Linear response theory of hydrodynamics with spin

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We use linear response techniques to develop the previously proposed relativistic ideal fluid limit with a non-negligible spin density. We confirm previous results and obtain expressions for the microscopic transport coefficients using Kubo-like formulae and build up the effective field theory from the computed correlation functions. We also confirm that polarization makes vortices acquire an effective mass via a mechanism similar to the Anderson-Higgs mechanism in superconductors. As speculated earlier, this could stabilize the ideal hydrodynamic limit against fluctuation-driven vortices.

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I. INTRODUCTION

An interesting problem in fundamental relativistic fluid dynamics is the form the ideal fluid will have when the microscopic medium has a non-zero polarization density \cite{1-3}. This is a challenging problem even at an intuitive level because several characteristics associated with ideal fluids such as isotropy and conservation of circulation, will not apply when spin density is non-zero. Indeed, several approaches have been tried \cite{4-8}, with a consensus on even the fundamental dynamics still lacking.

One virtue of the Lagrangian approach \cite{1-3} is that it allows us to start from local equilibrium as an assumption and build up the lagrangian from the free energy. This has allowed us to obtain several results in an intuitive way, such as the necessity of parallelism between spin and angular momentum \cite{1} and the necessity of dissipation for a causal theory \cite{2, 3}. However, this is an effective theory approach which, by definition, is independent of any link with microscopic theory.

In this work, we try to make such a link by reformulating the previous theory in terms of linear response and correlators. This also allows for the analysis of the response of the bulk hydrodynamic evolution to microscopic fluctuations and correlations.

Our Lagrangian, following \cite{9}, contains the information of the equation of state and entropy current in terms of the field $\phi_I$ of the Lagrangian coordinates of the fluid element $\phi_I$. The entropy of the volume element is then the volume of the element

$$ b = \left( \det \theta_{IJ} \partial_{[I} \phi_J \partial^{I]} \right)^{1/2} \quad (1) $$

in the absence of chemical potentials this is the only degree of freedom possible. Including the polarization tensor $y_{\mu\nu}$ is similar to including a chemical potential which however transforms as a vector in the co-moving frame \cite{1, 2}. Since spin density is not conserved, $y_{\mu\nu}$ is an auxiliary field interacting with $b$ via the equation of state.

Note that, because of this, while $y_{\mu\nu}$ breaks local isotropy explicitly, but does not vanish at thermodynamic equilibrium. Indeed, in the case where vortical susceptibility is calculated explicitly \cite{1}, the expression for magnetic and vortical susceptibility parallel each other, suggesting the dynamics is the same up to $C$ symmetry. In a fluid with no chemical potential one expects the spin alignment will not produce a magnetic field (since the magnetic moment of particles and antiparticles is opposite), but it will break isotropy and take angular momentum out of vorticity and vice-versa.

For a well-defined local equilibrium vorticity and polarization need to be parallel \cite{1}, i.e.

$$ y_{\mu\nu} = \chi (b, \omega_{\mu\nu}) \omega_{\mu\nu}, \quad (2) $$

where the relativistic vorticity \cite{10} includes the enthalpy $w$

$$ \omega_{\mu\nu} = 2 \nabla_{[\mu} w_{\nu]} \quad (3) $$

$$ = 2 w \left( \nabla_{[\mu} u_{\nu]} - \dot{u}_{[\mu} u_{\nu]} + u_{[\mu} \nabla_{\nu]} \ln w \right) \simeq \beta \nabla_{[\mu} u_{\nu]} $$

In \cite{1} we have shown that this Lagrangian leads to three conservation law type equations

$$ J^\mu_I = 4 c \partial_\nu \left\{ F^\nu \left( \chi (\chi + 2 \partial_{[\mu} \chi) \omega_{\alpha\beta} g^\alpha \nu P^I_{\nu} \beta] \right) \right\} - F^\nu \left[ u_{\rho} P^I_{\rho} \left( 1 - c g^2 - 2 c \beta \omega^2 \partial_{\mu} \chi \right) \right] - 2 c \left( \chi + 2 \omega^2 \partial_{[\mu} \chi \right) F^\nu \times \right\} $$
\[
\chi \omega^2 - \frac{1}{b} y_{\rho \sigma} (u_\alpha \partial^\rho K^\sigma - u_\alpha \nabla^\rho K^\sigma) \left\{ P_\sigma^\mu - \frac{1}{6b} y_{\rho \sigma} \varepsilon^{\mu \rho \alpha \beta} \varepsilon_{IJK} \nabla^\sigma \partial_\alpha \phi^J \partial_\beta \phi^K \right\}. \tag{4}
\]

This is the “ideal hydrodynamic limit with polarization”, the equation of motion of a fluid with spin density where local equilibrium is reached instantaneously.

However, as shown in [2, 3] this equation produces non-causal perturbations. Causality means equation 3 can only be achieved as a relaxation asymptotic limit,

\[
\tau_Y \partial_t \delta Y_{\mu \nu} + \delta Y_{\mu \nu} = \chi(b, w^2)\omega_{\mu \nu} \tag{5}
\]

Where non-equilibrium “Magnon” tensor \(Y_{\mu \nu}\) relaxes to the equilibrium value. Equation 4, analogously to other Maxwell-Cattaneo cases, needs to be updated with \(Y_{\mu \nu}\) as an additional degree of freedom.

The next three sections will link these results to the more traditional linear response theory. A correlation function for \(J^\mu_I, T_{\mu \nu}\) and \(Y_{\mu \nu}\) will be derived. A fluctuation dissipation relation linking \(\chi\) and \(\tau_Y\) will also be derived.

II. LINEAR RESPONSE ANALYSIS

To make a link to microscopic theory, we use the assumption of local equilibrium and linear response theory. While hydrodynamics is highly non-linear, one assumes that any microscopically-driven perturbation starts off in the linearized stage from the equilibrium state, hence its growth rate can be approximated by a linear response function. This is also equivalent to assuming these changes develop slowly enough to be considered “adiabatic”, so local thermalization can be assumed at any moment in its evolution. Then, the coefficient of the constitutive relation are given by taking the low energy and long wavelength limit for the correlation function of operators in different points in the space and time. Since the system should not, in this regime, distinguish weather the deviation came from either an external disturbance or natural fluctuation, the transport coefficient must be the same for both linear response and correlation function.

A. The magnon field \(Y_{\mu \nu}\)

In this spirit, building on [1, 3] let the vorticity-fluctuation coupling be given by the interacting picture Hamiltonian \(H_I\) [11]

\[
H_I(t) = \int d^3 x Y^{\mu \nu}(t, \vec{x}) \omega_{\mu \nu}(t, \vec{x}) \tag{6}
\]

where the vorticity \(\omega_{\mu \nu}\) is treated as a classical source to the hermitian polarization operator \(Y^{\mu \nu}(t, \vec{x})\). The unitary operator generating the temporal evolution reads

\[
U(t, t_0) = \mathcal{T} \left( e^{-\int_{t_0}^t H_I(t') dt'} \right) \tag{7}
\]

Where \(\mathcal{T}\) is a time-ordering product. In our case, the bounded chronological product of operator remains unchanged because only linear approximation will be present and a full treatment is out of scope of this work, too much laborious and nonphysical realizable. The first question relates how \(\omega_{\mu \nu}\) affects the field operator \(Y^{\mu \nu}\), so the main objective relies on comprehend deviation from local equilibrium. At this point, the proper density matrix in Heisenberg picture

\[
\rho = U(t, t_0) \rho_0 U^\dagger(t, t_0), \quad \rho = \frac{1}{Z} \sum_\alpha e^{-\beta H_\alpha - \omega \cdot \vec{J}/T} \tag{8}
\]

