Taylor Spectrum for Modules over Lie Algebras

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Abstract. In this paper we generalize the notion of the Taylor spectrum to modules over an arbitrary Lie algebra and study it for finite-dimensional modules. We show that the spectrum can be described as the set of simple submodules in the case of nilpotent and semisimple Lie algebras. We also show that this result does not hold for solvable Lie algebras and obtain a precise description of the spectrum in the case of Borel subalgebras of semisimple Lie algebras.

Key words: Taylor spectrum, Lie algebra cohomology.

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1. Introduction

In 1970 Taylor [16] introduced the notion of the joint spectrum for a finite tuple of commuting operators \( T = (T_1, \ldots, T_n) \) on a Banach space \( V \). It was defined as the set of points \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n \) such that a certain Koszul complex associated to \( \lambda \) is not exact. The Taylor spectrum of a single operator coincides with the classical spectrum. The same year, Taylor established the existence of a holomorphic functional calculus in a neighborhood of the spectrum [17]. In 1972 he proposed a framework for a noncommutative functional calculus [18], but the notion of spectrum for noncommuting tuples of operators had not yet been developed.

The first step in this direction was made by Fainshtein in [11]. He generalized the Taylor spectrum to tuples of operators generating a finite-dimensional nilpotent Lie algebra. In [3] Boasso and Larotonda independently introduced the Taylor spectrum for representations of solvable Lie algebras. It is given by

\[
\sigma^BL_g(V) = \{ \lambda \in (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^* : \text{Tor}^{[\mathfrak{g}]}_k(V, \mathbb{C}_\lambda) \neq 0 \text{ for some } k \in \mathbb{Z}_{\geq 0} \},
\]  

(1)

where \( \mathfrak{g} \) is a solvable Lie algebra and \( V \) is a right Banach \( \mathfrak{g} \)-module.

The spectrum \( \sigma^BL \) is equivalent to that defined by Fainshtein in the nilpotent case. Moreover, we can think about commuting tuples \( T = (T_1, \ldots, T_n) \) as representations of an \( n \)-dimensional Abelian Lie algebra, and it turns out that the Boasso–Larotonda spectrum for Abelian Lie algebras is equivalent to the original Taylor spectrum.

The Taylor spectrum for Lie algebras has been extensively studied by many authors. A variety of classical theorems were generalized, such as the projection property and various kinds of the spectral mapping theorem. Details can be found in the monograph [1] and in a series of works by Dosi (see, e.g., [7]–[9]). In 2010 Dosi established the existence of a noncommutative holomorphic functional calculus in a neighborhood of the spectrum for supernilpotent Lie algebras of operators [10].

In this paper we introduce the notion of Taylor spectrum for modules over an arbitrary finite-dimensional Lie algebra and study it for finite-dimensional modules. Let \( \hat{\mathfrak{g}} \) be the set of isomorphism classes of finite-dimensional simple \( \mathfrak{g} \)-modules. The spectrum of a \( \mathfrak{g} \)-module is a subset of \( \hat{\mathfrak{g}} \) defined in terms of the nonvanishing property of some homology groups (see Definition 3.1).

The article is organized as follows. In Section 2 we recall the necessary facts about Lie algebra cohomology. In Section 3 we give the definition of the spectrum and study its properties. Theorem 3.5 gives a complete description of the spectrum of finite-dimensional modules over a semisimple Lie algebra.
algebra. In this case, the spectrum coincides with the set of isomorphism classes of simple submodules. This result holds even for Banach modules, as shown in Corollary 3.7. We conclude the section with Proposition 3.8, which describes the behavior of the spectrum under Lie algebra extensions.

Section 4 is devoted to the spectrum for solvable Lie algebras. This case has been studied previously (see [1, 3]), but mostly for infinite-dimensional modules. We explore the spectrum for finite-dimensional modules. It turns out that the spectrum is hard to compute even for the trivial module. An approximation is given by Theorem 4.1, which is a reformulation in terms of the Taylor spectrum of an earlier result by Millionshchikov [15]. We describe completely the spectrum in the case of nilpotent Lie algebras (Theorem 4.4) and Borel subalgebras of semisimple Lie algebras (Theorem 4.8).

2. Preliminaries

2.1. Notation. In this article all algebras, including Lie algebras, are defined over the algebraically closed field \( \mathbb{K} \) of characteristic zero. We use the notation \( U\mathfrak{g} \) for the enveloping algebra of a Lie algebra \( \mathfrak{g} \). We denote by \( \mathfrak{g}-\text{mod} \) and \( \text{mod-}\mathfrak{g} \) the categories of left and right \( \mathfrak{g} \)-modules, respectively. Likewise, we write \( \hat{g} \) for the set of isomorphism classes of simple finite-dimensional \( \mathfrak{g} \)-modules and \( \mathbb{K} \) for the trivial module. If \( S \) is a simple \( \mathfrak{g} \)-module, we also denote by \( S \in \hat{g} \) its isomorphism class. This will not lead to a confusion.

2.2. Functors on categories of modules. In the rest of this section we denote by \( \mathfrak{g} \) an arbitrary finite-dimensional Lie algebra. Let \( V \) be a \( \mathfrak{g} \)-module. We define the vector spaces

\[
V^g = \{ v \in V : g \cdot v = 0 \forall g \in \mathfrak{g} \},
\]

called the spaces of invariants, and

\[
V_g = V/\mathfrak{g}V,
\]
called the space of coinvariants. The assignments \( V \mapsto V^g \) and \( V \mapsto V_g \) define functors from \( \mathfrak{g}-\text{mod} \) (or \( \text{mod-}\mathfrak{g} \)) to the category of vector spaces. These functors are isomorphic to \( \text{Hom}_{U\mathfrak{g}}(\mathbb{K}, V) \) and \( \mathbb{K} \otimes_{U\mathfrak{g}} V \), respectively.

