Non-relativistic metrics from back-reacting fermions

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Abstract

It has recently been pointed out that under certain circumstances the back-reaction of charged, massive Dirac fermions causes important modifications to AdS\textsubscript{2} spacetimes arising as the near-horizon geometry of extremal black holes. In a WKB approximation, the modified geometry becomes a non-relativistic Lifshitz spacetime. In three dimensions, it is known that integrating out charged, massive fermions gives rise to gravitational and Maxwell Chern–Simons terms. We show that Schrödinger (warped AdS\textsubscript{3}) spacetimes exist as solutions to a gravitational and Maxwell Chern–Simons theory with a cosmological constant. Motivated by this, we look for warped AdS\textsubscript{3} or Schrödinger metrics as exact solutions to a fully back-reacted theory containing Dirac fermions in three and four dimensions. We work out the dynamical exponent in terms of the fermion mass and generalize this result to arbitrary dimensions.

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1. Introduction

In recent times, there has been a concerted effort to find real world applications of the AdS/CFT correspondence. By extending the rules of the correspondence to finite temperature, there are now well-established prescriptions to calculate the retarded Green’s functions which enable us to extract transport properties of interesting physical systems. Although none of these systems, strictly speaking, describe the real world, it is hoped that by studying these problems, one will gain intuition about real physical systems which are for example in the same universality class. On the one hand, AdS/CFT methods have found applications to particle physics, e.g. the viscosity of the strongly coupled quark gluon plasma \cite{1}. On the other, these methods have been extended to studying table-top condensed matter physics \cite{2}. It is fair to say that applications of gauge/gravity duality will yield many more surprises.
One application that has garnered much attention lately is the computation of the fermion Green’s functions using AdS/CFT which yield non-Fermi liquid-type behaviour [3]. In the simplest example, one starts with a Reissner–Nordström AdS black hole and considers the Dirac equation in this background. In the zero temperature limit, the black hole becomes extremal and the near-horizon geometry contains an AdS$_2$ factor. This AdS$_2$ is thought to play an important role in the existence of a Fermi surface. The resulting fermion Green’s function shows non-Fermi liquid-type behaviour which is exciting in the context of connections with condensed matter systems, e.g. high-$T_c$ superconductivity. However, in the zero temperature limit, the black hole and hence the corresponding dual field theory still has a large entropy which is far from what is expected in the realistic situations.

Recently in [4], it has been pointed out that in certain regimes the back-reaction of the fermions cannot be ignored. In fact, the AdS$_2$ geometry has been argued to be modified into a ‘Lifshitz’ [5] type and the associated entropy enigma is thought to disappear since the horizon area is now zero. Motivated by this, we examine the problem of back-reaction of Dirac fermions in (2 + 1)- and (3 + 1)-dimensional gravitational theories in the context of AdS/CFT in some detail. The (2 + 1)-dimensional bulk theory will describe (1 + 1)-dimensional field theories. In this context interesting non-Luttinger-type behaviour has been found in [6]. The (3 + 1)-dimensional bulk theory will describe interesting (2 + 1)-dimensional field theories which are of relevance to condensed matter physics related to high-$T_c$ superconductivity. We will restrict our attention to the simplest case where only the metric, a $U(1)$ gauge field and a charged, massive Dirac fermion are involved.

We begin with the discussion of a (2 + 1)-dimensional bulk theory. It is known that [7–9] when one integrates out a charged fermion in 2 + 1 dimensions, the effective action that one is led to contains the Maxwell and gravitational Chern–Simons terms. Gravitational Chern–Simons theories have been the subject of much attention recently due to the connection with topologically massive gravity (TMG)[10]. In TMG null warped AdS$_3$ have already been found. The new finding in our case is their existence in the presence of the $U(1)$ Chern–Simons term $A \wedge F$. We show that in addition to the usual asymptotically AdS$_3$ spacetime, one also gets Schrödinger spacetimes [11] whose dynamical exponent depends on the Maxwell Chern–Simons or the gravitational Chern–Simons terms. Similar solutions (called warped AdS$_3$) have already been reported in [12] in the presence of the gravitational Chern–Simons terms. Inspired by this finding, we look for Schrödinger spacetimes as exact solutions to back-reacted Dirac fermions. Since fermions obey the Pauli exclusion principle, they cannot be treated classically. We will treat the system of gravity, gauge field and fermions in a semi-classical manner following [13, 14]. In this approach, the fermion stress tensor and current appearing in the equations of motion are evaluated as expectation values in some state $|Q\rangle^3$. We will consider $|Q\rangle$ to be the state made from $N$ fermions. It will turn out that the role of $\hbar$ is played by $1/Nq$ so that for a fixed charge, taking the large $N$ limit will correspond to making $\hbar$ small which is needed for a semi-classical approximation to make sense. The solution to the full problem in a self-consistent manner in general is a very difficult task. Typically one resorts to some approximation as in [14] where a WKB approximation is used to derive the Oppenheimer–Volkoff equations. Quite remarkably, our problem will turn out to be amenable to an exact solution! The key ingredient that makes this possible is the fact that the fermion couples to the gauge field. The equations of motion then lead to certain important constraints which make an exact solution possible. In arbitrary bulk dimensions $D$, we will show that the dynamical exponent is given by

$$z = \pm 2mL - (D - 2),$$

(1.1)

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3 This method is used quite routinely in nuclear physics. See, e.g., [15].
where \( L \) is the length scale entering in the definition of the cosmological constant in the usual way. Lifshitz metrics are known to emerge as a WKB approximation \([4]\). It is not known if they are exact solutions to a system of self-gravitating fermions as in the case of the Schrödinger metrics. In the \((2+1)\)-dimensional case, as we will show, Lifshitz metrics are not exact solutions to the equations of motion. However, in higher dimensions we expect there to be more general ways of solving the equations and it could be that other interesting solutions exist. We leave this interesting question as an open problem\(^4\).

This paper is organized as follows. In section 2, we consider Chern–Simons theories in 2+1 dimensions and show that Schrödinger-type metrics with specific dynamical exponents exist as solutions to the system of equations. Motivated by this, we ask if similar metrics exist as fully back-reacted solutions to Dirac fermions in section 3. We work out the dynamical exponent in arbitrary dimensions, i.e., where the original metric would have been \( \text{AdS}_2 \times \mathbb{R}^{D-2} \). We conclude with open questions in section 4. In appendix A we spell out our conventions and in appendix B, we derive spin connections for a certain class of metrics, which includes, AdS, Schrödinger and Lifshitz metrics and set up the Dirac equation in these backgrounds. In appendix C we consider the stress tensor for a Dirac fermion\(^5\) in a background containing an AdS\(_2\) factor. We denote the charge of the fermion by \( q \) and mass by \( m \). The AdS\(_2\) arises as the near horizon geometry of some extremal Reissner–Nordström black hole. In particular, a gauge field \( A_\mu \) needs to be turned on for Einstein’s equations to be satisfied. We consider a charged, massive Dirac fermion in this background as a probe and work out the stress tensor. We find that quite generally an off-diagonal component of the stress tensor is turned on. We show that when \( m < \sqrt{2q^2 + 1} \), the back-reaction of fermions becomes important. In appendix D, we study a quantum mechanical toy model in path-integral formalism which corresponds to coupling of macroscopic fermi gas to a bosonic harmonic oscillator. We integrate out the fermions and obtain an effective action for the boson. We show that when there are a large number of fermions, the back-reaction of fermions is well approximated by replacing fermion number operators by their expectation values evaluated in the \( N \) fermion state.

2. Schrödinger spacetimes in Chern–Simons theories

We will begin with the discussion of \((2+1)\)-dimensional Einstein–Maxwell–Chern–Simons theory. Our motivation for considering this theory came from the observation that for a probe fermion in AdS\(_2\) geometry, off-diagonal components of the stress tensor are non-zero. (This has been discussed in detail in appendix C.) As a result it may be expected that any non-relativistic modification of the metric will be of the Schrödinger type rather than Lifshitz. As explained in the introduction, on integrating out fermions in \(2+1\) dimensions, one gets Maxwell and/or gravitational Chern–Simons theories. Rather than looking at the more complicated problem of back-reaction of fermions, we wish to begin by investigating if Schrödinger type solutions exist for a Maxwell and/or gravitational Chern–Simons theories. Consider the action given in equation (A.1). The equations of motion for the metric read

\[
R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = - \frac{1}{L^2} g_{\alpha\beta} - \frac{1}{2} \left( F_{\alpha\gamma} F_{\beta}^{\gamma} - \frac{1}{4} F^2 g_{\alpha\beta} \right) + \mu G C_{\alpha\beta} = 0, \tag{2.1}
\]

where \( C^{\mu\nu} = \epsilon^{\mu\alpha\beta} \nabla_\alpha \left( R^{\nu}_{\rho} - \frac{1}{4} \delta^{\nu}_{\rho} R \right) \) is the Cotton tensor. The equations of motion for the gauge field read

\[
\nabla_\mu F^{\mu\nu} + 2\mu \rho \epsilon^{\nu\alpha\rho} F_{\alpha\rho} = 0. \tag{2.2}
\]

\(^4\) The fermionic stress tensor (equation (7.43)) in \([4]\) can be shown not to lead to Schrödinger spacetimes.