Linearizing from the equilibrium expectation value we get

\[
\langle Y^{\mu \nu}(t, \vec{x}) \rangle_\omega = \langle \rho_0(t) U^\dagger(t, t_0) Y^{\mu \nu}(t, \vec{x}) U(t, t_0) \rangle \approx \langle Y^{\mu \nu}(t, \vec{x}) \rangle_{\omega=0} + i \int dt' \int d^3 \vec{x}' \langle [Y^{\mu \nu}(t, \vec{x}), Y^{\alpha \beta}(t', \vec{x}')] \rangle_{eq} \omega_{\alpha \beta}(t', \vec{x}') \tag{9}
\]
with \( \langle \delta Y^{\mu\nu} \rangle = (Y^{\mu\nu}(t, \vec{x}))_{\omega} - (Y^{\mu\nu}(t, \vec{x}))_{\omega=0} \), where \( (Y^{\mu\nu}(t, \vec{x}))_{\omega=0} \) is classical average at local equilibrium, before the vortex (source) switched on. The induced polarization reads

\[
\langle \delta Y^{\mu\nu} \rangle = i \int_{-\infty}^{t} dt' \int d^3 \vec{x}' e^{i\epsilon t'} \Theta(t-t') \langle [Y^{\mu\nu}(t, \vec{x}), Y^{\alpha\beta}(t', \vec{x}')] \rangle_{eq} \omega_{\alpha\beta}(t', \vec{x}')
\]

(10)

In order to our Kubo’s formula still physical reasonable, we assume an adiabatic change in local variable

\[
\omega^{\mu\nu}(\vec{x}', t') = e^{\epsilon t'} \Theta(-t') \omega^{\mu\nu}(\vec{x}), \quad t < 0 \\
\omega^{\mu\nu}(\vec{x}', t') = 0, \quad t > 0
\]

(11)

Note that, a factor \( e^{\epsilon t} \) is added upon integral for the sake of smearing out any short-range fluctuations. The constant \( \epsilon \), in the physical sense, settle down an adiabatic evolution of spin-orbit balance until the transport process archives a thermalization (maximum entropy). In addition, in the absence of spontaneous symmetry breaking (see conclusion of [3]) the \( \frac{1}{\tau} \) conducts the transition rate in which the spins interacting with each other \( \epsilon \sim \tau \). In this case, the linear approximation remains valid and the physical meaning of \( \epsilon \) in the source in Eq. 11 is a rate of increasing \( \omega^{\mu\nu} \), or density of spin aligned with vortex. Applying the equation 11, we get the expectation value of correlation function between two operator in different point of space-time in equilibrium ensemble

\[
\langle \delta Y^{\mu\nu} \rangle = i \int_{-\infty}^{t} dt' \int d^3 \vec{x}' e^{i\epsilon t'} \Theta(t-t') \langle [Y^{\mu\nu}(t, \vec{x}), Y^{\alpha\beta}(t', \vec{x}')] \rangle_{eq} \omega_{\alpha\beta}(t', \vec{x}') \\
= i \int_{-\infty}^{0} dt' \int d^3 \vec{x}' e^{i\epsilon t'} \Theta(t-t') \langle [Y^{\mu\nu}(t, \vec{x}), Y^{\alpha\beta}(t', \vec{x}')] \rangle_{eq} \omega_{\alpha\beta}(t', \vec{x}') \\
= i \int_{-\infty}^{0} dt' \int d^3 \vec{x}' e^{i\epsilon t'} \Theta(t-t') \langle [Y^{\mu\nu}(t, \vec{x}), Y^{\alpha\beta}(t', \vec{x}')] \rangle_{eq} \omega_{\alpha\beta}(t', \vec{x}')
\]

(12)

The commutator above expresses the retarded green function defined as

\[
G_{Y^{\mu\nu}, Y^{\alpha\beta}}^R(\vec{x}', \vec{x}, t, t') \equiv -i\Theta(t-t') \langle 0 | Y^{\mu\nu}(\vec{x}, t), Y^{\alpha\beta}(\vec{x}', t') | 0 \rangle_{eq}
\]

(13)

The polarization-polarization correlation function comprehends the amount of fluctuation correlated in space and time and the heaviside step function \( \Theta(t-t') \) assures causality principle. We shall rewrite the eq. (12) as that of a single “Retarded” Greens function \( G^R \)

\[
\langle \delta Y^{\mu\nu} \rangle = i \int_{-\infty}^{0} dt' \int d^3 \vec{x}' e^{i\epsilon t'} G_{Y^{\mu\nu}, Y^{\alpha\beta}}^R(\vec{x}', \vec{x}, t-t') \omega_{\alpha\beta}(t', \vec{x}') \\
= i \int_{-\infty}^{0} dt' \int d^3 \vec{x}' e^{i\epsilon t'} G_{Y^{\mu\nu}, Y^{\alpha\beta}}^R(\vec{x}', \vec{x}, t-t') \omega_{\alpha\beta}(t', \vec{x}')
\]

(14)

The treatment, played here, is not the only possible but concedes the transport coefficient by natural fluctuations from the pole of hydrodynamic modes. As it is well known, the constitutive coefficient cannot be explained by hydrodynamic theory but rather microscopic one. The next subsection will make an explicit link.

**B. Susceptibility**

A thermodynamic identity rises up when average of polarization disturbance relates linearly by vortex.

\[
\langle Y^{\mu\nu} \rangle = \chi(b, \omega^2) \omega^{\mu\nu}
\]

(15)

Susceptibility, \( \chi(b, \omega^2) \), tell us how polarization behavior is induced from small perturbation around equilibrium and after turning off \( \omega^{\alpha\beta} \). We obtain the susceptibility by taking the hydrodynamic regime, where the comoving coordinates are perturbed against the hydrostatic coordinates \( X_I \).

\[
\phi_I = X_I + \pi_I
\]

(16)

given a wavenumber \( k_I \), \( \pi_I \) can be separated into a sound-wave and a vortex part

\[
\pi_I = \pi_I^T(k_I) + \pi_I^L(k_I) \quad , \quad k_I \pi_I^T = 0 \quad , \quad k_I \pi_I^L = |k||\pi|
\]

(17)
then the susceptibility becomes

$$
\chi = \lim_{k \to 0} \left( [\langle \pi^T_a \pi^T_b \rangle - \langle \pi_a^T \pi_b^T \rangle] + [\langle \pi_a^L \pi_b^L \rangle - \langle \pi_a^L \pi_b^L \rangle] \right) / V
$$

where $V$ is the volume of phase space in the thermodynamic limit. The limit of small frequency

$$
\chi(b, \omega^2) = \frac{\delta Y^{\mu\nu}}{\delta \omega^{ab}} \bigg|_{\omega=0}
$$

where $\chi(b, \omega^2)$ (thermodynamic derivative) is a statistic thermodynamic quantity. $\chi_{ab}$ is symmetric matrix and its diagonal forms turns out when aligned with axial rotation axes. The susceptibility is an analytical function, which the poles lies below the real axis, thus we may split up

$$
\chi(\omega + i\epsilon, \vec{k}) = \chi'(\omega, \vec{k}) + i\chi''(\omega, \vec{k})
$$

the imaginary $\chi''$ (absorptive or dissipative) and real $\chi'$ (symmetric in time) part. $\det(\chi) \geq 0$.

The relation of imaginary part of susceptibility with retarded green function

$$
\langle 0 | Y^{\mu\nu}(\vec{x}, t), Y^{\alpha\beta}(\vec{x}', t') | 0 \rangle_{eq} = \int d\omega d\vec{k} \frac{\chi''(\omega, \vec{k})}{\omega} e^{i(\omega(t-t')-\vec{k}(\vec{x}-\vec{x}'))}
$$

provides a linking between a linear response described by a linearized hydrodynamic and correlation function. The $\chi''$ is responsible for all information on commutator of polarization. From the definition of Green function, we have the Kramers-Konig relation [11] linking the retarded Greens function $G_R$ to the advanced one $G_A$

$$
Re[G_R] = Re[G_A], \quad Im[G_R] = -Im[G_A]
$$

The spectral representation is

$$
G^{R/A}_{ab}(\omega, \vec{k}) = \int \frac{d\omega'}{2\pi} \rho^{YY'}(\omega, \vec{k}) \frac{\chi''(\omega, \vec{k})}{\omega - \omega' + i\epsilon}
$$

The spectral density $\rho^{YY'}$ is non negative and a real function containing the density of state at frequency $\omega$ and $\vec{k}$.

