Total Roman {3}-Domination: The Complexity and Linear-Time Algorithm for Trees

Xinyue Liu, Huiqin Jiang, Pu Wu and Zehui Shao

Institute of Computing Science and Technology, Guangzhou University, Guangzhou 510006, China; xinyue050420@outlook.com or 2111906061@e.gzhu.edu.cn (X.L.); hq.jiang@hotmail.com or 2111906061@e.gzhu.edu.cn (H.J.); puwu1997@126.com or 2111806056@e.gzhu.edu.cn (P.W.)

Abstract: For a simple graph $G = (V, E)$ with no isolated vertices, a total Roman $\{3\}$-dominating function (TR3DF) on $G$ is a function $f : V(G) \to \{0, 1, 2, 3\}$ having the property that (i) $\sum_{v \in N(v)} f(w) \geq 3$ if $f(v) = 0$; (ii) $\sum_{v \in N(v)} f(w) \geq 2$ if $f(v) = 1$; and (iii) every vertex $v$ with $f(v) \neq 0$ has a neighbor $u$ with $f(u) \neq 0$ for every vertex $v \in V(G)$. The weight of a TR3DF $f$ is the sum $f(V) = \sum_{v \in V(G)} f(v)$ and the minimum weight of a total Roman $\{3\}$-dominating function on $G$ is called the total Roman $\{3\}$-domination number denoted by $\gamma_{\{3\}}(G)$. In this paper, we show that the total Roman $\{3\}$-domination problem is NP-complete for planar graphs and chordal bipartite graphs. Finally, we present a linear-time algorithm to compute the value of $\gamma_{\{3\}}(G)$ for trees.

Keywords: dominating set; total roman $\{3\}$-domination; NP-complete; linear-time algorithm

1. Introduction

Let $G = (V, E)$ be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. For every vertex $v \in V$, the open neighborhood $N_G(v) = N(v) = \{u \in V(G) : uv \in E(G)\}$ and the closed neighborhood $N_G[v] = N[v] = N(v) \cup \{v\}$. We denote the degree of $v$ by $d_G(v) = d(v) = |N_G(v)|$. A vertex of degree one is called a leaf and its neighbor is a support vertex, and a support vertex is called a strong support if it is adjacent to at least two leaves. Let $S_0$ be a star with order $n$. A tree $T$ is an acyclic connected graph. $G = (G_1 \cup G_2)$ is a union graph $G$ such that $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$.

Given a graph $G$ and a positive integer $k$, assume that $f : V(G) \to \{0, 1, 2, ... , k\}$ is a function, and suppose that $(V_0, V_1, ..., V_k)$ is the ordered partition of $V$ induced by $f$, where $V_i = \{v \in V(G) : f(v) = i\}$ for $i \in \{0, 1, ..., k\}$. Then we can write $f = (V_0, V_1, ..., V_k)$ and $\omega_f(V(G)) = \sum_{v \in V(G)} f(v)$ is the weight of a function $f$. A subset $S$ of a vertex set $V(G)$ is a dominating set of $G$ if for every vertex $v \in V(G) \setminus S$, there exists a vertex $w \in S$ such that $uv$ is an edge of $G$. The domination number of $G$ denoted by $\gamma(G)$ is the smallest cardinality of a dominating set $S$ of $G$ [1]. A function $f : V(G) \to \{0, 1\}$ is called a dominating function (DF) on $G$ if every vertex $u$ with $f(u) = 0$ has a vertex $v \in N(u)$ such that $f(v) = 1$ [2]. The dominating set problem (DSP) is to find the domination number of $G$, which has been deeply and widely studied in recent years [3–7].

A subset $S$ of a vertex set $V(G)$ is a total dominating set of $G$ if $\bigcup_{v \in S} N(v) = V(G)$. The total domination number of $G$ denoted by $\gamma_t(G)$ is the smallest cardinality of a total dominating set $S$ of $G$ [8]. The dominating function of $G$ denoted by $\gamma_i(G)$ is the smallest cardinality of a total dominating set $S$ of $G$ [9]. The literature on the subject of total domination in graphs has been surveyed and provided in detail in a recent book [9]. Moreover, Michael A. Henning et al. presented a survey of selected recent results on total domination in graphs [10].

