A Relation-Theoretic Matkowski-Type Theorem in Symmetric Spaces

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Abstract: In this paper, we present a fixed-point theorem in \( \mathcal{R} \)-complete regular symmetric spaces endowed with a locally \( T \)-transitive binary relation \( \mathcal{R} \) using comparison functions that generalizes several relevant existing results. In addition, we adopt an example to substantiate the genuineness of our newly proved result. Finally, as an application of our main result, we establish the existence and uniqueness of a solution of a periodic boundary value problem.

Keywords: fixed points; symmetric spaces; binary relations; \( T \)-transitivity; regular spaces

MSC: 47H10; 54H25

1. Introduction

In 1922, one of the most pivotal results in analysis was proved by Banach [1] in his doctoral thesis, which asserts that every contraction mapping on a complete metric space admits a unique fixed point. This principle continues to inspire generations of researchers in metric fixed-point theory. Thus far, this classical result has been generalized and improved in various ways, and by now, there exists an extensive literature on and around this premier result. Over the last several decades, there have been many interesting generalizations of this classical result in various directions.

There exist several extensions of the Banach contraction principle to various spaces obtained by lightening the underlying involved metric conditions. In doing so, we are in receipt of several spaces, namely: rectangular metric spaces, generalized metric spaces, partial metric spaces, \( b \)-metric spaces, partial \( b \)-metric spaces, symmetric spaces, quasimetric spaces, quasi-partial metric spaces, and many more. In 1976, Cicchese [2] established the first ever fixed-point theorem in the framework of symmetric spaces. The idea of such spaces was coined by Wilson [3] by relaxing the triangle inequality from metric conditions. By now, there exists a considerable literature on fixed-point theory in symmetric spaces. For work of this kind, one can be referred to [4–11].

On the other hand, there have been various generalizations that were obtained by varying the class of contractions (e.g., see [12–14]). In 2004, Ran and Reurings [15] obtained a very useful generalization of the Banach fixed-point theorem in a partially ordered metric space by taking a relatively weaker contraction condition that is required to hold only on those elements that were comparable in the underlying ordering. In doing so, they were essentially motivated by Turinici [16]. This result was further generalized by Nieto and Rodríguez-López in [17,18] in 2005 and 2007, respectively. Subsequently, in 2015, Alam and Imdad [19] furnished a natural extension of the Banach contraction principle in a complete metric space endowed with a binary relation that generalizes all of the above-mentioned results [15,17,18].

The existing literature contains several results on nonlinear contractions, which were initiated by Browder [13] and were followed by similar works by Boyd and Wong [14].
and Matkowski [20]. In 2014, Bessenyei and Páles [21] extended Matkowski’s result in symmetric spaces, which required an additional regularity condition.

The intent of this paper is to prove a relation-theoretic version of a theorem due to Bessenyei and Páles [21]. In doing so, we are essentially motivated by [15, 17–19].

2. Preliminaries

In this section, we recall some definitions, propositions, and lemmas that will be utilized in our subsequent discussions. The following are taken from Wilson’s paper [3] on symmetric space. Throughout the paper, \( \mathbb{R}, \mathbb{R}^+, \mathbb{N}, \mathbb{N}_0, \) and \( \mathbb{Q} \) denote the sets of reals, nonnegative reals, natural numbers, whole numbers, and rational numbers respectively.

**Definition 1.** Let \( X \) be a nonempty set and let \( d : X \times X \to \mathbb{R}^+ \) be a mapping satisfying the following axioms: for each \( a, b \in X \),

(i) \( d(a, b) = 0 \) if and only if \( a = b \);

(ii) \( d(a, b) = d(b, a) \).

Then, \( d \) is called symmetric on \( X \) and the pair \( (X, d) \) is a symmetric space.

In such spaces, the notions of convergent and Cauchy sequences are as usual.

- A sequence \( (x_n) \subset X \) is said to converge to \( x \in X \) if \( \lim_{n \to \infty} d(x_n, x) = 0 \).
- A sequence \( (x_n) \subset X \) is said to be Cauchy if for each \( \varepsilon > 0 \), there exists \( N \in \mathbb{N} \) such that \( d(x_n, x_m) < \varepsilon \) \( \forall n, m \geq N \).

