WEIGHTED PLURICOMPLEX ENERGY

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Abstract. We study the complex Monge-Ampère operator on the classes of finite pluricomplex energy $E_\chi(\Omega)$ in the general case ($\chi(0) = 0$ i.e. the total Monge-Ampère mass may be infinite). We establish an interpretation of these classes in terms of the speed of decrease of the capacity of sublevel sets and give a complete description of the range of the operator $(dd^c)^n$ on the classes $E_\chi(\Omega)$.

1. Introduction

Let $\Omega \subset \mathbb{C}^n$ be a bounded hyperconvex domain, i.e. a connected, bounded open such that there exists a negative plurisubharmonic $\rho$ such that $\{z \in \Omega; \rho(z) < -c\} \subseteq \Omega$, $\forall c > 0$. Such a function $\rho$ is called an exhaustion function. We let $PSH(\Omega)$ denote the cone of plurisubharmonic functions (psh for short) on $\Omega$ and $PSH^-(\Omega)$ denote the subclass of negative functions.

In two seminal papers [12], [13], U.Cegrell has introduced and studied the complex Monge-Ampère operator $(dd^c)^n$ on special classes of unbounded plurisubharmonic functions in $\Omega$, called Energy classes. In [6], a formalism developed in [19], was used to give a unified treatment of all these classes in the case of finite total Monge-Ampère mass. Here, we continue our study in a more general context. Given an increasing function $\chi : \mathbb{R}^- \to \mathbb{R}^-$, we consider the set $E_\chi(\Omega)$ of plurisubharmonic functions of finite $\chi$-weighted Monge-Ampère energy. These are the functions $u \in PSH(\Omega)$ such that there exists a decreasing sequence $u_j \in E_0(\Omega)$ with limit $u$ and

$$\sup_{j \in \mathbb{N}} \int_{\Omega} (-\chi) \circ u_j (dd^c u_j)^n < +\infty,$$

where $E_0(\Omega)$ is the cone of all bounded plurisubharmonic functions $\varphi$ defined on the domain $\Omega$ with finite total Monge-Ampère mass and $\lim_{z \to \zeta} \varphi(z) = 0$, for every $\zeta \in \partial \Omega$. When $\chi(t) = -(-t)^p$ (resp. $\chi(t) = -1 - (-t)^p$), $E_\chi(\Omega)$ is the class $E^p(\Omega)$ (resp. $\mathcal{F}^p(\Omega)$ ) studied by U.Cegrell in [12].

The classes $E_\chi(\Omega)$ have very different properties, depending on whether $\chi(0) = 0$ or $\chi(0) \neq 0$, $\chi(-\infty) = -\infty$ or $\chi(-\infty) \neq -\infty$, $\chi$ is convex or concave. If the function $\chi$ is convex, or concave, then the class $E_\chi(\Omega)$ is subset of a natural family of psh functions introduced by U. Cegrell in [14] (cf section 4). In particular, we have

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**Proposition A.** Let $\chi : \mathbb{R}^- \rightarrow \mathbb{R}^-$ be a convex, or concave, increasing function such that $\chi(-\infty) = -\infty$ and $\chi(0) = 0$. Then
\[ \mathcal{E}_\chi(\Omega) \subset \mathcal{N}^\omega(\Omega). \]
In particular the Monge-Ampère measure $(dd^c u)^n$ of a function $u \in \mathcal{E}_\chi(\Omega)$ is well defined and does not charge pluripolar sets. More precisely,
\[ \mathcal{E}_\chi(\Omega) = \left\{ u \in \mathcal{N}(\Omega) / \chi \circ u \in L^1((dd^c u)^n) \right\}. \]
Many properties follow from an interpretation of these classes in terms of speed of decrease of the capacity of sublevel sets:

**Proposition B.** If $\chi$ is an increasing convex function, then we have
\[ \mathcal{E}_\chi(\Omega) = \left\{ \varphi \in PSH^-(\Omega) / \int_0^{+\infty} t^n \chi'(-t) \text{Cap}_\Omega(\{ \varphi < -t \}) dt < +\infty \right\}. \]
Here Cap$_\Omega(\cdot)$ denotes the Monge-Ampère capacity introduced by E. Bedford and B.A. Taylor [3]. This yields in particular several properties: the classes $\mathcal{E}_\chi(\Omega)$ are convex, stable under taking the maximum.

In section 5, we study the range of the complex Monge-Ampère operator on the classes $\mathcal{E}_\chi(\Omega)$ in the case when the function $\chi$ is convex. Given a positive Borel measure $\mu$ on $\Omega$, we have:

**Theorem C.** Let $\chi : \mathbb{R}^- \rightarrow \mathbb{R}^-$ be an increasing convex function such that $\chi(-\infty) = -\infty$. Then there exists a unique function $\varphi \in \mathcal{E}_\chi(\Omega)$ such that $\mu = (dd^c \varphi)^n$ if and only if there exists a constant $C > 0$ such that
\[ \int_{\Omega} -\chi \circ u d\mu \leq C \max \left( 1, \left( \int_{\Omega} -\chi \circ u (dd^c u)^n \right)^{\frac{1}{n}} \right), \quad \forall u \in \mathcal{E}_0(\Omega). \]

The proof of this theorem remains valid when $\chi(t) = -(-t)^p$ for $p > 0$, which yields a simple proof of the main theorem in [12].

In section 6, using results from [16] and [23], we prove that, for almost all weights $\chi$, the functions of the classes $\mathcal{E}_\chi(\Omega)$ admit global subextension with logarithmic growth and local subextension with finite $\chi$-energy.

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2. **The class $\mathcal{F}(\Omega)$**

In this section we give some properties of the U.Cegrell class $\mathcal{F}(\Omega)$. The main tool will be the capacity estimate of the sublevel sets of psh functions. The Monge-Ampère capacity has been introduced and studied by E.Bedford and A.Taylor in [3]. Given $K \subset \Omega$ a Borel subset, its Monge-Ampère capacity relatively to $\Omega$ is defined by
\[ \text{Cap}_\Omega(K) := \sup \left\{ \int_K (dd^c u)^n ; u \in PSH(\Omega), -1 \leq u \leq 0 \right\}. \]
Let recall some U.Cegrell’s classes. The class \( E(\Omega) \) is the set of plurisubharmonic functions \( u \) such that for all \( z_0 \in \Omega \), there exists a neighborhood \( V_{z_0} \) of \( z_0 \) and \( u_j \in E_0(\Omega) \) a decreasing sequence which converges towards \( u \) in \( V_{z_0} \) and satisfies \( \sup_j \int_{\Omega}(dd^cu_j)^n < +\infty \). U.Cegrell has shown [13] that the operator \((dd^cu)^n\) is well defined on \( E(\Omega) \) and continuous under decreasing limits. The class \( E(\Omega) \) is stable under taking maximum and it is the largest class with these properties (Theorem 4.5 in [13]). The class \( E(\Omega) \) has been further characterized by Z.Blocki [8], [9].

The class \( F(\Omega) \) is the “global version” of \( E(\Omega) \): a function \( u \) belongs to \( F(\Omega) \) iff there exists a decreasing sequence \( u_j \in E_0(\Omega) \) converging towards \( u \) in all of \( \Omega \), which satisfies \( \sup_j \int_{\Omega}(dd^cu_j)^n < +\infty \).

The class \( F^a(\Omega) \) is the set of functions \( u \in F(\Omega) \) whose Monge-Ampère measure \((dd^cu)^n\) is absolutely continuous with respect to capacity i.e. it does not charge pluripolar sets. Similarly, \( E^a(\Omega) \) is the set of functions \( u \in E(\Omega) \) whose Monge-Ampère measure \((dd^cu)^n\) vanishes on pluripolar sets.

**Lemma 2.1.** Fix \( \varphi \in \mathcal{F}(\Omega) \). Then for all \( s > 0 \) and \( t > 0 \),

\[
(2.1) \quad t^n \text{Cap}_\Omega(\varphi < -s - t) \leq \int_{\{\varphi < -s\}} (dd^c\varphi)^n \leq s^n \text{Cap}_\Omega(\varphi < -s).
\]

Therefore \( u \in F(\Omega) \) if and only if \( \limsup_{s \to 0} s^n \text{Cap}_\Omega(u < s) = +\infty \). In particular, if \( u \in F(\Omega) \) then

\[
\int_{\Omega} (dd^cu)^n = \lim_{s \to 0} s^n \text{Cap}_\Omega(u < s)
\]

and

\[
\int_{\{u = -\infty\}} (dd^cu)^n = \lim_{s \to +\infty} s^n \text{Cap}_\Omega(u < s).
\]

Note that the complex Monge-Ampère measure of a psh function \( u \) on \( \Omega \) does not charge pluripolar sets if and only if it puts no mass on \( (u = -\infty) \) (cf. [7]). Thus \( u \in F^a(\Omega) \) if and only if \( \lim_{s \to +\infty} s^n \text{Cap}_\Omega(u < s) = 0 \).

