On the locus formed by the maximum heights of projectile motion with air resistance

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Abstract

We present an analysis on the locus formed by the set of maxima of the trajectories of a projectile launched in a medium with linear drag. Such a place, the locus of apexes, is written in terms of the Lambert W function in polar coordinates, confirming the special role played by this function in the problem. To characterize the locus, a study of its curvature is presented in two parameterizations, in terms of the launch and the polar angles. The angles of maximum curvature are compared with other important angles in the projectile problem. As an addendum, we find that the synchronous curve in this problem is a circle as in the drag-free case. The presentation is suitable for advanced undergraduate students.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

An amazing characteristic of some old-fashioned problems is how they have endured. Projectile motion is one of them. Being one of the main problems taught in elementary physics, variations and little known facts about it appear throughout the physics literature of the 21st century. A search on the web or in the Science Citation Index proves this fact. Some of the recent studies deal with the problem of air resistance in projectile motion, and its pedagogical character make it an excellent example to introduce the Lambert W function, a special function [1]. The Lambert W function is involved in many interesting problems for physicists and engineers, from the solution of the jet fuel problem to epidemics [1] or even helium atom eigenfunctions [2]. One of these problems is the solution for the range \( R \) in cases when the air resistance has the form \( \vec{f} = -mb\vec{v} \) [3, 4].

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The analysis of motion in the presence of a drag force is part of standard classical mechanics textbooks [5]. The problem of projectile motion under gravity with a linear drag force is solvable using direct integration, and offers an example to explore, with students, the properties of motion in the presence of air. The level of mathematical skill required for students to understand the present exposition is equal to that required in understanding Marion’s book.

In this paper we analyse the little known fact of the locus formed by the maxima of all the projectile trajectories at a launch angle $\alpha$ and in the presence of a drag force proportional to the velocity; we shall denote this locus as $C_m(\varepsilon)$.

The resulting locus becomes a Lambert $W$ function of the polar coordinate $r(\theta)$ departing from the origin. This problem arises as a natural continuation from the fact that in the drag-free case, such a locus is an ellipse [6–8] with a universal eccentricity $e = \sqrt{3}/2$ [6].

The paper is organized as follows. In section 2 the set of maxima for projectile trajectories moving in the presence of air resistance is presented. In section 3 we find a closed form, in polar coordinates, to express such a locus, $C_m$. In section 4 we present a numerical evaluation of the curvature of $C_m$ using the polar angle and the launch angle as parameterizations. Additionally, we demonstrate in section 5 that the synchronous curve is a circle as in the drag-free case, and in section 6 we conclude.

2. The projectile problem with air resistance

Several approximations to consider air resistance exist in the literature, the simplest of which is the linear case. This problem admits a full solution by direct integration but requires the use of special functions and a certain knowledge of the properties of implicit functions. Hence, this case is a good example where the formal solution is easy to obtain for the students but presents a certain amount of difficulty to obtain the desired solution, for instance the range [3]. Furthermore, it is a challenge for the students to find out, in an apparently solvable problem, the correct answers.

In the case of linear drag, the force is given by

$$\vec{F} = -mb\vec{v} - mg\hat{j},$$

where $m$ is the mass of the projectile and $b$ is the drag coefficient. The unit of $b$ is s$^{-1}$. The solutions are obtained through direct integration of (1) yielding

$$x(t) = \frac{u_0}{b} \left[ 1 - \exp(-bt) \right],$$

$$y(t) = \frac{u_0 + g/b}{b} \left[ 1 - \exp(-bt) \right] - gt/b.$$  

We used the initial conditions $x(0) = y(0) = 0$, and $u_0 = V_0 \cos \alpha$ and $w_0 = V_0 \sin \alpha$.

For the same initial speed $V_0$ these solutions are the function of the launch angle $\alpha$, and the locus formed by the apexes is obtained if time is eliminated between the solutions (2) and (3) giving

$$y(x) = \frac{u_0 + g/b}{u_0} x - \frac{g}{b^2} \ln \left( 1 - \frac{bx}{u_0} \right),$$

and considering the value at the maximum, via $dy/dx = 0$. The corresponding solution is

$$x_m = \frac{\cos \alpha \sin \alpha}{1 + \varepsilon \sin \alpha},$$

1 To our knowledge, the description of this locus for a linear drag force has not been cited in the literature.
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Figure 1. Locus $C_m(\varepsilon)$ formed by the apexes of all the projectile trajectories (continuous blue line) given by (5) and (6) in rectangular coordinates or by (11), the last one expresses $C_m$ in polar coordinates and in terms of the Lambert $W$ function. The dashed red line is the ellipse of eccentricity $e = \sqrt{3}/2$, which represents the drag-free case, i.e. $C_m(0)$. The parameters are $V_0 = 10$ and $\varepsilon = 0.1$.

