Variational procedure for higher-derivative mechanical models in a fractional integral

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received 31 July 2019; accepted in final form 30 March 2020
published online 13 April 2020

PACS 02.30.Hq – Ordinary differential equations
PACS 02.30.Xx – Calculus of variations
PACS 03.50.–z – Classical field theories

Abstract – We present both the Lagrangian and Hamiltonian procedures to treat higher-derivative equations of motion for mechanical models by adopting the Riemann-Liouville fractional integral to formulate their respective actions. Our focus is the possible interplay between fractionality and a dynamics based on higher derivatives. We point out and discuss the efficacy and difficulties of this approach. We also contemplate physical and geometrical interpretations and present details of the inspection we carry out by considering an explicit situation, that of a higher-derivative harmonic oscillator. Additionally, we have also used a recent proposal of a variational approach with local deformed derivatives. In this context, we have derived a complete set of linear and non-linear equations for a Pais-Uhlenbeck–type oscillator.

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Introduction. – The deeper comprehension of non-linearities has been inspired by the discovery and insight of a new phenomenon known as dynamical chaos. It is easy to understand the reasons for the flourishing of the area, since any typical system with more than one degree of freedom may exhibit chaotic motion for some initial conditions. We do not know yet what the measure of chaotic trajectories is, but it seems to be rather non-trivial. This renders the study of chaos of primary importance for building up models of dynamical processes in Nature [1]. At a very fundamental level, most phenomena of Nature are strongly dominated by non-linear effects.

Fractional calculus (FC) is one of the possible generalizations of classical calculus. It has been used in several fields of science. FC provides an interesting, and sometimes unexpected, redefinition of the mathematical tools, and it seems very useful to approach anomalous and fractional systems; in particular, we can cite the continuous time random walk scheme as an illustrative physical example, where, within the fractional approach, it is possible to include external “fields” in a straightforward manner. Also, the consideration of transport in the phase space spanned by both position and velocity coordinates is possible within the same formalism. Moreover, the calculation of boundary value problems is analogous to the procedure adopted to study the corresponding standard equations [2–5]. Other important applications may be found by investigating response functions, where a great deal of problems has been reported on the phenomenon of non-exponential and power-law relaxation, which is typically observed in complex systems such as dielectric and ferroelectric materials, polymers and others. The FC approach to describe dynamical processes in disordered or complex systems, such as relaxation or dielectric behavior in polymers or photo bleaching recovery in biologic membranes, has proved to be an extraordinarily successful tool.
authors have proposed different fractional relaxation models to study filled polymer networks and investigate the dependence of the decisive occurring parameters on the filler content\cite{6,7}. The investigation of exactly solvable fractional models of linear viscoelastic behavior is another successful field of application. In recent years, both phenomenological and molecular-based theories for the study of polymers and other viscoelastic materials came up with integral or differential equations of fractional order. Some current models of viscoelasticity based on FC are usually derived from the Maxwell model by replacing the first-order derivative \((d/dt)\) by its corresponding fractional version \((d^\alpha/dt^\alpha)\) \cite{8}, where \(\alpha\) is non-integer.

Recently, a different approach has been investigated, in which a variational principle with an embedded deformative derivative DD \cite{9}. This approach gives a good description of dissipative systems and is convenient to yield linear and non-linear equations in a very consistent way, for an improper Riemann integral is part of the DD algebra\footnote{More details shall be provided in a paper in progress.}.

The study of systems whose Lagrangians exhibit higher derivatives was proposed long ago \cite{10}; models with higher derivatives, for instance, where \(L(q^{(3)},\dot{q},\ddot{q},q)\) or cases of even higher derivatives may appear in many different areas of physics. At a first glance, their interest might appear to be merely academic, since most of the Lagrangian models in physics are based only on kinetic terms built up with first derivatives (velocities), yielding second-order differential equations of motion. On the other hand, theories and models described by high-order derivative Lagrangians may be relevant in diverse contexts; for instance \cite{10,11}. Such formulations exhibit interesting properties such as the improvement of the convergence of momentum-space loop integrals associated to Feynman graphs in the case of quantum field-theoretic models \cite{12}. Higher-derivatives models, on the other hand, impose undesirable consequences on the stability of their dynamics even at the classical level, once their energy is not in general bounded from below. At the quantum level, this immediately leads to the non-existence of a ground state, appearance of tachyonic and fractals, heat transfer, biology, electronics, signal processing, robotics, system identification, genetic algorithms, percolation, modeling and identification, telecommunications, chemistry, irreversibility, control systems as well as engineering and economy including finance \cite{14,15}. The generalization of the concept of derivative with non-integer values goes back to the beginning of the theory of differential calculus. Nevertheless, the development of the theory of FC is due to contributions of many mathematicians such as Euler, Liouville, Riemann, and Letnikov \cite{16–19}.

