From Domain Wall to Overlap in 2+1d

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Abstract: The equivalence of domain wall and overlap fermion formulations is demonstrated for lattice gauge theories in 2+1 spacetime dimensions with parity-invariant mass terms. Even though the domain wall approach distinguishes propagation along a third direction with projectors $\frac{1}{2}(1 \pm \gamma_3)$, the truncated overlap operator obtained for finite wall separation $L_s$ is invariant under interchange of $\gamma_3$ and $\gamma_5$. In the limit $L_s \to \infty$ the resulting Ginsparg-Wilson relations recover the expected $U(2N_f)$ global symmetry up to $O(a)$ corrections. Finally it is shown that finite-$L_s$ corrections to bilinear condensates associated with dynamical mass generation are characterised by whether even powers of the symmetry-breaking mass are present; such terms are absent for antihermitian bilinears such as $i\bar{\psi}\gamma_3\psi$, markedly improving the approach to the large-$L_s$ limit.

Keywords: Lattice Gauge Field Theories, Field Theories in Lower Dimensions, Global Symmetries

1 Introduction

Relativistic fermions moving in 2 spatial dimensions are the focus of much attention, in part due to the stability of Dirac points in graphene and surface states of topological band insulators when the underlying Hamiltonian is symmetric under time reversal and spatial inversion (see, eg. [1]). Even in this case a gap may develop at the Dirac points in the presence of interactions. The corresponding issue in quantum field theory is the stability of the vacuum with respect to spontaneous generation of a parity-invariant bilinear condensate of the form $\langle \bar{\psi}\Gamma_i\psi \rangle \neq 0$. Since the transition to a gapped phase generically occurs for strong interactions, it defines a quantum critical point (QCP) [2]; the phase diagram for planar fermionic systems with various interactions and characterisation of possible QCPs as a function of the number of fermion species $N_f$ remain open questions [3].

To date there have been many lattice field theory simulations probing QCPs using the staggered fermion formulation [4] (a notable recent exception employs the SLAC derivative [5]); $N$ staggered fermions describe $N_f = 2N$ continuum flavors each having 4 spinor components [6], with global symmetry group $U(N) \otimes U(N)$ spontaneously broken by a parity-invariant mass to $U(N)$. However, because there are two matrices $\gamma_3$ and $\gamma_5$ which anticommute with the kinetic operator, the correct continuum symmetry
breaking is \( U(2N_f) \rightarrow U(N_f) \otimes U(N_f) \). For the strongly-interacting continuum limit at a QCP, there is no reason \textit{a priori} to expect the correct symmetry-breaking pattern to be recovered.

For this reason the properties of domain wall fermions, which purportedly more faithfully reproduce continuum symmetries, were explored for \( 2+1+1 \) in Ref. \[7\]. In particular bilinear condensates and meson correlators constructed from distinct spinor combinations, but which should yield identical results in a U(2)-invariant theory, were investigated as a function of the extent \( L_s \) of the “third” direction separating the domain walls. Numerical results obtained in the context of quenched non-compact QED\(_3\) with variable coupling strength support U(2) symmetry being restored as \( L_s \rightarrow \infty \). In 2+1d the Ginsparg-Wilson relation specifying the optimal requirements for lattice fermions to avoid species doubling while retaining as much of the continuum global symmetry as possible \[8\] generalises to a set of three relations (since chiral rotations are now specified by an element of U(2) rather than U(1)). These were set out in \[7\], along with the specification of an overlap Dirac operator \( D_{ov} \) \[9\] defined in 2+1d in which realises them. As it must, \( D_{ov} \) has equivalent properties under the U(2) rotations generated by \( \gamma_3 \) and \( \gamma_5 \).

In the domain wall approach, the 2+1d fields \( \psi, \bar{\psi} \) are defined in terms of surface states fields \( \Psi_\pm, \bar{\Psi}_\pm \) which are approximately localised on the walls and are \( \pm \) eigenstates of \( \gamma_3 \) \[10\]. Some questions which remain unanswered in \[7\] are: the extent to which the domain wall formulation, in which propagation along the direction separating the walls is governed by \( \gamma_3 \), can maintain the equivalence between \( \gamma_3 \) and \( \gamma_5 \) rotations for finite \( L_s \); the reason for \( O(a) \) violations of U(2) symmetries even in the overlap limit \( L_s \rightarrow \infty \); and a better understanding of why finite-\( L_s \) corrections are minimised by choosing \( i\langle \bar{\psi}\gamma_3\psi \rangle \), rather than \( \langle \bar{\psi}\psi \rangle \), as the bilinear condensate to focus on. In this brief technical Letter I outline how the overlap operator is recovered in the \( L_s \rightarrow \infty \) limit of the domain wall formulation using a by now familiar sequence of matrix algebra operations. In particular, it will prove possible to extend the key results on the equivalence of \( \gamma_3 \) and \( \gamma_5 \) to a truncated overlap operator defined by domain wall fermions with finite \( L_s \). As well as providing a firm conceptual foundation for domain wall fermions and their symmetry properties in 2+1d, the proof sheds light on each of these outstanding issues.

