SEMICLASSICAL ANALYSIS IN THE LIMIT CIRCLE CASE

D. R. YAFAEV

To the memory of Serezha Naboko

Abstract. We consider second order differential equations with real coefficients that are in the limit circle case at infinity. Using the semiclassical Ansatz, we construct solutions (the Jost solutions) of such equations with a prescribed asymptotic behavior for \( x \to \infty \). It turns out that in the limit circle case, this Ansatz can be chosen common for all values of the spectral parameter \( z \). This leads to asymptotic formulas for all solutions of considered differential equations, both homogeneous and non-homogeneous. We also efficiently describe all self-adjoint realizations of the corresponding differential operators in terms of boundary conditions at infinity and find a representation for their resolvents.

1. Introduction

1.1. Setting the problem. We consider a second order differential equation

\[-(p(x)u'(x))' - q(x)u(x) = zu(x), \quad p(x) > 0, q(x) = \bar{q}(x), \quad x \in \mathbb{R}_+, \ z \in \mathbb{C}, \ (1.1)\]

under some minimal regularity assumptions on coefficients \( p(x) \) and \( q(x) \) guaranteeing that its solutions, as well as their derivatives, are continuous on \([0, \infty)\). We are interested in the limit circle (LC) case at infinity where all solutions are in \( L^2(\mathbb{R}_+) \) for all \( z \in \mathbb{C} \). Equation (1.1) is known as the Schrödinger equation if \( p(x) = 1 \), but we keep the same term in the general case. We write \(-q(x)\) because under our assumptions \( q(x) \to +\infty \) as \( x \to \infty \).

Our analysis relies on a construction of solutions \( f_z(x) \) of equation (1.1) distinguished by their asymptotics

\[f_z(x) = (p(x)q(x))^{-1/4} e^{i \Xi(x)} (1 + o(1)), \quad z \in \mathbb{C}, \ x \to \infty, \quad (1.2)\]

where

\[\Xi(x) = \int_{x_0}^{x} (q(y)/p(y))^{1/2} dy\]

\((x_0 \text{ is an arbitrary fixed number})\). Solutions \( f_z(x) \) are known as the Jost solutions of equation (1.1). Relation (1.2) shows that the leading terms of asymptotics of the Jost
solutions do not depend on \(z \in \mathbb{C}\). This fact is specific for the LC case. The functions \(f_z(x)\) and \(\bar{f}_z(x)\) satisfy the same equation (1.1) and are linearly independent so that an arbitrary solution of equation (1.1) is their linear combination. We require that
\[
\int_{x_0}^{\infty} (p(x)q(x))^{-1/2} dx < \infty
\]
whence \(f_z \in L^2(\mathbb{R}_+)\) for all \(z \in \mathbb{C}\). Thus, according to (1.2), we are in the LC case.

1.2. Limit point versus limit circle. The Weyl limit point/circle theory states that differential equation (1.1) always has a non-trivial solution in \(L^2(\mathbb{R}_+)\) for \(\text{Im}\ z \neq 0\). This solution is either unique (up to a constant factor) or all solutions of (1.1) belong to \(L^2(\mathbb{R}_+)\). The first instance is known as the limit point (LP) case and the second one – as the limit circle (LC) case. Consistent presentations of the Weyl theory can be found, for example, in the books [1], Chapter IX, [2], Chapter XIII and [6], Chapter X.1.

Our goal is to study self-adjoint operators in the space \(L^2(\mathbb{R}_+)\) associated with a differential operator
\[
(Hu)(x) = -(p(x)u'(x))' - q(x)u(x).
\]
The operator \(H\) defined on a domain \(C_0^\infty(\mathbb{R}_+)\) is symmetric in \(L^2(\mathbb{R}_+)\), but to make it self-adjoint, one has to add boundary conditions at \(x = 0\) and, eventually, for \(x \to \infty\). The boundary condition at the point \(x = 0\) looks as
\[
u'(0) = \alpha u(0) \quad \text{where} \quad \alpha = \bar{\alpha}.
\]
The value \(\alpha = \infty\) is not excluded. In this case (1.4) should be understood as the equality \(u(0) = 0\). We always require condition (1.4), fix \(\alpha\) and do not keep track of \(\alpha\) in notation.

Let us define a symmetric operator \(H_{00}\) by an equality \(H_{00}u = Hu\) on a domain \(\mathcal{D}(H_{00})\) consisting of smooth functions \(u(x)\) satisfying boundary condition (1.4) and such that \(u(x) = 0\) for sufficiently large \(x\). The operator \(H_{00}\) is essentially self-adjoint if and only if the LP case occurs (see, e.g., Theorem X.7 in the book [6]). In the LC case, \(H_{00}\) has a one parameter family \(H_\omega\) (here \(\omega\) is a point on the unit circle \(\mathbb{T} \subset \mathbb{C}\)) of self-adjoint extensions distinguished by some conditions for \(x \to \infty\). Their description can be performed in different terms. Our analysis relies on asymptotic formula (1.2). Alternative possibilities are briefly discussed in Sect. 4.5.

1.3. Plan of the paper. The existence of the Jost solutions is proven in Sect. 2. Another important element of our approach is a construction of an operator \(R(z)\) (we call it the quasiresolvent) playing the role of the resolvent for the maximal operator \(H_{\text{max}} := H_{00}^*\) (the adjoint of \(H_{00}\)). We emphasize that \(R(z)\) is an analytic function of \(z \in \mathbb{C}\). The operator \(R(z)\) is constructed in Sect. 3. This section is close to Sect. 2 of paper [12].
In Sect. 4 we show (see Theorem 4.1) that all functions $u \in \mathcal{D}(H_{\text{max}})$ have asymptotic behavior
\begin{equation}
  u(x) = \left( p(x) q(x) \right)^{-1/4} \left( s_+ e^{i \Xi(x)} + s_- e^{-i \Xi(x)} + o(1) \right), \quad x \to \infty,
\end{equation}
with some coefficients $s_\pm = s_\pm(u) \in \mathbb{C}$. We distinguish a set $\mathcal{D}(H_\omega) \subset \mathcal{D}(H_{\text{max}})$ by the condition
\begin{equation}
  s_+(u) = \omega s_-(u) \quad \text{where} \quad |\omega| = 1.
\end{equation}
Asymptotic coefficients $s_\pm(u)$ in formula (1.5) play the role of boundary values $u(0)$ and $u'(0)$ for functions $u$ in the local Sobolev space $H^2_{\text{loc}}$, and equality (1.6) plays the role of boundary condition (1.4).

