Localization of five-dimensional Elko Spinors on dS/AdS Thick Branes

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Abstract. Different from the Dirac spinor, the localization of a five-dimensional Elko spinor on a brane with codimension one is very difficult because of the special structure of the Elko spinor. By introducing the coupling between the Elko spinor and the scalar field generating the brane, we have two mechanisms for localization of the zero mode of a five-dimensional Elko spinor on a brane. One is the Yukawa-type coupling and the other is the non-minimal coupling. In this paper, we investigate the localization of the Elko zero mode on de Sitter and Anti-de Sitter thick branes with the two localization mechanisms, respectively. It shows that both the mechanisms can realize the localization. The forms of the coupling functions of the scalar field in the two mechanisms have similar properties respectively and they play similar role for the localization.

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1 Introduction

There are many classical problems which the Standard Model (SM) can not interpret sufficiently, such as the hierarchy problem [1,2,3,4,5,6], cosmological problem [7,8,9,10,11,12,13,14] and the nature of dark matter and dark energy [15,16,17,18]. Because extra dimension and brane-world theories can provide new mechanisms to solve these problems, they have attracted more and more attention since the famous Arkani-Hamed, Dimopoulos and Dvali (ADD) [1] and Randall-Sundrum (RS) brane-world models [2,19] were presented. Unlike early models where the brane is a geometric hypersurface embedded in a higher dimensional spacetime, in a more realistic brane model the brane should have thickness and inner structure. Such thick branes can be generated by bulk matter fields (mostly scalar fields) [20,21,22,23,24,25,26,27,28,29,30,31,32,33,34,35,36,37,38,39,40], or can be realized in pure gravities [41,42,43] (see Refs. [29,44] for more detailed introduction about thick brane models).

In the braneworld scenario, an important and interesting issue is to investigate the mechanism by which the Kaluza-Klein (KK) modes of various fields could be localized on the brane. These KK modes contain the information of extra dimensions. Especially, the zero modes of matter fields on the brane stand for the four-dimensional massless particles, and they can rebuild the four-dimensional SM at low energy. A lot of work about the localization of various matter fields on branes have been done [15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30,31,32,33,34,35,36,37,38,39,40,41,42,43,44,45,46,47,48,49,50,51,52,53,54,55,56,57,58,59,60,61,62,63,64,65,66,67,68].

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On the other hand, Elko spinor, which is named from the eigenspinor of the charge conjugation operator, has attracted continuing interest since it was introduced by Ahluwalia and Grumiller in 2005 [69,70]. It is a new spin-1/2 quantum field which satisfies the Klein-Gordon (KG) equation rather than the Dirac one, and only interacts with itself, Higg fields and gravity [69,70,71,72,73,74,75,76]. As a candidate of dark matter, it was widely investigated in particle physics [69,70,71,72,73,74,75,76], cosmology [77,78,79,80,81,82,83,84,85,86] and mathematical physics [87,88,89,90,91,92,93].

In Refs. [94,95,96], the localization of the zero mode of a five-dimensional Elko spinor on various Minkowski branes has been considered. A coupling mechanism should be introduced in order to localize the Elko zero mode on a brane. The first choice is the Yukawa-type coupling $-\eta F(\phi \text{ or } R) \lambda \lambda$ between the five-dimensional Elko spinor $\lambda$ and the background scalar field $\phi$ [94,95] or the Ricci scalar $R$ [96]. Here $F$ is a function of the background scalar field or...
the Ricci scalar and $\eta$ is the coupling constant. Recently, another localization mechanism, the non-minimal coupling $f(\phi)\mathcal{L}_{\text{Elko}}$ between the Elko spinor and the background scalar field, has been investigated in Ref. [77]. Here $\mathcal{L}_{\text{Elko}}$ is the Lagrangian of the Elko spinor and $f(\phi)$ is a function of the background scalar field. It has been shown that by introducing an auxiliary function $K(z)$, the general expression of the Elko zero mode $\alpha_0$ and the scalar function $f(\phi)$ could be obtained. And different forms of $K(z)$ will lead to different solutions of the zero mode and the scalar function. Thus, a non-minimal coupling can also provide the possibility of localizing the Elko zero mode. By now, all of the investigations about the localization of a five-dimensional Elko spinor were concerned with Minkowski branes. As we know, the properties of de Sitter (dS) and Anti-de Sitter (AdS) branes are very different from those of Minkowski branes, and the results of the localization of matter fields on dS and AdS branes are different compared to those on Minkowski branes. Thus, the localization of a five-dimensional Elko spinor on dS and AdS branes is an interesting topic. At the same time, there are two kinds of localization mechanisms. What are the differences and similarities between them also attract our great interest. We believe that investigating the differences and similarities between them will be helpful to further explore the new localization mechanism and expand the possibility of localizing fields.

This paper is organized as follows. We first review the Yukawa-type and non-minimal couplings in Sec. 2. Then in Sec. 3, we consider the localization of the zero mode of a five-dimensional Elko spinor on another AdS thick brane. Finally, a brief conclusion is given in Sec. 5.

2 Review of localization mechanisms

In this section, we review the two localization mechanisms for a five-dimensional Elko spinor in a thick brane model, namely, the Yukawa-like coupling and the non-minimal coupling between the Elko spinor and the background scalar field generating the thick brane.

The line-element is generally assumed as

$$ds^2 = e^{2A(y)}g_{\mu\nu}dx^\mu dx^\nu + dy^2,$$

where the warp factor $e^{2A(y)}$ is a function of the extra dimension $y$. By performing the following coordinate transformation

$$dz = e^{-A(y)}dy,$$

the metric (1) is transformed as

$$ds^2 = e^{2A}(g_{\mu\nu}dx^\mu dx^\nu + dz^2),$$

which is more convenient for discussing the localization of gravity and various matter fields.

