DYNAMICAL DELOCALIZATION FOR DISCRETE MAGNETIC RANDOM SCHRÖDINGER OPERATORS

SIMON BECKER AND RUI HAN

ABSTRACT. We study discrete magnetic random Schrödinger operators on the square and honeycomb lattice under weak disorder. We show that there is, in the case of the honeycomb lattice with magnetic flux close to any rational, both strong dynamical localization and delocalization close to the conical point. We obtain similar results for the discrete random Schrödinger operator on the $\mathbb{Z}^2$-lattice with weak magnetic fields, close to the bottom and top of its spectrum. As part of this analysis, we give a rigorous derivation of the quantum Hall effect for both models derived from the density of states for which we obtain an asymptotic expansion in the disorder parameter. The expansion implies (leading order in the disorder parameter) universality of the integrated density of states. We also show that on the hexagonal lattice the Dirac cones occur for any rational magnetic flux.

Figure 1. Full Hofstadter butterfly for honeycomb lattice.-Different colours indicate different Hall conductivities.
1. Introduction and statement of results

It has been both experimentally and theoretically observed [M06, BTBB07, Z06] that Anderson localization in mildly disordered graphene (in magnetic fields) is suppressed and the material remains metallic under weak disorder.

To understand such phenomena, we study in this article discrete random Schrödinger operators, the tight-binding limits of continuous random Schrödinger operators, under weak disorder in weak magnetic fields on the $\mathbb{Z}^2$ lattice $\Lambda$ and for magnetic fluxes close to rationals on the honeycomb lattice $\Lambda$:

\[
(H^h_{\Lambda, \omega} u)(\gamma) := -\frac{1}{4} \left( e^{ih\gamma_2/2} u(\gamma + \vec{b}_1) + e^{-ih\gamma_2/2} u(\gamma - \vec{b}_1) \\
+ e^{-ih\gamma_1/2} u(\gamma + \vec{b}_2) + e^{ih\gamma_1/2} u(\gamma - \vec{b}_2) \right) + \lambda V_\omega(\gamma) u(\gamma)
\]

\[
(H^h_{\tilde{\Lambda}, \omega} u)(v) := -\frac{1}{3} \left( \sum_{\vec{e} \in \tilde{E}, i(\vec{e}) = v} e^{-iA\vec{e}} u(t(\vec{e})) + \sum_{\vec{e} \in \tilde{E}, t(\vec{e}) = v} e^{iA\vec{e}} u(i(\vec{e})) \right) + \lambda V_\omega(v) u(v).
\]

where $V_\omega$ is i.i.d. random potential on the lattice $\Lambda$. For the precise definitions of these operators, see Section 2.2.

The part of the energy spectrum of graphene, modeled here by the discrete operator on the honeycomb lattice, that is relevant for most of its remarkable physical properties, is the energy spectrum close to the conical points, the so-called Dirac points at energy zero, see Fig. 4. The existence of Dirac points for the tight-binding graphene model in the absence of magnetic field is known since [W47]. A recent significant result [FW12] shows the existence of Dirac points for the continuous non-magnetic Schrödinger operators with honeycomb lattice potentials. In Theorem 2, we prove the existence of Dirac cones at energy zero for the tight-binding model for any rational magnetic flux. We then show that in a neighbourhood of this energy both dynamical localization, implying Anderson localization, as well as dynamical delocalization occurs for perturbations of any rational flux under weak disorder. We refer to the beginning of Section 5.1 for precise definitions. This allows us to study transport properties of graphene, see also [Pe10] for the non-magnetic case and [GS06] for the magnetic case. We verify similar properties near the top and bottom of the spectrum of the discrete magnetic Anderson model on $\mathbb{Z}^2$ with weak magnetic fields, see Fig. 3.

To keep our presentation clear, we present the results for both lattices with small magnetic fields below. For the results on magnetic perturbations of non-trivial rational fluxes on the hexagonal lattice, we refer the readers to Theorem 4.
Theorem 1 (Dyn. Delocalization; Small fields). Let the magnetic flux \( h > 0 \), through a fundamental domain of the underlying lattice (\( \mathbb{Z}^2 \) or hexagonal), be sufficiently small such that there exists a family of disjoint disorder-broadened Landau bands, defined in (3.44). If the disorder is sufficiently small such that the Hall conductivity jumps in each Landau band (Proposition 1.1), there exists in each Landau band (at least) one energy that belongs to the region of dynamical delocalization.

The spectral properties of the discrete magnetic Laplacian (DML) on \( \mathbb{Z}^2 \), and of the almost Mathieu operator (AMO), have been extensively studied over the past forty years, see for instance a survey [MJ17] and some recent advancements [AYZ17, JL18, JK19]. Significant progress on the location of the spectrum has been made for magnetic Schrödinger operators using semiclassical analysis [HS88, HS89, HS90b, W94]. In two preceding articles [BHJ18, BZ19], by the authors, this study was extended to spectral properties and the density of states (DOS) of the magnetic Schrödinger operator on the honeycomb lattice -but without disorder. It was shown in [BZ19, Theorem 1] that the DOS for the magnetic Schrödinger operator on the honeycomb quantum graph close to the conical point- is concentrated at so-called relativistic Landau levels.

The spectral analysis in [BHJ18] showed that for the DML on the hexagonal lattice, close to the conical point, there is no point spectrum, as the analogy to the magnetic two-dimensional Dirac operator suggests. Instead, the spectrum of the DML on the honeycomb lattice is either absolutely continuous (a.c.) band spectrum or singular continuous (s.c.) and a Cantor set of Lebesgue measure zero, depending on the arithmetic properties of the magnetic flux through a single honeycomb.

Our strategy to analyze the metallic and insulating regimes is as follows: First, we locate the spectrum of the random operators using semiclassical analysis. This is done by deriving an expansion of the DOS stated in Theorem 3. Our result implies that besides a shift of the Landau levels [M06], the integrated density of states is to the first two leading orders invariant under small disorder. We then conclude, using semiclassical techniques as in [BZ19], that there are spectral gaps between Landau levels, see Proposition 3.6, to show that the quantum Hall effect (QHE) is invariant under weak disorder.

For general Fermi energies, the Chern number of the Fermi projection in both models does not possess closed-form expressions and can only be computed numerically from the TKNN formula [TKNN82, AEG14], see Figures 1 and 6. However, semiclassical arguments allow us to compute the Chern number of the spectrum close to the spectral edges. The result we obtain is in agreement with the experimental results for graphene [Z05].
Proposition 1.1 (QHE under weak disorder; Small magnetic fields). For sufficiently small magnetic flux \( h > 0 \), there are spectral gaps between disorder-broadened Landau bands up to some non-trivial disorder parameter \( \lambda_0 > 0 \). In particular, the Hall conductivity \( c_H \) with Fermi energy \( \mu \) in spectral gaps between the disorder-broadened Landau bands \( B^h_{\lambda, n} \) for \( \lambda \in [0, \lambda_0(h)] \) coincides with the Hall conductivity in the case of no disorder

\[
\begin{align*}
    c_H(H^h_{\mathbf{n}, \lambda, \omega}, \mu) &= \frac{\pi}{2\pi}, \quad \mu \text{ between } B^h_{\mathbf{n}, \lambda, n} \text{ and } B^h_{\mathbf{n}, \lambda, n+1} \text{ with } 1 \leq n \leq N_\mathbf{n}(h, \lambda_0) \\
    c_H(H^h_{\mathbb{O}, \lambda, \omega}, \mu) &= \begin{cases} \\
        \frac{2n+1}{2\pi}, \quad \mu \text{ between } B^h_{\mathbb{O}, \lambda, n} \text{ and } B^h_{\mathbb{O}, \lambda, n+1} \text{ with } 0 \leq n \leq N_\mathbb{O}(h, \lambda_0) \\
        \frac{2n-1}{2\pi}, \quad \mu \text{ between } B^h_{\mathbb{O}, \lambda, n-1} \text{ and } B^h_{\mathbb{O}, \lambda, n} \text{ with } -N_\mathbb{O}(h, \lambda_0) \leq n \leq 0. 
    \end{cases}
\end{align*}
\] (1.1)

We then show that the discrete magnetic random Schrödinger operators undergo metal/insulator transitions, using the framework of Germinet-Klein [GK01] and Klein-Germinet-Schenker [GKS04]. More precisely, we prove dynamical localization away from the Landau levels and the existence of (at least one) mobility edge at the Landau levels.

Finally, we show that on the honeycomb lattice there exist Dirac cones for all rational magnetic fields.

Theorem 2. For any flux \( \phi = \frac{2\pi p}{q} \in 2\pi \mathbb{Q} \), the operator \( H^\phi_\mathbb{O} \) possesses Dirac points at energy zero.

With this observation we study the quantum Hall effect and the existence of mobility edges for magnetic perturbations of any rational flux, too.

As an immediate consequence, our analysis shows that Simon’s 2nd problem (localization throughout the spectrum in two dimensions for the Anderson model) [S00] is unstable under arbitrary small constant magnetic perturbations if the disorder is suitably small, too.

The discrete models, studied in this article, are the semiclassical limit of Schrödinger operators in continuous space [HKL16, K95, FLW16], see also [FW12, DFW18, D18, D19] for related results. It would be interesting to study transport properties directly for continuous magnetic Schrödinger operator with periodic electric potential under disorder as well.

Finally, we hope to be able to extend this study of metal/insulator transitions to many-body systems using recent advances on the quantum Hall effect for many-body systems [GMP12, GMP18, BBDF18, HM15]. A thorough understanding of one-body systems and the proof of the existence of spectral gaps are a likely prerequisite to obtain a similar result for many-body systems.
Acknowledgements. The first author is supported by the UK Engineering and Physical Sciences Research Council (EPSRC) grant EP/L016516/1 for the University of Cambridge Centre for Doctoral Training. The second author is partially supported by NSF-DMS-1800689. The Cambridge Centre for Analysis is gratefully acknowledged (S.B). The first author is grateful to Gian-Michele Graf for bringing the Streda formula to his attention. Helpful remarks and discussions with Svetlana Jitomirskaya, Hermann Schulz-Baldes and Maciej Zworski are gratefully acknowledged as well.

Notation. \( B_\ell (r) \) is the ball of radius \( r \) centred at \( x \). We write \( f_\alpha = \mathcal{O}_\alpha (g)_H \) for \( \| f \|_H \leq C_\alpha g \) and \( f = \mathcal{O}(h^\infty)_H \) means that for any \( N \) there exists \( C_N \) such that \( \| f \|_H \leq C_N h^N \). We write \( \langle x \rangle := \sqrt{1 + |x|^2} \). \( \mathcal{U} (\mathcal{H}) \) are the unitary operators on a Hilbert space \( \mathcal{H} \). The symbol class \( \mathcal{S}_{ho} \), of possibly matrix-valued symbols, is defined as

\[
\mathcal{S}_{ho} := \{ a(\cdot, h) \in C^\infty (T^* \mathbb{R}) : \forall \alpha \in \mathbb{N}_0^2 \exists C_\alpha > 0 \ \forall h \in [0, h_0] : \ |\partial^{\alpha} a(\cdot, h)| \leq C_\alpha \}.
\]

We write \( a \sim \sum_{j=0}^\infty a_j h^j \) to denote an asymptotic expansion of symbols, cf. [Zw12, 4.4.2] where \( a_j \in \mathcal{S} \), with

\[
\mathcal{S} := \{ a \in C^\infty (T^* \mathbb{R}) ; \forall \alpha \in \mathbb{N}_0^2 \exists C_\alpha > 0 : |\partial^{\alpha} a| \leq C_\alpha \}
\]

and denote the class of symbols allowing such an expansion by \( \mathcal{S}^3 \). The standard basis vectors of \( \ell^2 (\mathbb{Z}^2) \) are for \( \gamma \in \mathbb{Z}^2 \) denoted by \( \delta_\gamma := (\delta_{\gamma, \gamma'})_{\gamma'} \) and occasionally by \( \vec{e}_i \) if the Hilbert space is finite-dimensional. \( \mathcal{L} (X, Y) \) are the bounded linear operators between normed spaces \( X, Y \). \( \mathbb{E} \) and \( \text{Var} \) denote expectation and variance. The semiclassical Weyl quantization of a symbol \( a \in \mathcal{S}_h (T^* \mathbb{R}) \) is for suitable functions \( u \) defined as

\[
\left( \text{Op}_h^w (a) u \right)(x) := \left( a^w (x, hp_x, h) u \right)(x) := \frac{1}{2\pi h} \int_\mathbb{R} \int_\mathbb{R} e^{\frac{i}{\hbar} \langle x-y, \xi \rangle} a \left( \frac{x+y}{2}, \xi, h \right) u(y) \ dy \ d\xi.
\]

Here, \( p_x := -i \frac{d}{dx} \). Conversely, we write \( \sigma \left( \text{Op}_h^w (a) \right) := a \) to denote the Weyl symbol of a \( \Psi DO \) and \( \sigma_0 \left( \text{Op}_h^w (a) \right) \) for the principal symbol. Analogously, higher order symbols are denoted by \( \sigma_k \), respectively. The semiclassical wavefront set is denoted by \( \text{WF}_h \), see [Zw12, Sec.8.4]. We also write \( \mathbb{Z}^2_+ := (2\pi \mathbb{Z})^2 \). For a subset \( I \subset \mathbb{R} \) we denote by \( \oint_I \) a contour integral over a path in the complex plane that encloses \( I \) sufficiently close. The meaning of sufficiently close will be obvious from context.

The spectrum of an operator \( T \) is denoted by \( \Sigma (T) \). We sometimes use the convention \( h := \frac{\hbar}{2\pi} \) where \( \hbar \) is the magnetic flux (thus this notation should not be confused with Planck’s constant). The \( p \)-th Schatten class is denoted by \( \mathcal{L}^p \). The symplectic form on \( \mathbb{R}^2 \) is denoted by \( \sigma_{\text{symp}} (\gamma, \delta) := \gamma_1 \delta_2 - \delta_1 \gamma_2 \). Finally, we use Wirtinger derivatives

\[
D_\xi := \frac{1}{2} (p_x - i \partial_y) \quad \text{and} \quad D_\eta := \frac{1}{2} (p_x + i \partial_y)
\]

where we recall that \( D_\xi f \) is nothing but the derivative of a holomorphic function \( f \). In particular, holomorphic functions satisfy \( D_\xi f = 0 \) by the Cauchy-Riemann equations. \( \mathcal{S} (\mathbb{R}^n) \) are the sequences that decay faster than any polynomial power. We also write \( \mathcal{S} (\mathbb{R}^n) \) or \( \mathcal{S} (\mathbb{C}^n) \) for the Schwartz
functions on $\mathbb{R}^n$ or $\mathbb{C}^n$. We also define for one of the two lattices $\Lambda$ we study in this article, the truncated sets

$$\Lambda_L := \left\{ y \in \mathbb{R}^2; y = \gamma_1 \vec{b}_1 + \gamma_2 \vec{b}_2 + [y] + \text{for } \gamma \in \{-L, \ldots, L\}^2 \text{ and } [y] \in W_\Lambda \right\}$$

where $\vec{b}_1$ and $\vec{b}_2$ are the basis vectors of the lattice and $W_\Lambda$ a fundamental domain.

2. LATTICES AND DISCRETE RANDOM SCHROEDINGER OPERATORS

2.1. Geometry of lattices. The $\mathbb{Z}^2$ lattice $\square$, see Fig. 2a. The square lattice $\Lambda_\square := \mathbb{Z}^2$ is spanned by basis vectors $\vec{b}_\square,1 := (1, 0), \vec{b}_\square,2 := (0, 1)$ and its fundamental cell $W_{\Lambda_\square}$ consists of just the vertex $r_0 := (0, 0)$. Although we do not study operators on the associated graph, we also introduce the set of edges $E_\square$ on the square graph consisting of the two edges

$$\vec{f}_\uparrow := \text{conv} \left( \{ r_0, (1, 0) \} \right) \setminus \{ r_0, (1, 0) \},$$

$$\vec{f}_\rightarrow := \text{conv} \left( \{ r_0, (0, 1) \} \right) \setminus \{ r_0, (0, 1) \}$$

and translations thereof by basis vectors $\vec{b}_\square,1, \vec{b}_\square,2$. To orient the graph, we also define a map $i : E_\square \to \Lambda_\square$ by $i(\vec{f}_\uparrow) := i(\vec{f}_\rightarrow) := r_0$ and extend it to all edges by translation

$$i(\vec{f}_\uparrow + \gamma) = i(\vec{f}_\rightarrow + \gamma) = r_0 + \gamma \text{ for } \gamma \in \mathbb{Z}^2.$$

Let us now turn to the hexagonal lattice:

The hexagonal lattice $\odot$, see Fig. 2b. The hexagonal lattice $\Lambda_\odot$ is obtained by
translating its fundamental cell $W_{\Lambda^\circ}$, consisting of vertices

$$r_0 := (0, 0), \quad r_1 := \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

along the basis vectors of the lattice. The basis vectors are

$$\vec{b}_{\circ,1} := \left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right) \quad \text{and} \quad \vec{b}_{\circ,2} := \left(0, \sqrt{3}\right).$$

As in the case of the $\mathbb{Z}^2$ lattice, we also introduce auxiliary edges

$$\vec{f} := \text{conv} (\{r_0, r_1\} \setminus \{r_0, r_1\},$$

$$\vec{g} := \text{conv} (\{r_0, (-1, 0)\} \setminus \{r_0, (0, -1)\},$$

$$\vec{h} := \text{conv} \left(\{r_0, \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)\}\setminus \{r_0, \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)\},$$

and define the set of all edges $\mathcal{E}_{\circ}$ as the set of all translates of these three edges along the basis vectors $\vec{b}_{\circ,1}, \vec{b}_{\circ,2}$ of the hexagonal lattice.

We call translates of $r_0$ by basis vectors $\vec{b}_{\circ,1}, \vec{b}_{\circ,2}$ initial vertices $\Lambda_i^\circ$ whereas translates of $r_1$ will be referred to as terminal vertices $\Lambda_t^\circ$. Moreover, we consider maps $i : \mathcal{E}_{\circ} \to \Lambda^\circ$ and $t : \mathcal{E}_{\circ} \to \Lambda^\circ$ that map edges to the respective initial or terminal vertex they contain.

In the sequel, we will use the isomorphism $\ell^2(\Lambda^\circ) \simeq \ell^2(\mathbb{Z}^2; \mathbb{C}^2)$ as the honeycomb has two basis vectors and two vertices in its fundamental domain. More generally, any lattice with $\Lambda$ spanned by two basis vectors with $n$ vertices in its fundamental domain satisfies $\ell^2(\Lambda) \simeq \ell^2(\mathbb{Z}^2; \mathbb{C}^n)$.

2.2. Discrete random Schrödinger operators. We consider a constant magnetic field. The vector potential $A$ is a one form on $\mathbb{R}^2$ and the magnetic field is given by $B = dA$. For homogeneous magnetic fields

$$B := B \, dx_1 \wedge dx_2$$

we can choose a symmetric gauge for the vector potential $A$ such that

$$B = dA, \quad A = \frac{1}{2} B (-x_2 \, dx_1 + x_1 \, dx_2).$$

The discrete magnetic Laplacians (DMLs) with single-site disorder are then defined as follows: First, we take the scalar potential $A_{\vec{e}} \in C^\infty(\vec{e})$ along edges $\vec{e} = e_1 \, dx_1^* + e_2 \, dx_2^*$ of the respective graph, where $dx_j(dx_i^*) = \delta_{ij}$ is defined by evaluating the 1-form on the graph along the vector field generated by the respective edge $\vec{e}$:

$$A_{\vec{e}}(t) := A \left( i(\vec{e}) + t\vec{e} \right) (e_1 \, dx_1^* + e_2 \, dx_2^*) = A \left( i(\vec{e}) \right) (e_1 \, dx_1^* + e_2 \, dx_2^*).$$

The quantities $A_{\vec{e}}$ on the square lattice are given by

$$A_{\vec{f}^* + \gamma_1 \vec{b}_{\bullet,1}^* + \gamma_2 \vec{b}_{\bullet,2}^*} = \gamma_1 \gamma_2 \text{ and } A_{\vec{f}^* + \gamma_1 \vec{b}_{\bullet,1}^* + \gamma_2 \vec{b}_{\bullet,2}^*} = -\gamma_1 \gamma_2$$

where $\gamma_1$ and $\gamma_2$ are the magnetic fluxes through the respective edges.
and the quantities $A_\mathcal{O}$ on the hexagonal lattice are explicitly given by

\begin{align}
A_{\mathcal{O}}^{f+\gamma_1 b_{\mathcal{O},1}+\gamma_2 b_{\mathcal{O},2}} &= \frac{\hbar \omega}{6}(\gamma_1 - \gamma_2), \\
A_{\mathcal{O}}^{g+\gamma_1 b_{\mathcal{O},1}+\gamma_2 b_{\mathcal{O},2}} &= \frac{\hbar \omega}{6}(\gamma_1 + 2\gamma_2), \text{ and} \\
A_{\mathcal{O}}^{h+\gamma_1 b_{\mathcal{O},1}+\gamma_2 b_{\mathcal{O},2}} &= -\frac{\hbar \omega}{6}(2\gamma_1 + \gamma_2)
\end{align}

(2.9)

where the magnetic flux for either lattice is defined as

$$h_{\mathcal{O}} := B$$ and $$h_{\mathcal{O}} := \frac{B}{|b_1 \wedge b_2|} = \frac{3\sqrt{3}}{2}B.$$ (2.10)

From this point on, we may suppress the dependence on the lattices in some notations if there is no ambiguity or if the results hold for both lattices.

We now define the discrete magnetic random Schrödinger operators:

**Definition 2.1** (Discrete magnetic Schrödinger operators). We define discrete magnetic random Schrödinger operators $H_{\mathcal{O}}^h \in \mathcal{L}(\ell^2(\Lambda_{\mathcal{O}}))$ and $H_{\mathcal{O}}^h \in \mathcal{L}(\ell^2(\Lambda_{\mathcal{O}}))$ on the square $\mathcal{O}$, using (2.10), and hexagonal $\mathcal{O}$ lattice, using (2.9), respectively

\begin{align}
(H_{\mathcal{O},\lambda,\omega}^h u)(\gamma) &= \frac{1}{4} \left( e^{ih\gamma_1/2}u(\gamma + \vec{b}_1) + e^{-ih\gamma_2/2}u(\gamma - \vec{b}_1) \\
&+ e^{-ih\gamma_1/2}u(\gamma + \vec{b}_2) + e^{ih\gamma_2/2}u(\gamma - \vec{b}_2) \right) + \lambda V_\omega(\gamma)u(\gamma) \\
(H_{\mathcal{O},\lambda,\omega}^h u)(v) &= \frac{1}{3} \left( \sum_{\vec{e} \in \mathcal{E}_{\mathcal{O}}, i(\vec{e}) = v} e^{-iA_\mathcal{O}u(t(\vec{e}))} + \sum_{\vec{e} \in \mathcal{E}_{\mathcal{O}}, i(\vec{e}) = v} e^{iA_\mathcal{O}u(i(\vec{e}))} \right) + \lambda V_\omega(v)u(v)
\end{align}

(2.11)

where the parameter $\lambda > 0$ measures the disorder strength. The random potential satisfies $V_\omega(v) = \omega(v)$, where $\{\omega(v)\}_{v \in \Lambda}$ is a family of i.i.d with common probability distribution $\nu$ of compact support on $\mathbb{R}$. We write $(\Omega, \mathbb{P})$ the underlying probability space, and $\mathbb{E}$ the expectation.

