Further functional determinants

J.S.Dowker* and J.S.Apps

Department of Theoretical Physics,
The University of Manchester, Manchester, England.

Abstract
Functional determinants for the scalar Laplacian on spherical caps and slices, flat balls, shells and generalised cylinders are evaluated in two, three and four dimensions using conformal techniques. Both Dirichlet and Robin boundary conditions are allowed for. Some effects of non-smooth boundaries are discussed; in particular the 3-hemiball and the 3-hemishell are considered. The edge and vertex contributions to the $C_{3/2}$ coefficient are examined.

January 1995

*Dowker@a3.ph.man.ac.uk
1. Introduction

This paper is a continuation of our evaluation of the functional determinant of the (conformal) Laplacian on various regions of the $d$-sphere and Euclidean $d$-space. The general idea is to use the behaviour of the functional determinant under conformal transformations. For the present, this technique is limited to $d \leq 4$ because of the unavailability of the requisite conformal anomaly, $\zeta(0)$, – essentially the $C_{d/2}$ coefficient in the short-time expansion of the heat kernel. The availability of this coefficient is even further restricted if the boundary is only piecewise smooth; then, only $C_1$ is known completely.

The object of the present work is to extend the results described in [1–3] to higher dimensions and to other regions such as spherical slices and shells. We also wish to make progress with the relevant heat-kernel coefficient ($C_{3/2}$) for a piecewise smooth boundary. The general form of the coefficients in this case is an open problem.

The expressions derived for the functional determinants may have some specific quantum field theoretic application. Caps and balls, for example, arise in discussions of quantum cosmology [4–6]. There are also applications to statistical mechanics in connection with finite size effects, e.g. [7] and to conformal field theory, e.g. [8]. In mathematics, the critical points of the determinant have some interest, e.g. [9,10]. For example, the uniformisation theorem can be proved using this approach, [11].

There are several relevant calculations of functional determinants. We mention only those by Aurell and Salomonsen on certain simplicial decompositions, [12,13], those by Branson and Ørsted, [14], and those by Elizalde, [15].

2. Basic equations.

The functional determinant of the positive elliptic operator $D$ is defined in terms of the $\zeta$–function of $D$ by the usual relation,

$$\ln \det D = -\zeta'(0).$$

For a conformally invariant (scalar) field theory, the conformal behaviour under infinitesimal Weyl rescalings, $g \rightarrow \bar{g} = \exp(-2\omega)g$, is controlled by the conformal anomaly, $\zeta(0)$. Integrated along a conformal family of metrics, this anomaly yields the finite change in the functional determinant (or, equivalently, the effective action $W = \frac{1}{2} \ln \det D$),

$$\frac{1}{2} \ln \frac{\det D}{\det \bar{D}} = W[\bar{g}, g].$$

(1)
In two dimensions, the cocycle function $W[\bar{g}, g]$ has been given by Lüscher, Symanzik and Weiss [16], Polyakov [17] (when the boundary is empty) and by Alvarez [18]. The Dirichlet three-dimensional expression can be found in [19] (after correction) as can the four-dimensional one. This last, when there is no boundary, has been known for some time. Recently, Branson and Gilkey [20] have given the four-dimensional result for more general differential operators and also for Robin boundary conditions (see also [21]). Some related results are given by Dettki and Wipf [22].

For completeness, and to correct some errors, the forms of $W[\bar{g}, g]$ are given here following [19].

In three dimensions, and for Dirichlet conditions,

$$W^D[\bar{g}, g] = \frac{1}{1536\pi} \int_{\partial M} \left[ (6\text{tr}(\kappa^2) - 3\kappa^2 - 16\hat{\Delta}_2\omega - 4\hat{R})\omega 
+ 30\kappa N + 18N^2 - 24n^\mu n^\nu \omega_{\mu\nu} \right], \tag{2}$$

while in four dimensions

$$W^D[\bar{g}, g] = \frac{1}{2880\pi^2} \int_M \left[ (|\text{Riem}|^2 - |\text{Ric}|^2 + \Delta_2 R)\omega - 2R_{\mu\nu}\omega^\mu\omega^\nu 
- 4\omega^\mu\omega_\mu \Delta_2\omega + 2(\omega^\mu\omega_\mu)^2 + 3(\Delta_2\omega)^2 \right]
+ \frac{1}{5760\pi^2} \int_{\partial M} \left[ \left( \frac{320}{21}\text{tr}(\kappa^3) - \frac{88}{7}\kappa\text{tr}(\kappa^2) + \frac{40}{21}\kappa^3 - 4R_{\mu\nu}\chi^{\mu\nu} 
- 4\kappa R_{\mu\nu} n^\mu n^\nu + 16R_{\mu\nu\rho\sigma} n^\mu n^\rho \chi^{\nu\sigma} - 2n^\mu \partial_\mu R \right) 
- N \left( \frac{12}{7}\kappa^2 - \frac{60}{7}\kappa\text{tr}(\kappa^2) - 12\Delta_2\omega + 8\omega^\mu\omega_\mu \right) 
+ \frac{4}{7}N^2\kappa + \frac{16}{21}N^3 + 24\kappa \Delta_2\omega - 4\chi^{\mu\nu} \omega_\mu \omega_\nu - \right]
20\kappa \omega^\mu \omega_\mu - 30n^\mu \partial_\mu (\Delta_2\omega - \omega^\nu \omega_\nu) \right]. \tag{3}$$

The normal to $\partial M$, $n^\mu$, points inwards and $\omega_\mu = \partial_\mu \omega$, $N = n^\mu \omega_\mu$, $\omega_{\mu\nu} = \omega_{||\mu\nu}$. The curvature conventions are those of Hawking and Ellis. Various equivalent forms can be found upon partial integration or use of the Gauss-Codazzi equations.

We shall also be concerned with Robin boundary conditions (sometimes called Neumann by the mathematicians). The corresponding cocycle function in three
dimensions is

\[ W^R[\bar{g}, g] = -W^D[\bar{g}, g] + \frac{1}{256\pi} \int_{\partial M} \left[ 2\text{tr} (\kappa^2) + \kappa^2 + 32\psi^2 - 16\psi\kappa \right] \omega \\
+ 2(3\kappa - 8\psi)n_\mu \omega_\mu + 2(n_\mu \omega_\mu)^2. \] (4)

The four dimensional expression is more lengthy,

\[ W^R[\bar{g}, g] = W^D[\bar{g}, g] \\
- \frac{1}{2520\pi^2} \int_{\partial M} \left[ 2(\text{tr} (\kappa^3) - \kappa\text{tr} \kappa^2 + \frac{2}{9}\kappa^2) \omega \\
+ 2(\text{tr} (\kappa^2) - \frac{2}{3}\kappa^2) N - 2\kappa N^2 - 2N^3 \\
+ 105 \left( 4\psi'^3 \omega - 2\psi'^2 N + \frac{1}{3}\psi' (\nabla_2 \omega - \omega_\mu \omega_\mu) \right) \\
- 7(\kappa^2 - 3\text{tr} \kappa^2) (\psi' \omega - \frac{1}{6} N) + 7\psi' N (2\kappa + 3N) \\
- \frac{35}{4} (3n_\nu \partial_\mu - 2\kappa)(\nabla_2 \omega - \omega_\mu \omega_\mu) \right]. \] (5)

where \( \psi' = \psi - \kappa/3. \)

An indication of how these equations are derived is given in section 10. We note that they satisfy the required symmetry under interchange of \( g \) and \( \bar{g}. \)

3. Conformal transformations.

The conformal transformations under consideration are those between the sphere, \( S^{d+1} \), Euclidean space, \( \mathbb{R}^{d+1}(\sim \mathbb{R}^+ \times S^d) \) and the cylinder, \( \mathbb{R} \times S^d \).

Our general strategy is guided by the fact that it is easier to calculate the functional determinant on the sphere and the cylinder, than on the Euclidean ball.

In the present paper we wish to extend some of the results of [2] to three and four dimensions. This involves the equatorial stereographic projection, \( S^{d+1} \to \mathbb{R}^{d+1} \), expressed by giving the sphere metric in the conformally-flat form,

\[ d\sigma_{d+1}^2 = \frac{4}{(1 + r^2)^2} dr^2. \] (6)

We shall determine the functional determinant on a 3-ball and a 4-ball, and, by an inverse projection, on a spherical cap. Then we turn to Euclidean spherical shells, which arise on conformally transforming the generalised cylinder (the ‘Einstein
Universe’), $\mathbb{R} \times S^d$, to Euclidean space. The standard conformally-flat metric on the cylinder is

$$ds^2 = d\tau^2 + d\sigma_d^2 = e^{-2\tau}(dr^2 + r^2d\sigma_d^2)$$

with $r = \exp \tau$. An inverse stereographic projection would take such a shell to a slice of the sphere, $S^{d+1}$.

