Short Distance Modification of a Gravitational System and its Optical Analog

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Abstract

Motivated by developments in string theory, such as T-duality, it has been proposed that the geometry of spacetime should have an intrinsic minimal length associated with it. This would modify the short distance behavior of quantum systems studied on such a geometry, and an optical analog for such a short distance modification of quantum system has also been realized by using non-paraxial non-linear optics. As general relativity can be viewed as an effective field theory obtained from string, it is expected that this would also modify the short distance behavior of general relativity. Now the Newtonian approximation is a valid short distance approximation to general relativity, and Schrodinger-Newton equation can be obtained as a non-relativistic semi-classical limit of such a theory, we will analyze the short distance modification of Schrodinger-Newton equation from an intrinsic minimal length in the geometry of spacetime. As an optical analog of the Schrodinger-Newton equation has been constructed, it is
possible to optically realize this system. So, this system is important, and we will numerical analyze the solutions for this system. It will be observed that the usual Runge-Kutta method cannot be used to analyze this system. However, we will use a propose and use a new numerical method, which we will call as the two step Runge-Kutta method, for analyzing this system.

**Keywords:** short distance modification, generalized uncertainty principle (GUP), Schrodinger-Newton equation, Runge-Kutta method

1 Introduction

It is known that general relativity is an effective field theory approximation to some more fundamental theory, such as the string theory. So, we would expect some features of this fundamental theory to modify the short distance behavior of the general relativity. An interesting feature of string theory is that there is an intrinsic minimal length scale in string theory, and it is not possible to define the geometry of spacetime below that length scale [1]–[5]. This is because strings have an extended structure, and the fundamental string is the smallest probe in perturbative string theory. So, it is not possible to probe the geometry of spacetime below the string length scale . Thus, the string length acts, which is given by \( l_s = \alpha' \) as a minimum measurable length in string theory. Even thought to analyze non-perturbative effects in string theory, we also need to consider D0-brane, and D0-branes are point like objects, it can still be argued that there would exist a minimal length in non-perturbative string theory. In fact, it has been demonstrated that an intrinsic minimal length \( l_{\text{min}} \) exists even if D-branes scattering is considered [6]–[7]. Here this minimal length \( l_{\text{min}} \) is related to string length \( l_s \) as \( l_{\text{min}} = l_s g_s^{1/3} \), where \( g_s \) is the string coupling constant [7].

It can be argued from T-duality that such a minimal length scale would naturally exist in string theory. This can be done by analyzing the total energy of the quantized string. Now if we consider one additional dimension compactification on a radius \( R \), then this total energy would depend on the excitation \( n \) and winding number \( w \) [6]. Now then under T-duality, it is possible to interchange the excitation \( n \) and the winding number \( w \), such that \( R \rightarrow l_s^2 / R \), and \( n \rightarrow w \). This implies that it is not possible to describe string theory below \( l_s \). This is because the description of string below \( l_s \) is the same as the description above it. So, using T-duality it can be argued that the geometry of spacetime in string theory has an intrinsic length associated with it [6]. It may be noted that the T-duality has also been used to analyze the Green’s function for the the center of mass of the string using an effective
path integral [8]-[9]. It has been observed that there a minimal length also exists in this Green’s function, and it is not possible to probe length scales below that minimal length in this formalism [8]-[9]. So, a minimal length exists in string theory because of the T-duality. Now as general theory of relativity is a low energy effective field theory approximation of string theory, it can be argued that the short distance behavior of general relativity should be modified such that there is an intrinsic minimal length scale associated with it. It may be noted that such a minimal length also occurs in other branches of quantum gravity such as the loop quantum gravity [10].

