ON PRESENTATIONS OF GENERALIZATIONS OF BRAIDS WITH FEW GENERATORS

V. VERSHININ

Abstract. In his initial paper on braids E. Artin gave a presentation with two generators for an arbitrary braid group. We give analogues of this Artin’s presentation for various generalizations of braids.

The diverse aspects of presentations of braid groups and their generalizations continue to attract attention [12], [5], [15]. The canonical presentation of the braid group $Br_n$ was given by E. Artin [1] and is well known. It has the generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ and relations

\[
\begin{align*}
\sigma_i \sigma_j &= \sigma_j \sigma_i, \text{ if } |i-j| > 1, \ i,j = 1, ..., n-1; \\
\sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}, \ i = 1, ..., n-2.
\end{align*}
\]

There exist other presentations of the braid group. J. S. Birman, K. H. Ko and S. J. Lee [5] introduced the presentation with generators $a_{ts}$ with $1 \leq s < t \leq n$ and relations

\[
\begin{align*}
a_{ts} a_{rq} &= a_{rq} a_{ts} \text{ for } (t-r)(t-q)(s-r)(s-q) > 0, \\
a_{ts} a_{sr} &= a_{tr} a_{ts} = a_{sr} a_{tr} \text{ for } 1 \leq r < s < t \leq n.
\end{align*}
\]

The generators $a_{ts}$ are expressed in the canonical generators $\sigma_i$ as follows:

\[
a_{ts} = (\sigma_{t-1} \sigma_{t-2} \cdots \sigma_{s+1}) \sigma_s (\sigma_{s+1}^{-1} \cdots \sigma_{t-2}^{-1} \sigma_{t-1}^{-1}) \text{ for } 1 \leq s < t \leq n.
\]

An analogue of the Birman-Ko-Lee presentation for the generalizations of braids (namely, for the singular braid monoid) was obtained in [15].

In the initial paper [1] Artin gave another presentation of the braid group, with two generators, say $\sigma_1$ and $\sigma$, and the following relations:

\[
\begin{align*}
\sigma_1 \sigma_1^{-i} &= \sigma_1 \sigma_i \sigma_1^{-i} \sigma_1 \text{ for } 2 \leq i \leq n/2, \\
\sigma^n &= (\sigma \sigma_1)^{n-1}.
\end{align*}
\]

The connection with the canonical generators is given by the formulae:

\[
\sigma = \sigma_1 \sigma_2 \cdots \sigma_{n-1},
\]

\[
\sigma_{i+1} = \sigma_i \sigma_1 \sigma_i^{-1}, \ i = 1, \ldots n - 2.
\]

This presentation was also discussed in the book by H. S. M. Coxeter and W. O. J. Moser [8].

It is interesting to obtain the analogues of the presentations of the type [1] for various generalizations of braids [6], [9], [2], [4], [11], [14].

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Let us consider the braid group of a sphere $Br_n(S^2)$. It has the presentation with generators $\delta_i$, $i=1,...,n-1$ and relations:

\[
\begin{align*}
\delta_i \delta_j &= \delta_j \delta_i, \text{ if } |i - j| > 1, \\
\delta_i \delta_{i+1} \delta_i &= \delta_{i+1} \delta_i \delta_{i+1}, \\
\delta_1 \delta_2 ... \delta_n & = 1.
\end{align*}
\]

This presentation was find by O. Zariski [16] in 1936 and rediscovered later by E. Fadell and J. Van Buskirk [10] in 1961. From this presentation one can easily obtain the presentation with two generators $\delta_1, \delta$ and the following relations:

\[
\begin{align*}
\delta_i \delta_1 \delta_i^{-1} &= \delta_i \delta_1 \delta_i^{-1} \delta_1 \text{ for } 2 \leq i \leq n/2, \\
\delta^n &= (\delta \delta_1)^{n-1}, \\
\delta^n (\delta_1 \delta^{-1})^{n-1} &= 1.
\end{align*}
\]

