Large Flocks of Small Birds: On the Minimal Size of Population Protocols

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Abstract

Population protocols are a well established model of distributed computation by mobile finite-state agents with very limited storage. A classical result establishes that population protocols compute exactly predicates definable in Presburger arithmetic. We initiate the study of the minimal amount of memory required to compute a given predicate as a function of its size. We present results on the predicates $x \geq n$ for $n \in \mathbb{N}$, and more generally on the predicates corresponding to systems of linear inequalities. We show that they can be computed by protocols with $O(\log n)$ states (or, more generally, logarithmic in the coefficients of the predicate), and that, surprisingly, some families of predicates can be computed by protocols with $O(\log \log n)$ states. We give essentially matching lower bounds for the class of 1-aware protocols.

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1 Introduction

Population protocols \cite{4} are a model of distributed computation by anonymous, identical, and mobile finite-state agents. Initially introduced to model networks of passively mobile sensors, they also capture the essence of distributed computation in trust propagation or chemical reactions, the latter under the name of chemical reaction networks (see e.g. \cite{18}). Structurally, population protocols can also be seen as a special class of Petri nets or vector addition systems \cite{11}.

Since the agents executing a protocol are anonymous and identical, its global state—called a configuration—is completely determined by the number of agents at each local state. In each computation step, a pair of agents, chosen by an adversary subject to a fairness condition stating that any repeatedly reachable configuration is eventually reached, interact and move to new states according to a joint transition function. In a closely related model, the adversary chooses the pair of agents uniformly at random.

A protocol computes a boolean value for a given initial configuration if in all fair executions all agents eventually agree to this value—so, intuitively, population protocols compute by reaching consensus. Given a set of initial configurations, the predicate computed by a protocol...
is the function that assigns to each configuration $C$ the boolean value computed by the protocol starting from $C$.

Much research on population protocols has focused on their expressive power, i.e., the class of predicates computable by different classes of protocols (see e.g. [3, 6, 13, 16, 7]). In a famous result [6], Angluin et al. have shown that predicates computable by population protocols are exactly the predicates definable in Presburger arithmetic. There is also much work on complexity metrics for protocols. The main two metrics are the runtime of a protocol—defined for the model with a randomized adversary as the expected number of pairwise interactions until all agents have the correct output value—and its state space size, e.g. the number of states of each agent. In [5], Angluin et al. show that every Presburger predicate is computed with high probability by a population protocol with a leader—a distinguished auxiliary agent that assumes a specific state in the initial configuration irrespective of the input — in $O(n \log^4 n)$ interactions in expectation, where $n$ is the number of agents of the initial configuration. Several recent papers study time-space trade-offs for specific tasks, like electing a leader [10], or for specific predicates, like majority [2, 1, 9].

In this paper we study the state space size of protocols as a function of the predicate they compute. In particular, we are interested in the minimal number of states needed to evaluate systems of linear constraints (a large subclass of the predicates computed by population protocols) as a function of the number of bits needed to describe the system. To the best of our knowledge, this question has not been considered so far. We study the question for protocols with and without leaders. Our results show that protocols with leaders can be exponentially more compact than leaderless protocols.

In order to introduce our results in the simplest possible setting, in the first part of the paper we focus on the family of predicates $\{x \geq n : n \in \mathbb{N}\}$. These predicates specify the well-known flock-of-birds problem [4], in which tiny sensors placed on birds have to reach consensus on whether the number of sick birds in a flock exceeds a given constant. The minimal number of states for computing $x \geq n$ formalizes a very natural question about emerging behavior: How many states must agents have in order to exhibit a “phase transition” when their number reaches $n$? The standard protocol for the predicate $x \geq n$ (see Example 1) has $n+1$ states. We also give a lower bound:

(1) There exists a family $\{P_n : n \in \mathbb{N}\}$ of leaderless population protocols such that $P_n$ has $O(\log_2 n)$ states and computes the predicate $x \geq n$ for every $n \in \mathbb{N}$.

We also give a lower bound:

(2) For every family $\{P_n : n \in \mathbb{N}\}$ of leaderless population protocols such that $P_n$ computes $x \geq n$, there exist infinitely many $n$ such that $P_n$ has at least $(\log n)^{1/4}$ states.

However, this bound is only existential (“there exists infinitely many $n$” instead of “for all $n$”). Moreover, it follows from a counting argument that does not provide any information on the values of $n$ realizing the bound. Is there a poly-logarithmic universal bound? We show that, surprisingly, the answer is negative:

(3) There exists a family $\{P_n : n \in \mathbb{N}\}$ of population protocols with two leaders, and values $c_0 < c_1 < \ldots \in \mathbb{N}$, such that $P_n$ has $O(\log \log c_n)$ states and computes the predicate $x \geq c_n$ for every $n \in \mathbb{N}$. 

Protocol size for the flock-of-birds problem. In a first warm-up phase we exhibit a family of leaderless protocols with only $O(\log n)$ states. More precisely, we prove:

(1) There exists a family $\{P_n : n \in \mathbb{N}\}$ of leaderless population protocols such that $P_n$ has $O(\log_2 n)$ states and computes the predicate $x \geq n$ for every $n \in \mathbb{N}$.

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(3) There exists a family $\{P_n : n \in \mathbb{N}\}$ of population protocols with two leaders, and values $c_0 < c_1 < \ldots \in \mathbb{N}$, such that $P_n$ has $O(\log \log c_n)$ states and computes the predicate $x \geq c_n$ for every $n \in \mathbb{N}$.
Observe that in these protocols the “phase transition” occurs at \( x = c_n \), even though no agent has enough memory to index a particular bit of \( c_n \).

Can one go even further, and design \( O(\log \log \log n) \) protocols? We show that the answer is negative for 1-aware protocols. Both the standard protocol for \( x \geq n \) and the families of (1) and (3) have the following, natural property: If the number of agents is greater than or equal to \( n \), then the agents not only reach consensus \( 1 \), they also eventually know that they will reach this consensus. We say that these protocols are 1-aware.

We obtain lower bounds for 1-aware protocols that essentially match the upper bounds of (1) and (3):

(4) Every leaderless, 1-aware population protocol computing \( x \geq n \) has at least \( \log_3 n \) states.

(5) Every 1-aware protocol (leaderless or not) computing \( x \geq n \) has at least \((\log \log n)/151\)^{1/9} states.

**Protocols for systems of linear inequalities.** In the second part of the paper we show that our results can be extended to other predicates. First, instead of the simple predicate \( x \geq c \), every agent has enough memory to index a particular bit of \( c \).

Observe that in these protocols the “phase transition” occurs at \( x = c_n \), even though no agent has enough memory to index a particular bit of \( c_n \).

Finally, in the most involved construction of the paper, we show that the same applies to leaderless systems of linear inequalities. Note that the standard conjunction construction, which produces a protocol for \( \phi_1 \land \phi_2 \) from protocols computing predicates \( \phi_1 \) and \( \phi_2 \), cannot be applied because it would lead to exponentially large protocols.

(7) There is a protocol with at most \( O((\log m + n)(m + k)) \) states and \( O(m(\log m + n)) \) leaders that computes \( Ax \geq c \), where \( A \in \mathbb{Z}^{m \times k} \) and \( n \) is the size of the largest entry in \( A \) and \( c \).

**Structure of the paper.** Section 2 introduces basic definitions, protocols with and without leaders, and a simple construction with an involved correctness proof showing how to simulate protocols with \( k \)-way interactions by standard protocols with binary interactions. Sections 3 to 6 present our bounds on the flock-of-birds predicates, and Section 7 the bounds on systems of linear inequalities. Due to space constraints, some proofs are deferred to the appendix.

## 2 Preliminaries

**Numbers.** Let \( n \in \mathbb{N}_{\geq 0} \). The logarithm in base \( b \) of \( n \) is denoted by \( \log_b n \). Whenever \( b = 2 \), we omit the subscript. We define \( \text{bits}(n) \) as the set of indices of the bits occurring in the binary representation of \( n \), e.g. \( \text{bits}(13) = \{0, 2, 3\} \) since 13 = 11012. The size of \( n \), denoted \( \text{size}(n) \), is the number of bits required to represent \( n \) in binary. Note that \( |\text{bits}(n)| \leq \text{size}(n) = |\log n| + 1 \).

**Multisets.** A multiset over a finite set \( E \) is a mapping \( M : E \rightarrow \mathbb{N} \). The set of all multisets over \( E \) is denoted \( \mathbb{N}^E \). For every \( e \in E \), \( M(e) \) denotes the number of occurrences of \( e \) in \( M \), and for every \( E' \subseteq E \) we define \( M(E') \overset{\text{def}}{=} \sum_{e \in E'} M(e) \). The support and size of \( M \) are defined respectively as \( [M] \overset{\text{def}}{=} \{ e \in E : M(e) > 0 \} \) and \( |M| \overset{\text{def}}{=} \sum_{e \in E} M(e) \). Addition and comparison are extended to multisets componentwise, i.e. \( (M + M')(e) \overset{\text{def}}{=} M(e) + M'(e) \) for every \( e \in E \), and \( M \leq M' \overset{\text{def}}{=} M(e) \leq M'(e) \) for every \( e \in E \). We define multiset difference as \( (M \ominus M')(e) \overset{\text{def}}{=} \max(M(e) - M'(e), 0) \) for every \( e \in E \). The empty multiset is denoted \( \emptyset \).
We sometimes denote multisets using a set-like notation, e.g. \( \{ f, 2 \cdot g, h \} \) is the multiset \( M \) such that \( M(f) = 1, M(g) = 2, M(h) = 1 \) and \( M(e) = 0 \) for every \( e \in E \setminus \{ f, g, h \} \).

**Population protocols.** We introduce a rather general model of population protocols, allowing for interactions between more than two agents and for leaders. A \( k \)-way population protocol is a tuple \( \mathcal{P} = (Q,T,I,L,O) \) such that

- \( Q \) is a finite set of states,  
- \( T \subseteq \bigcup_{2 \leq i \leq k} Q^i \times Q^i \) is a set of transitions,  
- \( I \subseteq Q \) is a set of initial states,  
- \( L \in \mathbb{N}^Q \) is a set of leaders, and  
- \( O : Q \to \{0,1\} \) is the output mapping.

We assume throughout the paper that agents can always interact, i.e., that for every pair of states \( (p,q) \), there exists a pair of states \( (p',q') \) such that \( ((p,q),(p',q')) \in T \).

A configuration of \( \mathcal{P} \) is a multiset \( C \in \mathbb{N}^Q \) such that \( |C| > 0 \). Intuitively, \( C \) describes a non-empty empty collection containing \( C(q) \) agents in state \( q \) for every \( q \in Q \). We denote the set of configurations over \( E \subseteq Q \) by \( \text{Pop}(E) \). A configuration \( C \) is initial if \( C = D + L \) for some \( D \in \text{Pop}(I) \). So, intuitively, leaders are distinguished agents that are present in every initial configuration. The number of leaders of \( \mathcal{P} \) is \( |L| \). We say that \( \mathcal{P} \) is leaderless if it has no leader, i.e. if \( L = 0 \). We discuss protocols with and without leaders later in this section.

Let \( t = ((p_1, p_2, \ldots, p_i), (q_1, q_2, \ldots, q_i)) \) be a transition. To simplify the notation, we denote \( t \) as \( p_1, p_2, \ldots, p_i \rightarrow q_1, q_2, \ldots, q_i \). Intuitively, \( t \) describes that \( i \) agents at states \( p_1, \ldots, p_i \) may interact and move to states \( q_1, \ldots, q_i \). The preset and postset of \( t \) are respectively defined as \( \text{pre}(t) \overset{\text{def}}{=} \{ p_1, p_2, \ldots, p_i \} \) and \( \text{post}(t) \overset{\text{def}}{=} \{ q_1, q_2, \ldots, q_i \} \). We extend presets and postsets to sets of transitions, e.g. \( \text{pre}^* \overset{\text{def}}{=} \bigcup_{t \in T} \text{pre}(t) \) and \( \text{post}^* \overset{\text{def}}{=} \bigcup_{t \in T} \text{post}(t) \).

We say that \( t \) is enabled at \( C \in \text{Pop}(Q) \) if \( C \geq \text{pre}(t) \). If \( t \) is enabled at \( C \), then it can occur, in which case it leads to the configuration \( C' = (C \cap \text{pre}(t)) + \text{post}(t) \). We denote this by \( C \xrightarrow{t} C' \). We say that \( t \) is silent if \( \text{pre}(t) = \text{post}(t) \). In particular, if \( t \) is silent and \( C \xrightarrow{t} C' \), then \( C = C' \). We write \( C \xrightarrow{t} C' \) if \( C \xrightarrow{t} C' \) for some \( t \in T \). We write \( C \xrightarrow{t_1, t_2, \ldots, t_k} C' \) if there exist \( C_0, C_1, \ldots, C_k \in \text{Pop}(Q) \) and \( t_1, t_2, \ldots, t_k \in T \) such that \( C = C_0 \xrightarrow{t_1} C_1 \xrightarrow{t_2} \cdots C_k = C' \.

We write \( C \Rightarrow C' \) if \( C \xrightarrow{t_1, t_2, \ldots, t_k} C' \) for some \( \sigma \in T^* \). We say that \( C' \) is reachable from \( C \) if \( C \Rightarrow C' \).

The support of a sequence \( \sigma = t_1 t_2 \cdots t_n \in T^* \) is \( [\sigma] \overset{\text{def}}{=} \{ t_i : 1 \leq i \leq n \} \).

**Example 1.** The flock-of-birds protocol mentioned in the introduction is formally defined as \( \mathcal{P}_n = (Q,T,I,L,O) \) where \( Q = \{0,1,\ldots,n\} \), \( I = \{1\} \), \( L = 0 \), \( O(a) = 1 \iff a = n \), and \( T \) consists of the following transitions:

\[
\begin{align*}
  s_{a,b} : a, b &\rightarrow 0, \min(a + b, n) & \text{for every } 0 \leq a, b < n, \\
  t_a : a, n &\rightarrow n, n & \text{for every } 0 \leq a \leq n.
\end{align*}
\]

\( \mathcal{P}_n \) is 2-way and leaderless. Intuitively, it works as follows. Each agent stores a number. When two agents meet, one agent stores the sum of their values and the other one stores 0. Sums cap at 0. Once an agent reaches 0, all agents eventually get converted to 0. To illustrate the above definitions, observe that \( \text{pre}(s_{2,3}) = \{2, 3\} \), \( \text{post}(s_{2,3}) = \{2, 3\} \) and \( \text{post}(t_2) = \{n, n\} \).

Configuration \( \{1, 1, 1\} \) is initial, but \( \{1, 0, 2\} \) is not. We have \( \{1, 1, 1\} \xrightarrow{t_1} \{1, 0, 2\} \xrightarrow{t_2} \{1, 2, 2\} \xrightarrow{t_2} \{2, 2, 2\} \) or more concisely \( \{1, 1, 1\} \xrightarrow{t_2 t_2} \{2, 2, 2\} \) where \( \sigma = s_{1,1} t_1 t_1 \).

**Computing with population protocols.** An execution \( \pi \) is an infinite sequence of configurations \( C_0 C_1 \cdots \) such that \( C_0 \Rightarrow C_1 \Rightarrow \cdots \). We say that \( \pi \) is fair if for every
configuration $D$ the following hold:

If $\{i \in \mathbb{N} : C_i \rightarrow D\}$ is infinite, then $\{i \in \mathbb{N} : C_i = D\}$ is infinite.

In other words, fairness ensures that a configuration cannot be avoided forever if it can be reached infinitely often along $\pi$. We say that a configuration $C$ is a consensus configuration if $O(p) = O(q)$ for every $p, q \in [C]$. If a configuration $C$ is a consensus configuration, then its output $O(C)$ is the unique output of its states, otherwise it is $\perp$. An execution $\pi = C_0C_1 \cdots$ stabilizes to $b \in \{0, 1\}$ if $O(C_i) = O(C_{i+1}) = \cdots = b$ for some $i \in \mathbb{N}$. The output of $\pi$ is $O(\pi) \overset{\text{def}}{=} b$ if it stabilizes to $b$, and $O(\pi) \overset{\text{def}}{=} \perp$ otherwise. A consensus configuration $C$ is stable if every configuration $C'$ reachable from $C$ is a consensus configuration such that $O(C') = O(C)$. It can easily be shown that a fair execution stabilizes to $b \in \{0, 1\}$ if and only if it contains a stable configuration whose output is $b$.

