Pointed computations and Martin-Löf randomness

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For Barry, for whom the magnificent incomputability of the world was a deeply held belief.
The only response? An attempt at understanding this chaos at a higher order.

Abstract. Schnorr showed that a real $X$ is Martin-Löf random if and only if $K(X \upharpoonright n) \geq n - c$ for some constant $c$ and all $n$, where $K$ denotes the prefix-free complexity function. Fortnow (unpublished) and Nies, Stephan and Terwijn [NST05] observed that the condition $K(X \upharpoonright n) \geq n - c$ can be replaced with $K(X \upharpoonright r_n) \geq r_n - c$, for any fixed increasing computable sequence $(r_n)$, in this characterization. The purpose of this note is to establish the following generalisation of this fact. We show that $X$ is Martin-Löf random if and only if $\exists c \forall n K(X \upharpoonright r_n) \geq r_n - c$, where $(r_n)$ is any fixed pointedly $X$-computable sequence, in the sense that $r_n$ is computable from $X$ in a self-delimiting way, so that at most the first $r_n$ bits of $X$ are queried in the computation of $r_n$. On the other hand, we also show that there are reals $X$ which are very far from being Martin-Löf random, but for which there exists some $X$-computable sequence $(r_n)$ such that $\forall n K(X \upharpoonright r_n) \geq r_n$.

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1 Introduction

A well known result of Schnorr (see Chaitin [Cha75]) is that Martin-Löf’s notion of algorithmic randomness from [ML66] can be expressed in terms of incompressibility with respect to prefix-free machines. In particular, a real \(X\) is Martin-Löf random if and only if \(\exists c \forall n K(X \upharpoonright n) > n - c\), where \(K\) denotes the prefix-free complexity function. The latter condition says that there exists a constant \(c\) such that all the initial segments of \(X\) are \(c\)-incompressible (in a prefix-free sense).

As reported in Downey and Hirschfeldt [DH10, Proposition 6.1.4], Fortnow (unpublished) and Nies, Stephan and Terwijn [NST05] showed that Schnorr’s characterisation remains valid if we replace the condition \(\exists c \forall n K(X \upharpoonright n) > n - c\) with \(\exists c \forall n K(X \upharpoonright r_n) > r_n - c\), where \((r_n)\) is any computable increasing sequence.

In this note we consider the extent to which this fact can be generalised to incomputable increasing sequences \((r_n)\). It is well known that there are reals which are not Martin-Löf random, yet have infinitely many 0-incompressible initial segments. Hence this characterisation does not hold for arbitrary increasing sequences \((r_n)\). We consider the case when \((r_n)\) is computable from \(X\). Our main result is that if \((r_n)\) is computable from \(X\) in a certain restricted way, then \(X\) is Martin-Löf random if and only if \(\exists c \forall n K(X \upharpoonright r_n) > r_n - c\). On the other hand we show how to construct reals \(X\) and \(X\)-computable sequences \((r_n)\) such that the above equivalence fails, so we have \(\forall n K(X \upharpoonright r_n) \geq r_n\) but \(X\) is very far from being Martin-Löf random.

1.1 Pointed computability

Our main result is that if \(X\) computes \((r_n)\) in a certain natural fashion, then \(\exists c \forall n K(X \upharpoonright r_n) > r_n - c\) is a sufficient and necessary condition for the Martin-Löf randomness of \(X\). In this section we formalise the notion of oracle-computability required in order for this equivalence to hold, which we call pointed computability.

A Turing functional \(\Phi\) can be thought of as a machine which takes as inputs a number \(n\) and a program \(\sigma\), and either halts on these inputs producing a number \(\Phi^\sigma(n)\) as output, or else diverges. The consistency of \(\Phi\) requires that if \(\rho_0 \subseteq \rho_1\) and \(\Phi^{\rho_0}(n) \downarrow\), then the computation \(\Phi^{\rho_1}(n)\) is identical to that for \(\Phi^{\rho_0}(n)\), yielding the same output. Then \(\Phi^X(n)\) can be defined as \(\lim_s \Phi^{X \upharpoonright s}(n)\). Without loss of generality, given a Turing functional \(\Phi\), a string \(\rho\) and a number \(n\), we may assume that \(\Phi^\rho(n) \downarrow\) implies \(|\rho| \geq n\) and \(|\rho| \geq \Phi^\rho(n)\). This is a standard convention and it is not hard to see that if \(\Psi^X = Z\) for two reals and a Turing functional \(\Psi\) which might not obey the convention, then there exists a Turing functional \(\Phi\) which does obey the stated convention and for which \(\Phi^X = Z\).

