Flat slice Hamiltonian formalism for dynamical black holes

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(Dated: May 2, 2019)

We give a Hamiltonian analysis of the asymptotically flat spherically symmetric system of gravity coupled to a scalar field. This 1+1 dimensional field theory may be viewed as the "standard model" for studying black hole physics. Our analysis is adapted to the flat slice Painleve-Gullstrand coordinates. We give a Hamiltonian action principle for this system, which yields an asymptotic mass formula. We then perform a time gauge fixing that gives a Hamiltonian as the integral of a local density. The Hamiltonian takes a relatively simple form compared to earlier work in Schwarzschild gauge, and therefore provides a setting amenable to full quantisation.

PACS numbers: 04.60.Ds

I. INTRODUCTION

It may be argued that the central unsolved problem in black hole physics is a full quantization of the gravity-scalar field system in spherical symmetry. All the puzzles associated with black holes originated from studying this system classically and semi-classically [1, 2, 3, 4]. The extra step to full quantization has never been attempted, mainly due to the intractability of the Hamiltonian system. If this could be accomplished, we would have a complete scenario for studying the formation and evaporation of Schwarzschild black holes in a fully quantum dynamical setting, and in the simplest "no frills" context.

Although there has been some progress in this area in string theory, it is restricted to highly special black holes in derived supergravity models [5]. In the loop quantum gravity program, the work on this problem has so far been restricted to "isolated" horizons [6]. As the name suggests, these horizons are not appropriate for studying the full dynamics of
matter and gravity, where horizons can form and evolve in response to matter flows.

The first attempt at starting a quantization of the gravity-scalar field model in four spacetime dimensions dates back to the seventies, when the Hamiltonian theory was worked out in a parametrization adapted to Schwarzschild coordinates by Berger et. al. (BCNM) \cite{7}, and later clarified by Unruh \cite{2}. In this setting a reduced Hamiltonian for the scalar field was obtained in a particular gauge. This Hamiltonian was complicated enough that quantization was effectively untenable. (Unruh comments on this by saying “I present it here in the hope that someone else may be able to do something with it.” \cite{2}). The problem has since remained largely unaddressed, except for related work on shell collapse \cite{8, 12}, and a geometrodynamical quantization of the (vacuum) Schwarzschild black hole \cite{9, 10}.

In this paper we reanalyse the spherically symmetric gravity-scalar field system. The main new ingredient is the application of the Arnowitt-Deser-Misner (ADM) Hamiltonian formulation to coordinates adapted to the flat slice Painleve-Gullstrand (PG) coordinates \cite{11, 12, 13, 14, 15}, with a time gauge fixing. The black hole metric in these coordinates is given by

\[ ds^2 = -dt^2 + \left( dr + \sqrt{\frac{2M}{r}} dt \right)^2 + r^2 d\Omega^2. \]  (1)

The spatial metric \( e_{ab} \) given by the constant \( t \) slices is flat, and the extrinsic curvature of the slices is

\[ K_{ab} = -\sqrt{\frac{2M}{r^3}} \left( e_{ab} - \frac{3}{2} s_a s_b \right), \]  (2)

where \( s^a = x^a / r \) for Cartesian coordinates \( x^a \) \cite{16}.

The black hole mass information is contained only in the extrinsic curvature, which in the canonical ADM variables \( (q_{ab}, \tilde{\pi}^{ab}) \) determines the momenta \( \tilde{\pi}^{ab} \) conjugate to the spatial metric \( q_{ab} \). In this form, the mass formula is necessarily different from the ADM mass integral, since the spatial slices are flat.

The PG coordinates motivate the following prescription for the falloff conditions for asymptotic flatness:

\[ q_{ab} = e_{ab} + \frac{f_{ab}(\theta, \phi)}{r^{3/2+\epsilon}} + \mathcal{O}(r^{-2}) \]  (3)

\[ \pi^{ab} = \frac{g^{ab}(\theta, \phi)}{r^{3/2}} + \mathcal{O}(r^{-3/2-\epsilon}), \]  (4)

where \( \epsilon > 0 \), \( f^{ab}, g^{ab}, h^{ab} \) are symmetric tensors, \( \pi^{ab} = \tilde{\pi}^{ab} / \sqrt{q} \), and \( q = \det(q_{ab}) \). In this definition it is manifest that the leading terms correspond to the black hole solution in PG.
coordinates. The form of the next terms containing $\epsilon$ is necessitated by the requirement of a well defined action principle. It is useful to compare these with the conditions motivated by the Schwarzschild coordinates, where the leading order terms in the metric and extrinsic curvature are $1/r$ and $1/r^2$, respectively [17, 18].

