Refactorisation of the Dirichlet convolution

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Abstract. We present a new way to factor the dirichlet convolution for completely multiplicative functions which led us to constructing a ring that arise from the operations involved in the factorisation. We will conclude by some identities that was found during this work.

An application of the results gives us a generalisation of the following Hardy formula:

\[ \zeta(x)^2 = \zeta(2x) \sum_{m=1}^{+\infty} \frac{2^{\omega(m)}}{m^x} \]

which is:

\[ |\zeta(z)|^2 = \zeta(2x) \sum_{m=1}^{+\infty} \frac{1}{m^x} 2^{\omega(m)} \prod_{\omega(m)} \cos(y \ln(p^{v_p(m)})) \]

with:
- \( x > 1 \) in Hardy's formula
- \( z \) be a complex number with \( z = x+iy \) and \( \Re(z) > 1 \)
- \( \omega(m) \): number of unique primes in \( m \)
- \( v_p(m) \): power of the prime \( p \) in \( m \)
- \( \mathbb{P} \): set of the prime number

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1. Introduction

We will begin by writing some definitions that will be used throughout this article, then we will state the results obtained starting with what we see as the most important ones. In the last sections, we will prove our claims and discuss possible applications and future investigations over this work.

**Definition 1.1.** The zeta series was discovered by Leonhard Euler and is defined as follows:

\[ \forall s \in \mathbb{C}, \; \Re(s) > 1 \; \zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s} \]

The series \( \sum_{n=1}^{+\infty} \frac{1}{n^s} \) does not converge for \( \Re(s) \leq 1 \), however we can define an extension (unique in the sense of the analytic extension) on the complex plane, except in \( s = 1 \).

This analytic extension is what is called the zeta function \([1][8][11]\).

**Definition 1.2.** Euler has also established the "Eulerian" product associated with the zeta function:

\[ \forall s \in \mathbb{C}, \; \Re(s) > 1 \; \zeta(s) = \prod_{p \in \mathbb{P}} \frac{1}{1 - \frac{1}{p^s}} \]

To understand more easily the importance of this formula, it is more appropriate to write it as follows:

\[ \forall s \in \mathbb{C}, \; \Re(s) > 1 \; \zeta(s) = \frac{1}{1 - \frac{1}{2^s}} \prod_{p \in \mathbb{P}} \frac{1}{1 - \frac{1}{p^s}} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \ldots \]

The idea is that the formula \( \sum_{n=1}^{+\infty} \frac{1}{n^s} \) presents the integers as a sequence of ordered numbers separated by a fixed step (additive, metric and ordered vision), while the formula \( \prod_{p \in \mathbb{P}} \frac{1}{1 - \frac{1}{p^s}} \) is an integer combination of different powers of prime numbers (multiplicative and combinatory vision).

**Definition 1.3.** Riemann proved \([1]\) that the zeta function satisfies the following functional equation:
\[ \forall s \in \mathbb{C} - \{1\} : \zeta(s) = 2^s \pi^{s-1} \sin \left( \frac{\pi s}{2} \right) \Gamma(1-s) \zeta(1-s) \]

**Remark 1.1.** This result expresses a relation between the values \( \zeta(s) \) and \( \zeta(1-s) \), and more particularly, it indicates that the set of zeros of the zeta function is symmetrical with respect to the line \( \Re(s) = \frac{1}{2} \).

The Riemann Hypothesis describes a pattern, a regularity that prime numbers are supposed to follow. This pattern is the fact that they are, at least in part, distributed pseudo randomly. Knowing this information allows us to better frame them.

One of the strategies that mathematicians have applied to try to answer this question is the generalization [3]: They tried to define several classes of functions having the same fundamental properties of the zeta function (Eulerian product, functional equation, Riemann hypothesis ...) and sought to understand these new functions. This allows to focus only on the important properties related to the Riemann hypothesis.

Several classes of functions have been defined, some of these courses are at the heart of a major research program such as Polya-Hilbert program and The Langlands program.

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3. **Definitions**

We will start the definitions by recalling one of the most important family of functions that arise in many of the problems in number theory

**Definition 3.1.** The set of multiplicative function is defined by the following properties:

\[ F : \mathbb{N}^* \rightarrow \mathbb{C} \]
\[ F(1) = 1 \]
\[ (a, b) = 1 \Rightarrow F(ab) = F(a)F(b) \]

We also often encounter a smaller family of functions \( \mathbb{M}_c \), yet more simple than the multiplicative functions of the set \( \mathbb{M} \)

**Definition 3.2.** The set of completely multiplicative function $M_c$ is described by:

$$F \in M_c \iff F \in M, \quad a, b \in \mathbb{N}^* : F(ab) = F(a)F(b)$$

**Remark 3.1.** The set $M_c$ is stable by multiplication, which is useful for building structures over it. $M_c$ family contains many functions that are not particularly linked with number theory as the constraints defining the set are general. As an example, we could give the exponential $x \to \exp(x)$, and the module $x \to |x|$. Conversely, the set $M$ contains many important functions in arithmetic, one of the most useful in analytic number theory being Dirichlet’s characters.

**Remark 3.2.** In this paper, we will use the following notation $\forall a, b \in \mathbb{N}^* (a, b)$ refer to the grand commun divisor of $a$ and $b$.

**Definition 3.3.** The omega function $\omega$ is defined as follow:

$$\forall n \in \mathbb{N}^* \omega(n) = \sum_{p \mid n} 1$$

**Definition 3.4.** A Dirichlet character is a multiplicative function that verify:

$$\chi : \mathbb{N} \rightarrow \mathbb{C}$$

$$\forall a, b \in \mathbb{N} \Rightarrow \chi(ab) = \chi(a)\chi(b)$$

$$\exists k \in \mathbb{N}^*, \forall n \in \mathbb{N} \chi(n) = \chi(n+k)$$

$$\forall n \in \mathbb{N}^* (n, k) > 1 \Rightarrow \chi(n) = 0$$

**Remark 3.3.** A Dirichlet character define a multiplicative function.

**Definition 3.5.** The generative dirichlet serie associated with a multiplicative function $F$ is defined as:

$$\forall F \in M, \forall s \in \mathbb{C}, \Re(s) > \sigma : D(F, s) = \sum_{n=1}^{\infty} \frac{F(n)}{n^s}$$

where $\sigma$ define a region of the complex plan $\mathbb{C}$ where the serie converge.

The dirichlet characters define a family of functions that generalized some of the specificities of the rieman zêta function, and thus gave rise to a new more general conjecture than Riemann hypothesis.

**Definition 3.6.** The L series are the Dirichlet generative series associated with a dirichlet character:

$$\forall s \in \mathbb{C}, \Re(s) > 1 : L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

**Definition 3.7.** The L functions are the analytical extention of the L series.
Remark 3.4. The Riemann zeta function is the L serie for the character 1.

\[ \forall n \in \mathbb{N} : \chi(n) = 1 \]

Definition 3.8. The generalised riemann hypothesis (GRH) is the assertion that the following proposition is true for all L functions:

\[ \forall s \in \mathbb{C}, 0 < \Re(s) < 1 : L(\chi, s) = 0 \Rightarrow \Re(s) = \frac{1}{2} \]

Remark 3.5. The set of complex number that have 0 < \Re(s) < 1 is often called the critical band and the ligne of complexe number defined by \Re(s) = \frac{1}{2} is called the critical ligne.

We will now present some operations that has been studied in this work and its associated ring.

Definition 3.9. We introduct the weight functions \( W \) as follow:

\[ W : \mathbb{N}^* \rightarrow \mathbb{C} \]

Definition 3.10. We define the operation \( \Box_w \), which we will call the W-convolution:

\[ \forall F, G \in \mathbb{M}, \forall m \in \mathbb{N}^* : [F \Box_w G](m) = \sum_{ab=m} F(a)G(b)W(a,b) \]

Remark 3.6. For notation purpuses, we will write this notation simply \( \Box \).

Definition 3.11. We define the operation (the multiplication) \( \times \) as follow:

\[ \forall F, G \in \mathbb{M}, \forall m \in \mathbb{N}^* : [F \times G](m) = F(m)G(m) \]

Here we will define what we mean by a derivation over a ring.