We also define two functors \( \Box^* : \mathfrak{g}-\text{mod}^{\text{op}} \to \mathfrak{g}-\text{mod} \) and \( \Box : \mathfrak{g}-\text{mod} \to \text{mod-}\mathfrak{g} \) as follows. The duality functor \( \Box^* \) maps a \( \mathfrak{g} \)-module \( V \) to its dual vector space with the left \( \mathfrak{g} \)-action given by

\[
(g \cdot f)(v) = -f(g \cdot v) \quad \text{for all } f \in V^*, v \in V, g \in \mathfrak{g}.
\]

The antipode functor \( \Box \) maps \( V \) to itself as a vector space but with the right \( \mathfrak{g} \)-action given by

\[
v \cdot g = -g \cdot v \quad \text{for all } v \in V, g \in \mathfrak{g}.
\]

The antipode functor defines an equivalence of the categories \( \mathfrak{g}-\text{mod} \) and \( \text{mod-}\mathfrak{g} \). We also denote by \( \Box^* \) and \( \Box \) the functors on the category of right \( \mathfrak{g} \)-modules defined in the same way. It is easy to see that \( (\Box^*)^\circ \) is naturally isomorphic to the identity functor. However, \( (\Box)^* \) is naturally isomorphic to the identity functor only on the subcategory of finite-dimensional modules.

Another important functor is \( \Box \otimes_{\mathbb{K}} \Box : \text{mod-}\mathfrak{g} \times \mathfrak{g}-\text{mod} \to \mathfrak{g}-\text{mod} \). If \( V \in \text{mod-}\mathfrak{g} \) and \( W \in \mathfrak{g}-\text{mod} \), then \( V \otimes_{\mathbb{K}} W \) is the tensor product of \( V \) and \( W \) as vector spaces with the action of \( \mathfrak{g} \) defined by

\[
g \cdot (v \otimes w) = v \otimes (g \cdot w) - (v \cdot g) \otimes w \quad \text{for all } w \in W, v \in V, g \in \mathfrak{g}.
\]

For \( V, W \in \mathfrak{g}-\text{mod} \) (in \( \text{mod-}\mathfrak{g} \)), we denote by \( V \otimes_{\mathbb{K}} W \) the left (respectively, right) \( \mathfrak{g} \)-module \( V^\circ \otimes_{\mathbb{K}} W \) (respectively, \( V \otimes_{\mathbb{K}} W^\circ \)).

Given \( V \in \mathfrak{g}-\text{mod} \) and a simple \( \mathfrak{g} \)-module \( S \in \hat{g} \), we write \( V_S \) for the \( \mathfrak{g} \)-module \( S \otimes_{\mathbb{K}} V \). If \( S \) is one-dimensional, then it can be described by a character \( \lambda \in (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^* \). In this case, we write
associativity of tensor product to obtain a natural isomorphism of the functors 

\[ g \cdot s = \lambda(g)s \quad \text{for all } s \in K_\lambda \text{ and } g \in g. \]

We also use the notation \( V_{-s} \) for the module \( S^s \otimes_K V \). If \( S \) is the one-dimensional \( g \)-module described by a character \( \lambda \), then \( V_{-s} \) is isomorphic to \( V_{-\lambda} \).

2.3. Lie algebra (co)homology. In this subsection we briefly recall the necessary definitions and properties of Lie algebra cohomology. For details, we refer the reader to any related textbook (see, e.g., [19; Chapter 7]).

Definition 2.1. Let \( V \) be a left \( g \)-module. For all \( k \in \mathbb{Z}_{\geq 0} \), the \( k \)th homology of \( g \) with coefficients in \( V \) is defined as

\[ H_k(g, V) = \text{Tor}_k^{U_g}(K, V). \]

Dually, the \( k \)th cohomology is defined as

\[ H^k(g, V) = \text{Ext}_k^{U_g}(K, V). \]

(Co)homologies are functors from the category \( g\text{-mod} \) to the category of vector spaces. They can be computed by means of the Chevalley–Eilenberg free resolution of the trivial \( g \) module \( K \) (see [19; Definition 7.7.1]). In degree \( k \) it has the module \( F_k = U_g \otimes_K \Lambda^k g \) with differential given by

\[
d(u \otimes g_1 \wedge \cdots \wedge g_p) = \sum_{i=1}^p (-1)^{i+1} u g_i \otimes g_1 \wedge \cdots \wedge \hat{g}_i \wedge \cdots \wedge g_p + \sum_{i<j} (-1)^{i+j} u \otimes [g_i, g_j] \wedge \cdots \wedge \hat{g}_i \cdots \wedge \hat{g}_j \cdots \wedge g_p,
\]

where \( u \in U_g \) and \( g_i \in g \). We need the following fact about cohomology.

Lemma 2.2. Let \( V \in \text{mod-}g \) and \( W \in \text{g-mod} \). Then

\[ \text{Tor}_k^{U_g}(V, W) \cong \text{Tor}_k^{U_g}(K, V \otimes_K W) = H_k(g, V \otimes_K W) \]

for all \( k \in \mathbb{Z}_{\geq 0} \).

