On structure of solutions of 1-dimensional 2-body problem in Wheeler-Feynman electrodynamics

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Summary. — The problem of 1-dimensional ultra-relativistic scattering of 2 identical charged particles in classical electrodynamics with retarded and advanced interactions is investigated.

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Introduction

One of the unsolved problems in theoretical physics is Hamiltonian and quantum description of discrete relativistic systems with retarded and advanced interactions. Such systems result from exclusion of field freedom degrees in field theories: while expressing fields by sources in classical field theories, or performing integration by fields in generating functionals of quantum field theories (carried out at definite boundary conditions) finite-dimensional theories arise, describing the motion of sources.

Field exclusion in classical electrodynamics leads to Wheeler-Feynman system [1] — a relativistic theory with finite number of freedom degrees, whose action is Poincare-invariant functional of world lines of charges. This theory is equivalent to classical electrodynamics under the following boundary condition: field emitted by a charge is completely absorbed by other charges, so that outside some sphere (of radius greater than the Universe one) the resulting field of charges vanishes. The given formulation of electrodynamics has the following properties:

1. In Wheeler-Feynman electrodynamics a field created by a point charge is a half-sum of retarded and advanced potentials (dislike the standard approach, which uses only the retarded potentials). Consequently, the time reversion symmetry in this theory evinces not only in the action, but also in solutions of equations of motion.

2. T-nonsymmetric effects like radiative damping arise in multibody problems and have thermodynamic nature [1].

3. As the classical theory has finite number of freedom degrees, one should expect that the corresponding quantum theory has no divergencies.
Canonical quantization of Wheeler-Feynman electrodynamics has not been performed, because Hamiltonian formulation of this theory was unknown. Moreover, Lagrange equations of motion in Wheeler-Feynman electrodynamics are not ordinary differential equations, but belong to a poorly investigated class of functional equations (differential equations with deviating arguments [2]), that has no elaborate approaches to solution: neither general analytical methods, nor steady numeric techniques. Existence and uniqueness of solutions for this class of equations have not been studied thoroughly. That’s why a general structure of solutions of classical electrodynamics in Wheeler-Feynman formulation is still unknown.

Particular results obtained in this domain are: solution uniqueness theorem for large distances between charges [3]; exact solution of the problem for circular motion of charges in the theory with retarded and advanced interactions [4], and the theory with retarded interaction and radiative reaction [5]; exactly solvable modification of the problem [6], with one charge influenced by retarded interaction, and the other — by advanced one; numerical solution of 1-dimensional 2-body problem [7], extended to velocities \( v < 0.9545c \).

Here we will continue the solution of the last problem to greater velocities. Besides that we will find new solutions in the range of \( v < 0.9545c \) (along with those in [7]). Hamiltonian formulation of the theory constructed in [8] is used for the problem solution.

The paper has the following structure: the first section describes the methods under the use, the second one presents the obtained results. In three appendices boundary effects and limiting regimes are examined, and also a list of additional questions interesting for further investigation is given.

1. – Methods

0. We will consider one-dimensional scattering of two relativistic particles of equal charges and masses. The methods used allow to study scattering of particles of different masses, however, the case of identical particles is of the most interest due to additional symmetries the problem acquires.

We use the following system of units: light velocity \( c = 1 \), classical radius of a particle \( e^2/mc^2 = 1 \) (all distances are measured in classical radii).

1. A method described in [8] is used for the problem solution. The clue idea of the approach is a choice of special parametrization of particles world lines, in which the problem gets the following formulation (fig.1). Let’s consider a system of particles \( x_n, y_n \), moving in 2-dimensional space-time so, that they are always located in vertices of a polygonal line assembled of light rays. It’s enough to set a motion of particles \( x \), and define \( y \) trajectories as \( y_n^+ (\tau) = x_n^+ (\tau) \), \( y_n^- (\tau) = x_n^{−1} (\tau) \) (in light coordinates \( x^\pm = x^0 \pm x^1 \)). The motion is defined by a system of differential equations for \( x_n^\pm (\tau) \) (simultaneous by \( \tau \)), which is equivalent to the original one with deviating arguments. Then additional conditions are imposed onto the particles trajectories, ensuring their sewing together to the whole paths of \( x \) and \( y \) particles. Therefore, this approach reduces the problem to a set of ordinary differential equations, that defines particles evolution, and a set of equations for initial conditions and evolution time of the form \( F(X) = 0 \), that provides trajectories sewing together.
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It’s possible to restrict calculations to a finite number of trajectories $x_n, n = 1...N$. This possibility follows from exponential increase of distance between particles with large $n$: $|x_n - y_n| \sim g^{2n}$, $q = (1 + v)/(1 - v)$, $v$ — the velocity of the particles in center-of-mass frame (CMF); and starting from some $N$, their interaction becomes negligible. As $q \to \infty$ while $v \to 1$, for large velocities few ($N = 2, 3$) steps of the light stairway are sufficient to move the particles out of the interaction region. Thus, this method is especially convenient in ultra-relativistic case (for small velocities more equations should be considered).

