THE REGULARITY THEORY FOR THE DOUBLE OBSTACLE PROBLEM
FOR FULLY NONLINEAR OPERATOR

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ABSTRACT. In this paper, we prove the existence and uniqueness of $W^{2,p}$ ($n < p < \infty$) solutions of a double obstacle problem with $C^{1,1}$ obstacle functions. Moreover, we show the optimal regularity of the solution and the local $C^1$ regularity of the free boundary. In the study of the regularity of the free boundary, we deal with a general problem, the no-sign reduced double obstacle problem with an upper obstacle $\psi$, $F(D^2u, x) = \chi_{\Omega(u) \cap \{u < \psi\}} + F(D^2\psi, x) \chi_{\Omega(u) \cap \{u = \psi\}}$, $u \leq \psi$ in $B_1$, where $\Omega(u) = B_1 \setminus (\{u = 0\} \cap \{Vu = 0\})$.

1. Introduction

Obstacle problems with a single obstacle appear in various fields of study such as porous media, elasto-plasticity, optimal control, and financial mathematics, see [8, 4]. The regularity of the solutions and the free boundaries of the problems have been actively studied by [3, 6, 11, 13, 2].

The double obstacle problem, which is the obstacle problem with two obstacles, originates in the study of optimal investment problems with transaction costs, the game of tug-of-war, and semiconductor devices, (see [18] and the references therein). Recently, global homogeneous solutions to the double obstacle problem, with homogeneous obstacles was considered by [1] and the regularity of the free boundaries of the double obstacle problem for Laplacian was obtained by [15].

In this paper, we discuss the regularity of the solution and the free boundary for the double obstacle problem of the fully nonlinear operator. Precisely, we prove the existence and uniqueness of $W^{2,p}$ ($n < p < \infty$) solutions of double obstacle problem for the fully nonlinear operator in a domain $D \subset \mathbb{R}^n$,

\[
\begin{align*}
F(D^2u, x) &\geq 0, & \text{in } \{u > \phi_1\} \cap D, \\
F(D^2u, x) &\leq 0, & \text{in } \{u < \phi_2\} \cap D, \\
\phi_1(x) &\leq u(x) \leq \phi_2(x) & \text{in } D, \\
u(x) & = g(x) & \text{on } \partial D,
\end{align*}
\]

(FB)

with $\phi_1, \phi_2 \in C^{1,1}(\overline{D})$, $\partial D \in C^{2,\alpha}$, $g \in C^{2,\alpha}(\overline{D})$ and $\phi_1 \leq g \leq \phi_2$ in $\partial D$. The optimal ($C^{1,1}$) regularity of the solution $u$ to (FB) is also obtained. Moreover, we have $C^1$ regularity of the free boundary $\partial\{u = \psi_1\}$ of (FB), by studying the regularity of the free boundary for a general problem $\{FB_{\text{no sign local}}\}$, which contains a reduced version $\{FB_{\text{local}}\}$ of (FB).

2010 Mathematics Subject Classification. Primary 35R35, 35B65.

Key words and phrases. free boundary problem, obstacle problem, double obstacle problem, fully nonlinear operator.
Specifically, by subtracting the lower obstacle $\phi_1$ from the solution $u$, the problem $(FB)$ is reduced to the double obstacle problem with the zero lower obstacle:

$$F(D^2u, x) = f(\chi_{[0,c_u<0]} + F(D^2\psi, x)\chi_{[0,c_u=0]}), \quad 0 \leq \psi \leq u \quad \text{in } B_1,$$

with $\psi \in C^{1,1}(B_1) \cap C^2(\{|\psi| > 0\})$, $f \in C^{0,1}(B_1)$, see Subsection 1.2 for more detail. Furthermore, we consider a general problem $(FB_{\text{nosign local}})$ of $(FB_{\text{local}})$, which relaxes the sign condition of $u$ in $(FB_{\text{local}})$ (i.e., the solution could be below the lower zero obstacle):

$$F(D^2u, x) = f(\chi_{\Omega \cap [u<\psi]} + F(D^2\psi, x)\chi_{\Omega \cap [u=\psi]}), \quad u \leq \psi \quad \text{in } B_1,$$

where $\Omega(u) := B_1 \setminus \{(u = 0) \cap \{\nabla u = 0\}\}$ and $f \in C^{0,1}(B_1)$, with the upper obstacle function

$$\psi \in C^{1,1}(B_1) \cap C^2(\Omega(\psi)), \quad \Omega(\psi) := B_1 \setminus \{(\psi = 0) \cap \{\nabla \psi = 0\}\}.$$

By obtaining the regularity of the free boundary $\Gamma(u) := \partial \Omega(u) \cap B_1$ of $(FB_{\text{nosign local}})$, we have the regularity of the free boundary $\partial(u = \psi_1)$ for $(FB)$ as a corollary, see Corollary 1.5.

The result of the problem $(FB_{\text{nosign local}})$ is a generalization of the theory for the no-sign single obstacle problem ($\psi = \infty$ in $(FB_{\text{nosign local}})$) studied in [6, 7]. Moreover, it is an extension for the result of the problem for Laplacian in [15].

1.1. Methodology and contents. The main idea to have the regularity of the free boundary, $\Gamma(u)$ of $(FB_{\text{nosign local}})$, which corresponds to $\partial(u = \psi_1)$ in $(FB)$, is considering the upper obstacle $\psi$ as a solution of the single obstacle problem, $(FB_{\text{local}})$ with $\psi = \infty$. Additionally, we apply the method of blowup to the upper obstacle $\psi$ with the thickness assumption of the zero set of $\psi$ at 0, which means that the zero set near the free boundary point 0 is sufficiently large in some sense, see Subsection 1.5. Then, the blowup $\psi_0$ of the upper obstacle $\psi$ is of the form $c(x_+^*)^2$, $c > 0$ and it is crucially used to have the regularity of the free boundary.

The main difficulty to have the regularity of the free boundary is the lack of monotonicity formulas, used in the problem for Laplacian in [15]. Precisely, in the paper, by using the formulas, we have the classification of global solutions, a global solution of $(FB_{\text{nosign local}})$ in whole domain $\mathbb{R}^n$ is of the form $c(x_+^*)^2$, $c > 0$. However, it is not applicable for the fully nonlinear case due to the nonlinearity.

Hence, for the fully nonlinear operator, we focus on the fact that the global solution $u$ is zero in a half-space $\{x_n \leq 0\}$. Then, the optimal $C^{1,1}$ regularity for $u$ implies that $\partial u / x_n$ is finite in $\mathbb{R}^n$. Therefore, we prove that $\partial u / x_n$ is identically zero in $\mathbb{R}^n$ for any direction $e \in S^{n-1} \cap e_n^\perp$, which implies that $u$ is an one-dimensional function and it is of the form $c(x_+^*)^2$, for a positive constant $c$. It is noticeable that similar arguments for the second derivative have been introduced in [16], and the one for the first derivative as above has been considered in [10] in the study of the free boundary near the fixed boundary.

Now, we summarize the contents of this paper. In Subsection 2.1, we have the existence and uniqueness of the $W^{2,p}$ ($1 < p < \infty$) solution of $(FB)$ by using a penalization method, Proposition 2.1. Since the obstacles $\phi_1$ and $\phi_2$ have regularity, we consider the penalization method with bounded penalty term. In
Subsection 2.2 we have the optimal regularity of the solution of \( \{FB\} \) Proposition 2.2 by using the quadratic growth of the solution of \( \{FB_{\text{local}}\} \), Proposition 2.2.

In Subsection 3.2, we obtain the classification of global solutions Proposition 3.3, the global solution \( u \) to \( \{FB_{\text{no sign local}}\} \) with the upper obstacle \( \psi = c(x_n^+)^2 \) is also of the form \( c_1(x_n^+)^2 \), for some \( c, c_1 > 0 \), by using the argument discussed in the previous paragraph.

Therefore, in Subsection 3.3 we prove the directional monotonicity and the proof of the regularity for the free boundaries, Theorem 1.4 using the methods considered in [13][19] and references therein.

Remark 1.1. The reason to set the regularity of the obstacle functions \( \phi_1, \phi_2 \) in \( \{FB\} \), and \( \psi \) in \( \{FB_{\text{local}}, FB_{\text{no sign local}}\} \) to \( C^{1,1} \) is closely related to the main idea introduced in the first paragraph of the previous subsection. Indeed, to apply the method of blowup to the upper obstacle \( \psi \), the regularity of \( \psi \) should be at least \( C^{1,1} \). Furthermore, the thickness assumption of the zero set of \( \psi \) means that the region where the equation \( F(D^2 \psi) = 0 \) satisfies \((\psi = 0)\) is sufficiently large. Hence, \( D^2 \psi \) should not be continuous, and therefore, the regularity of the upper obstacle \( \psi \) should not be better than \( C^{1,1} \).

1.2. Reduction of \( \{FB\} \). By subtracting the lower obstacle \( \phi_1 \) from the solution \( u \), we reduce the problem \( \{FB\} \) to the double obstacle problem with zero lower obstacle. Specifically, we define \( F(M,x) := F(M + D^2 \phi_1, x) - F(D^2 \phi_1, x) \) and \( v := u - \phi_1 \), where \( u \) is a \( W^{2,p} \) \((n < p < \infty)\) solution of \( \{FB\} \). Then,

\[
\tilde{F}(D^2 v, x) = F(D^2 u, x) - F(D^2 \phi_1, x) \\
= -F(D^2 \phi_1, x) \chi_{[\phi_1 < u < \phi_2]} + (F(D^2 \phi_2, x) - F(D^2 \phi_1, x)) \chi_{[\phi_1 < u = \phi_1]} \\
= -F(D^2 \phi_1, x) \chi_{[0 < \psi < \phi_2 - \phi_1]} + \tilde{F}(D^2 (\phi_2 - \phi_1), x) \chi_{[\psi = \phi_2 - \phi_1]}.
\]

By replacing \( f = -F(D^2 \phi_1, x) \), \( \psi = \phi_2 - \phi_1 \) and reusing \( v = u - \phi_1 \) by \( u \), \( \tilde{F}(M, x) = F(M + D^2 \phi_1, x) - F(D^2 \phi_1, x) \) by \( F(M, x) \), \( u \) is a \( W^{2,p} \) solution of

\[
F(D^2 u, x) = f \chi_{[0 < \psi < \phi_1]} + F(D^2 \psi, x) \chi_{[\psi = \phi_1]} \quad \text{a.e. in } D, \quad (1)
\]

with \( 0 \leq u \leq \psi \) in \( D \), \( f \in L^\infty(D) \), and \( \psi \in C^{1,1}(\overline{D}) \). Since we discuss the local regularity of the free boundaries, we consider a local form \( \{FB_{\text{local}}\} \) of (1).
1.3. Notations. We will use the following notations throughout the paper.