Given a Turing functional \(\Phi\), a number \(n\) and strings \(\rho_0, \rho_1\), the consistency property says that if \(\Phi^{\rho_0}(n) \neq \Phi^{\rho_1}(n)\) then we must have that \(\rho_0 \not| \rho_1\), i.e. the finite oracles differ at a digit which is less than \(\min(|\rho_0|, |\rho_1|)\). The following definition is based on a more stringent consistency requirement.
Definition 1.1 (Pointed computations). Given a real $X$, we say that a sequence $(r_n)$ is pointedly $X$-computable if $(r_n)$ is (strictly) increasing and there exists a Turing functional $\Phi$ such that $\Phi^X_{\downarrow n}(n) \downarrow r_n$ for each $n$.

The latter condition in Definition 1.1 says that some oracle Turing machine computes each $r_n$ from $X$ with oracle-use bounded above by $r_n$. Note that, without loss of generality, we may assume that the oracle-use in this computation is exactly $r_n$.

As a typical example of pointed computations (suppressing the monotonicity requirement for now), consider an oracle machine which starts on input $n$ by reading increasingly longer initial segments of the oracle tape, and eventually stops after $s > n$ steps, with output $s$. If, for a given $X$, such a machine converges for every $n$, then it produces a pointed computation in the sense of Definition 1.1. Another example is the settling time of a non-computable c.e. set $A$: let $r_0 = 0$ and for each $n$ let $r_{n+1}$ be the least number which is larger than $r_n$ and such that $A \uparrow_{r_n} = A \uparrow_{r_{n+1}}$ for all $s \geq r_{n+1}$. Then $(r_i)$ is pointedly $A$-computable and non-computable.

Later we will note that there are weaker notions of computability which suffice for our characterization of Martin-Löf randomness. One such notion is the condition that $K(X \uparrow_{r_n}) = O(1)$ for all $n$ and all $\tau \supseteq X \uparrow_{r_n}$. Note that the latter is the non-uniform version of the notion of Definition 1.1.

1.2 Our results

Our main result is the following, which we prove in Section 2.

Theorem 1.2 (Randomness condition). Suppose that $(r_n)$ is pointedly computable from a real $X$. Then $X$ is Martin-Löf random if and only if $\exists c \forall n K(X \uparrow_{r_n}) > r_n - c$.

It is well-known that there are reals which are not Martin-Löf random, yet have infinitely many incompressible initial segments. Hence Theorem 1.2 does not hold if we simply waive the requirement that $(r_n)$ is pointedly computable from $X$. One may ask, however, if Theorem 1.2 continues to hold if we merely require that $(r_n)$ is computable from $X$ and not that it is pointedly $X$-computable. It is not surprising that the latter question has a negative answer. One way to exhibit an example witnessing this fact, is to construct a real with infinitely many incompressible initial segments, which computes the halting problem and is not Martin-Löf random. Since the prefix-free complexity function is computable from the halting problem $\emptyset'$, given such an oracle $X$ we can effectively find infinitely many $t$ such that $K(X \uparrow_t) \geq t - c$. This gives the following fact.

Proposition 1.3. Suppose that $X$ computes the halting problem, $X$ is not Martin-Löf random and there exists some constant $c$ and infinitely many $n$ such that $K(X \uparrow_n) \geq n - c$. Then $X$ computes an increasing sequence $(r_n)$ such that $K(X \uparrow_{r_n}) \geq r_n - c$ for all $n$. 

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In Section 3 we present two ways of constructing oracles $X$ which have the properties mentioned in Proposition 1.3, thus establishing the following.

**Theorem 1.4.** There exists a real $X$ and an $X$-computable increasing sequence $(r_n)$, such that $r_n < K(X \upharpoonright r_n)$ for all $n$, and $X$ is not Martin-Löf random.

Our first construction of such $X$ involves starting from a Martin-Löf random $Y$ which computes the halting problem, and inserting zeros at certain places, thus causing $X$ to be non-random, while preserving its ability to calculate lengths at which its initial segments have high prefix-free complexity. The second construction of such an oracle $X$ is more flexible, and gives a real which is highly non-random, in the sense that its characteristic sequence has a computable subsequence of zeros.

