Gravitational Waves in Minkowski Spacetime Background

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Abstract. One contribution of Einstein’s equation is the existence of gravitational waves. The weak gravitational waves are solutions of the linearized Einstein’s equation around a Minkowski spacetime. By using a certain coordinate freedom, waves propagate at the speed of light and propagating plane waves in z-direction gives two polarization states. The geodesic equation gives an oscillating proper distance of two rest particles when gravitational wave passes, while for proper time, no time dilatation meaning each particle has the same proper time. The fluctuation effect of gravitational waves causes ripples in spacetime making each particle feels a tidal acceleration.

1. Introduction

Gravitational interaction is one of the four known natural interactions. In the Isaac Newton’s perspective, gravitation is an attraction force between two objects. But according to Einstein’s theory, gravitation is a curved spacetime symbolized by coupling the Einstein’s tensor with the energy-momentum tensor. The outstanding of Einstein’s theory gave remarkable results. Detection gravitational waves using LIGO and Virgo detector [1,2] has a tremendous impact on physics, astrophysics and cosmology while the effect of gravitation on braneworlds model in the area of localization fields is discussed in Refs. [3-8]. In the following, we review the appearance of gravitational waves and contrast the corresponding metric with some general forms of familiar metrics. The review is based on Ref. [9].

The perturbed Minkowski spacetime is characterized by the metric tensor

\[ g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \]  \hspace{1cm} (1)

where \( \eta_{\mu\nu} = \text{diag} (-1,1,1,1) \) is the Minkowski tensor and \( |h_{\mu\nu}| \ll 1 \) represent small perturbation around the background Minkowski spacetime. Using Lorentz transformation \( x'\mu = \Lambda^\mu_\nu x^\nu \) gives

\[ g'_{\mu\nu} = \Lambda^\alpha_\mu \Lambda^\beta_\nu g_{\alpha\beta} \]

meaning \( h_{\mu\nu} \) can be treated as second rank tensor

\[ h'_{\mu\nu} = \Lambda^\alpha_\mu \Lambda^\beta_\nu h_{\alpha\beta}, \]  \hspace{1cm} (2)
The linearized Riemann tensors
\[ R_{\alpha\beta\mu\nu} = \frac{1}{2} \left[ \partial_\mu \partial_\nu h_{\alpha\beta} - \partial_\mu \partial_\alpha h_{\beta\nu} - \partial_\nu \partial_\beta h_{\alpha\mu} + \partial_\nu \partial_\alpha h_{\beta\mu} \right], \]  
(3)
give the linearized Einstein’s equation
\[ \partial^\gamma \partial_\nu h^{\mu\sigma} + \partial^\sigma \partial_\nu h^{\mu\gamma} - \partial^\alpha \partial_\nu h^{\mu\gamma} - \partial^\gamma \partial_\nu h - \eta^{\gamma\sigma} \partial_\nu \partial_\beta h^{\mu\beta} + \eta^{\gamma\sigma} \partial_\mu \partial_\nu h = 8\pi G T^{\gamma\sigma}, \]  
(4)
where \( h = \eta^{\gamma\sigma} h_{\gamma\sigma} \) is the trace of \( h_{\gamma\sigma} \). \( G \) is the gravitational constant, and \( T^{\gamma\sigma} \) are the energy-momentum tensors of matter. In terms the trace-reversed metric perturbation
\[ \tilde{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2} \eta_{\alpha\beta} h, \]  
(5)
the Einstein’s equation reads
\[ -\partial^\mu \partial_\mu \tilde{h}^{\gamma\sigma} + \partial^\gamma \partial_\mu \tilde{h}^{\mu\sigma} + \partial^\sigma \partial_\mu \tilde{h}^{\mu\gamma} - \eta^{\gamma\sigma} \partial_\mu \partial_\nu \tilde{h}^{\mu\nu} = 16\pi G T^{\gamma\sigma}. \]  
(6)

A general infinitesimal coordinate transformation \( x'^\mu \rightarrow x^\mu + \xi^\mu (x^\nu) \) where \( \xi^\mu \) are infinitesimal vector fields leads to metric perturbation and trace-reversed metric perturbation changes according to.
\[ h'_{\mu\nu} = h_{\mu\nu} - \xi_{\nu,\mu} - \xi_{\mu,\nu}, \tag{7a} \]
\[ \tilde{h}'_{\mu\nu} = \tilde{h}_{\mu\nu} - (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu) + \eta_{\mu\nu} \partial_\alpha \xi^\alpha, \tag{7b} \]
while the linearized Riemann tensor does not change, \( R'_{\alpha\beta\mu\nu} = R_{\alpha\beta\mu\nu} \). Equations (7) remind us with a standard gauge transformation \( A'^\mu = A^\mu - \partial^\mu \xi \) in which the gauge field \( A^\mu \) has some degrees of freedom. Considering the trace-reversed metric perturbation, or equivalently the metric perturbation, as a gauge field one may take a Lorentz gauge condition.
\[ \partial_\mu \tilde{h}^{\mu\nu} = 0 \]  
(8)