The space is said to be complete if every Cauchy sequence converges. For an open ball centered at \( p \) with radius \( r \), the notation \( B(p, r) \) is used. The diameter of \( B(p, r) \) is the supremum of distances taken over the pairs of points of the ball. The topology of such spaces is the topology induced by the open balls.

Because of the unavailability of the triangle inequality, the following problems are obvious:

- There is nothing to assure that limits are unique (thus, the space need not be Hausdorff);
- A convergent sequence need not be a Cauchy sequence;
- The mapping \( d(a, \cdot) : X \to \mathbb{R} \) need not be continuous.

**Definition 2.** Consider a symmetric space \( (X, d) \). A function \( \psi : \mathbb{R}^2_+ \to \mathbb{R}^+ \) is a triangle function [21] for \( d \) if the following hold:

(i) \( \psi(u, v) = \psi(v, u) \forall u, v \in \mathbb{R}^+; \)

(ii) \( \psi \) is monotone increasing in both of its arguments;

(iii) \( \psi(0, 0) = 0; \)

(iv) \( d(x, y) \leq \psi(d(x, z), d(y, z)) \forall x, y, z \in X \) for all \( x, y, z \in X \).

It has been shown in [21] that every symmetric space \( (X, d) \) admits a unique triangle function \( \Phi_d \), which has the property that if \( \psi \) is any other triangle function for \( d \), then \( \Phi_d \leq \psi \). Such a triangle function \( \Phi_d \) is called the basic triangle function.

**Definition 3.** A symmetric space \( (X, d) \) is called a regular space if the basic triangle function with respect to the symmetric \( d \) is continuous at \((0, 0)\).

Throughout this paper, we shall restrict our attention to regular spaces only. The utility of such spaces is enlightened by the next important result.

**Lemma 1** ([21]). The topology of a regular symmetric space is Hausdorff. A convergent sequence in a regular symmetric space has a unique limit and it has the Cauchy property. Moreover, a symmetric space \( (X, d) \) is regular if and only if
Theorem 1. If \((X, d)\) is a complete regular symmetric space and \(\varphi\) is a comparison function, then every \(\varphi\)-contraction on \(X\) has a unique fixed point.

3. Relation-Theoretic Notions and Related Results

Definition 5 ([22]). Let \(X\) be a nonempty set. A subset \(R\) of \(X \times X\) is called a binary relation on \(X\). For \(x, y \in X\), we say that \(x\) is related to \(y\), or in other words, \(x\) relates to \(y\) under \(R\). Sometimes, we write \(xRy\) instead of \((x, y) \in R\). If \((x, y) \notin R\), we say \(x\) is not related to \(y\).

Definition 6. Let \(R\) be a binary relation on a nonempty set \(X\) and \(x, y \in X\). We say that \(x\) and \(y\) are \(R\)-comparative if either \((x, y) \in R\) or \((y, x) \in R\). When \(x\) and \(y\) are \(R\)-comparative, we write it as \([x, y] \in R\).

Proposition 1 ([19]). If \((X, d)\) is a symmetric space, \(R\) is a binary relation on \(X\), \(T\) is a self-mapping on \(X\), and \(\varphi: \mathbb{R}^+ \to \mathbb{R}^+\) is a comparison function, then the following conditions are equivalent:

(i) \(d(T(x), T(y)) \leq \varphi(d(x, y)) \forall (x, y) \in R\);

(ii) \(d(T(x), T(y)) \leq \varphi(d(x, y)) \forall [x, y] \in R\).

The proof is simple and follows from symmetry of \(d\).

Definition 7. A binary relation \(R\) defined on a nonempty set \(X\) is called

- reflexive if \((x, x) \in R \forall x \in X\);
- transitive if \((x, y) \in R\) and \((y, z) \in R\) implies \((x, z) \in R\);
- complete, connected, or dichotomous if \([x, y] \in R \forall x, y \in X\).

Definition 8 ([19]). Let \(X\) be a nonempty set endowed with a binary relation \(R\). A sequence \((x_n) \subset X\) is called \(R\)-preserving if \((x_n, x_{n+1}) \in R \forall n \in \mathbb{N}\).