Another generalization of braids is the Baez–Birman monoid $SB_n$ which is also called as singular braid monoid [2], [4]. It is defined as a monoid with generators $\sigma_i, \sigma_i^{-1}, x_i, i = 1, \ldots, n-1$, and relations

\[
\begin{align*}
\sigma_i \sigma_j &= \sigma_j \sigma_i, \text{ if } |i - j| > 1, \\
x_i x_j &= x_j x_i, \text{ if } |i - j| > 1, \\
x_i \sigma_j &= \sigma_j x_i, \text{ if } |i - j| \neq 1, \\
\sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}, \\
\sigma_i \sigma_{i+1} x_i &= x_{i+1} \sigma_i \sigma_{i+1}, \\
\sigma_{i+1} \sigma_i x_{i+1} &= x_i \sigma_{i+1} \sigma_i, \\
\sigma_i \sigma_i^{-1} &= \sigma_i^{-1} \sigma_i = 1.
\end{align*}
\]

In pictures, $\sigma_i$ corresponds to the canonical generator of the braid group and $x_i$ represents an intersection of the $i$th and $(i+1)$st strand as in Figure 1. The singular braid monoid on two strings $SB_2$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}^+$. 

Motivation for the introduction of this object was the Vassiliev – Goussarov theory of finite type invariants.

Let $F_n$ be a free group on $n$ generators $x_1, \ldots, x_n$ and Aut $F_n$ its automorphism group. The braid-permutation group $BP_n$, considered by R. Fenn, R. Rimányi and C. Rourke [11], is the
subgroup of $\text{Aut} F_n$, generated by two sets of the automorphisms: $\sigma_i$

\[
\begin{array}{l}
    x_i \mapsto x_{i+1}, \\
    x_{i+1} \mapsto x_{i+1}^{-1} x_i x_{i+1}, \\
    x_{j} \mapsto x_{j}, j \neq i, i + 1,
\end{array}
\]

and $\xi_i$:

\[
\begin{array}{l}
    x_i \mapsto x_{i+1}, \\
    x_{i+1} \mapsto x_{i}, \\
    x_{j} \mapsto x_{j}, j \neq i, i + 1.
\end{array}
\]

R. Fenn, R. Rimányi and C. Rourke proved [11] that this group is given by the set of generators: $\{\xi_i, \sigma_i, \ i = 1, 2, \ldots, n - 1\}$ and relations:

\[
\begin{align*}
    \xi^2_i &= 1, \\
    \xi_i \xi_j &= \xi_j \xi_i, \text{ if } |i - j| > 1, \\
    \xi_i \xi_{i+1} \xi_i &= \xi_{i+1} \xi_i \xi_{i+1}.
\end{align*}
\]

The symmetric group relations

\[
\begin{align*}
    \sigma_i \sigma_j &= \sigma_j \sigma_i, \text{ if } |i - j| > 1, \\
    \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}.
\end{align*}
\]

The braid group relations

\[
\begin{align*}
    \sigma_i \xi_j &= \xi_j \sigma_i, \text{ if } |i - j| > 1, \\
    \xi_i \xi_{i+1} \sigma_i &= \sigma_{i+1} \xi_i \xi_{i+1}, \\
    \sigma_i \sigma_{i+1} \xi_i &= \xi_{i+1} \sigma_i \sigma_{i+1}.
\end{align*}
\]

The mixed relations

R. Fenn, R. Rimányi and C. Rourke also gave a geometric interpretation of $BP_n$ as a group of welded braids.

Let us obtain a presentation of the singular braid monoid and the braid-permutation group analogous to [11]. If we add the new generator $\sigma$, defined by (2) to the set of generators of $SB_n$ then the following relations hold

\[
x_{i+1} = \sigma^i x_{i} \sigma^{-i}, \quad i = 1, \ldots, n - 2.
\]

This gives a possibility to get rid of $x_i$, $i \geq 2$.