Let us discuss the properties of correlation function $\langle \phi(t) \phi(t') \rangle = \langle \phi(t) \phi(-t) \rangle$. These functions are independent of details configuration of the system unless an external field is applied. The symmetry under time reversal manifests in the form of $\langle \phi(t) \phi(t') \rangle = \langle \phi(t') \phi(t) \rangle$. In our picture of fluctuation, we have to know how the fields changed under reverse of velocity due to switch direction of external rotating frame from $\Omega$ to $-\Omega$. From this perspective, any even combination of longitudinal ($\pi_L$) or transverse ($\pi_T$) are unaffected by reversal $\Omega$ and thus the retarded green function exhibits the following property

$$
G^R_{ab}(t, \Omega) = G^R_{ba}(t, -\Omega) = G^R_{ab}(-t, -\Omega) = G^R_{ab}(-t, \Omega),
$$

The general formulation is $G^R_{ab}(\omega, \vec{k}, \Omega) = \eta_1 \eta_2 G^R_{ab}(\omega, \vec{k}, \vec{\Omega})$, where $S = diag(\eta_1, \eta_2, \ldots) = diag(1, 1, \ldots)$ whereas the kinetic coefficients takes the form

$$
\gamma_{ik}(\Omega) = \gamma_{ki}(-\Omega)
$$

The Hamiltonian under time reversal operator reads

$$
\Theta H(\omega) \Theta^{-1} = H(-\omega)
$$

If we simultaneously change the signal of all charge, the current and $\Omega$ remain in the same direction. There is no charge conjugation so $J^T_b \to J^T_b$ and up to first order it is easy to see that $J^T_b \sim k^\mu \omega_1 \theta^b_0$ is consistent with this
symmetry. To develop the necessary tool for the evaluation of average at disturbed system, we rewrite the locally conserved Noether current at a variational principle

$$\frac{\delta L}{\delta \dot{\omega}^\mu} \equiv -J^\mu$$  \hspace{1cm} (27)

where $\dot{\omega} = \epsilon^{\alpha\mu\nu} \dot{\omega}_{\mu\nu}$. Being $\omega_{\mu\nu}$ the vorticity field. The conserved current Eq. 4

We want to investigate how the gradient of hydrodynamical variable disappear by an external disturbance. This allows us to introduce the systematic of Kubo formulae for current. The general linear transformation reads

$$\langle J^\mu_i(t, \vec{x}) \rangle_\omega - \langle J^\mu_i(t, \vec{x}) \rangle_0 = \int d^3x' dt' \langle [J_i(t, \vec{x}), J_j(t', \vec{x}')] \rangle \delta^\mu_{ij}$$

$$\langle J^\mu_i(t, \vec{x}) \rangle_0 = 2 e \chi^2 F'(b_\omega) b_\omega \omega^2 \delta^\mu_i$$  \hspace{1cm} (28)

$$\langle J^\mu_i(t, \vec{x}) \rangle_\omega = 2 e \chi^2 F' \omega^2 \delta^\mu_i + \frac{\chi''(b_\omega \omega^2)}{\omega}$$

The main consequence is that $\langle Y^{\mu\nu} \rangle_{\omega=0}$ refers to the microscopic quantum operator average before switched on the background field. It is suggestive the non-vanishing of the vacuum expectation value of classical field is similar to derivation of Higgs mechanism, so that we can roughly say the mass term $M^2 = (\chi^2 \sqrt{F'(b_\omega)b_\omega})^{-1}$. One can use the background value of Lagrangian $F_\omega(b_\omega = 1) = w_0$ and consider that $w_0$ multiply every term in the current. The effective mass from dissipative interaction between the spin and cell-fluid under the presence of rotation is $M^2 = (\chi^2)^{-1}$. The $\hat{\omega}^\mu$ is not an elementary field but a product of them and plays a role of source. Thus, polarization realizes Landau’s original observation [12] that to stabilize hydrodynamics vortices must have a mass gap (this turned out to describe superfluidity, but not ordinary fluids). In contrast, [13] conjectured that there is no stable quantum theory of fluids and [26] argued that for such a theory to exist only conserved observables are allowed.

At this point, the applied vortex has a similar form as the magnetic field potential in electromagnetic theory. Although the vortex has no propagation, we can treat the flow as a gauge covariant derivative which performs an infinitesimal rotation in each fluid. It is easy to see how local translational symmetry connects with Euler and Lagrangian picture, thus following this line of thought, the lagrangian density appears invariant under global $SO(3)$ symmetry (complex phases are irrelevant in the many particle per unit volume limit [1]) due to gauge covariant scenario of unperturbed system and over the EFT language its comes from diffeomorphism invariance. So long as there is no dissipation, the vortex behaves exactly as a gauge field, with the covariant derivative “propagating” the gauge element along the fluid. This symmetry also keeps the vortex massless [13].

In this respect, as shown in more detail in section IV A, polarization acts as a Higgs mechanism [14] giving the vortex a "mass" related to $\chi$. Unlike the usual Higgs mechanism, however, Gauge symmetry does not allow us to remove Ostrogradski’s instabilities even in the linearized limit [2] (in electromagnetism, they do show up away from this limit [15]). Dissipation therefore becomes necessary [3]. In the next section, we shall see the effect it will have in correlation functions.

C. Relaxation time

As shown in [2, 3], the polarization responding immediately to small external field according to Eq. 15 generally violates causality. As consequence, polarization cannot appear immediately when a vorticity field is turned on. We have to specify a "minimal" time delay, in analogy with the Maxwell-Cattaneo equation [16, 17].

Adjusting the previous formalism to this realization, the Hermitian operator has to decay in absence of a vorticity field with a characteristic time scale $\tau_\gamma$. Equation 5, derived in [3] would then appear by linking the expectation value of the magnon to a classical vorticity source

$$\frac{\partial}{\partial t} \langle Y^{\mu\nu} \rangle = -\frac{1}{\tau_\gamma} [Y^{\mu\nu} + \chi(\omega^2 b_\omega) \omega_{\mu\nu}]$$  \hspace{1cm} (29)

Differentiating this equation in relation to time, we can write the equation of motion. According to conservation of total angular momentum at thermodynamic equilibrium, $\partial_\Lambda S^{\lambda\mu\nu} = -2T^{\mu\nu}_A$, where $S^{\lambda\mu\nu}$ is the spin tensor and $T^{\mu\nu}_A$ antisymmetric tensor, and a partial integration substitution from Laplace transform. We get
the initial condition with no polarization

the underbrace defines the linearized limit where \( \vec{\pi} \) is perpendicular to the vorticity plane defined by \( \vec{\omega} \), assuming the initial condition with no polarization \( \langle Y^{\mu\nu}(0, |\vec{k}|) \rangle = \chi \omega^{\mu\nu}(k) \)

\[
\frac{\partial}{\partial t^2} \langle Y^{\mu\nu}(\vec{k}, t) \rangle = (iz + \gamma z^2)^{-1} \left( (1 - iz\gamma) \chi(b_o, 0) \omega^{\mu\nu}(\vec{k}, 0) - 4w_0\chi^2(b_o, 0)g^{\rho\mu}g^{\sigma\nu} \int_0^\infty dt e^{iz(t - t')} \partial_\beta \omega^{\rho\sigma}(\vec{k}, t') \right)
\]

(30)

III. CORRELATION FUNCTIONS FROM FUNCTIONAL METHODS

In the previous section, we saw that the Green-Kubo relation can be defined by hydrodynamic variable which attained the transport equation from conservation law. However, hydrodynamics with polarization cannot [1] entirely be written in terms of such laws. To get the more general correlation functions allowed within the theory it is easier to use functional methods.

