On Problems of the Lagrangian Quantization of $W_3$-gravity

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Abstract

We consider the two-dimensional model of $W_3$-gravity within Lagrangian quantization methods for general gauge theories. We use the Batalin–Vilkovisky formalism to study the arbitrariness in the realization of the gauge algebra. We obtain a one-parametric non-analytic extension of the gauge algebra, and a corresponding solution of the classical master equation, related via an anticanonical transformation to a solution corresponding to an analytic realization. We investigate the possibility of closed solutions of the classical master equation in the $Sp(2)$-covariant formalism and show that such solutions do not exist in the approximation up to the third order in ghost and auxiliary fields.

1. Introduction

The two-dimensional model of $W_3$-gravity proposed in [1] is an example of an irreducible gauge theory with an open algebra of gauge transformations. The covariant quantization of this model in the Batalin–Vilkovisky (BV) formalism [2] was considered in [1, 3, 4]. In particular, in [3, 4] a closed solution of the classical master equation (CME) was obtained. In [3], the arbitrariness of solutions was discussed in terms of anticanonical transformations [5], which effectively describe the arbitrariness in the gauge algebra.

In this paper, we continue to investigate the aspects of Lagrangian quantization of the model [1], namely, we extend the study of the gauge algebra, using solutions of CME in the BV formalism, and analyze the possibility of finding a closed solution of CME in the $Sp(2)$-covariant quantization [6].

Our interest in the Lagrangian quantization of the model is due to its peculiarities at the classical level. On the one hand, the Hamiltonian structure of the model does not conform to the assumptions [7] that guarantee the applicability of some general statements established in the theory of constraint systems. On the other hand, the structure of extremals and gauge generators in the Lagrangian formulation leads to a freedom in the definition of the structure functions that admits representations with non-analytic structure functions.

Since the model admits a closed solution of CME, this allows one to study the arbitrariness in the structure functions using anticanonical transformations of solutions. We obtain a realization of the gauge algebra depending on a free parameter and corresponding, for non-vanishing values of this parameter, to non-analytic realizations of the structure functions. The solution of CME constructed in [3] corresponds to an analytic representation of the gauge algebra. We obtain a closed solution of CME in the case of non-analytic structure functions. This solution is related to [3] via an anticanonical transformation and contains structure functions of the third level. We notice that the arbitrariness in the gauge algebra found in [3] is the unique arbitrariness, with the given set of generators, that preserves analyticity and ensures the absence of structure functions beyond the second level.

The fact that $W_3$-gravity has a closed solution [3] of CME in the BV formalism also explains our interest in the $Sp(2)$-covariant quantization of this model. Note that the solution [3] is based on a choice of structure functions.

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of the gauge algebra where the openness is described by a single structure function, being also linear in the fields. The choice [3] provides the simplest realization of the gauge algebra in the analytic class [3], for which one could attempt to find a closed solution of CME in the Sp(2)-covariant formalism despite the considerably more complicated structure of the corresponding approximated solutions [3].

Using the general results for approximated solutions [5] in the Sp(2)-covariant formalism and the realization of the gauge algebra [3], we show that a closed solution of CME for the correspondingly approximated solutions [8].

In Sect. 2, we discuss the peculiarities of the model in the Lagrangian and Hamiltonian formulations. In Sect. 3, we solve the gauge algebra at the second level, present solutions of CME corresponding to different choices of the gauge algebra and discuss the results from the viewpoint of anticanonical transformations. In Sect. 4, we consider the model within the Sp(2)-covariant formalism. In the Appendix, we present the details of the Sp(2)-covariant calculations.

2. Peculiarities of the Lagrangian and Hamiltonian formulations of the model

The model of \( W_3 \)-gravity [1, 3, 4] is described by the classical action of the form

\[
S_0 = \int \mathcal{L} \, dx^2, \quad \mathcal{L} = \frac{1}{2} \phi' \dot{\phi} - \frac{1}{2} h \phi'^2 - \frac{1}{3} B \phi'^3. \tag{1}
\]

Here, the bosonic fields \((\phi, h, B)\) are defined on a two-dimensional space with coordinates \(x = (x^+, x^-)\), and the following notation is used:

\[
\phi' \equiv \partial \phi = \frac{\partial \phi}{\partial x^+}, \quad \dot{\phi} \equiv \partial \phi = \frac{\partial \phi}{\partial x^-}, \quad d^2x = dx^+ dx^-.
\]

The equations of motion read

\[
\begin{align*}
\frac{\delta S_0}{\delta \phi} &= (-\dot{\phi} + h\phi' + B\phi'^2)' = 0, \\
\frac{\delta S_0}{\delta h} &= -\frac{1}{2} \phi'^2 = 0, \quad \frac{\delta S}{\delta B} = -\frac{1}{3} \phi'^3 = 0, \tag{2}
\end{align*}
\]

and obviously imply

\[\phi' = 0.\]

The action (1) is invariant under gauge transformations of the form [1, 4]

\[
\begin{align*}
\delta \phi &= \phi' \epsilon + \phi'^2 \lambda, \\
\delta h &= \dot{\epsilon} - h \epsilon' + h' \epsilon + \phi'^2 (B' \lambda - B \lambda'), \\
\delta B &= B' \epsilon - 2B \epsilon' + \lambda - h \lambda' + 2h' \lambda
\end{align*} \tag{3}
\]

with the bosonic parameters \(\xi^a = (\epsilon, \lambda)\). Denoting \(A^i = (\phi, h, B)\), and \(\delta A^i = R^i_\alpha(A) \xi^\alpha\), we have the following identification of the gauge generators \(R^i_\alpha\):

\[
\begin{align*}
R^\phi_\alpha &= (\phi', \phi'^2), \\
R^h_\alpha &= \left( \partial - h \partial + h', \phi'^2 (B' - B \partial) \right), \\
R^B_\alpha &= (B' - 2B \partial, \partial h \partial + 2h'). \tag{4}
\end{align*}
\]

The generators \((R^h_\alpha, R^B_\alpha)\) do not vanish on shell, and therefore are non-trivial. The generator \(R^\phi_\alpha\) vanishes on shell. However, it cannot be presented as an action of a local operator on the extremals. Thus, we encounter a non-typical case. As a consequence, the gauge algebra of the model may contain non-analytic structure functions. In the following section, we shall see that this is indeed the fact.

It should be noted that the properties of the quadratic approximation defined by the action \(S_0^{(2)} = S_0 |_{h=B=0}\). In
particular, the quadratic theory is not a gauge one. The obvious reparametrization invariance of the latter model belongs to a wider class of symmetries, having the form

$$\delta \phi = \sum_{s=1}^{m} (k_s \phi^s - \tilde{k}_s \phi^s - 1), \quad k_s = k_s(x^+) = k_s(x^-). \quad (5)$$

Particular cases of (5), namely, $s = 2$ and $\tilde{k}_s = 0$, have been discussed in [9]. The presence of restrictions $k_s = \tilde{k}_s = 0$ on the parameters does not allow one to treat (5) as gauge transformations. Another argument in favour of this interpretation can be found in the Hamiltonian formalism, which does not have first-class constraints, and thus non-trivial gauge invariance is absent.

One may expect that the properties of the complete theory and its quadratic approximation should also be essentially different in the Hamiltonian formulation. Namely, the constraint structure of the complete theory and that of its quadratic approximation must be different. To analyze this problem in more detail, let us construct the Hamiltonian formulation of the theory under consideration. We select $x^-$ to be the time variable. In this case, the corresponding Hessian matrix has a constant rank on shell.

There are three primary constraints:

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = \frac{1}{2} \phi' \rightarrow \Phi^{(1)}_1 = p_\phi - \phi'/2 = 0,$$

$$p_h = \frac{\partial L}{\partial \dot{h}} = 0 \rightarrow \Phi^{(1)}_2 = p_h = 0,$$

$$p_B = \frac{\partial L}{\partial \dot{B}} = 0 \rightarrow \Phi^{(1)}_3 = p_B = 0.$$

The total Hamiltonian reads

$$H^{(1)} = H + \lambda_i \Phi^{(1)}_i, \quad H = \int \left( \frac{1}{2} h \phi'^2 + \frac{1}{3} B \phi'^3 \right) dx^+,$$

The consistency conditions for the primary constraints imply the secondary constraint $\Phi^{(2)} = \phi' = 0$ and define one of $\lambda$'s, namely, $\lambda_1 = 0$. No more secondary constraints appear, and $\lambda_2, \lambda_3$ remain undetermined. An equivalent complete set of constraints can be written as

$$\Phi^{(1)}_2 = p_h, \quad \Phi^{(1)}_3 = p_B,$$

$$\Phi^{(1)}_1 = p_\phi, \quad \Phi^{(2)} = \phi'. \quad (6)$$

Here, (6) are first-class constraints, while (7) are second-class constraints.

One can easily see that in the Hamiltonian formulation of the quadratic theory with the action $S_0^{(2)}$ there exists only one primary constraint $\Phi^{(1)} \equiv p_\phi - \phi'/2 = 0$, with $H^{(1)} = \lambda \Phi^{(1)}$. The constraint $\Phi^{(1)}$ is a second-class one

$$\left\{ \Phi^{(1)}(x^+_1), \Phi^{(1)}(x^+_2) \right\} = -\delta'(x^+_1 - x^+_2).$$

It is obvious that the constraints of the quadratic and complete theories have a different structure: the constraints of the complete theory are not the constraints of the quadratic theory with non-linear corrections to them. This peculiarity of the model (11) does not conform to the assumptions (7) which guarantee the applicability of general statements established in the theory of constraint systems. This means that the Hamiltonian quantization of the given model may encounter difficulties. Note that the choice of the time variable as $x^+$ leads to a Hessian matrix whose rank is not constant in the vicinity of the zero point, being a natural point of consideration in the perturbation theory.

It should be noted that the model with the action (11) and generators (11) satisfies the conditions that guarantee the applicability of Lagrangian quantization [2]. Indeed, there exists a stationary point ($\phi' = 0$) in whose neighborhood the action and gauge generators are analytic. Besides, at the stationary point there hold the rank conditions

$$\text{rank} \left\| S_{0,ij}(A) \right\|_{A=A_0} = n - m, \quad \text{rank} \left\| R^\alpha_i(A) \right\|_{A=A_0} = m, \quad (8)$$

where $A_0^\alpha$ denotes the stationary point; rank is understood with respect to the discrete indices $i = 1, ..., n$, $\alpha = 1, ..., m$; and $S_{0,ij}(A)$ is given by

$$S_{0,ij}(A) \equiv \frac{\delta^2 S_0(A)}{\delta A^i \delta A^j}.$$

Here and elsewhere, we use the notation $\delta'(x) = \partial \delta(x)$.\footnote{Here and elsewhere, we use the notation $\delta'(x) = \partial \delta(x)$.}
Calculating the matrices \(\|S_{0,ij}(A)\|_{A=A_0}\) and \(\|R^A_{0,ij}(A)\|_{A=A_0}\) in the model (1), (4), we find
\[
\|S_{0,ij}(A)\|_{A=A_0} = \begin{pmatrix} -\partial \bar{\partial} + \partial (h \partial) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \delta(x_1 - x_2)
\]
and
\[
\|R^A_{0,ij}(A)\|_{A=A_0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -h \partial + h' & B' - 2B \partial \\ 0 & 0 & -h \partial + 2h' \end{pmatrix},
\]
where the rows and columns of (9) are labeled by \((\phi, h, B)\), while the rows and columns of (10) are labeled by \((\epsilon, \lambda)\) and \((\phi, h, B)\), respectively. From (9) and (10), it follows that the conditions (8) and (9) are satisfied, with \(n = 3, m = 2\). The fulfillment of the rank conditions (8) implies that the set of gauge generators (3) is complete and linearly independent (irreducible) generators. Thus, the model of \(W_3\)-gravity can be quantized using Lagrangian methods for theories with linearly independent (irreducible) generators.

