On the Direct Product of Intuitionistic Fuzzy Topological d-algebra

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ABSTRACT
We applied the direct product concept on the notation of intuitionistic fuzzy semi d-ideals of d-algebra with the investigation of some theorems. Also, we studied the notation of direct product of intuitionistic fuzzy topological d-algebra, with the notation of relatively intuitionistic continuous mapping, on the direct product of intuitionistic fuzzy topological d-algebra.

Keywords: direct product, topological d-algebra, semi d-ideal, intuitionistic set, d-algebra.

1. Introduction
A d-algebra is the classes of abstract algebra introduced by Negger and Kim [1] as a useful generalization of BCK-algebra. While the idea of fuzzy set, introduced by Zadeh [2] and Atanassov [3] generalized it to the concept of intuitionistic fuzzy set. Jun et al. [4] applied this notion on d-algebra. In another line, Abdullah and Hassan [5] studied the concept of semi d-ideal on d-algebra. After that, Hasan [6] introduced the concept of intuitionistic fuzzy semi d-ideals. Here, we applied the direct product concept on the notation of intuitionistic fuzzy semi d-ideals of d-algebra, with several interesting results. We also studied the notation of the direct product of intuitionistic fuzzy topological d-algebra.

2. Preliminaries
We will offer here some basic concepts which we need for this study.

Definition (2.1): [1] A d-algebra is a non-empty set \( H \) with a constant \( 0 \) and a binary operation \( * \) with the conditions below:

i. \( v * v = 0 \)

ii. \( 0 * v = 0 \)

iii. \( v * u = 0 \) and \( u * v = 0 \), which implies that \( v = u \), such that \( v, u \in H \). We will refer to \( v * u \) by \( vu \) and \( v \leq u \) iff \( vu = 0 \).

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Every $H$ or $G$ will denote for a d-algebra in this paper.

**Definition (2.2)**: [5] We define the semi d-ideal of $H$ as a subset $V \neq \emptyset$ of $H$ with:

(i) $v, u \in V$ implies $vu \in V$ ,
(ii) $vu \in V$ and $u \in V$ implies $v \in V$, $\forall \ v, u \in H$

**Definition (2.3)**: [2] A fuzzy set $\omega$ in any set with $H \neq \emptyset$ is a function $\omega: H \rightarrow [0,1]$. Also, for all $t \in [0,1]$, the set $\omega_t = \{v \in H, \omega(v) \geq t\}$ is a level subset of $\omega$.

**Definition (2.4)**: [7] We define a fuzzy set $\omega$ as fuzzy d-subalgebra with the following condition:

For any $v \in H$, $\forall u \in H$, $\omega(vu) \geq \min\{\omega(v), \omega(u)\}$.

**Definition (2.5)**: [6] We call the fuzzy set $\omega$ as a fuzzy semi-d-ideal if these conditions hold:

$(FI_1) \omega(vu) \geq \min\{\omega(v), \omega(u)\}$ and $(FI_2) \omega(v) \geq \min\{\omega(v), \omega(u)\}$, for all $v, u \in H$.

**Definition (2.6)**: [3] A function $S$ in $H$ is called intuitionistic fuzzy set, with the form $S = \{<x, \alpha_S(v), \beta_S(v): v \in H\}$, such that $\alpha_S: H \rightarrow [0,1]$, $\beta_S: H \rightarrow [0,1]$, is the membership degree $\alpha_S(v)$ and non-membership degree $\beta_S(v)$ $\forall \ v \in H$ to the set $S$, and $0 \leq \alpha_S(v) + \beta_S(v) \leq 1$, $\forall \ v \in H$.

We will use $S = \{<\alpha_S, \beta_S > \}$ instead of $S = \{<v, \alpha_S(v), \beta_S(v): v \in H\}$ and call it IFS for short.

**Definition (2.7)**:[8] Let $f: H \rightarrow G$ be a mapping. If $S = \{<u, \alpha_S(u), \beta_S(u): u \in G\}$, then $f^{-1}(S)$ is the IFS in $H$ defined by:

$$f^{-1}(S) = \{<v, f^{-1}(\alpha_S(v)), f^{-1}(\beta_S(v)) : v \in H\}$$

Also, if $D = \{<v, \alpha_D(v), \beta_D(v) : v \in H\}$ is an IFS in $H$, then $f(D)$ is denoted by

$$f(D) = \{<u, f_{\sup}(\alpha_D(u)), f_{\inf}(\beta_D(u)) : u \in G\},$$

where

$$f_{\sup}(\alpha_D(u)) = \sup_{v \in f^{-1}(u)} \alpha_D(v) \text{ if } f^{-1}(u) \neq \emptyset,$$

and

$$f_{\inf}(\beta_D(u)) = \inf_{v \in f^{-1}(u)} \beta_D(v) \text{ if } f^{-1}(u) \neq \emptyset,$$

for each $u \in G$.

**Definition (2.8)**: [9] If $D$ is an IFS in $H$, then

(i) $\square D = \{<v, \alpha_D(v) : v \in H\} = \{<v, \alpha_D(v), \bar{\alpha_D(v)} : v \in H\} = \{<v, \alpha_D(v), \bar{\alpha_D(v)} : v \in H\}$

(ii) $\Diamond D = \{<v, 1 - \beta_D(v) : v \in H\} = \{<v, 1 - \beta_D(v), \beta_D(v) : v \in H\} = \{<v, \bar{\beta_D(v)}, \beta_D(v) : v \in H\}$

**Definition (2.9)**: [3] Let $C = \{<\alpha_C, \beta_C > \}$ and $S = \{<\alpha_S, \beta_S > \}$ are IFS of $H$, then the cartesian product

$C \times S = \{<\alpha_C \times \alpha_S, \beta_C \times \beta_S > \}$ of $H \times H$ is defined by the following:

$(\alpha_C \times \alpha_S)(a, b) = \min(\alpha_C(a), \alpha_S(b))$ and $(\beta_C \times \beta_S)(a, b) = \max(\beta_C(a), \beta_S(b)),$

where $(\alpha_C \times \alpha_S)(a, b) : H \times H \rightarrow [0,1]$ and $(\beta_C \times \beta_S)(a, b) : H \times H \rightarrow [0,1]$.