Definition 3.12. A derivation \( \nabla \) over \( \mathbb{M} \) is an operator how act over the set \( \mathbb{M} \) which respect:

\[ \forall F, G \in \mathbb{M} \ , \ \nabla(F \Box G) = \nabla(F) \Box \nabla(G) \]

\[ \forall F, G \in \mathbb{M} \ , \ \nabla(F \times G) = \nabla(F) \times G \Box F \times \nabla(G) \]

We will end this reminder of the definitions used here by recalling the Dirichlet ring of arithmetic functions:

Definition 3.13. The Dirichlet ring \( (\Omega, \ast, \cdot) \) of arithmetic functions is constructed as follows:

\[ F \in \Omega \iff F : \mathbb{N} \rightarrow \mathbb{C} \]

\[ \forall F, G \in \Omega : [F \ast G](n) = \sum_{ab=n} F(a)G(b) \]

\[ \forall F, G \in \Omega : [F \cdot G](n) = F(n)G(n) \]

Remark 3.7. The operation \( \ast \) used in the ring is called the Dirichlet convolution

\[ \forall F, G \in \Omega : [F \ast G](n) = \sum_{ab=n} F(a)G(b) \]
4. Results

We will present the results that was established throughout this work starting by what we sees as the most original ones.

Using this convolution and multiplication over the set of multiplicative functions $\mathcal{M}$, we have constructed the following result

**Theorem 4.1** (Ring unicity theorem).

$(\mathcal{M}, \circ, \times)$ is a commutative ring $\iff \forall a, b \in \mathbb{N}^* : W(a, b) = \begin{cases} 1 & \text{if } (a, b) = 1 \\ 0 & \text{if not} \end{cases}$

The operations used to define the ring $(\mathcal{M}, \circ, \times)$ could then be used to express the following equalities which allows us to give a new formula for the Dirichlet series

**Theorem 4.2** (Refactorisation theorem).

$\forall F, G \in \mathcal{M}_c : D(F \ast G, s) = D(F \times G, 2s) \times D(F \Box G, s)$

A possible reformulation could be:

**Proposition 4.1.** for $F$ and $G$ completely multiplicative:

$\forall F, G \in \mathcal{M}_c : D(F, s) \times D(G, s) = D(F \times G, 2s) \times D(F \Box G, s)$

We then have the following useful related result

**Proposition 4.2.**

$\forall F, G \in \mathcal{M}_c : D(F, s) \times D(G, s) = D(F \times G, 2s) \times D(F \Box G, s)$

This last proposition could have some direct applications as the $D(F \times G, 2s)$ section does not depend on $y$ ($s := x + iy$).

For example, it allows us to generalise one of Hardy’s formula [9]:

**Proposition 4.3.** $\forall z \in \mathbb{C}, \Re(z) > 1 :$

$$\zeta(z)\overline{\zeta(z)} = |\zeta(z)|^2 = \zeta(2x) \sum_{m=1}^{+\infty} \frac{1}{m^2} 2^{\omega(m)} \prod_{j=1}^{\omega(m)} \cos(y \ln(p_j^{v_{p_j}(m)}/m))$$

This formula makes it possible to deduce a known result of Hardy by taking $y = 0$:

$$\zeta(x)^2 = \zeta(2x) \sum_{m=1}^{+\infty} \frac{2\omega(m)}{m^2}$$

The two following identities independent was proven aside during this work, they do not make use of the previous unitary arithmetic framework

**Proposition 4.4.**

$\forall s \in \mathbb{C}, \Re(s) > 1 :$

$$\sum_{n=1}^{+\infty} \frac{1}{n^s} \sum_{p \in \mathbb{P}} \frac{1}{p^s} \frac{1}{\omega(np)} = \zeta(s) - 1$$
**Proposition 4.5.**
\[ \forall s \in \mathbb{C}, \Re(s) > 1 : \sum_{n=1}^{\infty} \frac{1}{n^s} \cdot \frac{1}{\omega(n)} + 1 \cdot \sum_{p \in \mathbb{P}} \frac{1}{p^s} + \sum_{p \nmid n} 1 = \zeta(s) - 1 \]

This result is about derivation operators over the ring \((M, \square, \times)\)

**Theorem 4.3 (Character derivation theorem).** Any derivation \(\nabla\) acting on the ring \((M, \square, \times)\) is null over all dirichlet characters:
\[ \forall \chi, \quad \nabla(\chi) = 0 \]

The following corollaries are some direct application of the ring operations

**Proposition 4.6.** for \(F\) and \(G\) completely multiplicative with \(\delta_1(1) = 1\) and \(\forall n \in \mathbb{N}^* - \{1\} : \delta_1(n) = 0 :\)
\[ \forall F, G \in M : F \times G = \delta_1 \Rightarrow D(F, s) \times D(G, s) = D(F \square G, s) \]

**Proposition 4.7.** Let \(A \in \mathbb{P}\) and \(\bar{A} \in \mathbb{P}\) be complementary in \(\mathbb{P}\), let \(F\) be an arithmetic completely multiplicative function:
\[ \forall F \in M : D(\mathbf{1}_A \times F, s) \times D(\mathbf{1}_{\bar{A}} \times F, s) = D(F, s) \]

**Remark 4.1.** This formula reflect the euler product, let’s take zêta as example:
\[ D(1, s) = \prod_{p \in \mathbb{P}} \frac{1}{1 - \frac{1}{p}} = \prod_{p \in A} \frac{1}{1 - \frac{1}{p}} \times \prod_{p \in \bar{A}} \frac{1}{1 - \frac{1}{p}} = D(\mathbf{1}_A \times \mathbf{1}, s) \times D(\mathbf{1}_{\bar{A}} \times \mathbf{1}, s) \]

**Proposition 4.8.** Identity in \((M, \square, \times)\):
\[ [F \square F](m) = [2^\omega(n) \times F](m) \]

**Proposition 4.9.**
\[ \forall n \in \mathbb{N} - \{0, 1\}, \forall a \in \mathbb{N} - \{0, 1\} : \sum_{k=1}^{\phi(n)} \chi_k(a) = \phi(n)^{\omega(a)} \cdot \delta_{n([Id, -1](a))} \]

**Remark 4.2.** For \(y \in \mathbb{R}\):
Note the functions \(n \in \mathbb{N}^* : n \to \cos(y \ln(n))\) and \(n \in \mathbb{N}^* : n \to \sin(y \ln(n))\) respectively Cosay and Sina\(_y\) because they play a similar role to the trigonometric functions inside the ring \((M, \square, \times)\):
\[ \text{Cosay}^2 \square \text{Sina}^2 = \mathbf{1} \]
This function will be used during the proof, it have a nice reformulation using the unitary ring operations:

\[ Q_\chi(l) = \left[ \delta_{l=1}(l) + \sum_{m,k \in \mathbb{N}, m \geq l, k \geq h} 2 \Re \left( \frac{\chi(m)}{m^{\alpha}} \overline{\chi(k)} \frac{1}{k^{\beta}} \right) \right] \]

\[ = [\cos_\alpha \times \Re \chi \Box \sin_\alpha \times \Im \chi][l] \]

Avec:

\[ \forall n \in \mathbb{N}^* \forall p \in \mathbb{P} : \Re \chi[p^n] = \Re(\chi(p^{p_\alpha(n)})) \]

\[ \forall n \in \mathbb{N}^* \forall p \in \mathbb{P} : \Im \chi[p^n] = \Im(\chi(p^{p_\beta(n)})) \]

This formulation could recall a scalar product between two complex values.

5. Existence and unicity of the ring \((\mathbb{M}, \Box, \times)\)

In this chapter, our goal is to build a specific ring of functions and prove its uniqueness across a range of similar possible structures. The operation used in the final ring has been studied in numerous articles such as \([12]\) \([2]\) \([10]\). We will begin with a few definitions, then we will progressively investigate what conditions the convolution used in the ring should meet.