The right hand inequality in (2.1) has proved by S.Kolodziej [22] when \( \varphi \) is bounded (see also [5] and [18] for the compact setting). For the convenience of the reader we give here a simple proof which uses the same idea.

**Proof.** Fix \( s, t > 0 \). Let \( K \subset \{ \varphi < -s - t \} \) be a compact subset. Then

\[
\text{Cap}_\Omega(K) = \int_{\Omega} (dd^cu^*_K)^n = \int_{\{\varphi < -s - t\}} (dd^cu^*_K)^n
\]

\[
= \int_{\{\varphi < -s + tu^*_K\}} (dd^cu^*_K)^n = \frac{1}{t^n} \int_{\{\varphi < v\}} (dd^c v)^n,
\]

where \( u^*_K \) is the relative extremal function of the compact \( K \) and \( v := -s + tu^*_K \). It follows from [7] that

\[
\frac{1}{t^n} \int_{\{\varphi < v\}} (dd^c v)^n \leq \frac{1}{t^n} \int_{\{\varphi < \max(\varphi, v)\}} (dd^c \max(\varphi, v))^n \leq \frac{1}{t^n} \int_{\{\varphi < max(\varphi, v)\}} (dd^c \varphi)^n \leq \frac{1}{t^n} \int_{\{\varphi < max(\varphi, v)\}} (dd^c \varphi)^n.
\]
Taking the supremum over all K’s yields the first inequality. For the right hand inequality, we have
\[
\int_{\{\varphi \leq -s\}} (dd^c \varphi)^n = \int_{\Omega} (dd^c \varphi)^n - \int_{\varphi > -s} (dd^c \varphi)^n
\]
\[
= \int_{\Omega} (dd^c \max(\varphi, -s))^n - \int_{\varphi > -s} (dd^c \max(\varphi, -s))^n
\]
\[
= \int_{\varphi \leq -s} (dd^c \max(\varphi, -s))^n \leq s^n \text{Cap}_\Omega \{\varphi \leq -s\}.
\]

It is known (see [12, 13]) that the class $\mathcal{F}(\Omega)$ has many properties. Namely it is a convex cone, stable under maximum: if $u \in \mathcal{F}(\Omega)$ and $v \in \text{PSH}^-(\Omega)$ then $\max(u, v) \in \mathcal{F}(\Omega)$ and if $u \in \mathcal{F}(\Omega)$ then $\limsup_{u \to \partial \Omega} u(z) = 0$. The subclass $\mathcal{F}^a(\Omega)$ satisfies the same properties. All these properties can be deduced easily from Lemma 2.1 using just some basic properties of the Monge-Ampère capacity.

The following corollary generalizes some result in [17].

**Corollary 2.2.** Fix $u \in \mathcal{F}(\Omega)$, and Let $h : (-\infty, 0] \to (-\infty, 0]$ be an increasing function such that $h(0) = 0$ and $h \circ u$ is psh. Then $h \circ u \in \mathcal{F}(\Omega)$ if and only if $h'(0^-) < \infty$. Furthermore $h \circ u \in \mathcal{F}^a(\Omega)$ if and only if $u \in \mathcal{F}^a(\Omega)$ or $h'(-\infty) = 0$. Moreover we have
\[
\int_{\Omega} (dd^c h \circ u)^n = (h'(0^-))^n \int_{\Omega} (dd^c u)^n.
\]
\[
\int_{(h \circ u = -\infty)} (dd^c h \circ u)^n = (h'(-\infty))^n \int_{(u = -\infty)} (dd^c u)^n.
\]
Here $h'(0^-) = \lim_{s \to 0^-} h(s)/s$ and $h'(-\infty) = \lim_{s \to +\infty} h(s)/s$.

U.Cegrell observed in [14] that if $u \in \mathcal{F}(\Omega)$ then $-(-u)^{1/n} \notin \mathcal{F}(\Omega)$. The corollary above state that $-(-u)^{1/n} \notin \mathcal{F}(\Omega), \forall \alpha < 1$.

We end up this section by extending some result in [24].

**Corollary 2.3.** Let $\Omega_1$ and $\Omega_2$ be two hyperconvex domains in $\mathbb{C}^n$ and $\mathbb{C}^p$ respectively. Suppose $u_1 \in \mathcal{F}(\Omega_1)$ and $u_2 \in \mathcal{F}(\Omega_2)$, then $\max(u_1, u_2) \in \mathcal{F}(\Omega_1 \times \Omega_2)$ and
\[
\int_{\Omega_1 \times \Omega_2} (dd^c \max(u_1, u_2))^{n+p} = \int_{\Omega_1} (dd^c u_1)^n \int_{\Omega_2} (dd^c u_2)^p,
\]
\[
\int_{(u_1 = -\infty) \times (u_2 = -\infty)} (dd^c \max(u_1, u_2))^{n+p} = \int_{(u_1 = -\infty)} (dd^c u_1)^n \int_{(u_2 = -\infty)} (dd^c u_2)^p.
\]

Moreover, $(dd^c \max(u_1, u_2))^{n+p}$ vanishes on the pluripolar subsets of $\Omega_1 \times \Omega_2$ if and only if $(dd^c u_1)^n$ (or $(dd^c u_2)^n$) vanishes on the pluripolar subsets of $\Omega_1$ (resp. of $\Omega_2$).

**Proof.** Observe that
\[
\{(z, w) \in \Omega_1 \times \Omega_2; \max(u_1(z), u_2(w)) \leq -s\} = \{z \in \Omega_1; u_1(z) \leq -s\} \times \{w \in \Omega_2; u_2(w) \leq -s\}.
\]
Then it follows from \[10\] that
\[
s^{n+p}\text{cap}_{\Omega_1 \times \Omega_2} \left\{ (z,w) \in \Omega_1 \times \Omega_2; \max(u_1(z), u_2(w) \leq -s) \right\} = 
\]
\[
s^n\text{cap}_{\Omega_1} \{ z \in \Omega_1; u_1(z) \leq -s \} s^p\text{cap}_{\Omega_2} \{ w \in \Omega_2; u_2(w) \leq -s \}.
\]
Hence the desired results follow by Lemma 2.1.

3. Capacity of sublevel set

It’s well known that if \(u \in PSH^-{\Omega}\) is any psh function then for every compact \(K \subset \Omega\) there is a constant \(C > 0\) such that
\[
\text{Cap}_{\Omega}\left\{ u < -s \right\} \cap K \leq \frac{C}{s}, \quad \forall \ s > 0.
\]
But if \(u \in \mathcal{E}(\Omega)\), the capacity of sublevel set decreases at least like \(s^{-n}\), i.e. for every compact \(K \subset \Omega\) there is a constant \(C > 0\) such that
\[
\text{Cap}_{\Omega}\left\{ u < -s \right\} \cap K \leq \frac{C}{s^n}, \quad \forall \ s > 0.
\]
In fact this is a necessary condition (cf Lemma 2.1) but not sufficient to get \(u \in \mathcal{E}(\Omega)\).

Indeed, let \(B \subset \mathbb{C}^n, n \geq 2\), the unit ball, we consider the psh function \(u(z) = -(-\log |z_1|)^{\frac{1}{n}}\). It’s clear that it satisfies the last condition but \(u \not\in \mathcal{E}(\Omega)\), cf [13], [15].

In this section, we show that if the capacity of sublevel set of a psh function decreases fast enough then its complex Monge-Ampère \((dd^c u)^n\) is well defined.

Denote by \(\mathcal{P}_n(\Omega)\) the space of all negative psh function \(u \in PSH^-{\Omega}\) such that
\[
\int_0^\infty s^{n-1}\text{Cap}_{\Omega}\left\{ u < -s \right\} \cap K)ds < \infty,
\]
for every compact \(K \subset \Omega\).

Bedford has introduced the following class (see [2]). Let \(\theta : \mathbb{R} \to \mathbb{R}\) be a decreasing function such that \(t \to -(-t\theta(-t))^{1/n}\) is an increasing and convex function on \([-\infty, 0]\) and
\[
(3.1) \quad \int_1^{+\infty} \frac{\theta(t)}{t} dt < +\infty.
\]
Define \(\mathcal{B}(\Omega)\) to be the class of negative function \(u \in PSH^-{\Omega}\) such that for any \(z_0 \in \Omega\) there exist a neighborhood \(\omega\) of \(z_0\), a negative psh function \(\psi\) and a decreasing function \(\theta\) satisfying (3.1) such that \(-(-\psi\theta(-\psi))^{1/n} \leq u\) on \(\omega\).

