Article

Maclaurin Coefficient Estimates of Bi-Univalent Functions Connected with the q-Derivative

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Abstract: In this paper we introduce a new subclass of the bi-univalent functions defined in the open unit disc and connected with a $q$-analogue derivative. We find estimates for the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in this subclass, and we obtain an estimation for the Fekete-Szegő problem for this function class.

Keywords: bi-univalent functions; Hadamard (convolution) product; coefficients bounds; $q$-derivative operator; differential subordination

MSC: Primary 05A30, 30C45; Secondary 11B65, 47B38

1. Introduction, Definitions and Preliminaries

Let $\mathcal{A}$ denote the class of normalized analytic functions in the unit disc $U := \{ z \in \mathbb{C} : |z| < 1 \}$ of the form

$$ f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad z \in U, $$

and let $\mathcal{S} \subset \mathcal{A}$ consisting on functions that are univalent in $U$.

If the function $h \in \mathcal{A}$ is given by

$$ h(z) = z + \sum_{k=2}^{\infty} c_k z^k, \quad z \in U, $$

then, the Hadamard (or convolution) product of $f$ and $h$ is defined by

$$ (f * h)(z) := z + \sum_{k=2}^{\infty} a_k c_k z^k, \quad z \in U. $$

The theory of $q$-calculus plays an important role in many areas of mathematical, physical, and engineering sciences. Jackson ([1,2]) was the first to have some applications of the $q$-calculus and introduced the $q$-analogue of the classical derivative and integral operators [3].
For $0 < q < 1$, the $q$-derivative operator [2] for $f \ast g$ is defined by

$$D_q (f \ast g) (z) := D_q \left( z + \sum_{k=2}^{\infty} a_k c_k z^k \right)$$

$$= \frac{(f \ast h) (z) - (f \ast h) (qz)}{z(1 - q)} = 1 + \sum_{k=2}^{\infty} [k, q] a_k c_k z^{k-1}, \ z \in U,$$

where

$$[k, q] := \frac{1 - q^k}{1 - q} = 1 + \sum_{j=1}^{k-1} q^j, \quad [0, q] := 0.$$

(3)

Using the definition formula (3) we will define the next two products:

(i) For any non negative integer $k$, the $q$-shifted factorial is given by

$$[k, q]! := \begin{cases} 1, & \text{if } k = 0, \\ \prod_{i=1}^{k} [i, q], & \text{if } k \in \mathbb{N} := \{1, 2, \ldots\}. \end{cases}$$

(ii) For any positive number $r$, the $q$-generalized Pochhammer symbol is defined by

$$[r, q]_k := \begin{cases} 1, & \text{if } k = 0, \\ \prod_{i=1}^{k} [r + i - 1, q], & \text{if } k \in \mathbb{N}. \end{cases}$$

For $\lambda > -1$ and $0 < q < 1$, we define the linear operator $\mathcal{H}_{h \lambda}^{\lambda,q} : A \rightarrow A$ by

$$\mathcal{H}_{h \lambda}^{\lambda,q} f(z) = \mathcal{M}_{h \lambda+1}(z) = z D_q (f \ast h) (z), \ z \in U,$$

where the function $\mathcal{M}_{h \lambda+1}$ is given by

$$\mathcal{M}_{h \lambda+1}(z) := z + \sum_{k=2}^{\infty} \frac{[\lambda + 1, q]_{k-1}}{[k - 1, q]!} z^k, \ z \in U.$$

A simple computation shows that

$$\mathcal{H}_{h \lambda}^{\lambda,q} f(z) := z + \sum_{k=2}^{\infty} \phi_{k-1} a_k z^k, \ z \in \mathbb{C}, \ (\lambda > -1, \ 0 < q < 1),$$

(4)

where

$$\phi_{k-1} := \frac{[k, q]!}{[\lambda + 1, q]_{k-1}} c_k, \ k \geq 2.$$

(5)

**Remark 1.** From the definition relation (4) we can easily verify that the next relations hold for all $f \in A$:

(i) $[\lambda + 1, q] \mathcal{H}_{h \lambda}^{\lambda,q} f(z) = [\lambda, q] \mathcal{H}_{h \lambda+1}^{\lambda,q} f(z) + q^{1 + \sqrt{1}} \left( \mathcal{H}_{h \lambda+1}^{\lambda,q} f(z) \right), \ z \in U$;

(ii) $T_{h \lambda}^{\lambda,q} f(z) := \lim_{q \rightarrow 1} \mathcal{H}_{h \lambda}^{\lambda,q} f(z) = z + \sum_{k=2}^{\infty} \frac{k!}{[\lambda + 1]_{k-1}} a_k c_k z^k, \ z \in U.$

(6)

**Remark 2.** Taking different particular cases for the coefficients $c_k$ we obtain the next special cases for the operator $\mathcal{H}_{h \lambda}^{\lambda,q}$. 