Since 1931, when Bauer \cite{20} showed that we cannot use the variational principle to obtain a single linear dissipative equation of motion with constant coefficients, a new horizon of possibilities was glimpsed. Nowadays, it has been observed that both in physics and mathematics the methodology necessary to understand new questions has changed towards more compact notations and powerful non-linear and qualitative methods. So, after Bauer’s corollary, Bateman \cite{21} proposed a procedure where multiple equations were obtained following from a Lagrangian. Riewe \cite{22} observed that using FC it was possible to obtain a formalism which could be used to describe both conservative and nonconservative systems. Namely, using this approach one can obtain the Lagrangian and Hamiltonian equations of motion also for non-conservative systems. In \cite{23} Agrawal studied a fractional variational problem. A fractal concept applied to quantum physics has been investigated and reported in \cite{24}.

This subject has recently been re-assessed in \cite{25} and the solution of a fractional Dirac equation (order \(\alpha = 2/3\)) was introduced in \cite{26}.

**Modified equations.** Let us start first defining an integral real where \(a \in \mathbb{R}\) and \(f\) is an integrable real function for \(t \geq a\) where we will consider a \(n\)-fold integral \cite{17}. If \(n \in \mathbb{N}\) and \(I^n_a f(t) = \int^t_a dt_1 \int^{t_1} a dt_2 \cdots \int^{t_{n-1}} a f(s)ds\), we can show using induction that \(I^n_a f(t) = \frac{1}{(n-1)!} \int^t_a (t-s)^{n-1}f(s)ds\). Where the Gamma function \(\Gamma(n) = (n-1)!\), is defined as \(\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt, x > 0\), then we can define the so-called Riemann-Liouville integral (RLI) of arbitrary order \(\alpha > 0\), and changing the variables \(s \rightarrow \tau\) and \(n \rightarrow \alpha\), where now \(\alpha \in \mathbb{R}\), \(I^n_a f(t) = \frac{1}{\Gamma(\alpha)} \int^t_a (t-\tau)^{\alpha-1}d\tau\).

Actually, fractional integral and fractional differentiation are generalizations of usual integer-order integral and differential calculus. However, there is some cloud on its physical and geometrical interpretations. Some papers
try to connect the fractional integral and derivatives with the fractal world, however it was shown that it is unequivocal [27,28].

It is well known that several definitions of fractional derivative and integral exist, for instance, Grunwald-Letnikov, Caputo, Weyl, Feller, Erdelyi-Kober and Riesz fractional derivatives as well as fractional Liouville operators, which have been popularized when fractional integration is performed in dynamical systems under study [13]. Following this idea, let us consider a smooth Lagrangian function. For any smooth path \( q: [a, b] \to M \) satisfying boundary conditions \( q(1) = q_1 \) and \( q(2) = q_2 \), we define the fractional action integral by \( S^\alpha[q](t) = \frac{1}{\Gamma(\alpha)} \int_a^t L(q(\tau), \dot{q}(\tau), \tau)(t - \tau)^{\alpha - 1}d\tau \), where \( \alpha \in (0, 1] \) and \( \dot{q} = dq/d\tau \) is the derivative with respect to the intrinsic time \( \tau \in (a,t) \) and \( t \in [a,b] \). Notice that the Lagrangian \( L \) in action \( S^\alpha[q](t) \) is weighted by \( \frac{1}{\Gamma(\alpha)}(t - \tau)^{\alpha - 1} \), and we can write the action in the following form: \( S^\alpha[q](t) = \int_a^t L[q(\tau), \dot{q}(\tau), \tau]d\tau \), where \( g_i(\tau) = \frac{1}{\Gamma(\alpha + 1)}(t^n - (t - \tau)^n) \). The function \( g_i(\tau) \) generates a scale property, and when we consider the plot of \( g_i \) for \( 0 \leq \tau \leq t \) in the \((g, \tau)\) plane we observe a standard like shadows on the walls [29] where the so-called memory effect is present. Typically, the complexity of systems is connected to the memory of long-range interactions by non-locality (spatial) effects. To understand the memory effect, one can argue that the long-term memory tells us that the evaluation of the state at a certain moment, \( t \), in a system, depends on values in moments from the past. In this way, a change in the output function value of a system depends not only on the input function values, but also on the whole history of evolution of the system. Here, the memory effect comes from the convolution of weight function, eq. (6), and the Lagrangian. The weight function \( (6) \) can be considered as a time-dependent probability that gives different probabilities to the different times. It also possible that a two-stage–like process [30] could be considered, e.g., if we consider a “short-memory” principle [18], in such a way that only the recent past of the system is taken into account. That is, if a whole time interval for the evolution of a system is \([a, l]\), then we can consider as a good approximation the interval \([t - l, t]\), where \( l \) is the memory length. This consideration, if applicable, can reduce the amount of numerical computational work [18]. Here, we will not go into details and analysis of this point for the oscillator under study, which will be left for further studies. For details about connection of memory effects, time scale and kernels, the reader can consult refs. [31,32].

