An optimization problem on the sphere

Andreas Maurer
Adalbertstr 55
D80799 München

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Abstract
We prove existence and uniqueness of the minimizer for the average geodesic distance to the points of a geodesically convex set on the sphere. This implies a corresponding existence and uniqueness result for an optimal algorithm for halfspace learning, when data and target functions are drawn from the uniform distribution.

1 Introduction
Let \( S^{n-1} \) be the unit sphere in \( \mathbb{R}^n \) with normalized uniform measure \( \sigma \) and geodesic metric \( \rho \), and let \( K \) be a proper convex cone with nonempty interior in \( \mathbb{R}^n \). We will show that the function \( \psi : S^{n-1} \rightarrow \mathbb{R} \) defined by

\[
\psi_K (w) = \int_{K \cap S^{n-1}} \rho (w, y) \, d\sigma (y)
\]

attains its global minimum at a unique point on \( S^{n-1} \). While existence of the minimum is straightforward, uniqueness seems surprisingly difficult to prove.

A similar problem has been considered in \[2\] and \[1\]. In these works the intention is to define a centroid, so integration is replaced by finite summation and \( \rho (w, y) \) replaced by \( \rho (w, y)^2 \). Since the problem is rather obvious, it appears likely that a proof of the above result exists somewhere in the literature and we just haven’t been able to find it.

2 Optimal halfspace learning
Our motivation to consider this problem arises in learning theory. Specifically we consider an experiment, where

1. A unit vector \( u \) is drawn at random from \( \sigma \) and kept concealed from the learner.
2. A sample \( x = (x_1, \ldots, x_m) \in (S^{n-1})^m \) is generated in \( m \) independent random trials of \( \sigma \).

3. A label vector \( y = u(x) \in \{-1, 1\}^m \) is generated according to the rule \( y_i = \text{sign}(\langle u, x_i \rangle) \), where \( \langle ., . \rangle \) is the euclidean inner product and \( \text{sign}(t) = 1 \) if \( t > 0 \) and \( -1 \) if \( t < 0 \). The \( \text{sign} \) of 0 is irrelevant, because it corresponds to events of probability zero.

4. The labeled sample \((x, y) = (x, u(x))\) is supplied to the learner.

5. The learner produces a hypothesis \( f(x, y) \in S^{n-1} \) according to some learning rule \( f : (S^{n-1})^m \times \{-1, 1\}^m \rightarrow S^{n-1} \).

6. An unlabeled test point \( x \in S^{n-1} \) is drawn at random from \( \sigma \) and presented to the learner who produces the label \( y = \text{sign}(\langle f(x, y), x \rangle) \).

7. If \( \text{sign}(\langle u, x_i \rangle) = y \) the learner is rewarded one unit, otherwise a penalty of one unit is incurred.

We now ask the following question: Which learning rule \( f \) will give the highest average reward on a very large number of independent repetitions of this experiment?

Evidently the optimal learning rule has to minimize the following functional:

\[
\Omega(f) = \mathbb{E}_{u \sim \sigma} \mathbb{E}_{x \sim \sigma} \mathbb{P}_x \{ \text{sign}(\langle f(x, u(x)), x \rangle) \neq \text{sign}(\langle u, x \rangle) \}.
\]

Now a simple geometric argument shows that for any \( v, u \in S^{n-1} \) we have

\[
\mathbb{P}_x \{ \text{sign}(\langle v, x \rangle) \neq \text{sign}(\langle u, x \rangle) \} = \rho(v, u) / \pi,
\]

relating the misclassification probability to the geodesic distance. For a labeled sample \((x, y) \in (S^{n-1})^m \times \{-1, 1\}^m\) we denote

\[
C(x, y) = \{ u \in S^{n-1} : u(x) = y \}.
\]

