Study on the dependence of the two-dimensional Ikeda model on the parameter

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ABSTRACT
Based on the property of solutions of the nonlinear differential equation, this paper focuses on the behavior of solutions to the two-dimensional Ikeda model, especially the dependence of the solutions on the parameter. The dependency relationship of the two-dimensional Ikeda model on the parameter is revealed by a large sample of proper numerical simulations. With the parameter varying from 0 to 1, the numerical solutions change from a point attractor to periodic solutions, then to chaos, and end up with a limit cycle. Furthermore, the route from bifurcation to chaos is shown to be continuous period-doubling bifurcations. The nonlinear structures presented by the solution of the two-dimensional Ikeda model indicate that, by setting different model parameters, one can test a new method that will be adopted to study atmospheric or oceanic predictability and/or stability. The corresponding test results provide some useful information on the ability of the new approach overcoming the impacts of strong nonlinearity.

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Introduction
To forecast a weather phenomenon or an ocean event, due to the effect of nonlinearity, the predictability of the phenomenon or event should be the first concern. A natural idea is to study practical dynamic models directly. However, considering the complexity of practical models, including high dimensionality and the complex interactions of the practical situation, this paper utilizes a simple idealized model, which can depict some important dynamic processes of the practical model, such as the nonlinear process to investigate the theory of, and methods for, predictability issues, and then apply the result to research on practical forecasting models (Mu and Duan 2003).

Actually, the Lorenz model has been used as an idealized model in forecast research, not only for its simplicity and strong nonlinearity, but also for its comprehensive analysis of the property and stability of the solution. Some results from predictability studies based on the Lorenz model have been applied to help predict practical situations (Ding and Li 2012; Zheng et al. 2012).

The Ikeda model was originally proposed by Kensuke Ikeda as a model of light going across a nonlinear optical resonator (ring cavity containing a nonlinear dielectric medium), and the two-dimensional Ikeda difference scheme is its most common form. As a nonlinear dynamic system model, the Ikeda model is less popular compared with the Lorenz model, despite being stronger.

Owing to the interactions of nonlinearity, characters including attractors, saddle points, and chaos usually appear in different regions of nonlinear dynamic systems. An attractor, which exists in the phase plane, is an important concept in dynamics, used to describe the convergent features of movement. In short, an attractor is a set to which every point or orbit nearby converges (Xi and Xi 2007). An attractor can be in different forms, such as a fixed point, a limit cycle, an integer-dimension torus, and a singular one. A saddle point is a singular point that is stable along a certain direction, but unstable along another (Fu and Fan 2011). The chaotic phenomenon, referring to a certain but unpredictable movement, is currently a key
issue in nonlinear dynamics. It shows randomness, but is
definitely a stochastic-like process from a deterministic sys-
tem (Niu 2007). Chaos is commonly bifurcated by the fixed
point, such as period-doubling bifurcation, fits-bifurcation,
hysteresis, and elusive salutation and bifurcation controlled
by double parameters. It is chaos that allowed Lorenz to
propose the predictability issue for atmospheric motion.

Stemler and Judd (2009) applied the shadowing filter
method to state estimation, preliminary data assimilation,
and predictability research in the two-dimensional Ikeda
model. In their numerical experiment, the parameter of the
Ikeda model was set to a fixed value, and relevant analysis
only focused on limited initial values. The result of the num-
erical experiment in Stemler and Judd (2009) indicates that
the system develops to chaos just right with the parameter
they selected, so the applicability of their method for other
situations needs further exploration. Chen, Chen, and Ma
(1990) analyzed the Ikeda instability of optical bistability
and derived the Ikeda equation in another way, revealing
the physical origin of the instability of optical bistability.
However, they focused only on the physical origin of the
Ikeda model, without further discussion of the solution of
the model. Xie and Zhang (2007) investigated the adaptive
synchronization of the Ikeda system and demonstrated the
effectiveness of this method, without analyzing the Ikeda
model. Chui et al. (2004) used Ikeda time series with variable
parameters to verify the effectiveness of the support vector
machine model they established in their article and noticed
that the solutions of the two-dimensional Ikeda model
change with the variation of the parameter. However, their
aim was to predict with the model instead of discussing the
dependence of the model on the parameter.

