American Basket and Spread Option Pricing by a Simple Binomial Tree

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In this article, we address the problem of valuing and hedging American options on baskets and spreads—that is, on portfolios consisting of both long and short positions. The main challenge here is dealing with multiple underlying assets: a situation where traditional methods such as binomial trees or Monte Carlo simulations lead to prohibitive computational time or memory requirements.

The key feature of our approach is constructing a simple two-dimensional binomial tree for the basket evolution. For that, we approximate the basket price process by a suitable geometric Brownian motion, shifted by an appropriate amount along the x-axis and potentially reflected over the y-axis. These adjustments to the GBM are necessary for dealing with negative basket values and possible negative skewness of basket distribution. This approximation is inspired by the generalized lognormal approach for European basket options introduced by Borovkova et al. [2007].

We match the basket volatility and build a single binomial tree for the basket evolution, which we use for valuing American (or any other path-dependent) options on the basket, calculating deltas and deciding on an early exercise. We evaluate our approach by comparing binomial tree option prices to those obtained by other methods, where possible. We show that our method performs remarkably well: Option prices obtained by our method coincide up to the second decimal with those obtained by a much more time-consuming full binomial tree. Furthermore, we evaluate the delta-hedging performance of our method and show that our hedge errors are comparable with those obtained for a single-asset American option. The advantages of our method are that it is simple, computationally extremely fast, and efficient, while providing accurate option prices and deltas.

AMERICAN BASKET OPTIONS

A basket option is an option whose underlying asset is a basket, that is, a portfolio of assets. Closely related to it is a spread option, where the underlying instrument is a spread, that is, the difference in prices of two or more assets. Basket and, particularly, spread options are very popular in commodity markets, where producers are often exposed to spreads between a raw material (e.g., crude oil, fuel, or soybeans) and end products (e.g., gasoline and heating oil, electricity, or soybean oil and soybean meal). Basket and spread options are traded over the counter as well as on exchanges and often have an American exercise feature.

It is well known that there is no closed-form solution for the price of an American option, even when it is written on a single asset exhibiting geometric Brownian motion. The most common method of valuing American options is the binomial tree method, introduced by Cox, Ross, and Rubinstein [1979]. Another way to value an American option is by Monte Carlo simulations, as suggested...
by Longstaff and Schwartz [2001]. However, when the underlying value of an American option depends on several assets, such as in the case of basket or spread options, both binomial and simulation methods quickly become either too slow or completely intractable. For example, to value an American basket option by a binomial tree method, a binomial tree must be replaced by a “multidimensional” binomial tree, also known as a binomial pyramid, which is feasible for two assets but becomes progressively less manageable for three or more assets, as the size of the tree grows exponentially with the number of underlying assets.

An alternative is the so-called implied binomial tree, introduced by Rubinstein [1994], which can be applied to American options on baskets. However, such baskets can contain only long positions, so this method is unable to cope with, for example, American spread options. The Longstaff and Schwartz method is also able, in principle, to cope with several underlying assets, but in this case it involves complicated Hermite polynomials. Moreover, as with all Monte Carlo methods, it becomes slow when the number of assets increases. So there is an obvious need for a fast, accurate, and simple method to value and hedge American basket and spread options. In this article, we provide such a method.

In Borovkova et al. [2007], a new approach was introduced for the valuing and hedging of European basket and spread options: the so-called generalized lognormal (GLN) method, which subsequently has been extended to Asian basket options. Inspired by the main ideas of this method, here we suggest an algorithm for building a single two-dimensional binomial tree for the basket evolution. With such a tree at hand, American (and any other path-dependent, e.g., Bermudan) basket options can be priced and hedged within a very short time.

First, we briefly summarize the GLN method as it was introduced in Borovkova et al. [2007] (applicable to a basket of futures) and provide its extension to baskets of other assets, such as stocks or currencies.

