ON THE TRANSPORT OPERATORS ARISING FROM
LINEARIZING THE VLASOV-POISSON OR EINSTEIN-VLASOV
SYSTEM ABOUT ISOTROPIC STEADY STATES

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Abstract. If the Vlasov-Poisson or Einstein-Vlasov system is linearized about
an isotropic steady state, a linear operator arises the properties of which are
relevant in the linear as well as nonlinear stability analysis of the given steady
state. We prove that when defined on a suitable Hilbert space and equipped
with the proper domain of definition this transport operator $T$ is skew-adjoint,
i.e., $T^* = -T$. In the Vlasov-Poisson case we also determine the kernel of this
operator.

1. Introduction. The Vlasov-Poisson and Einstein-Vlasov systems describe a self-
gravitating, collisionless gas in the framework of Newtonian mechanics or General
Relativity, respectively. For an important class of steady states of these systems
the particle density on phase space depends only on the local or particle energy,
i.e., $f_0(x, v) = \phi(E(x, v))$ with some given function $\phi$. The particle energy $E$ is a
$C^2$ function of position $x \in \mathbb{R}^3$ and velocity (or momentum) $v \in \mathbb{R}^3$, determined by
the steady state; details follow below. We study the linear transport operator

$$T f := \{f, E\} = \partial_x E \cdot \partial_x f - \partial_v E \cdot \partial_v f.$$  (1)

Here $\{\cdot, \cdot\}$ denotes the usual Poisson bracket, $\partial_x$ and $\partial_v$ are gradients with respect
to the indicated variable, and $\cdot$ denotes the Euclidean scalar product. The operator $T$
arises in the stability analysis of the steady state by linearization of the corres-
ponding system. It is skew-symmetric with respect to the $L^2$ scalar product on
the proper domain in phase space. As is amply documented in the literature, it
is in general not obvious how to realize such an operator as skew-adjoint, which is
the desired property from a functional analysis point of view, cf. [13, 13.4 Example]. Since $T$
has come up in various places in the literature [5, 7, 9] without the above question
having been properly addressed, we give a careful proof that $T$ is
skew-adjoint when defined on a suitable Hilbert space and equipped with the proper
domain. In addition, we determine the kernel of this operator in the Vlasov-Poisson
case; the analogous result in the relativistic case is open.

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We now discuss the two systems in more detail. As motivation and background we note that in astrophysics, a self-gravitating, collisionless gas is used to model galaxies or globular clusters, cf. [3]. In the context of Newtonian physics such an ensemble is described by the Vlasov-Poisson system:

\[
\begin{align*}
\partial_t f + v \cdot \partial_x f - \partial_x U \cdot \partial_v f &= 0, \\
\Delta U &= 4\pi \rho, \quad \lim_{|x| \to \infty} U(t, x) = 0,
\end{align*}
\]

(2)

Here \(t \in \mathbb{R}, x, v \in \mathbb{R}^3\) stand for time, position, and velocity, \(f = f(t, x, v) \geq 0\) is the density of the ensemble on phase space, \(\rho = \rho(t, x)\) is the corresponding spatial density, \(U = U(t, x)\) denotes the induced gravitational potential, and unless explicitly stated otherwise, integrals extend over \(\mathbb{R}^3\). For background on this system we refer to [12].

When describing the same physical situation in the general relativistic context the role of the potential is taken over by the Lorentz metric on spacetime. We restrict ourselves to the case of spherical symmetry and write the metric in Schwarzschild form

\[ds^2 = -e^{2\mu(t,r)}dt^2 + e^{2\lambda(t,r)}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2).\]

Here \(t \in \mathbb{R}\) is the time coordinate, \(r \in [0, \infty]\) is the area radius, i.e., \(4\pi r^2\) is the area of the orbit of the symmetry group \(\text{SO}(3)\) labeled by \(r\), and the angles \(\theta \in [0, \pi]\) and \(\phi \in [0, 2\pi]\) parametrize these orbits. The spacetime is required to be asymptotically flat with a regular center, which corresponds to the boundary conditions

\[\lim_{r \to \infty} \lambda(t, r) = \lim_{r \to \infty} \mu(t, r) = 0 = \lambda(t, 0).\]

(5)

We write \((x^a) = r(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)\), let \((p^a)\) denote the canonical momenta corresponding to the spacetime coordinates \((x^a) = (t, x^1, x^2, x^3)\), and define

\[v^a := p^a + (e^\lambda - 1) \frac{x \cdot p}{r} \frac{x^a}{r}\]

for \(a = 1, 2, 3\); as above \(\cdot\) stands for the Euclidean scalar product in \(\mathbb{R}^3\). As in the Newtonian case all particles have the same rest mass, normalized to unity. Then

\[p_0 = -e^\mu \sqrt{1 + |v|^2}, \text{ where } |v|^2 = v \cdot v.\]

The density function \(f = f(t, x, v) \geq 0\) is to be spherically symmetric, i.e., for any rotation \(A \in \text{SO}(3)\), \(f(t, x, v) = f(t, Ax, Av)\). In this set-up the Einstein-Vlasov system becomes

\[
\begin{align*}
\partial_t f + e^{\mu - \lambda} \frac{v}{\sqrt{1 + |v|^2}} \cdot \partial_x f &- \left(\partial_t \lambda \frac{x \cdot v}{r} + e^{\mu - \lambda} \partial_r \mu \sqrt{1 + |v|^2}\right) \frac{x}{r} \cdot \partial_v f = 0, \\
e^{-2\lambda}(2r\partial_r \lambda - 1) + 1 &= 8\pi r^2 \rho, \\
e^{-2\lambda}(2r\partial_r \mu + 1) - 1 &= 8\pi r^2 p,
\end{align*}
\]

(6)

\[\rho(t, r) = \rho(t, x) = \int \sqrt{1 + |v|^2} f(t, x, v) \, dv,\]

\[p(t, r) = p(t, x) = \int \left(\frac{x \cdot v}{r}\right)^2 f(t, x, v) \, \frac{dv}{\sqrt{1 + |v|^2}}.\]
For a detailed derivation of these equations we refer to [11]; note that we have written down only a closed subsystem of the over-determined Einstein-Vlasov system. For more background on the latter we refer to [1].

Both systems possess a plethora of steady states. An important class of spherically symmetric such states is obtained via the ansatz

\[ f_0 = \phi \circ E, \]

where \( \phi : \mathbb{R} \to [0, \infty] \) is a suitable ansatz function and \( E \) is the local or particle energy, i.e.,

\[ E(x, v) = \begin{cases} 
\frac{1}{2} |v|^2 + U_0(x), & \text{Newtonian case,} \\
-p_0 = e^{\mu_0(x)} \sqrt{1 + |v|^2}, & \text{relativistic case.}
\end{cases} \]

Here \( U_0 \) or \( \mu_0 \) is the time independent potential or metric component of the steady state, and it is easy to check that \( f_0 \) satisfies the corresponding Vlasov equation 2 or 6. In order to obtain a steady state one can substitute the ansatz 7 into the definition of \( \rho \) (and \( p \)) and try to solve the resulting non-linear problem for \( U_0 \) or \( \mu_0 \); we refer to [10] for sufficient conditions on the ansatz function \( \phi \) which guarantee that suitable steady states result from this procedure.

