Comments On Torsion and MacDowell-Mansouri gravity.

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We construct a generalization for the MacDowell-Mansouri formulation of gravity. New parameters are introduced into the action to include the non-dynamical Holst term, independently from the topological Nieh-Yan class. Finally, we consider the new parameters as fields and analyze the solutions coming from their equations of motion. The new fields introduce torsional contributions to the theory that modify Einstein’s equations.

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I. INTRODUCTION

The success of Loop Quantum Gravity (LQG) can be traced to the polynomial nature of Ashtekar formulation, this approach is independent of background structures in space time. It was pointed out in [1], that a formulation of gravity based on a pure real connection can be constructed, unfortunately the constraints in Barbero’s Hamiltonian formulation are more complicated (this is the price to pay for a real formulation). Immirzi [2], pointed out that
Barbero’s transformations can be generalized and a one parameter family of transformations can be constructed. Classically the Immirzi parameter $\beta$ appears as a free parameter with no physical significance, but is of significant relevance at the quantum level. A generalized Hilbert-Palatini (HP) action containing the Barbero-Immirzi parameter (BI) was proposed by Holst [3], from which Barbero’s results can be derived. The term added in the HP action by Holst does not affect, as expected, the classical equations of the theory. Furthermore, it was shown in [4] that the Holst action can be rewritten as the Nieh-Yan invariant by adding the torsion-torsion term. In [5], the authors pointed out the similarities between the Immirzi parameter and the $\theta$-ambiguity that arises in Yang-Mills theories. Pursuing this idea, [6, 7] offered a possibility to interpret the BI parameter along the same lines as the $\theta$-parameter. The constructions where made in Yang-Mills type theory of gravity, the MacDowell-Mansouri (MM) action [8] with a topological $\theta$-term.

In the middle of the 70’s MacDowell and Mansouri [8], proposed and action for general relativity, where instead of taking as the gauge group $SO(3, 1)$ they considered $SO(4, 1) \supset SO(3, 1)$ (or $SO(3, 2)$) (anti) de Sitter group. To obtain a dynamical theory it was necessary to break the symmetry explicitly from the action, obtaining the Palatini’s action, the Euler class and the cosmological constant. Since then, some attempts have appeared trying to maintain the full symmetry of the gauge group. One of them was proposed by Stelle and West [9], they introduced an auxiliary vector field $v$, which makes the action invariant under the full gauge group, but at some point the symmetry is broken by choosing a preferred direction on $v$. Another relatively new approach which maintained the full symmetry is due to Randono [10], there they consider a model with a multiplet of spinor fields and assume that there exists a preferred vacuum state where they are isotropic and homogeneous in space and time. This amounts to getting only two speculation values that are constant. These constants are related to the gravitational constant and the Immirzi parameter, and the action is reduced to the MacDowell-Mansouri action (MM). The $\theta$ term introduced by this formalism has a direct meaning when we consider vanishing torsion theories, it is related to the Barbero-Immirzi parameter [2]. This approach, even when is very powerful, for example, in constructing consistent supersymmetric extensions [7], shows us that the full symmetry gauge group must be broken, and even more, if we want to obtain the well known Immirzi parameter we have to consider zero torsion manifolds.

As we can see the MM approach to gravity has the potential to give new insight in to
some of the puzzles of quantum gravity. Is therefore the goal of this paper is to generalize the MM formulation. The work is arranged as follows, in Section II the notation is established. The gauge theory is presented in Section III. We present the action with torsion in Section IV. Finally Section V is devoted to conclusions and outlook.

II. PRELIMINARIES AND NOTATION

MacDowell-Mansouri theory is a gauge theory of gravity with the gauge group $G \supset SO(3,1)$, where $G$ depends on the sign of the cosmological constant

$$G = \begin{cases} SO(4,1), & \text{for } \Lambda > 0, \\ SO(3,2), & \text{for } \Lambda < 0. \end{cases}$$

(1)

The current observations suggest that $\Lambda > 0$, therefore we will use $SO(4,1)$, but following the same procedure, the $\Lambda < 0$ case is straightforward ($\Lambda = 0$ is MM gravity). Let $t_{ab}$ be the elements of $\mathfrak{so}(4,1)$ Lie algebra, where the indices take the values $a, b, \ldots = 0, 1, 2, 3, 5$ and satisfy

$$[t_{ab}, t_{cd}] = f_{ab}^{\quad ef} t_{ef} = 4 \eta_{ab,c} \eta_{d}^{\quad f} t_{ef},$$

(2)

where $\eta_{ab} = \text{diag}(-1, 1, 1, 1, 1)$ and $\eta_{ab,cd} = (1/2)(\eta_{ac}\eta_{bd} - \eta_{bc}\eta_{ad})$. The Cartan-Killing form $\tilde{\kappa}_{abcd}$ in the adjoint representation is

$$\tilde{\kappa}_{abcd} = \frac{1}{I_{ad}} \text{Tr}(t_{ab}t_{cd}) = f_{abef} g_{fcdh} e^{ef} = \eta_{abcd},$$

(3)

where $I_{ad} = -12$. From this we can construct an invariant action which coincides with the Pontrjagin class in $SO(4,1) \{11, 12\}$.

