Charged massive scalar field configurations supported by a spherically symmetric charged reflecting shell

Shahar Hod
The Ruppin Academic Center, Emeq Hefer 40250, Israel
and
The Hadassah Institute, Jerusalem 91010, Israel
(Dated: October 1, 2018)

The physical properties of bound-state charged massive scalar field configurations linearly coupled to a spherically symmetric charged reflecting shell are studied analytically. To that end, we solve the Klein-Gordon wave equation for a static scalar field of proper mass $\mu$, charge coupling constant $q$, and spherical harmonic index $l$ in the background of a charged shell of radius $R$ and electric charge $Q$. It is proved that the dimensionless inequality $\mu R < \sqrt{(qQ)^2 - (l + 1/2)^2}$ provides an upper bound on the regime of existence of the composed charged-spherical-shell-charged-massive-scalar-field configurations. Interestingly, we explicitly show that the discrete spectrum of shell radii $\{R_n(\mu, qQ, l)\}_{n=0}^{\infty}$ which can support the static bound-state charged massive scalar field configurations can be determined analytically. We confirm our analytical results by numerical computations.

I. INTRODUCTION

The influential no-hair theorems [1] have revealed the interesting fact that spherically symmetric asymptotically flat black holes cannot support static massive scalar field configurations in their exterior regions [2–4]. Motivated by this well known property of spherically symmetric static black holes, we have recently [5] extended this no-scalar-hair theorem to the regime of regular curved spacetimes. In particular, it was proved in [5] that spherically symmetric compact reflecting [7] stars cannot support regular self-gravitating neutral scalar field configurations in their exterior regions.

One naturally wonders whether this no-scalar-hair behavior [5] is a generic feature of compact reflecting objects? In particular, we raise here the following physically intriguing question: Can regular static charged massive scalar field configurations be supported by a compact spherically symmetric charged reflecting object? In order to address this interesting question, in this paper we shall study, using analytical techniques, the Klein-Gordon wave equation for a static linearized scalar field of proper mass $\mu$ and charge coupling constant $q$ in the background of a spherically symmetric charged reflecting shell of radius $R$ and electric charge $Q$.

Our results (to be proved below) reveal the fact that, for given parameters $\{\mu, q, l\}$ [8] of the charged massive scalar field, there exists a discrete set of shell radii $\{R_n(\mu, qQ, l)\}_{n=0}^{\infty}$ which can support the static bound-state charged massive-scalar-field configurations. In particular, as we shall explicitly show below, the regime of existence of these composed charged-spherical-shell-charged-massive-scalar-field configurations is restricted by the characteristic inequality $(qQ)^2 > (\mu R)^2 + (l + 1/2)^2$ [8, 10]. This relation implies, in particular, that spatially regular bound-state configurations made of neutral scalar fields [5] cannot be supported by a spherically symmetric compact reflecting object.

II. DESCRIPTION OF THE SYSTEM

We shall analyze the physical properties of a scalar field $\Psi$ of proper mass $\mu$ and charge coupling constant $q$ which is linearly coupled to a spherically symmetric charged shell of radius $R$. The shell is assumed to have negligible self-gravity:

$$M, Q \ll R,$$

where $\{M, Q\}$ are the proper mass and electric charge of the shell, respectively.

Decomposing the static scalar field $\Psi$ in the form [11]

$$\Psi(r, \theta, \phi) = \sum_{lm} e^{im\phi} S_{lm}(\theta) R_{lm}(r),$$

one finds that the spatial behavior of the radial scalar eigenfunctions $\{R_{lm}(r)\}$ in the spacetime region outside the charged spherical shell is governed by the ordinary differential equation [12, 16]

$$\frac{d}{dr} \left( r^2 \frac{dR_{lm}}{dr} \right) + U R_{lm} = 0,$$
where
\[ U = (qQ)^2 - (\mu r)^2 - K_l. \]  

(4)

Here \( K_l = l(l + 1) \) is the characteristic eigenvalue of the angular scalar eigenfunctions \( \{S_{lm}(\theta)\} \).

