Higher Derivative Corrections
to Non-Abelian Vortex Effective Theory

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Abstract

We give a systematic method to calculate higher derivative corrections to low-energy effective theories of solitons, which are in general nonlinear sigma models on the moduli spaces of the solitons. By applying it to the effective theory of a single BPS non-Abelian vortex in $U(N)$ gauge theory with $N$ fundamental Higgs fields, we obtain four derivative corrections to the effective sigma model on the moduli space $\mathbb{C} \times \mathbb{CP}^{N-1}$. We compare them with the Nambu-Goto action and the Faddeev-Skyrme model. We also show that Yang-Mills instantons/monopoles trapped inside a non-Abelian vortex membrane/string are not modified in the presence of higher derivative terms.
1 Introduction

Solitons are smooth localized solutions of nonlinear partial differential equations, and they are ubiquitous in diverse fields in physics and mathematics. Topological solitons are smooth localized solutions of field equations in quantum field theories, and their stability is ensured by topological charges. A particularly interesting class of topological solitons is Bogomol’nyi-Prasad-Sommerfield (BPS) solitons \([1]\), which saturate minimum energy bounds called the Bogomol’nyi bounds. They are the most stable solutions with given topological numbers and satisfy first order partial differential equations called BPS equations. Prominent examples of such BPS solitons are instantons in Euclidean Yang-Mills theory \([2]\), ‘t Hooft-Polyakov monopoles \([3]\) in the Bogomol’nyi limit \([1]\), and Abrikosov-Nielsen-Olesen (ANO) vortices \([4]\) at the critical coupling (a standard reference is \([5]\)). Recent examples of BPS topological solitons in gauge theories are BPS kinks (domain walls) in Abelian \([6]\) and non-Abelian \([7]\) gauge theories.

While they preserve half of supersymmetry and are quantum mechanically stable in supersymmetric gauge theories \([8]\) on one hand, one of the most important features of BPS solitons is that there exist no static forces among them and consequently a continuous family of configurations with degenerate energy is allowed. As a result, generic solutions of BPS solitons contain collective coordinates, called moduli parameters as integration constants. The moduli parameters of BPS solitons parametrize the space of whole solutions of BPS equations, called the moduli space. Although there is no static force among BPS solitons, they non-trivially scatter each other when they are moving. While such dynamics was studied by computer simulations, it is very difficult to study it by means of a fully analytic approach. The seminal idea of Manton \([9]\) is that when solitons move slowly, their dynamics can be described as geodesics on their moduli space. This low-energy approximation is now called the moduli, geodesic, or Manton approximation. After his work, the moduli spaces are recognized as the most important objects associated with BPS solitons \([3]\). The Atiyah-Hitchin metric on two BPS monopoles is one of the most prominent examples \([10]\). By examining geodesics on it, one can study the scattering of two BPS monopoles and find some interesting phenomena such as the right angle scattering in head on collisions. The moduli space of BPS monopoles was further studied in \([11, 12]\). The moduli space and soliton scattering were studied for ANO vortices in the BPS limit \([13, 14, 15, 16, 17, 18, 19]\) and for BPS domain walls \([20]\). More recently, the moduli space dynamics has been successfully applied even to 1/4 BPS composite solitons such as domain wall networks \([21]\) and strings stretched between parallel domain walls \([22]\). Although there has been a lot of remarkable progress in study of the moduli space of BPS solitons, the analysis of their dynamics has been restricted to the leading order of the low-energy (small-velocity) limit, except for a few examples \([23, 24]\).

Non-Abelian vortices were found recently in \(\mathcal{N} = 2\) supersymmetric \(U(N)\) QCD \([25, 26]\).
Although there has been a lot of progress in the study of non-Abelian vortices [27, 28], here we concentrate on their moduli space. A single non-Abelian vortex has moduli space $\mathbb{C} \times \mathbb{C}P^{N-1}$ where the former denotes the position and the latter, called orientational moduli, comes from the $SU(N)$ vacuum symmetry broken in the presence of the vortex [25, 26]. One of the most important facts about the orientational moduli is that confined monopoles and trapped Yang-Mills instantons in the Higgs phase are realized, respectively, as kinks [29] and lumps [30, 31] in the vortex world-sheet $\mathbb{C}P^{N-1}$ sigma model. This world-sheet effective theory provides a physical explanation for the relationship between BPS spectra in two-dimensional $\mathcal{N} = (2, 2)$ sigma models and four-dimensional $\mathcal{N} = 2$ supersymmetric QCD [32, 33]. The moduli space of multiple vortices with full moduli parameters was completely determined (without metric) in [34, 28, 35]; the moduli space for $k$ separated vortices is a $k$-symmetric product $(\mathbb{C} \times \mathbb{C}P^{N-1})^k/S_k$ of the single vortex moduli space [34] while the whole space is regular. General formula for the moduli space metric and its Kähler potential were given in [36]. The metric of the moduli space on the coincident vortices was found [37, 38] and was applied to low-energy dynamics of two non-Abelian vortices in head-on collision, reconnections of two non-Abelian cosmic strings [37] and flux matching of vortex-monopole composite [39]. Recently the moduli space metric of multiple non-Abelian vortices has been finally obtained for well-separated vortices [40], and their low-energy dynamics has been examined [41].

The purpose of this paper is to propose a systematic method to study dynamics of BPS solitons moving with higher velocities beyond the Manton approximation; we give a general scheme to calculate derivative corrections and apply it to obtain four-derivative corrections to the low-energy effective theory on a single non-Abelian vortex. In the spirit of the low-energy effective action, the leading order terms can be obtained by integrating out massive modes. The lowest order terms take the form of nonlinear sigma models with two derivative terms for light or massless fields, typically Nambu-Goldstone modes. A famous example is the chiral Lagrangian or chiral perturbation theory for pions which are the Nambu-Goldstone bosons of chiral symmetry breaking. The next-leading terms consist of four or higher derivative corrections, which are the focus of our study. In order to obtain four derivative terms, we solve equations for massive modes and eliminate them order by order. As a concrete example, we consider a single non-Abelian vortex. We obtain four-derivative corrections for the translational moduli $\mathbb{C}$, the orientational moduli (the $\mathbb{C}P^{N-1}$ model) and their mixing terms.