- \( C, C_0, C_1 \) generic constants
- \( \chi_E \) the characteristic function of the set \( E, (E \subset \mathbb{R}^n) \)
- \( E \) the closure of \( E \)
- \( \partial E \) the boundary of a set \( E \)
- \( |E| \) \( n \)-dimensional Lebesgue measure of the set \( E \)
- \( B_r(x), B_r \) \( \{ y \in \mathbb{R}^n : |y - x| < r \} \), \( B_r(0) \)
- \( \Omega(u), \Omega(\psi) \) see Equation [FB_{n,sign loca}]\( ^{1,2}\)
- \( \Lambda(u), \Lambda(\psi) \) \( B_1 \setminus \Omega(u), B_1 \setminus \Omega(\psi) \)
- \( \Gamma(u), \Gamma(\psi)(u) \) \( \partial \Lambda(u) \cap B_1, \partial(\psi = \psi) \cap B_1 \)
- \( \Gamma(\psi)(u) \) \( \Gamma(u) \cap \Gamma(\psi)(u) \) (the intersection of free boundaries)
- \( \partial_\nu, \partial_v \) first and second directional derivatives
- \( P_r(M), P_{\infty}(M) \) see Definition [2,1,1,5,2]
- \( \delta_\nu(u, x), \delta_\nu(u) \) see Definition [1,1,1]
- \( \mathcal{P}^+, \mathcal{P}^- \) Pucci operators
- \( S, \widetilde{S}, \tilde{S}, S^* \) the viscosity solution spaces for the Pucci operators

We refer to the book of Caffarelli-Cabrè [3], for the definitions of the viscosity solution, Pucci operators \( \mathcal{P}^+ \) and the spaces of viscosity solutions of the Pucci operators \( S(\lambda_0, \lambda_1, f), \widetilde{S}(\lambda_0, \lambda_1, f), \tilde{S}(\lambda_0, \lambda_1, f) \), and \( S^*(\lambda_0, \lambda_1, f) \).

1.4. Conditions on \( F = F(M, x) \). We assume that the fully nonlinear operator \( F(M, x) \) satisfies the following conditions:

1. **(F1)** \( F(0, x) = 0 \) for all \( x \in \mathbb{R}^n \).
2. **(F2)** \( F \) is uniformly elliptic with ellipticity constants \( 0 < \lambda_0 \leq \lambda_1 < +\infty \), that is
   \[
   \lambda_0|N| \leq F(M + N, x) - F(M, x) \leq \lambda_1|N|,
   \]
   for any symmetric \( n \times n \) matrix \( M \) and positive definite symmetric \( n \times n \) matrix \( N \).
3. **(F3)** \( F(M, x) \) is convex in \( M \) for all \( x \in \mathbb{R}^n \).
4. **(F4)**
   \[
   |F(M, x) - F(M, y)| \leq C|\|M\|-\|y\|^\alpha,
   \]
   for some \( 0 < \alpha \leq 1 \).
5. **(F4')**
   \[
   |F(M, x) - F(M, y)| \leq C(\|M\| + C_1)|x - y|^{\alpha},
   \]
   for some \( 0 < \alpha \leq 1 \).

**Remark 1.2.** We define oscillations of the fully nonlinear operator \( F \) in the variable \( x \) by

\[
\beta_F(x, x_0) := \sup_{M \in S([0])} \frac{|F(M, x) - F(M, x_0)|}{\|M\|}
\]

and

\[
\tilde{\beta}_F(x, x_0) := \sup_{M \in \tilde{S}} \frac{|F(M, x) - F(M, x_0)|}{\|M\| + 1}.
\]

For any fixed \( x_0 \), the condition (F4) implies that \( \beta_F \) and \( \tilde{\beta}_F \) are \( C^\alpha \) at \( x_0 \). Then, \( \beta_F \) and \( \tilde{\beta}_F \) satisfy the conditions for the \( W^{2,p} \) and \( C^{2,\alpha} \) estimates of viscosity solutions \( v \) to \( F(D^2v, x) = f(x) \), respectively (see Chapter 7 and 8 in [5] and [20]).

Hence, in Section 2 we assume that \( F \) satisfies (F4) and the \( W^{2,p} \) estimate is used in the proof of the existence and uniqueness of \( W^{2,p} \) solution \( (F_{n,sign loca}) \), Proposition 2.1, and \( C^{2,\alpha} \) estimate is used in the proof of optimal regularity of solution, Proposition 2.3.
In Section 3, we study the regularity of the free boundary for the reduced forms, $\nabla F_{\text{local}}$ and $\nabla F_{\text{nosign local}}$. If $F$ is the fully nonlinear operator of $\nabla F$, then $\tilde{F} = F(M + D^2 \tilde{g}, x) - F(D^2 \tilde{g}, x)$, introduced in Subsection 1.2 is the fully nonlinear operator in $\nabla F_{\text{local}}$ and $\nabla F_{\text{nosign local}}$. If $F(M, x)$ satisfies (F4), then $\tilde{F}(M, x)$ satisfies $(F4)'$ and $\tilde{F}(x, x_0)$ is $C^\alpha$ for the variable $x$ at fixed $x_0 \in \mathbb{R}^n$. Hence, we have the $C^{2, \alpha}$ estimate of viscosity solutions $v$ to $\tilde{F}(D^2 v, x) = f(x)$, and it is used in Lemma 3.4 to have that the blowup $u_0$ of $u$ of $\nabla F_{\text{nosign local}}$ is a global solution.

We note that, in Section 3 when we study the regularity of the free boundary for the reduced forms, $\nabla F_{\text{local}}$ and $\nabla F_{\text{nosign local}}$, we denote the fully nonlinear operator by $F$, instead of $\tilde{F}$. Hence, we assume $(F4)'$ for a fully nonlinear operator $F$, in Section 3.

1.5. Definitions. In this subsection, we define the rescaling, blowup, thickness of coincidence sets $\Lambda(u)$ and $\Lambda(u) \cap \Lambda(\psi)$, and solution spaces. These concepts are already discussed in the literature of the obstacle problem, e.g. [3, 4, 12, 19, 7, 15]. We introduce the concepts for $\nabla F_{\text{nosign local}}$, for the reader's convenience.

In order to find the possible configuration of the solution near the free boundary, the following blowup concept has been used heavily at [3, 8] and other references. For a $W^{2, p}$ solution, $u$, of $\nabla F_{\text{nosign local}}$ in $B_r$, we define the rescaling function of $u$ at $x_0 \in \partial \Lambda(u) \cap B_r$ with $\rho > 0$ as

$$u_\rho(x) = u_{\rho, x_0}(x) := \frac{u(x_0 + \rho x) - u(x_0)}{\rho^2}, \quad \text{for } x \in (B_r - x_0)/\rho.$$ 

By optimal $(C^{1,1})$ regularity of solution $u$ (Theorem 1.3), for any sequence $\rho_i \to 0$, there exists a subsequence $\rho_i$, of $\rho_i$, and $u_0 \in C^{1,1}_{\text{loc}}(\mathbb{R}^n)$ such that

$$u_{\rho_i} \to u_0$$ 

uniformly in $C^{1,\alpha}_{\text{loc}}(\mathbb{R}^n)$ for any $0 < \alpha < 1$.

The limit function $u_0$ is a blowup of $u$ at $x_0$.

**Definition 1.1.** (Thickness of the coincident set $\Lambda(u)$) We denote by $\delta_r(u, x)$ the thickness of $\Lambda(u)$ in $B_r(x)$, i.e.,

$$\delta_r(u, x) := \frac{MD(\Lambda(u) \cap B_r(x))}{r},$$

where $MD(A)$, the minimal diameter of subset $A$ of $\mathbb{R}^n$, is the least distance between two parallel hyperplanes containing $A$. We will use the abbreviated notation $\delta_r(u)$ for $\delta_r(u, 0)$.

To briefly explain the theory of the regularity of free boundary in Section 5 we define classes of local and global solutions of the problem.

**Definition 1.2.** (Local solutions) We say a $W^{2, p}$ function $u$ belongs to the class $P_r(M)$ ($0 < r < \infty$), if $u$ satisfies

(i) $F(D^2 u, x) = f \chi_{\Omega(u) \cap \{u < \psi\}} + F(D^2 \psi, x) \chi_{\Omega(u) \cap \{u = \psi\}}, \quad u \leq \psi \quad \text{in } B_r,$

(ii) $\|D^2 u\|_{L^p(B_r)} \leq M$,

(iii) $0 \in \Gamma(u),$

where $f \in C^{0, \alpha}(B_r)$ and $\psi \in C^{1,1}(B_r) \cap C^{2,\alpha}(\Omega(\psi))$.

**Definition 1.3.** (Global solutions) We say a $W^{2, p}$ function $u$ belongs to the class $P_\infty(M)$, if $u$ satisfies

(i) $F(D^2 u) = f \chi_{\Omega(u) \cap \{u < \psi\}} + F(D^2 \psi) \chi_{\Omega(u) \cap \{u = \psi\}}, \quad u \leq \psi \quad \text{in } \mathbb{R}^n$,

(ii) $F(D^2 \psi) = a \chi_{\Omega(\psi)}$ in $\mathbb{R}^n$, for a constant $a > 1$,

(iii) $\|D^2 u\|_{\infty, \mathbb{R}^n} \leq M$,

(iv) $0 \in \Gamma(u).$
1.6. Main Theorems. The purposes of the paper are to obtain the existence, uniqueness, and optimal regularity of the solution for the double obstacle problem and the regularity of the free boundary. The main theorems are as follows:

**Theorem 1.3** (Existence, uniqueness and optimal regularity). Assume $F$ satisfies (F1)-(F4). Then the following holds:

(i) For $n < p < \infty$, there exist $W^{2,p}$ solution $u$ of $\{FB\}$ with $\phi_1, \phi_2 \in C^{1,1}(\overline{D}), \partial D \in C^{2,\alpha}$, $g \in C^{2,\alpha}(\overline{D})$, and $\phi_1 \leq g \leq \phi_2$ in $D$.

