Probing Neutral Majorana Fermion Edge Modes with Charge Transport

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We propose two experiments to probe the Majorana fermion edge states that occur at a junction between a superconductor and a magnet deposited on the surface of a topological insulator. Combining two Majorana fermions into a single Dirac fermion on a magnetic domain wall allows the neutral Majorana fermions to be probed with charge transport. We will discuss a novel interferometer for Majorana fermions, which probes their $Z_2$ phase. This setup also allows the transmission of neutral Majorana fermions through a point contact to be measured. We introduce a point contact formed by a superconducting junction and show that its transmission can be controlled by the phase difference across the junction. We discuss the feasibility of these experiments using the recently discovered topological insulator Bi$_2$Se$_3$.

PACS numbers: 71.10.Pm, 74.45.+c, 03.67.Lx, 74.90.+n

Majorana fermions have attracted interest in condensed matter physics because their exotic non-Abelian quantum statistics form the basis for topological quantum computation. Potential electronic systems hosting Majorana fermions include the $\nu = 5/2$ quantum Hall state, the $p$-wave superconductor Sr$_2$RuO$_4$, and topological insulator/superconductor structures. In the $\nu = 5/2$ quantum Hall state, a Majorana bound state is associated with the charge $e/4$ quasiparticle, and gapless chiral Majorana fermions form the neutral sector of the edge states. Thanks to the $e/4$ charge, the quasiparticle’s non-Abelian statistics can be probed by measuring charge transport of the edge states. Recent experiments have shown evidence for the quasiparticle charge $e/4$, and there are now intense efforts to prove or disprove their non-Abelian nature.

Detecting Majorana fermions in superconductors is more challenging because they are electrically neutral. In this work, we propose two experiments to probe neutral Majorana fermion edge states predicted in superconductor/magnet/topological insulator structures. Our basic setup, shown in Fig. 1, involves a grounded superconductor surrounded by two magnets with opposite out-of-plane magnetization, which are both deposited on the surface of a topological insulator. The magnetic domain wall gives rise to chiral Dirac fermions that play the role of “leads” connecting the superconductor to the source and drain. An electron incident from the source splits into two Majorana fermions which take different paths around the edge of the superconductor and then recombine before going to the drain. We will show the source to drain conductance probes the interference of the Majorana fermions, forming a novel “$Z_2$ interferometer”. In addition, we will show that the transmission of Majorana fermions through a “point contact” formed by a Josephson junction between two superconductors can be measured, and that the transmission can be tuned by controlling the phase difference across the junction.

A topological insulator has gapless surface states that are topologically protected in the absence of time reversal or gauge symmetry breaking fields. Breaking time reversal symmetry either by an applied magnetic field or by depositing a magnetic material can open an energy gap leading to a novel surface quantum Hall effect with $\sigma_{xy} = \pm e^2/2h$. Depositing a superconductor on the surface leads, via the proximity effect, to a surface superconducting state that hosts Majorana fermions. In view of the recent experimental discoveries of topological insulator phases in Bi$_x$Sb$_{1-x}$ and Bi$_2$Se$_3$, the experimental study of these novel gapped phases is now possible.

The superconducting and magnetic phases of the surface states, as well as the gapless states at interfaces between them, can be described with the Bogoliubov de Gennes (BdG) formalism. The Hamiltonian is $H = \Psi \dagger \mathcal{H} \Psi /2$, where $\Psi = (\psi_1, \psi_1, \psi_1^\dagger, -\psi_1^\dagger)^T$ and

