On the asymptotic value of the choice number of complete multi-partite graphs

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Abstract

We calculate the asymptotic value of the choice number of complete multi-partite graphs, given certain limitations on the relation between the sizes of the different sides. In the bipartite case, we prove that if \( n_0 \leq n_1 \) and \( \log n_0 \gg \log \log n_1 \), then \( ch(K_{n_0,n_1}) = (1 + o(1)) \frac{\log_2 n_1}{\log_2 x_0} \), where \( x_0 \) is the unique root of the equation \( x - 1 - x^{k-1} = 0 \) in the interval \([1, \infty)\) and \( k = \frac{\log_2 n_1}{\log_2 n_0} \). In the multi-partite case, we prove that if \( n_0 \leq n_1 \ldots \leq n_s \), and \( n_0 \) is not too small compared to \( n_s \), then \( ch(K_{n_0,\ldots,n_s}) = (1 + o(1)) \frac{\log_2 n_s}{\log_2 x_0} \). Here \( x_0 \) is the unique root of the equation \( sx - 1 - \sum_{j=0}^{s-1} x^{k_j - 1} = 0 \) in the interval \([1, \infty)\), and for every \( 0 \leq i \leq s - 1 \), \( k_i = \frac{\log_2 n_s}{\log_2 n_i} \).

Key words: choice number.
1 Introduction

The choice number $ch(G)$ of a graph $G = (V, E)$ is the minimum number $k$ such that for every assignment of a list $S(v)$ of at least $k$ colors to each vertex $v \in V$, there is a proper vertex coloring of $G$ assigning to each vertex $v$ a color from its list $S(v)$. The concept of choosability was introduced by Vizing in 1976 [5] and independently by Erdős, Rubin and Taylor in 1979 [2]. It is also shown in [2] that the choice number of the complete bipartite graph $K_{n,n}$ satisfies $ch(K_{n,n}) = (1 + o(1)) \log_2 n$. The choice number of the complete multi-partite graph has been investigated by several researchers. Among the results: Alon [1] proved that the choice number of a complete $r$-partite graph with parts of size $m$ is $\Theta(r \log m)$, Kierstead [3] proved that the choice number of a complete $r$-partite graph with parts of size $3$ is $[(4r - 1)/3]$, and Reed and Sudakov [4] proved that if the number of parts $r$ in the complete $r$-partite graph on $n$ vertices is very large, i.e. $\frac{r}{n} = c$ for any constant $c > \frac{1}{2}$, then the choice number is $r$. In this paper we calculate the asymptotic value of the choice number of a general complete bipartite graph $K_{n_0,n_1}$ and then expand the result to the case of a complete multi-partite graph. We begin by proving (note that throughout this paper all logs are binary):

**Theorem 1** Let $2 \leq n_0 \leq n_1$ be integers, and let $n_0 = (\log n_1)^{\omega(1)}$. Denote $k = \frac{\log n_1}{\log n_0}$. Let $x_0$ be the unique root of the equation $x - 1 - x^{\frac{k+1}{k}} = 0$ in the interval $[1, \infty)$. Then $ch(K_{n_0,n_1}) = (1 + o(1)) \frac{\log n_1}{\log x_0}$.

As usual, $\omega(1)$ stands for a function tending to infinity arbitrarily slowly as its variable tends to infinity. Notice that for the case of equal parts (i.e., when $n_0 = n_1$), we have $k = 1, x_0 = 1$ and thus $ch(K_{n_0,n_0}) = (1 + o(1)) \log n_0$, matching (naturally) the above mentioned result of Erdős, Rubin and Taylor [2].

We will prove the theorem in two parts, showing first the upper bound and then the lower bound. In the graph $K_{n_0,n_1}$ we label the group of $n_0$
vertices by $V_0$ and the group of $n_1$ vertices by $V_1$.

2 The Upper Bound

Theorem 2 Let $2 \leq n_0 \leq n_1$ be integers. Denote $k = \frac{\log n_1}{\log n_0}$. Let $x_0$ be the unique root of the equation $x - 1 - x^{\frac{k-1}{k}} = 0$ in the interval $[1, \infty)$. Then $\text{ch}(K_{n_0,n_1}) \leq \lceil \frac{\log n_1}{\log x_0} \rceil + 1$.

Proof.

Lemma 2.1 If there exists a $p$, $0 \leq p \leq 1$, s.t. $n_0 p^r + n_1 (1-p)^r \leq 1$ then $\text{ch}(K_{n_0,n_1}) \leq r$.

Proof. We show that given, for each vertex $v \in V(K_{n_0,n_1})$, a set of colors $S(v)$ of size $r$, there is a proper vertex coloring of the graph, assigning to each vertex $v$ a color from $S(v)$.

We partition the set of all available colors $S = \bigcup_{v \in V} S(v)$ into two subsets $S_1$ and $S_0$ in the following manner: each color $c \in S$ is chosen randomly and independently with probability $p$ to be in $S_1$, and with probability $1-p$ to be in $S_0$. We will show that with positive probability the sets $S_0$ and $S_1$ chosen satisfy the condition: each vertex $v \in V_0$ has a color $c \in S(v)$ s.t. $c \in S_0$, and each vertex $v \in V_1$ has a color $c \in S(v)$ s.t. $c \in S_1$. Given such $S_0$ and $S_1$, we can color each vertex in $V_0$ with a color from $S_0$, and each vertex in $V_1$ with a color from $S_1$, and since $S_0 \cap S_1 = \emptyset$, we get a proper coloring.