\(^5\) Dirac systems in AdS spacetimes have also been studied in \([16]\).
It can be shown that the following Schrödinger (or warped AdS$_3$ [12]) solutions exist:

\begin{equation}
\text{ds}^2 = -r^2 \text{d}t^2 \mp 4r \text{d}t \text{d}x + \frac{\text{d}r^2}{4r^2},
\end{equation}

\begin{equation}
A_t = \frac{2}{z} [r^2 \psi (z - 1) (1 \mp \mu_G \pm 2 \mu_G z)]^{1/2},
\end{equation}

\begin{equation}
z = 0, 1, \mp 4 \mu_F, \frac{\mu_G \mp 1}{2 \mu_G}.
\end{equation}

To avoid clutter, we have set $L = 1$. Here $z$ is the standard dynamical exponent and $z = 1$ corresponds to the AdS solution, whereas $z = 0$ corresponds to a chiral wave AdS solution. The thing to note here is the existence of Schrödinger solutions when either or both of the Chern Simons terms exist. The existence of these solutions in the presence of the gravitational Chern–Simons term can be anticipated from the general discussion in [18]. The case $z = 2$ ($\mu_F = \mp 1/2$ or $\mu_G = \mp 1/3$) is called the null-Warped AdS$_3$ [12].

At this point it is worth noting that Chern–Simons terms are not invariant under parity (P) and time reversal (T) transformations but are invariant under the combined transformation PT. This symmetry is also shared by the Schrödinger background, which under P or T transformation swaps the sign of the $\text{d}t \text{d}x$ term but the metric is invariant under combined PT transformation. However, in the case of $z = 1$ the metric is diffeomorphic to the AdS metric and preserves P and T separately. This does not contradict the earlier conclusion because for $z = 1$, all components of the gauge field vanish and in that case it is natural to expect to recover P and T symmetry.

With a bit more work it can also be shown that a Lifshitz metric is not a solution to the above set of equations. We begin with the ansatz

\begin{equation}
\text{ds}^2 = -\frac{\text{d}t^2}{r^2} + \frac{\text{d}r^2 + \text{d}x^2}{r^2}, \quad A_t = \phi (r), \quad A_r = 0, \quad A_x = \chi (r).
\end{equation}

Then the $tt$ component of the metric equations of motion leads to $r^2 \psi' \phi'^2 + r^4 \chi'^2 = 0$ while the $xx$ component leads to $4 - 4 \psi^2 + 2r^2 \phi'^2 + r^4 \chi'^2 = 0$. Combining these two we are led to $z = \pm 1$. The solution $z = 1$ is the usual AdS. The choice $z = -1$ can be shown to lead to imaginary gauge fields. Thus we conclude that Lifshitz is not a solution to this system. We now turn to the more complicated problem of considering back-reaction due to fermions.

### 3. Schrödinger spacetimes from fermions

If we compute the stress tensor for a probe fermion in an AdS$_2$ background as in appendix C, we find that it contains off-diagonal components. This suggests that if their back-reaction is taken into account an exact background geometry of the form of a Lifshitz geometry, which is diagonal, is unlikely. Given, however, the intuition from flat space that fermions and the Chern–Simons terms are intimately related, and the similarity in the structure of the energy–momentum tensors of the gauge fields in the presence of the gauge Chern–Simons term and that of the fermions, one is tempted to conjecture that the back-reaction of charged fermions would also lead to Schrödinger spacetimes, exactly as shown to occur, in the previous section when the back-reaction of gauge fields in the presence of Chern–Simons terms is taken into account.

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6 This nomenclature is motivated by the obvious similarity of the metric with the standard Schrödinger spacetimes which involve additional spatial directions. Strictly speaking this is an abuse of nomenclature since in our case there are no spatial directions which transform under dilatations. The conformal boundary of such spacetimes has been discussed in detail in [17].
account. To be explicit, the full Lagrangian we will be considering is (total bulk spacetime dimensions is denoted by $D = d + 1$)

$$S = \frac{1}{2\ell_p^{d+1}} \int d^{d+1}x \sqrt{-g} \left[ R - 2\Lambda - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i \left( \bar{\psi} \Gamma^a \epsilon^a_\mu D_\mu \psi - m \bar{\psi} \psi \right) \right],$$

(3.1)

where $\Lambda = -\frac{d(d-1)}{2L^2}$, and the $\Gamma$-matrix convention is given in appendix B. Note here that the field is generally expressible as a sum over a complete set of eigenmodes

$$\psi(r, t, x) = \sum_k \left( a_k \psi_{a,k}(r, t, x) + b_k^\dagger \psi_{b,k}(r, t, x) \right),$$

$$\psi^\dagger(r, t, x) = \sum_k \left( a_k^\dagger \psi_{a,k}^\dagger(r, t, x) + b_k \psi_{b,k}^\dagger(r, t, x) \right),$$

(3.2)

where $a_k, a_k^\dagger, b_k, b_k^\dagger$ denote the anti-commuting creation and annihilation operators of the corresponding fermion and anti-fermion modes $\psi_{a/b,k}(r, t, x)$, respectively, $k$ denotes some general quantum numbers of the modes, each mode $\psi_{a/b,k}(r, t, x)$ a two-component spinor being a solution to the Dirac equation, and that to lowest order ignoring quantum fluctuations of the background we have

$$\{a_k, a_{k'}\} = \{b_k, b_{k'}\} = \hbar \delta_{k,k'},$$

(3.3)

while all other anti-commutations between operators vanish. Since the background geometry itself is strongly back-reacted by the fermions, in general it is slightly ambiguous to define what the vacuum would be. We assume the existence of such a stable vacuum such that

$$a_k|0\rangle = b_k|0\rangle = 0.$$  

(3.4)

Any general excited state would be of the form

$$|Q\rangle_{\text{general}} = \prod_i a_{k_i}^\dagger \prod_j b_{j_j}^\dagger |0\rangle,$$

(3.5)

although they are not generally the $N$-fermion ground state at zero temperature, and are thus not necessarily stable against decay. The Einstein equation is, in our conventions (appendix A),

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = \frac{d(d-1)}{2L^2} g_{\alpha\beta} - \frac{1}{2} \left( F_{\alpha\gamma} F^{\gamma}_{\beta} - \frac{1}{4} F^2 g_{\alpha\beta} \right) = \frac{1}{2} \langle Q | T^I_{\alpha\beta} | Q \rangle,$$

(3.6)

where $T^I_{\alpha\beta}$ is as defined in (C.3) and we reproduce here for convenience

$$T^I_{\alpha\beta} = \frac{1}{2} (-i \bar{\psi} \Gamma_\mu D_\nu \psi + i \bar{\psi} D_\nu (\Gamma_\mu) \psi).$$

(3.7)

This form of the stress tensor is given in [19] and to the best of our knowledge was first given in [13]. The covariant derivative $D_\mu$ is defined in equation (C.2). Maxwell’s equations are given by

$$\nabla_\nu F_{\mu\nu} = \langle Q | j_\mu | Q \rangle$$

$$j_\mu = -q \bar{\psi} \Gamma_\mu \psi.$$

(3.8)

Note that in the above expressions, the fermion stress tensor and electric currents that back-react on the background are really the expectation values of the corresponding operators, evaluated on the state $|Q\rangle$. At zero temperature, the ground state of $N$ fermions would correspond to piling up the various modes, starting from the bottom mode. Naively, both the currents and the fermion stress tensor would roughly take the form $\sum_k a_k^\dagger a_k |\psi_k(r, t, x)|^2$, whose expectation value in the state $|Q\rangle$ would involve a sum over contributions of modes that are occupied in $|Q\rangle$. However, here we should note that in general one is confronted with a diverging expectation value due to contributions even of unfilled modes. It is ambiguous to define normal-ordered operators in curved space, since vacuum energy gravitates as well
Here we approach the problem via the analytic continuation [19]. As we will see in the following section, the modes that are consistent with the background symmetry and Gauss’s law constraints are simply chiral plane waves along $t$ in Schrödinger space, with trivial radial dependence. Therefore, they contribute to the expectation value of the current or stress tensor via
\[ \langle Q | \sum_i a_i^\dagger a_i | Q \rangle \] simply as
\[ \sum_i \langle 0 | e^{\frac{i}{\hbar} p_i t} | 0 \rangle = N, \]
where $N$ is the number of modes. This contribution is precisely in the same way the unoccupied anti-fermion modes, of which we have infinitely many, will contribute via
\[ \langle Q | \sum_i b_i^\dagger b_i | Q \rangle = 1 + 2 + \cdots = \zeta(0) = -\frac{1}{2}, \]
where in the second equality we attempt to regularize it via analytic continuation. Adopting this approach would correspond to shifting $N$ by $-\frac{1}{2}$. Note that this is sub-leading in $N$, and in the large $N$ limit, which is the main focus of this paper, we will ignore this term.

The analysis that we present below is very similar in spirit to that in [14] for a system of self-gravitating fermions. The essential steps in this are as follows.

1. We want to find a particular set of fields involving the metric, gauge field and fermions such that it is an exact solution to the equations of motion given above.
2. In [14], a certain ansatz for the metric compatible with some pre-supposed symmetries is taken. The Dirac equation is solved in this background for a complete set of modes.
3. With these solutions, one works out the stress tensor $T_f$ entering Einstein’s equations, taking care of the antisymmetric nature of the fermions.
4. The full set of equations is now solved self-consistently.

In the regime of large quantum numbers and using a WKB approximation [14] recovers the Oppenheimer–Volkoff equations. In [4], a WKB approximation was used to obtain a Lifshitz solution. We will not make any such approximations in what we do. In addition to the metric and fermions as in [14] we also have a gauge field which makes the calculation somewhat more complicated. To keep life simple, we will begin with the ansatz for a Schrödinger spacetime with an undetermined dynamical exponent. Then we will ask if there exist a self-consistent set of solutions. To be compatible with the symmetries of the problem, we will assume that the gauge field is only dependent on the radial coordinate. At this stage, we note that the stress tensor for fermions has an $\hbar$. We will find that $1/Nq$ plays the role of $\hbar$ in what we do.

