Critical behavior in the rotating D-branes

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Abstract

The low energy excitation of the rotating D3-branes is thermodynamically stable up to a critical angular momentum density. This indicates that there is a corresponding phase transition of the $\mathcal{N}=4$ large $N$ super Yang-Mills theory at finite temperature. On the side of supergravity, we investigate the phase transition in the grand canonical ensemble and canonical ensemble. Some critical exponents of thermodynamic quantities are calculated. They obey the static scaling laws. Using the scaling laws related to the correlation length, we get the critical exponents of the correlation function of gauge field. The thermodynamic stability of low energy excitations of the rotating M5-branes and rotating M2-branes is also studied and similar critical behavior is observed. We find that the critical point is shifted in the different ensembles and there is no critical point in the canonical ensemble for the rotating M2-branes. We also discuss the Hawking-Page transition for these rotating branes. In the grand canonical ensemble, the Hawking-Page transition does not occur. In the canonical ensemble, however, the Hawking-Page transition may appear for the rotating D3- and M5-branes, but not for the rotating M2-branes.

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I. INTRODUCTION

Maldacena [1] has conjectured that there is a duality between the large $N$ limit of certain conformal field theories in various dimensions and supergravity on the product of anti-de Sitter spacetimes, spheres and other compact manifolds. Thus, some questions concerning large $N$ gauge theories may be answered via supergravity. For instance, one can calculate the correlation functions of the gauge theory [2,3], study thermodynamics and phase structure of the strong coupling gauge theory [4–10].

According to the Maldacena’s conjecture, thermodynamics of certain conformal field theories in the large $N$ limit can be described by thermodynamics of black holes in anti-de Sitter spacetimes. And the latter is found to be quite different from thermodynamics of black holes in flat spacetimes. Hawking and Page [11] found that the heat capacity is positive for large mass black holes, while is negative for small mass black holes. This indicates that there is a phase transition for the corresponding conformal field theory. In the high temperature phase, the black hole dominates the contribution to the conformal field theory; in the low temperature phase, the thermal anti-de Sitter space makes the dominant contribution. This phase transition has been interpreted as the confinement/deconfinement transition in the Super Yang-Mills theory [3].

As an explicit example of the Maldacena’s conjecture, the low energy fluctuations of $N$ coincident D3-branes are supposed to be described by $\mathcal{N}=4$ super Yang-Mills theory with gauge group $U(N)$ in four dimensions [12]. Indeed, the entropy of the near extremal black three-branes in the type IIB supergravity can be correctly reproduced by the super-Yang-Mills theory [13], apart from a numerical factor. With this in mind that the result in supergravity should correspond to that for the strong coupling gauge theory and the result in [13] was given for the free gauge theory, the disagreement therefore is not a matter. More recently, the coupling constant dependence in the thermodynamics of $\mathcal{N}=4$ super Yang-Mills theory has been investigated [14]. Considered the strong coupling limit, the result of the super Yang-Mills theory is supposed to be in agreement with that derived from supergravity.

In the past few months, much attention has been focused on the rotating D-branes. For instance, the rotating D-branes has been used to extend Witten’s QCD model [5] by introducing angular momentum parameters in order to decouple the unwanted Kaluza-Klein particles [15,16]. At zero temperature the Coulomb Branch of $\mathcal{N}=4$ super Yang-Mills theory is described in supergravity by multi-center solutions with D3-brane charge. At finite temperature and chemical potential the vacuum degeneracy is lifted, and minima of the free energy are shown to have a supergravity description as rotating black D3-branes [17]. In addition, Gubser [18] has found that the spinning black three-branes in the type IIB supergravity are thermodynamically stable up to a critical value of the angular momentum density. Inside the region of thermodynamic stability, the free energy from supergravity can be roughly reproduced by a naive model based on the free $\mathcal{N}=4$ super Yang-Mills theory on the world volume. The field theory model can correctly predict a limit on the angular momentum density. Recall that the angular momentum density corresponds to the density of R-charge of gauge field on the world volume. This implies that there exists a phase transition of the $\mathcal{N}=4$ super Yang-Mills theory at the critical value of the R-charge density.

In the present paper we will study the phase transition of the super Yang-Mills theory on the side of supergravity by looking at the critical behavior in the grand canonical ensemble.
and canonical ensemble, respectively. Some critical exponents related to thermodynamic quantities are obtained and they satisfy the static scaling laws. Further we derive the critical exponents of the correlation function of gauge field. We also discuss the Hawking-Page transition in the rotating black branes. The critical behavior in the rotating D3-branes will be discussed in the next section. In sections III and IV we extend this discussion to rotating M5-branes and M2-branes and investigate this kind of phase transition for the corresponding conformal fields on the world volumes, respectively. We present our main results in section V.

II. CRITICAL BEHAVIOR IN ROTATING D3-BRANES

A. The solution and thermodynamics

The rotating black three-branes in the type IIB supergravity may have three angular momentum parameters. Since we are interested in the critical behavior occurring at the critical angular momentum density, to show the scaling behavior, it is sufficient to consider an angular momentum parameter nonvanishing only. Such kind of solution has been given in [15,18,19]. The metric is

\[
ds^2 = \frac{1}{\sqrt{f}} \left( -h dt^2 + dx_1^2 + dx_2^2 + dx_3^2 \right) + \sqrt{f} \left[ \frac{dr^2}{h} - \frac{4ml \cosh \alpha}{r^4 \Delta f} \sin^2 \theta dt d\phi \\
+ r^2 (\Delta d\theta^2 + \bar{\Delta} \sin^2 \theta d\phi^2 + \cos^2 \theta d\Omega_3^2) \right],
\]

where

\[
f = 1 + \frac{2m \sinh^2 \alpha}{r^4 \Delta},
\]
\[
\Delta = 1 + \frac{l^2 \cos^2 \theta}{r^2},
\]
\[
\bar{\Delta} = 1 + \frac{l^2}{r^2} + \frac{2ml \sin^2 \theta}{r^6 \Delta f},
\]
\[
h = 1 - \frac{2m}{r^4 \Delta},
\]
\[
\bar{h} = \frac{1}{\Delta} \left( 1 + \frac{l^2}{r^2} - \frac{2m}{r^4} \right).
\]