Proof. First, the functors \( K \otimes_{U_g} (\square \otimes_K W) \) and \( \square \otimes_{U_g} W \) are naturally isomorphic. This is because, for any \( V \in \text{mod-}g \), we have

\[ K \otimes_{U_g} (V \otimes_K W) \cong (V \otimes_K W)_g = V \otimes_{U_g} W, \]

where the action on \( V \otimes_K W \) is given by formula (2). Thus, it suffices to show that if \( P_* \rightarrow V \) is a flat resolution of \( V \), then \( P_* \otimes_K W \) is a flat resolution of \( V \otimes_K W \).

By definition the flatness of \( P_k \) means the exactness of the functor \( P_k \otimes_{U_g} \square \). We use the associativity of tensor product to obtain a natural isomorphism of the functors \( (P_k \otimes_K W) \otimes_{U_g} \square \) and \( P_k \otimes_{U_g} (W \otimes_K \square) \). The latter functor is a composition of exact functors, so it is also exact. If follows that \( P_k \otimes_K W \) is flat, and we obtain a natural isomorphism \( \text{Tor}_k^{U_g}(V, W) \cong \text{Tor}_k^{U_g}(K, V \otimes_K W) \).

A useful variation of the Poincaré duality holds for finite-dimensional Lie algebras. Let \( n \) be the dimension of \( g \). We endow the one-dimensional vector space \( \Lambda^n g \) with the structure of a left \( g \)-module, which extends the adjoint action by the Leibniz rule. Namely,

\[ g \cdot g_1 \wedge \cdots \wedge g_n = \sum_{i=1}^n g_1 \wedge \cdots \wedge [g, g_i] \wedge \cdots \wedge g_n \]

for \( g, g_i \in g \).

161
Proposition 2.3 (Poincaré duality [14; Theorem 6.10]). For \(0 \leq k \leq n\) and a left \(\mathfrak{g}\)-module \(V\), there are vector space isomorphisms

\[
H^k(\mathfrak{g}, V^*) \cong H_k(\mathfrak{g}, V)^* \quad \text{and} \quad H^k(\mathfrak{g}, V) \cong H_{n-k}(\mathfrak{g}, (\bigwedge^n \mathfrak{g})^* \otimes_k V)
\]

natural in \(V\). Consequently,

\[
H^k(\mathfrak{g}, V^*) \cong H^{n-k}(\mathfrak{g}, (\bigwedge^n \mathfrak{g})^* \otimes_k V)^* .
\]

3. Taylor Spectrum of \(\mathfrak{g}\)-Modules

3.1. Definition and first properties. Let \(\mathfrak{g}\) be an arbitrary Lie algebra of dimension \(n\), and let \(V\) be a left \(\mathfrak{g}\)-module. Recall that \(\hat{\mathfrak{g}}\) is the set of isomorphism classes of simple finite-dimensional \(\mathfrak{g}\)-modules.

Definition 3.1. The Taylor spectrum of \(V\) is the subset of \(\hat{\mathfrak{g}}\) defined as

\[
\sigma_{\mathfrak{g}}(V) = \{S \in \hat{\mathfrak{g}}: \text{Tor}^U_k(\mathfrak{g}, S, V) \neq 0 \text{ for some } k \geq 0\}.
\]

We will occasionally use the notation \(\sigma(V)\) instead of \(\sigma_{\mathfrak{g}}(V)\) when it is clear what algebra \(\mathfrak{g}\) is meant. We will also write “spectrum” instead of “Taylor spectrum.”

Suppose that \(\mathfrak{g}\) is a solvable Lie algebra. By Lie's theorem, every simple \(\mathfrak{g}\)-module is one-dimensional. Thus, \(\hat{\mathfrak{g}}\) can be identified with the space \((\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^*\) of characters. In this case, the Taylor spectrum can be expressed in terms of the spectrum (1) defined by Boasso and Larotonda in [3]:

\[
\sigma_{\mathfrak{g}}(V) = -\sigma_{\mathfrak{g}}^{BL}(V^o).
\]

We may think about \(\sigma_{\mathfrak{g}}\) as a generalization of \(\sigma_{\mathfrak{g}}^{BL}\).

The spectrum has several equivalent useful characterizations.

Proposition 3.2. For an arbitrary \(S \in \mathfrak{g}\), the following conditions are equivalent:

1. \(S \in \sigma(V)\);
2. \(H_k(\mathfrak{g}, V_S) \neq 0\) for some \(k\);
3. \(H^k(\mathfrak{g}, \bigwedge^n \mathfrak{g} \otimes_k V_S) \neq 0\) for some \(k\);
4. \(S^* \otimes_k \bigwedge^n \mathfrak{g} \in \sigma(V^*)\).

Proof. (1) \(\iff\) (2). This follows immediately from Lemma 2.2.

(2) \(\iff\) (3). This is just the Poincaré duality (Proposition 2.3).

