Viscoelastic modes in a strongly coupled cold magnetized dusty plasma

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A generalized hydrodynamical model has been used to study low frequency modes in a strongly coupled, cold, magnetized dusty plasma. Such plasmas exhibit elastic properties due to strong correlations among dust particles and the tensile stresses imparted by the magnetic field. It has been shown that longitudinal compressional Alfven modes and elasticity modified transverse shear mode exist in such a medium. The features of these collective modes are established and discussed.

I. INTRODUCTION

Dusty plasmas are electron ion plasmas together with micron-sized dust grains that can carry several thousand elementary charges. The competition between the average Coulomb interaction energy between the dust particles and the average thermal energy is characterized by the Coulomb coupling parameter $\Gamma = q_d^2/(k_B T_d r)$ where $q_d$ is the charge on the dust grains, $r(\simeq n_d^{-1/3})$ is the average distance between them, $T_d$ is the temperature of the dust component and $k_B$ is the Boltzmann constant. The high, typically negative charge on the dust leads to large values of the Coulomb parameter $\Gamma \gg 1$ even at room temperature and the plasma is said to be in a strongly coupled state. A broad variety of systems in the astrophysical context such as interior of heavy planets, white dwarfs, neutron stars have matter in the strongly coupled state. In the laboratories, we have examples of strongly coupled plasmas produced for plasma processing and industrial applications, in semiconductor heterojunctions and in the laser implosion experiments with $\Gamma$ values in the range $1 - 100$. Both crystalline and fluid properties can coexist in a strongly coupled dusty plasma (or colloidal plasma). While in such a state, a plasma possess viscous properties typical of fluids as well as elastic properties similar to solids. The dependence of viscous and elastic coefficients of a medium on $\Gamma$ is known in the case of one component plasmas and Yukawa liquids. The well known weakly coupled ideal coulomb plasma (gas phase) characterized by $\Gamma \ll 1$ has no elastic property and trivial role of viscosity. At $\Gamma \sim 1$ viscosity comes into the system profoundly and as $\Gamma$ increases elastic property gradually becomes important as the plasma state switches over from the gaseous phase to liquid phase. When $\Gamma > \Gamma_c$ (beyond $\Gamma_c$ system becomes crystalline), viscosity disappears and only elasticity reigns over the system. So in the regime of $\Gamma$ from 1 to $\Gamma_c$ both viscosity and elasticity are of simultaneous concern and this property together is known as visco-elasticity. An analogous behavior occurs in the case of fluids made of large macromolecules such as polymer molecules that also exhibit viscoelastic property in contrast to fluids made of small molecules.

The presence of dust grains in a plasma with their charges and masses that are orders of magnitudes higher than that of ions give rise to new wave phenomena that are associated with longer time and length scales. In the strongly coupled state, the dusty plasma offers yet another advantage of studying wave phenomena that are typical of solids such as the transverse waves.

Using generalized hydrodynamic model, many authors had shown that longitudinal dust acoustic mode must be corrected by an additional term in the strongly coupled regime. They also found a new transverse shear-like mode which comes into effect due to the elastic property imparted to the medium due to the presence of dust particles. In a weakly coupled magnetized plasma, it is well known that the presence of magnetic field makes the medium elastic, enabling propagation of shear Alfven waves with Alfven speed $V_\alpha^2 = B^2/\mu_0 \rho_0$. Such kind of transverse waves do not exist in an un-magnetized plasma in the weakly coupled state. On the other hand, the presence of strong correlations among particles, leads to mechanical shear stresses that sustain the propagation of transverse waves like in solids. It is of interest to see the nature of the elastic mode that can propagate in a cold magnetized strongly coupled plasma. In this mode the effects of the elasticity due to the simultaneous presence of magnetic field as well as due to strongly coupled nature of the dust particles are taken into account. Basic equations for this study has been written in the framework of a generalized magnetohydrodynamic model to include viscoelastic effects. It is found that for both longitudinal and transverse perturbations, elasticity modified Alfven type modes can propagate that can be termed ‘magnetoelastic modes’. The dynamics of the outer crust of magnetized neutron stars consisting of elastic solid media permeated by frozen magnetic fields is likely to be governed by such modes.

