Egyptian Fractions with odd denominators

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Abstract

The number of solutions of the diophantine equation $\sum_{i=1}^{k} \frac{1}{x_i} = 1$, in particular when the $x_i$ are distinct odd positive integers is investigated. The number of solutions $S(k)$ in this case is, for odd $k$:

$$\exp \left( \exp \left( c_1 \frac{k}{\log k} \right) \right) \leq S(k) \leq \exp \left( \exp \left( c_2 k \right) \right)$$

with some positive constants $c_1$ and $c_2$. This improves upon an earlier lower bound of $S(k) \geq \exp \left( (1 + o(1)) \frac{\log 2 k^2}{2} \right)$.

1 Introduction

In this paper we study the number of solutions of the diophantine equation

$$(1.1) \quad \sum_{i=1}^{k} \frac{1}{x_i} = 1,$$

in particular, where the $x_i$ have some restrictions, such as all $x_i$ are distinct odd positive integers. Let us first review what is known for distinct positive

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integers, without further restriction: Let
\[
X_k = \{(x_1, x_2, \ldots, x_k) : \sum_{i=1}^{k} \frac{1}{x_i} = 1, \ 0 < x_1 < x_2 < \cdots < x_k\}.
\]
It is known that
\[
(1.2) \quad \exp \left( \exp \left( ((\log 2)(\log 3) + o(1)) \frac{k}{\log k} \right) \right) \leq |X_k| \leq c_0^{(\frac{3}{4}+\varepsilon)2^{k-3}},
\]
where \(c_0 = 1.264 \ldots\) is \(\lim_{n \to \infty} u_n^{1/2^n}\), \(u_n = 1, u_{n+1} = u_n(u_n + 1)\).

The lower bound is due to Konyagin [12], the upper bound due to Browning and Elsholtz [3]. Earlier results on the upper and lower bounds were due to Sándor [13] and Erdős, Graham and Straus (see [9], page 32).

The set of solutions has also been investigated with various restrictions on the variables \(x_i\). A quite general and systematic investigation of expansions of \(a/b\) as a sum of unit fractions with restricted denominators is due to Graham [10]. Elsholtz, Heuberger, Prodinger [7] gave an asymptotic formula for the number of solutions of (1.1), with two main terms, when the \(x_i\) are (not necessarily distinct) powers of a fixed integer \(t\).

Another prominent case is when all denominators \(x_i\) are odd. Sierpiński [16] proved that a nontrivial solution exists. It is known that for \(k = 9\) there are exactly 5 solutions, and for \(k = 11\), there are exactly 379,118 solutions (see [15, 2]). Chen, Elsholtz and Jiang [4] showed that for odd denominators \(x_i\) the number of solutions of (1.1) is increasing with a lower bound of \(\sqrt{2^{2k^2(1+o(1))}}\). Other types of restrictions on the denominator have been studied, e.g. by Croot [5] and Martin [11]. The number of solutions of the equation \(mn = \sum_{i=1}^{k} \frac{1}{x_i}\) have also been estimated by Elsholtz and Tao [8].

In this paper we take inspiration from the proof of Chen et al. [4] for odd denominators, and the proof of Konyagin [12] for lower bounds in the case of unrestricted \(x_i\). As Konyagin’s proof makes crucial use of ingenious identities, involving a lot of even numbers, it seems unclear whether one can generalize it to odd integers. Here is our main result:

**Theorem 1.1.** Let \(s \geq 1\) and let \(\{p_1, \ldots, p_s\}\) denote a set of primes, and let \(P = p_1 \cdots p_s\) be squarefree. Let \(k\) be sufficiently large. Moreover, if \(P\) is
even, let \( k \) be odd. Let

\[ X_{k,P} = \{ (x_1, x_2, \ldots, x_k) : \sum_{i=1}^{k} \frac{1}{x_i} = 1, \text{ with distinct positive } x_i \equiv \pm 1 \mod P \} . \]