$$\frac{\varepsilon^2 y_m}{\rho} = \varepsilon \sin \alpha - \ln(1 + \varepsilon \sin \alpha), \tag{6}$$

where we introduce the dimensionless perturbative parameter $\varepsilon \equiv bV_0/g$ and the dimensionless length $\rho = V_0^2/g$, noticing that $b^2/g$ can be expressed as $\varepsilon^2/\rho$. An alternative procedure consists of setting the derivative $dy/dt$ to zero to obtain the time of flight to the apex of the trajectory and evaluate the coordinates at that time. The points $(x_m, y_m)$ confirm the locus of the apexes $C_m(\varepsilon)$ for all the projectile trajectories as a function of the launch angle $\alpha$. In figure 1 we plot $C_m(\varepsilon)$, described by (5) and (6), for the drag-free case (in the dashed red line) and for $\varepsilon = 0.1$ (in the continuous blue line). Several projectile trajectories are plotted as thin black lines. The locus of apexes $C_m(\varepsilon)$ defined by $(x_m, y_m)$ is described parametrically by the launch angle $\alpha$ and changes for different values of $\varepsilon$. In the next section we find a description of $C_m(\varepsilon)$ in terms of polar coordinates and in a closed form using the Lambert $W$ function.

3. The locus $C_m$ as a Lambert $W$ function

In order to obtain an analytical closed form of the locus, we changed the variables to polar ones, i.e. $x_m = r_m \cos \theta_m$ and $y_m = r_m \sin \theta_m$. The selection of a description departing from that origin, instead of the centre or the focus of the ellipse, is because the resulting locus is no longer symmetric and the only invariant point is the launching origin. We have substituted the polar forms of $x_m$ and $y_m$ into equations (5) and (6), and rearranging terms it must be expressed as

$$\frac{r_m(\theta_m)}{\rho} \cos \theta_m \exp \left( -\varepsilon^2 \sin \theta_m \frac{r_m(\theta_m)}{\rho} \right) = \cos \alpha \sin \alpha \exp(-\varepsilon \sin \alpha). \tag{7}$$
The lhs depends on \( r_m \) and \( \theta_m \), meanwhile the rhs depends on \( \alpha \); however, the last angle is a function of \( \theta_m \) and reads

\[
\tan \theta_m = \frac{1}{\varepsilon^2} \left( \frac{\varepsilon \sin \alpha - \ln(1 + \varepsilon \sin \alpha)}{\cos \alpha \sin \alpha} \right) (1 + \varepsilon \sin \alpha),
\]

by making \( \tan \theta_m = y_m/x_m \) from (5) and (6).

In order to obtain \( \tilde{r}(\theta_m) \equiv r_m(\theta_m)/\rho \), we set

\[
f(\alpha(\theta_m)) \equiv \cos \alpha \sin \alpha \exp(-\varepsilon \sin \alpha),
\]

since (8) allows us to have, implicitly, \( \alpha(\theta_m) \). We shall return to this point later. Hence, we can write (7) as

\[
-\varepsilon^2 \sin \theta_m \tilde{r}(\theta_m) \exp(-\varepsilon^2 \sin \theta_m \tilde{r}(\theta_m)) = -\varepsilon^2 \tan \theta_m f(\alpha),
\]

where we multiplied both sides of (7) by \(-\varepsilon^2 \sin \theta_m \). Setting \( z = -\varepsilon^2 \tan \theta_m f(\alpha) \) and \( W(z) = -\varepsilon^2 \sin \theta_m \tilde{r}(\theta_m) \) in (10), it will have the familiar Lambert \( W \) function form, \( z = W(z) \exp(W(z)) \), from which we can obtain \( \tilde{r} \) as

\[
\tilde{r}(\theta_m) = -\frac{1}{\varepsilon^2 \sin \theta_m} W(-\varepsilon^2 \tan \theta_m f(\alpha)).
\]

It is important to note that the argument of the Lambert function in this equation is negative for all the values \( \varepsilon > 0 \). \( W(x) \) remains real in the range \( x \in [-1/e, 0) \) and has the branches denoted by 0 and \(-1[1] \). We select the principal branch, 0, since it is the bounded one; however, for values of \( \varepsilon > 1.1 \) there is a precision problem since the required argument values are near to \(-1/e \equiv -\exp(-1) \). It is important to stress that in (11) the independent variable is the angle \( \theta \), and it constitutes the parameterization of the curve \( C_m \).