The rotating black three-branes (2.1) have the horizon at \( r_+ \) determined by the equation, \( \bar{h} = 0 \), the horizon is

\[
r_+^2 = \frac{1}{2} \left( \sqrt{l^4 + 8m} - l^2 \right).
\]

From Eq. (2.1) we can directly read off the ADM mass and angular momentum per unit world volume

\[
3
\[ M = \frac{5\pi^3 m}{\kappa^2} \left( 1 + \frac{4}{5} \sinh^2 \alpha \right), \quad (2.3) \]

\[ J = \frac{2\pi^3 m}{\kappa^2} l \cosh \alpha. \quad (2.4) \]

Here \( \kappa^2 \) is related to the gravitational constant in ten dimensions through \( \kappa^2 = 8\pi G_{(10)}. \) Using the charge quantization condition [20], one may have the number \( N \) of the coincident D3-branes

\[ N = \frac{4\pi^{5/2} m}{\kappa} \sinh \alpha \cosh \alpha. \quad (2.5) \]

The angular velocity \( \Omega_H \) of the rotating black three-branes is

\[ \Omega = \frac{lr_+^2}{2m \cosh \alpha}. \quad (2.6) \]

The Hawking temperature and entropy of the rotating black three branes are

\[ T = \frac{r_+(2r_+^2 + l^2)}{4\pi m \cosh \alpha}, \quad (2.7) \]

\[ S = \frac{4\pi^4 m r_+}{\kappa^2} \cosh \alpha. \quad (2.8) \]

In fact, the entropy is the entropy density per unit world volume (throughout this paper all extensive quantities are taken as corresponding densities). These thermodynamic quantities obey the first law of black hole thermodynamics

\[ dM = T dS + \Omega dJ + \Phi dN, \quad (2.9) \]

where the chemical potential \( \Phi \) corresponding to the number \( N \) of the D3-branes \( \Phi = \pi^{1/2} \kappa^{-1} \sinh \alpha / \cosh \alpha. \) Since we are interested in the connection between the low energy excitation of the D3-branes and the \( N=4 \) large \( N \) super Yang-Mills theory, we have to hold \( N \) fixed and to consider the near-extremal configuration of the black three-branes. The near-extremal limit may be approached by taking \( m \to 0 \) and \( \alpha \to \infty, \) while \( N \) is kept as a constant. In this near-extremal limit, all physical quantities can be expressed in terms of the parameters \( m \) and \( l. \) Using \( e^{2\alpha} \approx \frac{\rho N}{m \pi^{5/2}}, \) these related thermodynamic quantities become

\[ E = 3\pi^3 \kappa^{-2} m, \]

\[ J = \pi^{7/4} \kappa^{-3/2} N^{1/2} m^{1/2} l, \]

\[ \Omega = \pi^{5/4} \kappa^{-1/2} N^{-1/2} m^{-1/2} l r_+^2, \]

\[ T = 2^{-1} \pi^{1/4} \kappa^{-1/2} N^{-1/2} m^{-1/2} (2r_+^3 + l^2 r_+), \]

\[ S = 2\pi^{11/4} \kappa^{-3/2} N^{1/2} m^{1/2} r_+. \quad (2.10) \]

Here \( r_+ \) is still given by Eq. (2.2) and \( E \) denotes the energy density of excitations which equals the ADM mass density of the black three-brane minus the mass density of the extremal one. It is easy to verify that these quantities in Eqs.(2.10) satisfy the first law of thermodynamics.
\[ dE = TdS + \Omega dJ. \] (2.11)

This rotating three-brane has the same quantum numbers as \( N \) coincident D3-brane with a density of R-charge on the world volume equal to the angular momentum density \( J \). For the D3-brane case, the R-symmetry group is precisely the rotating group SO(6) in the dimensions transverse to the brane. Thus the angular velocity \( \Omega \) conjugate to the angular momentum density can be called the voltage \([18]\). These thermodynamic quantities in Eqs. (2.10) should describe the thermodynamics of the \( \mathcal{N}=4 \) large \( N \) super Yang-Mills theory on the world volume.

**B. Grand canonical ensemble**

Since we keep the spatial volume of the world volume unchanged, to discuss the critical behavior of the conformal field theory on this world volume, it is convenient to use a grand canonical ensemble as done in a magnetic system \([22,23]\). In this ensemble, the appropriate thermodynamic potential is the Gibbs free energy, which is defined in this case as

\[ G = E - TS - \Omega J, \] (2.12)

and its infinitesimal variation is

\[ dG = -SdT - Jd\Omega. \] (2.13)

