Poisson Algebra of Differential Forms

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Abstract

We give a natural definition of a Poisson Differential Algebra. Consistency conditions are formulated in geometrical terms. It is found that one can often locally put the Poisson structure on the differential calculus in a simple canonical form by a coordinate transformation. This is in analogy with the standard Darboux’s theorem for symplectic geometry. For certain cases there exists a realization of the exterior derivative through a certain canonical one-form. All the above are carried out similarly for the case of a complex Poisson Differential Algebra. The case of one complex dimension is treated in detail and interesting features are noted. Conclusions are made in the last section.
1 Introduction

The motivation of the study of Poisson algebra of differential forms is two-fold. On
the one hand we want to understand differential calculi on quantum spaces from their
Poisson limits; on the other hand it is by itself of mathematical interest to generalize
the Poisson algebra to the differential calculus.

In Sec.2 we give the definition of Poisson differential algebra, which is motivated
by the Poisson limit of quantum differential calculi [1, 2, 3, 4]. Although the definition
seems to be quite natural, the consistency conditions turn out to be very restrictive. We
will see in Sec.3 that on a local symplectic patch the Poisson structure can always be
brought to a canonical form and the possible Poisson structures are labelled by a finite
number of parameters, among them, for example, are the $r$-matrices of the classical
Yang-Baxter equation. Equivalent Poisson Differential Algebras have their $r$-matrices
related to each other by similarity transformations. It is shown in Sec.4 that when the
torsion has a zero there exists a one-form realization of the exterior derivative [1, 2, 3, 4].
We extend the definition and results in Secs.2-4 to the case of complex manifolds in
Sec.5. We notice interesting cases where there is a natural Kähler form which gives a
covariantly constant metric and which is closely related to the one-form realization of
the exterior derivative. In Sec.6 we focus on one-dimensional complex manifolds and
find even stronger results than in the general case. The only allowed metrics are those
of a plane, a sphere and a Lobachevskian disk.

2 Definition of Poisson Differential Algebras

**Definition 1** For a differential calculus $\Omega$ generated by the coordinates \( \{x^\alpha\} \) and differ-
ential forms \( \{dx^\alpha\} \) on a differentiable manifold, a bilinear map \((\cdot, \cdot) : \Omega \otimes \Omega \rightarrow \Omega\) is
called a Poisson structure if for arbitrary $f, g, h \in \Omega$

1. It has the symmetry
   \[(f, g) = (-1)^{p(f)p(g)+1}(g, f),\]
   where the parity $p(f) = 0$ if $f$ is even and $p(f) = 1$ if $f$ is odd. (1)

2. The graded Jacobi identity holds
   \[(f, (g, h)) + (-1)^{p(f)(p(g)+p(h))}(g, (h, f)) + (-1)^{p(h)(p(f)+p(g))}(h, (f, g)) = 0.\] (2)

3. Elements in $\Omega$ act as derivation via the map on other elements
   \[(f, gh) = (f, g)h + (-1)^{p(f)p(g)} g(f, h).\] (3)

4. The Leibniz rule holds for the exterior derivative $d$
   \[d(f, g) = (df, g) + (-1)^{p(f)}(f, dg).\] (4)
5. \( (f, g) \) is a form of degree equal to the sum of the degrees of \( f \) and \( g \).

The differential algebra equipped with such a Poisson structure is called a Poisson differential algebra.

The first three properties can be derived if the Poisson structure is obtained from the limit of a graded commutator. They also resemble the properties of a graded universal enveloping algebra of a Lie algebra. The last two properties are motivated by the quantization of the differential calculus for a quantum space \([1, 2, 3, 4, 5, 6]\). By studying this Poisson differential algebra we will be able to reach a better understanding of the quantum differential calculus. It is remarkable that this simple definition of a Poisson structure on a differential calculus naturally leads to geometrical notions such as connection, torsion and curvature.

Assuming that the matrix \( P^{\alpha\beta} \) is invertible, the Poisson algebra is completely specified by the functions \( P^{\alpha\beta} \) and \( \Gamma_{\alpha\beta}^\gamma \) defined by

\[
(x^\alpha, x^\beta) = P^{\alpha\beta},
\]

\[
(x^\alpha, dx^\beta) = -P^{\alpha\gamma}\Gamma_{\gamma\delta}^\beta dx^\delta.
\]

\( P^{\alpha\beta} \) gives the usual Poisson structure on functions. Under the general coordinate transformation \( x \to x' = x'(x) \), it transforms as a rank-two tensor according to

\[
P^{\alpha\beta} \to P'^{\alpha\beta} = \frac{\partial x^\kappa}{\partial x'^\alpha} \frac{\partial x'^\beta}{\partial x^\gamma} P^{\gamma\delta}.
\]

