Critical behaviour of the 2d spin diluted Ising model via the equilibrium ensemble approach

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The equilibrium ensemble approach to disordered systems is used to investigate the critical behaviour of the two dimensional Ising model in presence of quenched random site dilution. The numerical transfer matrix technique in semi-infinite strips of finite width, together with phenomenological renormalization and conformal invariance, is particularly suited to put the equilibrium ensemble approach to work. A new method to extract with great precision the critical temperature of the model is proposed and applied. A more systematic finite-size scaling analysis than in previous numerical studies has been performed. A parallel investigation, along the lines of the two main scenarios currently under discussion, namely the logarithmic corrections scenario (with critical exponents fixed in the Ising universality class) versus the weak universality scenario (critical exponents varying with the degree of disorder), is carried out. In interpreting our data, maximum care is constantly taken to be open in both directions. A critical discussion shows that, still, an unambiguous discrimination between the two scenarios is not possible on the basis of the available finite size data.

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I. INTRODUCTION

Recent years have witnessed renewed efforts in the statistical physics community to understand phase transitions in simple disordered classical spin systems. As a matter of fact, these efforts have produced conflicting statements concerning the effects of disorder on critical phenomena, covering almost the complete spectrum of conceivable alternatives.

This holds in particular, but not exclusively, for two-dimensional (2d) disordered ferromagnetic (i.e., unfrustrated) Ising models. These models have been widely studied, both because the corresponding pure system is well understood and because they constitute a marginal case in the Harris criterion which assesses whether disorder constitutes a relevant or irrelevant perturbation for the critical behaviour of the pure system. For models of this type, the discussion currently appears to narrow down on two conflicting scenarios, namely the logarithmic corrections versus the weak universality scenario, though a broader spectrum of alternatives had been discussed earlier (the interested reader will find a comprehensive report on the literature up to approximately 1982 in an early review by Stinchcombe). We will describe these scenarios and discuss these (and related) results in greater detail later on.

The object of our study is the randomly spin diluted 2d Ising model. Our investigation is based on two main ingredients. First, we use the equilibrium ensemble approach to disordered systems to map the quenched system onto an equivalent thermodynamic equilibrium system in an enlarged phase space. Performing this mapping exactly would be tantamount to providing an exact solution to the original problem, which is clearly infeasible. So a scheme of approximations based on a moment matching idea is invoked. Second, we resort to conventional transfer matrix (TM) techniques to implement the method in practice on finite-width strips, analyzing finite-size results in the line of Nightingale’s phenomenological renormalization group scheme.

The purpose of the present paper is to provide details of our TM study as well as to include new material and to describe significant advances in the understanding of finite-size scaling (FSS) signatures in the presence of logarithmic corrections, which together have allowed us to boost the accuracy of our results considerably and to obtain a sounder appreciation of the subtleties that may emerge in the interpretation of the data.

The main and unexpected finding in Ref. 21 has been a continuous variation of the critical exponents α, β, γ and ν with the spin density ρ in a manner which was observed to comply with the idea of weak universality. That is, the exponent η describing the decay of critical correlations, and the magnetic exponent δ, as well as the ratios β/ν and γ/ν were found to be independent of ρ. The results were obtained by extrapolations of FSS data based on rather moderate strip-widths, and they were in complete quantitative and qualitative agreement with those of a Monte Carlo study by Kim and Patrascioiu. These results are in conflict with those supporting the logarithmic corrections scenario, where modifications of the relevant thermodynamical quantities at criticality appear through logarithmic terms in the reduced temperature, while the system is left in the same universality class (i.e. with the same critical exponents) as that of the pure 2d Ising model. These findings are almost all concerned with the ferromagnetic bond disordered situation, either theoretically, or via numerical and series expansion approaches, but the comparison with the spin diluted model is possible supposing – as it is generally believed to be the case – that the two systems are in the same universality class.

In discussing our results, we have to address two main issues. The first is concerned with the reliability of our method, which is based on an approximate description of quenched disorder. The evidence we have been able to compile does give us strong confidence in the validity of our approach. This granted, we turn to the second
issue: can – or more precisely to what extent can – our results provide evidence in favour of or against any of the conflicting scenarios so far advanced to describe the critical behaviour of 2$d$ disordered ferromagnetic Ising models? This is indeed a subtle question, and we devote almost two Sections (Secs. IV and V) to discussing it. It turns out that a considerable portion of the available finite-size data from simulations or TM studies – including those presented in our earlier study[18,21] as well as some new ones – may perhaps not allow to decide with sufficient confidence between the two most serious candidates. And we shall explain why. Larger system sizes would in any case be necessary in studies along those lines to permit taking sides.

Finally, since one of the aims of our study is to discriminate between these two contradicting scenarios, one of our main concerns is an open-minded attitude, perhaps not easily found in the literature, in the analysis of the numerical results: we take the greatest care not to select one of them a-priori and to fit our data to it, but we rather attempt to be open in both directions.

The remainder of our paper is organized as follows. In Secs. II A and II B, in order to set the scene and to fix our notation, we describe the model that is studied in the sequel and we briefly recall our methodological background, viz. the equilibrium ensemble approach to disordered systems (EEA). While Sec. II C presents some details related to the computational side of the problem, we reserve Sec. III for a brief review of the two above mentioned scenarios, and Sec. IV for a presentation and discussion of our main results. We describe a new finite-size procedure to extract the critical temperature of the model as a function of the degree of dilution, which enables us to draw an accurate approximation of the phase diagram. We then determine values for the critical exponents, for the central charge and for the specific heat of our approximating systems. We end our paper with a critical report of the available finite-size data obtained also by other authors, and with a comprehensive discussion both on the method and on the outcome of our investigation (Sec. V). An Appendix, mainly based on renormalization group results by Cardy and Ludwig,\textsuperscript{32,33} presents a systematic FSS analysis of thermodynamic quantities in the presence of logarithmic corrections.

II. SETTING THE SCENE

A. The system under investigation

The object of our investigation is the 2$d$ randomly spin diluted Ising model, described by the Hamiltonian

$$\mathcal{H}(\sigma|\kappa) = -J \sum_{\langle ij \rangle} k_i \sigma_i k_j \sigma_j - H \sum_i k_i \sigma_i \ .$$

(1)

Here, the first sum is over all nearest neighbour pairs $\langle ij \rangle$ of, say, a square lattice $\Lambda$, and the second over all sites $i$. The $\sigma_i$ denote Ising spins and the $k_i \in \{0,1\}$ are occupation numbers signifying whether in a disorder configuration $\kappa$ a site $i$ is occupied by a spin ($k_i = 1$) or not ($k_i = 0$). The $k_i$ are taken to be quenched random variables, i.e., they are fixed, and randomly chosen according to the probability

$$q(\kappa) = \prod_i \rho^{k_i}(1-\rho)^{1-k_i},$$

(2)

which simply requires each site to be occupied with probability $\rho$, and to be empty with probability $1-\rho$. Thus $\rho$ is the average density of spins in the system.

Our aim is to study the thermodynamics of the system, described by the (dimensionless) quenched free energy

$$f_q = -N^{-1} \langle \ln Z_N(\kappa) \rangle_q ,$$

(3)
i.e., the average of the system’s free energy over the distribution $q(\kappa)$ describing the statistics of the disorder configurations $\kappa$. Here $N$ denotes the system size, and $Z_N(\kappa)$ is the partition function of a system of size $|\Lambda| = N$ at fixed disorder configuration $\kappa$. In particular, we would like to locate the phase-transition line $T_c(\rho)$ below which the system is known to develop spontaneous ferromagnetic order in the thermodynamic limit $N \rightarrow \infty$, and to study the critical behaviour of the system at and around this line. Whereas the thermodynamics of the 1$d$ system is known exactly, an exact solution of the 2$d$ model is currently way beyond our abilities. Approximation methods, which may, however, well produce exact results in certain limiting cases are thus called for.

B. Methodical background: the equilibrium ensemble approach

The EEA to disordered systems, as well as its application to the 2$d$ spin diluted Ising model, have been accurately described elsewhere.\textsuperscript{28,21} Therefore we will not enter again into the details of the method, rather just concisely recall the main ideas.

The central idea is to treat the configurational degrees of freedom $k_i$ in a system with quenched disorder on the same footing as the dynamical variables proper – the spins, in the case at hand. That is, the phase space is enlarged to include the occupation-number configurations $\kappa$, and an additional term depending only on $\kappa$, the so-called dis t r a t ion po t ual $\phi(\kappa)$ is added to the Hamiltonian

$$\mathcal{H}(\sigma|\kappa) \rightarrow \mathcal{H}(\sigma, \kappa) = \mathcal{H}(\sigma|\kappa) + \phi(\kappa) \ ,$$

(4)

which is then determined in such a way that configuration averaging, as implied by Eq. (3), becomes part of the Gibbs average in the enlarged phase space.