In Section -II we present the generalized hydrodynamic model containing the Basic equations which supports viscoelastic stresses. In section III, we describe the analysis of mode dispersion by a linear stability analysis. In section IV, we have presented a short summary of this work.
II. MODEL AND BASIC EQUATIONS

We shall assume that the strongly coupled dusty plasma consists of the electrons and ions that are weakly coupled and highly charged dust particles with strong correlations among them. We also consider the effects of an external magnetic field $B = B_0 \hat{z}$ on such a strongly coupled plasma. The medium acquires an elastic property because of the tensile stresses exerted by the magnetic field lines as well as the strongly correlated dust grains. We wish to write down the magnetohydrodynamic equations for a strongly coupled magnetized dusty plasma with an intention to study the dispersion relations which exhibit coupling between low frequency waves that arise due to magnetic and solid-like stresses. We assume the characteristic wave frequency to be much smaller than the ion gyrofrequency, where dust dynamics is important. In such a situation the ion and electron inertial forces are much smaller than the corresponding Lorentz forces. Therefore equations of motion for the electron and ion fluids can be written as:

\begin{align}
0 &= -en_e(E + v_e \times B), \\
0 &= en_i(E + v_i \times B),
\end{align}

where, $n_{e,i}$ is the number density of electrons and ions fluid and $v_{e,i}$ is corresponding velocity. The electric and magnetic fields are denoted as $E$ and $B$ respectively. For the dust fluid we adopt the generalized magnetohydrodynamic model. Since a micron size dust grain may contain several thousand elementary charges, in principle dust fluid is highly viscous compared to electron and ion fluids, so we consider viscosity terms only in the context of the dust momentum equation. The momentum equation for the dust fluid is

\begin{align}
m_d \frac{d\mathbf{v}_d}{dt} &= Z e n_d (E + \mathbf{v}_d \times B) + \eta \nabla^2 \mathbf{v}_d + \left( \frac{\eta}{3} \right) \nabla (\nabla \cdot \mathbf{v}_d),
\end{align}

where, $n_d$ is the number density of dust fluid, $\mathbf{v}_d$ is the dust fluid velocity, $Z$ is the number of negative charges on a single dust particle, $\eta$ and $\xi$ are the dust shear viscosity and bulk viscosity coefficient respectively. The physical interpretation of the above equation is illustrated in Ref. 4.

Next we shall define the mass density, center of mass fluid flow velocity and current density for the bulk fluid. First mass density may be defined as $\rho_m = m_e n_e + m_i n_i + m_d n_d$. Since $m_e, m_i \ll m_d, \rho_m = \rho_d \approx m_d n_d$. Then bulk velocity $\mathbf{v} = (m_e n_e \mathbf{v}_e + m_i n_i \mathbf{v}_i + m_d n_d \mathbf{v}_d) / (m_d n_d) \approx \mathbf{v}_d$ and finally the current density is defined as $\mathbf{J} = e(n_i \mathbf{v}_i - n_e \mathbf{v}_e - Z n_d \mathbf{v}_d)$. The current density $\mathbf{J}$ related to the magnetic field $B$ through Ampere’s law is given by

\begin{align}
\nabla \times \mathbf{B} = \mu_0 \mathbf{J} = \mu_0 e(n_i \mathbf{v}_i - n_e \mathbf{v}_e - Z n_d \mathbf{v}_d).
\end{align}

Note here that we have neglected the displacement current in the above equation since we are interested in low frequency ($\omega \ll \omega_{pd}$) and long wavelength perturbations. By adding the equations (1), (2) and (3) and working in the MHD approximation for an viscous dusty plasma with infinite electrical conductivity and also with the quasineutrality condition $n_i \approx n_e + Z n_d$ we can write down the single fluid momentum equation of the bulk fluid as

\begin{align}
\rho_d \frac{d\mathbf{v}}{dt} &= \mathbf{J} \times \mathbf{B} + \eta \nabla^2 \mathbf{v} + \left( \frac{\eta}{3} \right) \nabla (\nabla \cdot \mathbf{v}).
\end{align}

