Towards the Anomalous Dimension to $\mathcal{O}(\Lambda_{QCD}/m_b)$ for Phase Space Restricted $\bar{B} \to X_u \ell \bar{\nu}$ and $\bar{B} \to X_s \gamma$

Michael Trott\textsuperscript{1,}\textsuperscript{*} and Alexander R. Williamson\textsuperscript{2,}\textsuperscript{†}

\textsuperscript{1}Department of Physics, University of California at San Diego, La Jolla, CA, 92093
\textsuperscript{2}Department of Physics, Carnegie Mellon University, Pittsburgh, PA, USA, 15213

Abstract

We examine the anomalous dimension matrix appropriate for the phase space restricted $\bar{B} \to X_u \ell \bar{\nu}$ and $\bar{B} \to X_s \gamma$ decay spectra to subleading nonperturbative order. The time ordered products of the HQET Lagrangian with the leading order shape function operator are calculated, as are the anomalous dimensions of subleading operators. We establish the renormalizability and closure of a subset of the non-local operator basis, a requirement for the establishment of factorization theorems at this order. Operator mixing is found between the operators which occur to subleading order, requiring the subleading operator basis be extended. We comment on the requirement for new shape functions to be introduced to characterize the matrix elements of these new operators, and the phenomenological consequences for extractions of $|V_{ub}|$.

\textsuperscript{*}Electronic address: mrtrott@physics.ucsd.edu
\textsuperscript{†}Electronic address: alexwill@andrew.cmu.edu
I. INTRODUCTION

Extracting the CKM parameter $|V_{ub}|$ is an important step in testing of the CKM description of CP violation in the $B$ meson system. Currently, the theoretically cleanest determinations of $|V_{ub}|$ come from inclusive semileptonic decays which are not sensitive to the details of hadronization; although recently an approach of extracting $|V_{ub}|$, utilizing $B \rightarrow \pi \pi$, has been advanced with a competitive error to inclusive methods. [1]

In inclusive extractions of $|V_{ub}|$, experimental cuts to exclude the charm background of $\bar{B} \rightarrow X_c \ell \bar{\nu}$ are imposed. This restricts the decay products to hadronic final states that have large energy $E_X \sim m_B$ and low invariant mass $M_X \sim \sqrt{m_B \Lambda_{QCD}}$. With these phase space restrictions the local OPE expansion [2] appropriate for sufficiently inclusive decays used to extract $|V_{cb}|$ [3], typically breaks down. [4]

As the local OPE, and the clean separation of scales that the local OPE represented in the analysis of $\bar{B} \rightarrow X_c \ell \bar{\nu}$, is no longer valid; a more involved theoretical approach is required to separate the scales relevant to these decays. Decay rates are expressed as convolutions of hard ($H$), jet ($J$) and soft physics ($S$) associated with the scales $m_b \gg \sqrt{\Lambda_{QCD}}$, in the following way,

$$d \Gamma = H \left( \frac{m_b}{\mu}, \alpha_s(\mu) \right) \int d \omega J \left( \frac{\sqrt{m_b \Lambda_{QCD}}}{\mu}, \alpha_s(\mu), \omega \right) S(\omega). \quad (1)$$

Although this factorization theorem has been proven diagrammatically [5] at leading order in $1/m_b$, it is not known to hold to all orders in the nonperturbative expansion. It has only recently been extended beyond leading nonperturbative order [6].

The systematic treatment of the nonperturbative corrections involve a two step matching procedure. One matches QCD onto the effective field theory of the intermediate scale, describing quarks and gluons with large energy and small offshellness, known as SCET [23, 24, 25], and uses the renormalization group evolution to run down to the soft scale. One then matches SCET onto the lightcone wavefunction of the $B$ meson, expressed in terms of HQET fields. One can also match directly from QCD onto HQET, a much simpler procedure at the cost of not summing the logarithms of the ratio of scales $\log \left( \frac{\sqrt{m_b \Lambda_{QCD}}/m_b}{m_B} \right)$ via SCET. In either case, the soft sector of the theory is expanded in terms of non-local operators. The leading order term in the $\Lambda_{QCD}/m_b$ expansion of the lightcone distribution function of the $B$ meson [7, 8] is known as the shape function.
At subleading order in the nonperturbative expansion, four additional non-local operators have been determined to be present \[9, 10, 11, 12, 13\], the matrix elements of which are referred to as subleading shape functions.

It is of some intrinsic interest to examine the renormalization of these non-local operators, as they are non-local and their renormalizability is not known \textit{a priori}. It is also important to know if this set of operators is complete in the error assigned to extractions of $|V_{ub}|$, and in considering the above factorization theorem beyond leading order. Examining the perturbative behavior of these subleading shape functions is also a necessary step to take to perform one loop matching calculations onto the soft sector. For these reasons we have determined the anomalous dimension to subleading nonperturbative order.

To this end, we have determined the contributions of the time ordered products of the subleading $\mathcal{L}_{HQET}$ with the leading order shape function, and examined the anomalous dimensions of the subleading nonperturbative operators. We establish the renormalizability and closure of a subset of the subleading non-local operators. We find that the known operator basis mixes with new operators, requiring that the subleading operator basis be extended. We also comment on the phenomenological consequences of these results.

\section{II. ANOMALOUS DIMENSION TO SUBLEADING ORDER}

\subsection{A. Notation}

We introduce two light-like vectors $n^\mu$ and $\bar{n}^\mu$ related to the velocity of the heavy quark by $v = \frac{1}{2}(n + \bar{n})$, and satisfying

$$n^2 = \bar{n}^2 = 0, \quad v \cdot n = v \cdot \bar{n} = 1, \quad n \cdot \bar{n} = 2.$$ \hspace{1cm} (2)

In the frame in which the $b$ quark is at rest, these vectors are given by $n^\mu = (1, 0, 0, 1)$, $\bar{n}^\mu = (1, 0, 0, -1)$ and $v^\mu = (1, 0, 0, 0)$. The projection of an arbitrary four-vector $a^\alpha$ onto the directions which are perpendicular to the lightcone is given by $a_{\perp}^\alpha = g_{\perp}^{\alpha\beta} a_\beta$, where

$$g_{\perp}^{\mu\nu} \equiv g^{\mu\nu} - \frac{1}{2} (n^\mu \bar{n}^\nu + n^\nu \bar{n}^\mu).$$ \hspace{1cm} (3)

We also define a perpendicular Levi-Civita tensor

$$\epsilon_{\perp}^{\alpha\beta} = \epsilon^{\alpha\beta\sigma\rho} v_\sigma n_\rho.$$ \hspace{1cm} (4)
We also use the projector \( P_+ = 1/2(1 + \not{p}) \) as well as the Dirac structure \( s^n = P_+ \gamma^n \gamma_5 P_+ \), so that \( v \cdot s = 0 \).

1. Distribution Notation

Rather than the usual definitions of the star distribution as given in Neubert and deFazio [15],

\[
\left( \frac{1}{x} \right)_* = \lim_{\beta \to 0} \left[ \frac{\theta(x - \beta)}{x} \delta(x - \beta) \log(x) \right],
\]

\[
\left( \frac{\log(x)}{x} \right)_* = \lim_{\beta \to 0} \left[ \frac{\theta(x - \beta)}{x} \log(x) + \frac{1}{2} \delta(x - \beta) \log^2(x) \right],
\]

we utilize the alternate notation, equivalent to the \( \mu \)-distribution’s in [19],

\[
\phi_n(x) \equiv \lim_{\beta \to 0} \left[ \frac{1}{n+1} \theta(x - \beta) \log^{n+1}(x) \right].
\]

This notation has a fairly easy correspondence to the usual star distribution notation

\[
\phi_0'(x) = \left( \frac{1}{x} \right)_*, \quad \phi_1'(x) = \left( \frac{\log(x)}{x} \right)_*.
\]

A useful identity given by analytic continuation is

\[
\frac{\theta(x)}{x^{1+\epsilon}} = -\frac{1}{\epsilon} \delta(x) + \phi_0'(x) - \epsilon \phi_1'(x) + \mathcal{O}(\epsilon^2)
\]

