A SPECTRAL VIEWPOINT OF THE LINKING FORM IN THE 3-TORUS

ADRIEN BOULANGER

Abstract. We compute the linking number between any two collections of homologically trivial oriented geodesics of the 3-torus endowed with the flat metric. As a corollary, we find a new formula relating the linking number and the intersection number of two collections of homologically trivial orbits of the geodesic flow over the flat 2-torus. Our method relies on spectral Theory of differential forms and on the De Rham-Vogel linking form.

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1. Introduction

1.1. Definitions and notations. Let $M$ be a connected closed manifold of dimension three. A **multicurve** is a finite set of pairwise disjoint oriented curves. Recall that a multicurve $\Gamma = (\gamma^i)_{i \in I}$ is **homologically trivial** if it is the boundary of some two chain $S_\Gamma$. To each pair $\Gamma$ and $\Upsilon = (\upsilon^j)_{j \in J}$ of homologically trivial multicurves we can attach a number which measures the way they link : the linking number. We refer to [1] chapter 3 section 4 and to [3] section 17 for an introduction to the linking number.

**Definition 1.1.** (see Figure 1.1) The **linking number** of two homologically trivial disjoint multicurves $\Gamma$ and $\Upsilon$, denoted by $\text{lk}(\Gamma, \Upsilon)$, is the algebraic intersection number of $\Upsilon$ with any 2-chain $S_\Gamma$ such that $\partial S_\Gamma = \Gamma$. 
Remark 1.2.

- Since $\Gamma$ is homologically trivial the intersection between $S_\Gamma$ and $\Upsilon$ does not depend on the choice of the surface $S_\Gamma$. Therefore the linking number is well defined.

- The linking number is symmetric with respect to the multicurves $\Gamma$ and $\Upsilon$, namely

\[ \text{lk}(\Gamma, \Upsilon) = \text{lk}(\Upsilon, \Gamma) \]

\[ S_\Gamma \]

\[ \Upsilon \]

\[ \Gamma \]

\[ \]

Figure 1. Here both collections $\Gamma$ and $\Upsilon$ are reduced to a single curve. Their linking number equals $\pm 1$ depending on the global orientation which has been chosen.

We will identify the subset $\mathbb{Z}^3 \subset \mathbb{R}^3$ to a subset of translations of $\mathbb{R}^3$ through the following canonical map

\[ \mathbb{Z}^3 \to \{ \text{translations of } \mathbb{R}^3 \} \]

\[ p \mapsto v + p \]

We will denote by $\mathbb{T}^3$ the corresponding flat torus $\mathbb{R}^3/\mathbb{Z}^3$. The goal of this paper is to compute the linking number between two homologically trivial multicurves consisting of geodesics of $\mathbb{T}^3$.

We first fix some notations. We parametrize geodesics of $\mathbb{T}^3$ as follows

\[ \begin{align*}
\gamma : \mathbb{R} & \to \mathbb{T}^3 \\
 t & \mapsto \begin{pmatrix}
\gamma_1 t + \nu_1 \\
\gamma_2 t + \nu_2 \\
\gamma_3 t + \nu_3
\end{pmatrix} \mod \mathbb{Z}^3
\end{align*} \]

where $\nu = \gamma(0) = \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} \in \mathbb{T}^3$ corresponds to a starting point of $\gamma$.

The vector $\gamma'(t) \in T_{\gamma(t)} \mathbb{T}^3$ could also be seen as an element of the Lie algebra canonically identified to $\mathbb{R}^3$ of $\mathbb{T}^3$. With such a description we have,

\[ \gamma'(t) = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix} \in \mathbb{R}^3 \]

which does not depend neither on $t$ nor on the starting point. To any geodesic of the three torus is so associated a vector $\gamma'$ of $\mathbb{R}^3$. The benefit of this parametrization is to make clear if the geodesic considered is closed or not, depending on whether its derivative belongs to the lattice $\mathbb{Z}^3$. Note that if the geodesic is closed any starting
point $\nu \in \operatorname{Im}(\gamma)$ could be chosen to describe the image of the geodesic, see Figure 2.

Figure 2. The cube identified by gluing its opposite faces with respect to the associate projection represents the three torus $\mathbb{T}^3$. The red curve $\Gamma$ has the vector $\Gamma' = (2, 2, 3)$ as derivative and passes through $0_{\mathbb{R}^3}$, which then could be chosen as the starting point of the curve.

A multicurve consisting of closed geodesics is called a multigeodesic. Note that the geodesics considered here are oriented by the paramatrization given by the equation (1.3).

Since we defined the three torus $\mathbb{T}^3$ as a quotient of $\mathbb{R}^3$ by a set of translations identified to $\mathbb{Z}^3$, its fundamental group is therefore canonically identified to $\mathbb{Z}^3 \subset \mathbb{R}^3$. This group being Abelian, the identification descends to its Abelianized group, itself identified to the first homology group with integer coefficients.

Through such an identification one can check that the derivative of a closed curve $\gamma' \in \mathbb{Z}^3$ is indeed the vector whose coefficients correspond to the integer homology class of the curve $\gamma$. Therefore, for the sake of notations, we will denote herafter by $[\gamma]$ the vector $\gamma'$. We thus have the following necessary and sufficient condition for a multigeodesic $\Gamma = (\gamma^i)_{i \in I}$ to be homologically trivial, namely:

$$\sum_{i \in I} [\gamma^i] = 0_{\mathbb{R}^3}.$$

Given to vector $u, v \in \mathbb{Z}^3$ we define the new vector $\beta^{u,v} \in \mathbb{Z}^3$ by:

- $\beta^{u,v}$ is orthogonal to both $u$ and $v$
- $\det(u, v, \beta^{u,v}) > 0$
- The euclidean norm $\|\beta^{u,v}\|$ of $\beta^{u,v}$ is minimal for the two first properties.

Given two closed geodesics $\gamma$ and $\nu$ we will abuse the notation by denoting $\beta^{\gamma,\nu}$ the vector $\beta^{[\gamma],[\nu]}$ defined through the two homology classe vectors $[\nu]$ and $[\gamma]$ associated to the derivatives of the curves $\nu$ and $\gamma$.

1.2. Main results. We can now state our main theorem:

**Theorem 1.4.** Given two homologically trivial multigeodesics $\Gamma = (\gamma^i)_{i \in I}$ and $\Upsilon = (\nu^j)_{j \in J}$ in the three torus, we have

$$\operatorname{lk}(\Gamma, \Upsilon) = \sum_{i \in I, j \in J} \det \left( [\gamma^i], [\nu^j], \frac{\beta^{\gamma^i,\nu^j}}{\|\beta^{\gamma^i,\nu^j}\|} \right) \frac{1 - 2 \langle \nu^j, \beta^{\gamma^i,\nu^j} \rangle}{2 \|\beta^{\gamma^i,\nu^j}\|}$$

where $\nu^j = \gamma^i(0) - \nu^j(0)$ and $[\alpha]$ denotes the unique representative in $[0, 1)$ of the real number $\alpha$ modulo $\mathbb{Z}$. 
Remark 1.6.
- our proof may be generalized to any \( n \)-dimensional torus. In fact, the notion of linking number remains well defined whenever one considers couples of manifolds of dimensions \( p \) and \( q \) satisfying \( p + q = n - 1 \).
- the formula (1.5) seems to depend on the parametrization we have chosen, we will clarify this point on the way of the proof through the remark 4.19.

The torus \( T^3 \) could be identified with the unitary tangent bundle of the two torus \( T^2 := \mathbb{R}^2 / \mathbb{Z}^2 \) denoted by \( UT^2 \) through the trivialisation

\[
UT^2 \to T^3 \quad u \mapsto ((x, y), \theta)
\]

where the point \((x, y) \in T^2\) is the based point of the vector \( u \) and \( \theta \) is the angle between this vector and the \( x \)-axis. With the unitary tangent bundle of a surface comes naturally a family of curves: the closed geodesics of the surface viewed as flow lines of the geodesic flow. Note that these curves are naturally oriented by the flow itself. In our case these curves of \( T^3 \) are still geodesics for the flat metric of \( T^3 \), see figure 3.

Remark 1.7. given any Riemannian manifold there is a metric on its unitary tangent bundle, the Sasaki metric, such that the geodesic of the base manifold viewed as curve of the unitary tangent bundle are still geodesics for this metric.