For each $v \in V_1$ the probability that a bad event occurs, i.e. that all the colors in $S(v)$ are chosen to be in $S_0$, is $(1-p)^r$. For each $v \in V_0$ the probability that a bad event occurs, i.e. that all the colors in $S(v)$ are chosen to be in $S_1$, is $p^r$. Therefore the expectation of the number of bad events that occur is $n_0 p^r + n_1 (1-p)^r \leq 1$. Since either $p > 0$ or $1-p > 0$, we can assume w.l.o.g. that $1-p > 0$. Then since, for example, the case in which all the colors in $S$ are chosen to be in $S_0$ happens with probability
(1 - p)^{|S|} > 0, and gives $n_1$ bad events, the case in which 0 events occur also happens with positive probability (otherwise the expectation would be greater than 1). Therefore we get the desirable partition.

**Lemma 2.2** Given $r$ s.t. $(\frac{1}{n_0})^{\frac{1}{r}} + (\frac{1}{n_1})^{\frac{1}{r}} \geq 1$, let $p = \frac{(\frac{1}{n_0})^{\frac{1}{r}}}{(\frac{1}{n_0})^{\frac{1}{r}} + (\frac{1}{n_1})^{\frac{1}{r}}}$. Then $n_0 p^r + n_1 (1 - p)^r \leq 1$.

**Proof.** If $p = \frac{(\frac{1}{n_0})^{\frac{1}{r}}}{(\frac{1}{n_0})^{\frac{1}{r-1}} + (\frac{1}{n_1})^{\frac{1}{r-1}}}$ then $(\frac{p}{1-p})^{r-1} = \frac{n_1}{n_0}$. Therefore
\[
n_0 p^r + n_1 (1 - p)^r = n_0 p^r + n_1 \left(\frac{n_0}{n_1}\right) p^{r-1} (1 - p) = n_0 p^{r-1}
= n_0 \left(\frac{(\frac{1}{n_0})^{\frac{1}{r-1}}}{(\frac{1}{n_0})^{\frac{1}{r-1}} + (\frac{1}{n_1})^{\frac{1}{r-1}}}\right)^{r-1} = \left(\frac{1}{(\frac{1}{n_0})^{\frac{1}{r-1}} + (\frac{1}{n_1})^{\frac{1}{r-1}}}\right)^{r-1}
\leq 1.
\]

All that remains now is to choose $r = r(n_0, n_1)$ satisfying the condition of Lemma 2.2. Let $r = \lceil \log n_1 \log x_0 \rceil + 1$. Then $r - 1 \geq \frac{\log n_1}{\log x_0}$, and hence $x_0 \geq n_1^{\frac{1}{r-1}}$. Since the function $f_k(x) = x - 1 - x^{\frac{k-1}{k}}$, where $k \geq 1$, is a monotonely increasing function in the interval $[1, \infty)$, and since $f_k(x_0) = 0$, it follows that $n_1^{\frac{1}{r-1}} \leq 1 + n_1^{\frac{k-1}{k}} = 1 + (\frac{n_1}{n_0})^{\frac{1}{r-1}}$ as required.

**3 The Lower Bound**

**Theorem 3** If $2 \leq n_0 \leq n_1$ are integers, and $n_0 = (\log n_1)^{\omega(1)}$, then $\text{ch}(K_{n_0, n_1}) \geq (1 - o(1)) \frac{\log n_1}{\log x_0}$, where $x_0$ is the unique root of the equation $x - 1 - x^{\frac{k-1}{k}} = 0$ in the interval $[1, \infty)$ and $k = \frac{\log n_1}{\log n_0}$.

**Proof.**

A cover of a hypergraph $H$ is a subset $M$ of the vertices of the hypergraph such that every hyperedge of $H$ contains at least one vertex of $M$. A minimum cover is a cover which has the least cardinality among all covers.
Let us generate the hypergraph $H_0$ created by the color lists of the vertices in $V_0$, i.e. the hypergraph whose vertices are the colors $\bigcup_{v \in V_0} S(v)$, and whose edges are the lists $S(v)$ for each $v \in V_0$. In the same way, we generate the hypergraph $H_1$ created by the color lists of the vertices in $V_1$.

For any $r$, if we wish to prove $ch(K_{n_0,n_1}) > r$, it is enough to show that there are parameters $t \geq r$ and $0 \leq l \leq t$ s.t. it is possible to choose for each vertex in $K_{n_0,n_1}$ a list of $r$ colors from $\{1, 2, \ldots, t\}$, and the lists chosen satisfy:

1. The minimum cover of the hypergraph $H_0$ created by the color lists of the vertices in $V_0$ (i.e. the minimum size of a set $L$ of colors s.t. for every $v \in V_0$, $S(v)$ contains at least one of the colors in $L$) is of cardinality at least $l$.

2. The minimum cover of the hypergraph $H_1$ created by the color lists of the vertices in $V_1$ is of cardinality at least $t - l + 1$.

If these conditions are satisfied, then when these color lists are assigned to the vertices of $K_{n_0,n_1}$, the graph cannot be properly colored. This is because at least $l$ colors are needed to color one side, and at least $t - l + 1$ to color the other. Since there are only $t$ colors in all, at least one color will be chosen by both sides – i.e., at least two vertices on opposite sides must be given the same color, implying that a proper coloring is not possible. Therefore, the choice number of the graph is greater than $r$.

**Lemma 3.1** If there exist parameters $t$ and $l$ such that $t \geq r$, $0 \leq l \leq t$ and

$$2^t e^{-\frac{(l)}{\binom{r}{t}} n_1} + 2^t e^{-\frac{(t-l)}{\binom{r}{t}} n_0} \leq 1 \quad (1)$$

then $ch(K_{n_0,n_1}) > r$.

**Proof.** It is easy to see that at least $l$ colors are required for a cover of the hypergraph $H_0$ created by the color lists of the vertices in $V_0$ if and only
if for each subset $C$ of size $t - l + 1$ of $\{1, 2, \ldots, t\}$ there is at least one $v \in V_0$ for which $S(v) \subseteq C$. In the same way, the minimum cover of the hypergraph $H_1$ created by the color lists of the vertices in $V_1$ is at least $t - l + 1$ if and only if for each subset $C$ of size $l$ of $\{1, 2, \ldots, t\}$ there is at least one $v \in V_1$ for which $S(v) \subseteq C$.

