Vectorial Prabhakar Hardy Type Generalized Fractional Inequalities under Convexity

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ABSTRACT

We present a detailed great variety of Hardy type fractional inequalities under convexity and $L^p$ norm in the setting of generalized Prabhakar and Hilfer fractional calculi of left and right integrals and derivatives. The radial multivariate case of the above over a spherical shell is developed in detail to all directions. Many inequalities are of vectorial splitting rational $L^p$ type or of separating rational $L^p$ type, others involve ratios of functions and of fractional integral operators.

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1. Background

This work is inspired by [3-11].

Here we consider the Prabhakar function (also known as the three parameter Mittag-Laffler function), (see [6], p. 97; [5])

\[ E_{\alpha, \beta}^\gamma(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{k! (\alpha k + \beta)} z^k, \tag{1} \]

where \( \Gamma \) is the gamma function; \( \alpha, \beta, \gamma \in \mathbb{R}: \alpha, \beta > 0, z \in \mathbb{R} \), and \( (\gamma)_k = \gamma(\gamma + 1) \ldots (\gamma + k - 1) \). It is \( E_{\alpha, \beta}^0(z) = \frac{1}{\Gamma(\beta)} \).

Here we follow [4].

Let \( a, b \in \mathbb{R}, a < b \) and \( x \in [a, b]; f \in C([a, b]) \). Let also \( \psi \in C^1([a, b]) \) which is increasing. The left and right Prabhakar fractional integrals with respect to \( \psi \) are defined as follows:

\[ (e_{\rho, \mu, a, \psi}^\gamma f)(x) = \int_a^b \psi'(t)(\psi(x) - \psi(t))^{\rho-1} E_{\rho, \mu}^\gamma \left[ \omega(\psi(x) - \psi(t))^{\gamma} \right] f(t) dt, \tag{2} \]

and

\[ (e_{\rho, \mu, b, \psi}^\gamma f)(x) = \int_c^b \psi'(t)(\psi(t) - \psi(x))^{\rho-1} E_{\rho, \mu}^\gamma \left[ \omega(\psi(t) - \psi(x))^{\gamma} \right] f(t) dt, \tag{3} \]

where \( \rho, \mu > 0; \gamma, \omega \in \mathbb{R} \).

Functions (2) and (3) are continuous ([4]).

Next, additionally, assume that \( \psi'(x) \neq 0 \) over \([a, b]\) and let \( \psi, f \in C^1([a, b]) \), where \( N = [\mu] \), (\([\cdot]\) is the ceiling of the number), \( 0 < \mu \notin \mathbb{N} \). We define the \( \psi \)-Prabhakar-Caputo left and right fractional derivatives of order \( \mu \) ([4]) as follows (\( x \in [a, b] \)):

\[ (c \, D_{\rho, \mu, a, \psi}^\gamma f)(x) = \int_a^x \psi'(t)(\psi(x) - \psi(t))^{\rho-1} E_{\rho, \mu}^\gamma \left[ \omega(\psi(x) - \psi(t))^{\gamma} \right] \left( \frac{1}{\psi'(t)} \right)^N f(t) dt, \tag{4} \]

and

\[ (c \, D_{\rho, \mu, b, \psi}^\gamma f)(x) = (-1)^N \int_x^b \psi'(t)(\psi(t) - \psi(x))^{\rho-1} E_{\rho, \mu}^\gamma \left[ \omega(\psi(t) - \psi(x))^{\gamma} \right] \left( \frac{1}{\psi'(t)} \right)^N f(t) dt. \tag{5} \]

One can write these (see (4), (5)) as

\[ (c \, D_{\rho, \mu, a, \psi}^\gamma f)(x) = (e_{\rho, \mu, a, \psi}^{\gamma-N} f^{(N)})(x), \tag{6} \]

and
\[
(C D^{N,\psi}_{p,\mu,a,b} - f)(x) = (-1)^N \left( e^{-\gamma \psi}_{p,N-\mu,a,b} f^{[N]}_{\psi}(x) \right),
\]
where
\[
f^{[N]}_{\psi}(x) = f^{(N)}(x) := \left( \frac{1}{\psi(x)} \right)^N f(x),
\]
\[\forall \ x \in [a,b].\]

Functions (6) and (7) are continuous on \([a,b]\).

Next we define the \(\psi\)-Prabhakar-Riemann Liouville left and right fractional derivatives of order \(\mu\) ([4]) as follows \((x \in [a,b], \ f \in C([a,b])):\)
\[
\left( \mathcal{D}_L^{\gamma,\psi}_{p,\mu,a,a+f} \right)(x) = \left( \frac{1}{\psi(x)} \right)^N \left( \int_a^x \psi'(t)(\psi(x) - \psi(t))^{N-\mu-1} E_{p,N-\mu}^\gamma \omega(\psi(x) - \psi(t)) \right) f(t) dt,
\]
and
\[
\left( \mathcal{D}_R^{\gamma,\psi}_{p,\mu,b,b-f} \right)(x) = \left( -\frac{1}{\psi(x)} \right)^N \left( \int_x^b \psi'(t)(\psi(t) - \psi(x))^{N-\mu-1} E_{p,N-\mu}^\gamma \omega(\psi(t) - \psi(x)) \right) f(t) dt.
\]
That is we have
\[
\left( \mathcal{D}_L^{\gamma,\psi}_{p,\mu,a,a+f} \right)(x) = \left( \frac{1}{\psi(x)} \right)^N \left( e^{-\gamma \psi}_{p,N-\mu,a,a} f \right)(x),
\]
and
\[
\left( \mathcal{D}_R^{\gamma,\psi}_{p,\mu,b,b-f} \right)(x) = \left( -\frac{1}{\psi(x)} \right)^N \left( e^{-\gamma \psi}_{p,N-\mu,b,b} f \right)(x),
\]
\[\forall \ x \in [a,b].\]

We define also the \(\psi\)-Hilfer-Prabhakar left and right fractional derivatives of order \(\mu\) and type \(0 \leq \beta \leq 1\) ([4]), as follows
\[
\left( \mathcal{D}^{\gamma,\beta,\psi}_{p,\mu,a,a+f} \right)(x) = e^{-\gamma \psi}_{p,\beta(N-\mu),a,a} \left( \frac{1}{\psi(x)} \right)^N e^{-\gamma(1-\beta)\psi}_{p,\beta(N-\mu),a,a} f(x),
\]
and
\[
\left( \mathcal{D}^{\gamma,\beta,\psi}_{p,\mu,b,b-f} \right)(x) = e^{-\gamma \psi}_{p,\beta(N-\mu),b,b} \left( -\frac{1}{\psi(x)} \right)^N e^{-\gamma(1-\beta)\psi}_{p,\beta(N-\mu),b,b} f(x),
\]
\[\forall \ x \in [a,b].\]
When $\beta = 0$, we get the Riemann-Liouville version, and when $\beta = 1$, we get the Caputo version.

We call $\xi = \mu + \beta(N - \mu)$, we have that $N - 1 < \mu \leq \mu + \beta(N - \mu) \leq \mu + N - \mu = N$, hence $[\xi] = N$.

We can easily write that

$$
\left( H_{D_{\psi}^{\beta,\gamma}} a, f \right)(x) = e^{-\psi(x)} D_{\psi}^{(1-\beta)} f(x),
$$

and

$$
\left( H_{D_{\psi}^{\beta,\gamma}} b, f \right)(x) = e^{-\psi(x)} D_{\psi}^{(1-\beta)} f(x),
$$

\forall \ x \in [a, b].

In this work we develop a great variety of fractional inequalities of Hardy type involving convexity and engaging the above exposed: $\psi$ -Prabhakar fractional left and right fractional integrals, the $\psi$ -Prabhakar-Caputo left and right fractional derivatives, the $\psi$ -Riemann-Liouville left and right fractional derivatives, and the $\psi$ -Hilfer-Prabhakar left and right fractional derivatives. The radial multivariate case of all of the above over a spherical shell is studied in full detail. We involve ratios of functions and of integral operators and we produce among others vectorial splitting rational $L_p$ inequalities, as well as separating rational $L_p$ inequalities.

2. Prerequisites

Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with positive $\sigma$-finite measures, and let $k : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be nonnegative measurable functions, $k(x, \cdot)$ measurable on $\Omega_2$, and

$$
K(x) = \int_{\Omega_2} k(x, y) d\mu_2(y), \text{ for any } x \in \Omega_1.
$$

We suppose that $K(x) > 0$ a.e. on $\Omega_1$ and by a weight function (shortly: a weight), we mean a nonnegative measurable function on the actual set. Let the measurable functions $g_i : \Omega_1 \to \mathbb{R}$, $i = 1, \ldots, n$, with the representation

$$
g_i(x) = \int_{\Omega_2} k(x, y) f_i(y) d\mu_2(y),
$$

where $f_i : \Omega_2 \to \mathbb{R}$ are measurable functions, $i = 1, \ldots, n$.

Denote by $\tilde{x} = x := (x_1, \ldots, x_n) \in \mathbb{R}^n$, $\tilde{g} := (g_1, \ldots, g_n)$ and $\tilde{f} := (f_1, \ldots, f_n)$.

We consider here $\Phi : \mathbb{R}_+^n \to \mathbb{R}$ a convex function, which is increasing per coordinate, i.e. if $x_i \leq y_i$, $i = 1, \ldots, n$, then

$$
\Phi(x_1, \ldots, x_n) \leq \Phi(y_1, \ldots, y_n).
$$

In [3], p. 588, we proved that

**Theorem 1** Let $u$ be a weight function on $\Omega_1$, and $k$, $K$, $g_i$, $f_i$, $i = 1, \ldots, n \in \mathbb{N}$, and $\Phi$ defined as above. Assume that the function $x \to u(x) \frac{k(x, y)}{K(x)}$ is integrable on $\Omega_1$ for each fixed $y \in \Omega_2$. Define $v$ on $\Omega_2$ by
Then
\[ \int_{\Omega_1} u(x) \Phi \left( \frac{g_1(x)}{K(x)}, \ldots, \frac{g_n(x)}{K(x)} \right) d\mu_1(x) \leq \int_{\Omega_2} v(y) \Phi \left( \|f_1(y)\|, \ldots, \|f_n(y)\| \right) d\mu_2(y), \] (20)

under the assumptions:

(i) \( f_i, \Phi(\|f_1\|, \ldots, \|f_n\|) \) are \( k(x, y) d\mu_2(y) \)-integrable, \( \mu_1 \)-a.e. in \( x \in \Omega_1 \), for all \( i = 1, \ldots, n \),

(ii) \( v(y) \Phi(\|f_1(y)\|, \ldots, \|f_n(y)\|) \) is \( \mu_2 \)-integrable.

**Notation 2** From now on we may write
\[ \tilde{g}(x) = \int_{\Omega_2} k(x, y) \tilde{f}(y) d\mu_2(y), \] (21)

which means
\[ (g_1(x), \ldots, g_n(x)) = \left( \int_{\Omega_2} k(x, y) f_1(y) d\mu_2(y), \ldots, \int_{\Omega_2} k(x, y) f_n(y) d\mu_2(y) \right). \] (22)

Similarly, we may write
\[ |\tilde{g}(x)| = \left| \int_{\Omega_2} k(x, y) \tilde{f}(y) d\mu_2(y) \right|, \] (23)

and we mean
\[ (|g_1(x)|, \ldots, |g_n(x)|) = \left( \left| \int_{\Omega_2} k(x, y) f_1(y) d\mu_2(y) \right|, \ldots, \left| \int_{\Omega_2} k(x, y) f_n(y) d\mu_2(y) \right| \right). \] (24)

We also can write that
\[ |\tilde{g}(x)| \leq \int_{\Omega_2} k(x, y) |\tilde{f}(y)| d\mu_2(y), \] (25)

and we mean the fact that
\[ |g_i(x)| \leq \int_{\Omega_2} k(x, y) |f_i(y)| d\mu_2(y), \] (26)

for all \( i = 1, \ldots, n, \) etc.

**Notation 3** Next let \( (\Omega_1, \Sigma_1, \mu_1) \) and \( (\Omega_2, \Sigma_2, \mu_2) \) be measure spaces with positive \( \sigma \)-finite measures, and let \( k_j : \Omega_1 \times \Omega_2 \rightarrow \mathbb{R} \) be a nonnegative measurable function, \( k_j(x, \cdot) \) measurable on \( \Omega_2 \) and
\[ K_j(x) = \int_{\Omega_2} k_j(x, y) d\mu_2(y), \quad x \in \Omega_1, \quad j = 1, \ldots, m. \] (27)
We suppose that \( K_j(x) > 0 \) a.e. on \( \Omega_1 \). Let the measurable functions \( g_{ji} : \Omega_1 \to \mathbb{R} \) with the representation

\[
g_{ji}(x) = \int_{\Omega_2} k_j(x,y) f_{ji}(y)d\mu_2(y),
\]

where \( f_{ji} : \Omega_2 \to \mathbb{R} \) are measurable functions, \( i = 1,\ldots,n \) and \( j = 1,\ldots,m \).

Denote the function vectors \( \tilde{g}_i := (g_{j1}, g_{j2}, \ldots, g_{jn}) \) and \( \tilde{f}_j := (f_{j1}, \ldots, f_{jm}) \), \( j = 1,\ldots,m \).

We say \( f_j \) is integrable with respect to measure \( \mu \), iff all \( f_{ji} \) are integrable with respect to \( \mu \).

We also consider here \( \Phi_j : \mathbb{R}^n_+ \to \mathbb{R}_+ \), \( j = 1,\ldots,m \), convex functions that are increasing per coordinate. Again \( u \) is a weight function on \( \Omega_1 \).

We make

**Remark 4** Following Notation 3, let \( F_j : \Omega_2 \to \mathbb{R} \cup \{-\infty, \infty\} \) be measurable functions, \( j = 1,\ldots,m \), with \( 0 < F_j(y) < \infty \) on \( \Omega_2 \). In (27) we replace \( k_j(x,y) \) by \( k_j(x,y)F_j(y) \), \( j = 1,\ldots,m \), and we have the modified \( K_j(x) \) as

\[
L_j(x) := \int_{\Omega_2} k_j(x,y)F_j(y)d\mu_2(y), \quad x \in \Omega_1.
\]

We assume \( L_j(x) > 0 \) a.e. on \( \Omega_1 \).

As new \( f_j \) we consider now \( \tilde{f}_j := \frac{f_j}{F_j} \), \( j = 1,\ldots,m \), where \( \tilde{f}_j = (f_{j1}, \ldots, f_{jm}) \); \( \tilde{g}_j = \left( \frac{f_{j1}}{F_j}, \ldots, \frac{f_{jm}}{F_j} \right) \).

Notice that

\[
g_{ji}(x) = \int_{\Omega_2} k_j(x,y) f_{ji}(y)d\mu_2(y) = \int_{\Omega_2} k_j(x,y)F_j(y) \left( \frac{f_{ji}(y)}{F_j(y)} \right)d\mu_2(y),
\]

\( x \in \Omega_1 \), all \( j = 1,\ldots,m \); \( i = 1,\ldots,n \).

