represented, as a set-valued functor, by an \( \mathcal{O}_k\)-algebra which is isomorphic to \( \mathcal{O}_k[[T_1, \ldots, T_d]] \) \((d \geq 0)\).

This condition is satisfied for example if: Note that under \((*)\), \( H^1(X, \mathcal{O}_X) \) is necessarily a free \( \mathcal{O}_k\)-module (cf. formula (7) below).

3.1 Review on formal exponential map (cf. [Laz], [Sch]).

3.1.1 Formal groups.

Recall that a (smooth \( d \)-dimensional commutative) formal group over \( \mathcal{O}_k \) is a covariant functor
\[
(\text{Art}/\mathcal{O}_k) \to \text{(Abelian groups)}
\]
whose underlying set-valued functor is isomorphic to the functor
\[
(A, m) \mapsto m^{\otimes d},
\]
which is pro-represented by \( \mathcal{O}_k[[T_1, \ldots, T_d]] \) and denoted by \( \text{Spf} \mathcal{O}_k[[T_1, \ldots, T_d]] \).

Group structure on \( \text{Spf} \mathcal{O}_k[[T_1, \ldots, T_d]] \) is equivalent to an \( \mathcal{O}_k\)-homomorphism
\[
\mathcal{O}_k[[T_1, \ldots, T_d]] \to \mathcal{O}_k[[T_1, \ldots, T_d]] \otimes_{\mathcal{O}_k} \mathcal{O}_k[[T_1, \ldots, T_d]].
\]
which is determined by \( d \) formal power series in \( 2d \) variables
\[
T_j \otimes 1, 1 \otimes T_j \quad 1 \leq j \leq d
\]

namely the images of \( T_i \). In the same manner, one finds that homomorphisms of formal groups (i.e. morphisms of group functors)
\[
\text{Spf} \mathcal{O}_k[[T]] \to \text{Spf} \mathcal{O}_k[[S]]
\]
are characterized by power series in variables \( \{T\} \) which are compatible with their group laws.
3.1.2 Tangent space.

There is a functor $T_0$, called “tangent space”, defined by

$$T_0 : \text{(Formal groups)} \rightarrow \text{(Free } \mathcal{O}_k\text{-modules)}$$

$$F \mapsto \text{Ker}[F(\mathcal{O}_k[\epsilon]) \rightarrow F(\mathcal{O}_k)]$$

where $\epsilon$ is an element subject to the only relation $\epsilon^2 = 0$. $T_0F$ has a module structure because $\mathcal{O}_k[\epsilon]$ is a “$\mathcal{O}_k$-module object” in the category of $\mathcal{O}_k$-algebras with a retraction to $\mathcal{O}_k$. Namely, the addition is

$$\mathcal{O}_k[\epsilon] \times_{\mathcal{O}_k} \mathcal{O}_k[\epsilon] \rightarrow \mathcal{O}_k[\epsilon]$$

$$a + b(\epsilon, 0) + c(0, \epsilon) \mapsto a + (b + c)\epsilon$$

while the multiplication by $a \in \mathcal{O}_k$ is

$$\mathcal{O}_k[\epsilon] \rightarrow \mathcal{O}_k[\epsilon]$$

$$\epsilon \mapsto a\epsilon.$$  

3.1.3 Ideal-valued points.

When $F$ is a formal group, $(A, m) \in (\text{Art}/\mathcal{O}_k)$ (or a pro-object of (Art/\mathcal{O}_k)) and $a \subset m$ is an ideal of $A$, we write $F(a)$ for $\text{ker}(F(A) \rightarrow F(A/a))$. It is an abelian group which is set-theoretically isomorphic to $a^{\otimes d}$ and its group law is given by the family of formal power series defining $F$.

3.1.4 Examples.

Assumption (*) states precisely that $\text{Pic}_{X/\mathcal{O}_k}$ is a formal group.

Another type of examples is formal groups associated to free $\mathcal{O}_k$-modules of finite type. Let $L$ be such a module. We define a formal group $L^+$ of dimension $d$ by

$$(A, m) \mapsto m \otimes L.$$

3.2 Comparison theorem.

We know “formal exponential isomorphism” (over $k$)

$$H^1(X, \mathcal{O}_X) \xrightarrow{\exp_{\text{formal}}} \text{Pic}_{X/\mathcal{O}_k}^\wedge$$

whose coefficients in degree 1 is determined by the $\mathcal{O}_k$-isomorphism

$$H^1(\mathcal{O}_X) \xrightarrow{\cong} T_0\text{Pic}_{X/\mathcal{O}_k}$$

induced by the split exact sequence

$$0 \rightarrow \epsilon\mathcal{O}_X[\epsilon] \xrightarrow{+1} \mathcal{O}_X^*[\epsilon] \rightarrow \mathcal{O}_X^* \rightarrow 1.$$
Note that for the definition of the formal exponential map, \( char(k) = 0 \) hypothesis is essential ("\( \mathbb{Q} \)-Theorem").