A population protocol $\mathcal{P} = (Q, T, I, L, O)$ is well-specified if for every initial configuration $C_0$, there exists $b \in \{0, 1\}$ such that every fair execution $\pi$ starting at $C_0$ has output $b$. If $\mathcal{P}$ is well-specified, then we say that it computes the predicate $\varphi : \text{Pop}(I) \rightarrow \{0, 1\}$ if for every $D \in \text{Pop}(I)$, every fair execution starting at $D + L$ has output $\varphi(D)$.

**Example 2.** Consider the protocol $\mathcal{P}_2$ defined in Example 1 (i.e., $n = 2$). We have $O(\{1, 1, 1\}) = 0$, $O(\{2, 2, 2\}) = 1$ and $O(\{1, 0, 2\}) = \perp$. The execution $\{1, 1, 1\} \rightarrow \{1, 0, 2\} \rightarrow \{1, 2, 2\} \rightarrow \{2, 2, 2\} \rightarrow \cdots$ is fair and its output is 1. However, the execution $\{1, 1, 1\} \rightarrow \{1, 0, 2\} \rightarrow \{1, 0, 2\} \rightarrow \cdots$ is not fair since $\{1, 0, 2\}$ occurs infinitely often and can lead to $\{2, 2, 2\}$ which does not occur.

**Leaders.** Intuitively, leaders are extra agents present in every initial configuration. Allowing a large number of leaders may help to compute predicates with fewer states. To illustrate this, consider the leaderless protocol of Example 1. It computes $x \geq n$ with $n + 1$ states. We describe a $2$-way protocol with only $4$ states, but $n$ leaders. It is an adaptation of the well-known basic majority protocol (see, e.g., [5]). Let $\mathcal{P}_n' = (Q, T, I, L_n, O)$ be the protocol where $Q \overset{\text{def}}{=} \{x, y, \pi, \overline{\pi}\}$, $I \overset{\text{def}}{=} \{x\}$, $L_n \overset{\text{def}}{=} \{\pi \cdot y\}$, $O(x) = O(\pi) \overset{\text{def}}{=} 1$, $O(y) = O(\overline{\pi}) \overset{\text{def}}{=} 0$, and where $T$ consists of the following transitions:

$$
\begin{align*}
&x, y \mapsto \pi, \overline{\pi}, &x, \overline{\pi} \mapsto x, \pi, &y, \pi \mapsto y, \overline{\pi}, &\pi, \overline{\pi} \mapsto \pi, \pi.
\end{align*}
$$

Informally, “active” agents in states $x$ and $y$ collide and become “passive” agents in states $\pi$ and $\overline{\pi}$. At some point, some active agents “win” and convert all passive agents to their output. It is known that this protocol is well-specified and computes the predicate $x \geq y$ when there are no leaders (i.e., if we set $L_n = \emptyset$). So, by initially fixing $n$ leaders in state $y$, $\mathcal{P}_n'$ computes $x \geq n$.

Thus, the predicate $x \geq n$ can be computed either with $O(n)$ states and no leaders, or with 4 states and $O(n)$ leaders. This indicates a trade-off between states and leaders, and one should avoid hiding all of the complexity in one of them. For this reason, we make these two quantities explicit in all of our results.

The reason for considering protocols with leaders is that, as we shall see, even a constant number of leaders demonstrably leads to exponentially more compact protocols for some predicates. Other papers have made similar observations with respect to other resource measures (see e.g. [5] [14]).

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1 This definition of fairness differs from the original definition of Angluin et al. [1], but is equivalent.
From $k$-way to 2-way protocols. In our constructions it is very convenient to use $k$-way transitions for $k > 2$. The following lemma shows that $k$-way protocols can be transformed into 2-way protocols by introducing a few extra states. Intuitively, a $k$-way transition is simulated by a chain of 2-way transitions. The first part of the chain “collects” $k$ participants one by one. First, two agents agree to participate, and one of them becomes “passive”, while the second “searches” for a third participant. This is iterated until $k$ participants are collected. In the second part, the last collected agent “informs” all passive agents, one by one, that $k$ agents have been collected; upon hearing this, the passive agents move to their destination states and become active again. To prevent faulty behavior when there are not enough agents, all transitions of the first part can be “reversed”, that is, the agent that is currently searching and the last collected agent can “revert” and “undo” the transition. While the construction is simple and intuitive, its correctness proof is very involved, because agents that reach their destination can engage in other interactions while other participants are still passive. The construction and the correctness proof are presented in Appendix A.

Lemma 3. Let $\mathcal{P} = (Q, T, I, L, O)$ be a well-specified $k$-way population protocol. For every $3 \leq i \leq k$, let $n_i$ be the number of $i$-way transitions of $\mathcal{P}$. There exists a 2-way population protocol $\mathcal{P}'$, with at most $|Q| + \sum_{3 \leq i \leq k} 3i \cdot n_i$ states, which is well-specified and computes the same predicate as $\mathcal{P}$.

3 Leaderless protocols for $x \geq n$

In this section, we consider leaderless protocols for the predicate $x \geq n$. We first show that the number of states required to compute this predicate can be reduced from the known $O(n)$ bound to $O((\log n)^{1/4})$, using a similar binary encoding as in [1]. Then we show an existential lower bound of $O((\log n)^{3/4})$.

A protocol with $O(\log n)$ states. We describe a leaderless size($n$)-way protocol $\mathcal{P}_n = (Q_n, T_n, I_n, 0, O_n)$ with size($n$) + 3 states that computes $x \geq n$. The states are $Q_n \overset{\text{def}}{=} \{0, 2^0, \ldots, 2^{\text{size}(n)}, n\}$ and the sole initial state is $I_n \overset{\text{def}}{=} \{2^0\}$. The output mapping is defined as $O_n(n) \overset{\text{def}}{=} 1$ and $O_n(q) \overset{\text{def}}{=} 0$ for every state $q \neq n$.

Before defining the set $T_n$ of transitions, we need some preliminaries. For every state $q \in Q_n$, let $\text{val}(q)$ denote the number $q$ stands for, i.e., $\text{val}(0) = 0$, $\text{val}(n) = n$, and $\text{val}(2^i) = 2^i$ for every $0 \leq i < \text{size}(n)$. Moreover, for every configuration $C$, let $\text{val}(C) \overset{\text{def}}{=} \sum_{q \in Q_n} \text{val}(q) \cdot C(q)$. A configuration $C$ is a representation of $m$ if $\text{val}(C) = m$. For example, the configuration $\{0, 2^1, 5 \cdot 2^3\}$ is a representation of $0 + 2^1 + 5 \cdot 2^3 = 42$. Observe that every initial configuration $C_0$ is a representation of $|C_0|$.

$T_n$ is the union of two sets $T_n^1$ and $T_n^2$. Intuitively, $T_n^1$ allows the protocol to reach from a representation of a number, say $m$, other representations of $m$. Formally, the transitions of $T_n^1$ are:

- $2^i \cdot 2^i \mapsto 2^{i+1} \cdot 0$ for every $0 \leq i < \text{size}(n)$
- $2^{i+1} \cdot 0 \mapsto 2^i \cdot 2^i$ for every $0 \leq i < \text{size}(n)$
- $\{2^i : i \in \text{bits}(n)\} \mapsto n, 0, \ldots, 0$ with $|\text{bits}(n)| - 1$ copies

The transitions of $T_n^2$ allow agents in state $n$ to “attract” all other agents to $n$. Formally, they are:

- $n, q \mapsto n, n$ for every $q \in Q_n$

Let us show that $\mathcal{P}_n$ computes $x \geq n$. Let $C_0 = \{m \cdot 2^0\}$. If $m < n$, then $C(n) = 0$ holds for every representation $C$ of $m$. Therefore, every configuration $C$ reachable from $C_0$
satisfies \( C(n) = 0 \) and, since \( n \) is the only state with output 1, the protocol stabilizes to 0. If \( m \geq n \), then it is possible to reach a representation \( C \) of \( m \) satisfying \( C(n) > 0 \), for example \( C = \{ n, (m - n) \cdot 2^i \} \). Since for every transition \( 2^i, 2^i \rightarrow 2^{i+1}, 0 \) the set \( T_n \) also contains the reverse transition \( 2^{i+1}, 0 \rightarrow 2^i, 2^i \), every representation \( C \) of \( m \) satisfying \( C(n) = 0 \) can reach a representation \( C' \) of \( m \) satisfying \( C'(n) > 0 \). Let \( \pi = C_0C_1C_2 \cdots \) be a fair execution. By fairness, there is some \( i \in \mathbb{N} \) such that \( C_i(n) > 0 \). Again by fairness, and because of \( T'_n \), there is also an index \( j \) such that \( C_k = (m \cdot n)^j \) for every \( k \geq j \), and so \( \pi \) stabilizes to 1.

Note that \( |Q_n| = \text{size}(n) + 3 \). Moreover, \( P_n \) has one \( \text{size}(n) \)-way transition. Thus, by Lemma 3 we obtain the following theorem:

\[ \textbf{Theorem 4.} \] There exists a family \( \{ P_0, P_1, \ldots \} \) of leaderless and 2-way population protocols such that \( P_n \) has at most \( 4|\text{log } n| + 7 \) states and computes the predicate \( x \geq n \).

An existential \((\text{log } n)^{1/4}\) lower bound. We show that every family \( \{ P_n \}_{n \in \mathbb{N}} \) of leaderless and 2-way protocols computing the family of predicates \( \{ x \geq n \}_{n \in \mathbb{N}} \) must contain infinitely many members of size \( \Omega((\text{log } n)^{1/4}) \). We call this an existential lower bound, contrary to a universal lower bound, which would state that \( P_n \) has size \( \Omega((\text{log } n)^{1/4}) \) for every \( n \geq 1 \).

\[ \textbf{Theorem 5.} \] Let \( \{ P_0, P_1, \ldots \} \) be an infinite family of leaderless and 2-way population protocols such that \( P_n \) computes the predicate \( x \geq n \) for every \( n \in \mathbb{N} \). There exist infinitely many indices \( n \) such that \( P_n \) has at least \( (\text{log } n)^{1/4} \) states.

\[ \text{Proof sketch.} \] The proof boils down to bounding the number \( d(m) \) of unary predicates computed by protocols with \( m \) states. The number of distinct sets of transitions, excluding silent ones, is bounded by \( 2^{m^2 - m^2} \). The number of possible initial states and output mappings are respectively \( m \) and \( 2^m \). Altogether, we obtain:

\[ d(m) \leq 2^{m^2 - m^2} \cdot m \cdot 2^m = 2^{m^2} \cdot \frac{2^m \cdot m}{2^m} \leq 2^{m^2}. \]

\[ \textbf{4 \ O(\text{log } n) \ protocol with leaders for some } x \geq n \]

The lower bound of Section [\ref{sec:lower-bound}] is not valid for every \( n \), it only ensures that, for some values of \( n \), protocols computing \( x \geq n \) must have a logarithmic number of states. We prove that, surprisingly, there is an infinite sequence \( n_1 < n_2 < \cdots \) of values that break through the logarithmic barrier: The predicates \( x \geq n_i \) can be computed by very small protocols with only \( O(\text{log log } n_i) \) states and two leaders. So, loosely speaking, a flock of birds can decide if it contains at least \( n_i \) birds, even though no bird has enough memory to store even one single bit of \( n_i \).

The result is based on a construction of [\ref{mayr1996}]. In this paper, Mayr and Meyer study the word problem for commutative semigroup presentations. Given a finite set \( A \) of generators, a presentation of a commutative semigroup generated by \( A \) is a finite set of productions \( S = \{ l_1 \rightarrow r_1, \ldots, l_m \rightarrow r_m \} \), where \( l_i, r_i \in A^* \) for every \( 1 \leq i \leq m \), satisfying:

- Commutativity: \( ab \rightarrow ba \in S \) for every \( a, b \in A \), and
- Reversibility: if \( l \rightarrow r \in S \), then \( r \rightarrow l \in S \).

Given \( \alpha, \beta \in A^* \), we say that \( \beta \) is derived from \( \alpha \) in one step, denoted by \( \alpha \rightarrow \beta \), if \( \alpha = \gamma l \delta \) and \( \beta = \gamma r \delta \) for some \( \gamma, \delta \in A^* \) and some \( r \rightarrow l \in S \). We say that \( \beta \) is derived from \( \alpha \) if \( \alpha \Rightarrow \beta \), where \( \Rightarrow \) is the reflexive transitive closure of the relation induced by \( \rightarrow \). Observe

\[ 2 \text{ In [\ref{mayr1996}], the elements of } S \text{ are written using uppercase letters. We use lowercase for convenience.} \]
that, by reversibility, we have $\alpha \overset{\pi}{\rightarrow} \beta$ iff $\beta \overset{\pi}{\rightarrow} \alpha$. Further, by commutativity we have $\alpha \overset{\pi}{\rightarrow} \beta$ iff $\pi(\alpha) \overset{\pi}{\rightarrow} \pi(\beta)$ for every permutation $\pi$ of $A$.

Mayr and Meyer study the following question: given a commutative semigroup presentation $S$ over $A$, and initial and final letters $s, f \in A$, what is the length of the shortest word $\alpha$ such that $s \overset{\alpha}{\rightarrow} f\alpha$? They exhibit a family of presentations of size $O(n)$ for which the shortest $\alpha$ has double exponential length $2^{2^n}$. More precisely, in [15, Sect. 6], they construct a family $\{S_n\}_{n \geq 1}$ of presentations over alphabets $\{A_n\}_{n \geq 1}$ satisfying the following properties:

1. $|A_n| = 14n + 10$, $|S_n| = 20n + 8$, and $\max\{|l|,|r|: l \rightarrow r \in S_n\} = 5$.
2. $\{s_n, f_n, b_n, c_n\} \subseteq A_n$ for every $n \geq 1$.
3. $s_n c_n \overset{\alpha}{\rightarrow} f_n a$ iff $\alpha = c_n b^{2^n}$ [15, Lemma 6 and 8].

To apply this result, for each $n \geq 1$ we construct a 5-way population protocol $P_n = (Q_n, T_n, I_n, L_n, O_n)$ with two leaders as follows:

- $Q_n \equiv A_n \cup \{x\}$ for some $x \notin A_n$.
- $T_n \equiv T_n^1 \cup T_n^2$, where:
  - $T_n^1$ contains a transition $\text{pad}(p)$ for every production $p = l \rightarrow r$ of $S_n$, obtained by “padding” $p$ with $x$ so that its left and right sides have the same length. For example, $\text{pad}(ab \rightarrow cd) = a, a, b \rightarrow c, d, x$, and $\text{pad}(a \rightarrow bc) = a, x \rightarrow b, c$.
  - $T_n^2 \equiv \{f_n, q \rightarrow f_n, f_n \mid q \in Q_n\}$.
- $I_n \equiv \{x\}$.
- $L_n \equiv (c_n, s_n)$, and
- $O_n(f_n) \equiv 1$ and $O_n(q) \equiv 0$ for every $q \neq f_n$.

Intuitively, $T_n^1$ allows $P_n$ to simulate derivations of $S_n$: a step $C \overset{\text{pad}(p)}{\rightarrow} C'$ of $P_n$ simulates a one-step derivation of $S_n$. We make this more precise. Given $\alpha \in A^*_n$ and $m \geq |\alpha|$, let $C_{\alpha,m}$ be the configuration of $P_n$ defined as follows: $C_{\alpha,m}(x) = m$, and $C_{\alpha,m}(a) = |\alpha|_a$ for every $a \in A_n$, where $|\alpha|_a$ is the number of occurrences of $a$ in $\alpha$. Further, given a configuration $C$ of $P_n$, let $\alpha_C$ be the element of $S_n$ given by $\alpha_C = a_1^{C(a_1)} \cdots a_m^{C(a_m)}$, where $a_1, \ldots, a_m$ is a fixed enumeration of $A_n$. We have:

**Lemma 6.** Let $\alpha, \beta \in A^*_n$ and let $C, C'$ be configurations of $P_n$.

(a) If $\alpha \overset{p_1 \cdots p_k}{\rightarrow} \beta$ in $S_n$, then for every $m \geq 4k$, $C_{\alpha,m} \overset{\text{pad}(p_1) \cdots \text{pad}(p_k)}{\rightarrow} C_{\beta,m'}$ in $P_n$ for some $m' \geq 0$.

(b) If $C \overset{\text{pad}(p_1) \cdots \text{pad}(p_k)}{\rightarrow} C'$ in $P_n$, then $\alpha_C \overset{p_1 \cdots p_k}{\rightarrow} \alpha_C'$ in $S_n$.

From Lemma 6 (1) and (3), the following can be shown:

**Theorem 7.** For every $n \in \mathbb{N}$, there is a 5-way protocol $P_n$ with at most $14n + 11$ states and at most $34n + 19$ transitions that computes the predicate $x \geq c_n$ for some number $c_n \geq 2^{2^n}$.

