In this paper we study the enstrophy transfers in helical turbulence using direct numerical simulation. We observe that the helicity injection does not have significant effects on the inertial-range energy and helicity spectra ($\sim k^{-5/3}$) and fluxes (constants). We also calculate the separate contributions to enstrophy transfers via velocity to vorticity and vorticity to vorticity channels. There are four different enstrophy fluxes associated with the former channel or vorticity stretching, and one flux associated with the latter channel or vorticity advection. In the inertial range, the fluxes due to vorticity stretching are larger than that due to advection. These transfers too are insensitive to helicity injection.

In this paper we derive formulas for mode-to-mode enstrophy transfers and their associated fluxes. These quantities arise due to the nonlinear interactions. In Sec. [1] we derive the expressions of mode-to-mode enstrophy transfers and the corresponding fluxes. In Sec. [11] after introducing the helical forcing that we use, the enstrophy fluxes are calculated from direct simulations of 3D turbulence using the pseudo-spectral code.
where transfers of enstrophy from modes $F$ is represented by

$$\nabla \cdot u = 0,$$

and $\omega$ is the vorticity. The vorticity $\omega = \nabla \times u$ obeys the following equation:

$$\frac{\partial \omega}{\partial t} = (\omega \cdot \nabla)u - (u \cdot \nabla)\omega + \nu \nabla^2 \omega + \nabla \times F,$$

where $F$ is an external force.

II. ENSTROPHY TRANSFERS AND FLUXES

The motion of an incompressible fluid is described by the Navier-Stokes equations

$$\frac{\partial u}{\partial t} = -(u \cdot \nabla)u - \nabla p + \nu \nabla^2 u + F,$$

$$\nabla \cdot u = 0,$$

where $u$ and $p$ are the velocity and pressure fields, $\nu$ the kinematic viscosity, and $F$ an external force.

The vorticity $\omega$ is given by

$$\nabla \times u,$$

with $k' + p + q = 0$. Using the incompressibility condition, we can show that

$$S_{\omega w}(k'|p, q) + S_{\omega w}(p|q, k') + S_{\omega w}(q|p, k') = 0,$$

where $S_{\omega w}$ represents the advection of vortices, thus involving enstrophy exchange among the vorticity modes $\omega(k')$, $\omega(p)$, and $\omega(q)$. On the other hand we have

$$S_{\omega w}(k'|p, q) + S_{\omega w}(p|q, k') + S_{\omega w}(q|k', p) \neq 0,$$

where $S_{\omega w}$ represents the stretching of vortices, thus either increasing or decreasing the enstrophy $E_\omega$. Incidentally, it implies that enstrophy is not a conserved quantity in 3D hydrodynamics.

Now we can split the above transfers into individual contributions from modes $p$ and $q$:

$$S_{\omega w}(k'|p, q) = S_{\omega w}(k'|p|q) + S_{\omega w}(k'|q|p)$$

$$S_{\omega w}(k'|p, q) = S_{\omega w}(k'|q|p) + S_{\omega w}(k'|p|q),$$

with

$$S_{\omega w}(k'|p|q) = -3[(k' \cdot u(q))(\omega(p) \cdot \omega(k'))],$$

$$S_{\omega w}(k'|q|p) = +3[(k' \cdot \omega(q))(u(p) \cdot \omega(k'))].$$

Here $S_{\omega w}(k'|p|q)$ and $S_{\omega w}(k'|q|p)$ denote two kinds of mode-to-mode enstrophy transfer, both being from $p$ to $k'$ with $q$ acting as a mediator. They arise due to advection and stretching respectively. Here we note that

$$S_{\omega w}(k'|p|q) = -S_{\omega w}(p|k'|q)$$

due to the incompressibility condition, $k \cdot u(k) = 0$; this result is consistent with [9]. Substitution of [13] and [14] into [5] yields

$$\frac{d}{dt} W(k') = \sum_p S_{\omega w}(k'|p|q) + \sum_p S_{\omega w}(k'|p|q)$$

$$- 2\nu k'^2 W(k') + k'^2 \Re\{u^*(k') \cdot F(k')\},$$

and $q = - (k' + p)$.