In the above equation apart from the viscous force on the righthand side there is also $\mathbf{J} \times \mathbf{B}$ force and to find the evolution of the magnetic field we need to find the electric field. For this first we add equations (1), (2) and then use quasineutrality condition ($n_i \approx n_e + Z n_d$) and the expression for current density to get the following form of the generalized Ohm’s law

\begin{align}
\mathbf{E} &= -\nabla \times \mathbf{B} - \frac{\mathbf{J} \times \mathbf{B}}{Ze n_d}.
\end{align}

Taking curl of the above equation and using Faraday’s law ($\nabla \times \mathbf{E} = -\partial \mathbf{B} / \partial t$), the time evolution of magnetic field for the bulk dusty plasma can be obtained as,

\begin{align}
\frac{\partial \mathbf{B}}{\partial t} &= \nabla \times (\mathbf{v} \times \mathbf{B}) + \frac{\nabla \times (\mathbf{J} \times \mathbf{B})}{Ze n_d},
\end{align}

where the first term in the right hand side is the convective term and the second one is the Hall term. In the limit of large magnetic Reynold’s number that is being considered here, the magnetic field lines can be assumed to be frozen in the dusty plasma and convected with the plasma fluid flow when the contributions of the Hall term are neglected.
The ratio of the Hall to the convection term can be estimated as \( \sim v_A/L_\omega \sim \delta_d/L \), where \( \omega_{cd} = ZeB_0/m_d \) and \( \delta_d = v_A/\omega_{cd} = c/\omega_{pd} \) are the dust cyclotron frequency and dust skin depth, and \( L \) and \( v_A = B_0/\sqrt{\mu_0\rho_{d0}} \) are the characteristic length and dust Alfvén velocity of the system. For waves with scale length \( L \gg \delta_d \), the Hall term can be neglected.

A strongly correlated dusty plasma system can be considered as a viscoelastic medium. In such a medium normal fluid like equations are modified due to the growing correlation between dust particles. Normal fluid viscosity coefficient in a viscoelastic medium becomes viscoelastic operator as described in detail in Frenkel’s book. We follow the same procedure and write the generalized equation of motion of dust fluid in a viscoelastic medium as

\[
\left(1 + \tau \frac{d}{dt}\right) \left[ \rho_d \frac{dv}{dt} - J \times B \right] = \eta \nabla^2 v + \left( \xi + \frac{\eta}{3} \right) \nabla(\nabla \cdot v),
\]

where \( \tau \) is the relaxation time of the medium. Eq. (8) can be considered as the generalized magnetohydrodynamic equation that contains viscoelastic effects. From the above equation it is clear that in the absence of viscoelastic effect the equation is simply Navier Stokes equation where kinetic pressure is replaced by the magnetic pressure. Therefore the limit \( \omega \tau \ll 1 \), for which equation (8) reduces to the standard magnetohydrodynamic equation describing magnetized plasmas can also be termed as hydrodynamic limit in analogy with the Navier-Stokes like equation mentioned before. Equations (4), (7) and (8) are magnetohydrodynamic equations describing low frequency phenomena in a strongly coupled, cold magnetized dusty plasma. Although derived in the context of a magnetized dusty plasma, these equations have a general appeal and can be utilized for investigations of strongly coupled magnetized fluids such as those occurring in astrophysical systems.