Theorem 4.4 shows that the restriction $H_\omega$ of the operator $H_{\text{max}}$ on $\mathcal{D}(H_\omega)$ is self-adjoint, and all self-adjoint extensions of the operator $H_{\text{min}}$ coincide with one of the operators $H_\omega$. Our proofs of these results are independent of the von Neumann formulas. Finally, we construct the resolvents of the operators $H_\omega$ in Theorem 4.5.

The construction of this paper is similar to the approach used in Sect. 3 of [11] in the case of Jacobi operators. Conditions on coefficients look completely differently for Jacobi and Schrödinger operators, but, in both cases, we are in the LC case and the leading terms of asymptotics of the corresponding Jost solutions do not depend on the spectral parameter. This implies that spectral properties of these two classes of operators are similar.

2. The semiclassical Ansatz

We here construct solutions of the Schrödinger equation (1.1) with asymptotics (1.2) for $x \to \infty$.

2.1. Regular solutions. To avoid inessential technical complications, we always suppose that $p \in C^1(\mathbb{R}_+)$, $q \in C(\mathbb{R}_+)$ and the functions $p(x)$, $q(x)$ have finite limits as $x \to 0$. We assume that $p(x) > 0$ for $x \geq 0$. The solutions of equation (1.1) exist, belong to $C^2(\mathbb{R}_+)$ and they have limits $u(+0) =: u(0)$, $u'(+0) =: u'(0)$. A solution $u(x)$ is distinguished uniquely by boundary conditions $u(0) = u_0$, $u'(0) = u_1$.

Recall that for arbitrary solutions $u$ and $v$ of equation (1.1) their Wronskian
\begin{equation}
  \{ u, v \} := p(x) (u'(x)v(x) - u(x)v'(x))
\end{equation}
does not depend on $x \in \mathbb{R}_+$. Clearly, the Wronskian $\{ u, v \} = 0$ if and only if the solutions $u$ and $v$ are proportional.

We introduce a couple of regular solutions of equation (1.1) by boundary conditions at the point $x = 0$:
\begin{equation}
  \begin{cases}
    \varphi_z(0) = 1, & \varphi_z'(0) = \alpha, \\
    \theta_z(0) = 0, & \theta_z'(0) = -p(0)^{-1}, \quad \text{if} \quad \alpha \in \mathbb{R}
  \end{cases}
\end{equation}
and
\[
\begin{cases}
\varphi_z(0) = 0, & \varphi_z'(0) = 1, \\
\theta_z(0) = p(0)^{-1}, & \theta_z'(0) = 0,
\end{cases}
\quad \text{if } \alpha = \infty.
\] (2.2)

Obviously, \(\varphi_z(x)\) (but not \(\theta_z(x)\)) satisfy boundary condition (1.4) and the Wronskian \(\{\varphi_z, \theta_z\} = 1\).

In the LC case all solutions of equation (1.1) are in \(L^2(\mathbb{R}_+)\). In particular,
\[
\varphi_z \in L^2(\mathbb{R}_+), \quad \theta_z \in L^2(\mathbb{R}_+) \quad \text{for all } \ z \in \mathbb{C}.
\] (2.3)

2.2. Jost solutions. The Jost solutions \(f_z(x)\) of the differential equation (1.1) are distinguished by their asymptotics (1.2) for \(x \to \infty\). They will be constructed in Theorem 2.1. Let us set
\[
a(x) = \left( p(x)q(x) \right)^{-1/4}, \quad \xi(x) = \sqrt{\frac{q(x)}{p(x)}} \quad \text{and} \quad \Xi(x) = \int_{x_0}^{x} \xi(y)dy
\] (2.4)
so that
\[
p(x)\xi(x)a^2(x) = 1.
\] (2.5)
Notation (2.4) will be used throughout the whole paper.

**Theorem 2.1.** Suppose that
\[
a \in L^2(x_0, \infty) \quad \text{and} \quad a(pa')' \in L^1(x_0, \infty)
\] (2.6)
for some \(x_0 > 0\). Then for all \(z \in \mathbb{C}\), equation (1.1) has a solution \(f_z(x)\) with asymptotics
\[
f_z(x) = a(x)e^{i\Xi(x)}(1 + o(1))
\] (2.7)
as \(x \to \infty\). If, additionally,
\[
\lim_{x \to \infty} p(x)a'(x)a(x) = 0,
\] (2.8)
then
\[
f_z'(x) = i\xi(x)a(x)e^{i\Xi(x)}(1 + o(1)), \quad x \to \infty.
\] (2.9)

**Remark 2.2.** In the leading particular case \(p(x) = 1\) conditions (2.6) mean that
\[
\int_{x_0}^{\infty} q(x)^{-1/2}dx < \infty \quad \text{and} \quad \int_{x_0}^{\infty} q(x)^{-1/4}|(q(x)^{-1/4})''|dx < \infty.
\]
Condition (2.8) reduces to \(q(x)^{-3/2}q'(x) = o(1)\) for \(x \to \infty\).

In a detailed notation, formula (2.7) coincides with (1.2). We call \(f_z(x)\) the Jost solution. We emphasize that the leading term of asymptotics of \(f_z(x)\) does not depend on \(z\). This is specific for the LC case. Otherwise a proof of Theorem 2.1 is relatively standard (cf. [5], Chapter 6). It relies on the fact that the Liouville-Green Ansatz
\[
A(x) = a(x)e^{i\Xi(x)}
\] (2.10)
satisfies equation (1.1) with a sufficiently good accuracy.
We start a proof of Theorem 2.1 with a multiplicative change of variable $s$

$$f_z(x) = A(x)\psi_z(x).$$

(2.11)

The following result will be obtained by a direct computation.

**Lemma 2.3.** Set

$$\rho_z(x) = a(x)(p(x)a'(x))' + za^2(x).$$

(2.12)

Then the equation

$$(p(x)f_z'(x))' + q(x)f_z(x) + zf_z(x) = 0$$

(2.13)

is equivalent to an equation

$$\psi''_z(x) + \left(2i\xi(x) - \frac{\xi'(x)}{\xi(x)}\right)\psi'_z(x) + \rho_z(x)\xi(x)\psi_z(x) = 0$$

(2.14)

for the function $\psi_z(x)$ defined by formula (2.11).