These falloff conditions provide the starting point for our analysis of the Hamiltonian dynamics for spherically symmetric metrics of the form

$$ds^2 = -f(r,t)^2 dt^2 + (dr + g(r,t)dt)^2 + r^2 d\Omega^2,$$

minimally coupled to a massless scalar field $\phi(r,t)$.

In the next section, we give a parametrization for the ADM variables $(q_{ab}, \pi^{ab})$ respecting these conditions to obtain a Hamiltonian theory. In Section III, we utilise a special time gauge fixing condition in which the reduced Hamiltonian takes a relatively simple form compared to the earlier works mentioned above. The setting therefore provides an arena in which a full quantization appears to be possible. The last section contains a brief comparison with the BCNM work, and a discussion of our approach to quantization. The present paper also provides the classical details which underlie the recent work on quantization by the authors [19, 20].

II. GRAVITY-SCALAR FIELD MODEL

A well defined variational principle for Einstein’s equations coupled to matter, and satisfying a specified class of boundary conditions, may be obtained by starting with the bulk Einstein-Hilbert action, or its canonical bulk ADM form. The first requirement is that all the terms in the action are well-defined. The second requirement is that the variation of the action be of a form such that all variational derivatives with respect to the field variables are well defined. As noted by Regge and Teitelboim [17, 18], this requires in general the addition of a surface term to the original action.

The phase space of the model is defined by prescribing a form of the gravitational phase space variables $q_{ab}$ and $\pi^{ab}$, together with falloff conditions in $r$ for these variables, and for the lapse and shift functions $N$ and $N^a$, such that the ADM 3+1 action for general relativity minimally coupled to a massless scalar field

$$S = \frac{1}{16\pi G} \int d^3 x dt \left[ \tilde{\pi}^{ab} q_{ab} + \tilde{P}_\phi \dot{\phi} - NH - N^a C_a \right]$$

(6)
is well defined. The constraints arising from varying the lapse and shift are

\[ \mathcal{H} = \frac{1}{\sqrt{q}} \left( \tilde{\pi}^{ab} \tilde{\pi}_{ab} - \frac{1}{2} \tilde{\pi}^2 \right) \sqrt{q} R(q) \]

\[ -8\pi G \left( \frac{1}{\sqrt{q}} \tilde{P}_\phi^2 + \sqrt{q} q^{ab} \partial_a \phi \partial_b \phi \right) = 0 \]  \hspace{1cm} (7)

\[ \mathcal{C}_a = D^c \tilde{\pi}_a^{bc} - 8\pi G \tilde{P}_\phi \partial_a \phi = 0, \]  \hspace{1cm} (8)

where \( \tilde{\pi} = \tilde{\pi}^{ab} q_{ab} \). The corresponding conditions for the matter fields \( \phi \) and \( \tilde{P}_\phi \) are determined by the constraint equations.

In this setting, the following parametrization for the 3-metric and conjugate momentum gives a reduction to spherical symmetry:

\[ q_{ab} = \Lambda(r, t)^2 s_a s_b + \frac{R(r, t)^2}{r^2} (e_{ab} - s_a s_b) \]  \hspace{1cm} (9)

\[ \tilde{\pi}^{ab} = \frac{P_\Lambda(r, t)}{2 \Lambda(r, t)} s^a s^b + \frac{r^2 P_R(r, t)}{4 R(r, t)} (e^{ab} - s^a s^b). \]  \hspace{1cm} (10)

Substituting these into the 3+1 ADM action shows that the pairs \( (R, P_R) \) and \( (\Lambda, P_\Lambda) \) are canonically conjugate variables. The reduced ADM 1+1 field theory action takes the form

\[ S_R = \frac{1}{4G} \int dt dr \left( P_R \dot{R} + P_\Lambda \dot{\Lambda} + P_\phi \dot{\phi} \right) \]

\[ -\frac{1}{4G} \int dt dr (NH + N^r C_r) \]

\[ + \text{surface term}, \]  \hspace{1cm} (11)

where we have performed the angular integral. The surface term is derived below. The Hamiltonian and diffeomorphism constraints are

\[ H = \frac{1}{R^2 \Lambda} \left[ \frac{1}{8} (P_\Lambda \Lambda)^2 - \frac{1}{4} (P_\Lambda \Lambda)(P_R R) \right] \]

\[ + \frac{2}{\Lambda^2} \left[ 2 R R'' \Lambda - 2 R R' \Lambda' - \Lambda^3 + \Lambda R^2 \right] \]

\[ + \left[ \frac{P_\phi^2}{2 L R^2} + \frac{R^2}{2 \Lambda} \phi'^2 \right]. \]  \hspace{1cm} (12)

\[ C_r = P_R R' - \Lambda P'_\Lambda + P_\phi \phi' = 0. \]  \hspace{1cm} (13)