Using Theorem 7 and Lemma 3 we obtain:

**Corollary 8.** There exists a family $\{P_0, P_1, \ldots\}$ of 2-way protocols with two leaders and a family $\{c_0, c_1, \ldots\}$ of natural numbers such that for every $n \in \mathbb{N}$ the following holds: $c_n \geq 2^{2^n}$ and protocol $P_n$ has at most $314 \log \log c_n + 131$ states and computes the predicate $x \geq c_n$.

## 5 Universal lower bounds for 1-aware protocols

To the best of our knowledge, all the protocols in the literature for predicates $x \geq n$, including those of Section 3 and Section 4, share a very natural property: if the number of agents is greater than or equal to $n$, then the agents not only eventually reach consensus 1, they also eventually know that they will reach this consensus. Let us formalize this idea:
Definition 9. A well-specified population protocol \( \mathcal{P} = (Q, T, I, L, O) \) is 1-aware if there is a set \( Q_1 \subseteq Q \setminus (I \cup L) \) of states such that for every initial configuration \( C_0 \) and every fair execution \( \pi = C_0 C_1 \cdots \)

1. if \( \pi \) stabilizes to 0, then \( C_j(Q_1) = 0 \) for every \( i \geq 0 \), and
2. if \( \pi \) stabilizes to 1, then there is some \( i \geq 0 \) such that \( C_j(Q \setminus Q_1) = 0 \) for every \( j \geq i \).

If in the course of an execution \( \pi \) an agent reaches a state of \( Q_1 \), then \( \pi \) cannot stabilize to 0 by (1), and so, since \( \mathcal{P} \) is well-specified, it stabilizes to 1; intuitively, at this moment the agent “knows” that the consensus will be 1. Further, if an execution stabilizes to 1, then all agents eventually reach and remain in \( Q_1 \) by (2), and so eventually all agents “know”.

Albeit seemingly restrictive, 1-aware protocols compute a significant subclass of predicates: monotonic Presburger predicates (see Appendix D for more details).

We say that a state \( q \) is coverable from a configuration \( C \) if \( C \xrightarrow{\pi} C' \) for some configuration \( C' \) such that \( C'(q) > 0 \). The fundamental property of 1-aware protocols is that, loosely speaking, consensus reduces to coverability:

Lemma 10. Let \( \mathcal{P} = (Q, T, \{x\}, L, O) \) be a 1-aware protocol computing a unary predicate \( \varphi \). We have \( \varphi(n) = 1 \) if and only if some state of \( Q_1 \) is coverable from \( \{n \cdot x\} + L \).

We show that for 1-aware protocols, the bounds of Sections 3 and 4 are essentially tight.

Leaderless protocols. We prove that a 1-aware, leaderless and 2-way protocol computing \( x \geq n \) has at least \( \log_3 n \) states. By Lemma 10 it suffices to show that some state of \( Q_1 \) is coverable from \( \{3^k \cdot q\} \), where \( q \) is the initial state. Proposition 11 below is the key to the proof. It states that for every finite execution \( C_1 \xrightarrow{\pi} C_2 \), there is \( C' \xrightarrow{\pi} C'_2 \) such that \( C'_1 \) has the same support as \( C_1 \) and is not too large, and \( C'_2 \) contains a “record” of all states encountered during the execution of \( \pi \) (this is the set \( [C_1] \cup \lceil \pi \rceil^* \)).

Let us define the norm of a configuration \( C \) as \( \|C\| \coloneqq \max\{C(q) : q \in [C]\} \). We obtain:

Proposition 11. Let \( \mathcal{P} = (Q, T, I, L, O) \) be a \( k \)-way population protocol and let \( C_1 \xrightarrow{\pi} C_2 \) be a finite execution of \( \mathcal{P} \). There exists a finite execution \( C'_1 \xrightarrow{\pi} C'_2 \) such that (a) \( \|C'_1\| = \|C_1\| \), (b) \( [C'_2] = [C_1] \cup \lceil \pi \rceil^* \), and (c) \( \|C'_2\| \leq (k + 1)^{|Q|} \).

Proposition 11 leads to:

Theorem 12. Every 1-aware, leaderless and 2-way population protocol \( \mathcal{P} = (Q, T, \{q_0\}, 0, O) \) computing \( x \geq n \) has at least \( \log_3 n \) states.

Proof. Let \( Q_1 \subseteq Q \) be the set of states from the definition of 1-awareness. Since \( L = 0 \), \( C_0 = \{n \cdot q_0\} \) is the smallest initial configuration with output 1, and by Lemma 10 the smallest initial configuration from which some state \( q_1 \in Q_1 \) is coverable. Let \( C_0 \xrightarrow{\pi} C \geq \{q_1\} \). Since \( q_1 \neq q_0 \), we have \( q_1 \in \lceil \pi \rceil^* \). By Proposition 11 and since \( \mathcal{P} \) is 2-way, \( q_1 \) is also coverable from \( C'_0 \) satisfying \( \|C'_0\| = \|C_0\| = \{q_0\} \) and \( \|C'_0\| = 3^{|Q|} \). Thus, \( C'_0 = \{3^{|Q|} \cdot q_0\} \). By minimality of \( n \), we get \( n \leq 3^{|Q|} \), and thus \( |Q| \geq \log_3 n \). ▶

Observe that the proof Theorem 12 uses the fact that \( \mathcal{P} \) is leaderless to conclude \( C'_0 = \{3^{|Q|} \cdot q_0\} \) from \( C'_0 = \|C_0\| \) and \( \|C_0\| = 3^{|Q|} \), which is not necessarily true with leaders.

---

3 We could also require the seemingly weaker property that eventually at least one agent “knows”. However, by adding transitions that “attract” all other agents to \( Q_1 \), we can transform a protocol in which some agent “knows” into a protocol computing the same predicate in which all agents “know”.
Protocols with leaders. In the case of protocols with leaders we obtain a lower bound from Rackoff’s procedure for the coverability problem of vector addition systems [17].

A vector addition system of dimension $k$ ($k$-VAS) is a pair $(A, v_0)$, where $v_0 \in \mathbb{N}^k$ is an initial vector and $A \subseteq \mathbb{Z}^k$ is a set of vectors. An execution of a $k$-VAS is a sequence $v_0 v_1 \ldots v_n$ of vectors of $\mathbb{N}^k$ such that each $v_{i+1} = v_i + a_i$ for some $a_i \in A$. We write $v_0 \rightarrow^{*} v_n$ and say that the execution has length $n$. A vector $v$ is coverable in $(A, v_0)$ if $v_0 \rightarrow^{*} v'$ for some $v' \geq v$. The size of a vector $v \in \mathbb{Z}^k$ is $\sum_{1 \leq i \leq k} \max(|v(i)|, 1)$. The size of a set of vectors is the sum of the size of its vectors. In [17] Rackoff proves:

> **Theorem 13** ([17]). Let $A \subseteq \mathbb{Z}^k$ be a set of vectors of size at most $n$ and dimension $k \leq n$, and let $v_0 \in \mathbb{N}^k$ be a vector of size $n$. For every $v \in \mathbb{N}^k$, if $v$ is coverable in $(A, v_0)$, then $v$ is coverable by means of an execution of length at most $2^{(3n)^8}$.

Using a standard construction from the Petri net literature, it can be shown that every 2-way protocol $P$ with $n$ states can be simulated by a VAS $V_P$ of size at most $12n^8$, where each execution of $P$ has a corresponding execution twice as long in $V_P$. Thus, by Theorem 13

> **Proposition 14.** Let $P = (Q, T, I, L, O)$ be a 2-way population protocol and let $q \in Q$. For every configuration $C$, if $q$ is coverable from $C$, then it is coverable by means of a finite execution of length at most $2^{(3m)^m} - 1$ where $m = 12|Q|^8$.

Using the above corollary, we derive:

> **Theorem 15.** Let $P$ be a 1-aware and 2-way population protocol. For every $n \geq 2$, if $P$ computes $x \geq n$, then $P$ has at least $(\log \log(n)/151)^{1/9}$ states.

### 6 Protocols for systems of linear inequalities

In Section 3 we have shown that the predicate $x \geq c$ can be computed by a leaderless protocol with $O(\log c)$ states. In this section, we will see that adding a few leaders allows to compute systems of linear inequalities. More formally, we show that there exists a protocol with $O((m+k) \cdot \log(dm))$ states and $O(m \cdot \log(dm))$ leaders computing the predicate $Ax \geq c$, where $A \in \mathbb{Z}^{m \times k}$, $c \in \mathbb{Z}^m$ and $d$ is the the largest absolute value occurring in $A$ and $c$.

There are three crucial points that make systems of linear inequalities more complicated than flock-of-birds predicates: (1) variables have coefficients, (2) coefficients may be positive or negative, and (3) they are the conjunction of linear inequalities. We will explain how to address the two first points by considering the special case of linear inequalities. We will then discuss how to handle the third point.

**Linear inequalities.** Note that the predicate $\sum_{1 \leq i \leq k} a_i x_i \geq c$ is equivalent to $\sum_{1 \leq i \leq k} a_i x_i + (1-c) > 0$. Therefore, it suffices to describe protocols for predicates of the form $\sum_{1 \leq i \leq k} a_i x_i + c > 0$. In order to make the presentation more pleasant, we will first restrain ourselves to the predicate $ax - by + c > 0$ for some fixed $a, b \in \mathbb{N}$ and $c \in \mathbb{Z}$. Such a predicate admits the difficult aspects, i.e. coefficients and negative numbers. Moreover, as we will see, handling more than two variables is not an issue.

Let us now describe a protocol $P_{lin}$ for the predicate $ax - by + c > 0$. The idea is to keep a representation of $ax - by + c$ throughout executions of the protocol. Let $n \overset{\text{def}}{=} \text{size}(\max(\log |a|, \log |b|, \log |c|, 1))$. As in Section 6, we construct states to represent powers of two. However, this time, we also need states to represent negative numbers:

$$Q^+ \overset{\text{def}}{=} \{+2^i : 0 \leq i \leq n\} \quad Q^- \overset{\text{def}}{=} \{-2^i : 0 \leq i \leq n\}.$$
We also need states \( X \overset{def}{=} \{x, y\} \) for the variables, and two additional states \( R \overset{def}{=} \{+0, -0\} \). The set of all states of \( P_{\text{lin}} \) is \( Q \overset{def}{=} X \cup Q^+ \cup Q^- \cup R \), and the initial states are \( I \overset{def}{=} X \).

Let us explain the purpose of \( R \). Intuitively, we would like to have the transitions:

\[
x \mapsto \{+2^i : i \in \text{bits}(a)\} \quad \text{and} \quad y \mapsto \{-2^i : i \in \text{bits}(|b|)\}.
\]

This way, every agent in state \( x \) (resp. \( y \)) could be converted to the binary representation of \( a \) (resp. \( b \)). Unfortunately, this is not possible as these transitions produce more states than they consume. This is where leaders become useful. If \( R \) initially contains enough leaders, then \( R \) can act as a reservoir of extra states which allow to “pad” transitions. More formally, let \( \text{rep}(z) : \mathbb{Z} \to \text{Pop}(Q \setminus X) \) be defined as follows:

\[
\text{rep}(z) \overset{def}{=} \begin{cases} 
\{+2^i : i \in \text{bits}(z)\} & \text{if } z > 0, \\
\{-2^i : i \in \text{bits}(|z|)\} & \text{if } z < 0, \\
\{0\} & \text{if } z = 0.
\end{cases}
\]

For every \( r \in R \), we add to \( P_{\text{lin}} \) the following transitions:

\[
\text{add}_{x,r} : x, r, r, \ldots, r \mapsto \text{rep}(a) \quad \text{and} \quad \text{add}_{y,r} : y, r, r, \ldots, r \mapsto \text{rep}(b).
\]

We set the leaders to \( L \overset{def}{=} \text{rep}(c) + \{4n + 2\} \cdot -0 \). We claim that \( 4n + 2 \) reservoir states are enough, we will explain later why. Now, the key idea of the construction is that it is always possible to put \( 2n \) agents back into \( R \). Thus, fairness ensures that the number of agents in \( X \) eventually decreases to zero, and then that the value represented over \( Q^+ \cup Q^- \) is \( ax - by + c \). We let the representations over \( Q^+ \) and \( Q^- \) “cancel out” until one side “wins”. If the positive (resp. negative) side wins, i.e. if \( ax - by + c > 0 \) (resp. \( ax - by + c \leq 0 \)), then it signals all agents in \( R \) to move to \( +0 \) (resp. \( -0 \)). To achieve this, for every \( 0 \leq i \leq n \), we add transition cancel, \( : +2^i, -2^i \mapsto +0, -0 \) to the protocol. Since bits of the positive and negative numbers may not be “aligned”, we follow the idea of Section \ref{section:reservoir} and add further transitions to change representations to equivalent ones:

\[
\begin{align*}
\text{up}^+_i &: +2^i, +2^i \mapsto +2^{i+1}, +0, \\
\text{down}^+_i,r &: +2^{i+1}, r \mapsto +2^i, +2^i, \\
\text{up}^-_i &: -2^i, -2^i \mapsto -2^{i+1}, -0, \\
\text{down}^-_i,r &: -2^{i+1}, r \mapsto -2^i, -2^i,
\end{align*}
\]

where \( 0 \leq i < n \) and \( r \in R \). Finally, for every \( 0 \leq i \leq n \), we add transitions to signal which side wins:

\[
\begin{align*}
\text{signal}^+_i &: +2^i, -0 \mapsto +2^i, +0, \\
\text{signal}^-_i &: -2^i, +0 \mapsto -2^i, -0.
\end{align*}
\]

Note that \(-0 \) “wins” over \(+0 \) because the predicate is false whenever \( ax - by + c = 0 \). It remains to specify the output mapping of \( P_{\text{lin}} \) which we define as expected, i.e. \( O(q) \overset{def}{=} 1 \) if \( q \in Q^+ \cup \{+0\} \), and \( O(q) \overset{def}{=} 0 \) otherwise.

Let us briefly explain why \( 4n + 2 \) reservoir states suffice. At any reachable configuration \( C \), transitions of the form \( \text{up}^+_i \) and \( \text{up}^-_i \) can occur until \( C(\pm 2^i) \leq 1 \) for every \( 0 \leq i < n \). Afterwards, at most \( 2n \) agents remain in these states. There can however be many agents in \( S = \{\pm 2^n, -2^n\} \). But, these two states represent numbers respectively larger and smaller than any coefficient, hence the number of agents in \( S \) can only grow by one each time a state from \( X \) is consumed. Overall, this means that \( C \overset{\rightarrow}{\longrightarrow} C' \) for some \( C' \) such that \( C'(R) \geq 2n \).
In order to handle more variables \( \{x_1, x_2, \ldots, x_k\} \), note that all we need to do is to set \( X = \{x_1, x_2, \ldots, x_k\} \) instead, and add transitions \( \text{add}_{x_i,r} \) for every \( 1 \leq i \leq k \) and \( r \in R \).

By applying Lemma \( \text{Lemma 12} \) on \( P_{\text{lim}} \), we obtain:

**Theorem 16.** Let \( a_1, a_2, \ldots, a_k, c \in \mathbb{Z} \) and let \( n = \text{size}(\max(|a_1|, |a_2|, \ldots, |a_k|, |c|, 1)) \).

There exists a 2-way population protocol, with at most \( 10kn \) states and at most \( 5n + 2 \) leaders, that computes the predicate \( \sum_{1 \leq i \leq k} a_i x_i + c > 0 \).

**Conjunction of linear inequalities.** We briefly explain how to lift the construction for linear inequalities to systems of linear inequalities. The details of the formal construction and proofs are a bit involved, and are thus deferred to Appendix \( \text{F} \). Let us fix some formal construction and prove the necessary results.

Let \( A \in \mathbb{Z}^{m \times k} \) and \( c \in \mathbb{Z}^m \). We sketch a protocol \( P_{\text{sys}} \) for the predicate \( Ax + c > 0 \). For every \( 1 \leq i \leq m \), we construct a protocol \( P_i \) for the predicate \( \sum_{j \leq k} A_{i,j} x_j + c_i > 0 \). Protocol \( P_i \) is obtained as presented earlier, but with some modifications. The largest power of two is picked as \( n \stackrel{\text{def}}{=} \text{size}(d) + \lceil \log 2m^2 \rceil \) where

\[
d \stackrel{\text{def}}{=} \max(1, \{|A_{i,j}| : 1 \leq i \leq m, 1 \leq j \leq k\}, \{|c_i| : 1 \leq i \leq m\}).
\]

The reason for this modification is that the number of agents, in a largest power of two, should now increase by at most \( 1/m \) each time an initial state is consumed, as opposed to \( 1 \).