To utilize the method in practice, one still needs an explicit representation for $\phi(\kappa)$. For spin or site diluted systems the following representation in terms of products of occupation numbers

$$-\beta \phi(\kappa) = \lambda_1 \sum_i (k_i - \rho) + \lambda_2 \sum_{\langle ij \rangle} (k_i k_j - \rho^2) + \ldots$$

$$+ \lambda_3 \sum_P \left( \prod_{i \in P} k_i - \rho^{|P|} \right) + \ldots$$

(5)
is always suitable. The first sum explicitly displayed in (5) is over all lattice sites, the second over all nearest neighbour pairs, the third over all elementary plaquettes (of size $|P|$) of the system, and so on.

It has been realized that the self-consistent constraint equations which determine the couplings $\lambda_\nu$ as a function of temperature $T$, magnetic field $H$ and spin density $\rho$ (Eqs. 4 of Ref. 21), can be written as necessary conditions describing the maximum of the equilibrium ensemble’s dimensionless free energy (per site) $f^\phi = -N^{-1}\ln Z^\phi = -N^{-1}\ln\sum_{\sigma,\kappa}\exp[-\beta\mathcal{H}^\phi(\sigma,\kappa)]$ with respect to the $\lambda_\nu$’s. They can therefore be written, and are in practice evaluated too, as

$$\frac{\partial f^\phi}{\partial \lambda_1} = \langle k_i \rangle_{\phi} - \rho = 0,$$

$$-\frac{\partial f^\phi}{\partial \lambda_2} = \langle k_i k_j \rangle_{\phi} - \rho^2 = 0, \ldots,$$

$$-\frac{\partial f^\phi}{\partial \lambda_P} = \left( \prod_{i \in P} k_i \right)_{\phi} - \rho^{|P|} = 0, \ldots. \tag{6}$$

Solving Eqs. (6) for every cluster of connected sites – i.e. exactly obtaining the quenched free energy via the EEA – is clearly infeasible. However, a systematic scheme of approximations may be put up by matching only a subset of the full set of moments of the equilibrium ensemble’s Gibbs distribution to the corresponding subset of moments of the problem with quenched disorder. This amounts to forcing $k_i$ correlations on larger and larger groups of lattice sites to coincide with those of the quenched system. The larger the number of constraints taken into account, the better a description of the fully quenched system is obtained.

In Ref. 21 four different approximating systems named (a) - (d) were studied, differing by the number of terms kept in (5), and thus by the set of constraints which is actually retained in (6). The present study is focused only on approximation (d): the first three contributions to the disorder potential are kept, i.e. single site, pair and square plaquette terms. In addition, due to the anisotropy of the strip geometry employed in the calculation (see Sec. II C below), pair occupancy parallel and perpendicular to the strip are treated as separate constraints, i.e. they are controlled by separate couplings $\lambda_\nu$ in the disorder potential which, in turn, have to be determined by two distinct equations of the type (6).

### C. Transfer Matrix Approach

The approach outlined above might – at least in principle – solve our problem to any desired degree of accuracy. However, in $d \geq 2$ not even the simplest approximating system is exactly solvable. To analyse the thermodynamics or the critical behaviour of our systems, we use the strip FSS techniques pioneered by Nightingale. A distinct advantage of the EEA in this context is that we are dealing with translationally invariant equilibrium systems with short range couplings at all levels of approximation within the moment matching scheme described above. FSS can therefore be implemented using conventional TM techniques for nonrandom systems.

Our strip is a $L \times L'$ square lattices, with the thermodynamic limit taken in the $L'$ direction, and periodic boundary conditions (unless otherwise stated) imposed in the perpendicular direction. The row to row transfer matrix $\Gamma$, constructed from the local Boltzmann factors, can be chosen to be symmetric for all systems (a) - (d) of Ref. 21, and is diagonalized for strips of fixed width $L$, with the appropriate set of constraints (Eqs. (6)).

The dimensionless free energy $f_L$ (per site) is related to the largest eigenvalue $\gamma_1$ of $\Gamma$ via

$$-f_L = L^{-1}\ln \gamma_1 \tag{7}$$

(to simplify notation, we omit stating the $L$ dependence of $\Gamma$ and its eigenvalues in what follows).

The constraint equations (6) have been formulated in terms of first derivatives of the free energy, and thereby in terms of first derivatives of $f_L$. As mentioned above, those derivatives are nothing but gradient information in the problem of maximizing $f_L$ (or minimizing $-f_L$) over the appropriate set of couplings. Notice that the transcendental nature of these equations necessitates an iterative numerical solution. Having in this way determined the (approximate) disorder potential, one may compute thermodynamic functions and, in particular, the correlation length of the order parameter fluctuations, which is needed in the phenomenological renormalization group scheme. The correlation length is given in terms of the largest and the second largest eigenvalue of the TM as

$$\xi_L^{-1} = -\ln \frac{\gamma_2}{\gamma_1} \tag{8}$$

Thermodynamic functions of interest are obtained by differentiating the free energy with respect to temperature or magnetic field. Here, care must be taken because of implicit field and temperature dependences in the couplings of the disorder potential. In other words, derivatives have to be performed along the solution-manifold of the appropriate set of constraint equations.

Introducing modified spins $S_i = k_i \sigma_i$ with values in $\{0, \pm 1\}$, in a single variable we encode, both, presence or absence of a spin (the occupation numbers being given by $k_i = |S_i|$), and the spin states. Since the dimension of $\Gamma$ grows as $3^L \times 3^L$, we have used two different strategies – with complementary strengths and weaknesses – to simplify the computational task of dealing with huge transfer matrices. The first is a group theoretical analysis which exploits the various symmetries of the matrix and effectively reduces its dimensionality. The second is the use of sparse matrix techniques.

We will not bother the reader by giving a detailed account of these techniques since, apart from the fact that they have been non-trivially specialized to deal with our particular system, they are not new (for the sparse matrix technique see, e.g., Ref. 30). We just notice that within either of them it is possible to identify the blocks $\Gamma^{(1)}$ and $\Gamma^{(2)}$, in the block diagonal representation of $\Gamma$,
that contain its two largest eigenvalues $\gamma_1$ and $\gamma_2$ respectively. $\Gamma^{(1)}$ is the block spanned by the space of eigenvectors transforming symmetrically under spin reversal, while $\gamma_2$ is found in the block spanned by the eigenvectors of $\Gamma$ which are anti-symmetric under spin reversal. In a similar spirit, one may enquire into the large distance behaviour of the $k_i$ correlation functions $G_k(r_{ij}) = \langle k_i k_j \rangle - \langle k_i \rangle \langle k_j \rangle$. This requires to locate a corresponding eigenvalue $\tilde{\gamma}_2$, which controls the asymptotic decay of $G_k(r)$. It is found as the second largest eigenvalue in the block corresponding to a representation of the symmetry group which is symmetric with respect to both external (spatial) and internal (spin) symmetries of the system. Indeed, whereas the spins $\sigma_i$ change sign under global spin reversal, entailing that the eigenvalue to be inserted in (8) for the computation of the spin-spin correlation length must belong to the antisymmetric block of $\Gamma$, the disorder variables $k_i = |S_i|$ do not.

The computation of second order derivatives of $\gamma_i$, needed to evaluate some of the relevant thermodynamical quantities, is feasible only in the group theoretical reduction approach. The second derivatives in fact require knowledge of the complete eigensystem, which is still far too large within the sparse matrix approach. In this case one has to resort to finite differences of first order derivatives to compute, e.g., the specific heat or the spin susceptibility. The advantage of the sparse-matrix approach, on the other hand, is that it can be pushed to considerably larger strip-widths than the group theory approach. We reach the value $L = 9$ with the latter and $L = 13$ with the former. Moreover, this approach does not exploit any symmetries beyond the spin reversal symmetry in zero magnetic field. It is therefore also applicable if one introduces an anti-ferromagnetic seam along the strip, a device which can be used to compute the domain wall free energy. We shall exploit this quantity below to considerably boost the precision of our FSS analysis.

III. THE TWO SCENARIOS

Before showing our results, it is time to briefly present the two main scenarios which have survived over a wider range of possibilities for the description of the critical behaviour of the 2d site diluted Ising model, but which are obviously mutually excluding.