Proposition 3.1. For any hyperconvex domain \(\Omega \subset \mathbb{C}^n\), we have \(\mathcal{B}(\Omega) \subset \mathcal{P}_n(\Omega)\). In particular, for any negative psh function \(v\) on \(\Omega\) and any \(0 < \alpha < 1/n\), \(-(-v)^\alpha \in \mathcal{P}_n(\Omega)\).

Proof. It follows from the definition of \(\mathcal{B}(\Omega)\) that, for any \(w \in \Omega\) and \(s > 0\)
\[
(3.2) \quad \{ u < -s \} \cap \omega \subset \{-(-\psi\theta(-\psi))^{1/n} < -s\} \cap \omega = \{-\psi\theta(-\psi) > s^n\} \cap \omega.
\]
Let \(\kappa\) be a function such that \(\kappa' = \theta\) and \(\kappa(0) = 0\). The function \(\kappa\) is concave. Hence
\[
\kappa(t) \geq t\theta(t), \ \forall t > 0,
\]
which together with (3.2) yield
\[
\int_{0}^{\infty} s^{n-1} \text{Cap}_{\Omega}(\{u \leq -s\} \cap \omega) ds \leq \int_{0}^{\infty} s^{n-1} \text{Cap}_{\Omega}(\{\kappa(-\psi) \geq s^{n}\} \cap \omega) ds
\]
\[
\leq C_{1} + \int_{1}^{\infty} s^{n-1} \text{Cap}_{\Omega}(\{\psi \leq -\kappa^{-1}(s^{n})\} \cap \omega) ds \leq C_{1} + C_{2} \int_{1}^{\infty} s^{n-1} \frac{1}{\kappa^{-1}(s^{n})} ds = C_{1} + C_{2} \int_{1}^{\infty} \frac{\theta(t)}{t} dt < \infty,
\]
which completes the proof. □

More generally, let us consider an increasing function \( h : \mathbb{R}^{-} \to \mathbb{R}^{-} \). Then we have:

**Proposition 3.2.** Suppose that \( h \) satisfies
\[
\int_{1}^{+\infty} \frac{(-h(-s))^{n-1}h'(-s)}{s} ds < +\infty.
\]
Then for any psh function \( u \in PSH^{-}(\Omega) \) such that \( h \circ u \in P_{n}(\Omega) \), we have \( h \circ u \in P_{n}(\Omega) \). Moreover, if \( h \) is convex, then \( h \circ PSH^{-}(\Omega) \subseteq P_{n}(\Omega) \).

The following lemma (cf [16]) will be useful later on.

**Lemma 3.3.** For any psh function \( u \in \mathcal{E}(\Omega) \), we have
\[
\int_{B} (dd^{c}u)^{n} \leq (\|u\|_{B})^{n} \text{Cap}_{\Omega}(B),
\]
provided that \( \|u\|_{B} = \sup_{B} |u| < \infty \).

**Proof.** Denote \( M = \sup_{B} |u| < \infty \), and fix \( \varepsilon > 0 \). Since \( B \subseteq \{u > -M - \varepsilon\} \), it follows from [7]
\[
\int_{B} (dd^{c}u)^{n} = \int_{B} (dd^{c}(\max(u, -M - \varepsilon)))^{n} < (M + \varepsilon)^{n} \text{Cap}_{\Omega}(B).
\]
Letting \( \varepsilon \to 0 \) yields the desired estimate. □

Here we will show that the complex Monge-Ampère operator is well defined in the space \( P_{n}(\Omega) \) and puts no mass on pluripolar sets.

**Theorem 3.4.** For every hyperconvex domain \( \Omega \subseteq \mathbb{C}^{n} \), we have
\[ P_{n}(\Omega) \subseteq \mathcal{E}^{n}(\Omega). \]

Conversely, if \( u \in \mathcal{E}(\Omega) \), then there exists an increasing convex function \( \chi : \mathbb{R}^{-} \to \mathbb{R}^{-} \) such that
\[
\int_{0}^{\infty} s^{n-1} \chi'(-s) \text{Cap}_{\Omega}(\{z \in K \mid u(z) \leq -s\}) ds < \infty,
\]
for all compact \( K \subseteq \Omega \).

**Proof.** The last statement is an immediate consequence of Corollary 4.4 in [7]. To prove the first one, fix \( u \in P_{n}(\Omega) \). It follows from [13] that there exists a decreasing sequence \( u_{j} \in \mathcal{E}^{0}(\Omega) \) such that \( \lim_{j} u_{j} = u \). Let \( B \subseteq \Omega \) be a ball and consider, for \( j \geq 1 \), the function \( \tilde{u}_{j} \) defined by
\[
\tilde{u}_{j}(z) := \sup\{v(z); v \in PSH^{-}(\Omega) \text{ and } v \leq u_{j} \text{ in } B\} \quad z \in \Omega.
\]
It’s clear that \( \tilde{u}_j \) decreases to \( u_B \) defined by

\[
u_B(z) = \sup\{v(z); \ v \in PSH^{-}(\Omega) \text{ and } v \leq u \text{ in } B \} \ \forall z \in \Omega.
\]

So, it’s enough to prove that

\[
sup_j \int_\Omega (dd^c \tilde{u}_j)^n < \infty.
\]

In fact, this a simple consequence of some precise estimate of the Monge-Ampère mass in terms of capacity of sublevel set which can be stated as follows. There exists a constant \( C = C(n) \) depending only in \( n \) such that

\[
\int_K (dd^c \varphi)^n \leq C \int_0^{+\infty} s^{n-1} \text{Cap}_\Omega(K \cap \{ \varphi \leq s \}) ds,
\]

for any negative bounded psh function \( \varphi \) and any Borel subset \( K \subseteq \Omega \).

Indeed, it follows from Lemma 3.3

\[
\int_K (dd^c \varphi)^n = \sum_{k=+\infty}^{k=-\infty} \int_{K \cap \{2^{k-1} \leq \varphi \leq 2^k\}} (dd^c \varphi)^n \leq \sum_{k=+\infty}^{k=-\infty} 2^{-kn} \text{Cap}_\Omega(K \cap \{-\varphi < 2^k\}) \leq C \sum_{k=+\infty}^{k=-\infty} \int_{2^{k-1}}^{2^k} ns^{n-1} \text{Cap}_\Omega(K \cap \{-\varphi \geq s\}) ds \leq C \int_0^{\infty} s^{n-1} \text{Cap}_\Omega(K \cap \{\varphi \leq -s\}) ds.
\]

Now, we apply the estimate (3.6) to \( \tilde{u}_j \), to get

\[
\int_\Omega (dd^c \tilde{u}_j)^n = \int_B (dd^c \tilde{u}_j)^n \leq C \int_0^{\infty} s^{n-1} \text{Cap}_\Omega(K \cap \{u \leq -s\}) ds < \infty.
\]

Which prove that \( u_B \in \mathcal{F}(\Omega) \) and therefore \( u \in \mathcal{E}(\Omega) \). Since the Monge-Ampère capacity \( \text{Cap}_\Omega(\cdot) \) vanishes on pluripolar sets, it follows that \( u_B \in \mathcal{E}(\Omega) \) and then \( u \in \mathcal{E}^a(\Omega) \).

**Corollary 3.5.** For any hyperconvex domain \( \Omega \subseteq \mathbb{C}^n \), we have \( \mathcal{B}(\Omega) \subset \mathcal{E}^a(\Omega) \), i.e. for any function \( u \in \mathcal{B}(\Omega) \), the complex Monge-Ampère measure \( (dd^c u)^n \) is well defined and puts no mass on the pluripolar sets.

If \( h: \mathbb{R}^- \to \mathbb{R}^- \) is an increasing convex function satisfying the condition (3.3), then \( h \circ \text{PSH}^{-}(\Omega) \subset \mathcal{E}^a(\Omega) \).

In particular, for any \( 0 < \alpha < 1/n \), the psh function \( -(u)^\alpha \in \mathcal{E}^a(\Omega) \).

A similar result of the second statement, with a deferent proof, has been obtained recently by Z.Blocki in [11]. The author wishes to thank to the anonymous referee for sending him the recent paper [11].