We will write $(\Omega, \mathbb{P})$ for the underlying probability space, hence $\Omega = \times_{v \in \Lambda} \mathbb{R}$, and $\mathbb{P} = \times_{v \in \Lambda} \nu$. We define the shifts operators $\{T_\delta^\Omega\}_{\delta \in \mathbb{Z}^2}$ on $\Omega$ by

$$T_\delta^\Omega \omega(v) = \omega(v - \delta_1 \vec{b}_1 - \delta_2 \vec{b}_2).$$ (2.12)

The sample space $\Omega$ of the configuration space of impurities $(\Omega, \mathbb{P})$ is, without loss of generality, assumed to be compact, cf. [C94, p. 372f.] for details.

We then write $H^h := H_{\lambda=0,\omega}^h$ for the non-random DML.

2.3. Magnetic translations, regularized traces, and the density of states measure. We start our analysis by introducing discrete translation operators $T_\gamma$ with $\gamma \in \mathbb{Z}^2$ for $\psi \in \ell^2(\Lambda)$

$$T_\gamma \psi(v) := \psi(v - \gamma_1 \vec{b}_1 - \gamma_2 \vec{b}_2).$$ (2.13)
The magnetic Schrödinger operator $H^h$ does, in general, not commute with standard lattice translations $T_\gamma$, but with magnetic translations $T^h_\gamma$ instead. These operators and powers of them, do not commute with each other, if $T^h_{(0,1)}$ and $T^h_{(1,0)}$ generate the irrational ($h \in \mathbb{R}\setminus\mathbb{Q}$) rotation algebra. Magnetic translations $T^h_\gamma : \ell^2(\Lambda) \rightarrow \ell^2(\Lambda)$ are unitary operators of the form

$$T^h_\gamma \psi := u^h(\gamma) T_\gamma \psi, \quad \psi = (\psi_v)_{v \in \Lambda} \in \ell^2(\Lambda), \quad |u^h(\gamma)| = 1, \quad \gamma \in \mathbb{Z}^2$$

that satisfy the commutation relation

$$T^h_\gamma T^h_\delta = e^{ih\sigma_{\text{sym}}(\gamma,\delta)} T^h_\delta T^h_\gamma.$$  

(2.15)

On the square lattice we define magnetic translations as

$$(T^h_{(1,0)} u)(\gamma) = e^{-ih/2\gamma_2} u(\gamma - \vec{b}_1) \quad \text{and} \quad (T^h_{(0,1)} u)(\gamma) = e^{ih/2\gamma_1} u(\gamma - \vec{b}_2)$$

(2.16)

and set then $T^h_\gamma := (T^h_{(1,0)})^{\gamma_1} (T^h_{(0,1)})^{\gamma_2}$.

On the hexagonal lattice, the magnetic translations $T^h_\gamma : \ell^2(\Lambda_\triangle) \rightarrow \ell^2(\Lambda_\triangle)$ are unitary operators of the above form (2.14) with prefactors $(u^h(\gamma)_v)_{v \in \Lambda_\triangle}$ defined as follows: Let $\alpha(\gamma) = \frac{h}{6}(\gamma_1 - \gamma_2)$, then we can define $u^B(\gamma)_v = e^{i\frac{h}{2}\sigma_{\text{sym}}(\gamma,\delta)} u^B(\gamma)_v$ with $\star \in \{0, 1\}$ where $u^B(\gamma)_r = 1$ and $u^B(\gamma)_r = e^{i\alpha(\gamma)}$. This way, the magnetic translations on both lattices satisfy

$$H^h_{\Lambda,\omega} T^h_\gamma = T^h_\gamma H^h_{\Lambda,T^\omega_\gamma,\omega}.$$  

(2.17)

The functional calculus implies that for measurable $f : \mathbb{R} \rightarrow \mathbb{R}$

$$f(H^h_{\Lambda,\omega}) T^h_\gamma = T^h_\gamma f(H^h_{\Lambda,T^\omega_\gamma,\omega})$$

(2.18)
such that for the Schwartz kernels $f(H_{\lambda,\omega}^h)[x, y] := \langle \delta_x, f(H_{\lambda,\omega}^h)\delta_y \rangle$ on the diagonal

$$f(H_{\lambda,\omega}^h)[x, x] = f(H_{\lambda,\omega}^h)[x - \gamma_1 \vec{b}_1 - \gamma_2 \vec{b}_2, x - \gamma_1 \vec{b}_1 - \gamma_2 \vec{b}_2].$$

To study the density of states (DOS) of the model, we define, for a lattice $\Gamma \subset \mathbb{R}^2$ and operators $A \in \mathcal{L}(\ell^2(\Gamma, \mathbb{C}^n))$ given by $A(s)(\gamma) := \sum_{\beta \in \Gamma} A[\gamma, \beta]s(\beta)$ with possibly matrix-valued kernel $A[\gamma, \beta] \in \mathbb{C}^n \times n$, the regularized trace

$$\tilde{\operatorname{tr}}_{\Gamma}(A) := \lim_{r \to \infty} \frac{1}{|B_0(r)|} \sum_{\gamma \in \Gamma \cap B_0(r)} \operatorname{tr}_{\mathbb{C}^n} A[\gamma, \gamma]$$

provided the limit exists.

Birkhoff’s ergodic theorem implies the a.s. existence of the regularized trace

$$\tilde{\operatorname{tr}}_{\Lambda}(f(H_{\lambda,\omega}^h)) = \mathbb{E} \left( \frac{\sum_{x \in W_{\Lambda}} f(H_{\lambda,\omega}^h)[x, x]}{|b_1 \wedge b_2|} \right) = \frac{\operatorname{tr} \int_{W_{\Lambda}} f(H_{\lambda,\omega}^h)}{|b_1 \wedge b_2|},$$

where $|b_1 \wedge b_2|^{-1}$ normalizes the number of vertices per unit volume. By Riesz’s theorem one can then associate to the regularized trace a Radon measure $\rho_{H_{\lambda,\omega}^h}$, the DOS measure, and by the preceding discussion, this measure is a.s. non-random. Thus $\rho_{H_{\lambda,\omega}^h} =: \rho_{H_{\lambda}^h}$ a.s. and therefore $\int_{\mathbb{R}} f(x) \ d\rho_{H_{\lambda}^h}(x) = \tilde{\operatorname{tr}}_{\Lambda}(f(H_{\lambda,\omega}^h))$ a.s..
DYNAMICAL DELocalization FOR DIScrete SCHRöDINGER OPERATORS. 11

3. THE SEMICLASSICAL EXPANSION OF THE DOS

We study the DOS by investigating operators \( f(H^h_{\lambda,\omega}) \) using the functional calculus of Helffer–Sjöstrand [HS88]. We first recall that any function \( f \in C_c^\infty(\mathbb{R}) \) can be extended to functions \( \tilde{f} \in \mathcal{S}(\mathbb{C}) \) such that \( \tilde{f}|_{\mathbb{R}} = f \) and \( D_\omega \tilde{f} = \mathcal{O}(|\text{Im } z|^\infty) \). Such functions \( \tilde{f} \) are then called \textit{almost analytic} extensions of \( f \). One possible way of defining \( \tilde{f} \) is by

\[
\tilde{f}(x + iy) = \frac{1}{2\pi} \chi(y)\psi(x) \int_{\mathbb{R}} \chi(y \xi) \tilde{f}(\xi) e^{ixy + iy} d\xi,
\]

where \( \chi, \psi \in C_c^\infty(\mathbb{R}) \), \( \psi|_{\text{supp } f + (-1,1)} = 1 \), \( \chi|_{(-1,1)} = 1 \). ([DS99, see Chapter 8] for details. A more pedestrian, but also more restrictive, way of defining almost-analytic extensions, for smooth functions \( f \in C_c^\infty(\mathbb{R}) \), is for \( n \in \mathbb{N} \) by

\[
\tilde{f}(x + iy) = \left( \sum_{r=0}^{n} f^{(r)}(x) \frac{(iy)^r}{r!} \right) \zeta(x + iy)
\]

where \( \zeta(x + iy) := \chi(y/|x|) \), \( \chi \in C_c^\infty \), \( \chi|_{[-1,1]} = 1 \), \( \text{supp}(\chi) \subseteq [-2,2] \).

Differentiating (3.2), one finds that \( |D_\omega \tilde{f}(z)| = \mathcal{O}(|\text{Im } z|^n) \) which follows from

\[
D_\omega \tilde{f}(x + iy) = \sum_{r=0}^{n} f^{(r)}(x) \frac{(iy)^r}{r!} D_\omega \zeta(x + iy) + f^{(n+1)}(x) \frac{(iy)^n \zeta(x + iy)}{n!}.
\]

A similar computation shows that the quasi-analytic extension satisfies

\[
|D_\omega \tilde{f}(k)(z)| = \mathcal{O}(|\text{Im } z|^{n-k}).
\]

The almost-analytic extension enters then in the Helffer-Sjöstrand formula which states that for any self-adjoint operator \( P \),

\[
f(P) = \frac{1}{\pi} \int_{\mathbb{C}} D_\omega \tilde{f}(z)(P - z)^{-1} dm(z)
\]

where \( m \) is the Lebesgue measure on \( \mathbb{C} \). For discrete random Schrödinger operators (2.11) this yields by applying the regularized trace

\[
\tilde{\text{tr}}_\Lambda(f(H^h_{\lambda,\omega})) = \frac{1}{\pi} \int_{\mathbb{C}} D_\omega \tilde{f}(z)\tilde{\text{tr}}_\Lambda((H^h_{\lambda,\omega} - z)^{-1}) \ dm(z).
\]

3.1. Magnetic matrices.

**Definition 3.1** (Magnetic matrices). Let \( f_\omega(\gamma) \in C_c(\Omega \times \mathbb{Z}^2; \mathbb{C}^{n \times n}) \) at first, where \( \omega \in \Omega \) and \( \gamma \in \mathbb{Z}^2 \). We define magnetic matrices as discrete operators

\[
A^h(f_\omega) \in \mathcal{L}(\ell^2(\mathbb{Z}^2; \mathbb{C}^{n \times n})), \quad A^h(f_\omega) := \left( e^{-i\frac{1}{2} \sigma_{\text{sympl}}(\gamma, \delta)} f_{\Omega^h}(\gamma - \delta) \right)_{\gamma, \delta \in \mathbb{Z}^2}.
\]

These matrices act on $\ell^2(\mathbb{Z}^2; \mathbb{C}^n)$ by matrix-like multiplication

$$\left( A^h (f_\omega) u \right)_\gamma = \sum_{\delta \in \mathbb{Z}^2} \left( A^h (f_\omega) \right)_{\gamma, \delta} u_\delta. \quad (3.8)$$

For yet another set of discrete magnetic translation operators $\tau^h_\gamma$ on the $\mathbb{Z}^2$-lattice

$$\tau^h_\delta (f_\omega)(\gamma) := e^{-i \frac{\hbar}{2} \sigma_{\text{sym}}(\gamma, \delta)} f_{T^h_\gamma \omega}(\gamma - \delta), \quad (3.9)$$

we find, in analogy to (2.17), that magnetic matrices are covariant with respect to discrete magnetic translations (3.9)

$$\tau^h_\gamma A^h (f_\omega) = A^h (f_\omega) \tau^h_\gamma. \quad (3.10)$$

Moreover, translations (3.9) satisfy the Weyl commutation relations

$$\tau^h_\gamma \tau^h_\delta = e^{i \hbar \sigma_{\text{sym}}(\gamma, \delta)} \tau^h_\delta \tau^h_\gamma. \quad (3.11)$$

For $f, g \in C_c(\Omega \times \mathbb{Z}^2; \mathbb{C}^{n \times n})$ we introduce the product

$$\left( (f \#_h g) \right)_\omega(\gamma) := \sum_{z \in \mathbb{Z}^2} f_\omega(\gamma - z) g_{T^h_\gamma \omega z}(z) e^{-i \frac{\hbar}{2} \sigma_{\text{sym}}(\gamma, z)}$$

$$= \sum_{z \in \mathbb{Z}^2} f_\omega(z) g_{T^h_{-\gamma} \omega}(\gamma - z) e^{-i \frac{\hbar}{2} \sigma_{\text{sym}}(\gamma, z)}. \quad (3.12)$$

This product is reconcilable with multiplication of magnetic matrices

$$A^h (f \#_h g)_\omega u(\xi) = A^h (f_\omega)(A^h (g_\omega)(u))(\xi). \quad (3.13)$$

Moreover, defining the involution

$$f^*_\omega(\gamma) := f_{T^h_{-\gamma} \omega}(\gamma) \quad (3.14)$$

we see that the adjoint of a magnetic matrix is again given by a magnetic matrix

$$\langle A^h (f_\omega)(g), h \rangle = \langle g, A^h (f^*_\omega)(h) \rangle. \quad (3.15)$$

**Remark 1.** The preceding computations show that magnetic matrices are the $*$-representation of a C*-algebra $C_h$ which is the closure of functions $f \in C_c(\Omega \times \mathbb{Z}^2; \mathbb{C}^{n \times n})$ with composition (3.12) and involution (3.14) under the norm $\| f \| := \sup_{\omega \in \Omega} \| A^h (f) \|$. This defines a continuous field (as a function of $h$) of C*-algebra $C_h$, cf. [BES94, Sec. F], [ST12].
To connect operators $H^h_{\lambda,\omega}$ with magnetic matrices, we define symbols

$$a_{\bullet}(1,0) = a_{\bullet}(0,1) = a_{\bullet}(-1,0) = a_{\bullet}(0,-1) = \frac{1}{4},$$

and for the hexagonal lattice

$$a_\circ(0,0) := \frac{1}{3} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad a_\circ(1,0) := a_\circ(0,1) := \frac{1}{3} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$a_\circ(-1,0) := a_\circ(0,-1) := \frac{1}{3} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and $a(\eta) = 0$ otherwise. The random symbols are then defined as $a_{\lambda,\omega,\bullet}(\gamma) = a_{\bullet}(\gamma) + \lambda \delta_0(\gamma)V_\omega(0)$ or $a_{\lambda,\omega,\circ}(\gamma) = a_\circ(\gamma) + \lambda \delta_0(\gamma) \text{diag}(V_\omega(r_0), V_\omega(r_1))$.

**Lemma 3.2.** There exist unitary multiplication operators $U_{\bullet} : \ell^2(\mathbb{Z}^2; \mathbb{C}) \to \ell^2(\mathbb{Z}^2; \mathbb{C})$ and $U_\circ : \ell^2(\mathbb{Z}^2; \mathbb{C}^2) \to \ell^2(\mathbb{Z}^2; \mathbb{C}^2)$ such that

$$H^h_{\lambda,\omega,\bullet} = U_{\bullet} A^h(a_{\lambda,\omega,\bullet}) U_{\bullet}^*$$

and

$$H^h_{\lambda,\omega,\circ} = U_\circ A^h(a_{\lambda,\omega,\circ}) U_\circ^*.$$  \hspace{1cm} (3.17)

In particular, since operators $U$ are multiplication operators, we find

$$\tilde{\tau}_\lambda \left((H^h_{\lambda,\omega} - z)^{-1}\right) = |\bar{b}_1 \wedge \bar{b}_2|^{-1} \tau_{\mathbb{Z}^2} \left((A^h(a_{\lambda,\omega}) - z)^{-1}\right).$$ \hspace{1cm} (3.18)

**Proof.** The first equivalence on the $\mathbb{Z}^2$ lattice in (3.17) is obtained by first passing from the symmetric to the Landau gauge and then conjugating this operator by $W u(\gamma) := e^{-i \frac{\gamma_1}{2} \gamma_2 u(\gamma)}$. For the hexagonal lattice, the transformation is slightly more involved. We start by defining two unitary maps: The first one is $U_1 z := (\zeta_v z(v))_{v \in V(\Lambda_\circ)}$ with recursively defined factors

$$\zeta_{r_0} := 1, \quad \zeta_{\gamma_1 \bar{b}_1 + \gamma_2 \bar{b}_2 + r_1} := e^{i A_{\gamma_1 \bar{b}_1 + \gamma_2 \bar{b}_2 + r_1} + f} \zeta_{\gamma_1 \bar{b}_1 + \gamma_2 \bar{b}_2 + r_0}$$

$$\zeta_{(\gamma_1 + 1) \bar{b}_1 + \gamma_2 \bar{b}_2 + r_0} := e^{i \left(-A_{(\gamma_1 + 1) \bar{b}_1 + \gamma_2 \bar{b}_2 + r_0} + A_{\gamma_1 \bar{b}_1 + \gamma_2 \bar{b}_2 + r_0} + f\right)} \zeta_{\gamma_1 \bar{b}_1 + \gamma_2 \bar{b}_2 + r_0}$$

and

$$\zeta_{\gamma_1 \bar{b}_1 + (\gamma_2 + 1) \bar{b}_2 + r_0} := e^{i \left(-A_{\gamma_1 \bar{b}_1 + (\gamma_2 + 1) \bar{b}_2 + r_0} + A_{\gamma_1 \bar{b}_1 + \gamma_2 \bar{b}_2 + r_0} + f\right)} \zeta_{\gamma_1 \bar{b}_1 + \gamma_2 \bar{b}_2 + r_0}$$

and $U_2 : \ell^2(V(\Lambda_\circ)) \to \ell^2(\mathbb{Z}^2; \mathbb{C}^2)$, $U_2(z) (\gamma) := \left( z(r_0 + \gamma_1 \bar{b}_1 + \gamma_2 \bar{b}_2), z(r_1 + \gamma_1 \bar{b}_1 + \gamma_2 \bar{b}_2) \right)^T$.

The unitary transform is then $A^h(a_{\lambda,\omega,\circ}) = (U_1 U_2 W^*) H^h_{\lambda,\omega,\circ} (U_1 U_2 W^*)$, see also [BZ19, Lemma 3.3, 3.5].

**3.2. Reduction of DOS.** We now continue with the derivation of the DOS. For this, we consider a $\Psi$DO representation of (non-random) magnetic matrices. To start, we observe the following expansion of the regularized trace of the resolvent of the random operators in terms of the deterministic one. Recall that we write $H^h := H^h_{\lambda=0,\omega}$ for the non-random DML.
Lemma 3.3. The resolvent of the discrete random Schrödinger operator $H^h_{\lambda,\omega}$ satisfies

$$
\tilde{\text{tr}}_A \left( (H^h_{\lambda,\omega} - z)^{-1} \right) = \sum_{k=0}^{2} \frac{(-\lambda \mathbb{E}(V) D_z)^k}{k!} \tilde{\text{tr}}_A \left( (H^h - z)^{-1} \right) + \frac{\lambda^2}{2} \text{Var}(V) D_z \sum_{r \in W_A} \left( \text{tr} \left( \mathbb{1}(r) (H^h - z)^{-1} \right) \right)^2 + O \left( \lambda^3 \| (H^h - z)^{-1} \|^3 \| (H^h_{\lambda,\omega} - z)^{-1} \| \right).
$$

(3.20)

Proof. The resolvent identity then yields a second-order approximation in the disorder parameter $\lambda$.

$$(H^h_{\lambda,\omega} - z)^{-1} = (H^h - z)^{-1} - \lambda (H^h - z)^{-1} V_\omega (H^h - z)^{-1} + \lambda^2 (H^h - z)^{-1} V_\omega (H^h - z)^{-1} V_\omega (H^h - z)^{-1} + O \left( \lambda^3 \| (H^h - z)^{-1} \|^3 \| (H^h_{\lambda,\omega} - z)^{-1} \| \right).$$

(3.21)

We study second-order approximations in $\lambda$ since this is the leading-order level at which the random nature of the perturbation enters.\footnote{\label{footnote}The mathematical difficulty, that arises from second-order contributions on, is to separate stochastically independent and dependent potentials from each other and to analyze them individually.}

Taking regularized traces in (3.21) yields

$$
\tilde{\text{tr}}_A \left( (H^h_{\lambda,\omega} - z)^{-1} \right) = (1 - \lambda \mathbb{E}(V) D_z) \tilde{\text{tr}}_A \left( (H^h - z)^{-1} \right) + \lambda^2 \tilde{\text{tr}}_A \left( (H^h - z)^{-1} V_\omega (H^h - z)^{-1} V_\omega (H^h - z)^{-1} \right) + O \left( \lambda^3 \| (H^h - z)^{-1} \|^3 \| (H^h_{\lambda,\omega} - z)^{-1} \| \right).
$$

(3.22)

Interchanging derivatives and regularized traces is easily justified by (2.21). Equation (3.22) can be rewritten, by separating (independent) potentials on different vertices from the squares of potentials such that

$$
\tilde{\text{tr}}_A \left( (H^h - z)^{-1} V_\omega (H^h - z)^{-1} V_\omega (H^h - z)^{-1} \right) = |\vec{b}_1 \wedge \vec{b}_2|^{-1} \mathbb{E} \text{ tr} \left( \mathbb{1}_{W_A} (H^h - z)^{-1} V_\omega (H^h - z)^{-1} V_\omega (H^h - z)^{-1} \right) = |\vec{b}_1 \wedge \vec{b}_2|^{-1} \mathbb{E}(V)^2 \text{ tr} \left( \mathbb{1}_{W_A} (H^h - z)^{-3} \right) + |\vec{b}_1 \wedge \vec{b}_2|^{-1} \text{Var}(V) \sum_{r \in W_A} \text{ tr} \left( \mathbb{1}(r) (H^h - z)^{-2} \right) \text{ tr} \left( \mathbb{1}(r) (H^h - z)^{-1} \right).
$$

(3.23)
Here, we used since \((H^h - z)^{-1}[\gamma, \gamma] = (H^h - z)^{-1}[T_{\gamma} \gamma, T_{\nu} \gamma]\), cf. (2.14) and (2.19)

\[
\sum_{x_1, x_2 \in W_A, \gamma \in \mathbb{Z}} (H^h - z)^{-1}[x_1, T_{-\gamma} x_2](H^h - z)^{-1}[T_{-\gamma} x_2, T_{-\gamma} x_2](H^h - z)^{-1}[T_{-\gamma} x_2, x_1] \\
= \sum_{x_1, x_2 \in W_A, \gamma \in \mathbb{Z}} (H^h - z)^{-1}[T_{\gamma} x_1, x_2](H^h - z)^{-1}[x_2, x_2](H^h - z)^{-1}[x_2, T_{\gamma} x_1] \\
= \sum_{r \in W_A, v \in A} (H^h - z)^{-1}[r, r](H^h - z)^{-1}[r, v](H^h - z)^{-1}[v, r] \\
= \sum_{r \in W_A} \text{tr} \left( \mathbb{I}_r (H^h - z)^{-2} \right) \text{tr} \left( \mathbb{I}_r (H^h - z)^{-1} \right).
\]

(3.24)

Inserting this into (3.22) yields (3.23). □

We now continue expressing the regularized traces of discrete Schrödinger operators in terms of pseudodifferential operators. For vectors \(\vec{e}_1 := (1, 0)\) and \(\vec{e}_2 := (0, 1)\), the identity (3.11) reduces to

\[
\tau^{-h}_{\vec{e}_1} \tau^{-h}_{\vec{e}_2} = e^{-ih} \tau^{-h}_{\vec{e}_2} \tau^{-h}_{\vec{e}_1}.
\]

(3.25)

This is a version of the canonical commutation relation. In semiclassical Weyl quantization, the same commutation relation is satisfied by

\[
\text{Op}_h^w (e^{ix}) \text{Op}_h^w (e^{i\xi}) = e^{-ih} \text{Op}_h^w (e^{i\xi}) \text{Op}_h^w (e^{ix}).
\]

(3.26)

Rather than analyzing directly the discrete operators \(H^h := H_{\lambda=0, \omega}^h\) or \(A^h(a) := A^h(a_{\lambda=0, \omega})\), we use a pseudodifferential representation that we obtain from the following ∗-homomorphism \(\Theta : \mathcal{S}(\mathbb{Z}^2; \mathbb{C}^{n \times n}) \to \mathcal{L}(L^2(\mathbb{R}; \mathbb{C}^{n \times n})):\)

\[
\Theta(f) := \text{Op}_h^w(f(x, \xi)) = \sum_{\gamma \in \mathbb{Z}^2} f(\gamma) \text{Op}_h^w((x, \xi) \mapsto e^{i(\gamma, (x, \xi))})
\]

such that \(\Theta(f \#_h g) = \Theta(f) \circ \Theta(g)\).