Figure 3. The red curve on the left represents a closed geodesic in \( T^2 \). This curve lifts to the unitary tangent bundle to the red one on the right. In this example the curve corresponds to \( \theta = \arctan(2) \). The green and blue curves are two other geodesics.

Theorem 1.4 can be specified as follows for geodesics of \( T^2 \).

Corollary 1.8. Let \( \Gamma = (\gamma_i)_{i \in I} \) and \( \Upsilon = (\upsilon_j)_{j \in J} \) be two collections of geodesics of \( T^2 \). Their linking number is

\[
\text{lk}(\Gamma, \Upsilon) = \sum_{i \in I, j \in J} i(\gamma_i, \upsilon_j) \frac{1 - x_{i,j}}{2}
\]

where \( x_{i,j} \) is the unique determination of \( \theta_i - \theta_j \) in \([0, 2\pi]\) and \( i(\gamma_i, \upsilon_j) \) is the algebraic number of intersection between \( \gamma_i \) and \( \upsilon_j \).

P. Dehornoy in his PhD thesis, [6, section 2], computes the linking number of such collections using a very different method. He associates to a homologically trivial collection of geodesics a polygon in the plane and he proves that the linking number of two collections is given by the volume of Minkowski sum of the corresponding polygons. The benefit of the formula above is to highlight that the linking number
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is computable by looking at the intersection points on the base surface and thanks to the data of a function of the intersection angle. The author wonder how to extend this property to hyperbolic surfaces.

Gauss, in the nineteenth century, gave the following exact formula for the linking number of two disjoint closed curves in $\mathbb{R}^3$:

$$\text{lk}(\gamma, \nu) = \frac{1}{4\pi} \int_{[0,2\pi]} \int_{[0,2\pi]} \det \left( \gamma'(t), \nu'(s), \frac{\gamma(s) - \nu(t)}{||\gamma(s) - \nu(t)||} \right) dsdt$$

where $\gamma(s)$ and $\nu(t)$ are any two parametrisations of the curves $\gamma$ and $\nu$.

The formula suggests one could compute on any Riemannian manifold the linking number of two homologically trivial multicurves $\Gamma$ and $\Upsilon$ by integrating some differential form on the product $\Gamma \times \Upsilon$. Our main tool is:

**Definition 1.9.** A linking form is a $(1,1)$-differential form on the manifold $M \times M \setminus \Delta$, where $\Delta$ is the diagonal, such that for all pairs of homologically trivial multicurves $\Gamma$ and $\Upsilon$:

$$\text{lk}(\Gamma, \Upsilon) = \int_{\Gamma} \int_{\Upsilon} \Omega$$

We will use the Hodge-De Rham laplacian $\Delta^1$ acting on differential 1-forms to exhibit a particular linking form. The Hodge-De Rham operator admits a partial inverse; the Green operator $G^1$, see [5, chapter 4] for a comprehensive description. The Green operator verifies the following equation:

$$G^1 \circ \Delta^1 = \text{Id}$$

in restriction to the space of smooth differential forms orthogonal to those in the kernel of the Laplacian, called harmonic. Notice that the first homology group $H^1_{DR}(M, \mathbb{R})$ is identified with the harmonic differential forms by Hodge theory. Moreover, the operator $G^1$ is a kernel operator, meaning there exists an $L^2$ $(1,1)$-form $g^1(x,y)$ on $M \times M$, singular along the diagonal such that for all $L^2$ 1-form $\eta$ we have:

$$G^1(\eta)(x) = \int_{M} *_y g^1(x,y) \wedge \eta(y) dy$$

where $*_y$ denotes the Hodge operator acting on the second variable, see section 2.3 for more details. If we denote by $d_y$ the differential acting on the second factor we have:

**Theorem 1.10.** [5, page 148], T. Vogel [8, Theorem 3]

*Let $(M,g)$ a closed Riemannian manifold. The $(1,1)$-differential form:

$$\Omega(x,y) = *_y d_y g^1(x,y)$$

is a linking form.*

This theorem associates a linking form to each metric on $M$. The main result of Section 3 is a formula which relates the linking number with the spectral theory of the metric $g$ through this particular linking form. All differential forms of any degree can be decomposed with respect to a $L^2$ basis of eigenforms, denoted by $(\eta_k)_{k \in \mathbb{N}}$, associated to the Hodge De Rham laplacian acting on 1-form. By definition an eigenform satisfies the equation

$$\Delta^1 \eta_k = \lambda_k \eta_k$$
The set \((\lambda_k)_{k \in \mathbb{N}}\) is the **spectrum** of \(\Delta^1\) as an operator, see \([2,3]\) for more details. Having an explicit basis of eigenforms allows to write down an almost explicit formula for the linking form.

**From now, we will use the convention** \(\frac{1}{\lambda_k} d\eta_k = 0\) **if the differential form** \(\eta_k\) **is closed.**

**Theorem 1.11.** Let \((M, g)\) a closed connected three manifold. If \(\Omega = * y d y g^1\) denotes the associated linking form we have the following formula :

\[
\int \Omega = \lim_{t \to 0} \sum_{k \geq 0} e^{-\lambda_k t} \int_{\gamma} \eta_k \int_{\nu} * \left( \frac{d\eta_k}{\lambda_k} \right)
\]

where \(\gamma\) and \(\nu\) are two 1-dimensional sub-manifold of \(M\).

This formula is the main tool to prove **Theorem 1.4**, combined with the fact that the spectral theory is well known for the flat torus \(T^3\).

**Remark 1.13.**
- Formula (1.12) still makes sense whenever \(\gamma\) and \(\nu\) are not homologically trivial, the result is essentially analytic.
- In Section 4 we will show that eventually the following series makes sense for any two geodesics \(\gamma\) and \(\nu\) in the three torus

\[
\sum_{k \in \mathbb{N}} \int_{\gamma} \eta_k \int_{\nu} * \left( \frac{d\eta_k}{\lambda_k} \right)
\]

This series turns out to be the limit of the the right side of equation (1.12), which is also the limit term by term of the series. A natural question is thus to ask whether or not the series (1.14) still makes sense for any closed Riemannian manifolds and for any two closed curves \(\gamma\) and \(\nu\)?

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## 2. Spectral theory and linking form

In this section we will describe how to relate the linking form with spectral theory through the lemma 2.18 proved in Subsection 2.3. The starting point toward such a connexion is the description of the linking form proposed in Theorem 1.10. The basics in analysis, recalled in subsection 2.1, will be usefull to understand how to build a linking form (the proof of Theorem 1.10 will be sketched in Subsection 2.2) as well to introduce the spectral theory we will need to prove the lemma 2.18. All the proofs involved in this section use the co-homological definition of the linking number. In fact, one can associated to a curve a closed 2-differential form : its Poincaré dual. When the curve is homologically trivial this differential form is exact. We recall some basics about this duality and we refer to \([3, page 230]\) for a comprehensive exposition of the details.

**Definition 2.1.** The **Poincaré dual** \(\omega_F\) of a multicurve \(\Gamma\) is a representative of the Thom class of the normal bundle of \(\Gamma\).

**Remark 2.2.** For all \(\epsilon > 0\) there exists a Poincaré dual with support contained in an \(\epsilon\) neighbourhood of the multicurve.
Let $\omega_{\Gamma}$ (resp. $\omega_{\Upsilon}$) be the Poincaré dual of the multicurves $\Gamma$ (resp. $\Upsilon$). If $\Gamma$ is homologically trivial there exist $\eta_{\Gamma}$ such that $d\eta_{\Gamma} = \omega_{\Gamma}$. The following Lemma is about to reformulate the definition of the linking number to this differential form formalism introduced above.

**Lemma 2.3.** Let $\Gamma$ and $\Upsilon$ two disjoint multicurves and $\omega_{\Gamma}, \omega_{\Upsilon}$ two Poincaré dual of them with disjoint support, the following holds:

\[(2.4) \text{lk}(\Gamma, \Upsilon) = \int_{M} \eta_{\Gamma} \wedge \omega_{\Upsilon}\]

**Remark 2.5.**
- One could think of $\omega_{\Upsilon}$ as the curve $\Upsilon$, $\eta_{\Gamma}$ as a surface $S_{\Gamma}$ bounded by $\Gamma$ and $\wedge$ - the wedge product - as the intersection.
- The fact that $\Gamma$ and $\Upsilon$ has to be disjoint as sets to define the linking number corresponds to the condition on the support of the Poincaré dual of those multicurves.

