Eguchi–Hanson metrics arising from Kähler-Einstein edge metrics

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Dedicated to Scott Wolpert on the occasion of his retirement

Abstract

Calabi–Hirzebruch manifolds are higher-dimensional generalizations of both the football and Hirzebruch surfaces. We construct a family of Kähler–Einstein edge metrics singular along two disjoint divisors on the Calabi–Hirzebruch manifolds and study their Gromov–Hausdorff limits when either cone angle tends to its extreme value. As a very special case, we show that the celebrated Eguchi–Hanson metric arises in this way naturally as a Gromov–Hausdorff limit. We also completely describe all other (possibly rescaled) Gromov–Hausdorff limits which exhibit a wide range of behaviors, resolving in this setting a conjecture of Cheltsov–Rubinstein. This gives a new interpretation of both the Eguchi–Hanson space and Calabi’s Ricci flat spaces as limits of compact singular Einstein spaces.

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1 Motivation

The main motivation for this work is a program of Cheltsov–Rubinstein concerning the small angle deformation of Kähler–Einstein edge (KEE) metrics, which in particular makes the following prediction [6, Conjecture 1.11]:

**Conjecture 1.1.** Suppose that $(X, D)$ is strongly asymptotically log Fano manifold with $D$ smooth and irreducible. Suppose that $\kappa := \inf\{\mathbb{N} \ni k \leq \dim X : (K_X + D)^k = 0\} \leq \dim X$, and that there exist KEE metrics $\omega_\beta, \beta \in (0, \epsilon)$ on $(X, D)$ for some $\epsilon > 0$. Then, $(X, D, \omega_\beta)$ converges in an appropriate sense as $\beta$ tends to zero to a generalized KE metric $\omega_\infty$ that is Calabi–Yau along its generic $(\dim X + 1 - \kappa)$-dimensional fibers.

In this article we actually treat a slightly more general situation where $D$ is allowed to have two disjoint smooth components $D = D_1 + D_2$, but the setting is essentially identical to that of Conjecture 1.1 since the angle $2\pi \beta_1$ along $D_1$ of the KEE metric $\omega_{\beta_1, \beta_2}$ actually determines the angle $2\pi \beta_2$ along $D_2$ and, importantly, vice versa.

A second motivation for this article is given by a prediction posed by two of the present authors in a previous work concerning the large angle limit of a family of Kähler–Einstein edge metrics constructed on the second Hirzebruch surface $\mathbb{F}_2$. On that surface let $D_1 := Z_{-2}$ denote the $-2$-curve and $D_2 := Z_2$ the smooth infinity section, a $2$-curve satisfying $Z_{-2} \cap Z_2 = \emptyset$. According to [14, Theorem 1.2] there exist for each $\beta_1 \in (0, 1)$ a unique Kähler–Einstein edge metric $\omega_{\beta_1, \beta_2}$ with angle $2\pi \beta_1$ along $Z_{-2}$ and angle $2\pi \beta_2 = 2\pi (2\beta_1 - 3 + \sqrt{9 + 12 \beta_1 - 12 \beta_1^2})/4$ along $Z_2$ and cohomologous to $\frac{1 + \beta_2}{1 - \beta_1} [Z_2] - [Z_{-2}]$. The following prediction was made [14, Remark 5.1]:

**Conjecture 1.2.** As $\beta_1$ tends to 1, an appropriate limit of $(\mathbb{F}_2, \omega_{\beta_1, \beta_2})$ converges to the Eguchi–Hanson metric.

The virtue of Conjecture 1.2 is that it proposes a remarkable new interpretation of the Eguchi–Hanson space from mathematical physics as an angle deformation limit of compact singular Einstein spaces with edge singularities.

The goal of the present article is to treat both conjectures and a bit more. We solve Conjecture 1.1 in the setting of Calabi–Hirzebruch manifolds as well as solve Conjecture 1.2. In fact we solve a generalized version of Conjecture 1.2 that interprets Calabi’s Ricci flat spaces as limits of compact KEE spaces. Moreover, due to the existence of two disjoint divisors there are also interesting limits to study that are not part of the above conjectures: these limits are studied in Theorems 2.4 and 2.8 below.

2 Results

Let $M$ be a compact Kähler manifold and $D = D_1 + \cdots + D_r$ a simple normal crossing divisor in $M$. A Kähler metric $\omega$ is said to have edge singularity along $D$ if $\omega$ is smooth on $M \setminus D$ and asymptotically equivalent to the model edge metric along $D$ [12, Definition 3.1]. The study of
Kähler–Einstein edge metrics dates back to Tian [15] where he considered applications of such metrics to algebraic geometry. Cheltsov–Rubinstein [6] initiated the program of studying small angle limits of Kähler–Einstein edge metrics. In previous works, two of us treated the Riemann surface footballs case and Hirzebruch surfaces case by first constructing Kähler–Einstein edge metrics on the manifolds using Calabi ansatz and then studying their limiting behaviors when the cone angles tend to 0 [14, 13]. In this paper, we consider a more general setting, Calabi–Hirzebruch manifolds. To construct these KEE metrics we use the standard Calabi ansatz in Section 4, generalizing [14]. The angle at either divisor $D_1 := Z_{n,k}$ or $D_2 := Z_{n,-k}$ then determines the angle on the other divisor which leads to two families of KEE metrics $\eta_{\beta_1}$ and $\xi_{\beta_2}$ on $\mathbb{F}_{n,k}$.

**Remark 2.1.** Very recently, Biquard–Guenancia completely solved the folklore case $\kappa = 1$ of Conjecture 1.1 [3] (that case of the conjecture seems to have been already conjectured by Tian and Mazzeo in the 90’s and then explicitly stated around 2009 by Donaldson [11, 10, 7]), and interestingly the Calabi ansatz makes an appearance in their proof as well. It would be interesting to explore whether some of the rather elementary ideas here can be combined with some of their deep estimates to attack the general case of Conjecture 1.1.

**Remark 2.2.** It is interesting to note that Abreu [1, §5] and subsequently Atiyah–LeBrun [2, (5.5)–(5.6)] also studied a family of Einstein metrics with orbifold/cone-edge singularities converging (roughly) to the Eguchi–Hanson metric. There are, however, important differences between their construction and ours: (i) their metrics are not Kähler, (ii) they work on the space $\mathbb{P}^2$ while we work on the second Hirzebruch surface $\mathbb{F}_2$ (which generalizes to $\mathbb{F}_{n,k}$ in higher-dimensions), (iii) the metrics they construct in the sequence are singular along a single $\mathbb{P}^1$ divisor instead of two $\mathbb{P}^1$ divisors, as in our construction, (iv) their limit metric is not precisely the Eguchi–Hanson metric, but rather its double cover (they obtain an angle of $4\pi$ along a $\mathbb{P}^1$ divisor). While our metrics are constructed directly from the Calabi ansatz as Kähler–Einstein edge metrics, Abreu constructs extremal (non-Einstein) Kähler metrics using toric formalism and then uses Derdzinki’s theorem to conclude that a conformal rescaling (dividing by the scalar curvature squared) will be Einstein (but not Kähler). LeBrun obtains the same family of metrics using a completely different method specific to dimension 4. One could take the double cover of our construction. That would have the effect of, roughly speaking, the curve being pushed out to infinity becoming smooth in the limit as the angle there would tend to $2\pi$ (instead of $\pi$), while the other angle tends to $4\pi$. But even then it is not clear how to relate our metrics that are Kähler–Einstein edge (and in particular Kähler) to the Abreu–Atiyah–LeBrun metrics, that are not Kähler.

Motivated by the previous remark, we pose:

**Problem 2.3.** Relate the Abreu–Atiyah–LeBrun metrics to ours.

A common feature for footballs, Hirzebruch surfaces, and Calabi–Hirzebruch manifolds is that there are two smooth disjoint divisors and hence two angle parameters $\beta_1, \beta_2$. In Section 3, we review the construction of Calabi–Hirzebruch manifolds [5, 9], denoted by $\mathbb{F}_{n,k}$ for $\mathbb{N} \ni n \geq 2$ and $k \in \mathbb{N}$, and define two (families of) model metrics. The first are Ricci-flat edge metrics on the total space of the (non-compact) line bundle $-kH_{\mathbb{P}^{n-1}}$, 

$$\omega_{\text{eh},n,k},$$

and an edge singularity of the angle $2\pi n/k$ along $Z_{n,k}$. The second are compact Kähler–Einstein edge spaces with positive Ricci curvature $(n+1)/k$ and an edge singularity of angle $2\pi/k$,

$$\omega_{\text{orb},n,k},$$

3
on the weighted projective space $\mathbb{P}^n(1, \ldots, 1, k)$. These two model spaces turn out to be the different Gromov–Hausdorff limits of large-angle limits of the KEE metrics we construct on the Calabi–Hirzebruch manifolds.

### 2.1 Large angle limits

The following result describes precisely the different large-angle limits that arise from these KEE metrics by either using different pointed limits or else parametrizing the angles in different ways. Note that $\beta_1$ ranges in $(0, n/k)$ and $\beta_2$ ranges in $(0, 1/k)$.

**Theorem 2.4.** Fix a base point $p$ on the zero section of $\mathbb{F}_{n,k}$ and $q$ on the infinity section. The pointed metric space $(\mathbb{F}_{n,k}, \eta_{\beta_1}, p)$ converges in the pointed Gromov–Hausdorff sense to $(-kH_{\mathbb{P}^{n-1}}, \omega_{\text{eh}, n,k}, p)$ as $\beta_1$ tends to $n/k$. On the other hand, $(\mathbb{F}_{n,k}, \xi_{\beta_2}, q)$ converges in the pointed Gromov–Hausdorff sense to $(\mathbb{P}^n(1, \ldots, 1, k), \omega_{\text{orb}, n,k}, q)$ as $\beta_2 \to 1/k$.

In particular, when $n = k = 2$, Theorem 2.4 resolves Conjecture 1.2. Theorem 2.4 amounts to saying that the famous Eguchi–Hanson metric from mathematical physics is the Gromov–Hausdorff limit of compact Kähler–Einstein edge metrics that we construct on the second Hirzebruch surface (see Remark 6.3 for more details or see Appendix B for another proof). More generally, when $n = k$, Theorem 2.4 recovers a family of Ricci-flat metrics on the total space of canonical bundle of $\mathbb{P}^{n-1}$ that was constructed by Calabi [4], once again as a limit of compact Einstein spaces.

Figure 1: The upper part shows the Kähler edge structure on the Calabi–Hirzebruch manifold $\mathbb{F}_{n,k}$. When $n = 2$, $\mathbb{F}_{2,k}$ is the $k$-th Hirzebruch surface and $Z_{2,\pm k}$ is a $\mp k$-curve. The lower part shows the different limits described in Theorem 2.4. Note that on the left, we get a non-compact limit, with $q$ (together with all of $Z_{n,-k}$) pushed-out to infinity. On the right we get a compact limit, with $p$ limiting (together with all of $Z_{n,k}$) to an isolated orbifold point.

**Remark 2.5.** It would be interesting to find physical interpretations of Theorem 2.4 in the case $n = k = 2$ of the Eguchi–Hanson metric.
The elementary proof of Theorem 2.4 is divided into two parts. In section 5, we study asymptotic behaviors of Kähler–Einstein edge metrics $\eta_{\beta_1}$ and $\xi_{\beta_2}$ when $\beta_1$, or respectively $\beta_2$, is close to $n/k$ or $1/k$. In section 6, we prove the convergence results by studying a family of ODEs that arises from the construction of $\eta_{\beta_1}$ and $\xi_{\beta_2}$.

**Remark 2.6.** The two families of Kähler–Einstein edge metrics $\eta_{\beta_1}$ and $\xi_{\beta_2}$ are related via a simple, but important, rescaling. In Theorem 2.4, we see that different limits arise for those two family of metrics. The normalization factor can be obtained by studying the asymptotic behavior of $\xi_{\beta_2}$: see Proposition 5.5 for details.

**Remark 2.7.** As noted before Theorem 2.4, $\beta_1$ ranges in $(0, n/k)$ and $\beta_2$ ranges in $(0, 1/k)$. In proving Theorem 2.4 for the family $\eta_{\beta_1}$, one obtains the asymptotic dependence of $\beta_2$ on $\beta_1$ in terms of a parameter that controls the length of the fibers in the Hirzebruch fibration—see (5.9). This shows that as $\beta_1$ tends to its maximal value $n/k$, $\beta_2$ tends to its maximal value as well, $1/k$, and the divisor $Z_{n,k}$ gets pushed-off to infinity. It thus comes a bit as a surprise that when we consider the family $\xi_{\beta_2}$ parametrized in terms of $\beta_2$, and we let $\beta_2$ tend towards $1/k$, while the parameter $\beta_1$ still tends towards its maximal value $n/k$ we get a completely different limiting behavior: instead of a non-compact limit we get a compact limit via a metric degeneration along $Z_{n,k}$. Thus, studying the two families is an essential feature of the setting and leads to two completely different Gromov–Hausdorff limits as in Theorem 2.4. It would be interesting to generalize this phenomenon to other settings.

### 2.2 Small angle limits

Next, we resolve Conjecture 1.1 in the setting of $\mathbb{F}_{n,k}$. Denote by $\widehat{\eta}_{\beta_1}$ and $\widehat{\xi}_{\beta_2}$ fiber-wise rescalings of $\eta_{\beta_1}, \xi_{\beta_2}$, respectively (see (7.7) and §7).

**Theorem 2.8.** Both $(\mathbb{F}_{n,k}, \eta_{\beta_1})$ and $(\mathbb{F}_{n,k}, \xi_{\beta_2})$ converge in the Gromov–Hausdorff sense to $(\mathbb{P}^{n-1}, k\omega_{FS})$ as $\beta_1$ or $\beta_2$ tends to 0. Moreover, as $\beta_1 \searrow 0$, $(\mathbb{F}_{n,k}, \widetilde{\eta}_{\beta_1}, p)$ converges in the pointed Gromov–Hausdorff sense to $(\mathbb{P}^{n-1} \times \mathbb{C}^*, \frac{k}{n}(n\pi_1^*\omega_{FS} + \pi_2^*\omega_{CY}))$, and similarly for $(\mathbb{F}_{n,k}, \widetilde{\xi}_{\beta_2}, q)$, where $p, q \in \mathbb{F}_{n,k}$ are as in Theorem 2.4.

A compendium of the limit theorems in [13, 14] and the present work is shown in Table 1.

| $n$ | $k$ | $\beta_1$ | $\beta_2$ | Sequence | Limit | Reference |
|-----|-----|------------|------------|----------|--------|----------|
| 1   | $k$ | $\beta_1 \searrow 0$ | $\beta_2(\beta_1) \equiv \beta_1$ | football metrics | $(\mathbb{C}, \omega_{FS})$ | folklore, [13, Thm 1.3] |
| 1   | $k$ | $\beta_1 \nearrow 1$ | $\beta_2(\beta_1) \equiv \beta_1$ | football metrics | $(\mathbb{P}, \omega_{FS})$ | folklore, [13, Thm 1.2] |
| 2   | $k$ | $\beta_1 \searrow 0$ | $\beta_2(\beta_1) = \beta_1 + O(\beta_1^2)$ | KEE metrics | $(\mathbb{P}, \omega_{FS})$ | [14, Thm 1.2] |
| 2   | $k$ | $\beta_1 \nearrow 1$ | $\beta_2(\beta_1) \searrow 1$ | rescaled KEE metrics | $(\mathbb{P}^2(1,1), \omega_{n,k})$ | Thm 6.4 |
| 2   | $k$ | $\beta_1 \nearrow 2$ | $\beta_2(\beta_1) \nearrow 1$ | KEE metrics | $(\mathbb{F}_{n,k}, \omega_{FS})$ | Thm 6.4 |
| 2   | $k$ | $\beta_1 \nearrow 1$ | $\beta_2(\beta_1) \nearrow 1$ | rescaled KEE metrics | $(\mathbb{P}^2(-1,1), \omega_{FS})$ | Thm 6.4 |
| 2   | $k$ | the metric degenerates on $Z_k$ | | KEE metrics | $(\mathbb{F}_{n,k}, \omega_{FS})$ | Thm 6.4 |
| 2   | $k$ | the metric degenerates on $Z_{2,k}$ | | KEE metrics | $(\mathbb{P}^2(1,1), \omega_{n,k})$ | Thm 6.4 |
| $n$ | $k$ | $\beta_1 \nearrow n/k$ | $\beta_2(\beta_1) \nearrow 1/k$ | KEE metrics | $(\mathbb{P}^{n-1}(1,1), \omega_{n,k})$ | Thm 6.4 |
| $n$ | $k$ | $\beta_1 \searrow 0$ | $\beta_2(\beta_1) = \beta_1 + O(\beta_1^2)$ | rescaled KEE metrics | $(\mathbb{P}^{n-1}(1,1), \omega_{n,k})$ | Thm 6.4 |
| $n$ | $k$ | $\beta_2(\beta_2) \nearrow n/k$ | $\beta_2 \nearrow 1/k$ | rescaled KEE metrics | $(\mathbb{P}^{n-1}(1,1), \omega_{n,k})$ | Thm 6.4 |
| $n$ | $k$ | $\beta_1(\beta_1) \nearrow n/k$ | $\beta_2 \nearrow 1/k$ | rescaled KEE metrics | $(\mathbb{P}^{n-1}(1,1), \omega_{n,k})$ | Thm 6.4 |
| $n$ | $k$ | $\beta_2(\beta_2) \nearrow n/k$ | $\beta_2 \nearrow 1/k$ | rescaled KEE metrics | $(\mathbb{P}^{n-1}(1,1), \omega_{n,k})$ | Thm 6.4 |

Table 1: Limits of Kähler–Einstein edge metrics on $\mathbb{F}_{n,k}$, the $k$-th Calabi–Hirzebruch manifold of dimension $n$. 
2.3 Organization

In Section 3, we first review the construction of Calabi–Hirzebruch manifolds and then define two families of model metrics respectively on the total space of line bundles $-kH_{\mathbb{P}^{n-1}}$ and the Calabi–Hirzebruch manifolds. In Section 4, we construct Kähler–Einstein edge metrics on the Calabi–Hirzebruch manifolds following Calabi ansatz. In Section 5, we study the asymptotic behaviors of Kähler–Einstein edge metrics by reducing it to the study of some ODEs. In Section 6, we consider the Gromov–Hausdorff limits of the Kähler–Einstein edge metrics on the Calabi–Hirzebruch manifolds and find out the model metrics in the limit. Theorem 2.4 is then proved in two parts: Theorem 6.1 and Theorem 6.4. The first statement of Theorem 2.8 about the non-rescaled limits is proved in Theorems 7.1 and 7.10. The second statement of Theorem 2.8 concerning rescaled limits and pointed limits is contained in Theorems 7.6 and 7.11. We also discuss the relation between Theorem 6.1 and Theorem 6.4 as mentioned in Remark 2.6 in Corollary 6.5.

