St-Polyform Modules and Related Concepts

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Abstract:
In this paper, we introduce a new concept named St-polyform modules, and show that the class of St-polyform modules is contained properly in the well-known classes; polyform, strongly essentially quasi-Dedekind and $\kappa$-nonsingular modules. Various properties of such modules are obtained. Another characterization of St-polyform module is given. An existence of St-polyform submodules in certain class of modules is considered. The relationships of St-polyform with some related concepts are investigated. Furthermore, we introduce other new classes which are; St-semisimple and $\kappa$-non St-singular modules, and we verify that the class of St-polyform modules lies between them.

Keywords: $\kappa$-nonsingular modules, Polyform modules, Semi-essential submodules, St-closed submodules, Strongly essentially quasi-Dedekind modules.

Introduction:
Throughout this paper, all rings are assumed to be commutative with a non-zero unity element, and all modules are unitary left $R$-modules. The notations $V \leq_\text{a} U$ and $V \leq_\text{sem} U$ mean that $V$ is an essential and semi-essential submodule of $U$ respectively. A submodule $V$ of $U$ is called essential if every non-zero submodule of $U$ has a non-zero intersection with $V$ (1, P.15). A submodule $V$ of $U$ is called semi-essential if every non-zero prime submodule of $U$ has a non-zero intersection with $V$ (2). A submodule $V$ of $U$ is called closed if $V$ has no proper essential extensions inside $U$ (1, P.18). Ahmed and Abbas introduced the concept of St-closed submodule, where a submodule $V$ of $U$ is said to be St-closed, if $V$ has no proper semi-essential extensions inside $U$ (3).

In this paper, we introduce and study a new class named St-polyform modules. This type of modules is contained properly in some classes of modules such as polyform, strongly essentially quasi-Dedekind and $\kappa$-non St-singular modules. An R-module $U$ is called polyform if for every submodule $V$ of $U$ and for any homomorphism $f: V \rightarrow U$, $\ker f$ is closed submodule in $U$ (4). A module $U$ is called strongly quasi-Dedekind, if $\text{Hom}_R(U, U) = 0$ for all semi-essential submodule $V$ of $U$ (5). An R-module $U$ is called $\kappa$-nonsingular, if for each homomorphism $f \in \text{End}(U)$ such that $\ker f$ is essential submodule of $V$, then $f = 0$ (6, P.95).

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We define in this work a proper class of $\kappa$-nonsingular modules named $\kappa$-non St-singular. We define St-polyform as follows: an R-module $U$ is called St-polyform, if for every submodule $V$ of $U$ and for every homomorphism $f: V \rightarrow U$, $\ker f$ is St-closed submodule in $V$. We verify that an St-polyform module is smaller than all of the classes: polyform, strongly quasi-Dedekind, $\kappa$-nonsingular and $\kappa$-non St-singular modules, see remark 2, proposition 30, proposition 40 and proposition 56. Beside that we give another generalization for St-polyform modules.

This work consists of three sections. In the first section we provide another characterization of St-polyform modules, we show that a module $U$ is St-polyform if and only if for each non-zero submodule $V$ of $U$ and for each non-zero homomorphism $f: V \rightarrow U$, $\ker f$ is not semi-essential submodule of $V$, see theorem 4. Also we present the main properties of St-polyform module, for example we show in proposition 7 the existence of St-polyform in certain class of modules, also we prove in the proposition 11; if $W \leq_\text{sem} V$ for every submodule $V$ of $U$ with $\text{Hom}_R(U, W) = 0$, then $U$ is a St-polyform module, and we show in the proposition 13 that a module $U$ is an St-polyform if its quasi-injective hull is St-polyform. In section two we investigate the relationships of St-polyform with polyform module and small polyform, where a submodule $V$ of $U$ is called small if $V + W \neq U$ for every proper submodule $W$ of $U$ (1, P.20). An R-module $U$ is called small polyform if for each non-
zero small submodule \( V \) of \( U \), and for each \( f \in \text{Hom}_R(V, U) \): \( \ker f \not\simeq V \) (4). Furthermore, we introduce another generalization for St-polyform module named essentially St-polyform module, and we show in theorem 26; the two concepts are equivalent under the class of uniform modules. The last section of this paper is devoted to study the relationships of St-polyform with other related concepts such as quasi-Dedekind and some of its generalizations as well as \( \kappa \)-nonsingular and Baer modules. We show that under certain condition an strongly essentially quasi-Dedekind module can be St-polyform, see theorem 31. Also, we give a partial equivalence between St-polyform and \( \kappa \)-nonsingular modules, see theorem 42. Moreover, other related concepts of St-polyform module are introduced which are St-semisimple, and \( \kappa \)-non St-singular modules.

**St-polyform modules:**

In this section, various properties and anther characterization for St-polyform modules are investigated. We start by the following definition.

**Definition 1:** An \( R \)-module \( U \) is called St-polyform, if for every submodule \( V \) of \( U \) and for any homomorphism \( f : V \rightarrow U \), \( \ker f \) is St-closed submodule in \( V \). A ring \( R \) is called St-polyform, if \( R \) is St-polyform \( R \)-module.

**Remark 2:** The St-polyform module is a proper class of polyform module. In fact if \( U \) is St-polyform module, then for every submodule \( V \) of \( U \) and for any homomorphism \( f : V \rightarrow U \), \( \ker f \) is St-closed submodule in \( V \). Since the class of closed submodule is greater than the class of St-closed submodule, thus \( \ker f \) is closed submodule in \( U \); hence \( U \) is a polyform module. On the other hand, not every polyform module is St-polyform for example; \( Z_5 \) as \( Z \)-module is clearly polyform module, but not St-polyform, since the identity homomorphism \( I : Z_5 \rightarrow Z_2 \) has zero kernel which is not St-closed submodule in \( Z_2 \) (3).

