Diffieties and Liouvillian Systems

Abdelkader Chelouah*

Laboratoire d’Informatique (LIX)
École Polytechnique
91128 Palaiseau Cedex, France
chelouah@lix.polytechnique.fr

Abstract

Liouvillian systems were initially introduced in [1] and can be seen as a natural extension of differential flat systems. Many physical non flat systems seem to be Liouvillian (cf [12, 11, 1, 10]). We present in this paper an alternative definition to this class of systems using the language of diffieties and infinite prolongation theory.

1 Introduction

Liouvillian systems were initially defined in the differential algebra setting. We give here a new formulation using the language of diffieties and infinite dimensional geometries. This mathematical framework is well suited to study Liouvillian systems. Recall that one of the main property of flat systems is that the variables of the system (state, inputs) can be directly expressed, without any integration of differential equations, in terms of the flat output and a finite number of its time derivative. Liouvillian systems share a similar property. To be able to derive the trajectories of a Liouvillian system, we also need some elementary integrations called quadratures. This can be illustrate through the following academic example

\[
\begin{align*}
\dot{x}_1 &= x_2 + x_4^2, \\
\dot{x}_2 &= x_3, \\
\dot{x}_3 &= u.
\end{align*}
\] (1)

It is quite easy to show that (1) is flat for \(i = 1, 2\) and not flat for \(i = 3\) (2). However, for \(i = 3\), the subsystem

\[
\begin{align*}
\dot{x}_2 &= x_3, \\
\dot{x}_3 &= u.
\end{align*}
\] (2)

*The author would like to thank Prof. Michel Fliess for helpful discussions and his constant support during this work.
is flat with a flat output \( y = x_2 \) and the trajectory of \( x_1 \) can be obtained by mean of an elementary integration \( x_1 = \int y + y^2 \).

For the sake of convenience, we first recall, in sections 2 and 3, some facts concerning the theory of diffieties and the Lie-Bäcklund approach to equivalence and flatness (cf [4, 5, 6, 7, 13]). In section 4, we define Liouvillian systems using the language of diffieties. Finally, we illustrate the class of Liouvillian through the concrete case of rolling bodies ([1], [10], [9]).

2 The language of diffieties

Let \( I \) be a countable set of cardinality \( \ell \), which may be finite or not, and \( \mathbb{R}^I \) the linear space of all real-valued functions \( x = (x^i) \) on \( I \). The space \( \mathbb{R}^I \) has the natural topology of the Euclidean space if \( I \) is finite and the Fréchet topology otherwise. The elements \( x^i, i \in I \), are called coordinates. For an open set \( U \subset \mathbb{R}^I \) we denote by \( C^\infty(U) \) the space of all real-valued functions on \( U \) that depend on finitely many coordinates and are smooth as functions of a finite number of variables. A chart on a set \( M \) is a 3-tuple \((U, \varphi, \mathbb{R}^I)\), where \( U \) is a subset of \( M \), \( \varphi \) is a bijection of \( U \) onto an open subset \( \varphi(U) \). The notions of smooth charts and smooth atlases can be defined as in the finite dimensional case. The set \( M \), equipped with an equivalence class of smooth atlases, is called a \( C^\infty \mathbb{R}^I \)-manifold. The number \( \ell \) does not depend on a chart \((U, \varphi, \mathbb{R}^I)\) and is called the dimension of the smooth manifold \( M \).

A diffiety is a pair \( \mathcal{M} = (M, CTM) \) where \( M \) is a \( C^\infty \mathbb{R}^I \)-manifold and \( CTM \) a finite dimensional involutive distribution on \( M \). The distribution \( CTM \) is called a Cartan distribution and its dimension the Cartan dimension of \( \mathcal{M} \). Local smooth sections of \( CTM \) are called Cartan fields. We are only concerned here with the case of ordinary diffieties, i.e., the dimension of \( CTM \) is equal to 1. For the sake of convenience, we use without distinction the notations \((M, CTM)\) and \((M, \partial_M)\) to denote the ordinary diffiety \( \mathcal{M} \), where \( \partial_M \) is a basis vector field of \( CTM \). Let \( \mathcal{M} = (M, CTM) \) be a diffiety with \( \dim CTM = 1 \). Let \((U, \varphi, \mathbb{R}^I)\) be a chart on \( M \) and \( \partial_M \) be a basis vector field of \( CTM \) on \( U \), then the 4-tuple \((U, \varphi, \mathbb{R}^I, \partial_M)\) is called a chart on \( \mathcal{M} \). We denote by \( \ker \partial_M \) the kernel of the linear map \( \partial_M : C^\infty(M) \to C^\infty(M) \), i.e.,

\[
\ker \partial_M = \{ \vartheta \in C^\infty(M) \mid \partial_M \vartheta = 0 \}.
\]

