General Low Energy Dynamics of Supersymmetric Monopoles

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ABSTRACT

We derive general low energy dynamics of monopoles and dyons in $N = 2$ and $N = 4$ supersymmetric Yang-Mills theories by utilising a collective coordinate expansion. The resulting new kind of supersymmetric quantum mechanics incorporates the effects of multiple Higgs fields, both in the $N = 2$ vector multiplet and hypermultiplets, having non-vanishing expectation values.

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1 Introduction

Super-Yang-Mills theories with extended supersymmetry have a rich spectrum of BPS monopole and dyon states. At weak coupling one can use semi-classical techniques to study their properties and one finds that the low-energy dynamics is governed by some kind of a supersymmetric quantum mechanics based on the moduli space of classical BPS monopole solutions.

Early work analyzed points in the classical moduli space of vacua of the field theory where only a single adjoint Higgs field is non-vanishing \[1, 2, 3, 4, 5, 6, 7, 8, 9\]. In this case the electric and magnetic charge vectors of the BPS dyons are proportional to each other and they preserve 1/2 of the supersymmetry. More recently it has been realised that when a second adjoint Higgs field is non-vanishing there is an interesting spectrum of BPS states with electric and magnetic charge vectors that are not parallel \[10, 11, 12, 13, 14, 15, 16\]. In theories with \(N = 4\) supersymmetry such BPS states preserve 1/4 of the supersymmetry, while in the theories with \(N = 2\) supersymmetry they still preserve 1/2.

In this more general situation it is becoming clear that the supersymmetric quantum mechanics that governs the low-energy dynamics includes potential terms. This has been studied in the \(N = 4\) theories in \[17, 18, 19, 20, 21\] and in \[22\] for pure \(N = 2\) super-Yang-Mills theories. The status of the derivation of these potential terms rests on two types of arguments. Firstly, rather direct arguments for the existence of a bosonic potential \[18, 19, 20, 21\] and secondly, indirect arguments for the fermions based on supersymmetry considerations \[18, 19, 20, 21, 22\].

Here we will improve upon the indirect arguments by showing that the supersymmetric quantum mechanics can in fact be derived using a more direct collective coordinate approach generalizing that of \[1, 2, 3\]. In addition to verifying the result of \[22\] for pure \(N = 2\) super-Yang-Mills theory this approach allows us to generalize to \(N = 2\) theories with hypermultiplets when the two adjoint Higgs fields in the \(N = 2\) vector multiplet are non-vanishing. The resulting supersymmetric quantum mechanics in this case both generalizes that of \[1, 3, 9\], which only considered a single Higgs field, and that of \[22\] which didn’t include hypermultiplets. As the \(N = 4\) theory is an \(N = 2\) theory with a single massless adjoint hypermultiplet, we also recover the supersymmetric quantum mechanics presented in \[18\] as a special case.

We will also consider how the supersymmetric quantum mechanics is modified when the scalars in the hypermultiplets acquire expectation values while maintaining a non-trivial Coulomb branch. In doing so we derive the general supersymmetric
quantum mechanics for $N = 4$ SYM theory presented in [21] when all six Higgs fields have non-vanishing expectation values, as well as making contact with the models of [23].

The supersymmetric quantum mechanics with potential terms that was presented in [22] and generalized here are new. We will show that they can be obtained by a non-trivial “Scherk-Schwarz” dimensional reduction of two-dimensional $(4,0)$ supersymmetric sigma models.

The plan of the rest of this paper is as follows: in section 2 we discuss pure $N = 2$ Super-Yang-Mills theory. We briefly recall that the general BPS equations consist of the usual BPS equations for a single Higgs field plus a secondary BPS equation. We next review some aspects of the geometry of the moduli space of solutions to the BPS equation for a single Higgs field that we use later. This section concludes by carrying out the collective coordinate expansion leading to the supersymmetric quantum mechanics of [22] that describes the low-energy monopole dynamics when the two adjoint Higgs fields have non-vanishing expectation values.

Section 3 generalizes the discussion to include matter fermions from hypermultiplets. The zero modes of the matter fermions gives rise to an Index bundle on the monopole moduli space. The effect of the second Higgs field is to introduce extra terms in the supersymmetric quantum mechanics constructed from a two-form on this bundle.

Section 4 generalizes to cases when scalars from the hypermultiplets also acquire expectation values in addition to the two adjoint Higgs fields in the $N = 2$ vector multiplet. The analysis covers the case that the hypermultiplets are in real representations of the gauge group. Since the $N = 4$ model is an $N = 2$ model with a single massless adjoint hypermultiplet, this analysis includes a derivation of the supersymmetric quantum mechanics presented in [21].

For the convenience of the reader Section 5 summarizes various aspects of the general dynamics and discusses the quantization. Section 6 briefly concludes. Finally, appendix A contains some technical calculations used in the text, while appendix B shows how the supersymmetric quantum mechanics that we derive can be obtained via non-trivial, “Scherk-Schwarz”, dimensional reduction.

2 Dynamics of Monopoles in Pure $N = 2$ SYM
2.1 BPS Equations

The pure $\mathcal{N} = 2$ super-Yang-Mills Lagrangian is given by

\[
L = -\text{Tr} \left\{ -\frac{1}{4} F_{MN} F^{MN} + \frac{1}{2} D_M \Phi^I D^M \Phi^I - \frac{1}{2} [\Phi^1, \Phi^2]^2 \\
- i\bar{\chi} \gamma^M D_M \chi + i\bar{\chi} [\Phi^1, \chi] - \bar{\chi} \gamma_5 [\Phi^2, \chi] \right\},
\]

(2.1)

where $\Phi^I, I = 1, 2$ denote the two real Higgs fields, $D_M \Phi^I = \partial_M \Phi^I + [A_M, \Phi^I]$, $\chi$ is a Dirac spinor and all fields are in the adjoint representation of the gauge group $G$. The anti-hermitian generators of the Lie algebra $\mathcal{G}$ are normalised so that $\text{Tr} t^a t^b = -\delta^{ab}$.

Our metric has mostly minus signature and $\gamma_5 = i\gamma_0 \gamma_1 \gamma_2 \gamma_3$. The classical vacuum satisfy $[\Phi^1, \Phi^2] = 0$ and thus $\phi^I$ lie in the Cartan subalgebra of $G$: $\Phi^I = \phi^I \cdot \mathbf{H}$. We will only consider vacua where the symmetry is maximally broken to $U(1)^r$ where $r$ is the rank of $G$. For a given vacuum we can define electric and magnetic charge two-vectors via

\[
Q^I_e = -\text{Tr} \int \hat{n} \cdot \vec{E} \Phi^I = \phi^I \cdot q,
\]

\[
Q^I_m = -\text{Tr} \int \hat{n} \cdot \vec{B} \Phi^I = \phi^I \cdot g,
\]

(2.2)

where the integration is over the asymptotic two-sphere with outward normal unit vector $\hat{n}$, and we have introduced the electric and magnetic charge vectors given by

\[
q = n^m_e \beta^m,
\]

\[
g = 4\pi n^m_m \beta^*_m,
\]

(2.3)

respectively, where $\beta^m$ are the simple roots and $\beta^*_m$ are the simple co-roots of $\mathcal{G}$, and $n^m_e$ are the topological winding numbers and $n^m_m$ are, in the quantum theory, the electric quantum numbers.

There is a classical bound on the mass given by [24, 13]

\[
M \geq \text{Max} \left[ |\tilde{Q}_e|^2 + |\tilde{Q}_m|^2 \pm 2(Q^1_e Q^2_m - Q^1_m Q^2_e) \right]^{1/2}.
\]

(2.4)

It can also be written in the form $\text{Max} |Z_\pm|$ where $Z_\pm = (Q^1_e \pm Q^2_m) + i(Q^1_m \mp Q^2_e)$. Only the charge $Z_-$ appears as a central charge in the $\mathcal{N} = 2$ supersymmetry algebra and BPS states preserving 1/2 of the supersymmetry satisfy $M = |Z_-|$ [25, 22]. A consequence of the bound (2.4) is that classical BPS solitons can only have charges that satisfy $|Z_-| \geq |Z_+|$. In subsequent sections we will mostly be concerned with BPS solitons.
The mass bound (2.4) is saturated when
\[ \vec{E} = \pm \bar{D}a, \]
\[ \vec{B} = \bar{D}b, \]
(2.5)
where we have defined the rotated Higgs fields via
\[ a = \cos \alpha \Phi^1 - \sin \alpha \Phi^2, \]
\[ b = \sin \alpha \Phi^1 + \cos \alpha \Phi^2, \]
(2.6)
and the angle \( \alpha \) is constrained to be
\[ \tan \alpha = \frac{Q_1 m}{Q_2 \pm Q_1 e}. \]
(2.7)
The second equation in (2.5) is the usual BPS equation for a single Higgs field and is referred to as the “primary BPS equation”. If we take static fields in the gauge \( A_0 = \mp a \), Gauss’ Law becomes the “secondary BPS equation” for the field \( a \):
\[ D^2 a + [b, [b, a]] = 0. \]
(2.8)
For a given solution of the primary BPS equation, the secondary BPS equation is exactly the same equation that is solved by gauge functions that generate zero modes about the original solution. For specified asymptotic behavior of \( a \) it has a unique solution. The solutions to the general equations can thus be viewed as electrically dressed solutions to the primary BPS equation. Finally we note that in terms of the vectors \( a, b \), the mass bound is given by
\[ M \geq \text{Max} \left| i(\pm a \cdot q + b \cdot g) + (b \cdot q \mp a \cdot g) \right| \]
\[ = \text{Max} \left| \pm a \cdot q + b \cdot g \right|. \]
(2.9)
where the second expression is obtained by noting that (2.7) can be recast as the constraint
\[ b \cdot q = \pm a \cdot g \]
(2.10)

### 2.2 Zero Modes

As we will discuss in the next subsection, the collective coordinate expansion is constructed about solutions of the ordinary BPS equation for a single Higgs field \( B_i = D_i \Phi \). It will be useful to summarize some aspects of the discussion of the geometry of the moduli spaces of solutions as presented in [26, 1]. We first define
a connection $W_\mu$ on $R^4$ that is translationally invariant in the four direction via $W_\mu = (A_i, \Phi)$. If $G_{\mu\nu}$ is the corresponding field strength then the BPS equations can be recast as self duality equations for $W_\mu$,

$$G_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} G_{\rho\sigma}. \quad (2.11)$$

Introducing the covariant derivative on $R^4$, $D_\mu = \partial_\mu + [W_\mu, ]$, we note that an infinitesimal gauge transformations on $(A_i, \Phi)$ can be recast in the form $\delta W_\mu(x) = D_\mu \Lambda$ if the gauge parameter $\Lambda(x)$ is restricted to be independent of $x^4$.

Denote the moduli space of solutions to the BPS equations within a given topological class $k$ by $\mathcal{M}_k$. A natural set of coordinates is provided by the moduli $z^m$ that specify the most general gauge equivalence class of solutions $W_\mu(x, z)$. The zero modes $\delta_m W_\mu$ about a given solution satisfy the linearized self-duality equation

$$D_{[\mu} \delta_m W_{\nu]} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} D_\rho \delta_m W_\sigma, \quad (2.12)$$

as well as

$$D_\mu \delta_m W_\mu = 0. \quad (2.13)$$

They can be used to construct a natural metric on $\mathcal{M}_k$ via:

$$g_{mn} = - \int d^3 x \text{Tr} (\delta_m W_{\mu} \delta_n W_{\mu}). \quad (2.14)$$

We see that (2.13) implies that the zero mode is orthogonal to gauge modes.