The motion is defined by a system of Hamiltonian equations for coordinates and momenta:

$$
\dot{x}_n^\pm = \frac{\partial H}{\partial p_n^\pm}, \quad \dot{p}_n^\pm = -\frac{\partial H}{\partial x_n^\pm}
$$

with Hamiltonian:

$$
H(x, p) = (1 0) g_N...g_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad g_n = \begin{pmatrix} r_n^+ r_n^- - 1 & r_n^+ \\ -r_n^- - 1 & -1 \end{pmatrix},
$$

$$
r_n^\pm = 2 \left( \frac{p_n^\pm}{m} + \frac{1}{x_{n+1}^\pm - x_n^\pm} \right), \quad n = 1...N - 1, \quad r_N^\pm = 2p_N^\pm / m.
$$

The Hamiltonian is a polynomial function of $r_n^\pm$ (linear for each $r_n^\pm$). It’s convenient to rewrite the equations completely in terms of $(x, r)$, see [8].

The Hamiltonian is a Dirac’s constraint [9], that briefly speaking means the following. As the Hamiltonian is conserved in evolution:

$$
\dot{H} = \dot{x}_n^\pm \frac{\partial H}{\partial x_n^\pm} + \dot{p}_n^\pm \frac{\partial H}{\partial p_n^\pm} = 0,
$$

phase trajectories lie on the surface $H = \text{Const}$. In Wheeler-Feynman electrodynamics only those trajectories, that lie on the surface $H = 0$, correspond to a physically meaningful evolution. This constraint, causing the fact that not all momenta are independent,
arises due to parametrical invariance of the action and is typical for all relativistic theories. E.g. in a theory of free relativistic particle analogous constraint has a form of a “mass shell condition” \( p^2 - m^2 = 0 \), identically satisfied on all trajectories, because

\[
p_\mu = m x_\mu / \sqrt{2} \text{ by definition.}
\]

The sewing conditions have the form

\[
\begin{align}
(3a) & \quad x_n^\mu (T) = x_{n+1}^\mu (0) ~ n = 1...N - 1, \\
(3b) & \quad p_n^\mu (T) = p_{n+1}^\mu (0) ~ n = 2...N - 2, \quad \mu = +, -; \\
& \quad p_1^+ (T) = p_2^+ (0) - m/2 \cdot (x_2^+ (0) - x_1^+ (0))^{-1}, \\
& \quad p_{N-1}^- (T) + m/2 \cdot (x_N^-(T) - x_{N-1}^- (T))^{-1} = p_N^- (0),
\end{align}
\]

for \( n = 2...N - 2 \) finite values of \((x_n^\mu, p_n^\mu)\) coincide with initial values of these quantities for the next trajectory\(^1\). The quantities \((x_n^\mu, p_n^\mu)_T\), determined by solving differential equations (1), are functions of initial data \((x_n^\mu, p_n^\mu)_0\) and “time” \( T \). So (3) is a system of nonlinear equations on initial data and \( T \).

It has been shown in [8] that conditions (3) alone do not provide smoothness of trajectories sewing. One more condition comes from the action minimum principle:

\[
\begin{align}
(4) & \quad u_n^\mu (T) = u_{n+1}^\mu (0), \quad u_1^\mu = \left( \frac{dy_1^-}{dy_n^-} \right)^{1/2} = \Psi_1^\mu / \Psi_2^\mu, \quad \Psi_n = g_n...g_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix},
\end{align}
\]

which is equivalent to smoothness of the \( y_n \) and \( y_{n+1} \) trajectories sewing together. This condition should be imposed for some \( n = 1...N - 2 \), and together with (3) it will provide smoothness of sewing together all \( x_n, y_n \) trajectories (except of the boundary ones: \( x_{1,2}, x_{N-1,N} \), see Appendix A).