As pointed by Wise\{12\}, in the description of rolling geometries \{13, 14\} the spacetime geometries relevant to gravity are of a special type called reductive geometry. In particular, $SO(4,1)$ is a reductive geometry, where as in any Klein geometry, we can write

$$\mathfrak{so}(4,1) \cong \mathfrak{so}(3,1) \oplus \mathfrak{so}(4,1)/\mathfrak{so}(3,1),$$

(4)

this isomorphism holds not only as vector spaces but as representations of Ad($SO(3,1)$).

To visualize this splitting we consider the fundamental representation of the de Sitter Lie algebra in the following basis

$$\left\{ -\frac{1}{2} \gamma^{[I}\gamma^{J]} , \frac{1}{2} \gamma^{5}\gamma^{K} \right\},$$

(5)
we have adopted the complex $4 \times 4$ matrix representation of the Clifford algebra, where these matrices satisfies the relations

$$\gamma^I \gamma^J + \gamma^J \gamma^I = 2\eta^{IJ}, \quad (6)$$

where $I, J, K, L$ are $\mathfrak{so}(3,1)$ Lie algebra valued indices, and as usual

$$\gamma^5 = \frac{i}{4!} \epsilon_{IJKL} \gamma^I \gamma^J \gamma^K \gamma^L = i\gamma^0 \gamma^1 \gamma^2 \gamma^3, \quad (7)$$

where $\epsilon_{0123} = 1 = -\epsilon^{0123}$ is the totally antisymmetric tensor. We find the relation among the generators of the de Sitter algebra in the adjoint representation and the generators in the fundamental representation

$$t_{ab} = \begin{pmatrix} t_{IJ} & t_{I5} \\ t_{5J} & 0 \end{pmatrix} = \frac{1}{2(4^{1/3})} \begin{pmatrix} -\gamma[I \gamma] & \gamma J \gamma^5 \\ \gamma^5 \gamma J & 0 \end{pmatrix}. \quad (8)$$

The Lie algebra brackets in the fundamental representation are given by

$$[t_{IJ}, t_{KL}] = -\frac{1}{2} \left( \eta_{[I|K|L]} M N - \eta_{[I|L|J]} M N + N \leftrightarrow M \right) t_{MN},$$

$$[t_{IJ}, t_{K5}] = -\frac{1}{2} \eta_{IJK} L t_{L5},$$

$$[t_{I5}, t_{J5}] = -\frac{1}{2} \eta_{IJK} K t_{KL}, \quad (9)$$

and the corresponding Cartan-Killing form

$$\kappa_{abcd} = \frac{1}{I_{\text{fun}}} \text{Tr}(t_{ab} t_{cd}) = \begin{pmatrix} \eta_{IJ,KL} & 0 \\ 0 & \kappa_{MN} \end{pmatrix}, \quad (10)$$

where we have defined $I_{\text{fun}} = -4^{2/3}/2$, and

$$\kappa_{MN} = \begin{pmatrix} \kappa_{M5N5} & \kappa_{M55N} \\ \kappa_{5M5N} & \kappa_{55MN} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \eta_{MN} & -\eta_{MN} \\ -\eta_{MN} & \eta_{MN} \end{pmatrix}. \quad (11)$$

We observe an orthogonal splitting invariant under $SO(3,1)$, of the Cartan-Killing form. Then

$$\mathfrak{so}(4,1) \cong \mathfrak{so}(3,1) \oplus \mathbb{R}^{3,1}, \quad (12)$$

as vector spaces instead of Lie algebras [12].