The logarithmic corrections scenario is based on the quantum field theory results by Dotsenko and Dotsenko\(^2\) and Shalaev,\(^3\) Shankar\(^4\) and Ludwig\(^5\) (the latter contributions correcting certain errors in the former). Though strictly valid only in the limit of weak disorder, they indicate that the presence of impurities affects the critical properties of the model only through a set of logarithmic corrections to the pure system behaviour. In particular, according to this picture, the correlation length of the infinite system close to the phase transition is expected to show the form

$$\xi_{\infty} \sim t^{-\nu} \left(1 + \tilde{g} \ln \left( \frac{1}{t} \right) \right)^{\tilde{\gamma}} ,$$

where $t$ is the reduced temperature $t = (T - T_c)/T_c \ll 1$ ($T_c$ being the critical temperature, which will depend on $\rho$), the exponents $\nu$ and $\tilde{\gamma}$ are respectively 1 and $1/2$, and $\tilde{g}$ is a non-negative constant, such that $\tilde{g} = 0$ in the pure case (usual power law behaviour recovered) and increases with increasing disorder. A similar behaviour holds for the magnetic susceptibility:

$$\chi_{\infty} \sim t^{-\gamma} \left(1 + \tilde{g} \ln \left( \frac{1}{t} \right) \right)^{\tilde{\gamma}} ,$$

with $\gamma = 7/4$ and $\tilde{\gamma} = 7/8$. For the specific heat, too, the exponent of the pure case $\alpha = 0$ is not modified by the introduction of disorder, but the simple logarithmic behaviour $C_{\infty} \sim \ln(1/t)$ is substituted by the double logarithmic singularity

$$C_{\infty} \sim \ln \left(1 + \tilde{g} \ln \left( \frac{1}{t} \right) \right) .$$

The behaviour of the critical spin-spin correlation function is instead predicted not to change in presence of impurities, thus its anomalous dimension $\eta$ retains its pure Ising value $\eta = 1/4$.

Recent numerical works\(^19\)–\(^21\) have provided evidence in contrast to what has just been presented above. These findings show quantities such as the susceptibility and the correlation length to display simple power-law singularities, at criticality, with exponents $\gamma$ and $\nu$ varying continuously with disorder in a way, however, that their ratio $\gamma/\nu$ is kept constant at the pure system value. According to this weak universality\(^31\) scenario, the ratio of exponents should not depend on disorder, and the specific heat was observed to saturate at a nondivergent value as $t \to 0$.

In a finite-size numerical investigation like ours it is also essential to translate the predictions of these scenarios into their finite-size counterparts. This is straightforward and well known in the case of power law singularities (and power law corrections to scaling), but less trivial within the logarithmic corrections scenario. These questions are dealt with in detail in the Appendix and, where they arise, in the following Section.

As already stressed in the Introduction, we have tried to look at our results with open mindedness: none of the two scenarios is chosen as a reference, we rather attempt to let our analysis to decide for those of the two which consistently fits the whole series of our data.

IV. MAIN RESULTS

A. Determination of the critical temperature

The 2d spin diluted Ising model exhibits a phase transition from a paramagnetic to a ferromagnetic low-temperature phase, provided the spin density $\rho$ is larger than the critical site percolation density $\rho_c \approx 0.592745$\(^34\). The model unfortunately lacks the possibility, known for its random-bond counterpart, to exactly determine the
critical temperature as a function of the degree of disorder from duality. This is one of the reasons which render the bond disordered version more attractive for a numerical study, i.e. the possibility of studying its critical behaviour by sitting exactly at \( T_c \). A very precise numerical determination of \( T_c \) is thus called for in our case, and it has indeed been reached by a joint analysis of the finite-size behaviour of the correlation length (as defined in Eq. (8)) and of the domain wall free energy (or interfacial tension), the latter obtained by comparing free energies of systems with periodic and antiperiodic boundary conditions in the direction perpendicular to the strip. Antiperiodic boundary conditions in a ferromagnetic system force an interface along the cylinder’s length. In practice, the interface is created by introducing an antiferromagnetic seam along the cylinder, i.e. by reversing in each row the bond \( J \to -J \) between two fixed spins (e.g. spins \( L \) and 1). The interface (domain wall) free energy per unit length \( \sigma_L \) is given by the difference in free energy between the system with periodic and antiperiodic boundary conditions, and reads

\[
\beta \sigma_L = -\ln \frac{\gamma_{1,abc}}{\gamma_1} \tag{12}
\]

with \( \gamma_{1,abc} \) the largest eigenvalue of the TM with antiperiodic boundary conditions (compare Eq. (12) to Eq. (8)).

Phenomenological renormalization\(^{29,30} \) predicts that the correlation length \( \xi_L \) scales as \( L \) at criticality. The correct formula reads

\[
\xi_L^{-1} = L^{-1} (A + B_L \rho \gamma_{yrrr} + \ldots) \tag{13}
\]

where the correction-to-scaling terms (the exponent \( \gamma_{yrrr} \) is related to the presence of irrelevant scaling fields and it is known to be \( \gamma_{yrrr} = -2 \) for the Ising model\(^{35} \)) split the exact crossing of the curves \( L/\xi_L \) versus \( \beta J \) at the critical point for different values of \( L \) into a sequence of distinct intersections of abscissae \( (\beta J)_L \) (see Fig. 1). Extrapolating this sequence to its asymptotic value provides an accurate determination of the critical temperature. The task is accomplished by a combination of different extrapolation techniques. They are basically: the Bulirsch and Stoer algorithm discussed by Henkel and Schütz;\(^{36} \) the three-point iterated fit method presented by Blöte and Nienhuis;\(^{37} \) a third method, close in spirit to the latter, which uses fitting procedures at all stages of extrapolation.\(^{38} \) These three different algorithms give accurate and consistent results, since they lead to extrapolated values which do not significantly differ from each other. They are also used in the following wherever the \( L \to \infty \) limit of a sequence needs to be extracted. Their comparison sets the error bars of our analysis.

The same scaling argument presented for \( \xi_L^{-1} \) can be applied also to the wall free energy \( \beta \sigma_L \sim L^{-1} \), and the corresponding sequence \( (\beta J)_L^{\text{int}} \) similarly analyzed.

The apparent “mirror symmetry” of the two sets of curves displayed in Fig. 1 suggests also a new method to extract \( \beta J \): the analysis of the sequence \( (\beta J)_L^{\text{int}} \) of mutual intersection points of the curves \( L/\xi_L \) and \( L/\beta \sigma_L \) versus \( \beta J \) for fixed values of \( L \). This sequence also converges to \( \beta J \) in the limit of large \( L \), but much faster than the previous sequences \( (\beta J)_L^{\text{int}} \) and \( (\beta J)_L^{\text{int}} \), and its extrapolation through the methods described above furnishes a much more precise determination of the critical temperature (upper half of Fig. 2). Already for the rather small value \( L = 6 \), the pure 2d Ising model value of \( (\beta J)_L^{\text{int}} \) differs from the exact critical value only after the 11th decimal digit. Neither this striking result nor the analysis of the \( (\beta J)_L^{\text{int}} \) sequence are, to our knowledge, present in the literature, and we regard them as remarkable findings on their own.

The whole machinery can be carried over to the diluted case. It is worth remembering that in our investigation the spin density \( \rho \) is treated as a parameter fixed to definite values in the range \( \rho_c \leq \rho \leq 1 \) to scan the phase diagram. The values of \( \rho \) analyzed are \( \rho = 0.95, 0.8, 0.75, 2/3 \).

For \( \rho \neq 1 \) the “mirror symmetry” of Fig. 1 is lost when the value of \( \rho \) substantially departs from the pure system value. Logarithmic terms would also appear in Eqs. (13) in case of the logarithmic corrections scenario\(^{32} \) (see the Appendix and the discussion in Sec. IV B below). In spite of that, the sequences \( (\beta J)_L^{\text{int}} \) still converge quite fast to their asymptotic value, again allowing extremely accurate determination of \( (\beta J)_L^{\text{int}}(\rho) \) (see lower half of Fig. 2) also for the lower values of \( \rho \), where closeness to the percolation threshold induces stronger finite-size deviations from the asymptotic behaviour of the quantities under investigation.

Fig. 3 presents the phase diagram of the 2d site diluted Ising model, showing the values of \( T_c(\rho)/J \) obtained in this way for strip-widths up to \( L = 13 \).

The critical slope of the curve \( T_c(\rho) \) versus \( \rho \) at \( \rho = 1 \) is
exactly known\textsuperscript{39} to be $S_c = T_c^{-1} dT_c / d\rho|_{\rho=1} = 2/[\ln(1 + \sqrt{2})(1 + \sqrt{2}/\pi)] = 1.564785\ldots$; from an additional calculation limited to size $L = 11$ at a value of $\rho$ very close to 1 and from a numerical evaluation of the derivative, we obtain an estimate which differs from the exact result by approximately 0.01%.

Let us finally point out that the method just described provides a very precise determination of the critical temperatures for each of our approximating systems. We make no claim, however, that these values coincide with those of the fully quenched system: for fixed $\rho$ they in fact turn out to be slightly different from system (a) to (d)\textsuperscript{21} and need not be the same as Monte Carlo data.\textsuperscript{14} The reason is that nonuniversal quantities, such as indeed the critical temperature or the value of the percolation threshold $\rho_c$, do depend on the approximating system chosen, while universal quantities, such as critical exponents, central charge, etc., do not. Our confidence in this statement, and consequently in the validity of our method, will be addressed in Sec. V.

**B. The exponent $\eta$ and the ratio $\gamma/\nu$**

Having determined with great precision the critical temperature of our model, we now turn to the evaluation of critical exponents.

