QUASICONFORMAL FEATURES AND FREDHOLM EIGENVALUES OF
CONVEX POLYGONS

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Abstract. An important open problem in geometric complex analysis is to find algorithms
for explicit determination of basic functionals intrinsically connected with conformal and
quasiconformal maps, such as their Teichmüller and Grunsky norms, Fredholm eigenvalues
and the quasireflection coefficient. This has not been solved even for convex polygons. This
case has intrinsic interest in view of the connection of such polygons with the geometry of
the universal Teichmüller space.

We provide a new approach, based on affine transformations of univalent functions.

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1. PREAMBLE AND RESULTS

1. Introductory remarks
The basic functionals intrinsically connected with conformal and quasiconformal maps
such as their Teichmüller and Grunsky norms, the first Fredholm eigenvalue, the quasire-
fection coefficient imply a deep quantitative characterization of the features of these maps.
Thus the problem to find the algorithms for explicit determination of these quantities is very
important but still remains open.

The following general result obtained in [10] by applying holomorphic motions solves this
problem for unbounded convex domains giving an explicit representation of functionals by
geometric characteristics of domains. Let $\mathbb{D} = \{z : |z| < 1\}$, $\mathbb{D}^* = \{z \in \hat{\mathbb{C}} : |z| > 1\}$.

**Theorem A.** For every unbounded convex domain $D \subset \mathbb{C}$ with piecewise $C^{1+\delta}$-smooth
boundary $L$ ($\delta > 0$) (and all its fractional linear images), the equalities

$$q_L = 1/\rho_L = \kappa(f) = \kappa(f^*) = k(f) = k(f^*) = 1 - |\alpha|$$

hold, where $f$ and $f^*$ denote the appropriately normalized conformal maps $\mathbb{D} \to D$ and $\mathbb{D}^* \to
D^* = \hat{\mathbb{C}} \setminus D$, respectively, $k(f)$ and $k(f^*)$ are the minimal dilatations of their quasiconformal
extensions to $\hat{\mathbb{C}}$, $\kappa(f)$ and $\kappa(f^*)$ stand for their Grunsky norms, and $\pi|\alpha|$ is the opening
of the least interior angle between the boundary arcs $L_j \subset L$. Here $0 < \alpha < 1$ if the
corresponding vertex is finite and $-1 < \alpha < 0$ for the angle at the vertex at infinity.

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The same is true for unbounded concave domains (the complements of convex ones) which do not contain \(\infty\); for those one must replace the last term by \(|\beta| - 1\), where \(\pi|\beta|\) is the opening of the largest interior angle of \(D\).

In particular, for any closed unbounded curve \(L\) with the convex interior which is \(C^{1+\delta}\) smooth at all finite points and has at infinity the asymptotes approaching the interior angle \(\pi\alpha < 0\), we have

\[
q_L = 1/\rho_L = 1 - |\alpha|.
\]

In contrast, there are bounded convex domains even with analytic boundaries \(L\) whose conformal mapping functions have different Grunsky and Teichmüller norms, and therefore, \(\rho_L < 1/q_L\).

2. Results and applications.

The aim of this paper is to provide the classes of bounded convex domains, especially polygons, for which these norms are equal and give explicitly the values of the associate curve functionals \(k(f),\ \varpi(f),\ q_L,\ \rho_L\).

Consider the class \(\Sigma^0\) of univalent functions \(F(z) = z + b_0 + b_1z^{-1} + \ldots\) mapping conformally the disk \(\mathbb{D}^*\) into \(\hat{\mathbb{C}} \setminus \{0\}\) and having quasiconformal extensions to \(\hat{\mathbb{C}}\) with \(F(0) = 0\). This collection naturally relates to the universal Teichmüller space \(\mathbf{T}\) (the space of quasisymmetric homeomorphisms \(h\) of the unit circle factorized by Möbius transformations) modeled by the Schwarzian derivatives

\[
S_F = (F''/F')' - (F''/F')^2/2
\]

of \(F \in \Sigma^0\) in \(\mathbb{D}^*\). Their inversions \(f(z) = 1/F(1/z)\) form the class \(S^0\) of univalent functions \(f(z) = z + \sum a_n z^n\) of univalent functions in the unit disk \(\mathbb{D}\) with quasiconformal extension to \(\mathbb{D}^*\) preserving \(z = \infty\), and \(\varpi(F) = \varpi(f)\).

One of the interesting questions is whether the equality of Teichmüller and Grunsky norms is preserved under the affine deformations \(g^c(w) = c_1 w + c_2 \overline{w} + c_3\) with \(c = c_2/c_1\) (as well as of more general maps) of quasidisks. In the case of unbounded convex domains, this follows from Theorem A. We establish this here for bounded domains \(D\). More precisely, we consider the maps \(g^c\) which are conformal in the complementary domain \(D^* = \hat{\mathbb{C}} \setminus \overline{D}\) and have in \(D\) a constant quasiconformal dilatation \(c\).

**Theorem 1.** For any function \(F \in \Sigma^0\) with \(\varpi(F) = k(F)\) mapping the disk \(\mathbb{D}^*\) onto the complement of a bounded domain (quasidisk) \(D\) and any affine deformation \(g^c\) of this domain, we have the equality

\[
\varpi(g^c \circ F) = k(g^c \circ F).
\]

Theorems 1 essentially increases the set of quasicircles \(L \subset \hat{\mathbb{C}}\) for which \(\rho_L = 1/q_L\) giving simultaneously the explicit values of these curve functionals. Even for quadrilaterals, this fact was known until now only for some special types of them (for rectangles \([10], [19 - 21]\) and for rectilinear or circular quadrilaterals having a common tangent circle \([26]\).