In the literature a different expansion is known: an expansion from the Nambu-Goto action [42] in powers of (the inverse of) the width of strings. The effective action of a string in the thin limit can be described by the Nambu-Goto action [43]. The finite-width correction in general takes the form of extrinsic curvature squared, which is called stiffness or rigidity of string [44].
Dynamics of a string with the correction term was studied [45]. The explicit calculation of the rigidity term was performed in the Abelian Higgs model by many authors in the context of cosmic strings and QCD strings [16]. This expansion is in powers of \( \#(\partial) - \#(X) \), where we denote the number of derivatives and fields as \( \#(\partial) \) and \( \#(X) \), respectively. The leading term, the Nambu-Goto action, is the zeroth order term in the series, and thus contains the same number of derivatives and fields. The next-leading term, the stiffness term, contains two more derivatives than the number of fields starting from \( O(\partial^6 X^4) \). One advantage of this width expansion is that even the leading term, the Nambu-Goto action, contains infinite number of derivatives so that it can describe large fluctuations or bends of a string. However the expansion in powers of \( \#(\partial) - \#(X) \) is not applicable for dynamics of moduli in general; for instance multiple solitons or even a single non-Abelian soliton with internal degrees of freedom. — On the other hand, our expansion is purely a derivative expansion commonly used in the literature of field theories. It is valid at low energy and even the leading term, the nonlinear sigma model, can describe dynamics of multiple solitons.

In the present paper, we will discuss the higher derivative corrections to the low-energy effective theory on a non-Abelian vortex world-volume. By a symmetry argument, one can easily write down the generic form of the higher derivative terms and find that the corrections cause instability for the lumps (sigma model instantons) which are identified with the Yang-Mills instantons trapped inside a vortex. According to the Derrick’s scaling argument, the size of a lump expands in the presence of the generic higher derivative terms. Since the lumps are responsible for non-perturbative effects in the vortex world-sheet effective theory, one may think that the correspondence between the BPS spectra in 2d and 4d [32, 33] would be modified by the higher derivative corrections. However, as we will see in Sec. 5.2, the lump solutions are still stable if the higher derivative terms have a specific form. In this paper, we will calculate the explicit form of the higher derivative terms and find that they do not modify the lump solutions.

This paper is organized as follows. We first review non-Abelian vortices and their moduli space in Sec. 2. After illustrating our method for the derivative expansion in a simple example of classical mechanics in Sec. 3, we develop a systematic method to obtain derivative corrections to the effective action for non-Abelian vortices in Sec. 4. In Sec. 5.1 we compare our result for the effective action of a single vortex with previously known models, the Nambu-Goto action and the Faddeev-Skyrme model [17]. In Sec. 5.2 we discuss that the four derivative terms do not modify the Yang-Mills instanton solutions trapped inside a non-Abelian vortex [30, 31]. In Sec. 6 we discuss higher order corrections to the vortex effective action in a mass deformed model and show that the classical BPS spectrum is not modified by the higher order corrections. Sec. 7 is devoted to summary and discussion.
2 The non-Abelian vortices

In this section, we briefly review the non-Abelian vortices and summarize the basic tools to describe the moduli space of BPS configurations.

Let us consider the $U(N)$ gauge theory in $(d+1)$-dimensional spacetime with gauge field $W_\mu$ and $N_F = N$ Higgs fields $H$ ($N$-by-$N$ matrix) in the fundamental representation. The Lagrangian of our model takes the form

$$\mathcal{L} = \text{Tr} \left[ -\frac{1}{2g^2} F_{\mu\nu} F^{\mu\nu} + D_\mu H (D^\mu H) - \frac{g^2}{4} (HH^\dagger - v^2 1_N)^2 \right],$$

where $g$ is the gauge coupling constant and $v^2$ is the Fayet-Iliopoulos parameter. We use the almost minus metric $\eta_{\mu\nu} = (+1, -1, -1, \cdots, -1)$. Although we can choose different gauge coupling constants for the $U(1)$ and $SU(N)$ parts of the gauge group, we set them equal for notational simplicity. Our notation for the covariant derivative and the field strength is

$$D_\mu H = (\partial_\mu + iW_\mu)H, \quad F_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu + i[W_\mu, W_\nu].$$

As is well known, the Lagrangian can be embedded into a supersymmetric theory with eight supercharges. The vacuum condition is solved by

$$H = v 1_N.$$  

This vacuum expectation value (VEV) of the Higgs field completely breaks the gauge symmetry, whereas the following $SU(N)_{C+F}$ diagonal symmetry remains unbroken

$$H \rightarrow U_C H U_F, \quad U_C = U_F \in SU(N)_{C+F}.$$  

Let us consider the non-Abelian vortices in this color-flavor locked Higgs vacuum. We assume that the vortices are localized in the $z = x_{d-1} + ix_d$ plane. The topological sectors of the field configurations are classified by the vorticity, i.e. the magnetic flux of the overall $U(1)$ gauge group

$$k \equiv -\frac{1}{2\pi} \int dz \wedge d\bar{z} \text{ Tr } F_{z\bar{z}}, \quad k \in \mathbb{Z}.$$  