These constraints are first class. The falloff conditions induced on the reduced variables by
\[ R = r + O(r^{-1/2-\epsilon}), \]
\[ P_R = Ar^{-1/2}/2 + O(r^{-1-\epsilon}), \] (14)
\[ \Lambda = 1 + O(r^{-3/2-\epsilon}), \]
\[ P_\Lambda = Ar^{1/2} + O(r^{-\epsilon}) \] (15)
\[ \phi = Br^{-1/2} + O(r^{-3/2-\epsilon}), \]
\[ P_\phi = Cr^{1/2} + O(r^{-\epsilon}), \] (16)

where \( A, B, C \) are constants. This means that the asymptotic region is not dynamical (as it should be since it is flat). The constant \( A \), which appears in the expressions for \( P_R \) and \( P_\Lambda \) above, will turn out to be captured by a surface integral (see below) and is proportional to the mass of the system. The above conditions together with the falloff conditions on the lapse and shift functions

\[ N^r = Ar^{-1/2} + O(r^{-1/2-\epsilon}) \]
\[ N = 1 + O(r^{-\epsilon}) \] (17)

ensure that \( S_R \) is well-defined. More explicitly, they ensure that the symplectic structure is well-defined, which means that the integral of the terms \( P_R \dot{R} \) etc. converges. Furthermore, they guarantee that \( H \) vanishes to first order, and has a falloff \( r^{-1-\epsilon} \) beyond leading order, as required for the action to be well-defined. The same holds for the diffeomorphism constraint. Note that the factor \( 1/2 \) in the leading order term in \( P_R \) ensures that to this order both the diffeomorphism and Hamiltonian constraints vanish. This corresponds to the black hole solution in PG coordinates, which motivated our falloff conditions in the first place. Taken together these observations guarantee that the bulk action is well-defined.

Consider now the variation of this action to see what surface terms need to be added to make the variational principle well-defined. Following \[ \text{[17]} \], we compute the variation \( \delta S_R \), see what surface terms arise in it, and identify the terms that vanish due to the falloff conditions; the ones that do not must be compensated for by adding a surface term to the starting bulk action. In our case surface terms arise from those terms in the action that contain \( r \) derivatives. These are the Ricci scalar, matter density, and the radial diffeomorphism terms.
The variation is

\[ 4G \delta S_R = \int dt dr \ (\text{terms giving eqns. of motion}) \]

\[ - \int dt \left[ N^r P_\phi + \frac{NR^2 \phi'}{\Lambda} \right] \delta \phi \]

\[ + \int dt \left[ N^r \Lambda \delta P_\Lambda + \frac{4NR' \Lambda^2}{\Lambda^2} \delta \Lambda \right] \]

\[ + \int dt \left[ 8 \left( \frac{NR}{\Lambda} \right)' - \frac{4NR'}{\Lambda} + \frac{4NRA'}{\Lambda^2} \right] \delta R \]

\[ - \int dt [N^r P_R] \delta R, \quad (18) \]

where the last four terms are differences of surface integrals evaluated at \( r = 0 \) and \( r = \infty \).

The variational principle is well-defined if each of these surface terms vanishes.

At \( r = \infty \) most of these vanish by virtue of the falloff conditions. This leaves two terms of order one. The one proportional to \( \delta \phi \) can be eliminated by requiring this variation to vanish at infinity. The other, proportional to \( \delta P_\Lambda \), is dealt with by adding a surface term at infinity to the original bulk action whose variation cancels the offending surface term from the variation. It is this term that captures the conserved asymptotic mass.

At \( r = 0 \) we do not impose falloff conditions on the phase space variables because there is no physical guidance for this. We thus require the usual prescription that the variations of the configuration variables vanish there:

\[ \delta \phi |_{r=0} = 0, \]

\[ \delta R |_{r=0} = 0, \]

\[ \delta \Lambda |_{r=0} = 0. \quad (19) \]

This leaves only the term proportional to \( \delta P_\Lambda \) at \( r = 0 \). To deal with this we require either the addition of a surface term with the opposite sign to the one at infinity, or the condition \( N^r (r = 0) = 0 \). The former would subtract from the asymptotic mass. Therefore for the vacuum solutions (\( \phi = 0 \) and \( P_\phi = 0 \)) in the gauge \( R = r \Lambda = 1 \), the sum of the surface terms at \( r = 0 \) and \( r = \infty \) would cancel. This suggests that we make the latter choice.

