The K-inductive Structure of the Noncommutative Fourier Transform

Samuel G. Walters

Dedicated to Canada on her 150th birthday

Abstract. The noncommutative Fourier transform of the irrational rotation C*-algebra is shown to have a K-inductive structure (at least for a large concrete class of irrational parameters, containing dense $G_δ$’s). This is a structure for automorphisms that is analogous to Huaxin Lin’s notion of tracially AF for C*-algebras, except that it requires more structure from the complementary projection.

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1. Introduction

In this paper we prove that the noncommutative Fourier transform of the irrational rotation C*-algebra $A_\theta$ has a K-inductive structure for at least a large class of irrationals $\theta$ (containing concrete dense $G_\delta$’s) – see Theorem 1.2. Let us explain this.

Let $\mathcal{B}$ denote the collection of C*-algebras (which we regard as building blocks) consisting of matrix algebras, matrix algebras over the unit circle, or finite direct sums of these. By a $\mathcal{B}$-type algebra we mean one that is C*-isomorphic to an algebra in the collection $\mathcal{B}$. For example, the Elliott-Evans structure theorem \cite{3} states that the irrational rotation C*-algebra $A_\theta$ can be approximated by unital C*-subalgebras of the form $M_q(C(T)) \oplus M_{q'}(C(T))$, which are in the class $\mathcal{B}$.

Definition 1.1. Let $\alpha$ be an automorphism of a unital C*-algebra $A$. We say that $\alpha$ is K-inductive if, for each $\epsilon > 0$ and each finite subset $S$ of $A$, there exist a finite number of $\mathcal{B}$-type building block C*-subalgebras $B_1, \ldots, B_n$ of $A$ with respective unit projections $e_1, \ldots, e_n$, such that

1. $B_j$ and $e_j$ are $\alpha$-invariant for each $j$,
2. $\|e_j x - xe_j\| < \epsilon$, $\forall x \in S$ and each $j$,
3. $e_j x e_j$ is to within distance $\epsilon$ from $B_j$, $\forall x \in S$ and each $j$,
4. $[e_1] + \cdots + [e_n] = [1]$ in $K_0(A^\alpha)$.

Here, $A^\alpha$ is the fixed point subalgebra of $A$ (i.e., the C*-orbifold under $\alpha$). Note that the equality of $K$-classes in condition (4) is stronger than simply requiring it to hold in $K_0(A)$. A projection $e$ is a matrix projection in $A$ when it is approximately central and is the unit of a subalgebra $B \cong M_n(\mathbb{C})$ (for some $n$), and the cut downs $exe$ are close to $B$ for each $x$ in any prescribed finite subset $S \subset A$.

The rotation C*-algebra (or noncommutative 2-torus) $A_\theta$ is the universal C*-algebra generated by unitaries $U, V$ enjoying the Heisenberg relation

$$VU = e^{2\pi i \theta} UV.$$  

The noncommutative Fourier transform (NCFT) of $A_\theta$ is the canonical order four automorphism (or symmetry) $\sigma$ given by

$$\sigma(U) = V^{-1}, \quad \sigma(V) = U.$$  \hspace{1cm} (1.1)

We will simply say ‘Fourier transform’ (and drop the adjective ‘noncommutative’). \footnote{The connection between this C*-Fourier transform and the classical Fourier transform $\hat{f}$ is aptly expressed in terms of the C*-inner product equation $\sigma(\langle f, g \rangle_B) = \langle \hat{f}, \hat{g} \rangle_B$ as in \cite{8} (but we will not need this fact here).} The Elliott Fourier Transform problem, which is still open, is the problem of determining the inductive limit structure of the Fourier transform.
K-inductive Structure

of the irrational rotation C*-algebra $A_\theta$ with respect to basic building blocks consisting of finite dimensional algebras and circle algebras. (Or, more generally, in terms of type I C*-subalgebras.)

The main result of this paper is the following theorem, where $G$ is any of the dense $G_\delta$ sets in $(0,1)$ constructed in Section 2.5 below.

**Theorem 1.2. (Structure Theorem)** For each irrational number $\theta$ in the dense $G_\delta$-set $G$, the noncommutative Fourier transform $\sigma$ of $A_\theta$ is a K-inductive automorphism with respect to matrix algebras. More specifically, for each $\varepsilon > 0$ and each finite subset $S$ of $A_\theta$, there are three $\mathcal{B}$-type building block matrix C*-subalgebras

$$B_1 \cong M_m(\mathbb{C}), \quad B_2 \cong M_n(\mathbb{C}), \quad B_3 = M\oplus \sigma(M)\oplus \sigma^2(M)\oplus \sigma^3(M), \quad M \cong M_\ell(\mathbb{C})$$

(for some integers $\ell, m, n$), with respective unit projections

$$e_1, \quad e_2, \quad f = g + \sigma(g) + \sigma^2(g) + \sigma^3(g)$$

where $g$ is the unit projection of $M$ with $g\sigma^j(g) = 0$ (for $j = 1, 2, 3$), such that

1. $B_1, B_2, B_3$ and $e_1, e_2$ are $\sigma$-invariant,
2. $\|e_jx - xe_j\| < \varepsilon$ and $\|gx - xg\| < \varepsilon$, $\forall x \in S$ and $j = 1, 2$,
3. $e_jx e_j$ and $gxg$ are to within distance $\varepsilon$ from $B_j$ and $M$, respectively, $\forall x \in S$ and each $j = 1, 2$.