**THE GLN APPROACH**

When valuing European basket options, we deal with the distribution of the basket value on a fixed date, such as the option’s maturity date $T$. In the Black–Scholes model, this distribution is lognormal. In the case of a basket of assets, this distribution is no longer lognormal, even when the distributions of all the assets in a basket are. It has been long known (e.g., observed already by Mitchell [1968]) that the sum of lognormal random variables can be well approximated by a lognormal random variable, whose moments can be matched to the moments of the sum. This lognormal approximation works very well in many areas of science, and certainly better than normal approximation. However, for spreads and baskets with negative and positive weights (i.e., portfolios containing long and short positions), the lognormal approximation is no longer possible, because such a basket can have negative values and the distribution of such a basket can be negatively skewed.

A three-parameter family of the so-called generalized lognormal distributions offers a convenient way out: A distribution from this family can have negative values (the so-called shifted lognormal), negative skewness (the negative lognormal) or both (the negative shifted lognormal). For an illustration of these distributions’ densities, see Exhibit 1.

A GLN distribution has not two but three parameters: scale $m$, shape $s$, and location (shift) $\tau$. To moment-match the distribution of a basket to a GLN distribution, we have to match not the first two but the first three moments of the basket. A GLN distribution can be used to approximate the distribution of a basket on any fixed date, by the following procedure.

First, assume that the $N$ assets in the basket are futures (e.g., commodity futures), evolving under the risk-neutral measure as drift free, correlated geometric Brownian motions:

$$dF_i(t) = \sigma_i F_i(t) dW_i(t), \quad i = 1, ..., N$$

with initial values $F_i(0)$, volatilities $\sigma_i$, and $dW_i(t) dW_j(t) = \rho_{ij} dt$. Let the portfolio weights be $a_i$, $i = 1, ..., N$ (these can be negative as well as positive). Then, on any fixed date $t$, the basket value is given by

$$B(t) = \sum_{i=1}^{N} a_i F_i(t)$$

and the first three moments of the basket distribution can be easily calculated:

$$M_1(t) = \mathbb{E}[B(t)] = \sum_{i=1}^{N} a_i M_1$$

$$M_2(t) = \mathbb{E}[B^2(t)] = \sum_{i=1}^{N} a_i^2 M_1$$

$$M_3(t) = \mathbb{E}[B^3(t)] = \sum_{i=1}^{N} a_i^3 M_1$$
\[ M_2(t) = E[B^2(t)] = \sum_{j=1}^{N} \sum_{i=1}^{N} a_j a_i F_i(0) F_j(0) e^{(p_j, p_i)} (4) \]

\[ M_3(t) = E[B^3(t)] = \sum_{k=1}^{N} \sum_{i=1}^{N} \sum_{j=1}^{N} a_k a_i a_j F_i(0) F_j(0) F_k(0) e^{[(p_i, p_j, p_k)]} \]

The basket skewness is given by

\[ \eta_{B(t)} = \frac{E[B(t) - EB(t)]^3}{s_{B(t)}^3} \]  \[ \text{(5)} \]

where \( s_{B(t)} = \sqrt{E[B^2(t) - (EB(t))^2]} \) is the standard deviation of the basket value at time \( t \). On the other hand, the first three moments for the shifted lognormal distribution are

\[ M_1 = \tau + \exp\left( m + \frac{1}{2} s^2 \right) \]  \[ \text{(7)} \]

\[ M_2 = \tau^2 + 2\tau \exp\left( m + \frac{1}{2} s^2 \right) + \exp\left( 2m + 2s^2 \right) \]  \[ \text{(8)} \]

\[ M_3 = \tau^3 + 3\tau^2 \exp\left( m + \frac{1}{2} s^2 \right) + 3\tau \exp\left( 2m + 2s^2 \right) + \exp\left( 3m + \frac{3}{2} s^2 \right) \]  \[ \text{(9)} \]

and for the negative shifted lognormal distribution, the moments \( M_i \) and \( M_3 \) are replaced by \( -M_i \) and \( -M_3 \). For the European basket options, we approximate the basket value at maturity \( T \) by the shifted or negative shifted lognormal distribution, depending on the sign of skewness. We set the first three moments of the appropriate lognormal distribution (shifted if \( \eta_{B(T)} > 0 \) and negative shifted if \( \eta_{B(T)} < 0 \) equal to the first three moments of the basket and numerically solve this system of equations for \( m, \sigma, \) and \( \tau \). The solution to this problem exists and is unique; for details, see Borovkova et al. [2007].