Based on these observations, functional differentiability of the action is guaranteed if we add the term

\[ - \int dt (N^r \Lambda P_\Lambda) |_{r=\infty} \quad (20) \]
to the original bulk action. We can then derive the evolution equations:

\[ \dot{R} = -N \frac{P_\Lambda}{4R} + N' R' \]  

(21)

\[ \dot{P}_R = N \left[ \frac{P_\Lambda^2}{4R^3} - \frac{P_R P_\Lambda}{4R^2} + \frac{P_\phi^2}{\Lambda R^3} - \frac{R' \phi'^2}{\Lambda} \right] 
- \left( \frac{4RN'N}{\Lambda^2} \right)' + \left( \frac{4RN}{\Lambda} \right)' + \left( N' P_R \right)' \]  

(22)

\[ \dot{\Lambda} = \frac{N}{4R^2 \Lambda} \left( P_\Lambda \Lambda^2 - \Lambda P_R \right) + (\Lambda N')' \]  

(23)

\[ \dot{P}_\Lambda = N \left( -\frac{P_\Lambda^2}{8R^2} + \frac{4RR''}{\Lambda^2} + 2 + 2(\Lambda')^2 - \frac{8RR'N'}{\Lambda^3} \right) 
- \left( \frac{4RR'N}{\Lambda^2} \right)' + \frac{N}{2\Lambda^2 R^2} \left( P_\phi^2 + R^4(\phi')^2 \right)' + N' P_\Lambda' \]  

(24)

\[ \dot{\phi} = N \frac{P_\phi}{\Lambda R^2} + N' \phi' \]  

(25)

\[ \dot{P}_\phi = \left( N \frac{\phi' R^2}{\Lambda} \right)' + \left( N' P_\phi \right)'. \]  

(26)

Note that the surface term (20) is the mass formula for the flat slice parametrization. Substituting the asymptotic forms of the variables (15) and (17) gives

\[ N' \Lambda P_\Lambda = A^2 + \mathcal{O}(r^{-\epsilon}), \]  

(27)

which shows that the parameter \( A \) contains conserved mass information. The relation between \( A \) and the conventional mass parameter \( M \) in (1) is obtained by comparing the \( \pi^{ab} \) obtained from (2) with our parametrization (10). This gives \( A = 4\sqrt{2M} \). We note also that our falloff conditions are preserved under this evolution, which ensures their consistency. This is easily seen by computing the left and right hand sides of the above evolution equations in the asymptotic regime.

### III. TIME GAUGE FIXING

We now gauge fix the theory defined above with the condition \( \Lambda = 1 \). It is second class with the Hamiltonian constraint. Demanding that it be preserved in time implies from (28) the relation

\[ N(P_\Lambda - RP_R) = -4R^2(N')' \]  

(28)
between the lapse $N$ and the shift $N^r$. As a result of the time gauge fixing the Hamiltonian constraint \( (12) \) must be imposed strongly. This gives a quadratic equation for $P_\Lambda$ in terms of the remaining variables. A comparison with the vacuum solution in fully gauge-fixed form (ie. with the coordinate fixing condition $R = r$), uniquely selects the positive root. This gives

$$P_\Lambda = P_R R + \sqrt{(P_R R)^2 - X},$$  \hspace{1cm} (29)

where

$$X = 16R^2(2RR'' - 1 + R'^2) + 16R^2 H_\phi$$  \hspace{1cm} (30)

and

$$H_\phi = \frac{P_\phi^2}{2R^2} + \frac{R^2}{2}\phi'^2.$$  \hspace{1cm} (31)

The positivity of the argument of the square root follows from the dominant energy condition for the massless scalar field.

The solution for the lapse function now reads

$$N = -\frac{4R^2(N^r)' \sqrt{(P_R R)^2 - X}}{(P_R R)^2 - X}.$$  \hspace{1cm} (32)

The reduced Hamiltonian equations for the remaining canonical variables $(R, P_R)$ and $(\phi, P_\phi)$ are obtained by substituting the gauge condition $\Lambda = 1$, the corresponding solution (29) of the Hamiltonian constraint, and the lapse equation (32) into the unfixed evolution equations (21)-(26), and into the radial diffeomorphism constraint. The gauge fixed equations are

$$\dot{R} = -\frac{N}{4R^3} \left( P_R R + \sqrt{(P_R R)^2 - X} \right) + N^r R'$$  \hspace{1cm} (33)