There is some positive constant \( c(P) \) such that the following holds:

\[ |X_{k,P}| \geq \exp \left( \exp \left( c(P) \frac{k}{\log k} \right) \right) . \]

The case \( P = 2 \) is the case of odd denominators:

**Corollary 1.2.** Let \( k \) be odd, and

\[ X_{k,\text{odd}} = \{ (x_1, x_2, \ldots, x_k) : \sum_{i=1}^{k} \frac{1}{x_i} = 1, \text{ with odd distinct positive } x_i \} . \]

There is some positive constant \( c \) such that the following holds:

\[ |X_{k,\text{odd}}| \geq \exp \left( \exp \left( c \frac{k}{\log k} \right) \right) . \]

For comparison, an upper bound of type \( \exp (\exp (c_2 k)) \) follows from the unrestricted case, see (1.2).

## 2 Proof

**Lemma 2.1.** Let \( P > 1 \) be a squarefree integer. Let \( \omega(n) \) denote the number of distinct prime factors of \( n \), and \( d(m) \) the number of divisors of \( n \). The following holds: \( \omega(P^m - 1) \geq d(m) - 6 \).

**Proof.** Due to a result of Bang, Zsigmondy, Birkhoff and Vandiver (see e.g. Schinzel [14]), it is known that for \( n > 6 \) the values of \( P^n - 1 \) have at least one primitive prime factor. (A prime factor of the sequence \( P^n - 1 \) is primitive if it divides \( P^n - 1 \), but does not divide any \( P^m - 1 \) with \( m < n \).)

Let \( m = m_1 m_2 \). For each divisor \( m_1 \) one has the factorization

\[ P^m - 1 = (P^{m_1} - 1)(P^{m_1 m_2 - m_1} + P^{m_1 m_2 - 2m_1} + \cdots + P^{m_1} + 1) , \]

hence the number of prime factors of \( P^m - 1 \) is at least the sum of the number of primitive prime factors of \( P^{m_1} - 1 \), for all possible divisors \( m_1 \) of \( m \). \( \square \)
Lemma 2.2. For $X \geq 3$, there exists a natural number $m < X$ such that $d(m) > \exp\left((\ln 2 + o(1)) \frac{\ln X}{\ln \ln X}\right)$ as $X \to \infty$.

This follows from a theorem of Wigert [17], but can also be seen directly. Let $P_r = \prod_{i=1}^r q_i$ be the product over the first primes, and choose $m = P_r$ if $P_r \leq X < P_{r+1}$. Then $d(m) = 2^r = \exp\left((\ln 2 + o(1)) \frac{\ln m}{\ln \ln m}\right) = \exp\left((\ln 2 + o(1)) \frac{\ln X}{\ln \ln X}\right)$. Taking the first $r$ odd primes, one can also find an odd number $m$ of this type.

Lemma 2.3. For every $a, b, n_0 \in \mathbb{N}$ the following holds: every positive integer can be written as a finite sum of distinct fractions of the form $\frac{1}{an+b}$, $n \geq n_0$.

This result with $n_0 = 0$ was originally proved by van Albada and van Lint [1]. The result for general $n_0$ easily follows by using the progression $a'n + b' = an + (an_0 + b), n \geq 0$.

As an easy consequence we have:

Lemma 2.4. There exist distinct positive integers $l_1, \ldots, l_{r_1}, m_1, \ldots, m_{r_2}, n_1, \ldots, n_{r_3}$, all larger than 1, in the residue class 1 mod $3P(P^2-1)$ such that the following holds:

$$\sum_{i=1}^{r_1} \frac{1}{l_i} = P - 2, \quad \sum_{i=1}^{r_2} \frac{1}{m_i} = 1, \quad \sum_{i=1}^{r_3} \frac{1}{n_i} = P,$$

If $P = 2$, then $r_1 = 0$, otherwise $r_1, r_2, r_3 > 0$. Moreover, it is clear that $r_2 \equiv 1 \mod P$.