2 From Domain Wall to Overlap

First we review the passage from the domain wall formulation of lattice fermions to the overlap operator. The corresponding treatment for 4d gauge theories is well-known \[11\]: here we follow closely the treatment of \[12\]. We begin from the 2 + 1d domain wall operator defined in \[7\], correcting an overall (unphysical) sign:

\[
S^{dw} = \sum_{x,y} \sum_{s,r} \bar{\Psi}(x,s) D(x,s|y,r) \Psi(y,r),
\]  

(1)
The fields $\Psi$, $\bar{\Psi}$ are four-component spinors defined in 2+1+1 dimensions, and

$$D(x, s|y, s') = \delta_{s, y} D_W(x|y) + \delta_{x, y} D_3(s|s'), \quad (2)$$

where the first term is the 2 + 1d Wilson operator defined on spacetime volume $V$

$$(D_W - M)_{x,y} = -\frac{1}{2} \sum_{\mu=0,1,2} \left[ (1 - \gamma_\mu)U_\mu(x)\delta_{x+\hat{\mu}, y} + (1 + \gamma_\mu)U_\mu(y)\delta_{x-\hat{\mu}, y} \right] + (3 - M)\delta_{x,y}, \quad (3)$$

and $D_3$ controls hopping along the dimension separating the domain walls at $s = 1$ and $s = L_s$, which we will refer to as the third direction:

$$D_3(s,s') = -[P_+ \delta_{s+1,s'}(1 - \delta_{s,L_s}) + P_- \delta_{s-1,s'}(1 - \delta_{s',1})] + \delta_{s,s'}, \quad (4)$$

where the projectors $P_{\pm} \equiv \frac{1}{2}(1 \pm \gamma_3)$. Following convention, in (3) we include interaction with a SU($N_c$) valued gauge connection field $U_\mu(x)$ located on the lattice links, noting in passing that some models relevant for 2+1d QCPs share the global $U(2N_f)$ symmetries of gauge theories.

Initially we supplement (1) with a hermitian mass term coupling fields on opposite walls:

$$m_h S_h = m_h \sum_x \bar{\Psi}(x, L_s) P_- \Psi(x, 1) + \bar{\Psi}(x, 1) P_+ \Psi(x, L_s). \quad (5)$$

The operator $D_W - M + D_3 + m_h S_h$ can be represented as a $L_s \times L_s$ matrix consisting of $4VN_c \times 4VN_c$ blocks:

$$D(m_h) =
\begin{bmatrix}
D_W - M + 1 & 0 & \cdots & +m_h \\
-1 & D_W - M + 1 & 0 \\
0 & -1 & \ddots \\
\vdots & & \ddots & -1 & D_W - M + 1 \\
0 & D_W - M + 1 & -1 \\
\vdots & & \ddots & 0 \\
&m_h & \cdots & -1 & D_W - M + 1 \\
\end{bmatrix} P_+ +
\begin{bmatrix}
D_W - M + 1 & -1 & \cdots & 0 \\
0 & D_W - M + 1 & -1 \\
\vdots & \cdots & \ddots & 0 \\
+m_h & \cdots & -1 & D_W - M + 1 \\
\end{bmatrix} P_- \quad (6)$$

Now define the cyclical shift operator $P_{s,s'} \equiv [\delta_{s-1,s'}(1 - \delta_{s,1}) + \delta_{s,1}\delta_{s',L_s}]P_- + \delta_{s,s'}P_+$ so that

$$DP =
\begin{bmatrix}
Q_+ & 0 & \cdots & Q_-C_- \\
Q_- & Q_+ & 0 & 0 \\
0 & Q_- & \ddots & \ddots \\
\vdots & \ddots & \ddots & 0 \\
0 & 0 & \ddots & Q_-Q_+ C_+ \\
\end{bmatrix} \quad (7)$$

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with
\[ Q_\pm = (D_W - M + 1)P_\pm - P_\mp; \]  
\[ C_\pm(m_h) = \frac{1}{2}(1 - m_h) \pm \frac{1}{2}(1 + m_h)\gamma_3 = P_\pm - m_hP_\mp. \]  
(8)  
(9)

Now define the block diagonal matrix \( Q = Q_+I_{4VN_c \times 4VN_c} \); it is important to note that \( Q_\pm \neq Q_\pm(m_h), \ P \neq P(m_h), \ \mathcal{P} \neq \mathcal{P}(m_h) \). With \( D \equiv Q^{-1}\mathcal{P} \), we deduce
\[
\det[D^{-1}(1)\tilde{D}(m_h)] = \det[D^{-1}D(m_h)],
\]  
(10)

where
\[
\tilde{D} = \begin{bmatrix}
1 & 0 & \cdots & -T^{-1}C_-
-1 & 1 & 0 & 0
0 & -1 & 1 & \vdots
\vdots & \ddots & \ddots & 0
0 & \cdots & -1 & C_+
\end{bmatrix},
\]  
(11)

with \( T = -Q_+^{-1}Q_+ \).