Thus, \((4N - 4)\) equations (3),(4) and \( H = 0 \) are imposed on \((4N + 1)\) unknowns \((x_n^\pm (0), p_n^\pm (0), T)\). Among the remaining 5 freedom degrees, 4 correspond to trivial transformations of solutions:

- 2 translations of solutions:
  \( x^\pm \rightarrow x^\pm + c^\pm; \)
- Lorentz transformations:
  \( x^+ \rightarrow x^+ c, \ x^- \rightarrow x^- / c; \quad p^+ \rightarrow p^+ / c, \ p^- \rightarrow p^- c; \)
- reparametrization (shift along the trajectories):
  \( (x, p)^{\pm}_0 \rightarrow (x, p)^{\pm}_{T+c}; \)

and one freedom degree corresponds to trajectories deformation in variation of relative velocity of the particles.

2. The trivial freedom degrees can be eliminated, e.g. setting \( x_1^+ (0) = 0 \), \( r_1^- (0) = Const \) — a value of this constant can be taken from Coulomb approximation (see below) and fixed in all further considerations. Applying translations and Lorentz transformation to a solution found, one can move it to another reference frame, e.g. CMF.

\(^1\) For \( n = 1, N - 1 \) two \( p_n^- \)-conditions disappear, see [8]. Also, in the action transformation (2)\(\rightarrow\)(4) in [8], carried out for finite trajectories, off-integral terms appear, resulting to asymptotically vanishing corrections to the remaining two conditions.
In order to fix reparametrization freedom degree, let’s add one more equation: \((x_1^+ - x_N^-)(0) = (x_N^+ - x_1^-)(T)\) in CMF, geometrical interpretation of which is clear from fig.2. Amongst all of parametrization fixing constraints this one is especially convenient, because it prevents a solution from “slipping off” along a trajectory and ensures that boundary points are distant from interaction region, both in the past, and in the future.

One more variable should be fixed in the problem, e.g. \(r_1^+(0)\), it will control particles velocities. The expression for \(v\) (initial velocity of particles \(x_1, y_1\) in CMF) has the form: \(v = 2(r_1^+(0)r_1^-(0))^{-1} - 1\), see [8]. It’s convenient to introduce a parameter \(\mu = (1 - v)^{-1}\) with range \([1, +\infty)\), and express \(r_1^+(0)\) by it.

Equation \(H = 0\) is linear for each \(r_n^\pm\) and can be solved explicitly (e.g. for \(r_2^+(0)\)). Thus, with the changes made, we have the system of \((4N - 4)\) equations \(F(\mu, X) = 0\) for \((4N - 4)\) unknowns \(X\), depending on one parameter \(\mu\).

Newton’s method was applied to resolve this system. Coulomb (non-relativistic) solution of the problem at \(v = 0\) was used as a starting point, it has the form (in the designations introduced):

\[
\begin{align*}
x_1^+(\tau) &= \frac{1}{2v^2} \cosh 2v\tau, \quad x_1^0(\tau) = \frac{1}{2v^3} (2v\tau + \cosh 2v\tau \sinh 2v\tau), \\
x_n^0 &= x_n^0(\tau_n), \quad \tau_n = n - 1 - N/2 \quad n = 1...N, \\
u_n^x &= \left(\frac{x_n}{x_n^+}\right)^{1/2} \quad n = 1...N, \quad u_n^y = \left(\frac{x_n + 1}{x_n^+}\right)^{1/2} \quad n = 1...N - 1, \\
r_n^+ &= -u_n^x - u_n^y \quad n = 1...N - 1, \quad r_n^- = -\frac{1}{u_n^x} - \frac{1}{u_n^y} \quad n = 2...N, \\
r_N^+ &= -u_N^x, \quad r_1^- = -\frac{1}{u_1^x}.
\end{align*}
\]

Then after a few iterations precise solution was found\(^{(2)}\). Then the \(\mu\) parameter was

\(^{(2)}\) The condition \(|F| < 10^{-5}\) has been used as a finish criterion; while far from solutions \(|F| \sim 10^2 ... 10^3\).
increased by $\Delta \mu$, and the solution found was used as a starting point for new value of $\mu$. The step $\Delta \mu$ was chosen adaptively: it was

- decreased by 2 times, if convergency has been lost;
- increased by 2 times, if solution has been found;

so $\Delta \mu$ has been automatically kept in an optimal region (3).