We now fix such a steady state and linearize the Vlasov-Poisson system about it. If we substitute \( f_0 + f \) for \( f \) and drop the quadratic term in \( f \) in the Vlasov equation 2, the linearized equation becomes

\[ \partial_t f + v \cdot \partial_x f - \partial_x U_0 \cdot \partial_v f + \ldots = 0, \]

where \( \ldots \) stands for a term which depends non-locally on \( f \); i.e., on the field induced by \( f \), and which is of no interest here. The transport term induced by the steady state is \( \mathcal{T} f = \{ f, E \} \). In a stability analysis, the operator typically acts on the weighted space \( L^2_\chi \left( S_0 \right) \), where \( S_0 = \{(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid f_0(x, v) > 0\} \) so that \( S_0 \) is the support of the steady state, a compact set, and \( \chi = 1/|\phi \circ E| \); throughout the paper functions are real-valued. The weight \( \chi \) arises naturally if one computes the second variation of the so-called energy-Casimir functional at a given steady state, and this second variation is used to define the distance measure for the stability estimate, cf. [6]; the results of the present paper remain true if we take \( \chi = 1 \).

The crucial point for our analysis is to give the proper weak formulation of the operator \( \mathcal{T} \) and define a domain for it such that it is not only skew-symmetric, but skew-adjoint. We will also determine the kernel of this operator.

If we proceed analogously for the Einstein-Vlasov system, the linearized equation reads

\[ \partial_t f + e^{\mu_0 - \lambda_0} \frac{v}{\sqrt{1 + |v|^2}} \cdot \partial_x f - e^{\mu_0 - \lambda_0} \sqrt{1 + |v|^2} \partial_x \mu_0 \cdot \partial_v f + \ldots = 0; \]

the analogous comment as in the Newtonian case applies to the term \( \ldots \). Here the linear transport operator equals \( e^{-\lambda_0} \mathcal{T} \). But at the same time the natural weight in the \( L^2 \) space on which the operator acts is \( \chi = e^{\lambda_0}/|\phi \circ E| \). Hence the true relativistic transport operator \( e^{-\lambda_0} \mathcal{T} \) on the properly weighted space is skew-adjoint iff \( \mathcal{T} \) is skew-adjoint on the space with weight \( \chi = 1/|\phi \circ E| \). Thus we keep to the operator \( \mathcal{T} \) as defined before, since this allows us to treat the Newtonian and the relativistic case largely in parallel.

In the context of both linear and nonlinear stability results the properties of the operator \( \mathcal{T} \) play a role at various places in the mathematics literature, cf. [5, 7, 9],...
but also in the physics literature, cf. [8], where \( \mathcal{T} \) appears both in the Newtonian and the relativistic case. We also refer to the second author’s master thesis [15].

In the next section we formulate the basic assumptions and main results of this paper. The skew-adjointness of \( \mathcal{T} \) is proven in Section 3, and in the Newtonian case its kernel is determined in Section 4.

2. Basic assumptions and main results.

(A1) Let \( \phi: \mathbb{R} \to [0, \infty[ \) be such that there exists a constant \( E_0 \) such that \( \phi \in C^1([-\infty, E_0]) \) with \( \phi(E) < 0 \) for \( E < E_0 \), and \( \phi(E) = 0 \) for \( E \geq E_0 \). In the Newtonian case, \( E_0 < 0 \), in the relativistic case, \( 0 < E_0 < 1 \).

(A2) Let \( f_0 = \phi \circ E \) be a steady state of the Vlasov-Poisson or Einstein-Vlasov system, where \( E \) is defined as in 8 and with corresponding, spherically symmetric potential \( U_0 \in C^2(\mathbb{R}^3) \cap C^2([0, \infty[) \) or metric components \( \lambda_0, \mu_0 \in C^2(\mathbb{R}^3) \cap C^2([0, \infty[) \). Let these satisfy the boundary conditions specified in 3 or 5 respectively.

At this point we identify \( U_0(x) = U_0(|x|) \) if \( U_0 \) is spherically symmetric. As shown in [10], many steady states satisfy these assumptions. In addition, \( U_0 \) and \( \mu_0 \) are strictly increasing as radial functions. The set where \( f_0 \) is positive is given by

\[
S_0 = \{ (x, v) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid E(x, v) < E_0 \}.
\]

Due to the cut-off energy \( E_0 \) in (A1) and the boundary condition for \( U_0 \) respectively \( \mu_0 \) at infinity \( S_0 \) is a bounded, spherically symmetric domain; note that the restriction on \( E_0 \) is such that by 8, \( E(x, v) \geq E_0 \) in both the Newtonian or the relativistic case, if \( |x| \) or \( |v| \) are sufficiently large. On functions \( f \in C^1(S_0) \) we define \( \mathcal{T}f \) as in 1. The characteristic flow of the first order differential operator \( \mathcal{T} \) is given by the Hamiltonian system

\[
\dot{x} = \partial_v E(x, v), \quad \dot{v} = -\partial_x E(x, v).
\]

For each \( (x, v) \in \mathbb{R}^3 \times \mathbb{R}^3 \) it has a global, unique solution \( t \mapsto (X, V)(t, x, v) \) such that \( (X, V)(0, x, v) = (x, v) \). For every \( t \in \mathbb{R} \) the map \( (X, V)(t, \cdot, \cdot) \) is a \( C^1 \) diffeomorphism on \( \mathbb{R}^3 \times \mathbb{R}^3 \) which is measure-preserving, i.e., \( \det \frac{\partial (X, V)}{\partial (x, v)}(t, x, v) = 1 \), since the divergence of the right hand side of 10 vanishes. The characteristic flow preserves the particle energy \( E \), in particular the set \( S_0 \) is invariant under the flow. Due to the spherical symmetry of the potential or metric of the steady state the quantity

\[
L(x, v) := |x \times v|^2,
\]

the modulus of angular momentum squared of the particle with coordinates \( (x, v) \in \mathbb{R}^3 \times \mathbb{R}^3 \), is also preserved by the characteristic flow. For \( f \in C^1(S_0) \) we can express the transport operator as

\[
\mathcal{T}f(x, v) = \frac{d}{dt} \left. \left[ f(X(t, x, v), V(t, x, v)) \right] \right|_{t=0}, \quad (x, v) \in S_0.
\]

**Remark 1.** The characteristic flow corresponding to the linearized Einstein-Vlasov system as written in 9 is not measure-preserving, due to the additional factor \( e^{-\lambda_0} \) in the right hand side of the corresponding characteristic system. However, this flow does preserve the weighted phase-space measure \( e^{\lambda_0(x)} dv \, dx \), which is then taken into account by the modified weight in the corresponding \( L^2 \) space on which \( e^{-\lambda_0} \mathcal{T} \) would act. We drop the factor \( e^{-\lambda_0} \) until the skew-adjointness of \( \mathcal{T} \) is proven and then restore it.
The following integration by parts formula shows that the transport operator $T$ is skew-symmetric with respect to weighted $L^2$ scalar products, at least when defined on smooth functions; it should be noticed that no boundary terms appear in this formula.

**Proposition 1.** Let $\chi \in C([-\infty, E_0])$. Then, for any $f, g \in C^1(S_0)$,

$$\int_{S_0} \chi \circ E f T g = - \int_{S_0} \chi \circ E T f g,$$

provided both integrals exist which for example is the case if $f \in C^1_{c}(S_0)$.

**Proof.** We abbreviate $z = (x, v) \in \mathbb{R}^3 \times \mathbb{R}^3$ and define $S_n = \{ z \in \mathbb{R}^6 \mid E(z) < E_0 - 1/n \}$. The properties of the characteristic flow and a change of variables imply that for every $t \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$\int_{S_n} \chi(E(z)) f((X, V)(t, z)) g((X, V)(t, z)) \, dz = \int_{S_n} \chi(E(z)) f(z) g(z) \, dz.$$

The restriction to $S_n$ and the properties of $f, g$, and $\chi$ guarantee that the left hand side can be differentiated with respect to $t$ under the integral. Thus

$$0 = \frac{d}{dt} \int_{S_n} \chi(E(z)) f((X, V)(t, z)) g((X, V)(t, z)) \, dz \bigg|_{t=0} = \int_{S_n} \chi(E(z)) (Tf(z) g(z) + f(z) Tg(z)) \, dz.$$

Hence the assertion follows with $S_n$ instead of $S_0$, and taking $n \to \infty$ completes the proof.