A quick way to see that this is true is that in our solution, the stress tensor from fermions is of the same order as that of the Maxwell fields and the Einstein tensor, not surprisingly since we found a fully back-reacted solution. However, the general form of the charged fermion stress tensor is roughly $T_f \sim Nq$, using the anti-commutation relation of the operators. Therefore, our solution naturally forces $Nq \sim 1/\hbar$. It is interesting to note here that for small $Nq$, it would imply that $\hbar$ is large and that a classical approximation would not stand. This conforms with our intuition that a classical approximation improves when the fermi gas becomes macroscopic. To find further support for our approach, we present a path-integral calculation of a toy model consisting of fermions coupled to bosons in appendix D. We found that quite generically in the limit of large fermion number $N$, our semi-classical treatment adopted here is a good approximation.

3.1. 2+1 dimensions

Like the Chern–Simons terms, massive fermions in 2+1 dimensions also break P and T symmetry. We therefore expect the result here to be similar to what we got in the presence of Chern–Simons terms. Anticipating that the back-reacted exact geometry is going to be given
by the Schrödinger metric, we start with the ansatz metric exactly as in the previous section, namely
\[ ds^2 = L^2 \left(-r^2 \, dt^2 - 4\epsilon r \, dr \, dx + \frac{dr^2}{4r^2}\right), \]  
where \( \epsilon = \pm 1 \), and leave the gauge fields and the index \( z \) to be determined by the equations of motion. The three sets of equations involved are the Maxwell equations, the Einstein equations and the Dirac equations, to be solved simultaneously. We will, unless otherwise specified, set \( L = 1 \) to avoid clutter in the equations.

To begin with, we compute the spin-connection necessary to build the Dirac equations. Choosing explicitly the vierbein as
\[ e^0 = r - z^2 (r z \, dt + 2\epsilon r \, dx), \quad e^1 = 2\epsilon r \, dx, \quad e^d = \frac{1}{2r} \, dr, \]
the corresponding spin-connection is
\[ \omega^{01} = \frac{z - 1}{2} \, dr, \quad \omega^{0d} = r^{-z} (r^2 \, dt + 2\epsilon r \, dx), \quad \omega^{1d} = r^{-z} (r^2 (z - 1) \, dt + 2\epsilon r \, dx). \]

Considering a particular mode of the Dirac spinor,
\[ \psi_{\omega,k}(r, t, x) = e^{-i\omega t - i k x} \left( \psi_{+,\omega,k}(r) \right) \]
the Dirac equation for a fermion of mass \( m \) takes the form
\[ -2i r^{-z/2} (\omega - q A_r(r)) \psi_{-,\omega,k}(r) - 2i r^{-z/2} (\omega - q A_r(r)) \psi_{+,\omega,k}(r) \]
\[ + \{1 + 2m - z + i r^{(-1+z/2)} (k - q A_x(r))\} \psi_{+,\omega,k}(r) - 4r \psi_{-,\omega,k}(r) = 0. \]  

It is simplest to start examining Maxwell’s equations (3.8). We shall begin by picking the gauge
\[ A_r = 0. \]
We will be looking at a radially symmetric solution such that the metric and the gauge fields contain only \( r \) dependence. Given our gauge choice, Maxwell’s equation for the \( A_r \) component becomes purely a constraint on the vanishing of the \( r \)-component of the fermion current. That gives, explicitly,
\[ j_r = -q \epsilon \left( \frac{\hat{\psi}_+^\dagger \hat{\psi}_- - \hat{\psi}_-^\dagger \hat{\psi}_+}{2r} \right), \]
where we have defined the operators \( \hat{\psi}_a \), to be distinguished from the modes \( \psi_{\pm,\omega,k} \):
\[ \psi = \left( \hat{\psi}_+, \hat{\psi}_- \right), \quad \hat{\psi}_a = \sum a_i e^{-i(\omega t - k x)} \psi_{a,\omega,k}^i + b_i^* e^{i(\omega t - k x)} \psi_{a,\omega,k}^i. \]
At the same time, to preserve the symmetry of the system, we keep the magnetic field zero, i.e. \( F_{tr} = 0 \); we only allow for
\[ A_t(r) = a_1^t, \]
for some constant $a_1^r$, which, from the gauge equations, further give
\begin{equation}
A_r'(r) + r A_r''(r) = -q \epsilon ((r - \frac{z}{2}) (Q | \hat{\psi}_- - \hat{\psi}_+)(\hat{\psi}_- - \hat{\psi}_+)) |Q\rangle = 0. \tag{3.20}
\end{equation}
To satisfy both requirements (3.17), (3.20), in general we have
\begin{equation}
\psi_{\pm,\omega,k}(r) = \psi_{\pm,\omega,k}^a(r), \quad \psi_{\pm,\omega,k}(r) = \psi_{\pm,\omega,k}^b(r), \tag{3.21}
\end{equation}
for each contributing mode occupied in the state vector $|Q\rangle$.

Now going back to the Dirac equations, given relation (3.21) the Dirac equations reduce to
\begin{equation}
r \left( 1 - 2m + z \right) \psi_{\pm,\omega,k}^{a,b} + 4r \psi_{\pm,\omega,k}^{a,b} \pm i r z (k - q A_x) \psi_{\pm,\omega,k}^{a,b} = 0. \tag{3.22}
\end{equation}
It is important to note that with relation (3.21), the $A_t$ and time-derivatives have completely dropped out from the Dirac equation. This implies that all the modes with different $\omega$ share the same radial wavefunction, which also means
\begin{equation}
\psi_{\pm,\omega,k}^{a,b} = \psi_{\pm,\omega,k}^{b,a}. \tag{3.23}
\end{equation}
Also the term proportional to $(k - q A_x)$ renders the two equations inconsistent. To obtain non-trivial solutions, it implies that
\begin{equation}
a_1^r = \frac{k}{q}, \tag{3.24}
\end{equation}
which cannot be satisfied generally for an arbitrary $k$-mode in a given background of some $a_1^r$. This implies that the modes consistent with the background are those with arbitrary $\omega$ but fixed $k$ such that (3.24) is satisfied. These solutions however are gauge equivalent to having $k = A_t = 0$, via the gauge transformation $\chi = -\frac{k}{q} x$, i.e. $\psi \rightarrow e^{i k x} \psi$, such that correspondingly $A_x \rightarrow A_x - \frac{k}{q}$. We will thus from now on set $A_x = k = 0$, i.e. in full generality, we have
\begin{equation}
\psi(t, r) = \left( \begin{array}{c} \psi_1^a(r) \\
\psi_1^b(r)
\end{array} \right) = \sum_i \left( a_\omega e^{-i \omega t} \Psi_\omega(r) + b_\omega e^{i \omega t} \Psi_\omega(r) \right), \tag{3.25}
\end{equation}
where we have
\begin{equation}
\psi_{\pm,\omega,0}^a = \psi_{\pm,\omega,0}^b = \Psi_\omega(r). \tag{3.26}
\end{equation}
To avoid complications, for the moment we treat the spectrum as discrete. This does not alter our results in any significant way even when the continuous limit is taken. We consider the back-reaction of a large number of fermions. This can be done by building explicitly a fermi gas by constructing the $N$-fermion state vector $|Q\rangle$:
\begin{equation}
|Q\rangle = \prod_i^N a_\omega^{\dagger} |0\rangle. \tag{3.27}
\end{equation}
Let us emphasize here that $N$ is the number of modes excited, which is infinity in a continuum limit for finite fermi energy. The fermion density distribution $\rho$ however is smooth. As we have demonstrated above, since modes of different $\omega$ have the same $r$-wavefunction, evaluating the expectation value of the stress tensor and the current on $|Q\rangle$ would be proportional to the term, schematically given by
\begin{equation}
\langle Q | \hat{\psi}_+ \hat{\psi}_- | Q \rangle = |\Psi_\omega(r)|^2 (Q | \sum_i^N a_\omega^{\dagger} a_\omega + \sum_i^\infty b_\omega^{\dagger} b_\omega | Q \rangle = \hbar \left( N - \frac{1}{2} \right) |\Psi_\omega(r)|^2, \tag{3.28}
\end{equation}
where the unfilled-anti-fermionic states contribution $bb^\dagger$ has been regularized as discussed in (3.9). Since they are sub-leading in $N$ after regularization, we will not include them explicitly
in the rest of our discussion. Note that in the expectation values of the operators, cross terms occurring in the product $\hat{\psi}^\dagger \hat{\psi}$ vanish, and as a result both the stress tensor and the currents have no $x$, $t$ dependence. Returning to the Dirac equation, one can then readily solve it to give

$$\Psi_+(r) = p_1 r^{\delta_p}, \quad \delta_p = \frac{1}{2}(-1 + 2m - z)$$

(3.29)

for some suitable overall constant normalization $p_1$. The Maxwell equation is left with one component determining $A_t$. The expectation of the $t$-component of the current is given by

$$\langle Q| j_t | Q \rangle = -qr^3 \langle Q | (|\hat{\psi}_-|^2 + |\hat{\psi}_+|^2) | Q \rangle = -2r^{z+2\delta_p} Nq\hbar |p_1|^2.$$

(3.30)

Substituting in the Maxwell equations we have

$$\hbar N|p_1|^2 qr^{\frac{3}{2}(z+4\delta_p)} = 2r A_t - 2r^2 A''_t = 0,$$

(3.31)

which gives

$$A_t = \frac{2N|p_1|^2 qr^{\frac{3}{2}(z+4\delta_p)}}{(z + 4\delta_p)^2} + c'_2 + c'_1 \log r$$

