ON LINEAR RELATIONS FOR DIRICHLET SERIES FORMED BY RECURSIVE SEQUENCES OF SECOND ORDER

CARSTEN ELSNER and NICLAS TECHNAU

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Abstract

Let $F_n$ and $L_n$ be the Fibonacci and Lucas numbers, respectively. Four corresponding zeta functions in $s$ are defined by

$$\zeta_F(s) := \sum_{n=1}^\infty \frac{1}{F_n^s}, \quad \zeta^{*}_F(s) := \sum_{n=1}^\infty \frac{(-1)^{n+1}}{F_n^s}, \quad \zeta_L(s) := \sum_{n=1}^\infty \frac{1}{L_n^s}, \quad \zeta^{*}_L(s) := \sum_{n=1}^\infty \frac{(-1)^{n+1}}{L_n^s}.$$

As a consequence of Nesterenko’s proof of the algebraic independence of the three Ramanujan functions $R(\rho)$, $Q(\rho)$, and $P(\rho)$ for any algebraic number $\rho$ with $0 < \rho < 1$, the algebraic independence or dependence of various sets of these numbers is already known for positive even integers $s$. In this paper, we investigate linear forms in the above zeta functions and determine the dimension of linear spaces spanned by such linear forms. In particular, it is established that for any positive integer $m$ the solutions of

$$\sum_{s=1}^m (t_s \zeta_F(2s) + u_s \zeta^{*}_F(2s) + v_s \zeta_L(2s) + w_s \zeta^{*}_L(2s)) = 0$$

with $t_s, u_s, v_s, w_s \in \mathbb{Q}$ ($1 \leq s \leq m$) form a $\mathbb{Q}$-vector space of dimension $m$. This proves a conjecture from the Ph.D. thesis of Stein, who, in 2012, was inspired by the relation $-2 \zeta_F(2) + \zeta^{*}_F(2) + 5 \zeta_L(2) = 0$. All the results are also true for zeta functions in $2s$, where the Fibonacci and Lucas numbers are replaced by numbers from sequences satisfying a second-order recurrence formula.

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1. Introduction and statement of the results

There are many results on series formed by powers of such integers which satisfy a linear recurrence relation of order two. By studying the algebraic character of sets of such $q$-series, surprising identities were found. For instance, in the case of Fibonacci numbers $F_n$, let

$$x := \sum_{n=1}^\infty \frac{1}{F_n^2}, \quad y := \sum_{n=1}^\infty \frac{1}{F_n^4}, \quad z := \sum_{n=1}^\infty \frac{1}{F_n^6}.$$

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Then we have the following nonlinear relation between four series (cf. [5]), namely
\[ \sum_{n=1}^{\infty} \frac{1}{F_n^8} = \frac{15y}{14} + \frac{1}{378(4x + 5)^2} \left( 256x^6 - 3456x^5 + 2880x^4 + 1792x^3z - 11100x^3 
+ 20160x^2z - 10125x^2 + 7560xz + 3136z^2 - 1050z \right). \]

Now, in order to look at the situation from a more conceptual point of view, let \( \alpha, \beta \) be complex algebraic numbers with \( |\beta| < 1 \) and \( \alpha \beta = -1 \). We define
\[ U_n := \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad V_n := \alpha^n + \beta^n \]
for \( n \geq 0 \). Both sequences, \( U_n \) and \( V_n \), satisfy the second-order recurrence formula
\[ X_{n+2} = (\alpha + \beta)X_{n+1} + X_n \quad (n \geq 0). \]

For \( \beta = (1 - \sqrt{5})/2 \), we get the Fibonacci numbers \( U_n = F_n \) and the Lucas numbers \( V_n = L_n = F_{n-1} + F_{n+1} \). Moreover, for positive integers \( s \), we introduce the series
\[ \Phi_{2s} := (\alpha - \beta)^{-2s} \sum_{n=1}^{\infty} \frac{1}{U_{2s}^n}, \quad \Phi_{2s}^* := (\alpha - \beta)^{-2s} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{U_{2s}^n}, \]
\[ \Psi_{2s} := \sum_{n=1}^{\infty} \frac{1}{V_{2s}^n}, \quad \Psi_{2s}^* := \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{V_{2s}^n}. \]

In this paper we focus on linear forms with rational coefficients in
\[ \{ \Phi_{2s}, \Phi_{2s}^*, \Psi_{2s}, \Psi_{2s}^* : s = 1, \ldots, m \} \]
and prove a conjecture of M. Stein on the dimension of the kernel of such forms; see (1-6) and Conjecture 1.1 at the end of this introductory section.

For a survey on irrationality, transcendence, and algebraic independence results for series involving reciprocal Fibonacci and Lucas numbers, we refer the reader to [9, Sections 1.1–1.3]. Here, we present an outline devoted to the most important interim results which finally led to the problem treated in this paper. At the beginning in 1989, André-Jeannin [1] proved the irrationality of the series
\[ \sum_{n=1}^{\infty} \frac{1}{F_n}, \quad \sum_{n=1}^{\infty} (-1)^n \frac{1}{F_n}, \quad \sum_{n=1}^{\infty} \frac{1}{L_n}, \quad \sum_{n=1}^{\infty} (-1)^n \frac{1}{L_n}, \]
where the underlying idea was inspired by Apéry’s proof of the irrationality of \( \zeta(3) \). Eight years later, Duverney et al. [4] succeeded in proving the transcendence of the numbers
\[ \sum_{n=1}^{\infty} \frac{1}{F_{2s}}, \quad \sum_{n=1}^{\infty} \frac{1}{L_{2s}}, \quad \sum_{n=1}^{\infty} \frac{1}{F_{2n-1}}, \quad \sum_{n=1}^{\infty} \frac{1}{L_{2n}} \quad (1-1) \]
for any positive integer \( s \). These results are derived using Nesterenko’s theorem on Ramanujan functions [8] as follows. Let \( K \) and \( E \) denote the complete elliptic integrals.
of the first and second kinds, respectively, with the modulus \( k \in \mathbb{C} \setminus \{0, \pm 1\} \), defined by
\[
K = K(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2 t^2)}}, \quad E = E(k) = \int_0^1 \sqrt{\frac{1-k^2 t^2}{1-t^2}} \, dt. \tag{1-2}
\]
Their relationships to the well-known Ramanujan functions
\[
P(z) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) z^n,
\]
\[
Q(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) z^n,
\]
\[
R(z) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) z^n,
\]
with
\[
\sigma_r(n) = \sum_{d|n} d^r \quad (r = 1, 2, \ldots)
\]
are given by
\[
P(q^2) = \left(\frac{2K}{\pi}\right)^2 \left(\frac{3E}{K} - 2 + k^2\right),
\]
\[
Q(q^2) = \left(\frac{2K}{\pi}\right)^4 (1 - k^2 + k^4),
\]
\[
R(q^2) = \left(\frac{2K}{\pi}\right)^6 \frac{1}{2} (1 + k^2)(1 - 2k^2)(2 - k^2),
\]
where \( K \) is linked to \( k \) via (1-2) and \( q = \exp(-\pi K(\sqrt{1-k^2}/K(k))) \). By Nesterenko’s theorem, for any algebraic number \( q \), the quantities \( P(q^2), Q(q^2), \) and \( R(q^2) \) are algebraically independent over \( \mathbb{Q} \) and so are the quantities \( K/\pi, E/\pi, \) and \( k \).

Now the series from (1-1) can be written as series of hyperbolic functions. Applying some identities from Zucker [10], one can express the latter series in terms of so-called \( q \)-series defined by
\[
A_{2j+1}(q) = \sum_{n=1}^{\infty} \frac{n^{2j+1} q^{2n}}{1-q^{2n}}, \quad B_{2j+1}(q) = \sum_{n=1}^{\infty} \frac{(-1)^n n^{2j+1} q^{2n}}{1-q^{2n}},
\]
\[
C_{2j+1}(q) = \sum_{n=1}^{\infty} \frac{n^{2j+1} q^n}{1-q^{2n}}, \quad D_{2j+1}(q) = \sum_{n=1}^{\infty} \frac{(-1)^n n^{2j+1} q^n}{1-q^{2n}},
\]
where \( q \) is a real algebraic number. Finally, these \( q \)-series can be expressed as polynomials in \( K/\pi, E/\pi, \) and \( k \), so that finally the transcendence of the series from (1-1) follows from the algebraic independence of \( K/\pi, E/\pi, \) and \( k \) over \( \mathbb{Q} \). For more details, see [9, Section 1.4].
This was the state when the first-named author of this paper started his joint work with Shimomura and Shiokawa on this subject. They extended it to problems on algebraic independence and dependence over \( \mathbb{Q} \). At the beginning, these authors only considered the Fibonacci and Lucas zeta functions, defined by

\[
\zeta_F(s) := \sum_{n=1}^{\infty} \frac{1}{F_n^s}, \quad \zeta_F'(s) := \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{F_n^s}, \quad \zeta_L(s) := \sum_{n=1}^{\infty} \frac{1}{L_n^s}, \quad \zeta_L'(s) := \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{L_n^s}.
\]

It was shown in [5] that the three numbers in each of the sets

\[
\{ \zeta_F(2), \zeta_F(4), \zeta_F(6) \}, \quad \{ \zeta_F'(2), \zeta_F'(4), \zeta_F'(6) \},
\]

\[
\{ \zeta_L(2), \zeta_L(4), \zeta_L(6) \}, \quad \{ \zeta_L'(2), \zeta_L'(4), \zeta_L'(6) \}
\]

are algebraically independent over \( \mathbb{Q} \) and that for any integer \( s \geq 4 \) each of the series \( \zeta_F(2s), \zeta_F'(2s), \zeta_L(2s) \), and \( \zeta_L'(2s) \) can be expressed as rational functions in the three series of the same type for \( s = 2, 4, 6 \), that is, for \( s \geq 4 \),

\[
\zeta_F(2s) \in \mathbb{Q}(\zeta_F(2), \zeta_F(4), \zeta_F(6)), \quad \zeta_F'(2s) \in \mathbb{Q}(\zeta_F'(2), \zeta_F'(4), \zeta_F'(6)),
\]

\[
\zeta_L(2s) \in \mathbb{Q}(\zeta_L(2), \zeta_L(4), \zeta_L(6)), \quad \zeta_L'(2s) \in \mathbb{Q}(\zeta_L'(2), \zeta_L'(4), \zeta_L'(6)).
\]