**Examples and Remarks 3:**

i. Simple module is not St-polyform module. The proof is similar as proving \( Z_2 \) is not St-polyform in remark 2.

ii. \( Z_6 \) is not St-polyform module. In fact there exists \( f : \langle 2 \rangle \rightarrow Z_6 \) defined by \( f(\overline{x}) = \overline{2x} \) \( \forall \overline{x} \in \langle 2 \rangle \). Note that \( \ker f = \langle 4 \rangle \), and \( \langle 4 \rangle \) is not St-closed submodule in \( Z_6 \).

iii. Epimorphic image of St-polyform module may not be St-polyform; for example \( Z_{10} \) is St-polyform, while \( \frac{Z_{10}}{\langle 2 \rangle} \cong Z_5 \). By i, \( Z_5 \) is not St-polyform.

iv. Monoform module need not be St-polyform. For example, \( Z_2 \) is a monoform \( Z \)-module, but it is not St-polyform as we seen in remark 2.

v. Uniform may not be St-polyform module, where a non-zero module \( U \) is called uniform if \( U \) every non-zero two submodules of \( U \) have non-zero intersection (1, P.85).

vi. \( Q \) as \( Z \) is not St-polyform. In fact \( Q \) is uniform module, hence it is semi-uniform, and the result follows by v.

vii. \( Z_6 \) is an St-polyform module, since every submodule of \( Z_6 \) is St-closed. So the kernel of any homomorphism from each submodule to \( Z_6 \) is St-closed. For the same argument \( Z_{10} \) is St-polyform.

viii. \( Z_{12} \) is not St-polyform \( Z \)-module.

ix. A submodule of St-polyform module may not be St-polyform, for example; by vii, \( Z_6 \) is an St-polyform module, but \( A = \langle 2 \rangle \subseteq Z_6 \) is not St-polyform, since \( A \) is simple module, which is not St-polyform as we showed in i.

The following theorem gives another characterization of St-polyform module.

**Theorem 4:** An \( R \)-module \( U \) is St-polyform, if and only if for each non-zero submodule \( V \) of \( U \) and for each non-zero homomorphism \( f : V \rightarrow U \), \( \ker f \) is not semi-essential submodule of \( V \).

**Proof:** \( \Rightarrow \) Assume that there exists a non-zero submodule \( V \) of \( U \) and a non-zero homomorphism \( f : V \rightarrow U \) such that \( \ker f \) is semi-essential submodule of \( V \). But \( \ker f \subseteq \text{Stc} V \), therefore \( \ker f = V \), hence \( f = 0 \) which is a contradiction. That is \( \ker f \not\simeq \text{Stem} V \).

\( \Leftarrow \) Suppose that there exists a submodule \( V \) of \( U \) and a homomorphism \( f : V \rightarrow U \) such that \( \ker f \) is not St-closed submodule in \( V \). By definition of St-closed, there exists a submodule \( W \) of \( V \) such that \( \ker f \subseteq \text{Stem} W \subseteq V \). Consider the homomorphism \( f \circ i : W \rightarrow U \). It is clear that \( f \circ i \neq 0 \), and since \( \ker f \subseteq W \), then \( \ker(f \circ i) \subseteq \text{Stem} W \). But this is contradict with our assumption, thus \( \ker f \) is St-closed submodule of \( V \).

The following examples are checked by using theorem 4.

**Examples 5:**

i. Any semi-uniform module is not St-polyform module, where a non-zero \( R \)-module \( U \) is called semi-uniform if every non-zero submodule has non-zero intersection with all prime submodules of \( U \).

**Proof i:** Let \( V \) be a non-zero submodule of \( U \), and \( f : V \rightarrow U \) be a non-zero homomorphism. Assume that \( U \) is St-polyform module, so \( \ker f \not\simeq \text{Stem} V \), hence \( \ker f \not\simeq \text{Stem} U \). But this contradicts the definition of semi-uniform module, thus \( U \) is not St-polyform.

\( \blacksquare \)
ii. Z is not St-polyform Z-module. In fact since Z is semi-uniform module, so the result follows by i.

iii. \(Z_4\) is not St-polyform module. In fact if we take \(V=Z_4\) in the theorem 4 as a submodule of itself, then there exists a homomorphism \(f \in \text{Hom}_R(Z_4,Z_4)\) defined by \(f(x)=2x \quad \forall x \in Z_4\), note that \(\ker f=2\) which is semi-essential submodule of \(Z_4\). Thus \(Z_4\) is not St-polyform module.

iv. \(Z \oplus Z_2\) is not St-polyform Z-module. To show that; assume there exist a submodule \(V=Z \oplus Z_2\) and a homomorphism \(f: V \rightarrow U\) defined by \(f(x,y)=(0,x)\), where \(x \in Z_2, y \in Z_2\).

Note that \(f \neq 0\), and \(\ker f=\{(x,y) \in V | f(x,y)=0\} = \{(x,y) \in V | \bar{x}=0\} = 2Z \oplus Z_2\), hence \(\ker f \not\subseteq \text{sem} V\). So \(Z \oplus Z_2\) is not St-polyform module.