The first statement has been also proved in [15] and [16].
4. The Weighted Energy Class

**Definition 4.1.** Let $\chi : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing function. We let $\mathcal{E}_\chi(\Omega)$ denote the set of all functions $u \in PSH(\Omega)$ for which there exists a sequence $u_j \in \mathcal{E}_0(\Omega)$ decreasing to $u$ in $\Omega$ and satisfying

$$\sup_{j \in \mathbb{N}} \int_{\Omega} (-\chi) \circ u_j (dd^c u_j)^n < \infty.$$ 

This definition clearly contains the classes of U.Cegrell:

- $\mathcal{E}_\chi(\Omega) = \mathcal{F}(\Omega)$ if $\chi$ is bounded and $\chi(0) \neq 0$;
- $\mathcal{E}_\chi(\Omega) = \mathcal{E}_p(\Omega)$ if $\chi(t) = -(-t)^p$;
- $\mathcal{E}_\chi(\Omega) = \mathcal{F}_p(\Omega)$ if $\chi(t) = -1 - (-t)^p$.

Let us stress that the classes $\mathcal{E}_\chi(\Omega)$ are very different whether $\chi(0) \neq 0$ (finite total Monge-Ampère mass) or $\chi(0) = 0$, $\chi(-\infty) = -\infty$ or $\chi(-\infty) \neq -\infty$, and $\chi$ is convex or concave.

The case $\chi(0) \neq 0$ was studied in [7], here we consider the general case $\chi(0) = 0$.

It is useful in practice to understand these classes through the speed of decreasing of the capacity of sublevel sets.

**Definition 4.2.**

$$\hat{\mathcal{E}}_\chi(\Omega) := \left\{ \varphi \in PSH^-(\Omega) / \int_0^{+\infty} t^n \chi'(\varphi < -t) \text{Cap}_\Omega(\{\varphi < -t\}) dt < +\infty \right\}.$$ 

The classes $\mathcal{E}_\chi(\Omega)$ and $\hat{\mathcal{E}}_\chi(\Omega)$ are closely related:

**Proposition 4.3.** The classes $\hat{\mathcal{E}}_\chi(\Omega)$ are convex and if $\varphi \in \hat{\mathcal{E}}_\chi(\Omega)$ and $\psi \in PSH^-(\Omega)$, then $\max(\varphi, \psi) \in \hat{\mathcal{E}}_\chi(\Omega)$.

One always has $\hat{\mathcal{E}}_\chi(\Omega) \subset \mathcal{E}_\chi(\Omega)$, while

$$\mathcal{E}_\chi(\Omega) \subset \hat{\mathcal{E}}_\chi(\Omega), \text{ where } \hat{\chi}(t) = \chi(t/2).$$

**Proof.** Cf Proposition 4.2. in [7].

**Corollary 4.4.** Let $\chi : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing function. If $u \in \mathcal{E}_\chi(\Omega)$ then $\limsup_{z \to \zeta} u(z) = 0$, $\forall \zeta \in \partial \Omega$.

**Proof.** In fact, we prove the following claim which has its self interest.

If a subset $E \subset \Omega$ has a “big contact” with the boundary $\partial \Omega$ of $\Omega$, then its Monge-Ampère Capacity is infinite. For instance, if $E = B \cap \Omega$, where $B$ is a ball centered at some point in $\partial \Omega$.

Indeed, let $K_j$ be an increasing sequence of regular compact subsets such that $E = \cup K_j$. The extremal function $u_{K_j} \in \mathcal{E}_0(\Omega)$ and decreases to the extremal function $u_E$. It’s clear that $u_E \notin \mathcal{F}(\Omega)$. Thus

$$\sup_j \text{Cap}_\Omega(K_j) = \sup_j \int_{\Omega} (dd^c u_{K_j})^n = +\infty.$$ 

Therefore $\text{Cap}_\Omega(E) = +\infty$.

Now, we prove the corollary. Assume that there exists a $\zeta_0 \in \partial \Omega$, such that $\limsup_{z \to \zeta_0} u(z) = \delta < 0$. This yields that there exists a small ball
centered at $\zeta_0$ such that $B \cap \Omega \subset \{u < \delta/2\}$. Then, it follows from the claim that

$$\text{Cap}_\Omega \{u < -s\} = +\infty, \forall s \leq -\delta/2,$$

which contradicts the fact $u \in \mathcal{E}_\chi(\Omega) \subset \hat{\mathcal{E}}\chi(\Omega)$.

\[ \square \]

**Theorem 4.5.** Let $\chi : \mathbb{R}^- \to \mathbb{R}^-$ be a convex, or concave, increasing function such that $\chi(-\infty) = -\infty$ and $\chi(0) = 0$. Then

$$\mathcal{E}_\chi(\Omega) \subset \mathcal{E}^a(\Omega).$$

Moreover, if $u \in \mathcal{E}(\Omega)$ and $v \in \mathcal{E}_\chi(\Omega)$ are such that $(dd^c v)^n \leq (dd^c u)^n$, then $u \leq v$.

**Proof.** Fix $u \in \mathcal{E}_\chi(\Omega)$, denote $u_j$ a defining sequence such that

$$\sup_j \int_\Omega -\chi(u_j)(dd^c u_j)^n < +\infty.$$

1)° If $\chi$ is convex: It clear that

$$\sup_j \int_\Omega -\chi(u_1)(dd^c u_j)^n \leq \sup_j \int_\Omega -\chi(u_j)(dd^c u_j)^n < +\infty.$$  

So it follows from [14] that $u \in \mathcal{E}(\Omega)$.

2)° If $\chi$ is concave: It follows from the concavity assumption that

$$-\chi(-j) = \chi(0) - \chi(-j) \leq j\chi'(-j).$$

Then for any compact $K \Subset \Omega$,

$$\int_0^{+\infty} t^{n-1}\text{Cap}_\Omega(\{\varphi < -t\} \cap K)dt \leq C_1 + C_2 \int_0^{+\infty} t^n\chi'(-t)\text{Cap}_\Omega(\{\varphi < -t\})dt < +\infty.$$  

Therefore, from Theorem 3.3, we deduce $u \in \mathcal{E}(\Omega)$.

Now we prove the second statement. In fact we will adapt the same idea as in the proof of Theorem 6.2 in [12] for the case $\mathcal{E}_p(\Omega)$. Fix $\rho \in \mathcal{E}_0(\Omega)$, not identically 0. We assume that $-1 \leq \rho < 0$.

First, we assume that $\chi$ is convex. Then for any $j \geq 1$, we have

$$(dd^c \max(v, j\rho))^n = 1\{v > j\rho\}(dd^c v)^n + 1\{v \leq j\rho\}(dd^c \max(v, j\rho))^n,$$

where $1_A$ is the characteristic function for the set $A$. By [20] there exist $g_j \in \mathcal{E}_0$ such that $(dd^c g_j)^n = 1\{v \leq j\rho\}(dd^c \max(v, j\rho))^n$. Thus $(dd^c (u+g_j))^n \geq (dd^c \max(v, j\rho))^n$. It follows from the comparison principle for bounded psh functions (see for example [3], [13]) that $u + g_j \leq \max(v, j\rho))^n$. Hence

$$u + \limsup_{j \to -\infty} g_j \leq v = \lim_{j \to -\infty} \max(v, j\rho).$$

Now it’s enough to prove that $\limsup_{j \to -\infty} g_j = 0$ a.e. Denote $w_m := (\sup_{k \geq m} g_k)^*$, we prove that $\int_\Omega -\chi(w_m)(dd^c w_m)^n = 0$, and this implies that
We claim that

\[
\int_{\Omega} -\chi(w_m)(dd^c w_m)^n \leq \int_{\Omega} -\chi(m\rho)(dd^c w_j)^n \leq \int_{\Omega} -\chi(m\rho)(dd^c g_j)^n
\]

\[
\leq \sup_{z \in \Omega} \frac{-\chi(m\rho(z))}{-\chi(j\rho(z))} \int_{\Omega} -\chi(j\rho)^{(v\leq j\rho)}(dd^c \max(v, j\rho))^n
\]

\[
\leq \sup_{z \in \Omega} \frac{-\chi(m\rho(z))}{-\chi(j\rho(z))} \sup_{j \geq m} \int_{\Omega} -\chi((\max(v, j\rho))(dd^c \max(v, j\rho))^n < +\infty.
\]

We claim that

\[
\lim_{j \to \infty} \sup_{z \in \Omega} \frac{-\chi(m\rho(z))}{-\chi(j\rho(z))} = 0.
\]

Indeed, for \(z \in \Omega\), put \(s = \rho(z)\). Assume, on the contrary, that

\[
(4.1) \quad \lim_{j \to \infty} \sup_{-1 \leq s \leq 0} \frac{-\chi(ms)}{-\chi(js)} > \delta > 0.
\]