The next step involves replacing our basket \( B(T) \) by \( B'(T) \), which equals either \( B(T - \tau) \) or \( -B(T - \tau) \) depending on the sign of \( \eta_{B(T)} \), and observing that \( B'(T) \) has the regular lognormal distribution. Now, the (European) basket option on \( B(T) \) can be valued as the option on \( B'(T) \) by the Black–Scholes formula by changing the strike \( X \) to \( X' = X - \tau \) and, in the case of negative lognormal approximation (\( \eta_{B(T)} < 0 \)), replacing a call by a put and vice versa.

Now, let the assets in the basket be stocks, whose prices at time \( t \) we denote as \( S_i(t) \), paying some known dividend yield \( q_i \), or currencies (in which case the dividend yield is replaced by the corresponding foreign interest rate \( r_i \)). In this case, under the risk-neutral measure, the asset prices follow the GBM process with drift:

\[ dS_i(t) = (r - q_i) S_i dt + \sigma_i S_i(t) dW_i(t), \quad i = 1, \ldots, N \]  \[ \text{(10)} \]

where again the initial values are \( S_i(0) \), volatilities are \( \sigma_i \), and \( dW_i(t)dW_j(t) = \rho_{ij} dt \). The first three moments
of the basket of stocks with weights $a_i, \ldots, a_N$ are now given by

$$M_i(t) = \sum_{j=1}^{N} a_j S_j(0)e^{-\frac{t}{\sigma^2}}$$

(11) and

$$M_2(t) = \sum_{i=1}^{N} \sum_{j=1}^{N} a_i a_j S_i(0)S_j(0)e^{(\rho_{ij} - \frac{\sigma_i^2 + \sigma_j^2}{2})t}$$

(12)

$$M_3(t) = \sum_{k=1}^{N} \sum_{j=1}^{N} \sum_{i=1}^{N} a_i a_j a_k S_i(0)S_j(0)S_k(0)e^{(\rho_{ij} + \rho_{jk} + \rho_{ki} - \frac{3}{2} \sigma_i^2 - \frac{3}{2} \sigma_j^2 - \frac{3}{2} \sigma_k^2)t}$$

(13)

where $\rho$ is the average annualized dividend yield of the basket, effective during the lifetime of the basket option, or an average foreign interest rate, for a currency baskets. This dividend yield can be computed (or estimated) in the same way as the dividend yield on a stock index.

The rest of the GLN method remains the same as for futures baskets: The basket skewness is computed by Equation (6) and the Moments (11–13) are matched to the first three moments of the appropriate approximating distribution. The option price is then calculated by shifting the strike and eventually replacing a call by a put and vice versa.

In the case of American options, we face an extra complication: We need to approximate not just the distribution of the terminal value of the basket $B(T)$ (or of the average value, as in the case of an Asian option), but the entire basket value process $(B(t))_{t \in [0,T]}$, from the option’s initiation $(t = 0)$ until the option’s maturity $T$. Here, we develop the method that does just that. The main idea of it is as follows: We replace the basket process $B(t)$ by either $B(t)+\tau(t)$ or $B(t)-\tau(t)$, depending on the sign of the basket’s skewness, and then approximate this process by an appropriate geometric Brownian motion. The next section explains these ideas in more detail.

**GBM APPROXIMATION AND THE BINOMIAL TREE**

The main difficulty in pricing an American basket option by the standard binomial method of Cox, Ross, and Rubinstein is the high dimensionality of the binomial tree. Here, we build a single binomial tree for the basket’s evolution, such as that routinely used in valuing a single-asset American option. Such a binomial tree can be used for pricing and hedging American (and other path-dependent, such as Bermudan) basket options.