(3.32)

for some constants $c'_i$. The log term would appear in the energy–momentum tensor as a lone term. We therefore set

$$c'_1 = 0.$$  

(3.33)

Returning to the Einstein equations, we inspect specifically the $tt$ component. Evaluating the LHS of equation (3.6), we get

$$R_{tt} - \frac{1}{2} g_{tt} R - g_{tt} = \frac{1}{2} F_{t\gamma} F_t^\gamma - \frac{1}{4} F_t^2 g_{tt} = 2(r^2(-1 + z) - r^2 A_t^2),$$

(3.34)

and the expectation value of the $tt$-component of the fermionic energy–momentum tensor evaluated on the state $|Q\rangle$:

$$\langle Q| T_{tt} | Q \rangle = -2r^3 \langle Q | \hat{\psi}_+ \partial_t \hat{\psi}_+ | Q \rangle - q A_t \langle Q | \hat{\psi}_+ \hat{\psi}_+ | Q \rangle$$

$$= -2r^{z+2\delta_p} \hbar \left( \sum_{i=1}^N |p_i|^2 \omega_i - q A_t \sum_{i=1}^N |p_i|^2 \right)$$

$$= -2r^{z+2\delta_p} |p_1|^2 \hbar \left( \sum_{i=1}^N \omega_i - qN \left( c'_2 - \frac{\hbar 2N|p_1|^2 qr^{\frac{3}{2}(z+4\delta_p)}}{(z + 4\delta_p)^2} \right) \right),$$

(3.35)

which, together with (3.32), are substituted into the Einstein equation (3.6). The $tt$-component of the equation is thus left with three different terms. The metric contributes a term that goes like $r^z$, and the gauge and fermion stress tensor contain two different terms: one contributes to a $r^{z+4\delta_p}$, and another term proportional to $qNc'_2 - \sum_{i=1}^N \omega_i$ goes like $\sim r^{z+2\delta_p}$. The only way to obtain a non-trivial solution is to switch this latter term off. Therefore, we have

$$c'_2 = \frac{\sum_{i=1}^N \omega_i}{qN} = \frac{\sum_{n=0}^{N-1} \frac{n\pi}{Nq} \frac{(N-1)N\pi}{2qNl}}{2q} \sim \frac{\omega_F}{2q},$$

(3.36)

where $l$ is the regulated size of the time direction. To obtain a non-vanishing spinor solution, one is forced to take $\delta_p = 0$. This relates the index $z$ to the fermion mass:

$$z = 2mL - 1,$$

(3.37)

Note that because of the way we defined the measure of the frequency sum in (3.25), the normalization $p_1$ should be $\omega$ independent to be consistent with the anti-commutation relations (3.3) and subsequently canonical anti-commutation relation between the $\psi$ and $\hat{\psi}$. 
where we have restored $L$. With that the Einstein equation is thus reduced simply to an algebraic equation constraining $p_1$

\[-2N^2\hbar^2|p_1|^4 q^2 + (z - 1)z^3 = 0,\]  

(3.38)

which readily yields

\[|p_1|^2 = \frac{(4mL - 1)(2mL - 1)^3}{\hbar Nq},\]  

(3.39)

where we have restored the radius $L$. However, $p_1$ is the normalization of our plane-wave solutions. It has to take a fixed value. The normalization of the wavefunction is determined by

\[
\int d^3 x \Psi_1^\dagger(r)\Psi_1(r) = |p_1|^2 V = 1,
\]

(3.40)

where $V$ is the regulated volume.

Therefore, relation (3.39) fixes the value of $N$ that would back-react to give rise to the particular geometry and electric fields, i.e.

\[
\frac{(4mL - 1)(2mL - 1)^3}{\hbar Nq} = \frac{1}{V},
\]

(3.41)

which gives

\[
\frac{Nq\hbar}{V} = \rho \hbar = ((4mL - 1)(2mL - 1)^3)^{1/2} \sim O(1),
\]

(3.42)

where $\rho$ is the charge density. The Einstein equation dictates that $\hbar \rho \sim 1$ which is expected from our intuitive argument presented at the beginning of the section.

Being a constant solution, the $t$-component of the fermion energy–momentum tensor

\[
T^f_{tr} = \frac{ir^2}{2} (\tilde{\psi}_{\#}^\dagger \psi_{\#}^* - \psi_{\#}^\dagger \tilde{\psi}_{\#}^*),
\]

(3.43)

which is a lone term in the Einstein equation also vanishes. Similarly replacing $m \rightarrow -m$ we will get $z = -2mL - 1$.

There are a few interesting limits to take. One could for example consider taking the limit $mL \rightarrow \frac{1}{2}$ such that $z \rightarrow 0$. Staring at (3.38), (3.39) it means a non-trivial fermion solution is only recovered for $q \rightarrow 0$ at the same rate. In this limit however the gauge component $A_t$ becomes a constant, but in fact an infinite constant since $c_2'$ is proportional to $1/q$. Remarkably this is precisely what happens in the gauge-gravitational Chern–Simons analysis when we take the limit $z \rightarrow 0$ and $\mu_F \rightarrow 0$ in equation (2.5). One could consider an alternate situation where $q = 0$. In this case the log term could have been allowed without entering into the Einstein equation through the fermion stress tensor. Indeed we checked that this is a solution, provided that $m = \frac{1}{2}$ and $z = 0, 1$. When $z = 1$ the fermion radial wavefunction is in fact not a constant but goes like

\[
\Psi_{\#}(r) \sim r^{-1}.
\]

(3.44)

When one compares with the gauge-gravitational Chern–Simons setting, where one analogously takes $\mu_F = 0$ and allows for the homogeneous solution $A_t \sim c_2' + c_1' \log r$, it turns out that $z = 0, 1$ are again solutions, provided that one has to take

\[
\mu_G(z - 1)z \sim c_1'^2
\]

(3.45)

in either case, i.e. $\mu_G$ curiously approaches infinity. There appears to be a one-to-one correspondence between the gauge-gravitational Chern–Simons theory and the Einstein–Maxwell–Dirac system, as would be expected if Chern–Simons terms are recovered by integrating out fermions. However our results suggest that the relationship between the value of the Chern–Simons couplings and the fermion mass and charge departs from that in flat space.
3.2. 3+1 dimensions and beyond

The procedure above can be simply generalized to higher dimensions. Consider for concreteness the case \( D = 3 + 1 \). In this case the metric ansatz we use is the 4D Schrödinger metric

\[
dx^2 = L^2 \left( -r^2 \frac{dt^2}{dz^2} - 4 \epsilon r \frac{dt}{dz} dx + r^2 \left( \frac{dr^2}{4r^2} + \frac{dz^2}{r^2} + \frac{dy^2}{r^2} - \frac{dz}{r} dy \right) \right),
\]

(3.46)

where we will again set \( L = 1 \) from now on. Using the same choice of vierbeins as in 2 + 1 with the addition of

\[
e^j = \sqrt{r} \, dy,
\]

(3.47)

where the hat denotes tangent coordinates. The spin-connections will be given again by (3.12), with only the new addition of

\[
\omega^3 = - \sqrt{r} \, dy.
\]

(3.48)

Choosing again the Dirac spinor to be

\[
\psi = \sum_{i,j} e^{-i(\omega t - k_1 x - k_2 y)} \hat{\psi}_{\omega, a, k_1, k_2}^X \hat{\psi}_{\omega, a, k_1, k_2}^X + e^{i(\omega t - k_1 x - k_2 y)} \hat{\psi}_{\omega, a, k_1, k_2}^Y \hat{\psi}_{\omega, a, k_1, k_2}^Y,
\]

(3.49)

where \( \chi \in \{1, 2, 3, 4\} \) are spinor polarization indices,

\[
\hat{\psi}_{\omega, a, k_1, k_2}^X = \left( \begin{array}{c} \hat{\psi}_{\omega, a, k_1, k_2}^{X, 1} (r) \\ \hat{\psi}_{\omega, a, k_1, k_2}^{X, 2} (r) \end{array} \right), \quad \hat{\psi}_{\omega, a, k_1, k_2}^Y = \left( \begin{array}{c} \hat{\psi}_{\omega, a, k_1, k_2}^{Y, 1} (r) \\ \hat{\psi}_{\omega, a, k_1, k_2}^{Y, 2} (r) \end{array} \right),
\]

(3.50)

and we take as before \( k_i = 0, i = 1, 2 \). For clear notations, we will omit the polarization index \( \chi \) in the following, only to look explicitly for the non-trivial spinor polarization that would solve the system of equations. Assuming only \( r \) dependence and going through a similar analysis as in the previous section leads to very simple results. Starting again with the Maxwell equations to obtain constraints on the spinors, the equation for \( A_r \) gives

\[
\hat{\psi}_{+}^2 \hat{\psi}_{-}^1 + \hat{\psi}_{+}^1 \hat{\psi}_{-}^2 - \hat{\psi}_{+}^1 \hat{\psi}_{+}^2 - \hat{\psi}_{+}^2 \hat{\psi}_{+}^1 = 0,
\]

(3.51)

where as in the previous section we again denote the components of the \( \psi \) field operator with a hat, to be distinguished from the modes. The simplest solution is again taking, for each mode,

\[
\hat{\psi}_{+, a, k_1, k_2}^+ = \hat{\psi}_{-, a, k_1, k_2}^- = 0.
\]

(3.52)