Then there exists a sequence \(s_j\) converging towards 0 such that \(\frac{-\chi(ms_j)}{-\chi(js_j)} > \delta > 0\). Since \(ms_j \to 0\), as \(j \to \infty\), it follows that \(js_j \to 0\), as \(j \to \infty\). Since \(\chi\) is convex, we have

\[
\frac{-\chi(ms_j)}{-\chi(js_j)} \sim \frac{ms_j}{js_j} = \frac{m}{j} \to 0, \text{ as } j \to +\infty,
\]

which contradicts (4.1). Therefore, the claim is proved. Hence \(\lim_{j \to \infty} g_j = 0, \text{ a.e.}\)

Now, if \(\chi\) is concave. We modify slightly the above proof. Indeed, since \(\chi\) is concave, the function \(\chi^{-1}(j\rho) \in E_0(\Omega)\) for any \(j > 0\). Then

\[
(dd^c \max(v, \chi^{-1}(j\rho)))^n = 1_{\{v > \chi^{-1}(j\rho)\}}(dd^c v)^n + 1_{\{v \leq \chi^{-1}(j\rho)\}}(dd^c \max(v, \chi^{-1}(j\rho)))^n.
\]

We consider the function \(g_j \in E_0(\Omega)\) satisfying

\[
(dd^c g_j)^n = 1_{\{v \leq \chi^{-1}(j\rho)\}}(dd^c \max(v, \chi^{-1}(j\rho)))^n.
\]

Then we repeat the same arguments as above. \(\square\)

Note that if \(u \in E_\chi(\Omega)\) is such that \(\int_{\Omega}(dd^c u)^n < +\infty\) then \(u \in F(\Omega)\). Therefore, by Lemma 2.1, the total mass \(\int_{\Omega}(dd^c u)^n\) depends only on the behavior of \(u\) near \(\partial\Omega\). Now, if \(\int_{\Omega}(dd^c u)^n = +\infty\) then \(\int_{\Omega}(dd^c \max(u, -s))^n = +\infty, \forall s > 0\), and since \(\int_{\{u = -s\}}(dd^c \max(u, -s))^n < +\infty\) (cf Lemma 3.3) it follows that \(\int_{\{u > -s\}}(dd^c u)^n = +\infty\) and \(\int_{\{u \leq -s\}}(dd^c u)^n < +\infty, \forall s > 0\).

**Lemma 4.6.** If \(u \in E_\chi(\Omega)\) then there exists a decreasing sequence \(u_j \in E_0(\Omega)\) with \(\lim u_j = u\) and

\[
\lim_{j \to \infty} \int_{\Omega} (-\chi) \circ u_j(dd^c u_j)^n = \int_{\Omega} (-\chi) \circ u(dd^c u)^n < +\infty.
\]

This result was proved by U.Cegrell (cf [12]) for the classes \(E_p(\Omega)\). The same proof still valid in the general context. For the convenience of the reader we give here the proof.
Proof. It follows from [20] that there exists, for each \( j \in \mathbb{N} \), a function \( u_j \in \mathcal{E}_0(\Omega) \) such that \((dd^c u_j)^n = 1_{\{u > j\rho\}}(dd^c u)^n\), where \( \rho \in \mathcal{E}_0(\Omega) \) any defining function for \( \Omega = \{\rho < 0\} \). Observe that \((dd^c u)^n \geq (dd^c u_{j+1})^n \geq (dd^c u_j)^n\). We infer from the comparison principle that \((u_j)\) is a decreasing sequence and \( \lim_j u_j = u \). The monotone convergence theorem thus yields

\[
\int_\Omega (-\chi) \circ u_j(dd^c u_j)^n = \int_\Omega (-\chi) \circ u_11_{\{u > j\rho\}}(dd^c u)^n \to \int_\Omega (-\chi) \circ u(dd^c u)^n < +\infty.
\]

The following capacity estimates of sublevel sets will be useful later on.

**Proposition 4.7.** Let \( \chi : \mathbb{R}^- \to \mathbb{R}^- \) be an increasing convex, or concave, function such that \( \chi(-\infty) = -\infty \) and \( \chi(0) = 0 \). Then

\[
\text{Cap}_\Omega(\{\varphi < -s\}) \leq \frac{1}{|s^n \chi(-s)|} \int_{\{\varphi < -s\}} -\chi(\varphi)(dd^c \varphi)^n, 
\]

for any \( s > 0 \) and any function \( \varphi \in \mathcal{E}_\chi(\Omega) \).

**Proof.** Follows from Lemma 2.1 by approximating \( \varphi \) by \( \varphi_j \in \mathcal{E}_0(\Omega) \) given by the lemma above. \( \square \)

**Proposition 4.8.** Let \( \chi : \mathbb{R}^- \to \mathbb{R}^- \) be an increasing convex, or concave, function such that \( \chi(-\infty) = -\infty \) and \( \chi(0) = 0 \). Then there exists a constant \( C = C(\chi) \) such that

\[
\text{Cap}_\Omega(\{\varphi < -s\}) \leq \frac{C}{s^n} \int_\Omega -\chi(\varphi)(dd^c \varphi)^n, \quad \forall s > 0, \forall \varphi \in \mathcal{E}_\chi(\Omega).
\]

**Proof.** First we give the proof in the case \( n = 2 \). Let \( K \in \{\varphi < -s\} \) be a compact subset, \( u_K \) denotes its relative extremal function. Choose \( \chi_1 : \mathbb{R}^- \to \mathbb{R}^- \) to be an increasing function such that \( \chi''_1 = \chi \) and \( \chi_1(0) = 0 \). Then

\[
(4.2) \quad dd^c \chi_1(\varphi) = \chi''_1(\varphi)d\varphi \wedge d^c \varphi + \chi'_1(\varphi)dd^c \varphi \leq \chi'_1(\varphi)dd^c \varphi,
\]

and

\[
(4.3) \quad -dd^c \chi_1'(\varphi) = -\chi'''_1(\varphi)d\varphi \wedge d^c \varphi - \chi'_1(\varphi)dd^c \varphi \leq -\chi(\varphi)dd^c \varphi.
\]

It follows from [13] that there exists a decreasing sequence \( \varphi_j \in \mathcal{E}_0(\Omega) \cap C(\Omega) \) such that \( \varphi_j \searrow \varphi \). Then integrating by part together with the previous
inequalities yield
\[
\int_K (dd^c u_K)^n \leq \int_K \frac{-\chi_1(\varphi/s)}{-\chi_1(-1)} (dd^c u_K)^n = \lim_j \int_K \frac{-\chi_1(\varphi_j/s)}{-\chi_1(-1)} (dd^c u_K)^n
\]
\[= \lim_j \frac{1}{-\chi_1(-1)} \int_{\Omega} -u_K dd^c -\chi_1(\varphi_j/s) \wedge (dd^c u_K)^{n-1}
\]
\[\leq \lim_j \frac{C}{s} \int_{\Omega} -u_K \chi_1'(\varphi_j/s) dd^c \varphi_j \wedge (dd^c u_K)^{n-1}
\]
\[\leq \lim_j \frac{C}{s} \int_{\Omega} \chi_1'(\varphi_j/s) dd^c \varphi_j \wedge (dd^c u_K)^{n-1}
\]
\[\leq \lim_j \frac{C}{s^2} \int_{\Omega} u_K dd^c \chi_1'(\varphi_j/s) \wedge dd^c \varphi_j \wedge (dd^c u_K)^{n-2}
\]
\[\leq \lim_j \frac{C}{s^2} \int_{\Omega} -\chi''(\varphi_j/s) (dd^c \varphi_j)^2 \wedge (dd^c u_K)^{n-2}
\]
\[= \frac{C}{s^2} \int_{\Omega} -\chi''(\varphi/s) (dd^c \varphi)^2 \wedge (dd^c u_K)^{n-2}.
\]

For the general case, we use the same arguments. Indeed, we consider an increasing function $\chi_1 : \mathbb{R}^- \to \mathbb{R}^-$ such that $\chi_1^{(n)} = \chi$ and $\chi_1(0) = 0$. Then, the repeated application of inequalities (4.2), (4.3) and the integration by part yields the desired estimate. $\square$

Hereafter, we will see that in fact, the classes $E_\chi(\Omega)$ live in some natural set of psh functions introduced by U.Cegrell in [14]. Let us recall its definition. Let $\Omega_j \subseteq \Omega$ be an increasing sequence of strictly pseudoconvex domains such that $\Omega = \bigcup \Omega_j$. Let $u \in E(\Omega)$ be given and put

\[u_{\Omega_j} := \sup \{ \varphi \in PSH(\Omega); \varphi \leq u \text{ on } \Omega \setminus \Omega_j \}.
\]

Then the sequence $u_{\Omega_j} \in E(\Omega)$ is increasing, so $\bar{u} := (\lim_j u_{\Omega_j})^* \in E(\Omega)$. The definition of $\bar{u}$ is independent of the choice of the sequence $\Omega_j$ and is maximal i.e. $(dd^c \bar{u})^n = 0$. $\bar{u}$ is the smallest maximal psh function above $u$.