The additional data required for a Poisson structure on \( \Omega \) is given by \( \Gamma_{\alpha\beta}^\gamma \), which transforms as a connection as a result of

\[
\Gamma_{\beta\gamma}^\alpha \to \Gamma'_{\beta\gamma}^\alpha = \frac{\partial x^\kappa}{\partial x'^\beta} \frac{\partial x'^\lambda}{\partial x^\gamma} \left( \frac{\partial x'^\alpha}{\partial x^\delta} \Gamma_{\gamma\delta}^\kappa - \frac{\partial^2 x^\alpha}{\partial x^\kappa \partial x^\gamma} \right).
\]

We define the connection one-form for a Poisson algebra by

\[
\Gamma^\alpha_\beta = dx^\gamma \Gamma_{\gamma\beta}^\alpha.
\]

Note that

\[
\tilde{\Gamma}_{\beta}^\alpha = \Gamma_{\beta\gamma}^\alpha dx^\gamma
\]

also transforms like a connection one-form but is different from the one above because in general the torsion

\[
T_{\beta\gamma}^\alpha = \Gamma_{\beta\gamma}^\alpha - \Gamma_{\gamma\beta}^\alpha
\]

does not vanish.

For a tensor \( U_{\beta\ldots}^{\alpha\ldots} \), the covariant derivative \( \nabla_\delta U_{\beta\ldots}^{\alpha\ldots} \) is defined by

\[
\nabla_\delta U_{\beta\ldots}^{\alpha\ldots} = \partial_\delta U_{\beta\ldots}^{\alpha\ldots} + \Gamma_{\delta\mu}^{\alpha} U_{\beta\ldots}^{\mu\ldots} - U_{\mu\ldots}^{\alpha\ldots} \Gamma_{\delta\beta}^{\mu},
\]

and similarly for the covariant derivative \( \tilde{\nabla} \) with respect to \( \tilde{\Gamma} \).

We will see later that the torsion plays an important role in the Poisson algebra.
3 Solutions of Poisson Algebras

The Poisson structure functions $P^{\alpha\beta}$ and $\Gamma^\gamma_{\beta\gamma}$ are greatly constrained by the Jacobi identities, the Leibniz rule and the other properties of a Poisson differential algebra. We will show in this section that on a symplectic local patch of the manifold the Poisson structure is completely characterized by a finite number of parameters.

The algebra of differential calculus on a local patch is generated by the functions $\{x^\alpha\}$ and one-forms $\{dx^\alpha\}$. We shall examine all Jacobi identities and Leibniz rules applied to all generators.

First of all $P^{\alpha\beta}$ has to satisfy the Jacobi identity for three functions

$$
\sum_{(\alpha,\beta,\gamma)} P^{\alpha\delta} \frac{\partial}{\partial x^\delta} P^{\beta\gamma} = 0,
$$

where $(\alpha,\beta,\gamma)$ indicates cyclic permutations of $\alpha, \beta, \gamma$.

In the following we assume that $P^{\alpha\beta}$ is invertible with the inverse $P_{\alpha\beta} (P_{\alpha\gamma} P^{\gamma\beta} = P^{\beta\gamma} P_{\gamma\alpha} = \delta^\beta_\alpha)$ giving the symplectic structure. Eq. (13) can be written as

$$
\sum_{(\alpha,\beta,\gamma)} \partial_\alpha P_{\beta\gamma} = 0,
$$

where $\partial_\alpha$ denotes $\frac{\partial}{\partial x^\alpha}$, or equivalently,

$$
dP = 0,
$$

where $P$ is the symplectic two-form

$$
P = \frac{1}{2} P_{\alpha\beta} dx^\alpha dx^\beta.
$$

Using Eq. (13), one can show that the Jacobi identity for two functions and one one-form requires that the Riemann curvature vanishes

$$
R^\alpha_\beta = \frac{1}{2} R^\alpha_{\beta\gamma\delta} dx^\gamma dx^\delta = 0,
$$

where

$$
R^\alpha_\beta = d\Gamma^\alpha_\beta + \Gamma^\alpha_\gamma \Gamma^\gamma_\beta.
$$

The vanishing of the curvature implies that the connection is a pure gauge. Hence there exists a square matrix of functions $M^{\alpha A}$ such that

$$
\Gamma^\alpha_{\beta\gamma} = M^{\alpha A} \partial_\beta M_{A\gamma},
$$

where $M_{A\alpha}$ is the inverse of $M^{\alpha A}$. Note that (19) defines $M^{\alpha A}$ up to an invertible constant matrix $N^A_B : M^{\alpha A}$ and $M^{\alpha B} N^A_B$ give the same connection.
The Leibniz rule for two functions

\[ d(x^\alpha, x^\beta) = (dx^\alpha, x^\beta) + (x^\alpha, dx^\beta) \]  

implies that

\[ \tilde{\nabla} P^{\alpha\beta} = dP^{\alpha\beta} + P^{\alpha\gamma} \tilde{\Gamma}^{\beta}_{\gamma} + \tilde{\Gamma}^{\alpha}_{\beta} P^{\gamma\beta} = 0. \]