In this ensemble, two thermodynamic variables are temperature \( T \) and the voltage \( \Omega \). Here the temperature \( T \) is just the Hawking temperature \([13]\) and the voltage \( \Omega \) is the angular velocity of the black three-brane. According to the definition \( C_\Omega = T \left( \frac{\partial S}{\partial T} \right)_\Omega \), we find that the heat capacity at the constant voltage,

\[ C_\Omega = 2\pi^{1/4} \kappa^{-3/2} N^{1/2} \frac{\sqrt{mr_+ (2r_+^2 + l^2)(3r_+^2 - l^2)}}{(r_+^2 + l^2)(2r_+^2 - l^2)}, \] (2.14)

is positive if \( l^2 < 2r_+^2 \) or \( l^2 > 3r_+^2 \), and negative between them; it is divergent at \( l^2 = 2r_+^2 \). Note from Eq. (2.2) that \( l^2 < 2r_+^2 \) is equivalent to the condition

\[ l^4/m < 8/3, \] (2.15)

which just corresponds to the stability condition \( \chi \equiv \frac{27\pi^2}{8N^2} \frac{l^4}{E^3} < 1/3 \) found in \([18]\), there to derive this condition Gubser has used the argument that the Hessian matrix of the entropy with respect to the energy and angular momentum densities has no positive eigenvalues. According to the duality between the conformal field theory and supergravity on anti-de Sitter spacetimes \([1]\), this indicates that the \( \mathcal{N}=4 \) large \( N \) super Yang-Mills theory has a phase transition, which is characterized by the R-charge. The critical point is at \( l^4/m = 8/3 \). Beyond this critical point, the heat capacity is negative. This should be pointed out that this does not mean that the gauge field enters the negative heat capacity phase, but implies that the description via the rotating black three-branes breaks down, as happens in the black holes in anti-de Sitter spaces \([3]\). At this critical point, the critical temperature \( T_c \) and critical voltage \( \Omega_c \) are
\[ T_c = 2\pi^{1/4}3^{-1/2}\kappa^{-1/2}N^{-1/2}l, \]
\[ \Omega_c = 2^{1/2}3^{-1/2}\pi^{5/4}\kappa^{-1/2}N^{-1/2}l, \]  
which satisfy \( \Omega_c/T_c = 1/\sqrt{2\pi} \).

Now we are going to investigate the critical behavior of the conformal field theory at this critical point. Before doing so, it is of some interest to compare this system with a magnetic system. As is well known, there is a continuous phase transition occurring from an ordered ferromagnetic state to a paramagnetic state in a magnetic system [22,23]. The critical point is at zero applied magnetic field \( H \) and at the critical temperature \( T_c \) the derivative of the magnetization \( M \) diverges. In this magnetic system, the variation of the Gibbs free energy is [22]

\[ dG = -SdT - MdH. \]  
Therefore, there is an analogy between our system and the magnetic system: \( M \leftrightarrow J, \) \( H \leftrightarrow \Omega \). Due to this analogy, one can define some critical exponents related to certain thermodynamic quantities. As usual, a critical exponent describing the behavior near the critical point of a general function \( f(x) \) is defined as

\[ \sigma = \lim_{\epsilon \to 0} \frac{\ln f(\epsilon)}{\ln \epsilon}, \quad \epsilon = \frac{x - x_c}{x_c}, \]  
where \( \sigma \) is called the critical exponent of the function \( f(x) \) and \( x_c \) is the critical value of variable \( x \). Following the definition of critical exponents for a magnetic system [22,23], for our system we define

\[ C_\Omega = T \left( \frac{\partial S}{\partial T} \right)_\Omega \sim \left\{ \begin{array}{ll} \epsilon_{\Omega}^{-\alpha} & \text{for } \epsilon_\Omega = 0 \\ \epsilon_{\Omega}^{\varphi} & \text{for } \epsilon_T = 0, \end{array} \right. \]  
for the heat capacity at the constant voltage, where \( \epsilon_T = \frac{T - T_c}{T_c} \) and \( \epsilon_\Omega = \frac{\Omega - \Omega_c}{\Omega_c} \). For the charge density

\[ J - J_c \sim \left\{ \begin{array}{ll} \epsilon_{\Omega}^{\delta} & \text{for } \epsilon_\Omega = 0 \\ \epsilon_{\Omega}^{\delta-1} & \text{for } \epsilon_T = 0. \end{array} \right. \]  
For the isothermal capacitance

\[ \chi_T = \left( \frac{\partial J}{\partial \Omega} \right)_T \sim \left\{ \begin{array}{ll} \epsilon_{\Omega}^{\gamma} & \text{for } \epsilon_\Omega = 0 \\ \epsilon_{\Omega}^{\gamma-1} & \text{for } \epsilon_T = 0. \end{array} \right. \]  
And for the entropy

\[ S - S_c \sim \left\{ \begin{array}{ll} \epsilon_{\Omega}^{1-\alpha} & \text{for } \epsilon_\Omega = 0 \\ \epsilon_{\Omega}^{\psi} & \text{for } \epsilon_T = 0. \end{array} \right. \]  

A remarkable feature of critical points is the so-called static scaling hypothesis [22], which asserts that near the critical point, the singular part of the Gibbs free energy is a generalized homogeneous function. For our case, we may write
\[ G(\lambda^p \epsilon_T, \lambda^q \epsilon_\Omega) = \lambda G(\epsilon_T, \epsilon_\Omega), \quad (2.23) \]

where \( p \) and \( q \) are two scaling parameters, and \( \lambda \) is an arbitrary constant. Thus, those critical exponents defined in Eqs. (2.19)-(2.22) can be expressed in terms of these two constants \( p \) and \( q \). With the help of Eq. (2.13), one may arrive at

\[ C_\Omega = -T \left( \frac{\partial^2 G}{\partial T^2} \right)_\Omega, \quad J = - \left( \frac{\partial G}{\partial \Omega} \right)_T, \quad \chi_T = - \left( \frac{\partial^2 G}{\partial \Omega^2} \right)_T, \quad S = - \left( \frac{\partial G}{\partial T} \right)_\Omega. \quad (2.24) \]