From Eq. (10), we recognize $\eta_{IJ,KL}$ as the Cartan-Killing form for $\mathfrak{so}(3,1)$, but in four dimensions we have another invariant form, the Levi-Civita tensor $\epsilon_{IJKL}$, and in the de Sitter
algebra is possible to obtain this form from $\gamma^5$,

$$\kappa_{abcd}^{(1)} = \frac{1}{I_{\text{fun}}} \text{Tr}(i\gamma^5 t_{ab} t_{cd}) = \frac{1}{2} \begin{pmatrix} \epsilon_{IJKL} & 0 \\ 0 & 0 \end{pmatrix}. \quad (13)$$

In order to construct this second invariant form, but we have to explicitly break the group symmetry, i.e., the presence of $(i\gamma^5)$ in the trace term, breaks the symmetry from $\mathfrak{so}(4,1)$ down to $\mathfrak{so}(3,1)$. The invariant constructed in Eq. (13) is often used in the MM formulation of gravity [9].

But a question arises, is it possible to recover the form $\eta_{IJ,KL}$ by means of breaking a symmetry? The answer is yes, and it can be done by defining the form

$$\kappa_{abcd}^{(2)} = \frac{1}{I_{\text{fun}}} \text{Tr}((i\gamma^5 t_{ab})(i\gamma^5 t_{cd})) = -\frac{1}{2} \begin{pmatrix} \eta_{IJ,KL} & 0 \\ 0 & 0 \end{pmatrix}. \quad (14)$$

We can use Eq. (14) and Eq. (13) to see if we can propose a MM generalization for gravity.

### III. GAUGE THEORY

Let us take the de Sitter group $SO(4,1)$ as our gauge group, a 4-dimensional oriented smooth manifold $\mathcal{M}$. To avoid spacetimes with bad causality properties we consider $\mathcal{M} = \mathbb{R} \times \Sigma$, $\Sigma$ is compact and without boundary, where $\mathbb{R}$ represents an evolution parameter and choose a principal $SO(4,1)$-bundle $P$ over $\mathcal{M}$. We define the connection $A$ which is a $\mathfrak{so}(4,1)$-valued 1-form on $\mathcal{M}$, $A_{\mu} t_{ab} dx^\mu$, where $t_{ab}$ are the generators of the $\mathfrak{so}(4,1)$ Lie algebra. In the fundamental representation the 1-form is

$$A_{\mu}^{ab} t_{ab} = A_{\mu}^{IJ} t_{IJ} + 2A_{\mu}^{I5} t_{I5}. \quad (15)$$

After imposing the following identification of the gauge field $A$ with the spin connection $\omega$ and with the tetrad $e$

$$A_{\mu}^{ab} = \begin{pmatrix} A_{\mu}^{IJ} & A_{\mu}^{I5} \\ A_{\mu}^{5J} & 0 \end{pmatrix} = \begin{pmatrix} \omega_{\mu}^{IJ} & -\frac{1}{l} e_{\mu}^{I} \\ \frac{1}{l} e_{\mu}^{J} & 0 \end{pmatrix}. \quad (16)$$

The covariant derivative $D$ of the de Sitter group acts over Lie algebra valued fields $\xi = \xi_{ab} t_{ab}$ as follows

$$D\xi = \left[ D\xi_{IJ} + \frac{1}{l} \left( e^{J} \wedge \xi^{I5} - e^{I} \wedge \xi^{J5} \right) \right] t_{IJ} + 2 \left[ D\xi^{I5} - \frac{1}{l} e^{J} \wedge \xi_{I5} \right] t_{I5}. \quad (17)$$
We used the covariant derivative in the Lorentz group, which it’s defined for a $\mathfrak{so}(3,1)$-Lie algebra valued field $\chi = \chi^{IJ} t_{IJ}$ as,

$$D\chi = (d\chi^{IJ} + \omega^{IK} \wedge \chi^J_K + \omega^{JK} \wedge \chi^I_K) t_{IJ}. \quad (18)$$

The field strength, as usual, is $F = dA + \frac{1}{2} [A, A]$, then

$$F \to \begin{cases} 
F^{IJ} = R^{IJ} - \frac{1}{12} e^I \wedge e^J, \\
F^5 = -\frac{1}{7} T^I, 
\end{cases} \quad (19)$$
in the fundamental representation, also

$$T^I = De^I = de^I + \omega^{IK} \wedge e_K, \quad (20)$$

$$R^{IJ} = d\omega^{IJ} + \omega^{IK} \wedge \omega^J_K, \quad (21)$$

where $R^{IJ}$ is the curvature and $T^I$ is the torsion, both in $\mathfrak{so}(3,1)$. Now we are in position to define the actions for the theory.