3. Extended gauge algebra of the model

In this section, we shall consider the gauge algebra of the model with the action (11) and the gauge generators (3). The gauge generators determine the first level of the gauge algebra, given by the corresponding Noether identities \(S_{0,ij}(A)R^A_{0,ij}(A) = 0\). At the second level, the gauge algebra coincides with the algebra of gauge generators with respect to their commutator, given by
\[
[\delta_1, \delta_2]A^i = \left[R^A_{\alpha,ij}(A)R^A_{\beta,j}(A) - R^A_{\beta,ij}(A)R^A_{\alpha,j}(A)\right]\xi_1^\beta \xi_2^\alpha
\]
where \(\delta_1, \delta_2\) correspond to gauge transformations with parameters \(\xi_1^\beta, \xi_2^\alpha\). Then, using the explicit form of gauge transformations (3) and identifying \(\xi_1^\beta = (\epsilon_1, \lambda_1), \xi_2^\alpha = (\epsilon_2, \lambda_2)\), we get
\[
[\delta_1, \delta_2]\phi = -\phi' \epsilon(1,2) - \phi''(1,2) \left[(\epsilon \lambda)(1,2) - (\epsilon \lambda)(2,1)\right] - 2\phi'' \lambda(1,2),
\]
\[
[\delta_1, \delta_2]h = -(\partial - h \partial + h')\epsilon(1,2) - \phi''(1,2) \left[(\epsilon \lambda)(1,2) - (\epsilon \lambda)(2,1)\right] - \phi''(\partial - h \partial + 3h' + 2\phi' B + 4\phi'' B)\lambda(1,2),
\]
\[
[\delta_1, \delta_2]B = -(B' - 2B\partial)\epsilon(1,2) - (\partial - h \partial + 2h')\left[(\epsilon \lambda)(1,2) - (\epsilon \lambda)(2,1)\right] - \phi''(1,2)B' - 4\phi' B + 2\phi''(1,2)B)\lambda(1,2),
\]
where we have used the notation
\[
\epsilon(1,2) = \epsilon_1 \epsilon_2' - \epsilon_1' \epsilon_2, \quad (\epsilon \lambda)(1,2) = \epsilon_1 \lambda_2' - 2\epsilon_1' \lambda_2, \quad (\epsilon \lambda)(2,1) = \lambda_1 \lambda_2' - 2\lambda_1' \lambda_2.
\]

To analyze the relations (11), we remind that in the bosonic case of a gauge theory with a complete set of generators the commutator of gauge generators has the general form (2)
\[
R^A_{\alpha,ij}(A)R^A_{\beta,j}(A) - R^A_{\beta,ij}(A)R^A_{\alpha,j}(A) = -R^A_{ij}(A)F^0_{\alpha \beta}(A) - S_{0,ij}(A)M^i_{\alpha \beta}(A),
\]
where \(F^0_{\alpha \beta}(A)\) and \(M^i_{\alpha \beta}(A)\) are structure functions, generally depending on the fields \(A^i\) and possessing the antisymmetry properties
\[
F^0_{\alpha \beta}(A) = -F^0_{\beta \alpha}(A), \quad M^i_{\alpha \beta}(A) = -M^i_{\beta \alpha}(A) = -M^i_{\alpha \beta}(A).
\]
The set of structure functions \(F^0_{\alpha \beta}\) and \(M^i_{\alpha \beta}\) defines the gauge algebra at the second level. If \(M^i_{\alpha \beta}(A) = 0\), the gauge algebra is closed. If \(M^i_{\alpha \beta}(A) \neq 0\), it is open.

Taking into account the general structure (13) of the algebra of generators and the explicit form (4) of gauge generators in the model of \(W_3\)-gravity, one gets from (11) the following structure functions \(F^1_{11}, F^2_{21}, F^3_{12}\):
\[
\epsilon(1,2) = F^1_{11} \epsilon_1 \epsilon_2, \quad (\epsilon \lambda)(1,2) = F^2_{21} \epsilon_1 \lambda_2, \quad (\epsilon \lambda)(2,1) = -F^2_{12} \epsilon_2 \lambda_1,
\]
namely,
\[
F^1_{11} = F^{(x)}_{\epsilon(y_1) \lambda(y_2)} = \delta(x - y_2) \delta'(x - y_1) - \delta(x - y_1) \delta'(x - y_2),
\]
\[
F^2_{21} = F^{(x)}_{\lambda(y_1) \epsilon(y_2)} = \delta(x - y_2) \delta'(x - y_1) - 2\delta(x - y_1) \delta'(x - y_2),
\]
\[
F^3_{12} = F^{(x)}_{\epsilon(y_1) \lambda(y_2)} = -\left[\delta(x - y_1) \delta'(x - y_2) - 2\delta(x - y_2) \delta'(x - y_1)\right].
\]
Obviously, these structure functions do not depend on the fields.

The structure functions related to the terms of the gauge algebra containing \( \lambda_{1,2} \) are not so easy to determine, since, besides \( R_\alpha^\lambda \), they also contain the structure functions \( M_{\alpha\beta}^{ij} \). By virtue of the relation \( \equiv \) for \( [\delta_1, \delta_2] \phi \), it is natural to suggest the following Ansatz for the remainder:

\[
2\phi'^3 \lambda_{1,2} = \left( R_1^\phi F_{22}^1 + R_2^\phi F_{22}^2 + \frac{\delta S_0}{\delta h} M_{22}^{\phi h} + \frac{\delta S_0}{\delta B} M_{22}^{\phi B} \right) \lambda_1 \lambda_2. \tag{16}
\]

According to (2) and (4), we can parameterize \( F_{22}^1, F_{22}^2, M_{22}^{\phi h}, M_{22}^{\phi B} \) as follows:

\[
F_{22}^1 \lambda_1 \lambda_2 = \alpha_1 \phi'^2 \lambda_{1,2}, \quad F_{22}^2 \lambda_1 \lambda_2 = \alpha_2 \phi' \lambda_{1,2},
\]

\[
M_{22}^{\phi h} \lambda_1 \lambda_2 = 2\beta_1 \phi' \lambda_{1,2}, \quad M_{22}^{\phi B} \lambda_1 \lambda_2 = 3\beta_2 \lambda_{1,2}, \tag{17}
\]

where the constant parameters \( \alpha_1, \alpha_2, \beta_1, \beta_2 \) must satisfy the relation

\[
\alpha_1 + \alpha_2 - \beta_1 - \beta_2 = 2. \tag{18}
\]

Returning with these results to the remainder of \( [\delta_1, \delta_2] h \) and \( [\delta_1, \delta_2] B \), we shall seek the structure functions in the form

\[
(\phi'^2 B' - 4\phi' \phi'' B - 2\phi'^2 B \delta) \lambda_{1,2} = \left( R_1^B F_{22}^1 + R_2^B F_{22}^2 + \frac{\delta S_0}{\delta \phi} M_{22}^{B \phi} + \frac{\delta S_0}{\delta h} M_{22}^{B h} \right) \lambda_1 \lambda_2. \tag{19}
\]

Within this Ansatz, the only structure coefficient left to determine is \( M_{22}^{B h} \). Let us represent it by a certain operator \( M(A_1, \partial, \partial) \), as follows:

\[
M_{22}^{B h} \lambda_1 \lambda_2 = M \lambda_{1,2}.
\]

Then, using the identification \( \equiv \) and the condition \( \equiv \), we find the following relations, which explicitly realize the second-level gauge algebra of \( W_3 \)-gravity:

\[
\alpha_1 = 1, \quad \alpha_2 = 0, \quad \beta_1 + \beta_2 = -1, \quad \phi'^2 M = 6\beta_2 \frac{\delta S_0}{\delta \phi}.
\]

Thus, the Ansatz (16), (17) determines the algebra with accuracy up to a free parameter, \( \beta_2 \equiv \beta \). Together with (16), we obtain the remaining non-vanishing structure functions of the second level:

\[
F_{22}^1 = \phi'^2 [\delta(x - y_2) \delta'(x - y_1) - \delta(x - y_1) \delta'(x - y_2)], \quad M_{22}^{\phi h} = -2(1 + \beta) \phi' \delta(x - y_1) \delta(y - y_2) \delta'(y - y_1) - \delta(y - y_1) \delta'(y - y_2)],
\]

\[
M_{22}^{\phi B} = 3\beta \delta(x - y) \delta(y - y_2) \delta'(y - y_1) - \delta(y - y_1) \delta'(y - y_2)], \quad M_{22}^{B h} = 6\beta \phi' - 2 \frac{\delta S_0}{\delta \phi} \delta(x - y) \delta(y - y_2) \delta'(y - y_1) - \delta(y - y_1) \delta'(y - y_2)]. \tag{20}
\]

If \( \beta \neq 0 \), then we have a realization of the algebra with a coefficient \( M_{22}^{B h} \) non-analytic at the stationary point of \( S_0 \). Under the requirement of analyticity (\( \beta = 0 \)) we obtain a realization of the gauge algebra of \( W_3 \)-gravity with the gauge generators \( R_\alpha^\lambda \) \( \equiv \) and the following non-vanishing structure functions \( F_{22}^1, \ldots, M_{22}^{\phi h} \):

\[
F_{11}^1 = \delta(x - y_2) \delta'(x - y_1) - \delta(x - y_1) \delta'(x - y_2),
\]

\[
F_{12}^1 = \phi'^2 [\delta(x - y_2) \delta'(x - y_1) - \delta(x - y_1) \delta'(x - y_2)],
\]

\[
F_{21}^1 = \delta(x - y_2) \delta'(x - y_1) - 2\delta(x - y_1) \delta'(x - y_2), \tag{21}
\]

and

\[
M_{22}^{\phi h} = 2\phi' \delta(x - y) \delta(y - y_2) \delta'(y - y_1) - \delta(y - y_2) \delta'(y - y_1). \tag{22}
\]

This is the realization of the gauge algebra of \( W_3 \)-gravity which was used in \( \equiv \) to construct a solution of the classical master equation. The realization \( \equiv \) provides a non-analytic extension of \( \equiv \).
We can see that the model of $W_3$-gravity belongs to the class of irreducible gauge theories with an open algebra of gauge generators.

To complete the definition of the gauge algebra, it is necessary to add also the set of structure functions at higher levels \[2\]. For irreducible theories, this set can be derived by using: (i) the Jacobi identity for the commutator of gauge transformations, (ii) the conditions of completeness and irreducibility, and (iii) the previous gauge relations, such as the Noether identities or relations \[13\]. In general, the gauge algebra consists of an infinite set of structure functions, which define an infinite number of structure relations.