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(i) For $c_k = \frac{(-1)^k - 1}{4^k(1 - 1)!} \Gamma(v + 1)$, $v > 0$, we obtain the operator $\mathcal{N}_{\alpha_1}^{\lambda, \alpha}$ studied by El-Deeb and Bulboaca [4]:
\[
\mathcal{N}_{\alpha_1}^{\lambda, \alpha} f(z) := z + \sum_{k=2}^{\infty} \frac{(-1)^k - 1}{4^k(1 - 1)!} \Gamma(v + 1) \cdot \frac{[k, q]!}{[\lambda + 1, q]_{k-1}} \alpha_k z^k,
\]
where
\[
\phi_k := \frac{(-1)^k - 1}{4^k(1 - 1)!} \Gamma(v + 1).
\]

(ii) For $c_k = \left(\frac{n + 1}{n + k}\right)^{\alpha}$, $\alpha > 0$, $n \geq 0$, we obtain the operator $\mathcal{M}_{\alpha_n}^{\lambda, \alpha} =: \mathcal{M}_{\alpha_n}^{\lambda, \alpha}$ studied by El-Deeb and Bulboaca [5]:
\[
\mathcal{M}_{\alpha_n}^{\lambda, \alpha} f(z) := z + \sum_{k=2}^{\infty} \left(\frac{n + 1}{n + k}\right)^{\alpha} \cdot \frac{[k, q]!}{[\lambda + 1, q]_{k-1}} \alpha_k z^k, \ z \in \mathbb{U};
\]

(iii) For $c_k = 1$ we obtain the operator $\mathcal{J}_{\alpha}^{\lambda}$ studied by Arif et al. [6], defined by
\[
\mathcal{J}_{\alpha}^{\lambda} f(z) := z + \sum_{k=2}^{\infty} \frac{[k, q]!}{[\lambda + 1, q]_{k-1}} \alpha_k z^k, \ z \in \mathbb{U};
\]

(iv) For $c_k = \frac{m^k - 1}{(k - 1)!} e^{-m}$, $m > 0$, we obtain the $q$-analogue of Poisson operator defined in [7] by:
\[
\mathcal{P}_{q}^{\lambda, m} f(z) := z + \sum_{k=2}^{\infty} \frac{m^k - 1}{(k - 1)!} e^{-m} \cdot \frac{[k, q]!}{[\lambda + 1, q]_{k-1}} \alpha_k z^k, \ z \in \mathbb{U};
\]

(v) For $c_k = \left[\frac{1 + \ell + \mu(k - 1)}{1 + \ell}\right]^{m}$, $\mu, \ell \geq 0$, we obtain the $q$-analogue of Prajapati operator defined in [8] by
\[
\mathcal{P}_{q, \ell}^{\lambda, m} f(z) := z + \sum_{k=2}^{\infty} \left[\frac{1 + \ell + \mu(k - 1)}{1 + \ell}\right]^{m} \cdot \frac{[k, q]!}{[\lambda + 1, q]_{k-1}} \alpha_k z^k, \ z \in \mathbb{U}.
\]

The Koebe one-quarter theorem ([9]) proves that the image of $\mathbb{U}$ under every univalent function $f \in \mathcal{S}$ contains the disk of radius $\frac{1}{4}$. Therefore, every function $f \in \mathcal{S}$ has an inverse $f^{-1}$ that satisfies
\[
f(f^{-1}(w)) = w, \quad (|w| < r_0(f), \ r_0(f) \geq \frac{1}{4}),
\]
where
\[
f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \ldots.
\]

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both $f$ and $f^{-1}$ are univalent in $\mathbb{U}$. Let $\Sigma$ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1). Note that the functions $f_1(z) = \frac{z}{1 - z}, f_2(z) = \frac{1 + z}{1 - z}, f_3(z) = -\log(1 - z)$, with their corresponding inverses $f_1^{-1}(w) = \frac{w}{1 + w}, f_2^{-1}(w) = \frac{2w - 1}{-w}, f_3^{-1}(w) = \frac{e^w - 1}{e^w}$, are elements of $\Sigma$ (see [10]). For a brief history and interesting examples in the class $\Sigma$ see [11]. Brannan and Taha [12] (see also [10]) introduced certain subclasses of the bi-univalent functions $\mathcal{S}_{\alpha}^{*}(\alpha)$ and $K(\alpha)$ of starlike and convex functions of order $\alpha$ $(0 \leq \alpha < 1)$, respectively (see [11]). Following Brannan and Taha [12],
a function \( f \in A \) is said to be in the class \( S_{\alpha}^{*} (a) \) of bi-starlike functions of order \( \alpha \) (\( 0 < \alpha \leq 1 \)), if each of the following conditions are satisfied:

\[
f \in \Sigma, \quad \text{with} \quad \left| \arg \frac{zf'(z)}{f(z)} \right| < \frac{\alpha \pi}{2}, \quad z \in \mathbb{U},
\]

and

\[
\left| \arg \frac{zg''(w)}{g'(w)} \right| < \frac{\alpha \pi}{2}, \quad w \in \mathbb{U},
\]

where the function \( g \) is the analytic extension of \( f^{-1} \) to \( \mathbb{U} \), given by

\[
g(w) = w - a_2 w^2 + \left( 2a_2^2 - a_3 \right) w^3 - \left( 5a_2^3 - 5a_2 a_3 + a_4 \right) w^4 + \ldots, \quad w \in \mathbb{U}. \tag{12}
\]

A function \( f \in A \) is said to be in the class \( K_{\alpha}^{*} (a) \) of bi-convex functions of order \( \alpha \) (\( 0 < \alpha \leq 1 \)), if each of the following conditions are satisfied:

\[
f \in \Sigma, \quad \text{with} \quad \left| \arg \left( 1 + \frac{zf''(z)}{f(z)} \right) \right| < \frac{\alpha \pi}{2}, \quad z \in \mathbb{U},
\]

and

\[
\left| \arg \left( 1 + \frac{zg''(w)}{g'(w)} \right) \right| < \frac{\alpha \pi}{2}, \quad w \in \mathbb{U}.
\]

The classes \( S_{\alpha}^{*} (a) \) and \( K_{\alpha}^{*} (a) \) of bi-starlike functions of order \( \alpha \) and bi-convex functions of order \( \alpha \) (\( 0 < \alpha \leq 1 \)), corresponding to the function classes \( S^* (a) \) and \( K (a) \), were also introduced analogously. For each of the function classes \( S_{\alpha}^{*} (a) \) and \( K_{\alpha}^{*} (a) \), they found non-sharp estimates on the first two Taylor-Maclaurin coefficients \( |a_2| \) and \( |a_3| \) [10,12].