First consider the action in the adjoint representation

$$I_{FF} [A] = \int_\mathcal{M} \text{Tr} F \wedge F = \int_\mathcal{M} F^{ab} \wedge F^{cd} \tilde{\kappa}_{abcd} = \int_\mathcal{M} F^{ab} \wedge F_{ab}. \quad (22)$$

This action corresponds to the Pontrjagin class of $SO(4,1)$, therefore is topological and does not give dynamical information. On the other hand, it is possible to calculate the same action but using the orthogonal decomposition of the Cartan-Killing form by means of the fundamental representation Eq. (10), then the action reads

$$I_{FF} = \int_\mathcal{M} F^{ab} \wedge F^{cd} \kappa_{abcd} = \int_\mathcal{M} R^{IJ} \wedge R_{IJ} - \frac{2}{12} \left[ R^{IJ} \wedge e_I \wedge e_J - T^I \wedge T_J \right], \quad (23)$$

where we can identify the Pontrjagin class for $SO(3,1)$ and the only closed 4-form invariant under local Lorentz rotations associated with the torsion of the manifold, the so-called Nieh-Yan class, which is given by

$$d(e^I \wedge T_I) = R^{IJ} \wedge e_I \wedge e_J - T^I \wedge T_I. \quad (24)$$

Again, the action is purely topological, this is something that we expected as the information that comes from the action doesn’t depend on the representation. Finally, from Eq. (22) and Eq. (23) we obtain the well-known result

$$\text{Pontrjagin}(SO(4,1)) = \text{Pontrjagin}(SO(3,1)) + \text{N.Y.}. \quad (25)$$
MacDowell and Mansouri [8] observed that in order to obtain a dynamical action, it is necessary to break the symmetry explicitly. Then the action proposed is

$$S_{MM}[A] = \int_M \text{Tr} \left( i \gamma^5 F \wedge F \right) = \int_M F^{ab} \wedge F^{cd} \kappa^{(1)}_{abcd},$$  \hspace{1cm} (26)$$

and from Eq.(13) get

$$S_{MM} = \int_M \frac{1}{2} R^{IJ} \wedge R_{IKKL} \epsilon_{IJKL} - \frac{1}{l^2} R^{IJ} \wedge e^K \wedge e^L \epsilon_{IJKL} + \frac{1}{2l^4} e^I \wedge e^J \wedge e^K \wedge e^L \epsilon_{IJKL}. \hspace{1cm} (27)$$

If $1/l^2 = \Lambda$ is the cosmological constant, then we identify the Euler class the Palatini’s action plus the cosmological constant in Eq.(27). From this action, we obtain two equations of motion, the zero torsion condition which means that allows the spin connection to be written in terms of the tetrad field, and when we substitute back into the second equation of motion, we arrive to the Einstein’s equations with cosmological constant.

In the background independent approaches to gravity, the starting point is the Holst action [3], which is written as a sum of the Palatini action plus the Holst term,

$$S_H = \int_M \epsilon_{IJKL} R^{IJ}(\omega) \wedge e^K \wedge e^L + \frac{1}{\gamma} R^{IJ}(\omega) \wedge e_I \wedge e_J. \hspace{1cm} (28)$$

One way to introduce the Holst action is to consider the approach given by Mercuri and Randono [10]. They construct an action invariant under $SO(4,1)$, with a preferred vacuum state, which leads to two constant expectation values which reduces to MacDowell-Mansouri action with a topological $\theta$-term

$$S = S_{MM} + \theta S_{FF}, \hspace{1cm} (29)$$

then if we consider the equation of motion $T^I = 0$ we obtain the Euler class plus the Holst term, so the Immirzi parameter is related to the $\theta$-term.

When non-vanishing torsion is present, Chandía and Zanelly [11] proposed torsional contributions to the chiral anomaly in the form of a Nieh-Yan term. Because of the importance of the Holst term and the Nieh-Yan class, we are interested in an action where these two terms are present as independent components.

For this purpose, we will use $\kappa_{abcd}^{(2)}$ from Eq.(14). Now consider the action

$$S_{PH} = \int_M F^{ab} \wedge F^{cd} \kappa^{(2)}_{abcd}$$

$$= \int_M -R^{IJ} \wedge R_{IJ} + \frac{2}{l^2} R^{IJ} \wedge e_I \wedge e_J, \hspace{1cm} (30)$$
then the action is a sum of the Pontrjagin class plus the Holst term. If we want to consider
the most general case, when a non-vanishing torsion is present, in the MM approach, it
is necessary to consider a linear combination of the three different actions that could be
constructed in the theory

\[ S_G = \mu S_{FF} + \nu S_{MM} + \rho S_{PH}, \]