Many scholars have worked on the two-dimensional
Ikeda model since it was proposed (Chen, Chen, and Ma
1990; Chui et al. 2004; Liu 2009; Stemler and Judd 2009; Xie
and Zhang 2007), but in most cases they used the model
to study the issue of concern, without comprehensive
and detailed discussion about the behavior of the model
solution. Usually, one only uses the Ikeda model with a
specific model parameter and initial condition to test the
effectiveness of a new method in terms of the estimation
of the model state, predictability and data assimilation
(Chui et al. 2004; Stemler and Judd 2009; Xie and Zhang
2007). The present study attempts to comprehensively
investigate the behavior of the Ikeda model solution by
using the theory of nonlinear dynamic systems.

The two-dimensional Ikeda model has stronger non-
linearity than the Lorenz model. Therefore, it can be an
alternative to the Lorenz model when testing the ability
of a new approach for overcoming the impacts of strong
nonlinearity. In order to use this model more effectively,
it is necessary to know the characteristics of the nonlin-
ear interactions, such as attractors, saddle points, and the
chaos demonstrated by the model solution. This is the
objective of the present study. Based on the behavior and
stability of the nonlinear differential equation solutions,
this study aims to investigate the two-dimensional Ikeda
model solution’s behavior, and reveal the dependency of
the solution on the model parameter by numerical experi-
ments.

The two-dimensional Ikeda model
The Ikeda model is a series of delay differential equations,
proposed by Kensuke Ikeda when he explored the non-
linear dielectric medium. Deduced through the Maxwell–
Bloch equations (Ikeda 1979), it can be simplified as
difference equations, as shown below:

$$ z_{t+1} = A + Bz_t e^{i(z_t^2 + C)} . $$

Here, $i$ is the imaginary unit; $t$ represents the time step; $z$
stands for the electric field inside the ring cavity; and $A$, $B$, $C$
are physical parameters. $A$ is related to the intensity of the
incident light and indicates laser light applied from the out-
side; $B$ is called the dissipation parameter, characterizing the
dissipation of the electric field; and $C$ indicates the laser light
applied from the outside. A two-dimensional case of Equation
(1) is the common two-dimensional Ikeda model:

$$ \begin{aligned}
  x_{t+1} &= 1 + \mu (x_t \cos \theta_t - y_t \sin \theta_t) \\
  y_{t+1} &= \mu (x_t \sin \theta_t + y_t \cos \theta_t) 
\end{aligned} $$

In Equation (2), the value range of the parameter $\mu$ is
[0, 1], and

$$ \theta_t = a - \frac{b}{1 + x_t^2 + y_t^2}. $$

The values of $a$ and $b$ are usually 0.4 and 6, respectively.
From the expression of the model we find that the trigono-
metric functions appear in Equation (2), and Equation
(3) is a fraction whose denominator includes two quadratic
components. Clearly, the two-dimensional Ikeda model
has fairly strong nonlinearity, which means the solutions of
the model will experience a fair amount of movement with
the variation of the parameter.

Behavior of the solutions and the dependence
on the parameters of the model
For the two-dimensional Ikeda model, Equation (2), it
seems it is difficult to obtain the corresponding analytical
solutions directly. Therefore, through a large sample of
numerical experiments, this paper attempts to achieve a
comprehensive and more accurate understanding of the
To explore the influence of different parameter values on the behavior of the model solutions, the internal $[0, 1]$ is

**Figure 1.** Solutions of the last 5000 steps when the model parameter $\mu = 0.402$ and 0.403.

**Figure 2.** Solutions of the last 5000 steps when $\mu = 0.500$, 0.600, and 0.640.

**Figure 3.** Solutions of the last 5000 steps when $\mu = 0.643$, 0.644, and 0.650.
equally divided into 1000 segments and the value of every 1/1000 equant point is assigned to the model parameter $\mu$. The initial value chosen in this paper is $(x_0, y_0) = (0.25, -0.325)$. For every value of the parameter, the model is integrated 10,000 steps along with time. This paper chooses and analyzes several representative situations, the results of which are shown in Figures 1–6. A more comprehensive statistical result of the numerical tests is given in Table 1. Since the observation of attractors may be affected by the irregular trajectory of points before the model reaches a plateau, in this paper the first five figures only show results of the last 5000 steps, when the model becomes stable.