Define $\mathcal{N}(\Omega) := \{ u \in E(\Omega); \bar{u} = 0 \}$. In fact, this class is the analogous of potentials for subharmonic functions. Also, denote $\mathcal{N}^+(\Omega) = \mathcal{E}^+(\Omega) \cap \mathcal{N}(\Omega)$.

**Proposition 4.9.** Let $\chi : \mathbb{R}^- \to \mathbb{R}^-$ be a convex, or concave, increasing function such that $\chi(-\infty) = -\infty$ and $\chi(0) = 0$. Then

\[E_\chi(\Omega) \subset \mathcal{N}^+(\Omega).
\]

In particular the Monge-Ampère measure $(dd^c u)^n$ of a function $u \in E_\chi(\Omega)$ is well defined and does not charge pluripolar sets. More precisely,

\[E_\chi(\Omega) = \{ u \in \mathcal{N}(\Omega) / \chi \circ u \in L^1((dd^c u)^n) \}.
\]

**Proof.** Fix $u \in E_\chi(\Omega)$ and $u_j \in E_0(\Omega)$ a defining sequence such that

\[\sup_j \int_{\Omega} -\chi(u_j)(dd^c u_j)^n < +\infty.
\]

It follows from the upper semi-continuity of $u$ that $-\chi(u)(dd^c u)^n$ is bounded from above by any cluster point of the bounded sequence $-\chi(u_j)(dd^c u_j)^n$. Therefore $\int_{\Omega} (-\chi \circ u)(dd^c u)^n < +\infty$, in particular $(dd^c u)^n$ does not charge the set $\{ \chi(u) = -\infty \}$, which coincides with $\{ u = -\infty \}$, since $\chi(-\infty) = -\infty$. \hfill $\square$
−∞. It follows from Theorem 2.1 in [7], \((dd^c u)^n\) does not charge pluripolar sets. Now it remains to prove that \(u \in \mathcal{N}(\Omega)\) i.e. the smallest maximal function above \(u\) is null. Let \(\tilde{u}\) be a such function. Then \(u \leq \tilde{u} \leq 0\), thus \(\tilde{u} \in \mathcal{E}_\chi(\Omega) \subset \mathcal{E}_\mathcal{H}(\Omega)\). It follows from Lemma 4.6 that there exists a decreasing sequence \(\tilde{u}_j \in \mathcal{E}_0(\Omega)\) with \(\lim \tilde{u}_j = \tilde{u}\) and

\[
\lim_{j \to -\infty} \int_{\Omega} (-\chi) \circ \tilde{u}_j (dd^c \tilde{u}_j)^n = \int_{\Omega} (-\chi) \circ \tilde{u} (dd^c \tilde{u})^n < +\infty.
\]

Hence Lemma 2.1 implies that \(\int_0^{+\infty} t^n \chi'(-t/4) \text{Cap}_{\Omega}(\{\tilde{u} < -t\}) dt = 0\), this yields that \(\tilde{u} = 0\).

To prove the last assertion, it remains to show the reverse inclusion

\[
\mathcal{E}_\chi(\Omega) \supset \{u \in \mathcal{N}(\Omega) / \chi \circ u \in L^1((dd^c u)^n)\}.
\]

This is an immediate consequence of Lemma 4.6.

Note that, unlike the case \(\chi(0) \neq 0\) with the class \(\mathcal{F}(\Omega)\) (cf [7]), we have

\[
\bigcap_{\chi(0)=0, \chi(-\infty)=-\infty} \mathcal{E}_\chi(\Omega) \subset \mathcal{N}(\Omega) \cap L^\infty(\Omega), \quad \text{and} \quad \bigcup_{\chi(0)=0, \chi(-\infty)=-\infty} \mathcal{E}_\chi(\Omega) \not\subset \mathcal{N}(\Omega) \cap L^\infty(\Omega).
\]

One can see [14] for examples of functions in the class \(\mathcal{N}(\Omega) \cap L^\infty(\Omega)\) which do not belong to any \(\mathcal{E}_\chi(\Omega)\).

Let \(\chi : \mathbb{R}^- \to \mathbb{R}^-\) be an increasing function. We say that \(\chi\) is admissible if and only if \(\chi\) is convex or concave and if there exists a constant \(M > 0\) such that

\[
(4.4) \quad \chi'(-2s) \leq M\chi'(-s), \quad \forall s > 0.
\]

Observe that any homogeneous function \(\chi(t) = -(-t)^p\) \(p \geq 1\), is admissible. Another example of admissible function which is not homogenous (cf [19]) is \(\chi(t) = -(-t)^p(\log(-t+\epsilon))^\alpha\), \(p \geq 1\) and \(\alpha > 0\).

**Proposition 4.10.** If \(\chi\) is an increasing admissible function, then we have

\[
\mathcal{E}_\chi(\Omega) = \left\{ \varphi \in \text{PSH}^-(\Omega) / \int_0^{+\infty} t^n \chi'(-t) \text{Cap}_{\Omega}(\{\varphi < -t\}) dt < +\infty \right\}.
\]

**Proof.** Follows easily from Lemma 2.1 and (4.4).

**Theorem 4.11.** Let \(\chi : \mathbb{R}^- \to \mathbb{R}^-\) be an admissible increasing function such that \(\chi(-\infty) = -\infty\) and \(\chi(0) = 0\). Fix \(u \in \mathcal{E}_\chi(\Omega)\) and set \(u^j = \max(u, -j)\).

Then for each Borel subset \(B \subset \Omega\),

\[
\lim_{j \to -\infty} \int_B (dd^c u^j)^n = \int_B (dd^c u)^n,
\]

and

\[
\int_B \chi(u^j)(dd^c u^j)^n \to \int_B \chi(u)(dd^c u)^n.
\]

Furthermore, if \(u_j\) is any decreasing sequence in \(\mathcal{E}_\chi(\Omega)\) converging to \(u\), then

\[
\lim_j \int_\Omega \chi(u_j)(dd^c u_j)^n = \int_\Omega \chi(u)(dd^c u)^n.
\]

The first statement, as we will see in the proof, still valid for all weight \(\chi\).
Proof. Let $B \subset \Omega$ be a Borel subset. If $\int_B (dd^c u)^n = +\infty$ then for any $j > 0$, $\int_B (dd^c u)^n = +\infty$. So we assume that $\int_B (dd^c u)^n < +\infty$. It follows from Lemma 3.3 and Proposition 4.1

$$\left| \int_B (dd^c u)^n - \int_B (dd^c u)^n \right| \leq \int_{\{u \leq -j\}} (dd^c u)^n + \int_{\{u \leq -j\}} (dd^c u)^n$$

$$\leq j^n \text{Cap}_\Omega(\{u < -j\}) + \int_{\{u < -j\}} \frac{-\chi(u)}{-\chi(-j)} (dd^c u)^n$$

$$\leq \frac{2^{n+1}}{-\chi(-j/2)} \int_{\{u < -j/2\}} -\chi(u)(dd^c u)^n, \text{ as } j \to +\infty.$$

The proof that $\chi \circ u^j (dd^c u)^n$ converges strongly towards $\chi \circ u (dd^c u)^n$ goes along similar lines, first observe that from Lemma 3.3, we have

(4.5) \[ \int_{\{u \leq -j\}} -\chi \circ u^j (dd^c u)^n = -\chi(-j) \int_{\{u \leq -j\}} (dd^c u)^n \]

(4.6) \[ \leq -\chi(-j)j^n \text{Cap}_\Omega(\{u < -j\}). \]

Since $\chi$ is an admissible function, it follows that there exists a constant $C > 1$ such that

$$-\chi(-2s) \leq -C \chi(-s), \forall s > 0$$

This yields

(4.7) \[ \lim_{j \to \infty} -\chi(-j)j^n \text{Cap}_\Omega(\{u < -j\}) \leq \lim_{j \to \infty} -C \chi(-j/2)j^n \text{Cap}_\Omega(\{u < -j\}) \]

$$\leq \lim_{j \to \infty} -2^{n+1}C \chi(-j)j^n \text{Cap}_\Omega(\{u < -2j\})$$

$$\leq \lim_{j \to \infty} 2^{n+1}C \int_{\{u \leq -j\}} -\chi(u)(dd^c u)^n = 0.$$

Then (4.6) and (4.7) together with Proposition 4.1 imply that

$$\lim_{j \to +\infty} \int_{\{u \leq -j\}} -\chi \circ u^j (dd^c u)^n = 0.$$

Hence the proof of the second statement is completed.