For each vertex $v$ in $K_{n_0,n_1}$, let $S(v)$ be a random subset of cardinality $r$ of $\{1, 2, \ldots, t\}$, chosen uniformly and independently among all $\binom{t}{r}$ subsets of cardinality $r$ of $\{1, 2, \ldots, t\}$. We wish to find an $r$ that guarantees that with positive probability:

1. For every subset $C$ of size $t - l + 1$ there is a vertex $v \in V_0$ s.t. $S(v) \subseteq C$, and
2. For every subset $C$ of size $l$ there is a vertex $v \in V_1$ s.t. $S(v) \subseteq C$.

To simplify the calculations, we will change Condition 1 above to the stronger condition that:

1. For every subset $C$ of size $t - l$ there is a vertex $v \in V_0$ s.t. $S(v) \subseteq C$.

For each fixed subset $C$ of cardinality $l$ of $\{1, 2, \ldots, t\}$ and each $v \in V_1$, the probability that $S(v) \not\subseteq C$ is $1 - \frac{\binom{t-l}{r}}{\binom{t}{r}} = 1 - \frac{\binom{t-l}{r}}{\binom{t}{r}}$. Since there are $n_1$ vertices in $V_1$ and $\binom{t}{r}$ subsets of cardinality $l$ of $\{1, \ldots, t\}$, and since the color groups of the vertices were chosen independently, the probability that there is a subset $C$ of size $l$ that does not contain $S(v)$ for any $v \in V_1$ is at most $\binom{t}{r}\left(1 - \frac{\binom{t-l}{r}}{\binom{t}{r}}\right)^{n_1} < 2^r e^{-\binom{t-l}{r}n_1}$. In a similar fashion, the probability that there is a subset $C$ of size $t - l$ that does not contain $S(v)$ for any $v \in V_0$ is at most $\binom{t-l}{r}\left(1 - \frac{\binom{t-l}{r}}{\binom{t}{r}}\right)^{n_0} < 2^r e^{-\binom{t-l}{r}n_0}$.

We are looking for an $r$ that guarantees that the probability that at least one of Conditions 1 and 2 does not hold is smaller than 1. Therefore it is enough to show the sum of these probabilities is smaller than 1, i.e., it is enough to show: $2^r e^{-\binom{t}{r}n_1} + 2^r e^{-\binom{t}{r}n_0} \leq 1$. ■
Before proceeding to find \( t \) and \( l \) that fit Lemma 3.1, we derive bounds on \( x_0 \) that will be useful at later stages of the proof.

**Lemma 3.2** \( 2 \leq x_0(k) < \max(k, e + 2) \)

**Proof.** We begin by showing that if \( k > e + 1 \), then \( x_0(k) < k \). Since \( f_k(x) = x - 1 - x^{\frac{k}{k+1}} \) is monotonely increasing, we need to show that \( f_k(k) > 0 \), or \( k - k^{\frac{k}{k+1}} - 1 > 0 \), or \( (k - 1)^{\frac{k}{k+1}} > k^k \). But the function \( h(x) = x^{\frac{1}{x}} \) is monotonely decreasing for \( x > e \). So if \( k > e + 1 \) then \( k - 1 > e \) and therefore \( (k - 1)^{\frac{k}{k+1}} > k^k \).

It can easily be seen that \( x_0 \) increases monotonely as a function of \( k \) (i.e. if \( k_2 \geq k_1, x_0(k_2) \geq x_0(k_1) \)). Therefore if \( k \leq e + 2 \), then \( x_0(k) \leq x_0(e + 2) < e + 2 \).

To prove the lower bound on \( x_0 \), observe that \( f_k(2) = 2 - 1 - 2^{\frac{k}{k+1}} = 1 - 2^{\frac{k}{k}} \leq 0 \) for every \( k \geq 1 \).

**Lemma 3.3** Let \( n_0 = (\log n_1)^{\omega(1)} \). Define \( r_0 = \frac{\log n_1}{\log x_0}, u = \frac{4\log\log n_1}{\log n_0} r_0 \) and \( r = r_0 - u \). Then \( r = (1 - o(1)) r_0 \), and for \( t = (\frac{n_1}{n_0})^{\frac{1}{r}} r^2 \) and \( l = t \frac{1}{(\frac{n_1}{n_0})^{\frac{1}{r}} + 1}, 2^l e^{\frac{-(t-l)}{\omega(1)} n_1} + 2^l e^{-\frac{(t-l)}{\omega(1)} n_0} \leq 1 \).

**Proof.** If \( n_0 = (\log n_1)^{\omega(1)} \) then \( \log\log n_1 \ll \log n_0 \), and therefore \( u = o(r_0), \) and \( r = (1 - o(1)) r_0 \), as required. From the fact that \( r = (1 - o(1)) r_0 \), it also follows that \( r = \omega(1) \). This is because \( x_0 < \max(k, e + 2) \), and therefore, if \( k \leq e + 2 \) then \( r_0 = \frac{\log n_1}{\log x_0} > \frac{\log n_1}{\log (e+2)} = \omega(1) \), and otherwise \( r_0 = \frac{\log n_1}{\log x_0} > \frac{\log n_1}{\log k} = \frac{\log n_1}{\log \log n_1} = \frac{\log n_1}{\log \log n_1 - \log \log n_0} \geq \frac{\log n_1}{\log \log n_1} = \omega(1) \). Hence \( r = (1 - o(1)) r_0 = \omega(1) \).

