HAMILTON PATHS AND CYCLES IN VERTEX-TRANSITIVE GRAPHS
OF ORDER 6p

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Abstract

It is shown that every connected vertex-transitive graph of order 6p, where \( p \) is a prime, contains a Hamilton path. Moreover, it is shown that, except for the truncation of the Petersen graph, every connected vertex-transitive graph of order 6p which is not genuinely imprimitive contains a Hamilton cycle.

Keywords: graph, vertex-transitive, Hamilton cycle, Hamilton path, automorphism group.

1 Introductory remarks

This paper deals with the existence of Hamilton paths and Hamilton cycles in connected vertex-transitive graphs of order 6p, where \( p \) is a prime. (Throughout this paper \( p \) will always denote a prime number.) The question whether every connected vertex-transitive graph contains a Hamilton path was posed by Lovász in 1969 (see [24]). So far no example giving a negative answer to this question has been found. Moreover, apart from the trivial example \( K_2 \), there are only four known connected vertex-transitive graphs, which do not contain a Hamilton cycle. These are the Petersen graph, the Coxeter graph and the truncations of these two graphs, that is the graphs obtained from them by replacing each vertex by a triangle. This supports the conjecture of Thomassen [7, 34] that only finitely many connected vertex-transitive graphs without a Hamilton cycle exist. On the other hand, Babai [5, 6] conjectured that infinitely many such graphs exist.

Despite the fact that these questions have been challenging mathematicians for almost forty years, only partial results have been obtained thus far. For instance, it is known that connected vertex-transitive graphs of orders \( kp \), where \( k \leq 5 \), \( p^j \), where \( j \leq 4 \), and \( 2p^2 \) contain a Hamilton path. Furthermore, for all of these families, except for the graphs of order 5p, it is also known that they contain a Hamilton cycle (except for the above mentioned Petersen and Coxeter graph), see [1, 10, 23, 27, 28, 29, 31, 32, 35]. The problem has also been considered for the subclass of Cayley graphs, resulting in a number of partial results (see for example [1, 12, 19, 22, 26, 37, 38]). Also, it is known that every connected vertex-transitive graph, other than the Petersen graph, whose automorphism group contains a transitive subgroup with a cyclic commutator subgroup of prime-power order, has a Hamilton cycle. The result was proved in [15] and it uses a results

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from a series of papers dealing with the same group-theoretic restrictions in the context of Cayley graphs [17, 26, 37].

The main object of this paper is to show that every connected vertex-transitive graph of order $6p$ contains a Hamilton path. This result represents a new building block of the project to show that all connected vertex-transitive graphs on up to 100 vertices have this property.

**Theorem 1.1** Every connected vertex-transitive graph of order $6p$, where $p$ is a prime, contains a Hamilton path. Moreover, with the exception of the truncation of the Petersen graph, every such graph which is not genuinely imprimitive contains a Hamilton cycle.

The paper is organized as follows. In Section 2 notions concerning this paper are introduced together with the notation and some auxiliary results that are needed in the subsequent sections. The rest of the paper is devoted to proving Theorem 1.1. As a vertex-transitive graph is either genuinely imprimitive, quasiprimitive or primitive, we divide our investigation depending on which of these three families the graph in question belongs to. The genuinely imprimitive graphs are considered in Section 3. The investigation of these graphs depends on the size of the corresponding blocks. As for the quasiprimitive and primitive graphs of order $6p$, they are known (see [18, 33]). Therefore, the existence of Hamilton paths (or cycles) in these graphs can (at least in general) be verified. This is done in Sections 4 and 5. Finally, the results are combined in Section 6, where the Theorem 1.1 is proved.

## 2 Notation and preliminary results

Throughout this paper graphs are finite, simple and undirected, and groups are finite, unless specified otherwise. Furthermore, a multigraph is a generalization of a graph in which we allow multiedges and loops. Given a graph $X$ we let $V(X)$ and $E(X)$ be the vertex set and the edge set of $X$, respectively. For adjacent vertices $u, v \in V(X)$ we write $u \sim v$ and denote the corresponding edge by $uv$. Let $U$ and $W$ be disjoint subsets of $V(X)$. The subgraph of $X$ induced by $U$ will be denoted by $X[U]$. Similarly, we let $[U, W]$ denote the bipartite subgraph of $X$ induced by the edges having one endvertex in $U$ and the other endvertex in $W$.

Given a transitive group $G$ acting on a set $V$, we say that a partition $B$ of $V$ is $G$-invariant if the elements of $G$ permute the parts, that is, blocks of $B$, setwise. If the trivial partitions $\{V\}$ and $\{\{v\} : v \in V\}$ are the only $G$-invariant partitions of $V$, then $G$ is said to be primitive, and is said to be imprimitive otherwise. In the latter case we shall refer to a corresponding $G$-invariant partition as to a complete imprimitivity block system, in short an imprimitivity block system, of $G$.

A graph $X$ is said to be vertex-transitive if its automorphism group, denoted by $\text{Aut}X$, acts transitively on $V(X)$. A vertex-transitive graph for which each transitive subgroup of its automorphism group is primitive is called a primitive graph. Otherwise it is called an imprimitive graph. If $X$ is imprimitive with an imprimitivity block system which is formed by the orbits of a normal subgroup of some transitive subgroup $G \leq \text{Aut}X$, then the graph $X$ is said to be genuinely imprimitive. If $X$ is imprimitive, but there exists no transitive subgroup $G$ of the automorphism group of $X$ having a nontransitive normal subgroup, then $X$ is said to be quasiprimitive. Note that if $\mathcal{B}$ is an imprimitivity block system of some vertex-transitive graph, then any two blocks $B, B' \in \mathcal{B}$ induce isomorphic vertex-transitive subgraphs.

The following simple observation about imprimitive groups of certain degrees will be useful latter on.
**Lemma 2.1** Let $G$ be an imprimitive permutation group of degree $mq$, $q$ a prime, with a complete imprimitivity block system $\mathcal{B}$ and let $H \leq G$ have $m$ orbits of length $q$. Let $S$ be an orbit of $H$ and let $B \in \mathcal{B}$ be such that $B \cap S \neq \emptyset$. Then one of the following holds:

(i) $|B \cap S| = 1$, in which case $|B \cap S'| = 1$ for every orbit $S'$ of $H$ which meets $B$, or

(ii) $B \cap S = S$, in which case $q$ divides $|B|$.

**Proof.** Let us first show that $|B \cap S|$ equals either to 1 or to $q$. Suppose there exist distinct points $u, v \in B \cap S$. As $S$ is of prime length $q$, there exists some $\varphi \in H$, mapping $u$ to $v$, such that the restriction of $\varphi$ to $S$, denoted by $\varphi|_S$, is of order $q$. Then the orbit of $\varphi$ containing $u$ coincides with $S$. As $u \varphi = v$ and $u, v \in B$, the block $B$ is fixed by $\varphi$. Consequently, $S \subseteq B$.

Suppose now that $B \cap S = \{u\}$ but $B \cap S' = S'$ for some orbits $S$ and $S'$ of $H$. In view of $B \cap S \neq S$, some element of $H$ moves the block $B$ to some other block. On the other hand (as $B \cap S' = S'$), every element of $H$ fixes $B$ setwise. This contradiction proves (i). That $q$ divides $|B|$ when $B \cap S = S$ is now clear.

Given a graph $X$ and a partition $\mathcal{P}$ of its vertex set we let the *quotient graph corresponding to $\mathcal{P}$* be the graph $X_\mathcal{P}$ whose vertex set equals $\mathcal{P}$ with $A, B \in \mathcal{P}$ adjacent if there exist vertices $a \in A$ and $b \in B$, such that $a \sim b$ in $X$.

Let $m \geq 1$ and $n \geq 2$ be integers. An automorphism of a graph is called $(m,n)$-semiregular if it has $m$ orbits of length $n$ and no other orbit. Let now $X$ be a graph admitting an $(m,n)$-semiregular automorphism $\rho$ and denote the set of the orbits of $\rho$ by $\mathcal{S}$. Let $S, S' \in \mathcal{S}$. Clearly, the graph $[S, S']$ is regular. We let $d(S, S')$ denote the valency of $[S, S']$. We let the quotient multigraph corresponding to $\rho$ be the multigraph $X_\rho$ whose vertex set is $\mathcal{S}$ and in which $S, S' \in \mathcal{S}$ are joined by $d(S, S')$ edges. Observe that $\mathcal{S}$ is a partition of $V(X)$, so we can also consider the quotient graph $X_\mathcal{S}$ which is precisely the underlying graph of $X_\rho$.

**Remark.** Note that if $G$ is as in Lemma 2.1 and $\varphi \in G$ is $(m, q)$-semiregular, then the subgroup $\langle \varphi \rangle$ has $m$ orbits of length $q$, and so Lemma 2.1 applies.

For the sake of completeness we state the following classical result which will be used throughout the paper.