The bound-state (spatially localized) charged massive scalar field configurations that we shall analyze below are characterized by asymptotically decaying eigenfunctions
\[ \Psi(r \to \infty) \sim \frac{1}{r} e^{-\mu r} \]  

(5)

at spatial infinity. In addition, the presence of the spherically symmetric charged reflecting shell at \( r = R \) dictates the boundary condition
\[ \Psi(r = R) = 0 \]  

(6)

for the characteristic scalar eigenfunctions.

In the next section we shall explicitly show that the radial differential equation (3), which determines the spatial behavior of the characteristic radial eigenfunctions \( \{R_{lm}(r)\} \) of the charged massive scalar fields in the background of the charged spherical shell, is amenable to an analytical treatment.

III. THE RESONANCE EQUATION FOR THE COMPOSED CHARGED-SPHERICAL-SHELL-CHARGED-MASSIVE-SCALAR-FIELD CONFIGURATIONS

As we shall now show, the characteristic radial equation (3) for the charged massive scalar eigenfunction \( R_{lm}(r) \) can be solved analytically. Defining the new radial function
\[ \psi_{lm} = r^{1/2} R_{lm} \]  

(7)

and using the dimensionless radial coordinate
\[ z = \mu r, \]  

(8)

one obtains the differential equation
\[ z^2 \frac{d^2 \psi}{dz^2} + z \frac{d\psi}{dz} - \left[ z^2 + (l + \frac{1}{2})^2 - (qQ)^2 \right] \psi = 0 \]  

(9)

for the characteristic radial scalar eigenfunction \( \psi \).

The general solution of the radial differential equation (9) can be expressed in terms of the modified Bessel functions (see Eq. 9.6.1 of [17])
\[ \psi(z) = A \cdot K_\nu(z) + B \cdot I_\nu(z), \]  

(10)

where
\[ \nu^2 = (l + \frac{1}{2})^2 - (qQ)^2 \]  

(11)

and \( \{A, B\} \) are normalization constants. The asymptotic large-\( r \) (large-\( z \)) behavior of the radial solution (10) is given by (see Eqs. 9.7.1 and 9.7.2 of [17])
\[ \psi(z \to \infty) = A \cdot \sqrt{\frac{\pi}{2 z}} e^{-z} + B \cdot \frac{1}{\sqrt{2 \pi z}} e^z. \]  

(12)

Taking cognizance of the boundary condition (5), which characterizes the asymptotic spatial behavior of the bound-state (spatially localized) scalar configurations, one deduces that the coefficient of the exploding exponent in (12) must vanish:
\[ B = 0. \]  

(13)
One therefore concludes that the bound-state configurations of the charged massive scalar fields in the background of the charged spherical shell are characterized by the radial eigenfunction

\[ \psi(r) = A \cdot K_\nu(\mu r) . \]  

(14)

Taking cognizance of Eq. (14) and the boundary condition (6) which is dictated by the presence of the spherically symmetric reflecting shell, one finds the characteristic resonance equation

\[ K_\nu(\mu R) = 0 \]  

(15)

for the composed static charged-spherical-shell-charged-massive-scalar-field configurations. Interestingly, as we shall show below, the resonance condition (15) determines the discrete set of shell radii \( \{ R_n(\mu, qQ, l) \}_{n=0}^{\infty} \) which can support the bound-state charged massive scalar field configurations.

In the next section we shall prove that the resonance condition (15) can only be satisfied in the bounded regime

\[ (qQ)^2 > (\mu R)^2 + (l + \frac{1}{2})^2. \]  

(16)

The necessary inequality (16), to be proved below, implies in particular that spatially regular static bound-state configurations made of neutral scalar fields cannot be supported by a spherically symmetric compact reflecting object.