In more detail,
\[
T = -[(D_W - M + 1)P_- - P_+]^{-1}[(D_W - M + 1)P_+ - P_-] = \begin{bmatrix} 1 - \gamma_3\frac{(D_W - M)}{2 + (D_W - M)} \end{bmatrix}^{-1} \begin{bmatrix} 1 + \gamma_3\frac{(D_W - M)}{2 + (D_W - M)} \end{bmatrix} = \frac{1 - H}{1 + H}
\]  
(12)

where the hermitian \( 4VN_c \times 4VN_c \) matrix \( H \) is defined
\[
H = -\gamma_3[2 + (D_W - M)]^{-1}[D_W - M] \equiv -\gamma_3A.
\]  
(13)

Hermiticity of \( H \) requires \( \gamma_3A\gamma_3 = A^\dagger \), which is the case for \( A \) defined by (3). Up to an unphysical sign and with \( \gamma_3 \) assuming the role played by \( \gamma_5 \) in \( 4d \) gauge theories, \( H \) is identical with the Shamir kernel [13].

Next observe that in the form (11), \( \tilde{D} = LDU \) with
\[
L = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
-1 & 1 & 0 & \vdots \\
0 & -1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1
\end{bmatrix}; \quad U = \begin{bmatrix}
1 & 0 & \cdots & -T^{-1}C_-
0 & 1 & 0 & -(T^{-1})^2C_-
\vdots & \ddots & \ddots & -(T^{-1})^3C_-
0 & \cdots & \cdots & 1
\end{bmatrix}
\]  
(14)

and
\[
D = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & 0 & \vdots \\
\vdots & 1 & \ddots & \vdots \\
0 & \cdots & C_+ - (T^{-1})^{L_s}C_-
\end{bmatrix},
\]  
(15)
Again, note $L \neq L(m_h)$, and $\det L = \det U = 1$. We conclude

$$\det[D(1)^{-1}D(m_h)] = \det[D(1)^{-1}D(m_h)] = \det[D_{L_s,L_s}(1)^{-1}D_{L_s,L_s}(m_h)],$$

where the $4VN_c \times 4VN_c$ matrix $D_{L_s,L_s}$ is the Schur complement of $D$:

$$D_{L_s,L_s}(m_h) = C_+ - (T^{-1})^{L_s}C_- = (1 + T)^{-1} \gamma_3 \frac{1}{2}
\left[(1 + m_h) - (1 - m_h)\gamma_3 \frac{1 - T}{1 + T}\right],$$

with $T \equiv T^{L_s}$. We now multiply both sides of (17) by $D_{L_s,L_s}^{-1}(1)$ to find that the combination of domain wall fermion determinants $\det[D(1)^{-1}D(m_h)]$ is the same as the determinant of the truncated overlap operator

$$D_{L_s}[H] = \frac{1}{2}
\left[(1 + m_h) - (1 - m_h)\gamma_3 \frac{1 - (1 - H)\gamma_3}{1 + (1 - H)\gamma_3} T^{L_s}\right],$$

$\equiv \frac{1}{2}
\left[(1 + m_h) - (1 - m_h)\gamma_3 \tanh(L_s \tan^{-1} H)\right].$ (19)

In order for the tanh function to be defined by a power series the second equality (19) requires $H$ to be a bounded operator, namely $|H| < 1$. The factor $D(1)^{-1}$ can be thought of as modelling Pauli-Villars boson fields which cancel the contributions of the fermions from the 4d bulk. Now, $\tanh(L_s \tan^{-1}(x))$ is an analytic approximation to the signum function $\text{sgn}(x)$ which becomes exact in the limit $L_s \to \infty$. So long as $H$ is hermitian and bounded, we therefore recover the overlap operator [9]:

$$\lim_{L_s \to \infty} D_{L_s} = D_{ov} = \frac{1}{2}
\left[(1 + m_h) - (1 - m_h)\gamma_3 \sgn\frac{D_W - M}{2 + (D_W - M)}\right],$$

$\equiv \frac{1}{2}
\left[(1 + m_h) + (1 - m_h)\frac{A}{\sqrt{A^2}}\right],$ (20)

where the unphysical nature of the sign of $\gamma_3$ is manifest. For $m_h \to 0$ (20) coincides with the 2+1d overlap operator given in [7].