(3) \(\iff\) (4). Again, by the Poincaré duality we have

\[
H^k(\mathfrak{g}, (\bigwedge^n \mathfrak{g})^* \otimes_k V_S)^* = H^{n-k}(\mathfrak{g}, (V_S)^*).
\]

Observe that \((V_S)^*\) is isomorphic to \(S \otimes_k V^* \cong \bigwedge^n \mathfrak{g} \otimes_k (\bigwedge^n \mathfrak{g})^* \otimes_k S \otimes_k V^* \cong \bigwedge^n \mathfrak{g} \otimes_k (V^*)^* - (S^* \otimes_k \bigwedge^n \mathfrak{g})^*\). Using the equivalence of (1) and (3), we obtain the desired result. \(\square\)

Lemma 3.3. Let \(V, V',\) and \(V''\) be \(\mathfrak{g}\)-modules such that there is a short exact sequence

\[
0 \to V' \to V \to V'' \to 0 .
\]

Then

1. \(\sigma_{\mathfrak{g}}(V) \subset \sigma_{\mathfrak{g}}(V') \cup \sigma_{\mathfrak{g}}(V'')\).
2. If the short exact sequence splits, i.e., \(V \cong V' \oplus V''\), then \(\sigma_{\mathfrak{g}}(V) = \sigma_{\mathfrak{g}}(V') \cup \sigma_{\mathfrak{g}}(V'').\)
Proof. The tensor product over $\mathbb{K}$ is exact. So, for any simple $\mathfrak{g}$-module $S$, the sequence

$$0 \to V'_S \to V_S \to V''_S \to 0$$

is exact. It induces the homology long exact sequence

$$\ldots \to H_k(V'_S) \to H_k(V_S) \to H_k(V''_S) \overset{\partial}{\to} H_{k-1}(V'_S) \to \ldots$$

If $H_k(V'_S) = H_k(V''_S) = 0$ for all $k$, then the $H_k(V_S)$ are zero as well. This proves (1).

To prove (2), observe that the connecting homomorphisms $\partial$ are all zero if the exact sequence splits. So, $H_k(V_S) = H_k(V'_S) \oplus H_k(V''_S)$ and $\sigma_{\mathfrak{g}}(V) = \sigma_{\mathfrak{g}}(V') \cup \sigma_{\mathfrak{g}}(V'')$. □

3.2. Semisimple Lie algebras. Let $\mathfrak{g}$ be a semisimple Lie algebra of dimension $n$. Note that $\wedge^n \mathfrak{g}$ is isomorphic to the trivial $\mathfrak{g}$-module. It follows from Proposition 3.2 that the Taylor spectrum of a $\mathfrak{g}$-module $V$ can be described as

$$\sigma_{\mathfrak{g}}(V) = \{S \in \hat{\mathfrak{g}}: H^k(\mathfrak{g}, V_S) \neq 0 \text{ for some } k\}.$$ 

The following lemma helps us to compute the spectrum for finite-dimensional $\mathfrak{g}$-modules.

Lemma 3.4 ([19; Theorem 7.8.9]). Let $S$ be a simple $\mathfrak{g}$-module. Assume that $S \not\cong \mathbb{K}$. Then

$$H^k(\mathfrak{g}, S) = 0 \text{ for all } k.$$

If the module is trivial, then, obviously, $H^0(\mathfrak{g}, \mathbb{K}) \cong \mathbb{K}$. The following theorem gives a complete description of the spectrum for semisimple Lie algebras.

Theorem 3.5. Let $V$ be a finite-dimensional $\mathfrak{g}$-module. Then the Taylor spectrum of $V$ coincides with the set of simple components of $V$:

$$\sigma_{\mathfrak{g}}(V) = \{S \in \hat{\mathfrak{g}}: S \text{ is isomorphic to a submodule of } V\}. \quad (3)$$

Proof. By Lemma 3.3(2) the spectrum of $V$ is the union of the spectra of its simple submodules, so that we may assume that $V$ is simple. Let $S$ be a simple $\mathfrak{g}$-module. Let $V_S = \bigoplus_{i=1}^m S_i$ be a simple decomposition of $V_S$. Then we have $H^k(\mathfrak{g}, V_S) \cong \bigoplus_{i=1}^m H^k(\mathfrak{g}, S_i)$. By Lemma 3.4 $H^k(\mathfrak{g}, V_S)$ is nonzero for some $k$ if and only if $S_i \cong \mathbb{K}$ for some $i$, that is, if and only if $(V_S)^\# \neq 0$. On the other hand, the module $V_S$ is isomorphic to $\text{Hom}_\mathbb{K}(S, V)$ and $(V_S)^\# \cong \text{Hom}_\mathbb{K}(S, V)$, so it is one-dimensional if $S \cong V$ and zero otherwise. It follows that $S$ lies in the spectrum of $V$ if and only if it is isomorphic to a submodule of $V$. □

In fact, formula (3) holds even for Banach $\mathfrak{g}$-modules (here we assume that $\mathbb{K} = \mathbb{C}$). To prove this, we need the following fact.

Proposition 3.6 ([1; Sec. 30, Corollary 5]). Any Banach module $V$ over a semisimple Lie algebra $\mathfrak{g}$ is the union of its finite-dimensional submodules.

In other words, $V$ is the colimit of the filtered diagram $\mathfrak{U}$ of its finite-dimensional submodules. Recall that the homology and the tensor product commute with filtered colimits. Thus, we can strengthen Theorem 3.5.

Corollary 3.7 ([1; Sec. 30, Corollary 5]). The Taylor spectrum of a Banach $\mathfrak{g}$-module $V$ coincides with the set of isomorphism classes of simple submodules of $V$.