The theory of conformal invariance\textsuperscript{40} relates the amplitude $A$ of the leading term in the FSS behaviour of $\xi$ (Eq. (13)) to the anomalous dimension of the spin-spin correlation function via $A = \pi\eta$, providing a simple method to extract $\eta$ just by extrapolating the sequence $L/\xi_L$ to $L \rightarrow \infty$. The same procedure can be applied as well to $L/\sigma_L$ since the amplitudes for the correlation length and

**FIG. 2.** The sequences of finite-size approximations to the critical inverse temperature $\beta J_L (\text{circles})$, $\beta J_L' (\text{squares})$ and $\beta J_L^{\text{int}} (\text{diamonds})$ as a function of strip-width $L$ for $\rho = 1$ (the pure Ising case) and for spin density $\rho = 0.75$. It is apparent that in both cases the last sequence converges faster than the other two to the critical inverse temperature of the infinite system represented by the dashed line. Here, as well as in the following pictures, no error bars are shown if smaller than the symbol’s size.

for the wall free energy are equal.

In the pure Ising case, $\eta$ is known to be 1/4. Our extrapolations unambiguously give the same value

$$\eta = \frac{1}{4}$$

for all the values of $\rho$ considered, with a maximum estimated error of 0.0004 (the error arising from the uncertainty in the location of the critical temperature being much smaller than the difference between final values from different extrapolation procedures). From this we can conclude that the exponent $\eta$ does not depend on disorder, a result which has been known already\textsuperscript{20,21,9,41,11} but without the precision shown above.

Note that the correlation length $\xi$ extracted from the ratio of TM eigenvalues (Eq. (8)) within the EEA refers to the decay of the average spin-spin correlation function in the disordered system. This is to be contrasted with the correlation length that describes the typical decay of the correlation function in a given disorder configuration, a quantity readily obtainable from a corresponding ratio of Lyapunov exponents within a random TM approach.\textsuperscript{41,11} Our identification may be confirmed by plotting $L/\xi_L$ versus $1/L^2$ (Fig. 4) and noticing that no corrections to a linear dependence smoothly extrapolating to 1/4 appear, as it could instead be expected for the behaviour of the typical correlation length (see in particular the discussion focusing on Fig. 1 of Ref. 41, and Ref. 11).

We would like to point out that the smallness of the estimated error of the exponent $\eta$ is of particular relevance. It can in fact be also read as a strong confirmation that our approximations are sufficient to place the model under study in the right universality class, and that universal quantities therefore turn out to be correctly reproduced.

This is confirmed by the behaviour of the magnetic susceptibility data obtained via the group theoretical re-
The scaled inverse correlation length $L/\pi\xi_L$, plotted as a function of $1/L^2$, for the pure Ising case (circles), for $\rho = 0.80$ (diamonds) and for $\rho = 2/3$ (squares). The lines just connect the displayed points. Note the vertical scale.

The finite size scaling analysis of the susceptibility $\chi$ predicts a leading $L$ dependence of the form $\chi_L \sim L^{\gamma/\nu}$, and the ratio $\gamma/\nu$ is extracted by analyzing the sequence

$$
\left( \frac{\gamma}{\nu} \right)_L = \frac{\ln(\chi_{L+1}/\chi_L)}{\ln(L+1/L)}.
$$

(15)

Corrections to scaling generally modify the pure power law, and an extrapolation to $L \to \infty$ is needed. Corrections may be again of pure power law form in $L$ (in the conventional situation arising from irrelevant scaling fields), or indeed by terms containing logarithms of $L$ (in the log-corrections scenario). In the latter case, if the largest available system size is too small, the data might still reflect a preasymptotic regime, and the very idea of extrapolating a sequence like (15) may lose its validity in that case. We refer the reader to the Appendix for a discussion of this point. We are however not able to detect the non-universal corrections predicted to be present in the preasymptotic regime of the log-corrections scenario. They are either absent, or too small to be detected (Eq. A16d). The same is true for $\eta$.

Extrapolating the sequence (15) provides the result

$$
\frac{\gamma}{\nu} = \frac{7}{4}
$$

(16)

for all the points studied in the present paper. The estimated error is around 0.001, bigger than for $\eta$ because a shorter sequence ($L_{\text{max}} = 9$) has been utilized. These data show that the ratio $\gamma/\nu$ is independent of the degree of dilution, a finding in complete accordance with the constancy of $\eta$ via the Fisher’s relation $\gamma/\nu = 2 - \eta$, which is thereby demonstrated to be satisfied with a precision of the order of a tenth of a per cent.

The observed constancy of $\eta$ and $\gamma/\nu$ is unfortunately not sufficient to discriminate between the two admissible scenarios, since both predict that the ratio retains the pure Ising value $7/4$, irrespective of disorder. It only shows that for these two quantities extrapolations are not noticeably misguided by logarithmic corrections, in case they are present.

C. The central charge

Conformal invariance provides a link between the values of the dimensionless free energy per site at criticality $f_L$ in a finite strip with periodic boundary conditions, and the central charge $c$ which characterizes the universality class of a conformally invariant model. The relation reads

$$
f_L \sim f_\infty - \frac{\pi c}{6L^2} + \ldots,
$$

(17)

with $f_\infty$ the nonuniversal bulk (infinite system) free energy per site. Eq. (17) is supposed to retain its validity also in a random system, provided $c$ is substituted by an effective central charge $c'$ (Ref. 33). The next-to-leading term, omitted in (17), differs depending on whether logarithmic corrections are present or not. In the latter case it is believed to depend on $L$ as $B_f L^{-2\gamma_{\text{irr}}} = B_f L^{-4}$ if we adopt the value $\gamma_{\text{irr}} = -2$ of Ref. 37 or, equivalently, that born out from the analysis of Sec. IV B above. In the former case, instead, Ludwig and Cardy derived the results presented in Eqs. (A18) of the Appendix. To avoid dealing with the $f_\infty$ term, we prefer to extract finite-size estimates of $c'$ through the formula

$$
c'_L = \frac{6}{\pi} (f_{L+1} - f_L) \frac{L^2(L+1)^2}{2L+1}
$$

(18)

and analyze them both by extrapolating the above sequence to $c'$, and possibly looking for finite-size corrections (next-to-leading terms).

The outcome of our analysis does not sensibly differ from what we obtained in the study of the exponent $\eta$. That is, while $c'$ appears to be clearly given by

$$
c' = \frac{1}{2},
$$

(19)

equal to the pure Ising value, for all the points studied (with a maximum error of 0.0003), we are not able to individuate any correction of the kind of those expressed by Eq. (A18). The plots of $c'_L$ versus $1/L^2$ are lines showing no appreciable bending.

The estimates $c'_L$ converge to their asymptotic value always from above, both in the pure as well as in the diluted case. This is at least a necessary condition for the reflection positivity property, which holds for unitary models but it might not hold for random models. The discussion on this point in Ref. 43 is indeed based on Eqs. (A18) which would predict a convergence from below, the first correction terms to the value $c' = 1/2$ being negative. But again, they are either absent, or too small to be discernible within our errors, or convergence from below sets in for system sizes beyond those accessible by our TM calculations.

D. The analysis of the correlation length

The correlation length provides us with another tool to push our investigation further: the study of the exponent
TABLE I. Various exponents for different values of spin density \( \rho \). In the second column the exponent \( \nu \) as extrapolated from Eq. (22). The third column shows the value of the ratio \( \alpha/\nu \) extracted from Eq. (25). The last column provides a test for the validity of the hyperscaling relation \( 2/\nu - \alpha/\nu = d = 2 \).

| \( \rho \)  | \( \nu \)       | \( \frac{\alpha}{\nu} \) | \( \frac{2}{\nu} - \frac{\alpha}{\nu} \) |
|-----------|---------------|---------------------------|-------------------------------------------|
| 0.95      | 1.054 ± 0.002 | -0.102 ± 0.008            | 1.999 ± 0.012                             |
| 8/9       | 1.113 ± 0.001 | -0.214 ± 0.007            | 2.011 ± 0.009                             |
| 0.80      | 1.15 ± 0.03   | -0.30 ± 0.02              | 2.04 ± 0.06                               |
| 0.75      | 1.18 ± 0.02   | -0.35 ± 0.02              | 2.05 ± 0.05                               |
| 2/3       | 1.23 ± 0.05   | -0.445 ± 0.005            | 2.07 ± 0.07                               |

\( \nu \), extracted, in addition to \( \eta \), from a FSS analysis of \( \xi \), can be of particular relevance since it is a single isolated exponent, therefore it is predicted to show a different behaviour in the two scenarios.

Our analysis follows general reasoning.\(^{29}\) A recent application to disordered Ising models has been provided by Aarão Reis \textit{et al.}\(^{12}\) Two quantities at our disposal can be subjected to the same investigation, the correlation length \( \xi \) as well as the domain free energy \( \beta \sigma \): both show the same asymptotic behaviour, though with different corrections to scaling.