The mathematical concept of Roman domination is originally defined and discussed by Stewart et al. [11] and ReVelle et al. [12]. A Roman dominating function (RDF) on graph $G$ is a function $f : V(G) \to \{0, 1, 2\}$ such that every vertex $v \in V(G)$ for which $f(u) = 0$ is adjacent to at least one vertex $u$ with $f(u) = 2$ [13]. The Roman domination number of
G is the minimum weight overall RDFs, denoted by $\gamma_R(G)$ [14]. On the basis of Roman domination, signed Roman domination [15], double Roman domination [16] and total Roman domination [17] have been proposed recently.

The total Roman dominating function(TRDF) on G is an RDF $f$ on G with an additional property that every vertex $v \in V (G)$ with $f(v) \neq 0$ has a neighbor $u$ with $f(u) \neq 0$. Let $\gamma_{TR}(G)$ denote the minimum weight of all TRDFs on G. A TRDF on G with weight $\gamma_{TR}(G)$ is called a $\gamma_{TR}(G)$-function. The conception of TRDF was first defined by Hossein Ahangar et al. [18]. In addition, Nicolás Campanelli et al. studied the total Roman domination number of the lexicographic product of graphs [17] and Chloe Lampman et al. presented some basic results of Edge-Critical Graphs [19].

The Roman $\{2\}$-dominating function (also named Italian domination) $f$ [20] introduced by Chellali et al. which is defined as follows: $f: V(G) \rightarrow \{0,1,2,3\}$ has the property that $\sum_{u \in N(v)} f(u) \geq 2$ for $f(v) = 0$ [21]. Chellali et al. proved that the Roman $\{2\}$-domination problem is NP-complete for bipartite graphs [21]. Hangdi Chen showed that the Roman $\{2\}$-domination problem is NP-complete for split graphs, and gave a linear-time algorithm for finding the minimum weight of Roman $\{2\}$-dominating function in block graphs [22]. As a generalization of Roman domination, Michael A. Henning et al. studied the relationship between Roman $\{2\}$-domination and dominating set parameters in trees [20].

A Roman $\{3\}$-dominating function(RDF) $f$ defined by Mojdeh et al. [23], which is defined as follows: $f: V(G) \rightarrow \{0,1,2,3\}$ has the property that for every vertex $v \in V(G)$ with $f(v) \in \{0,1\}$ and $\sum_{u \in N(v)} f(u) \geq 3$. Mojdeh et al. presented an upper bound on the Roman $\{3\}$-domination number of a connected graph G, characterized the graphs attaining upper bound and showed that the Roman $\{3\}$-domination problem is NP-complete, even restricted to bipartite graphs [23].

The total Roman $\{3\}$-domination [24] was studied recently. The total Roman $\{3\}$-dominating function(TRDF) on a graph G is an RDF on G with the additional property that every vertex $v \in V(G)$ with $f(v) \neq 0$ has a neighbor $u$ with $f(u) \neq 0$. The minimum weight of a total Roman $\{3\}$-dominating function on G denoted by $\gamma_{t(R3)}(G)$ is named the total Roman $\{3\}$-domination number of G. A $\gamma_{t(R3)}(G)$-function is a total Roman $\{3\}$-dominating function on G with weight $\gamma_{t(R3)}(G)$. Doost Ali Mojdeh et al. showed the relationship among total Roman $\{3\}$-domination, total domination, and total Roman $\{2\}$-domination parameters. They also presented an upper bound on the total Roman $\{3\}$-domination number of a connected graph G and characterized the graphs arriving this bound. Finally, they investigated that total Roman $\{3\}$-domination problem is NP-complete for bipartite graphs [24].

In this paper, we further investigate the complexity of total Roman $\{3\}$-domination in planar graphs and chordal bipartite graphs. Moreover, we give a linear-time algorithm to compute the $\gamma_{t(R3)}$ for trees which answer the problem that it is possible to construct a polynomial algorithm for computing the number of total Roman $\{3\}$-domination for trees [24].