2.1 Yukawa-like coupling

Firstly, we start with the action of a five-dimensional massless Elko spinor

$$S = \int d^5x\sqrt{-g}\mathcal{L}_{\text{Elko}},$$

$$\mathcal{L}_{\text{Elko}} = -\frac{1}{4}g^{MN}(\bar{\mathcal{D}}_M \lambda \mathcal{D}_N \lambda + \mathcal{D}_N \lambda \mathcal{D}_M \lambda) - \eta F(\phi) \bar{\lambda} \lambda,$$

where the last term is the Yukawa-like coupling with $F(\phi)$ a function of the background scalar field $\phi$, and $\eta$ is the coupling constant. In this paper, $M,N,\cdots = 0,1,2,3,5$ and $\mu,\nu,\cdots = 0,1,2,3$ denote the five-dimensional and four-dimensional spacetime indices, respectively. The covariant derivatives are

$$\mathcal{D}_M \lambda = (\partial_M + \Omega_M)\lambda, \quad \mathcal{D}_M \lambda = \partial_M \lambda - \bar{\lambda} \Omega_M,$$

where the spin connection $\Omega_M$ is defined as

$$\Omega_M = \frac{i}{2} (\epsilon_{AB} e_B^N \Gamma_{MN}^\rho - e_B^N \partial_M e_{AN}) S^{AB},$$

$$S^{AB} = \frac{i}{4}[\gamma^A, \gamma^B].$$
Here $e^A_M$ is the vierbein and satisfies the orthonormality relation $g_{M N} = e^A_M e^B_N g_{A B}$. $A, B \cdots = 0, 1, 2, 3, 5$ stand for the five-dimensional local Lorentz indices. So the non-vanishing components of the spin connection $\Omega_M$ are

$$\Omega_\mu = \frac{1}{2} \partial_\mu A \gamma_\mu \gamma_5 + \hat{\Omega}_\mu. \quad (7)$$

Here $\gamma_\mu$ and $\gamma_5$ are the four-dimensional gamma matrixes on the brane, and they satisfy $\{\gamma^\mu, \gamma^\nu\} = 2 \delta^{\mu \nu}$.

Then the equation of motion for the Elko spinor coupled with the scalar field is read as

$$\frac{1}{\sqrt{-g}} \mathcal{D}_M (\sqrt{-g} g^{M N} \mathcal{D}_N \lambda) - 2 \eta F(\phi) \lambda = 0. \quad (8)$$

By considering the metric (3) and using the non-vanishing components of the spin connection (7), the above equation can be rewritten as

$$\frac{1}{\sqrt{-g}} \hat{\mathcal{D}}_\mu (\sqrt{-g} g^{\mu \nu} \hat{\mathcal{D}}_\nu \lambda) + \left[ - \frac{1}{4} A^2 \hat{g}^{\mu \nu} \gamma_\mu \gamma_\nu \lambda 
+ \frac{1}{2} A^\prime \left( \frac{1}{\sqrt{-g}} \hat{\mathcal{D}}_\mu (\sqrt{-g} g^{\mu \nu} \gamma^{\nu 5} \lambda) + \hat{g}^{\mu \nu} \gamma^\mu \gamma^5 \hat{\mathcal{D}}_\nu \lambda \right) 
+ e^{-3 A} \partial_\mu (e^{3 A} \partial_\mu \lambda) \right] = 2 \eta e^{2 A} F(\phi) \lambda = 0. \quad (9)$$

Here $\hat{g}_{\mu \nu}$ is the induced metric on the brane, and $\hat{\mathcal{D}}_\mu \lambda = (\partial_\mu + \hat{\Omega}_\mu) \lambda$ with $\hat{\Omega}_\mu$ the spin connection constructed by the induced metric $\hat{g}_{\mu \nu}$. From $\mathcal{D}_\mu e^a_{\nu \nu} = 0$, we can get $\hat{\mathcal{D}}_\mu \hat{g}^{\mu \nu} = \mathcal{D}_\mu (\hat{e}^a_\mu \hat{e}^a_{\nu \nu}) = 0$. Thus, the above equation can be simplified as

$$\frac{1}{\sqrt{-g}} \hat{\mathcal{D}}_\mu (\sqrt{-g} g^{\mu \nu} \hat{\mathcal{D}}_\nu \lambda) - A^{\prime} \gamma^5 \gamma^\mu \hat{\mathcal{D}}_\mu \lambda - A^2 \lambda + e^{-3 A} \partial_\mu (e^{3 A} \partial_\mu \lambda) = 2 \eta e^{2 A} F(\phi) \lambda = 0. \quad (10)$$

Next, we introduce the following KK decomposition

$$\lambda_{\pm} = e^{-3 A/2} \sum_n \left( \alpha_n(z) \zeta_{\pm}^{(n)}(x) + \alpha_n(z) \tau_{\pm}^{(n)}(x) \right)$$

$$= e^{-3 A/2} \sum_n \alpha_n(z) \hat{\lambda}_{\pm}^{(n)}. \quad (11)$$

For simplicity, we omit the $\pm$ subscript for the $\alpha_n$ functions in the following. $\zeta_{\pm}^{(n)}(x)$ and $\tau_{\pm}^{(n)}(x)$ are linear independent four-dimensional Elko spinors, and they satisfy

$$\gamma^\mu \hat{\mathcal{D}}_\mu \zeta_{\pm}^{(n)}(x) = \mp i \kappa_{\pm}(x), \quad \gamma^\mu \hat{\mathcal{D}}_\mu \tau_{\pm}^{(n)}(x) = \pm i \tau_{\pm}(x), \quad (12)$$

$$\gamma^5 \zeta_{\pm}^{(n)}(x) = \pm \tau_{\pm}^{(n)}(x), \quad \gamma^5 \tau_{\pm}^{(n)}(x) = \mp \zeta_{\pm}^{(n)}(x). \quad (13)$$