(31)

where \( \mu, \nu, \rho \) are arbitrary constants (we have not introduced a global factor \( \kappa = -l^2/16\pi G \)
into the action \( S_G \), to simplify the calculations). Then the action reads

\[ S_G = \int_M \left( (\mu - \rho)R^{IJ} \wedge R_{IJ} + \frac{\nu}{2}R^{IJ} \wedge R^{KL} \epsilon_{IJKL} - \frac{2\mu}{l^2} \left( R^{IJ} \wedge e_I \wedge e_J - T^I \wedge T_I \right) \right) \]

\[ - \frac{\nu}{l^2} \left[ R^{IJ} \wedge \left( \epsilon_{IJKL} e^K \wedge e^L - \frac{2\rho}{\nu} e_I \wedge e_J \right) \right] + \frac{\nu}{2l^4} \epsilon_{IJKL} e^I \wedge e^J \wedge e^K \wedge e^L, \]

(32)

In this case the Immirzi parameter is associated with the quotient \( \rho/\nu \sim \gamma \).

The action Eq.(31) contains all the topological invariants in 4-dimensions, the Euler class,
the Pontrjagin class, the Nieh-Yan class as well as the dynamical part. The equations of
motion of the \( S_G \) action are

\[ D \left( \epsilon_{IJKL} e^K \wedge e^L - \frac{2\rho}{\nu} e_I \wedge e_J \right) = 0, \]

(33)

\[ -R^{IJ} \wedge e^K \epsilon_{IJKL} - \frac{2\rho}{\nu} R^I_L \wedge e_J + \Lambda e^I \wedge e^J \wedge e^K \epsilon_{IJKL} = 0, \]

(34)

they don’t look pretty different from the equation of motion for the Holst action with
cosmological constant term, and at first sight, it could be argued the need of the action
\( S_{PH} \). But, if we want to include torsion into the theory (we consider first a non-matter
contribution) by promoting the free parameters \( \mu, \nu, \rho \), as spacetime fields (check [15, 16] for
the case with the Immirzi parameter) then the three terms are relevant.

**IV. TORSIONAL ACTION**

Recently the Barbero-Immirzi parameter has been considered to be a field [15, 16]. In
order to generalize the action Eq.(31), let us consider the arbitrary constants as fields,
\( \mu = \mu(x), \nu = \nu(x), \rho = \rho(x) \) and \( \mu, \nu, \rho \in C^\infty(M) \) (where \( C^\infty \) denotes the space of \( C^\infty \)
functions on \( \mathcal{M} \) with compact support in \( \mathcal{M} \), then

\[
S_G = \int_\mathcal{M} (\mu(x) - \rho(x)) R^{I J} \wedge R_{I J} + \frac{1}{2} \nu(x) R^{I J} \wedge R^{K L} \epsilon_{I J K L} + \frac{2}{l^2} \mu(x) D(e^I \wedge T_I)
- \frac{1}{l^2} \nu(x) R^{I J} \wedge e^K \wedge e^L \epsilon_{I J K L} + 2 \frac{1}{l^2} \rho(x) e^{I J} \wedge e^K \wedge e^L.
\] (35)

It is important to note that in this work we will not consider matter contributions (for an action depending on these arbitrary fields, \( L_{\text{matter}} = L_{\text{matter}}(\mu, \nu, \rho) \) see \[15\]).

The equations of motion coming from the action \( S_G \) are

\[
\mu(x) \Rightarrow R^{I J} \wedge \left[ e_I \wedge e_J - \frac{l^2}{2} R_{I J} \right] = T^I \wedge T_I,
\]

\[
\rho(x) \Rightarrow R^{I J} \wedge \left[ e_I \wedge e_J - \frac{l^2}{2} R_{I J} \right] = 0,
\]

\[
\nu(x) \Rightarrow \frac{l^2}{2} R^{I J} \wedge R^{K L} \epsilon_{I J K L} = \left[ R^{I J} \wedge e^K \wedge e^L - \frac{1}{2l^2} e^I \wedge e^J \wedge e^K \wedge e^L \right] \epsilon_{I J K L},
\]

\[
\omega(x) \Rightarrow D(e^M \wedge e_N) = -\frac{2\rho \eta_{M N}^{IJ} + \nu \epsilon_{M N}^{IJ}}{4 \rho^2 + 4 \nu^2} (2D \mu \wedge [l^2 R_{I J} + e_I \wedge e_J] - 2D \rho \wedge [l^2 R^{K L} - e^K \wedge e^L] \epsilon_{I J K L}) + (2D \nu \wedge [l^2 R^{K L} - e^K \wedge e^L] \epsilon_{I J K L}),
\]

\[
e(x) \Rightarrow R^{I J} \wedge e^K \epsilon_{I J K L} - \frac{1}{l^2} e^I \wedge e^J \wedge e^K \epsilon_{I J K L} = \frac{2}{\nu} D \mu \wedge T_L - \frac{2\rho}{\nu} D T_L,
\] (36)