And then one has

\[ \alpha = 2 - \frac{1}{p}, \quad \beta = \frac{1 - q}{p}, \quad \delta = \frac{q}{1 - q}, \]
\[ \gamma = \frac{2q - 1}{p}, \quad \psi = \frac{1 - p}{q}, \quad \varphi = \frac{2p - 1}{q}. \quad (2.25) \]

Eliminating \( p \) and \( q \), one has a set of equalities among those critical exponents called scaling laws [22,23]

\[ \alpha + 2 \beta + \gamma = 2, \]
\[ \alpha + \beta(\delta + 1) = 2, \]
\[ \gamma(\delta + 1) = (2 - \alpha)(\delta - 1), \]
\[ \gamma = \beta(\delta - 1), \]
\[ (2 - \alpha)(\delta \psi - 1) + 1 = (1 - \alpha)\delta, \]
\[ \varphi + 2\psi - \delta^{-1} = 1. \quad (2.26) \]

It should be noticed that these equations are not independent.

Now we calculate the critical exponents to show that they satisfy the scaling laws (2.26). Using those thermodynamic quantities in Eqs. (2.10), we obtain

\[ \chi_T = \kappa^{-1} \pi^{1/2} N m r^{-2} \left( \frac{2r_+}{r_+^4} + \frac{5l^2 r_+^2}{r_+^4} - l^4 \frac{2r_+^2}{r_+^4} + l^2 \right). \quad (2.27) \]

According to the definitions (2.19)-(2.22), we find

\[ \alpha = \varphi = \beta = \psi = \gamma = \delta^{-1} = 1/2. \quad (2.28) \]

The critical exponent of the heat capacity at the constant voltage was also obtained in [18]. Note from Eq. (2.28) that a peculiar feature of these critical exponents is that they are all equal to 1/2. However, it is easy to verify that they indeed satisfy the static scaling laws (2.26).

An important physical quantity describing phase transitions is the correlation function, which in general has the form [22,24]

\[ G(r) \sim \frac{e^{-r/\xi}}{r^{d-2+\eta}}, \quad (2.29) \]
where $\xi$ is the correlation length and it diverges at the critical points, $\eta$ is the Fisher exponent, and $d$ is the dimensionality of space. Defining the critical exponent $\nu$ related to the correlation length as

$$\xi \sim \epsilon^{-\nu} \quad \text{for} \quad \epsilon_\alpha = 0,$$

one has a set of equalities related to $\nu$ and $\eta$:

$$d \nu = 2 - \alpha,$$

$$d - 2 + \eta = \frac{2d}{\delta + 1} = \frac{2d\beta}{2 - \alpha} = \frac{2\beta}{\nu},$$

$$(2 - \eta)\nu = \gamma,$$

$$d\frac{\delta - 1}{\delta + 1} = 2 - \eta.$$

Once again, these equations are not independent, either. According to the Maldacena’s conjecture, the correlation function of some conformal field theory can be calculated. However, the correlation function for the $\mathcal{N}=4$ super-Yang-Mills theory at finite temperature seems not yet to have been calculated. Now we are going to get the critical exponents $\nu$ and $\eta$ using the scaling laws (2.31). For the low energy physics on the D3-brane, its world volume is four dimensional, so we have $d = 3$. Thus one can get easily from Eqs. (2.31)

$$\nu = 1/2, \quad \eta = 1.$$

It would be interesting to verify the critical behavior of the correlation function using the naive field theory model suggested in [18].

For the static black three branes, it is known that its thermodynamics is always stable, which is indicated by the positive definite heat capacity. As a consequence, the gauge field described by the supergravity solution is always in the high temperature deconfinement phase. That is, the Hawking-Page phase transition does not take place in the static D3-branes. Because of the above instability of thermodynamics, it becomes interesting to investigate whether the Hawking-Page phase transition appears or not in the rotating D3-branes. Recall that the criterion of the Hawking-Page phase transition is the sign change of the Euclidean action $I$ of black hole solutions [11]. In the grand canonical ensemble, we have $I = T^{-1}G$. From Eqs. (2.10) we have

$$G = \frac{1}{2} \pi^3 \kappa^{-2} r_+ (r_+^2 + l^2),$$

from which it is easy to see that the Euclidean action is always negative. We conclude therefore that the Hawking-Page phase transition does not occur in the rotating D3-branes; as in the static D3-branes, the system is always in the high temperature deconfinement phase, despite the instability of thermodynamics.

### C. Canonical ensemble

In [22] it is proven that if a function with two variables $f(x, y)$ is a generalized homogeneous function, then the Lagrange transforms of $f(x, y)$ are also generalized homogeneous
functions. This result has the immediate consequence that if one of thermodynamic potentials is a generalized homogeneous function near the critical point, other thermodynamic potentials must also be generalized homogeneous functions \[22\]. For instance, for a magnetic system, if the Gibbs free energy has the scaling relation

\[G(\lambda^a \epsilon, \lambda^a H) = \lambda G(\epsilon, H),\]

the Helmholtz free energy then must have the scaling relation

\[F(\lambda^a \epsilon, \lambda^a M) = \lambda F(\epsilon, M),\]

with \(a_M = 1 - a_H\). In fact, this is a consequence of the equivalence of thermodynamic ensembles. In the previous subsection we have discussed the critical behavior in the grand canonical ensemble. In this subsection we do the same thing, but in the canonical ensemble. In this case, the appropriate thermodynamic function is the Helmholtz free energy defined as

\[F = E - TS, \quad \text{(2.34)}\]

and its variation is

\[dF = -SdT + \Omega dJ. \quad \text{(2.35)}\]