For an explicit example, we refer to the identity for \( \zeta_F(8) \) given at the beginning of this section. A few years later the authors described in [6] all subsets of

\[
\Gamma := \{ \zeta_F(2), \zeta_F(4), \zeta_F(6), \zeta_F'(2), \zeta_F'(4), \zeta_F'(6), \zeta_L(2), \zeta_L(4), \zeta_L(6), \zeta_L'(2), \zeta_L'(4), \zeta_L'(6) \}
\]

with either algebraically dependent or algebraically independent numbers. They proved that every four numbers in \( \Gamma \) are algebraically dependent over \( \mathbb{Q} \), whereas every two distinct numbers are algebraically independent over \( \mathbb{Q} \). Moreover, there are 198 of the 220 three-element subsets of \( \Gamma \) each containing algebraically independent numbers over \( \mathbb{Q} \). For the remaining 22 three-element subsets of \( \Gamma \), explicit algebraic relations are given. A complete list of these relations is contained in the appendix of [9]. With regard to the main result of this paper, we cite the only linear relation among them,

\[
-2\Phi_2 + \Phi_2^* + \Psi_2^* = 0. \quad (1-3)
\]

Note that

\[
\zeta_F(2s) = 5^s \Phi_{2s}, \quad \zeta_F'(2s) = 5^s \Phi_{2s}^*,
\]

\[
\zeta_L(2s) = \Psi_{2s}, \quad \zeta_L'(2s) = \Psi_{2s}^*.
\]

A more general result for the Fibonacci zeta function at positive even integers is presented in [7]. The authors established an algebraic independence criterion based on the nonvanishing of the Jacobian determinant of a quadratic system of polynomials, which is the kernel of the proof that for positive integers \( s_1 < s_2 < s_3 \) the values \( \zeta_F(2s_1), \zeta_F(2s_2), \) and \( \zeta_F(2s_3) \) are algebraically independent over \( \mathbb{Q} \) if and only if at least one of the numbers \( s_j \) is even.

In his Ph.D. thesis, Stein generalized all these investigations to the set

\[
\Omega := \{ \Phi_{2s_1}, \Phi_{2s_2}^*, \Psi_{2s_3}, \Psi_{2s_4}^*, | s_1, s_2, s_3, s_4 \in \mathbb{N} \}.
\]
Stein’s main results specify all subsets of $\Omega$ which are algebraically independent over $\mathbb{Q}$, namely, [9, Theorems 5.1, 5.3 and 5.4]: in particular, it turns out that any four (or more) numbers in $\Omega$ are algebraically dependent over $\mathbb{Q}$, see [9, Theorem 5.4], and that there are precisely 22 classes of three-element subsets that are algebraically independent over $\mathbb{Q}$. Additionally, any two elements of $\Omega$ are algebraically independent over $\mathbb{Q}$. Although he has settled the problem of the algebraic independence or dependence for subsets of $\Omega$ completely, Stein posed the problem on the linear independence and dependence of $m$-subsets of $\Omega$ with $m > 4$ (cf. [9, Section 5.3]). He gave two partial answers. For his first result in this direction, he denotes for any positive integer $s$ by $W_{2s} \in \Omega$ one of the numbers $\Phi_{2s}, \Phi_{2s}^*, \Psi_{2s},$ or $\Psi_{2s}^*$.

**Theorem 1.1** [9, Theorem 5.5]. Let $1 \leq s_1 < s_2 < \cdots < s_m$ be $m$ positive integers for $m \in \mathbb{N}$. Then the numbers $W_{2s_1}, \ldots, W_{2s_m}$ are linearly independent over $\mathbb{Q}(E/\pi, k)$.

For the second result, a positive integer $s$ is fixed and the four numbers $\Phi_{2s}, \Phi_{2s}^*, \Psi_{2s},$ and $\Psi_{2s}^*$ are considered.

**Theorem 1.2** [9, Theorem 5.6]. For any $s \geq 2$, the four numbers $\Phi_{2s}, \Phi_{2s}^*, \Psi_{2s},$ and $\Psi_{2s}^*$ are linearly independent over $\mathbb{Q}$, that is, the linear equation

$$t_s \Phi_{2s} + u_s \Phi_{2s}^* + v_s \Psi_{2s} + w_s \Psi_{2s}^* = 0$$

has no nontrivial solution $t_s, u_s, v_s, w_s \in \mathbb{Q}$. For $s = 1$, the general solution of (1-4) is

$$-2u\Phi_2 + u\Phi_2^* + u\Psi_2^* = 0 \quad (u \in \mathbb{Q}).$$

(1-5)

By setting $u = 1$, Stein regained in (1-5) the result from (1-3).

In his endeavour to construct subsets of $\Omega$ with more than three linearly independent elements, Stein found two examples (cf. [9, page 83]),

$$(-2u + v)\Phi_2 + u\Phi_2^* + (u - v)\Psi_2^* - 7v\Phi_4 + 8v\Phi_4^* + v\Psi_4 = 0 \quad (u, v \in \mathbb{Q})$$

and

$$-6(u + w)\Phi_2 + 6u\Phi_2^* + 6w\Psi_2^* + 6(u - v - w)\Phi_4 + 6v\Phi_4^* + 6(u - w)\Psi_4$$

$$+ 32(8u - v - 8w)\Phi_6 - 31(8u - v - 8w)\Phi_6^* + (-8u + v + 8w)\Psi_6^* = 0$$

$$(u, v, w \in \mathbb{Q}).$$

Using a computer-algebra system, Stein found more examples of such linear identities for larger subsets of $\Omega$. We define $V_m$ to be the set of all $(t_1, \ldots, t_{4m}) \in \mathbb{Q}^{4m}$ satisfying

$$\sum_{s=1}^{m} (t_{4s-3} \Phi_{2s} + t_{4s-2} \Phi_{2s}^* + t_{4s-1} \Psi_{2s} + t_{4s} \Psi_{2s}^*) = 0.$$
Conjecture 1.1 [9, Conjecture 5.1]. One has \( \dim_{\mathbb{Q}} V_m = m \) for every positive integer \( m \). Moreover, we have for \((t_1, \ldots, t_{4m}) \in V_m\),
\[
t_{4s} = 0 \quad \text{for} \ 2 \mid s \quad \text{and} \quad t_{4s-1} = 0 \quad \text{for} \ 2 \nmid s.
\]
\( (1-7) \)

The goal of this paper is to prove this conjecture.

Theorem 1.3. Conjecture 1.1 is true.

For the proof of this theorem, we exploit many auxiliary results which are already provided by the theory sketched above for the transcendence and algebraic independence results of the numbers from \( \Omega \). All these tools are contained in [9], so that we can always refer the reader to this Ph.D. thesis. The following lines outline the key steps in our proof and thus the structure of the present paper.

Organization of the paper. The reasoning of our approach can be outlined as follows. Firstly, we make use of the known explicit formulae for a number \( \xi \in \{ \Phi_{2s}, \Phi^*_{2s}, \Psi_{2s}, \Psi^*_{2s} \} \). Indeed, \( \xi \) can be written as a nonzero, multivariate polynomial \( P(\xi; E/\pi, K/\pi, k) \) in the algebraically independent numbers \( E/\pi, K/\pi, k \) given by the complete elliptic integrals in (1-2). All of this is detailed in Sections 2 and 3. Thereafter, we observe that \( P(\xi; E/\pi, K/\pi, k) \) is essentially a linear form in (precisely) one of four auxiliary polynomials \( \Theta_j^+, \Lambda_j^+ \). Then, by degree considerations and by analysing the structure of \( P(\xi; E/\pi, K/\pi, k) \) for all possible values of \( \xi \), the problem of characterizing that \( t \in V_m \) can be translated to investigating the vector space of \((a, b, c, d) \in \mathbb{Q}^4\) satisfying
\[
a\Theta_j^+ + b\Theta_j^- + c\Lambda_j^+ + d\Lambda_j^- = 0,
\]
that is, to determine linear dependences between the auxiliary polynomials, which is the content of Section 4. In Section 5, we use the fact that \( \Theta_j^+, \Lambda_j^+ \) originate from the Laurent series of the Jacobi elliptic functions to show that determining the linear dependences of \( \Theta_j^+, \Lambda_j^+ \) can be reduced to proving an identity between the Jacobi elliptic functions. Said identity is seen, after some manipulations, to be equivalent to an analogue (cf. Lemma 5.3) of the well-known doubling formula
\[
\sin(2z) = 2 \sin(z) \cos(z), \quad z \in \mathbb{C},
\]
for the Jacobi elliptic function \( \sin \). After these intermediate steps are done, we can give the proof of Theorem 1.3 for all sufficiently large \( m \) and deal with the finitely many remaining cases in the appendix.

2. Notation and preliminaries

Let \( s \geq 1 \) and let \( \xi \) denote one of the numbers from the set \( \Omega_s := \{ \Phi_{2s}, \Phi^*_{2s}, \Psi_{2s}, \Psi^*_{2s} \} \). We start by defining the notation needed to rewrite \( \xi \) in a way suitable to our purposes and we shall briefly describe the maxim of the rewriting. The given algebraic number \( \beta \) defines the sequences \( U_n \) and \( V_n \) for \( \Phi_{2j}, \Phi^*_{2j}, \Psi_{2j}, \Psi^*_{2j} \). We proceed, as an intermediate part, to argue that \( \beta \) also fixes \( E, K, k \) (cf. (1-2)). To see this, let us first define the following \( \theta \) functions. For a complex number \( q \) such that \(|q| < 1\), we let
\[\theta_2(q) = 2 \sum_{\nu=0}^{\infty} q^{(\nu+1/2)^2}, \quad \theta_3(q) = 1 + 2 \sum_{\nu=1}^{\infty} q^{\nu^2}.\]

Taking \(q = \beta^2\), we have, cf. [2, (4.2.1) to (4.2.4)] and [10, (1), (2) and (5)], \(k := \theta_2^2(q)/\theta_3^2(q)\), and \(E\) from (1-2) is fixed since \(k\) is. Thus, \(E, K, k\) are fixed (cf. (1-2)).