Next let’s check the overlap operator (20) has the expected weak-coupling limit. For link fields $U_\mu = 1$, and with lattice spacing set to unity, in momentum space $D_W = i \sum_\mu \gamma_\mu \sin p_\mu + \sum_\mu (1 - \cos p_\mu)$, implying propagator poles at $p_\mu \approx 0$ and near the Brillouin Zone corners $p_\mu \approx \pi$. At the origin $D_W \approx i \gamma_\mu p_\mu$ so

$$\text{sgn}(H) = \frac{H}{\sqrt{H^2}} \approx -\gamma_3 \frac{(i p - M)(2 - M)}{(2 - M) M} = -\gamma_3 \left[i p M - 1\right]$$

(21)

so that the overlap operator

$$D_{ov} \approx i p \frac{(1 - m_h)}{2M} + m_h.$$ (22)
Taking into account a benign wavefunction renormalisation, this is the propagator for a continuum species with mass proportional to $m_h$. By contrast near a doubler pole
\[ \tilde{p}_\mu = p_\mu - (i, j, k) \pi \approx 0, \quad i, j, k \in \{+1, -1\}, \]
\[ \sgn(H) \approx -\gamma_3 \frac{i\tilde{p} + (2n - M)}{(2n - M)} = -\gamma_3 \left[ \frac{i\tilde{p}}{(2n - M)} + 1 \right] \] (23)
with $n = |i| + |j| + |k|$, so the overlap is
\[ D_{ov} \approx 1 + \frac{(1 - m_h)}{2(2n - M)} i\tilde{p}. \] (24)
So long as $(2n - M)$ is not too small, the species has a mass of $O(1)$ in cutoff units, and decouples from low-energy physics.

### 3 Equivalence of $\gamma_3$ and $\gamma_5$

Despite the manifest independence of the overlap operator $D_{ov}$ (20) of which matrix $\gamma_3$ or $\gamma_5$ is used to define the hermitian argument $H$ of the signum function, for finite $L_s$ it remains unclear whether the distinction is important or not [7], since clearly the definition (4) of the domain wall operator $D_3$ distinguishes them. We can address this using the analytic approximation for signum (19).

First, the series expansion for $\tanh^{-1} H$ is well-defined since $H = \gamma_3 A$ is a bounded operator, ie. $|H| = M/(2 - M) < 1$ for $0 < M < 1$:
\[ \tanh^{-1} H = H + \frac{H^3}{3} + \frac{H^5}{5} + \cdots \] (25)
Each term is on odd power, so can be reexpressed using $\gamma_3 A \gamma_3 = A^\dagger$:
\[ H^{2n+1} = \gamma_3 A(A^\dagger A)^n. \] (26)
The signum approximation is then
\[ \tanh(L_s \gamma_3 A \sum_n b_n (A^\dagger A)^n) = \frac{\sinh(L_s \gamma_3 A \sum_n b_n (A^\dagger A)^n)}{\cosh(L_s \gamma_3 A \sum_n b_n (A^\dagger A)^n)} \] (27)
with $b_n = (2n + 1)^{-1}$. In the McLaurin series expansions of the hyperbolic functions on the RHS of (27), expansion of the argument yields a general term of the form
\[ L_s^m \left( \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_m}^{\infty} \right) \prod_{i=1}^{m} [b_{n_i} (\gamma_3 A)(A^\dagger A)^{n_i}] \] (28)

\footnote{For free fermions the most stringent limit on $M$ comes from the origin of momentum space. In practice on any finite lattice with antiperiodic temporal boundary conditions $M = 1$ is safe since $|H| = 1/\sqrt{5 - 4 \cos \frac{\pi}{L_t}} < 1$ for $L_t < \infty$.}
For the sinh series, $m$ is an odd integer so that the term in square brackets reads

$$
\left(\prod b_{n_i}\right)(\gamma_3 A)(A^\dagger)^{n_1}(\gamma_3 A)(A^\dagger A)^{n_2}\ldots(\gamma_3 A)(A^\dagger A)^{n_m} = \left(\prod b_{n_i}\right)(\gamma_3 A)(A^\dagger)^{n_1}(A^\dagger A)^{n_2+1}(A^\dagger A)^{n_3}\ldots(A^\dagger A)^{n_{m-1}+1}(A^\dagger A)^{n_m} = \left(\prod b_{n_i}\right)(\gamma_3 A)(A^\dagger A)^{\Sigma_i n_i+(m-1)/2}.
$$

(29)

For the cosh series $m$ is even and a similar argument gives the general term

$$
\left(\prod b_{n_i}\right)(A^\dagger A)^{\Sigma_i n_i+m/2}.
$$

The final step is to observe that $[(\gamma_3 A)^{-1}, (A^\dagger A)^n] = 0$ for any $n$; the RHS of (27) can therefore be manipulated to bring $\gamma_3 A$ to the left of all terms in the expansion, whereupon the $\gamma_3$ cancels in the expression (19) for the truncated overlap. Now using the fact that $\gamma_5$ has identical properties with respect to commutation with $A$, we can reverse all the steps to rewrite the truncated overlap operator

$$
D_{Ls}[H] = \frac{1}{2} \left[ (1 + m_h) + (1 - m_h)\gamma_5 \tanh(L_s \tanh^{-1} \gamma_5 A) \right].
$$

(30)

This establishes that the truncated overlap operator is equally blind to the distinction between $\gamma_3$ and $\gamma_5$ as the overlap (20).