**Proposition 2.2** [25, Theorem 3.4] Let $p$ be a prime and let $P$ be a Sylow $p$-subgroup of a permutation group $G$ acting on a set $\Omega$. Let $\omega \in \Omega$. If $p^n$ divides the length of the $G$-orbit containing $\omega$, then $p^n$ also divides the length of the $P$-orbit containing $\omega$.

The following proposition is a generalization of [25, Theorem 3.4].

**Proposition 2.3** Let $X$ be a vertex-transitive graph of order $mp$, where $m < p$, $p$ a prime, and let $G \leq \text{Aut}X$ be a transitive subgroup of automorphisms of $X$. Then there exists some $(m, p)$-semiregular automorphism $\rho$ of $X$, such that $\rho \in G$.

**Proof.** Since $G$ is transitive on $V(X)$ and $X$ is of order $mp$, the order $|G|$ of $G$ is divisible by $p$. Let $P$ be a Sylow $p$-subgroup of $G$. Since the length $l$ of an orbit of $P$ divides its order $|P|$, it can either be 1 or $p$ (recall that $m < p$). By Proposition 2.2, $p$ divides $l$ and thus $l = p$. Therefore $P$ has exactly $m$ orbits of length $p$. Following the proof of [25, Theorem 3.4] one can now show that there exists some $\rho \in P$ such that $\rho$ is $(m, p)$-semiregular.

The following lemma can be deduced from [14, Lemma 2].
Lemma 2.4 Let \( X \) be a vertex-transitive graph of order \( mq \), where \( q \) is a prime, let \( G \) be an imprimitive subgroup of automorphisms of \( X \) and let \( N \) be a normal subgroup of \( G \) with orbits of length \( q \). Then \( X \) has an \((m,q)\)-semiregular automorphism whose orbits coincide with the orbits of \( N \).

We now introduce the following notion of a lift of a path in a graph with a semiregular automorphism. Let \( X \) be a graph that admits an \((m,n)\)-semiregular automorphism \( \rho \). Let \( S = \{S_1, S_2, \ldots, S_m\} \) be the set of orbits of \( \rho \), let \( X_S \) be the corresponding quotient graph and let \( \varphi : X \rightarrow X_S \) be the corresponding projection. Let \( W = S_{i_1}S_{i_2} \ldots S_{i_k} \) be a path in \( X_S \). We let the lift of the path \( W \) be the set of all paths of \( X \) whose images under \( \varphi \) are \( W \). The following lemma is straightforward and is just a reformulation of [31, Lemma 5].

Lemma 2.5 Let \( X \) be a graph admitting an \((m,p)\)-semiregular automorphism \( \rho \), where \( p \) is a prime. Let \( C \) be a cycle of length \( k \) in the quotient graph \( X_S \), where \( S \) is the set of orbits of \( \rho \). Then, the lift of \( C \) either contains a cycle of length \( kp \) or it consists of \( p \) disjoint \( k \)-cycles. In the latter case we have \( d(S, S') = 1 \) for every edge \( SS' \) of \( C \).

A path of \( X \) which meets each of the vertices of \( X \) is called a Hamilton path of \( X \). A Hamilton cycle is defined in a similar way. The following classical result, due to Jackson [21], giving a sufficient condition for the existence of Hamilton cycles in 2-connected regular graphs will be used throughout this paper (Note that every connected vertex-transitive graph is 2-connected).

Proposition 2.6 [21, Theorem 6] Every 2-connected regular graph of order \( n \) and valency at least \( n/3 \) contains a Hamilton cycle.

The next result may be extracted from [16, Theorem 2.10].

Theorem 2.7 Let \( G \) be a transitive permutation group of degree \( 6p \), \( p \geq 5 \) a prime, with an imprimitivity block system \( B \) formed by a (proper, intransitive) minimal normal subgroup \( N \) of \( G \). Then \( N^B \) is simple for all blocks \( B \in B \).

We let \( \mathbb{Z}_n = \{0, 1, \ldots, n - 1\} \) denote the ring of integers modulo \( n \), and we let \( \mathbb{Z}_n^* \) be the multiplicative group of the units of \( \mathbb{Z}_n \).

In the subsequent sections some of the graphs will be represented in the Frucht’s notation [20]. For the sake of completeness we include the definition. Let \( X \) be a connected vertex-transitive graph of order \( mn \) admitting an \((m,n)\)-semiregular automorphism \( \rho \). Let \( S = \{S_i \mid i \in \mathbb{Z}_m\} \) be the set of orbits of \( \rho \). Denote the vertices of \( X \) by \( v_i^j \), where \( i \in \mathbb{Z}_m \) and \( j \in \mathbb{Z}_n \), in such a way that \( S_i = \{v_i^j \mid j \in \mathbb{Z}_n\} \) with \( v_i^j = v_i^j \rho^j \). Then \( X \) may be represented by the notation of Frucht [20] emphasizing the \( m \) orbits of \( \rho \) in the following way. The \( m \) orbits of \( \rho \) are represented by \( m \) circles. The symbol \( n/R \), where \( R \subseteq \mathbb{Z}_n \setminus \{0\} \), inside a circle corresponding to the orbit \( S_i \) indicates that for each \( j \in \mathbb{Z}_n \), the vertex \( v_i^j \) is adjacent to all the vertices \( v_i^{j+r} \), where \( r \in R \). When \( X(S_1) \) is an independent set of vertices we simply write \( n \) inside its circle. Finally, an arrow pointing from the circle representing the orbit \( S_i \) to the circle representing the orbit \( S_k \), \( k \neq i \), labeled by the set \( T \subseteq \mathbb{Z}_n \) indicates that for each \( j \in \mathbb{Z}_n \), the vertex \( v_i^j \in S_i \) is adjacent to all the vertices \( v_k^{j+t} \), where \( t \in T \). An example illustrating this notation is given in Figure 1.
3 Genuinely imprimitive graphs

Throughout this section let $X$ be a connected genuinely imprimitive graph of order $6p$, $p > 3$ a prime, admitting an imprimitive subgroup $G$ of Aut$X$ with a nontransitive minimal normal subgroup $N \triangleleft G$. Let the set of orbits of $N$ (and thus blocks for $G$) be denoted by $B$.

The task of showing that $X$ has a Hamilton path is divided into six different cases depending on the size of the blocks in $B$. Each of them is covered by a separate lemma (see Lemmas 3.1, 3.3, 3.4, 3.5, 3.6 and 3.7). If the size of blocks equals to $p$ or $6$ we in fact show that $X$ contains a Hamilton cycle.

Lemma 3.1 If the size of blocks in $B$ is 2 then $X$ has a Hamilton path.

Proof. Since $X_B$ is a connected vertex-transitive graph of order $3p$ it has a Hamilton cycle $C$. By Lemma 2.4 $X$ has a $(3p, 2)$-semiregular automorphism whose set of orbits equals $B$. Thus, by Lemma 2.5 the lift of $C$’s either contains a Hamilton cycle of $X$ or it contains a disjoint union of two cycles of length $3p$. Since $X$ is connected a Hamilton path exists in $X$. □

The following auxiliary lemma will be used in the proof of Lemma 3.3.

Lemma 3.2 If the size of blocks in $B$ is 3 and the quotient graph $X_B$ is isomorphic to the Petersen graph then $X$ has a Hamilton path.

Proof. Note that in this case $p = 5$. By Lemma 2.4 there exists a $(10, 3)$-semiregular automorphism $\varphi$ of $X$ whose orbit set equals $B$. Suppose there exist two disjoint 5-cycles in $X_B$ whose lifts both contain a 15-cycle. Then the connectedness of $X$ implies that $X$ has a Hamilton path. We can thus assume that no two such 5-cycles exist in $X_B$. We claim that this implies that for any two adjacent orbits $B, B' \in B$ of $\varphi$ we have $d(B, B') = 1$. Suppose this is not the case. It is easy to see that we then have two disjoint 5-cycles in $X_B$ such that each of them contains an edge corresponding to a multiedge in $X_\varphi$. But then Lemma 2.5 implies that the lifts of both of these two 5-cycles contain 15-cycles, a contradiction.

Note that in the case when $X(\langle B \rangle)$ is not an independent set for some (and thus all) $B \in B$ a Hamilton path exists in $X$. We can thus assume that $X(\langle B \rangle)$ is an independent set for all $B \in B$. Let $G$ denote the permutation group corresponding to the natural action of $G$ on $X_B$. Since the only transitive subgroups of the automorphism group of the Petersen graph are $S_5$, $A_5$ and $AGL(1, 5)$, the fact that $G$ is transitive implies, that a subgroup $H$ of $G$, which is isomorphic to $AGL(1, 5)$ or to $A_5$, exists. As we demonstrate below, each of these two cases lead to a contradiction, which shows that $X$ has a Hamilton path, as required.