The object of the paper is to introduce a new subclass of functions \( L_{\Sigma}^{q,\lambda} (\eta; h; \Phi) \) of the class \( \Sigma \), that generalize the previous defined classes. This subclass is defined with the aid of a general \( H_n^{\alpha q} \) linear operator defined by convolution products together with the aid of \( q \)-derivative operator. This new class extends and generalizes many previous operators as it was presented in Remark 2, and the main goal of the paper is to find estimates on the coefficients \( |a_2|, |a_3| \), and for the Fekete-Szegő functional for functions in these new subclasses.

These classes will be introduced by using the subordination and the results are obtained by employing the techniques used earlier by Srivastava et al. [10]. This last work represents one of the most important study of the bi-univalent functions, and inspired many investigations in this area including the present paper, while many other recent papers deal with problems initiated in this work, like [13–16], and many others. The novelty of our paper consists of the fact that the operator used by defining the new subclass of \( \Sigma \) is a very general operator that generalizes many earlier defined operators, it does not overlap with those studied in the above mentioned papers (that \( \Phi'(0) > 0 \) and \( \Phi(\mathbb{U}) \) is symmetric with respect to the real axis), while for the function \( \Phi \) from Definition 1 we did not assume any restrictions like in many other papers, excepting the fact that \( \Phi(0) = 1 \) is necessary for the subordinations (13) and (14).

If \( f \) and \( F \) are analytic functions in \( \mathbb{U} \), we say that \( f \) is subordinate to \( F \), written \( f(z) \prec F(z) \), if there exists a Schwarz function \( s \), which is analytic in \( \mathbb{U} \), with \( s(0) = 0 \), and \( |s(z)| < 1 \) for all \( z \in \mathbb{U} \), such that \( f(z) = F(s(z)) \), \( z \in \mathbb{U} \). Furthermore, if the function \( F \) is univalent in \( \mathbb{U} \), then we have the following equivalence ([17,18])

\[
f(z) \prec F(z) \Leftrightarrow f(0) = F(0) \quad \text{and} \quad f(\mathbb{U}) \subset F(\mathbb{U}).
\]

Throughout this paper we assume that \( \Phi \) is an analytic function in \( \mathbb{U} \) with \( \Phi(0) = 1 \) of the form

\[
\Phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots, \quad z \in \mathbb{U}.
\]
Now we define the following subclass of bi-univalent functions $\mathcal{L}^{q,\lambda}_\Sigma (\eta; h; \Phi)$:

**Definition 1.** If the function $f$ has the form (1) and $h$ is given by (2), the function $f$ is said to be in the class $\mathcal{L}^{q,\lambda}_\Sigma (\eta; h; \Phi)$ if the following conditions are satisfied:

$$f \in \Sigma, \text{ \ with \ } 1 + \frac{1}{\eta} \left( z D_a \frac{h^{q,\lambda}_a f(z)}{h^{q,\lambda}_a f(z)} - 1 \right) < \Phi(z),$$

(13)

and

$$1 + \frac{1}{\eta} \left( w D_a \frac{h^{q,\lambda}_a g(w)}{h^{q,\lambda}_a g(w)} - 1 \right) < \Phi(w),$$

(14)

with $\lambda > -1, 0 < q < 1$, and $\eta \in \mathbb{C} \setminus \{0\}$, where the function $g$ is the analytic extension of $f^{-1}$ to $\mathbb{U}$, and is given by (12).

**Remark 3.**

(i) Putting $q \to 1^-$ we obtain that

$$\lim_{q \to 1^-} \mathcal{L}^{q,\lambda}_\Sigma (\eta; h; \Phi) =: \mathcal{G}^{\lambda}_\Sigma (\eta; h; \Phi),$$

which represents the functions $f \in \Sigma$ that satisfy (13) and (14) for $H^{q,\lambda}_a$ replaced with $\mathcal{I}^{\lambda}_a$ (6).

(ii) Putting $c_k = \frac{(-1)^k}{k!} (k-1)! (v + 1)$, $\nu > 0$, we obtain the class $\mathcal{B}^{q,\lambda}_\Sigma (\eta, \nu; \Phi)$, that represents the functions $f \in \Sigma$ that satisfy (13) and (14) for $H^{q,\lambda}_a$ replaced with $N^{\nu,\lambda}_a$ (7).

(iii) Putting $c_k = \left( \frac{n + 1}{n + k} \right)^{\lambda}$, $\alpha > 0$, $n \geq 0$, we obtain the class $\mathcal{M}^{q,\lambda}_\Sigma (\eta, n, \alpha; \Phi)$, that represents the functions $f \in \Sigma$ that satisfy (13) and (14) for $H^{q,\lambda}_a$ replaced with $M^{q,\lambda}_a$ (9).

(iv) Putting $c_k = \frac{m^{k-1}}{(k-1)!} e^{-m}$, $m > 0$, we obtain the class $\mathcal{J}^{q,\lambda}_\Sigma (\eta, m; \Phi)$, that represents the functions $f \in \Sigma$ that satisfy (13) and (14) for $H^{q,\lambda}_a$ replaced with $J^{q,\lambda}_a$ (10).