Further, there exist $\sigma$-invariant unitaries $w, z$ in $A_\theta$ satisfying the equation

$$e_1 + we_2w^* + zfz^* = 1.$$  \hspace{1cm} (1.2)

This equation (1.2) is equivalent to condition (4) in the above definition since the orbifold $A^*_\theta$ has the cancellation property.

We may schematically display the K-inductive structure of the NCFT on $A_\theta$ in terms of building blocks as “• ⊕ • ⊕ ◦ ⊕ ◦ ⊕ ◦ ⊕ ◦” where each bullet • represents a $\sigma$-invariant matrix algebra and the open bullets ◦ are matrix algebras that are cyclically permuted by the NCFT.

The notion of K-inductive is a natural extension of Huaxin Lin’s notion of tracially AF for C*-algebras [4] to automorphisms. The one difference is that whereas tracially AF means that there are plenty of finite dimensional projections whose complements are equivalent to some projection in a prescribed hereditary C*-subalgebra, in the case of K-inductive the complement is required to have a rather specific structure – i.e., is required to be invariantly equivalent to other projections of a building block nature.

We believe that a similar result can be proved for any irrational $\theta$. Our choice of the classes $G$ of irrationals makes our computations far more accessible by avoiding number theoretic complications (and helps to make the paper shorter). A similar approach to that presented in this paper would probably also show that the cubic and hexic transforms of $A_\theta$ – namely the canonical order 3 and 6 automorphisms studied in [1], [2], [10] – are K-inductive automorphisms as well, with respect to matrix algebras possibly including circle
algebra building blocks. The hoped-for conclusion, then, is that all the canonical finite order automorphisms (the only orders being 2, 3, 4, and 6) are K-inductive automorphisms for all irrational $\theta$.

2. The Framework

2.1. Continuous Field of Fourier Transforms. We write $U_t, V_t$ for the continuous sections of canonical unitaries of the continuous field $\{A_t : 0 < t < 1\}$ of rotation C*-algebras such that $V_t U_t = e(t) U_t V_t$, where we’ve used the now common notation

$$e(t) := e^{2\pi it}. $$

(The unitaries $U_t, V_t$ generate $A_t$ for each $t$.) When dealing with a specific irrational rotation algebra $A_\theta$ we often write its unitary generators simply as $U, V$ instead of $U_\theta, V_\theta$. On the field $\{A_t\}$ there is a field of noncommutative Fourier transforms $\sigma_t$ given on the fiber $A_t$ by

$$\sigma_t(U_t) = V_t^{-1}, \quad \sigma_t(V_t) = U_t.$$

Often we omit the subscript on $\sigma_t$ and simply write $\sigma$ since there will be no risk of confusion.

2.2. Basic Matrix Approximation. We will use the following result from [9], Theorem 1.5 (a result that was originally rooted in [8]).

**Theorem 2.1.** ([9], Theorem 1.5.) Let $\theta > 0$ be an irrational number and $p/q > 0$ be a rational number in reduced form such that $0 < q(\theta - p) < 1$. Let $\sigma$ be the Fourier transform of $A_\theta$. Then there exists a Fourier invariant smooth projection $e$ in $A_\theta$ of trace $\tau(e) = q(\theta - p)$ and a Fourier equivariant isomorphism

$$\eta : eA_\theta e \to M_q \otimes A_{\theta'}$$

such that

$$\eta \sigma = (\Sigma \otimes \sigma') \eta$$

(2.1)

where $\Sigma$ and $\sigma'$ are Fourier transform automorphisms of $M_q$ and $A_{\theta'}$, respectively, given by

$$\Sigma(u) = v, \quad \Sigma(v) = u^*, \quad \sigma'(a) = b^*, \quad \sigma'(b) = a,$$

(2.2)

where $M_q = C^*(u,v)$ and $u, v$ are order $q$ unitary matrices with $vu = e(L_q)uv$, and $A_{\theta'}$ is generated by unitaries $a, b$ with $ba = e(\theta')ab$, where $\theta' = \frac{c\theta + d}{q\theta - p}$ is an irrational number in the $GL(2,\mathbb{Z})$ orbit of $\theta$. (Here, $c, d$ are integers such that $cp + dq = 1$.)

Furthermore, given a sequence of rational approximations $p/q$ of $\theta$ such that $0 < q(\theta - p) < \kappa < 1$ for some fixed number $\kappa$, the projection $e$ is a matrix projection: $e$ is approximately central and $\eta(eUe), \eta(eVe)$ are close to order $q$ unitary generators of $M_q$ and which is Fourier invariant – the approximations here go to 0 as $q \to \infty$.