This choice also easily ensures the vanishing of the source currents for \( A_x \) and \( A_y \). Subsequently the Einstein equation and the Maxwell equations would again imply

\[
A_r = A_x = A_y = 0.
\]

(3.53)

Fermions taking a constant value is forced to be a solution

\[
\psi_{\pm, a, k_1, k_2}^1 = p_\pm^1
\]

(3.54)

for some normalization constants \( p_\pm^1 \), and we arrive at

\[
A_r = c_1^r \log r + c_2^r + \frac{\langle Q | \sqrt{\hat{\psi}_{+}^1 \hat{\psi}_{-}^1 \hat{\psi}_{+}^2 \hat{\psi}_{-}^2} | Q \rangle}{z(1 + z)} | Q \rangle,
\]

(3.55)

where

\[
| Q \rangle = \prod_{i} a_{\omega, a, k_1}^i a_{\omega, a, k_2}^i | 0 \rangle, \quad \langle Q | \sqrt{\hat{\psi}_{+}^1 \hat{\psi}_{-}^1} | Q \rangle = N_{\pm} | p_{\pm}^1 |^2.
\]

(3.56)
and $a_{\pm \omega}^\dagger$ denotes the creation operators of the fermion modes $\psi_{\pm \omega}$, respectively. $N_{\pm}$ are the fermion numbers for the $\pm$ modes respectively. As in $d = 2 + 1$, the Einstein equation dictates that $c_1^2 = 0$ and

$$\left( N_+ |p_1^+|^2 + N_- |p_1^-|^2 \right) c_2^2 = \frac{1}{q} \left( \sum_i \omega_{+i} |p_i^+|^2 + \sum_j \omega_{-j} |p_j^-|^2 \right).$$  (3.57)

The two remaining Dirac equations for constant fermions cannot be satisfied simultaneously for arbitrary massive fermions when both $\psi_1^+$ and $\psi_1^-$ are non-vanishing. Explicitly, the remaining Dirac equations are given by

$$p_1^+ (2 + 2m + z) = 0, \quad p_1^- (2 - 2m + z) = 0.$$  (3.58)

We are thus left with three different possibilities.

3.2.1. Massless fermions. For massless fermions,

$$z = -2,$$  (3.59)

and the remaining Einstein equation is

$$\frac{3(12 - (Q|\hat{\psi}_1^+ \psi_1^+ + \hat{\psi}_1^+ \psi_1^+|^2)q^2)|Q}{2r^2} = 0,$$  (3.60)

which relates $N_-$ to $N_+$.

3.2.2. Massive fermions. For massive fermions, there are two independent sets of solutions:

$$p_1^+ = 0, \quad N_- |p_1^-|^2 = \pm \frac{2\sqrt{(9 + 18m + 8m^2)(m + 1)(2m + 1)^2}}{q\sqrt{4m + 3}}$$  (3.61)

and

$$z = 2mL - 2.$$  (3.62)

This solution applies when $m \neq -\frac{1}{2}$. When $m$ actually takes that value, $\psi_1^-$ becomes unconstrained because the Einstein equation vanishes without further constraint on $\psi_1^-$. Similarly one could have

$$p_1^- = 0, \quad N_+ |p_1^+|^2 = \pm \frac{2\sqrt{(9 - 18m + 8m^2)(m - 1)(2m - 1)^2}}{q\sqrt{4m - 3}}$$  (3.63)

where

$$z = -2mL - 2.$$  (3.64)

and $m \neq \frac{3}{2}$. Otherwise $N_+$ is unconstrained.

Here we have an interesting pattern to note. When $D = 2 + 1$, the resultant index $z = 2mL - 1$ and at $D = 3 + 1$ that gives, depending on chirality, $\pm 2mL - 2$. It turns out that exact solutions of charged constant spinor solutions and Schrödinger metrics can be found in arbitrary dimensions. In $D = 4 + 1$ for example we have

$$z = \pm 2mL - 3.$$  (3.65)

There is a clear pattern in the constraint on $z$ as we go to a higher dimension, namely

$$z = \pm 2mL - (D - 2).$$  (3.66)

The reason it gets shifted in this particular manner is that these constraints follow from requiring the coefficient multiplying $\psi$ in the Dirac equation to vanish, for the existence of
constant spinor solutions. The mass term enters in the same way for arbitrary dimensions, but for each additional dimension we get an extra spin-connection component, as illustrated in (3.48), which enters into the Dirac equation, after multiplying by the curved space gamma matrices, simply as a constant. Therefore as we move onto higher dimensions the constraint on \( z \) gets recurrently shifted by \(-1\), starting from the value when \( d = 2 + 1 \). The same chiral modes that are obtained in the previous section also appear here. However, in dimensions higher than \( d = 2 + 1 \), spinor dimensions increase such that the constraints from Maxwell’s equations become a much weaker restriction and it should then be possible generally to switch on momentum in other directions. As a result our solution would not be able to capture the back-reaction of a general state where modes with non-trivial momenta \( k \) are occupied.

3.3. Can there be an exact Lifshitz solution?

Given our analysis with the Chern–Simons theory and the form of the probe fermion stress tensor, it would be interesting to look for exact Lifshitz solutions. However, in the \( k_\mu = 0 \) case in 2+1 dimensions we can show that this is not possible. Note that in light of our discussion of the semi-classical analysis, this will only make sense if the charge \( q \) is large compared to the mass. The following should only be treated schematically and we leave a more rigorous analysis for future work.

We will again start with the Einstein–Maxwell–Dirac system in 2+1 dimensions and look for an exact solution to the equations of motion in the background given in equation (2.6). The set of equations are

\[
\Gamma^\nu D_\nu \psi - m \psi = 0, \quad \nabla^\nu F_{\mu \nu} = j_\mu, \quad (3.67)
\]

\[
R_{\alpha \beta} - \frac{1}{2} g_{\alpha \beta} R = \frac{d(d - 1)}{2 L^2} g_{\alpha \beta} - \frac{1}{2} \left( F_{\alpha \gamma} F_{\gamma \beta} - \frac{1}{4} F^2 g_{\alpha \beta} \right) = \frac{1}{2} T^f_{\alpha \beta}. \quad (3.68)
\]

Since we are looking for field configurations depending only on \( r \), the left-hand side of the \( A_r \) equation of motion vanishes trivially. This imposes a chirality constraint on the components of the Dirac fermion, namely

\[
\psi_+ = \psi_- . \quad (3.69)
\]

Inserting this condition back into the Dirac equation, we get two equations for \( \psi_+ \), which are inconsistent unless

\[
A_x = -r^{z-1} A_t. \quad (3.70)
\]

Using this condition the Dirac equation can be solved,

\[
\psi_+(r) = \sqrt{r} p_1 r^{-\frac{m}{2}}. \quad (3.71)
\]

We can then substitute this solution and the relation between \( A_x \) and \( A_t \) in the Einstein equation. The \( x-r \) component of the Einstein equation then implies that \( \psi_+ \) is a real function and as a result the integration constant \( p_1 \) is real. Having done this, we can solve the \( A_t \) component of the Maxwell equation, which yields

\[
A_t(r) = c_2 + \frac{c_1 r^{1-\epsilon}}{1-\epsilon} - \frac{2 p_1^2 q r^{1-2m}}{(1-2m)(2m-\epsilon)}. \quad (3.72)
\]

Finally we can solve for the remaining components of the Einstein equations. Although we have four equations, only two of them are independent. After having already determined fermion and gauge fields, these two equations are purely algebraic in nature and they

Note also that in all these solutions \( T^f_{\mu \nu} - T^\nu_{\mu} = 0 \).
constrain the remaining undetermined integration constants, $c_1$, $c_2$ and $p_1$ and the parameter $z$. Unfortunately, there is no consistent choice of these constants that can satisfy both equations. The same analysis can be applied generally for $k_μ ≠ 0$, which suggests that Lifshitz spacetimes are not the exact solution to the equations of motion. This however does not contradict the fact that in the WKB limit it becomes an approximate solution.

4. Discussion

Motivated by results of [4], we studied a system of self-gravitating charged Dirac fermions and showed that non-relativistic Schrödinger metrics exist as exact solutions. In 2+1 dimensions, this was anticipated by studying the three-dimensional Einstein–Maxwell–Chern–Simons system. Although the gauge Chern–Simons term does not couple to the metric, it affects the background by modifying the Maxwell equations. We found that in the presence of gauge and/or gravitational Chern–Simons term, one generically finds Schrödinger spacetime as a suitable background.

One may be worried that by treating the problem involving fermions semi-classically, we could be missing important higher order effects. For starters, the effects of quantum gravity are under control as we implicitly assume that the gravitational coupling is small. Next, one could be worried that since there are a large number of fermions, there could be an enhancement of higher loop effects by their presence. However, as demonstrated in a general toy model in appendix D, as long as the number of modes $N$ excited is large, the tree level contribution from contraction with the external state would be boosted by factors of $N$ relative to quantum loops at each order of perturbation in the boson–fermion interaction. This gives support to our semi-classical treatment.

Since in the flat space, (charged) fermions induce gauge and gravitational Chern–Simons terms through parity anomaly, the result obtained in the Einstein–Maxwell–Chern–Simons was a precursor to our expectation of back-reaction of fermions on AdS space. Armed with this result we studied fermions in the Schrödinger background in the Einstein–Maxwell theory without Chern–Simons terms, anticipating that the Schrödinger spacetime is the fully back-reacted geometry of the fermions in AdS space. We indeed found a consistent solution to the gravity–Maxwell–fermion system in the Schrödinger spacetime in the arbitrary dimension for the critical exponent $z = ±2mL − (D − 2)$, where $D$ is the number of bulk spacetime dimensions.