### III. LINEAR STABILITY ANALYSIS

Before going to the stability analysis it is useful to explain the equilibrium. For simplicity we have assumed that in equilibrium plasma is homogeneous. The homogeneous plasma is described by the constant variables \( \rho_d = \rho_0, v = 0, B = B_0 \hat{z} \). With the equilibrium mentioned above we perturbed the system with a small amplitude perturbations i.e. \( v = v_1(r, t), B = B_0 + B_1(r, t) \) and \( J = 0 + J_1(r, t) \) where all the variables with subscript one are perturbations. Linearizing Eqs. (4), (7) and (8) around the equilibrium mentioned above we have

\[
\nabla \times B_1 = \mu_0 J_1,
\]

\[
\frac{\partial B_1}{\partial t} = \nabla \times (v_1 \times B_0),
\]

\[
\left(1 + \tau \frac{\partial}{\partial t}\right) \left[ \rho_0 \frac{\partial v_1}{\partial t} - J_1 \times B_0 \right] = \eta \nabla^2 v_1 + \left( \xi + \frac{\eta}{3} \right) \nabla(\nabla \cdot v_1).
\]

We consider that a wave is propagating making an angle \( \theta \) with unperturbed magnetic field \( B_0 \) i.e. wave vector \( \mathbf{k} \) and \( B \) are in the same plane with wave vector \( \mathbf{k} = k_x \hat{x} + k_z \hat{z} \). Since the above equations are linear we can Fourier transform these equations assuming the solutions for the perturbed variables are in the form \( \sim \exp[-i(\omega t - \mathbf{k} \cdot \mathbf{r})] \). Here \( \omega \) is the frequency and \( \mathbf{k} \) is the wave vector of the mode under consideration. Substituting perturbed solutions in Eqs. (9) - (11) we find

\[
\mathbf{k} \times B_1 = -i\mu_0 J_1,
\]

\[
(1 - i\omega \tau)[-i\omega \rho_0 v_1 - J_1 \times B_0] = -\eta k^2 v_1 - \left( \xi + \frac{\eta}{3} \right) \mathbf{k}(\mathbf{k} \cdot v_1),
\]

\[
\omega B_1 = B_0(\mathbf{k} \cdot v_1) - (B_0 \cdot \mathbf{k})v_1.
\]

In the limit \( \omega \tau \gg 1 \), it is possible to get a dispersion relation that describes purely propagating modes without any dissipative damping. This is known as the kinetic limit as opposed to the hydrodynamic one. In the electrostatic limit \( B_1 = 0 \) with \( \omega \tau \gg 1 \), from Eq. (13) one can find both compressional mode with \( \mathbf{k} \cdot v_1 \neq 0 \) and shear mode with \( \mathbf{k} \times v_1 \neq 0 \). The velocity of the shear wave and the compressional wave are found to be \( v_{sh}^2 = \eta/\tau \rho_0 \) and...
\[ v_e^2 = (\xi + \eta)/\tau p_0 \] as investigated before. Eliminating \( \mathbf{B}_1 \) and \( \mathbf{J}_1 \) from above three equations \([12], [14]\) we have equations for \( \mathbf{v}_1 \) as

\[
[\omega^2 - v_A^2 k_z^2 - v_{sh}^2 k_z^2] \mathbf{v}_1 + [(v_{sh}^2 - v_e^2)\mathbf{k} - \hat{x} k_x v_A^2] (\mathbf{k} \cdot \mathbf{v}_1) + k_z v_A (\mathbf{v}_A \cdot \mathbf{v}_1) \mathbf{k} = 0. \tag{15}
\]

To find the dispersion relation we have taken two different kinds of polarization for the velocity vector \( \mathbf{v}_1 \). First let us take \( \mathbf{v}_1 = v_{1x} \hat{x} + v_{1z} \hat{z} \) which means the velocity vector is polarized in the \((x - z)\) plane i.e. in the plane where the propagation vector lies. From Eq.\([15]\), considering \( x \) and \( z \) components the dispersion equation in matrix form can be written as,

\[
\begin{pmatrix}
\omega^2 - v_A^2 k_z^2 - v_{sh}^2 k_z^2 & -(v_{sh}^2 - v_e^2) k_x k_z \\
-(v_{sh}^2 - v_e^2) k_x k_z & \omega^2 - v_{sh}^2 k_z^2 - v_e^2 k_z^2
\end{pmatrix}
\begin{pmatrix}
v_{1x} \\
v_{1z}
\end{pmatrix} = 0. \tag{16}
\]