In this ensemble the thermodynamic variables are the temperature \(T\) and the charge density \(J\). Other thermodynamic quantities can be expressed in terms of them. We note that the heat capacity at the constant charge

\[C_J = \left(\frac{\partial E}{\partial T}\right)_J = 12\pi^{11/4} \kappa^{-3/2} \eta^{1/2} m^{3/2} r^{-1} \frac{2r^2 + t^2}{2r^4 + 5l^2 r^2 - t^4}, \quad \text{(2.36)}\]

is positive when

\[l^4/m < (l^4/m)_c = \frac{19 + \sqrt{297}}{4} \approx 9.058; \quad \text{(2.37)}\]

negative as \(l^4/m > (l^4/m)_c\); and diverges at \((l^4/m)_c\). Comparing with Eq. \((2.15)\), it is interesting to observe that these two critical points are not in agreement with each other. At \(l^4/m = 8/3\), no special happens in the heat capacity \(C_J\). Also we can check that no first derivatives and second derivatives of the Helmholtz free energy diverge at \(l^4/m = 8/3\). This implies that in this ensemble \(l^4/m = 8/3\) is not a critical point. Instead \(l^4/m = 9.058\) is a critical point according to the definition of critical points: one of second derivatives of certain thermodynamic potential diverges there. Indeed two physical quantities \(C_J\) and \(\chi_T^{-1}\), which are second derivatives of the Helmholtz free energy with respect to \(T\) and \(J\), respectively, diverge at \(l^4/m = (l^4/m)_c\). At this critical point, the critical temperature is

\[T_c = 2^{-3/2} \pi^{1/4} \kappa^{-1/2} \eta^{-1/2} l x^{-1/4} (8 + x)^{1/2} ((8 + x)^{1/2} - x^{1/2})^{1/2}, \quad \text{(2.38)}\]

where \(x = (19 + \sqrt{297})/4\). Using corresponding definitions of critical exponents, that is, in Eqs. \((2.19)-(2.22)\) replacing \(C_\Omega\) by \(C_J\), \(\chi_T\) by \(\chi_J^{-1}\), \(J\) by \(\Omega\), and \(\epsilon_\Omega\) by \(\epsilon_J\), we find that these critical exponents have the same values as those in Eq. \((2.28)\), i.e.,
\[ \alpha = \varphi = \beta = \psi = \gamma = \delta^{-1} = 1/2, \]

and then they also satisfy the same scaling laws \((2.26)\).

This interesting result gives rise to the question whether these two ensemble are equivalent or not. From our results it indicates that these two ensembles are not equivalent to one another. The critical point \(l_4/m = 8/3 \approx 2.667 \) in the grand canonical ensemble is shifted to \(l_4/m \approx 9.058 \) in the canonical ensemble. Indeed there have been some arguments that thermodynamic ensembles are not equivalent in the self-gravitating system \([23,24]\). And the thermodynamic stability depends on the chosen environment (ensemble) \([27]\). Therefore, the stability boundary in \([18]\) is given in the grand canonical ensemble. In the canonical ensemble the stability boundary is \(l_4/m < 9.058 \). In both of two critical points, corresponding critical exponents are same and satisfy the static scaling laws. Furthermore, in the canonical ensemble, the Euclidean action is \(T^{-1}F\), from Eqs. \((2.10)\) we have

\[ F = -\frac{1}{2} \pi^3 \kappa^{-2} r_+^2 (r_+^2 - l^2). \]

This free energy is negative if \(r_+^2 > l^2\), that is \(l_4/m < 1\). But it becomes positive as \(l_4/m > 1\) and changes its sign at \(l_4/m = 1\). This indicates that the Hawking-Page phase transition may occur in the canonical ensemble.

### III. CRITICAL BEHAVIOR IN ROTATING M5-BRANES

In this section we discuss the critical behavior for the rotating M5-branes in M theory. Similar to the previous section, we also consider the case in which an angular momentum parameter does not vanish only. The rotating M5-brane solution has been found by Cvetic and Youm in \([21]\). The metric can be written down as

\[
\begin{align*}
\text{ds}_{11}^2 &= f^{-\frac{1}{2}}(-h dt^2 + dx_1^2 + \cdots + dx_5^2) + f^\frac{1}{2} \left[ \frac{dr^2}{h} + r^2 (\Delta d\theta^2 + \tilde{\Delta} \sin^2 \theta d\phi^2 \\
&+ \cos^2 \theta d\Omega_5^2) - \frac{4ml \cosh \alpha}{r^3 \Delta f} \sin^2 \theta dt d\phi \right],
\end{align*}
\]

where

\[
\begin{align*}
f &= 1 + \frac{2m \sinh^2 \alpha}{r^3 \Delta}, \\
\Delta &= 1 + \frac{l^2 \cos^2 \theta}{r^2}, \\
\tilde{\Delta} &= 1 + \frac{l^2}{r^2} + \frac{2ml \sin^2 \theta}{r^5 \Delta f}, \\
h &= 1 - \frac{2m}{r^3 \Delta}, \\
\tilde{h} &= \frac{1}{\Delta} \left[ 1 + \frac{l^2}{r^2} - \frac{2m}{r^3} \right].
\end{align*}
\]
Through a double dimensional reduction, the rotating black M5-brane solution can be reduced to a rotating black D4-brane solution in the type IIA supergravity. In the reduction, thermodynamics is kept unchanged. The rotating black M5-brane (3.1) has the horizon determined by the positive real root of the equation

\[ r_+^3 + l^2 r_+ - 2m = 0. \]  

(3.2)

Those thermodynamic quantities relevant to the rotating black M5-branes are readily obtained. The result is

\[ M = \frac{32\pi^2 m}{3\kappa^2} \left( 1 + \frac{3}{4} \sinh^2 \alpha \right), \]  