**Definition (2.10)**: [3] Let $C = \{<\alpha_C, \beta_C > \}$ and $S = \{<\alpha_S, \beta_S > \}$ are IFS of $H$, for any $t \in [0,1]$. The set

$U(\alpha_C \times \alpha_S, t) = \{(v, u) \in H \times H, (\alpha_C \times \alpha_S)(v, u) \geq t\}$

is called the upper level of $(\alpha_C \times \alpha_S)(v, u)$ and the set

$L(\beta_C \times \beta_S, t) = \{(v, u) \in H \times H, (\beta_C \times \beta_S)(v, u) \geq t\}$

is the lower level of $(\beta_C \times \beta_S)(v, u)$.

**Definition (2.11)**: [4] An IFS $D = \{<\alpha_D, \beta_D > \}$ in $H$ is called intuitionistic fuzzy d-algebra with the conditions $\alpha_D(vu) \geq \min(\alpha_D(v), \alpha_D(u))$ and $\beta_D(vu) \leq \max(\beta_D(v), \beta_D(u))$, for all $v, u \in H$.

**Definition (2.12)**: [10] An intuitionistic fuzzy semi d-ideal of $H$, shortly $IFSd - ideal$, is an IFS, where $D = \{<\alpha_D, \beta_D > \}$ in $H$ satisfies the following inequalities:

$(IFSd_1) \alpha_D(v) \geq \min(\alpha_D(vu), \alpha_D(u))$ and $(IFSd_2) \beta_D(v) \leq \max(\beta_D(vu), \beta_D(u))$

$(IFSd_3) \alpha_D(vu) \geq \min(\alpha_D(v), \alpha_D(u))$ and $(IFSd_4) \beta_D(vu) \leq \max(\beta_D(v), \beta_D(u))$, for all $v, u \in H$.

**Proposition (2.13)**: [4] Every IFS d-algebra $IFSd - ideal$ $D = \{<\alpha_D, \beta_D > \}$ of $H$ satisfies the inequalities $\alpha_D(0) \geq \alpha_D(v)$ and $\beta_D(0) \leq \beta_D(v)$, $\forall v \in H$.

3. Direct product of IFS d-ideal

We apply here the notation of direct product for intuitionistic set on intuitionistic fuzzy d-algebra and intuitionistic semi d-ideal.

**Proposition (3.1)**: Let $C = \{<\alpha_C, \beta_C > \}$ and $S = \{<\alpha_S, \beta_S > \}$ are $IFSd - ideal$ of $H$, then $C \times S = \{<\alpha_C \times \alpha_S, \beta_C \times \beta_S > \}$ is $IFSd - ideal$ of $H \times H$.

**Proof:** We know that for any $(a_1, b_1), (a_2, b_2) \in H \times H$, we have
\[ (a_c \times a_s)(a_1, b_1) = \min\{a_c(a_1), a_s(b_1)\} \geq \min\{\min\{a_c(a_1 a_2), a_c(a_2)\}, \min\{a_s(b_1 b_2), a_s(b_2)\}\} \]

\[= \min\{\min\{a_c(a_1 a_2), a_s(b_1 b_2)\}, \min\{a_c(a_2), a_s(b_2)\}\} \]

and

\[ (a_c \times a_s)((a_1, b_1), (a_2, b_2)) \]

Also, we have

\[ (a_c \times a_s)((a_1, b_1), (a_2, b_2)) = \min\{a_c(a_1, a_2), a_c(a_1, b_2), a_s(b_1, b_2)\} \]

\[ \geq \min\{\min\{a_c(a_1, a_2), a_c(a_1, b_2)\}, \min\{a_s(b_1, b_2)\}\} \]

Also, we have

\[ (a_c \times a_s)((a_1, b_1), (a_2, b_2)) = \max\{\beta_c(a_1 a_2), \beta_c(a_2), \beta_s(b_1 b_2)\} \]

\[ \leq \max\{\max\{\beta_c(a_1 a_2), \beta_s(b_1 b_2)\}, \max\{\beta_c(a_2), \beta_s(b_2)\}\} \]

**Proposition (3.2)**: Let \( C \times S = \langle a_c \times a_s, \beta_c \times \beta_s \rangle \) be an IFSD-d-ideal of \( H \times H \), then \((a_c \times a_s)((0,0), (a, b)) \geq (a_c \times a_s)(a, b) \) and \( (\beta_c \times \beta_s)(0,0) \leq \beta_c \times \beta_s)(a, b) \).

**Proof**: We know that \((a_c \times a_s)(0,0) = \min\{a_c(0), a_s(0)\} \geq \min\{a_c(a), a_s(b)\} = (a_c \times a_s)(a, b) \) and \((\beta_c \times \beta_s)(0,0) = \max\{\beta_c(0), \beta_s(0)\} \leq \max\{\beta_c(a), \beta_s(b)\} = (\beta_c \times \beta_s)(a, b) \).

**Proposition (3.3)**: Let \( C \times S = \langle a_c \times a_s, \beta_c \times \beta_s \rangle \) be an IFSD-d-ideal of \( H \times H \). If \( (a, b) \leq (x, y) \), then \((a_c \times a_s)(a, b) \geq (a_c \times a_s)(x, y) \) and \( (\beta_c \times \beta_s)(a, b) \leq (\beta_c \times \beta_s)(x, y) \).

**Proof**: Let \((a, b), (x, y) \in H \times H \) such that \((a, b) \leq (x, y) \). This implies that \((a, b)(x, y) = (0,0) \).