For the purpose of this paper, it is sufficient to know the structure of the gauge algebra up to the third level. Let us consider the Jacobi identity for the gauge transformations

$$[\delta_1, [\delta_2, \delta_3]] A^i + \text{cycl.perm.}(1, 2, 3) = 0.$$ 

Then, using differential consequences of the Noether identities $S_{0,i}(A) R^i_\alpha(A) = 0$, we obtain

$$\left( R^i_\gamma D^\gamma_{\alpha \beta \delta} + S_{0,k} Z^i_{\alpha \beta \delta} \right) \xi^{\beta \gamma} \xi^{\delta i} + \text{cycl.perm.}(1, 2, 3) = 0,$$  

(23)

with the following abbreviations:

$$D^\gamma_{\alpha \beta \delta} = \left( F^\gamma_{\alpha \alpha} F^\gamma_{\beta \beta} + F^\gamma_{\alpha \beta} \right) + \text{cycl.perm.}(\alpha, \beta, \delta),$$

$$Z^i_{\alpha \beta \delta} = \left( M^i_{\alpha \sigma} F^\sigma_{\beta \beta} + M^i_{\alpha \beta \jmath} R^j_{\beta \delta} - R^k_{\alpha \jmath} M^i_{\beta \delta} + R^i_{\alpha \jmath} M^k_{\beta \delta} \right) + \text{cycl.perm.}(\alpha, \beta, \delta),$$

where $D^\gamma_{\alpha \beta \delta}$ and $Z^i_{\alpha \beta \delta}$ are antisymmetric in the indices $(\alpha, \beta, \delta)$ and $(i, k)$. By virtue of the completeness and linear independence of the generators $R^i_\alpha$, the relation \[24\] is solved by \[2\]

$$D^\gamma_{\alpha \beta \delta} = S_{0,k} Q^\gamma_{\alpha \beta \delta}, \quad Z^i_{\alpha \beta \delta} = R^i_\alpha Q^\gamma_{\alpha \beta \delta} - R^k_{\gamma i} Q^\gamma_{\alpha \beta \delta} = S_{0,j} M^i_{\alpha \beta \delta},$$

(24)

where $Q^\gamma_{\alpha \beta \delta}$ and $M^i_{\alpha \beta \delta}$ are structure functions, antisymmetric in the indices $(\alpha, \beta, \delta)$ and $(i, j, k)$. For irreducible theories, the functions $Q^\gamma_{\alpha \beta \delta}$ and $M^i_{\alpha \beta \delta}$ define the structure of the gauge algebra at the third level.

To analyze the structure functions of $W_3$-gravity beyond the second level, it is convenient to apply the BV formalism \[2\]. Within this formalism, all structure relations can be collected into a solution of the classical master equation. This equation is formulated for the bosonic extended action $S = S(\phi, \phi^*)$. For irreducible theories, the action depends on the minimal set of classical and ghost fields $\phi^A = (A^i, C^\alpha)$, $\varepsilon(C^\alpha) = \varepsilon(\phi^A) + 1$, and the corresponding antifields $\phi^*_A = (A^*_i, C^*_\alpha)$, $\varepsilon(\phi^*_A) = \varepsilon(\phi^A) + 1$, with the following distribution of the ghost number:

$$\text{gh}(A^i) = 0, \quad \text{gh}(C^\alpha) = 1, \quad \text{gh}(\phi^*_A) = -1 - \text{gh}(\phi^A).$$

The classical master equation for the gauge algebra is defined by

$$\frac{\delta S}{\delta \phi^A} \frac{\delta S}{\delta \phi^*_A} = 0,$$

(25)

and is subject to the boundary condition

$$S|_{\phi^* = 0} = S_0(A).$$

A solution $S = S(\phi, \phi^*)$ can be sought as a Taylor series in the ghost fields $C^\alpha$,

$$S = S_0(A) + \sum_{n=1} S_n, \quad S_n \sim (C^n), \quad \varepsilon(S_n) = 0, \quad \text{gh}(S_n) = 0,$$

with the following result (see, e.g., \[10\]), considered in the bosonic case $\varepsilon(A^i) = \varepsilon(C^\alpha) = 0$, with accuracy up to the third order:

$$S(\phi, \phi^*) = S_0(A) + A^*_i R^i_\alpha C^\alpha - \frac{1}{2} \left( C^\alpha F^\gamma_{\alpha \beta} - \frac{1}{2} A^*_i A^*_j M^i_{\beta \gamma} \right) C^\beta C^\gamma$$

$$- \frac{1}{2} \left( C^*_i A^*_j Q^i_{\alpha \beta \gamma} - \frac{1}{6} A^*_i A^*_j A^*_k M^i_{\alpha \beta \gamma} \right) C^\gamma C^\beta C^\alpha + \cdots,$$

(26)
where $F^{\gamma}_{\alpha\beta}$, $M^{ij}_{\alpha\beta}$ and $Q^{\delta\gamma}_{\alpha\beta}$, $M^{ij\delta\gamma}_{\alpha\beta}$ are the structure functions of the gauge algebra at the second and third levels, respectively. In the case of $W_3$-gravity, we shall consider four solutions of CME, labeled by (a), (b), (c), (d).

(a) A closed solution of the form (26) for $W_3$-gravity can be constructed using non-trivial structure functions $F^{\alpha}_{\alpha\beta}$, (21), and $M^{ij}_{\alpha\beta}$, (22), in the minimal sector of the classical fields $A^i = (\phi, h, B)$ and the ghost fields $C^\alpha = (c, l)$:

$$S = S_0 + S_1 + \int d^2x \left[ c^* (c'c + \phi' h') + t^* (l'c + 2c'l) + 2\phi h^* \phi' l' \right],$$  

(27)

where the initial classical action $S_0$ is given by (1), and the action $S_1$ is determined by the gauge generators as

$$S_1 = \int d^2x \left[ \phi^* (\phi' c + \phi' h^* l') + h^* [c - h' c + h^* (B' l - B l')] + B^* \left( B' c - 2B c' + \dot{l} - h l' + 2h' l' \right) \right].$$

It follows from (27) that all structure functions of higher levels are equal to zero if one uses the realization of the gauge algebra in the form (4), (15), (21) and (22).

(b) It is not difficult to construct an action (26) that corresponds to the case of the gauge algebra with non-analytic structure functions (20). To this end, we remind that any anticanonical transformation of the field-antifield variables $\phi^A$, $\check{\phi}^A$, determined by

$$\check{\phi}^A = \frac{\delta X(\phi, \check{\phi}^*)}{\delta \phi^A}, \quad \phi^* = \frac{\delta X(\phi, \check{\phi}^*)}{\delta \check{\phi}^A},$$

with the generating functional $X = X(\phi, \check{\phi}^*)$, $\varepsilon(X) = 1$, $gh(X) = -1$, transforms solutions of CME into solutions.

Making an anticanonical transformation of (27) with the generating functional

$$X(\phi, \phi^*) = E(\phi, \phi^*) + 6\beta \int d^2x \phi^* h^* B^* \phi'^{-2} l' l,$$

where $E(\phi, \phi^*)$ is the generating functional of the identical transformation, we obtain an action with the second-level structure functions (15), (20),

$$S(\beta) = S_0 + S_1 + \int d^2x \left[ c^* (c'c + \phi'^2 l'l) + t^* (l'c + 2c'l) + 6\beta h^* B^* \phi'^{-2} \left( \phi^* - h^* \phi' - \phi'^2 h + \phi'^2 B - 2\phi \phi'' B \right)' l' l$$

$$- \phi^* \left( 3\beta B^* - 2(1 + \beta) h^* \phi' - 12\beta h^* B^* \phi'^{-2} \phi' \right)' l' l \right].$$

(28)

This action, which is also a closed solution of CME, coincides with the action (27) when $\beta = 0$.

The realizations of the second-level gauge algebra with analytic (21), (22) and non-analytic (20) structure functions are equivalent in the sense of the anticanonical transformation relating the corresponding solutions (27) and (28). At the same time, from (28) it follows that the case of the non-analytic realization of the gauge algebra leads to a more complicated gauge structure. Indeed, with non-vanishing $M^{\phi h B}$, one also obtains (non-analytic) structure functions at the third level (see (21) and (22)).

(c) It should be noted that the discussed arbitrariness in the choice of the gauge structure functions $F^{\gamma}_{\alpha\beta}$, $M^{ij}_{\alpha\beta}$ for $W_3$-gravity is by no means unique. Indeed, the condition of analyticity admits a freedom in the choice of the second-level structure functions. Namely, let us consider the action (3)

$$S(\alpha) = S_0 + S_1 + \int d^2x \left[ c^* (c'c + (1 - \alpha) \phi'^2 l'l) + t^* (l'c + 2c'l) + 2\alpha h^* \left( h^* - h'' h - 3B^* B' - 2BB'' \right)' l' l + 2(1 + \alpha) \phi^* h^* \phi' l' l \right],$$

(29)

obtained from (27) via anticanonical transformations with the generating functional

$$X(\phi, \phi^*) = E(\phi, \phi^*) - 2\alpha \int d^2x h^* c^* l' l,$$  

(30)
where $\alpha$ is a free parameter. The action $F^1_1$ satisfies CME with the same boundary condition and gauge generators, but it corresponds to another set of gauge structure functions $F^1_{\alpha\beta}$,

\[
F^1_1 = \delta(x - y_2)\delta'(x - y_1) - \delta(x - y_1)\delta'(x - y_2),
\]

\[
F^2_2 = (1 - \alpha)\delta'(x - y_2)\delta'(x - y_1) - \delta(x - y_1)\delta'(x - y_2),
\]

\[
F^2_1 = \delta(x - y_2)\delta'(x - y_1) - 2\delta(x - y_1)\delta'(x - y_2),
\]

and non-vanishing matrices $M^\alpha_{\beta\gamma}$,

\[
M^{\alpha h}_{22} = 2(1 + \alpha)\delta'(x - y) [\delta(y - y_1)\delta'(y - y_2) - \delta(y - y_2)\delta'(y - y_1)],
\]

\[
M^{\alpha h}_{2h} = 2\alpha (\bar{\partial}_x - \bar{\partial}_y - (\partial_x - \partial_y)h) \delta(x - y) [\delta(y - y_1)\delta'(y - y_2) - \delta(y - y_2)\delta'(y - y_1)],
\]

\[
M^{\alpha h}_{hB} = -2\alpha (3B' - 2B\partial_y) \delta(x - y) [\delta(y - y_1)\delta'(y - y_2) - \delta(y - y_2)\delta'(y - y_1)],
\]

depending on the fields $\phi, h, B$. The analytic choice of the second-level structure functions \([21], [22]\) is a particular case of \([31], [32]\), corresponding to $\alpha = 0$.

Note that the extended analytic realization \([31], [32]\) can be obtained from the Ansatz \([16]\) and the parameterization \([17]\), with $\alpha_1 = 1 - \alpha, \beta_1 = -1 - \alpha, \alpha_2 = \beta_2 = 0$. These values of the parameters are related to a modified Ansatz \([14]\), where $M^{\alpha B}_{22} = 0$, and a structure function $M^{2B}_{22}$ is included.

A remarkable property of the action \([24]\) is its dependence on the ghost fields $c, l$. For any value of $\alpha$, they enter the action only in the second order — in contrast to the action \([25]\), which depends on the ghost fields in the third order if $\beta \neq 0$.