The emergence of Schrödinger spacetime can also be seen perturbatively. Let us start with AdS$_3$ spacetime with the metric

$$ds^2 = L^2 \left( -4r \, dt \, dx + \frac{dr^2}{4r^2} \right)$$

and consider gauge field components $A_μ$ taking at most constant values. We can then solve the Dirac equation for a fermion with charge $q$ and mass $m$ in this background. The solution is

$$\psi(r, t, x) = r^{-\frac{mL}{2}} \exp(-i\omega t) \left( \begin{array}{c} 1 \\ i \end{array} \right).$$

We then look at the back-reaction of this solution on the background metric and gauge fields. Since we are treating this fermion solution perturbatively, the back-reaction will be tiny

10 It is important to note that the fermion gas we consider is a smooth density distribution, which alleviates Coulomb repulsion particularly along the $x$-direction, which otherwise would have made the gas unstable.

11 Fermion mass term, in fact, breaks parity symmetry explicitly in 2+1 dimensions. On the other hand in 3+1 dimensions it does not break parity.
(proportional to \( \hbar \)) but indicative of the effect of the macroscopic droplet of these fermions on the background. With this in mind we evaluate the fermion current and the energy–momentum tensor and feed it back into equations of motion of the background. The solution to equations with back-reaction of fermions generates the non-trivial gauge field component \( A_t \) and metric component \( g_{tt} \), which take the form

\[
A_t(r) = \hbar \left( c_2 + c_1 \ln r + \frac{4qr m^{-1/2}}{(2m - 1)^2} \right), \tag{4.3}
\]

\[
g_{tt} = \hbar \left( \frac{1}{(2m - 1)^2} \left( \frac{-4q^2 r^{2m-1}}{(2m^2 - 3m + 1)} + e_1 \ln r + e_2 r^{m-1/2} \ln r \right) \right), \tag{4.4}
\]

where \( c_i \) are constants of integration and the \( e_i \)'s are fixed in terms of \( c_i \) via the Einstein equation. In particular, setting \( c_1 = 0 \) sets both \( e_i = 0 \). In this limit we are actually reduced to the original solution we found previously in (3.10), (3.32) and (3.37). The fermions only appear differently because we have picked a different set of vierbeins to suit our purpose here starting off from an off-diagonal metric.

It is interesting to note that for \( z > 0 \) the Schrödinger backgrounds are horizon-free and as a result do not have macroscopic entropy associated with them. The phenomenon of disappearance of the horizon due to fermion back-reaction is similar to what was proposed in [4]. It would be interesting to find a dual description of our solutions. In the 2+1 dimensions we studied the back-reacted geometry supported either by the Chern–Simons terms or by fermions. As mentioned earlier, around flat backgrounds it is known that Chern–Simons terms are induced by one-loop effects in the fermionic theory [7, 8]. In the context of curved space we have two Chern–Simons terms, gauge and gravitational. The Schrödinger metric obtained in the Chern–Simons theory has (up to sign) two new exponents apart from the standard one corresponding to AdS space: \( z = \mp 4 \mu_F \) and \( z = (\mu_G \mp 1)/(2 \mu_G) \) (see (2.5)). On the other hand, in the fermionic theory we have only one exponent (up to a sign of the fermion mass term). It is tempting to conjecture that even in curved spacetimes, the fermions would induce Chern–Simons terms. It would be interesting to directly relate these two theories and compare their resulting exponent \( z \).

Let us make a brief comment on the Maxwell–Chern–Simons theory. Around flat spacetime it is possible to arrange the coefficient of the gauge Chern–Simons term and the number of fermion species to switch off the gravitational Chern–Simons term [9]. It turns out that this system has already been studied in [21, 22] and [23]. In [23], it was proved that there exist no black hole solutions in this theory. The existence of the Chern–Simons term in 2+1 dimensions implies the presence of a chiral anomaly in the (1+1)-dimensional theory [24]. The induced chiral anomaly in the boundary theory is cancelled by the chiral current in the boundary theory. In [22], it was found (after transforming to our conventions) that regular black hole solutions only exist if \( \mu_F \mu_G > 1/6 \). This is reminiscent of condition (C.11).

It has been shown in [4] that Lifshitz metrics are solutions when a WKB approximation is made. We leave it as an open problem to examine if they are also exact solutions. In light of our analysis using the Chern–Simons theory and the self-gravitating fermions, this is unlikely to occur in 2+1 dimensions. However in higher dimensions, there are more ways to solve the constraint \( j_r = 0 \) than in 2+1 dimensions. So it could very well be that both Lifshitz and Schrödinger metrics exist as exact solutions. It will be exciting to find new backgrounds from self-gravitating fermions. Of course in general, this is a very difficult problem.
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Appendix A. Conventions

We use the following conventions for three and four dimensions. The three-dimensional action with Chern–Simons terms is

\[
I_3 = \frac{1}{2\ell_p} \int d^3x \left[ \sqrt{-g} \left( R + \frac{2}{L^2} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) + L_{MCS} + L_{GCS} \right]
\]

(A.1)

where the Maxwell–Chern–Simons and gravitational Chern–Simons are given by

\[
L_{MCS} = \mu F^{a\beta\gamma} A_a F_{\beta\gamma},
\]

(A.2)

\[
L_{GCS} = \mu G^{\lambda\mu\nu} \Gamma^\alpha_{\lambda\mu\nu} \left( \partial_\mu \Gamma^\beta_{\alpha\nu} + \frac{2}{3} \Gamma^\beta_{\mu\nu} \Gamma^\gamma_{\alpha\gamma} \right).
\]

(A.3)

The four-dimensional action is

\[
I_4 = \frac{1}{2\ell^2_p} \int d^4x \left[ \sqrt{-g} \left( R + \frac{6}{L^2} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) \right].
\]

(A.4)

The Einstein equation is

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \frac{1}{2} T_{\mu\nu},
\]

(A.5)

where \( \Lambda \) is the cosmological constant in appropriate dimensions and \( T_{\mu\nu} \) is the matter stress energy tensor. The matter stress energy tensor with this choice of conventions is defined as

\[
T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}}.
\]

(A.6)

The generic asymptotically AdS metric is written as

\[
ds^2 = -\frac{r^2}{L^2} f(r) \, dt^2 + \frac{L^2}{r^2} \, dr^2 + \frac{r^2}{L^2} \, dx_i^2.
\]

(A.7)

In the absence of Chern–Simons terms in three dimensions it can be shown that (A.7) solves equations of motion to give a charged BTZ black hole solution [25–27] with

\[
f(r) = 1 - \frac{q^2 L^2}{4r^2} - \frac{q^2 L^2}{2r^2} \ln \frac{r}{r_0}, \quad A_t = q \ln \frac{r}{r_0},
\]

(A.8)

where \( r_0 \) is the location of the outer horizon. The condition for existence of two horizons is \( q^2 L^2 > 4r_0^2 \) and the extremality condition is \( r_s = q L/2 \).

In four dimensions, (A.7) solves the equations of motion to give a charged Reissner–Nordström black hole solution, with the behaviour of \( f(r) \) and \( A_t(r) \) given by

\[
f(r) = 1 - \frac{M}{r^3} + \frac{q^2 L^2}{4r^4}, \quad A_t(r) = q \left( \frac{1}{r} - \frac{1}{r_0} \right).
\]

(A.9)
In order to have double horizon \(27M^4 \geq 4q^6\) and the equality corresponds to extremal solution.

The near-horizon metric in the extremal limit is given by \(\text{AdS}_2 \times M_n\), where \(M_n\) is an \(n\)-dimensional manifold, which in our case is either \(R^d\) or \(S^d\). Our convention for the \(\text{AdS}_2\) metric in Poincaré coordinates is

\[
\text{ds}^2_{\text{AdS}_2} = -\frac{r^2}{L^2} \text{dt}^2 + \frac{L^2}{r^2} \text{dr}^2,
\]

where \(L\) is the radius of \(\text{AdS}_2\). In 2+1 dimensions, according to our conventions,

\[
A_t = \sqrt{2} L, \quad \tilde{L} = \frac{L}{\sqrt{2}},
\]

while in 3+1 dimensions,

\[
A_t = \sqrt{2} \frac{r}{L}, \quad \tilde{L} = \frac{L}{\sqrt{6}}.
\]

**Appendix B. Spin connection and the Dirac equation**

Consider a general metric of the form

\[
\text{ds}^2 = -g_{tt}(r) \text{dt}^2 + g_{rr}(r) \text{dr}^2 + g_{xx}(r) \text{dx}^2 + 2g_{tx}(r) \text{dt} \text{dx},
\]

with all metric components having only radial dependence. We will use \(\mu, \nu, \ldots\) to denote curved spacetime indices and \(i, j, \ldots\) to denote curved spatial indices. We will reserve the index ‘\(d\)’ for denoting a flat coordinate related to the curved index \(r\) by the vielbein. Flat space indices for boundary coordinates are denoted by \(a, b, \ldots\).

