Quark-Lepton Symmetry In Five Dimensions

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We construct a complete five dimensional Quark-Lepton symmetric model, with all fields propagating in the bulk. The extra dimension forms an \( S^1/Z_2 \times Z_2' \) orbifold with the zero mode fermions corresponding to standard model quarks localised at one fixed point. Zero modes corresponding to left(right)-chiral leptons are localised at (near) the other fixed point. This localisation pattern is motivated by the symmetries of the model. Shifting the right-handed neutrinos and charged leptons slightly from the fixed point provides a new mechanism for understanding the absence of relations of the type \( m_e = m_u \) or \( m_e = m_d \) in Quark-Lepton symmetric models. Flavour changing neutral currents resulting from Kaluza Klein gluon exchange, which typically arise in the quark sector of split fermion models, are suppressed due to the localisation of quarks at one point. The separation of quarks and leptons in the compact extra dimension also acts to suppress the proton decay rate. This permits the extra dimension to be much larger than that obtained in a previous construct, with the bound \( 1/R \gtrsim 30 \text{ TeV} \) obtained.

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I. INTRODUCTION

It is known that the evident differences between quarks and leptons may be no more than the low energy manifestation of a more symmetric underlying theory. In particular it has been shown that one may construct four dimensional models which are invariant under a discrete quark-lepton (QL) symmetry, whereby one interchanges all quarks and leptons in the Lagrangian \([1, 2, 3, 4, 5, 6, 7]\). This idea requires the introduction of leptonic colour, an \( SU(3) \) gauge group acting on a set of generalised leptons, and thus predicts new gauge bosons and fermions.

Recently the notion of a QL symmetry has been investigated in five dimensions \([8]\). In that work the scalar and gauge fields of the QL symmetric model were assumed to propagate in the bulk, whilst all quark and lepton fields were confined to a brane. The extra dimension was assumed to form an \( S^1/Z_2 \times Z_2' \) orbifold, the construction of which reduced the five dimensional QL symmetric gauge group to one of its subgroups. The model required an order \( 10^{11} \text{ GeV} \) cut-off to suppress the proton decay rate and the extra dimension was taken to be of order \( 1/R \sim 10^9 \text{ GeV} \).

In the present work we construct a complete five dimensional QL symmetric model, with all fields assumed to propagate in a five dimensional spacetime. Placing fermions in the bulk provides new ways of addressing the problems encountered in the previous construct. Provided that the Standard Model (SM) quarks and leptons are localised at (or near) different ends of the compact extra dimension it may be as large as \( 1/R \sim 30 \text{ TeV} \), permitting the observation of higher dimensional physics at future colliders.

A common problem encountered in QL symmetric models is the presence of tree level mass relations of the type \( m_e = m_u \) or \( m_e = m_d \). The higher dimensional framework provides a new way to understand the absence of such mass relations. The mass relations may be removed from the effective four dimensional theory if zero mode fermions corresponding to SM quarks and leptons have different profiles along the extra dimension. In our model this occurs as follows. All zero mode fermions corresponding to SM quarks are localised at one orbifold fixed point, with the varying widths of the fifth dimensional quark wavefunctions determining the degree of wavefunction overlap between left- and right-chiral SM quarks. This in turn determines the size of the effective four dimensional quark Yukawa couplings with the SM Higgs scalar. Leptons are localised at the opposite end of the extra dimension, with \( SU_L(2) \) doublet leptons localised at the fixed point and right-chiral neutrinos and charged leptons shifted slightly into the bulk. Thus the fifth dimensional wavefunction overlaps in the lepton sector tend to be reduced relative to that of the quark sector, motivating the flavour differences observed between quarks and leptons. By shifting the right-chiral neutrinos further into the bulk than the right-chiral charged leptons, neutrino Dirac masses may be suppressed below the electroweak scale.

The symmetries of the model motivate the localisation pattern of SM fermions outlined above. The localisation of quarks at one point in the extra dimension allows one to evade the flavour changing neutral current bounds on the size of the extra dimension which arise generically in the quark sector of split fermion models \([9]\). These bounds may be quite severe, requiring \( 1/R \) to be as large as \( 5000 \text{ TeV}^{-1} \) \([10]\). The flavour changing neutral current bounds which arise in the lepton sector are much weaker and permit the extra dimension to be relatively large.

We note that the concept of leptonic colour has recently been generalised in \([11]\) and studied within the context of unified theories in \([12, 13, 14, 15, 16, 17]\). Split fermions have also been employed to remove quark-lepton mass relations in a different context in \([18]\).
The layout of this paper is as follows. In Section II the gauge and scalar content of the model are discussed. Section III introduces fermions and we determine the orbifold parity assignments necessary to obtain a realistic zero mode fermion spectrum. The Yukawa sector of the model is introduced in Section IV whilst Section IX addresses the lifetime and cosmological signatures. The neutrino sector is considered separately in Section VI, with the neutrino sector considered separately in Section VIII. The size of the extra dimension is discussed in Section IX whilst Section X addresses the layout of this paper is as follows. In Section II the gauge and scalar content of the model are discussed. Section III introduces fermions and we determine the orbifold parity assignments necessary to obtain a realistic zero mode fermion spectrum. The Yukawa sector of the model is introduced in Section IV whilst Section IX addresses the lifetime and cosmological signatures. The neutrino sector is considered separately in Section VI, with the neutrino sector considered separately in Section VIII. The size of the extra dimension is discussed in Section IX whilst Section X addresses the lifetime and cosmological bounds on the mass of the lightest SM singlet neutrino. A discussion of the expected scale for new physics and some experimental signatures is provided in Section X and the paper concludes in Section XI.

II. THE GAUGE AND SCALAR SECTORS

The QL symmetric gauge group is

\[ G_{ql} = SU(3) \times SU_c(3) \times SU_L(2) \times U_X(1), \]

where \( SU(3) \) is the lepton colour group and \( SU_c(3) \) is the usual colour group for quarks. The action of the group \( SU_L(2) \) on the zero mode fermion spectrum will be identified with the usual weak group of the SM and \( X \neq Y \), where \( Y \) is the SM hypercharge. Under the discrete QL symmetry

the gauge fields transform as:

\[ G^M_c \leftrightarrow G^M_l, \quad W^M \leftrightarrow W^M, \quad C^M \leftrightarrow -C^M, \]

where \( G^M_c \) are the \( SU_c(3) \) gauge bosons, \( W^M \) are the weak bosons and \( C^M \) is the \( U_X(1) \) gauge boson. The five dimensional Lorentz index takes the values \( M = \mu, 5 \), where \( \mu \) is the \( 3 + 1 \) dimensional Lorentz index. The additional spatial dimension is taken as the orbifold \( S^1/Z_2 \times Z_2' \), whose coordinate is labelled as \( y \). The construction of the orbifold proceeds via the identification \( y \rightarrow -y \) under \( Z_2 \) and \( y' \rightarrow -y' \) under the \( Z_2' \) symmetry, where \( y' = y + \pi R/2 \). The physical region in \( y \) is given by the interval \( [0, \pi R/2] \). The Lagrangian is required to be invariant under the discrete \( Z_2 \times Z_2' \) symmetry, whose action in the space gauge fields is defined as follows:

\[ W_\mu(x^\mu, y) \rightarrow W_\mu(x^\mu, y) = PW_\mu(x^\mu, y)P^{-1}, \]

\[ W_5(x^\mu, y) \rightarrow W_5(x^\mu, y) = -PW_5(x^\mu, y)P^{-1}, \]

\[ W_\mu(x^\mu, y') \rightarrow W_\mu(x^\mu, y') = PW_\mu(x^\mu, y')P'^{-1}, \]

\[ W_5(x^\mu, y') \rightarrow W_5(x^\mu, y') = -P'W_5(x^\mu, y')P'^{-1}, \]

where \( W \) denotes a generic gauge field. We take \( P \) and \( P' \) to be trivial for the \( SU_c(3), SU_L(2) \) and \( U_X(1) \) gauge bosons. For the \( SU_L(3) \) gauge bosons we choose \( P = \text{diag}(1,1,1) \) and \( P' = \text{diag}(-1,1,1) \), which will reduce the leptonic colour symmetry to \( SU_L(2) \otimes U_X(1) \) in the four dimensional effective theory. We write the five dimensional \( SU_L(3) \) gauge bosons as

\[ G_l = T_\mu G_1^\mu \]

\[ = \begin{pmatrix} -\frac{2}{\sqrt{3}}G^8 & \sqrt{2}Y^1 & \sqrt{2}Y^2 \\ \sqrt{2}Y^1 & -\frac{4}{\sqrt{3}}G^3 + \frac{1}{\sqrt{3}}G^8 & \sqrt{2}G^3 \\ \sqrt{2}Y^2 & \sqrt{2}G^3 & -G^3 + \frac{1}{\sqrt{3}}G^8 \end{pmatrix}, \]

and find their \( Z_2 \times Z_2' \) parities to be

\[ Y^1, Y^2, Y_1^{11}, Y_1^{21} \rightarrow (+, -), \]

\[ Y_5^1, Y_5^2, Y_5^{11}, Y_5^{21} \rightarrow (-, +), \]

\[ G_1^{\mu}, G_1^\mu, \tilde{G}_1^\mu \rightarrow (+, +), \]

\[ G_5^8, G_5^3, \tilde{G}_5^3, \tilde{G}_5^1 \rightarrow (-, -). \]