And the four-dimensional Elko spinor $\hat{\lambda}^{(n)}$ should satisfy the K-G equation:

$$\frac{1}{\sqrt{-g}} \hat{\mathcal{D}}_\mu (\sqrt{-g} g^{\mu \nu} \hat{\mathcal{D}}_\nu \hat{\lambda}^{(n)}) = m^2_n \hat{\lambda}^{(n)}. \quad (14)$$

with $m_n$ the mass of the Elko spinor on the brane. Thus, we can get the following equations of motion for the Elko KK modes $\alpha_n$:

$$\alpha_n'' - \left( \frac{3}{2} A'' + \frac{13}{4} (A')^2 \right) \alpha_n + 2 \eta e^{2 A} F(\phi) - m^2_n + i m_n A' \alpha_n = 0. \quad (15)$$

For the purpose of getting the action of the four-dimensional massless and massive Elko spinors from the action of a five-dimensional massless Elko spinor with Yukawa-like coupling:

$$S_{\text{Elko}} = \int d^5 x \sqrt{-g} \left[ - \frac{1}{4} g^{M N} (\mathcal{D}_M \bar{\lambda} \mathcal{D}_N \lambda + \mathcal{D}_N \bar{\lambda} \mathcal{D}_M \lambda) - \eta F(\phi) \bar{\lambda} \lambda \right]$$

$$= \frac{1}{2} \sum_n \int d^4 x \left[ \frac{1}{2} g^{\mu \nu} (\mathcal{D}_\mu \bar{\lambda}^n \mathcal{D}_\nu \lambda^n + \mathcal{D}_\nu \bar{\lambda}^n \mathcal{D}_\mu \lambda^n) + m^2_n \bar{\lambda}^n \lambda^n \right]. \quad (16)$$
we should introduce the following orthonormality condition for \( \alpha_n \):

\[
\int \alpha_n^* \alpha_m dz = \delta_{nm}. \tag{17}
\]

For the Elko zero mode \( (m_0 = 0) \), Eq. (15) reads

\[
[-\partial_z^2 + V_0^Y(z)]\alpha_0(z) = 0, \tag{18}
\]

where

\[
V_0^Y(z) = \frac{3}{2} A'' + \frac{13}{4} A'^2 + 2\eta e^2A F(\phi). \tag{19}
\]

For this case, the orthonormality condition is given by

\[
\int \alpha_0^* \alpha_0 dz = 1. \tag{20}
\]

As we show in our previous work \cite{94}, there exist many similarities between the Elko field and the scalar field. For a five-dimensional free massless scalar field, the Schrödinger-like equation for the scalar zero mode \( h_0 \) \cite{49,94} can be read as

\[
[-\partial_z^2 + V_0] h_0 = [-\partial_z^2 + \frac{3}{2} A'' + \frac{9}{4} A'^2] h_0 \\
= \left[ \partial_z + \frac{3}{2} A' \right] \left[ -\partial_z + \frac{3}{2} A' \right] h_0 \\
= 0. \tag{21}
\]

The solution is given by \( h_0(z) \propto e^{pA(z)} \) and it satisfies the orthonormality relation for any brane embedded in a five-dimensional Anti-de Sitter (AdS) spacetime. However, the effective potential \( V_0 \) for the five-dimensional free massless Elko spinor is \cite{94}

\[
V_0(z) = \frac{3}{2} A'' + \frac{13}{4} A'^2 = \frac{3}{2} A'' + \frac{9}{4} A'^2 + A^2. \tag{22}
\]

The additional term \( A^2 \) prevents the localization of the zero mode.

When the Yukawa-like coupling is introduced, the coefficient numbers of \( A'' \) and \( A'^2 \) can be regulated:

\[
\frac{3}{2} A'' + \frac{13}{4} A'^2 + 2\eta e^2A F(\phi) = (pA')' + (pA')^2, \tag{23}
\]

where \( p \) is a real constant. From Eq. (23), the form of \( F(\phi) \) can be got

\[
F(\phi) = -\frac{1}{2\eta} e^{-2A} \left[ \left( p - \frac{3}{2} \right) A'' + \left( p^2 - \frac{13}{4} \right) A'^2 \right]. \tag{24}
\]

Then, Eq. (18) can be rewritten as

\[
[-\partial_z^2 + V_0^Y] \alpha_0 = [-\partial_z^2 + pA'' + p^2 A'^2] \alpha_0 \\
= [\partial_z + p A'] [-\partial_z + p A'] \alpha_0 \\
= 0, \tag{25}
\]

and the Elko zero mode reads

\[
\alpha_0(z) \propto e^{pA(z)}. \tag{26}
\]
2.2 Non-minimal coupling

On the other hand, for the non-minimal coupling, the action could be written as

\[ S = \int d^5x \sqrt{-g} f(\phi) \mathcal{L}_{\text{Elko}}, \]

\[ \mathcal{L}_{\text{Elko}} = -\frac{1}{2} g^{MN} \left( \mathcal{D}_M \lambda \mathcal{D}_N \lambda + \mathcal{D}_N \lambda \mathcal{D}_M \lambda \right). \]  \hfill (27)