A measurable function $f: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ is *spherically symmetric*, iff for every $A \in \text{SO}(3)$ the identity $f(Ax, Av) = f(x, v)$ holds for a. a. $(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3$; the exceptional set of measure zero may depend on $A$. Functions defined on $S_0$ will always be extended by 0 to $\mathbb{R}^3 \times \mathbb{R}^3$ so that spherical symmetry is also defined for them. The subspace of spherically symmetric functions in a certain function space is denoted by the subscript $r$, for example $C^1_{r,c}(S_0)$ denotes the space of compactly supported, spherically symmetric $C^1$ functions on the set $S_0$.

We now give the definition of the transport operator which will make it skew-adjoint.

**Definition 2.1.** (a) For $f \in L^1_{\text{loc},r}(S_0)$, $Tf$ exists weakly if there exists $\mu \in L^1_{\text{loc},r}(S_0)$ such that for every test function $\xi \in C^\infty_{c,r}(S_0)$,

$$\int_{S_0} \frac{1}{|\phi^\prime \circ E|} f T \xi = - \int_{S_0} \frac{1}{|\phi^\prime \circ E|} \mu \xi.$$

In this case $Tf := \mu$ weakly.

(b) The real Hilbert space

$$H := \left\{ f \in L^1_{\text{loc},r}(S_0) \mid \int_{S_0} \frac{1}{|\phi^\prime \circ E|} f^2 < \infty \right\}$$

is equipped with the scalar product

$$\langle f, g \rangle_H := \int_{S_0} \frac{1}{|\phi^\prime \circ E|} f g.$$
(c) The domain of $\mathcal{T}$ is defined as

$$D(\mathcal{T}) := \{ f \in H \mid \mathcal{T}f \text{ exists weakly and } \mathcal{T}f \in H \}.$$ 

We can now formulate our main results.

**Theorem 2.2.** The transport operator $\mathcal{T} : H \ni D(\mathcal{T}) \rightarrow H$ is skew-adjoint, i.e., $\mathcal{T}^* = -\mathcal{T}$, in both the Newtonian and the relativistic cases.

This theorem will be proven in the next section, but some immediate comments are in order.

**Remark 2.** (a) If $\mathcal{T}f$ exists weakly for some $f \in L^1_{\text{loc},r}(S_0)$, then it is uniquely determined a.e. on $S_0$.

(b) If $f \in C^1_{c,r}(S_0)$, then the weak and the classical definition 1 of $\mathcal{T}f$ coincide due to Proposition 1 and since $\mathcal{T}$ preserves spherical symmetry. In particular, $C^1_{c,r}(S_0) \subset D(\mathcal{T})$ so that the operator $\mathcal{T}$ in Theorem 2.2 is densely defined.

(c) If $\mathcal{T}f$ exists weakly for some $f \in L^1_{\text{loc},r}(S_0)$, then the formula in Definition 2.1 (a) holds for all test functions $\xi \in C^1_{c,r}(S_0)$. To see this we take such a function and define $\xi_k := J_k * \xi \in C^\infty_{c,r}(S_0)$, where $J \in C^\infty_c(B_1(0))$ is a mollifier with $\int J = 1$, and $J_k := k^6 J(k \cdot)$ for $k \in \mathbb{N}$. There exists a compact subset $K \subset S_0$ such that supp$\xi_k \subset K$ for $k$ sufficiently large, and $\xi_k \rightarrow \xi$. Then $\mathcal{T}\xi_k \rightarrow \mathcal{T}\xi$ as $k \rightarrow \infty$, uniformly on $K$. This implies the identity in Definition 2.1 (a) for $\xi$.

(d) In the relativistic case we restricted ourselves to spherically symmetric functions from the start, but also in the applications in the Newtonian case the skew-adjointness is needed on spherically symmetric functions.

**Theorem 2.3.** In the Newtonian case,

$$\ker \mathcal{T} = \{ f \in H \mid \exists g : \mathbb{R}^2 \rightarrow \mathbb{R} : f(x,v) = g(E(x,v),L(x,v)) \text{ a.e. on } S_0 \}.$$ 

This theorem will be proven in Section 4.

**Remark 3.** (a) Since $\mathcal{T}f = 0$ means that $f$ satisfies the stationary Vlasov equation with the potential $U_0$, Theorem 2.3 generalizes the fact that for spherically symmetric steady states of the Vlasov-Poisson system the density $f$ on phase space depends only on the quantities $E$ and $L$, a fact known as Jeans’ Theorem, cf. [2].

(b) In the relativistic case Jeans’ Theorem is false, cf. [14]. Since in the present paper we restrict ourselves to isotropic steady states, Theorem 2.3 might still be correct also in the relativistic case, but the proof given below does not work there.

3. **Proof of theorem 2.2—Skew-adjointness of $\mathcal{T}$.** In view of Proposition 1, the main tool for the proof of Theorem 2.2 is to approximate a function from $D(\mathcal{T})$ by smooth functions in such a way that the images under $\mathcal{T}$ converge as well. As a first step we exploit the fact that $\mathcal{T}E = \mathcal{T}L = 0$.

**Lemma 3.1.** (a) Let $f \in D(\mathcal{T})$ and $\chi \in C^1([0,\infty[)$ be such that $\chi \circ Lf, \chi \circ L\mathcal{T}f \in H$. Then $\chi \circ Lf \in D(\mathcal{T})$ with $\mathcal{T}(\chi \circ Lf) = \chi \circ L\mathcal{T}f$ weakly.

(b) Let $f \in D(\mathcal{T})$ and $\chi \in C([-\infty, E_0[)$ be such that $\chi \circ Ef, \chi \circ E\mathcal{T}f \in H$. Then $\chi \circ Ef \in D(\mathcal{T})$ with $\mathcal{T}(\chi \circ Ef) = \chi \circ E\mathcal{T}f$ weakly.
Proof. As to part (a), let $\xi \in C^1_{c,r}(S_0)$ be a test function. Since $\chi \circ L \in C^1_c(S_0)$, we know that $\chi \circ L \xi \in C^1_{c,r}(S_0)$ as well. Since $L$ is constant along characteristics, $TL = 0$ and therefore $T(\chi \circ L \xi) = \chi \circ L T \xi$ classically. Thus, by Definition 2.1,

$$
\int_{S_0} \frac{1}{|\phi' \circ E|} f \chi \circ L T \xi = \int_{S_0} \frac{1}{|\phi' \circ E|} f T(\chi \circ L \xi) = - \int_{S_0} \frac{1}{|\phi' \circ E|} T f \chi \circ L \xi.
$$

The proof of part (b) is exactly the same, provided $\chi \in C^1([-\infty, E_0[)$. If $\chi$ is only continuous we mollify $\chi$; we omit the details since below we need to apply the lemma with $\chi = |\phi'|$, and for all steady states of interest this function is in $C^1([-\infty, E_0[)$.

Corollary 1. Let $f \in D(T)$. Then for any test function $\xi \in C^1_{c,r}(S_0),$

$$
\int_{S_0} f T \xi = - \int_{S_0} T f \xi.
$$

Proof. We apply Lemma 3.1 to $\chi = |\phi'|$, which if necessary has to be cut to ensure integrability; due to the compact support of test functions we may restrict ourselves to a compact subset of $S_0$.