(3.3)

\[ J = \frac{16\pi^2 m l}{3\kappa^2} \cosh \alpha, \]  

(3.4)

\[ \Omega = \frac{lr_+}{2m \cosh \alpha}, \]  

(3.5)

\[ T = \frac{8\pi m \cosh \alpha}{3r_+^2 + l^2}, \]  

(3.6)

\[ S = \frac{32\pi^3 m r_+}{3\kappa^2} \cosh \alpha. \]  

(3.7)

where \( \kappa^2 = 8\pi G_{(11)} \) related to the gravitational constant in eleven dimensions. According to the charge quantization condition \[20\], the number \( N \) of the coincident M5-branes is

\[ N = 2^{10/3} \pi^{5/3} \kappa^{-2/3} m \cosh \alpha \sinh \alpha. \]  

(3.8)

These quantities trivially satisfy the first law of black hole thermodynamics

\[ dM = TdS + \Omega dJ + \Phi dN, \]

where the chemical potential \( \Phi = 2^{-1/3} \pi^{-4/3} \kappa^{-2/3} \sinh \alpha / \cosh \alpha \). As in the case of D3-branes, to connect the thermodynamics of black M5-branes and that of the conformal field theory on the world volume, we take the near-extremal limit. Taking \( m \to 0 \) and \( \alpha \to \infty \) and keeping \( N \) fixed, we obtain

\[ E = 20\pi^2 3^{-1} \kappa^{-2} m, \]

\[ J = 2^{7/3} 3^{-1} \pi^{7/6} \kappa^{-5/3} N^{1/2} m^{1/2} l, \]

\[ \Omega = 2^{2/3} \pi^{5/6} \kappa^{-1/3} N^{-1/2} m^{-1/2} l r_+, \]

\[ T = 2^{-4/3} \pi^{-1/6} \kappa^{-1/3} N^{-1/2} m^{-1/2} (3r_+^2 + l^2), \]

\[ S = 2^{10/3} 3^{-1} \pi^{13/6} \kappa^{-5/3} N^{1/2} m^{1/2} r_+. \]  

(3.9)

We first consider the thermodynamic stability of rotating M5-branes in grand canonical ensemble. The heat capacity at the constant angular velocity (voltage) is

\[ C_\Omega = 2^{10/3} 3^{-1} \pi^{13/6} \kappa^{-5/3} N^{1/2} m^{1/2} r_+ \frac{15r_+^4 + 3l^2 r_+^2 - 2l^4 3r_+^2 + l^2}{3r_+^4 + 4l^2 r_+^2 + l^4 3r_+^2 - l^2}. \]  

(3.10)

This heat capacity is positive when
\[ l^2 < 3r_+^2, \quad \text{that is,} \quad l^3/m < \sqrt{27}/2 \]  \hspace{1cm} (3.11)

or \( 2l^4 > 15r_+^4 + 3l^2r_+^2 \), negative between them, and diverges at \( l^2 = 3r_+^2 \). Therefore the thermodynamically stable boundary is \( l^3/m < \sqrt{27}/2 \) in this ensemble. At the critical point \( l^2 = 3r_+^2 \), the isothermal capacitance also diverges as

\[ \chi_T = \left( \frac{\partial J}{\partial \Omega} \right)_T = 2^{5/3} \pi^{1/3} \kappa^{-4/3} N m r_+^{-1} \left( r_+^4 + 4l^2r_+^2 - l^4 \right) 3r_+^4 + 4l^2r_+^2 + l^4 3r_+^2 - l^2. \]  \hspace{1cm} (3.12)

The critical temperature \( T_c \) and critical voltage \( \Omega_c \) are

\[ T_c = 2^{-5/6} 3^{1/4} \pi^{1/6} \kappa^{-1/3} N^{-1/2} l^{1/2}, \]
\[ \Omega_c = 2^{-7/6} 3^{1/4} \pi^{5/6} \kappa^{-1/3} N^{1/2} l^{1/2}, \]  \hspace{1cm} (3.13)

they obey \( \Omega_c/T_c = 2\pi/\sqrt{3} \). Furthermore, one can easily find that those critical exponents defined in Eqs. (2.19)-(2.22) in this case have the same value as those in Eq. (2.28). However, due to the fact that the low energy excitation of M5-branes should be described by a six dimensional \((0,2)\) conformal field theory in the large \( N \) limit \([\text{I}]\), the critical exponents related to the correlation function will be changed. Considering \( d = 5 \), we find from Eqs. (2.31)

\[ \nu = 3/10, \quad \eta = 1/3. \]  \hspace{1cm} (3.14)

We now turn to the canonical ensemble. In this ensemble, some second derivatives of the Helmholtz free energy are characterized by the heat capacity at constant angular momentum (R-charge) and the inverse isothermal capacitance \( \chi_T^{-1} \). Note that both the heat capacity

\[ C_J = 2^{13/3} 3^{-2} 5^{13/6} \kappa^{-5/3} N^{1/2} m^{3/2} \frac{3r_+^2 + l^2}{r_+^4 + 4l^2r_+^2 - l^4}, \]  \hspace{1cm} (3.15)

and \( \chi_T^{-1} \) diverge at

\[ l^4 = r_+^4 + 4l^2r_+^2. \]  \hspace{1cm} (3.16)

Obviously, this critical point is different from that in Eq. (3.11). That is, the critical point is shifted because of the different ensemble. However, once again, it is trivial to verify that at the critical point (3.16) corresponding critical exponents have the same values as those at the critical point (3.11). In addition, we find

\[ G = -\frac{2}{3} \pi^2 \kappa^{-2} (r_+^3 + l^2 r_+), \]  \hspace{1cm} (3.17)
\[ F = -\frac{2}{3} \pi^2 \kappa^{-2} (r_+^3 - 3l^2 r_+). \]  \hspace{1cm} (3.18)