So we can write

\[
\tilde{g}_i(x) = \int_{\Omega_2} \left( k_j(x,y)F_j(y) \right) \frac{f_{ji}(y)}{F_j(y)}d\mu_2(y), \quad j = 1,\ldots,m.
\]

We mention

**Theorem 5** ([3], p. 481) Here we follow Remark 4. Let \( \rho \in \{1,\ldots,m\} \) be fixed. Assume that the function
Theorem 6 ([3], p. 519) Here all as in Notation 3 and Remark 4. Assume that the functions \((j = 1, 2, \ldots, m \in \mathbb{N})\)

\[
x \mapsto \left( \frac{u(x)k_j(x, y)F_j(y)}{K_j(x)} \right)
\]

are integrable on \(\Omega_1\), for each fixed \(y \in \Omega_2\). Define \(W_j\) on \(\Omega_2\) by

\[
W_j(y) := \left( \int_{\Omega_1} \frac{u(x)k_j(x, y)}{K_j(x)} \, d\mu_1(x) \right) F_j(y) < \infty,
\]

on \(\Omega_2\).
Let \( p_j > 1: \sum_{j=1}^{m} \frac{1}{p_j} = 1 \). Let the functions \( \Phi_j : \mathbb{R}^n \to \mathbb{R}_+ \), \( j = 1, \ldots, m \), be convex and increasing per coordinate.

Then
\[
\int_{\Omega_1} u(x) \prod_{j=1}^{m} \Phi_j \left( \frac{g_j(x)}{L_j(x)} \right) d\mu_1(x) \leq \prod_{j=1}^{m} \left( \int_{\Omega_2} W_j(y) \Phi \left( \frac{f_j(y)}{F_j(y)} \right) d\mu_2(y) \right)^{\frac{1}{p_j}},
\]
under the assumptions:

(i) \( \frac{f_j(y)}{F_j(y)} \Phi_j \left( \frac{f_j(y)}{F_j(y)} \right) \) are both \( k_j(x, y) F_j(y) d\mu_2(y) \)-integrable, \( \mu_1 \)-a.e. in \( x \in \Omega_1 \), \( j = 1, \ldots, m \),

(ii) \( W_j \Phi_j \left( \frac{f_j(y)}{F_j(y)} \right) \) is \( \mu_2 \)-integrable, \( j = 1, \ldots, m \).

We make

**Remark 7** Let \( (\Omega_1, \Sigma_1, \mu_1) \) and \( (\Omega_2, \Sigma_2, \mu_2) \) be measure spaces with positive \( \sigma \)-finite measures, and let \( k : \Omega_1 \times \Omega_2 \to \mathbb{R} \) be nonnegative measurable functions, \( k(x, \cdot) \) measurable on \( \Omega_2 \), and

\[
K(x) = \int_{\Omega_2} k(x, y) d\mu_2(y), \text{ for any } x \in \Omega_1.
\]

We assume \( K(x) > 0 \) a.e. on \( \Omega_1 \) and the weight functions are nonnegative functions on the related set. We consider measurable functions \( g_i : \Omega_1 \to \mathbb{R} \), with the representation

\[
g_i(x) = \int_{\Omega_2} k(x, y) f_i(y) d\mu_2(y),
\]
where \( f_i : \Omega_2 \to \mathbb{R} \) are measurable functions, \( i = 1, \ldots, n \). Here \( u \) stands for a weight function on \( \Omega_1 \). So we follow Notation 3 for \( j = m = 1 \). We write here \( \vec{g} := (g_1, \ldots, g_n) \), \( \vec{f} := (f_1, \ldots, f_n) \).

We set
\[
\left\| \vec{f}(y) \right\|_\infty := \max \{ \| f_1(y) \|, \ldots, \| f_n(y) \| \},
\]
and
\[
\left\| \vec{f}(y) \right\|_q := \left( \sum_{i=1}^{n} |f_i(y)|^q \right)^{\frac{1}{q}}, q \geq 1.
\]

We assume that
\[ 0 < \left\| \tilde{f}(y) \right\|_q < \infty, \text{ a.e. on } (a, b), \]  

(37)

\[ 1 \leq q \leq \infty \quad \text{fixed.} \]

Let

\[ L_q(x) := \int_{\Omega_2} k(x, y) \left\| \tilde{f}(y) \right\|_q \, d\mu(y), \quad x \in \Omega_1, \]  

(38)

\[ 1 \leq q \leq \infty \quad \text{fixed.} \]

We assume \( L_q(x) > 0 \) a.e. on \( \Omega_1 \).

We further assume that the function

\[ x \mapsto \left\{ \frac{u(x)k(x, y)\left\| \tilde{f}(y) \right\|_q}{L_q(x)} \right\} \]  

(39)

is integrable on \( \Omega_1 \), for almost each fixed \( y \in \Omega_2 \).

Define \( W_q \) on \( \Omega_2 \) by

\[ W_q(y) := \left( \int_{\Omega_1} \frac{u(x)k(x, y)}{L_q(x)} \, d\mu(x) \right) \left\| \tilde{f}(y) \right\|_q < \infty, \]  

(40)

a.e. on \( \Omega_2 \).

Let

\[ \tilde{\gamma} := \left( \frac{f_1}{\left\| \tilde{f}(y) \right\|_q}, \frac{f_2}{\left\| \tilde{f}(y) \right\|_q}, \ldots, \frac{f_n}{\left\| \tilde{f}(y) \right\|_q} \right), \]  

(41)

i.e. \( \tilde{\gamma} = \frac{\tilde{f}}{\left\| \tilde{f}(y) \right\|_q} \).

Here \( \Phi : \mathbb{R}^n_+ \rightarrow \mathbb{R} \) is a convex and increasing per coordinate function.

We mention

**Theorem 8** ([3], p. 536) Let all here as in Remark 7. Then
\[
\int_{\Omega_1} u(x) \Phi \left( \frac{g(x)}{L^q(x)} \right) d\mu(x) \leq \int_{\Omega_2} W'_q(y) \Phi \left( \frac{f(y)}{\|f(y)\|_q} \right) d\mu_2(y), \tag{42}
\]

under the assumptions:

(i) \( \frac{\tilde{f}(y)}{\|\tilde{f}(y)\|_q} , \Phi \left( \frac{\tilde{f}(y)}{\|\tilde{f}(y)\|_q} \right) \) are both \( k(x,y)\|\tilde{f}(y)\| d\mu_2(y) \) -integrable, \( \mu_1 \) -a.e. in \( x \in \Omega_1 \),

(ii) \( W'_q(y) \Phi \left( \frac{\tilde{f}(y)}{\|\tilde{f}(y)\|_q} \right) \) is \( \mu_2 \) -integrable.

Theorem 8 comes directly from Theorem 1.

We will also use:

Let \( (\Omega_1, \Sigma_1, \mu_1), (\Omega_2, \Sigma_2, \mu_2) \) measure spaces with positive \( \sigma \)-finite measures, and \( k_i : \Omega_1 \times \Omega_2 \to \mathbb{R} \) are nonnegative measurable functions, with \( k_i(x,) \) measurable on \( \Omega_2 \), and measurable functions \( g_{ji} : \Omega_1 \to \mathbb{R} \) :

\[
g_{ji}(x) = \int_{\Omega_2} k_i(x,y) f_{ji}(y) d\mu_2(y),
\]

where \( f_{ji} : \Omega_2 \to \mathbb{R} \) are measurable functions, for all \( j = 1,2; i = 1,...,m \).

**Theorem 9** ([3], p. 552) Here \( 0 < f_{2i}(y) < \infty \), a.e., \( i = 1,...,m \). Assume that the functions \( (i = 1,...,m \in \mathbb{N}) \)

\[
\phi(x) = \left( \frac{u(x)k_i(x,y)f_{2i}(y)}{g_{2i}(x)} \right)
\]

are integrable on \( \Omega_1 \), for each fixed \( y \in \Omega_2 \); with \( g_{2i}(x) > 0 \), a.e. on \( \Omega_1 \).

Define \( \psi_i \) on \( \Omega_2 \) by

\[
\psi_i(y) = f_{2i}(y) \int_{\Omega_1} u(x) \frac{k_i(x,y)}{g_{2i}(x)} d\mu_1(x) < \infty,
\tag{43}
\]

a.e. on \( \Omega_2 \).

Let \( p_i > 1 : \sum_{i=1}^{m} \frac{1}{p_i} = 1 \). Let the functions \( \Phi_i : \mathbb{R}_+ \to \mathbb{R}_+, i = 1,...,m \), be convex and increasing. Then
\[
\int_{\Omega_1} u(x) \prod_{i=1}^{m} \Phi_i \left( \frac{g_{i_1}(x)}{g_{i_2}(x)} \right) d\mu_i(x) \leq \prod_{i=1}^{m} \left( \int_{\Omega_2} \psi_i(y) \Phi_i \left( \frac{f_{i_1}(y)}{f_{i_2}(y)} \right)^{\frac{1}{p_i}} d\mu_2(y) \right)^{\frac{1}{p_i}},
\]  
(44)

under the assumptions:

1. \( \frac{f_{i_1}(y)}{f_{i_2}(y)} \), \( \Phi_i \left( \frac{f_{i_1}(y)}{f_{i_2}(y)} \right)^{\frac{1}{p_i}} \) are both \( k_i(x,y)f_{i_2}(y)d\mu_2(y) \)-integrable, \( \mu_1 \)-a.e. in \( x \in \Omega_1 \),

2. \( \psi_i(y) \Phi_i \left( \frac{f_{i_1}(y)}{f_{i_2}(y)} \right)^{\frac{1}{p_i}} \) is \( \mu_2 \)-integrable, \( i = 1, \ldots, m \).

3. Main Results

We make

Remark 10 Here \( \rho_j, \mu_j, \gamma_j, \omega_j > 0 \); \( f_{ji} \in C([a,b]) \) and \( \psi \in C^1([a,b]) \) which is increasing; \( j = 1, \ldots, m \) and \( i = 1, \ldots, n \). Set

\[
\varphi_{j^+}(y) := \left\| e_{\rho_j,\mu_j,\omega_j,a,f_j}(y) \right\| := \max_{j=1,\ldots,n} \left\{ e_{\rho_j,\mu_j,\omega_j,a,f_j}(y) \right\},
\]  
(45)

and

\[
\varphi_{j^+}(y) := \left( \sum_{i=1}^{n} e_{\rho_j,\mu_j,\omega_j,a,f_j}(y) \right)^{\frac{1}{q}}, \quad q \geq 1;
\]  
(46)

\( y \in [a,b] \), which \( \varphi_{j^+} \) are continuous functions, \( j = 1, \ldots, m \). We have that

\[
0 < \varphi_{j^+}(y) < \infty \quad \text{in} \ [a,b],
\]  
(47)

\( j = 1, \ldots, m \); where \( 1 \leq q \leq \infty \) is fixed.

Here it is

\[
k_j^+(x,y) := k_j(x,y) = \begin{cases} 
\psi'(y)(\psi(x)-\psi(y))^{\gamma_j-1} E_{\rho_j,\mu_j}^{\gamma_j} [\omega_j (\psi(x)-\psi(y))^{\gamma_j}] a < y \leq x, \\
0, x < y < b,
\end{cases}
\]  
(48)

\( j = 1, \ldots, m \), and

\[
L_{jq}^+(x) := \int_a^x \psi'(y)(\psi(x)-\psi(y))^{\mu_j-1} E_{\rho_j,\mu_j}^{\mu_j} [\omega_j (\psi(x)-\psi(y))^{\mu_j}] \varphi_{j^+}(y) dy,
\]  
(49)

\( \forall \ x \in [a,b], 1 \leq q \leq \infty \).
We have that \( L^+_{jq}(x) > 0 \) on \([a,b]\).

Let \( \rho \in \{1, \ldots, m\} \) be fixed. The weight function \( u \) is chosen so that

\[
U_+^u(y) := \left( \prod_{j=1}^{m} \varphi_j^u(y) \right) \int_y^b \frac{u(x) \prod_{j=1}^{m} k_j^+(x,y)}{L^+_{jq}(x)} \, dx < \infty,
\]

(50)

\( \forall \ y \in [a,b] \), and that \( U_+^u \) is integrable on \([a,b]\).

A direct application of Theorem 5 gives:

**Theorem 11** It is all as in Remark 10. Here \( \Phi_j : \mathbb{R}^n_+ \rightarrow \mathbb{R}_+ \), \( j = 1, \ldots, m \), are convex functions increasing per coordinate. Then

\[
\int_y^b u(x) \prod_{j=1}^{m} \Phi_j \left( \frac{\gamma_j^{\rho_j\mu_j,\omega_j,b} - f_j(x)}{L^+_{jq}(x)} \right) \, dx \leq \left( \prod_{j=1}^{m} \int_y^b \int_y^b \int_y^b \Phi_j \left( \frac{f_j(y)}{\rho_j^{\rho_j\mu_j,\omega_j,b} - f_j(x)} \right) U_+^u(y) \, dy \right) \left( \prod_{j=1}^{m} \int_y^b \int_y^b \int_y^b \Phi_j \left( \frac{f_j(y)}{\rho_j^{\rho_j\mu_j,\omega_j,b} - f_j(x)} \right) U_+^u(y) \, dy \right).
\]

(51)

We make

**Remark 12** Here \( \rho_j, \mu_j, \gamma_j, \omega_j > 0 \); \( f_{ji} \in C\left([a,b]\right) \) and \( \psi \in C^1\left([a,b]\right) \) which is increasing; \( j = 1, \ldots, m \) and \( i = 1, \ldots, n \). Set

\[
\varphi_j^\infty(y) = \left\| \frac{\gamma_j^{\rho_j\mu_j,\omega_j,b} - f_j(y)}{L^+_{jq}(x)} \right\|_\infty = \max_{j=1,\ldots,m} \left\| \frac{\gamma_j^{\rho_j\mu_j,\omega_j,b} - f_j(y)}{L^+_{jq}(x)} \right\|_\infty,
\]

(52)

and

\[
\varphi_j^q(y) = \left\| \frac{\gamma_j^{\rho_j\mu_j,\omega_j,b} - f_j(y)}{L^+_{jq}(x)} \right\|_q = \left( \sum_{i=1}^{n} \left\| \frac{\gamma_j^{\rho_j\mu_j,\omega_j,b} - f_j(y)}{L^+_{jq}(x)} \right\|_q^q \right)^{\frac{1}{q}}, \quad q \geq 1;
\]

(53)

\( y \in [a,b] \), which \( \varphi_j^q \) are continuous functions, \( j = 1, \ldots, m \). We have also that

\[
0 < \varphi_j^q(y) < \infty \text{ in } [a,b],
\]

(54)

\( j = 1, \ldots, m \); where \( 1 \leq q \leq \infty \) is fixed.
\[ k_j(x,y) = \begin{cases} 
\psi'(y)(\psi(y) - \psi(x))^\alpha_j \left[ \frac{\omega_j(y)}{\omega_j(x)} \right] x \leq y < b, \\
0, a < y < x,
\end{cases} \quad (55) \]

\[ L_{jq}(x) := \int_x^b \psi'(y)(\psi(y) - \psi(x))^\alpha_j \left[ \frac{\omega_j(y)}{\omega_j(x)} \right] \phi_j(y) dy, \quad (56) \]

\[ \forall x \in [a,b], \quad 1 \leq q \leq \infty. \]

We have that \( L_{jq}(x) > 0 \) on \([a,b]\).