Now, once the first and second assertions are proved, we apply the same proof as that of Theorem 3.4 in [7] to show the last statement. \qed

We conclude this section with a characterization of bounded function in the classes $E_\chi(\Omega)$, extending Y. Xing’s main result in [25].

Proposition 4.12. Let $u \in E_\chi(\Omega)$. Then $u$ is bounded in the domain $\Omega$ if and only if there exist constants $A > 0$ and $B$ such that for any real $k < B$ with $\text{Cap}_\Omega(u < k) \neq 0$ we can find an increasing sequence $k \leq k_1 < k_2 < \cdots < k_s = B$ with $k_1 < k + 1$ and

$$\sum_{j=2}^{s} \left( \frac{\int_{\{u < k_j\}} (dd^c u)^n}{\text{Cap}_\Omega(u < k_{j-1})} \right)^{1/n} < A.$$
Proof. The necessary implication is obvious. To show the sufficient one, assume on the contrary that $u$ is unbounded. Then $\text{Cap}_\Omega(u < k) \neq 0$ for all $k < 0$. It follows from Lemma 2.1

$$B - 1 - k \leq \sum_{j=2}^{s} k_j - k_{j-1} \leq \sum_{j=2}^{s} \left( \frac{\int_{(u < k_j)}(dd^c u)^n}{\text{Cap}_\Omega(u < k_{j-1})} \right)^{1/n} < A.$$  

Hence $B - 1 - k \leq A$ for all $k < B$, which is impossible. The proof is complete. \qed

5. The range of the complex Monge-Ampère operator

The image of the complex Monge-Ampère operator acting on the classes $\mathcal{E}_p(\Omega)$, has been extensively studied by U.Cegrell. The main result of his study, achieved in [12], is given as follows. Given a positive measure $\mu$, then there exists a unique function $\varphi \in \mathcal{E}_p(\Omega)$ such that $\mu = (dd^c \varphi)^n$ if and only if there exists a constant $C > 0$ such that

$$(5.1) \quad \int_{\Omega} (-u)^p d\mu \leq C \left( \int_{\Omega} (-u)^p (dd^c u)^n \right)^{\frac{p}{n+p}}, \quad \forall u \in \mathcal{E}_0(\Omega).$$

Observe that this necessary and sufficient condition is equivalent to the following: The operator $u \rightarrow \int_{\Omega} (-u)^p d\mu$ is uniformly bounded on the compact “pseudo-ball” $\tilde{\mathcal{E}}_p(\Omega) := \{ u \in \mathcal{E}_p(\Omega); \int_{\Omega} (-u)^p (dd^c u)^n \leq 1 \}$. The following theorem extends U.Cegrell’s main result [12].

Theorem 5.1. Let $\chi : \mathbb{R}^- \to \mathbb{R}^-$ be an increasing convex function such that $\chi(-\infty) = -\infty$. The following conditions are equivalent:

(1) there exists a unique function $\varphi \in \mathcal{E}_\chi(\Omega)$ such that $\mu = (dd^c \varphi)^n$;

(2) there exists a constant $C_1 > 0$ such that

$$\int_{\Omega} -\chi \circ u d\mu \leq C_1, \quad \forall u \in \tilde{\mathcal{E}}_0(\Omega),$$

(3) there exists a constant $C_2 > 0$ such that

$$\int_{\Omega} -\chi \circ u d\mu \leq C_2 \max \left( 1, \left( \int_{\Omega} -\chi \circ u (dd^c u)^n \right)^{\frac{1}{n}} \right), \quad \forall u \in \mathcal{E}_0(\Omega).$$

Here $\tilde{\mathcal{E}}_0(\Omega) := \{ u \in \mathcal{E}_0(\Omega); \int_{\Omega} (-u)^p (dd^c u)^n \leq 1 \}$.

Proof. We prove that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1).

We start with (3) $\Rightarrow$ (1). It follows from [7] (see also Proposition 4.10) that the class $\mathcal{E}_\chi(\Omega)$ characterizes pluripolar sets. Then the assumption (5.3) on $\mu$ implies in particular that it vanishes on pluripolar sets. It follows from [13] that there exists a function $u \in \mathcal{E}_0(\Omega)$ and $f \in L^1_{\text{loc}}((dd^c u)^n)$ such that $\mu = f(dd^c u)^n$.

Consider $\mu_j := \min(f, j)(dd^c u)^n$. This is a finite measure which is bounded from above by the complex Monge-Ampère measure of a bounded function. It follows therefore from [20] that there exist $\varphi_j \in \mathcal{E}_0(\Omega)$ such that $(dd^c \varphi_j)^n = \min(f, j)(dd^c u)^n$. 

The comparison principle shows that $\varphi_j$ is a decreasing sequence. Set $\varphi = \lim_{j \to \infty} \varphi_j$. It follows from (5.3) that

$$
\int_{\Omega} -\chi(\varphi_j)(dd^c \varphi_j)^n \leq C_2 \max \left( 1, \left( \int_{\Omega} -\chi(\varphi_j)(dd^c \varphi_j)^n \right)^{1/n} \right).
$$

Hence

$$
\sup_j \int_{\Omega} -\chi(\varphi_j)(dd^c \varphi_j)^n \leq C_2^{n/n-1} < \infty.
$$

So it follows from Proposition 4.10 that

$$
\sup_j \int_{0}^{+\infty} t^n \chi'(t) \operatorname{Cap}_{\Omega}(\{\varphi < -t\})dt < +\infty,
$$

which implies that

$$
\int_{0}^{+\infty} t^n \chi'(t) \operatorname{Cap}_{\Omega}(\{\varphi < -t\})dt < +\infty.
$$

Then $\varphi \not\equiv -\infty$ and therefore $\varphi \in E_\chi(\Omega)$.

We conclude now by continuity of the complex Monge-Ampère operator along decreasing sequences that $(dd^c \varphi)^n = \mu$. The unicity of $\varphi$ follows from the comparison principle (Theorem 4.5).

Now, we prove (2) $\Rightarrow$ (3). Let $\psi \in \tilde{E}_0(\Omega)$, denote $E_\chi(\psi) := \int_{\Omega} -\chi(\psi)(dd^c \psi)^n$. If $\psi \in \tilde{E}_0(\Omega)$, i.e. $E_\chi(\psi) \leq 1$ then

$$
\int_{\Omega} -\chi(\psi)d\mu \leq C_1.
$$

If $E_\chi(\psi) > 1$. The function $\tilde{\psi}$ defined by

$$
\tilde{\psi} := \frac{\psi}{E_\chi(\psi)^{1/n}} \in \tilde{E}_0(\Omega).
$$

Indeed, from the monotonicity of $\chi$, we have

$$
\int_{\Omega} -\chi\left(\frac{\psi}{E_\chi(\psi)^{1/n}}\right)(dd^c \frac{\psi}{E_\chi(\psi)^{1/n}})^n \leq \frac{1}{E_\chi(\psi)} \int_{\Omega} -\chi(\psi)(dd^c \psi)^n = 1.
$$

It follows from (5.2) and the convexity of $\chi$

$$
\int_{\Omega} -\chi(\psi)d\mu \leq E_\chi(\psi)^{1/n} \int_{\Omega} -\chi\left(\frac{\psi}{E_\chi(\psi)^{1/n}}\right)d\mu \leq C_1 E_\chi(\psi)^{1/n}.
$$

Hence we get (3) with $C_2 = \max(1, C_1)$.

For the proof of the remaining implication (1) $\Rightarrow$ (2), we use the same idea as in [19]. Let $u \in \tilde{E}_0(\Omega)$ and $\varphi \in E_\chi(\Omega)$. Observe that for any $s > 0$, we have

$$
(u < -s) \subset (u < -\frac{s}{2}) \cup (\varphi < -\frac{s}{2}).
$$
Hence

\[(5.4) \quad \int_{\Omega} -\chi \circ u (dd^c \varphi)^n = \int_{0}^{\infty} -\chi'(-s) \int_{(u<s)} (dd^c \varphi)^n ds \]

\[\leq \int_{0}^{\infty} \chi'(-s) \int_{(u<\varphi-\frac{s}{2})} (dd^c \varphi)^n ds + \int_{0}^{\infty} \chi'(-s) \int_{(\varphi<\frac{s}{2})} (dd^c \varphi)^n ds \]

\[\leq 2 \int_{0}^{\infty} \chi'(-2s) \int_{(u<\varphi-s)} (dd^c \varphi)^n ds + 2 \int_{0}^{\infty} \chi'(-2s) \int_{(\varphi<-s)} (dd^c \varphi)^n ds. \]

The convexity of \(\chi\) yields that

\[(5.5) \quad \chi'(-2s) \leq M \chi'(-s), \quad \forall s > 0.\]

It follows by the comparison principle that, for all \(s > 0\)

\[(5.6) \quad \int_{(u<\varphi-s)} (dd^c \varphi)^n \leq \int_{(u<\varphi-s)} (dd^c u)^n. \]

Together (5.4), (5.5) and (5.6) imply that there exists a constant \(C\) independent of \(u\) such that \(\int_{\Omega} -\chi \circ u (dd^c \varphi)^n \leq C, \quad \forall u \in \mathcal{E}_0(\Omega). \)

Note that if \(\chi\) is homogenous, i.e. \(\chi(t) = -(t)^p\) with \(p > 0\), then the above theorem still valid, but we replace the assertion (3) by the following (3') there exists a constant \(C' > 0\) such that

\[(5.7) \quad \int_{\Omega} -\chi \circ u d\mu \leq C' \max \left(1, \left(\int_{\Omega} -\chi \circ u (dd^c u)^n\right)^{\frac{p}{n+p}}\right), \quad \forall u \in \mathcal{E}_0(\Omega), \]

which, thanks to the homogeneity, is equivalent to (5.1). In particular, this generalizes the U. Cegrell’s main theorem in [12] for \(p \geq 1\) and in [1] for \(0 < p \leq 1\).

