Depth-Independent Lower bounds on the Communication Complexity of Read-Once Boolean Formulas

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Abstract

We show lower bounds of $\Omega(\sqrt{n})$ and $\Omega(n^{1/4})$ on the randomized and quantum communication complexity, respectively, of all $n$-variable read-once Boolean formulas. Our results complement the recent lower bound of $\Omega(n/8^d)$ by Leonardos and Saks [LS09] and $\Omega(n/2^{\Omega(d \log d)})$ by Jayram, Kopparty and Raghavendra [JKR09] for randomized communication complexity of read-once Boolean formulas with depth $d$.

We obtain our result by “embedding” either the Disjointness problem or its complement in any given read-once Boolean formula.

1 Introduction

A read-once Boolean formula $f : \{0, 1\}^n \rightarrow \{0, 1\}$ is a function which can be represented by a Boolean formula involving AND and OR such that each variable appears, possibly negated, at most once in the formula. An alternating AND-OR tree is a layered tree in which each internal node is labeled either AND or OR and the leaves are labeled by variables; each path from the root to the any leaf alternates between AND and OR labeled nodes. It is well known (see eg. [HW91]) that given a read-once Boolean formula $f : \{0, 1\}^n \rightarrow \{0, 1\}$ there exists a unique alternating AND-OR tree, denoted $T_f$, with $n$ leaves labeled by input Boolean variables $z_1, \ldots, z_n$, such that the output at the root, when the tree is evaluated according to the labels of the internal nodes, is equal to $f(z_1 \ldots z_n)$. Given an alternating AND-OR tree $T$, let $f_T$ denote the corresponding read-once Boolean formula evaluated by $T$.

Let $x, y \in \{0, 1\}^n$ and let $x \land y, x \lor y$ represent the bit-wise AND, OR of the strings $x$ and $y$ respectively. For $f : \{0, 1\}^n \rightarrow \{0, 1\}$, let $f^\land : \{0, 1\}^n \times \{0, 1\}^n \rightarrow \{0, 1\}$ be given by $f^\land(x, y) = f(x \land y)$. Similarly let $f^\lor : \{0, 1\}^n \times \{0, 1\}^n \rightarrow \{0, 1\}$ be given by $f^\lor(x, y) = f(x \lor y)$. Recently Leonardos and Saks [LS09], investigated the two-party randomized communication complexity, denoted $R(\cdot)$, of $f^\land, f^\lor$ and showed the following. (Please refer to [KN97] for familiarity with basic definitions in communication complexity.)

**Theorem 1 ([LS09])** Let $f : \{0, 1\}^n \rightarrow \{0, 1\}$ be a read-once Boolean formula such that $T_f$ has depth $d$. Then

$$\max\{R(f^\land), R(f^\lor)\} \geq \Omega(n/8^d).$$

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In the theorem, the depth of a tree is the number of edges on a longest path from the root to a leaf. Independently, Jayram, Kopparty and Raghavendra [JKR09] proved randomized lower bounds of \(\Omega(n/2\Omega(d\log d))\) for general read-once Boolean formulas and \(\Omega(n/4^d)\) for a special class of “balanced” formulas.

It follows from results of Snir [Sni85] and Saks and Wigderson [SW86] (via a generic simulation of trees by communication protocols [BCW98]) that for the read-once Boolean formula with their canonical tree being a complete binary alternating AND-OR tree, the randomized communication complexity is \(O(n^{0.753\ldots})\), the best known so far. However in this situation, the results of [LS09, JKR09] do not provide any lower bound since \(d = \log_2 n\) for the complete binary tree. We complement their result by giving universal lower bounds that do not depend on the depth. Below \(Q(\cdot)\) represents the two-party quantum communication complexity.

Theorem 2 Let \(f : \{0, 1\}^n \to \{0, 1\}\) be a read-once Boolean formula. Then,
\[
\max \{R(f^\wedge), R(f^\vee)\} \geq \Omega(\sqrt{n}).
\]
\[
\max \{Q(f^\wedge), Q(f^\vee)\} \geq \Omega(n^{1/4}).
\]

Remark:
1. Note that the maximum in Theorem 1 and 2 is necessary since for example if \(f\) is the AND of the \(n\) input bits then it is easily seen that \(R(f^\wedge)\) is 1.
2. This fact is easy to observe for balanced trees, as also remarked in [LS09].

2 Proofs

In this section we show the proof of Theorem 2. We start with the following definition.

Definition 1 (Embedding) We say that a function \(g_1 : \{0, 1\}^r \times \{0, 1\}^r \to \{0, 1\}\) can be embedded into a function \(g_2 : \{0, 1\}^t \times \{0, 1\}^t \to \{0, 1\}\), if there exist maps \(h_a : \{0, 1\}^r \to \{0, 1\}^t\) and \(h_b : \{0, 1\}^r \to \{0, 1\}^t\) such that \(\forall x, y \in \{0, 1\}^r\), \(g_1(x, y) = g_2(h_a(x), h_b(y))\).