If we let $W_\mu(x, z)$ be a family of BPS monopole configurations, the zero modes are given by

$$\delta_m W_\mu = \partial_m W_\mu - D_\mu \eta_m, \quad (2.15)$$

where the gauge parameters $\eta_m(x, z)$ are chosen to satisfy (2.13). The gauge parameters $\eta_m$ define a natural connection on $\mathcal{M}_k$ with covariant derivative

$$s_m = \partial_m + [\eta_m, ], \quad (2.16)$$

and field strength

$$\phi_{mn} = [s_m, s_n]. \quad (2.17)$$

The pair $(W_\mu(x, z), \eta_m(x, z))$ defines a natural connection on $R^4 \times \mathcal{M}_k$. The components of the field strength are given by $G_{\mu\nu}$, $\phi_{mn}$ and the mixed components are given by

$$[s_m, D_\mu] = \delta_m W_\mu. \quad (2.18)$$
Note the following identities:

\[ s_m G_{\mu \nu} = 2D_{[\mu} \delta_m W_{\nu]}, \]
\[ D_\mu \phi_{mn} = -2s_{[a} \delta_{b]} W_\mu, \]
\[ \phi_{mn} = 2(D_\mu D_\mu)^{-1}[\delta_m W_\nu, \delta_n W_\nu]. \]  \hspace{1cm} (2.19)

The Christoffel connection associated with the metric \((2.14)\) can be written in the form:

\[ \Gamma_{mnk} = g_{ml} \Gamma_{lk} = -\int d^3x \text{Tr} (\delta_m W_\mu \delta_k \delta_n W_\mu). \]  \hspace{1cm} (2.20)

The hyper-Kähler structure on \(R^4\) gives rise to a hyper-Kähler structure on \(\mathcal{M}_k\). The three complex structures can be written

\[ J^{(s)}_{n m} = -g^{np} \int d^3x J^{(s)}_{\mu \nu} \text{Tr} (\delta_m W_\mu \delta_p W_\nu), \]  \hspace{1cm} (2.21)

and we note that

\[ J^{(s)}_{m n} \delta_n W_\mu = -J^{(s)}_{\mu n} \delta_m W_\nu. \]  \hspace{1cm} (2.22)

We now recall some aspects of the zero modes of the adjoint fermions. It is convenient to introduce hermitian Euclidean gamma matrices via

\[ \Gamma_i = \gamma_0 \gamma_i, \quad \Gamma_4 = \gamma_0, \]  \hspace{1cm} (2.23)

satisfying \(\{\Gamma_\mu, \Gamma_\nu\} = 2\delta_{\mu \nu}\) and define \(\Gamma_5 = \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4\). The fermion zero modes are time independent solutions of the Dirac equation in the presence of a BPS monopole and thus solve:

\[ \Gamma_\mu D_\mu \chi = 0. \]  \hspace{1cm} (2.24)

They are necessarily anti-chiral. The monopole breaks 1/2 of the supersymmetry and they unbroken supersymmetry can be used to pair the bosonic and fermionic zero modes via

\[ \chi_m = \delta_m W_\mu \Gamma_\mu \epsilon_+, \]  \hspace{1cm} (2.25)

where \(\epsilon_+\) is a c-number spinor that can be chosen to satisfy

\[ \epsilon_+^\dagger \epsilon_+ = 1, \quad J^{(3)}_{\mu \nu} = -i\epsilon_+^\dagger \Gamma_{\mu \nu} \epsilon_. \]  \hspace{1cm} (2.26)

Using (2.22) we deduce that the fermionic zero modes satisfy

\[ J^{(3)}_{m n} \chi_n = i\chi_m, \]  \hspace{1cm} (2.27)

and hence that two bosonic zero modes are paired with one fermionic zero mode, in accord with the Callias index theorem [27].
2.3 Bosonic Monopole Dynamics

The semi-classical quantization of BPS monopoles begins with a mode expansion of the fields about a given classical solution. For each zero mode one must introduce a collective co-ordinate. By ignoring all of the non-zero modes one obtains a description of the low-energy dynamics. For the case of a single Higgs field in pure $N = 2$ SYM this was carried out in detail in [1]. The resulting supersymmetric quantum mechanics is a consistent, i.e. supersymmetric, truncation of the full field theory dynamics. Here we generalize this derivation to include the effects of a second Higgs field having a non-vanishing expectation value.

Let us first consider the bosonic case. There have been a number of separate but related arguments that conclude that the effect of the second Higgs field, in an appropriate limit, is to give rise to a potential term that is the norm of a tri-holomorphic Killing vector on the moduli space [18, 19, 20, 21]. Let us paraphrase the arguments here in a way that is most useful to include fermions.

We begin by emphasising that we derive the low-energy dynamics of monopoles; dyons then emerge as particular excited states of the monopole dynamics. We thus begin with a given magnetic charge vector $g$ and fixed Higgs expectation values $\Phi^I$. Setting $q = 0$ then fixes the angle $\alpha$ (2.7) and hence specifies the fields $a, b$ defined in (2.6). It is important to notice that this means the expectation value $a$ is orthogonal to the magnetic charge,$$
a \cdot g = 0. \quad (2.28)$$
The collective coordinate expansion then begins with a static purely magnetic solution to the primary BPS equation $B_i = D_ib$. The dynamical effect of the second Higgs field is treated as a perturbation of this solution. The collective coordinate expansion can be considered to be an expansion in the number of time derivatives $n = n_\theta$. The equations of motion of the low-energy effective action will be of order $n = 2$ so we must ensure that a collective coordinate ansatz solves the equations of motion of the field theory to order $n = 0$ and $n = 1$. To incorporate the affects of the second Higgs field we will also assume that $a$ is of order $n = 1$. We next write the Lagrangian in terms of $b, a$ rather than $\Phi^1, \Phi^2$, respectively, to obtain

$$L = -\frac{1}{2} \Tr \left\{ -\frac{1}{2} F_{MN} F^{MN} + D_M a D^M a + D_M b D^M b + [a, b]^2 \right\}. \quad (2.29)$$

To order $n = 0$ the equations of motion are all solved for a time dependent solution to the primary BPS equation $W_\mu(x, z(t))$, with $W_4 = b$. At order $n = 1$ we need to solve the $A_0$ equation of motion, Gauss’s Law, and the $a$ equation of motion.
The former is solved, as usual, by setting \( A_0 = \dot{z}^m \eta_m \) and noting that the terms involving \( a \) are higher order. The order \( n = 1 \) equation of motion for \( a \) is simply the secondary BPS equation \( D_\mu D_\mu a = 0 \), since \( D_0 D_0 a \) is higher order. This equation has a unique solution for specified asymptotic behavior (expectation value) of \( a \). Since this is precisely the equation satisfied by the gauge parameter specifying the gauge-zero mode, \( D_\mu a \) must be a linear combination of gauge zero modes. More precisely we have

\[
D_\mu a = -G^m \delta_m W_\mu, \tag{2.30}
\]

where \( G^m \) is a linear combination of the \( r \) tri-holomorphic Killing vector fields \( \mathbf{K} \) on \( \mathcal{M}_k \) corresponding to the \( U(1)^r \) gauge transformations\[1]::

\[
G = a \cdot \mathbf{K}. \tag{2.31}
\]

Having solved the equations of motion to order \( n = 0, 1 \) we can substitute the ansatz into the field theory Lagrangian. After integrating over space we get

\[
S = \frac{1}{2} \int dt [\dot{z}^m \dot{z}^n g_{mn} - G^m G^n g_{mn}] - b \cdot g. \tag{2.32}
\]

Note that the corresponding energy admits a Bogomol’nyi bound, \( E \geq |\dot{z}^m G_m| + b \cdot g \), that is saturated when \( \dot{z}^m = \mp G^m \). States saturating this bound then have energy given by \( E = G^m G^n g_{mn} + b \cdot g \). Using our ansatz we next note that the electric field can be expressed via \( E_i = \dot{z}^m \delta_m W_i \). For configurations with \( \dot{z}^m = \mp G^m \) we have \( E_i = \pm D_i a \). Using the argument in \[17\] we can then show that the energy of these states can be recast in the form \( E = \pm a \cdot \mathbf{q} + b \cdot g \).

To relate this to the mass formula \( (2.9) \) it is helpful to first recall that the monopole moduli space splits into the product, modulo a discrete identification, of a centre of mass piece with a piece describing the relative motion of fundamental monopoles. Since the electric charge arising from the center of mass part is necessarily parallel to \( g \), we see that the electric excitation energy \( \pm a \cdot \mathbf{q} \) only captures the excitation energy due to relative electric charges. On the other hand centre of mass sector contribution to the electric energy can be written as \( (b \cdot \mathbf{q})^2 / 2b \cdot g \). Thus, in the moduli space approximation that began with \( a \cdot g = 0 \)[2] the electric energy of a BPS dyon splits cleanly into two pieces; \( \pm a \cdot \mathbf{q} \) arising from the electric energy of the relative sector and \( (b \cdot \mathbf{q})^2 / 2b \cdot g \) from the center of mass. We see that this is consistent with the \[1\] Note that the sign appearing in \( (2.30) \) is related to a choice of conventions for the signs of the Killing vectors \( \mathbf{K} \).

\[2\] Note that \( a \cdot g = 0 \) also implies that \( G^2 = 0 \) for the centre of mass part of the Lagrangian.
expansion of the first line of (2.9):

\[ M \simeq \mathbf{b} \cdot \mathbf{g} \pm \mathbf{a} \cdot \mathbf{q} + \frac{(\mathbf{b} \cdot \mathbf{q})^2}{2\mathbf{b} \cdot \mathbf{g}}, \quad (2.33) \]

2.4 Supersymmetric Monopole Dynamics

Let us now turn to a derivation of effective action when we include the fermions in the pure \( N = 2 \) super-Yang-Mills theory. It is again convenient to rewrite the pure \( N = 2 \) super Yang-Mills action in terms of \( a, b \). Noting that \((a, b)\) is a rotation of \((-\Phi^2, \Phi^1)\) we obtain

\[
L = -\operatorname{Tr} \left\{ -\frac{1}{4} F_{MN} F^{MN} + \frac{1}{2} D_M a D^M a + \frac{1}{2} D_M b D^M b - \frac{1}{2} [a, b]^2 
- i \tilde{\chi} \gamma^M D_M \chi + i \tilde{\chi} [b, \chi] + \tilde{\chi} \gamma_5 [a, \chi] \right\},
\]

with it understood that \( \chi \) has now been rotated by the angle \((\alpha - \pi/2)/2\). The collective coordinate expansion can now be considered to be an expansion in \( n = n_\theta + \frac{1}{2} n_f \), where \( n_f \) as the number of fermions. A low-energy ansatz for the fields should solve the equations of motion to order order \( n = 0, \frac{1}{2}, 1 \). By combining the ansatz for the case of a single Higgs field in [1] with the above ansatz for the bosonic case we are led to

\[
W_\mu = W_\mu (x, z(t)), \\
\chi = \delta_m W_\mu \Gamma^\mu \epsilon_+ \tilde{\lambda}^m (t), \\
A_0 = \hat{z}^m \eta_m - i \phi_{mn} \hat{\lambda}^m \tilde{\lambda}^n, \\
a = \bar{a} + i \phi_{mn} \hat{\lambda}^m \tilde{\lambda}^n, \quad (2.35)
\]

with

\[
D_\mu \bar{a} = -G^m \delta_m W_\mu. \quad (2.36)
\]