### 2.1. Spectral Theory

We recall some basic facts about the spectral theory of differential forms. For an introduction of the subject see for the exemple [4]. Let $(M, g)$ a closed connected Riemannian manifold of dimension $p$. The space of all differential forms is denoted by $\Omega^{\cdot}(M) = \bigoplus_{0 \leq k \leq p} \Omega^{k}(M)$. We denote by $*$ the Hodge operator and by $d$ the exterior derivative acting on $\Omega^{k}(M)$, abusing notations by omitting the degree $k$ of the space $\Omega^{k}(M)$ on which these operators are actually acting.

The Hodge operator is almost involutive in the following meaning

\[(2.6) * * = (-1)^{k(p-k)}\]

The Hodge star gives a natural scalar product defined on each $\Omega^{k}(M)$ by

\[<\alpha \cdot \beta> = \int_{M} \alpha \wedge * \beta,\]

The operator $d$ has a adjoint, $\delta$, for such a scalar product, which satifies :

\[<d\alpha \cdot \beta> = <\alpha \cdot \delta \beta>\]

Using the above equation and the Stokes formula one can easily prove

\[(2.7) \delta = (-1)^{p(k+1)+1} * d*\]

**Definition 2.8.** The Hodge de Rham laplacian $\Delta$ with respect to $g$ is the operator defined by :

\[\Delta = d\delta + \delta d\]

It acts on $\Omega^{*}(M)$ and stabilizes every $\Omega^{k}(M)$.

Since $d$ and $\delta$ are adjoint for the scalar product $< \cdot >$ defined above, the Laplacian $\Delta$ is self-adjoint for the same scalar product. Moreover, its restriction on $k$-forms, denoted by $\Delta^{k}$, is diagonalisable :

- there exists two sequences $(\eta^{k}_{n})_{n \in \mathbb{N}} \in \Omega^{k}(M)^{\mathbb{N}}$ and $(\lambda^{k}_{n})_{n \in \mathbb{N}} \in \mathbb{R}_{+}^{\mathbb{N}}$ such that $\Delta^{k} \eta^{k}_{n} = \lambda^{k}_{n} \eta^{k}_{n}$, $\|\eta^{k}_{n}\|_{L^{2}} = 1$
- if $\alpha \in \Omega^{k}(M)$ then $\alpha = \sum_{n \in \mathbb{N}} <\eta^{k}_{n} \cdot \alpha> \eta^{k}_{n}$.
A form $\alpha$ is said to be harmonic if it lies in the kernel of the Laplacian, meaning $\Delta \alpha = 0$. We denote by $\mathcal{H}^k$ the spaces of harmonics differential $k$-form, identified by a famous theorem of Hodge with the $k$-th group of De-Rham cohomology. The operator $\Delta^k$ is then generally non invertible, but it could be whether one takes care to restrain the study on a supplementary vector space of its kernel. To make it precise we introduce the $L^2$-orthogonal projector on the space of harmonic $k$-forms denoted by $\pi_{H^k}$ and the Green operator defined as follow.

**Definition 2.9.** A Green operator, denoted by $G^k$, associated to the laplacian $\Delta^k$ is an operator which solves the following equation on the space of smooth differential forms:

$$G^k \circ \Delta^k = \Delta^k \circ G^k = Id - \pi_{H^k}$$

Such operators always exist providing that the underlying manifold is closed, see [5, section 31].

**Remark 2.11.** The Green operator $G^k$ is uniquely defined up to the choice of its restriction to the space of harmonic form $\mathcal{H}^k$.

Let $\eta \in \Omega^*(M)$ be a differential form. We have two canonical inclusions of a manifold $M$ into the product $M \times M$ corresponding to both factors of this product which identify the manifold $M$ to each factors. We will denote by $\eta(x)$ (resp. $\eta(y)$) the differential form defined on $M \times M$ corresponding to the form $\eta$ on the first (resp. second ) factor which vanishes on every vectors coming from the second (resp. first ) factor.

Green’s operators are kernel operators: there exist $g^k \in L^2(\Omega^k(M \times M))$ such that for all $\alpha \in L^2(\Omega^k(M))$:

$$G^k(\alpha)_x = \int_{y \in M} \ast_y g^k(x,y) \land \alpha(y)$$

**Remark 2.13.** At the very end the Green kernel $g^k(x,y)$ is as a linear operator from the fibre of the fibre bundle of differential $k$-forms at $x$, $\Lambda^k(T^*_x M)$, to the fibre of the same fibre bundle at $y$. It is not obvious how to derive from such a linear map the (1,1)-form we mentioned above. Formally, a linear transformation $\Phi : E \rightarrow F$ could be written as an element of $E \otimes F^*$. If, moreover, one has a metric on $E$ the musical isomorphism gives a bijection between $E \otimes F^*$ and $E^* \otimes F^*$. Composing those algebraic identifications one can look at the family of linear transformation $g^k(x,y)$ as a tensorial product of two differential forms: one defined on $\Lambda^k(T^*_x M)$ and another defined on $\Lambda^k(T^*_y M)$. Any object which comes from such a construction is called a (1,1)-form.

2.2. **Linking form.** Let us recall Theorem 1.10

**Theorem (Vogel [8], De Rham [5])**. Let $(M, g)$ a closed Riemannian manifold. Then the following $(1,1)$-form:

$$\Omega = \ast_y d_y g^I$$

is a linking form i.e., for all pairs of disjoint homologically trivial cycles $\Gamma, \Upsilon \in M$ we have:

$$\text{lk}(\Gamma, \Upsilon) = \int_{\Gamma} \int_{\Upsilon} \Omega.$$
Sketch of proof: In Formula (2.4) of Lemma 2.3 the only object which is not a data of the problem is the primitive of the Poincaré dual of one of the two multicurves we had. The idea is to use Hodge Theory to get it. We start by recalling the Hodge splitting:

\[ \Omega^1(M) = \ker \Delta \oplus \text{Im} d \oplus \text{Im} \delta \]

In view of the equation (2.10) in Definition 2.9 given any exact form \( \omega \in \text{Im} d \), there exists an exact form \( \zeta = G(\omega) \) which solves the following equation:

\[ \Delta \zeta = d \delta \zeta + \delta d \zeta = \omega . \]

From the Hodge splitting the differential form \( \zeta \) has be closed and thus the term \( \delta d \zeta \) of the equation above vanishes:

\[ \Delta \zeta = d(\delta \zeta) = \omega . \]

Therefore, up to the choice of a metric \( g \), the differential form \( \delta G(\omega) \) is a canonical primitive of the differential form \( \omega \).

Now we can use this particular primitive in Lemma 2.3 and get

\[ \text{lk}(\Omega, \Upsilon) = \int_M \delta \circ G^2(\omega_T) \wedge \omega_T . \]

Using the fact that \( G^2 \) is a Kernel operator and the Fubini Theorem we obtain

\[ \text{lk}(\Gamma, \Upsilon) = \int_{M \times M} *_y \delta_y g^2 \wedge \omega_T(y) \wedge \omega_T(x) . \]

Moreover the Green kernel verifies the following identity see [5, page 132] :

\[ *_y \delta_y g^2 = *_y d_y g^1 . \]

We conclude by letting \( \omega_T \) (resp. \( \omega_T \)) tend to the integration current over the multicurve \( \Gamma \) (resp. \( \Upsilon \)), which gives the desired formula. □

Remark 2.15. We used the fact that the multicurves \( \Gamma \) and \( \Upsilon \) are disjoint. In order to let the Poincaré dual tend to the integration current, we have to be sure that points on the product \( \Gamma \times \Upsilon \) are far away from the diagonal, so that the Green kernel is smooth.

This particular linking form is built from a choice of a metric \( g \), then one can ask how it behaves under isometries. In fact, any isometry \( \Phi \) still acts on the product \( M \times M \setminus \Delta \) by the diagonal action \( \Phi \Delta \), and therefore it also acts on \( (1,1) \)-differential forms defined over such a product. The following lemma will be needed in the proof of Theorem 1.4:

Lemma 2.16. The linking form is invariant by the diagonal action of an isometry \( \Phi \):

\[ \Phi^* \Delta (\Omega) = \Omega \]

The proof consists to observe that the pull back by an isometry commutes with both the exterior derivative \( d \) and the Hodge star \( * \), and therefore with the Laplacian and the Green Operator.