(ii) For any compact set $K$ in $D$, we have

$$\|u\|_{C^{1,1}(K)} \leq M < \infty,$$

for some constant $M = M(\|u\|_{L^\infty(D)}, \|\phi_1\|_{C^{1,1}(D)}, \|\phi_2\|_{C^{1,1}(D)}, dist(K, \partial D)) > 0$.

**Theorem 1.4** (Regularity of free boundary). Assume $F \in C^1$ satisfies (F1)-(F3) and (F4)' and let $u \in P_1(M)$ with an upper obstacle $\psi$ such that

$$0 \in \partial \Omega(\psi), \lim_{x \to 0, x \in \Sigma(\psi)} F(D^2\psi(x), x) > f(0), \quad f \geq c_0 > 0 \text{ in } B_1,$$

and

$$\inf \{F(D^2\psi, x), F(D^2\psi, x) - f\} \geq c_0 > 0 \text{ in } \Omega(\psi).$$

Suppose

$$\delta_r(u, \psi) := \frac{MD(\Lambda(u) \cap \Lambda(\psi) \cap B_r)}{r} \geq \epsilon_0 \quad \text{for all } r < 1/4. \quad (2)$$

Then there is $r_0 = r_0(u, c_0, \|\nabla F\|_{L^\infty(B_{r_0}(0) \times (\overline{B_1} \cup \partial B_1))}, \|\nabla f\|_{L^\infty(\overline{B_1})}) > 0$, such that $\Gamma(u) \cap B_{r_0}$ is a $C^1$ graph.

Since Theorem 1.4 is for the reduced forms $\{FB_{\text{local}}\}$ and $\{FB_{\text{nonlocal}}\}$ with $f(M, x) = F(M + D^2\psi, x) - F(D^2\psi, x)$, where $F$ is the fully nonlinear operator in $\{FB\}$, the local regularity of the free boundary for $\{FB\}$ is obtained as a corollary.

**Corollary 1.5.** Let $u, \phi_1$, and $\phi_2$ be as in Theorem 1.3 and we assume that $\phi_2 - \phi_1 \in C^{2,1}(\overline{\Omega})$ and $\phi_1 \in C^{1,1}(D)$. Suppose that $0 \in \partial[u > \phi_1] \cap \partial[u < \phi_2]$.

$$0 \in \partial\{\phi_1 < \phi_2\}, \lim_{x \to 0, x \in \partial\{\phi_1 < \phi_2\}} F(D^2\phi_2(x), x) > 0, \quad -F(D^2\phi_1, x) \geq c_0 > 0 \text{ in } B_1,$$

and

$$\inf \{F(D^2\phi_2, x), F(D^2\phi_1, x), F(D^2\phi_2, x)\} \geq c_0 > 0 \text{ in } \{\phi_1 < \phi_2\}.$$

Suppose

$$\delta_r(\phi_2 - \phi_1, z) \geq \epsilon_0 \quad \text{for all } r < 1/4, z \in \partial\{\phi_1 < \phi_2\}$$

and

$$\delta_r(\phi_2 - \phi_1, z) \geq \epsilon_0 \quad \text{for all } r < 1/4.$$

Then, there is $r_0 = r_0(u - \phi_1, c_0, \|\nabla F\|_{L^\infty(B_{r_0}(0) \times (\overline{B_1} \cup \partial B_1))}, \|\nabla F\|_{L^\infty(\overline{B_1})}) > 0$, such that $\partial[u = \phi_1] \cap B_{r_0}$ is a $C^1$ graph.

**Remark 1.6.** We assume the thickness of $\Lambda(\psi)$ and $\Lambda(u)$ satisfies the assumption (2) in Theorem 1.4. Then, the assumption implies that

$$\delta_r(u_0, \psi_0) \geq \epsilon_0 \quad \text{for all } r > 0, \quad (3)$$

for any blowups $u_0$ and $\psi_0$ of $u$ and $\psi$ at 0, respectively. By (3), we have that the blowups $\psi_0$ of $\psi$ are the half-space type upper obstacle, $\psi_0 = \frac{1}{2}(x_1^+)^2$, in an appropriate system of coordinates, see e.g. Proposition 4.7 of [14]. Furthermore, (3) implies that the blowup
2. Existence, Uniqueness and Optimal Regularity

2.1. Existence, uniqueness of $W^{2,p}$ solution. For the single obstacle problem in [8, 13], the authors used an unbounded penalization term $\beta_\epsilon(z)$, such that $\beta_\epsilon(z)$ to $-\infty$, for $z < 0, \epsilon \to 0$. Then, $C^2$ regularity for obstacle function $\phi$ is needed to show that $\beta_\epsilon(u_\epsilon - \phi)$ is bounded, where $u_\epsilon$ is a solution of the penalization problem for the single obstacle problem with the obstacle function $\phi$. On the other hand, in this subsection, we consider a penalization problem (4) with a new penalty term $\beta_\epsilon$, whose $L^\infty$ norms are uniformly bounded by a constant that depends only on $C^{1,1}$ norms of the obstacle functions $\phi_1$ and $\phi_2$. Then, we have solutions $u_\epsilon$ of the penalization problem (4) such that $W^{2,\infty}$ norms of $u_\epsilon$ are uniformly bounded. Hence, there is a limit function $u_0$ of $u_\epsilon$ as $\epsilon \to 0$ in $W^{2,p}$ sense. Finally, we prove that the limit function $u_0$ of $u_\epsilon$ is the unique solution of (FB) with the obstacle functions $\phi_1 \in C^{1,1}$ and $\phi_2 \in C^{1,1}$.

**Proposition 2.1.** Assume $F$ satisfies (F1)-(F4). For $n < p < \infty$, there is a unique viscosity solution $u \in W^{2,p}(D)$ of (FB) with

$$
\|u\|_{W^{2,p}(D)} \leq C \left( \|F(D^2\phi_1, x)\|_{L^\infty(D)}, \|F(D^2\phi_2, x)\|_{L^\infty(D)} \right),
$$

where $\phi_1, \phi_2 \in C^{1,1}(D)$, $\partial D \in C^{2,\alpha}$, $g^{2,\alpha} \in C(D)$, and $\phi_1 \leq g \leq \phi_2$ on $\partial D$.

**Proof.** Let $\beta_1(z) \in C^{\infty}(\mathbb{R})$ be a function satisfying

$$
\begin{align*}
\beta_1(z) &= \begin{cases} 
\beta_1(-z) & \text{if } z < 0, \\
\beta_1(z) = 0 & \text{if } z > 0, \\
\beta_1(z) \leq 0 & \text{in } z \in \mathbb{R},
\end{cases}
\end{align*}
$$

and define $\beta_\epsilon(z) := \beta_1(z/\epsilon)$, for $\epsilon > 0$. We consider a penalization problem,

$$
\begin{align*}
&F(D^2u, x) = \beta_\epsilon(u - \phi_1) - \beta_\epsilon(\phi_2 - u) & \text{in } D, \\
&u(x) = g(x) & \text{on } \partial D.
\end{align*}
$$

By the $W^{2,p}$ regularity in [5] and [20], for each $v \in C^{0,\alpha}(D)$ ($0 < \alpha < 1$) there is a unique solution $w \in W^{2,p}(D)$ ($n < p < \infty$), of

$$
\begin{align*}
&F(D^2w, x) = \beta_\epsilon(v - \phi_1) - \beta_\epsilon(\phi_2 - v) & \text{in } D, \\
w(x) = g(x) & \text{on } \partial D,
\end{align*}
$$

with

$$
\|w\|_{W^{2,p}(D)} \leq \|w\|_{L^\infty(D)} + \|g\|_{W^{2,p}(D)} + \|\beta_\epsilon(v - \phi_1) - \beta_\epsilon(\phi_2 - v)\|_{L^p(D)},
$$

By the boundedness of $\beta_\epsilon$, we have
\[ \|u\|_{W^{2,p}(D)} \leq C_0, \]  

(5)

where \( C_0 \) is a constant which is independent for \( \epsilon \) and \( v \).

Let us consider a map \( S \) such that \( w = Sv \) for \( v \in C^{0,\alpha}(D) \). Since \( W^{2,p} \) space is compactly embedded in \( C^{0,\alpha} \), the boundedness of \( W^{2,p} \) norm of \( w \) implies that \( S|_{B_{C_0}} : B_{C_0} \rightarrow B_{C_0} \) is a compact map, where \( B_{C_0} \) is the \( C_0 \) ball centered at \( 0 \) in \( C^{0,\alpha}(D) \) and \( S|_{B_{C_0}} \) is the function \( S \) from \( B_{C_0} \) to \( B_{C_0} \) defined by \( S|_{B_{C_0}}(v) = S(v) \). Furthermore, the \( W^{2,p} \) estimate implies that \( S|_{B_{C_0}} \) is continuous. Hence, by Schauder’s fixed-point theorem, there is a function \( u_\epsilon \in B_{C_0} \) such that \( S|_{B_{C_0}}u_\epsilon = u_\epsilon \), i.e., there is \( u_\epsilon \in W^{2,p}(D) \) such that \( u_\epsilon \) is a solution of (4) and \( \|u_\epsilon\|_{W^{2,p}(D)} \leq C_0 \), where \( C_0 \) does not depend on \( \epsilon \). Then, there is a sequence \( \epsilon = \epsilon_i \rightarrow 0 \) and \( u \in W^{2,p}(D) \) such that

\[ u_\epsilon \rightarrow u \quad \text{weakly in } W^{2,p}(D), \quad n < p < \infty. \]

Thus, we have that \( \|u\|_{W^{2,p}(D)} \leq C_0 \) and

\[ u_\epsilon \rightarrow u \quad \text{uniformly in } D. \]

We claim that \( u \) is a solution of the double obstacle problem (4). First, we are going to prove that \( F(D^2u, x) \geq 0 \) in \( \{u > \phi_1\} \cap D \). Let \( x_0 \) be a point in \( \{u > \phi_1\} \cap D \) and let \( \delta = (u(x_0) - \phi_1(x_0))/2 \). Then, by the uniform convergence of \( u_\epsilon \) to \( u \), there is a ball \( B_r(x_0) \subset \{u > \phi_1\} \cap D \) and \( \epsilon_0 > 0 \) such that \( u_\epsilon - \phi_1 \geq \delta \) in \( B_r(x_0) \), for \( \epsilon < \epsilon_0 \). By the definition of \( \beta_\epsilon \), for \( \epsilon \leq \min\{\epsilon_0, \delta\} \), we have

\[ \beta_\epsilon(u_\epsilon - \phi_1) \equiv 0 \quad \text{and} \quad F(D^2u_\epsilon, x) \geq 0 \quad \text{in } B_r(x_0). \]

By the closedness of the family of viscosity solutions, Proposition 2.9 of [3], the uniform convergence of \( u_\epsilon \) to \( u \) implies that \( F(D^2u, x) \geq 0 \) in \( B_r(x_0) \). Since \( x_0 \in \{u > \phi_1\} \cap D \) is arbitrary, we obtain \( F(D^2u_\epsilon, x) \geq 0 \) in \( \{u > \phi_1\} \cap D \). We also have \( F(D^2u, x) \leq 0 \) in \( \{u < \phi_2\} \cap D \), from the same argument as above.