**Proof.** Differentiating (2.11) twice, we find that

$$(pf_z')' = pA\psi''_z + (2pA' + p'A)\psi'_z + (pA')'\psi_z.$$ Substituting this expression into (2.13) and dividing by $pA$

$$\psi''_z + \left(\frac{2A'}{A} + \frac{p'}{p}\right)\psi'_z + \frac{1}{p}\left(\frac{(pA')'}{A} + q + z\right)\psi_z = 0.$$ (2.15)

By definitions (2.10) and (2.11) the coefficient at $\psi'_z$ here equals

$$2\frac{A'}{A} + \frac{p'}{p} = \frac{2a'}{a} + 2i\xi + \frac{p'}{p} = 2i\xi - \frac{\xi'}{\xi}.\quad (2.16)$$

Next, we compute the coefficient at $\psi_z$ in (2.15). Using (2.5), we see that

$$pA' = (pa' + ia^{-1})e^{i\xi}$$

and

$$(pA')' = (pa')' - a^{-1}\xi e^{i\xi}$$

whence

$$A^{-1}(pA')' + q = a^{-1}(pa')'.$$ (2.17)

Substituting now expressions (2.16) and (2.17) into (2.15), we obtain equation (2.14) with the coefficient

$$\rho_z(x) = (p\xi)^{-1}(a^{-1}(pa')' + z).$$

In view of (2.5) this coincides with definition (2.12). □

Next, we reduce the differential equation (2.14) to a Volterra integral equation

$$\psi_z(x) = 1 + (2i)^{-1}\int_{x_0}^x \left(1 - e^{-2i\xi(y)}e^{2i\xi(y)}\right)\rho_z(y)\psi_z(y)dy, \quad x \geq x_0.$$ (2.18)
Lemma 2.4. Let assumptions \(2.6\) be satisfied. Then equation \(2.18\) has a unique solution \(\psi_z(x)\). This solution satisfies equation \(2.14\) and
\[
\psi_z(x) \to 1, \quad \psi_z'(x) = o(\xi(x)) \quad \text{as} \quad x \to \infty.
\] \hfill (2.19)

Proof. Note that \(\rho_z \in L^1(x_0, \infty)\) according to conditions \(2.6\). Therefore a bounded solution \(\psi_z(x)\) of equation \(2.18\) can be standardly constructed by iterations.

Let us check that \(\psi_z(x)\) satisfies equation \(2.14\). Differentiating \(2.18\), we see that
\[
\psi_z'(x) = \xi(x) e^{-2i\Xi(x)} \int_x^\infty e^{2i\Xi(y)} \rho_z(y) \psi_z(y) dy
\] \hfill (2.20)

and
\[
\psi_z''(x) = -\xi(x) \rho_z(x) \psi_z(x) + (\xi'(x) - 2i\xi^2(x)) e^{-2i\Xi(x)} \int_x^\infty e^{2i\Xi(y)} \rho_z(y) \psi_z(y) dy.
\] \hfill (2.21)

Substituting expressions \(2.20\) and \(2.21\) into the left-hand side of \(2.14\), we see that it equals zero. Relations \(2.19\) are direct consequences of \(2.18\) and \(2.20\). \hfill □

Now it is easy to conclude the proof of Theorem 2.1.

Proof. Define the function \(f_z(x)\) by formula \(2.11\). According to Lemma 2.3 it satisfies differential equation \(1.1\), and according to Lemma 2.4 it has asymptotics \(2.7\). Moreover, differentiating \(2.11\), we obtain
\[
f_z'(x) = i \xi(x) f_z(x) + a(x) e^{i\Xi(x)} \psi_z'(x) + a'(x) e^{i\Xi(x)} \psi_z(x).
\] As we have already seen, the first term on the right has asymptotics \(2.9\). It follows from relations \(2.19\) that the second and third terms are \(o(\xi(x) a(x))\) and \(o(a'(x))\), respectively. So, it remains to observe that \(o(a'(x)) = o(p(x)^{-1} a(x)^{-1}) = o(\xi(x) a(x))\) according to condition \(2.8\) and identity \(2.5\). \hfill □

Theorem 2.1 defines the Jost solutions for \(x > x_0\). Then the functions \(f_z(x)\) are extended to all \(x \geq 0\) as solutions of the differential equation \(1.1\).

Let us introduce the second solution \(\bar{f}_z(x)\) of differential equation \(1.1\). It follows from asymptotics \(2.7\), \(2.9\) and equality \(2.5\) that the Wronskian
\[
\{f_z, \bar{f}_z\} = 2i \lim_{x \to \infty} (p(x) \xi(x) a^2(x)) = 2i
\] \hfill (2.22)

so that these solutions are linearly independent.

We also observe that a solution \(f_z(x)\) of equation \(1.1\) is determined uniquely by conditions \(2.7\) and \(2.9\). Indeed, if \(\bar{f}_z(x)\) is another solution of equation \(1.1\) satisfying these conditions, then the Wronskian \(\{f_z, \bar{f}_z\} = 0\) so that \(\bar{f}_z(x) = c f_z(x)\) for some \(c \in \mathbb{C}\). This constant equals 1 according again to \(2.7\).
Thus, the following result is a direct consequence of Theorem 2.1.

**Proposition 2.5.** Under the assumptions (2.6) and (2.8) equation (1.1) is in the LC case (at infinity).

This result is not really new; cf. Theorem XIII.6.20 in the book [2].

2.3. **Arbitrary solutions of the homogeneous equation.** An arbitrary solution of the Schrödinger equation (1.1) is a linear combination of the Jost solutions $f_z(x)$ and $\bar{f}_z(x)$. In particular, this is true for regular solutions $\varphi_z(x)$ and $\theta_z(x)$ distinguished by the boundary conditions (2.1) or (2.2):

$$\varphi_z(x) = \sigma_+(z)f_z(x) + \sigma_-(z)\bar{f}_z(x) \quad (2.23)$$

and

$$\theta_z(x) = \tau_+(z)f_z(x) + \tau_-(z)\bar{f}_z(x), \quad (2.24)$$

where the coefficients $\sigma_\pm(z)$ and $\tau_\pm(z)$ can be expressed via the Wronskians:

$$2i\sigma_+(z) = \{\varphi_z, \bar{f}_z\}, \quad 2i\sigma_-(z) = -\{\varphi_z, f_z\},$$

$$2i\tau_+(z) = \{\theta_z, \bar{f}_z\}, \quad 2i\tau_-(z) = -\{\theta_z, f_z\}. \quad (2.25)$$

Observe that $\sigma_-(z) = \overline{\sigma_+(\bar{z})}$ and $\tau_-(z) = \overline{\tau_+(\bar{z})}$ (2.26) because $\varphi_z(x) = \overline{\varphi_z(x)}$ and $\theta_z(x) = \overline{\theta_z(x)}$. Of course, all coefficients $\sigma_\pm(z)$ and $\tau_\pm(z)$ are entire functions of $z$.

According to (2.23) and (2.24) the following result is a direct consequence of Theorem 2.1.