Here \( f(\phi) \) is a function of the background scalar field \( \phi \), which is only a function of the extra dimension \( z \) (or \( y \)). From the action (27) the following equation of motion can be got

\[ \frac{1}{\sqrt{-g f(\phi)}} \mathcal{D}_M (\sqrt{-g f(\phi)} g^{MN} \mathcal{D}_N \lambda) = 0. \]  \hfill (28)

By considering the metric (9) and using the non-vanishing components of the spin connection (7), we can rewrite Eq. (28) as:

\[ \frac{1}{\sqrt{-g}} \mathcal{D}_\mu (\sqrt{-g} g^{\mu\nu} \mathcal{D}_\nu \lambda) \]

\[ + \frac{1}{2} A'(1 \frac{1}{\sqrt{-g}} \mathcal{D}_\mu (\sqrt{-g} g^{\mu\nu} \gamma_5 \mathcal{D}_\nu \lambda) + \gamma^{\mu\nu} \gamma_5 \mathcal{D}_\nu \lambda)

\[ + e^{-3A} f^{-1}(\phi) \partial_z (e^{3A} f(\phi) \partial_z \lambda) \]

\[ = \frac{1}{\sqrt{-g}} \mathcal{D}_\mu (\sqrt{-g} g^{\mu\nu} \mathcal{D}_\nu \lambda) - A' \gamma_5 \gamma^\mu \mathcal{D}_\nu \lambda - A'^{2} \lambda

\[ + e^{-3A} f^{-1}(\phi) \partial_z (e^{3A} f(\phi) \partial_z \lambda) = 0. \]  \hfill (29)

In this case, we introduce the following KK decomposition:

\[ \lambda_\pm = e^{-3A/2} f(\phi)^{-1/2} \sum_n \left( \alpha_n(z) \varsigma^\pm_n(x) + \alpha_n(z) \tau^\pm_n(x) \right) \]

\[ = e^{-3A/2} f(\phi)^{-1/2} \sum_n \alpha_n(z) \tilde{\lambda}^\pm_n(x). \]  \hfill (30)

By noticing the linear independance of the \( \varsigma^\pm_n \) and \( \tau^\pm_n \)(\( \varsigma^\pm_0 \) and \( \tau^\pm_0 \)) and the K-G equation of the four-dimensional Elko spinor, the equation of motion of the KK mode \( \alpha_n \) can be got

\[ \alpha_n'' - \left( - \frac{1}{4} f^{-2}(\phi) f'^2(\phi) + \frac{3}{2} A' f^{-1}(\phi) f'(\phi) + \frac{1}{2} f^{-1}(\phi) f''(\phi) \right. \]

\[ + \left. \frac{3}{2} A'' + \frac{13}{4} (A')^2 - m_n^2 + i m_n A' \right) \alpha_n = 0. \]  \hfill (31)

For the Yukawa-like coupling case, by introducing the orthonormality conditions (17), we can get the action of the four-dimensional massless and massive Elko spinors from the action (27):

\[ S_{\text{Elko}} = \frac{1}{2} \int d^5x \sqrt{-g} f(\phi) g^{MN} (\mathcal{D}_M \tilde{\lambda} \mathcal{D}_N \lambda + \mathcal{D}_N \tilde{\lambda} \mathcal{D}_M \lambda) \]

\[ = -\frac{1}{2} \sum_n \int d^5x \left[ \frac{1}{2} g^{\mu\nu} (\mathcal{D}_\mu \tilde{\lambda}^n \mathcal{D}_\nu \tilde{\lambda}^n + \mathcal{D}_\nu \tilde{\lambda}^n \mathcal{D}_\mu \tilde{\lambda}^n) + m_n^2 \tilde{\lambda}^n \right]. \]  \hfill (32)

For the Elko zero mode with \( m_n = 0 \), Eq. (31) is simplified as

\[ [-\partial_z^2 + V_0^N(z)] \alpha_0(z) = 0, \]  \hfill (33)

where the effective potential \( V_0^N \) is given by

\[ V_0^N(z) = -\frac{1}{4} f^{-2}(\phi) f'^2(\phi) + \frac{3}{2} A' f^{-1}(\phi) f'(\phi) + \frac{1}{2} f^{-1}(\phi) f''(\phi) + \frac{3}{2} A'' + \frac{13}{4} A'^2. \]  \hfill (34)
and the Elko zero mode $\alpha_0(z)$ satisfies the orthonormality condition. By introducing three new functions $B(z)$, $C(z)$ and $D(z)$ satisfying

$$B(z) = \frac{f'(\phi)}{f(\phi)} = -3A' + \frac{A'^2}{C} - C - \frac{C'}{C},$$ \hspace{1cm} (35)

$$\partial_z D(z) = \frac{3}{2} A' + \frac{1}{2} B + C,$$ \hspace{1cm} (36)

the effective potential reads

$$V_0^N(z) = \frac{1}{4} B^2 + \frac{3}{2} A'B + \frac{1}{2} B' + \frac{3}{2} A'' + \frac{13}{4} A^2$$

$$= D'' + D'^2,$$ \hspace{1cm} (37)

and Eq. (33) can be reduced as

$$[-\partial_z^2 + V_0^N(z)]\alpha_0 = [-\partial_z^2 + D'' + D'^2]\alpha_0$$

$$= [\partial_z + D'] [-\partial_z + D'] \alpha_0$$

$$= 0.$$ \hspace{1cm} (38)

In addition, it will be convenient to define a new function $K(z)$:

$$K(z) = \frac{C'}{C} - C.$$ \hspace{1cm} (39)