Next, we prove that \( \phi_1 \leq u \leq \phi_2 \) in \( D \). Suppose that \( \{u < \phi_1\} \cap D \) is not empty and let \( x_0 \) be a point in \( \{u < \phi_1\} \cap D \). Then, by the uniform convergence of \( u_\epsilon \), there is a ball \( B_r(x_0) \) such that

\[ \beta_\epsilon(u_\epsilon - \phi_1) = -\max \{\|F(D^2\phi_1, x)\|_{L^p(D)}, \|F(D^2\phi_2, x)\|_{L^p(D)}\}, \beta_\epsilon(\phi_2 - u_\epsilon) \equiv 0 \quad \text{in } B_r(x_0) \]

and

\[ F(D^2u_\epsilon, x) \leq F(D^2\phi_1) \quad \text{in } B_r(x_0), \quad \text{for sufficiently small } \epsilon. \]

Consequently, \( F(D^2u, x) \leq F(D^2\phi_1) \) in \( \{u < \phi_1\} \cap D \). Moreover, the boundary condition \( \psi_1 \leq u = g \) on \( \partial D \) implies \( \{u < \phi_1\} \cap D \not= \emptyset \) and \( u \equiv \phi_1 \) on \( \partial(\{u < \phi_1\} \cap D) \). Hence, by the maximum principle, we have \( u \geq \phi_1 \) in \( \{u < \phi_1\} \cap D \) and it is a contradiction. The same method implies that \( \{u > \phi_2\} \cap D = \emptyset \) and \( \phi_1 \leq u \leq \phi_2 \) in \( D \). Hence, \( u \) is a solution of (4).

In order to prove the uniqueness, we suppose that there are two solutions \( u_1 \) and \( u_2 \) of (4) and \( \{u_1 < u_2\} \cap D \) is not empty. In \( \{u_1 < u_2\} \cap D \), the conditions \( \phi_1 \leq u_1 \leq \phi_2 \) and \( \phi_1 \leq u_2 \leq \phi_2 \) in \( D \) imply that \( \phi_2 > u_1 \) and \( u_2 > \phi_1 \) and we have \( F(D^2u_1, x) \leq 0 \leq F(D^2u_2, x) \) in \( \{u_1 < u_2\} \cap D \). Furthermore, by the boundary condition for \( u_1 \) and \( u_2 \), we have that \( u_1 \equiv u_2 \) on \( \partial(\{u_1 < u_2\} \cap D) \). Therefore, by the comparison principle, we have that \( u_1 \geq u_2 \) in \( \{u_1 < u_2\} \subset D \) and we arrive at a contradiction. \( \square \)
2.2. Optimal Regularity. In this subsection, we prove the optimal regularity of the double obstacle problem \( \text{FB} \) with \( C^{1,1} \) obstacles by using the reduced form of \( \text{FB}_{\text{red}} \). We will first prove the quadratic growth of the solution at the free boundary point.

Definition 2.1. For a positive constant \( c' \), let \( S(c') \) be a class of solutions \( u \in W^{2,n}(B_1) \) of
\[
F(D^2u, x) = f(x)\chi_{\{0 < u < c'\}} + F(D^2\psi, x)\chi_{\{c' \leq \psi\}}, \quad 0 \leq u \leq \psi \text{ in } B_1,
\]
with \( |f(x)|, |F(D^2\psi, x)|, |\psi| \leq c' \) in \( B_1 \) and \( 0 \in \Gamma(u) \).

Proposition 2.2 (Quadratic growth). Assume \( F \) satisfies (F1) and (F2). For any \( u \in S(c') \), we have
\[
S(r, u) := \sup_{x \in \overline{B}_r} u(x) \leq C_0 r^2,
\]
for a positive constant \( C_0 = C_0(c') \).

Proof. First, we show that there is a positive constant \( C_0 \) such that
\[
S(2^{-j-1}, u) \leq \max(C_0 2^{-2j}, 2^{-2}S(2^{-j}, u)) \quad \text{for all } j \in \mathbb{N} \cup \{0, -1\}.
\]
Suppose it fails, then, for each \( j \in \mathbb{N} \cup \{0, -1\} \), there exists \( u_j \in S \) such that
\[
S(2^{-j-1}, u_j) > \max(j2^{-2j}, 2^{-2}S(2^{-j}, u_j)).
\]

We consider
\[
\tilde{u}_j(x) := \frac{u(2^{-j}x)}{S(2^{-j-1}, u)} \quad x \in B_{2j}.
\]
Then, by the definition of \( \tilde{u} \) and \( S \),
\[
S(\tilde{u}_j, 1/2) = 1, \quad S(\tilde{u}_j, 1) = 4, \quad \text{and} \quad \tilde{u}_j(0) = 0.
\]
Since \( u \in S(c') \), by the condition (F1) and Proposition 2.13 of [5], we know that \( u \in S(\frac{1}{2}, n, \Lambda, c') \). Thus, the inequality \( (9) \) implies
\[
P^+(D^2\tilde{u}(x)) = \frac{2^{-2j}}{S(2^{-j-1}, u)} \cdot P^+(D^2u(2^{-j}x)) \geq -\frac{c'}{j}
\]
and
\[
P^{-}(D^2\tilde{u}(x)) = \frac{2^{-2j}}{S(2^{-j-1}, u)} \cdot P^{-}(D^2u(2^{-j}x)) \leq \frac{c'}{j},
\]
where \( P^\pm \) are Pucci operators, i.e., we obtain that \( \tilde{u} \in S(\lambda/n, \Lambda, c'/j) \). By Harnack inequality (Theorem 4.3 of [5]) and \( C^1 \) regularity (Proposition 4.10 of [5]), we know that \( \tilde{u}_j \to \tilde{u} \) in \( B_1 \), up to subsequences and
\[
\tilde{u} \in S(\lambda/n, \Lambda, 0) \quad \text{in } B_1,
\]
\( \tilde{u} \neq 0 \) in \( B_{1/2} \), and \( \tilde{u}(0) = 0 \). In other words, a nontrivial viscosity solution \( \tilde{u} \) has its minimum at an interior point. Hence, by the strong maximum principle, it is a contradiction.

Next, we claim that
\[
S(2^{-j}, u) \leq C_0 2^{-2j+2} \quad \text{for all } j \in \mathbb{N} \cup \{0\}.
\]
We may assume that \( C_0 > c'/4 \). Then, \( (10) \) holds for \( j = 0 \). Assume that \( (10) \) holds for \( j = j_0 \). By \( (8) \), we have the inequality \( (10) \) for \( j_0 + 1 \),
\[
S(2^{-(j_0+1)}, u) \leq \max(C_0 2^{-2j_0}, 2^{-2}S(2^{-j_0}, u)) \leq C_0 2^{-2j_0}.
\]
Thus, by the mathematical induction, we have \( (10) \) for all \( j \in \mathbb{N} \cup \{0\} \).
Proof. Let \( K \) be a compact set in \( D \) and \( \delta = \text{dist}(K, \partial D) \). Since \( u \in W^{2,p}(D) \), \( D^2 u = D^2 \phi_1 \) a.e. on \( |u = \phi_1| \) and \( D^2 u = D^2 \phi_2 \) a.e. on \( |u = \phi_2| \). Thus, it suffices to show that \( \|u\|_{W^{2,\infty}((\phi_1 < u < \phi_2) \cap K)} < +\infty \). Let \( x_0 \) be a point in \( \{\phi_1 < u < \phi_2\} \cap K \) and denote \( d(x_0) := \text{dist}(x_0, \partial u = \phi_2) \). We may assume that \( d(x_0) = \text{dist}(x_0, \partial u = \phi_1) \).

Case 1) \( 5d(x_0) < \delta \).

For \( v := u - \phi_1 \), we have that
\[
\tilde{F}(D^2 v, x) = -F(D^2 \phi_1, x)\chi_{\{\phi_1 < u < \phi_2\}} + \tilde{F}(D^2 (\phi_2 - \phi_1), x)\chi_{\{\phi_1 < \phi_2\}} \quad \text{in } D,
\]
where \( \tilde{F}(M, x) = F(M + D^2 \phi_1, x) - F(D^2 \phi_1, x) \), see Subsection 1.2.

Let \( y_0 \in \partial B_{d(x_0)}(x_0) \cap \{u = \phi_1\} \). Then \( B_{d(x_0)}(y_0) \subseteq B_{5d(x_0)}(x_0) \subseteq D \). Since \( \phi_1, \phi_2 \in C^{1,1}(\overline{D}) \), we know that \( \sigma(4dx + y^2)/(4d)^2 \) is in the solution space \( \mathcal{S}(c') \) for a positive number \( c' \) for the fully nonlinear operator \( F \) which also satisfies (F1) and (F2). Then, by Proposition 2.2, we obtain
\[
\|u - \phi_1\|_{L^\infty(B_{d(x_0)})} \leq C(\|\phi_1\|_{C^{1,1}(\overline{D}) \cap \{\phi_1 < \phi_2\}}, \|\phi_2\|_{C^{1,1}(\overline{D})}).
\]
Since \( F(D^2 u, x) = 0 \) in \( B_{d(x_0)}(x_0) \subseteq \{\phi_1 < u < \phi_2\} \), by \( C^{2,\alpha} \) estimate, we have that
\[
\|D^2(u - \phi_1)\|_{L^\infty(B_{d(x_0)})} \leq C \frac{\|u - \phi_1\|_{L^\infty(B_{d(x_0)})}}{d^2}.
\]
Thus, \( B_{d(x_0)} \subseteq B_{2d(y_0)} \) implies
\[
\|D^2(u - \phi_1)\|_{L^\infty(B_{d(x_0)})} \leq C(\|\phi_1\|_{C^{1,1}(\overline{D}) \cap \{\phi_1 < \phi_2\}}, \|\phi_2\|_{C^{1,1}(\overline{D})}).
\]
and
\[
\|D^2 u\|_{L^\infty(B_{d(x_0)})} \leq C(\|\phi_1\|_{C^{1,1}(\overline{D}) \cap \{\phi_1 < \phi_2\}}, \|\phi_2\|_{C^{1,1}(\overline{D})}).
\]

Case 2) \( 5d(x_0) > \delta \).