(d) One can prove that the arbitrariness in analytic structure functions described in \([3]\) is unique in the sense that it preserves the form of the action, being of second order in the ghost fields. Indeed, to preserve a given set of gauge generators, the generating functional of the anticanonical transformations must be at least of second order in the antifields and ghost fields. It can be verified by straightforward calculations that any simple form of such anticanonical transformations, except \([30]\), leads to an action depending on the ghost fields in the third order. For example, let us consider an anticanonical transformation with the generating functional

\[
X(\phi, \phi^\tau) = E(\phi, \phi^\tau) - 3\gamma \int d^2x B^* c^l l',
\]

which is similar to \([30]\). Then we obtain the action

\[
S(\gamma) = S_0 + S_1 + \int d^2x \left[ c^* (c' c + (1 - \gamma\phi')\phi'^2 l' l) + l^* (l' c + 2c l)
\right.
\]

\[
+ 3\gamma B^*(2BB'^* - (\bar{\partial} - h\partial + h^* h^*) l' l + \phi^* (2h^* + 3\gamma B^*) \phi'^* l' l + 3\gamma B^* c^* c l'),
\]

being of the third order in the ghost fields and containing the structure function $Q^{1B}_{22}$, which means that the Jacobi identity \([24]\) for the gauge generators closes only on shell.

The above discussion shows that for $W_3$-gravity there exists a class of “minimal” actions, terminating at $S_2$ and related by anticanonical transformations (cases a and c). Any other anticanonical transformation produces higher ghost contributions, starting at $S_3$ (case d), and leading also to non-analytic actions (case b). In what follows, we shall consider only analytic realizations of the gauge algebra.

4. $W_3$-gravity in the Sp(2)-covariant formalism

Let us consider the model of $W_3$-gravity in the framework of the Sp(2)-covariant extension \([3]\) of the BV quantization scheme. To this end, it is necessary to introduce the complete configuration space of fields $\phi^A$, constructed from the classical fields, as well as from the ghosts and auxiliary fields of the BV formalism, combined into completely symmetric tensors under the group Sp(2). Thus, for irreducible theories the fields $\phi^A$ are given by

\[
\phi^A = (A^\alpha, B^\alpha, C^{\alpha a}), \quad a = 1, 2,
\]

where the Sp(2)-doublets $C^{\alpha a}$ stand for the ghost-antighost pairs $(C^\alpha, \bar{C}^\alpha)$, while the Sp(2)-scalars $B^\alpha, \varepsilon(B^\alpha) = \varepsilon(\xi^\alpha)$, stand for the Lagrange multipliers known as Nakanishi–Lautrup fields. The fields $\phi^A$ are associated with the corresponding sets of antifields $\phi'^*_{An}$ and $\phi^*_{A}$,

\[
\varepsilon(\phi'^*_{An}) = \varepsilon(\phi^A) + 1, \quad \varepsilon(\phi^*_{A}) = \varepsilon(\phi^A).
\]
Thus, the antifields corresponding to \( C_a \) are given by
\[
\phi_A^a = (A_a^*, B_a^*, C_a^*) , \quad \bar{\phi}_A = (\bar{A}_a, \bar{B}_a, \bar{C}_a).
\]

The fields \( \phi^A \) and antifields \( \phi_A^* \), \( \bar{\phi}_A \) are ascribed the so-called new ghost number, denoted by “ngh” and subject to the following conditions:
\[
\text{ngh}(\phi_A^a) = -1 - \text{ngh}(\phi^A), \quad \text{ngh}(\bar{\phi}_A) = -2 - \text{ngh}(\phi^A),
\]
where the new ghost number of the fields in the case \( C \) is given by the rule
\[
\text{ngh}(A^i) = 0, \quad \text{ngh}(C^{\alpha a}) = 1, \quad \text{ngh}(B^\alpha) = 2.
\]

The basic object of the Sp(2)-covariant formalism \([6]\) is a bosonic functional \( S = S(\phi, \phi^*, \bar{\phi}) \) subject to the classical master equation
\[
\frac{\delta S}{\delta \phi^A} \frac{\delta S}{\delta \phi_{Aa}} + \epsilon_{ab} \phi_{Aa} \frac{\delta S}{\delta \phi_{A}} = 0 , \tag{34}
\]
with the boundary condition
\[
S\big|_{\phi=0, \phi^*=0} = S_0(A),
\]
where \( \epsilon_{ab} \) is a constant antisymmetric second-rank tensor, with \( \epsilon^{ac} \epsilon_{cb} = \delta_a^b \) and \( \epsilon^{12} = 1 \). The existence of solutions of CME in the Sp(2)-covariant formalism has been proved \([6]\) for both irreducible and reducible gauge theories of general kind. These solutions are sought as series in ghost and auxiliary fields \( (C, B) \), under the requirement of the new ghost number conservation:
\[
S = S_0 + \sum_{n=1} S_n, \quad \epsilon(S_n) = \text{ngh}(S_n) = 0, \quad S_n \sim (C)^{n-m}(B)^m, \quad 0 \leq m \leq n. \tag{35}
\]

For irreducible theories of general kind, an approximated solution of CME was found \([8]\) up to the third order in the powers of ghosts and auxiliary fields \( (C^{\alpha a}, B^\alpha) \). The approximation found in \([8]\) is completely determined by the structure functions up to the third level.

In the bosonic case \( \epsilon(A^i) = \epsilon(C^i) = 0 \), the solution found in \([8]\) has the form
\[
S(\phi, \phi^*, \bar{\phi}) = S_0(A) + A^{*}_{ia} R^i_a C^{\alpha a} + \bar{A}_i R^i_B B^\alpha - \epsilon_{ab} C_a^b B^a - \frac{1}{2} C_{ab} F_{ab} C^{ab} C_{ab} \\
+ \frac{1}{4} A^{*}_{ia} A^k_{jb} M_{ij}^{jk} C_{ab} C^{ab} + \frac{1}{2} \bar{A}_i R^i_B B^\alpha \epsilon_{ab} + \frac{1}{2} (2 \bar{C}_\gamma - B^\gamma) F_{ab} B^\alpha C^{ab} + \frac{1}{2} A^i_{ia} A_j M^{ij} B^\beta C^{ab} C^{ab} \\
+ \frac{1}{12} (2 \bar{C}_\alpha - B^\alpha) \left( F_{ab} F_{ab} + 2 F_{ab} R^l_{ab} \right) C^{\alpha c} C^{\alpha b} C^{ab} \epsilon_{ac} - \frac{1}{2} \bar{A}_i \bar{A}_j R^i_{ab} M_{ij}^{jk} B^\gamma C^{\beta i} C^{\beta j} C^{\alpha c} \epsilon_{ab}
\]
where we have assumed the absence of higher structure functions, as in the case of \( W_3 \)-gravity with the one-parametric family of analytic realizations of \( F_{ab} \) and \( M_{ij} \), given by \([41], [42]\).

Let us calculate the approximated expression \([16]\) in the case of \( W_3 \)-gravity with the simplest analytic choice of the second-level structure functions \([21], [22]\), corresponding to the zero value of the arbitrary parameter in \([11], [12]\). Denoting the ghosts and auxiliary fields in \( W_3 \)-gravity by \( C^{\alpha a} = (c^a, f^a) \) and \( B^\alpha = (u, v) \), we shall write down the contributions \( S_1, S_2, S_3 \) corresponding to \([36]\).

In the first order:
\[
S_1 = \int d^2 \phi \left[ \phi A^a (\phi^2 c^a + \phi^2 f^a) + h^a (\nabla_1 c^a - \phi^2 \nabla_1 (B^a)) + B_a \left( \nabla_2 u - 2 \nabla_1 (B^a) \right) \right] \\
+ \bar{\phi} (\phi^a u + \phi^a v) + \bar{h} \left( \nabla_1 u - \phi^a \nabla_1 (B^a) \right) + \bar{B} \left( \nabla_2 v - 2 \nabla_1 (B^a) \right) - \epsilon^{ab} c_{ab} u - \epsilon^{ab} f_{ab} v \right]. \tag{37}
\]

In the second order:
In the third order:

\[
S_3 = \int d^2 x \left\{ -\frac{1}{2} c_{ab} \left[ \nabla_1 (c^b c^a) + \phi'^2 \nabla_1 (l'^a l^a) \right] - \frac{1}{2} l_{ab}^* \left[ \nabla_2 (c^b l^a) + 2 \nabla_4 (l^b c^a) \right] \\
+ \left( \frac{1}{2} u^*_a - \bar{e}_a \right) \left[ \nabla_1 (c^a u) + \phi'^2 \nabla_1 (l^a u) \right] + \left( \frac{1}{2} v^*_a - \bar{l}_a \right) \left[ \nabla_2 (c^a v) + 2 \nabla_4 (l^a u) \right] \\
- \phi'^2 h^a \phi' \nabla_1 (l'^a l^a) + (\bar{h} \phi'^2 - \bar{\phi} h^a) \phi' \nabla_1 (l^a v) + \frac{1}{2} \varepsilon_{ab} \bar{\phi} (c^b + 2 \phi'^2 l^b) \left( \phi' + \phi'^2 l^a \right) \right] \\
+ \frac{1}{2} \varepsilon_{ab} \bar{h} \left[ 2 \phi' (\phi' c^a + \phi'^2 l^a) \nabla_1 (B l^a) - \phi'^2 \left( \nabla_2 (l^a - 2 \nabla_4 (B^a c^a)) \right) \right] \\
+ \nabla_1 \left[ (\nabla_1 c^a - \phi'^2 \nabla_1 (B l^a)) c^b \right] - \frac{1}{2} \varepsilon_{ab} \bar{B} \left[ \nabla_2 \left[ (\nabla_2 l^a - 2 \nabla_4 (B l^a)) c^b \right] \\
- \nabla_2 \left[ (\nabla_1 c^a - \phi'^2 \nabla_1 (B l^a)) l^b \right] \right] \right\}. \tag{38}
\]

In the third order:

\[
S_3 = \int d^2 x \left\{ -\frac{1}{6} c_{cd} \left( \frac{1}{2} u^*_a - \bar{e}_a \right) \left[ \nabla_1 \left[ (\nabla_1 c^a c^c) + \phi'^2 \nabla_1 (l'^a l^a) \right] \right] c^d \\
+ \phi'^2 \nabla_1 \left[ \left( \nabla_2 (c^a l^a) + 2 \nabla_4 (l^c c^a) \right) l^d \right] + 4 \phi' \left( \phi' c^a \right) \nabla_1 (l'^a l^d) \right] \\
+ \frac{1}{6} \varepsilon_{cd} \left[ \frac{1}{2} v^*_a - \bar{l}_a \right] \left\{ \nabla_2 \left[ (\nabla_1 (c^a c^c) + \phi'^2 \nabla_1 (l'^a l^a)) l^d \right] \\
- \nabla_2 \left[ (\nabla_2 (c^a l^a) + 2 \nabla_4 (l^c c^a)) c^d \right] - \bar{\phi} \varepsilon_{cd} \left[ \bar{h} \phi' \nabla_1 (l^a) \right] \left( c^d + 2 \phi' l^d \right) \right] \\
- \frac{1}{2} \varepsilon_{cd} h^a \nabla_1 \left[ \left( \phi' \nabla_1 (l^a) \right) c^d \right] - 2 \bar{h} \phi' \nabla_1 (l^a) \nabla_1 (B l^d) \right] \\
- \varepsilon_{cd} \bar{B} \nabla_2 \left[ (\phi' \nabla_1 (l^a)) l^d \right] + \frac{1}{3} \varepsilon_{cd} \phi^*_a \left[ \bar{h} \phi' \nabla_1 (l'^a l^a) \right] \left( c^d + 2 \phi' l^d \right) \right] \\
+ \frac{1}{3} \varepsilon_{cd} \left[ \nabla_1 \left[ (\phi' \nabla_1 (l'^a l^a)) c^d \right] - 2 \phi' \left[ \bar{h} \phi' \nabla_1 (l'^a l^a) \right] \nabla_1 (B l^d) \right] \\
+ \frac{1}{3} \varepsilon_{cd} h^a \nabla_2 \left[ \left( \phi' \nabla_1 (l^a l^a) \right) l^d \right] + \frac{2}{3} \varepsilon_{cd} \phi^*_a \left[ h^a \phi' \nabla_1 (l'^a l^a) \right] \left( c^d + 2 \phi' l^d \right) \right] \\
+ \frac{2}{3} \varepsilon_{cd} \bar{B} \nabla_2 \left[ \left( \phi' \nabla_1 (l^a l^a) \right) l^d \right] - \frac{1}{6} \varepsilon_{cd} (h^a \bar{\phi} + h^a \bar{\phi}) \left[ \phi' \nabla_1 \left[ (\nabla_2 (c^a l^a) \right] \\
+ 2 \nabla_4 (l^c c^a) \right) l^d \right] + 2 (\phi' c^a + \phi'^2 l^a) \nabla_1 (l^a l^d) \right] \right\}. \tag{39}
\]

In (37), (38), (39), we have used the following notation:

\[
\nabla_j (X Y) \equiv X Y' - j X' Y, \quad \nabla_j X \equiv \bar{X} - \nabla_j h(X), \tag{40}
\]

where \( j \) is a real number.

