WEAK STABILITY OF A LAMINATED BEAM

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ABSTRACT. In this paper, we consider the stability of a laminated beam equation, derived by Liu, Trogdon, and Yong [6], subject to viscous or Kelvin-Voigt damping. The model is a coupled system of two wave equations and one Euler-Bernoulli beam equation, which describes the longitudinal motion of the top and bottom layers of the beam and the transverse motion of the beam. We first show that the system is unstable if one damping is only imposed on the beam equation. On the other hand, it is easy to see that the system is exponentially stable if direct damping are imposed on all three equations. Hence, we investigate the system stability when two of the three equations are directly damped. There are a total of seven cases from the combination of damping locations and types. Polynomial stability of different orders and their optimality are proved. Several interesting properties are revealed.

1. Introduction. Several three-layer laminated beam and plate models were proposed in the late 1960's and early 1970's [8, 13, 10]. Later, the following general model was developed

\[
\begin{align*}
\rho_1 h_1 u_{1,t}^1 &= E_1 h_1 u_{1,xx}^1 + \tau, \\
\rho_3 h_3 u_{3,t}^3 &= E_3 h_3 u_{3,xx}^3 - \tau, \\
\rho h w_t &= G_1 h_1 (w_x + \phi_1)_x + G_3 h_3 (w_x + \phi_3)_x + h_2 \tau_x, \\
\rho_1 I_1 \phi_{1,t} &= E_1 I_1 \phi_{1,xx} + \frac{h_1}{2} \tau - G_1 h_1 (w_x + \phi_1), \\
\rho_3 I_3 \phi_{3,t} &= E_3 I_3 \phi_{3,xx} + \frac{h_3}{2} \tau - G_3 h_3 (w_x + \phi_3).
\end{align*}
\]

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Here, \( u_i, \phi_i, i = 1, 3 \) are the longitudinal displacement and shear angle of the \( i \)th layer (bottom and top layers); \( w \) is the transverse displacement of the beam; \( \tau \) is the shear stress in the core layer (\( i = 2 \)). The physical parameters represent the material properties. \( h_i, \rho_i, E_i, G_i, I_i > 0 \) are the thickness, density, Young’s modulus, shear modulus, and moments of inertia of the \( i \)th layer for \( i = 1, 2, 3 \), respectively, and \( \rho h = \rho_1 h_1 + \rho_2 h_2 + \rho_3 h_3 \).

When the rotatory inertia and the transverse shear of the bottom and top layers are neglected, equations (1.4) and (1.5) reduce to the familiar Euler-Bernoulli hypothesis \( \phi_1 - w_x = \phi_3 - w_x = 0 \). If we consider the core material to be linearly elastic, i.e., \( \tau = 2G_2\gamma \) with the shear strain

\[
\gamma = \frac{1}{2h_2}(-u_1 + u_3 + \alpha w_x),
\]

and \( \alpha = h_2 + (h_1 + h_3)/2 \), we then obtained the Rao-Nakra model [10],

\[
\begin{align*}
\rho_1 h_1 u_{1tt} &= E_1 h_1 u_{1xx} + \frac{G_2}{h_2}(-u^1 + u^3 + \alpha w_x), \\
\rho_3 h_3 u_{3tt} &= E_3 h_3 u_{3xx} - \frac{G_2}{h_2}(-u^1 + u^3 + \alpha w_x), \\
\rho h w_{tt} &= -EI w_{xxxx} + \frac{G_2\alpha}{h_2}(-u^1 + u^3 + \alpha w_x)_x,
\end{align*}
\]

(1.6) (1.7) (1.8)

where the new coefficients \( \rho h = \sum_{i=1}^3 \rho_i h_i \), \( EI = E_1 I_1 + E_3 I_3 \).

Furthermore, if the extensional forces in the top and bottom layers are also neglected, we obtain the Mead-Makrus model [8],

\[
\begin{align*}
\rho h w_{tt} &= -EI w_{xxxx} + \frac{G_2\alpha}{h_2}(-u^1 + u^3 + \alpha w_x), \\
2h_2\gamma_{xx} &= G_2d\gamma + \alpha w_{xxx}.
\end{align*}
\]

(1.9) (1.10)

which can be simplified into a six-order PDE for \( w \).

When the extensional motion of the bottom and top layers is neglected, we obtain the model proposed by Hansen and Spies [5].

Investigation on the qualitative properties of these models, such as stability and regularity of the solution, subject to certain damping mechanism started in the 1990’s. In [6], exponential stability was proved for the Mead-Makrus model (1.9)-(1.10) when the shear stress \( \tau \) and shear strain \( \gamma \) relation is assumed to be viscoelastic of Boltzmann type. When this relationship is of Kelvin-Voigt type, analyticity of the associated semigroup was proved by Hansen and Liu [4], which was further extended to the corresponding multi-layers beam and plate model by Allen and Hansen in [2, 1]. For the Rao-Nakra model (1.6)-(1.8), exponential stability was obtained when standard boundary damping is imposed on one end of the beam for all three displacements [9]. For the model in [5], exponential stability was proved [12] when structural damping and boundary damping are added, or when viscous damping are added to all three equations [11].

In this paper, we are interested in the stability of the Rao-Nakra model (1.6)-(1.8) with weaker damping. It is clear that exponential stability holds if all three displacements are damped by viscous or Kelvin-Voigt damping distributed over the spatial domain. On the other hand, the system could be unstable if only one displacement is damped, which will be proved in next section. Therefore, we consider
The initial condition is the case when two of the displacements are damped, i.e.,

\[
\begin{align*}
\rho_1 h_1 u_t^1 &= E_1 h_1 (u^1_x + a_1 u^1_{tx})_x + \frac{G_2}{h_2} (-u^1 + u^3 + \alpha w_x) - a_2 u_t^1, \\
\rho_3 h_3 u_t^3 &= E_3 h_3 (u^3_x + b_1 u^3_{tx})_x - \frac{G_2}{h_2} (-u^1 + u^3 + \alpha w_x) - b_2 u_t^3, \\
\rho h w_t &= -EI (w_{xx} + c_1 w_{txx})_x + \frac{G_2 \alpha}{h_2} (-u^1 + u^3 + \alpha w_x) - c_2 w_t,
\end{align*}
\]

where \(a_i, b_i, c_i \geq 0, i = 1, 2\) and only two of them not in the same equation are positive. At the boundary, the beam is hinged and the bottom and top layers satisfy Neumann boundary conditions. That is

\[
\begin{align*}
\begin{cases}
 u^1_x (0) = u^1_x (L) = 0, \\
u^3_x (0) = u^3_x (L) = 0, \\
w(0) = w(L) = w_{xx}(0) = w_{xx}(L) = 0,
\end{cases}
\end{align*}
\]

The initial condition is

\[
(u^1, u_t^1, u^3, u_t^3, w, w_t)(x, 0) = (u^1_0(x), u^1_0(x), u^3_0(x), w_0(x), w_0(x)).
\]

This paper is organized as follows. In Section 2, we present the semigroup setting of the system for well-posedness. Section 3 is devoted to show the polynomial stability of the system by the frequency domain method for seven different cases of damping locations and types. We continue to prove the optimality of the polynomial stability order in Section 4 by spectral analysis.

2. Preliminary and main results. Notice that \((u^1, u^3, w) = (C, C, 0)\) is a static solution to system (1.11)-(1.16) for any constant \(C\). In order to have a unique static solution, we shall choose a proper state space. Let

\[
L^2_0(0, L) = \{ u \in L^2(0, L) : \int_0^L u dx = 0, \}
\]

\[
H^1_+(0, L) = \{ u \in L^2(0, L) : u_x \in L^2(0, L) \}.
\]

Define

\[
H = (H^1_+(0, L) \times L^2_+(0, L))^2 \times H^2_0(0, L) \times L^2(0, L)
\]

with inner product

\[
\langle Y, Z \rangle_H = E_1 h_1 \langle y_1, z_1, x \rangle + \rho_1 h_1 \langle y_2, z_2 \rangle + E_3 h_3 \langle y_3, z_3, x \rangle + \rho_3 h_3 \langle y_4, z_4 \rangle + EI \langle y_{5,xx}, z_{5,xx} \rangle + \rho h \langle y_6, z_6 \rangle + \frac{G_2}{h_2} (-y_1 + y_3 + \alpha y_{5,x}, -z_1 + z_3 + \alpha z_{5,x})
\]

\[
= : Y, Z : \in H.
\]

for \(Y = (y_1, \cdots, y_6)^T, Z = (z_1, \cdots, z_6)^T \in H\). Hereafter, we use \(\langle \cdot, \cdot \rangle\) and \(\| \cdot \|\) to denote the usual \(L^2\) inner product and norm. Let \(v^1 = u^1_t, v^3 = u^3_t, y = w_t, \) and

\[
U = (u^1, v^1, u^3, v^3, w, y)^T.
\]

Define an operator \(A : D(A) \to H\) by
Theorem 2.2. If direct damping is only imposed on the beam equation, i.e.,
\[ a_2 = b_1 = b_2 = c_1 \cdot c_2 = 0, \text{ } c_1 + c_2 > 0, \text{ and } \frac{E_1}{\rho_1} = \frac{E_3}{\rho_3}, \]
then system (2.4) is unstable.