Proof. The filtered diagram $\mathfrak{U}$ of modules induces the filtered diagram $H_*(\mathfrak{g}, \mathfrak{U}_-) \otimes V$ of vector spaces. Moreover, $H_*(\mathfrak{g}, \mathfrak{U}_-)\otimes V$ is the colimit of $H_*(\mathfrak{g}, \mathfrak{U}_-)\otimes W$. Since all maps in $\mathfrak{U}$ are inclusions, so are the maps in $H_*(\mathfrak{g}, \mathfrak{U}_-)\otimes V$. Thus, $H_*(\mathfrak{g}, V_S)$ is nonzero if and only if $S \in \sigma(W)$ for some $W \in \mathfrak{U}$. This means that $S \in \sigma(V)$ if and only if $S$ is isomorphic to a submodule of $V$. □
3.3. Extensions of Lie algebras. Let \( \mathfrak{h} \) be an arbitrary Lie algebra, and let \( \lambda \) be a character of \( \mathfrak{h} \). By a one-dimensional extension of \( \mathfrak{h} \) we mean an exact sequence of Lie algebras

\[
0 \to \mathbb{K}_\lambda \to \mathfrak{g} \to \mathfrak{h} \to 0.
\]

Here \( \mathbb{K}_\lambda \) is an ideal of \( \mathfrak{g} \) and the commutator in \( \mathfrak{g} \) satisfies \([g, c] = \lambda(\pi(g))c\) for all \( g \in \mathfrak{g} \) and \( c \in \mathbb{K}_\lambda \).

The fundamental result of Lie algebra cohomology theory states that the isomorphism classes of such extensions are in one-to-one correspondence with the set \( H^2(\mathfrak{h}, \mathbb{K}_\lambda) = \text{Ext}^2_{\mathfrak{h}}(\mathbb{K}, \mathbb{K}_\lambda) \). We recall how to construct this correspondence. Let \( \xi \in \text{Hom}_\mathbb{K}(\wedge^2 \mathfrak{h}, \mathbb{K}_\lambda) \) be a cocycle. We define a Lie bracket on the vector space \( \mathfrak{g} = \mathfrak{h} \oplus \mathbb{K}_\lambda \) by

\[
[(h_1, c_1), (h_2, c_2)]_\xi = ([h_1, h_2], \lambda(h_1)c_2 - \lambda(h_2)c_1 + \xi(h_1 \wedge h_2))
\]

for all \( h_i \in \mathfrak{h} \) and \( c_i \in \mathbb{K}_\lambda \). It can be shown that this operation satisfies the axioms of a Lie bracket and that the extensions thus obtained are isomorphic only for cohomologous cocycles (see [19; Theorem 7.6.3]).

We adapt the techniques for computing cohomology of Lie algebra extensions developed in [12; Chap. II, Sec. 6]. In the rest of this section we write \( \mathfrak{g} \) for the one-dimensional extension of \( \mathfrak{h} \) represented by a cocycle \( \xi \in \text{Hom}_\mathbb{K}(\wedge^2 \mathfrak{h}, \mathbb{K}_\lambda) \). We also use the notation \( \mathfrak{c} \) for an ideal \( \mathbb{K}_\lambda \subset \mathfrak{g} \). Let \( V \) be an \( \mathfrak{h} \)-module. We denote by \( \mathfrak{V}^\pi \) the \( \mathfrak{g} \)-module which is obtained from \( V \) via the homomorphism \( \pi \). Note that if \( S \in \mathfrak{h} \) is an irreducible \( \mathfrak{h} \)-module, then \( S^\pi \) is also irreducible as a \( \mathfrak{g} \)-module. We use this to identify \( \mathfrak{h} \) with a subset of \( \mathfrak{g} \).

Assume that we have an \( \mathfrak{h} \)-module \( V \) and we want to compute the spectrum \( \sigma_\mathfrak{g}(V^\pi) \). The following proposition gives us some clues.

**Proposition 3.8.** Let \( \mathfrak{g} \) and \( V \) be as above. Then

1. \( \sigma_\mathfrak{g}(V^\pi) \subset \mathfrak{h} \subset \mathfrak{g} \);
2. if \( S \in \sigma_\mathfrak{g}(V^\pi) \), then either \( S \in \sigma_\mathfrak{h}(V) \) or \( S_{-\lambda} \in \sigma_\mathfrak{h}(V) \);
3. if the extension is central, i.e., \( \lambda = 0 \), then \( \sigma_\mathfrak{g}(V^\pi) = \sigma_\mathfrak{h}(V) \).

**Proof.** Let \( S \in \mathfrak{g} \setminus \mathfrak{h} \). It is easy to verify that \( \mathfrak{c}S = \{ c \cdot s : s \in S, c \in \mathfrak{c} \} \) is a submodule of \( S \). Since \( S \) is irreducible, it is either the zero submodule or the whole \( S \). If it is zero, then \( S \) is actually an \( \mathfrak{h} \)-module, which contradicts our assumption. Thus, \( \mathfrak{c}S = S \) and \( S_\mathfrak{c} = 0 \). The same argument can be used to show that \( S^\mathfrak{c} = 0 \).

The Hochschild–Serre spectral sequence \( E^2_{pq} = H_0(\mathfrak{c}, H_p(\mathfrak{h}, V_{-S})) \) converges to \( H_k(\mathfrak{g}, V_{-S}) \) (see [19; 7.5]). It suffices to prove that \( H_q(\mathfrak{c}, V_{-S}) = 0 \) for all \( q \geq 0 \). The only possible nonzero homology groups are \( H_0(\mathfrak{c}, V_{-S}) = (V_{-S})_\mathfrak{c} \cong (S^\mathfrak{c})_\mathfrak{c} \otimes_\mathbb{K} V = 0 \) and \( H_1(\mathfrak{c}, V_{-S}) = (V_{-S})^\mathfrak{c} \cong (S^\mathfrak{c})^\mathfrak{c} \otimes_\mathbb{K} V \), and they are zero, since \( V \) is \( \mathfrak{c} \)-invariant. Therefore, \( S \) is not in \( \sigma_\mathfrak{g}(V^\pi) \) by Proposition 3.2. We have proved (1).