Finally, we end with several appendices whose main purpose is to bridge the mathematical, or “Calabi’s” formalism, and the physics, or “Eguchi–Hanson” setting. These should make the article accessible to the physics audience, as well as introduce some of the physics motivation for the article to the mathematical audience. Appendices A and B provide a brief review on basic properties of Eguchi–Hanson metrics and explain how to understand Eguchi–Hanson metrics as Gromov–Hausdorff limits of Kähler–Einstein edge metrics. In Appendix C, we provide more examples of Kähler–Einstein edge metrics and their Gromov–Hausdorff limit metrics.

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This article is dedicated to Scott Wolpert that aside from his seminal mathematical contributions to complex geometry has been instrumental in forging the University of Maryland as a leader in that field, and has positively impacted the careers of all three authors as either graduate students or faculty in the department (and, in particular, in all likelihood this article would not have appeared had it not been for his constant support).

3 Model metrics

We first review the construction of Calabi–Hirzebruch manifolds [5, 9].

Definition 3.1. The Calabi–Hirzebruch manifold, denoted by $F_{n,k}$, where $n, k \in \mathbb{N}$, is defined as follows:

$$F_{n,k} := \mathbb{P}(-kH_{\mathbb{P}^{n-1}} \oplus \mathbb{C}_{\mathbb{P}^{n-1}}), \quad n, k \in \mathbb{N},$$

where $H_{\mathbb{P}^{n-1}}$ is the hyperplane bundle over $\mathbb{P}^{n-1}$ and $\mathbb{C}_{\mathbb{P}^{n-1}}$ is the trivial one.

An alternative way to define $F_{n,k}$ is by adding an infinity section to the blow up of $\mathbb{C}^n/\mathbb{Z}_k$ at the origin [14, Lemma 2.1].

In this section, we introduce two model metrics $\omega_{\text{eh},n,k}$ and $\omega_{\text{orb},n,k}$, that are defined on the total space of line bundle $-kH_{\mathbb{P}^{n-1}}$ and on the weighted projective space $\mathbb{P}^n(1, \ldots, 1, k)$, respectively. They will serve as the candidates of limit metrics in the latter sections.

Denote by $[Z_1 : \cdots : Z_n]$ the homogeneous coordinates on $\mathbb{P}^{n-1}$. Working on the chart $\{Z_i \neq 0\}$, we shall use the nonhomogeneous coordinates $z_j := Z_j/Z_i$ for all $j \neq i$. Denote
by \( w \) the coordinate along each fiber on \(-kH_{p,n-1}\) or \(\mathbb{F}_{n,k}\). Then \( w \in \mathbb{C} \) for \(-kH_{p,n-1}\) and \( w \in \mathbb{C} \cup \{\infty\} \) for \(\mathbb{F}_{n,k}\). In particular, we have two divisors on \(\mathbb{F}_{n,k}\): the zero section
\[
Z_{n,k} := \{ w = 0 \}
\]
and the infinity section
\[
Z_{n,-k} := \{ w = \infty \}.
\]
Consider the Hermitian norm on \(-kH_{p,n-1}\) (and also \(\mathbb{F}_{n,k}\))
\[
||(z_1, \ldots, \hat{z}_i, \ldots, z_n, w)|| := |w|^2 \left( 1 + \sum_{j \neq i} |z_j|^2 \right)^k,
\]
on the chart \( \{ Z_i \neq 0 \} \). Let \( s \) be the logarithm of this fiberwise norm, i.e.,
\[
s := \log |w|^2 + k \log \left( 1 + \sum_{j \neq i} |z_j|^2 \right).
\]
Next, we define model metrics that depend only on \( s \) on \(-kH_{p,n-1}\) and \(\mathbb{P}^n(1, \ldots, 1, k)\).

### 3.1 Non-compact case: model metrics on \(-kH_{p,n-1}\)

The first model metric, \(\omega_{\text{eh},n,k}\), is a Ricci-flat edge metric with edge singularity of angle \(2n\pi/k\) along the zero section \(Z_{n,k}\) of \(-kH_{p,n-1}\). In particular, when \( n = k \), this metric is smooth. Indeed, when \( n = k \), \(\omega_{\text{eh},n,k}\) coincides with the Ricci-flat metric on the canonical bundle of \(\mathbb{P}^{n-1}\) that was constructed by Calabi [4].

**Definition 3.2.** Let \( s \) be defined in (3.1). To define \(\omega_{\text{eh},n,k}\), introduce another coordinate \( \lambda \in (1, +\infty) \) such that
\[
\lambda = (e^{\frac{n}{k} s} + 1)^{\frac{1}{n}}.
\]
Thus \( \{ \lambda = 1 \} \) corresponds to the zero section \( Z_{n,k} = \{ w = 0 \} \) on \(-kH_{p,n-1}\). We define \(\omega_{\text{eh},n,k}\) by giving its potential function as follows:
\[
f(\lambda) := k\lambda + k \int \frac{1}{\lambda^n - 1} \, d\lambda,
\]
where the last term denotes an indefinite integral, and
\[
\omega_{\text{eh},n,k} := \sqrt{-1} \partial \bar{\partial} f(s).
\]
The metric \(\omega_{\text{eh},n,k}\) is a Ricci-flat metric on \(-kH_{p,n-1}\) with edge singularity of angle \(2n\pi/k\) along the zero section (See Theorem 6.1 for details). It can be seen as a generalization of Eguchi–Hanson metrics to higher dimensional manifolds (See Remark 3.3 for details). More precisely, in local coordinates,
\[
\omega_{\text{eh},n,k} = k(e^{\frac{n}{k} s} + 1)^{\frac{1}{n}} \pi^* \omega_{\text{FS}} + \frac{e^{\frac{n}{k} s}}{k(e^{\frac{n}{k} s} + 1)^{\frac{n-1}{n}}} \left( \sqrt{-1} \frac{dw \wedge d\bar{w}}{|w|^2} + \sqrt{-1} \alpha \wedge \bar{\alpha} + \sqrt{-1} \alpha \wedge \frac{d\bar{w}}{w} + \sqrt{-1} \frac{dw}{w} \wedge \bar{\alpha} \right),
\]
where
\[
\alpha = k \frac{\sum_{i \neq j} \bar{z}_i z_j}{1 + \sum_{i \neq j} |z_i|^2}, \quad \text{on the chart } \{z_j \neq 0\},
\]
and
\[
\pi_1([Z_1 : \cdots : Z_n], w) = [Z_1 : \cdots : Z_n]
\]
is the projection map from the total space \( -kH_{\mathbb{P}^{n-1}} \) to the base space \( \mathbb{P}^{n-1} \).

**Remark 3.3.** Fixing \( n = k = 2 \) in (3.2) and (3.3), we obtain that on \(-2H_{\mathbb{P}^1}\) the model metric \( \omega_{\text{eh},n,k} \) has the expression
\[
\sqrt{-1} \partial \bar{\partial} (2\lambda + \log(\lambda - 1) - \log(\lambda + 1)), \quad \lambda > 1.
\]
(3.4)

Set
\[
r := e^{\frac{1}{k} s}, \quad r > 0.
\]
(3.5)

Plugging (3.2) and (3.5) in (3.4), one finds \( \omega_{\text{eh},n,k} \) has the following expression on \(-2H_{\mathbb{P}^1}\):
\[
\sqrt{-1} \partial \bar{\partial} [\sqrt{r^4 + 1 + \log r^2} - \log(\sqrt{r^4 + 1} + 1)].
\]
(3.6)

(3.6) coincides with (A.11) in the Appendix where we put \( \epsilon = 1 \). In other words, when \( n = k = 2, \omega_{\text{eh},n,k} \) recovers the famous Eguchi–Hanson metric [8] that is constructed on the total space of \(-2H_{\mathbb{P}^1}\).

It is recalled in Proposition A.1 that the Eguchi–Hanson metric is Ricci-flat. The metric \( \omega_{\text{eh},n,k} \), which is also Ricci-flat but with edge singularities along \( Z_{n,k} \) when \( n \neq k \), can be seen as a generalization of Eguchi–Hanson metrics to the total space of \(-kH_{\mathbb{P}^{n-1}}\) for arbitrary \( n, k \in \mathbb{N}_{>0} \).

### 3.2 Compact case: model metrics on \( \mathbb{P}^n(1, \ldots, 1, k) \)

In this section, we introduce another model metric \( \omega_{\text{orb},n,k} \) that is defined on the weighted projective space \( \mathbb{P}^n(1, \ldots, 1, k) \). We first recall the construction of the weighted projective space and its orbifold structure.

**Definition 3.4.** For \( k \in \mathbb{N} \), consider the group action of \( \mathbb{C}^* \) on \( \mathbb{C}^{n+1} \setminus \{0\} \) given by
\[
\lambda \cdot (z_0, \ldots, z_{n-1}, z_n) = (\lambda z_0, \ldots, \lambda z_{n-1}, \lambda^k z_n), \quad \lambda \in \mathbb{C}^*, \quad (z_0, \ldots, z_{n-1}, z_n) \in \mathbb{C}^{n+1} \setminus \{0\}.
\]

Then the weighted projective space \( \mathbb{P}^n(1, \ldots, 1, k) \) is defined as the quotient of this group action:
\[
\mathbb{P}^n(1, \ldots, 1, k) := (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*.
\]

We use homogeneous coordinates on \( \mathbb{P}^n(1, \ldots, 1, k) \). A point \([x_0 : \cdots : x_{n-1} : x_n] \in \mathbb{P}^n(1, \ldots, 1, k)\) corresponds to an equivalence class in \( \mathbb{C}^{n+1} \setminus \{0\} \). More precisely,
\[
[x_0 : \cdots : x_{n-1} : x_n] := \{\lambda x_0, \ldots, \lambda x_{n-1}, \lambda^k x_n : \lambda \in \mathbb{C}^*\}.
\]
(3.7)

Consider the \( \mathbb{Z}_k \) action on \( \mathbb{C}^n \) such that the \( \mathbb{Z}_k \) orbit of a point \((z_1, \ldots, z_n) \in \mathbb{C}^n\) is
\[
\{e^{\frac{2\pi \sqrt{-1} \ell}{k}} z_1, \ldots, e^{\frac{2\pi \sqrt{-1} \ell}{k}} z_n : \ell = 0, \ldots, k - 1\}.
\]

Then near the point \([0 : 0 \cdots : 0 : 1] \in \mathbb{P}^n(1, \ldots, 1, k)\) the local structure is \( \mathbb{C}^n/\mathbb{Z}_k \). The next lemma shows the relation between \( \mathbb{P}^n(1, \ldots, 1, k) \) and \( \mathbb{F}_{n,k} \).
Lemma 3.5. The total space of the line bundle $kH_{\mathbb{P}^{n-1}}$ can be embedded in the weighted projective space $\mathbb{P}^n(1, \ldots, 1, k)$ and the complement is a single point $p = [0 : \cdots : 0 : 1] \in \mathbb{P}^n(1, \ldots, 1, k)$. Moreover, $F_{n,k}$ is the blow up of $\mathbb{P}^n(1, \ldots, 1, k)$ at $p$.

Proof. Since for any vector bundle $A$ and line bundle $L$ we have $\mathbb{P}(A \otimes L) = \mathbb{P}(A)$, it follows that $F_{n,-k}$ is biholomorphic to $F_{n,k}$ by taking $L = 2kH_{\mathbb{P}^{n-1}}$ in Definition 3.1 with the biholomorphism exchanging the zero and the infinity sections $Z_{n,k}$ and $Z_{n,-k}$. Thus, we identify $F_{n,k}$ as

$$F_{n,k} = \mathbb{P}(kH_{\mathbb{P}^{n-1}} \oplus \mathbb{C}_{\mathbb{P}^{n-1}}).$$

We first show the total space of $kH_{\mathbb{P}^{n-1}}$ can be naturally embedded in $\mathbb{P}^n(1, \ldots, 1, k)$.

Consider $\mathbb{P}^n(1, \ldots, 1, k)$ with homogeneous coordinate $[x_0 : \cdots : x_n]$ and embed $\mathbb{P}^{n-1}$ in $\mathbb{P}^n$ as $\{x_n = 0\}$. Recall the transition function at a point $[x_0 : \cdots : x_{n-1}] \in \mathbb{P}^{n-1}$ of the line bundle $kH_{\mathbb{P}^{n-1}}$ is given by

$$g_{ij}([x_0 : \cdots : x_{n-1}]) = \left(\frac{x_j}{x_i}\right)^k.$$  

(3.9)

Then the fiber of $kH_{\mathbb{P}^{n-1}}$ at an arbitrary point $[x_0 : \cdots : x_{n-1} : 0] \in \mathbb{P}^n(1, \ldots, 1, k)$ can be identified as the set of all $[x_0 : \cdots : x_{n-1} : \lambda]$ for $\lambda \in \mathbb{C}$, where $[x_0 : \cdots : x_{n-1} : \lambda]$ is defined in (3.7). By definition of the weighted projective space and (3.9), this is well-defined. Thus, $kH_{\mathbb{P}^{n-1}}$ can be naturally embedded in $\mathbb{P}^n(1, \ldots, 1, k)$ and the complement is the point $[0 : \cdots : 0 : 1]$. We have mentioned that the local structure of $\mathbb{P}^n(1, \ldots, 1, k)$ near this point is $\mathbb{C}^n/Z_k$.

Next, we blow up $\mathbb{P}^n(1, \ldots, 1, k)$ at $[0 : \cdots : 0 : 1]$. The exceptional divisor can be identified as adding an infinity section that is biholomorphic to $\mathbb{P}^{n-1}$ to the total space $kH_{\mathbb{P}^{n-1}}$. Combining this observation with (3.8), one realizes the manifold upstairs is the Calabi–Hirzebruch manifold $F_{n,k}$. We henceforth denote by $\pi$ the blow down map. $\square$

Now we are ready to define $\omega_{\text{orb},n,k}$. To do so, we first define another family of model metrics $\bar{\omega}_{\text{orb},n,k}$ on $F_{n,k}$. The metric $\bar{\omega}_{\text{orb},n,k}$ is a Kähler–Einstein edge metric with Ricci curvature $(n+1)/k$ and an edge singularity of angle $2\pi/k$ along $Z_{n,-k}$ that degenerates on $Z_{n,k}$. In particular, this metric is smooth along $Z_{n,-k}$ when $k = 1$ and collapses along $Z_{n,k}$. Indeed, when $k = 1$ this metric coincides with the Fubini–Study metric on $\mathbb{P}^n$. We will then define $\omega_{\text{orb},n,k}$ as the pull-back of the metric $\bar{\omega}_{\text{orb},n,k}$ under the blow up map.

Definition 3.6. Let $s$ be defined in (3.1). To define $\bar{\omega}_{\text{orb},n,k}$, introduce another coordinate $\nu \in (0, 1)$ such that

$$\nu = 1 - (e^{\frac{s}{k}} + 1)^{-1}.$$

Note that $\{\nu = 0\}$ and $\{\nu = 1\}$ correspond to the zero section $Z_{n,k}$ and the infinity section $Z_{n,-k}$ respectively. We define $\bar{\omega}_{\text{orb},n,k}$ by giving its potential function as follows:

$$g(\nu) := -k \log(1 - \nu), \quad \nu \in (0, 1),$$

and

$$\bar{\omega}_{\text{orb},n,k} := \sqrt{-1} \partial \bar{\partial} g(s).$$

Note that $\bar{\omega}_{\text{orb},n,k}$ is a degenerate Kähler–Einstein edge metric on $F_{n,k}$ with Ricci curvature $(n+1)/k$ and an edge singularity of the angle $2\pi/k$ along $Z_{n,-k}$, while $\bar{\omega}_{\text{orb},n,k}$ collapses along
\(Z_{n,k}\) (See Theorem 6.4 for details). More precisely, in local coordinates,
\[
\tilde{\omega}_{\text{orb},n,k} = \frac{k\tilde{e}^k}{e^k + 1} \tau_1 \omega_{\text{FS}} + \frac{e^k}{k(e^k + 1)} \left( \frac{\sqrt{-1} dw \wedge d\bar{w}}{|w|^2} + \sqrt{-1} \alpha \wedge \bar{\alpha} + \sqrt{-1} \alpha \wedge \frac{d\bar{w}}{w} + \sqrt{-1} \frac{dw}{w} \wedge \bar{\alpha} \right),
\]
where
\[
\alpha = k \sum_{i \neq j} \bar{z}_j dz_i \quad \text{on the chart } \{ z_j \neq 0 \},
\]
and
\[
\pi_1([Z_1 : \cdots : Z_n], w) = [Z_1 : \cdots : Z_n]
\]
is the projection map from \(F_{n,k}\) to the zero section \(Z_{n,k}\) identified as \(\mathbb{P}^{n-1}\).

**Definition 3.7.** For \(k > 1\), the model metric \(\omega_{\text{orb},n,k}\) on \(\mathbb{P}^n(1, \ldots, 1, k)\) is defined away from \([0 : \cdots : 0 : 1]\) as the pull-back of the metric \(\tilde{\omega}_{\text{orb},n,k}\) on \(F_{n,k} \setminus \{Z_{n,k}\}\) under the inverse of the blow up map. For \(k = 1\), we define \(\omega_{\text{orb},n,1}\) as the Fubini–Study metric on \(\mathbb{P}^n(1, \ldots, 1, 1) = \mathbb{P}^n\).

### 4 Constructions of Kähler–Einstein edge metrics

In this section, we aim to construct Kähler–Einstein edge metrics defined on the Calabi–Hirzebruch manifold \(F_{n,k}\) with edge singularities along the zero section \(Z_{n,k}\) and the infinity section \(Z_{n,-k}\) using Calabi ansatz. By Kähler–Einstein edge metrics we mean Kähler edge currents that satisfy the Kähler–Einstein equation on smooth locus. The reader may refer to [12] for detailed exposition of Kähler edge metrics.