Proof of Theorem. The idea employed in [4] and [12] is to write 1 as a sum of fractions where one denominator has a large number of divisors, and to split this fraction recursively into several fractions, where (at least) one of these has again a large number of divisors.

Here we show that it is possible to have, for any given $t \in \mathbb{N}$, the fraction $\frac{1}{P^t-1}$ as one of these fractions. Let us start with the trivial decomposition

$$1 = \frac{1}{P-1} + \frac{P-2}{P-1}.$$
In order to avoid that the denominator \( P - 1 \) occurs more than once we use Lemma 2.4 to write the integer \( P - 2 \) as a sum of distinct unit fractions, with \( l_i > 1 \): 
\[
P - 2 = \sum_{i=1}^{r_1} \frac{1}{l_i}.
\]

Next we observe that any fraction \( \frac{1}{P^{n-1}} \) can be decomposed to obtain a sum of unit fractions containing a) \( \frac{1}{P^{2n-1}} \) or b) \( \frac{1}{P^{n+1}-1} \).

\[
(a) \quad \frac{1}{P^n - 1} = \frac{1}{P^n + 1} + \frac{1}{P^{2n} - 1} + \sum_{i=1}^{r_2} \frac{1}{(P^{2n} - 1)m_i}.
\]

By Lemma 2.4
\[
1 = \sum_{i=1}^{r_2} \frac{1}{m_i}, \quad m_i \equiv 1 \mod 3P(P^2 - 1), m_i > 1 \text{ and distinct.}
\]

Note that all occurring denominators are distinct, with the possible exception that \( P^n + 1 = P^{2n} - 1 \) holds if \( P = 2, n = 1 \). In this case, one rewrites \( \frac{1}{P+1} = \frac{1}{3} = \sum_{i=1}^{r_2} \frac{1}{3m_i} \). These denominators have not been used before, as the \( l_i \) or \( m_i \) are congruent to 1 mod 3, whereas the new denominators \( 3m_i \) are not.

\[
(b) \quad \frac{1}{P^n - 1} = \frac{1}{P^{n+1} - 1} + \frac{P - 1}{(P^{n+1} - 1)(P^n - 1)} + \frac{P - 1}{(P^{n+1} - 1)}.
\]

Note that these three fractions are unit fractions, as the denominators are divisible by \( P - 1 \). These three fractions are distinct, unless \( n = 1 \). In this case the fraction \( \frac{1}{P^{2n-1}} \) occurs twice and one of these is rewritten as \( \frac{1}{P^{2n-1}} = \sum_{i=1}^{r_2} \frac{1}{(P^{2n} - 1)m_i} \). These denominators have not been used before, as the previous denominators \( l_i \) and \( m_i \) were by construction congruent to 1 mod \( P^2 - 1 \). Also, \( P^n + 1, P^{2n} - 1, (P^{2n} - 1)m_i \) are new.

For constructing a solution with \( \frac{1}{P^{t-1}} \) we write \( t \) in binary. The first binary digit is of course 1. For the positions \( i \geq 2 \) we perform two different types of steps, corresponding to (a) and (b) above:

1) If the \( i \)-th leading position is a 0, then we take the “doubling” a).
2) If the \( i \)-th leading position is a 1, then we first take the doubling a), followed by an “addition” b),
For example, if \( t = 53 = 110101_2 \) and starting from left to right:

\[
\begin{array}{cccccc}
  i = 1 | & 2 | & 3 | & 4 | & 5 | & 6 | \\
  1 | & 1 | & 0 | & 1 | & 0 | & 1 \\
  | & a & b | & a & | & a & b | & a & | & a & b \\
 n = 1 | & 2 & 3 & 6 & 12 & 13 & 26 & 52 & 53
\end{array}
\]

Generally, any integer \( t \) can be obtained in at most \( \frac{\log t}{\log 2} \) such steps a) or b). In other words, starting from \( n = 1 \) we can obtain a decomposition

\[
1 = \frac{1}{P^t - 1} + \sum_{i=1}^{k'-1} \frac{1}{x_i}
\]

with \( k' = O(r_1 + r_2 \log t + r_3) = O_P(\log t) \) unit fractions. Observe that all denominators have been rearranged to be distinct.