Let \( \rho \in \{1,...,m\} \) be fixed. The weight function \( u \) is chosen so that

\[ U_m^-(y) := \left( \prod_{j=1}^m \phi_j(y) \right)^{-1} \int_a^y \frac{u(x) \prod_{j=1}^m k_j(x,y)}{L_{jq}(x)} dx < \infty, \quad (57) \]

\[ \forall y \in [a,b], \text{ and that } U_m^- \text{ is integrable on } [a,b]. \]

A direct application of Theorem 5 gives:

**Theorem 13** It is all as in Remark 12. Here \( \Phi_j : \mathbb{R}_+^n \to \mathbb{R}_+ \), \( j = 1,...,m \), are convex functions increasing per coordinate. Then

\[ \int_a^b u(x) \prod_{j=1}^m \Phi_j \left( \frac{\psi'_j(x)\psi_j(x) - \phi_j(x)}{L_{jq}(x)} \right) dx \leq \int_a^b u(x) \prod_{j=1}^m \Phi_j \left( \frac{\psi'_j(x)\psi_j(x) - \phi_j(x)}{L_{jq}(x)} \right) L_{jq}(x) dy \quad (58) \]

We make

**Remark 14** Here \( j = 1,...,m; i = 1,...,n \). Let \( \rho_j, \mu_j, \omega_j > 0, \gamma_j < 0 \), and \( f_{ji} \in C^N_j([a,b]), N_j = [\mu_j], \mu_j \in \mathbb{N}; \theta := \max(N_1,...,N_m), \psi \in C^\theta([a,b]), \psi \) is increasing with \( \psi'(x) \neq 0 \) over \([a,b]\). Set

\[ f_{ji}^{N_j}(x) = \left( \frac{1}{\psi(x) dx} \right)^{N_j} f_{ji}(x), \quad x \in [a,b]. \]

Set

\[ \mathcal{L}_j^+(y) := \max_{j=1,...,m} \left\{ C_{\rho_j,\mu_j,\omega_j,a,f_{ji}}(y) \right\}, \quad (59) \]

and
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\[ q \lambda_{j^+}(y):= \left\| D^\gamma_{\rho_j, \mu_j, \omega_j, a^x + f_j(y)} \right\|_q := \left( \sum_{i=1}^n C D^\gamma_{\rho_j, \mu_j, \omega_j, a^x + f_j(y)} \right)^{\frac{1}{q}}, \quad q \geq 1; \]  

(60)
y \in [a, b], which all \( q \lambda_{j^+} \) are continuous functions, \( j = 1, \ldots, m \). We also have that

\[ 0 < q \lambda_{j^+}(y) < \infty \text{ in } [a, b], \]  

(61)

\( j = 1, \ldots, m \); where \( 1 \leq q \leq \infty \) is fixed.

Here it is

\[ c^k_j(x, y) := k_j(x, y) = \begin{cases} \psi'(y)(\psi(x) - \psi(y))^\gamma_{j^+ - \mu_j - 1} E_{\rho_j, \omega_j}^{\gamma_{j^+}} \left[ \omega_j(x) - \psi(y) \right] \lambda_{j^+}(y) dy, & a < y \leq x, \\ 0, & x < y < b, \end{cases} \]  

(62)

\( j = 1, \ldots, m \), and

\[ c^L_j(x) := \int_a^b \psi'(y)(\psi(x) - \psi(y))^\gamma_{j^+ - \mu_j - 1} E_{\rho_j, \omega_j}^{\gamma_{j^+}} \left[ \omega_j(x) - \psi(y) \right] \lambda_{j^+}(y) dy, \]  

(63)

\( \forall \ x \in [a, b], \ 1 \leq q \leq \infty, \ j = 1, \ldots, m \).

We have that \( c^L_j(x) > 0 \) on \([a, b]\).

Let \( \rho \in [1, \ldots, m] \) be fixed. The weight function \( u \) is chosen so that

\[ c^U_m(y) := \left( \prod_{j=1}^m \lambda_{j^+}(y) \right) \int_y^b \frac{u(x) \prod_{j=1}^m c^k_j(x, y)}{\prod_{j=1}^m c^L_j(x)} dx < \infty, \]  

(64)

\( \forall \ y \in [a, b], \) and that \( c^U_m \) is integrable on \([a, b]\).

A direct application of Theorem 11, see also (6), gives:

**Theorem 15** It is all as in Remark 14. Here \( \Phi_j : \mathbb{R}_+^n \rightarrow \mathbb{R}_+ \), \( j = 1, \ldots, m \), are convex functions increasing per coordinate. Then
\[
\int_a^b u(x) \prod_{j=1}^m \Phi_j \left( \frac{c D_{\rho_j, \mu_j, \omega_j, b}^{-j} f_j(x)}{L_{\rho_j}^-(x)} \right) dx \leq \int_a^b \prod_{j=1}^m \Phi_j \left( \frac{f_{j, y}^N(y)}{\nu_j \lambda_j^{-y}(y)} \right) dy \int_a^b \Phi_j \left( \frac{f_{\rho_j y}^N(y)}{\nu_j \lambda_j^{-y}(y)} \right) U_m(y) dy. \tag{65}
\]

We make

**Remark 16** Here \( j = 1, \ldots, m; i = 1, \ldots, n \). Let \( \rho_j, \mu_j, \omega_j > 0, \gamma_j < 0, \) and \( f_{j, y} \in C^N_j([a, b]), N_j = \left[ \mu_j \right], \mu_j \not\in \mathbb{N}; \theta := \max(N_1, \ldots, N_m), \psi \in C^\theta([a, b]), \psi \) is increasing with \( \psi'(x) \neq 0 \) over \([a, b] \). Set

\[
f_{j, y}^N(y) = \left( \frac{1}{\psi'(x)} \right)^N_j \frac{d f_{j, y}}{dx}, x \in [a, b] \].

Set

\[
\nu_j \lambda_j^{-y}(y) := \left\| D_{\rho_j, \mu_j, \omega_j, b}^{-j} f_j(y) \right\|_c = \max_{j=1, \ldots, m} \left\{ \left\| D_{\rho_j, \mu_j, \omega_j, b}^{-j} f_j(y) \right\|_c \right\},
\]

and

\[
\nu_j \lambda_j^{-y}(y) := \left\| D_{\rho_j, \mu_j, \omega_j, b}^{-j} f_j(y) \right\|_q = \left( \sum_{j=1}^n \left\| D_{\rho_j, \mu_j, \omega_j, b}^{-j} f_j(y) \right\|_q^q \right)^{\frac{1}{q}}, q \geq 1;
\]

\( y \in [a, b] \), which all \( \nu_j \lambda_j^{-} \) are continuous functions, \( j = 1, \ldots, m \). We also have that

\[
0 < \nu_j \lambda_j^{-}(y) < \infty \text{ in } [a, b],
\]

\( j = 1, \ldots, m, \) where \( 1 \leq q \leq \infty \) is fixed.

Here it is

\[
k_j^{-}(x, y) := k_j(x, y) = \begin{cases} 
\psi'(y) (\psi(y) - \psi(x))^N_j^{-1} E_{\rho_j, \mu_j, \omega_j, b}^{-j} [\omega_j (\psi(y) - \psi(x))^{p_j}] x \leq y < b, \\
0, a < y < x,
\end{cases}
\]

\( j = 1, \ldots, m, \) and

\[
L_{\rho_j}^-(x) := \int_a^b \psi'(y) (\psi(y) - \psi(x))^N_j^{-1} E_{\rho_j, \mu_j, \omega_j, b}^{-j} [\omega_j (\psi(y) - \psi(x))^{p_j}] \lambda_j^{-}(y) dy,
\]

\( \forall \ x \in [a, b], 1 \leq q \leq \infty, j = 1, \ldots, m \).

We have that \( \nu_j \lambda_j^{-}(x) > 0 \) on \([a, b] \).
Let \( \rho \in \{1, \ldots, m\} \) be fixed. The weight function \( u \) is chosen so that

\[
C U_m^-(y) := \left( \prod_{j=1}^{m} \lambda_j^-(y) \right) \int_a^y \frac{c k_j^-(x, y)}{\prod_{j=1}^{m} L_j^-(x)} \, dx < \infty,
\]

(71)

\( \forall \ y \in [a, b] \), and that \( C U_m^- \) is integrable on \([a, b]\).

A direct application of Theorem 13, see also (7), gives:

**Theorem 17** It is all as in Remark 16. Here \( \Phi_j : \mathbb{R}^+ \to \mathbb{R}^+ \), \( j = 1, \ldots, m \), are convex functions increasing per coordinate. Then

\[
\int_a^b u(x) \prod_{j=1}^{m} \Phi_j \left( \frac{c \left( \sum_{i=1}^{\rho} \int_a^b f_{ji}(x) \, dx \right)}{\prod_{j=1}^{\rho} \lambda_j^-(y)} \right) \, dy \leq \int_a^b \prod_{j=1}^{m} \Phi_j \left( \frac{\left( \sum_{i=1}^{\rho} \int_a^b f_{ji}(x) \, dx \right)}{\prod_{j=1}^{\rho} \lambda_j^-(y)} \right) \, dy.
\]

(72)

We make

**Remark 18** Here \( j = 1, \ldots, m \); \( i = 1, \ldots, n \). Let \( \rho_j, \mu_j, a_j > 0 \), \( \gamma_j < 0 \), and \( f_{ji} \in C([a, b]) \), \( N_j = \left[ \frac{\mu_j}{\rho_j} \right] \), \( \mu_j \notin \mathbb{N} \); \( \theta := \max(N_1, \ldots, N_m) \), \( \psi \in C^\theta([a, b]) \), \( \psi \) is increasing with \( \psi'(x) \neq 0 \) over \([a, b]\). Here \( 0 \leq \beta_j \leq 1 \) and \( \xi_j = \mu_j + \beta_j (N_j - \mu_j) \). We assume that \( \mathcal{D}_{\rho_j, \mu_j, a_j, a, f_{ji}}^{\gamma_j} \in C([a, b]) \), \( j = 1, \ldots, m \), \( i = 1, \ldots, n \). Set

\[
\mathcal{D}_{\rho_j, \mu_j, a_j, a, f_{ji}}^{\gamma_j} := \max_{j=1, \ldots, m} \left\{ \mathcal{D}_{\rho_j, \mu_j, a_j, a, f_{ji}}^{\gamma_j} \right\},
\]

(73)

and

\[
\mathcal{D}_{\rho_j, \mu_j, a_j, a, f_{ji}}^{\gamma_j} := \left( \sum_{i=1}^{n} \mathcal{D}_{\rho_j, \mu_j, a_j, a, f_{ji}}^{\gamma_j} \right)^\frac{1}{q}, \quad q \geq 1;
\]

(74)

\( y \in [a, b] \), which all \( q M_j^+ \) are continuous functions, \( j = 1, \ldots, m \). We also have that

\[
0 < q M_j^+(y) < \infty \text{ in } [a, b],
\]

(75)

\( j = 1, \ldots, m \); where \( 1 \leq q \leq \infty \) is fixed.

Here it is...
\[ p \kappa_j(x, y) := k_j(x, y) = \begin{cases} \psi'(y)(\psi(x) - \psi(y))^{\ell - \mu_j^{-1}} E_{\rho_j, \ell - \mu_j, j} \left[ \omega_j(x) - \psi(y) \right]^{\rho_j}, & a < y \leq x, \\ 0, & x < y < b, \end{cases} \]  
\( j = 1, \ldots, m, \) and

\[ p L_{jq}^+(x) := \int_a^x \psi'(y)(\psi(x) - \psi(y))^{\ell - \mu_j^{-1}} E_{\rho_j, \ell - \mu_j, j} \left[ \omega_j(x) - \psi(y) \right]^{\rho_j} M_j^+(y) dy, \]

\( \forall \ x \in [a, b], \ 1 \leq q \leq \infty. \)

We have that \( p L_{jq}^+(x) > 0 \) on \([a, b]\).

Let \( \rho \in \{1, \ldots, m\} \) be fixed. The weight function \( u \) is chosen so that

\[ p U_m^+(y) := \left( \prod_{j=1}^m M_j^+(y) \right) \int_y^b \frac{u(x) \prod_{j=1}^m p \kappa_j(x, y)}{\prod_{j=1}^m p L_{jq}^+(x)} \, dx < \infty, \]

\( \forall \ y \in [a, b], \) and that \( p U_m^+ \) is integrable on \([a, b]\).

A direct application of Theorem 11, see also (15), gives:

**Theorem 19** It is all as in Remark 18. Here \( \Phi_j : \mathbb{R}_+^n \rightarrow \mathbb{R}_+, \ j = 1, \ldots, m, \) are convex functions increasing per coordinate. Then

\[ \int_a^b u(x) \prod_{j=1}^m \Phi_j^\rho \left( \prod_{j=q}^\rho p L_{jq}^+(x) \right) dx \leq \prod_{j=1}^m \int_a^b \Phi_j^{\rho_j} \left( \prod_{j=q}^\rho p L_{jq}^+(x) \right) dy \]

\[ \left( \int_a^b \Phi_j^{\rho_j} \left( \prod_{j=q}^\rho p L_{jq}^+(x) \right) dy \right) \prod_{j=1}^m \Phi_j^\rho \left( \prod_{j=q}^\rho p L_{jq}^+(x) \right) dx. \]
Remark 20. Here \( j = 1, \ldots, m; \ i = 1, \ldots, n \). Let \( \rho_j, \mu_j, \omega_j > 0, \ \gamma_j < 0, \) and \( f_j \in C([a, b]), \ N_j = |\mu_j|, \ \mu_j \notin \mathbb{N}; \ \theta := \max(N_1, \ldots, N_n), \ \psi \in C^\theta([a, b]), \ \psi \) is increasing with \( \psi(x) \neq 0 \) over \( [a, b] \). Here \( 0 \leq \beta_j \leq 1 \) and 
\( \xi_j = \mu_j + \beta_j (N_j - \mu_j) \). We assume that \( D_{\rho_j, \mu_j, \omega_j}^{\gamma_j} f_j \in C([a, b]), \ j = 1, \ldots, m, \ i = 1, \ldots, n \). Set 
\[
M_{j-}(y) := \max_{i=1}^{m} \left\{ D_{\rho_j, \mu_j, \omega_j}^{\gamma_j} f_j(y) \right\}, \quad y \in [a, b], \quad \psi \psi > 0, \quad \text{all} \quad q M_{j-} \quad \text{are continuous functions,} \quad j = 1, \ldots, m. \quad \text{We also have that} 
\]
\[
0 < q M_{j-}(y) \leq \infty \quad \text{in} \quad [a, b], \quad j = 1, \ldots, m; \quad \text{where} \quad 1 \leq q \leq \infty \quad \text{is fixed.}
\]

Here it is
\[
\begin{align*}
\overline{p} k_j^-(x, y) &= k_j(x, y) = \left\{ \begin{array}{ll}
\xi_j^\gamma \psi(y)(\psi(y)-\psi(x))^{\xi_j-\mu_j} - 1 \ E_{\rho_j, \mu_j, \omega_j}^{\gamma_j} [\omega_j (\psi(y)-\psi(x))^{\gamma_j}], & x \leq y < b, \\
0, & a < y < x,
\end{array} \right.
\end{align*}
\]
\[
\overline{p} L_{jq}^- (x) := \int_x^b \overline{p} \psi'(y)(\psi(y)-\psi(x))^{\xi_j-\mu_j} - 1 \ E_{\rho_j, \mu_j, \omega_j}^{\gamma_j} [\omega_j (\psi(y)-\psi(x))^{\gamma_j}] M_{j-}(y)dy,
\]
\forall \ x \in [a, b], \ 1 \leq q \leq \infty.