Proof. Set
\[ u^1(x,t) = u^3(x,t) = e^{i\beta t} \phi(x), \text{ } w(x,t) = 0, \]
Substituting (2.7) into system (2.4), we obtain that
\[
\begin{align*}
\beta^2 \phi(x) + \frac{E_1}{\rho_1} \phi''(x) &= 0, \\
\phi'(0) &= \phi'(L) = 0,
\end{align*}
\]
(2.8)
Then, choosing \( \beta_n = \pm \frac{\sqrt{n} \pi}{\sqrt{m} L} \), we obtain
\[
\phi_n(x) = M \cos \frac{n \pi}{L} x.
\]
(2.10)
This implies that there are infinitely many eigenvalues \( \pm i \beta_n \) on the imaginary axis.

The above proof also leads to

**Corollary 2.1.** If \( a_1 = a_2 = b_1 = b_2 = 0 \) and \( \frac{E_1}{\rho_1} = \frac{E_3}{\rho_3} \), then system (2.4) subject to any damping in the beam equation is unstable.

This has motivated us to consider the cases where two of the equations in system (2.4) are directly damped. Our main results are summarized in the next theorem.

**Theorem 2.3.** Let \( e^{At} \) be the semigroup associated with system (2.4).

Case (I). When \( c_1 = c_2 = 0 \),

(i) \( a_1 = b_1 = 0, a_2, b_2 > 0 \), the semigroup \( e^{At} \) is polynomially stable of order \( \frac{1}{4} \);

(ii) \( a_1, b_1 > 0, a_2 = b_2 = 0 \), the semigroup \( e^{At} \) is polynomially stable of order \( \frac{1}{2} \);

(iii) \( a_1 = b_1 = 0, a_2, b_2 > 0 \), the semigroup \( e^{At} \) is polynomially stable of order \( \frac{1}{4} \).

Case (II). When \( a_1 = a_2 = 0 \),

(i) \( b_1 = c_1 = 0, b_2, c_2 > 0 \), when \( \frac{E_1}{\rho_1} = \frac{E_3}{\rho_3} \), i.e., the two wave equations have the same propagation speeds, the semigroup \( e^{At} \) is polynomially stable of order \( \frac{1}{4} \);

when \( \frac{E_1}{\rho_1} \neq \frac{E_3}{\rho_3} \), i.e., the two wave equations have different propagation speeds, the semigroup \( e^{At} \) is polynomially stable of order \( \frac{1}{2} \);

(ii) \( b_1 = c_2 = 0, b_2, c_1 > 0 \), when \( \frac{E_1}{\rho_1} = \frac{E_3}{\rho_3} \), the semigroup \( e^{At} \) is polynomially stable of order \( \frac{1}{2} \);

when \( \frac{E_1}{\rho_1} \neq \frac{E_3}{\rho_3} \), the semigroup \( e^{At} \) is polynomially stable of order \( \frac{1}{4} \);

(iii) \( b_1, c_1 > 0, b_2 = c_2 = 0 \), the semigroup \( e^{At} \) is polynomially stable of order \( \frac{1}{4} \);

(iv) \( b_1, c_2 > 0, b_2 = c_1 = 0 \), the semigroup \( e^{At} \) is polynomially stable of order \( \frac{1}{4} \).

Here, we say that the semigroup \( e^{At} \) is polynomially stable of order \( \frac{1}{k} \), if \( \forall U_0 \in D(A) \), there is a constant \( C > 0 \) such that the solution \( U \) of (2.4) satisfies
\[
\|U\|_H \leq \frac{C}{t^{\frac{k}{2}}} \|U_0\|_{D(A)}.
\]

**Theorem 2.4.** The orders of polynomial stability in Theorem 2.3 are optimal.

Our main tool is the frequency domain characterization of polynomial stability by Borichev and Tomilov.

**Theorem 2.5.** [3] Let \( H \) be a Hilbert space and \( A \) generates a bounded \( C_0 \)-semigroup in \( H \). Assume that
\[
i \mathbb{R} \subset \rho(A),
\]
(2.11)
and
\[
\sup_{|\beta| > 1} \frac{1}{|\beta|^k} \| (i \beta - A)^{-1} \| < +\infty, \quad \text{for some } k > 0.
\]
(2.12)
Then, there exists a positive constant $C > 0$ such that
\[
\|e^{tA}U_0\| \leq C\left(\frac{1}{7}\right)^{\frac{1}{2}}\|U_0\|_{D(A)}, \quad \forall t > 0,
\] (2.13)
for all $U_0 \in D(A)$.

To end this section, we give the following remark on our main results, which provides useful information to the design of laminated beam.

Remark 2.1.

(i). In general, direct dampings on both wave equations are more effective than on one wave equation and the beam equation.

(ii). When both wave equations are directly damped, Kelvin-Voigt damping induces faster energy decay rate than viscous damping. Moreover, the rate is unchanged when Kelvin-Voigt damping is replaced by viscous damping in one of the wave equations.

(iii). When the beam equation and one of the wave equations are directly damped, the rate stays the same when the wave speeds are different; and the rate improves otherwise. Moreover, it does not depend on whether Kelvin-Voigt damping or viscous damping is used.

3. Proof of Theorem 2.3. In this section, we shall prove Theorem 2.3. This amounts to verify conditions (2.11) and (2.12).

Proof. Assume that (2.11) is false, i.e., there is a $\lambda$ such that $i\lambda \in \sigma(A)$. Then there exist a sequence $\beta_n \to \lambda$ and a unit norm sequence $U_n = (u_1^n, v_1^n, u_3^n, v_3^n, w_n, y_n)^T \in D(A)$ such that
\[
(i\beta_n - A)U_n = F_n = o(1) \quad \text{in} \quad H,
\] (3.1)
i.e.,
\[
\begin{cases}
  i\beta u^1 - v^1 = f_1 = o(1), \quad \text{in} \quad H^1, \\
  i\beta v^1 - \frac{E_1}{\rho_1}(u_2^1 + a_1 v_1^1)_x - \frac{G_2}{\rho_1 h_1 h_2}(-u^1 + u^3 + \alpha w_x) + a_2 v^1 = f_2 = o(1), \\
  i\beta u^3 - v^3 = f_3 = o(1), \quad \text{in} \quad H^1, \\
  i\beta v^3 - \frac{E_3}{\rho_3}(u_2^3 + a_2 v_1^3)_x + \frac{G_2}{\rho_3 h_3 h_2}(-u^1 + u^3 + \alpha w_x) + b_2 v^3 = f_4 = o(1), \\
  i\beta w - y = f_5 = o(1), \quad \text{in} \quad H^2, \\
  i\beta y + \frac{EI}{\rho h}(w_{xx} + c_1 y_{xx})_{xx} - \frac{G_2 \alpha}{\rho h h_2}(-u^1 + u^3 + \alpha w_x)_x + c_2 y = f_6 = o(1).
\end{cases}
\] (3.2) (3.3) (3.4) (3.5) (3.6) (3.7)

The second, forth and sixth equations are in spaces $L^2_2, L^2_2$ and $L^2$, respectively. For convenience, we omit the subscript $n$ hereafter. From (3.1) and the dissipativeness of $A$ in (2.6),
\[
Re\langle AU, U \rangle_H = -\frac{a_1 E_1}{\rho_1} \|v_1^1\|^2 - \frac{b_1 E_3}{\rho_3} \|v_3^3\|^2 - \frac{c_1 EI}{\rho h} \|y_{xx}\|^2 - a_2 \|v^1\|^2 - b_2 \|v^3\|^2 - c_2 \|y\|^2 = o(1).
\] (3.8)

Substituting (3.2), (3.4), and (3.6) into (3.3), (3.5), and (3.7), respectively, we obtain
\[
-\beta^2 u^1 - \frac{E_1}{\rho_1}(u_2^1 + a_1 v_1^1)_x - \frac{G_2}{\rho_1 h_1 h_2}(-u^1 + u^3 + \alpha w_x) + a_2 v^1 = f_2 + i\beta f_1 = o(1),
\] (3.9)
\[ -\beta^2 u^1 - \frac{E_3}{\rho_3} (u_x^3 + b_1 v_x^3) + \frac{G_2}{\rho_3 h h_2} (-u^1 + u^3 + \alpha w_x) + b_2 v^3 = f_4 + i\beta f_3 = o(1), \quad (3.10) \]

\[ -\beta^2 w + \frac{EI}{\rho h} (w_{xx} + c_1 y_{xx}) - \frac{G_2 \alpha}{\rho h h_2} (-u^1 + u^3 + \alpha w_x) + c_2 y = f_6 + i\beta f_5 = o(1). \quad (3.11) \]

Then, the respective inner product of (3.9)-(3.11) with \( u^1, u^3, w \) yields

\[ -\|\beta u^1\|^2 + \frac{E_3}{\rho_1} \|u_x^1\|^2 - \frac{G_2}{\rho_1 h_1 h_2} (-u^1 + u^3 + \alpha w_x, u^1) + \frac{a_1 E_1}{\rho_1} \langle v_x^1, u_x^1 \rangle + a_2 (v^1, u^1) = o(1), \quad (3.12) \]

\[ -\|\beta u^3\|^2 + \frac{E_3}{\rho_3} \|u_x^3\|^2 - \frac{G_3}{\rho_3 h h_2} (-u^1 + u^3 + \alpha w_x, u^3) + \frac{b_1 E_3}{\rho_3} \langle v_x^3, u_x^3 \rangle + b_2 (v^3, u^3) = o(1), \quad (3.13) \]

\[ -\|\beta w\|^2 + \frac{EI}{\rho h} \|w_{xx}\|^2 + \frac{G_2 \alpha}{\rho h h_2} (-u^1 + u^3 + \alpha w_x, w_x) + \frac{c_1 EI}{\rho h} \langle y_{xx}, w_{xx} \rangle + c_2 (y, w) = o(1). \quad (3.14) \]

In what follows, we will show that the above will lead to a contraction \( \|U\|_H = o(1) \).