First we define the derivatives of these quantities with respect to temperature:

\[
\mu(t) = \frac{d\xi(t)}{dt},
\]

\[
\mu'(t) = \frac{d[\beta \sigma(t)]^{-1}}{dt}.
\]

A pure power-law divergence of \( \xi(t) \) at criticality like \( t^{-\nu} \) implies \( \mu(t) \sim t^{-\nu-1} \), which in turn translates into the finite-size prediction \( \mu_L \sim L^{1+1/\nu} \). The same relations hold respectively for \( \beta \sigma(t) \), \( \mu'(t) \) and \( \mu'_L \). From the \( \mu_L \) sequence (and similarly form the \( \mu'_L \) sequence), one obtains finite-size approximations \( \nu_L \) of the correlation length exponent via

\[

\nu_L^{-1} = \frac{\ln (\mu_{L+1}/\mu_L)}{\ln(L+1)/L)} - 1.
\]

Extrapolated values clearly appear to vary with \( \rho \), and to change continuously from the Ising value \( \nu = 1 \) to the percolation value \( \nu = 4/3 \),\(^{44}\) in a way shown in Table I. The large error bars on the values corresponding to stronger dilution are due to degradation in the quality of the fits, and even to the appearance of non-monotonicities in the sequence \( \nu_L \) versus \( L \) extracted from the domain wall free energy data, which prevent an extrapolation as precise as those for larger \( \rho \) values. The cause is probably the closeness of the percolation threshold, which requires a corresponding enlargement in system size in order to disentangle the system’s correct critical behaviour from crossover effects.

A similar data trend was extracted also from Monte Carlo simulations,\(^{20}\) and analogous variation with the strength of bond disorder was observed in Ref. 12. The variation of \( \nu \) as a function of \( \rho \) at constant \( \eta \) and constant \( \gamma/\nu \) would qualify the observed non-universality — if it must be upheld — to be of the \textit{weak} form.

On the other hand, one should be wary of the fact that preasymptotic logarithmic corrections might invalidate the very idea of extrapolating a \( \nu_L \) sequence of limited length; see (Eq. A16c). This was the attitude advocated by Aarão Reis \textit{et al.}, who did not give much weight to an analogous variation of \textit{their} extrapolated exponent \( \nu \) as a function of disorder. Rather they proposed to fit the \( L \) dependent \( \mu_L \) data to a form expected to hold in an intermediate (preasymptotic) range of \( L \) values within the logarithmic corrections scenario, viz.

\[
\mu_L \sim L^2(1 - A \ln L)^{2/\nu},
\]

which predicts \( (\mu_L/L^2)^2 \) to be linear in \( \ln L \). Note that this expression used by Aarão Reis \textit{et al.} was derived from heuristic considerations. It formally agrees with the true expression (A16b) only to first order in \( \ln L \), but it predicts the \textit{wrong} sign for the coefficient of the leading \( \ln L \) behaviour of the effective size-dependent correlation length exponent \( \nu_L \) (Eq. A16c).

In Fig. 5 we plot our results in the same form as Aarão Reis \textit{et al.} do, both for \( \mu_L \) and for \( \mu'_L \). The data show exactly the same trend as observed in Ref. 12, both with respect to system size \( L \) and with respect to the strength of the disorder. From this observation we derive additional strong confidence in the validity of our method. The pure system behaviour is soon substituted by an apparently linear term in \( \ln L \) for \( (\mu_L/L^2)^2 \) in the increasingly disordered system, as would be expected from (A16b). Notice, once more, the same asymptotic behaviour of \( \mu_L \) and \( \mu'_L \), with different corrections to scaling.

The data can, however, be consistently interpreted within a scenario of nonuniversal critical exponents as well. Indeed, for a power law divergence of the correlation length with \( \nu > 1 \), FSS would predict \( (\mu_L/L^2)^2 \sim L^{-\omega} \) with \( \omega = 2 - 2/\nu \) at sufficiently large \( L \) for the data plotted in Fig. 5. In the disorder range considered, \( \omega \) is a rather small quantity as the \( \nu \) extrapolated from Eq. (22) only slightly exceed 1. The data might therefore easily look like exhibiting a \( (1 - A \ln L) \) behaviour.
for the accessible range of system sizes, which was the form assumed in Ref. 12. This might however just be the result of an expansion to first order in the small quantity $\omega \ln L$ ($L^{-\omega} \sim 1 - \omega \ln L$). Upon closer inspection a curvature compatible with an $L^{-\omega}$ behaviour is indeed discernible in the figure for larger disorder. Using a fit against $L^{-\omega}$ we have another way of analyzing our data to determine $\nu$, and the values obtained this way are compatible with those obtained from extrapolation of the $\nu L$ data. Nonetheless, a curvature of the $(\mu L/L^2)^2$ data may occur also in the log-corrections scenario (depending on the value of $D_1$ in Eq. (A13)).

We can however affirm with certainty that neither Aarão Reis et al. nor we are seeing preasymptotic effects of logarithmic corrections in the data for $\nu L$, because they are increasing with $L$ (in our case at least for $\rho > 0.8$), whereas according to FSS they should decrease; see Eq. (A16c). Thus, if logarithmic corrections are present, they are for the available system sizes still masked by other corrections to scaling. Again, it seems that further investigation is required in order to be able to take sides on the critical behaviour of this system.

\section*{E. The specific heat}

The logarithmic correction scenario predicts for the specific heat the double logarithmic divergence of Eq. (11). This translates, at criticality, into the FSS prediction

$$ C_L \sim C_0 + C_1 \ln (1 + C_2 \ln L) , \quad (24) $$

where the pure system critical behaviour is recovered by the vanishing of $C_2$ (but with $C_1 C_2 = \text{const.}$) for $\rho = 1$. On the other hand, a scenario of nonuniversal critical exponents would lead to

$$ C_L \sim C_\infty + C_\infty L^{\alpha/\nu} . \quad (25) $$

Though $\alpha/\nu$ is a ratio of exponents, it is expected to vary with $\rho$, being directly related to $\nu$ via the hyperscaling relation $2 - \alpha = d \nu$, with $d$ the system’s dimensionality. Moreover, this relation imposes a severe constraint on $\alpha$: since $\nu$ is found to be greater than 1, $\alpha$ should be smaller than 0, in other words the specific heat should then turn out to be nondivergent, with its finite-size estimates saturating to a value $C_\infty$ at $T_c$.

We plot our data for the specific heat in the same form as in Ref. 13, i.e. against both $\ln L$ and $\ln \ln L$ (Fig. 6). The pure system divergence is well reproduced by the straight line the data show when plotted against $\ln L$, equivalently by the upward curvature when against $\ln \ln L$. Exactly as it happens in Ref. 13, as disorder is switched on and increased, it is apparent that both curves tend to bend downwards, the former markedly deviating from the latter instead approaching, a straight line (from top left to bottom right of Fig. 6). The law describing these finite size data has therefore to be at least less divergent with $L$ than $\ln L$. However, the same kind of behaviour must at moderate system sizes be expected in both scenarios (as we shall argue in greater detail in the concluding section). It is therefore difficult to conclude definitely in favour of Eq. (24): here more than elsewhere the need for significantly larger system sizes is particularly felt.

If we fit the data to a nondiverging size dependence, in accordance with Eq. (25), there is a consistency check to carry out, i.e. one may ask oneself whether the hyperscaling relation is verified or not, given the $\nu$ values from the correlation length. The fitting of the specific heat to the form (25) is performed first by taking all the points into account, then by reducing the data set by successively increasing the initial value of $L$ considered. In this way one can both check for the stability of such a fit and get a sequence of values of $\alpha/\nu$ which may be subjected to subsequent extrapolation procedures, when needed. The results of this analysis are shown in Table I. Its main outcome is the fact that, within a weak universality interpretation of the data, a small, seemingly systematic offset from hyperscaling, though still within estimated error bars, appears.

We have tried to discriminate between Eqs. (24) and (25) on the basis of a $\chi^2$ test. Although such a test appears to favour the log-corrections scenario by roughly an order of magnitude in $\chi^2$, the $\chi^2$ values for both alternatives are so small (typically $O(10^{-8})$ and $O(10^{-7})$ respectively for a choice of the data set) as to make it doubtful whether a meaningful model selection should be based on them. In particular, our data are not normally distributed random data (indeed they are deterministic), as instead strictly required by a reliable $\chi^2$ analysis.

\section*{V. DISCUSSION AND CONCLUSIONS}

In this paper we studied the critical behaviour of the 2$d$ Ising model with quenched random site dilution via the EEA. Our purpose was twofold: first we wanted to address again the question about the reliability of the method; second, we extended the study started in Ref. 21 to systems of larger sizes, also in view of the new results which have appeared in the literature since then.\textsuperscript{12–16}
order to allow perhaps a clearer discrimination between
the two contradicting pictures of the system’s critical be-
behaviour that survived so far, the logarithmic corrections
and the weak universality scenario.