4. Functional determinants on caps and balls.

The functional determinant on the 2-hemisphere has been determined by Weisberger and the general case is given in [3] (see also [20] for explicit three- and four-dimensional expressions). Therefore, for $d = 2$ and $d = 3$, (2) and (3) may be used to find the functional determinant on the 3- and 4-ball by means of the stereographic projection, (6).

It is possible to rescale the ball and project it back onto the sphere thereby giving a spherical cap, as in [2]. A corresponding application of (2) yields the cap functional determinant. The angle of the cap, $\theta$, and the radius of the ball, $a$, are related by $\theta = 2\tan^{-1}a$.

We take the opportunity of correcting the result of a transcription error in [2]. Equation (15) of this reference should be replaced by

$$W^{D}_{2\text{cap}}(\theta) - W^{D}_{2\text{hemisphere}} = -\frac{1}{3} \cos \theta - \frac{1}{6} \ln \tan \theta/2$$

with

$$W^{D}_{2\text{hemisphere}} = -\zeta_R'(1) - 2\zeta_R'(0),$$

and equation (16) by

$$W^{N}_{2\text{cap}}(\theta) - W^{N}_{2\text{hemisphere}} = \frac{1}{6} \cos \theta + \frac{1}{12} \ln(1 + \cos \theta) + \frac{5}{12} \ln(1 - \cos \theta),$$

with

$$W^{N}_{2\text{hemisphere}} = -\zeta_R'(1) + 2\zeta_R'(0).$$

The cocycle functions are

$$W^N[\bar{g}, g] = \frac{1}{6}(1 - \cos \theta) - \frac{1}{2} \ln(1 + \cos \theta) - \frac{1}{3} \ln 2$$

and

$$W^D[\bar{g}, g] = \frac{1}{6} \ln 2 - \frac{1}{3}(1 - \cos \theta).$$
Figs. 1 and 2 contain plots of (8) and (9) to replace those in [2].

Turning to three dimensions, the unit ball expression is found to be

\[ W^{D}_{3\text{ball}} = W^{D}_{3\text{hemisphere}} + \frac{7}{64} \]  \hspace{1cm} (12)

and for the cap,

\[ W^{D}_{3\text{cap}}(\theta) - W^{D}_{3\text{hemisphere}} = \frac{1}{48} \left( \ln \sin \theta + \frac{21}{4} \cos^2 \theta \right). \]  \hspace{1cm} (13)

This expression is symmetrical about \( \theta = \pi/2 \) (the hemisphere) where it has a local minimum. It tends to \(-\infty\) at the extremes, \( \theta = 0 \) and \( \theta = \pi \), and has maxima of \( \approx 0.0745 \) when \( \sin^2 \theta = 2/21 \) (\( \theta \approx 18^\circ, 162^\circ \)). A plot is given in Fig.3.

To complete the evaluation of \( W^{D}_{\text{cap}} \), the 3-hemisphere effective action is needed. This has been determined in [20,3] to be

\[ W^{D}_{3\text{hemisphere}} = \frac{3}{8} \zeta'_{R}(-2) - \frac{1}{4} \zeta'_{R}(-1) - \frac{1}{16} + \frac{1}{24} \ln 2 \approx -0.003682. \]  \hspace{1cm} (14)

The four-dimensional calculation is slightly more involved algebraically and yields firstly

\[ W^{D}_{4\text{ball}} = W^{D}_{4\text{hemisphere}} - \frac{1}{180} \ln 2 - \frac{17}{15120} \]  \hspace{1cm} (15)

followed by

\[ W^{D}_{4\text{cap}}(\theta) - W^{D}_{4\text{hemisphere}} = \frac{1}{180} \left( \frac{1}{168} \left( 1365 \cos \theta - 1399 \cos^3 \theta \right) + \ln \tan \theta/2 \right). \]  \hspace{1cm} (16)

A plot is shown in Fig.4.

To complete the numerics we need the four-hemisphere formula [3,20]

\[ W^{D}_{4\text{hemisphere}} = -\frac{1}{6} \zeta'_{R}(-3) + \frac{1}{4} \zeta'_{R}(-2) - \frac{1}{12} \zeta'_{R}(-1) - \frac{1}{516} \approx 0.003386. \]  \hspace{1cm} (17)

Equation (15) with (17) agrees with corollary 7.2 of reference [20].

Some Robin results, calculated using (4) and (5) with \( \psi = \psi_1 \) constant on the surface of the ball, are now presented, firstly for three dimensions,

\[ W^{R}_{3\text{ball}} = W^{R}_{3\text{hemisphere}} - \frac{5}{64} + \frac{1 - 2 \psi_1}{8}, \]  \hspace{1cm} (18)

\[ W^{R}_{3\text{cap}}(\theta) - W^{R}_{3\text{hemisphere}} = -\frac{1}{48} \left( 1 + 6(1 - 2 \psi_1)^2 \right) \ln \sin \theta - \frac{5}{64} \cos^2 \theta + \frac{1}{8} (1 - 2 \psi_1) \cos \theta. \]  \hspace{1cm} (19)
The value of $\psi$ on the rim of the cap is

$$\psi_c = \frac{1}{\sin \theta} \left( (\psi_1 - \frac{1}{2}) - \frac{1}{2} \cos \theta \right),$$

so

$$1 - 2\psi_1 = \cos \theta - 2\psi_c \sin \theta,$$

and that for the hemisphere

$$\psi_h = \psi_1 - \frac{1}{2}.$$

The geometrical choice is $\psi_1 = 1/2$, $\psi_c = \cot(\theta)/2$ or $\psi_h = 0$. In this case

$$W^R_{4\text{cap}}(\theta) - W^N_{3\text{hemisphere}} = -\frac{1}{48} \ln \sin \theta - \frac{5}{64} \cos^2 \theta$$

where the Neumann effective action on the 3-hemisphere is given in [3],

$$W^N_{3\text{hemisphere}} = -\frac{3}{8} \zeta'_R(-2) + \frac{1}{4} \zeta'_R(-1) + \frac{1}{16} + \frac{1}{12} \ln 2 \approx 0.067489.$$

The four dimensional expressions are

$$W^R_{4\text{ball}} = W^R_{4\text{hemisphere}} + \frac{1}{2160} - \frac{1}{180} \ln 2 + \frac{1}{12} (\psi - 1)^2 + \frac{1}{15} (\psi_1 - 1),$$

and

$$W^R_{4\text{cap}}(\theta) - W^R_{4\text{hemisphere}} =$$

$$-\frac{1}{180} \left( \frac{1}{24} (167 \cos^3 \theta - 165 \cos \theta) + \ln \tan(\theta/2) \right)$$

$$+ \frac{1}{15} (\psi_1 - 1) \cos^2 \theta + \frac{1}{12} (\psi_1 - 1)^2 \cos \theta + \frac{1}{6} (\psi_1 - 1)^3 \ln \sin \theta.$$

The relations between the values of $\psi$ are now,

$$\psi_1 - 1 = \psi_c \sin \theta - \cos \theta, \quad \text{and} \quad \psi_h = \psi_1 - 1.$$

For the geometrical choice, $\psi_c = \cot \theta$, i.e. $\psi_1 = 1$, we can use the expression for the Neumann 4-hemisphere effective action, [3],

$$W^N_{4\text{hemisphere}} = -\frac{1}{3} \zeta'_R(-3) - \frac{1}{2} \zeta'_R(-2) - \frac{1}{6} \zeta'_R(-1) - \frac{1}{288},$$

if numbers are required.
5. Symmetry.

It will be noticed that the $d = 3$ result, (13), is symmetrical about the hemisphere ($\theta = \pi/2$) while those for $d = 2$ and $d = 4$, (8) and (16), are antisymmetrical. It is possible to show that this is a general feature for odd and even dimensions. The proof is detailed here. Unless stated otherwise, all quantities refer to Dirichlet boundary conditions.

The form of the conformal change (1) is, in the present instance,

$$W_{\text{cap}}(\theta) - W_{\text{ball}}(a) = V_0^\theta + B_\theta$$

where $V$ and $B$ are volume and boundary integrals and we will take $0 \leq \theta \leq \pi/2$. The labels on $V$ are the limits of the colatitude integration.

For even dimensions,

$$W_{\text{slice}}(\theta, \theta') - W_{\text{shell}}(a, a') = V_{\theta}^{\theta'} + B_{\theta'} - B_\theta$$

since the normal to the boundary points into the domain, and is thus oppositely oriented on the two components (the inner and outer faces of the shell, or slice). For an odd-dimensional boundary, reversing the orientation reverses the sign of the integral.