It is easy to see that a similar result to this theorem applies for rational approximations of $\theta$ such that $0 < q(p - q\theta) < 1$ – simply by replacing $\theta$ by $1 - \theta$. 
and using the canonical isomorphism $A_{1-\theta} \cong A_{\theta}$ which canonically intertwines the Fourier transform.

Since $\theta$ will be fixed throughout the paper, we will write $e^+_q$ for the above canonical projection of trace $q^2\theta - pq$, since it has positive label (or ‘charge’), and write $e^-_q$ for the canonical projection of trace $pq - q^2\theta$ with negative label (see (2.4)). According to the last assertion of Theorem 2.1 we can have Fourier invariant matrix projections of both these types.

2.3. Covariant Projections. In [11] (but also somewhat evident in [8]) we showed that the Fourier invariant projections $e^+_q$ of Theorem 2.1 are instances of one and the same continuous field $E(t)$ of projections of the continuous field $\{A_t\}$ of rotation algebras such that $\tau(E(t)) = t$. The relation is canonically furnished by equation (2.3) for $e^+_q$ and (2.4) for its negative charge counterpart $e^-_q$.

Given an irrational number $t$ and integers $n,k$, where $n \neq 0$, one has the canonical unital $*$-morphism

$$\zeta_{n,t} : A_{n^2t-k} \to A_t, \quad \zeta_{n,t}(U_{n^2t-k}) = U^n_i, \quad \zeta_{n,t}(V_{n^2t-k}) = V^n_i.$$ 

This map clearly intertwines the Fourier transform

$$\sigma_t \zeta_{n,t} = \zeta_{n,t} \sigma_{n^2t-k}.$$ 

For rational approximations $\frac{p}{q} < \theta$ such that $0 < q^2\theta - pq < 1$, we have the projection

$$e^+_q = \zeta_{q,\theta} E(q^2\theta - pq) \in A_{\theta}$$ 

(2.3)

whose trace is $q^2\theta - pq$. This is what we mean by saying that $e^+_q$ is covariant (that it arises from the projection field $E(t)$ in a natural manner). We could also write down negatively charged projections\footnote{In the mathematical physics literature related to string theory and noncommutative geometry, the Connes-Chern number $-b^2$, for a projection of trace $ab - b^2\theta$, is referred to as the “charge” of the projection (or that of its associated instanton).} in $A_{\theta}$ defined by

$$e^-_b = v\zeta_{b,1-\theta} E(ab - b^2\theta)$$ 

(2.4)

of trace $ab - b^2\theta \in (0,1)$, where $v$ is the canonical isomorphism

$$v : A_{1-\theta} \to A_{\theta}, \quad v(U_{1-\theta}) = V_{\theta}, \quad v(V_{1-\theta}) = U_{\theta}.$$ 

(2.5)

We will need to use the parity automorphism $\gamma$ of $A_{\theta}$ defined by

$$\gamma(U) = -U, \quad \gamma(V) = -V$$

because it commutes with the Fourier transform\footnote{In fact, $\gamma$ is the only nontrivial of the toral action automorphisms that commutes with the Fourier transform $\sigma$.} and has the nice effect of flipping the signs of two of the topological invariants below (namely, $\psi_{11}, \psi_{22}$), while preserving the others.
2.4. **Topological Invariants.** With $U_\theta, V_\theta$ denoting the canonical unitaries satisfying

$$V_\theta U_\theta = e(\theta) U_\theta V_\theta$$

and the Fourier transform defined by

$$\sigma(U_\theta) = V_\theta^{-1}, \quad \sigma(V_\theta) = U_\theta$$

the following are the basic unbounded “trace” functionals defined on the canonical smooth dense $\ast$-subalgebra $A_\theta^\infty$:

$$
\begin{align*}
\psi_{10}^\theta(U_\theta^m V_\theta^n) &= e\left(-\frac{\theta}{4}(m+n)^2\right) \delta_2^{m-n}, \\
\psi_{11}^\theta(U_\theta^m V_\theta^n) &= e\left(-\frac{\theta}{4}(m+n)^2\right) \delta_2^{m-n-1}, \\
\psi_{20}^\theta(U_\theta^m V_\theta^n) &= e\left(-\frac{\theta}{2}mn\right) \delta_2^m \delta_2^n, \\
\psi_{21}^\theta(U_\theta^m V_\theta^n) &= e\left(-\frac{\theta}{2}mn\right) \delta_2^{m-1} \delta_2^n, \\
\psi_{22}^\theta(U_\theta^m V_\theta^n) &= e\left(-\frac{\theta}{2}mn\right) \delta_2^{m-n-1},
\end{align*}
$$

(2.6)

where $\delta_a^b$ is divisor delta function defined to be 1 if $a$ divides $b$, and 0 otherwise. These maps were calculated in [6] and used in [7], [8], [11]. (Sometime $\theta$ is omitted from the notation $\psi_{jk}^\theta$ when there is no risk of confusion.)