A real-valued \( C^\infty \) function \( \vartheta \) on \( M \) such that \( \vartheta \in \ker \partial_M \) is called a local first integral on \( \mathcal{M} \). A local first integral is said to be trivial if it is a constant (\( \theta \)). Let \( \phi : M \to N \) be a smooth mapping. As usual, we denote by \( \phi_* : TM \to TN \) the differential (or tangent) mapping of \( \phi \), where \( TM \) (resp. \( TN \)) is the tangent
bundle of $M$ (resp. $N$), and by $\phi^*: T^*N \to T^*M$ the dual differential mapping of $\phi$, i.e., the dual mapping of $\phi_*$, where $T^*M$ (resp. $T^*N$) cotangent bundle of $M$ (resp. $N$).

A smooth mapping $\phi: M \to N$ is called a Lie-Bäcklund morphism of a diffeity $\mathcal{M} = (M, CTM)$ into a diffeity $\mathcal{N} = (N, CTN)$, written $\phi: \mathcal{M} \to \mathcal{N}$, if it is compatible with the Cartan distributions $CTM$ and $CTN$, i.e., $\phi_*(CTM) \subset CTN$.

Example 2.1. Denote by $\mathbb{R}^m_\infty = \mathbb{R}^m \times \mathbb{R}^m \times \cdots$ the product of a countably infinite number of copies of $\mathbb{R}^m$. Consider the ordinary diffeity $\mathcal{F} = (F, CTF)$, where $F = \mathbb{R} \times \mathbb{R}^m_\infty$, and let $(U, \varphi, \mathbb{R} \times \mathbb{R}^m_\infty, \partial_F)$ be a chart on $\mathcal{F}$ with local coordinates $\{t, w_i^{(\nu)} | i = 1, \ldots, m; \nu \geq 0\}$ and basis Cartan field

$$\partial_F = \frac{\partial}{\partial t} + \sum_{i=1}^m \sum_{\nu \geq 0} w_i^{(\nu+1)} \frac{\partial}{\partial w_i^{(\nu)}}.$$ 

The diffeity $\mathcal{F}$, as above defined, is usually called trivial diffeity and plays a central role in the Lie-Bäcklund approach of flatness. On some occasions, we will use the short notation

$$\partial_F = \frac{\partial}{\partial t} + \sum_{\nu \geq 0} w^{(\nu+1)} \frac{\partial}{\partial w^{(\nu)}}.$$ 

to represent the basis Cartan field $\partial_F$.

3 Flat systems

A diffeity $\mathcal{M}$ is said to be (locally) of finite type if there exists a (local) Lie-Bäcklund submersion $\pi: \mathcal{M} \to \mathcal{F}$ such that the fibers are finite dimensional. The integer $m$ is called the (local) differential dimension of $\mathcal{M}$ (cf. [5]).

**Definition 3.1 ([5, 6]).** A system is a (local) Lie-Bäcklund fiber bundle $\sigma_M = (\mathcal{M}, \mathbb{R}, \lambda)$, where

(i) $\mathcal{M}$ is a diffeity of finite type where a Cartan field $\partial_M$ has been chosen once for all;

(ii) $\mathbb{R}$ is endowed with a canonical structure of a diffeity, with global coordinate $t$ and Cartan field $\partial/\partial t$;

(iii) $\lambda: \mathcal{M} \to \mathbb{R}$ is a Lie-Bäcklund submersion such that $\lambda_*(\partial_M) = \partial/\partial t$. 

$\blacksquare$
The system $\sigma_F = (\mathcal{F}, \mathbb{R}, \text{pr})$, where pr is the natural projection mapping $\text{pr}: \{t, w_i^{(\nu)}\} \to t$ and $\mathcal{F}$ a trivial diffiety, is called a trivial system.