The interior derivative estimate for \( u \in B_{\delta/4}(x_0) \) gives
\[
\|D^2 u\|_{L^\infty(B_{\delta/4}(x_0))} \leq C \frac{d^2}{\delta^2} \|u\|_{L^\infty(D)}.
\]
For the case \( d(x_0) = \text{dist}(x_0, \partial u = \phi_2) \), the same argument as above with \( \phi_2 - u \) implies the boundedness of the Hessian matrix of \( u \). Therefore, we obtain the optimal regularity of the solution \( u \) of \( \{F \}_{1.2} \). □

We note that the property for the classical single obstacle problem (i.e., \( \{F \}_{1.2} \)) with \( \phi_2 = \infty \) and \( F = \Delta \) is obtained in [3, 4]. The growth rate for the reduced single obstacle problem (\( \{F \}_{\text{local}} \)) with \( \phi_2 = \infty \) is discussed in [16], and the optimal \( (p/p - 1) \) growth rate for the p-Laplacian case is obtained in [12, 17].

For the case \( \{F \}_{\text{nonlocal}} \), with \( f \in C^{0,\alpha} \) and \( \psi \in C^{2,\alpha} \), the optimal regularity of the solutions is obtained by the theory in [9] (see [9] and Theorem 2.1 of [15] for more detail).
3. Regularity of the Free Boundary \( \Gamma(u) \)

In this section, we discuss the regularity of the free boundary of the double obstacle problem, \( F_{\text{Boussinesq local}} \). In Subsection 3.2, we show the classification of the global solution, which means that the global solution \( u \in P_{\infty}(M) \) with the upper obstacle \( \psi(x) = \frac{2}{r}(x_{n}^{+})^{2} \) is \( u = \frac{2}{r}(x_{n}^{+})^{2} \) or \( u = \frac{2}{r}(x_{n}^{-})^{2} \), see Proposition 3.4. In Subsection 3.3, we prove the proof of monotonocity of the local solution \( u \in P_{1}(M) \), Lemma 3.7. Then, we have that \( u \in P_{1}(M) \) is nonnegative in a small neighborhood \( B \) of \( 0 \), and \( u \) is a solution of the simple problem \( F_{\text{Boussinesq local}} \) in \( B \). Therefore, the blowup \( u_{0} \) of \( u \) should be \( \frac{2}{r}(x_{n}^{+})^{2} \), not \( \frac{2}{r}(x_{n}^{-})^{2} \), see Remark 3.2. In other words, we have the uniqueness of the blowup \( u_{0} = \frac{2}{r}(x_{n}^{+})^{2} \) of \( u \in P_{1}(M) \). Finally, we prove the regularity of the free boundary \( \Gamma(u) \), by using the direction monotonocity.

3.1. Non-degeneracy. In this subsection, we study the non-degeneracy of the solution \( u \in P_{1}(M) \), which is one of the important properties for solutions of obstacle problems. This property implies that \( 0 \) is also on the free boundary \( \Gamma(u_{0}) \), where \( u_{0} \) is a blow-up of \( u \) at \( 0 \in \Gamma(u) \), and \( \Gamma(u) \) has a Lebesgue measure zero.

**Lemma 3.1.** Assume \( F \) satisfies (F1) and (F2). Let \( u \in P_{1}(M) \). If \( f \geq c_{0} > 0 \) in \( B_{1} \) and \( F(D^{2}u, x) \geq c_{0} > 0 \) in \( \Omega(\psi) \), then

\[
\sup_{\partial B_{r}(x)} u \geq u(x) + \frac{c}{8\lambda_{1}n}r^{2} \quad x \in \Omega(\psi) \cap B_{1},
\]

where \( B_{r}(x) \subseteq B_{1} \).

**Proof.** Let \( x_{0} \in \Omega(u) \cap B_{1} \) and \( u(x_{0}) > 0 \). Since \( u \leq \psi \), we know that \( \{u = \psi\} = \{u = \psi\} \cap \{\nabla u = \nabla \psi\} \) and therefore, \( \Omega(u) \cap \{u = \psi\} \subseteq \Omega(\psi) \). By the assumptions for \( f \) and \( F(D^{2}u, x) \), we obtain \( F(D^{2}u, x) = f \chi_{\Omega(u)}(x)\psi + F(D^{2}u, x)\chi_{\Omega(u)}(x)\psi \geq c_{0} \) in \( \Omega(u) \). Thus, the uniformly ellipticity, (F2) in Definition 1.4 implies

\[
F(D^{2}w, x) \geq F(D^{2}u, x) - c \geq 0 \text{ on } B_{r}(x_{0}) \cap \Omega(u),
\]

where

\[
w(x) := u(x) - u(x_{0}) - \frac{c}{2\lambda_{1}n}|x - x_{0}|^{2}.
\]

Since \( w(x_{0}) = 0 \) and \( w(x) < 0 \) on \( \partial\Omega(u) \), the maximum principle on \( B_{r}(x_{0}) \cap \Omega(u) \) implies

\[
\sup_{\partial B_{r}(x) \cap \Omega(u)} w > 0 \quad \text{and} \quad \sup_{\partial B_{r}(x) \cap \Omega(u)} u \geq u(x) + \frac{c}{2\lambda_{1}n}r^{2}.
\]

Let \( x_{0} \in \Omega(u) \cap B_{1} \) and \( u(x_{0}) \leq 0 \). If there is a point \( x_{1} \in B_{r/2}(x_{0}) \) such that \( u(x_{1}) > 0 \). Then by the first case in the previous paragraph for \( x_{1} \) implies the non-degeneracy for \( x_{0} \).

If \( u(x) \leq 0 \) in \( B_{r/2}(x_{0}) \), then \( u(x) \equiv 0 \) in \( B_{r/2}(x_{0}) \) or \( u(x) < 0 \) in \( B_{r/2}(x_{0}) \), by the maximum principle. Since \( x^{0} \in \Omega(u) \), the second case is only possible and in the case, \( F(D^{2}u, x) \geq c \) in \( B_{r/2}(x_{0}) \). Then, it implies the non-degeneracy of \( u \) at \( x_{0} \).

For the case of \( x_{0} \in \partial \Omega(u) \cap B_{1} \), we take a sequence of points \( x_{j} \in \Omega(u) \) such that \( x_{j} \to x_{0} \) as \( j \to \infty \). By passing to the limit as \( j \) goes to \( \infty \), we have the desired inequality for \( x_{0} \in \Omega(u) \cap B_{1} \).

By the non-degeneracy and of the solution \( u \in P_{1}(M) \), we have the local porosity of \( \Gamma(u) \) and \( \Gamma(u) \) has Lebesgue measure zero, e.g. Section 3.2.1 of [19].
Remark 3.2. For $u \in P_1(M)$, by the definition of the rescalings and blowups and the non-degeneracy, we have $\sup_{B_r(u_0)} u_0 \geq \frac{1}{M} r^2$, for $r > 0$, where $u_0$ is a blowup of $u$ at $0$. Thus, the origin $0$ is on the free boundary $\Gamma(u_0)$ of $u_0$. On the other hand, in general, we do not know that $0 \in \Gamma^\psi(u)$ implies $0 \in \Gamma^\psi(u_0)$, for $u \in P_1(M)$.

However, in the same manner as the linear case in Remark 2.4 of [15], if we assume that $u$ is a non-negative function, then $v := \psi - u$ is a solution of

$$
\tilde{F}(D^2v, x) = \left( F(D^2\psi, x) - f \right) \chi_{\{0 < \psi < 1\}} + F(D^2\psi, x) \chi_{\{0 = \psi\}} \quad \text{in } B_1,
$$

where $\tilde{F}(M, x) := F(D^2\psi, x) - F(D^2\psi - M, x)$. Then, if we assume that $F(D^2\psi, x) - f \geq c$ and $F(D^2\psi, x) \geq c$ in $|\psi > 0|$, we have the non-degeneracy for $v$ which implies $0 \in \Gamma(\psi_0) = \Gamma^\psi(u_0)$ and $|\Gamma(\psi)| = |\Gamma^\psi(u)| = 0$.

Remark 3.3. We also note that $F(M, x) = F(D^2\psi, x) - F(D^2\psi - M, x)$ is a concave fully nonlinear operator. Thus, we can not apply the theory of the obstacle problem for the convex fully nonlinear operator in [13] to $F(M, x)$.

Precisely, it is uncertain that we can have Lemma 3.7 for $v$, which is that the blowup of $v$ at $x \in \Gamma(\psi) = \Gamma^\psi(u)$ near $0$ is of the form $\frac{c}{2}(x_i^+)^2$, for a positive constant $c$. Hence, in contrast with linear theory [15], we only have the regularity of the free boundary $\Gamma(u)$, not $\Gamma^\psi(u)$, see Theorem 1.3 and Corollary 1.5.

Lemma 3.4. Assume $F$ satisfies (F1)-(F3) and (F4)’. Let $u \in P_1(M)$ with an upper obstacle $\psi$ such that

$$
0 \in \partial \Omega(\psi), \quad \lim_{x \to 0, x \in \Omega(\psi)} F(D^2\psi, x) > f(0), \quad f \geq c_0 > 0 \text{ in } B_1,
$$

and

$$
F(D^2\psi, x) \geq c_0 > 0 \text{ in } \Omega(\psi).
$$

Then $u_0 \in P_\infty(M)$.