(v) Putting $c_k = \left[ \frac{1 + \mu (k-1)}{1 + \mu} \right]^m$, $m \in \mathbb{Z}$, $\ell \geq 0$, $\mu \geq 0$, we obtain the class $\mathcal{F}^{q,\lambda}_\Sigma (\eta, m, \ell, \mu; \Phi)$, that represents the functions $f \in \Sigma$ that satisfy (13) and (14) for $H^{q,\lambda}_a$ replaced with $F^{q,\lambda}_a$ (11).

**Remark 4.** If the function $h_*$ is given by

$$h_*(z) = \frac{z}{1 - z}, \ z \in \mathbb{U},$$

then $h_*$ has the form (2) with $c_k = 1, \ k \geq 2$, and according to (6) we have

$$\mathcal{I}^{0,\lambda}_h f(z) := \lim_{q \to 1^-} \mathcal{I}^{q,\lambda}_h f(z) = z + \sum_{k=2}^{\infty} \frac{k!}{(1)_{k-1}} a_k z^k = z + \sum_{k=2}^{\infty} \frac{k!}{(1)_{k-1}} a_k z^k = z f'(z), \ z \in \mathbb{U},$$

(15)

for all $f \in A$ of the form (1). Consider the function $f_*(z) = \frac{z}{1 - z} \in \Sigma$, and its inverse analytic extension on $\mathbb{U}$, $g_*(w) = \frac{w}{1 + w}$, let $\eta = 1$ and $\Phi_*(z) = \frac{1 + z}{1 - z}$. Using (15), the relations (13) and (14) become

$$1 + \frac{1}{\eta} \left( z \frac{\mathcal{I}_{0,\lambda}_h f_*(z)}{\mathcal{I}_{0,\lambda}_h f_*(z)}' - 1 \right) = 1 + \frac{z f''_*(z)}{f'_*(z)} = \Phi_*(z) \prec \Phi_*(z),$$
and
\[ 1 + \frac{1}{\eta} \left( \frac{w \left( T_n^0, g_n(w) \right)' - 1}{T_n^0, g_n(w)} \right) = 1 + \frac{w g_n''(w)}{g_n'(w)} = \Phi_\ast(-w) \prec \Phi_\ast(w). \]

Hence, using the notation of Remark 3 (i), we have \( \Phi^\circ_\lambda \left( \eta; \Phi \right) \neq \emptyset \) for some values of \( \lambda, \eta \), and some special choices of the functions \( h \) and \( \Phi \).

To prove our main results we need to use the following lemma:

**Lemma 1.** [19] [p. 172] If \( s(z) = \sum_{k=1}^{\infty} p_k z^k \) is a Schwarz function for \( z \in U \), then
\[ |p_1| \leq 1, \quad |p_k| \leq 1 - |p_1|^2, \quad k \geq 1. \]

2. **Coefficient Bounds for the Function Class** \( L^q_{\lambda}(\eta; h; \Phi) \)

Throughout this paper we are going to assume that \( \lambda > -1 \) and \( 0 < q < 1 \).

**Theorem 1.** If the function \( f \) given by (1) belongs to the class \( L^q_{\lambda}(\eta; h; \Phi) \), and \( \eta \in \mathbb{C}^* := \mathbb{C} \setminus \{0\} \), then
\[ |a_2| \leq \frac{|B_1| \sqrt{|B_1|}}{\sqrt{\frac{q}{\eta} [(1 + q)\phi_2 - \phi_1^2] B_1^2 - \frac{q^2}{\eta^2} B_2 \phi_1^2}}, \]
and
\[ |a_3| \leq \frac{|\eta| |B_1|}{q (q + 1) \phi_2} + \frac{\eta^2 |B_1|^2}{q^2 \phi_1^2}, \]
where \( \phi_{k-1}, k \in \{2, 3\} \), are given by (5).

**Proof.** If \( f \in L^q_{\lambda}(\eta; h; \Phi) \), from (13), (14), and the definition of subordination it follows that there exist two functions \( U \) and \( V \) analytic in \( U \) with \( U(0) = V(0) = 0 \) and \( |U(z)| < 1, |V(w)| < 1 \) for all \( z, w \in U \), such that
\[ 1 + \frac{1}{\eta} \left( z D_q \left( H_\lambda^q f(z) \right) - 1 \right) = \Phi(U(z)), \]  \( (16) \)
and
\[ 1 + \frac{1}{\eta} \left( w D_q \left( H_\lambda^q g(w) \right) - 1 \right) = \Phi(V(w)). \]  \( (17) \)

If \( U(z) = \sum_{k=1}^{\infty} u_k z^k \) and \( V(w) = \sum_{k=1}^{\infty} v_k w^k \), \( z, w \in U \), from Lemma 1 we have
\[ |u_k| \leq 1 \quad \text{and} \quad |v_k| \leq 1, \quad k \in \mathbb{N}. \]  \( (18) \)

Relations (16) and (17) lead to
\[ z D_q \left( H_\lambda^q f(z) \right) - 1 = \eta \left[ \Phi(U(z)) - 1 \right], \]  \( (19) \)
and
\[
\frac{w D_q \left( \mathcal{H}^L \mathcal{G}(w) \right)}{\mathcal{H}^L \mathcal{G}(w)} - 1 = \eta \left[ \Phi(V(w)) - 1 \right].
\] (20)