The time weighting acts as a time-dependent factor [26], and obviously when \( \alpha \to 1 \) we reobtain the usual functional \( S[q] = \int_a^t L(q(\tau), \dot{q}(\tau), \tau)d\tau \). The explicit time dependence introduced by the weight in the action that governs the dynamics in the regime of \( 0 < \alpha < 1 \) has the relevant consequence of violating the conservation of the energy of the system, which becomes clear if we invoke the Noether’s theorem. This means that, in the fractional scenario, the system behaves as a non-isolated system; fractionality is parametrising some sort of interaction of the system with an external source, whose microscopic details we do not know. There is therefore an exchange of energy between the mechanical system in consideration and some external agent represented by the fractionality. We shall be coming back to this point in the explicit example we are going to analyze in the fourth section. Constructing the functional variation of action \( \delta S^\alpha = 0 \), we obtain after some calculus that for, \( \frac{\partial S^\alpha}{\partial \dot{q}_i} - \frac{\partial}{\partial \tau} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial}{\partial \tau} \left( \frac{\partial L}{\partial q_i} \right) = 0, i = 1 \cdots n \). It can be noted now an intriguing fractional correction term. What does \( (1 - \alpha)/(t - \tau) \delta \) mean? Actually, the meaning is a time-dependent damping coefficient into dynamical equations where we can understand \( R = \frac{1}{(t - \tau)}L \) like a fractional Rayleigh dissipation function [33]. After these few words on the fractional formalism, we think that it is important to justify our choice of using the RL fractional technique instead of another very popular fractional framework: the Caputo time derivative CPT. The latter could be another way to attack the construction of fractional DB. Nevertheless, we understood that CPT is more appropriated for applications in several engineering problems due to the fact that it has a better relation with the Laplace transform and because the differentiation appears inside the integral. So, it smoothes the effects of noise and numerical differentiation. Another point is that Caputo’s definition is quite useful when we are treating the initial problem value, but this is not our goal. For our central purpose, the RL approach is more convenient than CPT. Having clarified our choice, we can write, from RLI, the following integral: \( \delta S^\alpha = \frac{1}{\Gamma(\alpha)} \int_a^t \left[ \delta \left( \frac{\partial L}{\partial \dot{q}_i} \right)(t - \tau)^{\alpha - 1} \right] d\tau = 0 \), so that \( \delta S^\alpha = \frac{1}{\Gamma(\alpha)} \int_a^t \left[ (\delta L)(t - \tau)^{\alpha - 1} + L(\delta(t - \tau)^{\alpha - 1}) \right] d\tau = 0 \), where \( L = \dot{p} \cdot \dot{q} - H(p, q, \tau) \) and the rest of the calculation is as standard as the variational calculus (VC) discussed in the text books. The modification is due to the FC formalism. However, we can readily deal with this additonal factor. Hence, after performing the variations on the Lagrangian (as in eq. (3)) and the damping factor, and isolating the coefficients for \( \delta \dot{q} \) and \( \delta \dot{p} \) that will be equal to zero, we can write a new set of perturbed equations, \( \dot{q}_i = \frac{\partial H}{\partial p_i}, \dot{p}_i = -\frac{\partial H}{\partial q_i} + p_i \left( \frac{\partial \Gamma}{\partial t} \right), \) which can be understood as the (fractional) Hamilton-Jacobi equations when this new action functional is considered. It is clear that when \( \alpha \to 1 \) our results will turn back to the usual case, as shown above. We showed in [34] that the quotient \( \frac{1}{(t - \tau)}p \) will be important in our fractional Dirac bracket formulation. The order \( \alpha \) will be directly related to the fractional approach. As commented before the presence of a fractional factor \( (1 - \alpha)/(t - \tau) \), is responsible for the generation of a time-dependent damping into the dynamics of the system, which is very useful to study models with smooth turbulence. Furthermore it is possible to establish a relationship between the fractional Rayleigh dissipation function and the EL equation, \( \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial}{\partial \tau} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial}{\partial \tau} \left( \frac{\partial L}{\partial q_i} \right) = 0, i = 1 \cdots n, \) where \( R \) is the fractional Rayleigh dissipation function,
given by \( R = L(\frac{t}{\tau}, \frac{q}{\tau}) \). Note that in the equation above, the dissipation function is part of the extended EL equation. The origin of the third term is non-standard and is due to fractional analysis and points to a time-dependent damping factor.