\( C(x, y) \) is thus the set of all hypotheses consistent with the labeled sample \((x, y)\). Observe that, given \( x \) and \( u \) there is exactly one \( y \) such that \( y = u(x) \), that is \( u \in C(x, y) \). We also have \( C(x, y) = K(x, y) \cap S^{n-1} \) where \( K(x, y) \) is the closed convex cone

\[
K(x, y) = \{ v \in \mathbb{R}^n : \langle u, y, x_i \rangle \geq 0, \forall 1 \leq i \leq m \}.
\]

We therefore obtain

\[
\Omega(f) = \pi^{-1} \mathbb{E}_{u \sim \sigma} \mathbb{E}_{x \sim \sigma} \rho(\langle f(x, u(x)), u \rangle) = \pi^{-1} \mathbb{E}_{x \sim \sigma} \sum_{y \in \{-1, 1\}^m} \mathbb{E}_{u \sim \sigma} \rho(\langle f(x, u(x)), u \rangle) 1_{C(x, y)}(u) = \pi^{-1} \mathbb{E}_{x \sim \sigma} \sum_{y \in \{-1, 1\}^m} \mathbb{E}_{u \sim \sigma} \rho(\langle f(x, y), u \rangle) 1_{C(x, y)}(u) = \pi^{-1} \mathbb{E}_{x \sim \sigma} \sum_{y \in \{-1, 1\}^m} \psi_{K(x, y)}(f(x, y)).
\]
If $K(x, y)$ has empty interior then the corresponding summand vanishes, so we can assume that $K(x, y)$ has nonempty interior. Clearly $-y_i x_i \notin K(x, y)$ for all example points, so $K(x, y)$ is a proper cone. Our result therefore applies and asserts the existence of a unique minimizer $f^*(x, y)$ of the function $\psi_{K(x, y)}$. The map $f^* : (x, y) \mapsto f^*(x, y)$ is then the unique optimal learning algorithm.

The map $f^*$ also has the symmetry property $f^*(Vx, y) = Vf^*(x, y)$ for any unitary $V$ on $\mathbb{R}^n$. This is so, because $\psi_{K(x, y)}(Vx, y) = \psi_{K(x, y)}(V^{-1}w)$, as is easily verified. We will also show, that the solution $f^*(x, y)$ must lie in the cone

$$\left\{ \sum_{i=1}^m \alpha_i y_i x_i : \alpha_i \geq 0 \right\}$$

and that $\psi_{K(x, y)}$ has no other local minima.

### 3 Proof of the main result

**Notation 1** $\rho(., .)$ is the geodesic distance and $\sigma$ the Haar measure on $S^{n-1}$. For $A \subseteq \mathbb{R}^n$ we denote $A_1 = \{ x \in A : \|x\| = 1 \} = A \cap S^{n-1}$. 'Cone' will always mean 'convex cone'. For $A \subseteq \mathbb{R}^n$ we denote

$$\hat{A} = \{ x : \langle x, v \rangle \geq 0, \forall v \in A \}.$$

This is always a closed convex set. A proper cone $K$ is one which is contained in some closed halfspace. For a set $A$ the indicator function of $A$ will be denoted by $1_A$.

**Lemma 2** Let $K$ be a closed cone

(i) If $w \notin K$ then there is a unit vector $z \in \mathbb{R}^n$ such that $\langle z, w \rangle < 0$ and $\langle z, y \rangle \geq 0$ for all $y \in K$.

(ii) $\hat{K} = K$.

(iii) Suppose that $K$ is proper and has nonempty interior, $w \in S^{n-1}$, $w \notin \hat{K} \cup (-\hat{K})$ and $\epsilon > 0$. Then there exists $z$ with $\|z\| = 1$ such that $-\epsilon < \langle z, w \rangle < 0$ and $\langle z, y \rangle > 0$ for all $y \in \hat{K} \setminus \{0\}$.