This relationship is valid when integrated against arbitrary functions \( f(x) \), where \( f \) is not singular at the origin. In general we can write the recursion relation

\[
\frac{\theta(x)}{x^{n+\epsilon}} = \frac{-1}{n-1+\epsilon} \frac{d}{dx} \left[ \frac{\theta(x)}{x^{n-1+\epsilon}} \right] \text{ for } n \geq 2.
\]

Several other useful properties of this function are (for some positive constant \( a \)):

\[
x \phi_0'(x) = \theta(x)
\]

\[
x \phi_0''(x) = \delta(x) - \phi_0'(x)
\]

\[
a \phi_0'(ax) = \phi_0'(x) + \delta(x) \log(a),
\]

as well as

\[
\int_{-\infty}^{a} f(x) \phi_0'(x) = \int_{0}^{a} \left( \frac{\theta(x)}{x} \right)_* f(x) + f(0) \log(a).
\]

For an \( a \) with dimension, such as \( a = \Lambda_{\text{QCD}} \), the above equation is suitably modified to obtain logs the dimensionless ratio \( a/\mu \).
B. Operators to subleading Order

At leading order a single non-local operator characterizes the nonperturbative physics,

\[ Q_0(\omega, \Gamma) = \bar{h}_v \delta(\omega + i n \cdot D) \Gamma h_v, \quad (12) \]

where the covariant derivative is \( D_\mu = \partial_\mu + i g_s A_\mu \).

The order \( \Lambda_{QCD}/m_b \) corrections to the \( \bar{B} \to X_u \ell \bar{\nu} \) and \( \bar{B} \to X_s \gamma \) decay spectra require the introduction of four additional non-local operators \[9, 10, 11, 12, 13\],

\[
\begin{align*}
    m_b Q_1^\mu(\omega, \Gamma) &= \bar{h}_v \{ iD_\perp^\mu, \delta(\omega + i n \cdot D) \} \Gamma h_v, \\
    m_b Q_2^\mu(\omega, \Gamma) &= \bar{h}_v [iD_\perp^\mu, \delta(\omega + i n \cdot D)] \Gamma h_v, \\
    m_b Q_3(\omega, \Gamma) &= \int d\omega_1 d\omega_2 \delta(\omega_1, \omega_2; \omega) \bar{h}_v \delta(\omega_2 + i n \cdot D) g_1^{\mu\nu} \{ iD_\perp^\mu, iD_\perp^\nu \} \delta(\omega_1 + i n \cdot D) \Gamma h_v, \\
    m_b Q_4(\omega, \Gamma) &= -\int d\omega_1 d\omega_2 \delta(\omega_1, \omega_2; \omega) \bar{h}_v \delta(\omega_2 + i n \cdot D) i\epsilon_{\perp}^{\mu\nu} [iD_\perp^\mu, iD_\perp^\nu] \delta(\omega_1 + i n \cdot D) \Gamma h_v,
\end{align*}
\]

where

\[
\delta(\omega_1, \omega_2) = \frac{\delta(\omega - \omega_1) - \delta(\omega - \omega_2)}{\omega_1 - \omega_2}. \quad (14)
\]

We define these operators rescaled by \( m_b \) for later convenience in the anomalous dimension. This rescaling should be noted when comparing to other work dealing with subleading shape functions. We also use the convention of labeling operators as \( Q_i \) operators when the Dirac structure is general, and referring to the operators as \( O_i \) and \( P_i \) when a particular Dirac structure is required.

We find that the operator basis must be extended beyond tree level to include, at least the following operator

\[
m_b \bar{Q}_1^\mu(\omega, \Gamma) = -2 \int d\omega_1 d\omega_2 \theta(\omega_1, \omega_2; \omega) K_2^\mu(\omega_1, \omega_2; \Gamma), \quad (15)
\]

where we have defined the following kernel and coefficient functions

\[
K_2^\mu(\omega_1, \omega_2; \Gamma) = \bar{h}_v \delta(\omega_1 + i n \cdot D) iD_\perp^\mu \delta(\omega_2 + i n \cdot D) \Gamma h_v, \quad (16)
\]

\[
\theta(\omega_1, \omega_2; \omega) = \frac{\theta(\omega - \omega_1) - \theta(\omega - \omega_2)}{\omega_1 - \omega_2}. \quad (17)
\]

The operator \( \bar{Q}_1 \) was originally defined in [9] by Mannel and Tackmann \[28, 29\] based on symmetry arguments and examining the endpoint of \( \bar{B} \to X_c \ell \bar{\nu} \) and taking the massless limit. We find that the operator is unambiguously required beyond tree level due to the mixing experienced with the original set of operators.
C. Operator Feynman rules

We use Feynman Gauge to calculate the anomalous dimension to subleading order as the usual choice of lightcone gauge introduces non physical poles in the calculation, for a review of the relevant issues see [30].

Below, we present the required one and two gluon Feynman rules. The two gluon Feynman rules are also required and used but are too lengthy to include here, they can be obtained from the authors upon request. The non-vanishing zero gluon Feynman rules in Feynman gauge are:

\[
\langle h_v(k) | Q_0(\omega, \Gamma) | h_v(k) \rangle = \delta(\omega + n \cdot k) \Gamma, \tag{18}
\]

\[
\langle h_v(k) | Q_1^\mu(\omega, \Gamma) | h_v(k) \rangle = 2 \frac{k^\mu}{m_b} \delta(\omega + n \cdot k) \Gamma, \tag{19}
\]

\[
\langle h_v(k) | Q_2^\mu(\omega, \Gamma) | h_v(k) \rangle = 2 \frac{k^\mu}{m_b} \delta(\omega + n \cdot k) \Gamma, \tag{20}
\]

The one gluon Feynman rule for the leading order operator is

\[
\langle h_v(k) A_0^\nu(\ell) | Q_0(\omega, \Gamma) | h_v(k) \rangle = -gT_a n^\nu \left( \frac{\delta_\perp(n \cdot \ell)}{n \cdot \ell} \right) \Gamma. \tag{21}
\]

The one gluon Feynman rules for single covariant derivative operators are:

\[
\langle h_v(k) A_0^\nu(\ell) | Q_1^\mu(\omega, \Gamma) | h_v(k) \rangle = -gT_a g_\perp^{\mu \nu} \delta_\perp(n \cdot \ell) \frac{\Gamma}{m_b} - gT_a n^\nu \left( 2 k + \ell \right)^\perp \left( \frac{\delta_\perp(n \cdot \ell)}{n \cdot \ell} \right) \frac{\Gamma}{m_b}, \tag{22}
\]

\[
\langle h_v(k) A_0^\nu(\ell) | \bar{Q}_1^\mu(\omega, \Gamma) | h_v(k) \rangle = -2 gT_a g_\perp^{\mu \nu} \left( \frac{\theta_\perp(n \cdot \ell)}{n \cdot \ell} \right) \frac{\Gamma}{m_b} + 2 gT_a n^\nu \ell_\perp^{\mu} \left( \frac{\theta_\perp(n \cdot \ell)}{(n \cdot \ell)^2} \right) \frac{\Gamma}{m_b}, \tag{23}
\]

\[
\langle h_v(k) A_0^\nu(\ell) | Q_2^\mu(\omega, \Gamma) | h_v(k) \rangle = gT_a g_\perp^{\mu \nu} \delta_\perp(n \cdot \ell) \frac{\Gamma}{m_b} - gT_a n^\nu \ell_\perp^{\mu} \left( \frac{\delta_\perp(n \cdot \ell)}{n \cdot \ell} \right) \frac{\Gamma}{m_b}. \tag{24}
\]