Here, \(\mathcal{S}(\mathbb{Z}^2; \mathbb{C}^{n \times n})\) are the \(\mathbb{C}^{n \times n}\)-valued functions that decay faster than any polynomial power on \(\mathbb{Z}^2\). We now define a regularized trace \(\widetilde{\text{tr}}\) for PDOs with periodic symbol such that \(\widetilde{\text{tr}}_{\mathbb{Z}^2}(A^h(f)) = \widetilde{\text{tr}}(\text{Op}_h^w(\hat{f}))\):

**Definition 3.4.** Let \(\hat{f} \in C^\infty(\mathbb{R}^2; \mathbb{C}^{n \times n})\) be \(\mathbb{Z}_\sigma^2\) periodic. Then we define the regularized trace

\[
\widetilde{\text{tr}}(\text{Op}_h^w(\hat{f})) := \int_{\mathbb{T}_\sigma^2} \text{tr}_{\mathbb{C}^n} \hat{f}(x, \xi) \frac{dx d\xi}{|\mathbb{T}_\sigma^2|}.
\]

(3.27)
We can express (3.6), by the $C^*$-homomorphism $\Theta$ and the trace identity, in terms of $\Psi$DOs

\[ Q^w(x, hp_x) := \frac{1}{2}(\cos(x) + \cos(hp_x)) \] and
\[ Q^w_\circ(x, hp_x) := \frac{1}{3} \begin{pmatrix} 0 & 1 + e^{ix} + e^{ihp_x} & 0 \\ 1 + e^{-ix} + e^{-ihp_x} & 0 & 0 \end{pmatrix}, \tag{3.28} \]

which are the semiclassical Weyl-quantizations of

\[ Q(x, \xi) := \tilde{a}(x, \xi) = \frac{\cos(x) + \cos(\xi)}{2} \]
and
\[ Q_\circ(x, \xi) := \tilde{a}_\circ(x, \xi) = \begin{pmatrix} 0 & \frac{1 + e^{ix} + e^{i\xi}}{3} & 0 \\ \frac{1 + e^{-ix} + e^{-i\xi}}{3} & 0 & 0 \end{pmatrix}. \tag{3.29} \]

In particular, the $C^*$-homomorphism $\Theta$ implies

\[ \tilde{\text{tr}}_{\mathbb{Z}^d} \left( (A^h(a) - z)^{-1} \right) = \tilde{\text{tr}} \left( (Q^w(x, hp_x) - z)^{-1} \right). \tag{3.30} \]

The trace on the right hand side is well-defined, as $(Q^w(x, hp_x) - z)^{-1}$ is again a $\Psi$DO with periodic symbol in $\mathcal{S}$ by the semiclassical Beal’s lemma [Zw12, Theorem 8.3], [HS88, Prop.5.1]. To conclude, we can express the DOS of $H^h_{\lambda,\omega}$ in terms of pseudodifferential operators (3.28) as follows:

**Proposition 3.5.** Let $f \in C^6_c(\mathbb{R})$ and $\tilde{f}$ be an almost analytic extension (3.2), then for $n = 1$, in case of the square, and $n = 2$, in case of the hexagonal lattice,

\[ \tilde{\text{tr}}_A(f(H^h_{\lambda,\omega})) = \frac{2}{\pi |b_1 \wedge b_2|} \sum_{k=0}^{\lambda \var(V)^k} \int_{\mathbb{C}} D_\mathbb{C} \tilde{f}(z) \tilde{\text{tr}} \left( (Q^w(x, hp_x) - z)^{-1} \right) dm(z) \]
\[ - \frac{\var(V)\lambda^2}{2\pi |b_1 \wedge b_2|} \sum_{i=1}^{n} \int_{\mathbb{C}} D_\mathbb{C} \tilde{f}(z) \tilde{\text{tr}} \left( (Q^w(x, hp_x) - z)^{-1} \right) dm(z) + O(\|f(5)\|_{L^\infty} \lambda^3). \tag{3.31} \]

**Proof.** By inserting (3.20) into the Helffer-Sjöstrand formula (3.6), we find

\[ \tilde{\text{tr}}_A(f(H^h_{\lambda,\omega})) = \frac{1}{\pi |b_1 \wedge b_2|} \int_{\mathbb{C}} D_\mathbb{C} \tilde{f}(z) \left( \sum_{k=0}^{\lambda \var(V)^k} \frac{(-\lambda \var(V)^k D_z)^k}{k!} \tilde{\text{tr}}_A \left( (H^h - z)^{-1} \right) \right) \]
\[ + \frac{\lambda^2 \var(V)}{2} D_z \sum_{r \in W_A} \left( \text{tr} \left( \mathbb{1}_{(r)} (H^h - z)^{-1} \right) \right)^2 + O(\lambda^3 |\text{Im}(z)|^{-4}) \] \tag{3.32} \]

Using $D_\mathbb{C} \tilde{f} = O(|\text{Im}(z)|^4)$, as in (3.3) for the almost-analytic extension, we can compensate the $|\text{Im}(z)|^{-4}$ singularity. To express the right-hand side in terms of $\Psi$DOs, rather than $H^h$, we use (3.18) and (3.30) which upon integration by parts yields (3.31).

Our main result on the DOS for small magnetic fields is stated in the following Theorem:
Theorem 3 (Semiclassical expansion of DOS). For small magnetic fields \( h > 0 \) and small disorder \( \lambda \) the DOS satisfies:

Square lattice (■): Let \( I \) be an interval \( I \subset [-1, -1 + \delta) \) or \( I \subset (1 - \delta, 1] \) for some \( \delta > 0 \) sufficiently small\(^6\) and \( f \in C^5_c(I) \), then for functions \( g_{\n,\n} \) (independent of \( \lambda \)), defined in (7.19),

\[
\hat{\tau}_\Lambda(f(H_{\n,\n,\omega}^h)) = \frac{h}{2\pi} \sum_{\n \in \mathbb{N}} f(\gamma(\n, h) + \lambda E(V))
\]

\[
- \frac{h \text{Var}(V)\lambda^2}{4\pi} \sum_{\n \in \mathbb{N}} \left( \frac{f''(\gamma(\n, h))}{2\pi} + f'(\gamma(\n, h))g_{\n,\n}(\gamma(\n, h), h) \right)
\]

\[
+ \mathcal{O}(\|f\|_{C^5}(\lambda^3 + h^\infty)) \ \text{a.s.,}
\]

with Landau levels \( \gamma(\n, h) = \kappa(nh, h) - 1 \) defined, for \( n \in \mathbb{N} \), by a Bohr-Sommerfeld condition

\[
F_{\n}(\kappa(\zeta, h), h) = \zeta + \mathcal{O}(h^\infty), \quad F_{\n}(s, h) \sim \sum_{j=0}^{\infty} h^j F_{\n,\n}(s), \quad F_{\n,\n} \in C^\infty(\mathbb{R}),
\]

\[
F_{0,\n}(s) = \frac{1}{2\pi} \int_{\gamma_s} \xi \, dx, \quad \gamma_s = \{(x, \xi) \in \mathbb{T}^2 : 2 - \cos(x) - \cos(\xi) = 2s\}, \quad F_{1,\n}(s) = \frac{1}{2},
\]

where \( \gamma_s \) is oriented clockwise in the \((x, \xi)\) plane.

Hexagonal lattice (○): Let \( I \) be an interval \( I \subset (-\delta, \delta) \) for some \( \delta > 0 \) sufficiently small\(^4\) and \( f \in C^5_c(I) \), then for functions \( g_{\n,\n} \), defined in (7.19),

\[
\hat{\tau}_\Lambda(f(H_{\n,\n,\omega}^h)) = \frac{h}{\pi |b_1 \wedge b_2|} \sum_{\n \in \mathbb{Z}} f(\gamma(\n, h) + \lambda E(V))
\]

\[
- \frac{h \text{Var}(V)\lambda^2}{2\pi |b_1 \wedge b_2|} \sum_{\n \in \mathbb{Z}} \left( \frac{f''(\gamma(\n, h))}{2\pi} + f'(\gamma(\n, h))g_{\n,\n}(\gamma(\n, h), h) \right)
\]

\[
+ \mathcal{O}(\|f\|_{C^5}(\lambda^3 + h^\infty)) \ \text{a.s.,}
\]

with Landau levels \( \gamma(\n, h) = \kappa(nh, h) \) satisfying \( \kappa(-\zeta, h) = -\kappa(\zeta, h) \), defined, for \( n \in \mathbb{Z} \), by a Bohr-Sommerfeld condition

\[
F_{\n}(\kappa(\zeta, h)^2, h) = |\zeta| + \mathcal{O}(h^\infty), \quad F_{\n}(s, h) \sim F_{0,\n}(s) + \sum_{j=2}^{\infty} h^j F_{j,\n}(s), \quad F_{j,\n} \in C^\infty(\mathbb{R}),
\]

\[
F_{0,\n}(s) = \frac{1}{4\pi} \int_{\gamma_s} \xi \, dx, \quad \gamma_s = \{(x, \xi) \in \mathbb{T}^2 : |1 + e^{ix} + e^{i\xi}|^2 = 9s\}, \quad F_{1,\n}(0) = 0,
\]

where \( \gamma_s \) is oriented clockwise in the \((x, \xi)\) plane.

\(^3\)This interval is located at the bottom/top of the spectrum in Figure 3.

\(^4\)This interval encloses energies around the Dirac points in Figure 4.
The proof of Theorem 3 is given at the end of this article in Section 7.

Remark 2. The different prefactor $h/2\pi$ for the square lattice compared with $h/\pi$ for the hexagonal lattice is due to the two-fold degeneracy of quasimodes on the hexagonal lattice (two Dirac cones and therefore two potential wells), cf. Fig. 4.

In particular, for functions $f$ whose first and second derivative vanishes at the Landau levels, the randomness only causes a shift of the Landau levels by $\lambda E(V)$. This can be thought of as a semiclassical universality result for the integrated density of states, if one takes $f$ to be (a smooth approximation of) an indicator function.

We start by showing that for small enough magnetic fields without disorder there exist spectral gaps between the Landau levels stated in Theorem 3. The presence of spectral gaps is crucial for the study of the quantum Hall effect, as the Hall conductivity remains unchanged as long as the Fermi energy stays inside a spectral gap.

From the Bohr-Sommerfeld condition stated in Theorem 3 in the absence of disorder, i.e. $\lambda \equiv 0$, we obtain to leading-order approximative Landau levels $z^{(1)}(h)$

$$F_{0,0}|I\left(z_{0,n}^{(1)}(h)\right) = nh, \quad \text{and} \quad F_{0,0}|I\left(z_{\phi,n}^{(1)}(h)\right) = |n|h,$$

where $F_0$ is the respective normalized phase space area in the Brillouin zone as stated in (3.34) and (3.36), and $I$ is the respective region of interest, i.e. the respective interval defined in Theorem 3. While approximate Landau levels $z_{\phi,n}^{(1)}(h)$ for the square lattice are uniquely defined by the first of the equations in (3.37), there are two solutions for...
the hexagonal lattice (because of the upper and lower cone, see Figure 4): Let us recall from Theorem 3 that the asymptotic expansion yields

$$
F_{\bullet}(z_{\bullet,n}(h), h) = F_{0,\bullet}(z_{\bullet,n}(h)) + O(h^2 z_{\bullet,n}(h), h) = n h + O(h^\infty),
$$

$$
F_{\circ}(z_{\circ,n}(h)^2) = F_{0,\circ}(z_{\circ,n}(h)^2) + O(h^2 z_{\circ,n}(h)^2) = |n| h + O(h^\infty),
$$

(3.38)

which gives for the leading-order approximations (3.37) of Landau levels

$$
z_{\bullet,n}(h) = z_{\circ,n}^{(1)}(h) + O(n h^3) + O(h^\infty)
$$

$$
z_{\circ,n}(h)^2 = z_{\circ,n}^{(1)}(h)^2 + O(|n| h^3) + O(h^\infty).
$$

(3.39)

Hence, by Taylor expansion, Landau levels are to leading order given by

$$
z_{\bullet,n}(h) = z_{\circ,n}^{(1)}(h) + O(n h^3) \text{ and } z_{\circ,0}(h) = 0 + O(h^\infty)
$$

$$
z_{\circ,n}(h) = z_{\circ,n}^{(1)}(h) + O\left(|n|^{\frac{1}{2}} h^\frac{5}{2}\right), n \neq 0.
$$

(3.40)

To make these expressions more concrete, we approximate the cross-section for the square lattice by using that

$$
\frac{\cos(x) + \cos(\xi)}{2} + 1 = \frac{(x - \pi)^2 + (\xi - \pi)^2}{4} + O(x^3 + \xi^3).
$$

Thus, $F_{0,\bullet}(s) = 2s + O(s^2)$ which yields for the Landau levels

$$
z_{\bullet,n}^{(1)}(h) = \left(\frac{n - \frac{1}{2}}{2}\right) h + O(n^2 h^2), \quad n \in \mathbb{N}.
$$

For the hexagonal lattice, we use that $|1 + e^{ix} + e^{i\xi}|^2/9$ vanishes at $(x, \xi) \in \mathbb{Z}_*^2 \pm (\frac{2\pi}{3}, -\frac{2\pi}{3})$, that is, at the Dirac points, see Figure 4.

In small neighbourhoods of $\pm(\frac{2\pi}{3}, -\frac{2\pi}{3})$ we can make a symplectic (and thus area-preserving) change of variables

$$
y = a(x + \xi), \quad \eta = b(\xi - x \pm \frac{4\pi}{3}), \quad 2ab = 1,
$$

and find that

$$
1 + e^{ix} + e^{i\xi} = c(\eta \mp iy) + O(y^2 + \eta^2),
$$

$$
1 + e^{-ix} + e^{-i\xi} = c(\eta \pm iy) + O(y^2 + \eta^2),
$$

(3.41)

where $c = 3^{\frac{1}{4}} 2^{-\frac{1}{2}}$ by choosing $a = \pm 2^{-\frac{1}{2}} 3^{-\frac{1}{4}}$ and $b = \pm 2^{-\frac{1}{2}} 3^{\frac{1}{4}}$. We thus conclude that for a Fermi velocity $v_F := \sqrt{2c}/3 = 3^{-3/4}$

$$
z_{\circ,n}^{(1)}(h) = v_F \text{sgn}(n) \sqrt{|n| h} + O(|n|h), \quad n \in \mathbb{Z}.
$$

**Proposition 3.6** (Spectral gaps between Landau levels). For small $h > 0$, the intersection of the region of interest $I$, in Theorem 3, with the spectrum of $H^h := H^h_{\lambda=0,\omega},$
\[\Sigma(H^h) \cap I, \text{ is contained in disjoint intervals defined by constants } C_{\bullet, n}, C_{O, n} > 0\]

\[B_{\bullet, n}(h) := [z_{\bullet, n}^{(1)}(h) - C_{\bullet, n}h^3, z_{\bullet, n}^{(1)}(h) + C_{\bullet, n}h^3], \quad n \in \{1, \ldots, N_{\bullet}(h)\}\]

\[B_{O, n}(h) := [z_{O, n}^{(1)}(h) - C_{O, n}h^{\frac{5}{2}}, z_{O, n}^{(1)}(h) + C_{O, n}h^{\frac{5}{2}}], \quad n \in \{-N_{O}(h), \ldots, N_{O}(h)\}.\]

Moreover, numbers \(N(h)\) have the property that \(\lim_{h \to 0} N(h) = \infty\).

**Proof.** Since the density of states measure is supported exactly where spectrum is, we conclude that the contribution to the DOS from the Landau levels, i.e. the first term on the right hand side of (3.33) and (3.35) is contained in closed Landau bands

\[B_{\bullet, n}(h) := [z_{\bullet, n}^{(1)}(h) - C_{\bullet, n}h^3, z_{\bullet, n}^{(1)}(h) + C_{\bullet, n}h^3], \quad n \in \mathbb{N}\]

\[B_{O, n}(h) := [z_{O, n}^{(1)}(h) - C_{O, n}h^{\frac{5}{2}}, z_{O, n}^{(1)}(h) + C_{O, n}h^{\frac{5}{2}}], \quad n \in \mathbb{Z}.\]

It remains to exclude spectrum of \(O(h^\infty)\)-size, see the error bounds in (3.33) and (3.35), outside intervals \(B_n\), possibly after modifying constants \(C_n\). This can be shown, using semiclassical techniques as in [BZ19, Prop.5.2]. To be precise, the Proposition in [BZ19] states that there exists an operator \(Q^w_0(x, hp_x)\) whose point spectrum for the hexagonal lattice around zero coincides with the Landau levels, such that if for \(\delta \in \text{nbhd}(0)\), and some fixed \(N_0\),

\[d(z, \Sigma(Q^w_0(x, hp_x))) > h^{N_0}\]

then the operator \(Q^w_0(x, hp_x)\), that is isospectral to \(H^h_O\), cf. [Sj89][Theo. 6.2], is also invertible for such \(z\). Hence, \(H^h_O\) does not possess any spectrum between the Landau bands. The same argument applies to the square lattice in a neighbourhood of \(\pm1\). □

The preceding Proposition implies that under small disorder, the closed Landau bands in the region of interest will broaden but are still non-overlapping since the decomposition \(H^h_{\lambda, \omega} = H^h + \lambda V_\omega\) implies

\[\Sigma(H^h_{\lambda, \omega}) \subset \{z \in \mathbb{R}; d(z, \Sigma(H^h)) \leq \lambda \|V\|_\infty\}.\]  \((3.44)\)

It follows from Proposition 3.6 and (3.44) that for sufficiently weak magnetic fields \(h > 0\) and small disorder \(\lambda \in (0, \lambda_0(h))\) there exist for \(H^h_{\lambda, \omega}\) finitely many (disorder-broadened) disjoint intervals \(B_{n, \lambda}(h) \supset B_n(h)\) with \(n \in \{1, \ldots, N_{\bullet, \lambda}(h)\}\), for the square lattice, or with \(n \in \{-N_{O, \lambda}(h), \ldots, N_{O, \lambda}(h)\}\), in case of the hexagonal lattice, such that

\[\Sigma(H^h_{\lambda, \omega}) \subset \cup_n B_{n, \lambda}(h) \quad \text{for all } \lambda \in (0, \lambda_0(h)),\]

where the union of \(n\) is taken over the respective sets.

Moreover, we assume without loss of generality that the disorder-broadened Landau bands are nested, i.e. for \(\nu \leq \lambda\) we have \(B_{n, \nu}(h) \subset B_{n, \lambda}(h)\).
4. Quantum Hall effect

4.1. The QHE without disorder. We start by studying the Quantum Hall effect in the absence of disorder using the DOS stated in Theorem 3 (we assume $\hbar \notin \mathbb{R} \backslash \mathbb{Q}$ in the following paragraph). We take Středa’s formula [S82] as the definition of the Hall conductivity:

**Definition 4.1** (Středa formula). For (possibly random) Schrödinger operators $H^{h}_{\lambda,\omega}$ with Fermi energy $\mu$ inside a gap $d(\mu, \Sigma(H^{h}_{\lambda,\omega})) > 0$ a.s. we define the Hall conductivity by the Středa formula

$$c_{H}(H^{h}_{\lambda,\omega}, \mu) := |\vec{b}_{1} \wedge \vec{b}_{2}| D_{h} \tilde{\text{tr}}_{\Lambda} \left( \mathbb{1}_{(-\infty, \mu]}(H^{h}_{\lambda,\omega}) \right). \quad (4.1)$$

The DOS is differentiable, since by (2.21) the right-hand side of

$$\tilde{\text{tr}}_{\Lambda}(\mathbb{1}_{I}(H^{h}_{\lambda,\omega})) = \frac{\mathbb{E} \text{ tr}_{W_{\Lambda}} \mathbb{1}_{I}(H^{h}_{\lambda,\omega})}{|\vec{b}_{1} \wedge \vec{b}_{2}|}$$

is differentiable. This follows from holomorphic functional calculus

$$\mathbb{1}_{I}(H^{h}_{\lambda,\omega}) = (2\pi i)^{-1} \oint_{I} (z - H^{h}_{\lambda,\omega})^{-1} dz,$$

as $H^{h}_{\lambda,\omega}$ depends analytically on $h$, i.e. $h \mapsto \mathbb{1}_{I}(H^{h}_{\lambda,\omega})$ is differentiable as long as $\partial I$ is in a spectral gap. Thus, $h \mapsto \tilde{\text{tr}}_{\Lambda}(\mathbb{1}_{I}(H^{h}_{\lambda,\omega}))$ is differentiable as well.