**Theorem 2.6.** Under the assumptions of Theorem 2.1 the solutions $\varphi_z(x)$ and $\theta_z(x)$ of equation (1.1) have asymptotics

$$\varphi_z(x) = a(x)(\sigma_+(z)e^{i\Xi(x)} + \sigma_-(z)e^{-i\Xi(x)} + o(1)) \quad (2.27)$$

and

$$\theta_z(x) = a(x)(\tau_+(z)e^{i\Xi(x)} + \tau_-(z)e^{-i\Xi(x)} + o(1)) \quad (2.28)$$

as $x \to \infty$. These asymptotic formulas can be differentiated in $x$; in particular,

$$\varphi'_z(x) = i\xi(a(x)(\sigma_+(z)e^{i\Xi(x)} - \sigma_-(z)e^{-i\Xi(x)} + o(1)) \quad (2.29)$$

In view of conditions (2.1) or (2.2) the Wronskian $\{\varphi_z, \theta_z\} = 1$. On the other hand, we can calculate this Wronskian using relations (2.22) and (2.23), (2.24). This yields an identity

$$2i(\sigma_+(z)\tau_-(z) - \sigma_-(z)\tau_+(z)) = 1, \quad \forall z \in \mathbb{C}. \quad (2.30)$$

Below, we need also the following fact.
Proposition 2.7. Under the assumptions of Theorem 2.1 we have an identity
\[ |\sigma_-(z)|^2 - |\sigma_+(z)|^2 = \text{Im} \int_0^\infty |\varphi_z(x)|^2 \, dx. \] (2.31)

Proof. Multiplying equation (1.1) for \( \varphi_z(x) \) by \( \bar{\varphi}_z(x) \), integrating and taking the imaginary part, we see that
\[ -\text{Im} \int_0^x (p(y)\varphi'_z(y)) \varphi_z(y) \, dy = \text{Im} z \int_0^x |\varphi_z(y)|^2 \, dy. \] (2.32)

It follows from asymptotic formulas (2.27) and (2.29) that
\[
p(x)\varphi'_z(x)\bar{\varphi}_z(x) = i(|\sigma_+(z)|^2 - |\sigma_-(z)|^2) + i(\sigma_+(z)\bar{\sigma}_-(z)e^{2i\Xi(x)} - \sigma_-(z)\bar{\sigma}_+(z)e^{-2i\Xi(x)}) + o(1)
\] where equality (2.5) has been used. Let us take the imaginary part of this expression. Then the second term on the right disappears. Substituting this expression into (2.32) and passing to the limit \( x \to \infty \), we arrive at (2.31). \( \square \)

2.4. Conditions on the coefficients. Let us discuss the assumptions of Theorem 2.1. The main condition is \( a \in L^2(\mathbb{R}_+) \). It requires that the product \( p(x)q(x) \to \infty \) sufficiently rapidly, roughly speaking, faster than \( x^2 \). The second inclusion (2.6) as well as condition (2.8) play auxiliary roles. They mean that \( p(x) \) does not grow too rapidly compared to \( a(x) \) and exclude too wild oscillations of the functions \( p(x) \) and \( a(x) \). For example, for the functions \( p(x) = x^\beta, q(x) = x^\gamma \) (for large \( x \)) conditions (2.6) and (2.8) are satisfied if
\[ \beta + \gamma > 2 \quad \text{and} \quad \beta - \gamma < 2. \] (2.33)

This implies that, necessarily, \( \gamma > 0 \), but it may be an arbitrary small number. Observe that \( \beta \to 2 \) if \( \gamma \to 0 \).

It is noteworthy that (2.33) allows negative \( \beta \) provided \( \gamma > 2 + |\beta| \). In particular, according to Proposition 2.5 for such \( \beta \) and \( \gamma \) the operator \( H_{\text{min}} \) is in the LC case. Condition on \( \gamma \) is very important here. Indeed, the results of [10] show that if \( q(x) = 0 \) and \( p(x) \to 0 \) very rapidly, then the corresponding Schrödinger operator is self-adjoint, its spectrum is absolutely continuous and coincides with \([0, \infty)\).

3. Schrödinger operators and their quasiresolvents

We refer to the books [3], §17, and [6], Sect. X.1, for background information on the theory of symmetric differential operators.
3.1. Minimal and maximal operators. We here consider differential operators (1.3) in the space \( L^2(\mathbb{R}_+) \). The scalar product in this space is denoted \( \langle \cdot, \cdot \rangle \); \( I \) is the identity operator.

Let us first define a minimal operator \( H_{00} \) by the equality \( H_{00}u = H_u \) on domain \( \mathcal{D}(H_{00}) \) that consists of functions \( u \in C^2(\mathbb{R}_+) \) such that \( u(x) = 0 \) for sufficiently large \( x \), limits \( u(+0) =: u(0), u'(+0) =: u'(0) \) exist and condition (1.4) is satisfied. Thus, the boundary condition (1.4) at \( x = 0 \) is included in the definition of the operator \( H_{00} \) so that its self-adjoint extensions are determined by conditions for \( x \to \infty \) only.

The closure of \( H_{00} \) will be denoted \( H_{\min} \). This operator is symmetric in the space \( L^2(\mathbb{R}_+) \), and under assumptions of this paper its domain \( \mathcal{D}(H_{\min}) \) can be described efficiently (see Proposition 4.3). The adjoint operator \( H^*_{\min} =: H_{\max} \) is again given by the formula \( H_{\max}u = H_u \) on a set \( \mathcal{D}(H_{\max}) \) consisting of functions \( u \) in the local Sobolev space \( H^2_{\text{loc}} \), satisfying boundary condition (1.4) and such that \( u \in L^2(\mathbb{R}_+) \) and \( Hu \in L^2(\mathbb{R}_+) \). In the LC case, the operator \( H_{\max} \) is not symmetric. Integrating by parts, we see that for all \( u, v \in \mathcal{D}(H_{\max}) \)

\[
\langle Hu, v \rangle - \langle u, Hv \rangle = \lim_{x \to \infty} p(x)(u(x)\bar{v}'(x) - u'(x)\bar{v}(x))
\]

(3.1)

where the limit in the right-hand side exists but is not necessarily zero.

Recall that

\[ H_{\min} = H_{\min}^{**} = H^*_{\max}. \]

The operator \( H_{\min} \) is self-adjoint if and only if the LP case occurs. In this paper we are interested in the LC case when

\[ H_{\min} \neq H_{\max} = H^*_{\min}. \]

Self-adjoint extensions \( H \) of the operator \( H_{\min} \) satisfy the condition

\[ H_{\min} \subset H = H^* \subset H^*_{\min} =: H_{\max}. \]

In the LC case the operators \( H_{\max} \) are not symmetric.