Now, \((a_c \times a_s)((a, b)) \geq \min\{a_c(a, b)(x, y), (a_c \times a_s)(a_1, b_2)\} \)

\[ \geq \min\{a_c(a_1, a_2), a_s(b_1, b_2)\} \geq (a_c \times a_s)((a, b)) \]

and \((\beta_c \times \beta_s)((a, b)) \leq \max\{\beta_c \times \beta_s)(a_1, a_2), a_c \times a_s)(a_1, b_2)\} \)

\[ \leq \max\{\beta_c \times \beta_s)(0,0), (\beta_c \times \beta_s)(x, y)\} \]

**Lemma (3.4)**: Let \( C \times S = \langle a_c \times a_s, \beta_c \times \beta_s \rangle \) be an IFSD-d-ideal of \( H \times H \). If \((a, b)(c, d) \leq (e, f) \) holds in \( H \times H \), then \((a_c \times a_s)(a, b) \geq \min\{a_c \times a_s)(c, d), (a_c \times a_s)(e, f)\) and \((\beta_c \times \beta_s)(a, b) \leq \max\{\beta_c \times \beta_s)(c, d), (\beta_c \times \beta_s)(e, f)\} \).

**Proof**: Let \((a, b), (c, d), (e, f) \in H \times H \) with \((a, b)(c, d) \leq (e, f) \). Then, \((a, b)(c, d)) (e, f) = (0,0) \).

\[(a_c \times a_s)(a, b) \geq \min\{a_c \times a_s)(a, b)(c, d), (a_c \times a_s)(c, d)\} \]

\[ \geq \min\{a_c \times a_s)\left(\left((a, b)(c, d)) (e, f)\right)\right), (a_c \times a_s)(c, d)\} \]

\[ \geq \min\{a_c \times a_s)(0,0), (a_c \times a_s)(e, f), (a_c \times a_s)(c, d)\} \]

\[ \geq \min\{a_c \times a_s)(e, f), (a_c \times a_s)(c, d)\} \).

\[(\beta_c \times \beta_s)(a, b) \leq \max\{\beta_c \times \beta_s)(a, b)(c, d), (\beta_c \times \beta_s)(c, d)\} \]

\[ \leq \max\{\beta_c \times \beta_s)(0,0), (\beta_c \times \beta_s)(e, f), (\beta_c \times \beta_s)(c, d)\} \]

\[ \leq \max\{\beta_c \times \beta_s)(e, f), (\beta_c \times \beta_s)(c, d)\} \). The proof is completed.
\textbf{Theorem (3.5)}: Let $C \times S = \langle \alpha_c \times \alpha_s, \beta_c \times \beta_s \rangle$ be an IFSd-ideal of $H \times H$, then for any $(a, b), (v_1, u_1), (v_2, u_2), \ldots, (v_n, u_n) \in H \times H$, such that \[
abla \left(\ldots \left(\left(\left((a, b), (v_1, u_1)\right) \left(v_2, u_2\right)\right), \ldots\right) \left(v_n, u_n\right)\right) = (0, 0),\]
implies that \[
\alpha_c \times \alpha_s(a, b) \geq \min\left\{\alpha_c \times \alpha_s(v_1, u_1), \beta_c \times \beta_s(v_2, u_2), \ldots, (\alpha_c \times \alpha_s)(v_n, u_n)\right\}\]
and \[
(\beta_c \times \beta_s)(a, b) \leq \max\left\{\beta_c \times \beta_s(v_1, u_1), (\beta_c \times \beta_s)(v_2, u_2), \ldots, (\beta_c \times \beta_s)(v_n, u_n)\right\}.
\]
Proof: We can obtain this directly from lemma 3.4 and theorem 3.5.

\textbf{Lemma (3.6)}: Let $C \times S = \langle \alpha_c \times \alpha_s, \beta_c \times \beta_s \rangle$ be an IFSd-ideal of $H \times H$, then $\square (C \times S) = \{\alpha_c \times \alpha_s, \overline{\alpha_c} \times \overline{\alpha_s} \} \square$ is an IFSd -ideal of $H \times H$.

Proof: We know that $(\alpha_c \times \alpha_s)(a, b) = \min\{\alpha_c(a), \alpha_s(b)\}$, therefore \[
1 - \left(\alpha_c \times \alpha_s\right)(a, b) = \min\{1 - \alpha_c(a), 1 - \alpha_s(b)\}.
\]
Thus, \[
(\alpha_c \times \alpha_s)(a, b) = 1 - \min\left\{\alpha_c(a), \alpha_s(b)\right\}, \text{ moreover we get } (\alpha_c \times \alpha_s)(a, b) = \max\left\{\alpha_c(a), \alpha_s(b)\right\}.
\]
Hence, $\square (C \times S) = \{\alpha_c \times \alpha_s, \overline{\alpha_c} \times \overline{\alpha_s} \}$ is an IFSd -ideal of $H \times H$.

\textbf{Lemma (3.7)}: Let $C \times S = \langle \alpha_c \times \alpha_s, \beta_c \times \beta_s \rangle$ be an IFSd-ideal of $H \times H$, then $\checkmark (C \times S) = \{\beta_c \times \beta_s, \overline{\beta_c} \times \overline{\beta_s} \}$ is an IFSd -ideal of $H \times H$.

Proof: We know that $(\beta_c \times \beta_s)(a, b) = \max\{\beta_c(a), \beta_s(b)\}$, therefore \[
1 - (\beta_c \times \beta_s)(a, b) = \max\{1 - \beta_c(a), 1 - \beta_s(b)\}.
\]
Thus, \[
(\beta_c \times \beta_s)(a, b) = 1 - \max\{\beta_c(a), \beta_s(b)\}. \text{ Moreover, we get } (\beta_c \times \beta_s)(a, b) = \min\{\overline{\beta_c}(a), \overline{\beta_s}(b)\}.
\]
Hence, $\checkmark (C \times S) = \{\beta_c \times \beta_s, \overline{\beta_c} \times \overline{\beta_s} \}$ is an IFSd -ideal of $H \times H$.

