Quivers for coloured planar $\phi^n$ theories at all loop orders for scattering processes from Feynman diagrams.

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Abstract

It is known how to construct the quivers for $\phi^3$ theories for all loop orders. Here we extend this further to the case of theories for $\phi^n$ theories, for any n. We also construct the quivers for mixed theories. We do this at all loop order.

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1 Introduction

Scattering amplitudes are observables of singular importance for many reasons. Computing them is one of the most important problems in physics. They are traditionally computed with the help of Feynman diagrams by doing an order by order summation of perturbations in the coupling strength. An alternate approach was developed by Nima Arkani-Hamed et al\cite{1}. For color planar scalar theories \cite{2} they start by drawing a dual polygon to the Feynman diagram.

![Diagonal dual to the internal propagator.](image)

**Figure 1:** The s-channel 2-2 scattering process and its dual

Here the sides of the polygon are dual to the external lines of the Feynman diagram and each diagonal is the dual to an internal propagator. Then you compute the top form for this geometry and the coefficient of that is the scattering amplitude. For example, in the case shown above, this is a 2-2 scattering in the s-channel. There is another diagram for the t-channel.

![Figure 2: The t-channel 2-2 scattering process and its dual](image)

There is an alternate way for computing the scattering amplitude of an n-point function, for any n, at one go without looking at all the channels one by one. This is only known at the tree level. This computation is done by starting with the quiver for any n-point tree level process and drawing its Auslander-Reiten walk \cite{43}.
On the left is the five point tree level Feynman diagram with its dual. On the right only the dual pentagon is shown. There are two types of nodes shown in the figure, dashes and dots. The dashes on the sides of the polygon are called frozen nodes and the dots are called unfrozen nodes. The quiver corresponding to this process is given by joining the two unfrozen nodes. One can assign a direction to the line joining the two nodes by selecting a clockwise or an anticlockwise direction. We have not shown that here.

To obtain the scattering amplitude we look at the Auslander-Reiten walk of this quiver and using the techniques in [43] compute the full scattering amplitude of any n-point process. It is known how to compute tree level and 1-loop quivers for $\phi^3$ theory from [30]. This was further extended to drawing quivers for $\phi^3$ theory at all loop levels upto winding number zero by [42]. In this paper we will attempt to further compute quivers for $\phi^n$ theories, for any n, including mixed cases, at all loop order for winding number zero.

In the next section we review the amplituhedron formalism for the colored planar scalar theories. Then we review the computation for quivers at all loops for colored $\phi^3$ theory as described in [42]. After that we juxtapose the two approaches of computing quivers and show the need for our approach. We also demonstrate the construction of quivers using our approach, step by step. Then we will see the construction of quivers for a few non-trivial cases of scattering processes. In the end we summarize and then indicate possible future directions.

2 A quick review of the ”Amplituhedron” formalism for scalar theories

In this section we summarize the results of [2].
2.1 Kinematic space

The Kinematic space $K_n$ of $n$ massless momenta $p_i$, $i=1,...,n$, is spanned by $^{n}C_2$ number of Mandelstam variables,

$$s_{ij} = (p_i + p_j)^2 = 2p_i.p_j \quad (2.1.1)$$

For space-time dimensions $d < n - 1$, all of them are not linearly independent and they are constrained by

$$\sum_{j=1; j\neq i}^{n} s_{ij} = 0, \quad i = 1,...,n \quad (2.1.2)$$

Therefore the dimension of $K_n$ reduces to

$$\text{dim}(K_n) = ^{n}C_2 - n = \frac{n(n-3)}{2} \quad (2.1.3)$$

For any set of particle labels $I \subset \{1, ..., n\}$

$$I \subset \{1, ..., n\} \quad (2.1.4)$$

one can define Mandelstamm variables as follows,

$$s_I = (\Sigma_{i \in I} p_i)^2 = \Sigma_{i,j \in I; i < j} s_{ij} \quad (2.1.5)$$

We always order the particles cyclically and define planar kinematic variables,

$$X_{i,j} = s_{i,i+1,...,j-1}; \quad 1 \leq i < j \leq n \quad (2.1.6)$$

Here

$$s_{i,i+1,...,j-1} = (p_i + p_{i+1} + ... + p_{j-1})^2 \quad (2.1.7)$$

which for massless particles is

$$2 (p_i.p_{i+1} + ... + p_{j-2}.p_{j-1}) \quad (2.1.8)$$

Then $X_{i,i+1} = X_{1,n} = 0$. The variables $X_{i,j}$ can be visualized as the diagonals between the $i^{th}$ and $j^{th}$ vertices of the corresponding n-gon.(See figure.)
These variables are related to the Mandelstam variables by the following relation

\[ s_{ij} = X_{i,j+1} + X_{i+1,j} - X_{i,j} - X_{i+1,j+1} \] (2.1.9)

In other words \(X_{i,j}\)'s are dual to \(\frac{n(n-3)}{2}\) diagonals of an \(n\)-gon made up of edges with momenta \(p_1, ..., p_n\). Each diagonal i.e., \(X_{ij}\) cuts the internal propagator of a Feynman diagram once. Thus there exists a one-one correspondence between cuts of cubic graphs and complete triangulations of an \(n\)-gon.