We have that \( \overline{p} L_{jq}^-(x) > 0 \) on \([a, b]\).

Let \( \overline{p} \in \{1, \ldots, m\} \) be fixed. The weight function \( u \) is chosen so that
\[
\overline{p} U_m^-(y) := \left( \prod_{j=1}^{m} M_{j-}(y) \right) \int_a^y \overline{p} \left( \prod_{j=1}^{m} \overline{p} k_j^-(x, y) \right) dx < \infty,
\]
\forall \ y \in [a, b], \ and \ that \ \overline{p} U_m^- \text{ is integrable on} \ [a, b].

A direct application of Theorem 13, see also (16), gives:
Theorem 21 It is all as in Remark 20. Here $\Phi_j : \mathbb{R}_+^n \to \mathbb{R}_+$, $j = 1, \ldots, m$, are convex functions increasing per coordinate. Then

$$\int_a^b u(x) \prod_{j=1}^m \Phi_j \left( H D_{\gamma_j, \mu_j, \alpha_j, b} f_j(x) \right) dx \leq \prod_{j=1}^m \Phi_j \left( M_j(y) \right) \quad (86)$$

$$\int_a^b u_j \prod_{j=1}^m \Phi_j \left( \frac{RL D_{\gamma_j, \mu_j, \alpha_j, b} f_j(y)}{q M_j(y)} \right) \, dy.$$ 

We make

Remark 22 The basic background here is as in Remark 10. Also $q \varphi_j^+(y)$, $1 \leq q \leq \infty$, $y \in [a, b]$ is as in (45), (46), (47); $k_j^+(x, y)$ is as (48) and $L^+_j(x)$ as in (49), where $x, y \in [a, b]$. Here it is

$$K_j^+(x) := K_j(x) = (\psi(x) - \psi(a))^\mu_j E_{\rho_j, \sigma_j, +1} \left[ \omega_j (\psi(x) - \psi(a))^\nu_j \right], \quad (87)$$

$$\forall \ x \in [a, b], \ j = 1, \ldots, m. \ \text{Indeed it is}$$

$$\frac{k_j^+(x, y)}{K_j^+(x)} = \left\{ \begin{array}{ll}
\chi_{[a, x]}(y) \psi'(y) (\psi(x) - \psi(y))^{\mu_j - 1} & \\
(\psi(x) - \psi(a))^{\nu_j} & \\
(\psi(x) - \psi(a))^{\nu_j}
\end{array} \right\} \left\{ \begin{array}{ll}
E_{\rho_j, \sigma_j, +1} \left[ \omega_j (\psi(x) - \psi(y))^\nu_j \right] & \\
E_{\rho_j, \sigma_j, +1} \left[ \omega_j (\psi(x) - \psi(a))^\nu_j \right]
\end{array} \right\}. \quad (88)$$

$$\forall \ x, y \in [a, b], \ j = 1, \ldots, m; \ \chi \ \text{is the characteristic function.}$$

We define $q W_j$ on $[a, b]$, with appropiate choice of weight function $u$, by

$$q W_j(y) := q \varphi_j^+(y) \left( \int_y^b u(x) k_j^+(x, y) \frac{1}{K_j^+(x)} \, dx \right) < \infty, \quad (89)$$

$$\forall \ y \in [a, b], \ \text{and that} \ q W_j \ \text{is integrable on} \ [a, b]; \ j = 1, \ldots, m.$$

A direct application of Theorem 6, see also (2), follows:

Theorem 23 It is all as in Remark 22. Let $p_j > 1; \sum_{j=1}^m \frac{1}{p_j} = 1$. Let the functions $\Phi_j : \mathbb{R}_+^n \to \mathbb{R}_+$, $j = 1, \ldots, m$, be convex and increasing per coordinate. Then
\[ \int_a^b u(x) \prod_{j=1}^m \Phi_j \left( \frac{e^{T_j^u x} e_{\rho_j, \mu_j, \omega_j, \alpha_j, \gamma_j} f_j(x)}{L_{\gamma_j}(x)} \right) dx \leq \prod_{j=1}^m \left( \int_a^b W_{\gamma_j}(y) \Phi_j \left( \frac{f_j(y)}{q \Phi_{\gamma_j}(y)} \right)^{\rho_j} dy \right)^{\frac{1}{\rho_j}}. \]  

(90)

We make

**Remark 24** The basic background here is as in Remark 12. Also \( q \varphi_{\gamma_j}(y) \), \( 1 \leq q \leq \infty \), \( y \in [a, b] \) is as in (52), (53), (54); \( k_j(x, y) \) is as (55) and \( L_{\gamma_j}(x) \) as in (56), where \( x, y \in [a, b] \). Here it is

\[ K_{\gamma_j}(x) := K_j(x) = (\psi(b) - \psi(x))^{\rho_j} \frac{E_{\rho_j, \mu_j}^x \left( \omega_j (\psi(b) - \psi(x))^{\rho_j} \right)}{E_{\rho_j, \mu_j}^x \left( \omega_j (\psi(b) - \psi(x))^{\rho_j} \right)}. \]  

(91)

\( \forall \ x \in [a, b], \ j = 1, \ldots, m \). Indeed it is

\[ \frac{k_j(x, y)}{K_{\gamma_j}(x)} = \left( \chi_{x, b-y}(y) \psi(y) \right)^{\rho_j-1} \left( \frac{E_{\rho_j, \mu_j}^x \left( \omega_j (\psi(y) - \psi(x))^{\rho_j} \right)}{E_{\rho_j, \mu_j}^x \left( \omega_j (\psi(y) - \psi(x))^{\rho_j} \right)} \right), \]  

(92)

\( \forall \ x, y \in [a, b], \ j = 1, \ldots, m \).

We define \( q W_{\gamma_j} \) on \( [a, b] \), with appropriate choice of weight function \( u \), by

\[ q W_{\gamma_j}(y) := q \varphi_{\gamma_j}(y) \left( \int_a^y u(x) \frac{k_j(x, y)}{K_{\gamma_j}(x)} dx \right) < \infty, \]  

(93)

\( \forall \ y \in [a, b] \), and that \( q W_{\gamma_j} \) is integrable on \( [a, b] \); \( j = 1, \ldots, m \).

A direct application of Theorem 6, see also (3), follows:

**Theorem 25** It is all as in Remark 24. Let \( p_j > 1; \sum_{j=1}^m \frac{1}{p_j} = 1 \). Let the functions \( \Phi_j : \mathbb{R}_+^m \to \mathbb{R}_+ \), \( j = 1, \ldots, m \), be convex and increasing per coordinate. Then

\[ \int_a^b u(x) \prod_{j=1}^m \Phi_j \left( \frac{e^{T_j^u x} e_{\rho_j, \mu_j, \omega_j, \alpha_j, \gamma_j} f_j(x)}{L_{\gamma_j}(x)} \right) dx \leq \prod_{j=1}^m \left( \int_a^b W_{\gamma_j}(y) \Phi_j \left( \frac{f_j(y)}{q \varphi_{\gamma_j}(y)} \right)^{\rho_j} dy \right)^{\frac{1}{\rho_j}}. \]  

(94)

We need

**Remark 26** The basic background here is as in Remark 14. Also \( q \lambda_{\gamma_j}(y) \), \( 1 \leq q \leq \infty \), \( y \in [a, b] \) is as in (59), (60), (61); \( C \ k_j^+(x, y) \) is as (62) and \( C \ L_{\gamma_j}^+(x) \) as in (63), where \( x, y \in [a, b] \). Here it is

\[ \int_a^b u(x) \prod_{j=1}^m \Phi_j \left( \frac{e^{T_j^u x} e_{\rho_j, \mu_j, \omega_j, \alpha_j, \gamma_j} f_j(x)}{L_{\gamma_j}(x)} \right) dx \leq \prod_{j=1}^m \left( \int_a^b W_{\gamma_j}(y) \Phi_j \left( \frac{f_j(y)}{q \varphi_{\gamma_j}(y)} \right)^{\rho_j} dy \right)^{\frac{1}{\rho_j}}. \]  

(90)
\[ C K^+_j(x) := K_j(x) = (\psi(x) - \psi(a))^{\eta_{j-1} - \mu_j} E_{\rho_j, N_{j-1}^{\eta_{j-1}}}^{-\tau_j} \left[ \omega_j (\psi(x) - \psi(a))^{\nu_j} \right] \]

\[ \forall \ x \in [a, b], \ j = 1, \ldots, m. \] Indeed it is

\[ \frac{\partial}{\partial K_j(x)} = \frac{1}{\eta_{j-1} - \mu_j} \left( \chi_{(a,b)}(y) \left( \psi(x) - \psi(y) \right)^{\eta_{j-1} - \mu_j - 1} \right) \frac{E_{\rho_j, N_{j-1}^{\eta_{j-1}}}^{-\tau_j} \left[ \omega_j (\psi(x) - \psi(y))^{\nu_j} \right]}{E_{\rho_j, N_{j-1}}^{-\tau_j} \left[ \omega_j (\psi(x) - \psi(a))^{\nu_j} \right]} \]

\[ \forall \ x, y \in [a, b], \ j = 1, \ldots, m. \]

We define \( C W_{j+} \) on \([a, b]\), with appropriate choice of weight function \( u \), by

\[ C W_{j+} (y) := \lambda_j (y) \left( \int_y^b u(x) \frac{C k_j(x,y)}{C K_j(x)} \, dx \right) < \infty, \]

\[ \forall \ y \in [a, b], \ \text{and that} \ C W_{j+} \ \text{is integrable on} \ [a, b]; \ j = 1, \ldots, m. \]

A direct application of Theorem 23, see also (6), follows:

**Theorem 27** It is all as in Remark 26. Let \( p_j > 1; \sum_{j=1}^m \frac{1}{p_j} = 1 \). Let the functions \( \Phi_j : \mathbb{R}_+^m \to \mathbb{R}_+ \), \( j = 1, \ldots, m \), be convex and increasing per coordinate. Then

\[ \int_a^b u(x) \prod_{j=1}^m \left( \frac{C D_{\rho_j, \mu_j, a_j} f_j(x)}{C L_{\eta_j}(x)} \right) \, dx \leq \prod_{j=1}^m \left( \int_{a_q^j} W_{j+} (y) \Phi_j \left( \frac{f_j^j(N_j)(y)}{\lambda_j(y)} \right)^{\frac{1}{p_j}} \, dy \right). \]

We need

**Remark 28** The basic background here is as in Remark 16. Also \( \lambda_j(y), \ 1 \leq q \leq \infty, \ y \in [a, b] \) is as in (66), (67), (68); \( C k_j^- (x, y) \) is as (69) and \( C L_{\eta_j}(x) \) as in (70), where \( x, y \in [a, b] \). Here it is

\[ C K_j^- (x) := K_j(x) = (\psi(b) - \psi(x))^{\eta_{j-1} - \mu_j} E_{\rho_j, N_{j-1}^{\eta_{j-1}}}^{-\tau_j} \left[ \omega_j (\psi(b) - \psi(x))^{\nu_j} \right] \]

\[ \forall \ x \in [a, b], \ j = 1, \ldots, m. \] Indeed it is

\[ \frac{\partial}{\partial K_j^-} = \frac{1}{\eta_{j-1} - \mu_j} \left( \chi_{(a,b)}(y) \left( \psi(y) - \psi(x) \right)^{\eta_{j-1} - \mu_j - 1} \right) \frac{E_{\rho_j, N_{j-1}^{\eta_{j-1}}}^{-\tau_j} \left[ \omega_j (\psi(y) - \psi(x))^{\nu_j} \right]}{E_{\rho_j, N_{j-1}^{\eta_{j-1}}}^{-\tau_j} \left[ \omega_j (\psi(b) - \psi(x))^{\nu_j} \right]} \]

\[ \forall \ x, y \in [a, b], \ j = 1, \ldots, m. \]
We define $c_q W_{j-}$ on $[a,b]$, with appropriate choice of weight function $u$, by

$$c_q W_{j-}(y) := q \lambda_{j-}(y) \left( \int_a^y u(x) c^{-1} k_{j-}(x,y) \right) < \infty,$$

(101)

$\forall \ y \in [a,b]$, and that $c_q W_{j-}$ is integrable on $[a,b]; \ j=1,...,m$.

A direct application of Theorem 25, see also (7), follows:

**Theorem 29** It is all as in Remark 28. Let $p_j > 1; \sum_{j=1}^m \frac{1}{p_j} = 1$. Let the functions $\Phi_j : \mathbb{R}^n \to \mathbb{R}_+$, $j = 1,...,m$, be convex and increasing per coordinate. Then

$$\int_a^b u(x) \prod_{j=1}^m \Phi_j \left( \int_a^b W_{j-}(y) \Phi_j \left( \frac{1}{p_j} \right) \right) dx \leq \prod_{j=1}^m \int_a^b W_{j-}(y) \Phi_j \left( \frac{1}{p_j} \right) dy.$$

(102)

We need

**Remark 30** The basic background here is as in Remark 18. Also $q M_{j+}(y)$, $1 \leq q \leq \infty$, $y \in [a,b]$ is as in (73), (74), (75); $p_j k^+(x,y)$ is as (76) and $p_j L_{jq}^-(x)$ as in (77), where $x,y \in [a,b]$. Here it is

$$p_j K_{j+}^-(x) := K_{j+}(x) = (\psi(x)-\psi(a))^\beta_j (-1)E_{\alpha_j,\beta_j+1} \left[ \omega_j (\psi(x)-\psi(a))^{\alpha_j} \right],$$

(103)

$\forall \ x \in [a,b], \ j=1,...,m$. Indeed it is

$$\frac{p_j k^+(x,y)}{p_j K^{+}_{j+}(x)} = \left( \chi_{(a,x)}(y) \psi'(y) (\psi(x)-\psi(y))^{\beta_j-\alpha_j-1} \right) \left( (\psi(x)-\psi(a))^{\beta_j-\alpha_j} \right),$$

(104)

$\forall \ x,y \in [a,b], \ j=1,...,m$.

We define $p_q W_{j+}$ on $[a,b]$, with appropriate choice of weight function $u$,

$$p_q W_{j+}(y) := q M_{j+}(y) \left( \int_a^y u(x) p q_k^+(x,y) \right) < \infty,$$

(105)

$\forall \ y \in [a,b]$, and that $p_q W_{j+}$ is integrable on $[a,b]; \ j=1,...,m$.