Case (I). When \( c_1 = c_2 = 0 \),

(i) \( a_1 = b_1 = 0, \quad a_2, b_2 > 0 \). From dissipation (3.8),

\[ \|v^1\| = o(1), \quad \|v^3\| = o(1). \quad (3.15) \]

Since \( \beta \) is finite, by (3.2) and (3.4), we also get

\[ \|u^1\| = o(1), \quad \|u^3\| = o(1). \quad (3.16) \]

Using (3.15)-(3.16) in (3.12)-(3.13), also by the boundedness of \( \|w_x\| \), it is easy to see that

\[ \|u_x^1\| = o(1), \quad \|u_x^3\| = o(1). \quad (3.17) \]

The inner product of (3.10) with \( w_x \) in \( L^2 \) leads to \( \|w_x\| = o(1) \), hence by the Poincaré inequality, \( \|w\| = o(1) \) and \( \|\beta w\| = o(1) \) since \( \beta \) is finite. Therefore, from (3.6) and (3.14),

\[ \|y\| = o(1), \quad \|w_{xx}\| = o(1). \quad (3.18) \]

We have reached the contradiction.

(ii) \( a_2 = b_2 = 0, \quad a_1, b_1 > 0 \). From dissipation (3.8),

\[ \|v_x^1\| = o(1), \quad \|v_x^3\| = o(1). \quad (3.19) \]

Since \( \beta \) is finite, by (3.2) and (3.4), then

\[ \|u_x^1\| = o(1), \quad \|u_x^3\| = o(1). \quad (3.20) \]

By the same reasoning in case (i), we can also get

\[ \|y\| = o(1), \quad \|w_{xx}\| = o(1). \quad (3.21) \]

(iii) \( a_1 = b_2 = 0, \quad a_2, b_1 > 0 \). From dissipation (3.8),

\[ \|v^1\| = o(1), \quad \|v_x^3\| = o(1). \quad (3.22) \]

It is straightforward to get \( \|u_x^1\| = o(1), \quad \|u_x^3\| = o(1), \quad \|y\| = o(1), \quad \|w_{xx}\| = o(1) \) by the same argument used in case (i).
Case (II). $a_1 = a_2 = 0$.

(i) $b_1 = c_1 = 0$, $b_2, c_2 > 0$. From dissipation (3.8),
\[ \|v^3\| = o(1), \quad \|y\| = o(1). \]  
Thus, since $\beta$ is finite, by (3.4) and (3.6)
\[ \|u^3\| = o(1), \quad \|w\| = o(1). \]  
In reference to (3.14) and the boundedness of $\|(-u^1 + u^3 + Hw_x)_x\|$, we have
\[ \|w_{xx}\| = o(1), \]  
which further leads to
\[ \|w_x\|^2 \leq C\|w\|\|w_{xx}\| = o(1). \]  
Now, (3.13) is simplified to
\[ \|u^3_x\| = o(1). \]  
In order to obtain estimate on $u^1$, we take $L^2$ inner product of (3.5) with $u^1$,
\[ \langle i\beta v^3 - \frac{E_3}{\rho_3} v^3_{xx} - \frac{G_2}{\rho_3 h_3 h_2} (-u^1 + u^3 + \alpha w_x) + b_2 v^3, u^1 \rangle = o(1). \]  
By (3.23) and (3.26)-(3.27), we can simplify (3.28) to
\[ \|u^1\| = o(1). \]  
where we have used
\[ |\langle u^3_{xx}^3, u^1 \rangle| \leq \|u^3_x\|\|u^1_x\| = o(1). \]  
Moreover, in reference of (3.2) and (3.12), we conclude that.
\[ \|u^3\| = o(1), \quad \|u^3_x\|^2 = o(1). \]  
(ii) $b_1 = c_2 = 0$, $b_2, c_1 > 0$. From dissipation (3.8),
\[ \|v^3\| = o(1), \quad \|y_{xx}\| = o(1). \]  
Since $\beta$ is finite, by (3.4) and (3.6)
\[ \|u^3\| = o(1), \quad \|w_{xx}\| = o(1). \]  
Repeating the analysis in (3.26)-(3.30) in the case (i), we can also get $\|u^3_x\| = o(1)$, $\|u^1\| = o(1)$ and $\|u^1_x\| = o(1)$.

(iii) $b_2 = c_2 = 0$, $b_1, c_1 > 0$. From dissipation (3.8),
\[ \|v^3\| = o(1), \quad \|y_{xx}\| = o(1). \]  
Since $\beta$ is finite, by (3.4) and (3.6)
\[ \|u^3_x\| = o(1), \quad \|w_{xx}\| = o(1). \]  
Repeat the analysis in (3.26)-(3.30) in the case (i), we also get $\|u^1\| = o(1)$ and $\|u^1_x\| = o(1)$.

(iv) $b_2 = c_1 = 0$, $b_1, c_2 > 0$. From dissipation (3.8),
\[ \|v^3\| = o(1), \quad \|y\| = o(1). \]  
Since $\beta$ is finite, by (3.4) and (3.6)
\[ \|u^3_x\| = o(1), \quad \|w\| = o(1). \]  
Repeating the analysis in (3.26)-(3.30) in the case (i), we can also get $\|u^1\| = o(1)$ and $\|u^1_x\| = o(1)$, $\|w_{xx}\| = o(1)$. For all the cases above we obtain $\|U\|_H = o(1)$ which contradicts with $\|U\|_H = 1$. Hence, we conclude that $i\mathbb{R} \subset \rho(A)$. 
Next, assume that (2.12) is false. Then there exist a sequence $\beta \to \infty$ and a unit sequence $U = (u^1, v^1, u^3, v^3, w, y) \in D(\mathcal{A})$ such that
\[
\beta^k \|(i\beta I - A)U\|_H = \|F\|_H = o(1),
\] (3.37)
i.e.,
\[
\begin{align*}
\beta^k(i\beta u^1 - v^1) &= f_1 = o(1), \quad \text{in } H^1_x, \quad \text{(3.38)} \\
\beta^k(i\beta v^1 - \frac{E_1}{\rho_1}(u^1_x + a_1 v^1_x) - \frac{G_2}{\rho_1 h_1 h_2}(-u^1 + u^3 + \alpha w_x) + a_2 v^1) &= f_2, \quad \text{(3.39)} \\
\beta^k(i\beta u^3 - v^3) &= f_3 = o(1), \quad \text{in } H^1_x, \quad \text{(3.40)} \\
\beta^k(i\beta v^3 - \frac{E_3}{\rho_3}(u^3_x + b_1 v^3_x) + \frac{G_2}{\rho_3 h_3 h_2}(-u^1 + u^3 + \alpha w_x) + b_2 v^3) &= f_4, \quad \text{(3.41)} \\
\beta^k(i\beta w - y) &= f_5 = o(1), \quad \text{in } H^2_x, \quad \text{(3.42)} \\
\beta^k(i\beta y + \frac{EI}{ph} (w_{xx} + c_1 y_{xx})_x - \frac{G_2 \alpha}{ph h_2}(-u^1 + u^3 + \alpha w_x)_x + c_2 y) &= f_6. \quad \text{(3.43)}
\end{align*}
\]

The second, forth and sixth equations equal to $o(1)$ in spaces $L^2_x$, $L^2_x$ and $L^2$ respectively. Again, we are going to show $\|U\|_H = o(1)$ for a contradiction. It follows from dissipation (3.8) and (3.37) that
\[
\Re \beta^k \langle AU, U \rangle_H = -\frac{a_1 E_1}{\rho_1} \|\beta^2 v^1_x\|^2 - \frac{b_1 E_3}{\rho_3} \|\beta^2 v^3_x\|^2 - \frac{c_1 EI}{ph} \|\beta^2 y_{xx}\|^2
\]
\[
- a_2 \|\beta^2 v^1\|^2 - b_2 \|\beta^2 v^3\|^2 - c_2 \|\beta^2 y\|^2 = o(1). \quad \text{(3.44)}
\]

Substituting (3.38), (3.40), and (3.42) into (3.39), (3.41), and (3.43), respectively, gives
\[
\begin{align*}
\beta^k(-\beta^2 u^1 - \frac{E_1}{\rho_1}(u^1_x + a_1 v^1_x)) - \frac{G_2}{\rho_1 h_1 h_2}(-u^1 + u^3 + \alpha w_x) + a_2 v^1) &= f_2 + i\beta f_1, \quad \text{(3.45)} \\
\beta^k(-\beta^2 u^3 - \frac{E_3}{\rho_3}(u^3_x + b_1 v^3_x)) + \frac{G_2}{\rho_3 h_3 h_2}(-u^1 + u^3 + \alpha w_x) + b_2 v^3) &= f_4 + i\beta f_3, \quad \text{(3.46)} \\
\beta^k(-\beta^2 w + \frac{EI}{ph} (w_{xx} + c_1 y_{xx})_x - \frac{G_2 \alpha}{ph h_2}(-u^1 + u^3 + \alpha w_x)_x + c_2 y) &= f_6 + i\beta f_5. \quad \text{(3.47)}
\end{align*}
\]