A. Reliability of the Method

As the description of quenched disorder within our
EEA is only approximate, we have to worry about how
good it actually is. Before gathering the different pieces
of evidence accumulated during our numerical study con-
fiming the validity of our approach, we would like to
address the question from a more general viewpoint. In
Ref. 21 the correlation length $\xi_k$ describing the asymp-
totic decay of the correlation function $G_k(r_{ij})$ of the
disorder degrees of freedom was introduced and studied. Its
finite-size estimates can be computed from the ratio of
two TM eigenvalues in complete analogy to (8) by replac-
ing $\gamma_2$ with $\tilde{\gamma}_2$, where $\tilde{\gamma}_2$ is the second eigenvalue of the
symmetric block of the transfer matrix $T$.

As regards the analysis of this new correlation length,
we have both bad and good news. The bad news is that
$\xi_k$ actually diverges at criticality in our different approx-
imating systems. This escaped our attention in Ref. 21:
while $\xi_k$ was observed never to exceed a few lattice spac-
ings for the accessible system size, its size dependence at $T_c$
had not been monitored. However, before con-
cluding from this that our systems provide only a rather
poor description of quenched disorder, we have also deter-
mined the behaviour of the ratio $R_L = \xi_k(T_c)/\xi_k(T_c)$
of these two correlation lengths at criticality, and the results
of this study may be taken for the good news. We
find that $R_L \to 1/8$ at large $L$, independently of $\rho$.
Since this limit represents nothing but the ratio $\eta/\eta_k$ of
the anomalous dimensions of the spin and occupation-
variable correlation functions at $T_c$, this implies that
$\eta_k = 2$, hence unusually large. The $k_1$-correlator is thus
‘almost summable’ at criticality (its sum has a logarith-
mic infrared divergence). Note that the ratio $G_k(r)/G(r)$
behaves as

$$\frac{G_k(r)}{G(r)} \sim \frac{1}{r^\gamma/4}$$  \hspace{1cm} (26)

and thus decays to zero, stating that $k_1$ correlations are
negligibly small if compared to $\sigma_1$ correlations at large
distances. This is why the system may be regarded as
effectively quenched despite remaining correlations be-
tween the $k_1$, and it may be regarded as the reason why
our results compare so favourably with those obtained via
more conventional approaches.

Indeed, both checks against exact results, wherever ob-
tainable, and direct comparison of our data with those
obtained by the random TM approach (which allows for
an in principle exact treatment of the disorder) increase
our confidence in the validity of our method beyond any
reasonable doubt. Among the exact results correctly re-
produced by our method we mention: i) data obtained
for a one-dimensional system;\textsuperscript{45} ii) the value of the
initial slope of the critical line in the phase diagram, re-
produced to within 0.01%; iii) the correct value of the
connectivity length exponent $\nu_p$ and of the crossover ex-
ponent $\phi = \nu_p/\nu = 1$ at percolation;\textsuperscript{21} v) the precision
with which the values of the exponent $\eta$, of the ratio
$\gamma/\nu$, and of the central charge are determined for all
the points investigated in the phase diagram. In addi-
tion, as mentioned above, the random TM data (ob-
tained in the case of bond disordered systems)\textsuperscript{12,13} ex-
hibit the same finite-size signature, and qualitatively
the same behaviour as regards their dependence on the dis-
order strength as those obtained in the present paper.

A definite advantage of our method over other numer-
ical methods dealing with disordered systems, is that it
does not suffer from non-self-averaging difficulties, and
provides directly through the ratio of TM eigenvalues
the average correlation length, not the typical one, such that
the successive analysis turns out to be more straightfor-
ward than, e.g., in Ref. 12.

The results discussed so far were all obtained within
approximating system (d). Comparative studies includ-
ing results also from the other systems (a)–(c) were pre-
sented in Ref. 21. They showed that systems (a)–(d)
appear to be in the same universality class as regards
their critical behaviour. Non-universal quantities, how-
ever, such as the critical temperature itself, or the value
of the percolation threshold $\rho_c$, do depend on the approx-
imating system, differing between those with and without
the plaquette constraint and those estimated by other nu-
merical works.\textsuperscript{14,34} The conclusion is that the description
of quenched disorder, though only approximate, appears
to be precise enough to put all our approximating systems
into the same universality class as the the fully quenched
system, and this is all that is required for the purpose of
the present study.

B. Discriminating Between the Scenarios

We now come to our second point, concerning the dis-
crimination between the two scenarios proposed for the
critical behaviour of the model considered in this paper.
Our view is that some of the FSS investigations which
have appeared in the literature – including the recent
ones\textsuperscript{8,10,12–14,16,20,21} – may perhaps not be as decisive on
this matter as their authors have tended to believe. We
shall now discuss some issues on which, we believe, clar-
ification or amendments are needed, presenting a critical
review of available material in the light of our own data
analysis.

The results of Refs. 20 and 21 have been interpreted as
providing evidence in favour of a weak universality sce-
nario. The former is a Monte Carlo investigation both at,
and off criticality of the site diluted system. On the basis
of $\chi^2$ data analyses, Kim and Patrasciou concluded that
their off-critical simulations of susceptibility and correla-
tion length were better described by modified power laws
than in terms of the log-corrections scenario. No analog-
ous discrimination was attempted for the specific heat,
and hence no check of hyperscaling was performed. The
main weakness of their off-critical simulations is that re-
duced temperatures are still sizeable, and the constancy
of the ratio $\gamma/\nu$ appears slightly less well satisfied than the individual errors on the values of $\gamma$ and $\nu$ would allow. Simulations at criticality yielded a specific heat very slowly increasing with system size, and arguments in favour of a saturation were advanced. The constancy of $\gamma/\nu$ was clearly shown and the validity of the Fisher relation confirmed, but – as noted above – this cannot be taken to support either scenario over the other.

Our own earlier TM strip-scaling results, too, were interpreted in terms of weak universality. Possible effects of logarithmic corrections in the effective size-dependent exponents $\nu_L$ were looked for, but were either absent or simply not discernible due to the rather moderate strip widths available in that study. Indeed, the rather limited system sizes may be taken as one of the weakest points of that investigation. Moreover, specific heats had not been computed and so hyperscaling within a varying exponents picture was not checked. Again, the observed constancy of $\gamma/\nu$ and $\eta$ may be regarded as a piece of evidence for the reliability of the method, but not in favour of either scenario, except in so far as implying, that if critical behaviour in the model were non-universal, the observed non-universality would have to be of the weak form.

Let us now turn to the recent investigations which have been taken to support the log corrections scenario. They are either of the TM strip-scaling type, or of the Monte Carlo type.

The strip scaling data of Aarão Reis et al., for the correlation length in the bond disordered system appeared to give values of $\nu$ slightly greater than that of the pure system, which the authors, however, discarded, attributing them to preasymptotic effects originating from logarithmic corrections. Their interpretation of the data in these terms does, indeed, describe the data rather well on the level of the $\mu_L$, but not on the level of the size dependent exponents $\nu_L$. Moreover, we recall our discussion of this point in Sec. IV D, according to which the power law picture provides a consistent interpretation of the data as well.

Turning to the critical specific heat data presented in Refs. 12 and 13, they were interpreted as “clearly suggesting a divergence in the thermodynamic limit”. This point was claimed to be strengthened by pushing system size up to $L = 18$ (and in less precise simulations even to impressive $L = 23$), and by noting that specific heat data appeared to be halfway between single- ($\ln L$) and double-logarithmic ($\ln \ln L$) logarithmic dependence on strip width, from which a divergence in the thermodynamic limit was inferred. One should, however, note that at moderate $L$ this kind of behaviour is to be expected in both scenarios if the correlation length exponent is close to 1 and, hence, $\alpha$ only slightly negative. The difference is that $C_L$, will continue to increase at least as fast as $\ln \ln L$ for all $L$, in the log-corrections scenario, whereas there will be a crossover to a growth which is slower than $\ln \ln L$ at $L_c \sim 1.083$ determined by Aarão Reis et al. for the $r = J_1/J_2 = 0.25$ case investigated in Ref. 13, one would have to locate the crossover length at $L_c \simeq 680$. Thus $L_{\text{max}} = 23$ is still much too small to allow concluding in favour of either scenario.

We have carried out $\chi^2$ data analyses on the raw data of Ref. 13, comparing the two scenarios, as we did on our own data. Power law fits and fits according to (24) show no significantly different quality. In particular, it seems that a double logarithmic law provides a better fit to the data selected from a window $[L_{\text{min}}, 23]$ with $L_{\text{min}}$ up to 8, while power laws give a smaller $\chi^2$ for the larger $L_{\text{min}}$ up to 13; for still larger $L$ the result of the fits becomes questionable for both hypotheses. Other kinds of fits have been tried, e.g. by selecting a movable window $[L_{\text{min}}, L_{\text{min}} + 7]$ and letting $L_{\text{min}}$ run over the data, but without any significant improvement. Only by discarding the larger $L$ values, which are still much too noisy (as it is obvious by a quick look at the derivative information plotted in Fig. 3 of Ref. 13), one gets values of $\chi^2$ slightly in favour of the double logarithmic form (24). Incidentally, however, the power-law fits lead to an estimate for $\alpha/\nu$ compatible via hyperscaling with the $\nu$ value reported in Ref. 10. This analysis cannot thus be seen as conclusive.