Because of (2.27) the complex fermionic Grassmann odd collective coordinates \( \tilde{\lambda}^m \) are not independent and satisfy

\[
- i \tilde{\lambda}^m J^{(3)m}_n = \tilde{\lambda}^n. \quad (2.37)
\]

Real independent \( \lambda^m \) can be defined via

\[
\lambda^m = \sqrt{2} \left( \tilde{\lambda}^m + (\tilde{\lambda}^m)^\dagger \right). \quad (2.38)
\]
If we ignore the shift in $a$ by $\bar{a}$, we have the ansatz for the case of a single Higgs field analysed in [1]. Hence after substituting into the action (2.34) the $\bar{a}$ independent terms lead to the supersymmetric quantum mechanics:

$$S = \frac{1}{2} \int dt [\dot{x}^m \dot{x}^n g_{mn} + ig_{mn} \lambda^m D_t \lambda^n] - b \cdot g, \quad (2.39)$$

where

$$D_t \lambda^m = \dot{\lambda}^m + \Gamma^m_{nk} \dot{z}^n \lambda^k. \quad (2.40)$$

Since the $\bar{a}$ dependent terms arising from $(D_0 \bar{a})^2$ in the action are again higher order than we are considering, we just need to focus on $D_\mu \bar{a} D_\mu \bar{a}$ and the Yukawa terms $\bar{\chi} \gamma_5 [a, \chi]$. The $\bar{a}$ dependent terms in the former are

$$\frac{1}{2} \text{Tr} D_\mu \bar{a} D_\mu \bar{a} + i \bar{\lambda}^m \lambda^n \text{Tr} D_\mu (\phi_{mn} D_\mu \bar{a}). \quad (2.41)$$

When we integrate over the spatial coordinates the second term vanishes and we are left with bosonic potential

$$- \frac{1}{2} G^m G^n g_{mn}. \quad (2.42)$$

as in the bosonic case. The $\bar{a}$ terms arising from the Yukawa term give rise to

$$i \int d^3 x \text{Tr} \chi^\dagger [\bar{a}, \chi], \quad (2.43)$$

which can be rewritten as

$$-2i \bar{\lambda}^m \lambda^n \int d^3 x \text{Tr} \delta_m W_\mu [\bar{a}, \delta_n W_\mu]$$

$$= -2i \bar{\lambda}^m \lambda^n \int d^3 x \text{Tr} \delta_m W_\mu (D_\mu s_n \bar{a} - s_n D_\mu \bar{a}), \quad (2.44)$$

where we have used (2.26) and (2.18), respectively. Using the fact that

$$s_m \bar{a} = G^m \phi_{mn}, \quad (2.45)$$

which can be proved by acting on both sides with $D^2$ and using the fact that $D^2$ has no zero modes, we note that the first term is a boundary term which vanishes. The remaining term can then be recast in the form

$$2i \bar{\lambda}^m \lambda^n G_{mn} = \frac{i}{2} \lambda^m \lambda^n G_{mn}. \quad (2.46)$$

In summary, the effect of the second field is thus to add the potential term (2.42) and the fermion bilinear (2.40) to the action (2.39) to obtain

$$S = \frac{1}{2} \int dt [\dot{x}^m \dot{x}^n g_{mn} + ig_{mn} \lambda^m D_t \lambda^n - G^m G^n g_{mn} - iD_m G_n \lambda^m \lambda^n] - b \cdot g. \quad (2.47)$$

We have thus derived the supersymmetric quantum mechanics that was first presented, based on supersymmetry considerations, in [22].


3 Inclusion of Matter Fermions

We now consider the low-energy dynamics of monopoles in $N=2$ Yang-Mills theories with hypermultiplets. This was first studied in [4, 5, 6] in the special case that only a single adjoint Higgs field has a non-trivial expectation value. The main new feature is that the matter fermions give rise to extra fermionic zero modes that provide a natural Index bundle over the moduli space of monopoles. The resulting supersymmetric quantum mechanics is coupled to this bundle. Here we will show that when the second adjoint Higgs field of the $N=2$ vector multiplet has a non-vanishing expectation value, this supersymmetric quantum mechanics is modified by terms constructed from a natural two-form on this bundle.

The massless hypermultiplet contribution to the Lagrangian is given by

$$L_H = \frac{1}{2} D_K M^\dagger D^K M + i \bar{\Psi} \gamma^K D_K \Psi - \bar{\Psi} (-i \Phi_1 - \gamma_5 \Phi_2) \Psi$$
$$+ M^{\dagger 1} \chi \Psi + \bar{\Psi} \chi M_1 + i M^{\dagger 2} \bar{\chi} \gamma_5 \Psi + i \bar{\Psi} \gamma_5 \chi^c M_2$$
$$+ \frac{1}{2} M^{\dagger} (\Phi^1 + \Phi^2) M + \frac{1}{8} (M^\dagger t^\alpha \tau_s M)^2,$$

where $M$ is a doublet of complex scalars $(M_1, M_2)^T$, $t^\alpha$ are anti-hermitian generators in the matter representation, $\tau_s$ are Pauli matrices, and $\chi^c$ is the charge conjugation of $\chi$ (defined precisely in section 4).

3.1 Zero Modes and the Index Bundle

Before discussing the effects of the second adjoint Higgs field, let us briefly discuss some of the geometry of the Index bundle defined by the fermion zero-modes. The fermion zero modes solve the Dirac equation in the background of a monopole configuration

$$\Gamma_\mu D_\mu \gamma_5 \Psi = 0,$$

and are chiral. Let $\Psi_A(x, z), A = 1 \ldots l$ be a basis of the fermion zero modes in monopole background specified by the moduli $z$ satisfying

$$\int d^3 x \Psi^\dagger_A \Psi_B \equiv \langle \Psi_A | \Psi_B \rangle = \delta_{AB},$$

where we have defined $\Psi_A^\dagger \equiv (\Psi_A)^\dagger$. It will be very useful to note the completeness relationship

$$|\Psi_A > \delta^{AB} < \Psi_B | + \Pi + \frac{1 - \Gamma_5}{2} = 1,$$
where the operator $\Pi$ projects onto the chiral non-zero modes and has the form
\begin{equation}
\Pi = \gamma_5 D_\mu D_\mu \gamma_5 \frac{1 + \Gamma_5}{2}.
\end{equation}

A connection on the Index bundle is defined by
\begin{equation}
A_{m\bar{A}B} = \langle \Psi_\bar{A} | s_m \Psi_B >.
\end{equation}

Using the results of section 2.2 and (3.4) one can show that the corresponding field strength can be written in the form
\begin{equation}
F_{mn\bar{A}B} = - s_m \Psi_\bar{A} | \Pi s_n \Psi_B > - s_n \Psi_\bar{A} | \Pi s_m \Psi_B > + \langle \Psi_\bar{A} | \phi_{mn} \Psi_B >.
\end{equation}

It is straightforward to see that the connection one-form is unitary and hence the structure group of the Index bundle is generically $U(l)$. The Index bundle thus admits a covariantly constant complex structure $I^{(3)}$ with Kähler form taken to be $I^{(3)}_{AB} = i \delta_{AB}$ (the superscript will be convenient in section 4). When the representation of the matter fermions in the gauge group is real or pseudo-real, however, the structure group is further restricted \cite{28, 23}. For the pseudo-real representation, the structure group of the bundle reduces to $O(l)$, while, for the real representation, the structure group reduces to a symplectic bundle $USp(l)$. A special case of the latter is the adjoint fermion zero modes that live in the co-tangent bundle of the moduli space which, being hyper-Kähler, is indeed symplectic. In this case note that the field strength $F$ is simply the Riemann curvature tensor.

### 3.2 Collective Coordinate Expansion

The collective coordinate expansion with two adjoint Higgs fields and matter fermions parallels what was done for the case of pure $N = 2$ SYM in section 2. We again first perform a chiral rotation to write the action in terms of $a, b$ which requires that we work with rotated fermions and matter fields. The ansatz for the vector multiplet fields is then given by
\begin{align}
W_\mu &= W_\mu(x, z(t)), \\
\chi &= \delta_m W_\mu \Gamma^\mu \epsilon^+_{\bar{\lambda} m}(t), \\
A_0 &= \bar{z}^m \eta_m - i \phi_{mn} \bar{\lambda}^m \bar{\lambda}^n + \frac{i}{D^2}(\Psi^{\dagger} t^\alpha \Psi t^\alpha), \\
a &= \bar{a} + i \phi_{mn} \bar{\lambda}^m \bar{\lambda}^n + \frac{i}{D^2}(\Psi^{\dagger} t^\alpha \Psi t^\alpha),
\end{align}

(3.8)
while for the matter fields it is given by

\[ \Psi = \psi^A(t) \Psi_A, \]

\[ M_1 = -\frac{2}{D^2}(\bar{\chi} \Psi), \]

\[ M_2 = -\frac{2i}{D^2}(\bar{\chi} \gamma_5 \Psi), \]

\[ (3.9) \]

where we have introduced the Grassmann odd complex collective coordinates \( \psi^A(t) \) for the matter fermion zero modes. This ansatz solves the equations of motion to order \( n = 0, 1/2, 1 \) and generalises that in \([5]\) by simply shifting the \( \bar{\alpha} \) field by a gauge function \( \bar{a} \) satisfying \((2.36)\).