The arguments in the proof of this theorem are extended almost straightforwardly to more general case:

**Theorem 2.** Let \(F \in \Sigma^0\) and \(\varpi(F) = k(F)\). Let \(h\) be a holomorphic map \(\mathbb{D} \to \mathbf{T}\) without critical points in \(\mathbb{D}\) and \(h(0) = S_F\). Denote by \(g^c\) the univalent solution of the Schwarzian equation \(S_g = h(c)\) on the domain \(F(\mathbb{D}^*)\). Then, for any \(c \in \mathbb{D}\), the composition \(g^c \circ F\) also satisfies \(\varpi(g^c \circ F) = k(g^c \circ F)\).
Note that by the lambda lemma for holomorphic motions, the map \( h \) determines a holomorphic disk in the ball of Beltrami coefficients on \( F(\mathbb{D}) \), which yields, together with assumptions of the theorem, that for small \( |c| \)

\[
g^c(w) = w + b_0^c + b_1^cw^{-1} + \ldots \quad \text{as} \quad w \to \infty
\]

with \( b_1^c \neq 0 \). This is an essential point in the proof.

The case of bounded convex polygons has an intrinsic interest, in view of the following negative fact underlying the features and contrasting Theorem A.

**Theorem 3.** There exist bounded rectilinear convex polygons \( P_n \) with sufficiently large number of sides such that

\[
\rho_{\partial P_n} < 1/q_{\partial P_n}.
\]

It follows simply from Theorem 1 that if a polygon \( P_n \), whose edges are quasiconformal arcs, satisfies \( \rho_{\partial P_n} = 1/q_{\partial P_n} \) then this equality is preserved for all its affine images. In particular, this is valid for all rectilinear polygons obtained by affine maps from polygons with edges having a common tangent ellipse (which includes the regular \( n \)-gons).

Theorem 3 naturally gives rise to the question whether the property \( \rho_{\partial P_n} = 1/q_{\partial P_n} \) is valid for all bounded convex polygons with sufficiently small number of sides.

In the case of triangles this immediately follows from Theorem 1 as well as from Werner’s result.

Noting that the affinity preserves parallelism and moves the lines to lines, one concludes from Theorem 1 that the equality \( \rho_{\partial P_4} = 1/q_{\partial P_4} \) holds in particular for quadrilaterals \( P_4 \) obtained by affine transformations from quadrilaterals which are symmetric with respect to one of diagonals and for quadrilaterals whose sides have common tangent outwardly ellipse (in particular, for all parallelograms and trapezoids). For the same reasons, it holds also for hexagons with axial symmetry having two opposite sides parallel to this axes.

In fact, Theorem 1 allows us to establish much stronger result answering the question positively for quadrilaterals.

**Theorem 4.** For every rectilinear convex quadrilateral \( P_4 \), we have

\[
x(F) = k(F) = \rho_{\partial P_4} = 1/q_{\partial P_4},
\]

where \( F \) is the appropriately normalized conformal map of \( \mathbb{D}^* \) onto \( P_4^* \).

## 2. BACKGROUND

We present briefly the needed notions and results underlying the above theorems adapting those to our case; for details see, e.g. [1], [6], [7], [12], [20].

1. **A glimpse at Grunsky inequalities and Fredholm eigenvalues.** Denote by \( \text{Belt}(\mathbb{D}) \) the unit ball of Beltrami coefficients \( \mu \) supported on \( \mathbb{D} \) and extended by zero to \( \mathbb{D}^* \), i.e.,

\[
\text{Belt}(\mathbb{D}) = \{ \mu \in L_\infty(\mathbb{C}) : \mu(z)|\mathbb{D}^* = 0, \|\mu\|_\infty < 1 \}
\]

and by \( w^\mu \) the solutions of the Beltrami equation \( \partial_zw = \mu\partial_zw \) on \( \mathbb{C} \) with the expansion \( w(z) = z + b_0 + b_1z^{-1} + \ldots \) in \( \mathbb{D}^* \).

The fundamental Grunsky theorem (extended to multiply connected domains by Milin [22]) states that a holomorphic function \( F(z) = z + \text{const} + O(1/z) \) in a neighborhood \( U_0 \) of
the infinite point is extended to a univalent function on the disk \( D^* \) if and only if it satisfies the inequality
\[
\left| \sum_{m,n=1}^{\infty} \sqrt{mn} \alpha_{mn} x_m x_n \right| \leq 1,
\]
where the Grunsky coefficients \( \alpha_{mn}(f) \) are determined by
\[
\log \frac{F(z) - F(\zeta)}{z - \zeta} = - \sum_{m,n=1}^{\infty} \alpha_{mn} z^{-m} \zeta^{-n}, \quad (z, \zeta) \in (D^*)^2,
\]
taking the principal branch of the logarithmic function, and \( x = (x_n) \) ranges over the unit sphere \( S(l^2) \) of the Hilbert space \( l^2 \) of sequences with \( \|x\|_2 = \sum_{n=1}^{\infty} |x_n|^2 \) (cf. [8]). The quantity
\[
\kappa(F) = \sup \left\{ \left| \sum_{m,n=1}^{\infty} \sqrt{mn} \alpha_{mn} x_m x_n \right| : x = (x_n) \in S(l^2) \right\}
\]
is called the Grunsky norm of the map \( F \).

It is dominated by the Teichmüller norm \( k(F) \) of this map, i.e., with the minimal dilatation among quasiconformal extensions of \( F \) onto \( \mathbb{D} \) (see [18], [16]); so,
\[
\kappa(F) \leq k(F) = \tanh \tau_T(0, S_F), \quad (5)
\]
where \( \tau_T \) denotes the Teichmüller distance on \( T \). The second norm is intrinsically connected with integrable holomorphic quadratic differentials on \( D \) (the elements of the subspace \( A_1 = A_1(\mathbb{D}) \) of \( L_1(\mathbb{D}) \) formed by holomorphic functions), while the Grunsky norm naturally relates to the abelian structure determined by the set of quadratic differentials
\[
A_1^2 = \{ \psi \in A_1 : \psi = \omega^2 \};
\]
having only zeros of even order on \( \mathbb{D} \). In terms of the pairing
\[
\langle \mu, \psi \rangle_D = \iint_D \mu(z) \psi(z) dxdy, \quad \mu \in L_\infty(\mathbb{D}), \quad \psi \in L_1(\mathbb{D}) \quad (z = x + iy),
\]
we have the following results characterizing the functions with \( \kappa(F) = k(F) \).