Since the operator \( H_{\min} \) commutes with the complex conjugation, its deficiency indices

\[ d_\pm := \dim \ker(H_{\max} - zI), \quad \pm \text{Im } z > 0, \]

are equal, i.e. \( d_+ = d_- =: d \), and, so, \( H_{\min} \) admits self-adjoint extensions. For an arbitrary \( z \in \mathbb{C} \), all solutions of equation (1.1) with boundary condition (1.4) are given by the formula \( u(x) = \Gamma\varphi_z(x) \) for some \( \Gamma \in \mathbb{C} \). They belong to \( \mathcal{D}(H_{\max}) \) if and only if \( \varphi_z \in L^2(\mathbb{R}_+) \). Therefore \( d = 0 \) if \( \varphi_z \not\in L^2(\mathbb{R}_+) \) for \( \text{Im } z \neq 0 \); otherwise \( d = 1 \).
3.2. Quasiresolvent of the maximal operator. Recall that in the LC case inclusions (2.3) are satisfied. Following [12], let us define, for all \( z \in \mathbb{C} \), a bounded operator \( \mathcal{R}(z) \) in the space \( L^2(\mathbb{R}_+) \) by the equality
\[
(\mathcal{R}(z)h)(x) = \theta_z(x) \int_0^x \varphi_z(y)h(y)dy + \varphi_z(x) \int_x^\infty \theta_z(y)h(y)dy.
\]
(3.2)

Actually, the operator \( \mathcal{R}(z) \) belongs to the Hilbert-Schmidt class. It depends analytically on \( z \in \mathbb{C} \) and \( \mathcal{R}(z)^* = \mathcal{R}(\bar{z}) \). We prove (see Theorem 3.1) that, in a natural sense, \( \mathcal{R}(z) \) plays the role of the resolvent of the operator \( H_{\text{max}} \). We call it the quasiresolvent of the operator \( H_{\text{max}} \).

Let us enumerate some simple properties of the operator \( \mathcal{R}(z) \). Differentiating definition (3.2), we see that
\[
(\mathcal{R}(z)h)'(x) = \theta'_z(x) \int_0^x \varphi_z(y)h(y)dy + \varphi'_z(x) \int_x^\infty \theta_z(y)h(y)dy
\]
(3.3)
for all \( h \in L^2(\mathbb{R}_+) \). In particular, it follows from relations (3.2) and (3.3) that
\[
(\mathcal{R}(z)h)(0) = \varphi_z(0)\langle h, \theta_z \rangle
\]
(3.4) and
\[
(\mathcal{R}(z)h)'(0) = \varphi'_z(0)\langle h, \theta_z \rangle
\]
(3.5)
where \( \varphi_z(0) \) and \( \varphi'_z(0) \) are defined by equalities (2.1) or (2.2).

A proof of the following statement is close to the construction of the resolvent for essentially self-adjoint Schrödinger operators.

**Theorem 3.1.** Let inclusions (2.3) hold true. For all \( z \in \mathbb{C} \), we have
\[
\mathcal{R}(z) : L^2(\mathbb{R}_+) \to \mathcal{D}(H_{\text{max}})
\]
(3.6) and
\[
(H_{\text{max}} - zI)\mathcal{R}(z) = I.
\]
(3.7)

**Proof.** Let \( h \in L^2(\mathbb{R}_+) \) and \( u(x) = (\mathcal{R}(z)h)(x) \). Boundary condition (1.4) is a direct consequence of relations (3.1) and (3.3). Differentiating (3.3), we see that
\[
(p(x)u'(x))' = (p(x)\theta'_z(x))' \int_0^x \varphi_z(y)h(y)dy
\]
\[+ (p(x)\varphi'_z(x))' \int_x^\infty \theta_z(y)h(y)dy + p(x)(\theta'_z(x)\varphi_z(x) - \theta_z(x)\varphi'_z(x))h(x).
\]
(3.8)
Since the Wronskian \( \{\varphi_z, \theta_z\} = 1 \), the last term in the right-hand side equals \(-h(x)\).

Putting now together together equalities (3.2) and (3.8) and using equation (1.1) for the functions \( \varphi_z(x) \) and \( \theta_z(x) \), we obtain the equation
\[
-(p(x)u'(x))' + q(x)u(x) - zu(x) = h(x)
\]
where \( h \in L^2(\mathbb{R}_+) \). Together with boundary condition (1.4) this implies that \( H_{\text{max}}u - zu = h \). This yields both (3.6) and (3.7). \( \square \)
Note that solutions $u(x)$ of differential equation (1.1) satisfying condition (1.4) are given by the formula $u(x) = \Gamma \varphi_z(x)$ for some $\Gamma \in \mathbb{C}$. Therefore we can state

**Corollary 3.2.** All solutions of the equation

$$(H_{\text{max}} - zI)u = h \quad \text{where} \quad z \in \mathbb{C} \quad \text{and} \quad h \in L^2(\mathbb{R}_+)$$

for $u \in \mathcal{D}(H_{\text{max}})$ are given by the formula

$$u = \Gamma \varphi_z + R(z)h \quad \text{for some} \quad \Gamma = \Gamma(z; h) \in \mathbb{C}. \quad (3.9)$$

An asymptotic relation for $(R(z)h)(x)$ is a direct consequence of definition (3.2) and condition (2.3):

$$(R(z)h)(x) = \theta_z(x) \langle h, \varphi_z \rangle + o(|\varphi_z(x)| + |\theta_z(x)|) \quad \text{as} \quad x \to \infty. \quad (3.10)$$

### 4. Self-adjoint extensions and their resolvents

Here we find an asymptotic behavior as $|x| \to \infty$ of all functions $u(x)$ in the domain of the maximal operator $H_{\text{max}}$. This allows us to give an efficient description of all self-adjoint extensions of the operator $H_{\text{min}}$.

#### 4.1. Domains of maximal operators

Recall that boundary condition (1.4) at $x = 0$ is included in our definition of the minimal operator $H_{\text{min}}$. Our goal now is to find a similar condition for $x \to \infty$ distinguishing self-adjoint extensions of $H_{\text{min}}$.

The starting point of our construction is asymptotic formula (1.5) for functions $u \in H_{\text{max}}$. Recall that the amplitude $a(x)$ and the phase $\Xi(x)$ were defined by formulas (2.4). The coefficients $\sigma_\pm(z)$, $\tau_\pm(z)$ are given by equalities (2.25), and the number $\Gamma(z; h)$ is determined by formula (3.9).

**Theorem 4.1.** Let the assumptions of Theorem 2.1 be satisfied. Then an arbitrary function $u \in \mathcal{D}(H_{\text{max}})$ has asymptotics (1.5) with some coefficients $s_\pm = s_\pm(u)$. They can be constructed by relations

$$s_+(u) = \Gamma(z; (H - zI)u)\sigma_+(z) + \langle (H - zI)u, \varphi_z \rangle \tau_+(z),$$

$$s_-(u) = \Gamma(z; (H - zI)u)\sigma_-(z) + \langle (H - zI)u, \varphi_z \rangle \tau_-(z) \quad (4.1)$$

where the number $z \in \mathbb{C}$ is arbitrary.