B. Enstrophy fluxes

Summing the previously discussed mode-to-mode enstrophy transfers over $p$ and $k'$ and depending on whether $p$ and $k'$ belong to the sphere of radius $k_0$ or not, we can define the five following enstrophy fluxes

$$\Pi_{\omega_0}^{\omega_0}(k_0) = \sum_{|p| \leq k_0} \sum_{|k'| \leq k_0} S_{\omega w}(k'|p|q)$$

$$\Pi_{\omega_p}^{\omega_0}(k_0) = \sum_{|p| \leq k_0} \sum_{|k'| > k_0} S_{\omega w}(k'|p|q)$$

$$\Pi_{\omega_k}^{\omega_0}(k_0) = \sum_{|p| > k_0} \sum_{|k'| \leq k_0} S_{\omega w}(k'|p|q)$$

$$\Pi_{\omega_p}^{\omega_k}(k_0) = \sum_{|p| > k_0} \sum_{|k'| > k_0} S_{\omega w}(k'|p|q)$$

$$\Pi_{\omega_k}^{\omega_k}(k_0) = \sum_{|p| > k_0} \sum_{|k'| > k_0} S_{\omega w}(k'|p|q).$$
where the superscript and subscript of $\Pi$ represent respectively the giver and the receiver modes, and the symbols $\leq$ and $\geq$ denote respectively the inside and outside of the sphere of radius $k_0$. Accordingly, $\Pi_{\omega \leq}^u(k_0)$ denotes the flux of enstrophy from all $x$ modes inside the sphere of radius $k_0$ to all $y$ modes outside the sphere. An illustration of these fluxes is given in Figure 1.

Customarily a turbulent flux is associated with some conserved quantity \cite{5}. For example, in two-dimensional hydrodynamic turbulence, the energy and enstrophy fluxes are connected to the corresponding conservation laws. The concept of turbulent flux could be generalized to a more complex scenario when a certain quantity is transferred from one field to another. Here, these fluxes are cross transfers among the two fields. Though enstrophy is not conserved, we define enstrophy fluxes of Eqs. (12) as enstrophy transfers from large/small scale velocity field to large/small vorticity field. The enstrophy flux $\Pi_{\omega \leq}^u(k_0)$ is related to the conservation law related to Eq. (9). Similar issues arise in magnetohydrodynamic turbulence where fluxes for kinetic and magnetic energies are defined even though kinetic and magnetic energies are not conserved individually. We define kinetic-to-magnetic energy fluxes as the energy transfers from large/small scale velocity field to from large/small scale magnetic field; such fluxes are crucial for dynamo action \cite{27,29}.

Summing \cite{16} over all modes outside the sphere of radius $k_0$, we obtain

$$\sum_{|k'|>k_0} \left( \frac{d}{dt} + 2\nu k'^2 \right) W(k') - k'^2 \Re \{ \mathbf{u}^*(k') \cdot \mathbf{F}(k') \} = (\Pi_{\omega \leq}^u + \Pi_{\omega \leq}^< + \Pi_{\omega \geq}^u)(k_0). \quad (22)$$

Performing a sum over all modes inside the sphere of radius $k_0$ yields

$$\sum_{|k'| \leq k_0} \left( \frac{d}{dt} + 2\nu k'^2 \right) W(k') - k'^2 \Re \{ \mathbf{u}^*(k') \cdot \mathbf{F}(k') \} = (\Pi_{\omega \leq}^u + \Pi_{\omega \leq}^> + \Pi_{\omega \geq}^u + \Pi_{\omega \geq}^>) (k_0). \quad (23)$$