After substituting this ansatz into the field theory action, the \( \bar{\alpha} \) independent terms give rise to the supersymmetric quantum mechanics presented in \([4, 5, 6]\):

\[ \mathcal{L} = \frac{1}{2} \left( g_{mn} \dot{z}^m \dot{z}^n + ig_{mn} \lambda^m D_t \lambda^n + i \psi^a D_t \psi^a + \frac{1}{2} F_{mnab} \lambda^m \lambda^n \psi^a \psi^b \right) - b \cdot g, \]

\[ (3.10) \]

where

\[ D_t \psi^a = \dot{\psi}^a + A_m \bar{z}^m \psi^b, \]

\[ (3.11) \]

and we traded off complex \( \psi^A \)'s in favor of real \( \psi^a \)'s (effectively this means we are embedding the \( U(l) \) bundle in an \( So(2l) \) bundle). The \( \bar{\alpha} \) dependent terms give rise to the potential terms presented in the last section plus an additional fermion bilinear. This latter term can be rewritten

\[ -i \psi^A \bar{\psi}^B T_{AB}, \]

\[ (3.12) \]

where we have defined \( \bar{\psi}^A \) as the complex conjugate of \( \psi^A \) and

\[ T_{AB} = \langle \Psi_A | \bar{a} \Psi_B \rangle. \]

\[ (3.13) \]

As \( T \) is anti-hermitian, in a real basis \((3.12)\) becomes \(-i \psi^a \bar{\psi}^b T_{ab}/2 \) with \( T_{ab} = -T_{ba} \). For hypermultiplets in general representations we cannot write \( T \) in a simpler form. However, as we will discuss it is crucial for consistency of the supersymmetric quantum mechanics that

\[ T_{AB;m} = F_{mnAB} G^m. \]

\[ (3.14) \]

To prove this we begin with

\[ \partial_m T_{AB} = \langle s_m \Psi_A | \bar{a} \Psi_B \rangle + \langle \Psi_A | (s_m \bar{a}) \Psi_B \rangle + \langle \Psi_A | \bar{a} s_m \Psi_B \rangle. \]

\[ (3.15) \]

Using \((3.4)\) the first term in \((3.15)\) can then be written

\[ \langle s_m \Psi_A | \Psi_C \rangle \delta^{CC} < \Psi_C | \bar{a} \Psi_B \rangle + \langle s_m \Psi_A | \Pi \bar{a} \Psi_B \rangle. \]

\[ (3.16) \]
The first term is \(-A_{m,q}d^{CC}T_{CB}\). Using the identity
\[
\not{D}_5 (a\, \not{\Psi} A - G_{m,s} \not{\Psi} A) = 0,
\]
which can be proven by acting with \(G_{m,s}\) on \(\not{D}_5 \not{\Psi} A = 0\), we can rewrite the second term as
\[
G^m < s_m \not{\Psi} A | \Pi s_n \not{\Psi} B > .
\]
The last term in (3.15) can be manipulated in a similar way. The second term can be rewritten using (2.45). Putting this together we deduce that
\[
\nabla_m T_{AB} = G_m \{ < s_m \not{\Psi} A | \Pi s_n \not{\Psi} B > - < \not{s}_{n} \not{\Psi} A | \Pi s_m \not{\Psi} B > + < \not{\Psi}_{A} | \phi_{mn} \not{\Psi} B > \}.
\]
Since the last term in braces is precisely the curvature \(F_{mnAB}\) we have established (3.14).

In conclusion the supersymmetric quantum mechanics describing the low-energy dynamics of monopoles in \(N = 2\) theories with matter when both adjoint Higgs fields are non-vanishing is given by
\[
\mathcal{L} = \frac{1}{2} \left( g_{mn} \dot{z}^m \dot{z}^n + ig_{mn} \lambda^m D_t \lambda^n - g^{mn} G_m G_n - iD_m G_n \lambda^m \lambda^n + i\psi^a D_t \psi^a + \frac{1}{2} F_{mnab} \lambda^m \lambda^n \psi^a \psi^b - iT_{ab} \psi^a \psi^b \right) - b \cdot g. \tag{3.20}
\]
The main new feature is the presence of the two-form \(T\) on the Index bundle. The action is invariant under the supersymmetry transformations
\[
\begin{align*}
\delta z^m &= -i\epsilon \lambda^m + i\epsilon_s J^{(s)m}_n \lambda^n, \\
\delta \lambda^m &= (\dot{z}^m - G^m)\epsilon + J^{(s)m}_n (\dot{z}^n - G^n)\epsilon_s - i\epsilon_s \lambda^k \lambda^n \not{J}^{(s)k} \Gamma^m_{ln} \\
\delta \psi^a &= -A_{m,a} \delta z^m \psi^b,
\end{align*}
\]
where \(\epsilon, \epsilon_s\) are constant one component Grassmann odd parameters, provided that in addition to the usual requirements that the moduli space is hyper-Kähler and that the field strength \(F\) is of type (1,1) with respect to all complex structures, the two form \(T\) satisfies (3.14). The action is also invariant under the following symmetry transformation generated by the tri-holomorphic Killing vector \(G\)
\[
\begin{align*}
\delta z^m &= k G^m, \\
\delta \lambda^m &= k G^m_n \lambda^n, \\
\delta \psi^a &= kT_{ab} \psi^b - A_{mb} \delta z^m \psi^b. \tag{3.22}
\end{align*}
\]
where \(k\) is a constant. In the case of \(N = 4\) supersymmetry, i.e. a single hypermultiplet in the adjoint representation, the bundle is the tangent bundle and \(T_{ab} = G_{a;b}\).

We have thus derived the action first presented in [18], which was obtained there via symmetry arguments.


### 3.3 Massive Matter Fields

Let us briefly consider the case that the hypermultiplets are massive\footnote{Early work on this issue can be found in \cite{29}.}. The relevant mass terms are given by\footnote{Since we work with the rotated fields $a, b$, we interpret $(m_R, m_I)$ to have been similarly rotated.}.

\begin{equation}
\label{eq:mass_terms}
m_R \bar{\Psi} \Psi - m_I i \bar{\Psi} \gamma_5 \Psi.
\end{equation}

Recall that the collective coordinate expansion begins by writing the field theory Lagrangian in terms of $a, b$. We can treat this term as a perturbation by taking the bare mass to be order $n = 1$, i.e., the same order of magnitude as $a$ and hence smaller than $b$. To leading order the Dirac equation for the matter fermions is then not modified from (3.2). Substituting our ansatz (3.9) into (3.23), we find that only $m_I$ part contributes;

\begin{equation}
\label{eq:m_I_contribution}
m_I \psi^A \bar{\psi} B \delta_{AB}.
\end{equation}

In terms of real fermions, $\psi^a, a = 1, \ldots, 2l$, we get

\begin{equation}
\label{eq:real_fermions}
\frac{i}{2} m_I \psi^a T_{ab}^{(3)} \psi^b.
\end{equation}

This term is naturally incorporated in the supersymmetric quantum mechanics (3.21) by adding it to $T_{ab}$, since the differential condition on $T$ allows a shift of $T$ by a covariantly constant piece.

When we quantise the supersymmetric quantum mechanics the term (3.24) will contribute a term $N_f m_I$ to the Hamiltonian where $N_f$ is the hypermultiplet fermion number. Recalling the discussion at the end of section 2.3, this will lead to the mass of the BPS states of the supersymmetric quantum mechanics being given by

\begin{equation}
\label{eq:BPS_mass}
M \simeq b \cdot g - a \cdot q + \frac{(b \cdot q)^2}{2b \cdot g} + N_f m_I,
\end{equation}

This result is in precise accord with the BPS mass formula arising from the general $\mathcal{N} = 2$ central charge formula. The latter can be written

\begin{equation}
\label{eq:general_formula}
M = \left| i(b \cdot g - a \cdot q + N_f m_I) + (b \cdot q + a \cdot g + N_f m_R) \right|,
\end{equation}

which reduces to (3.26) in the moduli space approximation in which $a \cdot g = 0$ and $m_R$ is neglected compared to $b$.\footnotetext{Since we work with the rotated fields $a, b$, we interpret $(m_R, m_I)$ to have been similarly rotated.}
4 More Potentials from the Matter Sector

In this section we analyse situations when one can turn on additional scalar vevs in the hypermultiplets while leaving the $U(1)$ gauge symmetries of the Coulomb phase intact. This will lead to additional potential terms in the supersymmetric low-energy dynamics of the monopoles. Considering the potential terms in the matter Lagrangian (3.1), we see that this is possible when the matter representation contains a zero-weight vector. Moreover it is only possible when the hypermultiplets are massless. A trivial example is when the hypermultiplets are in the adjoint representation. Less trivial examples are, for instance, symmetric tensors for $SO(k)$ and anti-symmetric tensors for $Sp(k)$.

We will further assume in this section that the representation is real. In this case the Index bundle associated with the matter fermions has a symplectic structure group and is equipped with three covariantly constant complex structures, $I^{(s)}$. A special case is when we have a single massless adjoint hypermultiplet, whose zero modes live in cotangent bundle with complex structures $I^{(s)} = J^{(s)}$, $s = 1, 2, 3$. The field theory is then $N = 4$ Yang-Mills theory, so our derivation of the low-energy dynamics will include a derivation, en-passant, of the effective action for $N = 4$ monopoles that was first presented, based on symmetry considerations, in [21].

4.1 Bosonic potential

The effect on the monopole dynamics of allowing the scalar fields $M$ to acquire expectation values is determined in a very similar manner to the treatment of the second adjoint Higgs field $a$ in sections 2 and 3. We regard the vevs of the two complex scalars $M$’s as a perturbation of order $n = 1$ and perform a perturbative expansion.

A low-energy ansatz that solves the equations of motion to order $n = 1$ is obtained by shifting the ansatz (3.9) via

$$
\Psi = \psi^A \Psi_A, \\
M_1 = \bar{M}_1 - \frac{2}{D^2} (\bar{\chi} \Psi), \\
M_2 = \bar{M}_2 - \frac{2i}{D^2} (\bar{\chi}^c \gamma_5 \Psi),
$$

(4.1)

where $\bar{M}_{1,2}$ are order $n = 1$ and solve the covariant Laplace equation in the monopole background

$$
D^2 \bar{M}_{1,2} = 0.
$$

(4.2)
The new terms that arise from this shift after substituting into the field theory action are either linear or quadratic in $\bar{M}_{1,2}$. The linear pieces generate fermionic bilinears and are discussed in the next subsection, while the quadratic pieces correspond to bosonic potential terms.

It will be convenient to exchange the two complex scalars $M_{1,2}$, for four real $H_i$’s via

$$M_1 = H_3 + iH_0,$$
$$M_2 = -H_1 + iH_2.$$  \hspace{1cm} (4.3)

and similarly exchange $\bar{M}_{1,2}$ for four real $\bar{H}_i$’s. Next note that, given (4.2), $\partial H_i \epsilon_+$ is a fermion zero mode and hence can be expanded in terms of our basis:

$$\partial H_i \epsilon_+ = -i\gamma^5\sqrt{2}K_i^A(z)\Psi_A.$$ \hspace{1cm} (4.4)

The quantities $K_i^A(z)$ define four sections on the dual of the Index bundle over the monopole moduli space.

After substituting the ansatz (4.1) into the field theory action and using (4.4) we find that the bosonic part of order $n=2$ that involves $H$ is given by

$$\frac{1}{2} \int d^3x \, (D_\mu \bar{H}_i)^\dagger (D_\mu H_i) = |K_i^A|^2 = \frac{1}{2}K_{ia}K_i^a,$$ \hspace{1cm} (4.5)

where we rewrote the complex quantities $K_i^A$ in terms of real quantities $K_i^a$ by expanding

$$K_i^A = \frac{1}{\sqrt{2}} (K_i^{2A-1} + iK_i^{2A})$$ \hspace{1cm} (4.6)

Since $i$ runs from 0 to 3, there could be four such bosonic potentials.

4.2 Fermion bilinear terms

After substituting (4.1) into (3.1) one finds that the fermionic bilinear terms arising from the kinetic terms of the $H$’s vanish. The non-zero fermionic bilinear terms arise from the Yukawa couplings in (3.1). Since the derivation is reasonably long, we point out here that the key results are given in (4.19) and (4.35).