It can be argued using black hole physics, that any theory of quantum gravity should have a minimum length at least of the order of Planck length $l_{PL}$ associated with it [11]-[12]. This is because energy needed to probe a region of spacetime below Planck scale is more than the energy needed to form a mini black hole in that region. However, it is possible for the minimal length scale to be larger than the Planck scale. Even in string theory, the minimal length scale, which is the string length scale, can be greater than the Planck scale. This is because the Planck length $l_{PL}$ can be expressed in terms of string length $l_s$ as $l_{PL} = g_s^{1/4}l_s$, where $g_s$ is the string coupling constant [6]. In fact, it has been argued that the minimal length scale can be several orders of magnitude greater than Planck length scale, and this can produce universal short distance corrections to all quantum mechanical systems [13]. As this minimal length scale can have low energy consequences, and so it can be phenomenologically fixed using low energy experiments [14]. In fact, it is possible to use opto-mechanical systems to measure such short distance modification of the quantum mechanics by the existence of a minimum measurable length scale [15]. However, it is also possible to test the Schrodinger-Newton equation experimentally using opto-mechanical systems [16]-[17]. The Schrodinger-Newton equation is a semi-classical approximation to quantum gravity, and it can be obtained as the non-relativistic limit of the Dirac equation and the Klein–Gordon equation with a classical Newtonian potential [18]. The Schrodinger-Newton equation has been used to describe interesting properties of gravitational systems at such short scales [19]-[23]. It is thought to be a valid semi-classical approximation as Newtonian gravity has been demonstrated to be a valid short distance approximation to general relativity to smallest scales at which general relativity has been tested, which is about 0.4mm [24].

As it is possible to test the Schrodinger-Newton experimentally using opto-mechanical systems [16]-[17], and it is also possible to use opto-mechanical systems to test short distance modification to quantum mechanical [15], it is important to analyze the short distance modification to Schrodinger-Newton equation. So, in this paper, we will study such a short distance modification
of the Schrodinger-Newton equation, and analyze this system numerically. We will demonstrate that the usual Runge-Kutta method cannot be used to obtain good numerical results due the existence of higher derivative terms. We will then propose a new two-step Runge-Kutta method, which will be demonstrated to be free from errors generated in the usual one-step Runge-Kutta method. Thus, the methods of this paper, can used to analyze other GUP deformed systems.

2 Schrodinger-Newton Equation and its Optical Analog

It is known that the general relativity cannot be quantized using the usual methods of quantum field theory, and even though we have various proposals for a quantizing general relativity, we do not still have a fully consistent quantum theory of quantum gravity. This has motivated the study of semi-classical quantum gravity, and in this approach, the gravitational field is treated as a background classical field, and the matter fields are treated quantum mechanically. Thus, in the semi-classical approximation, a Einstein tensor $G_{\mu\nu}$ is produced by quantum mechanical energy-momentum tensor for matter fields $\hat{T}_{\mu\nu}$. So, if $|\psi\rangle$ is the wave function of the matter fields, then we can write the Einstein equation in semi-classical approximation as

$$G_{\mu\nu} = \frac{8\pi G}{c^2} \langle \psi | \hat{T}_{\mu\nu} | \psi \rangle.$$  

(1)

It is known that till the smallest scales at which gravity has been tested (0.4 mm), the Newtonian gravity is a good approximation to general relativity [24]. Thus at small distances, we expect that the semi-classical approximation to be described by a Schrodinger-Newton equation, [25]-[30]

$$i\hbar \frac{\partial \psi}{\partial t} = H \psi = \frac{1}{2m} \hat{P}^2 \psi + m \Phi(R, t) \psi,$$  

(2)

where $H$ is the Hamiltonian operator of a system, with $m|\psi(R, t)|^2 = \rho(R, t)$ as the mass density and $\Phi(R, t)$ as the classical Newtonian potential. It may be noted that this potential also satisfies the Poisson equation

$$\nabla^2 \Phi(R, t) = -4\pi G m |\psi|^2.$$  

(3)

It is possible to construct a gravity analog for this system, and use it to analyze its properties. The gravity analogue are used for studying various gravitational effects, and this done using artificial systems. These systems
recreate some specific properties of the gravitational system, and can be experimentally realized in the laboratory \[31\]. These gravity analogs have been used to study analogous black holes, and analyze the analogous Hawking radiation in transonic fluid flows \[32\]. Such flows have been realized in various physical systems, such as flowing water \[33\], Bose-Einstein condensates \[34\]-\[35\] and nonlinear optics \[36\]-\[37\]. These gravity analogs are linear, but gravity is a very nonlinear theory. However, it is possible to realize the non-linearity in gravity analogs by using optical wave packets with thermal nonlinearity, and this system is mathematically equivalent to the Newton–Schrodinger equation \[38\]-\[39\]. This nonlinear gravity analog that can be constructed in laboratory experiments is based on the evolution of the amplitude \(E\), of an optical beam in a thermally focusing medium. This system is described by the following equation,

\[
i \frac{\partial E}{\partial z} + \frac{1}{2k} \nabla^2_2 E + k_0 \Delta n E = 0,
\]  