### 1.3 Related concepts and results from the literature

A central notion studied in this paper is that of a set $X$ which is able to compute a sequence of positions in its binary expansion where the corresponding initial segments are incompressible. Clearly every Martin-Löf random has this property, but there are also many reals with this property which are not Martin-Löf random. This notion might remind some readers of the *autocomplex reals* (see [KHMS06, KHMS11] or [DH10, Section 8.16]), which are the reals $X$ which compute a non-decreasing unbounded function $f$ such that $K(X \upharpoonright n) \geq f(n)$ for all $n$. Moreover, Theorem 1.2 has similarities to a result by Miller and Yu [MY10], which says that if $\sum_i 2^{-g(i)} < \infty$ and $g$ is computable from $X$ with identity oracle-use, then $\exists c \forall n K(X \upharpoonright n) \leq n + g(n) + c$. Note that the latter result can be seen as an extension of the following consequence of the Kraft-Chaitin inequality:

$$\text{if } g \text{ is computable and } \sum_i 2^{-g(i)} < \infty \text{ then there exists a constant } c \text{ such that for all } X \text{ and all } n \text{ we have } K(X \upharpoonright n) \leq n + g(n) + c. \quad (1.3.1)$$

It is clear that the result of Miller and Yu [MY10] is related to its special case (1.3.1) in the same way as our Theorem 1.2 is related to results originally obtained by Fortnow (unpublished) and Nies, Stephan and Terwijn [NST05], discussed earlier.

### 2 Proof of Theorem 1.2

We need to show that if $(r_n)$ is pointedly $X$-computable, then $X$ is Martin-Löf random if and only if $\exists c \forall n K(X \upharpoonright r_n) > r_n - c$.

The ‘only if’ direction in this statement is trivial, so it remains to show the ‘if’ direction. We prove the contrapositive. Assuming that $X$ is not Martin-Löf random and $(r_n)$ is pointedly $X$-computable, we show that for each constant $c$ there exists some $n$ such that $K(X \upharpoonright r_n) \leq r_n - c$. Let $\Phi$ be a Turing functional such that $\Phi^X(n) = r_n$ for all $n$, and such that for every $n, r, \rho$ such
that $\Phi^n(n) \downarrow = r$ we have $\Phi^{i\downarrow}(n) \downarrow$, $|\rho| > n$, $|\rho| \geq r$ and $\Phi^i(i) \downarrow$ for all $i < n$. Let $U$ be the underlying optimal universal prefix-free machine.

We define a prefix-free machine $M$ which makes use of the descriptions of $U$ as follows. For each vector $(\rho, \sigma, n, t)$, such that:

- $\Phi^i(i) \downarrow$ for all $i \leq n + 1$ and $\Phi^i(i) < \Phi^i(i + 1)$ for all $i < n + 1$;

- $t \in (\Phi^i(n), \Phi^i(n + 1))$ and $U(\sigma) = \rho \uparrow i$;

let $\tau$ be the digits of $\rho$ between digit $t$ and digit $\Phi^i(n + 1)$ and note that

$$U(\sigma) * \tau = \rho \uparrow \Phi^i(n + 1), \quad \Phi^{U(\sigma) * \tau}(n + 1) = \Phi^i(n + 1)$$

Let $M$ describe $\rho \uparrow \Phi^i(n + 1)$ with the string $\sigma * \tau$, defining $M(\sigma * \tau) = \rho \uparrow \Phi^i(n + 1)$. This completes the definition of $M$.