**Proposition 6:** A direct summand of St-polyform module is St-polyform.

**Proof:** Let \(U=U_1 \oplus U_2\) be a St-polyform module, where \(U_1\) and \(U_2\) are R-submodules of \(U\). Let \(V_1\) be a non-zero submodule of \(U_1\), and \(f: V_1 \rightarrow U_1\) be a non-zero homomorphism. Consider the following sequence:

\[ V_1 \rightarrow U_1 \rightarrow U_1 \oplus U_2 \]

where \(j\) is an injection homomorphism. Now, \(j \circ f: V_1 \rightarrow U_1\) and \(U_1\) is St-polyform, then \(\ker (j \circ f) \subseteq \text{sem} V_1\). Since \(\ker (j \circ f) = \{v_1 \in V_1 | (j \circ f)(v_1) = 0\} = \{v_1 \in V_1 | f(v_1) = 0\} = \ker f \oplus U_2\), then \(\ker f \oplus U_2 \subseteq \text{sem} U\). But \(U_2 \subseteq \text{sem} U_2\), thus \(\ker f \not\subseteq \text{sem} U_1\) (5, Lemma(1.18)). That is \(U_1\) is St-polyform.

The converse of proposition 6 is not true in general; for example each of \(Z_{10}\) and \(Z_6\) are St-polyform \(Z\)-modules; see 3iii., but \(Z_{10} \oplus Z_6\) is not St-polyform \(Z\)-module.

Recall that an R-module \(U\) is called Artinian if every descending chain of submodules in \(U\) is stationary (1,P.7). The following proposition indicates the existence of St-polyform submodules in certain class of modules.

**Proposition 7:** Every nonzero Artinian module has a submodule which is an St-polyform.

**Proof:** Let \(U\) be a non-zero Artinian module, and \(V\) be a submodule of \(U\). If \(V\) is St-polyform, then we are done. Otherwise there exists a submodule \(V_1\) of \(V\) and a homomorphism \(f_1: V_1 \rightarrow V\) with \(\ker f_1 \subseteq \text{St} V_1\) and \(\ker f_1 \subseteq \text{St} V_2\) for some proper submodule \(V_2\) of \(V_1\). Now, if \(V_1\) is St-polyform, then we are through, otherwise there exists a submodule \(V_3\) of \(V_2\) and a homomorphism \(f_2: V_3 \rightarrow V_2\) with \(\ker f_2 \subseteq \text{St} V_3\) and \(\ker f_2 \subseteq \text{St} V_4\) for some proper submodule \(V_4\) of \(V_3\). We continue in this manner until we arrive in a finite number of steps at a submodule which is an St-polyform submodule. Otherwise, we have an infinite descending chain \(V \supseteq V_1 \supseteq V_2 \supseteq \ldots\) of submodules of the module \(U\). But this is a contradiction, since \(U\) is Artinian. Therefore \(U\) contains an St-polyform submodule. ■

**Proposition 8:** Let \(U\) be an R-module. If either \(V_1\) or \(V_2\) are St-polyform modules, then \(V_1 \cap V_2\) is St-polyform module.

**Proof:** Assume that \(V_1\) is St-polyform module. Let \(V\) be a non-zero submodule of \(V_1 \cap V_2\), and let \(f: V \rightarrow V_1 \cap V_2\) be a non-zero homomorphism. Consider the following sequence:

\[ V \rightarrow V_1 \cap V_2 \rightarrow V_1 \]

Since \(V_1\) is a St-polyform module, then \(\ker (i \circ f) \not\subseteq \text{sem} V\). But \(\ker f = \ker (i \circ f)\), then \(\ker f \not\subseteq \text{sem} V\). That is \(V_1 \cap V_2\) is a St-polyform module. ■

Recall that an R-module \(U\) is called multiplication if for any \(f \in \text{End}_R(U)\), there exists an \(r \in R\) such that \(f(x) = rx \quad \forall x \in U\), where \(\text{End}_R(U)\) is the endomorphism ring of \(U\) (5).

**Proposition 9:** Let \(U\) be a faithful scalar R-module. Then \(R\) is an St-polyform ring if and only if \(\text{End}_R(U)\) is an St-polyform ring.

**Proof:** Since \(U\) is a faithful scalar module, then \(\text{End}_R(U) \cong R\) (7). So if \(R\) is an St-polyform module, then \(\text{End}_R(U)\) is polyform, and vice versa. ■

An R-module \(U\) is called multiplication for every submodule \(V\) of \(U\) there exists an ideal I of \(R\) such that \(V = IU\) (8, P.200).

**Corollary 10:** Let \(U\) be a finitely generated faithful and multiplication R-module. Then \(R\) is an St-polyform ring if and only if \(\text{End}_R(U)\) is St-polyform module.

**Proof:** Since \(U\) is finitely generated and multiplication, then \(U\) is a scalar module (7), and the result follows by proposition 9. ■

**Proposition 11:** Let \(U\) be an R-module. If \(W \subseteq \text{sem} V\) for every submodule \(V\) of \(U\), such that \(\text{Hom}_R(V/W, U) = 0\), then \(U\) is a St-polyform module.

**Proof:** Assume \(U\) is not St-polyform module, so there exists a submodule \(V\) of \(U\) and a non-zero homomorphism \(\alpha: V \rightarrow U\) such that \(\ker \alpha \subseteq \text{sem} U\).

Define \(\beta: V \rightarrow W / \ker \alpha\) by \(\beta(v+\ker \alpha) = \alpha(v) \quad \forall v\in V\), and \(\beta(v+\ker \alpha) = 0 \quad \forall v+\ker \alpha \not\in \text{ker} \beta\).

We can easily show that \(\beta\) is well defined and homomorphism. Since \(\alpha\) is a non-zero homomorphism, then \(\beta\) is also non-zero, thus \(\text{Hom}_R(V/W, U) \neq 0\). But this contradicts our assumption, therefore \(\ker \beta \not\subseteq \text{sem} U\).

**Proposition 12:** Let \(U\) be an R-module, and \(I\) be an ideal of \(R\) such that \(I \subseteq \text{ann}_R(U)\), then \(U\) is St-polyform R-module if and only if \(U\) is St-polyform \(R/I\) module.