A direct application of Theorem 23, see also (15), follows:
Theorem 31 It is all as in Remark 30. Here \( \Phi_j : \mathbb{R}^+_x \to \mathbb{R}_+ \), \( j = 1, \ldots, m \), are convex functions increasing per coordinate. Then

\[
\int_a^b u(x) \prod_{j=1}^m \Phi_j \left( \frac{\int \mathbb{D}_{\mathbf{x}, \mathbf{x}'} \gamma_j^\beta_j \, \phi_j^p \, \phi_j^{\mathbb{D}} \left( x \right) \mathbb{D}_{z_j, \mathbf{x}'} \gamma_j^\beta_j \, \phi_j^p \, \phi_j^{\mathbb{D}} \left( x \right) \mathbb{D}_{x_j, \mathbf{x}'} \gamma_j^\beta_j \, \phi_j^p \, \phi_j^{\mathbb{D}} \left( x \right) \right) \right) \, dx \leq \prod_{j=1}^m \int_a^b \mathcal{W}_{j^*}(y) \phi_j \left( \frac{\int \mathbb{D}_{\mathbf{x}, \mathbf{x}'} \gamma_j^\beta_j \, \phi_j^p \, \phi_j^{\mathbb{D}} \left( x \right) \mathbb{D}_{z_j, \mathbf{x}'} \gamma_j^\beta_j \, \phi_j^p \, \phi_j^{\mathbb{D}} \left( x \right) \mathbb{D}_{x_j, \mathbf{x}'} \gamma_j^\beta_j \, \phi_j^p \, \phi_j^{\mathbb{D}} \left( x \right) \right) \, dy \right) \prod_{j=1}^m \int_a^b \mathcal{W}_{j^*}(y) \phi_j \left( \frac{\int \mathbb{D}_{\mathbf{x}, \mathbf{x}'} \gamma_j^\beta_j \, \phi_j^p \, \phi_j^{\mathbb{D}} \left( x \right) \mathbb{D}_{z_j, \mathbf{x}'} \gamma_j^\beta_j \, \phi_j^p \, \phi_j^{\mathbb{D}} \left( x \right) \mathbb{D}_{x_j, \mathbf{x}'} \gamma_j^\beta_j \, \phi_j^p \, \phi_j^{\mathbb{D}} \left( x \right) \right) \, dy \right) \). (106)

We need

Remark 32 The basic background here is as in Remark 20. Also \( q \, M_{j^*}(y) \), \( 1 \leq q \leq \infty \), \( y \in [a, b] \) is as in (80), (81), (82); \( p \, k_j^-(x, y) \) is as in (83) and \( p \, L_{j^*}(x) \) as in (84), where \( x, y \in [a, b] \). Here it is

\[
p \, K_j^-(x) := K_j(x) = (\psi(b) - \psi(x))^p \, \phi_j^p \, \phi_j^{\mathbb{D}} \left( x \right) \mathbb{D}_{z_j, \mathbf{x}'} \gamma_j^\beta_j \, \phi_j^p \, \phi_j^{\mathbb{D}} \left( x \right) \mathbb{D}_{x_j, \mathbf{x}'} \gamma_j^\beta_j \, \phi_j^p \, \phi_j^{\mathbb{D}} \left( x \right) \right) \right) \). (107)

\( \forall \ x \in [a, b], \ j = 1, \ldots, m \). Indeed it is

\[
q \, W_{j^-}(y) := q \, M_{j^-}(y) \left( \int_a^b u(x) \, p \, k_j^-(x, y) \right) \, dx < \infty \), (109)

\( \forall \ y \in [a, b] \), and that \( p \, W_{j^-} \) is integrable on \([a, b]: j = 1, \ldots, m \).

A direct application of Theorem 25, see also (16), follows:

Theorem 33 It is all as in Remark 32. Here \( \Phi_j : \mathbb{R}^+_x \to \mathbb{R}_+ \), \( j = 1, \ldots, m \), are convex functions increasing per coordinate. Then

\[
\int_a^b u(x) \prod_{j=1}^m \Phi_j \left( \frac{\int \mathbb{D}_{\mathbf{x}, \mathbf{x}'} \gamma_j^\beta_j \, \phi_j^p \, \phi_j^{\mathbb{D}} \left( x \right) \mathbb{D}_{z_j, \mathbf{x}'} \gamma_j^\beta_j \, \phi_j^p \, \phi_j^{\mathbb{D}} \left( x \right) \mathbb{D}_{x_j, \mathbf{x}'} \gamma_j^\beta_j \, \phi_j^p \, \phi_j^{\mathbb{D}} \left( x \right) \right) \right) \, dx \leq \prod_{j=1}^m \int_a^b \mathcal{W}_{j^-}(y) \phi_j \left( \frac{\int \mathbb{D}_{\mathbf{x}, \mathbf{x}'} \gamma_j^\beta_j \, \phi_j^p \, \phi_j^{\mathbb{D}} \left( x \right) \mathbb{D}_{z_j, \mathbf{x}'} \gamma_j^\beta_j \, \phi_j^p \, \phi_j^{\mathbb{D}} \left( x \right) \mathbb{D}_{x_j, \mathbf{x}'} \gamma_j^\beta_j \, \phi_j^p \, \phi_j^{\mathbb{D}} \left( x \right) \right) \, dy \right) \prod_{j=1}^m \int_a^b \mathcal{W}_{j^-}(y) \phi_j \left( \frac{\int \mathbb{D}_{\mathbf{x}, \mathbf{x}'} \gamma_j^\beta_j \, \phi_j^p \, \phi_j^{\mathbb{D}} \left( x \right) \mathbb{D}_{z_j, \mathbf{x}'} \gamma_j^\beta_j \, \phi_j^p \, \phi_j^{\mathbb{D}} \left( x \right) \mathbb{D}_{x_j, \mathbf{x}'} \gamma_j^\beta_j \, \phi_j^p \, \phi_j^{\mathbb{D}} \left( x \right) \right) \, dy \right) \). (110)
We make

**Remark 34** Let \( f_i \in C([a, b]), i = 1, \ldots, n \), and \( \vec{f} = (f_1, \ldots, f_n) \). We set

\[
\|\vec{f}(y)\|_q := \max \|f_1(y), \ldots, f_n(y)\|_
\]

and

\[
\|\vec{f}(y)\|_q := \left( \sum_{i=1}^n |f_i(y)|^q \right)^{1/q}, q \geq 1; y \in [a, b].
\]

(111)

Clearly it is \( \|\vec{f}(y)\|_q \in C([a, b]), \) for all \( 1 \leq q \leq \infty \). We assume that \( \|\vec{f}(y)\|_q > 0 \), a.e. on \((a, b)\), for \( q \in [1, \infty) \) being fixed.

Let

\[
L_q^+(x) := \int_a^x k^+(x, y)\|\vec{f}(y)\|_q \, dy, x \in [a, b].
\]

(112)

\( 1 \leq q \leq \infty \) fixed.

We assume \( L_q^+(x) > 0 \) a.e. on \((a, b)\).

Here we considered

\[
k^+(x, y) := k(x, y) := \begin{cases} \psi^\prime(y)(\psi(x) - \psi(y))^{\alpha-1} E_{\psi,\psi}(\psi(x) - \psi(y))^{\alpha}, a < y \leq x, \\ 0, x < y < b, \end{cases}
\]

(113)

where \( \rho, \mu, \gamma, \omega > 0; \psi \in C^1([a, b]) \) which is increasing.

The weight function \( u \) is chosen so that

\[
W_q^+(y) := \|\vec{f}(y)\| \left( \int_y^b \frac{u(x)k^+(x, y)}{L_q^+(x)} \, dx \right) < \infty,
\]

(114)

a.e. on \((a, b)\) and that \( W_q^+ \) is integrable on \([a, b]\).

A direct application of Theorem 8 produces:

**Theorem 35** Let all as in Remark 34. Here \( \Phi: \mathbb{R}_+^n \to \mathbb{R} \) is a convex and increasing per coordinate function. Then
\[ \int_{a}^{b} u(x) \Phi \left( \frac{e^{\int_{a}^{x} f(t) \, dt}}{L_{q}(x)} \right) \, dx \leq \int_{a}^{b} W_{q}^{\ast}(y) \Phi \left( \frac{f(y)}{f(y)} \right) \, dy. \quad (115) \]

We make

**Remark 36** Let \( f_{i} \in C([a, b]), \ i = 1, \ldots, n, \) and \( \mathbf{f} = (f_{1}, \ldots, f_{n}) \). We set

\[
\left\| f(y) \right\|_{q} := \max \left\{ |f_{i}(y)|, \ldots, |f_{n}(y)| \right\},
\]

and

\[
\left\| f(y) \right\|_{q} := \left( \sum_{i=1}^{n} |f_{i}(y)|^{q} \right)^{\frac{1}{q}}, \ q \geq 1; \ y \in [a, b].
\] (116)

Clearly it is \( \left\| f(y) \right\|_{q} \in C([a, b]), \) for all \( 1 \leq q \leq \infty. \) We assume that \( \left\| f(y) \right\|_{q} > 0, \) a.e. on \( (a, b), \) for \( q \in [1, \infty] \)

being fixed.

Let

\[
L_{q}^{-}(x) := \int_{x}^{b} k^{-}(x, y) \left\| f(y) \right\|_{q} \, dy, \ x \in [a, b].
\] (117)

\[ 1 \leq q \leq \infty \] fixed.

We assume \( L_{q}^{-}(x) > 0 \) a.e. on \( (a, b). \)

Here we considered

\[
k^{-}(x, y) := k(x, y) := \begin{cases} 
\psi'(y)(\psi(y) - \psi(x))^{\mu - 1} E_{\rho, \mu}[\omega(\psi(y) - \psi(x))^{\rho}], & x \leq y < b, \\
0, & a < y < x,
\end{cases}
\] (118)

where \( \rho, \mu, \gamma, \omega > 0; \psi \in C^{1}([a, b]) \) which is increasing.

The weight function \( u \) is chosen so that

\[
W_{q}^{-}(y) := \left\| f(y) \right\|_{q} \left( \int_{a}^{y} \frac{u(x)k^{-}(x, y)}{L_{q}(x)} \, dx \right) < \infty,
\] (119)

a.e. on \( (a, b) \) and that \( W_{q}^{-} \) is integrable on \([a, b].\)

A direct application of Theorem 8 produces:

**Theorem 37** Let all as in Remark 36. Here \( \Phi : \mathbb{R}_{+}^{\ast} \to \mathbb{R} \) is a convex and increasing per coordinate function. Then
Next we deal with the spherical shell:

**Background 38** We need:

Let $N \geq 2$, $S^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}$ be the unit sphere on $\mathbb{R}^N$, where $|| \cdot ||$ stands for the Euclidean norm in $\mathbb{R}^N$. Also denote the ball $B(0,R) := \{x \in \mathbb{R}^N : |x| < R \} \subseteq \mathbb{R}^N$, $R > 0$, and the spherical shell

$$A := B(0,R_2) - B(0,R_1), 0 < R_1 < R_2.$$  \hfill (121)

For the following see [12, pp. 149-150], and [13, pp. 87-88].

For $x \in \mathbb{R}^N - \{0\}$ we can write uniquely $x = r\omega$, where $r = |x| > 0$, and $\omega = x/r \in S^{N-1}$, $|\omega| = 1$.

Clearly here

$$\mathbb{R}^N - \{0\} = (0, \infty) \times S^{N-1},$$  \hfill (122)

and

$$\overline{A} = [R_1, R_2] \times S^{N-1}. $$ \hfill (123)

We will be using

**Theorem 39** ([1, p. 322]) Let $f : A \to \mathbb{R}$ be a Lebesgue integrable function. Then

$$\int_{A} f(x)dx = \int_{S^{N-1}} \left( \int_{R_1}^{R_2} f(r\omega)r^{N-1}dr \right)d\omega.$$ \hfill (124)

So we are able to write an integral on the shell in polar form using the polar coordinates $(r, \omega)$.

We need

**Definition 40** Let $\rho, \mu, \gamma, w > 0$; $f \in C(\overline{A})$ and $\psi \in C^1([R_1, R_2])$ which is increasing. The left and right radial Prabhakar fractional integrals with respect to $\psi$ are defined as follows:

$$\left( e_{\rho, \mu, w, R_1}^\gamma f \right)(x) = \int_{R_1}^{x} \psi'(t)(\psi(r) - \psi(t))^{\mu-1} E_{\rho, \mu}^\gamma \left[ w(\psi(r) - \psi(t))^\nu \right] f(t\omega)dt,$$ \hfill (125)

and

$$\left( e_{\rho, \mu, w, R_2}^\gamma f \right)(x) = \int_{R_2}^{x} \psi'(t)(\psi(r) - \psi(t))^{\mu-1} E_{\rho, \mu}^\gamma \left[ w(\psi(r) - \psi(t))^\nu \right] f(t\omega)dt,$$ \hfill (126)

where $x \in \overline{A}$, that is $x = r\omega$, $r \in [R_1, R_2]$, $\omega \in S^{N-1}$.
Based on [1], p. 288 and [2, 4], we have that (125), (126) are continuous functions over \( \overline{A} \) when \( \mu \geq 1 \).

We make

**Remark 41** Let \( f_i \in C(\overline{A}) \), where the shell \( A \) is as in (121), \( i = 1, \ldots, n \), and \( f = (f_1, \ldots, f_n) \). We set

\[
\begin{align*}
\left\| \overline{f}(y) \right\|_\infty & := \max \{ f_i(y) \mid i = 1, \ldots, n \}, \\
\left\| \overline{f}(y) \right\|_q & := \left( \sum_{i=1}^{n} |f_i(y)|^q \right)^{1/q}, \quad q \geq 1; \quad y \in \overline{A}.
\end{align*}
\]

(127)

Clearly it is \( \left\| \overline{f}(y) \right\|_q \in C(\overline{A}) \), \( 1 \leq q \leq \infty \). One can write that

\[
\left\| \overline{f}(y) \right\|_q = \left\| \overline{f}(t\omega) \right\|_q, \quad 1 \leq q \leq \infty,
\]

(128)

where \( t \in [R_1, R_2] \), \( \omega \in S^{N-1} \); \( y = t\omega \), by Background 38.

We assume that \( \left\| \overline{f}(y) \right\|_q > 0 \) on \( \overline{A} \), \( 1 \leq q \leq \infty \) fixed.

Consider the kernel

\[
k^*_r(r,t) := k^*_r(r,t) := \chi^{r} \psi^{r}(t)(\psi^{r}(r) - \psi^{r}(t))^{\rho-1} E^{\psi^{r}}_{\rho,\mu} \left[ w(\psi^{r}(r) - \psi^{r}(t))^{\rho} \right].
\]

(129)

where \( \rho, \mu, \gamma, w > 0; \psi \in C^{1}([R_1, R_2]) \) which is increasing.

Let

\[
L^*_q(x) = L^*_q(r\omega) = \int_{R_1}^{R_2} k^*_r(r,t) \left\| \overline{f}(t\omega) \right\|_q dt,
\]

(130)

\( x = r\omega \in \overline{A}, \quad 1 \leq q \leq \infty \) fixed; \( r \in [R_1, R_2] \), \( \omega \in S^{N-1} \).

We have that \( L^*_q(r\omega) > 0 \) for \( r \in (R_1, R_2) \), for every \( \omega \in S^{N-1} \).

Here we choose the weight \( u(x) = u(r\omega) = L^*_q(r\omega) \).

Consider the function

\[
W^*_q(v) = W^*_q(t\omega) = \left\| \overline{f}(t\omega) \right\|_q \left( \int_{R_1}^{R_2} k^*_r(r,t) dr \right) < \infty,
\]

(131)

\( \forall \ t \in [R_1, R_2], \ \omega \in S^{N-1} \); and \( W^*_q(t\omega) \) is integrable over \( [R_1, R_2] \), \( \forall \ \omega \in S^{N-1} \).