Adding (22) and (23) and noticing that (15) implies $\Pi_{\omega \leq}^u + \Pi_{\omega \leq}^> + \Pi_{\omega \geq}^< + \Pi_{\omega \geq}^> = 0$, we obtain

$$\sum_{k'} \left( \frac{d}{dt} + 2\nu k'^2 \right) W(k') - k'^2 \Re \{ \mathbf{u}^*(k') \cdot \mathbf{F}(k') \} = (\Pi_{\omega \leq}^u + \Pi_{\omega \leq}^> + \Pi_{\omega \geq}^< + \Pi_{\omega \geq}^>)(k_0). \quad (24)$$

As the left hand side of (24) does not depend on $k_0$, it implies

$$\frac{d}{dk_0} (\Pi_{\omega \leq}^u + \Pi_{\omega \leq}^> + \Pi_{\omega \geq}^< + \Pi_{\omega \geq}^>) = 0 \quad (25)$$

which is valid at all times. That is, the following sum is independent of $k_0$.

Under a steady state, we obtain

$$\sum_{k'} 2\nu k'^2 W(k') - \sum_{k'_f \leq |k'| \leq k'_f} k'^2 \Re \{ \mathbf{u}^*(k') \cdot \mathbf{F}(k') \} = (\Pi_{\omega \leq}^u + \Pi_{\omega \leq}^> + \Pi_{\omega \geq}^< + \Pi_{\omega \geq}^>)(k_0), \quad (27)$$

where the forcing is employed to the wavenumber band $(k'_f, k'_f)$. A physical interpretation of the above equation is that the enstrophy injected by the external force is transferred to the enstrophy fluxes and the enstrophy dissipation.

III. SIMULATION METHOD AND RESULTS

The Navier-Stokes equations \cite{1} are solved numerically using a fully de-aliased, parallel pseudo-spectral code Tarang \cite{33,50} with fourth-order Runge-Kutta time stepping. For de-aliasing purpose, the 3/2-rule has been chosen \cite{37,38}. The viscosity is set to $\nu = 10^{-3}$ and the simulations are performed with a resolution of $512^3$.

A. Helical forcing

In Fourier space the velocity field satisfies

$$\frac{d}{dt} \mathbf{u}(k) = -i \sum_p (k \cdot \mathbf{u}(q)) \mathbf{u}(p) - \nu k^2 \mathbf{u}(k) + \mathbf{F}(k), \quad (28)$$

where $q = k - p$. Then the energy and helicity, which are defined as

$$E(k) = \frac{|\mathbf{u}(k)|^2}{2}, \quad H(k) = \frac{\mathbf{u}(k) \cdot \mathbf{\omega}(k)^*}{2}, \quad (29)$$
satisfy the following equations
\[
\frac{d}{dt} E(k') = - \sum_p 3 \{ (k' \cdot u(q))(u(p) \cdot u(k')) \} - 2\nu k^2 E(k') + \Re \{ u^*(k') \cdot F(k') \},
\]
\[
\frac{d}{dt} H(k') = \sum_p \Re \{ u(q) \cdot (\omega(p) \times \omega(k')) \} - 2\nu k^2 H(k') + \Re \{ \omega^*(k') \cdot F(k') \},
\]
where \( q = -(k' + p) \).
Following Carati et al. [22][40] and Teimuraz et al. [25][26], the forcing is taken to be of the form
\[
F(k) = \alpha u(k) + \beta \omega(k)
\]
such that
\[
\sum_{k^p_F < |k| \leq k^p_1} \Re \{ u^*(k) \cdot F(k) \} = \epsilon_E,
\]
and
\[
\sum_{k^p_H < |k| \leq k^p_1} \Re \{ \omega^*(k) \cdot F(k) \} = \epsilon_H.
\]
Here, the forcing is applied in the wavenumber band \((k^p_F, k^p_1]\), and \(\epsilon_E\) and \(\epsilon_H\) are the prescribed energy and helicity injection rates. From the above three equations \([32][34]\) we deduce that
\[
\alpha = \frac{1}{2} W_F \epsilon_E - H_F \epsilon_H, \quad \beta = \frac{1}{2} E_F \epsilon_H - H_F \epsilon_E,
\]
where \(E_F, H_F\) and \(W_F\) are respectively the energy, kinematic helicity and enstrophy in the forcing band,
\[
(E_F, H_F, W_F) = \sum_{k^p_F < |k| \leq k^p_1} (E(k), H(k), W(k)).
\]
In our simulations, the forcing band corresponds to \(k^p_F = 2\) and \(k^p_1 = 3\). The main advantage of using the forcing given by \([32][35]\) is that the injection rates of energy and helicity can be set independently. The realizability condition \(|H(k)| \leq k E(k)|\) directly comes from the definitions of energy and helicity and is therefore always satisfied, whatever the values of \(\epsilon_E\) and \(\epsilon_H\). For example even in the case \(\epsilon_E \ll \epsilon_H\), which was studied by Kessar et al. [22], the realizability condition is satisfied, reaching a state close to the maximal helical state \(|H(k)| = k E(k)|\).