One may expand the gauge bosons as a Fourier series in the compact extra dimension, with the \( Z_2 \times Z_2' \) parities constraining the series as usual. One has

\[ \psi_{(+,+)}(x^\mu, y) = \frac{2}{\sqrt{\pi R}} \left( \frac{1}{\sqrt{2}} \psi_{(+,+)}^{(0)}(x^\mu) + \sum_{n=1}^{\infty} \psi_{(+,+)}^{(n)}(x^\mu) \cos \frac{2ny}{R} \right), \]

\[ \psi_{(+,-)}(x^\mu, y) = \frac{2}{\sqrt{\pi R}} \sum_{n=1}^{\infty} \psi_{(+,-)}^{(n)}(x^\mu) \cos \frac{(2n-1)y}{R}, \]

\[ \psi_{(-,+)}(x^\mu, y) = \frac{2}{\sqrt{\pi R}} \sum_{n=1}^{\infty} \psi_{(-,+)}^{(n)}(x^\mu) \sin \frac{(2n-1)y}{R}, \]

\[ \psi_{(-,-)}(x^\mu, y) = \frac{2}{\sqrt{\pi R}} \sum_{n=1}^{\infty} \psi_{(-,-)}^{(n)}(x^\mu) \sin \frac{2ny}{R}, \]

where \( \psi \) represents a generic field. Thus the four dimensional charge 1/2 bosons \( Y_{1\mu}^1, Y_{1\mu}^2, Y_{1\mu}^{11} \) and \( Y_{1\mu}^{21} \) do not possess zero modes, with the \( n \)th mode possessing a mass of \( (2n-1)/R \).
cation the zero mode gauge group is
\[\mathcal{G}_{el} \to SU_l(2) \otimes SU_c(3) \otimes SU_L(2) \otimes U_X(1) \otimes U_X(1),\]
and the next stage of symmetry breaking requires
\[U_X(1) \otimes U_X(1) \to U_Y(1),\]
(7)
which shall be achieved by the usual Higgs mechanism. The scalar content necessary to break \(\mathcal{G}_{el}\) is
\[\chi \sim (3, 1, 1, 2/3), \quad \chi' \sim (1, 3, 1, 2/3),\]
(8)
with the action of the discrete QL symmetry given by,
\[\phi \leftrightarrow \hat{\phi}, \quad \chi \leftrightarrow \chi',\]
(9)
where \(\hat{\phi} = e^{i\phi^*}\) and \(e\) is the two dimensional anti-symmetric tensor. The \(Z_2 \times Z'_2\) parities of \(\phi\) are trivial whilst for \(\chi\) we have
\[\chi(x^\mu, y) \to \chi(x^\mu, -y) = P\chi(x^\mu, y),\]
\[\chi(x^\mu, y') \to \chi(x^\mu, -y') = -P'\chi(x^\mu, y'),\]
(10)
with \(P = \text{diag}(1, 1, 1)\) and \(P' = \text{diag}(-1, 1, 1)\). For \(\chi'\) we take
\[\chi'(x^\mu, y) \to \chi'(x^\mu, -y) = P\chi'(x^\mu, y),\]
\[\chi'(x^\mu, y') \to \chi'(x^\mu, -y') = P'\chi'(x^\mu, y'),\]
(11)
with \(P \text{ and } P'\) trivial. Under the symmetry reduction
\[SU_l(3) \to SU_l(2) \otimes U_X(1),\]
(12)
one has
\[\chi \to \chi_2 \oplus \chi_1,\]
(13)
where \(\chi_2 \sim (2, 1)\) and \(\chi_1 \sim (1, -2)\) have the \(Z_2 \times Z'_2\) parities
\[\chi_1 \to (+, +), \quad \chi_2 \to (+, -).\]
(14)
The zero mode for \(\chi_1\) may develop a VEV to break the gauge symmetry as follows:
\[SU_l(2) \otimes SU_c(3) \otimes SU_L(2) \otimes U_X(1) \otimes U_X(1) \downarrow SU_l(2) \otimes SU_c(3) \otimes SU_L(2) \otimes U_Y(1).\]
(15)
At this stage the hypercharge generator may be identified as
\[Y = X + \frac{1}{\sqrt{3}}T_8,\]
(16)
where \(T_8 = (1/\sqrt{3}) \times \text{diag}(-2, 1, 1)\) is a diagonal generator of \(SU_l(3)\). The final stage of symmetry breaking occurs when the neutral component of \(\phi\) develops the VEV \(u\) to give
\[SU_l(2) \otimes SU_c(3) \otimes SU_L(2) \otimes U_Y(1) \downarrow SU_l(2) \otimes SU_c(3) \otimes U_Q(1).\]
(17)
Assuming \(g_2^2 \gg g_3^2, g_7^2\), where \(g_l, g_Y, g_X\) is the \(SU_L(2)\) \([U_X(1)]\) coupling constant and \(g_s\) denotes the common \(SU_l(3)\) and \(SU_c(3)\) coupling constant, one may write the neutral gauge boson mass eigenvalues as \(8\)
\[M^{2(n)}_Y = \left(\frac{2n}{R}\right)^2,\]
\[M^{2(n)}_Z \simeq \frac{1}{2}(g^2 + g^2\mu)u^2 - g^2_3 X^2 + \left(\frac{2n}{R}\right)^2,\]
\[M^{2(n)}_Z \simeq \frac{2}{3}g^2_3 w^2 \left(1 + \frac{g^2_3}{3g^2_5}\right) + \left(\frac{2n}{R}\right)^2.\]
(18)
Note that the zero modes possess the same masses as the neutral gauge bosons in the minimal four dimensional QL symmetric model \([1, 4]\). These zero modes couple to fermions in exactly the same way as the neutral gauge bosons in the minimal QL symmetric model, making the phenomenology of these states identical to that of the neutral gauge bosons studied in \([4]\).

The zero modes consist of the massless photon, the \(Z\) boson with mass of order \(u\), the electroweak scale, and an additional neutral boson \(Z'\) with mass of order \(v\), the \(U_X(1) \otimes U_X(1)\) symmetry breaking scale. The phenomenological bound of \(M_{Z'} > 720\ \text{GeV}\) obtained in \([4]\) also applies to the zero mode \(Z'\) boson in the present model. Thus we obtain a lower bound on the \(U_X(1) \otimes U_X(1)\) symmetry breaking scale of \(w \geq 1\ \text{TeV}\), which is low enough to permit observation of the \(Z'\) boson at the LHC.

The Kaluza-Klein (KK) tower of charged 1/2 bosons possess the mass
\[M^{2(n)}_{Y_1} = M^{2(n)}_{Y_2} = \frac{1}{2}g^2_3 w^2 + \left(\frac{2n - 1}{R}\right)^2,\]
(19)
where the zero mode is absent. We shall discuss the bounds on \(R\) and the \(Y\) boson mass scales in Section \(\text{VIII}\).

The mass of the \(W\) bosons are given by
\[M^{2(n)}_W = \frac{1}{2}g^2_3 u^2 + \left(\frac{2n}{R}\right)^2,\]
(20)
with the zero mode corresponding to the usual \(W\) boson.

## III. FERMIION PARITIES

Whilst the five dimensional theory is vector-like and necessarily free from anomalies, we shall be introducing four dimensional chirality via the orbifold boundary conditions \([19]\).