The dispersion relation can be obtained equating the determinant of the matrix to zero which is given by

\[
\frac{\omega^2}{k^2} = \frac{1}{2} \left[ \left( v_A^2 + v_c^2 + v_{sh}^2 \right) \pm \frac{1}{2} \left[ \left( v_A^4 + (v_c^2 - v_{sh}^2)^2 \right) - 2v_A^2 (v_c^2 - v_{sh}^2) \cos 2\theta \right]^{1/2} \right] \tag{17}
\]

where \( \cos \theta = k_z/k \). In the limit of both \( \xi \) and \( \eta \) going to zero, Eq.\([17]\) reduces to the pure compressional Alfvén wave propagating in a cold plasma that is partly longitudinal and partly transverse. For \( \theta = 0 \), we obtain

\[
\begin{align*}
\omega^2 &= k^2 (v_{sh}^2 + v_A^2) \quad \text{for} \quad v_{1z} = 0 \\
&= k^2 v_c^2 \quad \text{for} \quad v_{1x} = 0
\end{align*} \tag{18}
\]

with the two modes being transverse and longitudinal respectively. When the direction of propagation perpendicular to the unperturbed magnetic field i.e. \( \theta = \pi/2 \), then we have

\[
\begin{align*}
\omega^2 &= k^2 (v_c^2 + v_A^2) \quad \text{for} \quad v_{1z} = 0 \\
&= k^2 v_{sh}^2 \quad \text{for} \quad v_{1x} = 0
\end{align*} \tag{19}
\]

The transverse component in this case is a purely mechanical shear mode independent of magnetic field since the Lorentz force vanishes in this case. The longitudinal component depends on the magnetic pressure as well as pressure due to viscous forces. In the general case when the propagation is oblique with respect to the magnetic field we get mixed modes that are partly transverse and partially longitudinal type with the polarization in the plane generated by the magnetic field and the propagation direction.

Next, we consider the velocity perturbation perpendicular to the direction of propagation vector i.e. \( \mathbf{v}_1 = v_{1y} \hat{y} \). From Eq.\([15]\) we have

\[
(\omega^2 - v_A^2 k_z^2 - v_{sh}^2 k_z^2) v_{1y} = 0. \tag{20}
\]

For \( v_{1y} \neq 0 \), a transverse mode propagates in the \( x - z \) plane with phase velocity

\[
v_p = \frac{\omega}{k} = \sqrt{(v_A^2 \cos^2 \theta + v_{sh}^2)} \tag{21}
\]

When \( \theta = 0 \) i.e., when the transverse shear wave is propagating along the unperturbed magnetic field(\( \mathbf{B}_0 \)), the phase velocity becomes

\[
v_p = \frac{\omega}{k} = \sqrt{(v_A^2 + v_{sh}^2)} \tag{22}
\]

In the absence of magnetic field, Eq.\([22]\) reduces to the dispersion relation for a purely elastic mode obtained in Ref.\([6]\). In the magnetohydrodynamic limit (pure fluid with \( \eta = 0 \)), the above mode reduces to the well known shear Alfvén wave. The dispersion relations for circularly polarized transverse shear Alfvén waves have been derived earlier. In the very low frequency limit \( \omega << \omega_{cd} \), such dispersion relations reduce to the linearly polarized waves described by Eq.\([22]\).

\section*{IV. CONCLUSIONS}

A generalized magnetohydrodynamic equation describing a strongly coupled, magnetized, cold dusty plasma has been set up. The equation is utilized to derive the dispersion relation that describes coupling between modes that
arise due to magnetic and viscoelastic stresses. In analogy with the magnetoacoustic modes that are sound-like modes that propagate in a magnetized fluid, the compressional and shear modes that propagate in a magnetized elastic fluid can be termed as ‘magnetoelastic modes’.

