ON AUTOMORPHISMS OF THE BANACH SPACE $\ell_\infty/c_0$

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Abstract. We investigate Banach space automorphisms $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ focusing on the possibility of representing their fragments of the form $T_{B,A} : \ell_\infty(A)/c_0(A) \to \ell_\infty(B)/c_0(B)$ for $A,B \subseteq \mathbb{N}$ infinite by means of linear operators from $\ell_\infty(A)$ into $\ell_\infty(B)$, infinite $A \times B$-matrices, continuous maps from $B^* = \beta B \setminus B$ into $A^*$, or bijections from $B$ to $A$. This leads to the analysis of general linear operators on $\ell_\infty/c_0$. We present many examples, introduce and investigate several classes of operators, for some of them we obtain satisfactory representations and for other give examples showing that it is impossible. In particular, we show that there are automorphisms of $\ell_\infty/c_0$ which cannot be lifted to operators on $\ell_\infty$ and assuming OCA+MA we show that every automorphism of $\ell_\infty/c_0$ with no fountains or with no funnels is locally, i.e., for some infinite $A,B \subseteq \mathbb{N}$ as above, induced by a bijection from $B$ to $A$. This additional set-theoretic assumption is necessary as we show that the continuum hypothesis implies the existence of counterexamples of diverse flavours. However, many basic problems, some of which are listed in the last section, remain open.

1. Introduction

The set-theoretic analysis of the Boolean algebra $\wp(\mathbb{N})/\text{Fin}$ of sets of the integers modulo finite sets, being far from concluded, has been quite successful. Some of the impressive results refer to the structure of automorphisms of the algebra which under CH can be quite pathologically complicated, as first observed by W. Rudin (Theorem 4.7 of [43]) but consistently may all be trivial, that is induced by an almost permutation of $\mathbb{N}$ (originally proved by S. Shelah in [44]). During the development of the theory, the Proper Forcing Axiom ([45]) and the Open Coloring Axiom (introduced in [46] by S. Todorcevic) have been established as tools not only implying the triviality of all automorphisms but also excluding other pathological mappings, for example embeddings of a big class of quotients into $\wp(\mathbb{N})/\text{Fin}$ ([50, 47, 23]).

These results have also a profound impact on more complex mathematical structures. For example, they directly imply that the question whether the only automorphisms of the Banach algebra $\ell_\infty/c_0$ are those induced by almost permutations of $\mathbb{N}$ is undecidable. Indirectly they recently inspired the research in $C^*$-algebras resulting in the undecidability of the structure of the automorphisms of the Calkin algebra of operators on the Hilbert space modulo the compact operators ([24, 37]).
The main focus of this paper is another natural question, namely, what is the impact of the combinatorics of \( \mathfrak{p}(\mathbb{N})/\text{Fin} \) on the automorphisms of \( \ell_\infty/c_0 \) considered as a Banach space\(^1\), in particular if the Open Coloring Axiom (OCA) or the Proper Forcing Axiom (PFA) can be successfully used in this context. At the moment the situation seems similar to that of the early stage of the research on \( \mathfrak{p}(\mathbb{N})/\text{Fin} \) and \( \mathbb{N}^* \): the usual axioms seem too weak to resolve many basic questions about the Banach space \( \ell_\infty/c_0 \) (\[8, 7, 49, 30\]), the continuum hypothesis provides some answers leaving a chaotic picture full of pathological objects obtained using transfinite induction (\[18, 9\]) and there is a hope (based, for example, on e.g., \[17\]) that alternative axioms like OCA, OCA+MA, PFA etc., would provide an elegant structural theory of automorphisms of \( \ell_\infty/c_0 \). This hope is not only based on the case of \( \mathfrak{p}(\mathbb{N})/\text{Fin} \) but some other cases as well (\[48, 35\]).

In order to explain our results we need to introduce some background and terminology. In the case of the Boolean algebra \( \mathfrak{p}(\mathbb{N})/\text{Fin} \) and its automorphism \( h \) the following conditions are equivalent for every two cofinite sets \( A, B \subseteq \mathbb{N} \):

- There is an isomorphism \( H: \mathfrak{p}(A) \to \mathfrak{p}(B) \) such that \( [H(C)]_{\text{Fin}} = h([C]_{\text{Fin}}) \) for all \( C \subseteq A \) (\( h \) lifts to \( \mathfrak{p}(\mathbb{N}) \));
- There is an isomorphism \( G: \text{FinCofin}(A) \to \text{FinCofin}(B) \) such that \( [\bigcup \{G(n): n \in C\}]_{\text{Fin}} = h([C]_{\text{Fin}}) \) for all \( C \subseteq A \) (\( h \) is induced by an almost automorphism of \( \text{FinCofin}(\mathbb{N}) \));
- There is a bijection \( \sigma: B \to A \) such that \( [\{n \in B: \sigma(n) \in C\}]_{\text{Fin}} = h([C]_{\text{Fin}}) \) for all \( C \subseteq A \) (\( h \) is trivial).

Another feature of liftings of automorphisms on \( \mathfrak{p}(\mathbb{N})/\text{Fin} \), i.e., homomorphisms of \( \mathfrak{p}(\mathbb{N}) \) satisfying the properties above is that

- Every isomorphism from \( \mathfrak{p}(A) \) and \( \mathfrak{p}(B) \) for \( A, B \subseteq \mathbb{N} \) infinite is continuous with respect to the product topologies on \( \{0, 1\}^A \) and \( \{0, 1\}^B \).

Moreover, if we identify points of \( \mathbb{N}^* \) with ultrafilters in \( \mathfrak{p}(\mathbb{N})/\text{Fin} \), the Stone duality gives that:

- for every homomorphism \( h \) of \( \mathfrak{p}(\mathbb{N})/\text{Fin} \) there is a continuous map \( \psi: \mathbb{N}^* \to \mathbb{N}^* \) such that
  \[
  \chi_{h([A]_{\text{Fin}}^* \cap \mathbb{N}^*)} = \chi_{A^*} \circ \psi
  \]
  for every \( A \subseteq \mathbb{N} \).

The corresponding notions for operators on \( \ell_\infty/c_0 \) are summarized in the following:

**Definition 1.1.** If \( T: \ell_\infty/c_0 \to \ell_\infty/c_0 \) is a linear bounded operator, \( A, B \subseteq \mathbb{N} \) cofinite, then we say that

1. \( T \) is liftable (can be lifted) if, and only if, there is a linear bounded \( S: \ell_\infty(A) \to \ell_\infty(B) \) such that for all \( f \in \ell_\infty \) we have
   \[
   T([f]_{c_0}) = [S(f)]_{c_0}
   \]
2. \( T \) is a matrix operator if, and only if, there is an operator \( S: c_0(A) \to c_0(B) \)
   given by a real matrix \( (b_{ij})_{i \in B, j \in A} \) such that for all \( f \in \ell_\infty(A) \) we have
   \[
   T([f]_{c_0}) = ([\sum_{j \in A} b_{ij} f(j)]_{i \in B})_{c_0}.
   \]

\(^1\)Recall that the Banach space \( \ell_\infty/c_0 \) is canonically isometric to the Banach space \( C(\mathbb{N}^*) \) of all continuous functions on the Stone space \( \mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N} \) of the Boolean algebra \( \mathfrak{p}(\mathbb{N})/\text{Fin} \). Hence, the link between \( \ell_\infty/c_0 \) and \( \mathfrak{p}(\mathbb{N})/\text{Fin} \) is canonical, however there are many more linear operators on \( \ell_\infty/c_0 \) than those induced by homomorphisms of the Boolean algebra \( \mathfrak{p}(\mathbb{N})/\text{Fin} \).
(3) $T$ is a trivial operator if, and only if, there is a nonzero real $r \in \mathbb{R}$, and a bijection $\sigma : B \to A$ such that for all $f \in \ell_\infty(A)$ we have

$$T([f]_{c_0}) = [rf \circ \sigma]_{c_0}.$$ 

(4) $T$ is canonizable\footnote{It would be reasonable to consider here also the possibility of having for all $f^* \in C(\mathbb{N}^*)$ the condition $T(f^*) = g f^* \circ \psi$, for some continuous nonzero $g \in C(\mathbb{N}^*)$. However, in the context of $\mathbb{N}^*$ all continuous functions are “locally constant” (8.2) so in the context of this paper there is no sense of introducing such a property.} along $\psi : \mathbb{N}^* \to \mathbb{N}^*$ if, and only if, $\psi$ is a surjective continuous mapping and there is a nonzero real $r$ such that for all $f^* \in C(\mathbb{N}^*)$ we have

$$\hat{T}(f^*) = rf^* \circ \psi.$$ 

In the case of liftable and matrix operators we will be using more complex phrases like automorphic liftable operator, embedding matrix operator etc., meaning that the operator is liftable or matrix respectively and it has the additional property.

In contrast with the case of $\wp(\mathbb{N})/\text{Fin}$ our results show that the relationships among these notions are far from equivalences: None of the implications or counterexamples to the reverse implications require additional set-theoretic axioms. The nontrivial parts of the above chart are the following facts:

- There are automorphisms of $\ell_\infty/c_0$ which are not liftable to a linear operator on $\ell_\infty$ \footnote{4.10};
- There are automorphisms of $\ell_\infty/c_0$ which are liftable but they are not matrix operators and none of their liftings are continuous on $B_\ell_\infty$ in the product topology \footnote{4.13};
- Automorphisms of $\ell_\infty/c_0$ which have liftings to $\ell_\infty$ continuous in the product topology are exactly the automorphic matrix operators \footnote{2.14}.

Note that the question of canonizing globally all automorphisms other than trivial is outright excluded by the clear fact that there are many matrices of isomorphisms on $c_0$ which are not matrices of almost permutations modulo $c_0$.

In the light of the above absolute results and the exclusion of the possibility of a global canonization or matricization we will concentrate on “local” versions of the above properties of the operators in the sense that they hold in some sense for copies of $\ell_\infty/c_0$ of the form $\ell_\infty(A)/c_0(A)$ for an infinite $A \subseteq \mathbb{N}$. Since the above properties depend on the link between $\ell_\infty/c_0$ and $\mathbb{N}^*$ or $\mathbb{N}$ we choose the approach of Drewnowski and Roberts from [18] which has functional analytic motivations and applications:

**Definition 1.2.** Suppose $A \subseteq \mathbb{N}$ is infinite. We define $P_A : \ell_\infty/c_0 \to \ell_\infty(A)/c_0(A)$ and $I_A : \ell_\infty(A)/c_0(A) \to \ell_\infty/c_0$ by

$$P_A([f]_{c_0}) = [f|_A]_{c_0(A)}, \quad I_A([g]_{c_0(A)}) = [g \cup 0_{\mathbb{N}\setminus A}]_{c_0}$$
for all \( f \in \ell_\infty \) and all \( g \in \ell_\infty(A) \). Suppose that \( T : \ell_\infty/c_0 \to \ell_\infty/c_0 \) is a linear bounded operator and \( A, B \subseteq \mathbb{N} \) two infinite sets. The localization of \( T \) to \((A, B)\) is the operator \( T_{B,A} : \ell_\infty(A)/c_0(A) \to \ell_\infty(B)/c_0(B) \) given by

\[
T_{B,A} = P_B \circ T \circ I_A.
\]

It was proved by Drewnowski and Roberts in [18] that for every operator \( T : \ell_\infty/c_0 \to \ell_\infty/c_0 \) and every infinite \( A \subseteq \mathbb{N} \) there is an infinite \( A_1 \subseteq A \) such that for all \([f]_{c_0} \in \ell_\infty(A_1)/c_0(A_1)\) we have \( T_{A_1,A_1}([f]_{c_0}) = [rf]_{c_0} \) for some real \( r \in \mathbb{R} \). However this does not exclude the possibility of \( T_{A_1,A_1} = 0 \), which actually is quite common. Thus the focus of this paper is to obtain localizations which are isomorphic embeddings or isomorphisms and the ultimate goal (not completely achieved) is to localize somewhere any automorphism to a canonical operator along a homeomorphism (which turned out to be impossible in ZFC by 6.5) and to a trivial automorphism under OCA + MA. However if one wants to iterate the use of several localization results (like in the case of [18]) it is useful to have right-local or left-local results and not just somewhere local results:

**Definition 1.3.** Suppose that \( T : \ell_\infty/c_0 \to \ell_\infty/c_0 \) is a linear bounded operator. Let \( \mathbb{P} \) be one of the properties “liftable”, “matrix operator”, “trivial”, “canonizable”.

1. We say that \( T \) is somewhere \( \mathbb{P} \) if, and only if, there are infinite \( A \subseteq \mathbb{N} \) and \( B \subseteq \mathbb{N} \) such that \( T_{B,A} \) has \( \mathbb{P} \).

2. We say that \( T \) is right-locally \( \mathbb{P} \) if, and only if, for every infinite \( A \subseteq \mathbb{N} \) there are infinite \( A_1 \subseteq A \) and \( B \subseteq \mathbb{N} \) such that \( T_{B,A_1} \) has \( \mathbb{P} \).

3. We say that \( T \) is left-locally \( \mathbb{P} \) if, and only if, for every infinite \( B \subseteq \mathbb{N} \) there are infinite \( B_1 \subseteq B \) and \( A \subseteq \mathbb{N} \) such that \( T_{B_1,A} \) has \( \mathbb{P} \).

To hope for isomorphic left-local properties one needs to assume that the image of \( T \) is big, for example that \( T \) is surjective. Similarly, for nontrivial right-local properties we need to assume that the kernel is small, for example that \( T \) is injective.

In contrast to the global versions, the local versions of the notions from Definition 1.1 behave like the Boolean counterparts:

**Proposition 1.4.** Suppose that \( T : \ell_\infty/c_0 \to \ell_\infty/c_0 \) is an automorphism. Then the following are equivalent

1. \( T \) is somewhere a liftable isomorphism,
2. \( T \) is somewhere an isomorphic matrix operator,
3. \( T \) is somewhere a liftable isomorphism with a lifting which is continuous in the product topology,
4. \( T \) is somewhere trivial.

**Proof.** The implication from (1) to (2) follows from 4.10, the equivalence of (2) and (3) is 2.13, the implication from (2) to (4) follows from 4.8, the fact that (4) implies (1) is clear. \( \Box \)

In fact the above equivalences hold (with the same proof) in the case of \( T \) being an isomorphic embedding and for right-localizations which are isomorphic embeddings. However, a surjective operator can be globally liftable but nowhere a matrix operator (4.12) or can be globally a matrix operator but nowhere trivial (4.6). Another reason why the above local notions make sense is the following:

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3By isomorphic embedding we mean an operator which is an isomorphism onto its closed range. Sometimes these operators are called bounded below.
Proposition 1.5. Suppose that $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ is a linear bounded operator and $A, B \subseteq \mathbb{N}$ are two infinite sets. Suppose that $T_{B,A}$ is canonical along a homeomorphism. Then, $T$ fixes a complemented copy of $\ell_\infty/c_0$ whose image under $T$ is complemented in $\ell_\infty/c_0$.

Proof. See the proof of Corollary 2.4 of [18]. □

In fact, the above proposition would also be true with the same proof if we weakened the hypothesis on $B$ from clopen to closed subset of $\mathbb{N}^*$ homeomorphic to $\mathbb{N}^*$. But to make sure that $A$ induces a subspace not just a quotient which is to be fixed we must insist on $A^*$ to be clopen. This approach in the context of other spaces $C(K)$ is quite fruitful for obtaining complemented copies of the entire $C(K)$ inside any isomorphic copy of the $C(K)$ (for example, for $C(K)$ with $K$ metrizable see [23], for $\ell_\infty$ see [22], and for $C([0, \omega_1])$ see [24], see problems in Section 7).

One should note, however, that the notion of e.g., somewhere trivial automorphism on $\ell_\infty/c_0$ has quite a different character than being somewhere trivial automorphism of $\phi(\mathbb{N})/Fin$, this is because the images of subspaces of the form $\{[f] \in \ell_\infty/c_0 : f|A = 0\}$ for $A \subseteq \mathbb{N}$ are usually not of the form $\{[f] \in \ell_\infty/c_0 : f|B = 0\}$ for $B \subseteq \mathbb{N}$, even if $T_{B,A}$ is trivial. Also, trivialization or canonization of $T_{B,A}$ does not yield any information about $T_{A,B}^{-1}$ as in the case of automorphisms of $\phi(\mathbb{N})/Fin$.

Having proven the equivalence of the local versions of the above notions one is left with deciding if automorphisms of $\ell_\infty/c_0$ are somewhere canonizable along homeomorphisms. If this happens their local structure is similar to that of homeomorphisms of $\mathbb{N}^*$ i.e., assuming OCA+MA they would be trivial and, for example, under CH not.

Canonization of automorphisms $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ (or corresponding $\hat{T} : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$) encounters, however, problems at least as difficult as understanding continuous maps defined on closed subsets of $\mathbb{N}^*$ with ranges in $\mathbb{N}^*$ (not only automorphisms of $\mathbb{N}^*$). To better understand why this is so, let us recall that linear bounded operators on $C(\mathbb{N}^*)$ can be represented as weakly* continuous mappings $\tau : \mathbb{N}^* \to M(\mathbb{N}^*)$ (see Theorem 1 in VI.7 of [19]), where $M(\mathbb{N}^*)$ denotes the Banach space of all Radon measures on $\mathbb{N}^*$ with the total variation norm identified by the Riesz representation theorem with the dual to $C(\mathbb{N}^*)$ with the weak* topology (see [14]). Often the points of $\mathbb{N}^*$ (identified with the Dirac measures) are sent by this map to measures that do not have atoms, and if they have atoms they may have many of them giving rise to partial multivalued functions into $\mathbb{N}^*$. One obtains $\tau(x)$ as $T^*(\delta_x)$ for each $x \in \mathbb{N}^*$ and the representation is given by

$$\hat{T}(f^*)(x) = \int f^* \ d\tau(x)$$

for every $f^* \in C(\mathbb{N}^*)$. The multifunctions, possibly of empty values, are given by

$$\varphi^T_\varepsilon(y) = \{x \in \mathbb{N}^* : |T^*(\delta_y)(\{x\})| \geq \varepsilon\}$$

for any $\varepsilon > 0$ or by $\varphi^T(y) = \bigcup_{\varepsilon > 0} \varphi^T_\varepsilon$. An equivalent condition for $T$ being somewhere canonizable along a homeomorphism is the existence of infinite $A, B \subseteq \mathbb{N}$ and a homeomorphism $\psi : B^* \to A^*$ such that

$$T^*(\delta_y)|A^* = r\delta_{\psi(y)}$$
for some nonzero \( r \in \mathbb{R} \), which in particular means that \( \varphi^T(y) \cap A^* = \{ \psi(y) \} \), or in other words that \( \psi \) is a homeomorphic selection from \( \varphi^T \). Right up front there could be two basic obstacles for the existence of such a selection, namely \( \bigcup_{y \in B^*} \varphi^T(y) \) could have empty interior or \( \{ y \in B^* : \varphi^T(y) \neq \emptyset \} \) could have empty interior for an infinite \( B \subseteq \mathbb{N}^* \). We call these obstacles (in stronger versions including nonatomic measures) fountains and funnels respectively and introduce two classes of operators (fountainless operators, Definition 3.13 and funnelless operators, Definition 3.18) for which by definition the above obstacles cannot arise, respectively, and we obtain some reasonable sufficient conditions for the canonization:

- Every automorphism on \( \ell_\infty/c_0 \) which is fountainless is left-locally canonizable along a quasi-open mapping \( 5.6 \);
- Every automorphism on \( \ell_\infty/c_0 \) which is funnelless is right-locally canonizable along a quasi-open mapping \( 5.8 \);

where quasi-open means that the image of every open set has nonempty interior \( 3.20 \). The second result is in fact a consequence of a study by G. Plebanek \[39\], however the proof of the first takes a considerable part of this paper. The possibility of obtaining these results is based on special properties of isomorphic embeddings and surjections. One ingredient is an improvement of a theorem of Cengiz (\"P\" in \[10\]) obtained by Plebanek (Theorem 3.3. in \[39\]) which guarantees that the range of \( \varphi^T/\|T\|\|T^{-1}\| \) covers \( \mathbb{N}^* \) if \( T \) is an isomorphic embedding. However, in this result the set of \( y \)'s where \( \varphi^T(y) \) is nonempty could be nowhere dense, so we exclude this possibility by assuming that \( T \) has no funnels. On the other hand we prove that if \( T \) is surjective, then either for each \( y \) the set \( \varphi^T(y) \) is nonempty or else there is a infinite \( A \subseteq \mathbb{N}^* \) such that \( \bigcup \{ \varphi^T(y) : y \in A^* \} \) is nowhere dense, the second possibility being excluded if \( T \) has no fountains.

Then one is still left with the problem of reducing a quasi-open map to a homeomorphism between two clopen sets. The results of I. Farah \[23\] allow us to conclude that OCA+MA implies that a quasi-open mapping defined on a clopen subset of \( \mathbb{N}^* \) and being onto a clopen subset of \( \mathbb{N}^* \) is somewhere a homeomorphism and so by results of Velickovic \[50\] it is somewhere induced by a bijection between two infinite subsets of \( \mathbb{N} \). Hence we obtain:

- (OCA+MA) Every fountainless automorphism of \( \ell_\infty/c_0 \) is left-locally trivial \( 6.4 \);
- (OCA+MA) Every funnelless automorphism of \( \ell_\infty/c_0 \) is right-locally trivial \( 6.4 \).

The continuum hypothesis shows that the above results are optimal in many directions. First, an obstacle to improving our above-mentioned ZFC selection results \( 5.6, 5.8 \) by replacing quasi-open to a homeomorphism between clopen sets is the following example:

- (CH) There is a fountainless and funnelless everywhere present isomorphic embedding globally canonizable along quasi-open map which is nowhere canonizable along a homeomorphism \( 6.10 \).

Here everywhere present is a weak version of a surjective operator \( P_A \circ T \neq 0 \) for any infinite \( A \subseteq \mathbb{N} \) see \[3.10\]. Automorphisms \( T \) have the property that \( P_A \circ T \) is everywhere present and \( T \circ I_A \) is an isomorphic embedding for any infinite \( A \subseteq \mathbb{N} \). Moreover we have the following:
• (CH) There is an automorphism of $\ell_\infty/c_0$ which is nowhere canonizable along a quasi-open map, in particular along a homeomorphism (6.5).

The above example is not a direct construction, but we have more concrete and slightly weaker examples (6.8) based on the existence in $\mathbb{N}^*$ of nowhere dense $P$-sets which are retracts of $\mathbb{N}^*$, due to van Douwen and van Mill ([14]). It is not excluded by our results (see Section 7) that consistently all isomorphic embeddings on $\ell_\infty/c_0$ are funnelless, however there are ZFC surjective operators which are not fountainless (3.3). And, of course, assuming CH there are well familiar nowhere trivial homeomorphisms of $\mathbb{N}^*$ which provide examples of globally canonizable operator which is nowhere liftable (6.9).

The are many basic problems concerning the automorphisms of $\ell_\infty/c_0$ left open, some of them are listed in Section 7. A breakthrough in developing the methods of direct applications of PFA in the space $\ell_\infty/c_0$ which was recently obtained by A. Dow in [17] may be especially useful in attacking these problems. The structure of the paper is as follows:

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Since any element of $C$ these spaces. However, not all continuous functions on $C$ linear operator $T$ an endomorphism respectively.

\[ \beta f \text{ - The element of } C(\beta N) \text{ which extends } f \in \ell_\infty \]
\[ \beta T \text{ - The operator from } C(\beta N) \text{ induced by an operator } T : \ell_\infty \to \ell_\infty \text{ which preserves } \beta 0 \text{ (i.e., } T[0] \subseteq 0) \text{ that is } \beta T(\beta f) = \beta(T(f)) \text{ for any } f \in \ell_\infty \]
\[ T^* \text{ - The operator on } C(\beta N) \text{ induced by an operator } T : \ell_\infty \to \ell_\infty \text{ which preserves } \beta 0 \text{ (i.e., } T[0] \subseteq 0) \text{ that is } T^*(f^*) = (T(f))^* \text{ for any } f \in \ell_\infty \]
\[ T \text{ - The operator from } C(\beta N) \text{ into itself which corresponds to } T : \ell_\infty/c_0 \to \ell_\infty/c_0, \text{ i.e., } \hat{T}(f^*) = g^* \text{ where } [g] = T([f]) \]
\[ h \text{ - The continuous selfmap of } N^* \text{ which corresponds via the Stone duality to an endomorphism } h \text{ of } N/Fin, \text{ i.e., } \hat{h}(x) = h^{-1}[x] \text{ when we identify points of } N^* \text{ with the ultrafilters of } N/Fin \]
\[ T_\psi \text{ - The operator } T_\psi : C(N^*) \to C(N^*) \text{ which maps } f \to f \circ \psi, \text{ for some continuous } \psi : N^* \to N^* \]

The remaining often used symbols are:
\[ M(N^*) \text{ - the Banach space of Radon measures on } N^* \text{ with the total variation norm, identified with the dual space to } C(N^*) \text{ or the dual space to } \ell_\infty/c_0 \text{ via the Riesz representation theorem} \]
\[ T^* \text{ - The dual or adjoint operator of } T, \text{ i.e., } T^*(\mu)(f) = \mu(T(f)). \text{ } T^* \text{ acts on the spaces of Radon measures if } T \text{ acts on a space of continuous functions} \]
\[ \delta_x \text{ - The Dirac measure concentrated on } x \]
2. Operators given by \(c_0\)-matrices. A linear operator \(R\) on \(\ell_\infty\) which preserves \(c_0\) (i.e., \(R[c_0] \subseteq c_0\)) defines, of course, an operator on \(c_0\). In the case of the Boolean algebra \(\wp(N)\), any Boolean automorphism preserves \(\text{FinCofin}(N)\) and its restriction to \(\text{FinCofin}(\mathbb{N})\) completely determines the automorphism. The analogous fact does not hold for linear automorphisms on \(\ell_\infty\), for example there are many distinct automorphisms of \(\ell_\infty\) which do not move \(c_0\) (see 2.15). However, the restrictions to \(c_0\) of operators on \(\ell_\infty\) which preserve \(c_0\) will play an important role, and in some cases will determine a given operator. So let us establish a transparent representation of operators on \(c_0\):

**Proposition 2.1.** \(R : c_0 \to c_0\) is a linear bounded operator if, and only if, there exists an \(N \times N\) matrix \((b_{ij})_{i,j \in N}\) such that

1. every row is in \(\ell_1\),
2. if we write \(b_i = (b_{ij})_j\), then \(\{\|b_i\|_{\ell_1} : i \in N\}\) is a bounded set,
3. every column is in \(c_0\),

and such that for every \(f \in c_0\) we have

\[
R(f) = \begin{pmatrix}
  b_{00} & b_{01} & \cdots \\
  b_{10} & b_{11} & \cdots \\
  \vdots & \vdots & \ddots
\end{pmatrix}
\begin{pmatrix}
  f(0) \\
  f(1) \\
  \vdots
\end{pmatrix}.
\]

**Proof.** Use the fact that \(c_0^* = \ell_1\) and put \(b_i = R^*(\delta_i)\), where \(\delta_i\) is the functional corresponding to the \(i\)-th coordinate for each \(i \in N\). \(\Box\)

This representation corresponds to representing endomorphisms of \(\text{FinCofin}(\mathbb{N})\) by finite-to-one functions from \(\mathbb{N}\) into itself. Such endomorphisms induce operators on \(c_0\) whose matrix satisfies the above characterization and where every row has one entry equal to 1 and the remaining entries equal to 0. Matrices define some operators on \(\ell_\infty\) as well, of course:

**Proposition 2.2.** Let \((b_{ij})_{i,j \in N}\) be a matrix and let \(b_i = (b_{ij})_j\) be the \(i\)-th row, for every \(i \in N\). Then,

\[
R(f) = \begin{pmatrix}
  b_{00} & b_{01} & \cdots \\
  b_{10} & b_{11} & \cdots \\
  \vdots & \vdots & \ddots
\end{pmatrix}
\begin{pmatrix}
  f(0) \\
  f(1) \\
  \vdots
\end{pmatrix},
\]

defines an linear bounded operator \(R : \ell_\infty \to \ell_\infty\) if, and only if, \(b_i \in \ell_1\), for all \(i \in N\), and \(\{\|b_i\|_{\ell_1} : i \in N\}\) is a bounded set.