Let us denote \( l_0 = l \) and \( l_1 = t - l \). Then \( t - l_i = t \frac{(n_1)}{n_0} \frac{1}{(\omega(1))^{t+1}} \), and \( 2^l e^{\frac{(t-l)}{\omega(1)} n_1} + 2^l e^{-\frac{(t-l)}{\omega(1)} n_0} = \sum_{i=0}^{1} 2^l e^{\frac{(t-l)}{\omega(1)} n_i} \). In order for this sum to be not greater than 1, it is enough to show that \( \frac{(t-l)}{\omega(1)} n_i \gg t \) for \( i = 0, 1 \). We begin by estimating \( \frac{(t-l)}{\omega(1)} n_i \).
Claim 3.4 \( \frac{(t-l_{i})}{(t_{i})} n_{i} > \frac{1}{2e} \left( \frac{n_{i}}{n_{0}} + 1 \right) \) for \( i = 0, 1 \).

Proof. \( \frac{(t-l_{i})}{(t_{i})} > \frac{(t-l_{i})}{(t_{i})} = \frac{(t-l_{i})}{(t_{i})} \left( 1 - \frac{t_{i}}{(t_{i})(t)} \right) > \frac{(t-l_{i})}{(t_{i})} \left( 1 - \frac{2t}{(t_{i})t} \right) \), where the last inequality is a result of \( r < \frac{t}{2} \).

Now since \( \frac{(t_{i})}{(t_{i})} = \frac{t_{i}}{(t_{i})} = \frac{r}{(n_{i})^{\frac{1}{r} + 1}} = \frac{r}{(n_{i})^{\frac{1}{r} + 1}} \leq \frac{r}{(n_{i})^{\frac{1}{r} + 1}} = \frac{1}{r} = o(1) \), and

\[ \frac{(t-l_{i})}{(t_{i})} = \frac{(t-l_{i})}{(t_{i})} = \frac{r}{(n_{i})^{\frac{1}{r} + 1}} = 1 \] we get (recalling that \( 1 - x \geq e^{-x} / 2 \) for \( 0 \leq x \leq 1/2 \)) \( \frac{(t-l_{i})}{(t_{i})} > \frac{(t-l_{i})}{(t_{i})} \frac{1}{2e^{x}} \). Therefore \( \frac{(t-l_{i})}{(t_{i})} n_{i} > \frac{(t-l_{i})}{(t_{i})} n_{i} \frac{1}{2e^{x}} \)

Hence in order to prove that (1) holds it is now enough to prove that \( \frac{n_{1}}{(n_{0})^{\frac{1}{r} + 1}} \gg t \).

Claim 3.5 \( \frac{n_{1}}{(n_{0})^{\frac{1}{r} + 1}} \gg t \).

Proof. \( \frac{n_{1}}{(n_{0})^{\frac{1}{r} + 1}} = \left( \frac{n_{1}}{(n_{0})^{\frac{1}{r} + 1}} \right)^{r} = \left[ \frac{\frac{n_{1}}{n_{0}}}{(n_{0})^{\frac{1}{r} + 1}} \left( \frac{n_{1}}{n_{0}} \right)^{\frac{1}{r} + 1} \right] \).

Since \( \frac{n_{1}}{(n_{0})^{\frac{1}{r} + 1}} = n_{1} \frac{\log n_{0}}{\log n_{0}} = n_{0} \frac{\log n_{0}}{\log n_{0}} = 1 \), we get

\( \frac{n_{1}}{(n_{0})^{\frac{1}{r} + 1}} = \frac{\frac{n_{1}}{n_{0}}}{(n_{0})^{\frac{1}{r} + 1}} = \frac{\frac{n_{1}}{n_{0}}}{(n_{0})^{\frac{1}{r} + 1}} = 1 \), where the last inequality follows from \( r < r_{0} \). So \( \frac{n_{1}}{(n_{0})^{\frac{1}{r} + 1}} = \left( \frac{n_{1}}{n_{0}} \frac{1}{r_{0}} + 1 \right)^{r} = n_{0} \frac{1}{r_{0}} + 1 \)

Let us now estimate \( t = \frac{n_{1}}{(n_{0})^{\frac{1}{r} + 1}} \). Observe that \( r^{2} < r_{0}^{2} = \frac{\log n_{1}}{\log n_{0}} \leq \log^{2} n_{1} \). Also,

\( \frac{n_{1}}{n_{0}} \frac{1}{r} = 2 \frac{\log n_{1} - \log n_{0}}{r} = 2 \left( \frac{1}{4 \log n_{1} - \log n_{0}} \right) (\log n_{1} - \log n_{0})^{\frac{1}{2} - \frac{1}{r}} \leq n_{0} \frac{1}{r_{0}} - \log n_{0} \frac{1}{r_{0}} \leq x_{0}^{1+o(1)} \).
where the last inequality stems from the assumption that \( n_0 = (\log n_1)^{\omega(1)} \).

Since \( x_0 = O(k) \), \( \left( \frac{n_0}{n_0} \right)^{\frac{1}{2}} \leq x_0^{1+o(1)} = (O(k))^{1+o(1)} = O((\log n_1)^{1+o(1)}) \). Therefore \( t = \left( \frac{n_1}{n_0} \right)^{\frac{1}{2}} r^2 = O((\log n_1)^{3+o(1)}) \ll \log^4 n_1 \).

This also ends the proof of Lemma 3.3, and therefore of the lower bound and of Theorem 1.