The functionals $\psi_{ij}$ are $\sigma$-invariant $\sigma$-traces and $\psi_{2j}$ are $\sigma^2$-traces. Recall that if $\alpha$ is an automorphism of an algebra $A$ (usually a pre-$C\ast$-algebra like $A_\theta^\infty$), by an $\alpha$-trace we understand a complex-valued linear map $\psi$ defined on $A$ satisfying the condition

$$\psi(xy) = \psi(x)\alpha(y)$$

for each $x,y$ in $A$; and we say that $\psi$ is $\sigma$-invariant when $\psi\sigma = \psi$. (Clearly, a $\sigma$-trace is automatically $\sigma$-invariant if its domain contains the identity, but a $\sigma^2$-trace need not be $\sigma$-invariant.) These unbounded linear functionals induce trace maps on the smooth $C\ast$-orbifold $A_\theta^{\infty,\sigma} = A_\theta^\infty \cap A_\theta^\sigma$, thereby inducing homomorphisms on $K$-theory $\psi_*: K_0(A_\theta^\sigma) \to \mathbb{C}$.

In [6] it was shown that $\{\psi_{10}, \psi_{11}\}$ is a basis for the 2-dimensional vector space of all $\sigma$-traces on $A_\theta^\infty$, and that $\{\psi_{20}, \psi_{21}, \psi_{22}\}$ is a basis for the 3-dimensional vector space of all $\sigma$-invariant $\sigma^2$-traces on $A_\theta^\infty$.

The unbounded traces $\psi_{jk}$ along with the canonical bounded trace $\tau$ comprise the associated Connes-Chern character group homomorphism for the fixed point algebra $A_\theta^\sigma$:

$$T: K_0(A_\theta^\sigma) \to \mathbb{C}^6, \quad T(x) = (\tau(x); \psi_{10}(x), \psi_{11}(x); \psi_{20}(x), \psi_{21}(x), \psi_{22}(x)).$$

For the identity one has $T(1) = (1; 1, 0, 1, 0, 0)$. It will be convenient to write

$$T(x) = (\tau(x); \text{Top}(x))$$

where

$$\text{Top}(x) := (\psi_{10}(x), \psi_{11}(x); \psi_{20}(x), \psi_{21}(x), \psi_{22}(x))$$

consists of the discrete topological invariants of $x$. Indeed, in view of [6] and [7], the values of the unbounded traces on projections, and on $K_0(A_\theta^\sigma)$, are quantized, with $\psi_{10}, \psi_{11}$ having range in the lattice subgroup $\mathbb{Z} + \mathbb{Z}(\frac{1}{2})$ of $\mathbb{C}$;
\[ \psi_{20}, \psi_{21} \text{ have range } \frac{1}{2} \mathbb{Z}; \text{ and } \psi_{22} \text{ has range } \mathbb{Z}. \] (Cf. Lemma \[2,3\] below which gives the topological invariants for the field \( E_t \).)

It is straightforward to check that the parity automorphism \( \gamma \) changes the signs of \( \psi_{11} \) and \( \psi_{22} \):

\[ \psi_{11} \gamma = -\psi_{11}, \quad \psi_{22} \gamma = -\psi_{22} \]

and it keeps the other \( \psi_{jk} \) unchanged. Thus,

\[ \text{Top}(\gamma(x)) = (\psi_{10}(x), -\psi_{11}(x); \psi_{20}(x), \psi_{21}(x), -\psi_{22}(x)). \]

The Connes-Chern map \( T \) was shown to be injective \([17]\) for a dense \( G_\delta \) set of irrationals \( \theta \), but since \( K_0(A_\theta^\sigma) \cong \mathbb{Z}^9 \) for all \( \theta \) by \([2]\) or \([5]\), \( T \) is injective for all irrational \( \theta \). This allows us to conclude that since \( A_\theta^\sigma \) has the cancellation property for any irrational \( \theta \), two projections \( e \) and \( e' \) in \( A_\theta^\sigma \) are unitarily equivalent by a \( \sigma \)-invariant unitary if and only if \( T(e) = T(e') \).

**Definition 2.2.** A projection \( f \) is called flat (or \( \sigma \)-flat), when it is an orthogonal sum of the form

\[ f = g + \sigma(g) + \sigma^2(g) + \sigma^3(g) \]

for some projection \( g \). We call such projection \( g \) a cyclic subprojection for \( f \) since it is orthogonal to its orbit under \( \sigma \) and its orbit sum gives \( f \).

Another reason we call \( f \) “flat” is because its topological invariants vanish:

\[ \text{Top}(f) = (0, 0; 0, 0, 0). \]

Indeed, if \( \psi \) is any of the two kinds of unbounded traces in \((2.6)\), we have (can assume \( g \) is smooth) \( \psi(g) = \psi(gg) = \psi(\sigma^j(g)g) = 0 \) for \( j = 1, 2 \), hence \( \psi(f) = 0 \).

In \([11]\) we proved that the topological invariants of the continuous section \( E(t) \) mentioned in Section 2.3 are given as follows.

**Lemma 2.3.** (\([11]\), Theorem 1.7.) The topological invariants of the projection section \( E(t) \) are

\[ \text{Top}(E(t)) = \left( \frac{1-i}{2}, \frac{1-i}{2}, 1, 1, 1 \right) \]

and its trace is \( \tau(E(t)) = t \) for each \( t \in (0, 1) \).