Next, we need the Jacobi elliptic functions (cf. [3]). Let \(sn(z, k) = sn(z)\) be the sine amplitude function obtained by inverting the meromorphic map

\[z \rightarrow \int_0^z \frac{dt}{\sqrt{(1 - t^2)(1 - k^2 t^2)}}.\]

Then, using Glaisher’s notation, let

\[ns^2(z) := \frac{1}{sn^2(z)}, \quad nc^2(z) := \frac{1}{1 - sn^2(z)},\]
\[nd^2(z) := 1 - k^2 sn^2(z), \quad dn^2(z) := \frac{1}{1 - k^2 sn^2(z)}.\]

Moreover, we need the following power series expansions.

\[ns^2(z) - \frac{1}{z^2} - \frac{1}{3}(1 + k^2) = \sum_{j \geq 1} c_j(k) z^{2j},\]
\[(1 - k^2)(nd^2(z) - 1) = \sum_{j \geq 1} d_j(k) z^{2j},\]
\[(1 - k^2)(nc^2(z) - 1) = \sum_{j \geq 1} e_j(k) z^{2j},\]
\[dn^2(z) - 1 = \sum_{j \geq 1} f_j(k) z^{2j}.\]

The coefficients \(c_j, d_j, e_j, f_j\) are polynomials in the elliptic modulus \(k^2\). Each such polynomial can be computed recursively; the recurrence formulae are consequences of nonlinear differential equations satisfied by \(ns^2(z), nd^2(z), nc^2(z),\) and \(dn^2(z)\); cf. Lemmas 3.1 to 3.4 in [9, pages 17–20] or [5]. Moreover, the auxiliary polynomials

\[\Theta^{-1}_{j-1}(k) := c_{j-1}(k) - d_{j-1}(k) =: \sum_{i=0}^{j} \alpha_{ji} k^{2i}, \quad (2-1)\]
\[\Theta^{+1}_{j-1}(k) := c_{j-1}(k) + d_{j-1}(k) =: \sum_{i=0}^{j} \beta_{ji} k^{2i}, \quad (2-2)\]
\[\Lambda^{-1}_{j-1}(k) := e_{j-1}(k) - f_{j-1}(k) =: \sum_{i=0}^{j} \gamma_{ji} k^{2i}, \quad (2-3)\]
\[\Lambda^{+1}_{j-1}(k) := e_{j-1}(k) + f_{j-1}(k) =: \sum_{i=0}^{j} \delta_{ji} k^{2i}. \quad (2-4)\]
for \( j \geq 2 \) play a fundamental role in stating the explicit formulae for \( \xi \in \Omega_j \). We note that for \( s \geq 1 \) any \( \xi \in \{ \Phi_{2^s}, \Phi_{2^s}^*, \Psi_{2^s}, \Psi_{2^s}^* \} \) can be written as

\[
\xi = P^{(s)}(k, \frac{K}{\pi}, \frac{E}{\pi}) = P_I^{(s)}(k, \frac{K}{\pi}) + P_{II}^{(s)}(k, \frac{K}{\pi}, \frac{E}{\pi}), \tag{2-5}
\]

where the multivariate polynomials \( P_I^{(s)}, P_{II}^{(s)} \) are defined precisely in the subsequent section. The idea behind this decomposition of \( \xi \) is that the second summand \( P_{II}^{(s)} \) is a multivariate polynomial in \( k, K/\pi, E/\pi \), which gathers four kinds of terms, namely, all terms that are either rational or are rational multiples of

\[
\left( \frac{2K}{\pi} \right)^2, \quad \left( \frac{2K}{\pi} \right)^2(2k^2 - 1), \quad \left( \frac{2K}{\pi} \right)^2(6E/K - 5 + 4k^2), \quad \left( \frac{2K}{\pi} \right)^2(\frac{2E}{K} - 1);
\]

note that

\[
\left( \frac{2K}{\pi} \right)^2 \frac{E}{K} = \left( \frac{2K}{\pi} \right) \left( \frac{2E}{K} \right).
\]

The first summand \( P_I^{(s)} \) is also a multivariate polynomial, however in \( k, K/\pi \) only, and gathers all rational multiples of the higher powers

\[
\left( \frac{2K}{\pi} \right)^{2j} k^{2i},
\]

where \( j \geq 2 \) and \( i \in \{0, 1, \ldots, j\} \).

Finally, we need the rational numbers \( a_j \) and \( b_j \) defined by the series expansions of the circular functions cosec\(^2\)\( z \) and sec\(^2\)\( z \), respectively, given by

\[
cosec^2 z = \frac{1}{z^2} + \sum_{j \geq 0} a_j z^{2j}, \quad a_j = \frac{(-1)^j(2j + 1)2^{2j+2}B_{2j+2}}{(2j + 2)!} \quad (j \geq 0)
\]

and

\[
sec^2 z = \sum_{j \geq 0} b_j z^{2j}, \quad b_j = \frac{(-1)^j(2j + 1)2^{2j+2}(2^{2j+2} - 1)B_{2j+2}}{(2j + 2)!} \quad (j \geq 0).
\]

Here, \( B_2 = 1/6, B_4 = -1/30, B_6 = 1/42, \ldots \) denote the Bernoulli numbers.

The explicit formulae for \( \xi \) involve coefficients \( \sigma_i(s) \), which are the elementary symmetric functions of the \( s - 1 \) numbers \(-1^2, -2^2, \ldots, -(s - 1)^2\) for \( s \geq 2 \). They are defined by \( \sigma_0(s) := 1 \) and, for \( i \geq 1 \) and \( j = 1, \ldots, s - 1 \), by

\[
\sigma_i(s) := (-1)^i \sum_{1 \leq r_1 < \ldots < r_i \leq s - 1} r_1^2 \cdots r_i^2.
\]

Now we state the explicit formulae for \( \xi \) in terms of \( k, K/\pi \) and \( E/\pi \); cf. [9, Section 3.2].
\[
\Phi_{2s} = \frac{1}{(2s-1)!} \left[ -\frac{(s-1)!^2}{24} \left( 1 - \left( \frac{2K}{\pi} \right)^2 \left( \frac{6E}{K} - 5 + 4k^2 \right) \right) \right. \\
+ \sum_{j=1}^{s-1} \sigma_{s-j}(s) (-1)^{(2j)!} \left( a_j - \left( \frac{2K}{\pi} \right)^{2j+2} \Theta_j^+(k) \right) \left. \right] \quad \text{(for } s \text{ even),} \quad (2-6)
\]

\[
\Phi_{2s} = \frac{1}{(2s-1)!} \left[ -\frac{(s-1)!^2}{24} \left( 1 - \left( \frac{2K}{\pi} \right)^2 \left( 1 - 2k^2 \right) \right) \right. \\
+ \sum_{j=1}^{s-1} \sigma_{s-j}(s) (-1)^{(2j)!} \left( a_j - \left( \frac{2K}{\pi} \right)^{2j+2} \Theta_j^+(k) \right) \left. \right] \quad \text{(for } s \text{ odd),} \quad (2-7)
\]

\[
\Phi_{2s} = \frac{1}{(2s-1)!} \left[ -\frac{(s-1)!^2}{24} \left( 1 - \left( \frac{2K}{\pi} \right)^2 \left( \frac{6E}{K} - 5 + 4k^2 \right) \right) \right. \\
- \sum_{j=1}^{s-1} \sigma_{s-j}(s) (-1)^{(2j)!} \left( a_j - \left( \frac{2K}{\pi} \right)^{2j+2} \Theta_j^-(k) \right) \left. \right] \quad \text{(for } s \text{ even),} \quad (2-8)
\]

\[
\Phi_{2s} = \frac{1}{(2s-1)!} \left[ -\frac{(s-1)!^2}{24} \left( 1 - \left( \frac{2K}{\pi} \right)^2 \left( \frac{6E}{K} - 5 + 4k^2 \right) \right) \right. \\
- \sum_{j=1}^{s-1} \sigma_{s-j}(s) (-1)^{(2j)!} \left( a_j - \left( \frac{2K}{\pi} \right)^{2j+2} \Theta_j^-(k) \right) \left. \right] \quad \text{(for } s \text{ odd),} \quad (2-9)
\]

\[
\Psi_{2s} = \frac{1}{(2s-1)!} \left[ -\frac{(s-1)!^2}{8} \left( 1 + \left( \frac{2K}{\pi} \right)^2 \left( 1 - \frac{2E}{K} \right) \right) \right. \\
+ \sum_{j=1}^{s-1} \sigma_{s-j}(s) (-1)^{(2j)!} \left( b_j - \left( \frac{2K}{\pi} \right)^{2j+2} \Lambda_j^-(k) \right) \left. \right] \quad \text{(for } s \text{ even),} \quad (2-10)
\]

\[
\Psi_{2s} = \frac{1}{(2s-1)!} \left[ -\frac{(s-1)!^2}{8} \left( \left( \frac{2K}{\pi} \right)^2 - 1 \right) \right. \\
+ \sum_{j=1}^{s-1} \sigma_{s-j}(s) (-1)^{(2j)!} \left( \left( \frac{2K}{\pi} \right)^{2j+2} \Lambda_j^+(k) - b_j \right) \left. \right] \quad \text{(for } s \text{ odd),} \quad (2-11)
\]

\[
\Psi_{2s} = \frac{1}{(2s-1)!} \left[ -\frac{(s-1)!^2}{8} \left( \left( \frac{2K}{\pi} \right)^2 - 1 \right) \right. \\
+ \sum_{j=1}^{s-1} \sigma_{s-j}(s) (-1)^{(2j)!} \left( \left( \frac{2K}{\pi} \right)^{2j+2} \Lambda_j^+(k) - b_j \right) \left. \right] \quad \text{(for } s \text{ even),} \quad (2-12)
\]

\[
\Psi_{2s} = \frac{1}{(2s-1)!} \left[ -\frac{(s-1)!^2}{8} \left( 1 + \left( \frac{2K}{\pi} \right)^2 \left( 1 - \frac{2E}{K} \right) \right) \right. \\
+ \sum_{j=1}^{s-1} \sigma_{s-j}(s) (-1)^{(2j)!} \left( b_j - \left( \frac{2K}{\pi} \right)^{2j+2} \Lambda_j^-(k) \right) \left. \right] \quad \text{(for } s \text{ odd).} \quad (2-13)
\]
3. Definition of $P^{(\xi)}_I$ and $P^{(\xi)}_{II}$