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**Proof:** Assume that \( U \) is an St-polyform \( R \)-module. Since \( I \subseteq \text{ann}_R(U) \), then it can be easily shown that \( \text{Hom}_R(V, U) = \text{Hom}_R(V, U) \) for each submodule \( V \) of \( U \), hence the result follows directly. \( \blacksquare \)

Recall that an \( R \)-module \( U \) is called injective if for every monomorphism \( f: A \to B \) where \( A \) and \( B \) be any \( R \)-modules, and for every homomorphism \( g: A \to U \), there exists a homomorphism \( h: B \to U \) such that \( h \circ f = g \) (8, P.33). A module \( U \) is called quasi-injective if it is \( U \)-injective \( R \)-module (8, P.83). The injective hull (quasi-injective hull) of a module \( U \) is defined as an injective (quasi-injective) module with essential extension of \( U \), it is denoted by \( E(U) \) (respectively \( \overline{U} \)) (8, P.39). Clark and Wisbauer in (9) proved that a module \( U \) is polyform if its quasi-injective hull is polyform. As analogue of that, we have the following result.

**Proposition 13:** Let \( U \) be an \( R \)-module. If the injective hull \( E(U) \) of \( U \) is St-polyform module, then \( U \) is St-polyform module.

**Proof:** Let \( V \) be a non-zero submodule of \( U \), and \( f: V \to U \) be a non-zero homomorphism. Suppose the converse is not true, that is \( \ker f \leq \text{sem} \ V \). Consider the following Fig. 1.

![Diagram](image)

**Figure 1. The diagram of injective the module**

where \( i: V \to E(V) \) and \( j: U \to E(U) \) are the inclusion homomorphisms. Since \( E(U) \) is injective, then there exists a non-zero homomorphism \( g: V \to U \) such that \( g \circ i = j \circ f \). It is clear that \( \ker (g \circ i) \subseteq \ker g \) and \( \ker f = \ker (j \circ f) \). Since \( E(U) \) is an St-polyform module, then \( \ker (g) \leq \text{sem} \ E(V) \). By definition of injective hull \( V \leq E(V) \), hence \( V \leq \text{sem} \ E(V) \), and by our assumption \( \ker f \leq \text{sem} \ V \), then by transitivity of semi-essential submodules \( \ker f \leq \text{sem} \ E(V) \) (2). On the other hand, clearly \( \ker f \leq \ker g \), therefore \( \ker g \leq \text{sem} \ E(V) \) (2), which is a contradiction. Therefore, \( \ker f \not\leq \text{sem} \ V \), i.e. \( V \) is an St-polyform module. \( \blacksquare \)

In example 3ix, we verified that a submodule of St-polyform may not be St-polyform. In the following proposition, we satisfy that under certain conditions.

**Corollary 14:** Let \( U \) be an injective and St-polyform module. If \( V \) is an essential submodule, then \( V \) is St-polyform module.

**Proof:** Since \( V \) is an essential submodule of \( U \), then \( E(V) = E(U) \) (10, Prop(2.22), P.45). But \( U \) is injective module, so \( U = E(U) \). This implies that \( E(V) = U \). Since \( U \) is St-polyform, then \( E(V) \) is St-polyform. The result follows by proposition 13. \( \blacksquare \)

Recall that a module over integral domain \( R \) is called divisible if \( U = E(U) \) (10, P.32).

**Corollary 15:** Let \( R \) be a division ring, and \( U \) be an St-polyform \( R \)-module. If \( V \) is essential submodule of \( U \), then \( V \) is an St-polyform module.

**Proof:** Since \( R \) is a division ring, then \( U \) is an injective module (10, P.30), and the result follows by corollary 14. \( \blacksquare \)

**Corollary 16:** If \( R \) is a division St-polyform ring, then each ideal of \( R \) is an St-polyform.

**Proof:** Let \( I \) be an ideal of \( R \). Since \( R \) is a division ring, then clearly every ideal of \( R \) is essential. On the other hand, since every module over division ring is an injective module (10, P.30), therefore \( I \) is injective. But \( R \) is an St-polyform ring, so by corollary 14, \( I \) is a St-polyform ideal. \( \blacksquare \)

**Corollary 17:** Let \( U \) be a divisible St-polyform module over P.I.D. If \( V \) is an essential submodule of \( U \), then \( V \) is St-polyform module.

**Proof:** Since \( U \) is divisible over P.I.D, then \( U \) is injective (10, Th(2.8), P.35). The result follows by corollary 14. \( \blacksquare \)

Recall that a commutative domain \( R \) is called Dedekind; if every non-zero ideal of \( R \) is invertible (10, P.36).

**Corollary 18:** Let \( U \) be a divisible module over Dedekind domain \( R \), and \( V \leq e U \). If \( U \) is a St-polyform module, then \( V \) is St-polyform.

**Proof:** Since Every divisible module over a Dedekind domain is injective (10, P.36), then by corollary 14, we are done. \( \blacksquare \)

**St-polyform and Polyform modules:**

In this section, we investigate the relationships of St-polyform module with polyform and small polyform modules. Besides that, we introduce another generalization for St-polyform modules.

In the previous section, we verified that the class of St-polyform modules is a proper subclass of polyform modules. In the following theorems, we use certain conditions under which St-polyform module can be polyform module. Before that; an \( R \)-module \( U \) is called fully prime if every proper submodule of \( U \) is prime (2).

**Theorem 19:** Let \( U \) be a fully prime \( R \)-module, then \( U \) is St-polyform if and only if \( U \) is a polyform module.