4 Generalization - Multi-Partite Graphs

We wish to estimate the choice number of a general \((s + 1)\)-partite graph \( K_{n_0, n_1, \ldots, n_s} \). In the graph \( K_{n_0, n_1, \ldots, n_s} \) we label the group of \( n_i \) vertices by \( V_i \), for each \( 0 \leq i \leq s \). Using a proof similar to that of the bipartite case, we will prove:

**Theorem 4** Let \( s \geq 1 \) be a fixed integer. Let \( 2 \leq n_0 \leq n_1 \leq \ldots \leq n_s \), and assume that \( n_0 = (\log n_s)^\alpha \), where \( \alpha \geq 2 \sqrt{\frac{\log n_s}{\log \log n_s}} \). For every \( 0 \leq i \leq s - 1 \) denote \( k_i = \frac{\log n_s}{\log n_i} \). Let \( x_0 \) be the unique root of the equation \( sx - 1 - \sum_{j=0}^{s-1} x^{-k_j} = 0 \) in the interval \([1, \infty)\). Then \( \text{ch}(K_{n_0, n_1, \ldots, n_s}) = (1 + o(1)) \frac{\log n_s}{\log x_0} \).

Observe that in the most basic case of equally sized parts (i.e. whenever \( n_0 = \ldots = n_s \)), we have \( x_0 = (s + 1)/s \), and thus \( \text{ch}(K_{n_0, n_0, \ldots, n_0}) = (1 + o(1)) \log n_0 / \log((s + 1)/s) \). Since \( \log((s + 1)/s) = \Theta(1/s) \), we recover the result of Alon [1] mentioned in the introduction.

Again we divide the proof into two parts – the upper bound and the lower bound.

5 The Upper Bound for Multi-Partite Graphs

**Theorem 5** Let \( 2 \leq n_0 \leq \ldots \leq n_s \) be integers, and let \( 0 < \epsilon < 1 \) be a constant. For every \( 0 \leq i \leq s - 1 \) denote \( k_i = \frac{\log n_s}{\log n_i} \). Let \( x_0 \) be the unique
root of the equation \((s + \epsilon) \cdot x - 1 - \sum_{j=0}^{s-1} x^{k_j-1} = 0\) in the interval \([1, \infty)\).

Define \(r = \lceil \log n_s / \log x_0 \rceil + 1\). Then \(ch(K_{n_0, \ldots, n_s}) \leq r\), for \(n_s\) large enough.

**Proof.**

**Lemma 5.1** If there exist \(p_0, \ldots, p_s\) such that \(0 \leq p_i \leq 1\) for every \(0 \leq i \leq s\), \(\sum_{i=0}^{s} p_i = 1\) and \(\sum_{i=0}^{s} n_i (1 - p_i)^r \leq 1\), then \(ch(K_{n_0, \ldots, n_s}) \leq r\).

**Proof.** The proof is identical to that of the bipartite case (Lemma 2.1), only this time we partition the set of all available colors into \(s+1\) sets, using the probabilities \(p_i\). A bad event for a vertex \(v \in V_i\) is one in which all the colors in \(S(v)\) are chosen to be in color groups other than \(S_i\), and it happens with probability \((1 - p_i)^r\). ■

**Lemma 5.2** Given \(r\) s.t. \(\sum_{i=0}^{s} n_i^{1 - \frac{1}{r-1}} \geq s^{\frac{1}{r-1}}\), let \(p_i = 1 - \frac{sn_i^{1 - \frac{1}{r-1}}}{\sum_{j=0}^{s} n_j^{1 - \frac{1}{r-1}}}\) for \(0 \leq i \leq s\). Then \(0 \leq p_i \leq 1\) for each \(0 \leq i \leq s\), \(\sum_{i=0}^{s} p_i = 1\), and \(\sum_{i=0}^{s} n_i (1 - p_i)^r \leq 1\).

**Proof.** In order for \(p_i\) to be non-negative, we must demand that for every \(0 \leq i \leq s\), \(\frac{sn_i^{1 - \frac{1}{r-1}}}{\sum_{j=0}^{s} n_j^{1 - \frac{1}{r-1}}} \leq 1\), or \(s \leq \sum_{j=0}^{s} (\frac{n_j}{n_i})^{\frac{1}{r-1}}\). But if \(s^{\frac{1}{r-1}} \leq \sum_{j=0}^{s} n_j^{\frac{1}{1-r}}\), then for every \(0 \leq i \leq s\), \(s^{\frac{1}{r-1}} \leq \sum_{j=0}^{s} n_j^{\frac{1}{1-r}} \leq \sum_{j=0}^{s} (\frac{n_j}{n_i})^{\frac{1}{1-r}}\). Also,

\[
\sum_{i=0}^{s} p_i = s + 1 - \sum_{i=0}^{s} (1 - p_i) = s + 1 - \sum_{i=0}^{s} \frac{s(n_i^{1 - \frac{1}{r-1}})}{\sum_{j=0}^{s} n_j^{1 - \frac{1}{r-1}}} = s + 1 - s = 1.
\]

If \(1 - p_i = \frac{sn_i^{1 - \frac{1}{r-1}}}{\sum_{j=0}^{s} n_j^{1 - \frac{1}{r-1}}}\) then \((\frac{1-p_i}{1-p_j})^{r-1} = \frac{n_j}{n_i}\). Therefore, for any \(i\),

\[
\sum_{j=0}^{s} n_j (1 - p_j)^r = n_i (1 - p_i)^r \sum_{j=0}^{s} (1 - p_j) = s \cdot n_i (1 - p_i)^{r-1}
\]

\[
= s \cdot n_i \left( \frac{sn_i^{1 - \frac{1}{r-1}}}{\sum_{j=0}^{s} n_j^{1 - \frac{1}{r-1}}} \right)^{r-1} = \left( \frac{s^{\frac{r}{r-1}}}{\sum_{j=0}^{s} n_j^{\frac{1}{r-1}}} \right)^{r-1} \leq 1.
\]
Let $r = \lceil \log x_0 \rceil + 1$. Then $r - 1 \geq \frac{\log n_s}{\log x_0}$, and thus $x_0 \geq n_s^{\frac{1}{r-1}}$.