Finally, the one gluon Feynman rules for two covariant derivative operators are as follows:

\[
\langle h_v(k) A_0^\nu(\ell) | Q_3(\omega, \Gamma) | h_v(k) \rangle = 2 gT_a \left( 2 k + \ell \right)^\perp \left( \frac{\delta_\perp(n \cdot \ell)}{n \cdot \ell} \right) - n^\nu k_\perp^{\mu} \left( \frac{\delta_\perp(n \cdot \ell)}{n \cdot \ell} \right) \frac{\Gamma}{m_b}, \tag{25}
\]

\[
+ 2 gT_a n^\nu \left( 2 k \cdot \ell_\perp + \ell_\perp^{\mu} \left( \frac{\delta_\perp(n \cdot \ell)}{(n \cdot \ell)^2} \right) \right) \frac{\Gamma}{m_b}, \tag{26}
\]

\[
\langle h_v(k) A_0^\nu(\ell) | Q_4(\omega, \Gamma) | h_v(k) \rangle = 2 gT_a \iota^\beta \ell_\perp^{\beta} \left( \frac{\delta_\perp(n \cdot \ell)}{n \cdot \ell} \right) \frac{\Gamma}{m_b}. \tag{27}
\]
where $\ell$ is the gluon momentum flowing out of the vertex, and the gluon carries Lorentz index $\nu$ and colour index $a$. We have also made the convenient definitions

$$\theta_{\pm}(x) = \theta(\omega + n \cdot k + x) \pm \theta(\omega + n \cdot k) \quad (22)$$

$$\delta_{\pm}(x) = \delta(\omega + n \cdot k + x) \pm \delta(\omega + n \cdot k). \quad (23)$$

### D. The Anomalous Dimension Matrix

The renormalization of the operators $Q_i(\omega, \Gamma)$ is performed in the usual fashion,

$$Q_i(\omega, \Gamma)_{\text{bare}} = \int d\omega' Z_{ij}(\omega', \omega, \bar{\mu}) Q_j(\omega', \bar{\mu}, \Gamma)_{\text{ren}}, \quad (24)$$

where $Z_{ij}(\omega', \omega, \bar{\mu})$ is a matrix of renormalization constants. The values of the elements of $Z_{ij}$ can be found by taking arbitrary partonic matrix elements of both sides, which at leading order gives $Z_{ij}^{(0)}(\omega', \omega, \bar{\mu}) = \delta_{ij} \delta(\omega - \omega')$.

To subleading order in $\alpha_s$ we have

$$\langle Q_i(\omega, \Gamma) \rangle^{(0)}_{\text{bare}} + \alpha_s \langle Q_i(\omega, \Gamma) \rangle^{(1)}_{\text{bare}} = \int d\omega' \left[ Z_{ij}^{(0)}(\omega', \omega, \bar{\mu}) + \alpha_s Z_{ij}^{(1)}(\omega', \omega, \bar{\mu}) \right]$$

$$\times \left[ \langle Q_j(\omega', \bar{\mu}, \Gamma) \rangle^{(0)}_{\text{ren}} + \alpha_s \langle Q_j(\omega', \bar{\mu}, \Gamma) \rangle^{(1)}_{\text{ren}} \right], \quad (25)$$

from which one obtains

$$\int d\omega' Z_{ij}^{(1)}(\omega', \omega, \bar{\mu}) \langle Q_j(\omega', \bar{\mu}, \Gamma) \rangle^{(0)} = \langle Q_i(\omega, \Gamma) \rangle^{(1)}_{\text{bare}} - \langle Q_i(\omega, \Gamma) \rangle^{(1)}_{\text{ren}}$$

$$= (\langle Q_i(\omega, \Gamma) \rangle^{(1)}_{\text{bare}})_{\text{div}} \quad (26)$$

where by $(\langle Q_i(\omega, \Gamma) \rangle^{(1)}_{\text{bare}})_{\text{div}}$, we refer to the UV divergent part of $\langle Q_i(\omega, \Gamma) \rangle^{(1)}_{\text{bare}}$. Because there are operators such as $Q_2$ and $Q_4$ which do not have a zero gluon form, we must consider matrix elements of Eq. (24) with at least one external gluon. These will be sufficient to identify the mixing of the various operators into $Q_2$ and $Q_4$. It should be noted that matrix elements with zero and one external gluon states are not sufficient in principle to determine the anomalous dimension matrix to subleading order. The operator

$$Q^{\mu \nu}(\omega_1, \omega_2, \Gamma) = h_v \left[ i D_{1\perp}^\mu, \delta(\omega_2 + in \cdot D) \right] \left[ i D_{1\perp}^\nu, \delta(\omega_1 + in \cdot D) \right] \Gamma h_v, \quad (27)$$

does not have a zero gluon or one gluon Feynman rule. Its nonvanishing Feynman rules start at two gluon external states. In this paper, we will not be calculating the two external gluon
diagrams necessary to find mixing into this operator, if any exists. We extract the anomalous
dimension matrix of the subleading operators by examining matrix elements containing one
perpendicularly polarized external gluon:

\[
\int d\omega' Z_{ij}^{(1)}(\omega', \omega, \bar{\mu}) \langle h_v(k) A_\perp | Q_i(\omega', \Gamma) | h_v(k) \rangle^{(0)}_{\text{ren}} = \langle h_v(k) A_\perp | Q_i(\omega, \Gamma) | h_v(k) \rangle^{(1)}_{\text{bare}} \div.
\]  

(28)

The non-perpendicular components of the gluon field were also examined but found to induce
no further mixing.

The mixing of \( Q_0 \) into the other operators is determined by calculating matrix elements
of this operator with insertions of the subleading HQET Lagrangian. Zero gluon matrix
elements are sufficient to find the mixing into \( Q_0 \), while one gluon matrix elements are
required for mixing into the remaining operators. Due to the spin symmetry violating
effects of the subleading HQET Lagrangian, the anomalous dimension of the \( P_i \) operators
can differ from that of the \( O_i \) operators.

The wavefunction renormalization of the bare operators expressed in terms of renormal-
ized fields are \( Q_i(\omega, \Gamma)_{\text{bare}} = Z_h Z_3^{n/2} Q_i(\omega, \Gamma) \) where \( n \) is the number of gluons in the operator,
and \( Q_i(\omega, \Gamma) \) is written in terms of renormalized fields. For diagrams with an external state
gluon we use the background field method to treat the external gluon as a classical field and
so we acquire no \( Z_3 \) factor due to the wavefunction renormalization of the gluon. [31]

E. Diagram Calculations

One Gluon Matrix Elements

The one gluon matrix elements are determined by calculating the diagrams shown in
Figure 1 for each operator. The external gluon in each of these diagrams is a background
field gluon and the external states are chosen to have perpendicular polarization. We utilize
dimensional regularization and the \( \overline{\text{MS}} \) scheme to regulate our divergences. To isolate and
remove the IR divergences in the calculation we keep all the particles off shell by retaining
factors of \( v \cdot k, v \cdot \ell \) and \( \ell^2 \), where \( \ell \) is the external gluon momentum.

To clearly illustrate the need to extend the operator basis we present the results for \( Q_1 \)
diagram by diagram. In general, for perpendicular polarized external gluons, only diagrams
(ac), (ad), (bc), (bd) and (dc) contribute to the amplitude. For diagram dc, the loop
FIG. 1: The one gluon diagrams which must be calculated for each operator.

The integrals to perform are as follows, with $c_1 \equiv (\alpha_s g_s T a g_{\perp}^{\mu\nu})/(4 \pi)$,

$$
\langle i A_{dc} \rangle^{(1)}_{\text{div}} = -\frac{c_1}{m_b} C_A (n \cdot \ell)^2 \mu^{4-d} \int \frac{d^d q}{(2\pi)^d} \frac{\delta_+(n \cdot \ell)}{n \cdot q (n \cdot \ell + n \cdot q)(q^2 + i\epsilon)((q + \ell)^2 + i\epsilon)}
$$

$$
+ \frac{c_1}{m_b} C_A (n \cdot \ell)^2 \mu^{4-d} \int \frac{d^d q}{(2\pi)^d} \frac{\delta(\omega + n \cdot k - n \cdot q) + \delta(\omega + n \cdot k + n \cdot \ell + n \cdot q)}{n \cdot q (n \cdot \ell + n \cdot q)(q^2 + i\epsilon)((q + \ell)^2 + i\epsilon)}
$$