(4)

where \(\nabla^2_2\) is the transverse two-dimensional Laplacian. Here we also have

\[
k = n_b \omega / c = n_b k_0,
\]  

(5)

where \(n_b\) is the background refractive. The nonlocal change in refractive-index \(\Delta n\) can be induced by heating the medium by a beam, and it can be expressed as

\[
\nabla^2_2 \Delta n = \alpha \beta \kappa^{-1} |E|^2,
\]  

(6)

where \(\kappa\) is the thermal conductivity, \(\beta\) is the thermo-optic coefficient, and \(\alpha\) is the absorption coefficient. This system can be used as a analog for the Newton–Schrodinger equation. The advantage of the optical analog of Newton–Schrodinger equation is that this system can be realized in laboratory, and its properties can be experimentally studied using such an analog.

### 3 Deformed Newton–Schrodinger Equation

It is important to analyze the short distance modification to the Schrodinger-Newton equation. This is because both the short distances effects from the Schrodinger-Newton equation \[16\]-\[17\], and short distance modification of quantum mechanics \[15\], can be measured opto-mechanical systems. Such a short distance modification to a quantum system occurs due to the existence of a minimal length scale in the geometry of spacetime, and it can be analyzed using a generalization of the uncertainty principle to a generalized
uncertainty principle (GUP) [40]-[44]. This generalization also deforms the Heisenberg algebra [45]-[48],

\[ [\hat{X}_i, \hat{P}_j] = i\hbar (\delta_{ij} + \beta \delta_{ij} \hat{P}^2 + 2\beta \hat{P}_i \hat{P}_j). \] (7)

This deformation of the Heisenberg algebra also deforms coordinate representation of the momentum operator [49]-[51]. It has been demonstrated that this deformed Heisenberg algebra satisfies the Jacobi identity, it is possible to demonstrate [13]-[14]

\[ [\hat{X}_i, \hat{X}_j] = 0 = [\hat{P}_i, \hat{P}_j]. \] (8)

We expect this short distance modification of the Heisenberg algebra to reduce to the usual deformation of the Heisenberg algebra, at low energies. So, if \( \hat{p}_i \) is the momentum at low energies, then we expect that

\[ \hat{p}_j = -i\hbar \frac{\partial}{\partial \hat{x}_j}, \] (9)

where \( \hat{x}_i \) is the coordinate conjugate to \( \hat{p}_i \), such that

\[ [\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij}. \] (10)

Now we can express the that at higher energies \( \hat{P}_j \), and the coordinate conjugate to it \( \hat{X}_i \psi(x) \) in terms of the low energy momentum and coordinates as [13]-[14]

\[ \hat{X}_i \psi(x) = \hat{x}_i \psi(x), \] (11)

\[ \hat{P}_i \psi(x) = \hat{p}_i (1 + \beta \hat{p}^2) \psi(x) \] (12)

This is because by using this representation, the first order in \( \beta \), [7] is satisfied. Here we neglect terms of order \( \beta^2 \) and higher. It is interesting to note that this deformation of the momentum operator has also been motivated by a non-anticommutative deformation of a supersymmetric field theory [52]. It is interesting to note that a GUP like deformation of the analog Newton-Schrodinger equation can also be performed, as it has been observed that a GUP like deformation of optical propagation of focused laser beams occurs in the non-paraxial nonlinear optics [53]-[54]. This is done by analyzing the propagation of light beyond the paraxial approximation, and then expanding to the first order. So, to the first order a simple non-paraxial system deforms to

\[ -(2k_0)^{-1} \partial_x^2 A \rightarrow -(2k_0)^{-1} \partial_x^2 A + (8k_0)^3 \partial_x^4 A, \] where \( A = \mathcal{E}e^{-k_0 z} \). This is the same deformation produced by GUP, if \( \beta = 3\lambda^2/8\hbar^2 \) [54].
Substituting \( \hat{X}_i, \hat{P}_i \) by \( x_i, p_i \), we can write the GUP deformed Schrodinger-Newton equation as (to the leading order in \( \beta \)) [55]-[58]

\[
\frac{i\hbar}{\partial t} \frac{\partial \psi}{\partial t} = H \psi = \frac{\hat{p}^2}{2m} \psi + \frac{\beta}{m} \hat{p}_4 \psi + \Phi(r,t) \psi.
\] (13)