On $\ell^{2}(\mathbb{Z}^{2})$ we define the rotation algebra $A_{h}$ as the operator norm closure

$$A_{h} := \left\{ T \in \mathcal{L}(\ell^{2}(\mathbb{Z}^{2}; \mathbb{C}^{n})); \exists k \in \mathbb{N}, c_{\gamma} \in \mathbb{C} : T = \sum_{|\gamma| \leq k} c_{\gamma} \tau_{h}^{\gamma} \right\}^{\|\cdot\|}. \quad (4.2)$$

Magnetic matrices introduced in Definition 3.1 form a $\ast$-representation of the irrational rotation algebra. We then focus on the subalgebra $A_{h}^{\infty} \subset A_{h}$ of magnetic matrices with rapidly decaying symbols, i.e. with coefficients in (4.2) that satisfy $(c_{\gamma}) \in \mathcal{S}(\mathbb{Z}^{2}; \mathbb{C})$.

The set $A_{h}^{\infty}$ is still a locally convex algebra equipped with standard seminorms inducing decay faster than any polynomial power $|c_{\gamma}|_{i} := \sup_{\gamma \in \mathbb{Z}^{2}} |(1 + |\gamma|)^{i} c_{\gamma}|_{\mathbb{C}^{n \times n}}$. Moreover, the inverse of a magnetic matrix $A^{h}(a) \in A_{h}^{\infty}$ is again a magnetic matrix [HS88, Prop. 5.1], i.e. we have for $z \notin \Sigma(A^{h}(a))$ that $(A^{h}(a) - z)^{-1} \in A_{h}^{\infty}$, again.$^{5}$

The smooth subalgebra $A_{h}^{\infty}$ is stable under holomorphic functional calculus [C94, Ch.3 App.C] which implies that Fermi projections of $A^{h}(a)$, are again elements of $A_{h}^{\infty}$, as long as $\mu \notin \Sigma(A^{h}(a))$

$$\mathbb{1}_{(-\infty, \mu]}(A^{h}(a)) = (2\pi i)^{-1} \oint_{\Sigma(A^{h}(a))} (z - A^{h}(a))^{-1} dz \in A_{h}^{\infty}. \quad \text{Equation } (3.25) \text{ shows that magnetic matrices satisfy the canonical commutation relation with } -h \text{ rather than } h.
The irrational rotation algebra $A_h^\infty$ possesses a unique normalized trace\footnote{since the weak closure of $A_h$ is a (hyperfinite) type $\Pi_1$ factor.} \cite[Prop. 2.3.2.4]{Sh94} which therefore agrees with the trace $\tilde{tr}$ we use in this article. The $K_0$ group of the irrational rotation algebra is given by $K_0(A_h) = \mathbb{Z} + \hbar \mathbb{Z}$ \cite{PV80a, PV80b}. Moreover, there exists a distinguished projection \cite{R81}, the so-called Powers-Rieffel projection $P_R$, which together with the identity generate the $K_0$ group. The inclusion of $K_0$ groups of the dense subalgebra $A_h^\infty$ into the one of $A_h$ is an isomorphism \cite[App. 3, Prop. 2a]{C85} which implies that the above results remain true for $A_h^\infty$ as well.

This implies that for any projection $P \in A_h^\infty$
\begin{equation}
\tilde{tr}_z(P) = \gamma_1 \tilde{tr}_2(id) + \gamma_2 \tilde{tr}_2(P_R) = \gamma_1 + \gamma_2 \hbar. \tag{4.3}
\end{equation}

In the language of noncommutative geometry our trace $\tau_0 := \tilde{tr}_z$ is called the 0-cocycle. For the quantum Hall effect the 2-cocycle $\tau_2$ with $a_0, a_1, a_2 \in A_h^\infty$ is of particular importance

$$\tau_2(a_0, a_1, a_2) := \tau_0(a_0(\delta_1(a_1)\delta_2(a_2) - \delta_2(a_1)\delta_1(a_2))) \tag{4.4}$$

with derivations

$$\delta_1(\tau^h_\gamma) := i\gamma_1 \tau^h_\gamma \quad \text{and} \quad \delta_2(\tau^h_\gamma) := i\gamma_2 \tau^h_\gamma. \tag{4.5}$$

In particular, we write $\Theta(a_0) := \tau_2(a_0, a_0, a_0)$ and will revisit $\Theta$ in the Kubo-Chern formula for the Hall conductance. It follows then from \cite[Cor. 16 in Ch. III Sec. 3]{C94} (see also \cite[p. 359]{C94}) that for any $a_0 \in K_0(A_h^\infty)$ one has

$$\Theta(a_0) = 2\pi i \gamma_2 \tag{4.6}$$

where $\gamma_2 \in \mathbb{Z}$ coincides with the eponymous integer in (4.3).

The semiclassical description of the DOS in Theorem 3 implies together with the results from the previous paragraph, the following Proposition’:

**Proposition 4.2** (Quantum Hall effect). Let $h > 0$ be small enough and consider zero disorder, i.e. $\lambda = 0$. The Hall conductivity is then in the spectral gaps between closed Landau bands (3.42) for the discrete Schrödinger operators $H^h$ given by

$$c_H(H^h(\mu), \mu) = \frac{\mu}{2\pi}, \quad \mu \text{ between } B_{\bullet,n} & B_{\bullet,n+1} \text{ with } n \in \{1, \ldots, N_{\bullet}(h)\} \quad \text{and}$$

$$c_H(H^h(\mu), \mu) = \begin{cases} 
\frac{2n+1}{2\pi}, \mu \text{ between } B_{\bigcirc,n} & B_{\bigcirc,n+1} \text{ with } 0 \leq n \leq N_{\bigcirc}(h) \\
\frac{2n-1}{2\pi}, \mu \text{ between } B_{\bigcirc,n-1} & B_{\bigcirc,n} \text{ with } 0 \geq n \geq -N_{\bigcirc}(h)
\end{cases} \tag{4.7}$$

**Proof.** We just have to find the integer-valued coefficients in (4.3) which we can obtain from the semiclassical expressions for the DOS in Theorem 3. Since Theorem 3 does not allow us immediately to study spectral projections $1_{I}(H^h_{\lambda,\omega})$ we use smooth cut-off functions $\tilde{1}_{I}(H^h_{\lambda,\omega})$ that coincide with the indicator function in the Landau bands.
therefore find in (4.3) that 
\[ \gamma_2 = 0 \]
and decay to zero in the spectral gaps (the DOS is supported on the spectrum, only). 

Theorem 3 implies that for Fermi energies \( \mu \) between Landau bands 
\[
\tilde{\text{tr}}_A(\mathbf{1}_{(-\infty, \mu]}(H^\mathbb{Z})) = \frac{\hbar}{2\pi} \sum_{n \in \mathbb{N}} \mathbf{1}_{(-\infty, \mu]}(z_n(h)) + O(h^\infty) \\
\tilde{\text{tr}}_A(\mathbf{1}_{[0, \mu]}(H^\mathbb{Z})) = \frac{\hbar}{\pi |\beta_1 \wedge \beta_2|} \sum_{n \in \mathbb{Z}} \mathbf{1}_{[0, \mu]}(z_n(h)) + O(h^\infty). \tag{4.8}
\]

Since the Hall conductivity is constant in spectral gaps and continuous in the magnetic field, the \( O(h^\infty) \) error term in Theorem 3 does not contribute to (4.3). We therefore find in (4.3) that \( \gamma_1 = 0 \) and

\[
\begin{align*}
\gamma_2 &= n, \quad \mu \text{ between } B_{\mathbb{Z}, n} & B_{\mathbb{Z}, n+1} \text{ with } n \in \{1, \ldots, N_{\mathbb{Z}}(h)\} \\
\gamma_2 &= \begin{cases} 
2n + 1, & \mu \text{ between } B_{\mathbb{Z}, n} & B_{\mathbb{Z}, n+1} \text{ with } 0 \leq n \leq N_{\mathbb{Z}}(h) \\
2n - 1, & \mu \text{ between } B_{\mathbb{Z}, n-1} & B_{\mathbb{Z}, n} \text{ with } 0 \geq n \geq -N_{\mathbb{Z}}(h).
\end{cases} \tag{4.9}
\end{align*}
\]

Let us recall how the Hall conductivity relates to the geometric framework of condensed matter physics [B84], see also [S83], following the construction in [C94, p.237+238]: We study the algebra \( \Omega^* := \mathcal{A}_h^\infty \otimes \wedge^\mathbb{C}^2 \). Using derivations (4.5), we can define the differentials
\[
\begin{align*}
d(a \otimes \alpha) &:= \delta_1(a) \bar{e}_1 \wedge \alpha + \delta_2(a) \bar{e}_2 \wedge \alpha \\
d(a_1 \otimes \bar{e}_1 + a_2 \otimes \bar{e}_2) &:= (\delta_1(a_2) - \delta_2(a_1)) \otimes \bar{e}_1 \wedge \bar{e}_2. \tag{4.10}
\end{align*}
\]

For forms of top degree there is the trace \( \int : \Omega^{\mathbb{Z}_2} \to \mathbb{C} \) given by \( \int a \otimes (\bar{e}_1 \wedge \bar{e}_2) = a_{00} \). Let \( p \in \mathcal{A}_h^\infty \) be a projection with module \( M^\infty := p \mathcal{A}_h^\infty \). For \( m \in M^\infty \) and \( a \in \mathcal{A}_h^\infty \) we define connections (Berry connections) \( \nabla_i : M^\infty \to M^\infty \)
\[
\nabla_i(\xi a) = \nabla_i(\xi) a + \xi \delta_i(a) := p \delta_i(\xi) a + \xi \delta_i(a), \quad i \in \{1, 2\}.
\]

The curvature tensor (Berry curvature), is then defined as \( R := [\nabla_1, \nabla_2] \otimes (\bar{e}_1 \wedge \bar{e}_2) \).

The first Chern number (Berry phase) is an invariant of the module, independent of the connection, defined by \( \text{Ch}(p) := (2\pi i)^{-1} \int R = (2\pi i)^{-1} \Theta(p) \).

With this vocabulary at hand, we now come to an equivalent second definition of the Hall conductivity:

**Definition 4.3** (Kubo-Chern formula). Let \( \mu \) be an energy in an a.s. spectral gap of \( \mathcal{A}_h(a_{\lambda, \omega}) \) with associated spectral projection \( P_A := \mathbf{1}_{(-\infty, \mu]}(\mathcal{A}_h(a_{\lambda, \omega})) \), then the conductivity tensor \( (\sigma_{jk})_{jk} \in \mathbb{C}^{2 \times 2} \) satisfies
\[
\sigma_{jk} := -i \, \text{tr}_{\mathbb{Z}^2} (P_A[[P_A, x_j], [P_A, x_k]]) = -i \mathbb{E} [\Theta(P_A)].
\]
The following Proposition states that the definitions of the Hall conductivity by the Kubo-Chern and Středa formula yield the same result and are the same for all equivalent versions of the (random) DML:

**Proposition 4.4.** Let $I$ be an interval such that $\partial I$ is in an a.s. spectral gap of $A^h(a_{\lambda,\omega})$ and let $P_A := \mathbb{1}_I(A^h(a_{\lambda,\omega}))$, then the Středa formula agrees with the off-diagonal conductivity in the Kubo-Chern formula

$$D_h \tilde{tr}_{Z^2}(P_A) = -i \tilde{tr}_{Z^2} (P_A[[P_A, x_1], [P_A, x_2]]) = -i \Theta(P_A).$$

Moreover, let $P_{H^h_{\lambda,\omega}}(I) := \mathbb{1}_I(H^h_{\lambda,\omega})$ be the Fermi projection of $H^h_{\lambda,\omega}$, the Kubo-Chern formulas of projections coincide for $X_i(\gamma_1 \vec{b}_1 + \gamma_2 \vec{b}_2 + r_j) := \gamma_i$

$$\tilde{tr}_A (P_H[[P_H, X_1], [P_H, X_2]]) = |\vec{b}_1 \land \vec{b}_2|^{-1} \tilde{tr}_{Z^2} (P_A[[P_A, x_1], [P_A, x_2]]). \quad (4.11)$$

**Proof.** The first part of the Proposition, follows from the noncommutative framework and a direct computation can be found in [ST12, Theorem 7]. The second part follows as $UH^h_{\lambda,\omega} = A^h(a_{\lambda,\omega})U$ for a unitary multiplication operator $U$, by Lemma 3.2,

$$|\vec{b}_1 \land \vec{b}_2| \tilde{tr}_A (P_H[[P_H, X_1], [P_H, X_2]])$$

$$= \mathbb{E} \ tr ((U^* \delta_0, P_H[[P_H, X_1], [P_H, X_2]] U^* \delta_0))$$

$$= \mathbb{E} \ tr_{C^n} ((\delta_0, P_A[[P_A, x_1], [P_A, x_2]]) \delta_0)$$

$$= \tilde{tr}_{Z^2} (P_A[[P_A, x_1], [P_A, x_2]]). \quad (4.12)$$

□

Finally, we shall use a third way of expressing the Hall conductivity using the relative index of projections. This representation is due to Avron, Seiler, and Simon [ASS94]. The version used here can be found in [AW15, Ch.14.5].

**Definition 4.5** (Index-theoretic formulation). Let $P_{\lambda,\omega}$ be an orthogonal projection on $\ell^2(\mathbb{Z}^2)$ satisfying the covariance relation $\tau^h_\gamma P_{\lambda,\tau^h_\gamma} = P_{\lambda,\omega} \tau^h_\gamma$ with translations (3.9) such that

$$\sum_{x \in \mathbb{Z}^2} |x| \left( \mathbb{E} |P_{\lambda,\omega}[0, x]|^3 \right)^{1/3} < \infty. \quad (4.13)$$

Using unitary operators $(U_a \psi)(x) := e^{-i\theta_a(x)} \psi(x)$ with $\theta_a(x) := \arg(x - a) \in (-\pi, \pi)$, the off-diagonal component of the conductivity tensor $\sigma_{1,2}$ is given by the almost sure and $a \in \mathbb{T}^*_a$ independent value of the relative index

$$2\pi \sigma_{1,2} = \text{ind}(P_{\lambda,\omega}, U_a P_{\lambda,\omega} U_a^*) = \mathbb{E} \ tr (P_{\lambda,\omega} - U_a P_{\lambda,\omega} U_a^*)^3$$

---

8The different sign compared with [ST12, (51)] is due to a different sign convention that we use for magnetic matrices.

9Here we use the obvious identification of $\mathbb{R}^2$ with $\mathbb{C}$. 
and coincides, if $P_{\lambda,\omega}$ is a spectral projection satisfying the conditions of Proposition 4.4, with the value given by the Kubo-Chern formula in Definition 4.3.

**Remark 3.** The index theoretic formulation implies that the Hall conductivity is integer-valued (up to the prefactor $(2\pi)^{-1}$) under disorder, too. This follows of course also from the Kubo-Chern formula using the approach presented in [BES94].

The index theoretic formulation of the Hall conductivity implies that the Hall conductivity is invariant, see Proposition 4.2, under mild disorder in the spectral gaps between closed disorder-broadened Landau bands:

**Proof of Proposition 1.1.** Consider a Fermi level $\mu$ between disorder-broadened Landau bands $B_{n,\lambda}$ and $B_{n+1,\lambda}$, i.e. $\mu$ is in a spectral gap of $A^h(a_{\lambda,\omega})$. We need to show that for Fermi projections $P_{\lambda,\omega} := 1_{(-\infty,\mu]}(A^h(a_{\lambda,\omega}))$ and $\lambda$ sufficiently close to zero, we have almost sure equality

$$\text{ind}(P_{\lambda,\omega}, U_a P_{\lambda,\omega} U_a^*) = \text{ind}(P_{0,\omega}, U_a P_{0,\omega} U_a^*).$$

(4.14)

By the resolvent identity and holomorphic functional calculus we find for the difference

$$P_{\lambda,\omega} - P_{0,\omega} = \frac{\lambda}{2\pi i} \oint_{(-\infty,\mu]} (A^h(a) - z)^{-1} V(A^h(a_{\lambda,\omega}) - z)^{-1} \, dz$$

which implies that $\lim_{\lambda \to 0} P_{\lambda,\omega} x = P_{0,\omega} x$ by dominated convergence, using the Combes-Thomas estimate stated in Lemma A.1 for the pointwise bound.

Let $T_{\lambda,\omega} = P_{\lambda,\omega} - U_a P_{\lambda,\omega} U_a^*$ be the difference operator, we then find

$$|\text{ind}(P_{\lambda,\omega}, U_a P_{\lambda,\omega} U_a^*) - \text{ind}(P_{0,\omega}, U_a P_{0,\omega} U_a^*)| = |\text{tr}(T_{\lambda,\omega}^3) - \text{tr}(T_{0,\omega}^3)|$$

$$\leq \left| \sum_{|\gamma| \leq n} \text{tr}_3 \langle \delta_{\gamma}, (T_{\lambda,\omega}^3 - T_{0,\omega}^3) \delta_{\gamma} \rangle \right| + \left| \sum_{|\gamma| > n} \text{tr}_3 \langle \delta_{\gamma}, (T_{\lambda,\omega}^3 - T_{0,\omega}^3) \delta_{\gamma} \rangle \right|. \quad (4.15)$$

(4.15)

It suffices to argue that for $\lambda$ small, the difference of indices is less than one almost surely to show (4.14). The first term on the right hand side is continuous in $\lambda$ by strong convergence and can therefore (for any fixed threshold $n$) be made arbitrarily small by taking $\lambda$ small enough. Thus, by Hölder’s inequality we find for the second term

$$\sup_{\lambda \in (0,\lambda_0)} \left| \sum_{|\gamma| > n} \langle \delta_{\gamma}, T_{\lambda,\omega}^3 \delta_{\gamma} \rangle \right| \leq \|T_{\lambda,\omega}\|_{L^3}^2 \|T_{\lambda,\omega} \delta_{|\gamma| > n}\|_{L^3}. \quad (4.16)$$

We can then use the elementary identity

$$|e^{-i\theta_{\alpha}(x)} - e^{-i\theta_{\alpha}(x+y)}| = |e^{-i\theta_{\alpha}(x)} - e^{-i\theta_{\alpha-y}(x)}| \leq \min \left\{ 2, \frac{|y|}{\sqrt{|x - \alpha||x + y - \alpha|}} \right\},$$
see [AW15, (14.24)], to estimate [AW15, Lemma 14.3 and (14.27)]

\[
\mathbb{E} \left\| T_{\lambda, \omega} \delta_{|\gamma|>n} \right\|_{L^3} \lesssim \sum_{y \in \mathbb{Z}^2} \mathbb{E} \left( \sum_{|x|>n} |T_{\lambda, \omega}[x + y, x]|^3 \right)^{1/3} \\
\lesssim \sum_{y \in \mathbb{Z}^2} \left( \sum_{|x|>n} \mathbb{E} |P_{\lambda, \omega}[x + y, x]|^3 |e^{-it\theta}(x+y) - e^{-it\theta}(x)|^3 \right)^{1/3} \\
\lesssim \sum_{y \in \mathbb{Z}^2} \left( \mathbb{E} |P_{\lambda, \omega}[y, 0]|^3 \right)^{1/3} \left( \sum_{|x|>n} |e^{-it\theta}(x+y) - e^{-it\theta}(x)|^3 \right)^{1/3} < \infty.
\]

(4.17)

The Combes-Thomas estimate in Lemma A.1 implies that (4.13) is uniformly bounded for \( \lambda \in (0, \lambda_0) \). This implies that the summand in (4.17) is uniformly bounded and by the dominated convergence theorem, this expression goes to zero as \( n \to \infty \). \( \Box \)

5. THE METAL/INSULATOR TRANSITION

5.1. Measures of transport. For our discussion of metal/insulator transitions, we first recall the definition of transport coefficients stated in [GK04]. Dynamical properties are studied using weighted norms

\[
M_{h_{\lambda, \omega}}^p(p, \zeta, t) = \left\| \langle x \rangle^{p/2} e^{-itH_{h_{\lambda, \omega}}^0} \zeta(H_{h_{\lambda, \omega}}) \delta_0 \right\|_{L^2}^2
\]

where \( \zeta \in C_{c,+}^\infty(\mathbb{R}) \) localizes to a fixed energy window. In particular, we say that at energies \( E \), \( H_{h_{\lambda, \omega}}^0 \) exhibits Hilbert-Schmidt localization if there is an open interval \( I \ni E \) such that for all \( \zeta \in C_{c,+}^\infty(I) \) and all \( p > 0 \)

\[
\mathbb{E} \left[ \sup_{t \in \mathbb{R}} M_{h_{\lambda, \omega}}^p(p, \zeta, t) \right] < \infty.
\]

The union of all such energies comprises the set \( \Sigma_{h,\text{loc}}^\lambda \). We also define expected time-Césaro averages

\[
M_h^p(p, \zeta, T) = \frac{1}{T} \int_0^T \mathbb{E} \left( M_{h_{\lambda, \omega}}^p(p, \zeta, t) \right) e^{-t/T} dt.
\]

The (lower) transport exponent is defined by

\[
\beta_h^p(p, \zeta) = \liminf_{T \to \infty} \frac{\log_+ M_h^p(p, \zeta, T)}{p \log(T)}, \quad \text{for } p > 0, \zeta \in C_{c,+}^\infty(\mathbb{R})
\]

and from this one defines the \( p \)-th local transport exponent

\[
\beta_h^p(p, E) = \inf_{I \ni E} \sup_{\zeta \in C_{c,+}^\infty(I)} \beta_h^p(p, \zeta) \in [0, 1].
\]
The local lower transport exponent is then defined as $\beta^h_\lambda(E) := \sup_{p>0} \beta^h_\lambda(p,E)$. The exponent $\beta^h_\lambda(E)$ is a measure of transport at energy $E$. This coefficient allows us to define two complementary regions, the (relatively open) region of dynamical localization or insulator region
\[ \Sigma^h_{DL} = \{ E \in \mathbb{R}; \beta^h_\lambda(E) = 0 \} \] (5.1)
that coincides with $\Sigma^h_{loc}$ [GK04, Theorem 2.8], and the (relatively closed) region of dynamical delocalization or metallic transport region
\[ \Sigma^h_{DD} = \{ E \in \mathbb{R}; \beta^h_\lambda(E) > 0 \} . \] (5.2)
An energy $E$ at which the transport coefficient $\beta^h_\lambda$ jumps from zero to a non-zero value is called a mobility edge.