The Monte Carlo study of site diluted system by Ballesteros et al. shows critical specific heat data reasonably well fitted (in terms of $\chi^2$) by a double logarithmic form, at least for low dilution. The same is true for a recent study of Selke et al. However, no power-law fitting is attempted for comparison. In earlier work on bond disordered systems, such power-law fits had been attempted, and were regarded inferior to double logarithmic ones. Note, however, that the maximum system sizes studied in Ref. 7 are for $r = 0.25$ still below, and for $r = 0.1$ above but still very close on a double logarithmic scale to the crossover lengths $L_c$ expected from the $\nu$ values reported in Ref. 10.

The $\nu_L$ data presented in Ref. 14 have, in our view, somewhat too large error bars to allow concluding with confidence that $\nu_L \rightarrow 1$ for large $L$ for all the values of $\rho$ also because Ballesteros et al. force the intercept through 1 rather than fitting it. The reasonably large values of $L$ reached in this investigation would probably set the system in the asymptotic regime (at least for the stronger disorder), and the FSS expression used by Ballesteros et al. is indeed $\nu_L = 1 + A'/\ln L$. Nonetheless the constant $A'$ in this expression should, strictly speaking, come out to be $1/2$, independently of the degree of disorder (see Eq. A17c), a property which their data and their fits do not respect.

In the off-critical Monte Carlo study of Talapov and Shchur, bond disorder was observed to lead to increased values for the magnetization and susceptibility exponents $\beta$ and $\gamma$, while the behaviour of the specific heat showed good agreement with the double logarithmic form (11). Together with the Rushbrooke equality $\alpha = 2\beta + \gamma = 2$, these findings embody an inconsistency from which the authors concluded that their increased values for $\beta$ and $\gamma$ cannot be asymptotic. On the other hand, it turns out that the specific heat data of Ref. 8 can be fitted equally well to a power law form with a value of $\alpha$ compatible
via the Rushbrooke relation with the increased values of \( \beta \) and \( \gamma \).\(^{46}\) Note also that the ratio \( \beta/\gamma \) in the disordered system is the same (to within a fraction of a percent) as in the pure system, as would be expected in a weak universality scenario, whereas individually \( \beta \) and \( \gamma \) change in the 6%-range.

The difficulties of FSS are avoided in the series expansion study of Roder et al.,\(^{15}\) their determination of the exponent \( \tilde{\gamma} \) in Eq. (10), which appears to saturate at \( 7/8 \) for sufficiently strong disorder, may perhaps be taken as the strongest piece of evidence currently available in favour of the validity of the logarithmic corrections scenario. Still, the analysis suffers from the relatively small length of the series, at least in some regions of the phase diagram, and one would wish this to be made more conclusive by including higher order terms, so as to reduce error bars. It is also worth noting at this point that their claim that \( \chi_L(T_c) \) is unaffected by logarithmic corrections cannot be upheld; see Eq. A15.

We now turn to the present investigation. First, by combining finite size signatures of correlation length and domain wall free energy, we have in particular been able to locate critical temperatures with extreme precision. Second, we have significantly enlarged our system sizes. In interpreting our data, maximum care was constantly taken to be open in both directions. Based on results of Cardy and Ludwig, a more systematic FSS analysis than in previous numerical studies has been performed. As we will presently show, neither way of looking at the data imply that the latter condition does not hold, at least for the weakly disordered ferromagnetic Ising model.

Considering the correlation length exponent \( \nu \), we observe that our FSS estimates converge to values continuously varying with \( \rho \). One might suspect, of course, that our extrapolations are misled because the algorithm would not pick up slowly varying logarithmic corrections. However, if we adopt this hypothesis, we find it not easy to reconcile with the fact that not even the slightest such effect is detectable in our extrapolations of \( \eta, \gamma/\nu \) and the central charge \( c \), even though similar, albeit smaller offsets must be expected to occur in these quantities as well, as is borne out by our analysis of FSS in the Appendix.

The fact that the critical specific heat data may be well fitted both to a double logarithmic behaviour (24) and to a cusp singularity (25) shows that it is difficult to discriminate with confidence between the two laws. However, we recall that a fit according to a modified power law (25) entails a weak violation of hyperscaling, though still within error bars.

In conclusion, as regards the two issues raised in this paper, we believe to have provided, if not a proof, satisfying arguments in favour of the trustworthiness and accuracy of the EEA to disordered systems. As to the second issue, we feel that further investigations would still be needed to provide a clearer and definite evidence in favour of either picture. The problems seen in the FSS analyses of the system are not exclusive to our results, but are – as we believe to have demonstrated – common to virtually all previous FSS data obtained so far.

Let us finally emphasize that we have no reasons to \textit{a priori} distrust the correctness of the theoretical picture that began to emerge through the work of Dotsenko and Dotsenko\(^2\) and the improved and corrected versions of Shalaev\(^3\), Shankar\(^4\) and Ludwig\(^5\). Still, in an ideal world, one would like to see this picture supported by numerical evidence much better than currently available.

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**APPENDIX: FSS WITH LOGARITHMIC CORRECTIONS**

In the present Appendix we collect the main results of the renormalization group (RG) analysis for the case where small amounts of disorder constitute a marginally irrelevant perturbation at the pure system’s fixed point, thus giving rise to logarithmic corrections, as in the field theoretical approach to the \textit{weakly disordered} 2d Ising model. We do this both for the sake of completeness, and to obtain sound understanding of the effects such corrections would have in a FSS analysis. This will turn out to be relevant not only in the asymptotic regime \( L \gg 1 \) but especially in the \textit{preasymptotic} regime as specified below.

We feel it particularly important to collect these results here, because the literature in the field abounds in heuristic derivations which have sometimes produced erroneous results and still more frequently even misconceptions as to what the effect of logarithmic corrections might be in the FSS signature of various thermodynamic functions and critical exponents.

The following is basically exploring results Ludwig and Cardy\(^{33}\) obtained in a replica approach to the problem, which are in turn based on earlier results on logarithmic corrections obtained by Cardy in a more general context.\(^32\)

If \( g \) denotes the marginally irrelevant coupling arising from disorder averaging in the (replicated) Hamiltonian,
and by \( \{u_n\} \) the set of other scaling fields at the pure systems critical point, the RG flow equations read

\[
\frac{dg}{dt} = -\pi bg^2 + O(g^3) \tag{A1}
\]

\[
\frac{du_n}{dt} = y_n u_n - 2\pi b_n g u_n + O(g^2 u_n) , \tag{A2}
\]

where the \( y_n \) are the eigenvalues of the linearized RG equations and the \( b_n \) are operator product expansion (OPE) coefficients, which can be formulated in terms of corresponding three-point functions.\(^{33}\)

Up to the order shown, these equations integrate to

\[
g(l) = \frac{g_0}{1 + \pi bg_0 l} \tag{A3}
\]

with \( g_0 = g(0) \)

\[
u(l) = u_n(0) e^{y_n l} (1 + \pi bg_0 l)^{-2b_n/b} \equiv u_n(0) \tilde{u}_n(l) . \tag{A4}
\]

Among the \( \{u_n\} \), the two relevant fields are \( u_\sigma \) which couples to the energy density and has RG eigenvalue \( y_\sigma = 1 \), and \( u_e \) coupling to the magnetization density with \( y_e = 15/8 \). One may thus put \( u_\sigma(0) = t \) and \( u_e(0) = h \).

In a finite system of linear extent \( L \) one may formally interpret \( L^{-1} \) as another relevant scaling field with RG eigenvalue 1.

In the context of the replica formula replication, the OPE coefficients and the coupling \( g \) are normalized such that the zero–replica limit of the product \( bg_0 \) and of the ratios \( b_n/b \) exist and are finite. The results of Ludwig and Cardy imply \( bg_0 = 8\Delta \) and \( b_\sigma/b = 1/4 \) in this limit, whereas \( b_\pi/b = 0 \). Here \( \Delta \) denotes the (bare) disorder strength: \( \Delta \propto \rho(1-\rho) \) for a randomly site diluted model.