The effective four dimensional gauge theory is free from anomalies if all anomalies cancel amongst the zero mode fermions \([20]\). After compactification the gauge group is given in equation \(6\). To cancel anomalies involving \(U_X(1)\) factors one requires both the SM leptons and extra coloured leptons to appear at the zero mode level. This may be achieved by doubling the fermion content of the QL symmetric model,
in analogy with the doubling required in five dimensional left-right symmetric models [21,22]. We denote the fermions by

\[
\begin{align*}
Q_i & \sim (1, 3, 2, 1/3), & L_i & \sim (3, 1, 2, -1/3), \\
U^c_i & \sim (1, 3, 1, -4/3), & E^c_i & \sim (3, 1, 1, 4/3), \\
D^c_i & \sim (1, 3, 1, 2/3), & N^c_i & \sim (3, 1, 1, -2/3),
\end{align*}
\]  

(21)

where the quantum numbers under \( G_{ql} \) are shown and \( i = 1, 2 \) labels distinct fermion multiplets. The action of the QL symmetry on the fermions is

\[
Q_i \leftrightarrow L_i, \quad U^c_i \leftrightarrow E^c_i, \quad D^c_i \leftrightarrow N^c_i.
\]

(22)

The transformation rules for a fermion field under the \( Z_2 \otimes Z'_2 \) discrete symmetries are

\[
F(x^\mu, y) \rightarrow F(x^\mu, -y) = \pm \gamma_5 P F(x^\mu, y), \\
F(x^\mu, y') \rightarrow F(x^\mu, -y') = \pm \gamma_5 P' F(x^\mu, y'),
\]

(23)

where \( F \) denotes a generic fermion field and the signs may be chosen independently for the distinct discrete symmetries. The matrices \( P \) and \( P' \) are identical to the ones used for the gauge sector in the last section, with the only non-trivial matrix being \( P' \) acting in lepton colour space. We demand that the lepton fields acquire the \( Z_2 \otimes Z'_2 \) quantum numbers:

\[
L_1 = \begin{pmatrix} L^1_{1L}(+, +) \\ L^1_{2L}(+, -) \\ L^1_{3L}(+, -) \\ L^1_{1R}(-, +) \\ L^1_{2R}(-, +) \\ L^1_{3R}(-, +) \end{pmatrix}, \quad L_2 = \begin{pmatrix} L^1_{1L}(+, -) \\ L^1_{2L}(+, +) \\ L^1_{3L}(+, +) \\ L^1_{1R}(-, -) \\ L^1_{2R}(-, -) \\ L^1_{3R}(-, -) \end{pmatrix},
\]

\[
N^c_1 = \begin{pmatrix} n^1_{1L}(+, +) \\ n^1_{1L}(+, -) \\ n^1_{1R}(-, -) \\ n^1_{1L}(+, -) \\ n^1_{1L}(+, +) \\ n^1_{1R}(-, +) \end{pmatrix}, \quad N^c_2 = \begin{pmatrix} n^1_{1L}(+, -) \\ n^1_{1L}(+, +) \\ n^1_{1R}(-, -) \\ n^1_{1L}(+, +) \\ n^1_{1L}(+, -) \\ n^1_{1R}(-, +) \end{pmatrix},
\]

\[
E^c_1 = \begin{pmatrix} e^1_{1L}(+, +) \\ e^1_{2L}(+, +) \\ e^1_{3L}(+, +) \\ e^1_{1R}(+, -) \\ e^1_{2R}(+, -) \\ e^1_{3R}(+, -) \end{pmatrix}, \quad E^c_2 = \begin{pmatrix} e^1_{1L}(+, -) \\ e^1_{2L}(+, -) \\ e^1_{3L}(+, -) \\ e^1_{1R}(+, +) \\ e^1_{2R}(+, +) \\ e^1_{3R}(+, +) \end{pmatrix}.
\]

Here the numerical superscripts \( 1, 2, 3 \) label the different lepton colours and the numerical subscripts \( 1, 2 \) label the different five dimensional fields. As \( SU_e(3) \) is not broken by the orbifold compactification the three quark colours all possess the same \( Z_2 \otimes Z'_2 \) parities. Suppressing the quark colour index, we enforce the following orbifold parities for quarks:

\[
Q_1 = (Q_{1L}(+, +), Q_{1R}(-, -)), \quad Q_2 = (Q_{2L}(+, -), Q_{2R}(-, +)),
\]

\[
U^c_1 = (u^c_{1L}(+, +), u^c_{1R}(-, -)), \quad U^c_2 = (u^c_{2L}(+, -), u^c_{2R}(-, +)),
\]

\[
D^c_1 = (d^c_{1L}(+, +), d^c_{1R}(-, -)), \quad D^c_2 = (d^c_{2L}(+, -), d^c_{2R}(-, +)).
\]

Only fermion fields with the orbifold parities \((+, +)\) appear at the zero mode level. We identify the zero modes of the fields \( L^1_{1L}, c^1_{1L}, Q_{1L}, u^c_{1L} \) with the SM fields \( L, c^L, Q, u^c \) and \( d^c \) respectively. The zero modes of the fields \( e^c_{1L,2L,3L}, e^c_{1R,2R,3R} \) and \( n^c_{1L,2L,3L} \) are the usual exotic leptons found in QL symmetric models (known as liptons in the literature [4]). The liptons form doublets under the remnant leptonic colour symmetry \( SU_l(2) \subset SU_l(3) \) and are confined into two particle bound states. These states form one of the key signatures of QL symmetric models, leading to an exotic spectrum of particles which may decay into SM fields by creation of electroweak gauge bosons [8,11]. We shall see below that the zero mode liptons acquire mass at the symmetry breaking scale \( w \) and may be observed at the LHC. Note that the zero mode liptons appear in different multiplets to the SM leptons. Consequently the lightest liptons will not couple directly to SM leptons via \( Y \)-boson exchange. This is one of the major phenomenological differences between the current model and that constructed in [8]. We explore the implications of this difference below.

The zero mode of \( n^c_{1L} \) is an additional neutrino which is sterile under the electroweak gauge group. This state will play the role of the usual SM singlet neutrino and we shall label it as \( \nu^c_\mu \). We shall discuss the mass of this state in Section [VII] and issues relating to its cosmological evolution in Section [IX].

### IV. YUKAWA COUPLINGS

The Yukawa Lagrangian must remain invariant under both gauge and orbifold parity transformations. It is convenient to split the five dimensional Yukawa Lagrangian into two portions, with the non-electroweak portion given by

\[
\begin{align*}
\mathcal{L}_{\text{non-ew}} &= \frac{1}{\sqrt{\Lambda}} \left\{ h_1[L^T_1 C_5^{-1} L_1 \chi + Q^T_1 C_5^{-1} Q_1 \chi'] + h_2[L^T_2 C_5^{-1} L_2 \chi + Q^T_2 C_5^{-1} Q_2 \chi'] + h'_1[N^T C_5^{-1} E^c \chi' + D^T C_5^{-1} U^c \chi''] + h'_2[N^T C_5^{-1} E^c \chi' + D^T C_5^{-1} U^c \chi''] + \text{H.c.} \right\}.
\end{align*}
\]

(24)

Here the \( h \)'s are Yukawa coupling matrices in flavour space, \( C_5 = \gamma^3 \gamma^5 \) is the five dimensional charge conjugation matrix and \( \Lambda \) is the cut-off. Contraction over \( SU_{e,1}(3) \) colour in-

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dices with the three dimensional anti-symmetric tensor $\epsilon^{\alpha\beta\gamma}$ is implied in \cite{24}. When the zero mode of $\chi$ develops a VEV $\langle \chi^{(0)} \rangle = w$ the $h_2$ and $h'_2$ terms in \cite{24} generate an order $w$ mass for the liptons. All SM fields remain massless at this stage of symmetry breaking.

The electroweak portion of the Yukawa Lagrangian is

\begin{equation}
\mathcal{L}_{\text{ew}} = \frac{1}{\Lambda} \left\{ \lambda_1 [E^\dagger C^{-1} L_1 \phi^d + U^\dagger C^{-1} Q_1 \phi^d] + \lambda_2 [E^\dagger C^{-1} L_2 \phi^d + U^\dagger C^{-1} Q_2 \phi^d] + \lambda_3 [N_1^\dagger C^{-1} L_1 \phi^d + D^\dagger C^{-1} Q_1 \phi^d] + \lambda_4 [N_2^\dagger C^{-1} L_2 \phi^d + D^\dagger C^{-1} Q_2 \phi^d] + \text{H.c.} \right\},
\end{equation}

where the $\lambda$’s are flavour space Yukawa coupling matrices. This Lagrangian generates fermion Dirac mass terms when $\phi^{(0)}$ develops a VEV, with the SM fermions acquiring mass through the $\lambda_1, \lambda_3$ terms. If the SM fermions have uniform profiles along the extra dimension, then the troublesome tree level mass relations $m_e = m_u$ and $m_\nu = m_d$ arise. However, if one is able to generate different five dimensional wavefunction profiles for quarks and leptons, the troublesome tree level relations will not persist in the effective four dimensional theory. We shall have more to say on this matter in Section \ref{sec:proton} but it will prove useful to consider the stability of the proton within our model first.