$$\dot{P}_R = \frac{N}{4R^3} \left( P_R^2 R^2 + P_R R \sqrt{(P_R R)^2 - X - X} \right)$$
$$+ N \left( \frac{P_\phi^2}{R^2} - R\phi'^2 \right) + 4R'N' - (4RN)'$$
$$+(N^r P_R)'$$  \hspace{1cm} (34)

$$\dot{\phi} = N \frac{P_\phi}{R^2} + N^r \phi'$$  \hspace{1cm} (35)

$$\dot{P}_\phi = (N\phi'R^2)' + (N^r P_\phi)'$$  \hspace{1cm} (36)

where it is understood that $N$ is given by (32). The remaining radial diffeomorphism constraint is

$$P_\Lambda' + P_R R' + P_\phi \phi' = 0,$$  \hspace{1cm} (37)
where $P_\Lambda$ is given by (29).

All these equations can be obtained from the gauge fixed reduced action

$$S_G^R = \int dt dr \left[ P_\phi \dot{\phi} + P_R \dot{R} - N^r (P_\Lambda' + P_R R' + P_\phi \phi') \right]$$

$$+ \int dt P_\Lambda (N^r)' .$$

(38)

The surface term may be combined with the bulk term to write the action in a form from which one can read off the gauge fixed Hamiltonian

$$H_G^R = \int_0^\infty [(N^r)' P_\Lambda + N^r (P_R R' + P_\phi \phi')] dr$$

$$= \int_0^\infty (N^r)' \left( R P_R + \sqrt{(P_R)^2 - X} \right) dr$$

$$+ \int_0^\infty N^r (P_R R' + P_\phi \phi') dr,$$

(39)

IV. DISCUSSION

The Hamiltonian (39) is a simpler expression than that obtained from the full time and coordinate gauge fixing given in [2], where Schwarzschild gauge is used. That Hamiltonian is

$$H_S = \int dr \left( \frac{\dot{\phi}^2}{4r^2} + r^2 \phi'^2 \right) \exp \left( \int_\infty^r S_\phi (r') dr' \right)$$

(40)

where

$$S_\phi (r) = \frac{P_\phi^2}{8r^3} + \frac{r \phi'^2}{2}.$$

(41)

It is apparent that the square root in (39) is easier to handle than the non-locality manifest in this formula.

The non-local term can be traced back to the fact that (40) is arrived at after both a time and radial coordinate gauge fixing. This may be seen in our formulation as well: Consider the radial gauge fixing $R(r, t) = r$. This leads to (i) fixing of the shift function $N^r$ by the condition

$$\dot{r} = 0 = (N^r)' r \left( 1 + \frac{r P_R}{\sqrt{(r P_R)^2 - X}} \right) + N^r$$

(42)

and (ii) fixing of $P_R$ by strong imposition of the remaining diffeomorphism constraint (37). The solution of (42) for $N^r$ is

$$N^r = \exp \left( - \int_\infty^r \frac{\sqrt{(r P_R)^2 - X}}{r \sqrt{(r P_R)^2 - X + r^2 P_R}} d\bar{r} \right),$$

(43)
which leads directly to a non-local term when substituted back into (39). Thus we learn that the complicated form of the fully gauge fixed Hamiltonian is traceable solely to the coordinate gauge fixing, but not the time gauge fixing. A partial gauge fixing in the diagonal parametrization of the metric \[2, 7\] would also lead to a simpler, local form of the reduced Hamiltonian.

For this reason, we propose that a quantization of this system be carried out in a partially gauge fixed setting, with only the time gauge fixed. Retaining a first class radial diffeomorphism constraint presents issues that are relatively easier to deal with; we have in fact already proposed \[19\] a kinematical quantization of this system in which the Hilbert space carries a representation of finite diffeomorphisms.

In summary, we have presented here in detail the canonical formalism for spherically symmetric gravity coupled to a scalar field adapted to the flat slice foliation. We have shown that all consistency conditions are satisfied. These include functional differentiability, surface terms, and the preservation under evolution of the falloff conditions at infinity.

Having obtained the reduced Hamiltonian \(39\), the next step in our program is to construct the corresponding operator using the techniques presented in \(19\). This work is to appear \(21\).

Acknowledgments

We are grateful to Jorma Louko for illuminating discussions and comments on an earlier draft of the paper. This work was supported in part by the Natural Science and Engineering Research Council of Canada.

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[16] The flat slice foliation covers the most interesting regions of spacetimes describing realistic matter collapse. As such it is especially suited for studying Hawking radiation from a fully quantum gravitational viewpoint, which is the central motivation for our work. Indeed, since no ”extended” spacetimes are known for dynamical matter collapse, the long time static Schwarzschild limit with its full Kruskal extension is an ”isolated” point in the solution space for generic spherically symmetric collapse.

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