The differential dimension of a system $\sigma_M = (\mathcal{M}, \mathbb{R}, \lambda)$, denoted $\text{dim diff } \sigma_M$, is the differential dimension of the associated diffiety $\mathcal{M}$.

**Definition 3.2 ([5])** Two systems $\sigma_M = (\mathcal{M}, \mathbb{R}, \lambda)$ and $\sigma_N = (\mathcal{N}, \mathbb{R}, \delta)$ are said to be (differentially) equivalent, written $\sigma_M \approx \sigma_N$, if and only if

(i) $\phi_*(\partial_M) = \partial_N$, where $\phi: \mathcal{M} \to \mathcal{N}$ is a Lie-Bäcklund isomorphism;

(ii) $\lambda = \phi^* \delta$.

A system $\sigma_M = (\mathcal{M}, \mathbb{R}, \lambda)$ is said to be (locally) differentially flat, or simply flat if it is (locally) equivalent to a trivial system. If $\{t, y_1^{(\nu)} \mid i = 1, \ldots, m; \nu \geq 0\}$ are local coordinates on $\mathcal{F}$ then $y = (y_1, \ldots, y_m)$ is called a flat or linearizing output.

### 4 Liouvillian Systems

A diffiety $\mathcal{S} = (S, \text{CTS})$ is called a subdiffiety of a diffiety $\mathcal{M} = (M, \text{CTM})$ if $S$ is a submanifold of $M$ and $\text{CTS} = T_S \cap \text{CTM}$, i.e., the natural embedding $\iota: \mathcal{S} \to \mathcal{M}$ is a Lie-Bäcklund immersion. The fiber bundle $T_SM$ denotes here the restriction of the vector bundle $TM$ on $S$, i.e.,

$$T_SM = \bigcup_{p \in S} T_pM.$$ 

The tangent mapping $\iota_*: TS \to TM$ is injective and the image $\iota_*(TS) \subset T_SM$. If $\mathcal{M}$ is of finite type, then clearly $\mathcal{S}$ is of finite type as well.

**Definition 4.1.** A system $\sigma_M = (\mathcal{M}, \mathbb{R}, \lambda)$ is said to be a differential extension of $\sigma_S = (\mathcal{S}, \mathbb{R}, \delta)$, denoted $\sigma_S/\sigma_M$ or $\sigma_S \subset \sigma_M$, if and only if

(i) $\mathcal{S}$ is a subdiffiety of $\mathcal{M}$;

(ii) the restriction $\iota^* \lambda = \delta$, where $\iota^*$ is the dual mapping of the natural embedding $\iota: \mathcal{S} \to \mathcal{M}$. 

---

Since we consider only diffieties of Cartan dimension 1, $\text{CTS} = \text{CT}_{S}M$ here.
Consider the differential extension \( \sigma_M/\sigma_S \), with \( \dim \text{diff } \sigma_M = m \). Since \( \mathcal{M} \) is of finite type, there exists a Lie-Bäcklund submersion \( \pi : \mathcal{M} \rightarrow \mathcal{F} \) such that its fibers are finite dimensional, say \( n \). Assume now that \( \sigma_M \) is not flat and \( \sigma_S \) is a flat with a flat output \( y = (y_1, \ldots, y_m) \). Define the canonical bundle morphism \( \rho : TM \rightarrow TM/TS \) that takes a vector \( \zeta \in T_pM, \ p \in M \), to its equivalence class \( \zeta + T_pS \) and let \( \tau : TM/TS \rightarrow M \) be the fiber bundle whose fibers \( \tau^{-1}(p) \), \( p \in M \), are finite dimensional. If \( \{t, \eta_1, \ldots, \eta_s, u_i^{(\nu)} \mid i = 1, \ldots, m; \nu \geq 0\} \) are local coordinates on \( \mathcal{F} \) then the Cartan distribution of \( \mathcal{F} \) is spanned by

\[
\partial_S = \frac{\partial}{\partial t} + \sum_{j=1}^{s} F_j \frac{\partial}{\partial \eta_j} + \sum_{i=1}^{m} \sum_{\nu \geq 0} u_i^{(\nu+1)} \frac{\partial}{\partial u_i^{(\nu)}},
\]

where \( F_j \) are \( C^\infty \) functions on \( S \). Using the short notation, \( \partial_S \) can be written under the form