3. With the help of these methods the solution was continued up to velocity $v = 0.937$. At this value of velocity Jacoby matrix of considering system degenerates (see fig. 3), and Newton’s method becomes inapplicable. In fact, the solution extrapolation enables to “jump over” this point and continue the solution (0) behind it. However, vanishing of the Jacobian indicates a change of topology for the set of solutions, therefore the vicinity of this point should be studied by other methods.

Fig. 3. Jacobian dependence on velocity.

Let’s extract in the Jacoby matrix $J_{ij} = \partial F_i / \partial X_j$ of size $K \times K$, $K = 4N - 4$, a submatrix $\tilde{J}$ of size $(K - 1) \times (K - 1)$, with determinant not-vanishing in the critical point. Place this non-degenerate block to bottom right matrix corner by renumerating the variables and equations. Now, fixing $(\mu, X_1)$, it’s possible to resolve $(K - 1)$ equations $F_i(\mu, X_1, X_j) = 0$, $i, j = 2...K$, for $(K - 1)$ unknowns $X_j$, using Newton’s method. Non-degeneracy of $\tilde{J}$ guarantees that the solutions found are isolated: for fixed $(\mu, X_1)$ in the vicinity of the solution $X_j$ there are no other ones. Further, we get $X_j(\mu, X_1)$ dependencies, carrying out step-by-step change of parameters $(\mu, X_1)$, similar to the described above. Substituting these dependencies to the remaining equation, we get a function of two variables $f(\mu, X_1) = F_1(\mu, X_1, X_j(\mu, X_1))$, a behavior of which is shown on the fig. 4 (in terms of variables $v$, $\Delta X_1 = X_1 - X_1^0$, where $X_1^0$ — solution (0)).

At $v < 0.937$ the equation $f(\Delta X_1) = 0$ has a single solution. In the critical point a tangent to the graph at zero is horizontal (that’s equivalent to $\det J = 0$). At $v > 0.937$ the tangent slope reverses its sign and two additional solutions appear. Thus, in the critical point bifurcation of solutions takes place: one solution splits into three.

(3) Other optimization methods are described in [10].
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Fig. 4. Dependence \( f(v, \Delta X_1) \). In the vicinity of the critical point this function defines a surface, known in catastrophe theory as Cayley surface [11].

Fig. 5. Trajectories shape.

Positions of the additional solutions are determined using dichotomy. Taking \( v \) far enough from the critical point one can continue the problem solution using Newton’s method.
2. – The results

1. Shapes of trajectories corresponding to found solutions are shown in fig.5. While passing critical point no irregularities of trajectories and graphs of velocities and accelerations are observed.

For the main branch (0) trajectories are symmetric with respect to spatial and temporal reflections \( P : x^1 \rightarrow -x^1 \) and \( T : x^0 \rightarrow -x^0 \).

For the additional branches (±) trajectories are not \( P \)- and \( T \)-symmetric, but transform to each other in these reflections. At the same time, these trajectories are \( PT \)-symmetric.

This effect: violation of solutions symmetry for symmetrically stated problem is related with non-uniqueness of solutions. Indeed, a symmetry of equations of motion (and initial data) under reflections means only that the reflected solution is also a solution (maybe another one), in other words, a set of solutions under reflections transits to itself. Only if a solution at the given initial data is unique (e.g. as for ordinary differential equations), one can conclude that the solution coincides with the reflected one, i.e. is symmetric. Thus, violation of \( P, T \)-symmetries for solutions is specific to differential equations with deviating arguments, which can have multiple solutions for the same initial data.

2. The dependencies of parameters \( d_m, t_m \) on \( v \) are shown in fig.6 (the definitions of parameters see in fig.5: \( d_m = \min x^1 - \max y^1 \) in CMF, \( t_m \) — temporal distance between the points, where extrema are reached).