We also replace each positive state \( q \in Q^+ \) of \( P_i \) by two states \( q_0 \) and \( q_1 \), its 0-copy and 1-copy. The reason behind this is that positive states should not necessarily have output 1. Indeed, one linear inequality may be satisfied while the other ones are not. Therefore, \(-0\) and each negative state \( q \in Q^- \) should be able to signal a 0-consensus to the positive states.

The transitions of the form \( \text{up}^+_j \), \( \text{down}^+_j \) and \( \text{cancel}_j \) are adapted accordingly.

Protocol \( P_{\text{sys}} \) is obtained as follows. First, subprotocols \( P_1, P_2, \ldots, P_m \) are put side by side. Their initial (resp. reservoir) states are merged into a single set \( X \) (resp. \( R \)). For every \( 1 \leq j \leq k \), transitions \( \text{add}_{x_j,r} \) of the \( m \) subprotocols are replaced by a single transition consuming \( x_j \), and enough reservoir states, and producing \( \text{rep}(A_{i,j}) \) in each subprotocol \( P_i \), where \( 1 \leq i \leq m \). The signal mechanisms are replaced by these new ones:

- the 0-copy of state \( +2^0 \) of all subprotocols can meet to convert \(-0\) to \(+0\),
- state \(+0\) can convert any positive state to its 1-copy,
- state \(-0\) or any negative state can convert \(+0\) to \(-0\), and any positive state to its 0-copy.

A careful analysis of the formal construction of \( P_{\text{sys}} \) combined with Lemma \( \text{Lemma 12} \) yields:

**Theorem 17.** Let \( A \in \mathbb{Z}^{m \times k}, c \in \mathbb{Z}^m \) and \( n = \text{size}(\max(1, \{|A_{i,j}| : 1 \leq i \leq m, 1 \leq j \leq k\}, \{|c_i| : 1 \leq i \leq m\})) \).

There exists a 2-way population protocol, with at most \( 27(\log m + n)(m + k) \) states and at most \( 14m(\log m + n) \) leaders, that computes the predicate \( Ax + c > 0 \).

7 Conclusion and further work

We have initiated the study of the state space size of population protocols as a function of the size of the predicate they compute. Previous lower bounds were only for single predicates, like the majority predicate \( x \leq y \), or for a variant of the model in which the number of states is a function of the number of agents.

There are many open questions. We conjecture that systems of linear inequalities can be computed by leaderless protocols with a polynomial number of states. A second, very intriguing question is whether the function \( f(n) \) giving the minimal number of states of a two-leader protocol computing \( x \geq n \) exhibits large gaps, i.e., if there are (families of) numbers \( c \) and \( c + 1 \) such that \( f(c) \) is exponentially larger than \( f(c + 1) \). A third question
is whether there exist protocols with $O(\log \log \log n)$ states for the flock-of-birds predicates $x \geq n$. Such protocols cannot be 1-aware, but they might exist. Their existence is linked to the long standing question of whether the reachability problem for reversible VAS (a model equivalent to the commutative semigroup representations of \[13\]) has the same complexity as reachability for arbitrary VAS (see \[12\] for a brief introduction).

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Throughout this appendix, we use the following notation for integer intervals: For $n, m \in \mathbb{N}$, $n \leq m$, we write $[n, m]$ to denote the set $\{n, n + 1, \ldots, m - 1, m\}$. Furthermore, by $[n]$ we denote the set $[1, n]$.

### A Proof of Lemma 3

Let $\mathcal{P} = (Q, T, I, L, O)$ be a $k$-way population protocol. We construct a 2-way population protocol $\mathcal{P}'$ from $\mathcal{P}$. For every transition $t : q_1, \ldots, q_k \mapsto r_1, \ldots, r_k$ where $k > 2$, we add new disabled states $D^t \overset{\text{def}}{=} \{d^t_1, \ldots, d^t_{k-2}\}$, active states $A^t \overset{\text{def}}{=} \{a^t_1, \ldots, a^t_{k-1}\}$ and backward states $B^t \overset{\text{def}}{=} \{b^t_2, \ldots, b^t_k\}$. Consider the following transitions, where $2 \leq \ell \leq k - 2$,

\[
\text{forth}^t_1 : q_1, q_2 \mapsto d^t_1, a^t_2 \quad \text{back}^t_1 : d^t_1, b^t_2 \mapsto r_1, r_2 \quad \text{success}^t : a^t_{k-1}, q^t_k \mapsto b^t_{k-1} r_k
\]

\[
\text{forth}^t_2 : a^t_1, q^t_{i+1} \mapsto d^t_2, a^t_{i+1} \quad \text{back}^t_2 : d^t_2, b^t_{i+1} \mapsto b^t_i, r_{i+1}.
\]

We define the inverse of a transition $t$ as $t^{-1} \overset{\text{def}}{=} \text{post}(t) \mapsto \text{pre}(t)$. We will replace every transition $t$ by the set of transitions $T^t \overset{\text{def}}{=} \text{Fwd}(t) \cup \text{Fwd}^{-1}(t) \cup \{\text{success}^t\} \cup \text{Bwd}(t)$ where

\[
\text{Fwd}(t) \overset{\text{def}}{=} \{\text{forth}^t_i : 1 \leq i \leq k - 1\}, \quad \text{Bwd}(t) \overset{\text{def}}{=} \{\text{back}^t_i : 1 \leq i \leq k - 1\},
\]

\[
\text{Fwd}^{-1}(t) \overset{\text{def}}{=} \{f^{-1} : f \in \text{Fwd}(t)\}.
\]

The transitions of $T^t$ are illustrated in Figure 1. Observe that a $k$-way transition $t$ can be simulated through the following sequence of 2-way transitions:

\[
\sigma_t \overset{\text{def}}{=} \text{forth}^t_1 \text{forth}^t_2 \cdots \text{forth}^t_{k-2} \text{success}^t \text{back}^t_{k-3} \cdots \text{back}^t_1.
\]

Intuitively, the transitions in Fwd($t$) temporarily “disable” all states of pre($t$). The index $i$ of the current active state $a_i$ keeps track of the progress that has been made in disabling the states of pre($t$). Once transition success$^t$ occurs, it is guaranteed that all states from pre($t$) have been disabled and, from this point, transition $t$ is simulated backward through the transitions of Bwd($t$), transforming disabled states into post($t$). Similarly, the index $i$ of the backward state $b_i$ keeps track of the progress that has been made in transforming disabled states into their respective states of post($t$). Note that a simulation attempt may be unsuccessful, e.g., because not all states from pre($t$) are initially present in the configuration. Unsuccessful attempts pose no problem as they can be undone by Fwd$^{-1}$(t).

Formally, $\mathcal{P}'$ is defined as $\mathcal{P}' \overset{\text{def}}{=} (Q', T', I, L, O')$ where

\[
Q' \overset{\text{def}}{=} Q \cup \bigcup_{t \in T} (D^t \cup A^t \cup B^t),
\]

\[
T' \overset{\text{def}}{=} \bigcup_{t \in T} T^t,
\]

\[
O'(q) \overset{\text{def}}{=} O(q) \text{ for every } q \in Q, \text{ and } O(d^t_i) = O(a^t_i) \overset{\text{def}}{=} O(q_i) \text{ and } O(b^t_i) \overset{\text{def}}{=} O(r_i) \text{ for every transition } t : q_1, q_2, \ldots, q_k \mapsto r_1, r_2, \ldots, r_k \text{ of } T.
\]

In the remainder of this appendix, we prove the following:

> **Lemma 3.** Let $\mathcal{P} = (Q, T, I, L, O)$ be a well-specified $k$-way population protocol. For every $3 \leq i \leq k$, let $n_i$ be the number of $i$-way transitions of $\mathcal{P}$. There exists a 2-way population protocol $\mathcal{P}'$, with at most $|Q| + \sum_{3 \leq i \leq k} 3i \cdot n_i$ states, which is well-specified and computes the same predicate as $\mathcal{P}$. 
The bound stated in Lemma 3 follows directly from the construction. Therefore, we must
only prove that $\mathcal{P}'$ computes the same predicate as $\mathcal{P}$. To facilitate the proof of Lemma 3,
we introduce a more fine-grained notion of “simulation” than mere equality of predicates.
Let $\mathcal{P}_1 = (Q_1, T_1, I_1, O_1)$ and $\mathcal{P}_2 = (Q_2, T_2, I_2, O_2)$ be two well-specified population
protocols. We say $\mathcal{P}_2$ simulates $\mathcal{P}_1$ if the following holds:
1. $Q_1 \subseteq Q_2$,
2. $I_1 = I_2$ and $L_1 = L_2$,
3. $O_1(q) = O_2(q)$ for every $q \in Q_1$,
4. $C \xrightarrow{t_1} C' \iff C \xrightarrow{t_2} C'$ for every $C, C' \in \text{Pop}(Q_1)$,
5. $\forall C \in \text{Pop}(Q_1), C' \in \text{Pop}(Q_2) : C \xrightarrow{t_2} C' \Rightarrow \exists C'' \in \text{Pop}(Q_1) : C' \xrightarrow{t_2} C'' \land C \xrightarrow{t_1} C''$.

Before proving Lemma 3, let us first show that the above notion of simulation indeed
implies equality of predicates:

\begin{proposition}
Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two well-specified protocols. If $\mathcal{P}_2$ simulates $\mathcal{P}_1$, then
$\mathcal{P}_1$ and $\mathcal{P}_2$ compute the same predicate.
\end{proposition}

\begin{proof}
Let $\pi_1$ and $\pi_2$ be fair executions of $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively, both starting from some
initial configuration $C_0 \in \text{Pop}(I_1) = \text{Pop}(I_2)$. Since $\mathcal{P}_1$ and $\mathcal{P}_2$ are well-specified, there exist
$b_1, b_2 \in \{0, 1\}$ such that $O_1(\pi_1) = b_1$ and $O_2(\pi_2) = b_2$. It remains to show that $b_1 = b_2$. By
fairness and Property 5 there exists some configuration $C \in \text{Pop}(Q_1)$ that occurs infinitely
often in $\pi_2$. By Property 3 Property 4 and well-specification of $\mathcal{P}_1$, configuration $C$ must
be stable in $\mathcal{P}_1$. Moreover, $C$ must be reachable from $C_0$ in $\mathcal{P}_1$ by Property 4. Thus, due
to well-specification of $\mathcal{P}_1$, we have $O_1(\pi_1) = O_1(C)$. By Property 3 we also know that
$O_1(C) = O_2(C)$ must hold. Consequently $b_1 = O(\pi_1) = O_1(C) = O_2(C) = O_2(\pi_2) = b_2$. \end{proof}

It remains to prove that if $\mathcal{P}$ is well-specified, then so is $\mathcal{P}'$, and that $\mathcal{P}'$ simulates
$\mathcal{P}$. We first show the latter. Properties 1–3 are clearly satisfied. To show the remaining
properties 4 and 5 fix some $n \in \mathbb{N}$, $C_0, C_1, \ldots, C_n \in \text{Pop}(Q')$ and $t_1, t_2, \ldots, t_n \in T'$ such
that $C_0 \in \text{Pop}(Q)$ and

$$C_0 \xrightarrow{t_1} C_1 \xrightarrow{t_2} \ldots \xrightarrow{t_n} C_n.$$ 

We define $H$ as the set of helper states of $Q'$, i.e.,

$$H \overset{\text{def}}{=} Q' \setminus Q.$$ 

Whenever an agent changes its state from $Q$ to $H$, the agent can be thought of as participating
in a simulation attempt of some $k$-way transition that was started at some point in time
$x \in [n]$. In order to make this association explicit, we annotate the helper states with
timestamps from $[n]$, i.e., we augment $H$ to $\hat{H} \overset{\text{def}}{=} H \times [n]$. We also augment every transition

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{gadget.png}
\caption{Gadget of 2-way transitions simulating the 3-way transition $q_1, q_2, q_3 \rightarrow r_1, r_2, r_3$. Circles
and squares depict respectively states and transitions.}
\end{figure}
The proof is by induction on all helper states labelled by the earliest one. \( t: \{q_1, q_2\} \rightarrow \{r_1, r_2\} \) of \( T' \) with timestamps \( x \in [n] \), i.e., \( t^x: \{q_1^x, q_2^x\} \rightarrow \{r_1^x, r_2^x\} \) where for every \( q \in Q' \), \( q^x \) is defined as:

\[
q^x \overset{\text{def}}{=} \begin{cases} q & \text{if } q \in Q, \\ (q, x) & \text{otherwise.} \end{cases}
\]

We now inductively define an execution \( \hat{C}_0 \overset{i_1}{\rightarrow} \ldots \overset{i_n}{\rightarrow} \hat{C}_n \) augmented by timestamps. Let \( \hat{C}_0 \overset{\text{def}}{=} C_0 \). For every \( i \in [n-1] \), let

\[
\begin{align*}
    a(i) & \overset{\text{def}}{=} \begin{cases} i & \text{if } i_t \subseteq Q, \\ \text{smallest } j \text{ s.t. } \text{pre}(i_t) \leq \hat{C}_{i-1} & \text{otherwise,} \end{cases} \\
    t_i & \overset{\text{def}}{=} t_{i_a(i)}, \\
    \hat{C}_i & \overset{\text{def}}{=} \hat{C}_{i-1} - \text{pre}(i_t) + \text{post}(i_t).
\end{align*}
\]

Intuitively, \( a(i) \) denotes the timestamp of the beginning of the simulation attempt which transition \( t_i \) belongs to. If \( t_i \) could belong to several simulation attempts, then we pick the earliest one.

For every \( x \in [n] \), let \( \hat{C}_i(x) \in \text{Pop}(H) \) denote the configuration resulting from extracting all helper states labelled by \( x \) from \( \hat{C} \), i.e., \( (\hat{C}_i(x))(h) = \hat{C}_i((h, x)) \) for every \( h \in H \).

\[\square\] Proposition 19. For every \( i \in [0, n] \) the following holds:

- \( \hat{C}_i \) is well-defined.
- If \( i \geq 0 \), then \( a(i) \) and \( t_i \) are well-defined.
- For every \( x \in [n] \), there exists a transition \( t \in T \); \( q_1, \ldots, q_n \rightarrow r_1, \ldots, r_n \) and some \( \ell < n \) such that if \( \hat{C}_i(x) \neq 0 \), then \( \hat{C}_i(x) = (d_1^i, d_2^i, \ldots, d_{\ell-1}^i, a_k^i) \) or \( \hat{C}_i(x) = (d_1^i, d_2^i, \ldots, d_{\ell-1}^i, b_k^i) \).
- \( \hat{C}_i \cap \text{Pop}(Q) = C_i \cap \text{Pop}(Q) \).

Proof. The proof is by induction on \( i \). Configuration \( \hat{C}_0 \) is clearly well-defined. Moreover, \( \hat{C}_0(x) = 0 \) for every \( x \in [n] \) and \( \hat{C}_0 = C_0 \), and hence the third and fourth points hold trivially.

Let \( i > 0 \) and assume the claim holds for all values smaller than \( i \). Let \( t: q_1, \ldots, q_k \rightarrow r_1, \ldots, r_k \in T \) be the transition that is simulated by \( t_i \), i.e. such that \( t_i \in T^t \). We make the following case distinction:

- Case 1: \( t_i = \text{forth}^t_i \). By definition of \( t_i \), we have \( (t_i) \subseteq Q \). Thus, \( a(i) \) and \( t_i \) are obviously well-defined. Note that \( \text{pre}(t_i) = \{q_1, q_2\} = \text{pre}(t_i) \leq C_{i-1} \). By induction hypothesis, \( \hat{C}_{i-1} \cap \text{Pop}(Q) = C_{i-1} \cap \text{Pop}(Q) \). In particular, this implies that \( (t_i) \leq \hat{C}_{i-1} \) which in turn implies that \( \hat{C}_i \) is well-defined. The third point holds since \( \hat{C}_i(i) = (d_1^i, a_k^i) \). The fourth point holds since

\[
\begin{align*}
\hat{C}_i \cap \text{Pop}(Q) & = ((\hat{C}_{i-1} \cap \text{pre}(t_i)) + \text{post}(t_i)) \cap \text{Pop}(Q) \\
& = ((\hat{C}_{i-1} \cap Q) \cap (\text{pre}(t_i) \cap Q)) + (\text{post}(t_i) \cap \text{Pop}(Q)) \\
& = ((C_{i-1} \cap Q) \cap (\text{pre}(t_i) \cap Q)) + (\text{post}(t_i) \cap \text{Pop}(Q)) \\
& = ((C_{i-1} \cap \text{pre}(t_i)) + \text{post}(t_i)) \cap \text{Pop}(Q) \\
& = C_i \cap \text{Pop}(Q).
\end{align*}
\]

- Case 2: \( t_i = \text{forth}^t_i \) for some \( 1 < \ell < k \). Recall that \( t_i: a_{\ell}^t, q_{\ell+1}^t \rightarrow d_{\ell}^t, a_{\ell+1}^t \). Since \( t_i \) is enabled at \( C_{i-1} \), we have that \( C_{i-1}(a_k^t) > 0 \). Thus, there exists some \( x \in [n] \) such that \( \hat{C}_{i-1}(a_k^t, x) > 0 \). Pick \( x \) as the smallest such number. By induction hypothesis, \( \hat{C}_{i-1}(x) = (d_1^i, \ldots, d_{k-1}^i, a_k^t) \) for some \( k < n \). Since \( \hat{C}_{i-1}(a_k^t, x) > 0 \), we must have
\( k = \ell \). Thus, \( \text{pre}(t_i^* \leq \hat{C}_{i-1} \). Now, observe that \( a(i) = x \), and hence that both \( a(i) \) and \( \hat{C}_i \) are well-defined. The third point holds since \( \hat{C}_i(x) = \{d_1^i, d_2^i, \ldots, d_k^i, a_{i+1}^i\} \). The proof of the fourth point is the same as in case 1.