For a given vorticity $k$, the tension (the energy per unit volume) of static configurations is bounded from below as

$$T = \int d^2x \text{ Tr } \left[ 4D_{z\bar{z}} H (D_{z\bar{z}} H)^\dagger + \frac{4}{g^2} iF_{z\bar{z}} - \frac{g^2}{4} (v^2 1_N - HH^\dagger) \right]^2 - 4D_{|z} (D_{\bar{z}} H H^\dagger) + 2v^2 iF_{z\bar{z}} \right] \geq 2\pi v^2 k,$$  

(2.6)
where we have assumed that $\mathcal{D}_i H \to 0$ at the infinity $|z| \to \infty$ so that the third term does not contribute to the energy. This Bogomol’nyi bound is saturated if the following equations are satisfied

$$\mathcal{D}_z H = 0, \quad iF_{zz} = \frac{g^2}{4}(v^2 1_N - H H^\dagger). \quad (2.7)$$

These equations are the BPS equations for non-Abelian vortices and describe the configurations of static BPS vortices which are minimal energy configurations in a given topological sector. Since the same equations can be derived by imposing the condition that one half of supersymmetry is preserved, Eq. (2.7) is also called 1/2 BPS equations. We can show that any of the BPS configurations is a solution of the following full equations of motion of the system

$$0 = \mathcal{D}_\mu \mathcal{D}^\mu H + \frac{g^2}{2}(HH^\dagger - v^2 1_N)H, \quad (2.8)$$

$$0 = \frac{2}{g^2} \mathcal{D}_\mu F_{\mu\nu} + i\left[H(D_\nu H)^\dagger - (D_\nu H)H^\dagger\right]. \quad (2.9)$$

In order to describe the moduli space of the BPS configurations, it is convenient to write the BPS solution as

$$H = v S^{-1} H_0, \quad W_\bar{z} = -iS^{-1}\bar{\partial}S, \quad (2.10)$$

where $H_0$ and $S$ are $N$-by-$N$ matrices. Then, the first BPS equation in Eq. (2.7) becomes

$$\partial_\bar{z} H_0 = 0. \quad (2.11)$$

Therefore all the entries of the $N$-by-$N$ matrix $H_0$ are arbitrary holomorphic polynomials. The matrix function $S \in GL(N, \mathbb{C})$ is determined from the second equation in Eq. (2.7), which can be rewritten into the following equation for $\Omega \equiv SS^\dagger$:

$$\frac{4}{g^2 v^2} \partial_\bar{z}(\Omega \partial_\bar{z} \Omega^{-1}) = H_0 H_0^\dagger \Omega^{-1} - 1_N. \quad (2.12)$$

For a given $H_0$, this equation can be viewed as a non-linear differential equation for $\Omega$. The boundary condition for $\Omega$ should be chosen so that the energy density vanishes at the spatial infinity $|z| \to \infty$. In general, this condition is solved by

$$\Omega \underset{|z| \to \infty}{\longrightarrow} H_0 H_0^\dagger. \quad (2.13)$$

The matrix $S$ can be determined from the solution $\Omega$ uniquely up to the unphysical $U(N)$ gauge transformation $S \to Su(x)$ (with $u(x) \in U(N)$). Solving Eq. (2.12) for an arbitrarily chosen $H_0(z)$, we can obtain a BPS vortex solutions through Eq. (2.10). In other words, the matrix $H_0(z)$ classifies all the BPS configurations. Hence the holomorphic matrix $H_0(z)$ is called “the moduli matrix” and Eq. (2.12) is called “the master equation” for vortices.
The matrices $H_0(z)$ and $\Omega$ are invariant under the original $U(N)$ gauge transformation. On the other hand, the original fields $H$ and $W_{\bar{z}}$ are invariant under “the $V$-transformations” defined by

$$H_0(z) \rightarrow V(z)H_0(z), \quad \Omega \rightarrow V(z)\Omega V(z)^\dagger, \quad V(z) \in GL(N, \mathbb{C}). \quad (2.14)$$

This is the gauge symmetry of the master equation Eq. (2.12) which does not change the physical quantities. Therefore, the solutions of the BPS equations are completely classified by the equivalence classes of the moduli matrix

$$H_0(z) \sim V(z)H_0(z). \quad (2.15)$$

This implies that the parameters contained in an appropriately gauge fixed moduli matrix $H_0$ can be interpreted as the moduli parameters of the BPS configurations.

Let us next discuss how the moduli matrix $H_0$ determines the topological sector of the corresponding solution. From Eq. (2.10), we can see that the magnetic flux of the overall $U(1)$ is given in terms of $\Omega$ by

$$\text{Tr} F_{z\bar{z}} = -i\partial_z \partial_{\bar{z}} \log \det \Omega. \quad (2.16)$$

By using the Stokes’ theorem and the boundary condition Eq. (2.13), we find that the vorticity Eq. (2.5) is given by

$$k = \frac{1}{4\pi i} \oint_{S_1^\infty} (dz \partial_z - d\bar{z} \partial_{\bar{z}}) \log \det(H_0 H_0^\dagger), \quad (2.17)$$

where $S_1^\infty$ denotes the clock-wise circle at infinity $|z| \rightarrow \infty$. This equation shows that the order of the polynomial $\det H_0(z)$ corresponds to the vorticity.