Suppose first that $H \cong AGL(1, 5)$. Then there exist two disjoint 5-cycles of $X_B$ interchanged by some element of $H$. The lift of each of them is thus a union of 3 disjoint 5-cycles. Hence, we can assume that the Frucht’s notation of $X$ is as in Figure 1. In view of our assumptions we have

$$a = c \text{ or } d = e, \quad b = d \text{ or } a = e, \quad c = e \text{ or } a = b, \quad a = d \text{ or } b = c, \quad b = e \text{ or } c = d.$$ 

As $X$ is connected, we cannot have $a = b = c = d = e$. With no loss of generality assume that $a \neq b$, and so $c = e$. Suppose first that $a = d$. Then $b \neq d$, and so $d = a = e = c$. The reader may check that then the vertices of $B_1$ are contained on precisely two 5-cycles, whereas the vertices of $B_0$ are contained on precisely four 5-cycles which is impossible in view of vertex-transitivity of $X$. Suppose then that $a \neq d$. Therefore, $b = c$ and thus also $d = e = c = b$. As above a contradiction to vertex-transitivity of $X$ is obtained.
Suppose now that \( H \cong A_5 \). We can assume that the Frucht’s notation of \( X \) is as in Figure 2 where the group \( H \) acts on \( X_\mathcal{B} \) in the obvious way. In view of the action of an automorphism of \( H \) whose action on \( X_\mathcal{B} \) corresponds to \((23)(45)\), we have \( e = 0 \). Furthermore, the element of \( H \) corresponding to \((12)(45)\) forces \( d = 0 \). Continuing in this way we find that \( c - f = 0 \), \( b - e = 0 \) and \( b + f = 0 \), which forces \( b = c = f = 0 \). However, this contradicts the connectedness of \( X \) and the proof is completed.

An \( n \)-bicirculant is a graph with a \((2, n)\)-semiregular automorphism. Every \( n \)-bicirculant \( X \) can be represented by a triple of subsets of \( \mathbb{Z}_n \) in the following way. Let \( \varphi \) be a \((2, n)\)-semiregular automorphism of \( X \), let \( U \) and \( W \) be the two orbits of \( \varphi \), and let \( u \in U \) and \( w \in W \). Let \( S = \{ s \in \mathbb{Z}_n \mid u \sim w^s \} \) be the symbol of the \( n \)-circuitual induced on \( U \) and let \( R \) be the symbol of the \( n \)-circuitual induced on \( W \) (relative to \( \varphi \)). Moreover, let \( T = \{ t \in \mathbb{Z}_n \mid u \sim w^t \} \). The ordered triple \([S, R, T]\) is the symbol of \( X \) relative to \((\varphi, u, w)\). Note that \( S = -S \) and \( R = -R \) are symmetric, that is, inverse-closed subsets of \( \mathbb{Z}_n \), and are independent of the particular choice of vertices \( u \) and \( w \).

In the rest of this section the well known wreath and Cartesian products of graphs will be encountered. To fix the notation, we include the definitions. For two graphs \( X \) and \( Y \) let \( X \wr Y \) denote the \emph{wreath product} of \( X \) by \( Y \), that is, the graph with vertex set \( V(X) \times V(Y) \) with two vertices \((a, u)\) and \((b, v)\) adjacent in \( X \wr Y \) if and only if either \( ab \in E(X) \) or \( a = b \) and \( uv \in E(Y) \). Note that the wreath product is sometimes referred to as the lexicographic product. The \emph{Cartesian product} \( X \square Y \) of graphs \( X \) and \( Y \) is the graph with vertex set \( V(X) \times V(Y) \), where two vertices \((a, u)\) and \((b, v)\) are adjacent in \( X \square Y \) if and only if either \( ab \in E(X) \) and \( u = v \), or \( a = b \) and \( uv \in E(Y) \).

**Lemma 3.3** If the size of blocks in \( \mathcal{B} \) is 3 then \( X \) has a Hamilton path.

**Proof.** By Lemma 2.4 there exists a \((2p, 3)\)-semiregular automorphism \( \varphi \) of \( X \) whose orbit set coincides with \( \mathcal{B} \). If the quotient graph \( X_\mathcal{B} \) is isomorphic to the Petersen graph, then Lemma 3.2 applies. We can thus assume that \( X_\mathcal{B} \) is not isomorphic to the Petersen graph. Therefore, \( X_\mathcal{B} \) has a Hamilton cycle \( C = B_0 B_1 \ldots B_{2p-1} B_0 \). In view of Lemma 2.5 we can assume that the lift of \( C \) consists of three disjoint \( 2p \)-cycles. So \( d(B_i, B_{i+1}) = 1 \) for all \( i \in \mathbb{Z}_{2p} \). Therefore, we can label the
vertices of $X$ by \( \{ u_i^j \mid i \in \mathbb{Z}_{2p}, j \in \mathbb{Z}_3 \} \) in such a way that $B_i = \{ u_i^j \mid j \in \mathbb{Z}_3 \}$ and that $u_i^j u_i^{j+1}$ is an edge of $X$ for every $i \in \mathbb{Z}_{2p}$ and $j \in \mathbb{Z}_3$. Moreover, we can assume that $X \langle B \rangle = 3K_1$ for all $B \in \mathcal{B}$ (otherwise $X$ contains a subgraph isomorphic to the Cartesian product $C_{2p} \square K_3$ which clearly has a Hamilton cycle).

There exists some $\psi \in N$ such that $\psi|_{B_0} = (u_0^0 u_0^1 u_0^2)$. By the above assumptions it is clear that $\psi|_{B} = (u_i^0 u_i^1 u_i^2)$ for all $i \in \mathbb{Z}_{2p}$. Therefore, we can assume that the automorphism $\varphi$ is in $N$. Note also that $N$ acts faithfully on each of its orbits $B \in \mathcal{B}$ and thus either $N \cong \mathbb{Z}_3$ or $N \cong S_3$. However, the latter case cannot occur, for then the Sylow 3-subgroup of $N$ is normal in $G$, contradicting the minimality of $N$.

By Proposition 2.3 a $(6, p)$-semiregular automorphism of $X$ exists if $p > 5$. We now show that such an automorphism exists also if $p = 5$. Suppose then that $X$ is of order 30. Let $P \leq G$ be a Sylow 5-subgroup of $G$. By Proposition 2.2 the lengths of its orbits are divisible by 5. Therefore, $P$ either has 6 orbits of length 5 or one orbit of length 25 and one orbit of length 5. However, a similar argument as in the proof of Lemma 2.1 shows that the latter case is impossible. So $P$ has 6 orbits of length 5. By Lemma 2.1 it follows that the group $H$ has two orbits of length 5 in its natural action on $X_B$. Thus an element $\psi \in P$ of order 5 is either $(6, 5)$-semiregular or it has 3 orbits of length 5 and 15 fixed points. In the latter case there exists some other element $\vartheta \in P$ such that none of the above 15 fixed points of $\psi$ is fixed by $\vartheta$. Hence either $\vartheta$ or $\vartheta \psi$ is $(6, 5)$-semiregular. This proves our claim that a $(6, p)$-semiregular automorphism of $X$ always exists. Let us denote it by $\rho$.

We claim that $\rho$ and $\varphi$ commute. Namely, since $N \cong \mathbb{Z}_3$, we have that $\rho^{-1} \varphi \rho$ is equal either to $\varphi$ or $\varphi^{-1}$. But $p$ is odd, so $\rho^{-1} \varphi \rho = \varphi^{-1}$ would imply $\rho^{-1} \varphi \rho^p = \varphi^{-1}$, which is clearly impossible as $\rho^p = 1$. Thus, $\varphi \rho = \rho \varphi$. Moreover, this element is of order 3$p$ and has precisely two orbits of length 3$p$ which implies that $X$ is a bicirculant. Let $[S, R, T]$ be one of its symbols corresponding to $\varphi \rho$, such that $0 \in T$. If there exists some $a \in T$ for which $\langle a \rangle = \mathbb{Z}_{3p}$, where $\langle a \rangle$ is the additive subgroup of $\mathbb{Z}_{3p}$ generated by $a$, then $X$ has a Hamilton cycle. Moreover, if $T$ contains an element of order $p$ and an element of order 3, then their difference generates $\mathbb{Z}_{3p}$, and so $X$ has a Hamilton cycle. We can therefore assume that $\langle T \setminus \{0\} \rangle$ is either empty or it is one of $\langle 3 \rangle$ or $\langle p \rangle$.

As $X \langle B \rangle$ is an independent set for each $B \in \mathcal{B}$, there is no element of order 3 in $S$ or in $R$. If $\langle S \rangle = \mathbb{Z}_{3p}$ and $\langle R \rangle = \mathbb{Z}_{3p}$, then the subgraphs induced on each of the orbits of $\varphi \rho$ are connected vertex-transitive graphs of order 3$p$, and so they both contain a Hamilton cycle. Clearly, $X$ has a Hamilton path in this case. With no loss of generality we can thus assume that $\langle S \rangle \neq \mathbb{Z}_{3p}$. This implies that $S = \emptyset$ or $\langle S \rangle = \langle 3 \rangle$. Suppose first that $S = \emptyset$. Then regularity of $X$ implies $R = \emptyset$ as well. By the above remarks on $T$, $X$ is not connected, a contradiction. Therefore, $\langle S \rangle = \langle 3 \rangle$.