**Proof.** Use the fact that \(\ell_1 \subseteq \ell_\infty\) and put \(b_i = R^*(\delta_i)\), where \(\delta_i\) is the functional corresponding to the \(i\)-th coordinate for each \(i \in N\). \(\Box\)
Definition 2.3.  
(i) We say that a matrix is a $c_0$-matrix if it satisfies conditions (1)-(3) of Proposition 2.1.
(ii) We say that a linear bounded operator $R : \ell_\infty \to \ell_\infty$ is given by a $c_0$-matrix if there exists a $c_0$-matrix $(b_{ij})_{i,j \in \mathbb{N}}$ such that

$$R(f) = \begin{pmatrix} b_{00} & b_{01} & \cdots \\ b_{10} & b_{11} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} f(0) \\ f(1) \\ \vdots \end{pmatrix},$$

for every $f \in \ell_\infty$.

Corollary 2.4. Suppose that $R : \ell_\infty \to \ell_\infty$ is a linear bounded operator which preserves $c_0$ and is given by

$$R(f) = \begin{pmatrix} b_{00} & b_{01} & \cdots \\ b_{10} & b_{11} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} f(0) \\ f(1) \\ \vdots \end{pmatrix},$$

where $(b_{ij})_{i,j \in \mathbb{N}}$ is a real matrix. Then $(b_{ij})_{i,j \in \mathbb{N}}$ is a $c_0$-matrix.

Proof. If such an operator on $\ell_\infty$ was not given by a $c_0$-matrix, then some of the columns of the corresponding matrix would not be in $c_0$ by 2.2 and by 2.1. Then the operator would not preserve $c_0$. □

Proposition 2.5. If a linear bounded operator $R : \ell_\infty \to \ell_\infty$ is given by a $c_0$-matrix, then $R = (R|c_0)**$.

Proof. Appendix 8.13 □

2.2. Falling and weakly compact operators. Let us recall the following characterization of weakly compact operators on $c_0$:

Theorem 2.6. Let $R : c_0 \to c_0$ be a linear bounded operator and let $(b_{ij})_{i,j \in \mathbb{N}}$ be the corresponding matrix. The following are equivalent:

1. $R$ is weakly compact.
2. $R^{**}[\ell_\infty] \subseteq c_0$.
3. $\|b_i\|_{l_1} \to 0$.

Proof. Appendix 8.14 □

Proposition 2.7. Let $R : \ell_\infty \to \ell_\infty$ be an operator given by a $c_0$-matrix. Then, $R$ is weakly compact if, and only if, $R[\ell_\infty] \subseteq c_0$.

Proof. If $R$ is weakly compact, then $R|c_0$ must be as well, and so by 2.6 we have $(R|c_0)**[\ell_\infty] \subseteq c_0$ but $(R|c_0)** = R$ by 2.5. In the other direction, use the fact that every operator defined on a Grothendieck Banach space into a separable Banach space is weakly compact (Theorem 1 (v) of [11]). □

Definition 2.8. A $c_0$-matrix operator $R : \ell_\infty \to \ell_\infty$ is called falling if, and only if, for every $\varepsilon > 0$ there is a partition $A_0, ..., A_{k-1}$ of $\mathbb{N}$ such that

$$\sum_{j \in A_m} |b_{ij}| < \varepsilon$$

for all $m < k$ and $i \in \mathbb{N}$ sufficiently large.
Proposition 2.9. Every operator on $\ell_\infty$ which is given by a $c_0$-matrix and is weakly compact is falling.

Proof. Use Theorem 2.6.

Proposition 2.10. There is a falling, non-weakly compact operator on $\ell_\infty$ given by a $c_0$-matrix.

Proof. Let $R : \ell_\infty \to \ell_\infty$ be given by the matrix

$$b_{ij} = \begin{cases} 1/(i+1), & \text{if } j \leq i \\ 0, & \text{otherwise} \end{cases}$$

for all $i \in \mathbb{N}$. By 2.6 this is not a weakly compact operator. Given $k \in \mathbb{N}$ if we consider $A_m = \{lk + m : l \in \mathbb{N}\}$, for $m < k$ then,

$$\sum_{j \in A_m} |b_{ij}| \leq \left(\frac{i+1}{k}\right) \left(\frac{1}{i+1}\right) = 1/k,$$

so the operator is falling.

2.3. Antimatrix operators. The behaviour opposite to operators given by a $c_0$-matrix is the subject of the following:

Definition 2.11. A linear bounded operator $R : \ell_\infty \to \ell_\infty$ will be called an antimatrix operator if, and only if, $R[c_0] = \{0\}$.

Using the isometry between $\ell_\infty$ and $C(\beta \mathbb{N})$, an operator $R$ on $\ell_\infty$ can be associated with an operator $\beta R$ on $C(\beta \mathbb{N})$ and these operators can be associated with weak* continuous functions from $\beta \mathbb{N}$ into the Radon measures $M(\beta \mathbb{N})$ on $\beta \mathbb{N}$ (see Theorem 1 in VI 7. of [19]). Since $\mathbb{N}$ is dense in $\beta \mathbb{N}$, such functions are determined by their values on $\mathbb{N}$. The following characterizations will be useful later on:

Lemma 2.12. Suppose $R : \ell_\infty \to \ell_\infty$ is a bounded linear operator such that $R[c_0] \subseteq c_0$. Then,

(a) $R$ is given by a $c_0$-matrix if, and only if, $R^*(\delta_n)$ is concentrated on $\mathbb{N}$ for all $n \in \mathbb{N}$, that is, $R^*(\delta_n) \in \ell_1$, $\forall n \in \mathbb{N}$.

(b) $R$ is an antimatrix operator if, and only if, $R^*(\delta_n)$ is concentrated on $\mathbb{N}^*$ for all $n \in \mathbb{N}$.

Proof. (a) Assume $R$ is given by a $c_0$-matrix $(b_{ij})_{i,j \in \mathbb{N}}$. Then, for every $f \in \ell_\infty$ we have $R^*(\delta_n)(f) = R(f)(n) = b_n(f)$, where $b_n$ is the $n$th row of $(b_{ij})_{i,j \in \mathbb{N}}$. So $R^*(\delta_n) = b_n$ and by definition of $c_0$-matrix, we have that $b_n \in \ell_1$.

Conversely, assume $R^*(\delta_n) \in \ell_1$. Let $M$ be the matrix formed by putting $R^*(\delta_n)$ as the $n$th row. Then, $R$ is induced by $M$. Moreover, since $R[c_0] \subseteq c_0$, we know that $M$ is a $c_0$-matrix.

(b) Suppose $R^*(\delta_n)$ is not concentrated on $\mathbb{N}^*$ for some $n \in \mathbb{N}$. Then, there exists an $m \in \mathbb{N}$ such that $R^*(\delta_n)(\{m\}) \neq 0$. Then, $R(\chi_{\{m\}})(n) = R^*(\delta_n)(\chi_{\{m\}}) \neq 0$. Therefore, $\chi_{\{m\}} \in c_0$ is a witness to the fact that $R[c_0] \neq \{0\}$, so $R$ is not an antimatrix operator.

Conversely, assume $R^*(\delta_n)$ is concentrated on $\mathbb{N}^*$, for every $n \in \mathbb{N}$. Fix $f \in c_0$. Then, for every $n \in \mathbb{N}$ we have $R(f)(n) = R^*(\delta_n)(\beta f) = \int_{\mathbb{N}} \beta f dR^*(\delta_n) = \int_{\mathbb{N}} \beta f dR^*(\delta_n) = 0$, because $\beta f[\mathbb{N}^*] = 0$. □
Thus a typical example of an antimatrix operator is one given by $R(f) = ((βf(x_i))_{i∈N})$ where $(x_i)_{i∈N}$ is any sequence of nonprincipal ultrafilters.

**Proposition 2.13.** If $R : ℓ_∞ → ℓ_∞$ is such that $R[c_0] ⊆ c_0$, then $R = S_0 + S_1$, where $S_0$ is given by a $c_0$-matrix and $S_1$ is an antimatrix operator.

**Proof.** As $R[c_0] ⊆ c_0$ there is a matrix $(b_{ij})_{i,j∈N}$ which satisfies $2.4$ Define $S_0$ as multiplication by this matrix, i.e. $S_0 = (R[c_0])^*$ by $2.5$. Now $S_1 = R - S_0$ is antimatrix, so we obtain the desired decomposition. □

### 2.4. Product topology continuity of operators

The importance of operators on $ℓ_∞$ given by $c_0$-matrices is expressed in the following theorem which exploits the fact that $ℓ_∞$ is the bidual space of $c_0$. In the theorem below, the weak$^*$ topology on $ℓ_∞$ is given by the duality $ℓ_1^* = ℓ_∞$ and $τ_p$ denotes the product topology in $ℝ^N$.

**Theorem 2.14.** Let $R : ℓ_∞ → ℓ_∞$ be a linear bounded operator. The following are equivalent:

1. $R = (R[c_0])^*$.
2. $R$ is given by a $c_0$-matrix.
3. $R$ is $w^*$-$w^*$-continuous and $R[c_0] ⊆ c_0$.
4. $R|B_{ℓ_∞^*} : (B_{ℓ_∞^*}, τ_p) → (ℓ_∞, τ_p)$ is continuous and $R[c_0] ⊆ c_0$.

**Proof.** Appendix 8.18 □

Thus, the nonzero antimatrix operators are discontinuous in the product topology. Such discontinuities are not, however, incompatible with being an automorphism or having a nice behaviour on $c_0$.

**Theorem 2.15.** There are discontinuous automorphisms of $ℓ_∞$ preserving $c_0$. There are different automorphisms on $ℓ_∞$ which agree on $c_0$. They can be the identity on $c_0$.

**Proof.** Let $(A_i)_{i∈N}$ be a partition of $ℕ$ into infinite sets. Let $x_i$ be any nonprincipal ultrafilter such that $A_i ∈ x_i$ for all $i ∈ ℕ$. For a permutation $σ : ℕ → ℕ$ define $R_σ(f)(n) = f(n) - βf(x_i) + βf(x_{σ(i)})$, where $i ∈ ℕ$ is such that $n ∈ A_i$. First note that $R_{σ^{-1}} ◦ R_σ = R_σ ◦ R_{σ^{-1}} = Id$ and so $R_σ$ is an automorphism. One verifies that $R_σ[c_0]$ is the identity for any permutation $σ$, in particular $R_σ - Id ≠ 0$ is antimatrix for any permutation $σ$ different than the identity and hence $R_σ$ is discontinuous by $2.13$. □

In this proof we really decompose $ℓ_∞$ as a direct sum $X ⊕ Y$, both factors necessarily isomorphic to $ℓ_∞$: the first of the functions constant on each set $A_i$ and the second of the functions equal to zero in each point $x_i$. Since the second factor contains $c_0$, the automorphisms of the first factor induce automorphisms of $ℓ_∞$ which do not move $c_0$. This lack of continuity is also present in homomorphisms of $φ(ℕ)$ (3.2.3. of [24]) but not its automorphisms.

### 3. Operators on $ℓ_∞/c_0$

#### 3.1. Ideals of operators on $ℓ_∞/c_0$

As usual by an (left, right) ideal we will mean a collection $ℐ$ of operators such that $T + S ∈ ℐ$ whenever $T, S ∈ ℐ$ and $S ◦ R, R ◦ S ∈ ℐ$ $(R ◦ S ∈ ℐ, S ◦ R ∈ ℐ)$ whenever $S ∈ ℐ$ and $R$ is any operator
on $\ell_\infty/c_0$. We say that an operator $T$ on $\ell_\infty/c_0$ factors through $\ell_\infty$ if, and only if, there are operators $R_1 : \ell_\infty/c_0 \to \ell_\infty$ and $R_2 : \ell_\infty \to \ell_\infty/c_0$ such that $T = R_2 \circ R_1$. It is clear that operators which factor through $\ell_\infty$ form a two-sided ideal. Also it is well known that weakly compact operators form a two-sided proper ideal (VI 4.5. of [19]). We introduce another class of operators:

**Definition 3.1.** An operator $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ is locally null if, and only if, for every infinite $A \subseteq \mathbb{N}$ there is an infinite $A_1 \subseteq A$ such that

$$T \circ I_{A_1} = 0.$$ 

Locally null should really be right-locally null, but left-locally null is just null, so there is no need of using the word “right”.

**Proposition 3.2.** Locally null operators form a proper left ideal which contains all weakly compact operators and all operators which factor through $\ell_\infty$.

**Proof.** It is clear that locally null operators form a proper left ideal.

Let us prove that every weakly compact operator on $\ell_\infty/c_0$ is locally null. We will use the fact that an operator $T$ on a $C(K)$-space is weakly compact if, and only if, $||T(f_n)|| \to 0$ whenever $(f_n)_{n \in \mathbb{N}} \subseteq C(K)$ is a bounded pairwise disjoint sequence (i.e., $f_n : f_m = 0$ for $n \neq m$) (see Corollary VI–17 of [12]).

Let $A \subseteq \mathbb{N}$ be infinite. Consider $\{A_\xi : \xi < \omega_1\}$, a family of almost disjoint infinite subsets of $A$. Notice that by the weak compactness of $T$ we have that the set of $\alpha \in \omega_1$ such that $T \circ I_{A_\alpha} \neq 0$ must be at most countable, so take $\alpha$ outside this set.

Now let $T = R_2 \circ R_1$ where $R_1 : \ell_\infty/c_0 \to \ell_\infty$ and $R_2 : \ell_\infty \to \ell_\infty/c_0$. Let $\mu_n = R_1^*(\delta_n)$. Let $A \subseteq \mathbb{N}$ be infinite. As the supports of $\mu_n$’s are c.c.c. and there are continuum many pairwise disjoint clopen subsets of $A^*$, there is an infinite $A_1 \subseteq A$ such that $|\mu_n|(A_1^*) = 0$ for every $n \in \mathbb{N}$. It follows that $R_1 \circ I_{A_1} = 0$, which completes the proof.

□

**Proposition 3.3.** There is a locally null operator on $\ell_\infty/c_0$ which factors through $\ell_\infty$ and is surjective. There is no surjective weakly compact operator.

**Proof.** Let $(x_n)_{n \in \mathbb{N}}$ be a discrete subset of $\mathbb{N}^*$. Define $R : \ell_\infty/c_0 \to \ell_\infty$ by $R([f]_{c_0}) = (f^*(x_n))_{n \in \mathbb{N}}$. It is well-known that the closure of $\{x_n : n \in \mathbb{N}\}$ in $\mathbb{N}^*$ is homeomorphic to $\beta \mathbb{N}$. So by the Tietze extension theorem $R$ is onto $\ell_\infty$. Furthermore, $Q \circ R$ is surjective, where $Q : \ell_\infty \to \ell_\infty/c_0$ is the quotient map. As no clopen subset $A^*$ of $\mathbb{N}^*$ is separable, below every infinite $A$ there is an infinite $A_1 \subseteq A$ such that no $x_n$ belongs to $A^*_1$. Then, $R \circ I_{A_1} = 0$ which proves that $R$ is locally null, and so $Q \circ R$ as well.

Weakly compact operators on an infinite dimensional $C(K)$ cannot be surjective because weakly compact subsets of an infinite dimensional Banach space have empty interior if the space is not reflexive. So countable unions of them are of the first Baire category, and in particular, the images of the balls cannot cover an infinite dimensional Banach space $C(K)$.

□

See [67] for more information on the ideal of locally null operators under CH.
3.2. Local behaviour of functions associated with the adjoint operator.

In general, for a linear bounded operator $T$ acting on the Banach space $C(K)$ for a compact $K$, the function which sends $x \in K$ to $\|T^*(\delta_x)\|$ is lower semicontinuous (e.g., Lemma 2.1. of [39]) and may be quite discontinuous.

**Proposition 3.4.** Suppose $F \subseteq \mathbb{N}^*$ is a nowhere dense retract of $\mathbb{N}^*$. There is a linear bounded operator $T$ on $C(\mathbb{N}^*)$ such that the function $\alpha : \mathbb{N}^* \to \mathbb{R}$ defined by

$$\alpha(y) = \|T^*(\delta_y)\|$$

for every $y \in \mathbb{N}^*$, is discontinuous in every point of $F$.

**Proof.** Define $T$ by putting

$$T(f) = f - f \circ r,$$

where $r : \mathbb{N}^* \to F$ is the retraction onto $F$. Then $T(f)(y) = f(y) - f(r(y))$ and so $T^*(\delta_y) = \delta_y - \delta_{r(y)}$. Hence $\alpha = 2\chi_{\mathbb{N}^* \setminus F}$. Since $F$ is nowhere dense, the set of discontinuities of $\alpha$ is $F$. \hfill $\square$

By Lemma 4.1. of [39], for every lower semicontinuous function, every $\varepsilon > 0$ and every open $U \subseteq \mathbb{N}^*$ there is an open $V \subseteq U$ such that the function’s oscillation on $V$ is smaller than $\varepsilon$. Hence, by [39] there is a dense open subset of $\mathbb{N}^*$ where the function which sends $y \in \mathbb{N}^*$ to $\|T^*(\delta_y)\|$ is locally constant. In the case of $\mathbb{N}^*$ we have not only the local stabilization of the values of $\|T^*(\delta_y)\|$ but the local stabilization of the Hahn decompositions of the measures $T^*(\delta_y)$:

**Lemma 3.5.** Suppose $T : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ is bounded linear and $B \subseteq \mathbb{N}$ is infinite. Then, there are an infinite $B_1 \subseteq_* B$, a real number $s$, and partitions $\mathbb{N} = C_n \cup D_n$ into infinite sets, such that for every $y \in B_1^*$ we have:

(i) $s = \|T^*(\delta_y)\|$  
(ii) if $T^*(\delta_y) = \mu^+ - \mu^-$ is the Jordan decomposition of the measure, then $\mu^-(C_n) < 1/4(n+1)$ and $\mu^+(D_n) < 1/4(n+1)$.

**Proof.** We construct by induction a $\subseteq$-decreasing sequence of infinite sets $(A_n)_{n \in \mathbb{N}}$, $y_n \in A_n^*$, and partitions $\mathbb{N} = C_n \cup D_n$ into infinite sets such that for every $n \in \mathbb{N}$ we have:

(1) $\sup\{\|T^*(\delta_y)\| : y \in A_n^*\} - \|T^*(\delta_{y_n})\| < 1/6(n+1)$
(2) $\|T^*(\delta_{y_n})\| - T^*(\delta_{y_n})(C_n^*) + T^*(\delta_{y_n})(D_n^*) < 2/6(n+1)$
(3) For all $y \in A_{n+1}^*$ we have $|T^*(\delta_y)(C_n^*) - T^*(\delta_{y_n})(C_n^*)| < 1/6(n+1)$ and $|T^*(\delta_y)(D_n^*) - T^*(\delta_{y_n})(D_n^*)| < 1/6(n+1)$.

This is arranged as follows. Put $A_0 = B$ and assume we have constructed $A_n$. Take $y_n \in A_n^*$ such that $\|T^*(\delta_{y_n})\| > \sup\{\|T^*(\delta_y)\| : y \in A_n^*\} - 1/6(n+1)$, and take a Hahn decomposition $\mathbb{N}^* = H_n^+ \cup H_n^-$ for the measure $T^*(\delta_{y_n})$. By the regularity, we may choose an infinite $C_n \subseteq \mathbb{N}$ such that $|T^*(\delta_{y_n})(H_n^+) - T^*(\delta_{y_n})(C_n^*)| < 1/6(n+1)$. If we put $D_n = \mathbb{N} \setminus C_n$, we obtain $|T^*(\delta_{y_n})(H_n^-) - T^*(\delta_{y_n})(D_n^*)| < 1/6(n+1)$. Therefore,

$$\|T^*(\delta_{y_n})\| = T^*(\delta_{y_n})(H_n^+) - T^*(\delta_{y_n})(H_n^-) < T^*(\delta_{y_n})(C_n^*) - T^*(\delta_{y_n})(D_n^*) + 2/6(n+1),$$

and so (2) holds.

By the weak$^*$ continuity of $T^*$, the set of points which satisfy the condition in (3) is an open neighbourhood of $y_n$, so we may take $A_{n+1} \subseteq_* A_n$ satisfying (3). This ends the induction.
Notice that $|T^*(\delta_y_n)(C_n^*)| \leq \|T\|$ for every $n \in \mathbb{N}$, and so there exists a convergent subsequence of $(T^*(\delta_y_n)(C_n^*))_{n \in \mathbb{N}}$. The same is true for the $D_n$’s and so we may assume that both of these sequences converge. Let us define

$$s^+ = \lim_{n \to \infty} T^*(\delta_{y_n})(C_n^*) \quad s^- = \lim_{n \to \infty} T^*(\delta_{y_n})(D_n^*).$$

Now let $B_1 \subseteq \mathbb{N}$ be infinite such that $B_1 \subseteq A_n$, for all $n \in \mathbb{N}$. We will show that for every $y \in B_1^*$ we have $\|T^*(\delta_y)|| = s$, where $s = s^+ - s^-$. Let us fix $y \in B_1^*$. Notice that from (3) we obtain that

$$s = \lim_{n \to \infty} (T^*(\delta_y)(C_n^*) - T^*(\delta_y)(D_n^*)) = \lim_{n \to \infty} T^*(\delta_y)(\chi_{C_n^*} - \chi_{D_n^*}).$$

Therefore, $s \leq \|T^*(\delta_y)||$. Now, by (1) and (2) the following holds for every $n \in \mathbb{N}$:

$$T^*(\delta_{y_n})(C_n^*) - T^*(\delta_{y_n})(D_n^*) > \|T^*(\delta_{y_n})|| - 2/6(n + 1) \geq \|T^*(\delta_y)|| - 1/2(n + 1).$$

Therefore, $s = \lim_{n \to \infty} T^*(\delta_{y_n})(C_n^*) - T^*(\delta_y)(D_n^*) \geq \|T^*(\delta_y)||$. This proves the first statement of the lemma.

To check that (ii) holds, let us fix $y \in B_1^*$ and the Jordan decomposition for the measure $T^*(\delta_y) = \mu^+ - \mu^-$. By going to a subsequence if necessary, we may assume that $|s - (T^*(\delta_{y_n})(C_n^*) - T^*(\delta_{y_n})(D_n^*))| < 1/6(n + 1)$, for every $n \in \mathbb{N}$. Observe that since $C_n^* \cup D_n^* = \mathbb{N}^*$, we have that

$$T^*(\delta_y)(C_n^*) - T^*(\delta_y)(D_n^*) = |T^*(\delta_y)(C_n^*)| + |T^*(\delta_y)(D_n^*)| - 2\mu^-(C_n^*) - 2\mu^+(D_n^*),$$

for every $n \in \mathbb{N}$. Then, by (3) we obtain

$$2(\mu^-(C_n^*) + \mu^+(D_n^*)) = s - (T^*(\delta_y)(C_n^*) - T^*(\delta_y)(D_n^*)) \leq s - (T^*(\delta_{y_n})(C_n^*) - T^*(\delta_{y_n})(D_n^*) - 2/6(n + 1)) < 1/6(n + 1) + 2/6(n + 1) = 1/2(n + 1).$$

Since both $\mu^-(C_n^*)$ and $\mu^+(D_n^*)$ are non negative, they are both strictly less than $1/4(n + 1)$ and (ii) is proved.

\[ \square \]

**Corollary 3.6.** Suppose $T : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ is bounded linear and $B \subseteq \mathbb{N}$ is infinite. Then, there are an infinite $B_1 \subseteq B$ and a Borel partition $\mathbb{N}^* = X \cup Y$ such that $X$ and $Y$ form a Hahn decomposition of $T^*(\delta_y)$, for every $y \in B_1^*$.

**Proof.** Let $B_1 \subseteq B$ and $C_n, D_n \subseteq \mathbb{N}$ be as in 3.3. Let $(n_k)_{k \in \mathbb{N}}$ be a strictly increasing sequence of positive integers such that $\frac{1}{4(n_k + 1)} < 1/2^k$, for $k \in \mathbb{N}$.

Let $F_i = \bigcap_{k \geq i} C_{n_k}^\ast \setminus X = \bigcup_{i \in \mathbb{N}} F_i$ and $Y = \mathbb{N}^* \setminus X$. Fix $y \in B_1^*$ and let $T^*(\delta_y) = \mu^+ - \mu^-$ be a Jordan decomposition of the measure. Since $F_i \subseteq F_{i+1}$ and $F_i \subseteq C_n^\ast$, for every $i \in \mathbb{N}$, by 3.3 we have that

$$\mu^-(F_i) \leq \mu^-(F_i) \leq \mu^-(C_i^\ast) < \frac{1}{4(n_i + 1)},$$

for every $i_0 \in \mathbb{N}$ and every $i \geq i_0$. Therefore, $\mu^-(F_i) = 0$, for every $i_0 \in \mathbb{N}$, and so $\mu^-(X) = 0$.