(21)

Plugging (19) into this equation we find

\[ \partial_\alpha G^A_{\beta} - \partial_\beta G^A_{\alpha} = 0, \]

(22)

where \( G^A_{\alpha} = P_{\alpha\beta} M^A_{\beta} \). This implies that locally we can find functions \( \{ \Phi^A \} \) such that

\[ G^A_{\alpha} = \partial_\alpha \Phi^A. \]

(23)

Hence

\[ M^{\alpha A} = P^{\alpha \beta} \partial_\beta \Phi^A. \]

(24)

The Leibniz rule for one function and one one-form follows from (21).

Using Eqs. (13) and (17), one can show after considerable calculations that the Jacobi identity for one function and two one-forms implies

\[ \nabla_\sigma (P^{\alpha \gamma} \tilde{R}^{\beta}_{\gamma \alpha \lambda}) = 0. \]

(25)

With the Leibniz rule (4) the Jacobi identity for three one-forms follows. It is easy to convince oneself that Eqs. (13), (17), (21) and (25) are all the conditions on \( P^{\alpha \beta} \) and \( \Gamma^\alpha_{\beta\gamma} \) that a Poisson differential algebra has to satisfy.

Now, we are going to prove that for any Poisson differential algebra with a symplectic two-form (16), there is a preferred local coordinate chart in which the Poisson structure is a quadratic form of the local coordinates (Eq. (45) below) and the connection coefficients can be expressed in terms of the same Poisson structure in a simple manner (Eq. (27) below). This is in some sense an analogue of the Darboux’ theorem for symplectic geometry (4).

Since \( M \) and \( P \) are both invertible matrices, \( G \) is also invertible. Hence by Eq. (24) \( \{ \Phi^A \} \) can be used as local coordinates. In this new coordinate system \( \{ \Phi^A \} \), we have

\[ M^{AB} = P^{AB}, \]

(26)

\[ \Gamma^A_{BC} = P^{AD} \partial_B P_{DC}, \]

(27)

\[ T^A_{BC} = P^{AD} \partial_D P_{BC}, \]

(28)

by Eqs. (13), (19) and (24).

It is interesting that in this coordinate system everything can be expressed in terms of \( P^{AB} = (\Phi^A, \Phi^B) \) alone. Note that due to the ambiguity in the matrix \( M^{\alpha A} \), one can perform the transformation

\[ \Phi^A \rightarrow N^A_B \Phi^B + V^A \]

(29)
on the coordinate $\Phi^A$ without changing the connection coefficient $\Gamma^\alpha_{\beta\gamma}$. Here $N^A_B$ and $V^A$ are constants.

Define a new basis of one-forms
\[ e_A \equiv M_{A\alpha} dx^\alpha = P_{AB} d\Phi^B. \] (30)

The connection vanishes in this basis
\[ \nabla^A = P^{AB} \partial_B \] (31)
and the torsion is
\[ T^B_{\ A} = \partial_A P^{BC}. \] (32)

Since we use the same kind of indices $(A, B, C, \cdots)$ for both bases \{\text{\textit{e}}_A\} and \{\text{\textit{d}}\Phi^A\}, one has to keep in mind which basis one is dealing with for a tensor with indices $(A, B, C, \cdots)$. We will say it explicitly whenever there could be a possible confusion.

It can be checked that
\[ (e_A, e_B) = -\tilde{R}_{AB} = -\frac{1}{2} \tilde{R}^{CD}_{AB} e_C e_D, \] (33)
where $\tilde{R}^{A}_{BCD}$ are the components of the curvature tensor in the \{\text{\textit{d}}\Phi^A\} bases,
\[ \tilde{R}^A_{BCD} = G^A_\alpha G^\beta_C G^\gamma_D G^\delta_B \tilde{R}^\alpha_{\beta\gamma\delta}, \] (34)
and $\tilde{R}^{CD}_{AB}$ comes from $\tilde{R}^A_{BCD}$ by raising and lowering indices with the $P^{AB}$,
\[ \tilde{R}^{CD}_{AB} = P_{AE} P^{CF} P^{DG} \tilde{R}^E_{BFG}. \] (35)
It satisfies
\[ \tilde{R}^{CD}_{AB} = \tilde{R}^{CD}_{BA} = -\tilde{R}^{DC}_{AB}. \] (36)

The basis \{\text{\textit{e}}_A\} worths special treatment because the Poisson brackets between $e_A$ and functions vanish
\[ (e_A, x^\alpha) = 0. \] (37)
Calculations in this basis could be much simpler than those in others. For example, to check the Jacobi identity for one function and two one-forms, it is sufficient to check
\[ (x^\alpha, (e_A, e_B)) = 0, \] (38)
which immediately gives
\[ \tilde{R}^{CD}_{AB} = \text{constants}. \] (39)

The same result can be obtained by expressing Eq.(25) in the basis \{\text{\textit{e}}_A\} using (17).