It may be noted that such a GUP deformed Schrodinger-Newton equation has been used to motivate various studies [55]-[58]. We will analyze the spherical symmetric solutions to this equation, as those solution have important physical applications. Thus, we can write (13) as

\[
-\frac{\hbar^2}{2mr^2} \frac{d}{dr} r^2 \left( \frac{d^2 \psi}{dr^2} \right) + \frac{\hbar^4 \beta}{m} \frac{d}{dr} \left( \frac{r^2}{dr} \frac{d^2 \psi}{dr^2} \right) + \Phi \psi = E \psi, \tag{14}
\]

\[
\frac{d}{r^2 dr} \frac{r^2 d\Phi}{dr} = 4\pi G m^2 \psi^2. \tag{15}
\]

We assume that \( \psi \) and \( \Phi \) approach zero as \( |r| \to \infty \) and are regular near the origin. Here we have obtained the short distance modification to the semi-classical gravity with an intrinsic minimal length in the geometry. Now we need to understand such a solution, and we shall numerically analyze such a solution.

### 4 Usual Runge-Kutta Method

In this section, we will analyze this system using the usual Runge-Kutta method. We will observe that there are several problems with the application of the usual Runge-Kutta method to this system. To analyze this system using the usual Runge-Kutta method, we redefine

\[
\psi = \zeta S, \quad E - \Phi = \xi V, \tag{16}
\]

where

\[
\zeta = \left( \frac{\hbar^2}{8\pi Gm^2} \right)^{1/2}, \quad \xi = \frac{\hbar^2}{2m}, \tag{17}
\]

and substitute them into the GUP deformed Schrödinger-Newton equation,

\[
- \frac{d^2}{r dr^2} (rS) - 2\hbar^2 \beta \frac{d^4}{r dr^4} (rS) = -VS, \tag{18}
\]

\[
\frac{d}{r^2 dr} \frac{r^2 dV}{dr} = -S^2. \tag{19}
\]
$S$ and $V$ have dimension (length)$^{-2}$, and the formulas (18) is invariant under a scale transformation:

$$(S, V, \beta, r) \rightarrow (\lambda^2 S, \lambda^2 V, \lambda^{-2} \beta, \lambda^{-1} r).$$

(20)

It may be noted that because of the rescaling freedom (20), one can fix $V_0 = 1$ and allow $S_0$ to vary, where $V_0$ and $S_0$ are the values of $V$ and $S$ at $r = 0$, respectively. Now, we can directly attempt to find the numerical solutions by using fourth-order Runge-Kutta NAG routine. Then, first, we need to determine the initial values for the different orders of the derivatives of $S$ and $V$. With bounded derivatives at $r = 0$, we assuming that

$$S = S_0 + C_i r^i, \quad V = 1 + D_i r^i.$$  

(21)

According to (18) and comparing the power of $r$, we have

\begin{align*}
2C_1 - 12\hbar^2 \beta C_3 &= 0, \\
6C_2 - 120\hbar^2 \beta C_4 &= -S_0, \\
12C_3 - 720\hbar^2 \beta C_5 &= -S_0 D_1 - C_1, \\
20C_4 - 1680\hbar^2 \beta C_6 &= -S_0 D_2 - C_2, \\
&\cdots
\end{align*}

\begin{align*}
2D_1 &= 0, \\
6D_2 &= -S_0^2, \\
12D_3 &= -2S_0 C_1, \\
20D_4 &= -2S_0 C_2, \\
&\cdots
\end{align*}

(22)

We can set $C_{2i+1}$ and $D_{2i+1}$ to be zero. Therefore, the first order derivative $S_0^{(1)}$ is zero, so is $V_0^{(1)}$. Since the highest derivative of $S$ in (18) is the fourth order, we still need to initialize the second and the third order derivative, $S_0^{(2)}$, $S_0^{(3)}$. It is obvious that $S_0^{(3)} = 0$. However, since $C_{2i}$ are not independent, we can not exactly solve the equations (22). If we assume that $\hbar^2 \beta$ very small, approximately, we have $S_0^{(2)} = 2C_2 = -\frac{S_0}{3}$.