The integrals are performed via the standard techniques of dimensional regularization with $d = 4 - 2\epsilon$, the $\overline{\text{MS}}$ renormalization scale $\tilde{\mu}^2 = 4 \pi \mu^2 e^{-\gamma_E}$, and the utilization of Eq.(8) and we have suppressed the lorentz and colour indicies. The UV poles obtained for this diagram for $Q_1$ after the integrals are performed and consideration of the symmetry factor are

$$
\langle i A_{dc} \rangle^{(1)}_{\text{div}} = \frac{c_1 C_A}{m_b \epsilon} \left( \frac{n \cdot \ell}{\tilde{\mu} (\omega + n \cdot k)} \phi_0(\frac{\omega + n \cdot k + n \cdot \ell}{\tilde{\mu}}) - \frac{n \cdot \ell}{\tilde{\mu} (\omega + n \cdot k + n \cdot \ell)} \phi_0(\frac{\omega + n \cdot k}{\tilde{\mu}}) \right)
$$

$$
+ \frac{c_1 C_A}{m_b \epsilon} \left( \delta_+(n \cdot \ell) \log(n \cdot \ell / \tilde{\mu}) \right).
$$

The results for diagrams ac and bc when inserting $Q_1$ are

$$
\langle i A_{ac} \rangle^{(1)}_{\text{div}} = \frac{c_1 C_A}{m_b \epsilon} \left( \frac{\delta(\omega + n \cdot k + n \cdot \ell)}{2 \epsilon} + \delta(\omega + n \cdot k + n \cdot \ell) \right)
$$
\[- \frac{c_1 C_A}{m_b \epsilon} \left( \log \left( \frac{n \cdot \ell}{\mu} \right) \delta(\omega + n \cdot k + n \cdot \ell) \right) + \frac{c_1 C_A}{m_b n \cdot \epsilon} \left( \frac{(\omega + n \cdot k)}{\omega + n \cdot k + n \cdot \ell} \theta \left( \frac{\omega + n \cdot k}{\mu} \right) - \theta \left( \frac{\omega + n \cdot k + n \cdot \ell}{\mu} \right) \right), \]

\[\langle i \mathcal{A}_{bc} \rangle_{\text{div}}^{(1)} = \frac{c_1 C_A}{m_b \epsilon} \left( \frac{\delta(\omega + n \cdot k)}{2 \epsilon} + \delta(\omega + n \cdot k) - \log \left( \frac{n \cdot \ell}{\mu} \right) \delta(\omega + n \cdot k) \right) + \frac{c_1 C_A}{m_b n \cdot \epsilon} \left( -\frac{(\omega + n \cdot k + n \cdot \ell)}{\omega + n \cdot k} \theta \left( \frac{\omega + n \cdot k + n \cdot \ell}{\mu} \right) + \theta \left( \frac{\omega + n \cdot k}{\mu} \right) \right). \]

Finally, the results for diagrams ad and bd for $Q_1$ insertions are

\[\langle i \mathcal{A}_{ad} \rangle_{\text{div}}^{(1)} = \frac{c_1 C_F}{m_b \epsilon} \left( \frac{\delta_+(n \cdot \ell)}{\epsilon} - \frac{2}{\mu} \phi_0^0 \left( \omega + n \cdot k + n \cdot \ell \right) - \frac{2}{\mu} \phi_0^0 \left( \omega + n \cdot k \right) \right) + \frac{c_1 C_A}{m_b \epsilon} \left( \frac{\delta(n \cdot \ell) + 2 \theta_+(n \cdot \ell/\mu)}{\delta_+(n \cdot \ell)} \right), \]

\[\langle i \mathcal{A}_{bd} \rangle_{\text{div}}^{(1)} = \frac{c_1 C_F}{m_b \epsilon} \left( \frac{\delta_+(n \cdot \ell)}{\epsilon} - \frac{2}{\mu} \phi_0^0 \left( \omega + n \cdot k + n \cdot \ell \right) - \frac{2}{\mu} \phi_0^0 \left( \omega + n \cdot k \right) \right) + \frac{c_1 C_A}{m_b \epsilon} \left( -\frac{\delta(n \cdot \ell) + 2 \theta_+(n \cdot \ell/\mu)}{\delta_+(n \cdot \ell)} \right). \]

The amplitudes combine to give the following UV poles

\[\langle i \mathcal{A}_{Q_1} \rangle_{\text{div}}^{(1)} = \frac{c_1 C_F}{m_b \epsilon} \left( \frac{2 \delta_+(n \cdot \ell)}{\epsilon} - \frac{4 \phi_0^0(n \cdot \ell/\mu)}{\mu} + \frac{c_1 C_A}{m_b \epsilon} \left( \delta_+(n \cdot \ell) + \frac{2 \theta_-(n \cdot \ell/\mu)}{n \cdot \ell} \right) \right). \]

Once the wavefunction renormalization terms are multiplicatively combined with the result, we express the amplitude in terms of renormalization matrix elements $d_i$ and the one gluon Feynman rules for the operators $Q_1^\mu$ and $\bar{Q}_1^\mu$ as follows

\[\langle i \mathcal{A}_{Q_1} \rangle_{\text{div}}^{(1)} = \frac{\alpha}{4 \pi} \int d\omega' \int d\omega \left[ d_1(\omega, \omega', \mu) \langle Q_1^\mu(\omega') \rangle^{(0)} - \langle \bar{Q}_1^\mu(\omega') \rangle^{(0)} \right], \]

where

\[d_1(\omega, \omega', \mu) = -\frac{2 C_F}{\epsilon^2} \delta(\omega - \omega') + \frac{2 C_F}{\epsilon} \delta(\omega - \omega') + \frac{4 C_F}{\mu} \phi_0^0 \left( \frac{\omega - \omega'}{\mu} \right), \]

\[d_4(\omega, \omega') = \frac{C_A}{\epsilon} \delta(\omega - \omega'). \]

The form of the mixing between $Q_1^\mu$ and $\bar{Q}_1^\mu$ deserves some comment. At zero gluon the matrix elements of these operators are identical causing this mixing to be undetermined for zero gluon external state diagrams, even though the zero gluon matrix elements of both
operators are nonzero, contrary to naive expectations. The contribution of the operator $Q_1^\mu$ to the renormalization matrix was also determined. We find that this operator mixes with itself, contributing a $d_1$ form to the matrix $Z_{SL}$. The antisymmetric operators $Q_2$ and $Q_4$ mix only with themselves and contribute diagonal factors of $d_1$ to the matrix of renormalization constants.

The inauspicious form of mixing between $Q_1^\mu$ and $Q_1^\mu$ is also present in the operator $Q_3$, however the corresponding analysis of $Q_3$ is more complicated and is still under investigation. Due to this complication in determining the full anomalous dimension matrix and the need for a two gluon calculation to determine the possible mixing with $Q^{\mu,\nu}(\omega_1, \omega_2, \Gamma)$ we present the results of our initial study of the anomalous dimension to subleading order in this paper and comment on the phenomenological consequences of the presented results. We collect our results in section III.

The $T$ products of $\mathcal{L}_{HQET}$ with $O_0$ and $P_0$

To find the mixing of the operators

\begin{align*}
O_0 &= \bar{h}_v \delta(\omega + in \cdot D) h_v, \\
P_\mu^0 &= \bar{h}_v \delta(\omega + in \cdot D) \gamma^\mu \gamma_5 h_v
\end{align*}

into the subleading operators, we must evaluate the time ordered products of the operators with the the subleading terms of the HQET Lagrangian ($\mathcal{L}_1$)

\begin{align*}
T_{O_0}(\omega) &= i \int d^4 x T \{ \mathcal{L}_1(x), O_0(0, \omega) \}, \\
T_{P_\mu^0}(\omega) &= i \int d^4 x T \{ \mathcal{L}_1(x), P_\mu^0(0, \omega) \}.
\end{align*}

We now explicitly refer to the Dirac structure of the operators. This is necessary due to the Dirac structure of the operators in the subleading Lagrangian. We treat the subleading Lagrangian as a single operator insertion for the purposes of our calculation. The different renormalization of the kinetic and chromomagnetic terms is accommodated by breaking the $T$ products up in to $T(O_0, O_k)$, $T(O_0, O_m)$, $T(P_\mu^0, O_k)$, $T(P_\mu^0, O_m)$ after the diagram calculations, where $O_k, O_m$ refer to the kinetic and chromomagnetic operators of the subleading Lagrangian.