#### 4.1 Constructions of \(\eta_\beta_1\) on \(F_{n,k}\)

Recall, the coordinate \(s\) is defined in (3.1), where \(w\) and \(z_j, j \neq i\) are coordinates we use when working on the chart \(\{Z_i \neq 0\} \subset F_{n,k}\). We seek a Kähler–Einstein edge metric
\[
\eta := \sqrt{-1} \partial \bar{\partial} f(s)
\]
for some smooth function \(f\) on \(F_{n,k}\). Define
\[
\tau(s) := f'(s), \quad s \in (-\infty, +\infty)
\]
\[
\varphi(s) := f''(s) = \tau'(s), \quad s \in (-\infty, +\infty).
\]

Let \(\pi_1\) and \(\pi_2\) be projections from \(F_{n,k}\) to the zero section \(Z_{n,k}\) and each fiber respectively, i.e.,
\[
\pi_1([Z_1 : \cdots : Z_n], w) = [Z_1 : \cdots : Z_n],
\]
\[
\pi_2([Z_1 : \cdots : Z_n], w) = w.
\]
Denote by \(\omega_{\text{FS}}\) the Fubini–Study metric on \(\mathbb{P}^{n-1}\) and define
\[
\omega_{\text{Cyl}} := \frac{\sqrt{-1} dw \wedge d\bar{w}}{|w|^2},
\]
\[
\alpha := \frac{k \sum_{j \neq i} \bar{z}_j dz_j}{1 + \sum_{j \neq i} |z_j|^2}, \quad \text{on the chart } \{ z_i \neq 0 \}.
\]
Then direct calculation (cf. [14, Section 3.3] for detailed computations in dimension \( n=2 \)) yields
\[
\eta = k\tau \pi_1^* \omega_{FS} + \varphi \left( \pi_2^* \omega_{Cyl} + \sqrt{-1} \alpha \wedge \bar{\alpha} + \sqrt{-1} \alpha \wedge \frac{dw}{w} + \sqrt{-1} \frac{dw}{w} \wedge \bar{\alpha} \right).
\tag{4.2}
\]

Positive definiteness of \( \eta \) then implies \( f' \geq 0 \) and \( f'' \geq 0 \) for \( s \in (-\infty, +\infty) \).

**Proposition 4.1.** Assume \( Z_{n,k} \) is non-collapsed, then \( \inf_{s \in \mathbb{R}} \tau(s) > 0 \). Moreover, \( \sup_{s \in \mathbb{R}} \tau(s) < +\infty \).

**Proof.** Assume by contradiction that \( \inf_{s \in \mathbb{R}} \tau(s) = 0 \). Then by (4.2), \( \eta \) is identically zero when restricted to \( Z_{n,k} \). However, we assume \( Z_{n,k} \) is non-collapsed. Thus we must have \( \inf_{s \in \mathbb{R}} \tau(s) > 0 \). To see \( \sup_{s \in \mathbb{R}} \tau(s) < +\infty \), we follow the arguments in [14, Lemma 3.2]. Indeed, restricting \( \eta \) to the fiber \( \{ Z_j = 0, \forall j \neq i \} \), in (4.2) we get
\[
\eta = \varphi \cdot \pi_2^* \omega_{Cyl} = \varphi \sqrt{-1} d w \wedge d \bar{w} |w|^2,
\tag{4.3}
\]
and also in this case \( s = \log |w|^2 \). Then we use the coordinate \( w = e^{s/2 + \sqrt{-1} \theta} \) on this fiber, and by (4.3) \( \eta \) restricted on the fiber gives a metric
\[
\eta = \frac{1}{2\varphi(\tau)} d\tau^2 + 2\varphi(\tau) d\theta^2.
\tag{4.4}
\]
Thus, the volume form on the fiber is \( d\tau \wedge d\theta \), and the volume of the fiber is \( 2\pi (\sup_{s \in \mathbb{R}} \tau(s) - \inf_{s \in \mathbb{R}} \tau(s)) \). Since for any Kähler edge metric the volume of a complex submanifold is finite, we conclude \( \sup_{s \in \mathbb{R}} \tau(s) < \infty \). \hfill \Box

By Proposition 4.1, we may rescale \( f \) by a positive constant such that \( \inf \tau = 1 \) and \( \sup \tau = T < \infty \). Thus, we henceforth assume that \( \tau \) ranges from \( [1, T] \).

**Proposition 4.2.** If \( \eta \) is a Kähler edge metric with conic singularities along \( Z_{n,k} \) and \( Z_{n,-k} \), then
\[
\varphi(1) = 0, \quad \frac{d\varphi}{d\tau}(1) = \beta_1,
\]
\[
\varphi(T) = 0, \quad \frac{d\varphi}{d\tau}(T) = -\beta_2,
\tag{4.5}
\]
if we denote by \( 2\pi \beta_1 \) and \( 2\pi \beta_2 \) respectively the angles of \( \eta \) along \( Z_{n,k} \) and \( Z_{n,-k} \).

**Proof.** Recall we assume \( \tau \) ranges from \([1, T]\). In particular, this implies that
\[
\lim_{s \to \pm \infty} \frac{d\tau}{ds} = 0,
\]
which combines with (4.1) show that
\[
\varphi(1) = \varphi(T) = 0.
\]
Next, we follow the arguments in the proof of [14, Proposition 3.3]. Indeed, it follows from [10, Theorem 1, proposition 4.4] that the potential function \( f \) of \( \eta \) has complete asymptotic
expansions near both \( w = 0 \) and \( w = \infty \). Near \( w = 0 \), the leading term in the expansion is \( |w|^{2\beta_1} \). More precisely, by (3.1),

\[
\varphi \sim C_1 + C_2 |w|^{2\beta_1} + (C_3 \sin \theta + C_4 \cos \theta)|w|^2 + O(|w|^{2+\epsilon})
\]

\[
= C_1 + C_2 e^{\beta_1 s} + (C_3 \sin \theta + C_4 \cos \theta) e^s + O(e^{(1+\epsilon)s}).
\]

We first find \( C_1 = 0 \) by the fact that \( \varphi(1) = \varphi(T) = 0 \) in (4.5). Moreover, the expansion can be differentiated term-by-term as \( |w| \to 0 \) or \( s \to -\infty \). As \( \varphi'(\tau) = \frac{\partial \varphi}{\partial s} d_s = \frac{\partial \varphi}{\partial s} / \varphi \), we obtain

\[
\varphi'(1) = \beta_1.
\]

The same arguments imply that \( \varphi'(T) = -\beta_2 \), where the minus sign comes from the fact that the leading term in this expansion is \( |w|^{-2\beta_2} = e^{-\beta_2 s} \).

**Definition 4.3.** A Kähler–Einstein edge metric \( \omega \) is a Kähler edge metric that has constant Ricci curvature away from the singular locus.

By Definition 4.3, \( \eta \) is a Kähler–Einstein edge metric if and only if it satisfies the following Kähler–Einstein edge equation:

\[
\text{Ric} \, \eta = \lambda \eta + (1 - \beta_1)[Z_{n,k}] + (1 - \beta_2)[Z_{n,-k}],
\]

where by \( \lambda \) we denote the Ricci curvature.

The next proposition shows that (4.6) is equivalent to an ODE satisfied by \( \tau \) and \( \varphi \) with boundary conditions given in (4.5).

**Proposition 4.4.** The Kähler edge metric \( \eta \) given in (4.2) satisfies the Kähler–Einstein edge equation (4.6) if and only if

\[
\lambda = \frac{n}{k} - \beta_1, \quad \beta_1 \in \left(0, \frac{n}{k}\right),
\]

\[
\varphi(\tau) = \frac{1}{k} \tau^n - \frac{1}{n+1}(\beta_1 - \frac{n}{k}) \frac{n+1}{\tau^n-1}, \quad \tau \geq 1.
\]

Moreover, \( \beta_2 \) and \( T \) are determined by \( \beta_1 \) such that \( T > 1 \) and \( \beta_2 > 0 \).

**Proof.** By (4.2), we calculate \( \text{Ric} \, \eta \):

\[
\text{Ric} \, \eta = -\sqrt{-1} \partial \bar{\partial} \log \eta^n
\]

\[
= -\sqrt{-1} \partial \bar{\partial} \log \left(k^{n-1} \tau^{n-1} \varphi(\pi^*_1 \omega_{FS})^{n-1} \wedge \pi^*_2 \omega_{Cyl}\right)
\]

\[
= (1 - \beta_1)[Z_{n,k}] + (1 - \beta_2)[Z_{n,-k}] + \left(n - k(n - 1) \frac{\varphi}{\tau} - k \frac{d \varphi}{d \tau} \right) \pi^*_1 \omega_{FS}
\]

\[
- \varphi \frac{d}{d \tau} \left((n - 1) \frac{\varphi}{\tau} + \frac{d \varphi}{d \tau}\right) \left(\pi^*_2 \omega_{Cyl} + \sqrt{-1} \alpha \wedge \bar{\alpha} + \sqrt{-1} \frac{d \bar{w}}{\bar{w}} + \sqrt{-1} \frac{d w}{w} \wedge \bar{\alpha}\right).
\]
Plugging (4.2) and (4.9) in (4.6), (4.6) is equivalent to
\[ n - k(n - 1)\frac{\varphi}{\tau} - k\frac{d\varphi}{d\tau} = \lambda k\tau, \]  
(4.10)
\[ -\varphi \frac{d}{d\tau}((n - 1)\frac{\varphi}{\tau} + \frac{d\varphi}{d\tau}) = \lambda \varphi. \]  
(4.11)

Now we first observe that (4.10) and (4.11) are equivalent since taking derivative of (4.10) with respect to \( \tau \) gives (4.11). Thus we conclude (4.6) is equivalent to the ODE (4.10) together with boundary conditions (4.5). Solving this ODE with boundary conditions gives (4.7) and (4.8). We need the assumption \( \beta_1 \in (0, n/k) \) to ensure the existence of \( T \) and \( \beta_2 \) that satisfy (4.5). Indeed, \( \beta_2 \) and \( T \) are determined by \( \beta_1 \) due to (4.5) and (4.8). More precisely, \( T \) should be a root of the polynomial that appears in the right hand side of (4.8) and \( -\beta_2 \) should be determined by the derivative \( \frac{d\varphi}{d\tau} \) at \( T \). Writing (4.8) as \( \varphi(\tau) = \frac{P(\tau)}{\tau^{n-1}} \), we factor \( P(\tau) \) as
\[ P(\tau) = (\tau - 1) \left[ \frac{1}{k} (\tau^{n-1} + \cdots + 1) + \frac{1}{n+1} \left( \beta_1 - \frac{n}{k} \right) (\tau^n + \cdots + 1) \right] = \frac{(\tau - 1)}{n + 1} \left[ (\beta_1 - \frac{n}{k}) \tau^n + \left( \frac{1}{k} + \beta_1 \right) (\tau^{n-1} + \cdots + 1) \right]. \]  
(4.12)

Under the assumption \( \beta_1 \in (0, n/k) \), one finds \( P(\tau) > 0 \) for \( 1 < \tau \ll 2 \) and \( P(\tau) < 0 \) for \( \tau \to \infty \). Thus, \( P(\tau) \) has at least one real root that is greater than 1. Denote by \( T \) the first root of \( P \) after 1. In particular, \( \frac{d\varphi}{d\tau}(T) < 0 \). By (4.5), this implies that \( \beta_2 > 0 \) as claimed.

By (4.1), there holds
\[ \frac{ds}{d\tau} = \frac{1}{\varphi(\tau)}. \]  
(4.13)

Combining (4.2), (4.13) and Proposition 4.4 altogether, we realize that given \( \beta_1 \in (0, n/k) \) we can construct a Kähler–Einstein edge metric \( \eta \) on \( \mathbb{F}_{n,k} \) using coordinates \( \tau \) and \( \varphi \). This Kähler–Einstein edge metric has Ricci curvature \( \lambda = n/k - \beta_1 \), and has edge singularities of angle \( 2\pi \beta_1 \) along \( Z_{n,k} \) and angle \( 2\pi \beta_2 \) along \( Z_{n,-k} \). Note that \( \beta_2 \) is determined by \( \beta_1 \). In other words, we can construct a family of Kähler–Einstein edge metrics on \( \mathbb{F}_{n,k} \) parametrized by \( \beta_1 \in (0, n/k) \). In Section 5, we study the asymptotic behavior of this family of metrics when \( \beta_1 \) approaches the two extremes: \( n/k \) or 0.

Recall, in the construction of \( \eta \) we rescale the metric such that \( \tau \) ranges from \([1, T]\). Note that \( T \) is determined by \( \beta_1 \). By Proposition 4.1, we may also rescale the metric such that \( \tau \) ranges from \([t, 1]\) for some \( 0 < t < 1 \). In such a way, we construct another family of Kähler–Einstein edge metrics on \( \mathbb{F}_{n,k} \) parametrized by \( \beta_2 \) in the remainder of this section. Such metrics can be seen as obtained after renormalizing metrics in the family that is parametrized by \( \beta_1 \). However, when studying their asymptotic behaviors they give rise to different limit metric.

### 4.2 Constructions of \( \xi_{\beta_2} \) on \( \mathbb{F}_{n,k} \)

Now consider a change of coordinate \( u := 1/w \). Then (3.1) can be written as
\[ s = -\log |u|^2 + k \log \left( 1 + \sum_{j \neq i}^{n} |z_j|^2 \right). \]
We still denote by $f(s)$ a smooth function on $\mathbb{F}_{n,k}$ and seek Kähler–Einstein edge metrics that have the form $\sqrt{-1} \partial \bar{\partial} f(s)$. Recall $\tau(s)$ and $\varphi(s)$ are defined in (4.1). Let

$$
\xi := \sqrt{-1} \partial \bar{\partial} f(s) = k \tau_1 \omega_{\text{FS}} + \varphi \left( \pi_2^* \omega_{\text{CY}} + \sqrt{-1} \alpha \wedge \bar{\alpha} \right),
$$

where $\pi_1$, $\pi_2$ and $\alpha$ are the same as those in (4.2). Recall by Proposition 4.1,

$$
0 < \inf_{s \in \mathbb{R}} \tau(s) < \sup_{s \in \mathbb{R}} \tau(s) < +\infty.
$$

We rescale the potential function $f$ such that for some $t > 0$,

$$
\sup_{s \in \mathbb{R}} \tau(s) = 1, \quad \inf_{s \in \mathbb{R}} \tau(s) = t.
$$

Assume $\xi$ is a Kähler edge metric on $\mathbb{F}_{n,k}$. Then, under the renormalization given by (4.15), Proposition 4.2 and Proposition 4.4, which respectively describe the boundary conditions for $\varphi$ and the ODE satisfied by $\varphi$, translate to the following two Propositions.

**Proposition 4.5.** Assume $\xi$ has edge singularities of angle $2\pi \beta_1$ and $2\pi \beta_2$ respectively along $Z_{n,k}$ and $Z_{n,-k}$. Recall $\varphi$ in (4.14). Then,

$$
\varphi(t) = 0, \quad \frac{d\varphi}{d\tau}(t) = \beta_1,
$$

$$
\varphi(1) = 0, \quad \frac{d\varphi}{d\tau}(1) = -\beta_2.
$$

**Proposition 4.6.** Under the same assumptions in Proposition 4.5, the Kähler edge metric $\xi$ satisfies the Kähler–Einstein edge equation if and only if

$$
\mu = \frac{n}{k} + \beta_2, \quad \beta_2 \in \left( 0, \frac{1}{k} \right),
$$

$$
\varphi(\tau) = \frac{1}{k} \tau^n - \frac{1}{n+1} \left( \frac{n}{k} + \beta_2 \right) \frac{\tau^{n+1} - 1}{\tau^n}, \quad \tau \in (t, 1),
$$

where by $\mu$ we denote the Ricci curvature of $\xi$. Moreover, $\beta_1$ and $t$ are determined by $\beta_2$.

By Propositions 4.5 and 4.6, we realize that given $\beta_2$ we can construct a family of Kähler–Einstein edge metrics on $\mathbb{F}_{n,k}$ parametrized by $\beta_2$ such that $\beta_1$ and $t$ are determined by $\beta_2$. This family of metrics can be obtained by renormalizing the family of metrics parametrized by $\beta_1$.

**5 Angle asymptotics**

In this section, we study the asymptotic behaviors of Kähler–Einstein edge metrics $\eta$ in (4.2) (respectively, $\xi$ in (4.14)) as $\beta_1$ (respectively, $\beta_2$) approaches either of its extremes.

We first study the limit behavior of $\eta$ when $\beta_1 \nearrow n/k$. It is enough to study the limiting behavior of $\varphi$ and $\tau$ thanks to Proposition 4.4.

**Proposition 5.1.** When $\beta_1$ is close to $n/k$, we have $T > 1$ and $T \to \infty$ as $\beta_1 \nearrow n/k$. 
Proof. Recall by (4.5), 1 and $T$ are both roots of $\varphi(\tau)$. By Proposition 4.4, $T > 1$. To prove $T \to \infty$ as $\beta_1 \uparrow n/k$, we write (4.8) as

$$\varphi(\tau) = \frac{1}{\tau^{n-1}} \cdot \frac{1}{n+1} (\beta_1 - \frac{n}{k})(\tau - 1)(\tau - \alpha_1) \cdots (\tau - \alpha_{n-1})(\tau - T),$$

(5.1)

where $\alpha_1, \ldots, \alpha_{n-1} \in \mathbb{C}$. Comparing (4.8) to (5.1), we have

$$(\tau - \alpha_1) \cdots (\tau - \alpha_{n-1})(T - \tau) = \tau^n + \frac{1 + k\beta_1}{k\beta_1 - n}(\tau^{n-1} + \cdots + 1).$$

By Vieta’s formulas,

$$(-1)^n \alpha_1 \cdots \alpha_{n-1} \cdot T = \frac{1 + k\beta_1}{k\beta_1 - n}, \quad (\beta_2) \quad (5.2)$$

$$\alpha_1 + \cdots + \alpha_{n-1} + T = \frac{1 + k\beta_1}{n - k\beta_1}. \quad (\beta_3) \quad (5.3)$$

By (4.8), $\varphi(\tau)$ has $n + 1$ (possibly complex) roots. Since

$$\varphi(\tau) \to \frac{1}{k} \frac{\tau^n - 1}{\tau^{n-1}}, \quad \text{as } \beta_1 \uparrow n/k,$$

we conclude that the $n$ roots of $\varphi$ except $T$ converge to the $n$th root of unity as $\beta_1 \uparrow n/k$. In particular, $|\alpha_1|, \ldots, |\alpha_{n-1}|$ converge to 1 as $\beta_1 \uparrow n/k$. However, by (5.2) we see

$$|\alpha_1| \cdots |\alpha_{n-1}| |T| \to +\infty, \quad \text{as } \beta_1 \uparrow n/k.$$ 

Thus we must have $T \to +\infty$ as $\beta_1 \uparrow n/k$. 