It follows immediately from the definition that $M$ is effectively calculable. Next we show that $M$ does not allocate two different strings the same description. Given two identical $M$-descriptions $\sigma_0 * \tau_0 = \sigma_1 * \tau_1$, since $U$ is a prefix-free machine we must have $\sigma_0 = \sigma_1$ and $\tau_0 = \tau_1$. By the construction of $M$, the string described in both cases is then $U(\sigma_0) * \tau_0$. In a similar manner, we may show that $M$ is a prefix-free machine. Suppose that $\sigma_0 * \tau_0 \subseteq \sigma_1 * \tau_1$ are two descriptions issued by $M$, resulting from the vectors $(\rho_0, \sigma_0, n_0, t_0)$, $(\rho_1, \sigma_1, n_1, t_1)$ respectively. Since $U$ is a prefix-free machine and $\sigma_0, \sigma_1$ are $U$-descriptions, we have $\sigma_0 = \sigma_1$, $t_0 = t_1$ and $\tau_0 \subseteq \tau_1$. Let $\sigma$ be $\sigma_0$ and let $t$ be $t_0$. Then $\rho_0 \subseteq \rho_1$ and hence

$$\Phi^{n_0}(n_0 + 1) = \Phi^{U(\sigma) * \tau_0}(n_0 + 1) = \Phi^{U(\sigma) * \tau_1}(n_0 + 1) = \Phi^{n_1}(n_0 + 1). \quad (2.0.1)$$

Thus $(\Phi^{n_0}(n_0), \Phi^{n_0}(n_0 + 1)] = (\Phi^{n_1}(n_0), \Phi^{n_1}(n_0 + 1)]$, and $t$ belongs to both intervals. This means that $n_0 = n_1$, because otherwise $t_1$ would have to belong to a different interval, not the one determined by $n_0$ and $\rho_1$, since the values of $\Phi$ are strictly monotone. This would be a contradiction by the choice of $(\rho_1, \sigma_1, n_1, t_1)$ in the definition of $M$ and the fact that $t_0 = t_1$ which was established earlier. So let $n = n_0 = n_1$. Since $U(\sigma) * \tau_0 = \rho_0 \uparrow \Phi^i(n_0 + 1)$ and $U(\sigma) * \tau_1 = \rho_1 \uparrow \Phi^i(n_0 + 1)$, by $(2.0.1)$ the strings $U(\sigma) * \tau_0$, $U(\sigma) * \tau_1$ have the same length, so $\tau_0 = \tau_1$, which shows that the two descriptions $\sigma_0 * \tau_0$, $\sigma_1 * \tau_1$ are identical. This completes the proof that $M$ is a prefix-free machine.

It remains to show that if $X$ is not Martin-Löf random, then for each $c$ there exists some $n$ with $K(X \uparrow_r) \leq r - c$. Let $d$ be a constant such that $K(\eta) \leq K_M(\eta) + d$ for all strings $\eta$. Given any constant $c$, since $X$ is not Martin-Löf random, there exists some $t > 0$ such that $K(X \uparrow t) \leq t - c - d$. Let $n$ be such that $t \in (r_n, r_{n+1})$. Then $M$ will describe $X \uparrow_{r_{n+1}}$ with $\sigma * \tau$

\footnote{An intuitive description of the argument that follows for showing that $\tau_0 = \tau_1$ is this: each of $\rho_0, \rho_1$ determine a sequence $\Phi^n(i), i \leq n_0 + 1$, although $\rho_1$ may be able to define $\Phi^i(i)$ for $i > n_0 + 1$. However $i \in (\Phi^n(n_0), \Phi^n(n_0 + 1)]$ so by the construction of $M$ we have that both $\tau_0, \tau_1$ equal the digits of $\rho_0$ (or equivalently $\rho_1$) between digit $t$ and digit $\Phi^n(n_0 + 1)$.}
where $\sigma$ is the shortest description of $X \upharpoonright t$ and the length of $\tau$ is $r_{n+1} - t$. The length of $\sigma$ is $K(X \upharpoonright t)$, which is at most $t - c - d$. We have:

$$K_M(X \upharpoonright r_{n+1}) \leq (t - c - d) + (r_{n+1} - t) = r_{n+1} - c - d.$$ 

So $K(X \upharpoonright r_{n+1}) \leq r_{n+1} - c$, as required.

### 3 Proof of Theorem 1.4

As discussed in the introduction, we present two different constructions of a real $X$ which computes an increasing sequence $(r_n)$ such that $K(X \upharpoonright r_n) > r_n$ for all $n$ and $X$ is not Martin-Löf random.