We start with the zero gluon diagrams. They are illustrated in Figure 2. The crosses in
FIG. 2: The diagrams which must be calculated for $Q_0$. The crosses represent insertions of the subleading HQET Lagrangian.

The diagrams represent the locations where one inserts the subleading HQET Lagrangian, given by

$$L_1 = \bar{h}_v \left( i D_{\perp} \right)^2 h_v - a(\bar{\mu}) \bar{h}_v \left( g \sigma_{\alpha \beta} C^{\alpha \beta} \right) \frac{4 m_b}{4 m_b} h_v. \quad (38)$$

The zero, one and two gluon Feynman rules for this Lagrangian, where we suppress the renormalization scale dependence of the $O_m$ operator, are

$$i L_1[0\text{-gluon}] = i \frac{k^2}{2 m_b} P_+$$

$$i L_1[1\text{-gluon}] = -i g T_a \frac{(2 k_\perp + l_\perp)^{\alpha}}{2 m_b} P_+ + i g T_a \frac{i \epsilon^{\rho \mu \nu \eta} v_\mu v_\rho s_\eta}{2 m_b}$$

$$i L_1[2\text{-gluon}] = i g^2 \{ T^{a_1}, T^{a_2} \} \frac{g^{\alpha_1 \alpha_2}}{2 m_b} P_+ + i g^2 [T^{a_1}, T^{a_2}] \frac{i \epsilon^{\alpha_1 \alpha_2 \rho \eta} v_\rho v_\eta}{2 m_b} \quad (39)$$

The UV divergent part of the sum of the subleading $L_{\text{HQET}}$ zero gluon results for both $O_0$ and $P_0$ (up to Dirac structure) is the same. Our result for the general Dirac structure operator $Q_0$, with indices suppressed, is

$$\langle h_v(k)|T_{(Q_0, O_0)}(\omega, \Gamma)|h_v(k)\rangle^{(1)}_{\text{div}} = \frac{\alpha_s}{4 m_b \pi} C_F \int d \omega' \frac{A_4 \delta(\omega - \omega')}{\epsilon} \langle h_v(k)|Q_0(\omega', \Gamma)|h_v(k)\rangle^{(0)} + \frac{\alpha_s}{4 \pi} C_F \int d \omega' \frac{3 \delta(\omega - \omega')}{\epsilon} v_\mu \langle h_v(k)|Q^{\mu}_1(\omega', \Gamma)|h_v(k)\rangle^{(0)}. \quad (40)$$

The latter term is the zero gluon matrix element of a modified $Q_1^\mu$ operator. The operator is modified to not have a perpendicular covariant derivative, but simply a $D^\mu$ in its definition. This term vanishes at this order due to the equations of motion, these results are combined with our operator anomalous dimensions in section III (It should be noted that the zero gluon calculation does not determine this mixing is with the $Q_1$ operator, as its zero gluon rule is identical to $\bar{Q}_1$. Consistency between zero and one gluon external state calculations determines this mixing to be with the modified $Q_1$ operator.)
There are many more one gluon diagrams than zero gluon diagrams, as illustrated in Figure 3. The diagrams explicitly given constitute half of the total number of diagrams that must be calculated for each of $O_0$ and $P_0$. The crosses represent insertions of the subleading HQET Lagrangian. The mirror diagram corresponding to each of the above diagrams is not shown.

The Dirac structure of the subleading Lagrangian force us to consider the Dirac structure
of these diagrams. Let us consider the one gluon diagrams of Figure 3 where the lightcone operator is \( Q_0 \). We will denote by \( \langle \ldots | A_R | \ldots \rangle \) and \( \langle \ldots | A_L | \ldots \rangle \) the amplitude of these diagrams and the amplitude of the mirror diagrams respectively. Because of the simple relationship between \( O_0 \) and \( P_0^\sigma \), the corresponding amplitudes for \( P_0^\sigma \) are \( \langle \ldots | s^\sigma A_R | \ldots \rangle \) and \( \langle \ldots | A_L s^\sigma | \ldots \rangle \).

Each of \( \langle A_R \rangle \) and \( \langle A_L \rangle \) can be decomposed into the two Dirac structures \( P_+ \) and \( s^\eta \), for example with a heavy quark target, with one gluon in the final external state:

\[
\langle h_v(k) A_\| | A_R | h_v(k) \rangle = \langle h_v(k) A_\| | A_R+P_+ | h_v(k) \rangle + \langle h_v(k) A_\| | A_R^\eta s_\eta | h_v(k) \rangle,
\]

and

\[
\langle h_v(k) A_\| | A_L | h_v(k) \rangle = \langle h_v(k) A_\| | A_L+P_+ | h_v(k) \rangle + \langle h_v(k) A_\| | A_L^\eta s_\eta | h_v(k) \rangle. \quad (41)
\]

Thus for operator \( O_0 \) we can write the total amplitude proportional to each of the Dirac structures after the calculations of the 40 diagrams required. The results for insertions of the \( O_k \) are:

\[
\langle h_v(k) A_\| | T_{(O_0, O_k)} | h_v(k) \rangle^{(1)}_{\text{div}} = \langle h_v(k) A_\| | (A_{R+} + A_{L+}) P_+ | h_v(k) \rangle^{(1)}_{\text{div}},
\]

\[
= - \frac{C_F \omega \alpha_s g_s}{\pi m_b \epsilon} n^\nu T_a \left( \frac{\delta_+ (n \ell)}{n \ell} \right) P_+ + \frac{3 C_F \omega \alpha_s g_s}{4 \pi m_b \epsilon} T_a \left( \nu^\nu \delta_+ (n \ell) - n^\nu (2 k \cdot v + \ell \cdot v) \left( \frac{\delta_+ (n \ell)}{n \ell} \right) \right) P_+. \quad (42)
\]

This result is easily matched, it is identical to the mixing form found in the zero gluon result with \( \Gamma = P_+ \), so that the mixing occurs with the operators \( O_0 \) and \( O_1 \):

\[
\langle h_v(k) A_\| | T_{(O_0, O_k)} | h_v(k) \rangle^{(1)}_{\text{div}} = \frac{\alpha_s}{4 \pi} C_F \int d\omega' \frac{4 \omega' \delta (\omega - \omega')}{\epsilon m_b} \langle h_v(k) A_\| | O_0(\omega') | h_v(k) \rangle^{(0)}_v + \frac{\alpha_s}{4 \pi} C_F \int d\omega' \frac{3 \delta (\omega - \omega')}{\epsilon} v_\mu \langle h_v(k) A_\| | O_1^\mu (\omega') | h_v(k) \rangle^{(0)}_v. \quad (43)
\]

The result of the T product with \( O_m \) is

\[
\langle h_v(k) A_\| | T_{(O_0, O_m)} | h_v(k) \rangle^{(1)}_{\text{div}} = \langle h_v(k) A_\| | [A_{R^\|} + A_{L^\|}] s_\eta | h_v(k) \rangle^{(1)}_{\text{div}},
\]

\[
= - \frac{C_A g_s \alpha T^a}{8 \pi m_b \epsilon} \left( \frac{i \epsilon^\nu \delta_\perp (n \ell) - i \epsilon_\perp n^\nu \delta_+ (n \ell)}{n \ell} \right). \quad (44)
\]

This result matches onto the one gluon rule for \( P_2 \), as expected by the symmetry of the single derivative operators:

\[
\langle h_v(k) A_\| | T_{(O_0, O_m)} | h_v(k) \rangle^{(1)}_{\text{div}} = - \frac{\alpha_s}{4 \pi} C_A \int d\omega' \frac{\delta (\omega - \omega')}{\epsilon} i \epsilon_\perp \langle h_v(k) A_\| | P_2^\mu (\omega') | h_v(k) \rangle^{(0)}_v.
\]