Since
\[
z D_q \left( \mathcal{H}^L f(z) \right) - 1 = q \phi_1 a_2 z + \left[ q(1 + q) \phi_2 a_3 - q \phi_1^2 a_2^2 \right] z^2 + \ldots,
\]
\[
w D_q \left( \mathcal{H}^L \mathcal{G}(w) \right) - 1 = -q \phi_1 a_2 w + \left[ q(1 + q) \phi_2 \left( 2a_2^2 - a_3 \right) - q \phi_1^2 a_2^2 \right] w^2 + \ldots,
\]
and
\[
\eta \Phi(U(z)) - 1 = \eta B_1 u_1 z + \eta \left( B_1 u_2 + B_2 u_2^2 \right) z^2 + \ldots,
\]
\[
\eta \Phi(V(w)) - 1 = \eta B_1 v_1 w + \eta \left( B_1 v_2 + B_2 v_2^2 \right) w^2 + \ldots.
\]

By equalization according the coefficients of \(z\) and \(w\) in (19) and (20), it follows that
\[
q \phi_1 a_2 = \eta B_1 u_1,
\] (21)
\[
q \left( 1 + q \right) \phi_2 a_3 - q \phi_1^2 a_2^2 = \eta \left( B_1 u_2 + B_2 u_2^2 \right),
\] (22)
\[
-q \phi_1 a_2 = \eta B_1 v_1,
\] (23)
\[
q \left( 1 + q \right) \phi_2 \left( 2a_2^2 - a_3 \right) - q \phi_1^2 a_2^2 = \eta \left( B_1 v_2 + B_2 v_2^2 \right).
\] (24)

Using (21) and (23) we obtain
\[
u_1 = -v_1.
\] (25)

Squaring (21) and (23), after adding relations, we get
\[
2q^2 \phi_2 \phi_1^2 = \eta^2 B_1 \left( u_1^2 + v_1^2 \right).
\] (26)

Adding (22) and (24) we have
\[
2q \left[ \left( 1 + q \right) \phi_2 - \phi_1^2 \right] a_2^2 = \eta \left[ B_1 \left( u_2 + v_2 \right) + B_2 \left( u_1^2 + v_1^2 \right) \right].
\]

From (26), replacing \(u_1^2 + v_1^2\) in the above equation, we have
\[
\left\{ 2q \left[ \left( 1 + q \right) \phi_2 - \phi_1^2 \right] \eta B_1 \right\} \left[ 2 \left( B_1 \left( u_2 + v_2 \right) \right) \right] a_2^2 = \eta^2 B_1 \left( u_2 + v_2 \right),
\]
that is
\[
a_2^2 = \frac{B_1^3 \left( u_2 + v_2 \right)}{2 \left\{ \frac{q}{\eta} \left( \left( 1 + q \right) \phi_2 - \phi_1^2 \right) \eta B_1 - \frac{q^2}{\eta^2} \phi_1^2 B_2 \right\}}.
\] (27)

Taking the absolute value of (27) and using the inequalities (18) we conclude that
\[
|a_2| \leq \frac{|B_1| \sqrt{|B_1|}}{\sqrt{\frac{q}{\eta} \left[ \left( 1 + q \right) \phi_2 - \phi_1^2 \right] B_1^2 - \frac{q^2}{\eta^2} B_2 \phi_1^2}},
\]
which gives the bound for \(|a_2|\) as we asserted in our theorem.
To find the bound for $|a_3|$, by subtracting (24) from (22), we get

$$2q(1 + q)\phi_2 \left( a_3 - a_2^3 \right) = \eta \left[ B_1 (u_2 - v_2) + B_2 \left( u_1^2 - v_1^2 \right) \right].$$  \hspace{1cm} (28)

Form (25), (26) and (28), we obtain

$$a_3 = \frac{\eta B_1 (u_2 - v_2)}{2q(1 + q)\phi_2} + \frac{\eta^2 B_2^2 (u_1^2 + v_1^2)}{2q^2 \phi_1^2}.$$  \hspace{1cm} (29)

Taking the absolute value of (29) and using the inequalities (18) we obtain

$$|a_3| \leq \frac{|\eta| |B_1|}{q(q + 1)\phi_2} + \frac{\eta^2 |B_1|^2}{q^2 \phi_1^2}.$$  

\[\Box\]

Putting $q \to 1^-$ in Theorem 1 we obtain the following corollary:

**Corollary 1.** If the function $f$ given by (1) belongs to the class $G^\alpha_k (\eta; h; \Phi)$ for $\eta \neq 0$, then

$$|a_2| \leq \frac{|B_1|}{\sqrt{|B_1|}} \sqrt{\frac{q \eta}{(1 + q)\psi_2 - \psi_1^2} \left[ B_1^2 - \frac{\eta^2 B_2^2}{\psi_1^2} \right]}.$$  

and

$$|a_3| \leq \frac{|\eta| |B_1|}{q(q + 1)\phi_2} + \frac{\eta^2 |B_1|^2}{q^2 \phi_1^2},$$

where $\phi_{k-1}, k \in \{2, 3\}$, are given by (5).

Taking $c_k = \frac{(-1)^{k-1} \Gamma(v + 1)}{4^{k-1}(k - 1)! \Gamma(k + v)}$, $v > 0$, in Theorem 1 we obtain the following special case:

**Corollary 2.** If $f \in B^\alpha_k (\eta, v; \Phi)$ is given by (1) and $\eta \neq 0$, then

$$|a_2| \leq \frac{|B_1|}{\sqrt{|B_1|}} \sqrt{\frac{q \eta}{(1 + q)\psi_2 - \psi_1^2} \left[ B_1^2 - \frac{\eta^2 B_2^2}{\psi_1^2} \right]}.$$  

and

$$|a_3| \leq \frac{|\eta| |B_1|}{q(q + 1)\phi_2} + \frac{\eta^2 |B_1|^2}{q^2 \phi_1^2},$$

where $\phi_{k-1}, k \in \{2, 3\}$, are given by (8).