Proof. Let $u_r$ and $\psi_r$ be sequences of the rescaling functions converging to blowups, $u_0$ and $\psi_0$, respectively. First, we claim that

$$
F(D^2\psi_0, 0) = F(D^2\psi(0), 0) \chi_{\Omega(\psi_0)} \text{ in } \mathbb{R}^n,
$$

where $\Omega(\psi_0) = \mathbb{R}^n \setminus \{ |\psi_0| = 0 \} \cap \{ |\nabla \psi_0| = 0 \}$ and $F(D^2\psi(0), 0) := \lim_{x \to 0, x \in \Omega(\psi)} F(D^2\psi(x), x)$. Let $x$ be a point in $\Omega(\psi_0)$. Then, by $C^{1,\alpha}_{\text{loc}}$ convergence of $\psi_r$ to $\psi_0$, we know that there exist $\delta > 0$ and $\eta_0$ such that $B_{\delta}(x) \subset \Omega(\psi_r)$, for all $i \geq \eta_0$. Then, by the definition of rescalings $\psi_r$, we have $r_i x \in \Omega(\psi)$. Furthermore, $\psi \in C^{2,\alpha}(\Omega(\psi))$ implies strong convergence of $\psi_r$ to $\psi_0$ in $C^{2,\beta}(B_{\delta}(x))$ for some $0 < \beta < \alpha$. Thus,

$$
F(D^2\psi_0(x), 0) = \lim_{i \to \infty} F(D^2\psi_r(x), r_i x) = \lim_{i \to \infty} F(D^2\psi(r_i x), r_i x) = F(D^2\psi(0), 0).
$$

Next, we prove that $u_0$ is a solution of

$$
F(D^2u_0, 0) = f(0) \chi_{\Omega(u_0) \cap \{u_0 < \psi_0\}} + F(D^2\psi(0), 0) \chi_{\Omega(u_0) \cap \{u_0 = \psi_0\}}, \quad u_0 \leq \psi_0 \text{ in } \mathbb{R}^n.
$$

The rescaling $u_r$ is a solution of

$$
F(D^2u_r, r_i x) = f(r_i x) \chi_{\Omega(u_r) \cap \{u_r < \psi_r\}} + F(D^2\psi_r, r_i x) \chi_{\Omega(u_r) \cap \{u_r = \psi_r\}}, \quad u_r \leq \psi_r \text{ in } B_{1/r_i},
$$
where $\Omega(u_\varepsilon) := B_1/\varepsilon \setminus (\{u_\varepsilon = 0\} \cap \{\nabla u_\varepsilon = 0\})$. Let $x$ be a point in $\Omega(u_\varepsilon) \cap \{u_\varepsilon < \psi_0\}$. Then, by $C^{1,\alpha}_{bc}$ convergence of $u_\varepsilon$ to $u_0$, there exist $\delta > 0$ and $i_0$ such that $B_\delta(x) \subset \Omega(u_\varepsilon) \cap \{u_\varepsilon < \psi_0\}$, for all $i \geq i_0$. Then

$$F(D^2u_\varepsilon(y)) = f(r,x) \text{ in } B_\delta(x).$$

Since $f \in C^{\alpha,\beta}(B_1)$, we have $C^{\alpha,\beta}$ estimates for $u_\varepsilon$, and we may assume strong convergence of $u_\varepsilon$ to $u_0$ in $C^{2,\beta}(B_0(x))$ for some $0 < \beta < \alpha$. Thus we have $|D^2u_\varepsilon(x)| \leq M$

$$F(D^2u_\varepsilon(x),0) = \lim_{i \to \infty} F(D^2u_\varepsilon(x),r,x) = \lim_{i \to \infty} f(r,x) = f(0) \geq c_0 > 0.$$

First, we observe that the zero set of $u_\varepsilon - \psi = \frac{\psi}{(x_n^+)^2} (\alpha > 1)$ and the thickness assumption (2), is $\frac{\psi}{(x_n^+)^2}$ or $\frac{\psi}{(x_n^-)^2}$.

Then, we have

$$\delta_r(u,\psi) \geq 0, \quad \text{for all } r > 0.$$

Therefore, we have that $|\delta_r(u,\psi)| \leq M$ and

$$\delta_r(u,\psi) \geq 0, \quad \text{for all } r > 0.$$

Then, we have

$$u(x) = \frac{1}{2}(x_n^+)^2 \quad \text{or} \quad u(x) = \frac{a}{2}(x_n^-)^2.$$

**Proof.** The condition $u \leq \psi = \frac{\psi}{(x_n^+)^2}$ on $\mathbb{R}^n$ implies that $u(x) \leq 0$ on $\{x_n < 0\}$. We claim that $\{x_n < 0\} \subset \Lambda(u)$. First, we suppose that $\partial \Omega(u) \cap \{x_n < 0\} \neq \emptyset$. Then, by non-degeneracy, (Lemma 3.1), we have that $\{u > 0\} \cap \{x_n < 0\} \neq \emptyset$ and we arrive at a contradiction. Next, we suppose that $\{x_n < 0\} \subset \Omega(u)$. Since $\{\psi = 0\} = \Lambda(\psi) = \{x_n \leq 0\}$, it is a contraction to $\delta_r(u,\psi) \geq 0$, for all $r > 0$.

Therefore, we have that $\{x_n < 0\}$ is contained in $\Lambda(u)$. Hence, $u = 0$ on $\{x_n \leq 0\}$ and $\partial_r u = 0$ on $\{x_n \leq 0\}$ for all $e \in S^{n-1} \cap e_n^+$, where $S^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$ and $e_n^+ := \{x \in \mathbb{R}^n : x \perp e\}$ for $e \in S^{n-1}$.
In order to have the conclusion, it suffices to show that $\partial_{e}u \equiv 0$ on $\mathbb{R}^{n}$, for any direction $e \in S^{n-1} \cap e_{n}^{\perp}$. Thus, we fix $e_{1} \in S^{n-1} \cap e_{n}^{\perp}$ and define

$$0 \leq \sup_{x \in \Omega \cap \{x_{n} > 0\}} \frac{\partial_{1}u(x)}{x_{n}} =: M_{0}. $$

By the optimal regularity and $\partial_{1}u \equiv 0$ on $\{x_{n} \leq 0\}$, we know that $M_{0}$ is finite. If we prove that $M_{0}$ is 0, we have that $\partial_{1}u \equiv 0$ on $\mathbb{R}^{n}$. Since the direction $e_{1}$ is arbitrary, we have that $\partial_{e}u = 0$ on $\{x_{n} \leq 0\}$, for all $e \in S^{n-1} \cap e_{n}^{\perp}$.

Arguing by contradiction, suppose $M_{0} > 0$. Since $\partial_{1}u \equiv 0$ on $(\Omega(u) \cap \{u < \psi\})^{c}$, we can take a sequence $x^{j} \in \Omega(u) \cap \{u < \psi\} \subset \{x_{n} > 0\}$ such that

$$\lim_{j \to \infty} \frac{1}{x_{n}^{j}} \partial_{1}u(x^{j}) = M_{0}. $$

Let $r_{j} := x_{n}^{j}$ and consider rescaling functions

$$u_{r_{j}}(x) := \frac{u((x')^{r_{j}} + r_{j}x)}{(r_{j})^{2}} \quad \text{and} \quad \psi_{r_{j}}(x) := \frac{\psi((x')^{r_{j}} + r_{j}x)}{(r_{j})^{2}} = \psi(x). $$

Then, $D^{2}u_{r}$ are uniformly bounded and $u_{r_{j}} \equiv 0$ on $\{x_{n} \leq 0\}$. Thus,

$$u_{r_{j}}(x) \to \bar{u}(x) \in C^{1,\alpha}_{\text{loc}}(\mathbb{R}^{n}) \quad \text{for any} \quad \alpha \in [0, 1),

$$

and $\bar{u}$ is a solution of

$$F(D^{2}u) = \chi_{\Omega(\bar{u}) \cap \{u < \psi\}} + a \chi_{\Omega(\bar{u}) \cap \{u = \psi\}}; \quad u \leq \psi \quad \text{a.e. in} \quad \mathbb{R}^{n}, $$

with the upper obstacle

$$\psi(x) = \frac{\bar{a}}{2}(x_{n}^{\perp})^{2}. $$

By the definition of $M_{0}$, for $x \in \{x_{n} > 0\}$,

$$\partial_{1}u_{r_{j}}(x) = \frac{\partial_{1}u((x')^{r_{j}} + r_{j}x)}{x_{n}} \leq M_{0}x_{n}. $$

Hence, we have $\partial_{1}\bar{u}(x) \leq M_{0}x_{n}$ on $\{x_{n} > 0\}$. Moreover,

$$\partial_{1}\bar{u}(e_{n}) = \lim_{j \to \infty} \partial_{1}u_{r_{j}}(e_{n}) = \lim_{j \to \infty} \frac{\partial_{1}u((x')^{r_{j}} + r_{j}e_{n})}{r_{j}} = \lim_{j \to \infty} \frac{\partial_{1}u(x^{j})}{x_{n}^{j}} = M_{0}. $$

If $e_{n} \in (\Omega(\bar{u}) \cap \{\bar{u} < \psi\})^{c}$, then $\partial_{1}\bar{u}(e_{n}) = 0$ and we arrive at a contradiction. Thus, $e_{n} \in \Omega(\bar{u}) \cap \{\bar{u} < \psi\}$. Let $\bar{\Omega}(\bar{u})$ be the connected component of $\Omega(\bar{u}) \cap \{\bar{u} < \psi\}$ containing $e_{n}$. By (13), we know that $\bar{\Omega}(\bar{u}) \subset \Omega(\bar{u}) \subset \{x_{n} > 0\}$.