As $X$ is regular, we have that $|S| = |R|$, and so either $\langle R \rangle = \langle 3 \rangle$ or $\langle R \rangle = \mathbb{Z}_{3p}$. In the former case the subgraph induced on each of the orbits of $\rho$ contains a $p$-cycle. Moreover, the facts that $\langle T \rangle \neq \mathbb{Z}_{3p}$ and $X$ is connected imply, that there exists some $a \in T$ of order 3, and so $a$ and 0 give rise to a 6-cycle of $X_{\rho}$. Therefore, $X$ has a Hamilton path in this case. We are left with the possibility $\langle R \rangle = \mathbb{Z}_{3p}$. In view of the fact that no element of order 3 exists in $R$, some $a \in R$ such that $\langle a \rangle = \mathbb{Z}_{3p}$ exists. We can assume that $a = 1$ (otherwise take $(\varphi \rho)^a$ instead of $\varphi \rho$). Since $\langle S \rangle = \langle 3 \rangle$, we have $3k \in S$ for some $k \in \{1, 2, \ldots, p-1\}$. Thus $X$ contains a subgraph isomorphic to the generalized Petersen graph $GP(3p, 3k)$ which has a Hamilton cycle (see [2]).

**Lemma 3.4** If the size of blocks in $\mathcal{B}$ is $p$ then $X$ has a Hamilton cycle.

**Proof.** The quotient graph $X_B$ is a connected vertex-transitive graph on 6 vertices. By Lemma 2.1 the blocks of $\mathcal{B}$ coincide with the orbits of some $(6, p)$-semiregular automorphism $\rho \in G$ of $X$. 

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which exists by Lemma 2.4. Let $S = \{S_i \mid i \in \mathbb{Z}_p\}$ denote the set of orbits of $\rho$ and denote the vertices of each $S_i$ with $u_i^j, j \in \mathbb{Z}_p$, where $u_i^j \rho = u_i^{j+1}$. The quotient graph $X_S = X_G$ is isomorphic to one of the following five graphs: $C_6, K_3 \square K_2, K_{3,3}, K_3 \bowtie 2 K_1$ or $K_6$ (these are the only connected vertex-transitive graphs on six vertices). It is easy to see that in all these cases for any edge $e$ of $X_S$ there exists a Hamilton cycle of $X_S$ containing $e$. Hence, by Lemma 2.5 we may assume that no multiedge exists in $X_S$. Moreover, we may label the orbits of $\rho$ in such a way that $S_i \sim S_{i+1}$ for every $i \in \mathbb{Z}_6$. If there exists a Hamilton cycle of $X_S$ whose lift contains a Hamilton cycle of $X$, there is nothing to prove. Therefore, we can assume that no such Hamilton cycle of $X_S$ exists. Consequently, we may assume that $u_i^j \sim u_{i+1}^j, i \in \mathbb{Z}_6$ and $j \in \mathbb{Z}_p$. Note also that we can assume that $X(S_i) = pK_1$ for all $i \in \mathbb{Z}_6$. Namely, if the subgraphs $X(S_i)$ are of valency 2, then a Hamilton cycle of $X$ exists by [3] Theorem 3.9, and if the subgraphs $X(S_i)$ are of valency at least 4, then [11] Theorem 4 implies that each of $X(S_i)$ is Hamilton-connected (that is, there exists a Hamilton path of $X(S_i)$ connecting any two vertices), and so a Hamilton cycle of $X$ clearly exists.

We distinguish five different cases depending on which of the five connected vertex-transitive graphs of order 6 the quotient graph $X_S$ is isomorphic to.

If $X_S \cong C_6$ then $S_iS_{i+1}$, where $i \in \mathbb{Z}_6$, are the only edges of $X_S$, and so $X$ is not connected, a contradiction.

Suppose that $X_S \cong K_3 \square 2 K_1$. Then we may assume that in addition to the edges $S_iS_{i+1}$, also $S_0S_4, S_1S_5, S_2S_6 \in E(X_S)$. Therefore,

$$E(X) = \{u_i^ju_{i+1}^j \mid i \in \mathbb{Z}_6, j \in \mathbb{Z}_p\} \cup \{u_i^j u_4^j, u_i^j u_2^j u_5^j, | j \in \mathbb{Z}_p\},$$

where $r_0 r_1 r_2 \in \mathbb{Z}_p$. Since $S_0S_4S_3S_1S_2S_5S_6$ and $S_0S_1S_3S_2S_5S_4S_0$ are Hamilton cycles of $X_S$, Lemma 2.5 implies that $r_0 - r_1 + r_2 = 0$ and $r_0 - r_2 - r_1 = 0$. Subtracting one of the equations from the other we get that $r_2 = 0$, and so $r_0 = r_1$. In view of the connectedness of $X$, we have $r_0 = r_1 \neq 0$. Then

$$u_0^0 u_4^{r_0} u_5^{r_0} u_0^{2r_0} \cdots u_0^{r_0} u_4^{0} u_5^{0} u_2^{0} u_3^{0} u_1^{0} u_2^{0} - r_0 u_3^{0} u_1^{r_0} - r_0 u_3^{0} u_1^{2r_0} \cdots u_2^{r_0} u_3^{r_0} u_1^{r_0} u_0^{0}$$

is a Hamilton cycle of $X$.

Suppose next that $X_S \cong K_{3,3}$. Hence we may assume that adjacencies in $X_S$ are $S_i \sim S_{i+1}$ and $S_i \sim S_{i+3}$, where $i \in \mathbb{Z}_6$. This implies that $E(X) = \{u_i^j u_{i+1}^j, u_i^j u_{i+3}^j \mid i \in \mathbb{Z}_6, j \in \mathbb{Z}_p\}$, where $r_i = -r_{i+3}$. Since $S_0S_3S_2S_1S_4S_5S_6, S_0S_3S_4S_5S_2S_1S_0$ and $S_0S_3S_2S_5S_4S_1S_0$ are Hamilton cycles of $X_S$, Lemma 2.5 implies that $r_0 + r_1 = 0, r_0 + r_5 = r_0 - r_2 = 0$ and $r_0 + r_2 + r_4 = r_0 + r_2 - r_1 = 0$. As $p \geq 5$, combining these equations we get that $r_i = 0$ for every $i \in \mathbb{Z}_p$, which contradicts the fact that $X$ is connected.

The remaining two cases ($X_S = K_3 \bowtie 2 K_1$ and $X_S = K_6$) are dealt with in a similar manner. We leave the details to the reader.

**Remark.** In the above proof a Hamilton cycle was shown to exist in $X$ using the following idea. When considering the possible arrangements of the edges of $X$, where the quotient graph $X_S$ has been given, the key factors are the connectedness of $X$ and Lemma 2.5. This way we find that either a Hamilton cycle of $X_S$ whose lift contains a Hamilton cycle of $X$ exists, or the structure of the edges of $X$ is completely determined in which case a Hamilton cycle of $X$ is easily found. The same approach will be used throughout this paper. The technical details will be left to the reader.
Lemma 3.5 If the size of blocks in $B$ is 6 then $X$ has a Hamilton cycle.

Proof. Note that $X_B$ is a connected $p$-circulant so it has a Hamilton cycle. Theorem 2.7 implies, that $N^B$ is simple of degree 6 for every $B \in \mathcal{B}$. The only two transitive simple groups of degree 6 up to permutation isomorphism are the alternating group $A_6$ and its subgroup isomorphic to $A_5$ (see [13]). They are both doubly transitive. Thus the subgraphs $X(B)$, $B \in \mathcal{B}$, are either all isomorphic to $K_6$ or they are all isomorphic to $6K_1$.

Suppose first that $X(B)$ is isomorphic to $K_6$ for all $B \in \mathcal{B}$. Then $X(B)$ is Hamilton connected for every $B \in \mathcal{B}$, and so a Hamilton cycle of $X$ clearly exists.

Suppose now that $X(B) = 6K_1$ for all $B \in \mathcal{B}$. Every simple subgroup of $A_6$ of order 60 is permutation isomorphic to $H = \langle (1\ 2\ 3\ 4\ 5), (1\ 2)(4\ 6) \rangle$ (see for example [13], Table 2.1). Thus for any $B \in \mathcal{B}$ and any vertex $v \in B$ we have some $\alpha \in N^B$ fixing $v$ and cyclically permuting the other five vertices of $B$. We claim that for any two adjacent blocks $B, B' \in \mathcal{B}$ the graph $[B, B']$ is isomorphic to $K_{6,6}$, to $K_{6,6} - 6K_2$ or to $6K_2$. Namely, suppose that a vertex $u \in B$ has at least two neighbors, say $v_1$ and $v_2$, in $B'$. By the above remarks there exists an automorphism $\alpha \in N$ fixing $u$ and permuting the other five vertices of $B$. We distinguish two different cases depending on the order $d$ of $\alpha|_{B'}$.