**Lemma 1.** [9], [16] For all \( F = F^\mu \in \Sigma^0 \),
\[
\kappa(F) \leq k = \frac{k + \alpha(F)}{1 + \alpha(F)}, \quad k = k(F),
\]
and \( \kappa(F) < k \) unless
\[
\alpha(F) := \sup \left\{ \|\langle \mu, \psi \rangle_D \| : \psi \in A_1^2, \|\psi\|_{A_1(\mathbb{D})} = 1 \right\} = \|\mu\|_\infty; \quad (6)
\]
the last equality is equivalent to \( \kappa(F) = k(F) \). Moreover, for small \( \|\mu\|_\infty \),
\[
\kappa(F) = \sup \|\langle \mu, \psi \rangle_D \| + O(\|\mu\|^2_\infty), \quad \|\mu\|_\infty \to 0,
\]
with the same supremum as in (6).

If \( \kappa(F) = k(F) \) and the equivalence class of \( F \) (the collection of maps equal to \( F \) on \( S^1 = \partial D^* \)) is a Strebel point, then the extremal \( \mu_0 \) in this class is necessarily of the form
\[
\mu_0 = \|\mu_0\|_\infty |\psi_0|/\psi_0 \quad \text{with} \quad \psi_0 \in A_1^2; \quad (7)
\]
Geometrically, (6) means the equality of the Carathéodory and Teichmüller distances on the image of the geodesic disk \( \mathbb{D}(\mu_0) = \{t\mu_0/\|\mu_0\|_\infty : t \in \mathbb{D}\} \) in the space \( \mathbf{T} \). For functions \( F \in \Sigma^0 \) holomorphic in the closed disk \( \mathbb{D}^* \), the relation (7) was also obtained by a different method in [21].

An important property of the Grunsky coefficients \( \alpha_{mn}(F) = \alpha_{mn}(S_F) \) is that these coefficients are holomorphic functions of the Schwarzians \( \varphi = S_F \) on the universal Teichmüller space \( \mathbf{T} \). Therefore, for every \( F \in \Sigma^0 \) and each \( \mathbf{x} = (x_n) \in S(l^2) \), the series

\[
h_{\mathbf{x}}(\varphi) = \sum_{m,n=1}^{\infty} \sqrt{mn} \alpha_{mn}(\varphi) x_m x_n
\]  

defines a holomorphic map of the space \( \mathbf{T} \) into the unit disk \( \mathbb{D} \), and \( \kappa(F) = \sup_{\mathbf{x}} |h_{\mathbf{x}}(S_F)| \).

The convergence and holomorphy of the series (8) simply follow from the inequalities

\[
\left| \sum_{m=j}^{M} \sum_{n=l}^{N} \sqrt{mn} \alpha_{mn} x_m x_n \right|^2 \leq \sum_{m=j}^{M} |x_m|^2 \sum_{n=l}^{N} |x_n|^2
\]

(for any finite \( M, N \)) which, in turn, are a consequence of the classical area theorem (see, e.g., [24, p. 61]).

Using Parseval’s equality, one obtains that the elements of the distinguished set \( A_1^2 \) are represented in the form

\[
\psi(z) = \frac{1}{\pi} \sum_{m+n=2}^{\infty} \sqrt{mn} x_m x_n z^{m+n-2}
\]  

with \( \mathbf{x} = (x_n) \in l^2 \) so that \( \|\mathbf{x}\|_2 = \|\psi\|_{A_1} \) (see [21]).

A crucial point here is that for a generic function \( F \in \Sigma^0 \) in (5) the strict inequality \( \kappa(F) < k(F) \) is valid; moreover, it holds on the (open) dense subset of \( \Sigma^0 \) in both strong and weak topologies (i.e., in the Teichmüller distance and in locally uniform convergence on \( D^* \)); see [9], [14], [17], [19], [20]. So it is important to know whether for a concrete function \( F \), we have \( \kappa(F) = k(F) \). This fact is deeply related to various topics in the complex geometry of the Teichmüller space theory, geometric complex analysis, Fredholm eigenvalues and boundary problems, operator theory, etc.

2. Quasireflections. The quasiconformal reflections (or quasireflections) represent a special case of topological orientation reversing involutions of the sphere \( S^2 = \hat{\mathbb{C}} = \mathbb{C} \cup \mathbb{C} \). Any quasireflection preserves pointwise fixed a quasicircle \( L \subset \hat{\mathbb{C}} \) interchanging its inner and outer domains (because, due to [12], any set \( E \subset S^2 \), which admits quasireflections, is necessarily located on a quasicircle with the same reflection coefficient).

One defines for each mirror \( E \) its reflection coefficient \( q_E = \inf \|\partial z f/\partial z f\|_\infty \) (taking the infimum over all quasireflections across \( E \)) and quasiconformal dilatation \( Q_E = (1 + q_E)/(1 - q_E) \geq 1 \). Due to [4], [12], [21],

\[
Q_E = (1 + k_E)^2/(1 - k_E)^2,
\]

where \( k_E = \inf \|\partial z f/\partial z f\|_\infty \) over all quasicircles \( L \supset E \) and all orientation preserving quasiconformal homeomorphisms \( f : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) with \( f(\mathbb{R}) = L \).
3. Fredholm eigenvalues. The Fredholm eigenvalues $\rho_n$ of a smooth closed Jordan curve $L \subset \hat{\mathbb{C}}$ are the eigenvalues of its double-layer potential, or equivalently, of the integral equation
\[
\frac{1}{\rho} u(z) + \frac{1}{\pi} \int_L u(\zeta) \frac{\partial}{\partial n_\zeta} \log \frac{1}{|\zeta - z|} \, ds_\zeta = h(z),
\]
which has has many applications (here $n_\zeta$ is the outer normal and $ds_\zeta$ is the length element at $\zeta \in L$).