$$\mathcal{H} = \tau^z \left[ v_F \hat{\mathbf{z}} \cdot \hat{\mathbf{\sigma}} \times (-i \nabla - e A) \right] \tau^+ + \Delta \tau^+ \tau^- + M \sigma_z.$$  (1)
Here $\psi$ and $\tilde{\psi}$ are electron operators of the surface states which are Kramers degenerate at $k = 0$. The first line in $\mathcal{H}$ describes the free surface states coupled to the vector potential $A$. $\bar{\sigma} = (\sigma^x, \sigma^y)$ are Pauli matrices, $v_F$ is the Fermi velocity and $\mu$ is the chemical potential. $\Delta \psi \bar{\psi}^\dagger + h.c.$ describes the superconducting proximity effect. Spatially uniform $\Delta$ gives a gapped excitation spectrum $E_k^\Delta = (\pm v|k| - \mu)^2 + \Delta^2$. $M \psi \bar{\sigma}_z \psi$ describes the Zeeman splitting due to the magnet. Spatially uniform $M$ gives $E_k^M = \sqrt{v^2|k|^2 + M^2} - \mu$, which is gapped when $M > \mu$. The BdG Hamiltonian has particle-hole symmetry, expressed by $\{\mathcal{H}, \mathcal{H}\} = 0$ where the particle-hole operator is $\Xi = \sigma^y \tau^y \xi^*$. The eigenstates $\xi_\pm$ with energy $\pm E$ obey $\xi_{-E} = \Xi \xi_E$, and only the $E > 0$ half of the spectrum represents independent excitations.

An interface between two half planes ($y > 0$ and $y < 0$) with different mass terms gives rise to gapless 1D domain-wall states. First consider a superconductor-magnet interface modeled by $\Delta = \Delta_0 \Theta(y)$ and $M = M_0 \Theta(-y)$. Solving (1), we find one chiral branch of bound states with a four component wavefunction $\xi_k(x,y)$ localized near $y = 0$, $\xi_{k=0}$ has zero energy and satisfies $\Xi \xi_0 = \xi_0$, which fixes its phase up to a $\pm$ sign. Using $k \cdot p$ theory the eigenstates for small $k$ are $\xi_k(x,y) = \exp(ikx) \xi_0(y)$ with energy $E(k) = hv_M k$, where $v_M = v_F \xi_0 |\tau_x \sigma_y \xi_0| = v_F \sqrt{1 - \mu^2/M^2_v}/(1 + \mu^2/\Delta_0^2)$. These define Bogoliubov operators $\gamma_k = \int dx dy \xi_k(x,y) \Psi(x,y)$ which satisfy $\gamma_k^\dagger = \gamma_{-k}$. The continuum operators $\gamma(x) = \int d \gamma_k e^{ikx}$ are Majorana fields, $\gamma^\dagger(x) = \gamma(x)$ obeying the low energy Hamiltonian $H = -i\hbar v_M \partial_y \gamma$. To model a magnetic domain wall we take $M = M_0 \text{sgn}(y)$. We find a gapless branch of chiral edge states between $\sigma_{xy} = \pm e^2/2h$. When expressed in the BdG formalism, two chiral branches of bound states with energy $E(k) \sim \hbar v_D k$ appear due to the double counting. For $E(k) > 0$, the two states have the form $f_k \otimes |\tau_z = 1\rangle$ and $\Theta f_{-k} \otimes |\tau_z = -1\rangle$. where $f_k(x,y)$ is a two component wavefunction in the $\sigma_z$ sector and $\Theta f = \sigma_y f^*$ is the time reversal operator. These correspond to the electron operators $c_{1,k}^\dagger$ and $c_{-k}$ respectively.

To analyze the device in Fig. 1, we employ the BCS mean field theory to calculate the transport current due to quasiparticles. This is justified because the superconducting order parameter at the surface inherits its phase from the bulk 3D superconductor, which behaves classically at low temperature. When the source is biased at a subgap voltage $V \ll \Delta_0$ the quasiparticles involved are exclusively the gapless Majorana fermion edge states.

An electron incident from the source can be transmitted to the drain as an electron, or converted to a hole by an Andreev process in which charge $2e$ is absorbed into the superconducting condensate. Before solving the general source to drain transmission problem we will show that the behavior at $E = 0$ follows from a simple argument. Scattering at the left tri-junction, where the incident Dirac fermion meets the superconductor, must transform an incident $E = 0$ electron $c_L^\dagger$ into a fermion $\psi$ built from the Majorana operators $\gamma_1$ and $\gamma_2$. The arbitrary sign of $\gamma_{1,2}$ allows us to choose $\psi = \gamma_1 + i\gamma_2$. Likewise, scattering at the right tri-junction transforms $\psi$ into a fermion in the right lead. This must be either $c_R^\dagger$ or $c_R$. A superposition of the two is not allowed because it is not a fermion operator. To determine which occurs, we observe that when the size of the superconductor shrinks continuously to zero, the left and right lead seamlessly connect to each other. Adiabatic continuity thus dictates that an incident $E = 0$ electron is transmitted as an electron, $c_L^\dagger \rightarrow c_R^\dagger$. When the ring encloses a quantized flux $\Phi = nh/2e$, this adiabatic argument breaks down. Instead odd $n$ introduces a branch cut for one of the Majorana modes, i.e. $\gamma_1 \rightarrow -\gamma_1$. Thus, when the ring encloses an odd number of flux quanta, $c_L^\dagger \rightarrow c_L$, and an incident $E = 0$ electron is converted to a hole.