The dependence \( d_m(v) \) for the main branch of solution has minimum at velocity \( v_m = 0.956 \), corresponding to \( d_m = 0.9075 \). This dependence (up to velocity value close to \( v_m \)) has been found in Andersen and von Baeyer work [7]. Overlaying the graph [7] to the one obtained here, we see their exact coincidence.

\textit{Note.} The following algorithm has been used in [7] for the problem solution. For a given path of particle \( x \) the forces acting on it from a particle \( y \) were calculated, assuming path of the \( y \) is a \textit{mirror reflected image} of path of the \( x \). Then the accelerations were integrated, corrections to \( x \) trajectories were found, and the described process was iterated. First of all, let’s note that this algorithm considerably uses assumption about \( P \)-symmetry of the trajectories, therefore it is only capable to obtain symmetrical solutions corresponding to the main branch (0). Then in order to find a solution the described process was performed at a fixed \( d_m \) value, which was diminished while moving to the range of greater velocities. The iterations converged till \( d_m = 0.9077 \) (that corresponds to \( v = 0.9545 \)) and diverged for less \( d_m \) values. This divergency was caused by proximity of a minimum of \( d_m(v) \) dependence — there are no solutions for \( d_m < 0.9075 \).

For \( v \rightarrow 1 \) solution in the main branch tends to Hill’s solution [12], for which the minimal distance between particles in CMF is \( d_m = 1 \). The trajectories are polygonal lines with vertices in points \((±0.5, ±0.5)\), see fig.5. There are \( \delta \)-like peaks on the accelerations graphs corresponding to these vertices (see top right graph in fig.7), one of them corresponds to retarded, and another one — to advanced interactions.

For (±) solutions \( d_m, t_m \rightarrow 0 \) for \( v \rightarrow 1 \). The trajectories in CMF tend to light rays emitted from the frame origin.
Fig. 6. $d_m(v)$ and $t_m(v)$ dependencies. [AB] — Andersen–von Baeyer solution, [H] — Hill’s solution.

Fig. 7. Graphs of accelerations $d^2x^1/(dx^0)^2$ in CMF. Dashed lines present contributions of retarded and advanced interactions.
Fig. 8. On the top: (+) solutions in various reference systems (dashed lines: $v = 0.95$, solid lines: $v \to 1$.) On the bottom: acceleration of particle $x$ in the system 2.

It’s interesting to investigate limiting shapes of the trajectories in reference systems, where one of the particles has zero velocity at $t \to \pm \infty$. There are two such systems (see fig.8), in one of them solution collapses to the frame origin, as in CMF. In the other system another regime is established. The trajectories have small fractures (breaks of smoothness), located in the points $A, A'$. These fractures are related with $\delta$-like acceleration peaks, which correspond to the interactions of one type (in fig.8 – retarded for $x$, and advanced for $y$). Interaction of another type has smooth dependence on time in ranges $CA, A'C'$ and defines trajectories shape in these intervals. It’s possible to give analytical description of this shape, see Appendix B.

3. Poincare-invariance of action [8] follows to appearance of Noether’s motion integrals, i.e. to the conservation of

$$\text{translations generators } P^\pm = \sum_i p_i^\pm$$

and boosts generator $M = \sum_i x_i^+ p_i^+ - x_i^- p_i^-.$

In this approach momenta $p_i^\pm$ are expressed via the $r_i^\pm$ values, which are uniquely defined by velocities $x_i^\pm$, and the $(x_{i+1}^\pm - x_i^\pm)^{-1}$ values, defined by distances between interacting particles, see (2),(5). The terms $(x_{i+1}^\pm - x_i^\pm)^{-1}$ correspond to interaction contribution to the integrals of motion, which is present in a standard approach [1] (emitted but still not absorbed field). The difference from the standard approach is that integrals of motion (6) comprise a sum over a sequence of points on the trajectories, linked by light stairway, which propagates both to the future and to the past of the system.

Conservation of integrals (6) has been used for the control of solutions accuracy in numerical analysis, see [10] for details.
4. The solutions $X(\mu)$ found here were put in the Internet for the convenience of readers and further problem study.

http://viswiz.gmd.de/~nikitin/fw_data/node1.html.