Corollary 2.17. If \( \Gamma \) and \( \Upsilon \) are two multicurves (not necessary homologically trivial) and if \( \Phi \) is an isometry of \( (M, g) \) then :

\[ \int \int_{\Gamma \times \Upsilon} *_y d_y g^1 = \int_{\Phi^{-1}(\Gamma) \times \Phi^{-1}(\Upsilon)} *_y d_y g^1 . \]
2.3. An $L^2$ formula for the linking number. To reduce the amount of notations introduced in the above subsection we will simply denote by $(\eta_k)$ (resp. $(\lambda_k)$) instead of $(\eta^1_k)$ (resp. $(\lambda^1_k)$) an orthonormal basis of one differential forms of eigenvectors of $\Delta^1$ (resp. the spectrum of $\Delta^1$).

To conclude this section we will prove the following formula which relates the spectral theory of $(M, g)$ and the linking number. This will be the first step toward the proof of Theorem 1.11.

**Lemma 2.18.** Let $\Gamma$ and $\Upsilon$ be two multicurves and $\omega_\Gamma$, $\omega_\Upsilon$ their Poincaré dual, then

$$\text{LK}(\Gamma, \Upsilon) = \sum_{k \in \mathbb{N}} \left( \int_M \omega_\Gamma \wedge * \left( \frac{d\eta_k}{\lambda_k} \right) \right) \left( \int_M \omega_\Upsilon \wedge \eta_k \right)$$

**Proof.** We will omit the the degree of the differentials forms our operator act on by denoting it $G$. In the proof of Theorem 1.10 Formula (2.14) gives :

$$\text{LK}(\Gamma, \Upsilon) = \int_M \delta G(\omega_\Gamma) \wedge \omega_\Upsilon = < \delta G(\omega_\Gamma) \cdot * \omega_\Upsilon >$$

Therefore by Parseval's identities :

$$\text{LK}(\Gamma, \Upsilon) = \sum_{k \in \mathbb{N}} < \delta G(\omega_\Gamma) \cdot \eta_k > < * \omega_\Upsilon \cdot \eta_k >$$

note that

$$< \delta G(\omega_\Gamma) \cdot \eta_k > = < G(\omega_\Gamma) \cdot d\eta_k >$$

which means that every term of the series (2.19) involving a closed differential form vanishes whenever the differential form $\eta_k$ is closed.

$$< \delta G(\omega_\Gamma) \cdot \eta_k > = < G(\omega_\Gamma) \cdot d\eta_k >$$

since $d$ and $\delta$ are adjoint

$$= < \omega_\Gamma \cdot G(d\eta_k) >$$

since $G$ is self adjoint

$$= < \omega_\Gamma \cdot dG(\eta_k) >$$

since $G$ and $d$ commute

$$= < \omega_\Gamma \cdot \frac{1}{\lambda_k} d\eta_k >$$

since $G(\eta_k) = \frac{\eta_k}{\lambda_k}$.

Recall that we use the convention $\frac{1}{\lambda_k} d\eta_k = 0$ if the differential form is closed. Finally we obtain

$$\text{LK}(\Gamma, \Upsilon) = \sum_{k \in \mathbb{N}} < \omega_\Gamma \cdot \frac{1}{\lambda_k} d\eta_k > < * \omega_\Upsilon \cdot \eta_k >$$

, therefore

$$\text{LK}(\Gamma, \Upsilon) = \sum_{k \in \mathbb{N}} \left( \int_M \omega_\Gamma \wedge * \left( \frac{d\eta_k}{\lambda_k} \right) \right) \left( \int_M \omega_\Upsilon \wedge \eta_k \right)$$

which proves the lemma. $\square$

3. Another formula for the linking number

In order to make computations one would like to "localize" the formula (2.19) of Lemma 2.18 and get the following :
A SPECTRAL VIEWPOINT OF THE LINKING FORM IN THE 3-TORUS

\[ \text{lk}(\Gamma, \Upsilon) = \sum_{k \in \mathbb{N}} \left( \int_{\Gamma} \left( \frac{d\eta_k}{\lambda_k} \right) \right) \left( \int_{\Upsilon} \eta_k \right). \]

by making the representative of the Poincaré dual \( \omega_{\Gamma} \) (resp. \( \omega_{\Upsilon} \)) converge to the integration current over the curves \( \Gamma \) (resp. \( \Upsilon \)). Unfortunately, this series does not make sense \textit{a priori}. In fact, \( \sum_{k \in \mathbb{N}} \frac{1}{\lambda_k} \) does not converge in general and the author does not know how to get a good control of the terms \( \left( \int_{\Gamma} \eta_k \right) \left( \int_{\Upsilon} \eta_k \right) \) as \( k \) tends to \( \infty \). In order to circumvent this difficulty we will use the heat kernel to “smooth” one of the two curves to get a good enough convergence. This will lead to Theorem \text{1.11} announced in the introduction:

**Theorem.** Let \((M, g)\) a closed Riemannian three manifold. If \( \Omega = \ast_y d_y g^1 \) denotes the associated linking form we have:

\[ \int_{\gamma \times \nu} \Omega = \lim_{t \to 0} \sum_{k > 0} e^{-\lambda_k t} \int_{\gamma} \eta_k \int_{\nu} \ast \left( \frac{d\eta_k}{\lambda_k} \right) \]

where \( \gamma \) and \( \nu \) are two disjoint 1-dimensional sub-manifolds of \( M \).

This diffusion process gives for every \( t > 0 \) a differential form which approximates the curve we chose to diffuse: the smallest \( t \), the better the curve is approximated by this differential form. This is essentially the goal of Subsection \text{3.2} to explain such a process. This done, we will be able to write down the series appearing in the right member of Equation (3.1) above, for each \( t > 0 \), as the outcome formula for the diffused curve. This is the goal of Subsection \text{3.3}. To conclude, we will show that letting \( t \) goes to 0 gives the left member of Equation (3.1).

**Remark 3.2.** What follows is essentially of an analytic nature, it is why both curves \( \gamma \) and \( \nu \) need not to be homologically trivial in the above theorem.

3.1. **The heat kernel and its properties.** The following definition is about the main tool we will use to smooth our curve.

**Definition 3.3.** The heat kernel, denoted by \( p^1_t \) associated to the Hodge De Rham laplacian acting on 1-differential forms is the kernel of the heat operator \( e^{-t\Delta^1} \), solving the following equation of unknown a family of 1-forms \( (\eta_t)_{t \in \mathbb{R}_+} \) :

\[ \Delta \eta_t = -\partial_t \eta_t \]

with initial condition \( \eta_0 \), a smooth differential form. Meaning that for all differential forms \( \eta_0 \) we have \( e^{-t\Delta^1}(\eta_0) \) converges to \( \eta_0 \) when \( t \) goes to 0 in the uniform norm topology.

It is well known, see [2, chapter 2.3 page 74] where the construction is precisely made, that these heat kernels always exist on compact Riemannian manifolds.

Let us denote by \( \| \cdot \| \) the \( C^1 \) norm of a differential form \( \omega \) over a manifold \( M \):

\[ \| \omega \|_{C^1} = \sup_{|\alpha| \leq 1} \sup_{x \in M} \| \nabla^\alpha \omega(x) \| \]

where \( \alpha \) is a multindex and \( |\alpha| \) is the length of that multindex and \( \nabla \) the covariant derivative.

This norm induces a norm on the space of linear transformation from the space of differential forms to itself and thus a norm on \((1, 1)\)-forms, that we will still denote
We are interested in $(1,1)$-forms singular over the diagonal of $M \times M$. For this kind of forms the previous norm cannot make sense. Let us denote by $\Delta_\epsilon$ an $\epsilon$ neighbourhood of the diagonal $\Delta \subset M \times M$. The following norm is more adapted to our set up. For the $(1,1)$-form $\eta(x) \otimes \beta(y)$ it is defined as

$$||\eta \otimes \beta||_{C^1 L_\epsilon} = \sup_{|\alpha| \leq l} \sup_{(x,y) \in M \times M \setminus \Delta_\epsilon} ||\nabla^\alpha \eta(x)|| \cdot ||\nabla^\alpha \beta||$$

We will need the following estimates:

**Theorem 3.5.** [2, page 87] Let $(M,g)$ be a compact manifold. For all $\epsilon > 0$ and all $N \in \mathbb{N}$ we have:

$$(3.6) \quad ||p_1^t||_{C^1} t \to 0 \mathcal{O}(t^N)$$

3.2. The diffused curve. Let $\gamma$ be a submanifold of dimension one, we will denote by $L^1(\Omega^1(M))$ the space of differential forms whose norm are summable with respect to the Lebesgue measure.