The most importance of using this gauge freedom is simplifying the linearized Einstein equation into:
\[ \eta^{\mu\nu} \partial_\mu \partial_\nu \tilde{h}^{\gamma\sigma} = -16\pi G T^{\gamma\sigma}, \]  
(9)
It turns out that \( \tilde{h}^{\gamma\sigma} \) represent gravitational waves propagating on a flat Minkowski background. The above equation also implies the local conservation of energy and momentum in linearized case, \( \partial_\beta T^{\alpha\beta} = 0 \). The plane-wave solution of above equation in a vacuum is,
\[ \tilde{h}^{\gamma\sigma} = Re[A^{\gamma\sigma} \exp(ik_\alpha x^\alpha)], \]  
(10)
where \( A^{\gamma\sigma} \) are second rank constant symmetric tensors, \( k_\alpha \) are 4-vector. Inserting the solution (10) into the Einstein equation (9) in vacuum, one finds.
\[ k^\beta k_\beta = 0; A^{\gamma\sigma} k_\gamma = 0, \]  
(11)
giving \( \omega^2 = \tilde{k} \cdot \tilde{k} \) and thus the gravitational wave propagates in vacuum with a speed of light \( c \). Since the gauge conditions (8) do not fully specify \( \xi_\mu \) we need some other conditions to specify the metric perturbation \( h_{\alpha\beta} \). The well-known transverse-traceless (TT) gauge corresponds to additional conditions \( A^\mu_{\ | \mu} = 0 \) and \( A_{\mu\alpha} = 0 \). For a wave travelling in the z-direction where \( k_x = k_y = 0, \omega = k_z = k \), the Lorentz gauge and \( A_{\mu\alpha} = 0 \) give \( A_{\mu3} = 0 \) and accordingly we have the wave of the form
\[ \tilde{h}^{\gamma\sigma} = Re \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{xx} & A_{xy} & 0 \\ 0 & A_{xy} & -A_{xx} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{i(kz - \omega t)}. \]  
(12)
Note that, the traceless condition $A^\mu_\mu = 0$ gives $A_{xx} = -A_{yy} \equiv h_+$ and $A_{xy} = A_{yx} \equiv h_\times$ is due to the symmetrical property of the metric. The wave has two different polarizations, the plus and cross polarizations:

$$h_+ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad h_\times \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ (13)

The most general solution of the linearized Einstein’s equation is a superposition of the solution (12). Since $\bar{h}^{\gamma\sigma}$ describes (reversed) metric changes, the gravitational wave (12) shows space deformation:

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} = \eta_{\alpha\beta} + \bar{h}_{\alpha\beta} + \frac{1}{2} \eta_{\alpha\beta} h = \eta_{\alpha\beta} + \bar{h}_{\alpha\beta} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + Re \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{xx} & A_{xy} & 0 \\ 0 & A_{xy} & A_{xx} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{i(kz-\omega t)}. \quad (14)$$

Note that for the above planewave $h = \bar{h} = A_{xx} + A_{xy} = 0$.

**Physical Significance of Gravitational Wave on a Free Particle**

The trajectory of a free particle is given by the following geodesic equation.

$$\frac{d^2 x^\mu}{d\tau^2} + u^\nu \Gamma^\mu_{\nu\nu} = 0,$$ (15)

where $x^\mu, \tau$ and $u^\alpha$ are, respectively, the particle position, the proper time, and the four-velocity. The particle is initially at rest; $u^\alpha = \delta^\alpha_t$ thus the initial acceleration of particle is

$$\frac{d^2 x^i}{d\tau^2} = -\Gamma^i_{00}. \quad (16)$$

For a linearized theory and TT-gauge the right-hand side of eq. (16) vanishes, showing no acceleration for a free particle in gravitational waves. A free particle initially at rest remains at rest. In the simple context, this means that the coordinate particle do not change. On the other hand, the proper distance between two particles both initially at rest with time-spatial coordinates are shown at $(t, x_1, 0, 0)$ and $(t, x_2, 0, 0)$ respectively, is given

$$\Delta s = \int_{x=x_1}^{x=x_2} \sqrt{g_{xx}} \, dx \approx \left(1 + \frac{A_{xx}}{2} \cos \omega t\right)(x_2 - x_1). \quad (17)$$