Because the Gibbs free energy is always negative, the Hawking-Page transition will not appear in the grand canonical ensemble; the conformal field is always in the high temperature phase. But the Helmholtz free energy will change its sign at \( r_+^2 = 3l^2 \). And hence the Hawking-Page transition may occur in the canonical ensemble for the rotating M5-branes.
IV. CRITICAL BEHAVIOR IN ROTATING M2-BRANES

In this section we consider the case of the rotating M2-branes. The rotating black M2-brane metric with a nonvanishing angular momentum is \[ ds_{11}^2 = f^{-2/3}(-h dt^2 + dx_1^2 + dx_2^2) + f^{1/3} \left[ \frac{dr^2}{h} + r^2(\triangle d\theta^2 + \tilde{\triangle} \sin^2 \theta d\phi^2 + \cos^2 \theta d\Omega_5^2) - \frac{4ml \cosh \alpha}{r^6 \Delta f} \sin^2 \theta dt d\phi \right], \] (4.1)

where

\[
\begin{align*}
    f &= 1 + \frac{2m \sinh^2 \alpha}{r^6 \Delta}, \\
    \Delta &= 1 + \frac{l^2 \cos^2 \theta}{r^2}, \\
    \tilde{\Delta} &= 1 + \frac{l^2}{r^2} + \frac{2ml \sin^2 \theta}{r^8 \Delta f}, \\
    h &= 1 - \frac{2m}{r^6 \Delta}, \\
    \tilde{h} &= \frac{1}{\Delta} \left( 1 + \frac{l^2}{r^2} - \frac{2m}{r^6} \right).
\end{align*}
\]

The horizon of the rotating black M2-brane (4.1) is determined by the equation \[ r_+^6 + l^2 r_+^4 - 2m = 0. \] (4.2)

Through a straightforward calculation, we have

\[
\begin{align*}
    M &= \frac{7\pi^4 m}{3\kappa^2} \left( 1 + \frac{6}{l} \sinh^2 \alpha \right), \quad (4.3) \\
    J &= \frac{2\pi^4 ml}{3\kappa^2} \cosh \alpha, \quad (4.4) \\
    \Omega &= \frac{l r_+}{2m \cosh \alpha}, \quad (4.5) \\
    T &= \frac{r_+^3 (3r_+^2 + 2l^2)}{4\pi m \cosh \alpha}, \quad (4.6) \\
    S &= \frac{4\pi^5 mr_+}{3\kappa^2} \cosh \alpha, \quad (4.7) \\
    N &= 2^{2/3} \pi^{10/3} \kappa^{-4/3} m \cosh \alpha \sinh \alpha. \quad (4.8)
\end{align*}
\]

which obey the first law of black hole thermodynamics

\[ dM = TdS + \Omega dJ + \Phi dN \]

with \( \Phi = 2^{1/3} \pi^{2/3} \kappa^{-2/3} \sinh \alpha / \cosh \alpha. \) Here \( \kappa^2 \) is related to the gravitational constant in eleven dimensions as in the preceding section. The thermodynamics of the low energy
excitations of the rotating M2-branes can also be obtained by taking the near-extremal limit: \( m \to 0 \) and \( \alpha \to \infty \), while keeping \( N \) fixed. In the limit, we reach

\[
E = 3^{-1} 4\pi^4 \kappa^{-2} m,
J = 2^{2/3} 3^{-1} \pi^{7/3} \kappa^{-4/3} N^{1/2} m^{1/2} l, \]
\[
\Omega = 2^{-2/3} \pi^{5/3} \kappa^{-2/3} N^{-1/2} m^{-1/2} l^4, \]
\[
T = 2^{-5/3} \pi^{2/3} \kappa^{-2/3} N^{-1/2} m^{-1/2} r_+^3 (3r_+^2 + 2l^2), \]
\[
S = 2^{5/3} 3^{-1} \pi^{10/3} \kappa^{-4/3} N^{1/2} m^{1/2} r_+. \quad (4.9)
\]

For the rotating M2-branes, from Eqs. (4.9) we have

\[
C_\Omega = 2^{5/3} 3^{-1} \pi^{10/3} \kappa^{-4/3} N^{1/2} m^{1/2} r_+^3 \frac{6r_+^4 + l^2 r_+^2 - 2l^4}{3^2 r_+^4 + 5l^2 r_+^2 + 2l^4 3r_+^2 - 2l^2}. \quad (4.10)
\]

It is easy to see that this heat capacity is positive as

\[
2l^2 < 3r_+^2, \quad \text{that is, } l^6 / m < 27 / 10, \quad (4.11)
\]

or \( 2l^4 > 6r_+^4 + l^2 r_+^2, \) negative between them, and diverges at \( 2l^2 = 3r_+^2 \). At this critical point, the isothermal capacitance diverges as well:

\[
\chi_T = 2^{4/3} \pi^{2/3} \kappa^{-2/3} N m r_+^3 \frac{r_+^2 + 2l^2}{3^2 r_+^4 + 5l^2 r_+^2 + 2l^2 3r_+^2 - 2l^2}. \quad (4.12)
\]

In this case, the critical temperature and critical voltage are

\[
T_c = 2^{1/3} 5^{-1/2} \pi^{2/3} \kappa^{-2/3} N^{-1/2} l^2, \]
\[
\Omega_c = 2^{5/6} 3^{-1/2} 5^{-1/2} \pi^{5/3} \kappa^{-2/3} N^{-1/2} l^2, \quad (4.13)
\]

that is, \( \Omega_c / T_c = \pi / \sqrt{6} \).