Using (17) and the following two identities:
\[ \sum_{(\beta, \gamma, \delta)} (R^\alpha_{\beta\gamma\delta} + \nabla_\beta T^\alpha_{\gamma\delta} - T^\alpha_{\beta\gamma} T^\kappa_{\gamma\delta}) = 0, \] (40)
\[ \tilde{R}^\alpha_{\beta\gamma\delta} - R^\alpha_{\beta\gamma\delta} = -\nabla_\gamma T^\alpha_{\delta\beta} - \nabla_\delta T^\alpha_{\beta\gamma} + T^\alpha_{\beta\kappa} T^\kappa_{\gamma\delta} + T^\alpha_{\gamma\kappa} T^\kappa_{\delta\beta} + T^\alpha_{\delta\kappa} T^\kappa_{\beta\gamma}, \] (41)
we find
\[ \tilde{R}^\alpha_{\beta\gamma\delta} = \nabla_\beta T^\alpha_{\gamma\delta}. \tag{42} \]
In the basis \( \{ e_A \} \) it is
\[ \tilde{R}^{CD}_{AB} = \partial_B T^C_A \tag{43} \]
which can be easily solved
\[ T^C_A = \tilde{R}^{CD}_{AB} \Phi^B + f^C_A, \tag{44} \]
where \( f^C_A = -f^D_A \) are constants. Now \( P^{AB} \) can be solved from (32)
\[ P^{AB} = (\Phi^A, \Phi^B) = \frac{1}{2} \tilde{R}^{AB}_{CD} \Phi^C \Phi^D + f^{AB}_C \Phi^C + g^{AB}, \tag{45} \]
where \( g^{AB} = -g^{BA} \) are constants.

We will call \( \{ \Phi^A \} \) the canonical coordinate system. If the torsion vanishes, the canonical coordinates will coincide with the Darboux coordinates up to the transformation (29), and the Poisson structure between function and forms will be trivial.

It is remarkable that in the canonical coordinate system all information about a Poisson algebra with invertible \( P^{AB} \) is encoded in the constants \( \tilde{R}^{AB}_{CD}, f^{AB}_C \) and \( g^{AB} \).

It is interesting that the curvature \( \tilde{R}^{AB}_{CD} \) must satisfy the classical Yang-Baxter equation. This can be seen as a result of the consistence of the Jacobi identity. Using the tensor product notation
\[ \tilde{R}_{12} = \tilde{R} \otimes 1, \tag{46} \]
\[ (\tilde{R}_{12})^{AB}_{CD} = \tilde{R}^{AB}_{CD}, \tag{47} \]
(45) can be written as
\[ (\Phi_1, \Phi_2) = \frac{1}{2} \tilde{R}_{12} \Phi_1 \Phi_2 + f_{123} \Phi_3 + g_{12}, \tag{48} \]
where matrix multiplication is implied. The Jacobi identity of three functions implies
\[ 0 = \sum_{(1,2,3)} (\Phi_1, (\Phi_2, \Phi_3)) \]
\[ = -\frac{1}{4} ([\tilde{R}_{12}, \tilde{R}_{13}] + [\tilde{R}_{12}, \tilde{R}_{23}] + [\tilde{R}_{13}, \tilde{R}_{23}]) \Phi_1 \Phi_2 \Phi_3 \]
\[ + \frac{1}{2} \sum_{(1,2,3)} (\tilde{R}_{23}(f_{124} \Phi_3 \Phi_4 + f_{134} \Phi_2 \Phi_4) + f_{234} \tilde{R}_{14} \Phi_1 \Phi_4) \]
\[ + \sum_{(1,2,3)} (\frac{1}{2} (\tilde{R}_{13} + \tilde{R}_{23}) g_{12} \Phi_3 + f_{234} f_{145} \Phi_5) \]
\[ + \sum_{(1,2,3)} (f_{234} g_{14}), \tag{49} \]
where \((1,2,3)\) stands for cyclic permutation of 1,2 and 3. Vanishing of the \( \Phi_1 \Phi_2 \Phi_3 \) term gives the classical Yang-Baxter equation,
\[ [\tilde{R}_{12}, \tilde{R}_{13}] + [\tilde{R}_{12}, \tilde{R}_{23}] + [\tilde{R}_{13}, \tilde{R}_{23}] = 0. \tag{50} \]
Other conditions can be obtained from the vanishing of the constant, $\Phi$ and $\Phi\Phi$ terms. Explicitly they are

\[
\begin{align*}
\sum_{(A,B,C)} (2\tilde{R}^{AB}_{FD} f^C_E + f^A_E \tilde{R}^{CF}_{DE}) &= 0, \\
\sum_{(A,B,C)} (\tilde{R}^{AB}_{ED} g^{CE} + f^A_E f^C_D) &= 0, \\
\sum_{(A,B,C)} f^D_B g^{CD} &= 0.
\end{align*}
\]

(51)  (52)  (53)

We will see in the next section that solutions with $f^A_B = 0$ is of particular interest.