It should be noticed that the form of $K(z)$ is arbitrary, and the forms of $C(z)$ and $B(z)$ are determined by the warp factor and any given $K(z)$. Now it is easy to get the zero mode $\alpha_0$:

$$\alpha_0(z) \propto e^{D(z)}$$

$$= \exp \left[ \frac{1}{2} \int_0^z \left( \frac{A'^2}{C} - \frac{C'}{C} + C \right) d\bar{z} \right]$$

$$= \exp \left[ \frac{1}{2} \int_0^z \left( \frac{A'^2}{C} - K \right) d\bar{z} \right]$$

$$= \exp \left[ \frac{1}{2} \int_0^z \frac{A'^2}{C} d\bar{z} \right] \exp \left[ -\frac{1}{2} \int_0^z K d\bar{z} \right],$$ \hspace{1cm} (40)

and the form of $f(\phi)$:

$$f(\phi(z)) = C \frac{e^{\int_0^z B(z) dz}}{C_1 - \int_0^z e^{\int_0^z K(z)dz} d\bar{z}},$$ \hspace{1cm} (42)

where $C_1$ is an arbitrary parameter. As we showed in our previous work, the role of $K(z)$ is similar to the auxiliary superpotential $W(\phi)$, which is introduced in order to solve the Einstein equations in thick brane models. For a given $K(z)$ the zero mode $\alpha_0$ is obtained by integrating Eq. (40). Then, the scalar field function $f(\phi(z))$ is determined by integrating Eq. (41). For different forms of $K(z)$, there exist different configurations of the zero mode $\alpha_0$ and function.
It gives us more choices and possibilities to study the localization of the Elko zero mode on the branes. Next we will consider the localization of the Elko zero mode with this two kinds of couplings on dS/AdS thick branes.

3 Localization of Elko zero mode on dS/AdS thick branes

In this section, we investigate the localization of the Elko zero mode with two kinds of couplings on single-scalar-field generated dS/AdS thick branes [26,66]. The system is described by the action

\[ S = \int d^4 x \sqrt{-g} \left( \frac{M_5}{4} R - \frac{1}{2} \partial \phi \partial \phi - V(\phi) \right), \]

where \( R \) is the five-dimensional scalar curvature and \( V(\phi) \) is the potential of the scalar field. For convenience, the fundamental mass scale \( M_5 \) is set to 1. The line element is described by Eq. (1) and the induced metrics \( \hat{g}_{\mu \nu} \) on the branes read

\[ \hat{g}_{\mu \nu} = \left\{ \begin{array}{ll}
-\hat{a}^2 dt^2 + e^{2\beta t} (dx_1^2 + dx_2^2 + dx_3^2) & \text{dS}_4 \text{ brane}, \\
-\hat{a}^2 dt^2 + e^{-2\beta t} (dx_1^2 + dx_2^2 + dx_3^2) & \text{AdS}_4 \text{ brane}.
\end{array} \right. \]

Here the parameter \( \beta \) is related to the four-dimensional cosmological constant of the dS or AdS brane by \( \Lambda_4 = 3\beta^2 \) or \( \Lambda_4 = -3\beta^2 \) [31,56,66]. By introducing the following scalar potential

\[ V(\phi) = \frac{3}{4} a^2 (1 + A_4) \left[ 1 + (1 + 3s)A_4 \right] \cosh^2 (b\phi) - 3a^2 (1 + A_4)^2 \sinh^2 (b\phi), \]

a brane solution can be obtained [26,66]:

\[ A(y) = -\frac{1}{2} \ln [as(1 + A_4) \sec y], \]

\[ \phi(y) = \frac{1}{b} \arcsinh (\tan y), \]

where \( y \equiv a(1 + A_4)y \). The parameters \( a, s \) and \( b \) are real with \( s \in (0, 1] \) and \( b = \sqrt{\frac{2(1 + A_4)}{3(1 + (1 + s)A_4)}} \). Note that the thick brane is extended in the range \( y \in \left( -\frac{\pi}{2a(1 + A_4)}, \frac{\pi}{2a(1 + A_4)} \right) \). By performing the coordinate transformation (2), we can get

\[ y = \frac{1}{a(1 + A_4)} \left[ 2 \arctan (e^{hz}) - \frac{\pi}{2} \right], \]

with \( h \equiv \sqrt{\frac{1 + A_4}{s}} \). It should be noticed that the range of the coordinate \( z \) will trend to infinite. By substituting the relation (18) into the solution (46) and (47), we can obtain the warp factor and scalar field in the coordinate \( z \) [66]:

\[ A(z) = -\frac{1}{2} \ln \left[ a^2 s(1 + A_4) \cosh^2 (hz) \right], \]

\[ \phi(z) = \frac{1}{b} \arcsinh [\sinh (hz)] = \frac{h}{b} z. \]

The warp factor \( e^{2A(z)} \) is convergent at boundary. When \( A_4 = 0 \), the above solution reduces to the flat brane one.