Since the coefficients of \( \exp_{\text{formal}} \) do not necessarily lie in \( \mathcal{O}_k \), \( \exp_{\text{formal}} \) is not necessarily a homomorphism of formal groups over \( \mathcal{O}_k \). Still, by a variant of Inverse Function Theorem, for some sufficiently large \( n \), \( \exp_{\text{formal}} \) maps \( H^1(\mathcal{X}, \mathcal{O}_X)^+((\pi^n)) \) into \( \text{Pic}_{\mathcal{X}/\mathcal{O}_k}(\pi^n) \) and it is bijective.

**Theorem 3.1.** For a sufficiently large \( n \), we have a commutative diagram

\[
\begin{array}{ccc}
H^1(\mathcal{X}, \mathcal{O}_X)^+((\pi^n)) & \xrightarrow{\exp_{\text{formal}}} & \text{Pic}_{\mathcal{X}/\mathcal{O}_k}(\pi^n) \\
\| & & \| \\
H^1(\mathcal{X}, \mathcal{O}_X)^{(n)} & \xrightarrow{\exp} & \text{Pic}(\mathcal{X})^{(n)}
\end{array}
\]

**Proof.** Fix a basis of \( H^1(\mathcal{X}, \mathcal{O}_X) \) and a coordinate of \( \text{Pic}_{\mathcal{X}/\mathcal{O}_k} \), so that the left hand sides and the right hand sides are respectively identified with the set \((\pi^n)^{\oplus d}\).

We use the following lemma.

**Lemma 3.2.** Let \( n > m + e/(p - 1) \). Then:

(i) The composite map

\[
H^1(\mathcal{X}, \mathcal{O}_X)^{(n)} \xrightarrow{\exp_{\text{formal}}} \text{Pic}(\mathcal{X})^{(n)}
\]

is defined by a family of formal power series over \( k \) (we allow \( n \) to increase by 1 if needed).

(ii) For any \( \nu \geq 1 \), if we choose sufficiently large \( \mu \geq \nu \), the map

\[
H^1(\mathcal{O}_X)^{(\mu)} / H^1(\mathcal{O}_X)^{(\mu + \nu)} \xrightarrow{\exp} \text{Pic}(\mathcal{X})^{(\mu)} / \text{Pic}(\mathcal{X})^{(\mu + \nu)}
\]

coincides with multiplication by the Jacobian matrix of \( \exp_{\text{formal}} \).

We observe that the lemma implies the theorem. First, it is easily seen that the family of power series in (i) is actually a homomorphism (over \( k \)) of the formal group laws \( H^1(\mathcal{X}, \mathcal{O}_X)^+ \) and \( \text{Pic}_{\mathcal{X}/\mathcal{O}_k} \) on \((\pi^n)^{\oplus d}\). Second, by \( \mathbb{Q} \)-Theorem (two homomorphisms of formal group laws over a torsion-free ring coincide if their first coefficients coincide), we are reduced to showing that the tangent maps of \( \exp_{\text{formal}} \) and \( \exp \) coincide. This is ensured by (ii).
Proof of the lemma. For (i), apply Proposition [LS] to the ring $O_k[[T_1, \ldots, T_d]]$, where each $T_i$ is an indeterminate. The images of $T_i$ under the map

\[(\pi^n[[T_1, \ldots, T_d]])^\oplus d \rightarrow H^1(O_X[[\{T\}]])(\pi)^n \exp \rightarrow \text{Pic}(X[[\{T\}]])(\pi)^n\]

give the desired family of power series over $k$, because every element $(a_1, \ldots, a_d)$ of $(\pi^n+1)^\oplus d$ lies in the image of $(\pi^n[[T_1, \ldots, T_d]])^\oplus d$ under the map induced by a local homomorphism $O_k[[\{T\}]] \rightarrow O_k$.

