Deterministic and stochastic aspects of the transition to turbulence

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Abstract. The purpose of this contribution is to summarize and discuss recent advances regarding the onset of turbulence in shear flows. The absence of a clear-cut instability mechanism, the spatio-temporal intermittent character and extremely long lived transients are some of the major difficulties encountered in these flows and have hindered progress towards understanding the transition process. We will show for the case of pipe flow that concepts from nonlinear dynamics and statistical physics can help to explain the onset of turbulence. In particular, the turbulent structures (puffs) observed close to onset are spatially localized chaotic transients and their lifetimes increase super-exponentially with Reynolds number. At the same time fluctuations of individual turbulent puffs can (although very rarely) lead to the nucleation of new puffs. The competition between these two stochastic processes gives rise to a non-equilibrium phase transition where turbulence changes from a super-transient to a sustained state.

Keywords: phase transitions into absorbing states (experiment), hydrodynamic instabilities, turbulence
1. Introduction

How turbulence first arises in simple shear flows has remained an open question for well over a century. Osborne Reynolds [1] was the first to observe that this transition depends on a dimensionless group, i.e. the Reynolds number, as well as on the amplitude of disturbances present in the system. Some of the leading theorists at the time (e.g. Lord Kelvin, Lord Rayleigh, Arnold Sommerfeld, Werner Heisenberg, Hendrik Antoon Lorentz [45]) attempted to probe the stability of pipe and related shear flows (i.e. channel and Couette flow) with essentially linear methods. After many unsuccessful attempts it has become clear (e.g. see [2]) that the occurrence of turbulence is unrelated to the linear stability of the laminar state, as Reynolds had already concluded from his experimental observations many years earlier. While pipe and Couette flow are believed to be stable for all Reynolds numbers, plane Poiseuille (i.e. channel) flow becomes unstable at a Reynolds number ($Re$) of about 5800. However, in the latter case turbulence is typically already observed at Reynolds numbers a little above 1000 and to hold the flow laminar up to the linear stability threshold actually requires considerable effort in experiments.

Transition in the type of flows discussed above is qualitatively different from the classical pictures for the transition to turbulence, which goes back to Landau and to Ruelle and Takens [3]. In both scenarios turbulence arises following a sequence of instabilities of the base flow (which hence becomes linearly unstable). While the Ruelle–Takens-type transition has been observed in several closed flows (e.g. [4]) it is noteworthy that even in this case the transition sequence explains only the onset of comparably low-dimensional chaotic motion, which is dynamically still far from the full spatio-temporal complexity encountered in turbulent flows. In open-shear flows such as pipes, all experimental observations show that the transition in contrast is rather abrupt, directly from laminar to turbulent. The latter type of transition has turned out to be far more difficult to understand.

In more recent years a new transition mechanism has been proposed based on dynamical systems concepts. Invariant solutions of the Navier–Stokes equations, such as periodic orbits or traveling waves are deemed to be ultimately responsible for the existence of the turbulent state. These new solutions\(^1\) arise as the Reynolds number is increased and, importantly, are entirely disconnected from the laminar flow. The proposition is then that chaotic and ultimately turbulent motion will arise following instabilities of these disconnected solutions (independent of the laminar state). Over the past two decades or so,

\(^{1}\) The first such solution was discovered for Couette flow by Nagata in 1990 [5].

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Figure 1. (Top) Isosurfaces of the streamwise vorticity of a puff at $Re = 2200$, taken from the segment denoted by a black horizontal bar in the bottom figure. Flow is from left to right. (Bottom) Space–time diagram for the reduction from $Re = 2800$ to 2200 in a frame co-moving with the mean flow. The local turbulence intensity $q(z) = \int \int (u_r^2 + u_\theta^2) r dr d\theta$ is plotted versus the axial position on a logarithmic color scale.