Conclusion

The analysis of 1-dimensional scattering of 2 charges of equal masses in Wheeler-Feynman electrodynamics has revealed splitting of solutions: the scattering is uniquely defined by asymptotic velocities of the charges for $v < 0.937c$; for $v > 0.937c$ there are 3 different solutions. Mirror reflection symmetry is violated in the splitting: one of three solutions is $P$-symmetric, the other two are not, but transit to each other under $P$-reflection. For limits of solutions at $v \to c$ analytical expressions have been derived.

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Appendix A.

Boundary effects

1. Sewing (3),(4) leads to appearance of fractures on the boundary parts of trajectories. A physical reason of this effect is the following.

The equations of motion [8] have been derived minimizing the action, defined for finite trajectories, see fig.9. Due to this fact the interaction between particles $y$ and $x$ switches on/off instantaneously; potential of $y$ particle field has breaks at the points $x_2, x_N$. These breaks correspond to $\delta$-like Lorentz force $F$, which causes trajectories fractures at the points $x_2, x_N$. Breaks of velocity at $x_2, x_N$ leads to breaks of second derivatives at $y_2, y_{N-1}$, and so on. Coefficient at $\delta$-function in $F$, which determines amplitude of these breaks, is proportional to $\eta^{-1}$ and vanishes while $\eta \to \infty$ ($N \to \infty$ or $v \to 1$). In the examined range of velocities the effects caused by instantaneous switching of interaction appear to be small (a break of velocity in fracture points is $\Delta v < 5 \cdot 10^{-4}$ for $v > 0.9$, $\Delta v < 5 \cdot 10^{-5}$ for $v > 0.95$, see [10]).

![Fig.9.](attachment://image.png)
2. Other schemes of interaction switching (e.g., continuation of trajectories by infinite straight lines) are able to remove the fractures. However, these techniques are less logical, as they do not follow from the action minimum principle for finite trajectories. Consideration of infinite trajectories in stairway parametrization is related with additional difficulties, see [8].

3. For $N = 3$ only the last 2 equations remain in (3b); in (4) $n = 1$. The trajectories $y_1, y_2$ are sewed together smoothly; fractures appear in sewing $x_1, x_2, x_3$ trajectories together.

For $N = 2$ conditions (3b) are absent, instead of (4) one should use a condition

$$u^x_1(T) - u^x_2(0) = \frac{u^y_1(T) u^y_2(0)}{(x^+_2 - x^+_1)(T)} + \frac{1}{(x^+_2 - x^+_1)(0)}, \quad u^x_1 = -\frac{1}{r_1}, \quad u^x_2 = -r^+_2$$

$x_1$ and $x_2$ trajectories are sewed together with a fracture, caused by action of $\delta$-like Lorentz forces from bounds of $y_1$ trajectory.

Appendix B.

Limiting regimes

1. A problem of particles motion, influenced by interaction of one type (e.g., retarded for $x$ and advanced for $y$) is exactly solvable [6]. Indeed, in light coordinates such a motion is described by differential equations, simultaneous by $\tau = x^- - y^-$, see fig.10a.

Fig.10.

The equations have the form:

$$\ddot{x}^+ = \frac{4\dot{y}^+ \dot{x}^{+3/2}}{(x^+ - y^+)^2}, \quad \ddot{y}^+ = -\frac{4\dot{x}^+ \dot{y}^{+3/2}}{(x^+ - y^+)^2}, \quad x^- = y^- = \tau,$$

(in the equations (6) from [8] only terms corresponding to considered interaction are kept). The solution is found by elementary methods:

$$x^+(u) = A(f(u) + g(u)), \quad y^+(u) = A(f(u) - g(u)), \quad x^-(u) = y^-(u) = Ah(u),$$

$$f(u) = \frac{1 + v^2}{1 - v^2} \cdot \frac{u}{1 - u^2} + \frac{1}{2} \ln \frac{1 + u}{1 - u}, \quad g(u) = \frac{2v}{1 - v^2} \left( \frac{1}{1 - u^2} + \frac{v^2}{1 - v^2} \right),$$

$$h(u) = \frac{u}{1 - u^2} + \frac{1}{2} \ln \frac{1 + u}{1 - u}, \quad A = \frac{(1 - v^2)^{3/2}}{4v^3}, \quad -1 < u < 1.$$ 