4. Introducing $m_3, m_5 \neq 0$

In [7] we exploited the possibility of U(2)-rotating the fields leaving the kinetic term unaltered while changing the form of the mass term. In terms of continuum fields defined in 2+1d the alternative but physically equivalent, antihermitian but parity-invariant mass terms are $im_3 \bar{\psi}\gamma_3 \psi$, $im_5 \bar{\psi}\gamma_5 \psi$. In the domain wall approach [5] is replaced by one of

$$
m_3 S_3 = im_3 \sum_x \bar{\Psi}(x, L_s)\gamma_3 P_- \Psi(x, 1) + \bar{\Psi}(x, 1)\gamma_3 P_+ \Psi(x, L_s); \quad (31)
$$

$$
m_5 S_5 = im_5 \sum_x \bar{\Psi}(x, L_s)\gamma_5 P_- \Psi(x, 1) + \bar{\Psi}(x, 1)\gamma_5 P_+ \Psi(x, 1). \quad (32)
$$

First consider a mass term $m_3 S_3$. The matrix manipulations outlined in Sec.2 leading to eqn. (7) go through as before, but with (11) replaced by

$$
C_{3\pm} = P_{\pm} \pm im_3 P_{\mp}, \quad (33)
$$

The Schur complement of $\tilde{D} = Q^{-1}DP$ is then

$$
D_{Ls, Ls}(m_h = 1)\frac{1}{2} \left[ (1 + im_3 \gamma_3) - \gamma_3 \frac{1 - T}{1 + T} (1 - im_3 \gamma_3) \right], \quad (34)
$$

implying a truncated overlap

$$
D_{Ls} = \frac{1}{2} \left[ (1 + im_3 \gamma_3) - \gamma_3 \tanh(L_s \tanh^{-1}(\gamma_3 A)) (1 - im_3 \gamma_3) \right], \quad (35)
$$
with $A$ still given by (13). An important technical point is that the passage from domain wall to overlap requires the Pauli-Villars matrix $D_{L_sL_s}(1) = (1 + \gamma_3)\gamma_3$ to continue to be defined with the hermitian mass term $1 \times S_h$. The overlap operator found in the limit $L_s \to \infty$ is thus

$$D_{ov} = \frac{1}{2} \left[(1 + im_3\gamma_3) + \frac{A}{\sqrt{A^*A}}(1 - im_3\gamma_3)\right]$$  \hspace{1cm} (36)$$

with $A$ defined in (13). In the weak coupling long wavelength limit

$$D_{ov} \approx i\frac{\gamma_3}{(1 - im_3\gamma_3)} + \frac{m_3}{2M}.$$  \hspace{1cm} (37)$$

This time there is an $O(a)$ term proportional to $\gamma_3$ not present in the continuum action, which cannot be absorbed by wavefunction rescaling. It seems highly plausible that this lies at the heart of the $O(a)$ departures from $U(2)$ symmetry observed when rotating fermion bilinears according to the remnant symmetries derived from the 3d Ginsparg-Wilson (GW) relations in Sec. 3 of [7].

Next consider the mass term $m_S S_5$. Even though this term differs from the other masses by coupling fields on the same domain wall to itself, rather than on opposite ones, the matrix manipulations of Sec. 2 still arrive at (7), with this time

$$C_{5\pm}^5 = P_{\pm} - im_5\gamma_5P_{\pm} = P_{\pm} - im_5\gamma_5,$$  \hspace{1cm} (38)$$

where the second step is crucial. The truncated overlap in this case is

$$D_{L_s}[H] = \frac{1}{2} \left[(1 + im_5\gamma_5) - \gamma_3\tanh(L_s\tanh^{-1} H)(1 - im_5\gamma_5)\right];$$  \hspace{1cm} (39)$$

however the considerations of Sec. 3 permit this to be rewritten

$$D_{L_s} = \frac{1}{2} \left[(1 + im_5\gamma_5) - \gamma_5\tanh(L_s\tanh^{-1}(\gamma_5A))(1 - im_5\gamma_5)\right].$$  \hspace{1cm} (40)$$

The complete equivalence between (41) and (35) is manifest.