to write the equation of motion for \( \omega \) we used the projector \( \mathcal{A} \) defined as follows

\[
\mathcal{A}_{I J K L} = \alpha \eta_{I J K L} + \beta \epsilon_{I J K L}
\] (37)

where \( \alpha, \beta \) are fields such that \( \alpha^2 \neq -4\beta^2 \). And its inverse

\[
(\mathcal{A}^{-1})_{I J K L} = \frac{1}{\alpha^2 + 4\beta^2}(\alpha \eta_{I J K L} - \beta \epsilon_{I J K L})
\] (38)

such that \( \mathcal{A} \mathcal{A}^{-1} = \mathcal{A}^{-1} \mathcal{A} = \eta_{I J K L} \).

Our main goal is to find the spin connection as function of the new fields and the tetrad field. When we substitute back into the equation of motion for the tetrad field, we obtain Einstein’s equation. For this purpose, let us analyze the equations of motion of the theory. From the equations for \( \mu \) and \( \rho \), we get two important results, one of them is related to the torsion as

\[
T^I \wedge T_I = 0.
\] (39)

The last equation involves non-linear differential equations in \( \omega \) and \( e \). Fortunately, there are two simple solutions for the Torsion, one is obtained by a direct decomposition into the
time component and the spacial components of the torsion-torsion product

\[ T^0 \wedge T_0 + T^1 \wedge T_1 + T^2 \wedge T_2 + T^3 \wedge T_3 = 0 \]  (40)

and considering one product depending of the others three.

The second family of solutions, we will consider a 2-form

\[ T^I = \alpha^I \wedge \beta, \]  (41)

where \( \alpha^I \) is a 1-form valued vector space, and \( \beta \) is a 1-form.

The second result is that \( R^{IJ} \wedge \epsilon^I \wedge \epsilon^J = \frac{l^2}{2} R^{IJ} \wedge R_{IJ} \), which can be integrated over the manifold to obtain

\[ \int_M R^{IJ} \wedge \epsilon^I \wedge \epsilon^J = \frac{l^2}{2} \int_M R^{IJ} \wedge R_{IJ}. \]  (42)

Then the Holst term is equal, on shell, to the Pontrjagin class of \( SO(3,1) \), this results was obtain by Liko [17] by means of the equation of motion for the tetrad field. In our case we have recovered that result only considering equation of motions for the new fields in a non vanishing torsion scheme. Before we end with the discussion for the second result, it is interesting to note that this could be rewritten, by means of Eq.(24), as follows

\[ d \left( e^I \wedge de^I + \frac{l^2}{2} \left( \omega^{IJ} \wedge d\omega_{IJ} + \frac{3}{2} \omega^{IJ} \wedge \omega^K \wedge \omega_{KJ} \right) \right) = 0, \]  (43)

then the torsion term and Chern-Simons class are related modulo a closed three form, so there is a cohomology class relation between them. An important observation is that if we consider the theory in three dimensions, this closed three form disappears.

From the equation of motion of \( \nu \) and \( e \), we get

\[ \frac{l^2}{2} R^{IJ} \wedge R^{KL} \epsilon_{IJKL} - \frac{1}{2l^2} e^I \wedge e^J \wedge e^K \wedge e^L \epsilon_{IJKL} = \frac{2}{\nu} D\mu \wedge T_L \wedge e^L - \frac{2\rho}{\nu} DT_L \wedge e^L. \]  (44)

The last equation relates the cosmological term and Euler class to the torsion contribution. If we consider the equation of motion of \( \omega \), multiply both sides by \( e^N \) and by using the equation of motion for the tetrad field, we find

\[ e \Phi^1_M d^4x = \Phi^2 \wedge DT_M + \Phi^3 \wedge T_M + e_M \wedge e^N \wedge T_N, \]  (45)

where \( e \) is the non-vanishing determinant of the tetrad and

\[ \Phi^1 = -\frac{\nu}{4} \partial_M \mu, \quad \Phi^2 = l^2 \left( \rho \left( \frac{2}{\nu} + 1 \right) D\nu \right), \quad \Phi^3 = -l^2 \left( \frac{\rho}{\nu} D\nu + \nu D\mu - D\rho \right) \wedge D\mu. \]  (46)
Therefore, we have two solutions for the spin connection

\[ \omega = \omega(\mu, \nu, \rho, e) \quad \text{and} \quad \omega = \omega(\alpha, \beta, e), \]

(47)

We consider as fundamental fields those appearing in the action, then we have to find a relation among \( \mu, \nu, \rho \) and \( \alpha, \beta \). In general we consider solutions of the form \( \alpha = \alpha(\mu, \nu, \rho) \) and \( \beta = \beta(\mu, \nu, \rho) \). Due to the complexity to find such relations, we will consider two different cases: first, the torsion equal to zero, the second one is to consider an ansatz for \( \alpha \) that simplify the equations.