**Modified variational principle for high-order systems.**

Higher-order fractional Lagrange equations of motion.

Our purpose is now to extend the fractional treatment to include systems with higher derivatives present. We shall focus on mechanical systems whose Lagrangians present second-order time derivatives. The interest on acceleration-dependent Lagrangians for mechanical systems relies on our claim that they may give us some clues on the properties of field-theoretic models whose classical dynamical equations are of fourth order in the time derivative. The extension to include orders higher than two at the Lagrangian level can be read as follows:

\[
\int \frac{\delta S}{\delta t} \cdot \left( \frac{d}{d\tau} \frac{\partial L}{\partial \dot{q}_i} \right) + \frac{d^2}{d\tau^2} \left( \frac{\partial L}{\partial \ddot{q}_i} \right) + \frac{1 - \alpha}{t - \tau} \frac{\partial L}{\partial \dot{q}_i} = 0, \quad i = 1 \cdots n.
\]

This extended version for the fractional EL equation has new time-dependent damping coefficients.

**Higher-order fractional Hamilton equations.** We shall now consider the associated Hamiltonian formulation for acceleration-dependent Lagrangian systems; once again, our starting point is the fractional action, \( S^\alpha(t) \), for systems with \( n \) degrees of freedom. The change of basis from \((q, \dot{q}, \ddot{q}, t)\) to the new coordinates \((q, p, \pi, t)\) is accomplished by the well-known Legendre transformation. So, we consider a function Hamiltonian \( H(q, p, \pi, t) \), and the fractional integral can be written now as \( S^\alpha(t) = \int [\sum_i p_i \dot{q}_i + \sum_j \pi_j \ddot{q}_j - H(p_i, \pi_i, q_i, \dot{q}_i, t)]d\tau \). Applying the variational principle yields the higher-order Hamilton equations as follows:

\[
\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial \dot{q}_i} + \frac{1 - \alpha}{t - \tau} p_i,
\]

\[
\ddot{q}_i = \frac{\partial H}{\partial \pi_i}, \quad \ddot{\pi}_i = -\frac{\partial H}{\partial \dot{q}_i} + \frac{1 - \alpha}{t - \tau} \pi_i.
\]

**A working example: higher-derivative harmonic oscillator.** – Our present task consists in analyzing a second derivative generalization of the harmonic oscillator in the version examined by Pais and Uhlenbeck [11]. We are however re-assessing it in the fractional integral framework. The classical and known Lagrangian is given by

\[
L = \frac{1}{2} \dot{q}^2 + \frac{1}{2} m \ddot{q}^2 - \frac{mg}{2} q^2, \quad \text{where} \quad m \text{ is the particle mass,} \quad \omega = \sqrt{\frac{g}{m}}, \text{and} \quad \beta \text{ is an arbitrary parameter with mass dimension equal to} \ (-2).
\]

In the PU model, \( \beta \) assumes the value \(-mg/(2\omega^2)\), where \( g \) is a small positive number \((0 < g < 1/4)\) playing the role of a coupling constant.