The least positive eigenvalue $\rho_L = \rho_1$ plays a crucial role and is naturally connected with conformal and quasiconformal maps. It can be defined for any oriented closed Jordan curve $L$ by
\[
\frac{1}{\rho_L} = \sup \frac{|D_G(u) - D_{G^*}(u)|}{D_G(u) + D_{G^*}(u)},
\]
where $G$ and $G^*$ are, respectively, the interior and exterior of $L$; $D$ denotes the Dirichlet integral, and the supremum is taken over all functions $u$ continuous on $\hat{\mathbb{C}}$ and harmonic on $G \cup G^*$. In particular, $\rho_L = \infty$ only for the circle.

An upper bound for $\rho_L$ is given by Ahlfors’ inequality
\[
\frac{1}{\rho_L} \leq q_L,
\]
where $q_L$ denotes the minimal dilatation of quasireflections across $L$. This inequality is equivalent to (4) and serves as a background for defining the value $\rho_L$. This value is intrinsically connected with the Grunsky operator, which is qualitatively expressed by the Kühnau-Schiffer theorem; it states that $\rho_L$ is reciprocal to the Grunsky norm $\kappa(f)$ of the Riemann mapping function of the exterior domain of $L$ (cf. [19], [25]).

4. Plurisubharmonicity of the Teichmüller metric. Due to the fundamental Gardiner-Royden theorem, the Kobayashi and Teichmüller metrics on Teichmüller spaces are equal. An essential strengthening of this theorem for the space $T$ established in [11] by applying the Grunsky coefficient technique yields

**Lemma 2.** [11] The infinitesimal forms of both metrics $K_T(\varphi, v)$ and $F_T(\varphi, v)$ on the tangent bundle $T(T)$ of $T$ are continuous logarithmically plurisubharmonic in $\varphi \in T$ and have constant holomorphic sectional curvature $\kappa_K(\varphi, v) = -4$.

In addition, these infinitesimal metrics are Lipshitz continuous (see [5]). The global distances (integrated forms of these metrics) are logarithmically plurisubharmonic in each of their variables on $T \times T$ (cf. [12]).

3. PROOF OF THEOREM 1

The proofs of all the above theorems essentially rely on the following deep fact stated as a conjecture in [17] and proven in [13].

**Proposition 1.** Any sequence of the functions $F_n \in \Sigma^0$ with $\kappa(F_n) = k(F_n)$ cannot converge locally uniformly in $\overline{D^*}$ to a function $F \in \Sigma^0$ with $\kappa(F) < k(F)$.

Thus it suffices to establish the assertion of the theorem for functions $F \in \Sigma^0$ which are holomorphic on the closed disk $\overline{D^*}$. Indeed, by the density theorem of [13], the Strebel points
\( \varphi = S^k|z|/\mu \in \mathbf{T} \) with \( \mu \in A_1^2 \) representing the functions \( F \in \Sigma^0 \) with equal Grunsky and Teichmüller norms are dense in \( \mathbf{B} \) norm in the set of all \( F \in \Sigma^0 \) with \( \kappa(F) = k(F) \). So one can pass to \( F^{\nu_r} \) with \( \mu_r(z) = k|\mu(rz)|/\mu(rz) \) taking \( r < 1 \) so that the homotopy disk \( \mathbb{D}(S_{F^{\nu_r}}) \) has no critical points in the annulus \( \{ r < |t| < 1 \} \). Then the equality (2) in the limit \( r \to 1 \) follows again by applying Proposition 1. We accomplish the proof of the theorem in three stages.

1°. First, we establish some auxiliary results characterizing the homotopy disk of a map with \( \kappa(F) = k(F) \).

Take the generic homotopy function

\[
F_t(z) = tF(z/t) = z + b_0t + b_1t^2z^{-1} + b_2t^3z^{-2} + \cdots : \mathbb{D}^* \times \mathbb{D} \to \hat{\mathbb{C}}.
\]

Then \( S_{F_t}(z) = t^{-2}S_F(t^{-1}z) \) and this point-wise map determines a holomorphic map \( \chi_F(t) = S_{F_t}(\cdot) : \mathbb{D} \to \mathbf{T} \) so that the homotopy disks \( \mathbb{D}(S_F) = \chi_F(\mathbb{D}) \) have only cuspidal critical points and foliate the space \( \mathbf{T} \). Note also that

\[
\alpha_{mn}(F_t) = \alpha_{mn}(F)t^{m+n},
\]

and if \( f(z) = 1/F(1/z) \) maps the unit disk onto a convex domain, then all level lines \( F(|z| = r) \) for \( z \in \mathbb{D}^* \) are starlike.

**Lemma 3.** If the homotopy function \( F_t \) of \( F \in \Sigma^0 \) satisfy \( \kappa(F_{t_0}) = k(F_{t_0}) \) for some \( 0 < t_0 < 1 \), then the equality \( \kappa(F_t) = k(F_t) \) holds for all \( |t| \leq t_0 \) and the homotopy disk \( \mathbb{D}(S_{F_t}) \) has no critical points \( t \) with \( 0 < |t| < t_0 \).

**Proof.** Take the univalent extension \( F_1 \) of \( F \) to a maximal disk \( \mathbb{D}_b^* = \{ z \in \hat{\mathbb{C}} : |z| > b \}, \ (0 < b < 1) \) and define

\[
F^*(z) = b^{-1}F_1(bz) \in \Sigma^0, \ |z| > 1.
\]

Its Beltrami coefficient in \( \mathbb{D} \) is defined by holomorphic quadratic differentials \( \psi \in A_1^2 \) of the form (9), and we have the holomorphic map

\[
h^x_b(S_{F_t}) = \sum_{m,n=1}^{\infty} \sqrt{mn} \alpha_{mn}(F^*)x_m^bx_n^b(bt)^{m+n}
\]

(11)

of the disk \( \mathbb{D}(S_{F_t}) \) into \( \mathbb{D} \). In view of our assumption on \( F \), the series (11) is convergent in some wider disk \( \{ |t| < a \} (a > 1) \).