**Definition 5.1.** We will start by introducing the weight functions \(W : \mathbb{N}^2 \rightarrow \mathbb{C}\)

This function is used as a weight in the following convolution

**Definition 5.2.** The \(W\) function define the operation \(\Box_w\), which we will call the \(W\)-convolution:

\[ \forall F, G \in \mathbb{M}, \forall m \in \mathbb{N}^* : [F \Box_w G](m) = \sum_{a \oplus b = m} F(a)G(b)W(a, b) \]

**Remark 5.1.** For notation purposes, we will write only \(\Box\).

We intent to construct a ring in this section, so let’s define it’s multiplication operation

**Definition 5.3.** We define the operation (the multiplication) \(\times\) as follow:

\[ \forall F, G \in \mathbb{M}, \forall m \in \mathbb{N}^* : [F \times G](m) = F(m)G(m) \]

Many intermediary results will be established before proving our main results, we will start by setting some tools that will show to be useful throughout the ring construction

**Definition 5.4.** We define the multiplicative indicatrice functions as follow:

\[ \forall S \subset \mathbb{P} \times \mathbb{N}^* : \mathbb{1}_S(p^n) = \begin{cases} 1 & \text{si } (p, n) \in S \cup \{1, 1\} \\ 0 & \text{sinon.} \end{cases} \]

We write also \(\mathbb{1}_s\) for \(s \in \mathbb{N}\), that we define as follow:

\[ \mathbb{1}_s = \mathbb{1}_S \iff s = \prod_{(p, n) \in S} p^n \]
**Lemma 5.1** (commutativity lemma). The operation $\sqcup$ is commutative if and only if the function $W(\cdot, \cdot)$ is commutative:

$$
\forall F, G \in \mathcal{M}, \forall m \in \mathbb{N}^* : F \sqcup G(m) = G \sqcup F(m) \iff \forall a, b \in \mathbb{N}^* : W(a, b) = W(b, a)
$$

**Proof.** Let’s start by the direct implication $\implies$:

$$
\mathbf{1}_a \sqcup \mathbf{1}_b(ab) = \sum_{nq=ab} \mathbf{1}_a(n) \mathbf{1}_b(q)W(n, q) = \mathbf{1}_a(a) \mathbf{1}_b(b)W(a, b) = W(a, b)
$$

On the other direction:

$$
\mathbf{1}_b \sqcup \mathbf{1}_a(ab) = \sum_{nq=ab} \mathbf{1}_b(n) \mathbf{1}_a(q)W(n, q) = \mathbf{1}_b(b) \mathbf{1}_a(a)W(b, a) = W(b, a)
$$

So we have the first direction.

Let’s establish now the reciprocal $\impliedby$, we suppose:

$$
\forall a, b \in \mathbb{N}^* : W(a, b) = W(b, a)
$$

We have, $\forall F, G \in \mathcal{M}, \forall m \in \mathbb{N}^* :$

$$
F \sqcup G(m) = \sum_{nq=m} F(n)G(q)W(n, q)
$$

$$
= \sum_{nq=m} F(n)G(q)W(n, q) + \sum_{nq=m} F(n)G(q)W(n, q) + \sum_{nq=m} F(n)G(q)W(n, q)
$$

$$
= \sum_{qn=m} G(q)F(n)W(q, n) + \sum_{qn=m} G(q)F(n)W(q, n) + \sum_{qn=m} G(q)F(n)W(q, n)
$$

$$
= \sum_{qn=m} G(n)F(q)W(n, q)
$$

$$
= G \sqcup F(m)
$$

So we have the reciprocal.

Which end the proof. $\square$

**Lemma 5.2** (stability lemma). $(\mathcal{M}, \sqcup)$ is stable if and only if the function $W$ is multiplicative over two variable in the following sense:

$$
\forall F, G \in \mathcal{M} : F \sqcup G \in \mathcal{M} \iff \forall a, b, c, d \in \mathbb{N}^*, (ab, cd) = 1 : W(a, b)W(c, d) = W(ac, bd)
$$

**Proof.** Let’s start by the direct implication $\implies$:

$$
\forall a_1, a_2, b_1, b_2 \in \mathbb{N}^* \text{ we set } : a_1a_2 = a \text{ and } b_1b_2 = b.
$$

Suppose $\gcd(a, b) = 1$.

$$
\mathbf{1}_{a_1b_1} \sqcup \mathbf{1}_{a_2b_2}(ab) = \sum_{nq=ab} \mathbf{1}_a(n) \mathbf{1}_b(q)W(n, q)
$$
We suppose:

\[ W(a_1a_2, b_1b_2) = W(a_1a_2, b_1b_2) \]

as \( f_{a_1b_1 \square b_{a_2b_2}} \) is a multiplicative function:

\[
\begin{aligned}
W(a_1a_2, b_1b_2)(ab) &= [f_{a_1b_1 \square b_{a_2b_2}}](a) \times [f_{a_1b_1 \square b_{a_2b_2}}](b) \\
&= \left[ \sum_{nq=a} f_{a_1b_1}(n) f_{a_2b_2}(q) W(n, q) \right] \times \left[ \sum_{nq=b} f_{a_1b_1}(n) f_{a_2b_2}(q) W(n, q) \right] \\
&= \left[ f_{a_1b_1}(a_1) f_{a_2b_2}(a_2) W(a_1, a_2) \right] \times \left[ f_{a_1b_1}(b_1) f_{a_2b_2}(b_2) W(b_1, b_2) \right] \\
&= W(a_1, a_2) W(b_1, b_2)
\end{aligned}
\]

So we have the first direction:

\[ W(a_1a_2, b_1b_2) = W(a_1, a_2) W(b_1, b_2) \]

Let’s establish now the reciprocal \( \iff \):

We suppose:

\[ \forall a, b, c, d \in \mathbb{N}^*, (ab, cd) = 1 : W(a, b) W(c, d) = W(ac, bd) \]

We have:

\[ \forall F, G \in \mathbb{M}, \forall a, b \in \mathbb{N}^*, (a, b) = 1 : [F \circ G](ab) = \sum_{nq=ab} F(n) G(q) W(n, q) \]

On the other direction:

\[ [F \circ G](a) \times [F \circ G](b) = \left[ \sum_{n_1n_2=a} F(n_1) G(n_2) W(n_1, n_2) \right] \times \left[ \sum_{q_1q_2=b} F(q_1) G(q_2) W(q_1, q_2) \right] \]

\[ = \sum_{n_1n_2=a} F(n_1) G(n_2) W(n_1, n_2) \sum_{q_1q_2=b} F(q_1) G(q_2) W(q_1, q_2) \]

\[ = \sum_{n_1n_2=a} \sum_{q_1q_2=b} F(n_1n_2) G(n_2q_2) W(n_1q_1, n_2q_2) \]

We have to prove:

\[ \sum_{nq=a_1a_2b_1b_2} F(n) G(q) W(n, q) = \sum_{n_1n_2=a} \sum_{q_1q_2=b} F(n_1q_1) G(n_2q_2) W(n_1q_1, n_2q_2) \]

it is equivalent to prove, knowing that \( a_1a_2 = a \), \( b_1b_2 = b \) and \( (a, b) = 1 \) that the variable change is bijective.

Étant donné \( n_1, n_2, q_1, q_2 \in \mathbb{N}^* \) qui vérifie \( n_1n_2 = a \) et \( q_1q_2 = b \).

Let’s set \( n = n_1q_1 \) and \( q = n_2q_2 \), we have the following equality:
\[ F(n_1 q_1)G(n_2 q_2)W(n_1 q_1, n_2 q_2) = F(n)G(q)W(n,q) \]

where \( nq = ab \).

Invertly, knowing that \( n, q \in \mathbb{N}^* \), we verify \( nq = ab \).

We have the following equality:
\[
F(n)G(q)W(n,q) = F((n,a),(b))G((q,a),(q,b))W((n,a),(b), (q,a),(q,b))
\]

where \( (n,a),(q,a) = a \) and \( (n,b),(q,b) = b \).