4 Properties of Poisson Algebras

Define

$$\xi = -e_A \Phi^A.$$  (54)

Then

$$(\xi, f) = df$$  (55)

for an arbitrary function $f$. One can check that

$$(\xi, dx^\alpha) = -\frac{1}{2} M^{\alpha A} f^C_D e_C e_D.$$  (56)

Therefore the one-form $\xi$ will become a one-form realization of the exterior derivative $d\omega$ for an arbitrary differential form $\omega$ if $f^A_B = 0$.

Note that sometimes it is possible to make a transformation (29) so that in the new coordinate system $f^A_B = 0$ and a one-form realization (54) exists. The existence of such a $\Phi$-coordinate system is equivalent to the existence of a zero of the torsion tensor. This can be seen from the following expression of $P^{AB}$:

$$P^{AB} = \frac{1}{2} \tilde{R}^{AB}_{CD} \Phi^C \Phi^D + T^{AB}_C(0) \Phi^C + P^{AB}(0),$$  (58)

where $T^{AB}_C(0)$ and $P^{AB}(0)$ are the torsion tensor and Poisson tensor evaluated at the origin $\Phi^A = 0$. Since the transformation (29) can translate the origin to any other point on the patch, if the torsion has a zero one can always choose the new origin to reside on that point. In addition, Eq. (58) shows that the maps from the space of all $\Phi$-coordinate systems, where an element is labelled by $(N^A_B, V^A)$, to the coefficients $\tilde{R}^{AB}_{CD}, f^A_B, g^{AB}$ of the Poisson tensor are the same as the composite of a rotation by $N^A_B$ with the tensors $\tilde{R}^{AB}_{CD}, T^{AB}_C, P^{AB}$ as maps from the local patch to the values of those tensors.

While in general

$$d\xi = (\frac{1}{2} f^A_C \Phi^C + g^{AB}) e_A e_B,$$  (59)
when \( f_C^{AB} = 0 \) it is tempting to interpret \( d\xi \) as the length element squared, \( g^{AB} \) as the metric and \( e_A \) as the vielbein. The tensor \( g^{AB} \) defined this way is covariantly constant with respect to the connection \( \Gamma \). But since \( g^{AB} \) is antisymmetric and \( d\xi, e_A \) are forms, in fact \( d\xi \) is more like the Kähler two form for a complex manifold.

As a result of (31), a metric \( h_{\alpha\beta} \) is covariantly constant, 
\[
\nabla h_{\alpha\beta} = 0
\]
if and only if its components \( h^{AB} \) in the \( \{e_A\} \) basis are constant.

Before going into the next topic, we remark that if a covariantly constant metric \( h \) is desired on the manifold, then (60) and (21) determine the connection to be
\[
\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} P_{\beta\delta} h_{\gamma} (h^{\epsilon\delta} \partial_{\kappa} P^{\alpha\delta} + h^{\alpha\kappa} \partial_{\kappa} P^{\beta\delta} - h^{\delta\kappa} \partial_{\kappa} P^{\alpha\beta} + P^{\epsilon\kappa} \partial_{\kappa} h_{\gamma}^{\alpha\delta} - P^{\delta\kappa} \partial_{\kappa} h_{\gamma}^{\alpha\epsilon} - P^{\epsilon\kappa} \partial_{\kappa} h_{\gamma}^{\beta\delta} + P^{\delta\kappa} \partial_{\kappa} h_{\gamma}^{\beta\epsilon})
\]
without using any other relations. This result may be relevant to certain generalizations of Riemannian geometry as Eq.s (60) and (21) relate \( h \) and \( P \) to a covariantly constant complex Hermitian tensor.

5 Complex Poisson Differential Algebras

All of the above can be easily specialized for the case of a complex differentiable manifold,

**Definition 2** For a differential calculus \( \Omega \) generated by the coordinates \( \{x^i, x^\bar{i}\} \) and differential forms \( \{dx^i, dx^\bar{i}\} \) on an \( N \) dimensional complex manifold, the map \((\cdot, \cdot) : \Omega \otimes \Omega \to \Omega\) is called a Poisson structure on \( \Omega \) if for arbitrary \( f, g, h \in \Omega \), it satisfies properties 1 to 3 of Definition 1 and also

4. The Leibniz rule holds for the holomorphic and anti-holomorphic exterior derivatives \( \delta, \bar{\delta} \):
\[
\delta (f, g) = (\delta f, g) + (-1)^{p(f)} (f, \delta g),
\]
\[
\bar{\delta} (f, g) = (\bar{\delta} f, g) + (-1)^{p(f)} (f, \bar{\delta} g).
\]

5. The holomorphic and anti-holomorphic degrees of the form \((f, g)\) are equal to the sums of those of \( f \) and \( g \).