**Definition 3.7.** We will call the **diffused curve** of $\gamma$, denoted by $e^{-t\Delta^1}(\gamma)$ the following family of operators:

$$e^{-t\Delta^1}(\gamma) : L^1(\Omega^1(M)) \to \mathbb{R} \quad \beta \mapsto \int_\gamma e^{-t\Delta^1}(\beta)$$

The definition comes with the following lemma, which makes precise how a diffused curve actually approximates the integration current over the curve $\gamma$:

**Lemma 3.9.** Let $\beta \in L^1(\Omega^1(M))$ continuous over a neighbourhood $U$ of the curve $\gamma$. Then the following holds:

$$e^{-t\Delta^1}(\gamma)(\beta) \overset{t \to 0}{\longrightarrow} \int_U \beta$$

**Proof:** the very definition of the heat kernel gives:

$$e^{-t\Delta^1}(\gamma)(\beta) = \int_\gamma \left( \int_M p_1^t(x,y) \wedge * \beta(y) dy \right) dx$$

The estimate (3.6) and the fact that $\beta$ is $L^1$ gives

$$|e^{-t\Delta^1}(\gamma)(\beta) - \int_U p_1^t(x,y) dx \wedge * \beta(y) dy| \overset{t \to 0}{\longrightarrow} 0$$

The previous equation shows that the limit value of the left member only depends on the class of $L^1$ differential form $\beta$ which are equal on a neighbourhood of the curve $\gamma$. Because $\beta$ is continuous on such a neighbourhood by hypothesis one can choose a continuous representative in its class. Using the uniform convergence given by the definition/theorem defining the heat kernel one has:

$$\int_U p_1^t(x,y) \wedge * \beta(y) dy \overset{t \to 0}{\longrightarrow} \beta(x)$$

uniformly to its limit (as a differential form defined over $U$). Because of this uniform convergence one is allowed to permute the limit and the integral over the curve $\gamma$:
\[ \int_{\gamma} \int_{U} p_1^t(x,y) \wedge * \beta(y)dy \xrightarrow{t \to 0} \int_{\gamma} \beta \]

which concludes the proof. \[\square\]

Given a curve \( \nu \) we want to apply Lemma 3.9 for the following 1-differential:

\[ \omega_{\nu}(y) := \int_{\nu} \Omega(x,y)dx \]

where \( \Omega = *_1d_1g_1 \) is the linking form. In order to use this lemma we need to show this form is \( L^1 \) and that it is continuous on a neighbourhood of \( \gamma \). The second point is automatically satisfied since \( \nu \) and \( \gamma \) have disjoint supports, the linking form being smooth away from the diagonal. The fact that this form is \( L^1 \) is an outcome of the following estimate, see De Rham [5, page 134]:

\[ ||\Omega(x,y)|| = O\left(\frac{1}{r^2(x,y)}\right) \]

where \( r(x,y) \) is the distance between \( x \) and \( y \). Which guarantees it belongs to the space \( L^1 \), because \( M \) has dimension three. Thus, Lemma 3.9 gives

**Corollary 3.10.** For all pairs of 1-submanifold \( \gamma \) and \( \nu \) of \( M \) we have:

\[ e^{-t\Delta_1}(\gamma)(\omega_{\nu}) \xrightarrow{t \to 0} \int_{\gamma} \int_{\nu} \Omega \]

3.3. **Approximate series.** We are now ready to prove Formula (3.1). Thank to the corollary 3.10 one has that the following tends when \( t \to 0 \) to the right member of the equation (3.1):

\[ e^{-t\Delta_1}(\gamma)(\omega_{\nu}) = \int_{\gamma} e^{-t\Delta_1}(\omega_{\nu}) \]

It remains though to recover the series appearing in the right member of the series (3.1) from the equation above. The series comes from the Plancherel Formula applicable whenever the differential form belongs to \( L^2 \), which is not the case for a integration current. To regularize everything we will use the semi group property of the heat operator, namely:

\[ \int_{\gamma} e^{-t\Delta_1}(\omega_{\nu}) = \int_{\gamma} e^{-\frac{t}{2}\Delta_1} e^{-\frac{t}{2}\Delta_1}(\omega_{\nu}) \]

Using the definition of a kernel we get

\[ \int_{\gamma} e^{-t\Delta_1}(\omega_{\nu}) = \int_{\gamma} \left[ \int_{M} p_1^t(x,y) \wedge * \left( e^{-\frac{t}{2}\Delta_1}(\omega_{\nu})(y) \right) \right]dx \]

By Fubini’s Theorem we have

\[ \int_{\gamma} \left[ \int_{M} p_1^t(x,y) \wedge * \left( e^{-\frac{t}{2}\Delta_1}(\omega_{\nu})(y) \right) \right]dx = \int_{M} \left[ \int_{\gamma} p_1^t(x,y)dx \right] \wedge * \left( e^{-\frac{t}{2}\Delta_1}(\omega_{\nu})(y) \right) \]

Rewriting the equation above as a \( L^2 \) scalar product, we have:
\[ e^{-t\Delta^1(\gamma)}(\omega_{\nu}) = \left\langle \left\lbrack \int_{\gamma} p_{\frac{1}{2}}(x,y)dx \right\rbrack \cdot e^{-\frac{t}{2}\Delta^1(\omega_{\nu})} \right\rangle \]

Using the Plancherel Formula one get:

\[ e^{-t\Delta^1(\gamma)}(\omega_{\nu}) = \sum_{k \in \mathbb{N}} \left\langle \left\lbrack \int_{\gamma} p_{\frac{1}{2}}(x,y)dx \right\rbrack \cdot \eta_k \right\rangle \left\langle e^{-\frac{t}{2}\Delta^1(\omega_{\nu})} \cdot \eta_k \right\rangle \tag{3.11} \]

It remains to identify the sum above with the right term of the equation (3.1) . In fact we show that both following identities

\[ e^{-\frac{\lambda_k t}{2}} \int_{\gamma} \eta_k = \left\langle \left\lbrack \int_{\gamma} p_{\frac{1}{2}}(x,y)dx \right\rbrack \cdot \eta_k \right\rangle \tag{3.12} \]

and

\[ e^{-\frac{\lambda_k t}{2\lambda_k}} \int_{\nu} * d\eta_k = \left\langle e^{-\frac{t}{2}\Delta^1(\omega_{\nu})} \cdot \eta_k \right\rangle \tag{3.13} \]

holds. This proves that each terms of the series appearing in (3.1) are equal to those of the Plancherel formulae (3.11) above.

We first compute (3.12) :

\[ \left\langle \left\lbrack \int_{\gamma} p_{\frac{1}{2}}(x,y)dx \right\rbrack \cdot \eta_k \right\rangle = \int_M \left\lbrack \int_{\gamma} p_{\frac{1}{2}}(x,y)dx \right\rbrack \wedge * \eta_k(y) \]

Using Fubini’s Theorem again we get

\[ \left\langle \left\lbrack \int_{\gamma} p_{\frac{1}{2}}(x,y)dx \right\rbrack \cdot \eta_k \right\rangle = \int_{\gamma} \left[ \int_M p_{\frac{1}{2}}(x,y)dx \wedge * \eta_k(y) \right] = \int_{\gamma} e^{-\frac{t}{2}\Delta^1(\eta_k)} \]

and then, because \( \eta_k \) is an eigenform of eigenvalue \( \lambda_k \) we obtain

\[ \left\langle \left\lbrack \int_{\gamma} p_{\frac{1}{2}}(x,y)dx \right\rbrack \cdot \eta_k \right\rangle = e^{-\frac{t}{2}\lambda_k} \int_{\gamma} \eta_k \tag{3.14} \]

Now we show that the second equation (3.13) also holds. We start off the scalar product

\[ \left\langle e^{-\frac{t}{2}\Delta^1(\omega_{\nu})} \cdot \eta_k \right\rangle = \int_M e^{-\frac{t}{2}\Delta^1(\omega_{\nu})} \wedge * \eta_k \]