In this case, the following choice of vielbeins is suitable for both diagonal metrics and Schrödinger metrics:

\[
e^0 = \frac{\sqrt{g_{tt}}}{} \text{dt} - \frac{g_{tx}}{\sqrt{g_{tt}}} \text{dx}, \quad e^d = \sqrt{g_{tt}} \text{dr}
\]

\[
e^1 = \sqrt{g_{tt}g_{xx} + g_{tx}g_{tx}} \text{dx}, \quad e^a = \sqrt{g_{ii}} \text{dx}^i, \quad a = 2, 3, \ldots.
\]

It is straightforward to derive components of spin connections from them and they are given by

\[
\omega_{d0} = \frac{1}{2} \frac{g_{tx}'}{\sqrt{g_{tt}g_{rr}}} \text{dt} - \frac{1}{2} \frac{g_{tx}}{\sqrt{g_{tt}g_{rr}}} \frac{g_{tx}'}{g_{tt}} \text{dx},
\]

\[
\omega_{d1} = \frac{1}{2\sqrt{(-\det g)}g_{tt}} \left[ (g_{tt}g_{xx}'(r) + g_{tx}g_{tx}'(r)) \text{dx} + (g_{tt}g_{tx}'(r) - g_{tx}g_{tt}'(r)) \text{dr} \right],
\]

\[
\omega_{di} = \frac{g_{ti}'}{2\sqrt{g_{rr}g_{tt}}} \text{dx}^i,
\]

where the metric components with primes are derivatives with respect to their argument. Let us now look at the Dirac equation in different backgrounds. The Dirac equation for a particle with mass \(m\) and charge \(e\) takes the form

\[
(\slashed{D} - m)\psi = 0,
\]

where

\[
\slashed{D} = \Gamma^\mu e^\mu_\nu \left( \partial_\nu + \frac{1}{2} \omega_{\nu\rho\sigma} \Gamma^\rho_\sigma + i q A_\nu \right).
\]
To solve the Dirac equation we will use the following $\Gamma$-matrix convention for 2+1 dimensions:

$$
\Gamma^1 = \sigma_3, \quad \Gamma^0 = i\sigma_2, \quad \Gamma^d = \sigma_1.
$$

(B.8)

Our (3+1)-dimensional $\Gamma$-matrix convention is

$$
\Gamma^d = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \Gamma^a = \begin{pmatrix} 0 & \sigma^a \\ \sigma^a & 0 \end{pmatrix}, \quad a = 0, 1, 2
$$

(B.9)

where $I$ is the $2 \times 2$ identity matrix and $\sigma^a$ are Pauli matrices.

### Appendix C. Stress tensor of fermionic probes in AdS

We will consider fermions in curved space. We shall begin presenting explicitely the fermionic contribution to the stress tensor [19]. The quadratic action for a charged massive Dirac spinor is

$$
S_{\text{Dirac}} = \frac{1}{2\ell^d} \int d^{d+1}x \sqrt{-g} (\bar{\psi} \Gamma^\mu D_\mu \psi - m \psi \bar{\psi})
$$

(C.1)

where

$$
D_\mu \psi = \left( \partial_\mu + \frac{i}{2} \omega^{ab}_{\mu} \Gamma_{ab} + i A_\mu \right) \psi, \quad \bar{\psi} \tilde{D}_\mu = \left( \partial_\mu \bar{\psi} - \frac{i}{2} \omega^{ab}_{\mu} \bar{\psi} \Gamma_{ab} - i q A_\mu \bar{\psi} \right).
$$

(C.2)

The corresponding stress tensor is

$$
T^f_{\mu\nu} = \frac{1}{4} (-i \bar{\psi} \Gamma_\mu D_\nu \psi + i \bar{\psi} \tilde{D}_\nu (\Gamma_\mu \psi))
$$

(C.3)

and symmetrization is denoted by $V_{(\mu \nu) V_{\nu}} = \frac{1}{2} (V_{\mu} V_{\nu} + V_{\nu} V_{\mu})$.

#### C.1. AdS$_2 \times R$

The background is, following our conventions,

$$
ds^2_{\text{AdS}2 \times R} = -\frac{r^2}{\tilde{L}^2} dr^2 + \frac{\tilde{L}^2}{r^2} dr^2 + dx^2, \quad A_t = \sqrt{\frac{2}{L}},
$$

(C.4)

and the only non-vanishing spin connection is

$$
\omega_{10} = \frac{r}{\tilde{L}^2} dt.
$$

(C.5)

Taking a Dirac spinor of the form

$$
\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix},
$$

(C.6)

and assuming only $r$ dependence, the Dirac equation is given explicitly by

$$
2r \psi_\pm' + 2iq\sqrt{2}\psi_\pm + (1 \mp 2m\tilde{L})\psi_\pm = 0.
$$

(C.7)

The solution is simply

$$
\psi_\pm = r^{-\frac{1}{2} \pm \delta} \begin{pmatrix} p_1^\pm \\ p_2^\pm \end{pmatrix}, \quad p_2^\pm = \pm \frac{ip_1^\pm (\delta \mp m\tilde{L})}{\sqrt{2}Lq}, \quad \delta = \tilde{L} \sqrt{m^2 - 2q^2}
$$

(C.8)

12 Note that [19] use the mostly negative convention while we follow the more standard GR convention of mostly positive.

for some constants $p_1^\pm$. Switching on one of the two solutions leads to

$$
T_{tt}^f = \left| p_1^\pm \right|^2 \frac{\sqrt{2} m (m L \mp \delta)}{L^3 q} r^{1 \pm 2 \delta},
$$

$$
T_{ix}^f = \pm \left| p_1^\pm \right|^2 \frac{m L (2 \delta \mp 1) + \delta \mp 2 \delta^2}{2 \sqrt{2} q L^3}.
$$

(C.9)

Note the appearance of $T_{it}$ components. The $rr$ component contributes only from cross terms when both solutions are switched on. Note that the Ricci tensor for the background metric is simply

$$
R_{tt} = \frac{r^2}{L^4}, \quad R_{rr} = -\frac{1}{r^2}.
$$

Therefore, in the near-horizon limit $r \rightarrow 0$, the solution scaling with $r^{-(1/2 + \delta)}$ would eventually become dominant over the background such that the probe approximation breaks down. For the other solution that scales as $r^{-(1/2 - \delta)}$, however, the probe approximation is good all the way to the horizon if $\delta > 1$, which gives

$$
m^2 > 2q^2 + 1.
$$

(C.11)

When the mass is smaller than $2q^2 + 1$, back-reaction in the near-horizon region cannot be ignored, with the likely result of destroying the horizon altogether. Therefore in the large mass and charge limit we recover the assertion in [4] that roughly the horizon is destroyed whenever $eq > m$, where $e = \sqrt{2}$ is the background electric field. Note also that these solutions independently satisfy

$$
T_{tt}^r - T_{rr}^r = -\left| p_1^\pm \right|^2 \frac{\sqrt{2} m (m L \mp \delta)}{L q} r^{-1 \pm 2 \delta} < 0.
$$

(C.12)

C.2. AdS$_2 \times R^2$

Let us repeat the exercise in the previous section in AdS$_2 \times R^2$. A static solution of a 4-component Dirac fermion would be given by

$$
\psi = \psi_r r^{-\frac{1}{2} + \delta} + \psi_1 r^{-\frac{1}{2} - \delta}, \quad \delta = L \sqrt{m^2 - 2q^2},
$$

where

$$
\psi_\pm = \begin{pmatrix} \psi_1^\pm \\ \psi_2^\pm \end{pmatrix}, \quad \psi_2^\pm = \pm \frac{(\frac{1}{L} \mp m)}{\sqrt{2} q L} (\sigma^1 \psi_1^\pm),
$$

(C.13)

where $\psi_\pm$ are two-component constant spinors and $\sigma_i$ are the Pauli matrices. The non-vanishing contribution of $\psi_\pm$ to the energy–momentum tensor is given by

$$
T_{tt}^f = \psi_\pm \sigma_1^\pm \frac{r^{1 \pm 2 \delta} (L^2 (m^2 + 2q^2) \mp 2L q \delta \mp \delta^2)}{\sqrt{2} L^3 q}
$$

$$
T_{ix}^f = \psi_\pm \sigma_3^\pm \frac{r^{1 \pm 2 \delta} (L^2 (m^2 - 2q^2) + \delta \mp 1 \delta + L (m \mp 2m \delta))}{2 \sqrt{2} L^3 q}
$$

$$
T_{iy}^f = -\psi_\pm \sigma_2^\pm \frac{r^{1 \pm 2 \delta} (L^2 (m^2 - 2q^2) + \delta \mp 1 \delta + L (m \mp 2m \delta))}{2 \sqrt{2} L^3 q}.
$$

(C.15)
As in the previous section, the energy–momentum clearly illustrates again that back-reaction is severe in the near-horizon region generally for \( \sqrt{2q} > m \). Note that here

\[
T_i' - T_r' = -\psi_1^\dagger \psi_\pm \frac{r^{1+2\delta} (L^2 (m^2 + 2q^2) \mp 2LM\delta + \delta^2)}{\sqrt{2} \sqrt{2q} L} < 0. \tag{C.18}
\]

C.3. \( \text{AdS}_2 \times S^2 \)

An entirely parallel story goes through if one were to replace \( R^2 \) in the previous subsection by \( S^2 \). Namely

\[
\psi = \psi_+ r^{-\frac{i}{2} \delta} + \psi_- r^{-\frac{i}{2} - \delta}, \quad \delta = L\sqrt{m^2 - 2q^2}, \tag{C.19}
\]

where, however even the lowest mode contains non-trivial angular dependence,

\[
\psi_\pm = \frac{1}{\sin \theta} \left( \psi_1^\dagger \psi_\pm \frac{1}{\sqrt{2q} L} \right), \quad \psi_2^\pm = \pm \left( \frac{\delta}{r} \mp \frac{m}{2} \right) (\sigma_1 \psi_1^\dagger \psi_\pm). \tag{C.20}
\]

The energy–momentum tensor here is similarly

\[
T_{\alpha\beta} = \csc \theta \psi_\dagger^\mu \psi_\mu \frac{r^{1+2\delta} (L^2 (m^2 + 2q^2) \mp 2LM\delta + \delta^2)}{\sqrt{2} \sqrt{2q} L} \tag{C.21}
\]

\[
T_{\alpha\phi} = \csc \theta \psi_\dagger^\mu \sigma_\mu \psi_\mu \frac{r^{2+2\delta} (L^2 (m^2 - 2q^2) + (\delta \mp 1)\delta + \tilde{L}(m \mp 2m\delta))}{2\sqrt{2} \sqrt{2q} L} \tag{C.22}
\]

\[
T_{\alpha\phi} = \frac{-2\sqrt{2} \cos \theta}{8qr} T_{\alpha\phi} = \psi_\dagger^\mu \sigma_\mu \psi_\mu \frac{r^{2+2\delta} (L^2 (m^2 - 2q^2) + (\delta \mp 1)\delta + \tilde{L}(m \mp 2m\delta))}{2\sqrt{2} \sqrt{2q} L}. \tag{C.23}
\]

The null energy condition is the same as equation (C.18).