The fractional EL equation obtained from the Lagrangian (1) can be read as follows:

\[
\dot{q}^{(4)}(t) = \frac{2(1 - \alpha)}{(t - \tau)} \dot{q}^{(3)}(t) + \frac{(1 - \alpha)(2 - \alpha)}{(t - \tau)^2} \dot{q}(t) - \frac{m}{2\beta} \omega^2 \dot{q}(t) - \frac{m}{2\beta} \omega^2 \dot{q}(t).
\]

Obviously, if \( \alpha = 1 \), we recover the known integer PU equation of motion. Unfortunately, we could not get an analytic solution to eq. (2). Therefore, the remainder of our discussion shall be carried out using numerically methods with the purpose of studying its dynamical behavior and thus establish comparisons between the results of the fractional-order systems and their integer-order counterparts. Since \( t \) in eq. (2) is the time for some observer in a particular reference frame, without loss of generality, we can set \( t = 0 \).

We go one step further appropriately choosing initial conditions, namely, \( q(\tau = 10^{-4}) = 1, \dot{q}(\tau = 10^{-4}) = 1, \ddot{q}(\tau = 10^{-4}) = \dddot{q}(\tau = 10^{-4}) = 0 \).

The results of the numerical simulations of (3) are plotted below, with different values of \( \alpha \) and \( g \). The graph in fig. 1 compares the curves obtained with \( \alpha \leq 1 \) and the one in fig. 2 compares the curves obtained with \( \alpha \geq 1 \).

Likewise, fig. 3 depicts the behavior of eq. (2) for values of \( g \). In fig. 3, we set the value of \( \alpha \) and plot different solutions of eq. (2) corresponding to the different values of \( g \). On the other hand, for values of \( g > 0.25 \), the solution of eq. (2) behaves quite differently from the previous ones. Figures 4 and 5 illustrate this fact. In all this figures, \( q(\tau) \),
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Fig. 2: The running of $q(\tau)$, with $m = 1$, $\omega = 10$, $g = 0.24$. Values of $\alpha \geq 1$.

Fig. 3: The running of $q(\tau)$, with $m = 1$, $\omega = 10$, $\alpha = 0.9$, $0 < g < 0.25$.

$t$, $m$ and $\omega$ are expressed in the international unit system while $g$ and $\alpha$ are dimensionless.

It has already been noted that the amplitude of oscillation of the system increases with time. Thus, it is also our interest to discuss this behavior by focusing on the Hamiltonian, which now should not be conserved, since the Lagrangian, in the fractional regime, displays an explicit time dependence thorough the time-dependent weight factor. By writing the fractional action integral for Lagrangian (1), we obtain

$$S^\alpha[q](t) = \frac{1}{\Gamma(\alpha)} \int \left( \beta q^2 + \frac{m^2}{2} q^2 - \frac{m \omega^2}{2} q^2 \right) (t - \tau)^{\alpha - 1} d\tau.$$  

Let us write down the following action:

$$S^{\alpha'}[q](t) = \frac{1}{\Gamma(\alpha)} \int \left( \gamma C^2 + \ddot{q} C + \frac{m^2}{2} q^2 - \frac{m \omega^2}{2} q^2 \right) (t - \tau)^{\alpha - 1} d\tau,$$

where $C$ is a new coordinate which depends on the variable $\tau$; $\gamma$ is a constant.

The important point to remember here is that two different Lagrangians are classically equivalent, once they can transformed into one another by local (in the time-like sense) coordinate redefinitions. By taking the action (7), we can readily obtain the equation of motion for the $C$-coordinate, which reads $C = -\frac{\gamma}{\beta} q$.