**Theorem 1.** The singular braid monoid $SB_n$ has a presentation with generators $\sigma_1, \sigma_1^{-1}, \sigma, \sigma^{-1}$ and $x_1$ and relations

\[
\begin{align*}
    \sigma_1 \sigma^i \sigma_1 \sigma^{-i} &= \sigma^i \sigma_1 \sigma^{-i} \sigma_1 \quad \text{for } 2 \leq i \leq n/2, \\
    \sigma^n &= (\sigma \sigma_1)^{n-1}, \\
    x_1 \sigma^i \sigma_1 \sigma^{-i} &= \sigma^i \sigma_1 \sigma^{-i} x_1 \quad \text{for } i = 0, 2, \ldots, n - 2, \\
    x_1 \sigma^i x_1 \sigma^{-i} &= \sigma^i x_1 \sigma^{-i} x_1 \quad \text{for } 2 \leq i \leq n/2, \\
    \sigma^n x_1 &= x_1 \sigma^n, \\
    x_1 \sigma \sigma_1 \sigma^{-1} \sigma_1 &= \sigma \sigma_1 \sigma^{-1} \sigma_1 x_1 \sigma^{-1}, \\
    \sigma_1 \sigma_1^{-1} &= \sigma_1^{-1} \sigma_1 = 1, \\
    \sigma \sigma^{-1} &= \sigma^{-1} \sigma = 1.
\end{align*}
\]
Proof. We follow the original Artin’s proof [1] and we begin with the presentation of $SB_n$ using the generators $\sigma_i, \sigma_i^{-1}, x_i, i = 1, \ldots, n - 1$, and relations (4). Then we add the new generators $\sigma, \sigma^{-1}$, relation (2) and the following relations

$$\sigma\sigma^{-1} = \sigma^{-1}\sigma = 1.$$ 

Consider $\sigma x_i$. Using the braid relations in the same way as Artin considered $\sigma\sigma_i$ we have

$$\sigma x_i = \sigma_1 \ldots \sigma_{n-1} x_i = \sigma_1 \ldots \sigma_i \sigma_i+1 \sigma_i+2 \ldots \sigma_{n-1} =$$

$$= \sigma_1 \ldots \sigma_{i+1} \sigma_i+1 \sigma_{i+2} \ldots \sigma_{n-1} = x_{i+1} \sigma.$$ 

We arrive thus at relations (3) and (5). Now we can get rid from the fifth relation in (4). First of all using (3) and (5) it is reduced to the case $i = 1$, which is considered as follows: $x_1$ commutes with $\sigma^2 \sigma^{-1}$:

$$x_1 \sigma^2 \sigma^{-1} \sigma^1 = \sigma^2 \sigma^{-1} \sigma x_1 = \sigma^2 \sigma^{-1} x_2 \sigma.$$ 

So, we obtain

$$x_1 \sigma^2 \sigma^{-1} = \sigma^2 \sigma^{-1} x_2.$$ 

This is equivalent to the fifth relation in (4) for $i = 1$. Using (3) and (4) the sixth relation in (4) is reduced to the case $i = 1$, which is the sixth relation in (5).

The third and forth relations in (6) are easy consequences of the corresponding relations in (4) and (3) and (5). The fifth relation in (6) is a consequence of the definition of $\sigma$ and the singular braid relations (4). The third relation in (4) is obtained from the forth and fifth relations in (4) in the same way as Artin obtained the commutation of $\sigma_i$ and $\sigma_j$ from the relations in (4). Essentially, Artin used the fact that it follows from the second relation in (4) that $\sigma^n$ is in the center of $Br_n$. Here we need the fifth relation in (4) to have this fact in $SB_n$. The third relation in (4) in the new generators is rewritten as follows

$$\sigma^i x_1 \sigma^{-i} \sigma^1 \sigma^{-j} = \sigma^j \sigma_1 \sigma^{-j} \sigma^i x_1 \sigma^{-i},$$

what is equivalent to

$$x_1 \sigma^j \sigma^{-i} \sigma^1 \sigma^{-j} = \sigma^j \sigma_1 \sigma^{-j} x_1.$$

If $j > i$ then this is exactly the third relation in (6), if $j < i$ then it follows from the third relation in (6) by conjugation by $\sigma^n$ and using the commutation of $\sigma^n$ with $x_1$. 

For the case of the braid-permutation group $SB_n$ we also add the new generator $\sigma$, defined by (2) to the set of standard generators of $BP_n$, then relations (3) and the following relations hold

$$\xi_{i+1} = \sigma^i \xi_1 \sigma^{-i}, \quad i = 1, \ldots, n - 2.$$ 

This gives a possibility to get rid of $\xi_i$ as well as of $\sigma_i$ for $i \geq 2$.