Consider a basket option maturing at time $T$. At any time $t \in [0,T]$, we can compute the first three moments and the skewness of the basket and approximate its distribution by an appropriate generalized lognormal distribution using the moment-matching procedure outlined above. Note that the parameters of this distribution, namely $m(t), s(t)$, and $\tau(t)$, vary in time (because the second and the third moments of the basket $M_2 = M_2(t)$ (Equation (8)) and $M_3 = M_3(t)$ (Equation (9)) are both functions of time). However, the type of the approximating distribution (shifted or negative shifted lognormal) remains the same throughout the option’s lifetime. This is because as $t$ increases, the absolute value of the basket skewness increases, so the sign remains the same. This can be seen by expressing the basket skewness via the first three moments $M_1(t), M_2(t), M_3(t)$ and examining the moments’ rate of growth with increasing $t$. We illustrate it numerically on the example of the following two baskets (both are spreads between two assets): $\text{Basket 2: } F_0 = [100;120]; \sigma = [0.2;0.3]; \rho_{1,2} = 0.9; a = [-1;1], \text{ and }$ $\text{Basket 3: } F_0 = [150;100]; \sigma = [0.3;0.2]; \rho_{1,2} = 0.7; a = [-1;1]$. $\text{The skewness of Basket 2 is positive and that of Basket 3 is negative at all times } t \in [0,T] \text{ (we took } T = 1 \text{ year, or 250 trading days). For both baskets, the skewness grows in absolute value as } t \text{ increases, which is shown in Exhibit 2.}$

So, if the basket skewness is positive, its distribution is always of the type shifted lognormal. If, at any time $t$, we shift the basket along the $x$-axis by the appropriate amount $-\tau(t)$, we obtain a basket whose distribution is regular lognormal with parameters $(m(t), s(t))$. Similarly, if the skewness is negative, the approximating distribution is negative shifted lognormal, and we must reflect the basket over the $y$-axis and shift it by $-\tau(t)$ to obtain the regular lognormal distribution with the same parameters.

Recall that a geometric Brownian motion at any fixed time also has a lognormal distribution. We use this fact to choose an appropriate approximating geometric Brownian motion, which we denote $B'(t)$. This GBM does not approximate the evolution of the basket $B'(t)$.
is the average dividend

\[ B(t) - r(t) \quad \text{or} \quad -B(t) - \tau(t) \quad (14) \]

This is close in spirit to the so-called displaced diffusions introduced by Rubinstein [1983]. Assume that the approximating GBM \( B(t) \) satisfies the following stochastic differential equation:

\[ \frac{dB^*(t)}{B^*(t)} = \mu^*dt + \sigma^*dW(t) \quad (15) \]

where \( W(t) \) is the standard Brownian motion. The drift and volatility of the approximating geometric Brownian motion must be matched to the basket’s drift and volatility, for which we proceed as follows.

Recall that the parameters of the lognormal distribution corresponding to \( B^*(t) \) \((m^*(t), s^*(t))\) can be expressed via the diffusion parameters \( \mu^*, \sigma^* \), time \( t \), and the initial value of the basket \( B^*(0) \) as follows:

\[
m^*(t) = \ln B^*(0) + \left( \mu^* - \frac{1}{2} \sigma^*{}^2 \right) t
\]

\[
s^*(t) = \text{Var}(\ln B^*(t)) = \sigma^*{}^2 t
\]

(16)

So, for each fixed \( t \), we match the first three moments of the basket to the first three moments of the corresponding lognormal distribution and find the parameters \( m(t), s(t), \tau(t) \), in the same way as in Borovkova et al. [2007]. The diffusion parameters \( \mu^*, \sigma^* \) are then calculated from Equation (16). These parameters turn out to be practically constant in time, as we shall demonstrate below.

If the risk-free drifts of all assets in the basket are the same \( \text{e.g., if all assets are futures or stocks paying no dividends} \), an important simplification can be made. Note that in this case, the drift of the basket \( B(t) \) under the risk-free measure is the same as that for individual assets. This follows from the fact that

\[
dB(t) = \sum_{i=1}^{N} a_i dS_i(t) = rB(t)dt + \sum_{i=1}^{N} a_i \sigma_i S_i(t)dW_i(t)
\]

for a basket of stocks or

\[
dB(t) = \sum_{i=1}^{N} a_i \sigma_i I_i(t)dW_i(t)
\]

for a futures basket. The fact that the drift is known significantly simplifies our calculations and deals with the well-known problem that historical data give an inaccurate estimate of the mean of a returns process. By setting the drift parameter \( \mu^* \) equal to a known value a priori, we reduce the number of unknown parameters. For dividend-paying stocks, this can be achieved by assuming that, as for a stock index, the risk-free drift of the basket is \( \mu^* = r - \bar{q} \), where \( \bar{q} \) is the average dividend yield.