Conversely, for arbitrary $s_+, s_- \in \mathbb{C}$, there exists a function $u \in \mathcal{D}(H_{\text{max}})$ such that asymptotics (1.5) holds.

**Proof.** According to Corollary 3.2 a function $u \in \mathcal{D}(H_{\text{max}})$ admits representation (3.9) where the operator $R(z)$ is defined by equality (3.2). In view of relation (3.10) and asymptotics (2.27), (2.28) we have

$$(R(z)h)(x) = a(x)\left(\tau_+(z)e^{i\Xi(x)} + \tau_-(z)e^{-i\Xi(x)}\right)\langle h, \varphi_z \rangle + o(a(x)), \quad x \to \infty, \quad (4.2)$$
for all functions $h \in L^2(\mathbb{R}_+)$. Therefore it follows from (3.9) that
\[
u(x) = a(x)\Gamma(z; (H - zI)u)\left(\sigma_+(z)e^{i\Xi(x)} + \sigma_-(z)e^{-i\Xi(x)}\right) \\
+ a(x)\left(\tau_+(z)e^{i\Xi(x)} + \tau_-(z)e^{-i\Xi(x)}\right)((H - zI)u, \varphi_\varepsilon) + o(a(x))
\]
as $x \to \infty$. This yields relation (1.5) with the coefficients $s_\pm$ defined by (4.1).

Conversely, given $s_+$ and $s_-$ and fixing some $z \in \mathbb{C}$, we consider a system of equations
\[
s_+ = \Gamma\sigma_+(z) + \langle h, \varphi_\varepsilon \rangle \tau_+(z), \\
s_- = \Gamma\sigma_-(z) + \langle h, \varphi_\varepsilon \rangle \tau_-(z).
\]
for $\Gamma$ and $\langle h, \varphi_\varepsilon \rangle$. According to (2.30) the determinant of this system is not zero so that $\Gamma$ and $\langle h, \varphi_\varepsilon \rangle$ are uniquely determined by $s_+$ and $s_-$. Then we take any $h$ such that its scalar product with $\varphi_\varepsilon$ equals the found value of $\langle h, \varphi_\varepsilon \rangle$. Finally, we define $u$ by formula (3.9). Asymptotics as $x \to \infty$ of $\varphi_\varepsilon(x)$ and $\langle \mathcal{R}(z)h \rangle(x)$ are given by formulas (2.27) and (1.2), respectively. In view of equations (4.3) this leads to asymptotics (1.5).

Theorem 4.1 yields a mapping $\mathcal{D}(H_{\text{max}}) \to \mathbb{C}^2$ defined by the formula
\[
u \mapsto (s_+(u), s_-(u)).
\]
The construction of Theorem 4.1 depends on the choice of $z \in \mathbb{C}$, but this mapping is defined intrinsically. In particular, we can set $z = 0$ in all formulas of Theorem 4.1.

Note that mapping (4.4) is surjective. We also observe that (4.4) plays the role of a mapping $u \mapsto (u(0), u'(0))$ for functions in the local Sobolev class $H^2_{\text{loc}}$.

Under the assumptions of Theorem 2.1 the right-hand side of (3.1) can be expressed in terms of the coefficients $s_+$ and $s_-$. 

**Proposition 4.2.** For all $u, v \in \mathcal{D}(H_{\text{max}})$, we have an identity
\[
\langle H_{\text{max}}u, v \rangle - \langle u, H_{\text{max}}v \rangle = 2i\left(s_+(u)s_+(v) - s_-(u)s_-(v)\right).
\]

**Proof.** Let us proceed from equality (3.1). It follows from formula (1.5) that
\[-ip(x)u'(x)\ddot{v}(x) = \left(s_+(u)e^{i\Xi(x)} - s_-(u)e^{-i\Xi(x)}\right)\left(s_+(v)e^{-i\Xi(x)} + s_-(v)e^{i\Xi(x)}\right) + o(1) \\
= s_+(u)s_+(v) - s_-(u)s_-(v) + s_+(u)s_+(v)e^{2i\Xi(x)} - s_-(u)s_+(v)e^{-2i\Xi(x)} + o(1),
\]
and, similarly,
\[ip(x)u(x)\ddot{v}(x) = \left(s_+(u)e^{i\Xi(x)} + s_-(u)e^{-i\Xi(x)}\right)\left(s_+(v)e^{-i\Xi(x)} - s_-(v)e^{i\Xi(x)}\right) + o(1) \\
= s_+(u)s_+(v) - s_-(u)s_-(v) + s_+(u)s_+(v)e^{2i\Xi(x)} + s_-(u)s_+(v)e^{-2i\Xi(x)} + o(1).
\]
Let us take the sum of the last two expressions and observe that the terms containing $e^{2i\Xi(x)}$ and $e^{-2i\Xi(x)}$ cancel each other. This yields
\[-ip(x)(u'(x)\ddot{v}(x) - u(x)\ddot{v}'(x)) = 2s_+(u)s_+(v) - 2s_-(u)s_-(v) + o(1).
\]
Passing here to the limit \( n \to \infty \) and using equality (3.1), we obtain identity (4.5). □

We can now characterize the set \( \mathcal{D}(H_{\min}) \).

**Proposition 4.3.** A vector \( v \in \mathcal{D}(H_{\max}) \) belongs to \( \mathcal{D}(H_{\min}) \) if and only if \( v(x) = o(a(x)) \) as \( x \to \infty \), that is,

\[
 s_+(v) = s_-(v) = 0. \tag{4.6}
\]

**Proof.** A vector \( v \) belongs to \( \mathcal{D}(H_{\max}^*) \) if and only if

\[
 \langle H_{\max}u, v \rangle = \langle u, H_{\max}v \rangle \tag{4.7}
\]

for all \( u \in \mathcal{D}(H_{\max}) \). According to Proposition 4.2 equality (4.7) is equivalent to

\[
 s_+(u)s_+(v) - s_-(u)s_-(v) = 0. \tag{4.8}
\]

This is of course true if (4.6) is satisfied. Conversely, if (4.8) is satisfied for all \( u \in \mathcal{D}(H_{\max}) \), we use that according to Theorem 4.1 the numbers \( s_+(u) \) and \( s_-(u) \) are arbitrary. This implies (4.6). □

This result shows that (4.4) considered as a mapping of the factor space \( \mathcal{D}(H_{\max})/\mathcal{D}(H_{\min}) \) onto \( \mathbb{C}^2 \) is injective.