Remark 4. [GK04, Theorem 2.10] implies that in two dimensions, the random Schrödinger operator $H^h_{\lambda,\omega}$ has the property that for all $E \in \mathbb{R}$ for which the transport exponent is positive $\beta^h_\lambda(E) > 0$, the coefficient satisfies already $\beta^h_\lambda(E) > 1/4$.

Fix $\varepsilon > 0$ and let $T$ be the multiplication operator by $\langle x \rangle^{1+\varepsilon}$. The random measure of $H^h_{\lambda,\omega}$ is defined for Borel sets $B \subset \mathbb{R}$ by $\mu_{\lambda,\omega}(B) := \| T^{-1} \mathbb{1}_B(H^h_{\lambda,\omega}) \|^2_{L^2}$, is supported on the spectrum of $H^h_{\lambda,\omega}$, such that $\mu_{\lambda,\omega}(B) < \infty$ if $B \subset \Sigma(H^h_{\lambda,\omega})$ is bounded.

The multiscale analysis in [GK06] has strong implications on energies in the region of dynamical localization that the authors call summable uniform decay of eigenfunction correlations (SUDEC), [GK06, Cor. 3] which we resume in the following Lemma:

Definition 5.1 (SUDEC). For a bounded interval $I$ with $\overline{I} \subset \Sigma^h_{DL}(H^h_{\lambda,\omega})$, we say that $H^h_{\lambda,\omega}$ exhibits SUDEC in $I$ if the spectrum of $H^h_{\lambda,\omega}$ is a.s. pure point and for each eigenvalue $E_{n,\omega,\lambda} \in I$ there is an ONB $\{ \phi_{n,j,\lambda,\omega} \}_{j \in \{1,...,\nu_{n,\lambda,\omega}\}}$ of the finite-dimensional eigenspace $\ker (H^h_{\lambda,\omega} - E_{n,\omega,\lambda})$ such that for any $\xi \in (0,1)$ there is $C_{I,\lambda,\omega,\xi} > 0$ such that
\[ \| \phi_{n,i,\lambda,\omega}(x) \| \| \phi_{n,j,\lambda,\omega}(y) \| \leq C_{I,\lambda,\omega,\xi} \sqrt{\alpha_{n,i,\lambda,\omega}} \sqrt{\alpha_{n,j,\lambda,\omega}} \langle x \rangle^{1+\varepsilon} \langle y \rangle^{1+\varepsilon} e^{-|x-y|^{\xi}}. \] (5.3)
Moreover, $\sum_{n \in \mathbb{N}, j \in \{1,...,\nu_{n,\lambda,\omega}\}} \alpha_{n,j,\lambda,\omega} = \mu_{\lambda,\omega}(I)$.

Remark 5. Up to a change of lattice and thus of constants, SUDEC for $H^h_{\lambda,\omega}$ holds if and only if it holds for $A^h(a_{\lambda,\omega})$.

5.2. Dynamical delocalization. We now turn to the proof of Theorem 1 showing that between disjoint disorder-broadened Landau bands there exists a mobility edge.

We study covariant projections that satisfy the following condition:

Definition 5.2 (P). A covariant projection on $\ell^2(\mathbb{Z}^2; \mathbb{C}^n)$ is said to satisfy condition (P) if for constants $\xi \in (0,1)$, $k > 0$, and $K_p < \infty$ the following bound holds
\[ \| P[0,x] \| = \| \langle \delta_0, P\delta_x \rangle \| \leq K_p(x)^k e^{-|x|^{\xi}}. \]
Clearly, for covariant eigenprojections \( P_{\lambda, \omega} := 1_{E_n, \omega, \lambda} (A^h(a_{\lambda, \omega})) \) on a single energy, (SUDEC) implies \((P)\) with \( k = 1 + \varepsilon \) and
\[
K_P := C_{I, \xi, \omega, \lambda} \sum_{i=1}^{\nu_{n, \lambda, \omega}} \alpha_{n, i, \lambda, \omega}.
\] (5.4)

The index formulation of the Hall conductivity implies immediately by the cyclicity of the trace that if \( P \) is a covariant finite-rank projection satisfying (4.13) then
\[
\text{ind}(P_{\lambda, \omega}, U_a P_{\lambda, \omega} U_a^*) = \text{tr} (P_{\lambda, \omega} - U_a P_{\lambda, \omega} U_a^*) = 0.
\] (5.5)

Moreover, for two orthogonal covariant projections satisfying sufficient decay properties one finds that [BES94, Sec.E Lem.12] for \( \Theta \) as in Definition 4.3
\[
\Theta(P + Q) = \Theta(P) + \Theta(Q).
\] (5.6)

**Lemma 5.3.** Let \( P \) be a covariant projection satisfying condition \((P)\). Then the quantity \( \Theta(P) \) is finite and is bounded for any \( \xi \in (0, 1) \) by a finite constant \( C_{\xi, \kappa} > 0 \)
\[
\|E\langle \delta_0, P[[P, x_1], [P, x_2]] \delta_0 \rangle\| \leq K_P C_{\xi, \kappa}.
\]

**Proof.** Condition \((P)\) implies the following bound
\[
\|E\langle \delta_0, P[[P, [x_1, P], P, x_2]] \delta_0 \rangle\| \leq \|E\langle [[x_1, P], P] \delta_0 \rangle\| \|E\langle [P, x_2, P] \delta_0 \rangle\| \leq \|E\langle \delta_0, P \delta_0 \rangle\|^2 \|E\langle x_1 P \delta_0 \rangle\|^2 \|E\langle x_2 P \delta_0 \rangle\|^2 \leq \sum_{x \in \mathbb{Z}^2} \|x\|_{\mathbb{C}^n}^2 \|E\langle \delta_0, x \delta_0 \rangle\|^2 \leq K_P^2 \sum_{x \in \mathbb{Z}^2} \|x\|_{\mathbb{C}^n}^{2(1+k)} e^{-2\|x\|_\xi} \leq K_P^2 C_{\xi, \kappa}^2.
\] (5.7)

\( \square \)

**Proposition 5.4.** Discrete random Schrödinger operators \( H^k_{\lambda, \omega} \) satisfy the conditions of the multiscale analysis in [GK01]. In particular, they satisfy (SUDEC) in regions of strong dynamical localization.

**Proof.** The Simon-Lieb inequality (SLI) follows for the discrete operators directly from the resolvent identity, see also [Ki07, Sec. 5.3] and the \textit{geometric resolvent identity} discussed there. The exponential decay inequality (EDI) is of similar flavor and straightforward in the discrete case, as discussed for the Anderson model in [DK89, Proof of Lemm. 3.1]. Since potentials at different vertices are independent, the independence at distance (IAD) assumption is clearly satisfied. The average number of eigenvalues estimate (NE) and Wegner estimate (W) are similar to the non-magnetic Anderson model, see [Ki07, Sec. 5.5] for detailed discussions. The strong generalized eigenfunction expansion (SGEE) follows also immediately from the Combes-Thomas estimate and a short proof is stated in Lemma A.2.

\( \square \)
Proof of Theorem 1. We can now finish the proof of Theorem 1 and assume that $H^h_{\lambda,\omega}$ would have only spectrum belonging to the insulating part of the spectrum. For an interval $I = [\lambda_1, \lambda_2]$ where $\lambda_1$ is in one spectral gap between disorder-broadened Landau bands and $\lambda_2$ in another such gap, it follows for $E_{\lambda,\omega}$ the set of eigenvalues of $H^h_{\lambda,\omega}$ in $I$ and $E_{\lambda,\omega} = \bigcup_{m\in\mathbb{N}} M_m$ with $M_m$ a subset of $E_{\lambda,\omega}$ of cardinality min $\{m, \dim(\text{ran}(1_{I}(H^h_{\lambda,\omega})))\} = \sum_{E_n,\lambda,\omega \in M_m} \Theta(1_{E_n,\lambda,\omega}(A^h_{\lambda,\omega}))) = 0$ (5.8)

which vanishes by letting $m \to \infty$ due to (SUDEC) and (5.4). Hence, the Hall conductivity must not jump for operators $H^h_{\lambda,\omega}$ which contradicts the findings of Proposition 1.1. □

Remark 6. To prove delocalization, the type of disorder was in so far irrelevant, as we only assumed the disorder to be small. Other discrete models to which this argument applies are discussed in [GK01, Remark 3.13].

5.3. Dynamical localization. We complete the analysis of discrete random Schrödinger operators by stating a short localization proof: We show that the spectral gaps of the DML between Landau levels can be filled with spectrum that belongs to the insulating part of the spectrum. That is, discrete magnetic Schrödinger operators can have a lot of spectrum that belongs to the insulating region. Since the IDS remains unaffected by the disorder to leading order, cf. Theorem 3, the DOS cannot have much mass away from the Landau bands. To establish the claim on the insulating region, we keep the support of the random potential fixed to some interval $[-\nu, \nu]$, creating thereby much spectrum away from the Landau levels, but rescale the probability distribution function (PDF) such that realizations of the random potential away from zero are unlikely by taking large parameters $m$ in the following Proposition:

Proposition 5.5. Let $\rho \in L^1(\mathbb{R})$ be an a.e. strictly positive PDF on $\mathbb{R}$ with $|\rho(x)| = O(x^{-\gamma})$ for some $\gamma > 1$. For some fixed $\nu$, we define the compactly supported density $\rho_{\nu,m}(x) := c_{\nu,m} m \rho(mx) 1_{[-\nu,\nu]}(x)$ where $c_{\nu,m} = \left(\int_{-\nu}^{\nu} \rho(y) dy\right)^{-1}$ is the normalization constant. Then, for $H^h_{\lambda,\omega}$ with PDF $\rho_{\nu,m}$ it follows that for Landau bands $B_n(h)$

$$\{E \in \mathbb{R}; E \text{ between } B_{n-1}(h) \text{ and } B_n(h)\} \subset \Sigma^{DL}_\lambda \text{ a.s.}$$

The proof is given in Appendix A.

6. Honeycomb structures with flux close to a rational

Hitherto, we studied the case of small magnetic flux $h > 0$ on both the square and hexagonal lattice. We will now continue by studying small magnetic perturbations
Figure 6. Hall conductivity (coloured) as a function of magnetic flux $h \in [0, 2\pi]$ (horizontal axis) and energy (vertical axis). Dark region do not carry spectrum. Different colours represent different conductivities.

of rational magnetic fluxes $2\pi p/q$ for the hexagonal lattice, see [HS88] for a similar analysis in case of Harper’s model. This study is inherently connected with self-similarity in the Hofstadter butterfly, see Fig.6, and the occurrence of magnetic mini-bands [C14]. We start by showing the existence of Dirac cones for rational flux $\phi = 2\pi p/q$ for $H_{\phi}^\Lambda$ at energy level 0. In the sequel, we write $\phi$ for the magnetic flux and use the variable $h$ to denote small perturbations thereof.

6.1. Dirac points. For magnetic flux $\phi = 2\pi p/q$, $H_{\phi}^\Lambda$ is a periodic operator. Let $k = (k_1, k_2) \in \mathbb{T}^*_2$, and let $H_{\phi}^\Lambda(k)$ be the operator $H_{\phi}^\Lambda$ on $\ell^2(\Lambda)$ subject to the pseudo-periodic condition:

$$z(\gamma + q\vec{b}_l, r_j) = e^{ik_l}z(\gamma, r_j), \quad j, l = 1, 2$$

where $\{\vec{b}_1, \vec{b}_2\}$ is the basis vector of $\Lambda$ and $\{r_0, r_1\}$ are the vertices in the fundamental domain $W_\Lambda$. 
We say that an energy \( E \) corresponding to some quasi-momentum \( \tilde{k} \) in the dispersion surface of \( H_\phi \) is a \textit{Dirac point}, if in a neighbourhood of such quasi-momentum, for some positive \( c > 0 \), there are two distinct branches of eigenvalues \( F_\pm(H_\phi^\tilde{k}(k)) \) such that
\[
F_\pm(H_\phi^\tilde{k}(k)) = E \quad \text{and} \quad F_\pm(H_\phi^\tilde{k}(k)) - E = \pm c|k - \tilde{k}| + \mathcal{O}(|k - \tilde{k}|^2).
\]

Next we will present the proof of Theorem 2.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{dispersion_surface.png}
\caption{Dispersion surface of \( H_\phi \). The Dirac cones at energy level zero persist for magnetic flux \( \phi = \pi \).}
\end{figure}

\textbf{Proof of Theorem 2.} The proof is built on some results of [HKL16]. Recall \( H_\phi \) is a tight-binding Schrödinger operator with flux \( \phi \) on the hexagonal lattice, acting on \( \ell^2(\mathbb{Z}^2, \mathbb{C}) \).

The Floquet matrix of \( H_\phi(k) \) is
\[
M_\phi(k) = \frac{1}{3} \begin{pmatrix}
I_q + e^{-ik_1}J_{p,q} + e^{-ik_2}K_q & I_q + e^{ik_1}J_{p,q} + e^{ik_2}K_q \\
0 & 0
\end{pmatrix} =: \begin{pmatrix}
0 & A \\
A^* & 0
\end{pmatrix},
\]
where \( J_{p,q} \) and \( K_q \) are \( q \times q \) matrices, which are defined as
\[
J_{p,q} = \text{diag}(\{e^{i(j-1)\phi}\}^q_{j=1}),
\]
and
\[
(K_q)_{jk} = \begin{cases}
1 & \text{if } k \equiv j + 1(\text{mod } q) \\
0 & \text{otherwise}.
\end{cases}
\]
The solutions of the characteristic equation \( \det(M_\phi(k) - \lambda) = 0 \) are the Floquet eigenvalues of \( H_\phi(k) \), which we label in increasing order:

\[
F_1(k) \leq F_2(k) \leq \cdots \leq F_{2q}(k).
\]

Take \( B_j := \cup_{k \in \mathbb{T}_2^*} F_j(k), 1 \leq j \leq 2q, \) to be the \( j \)-th spectral band of \( H_\phi \). The following was shown in [HKL16].

**Proposition 6.1.** We have

- \( \{B_j\}_{j=1}^{2q} \) are non-overlapping.
- \( B_q \cap B_{q+1} = \{0\} \).

The set \( S_j := \{(k, F_j(k)) : k \in \mathbb{T}_2^*\} \) is called the \( j \)-th dispersion surface.

Taking the square of \( M_\phi(k) \), we arrive at

\[
M_\phi^2(k) = \begin{pmatrix} A \mathcal{A}^* & 0 \\ 0 & \mathcal{A}^* A \end{pmatrix} = \frac{1}{9} \begin{pmatrix} 3I_q + M_T(k) & 0 \\ 0 & 3I_q + \hat{M}_T(k) \end{pmatrix},
\]

(6.5)

where

\[
\hat{M}_T(k) = e^{ik_1 J_{p,q}} + e^{-ik_1 J^*_{p,q}} + e^{ik_2 K_q} + e^{-ik_2 K^*_q} + e^{i(k_1-k_2) K^*_q J_{p,q}} + e^{-i(k_1-k_2) J^*_{p,q} K_q},
\]

(6.6)

and for \( M_T(k) \) one just exchanges \( J_{p,q} \) and \( K_q \). Furthermore, \( M_T(k) \) and \( \hat{M}_T(k) \) have the same non-zero eigenvalues. Let us denote the eigenvalues of \( M_T(k) \) by \( \{E_j(k)\}_{j=1}^q \), where each \( E_j \) is an analytic function in \( k \), note that we do not arrange them in increasing order here. Clearly we have

\[
\det(M_T(k) - \lambda) = \prod_{j=1}^q (E_j(k) - \lambda).
\]

(6.7)

By (6.5), \( M_T(k) + 3I_q \) is positive semidefinite, hence \( E_j(k) \geq -3 \) for \( 1 \leq j \leq q \), and the following holds:

\[
\left\{ F_m(k) \right\}_{m=q+1}^{2q} = \left\{ \frac{1}{3} \sqrt{E_j(k) + 3} \right\}_{j=1}^q \quad \text{and} \quad \left\{ F_m(k) \right\}_{m=1}^q = \left\{ -\frac{1}{3} \sqrt{E_j(k) + 3} \right\}_{j=1}^q.
\]

(6.8)

By Proposition 6.1, one concludes that \( -3 \in \bigcup_{j=1}^q \cup_{k \in \mathbb{T}_2^*} E_j(k) \). Without loss of generality, let

\[
E_1(\tilde{k}) = -3.
\]

(6.9)

Since the bands are non-overlapping, \( E_1(\tilde{k}) \) must be a single eigenvalue, hence for \( 2 \leq j \leq q \), we have \( E_j(\tilde{k}) > -3 \). Now, since \( -3 \) is the minimal value of \( E_1 \), we have

\[
\frac{\partial E_1}{\partial k_m}(\tilde{k}) = 0 \quad \text{for} \; m = 1, 2.
\]

(6.10)
The following Chambers formula was derived in [HKL16], see similar formulas in [AEG14].

**Proposition 6.2.** We have
\[ \det(M_T(k) - \lambda) = f_{p,q}(\lambda) + 2(-1)^{q+1}(\cos q k_1 + \cos q k_2 + (-1)^{q+1} \cos(q(k_1 - k_2))), \]  
(6.11)
where \( f_{p,q}(\lambda) \) is a polynomial in \( \lambda \) (independent of \( k \)) with leading coefficient \((-1)^q\).

Clearly, this proposition yields that
\[ \det(M_T(k_1, k_2) - \lambda) = \det(M_T(k_1 + 2\pi/q, k_2) - \lambda) = \det(M_T(k_1, k_2 + 2\pi/q) - \lambda), \]  
and
\[ \det(M_T(k_1, k_2) - \lambda) = \det(M_T(-k_1, -k_2) - \lambda). \]

Hence, we can restrict our attention to
\[ (k_1, k_2) \in \left[0, \frac{\pi}{q}\right] \times \left[-\frac{\pi}{q}, \frac{\pi}{q}\right]. \]

In the following, we denote
\[ 2(-1)^q(\cos q k_1 + \cos q k_2 + (-1)^{q+1} \cos(q(k_1 - k_2))) := g_q(k) \]  
(6.12)
for simplicity. A direct consequence of Chambers’ formula (6.11) is that
\[ \bigcup_{k \in \mathbb{T}_2} \Sigma(M_T(k)) = \{ \lambda : \min_{k \in \mathbb{T}_2} g_q(k) \leq f_{p,q}(\lambda) \leq \max_{k \in \mathbb{T}_2} g_q(k) \}. \]  
(6.13)

Use the fact that the energy \(-3\) is the bottom of the spectrum \( \bigcup_{k \in \mathbb{T}_2} \Sigma(M_T(k)) \), we have
\[ f_{p,q}(-3) = \max_{k \in \mathbb{T}_2} g_q(k). \]  
(6.14)

Simple computations show that
\[ \max_{k \in \mathbb{T}_2} g_q(k) = 3. \]  
(6.15)

Furthermore, for even \( q \), the maximum is attained at
\[ qk \in \{(\pi/3, -\pi/3), (-\pi/3, \pi/3)\} + 2\pi\mathbb{Z}^2, \]  
(6.16)
and for odd \( q \), the maximum is attained at
\[ qk \in \{(2\pi/3, -2\pi/3), (-2\pi/3, 2\pi/3)\} + 2\pi\mathbb{Z}^2. \]  
(6.17)

Plugging \( k = \tilde{k} \) and \( \lambda = -3 \) into (6.11), using (6.7) and the fact that \( E_1(\tilde{k}) = -3 \), we have
\[ 0 = \prod_{j=1}^q (E_j(\tilde{k}) + 3) = \det(M_T(\tilde{k}) + 3) = f_{p,q}(-3) - g_q(\tilde{k}). \]  
(6.18)
Hence we have
\[
\tilde{k} = \left( \frac{\pi}{3q}, -\frac{\pi}{3q} \right) \text{ for even } q, \text{ and } \tilde{k} = \left( \frac{2\pi}{3q}, -\frac{2\pi}{3q} \right) \text{ for odd } q. \tag{6.19}
\]

Differentiating (6.7) w.r.t. \( k_j, j = 1, 2, \) and taking (6.11) into account, we have
\[
\begin{align*}
2q(-1)^{q+1}(-\sin qk_1 + (-1)^q \sin q(k_1 - k_2)) &= \sum_{m=1}^{q} \frac{\partial E_m}{\partial k_1}(k) \prod_{j=1, j \neq m}^{q} (E_j(k) - \lambda) \\
2q(-1)^{q+1}(-\sin qk_2 - (-1)^q \sin q(k_1 - k_2)) &= \sum_{m=1}^{q} \frac{\partial E_m}{\partial k_2}(k) \prod_{j=1, j \neq m}^{q} (E_j(k) - \lambda)
\end{align*}
\]

Differentiating (6.20) again w.r.t. \( k_j, j = 1, 2, \) we have
\[
\begin{align*}
2q^2(-1)^{q+1}(-\cos qk_1 + (-1)^q \cos q(k_1 - k_2)) &= \sum_{m, \ell=1}^{q} \frac{\partial^2 E_m}{\partial k_1^2}(k) \sum_{m \neq \ell} \frac{\partial E_m}{\partial k_1}(k) \prod_{j=1, j \neq m, \ell}^{q} (E_j(k) - \lambda) \\
&\quad + \sum_{m=1}^{q} \frac{\partial^2 E_m}{\partial k_1 \partial k_2}(k) \prod_{j=1, j \neq m}^{q} (E_j(k) - \lambda) \\
2q^2 \cos q(k_1 - k_2) &= \sum_{m, \ell=1}^{q} \frac{\partial^2 E_m}{\partial k_1 \partial k_2}(k) \prod_{j=1, j \neq m, \ell}^{q} (E_j(k) - \lambda) \\
&\quad + \sum_{m=1}^{q} \frac{\partial^2 E_m}{\partial k_1 \partial k_2}(k) \prod_{j=1, j \neq m}^{q} (E_j(k) - \lambda) \\
2q^2(-1)^{q+1}(-\cos qk_2 + (-1)^q \cos q(k_1 - k_2)) &= \sum_{m, \ell=1}^{q} \frac{\partial^2 E_m}{\partial k_2^2}(k) \sum_{m \neq \ell} \frac{\partial E_m}{\partial k_2}(k) \prod_{j=1, j \neq m, \ell}^{q} (E_j(k) - \lambda) \\
&\quad + \sum_{m=1}^{q} \frac{\partial^2 E_m}{\partial k_1 \partial k_2}(k) \prod_{j=1, j \neq m}^{q} (E_j(k) - \lambda)
\end{align*}
\]