Definition 9 ([19]). Let \(X\) be a nonempty set and let \(T\) be a self-mapping on \(X\). A binary relation \(R\) on \(X\) is called \(T\)-closed if, for any \(x, y \in X\),

\[(x, y) \in R \Rightarrow (Tx, Ty) \in R.\]

Definition 10 ([23]). Let \(X\) be a nonempty set and let \(T\) be a self-mapping on \(X\). A binary relation \(R\) on \(X\) is said to be \(T\)-transitive if, for any \(x, y, z \in X\),

\[(Tx, Ty), (Ty, Tz) \in R \Rightarrow (Tx, Tz) \in R.\]

Definition 11 ([22]). Let \(X\) be a nonempty set endowed with a binary relation \(R\) and \(E \subset X\). The restriction of \(R\) to \(E\), denoted as \(R|_E\), is the set \(R \cap E^2\). Indeed, \(R|_E\) is a relation on \(E\) induced by \(R\).

Definition 12 ([23]). A binary relation \(R\) on a nonempty set \(X\) is called locally transitive if, for each \(R\)-preserving sequence \((x_n) \subset X\) with range \(E = \{x_n\}_{n \in \mathbb{N}_0}\), the binary relation \(R|_E\) is transitive.
Definition 13 ([23]). Let $X$ be a nonempty set and let $T$ be a self-mapping on $X$. A binary relation $\mathcal{R}$ on $X$ is called locally $T$-transitive if, for each (effectively) $\mathcal{R}$-preserving sequence $(x_n) \subset T(X)$ with range $E = \{x_n\}_{n \in \mathbb{N}_0}$, the binary relation $\mathcal{R}|_E$ is transitive.

Definition 14 ([24]). Let $X$ be a nonempty set and let $\mathcal{R}$ be a binary relation on $X$. For $x, y \in X$, a path of length $k$ (where $k$ is a natural number) in $\mathcal{R}$ from $x$ to $y$ is a finite sequence $\{x_0, x_1, x_2, \ldots, x_k\} \subset X$ satisfying the following conditions:

(i) $x_0 = x$ and $x_k = y$;
(ii) $(x_i, x_{i+1}) \in \mathcal{R}$ for each $i$ $(0 \leq i \leq k - 1)$.

Definition 15 ([23]). Let $X$ be a nonempty set and let $\mathcal{R}$ be a binary relation on $X$. A subset $E$ of $X$ is called $\mathcal{R}$-connected if, for each pair $x, y \in X$, there exists a path (in $\mathcal{R}$) from $x$ to $y$.

Definition 16 ([23]). Let $X$ be a nonempty set and let $\mathcal{R}$ be a binary relation on $X$. A subset $E$ of $X$ is called $\mathcal{R}$-$S$-connected if, for each pair $x, y \in X$, there is a finite sequence $\{x_0, x_1, x_2, \ldots, x_k\} \subset X$ satisfying the following conditions:

(i) $x_0 = x$ and $x_k = y$;
(ii) $[x_i, x_{i+1}] \in \mathcal{R}$ for each $i$ $(0 \leq i \leq k - 1)$.

Now, we define the analogue of the notion of $d$-self-closedness in metric space due to [23] in the framework of symmetric spaces.

Definition 17. Let $(X, d)$ be a symmetric space. A binary relation $\mathcal{R}$ defined on $X$ is called $d$-self-closed if, for any $\mathcal{R}$-preserving sequence $(x_n)$ converging to $x$, there exists a subsequence $(x_{n_k})$ of $(x_n)$ with $(x_{n_k}, x) \in \mathcal{R}$.

We will use the following notations in this paper:

- $F(T) := \{x \in X \mid T(x) = x\}$;
- $X(T, \mathcal{R}) := \{x \in X \mid (x, Tx) \in \mathcal{R}\}$.

4. Main Result

In an attempt to prove a relation-theoretic version of Matkowski’s theorem [20] in symmetric spaces, we prove the following.

Theorem 2. Let $(X, d)$ be a regular symmetric space, $\mathcal{R}$ a binary relation on $X$, and $T$ a self-mapping on $X$. Suppose that the following conditions hold:

(a) $(X,d)$ is $\mathcal{R}$-complete;
(b) $\mathcal{R}$ is $T$-closed and locally $T$-transitive;
(c) $T$ is either $\mathcal{R}$-continuous or $\mathcal{R}$ is $d$-self-closed;
(d) $X(T, \mathcal{R})$ is nonempty;
(e) There is a comparison function $\varphi$ such that

$$d(Tx, Ty) \leq \varphi(d(x, y)) \quad \forall (x, y) \in \mathcal{R}.$$ 

Then, $T$ has a fixed point.