6. Hermiticity:
\[
(f, g)^\ast = (-1)^{p(f)p(g)} (g^\ast, f^\ast),
\]
where the \( \ast \) operation on \( \Omega \) is the complex conjugation together with an inversion of ordering in a product:
\[
(f g)^\ast = g^\ast f^\ast = (-1)^{p(f)p(g)} f^\ast g^\ast.
\]

A complex differential algebra equipped with such a Poisson structure is called a complex Poisson differential algebra.
Using the notation \( \alpha \in I \cup \bar{I} \), where \( I = \{i, j, k, \ldots\} \) is the set of holomorphic indices and \( \bar{I} = \{\bar{i}, \bar{j}, \bar{k}, \ldots\} \) is the set of anti-holomorphic indices. Assuming that the matrix \( P^{\alpha \beta} \) is invertible, the complex Poisson differential algebra is defined by the functions \( P^{\alpha \beta}, \Gamma_{\beta \gamma}^{\alpha} \) as in (5) and (6), where \( \{x^\alpha\} = \{x^i, x^{\bar{i}}\} \).

Vanishing of the curvature \( R^{\alpha \beta \gamma \delta} = 0 \) implies that connection is of the form (19), which can be solved as

\[
M_{A\alpha}(0) = M_{A\beta}(0)(P \exp \int_0^1 \Gamma)_{\beta}^{\alpha},
\]

where \( M_{A\alpha}(0) \) is the value of \( M_{A\alpha} \) at a certain point where the path for the line integral \( \int_0^1 \) begins, \( P \) stands for path ordering and \( \Gamma \) is the matrix of one-forms \( \Gamma_{\beta}^{\alpha} \). \( M_{A\alpha}(0) \) can be any invertible constant matrix. We choose it to be block-diagonal

\[
M_{A\alpha}(0) = \begin{pmatrix} M_{ai} & 0 \\ 0 & M_{\bar{a}i} \end{pmatrix},
\]

where we split the index \( A \) into \( a \) and \( \bar{a} \), with no implication on the holomorphicity.

The new input property 5 implies that the following components of the connection vanishes

\[
\Gamma_{\alpha k}^{j} = \Gamma_{\alpha k}^{j} = 0
\]

so that \( \Gamma \) is also block-diagonal. It follows from (66) that \( M_{A\alpha} \) must be block-diagonal as well.

The basis of one-forms (30) now splits into the holomorphic part \( \{e_a\} \) and the antiholomorphic part \( \{e_{\bar{a}}\} \), where

\[
e_a = M_{ai}dx^i, \quad e_{\bar{a}} = M_{\bar{a}i}dx^{\bar{i}}
\]

satisfy

\[
(e_a, x^i) = (e_{\bar{a}}, x^{\bar{i}}) = 0.
\]

For arbitrary nondegenerate Hermitian constant matrix \( h^{ab}, h^{a\bar{b}}e_ae_{\bar{b}} \) is an admissible Kähler two-form [8] which gives a covariantly constant metric and is central to all functions.

It follows from property 5 and (33) that \( \tilde{R}^{AB}_{CD} \) vanishes unless the number of barred (unbarred) superscripts equals the number of barred (unbarred) subscripts. For example,

\[
\tilde{R}^{d\bar{e}}_{ab} = \tilde{R}^{d\bar{e}}_{\bar{a}b} = 0.
\]

Given the matrix \( M^{\alpha A} \) one can find \( \Phi^A \) up to additive constants from (24). \( \{\Phi^A\} \) is split into \( \{\phi^a\} \) and \( \{\phi^{\bar{a}}\} \). In terms of these functions \( \{\phi^a, \phi^{\bar{a}}\} \), the Poisson structure is particularly simple. For example,

\[
P^{ab} = (\phi^a, \phi^{\bar{b}}) = \frac{1}{2} \tilde{R}^{a\bar{b}}_{cd} \phi^c \phi^{\bar{d}} + f^{ab}_{\bar{c}} \phi^c + g^{ab}
\]