As an example, let us consider the case of single vortex configurations. The simplest single vortex solution can be obtained by embedding the single Abelian vortex solution in the upper-left corner of the $N$-by-$N$ matrices $H$ and $W_{\bar{z}}$

$$H = \begin{pmatrix} H^{AN0} & 0 \\ 0 & U_{N-1} \end{pmatrix}, \quad W_{\bar{z}} = \begin{pmatrix} W^{AN0}_{\bar{z}} & 0 \\ 0 & 0_{N-1} \end{pmatrix}. \quad (2.18)$$

The other single vortex solutions can be obtained by acting on the embedded solution with the $SU(N)_{\text{C+F}}$ symmetry

$$H \rightarrow U^\dagger H U, \quad W_{\bar{z}} \rightarrow U^\dagger W_{\bar{z}} U, \quad U \in SU(N)_{\text{C+F}}. \quad (2.19)$$

Since the vortex embedded into the upper-left corner breaks $SU(N)_{\text{C+F}} \rightarrow SU(N-1) \times U(1)$, the internal vortex moduli space is the complex projective space

$$\mathbb{C}P^{N-1} \cong \frac{SU(N)}{SU(N-1) \times U(1)}. \quad (2.20)$$
By modding out the unbroken $SU(N-1) \times U(1)$, we can fix the unitary matrix $U$ as

$$U = \begin{pmatrix} 1 & -\vec{b}^\dagger \\ 0 & 1_{N-1} \end{pmatrix} \begin{pmatrix} X_\frac{1}{2} & 0 \\ 0 & Y^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \vec{b} & 1_{N-1} \end{pmatrix}, \tag{2.21}$$

where the parameters $\vec{b} = (b^1, \cdots, b^{N-1})$ can be interpreted as the inhomogeneous coordinates of $\mathbb{C}P^{N-1}$ and $X$ and $Y$ are given by

$$X \equiv 1 + \vec{b}^\dagger \cdot \vec{b}, \quad Y \equiv 1_{N-1} + \vec{b} \otimes \vec{b}^\dagger. \tag{2.22}$$

The physical meaning of the orientational moduli can be seen from the magnetic flux

$$F_{z\bar{z}} = F_{z\bar{z}}^{\text{ANO}} \times \frac{1}{1 + |\vec{b}|^2} \begin{pmatrix} 1 & -\vec{b}^\dagger \\ -\vec{b} & \vec{b} \otimes \vec{b}^\dagger \end{pmatrix}. \tag{2.23}$$

Thus the orientational moduli $\vec{b}$ determines the $U(1)$ generator in which the vortex magnetic flux is embedded.

The moduli matrix $H_0$ and the solution of the master equation $\Omega$ corresponding to the single vortex configurations are given by

$$H_0(z) = V \begin{pmatrix} z - Z & 0 \\ 0 & 1_{N-1} \end{pmatrix} U, \quad \Omega = V \begin{pmatrix} e^\psi & 0 \\ 0 & 1_{N-1} \end{pmatrix} V^\dagger, \tag{2.24}$$

where $\psi$ is a real profile function satisfying

$$\frac{4}{g^2 v^2} \partial_z \partial_{\bar{z}} \psi = 1 - |z - Z|^2 e^{-\psi}, \tag{2.25}$$

with the boundary condition $\psi \to \log |z - Z|^2$. The complex parameter $Z$ in the moduli matrix $H_0$ can be interpreted as the position moduli of the vortex. Although the holomorphic matrix $V(z)$ can be an arbitrary element of $GL(N, \mathbb{C})$, it will be convenient to use the matrix of the form

$$V(z) = \begin{pmatrix} X^{-\frac{1}{2}} & 0 \\ 0 & Y^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 1 & (z - Z) \vec{b}^\dagger \\ 0 & 1_{N-1} \end{pmatrix}. \tag{2.26}$$

This $V$-transformation is chosen so that the moduli matrix $H_0(z)$ takes the form

$$H_0(z) = \begin{pmatrix} z - Z & 0 \\ \vec{b} & 1_{N-1} \end{pmatrix}. \tag{2.27}$$

The important point is that the $V$-transformation is completely fixed so that the moduli matrix is holomorphic not only in $z$ but also in the moduli parameters $Z$ and $\vec{b}$. In general, there exists $\text{\footnote{The square root of the matrix $Y$ is given by $Y^{\pm \frac{1}{2}} = (1 - P) + (1 + |\vec{b}|^2)^{\pm \frac{1}{2}} P$ with $P = |\vec{b}|^{-2} \vec{b} \otimes \vec{b}$.}}$
such a fixed form of the moduli matrix in each coordinate patch of the moduli space. The $V$-
transformation between a pair of the fixed moduli matrices induces a coordinate transformation
from one patch to another. For instance, in the case of $N = 2$, there are two fixed forms

$$H_0 = \begin{pmatrix} z - Z & 0 \\ b & 1 \end{pmatrix}, \quad H'_0 = \begin{pmatrix} 1 & b' \\ 0 & z - Z \end{pmatrix}. \quad (2.28)$$

These two matrices are related by

$$H'_0 = V H_0, \quad V = \begin{pmatrix} 0 & b' \\ -b & z - Z \end{pmatrix}, \quad b' = \frac{1}{b}. \quad (2.29)$$

Thus, the induced coordinate transformation is the standard transition map between the in-
homogeneous coordinates of $\mathbb{C}P^1$. As this example shows, the coordinate transformations are
holomorphic, so that the moduli space is a complex manifold.

For a general winding number $k$, we can fix generic moduli matrices to the following form

$$H_0(z) = \begin{pmatrix} (z - Z_1)(z - Z_2) \cdots (z - Z_k) \\ \bar{b}_1 e_1(z) + \bar{b}_2 e_2(z) \cdots + \bar{b}_k e_k(z) \end{pmatrix}, \quad e_I(z) = \prod_{J \neq I} \frac{z - Z_J}{Z_I - Z_J}, \quad (2.30)$$

The parameters $Z_I$ and $\bar{b}_I$ ($I = 1, \cdots, k$) are position and orientational moduli of $I$-th vortex
which cover a local coordinate patch of the moduli space of the vortices $\mathcal{M}_k$. As in the case of the
single vortex, the moduli matrix $H_0(z)$ is holomorphic with respect to the moduli parameters.
This fact will be important when we derive general formulas for the second and fourth order
effective Lagrangian in section 4.