We shall characterize long-distance dynamics properties by a small fluctuation from off-diagonal metric around a static background. We are essentially updating, in the polarization context discussed in [3], the analysis made in [18] and applied to relativistic hydro in [19, 20]. The green’s functions of the conserved currents, in our case the energy-momentum tensor and the \( J^I \) will be

\[
G_{\rho \sigma}^{R}(\vec{x}) = -\frac{\delta J^\rho(\vec{x})}{\delta \omega_\sigma(0)}|_{h=0=0}, \quad G_{T^{\mu \nu} J^I}(x) = -\frac{\delta T^{\mu \nu}(\vec{x})}{\delta \omega_\sigma(0)}|_{h=0=0},
\]

(31)

where \( T^{\mu \nu}(x) \equiv \sqrt{-g} \langle T^{\mu \nu}(x) \rangle_{\omega, g} \) and \( J(x) \equiv \sqrt{-g} \langle J^I(x) \rangle_{\omega, g} \)

The covariant derivative provide the interplay between out-of-equilibrium hydrodynamic variable and the metric and vorticity sources

\[
\nabla_\mu u^\nu = \partial_\mu u^\nu + \frac{1}{2} u^\rho (\partial_\mu h_\beta \rho + \partial_\rho h_\beta \mu - \partial_\beta h_{\mu \rho})u^\rho_0
\]

The higher orders stress tensor, not symmetrized, for a generic system is (summation in indices \( I \) of \( \phi_I \) omitted for brevity)

\[
T^{\mu \nu} = \left\{ \frac{\partial L}{\partial (\partial_\mu \phi)} - \partial_\beta \frac{\partial L}{\partial (\partial_\mu \partial_\beta \phi)} + \partial_\gamma \partial_\sigma \frac{\partial L}{\partial (\partial_\mu \partial_\beta \partial_\gamma \partial_\sigma \phi)} - \cdots \right\} \partial^\nu \phi + \left\{ \frac{\partial L}{\partial (\partial_\mu \partial_\beta \phi)} - \partial_\gamma \frac{\partial L}{\partial (\partial_\mu \partial_\beta \partial_\gamma \phi)} + \cdots \right\} \partial^\nu \phi + \cdots - \eta^{\mu \nu} L
\]

(33)

we note that we use the Canonical rather than the symmetric (Belinfante-Rosenfeld) form of the tensor to keep track of the diffeomorphism-dependent components that couple to \( g_{\mu \nu} \) [21]. The integral \( \int d^3 x T^{00} \) will of course be independent of pseudo-gauge transformations as expected at the level of the partition function [22].

The conserved current from Noether theorem of a space-time or internal symmetries for higher order fields is

\[
J^\rho = i\epsilon \left\{ \frac{\partial L}{\partial (\partial_\mu \phi)} - \partial_\beta \frac{\partial L}{\partial (\partial_\mu \partial_\beta \phi)} + \cdots \right\} \phi + \left\{ \frac{\partial L}{\partial (\partial_\mu \partial_\beta \phi)} + \cdots \right\} \partial^\beta \phi
\]

(34)

It is possible to write the exact formula for the density Lagrangian of polarized fluid where

\[
\mathcal{L} = F(b, y) = F(b(1 - c y_{\mu \nu} y^{\mu \nu})), \quad y_{\mu \nu} = \chi(b, \omega_{\mu \nu} \omega^{\mu \nu})\omega_{\mu \nu}, \quad \omega_{\mu \nu} = \nabla_{[\mu} u_{\nu]},
\]

(35)
Thus the stress tensor not symmetrized is

\[ T^{\mu\nu} = F^\rho \left\{ u_\rho P_1^{\mu\nu} \left( 1 - cy^2 - 2cb \omega^{\alpha\beta} \partial_\beta \chi \right) \right\} \partial^\nu \phi^I + \]

\[ 2cF^\rho (\chi + 2\omega^2 \partial_\omega \chi) \left[ \chi \omega^2 u_\rho - \frac{1}{6} g^\rho_\sigma \left( K^\rho_\sigma - u_\alpha \nabla^\rho K^\alpha \right) \right] P^{\mu\nu} - \frac{1}{60} g^\rho_\sigma \varepsilon^{\rho\mu\alpha\beta} \epsilon_{IJK} \nabla^\rho \alpha \partial_\beta \phi^I \partial_\gamma \phi^K \right\} \partial^\nu \phi^I + \]

\[ - \partial_\beta \left( 4cF^\rho \left( \chi + 2\omega^2 \partial_\omega \chi \right) \right) \partial^\nu \phi^I + \left\{ 4cF^\rho \left( \chi + 2\omega^2 \partial_\omega \chi \right) \right\} \partial^\nu \phi^I \]

and the current, we reproduce the earlier result, the conserved current of Eq. 4

\[ \text{A. Stress tensor perturbations} \]

The basic idea is to expand the variational of stress and current of the eqs. 31 in terms of the following projectors

\[ g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu}, \quad u_\mu \simeq \delta_0^{\mu} \left( 1 + \frac{1}{2} \pi^2 \right) + \delta_1^{\mu} \left( -\frac{\pi}{2} + \partial^\nu \phi^I \right), \quad \omega^2 \simeq - (\partial_{\mu} \pi) \cdot (\partial^\nu \pi) - [\partial \pi \cdot \partial \pi] \]

The Ward identities [23, 24]

\[ \partial_\nu G_{T^\mu T^\nu} (x) - \partial_\mu G_{T^\nu T^\nu} (x) = -\omega, \quad \partial_0 G_{T^\mu T^0} (x) - \partial_j G_{T^0 T^j} (x) = k_i p \]

\[ G_{T^\mu T^0} (x) > G_{T^\nu T^\nu} (x) \]

can be used to derive all components from one, which comes from the variational principle for the principal modes.

\[ G_{T^\mu T^0} (x) = -2 \delta T^{00} (x) \left|_{\omega = h = 0} \right. = w_0 \left[ i\omega k^2 - i\chi^2 \omega^3 k^2 + i\chi^4 \omega^5 k^2 \right] \]

\[ \chi^2 (\omega^4 + k^2 \omega^2) - \omega^2 + c_2^2 k^2 + i2w_0 \chi^2 \omega k^2 \]

while the other correlators, as expected, agree with the Ward identity Eq. 38

\[ G_{T^\mu T^0} (x) = -2 \delta T^{00} (x) \left|_{\omega = h = 0} \right. = w_0 \left[ i\omega k^2 - i\chi^2 \omega^3 k^2 + i\chi^4 \omega^5 k^2 \right] \]

\[ \chi^2 (\omega^4 + k^2 \omega^2) - \omega^2 + c_2^2 k^2 + i2w_0 \chi^2 \omega k^2 \]

\[ G_{T^0 T^0} (x) = -2 \delta T^{00} (x) \left|_{\omega = h = 0} \right. = 0 \]

\[ G_{T^0 T^0} (x) = -2 \delta T^{00} (x) \left|_{\omega = h = 0} \right. = \left[ \frac{-2i\omega^3 - 2i\chi^4 (\omega^7 - \omega^5 k^2 + \omega^3 k^4) + 2i\chi^2 (\omega^5 - \omega^3 k^2)}{(4 \omega^2 - k^2 \omega^2)} - \omega^2 \right] \]

\[ -2i\omega_0 \omega + 2i\omega_0 \chi^2 (\omega^3 - \omega k^2 / 2) \]

If \( \chi \to 0 \), we will recover the correlators derived previously for the theory without polarization [20]

\[ G_{T^0 T^0} (x) = -2 \delta T^{00} (x) \left|_{\omega = h = 0} \right. = i\omega_0 \chi^2 \omega^3 \]

The contribution of the backreaction terms, \( \mathcal{O}(\pi^2) \) will be explored in more detail in section IV

\[ \epsilon = -F(b_o) - b_o F_o (b_o) [\partial \pi] + \mathcal{O}(\pi^7), \quad p = F(b_o) - b_o F_o (b_o) + [\partial \pi] F_{bb} (b_o) + \mathcal{O}(\pi^2) \]
B. Current and vorticity perturbations