Next, take $L^2$ inner product of (3.45)-(3.47) with $u^1, u^3, w$, respectively. Since $\|\beta u^1\|, \|\beta u^3\|, \|\beta w\|$ are equivalent to $\|v^1\|, \|v^3\|, \|y\|$ which are bounded, we have
\[
\begin{align*}
-\beta^{k+2} \|u^1\|^2 + \beta^k \frac{E_1}{\rho_1} \|u^1_x\|^2 - \beta^k &\frac{G_2}{\rho_1 h_1 h_2}(-u^1 + u^3 + \alpha w_x, u^1) + \beta^k a_2 \langle v^1, u^1 \rangle = o(1), \quad \text{(3.48)} \\
+\beta^k \frac{E_1 a_1}{\rho_1} \langle v^1_x, u^1_x \rangle = \langle f_2 + i\beta f_1, u^1 \rangle = \langle f_2, u^1 \rangle + \langle i f_1, \beta u^1 \rangle = o(1), \quad \text{(3.48)} \\
-\beta^{k+2} \|u^3\|^2 + \beta^k \frac{E_3}{\rho_3} \|u^3_x\|^2 - \beta^k &\frac{G_2}{\rho_3 h_3 h_2}(-u^1 + u^3 + \alpha w_x, u^3) + \beta^k b_2 \langle v^3, u^3 \rangle = o(1), \\
+\beta^k \frac{E_3 b_1}{\rho_3} \langle v^3_x, u^3_x \rangle = \langle f_4 + i\beta f_3, u^3 \rangle = \langle f_4, u^3 \rangle + \langle i f_3, \beta u^3 \rangle = o(1), \quad \text{(3.49)} \\
-\beta^{k+2} \|w\|^2 + \beta^k \frac{EI}{ph} \|w_{xx}\|^2 - \beta^k &\frac{G_2 \alpha}{ph h_2}((-u^1 + u^3 + \alpha w_x)_x, w) + \beta^k c_2 \langle y, w \rangle \\
+\beta^k \frac{C_1 EI}{ph} \langle y_{xx}, w_{xx} \rangle = \langle f_6 + i\beta f_5, w \rangle = \langle f_6, w \rangle + \langle i f_5, \beta w \rangle = o(1). \quad \text{(3.50)}
\end{align*}
\]
Then, it follows from (3.38) and (3.40) that
\[ \|\beta^{1+1} v^1\| = o(1), \quad \|\beta^{1+1} v^3\| = o(1). \] (3.52)

Since \(\|\beta w\|\) and \(\|w_{xx}\|\) are bounded, \(\|\beta^\frac{1}{2} w_x\|\) is also bounded by interpolation. Applying (3.51)-(3.52) to (3.48) yields
\[ \frac{E_1}{\rho_1} \beta^k \|u^1_x\|^2 - \frac{G_2 \alpha}{\rho_1 h_1 h_2} \beta^k \langle w_x, u^1 \rangle = o(1). \] (3.53)

Dividing the above equation by \(\beta^{\frac{1}{2}+2}\) for \(k \geq 3\), we obtain
\[ \frac{E_1}{\rho_1} \|\beta^{\frac{1}{2}+\frac{1}{2}} u^1_x\|^2 - \frac{G_2 \alpha}{\rho_1 h_1 h_2} \langle\beta^\frac{1}{2} w_x, \beta^{\frac{1}{2}+1} u^1 \rangle = o(1), \] (3.54)
i.e.,
\[ \|\beta^{\frac{1}{2}+\frac{1}{2}} u^1_x\| = o(1). \] (3.55)

Similarly, we also have
\[ \|\beta^{\frac{1}{2}+\frac{1}{2}} u^3_x\| = o(1). \] (3.56)

In order to estimate the \(w\) term, we take the \(L^2\) inner product of (3.45) with \(\beta^\frac{1}{2} w_x\) to get
\[ \langle i \beta^{k+1} v^1 - \frac{E_1}{\rho_1} \beta^k u^1_{xx} - \frac{G_2}{\rho_1 h_1 h_2} \beta^k (-u^1 + \alpha w_x) - \frac{E_1 a_1}{\rho_1} \beta^k v^1_{xx} + a_2 \beta^k v^1, \beta^\frac{1}{2} w_x \rangle = o(1). \] (3.57)

Dividing (3.57) by \(\beta^{\frac{1}{2}+2}\) and integrating by parts, we can rewrite it as
\[ \frac{E_1}{\rho_1} \langle \beta^{\frac{1}{2}-1} u^1_x, w_{xx} \rangle - \frac{G_2 \alpha}{\rho_1 h_1 h_2} \langle \beta^{\frac{1}{2}-1} w_x, \beta^\frac{1}{2} w_x \rangle = o(1). \] (3.58)

For \(k = 3\), (3.58) becomes
\[ \frac{E_1}{\rho_1} \langle \beta u^1_x, w_{xx} \rangle - \frac{G_2 \alpha}{\rho_1 h_1 h_2} \|\beta^\frac{1}{2} w_x\|^2 = o(1). \] (3.59)

As \(\|w_{xx}\| = O(1)\) and \(\|\beta u^1_x\| = o(1)\) which is guaranteed by (3.55) for \(k = 3\), we conclude that \(\|\beta^\frac{1}{2} w_x\| = o(1)\). From (3.43), it is clear that \(\|\beta^{-1} w_{xxx}\|\) is bounded since \(c_1 = 0\), so is \(\|\beta^{-\frac{1}{2}} w_{xxx}\|\) by interpolation. Therefore,
\[ \langle \beta^\frac{1}{2} w_x, \beta^{-\frac{1}{2}} w_{xxx} \rangle = -\|w_{xx}\|^2 = o(1). \] (3.60)

Finally, (3.50) is now reduced to
\[ -\|\beta w\|^2 + \frac{EI}{\rho h} \|w_{xx}\|^2 = o(1), \] (3.61)

This leads to \(\|\beta w\| = o(1)\), which further implies that \(\|y\| = o(1)\). We have arrived the promised contradiction for \(k = 3\).
Combining it with (3.38) and (3.40) yields
\[ \| \beta^{\frac{1}{2}} v_1^1 \| = o(1), \quad \| \beta^{\frac{1}{2}} v_2^3 \| = o(1). \] (3.62)

By the Poincaré inequality,
\[ \| \beta^{\frac{1}{2}} v_1^1 \| = o(1), \quad \| \beta^{\frac{1}{2}} v_2^3 \| = o(1), \quad \| \beta^{\frac{1}{2}} + u_1^1 \| = o(1) \text{ and } \| \beta^{\frac{1}{2}} + u_2^3 \| = o(1). \] (3.64)

Next, the $L^2$ inner product of (3.52) with $\beta^{\frac{1}{2}} w_x$ gives
\[ - \beta^k \langle \beta^{\frac{1}{2}} u_3^3, \beta w \rangle - \frac{E_3}{\rho_3} \langle \beta^{\frac{1}{2}} u_3^3, w_{xx} \rangle - \frac{G_2}{\rho_3 h_3 h_2} \langle \beta^{\frac{1}{2}} (-u_1^1 + u_3^3 + \alpha w_x), w_x \rangle - \frac{b_1 E_3}{\rho_3} \langle \beta^{\frac{1}{2}} v_3^3, w_{xx} \rangle = o(1). \] (3.66)

For $k = 2$, (3.66) leads to
\[ \| \beta^{\frac{1}{2}} w_x \| = o(1). \]

This leads to the promised contradiction for $k = 2$.

(iii) $a_1 = b_2 = 0$, $a_2, b_1 > 0$. From dissipation (3.44),
\[ \| \beta^{\frac{1}{2}} v_1^1 \| = o(1), \quad \| \beta^{\frac{1}{2}} v_2^3 \| = o(1). \] (3.67)

Combining it with (3.38) and (3.40),
\[ \| \beta^{\frac{1}{2}} + u_1^1 \| = o(1), \quad \| \beta^{\frac{1}{2}} + u_2^3 \| = o(1). \] (3.68)

By the Poincaré inequality,
\[ \| \beta^{\frac{1}{2}} v_3^3 \| = o(1), \quad \| \beta^{\frac{1}{2}} + u_1^1 \| = o(1). \] (3.69)

Then, repeat the argument after (3.64) in case (ii). Thus,
\[ \| u_x^1 \| = o(1), \quad \| \beta w \| = o(1), \quad \text{and } \| w_{xx} \| = o(1). \]

We have arrived the promised contradiction for $k = 2$.