For fermions in a real representation of the gauge group, it is often convenient to introduce symplectic Majorana fermions: $\bar{\Psi}$ and $\bar{\chi}$ are each a doublet of Dirac spinors defined by

$$\bar{\chi} = \left( \begin{array}{c} \chi \\ -i\gamma_5\chi^c \end{array} \right), \hspace{1cm} \bar{\Psi} = \left( \begin{array}{c} \Psi \\ -i\gamma_5\Psi^c \end{array} \right).$$ \hspace{1cm} (4.7)
The Yukawa terms in (3.1) can then be written compactly as

\[ i \int d^3 x \tilde{\Psi} \tau_i \tilde{\chi} H_i, \]  

(4.8)

where \( \tau_i = (1, -i\tau_s) \) and \( H_i \) are real. The charge-conjugation of the spinor, \( \chi \), is defined as

\[ \chi^c \equiv C \tilde{\chi}^T = C(\gamma^0)^T \chi^* \]  

(4.9)

and similarly for \( \Psi^c \), where the charge-conjugation matrix \( C \) satisfies,

\[ CC^* = -1, \quad C \gamma^M = -\gamma^M C. \]  

(4.10)

It follows that \( C \Gamma^T = -\Gamma C \).

Accordingly, the zero mode ansatz for \( \tilde{\chi} \) is given by

\[ \tilde{\chi} = \left( \tilde{\lambda}^m \delta_m \gamma^\mu \Gamma^\mu \epsilon_+ \right), \]  

(4.11)

where \( \epsilon'_+ \equiv C \epsilon^*_+ \). Because \( \chi \) (and \( W \)) is in a real representation of the gauge group, complex conjugated zero modes can expressed as a linear combination of original zero modes:

\[ \delta_m \gamma^\mu \Gamma^\mu \epsilon_+ = C_m^k \delta_k \gamma^\mu \Gamma^\mu \epsilon_+. \]  

(4.12)

By a basis redefinition, the matrix \( C \) can be chosen to be anti-symmetric and unitary so that \( C^2 = -1 \). By taking the complex conjugate of the expression \( J^{(3)}_m \chi_k = i\chi_m \), it follows that \( C \) anticommutes with \( J^{(3)} \);

\[ C J^{(3)} = -J^{(3)} C. \]  

(4.13)

This matrix \( C \) generates a second complex structure on the moduli space which we will also denote by \( J^{(2)} \). Defining \( J^{(1)} = J^{(2)} J^{(3)} \) we obtain the hyper-Kähler structure of the monopole moduli space (which can be taken to be the same as (2.21) by an appropriate choice of complex structures on \( R^4 \)). We can use (4.12) to give an alternate expression for the zero mode ansatz of \( \tilde{\chi} \) where the roles of \( \epsilon \) and \( \epsilon'_+ \) are exchanged,

\[ \tilde{\chi} = \left( -\tilde{\lambda}^m C_m^k \delta_k \gamma^\mu \Gamma^\mu \epsilon'_+ \right). \]  

(4.14)

Using \( \delta_m \gamma^\mu = [s_m, D_\mu] \), we thus find two possible expressions for \( \chi H \)

\[ \chi \bar{H}_i = \tilde{\lambda}^m s_m \bar{\phi} \bar{H}_i \epsilon_+ + \ldots \]

\[ = -\tilde{\lambda}^m C_m^k s_k \bar{\phi} \bar{H}_i \epsilon'_+ + \ldots, \]  

(4.15)
and also for $-i\gamma_5\chi^c H$

$$-i\gamma_5\chi^c H = \tilde{\lambda}_m s_m \mathcal{D} \tilde{H}_i \epsilon^i_+ + \ldots$$
$$= \tilde{\chi}^m C_m k s_k \mathcal{D} \tilde{H}_i \epsilon^i_+ + \ldots$$

(4.16)

The ellipsis denote terms of the form $\mathcal{D}(\ldots)$, which do not contribute any new terms in the low energy dynamics, once we use the Dirac equation for $\Psi$ and the fact that the zero modes of $\Psi$ are chiral with respect to $\Gamma_5$. They will be ignored subsequently.

The matter fermion zero mode ansatz $\tilde{\Psi}$ takes the form

$$\tilde{\Psi} = \begin{pmatrix} \psi^A \Psi_A^A \\ -i\gamma_5 \psi^c \Psi^c_A \end{pmatrix}.$$  

(4.17)

We also have

$$\mathcal{D} \tilde{H}_i \epsilon^i_+ = -i\gamma_5 \sqrt{2} K_i^A(z) \Psi_A,$$
$$\mathcal{D} \tilde{H}_i \epsilon^i_+ = -\sqrt{2} K_i^c(z) \Psi^c_A,$$

(4.18)

where the second equation is derived from the first (which is just (4.4)).

When determining the contributions from the Yukawa couplings, one has a choice of writing out the zero modes $\chi_m$ as $\delta_m W_{\mu} \Gamma^\mu \epsilon^i_+$ or equivalently as $C_m^k \delta_k W_{\mu} \Gamma^\mu \epsilon^i_+$. Of course the answer should not depend on such choices, but the expression one gets does depend on the choices. In fact, we also could rewrite the same expression based on a different $\epsilon$ associated with different complex structures such as $\epsilon_+ + i \epsilon^i_+$. This redundancy of expressions gives us a very important constraint on the quantities $K_i$. As will be shown in Appendix, it implies a holomorphicity condition on $K_i$’s;

$$(J^{(s)} \nabla) (I^{(s)} K_i) = \nabla K_i,$$

(4.19)

for $s = 1, 2, 3$. $s$ labels the three complex structure on the tangent and the Index bundles. This fact will be used crucially in the derivation of fermion bilinears.

In the following we are going to switch between the above two expansions, so that $\Psi$ is always paired up with $\mathcal{D} \tilde{H}_i \epsilon^i_+$ while $\Psi^c$ is always paired up with $\mathcal{D} \tilde{H}_i \epsilon^i_+$. This can be achieved by using the first line of (4.15) and (4.16) for the Yukawa terms involving $\tilde{H}_0$ and $\tilde{H}_3$, and using the second line for the Yukawa terms involving $\tilde{H}_1$ and $\tilde{H}_2$.

Consider first the Yukawa terms containing $\tilde{H}_0$. One term is

$$i\sqrt{2} \int d^3 x (\psi^A) \Psi_A^A (\tilde{\chi}^m s_m) \Gamma_5 K_0^B(z) \Psi_B + \ldots,$$

(4.20)

where the ellipses denote the second term arising from the charge conjugate. The two terms can then be written

$$-i\sqrt{2}(\tilde{\chi}^m \nabla_m) (K_{0A} \psi^A) - i\sqrt{2}(\tilde{\chi}^m)^i \nabla_m) (K_{0A} \psi^A)$$

(4.21)
For the Yukawa terms containing $\bar{H}_3$, one gets the same two terms multiplied by $-i$ and $i$, respectively, to give

$$-\sqrt{2}(\bar{\lambda}^m \nabla_m)(K_{3\bar{A}}\psi^{\bar{A}}) + \sqrt{2}((\bar{\lambda}^m)^\dagger \nabla_m)(K_{3\bar{A}}\psi^{\bar{A}}). \quad (4.22)$$

These expressions can be recast in a more useful form using the fact that

$$((\bar{\lambda}^m)^\dagger \nabla_m)(K_{0\bar{A}}\psi^{\bar{A}}) = 0 = ((\bar{\lambda}^m)^\dagger \nabla_m)(K_{3\bar{A}}\psi^{\bar{A}}). \quad (4.23)$$

This can be derived using (4.19) and the fact that the relation among $\bar{\lambda}$’s (2.37) implies

$$\bar{\lambda}^m \nabla_m = \frac{1}{2} \bar{\lambda}^m \left( \delta^k_m - i J_m^{(3)k} \right) \nabla_k,$$

$$(\bar{\lambda}^m)^\dagger \nabla_m = \frac{1}{2} (\bar{\lambda}^m)^\dagger \left( \delta^k_m + i J_m^{(3)k} \right) \nabla_k. \quad (4.24)$$

The operators $(1 \mp iJ^{(3)})\nabla$ are holomorphic and anti-holomorphic covariant derivatives, so $\bar{\lambda}^m \nabla_m$ is composed of holomorphic derivatives only, while $(\bar{\lambda}^m)^\dagger \nabla_m$ is composed of anti-holomorphic derivatives only. Using this the terms arising from the $\bar{H}_0$ Yukawa term can be written

$$-i\sqrt{2}(\bar{\lambda}^m + (\bar{\lambda}^m)^\dagger)\nabla_m(K_{0\bar{A}}\psi^{\bar{A}} + K_{0\bar{A}}\psi^{\bar{A}}), \quad (4.25)$$

while those from the $\bar{H}_3$ Yukawa term become

$$\sqrt{2}(-\bar{\lambda}^m + (\bar{\lambda}^m)^\dagger)\nabla_m(K_{3\bar{A}}\psi^{\bar{A}} + K_{3\bar{A}}\psi^{\bar{A}}). \quad (4.26)$$

We next use (2.38) to write the expressions in terms of the real and independent $\lambda$’s to get

$$-i(\lambda^m \nabla_m)(K_{0\bar{A}}\psi^{\bar{A}} + K_{0\bar{A}}\psi^{\bar{A}}) + i(\lambda^m J_m^{(3)k}\nabla_k)(K_{3\bar{A}}\psi^{\bar{A}} + K_{3\bar{A}}\psi^{\bar{A}}). \quad (4.27)$$

As the final step, we trade off complex $K$’s and $\psi$’s in favor of real ones and find

$$-i(\lambda^m \nabla_m)K_{0\bar{A}}\psi^{\bar{A}} + i(\lambda^m J_m^{(3)k}\nabla_m)K_{3\bar{A}}\psi^{\bar{A}}, \quad (4.28)$$

as the fermion bilinears arising from $\bar{H}_0$ and $\bar{H}_3$ Yukawa terms.

The action of $-i\tau_{1,2}$ exchanges $\chi$ and $-i\gamma_5\chi^c$, so the $\bar{H}_{1,2}$ Yukawa terms are a bit different. Expanding $\chi$ in terms of $C_m^k \delta_k W_{\mu} \Gamma^\mu \epsilon'_+$ instead, we find

$$-\sqrt{2}(\bar{\lambda}^m)^\dagger C_m^k \nabla_k(K_{1\bar{A}}\psi^{\bar{A}}) + \sqrt{2}((\bar{\lambda}^m)C_m^k \nabla_k)(K_{1\bar{A}}\psi^{\bar{A}}). \quad (4.29)$$
and
\[ i\sqrt{2}(\tilde{\lambda}^m)^\dagger C_m^k \nabla_k (K_{2A}^A) + i\sqrt{2}(\lambda^m C_m^k \nabla_k (K_{2A}^A)). \] (4.30)

Since \( J^{(3)}C = -C J^{(3)} \), the (anti-)holomorphic covariant derivatives are now paired with \( \tilde{\lambda}^i \)'s (\( \tilde{\lambda}^i \)'s). As in the case of \( \tilde{H}_{0,3} \) Yukawa terms, we can complete the above expression by adding appropriate (anti-)holomorphic derivatives of \( K_{1,2} \) (\( K_{1,2}^* \)). The end result is,
\[ i\lambda^m (C J^{(3)})^k m \nabla_k K_{1a} \psi^a + i\lambda^m C_m^k \nabla_k K_{2a} \psi^a, \] (4.31)

which can be rewritten as
\[ i\lambda^m J_m^{(1)k} \nabla_k K_{1a} \psi^a + i\lambda^m J_m^{(2)k} \nabla_k K_{2a} \psi^a, \] (4.32)

where we use the fact that \( C \) is identified with a second complex structure, \( J^{(2)} \), and that \( J^{(2)} J^{(3)} \) becomes yet another complex structure, \( J^{(1)} \), completing the triplet of complex structures necessary for the hyper-Kähler geometry.