First we consider the case when the torsion vanishes and its covariant derivative, i.e. \( T^I = 0 \) and \( DT^I = 0 \), leading to \( \omega = \omega(e) \) and when we substitute back into the torsional modified Einstein’s equation, we have the usual equation in vacuum. We note that \( \alpha \) or \( \beta \) is identically zero, but it doesn’t matter which we take as zero, as the torsion vanish. Then from equation (45) we find

\[ \nu(x) \partial_M \mu(x) = 0, \]

(48)

then we have two solutions \( \nu = 0 \) for all \( x \in M \) or \( \mu \) is nonzero constant. The first is trivial, so we consider the second case. From the equation of motion for \( \omega \) we find

\[ (-2D\rho(x)\eta_{IJKL} + D\nu(x)\epsilon_{IJKL}) \wedge [l^2 R^{KL} - e^K \wedge e^L] = 0, \]

(49)

but the last equations involves the two independent \( \mathfrak{so}(3, 1) \) metric forms, each one multiplied by a independent field, so it implies that \( \nu, \rho \) are constants. Even if the new arbitrary fields are constants, it is interesting to note that their equations of motion are still valid so we have (42), and from equation (44), we have

\[ R^{IJ} \wedge R^{KL} \epsilon_{IJKL} = \Lambda^2 e^I \wedge e^J \wedge e^K \wedge e^L \epsilon_{IJKL}, \]

(50)

where \( \Lambda^2 = 1/l^4 \). The last equation could be integrated over the space-time manifold giving

\[ \Lambda^2 = \frac{\int_M R^{IJ} \wedge R^{KL} \epsilon_{IJKL}}{\int_M e^I \wedge e^J \wedge e^K \wedge e^L \epsilon_{IJKL}}, \]

(51)

but the Euler class is an integer that depends on the topology, so if we increase the volume of the space-time under consideration, we observe that the cosmological constant can be fixed.

Now let us consider a case with non-vanishing torsion, introducing the ansatz

\[ \alpha^I = e^I, \]

(52)
and using Eq.(47), for a consistent theory, we must find a relation for $\beta$ as $\beta = \beta(\mu, \nu, \rho, e)$. For this, we consider the torsion equations, on one side

$$T^I = \alpha^I \wedge \beta = e^I \wedge \beta,$$

(53)
on the other side, from the equation (45), the ansatz implies

$$\Phi^I_M = D^K_M \beta_K,$$

(54)

where we have defined the differential operator $D$ as

$$D^K_M = -\epsilon^{IJK}_M \Phi^2_I \partial_J - \epsilon^{IJK}_M \Phi^2_{IJ},$$

(55)

then the formal solution for $\beta$ is

$$\beta_K = \beta^0_K + \int G^K_I(x, x') \Phi^I_J(x') d^4x',$$

(56)

where $G^K_I$ is the Green function and $\beta^0_K$ is the solution for the homogeneous equation (for a good introduction see appendix A of [18], for a more formal reference see [19]). So we observe that it is possible to find $\beta = \beta(\mu, \nu, \rho)$, so the ansatz for the torsion is consistent.

We don’t need the explicit form of $\beta$ as we can work with the torsion term.

Consider the equation Eq.(44) and by using $DT^I = -e^I \wedge d\beta$, we obtain Eq.(51), i.e., we can fix the cosmological constant value. Finally the last equation to analyze is Einstein’s equation coming from the variation from the tetrad field,

$$R^I_J \wedge e^K \wedge e^L \epsilon_{IJKL} - \Lambda e^I \wedge e^K \wedge e^L \epsilon_{IJKL} = 0,$$

(57)

at first sight, it looks exactly like the Einstein’s equation, but this is not completely true, the spin connection now depends on the fields variables $\mu, \nu, \rho$, also from (53), the spin connection can be written as

$$\omega = \omega(e) + \omega(\beta(\mu, \nu, \rho)),$$

(58)

and Einstein’s equations are modified as follows

$$R^I_J(\omega(e)) \wedge e^K \wedge e^L \epsilon_{IJKL} - \Lambda e^I \wedge e^K \wedge e^L \epsilon_{IJKL} =$$

$$R^I_J(\omega(\beta)) \wedge e^K \wedge e^L \epsilon_{IJKL} + R^I_J(\omega(\beta), \omega(e)) \wedge e^K \wedge e^L \epsilon_{IJKL},$$