From these two lemmas, it is not difficult to verify that the following theorem is valid.

\textbf{Theorem (3.8)}: If $C = \langle \alpha_c, \beta_c \rangle$ and $S = \langle \alpha_s, \beta_s \rangle$ is an IFSd-ideal of $H$, then $\square (C \times S)$ and $\checkmark (C \times S)$ are IFSd -ideal of $H \times H$.

\textbf{Theorem (3.9)}: Let $C = \langle \alpha_c, \beta_c \rangle$ and $S = \langle \alpha_s, \beta_s \rangle$ are IFS of $H$, then $C \times S$ is IFSd -ideal of $H \times H$ if and only if $\forall t \in [0, 1], U(\alpha_c \times \alpha_s, t) \neq \emptyset$ and $L(\beta_c \times \beta_s, r) \neq \emptyset$ for any $r, t \in [0, 1]$. Let $(v_1, u_1), (v_2, u_2) \in H \times H$, such that $(v_1, u_1)(v_2, u_2) \in U(\alpha_c \times \alpha_s, t)$ and $(v_2, u_2) \in U(\alpha_c \times \alpha_s, t)$, then $(\alpha_c \times \alpha_s)((v_1, u_1)(v_2, u_2)) \geq t$ and $(\alpha_c \times \alpha_s)((v_2, u_2)) \geq t$, which implies that $(\alpha_c \times \alpha_s)((v_1, u_1)) \geq \min\{\alpha_c \times \alpha_s((v_1, u_1)(v_2, u_2)), (\alpha_c \times \alpha_s)((v_2, u_2))\} \geq t$, so that $(\alpha_c \times \alpha_s)((v_1, u_1)) \geq t$. Also, let $(v_1, u_1), (v_2, u_2) \in U(\alpha_c \times \alpha_s, t)$. Then \[
(\alpha_c \times \alpha_s)((v_1, u_1)) \geq t \text{ and } (\alpha_c \times \alpha_s)((v_2, u_2)) \geq t.
\]
But $(\alpha_c \times \alpha_s)((v_1, u_1)) \geq \min\{\alpha_c \times \alpha_s((v_1, u_1)), (\alpha_c \times \alpha_s)((v_2, u_2))\} \geq t$, so $(v_1, u_1)(v_2, u_2) \in U(\alpha_c \times \alpha_s, t)$. Therefore, $U(\alpha_c \times \alpha_s, t)$ is semi d-ideal in $H \times H$.

Let $(v_1, u_1), (v_2, u_2) \in H \times H$ such that $(v_1, u_1)(v_2, u_2) \in L(\beta_c \times \beta_s, r)$ and $(v_2, u_2) \in L(\beta_c \times \beta_s, r)$, then $(\beta_c \times \beta_s)((v_2, u_2)) \leq r$, and $(\beta_c \times \beta_s)((v_2, u_2)) \leq r$. Then $(\beta_c \times \beta_s)((v_2, u_2)) \leq \max\{\beta_c \times \beta_s((v_1, u_1)(v_2, u_2)), (\beta_c \times \beta_s)((v_2, u_2))\} \leq r$, so that $(v_1, u_1) \in L(\beta_c \times \beta_s, r)$. Also, let $(v_1, u_1), (v_2, u_2) \in L(\beta_c \times \beta_s, r)$. Then, $(\beta_c \times \beta_s)((v_1, u_1)) \leq r$ and $(\beta_c \times \beta_s)((v_2, u_2)) \leq r$.

Then $(v_1, u_1)(v_2, u_2) \in L(\beta_c \times \beta_s, r)$. Hence, $L(\beta_c \times \beta_s, r)$ is semi d-ideal in $H \times H$.

In a converse way, assume that for any $r, t \in [0, 1], U(\alpha_c \times \alpha_s, t)$ and $L(\beta_c \times \beta_s, r)$ are empty or semi d-ideal of $H \times H$. And $\forall (v_1, u_1) \in H \times H$ Let $(\alpha_c \times \alpha_s)((v_1, u_1)) = t$ and $(\beta_c \times \beta_s)((v_1, u_1)) = r$. Then $(v_1, u_1) \in U(\alpha_c \times \alpha_s, t) \cap L(\beta_c \times \beta_s, r) \neq \emptyset$.

Since $U(\alpha_c \times \alpha_s, t)$ and $L(\beta_c \times \beta_s, r)$ are semi-d-ideal, if there exist $(p_1, q_1), (p_2, q_2) \in H \times H$ such that $\alpha_c \times \alpha_s((p_1, q_1)) < \min\{\alpha_c \times \alpha_s((p_1, q_1)(p_2, q_2)), (\alpha_c \times \alpha_s)((p_2, q_2))\}$, then by taking \[
t_0 = \frac{1}{2}\left((\alpha_c \times \alpha_s)((p_1, q_1)) + \min\{\alpha_c \times \alpha_s((p_1, q_1)(p_2, q_2)), (\alpha_c \times \alpha_s)((p_2, q_2))\}\right),
\]
we have $(\alpha_c \times \alpha_s)((p_1, q_1)) < t_0 < \min\{\alpha_c \times \alpha_s((p_1, q_1)(p_2, q_2)), (\alpha_c \times \alpha_s)((p_2, q_2))\}$.

Hence, $(p_1, q_1) \notin U(\alpha_c \times \alpha_s, t_0)$, $(p_1, q_1)(p_2, q_2) \notin U(\alpha_c \times \alpha_s, t_0)$ and $(p_2, q_2) \notin U(\alpha_c \times \alpha_s, t_0)$.

That is, $U(\alpha_c \times \alpha_s, t_0)$ is not semi d-ideal, which is a contradiction.
Now, suppose that \((\alpha_c \times \alpha_s)((p_1, q_1), (p_2, q_2)) < \min\{(\alpha_c \times \alpha_s)((p_1, q_1), (\alpha_c \times \alpha_s)((p_2, q_2))\}.