### 3 Preliminary: a particle in $\mathbb{R}^n$

Before studying higher derivative corrections to the vortex effective action in field theories, we
first explain our basic strategy in a simple system of a particle in $\mathbb{R}^n$ with the following Lagrangian

$$L = \frac{m}{2} \dot{x} \cdot \dot{x} - V(x), \quad x(t) = (x^1, x^2, \cdots, x^n). \quad (3.1)$$

In a minimum energy configuration, the particle stays at the bottom of the potential $V$ where
the gradient of $V$ vanishes

$$\text{grad} V = 0. \quad (3.2)$$
Now let us assume that the potential $V$ has flat directions which are parameterized by $\phi^i$. Then we can define “the moduli space” of the minimum energy configurations by

$$\mathcal{M} = \{ x^{(0)}(\phi^i) \in \mathbb{R}^n \mid \text{grad } V = 0 \} ,$$

(3.3)

and $\phi^i$ can be interpreted as “the moduli parameters”. Since one can shift the particle to any points on the bottom of the potential without loss of energy, we can assume that the particle slowly moves along the moduli space $\mathcal{M}$ for sufficiently small excitation energy. This motion of the particle can be represented by the moduli parameters $\phi^i(t)$ which weakly depend on the time $t$, that is

$$x(t) = x^{(0)}(\phi^i(t)).$$

(3.4)

Then, the low energy effective Lagrangian of the particle can be obtained by substituting Eq. (3.4) into the original Lagrangian Eq. (3.1)

$$L_{\text{eff}} = L_{\text{eff}}^{(0)} + L_{\text{eff}}^{(2)} = -V(x^{(0)}) + \frac{m}{2} \dot{x}^{(0)} \cdot \dot{x}^{(0)},$$

(3.5)

where $L_{\text{eff}}^{(0)} = -V(x^{(0)})$ is the constant value of the potential at the bottom. The second order effective Lagrangian $L_{\text{eff}}^{(2)}$ can be rewritten by using the metric $g_{ij}$ on the moduli space $\mathcal{M}$ as

$$L_{\text{eff}}^{(2)} = \frac{m}{2} g_{ij} \dot{\phi}^i \dot{\phi}^j .$$

(3.6)

The metric is given by the inner products of the basis $\{ \Phi_i \}$ of “the zero modes”

$$g_{ij} \equiv \Phi_i \cdot \Phi_j , \quad \Phi_i \equiv \frac{\partial}{\partial \phi^i} x^{(0)} .$$

(3.7)

The effective equations of motion for the moduli parameters take the form of the geodesic equation

$$\ddot{\phi}^i + \Gamma^i_{jk} \dot{\phi}^j \dot{\phi}^k = 0,$$

(3.8)

where the connection is given by

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} (g_{lj,k} + g_{lk,j} - g_{jk,l}) = g^{il} \Phi_l \cdot \frac{\partial}{\partial \phi^j} \Phi_k .$$

(3.9)

This approximation is valid if the velocity of the particle is sufficiently small so that the time derivative is much less than the typical mass scale of the massive modes determined from the Hessian matrix $H$ of the potential $V$ defined by

$$[H(x^{(0)})]_{ab} = \frac{\partial^2 V}{\partial x^a \partial x^b} \bigg|_{x = x^{(0)}} .$$

(3.10)
Next, let us consider higher derivative corrections to the effective Lagrangian by taking the massive modes into account. To this end, we first introduce a correction to (3.4) by adding small fluctuations to “the slowly moving background” $x^{(0)}(\phi(t))$ as

$$x(t) = x^{(0)}(\phi(t)) + \delta x(t). \quad (3.11)$$

Since the motion of the particle along the flat direction is already represented by the moduli parameters $\dot{\phi}(t)$, we impose the following condition to avoid the double-counting of the degrees of freedom of the zero modes:

$$\delta x \cdot \Phi = 0. \quad (3.12)$$

This means that the fluctuations $\delta x$ is orthogonal to the tangent space of $\mathcal{M}$, namely the fluctuation $\delta x$ contains only massive modes. Then, the original Lagrangian Eq. (3.1) can be rewritten as

$$L = L_{\text{eff}}^{(0)} + L_{\text{eff}}^{(2)} + \delta L + \lambda^i(\delta x \cdot \Phi_i), \quad (3.13)$$

$$\delta L = m \dot{x}^{(0)} \cdot \delta x + \frac{m}{2} \dot{x} \cdot \delta x - \frac{1}{2} \dot{x} \mathbf{H} \delta x + \cdots, \quad (3.14)$$

where we have introduced the Lagrange multipliers $\lambda^i$ to impose the constraint Eq. (3.12). Note that there is no linear term in the Taylor expansion of the potential since the background satisfies $\text{grad}\, V(x^{(0)}) = 0$. Now let us consider the expansion of $\delta x$ with respect to the time derivative $\partial_t$

$$\delta x = x^{(2)} + x^{(4)} + \cdots, \quad x^{(n)} \sim \mathcal{O}(\partial^n t). \quad (3.15)$$

Note that $\delta x$ can have only terms with even numbers of the time derivatives due to the reflection symmetry $t \rightarrow -t$. There is no zeroth order term since the fluctuation $\delta x$ vanishes for the static configurations. Correspondingly, the Lagrangian $\delta L$ can also be expanded as

$$\delta L = L^{(4)} + L^{(6)} + \cdots. \quad (3.16)$$

The fourth order Lagrangian contains the background $x^{(0)}$ and the second order fluctuation $x^{(2)}$

$$L^{(4)} = m \dot{x}^{(0)} \cdot \ddot{x}^{(2)} - \frac{1}{2} \dot{x}^{(2)} \mathbf{H} \dot{x}^{(2)}. \quad (3.17)$$