\[
\partial_S = \frac{\partial}{\partial t} + F^1 \frac{\partial}{\partial \eta} + \sum_{\nu \geq 0} u^{(\nu+1)} \frac{\partial}{\partial u^{(\nu)}},
\]

with \( F^1 = (F_1^1, \ldots, F_s^1) \). A local smooth section \( \zeta \) of \( TM/TS \) is given by

\[
\zeta = \sum_{j=1}^{d=n-s} F^2_j \frac{\partial}{\partial \xi_j} = F^2 \frac{\partial}{\partial \xi},
\]

where \( F^2_j \) are \( C^\infty \) functions on \( M \), \( F^2 = (F_1^2, \ldots, F_d^2) \), with

\( \{t, \xi_1, \ldots, \xi_d, \eta_1, \ldots, \eta_s, u^{(\nu)}_i \mid i = 1, \ldots, m; \nu \geq 0\} \)

local coordinates on \( \mathcal{M} \).

**Definition 4.2.** Let \( \sigma_M \) be a differential extension of a flat system \( \sigma_S \) and \( y \) a given flat output of \( \sigma_S \). Then, \( \sigma_S \) is called a flat subsystem of \( \sigma_M \) and the flat output \( y \) of \( \sigma_S \), a partial linearizing output of \( \sigma_M \). If, in addition, that flat output \( y \) is such that \( d = \dim \tau^{-1}(p), \) with \( p \in M \) and \( \tau : TM/TS \rightarrow M \) the aforementioned fiber bundle, is minimal, then \( d \) is called the defect, \( \sigma_S \) a maximal flat subsystem and \( y \) a maximal linearizing output of \( \sigma_M \).

Consider now the classical dynamics

\[
\dot{x} = F(x, u), \quad (x, u) \in X \times U \subset \mathbb{R}^n \times \mathbb{R}^m,
\]

where \( x = (x_1, \ldots, x_n), \ u = (u_1, \ldots, u_m) \) and \( F = (F_1, \ldots, F_n) \) is a \( n \)-tuple of \( C^\infty \) functions on \( X \times U \). To \( \mathbb{M} \) we can associate a diffiety \( \mathcal{M} = (\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m, \partial_M) \) with local coordinates \( \{t, x_1, \ldots, x_n, u_i^{(\nu)} \mid i = 1, \ldots, m; \nu \geq 0\} \) and Cartan field

\[
\partial_M = \frac{\partial}{\partial t} + \sum_{j=1}^{n} F_j \frac{\partial}{\partial x_j} + \sum_{i=1}^{m} \sum_{\nu \geq 0} u_i^{(\nu+1)} \frac{\partial}{\partial u_i^{(\nu)}}.
\]
A subsystem of \([3]\) is given by a diffeity \(\mathcal{S} = (S, \partial_S)\), with local coordinates
\[
\{t, \eta_1, \ldots, \eta_s, u_i^{(\nu)} | i = 1, \ldots, m ; \nu \geq 0\}
\]
and a basis Cartan field
\[
\partial_S = \frac{\partial}{\partial t} + \sum_{j=1}^s F_j^1(\eta, u) \frac{\partial}{\partial \eta_j} + \sum_{i=1}^m \sum_{\nu \geq 0} u_i^{(\nu+1)} \frac{\partial}{\partial u_i^{(\nu)}},
\]
where \(\eta = (\eta_1, \ldots, \eta_s) \in X^1 \subset \mathbb{R}^s\) and \(F_j^1\) are \(C^\infty\) functions on \(X^1 \times U\). A local section \(\zeta\) of \(TM/TS\) is given by
\[
\zeta = \sum_{j=1}^{d-n-s} F_j^2(\eta, \xi, u) \frac{\partial}{\partial \xi_j} = F^2(x, \xi) \frac{\partial}{\partial \xi},
\]
where \(\xi = (\xi_1, \ldots, \xi_d) \in X^2 \subset \mathbb{R}^d\), \(F^2 = (F_1^2, \ldots, F_d^2)\) and \(F_j^2\) are \(C^\infty\) functions on \(X^1 \times X^2 \times U = X \times U\). The vector \(\xi\) represents only the complement of \(\eta\) (by renumbering the \(x_i\)'s if needed) to form the vector \(x\), i.e., \(x = (\eta, \xi)\). We can assume, in the sequel, that coordinates \(\eta\) and \(\xi\) are given by the projection mappings \(pr_1\) and \(pr_2\)
\[
pr_1 : \{t, x_1, \ldots, x_n, u_i^{(\nu)}\} \rightarrow \{t, \eta_1, \ldots, \eta_s, u_i^{(\nu)}\}
pr_2 : \{t, x_1, \ldots, x_n, u_i^{(\nu)}\} \rightarrow \{t, \xi_1, \ldots, \xi_d\}
\]
Thus, if \(\sigma_M\) is a differential extension of a flat system \(\sigma_S\), then dynamics \([3]\) admits the following decomposition
\[
\dot{x} = \left(\begin{array}{c}
\dot{\eta} \\
\ldots \\
\dot{\xi}
\end{array}\right) = \left(\begin{array}{c}
F^1(\eta, u) \\
\ldots \\
F^2(\eta, \xi, u)
\end{array}\right).
\]