2. Complexity

In this section, we study the complexity of total Roman $\{3\}$-domination of graph. We show that the total Roman $\{3\}$-domination problem is NP-complete for planar graphs and chordal bipartite graphs. Consider the following decision problem.

**Total Roman $\{3\}$-Domination Problem TR3DP.**

**Instance:** Graph $G = (V, E)$, and a positive integer $m$.

**Question:** Does $G$ have a total Roman $\{3\}$-function with weight at most $m$?

Please note that the dominating set problem is NP-complete for planar graphs [25] and chordal bipartite graphs [26]. We show the NP-completeness results by reducing the well-known NP-complete problem, dominating set, to TR3D.
Let $G$ be a graph on $n$ vertices. Let $T_v$ be the tree with $V(T_v) = \{v, v_a, v_b, v_c, v_d, v_e, v_f, v_p, v_q\}$, $E(T_v) = \{v v_a, v v_c, v v_e, v v_f, v v_p, v v_q, v v_d v_p, v v_d v_q\}$, as depicted in Figure 1.

![Figure 1. The tree $T_v$.](image)

Let $G'$ be the graph obtained by adding edges between $v' \in T_{v'}$ and $v'' \in T_{v''}$ if $v'v'' \in E(G)$ from the union of the trees $T_v$ for $v \in V(G)$. Please note that $|V(G')| = n \times |V(T_v)| = 9n$ and $|E(G')| = |E(G)| + n \times |E(T_v)| = |E(G)| + 8n$.

**Lemma 1.** If $G$ is a planar graph or chordal bipartite graph, so is $G'$.

**Lemma 2.** ([24]) Let $S_n$ be a star with $n \geq 3$, then $\gamma_{T_3(D_3)}(S_n) = 4$.

**Lemma 3.** Let $g$ be a TR3DF of $G$. If $v$ is a strong support vertex of $G$, then $\omega_g(N[v]) \geq 4$.

**Proof of Lemma 3.** Let $v_1, v_2, \ldots, v_k$ be leaves of $v$ with $k \geq 2$. Since $g(N[v]) \geq 3$ for $i \in \{1, 2, \ldots, k\}$, we have $g(v_i) \geq 3 - g(v)$ for $i \in \{1, 2, \ldots, k\}$. Then $\omega_g(N[v]) = g(v) + \sum_{i \in \{1, 2, \ldots, k\}} g(v_i) \geq g(v) + g(v_1) + g(v_2) \geq 6 - g(v)$. If $g(v) \leq 2$, it is clear that $\omega_g(N[v]) \geq 4$. If $g(v) = 3$, there exists a vertex $u \in N(v)$ with $g(u) \neq 0$. Then $\omega_g(N[v]) \geq 4$. □

**Lemma 4.** If $f$ is a DF of $G$ with $\omega_f(G) \leq \ell$, then there exists a TR3DF $g$ of $G'$ with $\omega_g(G') \leq \ell + 8n$.

**Proof of Lemma 4.** For each $v \in V(G)$, we define $g$ as follows: $V(T_v) \rightarrow \{0, 1, 2, 3\}$, $g(v_a) = g(v_b) = 1, g(v_c) = g(v_d) = 3, g(v) = f(v), g(x) = 0$ otherwise. It is clear that $g$ is a TR3DF of $G'$. Therefore we have that $\omega_g(G') = \omega_f(G) + 8n \leq \ell + 8n$. □

**Claim 1.** Let $g$ be a TR3DF of $G'$, then $\omega_h(T_v') \geq 8$.

**Proof of Claim 1.** By Lemmas 2, 3 and definition, we have that $\omega_h(N[v]) \geq 4$ and $\omega_h(N[v]) \geq 4$. Since $N(v) \cap N(v') = \emptyset$, then we can reduce $\omega_h(T_v') = \omega_h(N[v]) + \omega_h(N[v']) \geq 8$. □

**Claim 2.** If there exists a TR3DF $h$ of $G'$ with $h(v_a) + h(v_b) \geq 3$ for $v_a, v_b \in V(T_v)$, then there exists a TR3DF $g$ of $G'$ such that $\omega_g(G') \leq \omega_h(G')$ and $g(v_a) + g(v_b) \leq 2$.