The total \( T_{(P_0, O_k)} \) and \( T_{(P_0, O_m)} \) amplitudes can be written as

\[
\langle h_v(k) A_\| | T_{(P_0, O_k)} | h_v(k) \rangle = \langle h_v(k) A_\| | s^\sigma A_R+ | h_v(k) \rangle + \langle h_v(k) A_\| | s^\sigma A_L+ | h_v(k) \rangle,
\]

\[
\langle h_v(k) A_\| | T_{(P_0, O_m)} | h_v(k) \rangle = \langle h_v(k) A_\| | s^\sigma s_\eta (A_{R^\|})_\eta | h_v(k) \rangle + \langle h_v(k) A_\| | s^\sigma s_\eta (A_{L^\|})_\eta | h_v(k) \rangle. \quad (45)
\]
Using the decomposition

\[ s^\sigma s^\rho = i \epsilon^{\sigma \rho \mu \phi} s_\eta v_\phi - (g^{\sigma \rho} - v^\sigma v^\rho) P_+, \] (46)

we can decompose in terms of the pieces proportional to \( P_+ \) and \( s^\eta \) for these amplitudes. For \( T^\sigma_{(P, O_k)} \) and \( T^\sigma_{(P_0, O_m)} \) we find the following mixing

\[
\langle h_v(k) A^\nu_a | T^\sigma_{(P_0, O_k)} | h_v(k) \rangle^{(1)}_{\text{div}} = \langle h_v(k) A^\nu_a | s^\sigma (A_{R+} + A_{L+}) | h_v(k) \rangle,
\]

\[
= \frac{\alpha_s}{4 \pi} C_F \int d\omega' \frac{4 \omega' \delta(\omega - \omega')}{\epsilon m_b} \langle h_v(k) A^\nu_a | P_0(\omega')^\sigma | h_v(k) \rangle^{(0)} + \frac{\alpha_s}{4 \pi} C_F \int d\omega \frac{3 \delta(\omega - \omega')}{\epsilon} v_\mu \langle h_v(k) A^\nu_a | P_1^\mu(\omega')^\sigma | h_v(k) \rangle^{(0)},
\]

\[
\langle h_v(k) A^\nu_a | T^\sigma_{(P_0, O_m)} | h_v(k) \rangle^{(1)}_{\text{div}} = (v^\sigma v^\mu - g^\sigma \mu) \langle h_v(k) A^\nu_a | ((A_{R_0} + (A_{L_0})_\mu) P_+ | h_v(k) \rangle
\]

\[
+ \langle h_v(k) A^\nu_a | [i \epsilon^{\sigma \eta \mu \phi} v_\mu \delta(\omega - \omega')] (A_{R_0})_\eta (A_{L_0})_\eta^\sigma | h_v(k) \rangle,
\]

\[
= \frac{\alpha_s}{4 \pi} C_A \int d\omega' \frac{\delta(\omega - \omega')}{\epsilon} \langle h_v(k) A^\nu_a | O_{2_\mu}(\omega') | h_v(k) \rangle
\]

\[
- \frac{\alpha_s}{4 \pi} C_A \int d\omega \frac{\delta(\omega - \omega')}{\epsilon} \left[ (v^\sigma - n^\sigma) n^\eta (g^\sigma \eta - v^\sigma v^\eta) \right] \times
\]

\[
(\langle h_v(k) A^\nu_a | P_1^\mu(\omega') | h_v(k) \rangle^{(0)} - \langle h_v(k) A^\nu_a | P_1^\mu(\omega') | h_v(k) \rangle^{(0)}) \times
\]

\[
\langle h_v(k) A^\nu_a | P_1^\mu(\omega') | h_v(k) \rangle^{(0)} - \langle h_v(k) A^\nu_a | P_1^\mu(\omega') | h_v(k) \rangle^{(0)}
\]

\[
\langle h_v(k) A^\nu_a | T^\sigma_{(P_0, O_m)} | h_v(k) \rangle^{(1)}_{\text{div}} = \frac{\alpha_s}{4 \pi} C_A \int d\omega \frac{\delta(\omega - \omega')}{\epsilon} \langle h_v(k) A^\nu_a | O_{2_\mu}(\omega') | h_v(k) \rangle
\]

\[
- \frac{\alpha_s}{4 \pi} C_A \int d\omega \frac{\delta(\omega - \omega')}{\epsilon} \left[ (v^\sigma - n^\sigma) n^\eta (g^\sigma \eta - v^\sigma v^\eta) \right] \times
\]

\[
(\langle h_v(k) A^\nu_a | P_1^\mu(\omega') | h_v(k) \rangle^{(0)} - \langle h_v(k) A^\nu_a | P_1^\mu(\omega') | h_v(k) \rangle^{(0)}).
\]

III. RESULTS

A. Leading Nonperturbative Order

The order \( \alpha_s \) perturbative and leading order non-perturbative anomalous dimension matrix has been calculated by a variety of authors \[14, 19\]. Our results agree with theirs, and in the basis

\[
\{O_0, P_0\}
\]

the perturbative expansion is given by

\[
Z^{(0)}(\omega, \omega') = \begin{bmatrix} \delta(\omega - \omega') & 0 \\ 0 & \delta(\omega - \omega') \end{bmatrix},
\]

\[
Z^{(1)}(\omega, \omega', \bar{\mu}) = \frac{\alpha_s(\bar{\mu})}{4\pi} \begin{bmatrix} d_1(\omega, \omega', \bar{\mu}) & 0 \\ 0 & d_1(\omega, \omega', \bar{\mu}) \end{bmatrix},
\]

\[
\text{with}
\]

\[
15
\]
The distribution \( d_1(\omega, \omega', \bar{\mu}) \) is the combination of the operator and wavefunction renormalization counter terms, given by

\[
d_1(\omega, \omega', \bar{\mu}) = -\frac{2 C_F}{\epsilon^2} \delta(\omega - \omega') + \frac{2 C_F}{\epsilon} \delta(\omega - \omega') + \frac{4 C_F}{\bar{\mu} \epsilon} \phi'_0 \left( \frac{\omega - \omega'}{\bar{\mu}} \right).
\] (50)

Recall, our initial expression related the bare and renormalized operators,

\[
Q_0(\omega, \Gamma)_{\text{bare}} = \int d\omega' Z(\omega', \omega, \bar{\mu}) Q_0(\omega', \bar{\mu}, \Gamma)_{\text{ren}}.
\] (51)

We differentiate this equation with respect to \( \bar{\mu} \) to obtain our renormalization group equation

\[
\bar{\mu} \frac{d}{d\bar{\mu}} Q_0(\omega, \bar{\mu}, \Gamma)_{\text{ren}} = -\int d\omega' \gamma(\omega', \omega, \bar{\mu}) Q_0(\omega', \bar{\mu}, \Gamma)_{\text{ren}}.
\] (52)

The anomalous dimension matrix is determined using the useful result for MS

\[
\gamma(g_s) = -2 \alpha_s \frac{dZ_1(\alpha_s)}{d\alpha_s},
\] (53)

where \( Z_1 \) is the coefficient of the \( 1/\epsilon \) poles. We find

\[
\gamma(\omega, \omega', \bar{\mu}) = \begin{bmatrix}
\gamma_1(\omega, \omega', \bar{\mu}) & 0 \\
0 & \gamma_1(\omega, \omega', \bar{\mu})
\end{bmatrix},
\] (54)

with,

\[
\gamma_1(\omega, \omega', \bar{\mu}) = -\frac{\alpha_s(\bar{\mu})}{\pi} C_F \left[ \delta(\omega - \omega') + \frac{2}{\bar{\mu}} \phi'_0 \left( \frac{\omega - \omega'}{\bar{\mu}} \right) \right].
\] (55)