3.1 Yukawa-like coupling

Firstly, we consider the Yukawa-like coupling mechanism for the localization of the Elko zero mode. According to Eqs. (23)-(26), (49) and (50), the Elko zero mode, the function \( F(\phi) \) and the effective potential \( V_0^Y \) are given by

\[ \alpha_0 \propto e^{\alpha A(z)} = (a^2 s(1 + A_4))^{-\frac{1}{2}} \text{sech}^2 (hz), \]

\[ F(\phi) = -\frac{h^2 a^2 s}{16\eta} (1 + A_4) \left[ 25 - 4p(2 + p) + (4p^2 - 13) \cosh (2b\phi) \right], \]

\[ V_0^Y = pA'' + p^2 A'^2 = -ph^2 \text{sech}^2 (hz) + p^2 h^2 \tanh^2 (hz). \]
Now the orthonormality condition reads
\[
\int \alpha_0^* \alpha_0 dz = \int \alpha_2^* \alpha_2 dz \propto \int (a^2 s(1 + A_4))^{-p} \text{sech}^2(hz) dz
= \frac{4p}{h \eta} (a^2 s(1 + A_4))^{-p} F_1 (p, 2p; 1 + p; -1) < \infty.
\] (54)

It requires \( p > 0 \). In Fig. 1 we plot the shapes of the zero mode, the effective potential \( V_Y^0(z) \) and the function \( F(\phi) \), which show that the Elko zero mode can be localized on the brane. Therefore, the Yukawa-like coupling mechanism can be successfully used to localize the zero mode of the Elko spinor on the dS/AdS thick brane. We can find that the effective potential \( V_Y^0(z) \) is a PT potential. The shape of \( F(\phi) \) has a minimum around \( \phi = 0 \) and diverges when \( \phi \to \infty \). As \( \phi \to \infty \) the boundary values of the warp factor \( e^{2A} \) and \( F(\phi) \) are just opposite because there exists a factor \( e^{-2A} \) in the expression of \( F(\phi) \) (24).

Fig. 1. The shapes of the Elko zero mode \( \alpha_0(z) \) (thick line), the effective potential \( V_Y^0(z) \) (dashed line) are drawn on the left and the shape of function \( F(\phi) \) (right) is drawn on the right. The parameters are set to \( h = b = p = \eta = a^2 s(1 + A_4) = 1 \).

### 3.2 Non-minimal coupling

Next, we focus on the non-minimal coupling mechanism. As shown in our previous work [97], there exist different configurations of the Elko zero mode and \( f(\phi) \) for different choices of \( K(z) \). In this paper we will consider two kinds of \( K(z) \) and investigate the localization of the Elko zero mode (40).

#### 3.2.1 \( K(z) = -kA' \)

Firstly, it is a natural choice to consider \( K(z) = -kA' \) with \( k \) a positive constant. It is easy to get
\[
C(z) = \frac{1}{C_1 \cosh(hz)^{-k} - \frac{\cosh(hz) F(z) \sqrt{-\sinh^2(hz)}}{h + k}}
\] (55)

Here \( F(z) = 2F_1 \left( \frac{1}{2}, \frac{1+k}{2}, \frac{3+k}{2}; \cosh^2(hz) \right) \). Especially, when \( k = 1 \), the form of \( C(z) \) will be reduced to
\[
C(z) = \frac{h \cosh(hz)}{h C_1 - \sinh(hz)}.
\] (56)
Here $C_1$ is an arbitrary constant and we always let $C_1 = 0$. Thus, the zero mode is rewritten as

$$\alpha_0(z) \propto e^{D(z)} \exp \left[ \frac{1}{2} \int_0^z A'^2 \, dz \right] \exp \left[ -\frac{1}{2} \int_0^z K \, dz \right] \propto \text{sech}(h z) \exp \left[ -\frac{1}{4} \text{sech}^2(h z) \right].$$

(57)

It is easy to check that the orthonormality condition

$$\int \alpha_0^* \alpha_0 \, dz = \int \alpha_0^2 \, dz \propto \int \text{sech}^2(h z) \exp \left[ -\frac{1}{2} \text{sech}^2(h z) \right] \, dz = 2 \sqrt{2} F(\sqrt{2}/2) < \infty$$

(58)

can be satisfied. Here, $F(z)$ gives the Dawson integral and $F(\sqrt{2}/2) = 0.512496$. For this case, the effective potential $V_0^N(z)$ is given by

$$V_0^N(z) = \frac{h^2}{32} (-2 - 5 \cosh(2h z) - 10 \cosh(4h z) + \cosh(6h z)) \, \text{sech}^6(h z).$$

(59)

We plot the shapes of the zero mode and the effective potential in Fig. 2 from which we can see that the zero mode (57) is localized on the brane and the effective potential is a PT-like potential. The function $f(\phi)$ reads

$$f(\phi(z)) = \frac{a^4 s^2}{h^2} (1 + A_4)^2 \cosh^3(\phi(z)) \, \tanh^2(\phi(z)) \exp \left[ -\frac{1}{2} \text{sech}^2(\phi(z)) \right],$$

(60)

and it is plotted in Fig. 2 with $h = b = a^2 s(1 + A_4) = 1$. It is obvious that the shape of $f(\phi)$ is similar to $F(\phi)$ in the previous subsection, and it has a minimum at the point of $\phi = 0$ and diverges at infinity. The boundary behaviour of $f(\phi)$ is also opposite to the warp factor, because there exists a factor $e^{-3A}$ in the expression of $f(\phi)$ (41).

![Fig. 2. The shapes of the Elko zero mode $\alpha_0(z)$ and the effective potential $V_0^N(z)$ are drawn on the left and the shape of function $f(\phi)$ is drawn on the right. The parameters are set to $h = b = a^2 s(1 + A_4) = 1$.](image)

### 3.2.2 $K(z) = k \phi$

Another natural choice for $K(z)$ is $K(z) = k \phi = k h z$ with positive $k$, for which the form of $C(z)$ is

$$C(z) = -\frac{e^{\frac{h^2}{2} z^2}}{\text{Erfi} \left( \sqrt{\frac{h^2}{2}} z \right)},$$