With this set-up, we solve the formulas (18). However, when we consider $\beta > 0$, due to positive iterations (we means that $S^{(4)}$ equals to $S^{(2)}$ multiplying a positive coefficient and plus something) which can easily lead to a divergence, we can not obtain stable and smooth solutions. Therefore, for test, we consider $\beta < 0$.

The numerical results for $\beta = -10^{66}$ and $\hbar^2 \beta = -1.112 \times 10^{-2}$ and those for $\beta = -10^{65}$ and $\hbar^2 \beta = -1.112 \times 10^{-3}$ are shown in Figure 1 and Figure
respectively. Here, we showed the first four bound-states for each $\beta$. The first plot in each subfigure shows the solutions for $S$ (blue line) and $V$ (red line). The second plot in each subfigure has the meaning of the infinite limit of $V$, named $A$; and the third plot is related to the rescaling factor $\lambda$. Following the same standard as that in [59], we fix the bound-states which mark transitions between solutions in which $(S, V) \to (+\infty, -\infty)$ and those in which $(S, V) \to (-\infty, -\infty)$. The normalization requirement is that the summation of the probability in the whole space is equal to 1. Therefore, one should have

$$1 = \int_0^\infty 4\pi r^2|\psi|^2dr = \int_0^\infty 4\pi(\lambda r)^2 \left|\frac{\zeta S}{\lambda^2}\right|^2 d(\lambda r) = \frac{4\pi\zeta^2 B}{\lambda},$$

(23)

where $B = \int_0^\infty r^2 |S|^2 dr$. Therefore, the rescaling parameter should be $\lambda = 4\pi\zeta^2 B$. To see the limit of $V$ at the infinity, $V$ can be expanded in powers of $r^{-1}$ (only to the first order):

$$V = A + \frac{B}{r} + \cdots,$$

(24)

where

$$A = V_0 - \int_0^\infty rS^2 dr.$$  

(25)

Therefore, the limit of $V$ at infinity is $A$. Since at the infinity, the potential energy is approximate zero, we can determine the energy eigenvalues by normalizing the $A$ and also multiplying the coefficient $\zeta$. Therefore, we can obtain the

$$E = \xi A = \frac{\xi A}{\zeta^4(4\pi B)^2}.$$ 

(26)

Even though we can obtain the numerical result for $\beta < 0$, however, there are two obstacles preventing us to obtain good results. The first one is the error of the initial value for the second order derivative of $S_0$. Because of the existence of the deformed term, the highest order of the derivative of $S$ is the fourth order. Using fourth-order Runge-Kutta NAG routine, we need to give the initial values for the derivatives. Since $C_{2i+1} = 0$, we know the first order and third order derivatives of the $S_0$ are both zero. The coefficient $C_2$ is equal to $-S_0/6$ plus corrections from the non-zero $\beta C_4$ (see (22)) and hence the second derivative will be $-S_0/3$ plus corrections by some factor proportional to $\hbar^2 \beta$ (leading term). However, one cannot write down the exact correction for $S_0^2$, since to solve the exact $C_4$, we need to know $C_6, C_8$ and so on. Therefore, when we set the second derivatives $S_0$ to be $-S_0/3$, 

9
Figure 1: The first four bound-state wave functions for deformed Hamiltonian with $\beta = -10^{66}$ and the $A, B$ value for each Wavefunction.