Using the map (11), we pull back the hyperbolic metric \( \lambda_0(t) = |dt|/(1-|t|^2) \) to the disk \( \mathbb{D}(S_{F_t}) \) (parametrized by \( t \)) and define on this disk the conformal metric \( ds = \lambda_{\tilde{\kappa}^b}(t)|dt| \) with

\[
\lambda_{\tilde{\kappa}^b}(t) = (h^x_b \circ \chi_{F_t})^*\lambda_{\mathbb{D}} = \frac{|\tilde{h}^x_b(t)||dt|}{1-|\tilde{h}^x(t)|^2}.
\]

(12)

of Gaussian curvature \(-4\) at noncritical points. In fact, this is the supporting metric at \( t = a \) for the upper envelope \( \lambda_\kappa = \sup_{x \in S(t)} \lambda_{\tilde{\kappa}^b}(t) \) of metrics (12) followed by its upper semicontinuous regularization \( u(t) = \lim \sup_{r \to t} u(t') \) (supporting means that \( \lambda_{\tilde{\kappa}^b}(a) = \lambda_\kappa(a) \) and \( \lambda_{\tilde{\kappa}^b}(t) < \lambda_\kappa(t) \) in a neighborhood of \( a \)).
The metric $\lambda_\kappa(t)$ is logarithmically subharmonic on $D$ and its generalized Laplacian
\[
\Delta u(t) = 4 \liminf_{r \to 0} \frac{1}{r^2} \left\{ \frac{1}{2\pi} \int_0^{2\pi} u(t + re^{i\theta})d\theta - \lambda(t) \right\}
\]
satisfies
\[
\Delta \log \lambda_\kappa \geq 4\lambda_\kappa^2
\]
(while for $\lambda_{\tilde{h}_x b}$ we have at its noncritical points $\Delta \log \lambda_{\tilde{h}_x b} = 4\lambda_{\tilde{h}_x b}^2$).

Note also that the Grunsky coefficients define on the tangent bundle $T(T)$ a new Finsler structure $F_\kappa(\varphi, v)$ dominated by the infinitesimal Teichmüller metric $F(\varphi, v)$. This structure generates on any embedded holomorphic disk $\gamma(D) \subset T$ the corresponding Finsler metric $\lambda_\gamma(t) = F_\kappa(\gamma(t), \gamma'(t))$ and reconstructs the Grunsky norm by integration along the Teichmüller disks:

Lemma 4. \cite{11} On any extremal Teichmüller disk $D(\mu_0) = \{ \phi_T(t\mu_0) : t \in D \}$ (and its isometric images in $T$), we have the equality
\[
\tanh^{-1}[\kappa(f^{\mu_0})] = \int_0^r \lambda_\kappa(t)dt.
\]

Taking into account that the disk $D(S_f)$ touches at the point $\varphi = Sfa$ the Teichmüller disk centered at the origin of $T$ and passing through this point and that the metric $\lambda_\kappa$ does not depend on the tangent unit vectors whose initial points are the points of $D(S_f)$, one obtains from Lemma 3 and the equality $\kappa(f_a) = \kappa(f_a)$ that also
\[
\lambda_\kappa(a) = \lambda_\kappa(a).
\]

We compare the metric $\lambda_{\tilde{h}_x b}$ with $\lambda_\kappa$ using Lemma 2 and Minda’s maximum principle given by

Lemma 5. \cite{23} If a function $u : D \to [-\infty, +\infty]$ is upper semicontinuous in a domain $D \subset \mathbb{C}$ and its (generalized) Laplacian satisfies the inequality $\Delta u(z) \geq Ku(z)$ with some positive constant $K$ at any point $z \in D$, where $u(z) > -\infty$, and if
\[
\limsup_{z \to \zeta} u(z) \leq 0 \quad \text{for all } \zeta \in \partial D,
\]
then either $u(z) < 0$ for all $z \in D$ or else $u(z) = 0$ for all $z \in D$.

Take a sufficiently small neighborhood $U_0$ of the point $t = a$, and let
\[
M = \{ \sup \lambda_\kappa(t) : t \in U_0 \};
\]
then in this neighborhood, $\lambda_\kappa(t) + \lambda_{\tilde{h}_x b}(t) \leq 2M$ and the function
\[
u = \log \frac{\lambda_{\tilde{h}_x b}}{\lambda_\kappa} = \log \lambda_{\tilde{h}_x b} - \log \lambda_\kappa
\]
satisfies
\[
\Delta u = 4(\lambda_{\tilde{h}_x b}^2 - \lambda_\kappa^2) \geq 8M(\lambda_{\tilde{h}_x b}^2 - \lambda_\kappa).
\]
The elementary estimate
\[
M \log(t/s) \geq t - s \quad \text{for } 0 < s \leq t < M
\]
(with equality only for \( t = s \)) implies that

\[
M \log \frac{\lambda_{\hat{h}_{k^1}}(t)}{\lambda_{\hat{c}}(t)} \geq \lambda_{\hat{h}_{k^1}}(t) - \lambda_{\hat{c}}(t),
\]

and hence,

\[
\mathbb{D}u(t) \geq 4M^2u(t).
\]

Lemma 5 and the equality (13) imply that the metrics \( \lambda_{\hat{h}_{k^1}}, \lambda_{\hat{c}} , \lambda_{\hat{c}} \) must be equal in the entire disk \( \mathbb{D}(S_F) \), which yields by Lemma 3 the equality

\[
\lambda_{\hat{c}}(F_r) = k(F_r) = \left| \sum_{m,n=1}^{\infty} \sqrt{mn} \alpha_{mn}(F_1)r^{m+n}x_m^*x_n^* \right|
\]

for all \( r = |t| \in (0,1) \) (with \( (x_n^*) \in S(P^2) \) depending on \( r \)) and that for any \( F \in \Sigma^0 \) with \( \lambda_{\hat{c}}(F) = k(F) \) its homotopy disk \( \mathbb{D}(S_F) \) has only a singularity at the origin of \( T \).