On the other hand, we have $Y \subseteq \mathbb{N}^* \setminus F_i = \bigcup_{k \geq i} D_{n_k}^\ast$, for every $i \in \mathbb{N}$. Therefore, $\mu^+(Y) \leq \sum_{k \geq i} \mu^+(D_{n_k}^\ast) < \sum_{k \geq i} \frac{1}{4(n_k + 1)} < \sum_{k \geq i} 1/2^k$, for every $i \in \mathbb{N}$. It follows that $\mu^+(Y) = 0$. \[ \square \]
Corollary 3.7. Suppose that $T : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ is a linear bounded operator and $B \subseteq \mathbb{N}$ is infinite. Then, there is an infinite $B_1 \subseteq B$ such that the functions which map $y \in B_1^*$ to the positive part, to the negative part, and to the total variation measure of the measure $T^*(\delta_y)$, respectively, are all weak* continuous. In particular, $T$ is left-locally a regular operator.

Proof. Let $B_1 \subseteq B$ and $s$ be as in 3.5. If $s = 0$, then the first part of the corollary is trivially true and $P_{B_1} \circ T = 0$ is positive. Otherwise, consider the operator $\frac{1}{s}T$. We have that $\|(\frac{1}{s}T)^*(\delta_y)\| = 1$ for all $y \in B_1^*$. Now apply the second part of Lemma 2.2 of [35] which says that on the dual sphere sending the measure $\mu$ to its total variation $|\mu|$ is weakly* continuous. Since the positive and the negative parts of $\mu$ can be obtained from $\mu$ and $|\mu|$, and $|s\mu| = s|\mu|$ for nonnegative $s$, using the weak* continuity of $T^*$ we conclude the first part of the corollary.

For the second part we will define two positive operators $T^+$ and $T^−$ such that $T^+ − T^− = P_{B_1} \circ T$. For every $y \in B_1^*$ we have that $(P_{B_1} \circ T)^*(\delta_y) = T^*(P_{B_1}^*(\delta_y)) = T^*(\delta_y)$. So for every $y \in B_1^*$ define

$$T^+(f)(y) = \int f \, d(T^*(\delta_y))^+, \quad T^−(f)(x) = \int f \, d(T^*(\delta_y))^−.$$

It is clear that $P_{B_1} \circ T = T^+ − T^−$. The linearity of $T^+$ and $T^−$ follows from general properties of the integral. To see that they are bounded, notice that for every $f \in C(\mathbb{N}^*)$ with $\|f\| \leq 1$ we have that $\|T^+(f)\| \leq \sup_{y \in B_1^*} (T^*(\delta_y))^+(\mathbb{N}^*)$. If $\{(T^*(\delta_y))^+(\mathbb{N}^*) : y \in B_1^*\}$ were unbounded, the set $F_n = \{y \in B_1^* : (T^*(\delta_y))^+(\mathbb{N}^*) \geq n\}$ would be nonempty for every $n \in \mathbb{N}$. But by the $w^*$-continuity of the map $y \mapsto (T^*(\delta_y))^+$, each $F_n$ is closed. Since the $F_n$‘s form a decreasing chain of nonempty closed sets, there exists $y \in \bigcap_{n \in \mathbb{N}} F_n$. But this is impossible.

The same argument shows that $T^−$ is bounded.

\[ \square \]

Definition 3.8. Let $X$, $Y$ be topological spaces. A function $\varphi : X \to \varphi(Y)$ is called upper semicontinuous if for every open set $V \subseteq Y$ the following set is open in $X$

$$\{x \in X : \varphi(x) \subseteq V\}.$$

Our main interest in multifunctions is related to the following

Definition 3.9. Suppose that $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ is a linear bounded operator and $\varepsilon > 0$. We define

$$\varphi^T_\varepsilon(y) = \{x \in \mathbb{N}^* : |T^*(\delta_y)((\{x\})| \geq \varepsilon\},$$

$$\varphi^T(y) = \bigcup_{\varepsilon > 0} \varphi^T_\varepsilon(y).$$

Proposition 3.10. There is a linear bounded $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ such that $\varphi^T_{1/2}$ is not upper semicontinuous.

Proof. Consider the operator from the proof of 3.4. We have $\varphi_{1/2}(y) = \emptyset$ if $y \in F$ and $\varphi_{1/2}(y) = \{r(y), y\}$ whenever $y \in \mathbb{N}^* \setminus F$. So for example taking $V = \mathbb{N}^* \setminus F$ we obtain that $\{y : \varphi_{1/2}(y) \subseteq V\} = F$ which is not open, but closed nowhere dense.

However we obtain the left-local upper semicontinuity:
Lemma 3.11. Let \( T : C(\mathbb{N}^*) \to C(\mathbb{N}^*) \) be a bounded linear operator and let \( B \subseteq \mathbb{N}^* \) be infinite. Then, there exists \( B_1 \subseteq B \) such that \( \varphi^T \mid B_1^* \) is upper semicontinuous for every \( \varepsilon > 0 \).

Proof. Let \( B_1 \subseteq B \) and \( C_n, D_n \) be as in 3.5. Fix \( y \in B_1^* \) and an open \( V \subseteq \mathbb{N}^* \) such that \( \varphi^T(y) \subseteq V \). Then, for every \( x \in \mathbb{N}^* \setminus V \) we find a clopen neighbourhood \( U_x \) of \( x \) as follows. First notice that \( |T^*(\delta_y)(\{x\})| < \varepsilon \). Let \( N_x \in \mathbb{N} \) be such that \( |T^*(\delta_y)(\{x\})| < \varepsilon - 1/N_x \). Now, using the regularity of the measure \( T^*(\delta_y) \), find \( U_x \) such that \( |T^*(\delta_y)(U_x)| < \varepsilon - 1/N_x \). We may assume that \( U_x \) is included in either \( C_{N_x}^* \) or \( D_{N_x}^* \).

Since \( \mathbb{N}^* \setminus V \) is compact, we may find \( x_0, \ldots, x_{k-1} \in \mathbb{N}^* \setminus V \) such that \( \mathbb{N}^* \setminus V \subseteq \bigcup_{i<k} U_{x_i} \). Using the weak*-continuity of \( T^* \), we now find a clopen neighbourhood of \( y \), say \( E^* \), which we may assume to be included in \( B_1^* \), such that for every \( z \in E^* \) we have

\[
|T^*(\delta_y)(U_{x_i})| < \varepsilon - 1/N_{x_i}, \text{ for each } i < k.
\]

This is possible because \( |T^*(\delta_y)(U_x)| \leq |T^*(\delta_y)(U_{x_i})| < \varepsilon - 1/N_{x_i} \).

We claim that for every \( z \in E^* \) and every \( x \in \bigcup_{i<k} U_{x_i} \) we have \( |T^*(\delta_z)(\{x\})| < \varepsilon \), that is \( \varphi^T_z(z) \subseteq \overline{V} \). So fix \( z \in E^* \) and \( x \in U_{x_i} \). Let \( T^*(\delta_z) = \mu^+_z - \mu^-_z \) be a Jordan decomposition of the measure. Notice that \( |T^*(\delta_z)(U_{x_i})| \leq |T^*(\delta_z)(U_{x_i})| + 2\mu^-_z(U_{x_i}) \) and \( |T^*(\delta_z)(U_{x_i})| \leq |T^*(\delta_z)(U_{x_i})| + 2\mu^+_z(U_{x_i}) \). So if \( U_{x_i} \subseteq C_{N_{x_i}}^* \), since \( \mu^-_z(U_{x_i}) \leq \mu^-_z(C_{N_{x_i}}^*) < 1/4(N_{x_i} + 1) \), we have that

\[
|T^*(\delta_z)(\{x\})| \leq |T^*(\delta_z)(U_{x_i})| + 2\mu^-_z(U_{x_i}) < \varepsilon - 1/N_{x_i} + 1/2(N_{x_i} + 1) < \varepsilon.
\]

If \( U_{x_i} \subseteq D_{N_{x_i}}^* \), we use the fact that \( \mu^+_z(U_{x_i}) \leq \mu^+_z(D_{N_{x_i}}^*) < 1/4(N_{x_i} + 1) \) to obtain the same result. \( \square \)

3.3. Fountains and funnels. The property of being locally null can be expressed using a topological property of \( T^* \).

Proposition 3.12. A bounded linear operator \( T : C(\mathbb{N}^*) \to C(\mathbb{N}^*) \) is locally null if, and only if, there is a nowhere dense set \( F \subseteq \mathbb{N}^* \) such that \( T^*(\delta_y) \) is concentrated on \( F \) for every \( y \in \mathbb{N}^* \).

Proof. Suppose \( T \) is locally null. If we set \( D = \bigcup \{ A^* : T \circ I_A = 0 \} \), then \( D \) is an open dense set. Suppose \( |T^*(\delta_y)|(D) > \varepsilon \) for some \( y \in \mathbb{N}^* \) and some \( \varepsilon > 0 \). By the regularity of the measure we may find a compact \( G \subseteq D \) such that \( |T^*(\delta_y)|(G) > \varepsilon \).

We may further find finitely many \( A_0, \ldots, A_{n-1} \subseteq \mathbb{N} \) such that \( T \circ I_{A_i} = 0 \) for all \( i < n \) and \( \sum_{i<n} |T^*(\delta_y)|(A_i^*) > \varepsilon \). Choose \( i < n \) such that \( |T^*(\delta_y)|(A_i^*) > \varepsilon/n \) and a function \( f \) with support included in \( A_i^* \) such that \( T^*(\delta_y)(f) > \varepsilon/n \). Then, \( T(f)(y) \neq 0 \), which contradicts the hypothesis. Therefore, \( T^*(\delta_y) \) is concentrated on \( F = \mathbb{N}^* \setminus D \), for every \( y \in \mathbb{N}^* \).

Conversely, suppose \( F \) is a nowhere dense set such that for every \( y \in \mathbb{N}^* \) the measure \( T^*(\delta_y) \) is concentrated on \( F \). Given an infinite \( A \subseteq \mathbb{N} \), take \( A_1 \subseteq A \) infinite such that \( A_1 \cap \mathbb{N} \setminus F = \emptyset \). Then, \( |T^*(\delta_y)|(A_1^*) = 0 \) and it follows that \( T \circ I_{A_1} = 0 \). \( \square \)

As in the previous proposition, many results in the following parts of the paper will show the important role played by nowhere dense sets of \( \mathbb{N}^* \) in the context of
operators on $C(\mathbb{N}^*)$. It is this fact that leads to the definitions of fountains, funnels and fountainless and funnelless operators:

**Definition 3.13.** An operator $T : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ is called fountainless or without fountains if, and only if, for every nowhere dense set $F \subseteq \mathbb{N}^*$ the set

$$G = \{ y \in \mathbb{N}^* : T^*(\delta_y) \text{ is nonzero and concentrated on } F \}$$

is nowhere dense. A fountain for $T$ is a pair $(F,U)$ with $F \subseteq \mathbb{N}^*$ nowhere dense and $U \subseteq \mathbb{N}^*$ open such that all the measures $T^*(\delta_y)$ for $y \in U$ are concentrated on $F$.

**Lemma 3.14.** Let $T$ be fountainless and let $B \subseteq \mathbb{N}$ be infinite. If $P_B \circ T$ is locally null, then $P_B \circ T = 0$.

**Proof.** By 3.12 there is a nowhere dense $F \subseteq \mathbb{N}^*$ such that for every $y \in B^*$ we have that $(P_B \circ T)^*(\delta_y)$, which is equal to $T^*(\delta_y)$, is concentrated on $F$. By 3.13 the set $G = \{ y \in B^* : T^*(\delta_y) \neq 0 \}$ is nowhere dense. But this means that for every $f \in C(\mathbb{N}^*)$ we have $T(f)(x) = 0$ if $x \in B^* \setminus G$. Since $B^* \setminus G$ is dense in $B^*$ we conclude that $P_B \circ T = 0$.

**Corollary 3.15.** Suppose that $T$ is locally null and has no fountains, then $T = 0$.

**Proof.** Put $B = \mathbb{N}$ in 3.14.

**Definition 3.16.** We say that an operator $T : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ is everywhere present if, and only if, for every infinite $B \subseteq \mathbb{N}$ we have that $P_B \circ T \neq 0$.

In the following lemma we obtain a kind of left dual to an improvement of a theorem of Cengiz ("P" in [10]) obtained by Plebanek (Theorem 3.3. in [39]) which implies that if $T$ is an isomorphic embedding then every $x \in \mathbb{N}^*$ is in $\varphi^T(y)$ for some $y \in \mathbb{N}^*$.

**Lemma 3.17.** Suppose that $T : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ is an everywhere present fountainless operator. Then, for every infinite $B \subseteq \mathbb{N}$ there exists an infinite $B_1 \subseteq B$ such that $\varphi^T(y) \neq \emptyset$, for every $y \in B_1$.

**Proof.** Given an infinite $B \subseteq \mathbb{N}$, let $B_1 \subseteq \mathbb{N}$ and $C_n$, $D_n \subseteq \mathbb{N}$ be as in 3.3. Suppose that $y_0 \in B_1$ is such that $\varphi^T(y_0) = \emptyset$. For every $n \in \mathbb{N}$ we find an open covering of $\mathbb{N}^*$ as follows. Given $x \in \mathbb{N}^*$, find by the regularity of the measure $T^*(\delta_{y_0})$ a clopen neighbourhood of $x$, say $U_x$, such that $|T^*(\delta_{y_0})(U_x)| < 1/(2n+1)$ and $U_x$ is included in either $C_n^*$ or $D_n^*$.

By the compactness of $\mathbb{N}^*$ obtain for each $n \in \mathbb{N}$ an open covering $\{U_{n,i} : i < j_n \}$ of $\mathbb{N}^*$ such that for each $i < j_n$ we have that

1. $|T^*(\delta_{y_0})(U_{n,i})| \leq |T^*(\delta_{y_0})|(U_{n,i}) < 1/(2n+1)$, and
2. either $U_{n,i} \subseteq C_n^*$ or $U_{n,i} \subseteq D_n^*$.

By the weak* continuity of $T^*$ there are open neighbourhoods $V_n$ of $y_0$ such that $|T^*(\delta_{y_0})(U_{n,i})| < 1/(2n+1)$ holds for all $y \in V_n$ and all $i < j_n$. Let $V^*$ be a clopen subset of $\bigcap_{n \in \mathbb{N}} V_n \cap B_1^*$ and consider the family $A \subseteq \varphi(\mathbb{N})$ of those sets $A$ such that for each $n \in \mathbb{N}$ we have $A^* \subseteq U_{n,i}$ for some $i < j_n$. We claim that $|T^*(\delta_{y_0})(A)| = 0$ for every $y \in V^*$ and every $A \in A$.

So fix $y \in V^*$, $A \in A$ and $n \in \mathbb{N}$. We will show that $|T^*(\delta_{y})(A^*)| < 1/(n+1)$. Let $T^*(\delta_{y}) = \mu^+ - \mu^-$ be a Jordan decomposition of the measure. By 3.3 we have
that \( \mu^-(C_n^*) < 1/(4(n+1)) \) and \( \mu^+(D_n^*) < 1/(4(n+1)) \). Assume without loss of generality that \( U_{n,i_n} \subseteq C_n^* \). Then,

\[
|T^*(\delta_y)|(A^*) \leq |T^*(\delta_y)|(U_{n,i_n}) = T^*(\delta_y)(U_{n,i_n}) + 2\mu^-(U_{n,i_n}) \\
\leq |T^*(\delta_y)| + 2\mu^-(C_n^*) \\
< 1/(n+1) + 2/4(n+1) = 1/(n+1)
\]

So the claim is proved.

Notice that this implies that \((P_V \circ T)(f) = 0\) for every \( f \in C(\mathbb{N}^*) \) whose support is included in \( A^* \), for some \( A \in \mathcal{A} \). Therefore, if \( A \) is a dense family, by \ref{prop:general} we would have that \((P_V \circ T)(g) = 0\) for all \( g \in C(\mathbb{N}^*) \), but this would contradict the hypothesis that \( T \) is everywhere present.

We prove that \( \mathcal{A} \) is a dense family. For a fixed infinite \( E_0 \subseteq \mathbb{N} \), we may define by induction a \( \subseteq_{\omega} \) decreasing sequence \((E_n)\) of infinite sets by choosing \( \emptyset \neq E_{n+1}^* \subseteq E_n^* \cap U_{n,i_n} \), for some \( i_n < j_n \) (this is possible because \( \{U_{n,i} : i < j_n\} \) is an open covering of \( \mathbb{N}^* \) for each \( n \in \mathbb{N} \)). Take \( A \) such that \( A \subseteq_{\omega} E_n^* \), for all \( n \in \mathbb{N} \). It is clear that \( A \subseteq_{\omega} E_0 \) and \( A \in \mathcal{A} \).

Let us introduce a dual notion to a fountain:

**Definition 3.18.** An operator \( T : C(\mathbb{N}^*) \to C(\mathbb{N}^*) \) is called funnelless or without funnels if, and only if, for every nowhere dense set \( F \subseteq \mathbb{N}^* \) there is a nowhere dense \( G \subseteq \mathbb{N}^* \) such that for all \( y \in F \) the measure \( T^*(\delta_y) \) is concentrated on \( G \). A funnel for \( T \) is a pair \((U,F)\) with \( F \subseteq \mathbb{N}^* \) nowhere dense and \( U \subseteq \mathbb{N}^* \) open such that there is no proper closed subset of \( U \) where all the measures \( T^*(\delta_y)|U \) for \( y \in F \) are concentrated.

### 3.4. Operators induced by continuous maps and nonatomic operators.

**Definition 3.19.** Suppose that \( \psi : \mathbb{N}^* \to \mathbb{N}^* \) is a continuous map. Then \( T_{\psi} : C(\mathbb{N}^*) \to C(\mathbb{N}^*) \) is given for every \( f \in C(\mathbb{N}^*) \) by

\[
T_{\psi}(f) = f \circ \psi.
\]

**Definition 3.20.** A continuous map \( \psi : \mathbb{N}^* \to \mathbb{N}^* \) is called quasi-open if, and only if, the image of every nonempty open set under \( \psi \) has nonempty interior.

**Proposition 3.21.** Suppose that \( \psi : \mathbb{N}^* \to \mathbb{N}^* \) is a continuous map. Then \( T_{\psi} \) is fountainless if, and only if, \( \psi \) is quasi-open.

**Proof.** Notice that for every \( y \in \mathbb{N}^* \) we have \( T_{\psi}^*(\delta_y) = \delta_{\psi(y)} \). Notice also that for every subset \( X \subseteq \mathbb{N}^* \) the following holds

\[
|\delta_{\psi(y)}|(X) \neq 0 \iff \psi(y) \in X \iff y \in \psi^{-1}[X].
\]

Therefore, if \( \psi \) is quasi-open and \( F \subseteq \mathbb{N}^* \) is nowhere dense, we have that \( \{y \in \mathbb{N}^* : |T_{\psi}^*(\delta_y)|(\mathbb{N}^* \setminus F) = 0\} = \psi^{-1}[F] \) is nowhere dense, and so \( T_{\psi} \) is fountainless. On the other hand if \( T_{\psi} \) is fountainless, consider \( \psi[U] \) where \( U \) is open. If \( \psi[U] \) were nowhere dense, then \( \{y \in \mathbb{N}^* : |T_{\psi}^*(\delta_y)|(\mathbb{N}^* \setminus \psi[U]) = 0\} = \psi^{-1}[\psi[U]] \) would be nowhere dense, which contradicts the fact that \( U \subseteq \psi^{-1}[\psi[U]] \).

\( \square \)
Proposition 3.22. Let $\psi : \mathbb{N}^* \to \mathbb{N}^*$ be a continuous map. Then $T_\psi$ is funnellness if, and only if, $\psi$ sends nowhere dense sets into nowhere dense sets.

Proof. Suppose $T_\psi$ is funnellness and let $F \subseteq \mathbb{N}^*$ be nowhere dense. Let $G \subseteq \mathbb{N}^*$ be nowhere dense such that $T_\psi^*(\delta_y)$ is concentrated on $G$ for every $y \in F$. Then, as in the proof of Proposition 3.21, we have that $F \subseteq \{ y \in \mathbb{N}^* : |T_\psi^*(\delta_y)|(\mathbb{N}^* \setminus G) = 0 \} = \psi^{-1}[G]$. 

Now suppose $\psi$ sends nowhere dense sets into nowhere dense sets and let $F \subseteq \mathbb{N}^*$ be nowhere dense. Then, $F \subseteq \psi^{-1}[\psi[F]] = \{ y \in \mathbb{N}^* : |T_\psi^*(\delta_y)|(\mathbb{N}^* \setminus \psi[F]) = 0 \}$, which means that $T_\psi^*(\delta_y)$ is concentrated on $\psi[F]$, for every $y \in F$. □

Definition 3.23. An operator $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ is nonatomic if, and only if, for every $y \in \mathbb{N}^*$ the measure $T^*(\delta_y)$ is nonatomic.

Proposition 3.24. Every positive nonatomic operator on $\ell_\infty/c_0$ is locally null.

Proof. Since for every $y \in \mathbb{N}^*$ the measure $T^*(\delta_y)$ has no atoms, by the regularity of $T^*(\delta_y)$ and the compactness of $\mathbb{N}^*$ we may find for each $n \in \mathbb{N}$ a finite open covering $(U_i(y,n))_{i,j \leq n}$ of $\mathbb{N}^*$ by clopen sets such that $|T^*(\delta_y)|(U_i(y,n)) < 1/(2(n+1))$ holds for all $i < j(y,n)$.

Now by the weak* continuity of $T^*$, we may choose for each $n \in \mathbb{N}$ an open neighbourhood $V_n(y)$ of $y$ such that for all $z \in V_n(y)$ we have

$$|T^*(\delta_z)|(U_i(y,n)) = |T^*(\delta_z)(U_i(y,n))| < 1/2(n+1)$$

for all $i < j(y,n)$. The first equality follows from the hypothesis that $T$ is positive.

We have thus constructed for each $n \in \mathbb{N}$ an open covering $\{V_n(y) : y \in \mathbb{N}^*\}$ of $\mathbb{N}^*$. By the compactness of $\mathbb{N}^*$, for each $n \in \mathbb{N}$ take $y_0(n), \ldots, y_{m(n)-1}(n) \in \mathbb{N}^*$ such that

$$\mathbb{N}^* \subseteq \bigcup_{l \leq m(n)} V_{n}(y_l).$$

Now consider the family $A$ of those sets $A \subseteq \mathbb{N}$ such that given $n \in \mathbb{N}$, for each $l < m(n)$ there is $i < j(y_l,n)$ such that $A^*$ is included in $U_i(y_l,n)$. As in the proof of Proposition 3.14 it is easy to see that $A$ is dense and that for every $z \in \mathbb{N}^*$ and every $A \in A$ we have that $|T^*(\delta_z)|(A^*) = 0$. Therefore if $f \in C(\mathbb{N}^*)$ is $A^*$-supported we have $T(f) = 0$, as required. □

4. Operators on $\ell_\infty/c_0$ and operators on $\ell_\infty$

4.1. Operators induced by operators on $\ell_\infty$.

Definition 4.1. Suppose that $R : \ell_\infty \to \ell_\infty$ is a linear operator which preserves $c_0$, then $[R] : \ell_\infty/c_0 \to \ell_\infty/c_0$ is a linear operator defined by

$$[R][f]_{c_0} = [R(f)]_{c_0},$$

for every $f \in \ell_\infty$. If $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ is a linear operator, then a lifting $R : \ell_\infty \to \ell_\infty$ is any linear operator such that $[R] = T$.

Note that our terminology is slightly different than the one used in the literature concerning the trivialization of endomorphisms of $\wp(\mathbb{N})/\text{Fin}$. This is due to the fact that we do not use nonlinear liftings of linear operators.

Lemma 4.2. Let $R_0, R_1 : \ell_\infty \to \ell_\infty$ be linear operators which preserve $c_0$. Then,

1. $[R_0 + \alpha R_1] = [R_0] + [\alpha R_1]$, for every real $\alpha$. 

\[(2) \, [R_1 \circ R_0] = [R_1] \circ [R_0].\]

**Proof.** Fix \( f \in \ell_\infty \). Then,
\[
[R_0+\alpha R_1][(f)_{c_0}] = [(R_0+\alpha R_1)(f)]_{c_0} = (R_0(f) + \alpha [R_1(f)])_{c_0} = ([R_0] + \alpha [R_1])[(f)_{c_0}]
\]
and \([R_1] \circ [R_0] [(f)_{c_0}] = [R_1][R_0(f)]_{c_0} = [R_1 \circ R_0][(f)_{c_0}]\).
\(\square\)

**Proposition 4.3.** Let \( R_0, R_1 : \ell_\infty \to \ell_\infty \) be linear operators which preserve \( c_0 \). Then,

1. If \([R_0] = 0\) then, \( R \) is weakly compact.
2. If \([R_0] = [R_1]\) then, \( R_0 - R_1 \) is weakly compact.

**Proof.** \([R_0] = 0\) means that the image of \( R_0 \) is included in \( c_0 \). However, \( \ell_\infty \) is a Grothendieck space and all operators from such spaces into separable spaces are weakly compact (Theorem 1 of [11]). For part (2) apply 4.2 and part (1) to \( R_0 - R_1 \).
\(\square\)

So there could be many liftings of the same operator but they all differ by a weakly compact perturbation. When we look at \( \ell_\infty/c_0 \) as \( C(\mathbb{N}^*) \), then liftings correspond to extensions.

**Lemma 4.4.** Let \( T : C(\mathbb{N}^*) \to C(\mathbb{N}^*) \) be liftable to \( R : C(\beta \mathbb{N}) \to C(\beta \mathbb{N}) \). Then, for every \( y \in \mathbb{N}^* \) we have
\[
R^*(\delta_y) |\mathbb{N}^* = T^*(\delta_y).
\]

**Proof.** If \( Q : C(\beta \mathbb{N}) \to C(\mathbb{N}^*) \) is the restriction map, then the dual of the lifting relation \( T \circ Q = Q \circ R \) is \( Q^* \circ T^* = R^* \circ Q^* \). Notice that \( Q^* \) acts on measures on \( \mathbb{N}^* \) by extending them to \( \beta \mathbb{N} \) with \( \mathbb{N} \) having measure zero. So for every \( y \in \mathbb{N}^* \) we have \( T^*(\delta_y) = (Q^* \circ T^*)(\delta_y)|\mathbb{N}^* = (R^* \circ Q^*)(\delta_y)|\mathbb{N}^* = R^*(\delta_y)|\mathbb{N}^* \).
\(\square\)

### 4.2. Local properties of liftable operators on \( \ell_\infty/c_0 \).

**Proposition 4.5.** If \( R : C(\beta \mathbb{N}) \to C(\beta \mathbb{N}) \) is a positive falling operator, then the operator \([R] : C(\mathbb{N}^*) \to C(\mathbb{N}^*) \) is nonatomic and locally null.