The details of calculations are presented in Appendix A, using the example of \( S_3 \). Contrary to the expectations motivated by the results of the BV formalism, the approximation given by (37), (38), (39) does not provide a closed solution of CME, which is shown in Appendix B.

Therefore, a closed solution must contain higher contributions \( S_n \). To tackle this problem, one has to deal with the task of finding further approximations for the Sp(2)-covariant extended action of general irreducible theories with open algebras. This problem remains open. However, since the complexity of the approximations will gradually increase, it is not evident that a finite number of contributions will be sufficient to provide a solution in the given example of \( W_3 \)-gravity. Concerning the possibility of a closed solution with a finite number of contributions, there is evidence that such a solution may require contributions up to \( S_6 \) (see Appendix B). In this case, the task of solving CME seems to be extremely difficult. One could have hoped that the problem might be attacked with the help of some
transformations analogous to anticanonical ones. The role of such transformations in the Sp(2)-covariant scheme is played by operator transformations of the quantum action [9]. However, it should be noted that we have already started from the simplest realization of the gauge algebra, which in the BV formalism immediately provides a closed solution.

As mentioned previously, the analyzed example of a gauge theory with an open algebra is relatively simple, and therefore in the case of more complicated theories with open algebras it is natural to expect more technical difficulties. Nevertheless, the possibility of sufficient conditions that ensure the existence of a closed solution of the Sp(2)-covariant CME for theories with open algebras is an interesting problem that deserves investigation.

Concluding, note that in the limit $B = t^a = v = 0$ the functional $S_0 + S_1 + S_2 + S_3$ coincides with the closed solution [11] of CME for the model of $W_2$-gravity [12], whose classical action is given by [11] in the limit $B = 0$, and whose gauge transformations are given by [8] in the limit $\lambda = 0$.

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**Appendix A**

In this Appendix, we shall calculate the contribution $S_3$, [89], using a technique of reducing the corresponding expression to gauge algebra operations. The same technique can be applied to the calculation of the previous contributions $S_1$, $S_2$, [87], [88].

Let us consider the part of (36), corresponding to $\bar{a}$, where

$$S_3 = \frac{1}{12} (2\bar{C}_{db} - B_{db}^c) (F^c_{\alpha\beta} F_{\beta\gamma}^\gamma + 2 F^\delta_{\alpha\beta,k} R_{\alpha\beta,k}^\gamma C_{\gamma\epsilon}^\epsilon C_{\epsilon\alpha}^\alpha \varepsilon_{ac} - \frac{1}{2} \bar{A}_i \bar{A}_j R_{\alpha\beta,k}^\gamma M_{\beta\gamma}^k M_{\gamma\epsilon}^\epsilon C_{\epsilon\alpha}^\alpha \varepsilon_{ab} - \frac{1}{12} \bar{A}_i^a \bar{A}_j^b \delta C_{\gamma\epsilon}^\epsilon (F_{\alpha\beta,k}^\gamma C_{\epsilon\alpha}^\alpha \varepsilon_{ab} - \frac{1}{2} \bar{A}_i \bar{A}_j R_{\alpha\beta,k}^\gamma M_{\beta\gamma}^k M_{\gamma\epsilon}^\epsilon C_{\epsilon\alpha}^\alpha \varepsilon_{ab}) - \frac{1}{12} \bar{A}_i^a \bar{A}_j^b \delta C_{\gamma\epsilon}^\epsilon (F_{\alpha\beta,k}^\gamma C_{\epsilon\alpha}^\alpha \varepsilon_{ab} - \frac{1}{2} \bar{A}_i \bar{A}_j R_{\alpha\beta,k}^\gamma M_{\beta\gamma}^k M_{\gamma\epsilon}^\epsilon C_{\epsilon\alpha}^\alpha \varepsilon_{ab})$$

(A.1)

To simplify the consideration of this functional, let us introduce a set of three constant Grassmann doublets $\xi_{(n)}$, $\bar{\xi}_{(n)}$, $\bar{\xi}_{(k)}$, for $n = 0, 1, 2, 3$, and a set of four bosonic parameters $\xi_{(n)}$, $\bar{\xi}_{(n)}$, $\bar{\xi}_{(k)}$, $\bar{\xi}_{(k)}$, by the rule

$$\xi_{(n)}^\alpha = (\xi_{(0)}^\alpha, \xi_{(1)}^\alpha), \quad \bar{\xi}_{(n)}^\alpha = \bar{\xi}_{(0)}^\alpha, \quad \bar{\xi}_{(k)}^\alpha = C^{\alpha a} \mu_{a(k)}.$$

(A.2)

With this notation, we have

$$S_3 = \varepsilon_{ab} \frac{\partial F}{\partial \mu_{b(2)}} \frac{\partial F}{\partial \mu_{a(1)}} + \varepsilon_{ab} \frac{\partial F}{\partial \mu_{c(3)}} \frac{\partial F}{\partial \mu_{b(2)}} \frac{\partial F}{\partial \mu_{a(1)}}$$

(A.3)

where $F$ and $F_a$ are given by

$$F = \frac{1}{2} \bar{A}_i \bar{A}_j M_{\alpha\beta,k}^\gamma M_{\beta\gamma}^k C_{\epsilon\alpha}^\alpha (R_{\gamma\epsilon}^\epsilon (S_{(2)}^\gamma), k)$$

and

$$F_a = \frac{1}{12} (2\bar{C}_{\gamma a} - B_{\gamma a}^c) \left[ F_{\alpha\beta,k}^\gamma M_{\beta\gamma}^k C_{\epsilon\alpha}^\alpha (R_{\gamma\epsilon}^\epsilon (S_{(2)}^\gamma), k) + \frac{1}{6} (A_{a(1)} \bar{A}_j \bar{A}_j M_{\alpha\beta,k}^\gamma C_{\epsilon\alpha}^\alpha (S_{(2)}^\gamma, k) + 2 A_a \bar{A}_j M_{\beta\gamma}^k M_{\epsilon\alpha}^\epsilon (S_{(2)}^\gamma, k) + 2 R_{\gamma\epsilon}^\epsilon (M_{\alpha\beta,k}^\gamma C_{\epsilon\alpha}^\alpha (S_{(2)}^\gamma, k), k) - \frac{1}{2} A_a \bar{A}_j \left[ M_{\beta\gamma}^k M_{\epsilon\alpha}^\epsilon (S_{(2)}^\gamma, k) + 2 R_{\gamma\epsilon}^\epsilon (M_{\alpha\beta,k}^\gamma C_{\epsilon\alpha}^\alpha (S_{(2)}^\gamma, k), k) \right].

The above expressions can be rewritten in the compact form

$$F = \frac{1}{2} \bar{A}_i \delta(\delta(2)A^i)|_{\delta A \to \delta A_{(1)}, 1}$$

(A.4)

and

$$F_a = \frac{1}{12} (2\bar{C}_{\gamma a} - B_{\gamma a}^c) \left[ \delta(\delta(2)A^i)|_{\delta A \to \delta A_{(1)}, 3} + 2 \delta(\delta(3)A^i)|_{\delta A \to \delta A_{(1), 2}} \right] + \frac{1}{6} A_{a(1)} \delta(\delta(3)A^i)|_{\delta A \to \delta A_{(1), 2}}$$

(A.5)

The derivatives are applied in the following order: $\frac{\partial F}{\partial \mu_{b(2)}} \frac{\partial F}{\partial \mu_{a(1)}} = \frac{\partial F}{\partial \mu_{b(2)}} \left( \frac{\partial F}{\partial \mu_{a(1)}} \right)$. 