#### 3.1 An ad hoc construction

One way to construct a real $X$ with the property of Proposition 1.3 is to start from a Martin-Löf random real $Y$ which computes the halting problem, and insert zeros into $Y$ in a way that does not change the fact that $0'$ is computable from the resulting oracle, but does ensure non-randomness. Recall that a Martin-Löf random real $Y$ which computes the halting problem exists by the Kučera-Gács theorem [Kuč85, Gác86]. In this construction we use the result of Chaitin [Cha75], which asserts that:

$$\text{if } Z \text{ is Martin-Löf random then } \lim_s (K(Z \upharpoonright s) - s) = \infty. \tag{3.1.1}$$

We also use the fact that for each real $Z$ which is Martin-Löf random and each string $\sigma$, the real $\sigma^* Z$ is Martin-Löf random, and the fact from [Cha87] that:

$$\text{if } Z \text{ is Martin-Löf random and } f \text{ is a partial computable function on strings, then if } f(Z \upharpoonright n) \downarrow \text{ for infinitely many } n, \text{ there are infinitely many } t \text{ such that } f(Z \upharpoonright t) \neq Z(t). \tag{3.1.2}$$

The reader may observe that a partial computable prediction rule $f$ as above which is always successful on $Z$ would give rise to a computable martingale which succeeds on $Z$, which we know is not possible for Martin-Löf random reals (e.g. see [DH10, Section 6.3]).

Let $\Phi$ be a functional via which $Y$ computes the complexity function $K$, i.e. such that $\Phi^Y(\sigma) = K(\sigma)$ for all $\sigma$. We form $X$ from $Y$ by inserting 0s at various positions, in a stage by stage process. The real $X$ is defined as the limit of a sequence $X_s$ and we form each $X_{s+1}$ from $X_s$ by inserting a 0 at position $t_{s+1}$, which means that we define $X_{s+1}(n) = X_s(n)$ for $n < t_{s+1}$, $X_{s+1}(n) = 0$ for $n = t_{s+1}$ and $X_{s+1}(n + 1) = X_s(n)$ for $n \geq t_{s+1}$. At stage 0 we define $X_0 = Y$, and (for convenience) define $t_0 = -1$. Now inductively suppose that we have performed stages $0, \ldots, k$, and that we have recorded $t_0, \ldots, t_k$. For any string $\tau \subset X_k$, let $\tau^*$ be the string which
results from removing all of the 0s that we have inserted during the stages \( \leq k \). At step \( k + 1 \) we search for \( \sigma \in X_k \) of length \( > t_k + 2 \) and \( \tau \in X_k \) with \( \sigma \subset \tau \) such that \( \Phi^\tau \) computes \( K(\sigma) \) and \( K(\sigma) > |\sigma| \). By (3.1.1), it follows that such \( \sigma \) and \( \tau \) exist. Then we define \( t_{k+1} = |\tau| \) and insert a 0 at position \( t_{k+1} \).

This completes the construction of \( X \) given \( Y \). From the construction it follows that \( X \) computes both \( Y \) and the sequence \( (t_k) \). This follows because \( X \) is able to retrace the construction. Inductively suppose that \( X \) has been able to retrace the construction up until the end of stage \( k \), and so knows the values \( t_0, \ldots, t_k \). Then, using the oracle for \( X \) we can perform the same search that was carried out at stage \( k + 1 \), but using \( X \) rather than \( X_k \): we search for \( \sigma \subset X \) of length \( > t_k + 2 \) and \( \tau \subset X \) with \( \sigma \subset \tau \) such that \( \Phi^\tau \) computes \( K(\sigma) \) and \( K(\sigma) > |\sigma| \). Since the next zero is inserted after \( \tau \), the result of the search is the same as when \( X_k \) was used during the construction at stage \( k + 1 \). Then \( t_{k+1} = |\tau| \), completing the induction step. Therefore \( X \) computes the halting set \( \theta' \). Moreover there is a partial computable function \( f \) such that for each \( \sigma \subset X \) we have \( f(\sigma) \downarrow \) if and only if \( \sigma = X \uparrow_{t_k} \) for some \( k \) (\( f \) uses \( \sigma \) as an oracle to try and retrace the construction and converges on \( \sigma \) if it is of length \( t_k \) for some \( k \) in the retraced construction). For each \( k \), the next digit \( t_k \) of \( X \) is a 0, so \( f \) is a partial computable prediction rule that succeeds on \( X \), which means that \( X \) is not Martin-Löf random. This completes the construction of a set \( X \) with the properties of Proposition 1.3.