**Proof.** By the definition (2.3), given \( \varepsilon > 0 \) we have a cofinite set \( B \subseteq \mathbb{N} \) and a partition \( \{A_1, \ldots, A_k\} \) of \( \mathbb{N} \) such that
\[
R^*(\delta_i)(\beta A_m) = |R^*(\delta_i)(\beta A_m)| < \varepsilon
\]
for every \( m \leq k \) and every \( i \in B \). As any \( \delta_y \), for \( y \in \mathbb{N}^* \), is in the weak* closure of \( \{\delta_n : n \in B\} \), it follows by the weak* continuity of \( R^* \) that \( R^*(\delta_y)(\beta A_m) < \varepsilon \), for every \( y \in \mathbb{N}^* \) and every \( m \leq k \). But by 1.4 and the positivity of \( R \) we have \([R]^*(\delta_y)(A_m^*) = R^*(\delta_y)(A_m^*) \leq R^*(\delta_y)(\beta A_m) < \varepsilon \), for every every \( y \in \mathbb{N}^* \) and every \( m \leq k \). As \( \{A_1^*, \ldots, A_k^*\} \) is a partition of \( \mathbb{N}^* \), we conclude that \([R]^*(\delta_y) \) is nonatomic for every \( y \in \mathbb{N}^* \). By 3.21 \( [R] \) is locally null.
\(\square\)

**Corollary 4.6.** There is a matrix operator \( T \) which has fountains and such that whenever \( T \circ I_A \neq 0 \), we have that \( T \circ I_A \) is not canonizable along any continuous map. In particular \( T \) is nowhere trivial.
\textit{Proof.} The operator $R$ from \[2.10\] is a non-weakly compact, positive, falling operator on $\ell_\infty$. Its range is not included in $c_0$ by \[2.7\] (actually, the characteristic function of a subset of $\mathbb{N}$ of positive density is sent to an element not in $c_0$). So $T = [R] \neq 0$.

On the other hand, by \[2.5\] we know that $T$ is locally null, so by \[3.15\] it follows that $T$ has fountains.

Now note that by \[1.4\] we have that $(T \circ I_A)^*(\delta_y) = T^*(\delta_y)|A^*$ is nonatomic or zero for every $y \in \mathbb{N}^*$, so the second part of the corollary follows. \hfill \Box

\textbf{Proposition 4.7.} \textit{If $R : \ell_\infty \to \ell_\infty$ is an antimatrix operator, then the operator $[R] : \ell_\infty/c_0 \to \ell_\infty/c_0$ factors through $\ell_\infty$ and so is locally null.}

\textit{Proof.} Let $\mu_n = R^*(\delta_n)$. Since $R$ is antimatrix (Definition \[2.11\]) we may consider $\mu_n$ as a measure on $\mathbb{N}$. Consider $S : \ell_\infty/c_0 \to \ell_\infty$ given by $S([f]_{c_0}) = (\mu_n(\beta f))_{n \in \mathbb{N}}$ for every $f \in \ell_\infty$, and $Q : \ell_\infty \to \ell_\infty/c_0$ the quotient map. $S$ is well-defined since the measures $\mu_n$ are null on $\mathbb{N}$. We have for every $f \in \ell_\infty$ that

\[(Q \circ S)([f]_{c_0}) = ([\mu_n(\beta f)]_{n \in \mathbb{N}})_{c_0} = [R(f)]_{c_0} = [R([f]_{c_0})],\]

so $(Q \circ S)$ is $[R]$. To conclude that $[R]$ is locally null use \[3.2\]. \hfill \Box

\textbf{Theorem 4.8.} \textit{If $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ is a matrix operator which is an isomorphic embedding, then it is right-locally trivial.}

\textit{Proof.} Let $R : \ell_\infty \to \ell_\infty$ be given by a $c_0$-matrix such that $[R] = T$. Let $(b_{ij})_{i,j \in \mathbb{N}}$ be the matrix corresponding to $R$. Let $M > 0$ be such that $\|T([f]_{c_0})\| \geq M \|\|f\|_{c_0}\|$, for every $f \in \ell_\infty \setminus c_0$. Notice that this condition is equivalent to the statement that $\limsup_{n \to \infty} |R(f)(n)| \geq M$, for every $f \in \ell_\infty$ such that $\limsup_{n \to \infty} |f(n)| = 1$.

Fix an infinite $\bar{A} \subseteq \mathbb{N}$.

\textbf{Claim:} $\lim_{i \to \infty} \max\{|b_{ij}| : j \in \bar{A}\} \neq 0$.

Assume otherwise. We will construct an $f \in \ell_\infty$ such that $\limsup_{n \to \infty} |f(n)| = 1$ and $\limsup_{n \to \infty} |R(f)(n)| < M$.

Let $m_i = \min\{k \in \mathbb{N} : \sum_{j \geq k} |b_{ij}| < 1/(i+1)\}$, for every $i \in \mathbb{N}$. We shall construct by induction two strictly increasing sequences of integers $(i_n)_{n \in \mathbb{N}}$ and $(j_n)_{n \in \mathbb{N}}$, with $j_n \in \bar{A}$ for every $n \in \mathbb{N}$. Let $i_0 = 0$ and $j_0 = 1$. If we have constructed $i_l$, $j_l$, for $l \leq n$, take $i_{n+1} > i_n$ such that $\max\{|b_{ij}| : j \in \bar{A}\} < \frac{1}{(n+2)^2}$, for every $i \geq i_{n+1}$; take $j_{n+1} \in \bar{A}$ such that $j_{n+1} > \max\{m_l : l < i_{n+1}\}$ and $j_{n+1} > j_n$.

Now let $f$ be the characteristic function of $\{j_n : n \in \mathbb{N}\}$ and let $N \in \mathbb{N}$ be such that $\frac{N}{(n+1)^2} < M/4$ and $1/N < M/4$. Fix $k \geq i_N$. Then, $k \geq N$ and also, $i_n \leq k < i_{n+1}$, for some $n \geq N$. Consider the following:

\[|R(f)(k)| = \left| \sum_{j \in \mathbb{N}} b_{kj} f(j) \right| \leq \sum_{j < m_k} |b_{kj} f(j)| + \sum_{j \geq m_k} |b_{kj}| \leq \sum_{j < m_k} |b_{kj} f(j)| + 1/k \leq \sum_{j < m_k} |b_{kj}| + 1/N \quad \text{(because $j_{n+1} > m_k$)} \leq n \cdot \max\{|b_{kj}| : j \in \bar{A}\} + 1/N \leq \frac{n}{(n+1)^2} + 1/N \quad \text{(because $k \geq i_n$)} < M/2 \]

This contradicts the definition of $M$, and so the claim is proved.

Let $\delta > 0$ and let $B_0 \subseteq \mathbb{N}$ be infinite such that $\max\{|b_{ij}| : j \in \bar{A}\} > \delta$, for every $i \in B_0$. 


We shall construct by induction three strictly increasing sequence of integers, 
\((i_n)_{n \in \mathbb{N}}, (j_n)_{n \in \mathbb{N}}, (k_n)_{n \in \mathbb{N}}\), satisfying the following for every \(n \in \mathbb{N}\):

1. \(|b_{i_n,j_n}| > \delta\)
2. \(j_n \in A\)
3. \(k_n \leq j_n < k_{n+1}\)
4. \(\sum_{j < k_n} |b_{i_n,j}| < \frac{1}{2(n+1)}\)
5. \(\sum_{j \geq k_{n+1}} |b_{i_n,j}| < \frac{1}{2(n+1)}\)

Let \(k_0 = 0\) and \(i_0 = \min(B_0)\). Let \(j_0 \in A\) be such that \(|b_{i_0,j_0}| > \delta\). Let \(k_1 > j_0\) be such that \(\sum_{j \geq k_1} |b_{i_0,j}| < 1\).

Assume we have constructed \(i_l, j_l\) and \(k_{l+1}\), satisfying 1–5 for every \(l \leq n\). Let \(N\) be such that \(\sum_{j < k_{n+1}} |b_{i,j}| < \min\{\delta, \frac{1}{2(n+2)}\}\), for every \(i \geq N\) (it exists because it is a \(c_0\)-matrix). Let \(i_{n+1} \in B_0 \setminus N\). Let \(j_{n+1} \in A\) be such that \(|b_{i_{n+1},j_{n+1}}| > \delta\) (it exists because \(i_{n+1} \in B_0\)). Notice that \(j_{n+1} \geq k_{n+1}\) because \(|b_{i_{n+1}}| < \delta\), for every \(j < k_{n+1}\). Let \(k_{n+2} > j_{n+1}\) be such that \(\sum_{j \geq k_{n+2}} |b_{i_{n+1},j}| < \frac{1}{2(n+2)}\). This ends the inductive construction.

Now, \(\delta < |b_{i_n,j_n}| \leq \sup\{b_{i,j} : i,j \in N\}\), for every \(n \in \mathbb{N}\). Therefore, by going to a subsequence we may assume that \(b_{i_n,j_n}\) converges to some \(r\) with \(|r| \geq \delta\). Let \(A = \{i_n : n \in \mathbb{N}\}\) and \(B = \{i_n : n \in \mathbb{N}\}\). Let \(\sigma : B \to A\) be given by \(\sigma(i_n) = j_n\), for each \(n \in \mathbb{N}\).

CLAIM: \((P_B \circ T \circ I_A)((f)|_{c_0(A)}) = [r \circ \sigma]_{c_0(B)}\), for every \(f \in \ell_\infty(A)\).

Note that what we need to show is that \(\lim_{n \to \infty} \|R(f)(i_n) - r(f)\|_{\ell_\infty(A)} = 0\), for every \(f \in \ell_\infty(A)\). So fix \(f \in \ell_\infty(A)\) and fix an arbitrary \(\varepsilon > 0\). Let \(M'\) be such that \(\|T'(\delta_n)\| \leq M'\), for every \(n \in \mathbb{N}\) (it exists by definition of \(c_0\)-matrix). Let \(N_0\) be such that \(|b_{i_n,j_n} - r| < \frac{\varepsilon}{3\|f\|}\), for all \(n \geq N_0\), and let \(N_1\) be such that \(1/(N_1 + 1) < \frac{\varepsilon}{3\|f\|}\). Then, for every \(n \geq N_0 + N_1\) we have

\[
|R(f)(i_n) - r(f)(\sigma(i_n))| = |\sum_{j \in N} b_{i_n,j} f(j) - r f(j_n)| \\
\leq \sum_{j < k_n} |b_{i_n,j} f(j)| + \sum_{k_n \leq j < k_{n+1}} |b_{i_n,j} f(j)| \\
+ \sum_{j \geq k_{n+1}} |b_{i_n,j} f(j)| + |b_{i_{n+1},j_n} f(j_n) - r f(j_n)| \\
< \frac{\|f\|}{2(n+1)} + 0 + \frac{\|f\|}{2(n+1)} + \|f\| |b_{i_n,j_n} - r| \\
< \frac{\varepsilon/3}{\|f\|} + \frac{\varepsilon/3}{\|f\|} < \varepsilon.
\]

This concludes the proof.

\[
\square
\]

**Corollary 4.9.** Every liftable isomorphic embedding \(T : \ell_\infty/c_0 \to \ell_\infty/c_0\) is right-locally trivial.

**Proof.** Since \(T\) is liftable, there exist \(R_0, R_1 : \ell_\infty \to \ell_\infty\) an antimatrix operator and one given by a \(c_0\)-matrix, respectively, such that \(T = [R_0 + R_1] = [R_0] + [R_1]\). Fix an infinite \(A \subseteq \mathbb{N}\). By \(\text{Lem} 7\) take an infinite \(A_0 \subseteq A\) such that \(T \circ I_{A_0} = [R_1] \circ I_{A_0}\). Then, \([R_1] \circ I_{A_0}\) is a matrix operator which is an isomorphic embedding, so by \(\text{Lem} 8\) there exist infinite \(A_1 \subseteq A_0\) and \(B \subseteq \mathbb{N}\) such that \(P_B \circ T \circ I_{A_1} = P_B \circ [R_1] \circ I_{A_1}\) is trivial.

\[
\square
\]

**Corollary 4.10.** Every liftable isomorphic embedding \(T : \ell_\infty/c_0 \to \ell_\infty/c_0\) is right-locally an isomorphic matrix operator.
Proof. By [3.9] it suffices to recall that a trivial operator is an isomorphic matrix operator.

Corollary 4.11. Let \( P \) be one of the following properties: isomorphically liftable, isomorphically matrix, trivial, canonizable along \( \psi \). Suppose that \( S : \ell_\infty/c_0 \to \ell_\infty/c_0 \) is locally null. If \( T : \ell_\infty/c_0 \to \ell_\infty/c_0 \) is right-locally \( P \) (left-locally \( P \), somewhere \( P \)), then \( S + T \) is right-locally \( P \) (left-locally \( P \), somewhere \( P \)).

Proof. First we will note that if the localization \( T_{B,A} \) of \( T \) to \((A,B)\) has \( P \), then for every infinite \( A' \subseteq A \) there is an infinite \( B' \subseteq B \) such that the localization \( T_{B',A'} \) of \( T \) to \((A',B')\) has \( P \).

In the case where \( T_{B,A} \) is isomorphically liftable, by [4.9] it is enough to notice that a trivial operator is isomorphically liftable. Similarly, if \( T_{B,A} \) is isomorphically matrix, by [4.8] it is enough to notice that a trivial operator is isomorphically matrix.

If \( T_{B,A} \) is trivial, it is enough to take \( B' = \sigma^{-1}[A'] \), where \( \sigma : B \to A \) is the bijection witnessing the triviality of \( T_{B,A} \). Similarly, if \( T_{B,A} \) is canonizable along \( \psi \), we take \( B' \subseteq B \) such that \( (B')^* = \psi^{-1}[(A')^*] \).

Now, given a localization \( T_{B,A} \) with property \( P \), take an infinite \( A' \subseteq A \) such that \( S \circ I_{A'} = 0 \). By the above, there exists \( B' \subseteq B \) such that \( T_{B',A'} = (S + T)_{B',A'} \) has \( P \).

If we do not assume that the operator is bounded below, then there is no hope of obtaining local trivialization anywhere:

Proposition 4.12. There is a surjective operator \( T : \ell_\infty/c_0 \to \ell_\infty/c_0 \) which is globally liftable but is nowhere a nonzero matrix operator.

Proof. Let \( (x_n)_{n \in \mathbb{N}} \) be a discrete sequence of nonprincipal ultrafilters and consider the typical antimatric operator \( R : \ell_\infty \to \ell_\infty \) given by \( R(f) = ((\beta f)(x_n))_{n \in \mathbb{N}} \). Let \( T = [R] \). By [3.3] we know that \( T \) is surjective. Suppose for some infinite \( A, B \subseteq \mathbb{N} \) there is \( S : \ell_\infty(A) \to \ell_\infty(B) \) given by a \( c_0 \)-matrix and such that \([S] = T_{B,A}\). Let us denote by \( R_{B,A} \) the operator which maps \( f \in \ell_\infty(A) \) into \( R(f \cup 0_{\mathbb{N}\setminus A})B \). By [4.3] we have that \( S - R_{B,A} \) is weakly compact, and since \( R \) is an antimatric operator we have that \( R|c_0 = 0 \), so \( S|c_0(A) \) is weakly compact. Therefore, by [2.6] and [2.4] we have that the image of \( S \) is included in \( c_0(B) \) and so \( T_{B,A} = [S] = 0 \).

4.3. Lifting operators on \( \ell_\infty/c_0 \). In the case of the Boolean algebra \( \wp(\mathbb{N})/\text{Fin} \), any endomorphism which can be lifted to a homomorphism of \( \wp(\mathbb{N}) \) is induced by a homomorphism of \( \text{FinCofin}(\mathbb{N}) \). However, in the case of \( \ell_\infty/c_0 \), like for \( \ell_\infty \) [2.15], there exist automorphisms which are not determined by its values on \( c_0 \):

Proposition 4.13. There are liftable operators such that all their liftings are discontinuous and are not induced by its action on \( c_0 \), i.e., are not matrix operators. Moreover, such operators can be automorphisms of \( \ell_\infty/c_0 \).

Proof. Let \( (A_i)_{i \in \mathbb{N}} \) be a partition of \( \mathbb{N} \) into infinite sets. For each \( i \in \mathbb{N} \), let \( x_i \) be any nonprincipal ultrafilter such that \( A_i \in x_i \). For a permutation \( \sigma : \mathbb{N} \to \mathbb{N} \), consider the automorphism \( R_\sigma : \ell_\infty \to \ell_\infty \) from the proof of [2.15] which is given by

\[
R_\sigma(f)(n) = f(n) - \beta f(x_i) + \beta f(x_{\sigma(i)}),
\]
where \( i \in \mathbb{N} \) is such that \( \ell \in A_i \). Recall that \( R_\sigma \circ R_{\sigma^{-1}} = Id_{\ell_{\infty}} \), so by 1.2 we have that \( [R_\sigma] \circ [R_{\sigma^{-1}}] = [Id_{\ell_{\infty}}] = Id_{\ell_{\infty}/c_0} \). It follows that the operators \([R_\sigma]\) are automorphisms of \( \ell_{\infty}/c_0 \).

Now suppose that \( S : \ell_{\infty} \to \ell_{\infty} \) is a continuous lifting of \([R_\sigma]\). By 2.13 the operator \( S \) is given by a \( c_0 \)-matrix, and by 4.3 we have that \( S-R_\sigma \) is a weakly compact operator into \( c_0 \). Note that \( R_\sigma|_{c_0} = Id_{c_0} \), therefore \( S|_{c_0} = Id_{c_0} + W \), where \( W : c_0 \to c_0 \) is the restriction of \( S - R_\sigma \) to \( c_0 \) and so is weakly compact. By 2.5 we have

\[
S = (S|_{c_0})^* = Id_{\ell_{\infty}} + W^\ast = Id_{\ell_{\infty}} + W^{\ast\ast},
\]

and so \( S^* = Id_{M(\beta N^*)} + U \), where \( U \) is weakly compact by the Gantmacher theorem. Therefore, \( Id_{M(\beta N^*)} + U - R_\sigma^* \) is a weakly compact operator, and so is \( Id_{M(\beta N^*)} - R_\sigma^* \). We will show that this is impossible by showing that the bounded sequence of measures \( (\delta_{x_i})_{i \in \mathbb{N}} \) is not mapped onto a relatively weakly compact set.

A simple calculation gives that \( R_\sigma^*(\delta_x) = \delta_x - \delta_{x_i} + \delta_{x_i(1)} \), if \( x \in A_1^* \). It follows that \( R_\sigma^*(\delta_{x_i}) = \delta_{x_i(1)} \), for every \( i \in \mathbb{N} \). So \( (Id_{M(\beta N^*)} - R_\sigma^*)(\delta_{x_i}) = \delta_{x_i} - \delta_{x_i(1)} \), which by the Dieudonné-Grothendieck theorem implies that \( Id_{M(\beta N^*)} - R_\sigma^* \) is not weakly compact unless \( \sigma \) moves only finitely many \( i \in \mathbb{N} \), as the sequence \( (x_i)_{i \in \mathbb{N}} \) is discrete.

Unlike in the case of the algebra \( \wp(\mathbb{N})/\mathcal{F} \), nonliftable automorphisms of \( \ell_{\infty}/c_0 \) exist in ZFC.

Before proving this we need one:

**Lemma 4.14.** Suppose \( R : \ell_{\infty} \to \ell_{\infty} \) preserves \( c_0 \). If \( R \) is not weakly compact, then \([R]\) is not weakly compact either.

**Proof.** If \( R \) is not weakly compact, then there is an infinite \( A \subseteq \mathbb{N} \) such that \( R \) restricted to \( \ell_{\infty}^0(A) = \{ f \in \ell_{\infty} : f(\mathbb{N} \setminus A) = 0 \} \) is an isomorphism onto its range (see Prop. 1.2. from 11 and Corollary VI-17 of 12). Consider \( X = R^{-1}[c_0] \), a closed subspace of \( \ell_{\infty} \) containing \( c_0 \). Note that \( X \cap \ell_{\infty}^0(A) \) is separable as \( R[X \cap \ell_{\infty}^0(A)] \subseteq c_0 \) and \( R \) is an isomorphism on \( \ell_{\infty}^0(A) \). By the standard argument using the Stone-Weierstrass theorem with respect to simple functions one can find a countable Boolean algebra \( \mathcal{B} \) of subsets of \( A \) such that \( X \cap \ell_{\infty}^0(A) \) is included in the closure of the span of \( \{ X_B : B \in \mathcal{B} \} \).

Let \( (D_\xi)_{\xi < \omega_1} \) be a family of pairwise almost disjoint infinite subsets of \( \omega_1 \). For each \( \xi < \omega_1 \) take \( x \in D_\xi \) and let \( E_\xi \) be infinite such that \( E_\xi \subseteq (\{ B^* : B \in \mathcal{B} \cap x \} \cap \{ N^* \setminus B^* : B \in \mathcal{B} \setminus x \}) \) (it exists by 8.1 and because \( \mathcal{B} \) is countable). Now take \( u_\xi, v_\xi \in E_\xi \) distinct. It follows that no element of \( \mathcal{B} \) separates any of the pairs \( (u_\xi, v_\xi) \). Therefore, \( \beta f(u_\xi) = \beta f(v_\xi) \) for every \( f \in X \cap \ell_{\infty}^0(A) \).

For every \( \xi < \omega_1 \) choose \( g_\xi \in \ell_{\infty}^0(A) \) with support in \( D_\xi \) such that \( \|g_\xi\| = 1 \) and \( g_\xi(u_\xi) = 1 \) and \( g_\xi(v_\xi) = -1 \). Notice that \( R(g_\xi) \notin c_0 \), for every \( \xi < \omega_1 \). This implies that \( \|R(g_\xi)\| \notin c_0 \). For every \( \xi < \omega_1 \), there exists \( n \in \mathbb{N} \) such that for infinitely many \( \xi < \omega_1 \) we have \( \|R(g_\xi)\| > 1/n \). Since the \( g_\xi \) are pairwise disjoint, \([R]\) is not weakly compact by Corollary VI-17 of 12. 

**Proposition 4.15.** Every weakly compact operator on \( \ell_{\infty}/c_0 \) with nonseparable range is nonliftable. Such operators exist.

**Proof.** Suppose that \( S : \ell_{\infty}/c_0 \to \ell_{\infty}/c_0 \) is weakly compact with nonseparable range and \( R : \ell_{\infty} \to \ell_{\infty} \) is such that \([R] = S \) and \( R \) preserves \( c_0 \). \( R \) must be weakly compact itself by 4.14. In particular, the image of the unit ball under \( R \) is weakly
compact. Since weakly compact subsets of $\ell_\infty$ are norm separable (Corollary 4.6 of [42]), we have that the image of $R$ is separable. But this implies that the image of $[R] = S$ is separable as well, contradicting the hypothesis.

Now we construct a weakly compact operator $S : \ell_\infty/c_0 \to \ell_\infty/c_0$ with nonseparable range which is weakly compact. The construction is based on the fact that $\ell_\infty/c_0$ contains an isometric copy of $\ell_2(2^n)$. This follows from the result of Avilés in [2] which states that the unit ball in $\ell_2(2^n)$ with the weak topology (equivalently weak$^*$ topology) is a continuous image of $A(2^n)^\beta$, where $A(2^n)$ is the one point compactification of the discrete space of size $2^n$. On the other hand, by Theorem 2.5 and Example 5.3. in [6] we have that $A(2^n)^\beta$ is a continuous image of $\mathbb{N}^\ast$. Hence $C(B_{\ell_2(2^n)})$ embeds isometrically into $C(\mathbb{N}^\ast)$ and so does $\ell_2(2^n)$. So let $S_1: \ell_2(2^n) \to \ell_\infty/c_0$ be an isomorphism onto its range.

To complete the construction, it is enough to take a surjective operator $S_2 : \ell_\infty/c_0 \to \ell_2(2^n)$ and consider $S = S_1 \circ S_2$. This is because any operator into a reflexive Banach space is weakly compact (Corollary VI.4.3 of [19]) and weakly compact operators form a two sided ideal (Theorem VI.4.5 of [19]).

The existence such of a surjective operator follows from the complementation of $\ell_\infty$ in $\ell_\infty/c_0$ and the existence of a surjective operator $T : \ell_\infty \to \ell_2(2^n)$ which was proved in [40] Proposition 3.4. and remark 2 below it. It is based on a construction of an isomorphic copy of $\ell_2(2^n)$ inside $\ell_\infty$ (proposition 3.4 of [40]). Once we have an isomorphic embedding $T : \ell_2(2^n) \to \ell_\infty^\ast$ we consider

$$T^* \circ J : \ell_\infty \to \ell_\infty^\ast \to \ell_2(2^n)^\ast,$$

where $J : \ell_\infty \to \ell_\infty^\ast$ is the canonical embedding. We have that $(T^* \circ J)^* = J^* \circ T^\ast = T$ using the reflexivity of $\ell_2(2^n)$ to identify it with $\ell_2(2^n)^\ast$. But $T$ is one-to-one with closed range, so $T^* \circ J$ must be onto as required.

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**Theorem 4.16.** There is an automorphism $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ which cannot be lifted to a linear operator on $\ell_\infty$.

**Proof.** Consider $T_1 = Id + S$ where $S$ is any weakly compact operator on $\ell_\infty/c_0$ from the previous proposition. Since $S$ is strictly singular, $T_1$ is a Fredholm operator of Fredholm index 0 (see Proposition 2.c.10 of [32]), i.e., its kernel is finite dimensional of dimension $n$ and its range is of the same finite codimension $n$. Since finite dimensional subspaces of Banach spaces are complemented we can write

$$T_1 : Ker(T_1) \oplus X \to \text{Range}(T_1) \oplus Y$$

where $Y$ is of finite dimension $n$ and $X$ has the same finite codimension $n$. Let $U : Ker(T_1) \to Y$ be an isomorphism and define $T : Ker(T_1) \oplus X \to \text{Range}(T_1) \oplus Y$ by $T(z, x) = (T_1(x), U(z))$. It follows that $T$ satisfies

$$T = T_1 + U = Id + S + U$$

Having null kernel and being surjective it is an automorphism of $\ell_\infty/c_0$. Now let us show that $T$ cannot be lifted to an operator on $\ell_\infty/c_0$. $S + U$ is weakly compact with nonseparable range as a sum of an operator with this property and a finite rank operator, so it cannot be lifted by the previous proposition. Since the sum of two liftable operators is liftable and $Id$ is liftable, it follows that $T$ is an automorphism which cannot be lifted.

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Proposition 4.17. If $T : \ell_\infty / c_0 \to \ell_\infty / c_0$ is canonizable along a homeomorphism $\psi : \mathbb{N}^* \to \mathbb{N}^*$ and $\psi$ is a nontrivial homeomorphism (i.e., it is not induced by a bijection of two cofinite subsets of $\mathbb{N}$), then $T$ is not liftable.