and

\[
P^{a\bar{b}} = (\phi^a, \phi^{\bar{b}}) = \tilde{R}^{a\bar{b}}_{cd} \phi^c \phi^{\bar{d}} + f^{ab}_{\bar{c}} \phi^c + f^{a\bar{b}}_{\bar{c}} \phi^{\bar{c}} + g^{a\bar{b}}.
\]
In general, \((\phi^a)^*\) is different from \(\phi^\bar{a}\). But using the Hermiticity of the Poisson structure (property 6) and (24) one sees that by choosing \(M_{Aa}(0)\) to be of the form
\[
M_{Aa}(0) = \begin{pmatrix} m_{ai} & 0 \\ 0 & m_{\bar{a}i} \end{pmatrix}
\] (74)
where \(m_{\bar{a}i}\) is defined as
\[
m_{\bar{a}i} \equiv -m^*_{ai}
\] (75)
and \(m\) is some invertible matrix, we get
\[
(\partial_z \Phi^a)^* = \partial_{\bar{z}} \Phi^\bar{a}, \quad (\partial_z \Phi^\bar{a})^* = \partial_{\bar{z}} \Phi^a
\] (76)
so that \(\Phi^A\) can be chosen to satisfy
\[
\phi^{a*} = \phi^{\bar{a}}.
\] (77)
In this case,
\[
(\tilde{R}^{AB}_{CD})^* = -\tilde{R}^{\bar{A}\bar{B}}_{CD},
\] (78)
where \(\tilde{A} = (\bar{a}, \bar{a}) = (\bar{a}, a)\), and
\[
(e_a)^* = -e_{\bar{a}}.
\] (79)

The coordinates \((\phi^a, \phi^{\bar{a}})\) are fixed up to transformations
\[
\phi^a \rightarrow n^b_a \phi^b + v^a
\] (80)
and its complex conjugation. Here \(n^b_a\) is an arbitrary invertible constant matrix.

We have shown that given a complex Poisson differential algebra one can always find a coordinate system satisfying all the properties (72), (73), (77) and (78).

From the point of view of classifying all complex Poisson differential algebras, one is interested in how many different complex structures there can be for a certain Poisson structure given in terms of \(\phi^a, \phi^{\bar{a}}\). On this matter we only comment that the condition (67) restricts the transformation \(x^\alpha \rightarrow \Phi^A\) to satisfy the partial differential equation
\[
\frac{\partial x^i}{\partial \phi^a} P^{ab} + \frac{\partial x^i}{\partial \phi^{\bar{a}}} P^{\bar{a}b} = 0
\] (81)
and inequivalent complex structures correspond to inequivalent holomorphic classes of solutions to this equation. The coordinates \((x^i, x^i = x^{i*})\) and \((y^i, y^i = y^{i*})\) are said to be in the same holomorphic class if \(\partial x^i/\partial y^j = 0\).

In the coordinate system \((\phi^a, \phi^{\bar{a}} \equiv \phi^{a*})\), the one-forms
\[
\eta \equiv -e_a \phi^a, \quad \bar{\eta} \equiv -e_{\bar{a}} \phi^{\bar{a}}
\] (82)
are holomorphic and antiholomorphic respectively. They satisfy
\[
\eta^* = -\bar{\eta}
\] (83)
and

\begin{align*}
(\eta, x^i) &= dx^i, & (\eta, x^\bar{i}) &= 0, \\
(\bar{\eta}, x^\bar{i}) &= dx^\bar{i}, & (\bar{\eta}, x^i) &= 0.
\end{align*}

(84) (85)

For those solutions with $f^{AB}_C = 0$ (possibly after a transformation (80)), we have the one-form realization for $\delta$ and $\bar{\delta}$:

$$
\delta \omega = (\eta, \omega), \quad \bar{\delta} \omega = (\bar{\eta}, \omega),
$$

(86)

for any $\omega \in \Omega$. For this case, the 2-form

$$
K \equiv \delta \eta = g^{ab} e_a e_b = \delta \bar{\eta}
$$

(87)

is central in the Poisson bracket

$$(K, x^\alpha) = (K, dx^\alpha) = 0$$

(88)

as a consequence of (84) and (85). It is not hard to check that

$$
\delta \eta = g^{ab} e_a e_b, \quad \bar{\delta} \bar{\eta} = g^{\bar{a}\bar{b}} e_{\bar{a}} e_{\bar{b}}.
$$

(89)

If $g^{ab} = g^{\bar{a}\bar{b}} = 0$

$$
\delta \eta = \bar{\delta} \bar{\eta} = 0,
$$

(90)

then at least locally

$$
\bar{\eta} = \bar{\delta} V
$$

(91)

for certain function $V$, and

$$
K = \delta \bar{\delta} V
$$

(92)

shows that it is a Kähler manifold.

We will see in the next section that in the case of one dimensional complex Poisson differential algebra, the admissible Kähler metrics are those of constant curvature spaces.