### 3.2 A refined construction

Here we construct the required \( X \) by finite extensions. This construction can be combined with other requirements. For example, \( X \) can be highly non-random, in the sense that it has a computable sequence of 0s. We need some facts from the theory of prefix-free Kolmogorov complexity. For each string \( \sigma \), let \( \sigma^\tau \) denote the shortest prefix-free description of \( \sigma \) (if there are many shortest descriptions, we consider the one which describes \( \sigma \) first). Also let \( K(\tau | \rho) \) denote the prefix-free complexity of \( \tau \) relative to string \( \rho \). The following is a relativized version of Chaitin’s counting theorem from [Cha75].

**Lemma 3.1** (Relativized counting theorem). There exists a constant \( c \) such that for all \( \sigma, r \) and all \( n > \sigma \):

\[
\left| \left| \tau \right| \sigma \leq \tau \land \tau \in 2^n \wedge K(\tau | \sigma^\tau) \leq |\tau| - r - K(\sigma) \right| \leq 2^{n-K(\sigma)+c-r-K(n | \sigma^\tau)}.
\]

**Proof.** Given \( \sigma \) we define \( F(n | \sigma^\tau) \) for \( n > |\sigma| \) to be the \( - \log \) of the weight of the prefix-free descriptions relative to \( \sigma^\tau \) which describe extensions of \( \sigma \) of length \( n \). Then since the relative prefix-free complexity \( K \) is a minimal information measure, there exists a constant \( c \) such that for all \( n, \sigma \),

\[
2^{-F(n | \sigma^\tau)} < 2^{-K(n | \sigma^\tau)+c}.
\]

(3.2.1)

We claim that the constant \( c \) has the property of the statement of the lemma. For a contradiction, suppose that this is not the case. Then for some \( n \) there are more than \( 2^{n-K(\sigma)+c-r-K(n | \sigma^\tau)} \) many
Corollary 3.2 is the tool we are going to use for our finite extension construction. The probability, for all \( |τ| \), of randomness deficiency sometimes called the \( 2^{n−K(σ)−c−r−K(σ | σ^*)} \).

\[ 2^{−F(n | σ)} > 2^{n−K(σ)+c−r−K(σ | σ^*)} \cdot 2^{−n+r+K(σ)} = 2^{c−K(n | σ^*)}, \]

which contradicts (3.2.1). This contradiction concludes the proof of the lemma.

Recall the symmetry of information fact from [Gác74, Cha75]:

\[ K(τ) + K(σ | τ^*) = K(σ) + K(τ | σ^*) + Ω(1) \]

where \( f = g + Ω(1) \) for two functions \( f, g \) means that \( |f(n) − g(n)| \) is bounded above. Since \( K(τ) = K(σ, τ) + O(1) \) for all strings \( σ, τ \), by the symmetry of information, Lemma 3.1 has the following corollary.

**Corollary 3.2** (Relativized counting, again). There exists a constant \( c \) such that

\[ \left| \left\{ τ \mid σ ⊆ τ ∧ τ ∈ 2^n ∧ K(τ) ≤ |τ| − r \right\} \right| ≤ 2^{n−K(σ)+c−r−K(σ | τ^*)−K(n | σ^*)} \]

for all \( σ, r \) and all \( n > σ \).

Corollary 3.2 is the tool we are going to use for our finite extension construction. The problem we face is, given a string \( σ \) to find an extension \( τ \) such that \( K(τ) > |τ| \). Since there are \( 2^{n−|σ|} \) many extensions of \( σ \) of length \( n \), by Corollary 3.2 it suffices to consider \( n \) such that

\[ 2^{n−K(σ)+c−r−K(σ | σ^*)} < 2^{n−|σ|}, \]

which means that \( |σ| + c < K(σ) + K(n | σ^*) \) so

\[ K(n | σ^*) > |σ| − K(σ) + c. \]

Such a number \( n \) clearly exists in the interval \([|σ|, |σ| + 2^{c+|σ|−K(σ)}]\). The quantity \( |σ| − K(σ) \) is sometimes called the *randomness deficiency* of \( σ \). We have shown that:

There exists a constant \( c \) such that each \( σ \) can be extended by less than \( 2^{c+|σ|} − 1 \) many bits to a string \( τ \) with \( K(τ) > |τ| \).

\[ (3.2.2) \]