This will allow us to calculate the topological invariants of the canonical projections \( e_q^+ \) and \( e_q^- \) of Theorem \([2,1]\) using the following lemma.

**Lemma 2.4.** (\([11]\), Lemma 3.2.) Let \( \theta_n = n^2 \theta - k \) where \( k, n \) are integers. The unbounded traces on \( A_{\theta_n} \) and \( A_\theta \) are related by \( \zeta_{n,\theta} \) according to the equations

\[
\begin{align*}
\psi_{10}^{\theta_n,\theta} & = \psi_{10}^{\theta_n} + i^{-k} \delta_2^{n} \psi_{11}^{\theta_n}, \\
\psi_{11}^{\theta_n,\theta} & = i^{-k} \delta_2^{n-1} \psi_{11}^{\theta_n}, \\
\psi_{20}^{\theta_n,\theta} & = \psi_{20}^{\theta_n} + (-1)^k \delta_2^{n} \psi_{21}^{\theta_n} + \delta_2^{n} \psi_{22}^{\theta_n}, \\
\psi_{21}^{\theta_n,\theta} & = (-1)^k \delta_2^{n-1} \psi_{21}^{\theta_n}, \\
\psi_{22}^{\theta_n,\theta} & = \delta_2^{n-1} \psi_{22}^{\theta_n}.
\end{align*}
\]
\[ \text{(Lemma 2.4) is in fact easy to check by directly working out both sides on generic unitaries } U^m_{\theta_0} V^n_{\theta_0}. \]

Combining these two lemmas and applying them to the canonical projection \( e^+_q = \zeta_{q,\theta} \mathcal{E}(q^2 \theta - pq) \), we obtain its topological invariants

\[
\text{Top}(e^+_q) = \left( \frac{1+i}{2}[1 + (-1)^{q/2} \delta^q_2], \ \frac{1+i}{2}(-1)^{pq} \delta^{q-1}_2, \ \frac{1}{2} + \frac{3}{2} \delta^q_2, \ \frac{1}{2}(-1)^p \delta^{q-1}_2, \ \delta^{q-1}_2 \right). \tag{2.7}
\]

Likewise, the topological invariants of the negatively charged canonical projection \( e^-_b \) of trace \( ab - b^2 \theta \) (where \( a, b \) are coprime) are

\[
\text{Top}(e^-_b) = \left( \frac{1+i}{2}[1 + (-1)^{b/2} \delta^b_2], \ \frac{1+i}{2}i^{-ab} \delta^{b-1}_2, \ \frac{1}{2} + \frac{3}{2} \delta^b_2, \ \frac{1}{2}(-1)^a \delta^{b-1}_2, \ \delta^{b-1}_2 \right). \tag{2.8}
\]

To check the latter invariants of \( e^-_b \), one uses the following relations between the unbounded traces and the canonical isomorphism \( \psi \) of (2.5):

\[
\psi^\theta_{10} \psi = (\psi^\theta_{11})^*, \quad \psi^\theta_{11} \psi = -i(\psi^\theta_{11})^* \\
\psi^\theta_{20} \psi = \psi^\theta_{20}, \quad \psi^\theta_{21} \psi = -\psi^\theta_{21}, \quad \psi^\theta_{22} \psi = \psi^\theta_{22}
\]

which are straightforward to check by working them out on the unitary elements \( U^m_{-\theta} V^n_{1-\theta} \). (Here, \( \psi^* \) is the Hermitian adjoint \( \psi^*(x) = \psi(x^*) \).)

To verify the first component of \( \text{Top}(e^-_b) \), for example, we compute:

\[
\psi^\theta_{10}(e^-_b) = \psi^\theta_{10} \psi \zeta_{b,1-\theta} \mathcal{E}(ab - b^2 \theta) = \psi^\theta_{10} \zeta_{b,1-\theta} \mathcal{E}(\hat{\theta})
\]

where we have written \( \hat{\theta} := ab - b^2 \theta = b^2(1 - \theta) - (b^2 - ab) \). By Lemma 2.4 with \( \theta \) replaced by \( 1 - \theta \), and by Lemma 2.3 we get

\[
\psi^\theta_{10} \zeta_{b,1-\theta} \mathcal{E}(\hat{\theta}) = \psi^\theta_{10} \mathcal{E}(\hat{\theta}) + i^{-(b^2 - ab)} \delta^b_{11} \psi^\theta_{11} \mathcal{E}(\hat{\theta}),
\]

\[
= \frac{1-i}{2} + i^{-(b^2 - ab)} \frac{\delta^b_{11}}{2} \frac{1-i}{2}
\]

since if \( b \) is odd the delta term vanishes and when \( b \) is even, \( a \) has to be odd so can be removed in the power of \(-1\). After conjugating we obtain \( \psi^\theta_{10}(e^-_b) \), as asserted. The other invariants of \( e^-_b \) in (2.8) are similarly checked.