**Proof of Claim 2.** By the definition of TR3DF, we have $\omega_h(N[v]) \geq 3$ and $\omega_h(N[v]) \geq 3$, then we have $\omega_h(T_v') \geq 9$.

If $h(v) = 0$, then we define $g : V(G') \rightarrow \{0, 1, 2, 3\}$ such that $g(v_e) = g(v_f) = g(v_p) = g(v_q) = 0, g(v_c) = g(v_b) = g(v_d) = 1, g(x) = g(x) = h(x)$ otherwise, seeing Figure 2. Therefore g is a TR3DF of $G'$ such that $g(v_a) + g(v_b) \leq 2$ and $\omega_g(G') = \omega_h(G')$.

If $h(v) \geq 1$, then we define $g : V(G') \rightarrow \{0, 1, 2, 3\}$ such that $g(v_e) = g(v_f) = g(v_p) = g(v_q) = 0, g(v_a) = g(v_b) = 1, g(v_c) = g(v_d) = 3, g(x) = h(x)$ otherwise. Therefore g is a TR3DF of $G'$ such that $g(v_a) + g(v_b) \leq 2$ and $\omega_g(G') \leq \omega_h(G')$. □
By Lemmas 1, 4, 5, the total Roman \(3\)-domination problem is NP-complete for planar graphs and chordal bipartite graphs.

**Lemma 5.** If \(g\) is a TR3DF of \(G\) with \(\omega_{g}(G') \leq \ell + 8n\), then there exists a DF \(f\) of \(G\) with \(\omega_{f}(G) \leq \ell\).

**Proof of Lemma 5.** By Claim 2, w.l.o.g, let \(g\) be a TR3DF of \(G'\) with \(g(v_a) + g(v_b) \leq 2\) for \(v_a, v_b \in V(T')\), \(v \in V(G)\). Define \(f : V(G) \to \{0, 1, 2, 3\}\) such that \(f(v) = g(v)\) if \(g(v) \leq 1\), and \(f(v) = 1\) if \(g(v) \geq 2\). For each vertex \(v \in V(G)\), since \(g(v_a) + g(v_b) \leq 2\), we have \(g(v) \geq 1\) or there exists a vertex \(u \in N(v) \cap V(G)\) such that \(g(u) \geq 1\). Therefore \(f\) is DSF of \(G\) and \(\omega_{f}(G) \leq \omega_{g}(G) - 8n \leq \ell\) by Claim 1. \(\square\)

**Theorem 1.** By Lemmas 1, 4, 5, the total Roman \(3\)-domination problem is NP-complete for planar graphs and chordal bipartite graphs.

### 3. A Linear-Time Algorithm for Total Roman \(3\)-Domination in Trees

In this section, we present a linear-time algorithm to compute the minimum weight of total Roman \(3\)-dominating function for trees. First, we define the following concepts:

**Definition 1.** Let \(u\) be a vertex of \(G\), and let \(F_{u,G}^{(i,j)}\) on \(G\) be a function \(f : V(G) \to \{0, 1, 2, 3\}\) having the property that (i) \(f(u) = i\), \(\sum_{w \in N(u)} f(w) \geq j\); (ii) \(\forall v \in V(G) \setminus \{u\}, \sum_{p \in N(v)} f(p) \geq 3\) if \(f(v) \leq 2\) and \(\sum_{p \in N(v)} f(p) \geq 1\) if \(f(v) = 3\).

**Definition 2.** The minimum weight overall \(F_{u,G}^{(i,j)}\) functions on \(G\) denoted by \(\gamma_{u,G}^{(i,j)}(u, G)\) is the \(F_{u,G}^{(i,j)}\) number of \(G\), and a \(\gamma_{u,G}^{(i,j)}(u, G)\)-function is an \(F_{u,G}^{(i,j)}(u, G)\) with weight \(\gamma_{u,G}^{(i,j)}(u, G)\).