Then, by taking:
\[t_0 = \frac{1}{2}((\alpha_c \times \alpha_s)((p_1, q_1), (p_2, q_2)) + \min\{(\alpha_c \times \alpha_s)((p_1, q_1), (\alpha_c \times \alpha_s)((p_2, q_2))\} ,\]
we have
\[(\alpha_c \times \alpha_s)((p_1, q_1), (p_2, q_2)) < t_0 < \min\{(\alpha_c \times \alpha_s)((p_1, q_1), (\alpha_c \times \alpha_s)((p_2, q_2))\} .\]

Hence, \((p_1, q_1), (p_2, q_2) \in U(\alpha_c \times \alpha_s, t_0)\), but \((p_1, q_1), (p_2, q_2) \not\in U(\alpha_c \times \alpha_s, t_0)\).

That is, \(U(\alpha_c \times \alpha_s, t_0)\) is not semi d-ideal, which is a contradiction.

Now, assume that \((p_1, q_1), (p_2, q_2) \in H \times H\) such that:
\[
\beta_c \times \beta_s((p_2, q_2)) > \max\{\beta_c \times \beta_s((p_1, q_1), (p_2, q_2)), \beta_c \times \beta_s((p_2, q_2))\},
\]
By taking \(r_0 = \frac{1}{2}(\beta_c \times \beta_s((p_1, q_1), (p_2, q_2)) + \max\{\beta_c \times \beta_s((p_1, q_1), (p_2, q_2)), \beta_c \times \beta_s((p_2, q_2))\}\),
then \(\max\{\beta_c \times \beta_s((p_1, q_1), (p_2, q_2)), \beta_c \times \beta_s((p_2, q_2))\} < r_0 < \beta_c \times \beta_s((p_1, q_1))\) and there are \((p_1, q_1), (p_2, q_2) \in L(\beta_c \times \beta_s, r_0)\) and \((p_2, q_2) \in L(\beta_c \times \beta_s, r_0)\), but \((p_1, q_1) \not\in L(\beta_c \times \beta_s, r_0)\), and this is a contradiction.

Also, if we take \((p_1, q_1), (p_2, q_2) \in H \times H\) such that:
\[
\beta_c \times \beta_s((p_1, q_1), (p_2, q_2)) > \max\{\beta_c \times \beta_s((p_1, q_1), (p_2, q_2)), \beta_c \times \beta_s((p_2, q_2))\},
\]
then, by taking \(r_0 = \frac{1}{2}(\beta_c \times \beta_s((p_1, q_1), (p_2, q_2)) + \max\{\beta_c \times \beta_s((p_1, q_1), (p_2, q_2)), \beta_c \times \beta_s((p_2, q_2))\}\),
we have \(\max\{\beta_c \times \beta_s((p_1, q_1), (p_2, q_2)), \beta_c \times \beta_s((p_2, q_2))\} < r_0 < \beta_c \times \beta_s((p_1, q_1))\) and \((p_2, q_2) \in L(\beta_c \times \beta_s, r_0)\), but \((p_2, q_2) \not\in L(\beta_c \times \beta_s, r_0)\), and this is a contradiction.

**Theorem (3.10)**: Let \(C \times S \subseteq \alpha_c \times \alpha_s, \beta_c \times \beta_s > \) be an IFSD \(\text{d-ideal of } H \times H\), then the sets
\[H_{\alpha_c \times \alpha_s} = \{(a, b) \in H \times H : \alpha_c \times \alpha_s(a, b) = \alpha_c \times \alpha_s(0, 0)\}
\]
and
\[H_{\beta_c \times \beta_s} = \{(a, b) \in H \times H : \beta_c \times \beta_s(a, b) = \beta_c \times \beta_s(0, 0)\}
\]
are semi d-ideal in \(H \times H\).

**Proof:** If we take \((a, b), (x, y) \in H \times H\), let \((a, b), (x, y) \in H_{\alpha_c \times \alpha_s}\) and \((x, y) \in H_{\alpha_c \times \alpha_s}\). Then,
\[\alpha_c \times \alpha_s((a, b), (x, y)) = \alpha_c \times \alpha_s(0, 0) = \alpha_c \times \alpha_s((a, x), y), \]
so
\[\alpha_c \times \alpha_s((a, b), (x, y)) \geq \min\{\alpha_c \times \alpha_s((a, b), (x, y)), \alpha_c \times \alpha_s((a, x), y)\} = \alpha_c \times \alpha_s(0, 0).
\]
Knowing that \(\alpha_c \times \alpha_s((a, b)) = \alpha_c \times \alpha_s(0, 0)\) (proposition (3.1)), thus \((a, b) \in H_{\alpha_c \times \alpha_s}\).

Let \((a, b), (x, y) \in H_{\alpha_c \times \alpha_s}\). Then, \(\alpha_c \times \alpha_s((a, b), (x, y)) = \alpha_c \times \alpha_s(0, 0)\), so \(\alpha_c \times \alpha_s((a, b), (x, y)) \geq \min\{\alpha_c \times \alpha_s((a, b), (x, y)), \alpha_c \times \alpha_s((a, x), y)\} = \alpha_c \times \alpha_s(0, 0)\) (proposition (3.3)).

Knowing that \(\alpha_c \times \alpha_s((a, b), (x, y)) = \alpha_c \times \alpha_s(0, 0)\) (proposition (3.3)), thus \((a, b) \in H_{\alpha_c \times \alpha_s}\).

Also, let \((a, b), (x, y) \in H_{\beta_c \times \beta_s}\) and \((x, y) \in H_{\beta_c \times \beta_s}\). Then, \(\beta_c \times \beta_s((a, b), (x, y)) = \beta_c \times \beta_s(0, 0)\), so \(\beta_c \times \beta_s((a, b), (x, y)) \leq \max\{\beta_c \times \beta_s((a, b), (x, y)), \beta_c \times \beta_s((a, x), y)\} = \beta_c \times \beta_s(0, 0)\) (proposition (3.3)).