2.5. Class of Irrationals \( \mathcal{G} \). We begin with any dense set \( D \) of rational numbers \( \frac{k}{m} \) in the open interval \( (0, \frac{1}{2}) \) where \( k, m \geq 1 \). We form the following integers

\[
n = 4mk + 1, \quad q = n^2, \quad s = n^2 + 4m^2, \quad p = 4k^2(2n + 1), \quad r = p + 2n - 3
\]

which can easily be checked to satisfy the modular equation

\[
ps - qr = 1 \tag{2.9}
\]

(for any \( k, m \)). Let \( \kappa_2, \kappa_1 \) be any fixed pair of positive numbers such that

\[
0 < \kappa_2 \leq \frac{1}{2} < \kappa_1 < 1, \quad 1 < \kappa_1 + \kappa_2. \tag{2.10}
\]
One checks (using (2.9)) directly that the following inequality
\[
\frac{r}{s} < \frac{pq - K_1}{q^2} < \frac{rs + K_2}{s^2} < \frac{p}{q}
\]
(2.11)
holds for large enough \(k\). The left inequality holds for large enough \(k\) (specifically for \(k\) such that \(\frac{q}{s} > K_1\), since \(\frac{q}{s} \to 1\) as \(k \to \infty\)). (Indeed, the left inequality holds for \(k\) such that \(4k^2 \geq K_1\).) The middle inequality holds for all \(k,m\) by virtue of (2.10), and the right inequality holds always since \(K_2 \leq \frac{1}{2}\).

It is easy to see that the difference \(\frac{p}{q} - \frac{2k}{m}\) goes to 0 for large \(k,m\), hence the set of rationals \(\{\frac{p}{q}: k,m \geq 1\}\) is dense in the open interval \((0,1)\), as also is the set \(\{\frac{r}{s}\}\).

We can extend slightly inequality (2.11) to the following
\[
\frac{2km - \frac{1}{2}}{m^2} < \frac{r}{s} < \frac{pq - K_1}{q^2} < \frac{rs + K_2}{s^2} < \frac{p}{q} < \frac{2k}{m}
\]
(2.12)
where \(\theta\) will be the type of irrational that we’ll be interested in. The leftmost and rightmost inequalities here can be checked to hold for all \(k,m\) since they follow from the equalities
\[
rm^2 - s(2km - \frac{1}{2}) = 4k^2m^2 + m^2 + 2km + \frac{1}{4}, \quad 2kq - pm = 4k^2m + 2k.
\]
Of course, the remaining inequalities in (2.12) hold for large enough \(k,m\) depending on choice of \(K_1, K_2\) satisfying (2.10).

The above leads to the construction of various dense \(G_\delta\) sets of irrational numbers \(\theta\) in \((0,1)\) for each choice of \(K_1, K_2\) satisfying (2.10) and choice of dense set \(D\) of rational numbers in \((0,\frac{1}{2})\). Such irrationals \(\theta\) possess infinitely many pairs of integers \(k,m\), and associated rational approximations \(\frac{2k}{m}, \frac{p}{q}, \frac{r}{s}\) satisfying (2.12). For example, based on the inner inequality in (2.12), one takes a countable intersection of the dense unions of open intervals
\[
\mathcal{G} = \mathcal{G}_{K_1,K_2} = \bigcap_{N \geq 1} \bigcup_{k,m \geq N} \left( \frac{pq - K_1}{q^2}, \frac{rs + K_2}{s^2} \bigg| \frac{km}{m} \in D \right).
\]
(2.13)
One could conceivably construct specific irrationals in the class \(\mathcal{G}\).

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\textsuperscript{4}The middle inequality yields the quadratic inequality \(K_2x^2 - x + K_1 > 0\), where \(x = q/s\). By (2.10), the quadratic is a decreasing function over the interval \([0,1]\) and is positive at the endpoints, so it is positive on \([0,1]\).
3. Proof of Structure Theorem

We begin the proof with the following lemma. If $B$ is a C*-subalgebra of $A$ and $x \in A$, we use the standard notation $d(x, B)$ for the norm distance between $x$ and $B$: $d(x, B) = \inf \{ \|x - y\| : y \in B\}$.

**Lemma 3.1.** Let $\theta > 0$ be an irrational number and $M, N$ positive coprime integers such that $0 < N(N\theta - M) < 1$. Then for each $t \in (0, 1) \cap (\mathbb{Z} + \mathbb{Z}\theta)$ such that

$$t < \frac{1}{4}(N\theta - M)$$

there exists a cyclic projection $h$ (i.e., $h\sigma^j(h) = 0$ for $j = 1, 2, 3$) of trace $\tau(h) = Nt$. If, in addition, there is a sequence of rationals $M/N$ such that $0 < N(N\theta - M) < \kappa < 1$ for some fixed $\kappa$, then for each $\varepsilon > 0$, there are $N, M$ large enough such that

1. $\|hU - Uh\| < \varepsilon$, $\|hV - Vh\| < \varepsilon$.
2. there is a matrix C*-subalgebra $\mathcal{M}$ of $A$ having $h$ has its unit such that $d(hU, \mathcal{M}) < \varepsilon$, $d(hV, \mathcal{M}) < \varepsilon$.