We next come to the most crucial step, which determines the number of solutions:

**Lemma 2.5.** Let \( \sum_{i=1}^{r_3} \frac{1}{m_i} = P \) (by Lemma 2.4).

a) For any divisor \( d | (P^t - 1) \) the following is an identity.

\[
\frac{1}{P^t - 1} = \frac{1}{P^t - 1 + Pd} + \sum_{i=1}^{r_3} \frac{1}{P^t - 1 + Pd} n_i.
\]

b) The number of divisors \( d | P^t - 1 \) with \( d \equiv 1 \mod P \) is at least \( 2^{\omega(P^t - 1)} \).

c) If \( d \equiv 1 \mod P \), then all denominators are \( \pm 1 \mod P \).

Part a) and c) are easy to verify. For part b) observe: For any \( P \) prime factors \( p_k \), being coprime to \( P \), there is at least one subset of these primes, whose product is \( 1 \mod P \). Indeed, the sequence \( a_1 = p_1, a_2 = p_1 p_2, ..., a_P = \prod_{k=1}^{P} p_k \) must have two members \( a_i, a_j \), say, which are equivalent modulo \( P \). Then \( \frac{a_j}{a_i} = \prod_{k=i+1}^{j} p_k \equiv 1 \mod P \). Therefore, the number of divisors \( d \equiv 1 \mod P \) is at least \( 2^{\omega(P^t - 1)} \). (Clearly, this argument can be refined (see e.g. [6]), but this would not improve our final result.) All solutions
produced in this way are distinct, as each solution has a unique denominator $P^t - 1 + Pd$. Moreover, as all these denominators are greater than $P^t$, and as in our application $t$ will be chosen large, these new denominators are greater than those that have been used before.

We choose $t$ as a product of the first primes. By Lemma 2.2 the number of divisors, and hence the number of solutions satisfies:

$$|X_{k,P}| \geq 2^{\omega(P^t - 1)/P} \geq 2^{(d(t) - 6)/P} \geq 2^{\exp\left(\frac{\log 2 + o(1)}{P} \log \log t\right)} \geq \exp\left(\exp\left(c(P)k/\log k\right)\right).$$

Recall that the number of fractions is $k = O_P(\log t)$.

Finally let us comment on the condition that $k$ is odd, (see statement of the Theorem), when $P$ is even. By multiplying equation (1.1) by its common denominator, and reducing modulo $P$ it is clear that this condition is necessary. The condition is also sufficient as in view of step a) we can replace one fraction by $r_2 + 2$ fractions. Again, by the same argument $r_2 \equiv 1 \mod P$, so that effectively we replace one fraction by 3 fractions (modulo $P$). Iterating this, we can reach any residue class modulo $P$, when $P$ is odd, and the odd residue classes, when $P$ is even. The number of extra fractions required is $O(P r_2) = O_P(1)$. This does not influence the overall result. In any case, the theorem is valid for sufficiently large $k \geq k_P$, with this necessary and sufficient congruence obstruction.

**Remark 2.6.** We have not worked out the constant $c(P)$. One may observe that $c(P)$ might be as small as $\frac{1}{r_2}$. To estimate $r_2$ one observes that

$$\sum_{i=1}^{r_2} \frac{1}{3P(P^2 - 1)} \geq \sum_{i=1}^{r_2} \frac{1}{m_i} \approx \frac{\log r_2}{3P(P^2 - 1)} > P \text{ must hold.}$$

Hence $r_2$ appears to be at least of exponential growth in $P$. Taking denominators $x_i$ only coprime to $P$, but not necessarily restricted to $x_i \equiv \pm 1 \mod 3P(P^2 - 1)$ would improve this constant $c(P)$.

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