Knowing that \(\beta_c \times \beta_s((a, b), (x, y)) = \beta_c \times \beta_s(0, 0)\) (proposition (3.3)), so we get \((a, b) \in H_{\beta_c \times \beta_s}\).

Let \((a, b), (x, y) \in H_{\beta_c \times \beta_s}\). Then, \(\beta_c \times \beta_s((a, b), (x, y)) = \beta_c \times \beta_s(0, 0)\), so \(\beta_c \times \beta_s((a, b), (x, y)) \leq \max\{\beta_c \times \beta_s((a, b), (x, y)), \beta_c \times \beta_s((a, x), y)\} = \beta_c \times \beta_s(0, 0)\) (proposition (3.3)).

Hence, from proposition (3.3), we get \(\beta_c \times \beta_s((a, b), (x, y)) = \beta_c \times \beta_s(0, 0)\). Then, \((a, b), (x, y) \in H_{\beta_c \times \beta_s}\). Thus, \(\beta_c \times \beta_s\) is semi d-ideal.

The next theorems are easy to prove.

**Theorem (3.11):** In a d-homorphism \(f: H \times H \rightarrow G \times G\), if \(C \times S\) is an IFSD \(\text{d-ideal of } G \times G\), then \(f^{-1}(C \times S)\) is an IFSD \(\text{d-ideal of } H \times H\).

**Theorem (3.12):** Let \(f: H \times H \rightarrow G \times G\) be an d-homorphism and let \(C \times S\) be a direct product of IFSD \(C\) and \(S\) in \(G \times G\). If \(f^{-1}(C \times S) =< \alpha_{f^{-1}(C \times S)}, \beta_{f^{-1}(C \times S)} >\) is an IFSD \(\text{d-ideal of } H \times H\), then \(C \times S =< \alpha_{C \times S}, \beta_{C \times S} >\) is an IFSD \(\text{d-ideal of } G \times G\).

4. **Direct product of Intuitionistic fuzzy topological d-algebra**

In this section, we apply the concept of direct product for intuitionistic set on the notation of intuitionistic fuzzy topological d-algebra with some theorems of continues maps.

**Definition (4.1)** [3]: An intuitionistic fuzzy topology (IFT shortly) on a non-empty set \(H\) is a family \(\mathfrak{S}\) of IFSS in \(H\) that satisfies:

\[
\begin{align*}
(IFT_0) & \quad 0, 1 \in \mathfrak{S}, \\
(IFT_1) & \quad N_1 \cap N_2 \in \mathfrak{S} \text{ for any } N_1, N_2 \in \mathfrak{S}, \\
(IFT_2) & \quad \bigcup_{i \in \Delta} \mathfrak{N}_i \in \mathfrak{S} \text{ for any family } \{N_i, i \in \Delta\} \subseteq \mathfrak{S}.
\end{align*}
\]

So, we call the pair \((H, \mathfrak{S})\) as an intuitionistic fuzzy topological space (IFTS shortly) and the IFS in \(\mathfrak{S}\) as an intuitionistic fuzzy open (shortly IFOS) .
If we have a map \( f : H \rightarrow G \) such that \((H, \mathcal{S})\), \((V, \vartheta)\) are two IFTS, then \( f \) is called intuitionistic fuzzy continuous (IFC) if the inverse image for any IFS in \( \vartheta \) being IFS in \( \mathcal{S} \). Also, if the image for any IFS in \( \mathcal{S} \) is an IFS in \( \vartheta \), then we call \( f \) as an intuitionistic fuzzy open (IFO). \([1]\)

**Definition (4.2) [10]:** For an IFS \( \mathcal{K} \) in an IFTS \((H, \mathcal{S})\), we say that the induced intuitionistic fuzzy topology (shortly IIFT) on \( \mathcal{K} \) is a family of IFSs in \( \mathcal{K} \) such that the intersection of it with \( \mathcal{K} \) is an IFOS in \( H \). The IIFTs are denoted by \( \mathcal{S}_{\mathcal{K}} \) and \((\mathcal{K}, \mathcal{S}_{\mathcal{K}})\) is an intuitionistic fuzzy subspace (IFS ub) of \((H, \mathcal{S})\).

**Definition (4.3) [10]:** Take \((\mathcal{K}, \mathcal{S}_{\mathcal{K}})\) and \((\mathcal{M}, \vartheta_{\mathcal{M}})\) as IFSub of IFTSs \((H, \mathcal{S})\) and \((G, \vartheta)\), respectively, with the mapping \( f : H \rightarrow G \) be a mapping. Then, \( f \) is a mapping \( \mathcal{K} \) into \( \mathcal{M} \) if \( f(\mathcal{K}) \subseteq \mathcal{M} \). Also, \( f \) is called relatively intuitionistic fuzzy continuous (RIFC) if, for any IFS \( V_{\mathcal{M}} \) in \( \vartheta_{\mathcal{M}} \), the intersection \( f^{-1}(V_{\mathcal{M}}) \cap \mathcal{K} \) is an IFS in \( \mathcal{S}_{\mathcal{K}} \); and \( f \) is called relatively intuitionistic fuzzy open (RIFO) if, for any IFS \( U_{\mathcal{K}} \) in \( \mathcal{S}_{\mathcal{K}} \), the image \( f(U_{\mathcal{K}}) \) is IFS in \( \vartheta_{\mathcal{M}} \).

**Proposition (4.4):** Let \((\mathcal{K} \times \mathcal{M}, \mathcal{S}_{\mathcal{K} \times \mathcal{M}})\) and \((F \times \mathcal{L}, \vartheta_{F \times \mathcal{L}})\) be direct products of IFSub of direct product of IFTSs \((H \times \mathcal{S})\) and \((G \times \vartheta)\), respectively, and let \( f : H \times H \rightarrow G \times G \) be an intuitionistic fuzzy continuous mapping, such that \( f(\mathcal{K} \times \mathcal{M}) \subseteq (F \times \mathcal{L}) \). Then, \( f \) is RIFC mapping of \((\mathcal{K} \times \mathcal{M})\) into \((F \times \mathcal{L})\).