Considering $c_k = \left( \frac{n + 1}{n + k} \right)^{\alpha}$, $\alpha > 0$, $n \geq 0$, in Theorem 1 we obtain the following result:

**Corollary 3.** If $f \in M^\alpha_k (\eta, n; \alpha; \Phi)$ is given by (1) and $\eta \neq 0$, then

$$|a_2| \leq \frac{|B_1|}{\sqrt{|B_1|}} \sqrt{\frac{q \eta}{(1 + q)^{\alpha}(\lambda + 1, \lambda)\!|| (\lambda + 1)\!| \left[ B_1^2 - \frac{\eta^2 B_2^2}{\psi_1^2} \right]^{\alpha}}.$$  

\[\Box\]
and
\[ |a_3| \leq \frac{\eta |B_1| [\lambda + 1, q]_2 (n + 3)^a}{q(q + 1) [3, q]! (n + 1)^a} + \frac{\eta^2 |B_1|^2 [\lambda + 1, q]^2 (n + 2)^{2a}}{q^2 (2, q)!^2 (n + 1)^{2a}}. \]

Putting \( c_k = \frac{m^{k-1}}{(k - 1)!} e^{-m} \), \( m > 0 \), in Theorem 1 we obtain the following special case:

**Corollary 4.** If \( f \in \mathcal{L}_{\Sigma}^{q, \lambda} (\eta, h; \Phi) \) is given by (1) and \( \eta \neq 0 \), then
\[
|a_2| \leq \frac{|B_1| \sqrt{|B_1|}}{\sqrt{\frac{2 \Sigma q^3 e^{-m} - \varkappa^2 m^2 e^{-2m}}{\lambda[\lambda + 1, q]}}} B_1^2 - \frac{q^2}{\eta^2} B_2 \frac{(2, q)!^2}{\lambda[\lambda + 1, q]} m^2 e^{-2m},
\]
and
\[
|a_3| \leq \frac{2 |\eta| |B_1| [\lambda + 1, q]_2}{q(q + 1) [3, q]! m^2 e^{-m}} + \frac{\eta^2 |B_1|^2 [\lambda + 1, q]^2}{q^2 (2, q)!^2 m^2 e^{-2m}}.
\]

3. **Fekete-Szegő Problem for the Function Class \( \mathcal{L}_{\Sigma}^{q, \lambda} (\eta, h; \Phi) \)**

**Theorem 2.** If the function \( f \) given by (1) belongs to the class \( \mathcal{L}_{\Sigma}^{q, \lambda} (\eta, h; \Phi) \) for \( \eta \neq 0 \), then
\[
|a_3 - \mu a_2^2| \leq |\eta| |B_1| (|M + N| + |M - N|), \tag{30}
\]
with
\[
M = \frac{(1 - \mu)}{2q} \frac{\eta B_1^2}{(1 + q) \phi_2 - \phi_1^2} \left| \eta B_1^2 - 2q^2 \phi_1^2 B_2 \right|, \quad N = \frac{1}{2q(1 + q) \phi_2}, \tag{31}
\]
where \( \mu \in \mathbb{C} \), and \( \phi_k \), \( k \in \{2, 3\} \), are given by (5).

**Proof.** If \( f \in \mathcal{L}_{\Sigma}^{q, \lambda} (\eta, h; \Phi) \), like in the proof of Theorem 1, from (25) and (28) we have
\[
a_3 - a_2^2 = \frac{\eta B_1 (u_2 - v_2)}{2q(1 + q) \phi_2}. \tag{32}
\]
Multiplying (27) by \( (1 - \mu) \) we get
\[
(1 - \mu) a_2^2 = \frac{(1 - \mu) \eta^2 B_1^3 (u_2 + v_2)}{2q (1 + q) \phi_2 - \phi_1^2} \left| \eta B_1^2 - 2q^2 \phi_1^2 B_2 \right|. \tag{33}
\]
Adding (32) and (33), it follows that
\[
a_3 - \mu a_2^2 = \eta B_1 \left[ (M + N) u_2 + (M - N) v_2 \right], \tag{34}
\]
where \( M \) and \( N \) are given by (31). Taking the absolute value of (34), from (18) we obtain the inequality (30).

**Remark 5.** Algebra shows that the inequality \( |M| \leq N \) is equivalent to
\[
|\mu - 1| \leq \left| 1 - \left( \frac{B_2 q}{B_1^2} \right) \frac{\phi_1^2}{\eta (1 + q) \phi_2} \right|. \]

From Theorem 2 we get the next:
If the function \( f \) given by (1) belongs to the class \( \mathcal{L}^{\Phi}_{\mathbb{C}}(\eta; h; \Phi) \) for \( \eta \neq 0 \), then
\[
|a_3 - \mu a_2^2| \leq \frac{|\eta||B_1|}{q(1 + q)|\Phi_2|},
\]
where \( \mu \in \mathbb{C} \), and
\[
|\mu - 1| \leq 1 - \left( \eta + \frac{B_2 q}{B_1} \right) \frac{\phi_2^2}{\eta(1 + q)|\Phi_2|},
\]
with \( \phi_k, k \in \{2, 3\} \), are given by (5).

Putting \( q \to 1^- \) in Theorem 2 we obtain the following corollary:

**Corollary 5.** If the function \( f \) given by (1) belongs to the class \( \mathcal{G}^{\lambda}_{\mathbb{C}}(\eta, h; \Phi) \) for \( \eta \neq 0 \), then
\[
|a_3 - \mu a_2^2| \leq |\eta||B_1|(\|M + N\| + |M - N|),
\]
with
\[
M = \frac{(1 - \mu) \eta B_1^2}{2(2\phi_2 - \phi_1^2) \eta B_1^2 - 2\phi_1^2 B_2^2}, \quad \text{and} \quad N = \frac{1}{4\phi_2^2},
\]
where \( \mu \in \mathbb{C} \), and \( \phi_k, k \in \{2, 3\} \), are given by (5).