Here, it is worthwhile to recall that, since $C$ behaves as an auxiliary coordinate (its dependence is purely algebraic), it is a perfectly legitimate procedure to replace it in the action (6) by its equation of motion (8). After this substitution, by comparing the actions (6) and (7), one gets the action

$$S^{\alpha'}[q](t) = \frac{1}{\Gamma(\alpha)} \int \left( -\frac{1}{4\beta} C^2 + \ddot{q} C + \frac{m^2}{2} q^2 - \frac{m \omega^2}{2} q^2 \right) (t - \tau)^{\alpha - 1} d\tau.$$
By integrating the second term by parts yields

\[ S^{\alpha}[q](t) = \frac{1}{\Gamma(\alpha)} \int \left[ \frac{1}{4\beta} C^2 - \frac{q C}{t - \tau} - \frac{m^2}{2} q^2 \right] \frac{d\tau}{t - \tau}. \]  

(4)

We now consider the transformation \( B = \sqrt{m}q - \frac{C}{m} \), where \( B \) is a new coordinate. By replacing eq. (11) in the action (10), the Lagrangian can be explicitly expressed only in terms of the new basis of coordinates, \( B \) and \( C \), namely \( L = \frac{1}{2} B^2 - \frac{1}{2} C^2 - \frac{1}{2} a^2 B^2 - \frac{1}{2} \left( \frac{1}{4} \mu^2 \right) C)C^2 - \frac{e^2}{\sqrt{m}} BC - \frac{e^2}{\sqrt{m} \tau} B - \frac{e^2}{\sqrt{m} \tau} C \), which shows it is manifestly time-dependent. The steps above have been included here to explicitly show why the Hamiltonian is not conserved. This happens due to the explicit time dependence of the Lagrangian rewritten after the coordinate redefinitions.

A variational approach via local deformed derivatives. – In this section, we re-assess some of the aspects considered in [35], where an extended variational formalism is proposed. We extend the VC by writing down a higher-order-derivative Lagrangian formulated in terms of local deformed derivatives DD. The Riemann improper integral can be presented in different versions with respect to the general VC [36]. Our goal here is to obtain, via deformed EL equations, complete non-linear equations of motion for a PU-type oscillator. A number of works pointing out the usefulness of such an approach may be found in the literature [9,37–39]. Basically, we start off by considering a piecewise form of DD, given as

\[ D^\alpha_a f(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon (t - a)^{1-\alpha}) - f(t)}{\epsilon}, \]

where we assume that \( 0 < \alpha < 1 \) and the option 3 in the work of ref. [35] is adopted. To obtain a complete higher-order Pais-Uhlenbeck equation, we shall proceed as follows: we first consider the functional \( J[y] = \int_a^b L(x, y, D^2_y y, a D^3_y y, a D^4 (D^2_y y))dx \) and, as is usually done, to get an extremum, we set \( \delta J = 0 \). So, the resulting (EL) equations come out as

\[ \frac{\partial L}{\partial y} - D^2 \left( \frac{\partial L}{\partial D^2_y y} \right) + D^3 \left( x - a \right)^{1-\alpha} \frac{\partial L}{\partial (a D^2_y y)} = 0. \]

To obtain a complete EL equation for the Pais-Uhlenbeck oscillator, we now consider the Lagrangian \( L = a \beta q^2 + \gamma (a D^2_y q^2)^2 + \xi (a D^2_y q^2)^2 + b \frac{m^2}{2} q^2 - c \frac{m^2}{2} q^2 \), where \( a, b, c \) are constants that can only assume the values 0 or 1. Here, \( \xi, \gamma \) are to be fixed in such a way to yield the adequate signs for the description of the correct dynamics.

The resulting linear equation reads as

\[ aq^{(4)} + \frac{\xi}{\beta} q^{(3)} - b \frac{m}{2} \beta q^{(2)} - \frac{\gamma q^{(1)}}{\beta} - c \frac{m^2}{2} q = 0. \]

Notice the presence of a dissipative term, \( \frac{\gamma q^{(1)}}{\beta} \). Here, one can also read the Masterov’s odd order of the Pais-Uhlenbeck oscillator [40], by turning off adequate terms.

Now, in order to obtain a non-linear equation for the Pais-Uhlenbeck oscillator, let us introduce an additional term into the previous Lagrangian:

\[ L = \kappa (\mu D^2_t q)((\mu D^2_t q)^2) + \kappa \mu (\mu - 1) q^{(3)} - b \frac{m}{2} \beta q^{(2)} - c \frac{m^2}{2} q^2 = 0. \]

With the inclusion of this term, the resulting equation of motion becomes

\[ aq^{(4)} + \frac{\xi}{\beta} q^{(3)} - b \frac{m}{2} \beta q^{(2)} - \frac{\gamma q^{(1)}}{\beta} - 2 \kappa \frac{d}{dt} \left( q^{(2)} \right) - \frac{m^2}{2} q = 0. \]

The term \(-2 \kappa \frac{d}{dt} \left( q^{(2)} \right)\) can be rewritten as \(-2 \kappa \mu (\mu - 1) q^{(3)} \). So, a new non-linear dissipative term appears that is now part of the non-linear equation of motion for the Pais-Uhlenbeck oscillator.