\[2\text{The derivatives are applied in the following order: } \frac{\partial F}{\partial \mu_{b(2)}} \frac{\partial F}{\partial \mu_{a(1)}} = \frac{\partial F}{\partial \mu_{b(2)}} \left( \frac{\partial F}{\partial \mu_{a(1)}} \right).\]
where $\delta_{(n)}$ stand for gauge variations with parameters $\xi^a_{(n)}$; the quantities $\tilde{\xi}^\gamma_{(m,n)}$, $\tilde{\xi}^{ij}_{(m,n),l}$ are defined by
\begin{equation}
\tilde{\xi}^\gamma_{(m,n)} = F_{\alpha\beta}^\gamma \xi^\alpha_{(m)} \xi^\beta_{(n)}; \quad \tilde{\xi}^{ij}_{(m,n),l} = F_{\alpha\beta} \xi^\alpha_{(m,n)} \xi^\beta_{(n)}, \quad \tag{A.6}
\end{equation}
the variations $\delta$ are understood as usual variations of the quantities $\delta_{(n)} A^i$ with respect to $A^i$, where $\delta A^i$ in the resultant expression $\delta(\delta_{(n)} A^i)$ are replaced by $\delta A^i_{(m,n)}$, $\delta^\alpha A^i_{(m,n)}$, having the form
$$
\delta A^i_{(m,n)} = \tilde{A}_j \xi^{ji}_{(m,n)}; \quad \delta^\alpha A^i_{(m,n)} = A^j_{(m,n)} \tilde{\xi}^{ji}_{(m,n)};
$$
in accordance with the definition of $\tilde{\xi}^{ij}_{(m,n)}$ and $\tilde{\xi}^{ij}_{(m,n),l}$:
\begin{equation}
\tilde{\xi}^{ij}_{(m,n)} \equiv M^{ij}_{\alpha\beta} \xi^\alpha_{(m)} \xi^\beta_{(n)}; \quad \tilde{\xi}^{ij}_{(m,n),l} \equiv M^{ij}_{\alpha\beta} \xi^\alpha_{(m)} \xi^\beta_{(n),l} \xi^\gamma; \quad \tag{A.7}
\end{equation}
To calculate the quantities $F$ and $F_a$ in the model of $W_3$-gravity, we remind that $A^i = (\phi, h, B)$, $\xi^\alpha = (\epsilon, \lambda)$, with the gauge transformations $\delta A^i = R^i_a \xi^a$, given by $[\mathbb{I}]$. The (non-vanishing) quantities $\xi^\alpha_{(m,n)}$, $\tilde{\xi}^{ij}_{(m,n)}$, corresponding to subsequent gauge transformations with the parameters $\xi^\alpha_{(m)}$ and $\xi^\alpha_{(n)}$, have the form
\begin{equation}
\tilde{\xi}^\alpha_{(m,n)} = F_{\beta\gamma} \xi^\beta_{(m)} \xi^\gamma_{(n)} = (\tilde{\epsilon}(m,n), \tilde{\lambda}(m,n)), \quad \tilde{\xi}^{ij}_{(m,n)} = M^{ij}_{\alpha\beta} \xi^\alpha_{(m)} \xi^\beta_{(n)} = (\tilde{\epsilon}^\phi_{(m,n)}, \tilde{\xi}^{ij}_{(m,n)}), \quad \tag{A.8}
\end{equation}
with
$$
\tilde{\epsilon}^\alpha_{(m,n)} = \epsilon^{(m)} \epsilon^{(n)} - \epsilon^{(m)} \epsilon^{(n)} + \phi^{(m)} (\lambda^{(m)} \lambda^{(n)} - \lambda^{(m)} \lambda^{(n)}), \quad \tilde{\lambda}^{(m,n)} = \epsilon^{(m)} \lambda^{(n)} - 2 \epsilon^{(m)} \lambda^{(n)} + 2 \lambda^{(m)} \epsilon^{(n)} - 2 \lambda^{(n)} \epsilon^{(m)}, \quad \tilde{\xi}^{ij}_{(m,n)} = - \theta \equiv (2 \phi (\lambda^{(m)} \lambda^{(n)} - \lambda^{(m)} \lambda^{(n)}), \quad \tag{A.9}
$$
which follows from the parameterization $[\mathbb{I}]$, $[\mathbb{I}]$, $[\mathbb{I}]$ of the second-level structure functions in the case $\alpha_1 = 1$, $\alpha_2 = 0$, $\beta_1 = -1$, $\beta_2 = 0$, corresponding to the choice of $F_{\alpha\beta}^\gamma$, $M^{ij}_{\alpha\beta}$ in the analytic form $[\mathbb{I}]$, $[\mathbb{I}]$. Using the above identifications, we are now able to calculate all the structures which enter the quantities $F$ and $F_a$:
\begin{equation}
\delta(\delta_{(l)} A^i_{(m,n)}), \quad \delta(\delta_{(l)} \tilde{A}_j), \quad \delta(\delta_{(l)} \tilde{A}_j), \quad \delta(\delta_{(l)} A^i_{(m,n)}), \quad \delta(\delta_{(l)} \tilde{A}_j, A^i_{(m,n)}), \quad \delta(\delta_{(l)} \tilde{A}_j, A^i_{(m,n)}), \quad \tag{A.10}
\end{equation}
The structure $\tilde{\xi}^\alpha_{(m,n),l}$ is given by
\begin{equation}
\tilde{\xi}^\alpha_{(m,n),l} = (\tilde{\epsilon}_{(m,n),l}, \tilde{\lambda}_{(m,n),l}), \quad \tag{A.11}
\end{equation}
where
$$
\tilde{\epsilon}^\alpha_{(m,n),l} = \tilde{\epsilon}^\alpha_{(m,n)} - \sum_{(m,n)} \tilde{\epsilon}^\alpha_{(m,n)} + \phi^{(m)} (\lambda^{(m)} \lambda^{(n)} - \lambda^{(m)} \lambda^{(n)}), \quad \tag{A.12}
$$
To calculate $\delta(\delta_{(l)} \tilde{\xi}^\alpha_{(m,n)})$, we notice that
\begin{equation}
\delta(\delta_{(l)} \tilde{\xi}^\alpha_{(m,n)}) = (\delta(\delta_{(l)} \tilde{\epsilon}_{(m,n),l}), \delta(\delta_{(l)} \tilde{\lambda}_{(m,n),l})), \quad \tag{A.13}
\end{equation}
where
\[
\delta (l) \tilde{\epsilon}^{(m,n)}_{(m,n,l)} = 2\phi' (\phi' \epsilon_{(l)} + \phi'^2 \lambda_{(l)})'(\lambda_{(m)} \lambda'_{(n)} - \lambda'_{(m)} \lambda_{(n)}), \quad \delta (l) \tilde{\lambda}_{(m,n)} = 0.
\]

Similarly, we determine the manifest form of \( \tilde{\xi}^{ij}_{(m,n,l)} \),
\[
\tilde{\xi}^{ij}_{(m,n,l)} = (\tilde{\xi}^{\phi h}_{(m,n,l)}, \tilde{\xi}^{\phi h}_{(m,n,l)}),
\]
\[
\tilde{\xi}^{\phi h}_{(m,n,l)} = -\phi^{\phi h}_{(b)} = -2\phi' (\epsilon_{(m)} \lambda'_{(n)} - 2\epsilon'_{(m)} \lambda_{(n)} - 2\lambda_{(m)} \epsilon'_{(n)}) \lambda'_{(l)} + 2\phi' (\epsilon_{(m)} \lambda'_{(n)} - 2\epsilon'_{(m)} \lambda_{(n)} - 2\lambda_{(m)} \epsilon'_{(n)})' \lambda_{(l)},
\]
and the structure \( \delta (l) \tilde{\xi}^{ij}_{(m,n)} \),
\[
\delta (l) \tilde{\xi}^{ij}_{(m,n)} = (\delta (l) \tilde{\xi}^{\phi h}_{(m,n)}, \delta (l) \tilde{\xi}^{\phi h}_{(m,n)}),
\]
which implies
\[
\delta (l) \tilde{\xi}^{\phi h}_{(m,n)} = -2(\phi' \epsilon_{(l)} + \phi'^2 \lambda_{(l)})' (\lambda_{(m)} \lambda'_{(n)} - \lambda'_{(m)} \lambda_{(n)}). \tag{A.13}
\]
To calculate the quantities
\[
\delta (\delta (l) A^*_{\alpha A} A^l)|_{\delta A \to \delta \lambda_{A_{(m,n)}}}, \quad \delta (\delta (l) \bar{A}^*_i A^l)|_{\delta A \to \delta \lambda_{A_{(m,n)}}}, \quad \delta (\delta (l) \bar{A}^*_l A^l)|_{\delta A \to \delta \lambda_{A_{(m,n)}}},
\]
we notice that the field variations are given by
\[
\tilde{\delta} A^i_{(m,n,l)} = \tilde{A}^i_{\alpha A} \tilde{\xi}^{\alpha}_{(m,n)} = \tilde{\phi}^{\phi h}_{(m,n)} + \tilde{h}^{\phi h}_{(m,n)} = (\tilde{h}^{\phi h}_{(m,n)}, \tilde{\phi}^{\phi h}_{(m,n)}, 0)
\]
\[
\delta^* A^i_{(m,n,l)} = A^i_{\alpha A} \delta^{\alpha}_{(m,n)} = \phi^{\phi h}_{(m,n)} + h^{\phi h}_{(m,n)} = (\phi^{\phi h}_{(m,n)}, \phi^{\phi h}_{(m,n)}, 0),
\]
and therefore
\[
\tilde{\delta} A^i_{(m,n,l)} = (\tilde{\delta} \phi_{(m,n)}, \tilde{\delta} h_{(m,n)}, \tilde{\delta} B_{(m,n)}) = 2\phi' (\lambda_{(m)} \lambda'_{(n)} - \lambda'_{(m)} \lambda_{(n)}) (h_0, -\phi_0, 0),
\]
\[
\delta^* A^i_{(m,n,l)} = (\delta^* \phi_{(m,n)}, \delta^* h_{(m,n)}, \delta^* B_{(m,n)}) = 2\phi' (\lambda_{(m)} \lambda'_{(n)} - \lambda'_{(m)} \lambda_{(n)}) (h^*_0, -\phi^*_0, 0). \tag{A.14}
\]
From this result, we can see that it is sufficient to consider the variation of the expressions \( \delta (l) (A^*_{\alpha A} A^l) \), \( \delta (l) (\bar{A}^*_i A^l) \) only with respect to the fields \( \phi \) and \( h \), since the variation of \( B \) is equal to zero. We have
\[
\delta (\delta (l) \phi) = (\delta \phi)' (\epsilon_{(l)} + 2\phi' \lambda_{(l)}),
\]
\[
\delta (\delta (l) \bar{h}) = -2\delta (\phi)' (\epsilon_{(l)} + 2\phi' \lambda_{(l)}),
\]
\[
\delta (\delta (l) B) = -2\delta (\phi)' \lambda_{(l)}.
\]
Repeating the above expressions the variations \( \delta A^i \) with \( \tilde{\delta} A^i_{(m,n,l)} \), \( \delta^* A^i_{(m,n,l)} \), and substituting their manifest form, we obtain
\[
\delta (\delta (l) A^*_{\alpha A} A^l)|_{\delta A \to \delta \lambda_{A_{(m,n)}}} = 2\phi^{\phi h}_{\alpha} [\tilde{h}^{\phi h} (\lambda_{(m)} \lambda'_{(n)} - \lambda'_{(m)} \lambda_{(n)})]' (\epsilon_{(l)} + 2\phi' \lambda_{(l)})
\]
\[
+ 2h^*_\alpha \{ \phi^{\phi h}_{\alpha} (\lambda_{(m)} \lambda'_{(n)} - \lambda'_{(m)} \lambda_{(n)}) \epsilon_{(l)} - [\phi^{\phi h}_{\alpha} (\lambda_{(m)} \lambda'_{(n)} - \lambda'_{(m)} \lambda_{(n)})]' \epsilon_{(l)}
\]
\[
+ 2\phi' [h^{\phi h}_{\alpha} (\lambda_{(m)} \lambda'_{(n)} - \lambda'_{(m)} \lambda_{(n)})]' (B' \lambda_{(l)} - B \lambda'_{(l)})
\]
\[
+ 2B^*_\alpha \{ \phi^{\phi h}_{\alpha} (\lambda_{(m)} \lambda'_{(n)} - \lambda'_{(m)} \lambda_{(n)}) \lambda_{(l)} - 2[\phi^{\phi h}_{\alpha} (\lambda_{(m)} \lambda'_{(n)} - \lambda'_{(m)} \lambda_{(n)})]' \lambda_{(l)} \}
\}
\]
Similarly, we have
\[
\delta (\delta (l) \bar{A}^*_i A^l)|_{\delta A \to \delta \lambda_{A_{(m,n)}}} = 2\phi^{\phi h}_{\alpha} [h^{\phi h}_{\alpha} (\lambda_{(m)} \lambda'_{(n)} - \lambda'_{(m)} \lambda_{(n)})]' (\epsilon_{(l)} + 2\phi' \lambda_{(l)})
\]
\[
+ 2\tilde{h} \{ \phi^{\phi h}_{\alpha} (\lambda_{(m)} \lambda'_{(n)} - \lambda'_{(m)} \lambda_{(n)}) \epsilon_{(l)} - [\phi^{\phi h}_{\alpha} (\lambda_{(m)} \lambda'_{(n)} - \lambda'_{(m)} \lambda_{(n)})]' \epsilon_{(l)}
\]
\[
+ 2\phi' [h^{\phi h}_{\alpha} (\lambda_{(m)} \lambda'_{(n)} - \lambda'_{(m)} \lambda_{(n)})]' (B' \lambda_{(l)} - B \lambda'_{(l)})
\]
\[
+ 2\tilde{B} \{ \phi^{\phi h}_{\alpha} (\lambda_{(m)} \lambda'_{(n)} - \lambda'_{(m)} \lambda_{(n)}) \lambda_{(l)} - 2[\phi^{\phi h}_{\alpha} (\lambda_{(m)} \lambda'_{(n)} - \lambda'_{(m)} \lambda_{(n)})]' \lambda_{(l)} \}
\}
\]
and
\[
\delta(\delta(\bar{A}, A^l)|_{\delta A \to \delta A_{(m,n)}} = 2 \hat{h} [\phi'(\lambda_{(m)} \lambda'_{(n)} - \lambda'_{(m)} \lambda_{(n)})'] (\epsilon_{(l)} + 2 \phi') \lambda_{(l)}) \\
+ 2 \hat{h} \left\{ \phi'[\lambda_{(m)} \lambda'_{(n)} - \lambda'_{(m)} \lambda_{(n)}] \epsilon'_{(l)} - [\phi'(\lambda_{(m)} \lambda'_{(n)} - \lambda'_{(m)} \lambda_{(n)})'] \epsilon_{(l)} \\
+ 2 \hat{h}[\phi'(\lambda_{(m)} \lambda'_{(n)} - \lambda'_{(m)} \lambda_{(n)})'] (B' \lambda_{(l)} - B \lambda_{(l)}) \right\} \\
+ 2 \hat{B} \left\{ \phi'(\lambda_{(m)} \lambda'_{(n)} - \lambda'_{(m)} \lambda_{(n)} \lambda'_{(l)} - 2 [\phi'(\lambda_{(m)} \lambda'_{(n)} - \lambda'_{(m)} \lambda_{(n)})'] \lambda_{(l)} \right\} \right\}.
\tag{A.16}
\]