**Proof.** (i) Let $B$ be an open ball containing $w$ such that $K \cap B = \emptyset$. Define

$$O = \{ \lambda x : x \in B, \lambda > 0 \}.$$

Then $K$ and $O$ are nonempty disjoint convex sets and $O$ is open. By the Hahn-Banach theorem ([1], Theorem 3.4) there is $\gamma \in \mathbb{R}$ and $z \in \mathbb{R}^n$ such that

$$\langle z, x \rangle < \gamma \leq \langle z, y \rangle, \forall x \in O, y \in K.$$
Choosing \( y = 0 \in K \) gives \( \gamma \leq 0 \), letting \( \lambda \to 0 \) in \( \langle z, \lambda w \rangle < \gamma \) shows \( \gamma \geq 0 \), so that \( \gamma = 0 \). The normalization is trivial.

(i) Trivially \( K \subseteq \hat{K}^\circ \). On the other hand, if \( w \notin K \) let \( z \) be the vector from part (i). Then \( z \in \hat{K} \) but \( \langle w, z \rangle < 0 \), so that \( w \notin \hat{K}^\circ \).

(ii) Since \( w \notin \hat{K} \) there exists \( x_1 \in K \) s.t. \( \langle w, x_1 \rangle < 0 \). Since the interior of \( K \) is nonempty, \( K \) is the closure of its interior (Theorem 6.3 in [3]), so we can assume \( x_1 \in \text{int}(K) \). Similarly, since \( w \notin \left(\hat{K}^\circ\right) \) we have \( -w \notin \hat{K} \), so there is \( x_2 \in \text{int}(K) \) with \( \langle -w, x_2 \rangle < 0 \), that is \( \langle w, x_2 \rangle > 0 \). Since the interior of \( K \) is convex it contains the segment \([x_1, x_2]\), so by continuity of \( \langle w, \cdot \rangle \) there is some \( x_0 \in \text{int}(K) \) with \( \langle w, x_0 \rangle = 0 \). Since \( K \) is a proper cone \( 0 \notin \text{int}(K) \) and we can assume that \( \|x_0\| = 1 \).

Let \( c > 0 \) be such that \( x' \in K \) whenever \( \|x_0 - x'\| \leq c \). We define

\[
z = (1 - \eta)^{1/2} x_0 - \eta^{1/2} w, \text{ where } 0 < \eta < \min \left\{ \frac{c^2}{1 + c^2}, c^2 \right\}.
\]

Since \( \langle w, x_0 \rangle = 0 \) it is clear that \( z \) is a unit vector. Also \( \langle w, z \rangle = -\eta^{1/2} > -c \), and for any \( y \in \hat{K} \) we have \( x_0 - cy \in K \), so \( \langle y, x_0 - cy \rangle \geq 0 \) and

\[
\langle y, z \rangle = (1 - \eta)^{1/2} \langle y, x_0 - cy \rangle + c \langle y, y \rangle - \eta^{1/2} \langle y, w \rangle \\
\geq (1 - \eta)^{1/2} c - \eta^{1/2} > 0.
\]

Theorem 3 Let \( K \subset \mathbb{R}^{n-1} \) be a closed proper cone with nonempty interior, \( g : [0, \pi] \to \mathbb{R} \) continuous and the function \( \psi : S^{n-1} \to \mathbb{R} \) defined by

\[
\psi(w) = \int_K g(\rho(w, y)) d\sigma(y).
\]

(i) \( \psi \) attains its global minimum on \( S^{n-1} \).

(ii) If \( g \) is increasing then every local minimum of \( \psi \) must lie in \( \hat{K} \cup (-\hat{K}) \) and every global minimum of \( \psi \) must lie in \( K \cap \hat{K} \).

(iii) If \( g \) is increasing and convex in \( [0, \pi/2] \) then the global minimum of \( \psi \) is unique and corresponds to the only local minimum outside \( -\hat{K} \).