**Proof:** \( \Leftarrow \) By remark 2.

\( \Rightarrow \) Assume that \( U \) is polyform module, and let \( V \) be a submodule of \( U \), and \( f: V \to U \) be a
homomorphism. Since U is polyform, then \( \ker f \) is closed submodule in U. But U is fully prime, then \( \ker f \) is an St-closed in U \((3)\), hence U is St-polyform. \[\]

Recall that an R-module U is called fully essential, if every semi-essential submodule of U is essential \((2)\).

**Theorem 20:** Let U be a fully essential R-module, then U is St-polyform if and only if U is a polyform module.

**Proof:** \(\Rightarrow\) By remark 2. \\
\(\Leftarrow\) Let V be a non-zero submodule of U, and \( f : V \rightarrow U \) be a non-zero homomorphism. Since U is polyform, then \( \ker f \not\subseteq V \). But U is fully essential; therefore, \( \ker f \not\subseteq V \) \((2)\), that is U is St-polyform module.

The following proposition shows that the class of St-polyform domain coincides with the class of polyform domain.

**Theorem 21:** An integral domain \( R \) is an St-polyform if and only if \( R \) is polyform domain.

**Proof:** \(\Rightarrow\) It is obvious. \\
\(\Leftarrow\) Assume that \( R \) is a polyform domain. Let \( I \) be a non-zero ideal of \( R \), and \( f : I \rightarrow R \) be a non-zero homomorphism. Since \( R \) is integral domain, then \( \text{ann}(I)=0 \); that is \( \text{ann}_R(I) \not\subseteq R \). Thus \( R \) is St-polyform.

Hadi and Marhoon in \((4)\) gave a generalization of polyform module as follows:

**Definition 22:** An R-module U is called small polyform module if for each non-zero small submodule \( V \) of \( U \), and for each non-zero homomorphism \( f : V \rightarrow U \); \( \ker f \not\subseteq V \).

**Remark 23:** Every St-polyform module is small polyform.

**Proof:** Since every St-polyform module is polyform, so the result follows directly. \[\]

Now, we need to introduce another class of polyform modules which is bigger than polyform modules.

**Definition 24:** An R module U is called essentially polyform module if for each non-zero proper essential submodule \( V \) of \( U \), and for each non-zero homomorphism \( f : V \rightarrow U \); \( \ker f \not\subseteq U \).

We can generalize St-polyform as follows:

**Definition 25:** An R module U is called essentially St-polyform module if for each non-zero proper essential submodule \( V \) of \( U \), and for each non-zero homomorphism \( f : V \rightarrow U \); \( \ker f \not\subseteq V \).

It is clear that every St-polyform module is essentially St-polyform, and every essentially St-polyform module is essentially polyform module. Furthermore, it should be noted that the polyform module lies between St-polyform and essentially St-polyform module.

The following theorem gives a partial equivalence between St-polyform and essentially St-polyform module.

**Theorem 26:** Let U be a uniform module, then U is St-polyform if and only if U is essentially St-polyform.

**Proof:** \(\Rightarrow\) It is straightforward. \\
\(\Leftarrow\) Assume that U is essentially St-polyform, and let \( V \) be a non-zero submodule of U, and \( f : V \rightarrow U \) be a non-zero homomorphism. Since U is a uniform module so \( V \leq u \). But U is essentially St-polyform; therefore, \( \ker f \not\subseteq V \); that is U is an St-polyform module.

By replacing uniform module by hollow and essential submodule by small, we have the following; and the proof is in a similar way.

**Proposition 27:** Let U be a hollow module, then U is St-polyform if and only if U is small St-polyform.

We can summarize the main results of this section by the following implications of modules:

\[\text{St-polyform} \Rightarrow \text{Polyform} \Rightarrow \text{Essentially St-polyform}\]

\[\text{Essentially St-polyform} \Downarrow \text{Essentially polyform}\]

**St-polyform and other related concepts:**

This section is devoted to study the relationships of St-polyform with some related concepts such as quasi-Dedekind and some of its generalizations, \( \kappa \)-nonsingular, injective, extending, Baer and \( \kappa \)-non St-singular module.

Recall that an R-module U is called quasi-Dedekind, if for every non-zero homomorphism \( f \in \text{End}(U) \), \( \ker f=0 \) \((11)\).

**Remark 28:** It is worth mentioning that St-polyform modules and quasi-Dedekind modules are independent; for example the \( Z \)-module \( Z_6 \) is St-polyform module see example 3vii, but not quasi-Dedekind. On the other hand, \( Z \) is quasi-Dedekind \((11)\), but not St-polyform, see example 5ii.

**Proposition 29:** Let U be a semi-uniform module. If U is St-polyform then U is a quasi-Dedekind module.

**Proof:** Assume that U is St-polyform module, and let \( f \in \text{End}(U) \). If \( V \) be a non-zero submodule of U, then we have the following sequence:

\[ V \rightarrow U \rightarrow U \]

Where \( i \) is the inclusion homomorphism. Suppose that \( \ker f \neq 0 \), since U is St-polyform. Note that \( f \circ i \neq 0 \). Since U is St-polyform, then \( \ker(i \circ f) \not\subseteq U \), hence \( \ker(i \circ f) \not\subseteq V \). But this is a contradiction since U is a semi-uniform module, thus \( \ker f = 0 \). \[\]
The converse of proposition 29 is not true in general, for example $Z_2$ is a quasi-Dedekind module, but not St-polyform.

Recall that an $R$-module $U$ is called strongly essentially quasi-Dedekind if for each non-zero homomorphism $f \in \text{End}_R(U)$, $\ker f \not\leq_{\text{sem}} U$ (5).