Here \( \Phi : R_+^n \to R \) is a convex and increasing per coordinate function. By (115) we obtain
\[ \int_{R_{1}}^{R_{2}} L_{q^{*}}^{r}(r \omega) \Phi \left( \frac{e^{r^{w}}_{p, \mu, w, R_{1}} + f(r \omega)}{L_{q^{*}}^{r}(r \omega)} \right) dr \leq \int_{R_{1}}^{R_{2}} W_{q^{*}}^{r}(t \omega) \Phi \left( \frac{f(t \omega)}{\|f(t \omega)\|_{q}} \right) dt, \quad (132) \]

\[ \forall \ \omega \in S^{N-1}. \]

Here we have \( R_{1} \leq r \leq R_{2}, \) and \( R_{1}^{N-1} \leq r^{N-1} \leq R_{2}^{N-1}, \) and \( R_{1}^{1-N} \leq r^{1-N} \leq R_{2}^{1-N}, \) also \( r^{N-1}r^{1-N} = 1. \) Thus by (132), we have

\[ \int_{R_{1}}^{R_{2}} L_{q^{*}}^{r}(r \omega) \Phi \left( \frac{e^{r^{w}}_{p, \mu, w, R_{1}} + f(r \omega)}{L_{q^{*}}^{r}(r \omega)} \right) r^{N-1} dr \leq \left( \frac{R_{2}}{R_{1}} \right)^{N-1} \int_{R_{1}}^{R_{2}} W_{q^{*}}^{r}(r \omega) \Phi \left( \frac{f(r \omega)}{\|f(r \omega)\|_{q}} \right) r^{N-1} dr, \quad (133) \]

\[ \forall \ \omega \in S^{N-1}. \]

Therefore it holds

\[ \int_{S^{N-1}} \left( \int_{R_{1}}^{R_{2}} L_{q^{*}}^{r}(r \omega) \Phi \left( \frac{e^{r^{w}}_{p, \mu, w, R_{1}} + f(r \omega)}{L_{q^{*}}^{r}(r \omega)} \right) r^{N-1} dr \right) d\omega \leq \left( \frac{R_{2}}{R_{1}} \right)^{N-1} \int_{R_{1}}^{R_{2}} \left( \int_{S^{N-1}} W_{q^{*}}^{r}(r \omega) \Phi \left( \frac{f(r \omega)}{\|f(r \omega)\|_{q}} \right) r^{N-1} dr \right) d\omega. \quad (134) \]

Using Theorem 39 we derive:

**Theorem 42** All as in Remark 41. Then

\[ \int_{A} L_{q^{*}}^{r}(x) \Phi \left( \frac{e^{r^{w}}_{p, \mu, w, R_{1}} + f(x)}{L_{q^{*}}^{r}(x)} \right) dx \leq \left( \frac{R_{2}}{R_{1}} \right)^{N-1} \int_{A} W_{q^{*}}^{r}(x) \Phi \left( \frac{f(x)}{\|f(x)\|_{q}} \right) dx, \quad (135) \]

where \( e^{r^{w}}_{p, \mu, w, R_{1}} + f(x) = \left( e^{r^{w}}_{p, \mu, w, R_{1}} + f_{1}(x), \ldots, e^{r^{w}}_{p, \mu, w, R_{1}} + f_{n}(x) \right) \) and coordinates are assumed to be continuous functions on \( A. \)

We make

**Remark 43** Let \( f_{i} \in C(A), \) where the shell \( A \) is as in (121), \( i = 1, \ldots, n, \) and \( f = (f_{1}, \ldots, f_{n}). \) We set

\[ \|f(y)\| = \max \|f_{i}(y)\|, \]

and

\[ \left\| \tilde{f}(y) \right\|_{q} = \left( \sum_{i=1}^{n} \|f_{i}(y)\|^{q} \right)^{\frac{1}{q}}, \quad q \geq 1; y \in \bar{A}. \quad (136) \]

Clearly it is \( \left\| \tilde{f}(y) \right\|_{q} \in C(\bar{A}), \) \( 1 \leq q \leq \infty. \) One can write that
where $t \in [R_1, R_2]$, $\omega \in S^{N-1}$; $y = t \omega$, by Background 38.

We assume that $\left\| \tilde{f}(y) \right\|_q > 0$ on $\overline{A}$, $1 \leq q \leq \infty$ fixed.

Consider the kernel

$$k^*_{\gamma}(r,t) = k(r,t) := x_{(r,t)}(\|\| \psi'(t)(\psi(t) - \psi(r))^{\mu - 1} E_{p,\mu}^{\omega} \| \left[ w(\psi(t) - \psi(r)) \right]^\rho \right\}$$

where $\rho, \mu, \gamma, w > 0$; $\psi \in C^1([R_1, R_2])$ which is increasing.

Let

$$L^*_{q^*}(x) = L^*_{q^*}(r\omega) = \int_{R_1}^{R_2} k^*_{\gamma}(r,t) \left\| f(t \omega) \right\|_q dt,$$

$x = r \omega \in \overline{A}$, $1 \leq q \leq \infty$ fixed; $r \in [R_1, R_2]$, $\omega \in S^{N-1}$.

We have that $L^*_{q^*}(r\omega) > 0$ for $r \in (R_1, R_2)$, for every $\omega \in S^{N-1}$.

Here we choose the weight $u(x) = u(r\omega) = L^*_{q^*}(r\omega)$.

Consider the function

$$W^*_{q^*}(y) = W^*_{q^*}(t \omega) = \left\| f(t \omega) \right\|_q \left( \int_{R_1}^{R_2} k^*_{\gamma}(r,t) dr \right) < \infty,$$

\forall $t \in [R_1, R_2]$, $\omega \in S^{N-1}$; and $W^*_{q^*}(t \omega)$ is integrable over $[R_1, R_2]$, $\forall$ $\omega \in S^{N-1}$.

Here $\Phi : R_+^n \rightarrow R$ is a convex and increasing per coordinate function. By (120) we obtain

$$\int_{R_1}^{R_2} L^*_{q^*}(r \omega) \Phi \left( \frac{e^{\gamma W_{q^*}}} {R_{q^*}(r \omega)} f(r \omega) \right) dr \leq \int_{R_1}^{R_2} W^*_{q^*}(t \omega) \Phi \left( \frac{f(t \omega)} {L^*_{q^*}(t \omega)} \right) dt,$$

\forall $\omega \in S^{N-1}$.

Here we have $R_1 \leq r \leq R_2$, and $R_1^{N-1} \leq r^{N-1} \leq R_2^{N-1}$, and $R_2^{1-N} \leq r^{1-N} \leq R_1^{1-N}$, also $r^{N-1} r^{1-N} = 1$. Thus by (141), we have

$$\int_{R_1}^{R_2} L^*_{q^*}(r \omega) \Phi \left( \frac{e^{\gamma W_{q^*}}} {L^*_{q^*}(r \omega)} f(r \omega) \right) r^{N-1} dr \leq \left( \frac{R_2}{R_1} \right)^{N-1} \int_{R_1}^{R_2} W^*_{q^*}(r \omega) \Phi \left( \frac{f(r \omega)} {L^*_{q^*}(r \omega)} \right) r^{N-1} dr,$$

\forall $\omega \in S^{N-1}$.
Therefore it holds

\[
\int_{S^{N-1}} \left[ \int_{R_1}^{R_2} \left( \frac{\int_{|r|}^{r} f(r') dr'}{L_{q}(r)} \right)^{N-1} dr \right] d\omega \leq \left( \frac{R_2}{R_1} \right)^{N-1} \int_{S^{N-1}} \left[ \int_{R_1}^{R_2} \left( \frac{f(r)}{L_{q}(r)} \right)^{N-1} dr \right] d\omega.
\] (143)

Using Theorem 39 we derive:

**Theorem 44** All as in Remark 43. Then

\[
\int_{A^c} \left( \frac{\int_{|r|}^{r} f(x) dr}{L_{q}(x)} \right) dx \leq \left( \frac{R_2}{R_1} \right)^{N-1} \int_{A^c} \left( \frac{f(x)}{L_{q}(x)} \right) dx,
\] (144)

where

\[
\frac{\int_{|r|}^{r} f(x) dr}{L_{q}(x)} = \left( \frac{\int_{|r|}^{r} f(x) dr}{L_{q}(x)} \right)_{\rho,\mu,w,R_2} \frac{f(x)}{L_{q}(x)}
\]

and coordinates are assumed to be continuous functions on \( \overline{A} \).

We need

**Definition 45** Let \( \rho, \mu, w > 0, \gamma < 0, \quad N = \lceil \mu \rceil, \quad \mu \in \mathbb{N}; \quad f \in C^N(\overline{A}) \) and \( \psi \in C^N([R_1, R_2]), \psi' (r) \neq 0, \quad \forall \quad r \in [R_1, R_2], \quad \psi' \) is increasing. We define the \( \psi \)-Prabhakar-Caputo radial left and right fractional derivatives of order \( \mu \) as follows (\( x \in \overline{A}; \quad x = r \omega, \quad r \in [R_1, R_2], \quad \omega \in S^{N-1} \))

\[
\left( \frac{\partial}{\partial r} \right)^{\psi} \frac{\int_{|r|}^{r} f(x) dr}{L_{q}(x)} = \left( \frac{\partial}{\partial r} \right)^{\psi} \frac{\int_{|r|}^{r} f(x) dr}{L_{q}(x)}
\]

\[
\left( \frac{\partial}{\partial r} \right)^{\psi} \frac{f^{[N]}(x)}{L_{q}(x)} = \left( \frac{\partial}{\partial r} \right)^{\psi} \frac{f^{[N]}(x)}{L_{q}(x)}
\]

where

\[
f^{[N]}(x) = f^{[N]}(r \omega) = \left( \frac{1}{\psi'(r)} \right)^{N} f(r \omega),
\] (146)

is the \( N \)th order \( \psi \)-radial derivative of \( f \),

and

\[
\left( \frac{\partial}{\partial r} \right)^{\psi} \frac{\int_{|r|}^{r} f(x) dr}{L_{q}(x)} = \left( \frac{\partial}{\partial r} \right)^{\psi} \frac{\int_{|r|}^{r} f(x) dr}{L_{q}(x)}
\]

\[
(-1)^N \int_{r}^{R_2} \left( \psi'(r) - \psi'(r) \right)^{N-1} \frac{\int_{|r|}^{r} f(x) dr}{\psi'(r)^N} f(t \omega) dt = (-1)^N \left( \frac{\partial}{\partial r} \right)^{\psi} \frac{f^{[N]}(x)}{L_{q}(x)},
\] (147)

\( \forall \quad x \in \overline{A} \).
In this work we assume that \( C^r D^\gamma_{\rho,\mu,w,R_1} f \) and \( C^r D^\gamma_{\rho,\mu,w,R_2} f \) are continuous functions over \( \overline{A} \).

We make

**Remark 46** Let \( \rho, \mu, w > 0, \gamma < 0, N = \lceil \mu \rceil, \mu \notin \mathbb{N}; f_i \in C^N(\overline{A}), i = 1, \ldots, n, \) and \( \overline{f} = (f_1, \ldots, f_n) \), and \( \psi \in C^N([R_1, R_2]), \psi'(r) \neq 0, \forall r \in [R_1, R_2] \), and \( \psi' \) is increasing. We follow Definition 45 and we set:

\[
\left\| f^{[\gamma]}(y) \right\|_{\infty} := \max \left\{ f^{[\gamma]}(y), \ldots, f^{[\gamma]}(y) \right\},
\]
and

\[
\left\| f^{[\gamma]}(y) \right\|_{q} := \left( \sum_{i=1}^{n} f^{[\gamma]}(y)^q \right)^{\frac{1}{q}}, q \geq 1; y \in \overline{A}.
\]

One can write that

\[
\left\| f^{[\gamma]}(y) \right\|_{q} = \left\| f^{[\gamma]}(t \omega) \right\|_{q}, 1 \leq q \leq \infty,
\]

where \( t \in [R_1, R_2], \omega \in S^{N-1}; y = t \omega \).

Notice that \( \left\| f^{[\gamma]}(y) \right\|_{q} \in C(\overline{A}), 1 \leq q \leq \infty. \)

We assume that \( \left\| f^{[\gamma]}(y) \right\|_{q} > 0 \) on \( \overline{A}, 1 \leq q \leq \infty \) fixed.

Consider the kernel

\[
C^r k^+(r,t) := k(r,t) := \chi_{(R_1, R_2)}(t) \psi'(t) (\psi(r) - \psi(t))^{N-\mu-1} E^{r,q}_{\rho,\mu,w} \omega(\psi(r) - \psi(t))^{r-1}
\]

Let

\[
C^r L_q(x) = C^r L_q(r \omega) = \int_{R_1}^{R_2} C^r k^+(r,t) \left\| f^{[\gamma]}(t \omega) \right\|_{q} dt,
\]

\( x = r \omega \in \overline{A}, 1 \leq q \leq \infty \) fixed; \( r \in [R_1, R_2], \omega \in S^{N-1}. \)

We have that \( C^r L_q(r \omega) > 0 \) for \( r \in (R_1, R_2), \forall \omega \in S^{N-1}. \)

Here we choose the weight \( u(x) = u(r \omega) = C^r L_q(r \omega). \)

Consider the function

\[
C^r W_q^+(y) = C^r W_q^+(r \omega) = \left\| f^{[\gamma]}(t \omega) \right\|_{q} \left( \int_{R_1}^{R_2} C^r k^+(r,t) dr \right) < \infty,
\]

\( \forall t \in [R_1, R_2], \omega \in S^{N-1}; \) and \( C^r W_q^+(r \omega) \) is integrable over \( [R_1, R_2], \forall \omega \in S^{N-1}. \)
Here, $\Phi : \mathbb{R}_+^n \rightarrow \mathbb{R}$ is a convex and increasing per coordinate function.

A direct application of Theorem 42, along with (145) follows:

**Theorem 47** All as in Remark 46. Then

$$
\int_A^C L_q^+(x) \Phi \left( \left( \frac{\mathcal{D}^{\tau \nu}_{\rho, \mu, w, R_1} f}{C L_q^+(x)} \right) \right) dx \leq \left( \frac{R_2}{R_1} \right)^{N-1} \int_A^C W_q^+(x) \Phi \left( \left( \frac{f^{(N)}_\nu(x)}{C L_q^+(x)} \right) \right) dx,
$$

(153)

where $\left( \mathcal{D}^{\tau \nu}_{\rho, \mu, w, R_1} f \right)(x) = \left( \mathcal{D}^{\tau \nu}_{\rho, \mu, w, R_1} f_1(x), \ldots, \mathcal{D}^{\tau \nu}_{\rho, \mu, w, R_1} f_n(x) \right)$ and the coordinates are assumed to be continuous on $\overline{A}$.

We make

**Remark 48** Let $\rho, \mu, w > 0$, $\gamma < 0$, $N = [\mu]$, $\mu \notin \mathbb{N}$; $f_i \in C^N(\overline{A})$, $i = 1, \ldots, n$, and $\mathcal{f} = (f_1, \ldots, f_n)$, and $\psi \in C^N([R_1, R_2])$, $\psi'(r) \neq 0$, $\forall \ r \in [R_1, R_2]$, and $\psi$ is increasing. We follow Definition 45 and we set:

$$
\left\| f^{[N]}_\nu(y) \right\|_q := \max \left\{ \left| f^{[N]}_1(y) \right|, \ldots, \left| f^{[N]}_n(y) \right| \right\}
$$

and

$$
\left\| f^{[N]}_\nu(y) \right\|_q := \left( \sum_{i=1}^n \left| f^{[N]}_i(y) \right|^q \right)^{\frac{1}{q}}, \quad q \geq 1; \ \ y \in \overline{A}.
$$

(154)

One can write that

$$
\left\| f^{[N]}_\nu(y) \right\|_q = \left\| f^{[N]}_\nu(t \omega) \right\|_q, \ \ 1 \leq q \leq \infty,
$$

(155)

where $t \in [R_1, R_2]$, $\omega \in S^{N-1}$; $y = t \omega$.