We plug in \( k = \tilde{k} \) and \( \lambda = -3 \). Using (6.9) and (6.10), we have
\[
\begin{align*}
2q^2(-1)^{q+1}(-\cos q\tilde{k}_1 + (-1)^q \cos q(\tilde{k}_1 - \tilde{k}_2)) &= \frac{\partial^2 E_1}{\partial k_1^2}(\tilde{k}) \prod_{j=2}^{q} (E_j(\tilde{k}) + 3) \\
2q^2 \cos q(\tilde{k}_1 - \tilde{k}_2) &= \frac{\partial^2 E_1}{\partial k_1 \partial k_2}(\tilde{k}) \prod_{j=2}^{q} (E_j(\tilde{k}) + 3) \tag{6.22}
\end{align*}
\]

\[
\begin{align*}
2q^2(-1)^{q+1}(-\cos q\tilde{k}_2 + (-1)^q \cos q(\tilde{k}_1 - \tilde{k}_2)) &= \frac{\partial^2 E_1}{\partial k_2^2}(\tilde{k}) \prod_{j=2}^{q} (E_j(\tilde{k}) + 3)
\end{align*}
\]

Hence the Hessian matrix
\[
\begin{align*}
D^2_{k_1, k_2} E_1(\tilde{k})
&= \frac{2q^2(-1)^q}{\prod_{j=2}^{q} (E_j(\tilde{k}) + 3)} \begin{pmatrix}
\cos q\tilde{k}_1 - (-1)^q \cos q(\tilde{k}_1 - \tilde{k}_2) & (-1)^q \cos q(\tilde{k}_1 - \tilde{k}_2) \\
(-1)^q \cos q(\tilde{k}_1 - \tilde{k}_2) & \cos q\tilde{k}_2 - (-1)^q \cos q(\tilde{k}_1 - \tilde{k}_2)
\end{pmatrix}
\tag{6.23}
\end{align*}
\]
Plugging in the values of \( \tilde{k} \), see (6.19), we see that the Hessian matrix for either case is the same:

\[
D^2_{k_1,k_2} E_1(\tilde{k}) = \frac{2q^2}{\prod_{j=2}^{q}(E_j(\tilde{k}) + 3)} \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix},
\]

which is a positive definite matrix. By doing symplectic change of variables

\[
y(k) = a (k_1 + k_2), \quad \eta(k) = b \left( k_2 - k_1 + \frac{4\pi}{3q} \right) \text{ if } q \text{ is odd, and}
\]

\[
y(k) = a (k_1 + k_2), \quad \eta(k) = b \left( k_2 - k_1 + \frac{2\pi}{3q} \right) \text{ if } q \text{ is even, where}
\]

\[
a = 2^{-1/2}3^{-1/4} \quad \text{and} \quad b = 2^{-1/2}3^{1/4},
\]

clearly \( \tilde{y} := y(\tilde{k}) = 0 \) and \( \tilde{\eta} := \eta(\tilde{k}) = 0 \). Let \( \tilde{E}_1(y,\eta) := E_1(k_1, k_2) \). One then checks that using (6.24)

\[
D^2_{y,\eta} \tilde{E}_1(0,0) = \left( \frac{\partial(k_1,k_2)}{\partial(y,\eta)}(0,0) \right)^T D^2_{k_1,k_2} E_1(\tilde{k}) \left( \frac{\partial(k_1,k_2)}{\partial(y,\eta)}(0,0) \right)
\]

\[
= \frac{\sqrt{3}q^2}{\prod_{j=2}^{q}(E_j(\tilde{k}) + 3)} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

with

\[
\left( \frac{\partial(k_1,k_2)}{\partial(y,\eta)}(0,0) \right) = \begin{pmatrix} 2^{-1/2}3^{1/4} & 2^{-1/2}3^{-1/4} \\ 2^{-1/2}3^{1/4} & -2^{-1/2}3^{-1/4} \end{pmatrix}.
\]

Thus, we have in new coordinates close to each well

\[
\tilde{E}_1(y,\eta) = -3 + \frac{\sqrt{3}q^2}{2 \prod_{j=2}^{q}(E_j(\tilde{k}) + 3)} (y^2 + \eta^2) + \mathcal{O}(\|(y,\eta)\|^3).
\]

This yields for the hexagonal lattice using (6.8) the Dirac cones

\[
F_{q+1}(\tilde{k}) = \frac{q}{3^{1/4}} \frac{1}{\sqrt{2 \prod_{j=2}^{q}(E_j(\tilde{k}) + 3)}} \|(y,\eta)\| + \mathcal{O}(\|(y,\eta)\|^2).
\]

6.2. **Semiclassical analysis close to any rational.** In this subsection, we use variables \((x,\xi)\) instead of \(k = (k_1, k_2)\) to emphasize the underlying phase space structure. For the study of magnetic fluxes \(\phi = 2\pi \frac{p}{q} + h\) with \(\gcd(p, q) = 1\), we use that [HS90a, Sec.1] there is a \(C^\ast\)-homomorphism mapping scalar-valued \(\Psi\)DOs with \(\mathbb{Z}_2^2\)-periodic Weyl symbol

\[
\text{Op}_\phi^w(\alpha) = \sum_{\gamma \in \mathbb{Z}^2} a_\gamma(\gamma) \text{Op}_\phi^w((x,\xi) \mapsto e^{i((x,\xi),\gamma)}).
\]
to matrix-valued \( \Psi \)DOs \( \text{Op}_h^w(\hat{\Phi}(a_\Box)) \) on \( L^2(\mathbb{R}, \mathbb{C}^2 \otimes \mathbb{C}^q) \) with symbols that are the Fourier transform of
\[
\Phi(a_\Box) = \left( e^{-i\gamma_1 \gamma_2 h/2} a_\Box(\gamma) \otimes \left[ (J_{p,q})^{\gamma_1} (K_q^*)^{\gamma_2} \right] \right)_{\gamma \in \mathbb{Z}^2}
\]
with \( J_{p,q} \) and \( K_q \) as in (6.3) and (6.4). Note that \( \gamma_1 \gamma_2 = 0 \) for any \( a_\Box(\gamma) \neq 0 \), hence
\[
\Phi(a_\Box) = \left( a_\Box(\gamma) \otimes \left[ (J_{p,q})^{\gamma_1} (K_q^*)^{\gamma_2} \right] \right)_{\gamma \in \mathbb{Z}^2}
\]
In particular, the \( C^* \)-homomorphism preserves regularized traces, up to constants,
\[
\tilde{\text{tr}} \left( \text{Op}_h^w(\hat{a}_\Box) \right) = \int_{\mathbb{T}_2^2} \text{tr}_{C^2} \left( \hat{a}_\Box(x, \xi) \right) \frac{dx \, d\xi}{|\mathbb{T}_2^2|} = a_\Box(0) = q^{-1} \tilde{\text{tr}} \left( \text{Op}_h^w(\hat{\Phi}(a_\Box)) \right)
\]
and, as follows by combining [KL14, Theo. 2.1] with [HS90a, 1.2], also spectra
\[
\Sigma(H^\phi) = \Sigma(\text{Op}_h^w(\hat{a}_\Box)) = \Sigma \left( \text{Op}_h^w \left( \hat{\Phi}(a_\Box) \right) \right).
\]
Recall that \( M_\Box = \hat{\Phi}(a_\Box) \), see (6.2). We conclude by (3.16),(3.18),(3.30), and (6.29) that for \( M_\Box^w(x, hp_x) = \text{Op}_h^w M_\Box \),
\[
\tilde{\text{tr}}_{a_\Box} \left( (H^\phi - z)^{-1} \right) = \tilde{\text{tr}} \left( \left( M_\Box^w(x, hp_x) - z \right)^{-1} \right) \frac{q |\vec{b}_1 \wedge \vec{b}_2|}{|z - \xi_0|}
\]
We are concerned with the analysis of this operator close to the Dirac energy \( E = 0 \). To analyze the spectrum of \( M_\Box^w(x, hp_x) \) close to energies \( E = 0 \), we want to focus on the two touching bands touching at \( E = 0 \), first.

The obstruction to do so, is that for rational flux \( 2\pi \frac{p}{q} \) the two bands touching at \( E = 0 \) may not be isolated from the rest of the spectrum, cf. Fig. 7. At first glance, this creates an obstruction to block-diagonalize the operator \( \text{Op}_h^w M_\Box \) at zero energy to leading order. A way to overcome this issue is explained in the following remark:

**Remark 7** (Isolating bands touching at Dirac energies). We recall that \( M_\Box \) vanishes only at points \( z_0 := (x_0, \xi_0) \) as defined in (6.16) or (6.17), respectively. To analyze the operator \( \text{Op}_h^w M_\Box \) in a neighbourhood of zero energy, it suffices therefore to consider an auxiliary operator with symbol
\[
\hat{M}_\Box(z) := \chi(z) M_\Box(z) + (1 - \chi(z)) M_\Box \left( 2\varepsilon \frac{(z - z_0)}{||z - z_0||} \right)
\]
where \( \chi \in C^\infty(\mathbb{R}^2) \) and \( \chi(z) = 1 \) in a neighbourhood of \( z_0 \) and 0 outside. The parameter \( \varepsilon \) is chosen small enough such that the two eigenvalues of \( M_\Box \left( 2\varepsilon \frac{(z - z_0)}{||z - z_0||} \right) \) that belong to the two bands which touch at the Dirac energies are distinct from all remaining eigenvalues of \( M_\Box \left( 2\varepsilon \frac{(z - z_0)}{||z - z_0||} \right) \). Such a parameter \( \varepsilon > 0 \) exists since the remaining bands of \( M_\Box \) are possible touching the two bands that make up the Dirac cones, but they are not intersecting, cf. Fig. 7.
This way, \( \text{Op}_h^w(\tilde{M}_\circ) \) and \( \text{Op}_h^w(M_\circ) \) coincide microlocally, i.e. for any \( \chi \in C_c^\infty(\text{nbhd}(z_0)) \) we have

\[
\left\| \text{Op}_h^w \chi \left( \text{Op}_h^w(\tilde{M}_\circ) - \text{Op}_h^w(M_\circ) \right) \text{Op}_h^w \chi \right\| = O(h^\infty),
\]

see e.g. [Zw12, Theo. 4.25]. For our subsequent analysis, we may therefore just assume without loss of generality that the two touching bands of \( M_\circ \) at zero energy are gapped from the rest of its spectrum.

To analyze \( \text{Op}_h^w M_\circ \), we recall a few properties about the matrix-valued symbol \( M_\circ \) first. Clearly, \( \cup_{(x,\xi)\in T^*_2} M_\circ(x,\xi) \) has band spectrum \( B_\ell = [\gamma_\ell, \delta_\ell] \), \( 1 \leq \ell \leq 2q \), and we denote associated energy eigenvalues by \( \mu_\ell(x,\xi) \). The \( q \)-th and \( q + 1 \)-st band always touch at the Dirac point, i.e., \( \delta_q = \gamma_{q+1} = 0 \) by Theorem 2. The phase space coordinates at which the \( q \)-th and \( q + 1 \)-st band touch are denoted by \( z_j := (x_j, \xi_j) \in T^*_2 \), where \( j \in \{1, \ldots, 2q^2\} \), i.e., \( \mu_q(z_j) = \mu_{q+1}(z_j) = 0 \). There are by (6.16) and (6.17) precisely \( 2q^2 \) such points in a single fundamental domain \( T^*_2 \). For the analysis close to individual conical points, we fix a sufficiently small \( \varepsilon > 0 \) and consider energies \( E \in I_\varepsilon = (-\varepsilon, \varepsilon) \). We define for such energies the phase space level set \( \Sigma_j(E) := \mu^{-1}_0(\text{nbhd}(z_j))(E) \subset T^*_2 \) for \( \ell \in \{q, q + 1\} \) here, close to a single potential well centred at \( z_j \) and the phase space area \( V_{j,\varepsilon} := \bigcup_{E \in I_\varepsilon} \Sigma_j(E) \) of all energies in the interval \( I_\varepsilon \).

Remark 7 allows us to make two simplifying coordinate changes near the conical points which we discuss now:

There exists a unitary operator \( U \) such that\(^{10}\) [HS90a, Prop.3.1.1 & Cor.3.1.2]

\[
U^* \text{Op}_h^w M_\circ U = \begin{pmatrix} \text{Op}_h^w M_{D,\circ} & \text{Op}_h^w M_{R,\circ} \end{pmatrix}_{\in C^{(2q-2)\times(2q-2)}} \begin{pmatrix} \text{Op}_h^w M_{D,\circ} & \text{Op}_h^w M_{R,\circ} \end{pmatrix}_{\in C^{(2q-2)\times(2q-2)}},
\]

where \( \text{Op}_h^w M_{D,\circ} = \begin{pmatrix} 0 & \text{Op}_h^w b^* \\ \text{Op}_h^w b & 0 \end{pmatrix} + O(h) \).

The subscript \( D \) stands for Dirac and \( R \) for rest, and the symbol \( b \) satisfies \( b(x,\xi) = v_F(\xi + ix) + O(\|x,\xi\|^2) \) where the Fermi velocity \( v_F \) satisfies by (6.8) and (6.28)

\[
v_F = \frac{q}{3^{3/4}} \frac{1}{3^{q-1} \prod_{j=q+2}^{2q} (F_j(k))}.
\]

For the pseudodifferential operator \( \text{Op}_h^w M_\circ = \begin{pmatrix} 0 & \text{Op}_h^w A \\ \text{Op}_h^w A^* & 0 \end{pmatrix} \), with \( A \) as in (6.2), we obtain by squaring the operator

\[
(\text{Op}_h^w M_\circ)^2 = \begin{pmatrix} \text{Op}_h^w A \text{Op}_h^w A^* & 0 \\ 0 & \text{Op}_h^w A^* \text{Op}_h^w A \end{pmatrix}.
\]

\(^{10}\)We assume here by a simple change of coordinates that the Dirac point is located at \( (x,\xi) = 0 \)
By supersymmetry it follows that away from 0 both operators $\text{Op}_h^w \mathcal{A} \text{Op}_h^w \mathcal{A}^*$ and $\text{Op}_h^w \mathcal{A}^* \text{Op}_h^w \mathcal{A}$ have the same spectrum. The principal symbols are

$$\sigma_0 (\text{Op}_h^w \mathcal{A} \text{Op}_h^w \mathcal{A}^*) = M_T(x, \xi) + 3I_q$$

and

$$\sigma_0 (\text{Op}_h^w \mathcal{A}^* \text{Op}_h^w \mathcal{A}) = \hat{M}_T(x, \xi) + 3I_q$$

with the notation as in (6.5). Let $Z(x, \xi)$ now be either $M_T(x, \xi) + 3I_q$ or $\hat{M}_T(x, \xi) + 3I_q$. The lowest eigenvalue of $Z(x, \xi)$ is given by a smooth scalar function $(x, \xi) \mapsto \nu(x, \xi) = |\mu_{q+1}(x, \xi)|^2$, see Remark 7. Thus, there are analytic unitary matrices $V$ separating the lowest eigenvalue from the rest of the matrix

$$(V^*ZV)(x, \xi) = \text{diag}(\nu(x, \xi), B(x, \xi)), \quad (6.36)$$

where by Remark 7 we may assume that $\inf_{(x, \xi) \in T^* R} |\Sigma(B(x, \xi)) - \nu(x, \xi)| > 0$ and $B(x, \xi) \in \mathbb{C}(q-1) \times (q-1)$.

Thus, as for the Dirac-type operator above, [HS90a, Prop. 3.1.1 & Corr. 3.1.2] imply since the lowest band of $Z$, described by $\nu$, is gapped from the rest of the spectrum, there is a unitary operator $U$ and symbols $\tilde{\nu}, \tilde{B}$ with asymptotic expansions in $\mathcal{S}$, such that

$$U^* (\text{Op}_h^w \mathcal{A} \text{Op}_h^w \mathcal{A}^*) U = \begin{pmatrix} \text{Op}_h^w \tilde{\nu} & 0 \\ 0 & \text{Op}_h^w \tilde{B} \end{pmatrix} + O_{L(L^2(\mathbb{R}))(h^\infty)}, \quad (6.37)$$

where $\sigma_0(\tilde{\nu}) = \nu$ and $\sigma_0(\tilde{B}) = B$.

The main result of this section, a semiclassical trace formula close to rational flux, is then stated in the following Theorem:

**Theorem 4** (Semiclassical DOS and QHE close to a rational). For small perturbations $h > 0$ and magnetic flux $\phi = 2\pi \frac{p}{q} + h$, the DOS of $H_\phi^\mathcal{O}$ admits the following expansion: Let $I$ be an interval $I \subset (-\delta, \delta)$ for some $\delta > 0$ sufficiently small${}^{11}$ and $f \in C_c^\alpha (I)$, then

$$\tilde{\text{tr}}_\Lambda (f(H_\phi^\mathcal{O})) = \frac{qh}{\pi |b_1 \wedge b_2|} \sum_{n \in \mathbb{Z}} f(z_n(h, p, q)) + O(||f||_{C^\alpha} h^\infty), \quad (6.38)$$

---

${}^{11}$This interval encloses energies around the Dirac points in Figure 4.
with Landau levels \( z_n(h, p, q) = \kappa(nh, h, p, q) \) satisfying \( \kappa(-\zeta, h, p, q) = -\kappa(\zeta, h, p, q) \), defined by a Bohr-Sommerfeld condition

\[
F(\kappa(\zeta, h, p, q)^2, h, p, q) = |\zeta| + \mathcal{O}(h^\infty), \quad F(s, h, p, q) = \sum_{j=0}^{\infty} F_j(s, p, q) h^j, \quad F(0, p, q) = 0,
\]
where

\[
F_0(s, p, q) := \int_{\nu(x, \xi) \in [0, s]} \frac{dx \, d\xi}{4\pi q^2} \quad \text{and} \quad F_1(s, p, q) := \frac{1}{2} - \frac{d}{d\zeta} \left|_{\zeta=s} \right. \int_{\nu(x, \xi) \in [0, \zeta]} \sigma_1(\tilde{\nu})(x, \xi) \frac{dx \, d\xi}{4\pi q^2}.
\]

(6.39)

With the Fermi velocity \( v_F \) defined in (6.33), \( z_n \) satisfies

\[
\begin{align*}
z_0 &= \mathcal{O}(h^\infty) \quad \text{and} \quad \nonumber \\
z_n &= \text{sgn}(n) v_F \sqrt{|n|} h + \mathcal{O}(h), \quad n \neq 0. \quad (6.40)
\end{align*}
\]

In addition, the spectrum of the magnetic Schrödinger operator around zero \( \Sigma(H_\phi) \cap I \) is contained in disjoint closed Landau bands \( B_{\phi,n}(h, p, q) \ni z_n(h, p, q) \) with spectral gaps

\[
d(B_{\phi,n}(h, p, q), B_{\phi,n+1}(h, p, q)) \geq C_{n, p, q} h \quad (6.41)
\]
for some constant \( C_{n, p, q} > 0 \). The Hall conductivity satisfies for Fermi energies \( \mu \)

\[
c_H(H_\phi, \mu) = \begin{cases} 
\frac{(2n+1)q}{2\pi}, & \mu \text{ betw. } B_{\phi,\lambda,n} \text{ and } B_{\phi,\lambda,n+1} \text{ with } 0 \leq n \leq N_{\phi}(h, \lambda_0) \\
\frac{(2n-1)q}{2\pi}, & \mu \text{ betw. } B_{\phi,\lambda,n-1} \text{ and } B_{\phi,\lambda,n} \text{ with } 0 \geq n \geq -N_{\phi}(h, \lambda_0).
\end{cases}
\]

(6.42)

**Remark 8** (Dynamical delocalization). In particular, using the results from subsection 5.2, we conclude from (6.41) that for sufficiently weak disorder, such that the (disorder-broadened) Landau bands remain non-overlapping, there exists at least one mobility edge inside each Landau band at which delocalization occurs.

7. Proofs

We now state the proof of Theorems 3 and 4 with several references to details that are already discussed in [BZ19, HS88].

**Proof of Thm. 3 & Thm. 4.** **Step 1: Quasimodes and Landau levels.** Quasimodes and Landau levels are constructed as eigenfunctions and eigenvalues to localized operators, i.e. operators that coincide microlocally, up to a constant shift of the spectrum, with \( \PsiDOs \) (3.28) in a neighbourhood of a single potential well. For the square lattice,
such a localized operator with discrete spectrum at the bottom of the potential well, see Fig. 3, is defined by the Weyl symbol

\[ Q_\bullet^0(x, \xi) := Q_\bullet(x, \xi) + 2 - \chi_\bullet(x, \xi), \]

where

\[ \chi_\bullet \in C^\infty_c(\mathbb{R}^2; [0, 1]), \quad \chi_\bullet(x, \xi) = \begin{cases} 1, & \| (x, \xi) - (\pi, \pi) \|_\infty < \frac{1}{10}, \\ 0, & \| (x, \xi) - (\pi, \pi) \|_\infty > \frac{1}{5}. \end{cases} \] (7.1)

Thus, \( \text{Op}_h^w Q_\bullet^0 - z \) is elliptic [Zw12, Sec. 4.7] for \( z \) in a small neighbourhood of zero and \( (x, \xi) \notin \text{nbhd}(\pi, \pi) \) where the neighbourhood depends on \( z \).

On the hexagonal lattice such a localized operator with discrete spectrum close to zero energy, the energy level of the Dirac points, see Fig. 4, is defined by the symbol

\[ M_\circ^0(x, \xi) := M_\circ(x, \xi) + \begin{pmatrix} (\chi_\circ(x, \xi) - 1)I_q & 0 \\ 0 & (1 - \chi_\circ(x, \xi))I_q \end{pmatrix}, \] (7.2)

\( \chi_\circ \in C^\infty_c(\mathbb{R}^2; [0, 1]), \chi_\circ(z) = \chi_\circ(-z), \)

where \( \chi_\circ(x, \xi) = 1 \) on all \( \bigcup_{j \in \{1, \ldots, 2q^2\}} \mathcal{V}_j \) for some \( \delta > 0 \) sufficiently small and vanishes outside of \( \mathbb{T}_2^2 \).