A partial triangulation of a regular \(n\)-gon is a set of non-crossing diagonals which do not divide the \(n\)-gon into \((n-2)\) triangles.
The associahedron of dimension $(n-3)$ is a polytope whose co-dimension $d$ boundaries are in one-to-one correspondence with partial triangulations by $d$ diagonals. The vertices represent complete triangulations and the $k$-faces represent $k$ partial triangulations of the n-gon. The total number of ways to triangulate a convex n-gon by non-intersecting diagonals is the $(n-2)$-eth Catalan number, $C_{n-2} = \frac{1}{n-1} 2^{n-4} C_{n-2}$.

We now define the planar scattering form. It is a differential form on the space of kinematic variables $X_{ij}$ that encodes information about on-shell tree-level scattering amplitudes of the scalar $\phi^3$ theory. Let $g$ denote a tree cubic graph with propagators $X_{ia,jb}$ for $a = 1, ..., n - 3$. For each ordering of these propagators, we assigns a value $sign(g) \in \pm$ to the graph with the property that flipping two propagators flips the sign. The form must have logarithmic singularities at $X_{ia,jb} = 0$. Therefore one assigns to the graph a
d log form and thus defines the planar scattering form of rank $(n-3)$:}

\[
\Omega_n^{n-3} := \Sigma_{\text{planar } g} \text{sign}(g) \wedge_{a=1}^{n-3} d\log X_{iaja} \quad (2.1.10)
\]

where the sum is over each planar cubic graph $g$. There are two sign choices for each graph, therefore there are many different scattering forms. One can fix the scattering form uniquely if one demands projectivity of the differential form, i.e. if one requires that the form should be invariant under local $GL(1)$ transformations $X_{ij} \to \Lambda(X)X_{ij}$ for any index pair $(i,j)$. We use projectivity to define an operation called mutation. Two planar sub-graphs $g$ and $g'$ are related by a mutation if we can obtain one from the other just by exchanging 4-point sub-graph channels, figure 6. As we can see $X_{ij}$ and $X_{i'j'}$ are the mutated propagators of graphs $g$ and $g'$ respectively. Let’s denote the rest of the common propagators as $X_{ibjb}$ with $b = 1, \ldots, n - 4$. Under $X_{ij} \to \Lambda(X)X_{ij}$ the scattering form becomes,

\[
\text{Figure 6: Two graphs related by a mutation given by an exchange of } X_{ij} \to X_{i'j'} \text{ in a 4-point sub-graph.} \]

\[
\begin{align*}
\text{sign}(g) \ d\log(\Lambda(X)X_{ij})\Lambda_{b=1}^{n-4} d\log X_{ibjb} & \quad + \ldots \\
\text{sign}(g') \ d\log(\Lambda(X)X_{i'j'})\Lambda_{b=1}^{n-4} d\log X_{ibjb} & \quad + \ldots
\end{align*}
\]

Collecting the $d\log \Lambda$ terms

\[
\begin{align*}
\text{sign}(g) \ d\log\Lambda(X)\Lambda_{b=1}^{n-4} d\log X_{ibjb} + \text{sign}(g') \ d\log\Lambda(X)\Lambda_{b=1}^{n-4} d\log X_{ibjb} \quad (2.1.13)
\end{align*}
\]
\[ = (\text{sign}(g) + \text{sign}(g')) \, d\log(\Lambda(X)X_{ij}\Lambda_{a=1}^{n-4} d\log X_{ia,jb}) \quad (2.1.14) \]

Since we demand projectivity, the \( \Lambda(X) \) dependence has to disappear, i.e. when
\[ \text{sign}(g) = -\text{sign}(g') \quad (2.1.15) \]
for each mutation.

### 2.2 The kinematic associahedron.

We described above how one gets an associahedron \( A_n \) inside the kinematic space \( \mathcal{K}_n \) but it is not evident how it should be embedded in \( \mathcal{K}_n \). Because \( A_n \) and \( \mathcal{K}_n \) are of different dimensionality, \( \dim(A_n) = (n - 3) \) and \( \dim(\mathcal{K}_n) = \frac{n(n-3)}{2} \) respectively, we have to impose constraints to embed \( A_n \) inside \( \mathcal{K}_n \).