**Definition 4.3.** A system \(\sigma_M\) is called a Piccard-Vessiot extension of a system \(\sigma_S\) if:

(i) \(\sigma_S\) is flat;

(ii) the Cartan field \(\partial_M\) is of the form
\[
\partial_M = A(\eta, u)\xi \frac{\partial}{\partial \xi} + \partial_S.
\]

(iii) \(\ker \partial_M = \ker \partial_S\).
A differential extension $\sigma_M/\sigma_S$ such that $\sigma_M$ is a Piccard-Vessiot extension of $\sigma_S$ is said to be a Piccard-Vessiot extension.

In the case of a Piccard-Vessiot system, (3) takes the form

$$
\dot{x} = \begin{pmatrix}
\dot{\eta} \\
\ldots \\
\dot{\xi}
\end{pmatrix} = \begin{pmatrix}
F^1(\eta, u) \\
\ldots \\
A(\eta, u)\xi
\end{pmatrix},
$$

(4)

where $A(\eta, u)$ is a $n \times n$ matrix of $C^\infty(X^1 \times U)$ functions.

**Proposition 4.1.** Let $\sigma_M/\sigma_S$ be a Piccard-Vessiot extension. Then, $\sigma_M$ is locally controllable.

**Proof.** Since $\sigma_S$ is flat, $\sigma_S \simeq \sigma_F$, hence $\ker \partial_M = \ker \partial_S = \mathbb{R}$, i.e., any local first integral of $\sigma_M$ is trivial, and it follows that $\sigma_M$ is locally controllable (cf [8]).

**Definition 4.4.** Let $\sigma_M$ be a differential extension of a flat system $\sigma_S$ and $y$ a flat output of $\sigma_S$. Then, $\sigma_M$ is said to be a Liouvillian extension of $\sigma_S$, or simply $\sigma_M/\sigma_S$ is a Liouvillian extension, if and only if there exists a nested chain of subsystems $\sigma_S = \sigma_{S_0} \subset \sigma_{S_1} \subset \cdots \subset \sigma_{S_d} = \sigma_M$, with $\sigma_{S_j} = (\mathcal{S}_j, \mathbb{R}, \delta_j)$ and $\mathcal{S}_j = (S_j, \partial_{S_j})$, such that, for $j = 1, \ldots, d$, $\ker \partial_{S_j} = \ker \partial_{S_{j-1}}$, where either

(i) $\partial_{S_j} = \alpha_j \partial/\partial \xi_j + \partial_{S_{j-1}}$, $\alpha_j \in C^\infty(S_{j-1})$,

(ii) $\partial_{S_j} = \alpha_j \xi_j \partial/\partial \xi_j + \partial_{S_{j-1}}$, $\alpha_j \in C^\infty(S_{j-1})$.

If $\sigma_S$ is maximal (resp. partial), i.e., $d$ is the defect of $\sigma_M$, then $\sigma_M$ is called Liouvillian system (resp. partial Liouvillian system) and $y$ Liouvillian output (resp. partial Liouvillian output).