(a) $S_0 = 1.0894125784434689486$, $A = -0.1065$, $B = 3.615$

(b) $S_0 = 0.8266388605282536961$, $A = -0.5918$, $B = 5.69$

(c) $S_0 = 0.7442822176954627796$, $A = -0.4529$, $B = 7.56$

(d) $S_0 = 0.70000260975984696099$, $A = -0.3783$, $B = 9.225$
Figure 2: The first four bound-state wave functions for deformed Hamiltonian with $\beta = -10^{65}$ and the $A, B$ value for each wave function.
there is an initial error for $S_0^{(2)}$ which is proportional to $\hbar^2 \beta$. The second obstacle is the error from the iterations. When we perform the numerical calculation, the coefficient $2\hbar^2 \beta$ in front of $S^{(4)}$ can be treated as 1 and meanwhile we multiply a coefficient $\frac{1}{2\hbar^2 \beta}$ to the lower orders. Therefore, we know that by performing iterations, there is a error proportional to $\frac{1}{2\hbar^2 \beta}$. Therefore, if we choose a small $\beta$, the error from the second obstacle would be significant; if we choose a big $\beta$, the error from the first one would be not ignorable. Moreover, as we before said, for $\beta \geq 0$, we even can not find a stable solution. Thus, there are various problems with the application of the usual Runge-Kutta method to this system.

5 Two-Step Runge-Kutta Method

In the previous we observe that the usual Runge-Kutta method could not be used to analyze this system. So, in this section, we will develop a new method, which we shall call a two-step Runge-Kutta method, and it will be demonstrated that this new two-step Runge-Kutta method can be used to analyze this system. There are various problems with the application of the usual Runge-Kutta method. The main idea behind this two-step Runge-Kutta method is to perform the numerical analysis in two steps. In the first step the usual the numerical method are used to solve the un-deformed theory, which can directly remove the error form the two obstacles discussed in the previous section. This is followed by a second step in which the numerical solutions obtained for the un-deformed theory are deformed by a perturbation generated from the GUP deformation of the original theory. We call this approach which is based on two steps, as a two-step Runge-Kutta method.

So, for a undeformed theory, we have

$$\frac{d^2}{dr^2}(rS) = -rVS, \quad (27)$$

$$\frac{d^2}{dr^2}(rV) = -rS^2. \quad (28)$$

The formulas (27) is invariant under a scale transformation:

$$(S, V, r) \rightarrow (\lambda^2 S, \lambda^2 V, \lambda^{-1}r). \quad (29)$$

Now because of the rescaling freedom (29), one can also fix $V_0 = 1$ and allow $S_0$ to vary. Then, following a similar derivation as that (22) in the deformed
Figure 3: The first four bound-state wave functions. $S_0$ is the approximate amplitude of the Wavefunction at $r = 0$. $A$ is approximate value of $V$ at infinity. $B$ is related to the normalization factor.

Therefore, the first derivatives of $S$ and $V$ at $r = 0$ are both zero. Using a standard fourth-order Runge-Kutta NAG routine, we can resolve the formulas (30). See figure 3 and we showed the solutions for the first four wave functions. The solutions for $S$ and $V$ are shown in the first plot of each subfigure; the limit $A$ of $V$ at infinity and $B$ are shown in the second and
the third plot of each subfigure, respectively. Then we can determine the
energy eigenvalues by normalizing the $A$ and also multiplying the coefficient $\zeta$. Therefore, we obtain

$$E = \xi A = \frac{\xi A}{\zeta^2 (4\pi B)^2}. \quad (31)$$

Now, let us consider the correction due to the perturbation term $H_1$. Since

$$H_1 = \frac{\beta}{m} \hat{p}^4, \quad H_0 = \frac{\hat{p}^2}{2m} + \Phi, \quad (32)$$

we have

$$H_1 = (4\beta m) \left[ H_0^2 + \Phi^2 - (H_0 \Phi + \Phi H_0) \right]. \quad (33)$$

At the infinity, the potential energy goes to zero, for specific eigenstates, we have

$$\Delta E_{0n} = \delta \langle \psi_n | H_1 | \psi_n \rangle = 4\beta m E_n^2. \quad (34)$$

Considering the numerical result we just obtained, we know that

$$\frac{\Delta E_{0n}}{E_{0n}} = 4\beta m E_n = 4m(\lambda^2 \beta) \left( \frac{A_{0n}}{\lambda^2} \right) = 2\hbar^2 \beta A_{0n}. \quad (35)$$

Let’s call this relative difference of the eigenvalue the two-step numerical difference. Since we also directly computed the numerical values of the eigenvalues, we can compare the numerical difference with the two-step numerical difference. The numerical difference is

$$\frac{\Delta E_{0n}}{E_{0n}} = \frac{E_n - E_{0n}}{E_{0n}} = \frac{A_n}{B_{0n}} - \frac{A_{0n}}{B_{0n}}. \quad (36)$$