Adding up all terms, we thus find the following set of fermion bilinears from the Yukawa terms,
\[ -i\lambda^m \nabla_m K_{0a} \psi^a + i \sum_{s=1}^{3} \lambda^m J_m^{(s)k} \nabla_k K_{sa} \psi^a. \] (4.33)

The identity (4.19) allows us to rewrite this as
\[ -i\lambda^m \nabla_m K_{0a} \psi^a - i \sum_{s=1}^{3} \lambda^m \nabla_m J_m^{(s)b} K_{sb} \psi^a. \] (4.34)

After combining with the bosonic potential terms derived in the last subsection, we find the supersymmetric potential terms arising from the matter Higgs fields having non-vanishing expectation values is given by
\[ -\frac{1}{2} \sum_{i=0}^{3} |K_i|^2 - i\lambda^m \nabla_m K_{0a} \psi^a - i \sum_{s=1}^{3} \lambda^m \nabla_m J_m^{(s)b} K_{sb} \psi^a. \] (4.35)

We will discuss the supersymmetry of the action including these extra potential terms in the next section.

5 Supersymmetric Low Energy Dynamics

For the convenience of the reader, this section summarises the general low-energy dynamics of monopoles in \( N = 2 \) Yang-Mills theories with hypermultiplets that we
have derived. We also discuss the quantisation. Firstly, the action is given by
\[
L = \frac{1}{2} \left( g_{mn} \dot{z}^m \dot{z}^n + i g_{mn} \lambda^m \dot{D}_t \lambda^n + i \psi^a \mathcal{D}^a \psi^b + \frac{1}{2} F_{mnab} \lambda^m \lambda^n \psi^a \psi^b \right. \\
- g^{mn} G_m G_n - i D_m G_n \lambda^m \lambda^n - i T_{ab} \psi^a \psi^b \\
- K^a_i K_{ia} - 2 i I^a_i b K_{ibm} \lambda^m \psi^a, \tag{5.1}
\]
where \( I^{(0)}_a = \delta^b_a \), and \( i \) runs from 0 to 3. The action is invariant under \( N = 4 \) supersymmetry transformations given by
\[
\delta z^m = -i \epsilon^m + i \epsilon_s J^{(s)m}_n \lambda^n, \\
\delta \lambda^m = (\dot{z}^m - G^m) \epsilon + J^{(s)m}_{n}(\dot{z}^n - G^n) \epsilon_s - i \epsilon_s \lambda^k \lambda^n J^{(s)l}_{k} \Gamma^m_{ln}, \\
\delta \psi^a = -A^a_m b \delta z^m \psi^b - \epsilon (I^{(i)})^a_b K^b_i - \epsilon_s (I^{(i)})^a_b (I^{(s)})^b_c K^c_i, \tag{5.2}
\]
where \( \epsilon, \epsilon_s \) are constant one component Grassmann odd parameters, provided that several differential constraints are met: The first is the well-known requirements that the moduli space is hyper-Kähler and the curvature \( F \) is of (1,1) type with respect to all three complex structures of the manifold. In addition \( G \) must be a tri-holomorphic Killing vector field, and the two form on the bundle \( T \) must satisfy
\[
T_{ab:m} = F_{mnab} G^m. \tag{5.3}
\]
The section \( K \)'s on the dual bundle must satisfy a holomorphicity condition
\[
(J^{(s)n}) (I^{(s)})_i K^i = \nabla K^i, \tag{5.4}
\]
for each \( s = 1, 2, 3 \), and must also be “preserved” under the translation by \( G \)
\[
G^m \nabla_m K_{ia} = T^b_a K_{ib}. \tag{5.5}
\]
An additional condition is that
\[
K^a_i I^{(s)}_{ab} K^b_j = 0. \tag{5.6}
\]
When the sections \( K \) are non-vanishing we also require
\[
(I^{(s)})^c_b T_{ca} = (I^{(s)})^c_a T_{cb}. \tag{5.7}
\]
We shall show in appendix A, that (5.5), (5.6) and (5.7) are indeed satisfied. The action is also invariant under the following symmetry transformation generated by the tri-holomorphic Killing vector field:
\[
\delta z^m = k G^m, \\
\delta \lambda^m = k G^m \lambda^n, \\
\delta \psi^a = k T^a_{b} \psi^b - A^a_{mb} \delta z^m \psi^b, \tag{5.8}
\]

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where \( k \) is a constant. This symmetry is responsible for the presence of a central charge in the superalgebra.

Let us summarize the origin of various terms.

- The first line contains the basic ingredient of the monopole dynamics in \( N = 2 \) Yang-Mills theories. The \( z^m \) are coordinates on the monopole moduli space with metric \( g_{mn} \). The \( \lambda \)'s take values in the tangent bundle while the \( \psi \)'s take values in the Index bundle of the matter fermions. Generically, this Index bundle has a unitary structure group, but for real or pseudo-real matter representations it is symplectic or orthogonal, respectively. All interactions are thus encoded in the geometry of the moduli space and of the Index bundles over it. These terms suffice if, up to \( U(1)_R \) rotation, a single adjoint Higgs field has a non-vanishing vacuum expectation value, and no other Higgs field does.

- The second line is necessary when the second adjoint Higgs field is turned on and is not proportional to the first in the Lie algebra space. This is possible for rank two or higher gauge groups. Extra information is contained in the tri-holomorphic Killing vector field \( G \), which is picked out by the adjoint Higgs expectation values via \( (2.30) \). The two-form \( T \) is defined via \( (3.13) \) and when the bare mass for the hypermultiplets is non-vanishing it includes a constant piece as discussed in section 3.3.

- The third line is necessary when scalar fields in real massless hypermultiplets get a vacuum expectation value and still preserves the unbroken \( U(1) \) gauge groups. In this case, the Index bundle is symplectic and admits three covariantly constant complex structures \( I^{(s)} \). The sections \( K \) must be \( G \)-invariant in the sense of Eq. \( (5.5) \), and must be holomorphic in the sense of Eq. \( (5.4) \). Their normalization is determined by the Higgs expectation values via \( (4.4) \).

An important special case of the above Lagrangian occurs when one has a single massless adjoint hypermultiplet. The field theory is then \( N = 4 \) Yang-Mills theory, and the \( \psi \)'s live in the tangent bundle. The above Lagrangian should then be the same as the complete monopole dynamics in \( N = 4 \) Yang-Mills theory, first presented in [21]. This can be seen easily by identifying \( K \)'s as the additional tri-holomorphic Killing vector fields on the moduli space and setting \( T_{ab} = G_{a,b} \).

\( ^5 \)That the \( K \)'s must be tri-holomorphic Killing vector fields arises from the additional supersymmetries.
To quantize the effective action we first introduce a frame $e^E_m$ and define $\lambda^E = \lambda^m e^E_m$ which commute with all bosonic variables. The remaining canonical commutation relations are then given by

\[
[z^m, p_n] = i\delta^m_n, \\
\{\lambda^E, \lambda^F\} = \delta^{EF}, \\
\{\psi^a, \psi^b\} = \delta^{ab}.
\] (5.9)

We can realize this algebra on spinors on the moduli space by letting $\lambda^E_m = \gamma^F / \sqrt{2}$, where $\gamma^F$ are gamma matrices. The states must also provide a representation of the Clifford algebra generated by the $\psi$'s. The supercovariant momentum operator defined by

\[
\pi_m = p_m - \frac{i}{4} \omega_m^{EF} [\lambda^E, \lambda^F] - \frac{i}{2} A_{iab} \psi^a \psi^b,
\] (5.10)

where $\omega_m^{EF}$ is the spin connection, then becomes the covariant derivative acting on spinors twisted in an appropriate way by $A$. Note that

\[
[\pi_m, \lambda^n] = i \Gamma_m^{nk} \lambda^k, \\
[\pi_m, \psi^a] = i A_m^{a} \psi^b, \\
[\pi_m, \pi_n] = -\frac{1}{2} R_{mnkl} \lambda^k \lambda^l - \frac{1}{2} F_{mnab} \psi^a \psi^b.
\] (5.11)

The supersymmetry charges take the form

\[
Q = \lambda^m (\pi_m - G_m) - \psi^a \sum_{i=0}^{3} (I^{(i)} K_i)_a, \\
Q_s = \lambda^m J^{(s)n} (\pi_n - G_n) - \psi^a \sum_{i=0}^{3} (I^{(i)} I^{(s)} K_i)_a.
\] (5.12)

The algebra of supercharges is given by

\[
\{Q, Q\} = 2(\mathcal{H} - \mathcal{Z}), \\
\{Q_s, Q_t\} = 2 \delta_{st}(\mathcal{H} - \mathcal{Z}), \\
\{Q, Q_s\} = 0.
\] (5.13)

where the Hamiltonian $\mathcal{H}$ and the central charge $\mathcal{Z}$ is given by

\[
\mathcal{H} = \frac{1}{2\sqrt{g}} \pi_m \sqrt{g} g^{mn} \pi_n + \frac{1}{2} G_m G^m + \frac{i}{2} \lambda^m \lambda^a D_m G_n \\
+ \frac{i}{2} \psi^a \psi^b T_{ab} - \frac{1}{4} F_{mnab} \lambda^a \lambda^b \psi^a \psi^b \\
+ \frac{1}{2} K^a_i K^i_a + i I^{(i)} K_{iab} \lambda^b \psi^a,
\]
\[
\mathcal{Z} = G^m \pi_m - \frac{i}{2} \lambda^m \lambda^a (D_m G_n) + \frac{i}{2} \psi^a \psi^b T_{ab}.
\] (5.14)
Note that the operator $i\mathcal{Z}$ is the Lie derivative $\mathcal{L}_G$ acting on spinors twisted by $T$.

Although the algebra of supercharges contains a central charge $\mathcal{Z}$ we see that the states will either preserve all four supersymmetries of the supersymmetric quantum mechanics if $\mathcal{H} = \mathcal{Z}$, or none. This is entirely consistent with the fact that the parent $N = 2$ field theory has a complex central charge and hence BPS states preserve 1/2 of the eight field theory supercharges, while generic states preserve none of the supersymmetry (of course the vacuum preserves all of the supersymmetry).

6 Conclusions

We have presented a detailed derivation of the effective action governing the low-energy dynamics of monopoles and dyons in $N = 2$ super-Yang-Mills theory with hypermultiplets. It is valid when both adjoint Higgs fields in the $N = 2$ vector multiplet have non-vanishing expectation values. We have thus derived the supersymmetric quantum mechanics presented in [22] and generalised it to include the effects of the hypermultiplet fermion zero modes.

Our dynamics is also valid for certain cases when it is possible to have Higgs fields in the hypermultiplets acquire expectation values while maintaining a non-trivial Coulomb branch. This situation arises when the matter representation contains a zero weight vector. Our derivation in section 4 analysed cases when the matter fields are in real representations. A special case of this is $N = 4$ super-Yang-Mills theory and we have thus derived the supersymmetric quantum mechanics of [21]. Note that a representation (of a hypermultiplet) does not have to be real to have a zero weight vector and it would be interesting to know what the dynamics is for this general case.