By differentiating $F(D^{2}\bar{u}) = 1$ on $\bar{\Omega}(\bar{u})$ with respect to $e_{1}$, we have $F_{ij}(D^{2}\bar{u})\partial_{ij}\partial_{1}\bar{u} = 0$ and $F_{ij}(D^{2}\bar{u})\partial_{ij}(\bar{u} - M_{0}x_{n}) = 0$ on $\bar{\Omega}(\bar{u})$. Thus, the strong maximum principle implies that

$$\partial_{1}\bar{u} = M_{0}x_{n} \quad \text{on} \quad \bar{\Omega}(\bar{u}) \subset \{x_{n} > 0\}. $$

If there exists $x \in \partial\bar{\Omega}(\bar{u}) \cap \{x_{n} > 0\}$, then $\partial_{1}\bar{u}(x) = 0 = M_{0}$ and we have a contradiction, i.e., we have $\partial_{1}\bar{u}(x) \equiv M_{0}$ on $\{x_{n} > 0\}$ implies $\bar{\Omega}(\bar{u}) = \{x_{n} > 0\}$, $\partial_{1}\bar{u} \equiv M_{0}x_{n}$ on $\{x_{n} > 0\}$ and

$$\bar{u}(x) = M_{0}x_{1}x_{n} + g(x_{2}, ..., x_{n}) \quad \text{on} \quad \{x_{n} > 0\}. $$
Since \( \tilde{u} \) is in \( C^{1,1}_\text{loc}(\mathbb{R}^n) \) and \( \tilde{u} \equiv 0 \) on \( \{x_n \leq 0\} \), we have
\[
\partial_n \tilde{u}(x) = M_0 x_1 + \partial_n g(x_2, ..., x_n) = 0 \quad \text{on } \{x_n = 0\}
\]
and it does not hold unless \( M_0 = 0 \) and \( \partial_n g(x_2, ..., 0) = 0 \). Hence, we arrive at a contradiction. \( \square \)

3.3. **Directional Monotonicity and proof of Theorem 1.4**

In this subsection, we show the directional monotonicity for solutions of \( (\text{FB}_\text{nosign local}) \) and the regularity of the solutions \( u \in P_1(M) \). We note that the argument for the linear case is discussed in \( \cite{15} \).

**Lemma 3.6.** Assume \( F \in C^1 \) satisfies (F1)-(F3) and (F4)' and let \( u \) be a solution of
\[
F(D^2 u, r x) = f(rx)\chi_{\Omega(u)\cap\{u<\psi\}} + F(D^2 \psi(rx), rx)\chi_{\Omega(u)\cap\{0<u=\psi\}}, \quad u \leq \psi \quad \text{in } B_1
\]
and assume that \( f(x) \geq c_0 > 0 \) in \( B_1 \). Suppose that we have
\[
C\partial_e \psi - \psi \geq 0, \quad C\partial_e u - u \geq -\epsilon_0 \quad \text{in } B_1,
\]
for a direction \( e \) and \( \epsilon_0 < c/64\lambda_1 n \). Then we obtain
\[
C\partial_e u - u \geq 0 \quad \text{in } B_{1/4},
\]
if \( 0 < r \leq r_0' \), for some
\[
r_0' = r_0'(C, \epsilon_0, \|\nabla F\|_{L^\infty(B_1)}, \|\nabla \psi\|_{L^\infty(B_1)}).
\]

**Proof.** By differentiating \( F(D^2 u, r x) = f(rx) \) with respect to the direction \( e \), we have
\[
F_{ij}(D^2 u(x), x)\partial_j\partial_i u(x) = \partial_i f(rx) - \partial_j \partial_i F(D^2 u(x), r x) \quad \text{in } \Omega(u) \cap \{u < \psi\}, \quad (14)
\]
where \( \partial_e F \) is the spatial directional derivative of \( F(M, x) \) in the direction \( e \).

Arguing by contradiction, suppose there is a point \( y \in B_{1/2} \cap \Omega(u) \cap \{u < \psi\} \) such that \( C\partial_e u(y) - u(y) < 0 \). We consider the auxiliary function
\[
\phi(x) = C\partial_e u(x) - u(x) + \frac{c_0}{4\lambda_1 n}|x - y|^2.
\]

By Proposition 2.13 of \( \cite{5} \) and the condition (F1) \( F(0, x) = 0 \) for all \( x \in \mathbb{R}^n \), we have \( u \in S(\lambda_0/n, \lambda_1, f(rx)) \) in \( \Omega(u) \cap \{u < \psi\} \) and moreover \( \cite{14} \) implies \( \partial e u \in S(\lambda_0/n, \lambda_1, r\partial_e f(rx) - r(\partial F)(D^2 u(x), r x)) \). Hence, we have
\[
\phi \in S(\lambda_0/n, \lambda_1, r\partial_e f(rx) - r(\partial F)(D^2 u(x), r x) - f(rx) + c_0/2) \quad \text{in } \Omega(u) \cap \{u < \psi\}.
\]
Furthermore, since there is a sufficiently small constant \( r_0 = r_0(C, \epsilon_0, \|\nabla F\|_{L^\infty(B_1)}, \|\nabla \psi\|_{L^\infty(B_1)}) > 0 \) such that
\[
Cr\partial_e f(rx) - Cr(\partial F)(D^2 u(x), r x) - f(rx) + c_0/2 \\
\leq Cr\|\nabla F\|_{L^\infty(B_1)} + Cr\|\nabla \psi\|_{L^\infty(B_1)} - f(rx) + c_0/2 \leq 0,
\]
for all \( r < r_0 \), we have
\[
\phi(x) \in S(\lambda_0/n, \lambda_1, 0) \quad \text{in } \Omega(u) \cap \{u < \psi\}, \quad \text{for all } r \leq r_0.
\]
By the minimum principle of \( \phi \) in \( B_{1/4}(y) \cap \Omega(u) \cap \{u < \psi\} \), we have
\[
\inf_{\partial B_{1/4}(y)\cap\{0<u=\psi\}} \phi \leq \phi(y) < 0.
\]
Moreover, \( C\partial_e \psi - \psi \geq 0 \) in \( B_1 \) implies that \( \phi \geq 0 \) on \( \partial (\Omega(u) \cap \{u < \psi\}) \). Thus, we obtain
\[
\inf_{\partial B_{1/4}(y)\cap\{0<u=\psi\}} \phi < 0 \quad \text{and} \quad \inf_{\partial B_{1/4}(y)\cap\{0<u=\psi\}} (C\partial_e u - u) < -\frac{c_0}{128\lambda_1 n}.
\]
Since $\varepsilon_0 < \frac{c}{8\lambda_1^2n}$, we have a contradiction. \hfill \Box

Lemma 3.7 (Directional monotonicity). Let $u, \psi, F$ be as in Theorem 1.3. Then for any $\delta \in (0, 1)$, there exists

$$r_0 = r_0(u, \psi, c_0 \|\nabla F\|_{L^\infty(\Omega)}, \|F\|_{L^\infty(\Omega)}, \|\nabla f\|_{L^\infty(B_1)}) > 0$$

such that

$$u \geq 0 \quad \text{in } B_{r_1}$$
$$\partial_x u \geq 0 \quad \text{in } B_{r_0} \quad \text{for any } e \in C_\delta \cap \partial B_1,$$

where

$$C_\delta = \{x \in \mathbb{R}^n : x_n > \delta |x'|, \quad x' = (x_1, ..., x_{n-1})\}.$$

Proof. We denote $u_r$ and $\psi_r$ by rescalings of $u$ and $\psi$, respectively. The thickness assumption \([3]\) implies that $\psi_0 = \frac{1}{2}(x_0^*)^2$, $(a > 1)$, in an appropriate system of coordinates, see e.g. Proposition 4.7 of [14]. Then, by Proposition 3.5, we know that $u_0$ is $\frac{1}{4}(x_0^*)^2$ or $\frac{1}{2}(x_0^*)^2$. Hence, for any $e \in C_\delta = \{x \in \mathbb{R}^n : x_n > \delta |x'|, x' = (x_1, ..., x_{n-1})\}$, we obtain

$$\delta^{-1}\partial_x \psi_0 - \psi_0 \geq 0 \quad \text{and} \quad \delta^{-1}\partial_x u_0 - u_0 \geq 0 \quad \text{in } \mathbb{R}^n.$$

By the $C^{1,a}$ convergence of $\psi_r$ and $u_r$ to $\psi_0$ and $u_0$, respectively, we obtain that

$$\delta^{-1}\partial_x \psi_r - \psi_r \geq -\varepsilon_0 \quad \text{and} \quad \delta^{-1}\partial_x u_r - u_r \geq -\varepsilon_0 \quad \text{in } B_1,$$

for $\varepsilon_0 < c/64\lambda_1^2n$ and $r < \tilde{r}_\delta(u, \psi)$. Then, by applying the directional monotonicity for the solution of the single obstacle, Lemma 13 of [9] to $\psi$, we have that $\delta^{-1}\partial_x \psi_r - \psi_r \geq 0$ in $B_{1/2}$ for all $r < r_\delta'(u, \psi, c_0 \|\nabla F\|_{L^\infty(\Omega)}, \|F\|_{L^\infty(B_1)})$. Furthermore, by Lemma 3.6 we have

$$\delta^{-1}\partial_x u_r - u_r \geq 0 \quad \text{in } B_{1/4},$$

for $0 < r \leq \tilde{r}_\delta = \tilde{r}_\delta(u, \psi, c_0 \|\nabla F\|_{L^\infty(\Omega)}, \|F\|_{L^\infty(B_1)})$.