We recover the drag-free result

\[
\tilde{r} = 2 \frac{\sin \theta_m}{1 + 3 \sin^2 \theta_m}
\]

when \( \varepsilon \to 0 \). An explanation of this unfamiliar form of an ellipse is given in appendix A, followed by a discussion about the \( \varepsilon \to 0 \) limit of expression (11) in appendix B.

Formula (11) exhibits the strong relationship between the Lambert \( W \) function and the linear drag force projectile problem, since not only the range is given as this function [3, 4]. The problem provides the opportunity to study the \( W \) function in polar coordinates, which is almost absent in the review of [1]. Even when it is possible to write the locus in terms of \( y_m(x_m) \), this form does not show the formal elegance of relation (11).

Now we return to (8) since we need to solve it explicitly in order to have the function \( \alpha(\theta_m) \). At first sight, this task is not trivial. A way to do the inversion is to expand the rhs in a Taylor series and then invert the series term by term [9]. We used \textit{Mathematica} to perform this procedure up to \( O(18) \), but the resulting series does not converge for values in the argument larger than 1. The reason is the small convergence ratio for the Taylor expansion of \( \arcsin(\cdot) \).

An easier way to perform the inversion is to evaluate \( \theta_m(\alpha) \) using (8) and plot the points \( (\theta_m(\alpha), \alpha) \); the result is shown in figure 2. Note that this method is exact in the sense that we can obtain as many pairs of numbers as we need; a function is, finally, a one-to-one relation between two sets of real numbers. This is important for students who frequently do not have the clear meaning of the expression of a function \( f : \mathbb{R} \to \mathbb{R} \), but within the precision of our rulers and clocks. They will deal with numerical calculations of complex problems.

Another task is to obtain the derivative \( d\alpha/d\theta \) since it will be needed in the following sections. To this end, we note that both functions increase monotonically and their derivatives are not zero, except at the interval end. Hence, we can use the inverse function theorem to obtain \( d\alpha/d\theta = (d\theta/d\alpha)^{-1} \). The result is shown in figure 3(a) and the second derivative in
Figure 2. The angle $\theta$ as a function of the launch angle $\alpha$ for two different values of $\varepsilon$ and their inverses.

Figure 3. (a) First and (b) second derivatives of $\alpha$ as a function of $\theta$ for various values of the parameter $\varepsilon$. Note that major changes occur for $\theta < \pi/4$.

The second derivative is calculated using an approximation to the slope to the function previously calculated and using 10 000 points in the interval $[0, \pi/2]$. A smaller number of points could be considered.

4. The curvature of $C_m$

4.1. Polar angle parameterization

In the drag-free situation, $C_m$ is an ellipse and its description is well known; however, in the presence of linear drag this is not the case. We do not expect that the locus could be a
conic section and henceforth we need to characterize it. It is usual to consider the curvature, radius of curvature or the length of arc in order to characterize a locus. In the present case we consider the curvature of \( C_m \) in both parameterizations, first with the polar angle \( \theta_m \) and second with the launch angle \( \alpha \). We left the calculation of the length of arc to a later work since the presentation became increasingly complex, and the goal of the present section is to understand \( C_m \) and to illustrate the way it can be done using the Lambert \( W \) function. Here and in the rest of the section we drop, for clarity, the subindex \( m \) in \( r_m \) and \( \theta_m \).

The corresponding formula for the curvature \( K \) for polar coordinates is [10]

\[
K = \frac{1}{\rho} \frac{\tilde{r}^2 + 2\tilde{r}_\theta^2 - \tilde{r}_{\theta\theta}}{\left(\tilde{r}^2 + \tilde{r}_\theta^2\right)^{3/2}},
\]

(13)
in order to use (11). Here the subindex \( \theta \) corresponds to a derivative with respect to that variable.