Let us define $P^{(\xi)}_I$ first. Fix $\xi \in \{\Phi_{2s}, \Phi_{2s}^+, \Psi_{2s}, \Psi_{2s}^+\}$ for $s \geq 1$. Then

$$P^{(\xi)}_I(k, \frac{K}{\pi}) := \sum_{j=1}^{s-1} \left(\frac{2K}{\pi}\right)^{2j+2} w^{(s)}_j P^{(\xi)}_j(k), \quad (3-1)$$

where $w^{(s)}_j$ is the rational ‘weight factor’

$$w^{(s)}_j := (-1)^j \frac{\sigma_{s-j-1}(s)}{2^{2j+3}(2s-1)!} (2j)! \quad (j = 1, \ldots, s-1).$$

The polynomial $P^{(\xi)}_j \in \mathbb{Q}[k^2]$ is given by

$$P^{(\xi)}_j := \left(P^{(\Phi_{2s})}_j, P^{(\Phi_{2s}^+)}_j, P^{(\Psi_{2s})}_j, P^{(\Psi_{2s}^+)}_j\right) := \begin{cases} \left(-\Theta_j^-, \Theta_j^+, -\Lambda_j^-, \Lambda_j^+\right) & \text{for } s \text{ even,} \\ \left(-\Theta_j^+, \Theta_j^-, \Lambda_j^+, -\Lambda_j^-\right) & \text{for } s \text{ odd} \end{cases} \quad (3-2)$$

for $j \in \{1, \ldots, s-1\}$. Let us now define $P^{(\xi)}_{II}$. Letting

$$\hat{w}_s := \frac{(s-1)!^2}{24(2s-1)!}$$

denote another rational weight factor,

$$P^{(\xi)}_{II}(k, \frac{K}{\pi}, \frac{E}{\pi}) := \hat{w}_s \left(\frac{2K}{\pi}\right)^2 \tilde{P}^{(\xi)}_0(k, \frac{K}{\pi}, \frac{E}{\pi}) + R^{(\xi)},$$

where $P^{(\xi)}_0$ is the ‘initial’ polynomial given by

$$\left(P^{(\Phi_{2s})}_0, P^{(\Phi_{2s}^+)}_0, P^{(\Psi_{2s})}_0, P^{(\Psi_{2s}^+)}_0\right) := \begin{cases} \left(\frac{6E}{K} - 5 + 4k^2, 2k^2 - 1, \frac{6E}{K} - 3, -3\right) & \text{for } s \text{ even,} \\ \left(2k^2 - 1, \frac{6E}{K} - 5 + 4k^2, 3, 3 - \frac{6E}{K}\right) & \text{for } s \text{ odd} \end{cases} \quad (3-3)$$

Moreover, $R^{(\xi)}$ is the rational number

$$R^{(\Phi_{2s})} := (-1)^{s+1} \hat{w}_s + \sum_{j=1}^{s-1} a_j w^{(s)}_j =: -R^{(\Phi_{2s})}, \quad (3-4)$$

$$R^{(\Psi_{2s})} := -3\hat{w}_s + (-1)^s \sum_{j=1}^{s-1} b_j w^{(s)}_j =: -R^{(\Psi_{2s})}. \quad (3-5)$$

4. Connection to linear dependences of $\Theta_j^\pm, \Lambda_j^\pm$

In this section, we demonstrate how the linear dependences in (1-6) translate to linear dependences of the auxiliary polynomials $\Theta_j^\pm, \Lambda_j^\pm$. We define

$$x_1 := 1, \quad x_2 := \left(\frac{2K}{\pi}\right)^2,$$
\[ x_3 := \left( \frac{2K}{\pi} \right)^2 (2k^2 - 1), \quad x_4 := \left( \frac{2K}{\pi} \right)^2 \left( \frac{6E}{K} - 5 + 4k^2 \right) \]

and, for \( 2 \leq j \leq m \), we let

\[
\begin{align*}
x_{4j-3} & := \left( \frac{2K}{\pi} \right)^{2j} \cdot \begin{cases} 
\Theta_{j-1}^-(k) & \text{for } j \text{ even}, \\
\Theta_{j-1}^+(k) & \text{for } j \text{ odd},
\end{cases} \\n\frac{1}{K} & := \begin{cases} 
\Theta_{j-1}^+(k) & \text{for } j \text{ even}, \\
\Theta_{j-1}^-(k) & \text{for } j \text{ odd},
\end{cases}
\end{align*}
\]

\[
\begin{align*}
x_{4j-1} & := \left( \frac{2K}{\pi} \right)^{2j} \cdot \begin{cases} 
\Lambda_{j-1}^-(k) & \text{for } j \text{ even}, \\
\Lambda_{j-1}^+(k) & \text{for } j \text{ odd},
\end{cases} \\n\frac{1}{K} & := \begin{cases} 
\Lambda_{j-1}^+(k) & \text{for } j \text{ even}, \\
\Lambda_{j-1}^-(k) & \text{for } j \text{ odd}.
\end{cases}
\end{align*}
\]

For the ease of exposition, we put \( P_{l}^{(\xi)} := P_{l}^{(\xi)}(k, K/\pi) \) and \( P_{l}^{(\xi)} := P_{l}^{(\xi)}(k, K/\pi, E/\pi) \) for any \( \xi \in \{ \Phi_{2s}, \Phi_{2s}^*, \Psi_{2s}, \Psi_{2s}^* \} \). We restate (1-6) by collecting all terms involving \( x_1, \ldots, x_4 \). Note that \( P_{l}^{(\xi)} \) involves a factor of \( (2K/\pi) \) to a power of at least four in front of each term; since the variables \( x_1, \ldots, x_4 \) involve, if at all, a factor of \( (2K/\pi) \) to a power of at most two, we can disregard in (2-5) the terms \( P_{l}^{(\xi)} \) for the terms leading to \( x_1, \ldots, x_4 \). Therefore, we conclude via the algebraic independence of \( k, K/\pi, E/\pi \) over \( \mathbb{Q} \) that

\[
\sum_{i=1}^{m} (P_{l}^{(\Phi_{2s})} t_{4i-3} + P_{l}^{(\Phi_{2s}^*)} t_{4i-2} + P_{l}^{(\Psi_{2s})} t_{4i-1} + P_{l}^{(\Psi_{2s}^*)} t_{4i}) = 0. \tag{4-1}
\]

Moreover, we rearrange the remaining part of (1-6), namely,

\[
\sum_{i=2}^{m} (P_{l}^{(\Phi_{2s})} t_{4i-3} + P_{l}^{(\Phi_{2s}^*)} t_{4i-2} + P_{l}^{(\Psi_{2s})} t_{4i-1} + P_{l}^{(\Psi_{2s}^*)} t_{4i}),
\]

first to

\[
\sum_{i=2}^{m} \sum_{j=2}^{i} \left( \frac{2K}{\pi} \right)^{2j} w_{j-1}^{(i)} (P_{j-1}^{(\Phi_{2s})} (k) t_{4i-3} + P_{j-1}^{(\Phi_{2s}^*)} (k) t_{4i-2} + P_{j-1}^{(\Psi_{2s})} (k) t_{4i-1} + P_{j-1}^{(\Psi_{2s}^*)} (k) t_{4i}).
\]

Then, by interchanging the order of summation, we deduce that

\[
\sum_{j=2}^{m} \sum_{i=j}^{m} \left( \frac{2K}{\pi} \right)^{2j} w_{j-1}^{(i)} (P_{j-1}^{(\Phi_{2s})} (k) t_{4i-3} + P_{j-1}^{(\Phi_{2s}^*)} (k) t_{4i-2} + P_{j-1}^{(\Psi_{2s})} (k) t_{4i-1} + P_{j-1}^{(\Psi_{2s}^*)} (k) t_{4i}) = 0.
\]

Again, by applying the algebraic independence of \( k, K/\pi \) over \( \mathbb{Q} \), we infer, for \( 2 \leq j \leq m \), that

\[
\sum_{i=j}^{m} \left( \frac{2K}{\pi} \right)^{2j} w_{j-1}^{(i)} (P_{j-1}^{(\Phi_{2s})} (k) t_{4i-3} + P_{j-1}^{(\Phi_{2s}^*)} (k) t_{4i-2} + P_{j-1}^{(\Psi_{2s})} (k) t_{4i-1} + P_{j-1}^{(\Psi_{2s}^*)} (k) t_{4i}) = 0. \tag{4-2}
\]

Note that (4-1) and (4-2) can be regarded as linear forms in \( x_1, \ldots, x_{4m} \). Now we define the matrix \( A := (a_{i,j}) \in \mathbb{C}^{m \times (4m)} \) in which each entry \( a_{i,j} \) represents the factor in front of \( x_it_j \) from (4-1) and (4-2).
Remark 4.1. For \( i \geq 5 \), the \( i \)th row of \( A \) starts with \( 4(i - 4) \) many zeros. For future reference, we record that the \( 4 \times 4 \) submatrix \((a_{i,j})_{1 \leq i,j \leq 4}\) of rank three is given by

\[
\begin{pmatrix}
1 & -1 & -3 & 3 \\
0 & 0 & 3 & 0 \\
24 & 1 & 0 & 0 \\
0 & 1 & 0 & -1
\end{pmatrix}
\]  

(4-3)

The key issue for our proof of Conjecture 1.1 is to analyse the kernel of \( A \). For this purpose, we exhibit a ‘quasi-periodicity’ property of the rows of \( A \) in the subsequent lemma. For illustrating the underlying structures, the reader can find an example, where \( m = 3 \), in the final section.