In Fig. 2, the time evolution of (a) the total energy dissipation \(2\nu \sum k^2 E(k)\), (b) the total helicity dissipation \(2\nu \sum k^2 H(k)\), and (c) the total enstrophy dissipation \(2\nu \sum k^2 W(k)\) are plotted for the same energy injection rate \(\epsilon_E = 0.1\) and three helicity injections \(\epsilon_H = 0, 0.1, 0.2\). We observe that a statistical steady state is reached for \(t \geq 10\). In this steady state we find that, on an average, \(2\nu \sum k^2 E(k) \approx \epsilon_E\) and \(2\nu \sum k^2 H(k) \approx \epsilon_H\) which is consistent with the prescribed forcing.

**B. Energy and helicity spectra and fluxes**

From \([30][31]\) and following steps analogous to \([5][14]\), the following expression for mode-to-mode energy and helicity transfers can be derived:
\[
S^E(k'|p|q) = -3 \{ (k' \cdot u(q))(u(p) \cdot u(k')) \},
\]
\[
S^H(k'|p|q) = +\Re \{ u(q) \cdot (\omega(p) \times \omega(k')) \},
\]
leading to the following flux definitions
\[
\Pi_{E,H}(k_0) = \sum_{|k'| > k_0} \sum_{|p| < k_0} S^E,H, (k'|p|q).
\]
In Fig. 3 the enstrophy and helicity spectra, normalized by respectively \(\epsilon_E\) and \(\epsilon_H\), show a spectral scaling close to \(k^{-\frac{5}{3}}\) in the inertial range, for the three injection rates of helicity \(\epsilon_H = 0, 0.1, 0.2\). The normalized fluxes shown in the inset are approximately flat in the inertial range, and again insensitive to \(\epsilon_H\).

In Fig. 3 the black dashed curves correspond to the analytical formula for spectra and fluxes derived by Pao [11] for the energy and that are here extended to helicity.
They take the following form
\[
\frac{E(k)}{\epsilon_E} = \frac{K_E}{\epsilon_E} k^{-5/3} \exp \left( -\frac{3}{2} K_E (k/k_d)^{4/3} \right),
\]
\[
\frac{H(k)}{\epsilon_H} = \frac{K_H}{\epsilon_H} k^{-5/3} \exp \left( -\frac{3}{2} K_H (k/k_d)^{4/3} \right),
\]
\[
\frac{\Pi_E(k)}{\epsilon_E} = \exp \left( \frac{3}{2} K_E (k/k_d)^{4/3} \right),
\]
\[
\frac{\Pi_H(k)}{\epsilon_H} = \exp \left( \frac{3}{2} K_H (k/k_d)^{4/3} \right),
\]
where \(K_E\) is Kolmogorov’s constant, and \(K_H\) is another non-dimensional constants, and \(k_d = (\epsilon_E/\nu^{3/4})^{1/4}\) is the Kolmogorov’s wavenumber. Note however that Pao’s spectrum slightly overestimates the dissipation range spectrum \([22]\), and we expect similar discrepancies for the kinetic helicity that may show up in high-resolution simulations. These issues may be related to intermittency and enhanced dissipation due to bottleneck effect \([43, 44]\).