A direct calculation on the drag-free \( \tilde{r}(\theta) \) of (12) yields

\[
K_0 = \frac{1}{2\sqrt{2}\rho} \frac{\left(5 - 3 \cos 2\theta\right)^6}{\left(1 + 3 \sin^2 \theta\right)^4 \left(47 - 60 \cos 2\theta + 21 \cos 4\theta\right)^{3/2}},
\]

(14)
which has a maximum at \( \theta = 1/2 \arctan(4/3) \approx 0.464 \). A graph of this result appears in figure 4 (red line). Note that this value is different from that we obtain when we evaluate \( \theta(\alpha = \pi/4) = 1/2 \), the launch angle of maximum range. The maximum curvature occurs at a smaller angle than the angle of maximum range. It is interesting to note that the angle \( 2\theta = \arctan(4/3) \) corresponds to a triangle whose sides fulfill the relation \( 3^2 + 4^2 = 5^2 \), a Pythagorean triple.

Using the numerical results for \( \alpha(\theta) \) from the previous section, it is possible to carry out the calculation of \( K \) (see figure 4) by performing the derivatives of \( \tilde{r} \) from (11) in a direct form and evaluating numerically the required values of \( \alpha \) and its derivatives. For the derivatives of \( W(z) \) we used the expressions [1]

\[
\frac{d}{dx} W(x) = \frac{W(x)}{x(1+W(x))}, \quad \text{and} \quad \frac{d^2 W(x)}{dx^2} = -\frac{\exp(-2W(x))(W(x)+2)}{(1+W(x))^3}.
\]

(15)

Figure 4. Curvature of \( C_m(\epsilon) \) using the polar angle as the parameter from (13). The red line indicates the corresponding result for the ellipse, (14). The maximum occurs at \( \theta = 1/2 \arctan(4/3) \), different from the value of the maximum range \( \alpha = \pi/4 \).
Using this method, we obtained good results for values of $\varepsilon$ up to $\approx 1$, but we need to calculate the arguments of the Lambert $W$ function near the limit $\varepsilon = -1/e$ for larger values of $\varepsilon$. The reliability of our numerical result was tested comparing the first and second derivatives of $r(\theta)$ with those corresponding to the ellipse.

As can be seen in figure 4, $K$ also presents a maximum in all the cases that can be calculated. We left to a later work the analysis of the maxima distribution as a function of the perturbative parameter, which is not the case for the curvature with $\alpha$ parameterization as we shall see in the next section.

4.2. Launch angle parameterization

For the launch angle parameterization of $C_m$ we shall use the expression [10]

$$\kappa = \frac{x'y'' - y'x''}{(x'^2 + y'^2)^{3/2}} \tag{16}$$

to calculate the curvature from the rectangular form of (5) and (6), where the primes denote the derivative with respect to the parameterization variable, $\alpha$ in this case.

A direct calculation yields

$$\kappa = \frac{\sqrt{2}}{\rho} \frac{P_1(\varepsilon)}{(\sqrt{P_2(\varepsilon)})^3} \times (1 + \varepsilon \sin \alpha)^2. \tag{17}$$

with

$$P_1(\varepsilon) = 16 + 6\varepsilon^2 - 8\varepsilon^2 \cos 2\alpha + 2\varepsilon^2 \cos 4\alpha + 30\varepsilon \sin \alpha - \varepsilon \sin 3\alpha + \varepsilon \sin 5\alpha, \tag{18}$$

and

$$P_2(\varepsilon) = 5 + 3 \cos 4\alpha + 3\varepsilon^2 - 4\varepsilon^2 \cos 2\alpha + \varepsilon^2 \cos 4\alpha + 10\varepsilon \sin \alpha - 5\varepsilon \sin 3\alpha + \varepsilon \sin 5\alpha. \tag{19}$$

In the limit $\varepsilon \to 0$, we recover the drag-free curvature

$$\kappa_0 = 16\sqrt{2} \frac{1}{\rho} (5 + 3 \cos 4\alpha)^{3/2}, \tag{20}$$

which has a maximum at $\alpha = \pi/4$ in the interval $\alpha \in [0, \pi/2]$, as expected. A plot of $\rho\kappa(\alpha)$ for several values of $\varepsilon$ beginning at zero and ending at $\varepsilon = 10$ appears in figure 5(a). The drag-free case appears in red. Note that both extremal values increase for increasing $\varepsilon$ value as $\rho\kappa(0) \approx (16 + 6\varepsilon - 6\varepsilon^2)$ and $\rho\kappa(\pi/2) \approx \varepsilon$. Note that for small $\varepsilon$, $\kappa$ crosses the drag-free curvature $\kappa_0$.