Note that $x^{(4)}$ does not contribute to the fourth order Lagrangian since $\text{grad}\, V(x^{(0)}) = 0$. The massive modes $x^{(2)}$ can be eliminated by solving their equation of motion

$$\mathbf{H} \dot{x}^{(2)} + m \ddot{x}^{(0)} = \lambda^i \Phi_i. \quad (3.18)$$

To determine the Lagrange multiplier $\lambda^i$, let us take the inner products of the both hand sides and the zero modes $\Phi_j$

$$m \Phi_j \cdot \dot{x}^{(0)} = g_{ij} \lambda^i, \quad (3.19)$$
where we have used the fact that the zero modes \( \{ \Phi_i \} \) satisfy

\[
\Phi_i H = \frac{\partial}{\partial \phi^i} \left[ \text{grad } V(x^{(0)}) \right] = 0. \tag{3.20}
\]

We can show that the Lagrange multiplier is proportional to the second order equation of motion Eq. (3.8)

\[
\lambda^i = mg^{ij} \Phi_j \cdot \ddot{x}^{(0)} = m \left( \ddot{\phi}^i + \Gamma^i_{kl} \dot{\phi}^k \dot{\phi}^l \right). \tag{3.21}
\]

Substituting back \( \lambda^i \) into Eq. (3.18), we obtain the following equations of motion for the massive modes

\[
H x^{(2)} = -m P \ddot{x}^{(0)}, \tag{3.22}
\]

where \( P \) is the projection matrix which project out the zero modes

\[
P \ddot{x}^{(0)} \equiv \ddot{x}^{(0)} - g^{ij} (\ddot{x}^{(0)} \cdot \Phi_j) \Phi_i, \tag{3.23}
\]

Since the right hand side of Eq. (3.22) does not contain the zero mode directions, the matrix \( H \) can be “inverted” as

\[
x^{(2)} = -m G \ddot{x}^{(0)}, \tag{3.24}
\]

where \( G \) is the matrix satisfying

\[
HG = P, \quad GP = PG = G, \quad G \Phi_i = 0. \tag{3.25}
\]

Substituting the solution for the massive modes Eq. (3.24) into Eq. (3.17), we obtain the following fourth order effective Lagrangian

\[
L^{(4)}_{\text{eff}} = \frac{1}{2} x^{(2)} H x^{(2)} = \frac{m^2}{2} \dot{\phi}^i \dot{\phi}^j \dot{\phi}^k \dot{\phi}^l \left( \frac{\partial \Phi_j}{\partial \phi^i} G \frac{\partial \Phi_l}{\partial \phi^k} \right). \tag{3.26}
\]

Let us see a simple example of a particle in \( \mathbb{R}^2 \). We assume that the potential is rotationally symmetric \( V = V(|x|) \) and has a minimum at \( |x| = r_0 \), namely \( V'(r_0) = 0 \) and \( V''(r_0) > 0 \). The static configurations are parameterized by the moduli parameter \( \theta \) as

\[
x^{(0)} = \begin{pmatrix} r_0 \cos \theta \\ r_0 \sin \theta \end{pmatrix}. \tag{3.27}
\]

The second order effective Lagrangian is given by

\[
L^{(2)}_{\text{eff}} = \frac{m}{2} \dot{x}^{(0)} \cdot \dot{x}^{(0)} = \frac{m}{2} r_0^2 \dot{\theta}^2. \tag{3.28}
\]
The corresponding equation of motion for the moduli parameter is $\ddot{\theta} = 0$ and describes the particle rotating around the circle at $|x| = r_0$. The matrices $H$ and $G$ are respectively given by

$$H = V''(r_0) \, P, \quad G = V''(r_0)^{-1} P,$$

where $P$ is the projection operator

$$P = \begin{pmatrix} \cos^2 \theta & \sin \theta \cos \theta \\ \sin \theta \cos \theta & \sin^2 \theta \end{pmatrix}.$$ (3.30)

From Eq. (3.26), we obtain the following fourth order Lagrangian

$$L_{\text{eff}}^{(4)} = \frac{(mr_0)^2}{2} V''(r_0)^{-1} \dot{\theta}^4.$$ (3.31)

Even if we take into account this fourth order Lagrangian, the equation of motion is not modified $\ddot{\theta} = 0$ and solved by $\theta = \omega t + \theta_0$. On the other hand, the relation between the angular velocity $\omega$ and the angular momentum $l$ is modified as

$$l = mr_0^2 \left[ 1 + 2m \omega^2 V''(r_0)^{-1} \right] \omega.$$ (3.32)

This is just because the rotation radius is increased by the centrifugal force as

$$r_0 \rightarrow r_0 + mr_0 \dot{\theta}^2 V''(r_0)^{-1}.$$ (3.33)

The shift of the rotation radius can be also seen in the solution for the massive mode

$$x^{(0)} + \delta x \approx x^{(0)} - m G \dot{x}^{(0)} = \left[ r_0 + mr_0 \dot{\theta}^2 V''(r_0)^{-1} \right] \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}.$$ (3.34)

Therefore, the higher derivative term gives the correction from the massive mode which is slightly shifted by the motion of the zero mode (see Fig. 1).

![Fig. 1: The shift of the rotation radius.](image-url)
4 The effective theory

4.1 Derivative expansion

In this section, we discuss the effective Lagrangian for non-Abelian vortices by generalizing the method of the derivative expansion discussed in the previous section. To deal with the vortex positions and orientations $Z_I, \vec{b}_I$ ($I = 1, \cdots, k$) on an equal footing, we combine them into a set of complex moduli parameters $\phi^i$ ($i = 1, \cdots, kN = \text{dim}_C \mathcal{M}_k$), and assume that the moduli matrix $H_0(z)$ is always holomorphic in $\phi^i$ like Eq. (2.30).