Gathering together the contributions \[A.10\] corresponding to \( F \) and \( F_a \) in \[A.4\], \[A.5\], we can calculate \( S_3 \) by differentiating \( F \) and \( F_a \) with respect to Grassmann parameters, according to \[A.2\], \[A.3\]. Using the notation \( C^a = (c^a, l^a) \), \( B^a = (u, v) \) and \[10\], we obtain the expression \[39\].

**Appendix B**

In this Appendix, we shall prove that the contributions \( S_1, S_2, S_3 \) are not sufficient to provide a closed solution to the classical master equation \[44\] of the Sp(2)-covariant formalism, where a solution is sought as an expansion \[35\]. To this end, note that for irreducible theories the contribution \( S_1 \) can be chosen in the form \[9\]
\[ S_1 = A_a^R R_a^c C^a + \bar{A}_B^c B^a - \varepsilon^{ab} C^a_{ab} B^a. \]

Then the higher contributions \( S_{n+1} \) for \( n \geq 1 \) can be determined by iterations:
\[ W^a S_{n+1} = F^a_{n+1}, \tag{B.1} \]
where \( W^a \) is a doublet of differential operators, which in the case \( \varepsilon^a = 0 \) has the form
\[
W^a = S_{0,1} + \delta_{A^a} + A_{i}^B R^i_{c} \delta_{C^a} + (\bar{A}_B^c - \varepsilon_c^{abc} C_a^c) \frac{\delta}{\delta B^a_{c}} + \varepsilon^{ab} \frac{\delta}{\delta C^c_{ab}} + \varepsilon^{abc} \frac{\delta}{\delta \phi^c_{a}},
\]
and the quantities \( F^a_{n+1} \) are given by
\[
F^a_{n+1} = - \frac{1}{2} (S_{n}, S_{(n)})^a_{n+1}, \quad S_{(n)} = S_0 + \sum_{k=1}^{n} S_k,
\]
with \(( , )_{n+1}^a\) being the \((n+1)\)-th order of the extended antibracket in powers of \( C^{a}, B^a \),
\[
(F, G)^a = \frac{\delta F}{\delta \phi^a} \frac{\delta G}{\delta \phi^a} - (-1)^{(\varepsilon(F) + 1)(\varepsilon(G) + 1)} \frac{\delta G}{\delta \phi^a} \frac{\delta F}{\delta \phi^a},
\]
having the obvious property
\[
(F, G)^a = - (-1)^{(\varepsilon(F) + 1)(\varepsilon(G) + 1)} (G, F)^a.
\]

Let us assume that the functional \( S_3 \), given by
\[
S = S_3 = S_0 + S_1 + S_2 + S_3,
\]
is a close solution of CME. Then the quantities \( F^a_{n+1} \) for \( n \geq 3 \) must vanish identically:
\[
S_{n+1} = 0 \Rightarrow W^a S_{n+1} = F^a_{n+1} = 0, \quad n \geq 3.
\]
Considering all possible \( F^a_{n+1} \), \( n \geq 3 \), we have
\[
F^a_4 = - \frac{1}{2} (S_3, S_3)^2 = -(S_1, S_3)_4^a - (S_2, S_3)_4^a - \frac{1}{2} (S_2, S_2)^2,
F^a_5 = - \frac{1}{2} (S_4, S_3)^2 = -(S_2, S_3)^5_4^a = -(S_2, S_3)^5_3^a = \frac{1}{2} (S_3, S_3)^5_2^a,
F^a_6 = - \frac{1}{2} (S_5, S_3)^2 = -(S_3, S_3)^6_5^a = - \frac{1}{2} (S_3, S_3)^6_3^a.
\]

Note that \( F^a_{n+1} \equiv 0, \quad n \geq 6 \). Thus, the quantity \( F^a_6 \) has the simplest structure, which involves only the contribution \( S_3 \). In what follows, we shall check if the condition \((S_3, S_3)^6_6 = 0 \) is fulfilled in the model of \( W_3 \)-gravity, which is necessary for \( S_3 \) to be a closed solution of CME.
Let us consider the expression \( A_{\alpha} \) for \( S_3 \). Then \( (S_3, S_3)_6^0 \), given by

\[
\frac{1}{2} (S_3, S_3)_6^0 = \frac{\delta S_3}{\delta A^i} \frac{\delta S_3}{\delta A^i} ,
\]

decomposes into the following orders in antifields:

\[
\vec{A}(2 \bar{C} - B^*), \quad (\vec{A})^2 A^i, \quad (\vec{A})^3 ,
\]

namely,

\[
\frac{1}{2} (S_3, S_3)_6^0 = \frac{1}{(12)^2} D_1^a - \frac{1}{(12)^2} D_2^a + \frac{1}{24} D_3^a, \quad D_n^a \sim (\bar{A})^n ,
\]

where

\[
D_1^a = \bar{A}_m (2 \bar{C}_{vd} - B^*_{vd}) (F_{\rho\gamma}^\alpha F_{\delta\sigma}^\lambda + 2 F_{\rho\gamma}^\alpha R_{\delta\sigma}^k), (2 R_{\beta, n}^l M_{\alpha\gamma}^m + 4 R_{\beta, n}^l M_{\alpha\gamma}^m_{ln} - M_{\beta, n}^l M_{\alpha\gamma}^m - 2 M_{\beta, n}^l R_{\alpha\gamma}^m C^{\beta h} C^{\alpha} C^{\rho p} C^{\sigma} C^{\rho p} C^{\delta} \varepsilon_{bc} \varepsilon_{pq} ,
\]

\[
D_2^a = A_{\alpha} A_{\beta, n} (2 R_{\beta, n}^l M_{\alpha\gamma}^m + 4 R_{\beta, n}^l M_{\alpha\gamma}^m - M_{\beta, n}^l M_{\alpha\gamma}^m + M_{\beta, n}^l R_{\alpha\gamma}^m C^{\beta h} C^{\alpha} C^{\rho p} C^{\sigma} C^{\rho p} C^{\delta} \varepsilon_{bc} \varepsilon_{pq} ,
\]

\[
D_3^a = \bar{A}_i \bar{A}_j \bar{A}_m (2 R_{\beta, n}^l M_{\alpha\gamma}^m + 4 R_{\beta, n}^l M_{\alpha\gamma}^m - M_{\beta, n}^l M_{\alpha\gamma}^m - M_{\beta, n}^l R_{\alpha\gamma}^m C^{\beta h} C^{\alpha} C^{\rho p} C^{\sigma} C^{\rho p} C^{\delta} \varepsilon_{bc} \varepsilon_{pq} .
\]

We can see that the quantity \( D_3^a \), given by the order \((\bar{A})^3\), has the simplest form. Therefore, in what follows we shall investigate the question whether \( D_3^a \) is equal to zero in the model of \( W_3 \)-gravity.

Before taking into account the manifest form of the gauge algebra of the given model, let us simplify the consideration by rewriting \( D_3^a \) in the following manner:

\[
D_3^a = \varepsilon_{bc} \varepsilon_{pq} \frac{\partial r}{\partial \mu_{\rho(5)}} \frac{\partial r}{\partial \mu_{\alpha(3)}} \frac{\partial r}{\partial \mu_{\beta(2)}} \frac{\partial r}{\partial \mu_{\gamma(1)}} D .
\]

where \( D \) is given by

\[
D = \bar{A}_i [(\delta(5), A^1_i), k (\bar{A} \tilde{A}^{ji})], l (2 (\delta(2), A^1_i), n (\bar{A} \tilde{A}^{mn}_{1,3}) - 4 (\delta(2), A_m A^m), n (\tilde{A}^{ml}_{1,3,2}) + \tilde{A}^{ml}_{1,3,2} + 2 \delta(1) (\bar{A} \tilde{A}^{ml}_{1,3,2})] .
\]

Here, \( \delta(\alpha) \) stand for gauge variations with parameters \( \xi^\alpha (\alpha) \), \( \alpha = 0, 1, \ldots , 9 \), defined by \( A_{\alpha} \), while the quantities \( \tilde{\xi}^{ij}_{1,3,2} \) are given by \( A_{\alpha} \), \( A_{\beta} \), \( A_{\gamma} \). Using this notation, we can rewrite the quantity \( D \) in a compact form:

\[
D = \bar{A}_i \delta \left( (\delta(5), A^1_i) \right)_{\delta A \rightarrow \delta A_{(4,0)}} \delta A^1_{(1,3,2)} = \delta \left( (\delta(5), A^1_i) \right)_{\delta A \rightarrow \delta A_{(4,0)}} \left( \delta A^1_{(1,3,2)} \right)_{\delta A \rightarrow \delta A_{(4,0)}} ,
\]

where the variations \( \delta \) are understood as usual variations with respect to the fields \( A^i \). To obtain \( D \), one has to perform two successive variations of the quantity \( \delta(5), \bar{A}^1_i \), replacing the corresponding variations \( \delta A^i \) in the resulting expressions \( \delta(\bar{A}^1_i) \) and \( \delta(\bar{A}^1_i) \) by the quantities \( \bar{A}^1_{(4,0)} \) and \( A^1_{(1,3,2)} \)

\[
\bar{A}^1_{(4,0)} = \bar{A}^1_{(4,0)} , \quad A^1_{(1,3,2)} = 2 (\delta(2), A^1_i) , \quad A^1_{(1,3,2)} = 4 (\delta(2), A^1_i) , \quad A^1_{(1,3,2)} + 2 \delta(1) (\bar{A}^1_i) .
\]