Proof. We may assume that $\hat{T} = T_\psi$, that is, the constant $r$ of Definition 4.14 is 1. Suppose $R : \ell_\infty \to \ell_\infty$ is a lifting of $T$. Let $R = R_1 + R_2$ (see 2.13), where $R_1$ is an operator given by a $c_0$-matrix and $R_2$ is an antimatrix operator.

Claim 1: There cannot exist disjoint functions $f, g \in \ell_\infty$ (i.e., such that $f \cdot g = 0$) and $\varepsilon > 0$ such that $\int [\beta f] dR_1^* (\delta_i), \int [\beta g] dR_1^* (\delta_i) > \varepsilon$ for infinitely many $i \in \mathbb{N}$.

Indeed, in such a case, we would find an infinite $B \subseteq \mathbb{N}$ such that for every $y \in \hat{B}^*$, we would have $\int [\beta f] d(R_1^*(\delta_y)), \int [\beta g] dR_1^* (\delta_y) \geq \varepsilon$. Since $T = T_\psi$, for all $y \in \mathbb{N}^*$ we have that $T^*(\delta_y) = \delta_\psi(y)$, which implies that either $\int f^* dT^*(\delta_y) = 0$ or $\int g^* dT^*_1 (\delta_y) = 0$. Therefore, we will obtain a contradiction if we can free $T^*$ from the influence of $R_2^*$ somewhere. By 2.10 and using an argument as in the proof of 3.9 we find an infinite $B \subseteq B$ and pairwise disjoint finite $F_i \subseteq \mathbb{N}$ such that $|R_1^*(\delta_i)(\beta \mathbb{N} \setminus F_i)| < \varepsilon / 3 \max \{\|f\|, \|g\|\}$ for all $i \in B$. This implies that $\int_{F_i} [\beta] dR_1^* (\delta_i), \int_{F_i} [\beta g] dR_1^* (\delta_i) \geq \varepsilon / 3$. Consider an uncountable almost disjoint family $\{B_\xi : \xi < \omega_1\}$ of subsets of $B$ and sets $A_\xi = \bigcup \{F_i : i \in B_\xi\}$, for $\xi < \omega_1$. We have that $\int_{\partial A_\xi} [\beta f] dR_1^* (\delta_i), \int_{\partial A_\xi} [\beta g] dR_1^* (\delta_i) \geq \varepsilon / 3$ for each $i \in B_\xi$, since the measures are concentrated on $\mathbb{N}$ (see 2.12). Now, the sets $A_\xi$ are pairwise disjoint and the measures $R_2^* (\delta_i)$ are concentrated on $\mathbb{N}^*$, so there is a $\varepsilon_0 < \omega_1$ such that $|R_2^* (\delta_i)(\beta A_{\varepsilon_0})| = 0$ for all $i \in \mathbb{N}$. So by 4.14 and by the weak* continuity of $R_1^*$, for every $y \in B_0^*$ we have

$$\int (f^* |A_{\varepsilon_0}) dT^*(\delta_y) = \int_{B_0^*} (\beta f) dR_1^*(\delta_y) + R_2^*(\delta_y) \geq \varepsilon / 3 + 0 = \varepsilon / 3.$$ 

A similar calculation works for $\beta g |A_{\varepsilon_0}$ which gives the desired contradiction since the restrictions of disjoint functions are disjoint. So the claim is proved.

Let $(b_{ij})_{i,j \in \mathbb{N}}$ be the matrix of $R_1$, i.e., $R_1^*(\delta_i) = \sum_{j \in \mathbb{N}} b_{ij} \delta_j$ for all $i \in \mathbb{N}$. Let $j_i \in \mathbb{N}$ be such that $b_{ij_i}$ has the largest absolute value among the numbers $(b_{ij}) : j \in \mathbb{N}$ for all $i \in \mathbb{N}$.

Claim 2: The $b_{ij_i}$’s are separated from 0.

Assume otherwise. Then, we can find an infinite $B \subseteq \mathbb{N}$ such that for $i \in B$ the numbers $b_{ij_i}$’s converge to 0. If the sequence $(\|b_i\|_{\ell_i})_{i \in B}$ is not separated from zero, then by 2.10 there would be an infinite $B_0 \subseteq B$ such that the map $f \mapsto R_1(f|B)$ is weakly compact and so $P_{B_0} \circ (R_1) = 0$ by 2.6 and 2.8. This would then imply by 4.7 that $P_{B} \circ T$ is locally null, which is impossible since $P_B \circ T_\psi$ is an automorphism on $\psi[(B')^*]$-supported functions. Therefore, the sequence $(\|b_i\|_{\ell_i})_{i \in B}$ is separated from zero.

By 3.9 there exist $\delta > 0$, an infinite $B_0 \subseteq B$ and finite $F_i \subseteq \mathbb{N}$ for $i \in B$ which are pairwise disjoint and such that $\sum_{j \in F_i} |b_{ij}| > \delta$ for all $i \in B_0$. Since $b_{ij_i}$’s converge to 0, one can partition each $F_i$ into $H_i$ and $G_i$ such that $\sum_{j \in G_i} |b_{ij}| > \delta / 4$ and $\sum_{j \in H_i} |b_{ij}| > \delta / 4$ for sufficiently large $i \in B$ (construct $G_i$ considering initial fragments $G_i(k)$ of $F_i$, for $k \leq |F_i|$ starting with $G_i(0) = \emptyset$ and increasing the previous fragment by one element. Since the jumps between $\sum_{j \in G_i(k)} |b_{ij}|$ and $\sum_{j \in G_i(k+1)} |b_{ij}|$ can be at most $|b_{ij_i}|$, which is eventually less than $\delta / 4$, we can obtain the required $G_i$ and $H_i = F_i \setminus G_i$ at some stage $k \leq |F_i|$. But then we can
define two disjoint functions, $f$ with support $\bigcup_{i \in B} G_i$ and $g$ with support $\bigcup_{i \in B} H_i$, which contradict claim 1. Therefore, the claim is proved.

Now consider the matrix $(c_{ij})_{i,j \in \mathbb{N}}$ such that $c_{ij} = b_{ij}$, for $i \in \mathbb{N}$, and all other entries are zero. Write $R_1 = R_3 + R_4$ where $R_3$ is induced by $(c_{ij})_{i,j \in \mathbb{N}}$ and $R_4 = R_1 - R_3$. If $R_4$ were not weakly compact, we would have that the norms of its rows do not converge to zero [2.6]. Then, using [8.9] and an argument analogous to that of claim 2 we can construct disjoint functions which contradict claim 1. Thus $[R_4]$ must be zero by [2.5] and [2.6] and so $[R_1] = [R_3]$. Therefore, we may assume that $R_1$ is given by a matrix of a function from $\mathbb{N}$ into $\mathbb{N}$, that is, all entries of the matrix are equal to zero except for the $b_{ij}$’s, which are separated from zero by some $\delta > 0$.

**Claim 3:** There are cofinite sets $A, B \subseteq \mathbb{N}$ such that $J : B \to A$ given by $J(i) = j_i$ is a bijection.

If $\{ j_i : i \in \mathbb{N} \}$ is coinfinitely, say disjoint from an infinite $A \subseteq \mathbb{N}$, then $[R_1] \circ I_A = 0$ which together with the fact that $[R_2]$ is locally null (see [4.7]) leads to a contradiction with the fact that $T$ is an automorphism. Of course $J$ cannot send infinite sets into one value, because it would give rise to a column of the matrix of $R_1$ which would not be in $c_0$, as the entries of the matrix are separated from 0, contradicting [2.1]. If there are infinitely many values of $J$ which are assumed on distinct integers, then there are two disjoint infinite sets $B_1 = \{ i_n^1 : n \in \mathbb{N} \} \subseteq \mathbb{N}$ and $B_2 = \{ i_n^2 : n \in \mathbb{N} \} \subseteq \mathbb{N}$ such that $j_{i_n^1} = j_{i_n^2}$. Define $J'(n) = j_{i_n^1} = j_{i_n^2}$ and put $f(i_n^1) = b_{j_{i_n^1}} j'(n) / b_{j_{i_n^2}} j'(n)$ and otherwise put the value of $f$ to be 0. Note that whenever $A' \subseteq \{ J'(n) : n \in \mathbb{N} \} = A$, we have that

$$\delta_{j_{i_n^1}} (R_1(\chi_{A'})) - f(i_n^1) \delta_{j_{i_n^1}} (R_1(\chi_{A'})) = 0,$$

for all $n \in \mathbb{N}$. Let $\eta : B_1' \to B_2'$ be the extension of the bijection from $B_1$ to $B_2$ sending $i_n^1$ to $i_n^2$ for all $n \in \mathbb{N}$. It follows that for every $A$-supported $g$ and every $x \in B_1'$ we have

$$(\delta_x - (\beta f)(\eta(x))) \delta_{\eta(x)}([R_1][g]) = 0.$$ 

Find $B_1' \subseteq B_1$ such that $\psi([B_1']^*) = [A']^*$ for some infinite $A' \subseteq A$ such that $[R_2] \circ I_{A'} = 0$ (by [14.7]). This can be achieved since $\psi$ is a homeomorphism. Considering the values $T_\psi(g')$ for $A'$-supported $g'$’s we obtain all functions supported by $B_1' \cup [B_1']$. However, $[R_2]$ on such $g'$’s is zero, so we obtain a contradiction since the values of $[R_1]$ on such functions have the above restrictions. The claim is proved.

Thus $R_1'$ is $T_\phi$ where $\phi : \mathbb{N}^* \to \mathbb{N}^*$ is a trivial homeomorphism of $\mathbb{N}^*$. Therefore, there is $x \in \mathbb{N}^*$ such that $\psi^{-1}(x) \neq \phi^{-1}(x)$. It follows that there are infinite $B_1, B_2, A \subseteq \mathbb{N}$ such that $A^* = \phi[B_1^*], A^* = \psi[B_2^*]$ and $B_1 \cap B_2 = \emptyset$. Using [4.7] take an infinite $A' \subseteq A$ such that $[R_2] \circ I_{A'} = 0$. Then, $T_\psi([\chi_{A'}])$ and $T_\psi([\chi_{A'}])$ have disjoint supports, so we cannot have $[R_1 + R_2] = T$, which completes the proof.

\[\square\]

5. **Canonizing operators acting along a quasi-open mapping**

In [13] it was proved that for a linear bounded operator $T$ on $\ell_\infty/c_0$ and an infinite $A \subseteq \mathbb{N}$ there is a real $r \in \mathbb{R}$ and an infinite $B \subseteq A$ such that

$$T(f)|B^* = rf$$

for every $B$-supported $f$. This gives, for example, that if $P_1$ and $P_2$ are complementary projections on $\ell_\infty/c_0$, then at least one of them canonizes as above for a
nonzero $r$, in other words we obtain a local canonization along the identity on $B^*$. However, a big disadvantage of this result is that in general we cannot guarantee that the constant $r$ is nonzero. If one works with an automorphism, this kind of result is of no use. For example, consider an infinite and cofinite set $D \subseteq \mathbb{N}$ and the bijection $\sigma : \mathbb{N} \to \mathbb{N}$ such that $\sigma[D] = \mathbb{N} \setminus D$, $\sigma[\mathbb{N} \setminus D] = D$ and $\sigma^2$ is the identity. Define an automorphism $T$ of $\ell_\infty/c_0$ by $T([f]_{c_0}) = [f \circ \sigma]_{c_0}$. The above result gives an infinite $B \subseteq \mathbb{N}$ such that $T([f]_{c_0})B = 0[f]_{c_0}$ for every $B$-supported $f \in \ell_\infty$, which looses much of the information. So in this section we embark on finding a surjective $\psi : B^* \to A^*$ along which $T$ may canonize with $r$ nonzero as required in Definition 1.1. Note that a potential obstacle for finding such a canonization would be if $\bigcup \phi T[N^*]$ were nowhere dense. Actually, we have examples such that $\bigcup \phi T[N^*]$ is nowhere dense and $T$ is surjective \cite{3, 12}. So it is natural to assume that the surjections we consider are fountainless and that embeddings are funnelless. Under these assumptions we obtain a quasi-open $\psi$ such that $T^*(\delta_\psi(\{\psi(x)\})) \neq 0$ holds locally, which is sufficient for the canonization by the following:

**Theorem 5.1.** Let $T : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ be a bounded linear operator and let $\tilde{A} \subseteq \mathbb{N}$ be infinite. If $\psi : \mathbb{N}^* \to \mathbb{N}^*$ is a quasi-open continuous function such that $A^* \subseteq \psi[N^*]$, then there exist $r \in \mathbb{R}$ and clopen sets $A^* \subseteq \tilde{A}^*$ and $B^* = \psi^{-1}[A^*]$ such that $T_{B, \tilde{A}}(f^*) = r(f^* \circ \psi)B^*$, for every $f \in \ell_\infty(\tilde{A})$.

**Proof.** Fix $\tilde{A}$ and $\psi$ as above.

**CLAIM:** There exists an infinite $A \subseteq \tilde{A}$ and a clopen $E^* \subseteq \psi^{-1}[A^*]$ such that for every $y \in E^*$ there exists $r_y \in \mathbb{R}$ satisfying

$$T^*(\delta_y)A^* = r_y \delta_{\psi(y)}.$$

Suppose this does not hold. We will construct recursively sequences $(A_\xi)_{\xi < \omega_1}$, $(D_\xi)_{\xi < \omega_1}$ and $(E_\xi)_{\xi < \omega_1}$ of infinite subsets of $\mathbb{N}$, and a sequence $(a_\xi)_{\xi < \omega_1}$ of nonzero reals such that

1. $A_\eta \subseteq_* \tilde{A}$ and $D_\xi \subseteq_* A_\xi \setminus A_{\xi+1}$, for every $\xi < \eta < \omega_1$;
2. $E_\eta \subseteq_* E_\xi$, for every $\xi < \eta < \omega_1$;
3. $E_\xi \subseteq \psi^{-1}[A^*]$, for all $\xi < \omega_1$;
4. either $T^*(\delta_\eta)(D_\xi^*) > a_\xi > 0$ for all $y \in E_{\xi+1}^*$, or $T^*(\delta_y)(D_\xi^*) < a_\xi < 0$ for all $y \in E_{\xi+1}^*$.

Let $A_0 = \tilde{A}$ and $E_0^* = \psi^{-1}[A_0^*]$. Let $\eta < \omega_1$ and suppose we have constructed $A_\xi$, $D_\xi$, $E_\xi$ and $a_\xi$ satisfying (1)--(4) for every $\xi < \eta$. If $\eta$ is a limit ordinal, take an infinite $E$ such that $E \subseteq_* E_\xi$ for every $\xi < \eta$. By hypothesis there exists a clopen $A_\eta^* \subseteq \psi[E]^*$. Put $E_\eta^* = \psi^{-1}[A_\eta^*] \cap E^*$. Now we may suppose we have $A_\xi$ and $E_\xi$ for every $\xi \leq \eta$, and $D_\xi$ and $a_\xi$ for every $\xi < \eta$.

Take an infinite $A_\eta^*$ such that $(A_\eta^*)^* \subseteq \psi[E_\eta^*]$. By our assumption, there exist $y \in \psi^{-1}[(A_\eta^*)^* \cap E_\eta^*]$ and $X \subseteq (A_\eta^*)^* \setminus \{\psi(y)\}$ such that $T^*(\delta_y)(X) \neq 0$. By the regularity of $T^*(\delta_\eta)$, there exists an infinite $D_\eta \subseteq_* A_\eta^*$ such that $\psi(y) \notin D_\eta^*$ and $T^*(\delta_y)(D_\eta^*) \neq 0$. Let $a_\eta$ be such that either $0 < a_\eta < T^*(\delta_y)(D_\eta^*)$ or $0 > a_\eta > T^*(\delta_y)(D_\eta^*)$.

By the weak* continuity of $T^*$, there exists a clopen neighbourhood of $y$ such that either $T^*(\delta_z)(D_\eta^*) > a_\eta$ for all $z \in V$, or $T^*(\delta_z)(D_\eta^*) < a_\eta$ for all $z \in V$. Finally, choose $A_{\eta+1} = A_\eta^* \setminus D_\eta$ and $E_{\eta+1}^* = \psi^{-1}[A_\eta^*] \cap V \cap E_\eta^*$ (notice that $y \in E_{\eta+1}^*$). This ends the construction.
Since \(|a_{\xi}| > 0\) for every \(\xi < \omega_1\), there must exist \(n \in \mathbb{N}\) and an infinite \(I \subseteq \omega_1\) such that \(|a_{\xi}| > 1/n\) for every \(\xi \in I\). Hence, we may choose \(\xi_0, \ldots, \xi_{k-1} \in I\), for some \(k \in \mathbb{N}\), such that \(a_{\xi_0}, \ldots, a_{\xi_{k-1}}\) are all of the same sign and \(\sum_{i<k} a_{\xi_i} > \|T^*\|\).

Assume \(\xi_0 > \xi_i\) for \(i < k\). Take \(y \in E_{\xi_0+1}^*\). Then, since the \(D_{\xi_i}^*\) are pairwise disjoint and since \(y \in E_{\xi_i+1}^*\), for every \(i < k\), we have

\[
|T^*(\delta_y)(\bigcup_{i<k} D_{\xi_i}^*)| = \left| \sum_{i<k} T^*(\delta_y)(D_{\xi_i}^*) \right| > \left| \sum_{i<k} a_{\xi_i} \right| > \|T^*\|.
\]

This contradiction proves the claim.

Therefore, for every \(A\)-supported \(f \in \ell_\infty\) and every \(y \in E^*\) we have \(T(f)(y) = T^*(\delta_y)(f) = r_y f(\psi(y))\). In particular, \(T(\chi_{A^*})(y) = r_y\), for every \(y \in E^*\). This means that the function \(y \mapsto r_y\) with domain \(E^*\) is continuous. Then, by \textbf{8.2} it must be constant on some clopen \(B^* \subseteq E^*\). This means that for some \(r \in \mathbb{R}\) we have \(T_{B,A}(f) = r(f \circ \psi)|B\), for every \(A\)-supported \(f \in \ell_\infty\). By going to a subset of \(A\) we may choose \(A\) and \(B\) so that \(\psi^{-1}[A^*] = B^*\)

\[ \square \]

**Theorem 5.2.** Let \(T : C(\mathbb{N}^+) \to C(\mathbb{N}^+ )\) be a bounded linear operator and let \(A \subseteq \mathbb{N}\) be infinite and \(F \subseteq \mathbb{N}^+\) be closed. If \(\psi : F \to \mathbb{N}^+\) is an irreducible continuous function, then there exist \(r \in \mathbb{R}\) and an infinite \(A \subseteq \hat{A}\) such that \(T(f^*)|\psi^{-1}[A^*] = r(f^* \circ \psi)|\psi^{-1}[A^*]\), for every \(A\)-supported \(f \in \ell_\infty\).

**Proof.** The proof is similar to that of \textbf{5.1} so we will skip identical parts. The main difference is that nonempty \(G_\delta\)'s of \(F\) do not need to have nonempty interior. However the irreducibility of the map onto \(\mathbb{N}^+\) gives through Lemma \textbf{8.3} that appropriate \(G_\delta\)'s have nonempty interior. Fix \(\hat{A}, F\) and \(\psi\) as above.

**Claim:** There exists an infinite \(A \subseteq \hat{A}\) such that for every \(y \in \psi^{-1}[A^*]\) there exists \(r_y \in \mathbb{R}\) satisfying

\[
T^*(\delta_y)|A^* = r_y \delta_{\psi(y)}.
\]

Suppose this does not hold. We will construct recursively sequences \((A_\xi)_{\xi < \omega_1}\) and \((D_\xi)_{\xi < \omega_1}\) of infinite subsets of \(\mathbb{N}\), and a sequence \((a_\xi)_{\xi < \omega_1}\) of nonzero reals such that

1. \(A_0 \subseteq A_\xi \subseteq \hat{A}\) and \(D_\xi \subseteq A_\xi \setminus A_{\xi+1}\), for every \(\xi < \eta < \omega_1\);
2. either \(T^*(\delta_y)(D_{\xi}^*) > a_\xi > 0\) for all \(y \in \psi^{-1}[A_{\xi+1}^*]\), or \(T^*(\delta_y)(D_{\xi}^*) < a_\xi < 0\) for all \(y \in \psi^{-1}[A_{\xi+1}^*]\).

Let \(A_0 = \hat{A}\). Let \(\eta < \omega_1\) and suppose we have constructed \(A_\xi, D_\xi\) and \(a_\xi\) satisfying \(1\)–\(2\) for every \(\xi < \eta\). If \(\eta\) is a limit ordinal, take an infinite \(A_\eta\) such that \(A_\xi \subseteq A_\eta\) for every \(\xi < \eta\). Now we may suppose we have \(A_\xi\) for every \(\xi \leq \eta\), and \(D_\xi, a_\xi\) and \(A_\xi\) for every \(\xi < \eta\).

By our assumption, there exist \(y \in \psi^{-1}[A_\eta^*]\) and \(X \subseteq A_\eta^* \setminus \{\psi(y)\}\) such that \(T^*(\delta_y)(X) \neq 0\). By the regularity of \(T^*(\delta_y)\), there exists an infinite \(D_\eta \subseteq A_\eta\) such that \(\psi(y) \not\in D_\eta^*\) and \(T^*(\delta_y)(D_\eta^*) \neq 0\). Let \(a_\eta\) be such that either \(0 < a_\eta < T^*(\delta_y)(D_\eta^*)\) or \(0 > a_\eta > T^*(\delta_y)(D_\eta^*)\).

By the weak* continuity of \(T^*\), there exists \(V\) a clopen neighbourhood of \(y \in F\) such that either \(T^*(\delta_z)(D_\eta^*) > a_\eta\) for all \(z \in V\), or \(T^*(\delta_z)(D_\eta^*) < a_\eta\) for all \(z \in V\). \(V\) may be assumed to be included in \(\psi^{-1}[A_\eta^* \setminus D_\eta^*]\). By the irreducibility of \(\psi\) and Lemma \textbf{8.3} there is an infinite \(A_{\eta+1} \subseteq \mathbb{N}\) such that \(\psi^{-1}[A_{\eta+1}^*] \subseteq V \subseteq \psi^{-1}[A_\eta^* \setminus D_\eta^*]\). In particular \(A_{\eta+1} \subseteq A_\eta \setminus D_\eta\) (note that \(y\) may
not belong to $\psi^{-1}[A^{*}_{l+1}]$). This ends the construction. We finish the proof of the claim as in Theorem 5.1.

Therefore, for every $A$-supported $f \in \ell_\infty$ and every $y \in \psi^{-1}[A^*]$ we have $T(f^*)(y) = T^*(\delta_y)(f) = r_y f(\psi(y))$. In particular, $T(\chi_{A^*})(y) = r_y$, for every $y \in \psi^{-1}[A^*]$. This means that the function $y \mapsto r_y$ with domain $\psi^{-1}[A^*]$ is continuous. Then, by [8.7] it must be constant on some clopen set of $F$ of the form $\psi^{-1}[B^*]$ for an infinite $B \subseteq A$. This means that for some $r \in \mathbb{R}$ we have $T(f) = r(f \circ \psi)|\psi^{-1}[B]$, for every $B$-supported $f \in \ell_\infty$.

\[\Box\]

5.1. Left-local canonization of fountainless operators.

**Lemma 5.3.** Suppose that $B \subseteq \mathbb{N}$ is infinite, $T : C(\mathbb{N}^*) \rightarrow C(\mathbb{N}^*)$ is fountainless and everywhere present. Then,

$$ F = \bigcup \{ \varphi^T(y) : y \in B^* \} $$

has nonempty interior.

**Proof.** Suppose $F$ is nowhere dense. Take $B_1 \subseteq B$ and $C_n$, $D_n \subseteq \mathbb{N}$ as in [3.5]. In view of applying [3.1] we will find a dense family $A$ such that $T(f^*)|_{B^*_1} = 0$ whenever the support of $f$ is included in an element of $A$. This would contradict the fact that $T$ is everywhere present.

Fix a nonempty clopen $U \subseteq \mathbb{N}^*$ disjoint from $F$. Notice that for every $y \in B^*_1$ the measure $T^*(\delta_y)$ has no atoms in $U$. Therefore, by the regularity of $T^*(\delta_y)$ and the compactness of $U$ we may find for each $n \in \mathbb{N}$ an open covering $(U_i(y,n))_{i < j(y,n)}$ of $U$ by clopen sets such that $|T^*(\delta_y)|(U_i(y,n)) < 1/(2n+1)$ holds for all $i < j(y,n)$. We may further assume that either $U_i(y,n) \subseteq C_n^*$ or $U_i(y,n) \subseteq D_n^*$, for each $n \in \mathbb{N}$ and each $i < j(y,n)$.

Now by the weak* continuity of $T^*$, we may choose for each $n \in \mathbb{N}$ an open neighbourhood $V_n(y)$ of $y$ such that for all $z \in V_n(y)$ we have

$$ |T^*(\delta_z)(U_i(y,n))| < 1/(2n+1) $$

for all $i < j(y,n)$.

We have thus constructed for each $n \in \mathbb{N}$ an open covering $\{V_n(y) : y \in B^*_1\}$ of $B^*_1$. By the compactness of $B^*_1$, for each $n \in \mathbb{N}$ take $y_0(n),...,y_{m(n)-1}(n) \in B^*_1$ such that

$$ B^*_1 \subseteq \bigcup_{l < m(n)} V_n(y_l). $$

Now consider the family $A_U$ of those sets $E \subseteq \mathbb{N}$ such that given $n \in \mathbb{N}$, for each $l < m(n)$ there is $i < j(y_l,n)$ such that $E^*$ is included in $U_i(y_l,n)$. We claim that if $E \in A_U$, then for every $E^*$- supported $f^* \in C(\mathbb{N}^*)$ we have that $T(f^*)|_{B^*_1} = 0$.

Fix $E \in A_U$ and $y \in B^*_1$. We show that for every $n \in \mathbb{N}$ we have $|T^*(\delta_y)|(E^*) < 1/(n+1)$. Let $T^*(\delta_y) = \mu^+ - \mu^-$ be a Jordan decomposition of the measure. By [3.5] we have that $\mu^-(C_n^*) < 1/4(n+1)$ and $\mu^+(D_n^*) < 1/4(n+1)$. By construction there exists $l < m(n)$ such that $y \in V_n(y_l)$, and by the definition of $A_U$, there exists $i < j(y_l,n)$ such that $E^* \subseteq U_i(y_l,n)$. We may assume without loss of generality
that \( U_i(y_l, n) \subseteq C^*_n \). From this and from \((5.1)\) above we obtain:

\[
|T^*(\delta_y)|(E^*) \leq |T^*(\delta_y)|(U_i(y_l, n)) = T^*(\delta_y)(U_i(y_l, n) + 2\mu^-(U_i(y_l, n)) \leq |T^*(\delta_y)|(U_i(y_l, n))) + 2\mu^-(C^*_n) < 1/(2(n + 1) + 2/4(n + 1))
\]

Therefore, \(|T^*(\delta_y)|(E^*) = 0\) for every \( y \in B^*_1 \). So if the support of \( f^* \in C(\mathbb{N}^*)\) is included in \( E^* \), we have that \( 0 = \int f^* dT^*(\delta_y) = T^*(\delta_y)(f^*) = T(f^*)(y) \), for every \( y \in B^*_1 \), as claimed.