Further,

$$W_{\text{ball}}(a) = W_{\text{ball}}(1) - \zeta_{\text{ball}}(0) \ln a$$

so we have

$$W_{\text{cap}}(\theta) + W_{\text{cap}}(\pi - \theta) = 2W_{\text{ball}}(1) + V_0^\theta + V_0^{\pi - \theta} + B_\theta + B_{\pi - \theta}$$

whence

$$(W_{\text{cap}}(\theta) - W_{\text{hemisphere}}) + (W_{\text{cap}}(\pi - \theta) - W_{\text{hemisphere}})$$

$$= (V_{\pi/2}^{\pi - \theta} + B_{\pi - \theta} - B_{\pi/2}) - (V_\theta^{\pi/2} + B_{\pi/2} - B_\theta)$$

$$= (W_{\text{slice}}(\pi/2, \pi - \theta) - W_{\text{shell}}(1, 1/a)) - (W_{\text{slice}}(\theta, \pi/2) - W_{\text{shell}}(a, 1))$$

$$= W_{\text{shell}}(a, 1) - W_{\text{shell}}(1, 1/a),$$

since $W_{\text{slice}}(\pi/2, \pi - \theta) = W_{\text{slice}}(\theta, \pi/2)$ by symmetry. Hence we arrive at

$$(W_{\text{cap}}(\theta) - W_{\text{hemisphere}}) - (W_{\text{cap}}(\pi - \theta) - W_{\text{hemisphere}}) = -\zeta_{\text{shell}}(0) \ln a = 0.$$
\( \zeta_{\text{shell}}(0) \) vanishes because the space is flat, and the boundary terms cancel, for the same reason as before. Therefore \( W_{\text{cap}}(\theta) - W_{\text{hemisphere}} \) is antisymmetrical under \( \theta \to \pi - \theta \), as stated.

For odd dimensions, there are no volume integrals, and the boundary is even dimensional. Hence

\[
W_{\text{cap}}(\theta) - W_{\text{ball}}(a) = B_\theta
\]

and

\[
W_{\text{slice}}(\theta, \theta') - W_{\text{shell}}(a, a') = B_\theta + B_{\theta'}
\]

so that

\[
W_{\text{cap}}(\theta) - W_{\text{cap}}(\pi - \theta) = -2 \zeta_{\text{ball}}(0) \ln a + B_\theta - B_{\pi - \theta}
\]

\[
= -2 \zeta_{\text{ball}}(0) \ln a + (B_\theta + B_{\pi/2}) - (B_{\pi/2} + B_{\pi - \theta})
\]

\[
= -2 \zeta_{\text{ball}}(0) \ln a + \left( W_{\text{slice}}(\theta, \pi/2) - W_{\text{shell}}(a, 1) \right)
\]

\[
- \left( W_{\text{slice}}(\pi/2, \pi - \theta) - W_{\text{shell}}(1, 1/a) \right)
\]

\[
= W_{\text{shell}}(1, 1/a) - W_{\text{shell}}(a, 1) - 2 \zeta_{\text{ball}}(0) \ln a.
\]

However \( \zeta_{\text{shell}}(0) = 2 \zeta_{\text{ball}}(0) \) since the boundary terms are equal and there are no volume terms. Therefore

\[
W_{\text{cap}}(\theta) - W_{\text{cap}}(\pi - \theta) = W_{\text{shell}}(1, 1/a) - W_{\text{shell}}(a, 1) - \zeta_{\text{shell}}(0) \ln a
\]

\[
= 0
\]

and \( W_{\text{cap}}(\theta) \) is symmetrical under \( \theta \to \pi - \theta \), in odd dimensions.

We finally note that the three-dimensional Robin result (19) is symmetrical about the hemisphere if the values of \( \psi \) on the cap and hemisphere are also reversed under \( \theta \to \pi - \theta \) while the four-dimensional expression (22) is antisymmetrical. Again this is a general feature if \( d \geq 3 \).

6. Functional determinants on shells and slices

Weisberger has obtained the effective action on the annulus (or 2-shell) by conformal transformation from the cylinder, \( I \times S^1 \).

\[
W_{\text{2shell}}^D = W_{I \times S^1}^D - \frac{1}{12} \ln \left( \frac{r_1}{r_2} \right)
\]

where \( r_1 \) and \( r_2 \) are the outer and inner radii of the shell.
A conformal transformation of the \((d + 1)\)-shell into the \((d + 1)\)-sphere gives the effective action on a \((d + 1)\)-slice (or spherical, spherical shell). A summary of our findings follows.

\[
W_{3\text{shell}}^D = W_{I \times S^2}^D + \frac{1}{96} (2 \ln(r_1 r_2) + 9),
\]

\[
W_{4\text{shell}}^D = W_{I \times S^3}^D + \frac{1}{720} \ln \left( \frac{r_1}{r_2} \right),
\]

\[
W_{2\text{slicer}}^D = W_{2\text{shell}}^D + \frac{2}{3} \sin \Theta \sin \Delta,
\]

\[
W_{3\text{slicer}}^D = W_{3\text{shell}}^D - \frac{21}{192} \left( 1 - \cos 2\Theta \cos 2\Delta \right) + \frac{1}{24} \ln \left( \cos \Delta + \cos \Theta \right),
\]

\[
W_{4\text{slicer}}^D = W_{4\text{shell}}^D - \frac{1}{30240} \sin \Theta \sin \Delta \left( 2730 - 1399(2 + 2 \cos 2\Theta \cos 2\Delta + \cos 2\Theta + \cos 2\Delta) \right)
\]

where \(\Theta\) is the colatitude of the midpoint of the slice and \(\Delta\) is its angular half-width.

We note the geometrical relations

\[
\begin{align*}
    r_1 &= \frac{\sin \Theta + \sin \Delta}{\cos \Delta + \cos \Theta}, \\
    r_2 &= \frac{\sin \Theta - \sin \Delta}{\cos \Delta + \cos \Theta}, \\
    r_1 r_2 &= \frac{\cos \Delta - \cos \Theta}{\cos \Delta + \cos \Theta}, \\
    L &= \ln \left( \frac{r_1}{r_2} \right).
\end{align*}
\]

As an example, for a slice symmetrical about the equator, \(\Theta = \pi/2\) and the cylinder length is \(L = \ln \left( (1 + \sin \Delta)/(1 - \sin \Delta) \right) = 2 \tanh^{-1}(\sin \Delta)\) so that, as \(\Delta \to \pi/2\), \(L \to \infty\) and \(\Delta \to 0\) implies \(L \to 0\).

\(W_{4\text{slicer}}^D\), (27), simplifies to

\[
W_{4\text{slicer}}^D = W_{I \times S^3}^D + \frac{1}{720} \ln \left( \frac{1 + \sin \Delta}{1 - \sin \Delta} \right) - \frac{1}{15120} \sin \Delta \left( 1365 - 1399 \sin^2 \Delta \right)
\]

and is plotted in Fig.5. It shows a minimum at \(\Delta \approx 46^\circ\).

The general Robin results are more complicated. Corresponding to (23),

\[
W_{3\text{shell}}^R = W_{I \times S^2}^R - \frac{1}{96} (2 \ln(r_1 r_2) + 3) + \frac{1}{8} \left( (1 - 2r_1 \psi_1) + (1 + 2r_2 \psi_2) \right) - \frac{1}{8} \left( (1 - 2r_1 \psi_1)^2 \ln r_1 + (1 + 2r_2 \psi_2)^2 \ln r_2 \right),
\]
where $\psi_1$ and $\psi_2$ are the $\psi$ values on the outer and inner shell boundaries respectively. For $\psi_1 = 1/2r_1$ and $\psi_2 = -1/2r_2$, this simplifies to

$$W_{3\text{shell}}^R = W_{I \times S^2}^R - \frac{1}{96}(2 \ln(r_1r_2) + 3).$$

The corresponding $\psi$ functions on $I \times S^2$ are $r_1\psi_1 - 1/2$ and $r_2\psi_2 - 1/2$, implying Neumann conditions for the geometrical choice.

Transforming the shell to a slice yields,

$$W_{3\text{slice}}^R = W_{3\text{shell}}^R + \frac{5}{64}(1 - \cos 2\Theta \cos 2\Delta) - \frac{1}{24} \ln \left( \cos \Delta + \cos \Theta \right)$$

$$- \frac{1}{8}((2\psi_\alpha \sin \theta_1 - \cos \theta_1)^2 \ln(1 + \cos \theta_1)$$

$$+ (2\psi_\beta \sin \theta_2 + \cos \theta_2)^2 \ln(1 + \cos \theta_2))$$

$$+ \frac{1}{8}((2\psi_\alpha \sin \theta_1 - \cos \theta_1)(1 - \cos \theta_1) - (2\psi_\beta \sin \theta_2 + \cos \theta_2)(1 - \cos \theta_2))$$

(30)

where $\psi_\alpha$ and $\psi_\beta$ are the $\psi$ values on the slice boundaries, at the angles $\theta_1$ and $\theta_2$ respectively. For $\psi_\alpha = \cot(\theta_1)/2$ and $\psi_\beta = \cot(\theta_1)/2$ only the first line of (30) survives and the boundary condition on the interval is the Neumann one.

$$W_{4\text{shell}}^R = W_{I \times S^3}^R - \frac{1}{6}((r_1\psi_1 - 1)^3 \ln r_1 + (r_2\psi_2 - 1)^3 \ln r_2)$$

$$+ \frac{1}{12}((r_1\psi_1 - 1)^2 - (r_2\psi_2 - 1^2))$$

$$+ \frac{1}{40}((r_1\psi_1 - 1) + (r_2\psi_2 - 1)) + \frac{1}{720} \ln \left( \frac{r_1}{r_2} \right).$$

(31)

For the geometrical choice this reduces to the final term which equals the Dirichlet value.