**Proof.** Consider the canonical Fourier invariant projection $e$ in $A$ of trace $\tau(e) = N(N\theta - M)$ given by Theorem [2.1] and corresponding isomorphism

$$\eta : eAe \to M_N \otimes A'_{\theta'}, \quad \text{where} \quad \theta' = \frac{c\theta + d}{N\theta - M}$$

and $c, d$ are integers such that $cM + dN = 1$. Write $t = m + n\theta$, for some integers $m, n$, and let $K = Mn + Nm$ and $L = dn - cm$. Then

$$K\theta' + L = \frac{(Mn + Nm)(c\theta + d) + (dn - cm)(N\theta - M)}{N\theta - M} = \frac{t}{N\theta - M} < \frac{1}{4}$$

so that $K\theta' + L$ is in $(0, \frac{1}{4}) \cap (\mathbb{Z} + \mathbb{Z}\theta')$. By Theorem 1.6 of [9]5, there exists a $\sigma'$-cyclic projection $h'$ in $A_{\theta'}$ of trace $K\theta' + L = \frac{t}{N\theta - M}$, where $\sigma'$ is the Fourier transform of $A_{\theta'}$ in Theorem [2.1]. This gives the cyclic projection

$$h := \eta^{-1}(I_N \otimes h')$$

of trace

$$\tau(h) = N(N\theta - M) \cdot \frac{t}{N\theta - M} = Nt.$$ 

Since the isomorphism $\eta$ is Fourier covariant, as expressed by [2.1], the projection $h$ is a cyclic subprojection of $e$.

To prove the second assertion of the lemma, assume we have an infinite sequence of rationals $M/N$ such that $0 < N(N\theta - M) < \kappa < 1$ for some fixed $\kappa$. In view of the second part of Theorem [2.1], given $\varepsilon > 0$ there is $N$ large enough so that $\eta(eUe)$ and $\eta(eVe)$ are to within $\varepsilon$ of some elements of the matrix algebra $M_N$. Then

$$\mathcal{M} := \eta^{-1}(M_N \otimes h') = h\eta^{-1}(M_N)h$$

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5One could also use Theorem 1.5 in [12].
is a matrix C*-subalgebra of $A_\theta$ with identity element $h$. (So the algebra $\mathcal{M}$ is cyclic under $\sigma$.) As $\eta(h) = I_N \otimes h'$ commutes with $M_N \otimes h'$, the cut downs $hU = hUeh$ and $hV = heVeh$ are to within $\epsilon$ of elements of $\mathcal{M}$, hence condition (2) holds, and $\|ex - xe\| < \epsilon$, for $x = U, V$. To see that $h$ is approximately central, let $x = U, V$ and write
\[ hx - xh = h(ex - xe) + (ex - xe)h + he - exe \]
so that from $\|ex - xe\| < \epsilon$ one gets $\|hx - xh\| < 2\epsilon + \|he - exe\|$. Further, since $\eta$ is an isometry we get
\[ \|he - exe\| = \|\eta(h)\eta(exe) - \eta(exe)\eta(h)\| \]
and since $\eta(exe)$ is to within $\epsilon$ of an element of $M_N \otimes 1$, with which $\eta(h)$ commutes, one gets $\|he - exe\| < 2\epsilon$. Therefore, $\|hx - xh\| < 4\epsilon$ and $h$ is approximately central. $\blacksquare$

**Remark 3.2.** We point out that the proof of this lemma can be modified slightly to give approximately central Fourier invariant projections $h$ of trace $Nt$ (with the $1/4$ factor removed from the hypothesis on $t$).

We now have the groundwork necessary in order to proceed with the proof of Theorem 1.2.

Fix an irrational $\theta$ in the class $\mathcal{G}$ given by (2.13).

The inequalities (2.12) give three rational convergents of $\theta$ and three respective numbers
\[ 0 < 2km - m^2 \theta < \frac{1}{2}, \quad 0 < pq - q^2 \theta < \kappa_1, \quad 0 < s^2 \theta - rs < \kappa_2. \]