### B. Subleading Nonperturbative Order

We have determined the matrix of renormalization constants at subleading nonperturbative order \( Z_{SL} \), excluding operators of class \( Q_3 \). If we order our \( Q_i \) operators as

\[
O_i = \{ O_0, T_{(O_0, O_k)}, T_{(O_0, O_m)}, O^\mu_i, \bar{O}^\mu_i, O^\mu_2, O_4 \},
\]

\[
P^\sigma_i = \{ P^\sigma_0, T^\sigma_{(P_0, O_k)}, T^\sigma_{(P_0, O_m)}, P^\sigma_{\mu_1}, P^\sigma_{\mu_2}, P^\sigma_{\mu_3}, P^\sigma_4 \},
\] (56)

the leading order term in the perturbative expansion in the basis \( \{ O_i, P_i \} \) is given in block form as

\[
Z_{SL}^{(0)}(\omega, \omega') = \begin{pmatrix}
\Gamma^0_{O_i, O_j}(\omega, \omega') & \Gamma^0_{O_i, P_j}(\omega, \omega') \\
\Gamma^0_{P_i, O_j}(\omega, \omega') & \Gamma^0_{P_i, P_j}(\omega, \omega')
\end{pmatrix}.
\] (57)
Where the entries in the matrices in the above expression with \( i, j = 0, 1, \ldots, 6 \) are given by

\[
\Gamma^0_{O_i, O_j}(\omega, \omega') = (\delta_{i,j} - \delta_{i,0}\delta_{j,0})\delta(\omega - \omega'),
\]

\[
\Gamma^0_{P_i, P_j}(\omega, \omega') = (\delta_{i,j} - \delta_{i,0}\delta_{j,0})\delta(\omega - \omega'),
\]

\[
\Gamma^0_{O_i, P_j}(\omega, \omega') = 0, \\
\Gamma^0_{P_i, O_j}(\omega, \omega') = 0.
\] (58)

While the \( \mathcal{O}(\alpha_s) \) term in the expansion is

\[
Z_{SL}^{(1)}(\omega, \omega', \bar{\mu}) = \frac{\alpha_s(\bar{\mu})}{4\pi} \begin{pmatrix}
\Gamma^1_{O_i, O_j}(\omega, \omega') & \Gamma^1_{O_i, P_j}(\omega, \omega') \\
\Gamma^1_{P_i, O_j}(\omega, \omega') & \Gamma^1_{P_i, P_j}(\omega, \omega')
\end{pmatrix}.
\] (59)

with the diagonal block matrices

\[
\Gamma^1_{O_i, O_j}(\omega, \omega') =
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
d_2(\omega, \omega', \bar{\mu}) & 0 & 0 & d_3^{\mu}(\omega, \omega') & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & d_4(\omega, \omega', \bar{\mu}) & d_5(\omega, \omega') & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & d_1(\omega, \omega', \bar{\mu}) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & d_1(\omega, \omega', \bar{\mu}) \\
\end{bmatrix},
\]

\[
\Gamma^1_{P_i, P_j}(\omega, \omega') =
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
d_2(\omega, \omega', \bar{\mu}) & 0 & 0 & d_3^{\mu}(\omega, \omega') & 0 & 0 & 0 \\
0 & 0 & 0 & -d_7^{\mu\nu}(\omega, \omega') & d_7^{\mu\nu}(\omega, \omega') & 0 & 0 \\
0 & 0 & 0 & d_4(\omega, \omega', \bar{\mu}) & d_5(\omega, \omega') & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & d_1(\omega, \omega', \bar{\mu}) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & d_1(\omega, \omega', \bar{\mu}) \\
\end{bmatrix}.
\]
The off diagonal block matrices are as follows

\[
\Gamma^1_{O_i, P_j}(\omega, \omega') = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -d_6^{\mu\sigma}(\omega, \omega') & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
\Gamma^1_{P_i, O_j}(\omega, \omega') = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & d_6^{\mu\sigma}(\omega, \omega') & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

The \(d_i(\omega, \omega', \tilde{\mu})\) distributions are given by

\[
\begin{align*}
d_2(\omega, \omega', \tilde{\mu}) &= \frac{4 C_F}{\epsilon} \frac{\omega'}{m_b(\tilde{\mu})} \delta(\omega - \omega'), \\
d_3^{\mu}(\omega, \omega') &= \frac{3 C_F}{\epsilon} \delta(\omega - \omega') v^{\mu}, \\
d_4(\omega, \omega', \tilde{\mu}) &= d_1(\omega, \omega', \tilde{\mu}) - \frac{C_A}{\epsilon} \delta(\omega - \omega'), \\
d_5(\omega, \omega') &= \frac{C_A}{\epsilon} \delta(\omega - \omega'), \\
d_6^{\mu\sigma}(\omega, \omega') &= -\frac{C_A}{2 \epsilon} (i \epsilon^\mu_{\perp\sigma}) \delta(\omega - \omega'), \\
d_7^{\mu\eta}(\omega, \omega') &= \frac{C_A}{2 \epsilon} (v^\sigma - n^\sigma) g^{\mu\eta} + n^\eta (g^{\sigma\mu} - v^\sigma v^\mu) \delta(\omega - \omega').
\end{align*}
\]

We directly determine the anomalous dimension matrix to subleading order using Eq. (53) to be the following

\[
\gamma_{SL}(\omega, \omega', \tilde{\mu}) = \begin{pmatrix}
\gamma_{O_i, O_j}(\omega, \omega', \tilde{\mu}) & \gamma_{O_i, P_j}(\omega, \omega', \tilde{\mu}) \\
\gamma_{P_i, O_j}(\omega, \omega', \tilde{\mu}) & \gamma_{P_i, P_j}(\omega, \omega', \tilde{\mu})
\end{pmatrix}.
\]
The diagonal entries of the anomalous dimension matrix are

$$
\gamma_{O_iO_j}(\omega, \omega', \bar{\mu}) = \left[ \begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\gamma_2(\omega, \omega', \bar{\mu}) & 0 & 0 & \gamma_3^\prime(\omega, \omega', \bar{\mu}) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \gamma_4(\omega, \omega', \bar{\mu}) & \gamma_5(\omega, \omega', \bar{\mu}) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \gamma_1(\omega, \omega', \bar{\mu}) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \gamma_1(\omega, \omega', \bar{\mu}) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \gamma_1(\omega, \omega', \bar{\mu}) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma_1(\omega, \omega', \bar{\mu}) \\
\end{array} \right],
$$

$$
\gamma_{P_iP_j}(\omega, \omega', \bar{\mu}) = \left[ \begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\gamma_2(\omega, \omega', \bar{\mu}) & 0 & 0 & \gamma_3^\prime(\omega, \omega', \bar{\mu}) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\gamma_4^\mu(\omega, \omega', \bar{\mu}) & \gamma_5^\mu(\omega, \omega', \bar{\mu}) & 0 & 0 & 0 \\
0 & 0 & 0 & \gamma_4(\omega, \omega', \bar{\mu}) & \gamma_5(\omega, \omega', \bar{\mu}) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \gamma_1(\omega, \omega', \bar{\mu}) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \gamma_1(\omega, \omega', \bar{\mu}) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \gamma_1(\omega, \omega', \bar{\mu}) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma_1(\omega, \omega', \bar{\mu}) \\
\end{array} \right].
$$

The off diagonal entries are as follows

$$
\gamma_{O_iP_j}(\omega, \omega', \bar{\mu}) = \left[ \begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\gamma_6^\mu(\omega, \omega') & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array} \right],
$$

$$
\gamma_{P_iO_j}(\omega, \omega', \bar{\mu}) = \left[ \begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \gamma_6^\mu(\omega, \omega') & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array} \right].
$$
With the following entries in the anomalous dimension matrix,

\[
\gamma_1(\omega, \omega', \bar{\mu}) = -\frac{\alpha_s(\bar{\mu})}{\pi} C_F \left[ \delta(\omega - \omega') + \frac{2}{\bar{\mu}} \phi_0 \left( \frac{\omega - \omega'}{\bar{\mu}} \right) \right],
\]

\[
\gamma_2(\omega, \omega', \bar{\mu}) = -2 \alpha_s(\bar{\mu}) \frac{\omega' C_F}{\pi m_b(\bar{\mu})} \delta(\omega - \omega'),
\]

\[
\gamma_3^H(\omega, \omega', \bar{\mu}) = -3 \alpha_s(\bar{\mu}) \frac{C_F}{2 \pi} \delta(\omega - \omega') \nu^\mu,
\]

\[
\gamma_4(\omega, \omega', \bar{\mu}) = \gamma_1(\omega, \omega', \bar{\mu}) + \frac{\alpha_s(\bar{\mu})}{\pi} C_A \delta(\omega - \omega'),
\]

\[
\gamma_5(\omega, \omega', \bar{\mu}) = -\frac{\alpha_s(\bar{\mu})}{\pi} C_A \delta(\omega - \omega'),
\]

\[
\gamma_6^{\mu \sigma}(\omega, \omega', \bar{\mu}) = \frac{\alpha_s(\bar{\mu})}{4 \pi} C_A (i \epsilon_{\mu \nu}^{\lambda}) \delta(\omega - \omega'),
\]

\[
\gamma_7^{\mu \sigma \eta}(\omega, \omega', \bar{\mu}) = -\frac{\alpha_s(\bar{\mu})}{4 \pi} C_A \left( (\nu^\sigma - n^\sigma) g^{\mu \eta} + n^\eta (g^{\mu \nu} - n^\eta n^\nu) \right) \delta(\omega - \omega').
\]  

(62)

IV. CONCLUSIONS

We have examined the anomalous dimension matrix appropriate for the phase space restricted \( \bar{B} \to X_u \ell \bar{\nu} \) and \( \bar{B} \to X_s \gamma \) decay spectra to subleading nonperturbative order. The effects of the time ordered products of the HQET Lagrangian with the leading order shape function operator were determined and the renormalizability and closure of a subset of the non-local operator basis used to describe these spectra, to subleading order, was established.