For the four-dimensional slice, we find

$$W_{4\text{slice}}^R - W_{4\text{shell}}^R =$$

$$- \frac{1}{4320} \sin \Theta \sin \Delta \left( 330 - 167(2 + 2\cos 2\Theta \cos 2\Delta + \cos 2\Theta + \cos 2\Delta) \right)$$

$$+ \frac{1}{6}((\psi_\alpha \sin \theta_1 - \cos \theta_1)^3 \ln(1 + \cos \theta_1) + (\psi_\beta \sin \theta_2 + \cos \theta_2)^3 \ln(1 + \cos \theta_2))$$

$$- \frac{1}{12}((\psi_\alpha \sin \theta_1 - \cos \theta_1)^2 \ln(1 - \cos \theta_1) + (\psi_\beta \sin \theta_2 + \cos \theta_2)^2 \ln(1 - \cos \theta_2))$$

$$- \frac{1}{15}((\psi_\alpha \sin \theta_1 - \cos \theta_1) \sin^2 \theta_1 + (\psi_\beta \sin \theta_2 + \cos \theta_2) \sin^2 \theta_2)$$

(32)

with the same notation as before.
7. Generalised cylinders

We now find the functional determinant on cylinders needed to complete the calculation on shells and slices according to the equations in the previous section. Having some interest in its own right, it is developed in more generality than is strictly necessary for present purposes.

The $\zeta$–function on the generalised cylinder, $I \times \mathcal{M}$, is firstly constructed as

$$\zeta_{I \times \mathcal{M}}^D(s) = \sum_{n=1}^{\infty} \sum_{\lambda} d_{\lambda} \frac{d_{\lambda}}{(\pi^2 n^2/L^2 + \lambda^2)^s}$$

(33)

where $L$ is the length of the interval $I$ and $d_{\lambda}$ is the degeneracy of the $\lambda$ eigenvalue of the scalar Laplacian $-\Delta_2 + \xi R$, $\xi = (d - 1)/4d$, on $\mathcal{M}$, conformal in $(d + 1)$-dimensions. For Neumann conditions, the $n$-summation runs from 0 upwards, excluding any overall zero modes.

It is convenient to extend the $n$-summation over negative integers to get

$$\zeta_{I \times \mathcal{M}}(s) = \frac{1}{2} \left( \zeta_{S^1 \times \mathcal{M}}(s) \mp \zeta_{\mathcal{M}}(s) \right)$$

where, up to a factor, the first $\zeta$–function in the brackets can be thought of as a thermal one. The signs refer to the boundary conditions on the interval.

Standard manipulations similar to those leading to Kronecker’s limit formula (e.g. [23–26]) give

$$\zeta'_{I \times \mathcal{M}}(0) = \frac{1}{2} \left( \zeta'_{R \times \mathcal{M}}(0) \mp \zeta'_{\mathcal{M}}(0) \right) - \sum_{m=1}^{\infty} \frac{1}{m} K^{1/2}(2mL)$$

(34)

where $K^{1/2}(z)$ is the kernel for the pseudo-differential operator $(-\Delta_2 + \xi R)^{1/2}$. Also

$$\zeta_{R \times \mathcal{M}}(s) = \frac{1}{\sqrt{4\pi}} \frac{\Gamma(s - 1/2)}{\Gamma(s)} \zeta_{\mathcal{M}}(s - 1/2).$$

For $\mathcal{M}$ we choose that portion, denoted $\mathcal{F}$, of $S^d$ which acts as a fundamental domain for the complete symmetry group of a regular $(d + 1)$–polytope classified by the degrees $d_i$, $(i = 1, 2, \ldots, d_{d+1} = 2)$.

As shown in [27], $\zeta_{\mathcal{F}}(-1/2)$ is finite and, in this case,

$$\zeta'_{R \times \mathcal{F}}(0) = -\zeta_{\mathcal{F}}(-1/2).$$

(35)

The Neumann ($\psi = 0$) and Dirichlet $\zeta$–functions on $\mathcal{F}$ are explicitly [27],

$$\zeta_{\mathcal{F}}(s) = \zeta_d(2s, a \mid d),$$
where the general definition of the Barnes ζ-function is
\[
\zeta_d(s, a \mid d) = \frac{i\Gamma(1 - s)}{2\pi} \int_L dz \frac{\exp(-az)(-z)^{s-1}}{\prod_{i=1}^d (1 - \exp(-d_i z))} \\
= \sum_{m=0}^\infty \frac{1}{(a + m d)^s}, \quad \text{Re } s > d,
\]
(36)
which shows that the eigenvalues are perfect squares,
\[
\lambda_n = (a + m d)^2,
\]
(37)
the degeneracies arising from coincidences. It is then easy to construct the kernel
\[K_{1/2}.\] It is evident in (36). The parameter \(a\) is \((d - 1)/2\) for Neumann conditions
and \(\sum d_i - (d - 1)/2\) for Dirichlet. There are no zero modes if \(d > 1\).

Equation (34) becomes
\[
\zeta'_{I \times \mathcal{F}}(0) = -\frac{1}{2} \zeta_d(-1, a \mid d) + \zeta'_d(0, a \mid d) - \sum_{m=1}^\infty e^{-2amL} \frac{1}{m} \prod_{i=1}^d \frac{1}{1 - q_i^m}
\]
(38)
where \(q_i = \exp(-2Ld_i)\).

Barnes has given formulae for \(\zeta_d(-n), n \in \mathbb{Z}\), in terms of generalized Bernoulli
functions. In particular
\[
\zeta_d(-1, a \mid d) = \frac{(-1)^d}{\prod_i d_i (d-1)!} B^{(d)}_{d+1}(a \mid d)
\]
which is needed in (35).

The derivative of the Barnes ζ–function is related to the multiple gamma function, [28]. A method of evaluation is contained in [1] and so we may assume that
the functional determinant on the generalised cylinder, \(I \times \mathcal{F}\), is known. Unfortunately, as mentioned earlier, because the heat-kernel coefficients are not extant in
the piecewise smooth case if \(d \geq 3\), the functional determinant cannot be conformally transformed unless \(\mathcal{F}\) is the complete sphere. Of course, in this case, it is
not necessary to invoke the full generality of the Barnes ζ–function. The formulae
could be derived directly and have been known for a long time.

The hemisphere degrees are all unity, \(d = 1\), and the ζ–function reduces to
\[
\zeta_d(s, a) = \frac{i\Gamma(1 - s)}{2\pi} \int_L \frac{e^{z(d/2-a)}(-z)^{s-1}}{2d \sinh^{d}(z/2)} dz \\
= \sum_{m=0}^\infty \left( \frac{m + d - 1}{d - 1} \right) \frac{1}{(a + m)^s}, \quad \text{Re } s > d,
\]
(39)
with \( a = (d + 1)/2 \) for Dirichlet (D) and \( a = (d - 1)/2 \) for Neumann (N) conditions on the hemisphere rim.

The summation form can be adjusted to the following expressions,

\[
\begin{align*}
\zeta_D^d (s) &= \frac{1}{(d-1)!} \sum_{m=1}^{\infty} \frac{(m+q-1)\ldots(m-q)}{m^s}, \\
\zeta_N^d (s) &= \frac{1}{(d-1)!} \sum_{m=1}^{\infty} \frac{(m+q)\ldots(m-q+1)}{m^s},
\end{align*}
\]

(40)

for odd dimensions \( (d = 2q + 1) \), while for even dimensions \( (d = 2q + 2) \)

\[
\begin{align*}
\zeta_D^d (s) &= \frac{1}{(d-1)!} \sum_{m=0}^{\infty} \frac{(m+q)\ldots(m-q)}{(m+1/2)^s}, \\
\zeta_N^d (s) &= \frac{1}{(d-1)!} \sum_{m=0}^{\infty} \frac{(m+q+1)\ldots(m-q+1)}{(m+1/2)^s}.
\end{align*}
\]

(41)

We therefore need the Stirling number expansions

\[(m+a)(m+a-1)\ldots(m+a-b+1) = \sum_{k=0}^{b} S^{(k)}(a,b) m^k = \sum_{k=0}^{b} T^{(k)}(a,b) (m+1/2)^k\]

which allow the series in (40), (41) to be written as sums of Riemann \( \zeta \)-functions so providing a practical continuation in a familiar fashion.