Since \( \omega_{\nu} \) belongs to \( L^1(\Omega^1(M)) \) we can apply the Fubini Theorem to get:
\[ \int_{M} e^{-\frac{t}{2}\Delta_1} (\omega_\nu) \wedge *\eta_k = \int_{M} \left( \int_{M} p^1_\tau (x, y) \wedge *\eta_k (x) \right) \wedge *\omega_\nu \]

But because \( \eta_k \) is an eigenform of eigenvalue \( \lambda_k \) we have

\[ \int_{M} p^1_\tau (x, y) \wedge *\eta_k (x) = e^{-\frac{t}{2}\Delta_1} (\eta_k) = e^{-\frac{t}{2}\lambda_k} \eta_k \]

and thus,

\[ \langle e^{-\frac{t}{2}\Delta_1} (\omega_\nu) \cdot \eta_k \rangle = e^{-\lambda_k \frac{t}{2}} \int_{M} \omega_\nu \wedge *\eta_k . \]

But again, because \( \Omega \) integrable we have

\[ \int_{M} \omega_\nu \wedge *\eta_k = \int_{\nu} \left[ \int_{M} \Omega (x, y) \wedge *\eta_k (y) dy \right] dx \]

But because \( \eta_k \) is an eigenform and by construction of the linking form we have :

\[ \int_{M} \Omega (\cdot, y) \wedge *\eta_k (y) dy = *d \left( \int_{M} g^1 (\cdot, y) \wedge *\eta_k (y) dy \right) \]

\[ = *dG (\eta_k) \]

The Green operator \( G \) commuting with the differential \( d \), every terms involving a closed differential form \( \eta_k \) vanishes. In particular, still using the convention \( \frac{1}{\lambda_k} d\eta_k = 0 \), if the differential form is closed we have

\[ *dG (\eta_k) = \frac{*d\eta_k}{\lambda_k} . \]

Therefore

\[ \int_{M} \omega_\nu \wedge \eta_k = \int_{\nu} * \left( \frac{d\eta_k}{\lambda_k} \right) \]

And thus,

\[ \langle e^{-\frac{t}{2}\Delta_1} (\omega_\nu) \cdot \eta_k \rangle = e^{-\lambda_k \frac{t}{2}} \int_{\nu} * \left( \frac{d\eta_k}{\lambda_k} \right) \]

which is the desired equation.

4. Linking number of geodesics in the torus \( \mathbb{T}^3 \)

The aim of this section is to prove Theorem 1.4, the main tool is the formula given by Theorem 1.11 :

\[ \int_{\gamma \times \nu} \Omega = \int_{\gamma \times \nu} \lim_{s \to 0} \Omega_s = \lim_{s \to 0} \sum_{k > 0} e^{-\lambda_k s} \int_{\gamma} \eta_k \int_{\nu} * \left( \frac{d\eta_k}{\lambda_k} \right) \]

We start the section by recalling definitions and notations of our Introduction that will be used to describe the different objects involved in the proof. A complete
description of the spectral theory of 1-differential forms will also be given. This done, we will compute the series

\[ \sum_{k>0} e^{-\lambda_k s} \int_{\gamma} \int_{\nu} \eta_k \left( \frac{d\eta_k}{\lambda_k} \right) \]

for all \( s > 0 \) and any given pair of geodesics \( \gamma, \nu \) not necessary homologically trivials. Eventually, in Subsection 4.3 we make the parameter \( s \) let to 0 to obtain :

\[ \int_{\gamma,\nu} \Omega = \frac{1}{2||\beta||} \det([\gamma], [\nu], \frac{\beta}{||\beta||})(1 - 2(\mu \cdot \beta)) \]

To conclude the proof, we consider two collections of homologically trivial geodesics in the three torus \( \Gamma = (\gamma_j)_{j \in I} \) and \( \Upsilon = (\nu_j)_{j \in J} \). The integration being linear with respect to both factors we have :

\[
\text{lk}(\Gamma, \Upsilon) = \int_{\Gamma \times \Upsilon} \Omega = \int_{\cup_{\gamma_i \times \cup_{\gamma_j}}} \Omega = \sum_{i \in I, j \in J} \int_{\gamma_i \times \gamma_j} \Omega = \sum_{i \in I, j \in J} \frac{1}{2||\beta||} \det([\gamma_i], [\nu_j], \frac{\beta^{i,j}}{||\beta||})(1 - 2(\mu^{i,j} \cdot \beta^{i,j}))
\]

which is the result announced in 1.4.

### 4.1. Notation and spectral theory of the three flat torus.

- We denote by a lower index the coordinates of a vector and by the upper index a member of a family of vectors. For instance, \( \gamma_j^i \) denotes the i-th coordinate of the j-th vector of a certain family of vectors.
- Given a vector \( v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in \mathbb{R}^3 \) we denote by \( v^* \) the differential form \( v_1 dx_1 + v_2 dx_2 + v_3 dx_3 \). By abuse of the notation we will also denote by \( v^* \) the harmonic form \( v_1 dx_1 + v_2 dx_2 + v_3 dx_3 \) on the 3-torus \( T^3 \).
- The euclidian scalar product of two vectors \( a \) and \( b \) in \( \mathbb{R}^3 \) is denoted by \( (a \cdot b) \) its associated euclidian norm by \( || \cdot || \) and the classical wedge product by \( \wedge \).

The next lemma aims at describing an eigenvector of the Laplacian acting on differential 1-forms on the 3-torus \( T^3 \) from the following data :

- A vector \( k = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} \in \mathbb{Z}^3 \)
- An orthonormal basis \((v^1, v^2, v^3)\) of \( \mathbb{R}^3 \).
- A function \( f \in \{\cos, \sin\} \)

**Lemma 4.4.** The differential form

\[ \eta(x) = \sqrt{2} f(2\pi (k \cdot x))(v^j)^* \]

is a normalized eigenform of the laplacian with associated eigenvalue \( \lambda = (2\pi ||k||)^2 \).
Proof. We start by showing they are normalized:

\[
||\eta||_{L^2} := \int_{T^3} \eta \wedge *\eta \\
= \int_{T^3} 2f^2(2\pi(k \cdot x))(v^i)^* \wedge *(v^i)^* \\
= \int_{T^3} 2f^2(2\pi(k \cdot x))\,d\text{vol} = 1
\]

Because of the relation \( f^2 = \frac{1+k}{2} \).

To prove those differential form are eigenvector of the laplacian one has to compute the following:

\[
\Delta \eta = (d\delta + \delta d)\eta
\]

we first compute the term \( d\delta \eta \). Since \( \delta = -*d* \) on differential 1-form in dimension three, see (2.6) one has:

\[
d\delta \eta = d(-*d*)\eta = -\sqrt{2}(d \ast d) \left( f(2\pi(k \cdot x)) \ast (v^i)^* \right).
\]

Because of the identity \( *((v^i)^*)^* = (v^i)^* \wedge (v^j)^* \) whenever \( (i,j,t) \) is a circular permutation of \( (1,2,3) \) we have

\[
d\delta \eta = -\sqrt{2}d \ast d \left( f(2\pi(k \cdot x)) \wedge (v^i)^* \wedge (v^j)^* \right)
\]

and then

\[
d\delta \eta = -\sqrt{2}d \ast d \left( 2\pi k_i f'(2\pi(k \cdot x)) ((v^i)^* \wedge (v^j)^* \wedge (v^j)^*) \right)
\]

\[
= -2\sqrt{2}\pi k_i d f'(2\pi(k \cdot x))
\]

\[
= -2\sqrt{2}\pi k_i d f''(2\pi(k \cdot x))
\]

\[
= -4\sqrt{2}\pi^2 (k_i^2 f''(2\pi(k \cdot x))dx_i + k_i k_j f'''(2\pi(k \cdot x))dx_j + k_i k_i f''(2\pi(k \cdot x))dx_i)
\]

The computation for \( \delta d\eta \) term is similar and gives

\[
\delta d\eta = -4\sqrt{2}\pi^2 (k_i^2 f''(2\pi(k \cdot x))dx_i + k_i^2 f''(2\pi(k \cdot x)))
\]

\[
+ 4\sqrt{2}\pi^2 (k_i k_j f'''(2\pi(k \cdot x))dx_j - k_i k_i f''(2\pi(k \cdot x))dx_i)
\]