As for the critical exponents at this critical point, once again, we find those critical exponents defined in Eqs. (2.19)-(2.22) are the same as those (2.28), which seems to show the universality of this kind of critical behaviors in the conformal field theory. Note that the low energy physics on the M2-brane should be described by a conformal field theory in three dimensions \[1,20\]. Thus \( d = 2 \) in this case, from Eqs. (2.31), we furthermore have the critical exponents of correlation function

\[
\nu = 3/4, \quad \eta = 4/3. \quad (4.14)
\]

Differing from the cases of D3-branes and M5-branes, we find that the heat capacity at constant angular momentum (R-charge) for the M2-branes

\[
C_J = 2^{11/3} 3^{-2} \pi^{10/3} \kappa^{-4/3} N^{1/2} m^{3/2} r_+^3 \frac{5r_+^2 + 2l^2}{r_+^2 + 2l^2}, \quad (4.15)
\]
is always positive. Also the inverse isothermal capacitance has not any divergent point, which can be see clearly from Eq. (1.12). This means that the Helmholtz free energy has not any singular point, at least till its second derivatives. Therefore in the canonical
ensemble the M2-brane is thermodynamically stable and there is no critical point. In this sense, it would be interesting to note that there exist phase transitions in the self-gravitating system considered in [26] in the microcanonical ensemble and conical ensemble, but not in the grand canonical ensemble, which also supports the point of view of the inequivalence of thermodynamic ensembles. As for the Hawking-Page phase transition, in this case, we have from Eqs. (4.9)

\[ G = -\frac{1}{3}\pi^4 \kappa^{-2} (r_+^6 + l^2 r_+^4), \]
\[ F = -\frac{1}{3}\pi^4 \kappa^{-2} r_+^6. \]

(4.16) (4.17)

They are always negative and therefore the Hawking-Page phase transition does not occur in both of the grand canonical ensemble and canonical ensemble. It is worthwhile to note the difference between the thermodynamics of rotating D3- and M5-branes and of the rotating M2-branes.

V. CONCLUSIONS

In this work we have investigated the thermodynamic stability, critical behavior near the stability boundary, and the Hawking-Page transition for the low energy excitations of rotating D3-branes, M5-branes and M2-branes in the grand canonical ensemble and canonical ensemble, respectively. In the superconformal field theory which characterizes the low energy excitations of the branes, the angular momentum is interpreted as electric charge under a subgroup of the R-symmetry group. Therefore the existence of the stability boundary characterized by the angular momentum implies the existence of phase transition of the superconformal field theory, characterized by the R-charge.

The rotating black D3-brane is thermodynamically stable up to a critical angular momentum density. This indicates that there is, according to the Maldacena’s conjecture, a corresponding phase transition for the \( \mathcal{N}=4 \) large \( N \) super Yang-Mills theory at finite temperature, which is characterized by the R-charge. We have studied this phase transition on the side of supergravity by calculating some related critical exponents. Although these critical exponents (2.28) are all equal to \( 1/2 \), indeed they are shown to satisfy the static scaling laws. Using the scaling laws related to the correlation function, we have also deduced the critical exponents \( \nu = 1/2 \) and \( \eta = 1 \) of the correlation function of the gauge field. Of interest we found is that the critical point is different in the grand canonical ensemble and canonical ensemble, although the corresponding critical exponents at both of these two critical points have the same values and satisfy the same static scaling laws. This result seems to imply that indeed thermodynamic ensembles concerning black holes are not equivalent and thermodynamic stability depends on the chosen ensemble [23,24].

The rotating M5-brane and M2-brane have been found to have similar stability boundary determined by a critical angular momentum value. This implies that there is also a phase transition for the corresponding conformal field theory [1,20]. Some critical exponents are the same as those for the rotating three branes, which shows the universality of this kind of phase transition for the rotating branes. But the critical exponents related to the correlation function is different due to the different dimensionality of the world volume.
Through the calculation of the Euclidean action of the rotating black branes, we have found that the Hawking-Page transition does not occur in the grand canonical ensemble, as in the static D-branes. In the canonical ensemble, however, the Helmholtz free energy may change its sign for the rotating D3- and M5-branes, but not for the M2-branes. This seemingly implies that the Hawking-Page phase transition may occur in this ensemble. In addition, it seems to be worth noting that in the canonical ensemble the Helmholtz free energy for the rotating M2-branes is regular and there is no any critical point. Another point is that heat capacity at constant angular velocity (voltage) is also positive if the angular momentum density is large enough [see (2.14), (3.10) and (4.10)]. We have not yet understood its meanings and its relation, if any, to the thermodynamic stability. It would be interesting to understand the shift of critical point in the different ensembles in the field theory model suggested in [18]. Also it would be of some significance to extend the Gubser’s method [18] to explain the entropy and stability boundary for the rotating black M5-branes and M2-branes. Moreover, it should be important to check whether the critical exponents of correlation function in the field theory model is in agreement with the values given in this paper or not.

Note Added. Since this paper was finished, there have been three related papers [28–30] appearing in the hep-th archive. Refs. [28,29] relate the rotating D3-, M2-, M5-branes to the charged AdS black hole solutions in gauged supergravity theories, and study some thermodynamic properties of the latter. Ref. [30] discusses the thermodynamic stability and the corresponding QCD model for rotating D3-, M5-, and M2-branes with multiple angular momentum parameters. An interesting result is that in the canonical ensemble there may exist critical points for the rotating M2-branes with multiple angular momentum parameters. As a result, we suspect that the Hawking-Page transition may also appear in the canonical ensemble of this system.

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