The zero mode fluctuations along the vortex world-volume are described by the vortex moduli $\phi^i$ promoted to fields which weakly depend on the world-volume coordinates

$$\phi^i \to \phi^i(x^\alpha), \quad (\alpha = 0, 1, \cdots, d - 2). \quad (4.1)$$

These moduli fields induce fluctuations of massive modes around the weakly fluctuating vortex background

$$H(x^\mu) = H^{(0)}(z, \bar{z}, \phi(x^\alpha)) + \delta H(x^\mu), \quad (4.2)$$
$$W_\bar{z}(x^\mu) = W^{(0)}_\bar{z}(z, \bar{z}, \phi(x^\alpha)) + \delta W_\bar{z}(x^\mu), \quad (4.3)$$
$$W_\alpha(x^\mu) = 0 + \delta W_\alpha(x^\mu), \quad (4.4)$$

where $H^{(0)}$ and $W^{(0)}_\bar{z}$ are the BPS vortex background Eq. (2.10) depending on the world-volume coordinates $x^\alpha$ through the moduli fields $\phi(x^\alpha)$. We assume that the excitation energy of the fluctuations are much less than the typical mass scale of the massive modes. Then we can expand the induced fields with respect to the derivative by assuming that

$$\partial_\alpha \ll gv. \quad (4.5)$$

Note that $gv$ is the unique mass scale controlling the mass of the bulk fields, inverse width of the vortex and the mass scale of the massive modes localized on the vortex world-volume. The induced fluctuations are expanded with respect to the derivative $\partial_\alpha$ as

$$\delta H = H^{(2)} + H^{(4)} + \cdots, \quad (4.6)$$
$$\delta W_\bar{z} = W^{(2)}_\bar{z} + W^{(4)}_\bar{z} + \cdots, \quad (4.7)$$
$$\delta W_\alpha = W^{(1)}_\alpha + W^{(3)}_\alpha + \cdots. \quad (4.8)$$

These fluctuations can be determined by solving the equations of motion (2.8) and (2.9) order-by-order. Note that odd (even) order equations of motion for $\delta H$ and $\delta W_\bar{z}$ ($\delta W_\alpha$) are trivial due to the reflection symmetry $x^\alpha \to -x^\alpha$. Then, the effective Lagrangian can be obtained by
eliminating the induced fluctuations from original Lagrangian (2.1) expanded with respect to the derivative $\partial_\alpha$

$$\mathcal{L} = \mathcal{L}^{(0)} + \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \cdots. \quad (4.9)$$

As we will see, the Lagrangian is quadratic in the induced fluctuations up to the fourth order in the derivative $\partial_\alpha$, so that we can use the linearized equations of motion to eliminate the induced fluctuations. In order to determine the fluctuations, we have to specify the boundary conditions for them. In the singular gauge, the vortex background takes the following form at the infinity $|z| \to \infty$

$$H^{(0)} = v1_N + \mathcal{O}(e^{-gv|z|}), \quad W^{(0)}_\bar{z} = \mathcal{O}(\bar{z}^{-1}). \quad (4.10)$$

Since this vortex background is independent of the zero mode fluctuations $\phi(x^\alpha)$ at infinity, the minimal excitations induced by the zero modes should vanish at infinity. Therefore, we impose the following boundary conditions for the induced fluctuations

$$\delta H \to 0, \quad \delta W_\mu \to 0, \quad \text{(in the singular gauge).} \quad (4.11)$$

### 4.2 General formulas for the effective Lagrangian

#### 4.2.1 The zeroth order effective Lagrangian

The zeroth order Lagrangian contains only the background fields

$$\mathcal{L}^{(0)} = \text{Tr} \left[ -\frac{4}{g^2} |F^{(0)}_{z\bar{z}}|^2 - 2\partial_z H^{(0)}(\partial_z H^{(0)})^\dagger - \frac{g^2}{4} (H^{(0)} H^{(0)})^\dagger - v^2 1_N \right]. \quad (4.12)$$

Substituting the solution (2.10) and using the boundary condition Eq. (2.13), we find that the zeroth order term $\mathcal{L}^{(0)}_{\text{eff}}$ of the low-energy effective Lagrangian is given by the sum of the tension of the individual vortices

$$\mathcal{L}^{(0)}_{\text{eff}} \equiv \int d^2 x \mathcal{L}^{(0)} = -kT, \quad T \equiv 2\pi v^2. \quad (4.13)$$

#### 4.2.2 The second order effective Lagrangian

The second order Lagrangian takes the form (2.25, 26) (see also 33, 48, 30, 34).

$$\mathcal{L}^{(2)} = \text{Tr} \left[ \frac{4}{g^2} F^{(1)}_{z\bar{z}} F^{\alpha(1)}_{z\bar{z}} + \mathcal{D}_\alpha H^{(0)}(\mathcal{D}^\alpha H^{(0)})^\dagger \right]. \quad (4.14)$$

where $F^{(1)}_{z\bar{z}}$ and $\mathcal{D}_\alpha H^{(0)}$ are given by

$$F^{(1)}_{z\bar{z}} = \partial_\alpha W^{(0)}_\bar{z} - \partial_\bar{z} W^{(1)}_\alpha + i [W^{(1)}_\alpha, W^{(0)}_\bar{z}], \quad \mathcal{D}_\alpha H^{(0)} = \partial_\alpha H^{(0)} + i W^{(1)}_\alpha H^{(0)}. \quad (4.15)$$
Note that the terms containing the second order fields \( H^{(2)} \) and \( W^{(2)} \) do not contribute to the second order Lagrangian since these terms are proportional to the background equations of motion

\[
0 = \text{Tr} \left[ H^{(2)} \left( \frac{\delta}{\delta H} \int d^{d+1}x \mathcal{L}^{(0)} \right) \text{BPS background} \right], \quad \cdots. \tag{4.16}
\]