With establish the equality, so we have the reciprocal direction.

\[ \square \]

**Lemma 5.3 (identity lemma).** Let's suppose the stability and commutativity of the structure \((\mathcal{M}, \circ)\), the identity element is the function \( \delta_1 = \mathbb{I}_\varnothing \):
\[ \exists E \in \mathcal{M} \forall F \in \mathcal{M} : F \circ E = E \circ F = F \iff \forall(n, p) \in \mathbb{N} \times \mathbb{P}, W(1, p^n) = 1, E = \delta_1 \]

**Proof.**

Using commutativity, we have:
\[ \exists E \in \mathcal{M} \forall F \in \mathcal{M} : F \circ E = E \circ F = F \iff \exists E \in \mathcal{M} \forall F \in \mathcal{M} : F \circ E = F \]

Using stability, we could restrict the proof to:
\[ \exists E \in \mathcal{M} \forall F \in \mathcal{M} : F \circ E = F \iff \forall F \in \mathcal{M}, \forall(n, p) \in \mathbb{N}^* \times \mathbb{P}, : [F \circ E](p^n) = F(p^n) \]

We then compute the following W-convolution:
\[
[F \circ E](p^n) = \sum_{nq = p^n} F(n)E(q)W(n,q)
\]
\[
= \sum_{l=n}^{nq} F(p^l)E(p^{n-l})W(p^l, p^{n-l})
\]

Let's fix \( p \in \mathbb{P} \) and proof the direct proposition by (strong) recurrence:
\[ \forall F \in \mathcal{M}, \forall n \in \mathbb{N}, : [F \circ E](p^n) = F(p^n) \implies \forall n \in \mathbb{N}^*, W(1, p^n) = 1 \text{ et } E = \delta_1 \]

Initial case: \( n = 0 \)
\[ [F \circ E](1) = W(1,1) \]

So we get
\[ W(1,1) = 1 \text{ et } E(1) = 1 \]

Recurrence hypothesis:
\[ \forall F \in \mathcal{M}, \exists n_0 \in \mathbb{N}, : [F \circ E](p^{n_0}) = F(p^{n_0}) \implies \forall n \in \mathbb{N}^*, W(1, p^{n_0}) = 1 \text{ et } E = \delta_1 \]

\( n+1 \) case:
\[
[F \circ E](p^{n+1}) = \sum_{l=0}^{l=n+1} F(p^l)E(p^{n+1-l})W(p^l, p^{n+1-l})
\]
\[
= F(p^n)E(1)W(p^{n+1}, 1) + F(1)E(p^{n+1})W(1, p^{n+1})
\]
\[
= W(p^{n+1}, 1)(F(p^{n+1}) + E(p^{n+1}))
\]
so :
\[ W(p^{n+1}, 1)(F(p^{n+1}) + E(p^{n+1})) - F(p^{n+1}) = F(p^{n+1})(W(p^{n+1}, 1) - 1) + E(p^{n+1})W(p^{n+1}, 1) \]
as neither the identity element \( E \), nor the weight function \( W \) does not depend on \( F \), this expression could be seen as a polynôme of \( F(p^{n+1}) \).
Let’s set \( A = W(p^{n+1}, 1) - 1 \) and \( B = E(p^{n+1})W(p^{n+1}, 1) \), we have :
\[ A \cdot F(p^{n+1}) + B = 0 \iff A = 0 \text{ et } B = 0 \]
so :
\[ W(p^{n+1}, 1) = 1 \text{ and } E(p^{n+1}) = 0 \]
so we have the first proposition.

\[ \text{Proof} \]

\[ \text{Lemma 5.4 (associativity lemma). Hypothesis : } (M, \circ) \text{ is stable and commu-
}

tative , the structure is associative \((M, \circ)\) is equivalent to :
\[ \forall F, G, H \in \mathbb{M} : \left[ [F \circ G] \circ H \right] = \left[ F \circ [G \circ H] \right] \iff \forall a, b, c \in \mathbb{N}^*, W(a, b)W(ab, c) = W(b, c)W(bc, a) \]

\[ \text{Proof. the direct proposition } \iff \text{ could be prooven using the indicatrice functions :} \]
\[ \forall a, b, c \in \mathbb{N}^* \mathbb{I}_a \circ [\mathbb{I}_b \circ \mathbb{I}_c](abc) = \sum_{efg=abc} \mathbb{I}_a(f)\mathbb{I}_b(g)\mathbb{I}_c(e)W(g, e)W(ge, f) \]
\[ = W(b, c)W(bc, a) \]
and
\[ \forall a, b, c \in \mathbb{N}^* [\mathbb{I}_a \circ \mathbb{I}_b] \circ \mathbb{I}_c(abc) = \sum_{efg=abc} \mathbb{I}_a(f)\mathbb{I}_b(g)\mathbb{I}_c(e)W(f, g)W(fg, e) \]
\[ = W(a, b)W(ab, c) \]
so we have :
\[ \forall a, b, c \in \mathbb{N}^* : W(a, b)W(ab, c) = W(b, c)W(bc, a) \]

the reciproc \( \iff \) :
\[ \forall F, G, H \in \mathbb{M}, \forall (n, p) \in \mathbb{N}^* \times \mathbb{P} : 
\]
\[ [F \circ G] \circ H(p^n) = \sum_{ab=p^n} [F \circ G](a)H(b)W(a, b) \]
\[ = \sum_{ab=p^n} \sum_{cd=a} F(c)G(d)W(c, d)H(b)W(cd, b) \]
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\[ \sum_{bcd = p^n} F(c)G(d)H(b)W(c, d)W(cd, b) \]

and:

\[
F \boxdot [G \boxdot H](p^n) = \sum_{ba = p^n} F(b)[G \boxdot H](a)W(b, a) \\
= \sum_{ba = p^n} F(b) \sum_{cd = a} G(c)H(d)W(c, d)W(b, a) \\
= \sum_{bcd = p^n} F(b)G(c)H(d)W(c, d)W(b, cd) \\
= \sum_{bcd = p^n} F(c)G(d)H(b)W(d, b)W(c, db) \\
= \sum_{bcd = p^n} F(c)G(d)H(b)W(d, b)W(db, c)
\]

but we have:

\[ \forall a, b, c \in \mathbb{N}^*: W(a, b)W(ab, c) = W(b, c)W(bc, a) \]

so:

\[
\sum_{bcd = p^n} F(c)G(d)H(b)W(c, d)W(cd, b) = \sum_{bcd = p^n} F(c)G(d)H(b)W(d, b)W(db, c)
\]

which gives us the direct proposition.

\[ \square \]

**Lemma 5.5 (multiplication properties).** Let’s note:

\[ \mathbb{M}, \times \] is stable, commutative, associative and admit 1 as the identity element.

**Proof.** Stability:

\[ \forall F,G \in \mathbb{M}, \forall a,b \in \mathbb{N}^2 : [F \times G](ab) = F(ab) \times G(ab) = F(a)G(a)F(b)G(b) = [F \times G](a) \times [F \times G](b) \]

so:

\[ \forall F,G \in \mathbb{M} : F \times G \in \mathbb{M} \]

Commutativity:

\[ \forall F,G \in \mathbb{M}, \forall m \in \mathbb{N}^* : [F \times G](m) = F(m) \times G(m) = G(m) \times F(m) = [G \times F](m) \]

so:

\[ \forall F,G \in \mathbb{M} : F \times G = G \times F \]

Associativity:

\[ \forall F,G,H \in \mathbb{M}, \forall m \in \mathbb{N}^* : [(F \times G) \times H](m) = [F \times G](m) \times H(m) = F(m)G(m)H(m) = [F \times (G \times H)](m) \]

so:

\[ \forall F,G,H \in \mathbb{M} : [(F \times G) \times H] = [F \times (G \times H)] \]
$1$ is the identity element:

$$\forall F \in M, \forall (n,p) \in \mathbb{N} \times \mathbb{P}: [F \times 1](p^n) = F(p^n) \times 1 = F(p^n)$$

so, by unicity of the identity element, $1$ is the identity element of $(M, \times)$.

which proves the theorem. \[\square\]

**Lemma 5.6.** Hypothesis:

$$\forall a, b, c, d \in \mathbb{N}^*, (ab, cd) = 1 : W(a, b)W(c, d) = W(ac, bd)$$

which could be generalised as:

$$\forall n \in \mathbb{N}, \forall a_0, a_1, \ldots, a_n \in \mathbb{N}^*, \forall b_0, b_1, \ldots, b_n \in \mathbb{N}^* | \forall i, j \in [0, n], (a_i b_i, a_j b_j) = 1$$

$$\prod_{i=0}^{n} W(a_i, b_i) = W(\prod_{i=0}^{n} a_i, \prod_{i=0}^{n} b_i)$$

**Proof.** we will proceed by recurrence:

Cas $n = 0$:

$$\prod_{i=0}^{n} W(a_i, b_i) = W(\prod_{i=0}^{n} a_i, \prod_{i=0}^{n} b_i) \iff 1 = W(1, 1)$$

the hypothesis of recurrence:

$$\exists n_0 \in \mathbb{N}, \forall a_0, a_1, \ldots, a_{n_0} \in \mathbb{N}^*, \forall b_0, b_1, \ldots, b_{n_0} \in \mathbb{N}^* | \forall i, j \in [0, n_0], (a_i b_i, a_j b_j) = 1$$

$$\prod_{i=0}^{n_0} W(a_i, b_i) = W(\prod_{i=0}^{n_0} a_i, \prod_{i=0}^{n_0} b_i)$$