(iv) If \( g \) is increasing, convex in \([0, \pi/2]\) and concave in \([\pi/2, \pi]\), then the global minimum of \( \psi \) is unique and corresponds to its only local minimum on \( S^{n-1} \).

Proof. (i) is immediate since \( S^{n-1} \) is compact and \( \psi \) is continuous.

(ii) Fix \( w \in S^{n-1}, w \notin \hat{K} \cup (-\hat{K}) \). We will first show that there can be no local minimum of \( \psi \) at \( w \). Let \( \epsilon > 0 \) be arbitrary and choose \( z \) as in the lemma (iii). The functional \( z \) divides the sphere \( S^{n-1} \) into two open hemispheres

\[
L = \{u : \langle z, u \rangle < 0\} \quad \text{and} \quad R = \{u : \langle z, u \rangle > 0\}.
\]
and an equator of $\sigma$-measure zero. Note that $w \in L$ and $\hat{K}_1 \subseteq R$. We can write

$$c = \min_{y \in \hat{K}_1} \langle y, z \rangle > 0,$$

since $\hat{K}_1$ is compact and $y \mapsto \langle y, z \rangle$ is continuous. With $V$ we denote the reflection operator which exchanges points of $L$ and $R$

$$V x = -\langle x, z \rangle z + (x - \langle x, z \rangle z).$$

$V$ is easily verified to an isometry and $V^2 = I$.

Suppose now that $u \in R$ and $Vu \in K$. We claim that $u$ is in the interior of $K$. Indeed, if $u' \in \mathbb{R}^n$ satisfies $\|u - u'\| < 2 \langle u, z \rangle c$, then for all $y \in \hat{K}_1$ we have

$$\langle u', y \rangle = \langle u, y \rangle - \langle u - u', y \rangle \geq \langle u, y \rangle - 2 \langle u, z \rangle c$$

$$\geq \langle u, y \rangle - 2 \langle u, z \rangle \langle z, y \rangle = \langle Vu, y \rangle \geq 0,$$

so $u' \in \left(\hat{K}\right)^\circ = K$, by part (ii) of the lemma. This establishes the claim and shows that $V(K) \cap R$ is contained in the interior of $K$. It follows that

$$\forall u \in R, 1_K(u) \geq 1_K(Vu). \quad (1)$$

Also $V(K) \cap R$ is relatively closed in $R$ while $\text{int}(K) \cap R$ is open in $R$. Since $R$ is connected they can only coincide if $V(K) \cap R = R$. But this is impossible, since then

$$L \cup R = V(V(K) \cap R) \cup (V(K) \cap R) \subseteq V(V(K \cap L)) \cup \text{int}(K)$$

$$= (K \cap L) \cup \text{int}(K) \subseteq K,$$

and $K$ is assumed to be a proper cone. So $V(K) \cap R$ is a proper subset of $\text{int}(K) \cap R$. The inequality $\Box$ is therefore strict on the nonempty open set $(\text{int}(K) \cap R) \setminus (V(K) \cap R)$.

Using isometry and unipotence of $V$ we now obtain

$$\psi(w) - \psi(Vw) = \int_R (g(\rho(w, u)) - g(\rho(Vw, u))) 1_K(u) d\sigma(u) +$$

$$+ \int_L (g(\rho(w, u)) - g(\rho(Vw, u))) 1_K(u) d\sigma(u)$$

$$= \int_R (g(\rho(w, u)) - g(\rho(Vw, u))) (1_K(u) - 1_K(Vu)) d\sigma(u)$$

$$> 0.$$
$w$. We conclude that $w$ cannot be a local minimum of $\psi$, which proves the first assertion of (ii).

If $w \notin K$ choose $z$ as in part (i) of the lemma and let $W$ be the isometry $Wx = -\langle x, z \rangle z + (x - \langle x, z \rangle z)$. The $W \in K$ we have $g(\rho(w, u)) > g(\rho(Ww, u))$, so $\psi(w) > \psi(Ww)$ and $w$ cannot be a global minimiser of $\psi$. So every global minimiser must be in $K \cap \left(\hat{K} \cap (-\hat{K})\right)$. Since $K_1 \cap (-\hat{K}_1)$ is obviously empty the second assertion of (ii) follows.