Now, notice that it is enough to show that \( \mathcal{A}_U \) is dense under \( U \), as in that case \( \mathcal{A} = \bigcup \{ \mathcal{A}_U : U \subseteq \mathbb{N}^* \setminus F, U \text{ clopen} \} \) is the dense family we are after.

To see that \( \mathcal{A}_U \) is dense under \( U \), fix an infinite \( E_{0,0} \subseteq \mathbb{N} \) such that \( E^*_{0,0} \subseteq U \). We define by induction a \( \subseteq \) decreasing sequence of infinite sets \( \{ E_{n,l} : n \in \mathbb{N}, l < m(n) \} \) ordered lexicographically. Suppose \( (n', l') \) is successor of \( (n, l) \). Since \( (U_i(y_l, n))_{l < j(y_l, n)} \) is an open covering of \( U \), we may choose \( i < j(y_l, n) \) such that \( E_{n,l} \cap U_i(y_l, n) \neq \emptyset \). Take \( y \neq E_{n',l'} \subseteq E_{n,l} \cap U_i(y_l, n) \). Finally, if we take an infinite \( E \) such that \( E \subseteq E_{n,l} \) for every \( n \in \mathbb{N}, l < m(n) \), then it is clear that \( E \subseteq E_{0,0} \) and \( E \in \mathcal{A}_U \).

\[ \square \]

**Lemma 5.4.** Let \( B \subseteq \mathbb{N} \) be infinite. Suppose \( T : C(\mathbb{N}^*) \rightarrow C(\mathbb{N}^*) \) is such that \( \varphi^T(y) \neq \emptyset \) for each \( y \in B^* \) and \( \bigcup \{ \varphi^T(y) : y \in \mathbb{N} \} \) has nonempty interior for every open \( V \subseteq \mathbb{N}^* \). Then, there is an infinite \( B_1 \subseteq B \) and \( \varepsilon > 0 \) such that

1. \( \varphi^T_{1/\varepsilon}(y) \neq \emptyset \) for each \( y \in B_1^* \), and
2. \( \bigcup \{ \varphi^T_{1/\varepsilon}(y) : y \in D^* \} \) has nonempty interior for every infinite \( D \subseteq B_1 \).

**Proof.** Suppose that (1) fails for all infinite \( B' \subseteq B \) and all \( \varepsilon > 0 \). Let \( B_0 \subseteq B \), \( C_n, D_n \subseteq \mathbb{N} \) be given by Lemma 3.3. We will construct by induction a \( \subseteq \) decreasing sequence \( (B_n)_{n \in \mathbb{N}} \) of infinite subsets of \( \mathbb{N} \) such that \( \varphi^T_{1/(n+1)}(y) = \emptyset \) for every \( y \in B^*_n \). If we then take any \( y \in \bigcap_{n \in \mathbb{N}} B^*_n \), we will have that \( \varphi^T(y) = \emptyset \), which contradicts our hypothesis.

Assume we have already constructed \( B_n \). Since we are assuming that (1) fails, there exists \( y \in B^*_n \) such that \( \varphi^T_{1/(n+1)}(y) = \emptyset \). This means that \( |T^*(\delta_y)|(\{ y \}) = |T^*(\delta_y)(\{ x \}| < 1/(2(n + 1)) \), for every \( x \in \mathbb{N}^* \).

By the regularity of the measure \( T^*(\delta_y) \) and by the compactness of \( \mathbb{N}^* \), we may cover \( \mathbb{N}^* \) by finitely many clopen \( (U_i)_{i < k} \) such that \( |T^*(\delta_y)(U_i) < 1/2(n + 1) \), for each \( i < k \). We may further assume that each \( U_i \) is included in either \( C^*_n \) or \( D^*_n \). Since \( T^* \) is weak* continuous, we may find an open neighbourhood of \( y \), say \( V \), such that for every \( z \in V \) we have \( |T^*(\delta_z)(U_i)| < 1/2(n + 1) \), for each \( i < k \). Take \( B_{n+1} \) such that \( y \in B^*_{n+1} \subseteq B^*_n \cap \mathbb{N} \). We claim that \( \varphi^T_{1/(n+1)}(z) = \emptyset \), for every \( z \in B^*_{n+1} \).

Fix any \( z \in B^*_{n+1} \) and take \( T^*(\delta_z) = \mu^+ - \mu^- \) a Jordan decomposition of the measure. Recall that by Lemma 5.3 we have that \( \mu^-(C^*_n) < 1/(4(n + 1)) \) and \( \mu^+(D^*_n) < 1/(4(n + 1)) \). Now take any \( x \in \mathbb{N}^* \). Let \( i < k \) be such that \( x \in U_i \) and assume without loss of generality that \( U_i \subseteq C^*_n \). Then,

\[
|T^*(\delta_z)(\{ x \})| \leq |T^*(\delta_z)(U_i)| \leq |T^*(\delta_z)(U_i)| + 2\mu^-(C^*_n) < 1/(n + 1),
\]

and the claim is proved. This finishes the proof of the first part, so let us assume that \( \varepsilon_0 > 0 \) and \( B_0 \subseteq B \) are such that (1) holds.
To prove the second part, let us assume that for every $B' \subseteq B_0$ and every $\varepsilon > 0$ there exists an infinite $D \subseteq B_0$ with $\bigcup \{ \varphi^T_\varepsilon(y) : y \in D^* \}$ nowhere dense. We may then find an $\subseteq^*$-descending sequence of infinite sets $(D_n)_{n \in \mathbb{N}}$ such that $D_n \subseteq B$ and $\bigcup \{ \varphi^T_{1/n}(y) : y \in D_n^* \}$ is nowhere dense, for every $n \in \mathbb{N}$. Let $V \subseteq \bigcap_{n \in \mathbb{N}} D_n^*$ be a nonempty open. Then, since $\varphi^T(y) = \bigcup_{n \in \mathbb{N}} \varphi^T_{1/n}(y)$, we have

$$\bigcup \{ \varphi^T(y) : y \in V \} = \bigcup_{n \in \mathbb{N}} \bigcup \{ \varphi^T_{1/n}(y) : y \in V \} \subseteq \bigcup_{n \in \mathbb{N}} \bigcup \{ \varphi^T_{1/n}(y) : y \in D_n^* \},$$

which is nowhere dense by $\blacksquare$. This contradicts the hypothesis of the lemma. Therefore, there exist an infinite $B_1 \subseteq B_0$ and $\varepsilon_1 > 0$ which satisfy (2).

The lemma holds for $B_1$ and $\varepsilon = \min \{ \varepsilon_0, \varepsilon_1 \}$.

$\blacksquare$

**Lemma 5.5.** Suppose that $T : C(\mathbb{N}^*) \to C(\mathbb{N}^*)$ is fountainless and everywhere present. Then, for every infinite $B \subseteq \mathbb{N}$ there is an infinite $B_1 \subseteq B$ and a continuous quasi-open $\psi : B_1^* \to \mathbb{N}^*$ such that

$$T^*(\delta_y)(\{\psi(y)\}) \neq 0$$

for all $y \in B_1$.

**Proof.** Since $T$ is fountainless and everywhere present, by $\blacksquare$ and $\blacksquare$ we know that the hypothesis of $\blacksquare$ are satisfied. So find $\varepsilon > 0$ and an infinite $B_0 \subseteq B$ such that $\varphi^T_\varepsilon(y) \neq \emptyset$ for every $y \in B_0$, and $\bigcup \{ \varphi^T_\varepsilon(y) : y \in D^* \}$ has nonempty interior for every infinite $D \subseteq B_0$. We may also assume that there exist $C_n, D_n \subseteq \mathbb{N}$ for every $n \in \mathbb{N}$ such that the statement in $\blacksquare$ holds for $B_0$ and $C_n, D_n$.

**Claim 1:** There exists an infinite $B_0^* \subseteq B_0$ and a finite collection $(V_i)_{i < k}$ of almost disjoint infinite subsets of $\mathbb{N}$ such that for every $z \in (B_0^*)^*$ we have that $\varphi^T_\varepsilon(z) \subseteq \bigcup_{i < k} V_i^*$ and $|\varphi^T_\varepsilon(z) \cap V_i^*| = 1$, for each $i < k$.

We will construct recursively a $\subseteq^*$-descending sequence $(A_n)$ of subsets of $\mathbb{N}$, $y_n \in A_n^*$, finite collections $(V_{n,i})_{i < k_n}$ of almost disjoint infinite subsets of $\mathbb{N}$ and open intervals $I_n \subseteq \mathbb{R}$, $i < k_n$, such that for every $n$ we have

1. $\varphi^T_\varepsilon(z) \subseteq \bigcup_{i < k_n} V_{n,i}^*$, for all $z \in A_n^* + 1$
2. For every $i < k_n$ there exists $j < k_n$ such that $V_{n+1,i} \subseteq V_n,j$
3. $|T^*(\delta_z)(V_{n,i})| \subseteq I_n^*$, for all $z \in A_n^* + 1$ and all $i < k_n$
4. Length$(I_n^*) = \varepsilon(2^{n+1}k_n)^{-1}$, for all $i < k_n$

Begin by noticing that for every $y \in \mathbb{N}^*$ the number of elements of $\varphi^T_\varepsilon(y)$ is finite, as it must be bounded by $|T|/\varepsilon$. Let $A_0 = B_0$ and fix any $y_0 \in A_0^*$ and let $(x_0^i : i < k_0)$ be an enumeration of $\varphi^T_\varepsilon(y_0)$ (note that $k_0 \geq 1$). Let $N_0 \in \mathbb{N}$ be such that $1/N_0 < \varepsilon/8k_0$. By the regularity of the measure $T^*(\delta_{y_0})$, we find for each $i < k_0$ a clopen neighbourhood $V_{0,i}^*$ of $x_0^i$ such that

$$|T^*(\delta_{y_0})(V_{0,i}^*)| < |T^*(\delta_{y_0})(\{x_0^i\})| + \varepsilon/8k_0.$$

We may assume that the $V_{0,i}$’s are almost disjoint and that each of them is almost included in either $C_{N_0}$ or $D_{N_0}$.

For each $i < k_0$ we define $I_0^i \subseteq \mathbb{R}$ to be the open interval with centre $|T^*(\delta_{y_0})(\{x_0^i\})|$ and radius $\varepsilon/4k_0$. By the above, $|T^*(\delta_{y_0})(V_{0,i}^*)| \subseteq I_0^i$, and as we shall see, $|T^*(\delta_{y_0})(V_{0,i}^*)|$ does so as well. Indeed, take $i < k_0$ and assume without loss of generality that $V_{0,i}^* \subseteq C_{N_0}$. If $T^*(\delta_{y_0}) = \mu^* - \mu^-$ is a Jordan decomposition of the measure, then

$$|T^*(\delta_{y_0})(V_{0,i}^*)| \leq T^*(\delta_{y_0})(V_{0,i}^*) + 2\mu^-(V_{0,i}^*) \leq |T^*(\delta_{y_0})(V_{0,i}^*)| + 2\mu^-(C_{N_0}^*).$$
From this it follows that \(|T^*(\delta_{y_n})|(V_{0,i}^*) - |T^*(\delta_{y_n})(V_{0,i}^*)| < 1/2(N_0 + 1) < \varepsilon/8k_0,
and so \(|T^*(\delta_{y_n})(V_{0,i}^*)| \in I_i^0.

By the upper semicontinuity of \(\varphi^T_\varepsilon\) (Lemma 3.11) and the weak*-continuity of \(T^*\), we now find a clopen neighbourhood of \(y_0\), say \(A^*_1\), which we may assume to be included in \(A^*_0\), such that for every \(z \in A^*_1\) we have

\[
\varphi^T_\varepsilon(z) \subseteq \bigcup_{i<k_0} V_{0,i}^*
\text{ and } \left|T^*(\delta_i)(V_{0,i}^*)\right| \in I_i^0, \text{ for each } i \leq k_0.
\]

If we have that \(|\varphi^T_\varepsilon(z) \cap V_{0,i}^*| = 1\), for every \(z \in A^*_1\) and for each \(i \leq k_0\), then the recursion stops and the claim is proved. Otherwise, choose \(y_i \in A^*_1\) a witness to this fact, and repeat the procedure to obtain open intervals \(I_i^1\) with centre \(|T^*(\delta_{y_n})(\{x_i^0\})|\) and radius \(\varepsilon/2k_1\), and clopen sets \(V_{1,i}^*\) such that both \(|T^*(\delta_{y_n})(V_{1,i}^*)|\) and \(|T^*(\delta_{y_n})(\{x_i^0\})|\) lie inside \(I_i^1\), for each \(i < k_1\). Notice that we may take each set \(V_{1,i}\) as a subset of one of the \(V_{0,j}\). Then, by the same argument using the upper semicontinuity of \(\varphi^T_\varepsilon\) and the weak*-continuity of \(T^*\) we obtain an infinite \(A^*_2 \subseteq A^*_1\) such that for every \(z \in A^*_2\) we have \(|\varphi^T_\varepsilon(z) \cap V_{0,i}^*| = 1\) and \(|T^*(\delta_i)(V_{0,i}^*)| \in I_i^1\), for each \(i < k_1\).

We claim that this process stops after finitely many steps. First notice that the failure to stop at step \(n\) is due to one of two reasons:

(a) there exists \(y_{n+1} \in A^*_n\) such that \(|\varphi^T_\varepsilon(y_{n+1}) \cap V_{0,i}^*| \geq 2\) for some \(i < k_n\)

(b) there exists \(y_{n+1} \in A^*_n\) such that \(|\varphi^T_\varepsilon(y_{n+1}) \cap V_{0,i}^*| = 0\) for some \(i < k_n\)

Notice also that once condition (a) fails, it continues to fail in subsequent steps. So we may assume that we first only check for condition (a), and only after it does not occur do we check for condition (b).

By condition (2) in the construction, we have that every time (a) occurs there exists \(i < k_0\) such that \(|\varphi^T_\varepsilon(y_{n+1}) \cap V_{0,i}^*| \geq 2\). So for each \(i < k_0\) consider \(m_i \in \mathbb{N}\) such that \(m_i \cdot \varepsilon \geq T^*(\delta_{y_n})(\{x_i^0\})\) and suppose (a) has occurred at \(n = \sum_{i < k_0} m_i\) many steps. Suppose that still (a) happens once more. Then, there exists \(i_0 < k_0\) and \(n_0 < \cdots < n_{m_{i_0}} = n\) such that \(|\varphi^T_\varepsilon(y_{n_{i_0}}) \cap V_{0,i_{0}}^*| \geq 2\), for every \(j \leq m_{i_0}\). Hence, if \(x_{0,i}^{n+1}, x_{1,i}^{n+1} \in \varphi^T_\varepsilon(y_{n+1}) \cap V_{0,i}^*\), for certain \(i_0 < k_n\), then \(|T^*(\delta_{y_{n+i}})(\{x_0^{n+1}\}) + \varepsilon \leq |T^*(\delta_{y_{n+i}})(\{x_0^{n+1}, x_1^{n+1}\})| \leq |T^*(\delta_{y_{n+1}})(V_{0,i_0}^*)|\) and we obtain

\[
|T^*(\delta_{y_{n+1}})(\{x_0^{n+1}\})| < \sup \frac{T^*(\delta_{y_{n+i}})}{\varepsilon} - \varepsilon < \frac{|T^*(\delta_{y_{n+i}})(\{x_0^{n-1}\}) + \varepsilon(2^{n+2}k_0)^{-1} - \varepsilon}{\varepsilon} < \frac{|T^*(\delta_{y_{n+i}})(\{x_0^{n-1}\}) + \varepsilon(2^{n+2}k_0)^{-1} - \varepsilon} {\varepsilon} \leq |T^*(\delta_{y_{n+i}})(\{x_0^{n-1}\})| - m_{i_0} \cdot \varepsilon + (\varepsilon \sum_{j < n+1}(2^{n+2}k_j)^{-1} - \varepsilon) < 0 - \varepsilon/2.
\]

From this contradiction we conclude that (a) can occur at most at \((\sum_{i < k_0} m_i)\) many steps.

Now assume that \(n_0\) is such that (a) does not hold at step \(n\), for all \(n \geq n_0\). Suppose the recursion does not stop at step \(n \geq n_0\). Assume without loss of generality that \(\varphi^T_\varepsilon(y_{n+1}) \cap V_{0,0}^* = \emptyset\). Therefore \(\varphi^T_\varepsilon(y_{n+1}) \subseteq \bigcup_{0 < i < k_n} V_{1,i}^*\). Since we
also have \([T^*(\delta_{y,n+1})](\{x^i_{n+1} : i < k_{n+1}\}) \in I^*_n\) for each \(i < k_n\), we obtain
\[
|T^*(\delta_{y,n+1})|(\{x^i_{n+1} : i < k_{n+1}\}) \leq \sum_{i < k_n} |T^*(\delta_{y,n+1})|(\{x^i_{n+1}\}) - |T^*(\delta_{y,n+1})|(\{x^0_{n,0}\}) \\
< \sum_{i < k_n} \sup I^n_i - \inf I^n_0 \\
\leq \sum_{i < k_n} |T^*(\delta_{y,n})|(\{x^n_i\}) + k_0 \varepsilon/(2^{n+2}k_0) \\
- (\varepsilon - \varepsilon/2k_0) \\
\leq |T^*(\delta_{y,n})|(\{x^n_i : i < k_n\}) - \varepsilon/2.
\]
The last inequality holds because \(k_n \geq 1\). Since \(|T^*(\delta_{y,n})|(\{x^n_i : i < k_n\}) \geq \varepsilon\), for every \(n\), we conclude that (b) cannot occur indefinitely. Hence the recursion must stop after finitely many steps.

**Claim 2:** There exists \(t_0 < k\) and an infinite \(B_1 \subseteq B_0^*\) such that \(V_{i_0}^* \cap \bigcup \{\phi^T_\varepsilon(y) : y \in D^*\}\) has nonempty interior for every infinite \(D \subseteq B_1\).

Suppose this is not the case. Then, we may find a sequence on infinite sets \(B_0^* = A_0 \supseteq A_1 \supseteq \cdots \supseteq A_k\) such that \(V_{i}^* \cap \bigcup \{\phi^T_\varepsilon(y) : y \in A_{i+1}^*\}\) is nowhere dense, for \(i < k\). By Claim 1, we know that \(\bigcup \{\phi^T_\varepsilon(y) : y \in A_k^*\} \subseteq \bigcup_{i < k} V_{i}^*\), and so \(\bigcup \{\phi^T_\varepsilon(y) : y \in A_k^*\} \subseteq \bigcup_{i < k} V_{i}^* \cap \bigcup \{\phi^T_\varepsilon(y) : y \in A_k^*\}\) is also nowhere dense. But this contradicts the choice of \(B_0\).

Now we may define \(\psi : B_1^* \rightarrow \mathbb{N}^*\) by \(\psi(y) = V_{i_0}^* \cap \phi^T_\varepsilon(y)\). It is clear that \(\psi\) is quasi-open by Claim 2, so we conclude the proof by showing that \(\psi\) is continuous. Let \(U \subseteq V_{i_0}^*\) be any open set. Since \(V_{i_0}^*\) is clopen and \(\phi^T_\varepsilon\) is upper semicontinuous, we have that \(\psi^{-1}[U] = B_1^* \cap \{y \in \mathbb{N}^* : \phi^T_\varepsilon(y) \subseteq U \cup \mathbb{N}^* \setminus V_{i_0}^*\}\) is open.

**Theorem 5.6.** Suppose that \(T : C(\mathbb{N}^*) \rightarrow C(\mathbb{N}^*)\) is a fountainless, everywhere present operator. Then, \(T\) is left-locally canonizable along a quasi-open map.

**Proof.** Fix an infinite \(B \subseteq \mathbb{N}\) and find using [5.4] an infinite \(B_1 \subseteq B\) and a quasi-open \(\psi : B_1^* \rightarrow \mathbb{N}^*\) such that
\[
T^*(\delta_\psi)(\{\psi(y)\}) \neq 0
\]
for all \(y \in B_1^*\). Now use [5.7] to find clopen sets \(A^* \subseteq \psi[B_1^*]\) and \(B_2^* \subseteq \psi^{-1}[A^*]\) and a real \(r \in \mathbb{R}\) such that \(T_{B_2,A}(f^*) = r(f^* \circ \psi)|B_2\) for every \(f \in \ell_\infty(A)\). It follows that \(T^*(\delta_\psi)|A^* = r\delta_\psi(y)\) for each \(y \in B_2^*\), and so \(0 \neq T^*(\delta_\psi)(\{\psi(y)\}) = r\). Hence \(T_{B_2,A}\) is canonizable along \(\psi\). □

### 5.2. Right-local canonization of funnelless automorphisms.

The main result of this section is a consequence of our generalization [5.1] of the Drewnowski-Roberts canonization lemma and the following result of Plebanek which is implicitly proved in Theorem 6.1 of [59].

**Theorem 5.7.** Suppose that \(T : C(K) \rightarrow C(K)\) is an automorphism. Then, there is a \(\pi\)-base \(\mathcal{U}\) of \(\mathbb{N}^*\) such that for every \(U \in \mathcal{U}\) there is a closed \(F \subseteq \mathbb{N}^*\) and a continuous surjection \(\psi : F \rightarrow \mathbb{U}\) such that
\[
|T^*(\delta_\psi)|\{\psi(y)\}) \neq 0
\]
for all \(y \in F\).

**Theorem 5.8.** Suppose that \(T : C(\mathbb{N}^*) \rightarrow C(\mathbb{N}^*)\) is a funnelless automorphism. Then, \(T\) is right-locally canonizable along a quasi-open map.

**Proof.** Fix an infinite \(A \subseteq \mathbb{N}\). Let \(\mathcal{U}\) be as in [5.7] for \(T\) and find \(U \in \mathcal{U}\) such that \(U \subseteq A^*\). Let \(F\) and \(\psi\) be as in [5.1] for \(U\). Let \(A_1 \subseteq \mathbb{N}\) be infinite such that \(A_1^* \subseteq U\).
and put $F_1 = \psi^{-1}[A^*_1] \subseteq F$. Let $F_2 \subseteq F_1$ be closed such that $\psi|F_2$ is irreducible and onto $A^*_1$ and hence quasi-open (relative to the subspace topology on $F_2$) by lemma 3.6.

As $T$ is funnelless (Definition 5.18) and $A^*_1$ is open, $F_2$ cannot be nowhere dense, so let $B^*$ be a nonempty clopen set included in $F_2$. Now $\psi : B^* \to A^*_1$ is a continuous, quasi-open map (as a restriction of a quasi-open map to a clopen subset of $F_2$) satisfying $T^*(\delta_y)(\{\psi(y)\}) \neq 0$ for each $y$ in $B^*$. Therefore, we can apply 6.1 to obtain an infinite $A_2 \subseteq A_1$, a clopen $B^*_1 \subseteq \psi^{-1}[A^*_2]$ and a real $r \in \mathbb{R}$ such that $T_{B_1,A_2}(f^*) = r(f^* \circ \psi)|B^*_1$, for every $f \in \ell_\infty(A_2)$. In particular we have that

$$T^*(\delta_y)(E^*) = T_{B_1,A_2}(\chi_{E^*})(y) = r(\chi_{E^*} \circ \psi)(y) = r\delta_{\psi(y)}(E^*)$$

for every infinite $E \subseteq A_2$ and every $y \in B^*_1$. It follows that for each $y \in B^*_1$ we have $T^*(\delta_y)|A^*_2 = r\delta_{\psi(y)}$, and so $0 \neq T^*(\delta_y)(\{\psi(y)\}) = r$. Having $r \neq 0$, we conclude that $T_{B_1,A_2}$ is canonizable along $\psi$.

6. The impact of combinatorics on the canonization and trivialization of operators on $\ell_\infty/c_0$

As expected based on the study of $\mathbb{N}^*$ (e.g., [15], [47], [50], [10], [15], [23]), the impact of additional set-theoretic assumptions on the structure of operators on $\ell_\infty/c_0$ is also very dramatic.

6.1. Canonization and trivialization of operators on $\ell_\infty/c_0$ under OCA+MA.

Recall from [23] that an ideal $\mathcal{I}$ of subsets of $\mathbb{N}$ is called c.c.c. over Fin if, and only if, there are no uncountable almost disjoint families of $\mathcal{I}$-positive sets. Dually a closed subset $F \subseteq \mathbb{N}^*$ is called c.c.c. over Fin if $A^*_\xi \cap F = \emptyset$ for some $\xi < \omega_1$ whenever $\{A^*_\xi : \xi < \omega_1\}$ is an almost disjoint family of infinite subsets of $\mathbb{N}$. The following theorem by I. Farah (3.3.3. and 3.8.1. from [23]) will be crucial in this subsection:

**Theorem 6.1 (OCA+MA ([23])).** Let $h : \varphi(\mathbb{N})/\text{Fin} \to \varphi(\mathbb{N})/\text{Fin}$ be a homomorphism. Then, there is an infinite $B \subseteq \mathbb{N}$, a function $\sigma : B \to \mathbb{N}$ and a homomorphism $h_2 : \varphi(\mathbb{N})/\text{Fin} \to \varphi(\mathbb{N}\setminus B)/\text{Fin}$ such that $h(A) = [\sigma^{-1}[A]] \cup h_2([A])$ for every $A \subseteq \mathbb{N}$, and $\text{Ker}(h_2)$ is c.c.c. over Fin.

The following is a topological reformulation of the above theorem:

**Theorem 6.2 (Proposition 7, [10](OCA+MA)).** Suppose $\psi : \mathbb{N}^* \to \mathbb{N}^*$ is a continuous mapping. Then, there exist an infinite $B \subseteq \mathbb{N}$ and a function $\sigma : B \to \mathbb{N}$ such that

$$\psi(x) = \sigma^*(x)$$

for all $x \in B^*$ and $F = \psi([\mathbb{N}\setminus B]^*)$ is a nowhere dense closed c.c.c. over Fin set.