**Theorem 2.** The braid-permutation group $BP_n$ has presentation with generators $\sigma_1, \sigma$, and $\xi_1$ and relations

$$\begin{align*}
\sigma_1 \sigma^i \sigma_1 \sigma^{-i} &= \sigma^i \sigma_1 \sigma^{-i} \sigma_1 \quad \text{for } 2 \leq i \leq n/2, \\
\sigma^n &= (\sigma \sigma_1)^{n-1}, \\
\xi_1 \sigma^i \sigma_1 \sigma^{-i} &= \sigma^i \sigma_1 \sigma^{-i} \xi_1 \quad \text{for } i = 2 \ldots n - 2, \\
\xi_1 \sigma^i \xi_1 \sigma^{-i} &= \sigma^i \xi_1 \sigma^{-i} \xi_1 \quad \text{for } i = 2 \ldots n - 2, \\
\xi_1 \sigma \xi^{-1} \sigma_1 &= \sigma \sigma_1 \sigma^{-1} \sigma_1 \xi_1 \sigma^{-1}, \\
\xi_1 \sigma_1 \sigma^{-1} \xi_1 &= \sigma_1 \sigma^{-1} \xi_1 \sigma_1 \sigma^{-1}, \\
\xi_1 &= 1.
\end{align*}$$

□
Let us consider the generalized braid groups in the sense of Brieskorn (or so the called Artin groups) \[6\]. It is easy to see that for the braid groups of type \(B_n\) from the canonical presentation with generators \(\sigma_i, i = 1, \ldots, n - 1\) and \(\tau\), and relations:

\[
\begin{align*}
\sigma_i \sigma_j &= \sigma_j \sigma_i, \text{ if } |i - j| > 1, \\
\sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}, \\
\tau \sigma_i &= \sigma_i \tau, \text{ if } i \geq 2, \\
\tau \sigma_1 \tau \sigma_1 &= \sigma_1 \tau \sigma_1 \tau,
\end{align*}
\]

we can obtain the presentation with three generators \(\sigma_1, \sigma\) and \(\tau\) and the following relations:

\[
\begin{align*}
\sigma_1 \sigma_1 \sigma_1 \sigma^{-i} &= \sigma^i \sigma_1 \sigma^{-i} \sigma_1 \text{ for } 2 \leq i \leq n/2, \\
\sigma^n &= (\sigma_1 \sigma_1)^{n-1}, \\
\tau \sigma_1 \sigma_1 \sigma^{-i} &= \sigma^i \sigma_1 \sigma^{-i} \tau \text{ for } 2 \leq i \leq n-2, \\
\tau \sigma_1 \sigma_1 \sigma_1 &= \sigma_1 \tau \sigma_1 \tau.
\end{align*}
\]

If we add the following relations

\[
\begin{align*}
\sigma_1^2 &= 1, \\
\tau^2 &= 1
\end{align*}
\]

to \((7)\) we then arrive at a presentation of the Coxeter group of type \(B_n\).

Similarly, for the braid groups of the type \(D_n\) from the canonical presentation with generators \(\sigma_i\) and \(\rho\), and relations:

\[
\begin{align*}
\sigma_i \sigma_j &= \sigma_j \sigma_i, \text{ if } |i - j| > 1, \\
\sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}, \\
\rho \sigma_i &= \sigma_i \rho, \text{ if } i = 1, 3, \ldots, n - 1, \\
\rho \sigma_2 \rho &= \sigma_2 \rho \sigma_2
\end{align*}
\]

we can obtain the presentation with three generators \(\sigma_1, \sigma\) and \(\rho\) and the following relations:

\[
\begin{align*}
\sigma_1 \sigma_1 \sigma_1 \sigma^{-i} &= \sigma^i \sigma_1 \sigma^{-i} \sigma_1 \text{ for } 2 \leq i \leq n/2, \\
\sigma^n &= (\sigma_1 \sigma_1)^{n-1}, \\
\rho \sigma_1 \sigma_1 \sigma^{-i} &= \sigma^i \sigma_1 \sigma^{-i} \rho \text{ for } i = 0, 2, \ldots, n-2, \\
\rho \sigma_1 \sigma_1 \sigma^{-1} \rho &= \sigma_1 \sigma^{-1} \rho \sigma_1 \sigma^{-1}.
\end{align*}
\]

If we add the following relations

\[
\begin{align*}
\sigma_1^2 &= 1, \\
\rho^2 &= 1
\end{align*}
\]

to \((8)\) we come to a presentation of the Coxeter group of type \(D_n\).