Moreover, if

(f) $F(T)$ is $\mathcal{R}$-$S$-connected, then $T$ has a unique fixed point.

Proof. As $X(T, \mathcal{R})$ is nonempty, let $x_0$ be such that $(x_0, Tx_0) \in \mathcal{R}$. If $Tx_0 = x_0$, then we are done. Suppose that $Tx_0 \neq x_0$. Since $(x_0, Tx_0) \in \mathcal{R}$ and $\mathcal{R}$ is $T$-closed, we obtain by induction that

$$(T^n x_0, T^{n+1} x_0) \in \mathcal{R} \forall n \in \mathbb{N}.$$
Construct the sequence \((x_n)\) of Picard iterates with initial point \(x_0\), i.e., \(x_n = T^n(x_0)\). So, \((x_n, x_{n+1}) \in \mathcal{R}\ \forall n \in \mathbb{N}_0\), i.e., the sequence is \(\mathcal{R}\)-preserving. As \(\mathcal{R}\) is locally \(T\)-transitive, we have \((x_n, x_m) \in \mathcal{R}\ \forall m > n\). Observe that the sequence \(d(x_n, x_{n+k})\) tends to zero for all fixed \(k \in \mathbb{N}\):

\[
d(x_n, x_{n+k}) = d(Tx_{n-1}, Tx_{n+k-1}) \leq \varphi(d(x_{n-1}, x_{n+k-1})) \leq \varphi^2(d(x_{n-2}, x_{n+k-2})) \leq \cdots \leq \varphi^n d(x_0, x_k) \to 0 \text{ as } n \to \infty.
\]

Now, we are going to prove that \((x_n)\) is a Cauchy sequence. Let \(\epsilon > 0\) be any positive number. As \((X, d)\) is regular, the basic triangle function \(\Phi_d\) is continuous at \((0,0)\). So, there exists a neighborhood \(U\) of the origin such that \(\Phi_d(u, v) < \varepsilon \forall (u, v) \in U\). In other words, \(\exists \delta > 0\) such that \(\Phi_d(u, v) < \varepsilon \forall u, v : 0 \leq u, v \leq \delta\). We take \(\delta < \varepsilon\). As \(\varphi\) is a comparison function, \(\varphi^n(t) \to 0 \forall t > 0\); so there exists \(N \in \mathbb{N}\) such that \(\varphi^N(\varepsilon) < \delta\). Set \(S = T^N\). We can see that

\[
d(Sx, Sy) = d(T^N x, T^N y) \leq \varphi^N d(x, y) \text{ when } (x, y) \in \mathcal{R}.
\]

Define \(n_k : d(x_n, T^k Sx_n) < \delta \ \forall n \geq n_k\) and set \(M = \max\{n_0, n_1, \ldots, n_N\}\).

If \(V = \{x_M, x_{M+1}, x_{M+2}, \ldots, x_{M+k}, \ldots\}\) then for any \(y \in B(x_M, \varepsilon) \cap V, y \neq x_M\), we have

\[
d(T^k Sx_M, T^k Sy) = d(ST^k x_M, ST^k y) \leq \varphi^N d(T^k x_M, T^k y) \text{ as } (T^k x_M, T^k y) \in \mathcal{R}
\]

\[
\leq \varphi^N \Phi_d(x_M, y) < \varphi^N d(x_M, y) < \varphi^N(\varepsilon) < \delta.
\]

So,

\[
d(T^k Sy, x_M) \leq \Phi_d(d(T^k Sy, T^k Sx_M), d(T^k Sx_M, x_M)) \leq \Phi_d(\delta, \delta) \forall k = 0, 1, 2, \ldots, N;
\]

\[
\implies d(T^k Sy, x_M) < \varepsilon, \forall k = 0, 1, 2, \ldots, N.
\]

and for \(y = x_M\), \(d(T^k Sx_M, x_M) < \delta < \varepsilon, \forall k = 0, 1, 2, \ldots, N\).