The case $n + 1$:

$$\prod_{i=0}^{n+1} W(a_i, b_i) = \prod_{i=0}^{n} W(a_i, b_i)W(a_{n+1}, b_{n+1})$$

[reccurence hypothesis] \[= W(\prod_{i=0}^{n} a_i, \prod_{i=0}^{n} b_i)W(a_{n+1}, b_{n+1})\]

$$[(a_{n+1} b_{n+1}, \prod_{i=0}^{n} a_i, \prod_{i=0}^{n} b_i) = 1] \iff W(\prod_{i=0}^{n+1} a_i, \prod_{i=0}^{n+1} b_i)$$

So we have the results we wanted. \[\square\]

**Lemma 5.7.** Hypothesis: $(M, \circ)$ is stable, commutative, and have an identity element.

The function $W$ could be written as:

$$\forall n, q \in \mathbb{N}^*: W(n, q) = \prod_{p \mid \gcd(n, q)} W(p^\nu((n, q) / p), 1) \prod_{p \not\mid \gcd(n, q)} W(p^\nu((n, q), p^\nu((n, q))))$$
Remark 5.2. This means that in this case, the function $W$ is entirely determined by the image of equal prime power pairs.

**Proof.** The $W$ function have, by hypothesis, the following property:

$$\forall a, b, c, d \in \mathbb{N}^*, \ (ab, cd) = 1 : W(a, b)W(c, d) = W(ac, bd)$$

We have $\left(\frac{m}{(n, q)}, ((n, q))^2\right) = 1$ so:

$$\forall n, q \in \mathbb{N}^* : W((n, q), (n, q)) = W\left(\frac{n}{(n, q)}, \frac{q}{(n, q)}\right)W((n, q), (n, q))$$

Let's compute the first part:

$$\forall n, q \in \mathbb{N}^* : W(\left(\frac{n}{(n, q)}, \frac{q}{(n, q)}\right), 1) = W(\frac{n}{(n, q)}, 1)W(1, \frac{q}{(n, q)}) = \prod_{p \mid (n, q)} W(\frac{p^{v_p(n)}}{p^{v_p(n, q)}}), 1)$$

Let's compute the second part:

$$\forall n, q \in \mathbb{N}^* : W((n, q), (n, q)) = \prod_{p \mid (n, q)} W(\frac{p^{v_p(n)}}{p^{v_p(n, q)}}, (n, q))$$

So:

$$\forall n, q \in \mathbb{N}^* : W(n, q) = \prod_{p \mid (n, q)} W(\frac{p^{v_p(n)}}{p^{v_p(n, q)}}, 1) \prod_{p \mid (n, q)} W(\frac{p^{v_p(n)}}{p^{v_p(n, q)}}, (n, q))$$

By hypothesis (the existence of the identity element), we have that:

$$\forall n, q \in \mathbb{N}^* : W((n, q), (n, q)) = \prod_{p \mid (n, q)} W(\frac{p^{v_p(n)}}{p^{v_p(n, q)}}, (n, q))$$

\[\Box\]

**Lemma 5.8 (distributivity lemma).** Hypothesis: Commutativity, stability, associativity of the two operations in the structure $(\mathbb{M}, \circ, \times)$.

In the structure $(\mathbb{M}, \circ, \times)$, the operation $\circ$ is distributive by $\times$ if and only if the $W$ function, the weight of the convolution, follow some conditions:

$$\forall F, G, H \in \mathbb{M} : [F \circ G] \times H = [F \times H] \circ [G \times H] \iff \forall a, b \in \mathbb{N}^* : W(a, b) = \left\{\begin{array}{ll} 1 & \text{if } (a, b) = 1 \\ 0 & \text{if not} \end{array}\right.$$
**Proof.** Let’s start by the first direction $\implies$:

$$\forall l, f \in \mathbb{N} + f = n, \forall (n, p) \in \mathbb{N}^* \times \mathbb{P} :$$

$$[\mathbb{I}_p \cap \mathbb{I}_p'] \times \mathbb{I}_p(p^n) = (\mathbb{I}_p \cap \mathbb{I}_p')[p^n] \times \mathbb{I}_p[p^n]$$

$$= \sum_{ab=p^n} \mathbb{I}_p(a) \mathbb{I}_p(b) W(a, b)$$

$$= W(p^l, p^f)$$

$$\forall l, f \in \mathbb{N} + f = n, \forall (n, p) \in \mathbb{N}^* \times \mathbb{P} :$$

$$[\mathbb{I}_p \cap \mathbb{I}_p'] \times [\mathbb{I}_p \cap \mathbb{I}_p'](p^n) = \sum_{ab=p^n} [\mathbb{I}_p \cap \mathbb{I}_p'](a)[\mathbb{I}_p \cap \mathbb{I}_p'](b) W(a, b)$$

$$= \sum_{ab=p^n} \mathbb{I}_p(a) \mathbb{I}_p'(a) \mathbb{I}_p(b) \mathbb{I}_p'(b) W(a, b)$$

$$= \mathbb{I}_p(p^n) \mathbb{I}_p(p^n) \mathbb{I}_p'(1) W(p^n, 1) + \mathbb{I}_p'(1) \mathbb{I}_p(p^n) \mathbb{I}_p'(p^n) W(1, p^n)$$

$$= [\mathbb{I}_p(p^n) + \mathbb{I}_p'(p^n)]W(p^n, 1)$$

**Case** $l.f = 0$:

we then have $l = n$ or $f = n$:

$$[\mathbb{I}_p(p^n) + \mathbb{I}_p'(p^n)]W(p^n, 1) = [\mathbb{I}_p(p^n) + \mathbb{I}_p'(p^n)]W(p^n, 1) = 1$$

**Case** $l.f \neq 0$:

we then have $l \neq 0, f \neq 0, l \neq n, f \neq n$ and:

$$[\mathbb{I}_p(p^n) + \mathbb{I}_p'(p^n)]W(p^n, 1) = [0 + 0]W(p^n, 1)$$

$$= 0$$

So to sum up:

$$W(p^l, p^f) = \begin{cases} 1 & \text{if } l.f = 0 \\ 0 & \text{if not} \end{cases}$$

In other words (equivalently):

$$W(p^l, p^f) = \begin{cases} 1 & \text{if } (p^l, p^f) = 1 \\ 0 & \text{if not} \end{cases}$$

Following the precedent result, this determine entirely the function, but the function:

$$\forall a, b \in \mathbb{N}^* : W(a, b) = \begin{cases} 1 & \text{if } (a, b) = 1 \\ 0 & \text{if not} \end{cases}$$

verify those conditions, so the direct sens.