In order to approximate functions in $D(T)$ in a suitable way we have to mollify them, and for technical reasons this needs to be done in coordinates which are adapted to spherical symmetry. For $(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3$ we define

$$
r := |x|, \quad w := \frac{x \cdot v}{r}, \quad L := |x \times v|^2;
$$

the radial velocity $w$ is defined only if $x \neq 0$. A function $f \in H$ is spherically symmetric in the sense defined above iff there exists a measurable function $f^r : [0, \infty[ \times \mathbb{R} \times [0, \infty[ \to \mathbb{R}$ such that

$$
f(x, v) = f^r(r, w, L) \quad \text{for a.e. } (x, v) \in \mathbb{R}^3 \times \mathbb{R}^3.
$$

In this case, $f^r$ is uniquely defined a.e. on $[0, \infty[ \times \mathbb{R} \times [0, \infty[$. In what follows an upper index $r$ will indicate that a spherically symmetric object is expressed in the variables $r, w, L$. In particular,

$$
E^r(r, w, L) = \begin{cases}
\frac{1}{2} w^2 + \frac{L}{2r^2} + U_0(r), & \text{Newtonian case,} \\
\frac{e^{\rho_0(r)}}{r} \sqrt{1 + w^2 + \frac{L}{r^2}}, & \text{relativistic case},
\end{cases}
$$

$$
S^r_0 = \{(r, w, L) \in [0, \infty[ \times \mathbb{R} \times [0, \infty[ | E^r(r, w, L) < E_0\},
$$

and

$$
T^r f = \partial_r E^r \partial_r f - \partial_r E^r \partial_w f,
$$

for $f \in C^1_c(S^r_0)$. The transport operator $T$ preserves spherical symmetry, and for every $\xi \in C^1_{c,r}(S_0)$ with the property that $\xi^r \in C^1_c(S^r_0),$

$$
T^r \xi^r = (T \xi)^r \text{ on } S^r_0;
$$
note that $\xi \in C^1_c(S^0_0)$ does not imply that $\xi^r \in C^1_c(S^0_0)$, since the support of functions in $C^1_c(S^0_0)$ has to be bounded away from $r = 0$ and $L = 0$. The integration-by-parts formula in Proposition 1 can be rewritten as

$$
\int_{S^0_0} \chi(E^r(r, w, L)) f(r, w, L) \langle T^r g(r, w, L) \rangle \, dr \, dw \, dL = -\int_{S^0_0} \chi(E^r(r, w, L)) \langle T^r f \rangle (r, w, L) g(r, w, L) \, dr \, dw \, dL
$$

(12)

for all $f, g \in C^1_c(S^0_0)$ and $\chi \in C([-\infty, E_0[)$; this follows from the spherical symmetry of the integrands and the change-of-variables formula, observing that $dv = \frac{r^2}{2} dL \, dw$ and $dx = 4\pi r^2 dr$. Similarly, Corollary 1 can be rewritten in the variables $r, w, L$, i.e., $12$ remains valid with $\chi = 1, g \in C^1_c(S^0_0)$, and $f$ replaced by the representative $f^r$ of a function $f \in D(T)$.

We can now prove the desired approximation result.

**Proposition 2.** Let $f \in D(T)$. Then there exists a sequence $(F_k)_{k \in \mathbb{N}} \subset C^\infty_c(S^0_0)$ such that $F_k^r \in C^\infty_c(S^0_0)$ for $k \in \mathbb{N}$ and

$$
F_k \to f \quad \text{and} \quad T F_k \to T f \quad \text{in} \quad H \quad \text{as} \quad k \to \infty.
$$

**Proof.** We split the proof into several steps.

**Step 1: Reduction to a compact support.** For each $k \in \mathbb{N}$ let $\chi_k \in C^\infty(\mathbb{R})$ be an increasing cut-off function such that

$$
\chi_k(s) = 0 \quad \text{for} \quad s \leq \frac{1}{2k}, \quad \chi_k(s) = 1 \quad \text{for} \quad s \geq \frac{1}{k}.
$$

For $(x, v) \in S_0$ and $k \in \mathbb{N}$ let

$$
f_k(x, v) := \chi_k(L(x, v)) \chi_k(E_0 - E(x, v)) f(x, v).
$$

The boundedness of $\chi_k$ together with the spherical symmetry of $E$ and $L$ ensure that $f_k \in H$. Hence by Lemma 3.1, $f_k \in D(T)$ with $T f_k = (\chi_k \circ L) (\chi_k \circ (E_0 - E)) (T f)$, and by Lebesgue’s dominated convergence theorem,

$$
f_k \to f \quad \text{and} \quad T f_k \to T f \quad \text{in} \quad H \quad \text{as} \quad k \to \infty.
$$

By applying the following arguments to $f_k$ for $k \in \mathbb{N}$ sufficiently large instead of to $f$, we may assume that $f^r$ has compact support in $S^0_0$ and that there exists $m \in \mathbb{N}$ such that for a.e. $(r, w, L) \in S^0_0$ with $f^r(r, w, L) \neq 0$,

$$
E^r(r, w, L) < E_0 - \frac{1}{m} < 0 \quad \text{and} \quad \bar{B}_\frac{1}{m}(r, w, L) \subset S^0_0.
$$

This also implies that $T f = 0$ a.e. on \{$(x, v) \in S_0 \mid |x| \leq \frac{1}{m} \vee L(x, v) \leq \frac{1}{m} \vee E(x, v) \geq E_0 - \frac{1}{m}$\}. Furthermore, $(T f)^r$ has compact support in $S^0_0$.

**Step 2: The approximating sequence.** We first introduce some terminology. We let $H^r := \{ f : S^0_0 \to \mathbb{R} \text{ measurable} \mid \| f \|_{H^r} < \infty \}$, where the norm $\| \cdot \|_{H^r}$ is induced by the scalar product

$$
\langle f, g \rangle_{H^r} := 4\pi^2 \int_{S^0_0} \frac{1}{|\phi'(E^r(r, w, L))|} f(r, w, L) g(r, w, L) \, dr \, dw \, dL.
$$

The space $L^2(S^0_0)$ is obtained by dropping the weight $1/|\phi' \circ E^r|$, but the factor $4\pi^2$ is included in the corresponding scalar product and norm. Hence the change-of-variables formula shows that the map $f \mapsto f^r$ is an isometric isomorphism of the spaces $L^2(S^0_0) \cong L^2(S^0_0)$ or $H \cong H^r$. 

Let $J \in C_c^\infty(B_1(0))$ be a three-dimensional mollifier, i.e., $B_1(0) \subset \mathbb{R}^3$, $\int_{\mathbb{R}^3} J = 1$, and $J \geq 0$. We define $J_k := k^3 J(k \cdot)$ for $k \in \mathbb{N}$. Due to the compact support of $f^r$ and $(T f)^r$ in $S_0^*$, standard mollifying arguments, a change of variables, and the fact that the weight $1/|\phi' \circ E|$ is a strictly positive, bounded, continuous function on any compact subset of $S_0^*$ it follows that

$$f^r \in L^2(S_0^*)\text{ and } J_k * f^r \to f^r \text{ in } L^2(S_0^*) \text{ and in } H^r \text{ as } k \to \infty,$$

$$(T f)^r \in L^2(S_0^*) \text{ and } J_k * (T f)^r \to (T f)^r \text{ in } L^2(S_0^*) \text{ and in } H^r \text{ as } k \to \infty.$$

**Step 3: Boundedness.** We want to prove that $(T(J_k * f^r))_{k \in \mathbb{N}} \subset L^2(S_0^*)$ is bounded, from which we can then obtain the weak convergence of a subsequence. The properties of the support of $f^r$, the boundedness of $S_0^*$, and the mean value theorem yield the existence of a constant $C_E \geq 1$ such that

$$|E^r(z) - E^r(z')| \leq C_E |z - z'| \text{ for } z, z' \in \bar{B}_{\frac{1}{m}}(\text{supp } f^r) \subset S_0^*;$$