The angles $\alpha^\ast$ at which $\kappa$ attain their maxima are obtained in the usual way and we need to solve, numerically or graphically, a large trigonometrical polynomial in $\alpha^\ast$. In figure 5(b) the calculated values of $\alpha^\ast$ as a function of $\varepsilon$ appear. This angle is between the optimal angle for the maximum range (red circles and blue crosses) and the angle for the greatest forward skew (dashed line) [12]:

$$\alpha_{\text{skew}} = \arcsin \left[ \frac{1}{3\varepsilon} \left( (D_+/2)^{1/3} + (D_-/2)^{1/3} - 2 \right) \right], \tag{21}$$

where $D_\pm = \pm 3\varepsilon\sqrt{3(27\varepsilon^2 - 32) + 27\varepsilon^2 - 16}$, valid for $\varepsilon > \sqrt{32/27}$. In figure 5(b) the optimal angles are drawn; the red circles indicate the exact result in terms of the Lambert $W$ function [4, 11]

$$\alpha_{\text{max,s}} = \arcsin \left[ \frac{\varepsilon}{\exp \left( W \left( \frac{\varepsilon}{e} \right) + 1 \right) - 1} \right]. \tag{22}$$

2 In [12], the author comments that for $\varepsilon = 1$ we obtain the special value $\alpha_{\text{skew}} = \sin^{-1}(1/\phi)$ with $\phi = (1 + \sqrt{5})/2$, the golden ratio. However, the solution that appears in the article is no longer valid for $\varepsilon = 1$. If the solution corresponds to another root of the equation, this special value corresponds to the case when the initial speed is equal to the limit speed $b/g$. This makes this fact much more intriguing.
and the approximated result [3]

$$\alpha_{\text{max}, W} = \frac{W(\varepsilon^2/\varepsilon)}{\varepsilon}. \quad (23)$$

Both expressions are equivalent for large $\varepsilon$ but differ at small $\varepsilon$, as expected.

Meanwhile the difference between these angles at a small perturbative parameter is unimportant; at large $\varepsilon$ the behaviour of the corresponding trajectories is different. One reason is the large asymmetry in the locus formed by the set of apexes. In figure 6 we plot $C_m(\varepsilon)$ and the corresponding trajectories for the different launch angles for $\varepsilon = 80$. The blue line corresponds to $C_m(\varepsilon)$; note that the maximum height is $y \sim 0.12$ in contrast to $y \sim 5.0$ for the drag-free case; however, this can be the case of a small friction parameter $b$ and large initial velocity $V_0$ giving a large $\varepsilon$ value. The black line represents the orbit launched at $\alpha^*$, the red line the corresponding one to attain the maximum range and the blue dashed line the orbit with maximum skewness.

5. The synchronous curve

In MacMillan’s book [7], the calculation of the synchronous curve was done for the drag-free case. This curve would be formed if many projectiles were fired simultaneously from the same point, each with a different launch angle and the same initial speed $V_0$. The locus will be a circle of radius $V_0t$ and centre at the point $(0, -\frac{1}{2}gt^2)$, i.e.

$$x^2 + \left(y + \frac{1}{2}gt^2\right)^2 = V_0^2t^2. \quad (24)$$

Here, we demonstrate that a circle is a synchronous curve in the linear drag case as well. Following [7], we eliminate the launch angle $\alpha$ from the position solutions; in the present case
they are (2) and (3). We write $\cos \alpha$ and $\sin \alpha$ and rearrange the terms to give

$$
\cos \alpha = \frac{b}{V_0} \frac{x}{1 - \exp(-b)} \quad \text{and} \quad \sin \alpha = \frac{b}{V_0} \frac{y - \frac{g}{2b} (1 - bt - \exp(-bt))}{1 - \exp(-bt)}.
$$

Substituting these expressions into the identity $\cos^2 \alpha + \sin^2 \alpha = 1$, we obtain

$$
x^2 + (y - y_c(t))^2 = R^2(t), \tag{25}
$$

with

$$
y_c(t) = \frac{g}{2b} (1 - bt - \exp(-bt)) \tag{26}
$$

the centre and

$$
R(t) = \frac{V_0}{b} (1 - \exp(-bt)) \tag{27}
$$

the radius. In order to recover the case where $b \to 0$, we consider a Taylor expansion for the exponential up to second order in the exponential in (26) and up to first order in (27). The fact that this circle exists in the presence of a drag force is remarkable.