**Remark 4.1.** According to the definition, a local section $\zeta_j$ of $T\mathcal{S}_j/T\mathcal{S}_{j-1}$ is given either by

(i) $\zeta_j = \alpha_j(\eta, \xi_1, \ldots, \xi_{j-1}, u) \partial/\partial \xi_j$ (hence $\dot{\zeta}_j = \alpha_j(\eta, \xi_1, \ldots, \xi_{j-1}, u)$ and $\zeta_j = \int \alpha_j$) or

(ii) $\zeta_j = \alpha_j(\eta, \xi_1, \ldots, \xi_{j-1}, u) \xi_j \partial/\partial \xi_j$ (hence $\dot{\zeta}_j = \alpha_j(\eta, \xi_1, \ldots, \xi_{j-1}, u) \xi_j$ and $\zeta_j = e^{\int \alpha_j}$).

Hence, Liouvillian extensions are extensions by integrals (i) or exponential of integral (ii), usually called extensions by quadratures.
Actually, it is easy to see that Liouvillian extensions are a particular case of Piccard-Vessiot extensions. For (ii) of definition 4.4, it is clear that \( \sigma_{S_j} \) is a Piccard-Vessiot extension of \( \sigma_{S_{j-1}} \). The extension by integral can be obtained by considering the Piccard-Vessiot extension \( \sigma_{S_j} \) of \( \sigma_{S_{j-1}} \) with Cartan field given by

\[
\partial_{S_j} = \xi_{j+1} \frac{\partial}{\partial \xi_j} + (\dot{\alpha}_j / \alpha_j) \xi_{j+1} \frac{\partial}{\partial \xi_{j+1}},
\]

with \( \dot{\alpha}_j = d\alpha_j / dt \) and \( \alpha_j \in C^\infty(S_j) \).

**Remark 4.2.** Notice that an arbitrary linearizing output \( y \) for \( \sigma_S \) does not necessarily give rise to a Liouvillian system. Therefore, the Liouvillian character of a system strongly depends on the choice of \( y \).

Let \( T_n(C^\infty(X^1 \times \mathbb{R}^m)) \) the set of \( n \times n \) lower triangular matrices with components in \( C^\infty(X^1 \times \mathbb{R}^m) \) and \( U_n(C^\infty(X^1 \times \mathbb{R}^m)) \) the subset of \( T_n(C^\infty(X^1 \times \mathbb{R}^m)) \) such that all the diagonal components are equals to 1.

**Theorem 4.1.** Consider the Piccard-Vessiot system (4). If \( A(\eta, u) \in U_d(C^\infty(X^1 \times \mathbb{R}^m)) \) then (4) is Liouvillian.

**Proof.** Let \( A = (a_{i,j})_{i,j=1,d} \in U_d(C^\infty(X^1 \times \mathbb{R}^m)) \), then

\[
\dot{\xi}_i = \sum_{j=1}^{i} a_{i,j} \xi_j, \quad \text{with} \quad a_{i,i} = 1, \ i = 1, \ldots, d.
\]

In particular,

\[
\dot{\xi}_1 = \xi_1,
\]

and it follows that \( \xi_1 \) is an exponential of integral. Next,

\[
\frac{\dot{\xi}_2}{\xi_1} = \frac{\dot{\xi}_2}{\xi_1} - \frac{\xi_2}{\xi_1} = a_{2,1},
\]

in the other words,

\[
\xi_2 = \xi_1 \int a_{2,1}.
\]

Finally, differentiating \( \xi_i / \xi_1, \ i = 1, \ldots, d, \) gives

\[
\frac{\dot{\xi}_i}{\xi_1} = a_{i,1} + a_{i,2} \frac{\xi_2}{\xi_1} + \ldots + a_{i,i-1} \frac{\xi_{i-1}}{\xi_1}.
\]
Now, making the appropriate induction assumption, we deduce that \( \xi_i, i = 2, \ldots, d \), can be obtained by mean of the integral

\[
\xi_i = \xi_1 \int a_{i,1} + a_{i,2} \frac{\xi_2}{\xi_1} + \ldots + a_{i,i-1} \frac{\xi_{i-1}}{\xi_1},
\]

which concludes the proof. □

**Theorem 4.2.** Consider the Picard-Vessiot system \([4]\). If \( A(\eta, u) \in T_d(C^\infty(X^1 \times \mathbb{R}^m)) \) then \([4]\) is Liouvillian. □

**Proof.** Let \( A = (a_{i,j})_{i,j=1,d} \in T_d(C^\infty(X^1 \times \mathbb{R}^m)) \), i.e.,

\[
\dot{\xi}_i = \sum_{j=1}^{i} a_{i,j} \xi_j, \quad i = 1, \ldots, d.
\]

and set

\[
\dot{c}_i = a_{i,i}, \quad i = 1, \ldots, d.
\]