We are ready to construct the required real \( X \) by finite extensions

\[ ρ_0 ⊂ τ_1 ⊂ ρ_1 ⊂ τ_2 ⊂ ··· \]

In this construction the lengths \( ℓ_i := |τ_i| \) will be computable while the strings \( ρ_i \) will be chosen so that \( K(ρ_i) > |ρ_i| \) for all \( i \). For each \( i \) the string \( τ_{i+1} \) will be the concatenation of \( ρ_i \) with a string \( 10 \ldots 0 \) such that \( |τ_i| = ℓ_i \). So for each \( i \) the string \( ρ_i \) will be uniformly computable from \( τ_i \); simply find the first \( 1 \) starting from position \( |τ_i| \) in \( τ_i \) and moving to the left, and if this \( 1 \) is at position \( t \) then \( ρ_i = τ_t \). Let \( c \) be the constant in (3.2.2).

Let \( ρ_0 \) be the empty string \( λ \), so that \( K(ρ_0) > |ρ_0| \). Let \( τ_1 \) be the string \( ρ_0 + 10 \) and let \( ℓ_1 = |ρ_0| + 2 = 2 \). Note that the last digit of \( τ_1 \) is a 0. By (3.2.2) there exists an extension \( ρ_1 \) of \( τ_1 \) such that \( |ρ_1| − |τ_1| < 2^{c+ℓ_1} − 1 \) and \( K(ρ_1) > |ρ_1| \). Then let \( τ_2 \) be the concatenation of \( ρ_1 \)
with a string $10\ldots0$ such that the length $\ell_2 := |\tau_2|$ is $\ell_1 + 2^{c+\ell_1}$. Note that $\tau_2$ is longer than $\rho_1$ by at least 2 bits, so the last digit of $\tau_2$ is a 0. Similarly, we can choose an extension $\rho_2$ of $\tau_2$ of less than $2^{c+\ell_2} - 1$ many additional bits such that $K(\rho_2) > |\rho_2|$. As before we let $\tau_3$ be the concatenation of $\rho_2$ with a string $10\ldots0$ such that the length of $\tau_3$ is $\ell_3 := \ell_2 + 2^{c+\ell_2}$. Note that $\tau_3$ is longer than $\rho_2$ by at least 2 bits, so the last digit of $\tau_3$ is a 0.

The construction continues similarly, thus defining the computable sequence of lengths $\ell_{n+1} = \ell_n + 2^{c+\ell_n}$ for each $n > 0$, where $\ell_1 = 2$, and the sequences (3.2.3) such that for all $i > 0$ we have $K(\rho_i) > |\rho_i|$, $|\tau_i| = \ell_i$ and the last digit of $\tau_i$ is 0. If we let $X$ be the infinite extensions of the strings (3.2.3) then we have $X(\ell_i - 1) = 0$ for all $i$, so $X$ is not Martin-Löf random. On the other hand $X \upharpoonright \ell_i = \tau_i$ for all $i$, and since $\tau_i$ uniformly computes $\rho_i$, we have that the sequence $(|\rho_i|)$ is $X$-computable. Finally $K(X \upharpoonright |\rho_i|) = K(\rho_i) > |\rho_i|$ for all $i$, which concludes the verification of the required properties of $X$.

4 Conclusion and discussion

We have generalized the criterion for Martin-Löf randomness by Fortnow (unpublished) and Nies, Stephan and Terwijn [NST05], which says that if $X$ has incompressible segments of a computable sequence of lengths $(r_n)$, then it is Martin-Löf random. We proved that the condition that $(r_n)$ is computable can be replaced by the weaker condition that $(r_n)$ is pointedly $X$-computable, in the sense that $r_n$ is uniformly computable from any extension of $X \upharpoonright r_n$. It is a simple exercise to extend our proof of this fact in order to replace the condition of pointed computation with a non-uniform version of it, namely that $K(r_n \mid \tau) = O(1)$ for all $n$ and all $\tau \supseteq X \upharpoonright r_n$. On the other hand, we showed that this condition is no longer sufficient for Martin-Löf randomness, if we merely require that $r_n$ be computable from $X$. It would be interesting to refine this analysis and find exactly what kind of computations are allowed of a sequence $(r_n)$ from an oracle $X$ such that the Martin-Löf randomness of $X$ is equivalent to the segments $X \upharpoonright r_n$ being incompressible.

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