## 5  Ginsparg-Wilson Relations

Whilst the previous two sections have established the equivalence of the domain wall formulation with respect to a discrete interchange of the matrices $\gamma_3$ and $\gamma_5$, in order to study restoration of the full $U(2N_f)$ symmetry it is more convenient to examine the overlap operator. Following (20,36), in the large-$L_s$ limit we can write Lagrangian densities in terms of “Ginsparg-Wilson” fields $\Psi, \bar{\Psi}$:

$$\mathcal{L}_h = \bar{\Psi}[D_{ov}^0 + m_h(1 - D_{ov}^0)]\Psi;$$  \hspace{1cm} (41)$$

$$\mathcal{L}_3 = \bar{\Psi}[D_{ov}^0 + im_3(1 - D_{ov}^0)\gamma_3]\Psi;$$  \hspace{1cm} (42)$$

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where $D_{ov}^0$ is the overlap operator for massless fermions

$$D_{ov}^0 = \frac{1}{2} \left[ 1 + \frac{A}{\sqrt{A^\dagger A}} \right].$$

(43)

In both cases there is an $O(a)$ correction to the expected continuum form, but as noted above for the hermitian mass case the correction can be absorbed into a harmless rescaling of the kinetic term. For the antihermitian case by contrast the correction is not of the same form as a term in the continuum Lagrangian, as first noted in (although (42) differs in detail from eq. (34) of that paper).

The reconciliation is made by first observing that the GW relation appropriate for the domain wall operator (2) is

$$\gamma_3 D_{ov}^0 + D_{ov}^0 \gamma_3 = 2 D_{ov}^0 \gamma_3 D_{ov}^0.$$  

(44)

As expected, there are further GW relations, first with $\gamma_5$ replacing $\gamma_3$ in (44), and also a rotation generated by $i\gamma_3\gamma_5$ which along with a simple global phase rotation completely specifies the U(2): [7]

$$\gamma_5 D_{ov}^0 + D_{ov}^0 \gamma_5 = 2 D_{ov}^0 \gamma_5 D_{ov}^0; \quad \gamma_3\gamma_5 D_{ov}^0 - D_{ov}^0 \gamma_3\gamma_5 = 0.$$  

(45)

The associated symmetry in the massless limit is then [14, 7]

$$\Psi \mapsto e^{i\alpha \gamma_3 (1 - D_{ov}^0)} \Psi; \quad \bar{\Psi} \mapsto \bar{\Psi} e^{i\alpha (1 - D_{ov}^0) \gamma_3}.$$  

(46)

Strictly speaking, therefore, symmetry under global U(2) rotations of local fields is only recovered as $a \to 0$, under the assumption that the overlap operator $D_{ov}^0$ is sufficiently localised in this limit.

Next, define projection operators as follows:

$$P_\pm = \frac{1}{2} (1 \pm \gamma_3); \quad \hat{P}_\pm = P_\pm D_{ov}^0 \gamma_3$$  

(47)

with the property $\hat{P}_\pm D_{ov}^0 = D_{ov}^0 P_\pm$ following from [14]. With projected fields $\Psi_\pm = P_\pm \Psi$, $\bar{\Psi}_\pm = \bar{\Psi} P_\pm$, we can write

$$\mathcal{L}^0 = \bar{\Psi}_+ D_{ov}^0 \Psi_+ + \bar{\Psi}_- D_{ov}^0 \Psi_-= \bar{\Psi} D_{ov}^0 \Psi;$$  

(48)

$$m_{h_S}^{GW} = m_h (\bar{\Psi}_- \Psi_+ + \bar{\Psi}_+ \Psi_-) = m_h \bar{\Psi} (1 - D_{ov}^0) \Psi;$$  

(49)

$$m_{3_S}^{GW} = im_3 (\bar{\Psi}_- \gamma_3 \Psi_+ + \bar{\Psi}_+ \gamma_3 \Psi_-) = im_3 \bar{\Psi} (1 - D_{ov}^0) \gamma_3 \Psi$$  

(50)

consistent with [14,42]. The extension to the terms involving $\gamma_5$ is trivial [7].

\footnote{Note that in order to recover the expressions for the antihermitian mass terms derived in [7] we should have chosen a matrix decomposition of $D(m_{3,5})$ with the projectors $P_\pm$ multiplying to the left rather than to the right as in [6].}
6 Bilinear Condensates

The freedom to specify variants of the parity-invariant mass term can be exploited in the study of the corresponding bilinear condensates defined via

\[ \langle \bar{\psi} \Gamma_i \psi \rangle = \frac{\partial \ln Z}{\partial m_i} = \left\langle \text{tr} M^{-1} \frac{\partial M}{\partial m_i} \right\rangle, \]  

(51)

where \( \det M \) is the part of the functional measure coming from the fermions. For a \( \text{U}(2) \)-invariant theory the condensates generated by the masses \( m_h, m_3, m_5 \) should all coincide, and indeed numerical evidence for this as \( L_s \to \infty \) was presented for quenched non-compact \( \text{QED}_3 \) [7]. A particular useful result was that finite-\( L_s \) corrections are minimised by choosing the mass term antihermitian. We parametrise these in terms of residuals \( \Delta_h, \epsilon_h, \epsilon_3, \epsilon_5 \) which vanish exponentially as \( L_s \to \infty \) by writing:

\[
\begin{align*}
\frac{1}{2} \langle \bar{\psi} \psi \rangle_{L_s} &= \frac{1}{2} \langle \bar{\psi} \gamma_3 \psi \rangle_{L_s \to \infty} + \Delta_h(L_s) + \epsilon_h(L_s); \\
\frac{i}{2} \langle \bar{\psi} \gamma_3 \psi \rangle_{L_s} &= \frac{i}{2} \langle \bar{\psi} \gamma_3 \psi \rangle_{L_s \to \infty} + \epsilon_3(L_s); \\
\frac{i}{2} \langle \bar{\psi} \gamma_5 \psi \rangle_{L_s} &= \frac{i}{2} \langle \bar{\psi} \gamma_3 \psi \rangle_{L_s \to \infty} + \epsilon_5(L_s).
\end{align*}
\]