$B_r(M) := \bigcup_{\varepsilon \in M} B_r(x)$ for $r > 0$ and some set $M \subset \mathbb{R}^3$. We choose $k \in \mathbb{N}$ such that $k > 2C_E m$. For every $z = (r, w, L) \in S_0^*$ with $(J_k * f^r)(z) \neq 0$, it then follows that $\tilde{B}_{1/(2m)}(z) \subset S_0^*$ and $E^r(z) < E_0 - 1/(2m)$. In particular, $J_k * f^r \in C^0_c(S_0^*)$ for $k > 2C_E m$. For these $k$ and $z \in \text{supp } (J_k * f^r)$ it follows that

$$[T^r(J_k * f^r)](z) = \partial_w E^r(z) \nabla_z J_k(z - z') - \partial_r E^r(z) \partial_r J_k(z - z') f^r(z') \text{ for } z' \in (r', w', L') \text{.}$$

Since $\partial_r E^r$ and $\partial_w E^r$ are Lipschitz on $\tilde{B}_{1/m}(\text{supp } f^r)$, the absolute value of the first term can be estimated by

$$\frac{C}{k} \int_{\tilde{B}_{1/k}(z)} |DJ_k(z - z')| |f^r(z')| \, dz' = \frac{C}{k} \int_{\tilde{B}_{1/k}(0)} |DJ_k(\tilde{z})| |f^r(z - \tilde{z})| \, d\tilde{z}$$

$$= C k^3 \int_{\tilde{B}_{1/k}(0)} |DJ_k(\tilde{z})| |f^r(z - \tilde{z})| \, d\tilde{z} = Ck^3 |DJ_k(\cdot)||f^r(\cdot)|(z),$$

where $C > 0$ depends on the support of $f$ and the fixed steady state $f_0$ and $DJ_k$ denotes the total derivative of $J_k$. For the second term we note that $J_k(z - \cdot) \in C^1_c(S_0^*)$ for $k > 2C_E m$, since $\text{supp } J_k(z - \cdot) \subset \tilde{B}_{1/(2m)}(z) \subset S_0^*$. By 12 and the comment after that equation we conclude that

$$\int_{S_0^*} [\partial_w E(z') \partial_r J_k(z - z') - \partial_r E(z') \partial_w J_k(z - z')] f^r(z') \, dz'$$

$$= -(T^r [J_k(z - \cdot)], f^r)_2 = (J_k(z - \cdot), (T f)^r)_2 = [J_k * (T f)^r](z).$$

Altogether, we get the estimate

$$\|T(J_k * f^r)\|_2 \leq \|J_k * (T f)^r\|_2 + Ck^3 \|DJ_k(\cdot)||f^r||_2$$

$$\leq \|J_k * (T f)^r\|_2 + Ck^3 \|f^r\|_2 \|DJ_k(\cdot)\|_1,$$
where we used Young’s inequality. Since $J_k \ast (Tf)' \to (Tf)'$ in $L^2(S_0)$ as $k \to \infty$, the first term is bounded. As to the second, we note that $\|DJ(k)\|_1 = k^{-3/2}\|DJ\|_1$, and the desired boundedness is proven.

**Step 4: Weak convergence.** Let $f_k' := J_k \ast f'$. Due to the previous step there exists a subsequence which by abuse of notation we again denote as $(f_k')_{k \in \mathbb{N}}$ and a limit $g' \in L^2(S_0)$ such that

$$T'f_k' \to g' \text{ in } L^2(S_0) \text{ as } k \to \infty.$$ 

We need to show that $g = Tf$, where for $(x, v) \in S_0$ with $x \times v \neq 0$, $g(x, v) := g' \left( |x|, \frac{x \cdot v}{|x|}, |x \times v| \right)$.

Let $\xi \in C^1_c(S_0)$ be an arbitrary test function. We have to ensure that $\xi' \in C^1_c(S_0)$, which can be achieved due to the compact support of $f'$ in $S_0$. From the properties of the support of $f_k'$ shown in Step 3,

$$(\chi_{2m} \circ L)(\chi_{2m} \circ (E_0 - E))T'f_k' = T'f_k'$$

if $k$ is sufficiently large. Let $\tilde{\xi} := (\chi_{2m} \circ L)(\chi_{2m} \circ (E_0 - E))\xi$ and note that $\tilde{\xi}' = C^1_c(S_0)$. In addition, $fT\tilde{\xi} = f T\xi$ and $g\tilde{\xi} = g\xi$ a. e. on $S_0$, where the latter follows from the properties of the support of $f_k'$ and the Du Bois-Reymond theorem. Thus, changing variables yields

$$\langle g, \xi \rangle_H = \langle g, \tilde{\xi} \rangle_H = \langle g', \tilde{\xi}' \rangle_{H'} = \lim_{k \to \infty} \langle T'f_k', \tilde{\xi}' \rangle_{H'} = -\lim_{k \to \infty} \langle f_k', T'\tilde{\xi}' \rangle_{H'} = -\lim_{k \to \infty} \langle f_k', T\xi' \rangle_{H'} = -\langle f, T\xi \rangle_{H'},$$

where we used 12 and the fact that due to the compact support of $\tilde{\xi}'$, $\langle \cdot, \tilde{\xi}' \rangle_{H'} \in L^2(S_0)'$.

**Step 5: Strong convergence.** By the previous step, $T'f_k' \to (Tf)'$ in $L^2(S_0)$ as $k \to \infty$. Mazur’s lemma implies that for every $k \in \mathbb{N}$ there exists $N_k \geq k$ and weights $c_{i_k}^J, \ldots, c_{N_k}^J \in [0, 1]$ with $\sum_{j=k}^{N_k} c_j^J = 1$ such that

$$T' \left( \sum_{j=k}^{N_k} c_j^J f_j' \right) = \sum_{j=k}^{N_k} c_j^J T'f_j' \to (Tf)' \text{ in } L^2(S_0) \text{ as } k \to \infty.$$ 

Let $F_k^J := \sum_{j=k}^{N_k} c_j^J f_j'$ for $k \in \mathbb{N}$. Then $F_k^J \to f'$ in $L^2(S_0)$ and $H'$ as $k \to \infty$. Also $F_k^J \in C^\infty_c(S_0)$ for $k$ sufficiently large. Finally,

$$F_k(x, v) := F_k^J \left( |x|, \frac{x \cdot v}{|x|}, |x \times v| \right) \text{ for } (x, v) \in S_0 \text{ with } x \times v \neq 0,$$

extended by 0 on $S_0$, defines a function $F_k \in C^\infty_c(S_0)$ for $k$ sufficiently large, due to the compact support of $F_k^J$. Changing variables yields

$$F_k \to f \text{ in } H \text{ and } T F_k \to T f \text{ in } L^2(S_0) \text{ as } k \to \infty,$$

and the support properties of the involved functions allow us to conclude that $T F_k \to T f$ in $H$ as well, which finishes the proof. 

Let us recall the definition of the adjoint operator $T^* : H \subset D(T^*) \to H$ of the operator $T$. Its domain of definition is

$$D(T^*) := \{ f \in H \mid \exists h \in H \forall g \in D(T) : \langle T g, f \rangle_H = \langle g, h \rangle_H \}.$$
For \( f \in D(T^*) \), \( T^*f := h \); note that \( T \) is a densely defined operator on the Hilbert space \( H \).