From eq. (22) describing the transverse shear waves, after using the appropriate expressions for $\eta$ and $\tau$, we obtain:

$$\omega^2 = k^2 a_d^2 \frac{\lambda_d^2}{\rho_d^2} \omega^2 f(\Gamma) + \frac{c^2}{a_d^2} \frac{\omega_d^2}{\omega_{pd}^2}$$

with

$$f(\Gamma) = \left[ 1 - \gamma_d \left( 1 + \frac{u}{3} + \frac{\Gamma}{9} \frac{\partial u}{\partial \Gamma} + \frac{4}{15} u(\Gamma) \right) \right]$$

where $a_d$ is the Wigner-Seitz radius, $\gamma_d$ is the adiabaticity constant, and $u(\Gamma)$ is the excess internal energy of the system. In the limit of $1 \leq \Gamma \leq 200$, $u(\Gamma) = -0.9\Gamma + 0.95\Gamma^{1/4} + 0.19\Gamma^{-1/4} - 0.81$. In obtaining the dispersion relations for the magnetoelastic modes, we have neglected the effects of dust-neutral collisions. The effect of dust-neutral collisions, that are important in many experimental situations can be incorporated by replacing $\omega^2$ by $\omega(\omega + i\nu_{dn})$, where $\nu_{dn}$ is the dust-neutral collision frequency. Collisional effects can thus be considered to be negligible when the following condition holds

$$\frac{\nu_{dn}}{\omega_{pd}} \ll k\lambda_{Dd} \sqrt{f(\Gamma)}.$$

For $\lambda_{Dd} \leq a_d$ and for $ka_d \approx 0.1$, collisions can be neglected. Since the dust-neutral collision frequency is proportional to neutral gas pressure, experimentally, the modes can be observed at low neutral gas pressures.

The thermal contribution from electrons, ions and dust particles described by a total plasma pressure $p$ is known to lead to magnetosonic dust modes with the effective pressure given by $p + B^2/\mu_0$ where the comparison is between plasma pressure and magnetic pressure terms. In the present work, we have considered only a comparison between mechanical stresses and magnetic stresses without considering the plasma pressure terms. For transverse type stresses, there is no contribution from the plasma pressure terms.

The presence of magnetic fields in a dusty plasma can alter the currents on the dust surface thereby changing the nature of the dust charging mechanisms. Theoretical estimates have shown that the value of dust charge in magnetic fields depends on the size of the dust particle relative to the ion and electron gyro-radii and for strong fields, the dust charge can be substantially larger than the values in the absence of or weak magnetic fields.

For the magnetic and elastic effects to be of comparable importance the values of magnetic field strength should be such as to satisfy

$$\frac{\lambda_d^2\omega_d^2}{c^2} f(\Gamma) \approx \frac{\omega_d^2}{\omega_{pd}^2}$$

This leads to a dust larmor radius $\rho_{Ld}$ given by $\rho_{Ld} \approx c/\omega_{pd}\sqrt{f(\Gamma)}$. This gives a region of parameter space where the density, magnetic field, temperature and the coupling parameter $\Gamma$ should satisfy $n\rho_{Ld} f(\Gamma) \approx B^2/\mu_0$. The value of dust charge enters the above condition on different parameters through the value of $\Gamma$. Any dependence of the dust charge on the value of the magnetic field strength should manifest through the value of $\Gamma$ for which no explicit relation is known yet. In strongly coupled laboratory plasmas, where the typical dust densities and temperatures are $\sim 10^5 \text{cm}^{-3}$, $3 - 5 \text{eV}$, and $\Gamma$ in the range $1 < \Gamma < 50$, the values of magnetic field where the magneto-elastic effects are important will lead to large dust Larmor radii and almost unmagnetized dust grains. For high density astrophysical plasmas such as those occurring in white dwarfs and neutron stars where there exist a wide range of magnetic field strengths, the combined action of Hooke’s elastic and Lorentz magnetic force has been suggested to consistently interpret the detected Quasi-periodic oscillations. In such scenario, the generalized magnetohydrodynamic equations incorporating both viscoelastic and magnetic contributions can be considered as an appropriate model for studying various wave and instabilities.

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