For odd \( d \),

\[
\begin{align*}
\zeta_D^d (s) &= \sum_{k=0}^{2q} S^{(k)}(q-1,2q) \zeta_R(s-k), \\
\zeta_N^d (s) &= \sum_{k=0}^{2q} S^{(k)}(q,2q) \zeta_R(s-k),
\end{align*}
\]

and for even \( d \),

\[
\begin{align*}
\zeta_D^d (s) &= \sum_{k=0}^{2q+1} T^{(k)}(q,2q+1) \zeta_R(s-k,1/2), \\
\zeta_N^d (s) &= \sum_{k=0}^{2q+1} T^{(k)}(q+1,2q+1) \zeta_R(s-k,1/2).
\end{align*}
\]

(42)
A few expressions are made explicit:

\[ \zeta_2(s) = \zeta_R(s - 1, 1/2) \pm \frac{1}{2} \zeta_R(s, 1/2), \]
\[ \zeta_3(s) = \frac{1}{2} (\zeta_R(s - 2) \mp \zeta_R(s - 1)), \]
\[ \zeta_4(s) = \frac{1}{6} (\zeta(s - 3, 1/2) \mp \frac{3}{2} \zeta_R(s - 2, 1/2) - \frac{1}{4} \zeta_R(s - 1, 1/2) \pm \frac{3}{8} \zeta_R(s, 1/2)), \]
\[ \zeta_5(s) = \frac{1}{24} (\zeta(s - 4) \mp 2 \zeta_R(s - 3) - \zeta_R(s - 2) \pm 2 \zeta_R(s - 1)), \]
\[ \zeta_6(s) = \frac{1}{5!} (\zeta_R(s - 5, 1/2) \mp \frac{5}{2} \zeta_R(s - 4, 1/2) - \frac{5}{2} \zeta_R(s - 3, 1/2) \pm \frac{25}{4} \zeta_R(s - 2, 1/2) + \frac{9}{16} \zeta_R(s - 1, 1/2) \mp \frac{45}{12} \zeta_R(s, 1/2)), \]
\[ \zeta_7(s) = \frac{1}{6!} (\zeta_R(s - 6) \mp 3 \zeta_R(s - 5) - 5 \zeta_R(s - 4) \pm 15 \zeta_R(s - 3) + 4 \zeta_R(s - 2) \mp 12 \zeta_R(s - 1)), \]

where the upper sign refers to Dirichlet conditions on the hemisphere rim and the lower one to Neumann. The full sphere expression is found simply by adding these two forms.

It is now straightforward to determine \( \zeta_\prime_d(0) \) and \( \zeta_\prime_d(-1) \) in terms of named zeta functions and the cylinder functional determinant then follows from (38), the final term being readily evaluated.

To complete the conformal derivation of the functional determinants on shells and slices, more explicit values will be exhibited in the two cases of numerical relevance here:

\[ \zeta_\prime_{I \times HS^2}(0) = \frac{1}{2} \zeta_R(-1) + \left( \frac{1}{24} (1 \mp 6) \ln 2 \mp \frac{1}{96} \right) - \frac{1}{4} \sum_{m=1}^{\infty} \frac{e^{\pm mL}}{m \sinh^2 mL} \]  
(44)

and

\[ \zeta_\prime_{I \times HS^3}(0) = -\left( \frac{1}{2} \zeta_R(-2) \mp \frac{1}{2} \zeta_\prime_R(-1) \right) - \frac{1}{480} - \frac{1}{8} \sum_{m=1}^{\infty} \frac{e^{\pm mL}}{m \sinh^3 mL} \]  
(45)

with \( q = \exp(-2L) \). The above expressions are for Dirichlet conditions on \( I \). For Neumann conditions the terms in large brackets are to be reversed in sign.

Therefore, cf [27] for the final term,

\[ W_{I \times S^2}^{D,N} = \mp \frac{1}{2} \zeta_\prime_R(-1) \mp \frac{1}{24} \ln 2 \mp \frac{1}{4} \sum_{m=1}^{\infty} \frac{\cosh mL}{m \sinh^2 mL} \]
\[ W_{I \times S^3}^{D,N} = \pm \frac{1}{2} \zeta_\prime_R(-2) + \frac{1}{480} + \frac{1}{8} \sum_{m=1}^{\infty} \frac{\cosh mL}{m \sinh^3 mL} \]  
(46)
which are required in (23) and (24). Now, of course, $N$ and $D$ refer to interval boundary conditions. For large $L$ the summations vanish exponentially while for small $L$ one gets the typical Planckian high temperature forms.

8. Non-smooth boundaries

Although $C_{3/2}$ is not known for non-smooth boundaries, it is possible to make a certain amount of progress since dimensions restricts the unknown contributions. For Dirichlet conditions, in $d$-dimensions, we conjecture that

$$C^{D}_{3/2} = \frac{\sqrt{\pi}}{192} \sum_i \int_{\partial M_i} (6 \text{tr} (\kappa_i^2) - 3\kappa_i^2 - 4\hat{R} + 12(8\xi - 1)R)$$

$$- \frac{\sqrt{\pi}}{24} \sum_{(ij)} \int_{E_{ij}} \left[ \lambda(\theta_{ij})(\kappa_i + \kappa_j) + \mu(\theta_{ij})(\kappa^{(i)} + \kappa^{(j)}) \right] + \sum_l \int_{V_l} \nu \quad (47)$$

where $\kappa^{(i)}$ and $\kappa^{(j)}$ are the traces of two extrinsic curvatures, $\kappa^{(i)}$ and $\kappa^{(j)}$, associated with the codimension-2 intersection, $E_{ij} = \partial M_i \cap \partial M_j$. $\kappa^{(i)}$ can also be interpreted as the extrinsic curvature of $\partial M_i \cap \partial M_j$ considered as a codimension-1 submanifold of $\partial M_i$ – part of the ‘boundary of a boundary’. $\kappa_i$ and $\kappa_j$ are the traces of the extrinsic curvatures, $\kappa_i$ and $\kappa_j$, of the boundary parts, $\partial M_i$ and $\partial M_j$ respectively. $\theta_{ij}$ is the dihedral angle between these two parts. (It can vary along the intersection.) The integrand, $\nu$, of the vertex (or ‘corner’) contribution is a function of the dihedral angles between those boundary parts meeting at the vertex.

It is a theorem that, for a flat ambient space, the extrinsic curvatures and the normal fundamental forms define an embedded submanifold up to Euclidean motions if the Codazzi-Mainardi and Ricci conditions are satisfied, e.g. [29] IV p 64. Thus, in general, we might expect the heat-kernel coefficients to depend on the normal fundamental 1-form, $\beta$, of $E$ considered as a codimension-2 submanifold of $\mathcal{M}$. This is ruled out immediately on dimensional grounds since it would have to occur in $C^{D}_{3/2}$ in the gauge invariant combination $\text{curl} \beta \cdot \text{curl} \beta$. We note that $\beta$ is conformally invariant.

Some restrictions follow from the requirement that $C^{D}_{3/2}$ be conformally invariant, for $\xi = \xi(d) = (d - 2)/4(d - 1)$, when $d = 3$. Applying the standard conformal transformations, we find the relation

$$2 \tan(\theta/2)\lambda(\theta) + \mu(\theta) = 1. \quad (48)$$
In two dimensions the $\kappa^{(i)}$ are zero and the result for the disc shows that $\lambda(\pi) = 0$, hence $\mu(\pi)$ is finite. If $\theta_{ij} = \pi$, $\kappa^{(i)} + \kappa^{(j)} = 0$. (When $\theta_{ij} = \pi$ the join between $\partial M_i$ and $\partial M_j$ is not necessarily smooth.)

To prove (48), the required conformal transformations are

$$(2\text{tr} \kappa_i^2 - \kappa_i^2) \rightarrow e^{2\omega}(2\text{tr} \kappa_i^2 - \kappa_i^2 - 2(d - 3)\kappa_i N_i - (d - 3)(d - 1)N_i^2) \quad (49)$$

$$\hat{R} \rightarrow e^{2\omega}(\hat{R} + 2(d - 2)\hat{\Delta}_2 \omega - (d - 2)(d - 3)\omega^i \omega_i) \quad (50)$$

where $N_i = n_i^\mu \omega_\mu$ and $n_i^\mu$ is the inward normal to $\partial M_i$.

We are looking for the change in $C_{3/2}$ to vanish when $d = 3$ hence anything with a factor of $(d - 3)$ can be ignored (for this calculation). Note $8\xi - 1 = (d - 3)/(d - 1)$. Therefore in the $\partial M$ part, only the $\hat{\Delta}_2 \omega$ term from (50) remains. This is

$$-\frac{\sqrt{\pi}}{24}(d - 2)\int_{\partial M_i} e^{(3-d)\omega} \hat{\Delta}_2 \omega.$$

Now integrate by parts once,

$$-\frac{\sqrt{\pi}}{24}(d - 2)(3 - d)\int_{\partial M_i} e^{(3-d)\omega} h^{ab} \partial_a \omega \partial_b \omega + \frac{\sqrt{\pi}}{24}(d - 2)\sum_j \int_{E_{ij}} e^{(3-d)\omega} N_i \quad (51)$$

where $N_i = n_i^\mu \omega_\mu$ and $n_i^\mu$ is the normal (to the edge $E_{ij}$) that points into the face $\partial M_i$. (Any necessary summations over $i$ have been dropped, and $h^{ab}$ is the intrinsic metric on $\partial M$.)