(52)

The numerically dominant residual is \( \Delta_h \), defined to be the imaginary component of \( i \langle \bar{\psi} \gamma_3 \psi \rangle \) evaluated on just the + component of \( \Psi \):

\[
i \langle \bar{\Psi}(1) \gamma_3 P_+ \Psi(L_s) \rangle = \frac{i}{2} \langle \bar{\psi} \gamma_3 \psi \rangle_{L_s} + i \Delta_h(L_s). \]

(53)

The imaginary contribution from the \( \Psi_- \) component has opposite sign and hence cancels even for finite \( L_s \).

In order to understand why \( \Delta_h \) only contributes for the hermitian condensate, first consider the continuum case with \( M = \mathcal{D} + m_h \):

\[
\langle \bar{\psi} \psi \rangle = \text{tr}(\mathcal{D} + m_h)^{-1} = \text{tr} \left[ \frac{1}{\mathcal{D}} \left( 1 - \frac{m_h}{\mathcal{D}} + \frac{m_h^2}{\mathcal{D}^2} - \frac{m_h^3}{\mathcal{D}^3} + \cdots \right) \right] = -4 \left[ \frac{m_h}{\mathcal{D}^2} + \frac{m_h^3}{\mathcal{D}^4} + \cdots \right]
\]

(54)

where we assume \( m_h \) is small enough to justify the binomial expansion. Since the trace over an odd number of gamma matrices is zero, all even powers of \( m_h \) vanish on taking the trace, which makes sense since \( \langle \bar{\psi} \psi \rangle \) should be an odd function of \( m_h \). The mass term \( m_3 \gamma_3 \) yields the same series:

\[
i \langle \bar{\psi} \gamma_3 \psi \rangle = \text{tr}(\mathcal{D} + im_3 \gamma_3)^{-1}i\gamma_3 = \text{tr} \left[ \frac{1}{\mathcal{D}} \left( 1 - \frac{im_3}{\mathcal{D}} \gamma_3 + \frac{m_3^2}{\mathcal{D}^2} - \frac{m_3^3}{\mathcal{D}^3} \gamma_3 + \cdots \right) \right] \frac{1}{\mathcal{D}}(i\gamma_3)
\]

\[
= -4 \left[ \frac{m_3}{\mathcal{D}^2} + \frac{m_3^3}{\mathcal{D}^4} + \cdots \right]
\]

(55)
where we have used $\gamma_3 \not= D \partial \not= \bar{D}$. This time the even powers vanish because they consist of products of an odd number of matrices $\gamma_\mu$ ($\mu = 0, 1, 2$) with $\gamma_3$, so are proportional to either $\text{tr} \gamma_\mu \gamma_3$ or $\text{tr} \gamma_3 \gamma_\mu$.

Now, for a theory with functional weight $\text{det} D L_s [H]$ the corresponding expression for $\langle \bar{\psi} \psi \rangle$ is

$$
\text{tr} M^{-1} M' = \text{tr} [1 - \gamma_3 \epsilon_{L_s} + m_h (1 + \gamma_3 \epsilon_{L_s})]^{-1} [1 + \gamma_3 \epsilon_{L_s}] \\
= \text{tr} \left[ \frac{1 + \gamma_3 \epsilon_{L_s}}{1 - \gamma_3 \epsilon_{L_s}} \right] \left[ 1 - m_h \frac{1 + \gamma_3 \epsilon_{L_s}}{1 - \gamma_3 \epsilon_{L_s}} \right] + m_h^2 \frac{1 + \gamma_3 \epsilon_{L_s}}{1 - \gamma_3 \epsilon_{L_s}}^2 + \cdots.
$$

(56)

Here $\epsilon_{L_s} [H] \equiv \tanh(L_s \tanh^{-1} H)$ is the finite-$L_s$ approximation to the signum function. Now,

$$
\frac{1 + \gamma_3 \epsilon_{L_s}}{1 - \gamma_3 \epsilon_{L_s}} = [1 - \gamma_3 \epsilon_{L_s}]^{-1} [1 + \epsilon_{L_s} \gamma_3]^{-1} [1 + \gamma_3 \epsilon_{L_s}] \\
= (1 - \epsilon_{L_s}^2 - [\gamma_3, \epsilon_{L_s}])^{-1} (1 + \epsilon_{L_s}^2 + \{\gamma_3, \epsilon_{L_s}\}).
$$