We illustrate the parameter reduction on the example of a futures basket. Instead of setting the parameters \( m^*(t), s^*(t) \) in Equation (16) equal to \( m(t), s(t) \) (found by the moment-matching procedure) and then deducing the diffusion parameters \( \mu^*, \sigma^* \), we use the fact that \( \mu^* = 0 \) for the futures basket and directly calculate the volatility \( \sigma^* \) from the basket’s first three moments. Analyzing the second and the third central moments of the basket, we can explicitly write down the expression relating \( \sigma^* \) (or, rather \( e^{\sigma^*} \)) to the moments \( M_1, M_2, M_3(t) \):

\[
\frac{1}{K} \left( M_2(t) - M_1(t)^2 \right) = \frac{\left( e^{\sigma^*} - 1 \right)^3}{\left( 2 - 3e^{\sigma^*} + e^{2\sigma^*} \right)^2}
\]

(17)
EXHIBIT 3
Volatility $\sigma^*$ of Baskets 2 and 3 vs. Time

| Time (days) | Volatility Parameter Estimates | Volatility Parameter Estimates |
|-------------|-------------------------------|-------------------------------|
|             | Volatility of Basket 2        | Volatility of Basket 3        |
|             |                               |                               |
| 0           | 0.31                          | 0.33                          |
| 50          | 0.32                          | 0.34                          |
| 100         | 0.33                          | 0.35                          |
| 150         | 0.34                          | 0.36                          |
| 200         | 0.35                          | 0.37                          |
| 250         | 0.36                          | 0.38                          |

which holds for all $t$. From this relation, it follows that $e^{y_{\text{B}}}$ equals the root $y_{\text{B}} > 1$, of the third degree polynomial $y^3 + 3y^2 - 4 - K = 0$, with $K = K(t) > 0$ the function of $M_{1r}, M_{2r}(t)$ and $M_{3r}(t)$, given above. It can be easily proven that such a root exists and is unique, and it can be efficiently calculated by a numerical algorithm. Having found this root, the volatility is calculated as $\sigma^* = \sqrt[3]{\frac{b_{\text{B}}}{y_{\text{B}}}}$.

To approximate the basket evolution by a constant-volatility geometric Brownian motion $B'(t)$ in Equation (15), the parameter $\sigma^* = \sqrt[3]{\frac{b_{\text{B}}}{y_{\text{B}}}}$ should not depend on time $t$ (this is essential for building a recombining binomial tree). As there is no explicit formula for the root $y_{\sigma^*}$ (and, hence, for $\sigma^*$), we are unable to analyze the functional form of $\sigma^*$ or prove that it does not depend on $t$ (in fact, it even might be untrue). So we conducted an extensive numerical study to empirically investigate the behavior of $\sigma^*$. It has shown that $\sigma^*$ is practically constant. Here, we illustrate it on the example of Baskets 2 and 3. We take an equidistant discrete partition of the interval $[0, T]$: $0 = t_0 < t_1 < t_2 < \cdots < t_{\tau} < t_{\tau+1} < \cdots < t_n = T$ and calculate $\sigma^*$ for each $t_i$, as described above. Exhibit 3 shows the graphs of $\sigma^*$ versus $t$.

For Basket 2, the volatility is constant, for Basket 3 there is a slight upward drift (the difference between the volatility value at $t = 0$ and $T = T$ is less than 0.2%); however, it is unclear whether this difference may be attributed to the computational error. Similar pictures emerge for any basket we consider. So, on the basis of our empirical observations, we can assume that the basket evolution can be well approximated by a GBM with a constant volatility. In practice, we can take $\sigma^*$ obtained at $t = T/2$, or compute it at all partition points $t$ and take the average. To calculate the shift parameter $\tau(t)$, note that

$$\text{Var}(B(t)) = M_2(t) - M_1^2 = (B(0) - \tau(t))^2(e^{\sigma^* t} - 1)$$

from which it follows that

$$\tau(t) = \frac{M_2(t) - M_1^2}{\left(e^{\sigma^* t} - 1\right)^2}$$

where $y_{\text{B}} > 1$ is the root of the third degree polynomial found above. We do not need to assume that the shift parameter $\tau(t)$ is constant, as in practice we would calculate the value of the shift at each relevant time instant $t$ and shift the basket by the correct parameter $\tau(t)$. However, in practice the shift parameter $\tau$ also turns out to be nearly constant, which is illustrated in Exhibit 4 for Baskets 2 and 3.