**Proof.** By the Stone duality every continuous $\psi : \mathbb{N}^* \to \mathbb{N}^*$ corresponds to a homomorphism $h : \varphi(\mathbb{N})/\text{Fin} \to \varphi(\mathbb{N})/\text{Fin}$ given by $h([A]) = [D]$, where $D^* = \psi^{-1}[A^*]$, for every $A \subseteq \mathbb{N}$. Let $B \subseteq \mathbb{N}$, $\sigma : B \to \mathbb{N}$ and $h_2$ be as in 6.1 for the homomorphism $h$. For every infinite $A \subseteq \mathbb{N}$ we have $h([A]) \cap [B] = [\sigma^{-1}[A]]$ by 6.1 and so $\psi^{-1}[A^*] \cap B^* = (\sigma^{-1}[A])^*$. Therefore, $\sigma^* = \psi|B^*$. For every infinite $A \subseteq \mathbb{N}$ we have $h([A]) \cap [\mathbb{N}\setminus B] = h_2([A])$ by 6.1 so for every $x \in (\mathbb{N}\setminus B)^*$ we have $\psi(x) = h_2^{-1}[\{A : A \in x\}]$ by the Stone duality. The set $F = \psi([\mathbb{N}\setminus B]^*)$ is closed and for every $x \in (\mathbb{N}\setminus B)^*$, the set $h_2^{-1}[\{A : A \in x\}]$ is disjoint from $\text{Ker}(h_2)$,
which is c.c.c. over Fin. Therefore, $F$ is c.c.c. over Fin and c.c.c. over Fin sets are nowhere dense. \hfill \Box

It turns out that quasi-open maps $\psi : B^* \to \mathbb{N}^*$ can be reduced to bijections between subsets of $\mathbb{N}$ assuming OCA+MA.

**Lemma 6.3 (OCA+MA).** Let $B \subseteq \mathbb{N}$ be infinite. Suppose that $\psi : B^* \to \mathbb{N}^*$ is a continuous quasi-open mapping. Then, there are infinite $B_1 \subseteq B$, $A \subseteq \mathbb{N}$ and a bijection $\sigma : B_1 \to A$ such that $\psi|B_1^* = \sigma^*$. In particular, $\psi|B_1^*$ is a homeomorphism.

**Proof.** By 6.2 there exist $B_0 \subseteq B$ and a function $\sigma : B_0 \to \mathbb{N}$ such that $\psi(x) = \sigma^*(x)$ for all $x \in B_0$.

Now, since $\psi$ is quasi-open, there exists an infinite $E \subseteq \mathbb{N}$ such that $E^* \subseteq \psi|B_0^* = \sigma^*[B_0]$. Therefore, $E \subseteq \sigma[B_0]$ and in particular the image of $\sigma$ is infinite and so there is an infinite $B_1 \subseteq B_0$ such that $\sigma|B_1$ is a bijection onto its image $A$. \hfill \Box

**Theorem 6.4 (OCA+MA).** If $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ is a funtainless everywhere present operator, then it is left-locally canonizable and so left-locally trivial. If $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ is a funnelless automorphism, then it is right-locally canonizable and so right-locally trivial.

**Proof.** Apply 6.2 and 6.3 to obtain left-local or right-local canonization along a quasi-open mapping, respectively. Now use 6.3 to conclude that this mapping is somewhere induced by a bijection. \hfill \Box

6.2. **Operators on $\ell_\infty/c_0$ under CH.** The continuum hypothesis is a strong tool allowing transfinite induction constructions in $\wp(\mathbb{N})/\text{Fin}$ which induce objects in $\ell_\infty/c_0$. Actually, a considerable part of this strength is included in a powerful consequence of Parovičenko’s theorem: if $X$ is zero-dimensional, locally compact, $\sigma$-compact, noncompact Hausdorff space of weight at most continuum, then $X^* = \beta X \setminus X$ is homeomorphic to $\mathbb{N}^*$ (1.2.6 of [33]). In this section we will often be using this result combined with the universal property of $\beta X$ for locally compact $X$, that every continuous function on $X$ into a compact space extends to $\beta X$.

**Theorem 6.5 (CH).** There is an automorphism $T : \ell_\infty/c_0 \to \ell_\infty/c_0$ which is nowhere canonizable along a quasi-open map on an open set, in particular along a homeomorphism.

**Proof.** Let $K$ be an uncountable compact zero-dimensional metrizable space with countably many isolated points $\{ x_m : m \in \mathbb{N} \}$ which form a dense open subspace of $K$. By the classical classification of separable spaces of continuous functions there is an isomorphism $S : C(2^\mathbb{N}) \to C(K)$.

Let $X = \mathbb{N} \times K$ and $Y = \mathbb{N} \times 2^\mathbb{N}$. Note that $X,Y$ satisfy the hypothesis of the topological consequence of Parovičenko’s theorem (1.2.6 of [33]) mentioned above, hence there are homeomorphisms $\pi : \mathbb{N}^* \to X^*$ and $\rho : Y^* \to \mathbb{N}^*$. Define $\hat{\tau} : K \to ||S|| B_{C(2^\mathbb{N})}$, by $\hat{\tau}(x) = S^*(\delta_x)$ for each $x \in K$, where the dual ball is considered with the weak$^*$ topology and identified with the Radon measures on $2^\mathbb{N}$. Define $\tau : X \to ||S|| B_{C(\beta Y)}$, by putting $\tau(n,x)$ to be the measure on $\beta Y$ which is zero on the complement of $\{ n \} \times 2^\mathbb{N}$ and is equal to the measure $\hat{\tau}(x)$ on $\{ n \} \times 2^\mathbb{N}$.

By the universal property of $\beta X$ there is an extension $\beta \tau : \beta X \to ||S|| B_{C(\beta Y)}$.

**Claim:** For each $t \in X^*$ the measure $\beta \tau(t)$ is concentrated on $Y^*$. 

Fix $t \in X^*$. Note that for every $n \in \mathbb{N}$ the set $\beta X \setminus \{ k \in \mathbb{N} : k \leq n \} \times K$ is a neighbourhood of $t$. Also, for $k > n$ if $x \in \{ k \} \times K$, then $\tau(x)(U) = 0$ for every Borel subset $U$ of $\{ n \} \times 2^\mathbb{N}$. This completes the proof of the claim by the weak* continuity of $\beta \tau$.

Now we can define $T : C(Y^*) \to C(X^*)$ by $T(f)(t) = \int f \, d(\beta \tau(t))$ for every $t \in X^*$. It is a well defined bounded linear operator by Theorem 1 in VI.7 of [19]. We will show that $T_{\pi} \circ T \circ T_\tau$ is an automorphism of $\mathbb{N}^*$ which is nowhere canonizable along a quasi-open map. For the former we need to prove that $T$ is an isomorphism and for the latter we need to prove that for every nonempty clopen sets $U \subseteq X^*$, $O \subseteq Y^*$ there is no quasi-open $\phi : U \to O$ such that $(\beta \tau(t))(O) = r \delta_{\phi(t)}$ for every $t$ in $U$ and some nonzero $r \in \mathbb{R}$.

To prove that $T$ is an isomorphism, note that one can define $R : C(\beta Y) \to C(\beta X)$ by $R(f)(x) = \int f \, d(\beta \tau(x))$ for every $x \in X$, and that $C(\beta Y)$ can be identified with the $\ell_\infty$-sum of $C(2^\mathbb{N})$ while $C(\beta X)$ can be identified with the $\ell_\infty$-sum of $C(K)$. $R$ sends the subspace corresponding to the $c_0$-sum of $C(2^\mathbb{N})$ into the subspace corresponding to the $c_0$-sum of $C(K)$ since the original $S$ is an isomorphism and $R$ is the $\ell_\infty$-sum of the operator $S$. It follows that $T$ is induced by $R$ modulo the subspaces corresponding to the $c_0$-sums. Moreover, one can note using the fact that $S$ is bounded below that elements outside the subspace corresponding to the $c_0$-sum of $C(K)$ are send by $R$ onto elements outside the subspace corresponding to the $c_0$-sum of $C(K)$. It follows that $T$ is nonzero on every nonzero element, i.e., is injective. The surjectivity of $T$ follows from the surjectivity of $R$ which follows from the surjectivity of $S$.

Now let us prove that $T$ is nowhere canonizable along a quasi-open mapping. Fix $U$, $O$ clopen subsets of $X^*$ and $Y^*$ respectively, and suppose $\phi$ is as above and quasi-open. Fix a clopen $V \subseteq \phi[U]$. Let $U'$ be a clopen subset of $X$ such that $\beta U' \cap X^* \subseteq U$. The set $E$ of integers $n$ such that $U_n = U' \cap (\{ n \} \times K) \neq \emptyset$ must be infinite. Since the isolated points $\{ x_m : m \in \mathbb{N} \}$ are dense in $K$, we may assume, by going to a subset of $U$, that $U_n = \{ x_{k_n} \}$ for all $n \in E$ and some $k_n \in \mathbb{N}$. Therefore,

$$U' = \bigcup_{n \in E} \{ n \} \times \{ x_{k_n} \}.$$

Let $V_n = V' \cap \{ n \} \times 2^\mathbb{N}$ for $n \in E$, where $\beta V' \cap Y^* = V$. Let $W_n \subseteq V_n$ be a nonempty clopen such that $\tilde{\tau}(x_{k_n})|W_n$ has its total variation less than $|r|/2$ which can be found since $2^\mathbb{N}$ has no isolated points. Consider

$$W = \bigcup_{n \in E} \{ n \} \times W_n.$$

Then, $|\beta \tau(n, x_{k_n})(W')| < |r|/2$ for any $W' \subseteq W$ and any $n \in E$. By the weak* continuity of $\beta \tau$ we have that $|\beta \tau(t)(W')| \leq |r|/2$ for any $t \in U$, but this shows that $\beta \tau(t)$ is not $r \delta_{\phi(t)}$ as required.

One concrete construction using the methods as above due to E. van Douwen and J. van Mill is a nowhere dense retract $F \subseteq \mathbb{N}^*$ which is homeomorphic to $\mathbb{N}^*$ and which is a $P$-set (see 1.4.3. and 1.8.1. of [33]). We will require the following:

Lemma 6.6. (CH) Let $F \subseteq \mathbb{N}^*$ be a nowhere dense $P$-set. The space $\{ f \in C(\mathbb{N}^*) : f|F = 0 \}$ is isomorphic to $C(\mathbb{N}^*)$. 

Proof. Fix a $P$-point $p \in \mathbb{N}^*$ which exists assuming CH by the results of [13]. Let $(A^*_\alpha : \alpha < \omega_1)$ and $(B^*_\alpha : \alpha < \omega_1)$ be sequences of strictly increasing clopen sets such that $\bigcap_{\alpha < \omega_1}(\mathbb{N}^* \setminus A^*_\alpha) = F$ and $\bigcap_{\alpha < \omega_1}(\mathbb{N}^* \setminus B^*_\alpha) = \{p\}$ (they exist because $F$ is a $P$-set and $p$ is a $P$-point).

Using the standard argument construct recursively one-to-one, onto functions $\sigma_\alpha : B_\alpha \to A_\alpha$ such that $\sigma_\alpha = \sigma_\beta|B_\alpha$ for all $\alpha < \beta < \omega_1$. Put $\psi_\beta = \sigma_\beta^* : B^*_\beta \to A^*_\beta$ which is the corresponding homeomorphism.

Note that if $f \in C(\mathbb{N}^*)$ is such that $f|F = 0$, then for every $n \in \mathbb{N}$ there exists $\alpha < \omega_1$ such that $\mathbb{N}^* \setminus A^*_\alpha \subseteq f^{-1}\{t \in \mathbb{R} : |t| < 1/(n+1)\}$. Therefore, for each such $f$ there exists $\alpha < \omega_1$ such that $f|(\mathbb{N}^* \setminus A^*_\alpha) = 0$. So define

$$S : \{f \in C(\mathbb{N}^*) : f|F = 0\} \to \{f \in C(\mathbb{N}^*) : f(p) = 0\}$$

by putting $S(f) = (f \circ \psi_\alpha) \cup_0 B_{\beta}$, where $\alpha$ is any countable ordinal such that $f|(\mathbb{N}^* \setminus A^*_\alpha) = 0$. It is well defined because the homeomorphisms extend each other, and it is clearly a linear isometry. Now it is enough to note that $\{f \in C(\mathbb{N}^*) : f(p) = 0\}$ is isomorphic to $C(\mathbb{N}^*)$. To see that this is the case, notice that this space is a hyperplane, and recall that all hyperplanes are isomorphic to each other in any Banach space (see exercises 2.6 and 2.7 of [20]). In the case of $C(\mathbb{N}^*)$ we have

$$C(\mathbb{N}^*) \sim C(\mathbb{N}^*) \oplus \ell_\infty \sim C(\mathbb{N}^*) \oplus \ell_\infty \oplus \mathbb{R} \sim C(\mathbb{N}^*) \oplus \mathbb{R}$$

and so all hyperplanes are isomorphic to the entire $C(\mathbb{N}^*)$. This completes the proof. \qed

Proposition 6.7. (CH) The collection of locally null operators is not a right ideal. Moreover, the right ideal generated by locally null operators is improper.

Proof. Let $F \subseteq \mathbb{N}^*$ be a nowhere dense retract of $\mathbb{N}^*$ homeomorphic to $\mathbb{N}^*$, $\psi_1 : \mathbb{N}^* \to F$ the witnessing retraction and $\psi_2 : \mathbb{N}^* \to F$ the homeomorphism. Note that $\psi_2^{-1} \circ \psi_1$ is a well defined continuous map from $\mathbb{N}^*$ onto itself, and so $T_{\psi_2^{-1} \circ \psi_1}$ is a well defined operator from $C(\mathbb{N}^*)$ into itself. $T_{\psi_2}$ is locally null because $F$ is nowhere dense and hence $T_{\psi_2}(f^*) = f^* \circ \psi_2$ is zero for every $f \in \ell_\infty$ such that $f^*|F$ is zero. But for every $f \in \ell_\infty$ we have

$$T_{\psi_2} \circ T_{\psi_2^{-1} \circ \psi_1}(f^*) = f^* \circ \psi_2^{-1} \circ \psi_1 \circ \psi_2 = f^*,$$

because $\text{Im}(\psi_2) = F$ and $\psi_1|F = Id_F$. This means that $T_{\psi_2} \circ T_{\psi_2^{-1} \circ \psi_1} = Id$, which is not locally null. Moreover, for any operator $S : \ell_\infty/c_0 \to \ell_\infty/c_0$ we have that $S = (T_{\psi_2} \circ T_{\psi_2^{-1} \circ \psi_1}) \circ S$, which is in the right ideal generated by locally null operators. \qed

Nowhere dense $P$-sets homeomorphic to $\mathbb{N}^*$ which are retracts give also more concrete (compared to [6.7] examples of automorphisms failing canonizability like in [6.4])

Example 6.8 (CH). There is an automorphism $T$ of $\ell_\infty/c_0$ with the following properties:

1. $T$ is not fountainless
2. $T$ is not left-locally canonizable along any continuous map.
3. $T^{-1}$ is not funnelless
4. $T^{-1}$ is not right-locally canonizable along any continuous map.
Proof. Let $F$ be a nowhere dense retract of $N^*$ which is a $P$-set and is homeomorphic to $N^*$. Let $\psi_1 : N^* \to F$ be the witnessing retraction. We will need one more additional property of $F$, namely that $\psi_1$ is not one-to-one while restricted to any nonempty clopen set. This can be obtained by modifying the construction of 1.4.3. of [33] by replacing $W(\omega_1 + 1)$ with the “zero-dimensional long line”, i.e., the space $K$ obtained by gluing Cantor sets inside every ordinal interval $[\alpha, \alpha + 1)$ for $\alpha < \omega_1$, obtaining a nonmetrizable subspace $K$ of the long line which contains $W(\omega_1 + 1)$ and has no isolated points. One takes $\tilde{\pi} : K \to K$ which collapses the entire $K$ to the point $\omega_1$, $X = N \times K$, and $\pi : X \to X$ given by $\pi(n, x) = (n, \tilde{\pi}(x))$. As in 1.4.3. of [33] one proves that $\beta\pi[X^*] \subseteq X^*$ and $\psi_1 = \beta\pi|X^*$ is the required retraction. The argument why $\psi_1$ is not a one-to-one while restricted to any clopen set is similar to the one from the proof of Theorem 2.1. from [34]: if $U \subseteq X^*$ clopen, it is of the form $\beta U' \cap X^*$ where

$$U' = \bigcup_{n \in \mathbb{E}} \{n\} \times U_n$$

for some infinite $E \subseteq \mathbb{N}$ and nonempty clopen sets $U_n \subseteq K$ (consider $\chi_U$ and the relation of $X$ to $\beta X$). But these nonempty open sets have at least two points $x_n, y_n$ as $K$ has no isolated points. Of course $\pi(n, x_n) = (n, \omega_1) = \pi(n, y_n)$. Consider $x = \lim_{n \in \mathbb{N}} x_n$ and $y = \lim_{n \in \mathbb{N}} y_n$ ($u$ is a nonprincipal ultrafilter in $\varphi(N)$) which can be easily separated, so $x \neq y$ and $x, y \in U$. However $\psi_1(x) = \lim_{n \in \mathbb{N}} \pi(n, x_n) = \lim_{n \in \mathbb{N}} \pi(n, y_n) = \psi_1(y)$.

We can decompose $C(N^*) = X \oplus Y$ where

$$X = \{g \circ \psi_1 : g \in C(F)\}, \quad Y = \{f \in C(N^*) : f|F = 0\}.$$
Proof. Let \( \psi : \mathbb{N}^* \to \mathbb{N}^* \) be nowhere trivial homeomorphism of \( \mathbb{N}^* \). The existence of such a homeomorphism is a folklore result, its first construction is implicitly included in [43]. By Propositions 3.21 and 3.22, \( T_\psi \) has no fountains nor funnels. It is not locally trivial because \( \psi \) is not trivial on any clopen set. \( \square \)

**Theorem 6.10** (CH). There is a quasi-open surjective map \( \psi : \mathbb{N}^* \to \mathbb{N}^* \) such that the images of nowhere dense sets under \( \psi \) are nowhere dense and it is not a bijection while restricted to any clopen set. Therefore, \( T_\psi \) is an everywhere present isomorphic embedding of \( \ell_\infty/c_0 \) into itself with no fountains and with no funnels which is nowhere canonizable along a homeomorphism.

Proof. It is enough to construct a quasi-open irreducible surjection \( \psi : \mathbb{N}^* \to \mathbb{N}^* \) which is not a homeomorphism when restricted to any clopen set and consider \( T_\psi \) by 8.6, 3.21 and 3.22. Let \( \tilde{\psi} : 2^\mathbb{N} \to 2^\mathbb{N} \) be an irreducible surjection which is not a bijection while restricted to any clopen subset of \( 2^\mathbb{N} \) (e.g., obtained via the Stone duality by taking a dense atomless subalgebra of the free countable algebra which is proper below any element, see 8.3). Consider \( X = \mathbb{N} \times 2^\mathbb{N} \) and \( \phi : X \to X \), given by \( \phi(n, x) = (n, \tilde{\psi}(x)) \). By a topological consequence of Parovićenko’s theorem (see Theorem 1.2.6. of [35]) \( X^* = \beta X \setminus X \) is homeomorphic to \( \mathbb{N}^* \). Moreover \( \beta \phi : \beta X \to \beta X \) sends \( X^* \) into \( X^* \).

To check that \( \psi = \beta \phi|X^* \) is irreducible take any clopen \( U \subseteq X^* \), which must be of the form \( U = \bigcup_{n \in E} \{ n \} \times U_n \) for some infinite \( E \subseteq \mathbb{N} \) and nonempty clopen sets \( U_n \subseteq K \) (consider \( \chi_U \) and the relation of \( X \) to \( \beta X \)). By the irreducibility of \( \tilde{\psi} \), there are clopen \( V_n \subseteq 2^\mathbb{N} \) such that \( \tilde{\psi}[2^\mathbb{N} \setminus U_n] \cap V_n = \emptyset \). So

\[
\beta \phi[U] \cap \beta \left( \bigcup_{n \in E} \{ n \} \times V_n \right) = \emptyset,
\]

which completes the proof of the irreducibility of \( \psi \). The argument why \( \psi \) is not a one-to-one while restricted to any clopen set is similar to the one from the proof of Example 6.8 \( \square \).

A similar example as above is constructed in the proof of Theorem 2.1 from [34] however it does not have the property of preserving nowhere dense sets.

7. Open problems and final remarks

In this section we mention some open problems and some observations related to them. This should not be considered as a full list of urgent open problems concerning the Banach space \( \ell_\infty/c_0 \), for example we do not touch problems related to the primariness of \( \ell_\infty/c_0 \) (see [18], [17], [28]) subspaces of \( \ell_\infty/c_0 \) ([7], [29], [49], [30]), \( \ell_\infty \)-sums ([18], [17], [8]) or extensions of operators on \( \ell_\infty/c_0 \) ([9], [4]).

**Problem 7.1.** Is it consistent (does it follow from PFA or OCA+MA) that every automorphism \( T : \ell_\infty/c_0 \to \ell_\infty/c_0 \) can be lifted modulo a locally null operator? That is, is every such operator of the form \( T = [R] + S \), where \( R : \ell_\infty \to \ell_\infty \) and \( S \) is locally null?
This is related to the fact that our ZFC nonliftable operator (see 4.16) is of the above form. A ZFC possibility of somewhere canonizing every isomorphic embedding is excluded by 6.10 or 6.5. As under PFA or OCA+MA canonization along a homeomorphism gives trivialization we may ask:

**Problem 7.2.** Is it consistent (does it follow from PFA or OCA+MA) that every isomorphic embedding $T : \ell_\infty/c_0 \rightarrow \ell_\infty/c_0$ is somewhere trivial?

**Problem 7.3.** Is it true in ZFC that for every isomorphic embedding $T : \ell_\infty/c_0 \rightarrow \ell_\infty/c_0$ there is an infinite $A$, a closed $F \subseteq \mathbb{N}^*$ and a homeomorphism $\psi : F \rightarrow A^*$ such that $T(f^*)|F = r(f^* \circ \psi)$ for $A$-supported $f$’s and some nonzero $r \in \mathbb{R}$?

The positive solution of this problem would give the positive solution to Problem 7.9.

**Problem 7.4.** Is it consistent (does it follow from PFA, or OCA+MA) that every automorphism of $\ell_\infty/c_0$ is somewhere trivial?

In other words we ask here if the hypothesis in 6.4 of $T$ being funnelless or fountainless is needed under PFA or OCA+MA. In principle there may not be any fountains or funnels of automorphisms of $\ell_\infty/c_0$ under these assumptions. The only examples we have of such phenomena are for automorphisms under CH (6.8).

Note that by Plebanek’s result 5.7 and by going to a subset of $F$ using 5.2 and 8.4 we may assume that there is an infinite $A \subseteq \mathbb{N}$, a closed $F \subseteq \mathbb{N}^*$ and a homeomorphism $\psi : F \rightarrow A^*$ such that $T(f^*)|F = f^* \circ \psi$ for every $A$-supported $f$. If we knew that $(A^*, F)$ is not a funnel i.e., that $F$ is not nowhere dense, we could use Farah’s result 6.2 as in the proof of 6.4 to obtain somewhere trivialization of $T$. We were not able to prove, however, a similar reduction for fountains, which could be more useful in the context of applying Farah’s result as then $F$ would be a continuous image of $A^*$ which is a copy of $\mathbb{N}^*$ so the domain of $\psi$ is as required in 6.2.

One strategy for proving that under OCA+MA automorphisms do not have funnels or fountains is to use the result of I. Farah 6.2 directly to prove that the sets $F$ appearing in potential funnels or fountains cannot be nowhere dense. For this we would need to know that (a) such $F$’s are not c.c.c. over Fin, and that (b) they (or their clopen sets) are continuous (or homeomorphic) images of $\mathbb{N}^*$. There are several related natural questions which we were unable to solve.

**Problem 7.5.** Suppose that $T : C(\mathbb{N}^*) \rightarrow C(\mathbb{N}^*)$ is an automorphism (an isomorphic embedding), $A \subseteq \mathbb{N}$ is infinite, $F \subseteq \mathbb{N}^*$ is closed nowhere dense and $\psi : F \rightarrow A^*$ is continuous irreducible such that $T(f^*)|F = f^* \circ \psi$ for each $A$-supported $f \in \ell_\infty$. Is it consistent (under OCA+MA, or PFA) that $F$ cannot be c.c.c. over Fin?

**Problem 7.6.** Suppose that $T : C(\mathbb{N}^*) \rightarrow C(\mathbb{N}^*)$ is an automorphism and $F \subseteq \mathbb{N}^*$ is closed nowhere dense and $A \subseteq \mathbb{N}$ infinite such that $(F, A^*)$ is a fountain for $T$ or $(A^*, F)$ is a funnel for $T$. Is it true or consistent (under OCA+MA, or PFA) that $F$ cannot be c.c.c. over Fin?

A more general problem is

**Problem 7.7.** Suppose that $F \subseteq \mathbb{N}^*$ is nowhere dense closed not c.c.c (or even homeomorphic to $\mathbb{N}^*$) which has the linear extension property. Is it consistent (under OCA+MA, or PFA) that $F$ cannot be c.c.c. over Fin?
Recall that $F \subseteq \mathbb{N}^*$ has the linear extension property if there is a linear bounded operator $T : C(F) \to C(\mathbb{N}^*)$ such that $T(f)|F = f$ for each $f \in C(F)$. The existence of such an operator is a weak version of the existence of a retraction from $\mathbb{N}^*$ onto $F$. Alan Dow in [17] developed new methods (which may be quite useful in the above context) proving that PFA implies that the cozero sets do not have the linear extension property. The last couple of problems is related to possible applications of canonizations of embeddings.

**Problem 7.8.** Is it consistent that every copy of $\ell_\infty/c_0$ inside $\ell_\infty/c_0$ is complemented?

**Problem 7.9.** Is it true or consistent that every copy of $\ell_\infty/c_0$ inside $\ell_\infty/c_0$ contains a further copy of $\ell_\infty/c_0$ which is complemented in the entire space?

One should note that under CH examples of uncomplemented copies of $\ell_\infty/c_0$ inside $\ell_\infty/c_0$ were constructed in [9]. They can also be obtained under CH from a superspace of $\ell_\infty/c_0$ obtained in [5] in which $\ell_\infty/c_0$ is not complemented.

8. Appendix

8.1. $C(\mathbb{N}^*)$.

**Lemma 8.1.** Every nonempty $G_\delta$ set in $\mathbb{N}^*$ has a nonempty interior.

*Proof.* See section 1.2 of [33].

**Lemma 8.2.** Suppose $f : \mathbb{N}^* \to \mathbb{R}$ is continuous and $r \in \mathbb{R}$ is a value of $f$ at some point. Then there is a clopen $A^* \subseteq \mathbb{N}^*$ such that $f|A^* \equiv r$.

*Proof.* $f^{-1}\{r\} = \bigcap_{n \in \mathbb{N}} f^{-1}\{t \in \mathbb{R} : r - 1/n < t < r + 1/n\}$ is a nonempty $G_\delta$ set. By [5.1] we obtain an infinite $A \subseteq \mathbb{N}$ such that $A^* \subseteq f^{-1}\{r\}$.