\section{Proton Decay}

At the five dimensional level, the lowest dimension non-renormalizable operator which leads to proton decay in our model is

\begin{equation}
\frac{F}{\Lambda^3} \epsilon^{\alpha\beta\gamma} Q_2^\dagger Q_2^\dagger \lambda_5 L_5^\dagger.
\end{equation}

where $\alpha, \beta, \gamma$ are quark colour indices, $\bar{\alpha}$ is the lepton colour index and $F$ is a dimensionless coupling constant. If the SM quarks and leptons have uniform profiles across the extra dimension, proton decay occurs in the effective four dimensional theory via the operator

\begin{equation}
\frac{F}{(\Lambda \pi R)^{1/2}} \frac{w}{\Lambda^3} \epsilon^{\alpha\beta\gamma} Q_2^\dagger \Sigma Q^\dagger \Sigma L^\dagger,
\end{equation}

where $Q$ and $L$ denote four dimensional quark and lepton fields respectively. Current experimental bounds require the lifetime of the proton to be in excess of $1.6 \times 10^{33}$ years \cite{23}. With $w \sim 1$ TeV and $\Lambda R \sim 100$ one requires $\Lambda \sim 5 \times 10^{10}$ GeV to suppress the proton decay rate.

It is possible to reduce the scale of the cut-off by localising quarks and leptons at different points in the extra dimension \cite{24}. If quarks and leptons are separated in the extra dimension, the diminished overlap of the quark and lepton wavefunctions in the fifth dimension serves to reduce the proton decay rate. To suppress the proton decay rate we shall localise quarks and leptons at different fixed points. To illustrate our idea it will suffice to consider an $S^1/\mathbb{Z}_2$ orbifold. Let us add a gauge singlet, bulk scalar field $\Sigma$ which transforms as

\begin{equation}
\Sigma(x^\mu, y) \rightarrow -\Sigma(x^\mu, -y) = -\Sigma(x^\mu, y),
\end{equation}

under the $\mathbb{Z}_2$ symmetry and has the potential

\begin{equation}
V(\Sigma) = \frac{\kappa}{4\Lambda^2} (\Sigma^2 - v^2)^2,
\end{equation}

where $\kappa (v')$ is a dimensionless (dimensionful) constant. The minimum of the potential clashes with the orbifold boundary conditions \cite{28} and results in the vacuum profile \cite{19} (see also \cite{22}),

\begin{equation}
\langle \Sigma \rangle(y) \approx \frac{v}{\sqrt{2\pi R}} \tanh[\kappa(\pi R - y)] \tanh[\kappa y],
\end{equation}

where $\xi^2 = \kappa^2 v^2/2$ and we have introduced $v = \sqrt{2\pi R} v'$. The points $y = \pm \pi R$ are identified under the orbifold construction. One can simplify the analysis when $\kappa v^2 (2\pi R)^2 \gg 1$ by treating the VEV profile of $\Sigma$ as a step function \cite{25},

\begin{equation}
\langle \Sigma \rangle(y) = \frac{v}{\sqrt{2\pi R}} h(y),
\end{equation}

where,

\begin{equation}
h(y) = \begin{cases} +1 & \pi R > y > 0 \\ -1 & -\pi R < y < 0. \end{cases}
\end{equation}

In five dimensions, gauge invariance permits the Yukawa couplings

\begin{equation}
- \frac{h_F}{\sqrt{\Lambda}} FFFF\Sigma,
\end{equation}

for all fermion fields, where $h_F$ is a Yukawa coupling constant. The shape of zero mode fermions in the extra dimension, when $h_F v > 0$, is subsequently given by \cite{25}

\begin{equation}
F_L^{(0)} = \sqrt{\frac{2|h_F v|}{1 - e^{-2|h_F v| \pi R}}} e^{-|h_F v| |y|},
\end{equation}

or via the replacement $y \rightarrow (\pi R - y)$ if $h_F v < 0$, demonstrating the localisation of zero mode fermions at different fixed points on the $S^1/\mathbb{Z}_2$ orbifold, depending on the sign of the coupling constant $h_F$.\]
We wish to suppress the proton decay rate by localising quarks and leptons at different fixed points. Let us take $\Sigma$ to be odd under the discrete QL symmetry \[26\]. The resulting Yukawa Lagrangian is
\[
\mathcal{L}_\Sigma = \frac{1}{\sqrt{\Lambda}} \sum_{i=1,2} \left\{ h_{D_i} [D_i'^2 - N_i'^2] + h_{U_i} [U_i'^2 - E_i'^2] + h_{Q_i} [Q_i^2 - L_i^2] \right\} \Sigma ,
\]
(35)
where $h_{U_i}$, $h_{D_i}$, and $h_{Q_i}$ are Yukawa coupling constants. We use an obvious notation with $F^2 = FF$ and we have suppressed family indices. If one takes all Yukawa couplings to be greater than zero, quarks and leptons are automatically localised at different fixed points.

After integrating over the extra dimension, the operator \[26\] produces a proton decay inducing operator in the four dimensional theory. The approximate form of this operator is
\[
\frac{f'}{\sqrt{\lambda R}} \frac{w v}{\Lambda^2} \exp \{- c v R\} \frac{Q^3 L}{\Lambda^2} ,
\]
(36)
where $f'$ and $c$ are dimensionless constants and $Q$ and $L$ are four dimensional quark and lepton fields respectively. For $\Lambda$ of order $100 - 500 \text{ TeV}$ and an order TeV QL symmetry breaking scale, the lower bound on the lifetime of the proton requires $v \pi R > 40$ if one takes $c$ to be order unity.

## VI. FERMION MASS

We have seen that one is able to understand the long lifetime of the proton if quarks and leptons are localised at different fixed points. In Section IV we noted that despite the Yukawa coupling relations induced by the QL symmetry in equation \[25\], one may understand the absence of relations of the type $m_e = m_u$ in the effective four dimensional theory if quarks and leptons have different wavefunction profiles in the extra dimension.

Inspection of equations (34) and (35) reveals that the methods employed to separate quarks and leptons in the extra dimension induce identical fourth dimensional wavefunction profiles for fermions related by the QL symmetry. Upon integrating over the fifth dimension the troublesome tree level mass relationships persist, despite quarks and leptons being localised at different fixed points. We could in principle add a second SM Higgs doublet to the model and generate enough parameter freedom to remove the unwanted mass relations \[1\]. However this will not allow us to understand the lightness of the known neutrinos relative to the electroweak scale.

In this section we shall apply a purely higher dimensional mechanism to remove the undesirable tree level mass relations. The full construction of a theory of flavour is beyond the scope of the present work. We shall sketch our ideas in what follows and to this end it is helpful to recall some key results obtained in previous studies involving split fermions.

The work of Arkani-Hamed and Schmaltz (AS) \[24\] demonstrated that four dimensional flavour could be addressed in terms of fifth dimensional wavefunction overlaps. AS spatially separated the left- and right-chiral fermion fields in an extra dimension, with the size of the spatial separation influencing the degree of wavefunction overlap. The amount of overlap then determined the size of the effective four dimensional Yukawa couplings to the SM Higgs. However one need not separate the left- and right-chiral fields to address flavour in terms of fifth dimensional wavefunction overlaps. Indeed one can localise all the fermions at one point in the extra dimension, provided the left- and right-chiral fermions have fifth dimensional wavefunctions with different widths \[29\].

We shall employ each of these mechanisms to realize flavour. In particular we shall localise all quark fields at one point in the extra dimension, with the different widths of the left- and right-chiral quark fields determining the effective four dimensional Yukawa couplings in the quark sector. For the leptons we shall separate the left- and right-chiral fields in the extra dimension, thus motivating the observed flavour differences between the quark and lepton sectors.

AS separated fermion fields by introducing distinct bulk Dirac mass terms for the different bulk fermions. In an orbifold theory, however, the fermion transformations necessary to introduce four dimensional chirality preclude bulk Dirac mass terms. One may localise fermions at non-fixed points in an orbifold theory by introducing a second localising scalar \[27\]. This works as follows \[28\]. With one localising scalar the chiral zero mode of a fermion $F$ is localised at one of the orbifold fixed points. The point of localisation is determined by the sign of the product $h_{F} v$ as mentioned already in Section VIII which amounts to the sign of $h_{F}$ when $v > 0$. If the field $F$ couples to a second bulk scalar with an opposite sign Yukawa coupling, the second scalar tends to localise the zero mode at the opposite fixed point. When one scalar is dominant the fermion is found localised at the fixed point preferred by that scalar. In general, however, an interplay between the two scalars results in a compromise which sees the fermion localised in the bulk. More technical details may be found in \[27\].