Proof: Let \((U_{2} \times V_{2})(F \times \mathcal{L})\) be IFS in \( \vartheta_{F \times \mathcal{L}} \), then there exists \( U \times V \in \vartheta \), such that

\[
(U_{2} \times V_{2})(F \times \mathcal{L}) = (U \times V) \cap (F \times \mathcal{L})
\]

Since \( f \) is IFC, so it follows that \( f^{-1}(U \times V) \) is an IFS in \( \mathcal{S} \). So \( f^{-1}((U_{2} \times V_{2})(F \times \mathcal{L})) \cap (\mathcal{K} \times \mathcal{M}) = f^{-1}((U \times V) \cap (F \times \mathcal{L})) \cap (\mathcal{K} \times \mathcal{M}) = f^{-1}(U \times V) \cap f^{-1}(F \times \mathcal{L}) \cap (\mathcal{K} \times \mathcal{M}) = f^{-1}(U \times V) \cap (\mathcal{K} \times \mathcal{M}) \) is IFS in \( \mathcal{S}_{\mathcal{K} \times \mathcal{M}} \). This completes the proof.

**Definition (4.5):** For any \( H \) and any order pair \((a, b)\) of \( H \times H \), we define the self-map \((a, b)_{r}\) of \( H \times H \) by \((a, b)_{r}((x, y)) = (x, y)(a, b)\) for all \((x, y) \in H \times H\).

**Definition (4.6) [10]:** For an IFT \( \mathcal{S} \) on \( H \), if \( \mathcal{K} \) is an IFd-algebra with IIFT \( \mathcal{S}_{\mathcal{K}} \), then we say that \( \mathcal{K} \) intuitionistic fuzzy topology (shortly d-algebra) (IFTd-algebra). For any \( h \in H \), the mapping \( h_{r} : (\mathcal{K}, \mathcal{S}_{\mathcal{K}}) \rightarrow (\mathcal{K}, \mathcal{S}_{\mathcal{K}}) \), \( x \rightarrow xh \) is relatively intuitionistic fuzzy continuous.

**Definition (4.7):** For an IFT \( \mathcal{S} \) on \( H \), if \( \mathcal{K}, \mathcal{M} \) are IFd-algebras with IIFTs \( \mathcal{S}_{\mathcal{K}}, \mathcal{S}_{\mathcal{M}} \), respectively. Then, \( \mathcal{K} \times \mathcal{M} \) is called a direct product of IFTd-algebra if for any \((a, b) \in H \times H \) the mapping \((a, b)_{r} : (\mathcal{K} \times \mathcal{M}, \mathcal{S}_{\mathcal{K} \times \mathcal{M}}) \rightarrow (\mathcal{K} \times \mathcal{M}, \mathcal{S}_{\mathcal{K} \times \mathcal{M}}) \), \((x, y) \rightarrow (x, y)(a, b)\) is relatively intuitionistic fuzzy continuous.

**Theorem (4.8):** Let \( \delta : H \rightarrow G \) be a d-homomorphism and \( \mathcal{S}, \vartheta \) be IFTs on \( H \) and \( G \), respectively, such that \( \mathcal{S} = \delta^{-1}(\vartheta) \). If \( \mathcal{K} \times \mathcal{M} \) is a direct product of IFTd-algebra in \( G \times G \), then \( \delta^{-1}(\mathcal{K} \times \mathcal{M}) \) is an IFTd-algebra in \( H \times H \).

Proof: Suppose that \((a, b) \in H \times H \) and let \( U_{1} \times V_{1} \) be IFS in \( \mathcal{S}_{\delta^{-1}(\mathcal{K} \times \mathcal{M})} \). We know that \( \delta^{-1} \) is an IFC mapping of \((H \times H, \mathcal{S})\) into \((G \times G, \vartheta)\), so we have from (4.4) that \( \delta \) is an IFC mapping of \((\delta^{-1}(\mathcal{K} \times \mathcal{M}), \mathcal{S}_{\delta^{-1}(\mathcal{K} \times \mathcal{M})}) \) into \((\mathcal{N} \times \mathcal{M}, \mathcal{S}_{\delta^{-1}(\mathcal{K} \times \mathcal{M})}) \). Note that there exists an IFS \( U_{2} \times V_{2} \) in \( \mathcal{S}_{\delta^{-1}(\mathcal{K} \times \mathcal{M})} \) such that \( \delta^{-1}(U_{2} \times V_{2}) = U_{1} \times V_{1} \). Then

\[
\alpha_{(a,b)_{r}^{-1}(U_{1} \times V_{1})}((x, y)) = \alpha_{U_{1} \times V_{1}}((x, y)(a, b)) = \alpha_{U_{1} \times V_{1}}((x, y)(a, b)) = \alpha_{U_{2} \times V_{2}}(\delta((x, y)(a, b))) = \alpha_{U_{2} \times V_{2}}(\delta((x, y)(a, b)))
\]

and

\[
\beta_{(a,b)_{r}^{-1}(U_{1} \times V_{1})}((x, y)) = \beta_{U_{1} \times V_{1}}((x, y)(a, b)) = \beta_{U_{1} \times V_{1}}((x, y)(a, b)) = \beta_{U_{2} \times V_{2}}(\delta((x, y)(a, b))) = \beta_{U_{2} \times V_{2}}(\delta((x, y)(a, b))).
\]
Since $\mathbb{R} \times \mathcal{M}$ is a direct product of IFTd-algebra in $\times G$, then we have the RIFC mapping $(b_1, b_2)_\gamma: (\mathbb{R} \times \mathcal{M}, \delta_{\mathbb{R} \times \mathcal{M}}) \to (\mathbb{R} \times \mathcal{M}, \delta_{\mathbb{R} \times \mathcal{M}})$, $(y_1, y_2) \to (y_1, y_2)(b_1, b_2)$, for every $(b_1, b_2)$ in $G \times G$. Hence,

$$
\alpha_{(a,b)\gamma^{-1}\theta}(x_1, y_1) = \alpha_{(a,b)\gamma}(\delta(x_1, y_1)) = \alpha_{(a,b)\gamma}(\delta(a, b)) = \alpha_{(a,b)\gamma}(\delta((a, b)))
$$

and

$$
\beta_{(a,b)\gamma^{-1}\theta}(x_1, y_1) = \beta_{(a,b)\gamma}(\delta(x_1, y_1)) = \beta_{(a,b)\gamma}(\delta((a, b)))
$$

Therefore, $(a, b)\gamma^{-1}\theta(U_1 \times V_1) = \delta^{-1}(\delta((a, b)\gamma(U_2 \times V_2)))$.