(iii) Now let $w_1, w_2 \in K_1$ with $w_1 \neq w_2$. Connect them with a geodesic in $\hat{K}_1$ and let $w^* \in K_1$ be the midpoint of this geodesic, such that $\rho(w_1, w^*) = \rho(w^*, w_2) = \rho(w_1, w_2)/2 \leq \pi/2$. We define a map $U$ by

$$Ux = \langle x, w^* \rangle w^* - (x - \langle x, w^* \rangle) w^*.$$ 

Geometrically $U$ is reflection on the one-dimensional subspace generated by $w^*$. Note that $w_2 = Uw_1$ and that $\rho(u, Uu) = 2\rho(u, w^*)$ if $\rho(u, w^*) \leq \pi/2$ and that $\rho(u, Uu) = 2\pi - 2\rho(u, w^*)$ if $\rho(u, w^*) \geq \pi/2$.

Take any $u \in K_1$. Since $w^* \in K_1$ we have $\rho(u, w^*) \leq 2\pi$, whence $\rho(u, Uu) = 2\rho(u, w^*)$. All the four points $w_1, w_2, u$ and $Uu$ have at most distance $\pi/2$ from $w^*$ and lie therefore together with $w^*$ on a common hemisphere. By the triangle inequality

$$2\rho(u, w^*) = \rho(u, Uu) \leq \rho(u, w_1) + \rho(w_1, Uu) = \rho(u, w_1) + \rho(Uw_1, UUu) = \rho(u, w_1) + \rho(w_2, u).$$

If $u$ does not lie on the geodesic through $w_1$ and $w_2$ and not at distance $\pi/2$ from $w^*$ strict inequality holds, and since $K_1$ has nonempty interior strict inequality holds on an open subset of $K_1$. If $g$ is increasing and convex in $[0, \pi/2]$ then dividing by 2, applying $g$ and integrating over $K_1$ we get

$$\psi(w^*) < (1/2)\left(\psi(w_1) + \psi(w_2)\right).$$

It follows that there can be at most one point in $\hat{K}_1$ where the gradient of $\psi$ vanishes, and this point, if it exists, must correspond to a local minimum. By (ii) this is the unique global minimum and the only local minimum outside $-\hat{K}$, which establishes (iii).

(iv) If $x_1, x_2 \in -\hat{K}_1$ and $x^* \in -\hat{K}_1$ is their midpoint, then for $u \in K$ we obtain, using $\rho(x_i, u) = \pi - \rho(-x_i, u)$ and a reasoning analogous to the above,

$$\rho(u, w^*) \geq (1/2)\left(\rho(u, x_1) + \rho(u, x_2)\right),$$

the inequality being again strict on a set of positive measure and preserved under application of a function $g$ which is increasing and concave in $[\pi/2, \pi]$, so that

$$\psi(w^*) > (1/2)\left(\psi(x_1) + \psi(x_2)\right).$$

It again follows that there can be at most one point in $-\hat{K}_1$ where the gradient of $\psi$ vanishes, and this point must now correspond to a local maximum. We conclude that $\psi$ has a unique local minimum which lies in $\hat{K}_1$. \blacksquare
Remark. An example of a function as in (iii) is \( g(t) = t^2 \), in which case the minimizer is the spherical mass centroid considered in [2] and [1]. Examples of functions as in (iv) are of course the identity function, in which case we obtain the result stated in the introduction. We could also set \( g(t) = 2(1 - \cos t) \), in which case the function reads

\[
\psi(w) = \int_{K_1} \|w - y\|^2 \, d\sigma(y).
\]

In this case uniqueness of the minimum can be established with much simpler methods.

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