Thus we see that \(T^k S\) maps \(V \cap B(x_M, \varepsilon)\) into itself. In particular, each iteration of \(S\) maps \(V \cap B(x_M, \varepsilon)\) into itself. Now, if \(n > M\) is any arbitrarily given natural number, i.e., \(n = Nk + p\) where \(k \in \mathbb{N}_0\) and \(0 \leq p < N\), then

\[
T^n S = T^{Nk+p} S = T^p S^{k+1},
\]

and hence,

\[
T^n S(V \cap B(x_M, \varepsilon)) = T^p S^{k+1}(V \cap B(x_M, \varepsilon)) = T^p S(V \cap B(x_M, \varepsilon)) \subset T^p S(V \cap B(x_M, \varepsilon)) \subset V \cap B(x_M, \varepsilon); \text{ as } 0 \leq p < N.
\]

Therefore, \(T^n S(x_M) \in B(x_M, \varepsilon) \forall n > M\), i.e., \(x_M+N+k \in B(x_M, \varepsilon) \forall k \in \mathbb{N}\). As the space is regular, \(\text{diam}(x_M, \varepsilon) \to 0\) when \(\varepsilon \to 0\), and from this, we conclude that the sequence \(\{x_n\}\) is a Cauchy sequence. The completeness of the space \((X, d)\) gives some element \(x \in X\) such that \(x_n \to x\).

Now, if \(T\) is \(\mathcal{R}\)-continuous, then \(T(x_n) \to T(x)\), i.e., \(x_{n+1} \to T(x)\). As the space is regular, we conclude that \(T(x) = x\), as the limit is unique in regular spaces. So, the limit of the sequence constructed above is a fixed point.
If \( R \) is \( d \)-self-closed, then there is a subsequence \((x_{n_k})\) of \((x_n)\) such that \([x_{n_k}, x] \in R \forall k \in \mathbb{N}_0\). So,
\[
ed(x, Tx) \leq \Phi_d(d(x, x_{n_k+1}), d(x_{n_k+1}, Tx)) \leq \Phi_d[d(x_{n_k+1}, x), \varphi(d(x_{n_k}, x))].
\]

Now, for \( \epsilon > 0 \), there exists \( \delta > 0 \) such that \( \Phi_d(u, v) < \epsilon \ \forall u, v : 0 \leq u, v \leq \delta \), and for \( \delta > 0 \), there exists \( K \in \mathbb{N} \) such that \( d(x_n, x) \leq \delta \ \forall n \geq K \). Therefore, if we take \( n_k \geq K \), we have
\[
ed(x, Tx) \leq \Phi_d(\delta, \varphi(\delta)) \leq \Phi_d(\delta, \delta) < \epsilon.
\]
So, \( T(x) = x \) i.e., \( x \) is a fixed point.

To show that \( T \) has a unique fixed point, let \( y \) be any other fixed point of \( T \). Now, \( F(T) \) is \( R^S \)-connected and \( x, y \in F(T) \); so, there is a finite sequence of elements \( \{z_0, z_1, z_2, \ldots, z_k\} \subset X \) satisfying the following conditions:

(i) \( z_0 = x, z_k = y \);
(ii) \( |z_i, z_{i+1}| \in R \) for each \( i \) \((0 \leq i \leq k - 1)\).

Now, as \( T \) is a \( \varphi \)-contraction on \( R \), \( d(Tz_i, Tz_{i+1}) \leq \varphi(d(z_i, z_{i+1})) \). Using induction, we get \( d(T^n z_i, T^n z_{i+1}) \leq \varphi^n d(z_i, z_{i+1}) \). We already have, for \( \epsilon > 0 \), \( \exists \delta > 0 \) such that \( \Phi_d(u, v) < \epsilon \ \forall u, v : 0 \leq u, v < \delta \).

Let \( \delta_1 = \delta \), define \( \delta_i (2 \leq i \leq k - 1) : \Phi_d(u, v) < \delta_{i-1} \ \forall u, v : 0 < u, v < \delta_i \), and set \( \alpha = \min\{\delta_1, \delta_2, \ldots, \delta_{k-1}\} \).