- **Case 3**: \( t_i = \text{success} \) or \( t_i = \text{back}^k \). The reasoning is analogous to the last case.

  For every \( i \in [n] \), we say that \( a(i) \) is successful if there exist \( j \in [n] \) and \( t \in T \) such that \( a(i) = a(j) \) and \( t_j = \text{success}^k \). It can be shown that index \( j \) must be unique. We denote this index \( j \) by \( s(i) \).

  We now state three useful propositions whose proofs are left to the reader. Let \( \text{Fwd}^{-1} \stackrel{\text{def}}{=} \bigcup_{t \in T} \text{Fwd}^{-1}(t) \).

  **Proposition 20.** For every \( i \in [n] \), the following holds:
  
  - If \( t_i \not\in \text{Fwd}^{-1} \) and \( Q \cap (\hat{t}_i)^* \neq \emptyset \), then \( s(i) \leq i \).
  - If \( Q \cap (\hat{t}_i)^* = \emptyset \), then \( s(i) \geq i \).

  **Proposition 21.** Let \( C_0, C \in \text{Pop}(Q) \) be such that \( C_0 \not\rightarrow_{\sigma^T} C \). The following holds:
  
  - \( C \) is reachable from \( C_0 \) in \( \mathcal{P}' \) without using transitions from \( \text{Fwd}^{-1} \).
  - There exist \( C_1, C_2, \ldots, C_n \in \text{Pop}(Q') \) and \( t_1, t_2, \ldots, t_n \in T' \) such that \( C_0 \xrightarrow{t_1} C_1 \xrightarrow{t_2} \cdots \xrightarrow{t_n} C_n = C \) such that, for every \( i \in [n] \), \( a(i) \) is successful in the augmented execution \( \hat{C}_i \xrightarrow{t_i} \hat{C}_1 \xrightarrow{t_2} \cdots \xrightarrow{t_n} \hat{C}_n \).

  **Proposition 22.** Let \( t \in T \) and \( \sigma \in \left( T' \setminus \{\text{forth}_1^i, (\text{forth}_1^i)^{-1}\} \right)^* \) and \( C, C' \in \text{Pop}(Q) \). If \( C \xrightarrow{\text{forth}_1^i} C' \), then \( C \not\rightarrow C' \).

  The following lemma shows that the execution order of two transitions belonging to different simulation attempts can be swapped under certain conditions:

  **Lemma 23.** Let \( i \in [n-1] \) be such that \( a(i) \) and \( a(i+1) \) are both successful simulation attempts satisfying \( s(i+1) < s(i) \). If \( t_i \not\in \text{Fwd}^{-1} \), then \( \hat{C}_{i-1} \xrightarrow{t_i+t_{i+1}} \hat{C}_{i+1} \).

  **Proof.** For the sake of contradiction assume \( C_{i-1} \xrightarrow{t_i+t_{i+1}} C_{i+1} \) does not hold. This entails that
  
  \[ \hat{t}_i \cap (\hat{t}_{i+1}) \neq \emptyset \quad \text{(1)} \]

  Moreover, since \( s(i+1) < s(i) \), we have that \( a(i+1) \neq a(i) \). Thus
  
  \[ \hat{t}_i \cap (\hat{t}_{i+1}) \cap \hat{H} = \emptyset \quad \text{(2)} \]

  Inequality (1) and Equality (2) combined then yield
  
  \[ Q \cap \hat{t}_i \cap (\hat{t}_{i+1}) \neq \emptyset \quad \text{(3)} \]

  Since \( t_i \not\in \text{Fwd}^{-1} \) by assumption, we obtain from Proposition 20 and Inequality (3) that \( s(i) \leq i \). Moreover, Inequality (3) and Proposition 20 imply that \( s(i+1) \geq i+1 \). Thus \( s(i) < s(i+1) \), which contradicts our initial assumption that \( s(i+1) < s(i) \).

  **Corollary 24.** Property 4 holds.
Proof. Fix some $C, C' \in \text{Pop}(Q)$ and let $P_1 = P$ and $P_2 = P'$. 

$\Rightarrow$) Assume $C \xrightarrow{*} C'$. We have to show that $C \xrightarrow{*} C'$ holds. We saw earlier how a single $k$-way transition of $P_1$ can be simulated via a sequence of 2-way transitions of $P_2$. Thus, $\xrightarrow{*} \subseteq \xrightarrow{*}$ and we are done.

$\Leftarrow$) Assume $C \xrightarrow{t_1} C_1 \xrightarrow{t_2} \ldots \xrightarrow{t_n} C'$ for some $t_1, t_2, \ldots, t_n \in T'$. Consider the augmented run $\hat{C} \xrightarrow{t_1} \hat{C}_1 \xrightarrow{t_2} \ldots \xrightarrow{t_n} \hat{C}'$. By Proposition 21 we may assume that $a(i)$ is successful and $t_i \not\in \text{Fwd}^{-1}$ for every $i \in \{1\}$.

Let $A \overset{\text{def}}{=} \{a(i) : i \in \{1\}\}$ be the set of successful simulation attempts and let $m \overset{\text{def}}{=} |A|$. By repeatedly applying Lemma 23 we can reorder the augmented execution such that $\hat{C} \xrightarrow{T_1} C_1 \xrightarrow{T_2} C_1' \xrightarrow{T_3} \ldots \xrightarrow{T_m} \hat{C}'$ for some $C'_i$, where each $T_i$ is a sequence of transitions that belong to exactly one of the successful simulation attempts, i.e.

$$T_i \in \{t^{(t)} : t \in T_i\}$$

for some $x \in A$ and $t \in T$. Observe that $C_i' \in \text{Pop}(Q)$ for every $i \in \{1\}$, and moreover $C_i \xrightarrow{T} C_{i+1}$ for every $i \in \{1\}$. By Proposition 22 each sequence $T_i$ corresponds to the successful simulation of some $k$-way transition that must be enabled at $C_{i-1}$. Thus $C \xrightarrow{*} C'$, which completes the proof for Property 4.

In order to show Property 5, we only need to show that every execution of $P'$ can be extended to an execution that ends up in a configuration without helper states. Validity of Property 5 then follows from Property 4. The following lemma proves a slightly stronger result.

Lemma 25. Let $C_0, C_1, \ldots, C_n \in \text{Pop}(Q')$ and let $t_1, t_2, \ldots, t_n \in T'$ be such that $C_0 \in \text{Pop}(Q)$ and $C_0 \xrightarrow{T} C_1 \xrightarrow{T} \ldots \xrightarrow{T} C_n$. There exists some $C' \in \text{Pop}(Q)$ such that $C_n \xrightarrow{P'} C'$ and $Q \cap [C_n] \subseteq [C']$.

Proof. Consider the augmented run $\hat{C}_0 \xrightarrow{T} \hat{C}_1 \xrightarrow{T} \ldots \xrightarrow{T} \hat{C}_n$. If $\hat{C}_n \in \text{Pop}(Q)$, then we are done. Otherwise every helper state in $[\hat{C}_n]$ is labelled by some simulation attempt. Let $x_1 \leq x_2 \leq \ldots \leq x_m$ be these simulation attempts, i.e. let 

$$\{x_1, x_2, \ldots, x_m\} = \left\{x \in \{1\} : (H \times \{a\}) \cap [\hat{C}_n] \neq \emptyset\right\}.$$

By Proposition 19 one of two cases must hold: either (1) $\hat{C}_n(x_i) = [d^*_i, \ldots, d^*_i, a^*_i]$ or (2) $\hat{C}_n(x_i) = [d^*_i, \ldots, d^*_i, b^*_i]$ for some $t < n$ and $t \in T$. For each attempt $x_i$, we construct a sequence of transitions $T(x_i)$ as follows:

Case 1. We construct $T(x_i) \overset{\text{def}}{=} (\text{forth}_{i-1}^{-1})^{x_i} \ldots (\text{forth}_{i-1}^{-1})^{x_i}$. In this case, the sequence $T(x_i)$ “undoes” the unsuccessful simulation attempt $x_i$.

Case 2. We construct $T(x_i) \overset{\text{def}}{=} (\text{back}_{i}^{*})^{x_i} \ldots (\text{back}_{i}^{*})^{x_i}$. In this case, $T(x_i)$ “completes” the successful simulation attempt $x_i$.

Observe that $\hat{C}_n \xrightarrow{\text{T}(x_i)} C'$ implies that $[C'] \cap (H \times \{a_i\}) = \emptyset$. Also, note that $T(x_k)$ and $T(x_i)$ can occur independently for $k \neq i$, as the presets of $t_i$ and $t_k$ contained in $T(x_i)$ and $T(x_k)$ are disjoint, and their presets solely contain helper states which are labelled by different simulation attempts:

$$\bullet t_i \cap \bullet t_k = \bullet t_i \cap Q = \bullet t_k \cap Q = \emptyset.$$

Thus, there exists some $C' \in \text{Pop}(Q)$ satisfying $\hat{C}_n \xrightarrow{\text{T}(x_1) \ldots \text{T}(x_m)} C'$ and $[\hat{C}_n] \cap Q \subseteq [C']$. 

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Let $\pi : \tilde{C}_0 \xrightarrow{T_1} \tilde{C}_1 \xrightarrow{T_2} \cdots \xrightarrow{T_n} \tilde{C}_n$. Execution $\pi$ can be "projected" by removing the timestamps of its configurations and transitions. By definition of augmented executions, this projection yields an execution from $C_0$ to $C'$ in $\mathcal{P}'$, which proves the claim. ▶

**Corollary 26.** Property 5 holds.

It remains to show that $\mathcal{P}'$ is well-specified if $\mathcal{P}$ is well-specified.

**Proposition 27.** If $\mathcal{P}$ is well-specified, then $\mathcal{P}'$ is also well-specified.

**Proof.** Let $\mathcal{P}$ be a well-specified $k$-way protocol. For contradiction assume the simulating protocol $\mathcal{P}'$ was not well-specified. This means either of two things must hold:

1. There exist two fair executions $\pi_1$ and $\pi_2$ starting in the same initial configuration and such that $O(\pi_1) \neq O(\pi_2)$.
2. There exists a fair execution $\pi$ starting in an initial configuration such that $O(\pi) = \bot$. We only show that the validity of the second claim leads to a contradiction. The proof can easily be adapted to arrive at a contradiction for the first claim. Assume there exists a fair execution $\pi = C_0C_1C_2\cdots$ of $\mathcal{P}'$ starting in some initial configuration $C_0$ and such that $O(\pi) = \bot$. Due to well-specification of $\mathcal{P}$, Proposition 5 and 4 and fairness, we know this execution will reach a configuration $C_i$ that is stable in $\mathcal{P}$. Let $i \in \mathbb{N}$ be the smallest such index. Moreover, let $j$ be the smallest index larger than $i$ such that $O(C_j) \neq O(C_i)$. Since $\pi$ does not stabilize, such an index $j$ must exist. Observe that whenever an agent changes from a non-helper state to a helper-state, or from a helper-state to a helper-state, outputs do not change. Thus, it must hold that $C_{j-1} \xrightarrow{\text{success}} C_j$ for some $t \in T$, for only in this case an agent changes from a helper state to some non-helper state $q$ of output $O(q) \neq O(C_i)$. By Lemma 25 there exists some configuration $C' \in \text{Pop}(Q^T)$ such that $C_i \xrightarrow{\sigma} C'$ and $q \in \{C'\}$. From this and by Property 4 we have $C_i \xrightarrow{\sigma} C'$. But $O(C') \neq O(C_i)$, which contradicts our assumption that $C_i$ is stable in $\mathcal{P}$.

**B Detailed proofs of Section 3**

**Theorem 5.** Let $\mathcal{P}_0, \mathcal{P}_1, \ldots$ be an infinite family of leaderless and 2-way population protocols such that $\mathcal{P}_n$ computes the predicate $x \geq n$ for every $n \in \mathbb{N}$. There exist infinitely many indices $n$ such that $\mathcal{P}_n$ has at least $(\log n)^{1/4}$ states.

**Proof.** We first show that for every finite family $\{\mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_n\}$ of 2-way population protocols computing the predicates $\{x \geq 0, x \geq 1, \ldots, x \geq n\}$ there exists $0 \leq j \leq n$ such that $\mathcal{P}_j$ has at least $(\log n)^{1/4}$ states. For this, we prove an equivalent statement: 2-way protocols with at most $m$ states can compute at most $2^{m^4}$ unary predicates.

Let $d(m)$ be the number of unary predicates computed by 2-way population protocols with at most $m$ states. Every protocol with less than $m$ states can be extended to a protocol with $m$ states computing the same predicate, and so in order to bound $d(m)$ it suffices to consider protocols with exactly $m$ states. Further, for the same reason, we only consider protocols containing all possible silent transitions, i.e., all transitions of the form $x, y \mapsto x, y$. Such a protocol is completely determined by its set of non-silent transitions, its initial state, and its output mapping. Since the number of sets of non-silent transitions is bounded by $2^{m^4-m^2}$, the number of initial states by $m$, and the number of output mappings by $2^m$, there are at most $2^{m^4-m^2} \cdot m \cdot 2^m$ such protocols. Altogether we obtain:

$$d(m) \leq 2^{m^4-m^2} \cdot m \cdot 2^m = 2^{m^4} \cdot \frac{2^m \cdot m}{2m^2} \leq 2^{m^4}.$$
Now we prove the theorem. Let \( \{ P_0, P_1, \ldots \} \) be an infinite family of 2-way protocols such that \( P_i \) computes \( x \geq i \) for every \( i \in \mathbb{N} \). By the above result, for every \( n \geq 0 \) there is \( j_n \leq n \) such that \( P_{j_n} \) has at least \((\log n)^{1/4} \geq (\log j_n)^{1/4}\) states. It remains to prove that the set \( \{ j_0, j_1, \ldots \} \) is infinite. Let \( m_i \) be the number of states of \( P_{j_i} \). Since \( \lim_{n \to \infty} m_i = \infty \), we can extract from the sequence \( m_0, m_1, \ldots \) a strictly increasing subsequence \( m_n, m_{n+1}, \ldots \). Thus, the indices \( j_0, j_1, \ldots \) are all distinct, and we are done.

\[ \square \]

### C Detailed proofs of Section 4

#### Lemma 6. Let \( \alpha, \beta \in A_n^* \) and let \( C, C' \) be configurations of \( P_n \).

(a) If \( \alpha \xrightarrow{p_1 \cdots p_k} \beta \in S_n \), then for every \( m \geq 4k \), \( C_{n,m} \xrightarrow{\text{pad}(p_1) \cdots \text{pad}(p_k)} C_{n,m'} \in P_n \) for some \( m' \geq 0 \).

(b) If \( C \xrightarrow{\text{pad}(p_1) \cdots \text{pad}(p_k)} C' \) \( \in P_n \), then \( \alpha C \xrightarrow{p_1 \cdots p_k} \alpha C' \in S_n \).

Proof. For (a), the only reason why \( \text{pad}(p_1) \cdots \text{pad}(p_k) \) could not occur from \( C_{n,m} \) is that this configuration may not have enough agents in state \( x \). By (1), the left hand side of every transition \( \text{pad}(p_i) \) removes at most 4 agents from state \( x \), and so \( \text{pad}(p_1) \cdots \text{pad}(p_k) \) can occur for any \( m \geq 4k \). Item (b) follows immediately from the definitions.