Suppose now that \( S \in \mathfrak{h} \), so that \( \mathfrak{c} \) acts on \( V_{-S} \) trivially. Then the \( \mathfrak{h} \)-modules \( H_0(\mathfrak{c}, V_{-S}) \) and \( H_1(\mathfrak{c}, V_{-S}) \) are the homology groups of the complex \( 0 \to \mathfrak{c} \otimes_\mathbb{K} V_{-S} \to V_{-S} \to 0 \), where the middle map is the identity map on \( \mathfrak{c} \cdot V_{-S} \). The spectral sequence collapses if and only if all the differentials are isomorphisms of vector spaces.

If \( d_k \) is not an isomorphism for some \( k \), then either \( H_{k-2}(\mathfrak{h}, V_{-S} \otimes_\mathbb{K} \mathbb{K}_\lambda) \) or \( H_k(\mathfrak{h}, V_{-S}) \) is nonzero. This means that \( S \) or \( S_{-\lambda} \) is in \( \sigma_\mathfrak{h}(V) \), which proves (2).

Suppose that \( \lambda = 0 \). Since \( S \cong S_{-\lambda} \), we have \( \sigma_\mathfrak{g}(V^\pi) \subset \sigma_\mathfrak{h}(V) \) by (2). Let \( S \in \mathfrak{h} \) be in \( \sigma_\mathfrak{h}(V) \). There is an integer \( k \geq 0 \) such that \( H_k(\mathfrak{h}, V_{-S}) \neq 0 \) and \( H_i(\mathfrak{h}, V_{-S}) = 0 \) for \( i < k \). The differential \( d_k : H_k(\mathfrak{h}, V_{-S}) \to H_{k-2}(\mathfrak{h}, V_{-S}) \) is not an isomorphism, whence \( H_k(\mathfrak{g}, V_{-S}) \neq 0 \). This proves (3). □
4. The Case of Solvable Lie Algebras

4.1. The trivial module. In this section \( g \) denotes an arbitrary solvable Lie algebra of dimension \( n \). By Lie’s theorem every simple \( g \)-module is one-dimensional. So we identify \( \hat{g} \) with the space \( (g/[g,g])^* \) of characters.

Let \( V \) be a finite-dimensional \( g \)-module. By the set of weights \( \omega(V) \subset \hat{g} \) we mean the set of diagonal matrix entries in a triangular basis for \( V \). It is independent of the choice of such a basis and can also be described as the set of one-dimensional subquotients of \( V \) (this is the Jordan–Hölder theorem). Consider the adjoint representation \( \text{ad} \ g \in g\text{-mod} \). The weights \( \omega(\text{ad} g) \) are called the Jordan–Hölder values of \( g \). We denote by \( 2\rho \) the sum of all Jordan–Hölder values with multiplicities. If \( (g_1, \ldots, g_n) \) is a triangular basis of \( \text{ad} g \), then we have

\[
g \cdot g_1 \wedge \cdots \wedge g_n = 2\rho(g)g_1 \wedge \cdots \wedge g_n
\]

for \( g \in g \). So, \( \Lambda^n g \) is a one-dimensional \( g \)-module with character \( 2\rho \).

**Theorem 4.1.** Let \( g \) be a solvable Lie algebra of dimension \( n \). If \( \lambda \in \hat{g} \) is in the spectrum \( \sigma_g(K) \), then it is the sum of at most \( n \) Jordan–Hölder values of \( g \). Moreover, if \( \lambda \in \sigma_g(K) \), then \( 2\rho + \lambda \) is also in \( \sigma_g(K) \).

**Proof.** The proof is by induction on \( n \). The theorem is trivial for \( n = 1 \). Assume that the statement holds for all solvable Lie algebras of dimension \( n-1 \). Choose any one-dimensional ideal \( \mathfrak{c} \) in \( g \) and denote the corresponding character by \( \mu \). By Proposition 3.8(2) we know that the spectrum \( \sigma_g(K) \) is a subset of \( \{0, \mu\} + \sigma_{g/\mathfrak{c}}(K) \). By the induction hypothesis any \( \nu \in \sigma_{g/\mathfrak{c}}(K) \) is the sum of at most \( n-1 \) Jordan–Hölder values of \( g/\mathfrak{c} \), which are also Jordan–Hölder values of \( g \). Since \( \mu \) is a Jordan–Hölder value of \( g \), we conclude that any element of the spectrum \( \sigma_g(K) \) is the sum of at most \( n \) Jordan–Hölder values of \( g \).

To prove the second assertion, we use Proposition 3.2. The trivial module is isomorphic to its dual, so if \( K_\lambda \) is in the spectrum of \( K \), then \( K_\lambda \otimes_K \Lambda^n g \cong K_{2\rho - \lambda} \) is also in the spectrum.

**Remark.** An equivalent statement was first obtained by Millionshchikov in [15; Theorem 3.1]. Theorem 4.1 is a reformulation of that result in terms of the Taylor spectrum.

Obviously, 0 is always in the spectrum, and hence so is \( 2\rho \). So, generally, there are often more than one element in the spectrum of \( K \).