### 4.2. Self-adjoint extensions

All self-adjoint extensions \( H_\omega \) of the operator \( H_{\min} \) are parametrized by complex numbers \( \omega \in \mathbb{T} \subset \mathbb{C} \). Let a set \( \mathcal{D}(H_\omega) \subset \mathcal{D}(H_{\max}) \) of vectors \( u \) be distinguished by condition (1.6).

**Theorem 4.4.** Let the assumptions of Theorem 2.1 be satisfied. Then for all \( \omega \in \mathbb{T} \), the operators \( H_\omega \) are self-adjoint. Conversely, every operator \( H \) such that

\[
 H_{\min} \subset H = H^* \subset H_{\max} \tag{4.9}
\]

equals \( H_\omega \) for some \( \omega \in \mathbb{T} \).

**Proof.** We proceed from Proposition 4.2. If \( u, v \in \mathcal{D}(H_\omega) \), it follows from condition (1.6) that \( s_+(u)s_+(v) = s_-(u)s_-(v) \). Therefore according to equality (4.3)

\[
 \langle H_\omega u, v \rangle = \langle u, H_\omega v \rangle \text{ whence } H_\omega \subset H_{\omega}^*. \]

If \( v \in \mathcal{D}(H_{\omega}^*) \), then \( \langle H_\omega u, v \rangle = \langle u, H_{\omega}^* v \rangle \] for all \( u \in \mathcal{D}(H_\omega) \) so that in view of (4.5) equality (4.8) is satisfied. Therefore \( s_-(u)(\omega s_+(v) - s_-(v)) = 0 \). Since \( s_-(u) \) is arbitrary, we see that \( \omega s_+(v) - s_-(v) = 0 \), and hence \( v \in \mathcal{D}(H_\omega) \).

Suppose that an operator \( H \) satisfies conditions (4.9). Since \( H \) is symmetric, it follows from Proposition 4.2 that equality (4.8) is true for all \( u, v \in \mathcal{D}(H) \). Setting here \( u = v \), we see that \( |s_+(v)| = |s_-(v)| \). There exists a vector \( v_0 \in \mathcal{D}(H) \) such that \( s_-(v_0) \neq 0 \) because \( H \neq H_{\min} \). Let us set \( \omega = s_+(v_0)/s_-(v_0) \). Then \( |\omega| = 1 \) and relation (1.6) is a direct consequence of (4.8). □
4.3. **Resolvent.** Now it is easy to construct the resolvent of the operator $H_\omega$ defined in the previous subsection. We previously note that, by definition (1.5),

$$s_\pm(\varphi_z) = \sigma_\pm(z), \quad s_\pm(\varphi_z) = \tau_\pm(z)$$

and $|\sigma_+(z)| \neq |\sigma_-(z)|$ for $\text{Im } z \neq 0$, by Proposition 2.7.

**Theorem 4.5.** Let the assumptions of Theorem 2.1 be satisfied. Then for all $z \in \mathbb{C}$ with $\text{Im } z \neq 0$ and all $h \in L^2(\mathbb{R}_+)$, the resolvent $R_\omega(z) = (H_\omega - zI)^{-1}$ of the operator $H_\omega$ is given by the equality

$$R_\omega(z)h = \gamma_\omega(z)\langle h, \varphi_z \rangle \varphi_z + R(z)h$$  \hspace{1cm} (4.10)

where

$$\gamma_\omega(z) = -\frac{\tau_+(z) - \omega \tau_-(z)}{\sigma_+(z) - \omega \sigma_-(z)}.$$  \hspace{1cm} (4.11)

**Proof.** According to Corollary 3.2 a vector $u = R_\omega(z)h$ is given by formula (3.9) where the coefficient $\Gamma$ is determined by condition (1.6). It follows from Theorem 4.1 that the function $u(x)$ has asymptotics (1.5) with the coefficients $s_\pm$ defined by relations (4.3). Thus, $u \in D(H_\omega)$ if and only if

$$\Gamma \sigma_+(z) + \tau_+(z)\langle h, \varphi_z \rangle = \omega (\Gamma \sigma_-(z) + \tau_-(z)\langle h, \varphi_z \rangle)$$

whence

$$\Gamma = -\frac{\tau_+(z) - \omega \tau_-(z)}{\sigma_+(z) - \omega \sigma_-(z)} \langle h, \varphi_z \rangle.$$  \hspace{1cm} (4.11)

Substituting this expression into (3.9), we arrive at formulas (4.10), (4.11). \hspace{1cm} \square

**Corollary 4.6.** The resolvents $R_\omega(z)$ belong to the Hilbert-Schmidt class if $\text{Im } z \neq 0$, whence the spectra of the operators $H_\omega$ are discrete.

This result is well known; see, e.g., Theorem 10 in Chapter VII of the book [3] or Theorem 5.8 in the book [9].

It follows from formula (4.11) that the spectra of the operator $H_\omega$ consist of the points $z$ where

$$\sigma_+(z) - \omega \sigma_-(z) = 0.$$  \hspace{1cm} (4.12)

Since the functions $\sigma_+(z)$ and $\sigma_-(z)$ are analytic, this again implies that the spectra of $H_\omega$ are discrete. Of course the roots $z$ of equation (4.12) lie on the real axis because $|\sigma_+(z)| \neq |\sigma_-(z)|$ for $\text{Im } z \neq 0$. This fact had of course to be expected since $z$ are eigenvalues of the self-adjoint operator $H_\omega$. We finally note that the discreteness of the spectrum of the operators $H_\omega$ is quite natural because their domains $D(H_\omega)$ are distinguished by boundary conditions at both ends of $\mathbb{R}_+$. Therefore $H_\omega$ acquire some features of regular operators.
4.4. Spectral measure. In view of the spectral theorem, Theorem 4.5 yields a representation for the Cauchy-Stieltjes transform of the spectral measure \( dE_\omega(\lambda) \) of the operator \( H_\omega \).

**Theorem 4.7.** Let inclusions \((2.3)\) hold. Then for all \( z \in \mathbb{C} \) with \( \text{Im} \, z \neq 0 \) and all \( h \in L^2(\mathbb{R}_+) \), we have an equality

\[
\int_{-\infty}^{\infty} (\lambda - z)^{-1} d(E_\omega(\lambda)h,h) = \gamma_\omega(z)|\langle \varphi_z,h \rangle|^2 + (\mathcal{R}(z)h,h). \tag{4.13}
\]

Recall that the operators \( \mathcal{R}(z) \) are defined by formula \((3.2)\). Therefore \( (\mathcal{R}(z)h,h) \) are entire functions of \( z \in \mathbb{C} \), and the singularities of the integral in \((1.13)\) are determined by the function \( \gamma_\omega(z) \). Thus, \((1.13)\) can be considered as a modification of the classical Nevanlinna formula (see his original paper [4] or, for example, formula (7.6) in the book [7]) for the Cauchy-Stieltjes transform of the spectral measure in the theory of Jacobi operators. We mention however that, for Jacobi operators acting in the space \( L^2(\mathbb{Z}_+) \), there is the canonical choice of a generating vector and of a spectral measure. This is not the case for differential operators in \( L^2(\mathbb{R}_+) \).