One natural choice is to demand all planar kinematic variables being positive,
\[ X_{ij} \geq 1 \leq i < j \leq n. \quad (2.2.16) \]

These are \( \frac{n(n-3)}{2} \) inequalities and thus cut out a big simplex \( \Delta_n \) inside \( \mathcal{K}_n \) which is still \( \frac{n(n-3)}{2} \) dimensional. Therefore we need \( \frac{n(n-3)}{2} - n - 3 = \frac{(n-2)(n-3)}{2} \) more constraints to embed \( A_n \) inside \( \mathcal{K}_n \). To do that we impose the following constraints,
\[ s_{ij} = -c_{ij}; 1 \leq i < j \leq n-1, \ |i - j| \geq 2 \quad (2.2.17) \]
where \( c_{ij} \) are positive constraints.

These constraints give a space \( \mathcal{H}_n \) of dimension \( (n-3) \) which is precisely the dimension of \( A_n \). The kinematic associahedron \( A_n \) can now be embedded in \( \mathcal{K}_n \) as the intersection of the simplex \( \Delta_n \) and the subspace \( \mathcal{H}_n \) as follows,
\[ A_n := \mathcal{H}_n \cap \Delta_n. \quad (2.2.18) \]

Once one has the associahedron in \( \mathcal{K}_n \) all one has to do is to obtain its canonical form \( \Omega(A_n) \).

Since an associahedron is a simple polytope one can directly write down its canonical form as follows,
\[ \Omega(A_n) = \Sigma_{\text{vertex}} \, Z \, \text{sign}(Z) \Lambda_{a=1}^{n-3} d\log X_{ia,jb} \quad (2.2.19) \]

Here for each vertex \( Z \), \( X_{ia,jb} = 0 \) denote its adjacent facets for \( a=1, \ldots, n-3 \). Now we claim that the above differential form 2.2.19 is identical to the pullback of the scattering form 2.1.10 in \( \mathcal{K}_n \) to the subspace \( \mathcal{H}_n \). We can justify this statement by the identification \( g \leftrightarrow Z \) and \( \text{sign}(g) \leftrightarrow \text{sign}(Z) \).
1. There is a one to one correspondence between vertices $Z$ and the planar cubic graphs $g$. Also, $g$ and its corresponding vertex have the same propagators $X_{i,a,j,a}$.

2. Let $Z$ and $Z'$ be two vertices related by mutation. Note that mutation can also be framed in the language of triangulation. Two triangulations are related by a mutation if one can be obtained from the other by exchanging exactly one diagonal. See figure 7.

Thus for $Z$ and $Z'$ vertices we have

$$\Lambda_{a=1}^{n-3} dX_{i,a,j,a} = -\Lambda_{a=1}^{n-3} dX'_{i,j,a}$$

which leads to the sign-flip rule $\text{sign}(Z) = -\text{sign}(Z')$.

\[ \text{Figure 7: Two triangles related by a mutation } X_{24} \rightarrow X_{13}. \]

Therefore one can construct the following quantity (an $(n-3)$-form) which is independent of $g$ on pullback.

$$d^{n-3}X := \text{sign}(g)\Lambda_{a=1}^{n-3} dX_{i,a,j,a}. \quad (2.2.21)$$

Substituting this in 2.2.19 one gets,

$$\Omega(A_n = \sum_{\text{planar } g} \frac{1}{\prod_{a=1}^{n-3} X_{i,a,j,a}} d^{n-3}X. \quad (2.2.22)$$
Here

\[ \mathcal{M}_n = \Sigma_{\text{planar}} g \frac{1}{\Pi_{a=1}^{n-3} X_{ia,j_a}} \]  

is the tree level n-point scattering amplitude for the cubic scalar theory.

3 Review of results of rules for construction of quivers for $\phi^3$ theory.

We will discuss the rules for constructing quivers for $\phi^3$ theory. These can be used to construct quivers at all loop orders at all particle numbers.

3.1 Feynman diagram-like rules for writing down quivers from Feynman diagrams

Rules:

1. Replace each internal line by a node as shown in the figure 8. Therefore the quiver of a 4-point scattering process is an $A_1$ quiver.

2. Associate a node to each internal line of a $\phi^3$ vertex, then you connect two nodes along each "vertex line" as shown in the figure. Thus three nodes associated to the three lines of the $\phi^3$ vertex have an associated quiver that is a triangle. (Ref. fig. 9).