**IV. THE DOMAIN OF EXISTENCE OF THE CHARGED MASSIVE SCALAR HAIR**

Using the boundary conditions (5) and (6), one concludes that the scalar eigenfunction \( \psi \), which characterizes the radial behavior of the charged massive scalar fields, must have (at least) one extremum point, \( z = z_{\text{peak}} \), outside the spherically symmetric charged reflecting shell. In particular, at this extremum point the radial scalar eigenfunction \( \psi \) is characterized by the relations

\[ \{ \frac{d\psi}{dz} = 0 \text{ and } \frac{d^2\psi}{dz^2} < 0 \} \text{ for } z = z_{\text{peak}}. \]  

(17)

Substituting (17) into (9), one finds the characteristic inequality

\[ z_{\text{peak}}^2 + (l + \frac{1}{2})^2 - (qQ)^2 < 0. \]  

(18)

Taking cognizance of (8) and using the inequality \( r_{\text{peak}} > R \), one finds from (18) that the composed charged-spherical-shell-charged-massive-scalar-field configurations are characterized by the inequality (16). In particular, this inequality sets the upper bound

\[ \mu R < \sqrt{(qQ)^2 - (l + \frac{1}{2})^2} \]  

(19)

on the radius of the central charged supporting shell.

For later purposes, it is important to point out that the inequality (19) [or equivalently, the inequality (16)] implies that the static charged massive scalar field configurations are characterized by the relation \( \nu^2 < 0 \) [see Eq. (11)], which implies

\[ i\nu \in \mathbb{R}. \]  

(20)

**V. BOUND-STATE RESONANCES OF THE CHARGED MASSIVE SCALAR FIELDS IN THE BACKGROUND OF THE CHARGED SPHERICAL SHELL**

The analytically derived resonance condition (15) for the composed static charged-spherical-shell-charged-massive-scalar-field configurations can easily be solved numerically. In particular, one finds that, for given parameters \( \{ \mu, q, l \} \) of the charged massive scalar field, there exists a discrete set of shell radii,

\[ \cdots R_2 < R_1 < R_0 \equiv R_{\text{max}} < \frac{\nu}{\mu}, \]  

(21)
TABLE I: Composed charged-spherical-shell-charged-massive-scalar-field configurations. We display, for various values of the dimensionless physical parameter $\nu$ [see Eq. (11)], the largest possible dimensionless radius $\mu R^{\text{max}}(\nu)$ of the charged reflecting shell which can support the static bound-state charged massive scalar field configurations. One finds that $\mu R^{\text{max}}(\nu)$ is a monotonically increasing function of $|\nu|$. Note, in particular, that the dimensionless radii of the charged supporting shells are bounded from above by the inequality $\mu R^{\text{max}}(\nu) < |\nu|$ [see Eqs. (11) and (19)].

| $\nu$   | $\mu R^{\text{max}}(\nu)$ |
|---------|-----------------------------|
| $i$     | 0.0640                      |
| $20i$   | 15.343                      |
| $40i$   | 33.955                      |
| $60i$   | 52.999                      |
| $80i$   | 72.244                      |
| $100i$  | 91.609                      |

which can support the static bound-state charged massive scalar field configurations. In Table I we display the largest possible dimensionless radius $\mu R^{\text{max}}(\nu)$ of the supporting charged shell for various values of the dimensionless physical parameter $\nu$ [21]. From the data presented in Table I one finds that $\mu R^{\text{max}}(\nu)$ is a monotonically increasing function of $|\nu|$. Note, in particular, that the values of the dimensionless supporting radii $\mu R^{\text{max}}(\nu)$ conform to the analytically derived upper bound [19].

As we shall now show, the characteristic resonance equation [15] for the composed static charged-shell-charged-field configurations can be solved analytically in the asymptotic regimes [22]

$$\mu R \ll 1$$

and [23, 25]

$$\mu R \gg 1.$$  

To this end, we first note that the resonance condition [15] can be expressed in the from (see Eq. 9.6.2 of [17])

$$I_{\nu}(\mu R) = I_{-\nu}(\mu R).$$

for the dimensionless physical quantity $\mu R$ in the small-radius regime (22). From Eqs. (11) and (26) one finds the expression [29, 30]

$$\mu R(n) = 2 \left( \frac{\Gamma(\nu)}{\Gamma(-\nu)} \right)^{1/2|\nu|} \times e^{-\pi(n + \frac{1}{2})^2/|\nu|}; \quad n \in \mathbb{Z}$$

for the discrete spectrum of shell radii which can support the static bound-state charged massive scalar field configurations in the small-radius regime (22).