Figure 1 shows the result of the last 5000 steps when $\mu = 0.402$ and 0.403. The abscissa and the
experiences a bifurcation from two convergence points to four, which means the cycle of the periodic solution changes from two to four. Analogously, the cycle of the solution turns into eight from four as μ varies from 0.600 to 0.640.

From Figure 3 it can be clearly seen that the cycle of the solutions changes from eight to sixteen as the value of μ changes from 0.643 to 0.645. When μ continues to increase to 0.650, the system will lose its periodic solution.

As shown in Figure 4, when μ = 0.900, obvious chaos appears in the system; and when μ = 1.000, the system presents a limit cycle. Results also show that the system gradually develops towards chaos when the value of μ varies from 0.646 to 0.700. As the value of μ varies from 0.700 to 0.902, the system has apparent chaos. Elusive saltation

ordinate represent two components of the state in the model (the same in subsequent figures). The result shows that when the value of μ is set between 0 and 0.402, the model solutions will evolve to one point after a certain period. That is to say, the system has a point attractor on this occasion. The number of the convergence point increases when the value of μ is equal to or greater than 0.403. In Figure 1, when μ varies from 0.402 to 0.403, the system changes from a point attractor to a periodic solution with a cycle of 2. Clearly, a bifurcation exists between 0.402 and 0.403, and the critical value of the parameter, namely the bifurcation value, is between the two values of μ.

Comparing the three sub-graphs of Figure 2, along with the variation of μ from 0.500 to 0.600 the model experiences a bifurcation from two convergence points to four, which means the cycle of the periodic solution changes from two to four. Analogously, the cycle of the solution turns into eight from four as μ varies from 0.600 to 0.640.

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Table 1. The relationship between solutions and the parameter.

| Value of μ | Behavior of the solution                        |
|-----------|-------------------------------------------------|
| 0–0.402   | Point attractor                                 |
| 0.403–0.591| Periodic solution (2)                           |
| 0.592–0.634| Periodic solution (4)                           |
| 0.635–0.643| Periodic solution (8)                           |
| 0.644–0.645| Periodic solution (16)                          |
| 0.646–0.649| Chaos (Elusive saltation)                       |
| 0.650–0.802| Chaos at first; then break away; finally, a new singular attractor |
| 0.803–0.808, 0.850, 0.854| Chaos at first; then break away; finally, a new singular attractor |
| 0.903–0.999| Chaos                                           |
| 1         | Limit cycle                                     |
in chaotic cases occurs when the value of $\mu$ varies from $0.803$ to $0.808$, $\mu = 0.850$ and $0.854$, as shown in Figure 5. Meanwhile, with variation of $\mu$ from $0.903$ to $0.999$, the system presents chaos at first, and then breaks away from chaos until a singular attractor appears in a new region (shown in Figure 6).

Combining the numerical test results with Figures 1–6, and consulting the theory of the behavior and the stability of the nonlinear differential equation solution, we obtain the relationship between the behavior of the two-dimensional Ikeda model solutions and the model parameter $\mu$, as shown in Table 1.

**Conclusion**

From the numerical experiment described in this paper we find that the solutions of the model change with the variation of the parameter. Moreover, small parametric variation may result in great differences of solutions in certain ranges. The conclusion above reflects the high dependence of the two-dimensional Ikeda model on the parameter. Specifically, when the parameter varies from 0 to 1, the numerical solutions change from a point attractor to periodic solutions, then to chaos, and finally become a limit cycle. According to the bifurcation theory of the behavior and stability of the nonlinear differential equation solutions, we conclude that the varying process of the solutions of the two-dimensional Ikeda model along with the variation of the parameter is actually a course from bifurcation to chaos through continuous period-doubling bifurcations. With the increase of the model parameter $\mu$, the numerical solutions of this model possess a range of nonlinear behavior, which perfectly corresponds to analyses of nonlinear theory. Its nonlinear characteristics presented by the movement from bifurcation to chaos make the model an alternative to the Lorenz model. In view of the results of this study, one can choose a special model parameter to verify the effectiveness of an estimation method for different characteristic solutions of the two-dimensional Ikeda model. Since the Ikeda model has stronger nonlinearity than the Lorenz model, it is more significant to study predictability problems based on the Ikeda model. Also, we can use the Ikeda model to explore the influence of model parameter error on data assimilation according to the critical parameter values obtained in this paper. In addition, both the accuracy of the bifurcation values and the transition of the model from chaos to a limit cycle deserve further research in the future.

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