CASE 1. $d = 5$. Then $\alpha|_{B'}$ also fixes a vertex $v$ of $B'$ and cyclically permutes the other five vertices of $B'$. With no loss of generality assume $v \neq v_1$. Applying $\alpha$ to the edge $uv_1$ we get that the valency of $u$ in $[B, B']$ is either 5 or 6, depending on whether $u$ is adjacent to $v$ or not. Since $B$ is the set of orbits of $N$, a simple counting argument shows that the subgraph $[B, B']$ is isomorphic either to $K_{6,6}$ or to $K_{6,6} - 6K_2$ as claimed.

CASE 2. $d \neq 5$. With no loss of generality we can assume that $d = 1$ (otherwise take an appropriate power of $\alpha$). Since $u$ has a neighbor in $B'$, every vertex of $B$ has a neighbor in $B'$. Let $u' \in B$, $u' \neq u$, have a neighbor $v$ in $B'$. Applying $\alpha$ to the edge $u'v$ we get that $v$ is adjacent to all the vertices of $B$ except possibly $u$. Thus $[B, B']$ is isomorphic either to $K_{6,6}$ or to $K_{6,6} - 6K_2$, which completes the proof of our claim.

Now let $B \in \mathcal{B}$. We claim that there exists a block $B'$, adjacent to $B$, such that $[B, B']$ is not isomorphic to $6K_2$. Namely, if this is not the case, then a contradiction to the connectedness of $X$ is obtained by an argument similar to the one of the above two paragraphs. Since $G$ acts transitively on $X$, there exists an element $\psi \in G$ cyclically permuting the $p$ blocks of $B$. With no loss of generality we can assume that $B' = B\psi$ (otherwise take an appropriate power of $\psi$). It follows that $B\psi^i \sim B\psi^{i+1}$ for all $i \in \mathbb{Z}_p$. It is now evident that $X$ has a Hamilton cycle. $\blacksquare$

Lemma 3.6 If the size of blocks in $B$ is $2p$ then $X$ has a Hamilton path.

Proof. Note that $X_B = K_3$ and that the group $G$ acts edge transitively in its natural action on $X_B$. Let $\mathcal{B} = \{B_i \mid i \in \mathbb{Z}_3\}$. Let $P \leq G$ be some Sylow $p$-subgroup of $G$. In view of Proposition 2.2 and the fact that $G$ has 3 blocks of size $2p$, $P$ has 6 orbits of length $p$. Denote them by $S = \{S_i \mid i \in \mathbb{Z}_6\}$. By Lemma 2.1 each block in $\mathcal{B}$ is a union of two orbits of $P$. With no loss of generality we can assume that $B_0 = S_0 \cup S_1$, $B_1 = S_2 \cup S_3$ and $B_2 = S_4 \cup S_5$.

By Proposition 2.3 there exists a $(6, p)$-semiregular automorphism $\rho$ of $X$ such that $\rho \in G$ whenever $p > 5$. We show that we can assume such an element to exist even if $p = 5$. To this end suppose that $p = 5$ and that $X$ does not contain a Hamilton path. In view of Proposition 2.6 the valency of $X$ is at most 9. Let $\rho \in G$ be an element of order 5, whose action on $B_0$ is $(2, 5)$-semiregular (which exists by Proposition 2.3). With no loss of generality assume that $\rho \in P$. The
two orbits of $\rho$ in $B_0$ thus coincide with $S_0$ and $S_1$. If $\rho$ is not $(6,5)$-semiregular, then we can assume that it fixes some vertex $u \in S_2$. Since $X_B = K_3$, the vertex $u$ has a neighbor in $B_0$ and thus its valency in $[B_0, B_1]$ is at least 5. As $[B_0, B_1]$ is regular and $G$ acts edge-transitively on $X_B$, the valency of $u$ in $[B_1, B_2]$ is at least 5 as well, contradicting the fact that $u$ has valency at most 9. Thus $\rho$ is $(6,5)$-semiregular, as required.

We can clearly assume that the orbit set of $\rho$ is $S$. In view of regularity of the bipartite graphs $[B, B'], B, B' \in B$, the subgraph $\overline{X}_S$ of $X_S$, which is obtained from $X_S$ by deleting the edges $S_0S_1$, $S_2S_3$, $S_4S_5$ (if they exist), is clearly one of the graphs $Y_i, i \in \{0, 1, 2, 3, 4\}$ of Figure 3. However, for each of the graphs $Y_i, i \geq 1$, the following holds: if there exists a multiedge of $X_\rho$, then there exists a Hamilton cycle of $X_S$ which contains an edge corresponding to a multiedge of $X_\rho$. By Lemma 2.5 we can thus assume that no multiedge exists in $X_\rho$, except possibly if $\overline{X}_S = Y_0$. In view of the regularity of $X$ the graphs $Y_3$ and $Y_4$ are then not possible.

Figure 3: All possibilities for the subgraph $\overline{X}_S$ of $X_S$ when $G$ has a complete block system $B$ of three blocks of size $2p$.

If $X(B_0)$ is a connected graph, then for each of its vertices there exists a Hamilton path of $X(B_0)$ starting at that vertex, so $X$ clearly has a Hamilton path in this case. We can thus assume that $X(B_0)$ is not connected. As it is a vertex-transitive graph, it is isomorphic to $2pK_1$, to $pK_2$ or it is a disjoint union of two isomorphic connected $p$-circulants. We consider each of the three cases separately.

CASE 1. $X(B_0) \cong 2pK_1$. As $X$ is connected, the quotient graph $X_S = \overline{X}_S$ is one of $Y_1$ or $Y_2$. If $X \cong Y_1$, then connectedness of $X$ and Lemma 2.5 imply that the lift of $Y_1$ contains a Hamilton cycle of $X$. It is easy to see that if $X_S \cong Y_2$, the connectedness of $X$ forces some Hamilton cycle of $X_S$, whose lift contains a Hamilton cycle of $X$, to exist. We leave the details to the reader.

CASE 2. $X(B_0) \cong pK_2$. It is clear that $[S_0, S_1] \cong pK_2$. Suppose first that $\overline{X}_S \cong Y_0$. In this case every edge of $X_S$ is contained on some Hamilton cycle of $X_S$, and so Lemma 2.5 implies that we can assume that no multiedge exists in $X_\rho$. If there exists a Hamilton cycle of $\overline{X}_S$ whose lift contains a Hamilton cycle of $X$, we are done. If not, the connectedness of $X$ implies that $X \cong C_{3p} \square K_2$, and so $X$ contains a Hamilton cycle. In the case when $\overline{X}_S$ is isomorphic to one of $Y_1$ and $Y_2$ one can easily see that the connectedness of $X$ forces some Hamilton cycle of $X_S$, whose lift contains a Hamilton cycle of $X$, to exist. The details are left to the reader.

CASE 3. $X(B_0)$ is isomorphic to a disjoint union of two isomorphic connected $p$-circulants. In view of connectedness of $X$ the quotient graph $X_S = \overline{X}_S$ is one of $Y_1$ or $Y_2$ and so it has a Hamilton cycle. As the six $p$-circulants are precisely the graphs $X(S_i)$, where $i \in \mathbb{Z}_6$, a Hamilton path exists in $X$. This completes the proof. \[\square\]
Lemma 3.7 If the size of blocks in $B$ is $3p$ then $X$ has a Hamilton path.

Proof. Note that $|B| = 2$ and $X_B = K_2$. Let us denote the two blocks of $B$ by $B$ and $B'$. We first show that in the case when $p = 5$ we can assume a $(6,5)$-semiregular automorphism $\rho$ of $X$, with $\rho \in G$, to exist. Suppose on the contrary that $X$ does not contain a Hamilton path and that no such $\rho \in G$ exists. By Proposition 2.3, the valency of $X$ is at most 9. Let $P \leq G$ be a Sylow 5-subgroup of $G$. In view of Proposition 2.2 and Lemma 2.1 $P$ has six orbits of length 5 on $X$. Denote them by $S_i$, $i \in \mathbb{Z}_6$. With no loss of generality assume that $S_i \subset B$ for $i = 0, 1, 2$. Proposition 2.3 implies that there exists some $\psi \in G$, such that $\psi|_B$ is $(3, 5)$-semiregular. With no loss of generality assume that $\psi$ is of order 5 and $\psi \in P$. The orbits of $\psi$ on $B$ are then $S_0$, $S_1$, and $S_2$. In view of our assumptions $\psi|_{B'}$ is not semiregular. Moreover, $\psi|_{B'} \neq Id$, as otherwise $\psi^{-1}\psi\alpha$ is $(6, 5)$-semiregular on $X$, where $\alpha \in G$ is such that $B\alpha = B'$. Thus $\psi$ has at least one orbit of length 5 on $B'$ and at least 5 fixed points. We can assume that this orbit of length 5 is $S_3$ and that the 5 fixed points are the vertices of $S_4$. As $X_B \cong K_2$, we can assume that $S_1 \sim S_4$. Since $S_1$ and $S_4$ are orbits of $P$, it is clear that $[S_1, S_4] = K_{5,5}$. Moreover, since $x$ has at most 9 neighbors, the valency of $[B, B']$ is 5, and so $|B, B'| = 3K_{5,5}$. Since $S_1$ is a subset of the block $B$, it is now clear that $S_1$ itself is a block for $G$. Lemma 2.1 implies that the block system arising from $S_1$ coincides with $\{S_i \mid i \in \mathbb{Z}_6\}$. Using the fact that $X$ is connected one can see that there exist adjacent vertices $u$ and $v$ of $B'$ such that $\psi$ fixes precisely one of them. But then the valency of $X$ is at least 10, a contradiction which proves our claim.

