Algorithmic entropy, thermodynamics, and game interpretation

Lev Sakhnovich

99 Cove ave., Milford, CT, 06461, USA
E-mail: lsakhnovich@gmail.com

Mathematics Subject Classification (2010): Primary 03D32;
Secondary 68P30, 54C30, 91A05

Keywords. Length of program, Gibbs ensemble, game theory, statistical physics.

Abstract

Basic relations for the mean length and algorithmic entropy are obtained by solving a new extremal problem. Using this extremal problem, they are obtained in a most simple and general way. The length and entropy are considered as two players of a new type of a game, in which we follow the scheme of our previous work on thermodynamic characteristics in quantum and classical approaches.

1 Introduction

Algorithmic information theory (AIT) is an important and actively studied domain of computer science (see, e.g., interesting results and numerous references in [1, 2, 5, 11, 12]). AIT can be interpreted in terms of statistical physics (SP) (see [1, 12, 13] and references therein). Let us introduce the corresponding notions from AIT and SP.

1. The set of all AIT programs corresponds to the set of energy eigenvectors from SP.

2. The length $\ell_k$ of an AIT program corresponds to the energy eigenvalue $E_k$ from SP. (Here and further $k \geq 1$.)
We denote by $P_k$ the probability that the length of the program is equal to $\ell_k$, i.e. $P_k = P(\ell = \ell_k)$. Next, we introduce the notions of the mean length $L$ (of programs) and of the entropy $S$:

$$L = \sum_k P_k \ell_k, \quad S = -\sum_k P_k \log P_k.$$  \hfill (1.1)

The connection between $L$ and $S$ we interpret in terms of game theory. The necessity of the game theory approach can be explained in the following way. The notion of Gibbs ensemble is introduced in AIT using an analogy with the second law of thermodynamics:

*Gibbs ensemble maximizes entropy on the set of programs, where the values $\{\ell_k\}$ and $L$ are fixed.*

So, the problem of a conditional extremum appears. But the corresponding equation for the Lagrange multiplier is transcendental and very complicated. Therefore, another argumentation is needed to find the basic Gibbs formulas. This problem exists also for the SP case (see [3, Ch.1, section 1] and [4, Ch.3, section 28]). In the present note we use our approach the extremal SP problem [7, 8, 9, 10] to treat also to the corresponding AIT problem. Namely, we fix the Lagrange multiplier $\beta = 1/kT$. That is, we fix the AIT analogue $T$ of the temperature from SP and introduce the compromise function $F = -\beta L + S$. Then the mean length $L$ and the entropy $S$ are two players of a game and the compromise result is the extremum point of the $F$. Finally, we note, that the AIT analogue of temperature was discussed by K.Tadaki [12]. He proved the following assertion:

If the temperature is a computable positive number bounded by 1, it can be interpreted as the *compression rate* in the AIT analogue of thermodynamic theory.

## 2 Connection between length and entropy, a game theoretical point of view

Let the lengths $\ell_k$ of the programs be fixed. Consider the mean length $L$ and the entropy $S$, which are given in (1.1). Note that $\sum_k P_k = 1$. Hence, $P_k$ can be represented in the form $P_k = p_k/Z$, where $Z = \sum_k p_k$. Our aim is to find the probabilities $P_k$. For that purpose we consider the function

$$F = \lambda L + S,$$  \hfill (2.1)
where \( \lambda = -\beta = -1/kT \).

**Fundamental Principle.** The function \( F \) defines a game between the mean length \( L \) and the entropy \( S \).

To find the stationary point of \( F \) we calculate

\[
\frac{\partial F}{\partial p_j} = \lambda \left( \frac{\ell_j}{Z} - \sum_{k=1}^{\infty} \ell_k p_k / Z^2 \right) - (\log p_j) / Z + \sum_{k=1}^{\infty} p_k \log p_k / Z^2.
\]  

(2.2)

It follows from (2.2) that the point

\[ p_k = e^{\lambda \ell_k}, \quad k = 1, 2, \ldots \]

(2.3)

is a stationary point. Moreover, the stationary point is unique up to a scalar multiple. Without loss of generality this multiple can be fixed as in (2.3).

By direct calculation we get in the stationary point (2.3) the equalities

\[
\frac{\partial^2 F}{\partial p_k^2} = -Z_k / (p_k Z^2) < 0, \quad Z_k := \sum_{j \neq k} p_j; \quad \frac{\partial^2 F}{\partial p_k \partial p_j} = 1 / Z^2 > 0, \quad j \neq k.
\]  

(2.4)

Relations (2.4) imply the following assertion.