A. The regime $\mu R \ll 1$ of small charged supporting shells

Using the small argument ($\mu R \ll 1$) approximation

$$I_{\nu}(z) \approx \frac{(z/2)^\nu}{\Gamma(\nu + 1)} \quad \text{for} \quad z \to 0$$

of the modified Bessel function (see Eq. 9.6.7 of [17]), one finds from (24) the characteristic resonance equation [28]

$$\left( \frac{1}{2\mu R} \right)^{2\nu} = - \frac{\Gamma(\nu)}{\Gamma(-\nu)}$$

for the dimensionless physical quantity $\mu R$ in the small-radius regime (22). From Eqs. (11) and (26) one finds the expression [29, 30]

$$\mu R(n) = 2 \left[ \frac{\Gamma(\nu)}{\Gamma(-\nu)} \right]^{1/2|\nu|} \times e^{-\pi(n + \frac{1}{2})^2/|\nu|}; \quad n \in \mathbb{Z}$$

for the discrete spectrum of shell radii which can support the static bound-state charged massive scalar field configurations in the small-radius regime (22).

B. The regime $\mu R \gg 1$ of large charged supporting shells

Using the uniform asymptotic expansion

$$I_{\nu}(\nu z) \approx \frac{e^{\nu \eta}}{\sqrt{2\pi \nu (1 + z^2)^{1/4}}}, \quad \text{with} \quad \eta = \sqrt{1 + z^2 + \ln[z/(1 + \sqrt{1 + z^2})]} \quad \text{for} \quad |\nu| \to \infty$$

of the modified Bessel function (see Eqs. 9.7.7 and 9.7.11 of [17]), one finds from (24) the characteristic resonance equation

$$\frac{xe^{\sqrt{1-x^2}}}{1 + \sqrt{1-x^2}} = (-i)^{1/2|\nu|}$$

(28)
for the physical quantity $\mu R$ in the large-radius regime \[23\], where here we have used the dimensionless parameter

$$x \equiv \frac{\mu R}{|\nu|}.$$  

(30)

Substituting \[31\]

$$x = 1 - \epsilon \quad \text{with} \quad 0 < \epsilon \ll 1 \quad (31)$$

into \[29\], one obtains the leading order equation \[24, 32\]

$$1 - \frac{4\sqrt{2}}{3} \cdot \epsilon^{3/2} + O(\epsilon^{5/2}) = 1 - \frac{\pi(1 + 4n)}{2|\nu|} + O(|\nu|^{-2}) \quad ; \quad n = 0, 1, 2, ... \quad (32)$$

for $\epsilon$, which yields

$$\epsilon = \left( \frac{3\pi}{8\sqrt{2} \cdot |\nu|} \right)^{2/3}. \quad (33)$$

Taking cognizance of Eqs. \[30\], \[31\], and \[33\], one finds

$$\mu R(n) = |\nu| \left[ 1 - \left( \frac{3\pi}{8\sqrt{2} \cdot |\nu|} \right)^{2/3} \right] \quad ; \quad n = 0, 1, 2, ... \quad (34)$$

for the discrete spectrum of shell radii which can support the static bound-state charged massive scalar field configurations in the large-radius regime \[23\].