**Definition 8.3.** A surjective map is called irreducible if, and only if, it is not surjective when restricted to any proper closed subset.

**Lemma 8.4.** If $\psi : K \to L$ is surjective and $K$ and $L$ are compact, then there is a closed $F \subseteq K$ such that $\psi|F : F \to L$ is irreducible.

**Lemma 8.5.** Suppose that $\psi : F \to \mathbb{N}^*$ is a continuous surjection, where $F \subseteq \mathbb{N}^*$ is a closed subset of $\mathbb{N}^*$. $\psi$ is irreducible if, and only if, $\{\psi^{-1}[A^*] : A \subseteq \mathbb{N}^*\}$ is a dense subalgebra of clopen subsets of $F$.

*Proof.* If $U \subseteq F$ were a clopen subset of $F$ such that $\psi^{-1}[A^*] \subseteq U$ does not hold for any infinite $A \subseteq \mathbb{N}$, then $\psi|(F \setminus U)$ is onto $\mathbb{N}^*$ contradicting the irreducibility. If $U \subseteq F$ were a clopen subset of $F$ such that $\psi|(F \setminus U)$ is onto $\mathbb{N}^*$, then $\psi^{-1}[A^*] \subseteq U$ cannot hold for any infinite $A \subseteq \mathbb{N}$.

**Lemma 8.6.** Irreducible maps are quasi-open and map nowhere dense sets onto nowhere dense sets.

*Proof.* Suppose that $\psi : F \to G$ is irreducible. If the interior of $\psi[U]$ is empty for some open $U \subseteq F$, then it means that $\psi[F \setminus U]$ is dense in $G$, but $\psi[F \setminus U]$ is compact, and so is equal to $G$ contradicting the irreducibility of $\psi$.

Now suppose that $K \subseteq F$ is nowhere dense whose image contains an open $U \subseteq G$. As $\psi^{-1}[U]$ is open, there is $V \subseteq \psi^{-1}[U]$ such that $V \cap K = \emptyset$ and so $\psi[V] \subseteq \psi[K]$. Note that $\psi[F \setminus V] = G$ contradicting the irreducibility of $\psi$.
Lemma 8.7. Suppose \( f : F \to \mathbb{R} \) is continuous, \( F \subseteq \mathbb{N}^* \) is compact and there is an irreducible map \( \psi : F \to \mathbb{N}^* \). Then, there is an infinite \( A \subseteq \mathbb{N} \) such that \( f(\psi^{-1}(A^*)) \) is constant.

Proof. Construct infinite \( A_n \subseteq \mathbb{N} \) such that \( A_n+1 \subseteq \mathcal{N} \) and intervals \( I_n \subseteq \mathbb{R} \) such that the diameter of \( I_n \) is less than \( 1/n \) and such that \( f(\psi^{-1}(A_n^*)) \subseteq I_n \). The irreducibility guarantees the recursive step through Lemma 8.5. If \( A \subseteq A_n \) for all \( n \), then \( f(\psi^{-1}(A^*)) \) is constant. \( \square \)

Lemma 8.8. Countable unions of nowhere dense sets in \( \mathbb{N}^* \) are nowhere dense.

Proof. Let \( F_n \subseteq \mathbb{N}^* \) be nowhere dense for every \( n \in \mathbb{N} \). We may assume each \( F_n \) to be closed. Fix an open set \( U \subseteq \mathbb{N}^* \). Let \( B_0 \subseteq U \setminus F_0 \) be a nonempty clopen and choose by induction \( B_{n+1} \subseteq B_n \setminus F_{n+1} \) nonempty clopen. Since there exists a nonempty clopen \( V \subseteq \bigcap_{n \in \mathbb{N}} B_n \subseteq U \setminus \bigcup_{k \in \mathbb{N}} F_k \), we know that \( U \nsubseteq \bigcup_{n \in \mathbb{N}} F_n \). \( \square \)

8.2. Operators on \( \ell_{\infty} \) preserving \( c_0 \).

Lemma 8.9. Let \( (b_{ij})_{i,j \in \mathbb{N}} \) be a \( c_0 \)-matrix. If \( J \subseteq \mathbb{N} \) is such that \( \left( \sum_{j \in J} |b_{ij}| \right)_i \notin c_0 \), then there exist \( \varepsilon > 0 \), an infinite set \( B \) and finite \( F \subseteq J \) for each \( n \in B \), such that

1. \( F_n \cap F_k = \emptyset \), for distinct \( n,k \in B \),
2. \( \sum_{j \in F_n} |b_{ij}| = |\sum_{j \in F_n} b_{ij}| > \varepsilon/4 \), for all \( i \in B \), and
3. \( \lim_{i \in B} \sum_{j \in \bigcup_{k \in B} F_k} |b_{ij}| = 0 \)

Proof. Let \( (b_{ij})_{i,j \in \mathbb{N}} \) be a \( c_0 \)-matrix and fix a \( J \subseteq \mathbb{N} \) as in the hypothesis. Since \( (b_{ij})_{i,j \in \mathbb{N}} \) is a \( c_0 \)-matrix, we know that for every \( k \in \mathbb{N} \) the sequence \( \left( \sum_{j \leq k} |b_{ij}| \right)_i \) converges to zero. Hence, \( J \) must be infinite. Let \( J = \{ j_n : n \in \mathbb{N} \} \) be the increasing enumeration of \( J \). By hypothesis, there exist \( \varepsilon > 0 \) and an infinite \( B \subseteq \mathbb{N} \) such that \( \sum_{n \in \mathbb{N}} |b_{ij_n}| > \varepsilon \), for all \( i \in B \).

We will carry out an inductive construction from where we will obtain the sequence \( (F_n) \) and the set \( B \). Let \( i_0 \) be the first element of \( B \) and \( m_0 = 0 \). Since \( \sum_{n \in \mathbb{N}} |b_{ij_n}| \) converges, we may choose \( m_1 > m_0 \) such that \( \sum_{n \geq m_1} |b_{ij_n}| < \varepsilon/2 \). Suppose for every \( i \leq k \) we have chosen \( i_i \) and \( m_{i+1} \) satisfying

(a) \( m_i < m_{i+1} \), \( i_i \in B \), and \( i_i-1 < i_i \),
(b) \( \sum_{n < m_i} |b_{ij_n}| < \frac{\varepsilon}{4(i+1)} \), and
(c) \( \sum_{n \geq m_{i+1}} |b_{ij_n}| < \frac{\varepsilon}{4(i+1)} \).

Since \( \left( \sum_{j < m_{i+1}} |b_{ij}| \right)_i \) converges to zero, we may choose \( i_{k+1} \in B \), such that \( i_{k+1} > i_k \) and for all \( i \geq i_{k+1} \) we have \( \sum_{n < m_{k+1}} |b_{ij_n}| < \frac{\varepsilon}{4(k+2)} \). Furthermore, since \( \sum_{n \in \mathbb{N}} |b_{ik_{k+1}j_n}| \) converges, we may choose \( m_{k+2} > m_{k+1} \) such that \( \sum_{n \geq m_{k+2}} |b_{ik_{k+1}j_n}| < \frac{\varepsilon}{4(k+2)} \). This finishes the inductive construction.

Notice that for every \( k \in \mathbb{N} \) we have

\[
\varepsilon < \sum_{n \in \mathbb{N}} |b_{ik_{k+1}j_n}| = \sum_{n < m_k} |b_{ik_{k+1}j_n}| + \sum_{m_k \leq n < m_{k+1}} |b_{ik_{k+1}j_n}| + \sum_{n \geq m_{k+1}} |b_{ik_{k+1}j_n}| \leq \frac{\varepsilon}{4(k+1)} + \sum_{m_k \leq n < m_{k+1}} |b_{ik_{k+1}j_n}| + \frac{\varepsilon}{4(k+1)} \leq \frac{\varepsilon}{4(k+1)} + |b_{ik_{k+1}j_n}| + \varepsilon/2.
\]

Hence, \( \sum_{m_k \leq n < m_{k+1}} |b_{ik_{k+1}j_n}| > \varepsilon/2 \). By splitting the sum \( \sum_{m_k \leq n < m_{k+1}} b_{ik_{k+1}j_n} \) into its positive and negative parts, we obtain \( F_k \subseteq \{ j_n \in \mathbb{N} : m_k \leq n < m_{k+1} \} \) such
that \(|\sum_{j\in F_{ik}} b_{ij}| > \varepsilon/4\). So by letting \(B = \{i_k : k \in \mathbb{N}\}\), we know that conditions (1) and (2) of the lemma are satisfied. To obtain (3), fix \(\delta > 0\) and take \(m \in \mathbb{N}\) such that \(\frac{\varepsilon}{2(m+1)} < \delta\). By construction we have that \(\bigcup_{i \neq k} F_{ij} \subseteq \{j_n \in \mathbb{N} : n < m_k \text{ or } n \geq m_{k+1}\}\), for every \(k \in \mathbb{N}\). So, in particular for every \(k > m\), we have

\[
\sum_{j \in \bigcup_{i \neq k} F_{ij}} |b_{ik,j}| \leq \sum_{n < m_k} |b_{ik,j_n}| + \sum_{n \geq m_{k+1}} |b_{ik,j_n}| < \frac{\varepsilon}{2(k+1)} < \delta.
\]

In the following propositions we list some facts leading to the proof of Theorem 8.10.

**Proposition 8.10.** Let \(S : \ell_1 \to \ell_1\) be a linear bounded operator if, and only if, there exists a matrix \((b_{ij})_{i,j \in \mathbb{N}}\) which induces \(S\) and is such that every column is in \(\ell_1\) and the set \(\{\|b_{ij}\|_{\ell_1} : j \in \mathbb{N}\}\) is bounded.

**Proof.** Let \(S : \ell_1 \to \ell_1\) be a linear bounded operator. Since \(\ell_1^* = \ell_\infty\), for each \(i \in \mathbb{N}\) there exists \((b_{ij})_{j \in \mathbb{N}} \in \ell_\infty\) such that \(S(a)(i) = S^*(\delta_i)(a) = \sum_{j \in \mathbb{N}} b_{ij}a_j\), for every \(a = (a_k)_{k \in \mathbb{N}} \in \ell_1\). In other words, \(S\) is given by the matrix \((b_{ij})_{i,j \in \mathbb{N}}\).

Note that the \(j\)-th column of the matrix is equal to \(S(\delta_j) \in \ell_1\). Moreover, since \(S\) is bounded, we have that \(S(\{\delta_j : j \in \mathbb{N}\}) = \{(b_{ij})_{i \in \mathbb{N}} : j \in \mathbb{N}\}\) is bounded in \(\ell_1\).

Conversely, suppose \((b_{ij})_{i,j \in \mathbb{N}}\) is a matrix such that every column \((b_{ij})_{i \in \mathbb{N}}\) is in \(\ell_1\) and the set \(\{\|b_{ij}\|_{\ell_1} : j \in \mathbb{N}\}\) is bounded. We claim that this matrix induces a linear bounded operator \(S : \ell_1 \to \ell_1\).

First, we shall prove that such an operator is well defined. Fix \(a = (a_k)_{k \in \mathbb{N}} \in \ell_1\). Start by noting that for every \(i \in \mathbb{N}\) we have that \((b_{ij})_{j \in \mathbb{N}}\) is bounded by hypothesis, and so is in \(\ell_\infty\). Hence, \(\sum_{j \in \mathbb{N}} b_{ij}a_j\) is convergent for every \(i \in \mathbb{N}\). Now, we need to show that the sequence \(\left(\sum_{j \in \mathbb{N}} b_{ij}a_j\right)_{i \in \mathbb{N}}\) is in \(\ell_1\).

In view of applying Theorem 8.43 of [1], note that by hypothesis \(\sum_{i \in \mathbb{N}} b_{ij}a_j\) is absolutely convergent for every \(j \in \mathbb{N}\), and there exists \(M \in \mathbb{N}\) such that \(\sum_{i \in \mathbb{N}} |b_{ij}| < M\) for every \(j \in \mathbb{N}\). Therefore,

\[
\sum_{j \in \mathbb{N}} \sum_{i \in \mathbb{N}} |b_{ij}a_j| = \sum_{j \in \mathbb{N}} |a_j| \sum_{i \in \mathbb{N}} |b_{ij}| \leq \sum_{j \in \mathbb{N}} |a_j|M < \infty.
\]

Hence, by the cited Theorem, we have that both iterated series \(\sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} b_{ij}a_j\) and \(\sum_{j \in \mathbb{N}} \sum_{i \in \mathbb{N}} b_{ij}a_j\) converge absolutely. In particular, we have that

\[
\sum_{i \in \mathbb{N}} \left| \sum_{j \in \mathbb{N}} b_{ij}a_j \right| \leq \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} |b_{ij}a_j| < \infty,
\]

so that \(S(a) \in \ell_1\).

Clearly, \(S\) is linear. Moreover, if we take \(a = (a_k)_{k \in \mathbb{N}} \in \ell_1\) such that \(\|a\|_{\ell_1} \leq 1\), then by a similar argument as above we have

\[
\|S(a)\|_{\ell_1} \leq \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} |b_{ij}a_j| = \sum_{j \in \mathbb{N}} \sum_{i \in \mathbb{N}} |b_{ij}a_j| \leq \sum_{j \in \mathbb{N}} |a_j|M \leq M.
\]

Therefore, \(S\) is bounded.

\(\square\)
In the following proposition we identify $\ell^*_1$ with $\ell_\infty$.

**Proposition 8.11.** Let $R : \ell_\infty \to \ell_\infty$ be a linear bounded operator. Then $R = S^*$ for some linear bounded operator $S : \ell_1 \to \ell_1$ if, and only if, $R$ is given by a matrix. Moreover, the matrix corresponding to the operator $S$ is the transpose of the matrix corresponding to $R$.

**Proof.** Assume $R = S^*$, for a linear bounded operator $S : \ell_1 \to \ell_1$. Let $M_S = (b_{ij})_{i,j \in \mathbb{N}}$ be the matrix corresponding to $S$. Since $S(\delta_j)$ is the $j$-th column of $M_S$, for every $f \in \ell_\infty$ we have that

$$R(f)(j) = S^*(f)(j) = (f \circ S)(\delta_j) = \sum_{i \in \mathbb{N}} b_{ij} f(i).$$

In other words, $R$ is given by the transpose of $M_S$.

Conversely, suppose $R$ is given by a matrix $M_R = (b_{ij})_{i,j \in \mathbb{N}}$. By [2.2] and [8.10] we have that the transpose of $M_R$ defines a linear bounded operator $S : \ell_1 \to \ell_1$. It only remains to show that $S^* = R$. So fix for this purpose $f \in \ell_\infty$. If we regard it as an element of $\ell^*_1$, then for every $i \in \mathbb{N}$ we have

$$S^*(f)(\delta_i) = f \circ S(\delta_i) = \sum_{j \in \mathbb{N}} f(j) b_{ij},$$

but on the other hand,

$$R(f)(i) = \sum_{j \in \mathbb{N}} f(j) b_{ij}.$$

\[ \square \]

In the following proposition we identify $c_0^*$ with $\ell_1$.

**Proposition 8.12.** Let $R : c_0 \to c_0$ be a linear bounded operator. Then, $R^*$ is induced by the transpose of the matrix corresponding to $R$.

**Proof.** Let $M_R = (b_{ij})_{i,j \in \mathbb{N}}$ and $M_{R^*} = (b'_{ij})_{i,j \in \mathbb{N}}$ be the matrices corresponding to $R$ and $R^*$, respectively. Observe that for every $i,j \in \mathbb{N}$ we have that $b'_{ij} = R^*(\delta_j)(\chi_{\{i\}}) = \delta_j(R(\chi_{\{i\}})) = b_{ji}$. Hence, $M_{R^*}$ is the transpose of $M_R$. \[ \square \]

**Proposition 8.13.** A linear bounded operator $R : \ell_\infty \to \ell_\infty$ is given by a $c_0$-matrix if, and only if, $R = (R|c_0)^{**}$.

**Proof.** Assume $R$ is given by a $c_0$-matrix $M$. Then, $R|c_0 \subseteq c_0$ and so $R|c_0 : c_0 \to c_0$ is well defined and is also induced by $M$. By propositions [8.11] and [8.12] we have that $(R|c_0)^{**}$ is also induced by $M$. In other words, $R = (R|c_0)^{**}$.

Conversely, assume $R = (R|c_0)^{**}$. Let $M$ be the matrix corresponding to $R|c_0 : c_0 \to c_0$. By propositions [8.11] and [8.12] we have that $R = (R|c_0)^{**}$ is also induced by $M$, which is a $c_0$-matrix. \[ \square \]

**Theorem 8.14.** Let $R : c_0 \to c_0$ be a linear bounded operator and let $(b_{ij})_{i,j \in \mathbb{N}}$ be the corresponding matrix. The following are equivalent:

1. $R$ is weakly compact.
2. $R^{**}[\ell_\infty] \subseteq c_0$.
3. $\|b_i\|_{\ell_1} \to 0$, where $b_i = (b_{ij})_{j \in \mathbb{N}}$. 


Lemma 8.16. Let $\text{Lemma 8.16.}$  

Since $\text{Since}$  

operator, say of $\text{operator, say of}$  

Proposition 8.15.  

So fix an open set $\text{So fix an open set}$ $U \subseteq \mathbb{R}$ corresponding to $x \in \mathbb{R}$  

Assume $\text{Assume}$  

Proof.  

The equivalence of of (1) and (2) is well-known (see exercise 3 of Chapter 3 in [13]).  

By $\text{By}$ $\text{Lemma 8.16.}$ we have that $R^{**}$ is given $(b_{ij})_{i,j \in \mathbb{N}}$. Therefore, for any $f \in \ell_\infty$ and any $i \in \mathbb{N}$ we have $|R^{**}(f)(i)| = |\sum_{j \in \mathbb{N}} b_{ij} \cdot f(j)| \leq \sum_{j \in \mathbb{N}} |b_{ij}| \|f\|_{\ell_\infty}$, and it is clear that (3) implies (2).  

For the converse, assume (3) does not hold. Then, by Lemma $\text{Lemma 8.16.}$ there exist $\epsilon > 0$, an infinite set $A \subseteq \mathbb{N}$ and finite $F_n \subseteq \mathbb{N}$ for each $n \in \mathbb{N}$, such that  

(i) $F_n \cap F_k = \emptyset$, for distinct $n,k \in A$,  

(ii) $\sum_{j \in F_n} |b_{ij}| = |\sum_{j \in F_n} b_{ij}| > \epsilon/4$, for all $i \in A$, and  

(iii) there is an $m \in \mathbb{N}$ such that  

$$\sum_{j \in \bigcup_{k \neq m} F_k} |b_{ij}| < \epsilon/8, \quad \text{for all } i > m.$$

Let $f \in \ell_\infty$ be such that $\text{supp}(f) \subseteq \bigcup_{n \in \mathbb{N}} F_n$ and $b_{ij} \cdot f(j) = |b_{ij}|$, for every $i \in A$ and every $j \in F_i$. Then, for every $i \in A \setminus m$ we have  

$$|R^{**}(f)(i)| = |\sum_{j \in \bigcup_{n \in \mathbb{N}} F_n} b_{ij} \cdot f(j)| \geq |\sum_{j \in F_i} b_{ij} \cdot f(j)| - |\sum_{j \in \bigcup_{k \neq m} F_k} b_{ij} \cdot f(j)| > \epsilon/4 - \epsilon/8 = \epsilon/8.$$  

Therefore, $R^{**}(f) \notin \ell_0$.  

$\square$

Proposition 8.15. Let $T : X^{**} \to X^{**}$. Then, $T = R^{**}$ for some $R : X \to X$, and only if, $T$ is $w^*-w^*$-continuous and $T[X] \subseteq X$.  

Proof. Assume $T = R^{**}$ for some $R : X \to X$. It is well known that the fact that $T$ is a dual operator implies that it is $w^*-w^*$-continuous. Since $R[X] \subseteq X$ and $R \subseteq R^{**}$, we have that $T[X] \subseteq X$.  

Conversely, suppose $T$ is $w^*-w^*$-continuous and $T[X] \subseteq X$. Then, $T$ is a dual operator, say of $S : X^* \to X^*$. It is sufficient to show that $S$ is $w^*-w^*$-continuous.  

So fix an open set $U \subseteq \mathbb{R}$ and an $x \in X$. We denote by $\hat{x} \in X^{**}$ the element corresponding to $x \in X$. Consider the preimage by $S$ of the $w^*$-open subbasic $\{g \in X^* : g(x) \in U\}$, for given $x \in X$ and $U \subseteq \mathbb{R}$ open:  

$$S^{-1}\{g \in X^* : g(x) \in U\} = \{h \in X^* : S(h)(x) \in U\} = \{h \in X^* : \hat{x} \in S(h) \subseteq U\} = \{h \in X^* : S^*(\hat{x})(h) \in U\}.$$  

Since $S^* = T$ and $T[X] \subseteq X$, if we put $v = S^*(\hat{x}) \in X$, then we have  

$$S^{-1}\{g \in X^* : g(x) \in U\} = \{h \in X^* : h(v) \in U\},$$  

which is clearly $w^*$-open.  

$\square$

Lemma 8.16. Let $X$ be a Banach space.  

(a) For every $x \in X$ the functional $F_x : X^* \to \mathbb{R}$ given by $F_x(f) = f(x)$ is $w^*$-continuous.  

(b) $T : X^* \to X^*$ is $w^*-w^*$-continuous if, and only if, for every $x \in X$ the operator $T_x : X^* \to \mathbb{R}$ given by $T_x(f) = T(f)(x)$ is $w^*$-continuous.
**Proposition 8.17.** In both $\ell_1$ and $\ell_\infty$ the product topology is coarser than the $w^*$-topology. Moreover, in both cases the converse is only true when restricted to a bounded subspace.

**Proof.** We prove the statement for $\ell_1$, the argument being the same for $\ell_\infty$.

Let $k$ be a positive integer and $I_i$ be an open interval for every $i < k$. We will show that the following basic open of the product topology $U = \{(b_j)_{j \in \mathbb{N}} \in \ell_1 : b_i \in I_i, \forall i < k\}$ is also $w^*$-open. So let $(a_j)_{j \in \mathbb{N}} \in U$ and take $\varepsilon > 0$ such that $(a_i - \varepsilon, a_i + \varepsilon) \subseteq I_i$, for each $i < k$. Consider the following $w^*$-open set

$$O = \{(b_j)_{j \in \mathbb{N}} \in \ell_1 : \sum_{j \in \mathbb{N}} |b_j - a_j| < \varepsilon, \forall i < k\}.$$ 

Clearly, $(a_j)_{j \in \mathbb{N}} \in O$ and $O \subseteq U$. So the product topology is coarser than the $w^*$-topology.

Now fix $M \in \mathbb{R}$ and let $Id : (B_{\ell_1}(M), \tau_{w^*}) \to (B_{\ell_1}(M), \tau_p)$ be the identity map, where $B_{\ell_1}(M) = \{a \in \ell_1 : \|a\| \leq M\}$ and $\tau_{w^*}, \tau_p$ are the weak* topology and the product topology, respectively. By the above, this is a continuous function, and since $B_{\ell_1}$ is $w^*$-compact, we know that it is actually a homeomorphism. So $\tau_{w^*}$ and $\tau_p$ coincide on every bounded set.

However, this is not the case everywhere. Indeed, fix any $(a_j)_{j \in \mathbb{N}} \in \ell_1$ and $\varepsilon > 0$, and let $f \in c_0$ be such that it is not eventually zero. We show that the $w^*$-open set $O = \{(b_j)_{j \in \mathbb{N}} \in \ell_1 : \sum_{j \in \mathbb{N}} |b_j - a_j| < \varepsilon\}$ is not open in the product topology. For every positive integer $k$ and every $\delta > 0$, let $A_k^\delta = \{(b_j)_{j \in \mathbb{N}} \in \ell_1 : |b_j - a_j| < \delta, \forall i < k\}$. Note that the family $\{A_k^\delta : k \in \mathbb{N} \setminus \{0\}, \delta > 0\}$ is a local basis for $(a_i)_{i \in \mathbb{N}}$ in the product topology. Hence, it is enough to show that $A_k^\delta \not\subseteq O$, for every $k$ and every $\delta$. So fix $k \in \mathbb{N} \setminus \{0\}$ and $\delta > 0$. Take $m > k$ such that $f(m) \neq 0$ and define $(b_j)_{j \in \mathbb{N}} \in \ell_1$ by putting $b_j = a_j$ for all $j \neq m$ and choosing $b_m$ such that $|f(m)| = \sum_{j \in \mathbb{N}} |b_j - a_j| < \varepsilon$. Then, $\varepsilon \leq (b_m - a_m)|f(m)| = |\sum_{j \in \mathbb{N}} (b_j - a_j)f(j)|$. Therefore, $(b_j)_{j \in \mathbb{N}} \in A_k^\delta \setminus O$. 

**Theorem 8.18.** Let $R : \ell_\infty \to \ell_\infty$ be a linear bounded operator. The following are equivalent:

1. $R = (R[c_0])^{**}$.
2. $R$ is given by a $c_0$-matrix.
3. $R$ is $w^*$-$w^*$-continuous and $R[c_0] \subseteq c_0$.
4. $R\restriction_{B_{\ell_\infty}} : (B_{\ell_\infty}, \tau_p) \to (\ell_\infty, \tau_p)$ is continuous and $R[c_0] \subseteq c_0$.

**Proof.** (1) $\iff$ (2) See Proposition 8.13

(1) $\iff$ (3) See Proposition 8.15

(3) $\iff$ (4) Suppose $R$ is $w^*$-$w^*$-continuous and $R[c_0] \subseteq c_0$. Fix a set $U \subseteq \ell_\infty$ open in the product topology. By Proposition 8.17 and by the $w^*$-$w^*$-continuity of $R$, we know that $R^{-1}[U] \cap B_{\ell_\infty}$ is open in the product topology.

Conversely, assume $R$ restricted to the unit ball is continuous in the product topology and $R[c_0] \subseteq c_0$. Then, by Proposition 8.17 and since $R$ is bounded, we have that $R$ restricted to the unit ball is $w^*$-$w^*$-continuous. Now consider for each $a \in \ell_1$ the functional $R_a : \ell_\infty \to \mathbb{R}$ defined by $R_a(x) = R(x)(a)$. By Lemma 8.16(a), we have that $R_a\restriction_{B_{\ell_\infty}} = F_a \circ R\restriction_{B_{\ell_\infty}}$ is $w^*$-$w^*$-continuous. Then, by Corollary 4.46
in [20] we have that $R_a$ is $w^*$-continuous, and since this is true for every $a \in \ell_1$, we know by Lemma 8.16(b) that $R$ is $w^*$-$w^*$-continuous. □

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