To obtain the scattering probabilities at finite energy $0 < E \ll \Delta$, we use the BdG formalism to solve the scattering problem in the limit that the size of the ring $L$ is much larger than the decay length of the Majorana edge states into the bulk, which is of order max($h v_D/\Delta_0$, $h v_F/M_0$). First consider the scattering at the left tri-junction. A $2 \times 2$ scattering matrix $S(E)$ relates the two incoming states in the left lead $|\tau_z = \pm 1\rangle$, which we denote $e$ and $h$ (for electron and a hole), to the two outgoing Majorana edge states $\xi_1$ and $\xi_2$ on the top and bottom of the ring, $(\xi_1, \xi_2)^T = S(E)(e,h)^T$. To simplify the notation, we have used the channel label to denote the amplitude of the scattering states in the corresponding channel. Particle-hole symmetry implies that $S(E) = S^*(E)\tau_x$. At $E = 0$, this property, along with unitary $S^\dagger S = 1$, allows $S$ to be chosen as

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & i & -i & 0 \end{pmatrix} ,$$

so that $c_L^\dagger \rightarrow (\gamma_1 - i\gamma_2)/\sqrt{2}$. Another solvable limit is when the BdG Hamiltonian has a mirror symmetry $\mathcal{H}(-y) = M^{-1} \mathcal{H}(y) M$ with $M = i \sigma_y$. The electron and hole channels are eigenstates of $M$ with eigenvalue $\pm i$, whereas the two Majorana fermion edge states are interchanged. This leads to (2) at any energy. To obtain the exact scattering matrix at $E \neq 0$ for a tri-junction without mirror symmetry requires solving the 2D scattering problem. Here we assume that Eq. (2) is a good approximation of the scattering matrix at low energy.

Next we study the propagation of the chiral Majorana fermion. When there is no magnetic flux, in the semiclassical limit the wavefunction at energy $E \ll \Delta_0$ can be approximated by $\xi(l,s) = \xi_0(s) \exp(ik(E)l)$, where $l$ parameterizes the length along the interface, $s$ parameterizes the distance perpendicular to the interface and $k(E) = E/v_M$. In the presence of a magnetic flux $\Phi = nh/2e$, the superconducting phase $\phi$ winds by $2\pi n$ around the ring accompanied by a vector potential

\[ A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \Phi \]
$A = \nabla \phi$. It is convenient to choose a gauge in which the spatial variation of $\phi$ is concentrated near the middle of the upper semi-circle. Away from this “scattering region”, the wavefunction is a free Majorana edge mode as before. This scattering problem can be solved with a $U(2)$ gauge transformation that eliminates the spatial variation of $\phi$ and the nonzero $A$. The wavefunction of the exact scattering state is simply the undisturbed wavefunction multiplied by $\exp[i\tau_\phi(l)/2]$. We conclude that in the presence of a magnetic flux $\Phi = nh/2e$ the chiral Majorana edge mode $\gamma_1$ acquires an additional phase shift $n\pi$ across the junction.