So, $(a, b)\gamma^{-1}\theta(U_1 \times V_2) \cap \delta^{-1}(\mathbb{R} \times \mathcal{M}) = \delta^{-1}(\delta((a, b)\gamma(U_2 \times V_2))) \cap \delta^{-1}(\mathbb{R} \times \mathcal{M})$ is an IFS in $\mathcal{M}$.

**Theorem (4.9):** For a $d$-homorphism $\delta: H \to G$ and $\delta_1, \delta$ being IFTs on $H$ and $G$, respectively, such that $(\delta_1) = \delta$. If $D \times C$ is a direct product of IFTd-algebra in $H \times H$, then $\delta(D \times C) \gamma$ is an IFTd-algebra in $G \times G$.

*Proof:* We need to show that the mapping $(b_1, b_2)_\gamma: (D \times C, \theta_{D \times C}) \to (D \times C, \theta_{D \times C})$. $(y_1, y_2) \to (y_1, y_2)(b_1, b_2)$ is relatively intuitionistic fuzzy continuous for every $(b_1, b_2)$ in $H \times H$. Let $(U_1 \times V_1)_{D \times C}$ be IFS in $\mathcal{M}$.

Then, there exists an IFS $U_2 \times V_2$ in $\mathcal{M}$ such that $(U_1 \times V_1)_{D \times C} = (U \times V) \cap D \times C$.

Since $\delta$ is one-one, it follows that $\delta((U_1 \times V_1)_{D \times C}) = \delta((U \times V) \cap D \times C) = \delta((U \times V)) \cap \delta(D \times C)$, which is an IFS in $\mathcal{M}$. This shows that $\delta$ is RIFC.

Let $(U_1 \times V_1)_{D \times C}$ be an IFS in $\mathcal{M}$. Since $\delta$ is surjective, so we have for every $(b_1, b_2)$ in $G \times G$, there exists $(a_1, a_2)$ in $H \times H$ such that $(b_1, b_2) = \delta((a_1, a_2))$. Hence,

$$
\alpha_{\delta^{-1}(b_1, b_2)\gamma^{-1}(U_2 \times V_2)_{D \times C}}((x, y)) = \alpha_{\delta^{-1}(b_1, b_2)\gamma^{-1}(U_2 \times V_2)_{D \times C}}((x, y))
$$

and

$$
\beta_{\delta^{-1}(b_1, b_2)\gamma^{-1}(U_2 \times V_2)_{D \times C}}((x, y)) = \beta_{\delta^{-1}(b_1, b_2)\gamma^{-1}(U_2 \times V_2)_{D \times C}}((x, y))
$$

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\[
\beta_{(a_1,a_2)}^{-1}(\delta^{-1}((U_2 \times V_2)_{\delta(D \times C)})) (x,y) = \beta_{(a_1,a_2)}^{-1}(\delta^{-1}((U_2 \times V_2)_{\delta(D \times C)})) (x,y).
\]

Therefore, \(\delta^{-1}((b_1,b_2))^{-1}((U_2 \times V_2)_{\delta(D \times C)}) = (a_1,a_2)^{-1}((U_2 \times V_2)_{\delta(D \times C)}) \).

By hypothesis, the mapping \((a_1,a_2) : (D \times C, \mathcal{S}_{D \times C}) \to (\delta(D \times C), \mathcal{S}_{D \times C}) \), \((x,y) \to (x,y)(a_1,a_2)\) is RIFC and \(\delta\) is RIFC map such that \(\delta : (D \times C, \mathcal{S}_{D \times C}) \to (\delta(D \times C), \mathcal{S}_{D \times C}) \).

Thus, \(\delta^{-1}((b_1,b_2))^{-1}((U_2 \times V_2)_{\delta(D \times C)}) \cap (D \times C) = (a_1,a_2)^{-1}((U_2 \times V_2)_{\delta(D \times C)}) \cap (D \times C) \) is an IFS in \(\mathcal{S}_{D \times C} \).

Since \(\delta\) is RIFO, then
\[
\delta\left(\delta^{-1}((b_1,b_2))^{-1}((U_2 \times V_2)_{\delta(D \times C)}) \cap (D \times C) \right) = (b_1,b_2)^{-1}((U_2 \times V_2)_{\delta(D \times C)}) \cap \delta((D \times C))
\]
is IFS in \(\delta_{D \times C} \). This completes the proof.

**Conclusions**

We showed in this paper that the definition of relatively intuitionistic fuzzy continuous has led us to define the notation of the direct product of intuitionistic fuzzy topological d-algebra. We also found that the homomorphism map \(\delta\) provides the notion that the primage for the direct product of intuitionistic fuzzy topological d-algebra is also a direct product of intuitionistic fuzzy topological d-algebra. Also, the image for the direct product of intuitionistic fuzzy topological d-algebra is a direct product of intuitionistic fuzzy topological d-algebra.

We believe that this work can enhance further studies in this field for the generation of direct products of finite and infinite intuitionistic fuzzy semi d-ideals on d-algebra as well as intuitionistic topological d-algebra. We hope that this work can impact upcoming research in this field or in other algebraic structures.

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