We will append more rules to the list along the way.
**Construction rule 1**

![Diagram](image1)

Process

Replace each internal line by one node

Quiver

Figure 8: Construction rule 1.

**Construction Rule 2**

![Diagram](image2)

Figure 9: Construction rule 2.
3.2 Tree level diagrams

Figure 10: The three "vertex lines" of the $\phi^3$ vertex.

Figure 11: Construction of quivers for tree level processes.
We have shown the construction of the quivers for the tree level processes for the 4, 5 and the n particle cases using our technique in figure 11.

For the 4 point case, there is a single internal line and therefore we get an $A_1$ quiver. For the 5-point there are two internal lines connected by one $\phi^3$ vertex. We join the nodes across the "vertex lines". We get the $A_2$ quiver. For an n-point case we have $n - 1$ nodes connected by $n - 2$ vertices. We connect all nodes all each vertex line and get the $A_{n-1}$ quiver.

### 3.3 1-loop diagrams

![Diagram](image)

**Figure 12:** Demonstration of construction rules 1 and 2 for scattering processes at 1-loop.
Here we show the construction of the quivers for 1-loop diagrams. In part 1 of figure 12, we have the box diagram for a 4 particle scattering. In the next figure we decompose the diagram into its 3-point vertices as shown on the top right diagram. We associate nodes to all internal lines and connect the nodes across all ”vertex lines”. We do the same thing for diagram 2 in the

**Figure 13:** Construction rules 3 and 4.
In part 3 of figure 13, we can see that we come across a circular construction in one part of the quiver. This leads us to another rule,

Rule 3: Closed loops in quivers cancel out.

In part 4 of the figure we have a tadpole. We need another rule for this.

Rule 4:
For each tadpole with its base, replace it by a ”Y” with its base, i.e. separate the tadpole from the rest of the diagram and cut its loop so that it becomes a ”Y”. Then associate two nodes to each of the open branches(two) of the ”Y”. Then we connect all nearby nodes with both these nodes.

In figure 14 part 5 we see another process. Here we again decompose the Feynman diagram of the process into its $\phi^3$ vertices. We cancel out the circular part and we get the square quiver.

In part 6 we have an interesting example of a tadpole with internal lines on both sides. We again decompose the Feynman diagram into its vertices and associate two nodes with branches of the tadpole after cutting the loop open. Then, as mentioned before, we connect nodes on both sides of the tadpole to both nodes associated with the ”Y” of the tadpole.
Figure 14: Number 5 is a demonstration of rule 3. Number 6 is a demonstration of rule 4.
4 Feynman-like rules for quivers of scattering processes for $\phi^n$ theories, $n \geq 4$.

4.1 Construction of quivers from polygons and n-angulations (The long route).

When writing down quivers for $\phi^3$ theories we fully triangulate the polygon and then mark the unfrozen nodes to get quivers [42]. This is done at all loop order for any number of particles. To get quivers for $\phi^n$ theories, for any n, we n-angulate the polygon such that each n-vertex is surrounded by n sides and diagonals.

![Diagram of a quiver with $\phi^4$ vertex surrounded by two diagonals and two sides](image)

**Figure 15:** The $\phi^4$ vertex surrounded by two diagonals and two sides. All four sides are dual to the four external lines.

For example, let us look at the $\phi^4$ case at 1-loop for 4 particles.

Here, one can see that both $\phi^4$ vertices are surrounded by 2 sides and 2 diagonals = 4-gon.

We look at another example. We look at the more complicated case for 5 particles where we have a $\phi^4$ vertex and a $\phi^5$ vertex, at 1 loop.
The $\phi^4$ vertex.

The singularity corresponding to the loop.

The $\phi^5$ vertex.

The scattering process.

Figure 16: Scattering amplitude for 5 particles where we have a $\phi^4$ vertex and a $\phi^5$ vertex, at 1 loop.

As can be seen, the $\phi^4$ vertex is surrounded by 2 sides of the polygon + 2 diagonals = 4. Similarly the $\phi^5$ vertex is surrounded by 3 sides + 2 diagonals = 5-gon.

Thus, an n-gon for a $\phi^n$ vertex.

Now we describe how to obtain quivers using the dual polygon and its n-angulations and look at an example.

1. Assign an unfrozen node where the diagonals and internal propagators intersect.

2. Assign frozen nodes where the externals lines and the sides of the polygon intersect.

3. The lines connecting the nodes are called edges. Connect all nodes such that the edges of the quiver never cross external lines.