The scattering amplitude of the ring is found by composing the scattering matrices:

$$
\begin{bmatrix}
\frac{e}{h} \\
\end{bmatrix}_R = S^{-1} \cdot \begin{bmatrix}
\frac{e^{-i\pi n+ikl}}{h} & \frac{0}{0} \\
\frac{e^{ikl}}{0} & \frac{e^{ikl}}{0}
\end{bmatrix} \cdot S \begin{bmatrix}
\frac{e}{h} \\
\end{bmatrix}_L. \tag{3}
$$

The current in the drain when the source is biased at voltage $V$ and the superconductor and drain are grounded is

$$
I = (-1)^n \frac{e}{h} \int_0^\infty dE \left[f(E-eV)-f(E+eV)\right] \cos \theta(E),
$$

where $f$ is the Fermi-Dirac distribution function and $\theta = k(l_1 - l_2) = ES\delta L/v_M$ is the relative phase between two paths of different lengths. Evaluating the integral we find

$$
I = (-1)^n \frac{e}{h} \frac{\pi k_BT \sin(eV\delta L/v_M)}{\sinh(\pi k_BT \delta L/v_M)}, \quad k_BT, eV \ll \Delta_0. \tag{5}
$$

At fixed bias, the current “oscillates” as a function of the discrete magnetic flux $nh/2e$, reflecting the Aharonov Bohm phase for Majorana fermions, which takes values $\pm 1$. Our device thus functions as a “$Z_2$ interferometer” for Majorana fermions. The “visibility” of these oscillations is suppressed below a temperature scale $k_BT\delta L \equiv h v_M/\delta L$ due to thermal averaging. In addition, at finite bias voltage the current oscillates as a function of $V$ with a period $2\pi k_BT\delta L/e$ due to the energy dependence of the relative phase. That the oscillation persists to high bias voltages without any damping is due to the absence of dephasing in our calculation. A similar situation occurs in the electronic Mach-Zehnder interferometer: the decay of the magnitude of interference oscillation at high bias voltage is attributed to dephasing processes $^2$. Sources of dephasing in our system include coupling of Majorana fermions with other degrees of freedom, as well as interactions between Majorana fermions. Since Majorana fermions are neutral, we expect environmental coupling to be weak. In addition, the lowest order local interaction term within the Majorana fermions is $\gamma(x)\partial_x \gamma(x)\partial_x \gamma(x)$, which involves spatial derivatives at sixth order and will be strongly suppressed at low temperature. Thus there is reason to expect the low temperature dephasing rate for the Majorana fermion edge states will be smaller than that of ordinary electrons.

We next study the transmission of Majorana fermions across a Josephson junction between two superconductors, shown in Fig. 2a. The junction plays the role of a point contact for Majorana fermions and can be characterized by a scattering matrix relating incoming and outgoing Majorana modes, $\gamma_i^{\text{out}} = S_{ij}^{\text{pc}}(E)\gamma_j^{\text{in}}$. Each superconductor is connected to a source and drain by chiral electron modes at magnetic domain walls. An incident electron from $S_1$ splits into two Majorana modes. One of the two is scattered by the junction, and has a probably amplitude $t = S_{11}^{\text{pc}}$ of being transmitted and recombining with its partner before going to $D_1$. Following the previous procedure, we calculate the scattering matrix relating an incident fermion at $S_1$ to an outgoing fermion at $D_1$ to obtain the current flowing to $D_1$ when $S_1$ is at voltage $V$ and the other leads are grounded.

$$
I = e \int_0^\infty dE[f(E-eV)-f(E+eV)] \text{Re}[i t(E)e^{i\theta(E)}], \tag{6}
$$

where $\theta(E)$ is the same as in (4). At $E = 0$ particle-hole symmetry constrains $S_{pc}$ to be a real $O(2)$ matrix describing the transmission $t = \cos \delta$ and reflection $r = \sin \delta$ such that $\gamma_i^{\text{out}} + i\gamma_i^{\text{out}} = e^{i\delta}(\gamma_i^{\text{in}} + i\gamma_i^{\text{in}})$. The zero bias, zero temperature conductance, $G = I_{D1}/V_{S1} = te^2/h$ directly measures the transmission of the neutral Majorana fermions at the junction.