**Definition 3.** Let \(\text{coil}(x)\) be a function defined as follows: \(\text{coil}(x) = \begin{cases} x, x \geq 0; \\ 0, x < 0. \end{cases}\)

**Lemma 6.** For any graph \(G\) with specific vertex \(u\), we have

\[
\gamma_{1,R3}(G) = \min\{\gamma_{1,R3}^{(0,3)}(u, G), \gamma_{1,R3}^{(1,2)}(u, G), \gamma_{1,R3}^{(2,1)}(u, G), \gamma_{1,R3}^{(3,1)}(u, G)\}.
\]

**Lemma 7.** Suppose \(T_1\) and \(T_2\) are trees with specific vertices \(v\) and \(u\), respectively. Let \(T_3\) be the tree with the specific vertex \(u\), which is obtained by joining a new edge \(uv\) from the union of \(T_1\) and \(T_2\), as depicted in Figure 3.
Algorithm 1 Counting \( \gamma_{(i,j)} \) in trees.

Input: A tree \( T \) with a tree ordering \([v_1, v_2, \ldots, v_n]\).

Output: The TR3D number \( \gamma_{(i,j)}(T) \) of \( T \).

1. for \( p = 1 \) to \( n \) do
2.  for \( i = 0 \) to 3, \( j = 0 \) to 3 do
3.    if \( i = 0 \) then
4.      \( \gamma_{(i,j)}(v_p) \leftarrow i; \)
5.    else
6.      \( \gamma_{(i,j)}(v_p) \leftarrow \infty; \)
7.  for \( p = 1 \) to \( n - 1 \) do
8.    let \( v_q \) be the parent of \( v_p \)
9.    for \( i = 0 \) to 3 and \( j = 0 \) to 3 do
10.       if \( i = 0 \) then
11.         \( \gamma_{(i,j)}(v_q) = \min \{ \gamma^{(i,0)}_s(v_p) + \gamma^{(i,0)}(v_q) \mid s = 0, 1, 2 \}; \)
12.       else
13.         \( \gamma_{(i,j)}(v_q) = \min \{ \gamma^{(i,0)(3-i-s)}(v_p) + \gamma^{(i,0)}(v_q) \mid s = 0, 1, 2, 3 \}; \)
14. return \( \min \{ \gamma^{(0,3)}(v_n), \gamma^{(1,2)}(v_n), \gamma^{(2,1)}(v_n), \gamma^{(3,1)}(v_n) \} \)
4. Conclusions

The total Roman \{3\}-domination problem was introduced and studied in [24], and it was proven to be NP-complete for bipartite graphs. In this paper, we prove that the total Roman \{3\}-domination problem is NP-complete for planar graphs or chordal bipartite graphs, and showed a linear-time algorithm for total Roman \{3\}-domination problem on trees. For the algorithmic aspects of the total Roman \{3\}-domination problem, designing exact algorithms or approximation algorithms on general graphs, or polynomial algorithms for total Roman \{3\}-domination problem on some special classes graphs deserve further research.

Author Contributions: Conceptualization, X.L., H.J. and Z.S.; writing, X.L. and Z.S.; review, H.J. and Z.S.; investigation: P.W. All authors have contributed equally to this work. All authors have read and agreed to the possible publication of the manuscript.

Funding: This work is supported by the Natural Science Foundation of Guangdong Province under Grant 2018A0303130115.

Conflicts of Interest: The authors declare no conflict of interest.