(59)

where we observe modifications to the original Einstein’s equation due to the presence of the new fields coupled to the original action. The dynamical behavior of these fields could be computed once we have the explicit solution for the $\beta$ field from (56), and by using the Bianchi’s identity in the Einstein’s equation of motion.
V. CONCLUSION AND OUTLOOK

We consider two different Cartan-Killing forms that can be derived from the Lie algebra in $\mathfrak{so}(4, 1)$, one in the adjoint representation and the other one is coming from the fundamental representation. We use a representation of the Lie algebra as a direct sum of the two vector spaces $\mathfrak{so}(4, 1) \cong \mathfrak{so}(3, 1) \oplus \mathbb{R}^{3,1}$ and the actions constructed are topological. The dynamics are obtained by explicitly breaking $\mathfrak{so}(4, 1)$ to $\mathfrak{so}(3, 1)$. In $\mathfrak{so}(3, 1)$, it is possible to find two Cartan-Killing forms $\epsilon^{IJKL}$ and $\eta_{IJKL}$, and identify two metrics coming from the broken sector and one related to the unbroken one. In MM models, one usually works with form $\epsilon^{IJKL}$, but to have a more general contribution to the dynamics we constructed the action from a linear combination of the Cartan-Killing forms. We obtain the Palatini action, the cosmological constant term, the Euler and Pontrjagin terms (as in MM) but also get the Nieh-Yan class independent from Holst term.

The introduction of three arbitrary parameters associated with each Cartan-Killing form (one of them related to the Immirzi’s parameter), and inspired by the recently proposed works on the treatment of the Immirzi parameter as a field [15, 16], we considered these free parameters as fields and calculated the dynamics coming from these new fields. As expected, we get a non zero torsion, that in general depends on the new fields. Another important result is that the Holst is related to the Pontrjagin class of $SO(3, 1)$.

We where able to calculate two solutions for the torsion. From the first one, we can relate the cosmological constant to the Euler class and the volume of the manifold.

More general torsion contributions can be introduced (i.e. fermion contributions, with and without supersymmetry). Finally we believe that these formulations might shed light on the significance of the Immirizi parameter.

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VI. APPENDIX A

We adopt the complex 4x4 matrix representation of the Clifford algebra, whose algebraic properties are defined by the relations

\[ \gamma^I \gamma^J + \gamma^J \gamma^I = 2\eta^{IJ} \quad \text{and} \quad \gamma^5 \gamma^I + \gamma^I \gamma^5 = 0. \]  

(60)

The identities

\[ \gamma^I \gamma_I = 4I \]  

(61)

\[ \gamma^I \gamma^J \gamma_I = -\gamma^J \]  

(62)

\[ \gamma^I \gamma^J \gamma^K \gamma_I = 4\eta^{JK} I \]  

(63)

\[ \gamma^I \gamma^J \gamma^K \gamma^L \gamma_I = -\gamma^L \gamma^K \gamma^J \]  

(64)

\[ \gamma^I \gamma^J \gamma^K = \eta^{IJ} \gamma^K + \eta^{JK} \gamma^I - \eta^{IK} \gamma^J + i\epsilon^{IJK} \gamma^5 \]  

(65)

\[ \gamma^5 \gamma^5 = I \]  

(66)

and the trace identities

\[ Tr(\gamma^I \ldots \gamma^{I_{2n+1}}) = 0 \quad n = 0, 1, \ldots \]  

(68)

\[ Tr(\gamma^5 \gamma^I \ldots \gamma^{I_{2n+1}}) = 0 \quad n = 0, 1, \ldots \]  

(69)

\[ Tr(\gamma^I \gamma^J) = 4 \eta^{IJ} \]  

(70)

\[ Tr(\gamma^5) = Tr(\gamma^5 \gamma^I \gamma^J) = 0 \]  

(71)

\[ Tr(\gamma^I \gamma^J \gamma^K \gamma^L) = 4 (\eta^{IJ} \eta^{KL} - \eta^{IK} \eta^{JL} + \eta^{IL} \eta^{JK}) \]  

(72)

\[ Tr(\gamma^5 \gamma^I \gamma^K \gamma^L) = -4i \epsilon^{IJKL} \]  

(73)

\[ Tr(\gamma^I \ldots \gamma^I_n) = Tr(\gamma^I_n \ldots \gamma^I_1) \]  

(74)

\[ Tr(\gamma^I \ldots \gamma^I_n) = Tr(\gamma^I_n \ldots \gamma^I_1) \]  

(75)

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