**Proposition 30**: Every St-polyform module is strongly essentially quasi-Dedekind.

**Proof**: Let $U$ be St-polyform module. Let $V$ be a non-zero submodule of $U$, and $f: V \to U$ be a non-zero homomorphism. By assumption $\ker f$ is not semi-essential submodule in $V$. In particular, all non-zero endomorphisms of $U$ have kernels which are not semi-essential in $U$, proving our assertion.

The converse of proposition 30 is not true in general, for example $Z_2$ is strongly essentially quasi-Dedekind module (5, Ex (1.11)) but not St-polyform as we saw in remark 2. In the following theorem we use a condition under which the converse is true.

**Proposition 31**: Let $U$ be a quasi-injective $R$-module then $U$ is $U$ is St-polyform if and only if $U$ is a strongly essentially quasi-Dedekind module.

**Proof**: (⇒) By proposition 30.

(⇐) Let $V$ be a non-zero submodule of $U$, and $f: V \to U$ be a non-zero homomorphism. Consider the following Fig. 2.

![Figure 2. The diagram of injective module U](image)

where $i: V \to U$ is the inclusion homomorphism. Since $U$ is quasi-injective, then there exists a homomorphism $g: U \to U$ such that $g \circ i = f$. Now, $g \in \text{End}(U)$ and $U$ is essentially quasi-Dedekind; therefore, $\ker g \not\leq_{\text{sem}} U$. But $\ker f \subseteq \ker g$, then by transitivity of semi-essential submodule, $\ker f \not\leq_{\text{sem}} U$ (2), and we are done.

In (3) Ahmed and Abbas proved that if every submodule of $U$ is St-closed, then every submodule of $U$ is direct summand. This motivates us to introduce the following.

**Definition 32**: An $R$ module $U$ is called $St$-semisimple if every submodule of $U$ is St-closed.

This concept is clearly a proper subclass of semisimple modules, and we can easily prove the following.

**Remark 33**: Every $St$-semisimple module is $St$-polyform module.

We think that the converse of the remark 33 is not true in general, but we cannot find example.

**Definition 34**: Let $U$ be an $R$-module. We define $St$-singular submodule as follows:

$$\{u \in U \mid \text{ann}_R(u) \leq_{\text{sem}} R\}$$

It is denoted by $\text{St-sing}(U)$. If $\text{St-sing}(U) = U$, then $U$ is called $St$-singular module, and $U$ is called non $St$-singular if $\text{St-sing}(U) = 0$.

**Example 35**: $Q$ as $Z$-module is non $St$-singular, where $Q$ is the set of all rational numbers, since $\text{St-sing}(Q) = 0$. For the same reason $Z$ is non $St$-singular $Z$-module.

**Proposition 36**: Let $U$ and $V$ be $R$-modules. If $\text{Hom}_R(V,U)=0$ for each $St$-singular module $V$, then $U$ is non $St$-singular module.

**Proof**: Consider the inclusion homomorphism $i: \text{St-sing}(U) \to U$. It is clear that $\text{St-sing}(U)$ is $St$-singular module, so by assumption $i=0$. But $i(\text{St-sing}(U)) = \text{St-sing}(U)$, therefore sing $U=0$. That is $U$ is non $St$-singular.

**Remark 37**: For any submodule $V$ of an $R$-module $U$, $\text{St-sing}(V)=\text{St-sing}(U)\cap V$.

**Proof**: It is clear that $\text{St-sing}(V) \subseteq \text{St-sing}(U)$, so the result follows directly.

**Remark 38**: By using remark 37, we can easily show that the class of $St$-singular module is closed under submodules.

Recall that an $R$-module $U$ is called $\kappa$-nonsingular, if for each $f \in \text{End}_R(U)$, $\ker f \not\leq e U$, then $f=0$ (6, P.95). In other words, for every non-zero homomorphism $f \in \text{End}_R(U)$, $\ker f \not\leq e U$. As example for this class of modules is $Z$-module $Z_p$, it is $\kappa$-nonsingular for every prime number $P$, since $Z_p$ is a simple module; therefore, all non-zero endomorphisms are automorphisms.

**Remark 39**: The concept of $\kappa$-nonsingularity is strictly weaker than the concept of nonsingularity for modules (6, Ex(4.1.10), P.96), where an $R$-module $U$ is called nonsingular if $Z(U)=0$, where $Z(U)\subseteq \{u \in U \mid \text{ann}_R(u) \leq e R\}$ (1, P.30).

**Proposition 40**: Every $St$-polyform module is $\kappa$-nonsingular.

**Proof**: Let $U$ be $St$-polyform module. Let $V$ be a non-zero submodule of $U$, and $f: V \to U$ be a non-zero homomorphism. By assumption, $\ker f \not\leq e V$. As we take $V=U$, then we obtain $f: U \to U$, and $\ker f \not\leq e U$. Since every essential submodule is semi-essential (2), then $\ker f \not\leq e U$, hence $U$ is $\kappa$-nonsingular.

The converse of proposition 40 is not true in general as the following examples show:
**Examples 41:**

1. Every simple module is $\kappa$-nonsingular (12), but not St-polyform, see example 3i.
2. The Z-module Q is nonsingular module, hence it is $\kappa$-nonsingular (12). But Q is not St-polyform module, see example 3vi.
3. The Z-module $U = Q \oplus Z_2$ is not St-polyform module. In fact if $V = Z \oplus (0)$ be a non-zero submodule of U. Let $f : V \to U$ be a map defined by $f(x, 0) = (0, x)$, where $x \in Z$. It is clear that $f$ is a non-zero homomorphism, then $\ker f = \{(x, 0) \in V \mid f(x, 0) = (0, 0)\} = 2Z \oplus (0)$. We can easily verify that $2Z \oplus (0) \leq_{\text{sem}} V$, hence U is not St-polyform module. On the other hand, U is $\kappa$-nonsingular Z-module (12).