So the \( c_i \)'s are integrals of elements of the flat subsystem \( \sigma_S \). For \( i = 1 \), we get

\[
\dot{\xi}_1 = a_{1,1} \xi_1,
\]

and it follows that \( \xi_1 = e^{c_1} \). Next,

\[
\left( \frac{\dot{\xi}_2}{\xi_1} \right) = \frac{\dot{\xi}_2}{\xi_1} - \frac{\dot{\xi}_1}{\xi_1} = a_{2,1} + (a_{2,2} - a_{1,1}) \frac{\xi_2}{\xi_1},
\]

in the other words,

\[
\xi_2 = \xi_1 e^{c_2 - c_1} \int a_{2,1} e^{c_1 - c_2}.
\]

Finally, differentiating \( \xi_i/\xi_1, i = 2, \ldots, d \), gives

\[
\left( \frac{\dot{\xi}_i}{\xi_1} \right) = a_{i,1} + a_{i,2} \frac{\xi_2}{\xi_1} + \ldots + a_{i,i-1} \frac{\xi_{i-1}}{\xi_1} + (a_{i,i} - a_{1,1}) \frac{\xi_i}{\xi_1}.
\]

Now, making the appropriate induction assumption, we deduce that \( \xi_i, i = 2, \ldots, d \), can be obtained by mean of the relation

\[
\xi_i = \xi_1 e^{c_{i-1} - c_1} \cdot \int (a_{i,1} + a_{i,2} \frac{\xi_2}{\xi_1} + \ldots + a_{i,i-1} \frac{\xi_{i-1}}{\xi_1}) e^{c_1 - c_i},
\]

which concludes the proof. □
5 The rolling bodies

Let us illustrate the class of Liouvillian through the concrete case of rolling bodies (see [1] for a more larger treatment). The kinematic equations of motion of the contact point between two bodies rolling on top of each other are given in geodesic coordinates by

\[
\begin{align*}
\dot{v}_1 &= u_1, \\
\dot{w}_1 &= \frac{1}{2} u_2, \\
\dot{v}_2 &= u_1 \cos \psi - u_2 \sin \psi, \\
\dot{w}_2 &= -\frac{1}{\cos v_1} (u_1 \sin \psi + u_2 \cos \psi), \\
\dot{\psi} &= \frac{B_{v_1} u_2 - C_{v_2} (u_1 \sin \psi + u_2 \cos \psi)}{B_{v_1} B_{v_2} - C_{v_1} C_{v_2}},
\end{align*}
\]

where \( B = B(v_1, w_1) \), \( C = C(v_2, w_2) \), \( B_{v_1} = \partial B/\partial v_1 \) and \( C_{v_2} = \partial C/\partial v_2 \). In the case of the well known plate ball problem, \( B \equiv 1 \) and \( C = \cos(v_2) \) and (5) takes the form

\[
\begin{align*}
\dot{v}_1 &= u_1, \\
\dot{w}_1 &= u_2, \\
\dot{v}_2 &= u_1 \cos \psi - u_2 \sin \psi, \\
\dot{w}_2 &= -\frac{1}{\cos v_1} (u_1 \sin \psi + u_2 \cos \psi), \\
\dot{\psi} &= \tan v_2 (u_1 \sin \psi + u_2 \cos \psi).
\end{align*}
\]

This system is Liouvillian and a Liouvillian output is given by (see [1] for the details)

\[
\begin{align*}
x &= v_1 - v_2 \cos \psi, \\
y &= w_1 + w_2 \sin \psi.
\end{align*}
\]

The idea to give a new formulation of Liouvillian systems within the mathematical framework of diffieties was actually motivated by the study of the rolling bodies system. As matter of fact, it is not clear whether this system is Liouvillian within the differential algebraic setting (cf [3]).