In addition, set \( M = \max\{N_1, N_2, \ldots, N_{k-1}\} \), where \( N_i : d(T^n z_i, T^n z_{i+1}) \leq \varphi^n d(z_i, z_{i+1}) < \alpha \ \forall n \geq N_i \). Hence, for \( n \geq M \), we have
\[
\begin{align*}
d(T^n z_{k-1}, T^n y) &< \alpha \\
d(T^n z_{k-2}, T^n y) &\leq \Phi_d[d(T^n z_{k-2}, T^n z_{k-1}), d(T^n z_{k-1}, T^n y)] \\
&\leq \Phi_d(\alpha, \delta_{k-1}) \leq \Phi_d(\delta_{k-1}, \delta_{k-1}) < \delta_{k-2} \\
d(T^n z_{k-3}, T^n y) &\leq \Phi_d[d(T^n z_{k-3}, T^n z_{k-2}), d(T^n z_{k-2}, T^n y)] \\
&\leq \Phi_d(\alpha, \delta_{k-2}) \leq \Phi_d(\delta_{k-2}, \delta_{k-2}) < \delta_{k-3} \\
&\vdots \\
d(T^n z_1, T^n y) &\leq \Phi_d[d(T^n z_1, T^n z_2), d(T^n z_2, T^n y)] \\
&\leq \Phi_d(\alpha, \delta_2) \leq \Phi_d(\delta_2, \delta_2) < \delta_1 \\
d(T^n x, T^n y) &\leq \Phi_d[d(T^n x, T^n z_1), d(T^n z_1, T^n y)] \\
&\leq \Phi_d(\alpha, \delta_1) \leq \Phi_d(\delta_1, \delta_1) < \epsilon.
\end{align*}
\]

Therefore, \( d(T^n x, T^n y) = d(x, y) = 0 \), i.e., \( x = y \). Hence, the fixed point of \( T \) is unique. \( \square \)

Now, we consider some special cases, where our result deduces some well-known results from the existing literature.

1. Under the universal relation \( R = X^2 \), our theorem deduces the result by M. Bessenyei and Z. Páles [21]. Clearly, under the universal relation, the hypotheses of our result hold trivially.

2. As every metric space is a symmetric space, the result of Alam and Imdad [19], which is a generalization of the classical Banach contraction principle, is yielded immediately. In this case, we take \( \varphi(t) = ct \) as the comparison function, where \( c \in [0, 1) \) is such that \( d(Tx, Ty) \leq c(d(x, y)) \ \forall xRy \).

3. The fixed-point result of Ran and Reurings [15] can be obtained from our result, as every partially ordered complete metric space is automatically a symmetric space, and the associated relation to the partial order satisfies all the hypotheses of our result if we take the comparison function \( \varphi \) as the same as the earlier case (2), i.e., \( \varphi(t) = ct \).
(4) The result of Neito and Rodríguez-López becomes a corollary of our result because of the same reasons as the earlier one. Notice that the $d$-self-closedness property is a generalization of the ICU (increasing-convergence upper bound) property.

Finally, we produce an illustrative example to substantiate the utility of our result, which does not satisfy the hypotheses of the existing results [1,15,17–19,21,23], but satisfies the hypotheses of our result, and hence has a fixed point.

**Example 1.** Let $X = \mathbb{R}$ and $d(x, y) = (x - y)^2$; then, $(X, d)$ is a complete regular symmetric space. Consider the binary relation 

$$R = \{(x, y) \in \mathbb{R}^2 : x \geq y \geq 0, x \in \mathbb{Q}\}.$$ 

We define a mapping $T : X \to X$ as follows:

$$T(x) = \begin{cases} 2x, & \text{if } x \leq 0, \\ \frac{x}{3}, & \text{if } x > 0. \end{cases}$$

We see that the self-mapping $T$ on $X$ is not a $\varphi$-contraction on the whole space $X$ for any comparison function $\varphi$. So, the result of Bessenyei and Páles [21] does not apply here. However, when we consider the elements $x, y$ such that $(x, y) \in R$, then $T$ is a $\varphi$-contraction on $\mathbb{R}$ for $\varphi(t) = \frac{t}{2}$, and all the other hypotheses of our result hold.

In addition, we see that the fixed-point results of [1,15,17–19,23] do not apply here, as the space is not a metric space.