Lemma 4.1. Let \( m \geq 3 \), consider in the matrix \( A \) four consecutive elements in the \( v \)th row, \( v = 1, \ldots, m - 2 \), as one vector, and let \( l = 1, \ldots, m - v - 1 \). Then there is a permutation \( \pi = \pi_{v,l} \) of the numbers \( \{4(v + l) + 1, \ldots, 4(v + l) + 4\} \) such that the vectors

\[
(a_{4+v,4+v+1}, a_{4+v,4+v+2}, a_{4+v,4+v+3}, a_{4+v,4+v+4})
\]

and

\[
(a_{4+v,\pi(4(v+l)+1)}, a_{4+v,\pi(4(v+l)+2)}, a_{4+v,\pi(4(v+l)+3)}, a_{4+v,\pi(4(v+l)+4)})
\]

are linearly dependent over \( \mathbb{Q} \).

Proof. Because \( 4 + v \geq 5 \), it suffices to consider only the \( a_{i,j} \) arising from (4-2). We observe that each \( \Theta_{r,s}^+, \Theta_{s,r}^-, \Lambda_{r,s}^+, \Lambda_{s,r}^- \) occurs exactly once in the term

\[
P^{(\Phi_{2i})}_{j-1}(k)t_{4i-3} + P^{(\Phi_{2j})}_{j-1}(k)t_{4i-2} + P^{(\Psi_{2i})}_{j-1}(k)t_{4i-1} + P^{(\Psi_{2j})}_{j-1}(k)t_{4i}
\]

for any \( j = 4 + v \) and any \( i = 4 + v, \ldots, m \). This defines the permutation \( \pi_{v,l} \) and, since the weight factor \( w_{j-1}^{(i)} \neq 0 \) from (4-2) does not depend on \( \xi \in \{\Phi_{2i}, \Phi_{2j}, \Psi_{2i}, \Psi_{2j}\} \), the lemma follows. \( \square \)

5. Linear dependences of \( \Theta_{j}^+, \Lambda_{j}^- \)

Due to Lemma 3.6 in the Ph.D. thesis [9] of Stein, there are explicit formulae for some coefficients of the auxiliary polynomials. We shall use these formulae.

Lemma 5.1. Let \( \Theta_{j}^+, \Lambda_{j}^- \) be as in (2-1) to (2-4). Then

\[
\alpha_{j,0} = a_{j-1} = \beta_{j,0}, \quad \gamma_{j,0} = b_{j-1} = \delta_{j,0}.
\]

By putting \( \kappa_{j-1} := (-1)^{j-1}2^{j-3}(2j - 2)! \), the coefficients of the quadratic terms are given by

\[
\alpha_{j,1} = \kappa_{j-1} - \frac{j}{2}a_{j-1} = \beta_{j,1} + 2\kappa_{j-1},
\]

\[
\gamma_{j,1} + 2\kappa_{j-1} = \kappa_{j-1} - \frac{j}{2}b_{j-1} = \delta_{j,1};
\]
the coefficients of the quartic terms, letting \( \hat{k}_{j-1} := j(4j - 7)/32 \), can be written as

\[
\alpha_{j,2} = \frac{k_{j-1}}{16}(7 - 8j - 2^{2j-1}) + \hat{k}_{j-1}a_{j-1},
\]
\[
\beta_{j,2} = \frac{k_{j-1}}{16}(-9 + 8j + 2^{2j-1}) + \hat{k}_{j-1}a_{j-1},
\]
\[
\gamma_{j,2} = \frac{k_{j-1}}{16}(-7 + 8j - 2^{2j-1}) + \hat{k}_{j-1}b_{j-1},
\]
\[
\delta_{j,2} = \frac{k_{j-1}}{16}(9 - 8j + 2^{2j-1}) + \hat{k}_{j-1}b_{j-1},
\]

and the coefficients at \( z^{2j} \) are given by

\[
\alpha_{j,j} = 2^{2j}a_{j-1}, \quad \beta_{j,j} = (2 - 2^{2j})a_{j-1}, \quad \gamma_{j,j} = \delta_{j,j} = 0.
\]

**Lemma 5.2.** For \( j \geq 1 \), the subspace

\[
S_j := \{ t := (t_1, \ldots, t_4) \in \mathbb{Q}^4 : \langle t, (\Theta_j^-, \Theta_j^+, -\Lambda_j^-, \Lambda_j^+) \rangle = 0 \}
\]

is spanned by \( v_j := (1 - 2^{2j+1}, 2^{2j+1}, 1, 0)^T \). In particular, for each even \( j \geq 2 \) the three quantities \( x_{4j-3}, x_{4j-2}, x_{4j-1} \), and \( x_{4j-3}, x_{4j-2}, x_{4j} \) for odd \( j \geq 3 \), respectively, are linearly dependent over \( \mathbb{Q} \).

**Proof.** The argument splits into two parts. Firstly, we show that any element of \( S_j \) is necessarily a rational multiple of \( v_j \) and then that, indeed, any such multiple is an element of \( S_j \).

(i) If \( t \in S_j \), then \( t \) is in the kernel of the matrix

\[
\Xi := \begin{pmatrix}
-\alpha_{j+1,0} & \beta_{j+1,0} & -\gamma_{j+1,0} & \delta_{j+1,0} \\
-\alpha_{j+1,1} & \beta_{j+1,1} & -\gamma_{j+1,1} & \delta_{j+1,1} \\
-\alpha_{j+1,2} & \beta_{j+1,2} & -\gamma_{j+1,2} & \delta_{j+1,2} \\
-\alpha_{j+1,j+1} & \beta_{j+1,j+1} & -\gamma_{j+1,j+1} & \delta_{j+1,j+1}
\end{pmatrix}.
\]

Using the explicit expressions provided by Lemma 5.1 for the matrix entries, a calculation (performed by Mathematica) yields that \( \Xi \) can be transformed to the following row echelon form:

\[
\begin{pmatrix}
1 & 0 & 2^{2j+1} - 1 & 0 \\
0 & 1 & -2^{2j+1} & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

Therefore, the kernel of \( \Xi \) is given by \( \mathbb{Q}(1 - 2^{2j+1}, 2^{2j+1}, 1, 0) \).

(ii) Let \( t := t(1 - 2^{2j+1}, 2^{2j+1}, 1, 0) \) for \( t \in \mathbb{Q} \). Now we proceed to show that the Euclidean scalar product \( \langle t, (\Theta_j^-, \Theta_j^+, -\Lambda_j^-, \Lambda_j^+) \rangle \) is indeed the zero polynomial in \( k \). This is equivalent to demonstrating that

\[
0 = (2^{2j+1} - 1)\Theta_j^- + 2^{2j+1}\Theta_j^+ - \Lambda_j^- = (2^{2j+2} - 1)c_j + d_j - e_j + f_j \quad (5-1)
\]
Substituting (5-6) and (5-7) into (5-3) and dividing the resulting identity by and 

They imply the formulae

From (121.00) on \cite[page 20]{3}, we recall the basic identities

We start with an algebraic identity which holds for all complex numbers \(\alpha\) and \(\beta\),

For a simpler notation, we omit throughout the proof of Lemma 5.3 the parentheses

\[ (k^2 - (1 - \text{dn}^2 z)^2) = k^4 \text{dn}^4 z + (1 - \text{dn}^2 z)(k^2 - 1 + \text{dn}^2 z)((k^2 - 1) - \text{dn}^4 z + (2 + k^2) \text{dn}^2 z). \]

(5-3)

From (121.00) on \cite[page 20]{3}, we recall the basic identities

They imply the formulae

and

Substituting (5-6) and (5-7) into (5-3) and dividing the resulting identity by \(k^4\),

\[ (1 - k^2 \text{sn}^4 z)^2 = \text{dn}^4 z + (k^2 - 1)\text{sn}^2 z \text{cn}^2 z - \text{sn}^2 z \text{cn}^2 z \text{dn}^4 z + (2 + k^2)\text{sn}^2 z \text{cn}^2 z \text{dn}^2 z. \]
Using again (5-4) and (5-5), we continue our calculations by

\[
(1 - k^2 \text{sn}^4 z)^2 = (\text{sn}^2 z + \text{cn}^2 z) \text{dn}^2 z + (k^2 - 1) \text{sn}^2 z \text{cn}^2 z - \text{sn}^2 z \text{cn}^2 z \text{dn}^2 z
\]

\[
- k^2 \text{sn}^2 z \text{dn}^2 z + (2 + k^2) \text{sn}^2 z \text{cn}^2 z \text{dn}^2 z
\]

\[
= (1 - k^2) \text{sn}^2 z (\text{dn}^2 z - \text{cn}^2 z) + \text{cn}^2 z \text{dn}^2 z (1 - \text{sn}^2 z \text{dn}^2 z)
\]

\[
+ (2 + k^2) \text{sn}^2 z \text{cn}^2 z \text{dn}^2 z.
\]

Since \( z \) is no integer multiple of \( K \), the number \( \text{sn}^2 z \text{cn}^2 z \text{dn}^2 z \) does not vanish. Thus, we obtain the identity

\[
\frac{(1 - k^2 \text{sn}^4 z)^2}{\text{sn}^2 z \text{cn}^2 z \text{dn}^2 z} = (1 - k^2) \left( \frac{1}{\text{cn}^2 z} - \frac{1}{\text{dn}^2 z} \right) + \left( \frac{1}{\text{sn}^2 z} - \text{dn}^2 z \right) + (2 + k^2).
\]

The left-hand side equals \( 4 \text{sn}^2 (2z) \), which follows from (124.01) on [3, page 24]. Hence, the identity of the lemma is proven for \( k \neq 0 \).