(57)

In the limit $L_s \to \infty$, $\epsilon_{L_s}^2 = 1$, and the long-wavelength weak coupling limit (21) gives

$$
\lim_{L_s \to \infty} \{\gamma_3, \epsilon_{L_s}\} = 2; \quad \lim_{L_s \to \infty} [\gamma_3, \epsilon_{L_s}] = -\frac{2i\phi}{M},
$$

(58)

so (57) $\approx 2M/i\phi$ and we are on the right track. However, for finite $L_s$ $1 - \epsilon_{L_s}^2$ is a real quantity, and now there is no reason for the terms in (56) corresponding to even powers of $m_h$ to necessarily vanish. Another way of saying this is that the form of $A$ defining $\epsilon_{L_s}$ dictates that it is no longer the case that even powers of $m_h$ are proportional to the trace over an odd number of gamma matrices. We conclude that the function $\langle \bar{\psi} \gamma_3 \psi (m_h) \rangle$ in general contains an even component, labelled $\Delta_h$ in (52), weakly dependent on $m_h$ as $m_h \to 0$ and only vanishing as $L_s \to \infty$.

Now repeat the exercise for the mass term $m_3 S_3$:

$$
\text{tr} M^{-1} M' = \text{tr} [1 - \gamma_3 \epsilon_{L_s} + i m_3 \gamma_3 (1 + \epsilon_{L_s} \gamma_3)]^{-1} i \gamma_3 [1 + \epsilon_{L_s} \gamma_3] \\
= \text{tr} \left[ 1 + i m_3 \frac{1 + \gamma_3 \epsilon_{L_s}}{1 - \gamma_3 \epsilon_{L_s}} \gamma_3 \right]^{-1} [1 + \gamma_3 \epsilon_{L_s}] (i \gamma_3)
$$

(59)

Now, from (57) and the considerations of Sec. 3, all the terms in the binomial expansion of the first factor in (59) can only contain $L_s$ dependence in terms of the form $(\gamma_3 \epsilon_{L_s})^p$, $\epsilon_{L_s}^p \gamma_3$ with $p, q$ integer, which have the property that $\text{tr} \gamma_3 (\gamma_3 \epsilon_{L_s})^p = \text{tr} \gamma_3 (\epsilon_{L_s})^{2q} = 0$. This implies that only odd powers of $m_3$ survive the trace. Hence $i \langle \bar{\psi} \gamma_3 \psi (m_3) \rangle$ is an odd function of $m_3$, and the dominant residual $\Delta_h$ is necessarily absent. For finite $L_s$ when the limiting forms (58) do not hold, we cannot exclude corrections which are odd functions of $m_3$, corresponding to the residual $\epsilon_3$ in (52).

Finally, the arguments of Sec. 3 then imply the identical property for the condensate $i \langle \bar{\psi} \gamma_3 \psi \rangle$, consistent with the numerical results of [7].
7 Summary

In Sec. [2] we showed that the 2+1d domain wall fermion formulation introduced in [7] coincides with the overlap operator in the limit $L_s \to \infty$, and, importantly not simply in the continuum limit as suggested in the abstract of that paper. Whilst the Dirac matrices $\gamma_3$ and $\gamma_5$ enter the domain wall formulation (1) in very different ways, it was shown in Sec. [3] that the resulting 2+1d truncated overlap operator (19,30) is blind to the distinction between them even for $L_s$ finite. There seems to be no obstruction to modelling $U(2N_f) \to U(N_f) \otimes U(N_f)$ symmetry breaking in lattice simulations of 2+1d fermions, so long as it is understood that the nature of the $a > 0$ corrections to continuum symmetry operations, encapsulated in the GW relations (44,45), and needed, say for identifying interpolating operators for Goldstone modes [7], is more complicated than for 4d gauge theories, as discussed in Sec. [5]. In particular the antihermitian mass term (50) consistent with the GW relations contains an $O(a)$ correction of a form not present in the continuum action. Ultimately, successful control of these corrections will depend on the locality properties of the overlap operator $D_{ov}$ [15], which is a dynamical question.

On the other hand, the freedom to formulate alternative mass terms in 2+1d leads to a potentially important computational saving; as shown in Sec. [6] finite-$L_s$ corrections to bilinear condensates may be classified by whether they are odd or even functions of the symmetry-breaking mass $m_i$, and the dominant even component $\Delta_h$ is absent for the antihermitian mass terms $S_3, S_5$, whose use in numerical simulations with finite $L_s$ thus seems preferred, while recalling from Sec. [4] that the correct formulation of the Pauli-Villars bulk correction $\det D_{L_s, L_s}^{-1}(1)$ requires the hermitian mass $1 \times S_h$.

Acknowledgements

This work was supported by a Royal Society Leverhulme Trust Senior Research Fellowship, and in part by STFC grant ST/L000369/1. I continue to benefit enormously from discussions with Tony Kennedy.

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