First, notice that system (4) is not under a suitable form to describe a system in the differential algebraic setting (the associated differential field extension is not finitely generated). However, using transformations \( \sigma = \tan(\psi/2) \) and \( \xi = \tan(v_2/2) \), (5) writes (plate ball case)

\[
\begin{align*}
\dot{v}_1 &= u_1, \\
\dot{w}_1 &= u_2, \\
\dot{\xi} &= \frac{1+\xi^2}{2(1+\sigma^2)} [(1-\sigma^2)u_1 - 2\sigma u_2], \\
\dot{\psi} &= \frac{\xi}{1-\xi^2} [2\sigma u_1 + (1-\sigma^2)u_2], \\
\dot{\sigma} &= \frac{1+\xi^2}{1-\xi^2} [2\sigma u_1 + (1-\sigma^2)u_2].
\end{align*}
\]
and becomes explicit and rational. The associated differential ideal is thus prime and leads to finitely generated differential field extension. Nevertheless, the Liouvillian output (7) takes now the form

\[ \begin{align*}
\tilde{x} &= v_1 - \frac{1}{1 + \sigma^2} \sigma^2 \arctan \xi, \\
\tilde{y} &= w_1 + \frac{1}{1 + \sigma^2} w_2. 
\end{align*} \tag{9} \]

If we denote by \( \mathbb{R}\langle v_1, w_1, \xi, w_2, \sigma, u_1, u_2 \rangle \) the differential field generated by \( \mathbb{R} \) and the variables \( \{v_1, w_1, \xi, w_2, \sigma, u_1, u_2\} \), and \( \mathbb{R}_e\langle v_1, w_1, \xi, w_2, \sigma, u_1, u_2 \rangle \) its algebraic closure, then \( \tilde{x} \) and \( \tilde{y} \) are not in \( \mathbb{R}\langle v_1, w_1, \xi, w_2, \sigma, u_1, u_2 \rangle \), and it follows that (9) is not a Liouvillian output for (8) in this context.

References

[1] A. Chelouah and Y. Chitour. On the motion planning of rolling surfaces. *Forum Mathematicum*, 15(5):727–758, 2003.

[2] B. Charlet, J. Lévine, and R. Marino. Sufficient conditions for dynamic state feedback linearization. *SIAM J. Control and Optimization*, 29:38–57, 1991.

[3] A. Chelouah. Extension of differential flat field and liouvillian systems. In *Proc. IEEE Conf. Dec. Contr.*, volume 5, pages 4268–4273, 1997. San Diego.

[4] M. Fliess, J. Lévine, P. Martin, and P. Rouchon. Linéarisation par bouclage dynamique et transformation de Lie-Bäcklund. *C. R. Acad. Sci. Paris*, I-317:981–986, 1993.

[5] M. Fliess, J. Lévine, P. Martin, and P. Rouchon. Deux applications de la géométrie locales des diffiétés. *Ann. Inst. Henri Poincaré*, 66(3):275–292, 1997.

[6] M. Fliess, J. Lévine, P. Martin, and P. Rouchon. Nonlinear control and diffieties, with an application to physics. *Contemporary Mathematics*, 219:81–92, 1998.

[7] M. Fliess, J. Lévine, P. Martin, and P. Rouchon. A Lie-Bäcklund approach to equivalence and flatness of nonlinear systems. *IEEE Trans. on Automatic Control*, 44(5):922–937, 1999.

[8] M. Fliess, J. Lévine, P. Martin, F. Ollivier, and P. Rouchon. A remark on nonlinear accessibility conditions and infinite prolongations. *Systems and Control Letters*, 31(2):77–83, 1997.
[9] B. Kiss. *Planification de trajetoires et commande d’une classe de systèmes mécaniques plats et Liouviliens*. PhD thesis, École des Mines de Paris, 2001.

[10] B. Kiss, J. Lévine, and B. Lantos. On motion planning for robotic manipulation with permanent rolling contacts. *Int. J. of Robotics Research*, 21(5-6):443–461, May 2002.

[11] Hebertt Sira-Ramírez. Soft landing on a planet: A trajectory planning approach for the Liouvillian model. In *Proc. American Contr. Conf.*, pages 2936–2940, 1999.

[12] Hebertt Sira-Ramírez, Rafael Castro-Linares, and Eduardo Licéaga-Castro. Regulation of the longitudinal dynamics of an helicopter: A Liouvillian systems approach. In *Proc. American Contr. Conf.*, pages 2752–2756, 1999. San Diego, California.

[13] V.V. Zharinov. *Geometrical aspect of partial differential equations*. World Scientific, Singapour, 1992.