Abbreviations

The following abbreviations are used in this manuscript:

- DF Dominating function
- DSP Dominating set problem
- TRDF Total Roman dominating function
- \text{R}3\text{DF} Roman \{3\}-domination
- TR3DF Total Roman \{3\}-domination

References

1. Mojdeh, D.A.; Firoozi, P.; Hasni, R. On Connected (\gamma, k)-critical Graphs. Australas. J. Comb. 2010, 46, 25–36.
2. Thulasiraman, K.; Swamy, M.N.S. Graphs: Theory and Algorithms; Wiley: Hoboken, NJ, USA 1992.
3. Stojmenovic, I.; Seddigh, M.; Zunic, J. Dominating Sets and Neighbor Elimination-Based Broadcasting Algorithms in Wireless Networks. IEEE Trans. Parallel Distrib. Syst. 2002, 13, 14–25. [CrossRef]
4. Haynes, T.W.; Henning, M.A. Domination in Graphs; CRC Press: Boca Raton, FL, USA, 1998.
5. Lund, C.; Yannakakis, M. On the Hardness of Approximating Minimization Problems. J. ACM 1994, 41, 960–981. [CrossRef]
6. Cockayne, E.J.; Dawes, R.M.; Hedetniemi, S.T. Total domination in Graphs. Networks 1980, 10, 211–219. [CrossRef]
7. Haynes, T.W.; Hedetniemi, S.; Slater, P. Fundamentals of Domination in Graphs; Marcel Dekker: New York, NY, USA, 1998.
8. Cockayne, E.J.; Kinnersley, W.B.; West, D.B.; Zamani, R. Extremal Problems for Game domination Number. SIAM J. Discret. Math. 2013, 27, 2090–2107. [CrossRef]
9. Cockayne, E.J.; Dreyer, P.A.; Hedetniemi, S.M.; Hedetniemi, S.T. Roman Domination in Graphs. Discret. Math. 2004, 278, 11–22. [CrossRef]
10. Chambers, E.W.; Dinitz, J.H.; Slamin, D. Extremal Problems for Roman Domination. SIAM J. Discret. Math. 2009, 23, 1575–1586. [CrossRef]
11. Abdollahzadeh, H.A.; Henning, A.M.; Löwenstein, C.; Zhao, Y.; Samodivkin, V. Signed Roman domination in Graphs. J. Comb. Optim. 2014, 27, 241–255. [CrossRef]
12. Beeler, R.A.; Haynes, T.W.; Hedetniemi, S.T. Double Roman domination. Discret. Appl. Math. 2016, 211, 23–29. [CrossRef]
13. Campanelli, N.; Kuziak, D. Total Roman domination in the Lexicographic Product of Graphs. Discret. Appl. Math. 2019, 263, 88–95. [CrossRef]
14. Abdollahzadeh, H.A.; Henning, A.M.; Samodivkin, V.; Yero, G.I. Total Roman domination in Graphs. Appl. Anal. Discret. Math. 2016, 10, 501–517. [CrossRef]
15. Lampman, C.; Mynhardt, K.; Ogden, S. Total Roman domination Edge-Critical Graphs. Involve J. Math. 2019, 12, 1423–1439. [CrossRef]
16. Henning, M.A.; Klostermeyer, W.F. Italian Domination in Trees. Discret. Appl. Math. 2017, 217, 557–564. [CrossRef]
21. Chellali, M.; Haynes, T.W.; Hedetniemi, S.T.; McRae, A.A. Roman \([2]\)-domination. *Discret. Appl. Math.* 2016, 204, 22–28. [CrossRef]

22. Chen, H.; Lu, C. A Note on Roman \([2]\)-domination Problem in Graphs. *arXiv* 2018, arXiv:1804.09338.

23. Mojdeh, D.A.; Volkmann, L. Roman \([3]\)-domination (Double Italian domination). *Discret. Appl. Math.* 2020. 283, 555–564. [CrossRef]

24. Shao, Z.; Mojdeh, D.A.; Volkmann, L. Total Roman \([3]\)-domination in Graphs. *Symmetry* 2020, 12, 268. [CrossRef]

25. Zverovich, I.E.; Zverovich, V.E. An Induced Subgraph Characterization of Domination Perfect Graphs. *J. Graph Theory* 1995, 20, 375–395. [CrossRef]

26. Müller, H.; Brandstädt, A. The NP-completeness of Steiner Tree and Dominating set for Chordal Bipartite Graphs. *Theor. Comput. Sci.* 1987, 53, 257–265. [CrossRef]