Operator mixing was found between the operators which occur to subleading order, requiring that the subleading operator basis be extended to include the operator \( \bar{Q}_1 \). This requires the introduction of new shape functions to characterize the decay spectra of \( B \to X_u \ell \bar{\nu} \) and \( B \to X_s \gamma \) beyond tree level. The mixing determined between the operators \( Q_1 \) and \( Q_1 \) is of the pernicious form that required a one gluon external state calculation to determine, despite the non vanishing zero gluon Feynman rules of the operators. We have also demonstrated that the possible mixing with the operator \( Q^\mu \nu(\omega_1, \omega_2, \Gamma) \) in a similar manner; with vanishing Feynman rules for zero and one gluon, requires a two gluon external state calculation to completely determine the anomalous dimension at subleading non perturbative order.

Mixing was also determined between the T product \( T_{(O_0, O_k)} \) and the leading order shape function, and the T product \( T_{(O_0, O_m)} \) was shown to lead to mixing between the \( P_i \) and \( O_i \).
operators at this order.

The anomalous dimension and running of the $Q_1^\mu$, $Q_2^\mu$ and $Q_4$ operators was shown to be identical to the leading order shape function $Q_0$.

This work can be built upon in a number of ways. The anomalous dimension of the operator $Q_3$ is under investigation by the authors to establish the closure at one loop of the set of subleading operators discussed in this paper. The anomalous dimension of the subleading four quark operators should be investigated to determine the full anomalous dimension matrix at subleading order. Once the full anomalous dimension is determined, Sudakov logarithms in the perturbative corrections to the subleading operators can be resummed, so that renormalization group improved calculations can be undertaken for the $\bar{B} \to X_u \ell \bar{\nu}$ and $\bar{B} \to X_s \gamma$ decay spectra to subleading nonperturbative order.

More important than the small effect that these corrections have directly on the extraction of $|V_{ub}|$, is the fact that this work establishes the renormalizability of a subset of the soft sector nonperturbative expansion beyond leading order. This is a necessary step in extending QCD factorization theorems beyond leading nonperturbative order, validating the factorization based approach used for the phase space restricted $\bar{B} \to X_u \ell \bar{\nu} \ell$ and $\bar{B} \to X_s \gamma$ decay spectra beyond leading nonperturbative order.

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[1] M. Christian Arnesen, Ben Grinstein, Ira Z. Rothstein and Iain W. Stewart, Phys. Rev. Lett. 95: 071802, (2005).
[2] J. Chay, H. Georgi, and B. Grinstein, Phys. Lett. B247 (1990) 399; M. Voloshin and M. Shifman, Sov. J. Nucl. Phys. 41 (1985) 120; I.I. Bigi et al., Phys. Lett. B293 (1992) 430; Phys. Lett
B297 (1993) 477 (E); I.I. Bigi et al., Phys. Rev. Lett. 71 (1993) 496; A.V. Manohar and M.B. Wise, Phys. Rev. D49 (1994) 1310; B. Blok et al., Phys. Rev. D49 (1994) 3356; A. F. Falk, M. Luke and M. J. Savage, Phys. Rev. D 49, 3367 (1994).

[3] C.W. Bauer, Z. Ligeti, M. Luke, A.V. Manohar and M. Trott, Phys. Rev. D 70, 094017 (2004).
[4] Michael Luke, ECONF C0304052: WG107,2003.
[5] G. P. Korchemsky and G. Sterman, Phys. Rev. Lett. B340, 96, (1994).
[6] Keith. S.M. Lee, and Iain W. Stewart, Nucl. Phys. B 721 325-406 (2005) [hep-ph/0409045]
[7] M. Neubert, Phys. Rev. D49(1994)3392; D49(1994)4623; I.I. Bigi et al., Int. J. Mod. Phys. A9 (1994) 2467.
[8] T. Mannel and M. Neubert, Phys. Rev. D50 (1994) 2037.
[9] C. W. Bauer, M. E. Luke and T. Mannel, Phys. Rev. D 68 07015 (2003) arXiv:hep-ph/0102089
[10] C. W. Bauer, M. Luke and T. Mannel, Phys. Lett. B543 261 (2002).
[11] M. Neubert, Phys. Lett. B 543 269 (2002).
[12] A. K. Leibovich, Z. Ligeti and M. B. Wise, Phys. Lett. B539 242 (2002).
[13] S. W. Bosch, M. Neubert and G. Paz, JHEP 11 073 (2004) [hep-ph/0409115]
[14] C. N. Burrell, M. Luke and A. R. Williamson Phys. Rev. D69 07015 (2004).
[15] F. de. Fazio and M. Neubert, JHEP 0906, 017 (1999).
[16] T. Mannel and S. Recksiegel, Phys. Rev. D60 114040 (1999).
[17] S.W. Bosch, B. O. Lange, M. Neubert and G. Paz, Nucl. Phys. B. 699,335-386 (2004).
[18] S.W. Bosch, B. O. Lange, M. Neubert and G. Paz, Phys. Rev. Lett. 93,221801 (2004).
[19] C. W. Bauer and A. V. Manohar, Phys. Rev. D. 70,034024 (2004) arXiv:hep-ph/0312109
[20] B. O. Lange and M. Neubert Phys. Rev. Lett. 91,102001 (2003) arXiv:hep-ph/0303082
[21] R. Akhoury and I. Z. Rothstein, Phys. Rev. D 54, 2349 (1996); A. K. Leibovich and I. Z. Rothstein, Phys. Rev. D 61, 074006 (2000); A. K. Leibovich, I. Low and I. Z. Rothstein, Phys. Rev. D 61, 053006 (2000); A. K. Leibovich, I. Low and I. Z. Rothstein, Phys. Rev. D 62, 014010 (2000); A. K. Leibovich, I. Low and I. Z. Rothstein, Phys. Lett. B 486, 86 (2000).
[22] M. Beneke, F. Campanario, T. Mannel, and B. D. Pecjak, JHEP 0506:07 (2005) hep-ph/0411395
[23] C. W. Bauer, S. Fleming and M. E. Luke, Phys. Rev. D 63, 014006 (2001).
[24] C. W. Bauer, S. Fleming, D. Pirjol and I. W. Stewart, Phys. Rev. D 63, 114020 (2001).
[25] C. W. Bauer, D. Pirjol and I. W. Stewart, Phys. Rev. D 65, 054022 (2002).
[26] A. H. Hoang, Z. Ligeti and Michael Luke hep-ph/0502134

[27] M. Neubert hep-ph/0411027

[28] Thomas Mannel and Frank. J. Tackman Phys. Rev. D 71, 034017 (2005).

[29] Frank. J. Tackman, Phys. Rev. D 72: 034036 (2005) hep-ph/0503095

[30] George Leibbrandt, Rev. Mod. Phys. 59 1067 (1987).

[31] L.F. Abbott Nucl. Phys. B 185 189 (1981).

[32] E. G. Floratos, D. A. Ross and C. T. Sachrajda Nucl. Phys. B 129 66 (1977).

[33] G. P. Korchemsky and A. V. Radyushkin Nucl. Phys. B 283, 342 (1987); I. A. Korchemskaya and G. P. Korchemsky Phys. Lett. B 287, 169 (1992).