**Case 2.** \( k = 0 \).

We have \( \text{sn} z = \sin z \), \( \text{cn} z = \cos z \) and \( \text{dn} z \equiv 1 \) (cf. (122.08) on [3, page 21]). Therefore, in the case \( k = 0 \), the identity in question reduces to the trigonometric formula

\[
\frac{4}{\sin^2 (2z)} = \frac{1}{\cos^2 z} + \frac{1}{\sin^2 z}.
\]

Obviously this is a consequence of \( \sin(2z) = 2 \sin z \cos z \), provided that \( z \) is no integer multiple of \( K = K(0) = \pi/2 \).

**6. Proof of Theorem 1.3**

**Proof.** We introduce the \( 4 \times 4 \) matrices

\[
R^{(2s)} := \begin{pmatrix}
R^{(\Phi_{2s})} & R^{(\Phi'_{2s})} & R^{(\Psi_{2s})} & R^{(\Psi'_{2s})} \\
0 & 0 & 0 & -3\hat{w}_s \\
0 & \hat{w}_s & -2\hat{w}_s & 0 \\
\hat{w}_s & 0 & \hat{w}_s & 0
\end{pmatrix}
\]

if \( s \) is even and

\[
R^{(2s)} := \begin{pmatrix}
R^{(\Phi_{2s})} & R^{(\Phi'_{2s})} & R^{(\Psi_{2s})} & R^{(\Psi'_{2s})} \\
0 & 0 & 3\hat{w}_s & 0 \\
0 & \hat{w}_s & 0 & 2\hat{w}_s \\
0 & \hat{w}_s & 0 & -\hat{w}_s
\end{pmatrix}
\]

for odd \( s \). Moreover, for \( j = 1, \ldots, m - 1 \), we recall that

\[
P_{j}^{(s)} := \begin{cases}
(\Theta_j^-, \Theta_j^+, -\Lambda_j^-, \Lambda_j^+) & \text{for } s \text{ even}, \\
(\Theta_j^+, \Theta_j^-, \Lambda_j^+, -\Lambda_j^-) & \text{for } s \text{ odd}.
\end{cases}
\]
The subsequent \((m + 3) \times (4m)\) matrix of central importance in our proof is

\[
A^{(m)}_0 := \begin{pmatrix}
R^{(2)} & R^{(4)} & R^{(6)} & R^{(8)} & \ldots & R^{(2m)} \\
R^{(2)} & P^{(2)}_1 & P^{(3)}_1 & P^{(4)}_1 & \ldots & P^{(m)}_1 \\
& w^{(2)}_1 & P^{(3)}_1 & P^{(4)}_1 & \ldots & P^{(m)}_1 \\
& & \mathbf{0} & P^{(3)}_2 & P^{(4)}_2 & \ldots & P^{(m)}_2 \\
& & & \mathbf{0} & P^{(3)}_3 & \ldots & P^{(m)}_3 \\
& & & & \ddots & \ddots & \ddots \\
& & & & & \mathbf{0} & P^{(m)}_m \\
& & & & & & w^{(m)}_{m-1} \mathbf{1}
\end{pmatrix}
\]

In the sequel, we transform \(A^{(m)}_0\) step by step by elementary matrix operations. After performing the \(i\)th step \((i \geq 1)\) on \(A^{(m)}_{i-1}\), we denote the resulting \((m + 3) \times (4m)\) matrix by \(A^{(m)}_i\).

**Step 1.** Using elementary column operations, we transform \(R^{(2)}\) into a row echelon form,

\[
\hat{R}^{(2)} := \begin{pmatrix}
0 & -1/8 & 1/24 & -1/24 \\
0 & 1/8 & 0 & 0 \\
0 & 0 & 1/24 & 0 \\
0 & 0 & 0 & 1/24
\end{pmatrix}
\]

**Step \(j\) (for \(2 \leq j \leq m\)).** By elementary column operations, Lemma 4.1 allows us to transform the \(j\)th row of the matrix \(A^{(m)}_1\) into the row

\[
(0, \ldots, 0, w^{(j)}_{j-1}, P^{(j)}_{j-1}, 0, \ldots, 0).
\]

After the \(m\)th step, we achieve the \((m + 3) \times (4m)\) matrix

\[
A^{(m)} := \begin{pmatrix}
\hat{R}^{(2)} & * & * & * & \ldots & * \\
& w^{(2)}_1 P^{(2)}_1 & 0 & 0 & \ldots & 0 \\
& & w^{(3)}_2 P^{(3)}_2 & 0 & \ldots & 0 \\
& & & \mathbf{0} & w^{(4)}_3 P^{(4)}_3 & \ldots & 0 \\
& & & & \ddots & \ddots & \ddots \\
& & & & & \mathbf{0} & P^{(m)}_m \\
& & & & & & w^{(m)}_{m-1} P^{(m)}_{m-1}
\end{pmatrix}
\]

where each asterisk in the first row of \(A^{(m)}\) represents a (possibly different) rational number. Thanks to Lemma 5.2, for each \(j = 1, \ldots, m - 1\), the three entries \(\Theta^{+}_j, -\Lambda^{-}_j, \Lambda^{+}_j\) in \(P^{(j+1)}_j\) are linearly independent over \(\mathbb{Q}\), whereas the four entries \(-\Theta^{+}_j, \Theta^{+}_j, -\Lambda^{-}_j, \Lambda^{+}_j\) are linearly dependent over \(\mathbb{Q}\). Moreover, we know that \(\text{rank}_{\mathbb{Q}}(\hat{R}^{(2)}) = 3\). This implies
that $A^{(m)}_m$ has at least $3m$ linearly independent columns. Due to the rank-nullity theorem,
\[
\dim_{\mathbb{Q}}(\text{kernel}(A^{(m)}_m)) = 4m - \text{rank}_{\mathbb{Q}}(A^{(m)}_m) \leq m \quad (m \geq 1).
\]
(6-1)
In order to prove the inverse inequality of (6-1), we apply induction with respect to $m \geq 6$. Moreover, we show at the same time that every vector $\nu \in \text{kernel}(A^{(m)}_m)$ satisfies (1-7). Note that in the case $m = 1$ we obviously have
\[
\dim_{\mathbb{Q}}(\text{kernel}(A^{(1)}_1)) = \dim_{\mathbb{Q}}(\text{kernel}(\mathbb{R}^{(2)})) = 4 - 3 = 1
\]
and that (1-7) holds. For $m = 2$ and $m = 3$, we refer the reader to the example in the appendix, where the truth of (1-7) can be deduced from the explicit solutions given by the formulae
\[
(-2u + v)\Phi_2 + u\Phi_5^2 + (u - v)\Psi_2^* - 7v\Phi_4 + 8v\Phi_4^* + v\Psi_4 = 0 \quad (u, v \in \mathbb{Q})
\]
and
\[
(-u - w)\Phi_2 + u\Phi_5^2 + w\Psi_2^* + (u - v - w)\Phi_4 + v\Phi_4^* + (u - w)\Psi_4
\]
\[
+ \left( \frac{128}{3}u - \frac{16}{3}v - \frac{128}{3}w \right)\Phi_6 + \left( -\frac{124}{3}u + \frac{31}{6}v + \frac{124}{3}w \right)\Phi_6^*
\]
\[
+ \left( -\frac{4}{3}u + \frac{1}{v} + \frac{4}{3}w \right)\Psi_6^* = 0 \quad (u, v, w \in \mathbb{Q});
\]
cf. [9, (5.16) and (5.17)]. The desired inequality $\dim_{\mathbb{Q}}(\text{kernel}(A^{(m)}_m)) \geq m$ and the specific properties from (1-7) for $m = 4, 5, 6$ can be verified by computer-assisted computations. For making this step more transparent, we have included the code for a MAPLE program, which, upon specifying $m$ at the start of the program (in the line above Output (1)), checks these cases. Now let the assertions be true for $m \geq 6$. Then there exist $m$ linearly independent (column) vectors from $\mathbb{Q}^{4m+4}$, in which the last four entries vanish. We proceed to construct a vector $u$ from the kernel of $A^{(m+1)}_0$, whose last four components do not vanish simultaneously. To this end, we find the components of $u$ from the bottom up in groups of four entries. In the sequel, we assume that $m$ is even; for odd $m$ we apply analogous arguments. By Lemma 5.2, the vector
\[
\tau_1 \Psi_{m} := \tau_1 (1 - 2^{2m+1}, 2^{2m+1}, 1, 0)^T,
\]
where $\tau_1$ is a parameter to be fixed in due time, can be applied to form the last four entries of $u$. This is the first step of the recursive construction of $u$. To proceed with step $i + 1$ (after step $i$), we rewrite (4-2) by grouping together terms containing $\Theta_{m-i-1}^\pm, \Lambda_{m-i-1}^\pm$ and bracketing out. We obtain the equation
\[
\omega^\Theta_{m-i-1}(\tau_1, \ldots, \tau_i, \eta_1) \cdot \Theta_{m-i-1}^+ + \cdots + \omega^\Lambda_{m-i-1}(\tau_1, \ldots, \tau_i, \eta_4) \cdot \Lambda_{m-i-1}^- = 0.
\]
The quantities

\[ \omega^{(R_{m-1})}(\tau_1, \ldots, \tau_i, \eta_1), \]
\[ \omega^{(R_{m-1})}(\tau_1, \ldots, \tau_i, \eta_2), \]
\[ \omega^{(L_{m-1})}(\tau_1, \ldots, \tau_i, \eta_3), \]
\[ \omega^{(L_{m-1})}(\tau_1, \ldots, \tau_i, \eta_4) \]

are linear forms in \( \tau_1, \ldots, \tau_i \) with rational coefficients. Next, given a parameter \( \tau_{i+1} \in \mathbb{Q} \), we can find numbers \( \eta_1, \ldots, \eta_4 \) such that for every \( i = 1, \ldots, m - 1 \), the vector identity

\[
(\omega^{(R_{m-1})}(\tau_1, \ldots, \tau_i, \eta_1), \ldots, \omega^{(L_{m-1})}(\tau_1, \ldots, \tau_i, \eta_4))^T = \tau_{i+1}(2^{2(m-i-1)+1}, 2^{2(m-i-1)+1} - 1, 0, 1)^T
\]

holds. Now we fix the parameters \( \tau_1, \ldots, \tau_m \). Let \( \underline{u} = (u_1, \ldots, u_{4m+4}) \). If the equation

\[
\sum_{i=1}^{m} R^{(2)} : (u_{4i+1}, \ldots, u_{4i+4})^T = 0 \quad (6-2)
\]

holds, then we let \((u_1, \ldots, u_4) \in \text{kernel}(R^{(2)})\). Otherwise, if (6-2) does not hold, we can choose \( \tau_1, \ldots, \tau_m \) in such a way that every nontrivial linear form with rational coefficients in the components on the left-hand side of (6-2) vanishes. In order to achieve \( \tau_1 \neq 0 \), we exploit the condition \( m \geq 6 \), since we need two parameters to cancel the first component, and one parameter for cancelling each of the three remaining components. This completes the proof by induction and shows that

\[ \dim_{\mathbb{Q}} (\text{kernel}(A_{m}^{(m)})) \geq m. \]