By summing both of them we obtain:

\[
\Delta \eta = -4\sqrt{2}\pi^2 (k_i^2 f''(2\pi(k \cdot x))dx_i + k_i^2 f''(2\pi(k \cdot x))dx_i + k_i^2 f''(2\pi(k \cdot x))dx_i)
\]

But because of the relation \( f'' = -f \):

\[
\Delta \eta = 4\pi^2(k_1^2 + k_2^2 + k_3^2)\eta
\]

which is the announced result. \( \square \)

Remark 4.8. The family consisting of all possible differential 1-forms exhibited before is generating as an family of \( L^2 \) differential forms but not free, since we have the trivial relations \( \cos(-k \cdot x) = \cos(k \cdot x) \) and \( \sin(-k \sin x) = -\sin(k \cdot x) \).
4.2. Computation of the approximate linking number. We recall the parametrization proposed in the introduction for two geodesics \( \gamma \) and \( \nu \):

\[
\begin{align*}
\gamma &: \mathbb{R}/\mathbb{Z} \to T^3 \\
\quad t \mapsto \begin{pmatrix} \gamma_1 t + \nu_1 \\ \gamma_2 t + \nu_2 \\ \gamma_3 t + \nu_3 \end{pmatrix}
\end{align*}
\begin{align*}
\nu &: \mathbb{R}/\mathbb{Z} \to T^3 \\
\quad t \mapsto \begin{pmatrix} \nu_1 t + \mu_1 \\ \nu_2 t + \mu_2 \\ \nu_3 t + \mu_3 \end{pmatrix}
\end{align*}
\]

where \( \gamma_i, \nu_j \in \mathbb{Z} \) and \( \mu_j, \nu_j \in [0, 1] \).

We start by computing a single term appearing in the series \( (4.5) \):

\[
(4.10) e^{-\lambda t} \left( \int_{\gamma} * d\eta \right) \left( \int_{\nu} \eta \right). \]

We first note that we can assume \( \nu = 0 \). In fact the translation \( \tau_\nu : x \mapsto x + \nu \) of \( \mathbb{R}^3 \) descends to an isometry of \( \mathbb{R}^3/\mathbb{Z}^3 \), Corollary 2.17 gives:

\[
\int_{\nu} \int_{\tau^{-1}(\nu)} \Omega = \int_{\tau^{-1}(\nu)} \int_{\tau^{-1}(\nu)} \Omega,
\]

where the geodesic \( \tau^{-1}(\nu) \) passes through zero now. We will still denote by \( \mu \) the new starting point, \( \mu_{\gamma,\nu} = \mu - \nu \), of the geodesic \( \nu \). We now use a particular choice of datum of 4.5 to make the computation easier. Up to the trivial relations observed in the remark 4.8 the associated family of eigenform of \( \mathbb{R}^3/\mathbb{Z}^3 \) will be an \( L^2 \) orthonormal basis. To start, the observation above shows the first term of the product \( (4.10) \) vanishes if the function \( f \) is a sinus:

\[
\int_{\gamma} \eta = \int_{[0,1]} \sqrt{2} \sin(2\pi k \cdot (\gamma(t)))(v^i)^*([\gamma]) dt = 0
\]

where \( C_1 \) and \( C_2 \) are two constant, possibly zero.

Now we choose \( \nu_2, \nu_3 \in \mathbb{R}^3 \) such that

\[
\left( v^1 = \frac{[\gamma]}{||[\gamma]||}, v^2, v^3 \right)
\]

is an orthonormal basis of \( \mathbb{R}^3 \). We now consider the family of eigenforms of 4.5 with the function \( a_f(x) = \cos x \) and the above orthonormal basis of \( \mathbb{R}^3 \):

\[
\eta_{k,i} = \sqrt{2} \cos(2\pi(k \cdot x))(v^i)^* \]

We continue by looking at the first term in the product \( (4.10) \) involving such a form

\[
\int_{\gamma} \eta_{k,i} = \int_{t=0}^{1} \cos(2\pi t(k \cdot [\gamma]))(v^i)^*([\gamma]) dt
\]

where \( (v^i)^*([\gamma]) = ([\gamma] \cdot v^i) = ||[\gamma]|| \delta_{i,1} \) by our choice of the basis \( (v^1, v^2, v^3) \). Therefore this integral is zero if one of the following occurs:

- \( (k \cdot [\gamma]) \neq 0 \)
- \( i \neq 1 \)
Therefore, when this integral does not vanish, the function \( \cos((2\pi(k \cdot [\gamma])t) \) is identically constant equal to one. The integral then becomes

\[(4.11) \int_{\gamma} \eta_{k,i} = \sqrt{2}||[\gamma]|| \]

To summarize, the differential forms giving a non-vanishing term in the series (4.2) correspond to

\[\eta_{k,1} = \sqrt{2} \cos(k \cdot x) \left( \frac{[\gamma]}{||[\gamma]||} \right)^{\ast} \]

where \( k \cdot [\gamma] = 0 \). In this case we have

\[\int_{\gamma} \eta_{k,1} = \sqrt{2}||[\gamma]|| \]

**From now we will denote the form \( \eta_{k,1} \)** by \( \eta_k \).

Let us now compute the second term of (4.10)

\[\int_{\nu} * d\eta_k.\]

We fist compute the differential form \(*d\eta_k :\)

\[*d\eta_k = *d \left( \sqrt{2} \cos(2\pi(x \cdot k))(v^1)^{\ast} \right)\]

\[= -2\sqrt{2}\pi \sin(2\pi(x \cdot k)) \ast (k_1 dx_1 \wedge (v^1)^{\ast} + k_2 dx_2 \wedge (v^1)^{\ast} + k_3 dx_3 \wedge (v^1)^{\ast})\]

\[= -2\sqrt{2}\pi \sin(2\pi(x \cdot k))(k \wedge v^1)^{\ast}\]

Finally we get:

\[\int_{\nu} * d\eta_k = \int_{t=0}^{1} -2\sqrt{2}\pi \sin(2\pi t([\nu] \cdot k) + 2\pi(\mu \cdot k)) (k \wedge v^1)^{\ast}([\nu]) dt\]

\[= -2\sqrt{2}\pi \det\left( \frac{[\gamma]}{||[\gamma]||}, [\nu], k \right) \int_{0}^{1} \sin(2\pi t([\nu] \cdot k) + 2\pi(\mu \cdot k)) dt\]

Notice that the \( \int_{\nu} * d\eta_k \) vanishes if one of the following occurs:

* [\gamma] and [\nu] are proportional
* \( (k \cdot [\nu]) \neq 0 \)

Moreover, when \( \int_{\nu} * d\eta_k \) is non zero we have:

\[(4.12) \int_{\nu} * d\eta_k = -2\sqrt{2}\pi \det\left( \frac{[\gamma]}{||[\gamma]||}, [\nu], k \right) \sin(2\pi(\mu \cdot k))\]

From (4.11) and (4.12) we get:

\[(4.13) \int_{\gamma} \eta_k \int_{\nu} * d\eta_k = -4\pi \det([\gamma], [\nu], k) \sin(2\pi(\mu \cdot k))\]

if \( k \in \text{Span}([\gamma], [\nu])^{\perp} \) and 0 if not.

A \( k \in \mathbb{Z}^3 \) is called non vanishing if it lies in \( \text{Span}([\gamma], [\nu])^{\perp} \). The following lemma brings us to the definition of \( \beta \) proposed in the introduction[1]:
Lemma 4.14. Let $b_1$ and $b_2$ two vectors of $\mathbb{Z}^3$. The group
\begin{equation}
G := \text{Span}(b_1, b_2) \perp \cap \mathbb{Z}^3
\end{equation}
is cyclic and generated by two elements denoted $\pm \beta$.

Proof. First, as a set, it is non empty because the vector $b_1 \wedge b_2$ is still in $\mathbb{Z}^3$ and orthogonal to both $b_1$ and $b_2$. This group is trivially contained in the following one
\[ \text{Span}(b_1, b_2) \perp \subset \mathbb{R}^3 \]
and thus embedded it in $\mathbb{R}$. But subgroup of $\mathbb{R}$ are dense or cyclic, according to whether $0$ is isolated or not. As a subset of $\mathbb{Z}^3$, the unit element of $G$ is therefore isolated and the group has to be cyclic. \qed

Applying this lemma to the couple $([\gamma], [v])$ one has the following description of non vanishing $k$:
\begin{equation}
\text{Span}([\gamma], [v]) \perp \cap \mathbb{Z}^3 = \{ k \beta, \ k \in \mathbb{Z} \}
\end{equation}
By a choice made for on the generator we can assume that $([\gamma], [v], \beta)$ is positively oriented.