many traveling waves and periodic orbits have been found in direct numerical simulations for shear flows (for reviews see [6]–[8]). In simulations of turbulence in small periodic domains remarkably close encounters to such unstable solutions have been found (e.g. [9]). While coherent structures resembling traveling waves have also been observed in turbulent flows in experiments [10]–[12] a main discrepancy remains: close to onset, turbulence is always confined to structures localized in the streamwise direction and surrounded by laminar flow, called puffs for the case of pipe flow. Invariant solutions in this $Re$ range on the other hand are usually periodic in this direction. Only very recently the first streamwise localized invariant solutions were discovered for pipe flow [13]. In addition, it could be shown that chaotic motion indeed originates from one such localized periodic orbit. Albeit, in this study the dynamics were limited to a symmetry subspace, it nevertheless illustrates how turbulence can arise from a simple invariant solution, unrelated to the laminar state, as proposed above. Also, this study could explain the origin of another property of localized puffs in pipe flow, which is their transient nature. How (and if) turbulence develops from a transient to a sustained state has been subject to much recent debate. As we will argue below, spatial aspects are crucial in the transition mechanism and eventually lead to a non-equilibrium phase transition giving rise to sustained turbulence. In the following we will discuss this transition in more detail and briefly review important recent results.

2. Discussion

In pipe flow, turbulent puffs (figure 1(top)) are typically observed in a Reynolds number regime of approximately $1700 \lesssim Re \lesssim 2300$. They result from perturbations of finite amplitude and in experiments, if no great care is taken, they will often result from
distortions that the flow experiences directly at the inlet. If the inlet is designed carefully
to avoid such disturbances, and if the pipe is sufficiently straight and smooth, flows
can be held laminar up to much higher $Re$ (the record in experiments is currently at
$Re = 100 \, 000$). For controlled studies of transition it is desirable to start with a laminar
flow and to study its response to a well-controlled disturbance, which, for example, can
be a jet of fluid injected for a brief period through a small hole in the pipe wall. If
the amplitude of such a perturbation is large enough then a turbulent puff is created.
While directly after the flow has been perturbed the dynamics depend on the nature
of the disturbance (the puff first has to develop), after 100 to 150 advective time units
(measured in $D/U$, where $D$ is the diameter and $U$ the mean velocity) the resulting puff
is independent of the perturbation that triggered it.

Curiously, in this Reynolds number regime the turbulence always remains localized.
Even if an extended (axially) part of the pipe is disturbed the flow will always arrange
itself in turbulent segments (i.e. puffs) which are about 5 to 10$D$ long (the turbulent core
excluding the leading edge) and interspersed by laminar fluid. Vorticity isosurfaces of a
turbulent puff at $Re = 2200$ are shown in figure 1(top). Figure 1(bottom) shows energy
levels in a space–time plot (time from bottom to top) of a simulation of an initially
($t = 0$) fully turbulent flow. Upon reduction of $Re$ to 2200, laminar regions (blue) appear
and turbulence becomes confined to localized regions, i.e. puffs (red vertical stripes in
figure 1(bottom)). The direct numerical simulations were carried out with a spectral code
[14]. Fourier modes in the axial and azimuthal direction and finite differences in the
radial direction are employed and the resolution chosen here is $48 \times (\pm 2048) \times (\pm 40)$
in the radial, axial, and azimuthal directions. The domain is $180D$ in the axial direction
($D$ being the diameter) with periodic boundary conditions. As pointed out in [15], at
Reynolds numbers close to 2000, turbulent puffs extract energy from the adjacent (actually
upstream) laminar parabolic flow. The plug-like turbulent profile in the central part of
the puff is unable to sustain turbulence (or rather to feed turbulence in the downstream
direction), consequently the turbulence intensity decreases along the leading edge of the
puff in the downstream direction and the flow eventually relaminarizes. Due to the action
of viscosity, the laminar profile now begins to recover its parabolic shape, so that at a
sufficient distance a second puff can be sustained. If on the other hand the distance between
puffs is too short, then the downstream puff will border onto fluid with a flatter plug-like
profile in the upstream direction. Consequently it cannot extract sufficient kinetic energy
from the flow upstream and decays as shown in [15]. The interaction distance between
two puffs is approximately $20D$ [16]. As a result turbulent puffs have a minimum spacing
of that same distance. While this argument qualitatively explains why fully turbulent
flow cannot be sustained at these low $Re$, the energetic aspects of this process are not
understood in full detail.