#### Theorem 7. For every \( n \in \mathbb{N} \), there is a 5-way protocol \( P_n \) with at most \( 14n + 11 \) states and at most \( 34n + 19 \) transitions that computes the predicate \( x \geq c_n \) for some number \( c_n \geq 2^{2^n} \).

Proof. We first show that \( P_n \) is well-specified. Let \( C_0 \) be an initial configuration. We make a case distinction on whether \( f_n \) is coverable from \( C_0 \) or not.

**Case 1: \( f_n \) is coverable.** Let \( \pi = C_0 C_1 \cdots \) be a fair execution. We claim that \( C_i(f_n) > 0 \) for infinitely many indices \( i \). The claim proves the case since fairness and transitions of \( T^2 \) ensure that all agents eventually remain in \( f_n \), and hence that \( O(\pi) = 1 \).

For the sake of contradiction, assume the claim does not hold. Let \( i \in \mathbb{N} \) be the minimal index such that \( C_i(f_n) = C_{i+1}(f_n) = \cdots = 0 \). If \( i = 0 \), then \( \pi \) only consists of transitions of \( T_0^1 \). By assumption, \( C_0 \xrightarrow{\alpha} C \) for some configuration \( C \) such that \( C(f_n) > 0 \). Note that \( C \) does not occur in \( \pi \). We make use of the reversibility property of \( S_n \). Since \( \alpha \xrightarrow{\beta} \alpha \) in \( S_n \), by Lemma 6 we have \( C_j \xrightarrow{\beta} C_0 \xrightarrow{\alpha} C \) for every \( j \in \mathbb{N} \), which contradicts \( \pi \) being fair. Therefore, we must have \( i > 0 \). Let \( \sigma_j \) be the sequence from \( C_{i-1} \) to \( C_j \) in \( \pi \), for every \( j \geq i \). Note that \( C_{i-1} \) only occurs finitely often in \( \pi \). Moreover, each \( \sigma_j \) contains transitions from \( T_1^1 \). Therefore, using reversibility again, we obtain \( C_j \xrightarrow{\beta} C_{i-1} \) for every \( j \geq i \). We derive a contradiction since, by fairness, \( C_{i-1} \) should occur infinitely often in \( \pi \).

**Case 2: \( f_n \) is not coverable.** Let \( \pi = C_0 C_1 \cdots \) be a fair execution. Suppose \( O(\pi) \neq 0 \). As \( f_n \) is the only state with output 1, there exists \( i \in \mathbb{N} \) such that \( C_i(f_n) > 0 \). Since \( C_i \) is reachable from \( C_0 \), state \( f_n \) is coverable from \( C_0 \). This is a contradiction and hence \( O(\pi) = 0 \).

It remains to prove that \( P_n \) computes \( x \geq c_n \) for some number \( c_n \geq 2^{2^n} \). By (3) and Lemma 6 state \( f_n \) is coverable from some initial configuration \( C_0 \). By the above case 1, \( O(C_0) = 1 \). Let \( C_0 \) be the smallest such configuration. By (3) and Lemma 6 we have \( |C_0| \geq 2^{2^n} \). Moreover, state \( f_n \) is coverable from every configuration larger than \( C_0 \). Thus, by the above case 1, we have \( O(C_0') = 1 \) for every initial configuration \( C_0' \) such that \( |C_0'| \geq |C_0| \). Therefore, the protocol computes the predicate \( x \geq |C_0| \) where \( |C_0| \geq 2^{2^n} \).

\[ \square \]
Corollary 8. There exists a family \( \{ P_0, P_1, \ldots \} \) of 2-way protocols with two leaders and a family \( \{ c_0, c_1, \ldots \} \) of natural numbers such that for every \( n \in \mathbb{N} \) the following holds: \( c_n \geq 2^2 \) and protocol \( P_n \) has at most 314 log log \( c_n \) + 131 states and computes the predicate \( x \geq c_n \).

Proof. Let \( n \in \mathbb{N} \) and let \( P'_n = (Q'_n, T^n_{T}, T^n_{O}, I'_n, L'_n, O'_n) \) be the protocol of Theorem 7. By applying Lemma 3 to \( P'_n \) we obtain a 2-way protocol \( P_n = (Q_n, T_n, I_n, L_n, O_n) \) such that:

\[ |Q_n| = |Q'_n| + 3 \cdot 5 \cdot |T_{T}^n| \leq (14n + 11) + (300n + 120) = 314n + 131, \]

\( P_n \) computes the same predicate as \( P'_n \), i.e. \( x \geq c_n \) for some \( c_n \geq 2^{2n} \). ◀

D Monotonic predicates and 1-awareness

In this section, we relate 1-aware protocols to monotonic predicates.

Definition 28. Let \( n \in \mathbb{N} \) and let \( \varphi \subseteq \mathbb{N}^n \) be an \( n \)-ary predicate. We say \( \varphi \) is monotonic if and only if \( (y \geq x \land \varphi(x)) \implies \varphi(y) \) for every \( x, y \in \mathbb{N}^n \).

Proposition 29. For every monotonic predicate \( \varphi \subseteq \mathbb{N}^n \) of arity \( n \in \mathbb{N} \) there exists a finite family of thresholds \( \{ c_1, \ldots, c_m \} \subseteq \mathbb{N}^n \) such that

\[ \varphi(x) \iff \bigvee_{1 \leq i \leq m} x_i \geq c_i. \]

Proof. By the very definition of monotonicity, the set \( \{ x : \varphi(x) \} \) is upwards-closed w.r.t. \( \leq \) and thus has a finite number \( m \) of minimal elements by Dickson’s lemma. Picking these minimal elements \( c_1, \ldots, c_m \) as the finite family of thresholds then yields the claim to be shown. ◀

Lemma 30. Let \( n \in \mathbb{N} \) and let \( \varphi \) be some \( n \)-ary predicate computable by a population protocol. Predicate \( \varphi \) is computable by a 1-aware protocol if and only if \( \varphi \) is monotonic.

Proof. We first show that if \( \mathcal{P} \) is 1-aware, then the predicate \( \varphi \) computed by \( \mathcal{P} \) is monotonic. Let \( C_0, C'_0 \) be initial configurations such that \( \varphi(C_0) \) holds and \( C_0 \leq C'_0 \). We must show that \( \varphi(C'_0) \) holds. Let \( Q_1 \subseteq Q \) be the subset of states that makes \( \mathcal{P} \) 1-aware. Since \( \varphi(C_0) \) holds, there exists \( q \in Q_1 \) and a configuration \( C \) such that \( C_0 \seq C \) and \( q \in [C] \). Since \( C'_0 \geq C_0 \), we have \( C'_0 \seq C' \) for some \( C' \geq C \). This implies that \( q \in [C'] \). By 1-awareness of \( \mathcal{P} \), we conclude that \( \varphi(C'_0) \) holds.

For the converse direction, assume \( \varphi \) is a monotonic predicate computable by a population protocol. By Proposition 29, we may assume \( \varphi \) is a finite disjunction of predicates of the form \( x \geq c_i \) for some thresholds \( c_i \). As threshold-predicates can be computed by 1-aware protocols and 1-aware protocols are closed under disjunction, \( \varphi \) is computable by a 1-aware protocol, and we are done.

E Detailed proofs of Section 5

Lemma 10. Let \( \mathcal{P} = (Q, T, \{ x \}, L, O) \) be a 1-aware protocol computing a unary predicate \( \varphi \). We have \( \varphi(n) = 1 \) if and only if some state of \( Q_1 \) is coverable from \( \langle n \cdot x \rangle + L \).

Proof. Let \( n \in \mathbb{N} \) and let \( C_0 \overset{\text{def}}{=} \langle n \cdot x \rangle + L \).

\[ \Rightarrow \) Let \( \pi = C_0C_1 \cdots \) be a fair execution. Since \( \mathcal{P} \) computes \( \varphi \), we have \( O(\pi) = \varphi(n) = 1 \). By condition (2) of the definition of 1-awareness, \( C_j(Q_1) > 0 \) for some \( j \in \mathbb{N} \). We are done since \( C_0 \seq C_j \).
$\Leftarrow$ We have $C_0 \rightarrow C_1 \rightarrow \cdots \rightarrow C_n = C$ for some configurations $C_1, C_2, \ldots, C_n$. Let $\pi = C_0C_1 \cdots C_n$ be any fair execution extending this finite sequence. By condition (1) of the definition of $1$-awareness, $O(\pi) = 0$, and hence $O(\pi) = 1$.

**Proposition 11.** Let $\mathcal{P} = (Q, T, I, L, O)$ be a $k$-way population protocol and let $C_1 \xrightarrow{\pi} C_2$ be a finite execution of $\mathcal{P}$. There exists a finite execution $\pi' \rightarrow C_1'$ such that (a) $|C_1'| = |C_1|$, (b) $|C_2'| = |C_1| \cup |\pi'|^*$, and (c) $|C_1'\pi| \leq (k + 1)^{|Q|}$.

**Proof.** Let $c \triangleq k + 1$. We prove a stronger claim: $C_1', C_2'$, and $\pi'$ can be chosen so that they satisfy (a), (b), and a stronger property: (d) there is a sequence $t_1, t_2, \ldots, t_m$ of transitions of $|\pi|$ such that $\pi' = t_1^{m-1}t_2^{m-2} \cdots t_m$. Since $\pi$ is $1$-aware, it involves at most $C$ agents, and so, since $C_1 = C_2$, the claim is satisfied by $\pi' \triangleq \epsilon$ and the configurations $C_1'$ and $C_2'$ such that for every $q \in Q$,

$$C_1'(q) \triangleq C_2'(q) \begin{cases} 1 & \text{if } q \in |C_1|, \\ 0 & \text{otherwise.} \end{cases}$$

Assume that $|\pi| > 0$ and that the claim holds for sequences of length less than $|\pi|$. There exist $\sigma \in T^*$, $t \in T$ and a configuration $D$ such that $\pi = \sigma t$ and $C_1 \xrightarrow{\pi} D \xrightarrow{t} C_2$. By induction hypothesis, there exists an execution $C''_1 \xrightarrow{\sigma''} D''$ such that (a) $|C''_1| = |C_1|$, (b) $|D''| = |C_1| \cup |\sigma|^*$, and (c) $|\pi''| = t_1^{m-1}t_2^{m-2} \cdots t_m$ for a sequence $t_1, t_2, \ldots, t_m$ of transitions of $|\pi|$ satisfying $m \leq |\{t_1, \ldots, t_m\}|$.

If $t^* \subseteq \{t_1, \ldots, t_m\}$, then we can take $C_1' \triangleq C''_1$, $C_2' \triangleq D''$, and $\pi' \triangleq \pi''$. So assume $t^* \not\subseteq \{t_1, \ldots, t_m\}$. Since $C_1 \xrightarrow{\pi} D$, we have $|D| \subseteq |C_1| \cup |\sigma|^*$, and so, since $|C_1| \cup |\sigma|^* = |D''|$, by (a') and (b'), we get $|D| \subseteq |D''|$. Thus, since $t$ is enabled at $D$ and, by the definition of $c$, it involves at most $c - 1$ agents, $t$ is also enabled in $(c - 1) \cdot D''$. Moreover, by (d') we have

$$c \cdot C''_1 \xrightarrow{t_1^{m-1}t_2^{m-2} \cdots t_m} c \cdot D''.$$

So, since $c > 1$, we obtain

$$c \cdot C_1' \xrightarrow{t_1^{m-1}t_2^{m-2} \cdots t_m} c \cdot D'' \xrightarrow{t} (D'' + E)$$

for some configuration $E$. Taking $C_1' \triangleq c \cdot C''_1$, $C_2' \triangleq D'' + E$, and $\pi' = t_1^{m-1}t_2^{m-2} \cdots t_m$, we have $C_1' \xrightarrow{\pi} C_2'$. We prove that $C_1'$, $C_2'$, and $\pi'$ satisfy (a), (b), and (d):

(a) We must show $|C_1'| = |C_1|$. It follows from $|C_1'| = |c \cdot C''_1| = |C''_1| \triangleq |C_1|$.

(b) We must show $|C_2'| = |C_1| \cup |\pi|^*$. It follows from

$$|C_2'| = |D'' + E| = |D''| \cup |E| \triangleq |C''_1| \cup |\pi|^* \cup |E| = |C''_1| \cup |\sigma|^* \cup t^* = |C''_1| \cup |\sigma|^* \cup |t|^* = |C''_1| \cup |\sigma|^* \triangleq |C_1| \cup |\pi|^*.$$

(d) We must show that $\pi' = t_1^{m-1}t_2^{m-2} \cdots t_m$, where $t_1, \ldots, t_m, t$ belong to $|\pi|$, and $m + 1 \leq |\{t_1, \ldots, t_m\}|$.

Since $t_1, \ldots, t_m$ belong to $|\sigma|$ by (d'), the transitions $t_1, \ldots, t_m, t$ belong to $|\sigma t| = |\pi|$. Further, we have $m + 1 \leq |\{t_1, \ldots, t_m\}| + 1$ by (d'), and $|\{t_1, \ldots, t_m\}| + 1 \leq |\{t_1, \ldots, t_m, t\}|$ because, by assumption, $t^* \not\subseteq \{t_1, \ldots, t_m\}$. □
**Proposition 14.** Let \( \mathcal{P} = (Q, T, I, L, O) \) be a 2-way population protocol and let \( q \in Q \). For every configuration \( C \), if \( q \) is coverable from \( C \), then it is coverable by means of a finite execution of length at most \( 2^{(3m)^m} \) where \( m = 12|Q|^8 \).

**Proof.** Let \( Q = \{q_1, q_2, \ldots, q_n\} \), and let \( b \) be a fresh symbol not contained in \( Q \). We associate to \( \mathcal{P} \) a set \( A \subseteq \mathbb{Z}^{(|Q|+|T|+1)} \). The set \( A \) contains two vectors \( v_1^a, v_2^a \) for every transition \( t \in T \), defined as functions \( Q \cup T \cup \{b\} \to \mathbb{Z} \) in the following way: \( v_1^a(q) = - \text{pre}(t)(q) \) for all \( q \in Q \), \( v_1^a(t') = 1 \) if \( t = t' \) else 0 for all \( t \in T \), and \( v_2^a(b) = -1 \); \( v_2^a(q) = \text{post}(t)(q) \) for all \( q \in Q \), \( v_2^a(t') = -1 \) if \( t = t' \) else 0, and \( v_2^a(b) = 1 \). Intuitively, \( v_1^a \) “removes” agents from their current states, and \( v_2^a \) “adds” them to their new states. It is easy to see that for every \( v \in \mathbb{N}^{(|Q|+|T|+1)} \) satisfying \( v(t) = 0 \) for every \( t \in T \) and \( v(b) = 1 \), the VAS \((A,v)\) simulates \( \mathcal{P} \) from the configuration \( C \) satisfying \( C(q) = v(q) \) for every \( q \in Q \). An occurrence of \( t \) in \( \mathcal{P} \) is simulated by first adding \( v_2^a \) and then \( v_1^a \). The \( b \)-component ensures that \( v_1^a \) always directly precedes \( v_2^a \). Since \( A \) contains \( 2|T| \) vectors of dimension \((|Q|+|T|+1)\) with entries taken from \{-2, -1, 0, 1, 2\}, its size is bounded by \( 12|Q|^8 \):

\[
\text{size}(A) = \sum_{v \in A} \text{size}(v) = \sum_{v \in A} \sum_{1 \leq i \leq (|Q|+|T|+1)} \text{size}(\max(|v(i)|, 1)) \leq \sum_{v \in A} 2 \cdot (|Q| + |T| + 1) = 4|T| \cdot (|Q| + |T| + 1) \leq 4|Q|^4 \cdot (|Q|^4 + |Q|^4 + |Q|^4) = 12|Q|^8
\]

By applying Theorem 13 on \( A \), we obtain the desired bound. \(\blacktriangleright\)

**Theorem 31.** Let \( \mathcal{P} \) be a 1-aware and 2-way population protocol. For every \( n \geq 2 \), if \( \mathcal{P} \) computes \( x \geq n \), then \( \mathcal{P} \) has at least \((\log \log(n)/151)^{1/3}\) states.

**Proof.** Let \( \mathcal{P} = (Q, T, \{q_0\}, L, O) \) be a 1-aware 2-way population protocol computing the predicate \( x \geq n \). Let \( q_0 \) be the only initial state of \( \mathcal{P} \), and let \( Q_1 \subseteq Q \) be the set of states of \( \mathcal{P} \) that make it 1-aware. By Proposition 14 some state \( q_1 \in Q_1 \) is coverable from \( \{n \cdot q_0\} + L \) by means of an execution \( \sigma \) of length \( 2^{(3m)^m} \) where \( m \leq 12|Q|^8 \).