Case (II). $a_1 = a_2 = 0$,

(i) $b_1 = c_1 = 0$, $b_2, c_2 > 0$. From the dissipation (3.44),
\[ \| \beta^{\frac{1}{2}} v_3^3 \|^2 = o(1), \quad \| \beta^{\frac{1}{2}} y \|^2 = o(1). \] (3.70)

Combining it with (3.40) and (3.42) yields
\[ \| \beta^{\frac{1}{2}} + u_3^3 \| = o(1), \quad \| \beta^{\frac{1}{2}} + w \| = o(1). \] (3.71)

Since $\| w_{xx} \|$ is bounded, by interpolation
\[ \| \beta^{\frac{1}{2}} + w_x \| \leq C \| \beta^{\frac{1}{2}} + w \| \| w_{xx} \| = o(1). \] (3.72)

Hence, (3.50) is simplified to
\[ \frac{EI}{\rho h} \| \beta^{\frac{1}{2}} w_{xx} \|^2 + \beta^k \frac{G_2 \alpha}{\rho h h_2} \langle u^1, w_x \rangle = o(1). \] (3.73)
Dividing (3.73) by $\beta^{\frac{3k-6}{4}}$ for $k \geq 2$ yields

$$\frac{EI}{\rho h} \| \beta^{\frac{3}{4}} w_{xx} \|^2 + \frac{G_2 \alpha}{\rho h^2} \langle \beta u_1, \beta^{\frac{3}{4}} w_x \rangle = o(1),$$

(3.74)

which implies that

$$\| \beta^{\frac{3}{4}} w_{xx} \|^2 = o(1).$$

(3.75)

Let's consider the cases that the bottom and top layers have same or different wave speeds.

When $\frac{E_1}{\rho_1} = \frac{E_3}{\rho_3}$, Taking $L^2$ inner product of (3.39) with $u^3$, and (3.41) with $u^1$ gives

$$\langle i \beta^{k+1} v^1, u^3 \rangle - \frac{E_1}{\rho_1} \beta^k \langle u^1_{xx}, u^3 \rangle - \frac{G_2}{\rho_1 h_1 h_2} \beta^k \langle -u^1 + u^3 + \alpha w_x, u^3 \rangle = o(1),$$

(3.76)

$$\langle i \beta^{k+1} v^3, u^1 \rangle - \frac{E_3}{\rho_3} \beta^k \langle u^3_{xx}, u^1 \rangle + \frac{G_2}{\rho_3 h_3 h_2} \beta^k \langle -u^1 + u^3 + \alpha w_x, u^1 \rangle + b_2 \beta^k \langle v^3, u^1 \rangle = o(1).$$

(3.77)

We subtract (3.76) from (3.77), then take the real part. This will cancel the first two terms in each of them. Thus,

$$\frac{G_2}{\rho_3 h_3 h_2} \beta^k \| u^1 \|^2 = \frac{G_2}{\rho_1 h_1 h_2} \| u^3 \|^2 + \frac{G_2}{\rho_1 h_1 h_2} \beta^k \text{Re} \langle w_x, u^3 \rangle - \frac{G_2}{\rho_3 h_3 h_2} \beta^k \text{Re} \langle w_x, u^1 \rangle - \beta^k \text{Re} \langle b_2 v^3, u^1 \rangle + o(1).$$

(3.78)

Take $k = 2$, by (3.70)-(3.72), we have

$$\frac{G_2}{\rho_3 h_3 h_2} \| \beta u^1 \|^2 = \frac{G_2}{\rho_1 h_1 h_2} \| u^3 \|^2 + \langle \frac{G_2}{\rho_1 h_1 h_2} w_x, \beta^2 u^3 \rangle - \langle \frac{G_2}{\rho_3 h_3 h_2} \beta w_x, \beta u^1 \rangle - b_2 \langle \beta v^3, \beta u^1 \rangle + o(1) = o(1).$$

(3.79)

It follows from (3.48) and (3.49) that

$$\| u^1_x \|^2 = o(1), \quad \| u^3_x \|^2 = o(1),$$

(3.80)

due to (3.70)-(3.72).

When $\frac{E_1}{\rho_1} \neq \frac{E_3}{\rho_3}$: Taking $L^2$ inner product of (3.41) with $u^1$ and dividing by $\beta^{\frac{3}{4}}$

$$\langle i \beta^{\frac{3}{4}} v^3, \beta u^1 \rangle - \frac{E_3}{\rho_3} \beta^{\frac{3}{4}} \langle u^3_{xx}, u^1 \rangle + \frac{G_2}{\rho_3 h_3 h_2} \beta^{\frac{3}{4}} \langle -u^1 + u^3 + \alpha w_x, u^1 \rangle + b_2 \beta^{\frac{3}{4}} \langle v^3, u^1 \rangle = o(1).$$

(3.81)

By (3.70)-(3.72), (3.53) becomes

$$\frac{E_3}{\rho_3} \| \beta^{\frac{3}{4}} u^3_x \|^2 - \langle \beta^{\frac{3}{4}} u^1, \beta^{\frac{3}{4}} u^3_x \rangle = o(1).$$

(3.82)

Dividing by $\beta^{\frac{3}{4}}$ for $k \geq 4$, we have

$$\frac{E_3}{\rho_3} \| \beta^{\frac{3}{4}} u^3_x \|^2 - \langle \beta u^1, \beta^{\frac{3}{4}} u^3_x \rangle = o(1),$$

(3.83)

i.e.,

$$\| \beta^{\frac{3}{4}} u^3_x \|^2 = o(1).$$

(3.84)

Take $k = 4$ and apply (3.70)-(3.72) and (3.84) to (3.81). This leads to

$$\frac{E_3}{\rho_3} \langle \beta^{\frac{3}{4}} u^3_x, u^1 \rangle - \frac{G_2}{\rho_3 h_3 h_2} \beta^{\frac{3}{4}} \| u^1 \|^2 - \frac{G_2 \alpha}{\rho_3 h_3 h_2} \beta^{\frac{3}{4}} \langle w_x, u^1 \rangle = o(1)$$

(3.85)
Therefore,
\[ \| \beta u^1 \| = o(1). \]  \hfill (3.86)

Finally, (3.48) gives
\[ \| u^2 \| = o(1). \]  \hfill (3.87)

We have arrived the promised contradiction.

(ii) \( b_1 = c_2 = 0, b_2, c_1 > 0 \). From the dissipation (3.44),
\[ \| \beta \frac{2}{3} u^3 \| = o(1), \quad \| \beta \frac{2}{3} y_{xx} \| = o(1). \]  \hfill (3.88)

Combining it with (3.40) and (3.42),
\[ \| \beta \frac{2}{3} + w^3 \| = o(1), \quad \| \beta \frac{2}{3} + w_{xx} \| = o(1). \]  \hfill (3.89)

By interpolation,
\[ \| \beta \frac{2}{3} + 1 w_x \|^2 \leq C \| \beta w \| \| \beta \frac{2}{3} + 1 w_{xx} \| = o(1). \]  \hfill (3.90)

Repeat the arguments in (3.76)-(3.87) in case (i). Then,
\[ \| \beta u^1 \| = o(1), \quad \| u^1 \| = o(1), \quad \| u^2 \| = o(1), \]
when \( \frac{E_1}{\rho_1} = \frac{E_3}{\rho_3} \) and \( k = 2 \), or when \( \frac{E_1}{\rho_1} \neq \frac{E_3}{\rho_3} \) and \( k = 4 \). We have arrived the promised contradiction.

(iii) \( b_2 = c_2 = 0, b_1, c_1 > 0 \). From the dissipation (3.44),
\[ \| \beta \frac{2}{3} u^3 \| = o(1), \quad \| \beta \frac{2}{3} y_{xx} \| = o(1). \]  \hfill (3.91)

Combining it with (3.40) and (3.42),
\[ \| \beta \frac{2}{3} + 1 u^3 \| = o(1), \quad \| \beta \frac{2}{3} + 1 w_{xx} \| = o(1). \]  \hfill (3.92)

By the Poincaré inequality and interpolation,
\[ \| \beta \frac{2}{3} u^3 \| = o(1), \quad \| \beta \frac{2}{3} + 1 u^3 \| = o(1), \quad \| \beta \frac{2}{3} + 1 w_{xx} \| = o(1). \]  \hfill (3.93)

The \( L^2 \) inner product of (3.41) with \( u^1 \) gives
\[ \langle i \beta \frac{2}{3} u^3, u^1 \rangle - \frac{E_3}{\rho_3} \beta \frac{2}{3} \langle u^3_{xx}, u^1 \rangle - \frac{G_2}{\rho_3 h_3 h_2} \beta \frac{2}{3} \langle -u^1 + u^3 + \alpha w_x, u^1 \rangle \\
- b_1 \frac{E_3}{\rho_3} \beta \frac{2}{3} \langle u^3_{xx}, u^1 \rangle = o(1). \]  \hfill (3.94)

Then, dividing it by \( \beta \frac{2}{3} \) to get
\[ \langle i \beta \frac{2}{3} u^3, \beta u^1 \rangle - \frac{E_3}{\rho_3} \beta \frac{2}{3} \langle u^3_{xx}, u^1 \rangle - \frac{G_2}{\rho_3 h_3 h_2} \beta \frac{2}{3} \langle -u^1 + u^3 + \alpha w_x, u^1 \rangle \\
+ b_1 \frac{E_3}{\rho_3} \beta \frac{2}{3} \langle u^3_{xx}, u^1 \rangle = o(1). \]  \hfill (3.95)

Applying (3.91) and (3.92), the above is further reduced to
\[ \frac{E_3}{\rho_3} \beta \frac{2}{3} \langle u^3_{xx}, u_x \rangle + \frac{G_2}{\rho_3 h_3 h_2} \beta \frac{2}{3} \| u^1 \|^2 - \beta \frac{2}{3} \frac{G_2 \alpha}{\rho_3 h_3 h_2} \langle w_x, u^1 \rangle = o(1). \]  \hfill (3.96)

Diving (3.96) by \( \beta \frac{2}{3} \) for \( k \geq 4 \), we obtain
\[ \frac{E_3}{\rho_3} \beta \frac{2}{3} + 1 u^3_{xx}, u_x \rangle \frac{G_2}{\rho_3 h_3 h_2} \beta \frac{2}{3} + 1 \| u^1 \|^2 + \frac{G_2 \alpha}{\rho_3 h_3 h_2} \beta \frac{2}{3} + 1 \langle w_x, u^1 \rangle = o(1). \]  \hfill (3.97)

Take \( k = 4 \), (3.97) leads to
\[ \| \beta u^1 \| = o(1). \]  \hfill (3.98)
Thus, by (3.48),
\[ \|u^1_x\| = o(1). \] (3.99)
We have arrived the promised contradiction.