(61)
where \( \bar{k} \equiv k \frac{h}{b} \) and Erfi \( \left( \sqrt{\frac{k}{2} z} \right) \) is the imaginary error function. Thus the zero mode reads

\[
\alpha_0(z) \propto \exp \left[ \frac{1}{2} h^2 \sqrt{\frac{\pi}{2k}} \mathcal{L}(z) - \frac{\bar{k} z^2}{4} \right].
\] (62)

where

\[
\mathcal{L}(z) \equiv \int_0^z \text{Erfi} \left( \sqrt{\frac{k}{2} \bar{z}} \right) \tanh^2(h \bar{z}) \exp \left[ -\frac{\bar{k} \bar{z}^2}{2} \right] d\bar{z}.
\] (63)

It can be verified that the function \( \mathcal{L}(z) \) approaches a constant as \( |z| \to \infty \), which shows that \( \alpha_0(|z| \to \infty) \propto \exp \left[ -\frac{\bar{k} z^2}{2} \right] \) and its orthonormality condition can be satisfied. Thus, the zero mode (62) can be localized on the brane, see Fig. 3. The effective potential \( V_0^N(z) \) and the corresponding function \( f(\phi) \) have slightly complex forms and we only show their shapes in Fig. 3. Here, we can find the effective potential is an infinite deep potential instead of a PT one. However, the shape of \( f(\phi) \) is still similar to the ones in the previous two subsections. Therefore, the functions \( F(\phi) \) and \( f(\phi) \) have similar properties and play similar roles, although they appear in different places in the actions.

![Fig. 3. The shapes of the Elko zero mode \( \alpha_0(z) \) (thick line), the effective potential \( V_0^N(z) \) (dashed line) are drawn on the left and the shape of function \( f(\phi) \) (right) is drawn on the right. The parameters are set to \( \bar{k} = h = b = a^2 s (1 + \Lambda_4) = 1 \).](image)

4 Localization of Elko zero mode on AdS thick brane with divergent warp factor

In this section, we will consider another kind of AdS thick brane model and investigate the localization of the Elko zero mode with two kinds of couplings. The warp factor in previous section is convergent at the boundaries of the extra dimension, but the one in this section will be divergent, for which the zero mode of a five-dimensional free scalar field can not be localized on this brane [56]. This difference will bring us some different and interesting results. The action of this system reads [56]

\[
S = \int d^5x \sqrt{-g} \left[ \frac{1}{2} R - \frac{1}{2} g^{MN} \partial_M \phi \partial_N \phi - V(\phi) \right],
\] (64)

where \( R \) is the five-dimensional scalar curvature. Note that \( M_5 \) is set to 2 here. The metric is described by (3) and the induced metric is \( \hat{g}_{\mu \nu} = e^{-2\beta x_3} (-dt^2 + dx_1^2 + dx_2^2 + dx_3^3) \) with \( A_4 = -3\beta^2 \). For the scalar potential

\[
V(\phi) = -\frac{3(1 + 3\delta)\beta^2}{2\delta} \cosh^{2(1-\delta)} \left( \frac{\phi}{\phi_0} \right),
\] (65)

a thick AdS brane solution was given in Refs. [24][56]:

\[
A(z) = -\delta \ln \left| \cos \left( \frac{\beta}{\delta} z \right) \right|,
\] (66)

\[
\phi(z) = \phi_0 \text{arcsinh} \left( \tan \left( \frac{\beta}{\delta} z \right) \right)
\] (67)
with
\[ \phi_0 \equiv \sqrt{3\delta(\delta - 1)}. \] (68)

Here, the range of the extra dimension is \(-z_b \leq z \leq z_b\) with \(z_b = \left| \frac{\delta\pi}{2} \right|\) and the parameter \(\delta\) satisfies \(\delta > 1\) or \(\delta < 0\). It was found that only when \(\delta > 1\), there exists a thick 3-brane which localizes at \(|z| \approx 0 [24,56]\). Thus, we only consider the case of \(\delta > 1\). In this case, the warp factor \(e^{2A(z)}\) is diverge at the boundaries \(z = \pm z_b\).

4.1 Yukawa-like coupling

With the Yukawa-like coupling and by substituting Eqs. (66) and (67) into (23)-(26), the Elko zero mode \(\alpha_0\), the function \(F(\phi)\) and the effective potential \(V_0^Y\) read
\[
\alpha_0 \propto e^{pA(z)} = \cos^{-p\delta} \left( \frac{\beta}{\delta} z \right),
\] (69)
\[
F(\phi) = \frac{\beta^2}{8\delta\beta} \left[ 6 - 4p + (6 + 13\delta - 4p(1+p\delta)) \sinh^2 \left( \frac{\phi}{\phi_0} \right) \right] \cosh^{-2\delta} \left( \frac{\phi}{\phi_0} \right),
\] (70)
\[
V_0^Y = pA'' + p^2 A'^2 = \frac{p\beta^2}{\delta} \sec^2 \left( \frac{\beta}{\delta} z \right) + p^2 \beta^2 \tan^2 \left( \frac{\beta}{\delta} z \right).
\] (71)

And the orthonormality condition requires \(p\delta < 0\):
\[
\int \alpha_0^*\alpha_0 \, dz = \int \alpha_0^2 \, dz
\]
\[
\propto \int \frac{\beta}{\delta} \cos^{-2p\delta} \left( \frac{\beta}{\delta} z \right) \, dz
\]
\[
= \frac{\sqrt{\pi} \Gamma \left( \frac{1}{2} - p\delta \right)}{\Gamma(1-p\delta)} < \infty.
\] (72)

Therefore, the zero mode can be localized on this AdS thick brane for any negative \(p\) by introducing the Yukawa-like coupling mechanism. We plot the zero mode, the effective potential \(V_0^Y(z)\) and the function \(F(\phi)\) in Fig. 4. The effective potential is an infinite deep potential. And unlike previous model, here, \(F(\phi)\) has the shape of a volcano, this is because the boundary behavior of the warp factor is changed compared to the previous section.