The transmission amplitude $t$ can be controlled by adjusting the phase difference $\phi$ of the Josephson junction. $t(\phi)$ depends on the geometry of the junction. We consider a simple model,

$$
H = (\gamma_1, \gamma_2)[ -iv_M \partial_x \gamma_2 + \lambda(x) \cos(\phi/2) \tau_3] (\gamma_1, \gamma_2)^T, \tag{7}
$$

When $\lambda(x) = \Delta_0$ for $x \in [0, L]$, $H$ describes superconductors weakly coupled by single electron tunneling at a point $^3$, $^23$, $^24$. When $\lambda(x) = \Delta_0$ for $x \in [0, L]$ and 0 otherwise, $H$ becomes the low energy theory of a line junction $^4$. The transmission amplitude at $E = 0$ in this
model is
\[ t(\phi) = 1 / \cosh[\zeta \cos(\phi/2)], \]
with \( \zeta = \int dx \lambda(x)/2v_M \). Fig. 2b shows \( t(\phi) \) for different values of \( \zeta \). At \( \phi = \pi \), the transmission is perfect. This is guaranteed by gauge invariance. When \( \phi \to \phi + 2\pi \) one of the Majorana edge modes changes sign so \( r(\phi) = -r(\phi + 2\pi) \). Thus, \( r(\phi) = 0 \) and \( t(\phi) = 1 \) for some \( \phi \in [0,2\pi] \). For a symmetric junction this occurs at \( \phi = \pi \).

For a weakly coupled point contact (Fig. 2b, right inset), \( t(\phi) \) is energy-independent, but is only weakly dependent on \( \phi \). For a long line junction, (Fig. 2b, left inset) \( t(\phi) \) varies over a wide range of values between 0 and 1, but has a very narrow peak \( \delta t \sim h v_M/\Delta L \). In addition, near the peak the transmission will be strongly energy dependent due to the small gap when \( \phi \sim \pi \).

It is desirable to engineer the size and geometry of the Josephson junction in between these two limits, so that \( t(\phi) \) has a well defined peak which can be probed by the low temperature conductance.

It is worthwhile to compare the superconducting point contact for Majorana fermions studied here with a point contact in the \( \nu = 5/2 \) quantum Hall effect. Our point contact is precisely equivalent to the neutral sector of the \( \nu = 5/2 \) point contact, which has been described in terms of the Ising boundary conformal field theory [24]. For \( \nu = 5/2 \), however, the physics is dominated by the backscattering of charge \( e/4 \) quasiparticles, which is analogous to quantum tunneling vortices across the superconductor in our system. Since the superconducting phase is essentially a classical variable, this process is strongly suppressed in a superconducting point contact. Thus, unlike the \( \nu = 5/2 \) problem, vortex backscattering does not lead to a crossover to the weak tunneling limit.

The recently discovered topological insulator Bi$_2$Se$_3$ [20, 21], which has a large bulk gap \( \sim 50 \) meV is a promising material to probe these states. Unlike Bi$_{1-x}$Sb$_x$, its surface states have a small Fermi surface that encloses a single Dirac point. Photoemission experiments reveal a Fermi velocity \( h v_F \sim 3 \) eV nm and a Fermi energy \( \mu \sim 3 \) eV relative to the Dirac point. The current materials are unintentionally doped, with the bulk Fermi energy in the conduction band. If the material can be compensated either by doping or gating, it is likely that the surface Fermi energy can be made much closer to the Dirac point. This is important because achieving the magnetic gapped state requires a field \( M > \mu \). Moreover, the \( k \cdot p \) theory predicts that the Majorana velocity \( v_M \) is suppressed when \( \Delta_0 \ll \mu \), reducing the temperature scale \( T_{\Delta L} \) required to observe the signature of Majorana fermions. Our model calculation gives \( v_M \sim v_F (\Delta_0/\mu)^2 \). Assuming a superconductor can be found that gives a proximity induced gap \( \Delta_0 \sim 1 \) meV, we require size \( L > h v_F/\Delta_0 \sim 3 \) nm.

If \( \mu \sim 1 \) meV and \( \Delta L \sim 1 \) \( \mu \) then \( T_{\Delta L} \sim 30 \) nK. \( T_{\Delta L} \) can be larger if the path difference \( \Delta L \) can be finely tuned.

To conclude, we have proposed experiments to probe the interference and transmission of neutral Majorana fermions with charge transport. We hope they offer a first step towards the more ambitious goal of detecting the non-Abelian statistics of individual Majorana bound states and using them for quantum computation.

In a recent preprint, Akhmerov, et al., independently studied an interferometer similar to Fig. 1. We thank Carlo Beenakker for an insightful discussion. This work was supported by NSF grant DMR-0605066 and ACS PRF grant 44776-AC10.

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