Since the function $g_{k_0,...,k_{s-1},\epsilon}(x) = (s + \epsilon) \cdot x - 1 - \sum_{j=0}^{s-1} x^{\frac{k_j - 1}{k_j}}$, where $k_j \geq 1$ for each $j$, is a monotonically increasing function in the interval $[1, \infty)$, and since $g_{k_0,...,k_{s-1},\epsilon}(x_0) = 0$, it follows that for $r$ large enough, or for $n_s$ large enough (see Lemma 6.2 below, and the beginning of the proof of Lemma 3.3),

\[ \frac{1}{s r - 1} n_s^{\frac{1}{r-1}} \leq (s + \epsilon) n_s^{\frac{1}{r-1}} \leq 1 + \sum_{j=0}^{s-1} n_s^{\frac{1}{r-1} \frac{k_j - 1}{k_j}} = 1 + \sum_{i=0}^{s-1} \left( \frac{n_s}{n_i} \right)^{\frac{1}{r-1}} \text{ as required.} \]

6 The Lower Bound for Multi-Partite Graphs

Theorem 6 Let $2 \leq n_0 \ldots \leq n_s$ be integers, and let $n_0 = (\log n_s)^\alpha$, where $\alpha \geq 2 \sqrt{\frac{\log n_s}{\log \log n_s}}$. For every $0 \leq i \leq s - 1$ denote $k_i = \frac{\log n_s}{\log n_i}$. Let $x_0$ be the unique root of the equation $s \cdot x - 1 - \sum_{j=0}^{s-1} x^{\frac{k_j - 1}{k_j}} = 0$ in the interval $[1, \infty)$. Then $\text{ch}(K_{n_0,...,n_s}) \geq (1 - o(1)) \frac{\log n_s}{\log x_0}$.

Proof. Similarly to the bipartite case, in order to prove $\text{ch}(K_{n_0,...,n_s}) > r$, it is enough to show that there are a $t \geq r$ and a sequence of $0 \leq l_i \leq t$ for which $\sum_{i=0}^{s} l_i = t$, s.t. it is possible to choose for each vertex in $K_{n_0,...,n_s}$ a list of $r$ of colors from $\{1, 2, \ldots t\}$, and the lists chosen satisfy the following $s$ conditions: For each $0 \leq i \leq s - 1$ the minimum cover of the hypergraph created by the color lists of the vertices in $V_i$ is of cardinality at least $l_i$, and the additional condition: the minimum cover of the hypergraph created by the color lists of the vertices in $V_s$ is of cardinality at least $l_s + 1$.

As in the bipartite case, if these conditions are satisfied, then by the pigeonhole principle at least 2 vertices in different groups must be given the same color, so the choice number is greater than $r$. 

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Lemma 6.1 If there exist a parameter \( t \geq r \) and a sequence of \( 0 \leq l_i \leq t \) for which \( \sum_{i=0}^{s} l_i = t \) and
\[
\sum_{i=0}^{s} 2^i e^{-\frac{(t-l_i)x_{n_i}}{r}} \leq 1
\]
then \( ch(K_{n_0,...,n_s}) > r \).

**Proof.** Similar to the bipartite case. \( \blacksquare \)

As in the bipartite case, we calculate bounds on \( x_0 \) that will help us later on.

Lemma 6.2 \( \frac{s+1}{s} \leq x_0 < \max(k_0, e + 2) \)

**Proof.** Since for every \( 0 \leq i \leq s, n_0 \leq n_i \), it follows that \( k_0 = \frac{\log n_s}{\log n_0} \geq \frac{\log n_s}{\log n_i} = k_i \). Therefore, for a given \( x \) in the range \([1, \infty)\), \( x^\frac{k_0-1}{k_0} \geq x^\frac{k_i-1}{k_i} \) for all \( i \), and \( f_{k_0,...,k_{s-1}}(x) = sx - 1 - \sum_{i=0}^{s-1} x^\frac{k_i}{k_i} \geq sx - 1 - sx^\frac{k_0-1}{k_0} = s(x-x^\frac{k_0-1}{k_0}) - 1 \geq x - x^\frac{k_0-1}{k_0} - 1 \) (note all these functions increase monotonically as functions of \( x \)). Therefore the root \( x_0 \) in the range \([1, \infty)\) of the first equation \( sx - 1 - \sum_{i=0}^{s-1} x^\frac{k_i}{k_i} = 0 \), which is our equation, is not greater than the root \( x_1 \) of the equation \( x - x^\frac{k_0-1}{k_0} - 1 = 0 \).

But the last equation is \( f_{k_0}(x) = 0 \), and we already know from the bipartite case that its root is smaller than \( \max(k_0, e + 2) \).

To prove the lower bound observe that \( f_{k_0,...,k_{s-1}}(\frac{s+1}{s}) = s + 1 - 1 - \sum_{j=0}^{s-1} (\frac{s+1}{s})^{\frac{k_j}{k_j}} \leq s - s = 0 \), and thus by monotonicity \( x_0 \geq \frac{s+1}{s} \). \( \blacksquare \)

Lemma 6.3 Let \( n_0 = (\log n_s)^{\alpha} \), where \( \alpha \geq 2 \sqrt{\frac{\log n_s}{\log \log n_s}} \). Define \( r_0 = \frac{\log n_s}{\log n_0} \), \( u = \frac{4 \log \log n_s}{\log n_0} r_0 \) and \( r = r_0 - u \). Then \( r = (1 - o(1))r_0 \), and for \( t = (\frac{1}{s} \sum_{j=0}^{s} (\frac{n_j}{n_0})^\frac{1}{2} - 1)^2 \) and \( t - l_i = \frac{s(n_j)^{1/2}}{\sum_{i=0}^{s} (n_j)^{1/2}} \), one has: \( 0 \leq l_i \leq t, \sum_{i=0}^{s} l_i = t \), and \( \sum_{i=0}^{s} 2^i e^{-\frac{(t-l_i)x_{n_i}}{r}} \leq 1 \), i.e., the assumptions of Lemma 6.1 are satisfied.
Proof. Since \( n_0 = (\log n_s)^{\omega(1)} \), it follows that \( r = (1 - o(1))r_0 \), as in the bipartite case. Also, again as in the bipartite case, from \( x_0 < \max(k_0, e + 2) \) it follows that \( r_0 = \omega(1) \), and therefore \( r = \omega(1) \).