The green function of an energy momentum current can be obtained by a perturbation of the metric

\[
G_{j^0 T^{00}}^R(x) = -\frac{\delta J^0(x)}{\delta h_{00}(0)}|_{h=\omega=0} = \left. w_0 \frac{2i\chi^4(\omega^5 k^2 + \omega^4 k^3) - i\chi^2(6\omega^4 k + 2\omega^2 k^3 - \omega^3 k^2 - 2\omega k^3 c^2_k) - 2ic^2_k k^3}{\chi^2(\omega^4 + k^2 \omega^2) - \omega^2 + c^4_k k^2}\right|_{h=\omega=0} \tag{46}
\]

\[
G_{j^0 T^{0\alpha}}^R(x) = -\frac{\delta J^0(x)}{\delta h_{0\alpha}(0)}|_{h=\omega=0} = \left. w_0 \frac{2i\chi^4(\omega^5 k + \omega^5 k^2) - i\chi^2(6\omega^5 + 2\omega^3 k^2 - \omega^4 k - 2\omega k^3 c^2_k) - 2ic^2_k \omega k^2}{\chi^2(\omega^4 + k^2 \omega^2) - \omega^2 + c^4_k k^2}\right|_{h=\omega=0} \tag{47}
\]

We need to clarify that the "interaction" of vorticity field with the current arises by the coupling on the plane of vortex instead of the axial vector \( \hat{w}^\mu = \epsilon^{\mu\alpha\beta}\omega_{\alpha\beta} \).

Equations 47 can be combined with the Ward-like identity coming from the conservation of both \( J \) and the energy-momentum tensor

\[
\partial_\mu G_{j^0 T^{00}}^R(x) - \partial_\mu G_{j^0 T^{0\mu}}^R(x) = 0 \tag{48}
\]

and the useful formulae

\[
G_{j_{\omega^0}}^R(x) = -G_{j_{\omega^0}}^R(x), \quad G_{j_{\omega^0}}^R(x) < G_{j_{\omega^0}}^R(x) \tag{49}
\]

to obtain information about all the vorticity field correlators

\[
G_{j_{\omega^0}^0}(x) = -\frac{\delta J^0(x)}{\delta \omega^{0z}(0)}|_{A=\omega=0} = \left. w_0 \frac{2i\chi^4 + 2i\omega k^2 c^2_k}{\omega^2 - c^4_k k^2}\right|_{A=\omega=0} \tag{50}
\]

\[
G_{j_{\omega^0}^z}(x) = -\frac{\delta J^0(x)}{\delta \omega^{0z}(0)}|_{A=\omega=0} = \left. w_0 \frac{4i\chi^4 - 4i\omega k^2 \omega}{\omega^2 - c^4_k k^2}\right|_{A=\omega=0} \tag{51}
\]

\[
G_{j_{\omega^0}^{\alpha\beta}}(x) = -\frac{\delta J^0(x)}{\delta \omega^{0\alpha\beta}(0)}|_{A=\omega=0} = \left. w_0 \frac{8c^4_k \omega^2 k}{\omega^2 - c^4_k k^2}\right|_{A=\omega=0} \tag{52}
\]

\[
G_{j_{\omega^0}^z}(x) = -\frac{\delta J^0(x)}{\delta \omega^{0z}(0)}|_{A=\omega=0} = \left. -\frac{6i\omega k^2 \omega^3}{\omega^2 + i0^+}\right|_{A=\omega=0} = -6w_0 i\omega \chi^2 \tag{53}
\]

\[
G_{j_{\omega^0}^\alpha}(x) = -\frac{\delta J^0(x)}{\delta \omega^{0\alpha}(0)}|_{A=\omega=0} = \left. 0\right|_{A=\omega=0} \tag{54}
\]

\[
G_{j_{\omega^0}^z}(x) = -\frac{\delta J^0(x)}{\delta \omega^{0z}(0)}|_{A=\omega=0} = \left. w_0 \frac{2i\chi^4 + 2i\omega^3}{\omega^2}\right|_{A=\omega=0} \tag{55}
\]

\[
G_{j_{\omega^0}^{\alpha\beta}}(x) = -\frac{\delta J^0(x)}{\delta \omega^{0\alpha\beta}(0)}|_{A=\omega=0} = \left. 0\right|_{A=\omega=0} \tag{56}
\]

\[
G_{j_{\omega^0}^z}(x) = -\frac{\delta J^0(x)}{\delta \omega^{0z}(0)}|_{A=\omega=0} = 2iw_0 \chi^2 \omega, \tag{57}
\]

\[
G_{j_{\omega^0}^\alpha}(x) = 0, \tag{58}
\]
This last equation is actually related to the no-anomaly condition, which might be broken in theories such as \cite{25} combining local equilibrium with chiral magnetic and vortical effects.

Similarly, the Ward identity for vorticity

\[
\partial_0 G_{T^{00, \omega \omega}}^R (x) - \partial_i G_{T^{00, \omega \omega}}^R (x) = 0
\]

yields the stress-vorticity correlators

\[
G_{T^{00, \omega \omega}}^R (x) = - \left. \frac{\delta T^{\mu \nu} (x)}{\delta \omega} \right|_{A=\omega=0} = \frac{2i \chi^2 \omega k^2 + ik^2 \omega^2 - c_s^2 \chi^2 k^3}{\omega^2 - c_s^2 k^2}
\]

\[
G_{T^{00, \omega \omega}}^R (x) = - \left. \frac{\delta T^{\mu \nu} (x)}{\delta \omega} \right|_{A=\omega=0} = \frac{2i \chi^2 \omega k^2 + i \chi^2 \omega^3 - i c_s^2 \omega k^2}{\omega^2 - c_s^2 k^2}
\]

C. Relaxation times

Equipped with our intuition on the Maxwell-Cattaneo in equation 29 we redefine in Fourier space polarization to an asymptotic state to which the fluid relaxes

\[
Y^{\mu \nu} = \frac{\mu^{\mu \nu}}{1 + i \omega \tau_Y} \Rightarrow \chi \rightarrow \frac{\chi}{1 + i \omega \tau_Y}
\]

To appreciate the power of this substitution, we should recall the hydrodynamics poles from polarizable fluid suffer from unphysical behavior as well as unstable one \cite{3} and relaxation (unlike Maxwell-Cattaneo, it is first order) is needed to stabilize the behavior. Let us focus in the transverse green function and applying the above transformation

\[
G_{T^{\tau s, T^{\tau s}}}^R (x) = 4iB \omega^3 \rightarrow \frac{i \omega \chi^2 \omega^3}{(1 + i \omega \tau_Y)^2}
\]

This correction turns the group velocity modes bounded and provides a stable solution. For instance, after enforcing causality the poles of eq. 39 corresponds to the same evaluated in \cite{3}. The transport coefficient from the imaginary part of retarded Green Function of eq. 39, we have

\[
\lim_{\omega \rightarrow 0} \frac{1}{\omega^3} ImG_{T^{\tau s, \omega \omega}}^R (\omega, \vec{k}) = w_0 \chi^2, \quad \lim_{\omega \rightarrow 0} \frac{\omega}{k^2} ImG_{T^{00, \omega \omega}}^R (\omega, \vec{k}) = w_0
\]

\[
\lim_{\omega \rightarrow 0} \frac{\partial \omega}{3} \lim_{k \rightarrow 0} - \frac{\partial^2}{2} G_{j^\tau 0}^R (x) = w_0 \chi^2, \quad \lim_{\omega \rightarrow 0} \frac{\omega}{\tau_Y} \lim_{k \rightarrow 0} \frac{\omega}{2} ImG_{j^{\tau \omega \omega}}^R (x) = \frac{w_0 \chi^2}{\tau_Y}
\]

as discussed in \cite{24}, \lim_{\omega \rightarrow 0} \lim_{k \rightarrow 0} \neq \lim_{k \rightarrow 0} \lim_{\omega \rightarrow 0}. The integral representation of retarded polarization-polarization correlator introduces variational of energy-momentum from metric or vorticity field perturbation \( S^{\gamma \lambda} \).

\[
\delta T_{\mu \nu} (t, x) = \int d\omega d^3 \vec{k} e^{-i \omega t + i \vec{k} \cdot \vec{x}} G_{\mu \nu, \gamma \lambda} (\omega, \vec{k}) S^{\gamma \lambda}
\]