Discussion and final considerations. – In this paper, we have applied the Riemann-Liouville integral based on the FALVA approach to obtain the fractional non-linear dynamics equation involving Lagrangians with derivatives of higher order. In particular, we have contemplated the example of the PU harmonic oscillator. The results presented in the fourth section point to interesting aspects of the interplay between fractionality and higher derivatives. In the literature concerning higher-derivative systems, for both mechanical and field-theoretic models, the issue of fractionality is not sufficiently discussed. We believe that this connection should be more deeply exploited, especially in the case of classical fields, expecting that interesting features of the interference between higher-derivative dynamics and fractionality could be applied to the study of higher-derivative gravity models. We intend to follow this path and, for that reason, we have initiated our endeavour with the study of mechanical models reported in the present contribution.

We may look at the PU oscillator as the mechanical counterpart of a Klein-Gordon field with the additional action term \( \beta (\partial_\mu \partial^\mu \phi)^2 \) whenever we consider the so-called mechanical reduction, that is, a reduction process to one dimension, corresponding to the regime in which all fields are spatially homogeneous, \( \partial_\mu \phi = 0 \); all fields keep only time dependence. In the mechanical regime, the Maxwell Lagrangian for the electromagnetic field, \((-1/4) F_{\mu \nu} F_{\mu \nu}\), reduces to the problem of a three-dimensional free particle. Actually, as already anticipated in the paragraph above, higher-derivative field theories may become relevant at a more fundamental level or as effective field models stemming from some physics at very high energy scales. So, our investigation in this contribution may be viewed as a
laboratory to start getting insights on the relationship between higher derivatives and fractionality, once we also expect that, at very tiny distances, space-time fractionality may start yielding physical effects. As is clearly seen from eq. (20), by setting \( \beta = 0 \), we recover the ordinary harmonic oscillator, but now with the damping \( g \), that accounts for fractionality. And this is indeed our main point in this work: higher derivatives induce the undesirable effect of growing amplitudes; on the other hand, the damping parameter associated to fractionality may tame this effect and, therefore, may transform non-physical excitations (quantum-mechanical ghosts) into unstable physical particles. This is the way we motivate the introduction of the \( g \) in connection with higher derivatives. And our claim in the present study is that the PU model may be adopted as a laboratory to start giving a feeling on the relationship between higher derivatives and fractionality.

At first, we have considered the case where \( \alpha < 1 \) and \( g \) are fixed. Surprisingly, fig. 1 shows that the system maintains its oscillatory characteristics and its amplitude is seen to grow indefinitely, in contrast to the PU solution, which has a variable amplitude, but with a fixed maximum value. For example, the smaller the value of \( \alpha \), the bigger the amplitude. For \( \alpha = 1 \) and in the limit with \( g \to 0 \), the system displays the expected behavior for a simple harmonic oscillator. On the other hand, for values of \( \alpha > 1 \), fig. 2 shows that the system oscillates with an amplitude that grows with the fractionality.

We have next switched to the case with fixed \( \alpha \) and \( g \) in the interval \((0, 0.25)\). Analysing fig. 3, it is possible to notice that the amplitudes of oscillations grow whenever \( g \) increases. Finally, for values of \( g > 0.25 \), not only the amplitudes increase drastically, but the system loses its oscillatory behavior, as is shown in fig. 5. At this point, we wish to stress the difference in the behavior of the system for values of \( g \) greater than 0.25 as compared with values smaller than 0.25. Figures 4 and 5 clearly illustrate this fact.

The main statement of this paper is that the system governed by eq. (3) exhibits an oscillatory motion with increasing amplitudes in the presence of a new time-dependent damping coefficient.

A comparative study and a detailed discussion of the equations of motion that follow from the FALVA approach is a natural follow-up of the results worked out in the fifth section, and we shall be reporting on this investigation in a forthcoming paper.

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