**Proof of Theorem 2.2.** First we observe that \( T \) is skew-symmetric, i.e., for all \( f, g \in D(T) \) it holds that
\[
\langle f, Tg \rangle_H = -\langle Tf, g \rangle_H.
\]
(13) This follows by approximating one of the two functions via Proposition 2 and then using Definition 2.1.

Now let \( f \in D(T) \). Then (13) implies that
\[
\langle Tg, f \rangle_H = -\langle g, Tf \rangle_H = \langle g, -Tf \rangle_H
\]
for all \( g \in D(T) \), i.e., \( f \in D(T^*) \) with \( T^*f = -Tf \), and hence \( -T \subset T^* \).

If on the other hand \( f \in D(T^*) \) and \( h \in H \) are such that \( \langle Tg, f \rangle_H = \langle g, h \rangle_H \) for all \( g \in D(T) \), then since \( C^1_{c,r}(S_0) \subset D(T) \) this implies that
\[
\langle f, T\xi \rangle_H = -\langle h, \xi \rangle_H
\]
for all test functions \( \xi \in C^1_{c,r}(S_0) \). By Definition 2.1, this means that \( f \in D(T) \) with \( Tf = -h \), i.e., \( T^* \subset -T \), and the proof is complete. \( \square \)

We conclude this section with some further remarks.

**Remark 4.** (a) As noted in the introduction the relevant operator in the general relativistic case is actually given as \( \tilde{T} = e^{-\lambda_0 T} \). Let us denote by \( \tilde{H} \) the Hilbert space \( H \) equipped with the scalar product
\[
\langle f, g \rangle_{\tilde{H}} := \int_{S_0} e^{\lambda_0 |\phi'|} f g.
\]
Then the transport operator \( \tilde{T} : \tilde{H} \supset D(T) \to \tilde{H} \) is skew-adjoint; note that \( D(\tilde{T}) = D(T) \).

(b) Theorem 2.2 remains correct without the assumption of spherical symmetry, i.e., if this symmetry assumption is removed from all the relevant function spaces. Indeed, the proof then becomes simpler, but as mentioned before, in the applications the operator \( T \) is defined on spherically symmetric functions, and Theorem 2.2 including this assumption is needed in the proof of Theorem 2.3.

(c) The relation (11) to the characteristic flow induced by (10) suggests an alternative route to study the operator \( T \). Since the characteristic flow is measure preserving,
\[
(U(s)f)(x,v) := f(X(s,x,v),V(s,x,v)), \quad (x,v) \in S_0, \; s \in \mathbb{R},
\]
defines a unitary \( C^0 \)-group on \( H \). By Stone’s theorem, this \( C^0 \)-group has a unique skew-adjoint infinitesimal generator \( \tilde{T} \) defined on the dense subset
\[
D(\tilde{T}) := \left\{ f \in H \mid \lim_{s \to 0} \frac{U(s)f - f}{s} \text{ exists in } H \right\}
\]
by
\[
\tilde{T}f := \lim_{s \to 0} \frac{U(s)f - f}{s}.
\]
Since \( T \) is skew-adjoint on \( H \) as well and \( T \) and \( \tilde{T} \) coincide on the dense subset \( C_{c,r}(\Omega_0) \), \( T = \tilde{T} \), in particular \( D(T) = D(\tilde{T}) \).
4. Proof of theorem 2.3—The kernel of \( T \). We begin this section by stressing that from now on all the arguments refer only to the Newtonian case, i.e., to the Vlasov-Poisson system. In that context a smooth, spherically symmetric solution \( f \) of the equation \( Tf = 0 \) depends only on the quantities \( E \) and \( L \), cf. [2]. Functions in \( D(T) \) need not be smooth, and it therefore seems natural to prove Theorem 2.3 by mollifying such functions like in the previous section. The mollification of a function in the kernel of \( T \) need not belong to the kernel anymore, but we will show that the distance between elements of an approximating sequence obtained by mollification and their projection onto the space of functions depending only on \( (E, L) \) tends to zero. In order to define this projection, we first analyze the solutions of the characteristic system 10 in the coordinates \((r, w, L)\). Since \( L \) is constant along characteristics, we treat it as a parameter. For fixed \( L > 0 \) the particle trajectories are governed by

\[
\dot{r} = w, \quad \dot{w} = -\psi'_L(r),
\]

where the effective potential \( \psi_L \) is defined as

\[
\psi_L(r) := U_0(r) + \frac{L}{2r^2}.
\]

We need the following properties of this effective potential; these results can be found in [6, 9], but for the sake of completeness, we include their proofs.

Lemma 4.1. (a) For any \( L > 0 \) there exists a unique \( r_L > 0 \) such that

\[
\min_{|0, \infty|} \psi_L(r) = \psi_L(r_L) < 0.
\]

The mapping \([0, \infty| \ni L \mapsto r_L\) is continuously differentiable.

(b) For any \( L > 0 \) and \( E \in [\psi_L(r_L), 0| \) there exist two unique radii

\[
0 < r_-(E, L) < r_L < r_+(E, L) < \infty
\]

such that \( \psi_L(r_\pm(E, L)) = E \). The functions

\[
\{(E, L) \in |\infty|, 0| \times 0, \infty| \ni \psi_L(r_L) < E \} \ni (E, L) \mapsto r_\pm(E, L)
\]

are continuously differentiable.

(c) For any \( L > 0 \) and \( E \in [\psi_L(r_L), 0|, \)

\[
r_+(E, L) < -\frac{M_0}{E},
\]

where \( M_0 := ||f'||_{L^1(0, \infty)} \) denotes the total mass of the steady state.

(d) For any \( L > 0 \), \( E \in [\psi_L(r_L), 0| \) and \( r \in [r_-(E, L), r_+(E, L)] \) the following estimate holds:

\[
E - \psi_L(r) \geq L \frac{(r_+(E, L) - r)(r - r_-(E, L))}{2r^2 r_-(E, L) r_+(E, L)}.
\]

Proof. First we note the \( \psi'_L(r) = U'_0(r) - L/r^3 = r^{-2}(m_0(r) - L/r) \), where \( m_0(r) := 4\pi \int_0^r s^2 \rho_0(s) \, ds \) is the mass within the ball of radius \( r \) for the given steady state. Hence \( \psi'_L(r) = 0 \) is equivalent to \( m_0(r) = L/r \). Since the mapping \([0, \infty| \ni r \mapsto m_0(r) = L/r \) is strictly increasing and

\[
\lim_{r \to 0} \left( m_0(r) - \frac{L}{r} \right) = -\infty, \quad \lim_{r \to \infty} \left( m_0(r) - \frac{L}{r} \right) > 0,
\]

there exists a unique radius \( r_L > 0 \) with \( \psi'_L(r_L) = 0 \) as well as \( \psi'_L(r) < 0 \) for \( 0 < r < r_L \) and \( \psi'_L(r) > 0 \) for \( r > r_L \). This monotonicity together with \( \lim_{r \to 0} \psi_L(r) = \infty \)
and \( \lim_{r \to \infty} \psi_L(r) = \lim_{r \to \infty} U_0(r) = 0 \) implies that \( \psi_L(r_L) \) is negative and the minimal value of \( \psi_L \) on \([0, \infty[\). Since
\[
\frac{d}{dr} \left( m_0(r) - \frac{L}{r} \right) = 4\pi r^2 \rho_0(r) + \frac{L}{r^2} > 0
\]
for all \( r > 0 \), the continuous differentiability follows by the implicit function theorem. This proves part (a), and we note that \( r_L = L/m_0(r_L) \).

As to part (b), the monotonicity of \( \psi_L \) together with its limit behavior as \( r \to 0 \) and \( r \to \infty \) yields the existence and uniqueness of \( r_\pm(E, L) \). Since \( \psi_L'(r) \neq 0 \) for \( r \neq r_L \), the implicit function theorem implies that the mapping \( (E, L) \mapsto r_\pm(E, L) \) is continuously differentiable.