### VI. NUMERICAL CONFIRMATION

It is of physical interest to verify the validity of the analytically derived formulas \[27\] and \[34\] for the discrete set of charged shell radii which can support the static bound-state charged massive scalar field configurations. In Table \[1\] we display the dimensionless discrete radii $\mu R^{\text{analytical}}$ of the charged reflecting shells as obtained from the analytically derived formula \[24\] in the small-radius regime \[22\]. We also display the corresponding dimensionless radii $\mu R^{\text{numerical}}$ of the charged shells as obtained from a direct numerical solution of the characteristic resonance equation \[15\] \[21\]. One finds a remarkably good agreement \[33\] between the approximated radii of the charged supporting shells [as calculated from the analytically derived formula \[27\]] and the corresponding exact radii of the charged shells [as obtained numerically from the analytically derived resonance equation \[15\]].

| Formula          | $\mu R(n = -1)$ | $\mu R(n = 0)$ | $\mu R(n = 1)$ | $\mu R(n = 2)$ | $\mu R(n = 3)$ | $\mu R(n = 4)$ |
|------------------|-----------------|----------------|----------------|----------------|----------------|----------------|
| Analytical [Eq. \(27\)] | 2.2888          | 1.2210         | 0.6514         | 0.3475         | 0.1854         | 0.0989         |
| Numerical [Eq. \(15\)] | 2.4245          | 1.2393         | 0.6541         | 0.3479         | 0.1855         | 0.0989         |

**TABLE II**: Composed charged-spherical-shell-charged-massive-scalar-field configurations with $\nu = 5i$ [see Eq. \(11\)]. We present the dimensionless discrete radii $\mu R$ of the central charged reflecting shells as calculated from the analytically derived formula \[27\]. We also present the corresponding dimensionless radii of the central charged shells as obtained from a direct numerical solution of the characteristic resonance equation \[15\]. One finds a remarkably good agreement \[33\] between the approximated radii of the charged reflecting shells [as calculated from the analytically derived formula \[27\]] and the corresponding exact radii of the charged shells [as obtained numerically from the analytically derived resonance condition \[15\]].

In Table \[11\] we present the dimensionless discrete radii $\mu R^{\text{analytical}}$ of the charged reflecting shells as deduced from the analytically derived formula \[34\] in the large-radius regime \[23\]. We also present the corresponding dimensionless radii $\mu R^{\text{numerical}}$ of the charged shells as obtained from a direct numerical solution of the resonance equation \[15\] \[21\]. Again, one finds a remarkably good agreement between the approximated radii of the charged reflecting shells [as calculated from the analytically derived formula \[34\]] and the corresponding exact radii of the charged shells [as obtained numerically from the analytically derived resonance condition \[15\]].
TABLE III: Composed charged-spherical-shell-charged-massive-scalar-field configurations with $\nu = 200i$ [see Eq. (11)]. We display the dimensionless discrete radii $\mu R$ of the central charged reflecting shells as calculated from the analytically derived formula (34). We also display the corresponding dimensionless radii of the charged supporting shells as obtained from a direct numerical solution of the resonance condition (15). One finds a remarkably good agreement between the approximated radii of the charged supporting shells [as calculated from the analytically derived formula (34)] and the corresponding exact radii of the charged shells [as obtained numerically from the analytically derived resonance condition (15)].

| Formula | $\mu R(n = 0)$ | $\mu R(n = 1)$ | $\mu R(n = 2)$ | $\mu R(n = 3)$ | $\mu R(n = 4)$ | $\mu R(n = 5)$ |
|---------|----------------|----------------|----------------|----------------|----------------|----------------|
| Analytical [Eq. (34)] | 194.82 | 184.86 | 177.60 | 171.37 | 165.77 | 160.59 |
| Numerical [Eq. (15)] | 189.32 | 181.56 | 175.36 | 169.99 | 165.17 | 160.75 |

The main results derived in this paper and their physical implications are as follows:

1. We have proved that the upper bound [see Eqs. (11), (19), and (20)]
   $$
   \mu R < |\nu| ; \quad iv \equiv \sqrt{(qQ)^2 - (l + \frac{1}{2})^2} \in \mathbb{R}
   $$
   on the dimensionless radius of the charged reflecting shell provides a necessary condition for the existence of composed static charged-spherical-shell-charged-massive-scalar-field configurations.