A key attribute of puffs is their highly chaotic dynamics. This gives rise to a loss of
memory and limits the prediction of the flow’s future evolution. To illustrate the sensitive
dependence on initial conditions we simulated two puffs with velocity fields that only
deviate by $10^{-10}$. As usual for chaotic systems, this deviation grows exponentially, and
as can be seen from the energy–time series shown in figure 2, the signals notably depart
and become completely unrelated after about 200 advective time units $D/U$ (the time the
puffs takes to travel 200 pipe diameters downstream). The loss of predictability becomes
Figure 2. Sensitive dependence on the initial conditions of a puff at $Re = 1850$. These two lines are the time traces of the kinetic energy $E_{3D}$ of two runs starting with two very close initial conditions separated by $\sim 10^{-10}$. While one decays after about 300 time units, the other persists. Eventually the second one will also decay (not shown in the figure) due to the transient nature of the puffs.

especially clear when one puff suddenly decays (green curve in figure 2) while the other continues unchanged (i.e. the average quantities remain unchanged).

It is a typical feature of puffs that they live for very long times and decay suddenly (see figure 2). Extensive statistical studies have shown that the survival probability is exponentially distributed and the decay is memoryless. This behavior is in line with the escape from chaotic repellers observed in lower-dimensional systems [21, 22]. Here a chaotic attractor turns into a chaotic saddle after an unstable periodic orbit within the attractor and one on the basin boundary collide (unstable–unstable pair bifurcation). Above the bifurcation point, chaotic transients persist for very long times before they eventually decay.

A similar scenario has recently been observed for pipe flow [13]. In this numerical study the dynamics were confined to a symmetry subspace (imposing a mirror and a 2-fold rotation symmetry with respect to the pipe axis). While this somewhat simplifies the dynamics, the flow still exhibits turbulent motion for large enough $Re$. By following the laminar turbulent boundary (the so-called edge state [17], which in this case is a localized periodic orbit) to lower $Re$ the saddle node bifurcation where the periodic orbits originates was reached (at $Re = 1430$). However, the upper branch of this saddle node is a stable periodic orbit (see figure 3) and again localized, and its length is comparable to that of puffs.

For increasing $Re$ the orbit first undergoes a secondary Hopf bifurcation, followed by the formation of a chaotic attractor. This is the point where chaotic motion originates in this subspace. The basin of the attractor increases rapidly with $Re$, until it reconnects with the unstable periodic orbit on the basin boundary. At this moment the chaotic dynamics turn into long-lived transients (presumably following an unstable–unstable
Figure 3. Stable periodic orbit in direct numerical simulations of pipe flow with an imposed 2-fold rotation and a reflection symmetry. (Note that this is necessarily also a numerical solution of the full Navier–Stokes equation.) This orbit arises in a saddle node bifurcation and is originally stable in the subspace. At higher $Re$ a bifurcation sequence leads to chaos and transient turbulent puffs. The periodic orbit is shown at $Re = 1490$ (where $Re = UD/\nu$, $\nu$ is the kinematic viscosity, $D$ the pipe diameter and $U$ the mean velocity). Isosurfaces in red/blue show velocity deviations of $\pm 0.1U$ from the laminar parabolic profile. Yellow/cyan mark isosurfaces of positive/negative streamwise (i.e. axial) vorticity. Upstream and downstream of the periodic orbit the velocity field quickly approaches the laminar parabolic profile.

pair bifurcation). The memoryless nature of the decay and the loss of predictability is a direct consequence of the sensitive dependence on initial conditions characteristic of deterministic chaos.