Let’s now check the indirect sens $\iff$:

$$\forall F, G, H \in \mathbb{M} \forall (n, p) \in \mathbb{N}^* \times \mathbb{P} :$$
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\[ [F \circ G] \times H(p^n) = \left[ \sum_{a,b \leq p^n \atop (a,b) = 1} F(a)G(b)W(a,b) \right]H(p^n) \]

\[ = [F(1)G(p^n)W(1,p^n) + F(p^n)G(1)W(p^n,1)]H(p^n) \]

\[ = G(p^n)H(p^n) + F(p^n)H(p^n) \]

\[ \forall F, G, H \in M \forall (n,p) \in \mathbb{N}^* \times P : \]

\[ [F \times H] \circ [G \times H](p^n) = \sum_{a,b = p^n \atop (a,b) = 1} F(a)H(a)G(b)H(b)W(a,b) \]

\[ = F(1)H(1)G(p^n)H(p^n)W(1,p^n) + F(p^n)H(p^n)G(1)H(1)W(p^n,1) \]

\[ = G(p^n)H(p^n) + F(p^n)H(p^n) \]

So we have the result we want to establish.

\[ \square \]

**Lemma 5.9 (inverse lemma).** Hypothesis: commutativity, stability, existence of the neutral element in the structure \((M, \circ, \times)\) and the distributivity of \(\circ\) over \(\times\). In the structure \((M, \circ, \times)\), for all element of \(M\), there exist a unique inverse according to the operation \(\circ\) defined as:

\[ \forall F \in M : [F \circ I_F] = \delta_1 \iff \forall n, p \in \mathbb{N}^* \times P : I_F(p^n) = -F(p^n) \]

**Proof.** Let’s compute:

\[ \forall F \in M \forall n, p \in \mathbb{N}^* \times P : [F \circ I_F](p^n) = \sum_{a,b = p^n \atop (a,b) = 1} F(a)I_F(b)W(a,b) \]

\[ = F(1)I_F(p^n) + F(p^n)I_F(1) \]

\[ = I_F(p^n) + F(p^n) \]

\[ = \delta_1(p^n) \]

\[ = 0 \]

So

\[ \forall F \in M \forall n, p \in \mathbb{N}^* \times P : I_F(p^n) = -F(p^n) \]

In the case \(n=0\):

\[ \forall F \in M : [F \circ I_F](1) = \sum_{a=1 \atop (a,b) = 1} F(a)I_F(b)W(a,b) \]

\[ = F(1)I_F(1) \]

\[ = \delta_1(1) \]

\[ = 1 \]

So

\[ I_F(1) = 1 \]
let’s sum up:

\[
\forall F \in \mathbb{M}\forall n, p \in \mathbb{N} \times \mathbb{P} : I_F(p^n) = \begin{cases} 1 \text{ if } n = 0 \\ -F(p^n) \text{ if not} \end{cases}
\]

which represent the same function here:

\[
\forall F \in \mathbb{M}\forall n \in \mathbb{N}^* : I_F(n) = F^{-1}(n) = (-1)^{\omega(n)}F(n)
\]

In this section, we present the main results that we obtained and concluded using the precedent lemmas:

**Theorem 4.1 (Ring unicity theorem).**

\[(\mathbb{M}, \circ, \times)\text{ is a commutative ring} \iff \forall a, b \in \mathbb{N}^* : W(a, b) = \begin{cases} 1 \text{ if } (a, b) = 1 \\ 0 \text{ if not} \end{cases}\]

**Proof.** by combining the precedent lemmas

\[(\mathbb{M}, \circ, \times)\text{ is a commutative ring}
\]

\[
\Rightarrow \begin{cases} 
\forall a, b \in \mathbb{N}^* : W(a, b) = W(b, a) \text{ ( commutativity lemma : 5.1) } \\
\forall a, b, c, d \in \mathbb{N}^*, (ab, cd) = 1 : W(a, b)W(c, d) = W(ac, bd) \text{ ( stability lemma : 5.2) } \\
\forall (n, p) \in \mathbb{N} \times \mathbb{P}, W(1, p^n) = 1 \text{ ( identity lemma : 5.3) } \\
\forall a, b, c \in \mathbb{N}^*, W(a, b)W(ab, c) = W(b, c)W(bc, a) \text{ ( associativity lemma : 5.4) } \\
\forall a, b \in \mathbb{N}^* : W(a, b) = \begin{cases} 1 \text{ if } (a, b) = 1 \\ 0 \text{ if not} \end{cases} \text{ ( distributivity lemma : 5.8) } \\
\end{cases}
\]

invertly (the checking follows by testing each of the properties):

\[
\forall a, b \in \mathbb{N}^* : W(a, b) = \begin{cases} 1 \text{ if } (a, b) = 1 \\ 0 \text{ if not} \end{cases} \Rightarrow (\mathbb{M}, \circ, \times) \text{ is a commutative ring}
\]

**6. Ring’s propositions proofs**

We will prove in this section many theorems previously stated, we will also present propositions that are closely related to them

**Theorem 4.2 (Refactoristion theorem).**

\[
\forall F, G \in \mathbb{M}_c : D(F \ast G, s) = D(F \times G, 2s) \times D(F \Box G, s)
\]

**Proof.**

\[
\forall F, G \in \mathbb{M}_c : D(F \ast G, s) = D(F, s) \times D(G, s)
\]
\[
\begin{align*}
  &= \left( \sum_{n=1}^{+\infty} \frac{F(n)}{n^s} \right) \times \left[ \sum_{q=1}^{+\infty} \frac{G(q)}{q^s} \right] \\
  &= \sum_{n=1}^{+\infty} \sum_{q=1}^{+\infty} \frac{F(n) G(q)}{n^s q^s} \\
  &= \sum_{m=1}^{+\infty} \frac{F(m) G(k)}{(m)^s (k)^s} \\
  &= \sum_{r=1}^{+\infty} \frac{F(r) G(r)}{r^{2s}} \sum_{m,k=1}^{+\infty} \frac{F(m) G(k)}{(m)^s (k)^s} \\
  &= \left[ \sum_{r=1}^{+\infty} \frac{F(r) G(r)}{r^{2s}} \right] \left[ \sum_{l=1}^{+\infty} \sum_{m,k=1}^{+\infty} \frac{F(m) G(k)}{(m)^s (k)^s} \right] \\
  &= \left[ \sum_{r=1}^{+\infty} \frac{F(r) G(r)}{r^{2s}} \right] \left[ \sum_{l=1}^{+\infty} \sum_{m,k=1}^{+\infty} \frac{F(m) G(k)}{l^s} \right] \\
  &= D(F \times G, 2s) \times D(F \square G, s)
\end{align*}
\]

The alternative version have a similar prove:

**Proposition 6.1.**

\[ \forall F, G \in \mathbb{M}_c : D(F, s) \times D(G, \mathcal{S}) = D(F \times G, 2s) \times D(F \times \text{Id}_{\mathbb{C}}^{-1} \square G \times \text{Id}_{\mathbb{C}}^y, x) \]

**Proof.**

\[ F, G \in \mathbb{M}_c \ \forall y \in \mathbb{C} : \]

\[
D(F, s) \times D(G, \mathcal{S}) = \left( \sum_{n=1}^{+\infty} \frac{F(n)}{n^s} \right) \times \left[ \sum_{q=1}^{+\infty} \frac{G(q)}{q^s} \right] \\
= \sum_{n=1}^{+\infty} \sum_{q=1}^{+\infty} \frac{F(n) G(q)}{n^s q^s} \\
= \sum_{m=1}^{+\infty} \sum_{k=1}^{+\infty} \frac{F(mp) G(kp)}{(mp)^s (kp)^s}
\]
Proposition 4.6. for F and G completely multiplicative with \( \delta_1(1) = 1 \) and \( \forall n \in \mathbb{N}^* - \{1\} : \delta_1(n) = 0 \):

\[
\forall F, G \in \mathbb{M} : F \times G = \delta_1 \Rightarrow D(F, s) \times D(G, s) = D(F \boxtimes G, s)
\]

**Proof.**

\[
\forall F, G \in \mathbb{M} : F \times G = \delta_1 \Rightarrow D(F \times G, s) = 1
\]

so:

\[
\forall F, G \in \mathbb{M} : D(F, s) \times D(G, s) = D(F \boxtimes G, s) \times D(F \times G, s) = D(F \boxtimes G, s)
\]

**\( \Box \)**

**Proposition 4.7.** Let \( A \in \mathbb{P} \) and \( \bar{A} \in \mathbb{P} \) be complementary in \( \mathbb{P} \), let \( F \) be an arithmetic completely multiplicative function:

\[
\forall F \in \mathbb{M} : D(I \times F, s) \times D(\bar{I} \times F, s) = D(F, s)
\]