Next, we argue that the spectrum of both \( \text{Op}_h^w Q_\bullet^0 \) and \( \text{Op}_h^w M_\circ^0 \) is indeed contained in discrete intervals around zero. To do so, we define another pair of symbols

\[ Q_\bullet^1(x, \xi) := Q_\bullet(x, \xi) + 2 \quad \text{and} \quad M_\circ^1(x, \xi) := M_\circ(x, \xi) + \text{diag}(-I_q, I_q). \] (7.3)

The two associated operators with upper index 1 are invertible close to zero and we have

\[ \text{Op}_h^w Q_\bullet^0 - z = (\text{Op}_h^w Q_\bullet^1 - z) \,(\text{id} + K_\bullet(z)) \quad \text{and} \quad \text{Op}_h^w M_\circ^0 - z = (\text{Op}_h^w M_\circ^1 - z) \,(\text{id} + K_\circ(z)) \] (7.4)

for some compact operators

\[ K_\bullet(z) = (\text{Op}_h^w Q_\bullet^1 - z)^{-1} \chi_\bullet^w \quad \text{for} \quad z \notin \Sigma(\text{Op}_h^w Q_\bullet^1) \quad \text{and} \]

\[ K_\circ(z) = (\text{Op}_h^w M_\circ^1 - z)^{-1} \text{diag}(\chi_\circ^w, -\chi_\circ^w) \quad \text{for} \quad z \notin \Sigma(\text{Op}_h^w M_\circ^1). \] (7.5)

By analytic Fredholm theory [Zw12, Theorem D.4] this implies the discreteness of the spectrum of \( Q_\bullet^0 \) and \( M_\circ^0 \) close to zero. Thus, there exists a family of eigenvalues and orthonormal eigenfunctions such that

\[ (\text{Op}_h^w Q_\bullet^0 - \kappa_\bullet(nh, h)) u_{n, \bullet} = 0 \quad \text{and} \quad (\text{Op}_h^w M_\circ^0 - \kappa_\circ(nh, h)) u_{n, \circ} = 0. \] (7.6)

Localized operators with upper index 0 have the property that their spectra for energies close to zero stay close to the spectra of operators \( \text{Op}_h^w Q_\bullet \) and \( \text{Op}_h^w M_\circ \), respectively. In fact, an immediate adaptation of the proof of [BZ19, Lemma 5.2] shows that after possibly shrinking the energy window around zero to some \( \varepsilon_1 \) with \( 0 < \varepsilon_1 < \varepsilon \) and
DYNAMICAL DELOCALIZATION FOR DISCRETE SCHRÖDINGER OPERATORS.

41

\[ z \in [0, \varepsilon_1] - i[-1, 1] \text{ such that } d(z, \Sigma(\text{Op}_h^w Q^0_\Box)) > h^n, \text{ for some arbitrary but fixed } n \in \mathbb{N}, \text{ there is } h_0 \text{ such that for } h \in (0, h_0), \]

\[ (\text{Op}_h^w Q^0_\Box - z)^{-1} = O_{L^2 \to L^2}(d(z, \Sigma(\text{Op}_h^w Q^0_\Box))^{-1}) \quad (7.7) \]

and the analogous result is true for \( M^w_\Box \) as well.

Since \( \text{Op}_h^w M^w_\Box \) and \( \text{Op}_h^w M^0_\Box \) in \( \bigcup_{j \in \{1, \ldots, 2q^2\}} V_{j, \delta} \) coincide microlocally we infer from (7.6) that

\[ (\text{Op}_h^w M^w_\Box - \kappa_\Box(nh, h)) u_{n, \Box} = O(h^\infty). \quad (7.8) \]

Thus, one has to find all such microlocal solutions with \( \text{WF}_h(u_{n, \Box}) \subset \bigcup_{j \in \{1, \ldots, 2q^2\}} V_{j, \delta} \).

Microlocal solutions \( (\text{Op}_h^w A \text{Op}_h^w A^* - \lambda) v = O(h^\infty) \) for \( z = \pm \lambda, u := (u_1, u_2) := (v, z^{-1} \text{Op}_h^w A^* v) \in \mathbb{C}^{2q}. \quad (7.9) \)

Since 0 is in the spectrum of \( H^h_\Box \) for all \( h \in [0, 2\pi] \) [BHJ18, Lemma 5.1], we have that \( 0 \in \Sigma(\text{Op}_h^w M_\Box) \) for all \( h \) by (6.30). Invoking now (7.7) for the hexagonal lattice, implies that there exists an eigenvalue \( O(h^\infty) \) to the localized operator \( \text{Op}_h^w M^0_\Box \).

We can now apply the following Bohr-Sommerfeld condition [HR84, HS90a, CdV05]:

Let \( H : T^*\mathbb{R} \to \mathbb{R} \) be a classical symbol with expansion \( H \sim \sum_{i=0}^{\infty} H_i h^i \) Moreover, we assume the principal symbol \( H_0 \) to satisfy

1. \( H_0(z) = 0 \) and \( (D^2 H_0)(z) > 0, \)
2. The set \( \{ \nu \in \mathbb{R}^2 : H_0(\nu) < \delta \} \) is compact and connected for some \( \delta > 0 \) sufficiently small.
3. \( H_0 \) is strictly positive and does not possess any other critical points, apart from \( z \) in a sufficiently small nbhd of \( z \).

Then, the spectrum of \( \text{Op}_h^w(H) \) close to zero is given by the Bohr-Sommerfeld condition

\[ F(E, h) = \sum_{j=0}^{\infty} F_j(E) h^j = nh \]

where the leading-order term is the Bohr-Sommerfeld term

\[ F_0(E) = \frac{1}{2\pi} \int_{\{H_0 \leq E\}} dx \, d\xi \]

and the subprincipal term \( F_1 \) includes the Maslov correction and the contribution from the subprincipal symbol \( H_1 \)

\[ F_1(E) = \frac{1}{2} - \frac{1}{2\pi} \frac{d}{ds} \bigg|_{s=E} \int_{\{H_0 \leq s\}} H_1(x, \xi) \, dx \, d\xi. \quad (7.10) \]
Expressions for higher-order terms $F_j$ with $j \geq 2$ can be found in [CdV05].

This immediately yields the Bohr-Sommerfeld condition for the square lattice (3.34), by applying it to the microlocally equivalent symbol $Q^0$ in (7.1), since the subprincipal is zero and therefore $F_1(E) = \frac{1}{2}$.

In case of the hexagonal lattice, we use that by (7.9) and (6.37) it suffices to study the quasimodes to the symbol $\tilde{\nu}$. Clearly, $\tilde{\nu}$ satisfies both assumptions (1) and (3) of the Bohr-Sommerfeld condition.

By using cut-off functions $\chi_{j,0}$ that are localized to neighbourhoods $V_{j,\delta}$ of a single well, the localized symbol

$$\tilde{\nu}_j(x,\xi) := \tilde{\nu}(x,\xi) + (1 - \chi_{j,0})(x,\xi)$$

satisfies then all three conditions of the Bohr-Sommerfeld rule which yields (6.39).

When $q = 1$ and $A$ is scalar, a direct computation of (7.10) shows that $F_1 = 0$ [BZ19-2]. This yields the Bohr-Sommerfeld condition stated in Theorem 3.

Finally for the analysis close to rationals, the asymptotics of Landau levels (6.40) and the presence of gaps (6.41) follow immediately from both (6.32) and (6.33), and the explicit spectral analysis of the 2D-magnetic Dirac operator, cf. [HS90a][Prop 3.6.1 and (3.6.22)].

**Step 2: The Grushin problem.** To prove the trace formulae, we fix one Landau level and take $z_1$ and $\varepsilon_0$ with

$$\{\kappa(nh,h)\}_n \cap [z_1 - 2\varepsilon_0 h, z_1 + 2\varepsilon_0 h] = \{\kappa(n_1 h,h)\}, \quad n_1 = n_1(z_1,h). \quad (7.11)$$

Since symbols $Q_\bullet$ and $M_\circ$ are $2\pi$-periodic, they possess infinitely many potential wells. Therefore, we introduce a translation operator $r_\gamma u(x) := e^{i\gamma z^2} u(x - \gamma_1)$ to define translations of the quasimodes $w_\gamma := r_\gamma u$ for $\gamma \in \mathbb{Z}^2$. We then define operators $R_+: L^2(\mathbb{R}, \mathbb{C}^m) \to \ell^2(\mathbb{Z}^2; \mathbb{C}^m)$ and $R_- : \ell^2(\mathbb{Z}^2; \mathbb{C}^n) \to L^2(\mathbb{R}, \mathbb{C}^m)$ by

$$\left(R_+ u_+\right)(\gamma) := \int_\mathbb{R} u_+(x) \overline{w_\gamma(x)} \, dx \in \mathbb{C}^m, \quad R_- u_-(x) := \sum_{\gamma \in \mathbb{Z}^2} w_\gamma(x)u_-(\gamma), \quad (7.12)$$

where

- $n = m = 1$ for the square lattice and
- $n = 2q^2$, $m = 2q$ on the hexagonal lattice close to the flux $2\pi p/q$, in which case

$$u_-(\gamma) = \left(u_1(\gamma) \ldots u_{2q^2}(\gamma)\right)^t \in \mathbb{C}^{2q^2} \text{ and } w_\gamma(x) = \left(w_1(\gamma) \ldots w_{2q^2}(\gamma)\right) \in \mathbb{C}^{2q \times 2q}.$$

This way, the following Grushin problem [BZ19, Prop. 5.4] is well-posed for $z \in (z_1 - \varepsilon_0 h, z_1 + \varepsilon_0 h) + i(-1,1)$, where $\mathcal{P}(z) := \text{Op}_h^w Q_\bullet - z$ for the square and $\mathcal{P}(z) :=$...
Op\^w_h M_\circ - z for the hexagonal lattice,

\[
\begin{pmatrix} \mathcal{P}(z) & R_- \\ R_+ & 0 \end{pmatrix}^{-1} = \begin{pmatrix} E(z, h) & E_+(z, h) \\ E_-(z, h) & E_-(z, h) \end{pmatrix}, \tag{7.13}
\]

Schur’s complement formula implies that

\[\mathcal{P}(z)^{-1} = E(z, h) - E_+(z, h)E_\pm(z, h)E_-(z, h)\]

where \(E_+, E_\pm\) can be approximated by

\[E_+^0 := R_-, \quad E_-^0 := R_+, \quad E_\pm^0 = (z - \kappa(hn_1, n_1))\delta_{\gamma,0}. \tag{7.14}\]

Here, \(E_\pm(\gamma) = E_\pm^0(\gamma) + \mathcal{O}(h^\infty(\gamma)^{-\infty})\) for \(|\text{Im}(z)| > h^m\), for some fixed \(m\), and

\[E_+(z, h)v_+(x) = \sum_{\gamma \in \mathbb{Z}^2} r_\gamma W_0(x)v_+(\gamma), \quad W_0 = w_0 + e_0, \quad e_0 = \mathcal{O}(h^\infty), \tag{7.15}\]

\[(E_-(z, h)v)(\gamma) = (v, r_\gamma W_-), \quad W_- = w_0 + c_0, \quad c_0 \in \mathcal{O}(h^\infty)\]

where the estimates follow as in [BZ19, Proof of Prop. 5.4]. Moreover, we define the function \(G(z, h) := \int_{\mathbb{T}^2} \sigma(E(z, h))(x, \xi) \frac{dx \, d\xi}{|\mathbb{T}^2|}\) which is holomorphic in \(z \in (z_1 - \varepsilon_0 h, z_1 + \varepsilon_0 h + i(-1, 1))\) [BZ19, (6.1)].

To study

\[J(z, h) = \int_{\mathbb{T}^2} \text{tr}_{\mathbb{C}^m} \sigma(E_+ E_-)(x, \xi) \frac{dx \, d\xi}{|\mathbb{T}^2|}\]

we define, for fixed \(M\), the approximation \(J_0\) for

\[z \in (z_1 - \varepsilon_0 h, z_1 + \varepsilon_0 h + i(-1, 1)), n = n_1(z_1, h), \quad |\text{Im} z| > h^M\]

by using approximations (7.14)

\[J_0(z, h) = \int_{\mathbb{T}^2} (z - \kappa(n_1 h, h))^{-1} \text{tr}_{\mathbb{C}^m} \sigma(E_+^0 E_-^0)(x, \xi) \frac{dx \, d\xi}{|\mathbb{T}^2|}. \tag{7.16}\]

Estimates (7.15) imply then that \(J(z, h) = J_0(z, h) + \mathcal{O}(h^\infty)\).

To find a more explicit expression for \(J_0\) we study the Schwartz kernel \(K\) of the operator \(E_+^0 E_-^0\) given by

\[K(x, y) = \sum_{\alpha \in \mathbb{Z}^2} E_+^0(x, \alpha)E_-^0(\alpha, y) = \sum_{\alpha} w_\alpha(x)w_\alpha(y)^*,\]

from which the symbol of the pseudodifferential operator, appearing in (7.16), can be derived from the Schwartz kernel

\[\sigma(E_+^0(z, h)E_-^0(z, h))(x, \xi) = \sum_{\alpha \in \mathbb{Z}^2} \int_{\mathbb{R}} w_\alpha(x - \frac{w}{2})w_\alpha^*(x - \frac{w}{2})e^{i\xi \cdot \omega} dw \]

\[= \sum_{\alpha \in \mathbb{Z}^2} \int_{\mathbb{R}} e^{i\xi \cdot \omega} w_\alpha(x - \frac{w}{2} - \alpha_1)w_\alpha(x + \frac{w}{2} - \alpha_1)^* dw.\]
Hence, we obtain for the integral over the Weyl symbol

\[
\int_{\mathbb{T}^2} \sigma(E^0_+(z,h)E^0_-(z,h))(x,\xi) \frac{dx d\xi}{4\pi^2} = \sum_\alpha \int_{\mathbb{T}^2} \int_{\mathbb{R}} e^{i w(\xi - \alpha_2)} w_0(x - \frac{w}{2} - \alpha_1)w_0(x + \frac{w}{2} - \alpha_1)^* dw \frac{dx d\xi}{4\pi^2}
\]

\[
= \int_{\mathbb{R}^2} \int_{\mathbb{R}} e^{i w_0(x - \frac{w}{2})} w_0(x + \frac{w}{2})^* dw \frac{dx d\xi}{4\pi^2} = \frac{h}{2\pi} \int_{\mathbb{R}} w_0(x)w_0(x)^* dx.
\]

This implies for \( J_0 \) as in (7.16)

\[
J_0(z, h) = \int_{\mathbb{T}^2} (z - \kappa(n_1 h, h))^{-1} \text{tr}_{\mathbb{C}^m} \sigma (E^0_+ E^0_-)(x,\xi) \frac{dx d\xi}{|\mathbb{T}^2|} = \frac{h(z - \kappa(n_1 h, h, p, q))^{-1}}{2\pi} \sum_{i=1}^m \sum_{j=1}^n \int_{\mathbb{R}} |\langle \tilde{e}_i, w_j(x) \rangle|^2 dx
\]

\[
= \frac{h(z - \kappa(n_1 h, h, p, q))^{-1}}{2\pi} \sum_{j=1}^n \int_{\mathbb{R}} |w_j(x)|^2 dx = \frac{hn}{2\pi} (z - \kappa(n_1 h, h, p, q))^{-1}.
\]

For the hexagonal lattice with magnetic flux \( h \), the reflection symmetry of the Dirac points located at quasimomenta \( \pm ((\frac{2\pi}{3}, -\frac{2\pi}{3})) \) implies that the eigenfunctions \( u_{n_1}^\pm = (u_{n_1,1}^\pm, u_{n_1,2}^\pm) = (u_{n_1,1}^\pm, u_{n_1,1}^\pm) \) satisfy

\[
\int_{\mathbb{R}} \|w_0(x)^* \tilde{e}_i\|^2 dx = \int_{\mathbb{R}} |u_{n_1,i}^+(x)|^2 + |u_{n_1,i}^-(x)|^2 dx = 1 + O(h^\infty).
\]

Taking the regularized trace and exhibiting leading-order contributions shows that for \( |\text{Im}(z)| > h^M \), with arbitrary \( M \), and \( |z - z_1| \leq \varepsilon h \) there are analytic functions

\[
g_{\|n_1}(z, h) := G(z, h) + \frac{h}{2\pi} \sum_{n \neq n_1} (z - z_n(h))^{-1},
\]

\[
g_{\circ,n_1}(i, z, h) := \langle \tilde{e}_i, G(z, h) \tilde{e}_i \rangle_{C^2} + \frac{h}{2\pi} \sum_{n \neq n_1} (z - z_n(h))^{-1},
\]

\[
g_{\circ,n_1}(z, h) := \frac{g_{\|n_1}(1, z, h) + g_{\|n_1}(2, z, h)}{2},
\]

\[
g_{\circ,n_1}(z, h, p, q) := \text{tr}_{\mathbb{C}^2} G(z, h, p, q) + \frac{hn}{2\pi} \sum_{n \neq n_1} (z - z_n(h, p, q))^{-1},
\]

(7.19)
such that we obtain [BZ19, Prop. 6.1]

\[
\tilde{\text{tr}} \left( (Q_w^w(x, \h p_x) - z)^{-1} \right) = \frac{\hbar}{2\pi} (z - z_{n_1,\sharp}(h))^{-1} + g_{\#, n_1} (z, h) + \mathcal{O}(\hbar^\infty),
\]

\[
\tilde{\text{tr}} \left( (\tilde{e}_i, (Q_w^w(x, \h p_x) - z)^{-1} \tilde{e}_i)_C^2 \right) = \frac{\hbar}{2\pi} (z - z_{n_1,\circ}(h))^{-1} + g_{\circ, n_1} (i, z, h) + \mathcal{O}(\hbar^\infty), \text{ and}
\]

\[
\tilde{\text{tr}} \left( (M_w^w(x, \h p_x) - z)^{-1} \right) = \frac{\hbar q^2}{\pi} (z - z_{n_1,\circ}(h))^{-1} + g_{\circ, n_1} (z, h, p, q) + \mathcal{O}(\hbar^\infty).
\]

(7.20)

We also observe for later that

\[
(\tilde{\text{tr}} (Q_w^w(x, \h p_x) - z)^{-1})^2 = -\frac{\hbar^2}{4\pi^2} D_z (z - z_{n_1,\sharp}(h))^{-1}
+ \frac{\hbar}{2\pi} (z - z_{n_1,\sharp}(h))^{-1} g_{\#, n_1} (z, h) + g_{\#, n_1} (z, h)^2 + \mathcal{O}(\hbar^\infty) \text{ and}
\]

\[
(\tilde{\text{tr}} (\tilde{e}_i, (Q_w^w(x, \h p_x) - z)^{-1} \tilde{e}_i)_C^2)^2 = -\frac{\hbar^2}{4\pi^2} D_z (z - z_{n_1,\circ}(h))^{-1}
+ \frac{\hbar}{2\pi} (z - z_{n_1,\circ}(h))^{-1} g_{\circ, n_1} (i, z, h) + g_{\circ, n_1} (i, z, h)^2 + \mathcal{O}(\hbar^\infty).
\]

(7.21)

**Step 3: Trace formulae.**

We can now assume that \( \text{Re}(z) \in (z_1 - \varepsilon h, z_1 + \varepsilon h) \) is close to a Landau level and apply (7.20), as analyticity of the resolvent \((Q_w^w(x, \h p_x) - z)^{-1}\) away from the Landau bands implies that there is no contribution from \( z \) outside these intervals (integration by parts in Helffer-Sjöstrand formula).

**Trace formulae in Thm. 3.** From (3.3), we have since \( f \in C^5(I) \) that \( D_x \tilde{f}(z) = \mathcal{O}(\|f\|_{C^5} |\text{Im}(z)|^4) \). By Proposition 3.5, we obtain, by writing the adjusted prefactors
for the hexagonal lattice in parenthesis [ ] and for the square lattice without parenthesis,

\[
\tilde{\text{tr}}_A(f(H^h_{\lambda,\omega})) = \frac{[2]^h}{2\pi^2|b_1 \wedge b_2|} \int_{\mathcal{C}} \sum_{k=0}^{2} \frac{\lambda^k \mathbb{E}(V)^k}{k!} \frac{D_{\tilde{\varphi}}f^{(k)}(z)}{k!} \sum_n (z - z_n(h))^{-1} \, dm(z)
- \frac{[2]^h \text{Var}(V)\lambda^2}{8\pi^3|b_1 \wedge b_2|} \sum_n \int_{\mathcal{C}} D_{\tilde{\varphi}}f^{(n)}(z)(z - z_n(h))^{-1} \, dm(z)
- \frac{[2]^h \text{Var}(V)\lambda^2}{2\pi^2|b_1 \wedge b_2|} \sum_n \int_{\mathcal{C}} D_{\tilde{\varphi}}(f'(z)g_n(z,h))(z - z_n(h))^{-1} \, dm(z)
+ \frac{1}{\pi} \int_{|\text{Im } z| < h^M} D_{\tilde{\varphi}}(z) \mathcal{O}(1 \text{ Im } z)^{-1} \, dm(z) + \mathcal{O}(\|f\|_{C^5}(\lambda^3 + h^\infty))
= \frac{[2]^h}{2\pi|b_1 \wedge b_2|} \sum_n \sum_{k=0}^{2} \frac{\lambda^k \mathbb{E}(V)^k}{k!} f^{(k)}(z_n(h)) + \mathcal{O}(\|f\|_{C^5}(\lambda^3 + h^3M + h^\infty))
- \frac{[2]^h \text{Var}(V)\lambda^2}{2\pi|b_1 \wedge b_2|} \sum_n \left( \frac{f'(z_n(h))}{4\pi} + f'(z_n(h))g_n(z_n(h),h) \right)
= \frac{[2]^h}{2\pi|b_1 \wedge b_2|} \sum_n f(z_n(h)) + \lambda \mathbb{E}(V) + \mathcal{O}(\|f\|_{C^5}(\lambda^3 + h^3M + h^\infty))
- \frac{[2]^h \text{Var}(V)\lambda^2}{2\pi|b_1 \wedge b_2|} \sum_n \left( \frac{f'(z_n(h))}{4\pi} + f'(z_n(h))g_n(z_n(h),h) \right).
\]

(7.22)

By taking \( M \) arbitrarily large the trace formulae (3.33) and (3.35) of Theorem 3 follow.