Notice that $\left\| f^{[N]}_\nu(y) \right\|_q \in C(\overline{A})$, $1 \leq q \leq \infty$.

We assume that $\left\| f^{[N]}_\nu(y) \right\|_q > 0$ on $\overline{A}$, $1 \leq q \leq \infty$ fixed.

Consider the kernel

$$
\mathcal{C} k^-\left( r, t \right) := k(r, t) := \mathcal{X}_{r, R_2}(t) \psi'(t) \left( \psi(t) - \psi(r) \right)^{N-\mu-1} E_{\rho, \mu, w}^{\gamma} \left[ w \left( \psi(t) - \psi(r) \right)^{\gamma} \right]
$$

(156)

Let

$$
\mathcal{C} L_q^+(x) = \mathcal{C} L_q^+(r \omega) = \int_{R_1}^{R_2} \mathcal{C} k^-\left( r, t \right) \left| f^{[N]}_\nu(t \omega) \right|_q dt,
$$

(157)

$x = r \omega \in \overline{A}$, $1 \leq q \leq \infty$ fixed; $r \in [R_1, R_2]$, $\omega \in S^{N-1}$. 

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We have that $^CL_q(r\omega)>0$ for $r \in (R_1, R_2]$, $\forall \omega \in S^{N-1}$.

Here we choose the weight $u(x)=u(r\omega)=^CL_q(r\omega)$.

Consider the function

$$^CW_q^-(y)=^CW_q^-(t\omega)=\left[\int_{R_1}^{\infty} \frac{|f|_q^{\nu}(t\omega)}{C L_q(x)} \left(\int_{R_1}^{R_2} k^{-}(r,t) dr\right)\right] < \infty,$$

$\forall \ t \in [R_1, R_2]$, $\omega \in S^{N-1}$; and $^CW_q^-(t\omega)$ is integrable over $[R_1, R_2]$, $\forall \ \omega \in S^{N-1}$.

Here $\Phi : R^m_+ \rightarrow R$ is a convex and increasing per coordinate function.

A direct application of Theorem 44, along with (147) follows:

**Theorem 49** All as in Remark 48. Then

$$\int_A\ A \Phi \left(\frac{\left(\int_0^1 r\rho R^\gamma R \mu R ,w R ,x R f\right)(x)}{\left(\int_0^1 r\mu R ,w R ,x R f\right)(x)}\right) dx \leq \left(\frac{R_2}{R_1}\right)^{N-1}\int_A\ A \Phi \left(\frac{\left(\int_0^1 r\rho R^\gamma R \mu R ,w R ,x R f\right)(x)}{\left(\int_0^1 r\mu R ,w R ,x R f\right)(x)}\right) dx,$$

where $\left(\int_0^1 r\rho R^\gamma R \mu R ,w R ,x R f\right)(x)=\left(\int_0^1 r\rho R^\gamma R \mu R ,w R ,x R f\right)(x), \ldots, \left(\int_0^1 r\rho R^\gamma R \mu R ,w R ,x R f\right)(x)$ and the coordinates are assumed to be continuous on $\bar{A}$.

We need

**Definition 50** Let $\rho, \mu, w > 0$, $\gamma < 0$, $N = [\mu]$, $\mu \in \mathbb{N}$, $f \in C(\bar{A})$ and $\psi \in C^\mu([R_1, R_2])$, $\psi \ (r) \neq 0$, $\forall \ r \in [R_1, R_2]$ and $\psi$ is increasing. The $\psi$-Prabhakar-Riemann Liouville left and right radial fractional derivatives of order $\mu$ are defined as follows (see also Definition 40)

$$\left(\int_0^1 r\rho R^\gamma R \mu R ,w R ,f\right)(x)=\left(\int_0^1 r\rho R^\gamma R \mu R ,w R ,f\right)(x),$$

and

$$\left(\int_0^1 r\rho R^\gamma R \mu R ,w R ,f\right)(x)=\left(\int_0^1 r\rho R^\gamma R \mu R ,w R ,f\right)(x),$$

$\forall \ x \in \bar{A}$; where $x = r\omega$, $r \in [R_1, R_2]$, $\omega \in S^{N-1}$.

In this work we assume that $\left(\int_0^1 r\rho R^\gamma R \mu R ,w R ,f\right), \left(\int_0^1 r\rho R^\gamma R \mu R ,w R ,f\right) \in C(\bar{A})$.

Next we define the $\psi$-Hilfer-Prabhakar left and right radial fractional derivatives of order $\mu$ and type $\beta \in [0,1]$, as follows ($\xi := \mu + \beta(N - \mu)$, see also Definition 40):
\( \left( \frac{H}{R} D^{\gamma,\beta}_\rho,\mu,w,R \right) f(x) = \left( \frac{H}{R} D^{\gamma,\beta}_\rho,\mu,w,R_1 \right) f(r \omega) := e^{-\gamma \beta \mu} \left( \frac{RL}{R} D^{(1-\beta)\mu}_\rho,\xi,\omega,w,R_1 \right) f(x), \right) \)  

and

\( \left( \frac{H}{R} D^{\gamma,\beta}_\rho,\mu,w,R_2 \right) f(x) = \left( \frac{H}{R} D^{\gamma,\beta}_\rho,\mu,w,R_2 \right) f(r \omega) := e^{-\gamma \beta \mu} \left( \frac{RL}{R} D^{(1-\beta)\mu}_\rho,\xi,\omega,w,R_2 \right) f(x), \)  

\( \forall \ x \in \overline{A}; \text{ where } x = r \omega, \ r \in [R_1,R_2], \ \omega \in S^{N-1}. \)

In this work we assume that \( \left( \frac{H}{R} D^{\gamma,\beta}_\rho,\mu,w,R \right) f, \left( \frac{H}{R} D^{\gamma,\beta}_\rho,\mu,w,R_2 \right) f \in C(\overline{A}). \)

We make

**Remark 51** Let \( \rho, \mu, w > 0, \gamma < 0, \ N = \lfloor \mu \rfloor, \ \mu \in \mathbb{N}; \ 0 \leq \beta \leq 1, \ \xi = \mu + \beta(N - \mu), \ f_i \in C(\overline{A}), \ i = 1, \ldots, n, \) and \( \psi \in C^N([R_1,R_2]), \ \psi'(r) \neq 0, \ \forall \ r \in [R_1,R_2], \) and \( \psi \) is increasing. We follow Definition 50, especially (162) and we set:

\[
\| \left( \frac{RL}{R} D^{(1-\beta)\mu}_\rho,\xi,\omega,w,R_1 \right) f(y) \|_q := \max \left\{ \left\| \left( \frac{RL}{R} D^{(1-\beta)\mu}_\rho,\xi,\omega,w,R_1 \right) f_i(y) \right\|, i = 1, \ldots, n \right\} \]

and

\[
\left\| \left( \frac{RL}{R} D^{(1-\beta)\mu}_\rho,\xi,\omega,w,R_1 \right) f(y) \right\|_q := \left( \sum_{i=1}^n \left\| \left( \frac{RL}{R} D^{(1-\beta)\mu}_\rho,\xi,\omega,w,R_1 \right) f_i(y) \right\|^q \right)^{1/q}, \ q \geq 1; \ y \in \overline{A}. \]

One can write that

\[
\left\| \left( \frac{RL}{R} D^{(1-\beta)\mu}_\rho,\xi,\omega,w,R_1 \right) f(y) \right\|_q = \left\| \left( \frac{RL}{R} D^{(1-\beta)\mu}_\rho,\xi,\omega,w,R_1 \right) f(t \omega) \right\|_q, \ 1 \leq q \leq \infty, \]

where \( t \in [R_1,R_2], \ \omega \in S^{N-1}, \ y = t \omega. \)

Notice that \( \left\| \left( \frac{RL}{R} D^{(1-\beta)\mu}_\rho,\xi,\omega,w,R_1 \right) f(y) \right\|_q \in C(\overline{A}), \ 1 \leq q \leq \infty. \)

We assume that \( \left\| \left( \frac{RL}{R} D^{(1-\beta)\mu}_\rho,\xi,\omega,w,R_1 \right) f(y) \right\|_q > 0 \) on \( \overline{A}, \ 1 \leq q \leq \infty \) fixed.

Consider the kernel

\[
p^{-} k^+(r,t) := k(r,t) := \chi_{(R_1,r]}(t)\psi'(t)(\psi(r) - \psi(t))^{\xi - \mu - 1} E^{-\beta \mu}_{\rho,\xi - \mu} w(\psi(r) - \psi(t))^p \]  

Let

\[
p^{-} L^q_q(x) = \left( \frac{RL}{R} D^{(1-\beta)\mu}_\rho,\xi,\omega,w,R_1 \right) f(t \omega) \right\|_q dt, \]

where \( x = t \omega, \ t \in [R_1,R_2], \ \omega \in S^{N-1}. \)
We have that \( pL_q^+(r \omega) > 0 \) for \( r \in (R_1, R_2) \), \( \forall \omega \in S^{N-1} \).

Here we choose the weight \( u(x) = u(r \omega) = pL_q^+(r \omega) \).

Consider the function
\[
P^q W_q^+(y) = pW_q^+(t \omega) = \left\| \frac{R}{\mathbb{L}} D_{\rho, \xi, w, R_1 + f}^{\gamma} (t \omega) \right\|_{A} \left( \int_{R_1}^{P} k^+(r, t) \, dr \right) < \infty, \tag{168}
\]
\( \forall \ t \in [R_1, R_2], \omega \in S^{N-1}; \) and \( pW_q^+(t \omega) \) is integrable over \([R_1, R_2], \forall \omega \in S^{N-1}\).

Here \( \Phi : \mathbb{R}_+^n \to \mathbb{R} \) is a convex and increasing per coordinate function.

A direct application of Theorem 42, along with (162) follows:

**Theorem 52** All as in Remark 51. Then
\[
\int_A P L_q^+(x) \Phi \left( \frac{H D_{\rho, \mu, w, R_1 + f}^{\gamma} (x)}{P L_q^+(x)} \right) \, dx \leq \left( \frac{R_2}{R_1} \right)^{N-1} \int_A P W_q^+(x) \Phi \left( \frac{H D_{\rho, \mu, w, R_1 + f}^{\gamma} (x)}{P D_{\rho, \mu, w, R_1 + f}^{\gamma} (x)} \right) \, dx, \tag{169}
\]

where \( \frac{H D_{\rho, \mu, w, R_1 + f}^{\gamma} (x)}{P L_q^+(x)} = \left( H D_{\rho, \mu, w, R_1 + f_1}^{\gamma} (x), \ldots, H D_{\rho, \mu, w, R_1 + f_n}^{\gamma} (x) \right) \) and the coordinates are assumed to be continuous on \( \overline{A} \).

We make

**Remark 53** Let \( \rho, \mu, w > 0, \gamma < 0, N = \lceil \mu \rceil, \mu \not\in \mathbb{N}; 0 \leq \beta \leq 1, \xi = \mu + \beta(N - \mu), f_i \in C(\overline{A}), i = 1, \ldots, n, \) and \( \psi \in C^N([R_1, R_2]), \psi(r) \neq 0, \forall \ r \in [R_1, R_2], \) and \( \psi \) is increasing. We follow Definition 50, especially (163) and we set:
\[
\left\| \frac{R}{\mathbb{L}} D_{\rho, \xi, w, R_2 - f}^{\gamma} (y) \right\|_q := \max \left\{ \left( \frac{R}{\mathbb{L}} D_{\rho, \xi, w, R_2 - f}^{\gamma} f_1 (y) \right)^q, \ldots, \left( \frac{R}{\mathbb{L}} D_{\rho, \xi, w, R_2 - f}^{\gamma} f_n (y) \right)^q \right\}, q \geq 1; y \in \overline{A}. \tag{170}
\]

and
\[
\left\| \frac{R}{\mathbb{L}} D_{\rho, \xi, w, R_2 - f}^{\gamma} (y) \right\|_q := \left( \sum_{i=1}^n \left( \frac{R}{\mathbb{L}} D_{\rho, \xi, w, R_2 - f}^{\gamma} f_i (y) \right)^q \right)^{1/q}, 1 \leq q \leq \infty. \tag{171}
\]
where \( t \in [R_1, R_2] \), \( \omega \in S^{N-1} \), \( y = t\omega \).

Notice that \( \left\| \frac{RL_D}{RL_D} \right\| y \in C(A), 1 \leq q < \infty \).

We assume that \( \left\| \frac{RL_D}{RL_D} \right\| y > 0 \) on \( A \), \( 1 \leq q < \infty \) fixed.

Consider the kernel
\[
P^{-}(r,t) := k(r,t) := \frac{1}{\omega} \chi_{(r,R_2)}(t)\psi'(t)(\psi(t) - \psi(r))^{\xi - \mu} E_{\mu,\xi}(w(\psi(t) - \psi(r))^r)
\]  (172)

Let
\[
P^-_q(x) = \int_{R_1}^{R_2} P^-_q(r\omega) = \int_{R_1}^{R_2} k^-(r,t) \left\| \frac{RL_D}{RL_D} \right\| y dt
\]  (173)

\( x = r\omega \in A \), \( 1 \leq q < \infty \) fixed; \( r \in [R_1, R_2] \), \( \omega \in S^{N-1} \).

We have that \( P^-_q(r\omega) > 0 \) for \( r \in (R_1, R_2) \), \( \forall \omega \in S^{N-1} \).

Here we choose the weight \( u(x) = u(r\omega) = P^-_q(r\omega) \).

Consider the function
\[
P^-_q(y) = \int_{R_1}^{R_2} P^-_q(r\omega) = \left( \int_{R_1}^{R_2} k^-(r,t) dt \right)^{-1} < 0
\]  (174)

\( \forall t \in [R_1, R_2] \), \( \omega \in S^{N-1} \); and \( P^-_q(r\omega) \) is integrable over \([R_1, R_2]\), \( \forall \omega \in S^{N-1} \).

Here \( \Phi : \mathbb{R}^n \to \mathbb{R} \) is a convex and increasing per coordinate function.

A direct application of Theorem 44, along with (163) follows:

**Theorem 54** All as in Remark 53. Then

\[
\int_A P^-_q(x) \Phi \left( \frac{RL_D}{RL_D} \right) dx \leq \frac{R_2}{R_1} \int_A P^-_q(x) \Phi \left( \frac{RL_D}{RL_D} \right) dx,
\]  (175)

where \( \left( RL_D \right) = \left( RL_D f_1, ..., RL_D f_n \right) \) and the coordinates are assumed to be continuous on \( A \).

We make

**Remark 55** Let \( f_j \in C([a,b]), j = 1,2; i = 1,...,m; \psi \in C([a,b]) \) which is increasing. Let also \( \rho_i, \mu_i, \gamma_i, \omega_i > 0 \)
and \( \{ e_{p_i, \mu_i, \gamma_i, a} f_{ji} \} \), \( x \in [a, b] \) as in (2). We assume here that \( 0 < f_{ji} \) on \( [a, b] \), \( i = 1, \ldots, m \).