We finally note an obvious fact: if \( \lambda \) is an eigenvalue of an operator \( H_\omega \), then corresponding eigenfunctions equal \( c\varphi_\lambda(x) \) where \( c \in \mathbb{C} \).

4.5. Concluding remarks. In the LC case, self-adjoint extensions of the operator \( H_{\text{min}} \) are traditionally described by the following procedure; see the classical books [1], Chapter IX, [3], §17, 18, and the recent monograph [7], Sect. 14.4, 15.3 and 15.4. Take any real functions \( \varrho_j \in D(H_{\text{max}}), \ j = 1, 2, \) such that

\[
\lim_{x \to \infty} p(x)(\varrho'_1(x)\varrho_2(x) - \varrho_1(x)\varrho'_2(x)) = 0,
\]

and set \( \kappa_s(x) = s\varrho_1(x) + \varrho_2(x) \) where \( s \in \mathbb{R} \), \( \kappa_\infty(x) = \varrho_1(x) \). Let a set \( D_s \subset D(H_{\text{max}}) \) be distinguished by the condition

\[
\lim_{x \to \infty} p(x)(u'(x)\kappa_s(x) - u(x)\kappa'_s(x)) = 0 \quad \text{for} \quad u \in D_s,
\]

and let \( \tilde{H}_s \) be the restriction of \( H_{\text{max}} \) on the set \( D_s =: D(\tilde{H}_s) \). Then the operators \( \tilde{H}_s \) are self-adjoint, and all self-adjoint extensions of the operator \( H_{\text{min}} \) coincide with one of the operators \( \tilde{H}_s \) for some \( s \in \mathbb{R} \cup \{\infty\} \). This construction does not look very efficient, in particular, because it depends on the choice of the functions \( \varrho_1, \varrho_2 \).

Another possibility is to use von Neumann formulas. They were conveniently adapted to operators commuting with the complex conjugation in the survey [8], Theorem 2.6, and then applied in this paper to Jacobi operators (see also Sect. 16.3 in [7]). Following [12], we briefly describe here this construction for Schrödinger operators \((1.3)\). Recall that \( \varphi_0(x) \) and \( \theta_0(x) \) are the solutions of equation \((1.1)\) where \( z = 0 \) satisfying conditions \((2.1)\) or \((2.2)\). We set \( \tilde{\theta}_0(x) = \omega(x)\theta_0(x) \) where \( \omega(x) \) is a smooth function such that \( \omega(x) = 0 \) for small \( x \) and \( \omega(x) = 1 \) for large \( x \).
Define operators $H^{(t)}$ as the restrictions of $H_{\text{max}}$ on direct sums
\[ D(H_{\text{min}}) + \{ t\varphi_0 + \tilde{\theta}_0 \} =: D(H^{(t)}) \quad \text{for} \quad t \in \mathbb{R} \quad (4.14) \]
and $D(H_{\text{min}}) + \{ \varphi_0 \} =: D(H^{(\infty)})$. Then the operators $H^{(t)}$ are self-adjoint, and all self-adjoint extensions of the operator $H_{\text{min}}$ coincide with one of the operators $H^{(t)}$ for some $t \in \mathbb{R} \cup \{ \infty \}$. A drawback of this construction is that, apparently, the set $D(H_{\text{min}})$ cannot be described efficiently without some assumptions on the coefficients $p(x)$ and $q(x)$.

Finally, we indicate a link between the operators $H^{(t)}$ and $H_\omega$ considered in this paper. Let $u \in D(H^{(t)})$ where $t \in \mathbb{R}$. According to Theorem 2.6 and Proposition 4.3 it follows from definition (4.14) that $u(x)$ has asymptotics (1.5) where
\[ s_+(u) = t\sigma_+(0) + \tau_+(0) \quad \text{and} \quad s_-(u) = t\sigma_-(0) + \tau_-(0). \]
Therefore relation (1.6) is satisfied with
\[ \omega = \frac{t\sigma_+(0) + \tau_+(0)}{t\sigma_-(0) + \tau_-(0)}. \quad (4.15) \]
If $u \in D(H^{(\infty)})$, then this equality holds true with $t = \infty$, that is, $\omega = \sigma_+(0)\sigma_-(0)^{-1}$. Note that $|\omega| = 1$ by virtue of identity (2.26). This shows that $u \in D(H_\omega)$; see the definition at the beginning of Sect. 4.2. Conversely, if $u \in D(H_\omega)$, then $u \in D(H^{(t)})$ with $t$ determined by (4.15).

REFERENCES

[1] E. A. Coddington and N. Levinson, Theory of ordinary differential equations, McGraw-Hill, New York, 1955.
[2] N. Dunford and J. T. Schwartz, Linear operators, part 2, Interscience Publishers, New York, London, Sydney, 1963.
[3] M. A. Naimark, Linear differential operators, Ungar, New York, 1968.
[4] R. Nevanlinna, Asymptotische Entwickelungen beschränkter Funktionen und das Stieltjesche Momentenproblem, Ann. Acad. Sci. Fenn. A 18, No. 5 (1922), 52 pp.
[5] F. W. J. Olver, Introduction to asymptotics and special functions, Academic Press, 1974.
[6] M. Reed and B. Simon, Methods of Modern Mathematical Physics II, Academic Press, 1975.
[7] K. Schmüdgen, Unbounded self-adjoint operators on Hilbert space, Springer, Dordrecht, Heidelberg, New York, London, 2012.
[8] B. Simon, The classical moment problem as a self-adjoint finite difference operator, Advances in Math. 137 (1998), 82-203.
[9] E. C. Titchmarsh, Eigenfunction expansions associated with second-order differential equations, vol. 1, Oxford, 1946.
[10] D. R. Yafaev, Spectral and scattering theory for differential and Hankel operators, Advances in Math. 308 (2017), 713-766.
[11] D. R. Yafaev, Self-adjoint Jacobi operators in the limit circle case, arXiv 2104.13609.
[12] D. R. Yafaev, Self-adjoint differential operators in the limit circle case, arXiv 2105.08641.
Univ Rennes, CNRS, IRMAR-UMR 6625, F-35000 Rennes, France, SPGU, Univ. Nab. 7/9, Saint Petersburg, 199034 Russia, and NTU Sirius, Olympiysky av. 1, Sochi, 354340 Russia

Email address: yafaev@univ-rennes1.fr