(iv) \( b_2 = c_1 = 0, \ b_1, c_2 > 0 \). From dissipation,
\[ \|\beta^k y\|^2 = o(1) , \quad (3.100) \]
By the Poincaré inequality,
\[ \|\beta^k v^3\|^2 = o(1). \] (3.101)
Then by (3.40) and (3.42)
\[ \|\beta^k+1 u^3\| = o(1), \quad \|\beta^k+1 v^3\|^2 = o(1), \quad \|\beta^k+1 w\| = o(1). \] (3.102)
Repeat (3.72)-(3.75) in case (i). We again obtain
\[ \|\beta^k v^4\|^2 = o(1). \]
Repeat (3.94)-(3.97) in case (iii). For \( k = 4 \), we also have
\[ \|\beta u^1\| = o(1), \quad \|v^1_x\| = o(1). \]
We have arrived the promised contradiction.

4. Proof of Theorem 2.4. In this section, we shall show that the orders of poly-
nomial stability given in Theorem 2.3 are optimal. This is accomplished by showing
that there is a branch of eigenvalues \( \lambda \) of \( A \) such that
\[ (\text{Im} \lambda)^k \text{Re} \lambda = O(1), \quad |\lambda| \to \infty \]
for the value of \( k \) in each case in Theorem 2.3.

Assume that the order of polynomial stability is \( k + \epsilon \) for any \( \epsilon > 0 \). Then,
\[ \sup_{|\eta| > 1} \frac{1}{|\eta|^{k+\epsilon}} \|(i\eta I - A)^{-1}\| \leq M. \]
By the resolvent identity, for \( |\xi \eta^{k+\epsilon}| < \frac{1}{2\pi \eta} \),
\[ \left( -\xi + i\eta I - A \right)^{-1} = (i\eta I - A)^{-1} \left( I - \xi (i\eta I - A)^{-1} \right)^{-1} \]
\[ = (i\eta I - A)^{-1} \left( I - \xi |\eta|^{k+\epsilon} \frac{(i\eta I - A)^{-1}}{|\eta|^{k+\epsilon}} \right)^{-1}. \]
Thus, there is a curve on the complex plane
\[ \Gamma^\xi_{k+\epsilon} = \left\{ Z = -\frac{\xi}{\eta^{k+\epsilon}} \pm i\eta \mid \eta \geq 1 \right\} \cup \left\{ -\xi \pm i\eta \mid |\eta| < 1 \right\} \]
for some \( \xi > 0 \), which divides the complex plane into two parts, one is on the
right side of the curve including the curve itself. We denote it by \( C^\xi_{k+\epsilon} \). The above
resolvent identity implies that
\[ C^\xi_{k+\epsilon} \subset \rho(A). \]
But, the eigenvalues we found will eventually fall into \( C^\xi_{k+\epsilon} \) as \( \text{Im} \lambda \to \infty \). A
contraction.

For the convenience and readability, set
\[ \rho_i = h_i = 1, \quad i = 1, 2, 3; \quad L = \pi, \quad E_3 = G_2 = \alpha = 1. \]
Then \( \rho h = 3 \). We denote \( EI \) by \( K \). \( E_1 \) is left alone since it will determine whether
the wave speeds of the bottom and top layers are same or not.
Proof. Let $\lambda \in \sigma_p(A)$, $U = (u^1, v^1, u^3, v^3, w, y)^T \in D(A)$ be the eigenfunction corresponding to $\lambda$. We consider the eigenvalue equation

$$AU = \lambda U,$$

that is

$$\begin{cases}
\lambda u^1 - v^1 = 0, \\
\lambda u^1 - E_1(u^1_x + a_1 v^1_x)_x - (-u^1 + u^3 + w_x) + a_2 v^1 = 0, \\
\lambda u^3 - v^3 = 0, \\
\lambda v^3 - (u^3_x + b_1 v^3_x)_x + (-u^1 + u^3 + w_x) + b_2 v^3 = 0, \\
\lambda w - y = 0, \\
\lambda y + \frac{K}{3}(w_{xx} + c_1 y_{xx})_{xx} - \frac{1}{3}(-u^1 + u^3 + w_x)_x + \frac{c_2}{3} y = 0.
\end{cases}$$

Eliminating $v^1, v^3, y$ in Equations (4.1)-(4.6), we get

$$\begin{cases}
(\lambda^2 + 1) u^1 + a_2 \lambda u^3 - E_1(1 + a_1 \lambda) u^1_{xx} - (u^3 + w_x) = 0, \\
(\lambda^2 + 1) u^3 + b_2 \lambda w^3 - (1 + b_1 \lambda) u^3_{xx} - (u^1 - w_x) = 0, \\
\lambda^2 w + \frac{K}{3}(1 + c_1 \lambda) w_{xxxx} + \frac{c_2}{3} \lambda w - \frac{1}{3}(-u^1 + u^3 + w_x)_x = 0.
\end{cases}$$

Since $U \in D(A)$, we set

$$v^1(x) = A \cos nx, \quad v^3(x) = B \cos nx, \quad w(x) = C \sin nx.$$  

Substituting (4.10) into (4.7)-(4.9), we obtain

$$\begin{cases}
\gamma_1 A - B - nC = 0, \\
-A + \gamma_2 B + nC = 0, \\
\frac{n}{3} A + \frac{n}{3} B + \gamma_3 C = 0.
\end{cases}$$

where

$$\begin{cases}
\gamma_1 = \lambda^2 + a_2 \lambda + E_1 n^2(1 + a_1 \lambda) + 1, \\
\gamma_2 = \lambda^2 + b_2 \lambda + n^2(1 + b_1 \lambda) + 1, \\
\gamma_3 = \lambda^2 + \frac{c_2}{3} \lambda + \frac{K}{3} n^4(1 + c_1 \lambda) + \frac{n^2}{3}.
\end{cases}$$

Therefore, $\lambda$ satisfies $\Delta(\lambda) = 0$, where

$$\Delta(\lambda) = \det \begin{pmatrix} \gamma_1 & -1 & -n \\ -1 & \gamma_2 & n \\ -\frac{n}{3} & \frac{n}{3} & \gamma_3 \end{pmatrix}.$$ 

Case (I). $c_1 = c_2 = 0$

We consider the asymptotic behavior of $\lambda$. As we all known, the eigenvalue $\pm i \lambda_0$ of a single Euler-Bernoulli beam without any damping distribute in the imaginary axis. Moreover, we can obtain that

$$\lambda_0 = \sqrt{\frac{Kn^4}{3} + \frac{n^2}{3}}.$$
Set $\lambda_n = i\lambda_0 + \epsilon_n$, $\epsilon_n = o(n^2)$. Then
\[
\begin{align*}
\gamma_1 &= -\frac{Kn^4}{3} + i\alpha_1 n^2\lambda_0 + n^2(E_1 - \frac{1}{3}) + i\alpha_2 n^2 + 1 + i\alpha_1 n^2\epsilon_n + 2i\lambda_0\epsilon_n + \epsilon_n^2,
\gamma_2 &= -\frac{Kn^4}{3} + ib_1 n^2\lambda_0 + 2n^2 + ib_2 n^2 + 1 + b_1 n^2 \epsilon_n + 2i\lambda_0 \epsilon_n + b_2 \epsilon_n + \epsilon_n^2,
\gamma_3 &= 2i\lambda_0 \epsilon_n + \epsilon_n^2.
\end{align*}
\]

(i) $a_1 = b_1 = 0$, $a_2$, $b_2 > 0$

Substituting (4.15)-(4.17) into $\Delta(\lambda) = 0$, we get
\[
\frac{\Delta(\lambda)}{\lambda_0 n^8} = \frac{2K^2\epsilon_n}{9} + \frac{F_1(\epsilon_n)}{9n^2 \lambda_0} + o(n^{-4}) = 0,
\]
where every term of $F_1(\epsilon_n)$ is a polynomial function of $\epsilon_n$ and $F_1(\epsilon_n)/\epsilon_n \to 0$, $n \to \infty$. Hence,
\[
\epsilon_n = \frac{1}{Kn^2 \lambda_0} i + \sigma_n,
\]
where $\sigma_n = o(n^{-4})$.