The structure of Green function allows us to extract singular modes whose dispersion relation determine the behavior of system in the hydrodynamic limit \( \omega, k \ll 1 \) for \( G_{00,00} \) and \( G_{01,01} \)

The longitudinal perturbation is

\[
\delta T^{00} \propto \int \frac{d^3 \vec{k}}{(2\pi)^3} e^{i \vec{k} \cdot \vec{y} + i c_s k t + ic_s k^3 \frac{\omega}{\omega^3}} = \int \frac{d^3 \vec{k}}{(2\pi)^3} e^{i \vec{k} \cdot \vec{y} + i c_s k t + ic_s k^3 \omega \omega^3}
\]
and the transverse one

$$\delta I_{0x} \propto \int \frac{d^3k}{(2\pi)^3} e^{-ikx} e^{-|t|/\chi - k^2/2x} \sim \frac{T^2}{(4\pi)^{3/2}} \left[ \frac{\chi^{5/2}}{((\frac{2}{\chi} + 1)|t|)^{3/2}} - \frac{\chi^{3/2}}{2(\frac{2}{\chi} + 1)^{3/2}(|t|)^{1/2}} + \ldots \right]$$

where $\frac{2}{\chi} > 1$ because of causality relation derived in [3]. Since we are most likely dealing with a non-renormalizable theory, where the cutoff is physical, we factorize the exponential $e^{-|t|/\chi}$ up to first order $1/\chi$. This exponential play an important role of mass in the dispersion relation. Higher order terms should be dropped in the infrared limit because they are irrelevant. The next leading term of the integral with the expansion of $cos(k^2\gamma t)$ above will be

$$\sim \frac{T^2 \chi^{2/3}}{((\frac{2}{\chi} + 1)|t|)^{3/2}}.$$ The correlator were symmetrized.

Note that the dependence of the correlator on time, with the long time tail is the same as was calculated in [19], using similar methods but applied to thermodynamic fluctuations. This is not so surprising: As noted earlier [1] fluctuations and polarization depend on the same dimension of operators, and hence it is natural to expect their correlation to scale similarly.

### IV. INTERACTIONS OF HYDRODYNAMIC MODES

#### A. One loop effective action

Now we try to compute the properties of hydrodynamics with spin using perturbative techniques. One can do this by means of an effective field theory one, where the symmetry and light degree of freedom are the essential insight to analyze the behavior of flow.

In this framework, we hope to shed some light on scenarios in which flow conditions are similar to physical experiments. As a starting point, we examine the structure of the full action setting by slow sound perturbations (\(y_\mu\)) and fast microscopic (\(y^{\mu\nu}\)) degrees of freedom. Here, we shall drop a full analysis and consider a weak interaction between them. So the local action reads

$$S[\phi^I, y^{\mu\nu}] \simeq S_0[\phi^I] + S_y[y^{\mu\nu}] + S_{\text{int}}[\phi^I, y^{\mu\nu}],$$

The \(S_0[\phi^I]\) encodes, alone, the general idea of "standard" hydrodynamic, while \(S_y[y^{\mu\nu}]\) regulates the dynamics of polarization variable. It should become dominant at the microscopic scale (distance between microscopic degree of freedom), which is also natural cutoff of hydrodynamic. Now, we will concentrate our efforts to work out on \(S_{\text{int}}\) sector. Conventionally, the couplings between light and heavy should be treated as small in order to not break down the perturbative expansion. Let us now integrate out the fast \(y\) sector

$$e^{S_{\text{eff}}[\phi^I]} \equiv \int \mathcal{D}y^{\mu\nu} e^{S[y^I, y^{\mu\nu}]}$$

The method, even though systematic, has the property to be somewhat involved. Using instantaneus equilibrium [1, 26] within the EFT formalism we can rewrite the effective Lagrangian as

$$Z = \int \mathcal{D}\pi_L \mathcal{D}\pi_T e^{i \int d^3 \phi dt L_{\text{total}} = L_0 + L_{\text{int}} + L_{\text{self-int}} }$$

The \(L_0\) are natural fluctuation from sound waves, entropy, and vortex modes, \(L_{\text{int}}\) is the interplay of flow and polarization, and \(L_{\text{self-int}}\) reproduces the higher fluctuation from polarization-polarization interaction. It is convenient to work at low temperature where its gradient is low enough to evaluate Feynman diagrams and to . We expand the action up to fourth order, using Eq. 16 and following [2], for linear terms of up to 2 gradients we have

$$L_0 \simeq 1 + A \left\{ [\partial \pi] - \frac{1}{2} [\partial \pi^T, \partial \pi] - \frac{1}{2} \dot{\pi}^2 \right\} + \left\{ \frac{1}{2} A + C \right\} [\partial \pi]^2.$$
where the constants $A, C$ are

$$c_s^2 = \frac{2F''(b)}{F'(b)} + 1, \quad A = T_0 F'(b_0), \quad C = \frac{1}{2} \frac{\beta^2}{b_0} F''(b_0) \quad \text{(73)}$$

where the enthalpy is $w_0 = -F_b(b_0), \ b_0 = 1$ and has the dimension $[M^4]$. Note that here $\partial\pi^T$ means the transpose of the matrix of $\pi$s rather than the transverse direction. The first term of Lagrangian 72 indicates the minimized potential by $w_0$, the second one vanishes by integration. The third and fourth are the well known free propagation of sound-waves and the fifth transverse (strong coupling) excitations of $\phi'$. Of course there are vortex-sound interactions, however, they must be neglected if we compared with other terms of Lagrangian $\sim \chi^2$. Inverting Eq. 72 will give the propagator for the sound-waves in momentum space $G_{\partial\pi,\partial\pi}$ [26] ($\partial\pi$ represents the general matrix $\partial_i\pi_j$, including both $\pi_T$ and $\pi_L$, $\pi$] is the trace, defining the sound-wave

$$G_{\partial\pi,\partial\pi} = \frac{i k^2}{w^2 - c_s^2 k^2}, \quad G_{\partial\pi,\partial\pi} = \frac{i w k_j}{w^2 - c_s^2 k^2}, \quad G_{\partial\pi,\pi_m} = i \delta_{lm} + \frac{i c_s^2 k_j k_m}{w^2 - c_s^2 k^2} \quad \text{(74)}$$

It can be seen by inspection that the infrared limit $w \to 0$ of transverse mode propagators diverges, confirming the potential instability of [13].

The interaction term between cell fluid and polarization $\sim F(-by^2)$.

$$\mathcal{L}_{\text{int}} \simeq w_0 \chi^2(b_0, 0) \left\{ \left[ (\partial\pi)[(\partial\pi \cdot \partial\pi) + (\partial\pi)(\partial\pi \cdot \partial\pi)] \right] (1 + c_s^2 + \frac{2\chi b}{\chi_{\text{self-int}}}) - \left[ \frac{1}{2} \frac{\pi'}{\pi} + \frac{1}{2} \frac{\pi'}{\pi} \partial\pi \partial\pi \right] \frac{f''}{f'} \left[ (\partial\pi \cdot \partial\pi) + (\partial\pi \cdot \partial\pi) \right] (1 + \frac{2\chi b}{\chi_{\text{self-int}}}) \right\}$$

$$\left[ (\partial\pi) \right] (\partial\pi \cdot \partial\pi + [\partial\pi \cdot \partial\pi]) \frac{f''}{f'} \left[ \frac{2\chi b}{\chi_{\text{self-int}}} + \frac{2\chi b}{\chi_{\text{self-int}}} + 1 \right] \left[ \frac{2\chi b}{\chi_{\text{self-int}}} + \frac{2\chi b}{\chi_{\text{self-int}}} \right]$$

$$\left\{ \left[ (\partial\pi \cdot \partial\pi + [\partial\pi \cdot \partial\pi]) (\partial\pi \cdot \partial\pi) - \pi \cdot \partial \pi \cdot \partial\pi \right] \right\} (1 + c_s^2 + \frac{2\chi b}{\chi_{\text{self-int}}}) \quad \text{(75)}$$

In the following paragraphs we shall link each line to Feynman diagrams. The “gluon” (spring-like) lines will be used for transverse modes, and the “photon” (wavy) lines for longitudinal. In the limit where polarization vanishes the former correspond to vortices and the latter to sound waves. Note that the Feynman diagrammatic structure does not change when the relaxation time $\tau$ is present. One must just update the vertex and propagator terms to those of Eq. 62, $\chi \to \chi (1 + w\tau)^{-1}$. This is reasonable since Maxwell-Cattaneo is a constitutive equation, not entering in Lagrangian but, through the dissipation-fluctuation theorem, the correlation function.