The last term in (51) is not conformally invariant at $d = 3$. We expect it to be cancelled by the changes in the edge integrals. For these, the new conformal transformations needed are

$$\kappa^{(i)} \rightarrow e^{\omega}(\kappa^{(i)} + (d - 2)N_{(i)}) \quad (52)$$

and

$$\text{dvol}_I \rightarrow e^{(2-d)\omega} \text{dvol}_I.$$

Also

$$\kappa_i \rightarrow e^{\omega}(\kappa_i + (d - 1)n_i^\mu \omega_\mu).$$

In terms of the dihedral angle $\theta$ there is the geometrical relation (all the $n$’s are normalised),

$$n_i^\mu \big|_E = \text{cosec}(\theta_{ij}) n_{(j)}^\mu - \cot(\theta_{ij}) n_{(i)}^\mu \quad (53)$$
so that

\[ (n^\mu_i + n^\mu_j)|_E = \tan(\theta_{ij}/2) (n^\mu_{(i)} + n^\mu_{(j)}). \]  

(54)

Requiring the nonconformally invariant terms to cancel at \( d = 3 \) yields (48).

Information about \( \lambda, \mu \) and \( \nu \) follows in traditional fashion by special case evaluation. In two dimensions, the \( \kappa^{(i)} \) are zero. The heat-kernel expansion can be derived without difficulty for a sector of a disc, in particular for half a disc, \( HD^2 \).

From this we find that

\[ \lambda(\pi/2) = -3. \]  

(55)

The expansion for the cylinder, \( I \times D^2 \), follows by trivial product and yields

\[ \mu(\pi/2) = 7 \]  

(56)

agreeing with (48).

These values are sufficient to determine the effective action when there are no corners and the contiguous boundary parts are perpendicular. Such will be the case for the cylinder, \( I \times \) hemisphere. The results of this evaluation are given in section 9.

**Corner contributions**

The expansion on the polygonal cylinder, \( I \times \) polygon, is easily deduced from that on the polygon and yields the particular values for the constant term in the heat-kernel expansion, [30,31],

\[ w(\pi/\theta, 2, 2) = \mp \frac{1}{96} \left( \frac{\pi}{\theta} - \frac{\theta}{\pi} \right), \]

where the notation is \( w(\pi/\theta_1, \pi/\theta_2, \pi/\theta_3) \) in terms of the three dihedral angles \( \theta_1, \theta_2 \) and \( \theta_3 \).

Other specific values for trihedral corner contributions in three dimensions have been evaluated in [32]. The constant term is

\[ w(3, 3, 2) = \mp 1/16, \quad w(3, 4, 2) = \mp 15/128, \quad w(3, 5, 2) = \mp 15/64. \]
The smeared coefficient

Conformal variation can be employed to determine the useful smeared coefficient

\[ C_k^{(d)}[g; f] \equiv \int_{\mathcal{M}} C_k^{(d)}(g, x, x) f(x) \]

by the formula (cf [19])

\[ C_k^{(d)}[g; \delta \omega] = -\frac{1}{d - k/2} \delta C_k^{(d)} [e^{-2\omega} g; 1]_{\omega=0} - 2C_{k-1}^{(d)}(g; J\delta \omega) \quad (57) \]

derived from the variation of the zeta function, e.g. [33]. \( J \) is the operator

\[ J = (d - 1)(\xi - \xi(d))\Delta_2. \]

A straightforward calculation yields

\[ C_{3/2}^{(d)}[g; f] = \frac{\sqrt{\pi}}{192} \sum_i \int_{\partial \mathcal{M}_i} \left[ (6\text{tr}(\kappa_i^2) - 3\kappa_i^2 - 4\hat{R} + 12(8\xi - 1)R) f + 30n_i^\mu f_{\mu} - 24n_i^\mu n_i^\nu f_{\mu\nu} \right] \\
- \frac{\sqrt{\pi}}{24} \sum_{(ij)} \int_{E_{ij}} \left[ \lambda(\theta_{ij}) (\kappa_i + \kappa_j) f + \mu(\theta_{ij}) (\kappa_i^{(i)} + \kappa^{(j)}) f + \frac{1}{2} (\mu(\theta_{ij}) + 5) (n_i^\mu + n_j^\mu) f_{\mu} \right] + \sum_i \int_{V_i} \nu f, \quad (58) \]

where \( f_\mu = \partial_\mu f \) and \( f_{\mu\nu} = f_{\mu\nu} \). The first line (i.e. the smooth expression) can be found in [34,19].

A small technical point is that the second part of the right-hand side of (57) removes the \( \xi \) dependence from the \( \Delta_2\omega \) term that results from the variation of the \( R \) term in the first part, and replaces it with \( \xi(d) \), confirming that the edge terms in (58) do not depend on \( \xi \).

9. Robin boundary conditions

The expression for the Robin \( C_{3/2} \) is given in [35], eqn. 5.11c, in the smooth case as

\[ C_{3/2}^R = -C_{3/2}^D + \frac{\sqrt{\pi}}{32} \int_{\partial \mathcal{M}} (2\text{tr}(\kappa^2) + \kappa^2 + 32\psi^2 - 16\psi\kappa) \quad (59) \]
where \( \psi \) is the function in the boundary condition \((n^\mu \partial_\mu \phi - \psi \phi)|_{\partial \mathcal{M}} = 0\) obeying the compensating conformal transformation

\[
\psi \rightarrow e^{\omega} (\psi + \frac{1}{2} (d - 2) n^\mu \omega_\mu).
\] (60)

It is readily checked that the second term in (59) is conformally invariant at \( d = 3 \).

A purely geometrical choice for \( \psi \) is, in each dimension [35,36]

\[
\psi = \frac{(d - 2)}{2(d - 1)} \kappa
\] (61)

and in this case the integrand in (59) becomes

\[
12 \text{tr} (\kappa^2) - 6(1 - 2(d - 3)^2)\kappa^2
\]
evaluating to the conformally invariant combination, \( 12 \text{tr} (\kappa^2) - 6\kappa^2 \), at \( d = 3 \).

For a non-smooth boundary, the Robin condition applied to each piece is

\[
(n^\mu \partial_\mu \phi - \psi_i \phi)|_{\partial \mathcal{M}_i} = 0.
\] (62)

Applying the condition directly to the intersection \( E_{ij} \),

\[
(n^\mu \partial_\mu \phi - \psi^{(i)} \phi)|_{E_{ij}} = 0
\] (63)

and the general relation,

\[
(\psi_i + \psi_j)|_I = \tan(\theta_{ij}/2)(\psi^{(i)} + \psi^{(j)})
\] (64)
follows from (54).

The geometrical choice (61) on each boundary piece reads,

\[
\psi_i = \frac{(d - 2)}{2(d - 1)} \kappa_i.
\] (65)

On \( E_{ij} \) there is another choice. We could set, referring to (52),

\[
\psi^{(i)} = \frac{1}{2} \kappa^{(i)}.
\] (66)

Relation (64) is not consistent with both (65) and (66) being satisfied. Presumably (65) is the physically relevant choice, inducing the boundary condition on \( I \) via

\[
\psi^{(i)} = (\csc(\theta_{ij}) \psi_j + \cot(\theta_{ij}) \psi_i)|_E.
\]
Given only one (arbitrary smooth) submanifold, then of course only intrinsically defined objects are available. It is an interesting problem to extend the usual theory of heat-kernel expansions to the case when the field satisfies some condition on a submanifold of any codimension.

If the geometric choice (65) is not made, then it would seem simplest to assume that $\psi$ is continuous over all of $\partial M$ so that $\psi^{(i)} = \psi^{(j)}$.

Taking the Dirichlet expression as exhibited in (47), there will be an additional edge term involving $\psi$,

$$C_{3/2}^R = -C_{3/2}^D + \frac{\sqrt{\pi}}{192} \sum_i \int_{\partial M_i} \left( 12\text{tr} (\kappa_i^2) + 6\kappa_i^2 + 192\psi_i^2 - 96\psi_i\kappa_i \right)$$

$$+ \frac{\sqrt{\pi}}{12} \sum_{(ij)} \int_{E_{ij}} \rho(\theta_{ij}) (\psi^{(i)} + \psi^{(j)}) .$$

Relation (64) shows that the final term is general enough.

This time, conformal invariance at $d = 3$ gives the relation

$$2 \tan(\theta/2) \lambda(\theta) + \mu(\theta) + \rho(\theta) = 1 .$$

Moss [37] has considered the Robin case (with $\psi$ constant which is adequate here) on the disc and hemisphere. The expressions show immediately that $\rho(\pi) = 0$. Extending his results to the two-dimensional hemidisc and the 3-cylinder reveals that

$$\lambda(\pi/2) = 9, \quad \mu(\pi/2) = -5, \quad \rho(\pi/2) = -12 ,$$

with checks.