Therefore, Proposition 2.3 implies that we can assume that a $(6, p)$-semiregular automorphism $\rho$ of $X$ such that $\rho \in G$ exists. Let $S = \{S_i \mid i \in \mathbb{Z}_6\}$ be the set of its orbits. By Lemma 2.1, each block in $B$ is a union of three orbits of $\rho$. With no loss of generality we can assume that $B = S_0 \cup S_1 \cup S_2$ and $B' = S_3 \cup S_4 \cup S_5$. In view of regularity of the bipartite graph $[B, B']$, the subgraph $\bar{X}_S$ of $X_S$, which is obtained from $X_S$ by deleting the edges between the orbits inside the blocks $B$ and $B'$ (if they exist), is clearly one of the graphs $Y_i$, $i \in \{0, 1, 2, 3, 4, 5\}$ of Figure 1. However, for each of the graphs $Y_i$, $i \geq 2$, the following holds: if there exists a multiedge of $X_\rho$, then there exists a Hamilton cycle of $X_S$ which contains an edge corresponding to a multiedge of $X_\rho$. By Lemma 2.5 we can thus assume that no multiedge exists in $X_\rho$ except possibly when $\bar{X}_S = Y_0$ or $\bar{X}_S = Y_1$. Regularity of $X$ then implies that $Y_4$ and $Y_5$ are not possible.

Figure 4: All possibilities for the subgraph $\bar{X}_S$ of $X_S$ when $G$ has a complete block system $B$ of two blocks of size $3p$.

If $X\langle B \rangle$ is a connected graph, then it contains a Hamilton cycle (as it is a vertex-transitive graph of order $3p$) and so $X$ has a Hamilton path in this case. We can thus assume that $X\langle B \rangle$ is not connected, and so it is isomorphic to $3pK_1$, to $pK_3$ or it is a disjoint union of three isomorphic connected $p$-circulants. We consider each of the three cases separately. The technical details of each of them are left to the reader.

Case 1. $X\langle B \rangle \cong 3pK_1$. As $X$ is connected, the quotient graph $X_S = \bar{X}_S$ is one of $Y_2$ and $Y_3$. If $X \cong Y_2 \cong C_6$, then connectedness of $X$ and Lemma 2.5 imply that the lift of $Y_2$ contains a
Hamilton cycle of $X$. If however $X_S \cong Y_3 \cong K_{3,3}$, then one can see that some Hamilton cycle of $X_S$, whose lift contains a Hamilton cycle of $X$, exists.

**Case 2.** $X(B) \cong pK_3$. Then of course also $X(B') \cong pK_3$. It is clear that each $K_3$ in $B$, $B'$ intersects all the orbits of $\rho$ in $B$ and $B'$, respectively. Suppose first that $X_S \cong Y_0$. Then every edge of $X_S$ is contained on some Hamilton cycle of $X_S$. Hence Lemma 2.3 implies that we can assume that no multiedge exists in $X_\rho$. If there exists a Hamilton cycle of $X_S$ whose lift contains a Hamilton cycle of $X$, we are done. If not, the connectedness of $X$ implies that $X \cong C_3p\square K_2$, and so $X$ contains a Hamilton cycle. If $X_S \cong Y_1$ then there exists a multiedge of $X_\rho$ that is contained in a Hamilton cycle of $X_S$, and so a Hamilton cycle of $X$ exists. Finally, if $X_S$ is isomorphic to $Y_2$ or to $Y_3$ it is easy to see that some Hamilton cycle of $X_S$, whose lift contains a Hamilton cycle of $X$, exists.

**Case 3.** $X(B)$ is isomorphic to a disjoint union of three isomorphic connected $p$-circulants. Then the quotient graph $X_S = X_\rho$ is one of $Y_2$ or $Y_3$, and so it has a Hamilton cycle. As the six $p$-circulants are precisely the graphs $X(S_i)$, where $i \in \mathbb{Z}_6$, a Hamilton path exists in $X$. This completes the proof of Lemma 3.7.

## 4 Quasiprimitive graphs

Throughout this section let $X$ denote a connected quasiprimitive graph of order $6p$. In [33] a complete characterization of quasiprimitive graphs of order $pqr$, where $p$, $q$ and $r$ are distinct primes, was given via the well known generalized orbital graph construction relative to certain simple groups having an imprimitive permutation representation of degree $pqr$. All the possible group actions are given in Tables A and B in [33, p. 298-299]. For our purposes (we require that $pqr = 6p'$) only a handful of group actions needs to be considered. They are given in Table 1. Note that only row 11 of Table 1 corresponds to an infinite family of actions giving rise to quasiprimitive graphs of order $6p$. Lemma 4.1 shows that each of the quasiprimitive graphs corresponding to an action from this infinite family has a Hamilton cycle. As for the other rows of Table 1 each case is investigated separately. More precisely, we consider all the possible generalized orbital graphs and study their structural properties (using program package MAGMA [9]) which allows us to easily find a Hamilton path. In fact, in all the graphs, except for the truncation of the Petersen graph, a Hamilton cycle is found.

Let $G$ be a group acting on the cosets of its subgroup $H$ in a natural way. We say that the set $O(G,H)$ of generalized orbital graphs (in short GOGs) of this action is a *minimal connected orbital graph set* for this action if each connected GOG corresponding to this action contains some graph of $O(G,H)$ as a spanning subgraph. As we are only interested in whether a given GOG contains a Hamilton path (or a Hamilton cycle) Proposition 2.6 implies that we can disregard the graphs from $O(G,H)$ whose valencies are at least $[G:H]/3$. We let the remaining set of GOGs be the set $R(G,H)$ of *relevant graphs* for this action. It is now clear that in order to show that each GOG corresponding to the above mentioned action of $G$ contains a Hamilton path (Hamilton cycle) we only need to show that each GOG of $R(G,H)$ has this property.

We now describe the method of obtaining $R(G,H)$ for the action of row 1 of Table 1 in full detail. The other actions are dealt with in a similar way, so we only give the relevant graphs and leave the details to the reader. Each relevant graph $X$ will be represented in a structural way given by some semiregular automorphism $\varphi$ of $X$ from which the existence of a Hamilton cycle will be clear (except for the truncation of the Petersen graph). In the case when $\varphi$ is $(6,p)$-semiregular...
Frucht’s notation under a (10, 3)-semiregular automorphism. Lemma 2.5 implies that the latter graph contains a Hamilton cycle.

Finally, the unions $U_i \cup U_j$, where $i, j \in \{2, 3, \ldots, 7\}$, give rise to five nonisomorphic connected graphs. These are the graphs $X_3$, $X_4$, $X_5$ and $X_6$ of Table 2 and the graph of Figure 7 given in Frucht’s notation under a (10, 3)-semiregular automorphism. Lemma 2.5 implies that in all these cases the graph in question has a Hamilton cycle.

Table 1: Actions giving rise to quasiprimitive graphs of order $6p$.

| row | $p$ | Action |
|-----|-----|--------|
| 1   | 5   | $A_5$ on cosets of $\mathbb{Z}_2$ |
| 2   | 7   | $A_7$ on cosets of $A_5$ |
| 3   | 11  | $\text{PSL}(2, 11)$ on cosets of $D_{10}$ |
| 4   | 7   | $\text{PSL}(3, 2)$ on cosets of $\mathbb{Z}_4$ |
| 5   | 7   | $\text{PSL}(3, 2)$ on cosets of $A_5$ |
| 6   | 13  | $\text{PSL}(3, 3)$ on cosets of $\mathbb{Z}_5^2 \rtimes \mathbb{Z}_8$ |
| 7   | 13  | $\text{PSL}(3, 3)$ on cosets of $\mathbb{Z}_5^2 \rtimes D_8$ |
| 8   | 13  | $\text{PSL}(3, 3)$ on cosets of $\mathbb{Z}_5^2 \rtimes Q_8$ |
| 9   | 31  | $\text{PSL}(3, 5)$ on cosets of $\mathbb{Z}_5^2 \rtimes (\mathbb{Z}_5 \rtimes \mathbb{Z}_4^2)$ |
| 10  | 5   | $A_6$ on cosets of $A_4$ |
| 11  | $\frac{k+1}{2}$ | $\text{PSL}(2, k)$ on cosets of $\mathbb{Z}_2^k \times \mathbb{Z}_{(k-1)/6}$ where $3 \mid \frac{k+1}{2}$ and $k = s^m$ |

Graphs corresponding to row 1 of Table 1. Note that these graphs are of order 30. In the action of $A_5$ on the cosets of $\mathbb{Z}_2$ we get that $\mathbb{Z}_2$ has 15 nontrivial suborbits, 7 of which are self-paired. Of the seven self-paired suborbits, six are of length 2 and one is of length 1. The non-self-paired suborbits are of length 2. Denote the 15 nontrivial suborbits by $U_i$, $i \in \{1, 2, \ldots, 15\}$, where $U_1$ is of length 1, $U_2$, $U_3$, $U_7$ are the self-paired suborbits of length 2 and $U_{2i}$ is paired with $U_{2i+1}$ for $i \in \{4, 5, 6, 7\}$.