Now we can build the binomial tree for our basket $B(t)$ by the following algorithm.

- Build the binomial tree for the value $B'(t)$, which follows a GBM: at each time step on the tree, the value $B'$ moves up to $B'u$ with probability $q$ or down to $B'd$ with probability $1-q$, where

  $$u = \exp\left[\sigma^* \sqrt{\Delta t}\right]$$
  $$d = \exp\left[-\sigma^* \sqrt{\Delta t}\right]$$
  $$q = \frac{e^{\mu^* \Delta t} - d}{u - d}$$

  where $\mu^* = 0$ for futures baskets, $\mu^* = r$ for non-dividend paying stocks, and $\mu^* = r - \frac{q}{1-q}$ for baskets of stocks paying an average dividend yield $q$.

- Translate the obtained binomial tree for $B'$ into the binomial tree for $B$ using equations

  $$B(t) = B'(t) + \tau(t)$$
  if the approximating distribution is shifted lognormal,
  $$B(t) = -B'(t) - \tau(t)$$
  if the approximating distribution is negative shifted lognormal,
This tree can be now used for valuing an American option on \( B(t) \), computing an option’s deltas at each tree node and deciding on early exercise.

**NUMERICAL STUDY**

We apply our binomial tree model to several basket options. We cover a wide range of baskets and spreads, in terms of the number of assets and correlations. We compare the obtained prices to those obtained by other existing models, wherever possible.

Basket 1: \( F_0 = [50;50] \); \( \sigma = [0.3;0.2] \); \( \rho_{1,2} = 0.6 \); \( a = [0.3;0.7] \); \( X = 50 \).

Basket 2: \( F_0 = [100;120] \); \( \sigma = [0.2;0.3] \); \( \rho_{1,2} = 0.9 \); \( a = [-1;1] \); \( X = 30 \).

Basket 3: \( F_0 = [150;100] \); \( \sigma = [0.3;0.2] \); \( \rho_{1,2} = 0.7 \); \( a = [-1;1] \); \( X = -40 \).

Basket 4: \( F_0 = [95;90;105] \); \( \sigma = [0.2;0.3;0.25] \); \( \rho_{1,2} = \rho_{1,3} = 0.9 \); \( \rho_{2,3} = 0.8 \); \( a = [1;0.8;-0.5] \); \( X = -30 \).

Basket 5: \( F_0 = [100;90;95] \); \( \sigma = [0.25;0.3;0.2] \); \( \rho_{1,2} = \rho_{1,3} = 0.9 \); \( \rho_{2,3} = 0.8 \); \( a = [0.6;0.8;-1] \); \( X = 40 \).

Basket 6: \( F_{0,i} = 50, i = 1, \ldots, 5 \); \( \sigma_i = 0.25 \); \( \rho_{i,j} = 0.9 \); \( a_i = 0.2 \); \( X = 55 \).

Basket 7: \( F_{0,i} = 50, i = 1, \ldots, 5 \); \( \sigma_i = 0.25 \); \( \rho_{i,j} = 0.2 \); \( a_i = 0.2 \); \( X = 55 \).

Basket 8: \( F_0 = [100;120] \); \( \sigma = [0.2;0.3] \); \( \rho_{1,2} = 0.2 \); \( a = [-1;1] \); \( X = 30 \).

Note that Baskets 6 and 7 are similar, the difference is only in the correlation coefficient (0.9 vs. 0.2), and the same holds for Baskets 2 and 8 (both are spreads). We assume the risk-free interest rate is 5% per annum and the time to expiry (\( T \)) is one year.

The American call and put basket option prices obtained by the single binomial tree method (SBT), proposed here, are given in Columns 5 and 8 in Exhibit 5. The number of time steps in this single binomial tree is 250 (i.e., \( \Delta t = 1/250 \) (year) = 1 trading day). The first thing to mention is that the computational effort for our method is minimal: For all baskets, option prices are obtained nearly instantaneously, on a reasonable PC (we used the PC with Intel Core 2 dual processor and Matlab for implementation). The maximum computational time was 17 seconds for a five-asset basket.