Proposition 5.2. As $\beta_1 \uparrow n/k$, the length of the path on each fiber between the intersection point of the fiber with $Z_{n,k}$ and that of the fiber with $Z_{n,-k}$ tends to infinity. In other words, $Z_{n,-k}$ gets pushed–off to infinity as $\beta_1 \uparrow n/k$ if we fix a base point on $Z_{n,k}$.

Proof. Restricted to each fiber, by (4.4), $\eta = \varphi \sqrt{-1} dw \wedge d\bar{w} / |w|^2$ gives a metric

$$g = \frac{1}{2\varphi(\tau)} d\tau^2 + 2\varphi(\tau) d\theta^2.$$

Up to some constant, the distance between $\{\tau = 1\}$ and $\{\tau = T\}$ is given by

$$\int_1^T \frac{1}{\sqrt{\varphi(\tau)}} d\tau = \int_1^T \frac{\tau^{n-1}}{\sqrt{\frac{n/k - \beta_1}{n+1} \cdot (\tau - 1)(\tau - \alpha_1) \cdots (\tau - \alpha_{n-1})(T - \tau)}} d\tau.$$ 

(5.4)

Recall in the proof of Proposition 5.1, we have shown

$$T \sim \frac{1 + k\beta_1}{n - k\beta_1} \sim \frac{1 + n}{n/k - \beta_1}, \quad \text{as } \beta_1 \uparrow n/k.$$ 

(5.5)

For any fixed $\epsilon > 0$, consider

$$\int_{T-\epsilon}^T \frac{\tau^{n-1}}{\sqrt{\frac{n/k - \beta_1}{n+1} \cdot (\tau - 1)(\tau - \alpha_1) \cdots (\tau - \alpha_{n-1})(T - \tau)}} d\tau.$$ 

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We can find $\beta_1$ close to $n/k$ such that

$$T - \epsilon > \frac{1}{2}T.$$  

Then in $[T - \epsilon, T]$, we have

$$\tau^{\frac{n-1}{2}} > (T - \epsilon)^{\frac{n-1}{2}} > \left(\frac{1}{2}T\right)^{\frac{n-1}{2}},$$

$$\sqrt{(\tau - 1)(\tau - \alpha_1)\cdots(\tau - \alpha_{n-1})} < \sqrt{2(\tau^n - 1)} < \sqrt{2T^n}.$$ 

By (5.5) and (5.6), there exists a constant $C > 0$ that is independent of $\epsilon$ such that

$$\int_{T - \epsilon}^{T} \frac{\tau^{\frac{n-1}{2}}}{\sqrt{\frac{n}{k} - \beta_1}} \cdot \frac{1}{\sqrt{(\tau - 1)(\tau - \alpha_1)\cdots(\tau - \alpha_{n-1})(T - \tau)}} d\tau > C \cdot \frac{T^{\frac{n-1}{2}}}{\sqrt{\frac{n}{k} - \beta_1} \cdot T^{\frac{n}{2}}} \int_{T - \epsilon}^{T} \frac{1}{\sqrt{T - \tau}} d\tau > C\epsilon.$$ 

Since $\epsilon$ is arbitrarily chosen, the integral in (5.4) diverges as $\beta_1 \nearrow \frac{n}{k}$. Thus, we have shown $Z_{n,k}$ gets pushed-off to infinity as $\beta_1 \nearrow \frac{n}{k}$ if we choose a base point on $Z_{n,k}$. \qed

**Proposition 5.3.** \(\lim_{\beta_1 \nearrow \frac{n}{k}} \beta_2(\beta_1) = \frac{1}{k}\).

**Proof.** We calculate

$$\frac{d\varphi}{d\tau} = \frac{1}{k} \frac{\tau^{n-2}}{(\tau - 1)^2} (n + \tau^n - 1) + \frac{1}{n+1} (\beta_1 - \frac{n}{k}) \frac{\tau^{n-2}}{(\tau - 1)^2} (2\tau^{n+1} + n - 1).$$

By $\varphi(T) = -\beta_2$ we have

$$\frac{1}{kT^n} (n + T^n - 1) + \frac{1}{n+1} (\beta_1 - \frac{n}{k}) \frac{1}{T^n} (2T^{n+1} + n - 1) = -\beta_2.$$ 

(5.7)

Recall $\varphi(T) = 0$, then we have

$$\frac{T^n - 1}{k} + \frac{1}{n+1} (\beta_1 - \frac{n}{k}) (T^{n+1} - 1) = 0.$$ 

(5.8)

Combining (5.7) and (5.8),

$$(\frac{1}{k} - \beta_2) T^n = \beta_1 + \frac{1}{k}.$$ 

(5.9)

Since $T \to \infty$ as $\beta_1 \nearrow \frac{n}{k}$, there holds $\beta_2 \to 1/k$ as $\beta_1 \nearrow \frac{n}{k}$. \qed

In Section 6, we use asymptotic behaviors discussed above to study the limit metric of $\eta$ as $\beta_1$ approaches $n/k$ or 0. In the remainder of this section, we focus on the family of metrics $\xi$ that is parametrized by $\beta_2$. Inspired by Proposition 5.3, we study the asymptotic behaviors of $\xi$ as $\beta_2 \nearrow 1/k$. 

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Proposition 5.4. When $\beta_2 \not\nearrow \frac{1}{k}$, as an analogue of (5.9), there holds

$$(\frac{1}{k} + \beta_1)t^n = \frac{1}{k} - \beta_2.$$ 

Proof. Let $\varphi(\tau)$ and $t$ be as in (4.16) and (4.17). We calculate

$$\frac{d\varphi}{d\tau} = \frac{1}{k} \tau^{n-2} (\tau^n + n - 1) - \frac{1}{n+1} \left( \frac{n}{k} + \beta_2 \right) \frac{\tau^{n-2}}{(\tau-1)^2} (2\tau^{n+1} + n - 1).$$

By the fact that $\frac{d\varphi}{d\tau}(t) = \beta_1$,

$$\beta_1 = \frac{1}{k t^n} (n + t^n - 1) - \frac{1}{n+1} \left( \frac{n}{k} + \beta_2 \right) \frac{1}{t^n} (2t^{n+1} + n - 1).$$

Combining this with the fact that $\varphi(t) = 0$, there holds

$$\beta_1 t^n = \frac{1}{k} (n + t^n - 1) - \frac{2(t^n - 1)}{k} - \frac{n}{k} - \beta_2$$

$$= \frac{1}{k} t^n + \frac{1}{k} - \beta_2,$$

which implies

$$\left( \beta_1 + \frac{1}{k} \right) t^n = \frac{1}{k} - \beta_2$$

as claimed.

Proposition 5.5. As $\beta_2 \not\nearrow \frac{1}{k}$, $t = O\left(\left(\frac{k}{n+1}\right)^{\frac{1}{n}} \left(\frac{1}{k} - \beta_2\right)^{\frac{1}{n}}\right)$. In particular, $t$ tends to 0 when $\beta_2 \not\nearrow \frac{1}{k}$. Moreover, the length of the path on each fiber between the intersection point of the fiber with $Z_{n,k}$ and that of the fiber with $Z_{n,-k}$ converges to a finite number as $\beta_2 \not\nearrow \frac{1}{k}$.

Proof. Recall $t$ is a root of

$$\varphi(\tau) = \frac{1}{k} \tau^n - \frac{1}{k n} \left( \frac{n}{k} + \beta_2 \right) \frac{\tau^{n+1} - 1}{\tau^n - 1} = 0.$$ 

Direct calculations yield

$$\frac{\varphi(\tau)}{\tau - 1} = \frac{1}{\tau^{n-1}} \left( - \frac{1}{n+1} \left( \frac{n}{k} + \beta_2 \right) \tau^n + \frac{1/k - \beta_2}{n+1} (\tau^n - 1 + \cdots + \tau + 1) \right).$$

Then it is easy to see that every root of $\varphi(\tau)$ except for $\tau = 1$, denoted by $\alpha$, satisfies that $|\alpha| = O((k/(n+1))^{1/n}(1/k - \beta_2)^{1/n})$ when $\beta_2 \not\nearrow 1/k$. In particular, as $\beta_2 \not\nearrow 1/k$, all roots of $\varphi(\tau) = 0$ except for $\tau = 1$ converge to 0. We conclude that $t \to 0$. To see that the length between $Z_{n,k}$ and $Z_{n,-k}$ converges to a finite number in the limit, one follows the arguments in the proof of Proposition 5.2 and Proposition C.4. □
6 Large angle limits

In this section, we first study the Gromov–Hausdorff limit of the family of metrics \( \eta \) when the parameter \( \beta_1 \) approaches \( n/k \).

**Theorem 6.1.** Fix an arbitrary base point \( p \) on \( Z_{n,k} \). As \( \beta_1 \nearrow n/k \), the Kähler–Einstein edge metric \((F_{n,k}, \eta_{\beta_1}, p)\) converges in the pointed Gromov–Hausdorff sense to a Ricci-flat Kähler edge metric \((-kH_{P^n-1}, \eta_\infty, p)\) described in (6.5). Moreover, away from \( Z_{n,k} \), one has \( \eta_{\beta_1} \xrightarrow{C^0_{\text{loc}}} \eta_\infty \) for any \( k \geq 0 \) on the total space of \(-kH_{P^n-1}\) (seen as an open subset in \( F_{n,k} \)), and the limit metric \( \eta_\infty \) coincides with the model metric \( \omega_{eh,n,k} \) defined in Section 3.

**Proof.** We use notation \( \eta_{\beta_1}, \tau(\beta_1) \) and \( \phi(\beta_1) \) to emphasize the dependence of metrics and coordinates on \( \beta_1 \). Combining (4.1) and (4.8), we have

\[
\frac{ds}{d\tau(\beta_1)} = \frac{1}{\varphi(\beta_1)} = \frac{\tau(\beta_1)^{n-1}}{\frac{1}{k}(\tau(\beta_1)^n - 1) + \frac{n}{n+1}(\tau(\beta_1)^{n+1} - 1)}. \tag{6.1}
\]

**Claim 6.2.** The pointwise limits of functions

\[
\tau_\infty(s) := \lim_{\beta_1 \nearrow n/k} \tau(\beta_1, s), \quad s \in (-\infty, +\infty),
\]

\[
\varphi_\infty(s) := \lim_{\beta_1 \nearrow n/k} \varphi(\beta_1, s), \quad s \in (-\infty, +\infty)
\]

exist. Moreover, \( \tau_\infty(s) \) and \( \varphi_\infty(s) \) are smooth in \( s \).

**Proof of the Claim.** By the continuous dependence of ODEs in (6.1) on the parameter \( \beta_1 \), the pointwise limit function \( \tau_\infty \) exists. Since \( \tau(\beta_1, s) \) is smooth in \( s \) for each \( \beta_1 \), \( \tau_\infty(s) \) is smooth in \( s \). By (4.8) and the existence of \( \tau_\infty(s) \), \( \varphi_\infty(s) \) also exists. Moreover, \( \varphi_\infty(s) \) is smooth in \( s \) due to the smoothness of \( \tau_\infty(s) \). \( \Box \)

Thanks to Claim 6.2, \( \tau_\infty \) and \( \varphi_\infty \) satisfy

\[
\frac{ds}{d\tau_\infty} = \frac{\tau_\infty^{n-1}}{k(\tau_\infty^n - 1)}; \tag{6.2}
\]

\[
\varphi_\infty(\tau_\infty) = \frac{1}{k} \left( \frac{\tau_\infty^n - 1}{\tau_\infty^{n-1}} \right).
\]

Solving the first equation in (6.2), we get

\[
\tau_\infty(s) = (1 + e^{(s-C)\frac{k}{n}})^{1/n}, \quad \text{for some constant } C. \tag{6.3}
\]

By considering a change of coordinate \( w' = C'w \) for some appropriate \( C' \) in (3.1), we may choose \( C \) in (6.3) to be 0. Plugging (6.3) into the second equation in (6.2), one finds

\[
\varphi_\infty(s) = \frac{1}{k} \left( \frac{e^{s\frac{k}{n}}}{(e^{s\frac{k}{n}} + 1)^{\frac{n-1}{n}}} \right). \tag{6.4}
\]
Plugging (6.3) and (6.4) into (4.2), we obtain the convergence of Kähler–Einstein edge metric $\eta_{\beta_1}$ on any compact subsets of $\mathbb{F}_{n,k}$ in every $C^k$-norm to the following metric:

$$\eta_\infty := \lim_{\beta_1 \nearrow n/k} \eta_{\beta_1}$$

$$= k(1 + e^{s_2^2})^{1/n} \pi_1^* \omega_{FS} + \frac{1}{k} \cdot \frac{e^{s_2^2}}{(e^{s_2^2} + 1)^{n/k}} \left( \pi_2^* \omega_{CY} + \sqrt{-1} \alpha \wedge \bar{\alpha} \right).$$

Recall by (4.7), the Ricci curvature of $\eta_{\beta_1}$ is given by $\lambda_{\beta_1} = n/k - \beta_1$, which converges to 0 as $\beta_1 \nearrow n/k$. Thus, $\eta_\infty$ is a Ricci-flat Kähler edge metric on $-kH_{p_{n-1}}$. $\eta_\infty$ has edge singularity of angle $2n\pi/k$ along $Z_{n,k} \subset -kH_{p_{n-1}}$. Indeed, $\eta_\infty$ coincides with the model metric $\omega_{eh,n,k}$ defined in Section 3. To obtain the convergence in the pointed Gromov–Hausdorff sense, we first recall by Proposition 5.2 the distance between $Z_{n,-k}$ and $Z_{n,k}$ tends to infinity as $\beta_1 \nearrow n/k$. Once we choose a base point on $Z_{n,k}$. Since $\eta_{\beta_1}$ converges to $\eta_\infty$ on any compact geodesic balls centered at the base point, we conclude that $\eta_{\beta_1}$ converges in the pointed Gromov–Hausdorff sense to $\eta_\infty$ on $-kH_{p_{n-1}}$.

Remark 6.3. If we let $n = k = 2$ in Theorem 6.1, then by Remark 3.3 we obtain in the limit the Eguchi–Hanson metric with parameter $\epsilon$ set as 1 (see (A.11)). In other words, the Eguchi–Hanson metric arises as the pointed Gromov–Hausdorff limit of Kähler–Einstein edge metrics $\eta_{\beta_1}$ when $\beta_1 \nearrow 1$. This interesting observation has been conjectured in our previous work [14, Remark 5.1] and provided some of the motivation for the present article.

Next, we fix a base point on the infinity section $Z_{n,-k}$ to study the Gromov–Hausdorff limit of the family of Kähler–Einstein edge metrics $\xi$ on $\mathbb{F}_{n,k}$. In the limit, we obtain an orbifold Kähler–Einstein edge metric instead of the Ricci-flat edge metric obtained in Theorem 6.1.

From now on, we use $\xi_{\beta_2}$, $\tau(\beta_2)$ and $\varphi(\beta_2)$ to emphasize the dependence of metrics and coordinates on $\beta_2$. We consider the case $\beta_2 \nearrow 1/k$.

**Theorem 6.4.** Fix an arbitrary base point $p$ on the infinity section $Z_{n,-k}$. As $\beta_2 \nearrow 1/k$, the Kähler–Einstein edge metric $(\mathbb{F}_{n,k}, \xi_{\beta_2}, p)$ on $\mathbb{F}_{n,k}$ converges in the pointed Gromov–Hausdorff sense to an orbifold Kähler–Einstein edge metric $(\mathbb{P}^n(1, \ldots, 1, k), \xi_{\infty}, p)$ on the weighted projective space $\mathbb{P}^n(1, \ldots, 1, k)$ with an edge singularity of angle $2\pi/k$ along $Z_{n,-k}$. This limit metric coincides with the model metric $\omega_{orib,n,k}$ defined in Definition 3.7.

**Proof.** By similar notation and calculations as in the proof of Theorem 6.1, we have

$$\frac{ds}{d\tau(\beta_2)} = \frac{1}{\varphi(\beta_2)} = \frac{\tau(\beta_2)^{n-1}}{k_1(\tau(\beta_2)^n - 1) - \frac{n/k + \beta_2}{n+1}((\beta_2)^{n+1} - 1)}.$$

As $\beta_2 \nearrow 1/k$, we have

$$\frac{ds}{d\tau_\infty} = \frac{k}{\tau_\infty - \tau_\infty^2},$$

$$\varphi_\infty(\tau_\infty) = \frac{1}{k_1(\tau_\infty - \tau_\infty^2)}.$$
Solving (6.6) and considering a change of coordinate \( u' = C'u \) for some appropriate \( C \), we have
\[
\tau_\infty(s) = 1 - \frac{1}{e^{s/k} + 1}, \quad s \in (-\infty, +\infty),
\]
\[
\varphi_\infty(s) = \frac{1}{k} \frac{e^{s/k}}{(e^{s/k} + 1)^2}, \quad s \in (-\infty, +\infty).
\]
Thus, the limit metric on \( F_{n,k} \) is as follows:
\[
\tilde{\xi}_\infty = k \frac{e^{s/k}}{e^{s/k} + 1} \pi_1^* \omega_{FS} + \frac{1}{k} \frac{e^{s/k}}{(e^{s/k} + 1)^2} \left( \pi_2^* \omega_{Cyl} + \sqrt{-1} \alpha \wedge \bar{\alpha} \right.
\]
\[
+ \sqrt{-1} \alpha \wedge \frac{d\bar{w}}{w} + \sqrt{-1} \frac{dw}{w} \wedge \bar{\alpha} \left).
\]
Recall the Ricci curvature \( \mu_{\beta_2} \) is given in (4.17) by \( n/k + \beta_2 \) and converges to \( (n+1)/k \) in the limit. Thus \( \tilde{\xi}_\infty \) has Ricci curvature \( (n+1)/k \). Moreover, \( \tilde{\xi}_\infty \) has an edge singularity of angle \( 2\pi/k \) along \( Z_{n,-k} \). It degenerates on \( Z_{n,k} \) since \( \tau = 0 \) on \( Z_{n,k} \). Indeed, \( \tilde{\xi}_\infty \) coincides with the model metrics defined in Definition 3.6. Then by Definition 3.7, we denote by \( \xi_\infty \) the model metric on \( \mathbb{P}^n(1,\ldots,1,k) \) that is the pull-back of \( \tilde{\xi}_\infty \) under the blow up map.