5. Application to Ordinary Differential Equations

In this section, we study the existence and uniqueness of a first-order periodic boundary value problem as an application of our main fixed-point theorem.

Consider the first-order periodic boundary value problem

$$x'(t) = g(t, x(t)), t \in [0, \lambda]$$

$$x(0) = x(\lambda),$$

(1)

where $\lambda > 0$ and $g : [0, \lambda] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

We consider the space $X = C[0, \lambda]$ of all continuous functions on $[0, \lambda]$ under the symmetric given by

$$d(x, y) = \sup_{t \in [0, \lambda]} (x(t) - y(t))^2.$$

We define a relation $R$ on $X$ as

$$xRy \iff x(t) \leq y(t) \forall t \in [0, \lambda].$$

Now, we give the following definition, which will be useful in the subsequent theorem.

**Definition 18.** A function $z$ is said to be a lower solution of (1) if

$$z'(t) \leq g(t, z(t)) \text{ for } t \in [0, \lambda]$$

$$z(0) \leq z(\lambda).$$

**Theorem 3.** Consider problem (1) with $g : I \times \mathbb{R} \to \mathbb{R}$, a continuous function, and suppose that there exists some $k > 0$ such that for $s_1, s_2$ in $\mathbb{R}$ with $s_1 \geq s_2$,

$$0 \leq g(t, s_1) + ks_1 - g(t, s_2) + ks_2 \leq k \sqrt{\varphi(s_1 - s_2)^2},$$

where $\varphi$ is a comparison function. Then, the existence of a lower solution for (1) guarantees the existence of a unique solution of (1).
Problem (1) can be rewritten as
\[ x'(t) + kx(t) = g(t, x(t)) + kx(t), \quad t \in [0, \lambda] \]
\[ x(0) = x(\lambda). \]

This problem is equivalent to the integral equation
\[ x(t) = \int_0^\lambda G(t, s)[g(s, x(s)) + kx(s)]ds, \]
where
\[ G(t, s) = \begin{cases} \frac{e^{k(s-t)}}{\sqrt{1 - e^{k(t-s)}}}, & 0 \leq s \leq t \\ \frac{e^{k(t-s)}}{\sqrt{1 - e^{k(s-t)}}}, & 0 \leq t \leq s. \end{cases} \]

Consider the self-mapping \( T \) on \( X \) defined as
\[ (Tx)(t) = \int_0^\lambda G(t, s)[g(s, x(s)) + kx(s)]ds. \]

Here, it is apparent that a fixed point of \( T \) is, in fact, a solution of the above problem (1). Now, we will show that the hypotheses in Theorem 2 are satisfied.

To prove that the relation \( R \) is \( T \)-closed, take \( x, y \in X \) such that \( xRy \), i.e.,
\[ x(t) \leq y(t) \quad \forall t \in [0, \lambda]. \]

As \( y(t) \geq x(t) \), from the hypothesis, we obtain
\[ g(t, x(t)) + kx(t) \leq g(t, y(t)) + ky(t) \quad \forall t \in [0, \lambda]. \]

As \( G(t, s) > 0 \quad \forall t, s \in [0, \lambda] \), we have
\[ (Tx)(t) = \int_0^\lambda G(t, s)[g(s, x(s)) + kx(s)]ds \leq \int_0^\lambda G(t, s)[g(s, y(s)) + ky(s)]ds = (Ty)(t). \]

Hence, \( R \) is \( T \)-closed. In addition, for \( xRy \), we have
\[ \sqrt{d(Tx, Ty)} = \sup_{t \in [0, \lambda]} |(Tx)(t) - (Ty)(t)| \]
\[ \leq \sup_{t \in [0, \lambda]} \int_0^\lambda G(t, s)[g(s, x(s)) + kx(s) - g(s, y(s)) - ky(s)]ds \]
\[ \leq \sup_{t \in [0, \lambda]} \int_0^\lambda G(t, s)\sqrt{\varphi(y(s) - x(s))^2}ds \]
\[ \leq \sqrt{\varphi d(x, y)} \sup_{t \in [0, \lambda]} \int_0^\lambda G(t, s)ds \]
\[ = \sqrt{\varphi d(x, y)} \sup_{t \in [0, \lambda]} \frac{1}{e^{k\lambda} - 1} \left( \frac{1}{k} e^{k(\lambda+s-t)} \right)_{s=0}^{s=t} \]
\[ = \sqrt{\varphi d(x, y)} \sup_{t \in [0, \lambda]} \frac{1}{e^{k\lambda} - 1} (e^{k\lambda} - 1) \]
\[ = \sqrt{\varphi d(x, y)}. \]
Thus, we have 
\[ d(Tx, Ty) \leq \varphi d(x, y). \]

Hence, the required contraction condition (2) holds.