For the exceptional braid groups of types \(E_6 - E_8\) our presentations look similar to the presentation for the groups of type \(D\) \((8)\). We give it here for \(E_8\): it has three generators \(\sigma_1, \sigma\) and \(\omega\) and the following relations:

\[
\begin{align*}
\sigma_1 \sigma_1 \sigma_1 \sigma^{-i} &= \sigma^i \sigma_1 \sigma^{-i} \sigma_1 \text{ for } i = 2, 3, 4, \\
\sigma^8 &= (\sigma_1 \sigma_1)^7, \\
\omega \sigma_1 \sigma_1 \sigma^{-i} &= \sigma^i \sigma_1 \sigma^{-i} \omega \text{ for } i = 0, 1, 3, 4, 5, 6, \\
\omega \sigma^2 \sigma_1 \sigma^{-2} \omega &= \sigma^2 \sigma_1 \sigma^{-2} \omega \sigma^2 \sigma_1 \sigma^{-2}.
\end{align*}
\]
Similarly, if we add the following relations
\[
\begin{align*}
\sigma_1^2 &= 1, \\
\omega^2 &= 1
\end{align*}
\]
to (9) we arrive at a presentation of the Coxeter group of type \(E_8\).  

As for the other exceptional braid groups, \(F_4\) has four generators and it follows from its Coxeter diagram that there is no sense to speak about analogues of the Artin presentation (11). \(G_2\) and \(I_2(p)\) already have two generators and \(H_3\) has three generators. For \(H_4\) it is possible to diminish the number of generators from four to three and the presentation will be similar to that of \(B_4\).

We can summarize informally what we were doing. Let a group have a presentation which can be expressed by a “Coxeter-like” graph. If there exists a linear subgraph corresponding to the standard presentation of the classical braid group, then in the “braid-like” presentation of our group the part that corresponds to the linear subgraph can be replaced by two generators and relations (11). This re can be applied to the complex reflection groups whose “Coxeter-like” presentations is obtained in (7). For the series of the complex braid groups \(B(2e, e, r)\), \(e \geq 2\), \(r \geq 2\) which correspond to the complex reflection groups \(G(de, e, r)\), \(d \geq 2\) we take the linear subgraph with nodes \(\tau_2, \ldots, \tau_r\), and put as above \(\tau = \tau_2 \ldots \tau_r\). The group \(B(2e, e, r)\) have presentation with generators \(\tau_2, \tau, \sigma, \tau_2'\) and relations
\[
\begin{align*}
\tau_2 \tau_2^{-i} \tau_2^{-i} &= \tau_2 \tau_2^{-i} \tau_2^{-i} \text{ for } 2 \leq i \leq r/2, \\
\tau^r &= (\tau \tau_2)^{r-1}, \\
\sigma \tau_i \tau_2^{-i} &= \tau_i \tau_2^{-i} \sigma, \text{ for } 1 \leq i \leq r - 2, \\
\sigma \tau_2' \tau_2 \tau_2^{-1} \tau_2' &= \tau_2 \tau_2^{-1} \tau_2' \tau_2 \tau_2^{-1}, \\
\tau \tau_2^{-1} \tau_2' \tau_2 \tau_2^{-1} \tau_2' &= \tau_2 \tau_2 \tau_2^{-1} \tau_2 \tau_2 \tau_2^{-1}, \\
\tau_2 \sigma \tau_2' \tau_2 \tau_2^{-1} \tau_2' \tau_2 \tau_2^{-1} \cdots &= \sigma \tau_2' \tau_2 \tau_2^{-1} \tau_2' \tau_2 \tau_2^{-1} \cdots
\end{align*}
\]

If we add the following relations
\[
\begin{align*}
\sigma^d &= 1, \\
\tau_2^2 &= 1, \\
\tau_2'^2 &= 1
\end{align*}
\]
to (10) we come to a presentation of the complex reflection group \(G(de, e, r)\).