It is easily seen that if \(g_1\) can be embedded into \(g_2\) then the communication complexity of \(g_2\) is at least as large as that of \(g_1\).

Let us define the Disjointness problem \(\text{DISJ}_n : \{0, 1\}^n \times \{0, 1\}^n \to \{0, 1\}\) as \(\text{DISJ}_n(x, y) = \bigwedge_{i=1,\ldots,n}(x_i \lor y_i)\) (where the usual negation of the variables is left out for notational simplicity). Similarly the Non-Disjointness problem \(\text{NDISJ}_n : \{0, 1\}^n \times \{0, 1\}^n \to \{0, 1\}\) is defined as \(\text{NDISJ}_n(x, y) = \bigvee_{i=1,\ldots,n}(x_i \land y_i)\). We shall also use the following well-known lower bounds.

Fact 1 ([KS92, Raz92]) \(R(\text{DISJ}_n) = \Omega(n), R(\text{NDISJ}_n) = \Omega(n)\).

Fact 2 ([Raz03]) \(Q(\text{DISJ}_n) = \Omega(\sqrt{n}), Q(\text{NDISJ}_n) = \Omega(\sqrt{n})\).

Recall that for the given read-once Boolean formula \(f : \{0, 1\}^n \to \{0, 1\}\) its the canonical tree is denoted \(T_f\). We have the following lemma which we prove in Section 2.

Lemma 3 1. Let \(T_f\) have its last layer consisting only of AND gates. Let \(m_0\) be the largest integer such that \(\text{DISJ}_{m_0}\) can be embedded into \(f^\wedge\) and \(m_1\) be the largest integer such that \(\text{NDISJ}_{m_1}\) can be embedded into \(f^\vee\). Then \(m_0m_1 \geq n\).

2. Let \(T_f\) have its last layer consisting only of OR gates. Let \(m_0\) be the largest integer such that \(\text{DISJ}_{m_0}\) can be embedded into \(f^\wedge\) and \(m_1\) be the largest integer such that \(\text{NDISJ}_{m_1}\) can be embedded into \(f^\vee\). Then \(m_0m_1 \geq n\).
With this lemma, we can prove the lower bounds on \( \max \{ R(f^\top), R(f^\wedge) \} \) and \( \max \{ Q(f^\top), Q(f^\wedge) \} \) as follows. For an arbitrary read-once formula \( f \) with \( n \) variables, consider the sets of leaves

\[
L_{\text{odd}} = \{ \text{leaves in } T_f \text{ on odd levels} \}, \quad L_{\text{even}} = \{ \text{leaves in } T_f \text{ on even levels} \}
\]

At least one of the two sets has size at least \( n/2 \); without loss of generality, let us assume that it is \( L_{\text{odd}} \). Depending on whether the root is AND or OR, this set consisting only of AND gates or OR gates, corresponding to case 1 or 2 in Lemma \ref{lem:odd}. Then by the lemma, either \( \text{DISJ}_{\sqrt{n/2}} \) or \( \text{NDISJ}_{\sqrt{n/2}} \) can be embedded in \( f \) (by setting the leaves in \( L_{\text{even}} \) to 0's). By Fact \ref{fact:1} and \ref{fact:2} we get the lower bounds in Theorem \ref{thm:main}.

### 2.1 Proof of Lemma \ref{lem:odd}

We shall prove the first statement; the second statement follows similarly. We first prove the following claim.

**Claim 1** For an \( n \)-leaf \( (n > 2) \) alternating AND-OR tree \( T \) such that all its internal nodes just above the leaves have exactly two children (denoted the \( x \)-child and the \( y \)-child), let \( s(T) \) denote the number of such nodes directly above the leaves. Let \( m_0(T) \) be the largest integer such that \( \text{DISJ}_{m_0} \) can be embedded into \( f_T \) and \( m_1(T) \) be the largest integer such that \( \text{NDISJ}_{m_1} \) can be embedded into \( f_T \). Then \( m_0(T)m_1(T) \geq s(T) \).

**Proof:** The proof is by induction on depth \( d \) of \( T \). When \( n > 2 \), the condition of the tree makes \( d > 1 \), so the base case is \( d = 2 \).

**Base Case** \( d = 2 \): In this case \( T \) consists either of the root labeled AND with \( s(T) \) (fan-in 2) children labeled ORs or it consists of the root labeled OR with \( s(T) \) (fan-in 2) children labeled ANDs. We consider the former case and the latter follows similarly. In the former case \( f_T \) is clearly \( \text{DISJ}_{s(T)} \) and hence \( m_0(T) = s(T) \). Also \( m_1(T) \geq 1 \) as follows. Let us choose the first two children \( v_1, v_2 \) of the root. Further choose the \( x \) child of \( v_1 \) and the \( y \) child of \( v_2 \) which are kept free and the values of all other input variables are set to 0. It is easily seen that the function (of input bits \( x, y \)) now evaluated is \( \text{NDISJ}_1 \). Hence \( m_0(T)m_1(T) \geq s(T) \).