Let \( k \overset{\text{def}}{=} 2^{(3m)^m} \). Since \( \sigma \) removes at most \( k \) agents from state \( q_0 \), it is also enabled at the initial configuration \( C_0' \overset{\text{def}}{=} \{k \cdot q_0\} + L \). Further, since \( q_0 \notin Q_1 \) and \( q_1 \in Q_1 \), we have \( C_0' \overset{\sigma}{\rightarrow} C' \) for some configuration \( C' \) such that \( C'(q_1) > 0 \). By definition of 1-awareness, \( O(C_0') = 1 \), and thus since \( \mathcal{P} \) computes \( x \geq n \), we have \( k \geq n \).

Therefore, \( n \leq k \leq 2^{(3m)^m} \), which implies that \( n \leq 2^{(36|Q|^8)^{12}|Q|^8} = 2^{2 \log(36|Q|^8)|2|Q|^8} \), and in turn that \( \log \log(n) \leq \log(36|Q|^8) \cdot 12|Q|^8 \) for every \( n \geq 2 \). Note that \( \log(a) \leq \lambda \cdot a^{1/\lambda} \) for every \( a, \lambda \in \mathbb{N}_+ \). Thus, by taking \( a = 36|Q|^8 \) and \( \lambda = 8 \), we obtain \( \log \log(n) \leq 12 \cdot 8 \cdot 36^{1/8} \cdot |Q|^8 \leq 151|Q|^8 \), which implies that \( |Q| \geq (\log \log(n)/151)^{1/3} \). \(\blacktriangleright\)

**F Detailed proofs of Section 6**

Since linear inequalities are subsumed by systems of linear inequalities, we only give a proof sketch of Theorem 16 and we instead focus on proving Theorem 17 in details.
F.1 Linear inequalities

Theorem 16. Let \( a_1, a_2, \ldots, a_k, c \in \mathbb{Z} \) and let \( n = \text{size}(\max(|a_1|, |a_2|, \ldots, |a_k|, |c|, 1)) \). There exists a 2-way population protocol, with at most 10kn states and at most 5n + 2 leaders, that computes the predicate \( \sum_{1 \leq i \leq k} a_i x_i + c > 0 \).

Proof sketch. The bounds follow from the definition of \( \mathcal{P}_{\text{lin}} \) and Lemma \( [\text{3}] \). Let us sketch the correctness of \( \mathcal{P}_{\text{lin}} \). We associate a value to each state in the natural way, i.e., \( \text{val}(x_i) \overset{\text{def}}{=} a_i \), \( \text{val}(+2^i) \overset{\text{def}}{=} 2^i \), \( \text{val}(-2^i) \overset{\text{def}}{=} -2^i \) and \( \text{val}(0) \overset{\text{def}}{=} 0 \). Let

\[
X^+ \overset{\text{def}}{=} \{ x_i : 1 \leq i \leq k, a_i > 0 \} \quad \text{and} \quad X^- \overset{\text{def}}{=} \{ x_i : 1 \leq i \leq k, a_i < 0 \}.
\]

For every configuration \( C \), we let

\[
\text{val}^+(C) \overset{\text{def}}{=} \sum_{q \in Q^+ \cup X^+} \text{val}(q) \cdot C(q), \quad \text{val}^-(C) \overset{\text{def}}{=} \sum_{q \in Q^- \cup X^-} \text{val}(q) \cdot C(q),
\]

and \( \text{val}(C) \overset{\text{def}}{=} \text{val}^+(C) + \text{val}^-(C) \).

For every initial configuration \( C_0 \) and sequence \( C \rightarrow C' \), it can be shown that:

\[
\begin{align*}
\text{val}(C_0) &= \sum_{1 \leq i \leq k} a_i \cdot C_0(x_i) + c, \\
\text{val}^+(C) &\geq \text{val}^+(C'), \quad \text{val}^-(C) \leq \text{val}^-(C') \quad \text{and} \quad \text{val}(C) = \text{val}(C'), \\
C'(x) &= C(x) - \sum_{r \in R} |C_{\text{add}, r}| \sigma_{\text{add}, r} \quad \text{for every} \quad x \in X.
\end{align*}
\]

Using these facts, it is possible to show that the number of agents in the largest powers of 2 cannot grow too much, as otherwise the represented value would be too large or too small:

\[
C(+2^n) \leq 1 + \sum_{x \in X^+} \sum_{r \in R} |C_{\text{add}, r}| \sigma_{\text{add}, r} \quad \text{and} \quad C(-2^n) \leq 1 + \sum_{x \in X^-} \sum_{r \in R} |C_{\text{add}, r}| \sigma_{\text{add}, r}.
\]

Combining these observations, and by using transitions of the form \( \text{up}_i^+ \) and \( \text{up}_i^- \), it can be shown that

\[
\text{If} \ C_0 \text{ is initial and } C_0 \xrightarrow{\ast} C, \text{ then there exist } C' \text{ and } r \in R \text{ s.t. } C \xrightarrow{\ast} C' \text{ and } C'(r) \geq n.
\]

This implies that, in any fair execution, transitions of the form \( \text{add}_{x, r} \) can occur until the number of agents in \( X \) stabilizes to 0. Moreover, it implies that, in any fair execution, transitions of the form \( \text{down}_{x, r}, \text{down}_{l, r} \) and \( \text{cancel}_i \) can occur until the number of agents in \( Q^+ \) or \( Q^- \) stabilizes to 0. Finally, “signal” transitions ensure that every fair execution stabilizes to the right output.

F.2 Conjunction of linear inequalities

Let \( A \in \mathbb{Z}^{m \times k} \) and \( c \in \mathbb{Z}^n \). Let us now introduce in details the population protocol \( \mathcal{P}_{\text{sys}} = (Q, T, I, L, O) \) for the predicate \( Ax + c > 0 \). Let

\[
b_{\text{max}} \overset{\text{def}}{=} \max(1, \max\{|A_{i,j}| : i \in [m], j \in [k]\}, \max\{|c_i| : i \in [m]\})
\]

and \( n \overset{\text{def}}{=} \lceil \log 2m^2 \rceil + \text{size}(b_{\text{max}}) \). The following will later be crucial:

\[
2^n > 2m^2 \cdot b_{\text{max}}. \tag{4}
\]

The states of the protocol are defined as \( Q \overset{\text{def}}{=} X \cup Q^+ \cup Q^- \cup R \) where

\[
X \overset{\text{def}}{=} \{ x_1, x_2, \ldots, x_k \}, \quad R \overset{\text{def}}{=} \{ 0_0, 0_1 \}, \quad Q^+_j \overset{\text{def}}{=} \{ +2^i : i \in [0, n], \alpha \in \{ 0, 1 \} \}, \quad Q^- \overset{\text{def}}{=} \{ -2^i : i \in [0, n] \},
\]

\[
Q^+ \overset{\text{def}}{=} Q^+_1 \cup Q^+_2 \cup \cdots \cup Q^+_m, \quad Q^- \overset{\text{def}}{=} Q^-_1 \cup Q^-_2 \cup \cdots \cup Q^-_m.
\]
The initial states are defined as $I \equiv X$, and the output mapping as

$$O(q) \equiv \begin{cases} 1 & \text{if } q = 0 \text{, or } q = +2^i_{j,1} \text{ for some } i \in [0, n], j \in [m] \\ 0 & \text{otherwise}. \end{cases}$$

In order to define leaders and transitions, let us first give some definitions. Let $\text{rep}_{j}(d) : \mathbb{Z} \to \text{Pop}(Q \setminus X)$ be defined as follows:

$$\text{rep}_{j}(d) \equiv \begin{cases} i + 2^j_{j,\alpha} : i \in \text{bits}(d) & \text{if } d > 0, \\ i - 2^j_{j,\alpha} : i \in \text{bits}(|d|) & \text{if } d < 0, \\ 0 & \text{if } d = 0. \end{cases}$$

Let $\text{rep}(d) : \mathbb{Z}^m \to \text{Pop}(Q \setminus X)$ be defined as $\text{rep}(d) \equiv \text{rep}_1(d_1) + \text{rep}_2(d_2) + \ldots + \text{rep}_m(d_m)$. Leaders are defined as $L \equiv \text{rep}(e) + \left(5mn + 1\right) \cdot 0_o$.

It remains to describe the set of transitions $T$. It contains the following transitions which allow to change representations of numbers over $Q^+ \cup Q^-:

$$\text{up}_{i,j,\alpha,\beta} : +2^j_{j,\alpha}, +2^j_{j,\beta} \rightarrow +2^j_{j,\alpha} + 2^j_{j,\beta} \cdot 0_{\alpha,\beta}, +2^j_{j,\alpha} + 2^j_{j,\beta} \cdot 0_{\alpha,\beta}, +2^j_{j,\alpha} + 2^j_{j,\beta} \cdot 0_{\alpha,\beta} \rightarrow 0_{\alpha,\beta} \rightarrow 0_{\alpha,\beta}$$

$$\text{down}_{i,j,\alpha,\beta} : +2^j_{j,\alpha} + 2^j_{j,\beta} \cdot 0_{\alpha,\beta} \rightarrow +2^j_{j,\alpha} + 2^j_{j,\beta} \cdot 0_{\alpha,\beta}, +2^j_{j,\alpha} + 2^j_{j,\beta} \cdot 0_{\alpha,\beta} \rightarrow -2^j_{j,\alpha} - 2^j_{j,\beta}$$

where $i \in [0, n - 1], j \in [m]$ and $\alpha, \beta \in \{0, 1\}$. It contains the following transitions to cancel out equal numbers:

$$\text{cancel}_{i,j,\alpha} : +2^j_{j,\alpha} - 2^j_{j,\alpha} \rightarrow 0_{\alpha} \rightarrow 0_{\alpha}$$

where $i \in [0, n], j \in [m]$ and $\alpha \in \{0, 1\}$. It contains the following transitions to signal false and true consensus:

$$\text{false}_{i,j}^+: 0_{\alpha} + 2^j_{j,1} \rightarrow 0_{\alpha} + 2^j_{j,\alpha} \text{ and } \text{false}_{i,j}^- : -2^j_{j,\alpha} \rightarrow -2^j_{j,\alpha} \rightarrow 0_{\alpha} \rightarrow 0_{\alpha}$$

$$\text{true} : +2^i_{i,0,0} + 2^i_{i,0,1} + \ldots + 2^i_{i,m,0} \rightarrow +2^i_{i,0,0} + 2^i_{i,0,1} + \ldots + 2^i_{i,m,1} \rightarrow 0_{1}$$

where $i \in [0, n]$ and $j \in [m]$. Finally, it contains the following transitions to convert variables to their coefficients:

$$\text{add}_{j,\alpha} : x_j, 0_{\alpha}, 0_{\alpha}, \ldots, 0_{\alpha} \rightarrow \text{rep}(A_{\ast,j})$$

where $j \in [k], \alpha \in \{0, 1\}$, and $A_{\ast,j}$ is the $j^{th}$ column of $A$.

The rest of this appendix is dedicated to proving the correctness of $P_{\text{sys}}$. Before doing so, we need to introduce additional definitions. Let $\text{val} : Q \to \mathbb{N}$ be the function that associates a value to each state as follows:

$$\text{val}(x_j) \equiv \sum_{i \in [m]} A_{i,j}$$

for every $j \in [k]$,

$$\text{val}(+2^j_{j,\alpha}) \equiv 2^i$$

for every $i \in [0, n], j \in [m], \alpha \in \{0, 1\}$,

$$\text{val}(-2^j_{j,\alpha}) \equiv -2^i$$

for every $i \in [0, n], j \in [m]$,

$$\text{val}(0_o) \equiv \text{val}(0_i) \equiv 0.$$
We extend $\text{val}$ to configurations. For every $C \in \text{Pop}(Q)$ and every $i \in [m]$, let

$$
\text{val}_i^+(C) \overset{\text{def}}{=} \sum_{q \in Q_i^+} \text{val}(q) \cdot C(q), \quad \text{val}_i^+(C) \overset{\text{def}}{=} \sum_{i \in [m]} \text{val}_i^+(C),
$$

$$
\text{val}_i^-(C) \overset{\text{def}}{=} \sum_{q \in Q_i^-} \text{val}(q) \cdot C(q), \quad \text{val}_i^-(C) \overset{\text{def}}{=} \sum_{i \in [m]} \text{val}_i^-(C),
$$

$$
\text{val}_i(C) \overset{\text{def}}{=} \text{val}_i^+(C) + \text{val}_i^-(C) + \sum_{j \in [k]} A_{i,j} \cdot C(x_j)
$$

$$
\text{val}(C) \overset{\text{def}}{=} \sum_{i \in [m]} \text{val}_i(C).
$$

For every $j \in [k]$ and $\sigma \in T^*$, let $b_j \overset{\text{def}}{=} \sum_{i \in [m]} |A_{i,j}|$ and let $\text{num}_i(\sigma) \overset{\text{def}}{=} |\sigma|_{\text{add},i} + |\sigma|_{\text{add},i+1}$.

It is not so difficult to derive the following properties from the above definitions:

\textbf{Proposition 32.} Let $C_0, C, C' \in \text{Pop}(Q)$ and $\sigma \in T^*$ be such that $C_0$ is initial and $C \overset{\sigma}{\rightarrow} C'$.

The following holds for every $i \in [m]$:

(a) $\text{val}_i^+(C_0) = \max(e_i, 0)$ and $\text{val}_i^-(C_0) = \min(e_i, 0)$,

(b) $\text{val}_i(C') = \text{val}_i(C)$,

(c) if $C(X) = 0$, then $\text{val}_i^+(C) \geq \text{val}_i^+(C')$,

(d) if $C(X) = 0$, then $\text{val}_i^-(C) \leq \text{val}_i^-(C')$,

(e) $\text{val}_i^+(C') \leq \text{val}_i^+(C) + \sum_{j \in [k]} \text{num}_j(\sigma) \cdot b_j$,

(f) $\text{val}_i^-(C') \geq \text{val}_i^-(C) - \sum_{j \in [k]} \text{num}_j(\sigma) \cdot b_j$,

(g) $C'(X) = C(X) - \sum_{j \in [k]} \text{num}_j(\sigma)$.

From Proposition 32 we obtain the following useful proposition:

\textbf{Proposition 33.} Let $C_0, C \in \text{Pop}(Q)$ and $\sigma \in T^*$ be such that $C_0$ is initial and $C_0 \overset{\sigma}{\rightarrow} C$.

For every $i \in [m]$, the following holds:

$$
C(\{+2^n_{i,0}, -2^n_{i,1}\}) \leq (d + 1)/2m \quad \text{and} \quad C(-2^n_{i,2}) \leq (d + 1)/2m
$$

where $d = \sum_{j \in [k]} \text{num}_j(\sigma)$.