In order to show (c) we first note that for all \( r > 0 \),
\[
U_0(r) = -\frac{m_0(r)}{r} - 4\pi \int_r^\infty \rho_0(s) ds \\
\geq -\frac{1}{r} \left( m_0(r) + 4\pi \int_r^\infty \rho_0(s) ds \right) = -\frac{M_0}{r}.
\]
Hence every \( r > 0 \) with \( E - \psi_L(r) > 0 \) also satisfies
\[
E + \frac{M_0}{r} - \frac{L}{2r^2} > 0.
\]
This quadratic inequality implies that
\[
r_+(E, L) \leq \frac{L}{M_0 - \sqrt{M_0^2 + 2EL}} = -M_0 - \sqrt{M_0^2 + 2EL} < -\frac{M_0}{E},
\]
note that for \( 0 > E > \psi_L(r_L) \),
\[
M_0^2 + 2EL > M_0^2 - 2L \frac{M_0}{r_L} + \frac{L^2}{r_L^2} = (M_0 - m_0(r_L))^2 \geq 0.
\]
As to part (d) we define for \( r \in [r_-(E, L), r_+(E, L)] \),
\[
\xi(r) := E - \psi_L(r) - L \frac{(r_+(E, L) - r)(r - r_-(E, L))}{2r^2 r_-(E, L) r_+(E, L)}.
\]
Then the radial Poisson equation implies that
\[
\frac{d^2}{dr^2} [r \xi(r)] = -\frac{1}{r} \frac{d}{dr} [r^2 \psi'_{L}(r)] = -4\pi r \rho_0(r) \leq 0.
\]
The mapping \([r_-(E, L), r_+(E, L)] \ni r \mapsto r \xi(r) \in \mathbb{R}\) is therefore concave with \( \xi(r_\pm(E, L)) = 0 \). Hence \( \xi \geq 0 \) on the interval \([r_-(E, L), r_+(E, L)]\), and proof is complete.

We now consider an arbitrary, global solution \( \mathbb{R} \ni t \mapsto (r(t), w(t), L) \) of the characteristic system \( 14 \). Since the particle energy is conserved along characteristics, \( E = E(r(t), w(t), L) \) for all \( t \in \mathbb{R} \). We assume that \( L > 0 \) and \( E < 0 \), since other solutions are of no interest. For any \( t \in \mathbb{R} \),
\[
\psi_L(r_L) \leq \psi_L(r(t)) \leq \frac{1}{2} w^2(t) + \psi_L(r(t)) = E
\]
and thus \( r_-(E, L) \leq r(t) \leq r_+(E, L) \) by Lemma \( 4.1 \). Solving for \( w \) yields
\[
\dot{r}(t) = w(t) = \pm \sqrt{2E - 2\psi_L(r(t))}
\]
for \( t \in \mathbb{R} \). Therefore, \( r \) oscillates between \( r_-(E, L) \) and \( r_+(E, L) \), where \( \dot{r} = 0 \) is equivalent to \( r = r_\pm(E, L) \), and \( \dot{r} \) switches its sign when reaching \( r_\pm(E, L) \).
By applying the inverse function theorem and integrating, we obtain the following formula for the period of the $r$-motion, i.e., the time needed for $r$ to travel from $r_-(E, L)$ to $r_+(E, L)$ and back. For $L > 0$ and $\psi_L(r_L) < E < E_0$ let

$$T(E, L) := 2 \int_{r_-(E, L)}^{r_+(E, L)} \frac{dr}{\sqrt{2E - 2\psi_L(r)}}. \quad (15)$$

Since $E - \psi_L(r) > 0$ for $r_-(E, L) < r < r_+(E, L)$, this expression is well defined. Lemma 4.1 implies that $T(E, L)$ is finite:

$$T(E, L) = \sqrt{2} \int_{r_-(E, L)}^{r_+(E, L)} \frac{dr}{\sqrt{E - \psi_L(r)}} \leq 2 \int_{r_-(E, L)}^{r_+(E, L)} \frac{r \sqrt{r - (E, L)r_+(E, L)}}{\sqrt{L}} dr \leq 2 \int_{r_-(E, L)}^{r_+(E, L)} \frac{dr}{\sqrt{L}} \leq 2 \pi \frac{r^2_+(E, L)}{\sqrt{L}} \leq 2 \pi \frac{M^2_0}{E^2 \sqrt{L}}. \quad (16)$$

The projection onto the space of functions depending only on $E$ and $L$ is obtained by averaging over trajectories fixed by $(E, L)$. For fixed $(r, w, L) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ let $\mathbb{R} \ni t \mapsto (R, W)(t, r, w, L)$ be the unique global solution of the characteristic system 14 satisfying the initial condition $(R, W)(0, r, w, L) = (r, w)$. For $f \in H$ (extended by 0 to $\mathbb{R}^3 \times \mathbb{R}^3$) and $L > 0$, $\psi_L(r_L) < E < E_0$ we define its projection as

$$P f(E, L) := \int_0^1 f^r((R, W)(t T(E, L), r_-(E, L), 0, L), L) dt \quad \text{as a map from } H \text{ into } H, \quad (17)$$

Then $P f(E, L)$ is uniquely determined for a. e. $(E, L) \in \mathbb{R}^2$ satisfying $L > 0$ and $\psi_L(r_L) < E < E_0$, since

$$\int_{S_0} f(x, v) dx dv = 4\pi^2 \int_0^{E_0} \int_{\psi_L(r_L)}^T(E, L) P f(E, L) dE dL.$$

To obtain this identity we first change to the integration variables $(r, w, L)$ as explained in the derivation of 12, then change the order of integration from $(r, w, L)$ to $(L, w, r)$, change the integration variable $w$ to $E$, and finally we express the resulting integrand using 17. We want to interpret $P$ as a map from $H$ into $H$, so by slight abuse of notation we denote the function

$$S_0 \ni (x, v) \mapsto P f(E(x, v), L(x, v))$$

also by $P f$. Then for all $f, g \in H$,

$$\langle P f, g \rangle_H = 4\pi^2 \int_0^{E_0} \int_{\psi_L(r_L)}^T(E, L) \frac{T(E, L)}{\phi'(E)} P f(E, L) P g(E, L) dE dL = \langle f, P g \rangle_H,$$

in particular, $\langle P f, f \rangle_H = \|P f\|^2_H$, and hence $\|P f\|^2_H \leq \|f\|^2_H$ which means that $P$ maps $H$ into itself. Since $P P f = P f$, $P$ is the orthogonal projection onto the closed subspace

$$K_T := \{f \in H \mid \exists g: \mathbb{R}^2 \to \mathbb{R}: f(x, v) = g(E(x, v), L(x, v)) \text{ a. e. on } S_0\}$$

$$= \{f \in H \mid f(x, v) = P f(E(x, v), L(x, v)) \text{ a. e. on } S_0\}.$$
Projection operators similar to $\mathcal{P}$ have for example been used in [4, 5, 9].