The following proposition gives a partial equivalence between St-polyform and $\kappa$-nonsingular modules.

**Theorem 42:** Let $U$ be a fully essential quasi-injective module, then $U$ is St-polyform if and only if $U$ is $\kappa$-nonsingular provided that $\text{Hom}_R(V, U) \neq 0$.

**Proof:** Let $U$ be a fully essential module. If $U$ is $\kappa$-nonsingular, consider the following Fig. 3.

![Diagram](image_url)

where $i : V \to U$ is the inclusion homomorphism. Since $U$ is quasi-injective, then there exists a homomorphism $g : U \to U$ such that $g \circ i = f$. Now, $g \in \text{End}_R(U)$ and $U$ is $\kappa$-nonsingular, thus $g \leq \kappa U$. But $\ker f \leq \ker f$, thus $f \leq \kappa U$. Since $U$ is fully prime, then $f \leq \kappa_{\text{st}} U$.

**Corollary 43:** Let $U$ be a fully prime injective module. Then $U$ is an St-polyform module if and only if $U$ is $\kappa$-nonsingular.

**Proof:** Since every fully prime module is fully essential (2), and $\text{End}_R(U) \neq 0$, then the result follows by theorem 42.

**Lemma 44:** (11) If $U$ is an injective module, then $J(\text{End}_R(U)) = \{f \in \text{End}_R(U) \mid \ker f \leq \kappa U\}$.

**Corollary 45:** Let $U$ be a fully essential module. If $U$ is injective and $J(\text{End}_R(U)) = 0$, then $U$ is St-polyform.

**Proof:** Since $J(\text{End}_R(U)) = 0$, then it is clear that $U$ is $\kappa$-nonsingular. Since $\text{End}_R(U) \neq 0$, then by theorem 42 we are done.

The following theorem gives some useful relationships of St-polyform ring with some related concepts. Before that, we need the following characterization of essential submodules.

**Lemma 46:** (10, P.40) Let $U$ be an R-module. A submodule V of U is essential, if $\forall 0 \neq u \in U$, there exists $r \in R$ such that $0 \neq ru \in V$.

**Theorem 47:** Let $R$ be a fully essential quasi-injective ring. Consider the following statements:

1. $R$ is an St-semisimple ring
2. $R$ is an St-polyform ring.
3. $R$ is a $\kappa$-nonsingular ring.
4. $R$ is a polyform ring.
5. $R$ is a semiprime ring.
6. $R$ is a nonsingular ring.

Then: $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4), (5) \Rightarrow (4), (5) \Rightarrow (6) \Rightarrow (3), (6) \Rightarrow (4)$ and $(5) \Rightarrow (2)$.

**Proof:**

(2) $\Rightarrow (3)$ Since $U$ is fully essential quasi-injective, then by theorem 42 we are done.

(4) $\Rightarrow (3)$ By remark 39.

(6) $\Rightarrow (4)$ (6, P.95).

(5) $\Rightarrow (2)$ Assume that $R$ is not $\kappa$-nonsingular ring, so there exists a non-zero homomorphism $\varphi \in \text{End}_R(R)$ with $\ker \varphi \leq \text{sem} R$. If $\varphi \neq 0$ then there exists $0 \neq x \in R$ such that $\varphi(x) = tx \forall t \in R$. By lemma 46 there exists $0 \neq k \in R$ such that $0 \neq k \in \ker \varphi$. This implies that $0 = \varphi(xk) = x^2k$, hence $(xk)^2 = 0$. But $R$ is semiprime, therefore $xk = 0$ which is a contradiction. Thus $\varphi = 0$.

(5) $\Rightarrow (6)$ (1, Prop.1.27), P.35.

(6) $\Rightarrow (3)$ By remark 39.

(5) $\Rightarrow (2)$ Assume that $R$ is not St-polyform ring, so for each non-zero ideal I of R, there exists a homomorphism $f : I \to R$ such that $\ker f \leq \text{sem} R$. Since R is fully essential ring, then $\ker f \leq \kappa R$. The remain steps of the proof are similar of the direction $(5) \Rightarrow (3)$.

An R-module U is called extending, if every closed submodule of U is direct summand of U (8, P.118).

**Proposition 48:** Let $U$ be a fully essential module. If $U$ is an extending module, then $U$ is St-polyform module.

**Proof:** Let $0 \neq V \subseteq U$ and $f : V \to U$ be a non-zero homomorphism. Since U is an extending module, then $\ker f \leq \kappa U$, hence $\ker f \leq \kappa V$ (1, Prop.1.5, P.18). But $U$ is fully essential, thus $\ker f \leq_{\text{st}} U$, so we are done.

We need to give the following definition.
**Definition 49** (6, P.94): An R-module U is called Baer, if for every submodule V of U, \( \text{ann}_U(V) = \{f \} \), where \( f^2 = f \in \text{End}_R(U) \).

In order to verify the relation of St-polyform with Baer module, we need to introduce the following proposition.