Now, as there is some lower solution, say \( x_0 \in X \), we have 
\[ x_0'(t) \leq g(t, x_0(t)), \]
which can be rewritten as 
\[ x_0'(t) + kx_0(t) \leq g(t, x_0(t)) + kx_0(t) \text{ for } t \in [0, \lambda]. \]

Multiplying both the sides by \( e^{kt} \), we obtain 
\[ (x_0(t)e^{kt})' \leq [g(t, x_0(t)) + kx_0(t)]e^{kt} \text{ for } t \in [0, \lambda], \]
and thus, we get 
\[ x_0(t)e^{kt} \leq x_0(0) + \int_0^t [g(s, x_0(s)) + kx_0(s)]e^{ks}ds \text{ for } t \in [0, \lambda], \] (2)

which implies that 
\[ x_0(0)e^{k\lambda} \leq x_0(\lambda)e^{k\lambda} \leq x_0(0) + \int_0^\lambda [g(s, x_0(s)) + kx_0(s)]e^{ks}ds, \]
thereby yielding 
\[ x_0(0) \leq \int_0^\lambda \frac{e^{ks}}{e^{k\lambda} - 1}[g(s, x_0(s)) + x_0(s)]ds. \]

Using the above inequality (2), we get 
\[
x_0(t)e^{kt} \leq \int_0^t [g(s, x_0(s)) + x_0(s)]e^{ks}ds + \int_0^\lambda \frac{e^{ks}}{e^{k\lambda} - 1}[g(s, x_0(s)) + x_0(s)]ds \\
= \int_0^t [g(s, x_0(s)) + x_0(s)]\frac{e^{ks}}{e^{k\lambda} - 1}ds + \int_0^\lambda [g(s, x_0(s)) + x_0(s)]\frac{e^{ks}}{e^{k\lambda} - 1}ds \\
+ \int_0^\lambda \frac{e^{ks}}{e^{k\lambda} - 1}[g(s, x_0(s)) + x_0(s)]ds \\
= \int_0^t [g(s, x_0(s)) + x_0(s)]\frac{e^{k(s+\lambda)}}{e^{k\lambda} - 1}ds + \int_t^\lambda [g(s, x_0(s)) + x_0(s)]\frac{e^{ks}}{e^{k\lambda} - 1}ds. 
\]

Hence, 
\[ x_0(t) \leq \int_0^t \frac{e^{k(s+\lambda-t)}}{e^{k\lambda} - 1}[g(s, x_0(s)) + x_0(s)]ds + \int_t^\lambda \frac{e^{k(s-t)}}{e^{k\lambda} - 1}[g(s, x_0(s)) + x_0(s)]ds, \]
i.e., 
\[ x_0(t) \leq \int_0^\lambda G(t, s)[g(s, x_0(s)) + x_0(s)]ds = (Tx_0)(t). \]

Thus, the existence of some element \( x_0 \in X \) such that \( x_0RTx_0 \) is ensured.

To show that \( R \) is \( d \)-self-closed, let \( (x_n) \) be an \( R \)-preserving Cauchy sequence converging to \( x \in X \). As \( (x_n) \) is \( R \)-preserving, we have 
\[ x_0(t) \leq x_1(t) \leq x_2(t) \leq \cdots \leq x_n(t) \leq x_{n+1}(t) \leq \cdots \leq x(t) \quad \forall t \in [0, \lambda], \]
thereby yielding \( x_nRx \forall n \in \mathbb{N} \). Therefore, \( R \) is \( d \)-self-closed.

The remaining hypotheses of Theorem 2 also hold and are easy to check. Hence, \( T \)
possesses a fixed point in $X$.

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