The braid group \(B(d, 1, n)\), \(d > 1\), has the same presentation as the Artin – Brieskorn group of type \(B_n\), but if we add the following relations
\[
\begin{align*}
\sigma_1^2 &= 1, \\
\tau^d &= 1
\end{align*}
\]
to (7) then we arrive at a presentation of the complex reflection group \(G(d, 1, n)\), \(d \geq 2\).

For the series of braid groups \(B(e, e, r)\), \(e \geq 2\), \(r \geq 3\) which correspond to the complex reflection groups \(G(e, e, r)\), \(e \geq 2\), \(r \geq 3\) we take again the linear subgraph with the nodes \(\tau_2, \ldots, \tau_r\), and put as above \(\tau = \tau_2 \ldots \tau_r\). The group \(B(e, e, r)\) may have the presentation with
generators \( \tau_2, \tau, \tau'_2 \) and relations

\[
\begin{align*}
\tau_2 \tau^i \tau_2^{-i} & = \tau^i \tau_2 \tau^{-i} \tau_2 \quad \text{for } 2 \leq i \leq r/2, \\
\tau^r & = (\tau \tau_2)^{r-1}, \\
\tau'_2 \tau_2 \tau^{-1} \tau'_2 & = \tau \tau_2 \tau^{-1} \tau'_2 \tau \tau^{-1}, \\
\tau \tau_2 \tau^{-1} \tau'_2 \tau_2 \tau_2^{-1} \tau'_2 \tau_2 & = \tau \tau_2 \tau^{-1} \tau'_2 \tau \tau^{-1}, \\
\tau_2 \tau_2 \tau'_2 \tau_2 \tau_2 \tau'_2 \tau_2 \tau_2 & = \tau \tau_2 \tau_2 \tau_2^{-1} \tau'_2 \tau_2 \tau_2 \tau_2^{-1} \\
\tau_2 \tau_2 \tau_2 \tau_2 \tau_2 \tau_2 \tau_2 \tau_2 & = \tau \tau_2 \tau_2 \tau_2 \tau_2 \tau_2 \tau_2 \tau_2 \tau_2 & \text{e factors}
\end{align*}
\tag{11}
\]

If \( e = 2 \) then this presentation is the same as for the presentation for the Artin–Brieskorn

group of type \( D_r \) \([8]\). If we add the following relations

\[
\begin{align*}
\tau_2^2 & = 1, \\
\tau_2^{2^e} & = 1
\end{align*}
\]

to (11), then we obtain a presentation of the complex reflection group \( G(e, e, r) \), \( e \geq 2, r \geq 3 \).

As for the exceptional (complex) braid groups, it is reasonable to consider the groups \( Br(G_{30}) \),
\( Br(G_{33}) \) and \( Br(G_{34}) \) which correspond to the complex reflection groups \( G_{30}, G_{33} \) and \( G_{34} \).

The presentation for \( Br(G_{30}) \) is similar to the presentation \([7]\) of \( Br(B_4) \) with the last
relation replaced by the relation of length 5: the three generators \( \sigma_1, \sigma \) and \( \tau \) and the following
relations:

\[
\begin{align*}
\sigma_1 \sigma^2 \sigma_1 \sigma^{-2} & = \sigma^2 \sigma_1 \sigma^{-2} \sigma_1, \\
\sigma^4 & = (\sigma \sigma_1)^3, \\
\tau \sigma^i \sigma_1 \sigma^{-i} & = \sigma^i \sigma_1 \sigma^{-i} \tau \quad \text{for } i = 2, 3, \\
\tau \sigma_1 \tau \sigma_1 \tau & = \sigma_1 \tau \sigma_1 \tau \sigma_1.
\end{align*}
\tag{12}
\]

If we add the following relations

\[
\begin{align*}
\sigma_1^2 & = 1, \\
\tau^2 & = 1
\end{align*}
\]

to (12), then we obtain a presentation of complex reflection group \( G_{30} \).