Together with (6-1), we have proven that \( \dim_{\mathbb{Q}} (\text{kernel}(A_{m}^{(m)})) = m \) for \( m \geq 1 \). Additionally, we know that (1-7) holds by the construction of \( \underline{u} \). \( \Box \)

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**Appendix A. Computations for \( m = 3, 4, 5 \) and the matrix \( A_{0}^{(3)} \)**

The following MAPLE code performs the necessary computations for \( m = 3, 4, 5 \); note that one needs to change the value of \( m \) therein before each execution.
restart;

with(LinearAlgebra):

m := 2

a := j \rightarrow (-1)^j \cdot (2 \cdot j + 1) \cdot x^{2j+2} \cdot \text{bernoulli}(2 \cdot j + 2);

b := j \rightarrow (-1)^j \cdot (2 \cdot j + 1) \cdot x^{2j+2} \cdot (2j+2) \cdot \text{bernoulli}(2 \cdot j + 2);

\text{sigma} := (j,s) \rightarrow \text{piecewise}(s-j-1 = 0,1,(-1)^j \cdot \text{substitute}(x=0, \text{diff}(\text{expand}(\text{product}(x+h^2,h=1..s-1)),x(s-j-1))) / (s-j-1)!);

S := (j,s) \rightarrow \text{sigma}(j,s) \cdot (-1)^j \cdot (2j)!

\text{d} := s \rightarrow (s-1)^2 / 24 ; \quad e := s \rightarrow 1 / (2s-1)!

\text{H}[0] := s \rightarrow (1 - (s \mod 2)) \cdot \text{piecewise}(e(s) \cdot (-\text{d}(s) \cdot (x[1] - x[4]) + \text{add}(S(j,s) \cdot (a(j) \cdot x[1]) - (1 - (j \mod 2)) \cdot x[4j+2] - (j \mod 2) \cdot x[4j+1], j = 1,..s-1)) + (s \mod 2) \cdot (e(s) \cdot \text{d}(s) \cdot (x[1] + x[3]) + \text{add}(S(j,s) \cdot (a(j) \cdot x[1]) - (1 - (j \mod 2)) \cdot x[4j+2] - (j \mod 2) \cdot x[4j+1], j = 1,..s-1)))

\text{H}[1] := s \rightarrow (1 - (s \mod 2)) \cdot \text{piecewise}(e(s) \cdot (-\text{d}(s) \cdot (x[1] + x[3]) - \text{add}(S(j,s) \cdot (a(j) \cdot x[1]) - (1 - (j \mod 2)) \cdot x[4j+2] - (j \mod 2) \cdot x[4j+1], j = 1,..s-1)) + (s \mod 2) \cdot (e(s) \cdot \text{d}(s) \cdot (x[1] - x[4]) - \text{add}(S(j,s) \cdot (a(j) \cdot x[1]) - (1 - (j \mod 2)) \cdot x[4j+2] - (j \mod 2) \cdot x[4j+1], j = 1,..s-1)));

\text{H}[2] := s \rightarrow (1 - (s \mod 2)) \cdot \text{piecewise}(e(s) \cdot (-\text{d}(s) \cdot (x[4] - 2 \cdot x[3] - 3 \cdot x[1]) + \text{add}(S(j,s) \cdot (b(j) \cdot x[1]) - (1 - (j \mod 2)) \cdot x[4j+4] - (j \mod 2) \cdot x[4j+3], j = 1,..s-1)) + (s \mod 2) \cdot (e(s) \cdot (3 \cdot d(s) \cdot (x[2] \cdot x[1]) + \text{add}(S(j,s) \cdot (1 - (j \mod 2)) \cdot x[4j+3] + (j \mod 2) \cdot x[4j+4] - b(j) \cdot x[1]), j = 1,..s-1))

\text{H}[3] := s \rightarrow (1 - (s \mod 2)) \cdot \text{piecewise}(e(s) \cdot (-3 \cdot d(s) \cdot (x[2] \cdot x[1]) + \text{add}(S(j,s) \cdot (1 - (j \mod 2)) \cdot x[4j+3] - (j \mod 2) \cdot x[4j+4] - b(j) \cdot x[1]), j = 1,..s-1)) + (s \mod 2) \cdot (e(s) \cdot (-\text{d}(s) \cdot (x[4] - 2 \cdot x[3] - 3 \cdot x[1]) + \text{add}(S(j,s) \cdot (b(j) \cdot x[1]) - (1 - (j \mod 2)) \cdot x[4j+4] - (j \mod 2) \cdot x[4j+3], j = 1,..s-1)));

\text{f} := (k,s) \rightarrow \text{sum}(\text{piecewise}(k - 1 \mod 4 = w, 1, 0) \cdot H[w](s), w = 0..3) ;

\text{P} := \text{Matrix}\left(\begin{array}{cccccccccccc}
 4 & 4 & m & (i,j) \rightarrow & \text{diff}\left(f\left(j, \text{ceil}\left(\frac{j}{4}\right), x[i]\right), x[i]\right) ;
\end{array}\right);\text{Q} := \text{Matrix}\left(\begin{array}{cccccccccccc}
m - 1 & 4 & m & (i,j) \rightarrow & \text{diff}\left(f\left(j, \text{ceil}\left(\frac{j}{4}\right), x[k]\right), x[k], k = 4 \cdot i + 1 \cdot 4 \cdot i + 4\right) ;
\end{array}\right);

\text{M} := \text{Matrix}\left(m + 3, 4 \cdot m, \text{storage} = \text{sparse}\right); \quad \text{M}[1..4,1..4] := \text{P} ; \quad \text{M}[5..m + 3, 1..4] := \text{Q} ;

\text{LinearAlgebra}[\text{Rank}](\text{M}) ;

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Table 1. Initial values of $a_j, b_j,$ and $b_j$.

| $j$ | $a_j$ | $b_j$ | $c_j(k)$          | $d_j(k)$          |
|-----|-------|-------|-------------------|-------------------|
| 0   | $\frac{1}{3}$ | 1     | $-$               | $-$               |
| 1   | $\frac{1}{15}$ | 1     | $\frac{1}{15} (1 - k^2 + k^4)$ | $k^2 (1 - k^2)$ |
| 2   | $\frac{2}{189}$ | $\frac{2}{3}$ | $\frac{1}{189} (1 + k^2)(1 - 2k^2)(2 - k^2)$ | $-\frac{1}{3} k^2 (1 - k^2)(1 - 2k^2)$ |

Table 2. Initial values of $\Theta^-_j$.

| $j$ | $\Theta^-_j(k)$ | $\Theta^-_j(k)$          |
|-----|------------------|--------------------------|
| 1   | $\frac{1}{15} (16k^4 - 16k^2 + 1)$ | $\frac{1}{15} (1 + 14k^2 - 14k^4)$ |
| 2   | $\frac{2}{189} (2k^2 - 1)(32k^4 - 32k^2 - 1)$ | $-\frac{2}{189} (2k^2 - 1)(31k^4 - 31k^2 + 1)$ |

Table 3. Initial values of $e_j, f_j,$ and $\Lambda^\pm_j$.

| $j$ | $e_j(k)$ | $f_j(k)$ | $\Lambda^-_j(k)$ | $\Lambda^+_j(k)$ |
|-----|----------|----------|-------------------|-------------------|
| 1   | $1 - k^2$ | $-k^2$   | 1                 | $1 - 2k^2$        |
| 2   | $\frac{1}{3} (1 - k^2)(2 - k^2)$ | $\frac{1}{3} k^2 (1 + k^2)$ | $\frac{2}{3} (1 - k^2)$ | $\frac{2}{3} (k^4 - k^2 + 1)$ |

Firstly, we compute step by step all the expressions of the 12 functions $\Phi_{2j}$, $\Phi^*_j$, $\Psi_{2j}$, $\Psi^*_j$ for $j = 1, 2, 3$ using the formulae (2-6)–(2-13). Secondly, we decompose all these expressions into the polynomials $P_1(\xi)$ and $P_2(\xi)$ such that the elements of the matrix $A_0^{(3)}$ are obtained.