**Appendix D. A toy model justification for our semi-classical treatment**

Our discussion is largely based on the assumption that in the presence of a large fermi gas, one can treat the background classically, while the fermions propagate in this mean-field background quantum mechanically. We consider here a toy model where fermions are coupled to bosons. In the case where the fermion fields appear only up to the quadratic order, we can obtain the one-loop effective action of the bosons after integrating out the fermions in an \( N \)-fermion state. It turns out, in the large \( N \)-limit and to leading order in the boson–fermion coupling \( g \), that the effective equations of motion of the bosons are equivalent to the original one, except that the fermion operators are replaced by their expectation values evaluated on the \( N \)-fermion state. The toy model gives us more confidence in the methods we advocate in this paper. It is also noteworthy that the approach of treating back-reacting fermions on a mean-field background and obtaining a self-consistent set of solutions to all the equations of motion with an explicit built-in Fermi surface, is a well-known strategy in nuclear physics [15], which is equivalent in various limits to the Hartree–Fock approximation.

Our discussion here will follow closely [28]. The (Euclidean) action of our toy model at temperature \( T = \beta^{-1} \) in (0+1) dimension is as follows:

\[
S = S_b + S_f + S_{int}, \quad S_b = \frac{1}{2} \int_0^\beta d\phi \dot{\phi}^2 + m_b^2 \phi^2, \quad S_f + S_{int} = \int_0^\beta d\tau \sum_i b_i^\dagger b_i + (m_i - \mu_i) b_i^\dagger b_i - g \phi b_i^\dagger b_i, \tag{D.1}
\]
where we have introduced many species of fermionic fields $b_i$ and a corresponding chemical potential $\mu_i$, for each species, which are to be determined from some given occupations of the fermionic states. The partition function is given by
\[
Z = \int D[\phi] \prod_i D[ib_i^\dagger]D[b_i] \exp \left( \frac{-S}{\hbar} \right).
\] (D.2)

Since the action is quadratic in the fermions, the path integral can be readily done. One can express the fields as
\[
b_i = \sum_n e^{-i\omega_n^f \tau} b_{i,n}, \quad \phi = \sum_n e^{-i\omega_n^b \tau} \phi_n.
\] (D.3)

The $\omega_n$ here are thus Matsubara frequencies given by
\[
\omega_n^f = (2n + 1)\pi T, \quad \omega_n^b = 2n\pi T.
\] (D.4)

and $S_f + S_{\text{int}}$ can be rewritten as
\[
S_f + S_{\text{int}} = \sum_{i,m,n} ib_i^\dagger D_{mn}^i b_{j,n}, \quad D_{mn}^i = -i\beta ((-i\omega_n^b + \mu_i - m_i)\delta_{mn} - g\phi_{m-n}).
\] (D.5)

Integrating out the fermions one is then left with
\[
Z = \int D[\phi] \exp \left( -\frac{S_f}{\hbar} \right) \det \left( \frac{D}{\hbar} \right)
\times \int D[\phi] \exp \left( -\frac{S_b}{\hbar} \right) \det \left( 1 - \frac{g\phi_{m-n}}{-i\omega_n^b + \mu_i - m_i} \right).
\] (D.6)

To the lowest order in the coupling $g$, the fermion contribution to the free energy evaluates to [28], using various summation formulas
\[
\frac{F_f}{T} = -\ln Z_f = -\sum_{n,i} \ln[\beta((-i\omega_n^b + \mu_i - m_i))]
\times \frac{1}{2} \sum_i \left[ \beta^2 \left( \sum_n (\omega_n^b + (\mu_i - m_i))^2 \right) \right]
\times \sum_{i} \ln(1 + e^{-\beta(m_i-\mu)})
\] (D.7)

where we have dropped the contribution of vacuum energy which has to be regulated. We can determine the chemical potentials by
\[
\partial \mu_i / (T \ln Z) \sim \frac{1}{e^{\beta(m_i-\mu)} + 1} = N_i,
\] (D.8)

which is precisely the Fermi distribution. The ground state
\[
|Q\rangle = \prod_i |b_i^\dagger|0\rangle,
\] (D.9)

corresponding to a system of $N$ fermions with fermi energy $E_F$ can thus be obtained by picking
\[
\mu_i = E_F,
\] (D.10)

in which case the Fermi-distribution behaves like a step function, giving occupation $N_i = 1$ for all single particle states below $E_F$. Now returning to the full path integral, we have
\[
Z = \int D[\phi] \exp \left( -\frac{S_b}{\hbar} \right) \exp \left( \ln \det \left( \frac{D}{\hbar} \right) \right)
\times \int D[\phi] \exp \left( -\frac{S_b}{\hbar} \right) \exp \left( \sum_i \beta m_i + \ln(1 + e^{-\beta(m_i-\mu_i)}) - \ln \hbar \right).
\] (D.11)
where in the second line we have taken the leading \( g \) approximation, so that in the determinant of a matrix of the form
\[
\begin{pmatrix}
(1 + g\phi_0) & g\phi_1 & \ldots \\
g\phi_1 & (1 + g\phi_0) & \ldots \\
\vdots & \vdots & \ddots
\end{pmatrix},
\]
the leading \( g \) dependence came from the trace, which include only the diagonal components with \( \phi(m = 0) \). The off-diagonal components can only arise in higher \( g \) corrections.

Since in the zero-temperature limit \( \beta \to \infty \) the factor \( e^{-\beta(m_i - E_f)} \) approaches infinity for \( m_i < E_f \) and zero otherwise, we have
\[
Z \sim \int D[\phi] \exp\left(-\frac{S_b}{\hbar}\right) \exp\left(-\beta(m_i - E_f - g\phi_0)\right)
\]
\[
\sim \int D[\phi] \exp\left(-\frac{S_b}{\hbar}\right) \exp(-\beta(Ng\phi_0) + \cdots),
\]
(13)
where the vacuum energy that is supposed to be regulated, and other pieces independent of \( \phi \) are denoted in \( \cdots \). Note that to leading order in \( g \) the effective action of the bosons after integrating out the fermions is precisely as if we replace the fermion operators by their expectation values in the \( N \)-fermion state. For back-reaction on the bosons to be important, however, we need
\[
Ng \sim \frac{1}{\hbar}.
\]
(14)

Note also that the approximation we take in the partition function is equivalent to, from the perspective of perturbation theory via Feynman diagrams, only the tree diagrams, i.e. the perturbation operators are only contracting with external states. Consider for example the leading order \( g \) correction to any expectation values of some arbitrary operators \( O \) in the presence of the interaction terms
\[
\langle Q | O | \sum_i b_i^\dagger b_i | Q \rangle;
\]
(15)
we see that contraction of the \( b_i \)'s with the external creation operators in \( | Q \rangle \) results in a factor of \( N \), whereas contraction among the \( b_i \)'s, corresponding to loop diagrams is only of order 1 relatively. Similarly at order \( g^2 \), we have
\[
\frac{1}{2!} \int \int \sum_{i,j} \phi \phi \langle Q | O(b_i^\dagger b_j)(b_j^\dagger b_i) | Q \rangle,
\]
(16)
where contraction of the \( b \)'s with the external state would again give rise to an extra factor of roughly \( N(N - 1) - N^2 \) relative to contraction among the \( b \) and \( b^\dagger \). In the limit where \( N \) is large and \( g \) is small, the loop diagrams would be suppressed relative to tree diagrams.

To summarize, in the limit where \( Ng \sim \hbar^{-1} \), the back-reaction of the fermi gas on the bosons becomes significant, but at the same time we have the quantum loops under control because \( g \) is small. Particularly, the leading \( g \) effect on the bosons can be captured by replacing all the fermionic operators by their expectation values evaluated on the filled state \( | Q \rangle \) in the equations of motion of the bosons. In fact the coupling to the bosonic field \( g\phi b^\dagger b \) could have

13 Note that the combinatoric factor of \( 2! \) would appear in either internal contractions or contraction with the external state. We have also made use of the fact that states with different excitations are orthogonal to each other. Therefore, having chosen a particular contraction for the \( b_i \)'s on \( | Q \rangle \), the contraction of \( b_i \) with \( | Q \rangle \) is completely determined. This explains why there is only a factor of \( N(N - 1) \), rather than the square of it.

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been generalized to some arbitrary one of the form $gV(\phi)b^\dagger b$ without altering the conclusion, as long as $g$ is small. The toy model appears to justify the approach taken in this paper in including fermionic back-reactions on a bosonic background.

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