2. It was shown that, for given physical parameters $\{\mu, q, l\}$ of the charged massive scalar field, there exists a discrete set of charged shell radii $\{R_n(\mu, qQ, l)\}_{n=0}^{\infty}$ which can support the static spatially regular bound-state charged massive scalar field configurations. In particular, we have proved that the characteristic discrete set of supporting shell radii is determined by the analytically derived compact resonance condition $K_v(\mu R) = 0$ [see Eq. (13)].

3. We have explicitly shown that the physical properties of the composed static charged-shell-charged-massive-scalar-field configurations can be studied analytically in the asymptotic regimes of small [see Eq. (22)] and large [see Eq. (23)] charged supporting shells. In particular, we have provided compact analytical formulas for the discrete spectra of supporting shell radii in the asymptotic regimes $\mu R \ll 1$ [22] and $\mu R \gg 1$ [23] [see Eqs. (27) and (34), respectively].

4. Former analytical studies have revealed the interesting fact that spherically symmetric neutral reflecting stars share the no-scalar-hair property with asymptotically flat black holes [1, 4]. On the other hand, the results derived in the present paper have revealed the fact that compact charged reflecting objects have a much richer phenomenology. In particular, while the elegant no-hair theorems of Bekenstein and Mayo [31] (see also [33]) have shown that spherically symmetric asymptotically flat charged black holes cannot support static charged massive scalar field configurations, in the present work we have explicitly proved that charged reflecting shells can support static bound-state charged massive scalar field configurations in their exterior regions.

ACKNOWLEDGMENTS

This research is supported by the Carmel Science Foundation. I would like to thank Yael Oren, Arbel M. Ongo, Ayelet B. Lata, and Alona B. Tea for helpful discussions.

[1] J. E. Chase, Commun. Math. Phys. 19, 276 (1970); J. D. Bekenstein, Phys. Rev. Lett. 28, 452 (1972); C. Teitelboim, Lett. Nuovo Cimento 3, 326 (1972); J. D. Bekenstein, Physics Today 33, 24 (1980).
It is worth mentioning that it has recently been proved \[4\] that non-spherically symmetric rotating black holes can support stationary (rather than static) regular massive scalar field configurations in their exterior regions.

C. A. R. Herdeiro and E. Radu, Phys. Rev. Lett. 112, 221101 (2014); C. A. R. Herdeiro and E. Radu, Phys. Rev. D 89, 124018 (2014); C. A. R. Herdeiro and E. Radu, Int. J. Mod. Phys. D 23, 1442014 (2014); C. L. Benone, L. C. B. Crispino, C. Herdeiro, and E. Radu, Phys. Rev. D 90, 104024 (2014); C. Herdeiro, E. Radu, and H. Rúnarsson, Phys. Lett. B 739, 302 (2014); C. Herdeiro and E. Radu, Class. Quantum Grav. 32, 144001 (2015); C. A. R. Herdeiro and E. Radu, Int. J. Mod. Phys. D 24, 1542014 (2015); C. A. R. Herdeiro and E. Radu, Int. J. Mod. Phys. D 24, 1544022 (2015); J. C. Degollado and C. A. R. Herdeiro, Gen. Rel. Grav. 45, 2483 (2013); P. V. P. Cunha, C. A. R. Herdeiro, E. Radu, and H. F. Rúnarsson, Phys. Rev. Lett. 115, 211102 (2015); C. A. R. Herdeiro, E. Radu, and H. F. Rúnarsson, Phys. Rev. D 92, 084059 (2015); Y. Brihaye, C. Herdeiro, and E. Radu, Phys. Lett. B 760, 279 (2016).

S. Hod, to be published.

Here we have used the relation \[
\frac{\mu}{\hbar} \approx \frac{q}{\ell_s} \approx (\mu/\hbar) \leq 1
\]
for brevity. Here \(L_m(z)\) and \(K_n(z)\) are respectively the modified Bessel function of the first kind and the modified Bessel function of the second kind.

The numerically computed roots of the modified Bessel function \(K_0(x)\) [see the analytically derived resonance equation \[15\]] were obtained using the WolframAlpha Computational Knowledge Engine.