20. We may now investigate the action of affine deformations on the set of functions \( F \in \Sigma^0 \) with equal Grunsky and Teichmüller norms.

**Lemma 6.** For any affine deformation \( g^c \) of a convex domain \( D \) with expansion \( g^c(w) = w + b_0^c + b_1^c w^{-1} + \ldots \) near \( w = \infty \), we have

\[
b_1^c = \frac{S_{g^c}(\infty)}{6} = \frac{1}{6} \lim_{z \to \infty} w^4 S_{g^c}(w) \neq 0,
\]

and for sufficiently small \( |c| \) all composite maps

\[
W_{F,c}(z) = g^c \circ F(z) = z + \hat{b}_0^c + \hat{b}_1^c z^{-1} + \ldots, \quad F \in \Sigma^0
\]

also satisfy \( \hat{b}_1^c \neq 0 \).

**Proof.** One can assume that \( \Gamma = \partial D \) is a smooth curve. For small \( |c| \),

\[
g^c(w) = w - \frac{c}{\pi} \int_D \frac{dxdy}{z-w} + O(c^2),
\]

and by the Cauchy-Green formula,

\[
\iint_D \frac{dxdy}{z-w} = 2i \int_{\Gamma} \frac{zd\zeta}{z-w} = -\frac{2i}{w} \int_{\Gamma} z\zeta + O\left(\frac{1}{w^2}\right) = -\frac{1}{w} \int_D dxdy + O\left(\frac{1}{w^2}\right), \quad w \to \infty;
\]

hence, \( b_1 \neq 0 \), which proves (14).

The second assertion of the lemma follows from (14) and the equality

\[
6\hat{b}_1^c = \lim_{z \to \infty} z^4 S_{W_{F,c}}(z) = \lim_{z \to \infty} z^4 \left[ (S_{g^c} \circ F)(z)(F'(z)^2 + S_F(z)) \right]
\]

for any \( F \in \Sigma^0 \), which completes the proof.

Lemma 6 allows one to apply to compositions \( W_{F,c} \) the following result of Kühnau [19]:

**Lemma 7.** For any function \( F(z) = z + b_0 + b_1 z^{-1} + \cdots \in \Sigma(0) \) with \( b_1 \neq 0 \), the extremal quasiconformal extensions of the homotopy functions \( F_t \) to \( \mathbb{D} \) are defined for sufficiently small \( |t| \leq r_0 = r_0(F) \) \( (r_0 > 0) \) by nonvanishing holomorphic quadratic differentials, and therefore, \( \lambda_{\hat{c}}(F_t) = k(F_t) \).
It follows from Lemmas 6 and 7 that for any $F \in \Sigma^0$ and any affine transformation $g^c$ of domain $F(\mathbb{D})$ the homotopy functions

$$W_{F,c,t} = g^c \circ F_t$$

of maps (15) satisfy

$$\kappa(g^c \circ F_t) = k(g^c \circ F_t) \text{ for all } |t| \leq r_0(F, g^c) \ (r_0(F, g^c)).$$

Note also that since $F$ was chosen to be holomorphic on $\mathbb{D}^*$ and its homotopy disk has only a singularity at $t = 0$, the equality

$$S_{W_{F,c}}(z) = (S_{g^c} \circ F)(z)(F'(z)^2 + S_F(z))$$

implies that for sufficiently small $|c|$ the homotopy disk $\mathbb{D}(S_{W_{F,c}})$ of $W_{F,c}$ also has only a singularity at $t = 0$. Fix a such $c$.

Using the restrictions to $\mathbb{D}(S_{W_{F,c}})$ of the corresponding holomorphic maps

$$h_x(S_{W_{F,c}}) = \sum_{m,n=1}^{\infty} \sqrt{mn} \alpha_{mn}(W_{F,c})x_mx_n : \mathbb{T} \to \mathbb{D}$$

one obtains for $\hat{h}_x = h_x(S_{W_{F,c}}) \circ \chi_{W_{F,c}}$, the conformal metrics

$$\lambda_{\hat{h}_x}(t) = \frac{|\hat{h}'_x(t)||dt|}{1 - |\hat{h}_x(t)|^2}$$

of curvature $-4$. Their upper envelope $\lambda_\kappa(t)$ (followed by its upper semicontinuous regularization) is a subharmonic metric of generalized Gaussian curvature $\kappa_\lambda \leq -4$. It can be compared with the infinitesimal Kobayashi metric $\lambda_K$ on $\mathbb{D}(S_{W_{F,c}})$ similar to the previous step $1^0$ by applying Lemmas 3 and 4. This implies the equalities $\lambda_\kappa = \lambda_K$ on $\mathbb{D}(S_{W_{F,c}})$ and

$$\kappa(W_{F,c}) = k(W_{F,c}).$$

$3^0$. It remains to extend the last equality to all $c$ with $|c| < 1$. Noting that by the chain rule for Beltrami coefficients $\mu, \nu$ from the unit ball in $L_\infty(\mathbb{C})$,

$$w^\mu \circ w^\nu = w^\tau \text{ with } \tau = (\nu + \tilde{\mu})/(1 + \overline{\nu}\tilde{\mu})$$

and $\tilde{\mu}(z) = \mu(w^\nu(z))w_z^\nu/w_z^\nu$ (so for $\nu$ fixed, $\tau$ depends holomorphically on $\mu$ in $L_\infty$ norm), one can regard (16) as holomorphic functions of $c \in \mathbb{D}$ and construct in a similar way the corresponding Finsler metrics

$$\lambda_{h_x}(c) = \frac{|\hat{h}'_x(c)||dc|}{1 - |\hat{h}_x(c)|^2}, \ |c| < 1.$$ 

Now take the upper envelope $\lambda_\kappa(c)$ of these metrics and its upper semicontinuous regularization getting now a subharmonic metric of Gaussian curvature $\kappa_\lambda \leq -4$ on the nonsingular disk $\{|c| < 1\}$. One can repeat for this metric all the above arguments using the already established equality (17) for small $|c|$. The assertion of Theorem 1 follows again by applying Lemmas 4 and 5, which completes the proof of Theorem 1.