It is interesting that the low-energy dynamics of monopoles gives rise to supersymmetric quantum mechanics that have not been considered previously. We showed that they can be obtained by a non-trivial dimensional reduction of $(4,0)$ sigma models in two dimensions.

Finally, it would be interesting to use the effective action to study the BPS dyon spectrum in more general situations than have been considered so far. The most promising direction might be to generalise the approach of [31] using index theorems.

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Appendix A

In this appendix, we derive the conditions (4.19), (5.5) and (5.6) satisfied by the sections $K_i$ and also establish (5.7).

Holomorphicity Condition for $K_i$

First, we derive the holomorphicity condition (4.19). Since $\Psi$ is in a real representation, charge-conjugated zero modes can be expressed in terms of original zero modes as in the $\chi$ case,

$$\gamma_5 \Psi_A^c = i \tilde{C}_A^B \Psi_B,$$

where $\tilde{C}$ is an anti-symmetric unitary matrix with $\tilde{C}^2 = -1$. Then, using the expansions (4.17) and (4.18), the relations (4.15), (4.16) imply that

$$\tilde{\lambda}^m s_m K_i^A \tilde{C}_A^B \Psi_B + \ldots = \tilde{\lambda}^m C_m^n s_n K_i^A \Psi_A + \ldots,$$

where, for simplicity, we omitted the index $i$ in $K_i^A$ which plays no role in the appendix. Taking the inner product with $\Psi^\dagger_A \bar{\Psi}$, we find

$$\tilde{\lambda}^m \nabla_m K_i^A \tilde{C}_A^B \Psi_B + \ldots = \tilde{\lambda}^m C_m^n \nabla_n K_i^A \Psi_A + \ldots,$$

This is a nontrivial condition on $K$. Written in terms of the real quantities introduced in (4.6) and (2.38), it becomes

$$(1 + i J^{(3)}) \nabla I^{(2)} (1 - i J^{(3)}) K = -(1 + i J^{(3)}) J^{(2)} \nabla (1 + i I^{(3)}) K,$$

where $I^{(3)}$ is the third complex structure of the Index bundle which transforms the real part of $K$ into the imaginary part of $K$; $I^{(2)}$ is the second complex structure similar to $J^{(2)} = C$ in the $\chi$ case and has the block-diagonal form

$$I^{(2)} = \begin{pmatrix} \tilde{C} & 0 \\ 0 & -\tilde{C} \end{pmatrix},$$

when acting on $\begin{pmatrix} K_{2A-1} \\ K_{2A} \end{pmatrix}$. All the quantities are now real so the real and the imaginary parts of (A.4) should hold separately. In fact they reduce to the same condition

$$\nabla K = J^{(1)} \nabla I^{(1)} K + J^{(2)} \nabla I^{(2)} K - J^{(3)} \nabla I^{(3)} K.$$
Clearly (A.7) is consistent with the holomorphicity condition (4.19) but is not exactly the same. More conditions can be obtained by considering the fermionic zero modes associated with a complex structure other than \( J^{(3)} \). We first generalise (2.25), (2.26) by introducing \( c \)-number spinors \( \epsilon_s^{(s)} \), \( s = 1, 2, 3 \) satisfying

\[
\epsilon_+^{(s)} \epsilon_+^{(s)\dagger} = 1, \quad J_{\mu\nu}^{(s)} = -i\epsilon_+^{(s)\dagger} \Gamma_{\mu\nu} \epsilon_+^{(s)}, \quad J_{\mu\nu}^{(s)} \Gamma_{\nu} \epsilon_+^{(s)} = i\Gamma_{\mu} \epsilon_+^{(s)}, \quad (A.7)
\]

and denoting the corresponding zero modes as

\[
\chi_m^{(s)} = \delta_m W_{\mu} \Gamma_{\mu} \epsilon_+^{(s)}. \quad (A.8)
\]

The explicit form of \( \epsilon_+^{(s)} \) can be found in the following way. From the definition (4.12) of \( J^{(2)} = C \),

\[
\delta_m W_{\mu} \Gamma_{\mu} \epsilon_+^{(s)} = C_m \delta_n W_{\mu} \Gamma_{\mu} \epsilon_+^{(s)} = -J_{\mu\nu}^{(2)} \delta_m W_{\nu} \Gamma_{\mu} \epsilon_+, \quad (A.9)
\]

from which we find

\[
J_{\mu\nu}^{(2)} \Gamma_{\nu} \epsilon_+ = \Gamma_{\mu} \epsilon_+. \quad (A.10)
\]

(We will continue to omit the superscript label for quantities associated with \( J^{(3)} \).) Therefore \( \epsilon_+^{(2)} \) has the form

\[
\epsilon_+^{(2)} = e^{-i\pi/4} \sqrt{2} (\epsilon_+ - i\epsilon_+'), \quad (A.11)
\]

where the phase is chosen to simplify equations appearing below. \( \chi_m^{(2)} \) is then given by

\[
\chi_m^{(2)} = e^{-i\pi/4} \sqrt{2} (1 - iJ^{(2)})_m^n \chi_n \quad (A.12)
\]

With the definition \( \epsilon'_{+}^{(2)} = C \epsilon_{+}^{(2)^*} \), we can also expand the complex-conjugated zero modes in terms of \( \chi_n^{(2)} \),

\[
\delta_m W_{\mu} \Gamma_{\mu} \epsilon'_{+}^{(2)} = e^{i\pi/4} \sqrt{2} (-i + J^{(2)})_m^n \chi_n = (J^{(2)} J^{(3)})_m^n \chi_n^{(2)}, \quad (A.13)
\]

where the relation \( J_m^n \chi_n = i \chi_n \) is used. With \( J^{(1)} = J^{(2)} J^{(3)} \), the above equation corresponds to the counterpart of (4.12). A similar analysis can be repeated for \( J^{(1)} \), but we will omit the details.

Now let us consider the expansion of \( \mathcal{D} \bar{H}_t \epsilon_+^{(2)} \),

\[
\mathcal{D} \bar{H}_t \epsilon_+^{(2)} \equiv -i \sqrt{2} \gamma_5 R_t^{(2)A} \Psi_A. \quad (A.14)
\]
From the relation (A.11), it follows that $K_i^{(2)}$ is given by

$$K_i^{(2)} = e^{-i\pi/4} \frac{1}{\sqrt{2}} (K + i\bar{C}K_i^*).$$  \hspace{1cm} (A.15)$$

In terms of real quantities, this equation becomes

$$K_i^{(2)} = \frac{1}{2}(1 + I^{(2)})(1 + I^{(3)})K.$$  \hspace{1cm} (A.16)$$

With the expansion (A.14), the condition arising from (A.2) now takes the form

$$(1 + iJ^{(2)}) \nabla I^{(2)}(1 - iI^{(3)})K_i^{(2)} = -(1 + iJ^{(2)})J^{(1)}\nabla(1 + iI^{(3)})K_i^{(2)}. \hspace{1cm} (A.17)$$

Inserting (A.16) into (A.17) we obtain a condition for $K$,

$$\nabla K_i = J^{(1)}\nabla I^{(1)}K_i - J^{(2)}\nabla I^{(2)}K_i + J^{(3)}\nabla I^{(3)}K_i. \hspace{1cm} (A.18)$$

Performing a similar analysis for complex structures $J^{(s)}$ gives

$$\nabla K_i = J^{(s)}\nabla I^{(s)}K_i + J^{(t)}\nabla I^{(t)}K_i - J^{(u)}\nabla I^{(u)}K_i,$$

where $(s,t,u)$ is a cyclic permutation of $(1,2,3)$. Collectively, these condition implies the holomorphicity condition (4.19).

**Invariance of $K_i$’s under $G$**

To establish (5.5) consider the following integral:

$$\frac{1}{\sqrt{2}} \int d^3x \Psi_A^{\dagger} \gamma^0 G^m s_m \slashed{D} \bar{H}_i \epsilon_+.$$ \hspace{1cm} (A.20)$$

After substituting (4.18) and directly integrating, this becomes

$$G^m \nabla_m K_{i\bar{A}}. \hspace{1cm} (A.21)$$

Alternatively, we can commute $\slashed{D}$ through $G^m s_m$ and integrate by parts (noting that the surface term vanishes) to get

$$- \frac{1}{\sqrt{2}} \int d^3x D_\mu \Psi_A^{\dagger} \gamma^0 \Gamma_\mu G^m s_m \bar{H}_i \epsilon_+ + \frac{1}{\sqrt{2}} \int d^3x \Psi_A^{\dagger} \gamma^0 G^m [s_m , \slashed{D}] \bar{H}_i \epsilon_+. \hspace{1cm} (A.22)$$

The first term vanishes since $\gamma_5 \Psi_A$ is a zero mode while the second term can be written using (2.18) and (2.36) as

$$\frac{1}{\sqrt{2}} \int d^3x \Psi_A^{\dagger} \gamma^0 G^m \delta_m W_\mu \Gamma_\mu \bar{H}_i \epsilon_+ = -\frac{1}{\sqrt{2}} \int d^3x \Psi_A^{\dagger} \gamma^0 D_\mu \bar{a} \Gamma_\mu \bar{H}_i \epsilon_+,$$

(A.23)
which equals
\[
\frac{1}{\sqrt{2}} \int d^3x \Psi_A^\dagger \gamma^0 \bar{a} \partial \bar{H}_i \epsilon_+ = K_i^B \int d^3x \Psi_A^\dagger \bar{a} \Psi_B. \tag{A.24}
\]
Thus we find
\[
G^m \nabla_m K_{iA} = T_A^{\ B} \bar{K}_{iB}. \tag{A.25}
\]
Repeating the exercise for the charge-conjugated version, we find
\[
G^m \nabla_m K_{ia} = T_a^{\ b} \bar{K}_{ib}. \tag{A.26}
\]

Vanishing of \( \langle K_i | I^{(s)} | K_j \rangle \)

Consider the simplest case of \( I^{(3)} \). From the definition of the \( K \)'s, this inner product is equal to the integral
\[
-\frac{i}{2} \int d^3x \left( \epsilon_+^\dagger \partial \bar{H}_i \partial \bar{H}_j \epsilon_+ - (\epsilon_+')^\dagger \partial \bar{H}_i \partial \bar{H}_j \epsilon_+ \right). \tag{A.27}
\]
Since \( \partial^2 \epsilon_+ = \partial^2 \epsilon_+ = 0 \) this is a boundary integral given by
\[
-\frac{i}{2} \oint d\hat{n}_\mu \bar{H}_i D_\nu \bar{H}_j \left( \epsilon_+^\dagger \Gamma_\mu \Gamma_\nu \epsilon_+ - (\epsilon_+')^\dagger \Gamma_\mu \Gamma_\nu \epsilon_+ \right). \tag{A.28}
\]
Since \( \epsilon_+ \) and \( \epsilon_+ \) are normalized to unity, the symmetric part of \( \Gamma_\mu \Gamma_\nu \) in each term cancel. The anti-symmetric part is proportional to \( J^{(3)}_{\mu\nu} \) which is a complex structure on \( R^4 \);
\[
\oint d\hat{n}_\mu \bar{H}_i D_\nu \bar{H}_j J^{(3)}_{\mu\nu}. \tag{A.29}
\]
This integrand consists of angular covariant derivatives of \( \bar{H}_j \) contracted with \( \bar{H}_i \) as well as a term involving the action of the adjoint field \( b \) on \( \bar{H}_j \) again contracted with \( \bar{H}_i \). Let's work in the unitary gauge where the unbroken gauge \( U(1) \) generators are taken to be diagonal. In the asymptotic region the only surviving terms are then ordinary angular derivatives on \( \bar{H}_j \), since all the other terms are exponentially small and do not contribute to the surface integral.