We have shown that \( \tilde{\xi}_\infty \) is the limit of \( \xi_{\beta_2} \) as tensors in the pointwise smooth sense. Next, fix an arbitrary base point on \( Z_{n,-k} \). By Proposition 5.5 and the local smooth convergence result, we conclude that \( (\mathbb{P}^n(1,\ldots,1,k), \xi_{\beta_2}) \) is the limit of \( (\mathbb{P}^n_{n,k}, \xi_{\beta_2}) \) in the pointed Gromov–Hausdorff sense when \( \beta_2 \nearrow 1/k \). Moreover, the limit metric coincides with the model metric \( \omega_{orb,n,k} \) defined in Section 3.

As we pointed out in Section 5, the family of metrics \( \xi_{\beta_2} \) can be obtained by renormalizing the family of metrics \( \eta_{\beta_1} \). Comparing Theorem 6.1 to Theorem 6.4, we obtain different limit metrics for those two family of metrics. However, we show that after a proper normalization of \( \xi_{\beta_2} \), we obtain the same limit metric for both \( \xi_{\beta_2} \) and \( \eta_{\beta_1} \). The normalization factor is actually given by Proposition 5.5.

**Corollary 6.5.** Rescale the Kähler–Einstein edge metric \( \xi_{\beta_2} \) by \( ((n+1)/k)^{1/n}/(1/k - \beta_2)^{1/n} \), then the normalized metric converges in the pointed Gromov–Hausdorff sense to a Ricci-flat metric on \( -k \mathbb{P}^{n-1} \) when \( \beta_2 \nearrow 1/k \), where the base point is chosen from \( Z_{n,k} \). See the proof for a more precise explanation. Moreover, this Ricci-flat metric coincides with the one obtained in Theorem 6.1, i.e., the model metric \( \omega_{eh,n,k} \).

**Proof.** Consider a change of coordinate
\[
y(\beta_2) := \left( \frac{n+1}{k} \right)^{\frac{1}{n}} \cdot \frac{\tau(\beta_2)}{(1/k - \beta_2)^{\frac{1}{n}}}.
\]
By Proposition 5.5, the interval of definition of \( y(\beta_2) \) converges to \( [1, +\infty) \) as \( \beta_2 \nearrow 1/k \). The rescaled metric reads
\[
\left( \frac{n+1}{k} \right)^{\frac{1}{n}} \cdot \frac{\xi_{\beta_2}}{(1/k - \beta_2)^{\frac{1}{n}}}
\]
\[
= ky \pi_1^* \omega_{FS} + \left( \frac{n+1}{k} \right)^{\frac{1}{n}} \cdot \frac{\varphi(\beta_2)}{(1/k - \beta_2)^{\frac{1}{n}}} \left( \pi_2^* \omega_{Cyl} + \sqrt{-1} \alpha \wedge \bar{\alpha} - \sqrt{-1} \alpha \wedge \frac{d\bar{w}}{w} - \sqrt{-1} \frac{dw}{w} \wedge \bar{\alpha} \right).
\]
Recall
\[
\left(\frac{n+1}{k}\right)^{\frac{1}{n}} \cdot \frac{\varphi(\beta_2)}{(\frac{1}{k} - \beta_2)^{\frac{n}{k}}} = \left(\frac{n+1}{k}\right)^{\frac{1}{n}} \cdot \frac{1}{(\beta_2)^{n+1} - \frac{1}{k} + \beta_2)\left(\beta_2\right)^{n+1} - 1)}{(\beta_2)^{n+1} - \left(\frac{1}{k} - \beta_2\right)^{\frac{n}{k}}}.
\] (6.7)

Denote by \(y\) the coordinate in the limit. Letting \(\beta_2 \nearrow 1/k\), the right hand side of (6.7) converges to
\[
y^n - 1 - \frac{1}{ky^{n-1}}, \quad y \in [1, +\infty].
\]

Solving
\[
ds = \frac{1}{\varphi_{\beta_2}(\tau_{\beta_2})} \Rightarrow ds = \frac{1}{\varphi_{\beta_2}(\tau_{\beta_2})} \cdot \left(\frac{n+1}{k}\right)^{\frac{1}{n}} \cdot (n+1) = \frac{1}{\varphi_{\beta_2}(y_{\beta_2})},
\]
we obtain in the limit
\[
s = \frac{k}{n} \log(y^n - 1), \quad y \in (1, +\infty).
\] (6.8)

Thus the limit metric is given by
\[
\tilde{\xi_\infty} = k\pi_1^*\omega_{FS} + \frac{y^n - 1}{ky^{n-1}} \left(\pi_2^*\omega_{Cyl} + \sqrt{-1}\alpha \wedge \bar{\alpha} - \sqrt{-1}\alpha \wedge \frac{d\bar{u}}{u} - \sqrt{-1}\frac{du}{u} \wedge \bar{\alpha}\right),
\] (6.9)

where \(y\) and \(s\) satisfy (6.8). This limit metric is Ricci-flat. And it coincides with the limit metric in Theorem 6.1, i.e., the model metric \(\omega_{eh,n,k}\) defined in Section 3. Fix an arbitrary base point on \(Z_{n,k}\). By Proposition 5.5, the distance between \(Z_{n,k}\) and \(Z_{n,-k}\) tends to +\(\infty\) in the limit under the renormalized metric. Thus, \(Z_{n,-k}\) gets pushed-off to infinity in the limit. We obtain the pointed Gromov–Hausdorff convergence of \((\mathbb{F}_{n,k}, ((n+1)/k)^{1/n}\xi_{\beta_2}/(1/k - \beta_2)^{1/n})\) to \(-kH_{\mathbb{F}_n,n,k}\) with the metric obtained in (6.9).

\section{Small angle limits and fiberwise rescaling}

In this section we first consider the \(\beta_1 \searrow 0\) case for \(\eta_{\beta_1}\).

\textbf{Theorem 7.1.} As \(\beta_1\) tends to 0, \((\mathbb{F}_{n,k}, \eta_{\beta_1})\) converges in the Gromov–Hausdorff sense to \((\mathbb{P}^{n-1}, k\omega_{FS})\).

\textbf{Proof.} As \(\beta_1 \searrow 0\), by (4.8) we have
\[
\varphi_0 := \lim_{\beta_1 \searrow 0} \varphi_{\beta_1} = \frac{1}{\tau^{n-1}} \left(\left(1\right)^{\tau^n - 1} - \frac{n}{k(n+1)}(\tau^{n+1} - 1)\right) = \frac{1}{k(n+1)} \cdot \frac{1}{\tau^{n-1}}(-n\tau^{n+1} + (n+1)\tau^n - 1).
\]

Then we observe \(\varphi_0\) does not have any root greater than 1. Indeed, notice that
\[
-n\tau^{n+1} + (n+1)\tau^n - 1 = (\tau - 1)(1 + \cdots + \tau^{n+1} - n\tau^n)
= (\tau - 1)(1 - \tau^n + \tau^n - \cdots + \tau^{n+1} - \tau^n).
\]

Thus
\[
\varphi_0 := \lim_{\beta_1 \searrow 0} \varphi_{\beta_1} = \frac{1}{\tau^{n-1}} \left(\left(1\right)^{\tau^n - 1} - \frac{n}{k(n+1)}(\tau^{n+1} - 1)\right) = \frac{1}{k(n+1)} \cdot \frac{1}{\tau^{n-1}}(-n\tau^{n+1} + (n+1)\tau^n - 1).
\]

Therefore, \((\mathbb{F}_{n,k}, \eta_{\beta_1})\) converges in the Gromov–Hausdorff sense to \((\mathbb{P}^{n-1}, k\omega_{FS})\). \(\square\)
Thus \( \varphi_0 \) is always positive when \( \tau > 1 \). However, combining this fact with (5.1), we conclude that
\[
\lim_{\beta_1 \searrow 0} T = 1. \quad (7.1)
\]
Since \( \varphi(1) = \varphi(T) = 0 \), we have \( \varphi(\beta_1) \to 0 \) as \( \beta_1 \searrow 0 \). Since \( \tau \) ranges from 1 to \( T \), by (4.2) we conclude that as \( \beta_1 \searrow 0 \), \( \eta_{\beta_1} \) converges to \( k\tau^2 \omega_{\text{FS}} \). Thus we have shown \( (\mathbb{F}_{n,k}, \eta_{\beta_1}) \) converges in the Gromov–Hausdorff sense to \( (\mathbb{P}^{n-1}, k\omega_{\text{FS}}) \) when \( \beta_1 \searrow 0 \).

Roughly speaking, Theorem 7.1 says that as \( \beta_1 \searrow 0 \), the fibers collapse to the zero section. This motivates us to rescale the metric \( \eta_{\beta_1} \) along the fiber so that we can obtain a non-collapsed metric in the limit. We first need the following lemma.

**Lemma 7.2.** For \( \beta_1 > 0 \) and close to zero, \( T = T(\beta_1) = 1 + O(\beta_1) \).

**Proof.** By Proposition 4.4, \( T(\beta_1) \) is determined by \( \beta_1 \), and \( T(\beta_1) \) is the first root of the polynomial in (4.12). We rewrite the polynomial in (4.12) as
\[
P(\tau) = \frac{k\beta_1 - n}{k(n+1)} \left( \tau^n + \frac{k\beta_1 + 1}{k\beta_1 - n} (\tau^{n-1} + \cdots + 1) \right).
\]

Letting \( y = \tau - 1 \),
\[
P(y) = \frac{k\beta_1 - n}{k(n+1)} y \left( (y+1)^n + \frac{k\beta_1 + 1}{k\beta_1 - n} ((y+1)^{n-1} + \cdots + (y+1) + 1) \right)
= \frac{k\beta_1 - n}{k(n+1)} y^n + \cdots + y \left( n + \frac{k\beta_1 + 1}{k\beta_1 - n} (1 + \cdots + n - 1) + \frac{k(n+1)\beta_1}{k\beta_1 - n} \right)
= \frac{k\beta_1 - n}{k(n+1)} y Q(y) + \frac{k(n+1)\beta_1}{k\beta_1 - n},
\]
where \( Q(y) \) is a polynomial of degree \( n-1 \) whose coefficients depend on \( \beta_1 \) and whose constant term is
\[
Q(0) = n + \frac{k\beta_1 + 1}{k\beta_1 - n} (1 + \cdots + n - 1) = n \frac{(n-1)(k\beta_1 + 1)}{2(k\beta_1 - n)} = \frac{n + nk\beta_1 - n}{2(k\beta_1 - n)}. \quad (7.3)
\]

By Proposition 4.4, \( T - 1 \) is a root of the term in the parenthesis of the second equation in (7.2), i.e.,
\[
0 = (T - 1)Q(T - 1) + \frac{k(n+1)\beta_1}{k\beta_1 - n}.
\]
In particular, it follows that \( Q(T - 1) \neq 0 \) for small enough \( \beta_1 \). Thus, dividing we obtain
\[
T - 1 = \frac{k(n+1)\beta_1}{(-k\beta_1 + n)Q(T - 1)}.
\]
By (7.1) \( \lim_{\beta_1 \searrow 0} T = 1 \), and so \( \lim_{\beta_1 \searrow 0} Q(T - 1) = (n+1)/2 \) by (7.3). Altogether,
\[
T - 1 = \frac{2k}{n} \beta_1 + o(\beta_1), \quad (7.4)
\]
as claimed.

**Remark 7.3.** The last display generalizes [14, (5.1)] from the surface case \( n = 2 \) to any dimension. Note that in op. cit. \( n \) corresponds to our \( k \)
It follows that in the small angle limit, both angles approach zero at the same rate:

**Lemma 7.4.** For $\beta_1 > 0$ and close to zero, $\beta_2 = \beta_2(\beta_1) = \beta_1 + O(\beta_1^2)$.

*Proof.* Combining (7.1) and (5.9) it follows that $\lim_{\beta_1 \searrow 0} \beta_2 = 0$. Using this, and plugging (7.4), in (5.9) we find that

$$\beta_1 + \frac{1}{k} = \frac{1}{k} + 2\beta_1 - \beta_2 + o(\beta_1),$$

so $\beta_2 = \beta_1 + o(\beta_1)$, and so bootstrapping we obtain $\beta_2 = \beta_1 + O(\beta_1^2)$, as claimed. \hfill $\square$

Lemma 7.4 motivates treating the angles $2\pi \beta_1$ and $2\pi \beta_2$ on the same footing in the small angle regime, so that it reasonable to hope that under some appropriate rescaling the fibers converge to cylinders, as in [13, 14]. This is precisely what we prove next.

We change variable from $\tau \in (1, T)$ in (4.2) and (4.8) to

$$x := \tau - \frac{1 - \frac{k\beta_1}{n}}{\frac{k\beta_1}{n}},$$  

with $x \in \left(-\frac{1}{\beta_1}, \frac{1}{\beta_1} + O(1)\right)$ by (7.4). Note that $x = 0$ roughly corresponds to the middle section between $Z_{n,k}$ and $Z_{n,-k}$. By (4.8) and (7.5),

$$\varphi(x) = \frac{k}{2n} \beta_1^2 + \frac{k}{n} \beta_1^3 x + o(\beta_1^2), \quad x \in \left(-\frac{1}{\beta_1}, \frac{1}{\beta_1} + O(1)\right).$$  

Let $p \in \mathbb{F}_{n,k}$ be a fixed base point chosen from the section $\{x = 0\}$, which will serve as the base point we use later for pointed Gromov–Hausdorff convergence.

To find a fiberwise-rescaled limit, we next rescale the metric $\eta_{\beta_1}$ in (4.2) along each fiber, i.e., we define

$$\tilde{\eta}_{\beta_1} := k\tau \pi^*_1 \omega_{FS} + \frac{1}{\beta_1^2} \varphi \pi^*_2 \omega_{Cyl} + \varphi(\sqrt{-1} \alpha \wedge \bar{\alpha} + \sqrt{-1} \alpha \wedge \bar{d}w/w + \sqrt{-1} d\bar{w}/w \wedge \bar{\alpha}).$$  

**Remark 7.5.** This fiberwise rescaled metric is no longer Kähler. Indeed, since the metric $\eta_{\beta_1}$ in (4.2) is Kähler, i.e., $d\eta_{\beta_1} = 0$, there holds

$$d\tilde{\eta}_{\beta_1} = d\left(\eta_{\beta_1} - \left(1 - \frac{1}{\beta_1^2}\right) \varphi \pi^*_2 \omega_{Cyl}\right)$$

$$= -\left(1 - \frac{1}{\beta_1^2}\right) d\varphi \wedge \pi^*_2 \omega_{Cyl}$$

$$\neq 0.$$

**Theorem 7.6.** As $\beta_1 \searrow 0$, $(\mathbb{F}_{n,k}, \tilde{\eta}_{\beta_1}, p)$ converges in the pointed Gromov–Hausdorff sense to $(\mathbb{P}^{n-1} \times \mathbb{C}^*, \frac{k}{n}(n \pi^*_1 \omega_{FS} + \pi^*_2 \omega_{Cyl}), p)$.

*Proof.* We first show on compact subsets, the following pointwise convergence holds:

**Claim 7.7.** The restriction of $\tilde{\eta}_{\beta_1}$ to a fiber converges to a cylindrical metric pointwise on compact subsets. More precisely,

$$\lim_{\beta_1 \searrow 0} \frac{1}{\beta_1^2} \varphi \pi^*_2 \omega_{Cyl} = \frac{k}{n} \pi^*_2 \omega_{Cyl}.$$
Proof of the Claim. As shown in (4.4), the restriction of $\eta_{\beta_1}$ is given by
\[
\frac{1}{2\varphi(\tau)}d\tau^2 + 2\varphi(\tau)d\theta^2,
\]
thus the restriction of $\tilde{\eta}_{\beta_1}$ to a fiber, using the new coordinates (7.5) is given by
\[
\frac{k^2\beta_1^2}{2n^2\varphi(x)}dx^2 + \frac{2\varphi(x)}{\beta_1^2}d\theta^2. \tag{7.8}
\]
As $\beta_1 \searrow 0$, by (7.6), (7.8) converges pointwise on compact subsets to
\[
\frac{k}{n}dx^2 + \frac{k}{n}d\theta^2 = \frac{k}{n}\omega_{\text{Cyl}},
\]
as claimed.

By the collapsing arguments in the proof of Theorem 7.1 and the claim above, we have shown $\tilde{\eta}_{\beta_1}$ converges pointwise to $\frac{k}{n}(n\pi_1^*\omega_{FS} + \pi_2^*\omega_{\text{Cyl}})$ on compact subsets as $\beta_1 \searrow 0$. It remains to prove the pointed Gromov–Hausdorff convergence. Indeed, by arguments in the proof of Proposition 5.2, the distance between $Z_{n,k}$ and $\{x = 0\}$ and the distance between $Z_{n,-k}$ and $\{x = 0\}$ tend to infinity under the metric $\tilde{\eta}_{\beta_1}$. Thus in the limit $\beta_1 \searrow 0$, we get the product differential structure on $\mathbb{P}^{n-1} \times \mathbb{C}^*$ as claimed. Choosing the point $p$ as the base point, the pointwise convergence result implies that
\[
\lim_{\beta_1 \searrow 0} \tilde{\eta}_{\beta_1} = \frac{k}{n}(n\pi_1^*\omega_{FS} + \pi_2^*\omega_{\text{Cyl}})
\]
in the pointed Gromov–Hausdorff sense.

The $\beta_2 \searrow 0$ case for $\xi_{\beta_2}$ is similar to Theorem 7.1 and Theorem 7.6.