**Proof of Theorem 2.3, i.e., $\ker \mathcal{T} = K_\mathcal{T}$.** We first show the easy inclusion. Let $f \in K_\mathcal{T}$, i.e., for some $g : \mathbb{R}^2 \to \mathbb{R}$, $f(z) = g(E(z), L(z))$ a.e. on $S_0$; we abbreviate $z = (x, v)$. Since $E$ and $L$ are conserved along characteristics, we find that for every $\xi \in C^1_c(S_0)$,

$$
\int_{S_0} \frac{1}{|\phi'(E(z))|} f(z) \xi((X, V)(t, z)) \, dz = \int_{S_0} \frac{1}{|\phi'(E(z))|} f(z) \xi(z) \, dz.
$$

Thus

$$
0 = \frac{d}{dt} \left[ \int_{S_0} \frac{1}{|\phi'(E(z))|} f(z) \xi((X, V)(t, z)) \, dz \right] \bigg|_{t=0} = \int_{S_0} \frac{1}{|\phi'(E(z))|} f(z) \, \mathcal{T}_\mathcal{T}(\xi) \, dz.
$$

By Definition 2.1, this means that $f \in D(\mathcal{T})$ with $\mathcal{T} f = 0$, i.e., $f \in \ker \mathcal{T}$.

As to the other inclusion, let $f \in \ker \mathcal{T}$, i.e., $f \in D(\mathcal{T})$ with $\mathcal{T} f = 0$. As stated above, we will show $f \in K_\mathcal{T}$ by approximation. We will split this argument into several steps.

**Step 1: Reduction to a compact support.** As before, let $\chi_k \in C^\infty(\mathbb{R})$ be a smooth, increasing cut-off function with

$$
\chi_k(s) = 0 \text{ for } s \leq \frac{1}{2k}, \quad \chi_k(s) = 1 \text{ for } s \geq \frac{1}{k}
$$

for each $k \in \mathbb{N}$. Now set

$$
f_k(z) := \chi_k(L(z)) \chi_k(E_0 - E(z)) f(z)
$$

for $z \in S_0$ and $k \in \mathbb{N}$. Since $f_k \to f$ in $H$ as $k \to \infty$ and since $K_\mathcal{T}$ is closed, it suffices to show $f_k \in K_\mathcal{T}$ for every $k \in \mathbb{N}$ to conclude $f \in K_\mathcal{T}$. Thus, we assume that there exists $m \in \mathbb{N}$ such that for a.e. $z \in S_0$ with $f(z) \neq 0$ we have $L(z) \geq \frac{1}{m}$ and $E(z) \leq E_0 - \frac{1}{m} < 0$.

**Step 2: Approximation like in Proposition 2.** We can construct an approximating sequence $(F_k)_{k \in \mathbb{N}} \subset C^\infty_c(S_0)$ such that

$$
F_k \to f \text{ and } \mathcal{T}F_k \to \mathcal{T} f = 0 \text{ in } H \text{ as } k \to \infty,
$$

where

$$
\text{supp } F_k \subset \left\{ z \in S_0 \mid L(z) \geq \frac{1}{2m} \right\}, \quad k \in \mathbb{N}.
$$

Furthermore, $F_k \in C^\infty_c(S_0)$ for every $k \in \mathbb{N}$.

**Step 3: An auxiliary estimate.** In order to prove that the distance between $F_k$ and its projection $\mathcal{P}F_k$ tends to zero as $k \to \infty$ we first estimate the distance between a smooth function and its projection onto the space of functions depending only on $E$ and $L$. Let $\xi \in C^1_c(S_0)$ with $\xi^r \in C^1_c(S_0)$ be arbitrary, but fixed. We will use the abbreviation

$$
\zeta(t, E, L) := \xi^r((R, W)(t T(E, L), r_\infty(E, L), 0, L), L)
$$

for $t \in \mathbb{R}$, $L > 0$ and $\psi_L(r_L) < E < E_0$. For these $(E, L)$ we may therefore write

$$
\mathcal{P}\xi(E, L) = \int_0^1 \zeta(t, E, L) \, dt.
$$

By a slight abuse of notation, we denote the mapping

$$(x, v) \mapsto \mathcal{P}\xi(E(x, v), L(x, v)),$$
which is defined a.e. on \( S_0 \), by \( \mathcal{P}\xi \). By changing to \((t, E, L)\)-coordinates,

\[
\|\xi - \mathcal{P}\xi\|^2_H = 4\pi^2 \int_0^\infty \int_{\psi_L(r_L)} T(E, L) \left \| \frac{T(E, L)}{\phi'(E)} \right \| \int_0^1 |\zeta(t, E, L) - \int_0^t \zeta(s, E, L) ds|^2 dt dE dL \\
\leq 4\pi^2 \int_0^\infty \int_{\psi_L(r_L)} T(E, L) \left \| \frac{T(E, L)}{\phi'(E)} \right \| \int_0^1 |\zeta(t, E, L) - \zeta(s, E, L)|^2 ds dt dE dL,
\]

where we used the Cauchy-Schwarz inequality in the last step. To estimate the inner two integrals, we first consider the case \( s > t \):

\[
\int_0^1 \int_0^t |\zeta(t, E, L) - \zeta(s, E, L)|^2 ds dt = \int_0^1 \int_0^t |\partial_r \zeta(\tau, E, L)|^2 d\tau ds dt \\
\leq \int_0^1 \int_s^t |\partial_r \zeta(\tau, E, L)|^2 d\tau ds dt \leq \int_0^1 |\partial_r \zeta(\tau, E, L)|^2 d\tau,
\]

where we again used the Cauchy-Schwarz inequality. Estimating the part where \( s < t \) in a similar way, we obtain

\[
\|\xi - \mathcal{P}\xi\|^2_H \leq 8\pi^2 \int_0^\infty \int_{\psi_L(r_L)} T(E, L) \left \| \frac{T(E, L)}{\phi'(E)} \right \| \int_0^1 |\partial_r \zeta(\tau, E, L)|^2 d\tau dE dL.
\]

By definition

\[
\partial_r \zeta(\tau, E, L) = T(E, L) \langle T' \xi \rangle \langle (R, W)(\tau T(E, L), r_-(E, L), 0, L), L \rangle
\]

for \( \tau \in \mathbb{R}, L > 0 \), and \( \psi_L(r_L) < E < E_0 \). Using 16 we arrive at the estimate

\[
\|\xi - \mathcal{P}\xi\|^2_H \leq 32\pi^4 M_0^4 \int_0^\infty \int_{\psi_L(r_L)} T(E, L) \left \| \frac{T(E, L)}{\phi'(E)} \right \| \int_0^1 |(T\xi)'(\ldots)|^2 d\tau dE dL,
\]

where \( \ldots \) stands for \((R, W)(\tau T(E, L), r_-(E, L), 0, L), L\).

**Step 4: Conclusion.** We apply the estimate from the previous step to the elements of the approximating sequence. Due to the properties of their support we obtain the bound

\[
\frac{1}{E^4 L} \leq \frac{2m}{E_0^3}
\]

for all \( L > 0 \) and \( \psi_L(r_L) \leq E < E_0 \) for which there exists \( \tau \in \mathbb{R} \) and \( k \in \mathbb{N} \) such that

\[
0 \neq (TF_k)' \langle (R, W)(\tau T(E, L), r_-(E, L), 0, L), L \rangle.
\]

Using this bound and changing back into \((x, v)\)-coordinates we arrive at the estimate

\[
\|F_k - \mathcal{P}F_k\|^2_H \leq 64\pi^4 M_0^4 m \int_0^\infty \int_{\psi_L(r_L)} T(E, L) \left \| \frac{T(E, L)}{\phi'(E)} \right \| \int_0^1 |(TF_k)'(\ldots)|^2 d\tau dE dL \\
= 16\pi^2 M_0^4 m \int_0^\infty (TF_k)^2_H d\tau dE dL \to 0 \text{ as } k \to \infty.
\]

Since \( F_k \to f \), we obtain \( \mathcal{P}F_k \to f \) in \( H \) for \( k \to \infty \) as well. Since \( K_T \) is a closed subspace of \( H \) and \( (\mathcal{P}F_k)_{k \in \mathbb{N}} \subset K_T \), we conclude that \( f \in K_T \), and the proof is complete. □
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