Let us now investigate this idea in the QL symmetric framework, assuming an $S^1/Z_2$ orbifold again for simplicity. We introduce a second bulk scalar, $\sigma$, which transforms as
\[
\sigma(x^\mu, y) \rightarrow \sigma(x^\mu, -y) = - \sigma(x^\mu, y) ,
\]
(37)
under the orbifold $Z_2$ symmetry and transforms trivially under the QL symmetry. The Yukawa Lagrangian for $\sigma$ is
\[
\mathcal{L}_{\Sigma} = \frac{1}{\sqrt{\Lambda}} \sum_{i=1,2} \left\{ f_{D_i} [D_i'^2 + N_i'^2] + f_{U_i} [U_i'^2 + E_i'^2] + f_{Q_i} [Q_i^2 + L_i^2] \right\} \sigma .
\]
(38)
We achieved quark-lepton separation in the last section by demanding
\[
h_{U_i}, h_{D_i}, h_{Q_i} > 0.
\]
(39)
Let us further demand that
\[
f_{U_i}, f_{D_i}, f_{Q_i} > 0.
\]
(40)
so that all quark Yukawa couplings to $\Sigma$ and $\sigma$ are positive.

Consider first the effects of the second scalar $\sigma$ on the quark sector. Inspection of (35) and (38) reveals that both $\Sigma$ and $\sigma$ will attempt to localise the quarks at the same fixed point. Thus for all positive values of the Yukawa couplings, which we generically denote as $h_F$ and $f_F$, the quark fields will be localised at one fixed point. Note that the SM fields $Q$, $u_R$ and $d_R$ will each have, in general, different profiles in the extra dimension, allowing one to reproduce the necessary quark flavour spectrum along the lines of [9].

Let us now consider the lepton sector. This sector is more complicated because $\Sigma$ and $\sigma$ attempt to localise a given chiral zero mode lepton at different fixed points. We have shown in the next section, but for now let us estimate the degree of separation required between the field $L$ and the field $e_R$ to obtain an order MeV electron mass. It was shown in [27] that, in the decoupling limit of the two localising scalars, one may arrange zero mode fermions to be localised in the bulk of an orbifold theory with fifth dimensional profiles of the form

$$G^{(0)}(y) = N\exp\left\{ -\frac{k v^2}{2\Lambda}(y - y_m)^2 \right\}, \quad (41)$$

Here $N$ is a normalisation factor, $v$ denotes the VEV of one of the localising scalars and $y_m$ is the location of the wavefunction maxima. We show in Section [X] that we expect $v \geq 420$ TeV and use this lower bound in what follows. To leading order the dimensionless constant $k$ depends on the largest fermion-bulk scalar Yukawa coupling and the dimensionless scalar quartic self-coupling $k \approx h_F\sqrt{2\pi}$. Note that naive dimensional analysis permits values of $k \sim 24\pi^3/\sqrt{2} \sim 10^3$ if the underlying theory has strong couplings at the cutoff [25, 29]. However we shall be able to obtain acceptable suppression of the electroweak scale with significantly lower $k$ values [25].

The charged lepton mass terms arise from the couplings

$$\frac{\lambda}{\sqrt{\Lambda}} E_1^c C_5^{-1} L_1 \phi^1, \quad (42)$$

which lead to Dirac masses of the form

$$\frac{\lambda_1 u}{\sqrt{\Lambda}} \int_{0}^{\pi R} e_R^{(0)}(y)L^{(0)}(y)dy, \quad (43)$$

where $e_R^{(0)}(y)$ is the electron wavefunction in the extra dimension which takes the form of $G^{(0)}(y)$ in Equation (41). The localisation of the zero mode of the field $L$ is assumed to be dominated by the scalar $\Sigma$ such that $L^{(0)}(y)$ has the exponential form of equation (34).

Taking $\lambda, k \sim 1$ in (43) reveals that a charged lepton Dirac mass of order MeV is obtained if $e_R$ is localised a distance of $\pi R/15$ from the fixed point at which $L$ is localised. Thus one is not required to shift the zero modes corresponding to the SM right-chiral charged leptons far to obtain a realistic charged lepton spectrum. This slight shift from the fixed point should not produce an observable enhancement of the proton decay rate.

**VII. NEUTRINO MASSES**

Neutrinos acquire Dirac masses at the electroweak symmetry breaking scale via the coupling

$$\frac{\lambda}{\sqrt{\Lambda}} N_1^c C_5^{-1} L_1 \phi^1, \quad (44)$$

We wish to suppress the electroweak contribution to the neutrino mass by separating $\nu_R$ and $\nu_L$ in the extra dimension. Given that neutrino masses are required to be less than an eV it is clear that one must localise the SM neutrinos away from the quarks to suppress the proton decay mode $p \rightarrow \nu_L\pi$. We shall shift $\nu_R$ into the bulk to suppress the electroweak contribution to the neutrino mass, however the acceptable proximity of $\nu_R$ to the quarks is not as clear. We must consider the mass scale of this field to determine its acceptable proximity to the quarks, and this requires a consideration of the non-renormalizable contributions to the neutrino mass sector.

Consider the non-renormalizable operators which induce Majorana mass for the neutrinos in our model. The operator

$$\mathcal{O}_{\nu_L} = \frac{g}{\Lambda} (\chi^\dagger L_1 \phi^1)^2, \quad (45)$$

written in terms of five dimensional fields, with $g$ a dimensionless constant, has dimension ten and in the effective four dimensional theory induces the operator

$$\mathcal{O}^{eff}_{\nu_L} = \frac{g}{(\Lambda \pi R)^2} \frac{(\chi^{(0)} L^{(0)} \phi^{(0)})^2}{\Lambda^3}, \quad (46)$$

which has dimension seven. When $\chi^{(0)}$ and $\phi^{(0)}$ develop VEV’s, $\mathcal{O}^{eff}_{\nu_L}$ produces Majorana masses for the SM neutrinos $\nu_L$. These masses take the form

$$m_{\nu_L} = \frac{g}{(\Lambda \pi R)^2} \frac{(uw)^2}{\Lambda^3}. \quad (47)$$

We require $m_{\nu_L} \sim g \times 1$ eV so that this mass is in the right range to accommodate the atmospheric and solar neutrino oscillation data. The neutrinos $\nu_R$ also acquire mass at the non-renormalizable level. At the five dimensional level, one has the operator

$$\mathcal{O}_{\nu_R} = \frac{g'}{\Lambda^2} (\chi N_1^c)^2, \quad (48)$$

where $g'$ is a dimensionless coupling constant. This produces a Majorana mass for $\nu_R$ in the effective four dimensional theory,

$$m_{\nu_R} = \frac{g'}{(\Lambda \pi R)^2} \frac{w^2}{\Lambda}. \quad (49)$$
We intend to reduce the effective Dirac mass between $\nu_L$ and $\nu_R$ in the four dimensional theory below the electroweak scale by separating $L$ and $R$ in the extra dimension. To estimate the degree of suppression required of the effective Dirac mass we shall take $m_{\nu_R}$ to be $O(\text{GeV})$ (we shall see in Section[IX] that masses in this range are compatible with our framework). Consider the complete neutrino mass matrix in the Majorana basis for one generation,

$$
\begin{pmatrix}
m_{\nu_L} & m_D \\
m_D & m_{\nu_R}
\end{pmatrix},
$$

(50)

where $m_D$ is the effective four dimensional Dirac mass. Under the usual see-saw hierarchy

$$
m_{\nu_L} \ll m_D \ll m_{\nu_R},
$$

(51)

the mass eigenvalues take the approximate form

$$
m_1 \approx m_{\nu_L} - m_D^2/m_{\nu_R},
$$

(52)

$$
m_2 \approx m_{\nu_R}.
$$

(53)

The seesaw contribution to the light neutrino mass eigenvalue, $m_1^2/m_{\nu_R}$, is required to be order eV or less to ensure that $m_1$ is not too heavy. Hence one requires $m_D \lesssim 1\text{ keV}$ for $m_{\nu_R} \sim 1\text{ GeV}$ to ensure that $m_1^2/m_{\nu_R} \lesssim 10^{-1}\text{ eV}$. This requires $\nu_R$ to be localised a distance $\gtrsim \pi R/12$ from the boundary at which $L$ is localised. We show in Section[IX] that $m_{\nu_R}$ is required to be at least of order GeV. Thus proton decays into $\nu_R$ are kinematically forbidden and localising $\nu_R$ further into the bulk does not effect the stability of the proton, a result which holds even if $\nu_R$ is localised at the ‘quark end’ of the extra dimension.

VIII. THE SIZE OF THE EXTRA DIMENSION

We now consider the bounds on $R$, the compactification scale. In Section[III] it was shown that the mass of the lightest $Y$ boson has a contribution inversely proportional to $R$ (the usual KK contribution). In 4D [4] and 5D [8] QL models explored previously, bounds on the $Y$ boson mass have been obtained by considering the decay $\mu \rightarrow 3e$, which proceeds radiatively by the creation of virtual $Y$ bosons. This results in an approximate bound of $M_Y \gtrsim 5\text{ TeV}$. In [8] this led to a bound on $R$, however the current model requires a reevaluation of this bound.