The asymptotic behaviors of $t(\beta_2)$ and $\beta_1(\beta_2)$ when $\beta_2 \searrow 0$ are similar to those described in Lemma 7.2 and Lemma 7.4. We collected them as follows.

Lemma 7.8. For $\beta_2 > 0$ and close to zero,
\[
t = t(\beta_1) = 1 - \frac{2k}{n}\beta_2 + o(\beta_2).
\]

Lemma 7.9. For $\beta_2 > 0$ and close to zero,
\[
\beta_1 = \beta_1(\beta_2) = \beta_2 + O(\beta_2^2).
\]

Now we state the non-rescaling limit of $\xi_{\beta_2}$ as $\beta_2 \searrow 0$.

Theorem 7.10. As $\beta_2$ tends to 0, $(\mathbb{P}_{n,k}, \xi_{\beta_2})$ converges in the Gromov–Hausdorff sense to $(\mathbb{P}^{n-1}, k\omega_{FS})$.

Proof. By Proposition 4.5 and Proposition 4.6, $t$ tends to 1 as $\beta_2 \searrow 0$. The remaining proof is similar to that of Theorem 7.1.
To obtain a non-collapsed metric in the limit, we consider rescaling $\xi_2$ in the way of (7.7) and denote the rescaled metric by $\tilde{\xi}_2$:

$$\tilde{\xi}_2 := k \tau \pi^1_0 \omega_{FS} + \frac{1}{\beta_2} \varphi \pi^2_0 \omega_{CY} + \varphi (\sqrt{-1} \alpha \wedge \bar{\alpha} - \sqrt{-1} \alpha \wedge \frac{du}{u} - \sqrt{-1} du \wedge \bar{\alpha}).$$

Moreover, we consider a change of variable as (7.5):

$$u := \frac{\tau - 1 + \frac{k}{n} \beta_2}{\frac{k}{n} \beta_2}.$$

As before, we choose a fixed base point from the section $\{u = 0\}$.

**Theorem 7.11.** As $\beta_2 \downarrow 0$, $(E_n, \xi_2, q)$ converges in the pointed Gromov–Hausdorff sense to $(\mathbb{P}^{n-1} \times \mathbb{C}^*, \frac{k}{n} (n \pi^1_0 \omega_{FS} - \pi^2_0 \omega_{CY}), q)$.

**Proof.** By Lemma 7.8 and Lemma 7.9, we have similar asymptotic behaviors as Lemma 7.2 and Lemma 7.4 in the $\beta_2$ case. Then we can apply similar arguments as in the proof of Theorem 7.6. \qed

### A brief review on Eguchi–Hanson metrics

In this section, we give a brief review of the construction of Eguchi–Hanson metrics [8]. They are Ricci-flat Kähler metrics defined on the total space of the line bundle $-2H_{p1}$.

For $(x_1 + \sqrt{-1} y_1, x_2 + \sqrt{-1} y_2) \in \mathbb{C}^2$, the Hopf coordinates are defined as:

$$x_1 + \sqrt{-1} y_1 = \xi \cos \frac{\theta}{2} e^{\frac{\sqrt{-1}}{2} (\psi + \phi)},$$

$$x_2 + \sqrt{-1} y_2 = \xi \sin \frac{\theta}{2} e^{\frac{\sqrt{-1}}{2} (\psi - \phi)},$$

where $\xi \geq 0$, $\theta \in [0, \pi]$, $\psi \in [0, 4\pi]$ and $\phi \in [0, 2\pi]$. Define one-forms on $\mathbb{C}^2$ by

$$\sigma_1 := \frac{1}{\xi^2} (x_1 dy_2 - y_2 dx_1 + y_1 dx_2 - x_2 dy_1) = \frac{1}{2} (\sin \psi d\theta - \sin \theta \cos \psi d\phi),$$

$$\sigma_2 := \frac{1}{\xi^2} (y_1 dy_2 - y_2 dy_1 + x_2 dx_1 - x_1 dx_2) = \frac{1}{2} (-\cos \psi d\theta - \sin \theta \sin \psi d\phi),$$

$$\sigma_3 := \frac{1}{\xi^2} (x_2 dy_2 - y_1 dx_2 + x_1 dy_1 - y_1 dx_1) = \frac{1}{2} (d\psi + \cos \theta d\phi).$$

Direct calculations yield:

$$d\sigma_1 = 2 \sigma_2 \wedge \sigma_3,$$

$$d\sigma_2 = 2 \sigma_3 \wedge \sigma_1,$$

$$d\sigma_3 = 2 \sigma_1 \wedge \sigma_2.$$ \hfill (A.2)

The standard Euclidean metric on $\mathbb{C}^2$ can be written as

$$dx_1^2 + dy_1^2 + dx_2^2 + dy_2^2 = d\xi^2 + \xi^2 (\sigma_1^2 + \sigma_2^2 + \sigma_3^2).$$

The Eguchi–Hanson metric with parameter $\epsilon > 0$ is defined by

$$g_{EH, \epsilon} := \left(1 - \frac{\epsilon}{\xi^4}\right)^{-1} d\xi^2 + \xi^2 \left(\sigma_1^2 + \sigma_2^2 + \left(1 - \frac{\epsilon}{\xi^4}\right) \sigma_3^2\right), \quad \xi \geq \epsilon.$$ \hfill (A.3)
Proposition A.1. Eguchi–Hanson metrics defined in (A.3) are Ricci-flat.

Proof. We provide a proof by directly calculating connection forms and curvature forms of the metric. See Remark A.2 for another proof using Kähler forms of Eguchi–Hanson metrics. Consider a change of variable \( \zeta = \zeta(\xi) \) such that \( d\zeta = (1 - (\epsilon/\xi)^4)^{-1/2} d\xi \). Then we can write (A.3) in the form

\[
g_{\text{EH},\epsilon} = d\zeta^2 + f^2(\zeta)(g_0^2 + g_1^2 + g_2^2),
\]

where \( f = \xi \) and \( g = (1 - (\epsilon/\xi)^4)^{1/2} \). Consider an orthonormal basis

\[
(\omega^0, \omega^1, \omega^2, \omega^3) = (d\zeta, fg \sigma_3, f \sigma_1, f \sigma_2).
\]

Then \( g_{\text{EH},\epsilon} = \sum_{i=0}^3 (\omega^i)^2 \) and \( \{\omega^i\}_{i=0}^3 \) satisfy the following equations:

\[
d\omega^i = \omega^j \wedge \omega^i, \quad \text{for } i = 0, 1, 2, 3,
\]

\[
\omega^i_j + \omega^i_j = 0, \quad \text{for } i, j = 0, 1, 2, 3,
\]

where \( \{\omega^i\}_{i,j=0}^3 \) are connection forms with respect to \( \{\omega^i\}_{i=0}^3 \). Next let us determine connection forms. For \( i = 1 \), we have

\[
d\omega^1 = \frac{f'g + fg'}{fg} \omega^0 \wedge \omega^1 + \frac{2g}{f} \omega^2 \wedge \omega^3,
\]

where \( f' \) and \( g' \) denote the derivative with respect to \( \zeta \). Without loss of generality, we let

\[
\omega_0^1 = \frac{f'g + fg'}{fg} \omega^1,
\]

\[
\omega_2^1 = \frac{g}{f} \omega^3,
\]

\[
\omega_3^1 = -\frac{g}{f} \omega^2.
\]

(A.4)

It remains to find \( \omega_0^2, \omega_0^3 \) and \( \omega_2^5 \) due to the skew-symmetry of connection forms. By similar calculations we have

\[
d\omega^2 = \frac{f'g^2}{f} \omega^0 \wedge \omega^2 + \frac{2g}{fg} \omega^3 \wedge \omega^1,
\]

\[
d\omega^3 = \frac{f'g^2}{f} \omega^0 \wedge \omega^3 + \frac{2g}{fg} \omega^1 \wedge \omega^2.
\]

(A.5)

Thus, combining (A.4) and (A.5) we obtain

\[
\omega_0^2 = \frac{f'}{f} \omega^2,
\]

\[
\omega_0^3 = \frac{f'}{f} \omega^3,
\]

\[
\omega_2^3 = \frac{g^2 - 2}{fg}.
\]

(A.6)

Notice that

\[
f' = \frac{d\xi}{d\zeta} = (1 - (\epsilon/\xi)^4)^{1/2} = g,
\]

\[
g' = 2\epsilon^4 f^5.
\]
Combining (A.4), (A.6) and (A.7) we see

\[ \omega_0^1 = -\omega_2^3, \]
\[ \omega_2^1 = \omega_0^3, \]
\[ \omega_3^1 = -\omega_2^0. \]  

(A.8)

By (A.8) and the fact that \( R_i^j = d\omega_i^j - \omega_i^k \wedge \omega_k^j, \) for \( i, j = 0, 1, 2, 3, \) we obtain that the curvature forms also satisfy

\[ R_0^1 = -R_3^2, \]
\[ R_2^1 = R_3^0, \]
\[ R_3^1 = -R_2^0. \]

Finally, the Ricci-flatness comes from the formula \( \text{Ric}_{ij} = \sum_{k=0}^{3} R^k_{ikj} \) and the first Bianchi identity.

Introduce

\[ r^4 = \xi^4 - \epsilon^4, \quad \xi \geq \epsilon, \]

then (A.3) can be written as

\[ g_{EH,\epsilon} = \frac{r^2}{(\epsilon^4 + r^4)^{\frac{1}{2}}} (dr^2 + r^2 \sigma_2^2) + (\epsilon^4 + r^4)^{\frac{1}{2}} (\sigma_1^2 + \sigma_2^2), \quad r \geq 0. \] (A.9)

From (A.9) we are able to convert Eguchi–Hanson metrics into complex form by letting \( (z_1, z_2) \in \mathbb{C}^2 \) satisfy

\[ z_1 = r \cos \frac{\theta}{2} e^{\frac{\sqrt{-1}}{2}(\psi + \phi)}, \]
\[ z_2 = r \sin \frac{\theta}{2} e^{\frac{\sqrt{-1}}{2}(\psi - \phi)}. \] (A.10)

Denote by \( \omega_{EH,\epsilon} \) the Kähler form corresponding to \( g_{EH,\epsilon} \). Then by (A.9) we have in complex coordinates,

\[ \omega_{EH,\epsilon} = \sqrt{-1} \partial \bar{\partial} \left[ \sqrt{r^4 + \epsilon^4 + \log r^2 - \log(\epsilon^2 + r^4 + \epsilon^4)} \right] \]
\[ = \frac{\sqrt{-1} r^2}{\sqrt{r^4 + \epsilon^4}} (dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2) + \frac{\epsilon^4}{\sqrt{r^4 + \epsilon^4}} \sqrt{-1} \partial \bar{\partial} \log(r^2). \] (A.11)

(A.12)

Remark A.2. By calculating the Ricci form of \( \omega_{EH,\epsilon} \) as in (A.12), we can also derive the Ricci-
flatness of Eguchi–Hanson metrics. Indeed, from (A.12) we calculate

\[ \text{Ric}_{\omega_{\text{EH}, \epsilon}} = \text{Ric} \sqrt{-1} \left( \frac{r^2}{\sqrt{r^4 + \epsilon^4}} + \frac{\epsilon^4 |z_2|^2}{\sqrt{r^4 + \epsilon^4}} \right) dz_1 \wedge d\bar{z}_1 \]

\[ - \frac{\epsilon^4 z_2 \bar{z}_1}{\sqrt{r^4 + \epsilon^4}} dz_1 \wedge d\bar{z}_2 - \frac{\epsilon^4 z_1 \bar{z}_2}{\sqrt{r^4 + \epsilon^4}} dz_2 \wedge d\bar{z}_1 \]

\[ + \left[ \frac{r^2}{\sqrt{r^4 + \epsilon^4}} + \frac{\epsilon^4 |z_1|^2}{\sqrt{r^4 + \epsilon^4}} \right] dz_2 \wedge d\bar{z}_2 \]

\[ = -\sqrt{-1} \partial \bar{\partial} \log \left( \frac{r^2}{\sqrt{r^4 + \epsilon^4}} + \frac{\epsilon^4 |z_2|^2}{\sqrt{r^4 + \epsilon^4}} \right) \]

\[ - \frac{\epsilon^4 |z_1|^2 |z_2|^2}{(r^4 + \epsilon^4)r^8} \]

\[ = -\sqrt{-1} \partial \bar{\partial} \log 1 \]

\[ = 0. \]

From (A.9) one finds that \( g_{\text{EH}, \epsilon} \) is defined on \( \mathbb{C}^2 \) with possible singularity at \( r = 0 \). Since \( g_{\text{EH}, \epsilon} \) is invariant under the antipodal reflection, we have an induced metric on \( (\mathbb{C}^2 \setminus \{0\})/\mathbb{Z}_2 \) that admits no singularity. Consider the blow-up of \( (\mathbb{C}^2 \setminus \{0\})/\mathbb{Z}_2 \) at the origin, which is biholomorphic to the total space of the line bundle \(-2H_{\mathbb{F}_1}\), then a calculation in (B.4) below shows that \( \sigma_1^2 + \sigma_2^2 \) is the pull-back of the Fubini–Study metric from the exceptional divisor. Hence \( g_{\text{EH}, \epsilon} \) extends to a metric on the total space \(-2H_{\mathbb{F}_1}\) by letting \( r = 0 \) when restricting \( g_{\text{EH}, \epsilon} \) to the exceptional divisor.

**B Eguchi–Hanson metrics as Gromov–Hausdorff limits of Kähler–Einstein edge metrics**

In this section we give a direct proof of a special case of Theorem 6.1 when \( n = k = 2 \). We already know we will obtain Eguchi–Hanson metrics in the limit.

Recall in (4.2) we denote by \( \eta \) a Kähler–Einstein edge metric on Calabi–Hirzebruch manifolds \( \mathbb{F}_{n,k} \) that has the following form:

\[ \eta = k^r \pi_1^* \omega_{FS} + \varphi \left( \pi_2^* \omega_{\text{Cyl}} + \sqrt{-1} \alpha \land \bar{\alpha} + \sqrt{-1} \alpha \land \frac{dw}{w} + \sqrt{-1} \frac{d\bar{w}}{\bar{w}} \land \bar{\alpha} \right), \]  

(B.1)

where \( \pi_1, \pi_2 \) and \( \alpha \) are defined below (4.2).

From now on, we assume \( k = 2 \) and \( n = 2 \), i.e., consider the second Hirzebruch surface \( \mathbb{F}_2 \). To build a connection between \( g_{\text{EH}, \epsilon} \) and the Kähler edge metric \( \eta \) on \( \mathbb{F}_2 \), we first write \( \eta \) in terms of one forms introduced in (A.1).

Consider a change of coordinate \( w = v^2 \). The reason to do this is that \( w \) is the coordinate along each fiber of the line bundle \(-2H_{\mathbb{F}_1}\). Recall (A.10), then we have the following correspondence:

\[ z_1 = vz = r \cos \theta e^{\frac{\psi}{2} \sqrt{-1} (\psi + \phi)}, \]

\[ z_2 = v = r \sin \theta e^{\frac{\psi}{2} \sqrt{-1} (\psi - \phi)}. \]  

(B.2)
In particular,
\[ r^2 = |v|^2(1 + |z|^2) = |w|(1 + |z|^2). \]  

(B.3)

By the definition of \( \alpha \) in (4.2), (1, 1)-forms that appear in \( \eta \) are
\[ dz \wedge d\bar{z}, \quad \alpha \wedge \bar{\alpha}, \quad \alpha \wedge \bar{\alpha}, \quad \frac{4dv \wedge d\bar{v}}{|v|^2} = \frac{2d\bar{v}}{v} \wedge \alpha. \]

Let us first calculate \( \sigma_1^2 + \sigma_2^2 \) in terms of the coordinate \( z \) and \( v \). By (A.1) and (B.2) we have
\[ \sigma_1^2 + \sigma_2^2 = \frac{1}{4} d\theta^2 + \frac{1}{4} \sin^2 \theta d\phi^2 \]
\[ = \frac{1}{4} \left( -\frac{|z|dz}{(1 + |z|^2)z} - \frac{|z|d\bar{z}}{(1 + |z|^2)\bar{z}} \right)^2 + \]
\[ + \frac{1}{4} \left( \frac{2|z|}{1 + |z|^2} \right)^2 \left( \frac{1}{2\sqrt{-1}} \left( \frac{dz}{z} - \frac{d\bar{z}}{\bar{z}} \right) \right)^2 \]
\[ = \text{Re} \frac{dz \otimes d\bar{z}}{(1 + |z|^2)^2}. \]

(B.4)

For \( dr^2 \) and \( \sigma_3 \), By (A.1) and (B.2) we have
\[ 4r^2 dr^2 = \frac{r^4}{4} \left( \alpha + \bar{\alpha} + \frac{2dv}{v} + \frac{2d\bar{v}}{\bar{v}} \right)^2, \]
\[ \sigma_3^2 = \frac{1}{16} \left( \alpha - \bar{\alpha} + \frac{2dv}{v} - \frac{2d\bar{v}}{\bar{v}} \right)^2. \]

(B.5)

\[ dr^2 + r^2 \sigma_3^2 = \frac{r^2}{4} \text{Re} \left( \frac{4dv \otimes d\bar{v}}{|v|^2} + \alpha \otimes \bar{\alpha} + \alpha \otimes \frac{2d\bar{v}}{\bar{v}} + \frac{2dv}{v} \otimes \bar{\alpha} \right). \]

Denote by \( g_\eta \) the corresponding Riemannian metric on \(-2H_{p1}\) with respect to \( \eta \). Then combining (B.1), (B.4) and (B.5) we obtain
\[ g_\eta = 2\tau (\sigma_1^2 + \sigma_2^2) + \varphi \cdot \frac{4}{r^2} (dr^2 + r^2 \sigma_3^2). \]

(B.6)

For \( \tau \) and \( \varphi \) in (B.6), results in Section 4 apply after we fix \( n = k = 2 \) there. A key feature when \( n = 2 \) is the right hand side in (4.8) is a cubic polynomial, which is easy to handle. In other words, we will be able to derive more precise dependence of \( T \) and \( \beta_2 \) on \( \beta_1 \) comparing to results in Proposition 5.1 and Proposition 5.3. Indeed, this was done in the work of [14]. In this and the next section, we fix \( n = 2 \) and make use of several results obtained in [14].