The RG equations for the singular part of the free energy and the correlation length read

\[
f(t, h, g_0, L^{-1}) = e^{-dt} f(u_e(l), u_\sigma(l), g(l), e^l L^{-1}) \tag{A5}
\]

and

\[
\xi(t, h, g_0, L^{-1}) = e^l \xi(u_e(l), u_\sigma(l), g(l), e^l L^{-1}) . \tag{A6}
\]

Irrelevant scaling fields providing additional corrections to scaling have for simplicity been suppressed in these expressions. As usual, critical behaviour characteristic of the infinite system is obtained by considering \( f \) and \( \xi \) and their derivatives with respect to temperature (and field) at a scale \( l \) chosen such that \( u_e(l) = \pm 1 \). For \( |t| \ll 1 \) this gives

\[
e^{u_e^l} = \frac{1}{|t|} \left( 1 + \frac{\pi bg_0}{y_e} \ln \frac{1}{|t|} \right)^{2b_\pi/b} . \tag{A7}
\]

Inserting the values for the OPE coefficients reported in Ref. 33, and \( \nu = 1/y_e = 1 \) for the pure 2d Ising model, this gives the divergence of the correlation length at \( h = 0 \) to leading order

\[
\xi(t) \sim |t|^{-1} \left( 1 + 8\pi \Delta \ln \frac{1}{|t|} \right)^{1/2} . \tag{A8}
\]

The fact that \( b_\pi/b = 0 \) entails that magnetization and susceptibility will to leading order scale as pure powers of the correlation length, in agreement with earlier results of Shalaev.\(^{3} \)

Phenomenological renormalization is based on analyzing the finite size signatures of (A6) and of its temperature derivative

\[
\mu_L(t) = \frac{d}{dt} \xi(t, h, g_0, L^{-1}) = e^l \xi_e(l) u_e(l), u_\sigma(l), g(l), e^l L^{-1} , \tag{A9}
\]

where \( \xi_e \) designates the partial derivative of the correlation-length scaling function (A6) with respect to \( u_e(l) \). Finite size signatures at criticality \( (t = 0, h = 0) \) are obtained by analyzing these quantities at a scale \( e^l = L \), giving

\[
\xi_L^{-1} = L^{-1} \xi^{-1}(0, 0, g(ln L), 1) = e^{-1} \phi(g(ln L)) \tag{A10}
\]

and

\[
\mu_L = L^2 \left( 1 + 8\pi \Delta \ln L \right)^{-1/2} \xi(0, 0, g(ln L), 1) = L^2 \left( 1 + 8\pi \Delta \ln L \right)^{-1/2} \Phi_x(g(ln L)) \tag{A11}
\]

respectively. Eqs. (A10) and (A11) define the universal scaling functions \( \Phi_x \) and \( \Phi_x \) of the argument \( x \) which, in the zero-replica limit, is \( x = \Delta/(1 + 8\pi \Delta \ln L) \).

In the weak disorder limit or at large \( L \) this quantity is small, and one assumes that an expansion of these scaling functions in \( x \) exists. For \( \Phi \) one gets \( \Phi_x = \phi_0 + \phi_1 x + \phi_2 x^2 + O(x^3) \), the coefficients being known to be \( \phi_0 = \pi \eta \) from conformal invariance,\(^{40} \) and \( \phi_1 = 0 \) on account of the vanishing of \( b_\sigma \). Thus

\[
\xi_L^{-1} = L^{-1} \left[ \eta + \phi_2 \left( \frac{\Delta}{1 + 8\pi \Delta \ln L} \right)^2 \right] + O \left( \left( \frac{\Delta}{1 + 8\pi \Delta \ln L} \right)^3 \right) . \tag{A12}
\]

A similar expansion is expected to hold for \( \Phi_x \), i.e., \( \Phi_x = \phi_{x0} + \phi_{x1} x + O(x^2) \), so (with \( D_1 = \phi_{x1}/\phi_{x0} \))

\[
\mu_L = \phi_{x0} L^2 \left( 1 + 8\pi \Delta \ln L \right)^{-1/2} \times \left[ 1 + \frac{D_1 \Delta}{1 + 8\pi \Delta \ln L} + O \left( \left( \frac{\Delta}{1 + 8\pi \Delta \ln L} \right)^2 \right) \right] \tag{A13}
\]

though the coefficients \( \phi_{x0} \) and \( \phi_{x1} \) are to the best of our knowledge not known. Interest in this quantity derives from the fact that, within the phenomenological renormalization group scheme, the correlation length exponent \( \nu \) is obtained by computing the sequence of finite size approximants \( \nu_L \), defined by

\[
\nu_L^{-1} = \frac{d \ln \mu_L}{d \ln L} - 1 , \tag{A14}
\]

and by extrapolating this sequence to the large system limit. Eq. (22) is just the finite difference approximation to (A14).
Similarly, the FSS analysis for the susceptibility gives
\[ \chi_L = L^{\gamma / \nu} f_{\sigma_0}(0, 0, g(\ln L), 1) \]
\[ = L^{\gamma / \nu} \Phi_{\chi}(g(\ln L)) \quad (A15) \]
with the same conventions for the notation. The ratio \( \gamma / \nu \) is that of the pure system, i.e., \( \gamma / \nu = 7/4 \).
On account of the vanishing of \( b_4 \), no additional logarithmic terms above those following from the expansion of \( \Phi_{\chi} \) appear in (A15), in contrast to what happens for \( \mu_L \). As before, one assumes that an expansion of the form \( \Phi_{\chi}(x) = \phi_{\alpha_0} + \phi_{\alpha_1} x + \phi_{\alpha_2} x^2 + \mathcal{O}(x^3) \) exists, and defines the effective size dependent ratio \( (\gamma / \nu)_L \) by
\[ (\gamma / \nu)_L = \partial \ln \chi_L / \partial \ln L. \]
It will be useful to introduce the abbreviation \( E_i = \phi_{\chi i}/\phi_{\alpha 0} \) below.

For the interpretation of numerical data it is relevant to note results for these quantities both in the preasymptotic regime \( 8 \pi \Delta \ln L \ll 1 \) and in the asymptotic regime \( 8 \pi \Delta \ln L \gg 1 \).

(i) In the preasymptotic regime, they read
\[ \xi_{L}^{-1} = L^{-1} \left[ (\pi \eta + \phi_{2} \Delta^2 (1 - 16 \pi \Delta \ln L)) \right] \quad (A16a) \]
\[ \mu_{L} = \phi_{\alpha_0} L^2 (1 + D_1 \Delta - (1 + 2 D_1 \Delta)4 \pi \Delta \ln L) \quad (A16b) \]
\[ \nu_{L} = 1 + 4 \pi \Delta + (4 \pi \Delta^2) (1 + D_1/2 - 2 \ln L) \quad (A16c) \]
\[ \left( \frac{\gamma}{\nu} \right)_L = \frac{7}{4} - 8 \pi \Delta^2 E_1 + 8 \pi \Delta^3 (E_1^2 - 2 E_2) \]
\[ + 16 \pi E_1 \ln L \quad (A16d) \]
where we have kept the lowest order in \( \Delta \ln L \). Additional terms down by further factors of \( \Delta \) or \( \Delta \ln L \) are not shown. These results exhibit non-universal corrections.

(ii) In the asymptotic regime, on the other hand, one has
\[ \xi_{L}^{-1} = L^{-1} \left[ \pi \eta + \phi_{2} \left( \frac{1}{8 \pi \ln L} \right)^2 \right] + \mathcal{O} \left( \left( \frac{1}{\ln L} \right)^3 \right) \quad (A17a) \]
\[ \mu_{L} = \phi_{\alpha_0} L^2 (8 \pi \Delta \ln L)^{-1/2} \]
\[ \times \left[ 1 + \frac{D_1}{8 \pi \ln L} + \mathcal{O} \left( \left( \frac{1}{\ln L} \right)^2 \right) \right] \quad (A17b) \]
\[ \nu_{L} = 1 + \frac{1}{2 \ln L} + \mathcal{O} \left( \left( \frac{1}{\ln L} \right)^2 \right) \quad , \quad (A17c) \]
\[ \left( \frac{\gamma}{\nu} \right)_L = \frac{7}{4} - E_1 \frac{1}{8 \pi (\ln L)^2} + \mathcal{O} \left( \left( \frac{1}{\ln L} \right)^3 \right) \quad , \quad (A17d) \]
so the asymptotic corrections to \( \xi_{L}^{-1} \), \( \nu_{L} \) and \( (\gamma / \nu)_L \) turn out to be universal.

Ludwig and Cardy have also reported corresponding FSS expressions for the central charge, in both regimes. These are
\[ c = \frac{1}{2} - 128 \pi^3 \Delta^3 (1 - 24 \pi \Delta \ln L) \quad (A18a) \]
and
\[ c = \frac{1}{2} - \frac{1}{4} (\ln L)^{-3} + \mathcal{O} \left( \frac{1}{8 \pi \ln L} \right)^4 \quad , \quad (A18b) \]
respectively, where we have once more kept the lowest order in \( \Delta \ln L \) in the preasymptotic regime.

Additional terms further down by powers of \( L^{\nu_0} (1 + 8 \pi \Delta \ln L)^{-2 b_0 / \nu_0} \), with \( y_0 \approx -2 \), will appear due to irrelevant scaling fields.

Eqs. (A16) - (A18) are of great importance when analyzing FSS data, if logarithmic corrections are expected to be present. Especially in a strip-scaling approach such as ours, the preasymptotic results may turn out of particular relevance, since the maximum size available is such that one may not reach the asymptotic regime, which is likely to be true at least for weak disorder. This has, indeed, been the point of view emphasized by Aarão Reis et al.\(^12\)
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