We note from Section[III] that the leptons which form $SU_l(3)$ multiplets with SM leptons do not possess zero modes. Vertices in the effective 4D Lagrangian which couple SM leptons to leptons via $Y$ boson exchange will generally take the form

$$
\mathcal{L} \sim K_{1L'}Y^1L',
$$

(54)

where $Y$, $L'$ and $l$ denote $Y$ boson, lipton and SM lepton operators respectively and $K_{1L'}$ is a numerical factor representing the wavefunction overlap of $Y$, $l$ and $L'$ in the extra dimension. The lightest lipton that couples to a SM lepton $l$ via $Y$ exchange is one of the $n = 1$ KK liptons in the same $SU_l(3)$ multiplet as $l$. However the bulk scalars employed to localise chiral zero mode fermions in Sections[VI] and[VI] also alter the mass and 5D profiles of $n > 0$ KK modes. When one localising scalar is employed it has been shown that the low lying KK modes (with $n \neq 0$) also become localised, albeit with broader 5D wavefunctions [30]. Furthermore the odd and even KK modes tend to be localised at opposite fixed points on an $S^1/Z_2$ model. Given that we localise SM leptons at one boundary of the extra dimension, the $n = 1$ KK liptons will be found at the other boundary. Thus $K_{1L'} \sim \exp(-\nu_\pi R)$ for $n = 1$ liptons, rendering the associated vertex vanishingly small. Consequently the radiative decay $\mu \rightarrow 3e$ will occur only via the coupling of SM charged leptons to even $n$ KK mode liptons, with the dominant contribution arising from the $n = 2$ mode.

It is difficult to evaluate $K_{1L'}$ exactly for an $n = 2$ lipton $L'$ in our model, given the two localising scalars employed. This would require a determination of the fermion 5D wavefunction profiles for a two localising scalar scenario, which is beyond the scope of the present study. To approximate $K_{1L'}$, we use the one bulk scalar results of [30] where analytic expressions for the KK tower 5D wavefunctions were obtained. We find that the overlap is very small for a range of parameter values, with $K_{1L'} \sim 10^{-3}$ for a zero mode lepton $l$ and an $n = 2$ lipton $L'$. Assuming similar values for our model we find that the decay $\mu \rightarrow 3e$ is highly suppressed due to the factor of $K_{1L'}^2$ in the rate. The associated bound on $M_Y^{(1)}$, and thus $R$, is exceptionally weak.

A stronger bound on $R$ may be obtained by considering Flavour Changing Neutral Currents (FCNCs). It is a generic feature of models involving split fermions that FCNCs arise [10]. Zero mode gauge bosons possess uniform profiles in the extra dimension and thus couple uniformly to localised fermions. However the $n > 0$ KK mode gauge bosons are not uniform along the extra dimension and thus, in general, couple to different localised fermions with different strengths, resulting in FCNCs.

The most stringent bounds from FCNCs in split fermion models arise in the quark sector, where exchange of KK gluons can lead to bounds of $1/R > 5000\text{ TeV}$ [10]. We have localised all quarks at one fixed point, aiming to realise quark flavour through varying width 5D quark wavefunctions. This significantly reduces the bounds from FCNCs in the quark sector to $1/R \gtrsim 2 - 5\text{ TeV}$ [9]. In the lepton sector we have localised all left-chiral SM leptons at one fixed point, whilst the SM right-chiral leptons are localised near the fixed point, with different points of localisation for different fields. Thus only gauge bosons which couple to right-chiral fields, namely the hypercharge and $U_X(1) \subset SU_l(3)$ gauge bosons, will have markedly different couplings to different leptons and thus give rise to FCNCs in the lepton sector.

By considering the contribution of the KK tower for the electrically neutral $SU_L(2)$ gauge boson to $\mu \rightarrow 3e$ a bound of $1/R \gtrsim 30\text{ TeV}$ has previously been obtained [10]. In our model the KK tower for the $Z'$ gauge bosons will contribute to $\mu \rightarrow 3e$ and we expect this bound to apply, even though KK $SU_L(2)$ gauge bosons will not mediate this decay. Thus
we require $1/R \gtrsim 30$ TeV to avoid trouble with FCNCs in the model. We note that if all right-chiral SM leptons were localised at one point the lower bound of $1/R \geq 2 - 5$ TeV would apply, though this would have to be arbitrarily imposed on the model.

**IX. COSMOLOGY OF THE STERILE NEUTRINO**

We have seen in Section VII that the neutrinos will acquire Majorana masses at the non-renormalizable level. The cosmological density of the neutrino $\nu_R$ must be considered to ensure that this state does not spoil any of the successful features of the standard cosmological model. The neutrinos $\nu_R$ will remain in thermal equilibrium in the early universe. This results from annihilation’s involving $Z'$ bosons, namely:

$$\nu_R \bar{\nu}_R \leftrightarrow Z' \leftrightarrow l_l,$$  \hspace{1cm} (55)

where $l$ denotes a SM lepton. At some temperature the state $\nu_R$ will freeze out and the remaining cosmological abundance will contribute to the energy density of the universe. The perturbations to the usual scenario induced by $\nu_R$ depend on its mass and lifetime. It is known from 4D models that right-chiral SM leptons may decay via photon emission or via $SU_l(2) \subset SU_l(3)$ gluball emission.

Before we enter into the specifics, let us summarise the findings of this section for readers not interested in the more technical details. Below we show that in our 5D model the neutrino decays $\nu_R \rightarrow \nu_L \gamma$ is expected to dominate the decays involving gluball emission. Known cosmological bounds between the mass of heavy neutrinos and the lifetime for radiative decays then require $m_{\nu_R} \geq 100$ GeV. This result is obtained under the assumption that the neutrino Dirac mass $m_D$ is suppressed to keV energies by localising $\nu_R$ in the bulk. However the lightest $\nu_R$ effectively becomes stable if $m_D$ is suppressed by localising $\nu_R$ at the quark end of the extra dimension. In this case the usual bound for stable massive neutrinos applies and one requires $m_{\nu_R} \gtrsim 5$ GeV. Let us now consider these decay mechanisms in more detail.

**A. Radiative Heavy Neutrino Decays**

Consider the decay $\nu_R \rightarrow \nu_L \gamma$, where $\nu_L$ denotes a SM neutrino. In QL models this occurs radiatively via the creation of a $Y$ boson and a lipton (see Figure 1). The rate for these decays can be obtained by modifying existing results for similar neutrino decays. In particular reference [31] studied radiative neutrino decays of the type $\nu_{\mu} \rightarrow \nu_\mu \gamma$ when exotic leptons and gauge bosons exist. Equation (2) of that work identifies the rate for $\nu_{\mu} \rightarrow \nu_\mu \gamma$ to be

$$\Gamma(\nu_{\mu} \rightarrow \gamma \nu_\mu) = \frac{G_F^2 \alpha B^2 M_L^2 M_{\nu_\mu}^3}{128 \pi^4},$$  \hspace{1cm} (56)

where $G_F$ is the Fermi constant, $\alpha$ is the fine structure constant, $B$ is a numerical vertex factor and $M_L$ ($M_{\nu_\mu}$) is the mass of an exotic lepton (muon neutrino). This expression may be applied to the decay $\nu_R \rightarrow \gamma \nu_L$ in the 5D QL model by making the following replacements

$$\alpha \rightarrow \frac{1}{4} \alpha,$$  \hspace{1cm} (57)

$$M_{\nu_\mu} \rightarrow m_{\nu_R},$$  \hspace{1cm} (58)

$$B^2 G_F^2 \rightarrow K_1^2 K_2^2 G_Y^2 \sin^2 \theta / (\sqrt{2})^4.$$  \hspace{1cm} (59)

Here [57] is required as the $Y$ bosons and liptons carry electric charge $1/2$. The factor of $\sqrt{2}$ is due to the enhanced coupling of the $n > 0$ KK mode $Y$ boson and $G_Y$ is the equivalent of the Fermi constant for the $Y$ bosons,

$$\left( \frac{G_Y}{\sqrt{2}} \right)^2 = \left( \frac{g_5^2}{8 M_Y^{(1)} \alpha} \right)^2.$$  \hspace{1cm} (60)

The angle $\sin \theta$ represents the mixing between a lipton which couples to $\nu_R$ and a lipton which couples to $\nu_L$. This mixing is induced by electroweak symmetry breaking and vanishes in the limit $u \rightarrow 0$. The angle takes the approximate form

$$\sin \theta \approx \frac{K_3 u}{M_L},$$  \hspace{1cm} (61)

where $K_3$ is the 5D wavefunction overlap of the $\nu_R$ lipton with the $\nu_L$ lipton. We shall comment further on $K_3$ shortly. The low lying $n > 0$ KK mode liptons will have mass of order $v$, the bulk scalar symmetry breaking scale [30]. Note that the interplay of the $M_L^2$ factor in the numerator of (56) and the $M_L^{-1}$ dependence of $\theta$ renders the rate for $\nu_R \rightarrow \gamma \nu_L$ independent of $v$. We expect the $n = 2$ KK lipton to be the lightest state to contribute significantly to the decay $\nu_R \rightarrow \gamma \nu_L$. The $n = 1$ state is not localised at the same fixed point as the $n = 0$ chiral modes [30] and thus its contribution to the decay will be suppressed by the $K_3$ factor.