Since \( n = k = 2 \), we find \( \beta_1 \in (0, 1) \). We will study the asymptotic behaviors of Kähler–Einstein edge metrics \( \eta \) when \( \beta_1 \to 1 \). Recall [14, (4.20)]
\[ \beta_2 = \frac{1}{4} (2\beta_1 + \sqrt{3(3 - 2\beta_1)(1 + 2\beta_1)} - 3). \]

So we have \( \beta_2 \to \frac{1}{2} \) as \( \beta_1 \to 1 \). Recall, \( \tau \) ranges from \([1, T]\) and \( T \) is given by [14, (5.1)]
\[ T = 1 + 3 \sqrt{1 + \frac{1}{3} \beta_1 - \frac{1}{3} \beta_1^2 + 2\beta_1 - 1} \]
\[ = \frac{1}{4 - 4\beta_1}. \]
Thus, \( T \to +\infty \) as \( \beta_1 \to 1 \). Moreover, recall in (4.8) \( \varphi(\tau) \) is given by

\[
\varphi(\tau) = \frac{1}{2} \frac{\tau^2 - 1}{\tau} + \frac{1}{3} (\beta_1 - 1) \frac{\tau^3 - 1}{\tau}
\]

\[
= \frac{1}{3} (\beta_1 - 1)(\tau - 1)(\tau - \alpha_1)(\tau - T) / \tau, \quad \text{for } \tau \in [1, T],
\]

where \( \alpha_1 \) is given in [14, (5.2)] by \( \alpha_1 = 1 + 3 \frac{-\sqrt{1 + \frac{1}{3} \beta_1 - \frac{4}{3} \beta_1^2} + 2 \beta_1 - 1}{4 - 4 \beta_1} \) and \( \alpha_1 \) tends to \(-1\) as \( \beta_1 \to 1 \). Below we first show as \( \beta_1 \to 1 \), the divisor \( Z_2 \) gets pushed-off to infinity. This result is a special case of Proposition 5.2, and for the reader’s convenience we include a proof here.

**Proposition B.1.** The length of the path on each fiber between the intersection point of the fiber with \( Z_2 \) and that of the fiber with \( Z_{-2} \) tends to infinity as \( \beta_1 \to 1 \).

**Proof.** Restricted to the fiber \( \{z = 0\} \), by (B.1) \( \eta = \varphi \sqrt{-1} \text{d}w \wedge \text{d}\bar{w} / |w|^2 \) gives a metric

\[
g = \frac{1}{2 \varphi(\tau)} \text{d}\tau^2 + 2 \varphi(\tau) \text{d}\theta^2.
\]

Up to some constant, the distance between \( \{\tau = 1\} \) and \( \{\tau = T\} \) is given by

\[
\int_1^T \frac{1}{\sqrt{\varphi(\tau)}} \text{d}\tau = \int_1^T \frac{\sqrt{\tau}}{\sqrt{\frac{1}{3} (1 - \beta_1)(\tau - 1)(\tau - \alpha_1)(\tau - T)}} \sqrt{\tau} \text{d}\tau
\]

\[
= \frac{1}{\sqrt{\frac{1}{3} (1 - \beta_1)}} \int_1^T \frac{\sqrt{\tau}}{\sqrt{(\tau - 1)(\tau - \alpha_1)(T - \tau)}} \sqrt{\tau} \text{d}\tau
\]

\[
= \frac{1}{\sqrt{\frac{1}{3} (1 - \beta_1)}} \int_0^{T-1} \frac{\sqrt{\xi + 1}}{\sqrt{\xi (\xi + 1 - \alpha_1)(T - 1 - \xi)}} \text{d}\xi
\]

\[
= \int_0^{T-1} I \text{d}\xi.
\]

Near \( \xi = 0 \), terms \( \sqrt{\xi + 1} \) and \( \sqrt{\xi + 1 - \alpha_1} \) are uniformly bounded as \( \beta_1 \to 1 \). (B.8) satisfies

\[
\int_0^\epsilon I \text{d}\xi \leq C \cdot \frac{1}{\sqrt{1 - \beta_1}} \cdot \frac{1}{\sqrt{1 - \beta_1}} \cdot \sqrt{\xi_0^\epsilon},
\]

for some uniform constant \( C > 0 \) and any small \( \epsilon > 0 \). Thus the integration in (B.8) does not blow up near \( \xi = 0 \). Near \( \xi = T - 1 \), in (B.8) for any fixed \( \epsilon > 0 \), we can find \( \beta_1 \) close to 1 such that for \( \xi \in (T - 1 - \epsilon, T - 1) \), we have

\[
\sqrt{\xi + 1} \geq \sqrt{T - \epsilon} \geq \frac{1}{2} T,
\]

\[
\sqrt{\xi (\xi + 1 - \alpha_1)} \leq \sqrt{(T - 1)(T - \alpha_1)} \leq \sqrt{2} T.
\]

Then we have

\[
\int_{T-1-\epsilon}^{T-1} I \text{d}\xi \geq C \cdot \frac{1}{\sqrt{1 - \beta_1}} \cdot \frac{1 - \beta_1}{\sqrt{1 - \beta_1}} \cdot \int_{T-1-\epsilon}^{T-1} \frac{\text{d}\xi}{\sqrt{T - 1 - \xi}}
\]

\[
= C \cdot \frac{1}{\sqrt{1 - \beta_1}} \cdot \frac{1 - \beta_1}{\sqrt{1 - \beta_1}} \cdot \sqrt{\xi_0^\epsilon}
\]

(2.7)
for a uniform constant $C > 0$ and arbitrary $\epsilon > 0$. Since we can choose arbitrary large $\epsilon$ in (B.9), the integration in (B.8) does not converge as $T \to \infty$. Combining the discussions above we see that $\int_0^{T-1} I \, d\xi$ tends to $\infty$ as $\beta_1 \to 1$, i.e., $Z_{-2}$ gets pushed-off to infinity. \hfill \Box

From now on, we use $\beta_1$ as a subscript to emphasize the dependence on angles. By (B.1) and (B.7), on any compact subsets of $\mathbb{F}_2$ we have the following convergence in the $C^k$-norm for every $k$:

$$
\eta_{\infty} := \lim_{\beta_1 \to 1} \eta_{\beta_1}
= 2\tau_{\beta_1} \frac{\sqrt{-1} d\bar{z} \wedge d\bar{z}}{(1 + |z|^2)^2} + \frac{1}{2} \left( \tau_{\beta_1} - \frac{1}{\tau_{\beta_1}} \right) \left( \sqrt{-1} dw \wedge d\bar{w} + \sqrt{-1} \alpha \wedge \bar{\alpha} \right)
+ \sqrt{-1} \alpha \wedge \frac{d\bar{w}}{w} + \sqrt{-1} \left( \sqrt{1 - w_2^2} \right),
$$

(B.10)

Recall the Ricci curvature tensor of $\eta_{\beta_1}$ is given by

$$
\text{Ric} \eta_{\beta_1} = (1 - \beta_1)[C_1] + (1 - \beta_2)[C_2] + 2 \frac{\sqrt{-1} d\bar{z} \wedge d\bar{z}}{(1 + |z|^2)^2} - \sqrt{-1} d\bar{d} \log \tau_{\beta_1}
$$

$$
- \sqrt{-1} d\bar{d} \log \varphi_{\beta_1}.
$$

(B.11)

The Ricci curvature $\lambda_{\beta_1}$ of $\eta_{\beta_1}$ is given by $\lambda_{\beta_1} = 1 - \beta_1$. Notice that $\lambda_{\beta_1} \to 0$ as $\beta_1 \to 1$. Hence, by (B.11) and facts that $\varphi \to (r^2 - 1)/2r$ in the limit and $\tau$ is a function of $r^4 = |w|^2(1 + |z|^2)^2$ (recall (B.3)) we have

$$
2 \frac{\sqrt{-1} d\bar{z} \wedge d\bar{z}}{(1 + |z|^2)^2} - \sqrt{-1} d\bar{d} \log \tau_{\infty} - \sqrt{-1} d\bar{d} \log \varphi_{\infty} = 0,
$$

$$
\Rightarrow \tau_{\infty} \varphi_{\infty} = C|w|^2(1 + |z|^2)^2,
$$

$$
\Rightarrow \tau_{\infty} = C^{-\frac{1}{2}}(C + r^4)^{\frac{1}{2}}, \text{ for some constant } C > 0.
$$

(B.12)

Replacing $\tau$ and $\varphi$ in (B.6) using (B.12), we have

$$
\eta_{\infty} = 2C^{-\frac{1}{2}} \left( (C + r^4)^{\frac{1}{2}}(\sigma_1^2 + \sigma_2^2) + \frac{r^2}{(C + r^4)^{\frac{1}{2}}} (dr^2 + r^2 \sigma_3^2) \right).
$$

(B.13)

Comparing (B.13) to (A.9) we see $\eta$ converges on compact subsets to an Eguchi–Hanson metric as $\beta_1 \to 1$. Summarizing discussions above, we have shown the following result. It is a special case of Theorem 6.1 and provides a new way of understanding Eguchi–Hanson metrics.

**Theorem B.2.** Fix an arbitrary base point $p$ on the zero section. The Kähler–Einstein edge metric $(\mathbb{F}_2, \eta_{\beta_1}, p)$ on $\mathbb{F}_2$ converges in the pointed Gromov–Hausdorff sense to the following Eguchi–Hanson metric $(-2H_{p1}, \eta_{\infty})$, $p$ on $-2H_{p1}$ as $\beta_1 \to 1$:

$$
\eta_{\infty} = 2C^{-\frac{1}{2}} \left( (C + r^4)^{\frac{1}{2}}(\sigma_1^2 + \sigma_2^2) + \frac{r^2}{(C + r^4)^{\frac{1}{2}}} (dr^2 + r^2 \sigma_3^2) \right),
$$

where $C > 0$ is a constant.

**Proof.** The convergence on any compact subset of $-2H_{p1}$ of such $\eta$ to an Eguchi–Hanson metric in $C^k$-norm for every $k$ can be seen from (B.13). Combining this fact, the fact that $Z_{-2}$ gets pushed-off to infinity as $\beta_1 \to 1$ and choosing an arbitrary base point from the exceptional divisor $Z_2$, we obtain the convergence in the pointed Gromov–Hausdorff sense by considering convergence of $\eta$ to an Eguchi–Hanson metric on compact geodesic balls centered at the base point. \hfill \Box
C Examples of limit of Kähler–Einstein edge metrics

In this section, we fix $k = 1$ and $n = 2$. Then we follow Section 5 and Section 6 to give more concrete examples as limit of Kähler–Einstein edge metrics. In the previous work, we treated the case $\beta_1 \searrow 0$ and $\beta_1 \to 1$ [14]. In this section, we consider the several cases: $\beta_1 \not\nearrow 2$, $\beta_2 \not\nearrow 1$ with no rescaling and $\beta_2 \nearrow 1$ with rescaling. The asymptotic behaviors in such cases are summarized in Table 1.

Under the assumption $k = 1$ and $n = 2$, $\beta_1$ ranges from (0, 2). $\tau$ ranges from $[1, T]$. We will study the limiting behaviors of Kähler–Einstein edge metrics when $\beta_1 \to 2$.

By (4.8) $\varphi(\tau)$ satisfies

$$\varphi(\tau) = \frac{\tau^2 - 1}{\tau} + \frac{1}{3}(\beta_1 - 2)(\tau^3 - 1)$$

$$= \frac{1}{3}(\beta_1 - 2)(\tau - 1)(\tau - \alpha_1)(\tau - T)/\tau,$$

where $T$ and $\alpha_1$ satisfy [14, (5.1), (5.2)]

$$T = 1 + 3\sqrt{1 + \frac{2}{3}\beta_1 - \frac{1}{3}\beta_1^2 + \beta_1 - 1},$$

$$\alpha_1 = 1 + 3\sqrt{1 + \frac{2}{3}\beta_1 - \frac{1}{3}\beta_1^2 + \beta_1 - 1}.$$  \hfill (C.1)

Obviously $T \to +\infty$ and $\alpha_1 \to -1$ as $\beta_1 \to 2$. $\beta_2$ is given by [14, (4.20)]

$$\beta_2 = \frac{\beta_1 - 3 + 3\sqrt{1 + \frac{2}{3}\beta_1 - \frac{1}{3}\beta_1^2}}{2}.$$  \hfill (C.2)

Thus $\beta_2 \to 1$ as $\beta_1 \to 2$. More precisely, we have the following asymptotic behavior of $\beta_2(\beta_1)$:

**Lemma C.1.** For $\beta_1 < 2$ and close to 2, $\beta_2(\beta_1) = \frac{1}{3}\beta_1 + o(1)$.

The length of the path on each fiber between the intersection point of the fiber with $Z_1$ and that of the fiber with $Z_{-1}$ is given by the integration of $\varphi(\tau)$ from 1 to $T$. A similar calculation as in the proof of Proposition B.1 shows the following result.

**Proposition C.2.** The length of the path on each fiber between the intersection point of the fiber with $Z_1$ and that of the fiber with $Z_{-1}$ tends to infinity as $\beta_1 \to 2$.

Recall we assume Kähler–Einstein edge metrics on $F_1$ have the form as in (B.1). The Ricci curvature form of such Kähler–Einstein edge metrics, denoted by $\eta$, is given by (B.11). From now on, we denote by $\eta_{\beta_1}$, $\tau_{\beta_1}$ and $\varphi_{\beta_1}$ to emphasize the dependence of metrics and coordinates on $\beta_1$. Let us consider on any compact subsets of $F_1$,

$$\lim_{\beta_1 \to 2} \eta_{\beta_1} = \lim_{\beta_1 \to 2} \tau \left( \frac{\sqrt{-1}dz \wedge d\bar{z}}{(1 + |z|^2)^2} + \frac{\tau^2 - 1}{\tau} (\frac{\sqrt{-1}dw \wedge d\bar{w}}{|w|^2} + \sqrt{-1}dw \wedge \alpha + \sqrt{-1}\alpha \wedge \sqrt{-1}dw) \right)$$

$$= \tau \left( \frac{\sqrt{-1}dz \wedge d\bar{z}}{(1 + |z|^2)^2} + \frac{\tau^2 - 1}{\tau} \left( \frac{\sqrt{-1}dw \wedge d\bar{w}}{|w|^2} + \sqrt{-1}dw \wedge \alpha + \sqrt{-1}\alpha \wedge \sqrt{-1}dw \wedge \alpha \right) \right).$$

$$=: \eta_\infty$$
The Ricci curvature $\lambda_{\beta_1}$ of $\eta_{\beta_1}$ is given by $\lambda_{\beta_1} = 2 - \beta_1$. As $\beta_1 \to 2$, the Ricci curvature $\lambda_{\beta_1}$ tends to 0. Thus, the limit metric $\eta_\infty$ has Ricci curvature 0. Hence by (B.11), $\tau_\infty$ and $\varphi_\infty$ satisfy

$$2\sqrt{-1}dz \wedge d\bar{z} \over (1 + |z|^2)^2 - \sqrt{-1}\partial \bar{\partial} \log \tau_\infty - \sqrt{-1}\partial \bar{\partial} \log \varphi_\infty = 0,$$

$$\Rightarrow \tau_\infty \varphi_\infty = C|w|^4(1 + |z|^2)^2,$$

$$\Rightarrow \tau_\infty = (1 + C|w|^4(1 + |z|^2)^2)^{1\over 2}, \quad \text{for some constant } C > 0. \quad (C.2)$$

Thus we obtain the following theorem, which is a special case of Theorem 6.1.

**Theorem C.3.** Fix an arbitrary base point $p$ on $Z_1 \subset F_1$. As $\beta_1 \to 2$, the Kähler–Einstein edge metric $(F_1, \eta_{\beta_1}, p)$ on $F_1$ converges in the pointed Gromov–Hausdorff sense to a Ricci-flat metric $(-H_{p_1}, \eta_\infty, p)$ on $-H_{p_1}$ with conic singularity of angle $4\pi$ along $Z_1$.

**Proof.** By (C.2), the limit metric $\eta_\infty$ has the form

$$\eta_\infty = (1 + C|w|^4(1 + |z|^2)^2)^{1\over 2} \frac{\sqrt{-1}dz \wedge d\bar{z}}{(1 + |z|^2)^2} + C\frac{|w|^4(1 + |z|^2)^2}{(1 + C|w|^4(1 + |z|^2)^2)^{1\over 2}} \left( \frac{\sqrt{-1}dw \wedge d\bar{w}}{|w|^2} + \sqrt{-1}\alpha \wedge \bar{\alpha} + \sqrt{-1}\alpha \wedge \frac{dw}{w} \right) + \sqrt{-1}\frac{d\bar{w}}{w} \wedge \bar{\alpha},$$

for some constant $C > 0$. Thus $\eta_\infty$ has edge singularity of angle $4\pi$ along $Z_1$. For a fixed base point on $Z_1$, Proposition C.2 shows that $Z_1$ gets pushed-off to infinity in the limit. The remaining proof is the same as that of Theorem B.2. \qed

**C.1 Calculations in terms of $\beta_2$**

In this section, we choose a base point from the infinity section and then study the limit behavior of the Kähler–Einstein edge metrics. In the language of Section 5, we will consider the Kähler–Einstein edge metrics that are parametrized by $\beta_2$.

As in Section 5, we consider $u := 1/w$ as the fiber coordinate. Then $\{u = 0\}$ is the infinity section and $\{u = \infty\}$ is the zero section. We still define $s$ as follows:

$$s = \log(1 + |z|^2) - \log |u|^2.$$

Then as before $\{s = -\infty\}$ still corresponds to the zero section while $\{s = +\infty\}$ corresponds to the infinity section.

Denote now by $\xi$ the Kähler–Einstein edge metric that we seek on $F_1$. Assume $\xi = \sqrt{-1}\partial \bar{\partial} f(s)$ for some smooth function $f(s)$. As (4.14) we calculate

$$\eta = \sqrt{-1}\partial \bar{\partial} f(s) = \tau \pi^*_1 \omega_{FS} + \varphi \left( \frac{\sqrt{-1}du \wedge d\bar{u}}{|u|^2} + \alpha \wedge \bar{\alpha} - \alpha \wedge \frac{du}{u} - \frac{du}{u} \wedge \bar{\alpha} \right),$$

where $\alpha := \bar{z}dz/1 + |z|^2$, $\tau = f'(s)$ and $\varphi = f''(s)$ as before. As in Section 5, after a renormalization of the metric we may assume

$$\sup f'(s) = 1.$$

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We also assume \( f'(s) = t \), for some \( t > 0 \). In other words, \( \tau \) ranges from \([t, 1]\).