Here we consider the kernel

\[
k_i^+ (x, y) := k_i(x, y) = \begin{cases} \psi^+ (y)(\psi(x) - \psi(y))^{\mu_i - 1} E_{p_i, \mu_i}^* \left[ \omega_i (\psi(x) - \psi(y))^{\gamma_i} \right] & a < y \leq x, \\ 0, & x < y < b, \end{cases}
\]

(176)

\( i = 1, \ldots, m. \)

Choose weight \( u(x) \geq 0 \), so that

\[
\psi^+ (y) := f_{ji} (y) \int_y^b u(x) \frac{k_i^+(x, y)}{e_{p_i, \mu_i, \gamma_i, a} f_{ji} (x)} \, dx < \infty,
\]

(177)
a.e. on \( [a, b] \), and that \( \psi^+ \) is integrable on \( [a, b] \), \( i = 1, \ldots, m \).

Theorem 9 immediately implies:

**Theorem 56** All as in Remark 55. Let \( p_i > 1; \sum_{i=1}^m \frac{1}{p_i} = 1 \). Let the functions \( \Phi_i : R_+ \to R_+ \), \( i = 1, \ldots, m \), be convex and increasing. Then

\[
\int_a^b u(x) \prod_{i=1}^m \Phi_i \left( \left[ e_{p_i, \mu_i, \gamma_i, a} f_{ji} (x) \right] \right) \, dx \leq \prod_{i=1}^m \left( \int_a^b \psi^+ (y) \Phi_i \left( \left[ \frac{f_{ji} (y)}{f_{ji} (y)} \right]^{p_i} \right) \, dy \right)^{\frac{1}{p_i}}.
\]

(178)

We make

**Remark 57** Let \( f_{ji} \in C([a, b]) \), \( j = 1, 2; i = 1, \ldots, m; \psi \in C^1([a, b]) \) which is increasing. Let also \( \rho_i, \mu_i, \gamma_i, \omega_i > 0 \) and \( e_{p_i, \mu_i, \gamma_i, a} f_{ji} (x), x \in [a, b] \) as in (3). We assume here that \( 0 < f_{ji} \) on \( [a, b] \), \( i = 1, \ldots, m \).

Here we consider the kernel

\[
k_i^- (x, y) := k_i(x, y) = \begin{cases} \psi^+ (y)(\psi(x) - \psi(y))^{\mu_i - 1} E_{p_i, \mu_i}^* \left[ \omega_i (\psi(x) - \psi(y))^{\gamma_i} \right] & x \leq y < b, \\ 0, & a < y < x, \end{cases}
\]

(179)

\( i = 1, \ldots, m. \)

Choose weight \( u(x) \geq 0 \), so that
\[ \psi_i(y) := f_{2i}(y) \int_a^y u(x) \frac{k_i^-(x,y)}{E_{p_i,\mu_i,\alpha_i,b-f_{2i}}(x)} \, dx < \infty, \quad (180) \]

a.e. on \([a, b]\), and that \(\psi_i\) is integrable on \([a, b]\), \(i = 1, \ldots, m\).

Theorem 9 immediately implies:

**Theorem 58** All as in Remark 57. Let \(p_i > 1; \sum_{i=1}^m \frac{1}{p_i} = 1\). Let the functions \(\Phi_i : \mathbb{R} \to \mathbb{R}_+, i = 1, \ldots, m\), be convex and increasing. Then

\[ \int_a^b u(x) \prod_{i=1}^m \Phi_i \left( \frac{E_{p_i,\mu_i,\alpha_i,b-f_{2i}}(x)}{E_{p_i,\mu_i,\alpha_i,b-f_{2i}}(y)} \right) \, dx \leq \prod_{i=1}^m \left( \int_a^b \psi_i(y) \Phi_i \left( \frac{f_i(y)}{f_{2i}(y)} \right)^{p_i} \, dy \right)^{\frac{1}{p_i}}. \quad (181) \]

We make

**Remark 59** Let \(j = 1, 2; i = 1, \ldots, n\); \(\rho_i, \mu_i, \alpha_i > 0, \gamma_i < 0, N_i = \lceil \mu_i \rceil\), \(\mu_i \notin \mathbb{N}\); \(\theta := \max(N_1, \ldots, N_m)\), \(\psi \in C^\theta([a, b]), \psi'(x) \neq 0\) over \([a, b]\), \(\psi\) is increasing; \(f_{ji} \in C^N([a, b])\) and \(f_{ji}^{[N]}(x) = \left( \frac{1}{\psi(x)} \right)^{\gamma_i} f_{ji}(x), \forall x \in [a, b].\)

Here we consider the kernel

\[ C_k_i^+(x, y) := k_i(x, y) = \begin{cases} \psi'(y)(\psi(x) - \psi(y))^{N_i-\mu_i-1} E_{p_i,\mu_i,\alpha_i,b-f_{2i}}(\alpha_i(\psi(x) - \psi(y))^{\gamma_i}) \, a < y \leq x, \\ 0, \, x < y < b, \end{cases} \quad (183) \]

\(i = 1, \ldots, m.\)

Choose weight \(u \geq 0\), so that

\[ C_{\psi_i}(y) := f_{2i}^{[N]}(y)^{\frac{b}{y}} u(x) \left( C_{D_{p_i,\mu_i,\alpha_i,b-f_{2i}}}^+(x, y) \right) dx < \infty, \quad (184) \]

a.e. on \([a, b]\), and that \(C_{\psi_i}\) is integrable on \([a, b]\), \(i = 1, \ldots, m.\)
Theorem 56 immediately produces:

**Theorem 60** All as in Remark 59. Let \( p_i > 1; \sum_{i=1}^m \frac{1}{p_i} = 1 \). Let the functions \( \Phi_i : \mathbb{R}_+ \to \mathbb{R}_+, i = 1, \ldots, m \), be convex and increasing. Then

\[
\int_a^b u(x) \prod_{i=1}^m \phi_i \left( \frac{c \, D^{\gamma_i}_{\rho_i, \mu_i, \omega_i, a; x} f_i(x)}{c \, D^{\gamma_i}_{\rho_i, \theta_i, \omega_i, a; x} f_i(x)} \right) dx \leq \prod_{i=1}^m \int_a^c \psi_i(y) \phi_i \left( \frac{f_{i, \mu_i}^{N_i}(y)}{f_{i, \theta_i}^{N_i}(y)} \right) dy \left( \frac{1}{p_i} \right)^{1/p_i}. \tag{185}
\]

We make

**Remark 61** Let \( j = 1, 2; i = 1, \ldots, n \); \( \rho_i, \mu_i, \omega_i > 0 \), \( \gamma_i < 0 \), \( N_i = \lceil \mu_i \rceil \), \( \mu_i \notin \mathbb{N} \); \( \theta := \max(N_1, \ldots, N_m) \), \( \psi \in C^ \theta([a, b]) \), \( \psi'(x) \neq 0 \) over \([a, b]\), \( \psi \) is increasing; \( f_{ji} \in C^{N_i}([a, b]) \) and \( f_{ji}^{N_i}(x) = \left( \frac{1}{\psi'(x)} \right)^{N_i} f_{ji}(x) \) \( \forall \ x \in [a, b] \). Here

\[
\left( c \, D^{\gamma_i}_{\rho_i, \mu_i, \omega_i, a; b} f_{ji}(x) \right) = (-1)^{N_i} \left( c \, D^{\gamma_i}_{\rho_i, N_i; \mu_i, \omega_i, a; b} f_{ji}^{N_i}(x) \right), \tag{186}
\]

\( \forall \ x \in [a, b], j = 1, 2; i = 1, \ldots, m \).

We assume that \( 0 < f_{2i}^{N_i}(y) < \infty \) on \([a, b], i = 1, \ldots, m\).

Here we consider the kernel

\[
c_k_i(x, y) := k_i(x, y) = \begin{cases} \psi'(x) \left( \psi(y) - \psi(x) \right)^{N_i - 1} E^{\gamma_i}_{\rho_i, N_i; \mu_i, \omega_i} \left[ \psi_i(y) - \psi_i(x) \right]^{\gamma_i}, & y \leq b, \\ 0, & a < y < x, \end{cases} \tag{187}
\]

\( i = 1, \ldots, m \).

Choose weight \( u \geq 0 \), so that

\[
c_{\overline{\psi}_i}(y) := f_{2i}^{N_i}(y) \overline{u(x)} \left( c \, D^{\gamma_i}_{\rho_i, \theta_i, \omega_i, a; b} f_{2i}(x) \right) dx < \infty, \tag{188}
\]

a.e. on \([a, b]\), and that \( c_{\overline{\psi}_i} \) is integrable on \([a, b], i = 1, \ldots, m\).

Theorem 58 immediately produces:

**Theorem 62** All as in Remark 61. Let \( p_i > 1; \sum_{i=1}^m \frac{1}{p_i} = 1 \). Let the functions \( \Phi_i : \mathbb{R}_+ \to \mathbb{R}_+, i = 1, \ldots, m \), be convex and increasing. Then
\[
\int_a^b u(x) \prod_{i=1}^m \left( \frac{1}{\Phi_i} \left( \frac{\int_a^b C \psi_i(y) \Phi_i \left( \frac{f_{\psi_i}(y)}{f_{\psi_i}(y)} \right) dy}{\int_a^b C \psi_i(y) \Phi_i \left( \frac{f_{\psi_i}(y)}{f_{\psi_i}(y)} \right) dy} \right)^p \right) \frac{1}{p_i} \right) \right) \geq 0.
\] (189)

We make

**Remark 63** Let \( j = 1, 2; i = 1, \ldots, m; \rho_j, \mu_i > 0, \gamma_i < 0, N_i = \left[ \mu_i \right], \mu_i \notin \mathbb{N}; \theta := \max(N_i, \ldots, N_m), \psi \in C^0([a, b]), \psi'(x) \neq 0 \) over \([a, b], \psi' \) is increasing; \( f_{ji} \in C([a, b]) \). Let \( 0 \leq \beta_i \leq 1 \) and \( \xi_i = \mu_i + \beta_i(N_i - \mu_i), i = 1, \ldots, m \). We assume that \( RL D_{\rho_i, \xi_i, \alpha_i, a_0} f_{ji} \in C([a, b]) \) and \( 0 < RL D_{\rho_i, \xi_i, \alpha_i, a_0} f_{ji}(y) \) on \([a, b], i = 1, \ldots, m \). Here we have

\[
\left( RL D_{\rho_i, \xi_i, \alpha_i, a_0} f_{ji} \right)(x) = e^{-\gamma_i \beta_i \psi_i(x)} \prod_{i=1}^m \left( \frac{1}{\Phi_i} \left( \frac{f_{\psi_i}(y)}{f_{\psi_i}(y)} \right)^p \right) \frac{1}{p_i} \right) \right) \]

\forall x \in [a, b], j = 1, 2; i = 1, \ldots, m.

Here we consider the kernel

\[
p k_i^+(x, y) = \begin{cases} 
\psi_i(y)(\psi_i(x) - \psi_i(y))^\xi_i - \mu_i - 1 E^{-\gamma_i \beta_i \psi_i(x)} \left[ \mu_i \psi_i(x) - \psi_i(y) \right] a < y \leq x,
0, x < y < b,
\end{cases}
\]

\( i = 1, \ldots, m \).

Choose weight \( u \geq 0 \), so that

\[
p \psi_i(y) = \left( RL D_{\rho_i, \xi_i, \alpha_i, a_0} f_{ji} \right)(y) u(x) k_i^+(x, y) dx < \infty,
\]

\( a.e. \) on \([a, b] \), and that \( n \psi_i \) is integrable on \([a, b], i = 1, \ldots, m \).

Theorem 56 immediately produces:

**Theorem 64** All as in Remark 63. Let \( p_i > 1; \sum_{i=1}^m \frac{1}{p_i} = 1 \). Let the functions \( \Phi_i : \mathbb{R}_+ \to \mathbb{R}_+, i = 1, \ldots, m \), be convex and increasing. Then

\[
\int_a^b u(x) \prod_{i=1}^m \left( \frac{1}{\Phi_i} \left( \frac{\int_a^b C \psi_i(y) \Phi_i \left( \frac{f_{\psi_i}(y)}{f_{\psi_i}(y)} \right) dy}{\int_a^b C \psi_i(y) \Phi_i \left( \frac{f_{\psi_i}(y)}{f_{\psi_i}(y)} \right) dy} \right)^p \right) \frac{1}{p_i} \right) \right) \right) \]

We make
Remark 65 Let $j = 1, 2; i = 1, ..., m$; $\rho_i, \mu_i, \alpha_i > 0$, $\gamma_i < 0$, $N_i = \lceil \mu_i \rceil$, $\mu_i \notin \mathbb{N}$; $\theta := \max(N_1, ..., N_m)$, $\psi \in C^\theta([a, b])$, $\psi'(x) \neq 0$ over $[a, b]$, $\psi$ is increasing; $f_{ji} \in C([a, b])$. Let $0 \leq \beta_i \leq 1$ and $\xi_i = \mu_i + \beta_i(N_i - \mu_i)$, $i = 1, ..., m$. We assume that $R^L D_{\rho_i, \xi_i, \alpha_i, b}^{\gamma_i} f_{ji} \in C([a, b])$ and $0 < R^L D_{\rho_i, \xi_i, \alpha_i, b}^{\gamma_i} f_{ji}(y) < \infty$ on $[a, b]$, $i = 1, ..., m$. Here we have

$$
\left( R^L D_{\rho_i, \xi_i, \alpha_i, b}^{\gamma_i} f_{ji} \right)(x) = e^{-\gamma_i \beta_i y} \frac{R^L D_{\rho_i, \xi_i, \alpha_i, b}^{\gamma_i} f_{ji}}{R^L D_{\rho_i, \xi_i, \alpha_i, b}^{\gamma_i} f_{ji}}(x) \quad (194)
$$

$$\forall x \in [a, b], j = 1, 2; i = 1, ..., m.$$

Here we consider the kernel

$$p_k^i(x, y) := k_i(x, y) = \begin{cases}
\psi'(y)(\psi(y) - \psi(x))^{\xi_i - \mu_i - 1} e^{-\gamma_i \beta_i} \frac{\psi(x) - \psi(y)}{\xi_i} x \leq y < b, \\
0, a < y < x,
\end{cases} \quad (195)
$$

$i = 1, ..., m$.

Choose weight $\mu \geq 0$, so that

$$p^\mu \psi_i(y) := (R^L D_{\rho_i, \xi_i, \alpha_i, b}^{\gamma_i} f_{ji})(y) \int_a^x \frac{u(x) p_k^i(x, y)}{R^L D_{\rho_i, \xi_i, \alpha_i, b}^{\gamma_i} f_{ji}}(x) dx < \infty, \quad (196)$$

a.e. on $[a, b]$, and that $p^\mu \psi_i$ is integrable on $[a, b]$, $i = 1, ..., m$.

Theorem 58 immediately produces:

**Theorem 66** All as in Remark 65. Let $p_i > 1; \sum_{i=1}^m \frac{1}{p_i} = 1$. Let the functions $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, ..., m$, be convex and increasing. Then

$$\int_a^b u(x) \prod_{i=1}^m \Phi_i \left( \left( R^L D_{\rho_i, \xi_i, \alpha_i, b}^{\gamma_i} f_{ji} \right)(x) \right) dx \leq \prod_{i=1}^m \int_a^b \frac{p_i}{R^L D_{\rho_i, \xi_i, \alpha_i, b}^{\gamma_i} f_{ji}}(y) \Phi_i \left( \left( R^L D_{\rho_i, \xi_i, \alpha_i, b}^{\gamma_i} f_{ji} \right)(y) \right) dy \Bigg|_{y=a}^{y=b} \frac{1}{p_i}. \quad (197)$$

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