We claim that $u_r = 0$ in $\{x_n < -1/8 \} \cap B_{1/4}$. By the $C^{1,a}$ convergence of $u_r$ to $u$, we may assume that $\|u_r - u_0\|_{L^1(B_{1/4})} \leq \frac{c_1}{|x_0|} \times \frac{1}{128}$ for $0 < r \leq \tilde{r}_\delta$. Let $x_0$ be a point in $\{x_n < 0\} \cap \Omega(u_r) \cap B_{1/4}$. By the non-degeneracy, Lemma 3.1, we have

$$\sup_{\partial B_r(x_0)} u_r \geq u_r(x_0) + \frac{c}{8\lambda_1^2n} \rho^2,$$

(16)

where $\rho := |(x_0)_n|$. Since $u_0 = 0$ in $[x_n \leq 0]$ implies $\|u_r - u_0\|_{L^1(\Omega_n)} \leq \frac{c}{8\lambda_1^2n} \rho^2$, by (16), we have $\frac{c}{8\lambda_1^2n} \rho^2 \leq \frac{c}{8\lambda_1^2n} \times \frac{1}{128}$ and $\rho \leq \frac{1}{8}$. Therefore, $\{x_n < 0\} \cap \Omega(u_r) \cap B_{1/4} \subset \{-1/8 < x_n < 0\}$. Let $x_0$ be in $\{x_n < -1/8\} \cap B_{1/4}$. By the definition of scaling functions $u_r$, we have $u \geq 0$ in $B_{r_1}$, for $r_1 = \frac{1}{16}$. Furthermore, (15) implies that

$$\partial_x u \geq 0 \quad \text{in } B_{r_\delta} \quad \text{for any } e \in C_\delta \cap \partial B_1,$$

for $r_\delta = \frac{1}{4r_1} < r_1 \delta < 0, 1$. \hfill \Box
Lemma 3.8. Let \( u, \psi, F \) be as in Theorem 1.4. Then there exists
\[
r_1 = r_1(u, \psi, c_0, \|\nabla F\|_{L^\infty(B_{d}\times B_1)}, \|F\|_{L^\infty(B_d\times B_1)}, \|\nabla f\|_{L^\infty(B_1)}) > 0
\]
such that \( u \) is a solution of
\[
F(D^2 u, x) = f\chi_{\{0 < u < \psi\}} + F(D^2 \psi, x)\chi_{\{0 < u = \psi\}}, \quad 0 \leq u \leq \psi \quad \text{in } B_{r_1}.
\]
Moreover, if \( u_0 \) and \( \psi_0 \) are blowup functions of \( u \) and \( \psi \) at \( 0 \), respectively, then in an appropriate system of coordinates,
\[
\psi_0(x) = \frac{a}{2}(x_n^+)^2 \quad \text{and} \quad u_0(x) = \frac{1}{2}(x_n^+)^2.
\]
\[\]
**Proof.** By Lemma 3.7 there is \( r_1 = r_1(u, \psi, c_0, \|\nabla F\|_{L^\infty(B_{d}\times B_1)}, \|F\|_{L^\infty(B_d\times B_1)}, \|\nabla f\|_{L^\infty(B_1)}) > 0 \) such that \( u \geq 0 \) in \( B_{r_1} \). Hence \( u \) is a solution of
\[
F(D^2 u, x) = f\chi_{\{0 < u < \psi\}} + F(D^2 \psi, x)\chi_{\{0 < u = \psi\}}, \quad 0 \leq u \leq \psi \quad \text{in } B_{r_1}.
\]
and \( v := \psi - u \) is a solution of
\[
\bar{F}(D^2 v, x) = (F(D^2 \psi, x) - f)\chi_{\{0 < u < \psi\}} + F(D^2 \psi, x)\chi_{\{0 < u = \psi\}}, \quad 0 \leq v \leq \psi \quad \text{in } B_{r_1},
\]
where \( \bar{F}(M, x) := F(D^2 \psi, x) - F(D^2 \psi - M, x) \). Since \( 0 \leq v \leq \psi \), we have that \( \{v > 0\} \subset \{\psi > 0\} = \Omega(\psi) \). Thus, \( \min\{F(D^2 \psi, x), F(D^2 \psi, x) - f\} \geq c_0 > 0 \) in \( \Omega(\psi) \) implies
\[
\bar{F}(D^2 v, x) = (F(D^2 \psi, x) - f)\chi_{\{0 < u < \psi\}} + F(D^2 \psi, x)\chi_{\{0 < u = \psi\}} \geq c_0 > 0 \quad \text{in } \{v > 0\}.
\]
Thus, by the same argument in Lemma 3.1 we have the non-degeneracy for \( v \),
\[
\sup_{\partial B_{r_1}(x)} v \geq v(x) + \frac{\lambda}{8n}\quad x \in \overline{\Omega(v)} \cap B_{r_0},
\]
for \( B_{r_1}(x) \in B_{r_0} \). This implies \( 0 \in \Gamma(v_0) = \Gamma^{\psi_n}(u_0) \), where \( v_0 \) is a blowup functions of \( v \) at \( 0 \) such that \( v_0 = \psi_0 - u_0 \), see Remark 3.2. Consequently, we have
\[
\psi_0(x) = \frac{a}{2}(x_n^+)^2 \quad \text{and} \quad u_0(x) = \frac{1}{2}(x_n^+)^2
\]
in an appropriate system of coordinates. \( \square \)

By the uniqueness of the blowup for the single obstacle problem, we have the uniqueness of blowup for \( \psi \), i.e., for any sequence \( \lambda \to 0 \),
\[
\psi_\lambda \rightarrow \psi_0 = \frac{1}{2}(x_n^+)^2 \quad \text{in } C^{1,\alpha}_{loc}(\mathbb{R}^n),
\]
in an appropriate system of coordinates. Then, the uniqueness of the blowup for \( u \) directly follows from Lemma 3.8.

**Proposition 3.9** (Uniqueness of blowup). Let \( u, \psi, F \) be as in Theorem 1.4. Then the blowup function of \( u \) at \( 0 \) is unique, i.e., in an appropriate system of coordinates, for any sequence \( \lambda \to 0 \),
\[
u_\lambda \rightarrow \nu_0 = \frac{1}{2}(x_n^+)^2 \quad \text{in } C^{1,\alpha}_{loc}(\mathbb{R}^n)
\]
as \( \lambda \to 0 \).

In the following lemma, we have that the blowups of \( u \) for any points \( x \) near \( 0 \) are also half-space functions.

**Lemma 3.10.** Let \( u, \psi, F \) and \( r_1 \) be as in Theorem 1.4. Then there is \( r_1' = r_1'(u, \psi) > 0 \) such that the blowup function of \( u \) at \( x \in \Gamma(u) \cap B_{r_1} \) are half-space functions.
Proof. By the directional monotonicity for $u$ and $\psi$ (Lemma 3.7 and 3.8), we have that, for any $\delta \in (0, 1]$, there exists

$$r_\delta = r_\delta(u, c_0, \|\nabla F\|_{L^\infty(B_{2\delta} \times B_1)}, \|\nabla f\|_{L^\infty(B_{2\delta} \times B_1)}, \|\nabla F\|_{L^\infty(B_{2\delta} \times B_1)}) > 0$$

such that $r_\delta \geq r_\delta' = r_\delta'(u, \psi) > 0$ and

$$\psi, u \geq 0 \quad \text{in } B_{r_\delta'}, \quad \partial_\nu \psi, \partial_\nu u \geq 0 \quad \text{in } B_{r_\delta'} \quad \text{for any } \nu \in C_0.$$

Then, the free boundaries $\partial \{u = 0\} \cap B_{r_\delta} = \Gamma(u) \cap B_{r_\delta}, \partial \{\psi = 0\} \cap B_{r_\delta}$ are represented by Lipschitz functions.

Let $x^0$ be a point in $\Gamma(u) \cap B_{r_\delta}$ and assume that there exists $r_0 > 0$ such that

$$\{u = \psi\} \cap B_r(x^0) \neq \emptyset \quad \text{for all } r < r_0.$$ 

Then, there is a sequence of points $x^j$ such that $x^j \in \{u = \psi\}$ and $x^j \to x^0$ as $j \to \infty$. Thus,

$$\psi(x^j) = u(x^j) \to 0 \quad \text{as } j \to \infty, \quad \text{and } \quad x^j \in \{\psi = 0\}.$$

Since $u$ is nonnegative in $B_{r_\delta'}$, we have $0 \leq u \leq \psi$ and $\{\psi = 0\} \subset \{u = 0\}$ in $B_{r_\delta'}$. Thus, $x^0 \in \Gamma(u) \cap B_{r_\delta'}$ implies $x^0 \in \partial \{\psi = 0\}$. Furthermore, by the Lipschitz regularity of the zero set of $\psi$, $\{\psi = 0\}$ and the positivity of $u$ ($0 \leq u \leq \psi$), we obtain

$$\delta_r(u, \psi, x_0) = \delta_r(\psi, x_0) \geq \epsilon_0 \quad \text{for all } r < 1/4.$$ 

Then, by classification of blowup, Proposition 3.5, we know that the blowup of $u$ at $x^0$ is a half-space solution, which means that it is of the form $\frac{c}{2}(\frac{x}{d_n})^2$, for a positive constant $c$.

Next, we assume that, for $x^0 \in \Gamma(u) \cap B_{r_\delta'}$, there exists $r_0 > 0$ such that

$$\{u = \psi\} \cap B_r(x^0) = \emptyset.$$ 

Then $u$ is a solution of

$$F(D^2 u, x) = f(x_{u=0}), \quad u \geq 0 \quad \text{in } B_{r_0}(x^0).$$ 

On the other hand, Lipschitz regularities of $\Gamma(u)$ implies the thickness assumption for $u$ near $x_0$. Then, the blowup function of $u$ at 0 is a half-space solution. \hfill \Box

By using lemmas in this subsection and arguments in [13, 19, 9, 15], we prove one of the main theorems of the paper, Theorem 1.2.

Proof of Theorem 1.2. The directional monotonicity for $u$, Lemma 3.7 implies that the free boundary $\Gamma(u) \cap B_{1/2}$ is represented as a graph $x_n = f(x')$ for Lipschitz function $f$ and the Lipschitz constant of $f$ is less than $\delta$ in $B_{1/2}$. Since $\delta > 0$ can be chosen arbitrary small, we have a tangent plane of $\Gamma(u)$ and the normal vector $e_n$ at 0. By Lemma 3.10, for any $z \in \Gamma(u) \cap B_{1/2}$, we know that the blowup is the half-space solution and there is a tangent plane for $z \in \Gamma(u) \cap B_{1/2}$ with tangent vector $v_z$. By Lemma 3.7, for $z \in \Gamma(u) \cap B_{r_\delta}$, we have $\|v_z - e_n\| \geq 0$ for any $e \in C_0$. Hence, $v_z$ is in $C_1/\delta$ and then, for sufficiently small $\delta > 0$, $v_z$ is close to $e_n$. Specifically, we have that

$$|v_z - e_n| \leq C\delta, \quad z \in \Gamma(u) \cap B_{r_\delta}.$$ 

Therefore, $\Gamma(u)$ is $C^1$ at 0 and by the same argument, $\Gamma(u) \cap B_{r_\delta}$ is $C^1$. \hfill \Box
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Acknowledgement

We thank Henrik Shahgholian for helpful discussions on the double obstacle problem. Ki-Ahm Lee has been supported by the National Research Foundation of Korea (NRF) grant funded by the Korean government (MSIP) (No. NRF-2020R1A2C1A01006256). Ki-Ahm Lee also holds a joint appointment with the Research Institute of Mathematics of Seoul National University. This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (2020R1A6A3A01099425).

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