We need to show that for every \( i \), \( 0 \leq l_i \leq t \), or \( 0 \leq t - l_i \leq t \). Since \( t - l_i \) is obviously non-negative, we need to prove that \( t - l_i \leq t \), or \( \frac{s(\omega)}{\sum_{j=0}^{s} (\frac{n_j}{n_s})^r} \leq 1 \), or \( s \leq \sum_{j=0}^{s} (\frac{n_j}{n_s})^r \). Since \( n_0 \leq n_i \) for every \( i \), it is enough to show \( s \leq \sum_{j=0}^{s} (\frac{n_j}{n_s})^r \).

Since \( r_0 = \frac{\log n_s}{\log 2^n} \), we have: \( x_0 = n_s^{\frac{1}{r_0}} \), and so \( sn_s^{\frac{1}{r}} = 1 + \sum_{j=0}^{s-1} n_s^{\frac{1}{r_0}} n_j^{k_j-1} \). Therefore

\[
\sum_{j=0}^{s} \left( \frac{n_j}{n_s} \right)^r = \sum_{j=0}^{s} \left( \frac{1}{n_j} \right)^{r_0} n_0^{\frac{1}{r_0}} - \frac{n_j^{k_j-1}}{n_j^{k_j}} \sum_{j=0}^{s} \left( \frac{1}{n_j} \right)^{r_0} = s \frac{n_0^{\frac{1}{r_0}}}{n_s^{r_0}},
\]

so it is enough to show \( \frac{n_0^{\frac{1}{r_0}}}{n_s^{r_0}} \geq 1 \). But \( \frac{1}{r} - \frac{1}{r_0} = \frac{n_s}{r_0 n_s} \), so \( \frac{n_0^{\frac{1}{r_0}}}{n_s^{r_0}} = 2^{-\frac{1}{r} \log n_s^\frac{1}{r_0}} = 2^{-\frac{1}{r_0} \log n_s^\frac{1}{r_0}} = 2^{-\frac{1}{r} \log n_s^\frac{1}{r_0}} \). Also \( n_0^{\frac{1}{r}} = (\log n_s)^{\frac{1}{r}} = 2^{-\frac{1}{r} \log n_s^\frac{1}{r_0}} \). Therefore

\[
\frac{n_0^{\frac{1}{r_0}}}{n_s^{r_0}} = \left( 2^{\alpha \log \log n_s - \log n_s^\frac{1}{r_0}} \right)^{\frac{1}{r}} \geq 1,
\]

where the last inequality stems from the condition on \( \alpha \). Also,

\[
\sum_{i=0}^{s} l_i = (s + 1)t - \sum_{i=0}^{s} (t - l_i) = (s + 1)t - \sum_{i=0}^{s} t \frac{s(n_j)}{\sum_{j=0}^{s} (\frac{n_j}{n_s})^r} = st + t - st = t.
\]

All that is left for us to verify is that Condition (2) is fulfilled. The proof is, again, similar to the bipartite case.

Claim 6.4 \( \frac{(t-l_i)}{(t-r_0)} \frac{n_i}{\left( \sum_{j=0}^{s} (\frac{n_j}{n_s})^r \right)^{\frac{1}{2r_0}}} \) for \( 0 \leq i \leq s \).

Proof. We have: \( \frac{(t-l_i)}{(t-r_0)} > \frac{(t-l_i)}{(t-r)} = \left( \frac{t-l_i}{t} \right)^r \left( 1 - \frac{l_i}{t(t-l_i)} \right)^r \) \( > \left( \frac{t-l_i}{t} \right)^r \) \( \frac{s(n_j)}{\left( \sum_{j=0}^{s} (\frac{n_j}{n_s})^r \right)^{\frac{1}{2r_0}}} \), where the last inequality is a result of \( r < \frac{1}{2} \). By definition \( t-l_i = t - \frac{s(n_j)}{\left( \sum_{j=0}^{s} (\frac{n_j}{n_s})^r \right)^{\frac{1}{2r_0}}} \), so \( l_i = \frac{t}{\sum_{j=0}^{s} (\frac{n_j}{n_s})^r - s(n_j)^r} \), and \( \frac{l_i}{s-l_i} = \frac{\sum_{j=0}^{s} (\frac{n_j}{n_s})^r - s(n_j)^r}{s(n_j)^r} = \frac{1}{2} \sum_{j=0}^{s} (\frac{n_j}{n_s})^r - 1. \)
Now since \( \frac{\log r}{(t-l_i)_e} = \left( \frac{1}{s} \sum_{j=0}^{s} \left( \frac{n_j}{n_j} \right)^{\frac{1}{2}} \right)^{\frac{1}{r}} \leq \left( \frac{1}{s} \sum_{j=0}^{s} \left( \frac{n_j}{n_j} \right)^{\frac{1}{2}} \right)^{\frac{1}{r}} = \frac{r}{s} = o(1) \), we get \( \frac{(t-l_i)_e}{t} > \left( \frac{1-l_i}{t} \right)^{\frac{1}{2}} \). Hence
\[
\frac{(t-l_i)_e}{t} n_i > \left( \frac{1-l_i}{t} \right)^{r} n_i \frac{1}{2e^2} = \left( \frac{s \left( \frac{n_s}{n_s} \right)^{\frac{1}{2}}} {\sum_{j=0}^{s} \left( \frac{n_j}{n_j} \right)^{\frac{1}{2}}} \right)^{r} n_i \frac{1}{2e^2} = \frac{s^\alpha n_s}{\sum_{j=0}^{s} \left( \frac{n_j}{n_j} \right)^{\frac{1}{2}}} \cdot \frac{1}{2e^2}.
\]