As Baskets 1, 2, 3, and 8 consist of two assets, we can also compare the prices obtained by our method to those obtained by the two-dimensional binomial tree, or binomial pyramid (BP) method. Here, however, we divide the time to the option’s maturity (one year) into 150 time steps, not into 250 as in the single binomial tree method, to decrease the computational time (which certainly comes at the cost of accuracy). The computing time is in the range of 2.5–3 minutes (150–180 seconds), which is much more than for our single binomial tree method. As Basket 1 has only positive weights, the implied binomial tree approach of Rubinstein [1994] is also applicable, so here we can compare the prices obtained by our method to the implied binomial tree prices as well. As baskets 4, 5, 6, and 7 contain three or more assets, the multidimensional binomial tree method (binomial pyramid) can (in principle) deal with options on three or more assets, but this leads to a very large and computationally untractable binomial tree. We could not include the results of the BP method for three-asset baskets as we did not have enough memory and/or computing power to obtain these results within reasonable computing time.

Exhibit 5 shows that the SBT approach performs remarkably well in terms of option pricing (we see that for those baskets where comparison is possible). The prices obtained by our method are very close (the same...
up to the second decimal) to those obtained by the binomial pyramid method for baskets with positive and negative weights and for baskets with positive and negative skewness. In the case of a basket with positive weights, its performance is also comparable to the implied binomial tree method. However, our method is much faster than the binomial pyramid and more universal than the IBT, which can deal only with long-position baskets.

Next, we investigate how well our method performs with a limited number of time steps on the tree (e.g., 250) compared to a larger number of steps (1,000 or more), which in theory should provide much more accurate option prices. In our case this is possible, since the method builds only one simple binomial tree, which is very fast even for a very large number of time steps (for example, to evaluate a basket option with 2,500 time steps, we need just over one minute of computing time). So for all our test baskets, we compare the option prices obtained with 250; 1,000; 2,500; and 5,000 time steps. The results are shown in Exhibit 6.

It is clear from Exhibit 6 that already with a moderate number of steps (250, which corresponds to daily nodes on the tree), option prices are very close (within 0.1%) to more exact values, obtained with 5,000 time steps.

The differences between the true density and the generalized lognormal approximation are likely to be greatest in the tails of the distribution, which in turn would most affect far in- and out-of-the-money options. To see whether this is indeed the case, we also applied our method (and compared it to an arguably more precise binomial pyramid method) to options on Baskets 2 and 3 with strikes far in and out of the money. Exhibit 7 presents the results.

Exhibit 7 shows that the method does not suffer from inaccuracy in the tails: For all strikes, option prices obtained by the simple two-dimensional binomial tree are very close (differences are only in second decimals) to those obtained by the binomial pyramid, which should give a “true” representation of the tail probabilities of the basket. We did not consider strikes that are even further in or out of the money for two reasons: First, these are not representative of strikes actually traded in most option markets, and second, the prices of further out-of-the-money options are so close to zero that computational error becomes more significant than the model’s error.

Next, we investigate the delta-hedging performance of our method on the example of Baskets 1–5. For that, we generate many (10,000) paths of individual asset

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**Exhibit 5**

**American Basket Option Prices**

| Basket | Approx. Distr. | American Call | American Put |
|--------|----------------|---------------|--------------|
|        |                | BP            | IBT          | SBT          | tau | sigma |
| Basket 1 | shifted        | 3.967 3.970   | 3.966 3.966  | 3.968 3.958 | 3.966 | 1.069 | 0.212 |
| Basket 2 | shifted        | 4.380 4.368   | 4.368 14.078 | 14.068 35.717 | 0.360 |
| Basket 3 | neg. shifted   | 8.234 8.239   | 17.921 17.929 | 9.767    | -59.025 | 0.314 |
| Basket 4 | neg. shifted   | - - 7.680     | - - 7.195    | - - 31.966 | 0.15 |
| Basket 5 | shifted        | - - 6.861     | - - 9.767    | - - 50.090 | 0.200 |
| Basket 6 | shifted        | - - 2.611     | - - 7.459    | - - 1.277  | 0.236 |
| Basket 7 | shifted        | - - 1.125     | - - 5.981    | - - 2.612  | 0.153 |
| Basket 8 | shifted        | - - 10.521    | 20.092 20.208 | -167.155 | 0.202 |