\textbf{Proof.} Let $i \in [m]$. We only prove the first claim, the second one follows symmetrically. Let $S \overset{\text{def}}{=} \{+2^n_{i,0}, -2^n_{i,1}\}$. We must show that $C(S) \leq (d + 1)/2m$. For the sake of contradiction, suppose $C(S) > (d + 1)/2m$. We derive the following contradiction:

$$
\text{val}_i^+(C) \geq 2^n \cdot C(S)
$$

(by def. of $\text{val}_i^+$)

$$
> 2^n \cdot ((d + 1)/2m)
$$

(by assumption)

$$
= \frac{2^n + \sum_{j \in [k]} (2^n \cdot \text{num}_j(\sigma))}{2m}
$$

(by def. of $d$)

$$
= \frac{2m^2 \cdot b_{\max} + \sum_{j \in [k]} 2m^2 \cdot b_{\max} \cdot \text{num}_j(\sigma)}{2m}
$$

(by [4])

$$
= m \cdot b_{\max} + \sum_{j \in [k]} m \cdot b_{\max} \cdot \text{num}_j(\sigma)
$$

$$
\geq \text{val}_i^+(C_0) + \sum_{j \in [k]} m \cdot b_{\max} \cdot \text{num}_j(\sigma)
$$

(by Prop. 32(a))

$$
\geq \text{val}_i^+(C_0) + \sum_{j \in [k]} b_j \cdot \text{num}_j(\sigma)
$$

(by def. of $b_{\max}$ and $b_j$)

$$
\geq \text{val}_i^+(C)
$$

(by Prop. 32(e)).
The following proposition shows that it is always possible to convert at least \( mn \) agents back to a state of \( R \). This will later be useful in arguing that the number of agents in \( X \) can eventually be decreased to zero.

\[ \boxed{\text{Proposition 34.} \quad \text{Let } C_0, C \in \text{Pop}(Q) \text{ be such that } C_0 \text{ is initial. If } C_0 \xrightarrow{\ell} C, \text{ then there exist a configuration } C' \text{ and } \alpha \in \{0,1\} \text{ such that } C \xrightarrow{\ell+} C' \text{ and } C'(0_\alpha) \geq mn.} \]

**Proof.** If \( C(R) \geq 2mn \), then \( C' \xrightarrow{\ell+} C \) satisfies the claim by the pigeonhole principle. Therefore, assume \( C(R) < 2mn \). Let \( \sigma \in T^* \) be such that \( C_0 \xrightarrow{\ell} C \). Let

\[
U \overset{\text{def}}{=} \{ +\mathbf{2}_j, +\mathbf{2}_j, -\mathbf{2}_j : j \in [m] \} \quad \text{and} \quad V \overset{\text{def}}{=} (Q^+ \cup Q^-) \setminus U.
\]

We have

\[
C(V) = |C_0| - C(U) - C(X) - C(R) \quad \text{(by } |C| = |C_0|) \]

\[
> |C_0| - C(U) - C(X) - 2mn \quad \text{(by assumption)}
\]

\[
= |C_0| - C(U) - (C_0(X) - \sum_{j \in [k]} \text{num}_j(\sigma)) - 2mn \quad \text{(by Prop. 32(g))}
\]

\[
\geq 3mn + 1 - C(U) + \sum_{j \in [k]} \text{num}_j(\sigma) \quad \text{(by } C_0(R) \geq 5mn + 1) \]

\[
\geq 3mn + 1 - 2m \left( \frac{1 + \sum_{j \in [k]} \text{num}_j(\sigma)}{2m} \right) + \sum_{j \in [k]} \text{num}_j(\sigma) \quad \text{(by Prop. 33)}
\]

\[
= 3mn.
\]

Since \( C(V) > 3mn = |V| \), the pigeonhole principle implies that \( C(q) \geq 2 \) for some \( q \in V \). Therefore, a transition of the form \( \text{up}^+_{i,j,\alpha,\beta} \) or \( \text{up}^-_{i,j} \) can occur from \( C \), leading to a configuration \( D \) such that \( D(R) = C(R) + 1 \). If \( D(R) < 2mn \), then this argument can be repeated until a configuration \( C' \) such that \( C'(R) \geq 2mn \) is reached.

We now show that, in any fair execution, the number of agents in \( X \) eventually stabilizes to 0, and the value associated to each conjunct stabilizes to either some positive or some negative number.

\[ \boxed{\text{Proposition 35.} \quad \text{Let } \pi = C_0C_1 \cdots \text{ be a fair execution from an initial configuration } C_0. \text{ There exist } \ell \in \mathbb{N}, d_1^+, d_2^+, \ldots, d_m^+ \geq 0 \text{ and } d_1^-, d_2^-, \ldots, d_m^- \leq 0 \text{ such that for every } i \in [m], \text{ the following holds:}
1. \quad C_\ell(X) = C_{\ell+1}(X) = \cdots = 0,
2. \quad \text{val}^+_i(C_\ell) = \text{val}^+_i(C_{\ell+1}) = \cdots = d_i^+,
3. \quad \text{val}^-_i(C_\ell) = \text{val}^-_i(C_{\ell+1}) = \cdots = d_i^-,
4. \quad d_i^+ = 0 \text{ or } d_i^- = 0.} \]

**Proof.** For the sake of contradiction, assume there exist infinitely many indices \( i \) such that \( C_i(X) > 0 \). Let \( i \in \mathbb{N} \) be one of these indices. By Proposition 34, there exist \( D_i \in \text{Pop}(Q) \) and \( \alpha \in \{0,1\} \) such that \( C_i \xrightarrow{\ell} D_i \) and \( D_i(0_\alpha) \geq mn \). Hence, by definition of \( T \), there exists \( j \in [k] \) such that \( \text{add}_j,\alpha \) is enabled at \( D_j \). Since this holds for infinitely many indices, fairness implies that one transition of \( \{\text{add}_j,\alpha : j \in [k], \alpha \in \{0,1\}\} \) is taken infinitely often along \( \pi \). This is impossible since the number of agents in \( X \) cannot increase, and thus would eventually drop below zero. Therefore, there exists \( \ell \in \mathbb{N} \) such that \( C_\ell(X) = C_{\ell+1}(X) = \cdots = 0. \)
Let \( i \in [m] \). By Proposition 32, we have
\[
\text{val}^*_{\ell}(C_\ell) \geq \text{val}^*_{\ell}(C_{\ell+1}) \geq \cdots \geq 0, \\
\text{val}^{-}_{\ell}(C_\ell) \leq \text{val}^{-}_{\ell}(C_{\ell+1}) \leq \cdots \leq 0.
\]
Therefore, there exist \( \ell' \geq \ell \), \( d_1^+ \geq 0 \) and \( d_1^- \leq 0 \) such that
\[
\text{val}^*_{\ell'}(C_{\ell'}) = \cdots = d_1^+, \\
\text{val}^{-}_{\ell'}(C_{\ell'}) = \cdots = d_1^-.
\]
For the sake of contradiction, assume that \( d_1^+ \neq 0 \) and \( d_1^- \neq 0 \). Let \( J \subseteq [\ell', +\infty) \) be the set of all indices \( j \) such that \( D_j(Q_{\ell'}^+) > 0 \) and \( D_j(Q_{\ell'}^-) > 0 \). We may assume that \( J \) is infinite, as otherwise fairness would contradict \( 5 \) or \( 6 \). Let \( j \in J \). There exist \( \lambda, \lambda' \in [0, n] \) and \( \alpha \in \{0, 1\} \) such that \( D_j(+2^{m_\alpha}_{\lambda,\alpha}) > 0 \). Assume without loss of generality that \( \lambda \geq \lambda' \). The other case is proven symmetrically. Since \( D_j(0_{\beta}) \geq n \geq \lambda - \lambda' \) for some \( \beta \in \{0, 1\} \), the sequence
\[
down^+_{\lambda,\iota,\alpha,\beta} \cdot down^+_{\lambda-1,1,\alpha,\beta} \cdot \cdots \cdot down^+_{\lambda+1,1,\alpha,\beta} \cdot \cancel{\down^+_{\lambda,\iota,\alpha,\beta}}
\]
can occur from \( D_j \). The resulting configuration \( E_j \) is such that
\[
\text{val}^*_{\ell}(E_j) < \text{val}^*_{\ell}(D_j) \leq \text{val}^*_{\ell}(C_\ell) = d_1^+, \\
\text{val}^{-}_{\ell}(E_j) > \text{val}^{-}_{\ell}(D_j) \geq \text{val}^{-}_{\ell}(C_\ell) = d_1^-.
\]
Since \( \{E_{\ell'}, E_{\ell'+1}, \ldots\} \) is finite, fairness implies that one of these configurations occurs infinitely often along \( \pi \). This contradicts \( 5 \) and \( 6 \).

We are now ready to prove correctness of \( \mathcal{P}_{\text{sys}} \).

\textbf{Theorem 36.} \( \mathcal{P}_{\text{sys}} \) is well-specified and correct.

\textbf{Proof.} Let \( \pi : C_0 \xrightarrow{s_1} C_1 \xrightarrow{s_2} \cdots \) be a fair execution from an initial configuration \( C_0 \). By Proposition 35, there exist \( \ell \in \mathbb{N} \), \( d_1^+, d_2^+, \ldots, d_m^+ \geq 0 \) and \( d_1^-, d_2^-, \ldots, d_m^- \leq 0 \) such that for every \( i \in [m] \), the following holds:
\[
\begin{align*}
& (\text{a}) \quad C_\ell(X) = C_{\ell+1}(X) = \cdots = 0, \\
& (\text{b}) \quad \text{val}^*_{\ell}(C_\ell) = \text{val}^*_{\ell}(C_{\ell+1}) = \cdots = d_1^+, \\
& (\text{c}) \quad \text{val}^{-}_{\ell}(C_\ell) = \text{val}^{-}_{\ell}(C_{\ell+1}) = \cdots = d_1^-,
\end{align*}
\]
\[
\begin{align*}
& (\text{d}) \quad d_1^+ = 0 \text{ or } d_1^- = 0.
\end{align*}
\]

We first show well-specification. There are two cases to consider.

\textit{Case 1:} \( d_1^+ > 0 \) for every \( i \in [m] \). We claim that for every \( j \geq \ell \), configuration \( C_j \) can reach a configuration that contains some agent in state \( O_1 \). Let us argue that the validity of the claim concludes the case. By fairness, the claim implies that \( C_j(O_1) > 0 \) for infinitely many indices \( j \). Therefore, by fairness and transitions of the form \( \star, \star, \star, \star \), we have \( O(C_j) = 1 \) for infinitely many indices \( j \). By examining the presets and postsets of transitions from \( T \), we observe that any configuration whose output is 1 must be stable.

Let us now prove the claim. Let \( j \geq \ell \). By Proposition 34, there exist \( D_j \in \text{Pop}(Q) \) and \( \beta \in \{0, 1\} \) such that \( C_j \xrightarrow{s_1} D_j \) and \( D_j(0_{\beta}) \geq mn \). If \( \beta = 1 \), we are done. Thus, assume \( \beta = 0 \). Since \( d_1^+, d_2^+, \ldots, d_m^+ > 0 \), there exist \( \lambda_1, \lambda_2, \ldots, \lambda_m \in [0, n] \) and \( \alpha_1, \alpha_2, \ldots, \alpha_m \in \{0, 1\} \) such that \( D_j(+2^{m_\alpha}_{\lambda,\alpha}) > 0 \) for every \( i \in [m] \). Since \( D_j(0_{\beta}) \geq mn \), it is possible to construct a configuration \( E_j \in \text{Pop}(Q) \) and sequence \( w \in T^* \), made of transition “false” and transitions of the form \( \text{down}^+_{\star,\star,\star,\star} \), such that
Case 2: \(d^+ \equiv 0\) for some \(i \in [m]\). We claim that for every \(j \geq \ell\), configuration \(C_j\) can reach a configuration that contains some agent in state \(0_1\). Let us argue that the validity of the claim concludes the case. By fairness, the claim implies that \(C_j(0_1) > 0\) for infinitely many indices \(j\). Therefore, by fairness, transition “false” and transitions of the form \(false_\ast,\ast\), we have \(O(C_j) = 0\) for infinitely many indices \(j\). By examining the presets and postsets of transitions from \(T\), we observe that a configuration whose output is 0 can only reach a configuration whose output is not 0 through transition “true”. Since \(d^+ \equiv 0\), we have \(C_j(Q_i^+) = 0\) for every \(j \geq \ell\). Therefore, transition “true” is disabled at \(C_j\) for every \(j \geq \ell\).

Let us now prove the claim. Let \(j \geq \ell\). By Proposition 34, there exist \(D_j \in \text{Pop}(Q)\) and \(\beta \in \{0, 1\}\) such that \(C_j \xrightarrow{\beta} D_j\) and \(D_j(0_\beta) \geq mn\). If \(\beta = 0\), we are done. Thus, assume \(\beta = 1\). If \(C_j(Q^-_\ast) > 0\), then we are done, since a transition of the form \(false_\ast,\ast\) can occur, leading to a configuration \(E_j\) such that \(E_j(0_\alpha) > 0\). Therefore, assume \(C_j(Q^-_\ast) = 0\). Since \(C_j(0_\ast) > 0\), the prefix \(\sigma_1\sigma_2 \cdots \sigma_j\) must contain the transition “true”. Thus, there exists \(j' < j\) such that \(C_j'(Q^+_\ast) > 0\). Let \(j'\) be the largest such index. Transition \(\sigma_{j'+1}\) must be of the form cancel, \(\ast,\ast\). Therefore, \(C_{j'+1}(0_\ast) > 1\). By inspection of \(T\), we observe that “true” is the only transition that can decrease the number of agents in \(0_\ast\). By maximality of \(j'\), we have \(C_{j'+1}(Q^+_i) = C_{j'+2}(Q^+_i) = \cdots = 0\). Thus transition “true” cannot occur, and hence \(C_j(0_\ast) > 0\).

We are done proving well-specification. To conclude the proof, let us argue that \(P_{sys}\) indeed computes the predicate \(Ax + c > 0\). Let \(j \geq \ell\) be such that \(C_j\) is stable. For every \(i \in [m]\), we have

\[
\begin{align*}
c_i + \sum_{j \in [k]} A_{i,j} \cdot C_0(x_j) &= \text{val}_i(C_0) \quad \text{(By Prop. 32(a))} \\
&= \text{val}_i(C_j) \quad \text{(By Prop. 32(b))} \\
&= \text{val}^+_i(C_j) + \text{val}^-_i(C_j) + 0 \\
&= d^+_i + d^-_i.
\end{align*}
\]

Recall that \(d^+_i \geq 0\) and \(d^-_i \leq 0\) for every \(i \in [m]\). If \(Ax + c > 0\) holds, then we must have \(d^+_i > 0\) for every \(i \in [m]\). Therefore, case 1 holds, and hence \(O(\pi) = O(C_j) = 1\), which is correct. If \(Ax + c > 0\) does not hold, then we must have \(d^-_i = 0\) for some \(i \in [m]\). Therefore, case 2 holds, and hence \(O(\pi) = O(C_j) = 0\), which is also correct.

We may now prove the theorem from the main text:

**Theorem 17.** Let \(A \in \mathbb{Z}^{m \times k}\), \(c \in \mathbb{Z}^m\) and \(n = \text{size}(\max(1, \{|A_{i,j}| : 1 \leq i \leq m, 1 \leq j \leq k\}), \{|c_i| : 1 \leq i \leq m\})\). There exists a 2-way population protocol, with at most \(27(\log m + n)(m + k)\) states and at most \(14m(\log m + n)\) leaders, that computes the predicate \(Ax + c > 0\).

**Proof.** The value \(n\) which occurs in the statement of the theorem differs from the \(n\) defined in this appendix. To avoid any confusion, let us rename the latter as \(\ell\), i.e. \(\ell \equiv \lfloor \log 2m^2\rfloor + \text{size}(b_{\text{max}})\). Protocol \(P_{sys}\) has \(|Q| = 3m(\ell + 1) + k + 2\) states. Among these states, one transition is \((m + 1)\)-way and \(k\) transitions are \(\ell\)-way. By applying Lemma 3, we obtain a 2-way population protocol \(P'_{sys}\) which computes the same predicate as \(P_{sys}\) and whose
number of states $|Q'|$ is bounded as follows:

$$|Q'| = |Q| + 3(m + 1) + 3k\ell$$  
(By Lemma [3])

$$= [3m(\ell + 1) + k + 2] + 3(m + 1) + 3k\ell$$  
(by the size of $Q$)

$$= [3m\ell + 3m + k + 2] + [3m + 3] + 3k\ell$$

$$= 3m\ell + 6m + 3k\ell + k + 5$$

$$\leq 3m\ell + 6m\ell + 3k\ell + k\ell + 5k\ell$$

$$= 9m\ell + 9k\ell$$

$$= 9\ell(m + k)$$

By def. of $1\leq 9([\log 2m^2] + \text{size}(b_{\text{max}}))(m + k)$

$$= 9([\log 2 + 2 \log m] + \text{size}(b_{\text{max}}))(m + k)$$

$$= 9([1 + 2 \log m] + \text{size}(b_{\text{max}}))(m + k)$$

$$\leq 9[2 + 2 \log m + \text{size}(b_{\text{max}}))(m + k)$$

$$= 9(2 + 2 \log m + n)(m + k)$$  
(by $n = \text{size}(b_{\text{max}}))$

$$\leq 9(3 \log m + 3n)(m + k)$$

$$= 27(\log m + n)(m + k).$$  
Moreover, the number of leaders of $P'_{\text{sys}}$ is the same as for $P_{\text{sys}}$, namely

$$|L| = 5m\ell + 1 + |\text{rep}(c)|$$  
(by def. of $L$)

$$= 5m[\log 2m^2 + \text{size}(b_{\text{max}})] + 1 + |\text{rep}(c)|$$  
(by def. of $L$)

$$\leq 5m[2 \log m + 2 + \text{size}(b_{\text{max}})] + 1 + |\text{rep}(c)|$$  
(by $[\log 2m^2] \leq 2 \log m + 2$)

$$\leq 5m[2 \log m + 2 + n] + 1 + mn$$  
(by $n = \text{size}(b_{\text{max}})$ and $|\text{rep}(c)| \leq m \cdot \text{size}(b_{\text{max}}))$

$$= [10m \log m + 10m + 5mn] + 1 + mn$$

$$= 10m \log m + 10m + 6mn + 1$$

$$= 10m \log m + 4m + 6m + 6mn + 1$$

$$\leq 10m \log m + 4m \log m + 6mn + 6mn + mn$$

$$= 14m \log m + 13mn$$

$$\leq 14m(\log m + n).$$