Therefore in order to prove that (2) holds it is now enough to prove that
\[
\frac{s^\alpha n_s}{\sum_{j=0}^{s} \left( \frac{n_j}{n_j} \right)^{\frac{1}{2}}} \gg t \text{ (assuming } s \text{ is constant).}
\]

**Claim 6.5** \( \frac{s^\alpha n_s}{\sum_{j=0}^{s} \left( \frac{n_j}{n_j} \right)^{\frac{1}{2}}} \gg t. \)

**Proof.** We have:
\[
\frac{s^\alpha n_s}{\sum_{j=0}^{s} \left( \frac{n_j}{n_j} \right)^{\frac{1}{2}}} = \left( \frac{n_s^{\frac{1}{r}}}{\sum_{j=0}^{s} \left( \frac{n_j}{n_j} \right)^{\frac{1}{2}}} \right)^{\frac{1}{r}} \geq \left( \frac{n_s^{\frac{1}{r}}}{\sum_{j=0}^{s} \left( \frac{n_j}{n_j} \right)^{\frac{1}{2}}} \right)^{\frac{1}{r}} \geq \left( \frac{n_s^{\frac{1}{r}}}{\sum_{j=0}^{s} \left( \frac{n_j}{n_j} \right)^{\frac{1}{2}}} \right)^{\frac{1}{r}} \geq \left( \frac{n_s^{\frac{1}{r}}}{\sum_{j=0}^{s} \left( \frac{n_j}{n_j} \right)^{\frac{1}{2}}} \right)^{\frac{1}{r}}.
\]

Since \( \frac{s^x}{\sum_{j=0}^{s} \left( \frac{n_j}{n_j} \right)^{\frac{1}{2}}} = \frac{s^x}{\sum_{j=0}^{s} \left( \frac{n_j}{n_j} \right)^{\frac{1}{2}}} = 1 \), we get \( \frac{s^\alpha n_s}{\sum_{j=0}^{s} \left( \frac{n_j}{n_j} \right)^{\frac{1}{2}}} = \left( \frac{n_s^{\frac{1}{r}}}{\sum_{j=0}^{s} \left( \frac{n_j}{n_j} \right)^{\frac{1}{2}}} \right)^{\frac{1}{r}} \). Now,
\[
\frac{s^x}{\sum_{j=0}^{s} \left( \frac{n_j}{n_j} \right)^{\frac{1}{2}}} = \frac{s^x}{\sum_{j=0}^{s} \left( \frac{n_j}{n_j} \right)^{\frac{1}{2}}} = \frac{s^x}{\sum_{j=0}^{s} \left( \frac{n_j}{n_j} \right)^{\frac{1}{2}}} \geq \frac{s^x}{\sum_{j=0}^{s} \left( \frac{n_j}{n_j} \right)^{\frac{1}{2}}}.
\]

where the last inequality is a result of \( n_i \geq n_0 \) for all \( 1 \leq i \leq s \) and of \( r < r_0 \). So
\[
\sum_{j=0}^{s} \left( \frac{n_j}{n_j} \right)^{\frac{1}{2}} \geq n_0 \frac{1}{r_0} \text{, and } \frac{s^\alpha n_s}{\sum_{j=0}^{s} \left( \frac{n_j}{n_j} \right)^{\frac{1}{2}}} \geq \left( \frac{n_0}{r_0} \right)^{\frac{1}{r}} = n_0^{1-\frac{1}{r_0}} = n_0 \frac{1}{r_0} = \frac{\log n_0}{\log n_0} = \log^4 n_s.
\]

Let us now estimate \( t = \left( \frac{1}{s} \sum_{j=0}^{s} \left( \frac{n_j}{n_j} \right)^{\frac{1}{2}} \right)^{\frac{1}{r}} \). First, \( r^2 < r_0^2 = \left( \frac{\log n_s}{\log x_0} \right)^2 \leq \left( \frac{\log n_s}{\log x_0} \right)^2 = C \log^2 n_s \) where \( C = C(s) \) is a constant. Second,
\[
\left( \frac{n_s}{n_0} \right)^{\frac{1}{r}} = 2 \frac{\log n_0}{\log n_0} \frac{1}{r} \leq 2 \frac{\log n_0}{\log n_0} \frac{1}{r} = x_0 \frac{1+o(1)}{4 \log \log n_0} \leq x_0 \frac{1+o(1)}{4 \log \log n_0}.
\]

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where the last inequality stems from the assumption that $n_0 = (\log n_s)^{\omega(1)}$.

Since $x_0 = O(k_0)$, we get: $(\frac{n}{n_0})^{\frac{1}{r}} \leq x_0^{1+o(1)} = (O(k_0))^{1+o(1)} = O((\log n_s)^{1+o(1)})$.

Therefore

$$t = \left( \frac{1}{s} \sum_{j=0}^{s} \left( \frac{n_s}{n_j} \right)^{\frac{1}{r}} - 1 \right) r^2 = \left( \frac{1}{s} \sum_{j=0}^{s-1} \left( \frac{n_s}{n_j} \right)^{\frac{1}{r}} - \frac{s - 1}{s} \right) r^2$$

$$\leq \left( \frac{1}{s} \sum_{j=0}^{s-1} \left( \frac{n_s}{n_j} \right)^{\frac{1}{r}} \right) r^2 \leq \frac{1}{s} \frac{n_s}{n_0} \frac{1}{r} r^2 = \left( \frac{n_s}{n_0} \right)^{\frac{1}{r}} r^2 = O((\log n_s)^{3+o(1)})$$

$$\ll \log^4 n_s .$$

This also ends the proof of Lemma 6.3, and therefore of the lower bound of the multi-partite case and of Theorem 4.
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