It can be concluded that the proper distance between the two particles oscillate when the gravitational wave passes while for the proper time between two events at the same place, i.e. $(t_1, x_0, y_0, z_0)$ and $(t_2, x_0, y_0, z_0)$, is given by

$$\tau = \int_{t_1}^{t_2} \sqrt{g_{00}} \, dx^0 = \int_{t_1}^{t_2} \sqrt{1} \, dt = t_2 - t_1, \quad (18)$$

From the above equation, it turns out that no time dilatation between the two events happens when the gravitational waves pass in the Minkowski spacetime background.

The influence of gravitational fields causes each particle to feel a tidal gravitational force acting perpendicular to the wave propagation direction. The force exerted by the curvature of spacetime gives the following tidal acceleration

$$a^\alpha = -R^\alpha_{\mu\nu\sigma} u^\sigma u^\mu n^\nu, \quad (19)$$
where $n^\nu$ is four-vector, and $u^\sigma = \delta_t^\sigma$ is initial four-velocity. The only nonzero component of the four-velocity is the time component,

$$a^\sigma = -R^\sigma_{\mu\nu} n^\mu - R^\sigma_{\sigma t t} n^\nu,$$

(20)

giving

$$a^x = A_{xx} Sn^x + A_{xy} Sn^y,$$

(21a)

$$a^y = A_{xy} Sn^x - A_{xx} Sn^y,$$

(21b)

where $S = \frac{\omega^2}{2} \text{Re}[e^{i(\omega t - \omega z)}]$ is oscillating factor. The tidal acceleration may distort a particle and it can be dramatic for other particles that experience much larger differential or tidal force due to gravitational waves.

**Metric describing gravitational ripples in spherical and polar coordinates**

According to eq. (14), it can be developed metric describing gravitational waves in spherical coordinates as bellow

$$ds^2 = -dt^2 + dr^2[(\cos^2 \theta + \sin^2 \theta (1 + A_{xx} \cos(kz - \omega t) \cos 2\varphi))] + r^2 d\theta^2[\sin^2 \theta + \cos^2 \theta (1 + A_{xx} \cos(kz - \omega t)) (\cos 2\varphi + \sin 2\varphi)] + r^2 d\varphi^2 \sin^2 \theta (1 - A_{xx} \cos(kz - \omega t)) (\cos 2\varphi + \sin 2\varphi) + 2r dr d\theta \sin \theta \cos \theta A_{xx} \cos(kz - \omega t) (\cos 2\varphi - \sin 2\varphi)(2r dr d\varphi \sin^2 \theta + 2r^2 d\theta d\varphi \sin \theta \cos \theta)$$

(22a)

while for polar coordinates read

$$ds^2 = -dt^2 + dr^2(1 + A_{xx} \cos(kz - \omega t)) (\cos 2\theta + \sin 2\theta)) + r^2 d\theta^2(1 - A_{xx} \cos(kz - \omega t)) (\cos 2\theta + \sin 2\theta)) + 2r dr d\theta A_{xx} \cos(kz - \omega t) (\cos 2\theta - \sin 2\theta)$$

(22b)

Both metrics eq. (22a-22b) describing gravitational ripples have non-simple forms.

2. Conclusion

The linearized Einstein equation, eq. (4) describe weak gravitational waves around Minkowski spacetime with the perturbed metric is shown in eq. (1). Using specific gauge transformation and the trace-reversed metric perturbation (5) lead to change both metric perturbation eq. (7a) and trace-reversed metric perturbation eq. (7b) giving the simple expression linearized Einstein equation (9) with a Lorentz gauge condition (8). The plane-wave solution of eq. (9) in vacuum is shown in eq. (10) representing that the gravitational wave propagates with a speed of light $c$. Finally using the well-known transverse-traceless (TT) gauge and considering the wave travels in the $z$-direction give the form of wave in eq. (12) representing that the wave has two different polarizations, the plus and cross polarizations, eq. (13).

The effect of gravitational waves on a free particle initially rest gives no acceleration, eq. (16) meaning that the coordinate particle do not change. On the other hand, the proper distance between the two particles oscillate when the gravitational wave passes, eq. (17) while for the proper time between two events gives no time dilatation eq. (18). The influence of fluctuation gravitational waves causes ripples in spacetime making each particle feels a tidal acceleration, eq. (21a-21b) which can distort a particle. Finally the metric describing gravitational ripples in spherical and polar coordinates are shown in eq. (22a) and (22b) respectively.

**References**

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