4. PROOF OF THEOREM 3
Take a function \( F_0(z) = z + b_0 + b_m z^{-m} + \cdots \in \Sigma^0 \) with \( m > 1 \), \( F_0(0) = 0 \), mapping the disk \( \mathbb{D}^* \) onto a domain \( D \) with smooth boundary, hence having Teichmüller quasiconformal extension to \( \hat{\mathbb{C}} \) with Beltrami coefficient

\[
\mu_0(z) = k_0|\psi(z)|^p/\psi(z), \quad |z| < 1,
\]

where \( \psi_0 \in A_1(\mathbb{D}) \) is of the form

\[
\psi_0(z) = c_p z^p + c_{p+1} z^{p+1} + \cdots \quad c_p \neq 0, \quad p \text{ odd}
\]

(then \( \kappa(F_0) < k(F_0) = k_0 \)) and such that also its homotopy functions

\[
F_{0t}(z) = tF_0(z/t), \quad z \in \mathbb{D}^*, \quad 0 \leq t < 1,
\]

satisfy

\[
\kappa(F_{0t}) < k(F_{0t}) \quad \text{for} \quad 0 < t < 1 \quad (18)
\]

(one can use, for example, the map \( F_0(z) \) conformal in \( \mathbb{D}^* \) and having on the unit disk the Beltrami coefficient \( \mu_0(z) = k_0|z|^p/z^p \) with odd \( p \)). Its inversion \( f_0(z) = 1/F_0(1/z) = z-b_0z^2+\ldots \) maps, by the well-known geometric properties of univalent functions, the disks \( \mathbb{D}_r = \{|z| < r\}, \quad r < 1 \), images \( F_0(\mathbb{D}_r) \) of the disks \( \mathbb{D}_r = \{|z| < r\} \) onto convex domains for all \( r \leq 2 - \sqrt{3} \), and

\[
\kappa(f_0) = \kappa(F_0) = k(f_0).
\]

Take a fixed \( r < 2 - \sqrt{3} \) so that \( \|S_{f_0}\|_B < 2 \) and a dense subset \( E = \{z_1, z_2, \ldots, z_n, \ldots\} \) on the unit circle \( S^1 = \partial \mathbb{D} \). Now consider the convex rectilinear polygons \( P_n \) located in the interior of quasicircle \( L_0 = f_{0r}(S^1) \) with vertices \( f_{0r}(z_1), \ldots, f_{0r}(z_n) \) on \( L_0 \), and let \( F_{P_n} \) be an appropriately normalized conformal map of \( \mathbb{D}^* \) onto the complement of \( P_n \). Then, by (18) and Proposition 1, there exists a natural \( n_0 \) such that

\[
\kappa(F_{P_n}) < k(F_{P_n}) \quad \text{for any} \quad P_n \quad \text{with} \quad n > n_0.
\]

In view of invariance of both sides of (10) under the Möbius maps of \( \hat{\mathbb{C}} \) and by the Kühnau-Schiffer theorem the last inequality is equivalent to (3). This completes the proof of Theorem 3.

5. PROOF OF THEOREM 4

In view of Proposition 1, it sufficed to prove the theorem for bounded quadrilaterals \( P_4 = A_1A_2A_3A_4 \) with vertices \( A_j \) (ordered according to positive direction of \( \partial P_4 \)) such that that the line in \( P_4 \) drawn from the vertex \( A_1 \) parallel to the opposite edge \( A_2A_3 \) separates this edge from the remaining vertex \( A_4 \).

Fix such a quadrilateral \( P_4^0 = A_1^0A_2^0A_3^0A_4^0 \) and consider the collection \( \mathcal{P}^0 \) of quadrilaterals \( P_4 = A_1^0A_2^0A_3^0A_4 \) with the same first three vertices and variable \( A_4 \); the corresponding \( A_4 \) runs over a subset \( E \) of the trice punctured sphere \( \hat{\mathbb{C}} \setminus \{A_1^0, A_2^0, A_3^0\} \).

The conformal map \( F \) of the disk \( \mathbb{D}^* \) onto the complementary domain \( \overline{P_4} = \hat{\mathbb{C}} \setminus \overline{P_4} \) is represented by the Schwarz-Christoffel integral

\[
F(z) = d_1 \int_0^z \prod_{j=0}^{4} (\zeta - e_j)^{\alpha_j - 1} \frac{d\zeta}{\zeta^2} + d_0,
\]

where \( e_j = F^{-1}(A_j) \in S^1, \pi\alpha_j \) is the interior angle at \( A_j \) for \( P_4^* \), and \( d_0, d_1 \) are two complex constants. Let \( F^0 \) denote the conformal map for the complement of \( P_4^0 \).
One obtains from the general properties of quasiconformal maps and (19) that the logarithmic derivatives \( b_F = (\log F')' = F''/F' \) of maps \( F \) defining the quadrilaterals \( P_4 \in \mathcal{P}^0 \) are (for a fixed \( z \)) real analytic functions of \( t = A_4 \). Passing to their Schwarzians

\[
S_F = b_F - \frac{1}{2} b_F^2 \in \mathcal{T}
\]

one can find a smooth real arc \( \Gamma = b(t) \subset \mathcal{T} \) containing the point \( S_{F_0} \) and the points corresponding to trapezoids; here \( b \) denotes the map \( t = A_4 \rightarrow S_F \).