The smeared coefficient follows after a conformal transformation,

$$C_{3/2}^R[g; f] = -C_{3/2}^D[g; f] + \frac{\sqrt{\pi}}{32} \sum_i \int_{\partial M_i} \left( 2\text{tr} (\kappa_i^2) + \kappa_i^2 + 32\psi_i^2 - 16\psi_i\kappa_i \right) f$$

$$+ \frac{\sqrt{\pi}}{24} \sum_{(ij)} \int_{E_{ij}} \rho(\theta_{ij}) \left( 2(\psi^{(i)} + \psi^{(j)}) f + (n^\mu_{(i)} + n^\mu_{(j)}) f_\mu \right)$$

where $C_{3/2}^D[g; f]$ is given by (58).
10. The cocycle function

In the conformally invariant case, there are two equivalent ways of deriving the cocycle function, $W[\bar{g}, g]$. One consists of integrating the conformal anomaly equation,

$$\delta W[\bar{g}] = \frac{1}{(4\pi)^{3/2}} C^{(3)}_{3/2}[\bar{g}; \delta \omega]$$

and the other involves a finite transformation in general dimensions, the formula being

$$W[e^{-2\omega}g, g] = \lim_{d \to 3} \left(4\pi\right)^{-d/2} \frac{C^{(d)}_{3/2}[e^{-2\omega}g] - C^{(d)}_{3/2}[g]}{d - 3}.$$  

(72)

For smooth boundaries, equation (2) gives the result, which we will now refer to as $W^D_S[\bar{g}, g]$. The additional effects of edges and corners is contained in

$$W^D[\bar{g}, g] = W^D_S[\bar{g}, g] - \frac{1}{384\pi} \sum_{(ij)} \int_{E_{ij}} \left[ 2(\lambda(\kappa_i + \kappa_j) + \mu(\kappa^{(i)} + \kappa^{(j)}))\omega - (5 + \mu) (n^\mu_{(i)} + n^\mu_{(j)}) \omega_\mu + 4\omega(n^\mu_{(i)} + n^\mu_{(j)}) \omega_\mu \right] + \sum_l \int_{V_l} \nu \omega.$$  

(73)

The Robin cocyle function is determined to be

$$W^R[\bar{g}, g] = -W^D[\bar{g}, g] + \frac{1}{256\pi} \sum_i \int_{\partial M_i} \left[ (2\text{tr}(\kappa_i^2) + \kappa_i^2 + 32\psi_i^2 - 16\psi_i \kappa_i)\omega + 2(3\kappa - 8\psi)n^\mu_{(i)} \omega_\mu + 2(n^\mu_{(i)} \omega_\mu)^2 \right]$$

$$+ \frac{1}{384\pi} \sum_{(ij)} \int_{E_{ij}} \left[ 2\rho(\psi^{(i)} + \psi^{(j)})\omega - \rho(n^\mu_{(i)} + n^\mu_{(j)}) \omega_\mu \right]$$

(74)

where $W^D$ is given by (73)

These formulae, together with the values (55), (56) and (69) can be used to find the effective action on a 3-hemiball from that on a quarter 3-sphere and also
on a 3-hemishell from that on the cylinder $I \times 3$-hemisphere. We find

\[
\begin{align*}
W_{3\text{hemiball}}^D &= W_{\frac{1}{4}-3\text{sphere}}^D + \frac{1}{384} (53 - 4 \ln 2) + \frac{1}{48} \ln a \\
W_{3\text{hemicap}}^D &= W_{I \times 2\text{hemisphere}}^D + \frac{1}{96} \ln \left(\frac{r_1}{r_2}\right) + \frac{9}{192} \\
W_{3\text{hemishell}}^D &= W_{I \times 2\text{hemisphere}}^D + \frac{1}{96} \left[ 2 \ln(\cos \Theta + \cos \Delta) - 16 \sin \Theta \sin \Delta \\ &\quad + \frac{21}{4} (\cos 2\Theta \cos 2\Delta - 1) \right] \\
W_{3\text{hemislice}}^D &= W_{3\text{hemishell}}^D + \frac{1}{96} \left[ 2 \ln(\cos \Theta + \cos \Delta) - 16 \sin \Theta \sin \Delta \\ &\quad + \frac{21}{4} (\cos 2\Theta \cos 2\Delta - 1) \right]
\end{align*}
\]  

(75)

with the same geometrical relations as before, (28). For space reasons, the corresponding Robin expressions are not given.

Since $W_{I \times 2\text{hemisphere}}$ is contained in (44), the only unknown quantity on the right-hand side of (75) is $W_{\frac{1}{4}-3\text{sphere}}$. This can be obtained by the methods of [1,3].

10. The quartersphere effective action

In this case, there are two perpendicular reflecting hyperplanes and all the degrees are unity, except for $d_1 = 2$. The $\zeta$–function reduces to

\[
\zeta_{QS}(s, a) = \sum_{m,n=0}^{\infty} \left( \frac{m + d - 2}{d - 2} \right) \frac{1}{(a + 2n + m)^{2s} - \alpha^2}, \quad \text{Re } s > d,
\]  

(76)

where $\alpha = 1/2$ for conformal coupling in $d$-dimensions. This time $a = (d + 3)/2$ for Dirichlet (D) and $a = (d - 1)/2$ for Neumann (N) conditions. Again, for brevity, only the Dirichlet forms are exposed.

The techniques of [1,3], applied to (76) yield expressions involving the Barnes $\zeta$–function. One finds for the relevant quantity, (cf eqn.(23) of [3]),

\[
\zeta_{QS}'(0) = \zeta_d'(0, d/2 + 1) + \zeta_d'(0, d/2 + 2) - \sum_{r=1}^{u} \frac{1}{2^{2r}} N_{2r}(d) \sum_{k=0}^{r-1} \frac{1}{2k + 1}.
\]  

(77)

Here $u$ equals $d/2$ if $d$ is even, and $(d - 1)/2$ if $d$ is odd. $N$ is the residue of the Barnes $\zeta$–function,

\[
\zeta_d(s + r, a) \to \frac{N_r(d)}{s} \quad \text{as } s \to 0,
\]  

(78)
where, in this case,

$$
\zeta_d(s, a) = \frac{i \Gamma(1 - s)}{2\pi} \int L e^{z(d/2+1/2-\alpha)}(-z)^{s-1} \frac{dz}{d \sinh \frac{z}{2} \sinh(z)}
= \sum_{m,n=0}^{\infty} \left( \frac{m + d - 2}{d - 2} \right) \frac{1}{(a + 2n + m)^s}, \quad \text{Re} \ s > d. \tag{79}
$$

$N$ depends on $a$ and is given by a generalized Bernoulli polynomial. In the present case it is easiest to find the residue directly from the integral form of $\zeta_d$.

For $d = 3$ a standard evaluation gives

$$
\sum_{r=1}^{u} \frac{1}{2^{2r} \pi^r} \sum_{k=0}^{r-1} \frac{1}{2k + 1} = \frac{1}{4} N_2(3) = -\frac{1}{8}. \tag{80}
$$

To complete (77), it is necessary to calculate the first two terms on the right-hand side. Noting that $a = d/2 + 1$ or $a = d/2 + 2$, it is convenient to adjust the summation over $m$ in (79) to allow for the extra 1 or 2 and to keep the lower limit equal to 0. Then

$$
\zeta_d(s, \frac{d}{2} + 1) + \zeta_d(s, \frac{d}{2} + 2) = 2S_d(s) - \sum_{n=0}^{\infty} \left[ \frac{2}{(d/2 + 2n)^s} + \frac{d - 1}{(d/2 + 1 + 2n)^s} \right]
= 2S_d(s) - 2^{-s} \left( 2\zeta_R(s, \frac{d}{4}) + (d - 1)\zeta_R(s, \frac{d}{4} + \frac{1}{2}) \right) \tag{81}
$$

where, after rearrangement, the summation, $S_d(s)$, is

$$
S_d(s) = \sum_{m,n=0}^{\infty} \left( \frac{m + d - 2}{d - 2} \right) \frac{1}{(d/2 + 2n + m)^s}
= \frac{1}{(d - 2)!} \sum_{n=0, m=1}^{\infty} \frac{(m + q - 1) \ldots (m - q)}{(2n + m)^s}, \tag{82}
$$

for even dimensions ($d = 2q + 2$), while for odd dimensions, ($d = 2q + 3$),

$$
S_d(s) = \frac{1}{(d - 2)!} \sum_{n, m=0}^{\infty} \frac{(m + q) \ldots (m - q)}{(2n + m + 1/2)^s}. \tag{83}
$$

To reduce the summations further, we introduce, as in [1], the residue classes $m = 2l + p$ with $0 \leq l \leq \infty$ and $p = 0, 1$, treating the different $p$ values separately
(m even and m odd). The denominator functions read \(2(l + n) + \) constant, so we set \(N = l + n\) and effect the sum over \(\nu = l - n\) algebraically using

\[
\sum_{n,m=0}^{\infty} = \sum_{N=0}^{\infty} \sum_{\nu=-N}^{N}.
\]