**Exhibit 6**

**American Basket Option Prices, Increasing Number of Time Steps**

| Basket | 250 | 1000 | 2500 | 5000 | 250 | 1000 | 2500 | 5000 |
|--------|-----|------|------|------|-----|------|------|------|
| Basket 1 | 3.9656 | 3.9657 | 3.9653 | 3.9651 | 3.9653 | 3.9657 | 3.9653 | 3.9651 |
| Basket 2 | 4.3677 | 4.3723 | 4.3725 | 4.3734 | 14.0676 | 14.0714 | 14.0709 | 14.0717 |
| Basket 3 | 8.2387 | 8.2377 | 8.2357 | 8.2352 | 17.9288 | 17.9274 | 17.9254 | 17.9250 |
| Basket 4 | 7.6800 | 7.6718 | 7.6727 | 7.6724 | 7.1954 | 7.1876 | 7.1885 | 7.1882 |
| Basket 5 | 6.8606 | 6.8557 | 6.8549 | 6.8546 | 9.7666 | 9.7611 | 9.7603 | 9.7599 |
| Basket 6 | 2.6108 | 2.6110 | 2.6104 | 2.6102 | 7.4587 | 7.4584 | 7.4578 | 7.4576 |
| Basket 7 | 1.1246 | 1.1244 | 1.1241 | 1.1239 | 5.9812 | 5.9808 | 5.9805 | 5.9803 |
| Basket 8 | 10.5206 | 10.5189 | 10.5147 | 10.5160 | 20.2084 | 20.2063 | 20.2021 | 20.2034 |
prices (which follow correlated GBMs) of the length of one year, and for each path, we daily delta hedge an American option on the basket. The deltas with respect to the basket value are computed in a standard way and the deltas with respect to each asset are simply the products of the basket delta and the corresponding asset’s weight in the basket. We then collect hedging errors (equal to the difference between the option price and the hedging cost) and report their average characteristics in Exhibit 8.

For all baskets, except Basket 1, the absolute values of relative hedge errors are in the range of 3%–13% (negative hedge error means we are actually making money by delta-hedging an option), which is an acceptable range in practice for daily hedging. The only exception is Basket 1, where the hedge error is around 30% of the option price. However, if we compare this with the hedge error obtained for the single-asset American option with the same volatility as the Basket 1, we see that this hedging error is the same as for the basket American option (bottom line in Exhibit 8). So, in this case it is clear that the binomial method is not very accurate, regardless of whether we hedge a basket or a regular American option.

**CONCLUSIONS AND FUTURE WORK**

In this article, we have presented a simple, very fast, and computationally efficient method for valuing path-dependent options on baskets containing both short and long positions, such as spreads. Ours is essentially a binomial tree method, whereby we build a single two-dimensional binomial tree for the basket evolution. The method is based on the GBM approximation of the basket price process, shifted along the x-axis and potentially reflected over the y-axis. This allows us to deal with negative values and negative skewness of the basket distribution.

The main attraction of our method is that it reduces the high dimensionality of the full binomial tree that is normally needed for this problem; consequently, it is extremely fast and easy to implement. The numerical study shows that the method performs very well in terms of option pricing (also for far from ATM strikes) and delta-hedging: option prices obtained by our method coincide up to the second decimal with the prices obtained by the full binomial tree, which is extremely time consuming. The proposed method is very general in that it
can deal with any basket (potentially with very many assets), without compromising computational time and rigor. Any combination of basket weights is possible (positive as well as negative), and the method can value and delta hedge any path-dependent basket option.

Our continuing work in this area involves applying the method to real-world data for American basket and index options and including jumps into the individual assets’ dynamics, which would accommodate spark spread options or options on any other assets prone to price jumps.

ENDNOTES

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1Most exchange-traded commodity basket and spread options are written on baskets of futures; basket and spread options on spot commodities are rare and traded only over the counter.

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