4. Connect all nodes such that the edges of the quiver never cross internal propagators.

5. Connect all nodes such that the edges of the quiver never cross diagonals.
Figure 17: Illustration of quiver construction for 4 point 1-loop scattering process for $\phi^4$ theory.
6. Note down all the unfrozen nodes and the connections between them. Ignore all edges involving unfrozen nodes.

\[ \text{Connected unfrozen nodes.} \quad = \quad \text{The quiver.} \]

**Figure 18:** All unfrozen nodes and the edges between them and the final quiver.

### 4.2 The need for the new approach.

[43] have shown that at the tree level, the scattering amplitude for any number of particles for any interaction $\phi^n$ can be computed using a computer program. It would be ideal if such a program can be made for the entire process. The traditional method is to use the Feynman diagrammatic expansion. The approach by [43] only addresses the second half of the problem of computing the scattering forms once the quivers are known. To do their computation, the old approach of quiver construction, is to draw the dual polygon, n-angulate, draw all nodes, frozen and unfrozen, and connect them. This is a long and an arduous procedure which also does not seem to yield to an algorithmic (in the sense of being automated using a computer program) approach. The new rules based approach might possibly be implemented in a computer program.

Another point of interest, if both the approaches of computing the quivers are implemented using a computer program, the latter approach has a much lesser number of steps. This means the latter approach is more efficient computationally and would yield results much faster.

### 4.3 The Feynman-like rules for constructing quivers.

Now we list out a few basic structures using which all quivers for any theory, $\phi^n$ for $n \geq 4$, at all loop order, can be constructed.

$n = 3$ is a special case about which we will make a few comments later.

We will now draw the basic structures. Then we will list out the rules for drawing quivers from Feynman diagrams without dual polygons and n-
angulations. A step by step construction of quivers will be demonstrated after that.

4.4 The rules.

1. Assign an unfrozen node to all internal propagators.

2. We look at the two figures below. Figure 20 is a 5-point scattering amplitude at two loops. There is no external line between unfrozen

\textbf{Figure 19:} Basic structures using which all quivers for any theory, $\phi^n$ for $n \geq 4$, at all loop order, can be constructed.
nodes A and B. Therefore, they can be connected. Figure 21 is a 5-point scattering amplitude at two loops. There is an external line between unfrozen nodes A and B. Therefore, they cannot be connected. This implements rule 3.

Figure 20: A 5-point scattering amplitude at two loop. No external line between unfrozen nodes A and B. Therefore, they can be connected.
Figure 21: A 5-point scattering amplitude at two loop. There is an external line marked 'C' between the unfrozen nodes A and B. This prohibits the connection between the unfrozen nodes 'A' and 'B'.

3. As shown in figure 22, the edge of the quiver crossing the internal propagator at the point A' is prohibited. This implements rule 4.

4. Again, as shown in figure 22, the edge of the quiver connecting two non-adjacent nodes is prohibited. This implements rule 5.

Figure 22: A 5-point scattering amplitude at four loop.
Using the above heuristic rules we believe we can write down the quivers for any scattering process, for any number of particles, for any theory $\phi^n$, $n \geq 4$, at all loop order.

4.5 The $\phi^3$ case.

In section 3.3, in the $\phi^3$ case, we saw the need for a special rule, Rule 4 for tadpoles. We find that there is no need for such a rule in any other case. This rule for tadpoles is specific only to this case.

5 Summary and Conclusions

In this paper, we begin by explaining the need for the computation of the acattering amplitude of any process, namely, that they are the only observables of quantum gravity in asymptotically flat space-time. Then we explain the need of a new approach to the problem of computing scattering amplitudes, that, a first principles derivation of the fundamental analyticity properties encoding unitarity and causality of the theory of scattering amplitudes, failed. In the next section we introduce the ”Amplituhedron” approach started by Nima Arkani-Hamed et al and its extension to coloured scalar theories. Then we expand on the problem of computing scattering amplitudes for coloured scalar theories. In section 3 we review the rules for getting quivers for the $\phi^3$ case for any particle number at any loop order. In next section we list the rules of construction for quivers for $\phi^n$ theories, $n \geq 4$. It can be easily seen from here that this program can be generalized to all loop orders for any number of particles.

Future directions

[43] have computed the scattering amplitude for tree level processes for any number of particles. They start with the quiver of the tree level process, write down its Auslander-Reiten walk and then compute the amplitude. Here the quiver is 1-dimensional and the Auslander-Reiten walk is 2-dimensional. For computing amplitudes in any theory $\phi^n$ for 1-loop and above, the quiver is 2-dimensional, therefore the AR walk will be 3-dimensional. We hope to address that in our next paper and compute the scattering amplitude thereof. Also, note that the AR walk is 3-dimensional and no more for any loop order for any theory.
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