**Example 4.2.** Let \( g \) be a 3-dimensional solvable Lie algebra with basis \( e_1, e_2, e_3 \) and commutator given by \( [e_1, e_2] = e_2, [e_2, e_3] = 0, \) and \( [e_1, e_3] = \lambda e_3 \) for some \( \lambda \in \mathbb{K} \). The space of characters is one-dimensional, so we identify it with \( \mathbb{K} \) by the evaluation at \( e_1 \). Then \( \sigma_g(K) = \{0, 1, \lambda, 1 + \lambda\} \). Indeed, 0 and \( 2\rho = 1 + \lambda \) are always in the spectrum. Take \( \mu = \lambda \) or \( \mu = 1 \). From the Chevalley–Eilenberg resolution we know that \( H^1(g, K_{-\mu}) \) can be identified with the quotient of \( \ker(\mu) \subset g \) by the subspace generated by all elements of the form \( [x, y] - \mu(x)y + \mu(y)x \) for \( x, y \in g \). It is easy to check that \( \ker(\mu) \) is two-dimensional, while this subspace is only one-dimensional, so that, for 1 and \( \lambda \), the first homology groups are nonvanishing.

4.2. Nilpotent Lie algebras. The following well-known fact of representation theory for nilpotent Lie algebras is crucial for the computation of the Taylor spectrum.

**Lemma 4.3.** Let \( V \) be a finite-dimensional indecomposable module over a nilpotent Lie algebra \( g \). Then the set \( \omega(V) \) of weights consists of one element.

**Proof.** See [4; Chap. VII, Proposition 9].

We refer to modules with only one weight as monoweighted. Lemma 4.3 shows that any finite-dimensional module over a nilpotent Lie algebra can be decomposed into a sum of monoweighted submodules. We are ready to formulate the main result.
Theorem 4.4. Let $\mathfrak{g}$ be a nilpotent Lie algebra, and let $V$ be a finite-dimensional $\mathfrak{g}$-module. Then the spectrum $\sigma_{\mathfrak{g}}(V)$ coincides with the set of isomorphism classes of one-dimensional submodules of $V$.

Proof. The spectrum of the module $V$ is the union of the spectra of its indecomposable submodules by Lemma 3.3(2), so we may assume that $V$ is indecomposable. Then, by Lemma 4.3, $V$ is monoweighted. If $\mu$ is the only weight of $V$, then $\omega(V_{-\mu}) = \{0\}$ and $\sigma_{\mathfrak{g}}(V_{-\mu}) = \sigma_{\mathfrak{g}}(V) - \mu$, so we may additionally assume that $\omega(V) = \{0\}$. The proof is by induction on the dimension of $V$.

By Engel’s theorem all Jordan–Hölder values of $\mathfrak{g}$ are zero. Using Theorem 4.1, we conclude that the assertion holds for $V \cong \mathbb{K}$. If $\lambda$ is a character of $\mathfrak{g}$, then $\sigma_{\mathfrak{g}}(\mathbb{K}_\lambda) = \sigma_{\mathfrak{g}}(\mathbb{K}) + \lambda$. This means that the assertion holds for all one-dimensional modules.

Suppose now that $\dim V = m$ and that the theorem holds for modules of dimension less than $m$. Choose a one-dimensional trivial submodule of $V$. We have a short exact sequence of modules

$$0 \to \mathbb{K} \to V \to V/\mathbb{K} \to 0.$$ 

By Lemma 3.3(1) $\sigma(V) \subset \sigma(\mathbb{K}) \cup \sigma(V/\mathbb{K})$. But $\sigma(\mathbb{K}) = \sigma(V/\mathbb{K}) = \{0\}$ by the induction hypothesis. On the other hand, $0$ is in the spectrum of $V$, since $H_0(\mathfrak{g}, V) = V_0 \neq 0$. This completes the proof.

\[ \square \]

Corollary 4.5. Let $\mathfrak{g}$ be a nilpotent Lie algebra, and let $V' \subset V$ be finite-dimensional $\mathfrak{g}$-modules. Then $\sigma_{\mathfrak{g}}(V) = \sigma_{\mathfrak{g}}(V') \cup \sigma_{\mathfrak{g}}(V/V')$. If $W$ is another finite-dimensional $\mathfrak{g}$-module, then $\operatorname{Hom}_\mathfrak{g}(V, W)$ is nonzero if and only if $\sigma_{\mathfrak{g}}(V) \cap \sigma_{\mathfrak{g}}(W) \neq \emptyset$.

Proof. Obviously, $\omega(V) = \omega(V') \cup \omega(V/V')$. From Theorem 4.4 and Lemma 4.3 it is easy to see that $\sigma_{\mathfrak{g}}(V) = \omega(V)$, which proves the first assertion.

Suppose now that we have a nontrivial homomorphism $\varphi: V \to W$. It maps the one-dimensional submodules of $V/\ker \varphi$ to one-dimensional submodules of $W$. Hence $\sigma_{\mathfrak{g}}(V) \supset \sigma_{\mathfrak{g}}(V/\ker \varphi) \subset \sigma_{\mathfrak{g}}(W)$ and the intersection of spectra is nonempty.

On the other hand, if $\lambda \in \sigma_{\mathfrak{g}}(V) \cap \sigma_{\mathfrak{g}}(W)$, then there exists a surjective homomorphism $V \to \mathbb{K}_\lambda$ and an inclusion $\mathbb{K}_\lambda \to W$. The composition is a nontrivial homomorphism $V \to W$.

\[ \square \]

4.3. Borel subalgebras of semisimple Lie algebras. In this section we use the terminology of the theory of semisimple Lie algebras. We refer the reader to [13; Chap. 2] for details.