The perturbation breaks down when the energy $E^2 \sim \chi^{-2}$ where the dimension of $[\chi] = M^{-1}$ and $\pi = M^{-1}$. We will analyze the leading terms in accordance with $E \ll \chi^{-1}$. The strength interaction for a typical tree level process is $\mathcal{M}_{TT \to TT} \sim F^2_{\text{max}}$. The Feynman rules: each vertex a $w_0\chi^2$, $1/w_0$ internal line, and $1/\sqrt{w_0}$ external line. We extract fourth order terms whose tree level diagram has one legs as longitudinal perturbation propagation $\sim (\partial\pi)\mathcal{O}(\pi^4)$ by symmetry arguments, and terms without spatial derivative must vanish since they do not contribute on vortex amplitude but for longitudinal one. The representation of the first line of Eq. 75 corresponds to the Feynman Diagram in Fig 1

the vertical and horizontal lines are space and time arrow. The process is similar to Rayleigh scattering in Electrodynamics where a non-linear physical object (vortex) absorbs and emits one or more sound waves by a non-linear harmonic process. If we pay more attention, we may realize that only unsteady vortex rather than stationary one perpendicular to scattering plane provide variation of longitudinal velocity, so that vortex can participate of this process where the emitted sound wave has the same frequency of the incident one. The second and third line of Eq. 75
Feynman diagram language are in Fig. 2 (note that there is no backscattering process as at temperature fluctuation.)

[27] analyzed the production of vortices by soundwaves and associated sound wave scattering, illustrated in Fig. 2. The first diagram indicates the cross section between sound waves and vorticity distribution of a turbulent flow. The vertical and horizontal lines determine the space and time arrow. The interaction of vortex and sound describes too convection of weak vorticity fluctuation. In [28], it was shown that the absence of the energy gap and propagation velocity of [13] lead to a badly divergent loop. As is shown here, as hypothesized in [1, 3] the divergence is mitigated by polarization.

Note that there are some variations at the orientations of these diagrams as t,u - channel due to the lack of Lorentz covariance in the IJ indices, see [13] for a discussion on this.

The sound-vorticity coupling arises when non-linear perturbative hydrodynamical variables are taken into accounting at equations of motion of fluid, and produces vortex and sound waves. Furthermore, it is clear that the interaction is different for both of them, in other words, the vorticity production will be more favorable than sound wave since the external field is rotational (Fig. 3). The last but no least, the coupling affects convection, expansion and stretching of vorticity configuration. The second-to-last and last lines of Eq.75 describe the interaction of third order in \( \pi \) and gives the leading radiative order diagrams.

Vorticity-Vorticity interaction has been analyzed by [29] and also its generation of vorticity and sound.. Here, we describe in detail the effect of the integrated-out spin-spin interaction, the respective orders of which are \( \sim \sqrt{u_0^{-1}} \chi^2 \) and \( \sim u_0^{-1} \chi^4 \).

In order to keep track the real phenomena without lost the generality, we take upon the Lagrangian 35 only the
spin interactions with the background $b_0$, in other words, $\sim F(-b_0y^2(\pi))$, thus the spin interacts with the vacuum of fluid (hydrostatic configuration) when the physical coordinate are aligned with the comoving one at given pressure. After turning on the external field. The vacuum is invariant under $SO(2)$ and its original symmetry was $SO(3)$. The $\pi^x$ and $\pi^y$ eat two Goldstone boson and become massive, whereas the $\pi^z$ remains massless. The Lorentz boost is broken. The self-interacting lagrangian up to fourth order is Fig. 4.

While this theory is non-renormalizeable, and it is spacetime diffeomorphism that are broken, the above process (some directions of the Goldsone boson are “eaten” via interaction with the consensate) has a similarity to the Higgs mechanism [14].

The third line onwards of Eq. 75 represents the fourth order interaction. The tree-level and one loop amplitude are

It is useful to consider diagrams for the following correlator at $O(w^{-2}\chi^4(b_0,0))$ and $O(w^{-2}\chi^2(b_0,0))$ to self-interaction and vortex and “fluid” interaction, respectively.

$$\langle y^{ij}(x_1,t_1), y^{pq}(x_2,t_2) \rangle = \langle \partial^i \dot{\pi}^j, \partial^p \dot{\pi}^q \rangle - \langle \partial^i \pi^j, \partial^p \pi^q, \partial \pi^i \rangle + \ldots$$

(76)

$$\langle bu^0(x_1,t_1), y^{ij}(x_2,t_2) \rangle = \langle y^{ij}(x_2,t_2) \rangle + \langle [\partial \pi^j], \partial^i \pi^l \rangle + \frac{1}{2} \langle [\partial \pi^2], \pi^l \partial^j \partial^i \pi^l \rangle + \ldots$$

(77)

This is a diagrammatic way to understand how the divergence in Eq. 74 is regularized by polarization. The terms of $O(w^{-2}\chi^2(b_0,0))$, Fig 5 converts sound waves into vortices, and is peculiar to mediums with a spin. The extra gradient terms, when included in the propagator definition Eq. 74, will lead to higher powers of $w$ in the numerator which will cancel the divergence. Equation 83 does this explicitly.

We close by noting that the pressure generation by small vorticity fluctuation is irrelevant since the former depends upon the later by a higher order at $v(t, \vec{x})$, by EFT language $\sim c^2_s F_{b0}[\partial \pi]^2$ field fluctuation.

In the next subsection, we will use the diagrammating insights obtained here together with the results of section III to compute the interaction part of the propagator explicitly.

**B. Propagator expansion due to hydrodynamic interactions**

In this subsection, we use our intuition from the previous section and the calculations in section III to extend the dissipative phenomena investigated to higher order interactions. In doing so, we examine the effect on transport
coefficient of hydrodynamic fluctuation-generated collective excitations. We start from the linearization of Eq. 36

\[ T_{\mu\nu} \sim 2F'(b_0)\chi^2(b_0,0)\left(\frac{\partial}{\partial \phi} + \frac{1}{2}H_{IJ}\frac{\partial}{\partial \phi}\right)T_{\mu\nu} + 2F'(b_0)\chi(b_0,0)y_{\rho\nu}\delta^0_\mu + \frac{1}{3}F'(b_0)y_{\rho\nu}L_{\mu\nu} + \ldots \]

\[ J^\mu_I \sim F'(b_0)\chi(b_0,0)\partial_\rho w^\rho_{IJ} - 2F'(b_0)(y^2 - b_0 \chi \partial_\rho \chi^2)\delta^\mu_I \ldots \]  

where we used the projector

\[ H^{IJ}_{KL} = (\delta^{IJ}\delta_{KL} + \delta^K_L\delta^I_J) \]

the second order contribution to the propagators, labeled by \(G^{(2)}\) henceforward, are

\[ G^{(2)}_{J_L,J_R}(t,\vec{x}) = \frac{1}{w^2_0}\langle \pi_m(t,\vec{x})\pi_n(t,\vec{x})\rangle_{eq} = \frac{1}{w^2_0}\langle \pi_m(t,\vec{x})\pi_n(0)\pi_p(0)\rangle_{eq}, \]

\[ G^{(2)}_{T_L,T_R}(t,\vec{x}) = \frac{1}{w^2_0}H^{ij}_{mn}H^{kl}_{qp}\langle \pi_m(t,\vec{x})\pi_n(t,\vec{x})\pi_q(0)\pi_p(0)\rangle_{eq} = H^{ij}_{mn}H^{kl}_{qp}\frac{1}{w^2_0}\langle \pi_m(t,\vec{x})\pi_q(0)\pi_p(0)\rangle_{eq}\]

The displacement before the vortical field has been applied is equal for reason of symmetry. In other words, the 

\(\langle \pi_L(x)\pi_T(0)\rangle = \langle \pi_T(x)\pi_L(0)\rangle\) in the Lagrangian description due to \(SO(3)\) for a perfect fluid.