6 One Dimensional Case

For the case of one complex dimension, the Poisson structure is completely determined by the two functions $P, S$ which appear in

$$(z, \bar{z}) = P, \quad (z, dz) = S dz,$$

(93)

where $P$ is real and $S$ is holomorphic due to properties 4 and 6. All other relations of the Poisson structure can be derived by using the Leibniz rule. The Jacobi identity

$$
0 = ((z, \bar{z}), dz) + ((\bar{z}, dz), z) + ((dz, z), \bar{z})
$$

(94)

$$
= -P[\partial(P^{-1}S) - \partial(P^{-1}\bar{\partial}P)]
$$

11
implies that
\[ S = (\bar{\partial} + \bar{\alpha})P \]  \hspace{1cm} (95)
for some \( \bar{\alpha} = \bar{\alpha}(\bar{z}) \). Now, under a holomorphic coordinate transformation \( z \to z' = f(z) \),
\[ P' = (\partial f)(\bar{\partial f})P, \]  \hspace{1cm} (96)
\[ S' = (\partial f)S, \]  \hspace{1cm} (97)
\[ \bar{\partial}' = (\bar{\partial f})^{-1}\bar{\partial} \]  \hspace{1cm} (98)
and so
\[ \alpha' = (\partial f)^2[(\partial f)\alpha - \partial^2 f]. \]  \hspace{1cm} (99)
In particular, one can pick a coordinate transformation to make \( \alpha' = 0 \). Omitting the primes, we have just shown that one can always make a holomorphic coordinate transformation and arrive at the relation
\[ S = \bar{\partial}P, \]  \hspace{1cm} (100)
in a particular coordinate system. Because \( P \) should be real, it must be
\[ P = az\bar{z} + bz + \bar{b}\bar{z} + c, \]  \hspace{1cm} (101)
where the constant coefficients \( a, c \) are real and \( b \) is complex.

In the case of higher dimensions, the coordinate system \( \{\phi^a, \phi^{\bar{a}}\} \), which gives the quadratic form of the Poisson structure, are generically not holomorphic nor antiholomorphic functions of \( z^i, \bar{z}^i \) in which the complex structure of the manifold is defined. It is remarkable that in the case of one dimension, holomorphic transformation is enough to bring oneself to the canonical coordinate system.

For the metric to be covariantly constant, the only admissible Kähler form (up to normalization) is
\[ K = dzd\bar{z}h \]  \hspace{1cm} (102)
with the metric \( h \) given by
\[ h = P^{-2}. \]  \hspace{1cm} (103)
\( K \) is central to functions and forms.

Performing a fractional transformation
\[ z \to \frac{\alpha z' + \beta}{\gamma z' + \delta}, \]  \hspace{1cm} (104)
we have
\[ K = dz'd\bar{z}'h', \]  \hspace{1cm} (105)
with
\[ h' = \frac{|\alpha\delta - \beta\gamma|^2}{(Az'\bar{z}' + Bz' + B\bar{z}' + C)^2} \]  \hspace{1cm} (106)
and

\[
\begin{pmatrix}
A & B \\
\bar{B} & C
\end{pmatrix} = \begin{pmatrix}
\alpha & \gamma \\
\beta & \delta
\end{pmatrix} \begin{pmatrix}
a & b \\
\bar{b} & c
\end{pmatrix} \begin{pmatrix}
\bar{\alpha} & \bar{\beta} \\
\bar{\gamma} & \bar{\delta}
\end{pmatrix}.
\] (107)

One can always choose the fractional transformation with \(\alpha\delta - \beta\gamma \neq 0\) such that \(B = 0\) and so

\[
h' = \frac{|\alpha\delta - \beta\gamma|^2}{(Az'\bar{z} + C)^2}.
\] (108)

This is nothing but the most general form of a metric for a one dimensional Hermitian surface with constant Gaussian curvature (for the Levi-Civita connection). For \(A = 0\) or \(C = 0\) (do \(z'' = 1/z'\) in this case), \(h'\) gives the metric of a plane. For \(A \neq 0\), the metric is that of a sphere (resp. Lobachevskian plane) if \(A\) and \(C\) are of the same sign (resp. opposite sign). All three cases are actually Kähler manifolds. It is interesting to note that all connected, simply connected one-dimensional complex manifolds are biholomorphic to one of the three cases or a quotient of them over an isometric automorphism.

If \(a \neq 0\) one can always make a translation for \(z, \bar{z}\) so that \(b = \bar{b} = 0\) in the new coordinate system. It can be checked that in the new coordinate system the holomorphic and antiholomorphic exterior derivatives are realized by

\[
\eta = -\bar{z}P^{-1}dz, \quad \bar{\eta} = zP^{-1}d\bar{z},
\] (109)

which gives an admissible Kähler form described above by (87).

7 Conclusion

We found in this paper that the generalization of the Poisson structure to the differential calculus, or the Poisson limit of the differential calculus on a quantum space, has interesting geometrical structure in itself. We proved an analogue of Darboux’s theorem for the present case of a Poisson Differential Algebra. We also studied the compatibility of the complex structure with the Poisson structure on complex manifolds. In the case of one complex dimension the natural metrics are the ones whose Levi-Civita connections give constant curvatures. It is natural to ask if one can generalize the results for one complex dimension to higher dimensions. It will also be interesting to see if there exist a canonical procedure to quantize a Poisson differential algebra. We leave these questions for a future publication.

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