In the rest of the section $\mathfrak{s}$ is a semisimple Lie algebra and $\mathfrak{g}$ is a Borel subalgebra of $\mathfrak{s}$. It is known that $\mathfrak{g}$ is isomorphic to a semidirect product $\mathfrak{h} \ltimes \mathfrak{n}$, where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{s}$ and $\mathfrak{n} = [\mathfrak{g}, \mathfrak{g}]$. The positive roots in this case are simply the nonzero Jordan–Hölder values of $\mathfrak{g}$. We use the notation $W(\Delta)$ for the Weyl group. The length of an element $w \in W$ is denoted by $l(w)$.

The aim of this section is to describe the spectrum of irreducible $\mathfrak{s}$-modules considered as $\mathfrak{g}$-modules. First, we describe the cohomology of such modules over $\mathfrak{n}$ by using Kostant’s theorem. Then, we use the formula for the cohomology of a semidirect product to compute the cohomology over $\mathfrak{g}$.

Let $V$ be an $\mathfrak{s}$-module of highest weight $\lambda$. The cohomology groups $H^k(\mathfrak{n}, V)$ acquire the natural structure of an $\mathfrak{h}$-module after the canonical identification $\mathfrak{h} = \mathfrak{g}/\mathfrak{n}$. Kostant’s theorem gives an explicit formula for these modules.

Lemma 4.6 (Kostant’s theorem [14; Theorem 6.12]). As an $\mathfrak{h}$-module, $H^k(\mathfrak{n}, V)$ is the sum of one-dimensional modules with weights

$$w(\lambda + \rho) - \rho, \quad w \in W(\Delta), \quad l(w) = k,$$

where $\rho$ is the half-sum of positive roots (or, equivalently, the Jordan–Hölder values).

We say that an Abelian Lie algebra $\mathfrak{h}$ acts torally on an $\mathfrak{h}$-module $W$ if $W$ is a direct sum of one-dimensional submodules. If $\mathfrak{h}$ is a Cartan subalgebra of a semisimple Lie algebra $\mathfrak{s}$, then $\mathfrak{h}$ acts
torally on any finite-dimensional $\mathfrak{s}$-module. If $\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{n}$ is the Borel subalgebra of $\mathfrak{s}$, then $\mathfrak{h}$ also acts torally on $\mathfrak{n}$ and on any $\mathfrak{s}$-module.

**Lemma 4.7.** Let $\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{n}$ be a semidirect product such that $\mathfrak{h}$ acts torally on $\mathfrak{n}$ and on a $\mathfrak{g}$-module $V$. Then the cohomology groups of $V$ are given by

$$H^k(\mathfrak{g}, V) = \bigoplus_{p+q=k} \bigwedge^p \mathfrak{h}^* \otimes_K H^q(\mathfrak{n}, V)^{\mathfrak{h}}.$$

**Proof.** This is a special case of [6; Theorem 4].

The following theorem gives a complete description of the Taylor spectrum for Borel subalgebras in the special case of modules being restrictions of modules over a semisimple algebra.

**Theorem 4.8.** Let $\mathfrak{s}$ be a semisimple Lie algebra, and let $V$ be a simple $\mathfrak{s}$-module of highest weight $\lambda$. Let $\mathfrak{g}$ be a Borel subalgebra of $\mathfrak{s}$. Then the Taylor spectrum of $V$ considered as a $\mathfrak{g}$-module is given by

$$\sigma_\mathfrak{g}(V) = \{\rho + w(\lambda + \rho) : w \in W(\Delta)\},$$

where $\rho$ is the half-sum of positive roots and $W(\Delta)$ is the Weyl group.

**Proof.** By Proposition 3.2 we have $\nu \in \sigma_\mathfrak{g}(V)$ if and only if $H^k(\mathfrak{g}, \bigwedge^n \mathfrak{g} \otimes_K V_{-\nu})$ is nonzero for some $k$. The module $\bigwedge^n \mathfrak{g}$ is isomorphic to $\mathbb{K}_{2\rho}$, so that $\bigwedge^n \mathfrak{g} \otimes_K V_{-\nu} \cong V_{2\rho-\nu}$. By Lemma 4.7 we have

$$H^k(\mathfrak{g}, V_{2\rho-\nu}) = \bigoplus_{p+q=k} \bigwedge^p \mathfrak{h}^* \otimes_K H^q(\mathfrak{n}, V_{2\rho-\nu})^{\mathfrak{g}} = \bigoplus_{p+q=k} \bigwedge^p \mathfrak{h}^* \otimes_K (H^q(\mathfrak{n}, V)_{2\rho-\nu})^{\mathfrak{g}}.$$

The second equation follows from the fact that the shift by a character $2\rho - \lambda$ commutes with the cohomology of $\mathfrak{n}$. This is because tensoring with $\mathbb{K}_{2\rho-\lambda}$ is exact and commutes with taking $\mathfrak{n}$-invariants.

It follows that $\nu \in \sigma_\mathfrak{g}(V)$ if and only if $(H^q(\mathfrak{n}, V)_{2\rho-\nu})^{\mathfrak{g}}$ is nonzero for some $q$. By Lemma 4.6 $H^q(\mathfrak{n}, V)_{2\rho-\nu}$ is the sum of one-dimensional modules with weights $w(\lambda + \rho) + \rho - \nu$, where $w \in W$ is of length $l(w) = q$. We conclude that $\nu \in \sigma_\mathfrak{g}(V)$ if and only if $\nu = w(\lambda + \rho) + \rho$ for some $w \in W(\Delta)$.

**Remark.** A formula equivalent to (4) was obtained for the trivial module in [5; Sec. 4.4]. Theorem 4.8 generalizes it to arbitrary $\mathfrak{s}$-modules and gives a description in terms of the Taylor spectrum.

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