Since \( \bar{H}_j \) must solve the ordinary 3-dimensional Laplace equation at large \( r \) its asymptotic form is given by
\[
\bar{H}_j = \langle \bar{H}_j \rangle + \sum_{lm} c_{lm} Y_{lm} \frac{r}{r^{l+1}} + \cdots, \tag{A.30}
\]
where \( c_{lm} \) are constant vectors, \( Y_{lm} \) are the 3-dimensional spherical harmonics and the ellipsis denotes terms that are exponentially small in large \( r \). Since the coefficient
of the leading 1/r piece, \( Y_{00} \), is a constant, the boundary integral vanishes on the asymptotic two-sphere. Similar consideration starting with different \( \epsilon_+ \) as in the derivation of the holomorphicity condition above leads us to

\[
\langle K_i | I^{(s)} | K_j \rangle = 0. \quad (A.31)
\]

for \( s = 1, 2, 3 \).

Establishing \((I^{(s)})^c_b T_{cab} = (I^{(s)})^c_a T_{cb}\)

For \( I^{(3)} \) this condition is equivalent to the statement that \( T_{AB} = T_{AB} = 0 \) which is true by definition. For \( I^{(2)} \) consider the term with \( a, b \) both being holomorphic indices \( A, B \). We then have

\[
I^{(2)C}_B T_{CA} = \int d^3 x \Psi_\rho^L \tilde{a} \Psi_A = \int d^3 x \psi_\rho^T C \tilde{a} \psi_A \quad (A.32)
\]

where we have used \((A.1)\). This is symmetric in \( A, B \) since both \( C \) and the group generators are anti-symmetric. Other components and \( I^{(1)} \) can be dealt with similarly.

### Appendix B

The supersymmetric quantum mechanics \((5.1)\), which generalises that presented in \([22]\), is as far as we know new. We show here that it can be obtained from a non-trivial, “Scherk-Schwarz”, dimensional reduction of a two-dimensional sigma model with \((4,0)\) supersymmetry.

Let \((\sigma^0, \sigma^1, \theta^+)\) be coordinates of two-dimensional \((1,0)\) superspace and consider the following action

\[
\mathcal{S} = \frac{1}{2} \int d^2 \sigma d\theta^+ \left[ i D_+ z^m \partial_- z^n g_{mn} + \psi^a_- \nabla_+ \psi^b_- h_{ab} \right]. \quad (B.1)
\]

Here \( \sigma^\# = (\sigma^0 + \sigma^1)/2 \), \( \sigma^\# = (\sigma^0 - \sigma^1)/2 \) and \( D_+ = \partial_{\theta^+} - i \theta^+ \partial_{\phi^+} \). The scalar superfield \( z^m \) is a map from \((1,0)\) superspace to a target \( \mathcal{M} \) and the Grassmann odd superfield \( \psi^a_- \) takes values in a vector bundle over \( \mathcal{M} \). \( h_{ab} \) is a fiber metric satisfying \( \nabla_i h_{ab} = 0 \) and \( \nabla_+ \psi^a_- = D_+ \psi^a_- + A_m^a b D_+ z^m \psi^b_-, \) where \( A \) is a connection on the vector bundle.

The component form of the action can be obtained by first expanding the superfields via

\[
\begin{align*}
  z^m &= z^m + i \theta^+ \lambda^m_+ \\
  \psi^a_- &= \psi^a_- + \theta^+ f^a_-
\end{align*}
\]

\[(B.2)\]
After eliminating the auxiliary fields $f$ via their equations of motion we obtain

$$S = \frac{1}{2} \int d^2 l [ \partial_\pm \partial_{\pm} g_{mn} + i \lambda_+^m \nabla_+ \lambda_+^n g_{mn} + i \psi_-^a \nabla_- \psi_-^b h_{ab} + \frac{1}{2} F_{mnab} \lambda_+^m \lambda_+^n \psi_-^a \psi_-^b ],$$

(B.3)

where $F$ is the curvature of the connection $A$ and $\nabla_+$ and $\nabla_-\neq$ are the covariantization of $\partial_+$ and $\partial_-\neq$, respectively, with the pull back of the Christoffel symbols.

Let us suppose that the target manifold is hyper-Kähler and that the connection is tri-holomorphic so that the sigma model admits an extended $(4,0)$ supersymmetry. Suppose in addition that the action is invariant under the symmetry transformations generated by a tri-holomorphic Killing vector field $G^m$:

$$\delta z^m = k G^m, \quad \delta \lambda_+^m = -G^m \lambda_+^n, \quad \delta \psi_-^a = -T_{a}^{b} \psi_-^b + A_{m}^{a} \delta z^m \psi_-^b, \quad \delta \lambda_+^m = -G^m \lambda_+^n, \quad \delta \psi_-^a = -T_{a}^{b} \psi_-^b + A_{m}^{a} \delta z^m \psi_-^b,$$

(B.4)

where $k$ is a constant and the tensor $T_{ab} = -T_{ba}$ must satisfy

$$G^k F_{knab} = -T_{abm}, \quad \text{(B.5)}$$

which determines $T$ up to covariantly constant terms.

Ordinary dimensional reduction to a supersymmetric quantum mechanics is implemented by assuming that all of the fields are independent of the coordinate $\sigma^1$. Scherk-Schwarz reduction is achieved by demanding the weaker condition that the Lagrangian is independent. Using the invariance under the symmetry transformations (B.4) this can be achieved by letting the $\sigma^1$ dependence of the fields be given by

$$\partial_{\sigma^1} z^m = -G^m, \quad \partial_{\sigma^1} \lambda_+^m = -G^m \lambda_+^n, \quad \partial_{\sigma^1} \psi_-^a = -T_{a}^{b} \psi_-^b + A_{m}^{a} \psi_-^b.$$

(B.6)

After integrating over $\sigma^1$ one then obtains the following action

$$S = \frac{1}{2} \int dt [ \dot{z}^m \dot{z}^n g_{mn} - G^m G^m g_{mn} + i \lambda_+^m D_t \lambda_+^n g_{mn} + i \lambda_+^m \lambda_+^n G_{mn} + \frac{1}{2} F_{mnab} \lambda_+^m \lambda_+^n \psi_-^a \psi_-^b ]. \quad \text{(B.7)}$$

Identifying $\lambda_+$’s with $\lambda$’s, and $\psi_-$’s with $\psi$’s, we recover precisely the effective action (B.20) that describes the dynamics of monopoles with fermionic contributions from the hypermultiplets.
To obtain the supersymmetric quantum mechanics when the hypermultiplets have non-zero expectation values, we generalise the above construction\[4\] by performing Scherk-Schwarz reduction on a (4,0) model with potential \[30\]. To do this we add a (1,0) supersymmetric term,

\[
\Delta S = \frac{1}{2} \int d^2 \sigma d\theta^+ 2v_a \psi^a, \tag{B.8}
\]
to (B.1) where \(v_a\) is a section of the dual of the Index bundle. The combined quantum mechanics action is invariant under the symmetry transformations (B.4) provided that in addition to (B.5) the section satisfies

\[
v^a T_{ab} + v_{b;k} G^k = 0. \tag{B.9}
\]

Since the action (B.8) does not contain any derivatives, the Scherk-Schwarz reduction is equivalent to ordinary dimensional reduction. After eliminating the auxiliary fields it leads to

\[
\Delta S = \frac{1}{2} \int dt[-v_a v_b h^{ab} + 2iv_{a;m}\lambda^m \psi^a]. \tag{B.10}
\]

The combined action is automatically invariant under an \(N = 1\) supersymmetry. The extended \(N = 4\) supersymmetry of (B.7) will extend to that of the combined action under suitable conditions on the section \(v^a\). The supersymmetry transformations are

\[
\begin{align*}
\delta z^m &= -i \epsilon \lambda^m + i \epsilon_s J^{(s)m}_n \lambda^n, \\
\delta \lambda^m &= (z^m - G^m) \epsilon + J^{(s)m}_{n}(\dot{z}^n - G^n) \epsilon_s - i \epsilon_s \lambda^k \lambda^n J^{(s)l}_k \Gamma^m_{ln}, \\
\delta \psi^a &= -A^{a}_{m} b \delta z^{m} \psi^{b} + \epsilon v^{a} + \epsilon_s t^{a}_{(s)},
\end{align*}
\tag{B.11}
\]

provided that the sections \(t^{a}_{(s)}, s = 1, 2, 3\) can be found satisfying

\[
\begin{align*}
(v_a t^{a}_{(s)})_{;m} &= 0, \\
J^{(s)n}_{m} \nabla^n t^{a}_{(s)} &= -\nabla^m t^{a}_{(s)}, \\
G^m t^{a}_{(s);n} &= T_{ab} t^{b}_{(s)},
\end{align*}
\tag{B.12}
\]

where \(J^{(s)}\) are the three complex structures on the target manifold. Note that these conditions imply that the norm of \(v\) and those of the \(t\)’s differ only by a constant.

Consider now the particular case that the bundle associated with the fermionic variables \(\psi^a\) has the structure group \(Sp(n)\). In this case there exists four covariantly

---

\[6\] Another generalization, is to reduce a model with torsion \(H = db\). In the case that the Lie-derivative with respect to \(G\) of the two-form \(b\) vanishes the Scherk-Schwarz reduction proceeds in a straightforward manner. We do not present any details here as there is no obvious application to monopole dynamics.
constant rank-two tensors; the identity $I^{(0)}$, and the three complex structures $I^{(s)}$.

Let us write the section in terms of four vector fields $K_i$ via

$$v^a = - \sum_{i=0}^{3} (I^{(i)})^a_b K^b_i.$$  \hspace{1cm} (B.13)

A solution for the three related sections $t^a_{(s)}$ is

$$t^a_{(s)} = - \sum_{i=0}^{3} (I^{(i)})^a_b (I^{(s)})^b_c K^c_i,$$  \hspace{1cm} (B.14)

providing that the $K_i$’s satisfy

$$J_{m}^{(s)k} \nabla_k (I^{(s)}_a^b K_{ib}) = \nabla_m K_{ia} \quad \text{(no sum on $s$)},$$

$$K^a_i I^{(s)}_{ab} K^b_j = \text{constant},$$

$$G^a K_{ia;}^n = T_{ab} K^b_i,$$

$$(I^{(s)})^c_a T_{ca} = (I^{(s)})^c_a T_{cb}.$$  \hspace{1cm} (B.15)

Note that the first equation implies that (anti-)holomorphic covariant derivative of (anti-)holomorphic part of $K_i$ vanishes. If we make further assumption the constants $K^a_i I^{(s)}_{ab} K^b_j$ actually vanishes, the supersymmetric potential terms are given by

$$- \frac{1}{2} \sum_{i=0}^{3} \left( K^a_i K_{ia} + 2i I^{(i)}_a^b K^b_{ib} \lambda^m \psi^a \right),$$  \hspace{1cm} (B.16)

which are precisely those arising from hypermultiplet vacuum expectation values that we established in section 4.

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