Taking \( c_k = \frac{(-1)^{k-1} \Gamma(v + 1)}{4^{k-1}(k - 1)! \Gamma(k + v)} \), \( v > 0 \) in Theorem 2, we obtain the following special case:

**Corollary 6.** If the function \( f \) given by (1) belongs to the class \( \mathcal{B}^{\lambda}_{\mathbb{C}}(\eta, \nu; \Phi) \) for \( \eta \neq 0 \), then
\[
|a_3 - \mu a_2^2| \leq |\eta||B_1|(\|M + N\| + |M - N|),
\]
with
\[
M = \frac{(1 - \mu) \eta B_1^2}{2q[(1 + q)\psi_2 - \psi_1^2]\eta B_1^2 - 2q^2 \psi_1^2 B_2^2}, \quad \text{and} \quad N = \frac{1}{2q(1 + q)\psi_2^2},
\]
where \( \mu \in \mathbb{C} \), and \( \phi_k, k \in \{2, 3\} \), are given by (8).

Considering \( c_k = \left( \frac{n + 1}{n + k} \right)^\alpha \), \( \alpha > 0 \), \( n \geq 0 \) in Theorem 2, we obtain the next result:

**Corollary 7.** If the function \( f \) given by (1) belongs to the class \( \mathcal{M}^{\lambda}_{\mathbb{C}}(\eta, n, \alpha; \Phi) \) for \( \eta \neq 0 \), then
\[
|a_3 - \mu a_2^2| \leq |\eta||B_1|(\|M + N\| + |M - N|),
\]
with
\[
M = \frac{(1 - \mu) \eta B_1^2}{2qR_2(n, \alpha, \lambda, q)\eta B_1^2 - 2q^2 \left( \frac{[2q]!}{[\lambda + 1, q]} \right)^2 \left( \frac{n + 1}{n + 2} \right)^{2\alpha} B_2^2},
\]
and
\[
N = \frac{(n + 3)^\alpha [\lambda + 1, q]^2}{2q(1 + q)(n + 1)^\alpha [3, q]^r},
\]
where \( \mu \in \mathbb{C} \), and
\[
R_2(n, \alpha, \lambda, q) = (1 + q) \frac{[3, q]!}{[\lambda + 1, q]^2} \left( \frac{n + 1}{n + 3} \right)^\alpha - \left( \frac{[2, q]!}{[\lambda + 1, q]} \right)^2 \left( \frac{n + 1}{n + 2} \right)^{2\alpha}.
\]
If we take \( c_k = \frac{m^{k-1}}{(k-1)!} e^{-m}, m > 0 \) in Theorem 2, we get the next special case:

**Corollary 8.** If the function \( f \) given by (1) belongs to the class \( \mathcal{T}_{\Sigma}^{\mu, \lambda} (\eta, m; \Phi) \) for \( \eta \neq 0 \), then

\[
|a_3 - \mu a_2^2| \leq |\eta| |B_1| (|M + N| + |M - N|),
\]

with

\[
M = \frac{(1 - \mu) \eta B_1^2}{2qR(m, \lambda, q)\eta B_1^2 - 2q^2 \left( \frac{|2q|!}{|\lambda + 1|!} \right)^2 m^2 e^{-2m} B_2},
\]

and

\[
N = \frac{[\lambda + 1, q]_2}{q(1 + q)m^2 e^{-m} [3, q]!},
\]

where \( \mu \in \mathbb{C} \), and

\[
R(m, \lambda, q) = (1 + q) \frac{|3, q|!m^2 e^{-m}}{2[\lambda + 1, q]_2} - \left( \frac{[2, q]!}{[\lambda + 1, q]} \right)^2 m^2 e^{-2m}.
\]

We will give a few applications of the above results obtained for special choices of the function \( \Phi \), as follows.

1. The circular function \( \Phi(z) = \frac{1 + Az}{1 + Bz} \) \((-1 < B < A \leq 1) \) is convex in \( U \) and

\[
\Phi(U) = \left\{ w \in \mathbb{C} : \left| w - \frac{1 - AB}{1 - B^2} \right| < \frac{A - B}{1 - B^2} \right\}, \text{ if } -1 < B < A \leq 1,
\]

\[
\Phi(U) = \left\{ w \in \mathbb{C} : \text{Re } w > \frac{1 - A}{2} \right\}, \text{ if } -1 = B < A \leq 1.
\]

Since \( B_1 = A - B \) and \( B_2 = B(B - a) \), replacing this function in Theorem 1 and Theorem 2 we obtain the next example:

**Example 1.** If \( f \in \mathcal{L}_{\Sigma}^{\mu, \lambda} (\eta; h; \frac{1 + Az}{1 + Bz}) \) is given by (1) and \( \eta \neq 0 \), then

\[
|a_2| \leq \frac{|A - B| \sqrt{|A - B|}}{\sqrt{\left\{ \frac{q}{\eta} \left( 1 + q \right) \Phi_2 - \Phi_1^2 \right\} (A - B)^2 - \frac{q^2}{\eta^2} B (B - A) \Phi_1^2}},
\]

\[
|a_3| \leq \frac{||A - B|}{q(q + 1) \Phi_2} + \frac{\eta^2 |A - B|^2}{q^2 \Phi_1},
\]

and

\[
|a_3 - \mu a_2^2| \leq |\eta| |A - B| (|M + N| + |M - N|),
\]

with

\[
M = \frac{(1 - \mu) \eta (A - B)^2}{2q \left( \frac{1 + q}{1 + 1|} \right)^2 (A - B)^2 - 2q^2 \Phi_1^2 B (B - A)}, \text{ and } N = \frac{1}{2q(1 + q)\Phi_2},
\]

where \( \Phi_{k-1}, k \in \{2, 3\}, \) are given by (5).
Remark 6. For the special values \( A = 1 - 2\beta \) and \( B = -1 \) \((0 \leq \beta < 1)\), the above example yields to the next special case: if \( f \in \mathcal{L}_2^{b,\lambda} (\eta; h; \frac{1+(1-2\beta)z}{1-z}) \) is given by (1) and \( \eta \neq 0 \), then

\[
|a_2| \leq \frac{2\sqrt{2} (1-\beta)^2}{\sqrt{\left| \frac{4\eta [(1+q)\phi_2 - \phi_1^2] (1-\beta)^2 - \frac{2\eta^2}{\eta^2} (1-\beta) \phi_1^2} \right|}},
\]

\[
|a_3| \leq \frac{2|\eta| (1-\beta)}{q(q+1)\phi_2} + \frac{4\eta^2 (1-\beta)^2}{q^2\phi_1^2},
\]

and

\[
|a_3 - \mu a_2^2| \leq 2|\eta| (|\lambda + N| + |\lambda - N|),
\]

with

\[
M = \frac{4\eta (1-\mu) (1-\beta)^2}{8\eta q [(1+q)\phi_2 - \phi_1^2] (1-\beta)^2 - 4q^2\phi_1^2 (1-\beta)}, \quad \text{and} \quad N = \frac{1}{2q(1+q)\phi_2},
\]

where \( \phi_{k-1}, k \in \{2,3\} \), are given by (5).

2. Let consider the binomial function \( \Phi(z) = (1+z)^a, z \in \mathbb{U} \), with \( a \in \mathbb{C}^* \), where the power is considered at the principal branch, that is \( \Phi(0) = 1 \). Since

\[
\Phi(z) = (1+z)^a = 1 + \sum_{n=1}^{\infty} \frac{a(a-1)\cdots(a-n+1)}{n!} z^n, \quad z \in \mathbb{U},
\]

it follows that \( B_1 = a \) and \( B_2 = \frac{a(a-1)}{2} \). Replacing this function in Theorems 1 and 2 we get:

Example 2. If \( f \in \mathcal{L}_2^{b,\lambda} (\eta; h; (1+z)^a) \) is given by (1) and \( \eta \neq 0 \), then

\[
|a_2| \leq \frac{|a| |a|}{\sqrt{\left| \frac{\eta \left[ (1+q)\phi_2 - \phi_1^2 \right] a^2 - \frac{q^2}{2\eta^2} a(a-1)\phi_1^2} \right|}},
\]

\[
|a_3| \leq \frac{|\eta| |a|}{q(q+1)\phi_2} + \frac{\eta^2 |a|^2}{q^2\phi_1^2},
\]

and

\[
|a_3 - \mu a_2^2| \leq |\eta| |a| (|\lambda + N| + |\lambda - N|),
\]

with

\[
M = \frac{(1-\mu) \eta a^2}{2q [(1+q)\phi_2 - \phi_1^2] \eta a^2 - q^2\phi_1^2 a(a-1)}, \quad \text{and} \quad N = \frac{1}{2q(1+q)\phi_2},
\]

where \( \phi_{k-1}, k \in \{2,3\} \), are given by (5).

3. For the function \( \Phi(z) = \left( \frac{1+z}{1-z} \right)^{\sigma}, z \in \mathbb{U} \), with \( \sigma \in \mathbb{C}^* \), where the power is considered at the principal branch, that is \( \Phi(0) = 1 \), we have \( B_1 = 2\sigma \) and \( B_2 = 2\sigma^2 \). Therefore, from Theorems 1 and 2 we deduce the following example:
Example 3. If $f \in \mathcal{L}_{\Sigma}^{\alpha, \lambda} (\eta; h; (1 + z)^{\alpha})$ is given by (1) and $\eta \neq 0$, then

$$|a_2| \leq \frac{2\sqrt{2}|\sigma|}{\sqrt{|\sigma|}} \left( \frac{4\eta}{\eta} \left( (1 + q)\phi_2 - \phi_1^2 \right) \sigma^2 - \frac{2\eta^2}{\eta^2} \sigma^2 \phi_1^2 \right),$$

and

$$|a_3| \leq \frac{2|\eta||\sigma|}{q(q + 1)\phi_2} + \frac{4\eta^2|\sigma|^2}{q^2\phi_1^2},$$

with

$$\frac{4(1 - \mu)\eta\sigma^2}{8q\eta \left( (1 + q)\phi_2 - \phi_1^2 \right) \sigma^2 - 4q^2\phi_1^2\sigma^2}, \quad \text{and} \quad N = \frac{1}{2q(1 + q)\phi_2},$$

where $\phi_{k-1}, k \in \{2, 3\}$, are given by (5).

Remark 7. We mention that all the above estimations for the coefficients $|a_2|$, $|a_3|$, and Fekete-Szegő problem for the function class $\mathcal{L}_{\Sigma}^{\alpha, \lambda} (\eta; h; \Phi)$ are not sharp. To find the sharp upper bounds for the above functionals, it still is an interesting open problem, as well as for $|a_n|$, $n \geq 4$.

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