Note that the regime \[22\] corresponds to charged spherical shells whose radii are much smaller than the characteristic Compton length \(\hbar/\mu\) of the charged massive scalar field.

Note that the regime \[23\] corresponds to charged spherical shells whose radii are much larger than the characteristic Compton length \(\hbar/\mu\) of the charged massive scalar field.

Note that the strong inequality \(\mu R \gg 1\) corresponds to \(|\nu| \gg 1\) [see Eqs. \(11\) and \(19\)].

It is well noting that the Schwinger quantum pair-production mechanism \[26\] in the electric field of the charged shell is exponentially suppressed in the \(E_{\text{shell}} \ll E_e \equiv \mu^2 q^2/\hbar^2\) regime \[26, 27\] (here \(E_{\text{shell}} = Q/R^2\) is the electric field at the surface of the charged shell). Taking cognizance of the inequality \[19\], one deduces that this quantum effect is exponentially suppressed in the \(\mu R < |qQ| \ll (\mu R)^2\) regime. These two inequalities imply the strong inequality \(\mu R \gg 1\).

F. Sauter, Z. Phys. 69, 742 (1931); W. Heisenberg and H. Euler, Z. Phys. 98, 714 (1936); J. Schwinger, Phys. Rev. 82, 664 (1951).

M. A. Markov and V. P. Frolov, Teor. Mat. Fiz. 3, 3 (1970); W. T. Zaanen, Nature 247, 531 (1974); B. Carter, Phys. Rev. Lett. 33, 558 (1974); G. W. Gibbons, Comm. Math. Phys. 44, 245 (1975); L. Parker and J. Tiomno, Astrophys. Journ. 178, 809 (1972); T. Damour and R. Ruffini, Phys. Rev. Lett. 35, 463 (1975); S. Hod, Phys. Rev. D 59, 024014 (1999) arXiv:gr-qc/9906004; S. Hod and T. Piran, Phys. Rev. D 65, 024019 (1998) arXiv:gr-qc/9801060.

Here we have used Eq. 6.115 of \[17\].

Here we have used the relation \(-1 = e^{-i(1+2\pi n)}\), where the resonance parameter \(n\) is an integer.

Taking cognizance of the relation \[20\] and using Eq. 6.123 of \[17\], one deduces that the ratio \(\Gamma(\nu)/\Gamma(-\nu)\) is a complex number of the form \(e^{i\phi}\), where \(\phi \in \mathbb{R}\). This implies \(\Gamma(\nu)/\Gamma(-\nu)|^{1/2_i}\nu \in \mathbb{R}\). One therefore concludes that, for imaginary values of the dimensionless parameter \(\nu\) [see Eqs. \(11\) and \(20\)], the dimensionless radii \(\mu R(n)\) of the charged supporting shells as obtained from the analytically derived formula \[27\] are purely real numbers.

Note that \(\mu R < |\nu|\) [see Eqs. \(11\) and \(19\)], which implies \(x < 1\) (or equivalently, \(\epsilon > 0\)).

Here we have used the relation \(-1 = e^{-i(1+2\pi n)}\), where the resonance parameter \(n\) is an integer.

Interestingly, from the data presented in Table \(1\) one finds that the agreement between the numerical data as obtained...
from a direct numerical solution of the resonance equation (15) and the analytical formula (27) is quite good even in the regime $\mu R = O(1)$. This finding is quite surprising since the analytically derived formula (27) for the discrete spectrum of charged shell radii which can support the static bound-state charged massive scalar field configurations is formally valid in the restricted regime $\mu R \ll 1$ [see Eqs. (22) and (25)].

[34] A. E. Mayo and J. D. Bekenstein, Phys. Rev. D 54, 5059 (1996).
[35] S. Hod, Phys. Lett. B 713, 505 (2012); S. Hod, Phys. Lett. B 718, 1489 (2013) [arXiv:1304.6474]; S. Hod, Phys. Rev. D 91, 044047 (2015) [arXiv:1504.00009].