**Proposition 50:** Every Baer quasi-injective module is polyform.

**Proof:** Let V be a non-zero submodule of U, and \( f: V \rightarrow U \) be a non-zero homomorphism. Suppose the converse; that is \( \ker f \leq e \cdot V \). Consider the following Fig. 4:

![Diagram](image)

**Figure 4. The diagram of injective module U**

where \( i: V \rightarrow U \) is the inclusion homomorphism. Since U is quasi-injective, then there exists a homomorphism \( g: U \rightarrow V \) such that \( g \circ i = f \). Now, \( g \in \text{End}_R(U) \) and U is Baer, so \( \ker g = \text{ann}_U g = e \cdot U \).\( e = e \cdot U \) and \( S = \text{End}_R(U) \). This implies that \( \ker g \) is a direct summand of \( U \). Since \( \ker (i \circ g) \leq \ker g \), then \( \ker f = \text{ann}_U f = e \cdot V \), and \( S = \text{End}_R(U) \). This implies that \( \ker g \) is a direct summand of \( V \). On the other hand, \( \ker f \leq e \cdot V \), therefore \( \ker f = e \cdot V \), hence \( f = 0 \) which is a contradiction with assumption, thus \( \ker f \leq e \cdot V \).

**Corollary 51:** For a fully prime (or fully essential) module, every Baer quasi-injective module is St-polyform.

**Proof:** Since in the class of fully prime (or fully essential) modules the concept of essential submodules coincides with the concept of semi-essential, so the proof is in similar of the proposition 50.

**Proposition 52:** Let U be an extending module. If U is St-polyform, then U is a Baer module.

**Proof:** Since U is St-polyform, then by proposition 40, U is \( \kappa \)-nonsingular. On the other hand, U is extending, so U is a Baer module (6, Lemma(4.1.17), P.97).

**Theorem 53:** Let U be an quasi-injective module. Consider the following statements:

1. U is an St-polyform module.
2. U is a \( \kappa \)-nonsingular module.
3. U is a Baer module.
4. U is a polyform module.

Then: (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3) \( \Leftrightarrow \) (4), and if U is fully prime then (4) \( \Rightarrow \) (1).

**Proof:** (1) \( \Rightarrow \) (2) By proposition 40.

(2) \( \Rightarrow \) (3) Since U is quasi-injective, so clearly U is extending. But U is St-polyform, thus U is a Baer module (6, Lemma(4.1.17), P.97).

(3) \( \Leftrightarrow \) (4) Since U is Baer and quasi-injective, then by proposition 50, U is polyform. Conversely: Since U is polyform, then U is \( \kappa \)-nonsingular (6, Prop(4.1.5), P.95). But U is quasi-injective; therefore, U is extending. So U is \( \kappa \)-nonsingular and extending, this implies that U is a Baer module (6, Lemma(4.1.17), P.97).

(4) \( \Rightarrow \) (1) Since U is fully prime, then by theorem 20, we are done.

Now we introduce a subclass of \( \kappa \)-nonsingular module.

**Definition 54:** An R-module U is called \( \kappa \)-non St-singular, if for any non-zero homomorphism \( f \in \text{End}_R(U) \) \( \ker f \leq \text{sem}_V \), then \( f = 0 \). In other words, for every non-zero homomorphism \( f \in \text{End}_R(U) \) \( \ker f \leq \text{sem} U \).

**Remark 55:** Every \( \kappa \)-non St-singular R-module is \( \kappa \)-nonsingular.

**Proof:** Let \( f \in \text{End}_R(U) \) be a non-zero homomorphism. Since U is a \( \kappa \)-non St-singular module, then \( \ker f \leq \text{sem} U \), hence \( \ker f \leq \text{sem} U \). Thus U is \( \kappa \)-non St-singular module.

The converse of remark 55 is true under certain condition as the following proposition shows.

**Proposition 56:** Let U be a fully essential module, then U is \( \kappa \)-non St-singular module if and only if U is \( \kappa \)-nonsingular.

**Proof:** \( \Rightarrow \) By remark 55.

\( \Leftarrow \) Assume that U is a \( \kappa \)-nonsingular module. Let V be a non-zero submodule of U, and \( f \in \text{End}_R(U) \) be a non-zero homomorphism, so \( \ker f \leq \text{sem} V \). Since U is a fully essential module, then \( \ker f \leq \text{sem} V \) and we are done.

**Proposition 57:** Every St-polyform module is \( \kappa \)-non St-singular module.

**Proof:** It is similar of the proof of the proposition (40), but in this proposition we use the transitive property of semi-essential submodules (2), instead of the generalized property of semi-essential submodules.

We end this work by the following.

**Remark 58:** We can summarize the main results which were introduced in last section about the relationships of the St-polyform module with related concepts as follows:

St-polyform \( \Rightarrow \) strongly essentially quasi-Dedekind

St-polyform \( \Rightarrow \)polyform \( \Rightarrow \)\( \kappa \)-nonsingular
مقاس بوليفورم من النمط St- والمفاهيم ذات العلاقة

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الخلاصة:
في هذا البحث قمنا بطرح نوع جديد من المفاهيم أطلقنا عليه اسم مقاس بوليفورم من النمط St- والذي برهنا أنه محتوى فعلياً في بعض أصناف المقاسات المعروفة، مثل مقاس بوليفورم، مقاس كواسي ديدكند واسع بقوة والمقياس غير الشاذ من النمط 𝜅-، فقنا بالتحقق في هذا البحث من مجموعة من الخواص الأساسية لمقاس بوليفورم من النمط St-، وأعطينا تشخيصاً آخر له. كما تم البرهنة على وجود مقاس بوليفورم من النمط St- كمقاس جزئي في أصناف معينة من المقاسات، وكذلك درسنا علاقة المقاس بوليفورم من النمط St-، الذي ينتمي إلى النمط St-، بالمقياس الشبه البسيط من النمط St-، ونبرهنا أن المقاس بوليفورم من النمط St- يقع بينهما.

الكلمات المفتاحية: المقياسات غير الشاذة من النمط 𝜅-, المقياسات الجزئية شبه الواسعة، المقياسات الجزئية المغلقة من النمط St-، المقياسات الجزئية شبه الديدكندية الواسعة بقوة.