The factors $K_1$ and $K_2$ in equation (59) represent the wave function overlap between $n = 0$ chiral mode neutrinos, the $n = 2$ lipton which couples to this neutrino and the $n = 1$ Y

![FIG. 1: The graphs for $\nu_R \rightarrow \nu_L + \gamma$ in quark-lepton symmetric models.](image-url)
gauge boson. They take the form

\[
K_1 \sim \int_0^{\pi R/2} \nu_R^{(0)}(y)L^{(2)}(y) \cos \frac{y R}{\pi} dy,
\]

\[
K_2 \sim \int_0^{\pi R/2} L^{(0)}(y)\nu_R^{(2)}(y) \cos \frac{y R}{\pi} dy,
\]

(62)

where \( \nu_R^{(0)}(y) \) and \( L^{(0)}(y) \) are zero mode SM singlet neutrino and electroweak lepton doublet 5D wavefunctions respectively. \( L^{(2)} \) represents an \( n = 2 \) lipton 5D wavefunction and the \( Y \) boson fifth dimensional profile is given by the cosine factor. The absence of analytic expressions for the 5D fermion KK tower profiles in two bulk scalar models and the dependence on free parameters makes it difficult to determine these factors exactly. We approximate these using the results obtained for the one localising scalar case in [30] and find that typical values are \( K_{1,2} \sim 10^{-3} \).

To estimate \( K_3 \) we consider the case where \( \nu_R \) and \( \nu_L \) are localised by one bulk scalar, but the potential trapping \( \nu_R \) in the bulk has a minimum shifted away from the fixed point. Noting that \( \nu_R \) is shifted some distance away from the fixed point such that the effective Dirac mass coupling \( \nu_R \) and \( \nu_L \) in the 4D theory is \( m_D \) and using the analytic \( n = 2 \) KK mode wavefunctions of [30] we find that typically \( K_3 m \sim 10^2 m_D \).

The Dirac mass coupling the \( n = 2 \) liptons is expected to be larger than the associated \( n = 0 \) mode Dirac mass \( m_D \) as the 5D wavefunctions for the higher KK modes generally have broader wavefunctions in the extra dimension than the zero modes.

Putting all this together we find that

\[
\tau(\nu_R \to \gamma \nu_L) = 5 \times 10^{-11} (s GeV) \frac{M_{Y}^{(4)(1)}}{m_D^2 m_3^2}. \quad (63)
\]

Taking \( m_D \sim 1 \text{ keV} \) and \( 1/R \sim 30 \text{ TeV} \) gives \( \tau \sim 10^{22} s \) \( (\tau \sim 10^{19} s) \) for \( m_{\nu_R} = 10^2 \text{MeV} \) \( (1 \text{ GeV}) \). Such long lived neutrinos would still be decaying and would contribute to the diffuse photon background. Studies of the diffuse photon background require heavy neutrinos to satisfy the bound

\[
\tau \geq (10^{25} s \text{ GeV}^2) \times m^{-2} \quad \text{(64)}
\]

if their lifetime exceeds \( \tau \geq 3 \times 10^{17} s \). Thus one requires \( \tau \geq 10^{27} s \) for \( m_{\nu_R} = 10^2 \text{MeV} \) \( (1 \text{ GeV}) \), which is not satisfied by the radiative \( \nu_R \) decays. Note that increasing \( m_{\nu_R} \) decreases the radiative lifetime of \( \nu_R \). For \( m_{\nu_R} \geq 10 \text{ GeV} \) one finds that \( \tau \leq 10^{17} s \). If the lifetime of a heavy neutrino lies in the range \( t_{\text{rec}} \leq \tau \leq t_U \), where \( t_{\text{rec}} \) \( (t_U) \) is the time of recombination \( (\text{age of the universe}) \), a different relationship between the neutrinos mass and lifetime must be satisfied. This relationship takes the form [32]

\[
m_{\nu_R} \gtrsim 8 \times 10^{-3} s^{-1/3} \text{GeV}. \quad (65)
\]

Using [63] gives

\[
m_{\nu_R} \gtrsim (8 \times 10^{-3})^{1/2} \left( \frac{5 \times 10^{-11} M_{Y}^{(1)}}{m_D^2} \right)^{1/6} \quad (66)
\]

where all masses should be given in GeV. Using the lower bound for \( 1/R \) and \( m_D = 1 \text{ keV} \) \( (100 \text{ keV}) \) gives \( m_{\nu_R} \gtrsim 170 \text{ GeV} \) \( (40 \text{ GeV}) \) so that an order \( 10 \text{ GeV} \) \( \nu_R \) is disallowed. For masses \( m_{\nu_R} \geq 10^2 \text{ GeV} \) the bound [66] may be satisfied and thus consideration of the radiative decay of \( \nu_R \) into light neutrinos in a cosmological context leads to the bound \( m_{\nu_R} \geq 10^2 \text{ GeV} \). As mentioned already, the neutrino \( \nu_R \) may also decay via emission of an \( SU_l(2) \subset SU(3) \) glueball in QL symmetric models. Before concluding that cosmological considerations demand \( m_{\nu_R} \gtrsim 10^2 \text{ GeV} \) we must further investigate this decay mode.

### B. Glueball Mediated Heavy Neutrino Decays

The interaction eigenstate right-chiral neutrino is an \( SU_l(2) \) singlet and as such doesn’t couple directly to the \( SU_l(2) \) gluons (henceforth referred to as ‘stickions’, adopting the notation of [13]). However the physical mass eigenstate \( \nu_R \) does couple to the stickions. Recall that \( SU_l(2) \) is predicted to be unbroken symmetry and is thus expected to be confining. Stickon exchange leads to bound state fermions formed by the leptons and the \( Y \) bosons. Interestingly these bound state fermions possess the same quantum numbers as the SM leptons [33]. Consequently they mix with the known leptons and the resulting physical leptons contain small admixtures of states which couple to the stickions.

Of importance to us is the mixing of \( \nu_R \) with the SM singlet bound state fermions (henceforth bound state neutrinos). It has been shown that the neutrino-bound state neutrino mixing leads to heavy neutrino decays of the type \( \nu_R \to G \nu_L \), where \( G \) is an \( SU_l(2) \) glueball (henceforth a stickball). The stickballs have a mass set by the \( SU_l(2) \) strong interaction scale \( \Lambda_{SU_l(2)} \), which has been estimated in [33] to be of order \( 10 \text{ MeV} \). The lightest stickball is expected to be a Lorentz scalar \( G_s \) and the neutrino \( \nu_R \) may decay by emitting a real stickball if \( m_{\nu_R} > \Lambda_{SU_l(2)} \). If \( m_{\nu_R} < \Lambda_{SU_l(2)} \) the stickball must be virtual and the decay mode is \( \nu_R \to 3 \nu_L \).

The mixing between \( \nu_R \) and the bound state neutrinos was quantified in [33] in terms of two mixing angles \( \theta_{1,2} \). It was found that a maximal value of \( \sin^2 \theta_{1,2} \sim 10^{-6} \) is expected for 4D QL models. The rate for \( \nu_R \to G_s \nu_L \) then depends on \( \sin^2 \theta_{1,2} \).