Since \( \mathcal{T} \) is a domain, there is a tubular neighborhood containing \( \Gamma \) therefore, \( \Gamma \) is located on some nonsingular holomorphic disk of the form \( \Omega_0 = F(G_0) \subset \mathcal{T} \), where \( G_0 \) is a simply connected planar domain containing the set \( E \). This disk is not geodesic in the Teichmüller-Kobayashi metric on \( \mathcal{T} \) and does not pass through the basepoint \( \varphi = 0 \) of this space, but one can apply to it the same arguments as in the proof of Theorem 1 constructing similar to (8) the holomorphic maps

\[
h_x(\varphi) = \sum_{m,n=1}^{\infty} \sqrt{m \cdot n} \alpha_{mn}(F) x_m x_n : \mathcal{T} \rightarrow \mathbb{D} \quad (\varphi = S_F, \ x \in S(\mathbb{I}^2)).
\]

The restrictions of these maps to the disk \( \Omega_0 \) determine (again by pulling back the hyperbolic metric of the disk) the corresponding conformal metrics

\[
\lambda_{\tilde{h}_x}(t) = \frac{\|\tilde{h}_x'(t)\| dt}{1 - |\tilde{h}_x(t)|^2},
\]

and their upper semicontinuous envelope \( \lambda_x \) is a subharmonic metric on \( \Omega_0 \) of (generalized) Gaussian curvature \( \kappa(\lambda_x) \leq -4 \).

Noting that the collection \( \mathcal{P}^0 \) contains the trapezoids, for which we have the equalities (4) by Theorem 1 (and consequently, the infinitesimal equality (13) at the corresponding points \( t \)), one again obtains by applying Lemma 4 that the constructed metric \( \lambda_x \) must coincide at all points of \( \Omega_0 \) with the dominant infinitesimal Teichmüller-Kobayashi metric \( \lambda_K \) of \( \mathcal{T} \). Together with Lemma 5, this provides the global equalities (4) for all points of the disk \( \Omega_0 \), which yields the assertion of Theorem 4 for a given quadrilateral \( P_4^0 \).

### 6. ADDITIONAL REMARKS

1. Generically in Lemma 7, \( r_0(F) < 1 \); this is caused by the critical points of the disk \( \mathbb{D}(S_F) \) and circular symmetry of both infinitesimal metrics \( \lambda_x \) and \( \lambda_K \) on this disk.

2. Another reason why the convex polygons are interesting for quasiconformal theory is their close geometric connection with the geometry of universal Teichmüller space. The relations of type (1) are valid for bounded convex rectilinear polygons \( P_n \) in the following truncated form. Denoting the vertices of \( P_n \) by \( A_j \) and their interior angles by \( \pi \alpha_j \) (\( j = 1, \ldots, n \)), one represents the conformal map \( f_n \) of the upper half-plane \( H = \{ z : \operatorname{Im} z > 0 \} \) onto \( P_n \) by the Schwarz-Christoffel integral

\[
f_n(z) = d_1 \int_0^z (\xi - a_1)^{\alpha_1 - 1} (\xi - a_2)^{\alpha_2 - 1} \cdots (\xi - a_n)^{\alpha_n - 1} d\xi + d_0,
\]
(with \( a_j = f_n^{-1}(A_j) \in \mathbb{R} \) and complex constants \( d_0, d_1 \)). Its Schwarzian derivative is given by

\[
S_{f_n}(z) = b_{f_n}'(z) - \frac{1}{2} b_{f_n}''(z) = \sum_{j=1}^{n} \frac{C_j}{(z-a_j)^2} - \sum_{j,l=1}^{n} C_{jl}(z-a_j)(z-a_l),
\]

where \( C_j = \alpha_j - 1 - (\alpha_j - 1)^2/2 < 0 \), \( C_{jl} = (\alpha_j - 1)(\alpha_l - 1) > 0 \). It is a point of the universal Teichmüller space \( T \) modeled as a bounded domain in the space \( B(H) \) of hyperbolically bounded holomorphic functions on \( H \) with norm \( \| \varphi \|_B = \sup_H |z - \overline{z}|^2 |\varphi(z)| \).

Denote by \( r_0 \) the positive root of the equation

\[
\frac{1}{2} \left[ \sum_{j=1}^{n} (\alpha_j - 1)^2 + \sum_{j,l=1}^{n} (\alpha_j - 1)(\alpha_l - 1) \right] r^2 - \sum_{j=1}^{n} (\alpha_j - 1) r - 2 = 0,
\]

and put \( S_{f_{n,t}} = b_{f_n}' - b_{f_n}'',/2, t > 0 \). Then we have

**Proposition 2.** [15] For any convex polygon \( P_n \), the Schwarzians \( rS_{f_{n,r}} \) define for any \( 0 < r < r_0 \) a univalent function \( w_r : H \to \mathbb{C} \) whose harmonic Beltrami coefficients \( \nu_r(z) = -r(2y^2)S_{f_{n,r}}(\overline{z}) \) is extremal in its equivalence class, and

\[
k(w_r) = \nu_r(w_r) = \frac{r}{2} \| S_{f_{n,r}} \|_B.
\]

By the Ahlfors-Weill theorem [3], every \( \varphi \in B(H) \) with \( \| \varphi \|_B < 1/2 \) is the Schwarzian derivative \( S_W \) of a univalent function \( W \) in \( H \), and this function has quasiconformal extension onto the lower half-plane \( H^* = \{ z : \text{Im } z < 0 \} \) with Beltrami coefficient of the form

\[
\mu_\varphi(z) = -2y^2 \varphi(\overline{z}), \quad \varphi = S_f(z = x + iy \in H)
\]

called harmonic. Proposition 2 yields that any \( w_r \) with \( r < r_0 \) does not admit extremal quasiconformal extensions of Teichmüller type, and in view of extremality of harmonic coefficients \( \mu_{S_{w_r}} \) the Schwarzians \( S_{w_r} \) for some \( r \) between \( r_0 \) and 1 must lie outside of the space \( T \); so this space is not a starlike domain in \( B \).

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