Now, substituting (4.17)-(4.18) into $\Delta(\lambda) = 0$ again, we obtain that
\[
\frac{2K^2 \sigma_n i}{9} - \frac{K}{9} (3E_1 + 1) \left( -\frac{1}{Kn^2 \lambda_0} + \frac{2i\sigma_n}{n^2} \right) + \frac{2K}{3n^4} (a_2 + b_2) \left( \frac{1}{Kn^2} i + \lambda_0 \sigma_n \right) - \frac{1}{3} (a_2 + b_2) i \frac{1}{9} (3E_1 + 1) - \frac{1}{n^4 \lambda_0} + O(n^{-8}) = 0.
\]

Then, we have
\[
\sigma_n = -\frac{3}{2K^2 n^6} (a_2^2 + b_2^2) + Q(n)i,
\]
where
\[
Q(n) = \frac{1}{2K^2} (3E_1 + 1) \frac{1}{n^4 \lambda_0}.
\]

Therefore, in this case one branch of spectra of $A$ has the asymptotic expression as following:
\[
\lambda_n = -\frac{3(a_2^2 + b_2^2)}{2K^2 n^6} + i \left( \lambda_0 + \frac{1}{Kn^2 \lambda_0} + Q(n) \right) + o\left( \frac{1}{n^6} \right) = O(n^{-6}) + iO(n^2).
\]

(ii) $a_1$, $b_1 > 0$, $a_2 = b_2 = 0$

Substituting (4.15)-(4.17) into $\Delta(\lambda) = 0$, we get
\[
\frac{\Delta(\lambda)}{\lambda_0 n^8} = 2i \left( -\frac{K}{3} + \frac{a_1 E_1 i \lambda_0}{n^2} \right) \left( -\frac{K}{3} + \frac{b_1 i \lambda_0}{n^2} \right) \epsilon_n - 2i \epsilon_n \left( \frac{2K}{3n^2} - \frac{a_1 E_1 \lambda_0 i}{n^4} \right) + (E_1 - \frac{1}{3}) \frac{Kn^4}{3n^2} - b_1 \lambda_0 i \right) + F_2(\epsilon_n) + \frac{2K}{9n^2 \lambda_0} - \frac{b_1 + a_1 E_1}{3n^4} - \frac{1 + 3E_1}{9n^4 \lambda_0} + o(n^{-6}) = 0.
\]

where every term of $F_2(\epsilon_n)$ is a polynomial function of $\epsilon_n$ and $F_2(\epsilon_n)/(n^{-2}\epsilon_n) \to 0$, $n \to \infty$. Hence,
\[
\epsilon_n = -\frac{1}{6n^4} \left( \frac{a_1 E_1}{m_1} + \frac{b_1}{m_2} \right) + \frac{K}{18\lambda_0 n^2} \left( \frac{1}{m_1} + \frac{1}{m_2} \right) i + o(n^{-4}).
\]
where
\[ m_1 = \frac{K^2}{9} + \frac{a_1^2 E_1^2 \lambda_0^2}{n^4}; \quad m_2 = \frac{K^2}{9} + \frac{b_1^2 \lambda_0^2}{n^4}. \] (4.22)

Therefore, in this case, one branch of eigenvalues of \( A \) has the asymptotic expression as following:
\[ \lambda_n = -\frac{1}{6n^2} \left( \frac{a_1}{m_1} + \frac{b_1}{m_2} \right) + i \lambda_0 + \frac{K}{18 \lambda_0 n^2} \left( \frac{1}{m_1} + \frac{1}{m_2} \right) + o(n^{-4}) = O(n^{-4}) + iO(n^2). \] (4.23)

where \( \lambda_0, m_1, m_2 \) denote as (4.14) and (4.22), respectively.

(iii) \( a_1 = b_2 = 0, a_2, b_1 > 0 \)

Substituting (4.15)-(4.17) into \( \Delta(\lambda) = 0 \), we get
\[ \frac{\Delta(\lambda)}{\lambda_0 n^8} = -2i \epsilon_n \frac{K}{3} \left( \frac{K}{3} + \frac{\lambda_0}{n^2} \right) + \frac{1}{2K n^2 \lambda_0} + F_3(\epsilon_n) - \frac{1}{3n^2 \lambda_0} \left( \frac{K}{3} + \frac{\lambda_0}{n^2} \right) + o(n^{-4}) = 0, \]

where every term of \( F_3(\epsilon_n) \) is a polynomial function of \( \epsilon_n \) and \( F_3(\epsilon_n)/\epsilon_n \to 0, n \to \infty \). Hence,
\[ \epsilon_n = -\frac{1}{3m_2 n^4} + i \left( \frac{1}{2K n^2 \lambda_0} + \frac{K}{18m_2 n^2 \lambda_0} \right) + o(n^{-4}). \] (4.24)

where \( m_2 \) is defined in (4.22).

Therefore, in this case, one branch of eigenvalues of \( A \) has the asymptotic expression as following:
\[ \lambda_n = -\frac{b_1}{3m_2 n^4} + i(\lambda_0 + \frac{1}{2K n^2 \lambda_0} + \frac{K}{18m_2 n^2 \lambda_0}) + o(n^{-4}) = O(n^{-4}) + iO(n^2). \] (4.25)

Case (II). \( a_1 = a_2 = 0 \)

Similar to case (I), we will consider the asymptotic behavior of \( \lambda \). Set
\[ \lambda_n = i \lambda_0 + \epsilon_n, \quad \epsilon_n = o(n), \]

where
\[ \lambda_0 = \sqrt{E_1 n^2 + 1}. \] (4.26)

Then
\[
\begin{align*}
\gamma_1 &= 2i \lambda_0 \epsilon_n + \epsilon_n^2, \\
\gamma_2 &= b_1 n^2 i \lambda_0 + (1 - E_1) n^2 + b_2 \lambda_0 + b_1 n^2 \epsilon_n + 2 \lambda_0 \epsilon_n + \epsilon_n^2 + b_2 \epsilon_n, \\
\gamma_3 &= \frac{c_1 K n^4 i \lambda_0}{3} + \frac{K n^4}{3} + \frac{2}{3} n^2 + \frac{c_2 \lambda_0}{3} + \frac{c_1 K n^4}{3} \epsilon_n + \frac{2i \lambda_0 \epsilon_n}{3} + \frac{c_2 \epsilon_n}{3} + \epsilon_n^2 - 1.
\end{align*}
\] (4.27)

(i) \( b_1 = c_1 = 0, b_2, c_2 > 0 \)

When \( E_1 \neq 1 \), substituting (4.27)-(4.29) into \( \Delta(\lambda) = 0 \), we get
\[ \frac{\Delta(\lambda)}{\lambda_0 n^6} = (1 - E_1) \frac{2K i \epsilon_n}{3} + G_1(\epsilon_n) - (1 - E_1) \frac{1}{3 \lambda_0 n^2} - \frac{K}{3 \lambda_0 n^2} + o(n^{-3}) = 0, \]

where every term of \( G_1(\epsilon_n) \) is a polynomial function of \( \epsilon_n \) and \( G_1(\epsilon_n)/\epsilon_n \to 0, n \to \infty \). Hence,
\[ \epsilon_n = -\left( \frac{1}{2K} + \frac{1}{2(1 - E_1)} \right) \frac{1}{\lambda_0 n^2} i + \tilde{\sigma}_n, \] (4.30)

where \( \tilde{\sigma}_n = o(n^{-3}) \).
Now, substituting (4.27)-(4.30) into $\Delta(\lambda) = 0$ again, we obtain that
\[
(1 - E_1)\frac{2 Ki \sigma_n}{3} - \frac{b_2}{3n^4} + \frac{2 b_2 K}{3} \left( \frac{1}{2K} + \frac{1}{2\lambda_0 n^2 (1 - E_1)} \right) \frac{1}{n^4} - \frac{\lambda_0 \sigma_n}{n^2} + o(n^{-4}) = 0.
\]
Then, we have
\[
\sigma_n = -\frac{b_2}{2(1 - E_1)^2 n^4} + o(n^{-4}).
\]
Therefore, in this case, one branch of eigenvalues of $\mathcal{A}$ has the asymptotic expression as following:
\[
\lambda_n = -\frac{b_2}{2(1 - E_1)^2 n^4} + i \left( \lambda_0 - \left( \frac{1}{2K} + \frac{1}{2(1 - E_1)} \right) \frac{1}{\lambda_0 n^2} \right) + o(n^{-4})
\]
\[
= O(n^{-4}) + iO(n). \tag{4.31}
\]
where $\hat{\lambda}_0$ denotes as (4.26).

When $E_1 = 1$, substituting (4.27)-(4.29) into $\Delta(\lambda) = 0$, we get
\[
\Delta(\lambda) = \frac{2 b_2 c_2 Ki \epsilon_n}{3} + H_2(\epsilon_n) - \frac{c_2 Ki}{3 \lambda_0^2} + o(n^{-2}) = 0, \tag{4.32}
\]
where every term of $H_2(\epsilon_n)$ is a polynomial function of $\epsilon_n$ and $H_2(\epsilon_n)/\epsilon_n \to 0$, $n \to \infty$. Hence,
\[
\epsilon_n = -\frac{1}{2b_2 \lambda_0^2} + o(n^{-2}). \tag{4.33}
\]
Therefore, in this case, one branch of eigenvalues of $\mathcal{A}$ has the asymptotic expression as following:
\[
\lambda_n = -\frac{1}{2b_2 \lambda_0^2} + i \hat{\lambda}_0 + o(n^{-2}) = O(n^{-2}) + iO(n). \tag{4.34}
\]
where $\hat{\lambda}_0$ denotes as (4.26).