While the classical picture of turbulence is that of a chaotic attractor, dating back to the landmark paper of Ruelle and Takens, this approach only takes the temporal dynamics into account and neglects the spatial complexity, which is however intrinsic to turbulent flows. The importance of spatial aspects and the spatio-temporal intermittent character of the turbulence transition have been emphasized in studies of model system [18, 20]. Also the chaotic attractor hypothesis was questioned in the 1980s by Crutchfield and Kaneko [19], who proposed that chaotic super-transients are more relevant to turbulence. A first observation supporting this view came from direct numerical simulations of pipe flows where transients were observed [23]. A number of later studies were concerned with long-lived transients at low Reynolds numbers in pipe flow [24]–[30] and found that the decay is a memoryless process with a characteristic lifetime $\tau$ which is a function of $Re$. There was however no consensus whether or not the lifetimes of individual puffs became infinite or remained finite. A number of studies proposed that the turbulent flow decay rates (inverse characteristic lifetimes) scale as $\tau^{-1} \sim (Re_c - Re)$ [14, 24, 25] or at least as a power law $\tau^{-1} \sim (Re_c - Re)^n$ [14]. This view was questioned by Hof [26], who found that instead $\tau^{-1} \sim \exp(Re)$, implying that turbulence remains transient. In a refined study for much longer observation times than in any previous study (spanning almost eight orders of magnitude in time) $\tau^{-1}$ was found to scale super-exponentially with $Re$ [28]. This scaling was confirmed by another experiment [30] and in direct numerical simulations

\[ 2 \] The super-exponential scaling has been related to extreme statistics theory [31].
[29]. It is remarkable that based on their studies of spatially coupled maps, Crutchfield and Kaneko had not only proposed that fluid turbulence could evolve around spatially coupled transients but also that the lifetimes under certain conditions (type-2 super-transients) are memoryless and increase super-exponentially with system size. While it has been argued that an increase in $Re$ in turbulent flows is analogous to an increase in system size, in coupled chaotic maps this correspondence is however not entirely clear. The lifetime studies in pipe flow were carried out for single turbulent puffs (not many spatially coupled ones) and the puff size hardly changes with $Re$. One may argue that the smallest scales of turbulence decrease with $Re$ and hence the system size based on this smallest scale increases. Nevertheless, the Reynolds number range over which the lifetime increase is observed is relatively small ($1700 < Re < 2050$) and hence this size effect will be only very moderate.

A more direct analogy can be drawn to another model system of coupled chaotic maps, which is motivated by bistable excitable media [32]. A key difference from the above-mentioned map models is that here the susceptibility of a laminar site to perturbations from neighboring chaotic sites (and hence the minimum perturbation amplitude to trigger chaotic dynamics at a laminar site), decreases with $R^{-1}$ (where $R$ is a control parameter analogous to the Reynolds number). This model input reflects experimental observations of pipe flow where the minimum perturbation amplitude to trigger turbulence was found to scale with $Re^{-1}$ [33, 10, 34]. In the model, just like in the experiments, localized excited states, i.e. puffs, with transient lifetimes are observed, and with an increase in the control parameter the lifetimes of individual puffs scale faster than exponential and hence remain transient.

Figure 4. Puff-splitting process. At the puff leading edge vortices are shed in the downstream direction. While normally they decay, occasionally in rare cases they manage to escape beyond the interaction distance of the upstream puff. The first puff and the vortex patch are now separated by a region of laminar fluid and the vortex patch grows to a new puff. The isosurfaces correspond to the axial vorticity component (positive in blue, negative in yellow).
Figure 5. The intersection point between characteristic lifetimes and splitting times determines the onset of sustained turbulence in pipe flow. To the right of the intersection point, on average, new puffs are created faster (by puff-splitting) than existing puffs decay; hence (in the thermodynamic limit) spatio-temporally intermittent turbulence persists. The data is taken from [28, 36]. Note that both the decay and the splitting process are memoryless and described by characteristic timescales (see original papers for details).