For (ii), we use the diagram

\[
\begin{array}{ccccccc}
0 & \rightarrow & \epsilon \mathcal{O}_X[\epsilon] & \rightarrow & \mathcal{O}_X^*[\epsilon] & \rightarrow & \mathcal{O}_X^* & \rightarrow & 0 \\
\epsilon \rightarrow \pi^\mu & & & & & & & & \\
0 & \rightarrow & \pi^\mu \mathcal{O}_X/(\pi^\mu+\nu) & \exp(-) & (\mathcal{O}_X/(\pi^\mu+\nu))^* & \rightarrow & (\mathcal{O}_X/(\pi^\mu))^* & \rightarrow & 0
\end{array}
\]

where $\epsilon$ is subject to the only relation $\epsilon^2 = 0$, and which is commutative if $\mu$ is large enough to satisfy $\exp(\pi^\mu T) \equiv 1 + \pi^\mu T \pmod{\pi^\mu+\nu}$. This gives rise to the commutative diagram (let further $\mu > m + e/(p-1))$

\[
\begin{array}{ccccccc}
H^1(X, \mathcal{O}_X) & \exp_{\text{formal}} & T_0 \text{Pic}_X/O_k^\wedge \\
\times \pi^\mu & & & & & & \uparrow \\
H^1(\mathcal{O}_X/(\pi^\mu+\nu)) & \rightarrow & H^1(\mathcal{O}_X/(\pi^\mu+\nu))^{(\mu)} & \exp & \text{Pic}(X/(\pi^\mu+\nu))^{(\mu)} & \rightarrow & \text{Pic}(X/(\pi^\mu))^{(\mu)} \\
\rightarrow & & & & & & \uparrow (**) \\
H^1(S/O_k)/(\pi^\mu+\nu) & \rightarrow & H^1(\mathcal{O}_X/(\pi^\mu+\nu))^{(\mu)} & \rightarrow & \text{Pic}(X/(\pi^\mu+\nu))^{(\mu)} & \rightarrow & \text{Pic}(X/(\pi^\mu))^{(\mu)}
\end{array}
\]

where “$\exp_{\text{formal}}$” coincides with multiplication by $Jac(\exp_{\text{formal}})$ when the source and the target are identified with $O_k^{\oplus d}$, and the identity (**) is due to the assumption (*) at the beginning of this section. This shows (ii). Thus the proof of the theorem has been completed.

References

[Abh] S. Shreeram Abhyankar. Resolution of singularities of arithmetical surfaces. 1965, Arithmetical Algebraic Geometry (Proc. Conf. Purdue Univ., 1963) pp.111–152 Harper & Row, New York.

[Con] B. Conrad. Deligne’s Notes on Nagata Compactifications. J. Ramanujan Math. Soc. 22 (2007), 205–257
[EGA] A. Grothendieck, J. Dieudonné. Éléments de Géométrie Algébrique III Étude cohomologique des faisceaux cohérents. Publications Mathématiques, no.11, 1961 and no.17, 1963.

[Ful] W. Fulton. Intersection Theory, 2nd edition. Springer-Verlag, Berlin, 1998.

[Il] L. Illusie. Grothendieck’s existence theorem in formal geometry. With a letter (in French) of Jean-Pierre Serre. In: Fundamental algebraic geometry, 179–223, (Math. Surveys Monogr., 123) Amer. Math. Soc. Providence, RI, 2005.

[Gab] O. Gabber. On Space-filling Curves and Albanese Varieties. Geometric and Functional Analysis, vol.11 (2001) pp.1192–1200

[JS] U. Jannsen and S. Saito. Bertini Theorems and Lefschetz Pencils over Discrete Valuation Rings, with Applications to Higher Class Field Theory. J. Algebraic Geometry 21 (2012) 683-705.

[Laz] M. Lazard. Commutative Formal Groups. Lecture Notes in Mathematics 443, Springer-Verlag, 1975.

[Mat] A. Mattuck. Abelian varieties over $p$-adic ground fields. Ann. of Math. 62(1955), 92–119.

[Mil] J. S. Milne. Abelian Varieties. In: Cornell, G., Silverman, J.H. (eds) Arithmetic Geometry, pp. 103–150 New York, Springer, 1986.

[Ray] M. Raynaud. Spécialisation du foncteur de Picard. Inst. Hautes Etudes Sci. Publ. Math. No. 38, 1970, 27–76.

[SS] S. Saito, R. Sujatha. A finiteness theorem for cohomology of surfaces over $p$-adic fields and an application to Witt groups. $K$-theory and algebraic geometry: connections with quadratic forms and division algebras (Santa Barbara, CA, 1992), 403–415, Proc. Sympos. Pure Math., 58, Part 2, Amer. Math. Soc., Providence, RI, 1995.

[Sch] M. Schlessinger. Functors of Artin Rings. Trans. Amer. Math. Soc. 130, 1968, 208–222.