**Induction Step** \( d > 2 \): Assume the root is labeled AND (the case when the root is labeled OR follows similarly). Let the root have \( r \) children \( v_1, \ldots, v_r \) which are labeled OR and let the corresponding subtrees be \( T_1, \ldots, T_r \) rooted at \( v_1, \ldots, v_r \) respectively. Let without loss of generality the first \( r' \) (with \( 0 \leq r' \leq r \)) of these trees be of depth 1 in which case the corresponding \( s(\cdot) = 0 \). It is easily seen that

\[
s(T) = r' + \left( \sum_{i=r'+1}^{r} s(T_i) \right).
\]

For \( i > r' \), we have from the induction hypothesis that \( m_1(T_i)m_0(T_i) \geq s(T_i) \).

It is clear that \( m_0(T) \geq \sum_{i=1}^{r} m_0(T_i) \), since we can simply combine the Disjointness instances of the subtrees. Also we have \( m_1(T) \geq \max \{ m_1(T_{r'+1}), \ldots, m_1(T_r), 1 \} \), because we can either take any one of the subtree instances (and set all other inputs to 0), or at the very least can pick a pair of \( x, y \) leaves (as in the base case above) and fix the remaining variables appropriately to
realize a single AND gate which amounts to embedding NDISJ 1. Now,

\[ m_0(T) m_1(T) \geq \left( \sum_{i=1}^{r} m_0(T_i) \right) \cdot (\max\{m_1(T_1), \ldots, m_1(T_r), 1\}) \]

\[ \geq r' + \left( \sum_{i=r'+1}^{r} m_0(T_i) m_1(T_i) \right) \]

\[ \geq r' + \left( \sum_{i=r'+1}^{r} s(T_i) \right) = s(T). \]

Now we prove Lemma 3. Let us view \( f^\lor : \{0, 1\}^{2n} \to \{0, 1\} \) as a read-once Boolean formula, with input \( (x, y) \) of \( f^\lor \) corresponding to the \( x \)- and \( y \)-children of the internal nodes just above the leaves. Note that in this case \( T_{f^\lor} \) satisfies the conditions of the above claim and \( s(T_{f^\lor}) = n \). Hence the proof of the first statement in Lemma 3 finishes.

3 Concluding Remarks

1. The randomized communication complexity varies between \( \Theta(n) \) for the Tribes n function (a read-once Boolean formula whose canonical tree has depth 2) \cite{IKS03} and \( O(n^{0.753 \ldots}) \) for functions corresponding to completely balanced AND-OR trees (which have depth log n).

It will probably be hard to prove a generic lower bound much larger than \( \sqrt{n} \) for all read-once Boolean formulas \( f : \{0, 1\}^n \to \{0, 1\} \), since the best known lower bound on the randomized query complexity of every read-once Boolean formula is \( \Omega(n^{0.51}) \) \cite{HW91} and communication complexity lower bounds immediately imply slightly weaker query complexity lower bounds (via the generic simulation of trees by communication protocols \cite{BCW98}).

2. Ambainis et al. \cite{ACR+07} show how to evaluate any alternating AND-OR tree \( T \) with \( n \) leaves by a quantum query algorithm with slightly more than \( \sqrt{n} \) queries; this also gives the same upper bound for the communication complexity of \( \max\{Q(f^\lor), Q(f^\land)\} \). On the other hand, it is easily seen that the parity of \( n \) bits can be computed by a formula of size \( O(n^2) \) involving \( \text{AND, OR} \). Therefore it is easy to show that the function \( \text{Inner Product modulo 2} \ i.e. \) the function \( IP_m : \{0, 1\}^m \times \{0, 1\}^m \to \{0, 1\} \) given by \( IP_m(x, y) = \sum_{i=1}^{m} x_i y_i \bmod 2 \), with \( m = \sqrt{n} \) can be reduced to the evaluation of an alternating AND-OR tree of size \( O(n) \) and logarithmic depth. Since it is known that \( Q(IP_\sqrt{n}) = \Omega(\sqrt{n}) \) \cite{CdNT99}, we get an example of an alternating AND-OR tree \( T \) with \( n \) leaves and \( \log n \) depth such that \( Q(f^\lor) = \Omega(\sqrt{n}) \). Since the same lower bound also holds for shallow trees such as \( \text{OR} \), hence \( \Theta(\sqrt{n}) \) might turn out to be the correct bound on \( \max\{Q(f^\lor), Q(f^\land)\} \) for all alternating AND-OR trees \( T \) with \( n \) leaves regardless of the depth.

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