The trajectories are symmetric w.r.t. frame origin (are conserved under the transformation $(x,y)^\pm \rightarrow -(y,x)^\pm \Leftrightarrow u \rightarrow -u$). Solution is given for CMF, $v$ is asymptotic velocity of the
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particles in CMF. General solution can be obtained from (B.1), applying arbitrary Poincare transformations:

\[(x, y)^+ \rightarrow (x, y)^+ \cdot C + D^+, \quad (x, y)^- \rightarrow (x, y)^- / C + D^- .\]

The solutions (B.1) have the following important feature: for \( v \rightarrow 1 \) in zero velocity (laboratory) frame, depicted in fig.10b, the trajectories have polygonal shape. Interaction localizes in a vicinity of fractures \( A, A' \). Sewing this solution and \( P \)-reflected one together in points \( B, B' \), a limiting Hill’s solution for the problem with interaction of both types is obtained, see fig.10c.

2. Similarly one can obtain analytical description of limiting trajectories, presented in fig.8.

(Fig.11)

Let’s consider extremal trajectories for interaction of the 1-st type (fig.11a) and take into account contribution of interaction of the 2-nd type. The trajectories regions, where particles move at light velocity, give no contribution to the interaction of the 2-nd type, because this interaction is suppressed by the factor \((1 - v)/(1 + v) \rightarrow 0 \) for \( v \rightarrow 1 \) (see (6) in [8]). In the internal intervals \( CA, A'C' \) the interaction of the 2-nd type does not vanish, these trajectory regions have the shape, described by eq.(B.1), transformed by \( P \)-reflection \((x, y)^\pm \rightarrow (y, x)^\pm \) and transformations (B.2). Outer intervals are straight and are sewed with internal ones smoothly in points \( C, C' \), while in points \( A, A' \) there are breaks of slope, of the same amplitudes as those in fig.11a.

Thus, on internal intervals the solution has the form

\[(B.3) \quad x^- = A(f - g)/C, \quad y^- = A(f + g)/C, \quad x^+ = y^+ = AhC.\]

(We have excluded translations \( D^\pm \) from transformations (B.2) and further consider solutions symmetric w.r.t. frame origin. A more thorough analysis shows that there are no solutions with \( D^\pm \neq 0 \). Requirements of sewing with straight trajectory regions have the form:

\[(B.4) \quad x^+(u) = 1, \quad x^-(u) = 0, \quad \frac{dx^+(u)}{dx^-}(u) = 1 \quad \text{(point A in fig.11b)} \quad \Leftrightarrow \]

\[ C = 1/A(v)h(u), \quad f(v, u) - g(v, u) = 0, \quad G = \frac{1 - v^2}{(1 - vu)^2A(v)^2h(u)^2} - 1 = 0.\]

The function \( x^-(u) \) in the interval \(-1 < u < 1\) monotonically increases from \(-\infty\) to \(+\infty\), thus the second equation in (B.4) can be unambiguously resolved for \( u \) (with \( v \) fixed). Finding this solution numerically and substituting it to the third equation, we plot \( G(v) \) function graph, see fig.12.
The equation \( G(v) = 0 \) has the following solutions\(^4\):

1) \( v \to 1 \) corresponds to Hill’s solution.
2) \( v = 0.6087622455403174 \) \((u = 0.867439322472601, C = 0.5948623884508324)\) corresponds to limit of solution (+). In order to transform this solution to laboratory reference frame fig.11c, let’s define the coefficient

\[
\tilde{C} = \left( \frac{dx^+}{dx^-} (\tilde{u}) \right)^{-1/2}, \quad \text{where } x^+ (\tilde{u}) = -1 \text{ corresponds to point } C \text{ on fig.11b.}
\]

Solving the last equation, we find \( \tilde{u} = -0.8674393224726012, \tilde{C} = 3.2378655431429 \). Carrying out Lorentz transformations with coefficient \( \tilde{C} \) on the trajectories, we get limiting solution, depicted in top right of fig.8.

3) \( v = 0.9437848540619277 \) \((u = 0.9998999343906364, C = 0.1703458543931005)\), the solution is presented in fig.13. In the reference system fig.11b this solution is close to Hill’s one. Small (1\%) discrepancies of these solutions in the interaction region are observed only in vicinities of points \( C, C' \) (also there is a small discrepancy of the solutions far from the interaction region, caused by difference of velocities \( v_x \) for these solutions). In the laboratory reference system fig.11c non-trivial limiting shape establishes for this solution as well (correspondent coefficient \( \tilde{C} = 34.5179309569148 \)). For transition to reference systems, where \( v_x \to 1 \), coefficient \( \tilde{C} \to 0 \)

\(^4\) Equations (B.4) are satisfied with \( 10^{-15} \) accuracy on solution 2 and \( 10^{-14} \) accuracy on 3.
is required. This, particularly, means that in CMF at \( v \to 1 \) this solution collapses to the frame origin.