Even though these two correlation functions may appear similar, it is suggestive to emphasize that both of them come

from different variation. Besides that, since the fluctuations are Gaussian, we can order the operator in correlation

function in such a way

\[ G^{(2)}_{T_L,T_R}(\omega,\vec{k}) = \frac{2}{w^2_0}H^{ij}_{mn}H^{kl}_{qp}\int \frac{d\omega'}{2\pi} \frac{d^3k'}{2\pi^3} G^{R(1)}_{T^0_{mn},T^0_{pq}}(\omega',\vec{k}') G^{R(1)}_{T^0_{pq},T^0_{mn}}(\omega',\vec{k}') = \ldots + \mathcal{O}(M^4) + \ldots \]

where \(M\) is an ultraviolet cutoff and we use the approximate process of symmetrization of imaginary green function
described in [20].

\[ G^{R}_{T^0_{mn},T^0_{pq}}(\omega,\vec{k}) = \frac{(w_0T)}{2}\left[\left(\delta^{ij} - \frac{k^i k^j}{k^2}\right)\frac{16\chi^4(\omega^4 k^2 - \omega^2 k^4) - 2\chi^2(\omega^4 - k^2\omega^2)}{4\chi^2(\omega^4 - k^4\omega^2) - \omega^2} + \left(\frac{k^i k^j}{k^2}\right)\frac{\omega^2 - \chi^2(3\omega^4 + 2\omega^2 k^2 k^2) + \chi^4(3\omega^6 + 2\omega^4 k^4)}{\chi^2(\omega^4 + k^2\omega^2) - \omega^2 + \epsilon^2 k^2}\right] \]

When interactions are turned on the \(SO(3)\) symmetry breaks down and the \(\pi\) becomes massive. The transverse part
becomes strongly dependent of \(\chi\). If \(\chi \rightarrow 0\), then it yields \(G_{\partial x,\partial x} \rightarrow 0\). The transverse part may be then simplified

\[ G^{R}_{T^0_{mn},T^0_{pq}} = \frac{w_0}{2} \left(1 - \frac{(k^i)^2}{k^2}\right) \frac{\left((\omega^2 + k^2)^2\chi^2 k^2 + (\omega^2 - k^2)^2\chi^2\right)}{\left((\omega^2 + k^2)^2 - 1\right)\chi^2} \]

It is easy to see the denominator part of retarded green function above corresponds to \(p^2 - m^2\), where the mass

\(m \sim (2\chi)^{-1}\). We must remember that, because of the symmetries of fluid dynamics our green function will never look
like as green function of ”free particle” but rather as interacting green function, where the numerator is different from \(i\).
As we have justified previously the propagator is, in the infrared limit, that of a massive vector particle, thereby
realizing the effective vortex mass conjectured in [12].

\[ G^{R}_{T^0_{mn},T^0_{pq}} = \frac{w_0}{2} \left(1 - \frac{(k^i)^2}{k^2}\right) \frac{-1(\omega^2 - k^2)}{(\omega^2 - k^2)^2 - \frac{1}{\chi^2}} \]

Note the similarity of this propagator to one of a massive vector particle, confirming the analogy with the Higgs
mechanism argued for in the previous sub-section.
Eq. 83 can be used in Eq. 80 to obtain the propagators of all the other components of the energy-momentum tensor. Using dimensional regularization we obtain, in the $k = 0$ frame

$$G^{R(2)}_{T_{ij} T_{kl}}(\omega, \mathbf{k} = 0) = \frac{T^2 \chi^4 H_{ij}^{kl}}{(4\pi)^2 \mu^2} \left[ \frac{\lambda}{\epsilon}(2M^2 + M^2 p^2 + M^4) + \frac{2}{3}(p^4 - M^2 p^2 - 3M^4) + \int_0^1 dx(3a^4 + 6a^2 x^2 p^2 + x^4 p^4)ln(a^2/\mu^2) \right] \tag{84}$$

where $a = p^2(x - 1) + M^2$, $1/\epsilon = 1/\epsilon + \gamma_\epsilon + \mathcal{O}(\epsilon)$, $\mu^2 = 1 + 2\epsilon n\mu$. After algebra manipulations, the cutoff-independent part of the propagator reduces to

$$G^{R(2)}_{T_{ij} T_{kl}}(w) = \frac{T^2 H_{ij}^{kl}}{(4\pi)^2} (-1 - \frac{2}{3}\chi^2 + (6 + 2\omega^2\chi^2 + \frac{\chi^4}{4})ln(\frac{1}{\chi^2\mu^2})) \tag{85}$$

which remains finite for any non-zero $\chi^2$

V. DISCUSSION AND CONCLUSION

We used some common effective field theory techniques to study the correlation function of hydrodynamic variables. In the section II, the green functions arising from the linear response theorem established an easy way to look for dynamical of dissipative system. Thus, we could determine from local transport process and total angular momentum of Noether’s current the transport coefficient of polarization $\chi$ and $\tau_y$.

In order to generalize to correlation functions of non-conserved quantities, we use variational approach where the background perturbation of the vortical field and the metric produce all possible fluctuations in according with the symmetry of the system. The method regarded operators in different points in space and time to provide the form of polarization-polarization correlator in an equilibrium ensemble.

These green functions contain information on how the presence of external vortex field modify the structure of our hydrodynamic with spin, which as we find can give an effective mass to the vortices. In dealing with this cases is interesting to find out what are the connections between fluctuation and symmetry. First, let us mention the fluid equations of motion are changed somewhat in the presence of external field and so the symmetry under reversal time, homogeneous and isotropic are no longer valid. Initially, we have $h_{xx} = h_{yy} = h_{zz}$ due to $SO(3)$ group and after turning on the source, the pressure changes from the usual form of ideal fluid. Therefore, the direction along the axial rotation axes is $P_{\perp} = \frac{1}{2}T^a = \frac{1}{2}(T^x + T^y)$, whereas the perpendicular direction corresponds to $P_{\parallel} = \frac{1}{2}T^z$. Each perturbation relates different process of such a system, the transverse direction contributes to aligning of spins (polarization), while the longitudinal one contributes on convective and viscous process.

This sums up the linearized analysis of hydrodynamics with intrinsic polarization, provided the only dissipation is given by the magnon relaxation necessary to preserve causality (Section II C). An obvious extension is to include the microscopic shear and bulk viscosities, and study their interplay with the transport coefficients examined in this section. This could be done within a Schwinger-Keldysh formalism [30, 31], and will have to be left to a forthcoming work.

Another issue left for further work is the discussion of Gauge symmetry, which is in practice the interchange between angular momentum (carried by vorticity) and polarization (carried by $y_{\mu\nu}$ and $Y_{\mu\nu}$). Mathematically, this would be achieved by making sure all correlation functions are gauge-covariant and transform in the same way. Earlier literature [32] showed that this cannot be achieved so easily, and indeed the left hand side of Eq.s such as 9 would have different transformation properties from the right hand side. As [32] suggests, non-hydrodynamic modes might be necessary to resolve this ambiguity.

In conclusion, in this work we investigated the linear response and fluctuation-dissipation properties of the theory developed in [1–3]. We found that these early results are consistent with linear response theory, derived fluctuation-dissipation relations and built an effective lagrangian to one-loop, confirming that microscopic polarization can act as a “Higgs mechanism” for vorticity, giving a mass to vortices and stabilizing the theory. This confirms the intuition of [12], and could lead to a stable fully quantum theory without the additional conjectures enunciated in [26]. Phenomenological applications of this theory to heavy ion collisions and cosmology can now be developed.

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