As for the groups \( Br(G_{33}) \) and \( Br(G_{34}) \), we give here the presentation for the letter one
because the “Coxeter-like” graph for \( Br(G_{33}) \) has one node less in the linear subgraph (discussed
earlier) than that of \( Br(G_{34}) \). This presentation has the three generators \( s, z \) \((z = stuwx \text{ in}
the reflection generators) and \( w \) and the following relations:

\[
\begin{align*}
sz^i sz^{-i} & = z^i sz^{-i} s \quad \text{for } i = 2, 3, \\
\tau s^6 & = (zs)^5, \\
wz^i sw^{-i} & = z^i sz^{-i} w \quad \text{for } i = 0, 3, 4, \\
wz^i sz^{-i} w & = z^i sz^{-i} wz^i sz^{-i} \quad \text{for } i = 1, 2, \\
wz^2 sz^{-2} wzsz^{-1} wz^2 sz^{-2} & = zsz^{-1} wz^2 sz^{-2} wzsz^{-1} w.
\end{align*}
\tag{13}
\]

The same way if we add the following relations

\[
\begin{align*}
s^2 & = 1, \\
\tau^2 & = 1
\end{align*}
\]

to (13), then we come to a presentation of the complex reflection group \( G_{34} \).

We can obtain presentations with few generators for the other complex reflection groups
using the already observed presentations of the braid groups. For \( G_{25} \) and \( G_{32} \) we can use
the presentations (1) for the classical braid groups $Br_4$ and $Br_5$ with the only one additional relation

$$\sigma_1^3 = 1.$$ 

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REFERENCES

[1] E. Artin, Theorie der Zöpfe, Abh. Math. Semin. Univ. Hamburg, 1925, v. 4, 47–72.
[2] J. C. Baez, Link invariants of finite type and perturbation theory. Lett. Math. Phys. 26 (1992), no. 1, 43–51.
[3] D. Bessis, J. Michel, Explicit presentations for exceptional braid groups, arXiv:math.GR/0312191 v1 9 Dec 2003.
[4] J. S. Birman, New points of view in knot theory. Bull. Amer. Math. Soc. 1993, 28, No 2, 253–387.
[5] J. S. Birman; K. H. Ko; S. J. Lee, A new approach to the word and conjugacy problems in the braid groups. Adv. Math. 139 (1998), no. 2, 322–353.
[6] E. Brieskorn, Sur les groupes de tresses [d’après V. I. Arnol’d]. Séminaire Bourbaki, 24ème année (1971/1972), Exp. No. 401, pp. 21–44. Lecture Notes in Math., Vol. 317, Springer, Berlin, 1973.
[7] M. Broué, G. Malle, R. Rouquier Complex reflection groups, braid groups, Hecke algebras. J. Reine Angew. Math. 500, 127-190 (1998).
[8] H. S. M. Coxeter, W. O. J. Moser Generators and relations for discrete groups. 3rd ed. Ergebnisse der Mathematik und ihrer Grenzgebiete. Band 14. Berlin-Heidelberg-New York: Springer-Verlag. IX, 161 p. (1972).
[9] P. Deligne, Les inmeubles des groupes de tresses généralisés. Invent. Math. 17 (1972), 273–302.
[10] E. Fadell and J. Van Buskirk, The braid groups of $E^2$ and $S^2$. Duke Math. J. 29 1962 243–257.
[11] R. Fenn; R. Rimányi; C. Rourke, The braid-permutation group. Topology 36 (1997), no. 1, 123–135.
[12] V. Sergiescu, Graphes planaires et présentations des groupes de tresses, Math. Z. 1993, 214, 477–490.
[13] G. C. Shephard, J. A. Todd Finite unitary reflection groups. Can. J. Math. 6, 274-304 (1954).
[14] V. V. Vershinin, Braid groups and loop spaces. Russ. Math. Surv. 54, No.2, 273-350 (1999); translation from Usp. Mat. Nauk 54, No.2, 3-84 (1999).
[15] V. V. Vershinin On the singular braid monoid, arXiv:math.Gr/0309339 v1 20 Sep 2003. 12 pages.
[16] O. Zariski, On the Poincare group of rational plane curves, Am. J. Math. 1936, 58, 607-619.

Département des Sciences Mathématiques, Université Montpellier II, Place Eugène Bataillon, 34095 Montpellier cedex 5, France

E-mail address: vershini@math.univ-montp2.fr

Sobolev Institute of Mathematics, Novosibirsk, 630090, Russia

E-mail address: versh@math.nsc.ru