In the 5D theory additional factors arise. Again, because \( Y \) boson exchange only couples \( \nu_R \) to \( n > 0 \) KK mode leptons a factor of \( K_l^2 \) must be included. Also the study of [33] assumed Dirac neutrinos. We have Majorana neutrinos and the decay rate for stickball emission will depend on the mixing between \( \nu_L \) and \( \nu_R \), which takes the standard seesaw form of \( \sin \theta_s \sim m_D/m_{\nu_R} \) in our model. Putting this together we find that the replacement

\[
\sin^2 \theta_{1,2} \to K_l^2 \sin^2 \theta_{1,2} \left( \frac{m_D}{m_{\nu_R}} \right), \quad (67)
\]

is required to utilise the results of the 4D study. In [33] it was found that the lifetime for a 17 keV neutrino to decay by stickball emission could be \( \lesssim 10^9 s \). Employing the replacement [67] one finds that the lifetime for stickball emission is...
\leq 10^{25} \text{s for a heavy neutrino with mass } m_{\nu_R} = 10^2 \text{ MeV in the 5D model. The lifetime is inversely proportional to } m_{\nu_R} \text{ so that larger mass values decrease the lifetime. However the dominant decay mode will be } \nu_R \rightarrow \nu_L \gamma \text{ for } m_{\nu_R} > 10^5 \text{ MeV, so that the bound of } m_{\nu_R} \geq 10^2 \text{ GeV still applies.}

Interestingly both the stickball and photon decay modes for } \nu_R \text{ depend on the Dirac mass coupling } \nu_L \text{ and } \nu_R. \text{ For } m_D \rightarrow 0 \text{ these decays do not occur and } \nu_R \text{ becomes stable. In this limit it is important to ensure that the density of } \nu_R \text{ particles which reach at the freeze out temperature of } T_* \sim m_{\nu_R}/15 \text{ does not overclose the universe. This leads to the well known Lee-Weinberg bound of } m_{\nu_R} \geq 5 \text{ GeV for a massive Majorana neutrino. This is much weaker than the bound obtained when considering the radiative decay. In the present model the limit } m_D \rightarrow 0 \text{ corresponds to the limit in which } \nu_R \text{ is localised at the opposite boundary to } \nu_L. \text{ We note that this limit does not lead to rapid proton decays of the type } p \rightarrow \nu_R \pi \text{ as kinematic considerations of any neutrino which satisfies the Lee-Weinberg bound will preclude proton decay via this channel. Given that this is the lowest bound on the mass of } \nu_R \text{ we will consider this scenario in what remains. This setup has the added advantage of allowing for an improved understanding of the hierarchy between quark and lepton masses in a QL symmetric framework, a matter which is currently under investigation [24].}

X. BOUNDS ON THE REMAINING SCALES AND EXPERIMENTAL SIGNATURES

Having obtained the lower bound on the mass of } \nu_R \text{ in the preceding section we may now consider the bounds on the remaining scales in the theory. The QL symmetry breaking scale must satisfy the bound } w \geq 1 \text{ TeV, due to leptonic annihilations involving the } Z' \text{ bosons. If we take } 1/R = 30 \text{ TeV in accordance with the lower bound obtained by considering KK contributions to FCNCs, then the lower bound on } m_{\nu_R} \text{ and the upper bound of } 1 \text{ eV on } m_{\nu_R} \text{ translate into lower bounds on } w \text{ and } \Lambda. \text{ Using (47) and (49) gives:}

\begin{align*}
w & = \left( \frac{u^4 m_{\nu_R}^5 \pi R}{m_{\nu_L}^2} \right)^{1/6}, \\
\Lambda & = \left( \frac{m_{\nu_R}^2 \pi R}{m_{\nu_L}^2} \right)^{1/3},
\end{align*}

so that the lower bound on } m_{\nu_R} \text{ translates into the bounds}

\begin{align*}
w & \geq 5 \text{ TeV}, \\
\Lambda & \geq 460 \text{ TeV},
\end{align*}

for } 1/R = 30 \text{ TeV. Furthermore the demand that the proton does not decay too rapidly translates into a bound on } v, \text{ which we find to be } v \geq 420 \text{ TeV. This enables us to summarise the mass scales for the exotic fields in our construction and some of the associated phenomenology.}

With } 1/R = 30 \text{ TeV and } m_{\nu_R} = O(\text{GeV}) \text{ in accordance with the lower bounds, we find the zero mode } Z' \text{ boson mass to be } M_{Z'}^{(0)} = 5 \text{ TeV. The heaviest liptons acquire an order } w \text{ mass and all liptons will appear at energies } \lesssim w. \text{ These liptons do not couple directly to the known leptons via } Y \text{ exchange. They appear in different } SU(3) \text{ multiplets and the } Z_2 \times Z_2' \text{ parities preclude a direct coupling of the type } Y L_1 L_2. \text{ This is a major distinction between this model and previous QL symmetric constructs.}

Assuming order one Yukawa couplings the lightest liptons will possess masses less than } w. \text{ The lower bound of } w \geq 5 \text{ TeV permits these liptons to appear at TeV energies and thus these states may be observed at the LHC. Although they do not couple directly to leptons via } Y \text{ boson exchange they will couple with the known fermions through electroweak interactions and through } Z' \text{ exchange. Thus the } n = 0 \text{ liptons may appear at the LHC via interactions of the type } pp \rightarrow \gamma, Z \rightarrow LL \text{ etc. A key signature for this construct would be the appearance of liptons at the LHC. One could then discriminate this model from other QL symmetric models by studying the coupling of the liptons to leptons at the proposed ILC. The liptons of this model do not couple to leptons via } Y \text{ boson exchange and thus only electroweak and } Z' \text{ interactions would couple } e^+ e^- \text{ pairs to the liptons.}

We remind the reader that the unbroken } SU(2) \text{ symmetry confines liptons into two particle bound states. Having the lightest liptons in different } SU(3) \text{ multiplets to the SM leptons does not alter the stability of the lightest lipton bound states. These decay via the electromagnetic or weak interactions. As the electroweak bosons have uniform profiles in the fifth dimension their couplings to the liptons are the same as the 4D case. Thus the lifetimes of the lightest confined liptons are the same as the 4D case and these bound states present no cosmological concern.}

The liptons which do couple to the SM leptons are } n > 0 \text{ members of KK towers. These states possess mass of order } v \sim 420 \text{ TeV. Their couplings to SM leptons via } Y \text{ exchange are highly suppressed due to the localisation methods employed in this work, thus the associated phenomenology (like the rare decay } \mu \rightarrow 3e \text{) is also suppressed.}

Whilst new physics may appear at TeV energies in this model, it is not until energies of order 30 TeV that the higher dimensional nature of the theory reveals itself. At these scales the KK excitations for the neutral gauge bosons } Z \text{ and } Z', \text{ the photon, the gluons the } W \text{ bosons appear. The } Y \text{ bosons also appear at this scale, though at these energies they will only manifest themselves in precision experiments through couplings to the other gauge bosons.}

XI. CONCLUDING REMARKS

We have constructed a complete five dimensional QL symmetric model. This model differs from a previous five dimensional QL symmetric model [8] in that all fermions are assumed to propagate in the bulk. Placing fermions in the bulk provides the following advantages advantages over the earlier framework. Namely:

- The longevity of the proton is readily understood by localising quarks and leptons at (or near) different fixed points in the extra dimension.
The extra dimension may be as large as $R=1/(30 \text{ TeV})$ allowing the phenomenology associated with the KK towers of the gauge sector to be observed at future colliders.

The higher dimensional framework allows one to understand the absence of mass relations of the type $m_e = m_u$ or $m_e = m_d$ in a QL symmetric framework due to the different profiles of quark and lepton wavefunctions in the extra dimension.

The five dimensional model permits a purely higher dimensional mechanism whereby one may suppress the neutrino mass scale relative to the electroweak scale by spatially separating left- and right-chiral neutrino fields.

The bounds on the mass of $\nu_R$ depend on its localisation point. We find that $m_{\nu_R} \geq 100 \text{ GeV}$ is required if $\nu_R$ is localised near $\nu_L$, whilst $\nu_R$ can be of order GeV if it is localised at the ‘quark end’ of the extra dimension.

The model as it stands has some features which are introduced in a somewhat arbitrary fashion and it would be pleasing to uncover deeper reasons for their implementation. In particular it would be satisfying to understand why the five dimensional fermion $L_1$ couples more strongly to the bulk scalar $\Sigma$ than do $N_{11}^R$ and $E_{11}^R$. It would also be pleasing to discover a connection between the fermions that undergo $SU(3)$ symmetry breaking and the fermions which couple with opposite sign Yukawa coupling constants to the bulk scalars $\Sigma$ and $\sigma$. In four dimensional QL symmetric models one starts with two sets of fermions which are indistinguishable at high energies. At low energies leptons are defined as those fermions which experience $SU(3)$ symmetry breaking. In our five dimensional model there are two independent features which distinguish quarks from leptons. Leptons are defined to be those fermions which experience $SU(3)$ symmetry breaking and couple to $\sigma$ and $\Sigma$ with opposite signs. It would be pleasing to develop a mechanism which ensures that the fermions which undergo $SU(3)$ symmetry breaking must also couple to the two bulk scalars with opposite signs.

We note also that some interesting steps towards understanding flavour in a five dimensional Left-Right symmetric model have been made in [35]. An intriguing direction for further study would be to attempt to combine the methods employed in [35] with those of the present study.

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