To calculate the quantity \( D \) explicitly, we remind that in the model of \( W_3 \)-gravity we have \( A^i = (\phi, h, B) \), \( \xi^\alpha = (\psi, \lambda) \). The gauge transformations \( \delta A^i = R^a_{\alpha} \xi^\alpha \) are given by \( A_{\alpha} \) and the (non-vanishing) quantities \( \xi^\alpha (\alpha), \xi^\alpha_{1,3,2} \), corresponding to subsequent gauge transformations with the parameters \( \xi^\alpha (\alpha) \) and \( \xi^\alpha_{1,3,2} \), have the form \( A_{\alpha} \).
The quantity $\delta (\delta (5, \tilde{A}, A^i) |_{\delta A} \rightarrow \delta A_{(4,0)})$ is given by (A.10). Taking a variation of this expression, we find

$$\delta \left( \delta (5, \tilde{A}, A^i) |_{\delta A} \rightarrow \delta A_{(4,0)} \right) = 2\bar{\delta} h (\delta \phi) (\lambda (4) \lambda (0) - \lambda (4) \lambda (0)) (\epsilon (5) + 2\delta \phi \lambda (5))$$

$$+ 2\bar{\delta} h (\delta \phi) (\lambda (4) \lambda (0) - \lambda (4) \lambda (0)) \delta \phi \lambda (5) + 2\bar{\delta} h (\delta \phi) (\lambda (4) \lambda (0) - \lambda (4) \lambda (0)) \epsilon (5)$$

$$- [\bar{\delta} \phi (\lambda (4) \lambda (0) - \lambda (4) \lambda (0)) \delta \phi \lambda (5) + 4\delta \phi \lambda (5)] (B' \lambda (5) - B \lambda (5))$$

$$+ 4\delta \phi \lambda (5) ([\delta \phi \lambda (5) - (\delta B) \lambda (5)])$$

$$= 2\bar{\delta} h (\delta \phi) (\lambda (4) \lambda (0) - \lambda (4) \lambda (0)) \lambda (5)\{ [\delta \phi (\lambda (4) \lambda (0) - \lambda (4) \lambda (0)) \lambda (5) - 2\delta (\delta \phi) (\lambda (4) \lambda (0) - \lambda (4) \lambda (0)) \lambda (5)\}.$$  

(B.2)

Notice that the above expression does not contain any variations with respect to the field $\phi$. Therefore, in what follows it is sufficient to find only two components of the variation $\delta A_{(1,3,2)}$ needed to calculate the quantity $D$, namely, $\delta \phi (1,3,2)$ and $\delta B (1,3,2)$. With this in mind, we observe that in order to construct $\delta A_{(1,3,2)}$ we need to determine the following set of quantities:

$$\delta (2) A_{(i)} (k), \tilde{A} \tilde{\epsilon}^{i(k)} (1,3), \delta (2) \bar{A}_{(k)} A^{k}, \tilde{A} \tilde{\epsilon}^{i(k)} (1,3,2), \delta (1) (\bar{A} \tilde{\epsilon}^{i(k)} (3,3)),$$

where $i = (\phi, B), j = (\bar{\phi}, h, B)$, $\phi, B$. These quantities can be easily calculated using (17a), (A.9), (A.12).

We have

$$\delta \phi (1,3,2) = 2\delta (2) \phi_{,k} (A \tilde{\epsilon}^{i(k)} (1,3)) - 4\delta (2) \bar{A}_{k} A^{k} (A \tilde{\epsilon}^{i(k)} (1,3)) = 4\delta (2) \bar{A}_{k} A^{k} (A \tilde{\epsilon}^{i(k)} (1,3,2)) + 2\delta (3) (A \tilde{\epsilon}^{i(k)} (3,3))$$

$$= 4\bar{\delta} h (\lambda (1) \lambda (3) - \lambda (1) \lambda (3)) (\epsilon (2) + 2\phi \lambda (2)) - 8\phi (\lambda (1) \lambda (3) - \lambda (1) \lambda (3))$$

$$\times [\bar{\delta} \phi (\lambda (1) \lambda (3) - \lambda (1) \lambda (3)) \lambda (3) - 2\epsilon (1) \lambda (3)] - 4\bar{\delta} h (\lambda (1) \lambda (3) - \lambda (1) \lambda (3)) [\lambda (1) \lambda (3) - \lambda (1) \lambda (3)]$$

$$+ 2\delta (2) B (k) \lambda (k) (3) \lambda (1) \lambda (3) - \lambda (1) \lambda (3) \lambda (3) - \lambda (1) \lambda (3) \lambda (3) - \lambda (1) \lambda (3) \lambda (3),$$

$$\delta B (1,3,2) = 2\delta (2) B (k) \lambda (k) (3) \lambda (1) \lambda (3) - \lambda (1) \lambda (3) \lambda (3) - \lambda (1) \lambda (3) \lambda (3) - \lambda (1) \lambda (3) \lambda (3).$$

Now, we are able to calculate the quantity $D$, obtained by substituting the above variations $\delta \phi (1,3,2)$ and $\delta B (1,3,2)$ into (B.2). With this in mind, to simplify the consideration, we will first analyze the general structure of $D$ as regards its expansion in antifields. We can write symbolically

$$D = \left( \delta h \bar{\phi} B, \delta (2) \phi_{,k} + \bar{\delta} \phi \lambda \right) \lambda (5, \tilde{A}, A^i) \lambda (4,0),$$

which implies that the expression for $D$ decomposes into the following orders in antifields:

$$\delta (2) \phi_{,k} + \bar{\delta} \phi \lambda, \delta (2) \phi_{,k} + \bar{\delta} \phi \lambda, \delta (2) \phi_{,k} + \bar{\delta} \phi \lambda,$$

$$\delta (2) \phi_{,k} + \bar{\delta} \phi \lambda, \delta (2) \phi_{,k} + \bar{\delta} \phi \lambda, \delta (2) \phi_{,k} + \bar{\delta} \phi \lambda.$$

From the manifest form of $\delta \phi (1,3,2), \delta B (1,3,2)$ and $\delta \left( \delta (5, \tilde{A}, A^i) |_{\delta A} \rightarrow \delta A_{(4,0)} \right)$, one can observe that the contribution $\Delta (\bar{B})^2$ has the simplest form, which we shall denote by $D_{\bar{B}B^2}$. This contribution is given by

$$D_{\bar{B}B^2} = D^|_{\bar{h} = 0},$$

and therefore to calculate $D_{\bar{B}B^2}$ it is sufficient to consider the limits

$$\Delta (\delta (5, \tilde{A}, A^i) |_{\delta A} \rightarrow \delta A_{(4,0)}) |_{\bar{h} = 0}, \delta \phi (1,3,2) |_{\bar{h} = 0}, \delta B (1,3,2) |_{\bar{h} = 0}.$$

We have

$$D_{\bar{B}B^2} = \delta \left( \delta (5, \tilde{A}, A^i) |_{\delta A} \rightarrow \delta A_{(4,0)}) \right) |_{\delta A} \rightarrow \delta A_{(1,3,2)}, \bar{h} = 0,$$

with

$$\delta (\delta (5, \tilde{A}, A^i) |_{\delta A} \rightarrow \delta A_{(4,0)}) |_{\bar{h} = 0} = -2(\delta \phi) [\bar{\delta} (\lambda (4) \lambda (0) - \lambda (4) \lambda (0)) (3B \lambda (5) + 2B' \lambda (5))],$$

$$\delta \phi = \delta \phi (1,3,2) \bar{h} = -8\phi (\lambda (1) \lambda (3) - \lambda (1) \lambda (3)) (3B \lambda (2) + 2B' \lambda (2))'. $$

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where we have used integration by parts in the expression containing $\tilde{A}, A^i$. Thus, we obtain

$$ D_{\tilde{A}B^2} = 16\phi'(\lambda_0^2\lambda_3^2 - \lambda_0^3\lambda_3^2)(3\tilde{B}\lambda_2^2 + 2B^1\lambda_2^2)'. $$

The contribution $D_{\tilde{A}B^2}$ to $D$ determines the corresponding order in antifields in the quantity $D^a_3$, which represents the order $(\tilde{A})^3$ in $F^a_6$. Since $D_{\tilde{A}B^2}$ is related to $D$ by the limit $\hbar = 0$, we have

$$ D^a_3|_{\hbar = 0} = \varepsilon_{bc\varepsilon_{pq}} \frac{\partial_r}{\partial \mu_{r(5)}} \frac{\partial_r}{\partial \mu_{r(4)}} \frac{\partial_r}{\partial \mu_{r(3)}} \frac{\partial_r}{\partial \mu_{r(2)}} \frac{\partial_r}{\partial \mu_{r(1)}} D_{\tilde{A}B^2}, $$

with $\lambda_0 = v$, $\lambda_0(n) = \iota^n a(n)$. Taking derivatives with respect to the Grassmann parameters, and using integration by parts in the resultant expression, we obtain

$$ D^a_3|_{\hbar = 0} = 16\tilde{\phi}'[\lambda^1(\tilde{l}^a)^2 - (\tilde{l}^a)^2][3\tilde{B}(l^a)' + 2B^1(l^a)']'(l^a)'v - (l^a)'v][3\tilde{B}(l^a)' + 2B^1(l^a)']v \varepsilon_{bc\varepsilon_{pq}}. $$

One can show that $D^a_3|_{\hbar = 0}$ does not vanish identically. Since this fact is not evident in the $Sp(2)$-covariant form, we shall write the dummy indices $b, c, p, q$ manifestly in terms of the values 1, 2, denoting $l^1 = l$, $l^2 = l'$, and taking into account $\varepsilon_{12} = -1$. Let us also fix the free index as $a = 1$.

Using the cancellation of some terms containing squares of $l$, $l'$, we can represent $D^a_3|_{\hbar = 0}$ as follows:

$$ D^a_3|_{\hbar = 0} = -16\tilde{\phi}'(3\tilde{B}l' + 2B^1l')'[\lambda(l' - l)](3\tilde{B}(l' - l)') + (l' - l)\phi'(3\tilde{B}l' + 2B^1l') D^{12}, $$

where

$$ D^{12} = (l' - l)\tilde{B}l'v. $$

From this representation of $D^a_3|_{\hbar = 0}$ it is not yet evident if this quantity vanishes or not.

To examine the structure of $D^a_3|_{\hbar = 0}$ in more detail, let us represent $D^a_3|_{\hbar = 0}$ in the form

$$ D^a_3|_{\hbar = 0} = -16\tilde{\phi}(A + B), $$

where

$$ A \equiv (l' - l)[\phi'(3\tilde{B}l' + 2B^1l')]'D^{12}, $$

$$ B \equiv (l' - l)\phi'(3\tilde{B}l' + 2B^1l')D^{12}. $$

Note that $D^a_3|_{\hbar = 0}$ vanishes if and only if $A + B = 0$ because $\tilde{\phi}$ is an arbitrary field.

We can see that $A = 0$. Indeed, taking into account the manifest form of $D^{12}$, we have

$$ A \sim (l' - l)(l' - l) = (l')^2 - (l')^2 \equiv 0. $$

Let us consider the quantity $B$. Using the manifest form of $D^{12}$ and taking into account the cancellation of terms containing squares of $l$, $l'$, $l''$, we obtain

$$ B = \phi'(l''[4v\tilde{B}B'' - 10vB' - 15v'\tilde{B}B']l'' - (6vB' - 9vB)\tilde{B}l'' - 6vB^2l'' - 6v'\tilde{B}^2l'']. $$

This expression does not vanish identically, and therefore $A + B \neq 0$.

Note that since $(S_3, S_3)_6^a$ makes a non-vanishing contribution to the r.h.s. of the equation $W^aS_6 = F^a_6$ in (B.1) the contribution $S_6$ may be non-vanishing.

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