To sum up, the only terms of the series (4.12) which do not vanish are associated to the differential forms:
\[ \eta_{(k, \beta)} = \sqrt{2} \cos \left( (k \beta) \cdot x \right) \left( \frac{[\gamma]}{||[\gamma]||} \right)^* \]
thus indexed by any integer $k$.

From now we will denote the form by $\eta_k$ the form $\eta_{(k, \beta)}$.

Equation (4.13) becomes:
\begin{equation}
\int_{v} \eta_k \int_{\gamma} * d\eta_k = -4\pi n \det([\gamma], [v], \beta) \sin(2\pi (k(\mu \cdot \beta)))
\end{equation}
with $k \neq 0$. The corresponding value of the product (4.10) therefore becomes:
\[ e^{-\lambda_k s} \int_{v} \eta_k \int_{\gamma} * d\eta_k = e^{-(2\pi ||[\beta]||)^2 s n^2} \pi k ||[\beta]||^2 4\pi \det([\gamma], [v], \beta) \sin(2\pi (k(\mu \cdot \beta))) \]
However, as it is indicated in the remark 4.18 both differential forms $\eta_k$ and $\eta_{-k}$ are the same. We will thus consider $k$ to be positive from now.

\begin{equation}
- \sum_{k>0} e^{-(2\pi ||[\beta]||)^2 s n^2} \pi k ||[\beta]||^2 \det([\gamma], [v], \beta) \sin(2\pi k(\mu \cdot \beta))).
\end{equation}

Remark 4.19. We noticed in Remark 1.6 following Formula 1.5 in Theorem 1.4 that the quantity $\sin(2\pi k(\mu \cdot \beta)))$ involved both in Formulas 4.18 and 1.5 is not obviously well defined. In fact, ultimately, $\mu$ depends of a choice of a starting point coming from a parametrisation of the geodesic $v$. Therefore, it is not clear whether this scalar product is well defined in $\mathbb{R}/\mathbb{Z}$ (equivalently, to have a well defined representative in $[0,1)$). Let us check this does not depend on the choice made about $\mu$. An other choice of a starting point $\mu_2 \in v$ satisfies
\[ \mu_2 - \mu = t[v] + \alpha \]
for some \( t \in \mathbb{R} \) and \( \alpha \in \mathbb{Z}^3 \). Therefore:

\[
(\mu_2 \cdot \beta) = (\mu_2 - \mu + \mu \cdot \beta) = (\mu \cdot \beta) + (\alpha \cdot \beta)
\]

because \( \beta \in [v]_1 \). Reducing the above equation modulo \( \mathbb{Z} \) gives

\[
(\mu_2 \cdot \beta) = (\mu \cdot \beta)
\]

since \( (\alpha \cdot \beta) \in \mathbb{Z} \).

4.3. A family of functions which converge uniformly. If we were allowed to make \( t \to 0 \) terms by terms in the series (4.18) we would get:

\[
-C \sum_{k>0} \frac{1}{k} \sin(2\pi kx)
\]

with \( C = \frac{1}{||\beta||} \det([\gamma], [v], [\beta]) \) and \( x = (\mu \cdot \beta) \).

One can recognize the Fourier expansion of the following function:

\[
(4.20) \quad x \mapsto \left\{ \begin{array}{ll}
0 & \text{if } x = 0 \\
\frac{\pi}{2}(1 - 2x) & \text{on } (0,1)
\end{array} \right.
\]

extented on \( \mathbb{R} \) by parity and 1-periodicity. The series becomes:

\[
(4.21) \quad \int_{\gamma \times \nu} \Omega = \frac{1}{2||\beta||} \det([\gamma], [v], [\beta]) (1 - 2(\mu \cdot \beta))
\]

Which is precisely the desired formula.

Let us now justify the previous computation. The following lemma which is proved by the use of an Abel’s transform is the key

**Lemma 4.22.** Let \( a_k(t) \) and \( b_k(t) \) two sequences of functions defined on an interval \( I \) which contains 0 such that

1. \( \left( \sum_{k \leq n} a_k(t) \right) \) is uniformly bounded in \( t \)
2. The sequence of functions \( b_k(t) \) are decreasing when \( t \) is fixed and tends uniformly (in the variable \( n \)) to 0.

Then the series of function \( \sum_{k \in \mathbb{N}} a_k(t)b_k(t) \) converges uniformly over \( I \)

If we set \( a_k(t) = \sin(2\pi kx) \) and \( b_k(t) = \frac{e^{-\alpha k^2}}{k} \) one can easily check that the assumptions of the preceding lemma are satisfied on the interval \([0, \infty]\) for all \( x, \alpha \in \mathbb{R}^+ \). Thus the series of function:

\[
\sum_{k>0} \frac{e^{-\alpha k^2}}{k} \sin(2\pi kx)
\]

converges uniformly. We are then allowed to permute the limit with the sum:

\[
\lim_{t \to 0} \sum_{k>0} \frac{e^{-\alpha k^2}}{k} \sin(2\pi kx) = \sum_{k>0} \lim_{t \to 0} \frac{e^{-\alpha k^2}}{k} \sin(2\pi kx) = \sum_{k>0} \frac{\sin(2\pi kx)}{k}
\]

Which is the desired result.
4.4. **Linking number and dynamical system.** Particularly interesting collections of geodesics multicurves arise from the dynamic of geodesic flow of the two flat torus. See [7] and [6] for motivations and results in this direction. Here we focus on the geodesic flow on the unitary tangent bundle of the flat two torus. Topologically it is a three torus. Moreover the geodesic flow is explicit here

\[
(4.23) \quad \Phi_t : T^3 \to T^3 \quad (x, y, \theta) \mapsto (x + t \cos \theta, y + t \sin \theta, \theta)
\]

Periodic orbits of this dynamical system are also geodesics for the flat torus on \( T^3 \)

\[
(4.24) \quad \gamma : S^1 \to T^3 \quad t \mapsto (x + t \cos \theta, y + t \sin \theta, \theta)
\]

with \( \theta \) such that \( \arctan \theta \in \mathbb{Q} \). Note that the orbits are naturally oriented by the flow. The following corollary relates the linking number and the intersection number of the geodesics seen as curves on the 2-torus.

**Corollary 4.25.** If \( \Gamma = \gamma^i \) and \( \Upsilon = \upsilon^j \) are two collections of geodesics of \( T^2 \). They link according the following formula :

\[
\text{lk}(\Gamma, \Upsilon) = \sum_{i,j} i(\gamma^i, \upsilon^j) \frac{1 - x_{i,j}}{2}
\]

where \( x_{i,j} \) is the determination of \( \theta_i - \theta_j \) in \([0, 2\pi] \) and \( i(\gamma^i, \upsilon^j) \) is the algebraic intersection between the curves \( \gamma^i \) and \( \upsilon^j \) on the two torus.

**Proof.** Periodic orbits of the geodesic flow stay in the leaf \( \theta \) constant see (4.23) the vectors \([\gamma^i]\) and \([\upsilon^j]\) then belongs to \(\mathbb{R}^2\). We then get that the vector \( \beta^{i,j} \) defined in (4.16) is

\[
\beta^{i,j} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\]

for all \((i, j)\). This implies that \( ||\beta^{i,j}|| = 1 \), moreover the term \( \det( [\gamma^i], [\upsilon^j], \beta^{i,j}) \) then becomes \( \det_{\mathbb{R}^2}([\gamma^i], [\upsilon^j]) \) which is precisely the intersection number between the curves \( \gamma^i \) and \( \upsilon^j \) in the two torus. The quantity \((\beta^{i,j} \cdot \mu^{i,j})\) is the difference of angles between the two and thus \((\pi - (\mu^{i,j} \cdot \beta^{i,j})) = (\pi - (x_{i,j})) \) and the formula is proved. \( \square \)

**Remark 4.26.** It is not obvious this formula is the same that the one proposed by P. Dehornoy in [5] which involves Minkowski sum of planar polygon. It would be interesting to have a direct proof of the equivalence between both formulas.
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