For arbitrary dimensions, some general rearrangement of the numerators is needed so, for rapidity, attention is restricted to \(d = 3\), i.e. \(q = 0\), when the numerator in (83) is simply \(m^2 = \nu^2 + 2\nu(N + p) + 2Np + p^2 + N^2\). The term odd in \(\nu\) will sum to zero leaving,

\[
S_3(s) = \sum_{N=0,p=0,1}^{\infty} \frac{(2N + 1)(p^2 + 2Np + N^2 + N(N + 1)/3)}{(2N + p + 1/2)^s}
\]

\[
= \frac{2^{-s}}{12} \left(4\zeta_R(s - 2, \frac{1}{4}) + 4\zeta_R(s - 2, \frac{3}{4}) - \zeta_R(s - 1, \frac{1}{4}) - \zeta_R(s - 1, \frac{3}{4})\right)
\]

\[
= \frac{2^{-s}}{12} \left[4Z(s - 2, \frac{1}{4}) - Z(s - 1, \frac{1}{4})\right],
\]

where

\[
Z(s, x) = \sum_{m=-\infty}^{\infty} \frac{1}{|x + m|^s} = \zeta_R(s, x) + \zeta_R(s, 1 - x)
\]

is an Epstein \(\zeta\)-function, which is usually more convenient for manipulation. We have the relation

\[
2^{-s}Z(s, 1/4) = \zeta_R(s, 1/2) = (2^s - 1)\zeta_R(s).
\]

At \(d = 3\) the right-hand side of (81) becomes, finally,

\[
\frac{2^{-s}}{6} \left[4Z(s - 2, \frac{1}{4}) - Z(s - 1, \frac{1}{4}) - 12Z(s, \frac{1}{4})\right] + 2^{1+s}
\]

\[
= \frac{1}{6} \left[4\zeta_R(s - 2, \frac{1}{2}) - \zeta_R(s - 1, \frac{1}{2}) - 12\zeta_R(s, \frac{1}{2})\right] + 2^{1+s}.
\]

Combining the derivative at \(s = 0\) with (80) according to (77) gives

\[
W^{D}_{\frac{1}{2}+\text{3sphere}} = -\frac{1}{2} \zeta'_{QS}(0) = \frac{1}{4}\zeta'_R(-2) - \frac{1}{24}\zeta'_R(-1) - \frac{4}{3}\ln 2 - \frac{1}{16} \approx -0.987416
\]

needed in (75).

The calculation can be performed for any dimension \(d\) and also for the case when there are \(q\) hyperplanes inclined at \(\pi/q\).
12. Conclusion

Several extensions of this analysis can be envisaged. Higher dimensional bundles, such as spinors and vectors, could be considered. This would be connected with the recent work on mixed boundary conditions [38–40].

A question we have not been able to resolve is the evaluation of the $\lambda$, $\mu$ and $\nu$ functions in the expression for the heat-kernel coefficient $C_{3/2}$ for general dihedral angles. There is continuing interest in the calculation of such coefficients and it seems important to the present authors to extend the analysis to the piecewise smooth case, cf [41].

It should also be possible to calculate the functional determinants on balls directly using properties of Bessel functions, cf [4]. The heat-kernel coefficients on balls have been computed by Stewartson and Waechter [42] in one dimension, Waechter [43] in two, and Kennedy [44] up to five. We also mention the more recent work of Moss [37] and Bordag and Kirsten [45]. The work of Berry and Howls, [46], extends that of Stewartson and Waechter to obtain many more terms.

A technical problem of some value is the evaluation of the heat-kernel coefficients and functional determinants on particular manifolds for the general Robin case. Some results along these lines have been obtained by Moss [37] but could be extended. We note here that normalisable zero modes might exist for specific values of $\psi$ which would affect the calculation of the effective action.

References

1. J.S.Dowker Comm. Math. Phys. 162 (1994) 633.
2. J.S.Dowker Class. Quant. Grav. 11 (1994) 557.
3. J.S.Dowker J. Math. Phys. 35 (1994) 4989; erratum ibid, Feb.1995.
4. A.O.Barvinsky, Yu.A.Kamenshchik and I.P.Karmazin Ann. Phys. 219 (1992) 201.
5. Yu.A.Kamenshchik and I.V.Mishakov Int. J. Mod. Phys. A7 (1992) 3265.
6. P.D.D’earth and G.V.M.Esposito Phys. Rev. D43 (1991) 3234.
7. A.Capelli and A.Costa Nucl. Phys. B314 (1989) 707.
8. S.Guraswamy, S.G.Rajeev and P.Vitale $O(N)$ sigma-model as a three dimensional conformal field theory, Rochester preprint UR-1357.
9. K.Richardson J. Func. Anal. 122 (1994) 52.
10. T.P.Branson, S.-Y. A.Chang and P.C.Yang Comm. Math. Phys. 149 (1992) 241.
11. B.Osgood, R.Phillips and P.Sarnak *J. Func. Anal.* **80** (1988) 148.
12. E.Aurell and P.Salomonson *Comm. Math. Phys.* **165** (1994) 233.
13. E.Aurell and P.Salomonson *Further results on functional determinants of laplacians on simplicial complexes* [hep-th/9405140].
14. T.P.Branson and B.Ørsted *Proc. Am. Math. Soc.* **113** (1991) 669.
15. E.Elizalde, *J. Math. Phys.* **35** (1994) 3308.
16. M.Lüsher, K.Symanzik and P.Weiss *Nucl. Phys.* **B173** (1980) 365.
17. A.M.Polyakov *Phys. Lett.* **103** (1981) 207.
18. O.Alvarez *Nucl. Phys.* **B216** (1983) 125.
19. J.S.Dowker and J.P.Schofield *J. Math. Phys.* **31** (1990) 808.
20. P.B.Gilkey and T.P.Branson *Trans. Am. Math. Soc.* **344** (1994) 479.
21. J.P.Schofield Ph.D. Thesis, University of Manchester, 1991.
22. A.Dettki and A.Wipf *Nucl. Phys.* **B377** (1992) 252.
23. P.Epstein *Math. Ann.* **56** (1903) 615.
24. J.S.Dowker and G.Kennedy *J. Phys.* **A11** (1978) 895.
25. G.Kennedy *Phys. Rev.* **D23** (1981) 2884.
26. C.Itzykson and J.-B.Zuber *Nucl. Phys.* **B275** (1986) 580.
27. Peter Chang and J.S.Dowker *Nucl. Phys.* **B395** (1993) 407.
28. E.W.Barnes *Trans. Camb. Phil. Soc.* **19** (1903) 374.
29. M.Spivak *Differential Geometry* vols III, IV, Publish or Perish, Boston, 1975.
30. R.K.Pathria *Suppl. Nuovo Cim.* **4** (1966) 276.
31. H.P.Baltes *Phys. Rev.* **A6** (1972) 2252.
32. J.S.Dowker *Heat-kernels and polytopes* To be published
33. J.S.Dowker *Phys. Rev.* **D39** (1989) 1235.
34. T.P.Branson and P.B.Gilkey *Comm. Partial Diff. Equations* **15** (1990) 245.
35. G.Kennedy, R.Critchley and J.S.Dowker *Ann. Phys.* **125** (1980) 346.
36. G.Kennedy PhD thesis Manchester (1978).
37. I.Moss *Class. Quant. Grav.* **6** (1989) 659.
38. D.V.Vassilevich. *Vector fields on a disk with mixed boundary conditions* gr-qc /9404052.
39. I.Moss and S.Poletti *Phys. Lett.* **B333** (1994) 326.
40. G.Esposito, A.Y.Kamenshchik, I.V.Mishakov and G.Pollifrone *Phys. Rev.* **D50** (1994) 6329.
41. J.Cheeger *J. Diff. Geom.* **18** (1983) 575.
42. K.Stewartson and R.T.Waechter *Proc. Camb. Phil. Soc.* **69** (1971) 353.
43. R.T.Waechter *Proc. Camb. Phil. Soc.* **72** (1972) 439.
44. G.Kennedy *J. Phys.* **A11** (1978) L173.
45. M.Bordag and K.Kirsten *Heat-kernel coefficients of the Laplace operator on the 3-dimensional ball* [hep-th/9501064].
46. M.V.Berry and C.J.Howls *Proc. Roy. Soc.* **A447** (1994) 527.
FIGURE CAPTIONS

Fig.1. Difference between the 2-cap and 2-hemisphere effective actions plotted against the colatitude of the cap rim, for Dirichlet boundary conditions.

Fig.2. Difference between the 2-cap and 2-hemisphere effective actions plotted against the colatitude of the cap rim, for Neumann boundary conditions.

Fig.3. Difference between the 3-cap and 3-hemisphere effective actions plotted against the colatitude of the cap rim, for Dirichlet boundary conditions.

Fig.4. Difference between the 4-cap and 4-hemisphere effective actions plotted against the colatitude of the cap rim, for Dirichlet boundary conditions.

Fig.5. The effective action, $W$, on an equatorial spherical 4-slice plotted against half its angular width, $\Delta$. (Dirichlet boundary conditions.)