The comparison for $\beta = -10^{66}$ is shown in table [1]. The comparison for $\beta = -10^{65}$ is shown in table [2]. Considering the tables [1] and [2], we can see that for the first modes, the difference between the numerical $\Delta E_{01}$ and its two-step numerical value are quite small than other modes. At least, the numerical value and the theoretic value are in the same order of the magnitude. For the other modes, the numerical results are quite bad. The higher level of mode, the bigger difference. This might be because of the reason that for the higher modes, the initially error for $S^{(2)}_0$ would be amplified larger. Moreover, if we restrict $\beta$ to satisfy the bound from the experiment, $\beta$ would be positive and the absolute value of $\beta$ should be even small than what we chose. For this case, the numerical approach can not give us a solution. However, the two-step numerical approach can be suitable to general $\beta$. Therefore, we see that the two-step numerical approach has more applications. Thus, the two-step numerical approach resolves the problems with the usual Runge-Kutta method.
Table 1: $\beta = -10^{66}$.

|   | $\frac{\Delta E_{0n}}{E_{0n}}$ (numerical) | $\frac{\Delta E_{0n}}{E_{0n}}$ (semi-theoretic) | difference |
|---|------------------------------------------|------------------------------------------|-----------|
| 1 | 0.07437 | 0.02378 | 212%  |
| 2 | 0.18585 | 0.01690 | 1000% |
| 3 | 0.26474 | 0.01481 | 1687% |
| 4 | 0.31501 | 0.01372 | 2195% |

Table 2: $\beta = -10^{65}$.

|   | $\frac{\Delta E_{0n}}{E_{0n}}$ (numerical) | $\frac{\Delta E_{0n}}{E_{0n}}$ (semi-theoretic) | difference |
|---|------------------------------------------|------------------------------------------|-----------|
| 1 | 0.07437 | 0.002378 | 307%  |
| 2 | 0.18585 | 0.001690 | 1692% |
| 3 | 0.26474 | 0.001481 | 3254% |
| 4 | 0.31501 | 0.001372 | 4725% |

6 Conclusion

It is known that because of T-duality, the spacetime geometry in string theory has an intrinsic minimal length associated with it. So, it is expected that the short distance behavior of general relativity should also be modified in such a way that there is an intrinsic minimal length associated with it. It is also possible to have an optical analog for such short distance modification. So, in this paper, we have analyzed a short distance deformation of a semi-classical gravitational system with an intrinsic minimal length. In this system the gravitational field was treated as a classical field, and it was sourced by quantum mechanical matter fields. As such a system would be described by the Schrodinger-Newton equation, we analyzed its short distance modification by analyzing the GUP deformation of the Schrodinger-Newton equation. As the optical analog of Schrodinger-Newton equation means that such a system can be studied in laboratory using its optical analog. It was observed that the usual fourth-order Runge-Kutta method did not work for such a system. This motivated us to propose a new two-step Runge-Kutta method for analyzing this system. In this two-step Runge-Kutta method, the numerical analysis was perform in two steps. In the first step, the usual the numerical method were used to obtain the solution for the un-deformed theory. This is followed by a second step, and in that second step, the nu-
merical solutions obtained for the un-deformed theory were deformed by a perturbation. This perturbation was generated from the GUP deformation of the original theory. It was observed that this two-step Runge-Kutta method resolved the problems associated with the one step Runge-Kutta method.

This method can be used for studying other similar physical systems. It is expected that the GUP deformation of Schrodinger equation with any potential will have the same problems associated the GUP-deformation of the Schrodinger-Newton equation. Thus, it would not be possible to use the usual Runge-Kutta method for analyzing such a system. However, the two-step Runge-Kutta method, we proposed in this paper can be easily used for analyzing such a system. It may be noted that it is possible to consider a different from of the deformation of the uncertainty principle [60]. This deformation of the uncertainty principle produces a linear term derivative in the Schrodinger equation. It would be interesting to perform such a deformation of the Schrodinger-Newton equation and analyze the consequences of such a deformation. It would also be interesting to analyze if the usual Runge-Kutta method or the two-step Runge-Kutta method proposed in this paper, can be used to analyze such a deformation of the uncertainty principle. It would also be interesting to find an optical analog for such a deformation.

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