We are interested in the following approximately central canonical matrix projections
\[ e_m, \quad e_q^-, \quad e_s^+ \]
with respective traces $2km - m^2 \theta$, $pq - q^2 \theta$, $s^2 \theta - rs$. From (2.7) and (2.8) we obtain the topological invariants of the last two to be
\[
\text{Top}(e^-_q) = \left(\frac{1+i}{2}, \frac{1+i}{2}i^{-pq}; \frac{1}{2}, \frac{1}{2}(-1)^p, 1\right) \quad (3.1)
\]
\[
\text{Top}(e^+_s) = \left(\frac{1-i}{2}, \frac{1-i}{2}i^{-rs}; \frac{1}{2}, \frac{1}{2}(-1)^r, 1\right) \quad (3.2)
\]
as $q$ and $s$ are odd. Since $p \equiv 0 \mod 4$ and $rs \equiv -1 \mod 4$ (see first paragraph of Section 2.5), these become
\[
\text{Top}(e^-_q) = \left(\frac{1+i}{2}, \frac{1+i}{2}; \frac{1}{2}, \frac{1}{2}, 1\right) \quad (3.3)
\]
\[
\text{Top}(e^+_s) = \left(\frac{1-i}{2}, \frac{1-i}{2}; \frac{1}{2}, -\frac{1}{2}, 1\right). \quad (3.4)
\]
Taking the parity $\gamma$ of $e_s^+$ gives
\[
\text{Top}(\gamma e^+_s) = \left(\frac{1-i}{2}, -(\frac{1+i}{2}); \frac{1}{2}, -\frac{1}{2}, -1\right) \quad (3.5)
\]
and adding gives
\[
\text{Top}(e^-_q) + \text{Top}(\gamma e^+_s) = (1,0;1,0,0) = \text{Top}(1). \]
Therefore
\[ T(1) - T(e_q^-) - T(\gamma e_s^+) = (\tau_0; 0, 0, 0, 0) \] (3.6)
where the trace value \( \tau_0 \) here is
\[ \tau_0 = 1 - (pq - q^2 \theta) - (s^2 \theta - rs) = (1 + rs - pq) - (s^2 - q^2) \theta. \]
Computing these in terms of the parameters \( k, m \), one gets
\[ s^2 - q^2 = 8m^2(2m^2 + n^2) = 8m^2(16k^2m^2 + 8km + 2m^2 + 1) = 4m^2B \]
and
\[ 1 + rs - pq = 4m^2(64k^3m + 8km + 24k^2 - 1) = 4m^2A \]
where
\[ A = 64k^3m + 8km + 24k^2 - 1, \quad B = 2(16k^2m^2 + 8km + 2m^2 + 1). \]
Thus, we can write
\[ \tau_0 = 4m^2(A - B \theta). \]
We now claim that \( \tau_0 \) is the trace of an approximately central flat projection
\[ f = g + \sigma(g) + \sigma^2(g) + \sigma^3(g) \]
whose cyclic subprojection \( g \) is approximately central as well. First, it is straightforward to check that
\[ 2ks - m(r + 4) = 4k^2m + 2k - 3m > 0 \]
is positive (for all \( k, m \geq 1 \)), and that one has the equality
\[ sA - Br = 1. \]
These give the inequality
\[ \frac{4mA - 2k}{4mB - m} < \frac{r}{s} < \theta \]
from which we get
\[ t := m(A - B \theta) < \frac{1}{A}(2k - m \theta) < 1. \] (3.7)
To be sure that \( A - B \theta > 0 \), in view of (2.12) it is enough to see that
\[ \theta < \frac{rs + \kappa_2}{s^2} < \frac{A}{B}. \]
Cross multiplying the last inequality here reduces it to \( \kappa_2 < \frac{s^2}{B} \) (again using \( sA - Br = 1 \)) which holds since \( \kappa_2 \leq \frac{1}{2} \) and \( \frac{1}{2} < \frac{s}{B} \) follows from \( 2s = B + 4m^2 \).

To establish the claim just made, apply Lemma 3.1 with
\[ N(N(1 - \theta) - M) = m(2k - m \theta) = \tau(e_m^-) \]
i.e., with \( N = m \) and \( M = m - 2k \), and with \( t = m(A - B \theta) \). The hypothesis of this lemma that \( t < \frac{1}{A}(2k - m \theta) \) has already been checked in (3.7). Therefore, by Lemma 3.1 there exists an approximately central cyclic projection \( g \) of trace
\[ \tau(g) = mt = m^2(A - B \theta). \]
The second part of Lemma 3.1 (where the “$\kappa$” there can be taken to be $\frac{1}{2}$ in view of the inequalities (2.12) relating $\theta$ and $2k/m$) gives the matrix cut down approximation for $g$. The corresponding flat projection is then

$$f = g + \sigma(g) + \sigma^2(g) + \sigma^3(g)$$

with trace

$$\tau(f) = 4m^2(A - B\theta) = \tau_0.$$ 

Therefore (3.6) becomes

$$T(e^-_q) + T(\gamma e^+_s) + T(f) = T(1),$$

where all the underlying projections $e^-_q, e^+_s, g, f$ are approximately central matrix projections. Since the Connes-Chern map $T$ is injective we get the following equality of classes in $K_0(A^\sigma_\theta)$

$$[e^-_q] + [\gamma e^+_s] + [f] = [1]$$

as required by Definition 1.1. Since the orbifold C*-algebra $A^\sigma_\theta$ has the cancellation property, this equation of $K$-classes gives equation (1.2) of Theorem 1.2 for some Fourier invariant unitaries $w, z$ – namely, $e^-_q + w\gamma e^+_s w^* + zfz^* = 1$.

This completes the proof of Theorem 1.2 that the Fourier transform $\sigma$ is K-inductive on the irrational rotation C*-algebra $A_\theta$.

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DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF NORTHERN B.C., PRINCE GEORGE, B.C. V2N 4Z9, CANADA.

E-mail address: Samuel.Walters@unbc.ca

URL: http://hilbert.unbc.ca