**Proof.** Let \( A \in \mathbb{P} \) and \( \bar{A} \in \mathbb{P} \) be complementary in \( \mathbb{P} \), let \( F \) an arithmetic completely multiplicative function:

\[
D(I \times F, s) \times D(\bar{I} \times F, s) = D([I \times F] \times [I \times F], s) \times D([\bar{I} \times F] \times [\bar{I} \times F], s)
\]

\[
= \sum_{m=1}^{+\infty} \sum_{k=1}^{+\infty} \frac{F(mp)}{(mp)^{x+iy}} \frac{G(kp)}{(kp)^{x-iy}}
\]

\[
= \sum_{p=1}^{+\infty} \frac{F(p)G(p)}{(p)^{2x}} \sum_{m\cdot k=1}^{+\infty} \left[ \frac{F(m)}{m^{x+iy}} \frac{G(k)}{k^{x-iy}} \right]
\]

\[
= \sum_{p=1}^{+\infty} \frac{F(p)G(p)}{p^{2x}} \sum_{l=1}^{+\infty} \left[ \sum_{m\cdot k=1}^{+\infty} \frac{F(m)}{m^{x+iy}} \frac{G(k)}{k^{x-iy}} \right]
\]

\[
= \sum_{p=1}^{+\infty} \frac{F(p)G(p)}{p^{2x}} \sum_{l=1}^{+\infty} \left[ \sum_{m\cdot k=1}^{+\infty} \frac{F(m)}{m^{x+iy}} \frac{G(k)}{k^{x-iy}} \right]
\]

\[
= \sum_{p=1}^{+\infty} \frac{F(p)G(p)}{p^{2x}} \sum_{l=1}^{+\infty} \left[ \sum_{m\cdot k=1}^{+\infty} \frac{F(m)}{m^{x+iy}} \frac{G(k)}{k^{x-iy}} \right]
\]

\[
= D(F \times G, 2x) \times D(\frac{F}{1d_{l}^{vy}} \boxtimes G, \frac{1d_{e}^{vy}}{l^{x}})
\]

\[
= \sum_{p=1}^{+\infty} \frac{F(p)G(p)}{p^{2x}} \sum_{l=1}^{+\infty} \left[ \sum_{m\cdot k=1}^{+\infty} \frac{F(m)}{m^{x+iy}} \frac{G(k)}{k^{x-iy}} \right]
\]

\[
= D(F \times G, 2x) \times D(\frac{F}{1d_{l}^{vy}} \boxtimes G, \frac{1d_{e}^{vy}}{l^{x}})
\]

\[
= \sum_{p=1}^{+\infty} \frac{F(p)G(p)}{p^{2x}} \sum_{l=1}^{+\infty} \left[ \sum_{m\cdot k=1}^{+\infty} \frac{F(m)}{m^{x+iy}} \frac{G(k)}{k^{x-iy}} \right]
\]

\[
= D(F \times G, 2x) \times D(\frac{F}{1d_{l}^{vy}} \boxtimes G, \frac{1d_{e}^{vy}}{l^{x}})
\]

\[
= \sum_{p=1}^{+\infty} \frac{F(p)G(p)}{p^{2x}} \sum_{l=1}^{+\infty} \left[ \sum_{m\cdot k=1}^{+\infty} \frac{F(m)}{m^{x+iy}} \frac{G(k)}{k^{x-iy}} \right]
\]
\[ D(\mathbb{1}_A \times F \times \mathbb{1}_A \times F, s) \times D([\mathbb{1}_A \times F] \cap [\mathbb{1}_A \times F], s) \]
\[ = D(\mathbb{1}_A \times F \times \mathbb{1}_A \times F, s) \times D([\mathbb{1}_A \times \mathbb{1}_A] \times F, s) \]
\[ = D(\mathbb{1}_\emptyset \times F \times F, s) \times D(\mathbb{1} \times F, s) \]
\[ = D(\mathbb{1}, s) \]

**Proposition 4.8.** Identity in \((M, \Box, \times)\):

\[ [F \Box F](m) = [2^{\omega(m)} \times F](m) \]

**Proof.**

\[ [F \Box F](m) = \prod_{p | m} F(p^{v_p(m)}) + F(p^{v_p(m)}) = [2^{\omega(m)} \times F](m) \]

**Proposition 6.2.** Identity in \((M, \Box, \times)\):

\[ \text{Id}_c \Box \mathbb{1}(m) = \hat{\sigma}(m) \]

here the sigma is the sum of the prime divisors with there complementary.

**Proof.**

\[ \text{Id}_c \Box \mathbb{1}(m) = \prod_{p | m} p^{v_p(m)} + 1 = \sum_{a|b=m \atop (a,b)=1} a = \hat{\sigma}(m) \]

**Proposition 6.3.** Identity in \((M, \Box, \times)\):

\[ \phi = \text{Id}_c \times \left[ \mathbb{1} \Box \left( -1 \right)^{\omega(\text{rad})} \right] \]

**Proof.** The rewriting of the function \(\phi\) are derived simply from the following formula :

\[ \phi(n) = n \times \prod_{i=1}^{\omega(n)} \left( 1 - \frac{1}{p_i} \right) \]

**Proposition 4.9.**

\[ \forall n \in \mathbb{N} \setminus \{0,1\}, \forall a \in \mathbb{N} \setminus \{0,1\} : \quad \Box_{k=1}^{\phi(n)} \chi_k(a) = \phi(n)^{\omega(a)} \cdot \delta_{n\mid[\text{Id}_c-1](a)} \]
\[
\forall n \in \mathbb{N} - \{0, 1\} \forall a \in \mathbb{N} - \{0, 1\} : \\
\left[ \prod_{k=1}^{\phi(n)} \chi_k \right](a) = \prod_{p | a} \phi(n) \cdot \sum_{k=1}^{\phi(p^r(a))} \chi_k(p^r(a)) \\
= \prod_{p | a} \phi(n) \cdot \delta_{p^r(a) \equiv 1 \mod n} \\
= \prod_{p | a} \phi(n) \cdot \delta_{p^r(a) - 1 \equiv 0 \mod n} \\
= \phi(n)^{\omega(a)} \cdot \delta_{\prod_{p | a} p^r(a) - 1 \equiv 0 \mod n} \\
= \phi(n)^{\omega(a)} \cdot \delta_{|d_\omega - 1|(a) \equiv 0 \mod n} \\
= \phi(n)^{\omega(a)} \cdot \delta_{n ||d_\omega - 1|(a)} \\
\]

\begin{proof}
\end{proof}

**Proposition 4.4.**

\[\forall s \in \mathbb{C}, \Re(s) > 1 : \sum_{n=1}^{\infty} \frac{1}{n^s} \cdot \sum_{p \in \mathbb{P}} \frac{1}{p^s} \frac{1}{\omega(np)} = \zeta(s) - 1\]

\begin{proof}
\end{proof}
Proposition 4.5.

\( \forall s \in \mathbb{C}, \Re(s) > 1 : \sum_{n=1}^{\infty} \frac{1}{n^s} \cdot \frac{1}{\omega(n) + 1} \cdot \left( \sum_{p \in \mathbb{P}} \frac{1}{p^s} \right) + \sum_{p \mid n} \frac{1}{p^s} = \zeta(s) - 1 \)

Proof.

\[
\zeta(s) - 1 = \sum_{n=1}^{\infty} \frac{1}{n^s} \cdot \frac{1}{\omega(n) + 1} \cdot \sum_{p \in \mathbb{P}} \frac{1}{p^s} \cdot \frac{1}{\omega(np) + 1} = \sum_{n=1}^{\infty} \frac{1}{n^s} \left[ \sum_{p \mid n} \frac{1}{p^s} \cdot \frac{1}{\omega(n) + 1} \cdot \sum_{p \in \mathbb{P}} \frac{1}{p^s} \cdot \frac{1}{\omega(n) + 1} \right]
\]

7. Derivation over the ring \((\mathbb{M}, \square, \times)\)

Using the previously constructed ring \((\mathbb{M}, \circ, \times)\), we will in this section prove a result of the derivation operators that operate on this space

Definition 7.1. An element \( F \in \mathbb{M} \) is said to be idempotent if:

\[ F \times F = F \]

Definition 7.2. A Dirichlet principle character \( \chi \) is a Dirichlet character of period \( k \in \mathbb{N}^* \) with the following property:

\[ \chi(n) = \begin{cases} 1 & \text{if } (n, k) = 1 \\ 0 & \text{if not} \end{cases} \]

Proposition 7.1. A Dirichlet principle character is always idempotent:

\[ \chi \times \chi = \chi \]
**Proof.** A dirichlet character is a completely multiplicative function, so it is characterised by its value over the primes. As the possible value of any principal dirichlet character are 0 or 1, we obtain the idempotence property

$$\forall p \in \mathbb{P}, \chi(p)^2 = \chi(p)$$

which give us its idempotent identity:

$$\chi \times \chi = \chi$$

**Proposition 7.2.** For all modulus $k \in \mathbb{N}$, all dirichlet character set to the power $\phi(n)$ are equal to a principal dirichlet character:

$$\chi(n)^{\phi(k)} = \begin{cases} 1 & \text{if } (n, k) = 1 \\ 0 & \text{if not} \end{cases}$$