The unions $U_{2i} \cup U_{2i+1}$, where $i \in \{4, 5, 6, 7\}$, give rise to three nonisomorphic graphs, one of which is disconnected (with no loss of generality assume that this graph corresponds to $U_{14} \cup U_{15}$). The other two are given in Frucht’s notation under a (5, 6)-semiregular automorphism in Figure 5.

Using an argument similar to the one in the proof of Lemma 2.5 one can see that these two graphs both contain a Hamilton cycle.

It turns out that the graph arising from $U_1 \cup U_{14} \cup U_{15}$ is still disconnected. The graphs arising from $U_i \cup U_{14} \cup U_{15}$, where $i \in \{2, 3, \ldots, 7\}$, are all connected and isomorphic either to $X_1$ or to $X_2$ of Table 2 and so Lemma 2.5 implies that a Hamilton cycle exists in $X$. Therefore, we now only have to consider the GOGs arising from unions of some suborbits from $\{U_1, U_2, \ldots, U_7\}$.

For every $i \in \{1, 2, 3, \ldots, 7\}$ the graph arising from the suborbit $U_i$ is disconnected, whereas the graph arising from $U_1 \cup U_i$, $i \in \{2, 3, \ldots, 7\}$, is connected and isomorphic either to the truncation of the Petersen graph, or to the graph of Figure 6 given in the Frucht’s notation under a (10, 3)-semiregular automorphism. Lemma 2.5 implies that the latter graph contains a Hamilton cycle.

Finally, the unions $U_i \cup U_j$, where $i, j \in \{2, 3, \ldots, 7\}$, give rise to five nonisomorphic connected graphs. These are the graphs $X_3$, $X_4$, $X_5$ and $X_6$ of Table 2 and the graph of Figure 7 given in Frucht’s notation under a (10, 3)-semiregular automorphism. Lemma 2.5 implies that in all these cases the graph in question has a Hamilton cycle.
We have now clearly considered all the relevant graphs $\mathcal{R}(A_5, \mathbb{Z}_2)$. Note also, that each GOG corresponding to the action of $A_5$ on the cosets of $\mathbb{Z}_2$ which contains the truncation of the Petersen graph as a proper spanning subgraph contains a Hamilton cycle. We can thus conclude that each connected GOG arising from the action of $A_5$ on the cosets of $\mathbb{Z}_2$, except for the truncation of the Peterson graph, contains a Hamilton cycle.

Graphs corresponding to row 2 of Table 1: The relevant graphs are given in Table 3 and so it is clear that each GOG arising from this action contains a Hamilton cycle.

Graphs corresponding to row 3 of Table 1: The relevant graphs are given in Table 4 and so it is clear that each GOG arising from this action contains a Hamilton cycle.

Graphs corresponding to row 4 of Table 1: The relevant graphs are given in Table 5 and Figure 8 and so it is clear that each GOG arising from this action contains a Hamilton cycle.

Graphs corresponding to row 5 of Table 1: The relevant graphs are given in Table 6 and so it is clear that each GOG arising from this action contains a Hamilton cycle.

Graphs corresponding to row 6 of Table 1: It turns out that $\mathcal{R}(G, H) = \emptyset$ in this case, and so each GOG arising from this action contains a Hamilton cycle.

Graphs corresponding to row 7 of Table 1: The relevant graphs are given in Table 7 and so it is clear that each GOG arising from this action contains a Hamilton cycle.
Graphs corresponding to row 8 of Table 1: It turns out that $R(G, H) = \emptyset$ in this case, and so each GOG arising from this action contains a Hamilton cycle.

Graphs corresponding to row 9 of Table 1: The relevant graphs are given in Table 8 and so it is clear that each GOG arising from this action contains a Hamilton cycle.

Graphs corresponding to row 10 of Table 1: The relevant graphs are given in Table 9 and Figure 9, and so it is clear that each GOG arising from this action contains a Hamilton cycle.

Graphs corresponding to row 11 of Table 1: Lemma 4.1 below implies that each of the corresponding graphs contains a Hamilton cycle.

Lemma 4.1 Let $X$ be a graph corresponding to the action of row 11 of Table 1. Then $X$ contains a Hamilton cycle.

Proof. From [33, Table B and Section 4] we can extract that the action of $G = \text{PSL}(2, k)$ on the cosets of $\mathbb{Z}_k \times \mathbb{Z}_{(k-1)/6}$ (the action of row 11 of Table 1) gives rise to a vertex-transitive graph $X$ on which $G$ has a complete block system $\mathcal{B}$ of $k + 1 = 2p$ blocks of size 3 with block stabilizer $G_{\mathcal{B}} \cong \mathbb{Z}_k \times \mathbb{Z}_{(k-1)/2}$. Moreover, the permutation group $\bar{G}$ corresponding to the natural action of $G$ on $X_{\mathcal{B}}$ is doubly transitive, and so $X_{\mathcal{B}}$ is isomorphic to the complete graph $K_{2p}$ and the bipartite graphs $[B, B']$, where $B, B' \in \mathcal{B}$, are all isomorphic. Note also, that $p \geq 7$.

Since 3 divides $\frac{k-1}{2}$ and $p = \frac{k+1}{2}$ it is clear that $p \equiv 1 \pmod{3}$. Let $P$ be a Sylow 3-subgroup of $G = \text{PSL}(2, k)$ and let $\bar{P}$ denote the permutation group corresponding to the natural action of $P$ on $X_{\mathcal{B}}$. Since $\bar{P}$ is a 3-group and $2p \equiv 2 \pmod{3}$, there exist $B_0, B_1 \in \mathcal{B}$ which are fixed by $\bar{P}$. By Proposition 2.2, however, $P$ acts transitively on each of the two blocks $B_0$ and $B_1$. Since $X_{\mathcal{B}}$ is a complete graph, there exist adjacent vertices $u \in B_0$ and $v \in B_1$. Let $\varphi \in P$ be an automorphism which does not fix $u$. If it fixes $v$, then $[B_0, B_1]$ is a complete bipartite graph $K_{3,3}$, and so $X$ is of valency at least $6p - 3$, in which case Proposition 2.6 applies. We can therefore assume that $\varphi$ does not fix $v$. Then $[B_0, B_1]$ contains $3K_2$ as a subgraph. If $[B_0, B_1]$ is not isomorphic to $3K_2$ or if $X(B_0)$ is not an independent set, then the valency of $X$ exceeds $2p$, and we can again apply Proposition 2.6.

We can now assume that $[B, B'] \cong 3K_2$ and $X(B) = 3K_1$ for all $B, B' \in \mathcal{B}$. As $p \geq 7$, Proposition 2.3 implies that a $(6, p)$-semiregular automorphism $\rho$ of $X$, where $\rho \in G$, exists.
Denote its orbits by \( S = \{ S_i \mid i \in \mathbb{Z}_6 \} \). By Lemma 2.1, we have \( |S_i \cap B| \in \{0, 1\} \) for all \( i \in \mathbb{Z}_6 \) and \( B \in \mathcal{B} \). It is clear that we can then assume that \( \mathcal{A} = S_0 \cup S_1 \cup S_2 \) is a union of \( p \) blocks from \( \mathcal{B} \) and that \( \mathcal{A}' = S_3 \cup S_4 \cup S_5 \) is a union of the other \( p \) blocks from \( \mathcal{B} \). In view of our assumptions each vertex in \( \mathcal{A} \) has \( p \) neighbors in \( \mathcal{A}' \) and vice versa. Suppose there exists an orbit \( S_i \), with no loss of generality assume it is \( S_0 \), such that \( X(S_0) = K_p \) and \( S_0 \) is adjacent to only one of the orbits from \( \mathcal{A}' \), say to \( S_3 \). This implies that \( [S_0, S_3] = K_{p,p} \). Note that the vertices of \( S_0 \) are characterized by the fact that they are adjacent to all the vertices of \( S_0 \cup S_3 \) (except to itself). Moreover, as \( X \) is connected, each vertex of \( S_3 \) has at least one neighbor outside \( S_0 \cup S_3 \). It is now clear that \( S_0 \) is a block of imprimitivity for \( G \). But this implies that the quotient graph corresponding to the imprimitivity block system arising from \( S_0 \) is a vertex-transitive graph of order 6 which contains a Hamilton cycle. It is now clear that \( X \) also has a Hamilton cycle.