Fig. 4. The shapes of the Elko zero mode \(\alpha_0(z)\) (thick line), the effective potential \(V_0^Y(z)\) (dashed line) are drawn on the left and the shape of function \(F(\phi)\) (right) is drawn on the right. For visibility, here \(\alpha_0(z)\) has been magnified 100 times. The parameters are set to \(\eta = -1, \delta = \beta = 2, \) and \(p = -5\).
4.2 Non-minimal coupling

Finally, we turn to the non-minimal coupling mechanism by considering $K(z) = kA'$ with $k$ a positive constant. Note that the sign in front of $k$ is just opposite to the one in the previous section. When $k = \frac{\Delta}{\delta}$, the function $C(z)$ in (42) has the following form

$$C(z) = \frac{\bar{k} \sec(\bar{k}z)}{\ln \left( \frac{\cos(\bar{k}z) - \sin(\bar{k}z)}{\cos(\bar{k}z) + \sin(\bar{k}z)} \right)},$$

(73)

where $\bar{k} \equiv \frac{\beta}{\delta}$ and the parameter $C_1$ is set to be zero. Then the zero mode reads

$$\alpha_0(z) \propto \cos^+(\bar{k}z) \exp \left[ \frac{1}{2} k \delta^2 \int_0^z \sin(\bar{k}z) \tan(\bar{k}z) \ln \left( \frac{\cos(\bar{k}z) - \sin(\bar{k}z)}{\cos(\bar{k}z) + \sin(\bar{k}z)} \right) d\bar{z} \right].$$

(74)

It is easy to check that the above $\alpha_0(z)$ vanishes when $z \to z_b$ and the orthonormality condition for the Elko zero mode $\int \alpha_0^* \alpha_0 dz = 1$ can be satisfied. Therefore, the zero mode is localized on the brane (see Fig. 5). The effective potential is an infinite deep one and the function $f(\phi)$ has a closed form but we only show their shapes in Fig. 5. It can be found that all of the functions $F(\phi)$ and $f(\phi)$ have a minimum round $\phi = 0$ in both thick brane models.

Fig. 5. The shapes of the Elko zero mode $\alpha_0(z)$ (thick line), the effective potential $V^N_0(z)$ (dashed line) are drawn on the left and the shape of function $f(\phi)$ (right) is drawn on the right. For visibility, here $\alpha_0(z)$ has been magnified 5 times. The parameters are set to $\beta = \delta = 2$.

5 Conclusion and discussion

In this paper, we introduced two localization mechanisms to investigate the localization of the zero mode of a five-dimensional Elko spinor on dS/AdS thick branes. Firstly, we reviewed the two localization mechanisms, i.e, the Yukawa-type coupling and the non-minimal coupling. It showed that in order to obtain the Elko zero mode on a brane, the form of $F(\phi)$ in the Yukawa-type coupling mechanism is determined by the warped factor (i.e., $F(\phi) = -\frac{1}{2}\eta e^{-2A} \left[p - \frac{3}{2} A'' + \left(p^2 - \frac{13}{4}\right) A'^2\right]$), and the function $f(\phi)$ in the non-minimal coupling mechanism is determined by introducing an auxiliary function $K(z)$. Then, we considered two kinds of curved thick brane models and investigated the localization of the Elko zero mode with the two kinds of localization mechanisms.

In the first brane model, we considered the dS/AdS thick brane generated by a single scalar field. The results showed that for the Yukawa-type coupling, the zero mode can be localized on the brane under the condition $p > 0$. It is interesting to note that the factor $e^{-2A}$ in the expression of $F(\phi)$ leads to the result that the boundary behavior of $F(\phi)$ is opposite to the warp factor $e^{2A}$. Thus the shape of $F(\phi)$ diverges when $\phi \to \infty$ because of the convergent warp factor. For the non-minimal coupling mechanism, by introducing two different forms of the auxiliary function $K(z)$, the zero mode can be confined on the brane. And the coupling functions $f(\phi)$ with two different forms of $K(z)$ are similar to the $F(\phi)$ for the Yukawa-type coupling in this brane system. Therefore, these two coupling functions
have similar properties and play similar roles although they appear in different places in the action of five-dimensional Elko spinor. This result may help us to explore a new localization mechanism and expand the possibility of localizing Elko spinor.

Next, another AdS thick brane with divergent warp factor was considered. For the Yukawa-type coupling case, the zero mode can be localized on this AdS thick brane for any negative $p$, which is just opposite to the condition in the previous thick brane model. Because of the divergent warp factor in this brane system, the shape of $F(\phi)$ seems to be a volcano. For the non-minimal coupling case with the given auxiliary function $K(z)$, we obtained the localized zero mode on the brane and the coupling function $f(\phi)$. The function $f(\phi)$ still has a closed form. An interesting result is that both $F(\phi)$ and $f(\phi)$ always have a minimum around $\phi = 0$ no matter what the warp factor is.

In this paper, we gave the expressions of the coupling functions in order to localize the zero mode of a five-dimensional Elko spinor on curved thick branes. We found that the effective potential of the Schrödinger-like equation satisfied by the zero mode is a PT potential or an infinite deep potential. However, it should be noticed that the equation satisfied by the massive KK modes is a complex one, which is quite different from that satisfied by the zero mode. Thus the effective potential function in this paper is not applicable for the massive KK modes. And we cannot judge whether there exists a massive KK mode although the effective potential is PT-like or infinite. In the future, we will investigate the localization of the massive Elko KK modes on different thick branes.

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