6. Subextension in the class \(\mathcal{E}_\chi\)

Here we will show that functions in the classes \(\mathcal{E}_\chi(\Omega)\) admit subextension. We need to recall the usual Lelong class of psh functions. Let \(\gamma > 0\) be a positive real. Then

\[\mathcal{L}_\gamma(\mathbb{C}^n) := \left\{ \varphi \in PSH(\mathbb{C}^n); \limsup_{r \to +\infty} \max_{|s|=r} \frac{\varphi(z)}{log r} \leq \gamma \right\}. \]

**Proposition 6.1.** Let \(\chi : \mathbb{R}^- \to \mathbb{R}^-\) be an increasing function such that \(\chi(-\infty) = -\infty\) and

\[\int_{-\infty}^{+\infty} \frac{1}{s|\chi'(s)|^{1/n}} ds < +\infty. \]

Then for any function \(\varphi \in \mathcal{E}_\chi(\Omega)\) and any \(\varepsilon > 0\), there exists a function \(U_\varepsilon \in \mathcal{L}_\varepsilon(\mathbb{C}^n)\) such that \(U_\varepsilon \leq \varphi\) on \(\Omega\).

**Proof.** Define the function \(h(s) := \text{Cap}_\Omega(\{u < -s\})\). It follows from the proof of Theorem 4.11 that

\[\text{Cap}_\Omega(\{u < -s\}) \leq \frac{2^n}{s^n|\chi(-s/2)|^{1/n}} \int_{\{u \leq -s/2\}} -\chi(u)(dd^c u)^n. \]
Then
\[ \int_{\infty}^{\infty} h(s)^{1/n} ds \leq 2 \left( \int_{\Omega} -\chi(u)(dd^{c} u)^n \right)^{\frac{1}{n}} \int_{\infty}^{\infty} \frac{1}{s^{1/n}} ds < +\infty. \]

Hence the assertion follows from Theorem 4.1 in [16]. \( \square \)

**Theorem 6.2.** Let \( \Omega \subset \tilde{\Omega} \subset \mathbb{C}^n \) be hyperconvex domains. Let \( \chi : \mathbb{R}^- \to \mathbb{R}^- \) be an increasing function such that \( \chi(-\infty) = -\infty \). If \( u \in \mathcal{E}_{\chi}^{\prime}(\Omega) \), then there exists \( \tilde{u} \in \mathcal{E}_{\chi}^{\prime}(\tilde{\Omega}) \) such that \( \tilde{u} \leq u \) on \( \Omega \), \( (dd^{c} \tilde{u})^n \leq (dd^{c} u)^n \) on \( \Omega \) and \( E_{\chi}(\tilde{u}) \leq E_{\chi}(u) \).

**Proof.** With slightly different notations, the proof is identical to that in the case \( \mathcal{E}_p(\Omega) \). We refer the reader to [23] for details. \( \square \)

**References**

[1] P. Ahag & R. Czyz R. & H. H. Pham: Concerning the energy class \( \mathcal{E}_p \) for \( 0 < p < 1 \). Ann. Polon. Math. 91 (2007), 119-130.

[2] E. Bedford: Survey of pluripotential theory, Several Complex Variables, Mittag-Leffler institute 1987-88, Math. Notes, 38(1993), 48-95, Princeton Uni. Press, Princeton.

[3] E. Bedford & B. A. Taylor: A new capacity for plurisubharmonic functions. Acta Math. 149 (1982), no. 1-2, 1-40.

[4] E. Bedford & B. A. Taylor: Fine topology, \( \mathcal{S}i\mathcal{P}o\mathcal{V} \) boundary, and \( (dd^{c} f)^n \). J. Funct. Anal. 72 (1987), no. 2, 225-251.

[5] S. Benelkourchi: A note on the approximation of plurisubharmonic functions. C. R. Math. Acad. Sci. Paris, 342 (2006), 647–650.

[6] S. Benelkourchi & V. Guedj & A. Zeriahi: A priori estimates for weak solutions of complex Monge-Ampère equations., Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5), Vol. VII(2008), 81–96.

[7] S. Benelkourchi & V. Guedj & A. Zeriahi: Plurisubharmonic functions with weak singularities, Proceedings from the Kiselmanfest, 2006. Acta Universitatis Upsaliensis, Proceedings of the conference in honor of C. Kiselman (Kiselmanfest, Uppsala, May 2006) (in press).

[8] Z. Błocki: On the definition of the Monge-Ampère operator in \( \mathbb{C}^2 \). Math. Ann. 328 (2004), no. 3, 415–423.

[9] Z. Błocki: The domain of definition of the complex Monge-Ampère operator. Amer. J. Math. 128 (2006), no. 2, 519–530.

[10] Z. Błocki: Equilibrium measure of a product subset of \( \mathbb{C}^n \). Proc. Amer. Math. Soc. 128 (2000), no. 12, 3595–3599.

[11] Z. Błocki: Remark on the definition of the complex Monge-Ampère operator. Preprint [http://www.mitag-leffler.se/preprints/0708s/info.php?id=02](http://www.mitag-leffler.se/preprints/0708s/info.php?id=02)

[12] U. Cegrell: Pluricomplex energy. Acta Math. 180 (1998), no. 2, 187–217.

[13] U. Cegrell: The general definition of the complex Monge-Ampère operator. Ann. Inst. Fourier (Grenoble) 54 (2004), no. 1, 159–179.

[14] U. Cegrell: A general Dirichlet problem for of the complex Monge-Ampère operator, Ann. Polon. Math. 94 (2008), 131-147.

[15] U. CEGRELL: Explicit calculation of a Monge-Ampere measure, Actes des rencontres d’analyse complexe, 25-28 Mars 1999. Edited by Gilles Raby and Frédéric Symesak. Atlantique. Université de Poitiers, 2000.

[16] U. Cegrell & S. Kołodziej & A. Zeriahi: Subextension of plurisubharmonic functions with weak singularities. Math. Z. 250 (2005), no. 1, 7–22.

[17] J.-P. Demailly: Monge-Ampère operators, Lelong numbers and intersection theory. Complex analysis and geometry, 115–193, Univ. Ser. Math., Plenum, New York (1993).
[18] P. EYSSIDIEUX & V. GUEDJ & A. ZERIAHI: Singular Kähler-Einstein metrics. J. Amer. Math. Soc., to appear.

[19] V. GUEDJ & A. ZERIAHI: The weighted Monge-Ampère energy of quasi-plurisubharmonic functions. J. Funct. Anal. 250 (2007), no. 2, 442–482.

[20] S. KOLODZIEJ: The range of the complex Monge-Ampère operator. Indiana Univ. Math. J. 43 (1994), no. 4, 1321–1338.

[21] S. KOLODZIEJ: The complex Monge-Ampère equation. Acta Math. 180 (1998), no. 1, 69–117.

[22] S. KOLODZIEJ: The complex Monge-Ampère equation and pluripotential theory. Mem. Amer. Math. Soc. 178 (2005), no. 840, x+64 pp.

[23] H. H. PHAM: Pluripolar sets and the subextension in Cegrell’s classes, Complex Var. Elliptic Equ. 53 (2008), no. 7, 675–684.

[24] J. WIKLUND: Topics in pluripotential theory. Doctoral Thesis No. 30, 2004, Umeå university, Sweden.

[25] Y. XING: Complex Monge-Ampère measures of plurisubharmonic functions with bounded values near the boundary. Canad. J. Math. 52 (2000), no. 5, 1085–1100.

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