Our hypothesis: for large \( v \) the solution (0) undergoes one more bifurcation. As a result 3 solutions appear, one of them tends to Hill’s solution at \( v \to 1 \), while the others to new found solution and it’s \( P \)-adjoint. The form of \( \text{det} J(v) \) dependence (see fig.13) conforms to existence of root near \( v = 0.9995 \). The presented techniques can not continue the solution to velocities \( v > 0.999 \), due to computational difficulties described in [10].

APPENDIX C.

Proposal for further investigation

The main result of our work consists in the discovery of a fact, that the system considered has the same number of freedom degrees as it’s non-relativistic analog (a system of two particles on a line, interacting via potential forces).

In both cases the shape of the trajectories on a plane \((x, t)\) depends only on particles asymptotic velocities (in our problem is defined by it uniquely for \( v < 0.937 \) and three-wayly for \( v > 0.937 \)). Phase space of Hamiltonian theory in the studied problem is essentially 4-dimensional, as well as in non-relativistic case. The phase space structure (both topological, and symplectic), however, for these cases considerably differs. The scheme for construction of reduced phase space for the considered problem looks like the following.

The set of constraints (including \( H = 0 \) and sewing requirements) extracts 5-dimensional surface in extended phase space \((x, p)\). This surface looks like \( \mathcal{S}_5 = \Gamma_1 \times \mathcal{P}_3 \times \mathcal{T}_1 \), where \( \Gamma_1 = X(\mu) \) — the found branchy curve, \( \mathcal{P}_3 \) — Poincare group, \( \mathcal{T}_1 \) — group of shifts along the trajectories. Reduction of symplectic form \( dp^\mu \wedge dx^\mu \) from extended phase space onto this surface follows to symplectic form, degenerate along the \( \mathcal{T}_1 \) (solutions shifts are generated by Hamiltonian \( H \), included into the full set of constraints). Factorization \( \mathcal{S}_5 \) by \( \mathcal{T}_1 \) (or imposition of gauges like \( \tau \sim x^1 \)) gives 4-dimensional phase space \( \mathcal{S}_4 = \Gamma_1 \times \mathcal{P}_3 \), where the reduced symplectic form defines Poisson brackets.

In further investigations of the given problem it’s interesting to carry out:
1. Calculation of Poisson brackets for independent variables and construction of canonical basis in the obtained Hamiltonian mechanics.
2. A study of non-symmetrical problem \( m_1 \neq m_2 \). The fact that the branches \( X^{(0, \pm)}(\mu) \) of found solution are sewed together in one point of phase space is related to the symmetries of the problem. For \( m_1 \neq m_2 \) in a vicinity of critical point recombination and formation of disjoint branches of solution are possible (see fig.14).
3. Calculation of second variation of the action to investigate it’s extrema. Necessity of such investigation consist in the following. Let’s consider smooth function \( f : \mathbb{R}^n \to \mathbb{R} \), which has \( k > 1 \) extrema in \( \mathbb{R}^n \). In this case all extrema can not be minima, there should be extrema of

\[ \text{http://viswiz.gmd.de/~nikitin/fw_data/node1.html} \]
other type among them (maxima or saddles). One should expect that not all of found solutions of the problem correspond to minimum of action, there should be extrema of other types as well.

4. Investigation of system responses to external perturbations and analysis of causal properties of the system. As advanced interactions can lead to violation of causality principle, it’s interesting to answer the following question: can an observer related to one of the trajectories detect the external influence on it before it takes place? In the case of causality violation one should estimate the magnitude of the effect.

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d) \( v(t) \)

\( x_1 \)

\( x_2 \)

\( x_3 \)

\( v = 0.9 \)

\( (+) v = 0.95 \)

\( (0) v = 0.98 \)
a) $v = 0.937$
$X$

$\Delta \mu$

$\mu$
b) $v=0.937$
c) $v=0.937$