(ii) $b_1 = c_2 = 0$, $b_2$, $c_1 > 0$

When $E_1 \neq 1$, substituting (4.27)-(4.29) into $\Delta(\lambda) = 0$, we get
\[
\Delta(\lambda) = 2i \epsilon_n (1 - E_1) \frac{c_1 Ki \lambda_0}{3} + G_2(\epsilon_n) - \frac{c_1 Ki}{3n^2 \lambda_0} + o(n^{-3}) = 0, \tag{4.35}
\]
where every term of $G_2(\epsilon_n)$ is a polynomial function of $\epsilon_n$ and $G_2(\epsilon_n)/\epsilon_n \to 0$, $n \to \infty$. Hence,
\[
\epsilon_n = \hat{\epsilon}_n i + \hat{\sigma}_n, \tag{4.36}
\]
where $\hat{\sigma}_n = o(n^{-3})$ and
\[
\hat{\epsilon}_n = -\frac{1}{2(1 - E_1)n^2 \lambda_0}. \tag{4.37}
\]
Now, substituting (4.27)-(4.29) and (4.36) into $\Delta(\lambda) = 0$ again, we obtain that
\[
2i(1 - E_1) \frac{c_1 Ki}{3} (\hat{\epsilon}_n i + \hat{\sigma}_n) + 2i(\hat{\epsilon}_n i + \hat{\sigma}_n)[(1 - E_1) \frac{K}{3 \lambda_0} - \frac{b_2 c_1 K \lambda_0}{3n^2}] \]
\[
- \frac{c_1 Ki}{3n^2 \lambda_0} - \frac{1}{3n^2 \lambda_0^2} (1 - E_1) - \frac{K}{3n^2 \lambda_0^2} + o(n^{-4}) = 0.
\]
Then, we have
\[
\hat{\sigma}_n = -\frac{1}{2n^2 \lambda_0^2} - \frac{b_2}{2(1 - E_1)^2 n^4} + o(n^{-4}).
\]
Therefore, in this case, one branch of eigenvalues of $A$ has the asymptotic expression as following:

$$\lambda_n = -\frac{1}{2n^2\dot{\lambda}_0^2} - \frac{b_2}{2(1 - E_1)n^4} + i(\dot{\lambda}_0 - \frac{1}{2(1 - E_1)n^2\lambda_0}) + o(n^{-4})$$

$$= O(n^{-4}) + iO(n),$$  

where $\dot{\lambda}_0$ denotes as (4.26).

When $E_1 = 1$, substituting (4.27)-(4.29) into $\Delta(\lambda) = 0$, we get

$$\frac{\Delta(\lambda)}{\lambda_0^2 n^4} = 2\left[ \frac{b_2 Ki}{3} + \frac{2K\dot{\epsilon}_n i}{3}\right] \dot{\epsilon}_n + H_1(\dot{\epsilon}_n) - \frac{K}{3\lambda_0^2} + o(n^{-2}) = 0,$$

where every term of $H_1(\dot{\epsilon}_n)$ is a polynomial function of $\dot{\epsilon}_n$ and $H_1(\dot{\epsilon}_n)/\dot{\epsilon}_n \to 0$, $n \to \infty$. Hence,

$$\dot{\epsilon}_n = -\frac{1}{2b_2\lambda_0^2} + o(n^{-2}).$$  

(4.39)

Therefore, in this case, one branch of eigenvalues of $A$ has the asymptotic expression as following:

$$\lambda_n = -\frac{1}{2b_2\lambda_0^2} + i\dot{\lambda}_0 + o(n^{-2}) = O(n^{-2}) + iO(n)$$

(4.40)

where $\dot{\lambda}_0$ denotes as (4.26).

(iii) $b_2 = c_2 = 0$, $b_1$, $c_1 > 0$

Substituting (4.27)-(4.29) into $\Delta(\lambda) = 0$, we get

$$\frac{\Delta(\lambda)}{\lambda_0^2 n^6} = \frac{2ib_1 c_1 K\dot{\epsilon}_n}{3} + G_3(\dot{\epsilon}_n) - \frac{b_1 i}{3n^2\lambda_0^2} - \frac{c_1 Ki}{3\lambda_0^2 n^2} + o(n^{-4}) = 0,$$  

(4.41)

where every term of $G_3(\dot{\epsilon}_n)$ is a polynomial function of $\dot{\epsilon}_n$ and $G_3(\dot{\epsilon}_n)/\dot{\epsilon}_n \to 0$, $n \to \infty$. Hence,

$$\dot{\epsilon}_n = -\frac{1}{2c_1 K\lambda_0^2 n^2} - \frac{1}{2b_1 \lambda_0^2 n^2} + o(n^{-4}),$$  

(4.42)

Therefore, in this case, one branch of eigenvalues of $A$ has the asymptotic expression as following:

$$\lambda_n = -\frac{1}{2c_1 K\lambda_0^2 n^2} - \frac{1}{2b_1 \lambda_0^2 n^2} + i\dot{\lambda}_0 + o(n^{-4}) = O(n^{-4}) + iO(n),$$

(4.43)

where $\dot{\lambda}_0$ denotes as (4.26).

(iv) $b_2 = c_1 = 0$, $b_1$, $c_2 > 0$

Substituting (4.27)-(4.29) into $\Delta(\lambda) = 0$, we get

$$\frac{\Delta(\lambda)}{\lambda_0^2 n^6} = -\frac{2\dot{\epsilon}_n b_1 K}{3} + G_4(\dot{\epsilon}) - \frac{b_1 i}{3\lambda_0 n^2} + o(n^{-3}) = 0,$$  

(4.44)

where every term of $G_2(\dot{\epsilon}_n)$ is a polynomial function of $\dot{\epsilon}_n$ and $G_2(\dot{\epsilon}_n)/\dot{\epsilon}_n \to 0$, $n \to \infty$. Hence,

$$\dot{\epsilon}_n = \dot{\epsilon}_n i + \dot{\sigma}_n,$$  

(4.45)

where $\dot{\sigma}_n = o(n^{-3})$ and

$$\dot{\epsilon}_n = -\frac{1}{2K\lambda_0 n^2}.$$  

(4.46)
Now, substituting (4.27)-(4.29) and (4.45) into $\Delta(\lambda) = 0$ again, we obtain that
\[
2i(\tilde{\epsilon}_n + \tilde{\sigma}_n) - \frac{b_1 Ki_3}{3} + \frac{2}{3}\tilde{\lambda}_0 - \frac{\tilde{\lambda}_0}{3n^2\lambda_0^2} - \frac{1}{3n^2\lambda_0^2}(1 - E_1)
\]
\[-\frac{b_1 i}{3\tilde{\lambda}_0 n^2} + o(n^{-4}) = 0.
\]
Then, we have
\[
\tilde{\sigma}_n = -\frac{1}{2b_1 n^2\lambda_0^2} + o(n^{-4}).
\]
Therefore, in this case, one branch of eigenvalues of $A$ has the asymptotic expression as following:
\[
\lambda_n = \frac{1}{2b_1 n^2\lambda_0^2} + i\tilde{\lambda}_0 - \frac{1}{2K\lambda_0 n^2} + o(n^{-4}) = O(n^{-4}) + iO(n),
\]
where $\tilde{\lambda}_0$ denotes as (4.26).

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REFERENCES

[1] A. A. Allen and S. W. Hansen, Analyticity of a multilayer Mead-Markus plate, Nonlinear Anal., 71 (2009), e1835–e1842.
[2] A. A. Allen and S. W. Hansen, Analyticity and optimal damping for a multilayer Mead-Markus sandwich beam, Discrete Contin. Dyn. Syst. Ser. B., 14 (2010), 1279–1292.
[3] A. Borichev and Y. Tomilov, Optimal polynomial decay of functions and operator semigroups, Math. Ann., 347 (2010), 455–478.
[4] S. W. Hansen and Z. Liu, Analyticity of semigroup associated with a laminated composite beam, Control of Distributed Parameter and Stochastic Systems (Hangzhou, 1998), Kluwer Acad. Publ., Boston, MA, 1999, 47–54.
[5] S. W. Hansen and R. Spies, Structural damping in a laminated beam due to interfacial slip, J. Sound and Vibration, 204 (1997), 183–202.
[6] Z. Liu, S. A. Trogdon, and J. Yong, Modeling and analysis of a laminated beam, Math. Comput. Modeling, 30 (1999), 149–167.
[7] Z. Liu and S. Zheng, Semigroup Associated with Dissipative System, Res. Notes Math., Vol 394, Chapman & Hall/CRC, Boca Raton, 1999.
[8] D. J. Mead and S. Markus, The forced vibration of a three-layer, damped sandwich beam with arbitrary boundary conditions, J. Sound Vibr. (2), 10 (1969), 163–175.
[9] A. Özkan Özer and S. W. Hansen, Uniform stabilization of a multilayer Rao-Nakra sandwich beam, Evol. Equ. Control Theory, 2 (2013), 695–710.
[10] Y. V. K. S Rao and B. C. Nakra, Vibrations of unsymmetrical sanwich beams and plates with viscoelastic cores, J. Sound Vib. (3), 34 (1974), 309–326.
[11] C. A. Raposo, Exponential stability of a structure with interfacial slip and frictional damping, Applied Math. Letter, 53 (2016), 85–91.
[12] J. M. Wang, G. Q. Xu and S. P. Yung, Stabilization of laminated beams with structural damping by boundary feedback controls, SIAM Control Optim., 44 (2005), 1575–1597.
[13] M. J. Yan and E. H. Dowell, Governing equations for vibratory constrained-layer damping sandwich plates and beams, J. Appl. Mech. (4), 39 (1972), 1041–1046.

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