Now we calculate the Ricci curvature form \( \eta \) using coordinates \( u \) and \( z \):

\[
\text{Ric} \, \eta = -\sqrt{-1} \partial \bar{\partial} \log \eta^2 = -\sqrt{-1} \partial \bar{\partial} \log \frac{\tau \varphi}{|u|^2 (1 + |z|^2)^2} = (1 - \beta_1)[Z_1] + (1 - \beta_2)[Z_{-1}] + 2\pi^* \omega_{FS} - \sqrt{-1} \partial \bar{\partial} \log \tau - \sqrt{-1} \partial \bar{\partial} \log \varphi.
\]

Denote the Ricci curvature by \( \mu \). The Kähler–Einstein edge equation

\[
\text{Ric} \, \eta = \mu \eta + (1 - \beta_1)[Z_1] + (1 - \beta_2)[Z_{-1}]
\]

is equivalent to

\[
2 - \varphi - \varphi/\tau = \mu \tau, \quad \tau \in [t, 1]. \tag{C.3}
\]

Note that (C.3) gives us the same ODE derived in [14, (4.12)]. Now let us determine boundary conditions satisfied by \( \varphi(\tau) \). The same arguments in [14, Proposition 3.3] give us that

\[
\varphi(t) = \varphi(1) = 0, \quad \varphi'(t) = \beta_1, \quad \varphi'(1) = -\beta_2. \tag{C.4}
\]

Plugging boundary conditions in (C.3) implies that

\[
\mu = 2 + \beta_2.
\]

The solution to (C.3) is

\[
\varphi(\tau) = \frac{\tau^2 - 1}{\tau} + \frac{2 + \beta_2}{3} \frac{1 - \tau^3}{\tau}. \tag{C.5}
\]

Combining (C.4) and (C.5), we obtain the dependence of \( t \) and \( \beta_1 \) on \( \beta_2 \):

\[
t = \frac{1 - \beta_2 + \sqrt{(\beta_2 - 1)(-3\beta_2 - 9)}}{2(2 + \beta_2)},
\]

\[
\beta_1 = \frac{3}{2} + \frac{1}{2} \beta_2 - \frac{1}{2} \sqrt{(1 - \beta_2)(3\beta_2 + 9)}. \tag{C.6}
\]

Next, inspired by the result in Theorem C.3, we study the limiting behavior of Kähler–Einstein edge metrics when \( \beta_2 \) tends to 1.

**Proposition C.4.** The length of the path on each fiber between the intersection point of the fiber with \( Z_1 \) and that of the fiber with \( Z_{-1} \) converges to a finite number as \( \beta_2 \to 1 \).

**Proof.** As shown in Proposition B.1, when restricted to the fiber \( \{z = 0\} \), the distance between \( \{\tau = t\} \) and \( \{\tau = 1\} \) is given by

\[
\int_t^1 \frac{1}{\sqrt{\varphi(\tau)}} \, d\tau = \int_t^1 \frac{\sqrt{\tau}}{(2 + \beta_2)/3 \sqrt{(1 - \tau)(\tau - \alpha_1)(\tau - t)}} \, d\tau. \tag{C.7}
\]

It remains to show the integral in (C.7) converges uniformly as \( \beta_2 \to 1 \). Near \( \tau = 1 \), \( \sqrt{\tau}/(\tau - \alpha_1)(\tau - t) \) in (C.7) is uniformly bounded as \( \beta_2 \to 1 \). Thus for \( \epsilon > 0 \),

\[
\int_{1-\epsilon}^1 \frac{\sqrt{\tau}}{(2 + \beta_2)/3 \sqrt{(1 - \tau)(\tau - \alpha_1)(\tau - t)}} \, d\tau \leq C \int_{1-\epsilon}^1 \frac{1}{\sqrt{1 - \tau}} \, d\tau < \infty
\]
for some uniform constant $C > 0$. In other words, the integral (C.7) does not blow up near $\tau = 1$. It remains to study (C.7) near $\tau = t$. We consider a change of coordinate $\xi := \tau - t$. Then for $\epsilon > 0$,

$$
\int_{t}^{t+\epsilon} \frac{\sqrt{\tau}}{(2 + \beta_2)/(3\sqrt{(1 - \tau)(\tau - \alpha_1)}(\tau - t))} \, d\tau \leq C \int_{0}^{\epsilon} \frac{\sqrt{\xi + t}}{\sqrt{\xi + t - \alpha_1}},
$$

for some uniform constant $C > 0$. Since

$$
\lim_{\xi \to 0} \frac{\sqrt{\xi + t}}{\xi + t - \alpha_1} = \frac{\sqrt{t}}{\sqrt{t - \alpha_1}} \leq C,
$$

for some uniform constant $C > 0$ when $\beta_2 \to 1$, we conclude that the integral (C.7) converges as $\int_{0}^{t} 1/\sqrt{\xi}$ near $\tau = t$. Hence we have finished the proof.

From now on, we denote by $\xi_{\beta_2}$, $\tau_{\beta_2}$ and $\varphi_{\beta_2}$ to indicate the dependence of metrics and coordinates on $\beta_2$.

**Theorem C.5.** Fix an arbitrary base point $p$ on $Z - 1 \subset F_1$. As $\beta_2 \to 1$, the Kähler–Einstein edge metric $(F_1, \xi_{\beta_2}, p)$ on $F_1$ converge in the pointed Gromov–Hausdorff sense to the Fubini–Study metric $(\mathbb{P}^2, \omega_{FS}, p)$. We will show that $\xi_{\beta_2}$ converges pointwise smoothly to a degenerate metric tensor that is the pull-back of the Fubini–Study metric under the blow-up map on $F_1$.

**Proof.** Denote by $\xi_\infty$, $\tau_\infty$ and $\varphi_\infty$ the metric and coordinates in the limit when $\beta_2 \to 1$. To find a relation between $\tau_\infty$, $\varphi_\infty$ and $s$, consider the ODE satisfied by $s$ and $\tau_{\beta_2}$:

$$
\frac{ds}{d\tau_{\beta_2}} = \frac{1}{\varphi_{\beta_2}} = \frac{\tau_{\beta_2}}{(2 + \beta_2)(1 - \tau_{\beta_2})^3 + \tau_{\beta_2}^2 - 1}. \quad (C.8)
$$

Letting $\beta_2 \to 1$ in (C.8), we obtain (up to a constant that can be chosen to be 0)

$$
s = \log \frac{\tau_{\infty}}{1 - \tau_{\infty}}, \quad \tau_{\infty} \in (0, 1), \quad (C.9)
$$

where $\tau_{\infty}$ ranges from $(0, 1)$ since $t \to 0$ as $\beta_2 \to 1$. Obviously there holds

$$
\varphi_{\infty} = \tau_{\infty}(1 - \tau_{\infty}). \quad (C.10)
$$

Recall $s = \log(1 + |z|^2) - \log|u|^2$. Combining (C.9) and (C.10), the limit metric $\xi_\infty$ has the form:

$$
\xi_\infty = \tau_{\infty} \pi_1^* \omega_{FS} + \varphi_{\infty} \left( \pi_1^* \omega_{CY} + \sqrt{-1} \alpha \wedge \bar{\alpha} - \sqrt{-1} \alpha \wedge \frac{du}{u} - \sqrt{-1} \frac{du}{u} \wedge \bar{\alpha} \right)
$$

$$
= \frac{1 + |z|^2}{|u|^2 + 1 + |z|^2} \pi_1^* \omega_{FS} + \frac{1 + |z|^2}{(|u|^2 + 1 + |z|^2)^2} \sqrt{-1} du \wedge d\bar{u}
$$

$$
+ \frac{|u|^2(1 + |z|^2)}{(|u|^2 + 1 + |z|^2)^2} \left( \sqrt{-1} \alpha \wedge \bar{\alpha} - \sqrt{-1} \alpha \wedge \frac{du}{u} - \sqrt{-1} \frac{du}{u} \wedge \bar{\alpha} \right). \quad (C.11)
$$

Next, we derive the explicit formula for $\xi_\infty$. Recall,

$$
\alpha = \frac{dz}{1 + |z|^2}. \quad (C.12)
$$
Thus by (C.13) and by calculations:

$$\xi_{\infty} = \frac{1 + |z|^2}{|u|^2 + 1 + |z|^2} \cdot \sqrt{-1} \bar{d}z \wedge d\bar{z} + \frac{1 + |z|^2}{(|u|^2 + 1 + |z|^2)^2} \sqrt{-1} du \wedge d\bar{u}$$

$$+ \frac{|u|^2(1 + |z|^2)}{(|u|^2 + 1 + |z|^2)^2} \left( \sqrt{-1} \left( \frac{|z|^2}{(1 + |z|^2)^2} dz \wedge d\bar{z} - \sqrt{-1} \frac{\bar{z}u}{1 + |z|^2} dz \wedge d\bar{u} - \sqrt{-1} \frac{\bar{u}z}{1 + |z|^2} du \wedge d\bar{z} \right) \right)$$

$$= \frac{1 + |u|^2}{(1 + |u|^2 + |z|^2)^2} dz \wedge d\bar{u} - \sqrt{-1} \frac{\bar{u}z}{(1 + |u|^2 + |z|^2)^2} du \wedge d\bar{z}$$

$$= \sqrt{-1}\partial\bar{\partial}\log(1 + |u|^2 + |z|^2).$$

(C.13)

This limit metric does not have singularity along $Z_{-1}$ since $\beta_2 \to 1$ in the limit. Moreover, $\xi_{\infty}$ has Ricci curvature 3 since $\mu_{\beta_2}$ tends to 3 when $\beta_2 \to 1$. Moreover, the metric $\xi_{\infty}$ degenerates along $Z_1 = \{ u = \infty \}$ as $\xi_{\infty} \to 0$ as $|u| \to +\infty$.

We next show that $\xi_{\infty}$ is the pull-back of the Fubini–Study metric $\omega_{FS}$ on $\mathbb{P}^2$ under the blow-down map $\pi : \mathbb{F}_1 \to \mathbb{P}^2$. Figure 2 shows the blow up of $\mathbb{P}^2(1, 1, k)$ at $p$ giving rise to $\mathbb{F}_1$. To see this, we regard $\mathbb{F}_1$, which is the blow-up of $\mathbb{P}^2$ at one point $p$ (WLOG assuming $p = [1 : 0 : 0]$), as the variety embedded in $\mathbb{P}^2 \times \mathbb{P}^1$:

$$\mathbb{F}_1 = \{(x_0 : x_1 : x_2, [y_0 : y_1]) \in \mathbb{P}^2 \times \mathbb{P}^1 : x_1 y_1 = x_2 y_0\}.$$

Then the blow-down map $\pi$ is given by:

$$\pi : \mathbb{F}_1 \to \mathbb{P}^2$$

$$([x_0 : x_1 : x_2, [y_0 : y_1]) \mapsto [x_0 : x_1 : x_2].$$

Restricted on the chart $\{x_1 \neq 0\}$, there hold:

$$u = \frac{x_0}{x_1}, \quad z = \frac{x_2}{x_1}$$

and

$$\pi(u, z) = (u, z) \in \mathbb{P}^2.$$  \hspace{1cm} (C.14)

Recall the Fubini–Study metric on $\mathbb{P}^2$ (restricted to the chart $\{x_1 \neq 0\}$) has the formula

$$\omega_{FS} = \sqrt{-1} \log(1 + |u|^2 + |z|^2).$$

Thus by (C.13) and (C.14) we have shown,

$$\xi_{\infty} = \pi^* \omega_{FS},$$

confirming that $\xi_{\infty}$ degenerates along the exceptional curve $Z_1$.

We have shown $\xi_{\beta_2}$ converges on any compact subset of $\mathbb{F}_1$ to $\xi_{\infty}$. Now fix an arbitrary base point on $Z_{-1} \subset \mathbb{F}_1$. By Proposition C.4, the length between $Z_1$ and $Z_{-1}$ remains to be finite when $\beta_2 \to 1$. Thus $(\mathbb{F}_1, \xi_{\beta_2}, p)$ converges in the pointed Gromov–Hausdorff sense to $(\mathbb{P}^2, \omega_{FS}, p)$ when $\beta_2 \to 1$. \hfill $\square$

Next, we consider the limiting behavior of properly renormalized Kähler–Einstein edge metrics on $\mathbb{F}_1$ when $\beta_2 \to 1$.  

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Lemma C.6. Rescaling the metric $\xi$ by a factor $2(2 + \beta_2)/(1 - \beta_2 + \sqrt{(1 - \beta_2)(3\beta_2 + 9)})$, the interval of definition of $\tau$ will change from $[t, 1]$ to $[1, T]$ as in (C.1).

Proof. When we calculate in terms of $\beta_1$, we have

$$\beta_2 = \frac{\beta_1 - 3 + 3\sqrt{1 + \frac{2}{3}\beta_1 - \frac{1}{3}\beta_1^2}}{2}, \tag{C.15}$$

$$T = 1 + 3\sqrt{1 + \frac{2}{3}\beta_1 - \frac{1}{3}\beta_1^2 + \beta_1 - 1} \frac{4 - 2\beta_1}{4 - 2\beta_1}. \tag{C.16}$$

When we calculate in terms of $\beta_2$, we have

$$\beta_1 = \frac{3}{2} + \frac{1}{2}\beta_2 - \frac{1}{2}\sqrt{(\beta_2 - 1)(-3\beta_2 - 9)}, \tag{C.17}$$

$$t = \frac{1 - \beta_2 + \sqrt{(\beta_2 - 1)(-3\beta_2 - 9)}}{2(2 + \beta_2)}. \tag{C.18}$$

Direct calculation shows (C.15) and (C.17) are equivalent. Combining (C.15) and (C.16), we have

$$T = \frac{2(2 + \beta_2)}{4 - 2\beta_1}. \tag{C.19}$$

Combining (C.17) and (C.18), we have

$$t = \frac{4 - 2\beta_1}{2(2 + \beta_2)}. \tag{C.20}$$

Thus, $T = 1/t$ and we can rescale the metric $\eta$ by the factor $1/t$ to change the domain of $\tau$ from $[t, 1]$ to $[1, T]$. \hfill \Box

Inspired by Lemma C.6, we normalize $\xi_{\beta_2}$ by the factor $\sqrt{3}/\sqrt{1 - \beta_2}$ and study its limiting behavior when $\beta_2$ tends to 1.

Theorem C.7. Rescaling the metric $\xi_{\beta_2}$ by $\sqrt{3}/\sqrt{1 - \beta_2}$, then as $\beta_2 \to 1$, the renormalized metric converges in the pointed Gromov–Hausdorff sense to a Ricci-flat metric on $-\mathbb{H}_1$, where the base point is chosen from $\mathbb{Z}_1$. See the proof for an explicit explanation. This metric coincides with the one obtained in Theorem C.3.
Proof. Consider the change of coordinate $y_{\beta_2} := \sqrt{3} \tau_{\beta_2}/\sqrt{1 - \beta_2}$ in the following ODE:
\[
\frac{ds}{d\tau_{\beta_2}} = \frac{1}{\varphi_{\beta_2}(\tau_{\beta_2})}.
\]
In the following equations, we omit the subscript $\beta_2$:
\[
\frac{ds}{dy} \frac{dy}{d\tau} = \frac{1}{\varphi(y)} \Rightarrow \frac{ds}{dy} = \frac{\sqrt{3}}{3} \cdot \frac{(1 - \beta_2)y}{(\sqrt{1 - \beta_2 y})^2 - 1 + \frac{2 + \beta_2}{3}(1 - (\sqrt{1 - \beta_2 y})^3)}.
\]
By an abuse of notation, still denote by $y$ the coordinate in the limit. As $\beta_2 \to 1$, there holds
\[
\frac{ds}{dy} = \frac{y}{y^2 - 1}, \quad y \in (1, +\infty), \tag{C.19}
\]
where the range of $y$ comes from (C.18). Recall the renormalized metric $\sqrt{3} \xi_{\beta_2}/\sqrt{1 - \beta_2}$ reads
\[
\frac{\sqrt{3} \xi_{\beta_2}}{\sqrt{1 - \beta_2}} = \frac{\sqrt{3} \tau_{\beta_2}}{\sqrt{1 - \beta_2}} \pi^*_1 \omega_{FS} + \frac{\sqrt{3} \varphi_{\beta_2}}{\sqrt{1 - \beta_2}} \left( \pi^*_2 \omega_{\text{Cyl}} + \sqrt{-1} \alpha \wedge \bar{\alpha} - \sqrt{-1} \alpha \wedge \frac{d\bar{u}}{u} \right)
\]
\[
-\sqrt{-1} \frac{d\bar{u}}{u} \wedge \alpha \right). \tag{C.20}
\]
Combining (C.19) and (C.20) we obtain the limit metric
\[
\tilde{\xi}_\infty = \sqrt{e^{2s} + 1} \pi^*_1 \omega_{FS} + \frac{e^{2s}}{\sqrt{e^{2s} + 1}} \left( \pi^*_2 \omega_{\text{Cyl}} + \sqrt{-1} \alpha \wedge \bar{\alpha} - \sqrt{-1} \alpha \wedge \frac{d\bar{u}}{u} \right)
\]
\[
-\sqrt{-1} \frac{d\bar{u}}{u} \wedge \alpha \right).
\]
Note that this limit metric is Ricci-flat. It coincides with the metric obtained in Theorem C.3 if we choose $C = 1$ there. We have shown the renormalized Kähler–Einstein edge metrics converge to $\xi_\infty$ in smooth local sense. Next, fix a base point from the zero section $Z_1$. Then by Proposition C.4, we conclude that the infinity section $Z_{-1}$ gets pushed-off to infinity in the limit. Combining this fact with the local smooth convergence, we conclude the pointed Gromov–Hausdorff limit of $(F_1, \sqrt{3} \eta_{\xi_2}/\sqrt{1 - \beta_2})$ is $(-H_{F^1}, \xi_\infty)$.

\[ \square \]

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