We can thus assume that for each \( S_i \in S \) the following holds: if \( X(S_i) = K_p \) then the valency of \( S_i \) in the subgraph \( Y = [\mathcal{A}, \mathcal{A}] \) of \( X_S \) is at least two. Note also that if \( S_j \) is the only neighbor of \( S_i \) in \( Y \), then \( [S_i, S_j] = K_{p,p} \), and so \( S_i \) is the only neighbor of \( S_j \) in \( Y \) as well. We distinguish two cases depending on whether the graph \( Y \) contains a vertex of valency 1 or not.

**Case 1.** There exists a vertex of \( Y \) of valency one. With no loss of generality assume that the only neighbor of \( S_0 \) in \( \mathcal{A}' \) is \( S_3 \). We distinguish two cases depending on the valency \( d \) of \( S_1 \) in \( Y \).

If \( d = 1 \), say \( S_1 \sim S_4 \), then the valency of \( S_2 \) in \( Y \) is also 1, and so \( S_2 \sim S_5 \). In view of the above remarks each of \( S_i \in A \) has at least one neighbor inside \( A \), and the same holds for \( \mathcal{A}' \).

We can thus assume that \( S_0 \sim S_1, S_0 \sim S_2 \) and \( S_3 \sim S_4 \). Moreover, \( S_5 \) is adjacent to one of \( S_3 \) and \( S_4 \). If \( S_5 \sim S_3 \), then \( S_0S_1S_4S_5S_2S_0 \) is a Hamilton cycle of \( X_S \) which contains an edge corresponding to a multiedge of \( X_\rho \), so Lemma 2.5 applies. Suppose then that \( S_5 \not\sim S_3 \), and so \( S_5 \sim S_4 \). Note that this also implies that \( X(S_3) \neq pK_1 \) (otherwise the valency of the vertices of \( S_4 \) exceeds \( 2p - 1 \)). It is clear that then a Hamilton path of \( [S_0, S_3] \) with endvertices in \( S_0 \) exists. As \( [S_1, S_4] \cong [S_2, S_5] \cong K_{p,p} \), \( S_0 \sim S_1 \) and \( S_0 \sim S_2 \), the existence of a Hamilton cycle of \( X \) is evident.

If \( d > 1 \), then clearly \( [S_1 \cup S_2, S_4 \cup S_5] \cong K_{2,2} \). As the valency of \( S_0 \) in \( Y \) is one, we have \( X(S_0) \not\cong K_p \), and so \( S_0 \) is adjacent to at least one of \( S_1, S_2 \). Similarly, \( S_3 \) is adjacent to at least one of \( S_4, S_5 \). It is easy to see that a Hamilton cycle of \( X_S \) containing the edge \( S_0S_3 \) exists in this case, so Lemma 2.5 applies.

**Case 2.** No vertex of valency 1 exists in \( Y \). It is straightforward to check that in this case a Hamilton cycle of \( X_S \) containing an edge corresponding to a multiedge of \( X_\rho \) exists, and so Lemma 2.5 applies. We leave the details to the reader.

In view of the fact that the connected vertex-transitive graphs of orders \( 4p \) and \( 2p^2 \) contain a Hamilton cycle (except for the Coxeter graph) (see [23, 28]), the results of this section imply that the following proposition holds.

**Proposition 4.2** A connected quasiprimitive graph of order \( 6p \), \( p \) a prime, which is not isomorphic to the truncation of the Petersen graph, contains a Hamilton cycle.
### Table 2: Relevant graphs corresponding to the action of row 1 of Table 1

|   | 2 | 2 | 2 | 2 | 2 |
|---|---|---|---|---|---|
| X | 20 | 20 | 20 | 20 | 20 |
| p | 14 | 14 | 14 | 14 | 14 |
| X | 20 | 20 | 20 | 20 | 20 |
| p | 14 | 14 | 14 | 14 | 14 |

### Table 3: Relevant graphs corresponding to the action of row 2 of Table 1

|   | 2 | 2 | 2 | 2 | 2 |
|---|---|---|---|---|---|
| X | 20 | 20 | 20 | 20 | 20 |
| p | 14 | 14 | 14 | 14 | 14 |
| X | 20 | 20 | 20 | 20 | 20 |
| p | 14 | 14 | 14 | 14 | 14 |

### Table 4: Relevant graphs corresponding to the action of row 3 of Table 1

|   | 2 | 2 | 2 | 2 | 2 |
|---|---|---|---|---|---|
| X | 20 | 20 | 20 | 20 | 20 |
| p | 14 | 14 | 14 | 14 | 14 |
| X | 20 | 20 | 20 | 20 | 20 |
| p | 14 | 14 | 14 | 14 | 14 |
Table 5: Relevant graphs corresponding to the action of row 4 of Table 1

Table 6: Relevant graphs corresponding to the action of row 5 of Table 1

Table 7: Relevant graph corresponding to the action of row 7 of Table 1

Table 8: Relevant graphs corresponding to the action of row 9 of Table 1

Table 9: Relevant graphs corresponding to the action of row 10 of Table 1

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5 Primitive graphs

Throughout this section let $X$ denote a primitive graph of order 6$p$. In [18] the complete characterization of possible primitive graphs of order 2$pq$, where $p$ and $q$ are distinct odd primes, was given. Extracting the information about graphs of order 6$p$ we find that the only primitive graphs of order 6$p$, $p$ a prime, are the ones arising from the actions given in Table 10. Below we show that each of the corresponding graphs has a Hamilton cycle. We let the GOGs and the relevant graphs corresponding to some action be defined as in Section 4.

| row | $p$ | Action of Aut$X$       |
|-----|-----|------------------------|
| 1   | 17  | PSL(2,17) on cosets of $S_4$ |
| 2   | 11  | $S_{12}$ on pairs       |
| 3   | 31  | PSL(3,5) on cosets of $P_{1,2}$ |

Table 10: Primes $p$ for which there exists a graph $X$ on 6$p$ vertices such that Aut$X$ and all vertex-transitive subgroups of Aut$X$ act primitively on $X$.

Graphs corresponding to row 1 of Table 10: The relevant graphs are the so called $H$-graph (see [8]), which by [8] has a Hamilton cycle, and the graphs isomorphic to one of the graphs $X_1$, $X_2$, $X_3$, $X_4$, $X_5$ and $X_6$ of Table 11. It is therefore clear that each GOG arising from this action contains a Hamilton cycle.

Graphs corresponding to row 2 of Table 10: Note that $X$ is of order 66. If $\{1,2\} \sim \{i,j\}$, where $\{1,2\} \cap \{i,j\} = \emptyset$ then the valency of $X$ is at least 45, so Proposition 2.6 applies. Therefore, the neighbors set of $\{i,j\}$ contains a Hamilton cycle. We let the GOGs and the relevant graphs corresponding to some action be defined as in Section 4.

Graphs corresponding to row 3 of Table 10: The relevant graphs are isomorphic to the graphs of Table 8 and so it is clear that each GOG arising from this action contains a Hamilton cycle.

| $p$ | $\nu(X)$ | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | $X_7$ |
|-----|----------|-------|-------|-------|-------|-------|-------|-------|
| 2   | 2        | 163   | 163   | 163   | 163   | 163   | 163   | 163   |
| 3   | 3        | 163   | 163   | 163   | 163   | 163   | 163   | 163   |
| 5   | 5        | 163   | 163   | 163   | 163   | 163   | 163   | 163   |
| 7   | 7        | 163   | 163   | 163   | 163   | 163   | 163   | 163   |
| 11  | 11       | 163   | 163   | 163   | 163   | 163   | 163   | 163   |
| 13  | 13       | 163   | 163   | 163   | 163   | 163   | 163   | 163   |
| 17  | 17       | 163   | 163   | 163   | 163   | 163   | 163   | 163   |
| 19  | 19       | 163   | 163   | 163   | 163   | 163   | 163   | 163   |
| 23  | 23       | 163   | 163   | 163   | 163   | 163   | 163   | 163   |
| 29  | 29       | 163   | 163   | 163   | 163   | 163   | 163   | 163   |
| 31  | 31       | 163   | 163   | 163   | 163   | 163   | 163   | 163   |

Table 11: Relevant graphs corresponding to the action of row 1 and row 2 of Table 10.
The results of this section imply that the following proposition holds.

**Proposition 5.1** *A primitive graph of order $6p$, $p$ a prime, contains a Hamilton cycle.*

## 6 The proof of the main theorem

**Proof of Theorem 1.1**: In view of the results from [23, 28], we can assume that $p \geq 5$. If $X$ is not genuinely imprimitive, then either Proposition 4.2 or Proposition 5.1 applies. If $X$ is genuinely imprimitive, then apply one of Lemma 3.1, Lemma 3.3, Lemma 3.4, Lemma 3.5, Lemma 3.6 and Lemma 3.7, depending on the size of the corresponding blocks. ■

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