**Proof.** Using the Fermat–Euler theorem, we have:

$$(n, k) = 1 \Rightarrow n^{\phi(k)} \equiv 1 \pmod{k}$$

Which allow us to compute:

$$\chi(n^{\phi(k)}) = \chi(1 + mk) = \chi(1 + mk) = \begin{cases} 1 & \text{if } (n, k) = 1 \\ 0 & \text{if not} \end{cases}$$

Using the completely multiplicative property of $\chi$ as a dirichlet character:

$$\chi(n^{\phi(k)}) = \chi(n)^{\phi(k)}$$

So we have:

$$\chi(n)^{\phi(k)} = \chi(n^{\phi(k)}) = \begin{cases} 1 & \text{if } (n, k) = 1 \\ 0 & \text{if not} \end{cases}$$

**Proposition 7.3.** Any derivation acting on the ring $(\mathbb{M}, +, \times)$ is null over all principal dirichlet characters:

$$\chi \text{ is principal } \Rightarrow \nabla(\chi) = 1$$

**Remark 7.1.** For notation purposes, we will write the function $1 \emptyset$ as 0 in this section.

**Proof.** Let’s take a principal dirichlet character $\chi$, by the derivation operator, we have:

$$\nabla(\chi^2) = 2 \times \chi \times \nabla(\chi)$$

The principal dirichlet character are idempotent, so we have the following identity

$$\nabla(\chi^2) = \nabla(\chi)$$

Which gives us

$$\nabla(\chi) = 2 \times \chi \times \nabla(\chi) = 0$$

$$\Rightarrow [2 \times \chi - 1] \times \nabla(\chi) = 0$$
Knowing that a multiplicative function is characterised by its value over prime powers, we have:

\[ \forall p \in \mathbb{P}, \forall n \in \mathbb{N}^*, \chi(p^{v_p(n)}) = 0 \text{ or } \chi(p^{v_p(n)}) = 1 \Rightarrow \nabla(\chi(p^{v_p(n)})) = 0 \]

Which gives us that for all principal dirichlet characters:

\[ \nabla(\chi) = 0 \]

**Theorem 4.3 (Character derivation theorem).** Any derivation \( \nabla \) acting on the ring \((M, \circ, \times)\) is null over all dirichlet characters:

\[ \forall \chi, \nabla(\chi) = 0 \]

**Proof.** Let’s define the set of zeros of a dirichlet character

\[ Z_\chi = \{ a^n | n \in \mathbb{N}^*, a \in \mathbb{P}, \chi(a) = 0 \} \]

over which we could build an indicatrice function \( 1_{Z_\chi} \). We have constructed the set \( Z_\chi \) to fill the zeros, and only the zeros, of the dirichlet character \( \chi \), so we have:

\[ \forall n \in \mathbb{N}^*, \forall p \in \mathbb{P}, [\chi(p^n) = 0 \text{ and } 1_{Z_\chi}(p^n) \neq 0] \text{ or } [\chi(p^n) \neq 0 \text{ and } 1_{Z_\chi}(p^n) = 0] \]

note that this or is exclusive, which gives us

\[ \forall n \in \mathbb{N}^*, \forall p \in \mathbb{P}, (\chi \circ 1_{Z_\chi})(p^n) \neq 0 \]

and

\[ \chi \times 1_{Z_\chi} = 0 \]

by the derivation operator rules, we have:

\[ \nabla(\chi \circ 1_{Z_\chi} \circ (-1_{Z_\chi})) = \nabla(\chi \circ 1_{Z_\chi}) = \nabla(-1_{Z_\chi}) \]

\[ \nabla(\chi \circ 1_{Z_\chi})^{\phi(n)}(p^n) = \prod_{k=1}^{\phi(n)} \left(1_{Z_\chi}^{\phi(n)}(p^n) \right) \times \chi^{\phi(n) - k} \]

so the element \( (\chi \circ 1_{Z_\chi})^{\phi(n)} \) is an idempotent element

we could compute the derivation of this idempotent element in two way:

\[ \nabla((\chi \circ 1_{Z_\chi})^{\phi(n)}) = 0 \]

\[ \nabla(\chi \circ 1_{Z_\chi})^{\phi(n)} = \phi(n) \times (\chi \circ 1_{Z_\chi})^{\phi(n)-1} \times \nabla(\chi \circ 1_{Z_\chi}) \]

combining the two equaltions we obtain

\[ (\chi \circ 1_{Z_\chi}) \times \nabla(\chi \circ 1_{Z_\chi}) = 0 \]

we have also that

\[ (\chi \circ 1_{Z_\chi})(p^n) \neq 0 \]
which give us
\[ \nabla (\chi \cdot \mathbb{1}_{Z_\chi}) = 0 \]
we also know that \( \mathbb{1}_{Z_\chi} \) is an idempotent element, so we have
\[ \nabla (\mathbb{1}_{Z_\chi}) = 0 \]
which allow us to simplify the equation
\[ \nabla (\chi \cdot \mathbb{1}_{Z_\chi}) = \nabla (\chi) = \nabla (\mathbb{1}_{Z_\chi}) = \nabla (\chi) = 0 \]
so we have
\[ \nabla (\chi) = 0 \]

8. Conclusion

In this work, we have explored a possible factorization of some Dirichlet series and how to generalize it by constructing the ring \((\mathbb{M}, \square, \times)\) of multiplicative functions. We also proved that we can not get a similar ring by changing the weight of the convolution \(\square\) used, the indicator of the pairs of coprime integers is the only weight that generates this ring.

Several previous works have studied \(\square\) convolution as a product [4] [6] [7] [13] [5]. It is even called the unitary product in some references, but we use it in this article as a sum in the ring \((\mathbb{M}, \square, \times)\) which has led us to different structures.

This allowed us to have:
- A factorization of the Dirichlet series in some cases
- The zeta function being a special case of the dirichlet series, a generalization of a Hardy formula has been established
- Several functional identities internal to the ring have been found
- A general result on derivations and Dirichlet characters

9. Appendix

In this section we will build a vector space based on the ring \((\mathbb{M}, \circ, \cdot)\) that could help us better understand some identities

Definition 9.1. We define the operation \(\circ\) as an external operation:
\[ \forall F \in \mathbb{M}, \forall \lambda \in \mathbb{C}, \forall n \in \mathbb{N}^* \quad (\lambda \circ F)(n) = \prod_{p|n} \lambda \cdot F(p^{\nu_p(n)}) \]

Proposition 9.1. \((\mathbb{M}, \circ, \cdot)\) is a \(\mathbb{C}\)-vectorial space

Proof. We have that \((\mathbb{M}, \circ)\) is a commutative group, and the operation \(\circ\) verifies the following assertions:
\[ \forall F, G \in \mathbb{M}, \forall \lambda, \mu \in \mathbb{C} : \]
\[ \lambda \circ (F \circ G) = (\lambda \circ F) \circ (\lambda \circ G) \]
\[ (\lambda \cdot \mu) \circ F = (\lambda \circ F) \circ (\mu \circ F) \]
\[ (\lambda \cdot \mu) \circ F = \lambda \circ (\mu \circ F) \]
\[ 1 \circ F = F \]
which give that \((M, \circ, \circ)\) is a \(\mathbb{C}\)-vectorial space

**Proposition 9.2.** \((M, \circ, \circ)\) is also \(\mathbb{R}\)-vectorial space by taking the same definition of the external operation over the field \(\mathbb{R}\)

**Proof.** The proof on \(\mathbb{C}\) hold for \(\mathbb{R}\) by modifying the set of the external operation

**Remark 9.1.** \((M, \circ, \circ)\) over \(\mathbb{R}\) is a vector subspace of \((M, \circ, \circ)\) over \(\mathbb{C}\)

**Theorem 9.1.** \(M_{\mathbb{C}} = M_\mathbb{R} \oplus iM_\mathbb{R}\)

**Remark 9.2.** This could be useful to project element over the vector subspace \(M_\mathbb{R}\).

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