C. Results of enstrophy fluxes

The enstrophy fluxes given in \([17, 21]\) have been calculated for \(\epsilon_H = 0, 0.1, 0.2\). They are found to be insensitive to \(\epsilon_H\). Therefore only the results corresponding to \(\epsilon_H = 0\) are presented in Fig. 4. The manifestation of this independence is consistent with many numerical results which showed that the forward flux of kinetic energy is also insensitive to kinetic helicity injection at large scale \([45]\). Crucial changes can be expected when significant imbalance between helical modes of different signs is imposed by helicity forced over a wide range of scales \([22]\).

The flux \(\Pi_u^\omega\) is positive suggesting a direct cascade of enstrophy. In the inertial range it obeys a \(k^2\) scaling law. However, we observe that in the inertial range, \(\Pi_u^\omega\) is larger than \(\Pi_u^\omega\), showing that the enstrophy flux due to stretching is larger than the one due to advection. We also observe that \(\Pi_u^\omega\) is always negative, indicating that the small-scale velocity fluctuations squeeze (not stretch) the large scale vorticity. It also obeys a \(k^2\) scaling law in the inertial range.

The flux \(\Pi_u^\omega(k_0)\) is an accumulated sum of all the enstrophy transferred from \(u^\omega\) to \(\omega^\omega\) up to the radius \(k_0\). All the wavenumber shells have positive sign for these transfer, hence \(\Pi_u^\omega(k_0)\) is an increasing function of \(k_0\). Note that \(\Pi_u^\omega(\infty)\) is the net enstrophy transfer. The complementary flux, \(\Pi_u^\omega(k_0)\), has an opposite behaviour; that is, it decreases with \(k_0\). All the fluxes other than \(\Pi_u^\omega(k_0)\) vanish for \(k_0 > k_d\) because the fluctuations vanish in this range.

It is found that \((\Pi_u^\omega + \Pi_u^\omega + \Pi_u^\omega + \Pi_u^\omega)(k_0) \approx 157.3\), independently of \(k_0\), to a precision of about \(10^{-10}\). This value is comparable to the enstrophy dissipation which is about \(160.9\) (Fig. 2); the difference between the two quantities is \(\sum_{k_0^\omega < |k| < k_0^\omega} \sum_{|k'| < k_0^\omega} |k'|^2 \Re(u^k \cdot F(k')) \approx 3.6\); these computations are consistent with \([27]\).

IV. CONCLUSIONS

Using direct numerical simulations and varying the injection rate of helicity we find that injecting helicity does not change the results in terms of energy and helicity spectra and fluxes, and also in terms of enstrophy fluxes. The energy and helicity spectra and fluxes follow rather well the formula of Pao \([41]\) that we extended to helicity. Of course, some correction due to intermittency and
bottleneck effect should be added, but these issues are beyond the scope of the present paper. These results are consistent with the predictions that the inertial range properties of hydrodynamic turbulence is independent of kinetic helicity [13, 24].

The main objective of this paper is to introduce the derivation of mode-to-mode enstrophy transfers using which we compute the enstrophy fluxes. There are five different enstrophy fluxes. Four of them, \( \Pi_{u<}^{\omega<}, \Pi_{u<}^{\omega>}, \Pi_{u>}^{\omega<}, \Pi_{u>}^{\omega>} \), correspond to vorticity stretching, the fifth one \( \Pi_{u>}^{\omega<} \) is due to the advection of vorticity by the flow. It is remarkable that in the inertial range \( \Pi_{u>}^{\omega<} \) is larger than \( \Pi_{u>}^{\omega>} \) implying that the enstrophy flux due to vorticity stretching is larger than the one due to advection of vorticity. The sum of the first four fluxes corresponds to the contribution from all velocity modes to all vorticity modes and therefore should not depend on the wavenumber, that we confirm numerically with a precision better than \( 10^{-10} \).

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