6. Conclusions

We obtained an explicit form for the locus $C_m$, composed of the set of maxima of all the trajectories of a projectile launched at an initial velocity $V_0$ and in the presence of a linear drag force, $-mb\vec{v}$, i.e. $C_m$ is the locus of the apexes. In polar coordinates, $C_m$ is written in terms of the principal branch of the Lambert $W$ function for negative values. This represents the parameterization of the curve by the polar angle $\theta_m$ only and gives $C_m$ in a closed form exhibiting the strong relationship between the Lambert $W$ function and the linear drag problem. The curvature of $C_m$ is calculated for different values of the dimensionless parameter $\varepsilon \equiv bV_0/g$ in two parameterizations. The first one, the polar parameterization, shows a maximum that slightly departs from the drag-free case in $\theta = 1/2 \arctan(4/3)$. A wider exploration of the functional dependence with respect to $\varepsilon$ is pending due to numerical accuracy in the calculation.

**Figure 6.** Orbits launched at different angles and the corresponding locus $C_m(\varepsilon)$ for $\varepsilon = 80$. See text for explanation.
of the Lambert $W$ function near the limit at $x = -1/e$. In the case of a parameterization using the launch angle $\alpha$ there is no such restriction. In this case the curvature is calculated for a wide range of the parameter $\varepsilon$, yielding maximum at angle values larger than those corresponding to the maximum range. A comparison with the maximum skewness angle [12] is also done and the difference is larger than the previous one. As an addendum, we demonstrate that the synchronous curve in this case is a circle, as in the drag-free case. Besides the results obtained in this work, the increasing use of built-in special functions, as well as an extensive use of numerical results for solving problems, must be part of the science bachelors’ curricula.

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Appendix A. Polar form of an ellipse with the origin at the bottom

The ellipse canonical form or the polar form with the origin considered in one of the foci are standard knowledge. In the present case, however, we are required to consider the origin of the coordinates located in the bottom of the ellipse, since in the presence of a drag force the launching origin is the only invariant point where we change the drag force value. To obtain the ellipse form, we depart from the drag-free solutions at the locus of the apexes:

$$x_m = \rho \sin \alpha \cos \alpha, \quad \text{and} \quad y_m = \frac{\rho}{2} \sin^2 \alpha, \quad (A.1)$$

where $\rho \equiv \frac{V_0^2}{g}$. With the help of the trigonometric relations $\sin 2\alpha = 2 \sin \alpha \cos \alpha$ and $2 \sin^2 \alpha = 1 - \cos 2\alpha$, we can transform the above equations into

$$\sin 2\alpha = \frac{2x_m}{\rho}, \quad \text{and} \quad \cos 2\alpha = 1 - \frac{4y_m}{\rho}. \quad (A.2)$$

Taking the squares in both expressions, summing them and arranging terms, we arrive at

$$\frac{4r_m}{\rho} \left( \frac{r_m}{\rho} (1 + 3 \sin^2 \theta_m) - 2 \sin \theta_m \right) = 0, \quad (A.3)$$

where we used the polar coordinates $r_m$ and $\theta_m$. The solutions are $r_m = 0$ and

$$r_m(\theta_m) = \frac{2\rho \sin \theta_m}{1 + 3 \sin^2 \theta_m}. \quad (A.4)$$

The second one is the required form for the ellipse.

Appendix B. Drag-free limit for $\tilde{r}$

In order to obtain the drag-free limit for the locus $C_m$ given in (11), we note that

$$\tan \theta = (1/2) \tan \alpha \quad (B.1)$$

and that $f(\alpha) = \sin \alpha(\theta_m) \cos \alpha(\theta_m)$. The first expression is obtainable from the drag-free solutions (A.1), and the second is obtained by setting $b \to 0$ in (9).
The expansion in a power series of the Lambert $W$ function up to first order is simply the identity \([1]\) and hence
\[
\tilde{r}(\theta_m) = -\frac{1}{\epsilon^2 \sin \theta_m} W(-\epsilon^2 \tan \theta_m f(\alpha)) \approx -\frac{1}{\epsilon^2 \sin \theta_m} (-\epsilon^2 \tan \theta_m f(\alpha)) = 2 \sin \theta \cos^2 \alpha \cos^2 \theta_m,
\] (B.2)
where we used relation \((B.1)\) to obtain the last line. Using the trigonometric identity \(\sec^2 \alpha - \tan^2 \alpha = 1\) and \((B.1)\), we obtain
\[
\frac{\cos^2 \theta_m}{\cos^2 \alpha} = 1 + 3 \sin^2 \theta_m.
\] (B.3)
Using this result in the expression of \(\tilde{r}\) we obtain the desired result \((12)\).

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