Consequently in pipe flow the increase in the temporal complexity alone does not lead to sustained turbulence. As proposed by Moxey and Barkley [35], and later explicitly shown by Avila et al [36], turbulence becomes sustained by a spatial growth process called puff-splitting. Puff-splitting is commonly observed at Reynolds numbers of around 2300 [37, 38] and while here turbulence is still confined to puffs typically 5 to 10D in length, puff sizes fluctuate and can occasionally reach larger values. In these instances, in a small number of cases, a segment of turbulent fluid at the leading edge of the puff can escape further downstream beyond the puff–puff interaction distance and a new puff develops here (see figure 4). This splitting process leads to an increase in the turbulent fraction. As shown by Avila et al [36], puff-splitting is also intrinsically memoryless and can already be found at much lower Re, as previously expected. The characteristic time for such an event to occur decreases super-exponentially with Re. The argument for turbulence to become sustained is now straightforward. If the characteristic time for turbulent puffs to decay is smaller than the time for new puffs to be created (i.e. by splitting), then turbulence will eventually decay. If on the other hand new puffs are created faster than existing ones decay, then in the thermodynamic limit turbulence becomes sustained. The critical Reynolds number where turbulence changes from a transient to a sustained state can be estimated by the intersection point of the characteristic timescales of the two processes shown in figure 5. It should be noted that in any experiment, because of the finite system size, turbulence in principle remains transient. In practice, however, lifetimes increase extremely fast so that above the critical point the probability to still observe decay events becomes vanishingly small.
This transition is analogous to non-equilibrium phase transitions and, as we will argue below, bears close resemblance to directed percolation (DP) and related contact processes. First speculations about a possible connection between transition in linearly stable shear flows and DP date back to Pomeau in 1986 [39]. Just like in DP, pipe flow has a unique absorbing state, which is the laminar flow. Once turbulence has decayed, the flow cannot by itself return to turbulence unless it is disturbed from the outside. Recent studies of puff decay and splitting [28, 36] also suggest that the interaction is only short range, which is another requirement for DP [40]. No observations indicate that a localized puff would create a second one in a part of the pipe not adjacent to it (i.e. more than 25 $D$ or so away). Equally, it has never been observed that puffs would influence the lifetimes of other puffs that are sufficiently far away. These recent studies also infer that if such a relation to DP exists, a single unit (i.e. a lattice point in DP) must correspond to a turbulent puff and not, for example, to a single vortex. To explore this analogy further hence requires much larger system sizes, allowing one to follow the evolution of many turbulent puffs (spots). While a number of earlier studies (e.g. [41]) looked at such aspects retrospectively, the system sizes used were too small. In a numerical investigation [42] of plane Couette we consequently chose a narrow but very long domain so that a large number of turbulent stripes (analogous structure to puffs in pipes) could be accommodated. This study gave more direct evidence that the transition is indeed a non-equilibrium continuous phase transition. Here, just like in pipe flow, super-exponential lifetime and splitting statistics were observed for single stripes, and a critical point for the onset of sustained turbulence could be determined in the same manner as described above. The much shorter timescales at the intersection point between decay and splitting curves allowed us to resolve size distributions at this point. The distributions of laminar gaps exhibit scale invariance, supporting the proposition of a continuous phase transition. Also here the same transition scenario between transient localized chaos and sustained spatio-temporal intermittent chaos was found.

Analogies to DP have also been explored in recent theoretical studies, e.g. [32, 43, 44] In particular, for the coupled map model presented in [32], close agreement was found to the experimental results for pipe flow: the turbulent state becomes sustained when the splitting outweighs the decay of individual puffs. In addition, the critical exponent for the increase of the turbulent fraction above onset was found to agree well with the universal one for DP in $1 + 1$ dimensions.

To our knowledge, to date there has only been a single experimental confirmation of DP, which has been for liquid crystal electro-convection [46]. Hence answering the question whether also the transition to shear flow turbulence falls into this class would not only be a very important contribution for fluid dynamics but also for statistical physics. While at present a final answer to the question whether the laminar turbulence transition is a non-equilibrium phase transition in accordance with DP is outstanding, experiments and numerical simulations to clarify this question are underway. One of the main challenges here is to resolve the extremely long timescales relevant in the vicinity of the transition point (note that characteristic splitting and decay times in pipes correspond to almost $10^8$ advective time units!). In this time puffs travel a distance corresponding to $10^8$ pipe diameters. Equally, an accuracy in the Reynolds number of about 0.1% is required, setting a further challenge for experiments. A further open issue is the transition from spatially
intermittent turbulence (i.e. puffs) to expanding space-filling turbulent structures, which takes place somewhere between Reynolds numbers 2300 and 3000.

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