The expressions for $x_1, \ldots, x_{12}$ now follow from the formulae at the beginning of Section 4 and from Tables 2 and 3. Hence, we obtain with (2-6)–(2-13) and, noting that $P_1(\xi) = 0$ for every $\xi \in \Omega_1$, the following formulae.

$$
\Phi_2 = P_2^{(\Phi_2)} = \frac{1}{24} (1 - \left(\frac{2K}{\pi}\right)^2 (1 - 2k^2))
$$

$$
= \frac{1}{24} (x_1 + x_3),
$$
Table 4. Initial values of $w_j$ and $\sigma_j$.

| $\hat{w}_s$ | $w_j^{(s)}$ | $\sigma_{s-1}(s)$ | $\alpha_{s-1}(s)$ |
|-------------|-------------|--------------------|-------------------|
| $s = 1$     | $\frac{1}{24}$ | $-\frac{1}{24}$   | $-\frac{1}{24}$  |
|             | $\frac{1}{144}$ | $\frac{1}{96}$    | $1\frac{1}{96}$  |
| $s = 3$     | $\frac{1}{720}$ | $\frac{1}{384}$   | $-\frac{5}{192}$ |

\[
\Phi'_4 = \mathbf{P}_{l}^{(\Phi'_4)} = \frac{1}{1440} \left( \frac{2K}{\pi} \right)^4 \left( 6E - 5 + 4k^2 \right) - 1 = \frac{1}{24} (x_1 + x_3) + \frac{1}{96} (x_1 - x_6) = \frac{11}{1440} x_1 + \frac{1}{144} x_3 - \frac{1}{96} x_6,
\]

\[
\Psi'_2 = \mathbf{P}_{l}^{(\Psi'_2)} = \frac{1}{8} \left( \frac{2K}{\pi} \right)^2 (1 - \frac{2E}{K} + 1) = \frac{1}{24} (3x_1 + 2x_3 - x_4),
\]

\[
\Phi'_4 = \mathbf{P}_{H}^{(\Phi'_4)} = \frac{1}{1440} \left( \frac{2K}{\pi} \right)^4 \left( 6E - 5 + 4k^2 \right) - 1 = \frac{1}{24} (x_1 + x_3) + \frac{1}{96} (x_1 - x_6) = \frac{11}{1440} x_1 + \frac{1}{144} x_3 - \frac{1}{96} x_6,
\]

\[
\Psi'_2 = \mathbf{P}_{H}^{(\Psi'_2)} = \frac{1}{8} \left( \frac{2K}{\pi} \right)^2 (1 - \frac{2E}{K} + 1) = \frac{1}{24} (3x_1 + 2x_3 - x_4),
\]
\[ \Psi_4 = P^{(\Psi_4)}_I + P^{(\Psi_4)}_{II} = \frac{1}{96} \left( \frac{2K}{\pi} \right)^4 - 1 + \frac{1}{48} \left( \frac{2K}{\pi} \right)^2 \left( \frac{2E}{K} - 1 \right) - 1 = \frac{1}{144} (x_4 - 2x_3 - 3x_1) + \frac{1}{96} (x_7 - x_1) = -\frac{1}{32} x_1 - \frac{1}{72} x_3 + \frac{1}{144} x_4 + \frac{1}{96} x_7, \]

\[ \Psi_6 = P^{(\Psi_6)}_I + P^{(\Psi_6)}_{II} = \frac{1}{96} \left( \frac{2K}{\pi} \right)^4 (1 - 2k^2) + \frac{1}{48} \left( \frac{2K}{\pi} \right)^2 \] 

\[ \Phi_6 = P^{(\Phi_6)}_I + P^{(\Phi_6)}_{II} \]

\[ = \frac{1}{720} (1 - \left( \frac{2K}{\pi} \right)^2 (1 - 2k^2) + \frac{1}{5760} (1 - \left( \frac{2K}{\pi} \right)^2 (1 + 14k^2 - 14k^4) + \frac{1}{640} \left( \frac{2}{189} \right)^6 \left( \frac{2}{189} \right) (2k^2 - 1)(31k^4 - 31k^2 + 1)) = \frac{1}{720} (x_1 + x_3) + \frac{1}{384} (\frac{1}{15} x_1 - x_6) + \frac{1}{640} \left( \frac{2}{189} \right) x_1 - x_0 \]

\[ = \frac{191}{120960} x_1 + \frac{1}{720} x_3 - \frac{1}{384} x_6 - \frac{1}{640} x_0, \]

\[ \Phi_6^* = P^{(\Phi_6^*)}_I + P^{(\Phi_6^*)}_{II} \]

\[ = \frac{1}{720} \left( \frac{2K}{\pi} \right)^2 \left( \frac{6E}{K} - 5 + 4k^2 - 1 \right) - \frac{1}{5760} (1 - \left( \frac{2K}{\pi} \right)^4 (16k^4 - 16k^2 + 1)) - \frac{1}{640} \left( \frac{2}{189} \right)^6 \left( \frac{2}{189} \right) (2k^2 - 1)(32k^4 - 32k^2 - 1)) \]

\[ = \frac{1}{720} (x_4 - x_1) - \frac{1}{384} (\frac{1}{15} x_1 - x_5) - \frac{1}{640} \left( \frac{2}{189} \right) x_1 - x_10 \]

\[ = -\frac{191}{120960} x_1 + \frac{1}{720} x_4 + \frac{1}{384} x_5 + \frac{1}{640} x_{10}, \]

\[ \Psi_6 = P^{(\Psi_6)}_I + P^{(\Psi_6)}_{II} \]

\[ = \frac{1}{240} \left( \frac{2K}{\pi} \right)^2 (1 - \frac{1}{384} \left( \frac{2K}{\pi} \right)^4 (1 - 2k^2) - 1) + \frac{1}{960} \left( \frac{2K}{\pi} \right)^6 \left( k^4 - k^2 + 1 \right) - 1 \]

\[ = \frac{1}{240} (x_2 - x_1) + \frac{1}{384} (x_8 - x_1) + \frac{1}{640} \left( x_{11} - \frac{2}{3} x_1 \right) - \frac{1}{128} x_1 + \frac{1}{240} x_2 + \frac{1}{384} x_8 + \frac{1}{640} x_{11}, \]
We represent these 12 identities, which express each $\xi \in \Omega_1 \cup \Omega_2 \cup \Omega_3$ (whereas $\Omega_s$ was defined in Section 2) as linear forms in terms of $x_1, \ldots, x_{12}$, by the matrix

$$
\begin{pmatrix}
  x_1 & -x_1 & -x_1 & x_1 & -11x_1 & 11x_1 & -x_1 & x_1 & 191x_1 & -191x_1 & -x_1 & x_1 \\
  24 & 24 & 8 & 8 & 1440 & 1440 & 32 & 32 & 120960 & 120960 & 128 & 128 \\
  0 & 0 & x_2 & 0 & 0 & 0 & 0 & -x_2 & 0 & 0 & x_2 & 0 \\
  \frac{8}{24} & \frac{8}{24} & 12 & 12 & 144 & 144 & 72 & 72 & 720 & 720 & 360 & 360 \\
  0 & x_3 & 0 & x_3 & -x_3 & 0 & x_3 & 0 & 0 & x_3 & 0 & 0 \\
  \frac{24}{24} & \frac{24}{24} & 144 & 144 & 144 & 144 & 0 & 0 & 720 & 720 & 720 & 720 \\
  0 & 0 & 0 & x_4 & -x_4 & x_4 & -x_4 & -x_4 & x_4 & x_4 & -x_4 & -x_4 \\
  \frac{96}{24} & \frac{96}{24} & 96 & 96 & 96 & 96 & 384 & 384 & 384 & 384 & 384 & 384 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x_9 & x_10 & x_11 & -x_{12} \\
  \frac{640}{24} & \frac{640}{24} & 640 & 640 & 640 & 640 & 640 & 640 & 640 & 640 & 640 & 640
\end{pmatrix}
$$

The 12 columns correspond to the 12 $\xi$-functions $\Phi_2, \ldots, \Psi_6^\ast$. Row 1 to row 4 correspond to the linearly independent parameters $x_1, \ldots, x_4$, whereas row 5 and row 6 represent the sets of parameters $\{x_5, x_6, x_7, x_8\}$ and $\{x_9, x_{10}, x_{11}, x_{12}\}$, respectively. Here, every parameter from one set is linearly independent over $\mathbb{Q}$ from the parameters of the other set. This follows from the fact that the parameters $x_5, \ldots, x_8$ are provided with the factor $(2K/\pi)^4$ and the parameters $x_9, \ldots, x_{12}$ with the factor $(2K/\pi)^6$. Therefore, with regard to rank considerations, we simplify the above matrix by replacing each $x_1, \ldots, x_4$ in row 1 to row 4 by 1, and by removing the factors $(2K/\pi)^4$ and $(2K/\pi)^6$ from $x_5, \ldots, x_8$ and $x_9, \ldots, x_{10}$, respectively. Using Tables 1–4 as well as the vectors

$$
P^{(2)}_1 = (-\Theta_1^\ast, \Theta_1^+, -\Lambda_1^\ast, \Lambda_1^+),$$

$$
P^{(3)}_1 = (-\Theta_1^\ast, \Theta_1^+, -\Lambda_1^\ast),$$

$$
P^{(3)}_2 = (-\Theta_2^\ast, \Theta_2^+, -\Lambda_2^\ast),$$

$$
P^{(3)}_3 = (-\Theta_3^\ast, \Theta_3^+, -\Lambda_3^\ast).$$
we obtain the matrix $A^{(3)}_0$ given by

$$
\begin{pmatrix}
1 & -1 & -1 & 1 & -11 & 11 & -1 & 1 & 191 & -191 & -1 & 1 \\
24 & 24 & 8 & 8 & 1440 & 1440 & 32 & 32 & 120960 & 120960 & 128 & 128 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 48 & 0 & 0 & 240 \\
1 & 0 & 0 & 1 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & 1 \\
24 & 12 & 144 & 72 & 720 & 0 & 0 & 1 & 360 \\
0 & 1 & 0 & 1 & 0 & 1 & -1 & 0 & 1 & 0 & 1 & 0 \\
24 & 24 & 144 & 144 & 720 & 0 & 0 & 1 & 720 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 640 & 640 & 640 & 640
\end{pmatrix},
$$

which (with a simpler notation) corresponds to

$$A^{(3)}_0 = \begin{pmatrix} R^{(2)} & R^{(4)} & R^{(6)} \\ 0 & w^{(2)}_1 p^{(2)}_1 & 0 \\ 0 & 0 & w^{(3)}_2 p^{(3)}_2 \end{pmatrix}.$$

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CARSTEN ELSNER, University of Applied Sciences, Freundallee 15, D-30173 Hanover, Germany
e-mail: carsten.elsner@fhdw.de

NICLAS TECHNAU, Department of Mathematics, University of York, Heslington, York, YO10 5DD, UK
and
Technische Universität Graz, Institut für Analysis und Zahlentheorie, Steyrergasse 30/II, A-8010 Graz, Austria
e-mail: niclas.technau@york.ac.uk, technau@math.tugraz.at