**Corollary 2.1** The stationary point (2.3) is a maximum point of the function \( F \).

**Proof.** We shall use the following result (see [6, Ch.7, Problem 7]):

\[
\det \begin{bmatrix}
  r_1 & a & a & \ldots & a \\
  b & r_2 & a & \ldots & a \\
  b & b & r_3 & \ldots & a \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  b & b & \ldots & a & r_n \\
\end{bmatrix} = \frac{af(b) - bf(a)}{a - b},
\]

(2.5)

where

\[
f(x) = (r_1 - x)(r_2 - x)\ldots(r_n - x).
\]

(2.6)

In the case that \( a = b \) we have

\[
\det \begin{bmatrix}
  r_1 & a & a & \ldots & a \\
  a & r_2 & a & \ldots & a \\
  a & a & r_3 & \ldots & a \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  a & b & a & \ldots & r_n \\
\end{bmatrix} = -af'(a) + f(a).
\]

(2.7)
Using (2.4) and (2.7) we can calculate the Hessian $H_n(F)$ in the stationary point:

$$H_n(F) = Z^{-2n}[-f'(1) + f(1)], \quad (2.8)$$

where $f$ is given by (2.6) and $r_k = -Z_k/p_k < 0$. Rewrite (2.8) in the form

$$H_n(F) = (-Z)^{-n}\left(1 - \left(\sum_{k=1}^{n} p_k/Z\right)/\prod_{k=1}^{n} p_k\right)$$

to see that the relation $\text{sgn}(H_n(F)) = (-1)^n$ is true. Hence, the corollary is proved. □

So, we proved the proposition below.

**Proposition 2.1** The mean length and entropy satisfy relations

$$L = \sum_{k} \ell_k e^{\lambda_k} / Z, \quad (2.9)$$

$$S = -\sum_{k} (e^{\lambda_k} / Z) \log (e^{\lambda_k} / Z), \quad (2.10)$$

where $Z = \sum_{k} e^{\lambda_k}$.

Note that the basic relations (2.3), (2.9), and (2.10) are obtained by solving a new extremal problem. Namely, in the introduced function $F$ the parameter $\lambda$ is fixed instead of the length $L$, which is usually fixed.

**References**

[1] Baez J.C. and Stay M., *Algorithmic Thermodynamics*, arXiv:1010.2067, 2010 (Mathematical Structures in Computer Science to appear).

[2] Chaitin G. J., *Algorithmic information theory*, Cambridge Tracts in Theoretical Computer Science 1, Cambridge University Press, Cambridge, 1987.

[3] Feynman R.P., *Statistical Mechanics: a Set of Lectures*, Addison–Wesley, Reading, Massachusetts, 1972.
[4] Landau L.D. and Lifshits E.M., *Statistical Physics*, Pergamon Press, New York, 1968.

[5] Li M. and Vitányi P. M. B., *An introduction to Kolmogorov complexity and its applications* (3rd ed.), Texts in Computer Science, Springer, NY, 2008.

[6] Polya G. and Szegö G., *Aufgaben und Lehrsätze Aus der Analysis* (3rd ed., Vol. II), Springer, Berlin-New York, 1964.

[7] Sakhnovich L.A., *Comparison of Quantum and Classical Approaches in Statistical Physics*, Theor. Math. Phys. 123:3, 846-850, 2000.

[8] Sakhnovich L.A., *Comparison of Thermodynamic Characteristics of a Potential Well under Quantum and Classical Approaches*, Funct. Anal. Appl. 36:3, 205-211, 2002.

[9] Sakhnovich L.A., *Entropy and Energy, Non-extensive Statistical Mechanics*, arXiv:1103.1572, 2011.

[10] Sakhnovich L.A., *Comparison of Thermodynamics Characteristics in Quantum and Classical Approaches and Game Theory*, arXiv:10104717, v.2, 2011.

[11] Seibt P., *Algorithmic information theory. Mathematics of digital information processing*, Springer, Berlin, 2006.

[12] Tadaki K., *A statistical mechanical interpretation of algorithmic information theory: total statistical mechanical interpretation based on physical argument*, J. Phys.: Conf. Ser. 201 012006, 2010.

[13] Vos Post J., *Logic for Infinite Capitalists - Perfect Computers that Run Forever?* (http://hplusmagazine.com) to appear).