**Trace formula in Thm. 4.** Since \( f \) is now only assumed to be Hölder continuous, we require an additional approximation argument:

Let \( \psi \in C^\infty_c((0,1)) \) be a positive function with \( \int_\mathbb{R} \psi(s) \, ds = 1 \) and define \( \psi_h(s) := h^{-1}\psi(h^{-1}s) \) with \( f_h := f * \psi_h \). Moreover, we find \( \|f - f * \psi_{h}\|_{L^\infty} \leq \|f\|_{C^0} h^{\alpha M_0} \) and since the interval \( I \) can contain only \( \mathcal{O}(h^{-1}) \) many Landau levels, we have

\[
\sum_{|n| \leq C/h} \|f(z_n(h)) - f_h(z_n(h))\| = \mathcal{O}(\|f\|_{C^0} h^{\alpha M_0}).
\]

(7.23)

We observe that by (3.3) we have

\[
\|D_{\tilde{\varphi}}f_h(z)\|_{L^\infty} \leq \|f_h\|_{C^2} \|\text{Im}(z)\| = \mathcal{O}(\|f\|_{L^\infty} h^{-2M_0} |\text{Im}(z)|).
\]

(7.24)
We then use (7.24) and (6.29) for the hexagonal lattice to conclude that

\[
\tilde{\text{tr}}_\Lambda (f(h(H^h\lambda,\omega))) = \frac{qh}{\pi |b_1 \wedge b_2|} \int \mathbb{D}_\pi \tilde{f}_h(z) \sum_n (z - z_n(h))^{-1} \, dm(z) + \frac{1}{\pi} \int_{|\text{Im } z| < h^M} D_\pi \tilde{f}_h(z) O \left( |\text{Im } z|^{-1} \right) \, dm(z) + O \left( \| f_h \|_{L^\infty} h^\infty \right)
\]

(7.25)

Thus, we have from (7.23) that

\[
\tilde{\text{tr}}_\Lambda (f(H^h\lambda,\omega)) = \frac{qh}{\pi |b_1 \wedge b_2|} \sum_n f_h(z_n(h)) + O \left( \| f \|_{L^\infty} h^{M-2M_0} + \| f \|_{C^\alpha} h^{\alpha M_0} \right)
\]

(7.26)

which by choosing \( M = 3M_0 \) and \( M_0 \) arbitrarily large implies (6.38).

**Step 4: QHE and mobility edges for the hexagonal lattice.**

From (4.3) we conclude that for any Fermi projection \( P = \mathbb{1}_J(H^h_\omega) \) such that \( J \subset I \) with \( \partial J \) located inside a spectral gap of \( H^h_\alpha \) there are \( \gamma_1, \gamma_2 \in \mathbb{Z} \) such that

\[
\tilde{\text{tr}}_{\Lambda_\alpha} (P) = |\tilde{b}_1 \wedge \tilde{b}_2|^{-1} \left( \gamma_1 + \gamma_2 \left( \frac{p}{q} + \frac{h}{2\pi} \right) \right)
\]

(7.27)

The trace formula (6.38) on the other hand yields that

\[
\tilde{\text{tr}}_{\Lambda_\alpha} (P) = \frac{hq}{|b_1 \wedge b_2|} \sum_{n \in \mathbb{Z}} \mathbb{1}_J(z_n(h,p,q)) + O(h^\infty)
\]

(7.28)

Comparing coefficients (4.1) implies that the Hall conductivity, when gauged to be zero at zero energy, is given by (6.42) for sufficiently small \( h \).

**Appendix A. Multiscale analysis**

**Lemma A.1** (Combes-Thomas estimate). Let \( z \) be such that \( d \left( z, \Sigma \left( H^h_{\lambda,\omega} |_{\Lambda_L(x)} \right) \right) = \varepsilon \leq 1 \), then for any \( n, m \in \Lambda_L(x) \), with \( \Lambda_L(x) \) defined in (1.2), one has

\[
\left| \left( H^h_{\lambda,\omega} |_{\Lambda_L(x)} - z \right)^{-1} [m,n] \right| = O \left( \varepsilon^{-1} e^{-\frac{\pi}{2\varepsilon} ||n-m||_1} \right).
\]

(A.1)

**Proof.** The proof of (A.1) is a direct adaptation of [Ki07, Theorem 11.2].

**Lemma A.2** (SGEE). For \( \gamma > 1 + \lambda \| V \|_\infty \) and any \( \nu > 1 \) it follows that

\[
\text{tr} \left( (\bullet)^{-\nu} (H^h_{\lambda,\omega} + \gamma)^{-1} (\bullet)^{-\nu} \right) \leq C_\nu < \infty \text{ a.s. .}
\]
Proof. By the Combes-Thomas estimate stated above as Lemma A.1, which holds for some $\delta > 0$ since $\gamma \notin \Sigma(H^h_{\Lambda,\omega})$, we have for $W_{\Lambda_n} := W_{\Lambda} - n_1 \vec{b}_1 - n_2 \vec{b}_2$

$$\text{tr} \left( (\bullet)^{-\nu} (H^h_{\Lambda,\omega} + \gamma)^{-1} (\bullet)^{-\nu} \right)$$

$$= \sum_{n,m \in \mathbb{Z}^2} \text{tr} \left( \text{id}_{\mathcal{E}(W_{\Lambda_n})} (\bullet)^{-\nu} \mathbb{1}_{W_{\Lambda_n}} (H^h_{\Lambda,\omega} + \gamma)^{-1} \mathbb{1}_{W_{\Lambda_m}} (\bullet)^{-\nu} \right)$$

$$\leq C_\delta \sum_{n,m \in \mathbb{Z}^2} e^{-\delta/24} |n-m| \sup_{x \in W_{\Lambda_n}} | (\bullet)^{-\nu} | \sup_{x \in W_{\Lambda_m}} | (\bullet)^{-\nu} |$$

$$\lesssim C_\delta \sum_{n \in \mathbb{Z}^2} e^{-\delta/24} \sum_{m \in \mathbb{Z}^2} (1 + |n + m|)^{-\nu} (1 + |m|)^{-\nu}$$

$$\lesssim C_\delta \sum_{n \in \mathbb{Z}^2} e^{-\delta/24} \sum_{m \in \mathbb{Z}^2} (1 + |m|)^{-2\nu} < \infty$$

where we applied the Cauchy-Schwarz inequality in the last step to the inner series.

Proof of Prop. 5.5. We estimate the tail probability with respect to the new density

$$\rho_{\nu,m} \left( \{|x| \geq \varepsilon\} \right) = c_{\nu,m} \int_{[-\nu,v]\setminus[-\varepsilon,\varepsilon]} m \rho(mx) \ dx$$

$$\lesssim \int_{[-\nu,v]\setminus[-\varepsilon,\varepsilon]} \frac{m}{(mx)\gamma} \ dx = O((m\varepsilon)^{1-\gamma}).$$

Let us define the finite volume truncation $H^h_{\Lambda,\omega,\Lambda_L(x)} := H^h_{\Lambda,\omega}|_{\Lambda_L(x)}$ where $\Lambda_L(x)$ is defined in (1.2). Consider the set $\Sigma_\varepsilon(H^h) := \{ x \in \mathbb{R}; x \in [y - \varepsilon, y + \varepsilon], y \in \Sigma(H^h) \}$, containing the $\varepsilon$-broadened non-random spectrum of $H^h$. We have the following lower bound, with $I$ being the region of interest on the probability, using Bernoulli’s inequality $(1 - x)\alpha \geq 1 - \alpha x$ and the decay of the probability distribution,

$$\mathbb{P} \left( \Sigma(\mathbb{1}_I(H^h_{\Lambda,\omega,\Lambda_L(x)})) \subset \Sigma_\varepsilon(H^h) \right) \geq \mathbb{P} \left( |\lambda V_{\omega(v)}| \leq \varepsilon \text{ for } v \in \Lambda_L(x) \right)$$

$$\geq (1 - C(m\varepsilon)^{1-\gamma}|\Lambda_L(x)|) \geq 1 - \left( \frac{CL^2}{(m\varepsilon)^{\gamma-1}} \right)$$

where $C$ is allowed to change in the last line. We will use this estimate to infer that with high-probability an energy $E$ between $B_n(h) + 2\varepsilon$ and $B_{n+1}(h) - 2\varepsilon$ is in the resolvent set of $H^h_{\Lambda,\omega,\Lambda_L(x)}$ and has a distance $\varepsilon$ to the spectrum of $H^h$ for $m$ large enough.

Choosing $\varepsilon = \mu m^{-1}(3L^2)^{-1/4}$ in (A.4) with $\mu$ sufficiently large implies

$$\mathbb{P} \left( \Sigma(\mathbb{1}_I(H^h_{\Lambda,\omega,\Lambda_L(x)})) \subset \Sigma_\varepsilon(H^h) \right)$$

is arbitrarily close to 1, uniformly in $m$. Since for both the square and hexagonal lattice

$$L \lesssim |\{n \in \Lambda; \|n\|_1 \in [L - 1, L + 1]\}| \lesssim L$$

and

$$L^2 \lesssim |\{n \in \Lambda; \|n\|_1 \leq L/3\}| \lesssim L^2,$$
the Combes-Thomas estimate, stated in Lemma A.1, shows that for \( E \) between \( B_n(h) + 2\varepsilon \) and \( B_{n+1}(h) - 2\varepsilon \) with high probability

\[
\frac{L}{2}\frac{1}{3} \sum_{n,m \in \Lambda : \| n \|_1 \in [L-1,L+1], \| m \|_1 \leq L/3} \left| (H_{\lambda,\omega,\Lambda}^h(x) - E)^{-1}[n,m] \right| = O\left( L^{\frac{25}{9}} \varepsilon^{-1} e^{-C_1 \varepsilon L} \right). \tag{A.5}
\]

By the choice of \( \varepsilon \), this implies for sufficiently large \( L \geq L_0(m) \)

\[
L^{\frac{25}{9}} \varepsilon^{-1} e^{-C_1 \varepsilon L} \lesssim m L^{\frac{25}{9} - \frac{2}{\gamma - 1}} e^{-\frac{C_2}{m} L^{\frac{2}{\gamma - 1}}} < 1. \tag{A.6}
\]

In particular, choosing \( L_0(m) \propto m^{(1+\varepsilon')(\gamma-1)/(\gamma+1)} \) for some fixed \( \varepsilon' \in (0,(\gamma-1)/2) \) implies (A.6). This choice of \( L_0 \) ensures that also

\[
\lim_{m \to \infty} L_0(m)^{\frac{2}{\gamma - 1}} m^{-1} = \lim_{m \to \infty} m^{\frac{2(1+\varepsilon')-(1+\gamma)}{1+\gamma}} = 0.
\]

This implies by (A.6) that \( \varepsilon := \mu m^{-1}(3L^2)^{1/4} \) can be chosen arbitrarily small by taking \( m \) large enough such that by [GK03, Theorem 2.4]

\[
\{ E \in \mathbb{R} ; E \text{ between } B_{n-1}(h) \text{ and } B_n(h) \} \subset \Sigma_{\lambda}^{h,DL}.
\]

\[
\square
\]

REFERENCES

[AEG14] A. Agazzi, J.-P. Eckmann, G.-M. Graf, (2014). The Colored Hofstadter Butterfly for the Honeycomb Lattice, Journal of Statistical Physics 156(3).

[AJ09] A. Avila, and S. Jitomirskaya, (2009). The Ten Martini Problem, Annals of Mathematics, 170(1), pp. 303-342.

[AYZ17] A. Avila, J. You, Q. Zhou, (2017). Sharp phase transitions for the almost Mathieu operator. Duke Mathematical Journal, 166(14), pp.2697-2718.

[ASS94] J. Avron, R. Seiler, and B. Simon, (1994). Charge deficiency, charge transport and comparison of dimensions. Communications in Mathematical Physics, Volume 159, Issue 2, pp 399-422.

[AW15] M. Aizenman and S. Warzel, (2015). Random Operators: Disorder Effects on Quantum Spectra and Dynamics. AMS 2015.

[BBDF18] S. Bachmann, A. Bols, W. De Roeck and M. Fraas, (2018). Quantization of Conductance in Gapped Interacting Systems. Ann. Henri Poincaré 19, 695-708.

[BES94] J. Bellissard, A. van Elst, and H. Schulz-Baldes, (1994). Noncommutative geometry of quantum Hall effect. J. Math. Phys., Vol. 35, No. 10.

[B84] M. V. Berry, (1984). Quantal Phase Factors Accompanying Adiabatic Changes. Proceedings of the Royal Society A. 392 (1802): 45-57.

[BHLJ18] S. Becker, R. Han, and S. Jitomirskaya, Cantor spectrum of graphene in magnetic fields, Invent. Math. 218, 979-1041 (2019).

[BTBB07] J. H. Bardarson, J. Tworzydlo, P. W. Brouwer, and C. W. J. Beenakker, (2007). One-Parameter Scaling at the Dirac Point in Graphene, Phys. Rev. Lett. 99, 106801.

[BZ19] S. Becker and M. Zworski, (2019). Magnetic Oscillations in a Model of Graphene, Communications in mathematical physics.

[BZ19-2] S. Becker and M. Zworski, (2019). Magnetic Oscillations in a Model of Graphene-Addendum,
[C85] A. Connes, (1985). Non-commutative differential geometry, Publications Mathématiques de l’IHES, Volume 62, p. 41-144.

[C94] A. Connes, (1994). Noncommutative geometry, Academic Press, Inc., San Diego, CA.

[C14] X. Chen, J. Wallbank, A. Patel, M. Mucha-Kruczyski, E. McCann, and V. Fal’ko (2014). Dirac edges of fractal magnetic minibands in graphene with hexagonal moiré superlattices, Phys. Rev. B, 89, 7, 2014.

[CdV05] C. de Verdière (2005). Bohr-Sommerfeld Rules to All Orders, Annales Henri Poincare 6(5):925-936.

[D19] A. Drouot, (2019). The bulk-edge correspondence for continuous honeycomb lattices, arXiv:1901.06281.

[D18] A. Drouot, (2018). Characterization of edge states in perturbed honeycomb structures, arXiv:1811.08218.

[D18] A. Drouot, (2019). The bulk-edge correspondence for continuous dislocated systems, arXiv:1810.10603.

[DFW18] A. Drouot, C. L. Fefferman, and M. I. Weinstein (2019). Defect states for dislocated periodic media., arXiv:1901.06281.

[DK89] H. Dreifus and A. Klein. (1989). A new proof of localization in the Anderson tight binding model, Commun. Math. Phys., 124, 285-299.

[DS99] M. Dimassi and J. Sjöstrand, (1999). Spectral Asymptotics in the Semi-Classical Limit, Cambridge University Press.

[G06] G. Graf, (2006). Aspects of the integer quantum Hall effect, A Pure Mathematics: Volume 76, part 1: Spectral Theory and Mathematical Physics: A Festschrift in Honor of Barry Simon’s 60th Birthday.

[GK01] F. Germinet, A. Klein, (2001). Bootstrap Multiscale Analysis and Localization in Random Media, Communications in Mathematical Physics, Volume 222, Issue 2, pp 415-448.

[GK03] F. Germinet, A. Klein, (2003). Explicit finite volume criteria for localization in continuous random media and applications, GAFA, Geom. funct. anal. Vol. 13, 1201-1238.

[GK04] F. Germinet and A. Klein, (2004). A characterization of the Anderson metal-insulator transport transition. Duke Math. J. Volume 124, Number 2, 309-350.

[GKS04] F. Germinet, A. Klein, J. Schenker, (2007). Dynamical delocalization in random Landau Hamiltonians, Ann. of Math., 166, 215-244.

[GK06] F. Germinet and A. Klein, (2006). New Characterization of the Region of Complete Localization for Random Schrödinger Operators, Journal of Statistical Physics, Vol. 122, No. 1.

[GMP12] A. Giuliani, V. Mastropietro, M. Porta, (2012). Universality of conductivity in interacting graphene, Comm. Math. Phys. 311, 317-355.

[GMP18] A. Giuliani, V. Mastropietro, M. Porta, (2018). Quantization of the interacting Hall conductivity in the critical regime, arXiv:1803.11213

[GS06] V. P. Gusynin and S. G. Sharapov, (2006). Transport of Dirac quasiparticles in graphene: Hall and optical conductivities, Physical Review B 73, 245411.

[FH19] J. Fillman, R. Han, (2018). Discrete Bethe–Sommerfeld Conjecture for Triangular, Square, and Hexagonal Lattices, arXiv:1806.01988, J. d’Analyse Math., to appear.

[FS83] J. Fröhlich and T. Spencer, (1983). Absence of diffusion in the Anderson tight binding model for large disorder or low energy, Comm. Math. Phys. Volume 88, Number 2 (1983), 151-184.

[FW12] C. Fefferman and M. Weinstein, (2012). Honeycomb lattice potentials and Dirac points, J. Amer. Math. Soc. 25, 1169–1220.
[FLW16] C. Fefferman, J. Lee-Thorp, M. Weinstein, (2016). Honeycomb Schrödinger operators in the strong binding regime, Communications on Pure and Applied Mathematics Volume 71, Issue 6.

[HKL16] B. Helffer, P. Kerdelhúé, and J. Royo-Letelier, (2016). Chambers’s formula for the graphene and the Hou model with Kagome periodicity and applications, Annales Henri Poincaré, 17, Issue 4.

[HM15] M.B. Hastings and S. Michalakis, (2015). Quantization of Hall conductance for interacting-electrons on a torus. Commun. Math. Phys.334, 433-471.

[HR84] B. Helffer and D. Robert, (1984). Puits de potentiel généralisés et asymptotique semi-classique, Ann. Inst. H. Poincaré Phys. Théor. 41, 291-331.

[HS88] B. Helffer and J. Sjöstrand, (1989). Analyse semi-classique pour l’équation de Harper (avec application à l’équation de Schrödinger avec champ magnétique) Mém. Soc. Math. France (N.S.) 34.

[HS89] B. Helffer and J. Sjöstrand, (1989). Equation de Schrödinger avec champ magnétique et équation de Harper. in Schrödinger operators (Sønderborg, 1988), 118–197 Lecture Notes in Phys.345, Springer, Berlin.

[JK19] S. Jitomirskaya, I. Krasovsky, (2019). Critical almost Mathieu operator: hidden singularity, gap continuity, and the Hausdorff dimension of the spectrum arXiv:1909.04429.

[JL18] S. Jitomirskaya, W. Liu, (2018). Universal hierarchical structure of quasiperiodic eigenfunctions. Annals of Mathematics, 187(3), pp.721-776.

[HS90a] B. Helffer and J. Sjöstrand, (1990). Analyse semi-classique pour l’équation de Harper. II. Comportement semi-classique près d’un rationnel. Mém. Soc. Math. France (N.S.) 40, 1990.

[HS90b] B. Helffer and J. Sjöstrand, (1990). On diamagnetism and de Haas-van Alphen effect. Ann. Inst. H. Poincaré Phys. Théor. 52, 303–375.

[K95] P. Kerdelhúé, (1995). Équation de Schrödinger magnétique périodique avec symétrie d’ordre six: mesure du spectre II, Annales de l’I.H.P., section A, tome 62, n.2. p. 181-209.

[Ki07] W. Kirsch, (2007). An Invitation to Random Schrödinger operators. arXiv:0709.3707.

[KL14] P. Kerdelhúé, J. Royo-Letelier, (2014). On the low lying spectrum of the magnetic Schrödinger operator with kagome periodicity, Reviews in Mathematical Physics, Volume 26, Issue 10.

[L94] Y. Last, (1994). Zero measure spectrum for the almost Mathieu operator, Comm. Math. Phys.Volume 164, Number 2, 421-432.

[MJ17] C.A. Marx, S. Jitomirskaya (2017). Dynamics and spectral theory of quasi-periodic Schrödinger-type operators. Ergodic Theory and Dynamical Systems, 37(8), pp.2353-2393.

[M06] S. V. Morozov, K. S. Novoselov, M. I. Katsnelson, F. Schedin, L. A. Ponomarenko, D. Jiang, and A. K. Geim, (2006). Strong Suppression of Weak Localization in Graphene, Phys. Rev. Lett. 97, 016801.

[Pe10] N. Peres, (2010). The transport properties of graphene: an introduction, Review of Modern Physics 82(3).

[PV80a] M. Pimsner and D. Voiculescu, (1980). Exact sequences for K-groups and EXT-Groups of certain cross-product C*-algebras, J. Operator Theory (4), 93-118.

[PV80b] M. Pimsner and D. Voiculescu, (1980). Imbedding the irrational rotation C*-algebra into an af-algebra, J. Operator Theory (4), 201-210.

[R81] M. A. Rieffel, (1981). C*-algebras associated with irrational rotations, Pacific J. Math., Band 93.

[Sh94] M. Shubin, (1994). Discrete magnetic Laplacian, Comm. Math. Phys. Volume 164, Number 2, 259-275.
[S00] B. Simon, (2000). *Schrödinger operators in the twentieth century*, Journal of Mathematical Physics 41, 3523.

[S83] B. Simon, (1983). *Holonomy, the Quantum Adiabatic Theorem, and Berry’s Phase*, Phys. Rev. Lett., 51, 24, 2167-2170.

[Sj89] J. Sjöstrand, (1989). *Microlocal analysis for periodic magnetic Schrödinger equation and related questions*, in Microlocal Analysis and Applications J.-M. Bony, G. Grubb, L. Hörmander, H. Komatsu and J. Sjöstrand eds. Lecture Notes in Mathematics 1495, Springer. *Elementary linear algebra for advanced spectral problems*, Ann. Inst. Fourier 57, 2095–2141.

[S82] P. Středa, (1982). *Theory of quantised Hall conductivity in two dimensions*, J. Phys. C: Solid State Phys., 15.

[ST12] H. Schulz-Baldes and S. Teufel, (2013). *Orbital polarization and magnetization for independent particles in disordered media*, Communications in Mathematical Physics, Volume 319, Issue 3, pp 649-681.

[TKNN82] D.J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs, (1982). *Quantized Hall Conductance in a Two-Dimensional Periodic Potential*, Phys. Rev. Lett. 49, 405.

[W47] P.R. Wallace, The band theory of graphite, Phys. Rev., 71 (1947), p. 622.

[W94] W. M. Wang, (1984). *Asymptotic Expansion for the Density of States of the Magnetic Schrödinger Operator with a Random Potential*, Commun. Math. Phys. 172, 401-425.

[Zw12] M. Zworski, (2012). *Semiclassical analysis*, Graduate Studies in Mathematics 138 AMS.

[Z05] Y. Zhang, Y. Wen Tan, H. Stormer and P. Kim, (2005). *Experimental observation of the quantum Hall effect and Berry’s phase in graphene*, Nature volume 438, pages 201-204.

[Z06] K. Ziegler, (2006). *Robust Transport Properties in Graphene*, Phys. Rev. Lett. 97, 266802

E-mail address: simon.becker@damtp.cam.ac.uk

DAMTP, UNIVERSITY OF CAMBRIDGE, WILBERFORCE RD, CAMBRIDGE CB3 0WA, UK

E-mail address: rui.han@math.gatech.edu

GEORGIA TECH, 686 CHERRY STREET, ATLANTA, GA 30332, USA