The dynamical degrees of freedom in the second order Lagrangian are not only the zero modes \( \phi^i(x^\alpha) \) contained in the background fields \( (H^{(0)}, W^{(0)}_\bar{z}) \) but also the first order fluctuations \( W^{(1)}_\alpha \). We can eliminate \( W^{(1)}_\alpha \) by using the equations of motion

\[
\frac{4}{g^2} \left( D_z F^{(1)}_{\bar{z} \alpha} + D_{\bar{z}} F^{(1)}_{z \alpha} \right) = i \left[ H^{(0)} (D_\alpha H^{(0)})^\dagger - D_\alpha H^{(0)} H^{(0)^\dagger} \right]. \tag{4.17}
\]

By using the fact that the moduli matrix \( H_0(z) \) is holomorphic in the moduli parameters \( \phi^i \) and the matrix \( \Omega \) is the solution of the master equation Eq. (2.12), we can check that the solution is given by

\[
W^{(1)}_\alpha = i(\delta_\alpha S^\dagger S^{\dagger-1} - S^{-1}\delta_\alpha^\dagger S), \tag{4.18}
\]

where we have defined the differential operators \( \delta_\alpha \) and \( \delta_\alpha^\dagger \) by

\[
\delta_\alpha \equiv \partial_\alpha \phi^i \frac{\partial}{\partial \phi^i}, \quad \delta_\alpha^\dagger \equiv \partial_\alpha \bar{\phi}^i \frac{\partial}{\partial \bar{\phi}^i}. \tag{4.19}
\]

The solution Eq. (4.18) satisfies the boundary condition Eq. (4.11) since the asymptotic form of the matrix \( S \) in the singular gauge is \( S \to H_0(z, \phi^i) \) and the moduli matrix \( H_0(z, \phi^i) \) is holomorphic with respect to \( \phi^i \). Substituting the solution (4.18) into the second order Lagrangian (4.14), we obtain the following formal expression of the second order effective Lagrangian

\[
\mathcal{L}_{\text{eff}}^{(2)} = v^2 \int d^2x \delta_\alpha^\dagger \text{Tr} \left[ \delta_\alpha H_0 H_0^\dagger \Omega^{-1} \right] = g_{ij} \partial_\alpha \phi^i \partial_\alpha \bar{\phi}^j. \tag{4.20}
\]

This effective Lagrangian gives a natural Kähler metric on the moduli space of vortices

\[
g_{ij} \equiv v^2 \int d^2x \frac{\partial}{\partial \phi^i} \text{Tr} \left[ \frac{\partial H_0}{\partial \phi^i} H_0^\dagger \Omega^{-1} \right]. \tag{4.21}
\]

Note that the above formulas for the first order solution (4.18) and the moduli space metric are not invariant under the generic \( V \)-transformation Eq. (2.14). However they are invariant under the \( V \)-transformation respecting the holomorphy of \( H_0(z) \) as we pointed out at the end of Sec. 2.

In order to obtain the explicit form of the moduli space metric, one needs to know the solution of the master equation \( \Omega \). However, no analytic solution of \( \Omega \) has been known even for the minimal winding vortex, and hence it is quite difficult to obtain the explicit form of

\[^2\text{Note that the orthogonality condition between zero modes and fluctuations is trivial for } W^{(1)}_\alpha \text{ since } W_\alpha = 0 \text{ in the vortex background.}\]
the effective Lagrangian in general. Nevertheless, we can obtain an exact form of the second order effective Lagrangian for the single vortex since the degrees of freedom $Z, \vec{b}$ in this case are nothing but Nambu-Goldstone zero modes [30]. By substituting the solution Eq. (2.24) and using the boundary condition for the profile function $\psi \to \log |z - Z|^2$, we can show that

$$L^{(2)}_{\text{eff}} = \frac{T}{2} \partial_\alpha Z \partial^\alpha \bar{Z} + \frac{4\pi}{g^2} g_{ij}^{\text{FS}} \partial^\alpha b^i \partial_\alpha \bar{b}^j, \quad (4.22)$$

where $g_{ij}^{\text{FS}}$ is the Fubini-Study metric on $\mathbb{C}P^{N-1}$

$$g_{ij}^{\text{FS}} \equiv \frac{\partial}{\partial b^i} \frac{\partial}{\partial \bar{b}^j} \log(1 + |\vec{b}|^2). \quad (4.23)$$

Another example of analytic moduli space metric for higher winding vortices has been obtained for well-separated vortices [40].

Note the moduli space metric can be rewritten as inner products of zero modes as in the case of the example discussed in the previous section (see Eq. (3.7)). Before closing this subsection, let us review the zero modes and the derivative operator defining them [25]. The physical zero modes in a BPS background are defined as the solutions of the linearized BPS equations for the fluctuations around the background. With a suitable gauge fixing condition, the linearized BPS equations can be written as

$$\Delta \Phi \equiv \begin{pmatrix} iD^f_z & -\frac{g}{2} H^{(0)}_r \\ \frac{g}{2} H^{(0)\dagger}_r & iD^a_z \end{pmatrix} \begin{pmatrix} \delta H \\ \frac{g}{2} \delta W_z \end{pmatrix} = 0, \quad (4.24)$$

where the subscript $r$ denotes the fact that $H^{(0)}$ acts as right multiplication and $D^f_z$ and $D^a_z$ are covariant derivatives with the zeroth order gauge field which act on the fundamental and adjoint fields, respectively. There exists one zero mode $\Phi_i$ for each moduli parameter $\phi^i$ and the set $\{\Phi_i\}$ ($i = 1, \cdots, \dim_{\mathbb{C}} \mathcal{M}$) forms a basis of the zero modes. In terms of $\Omega, S$ and $H_0$, the basis $\Phi_i$ can be written as

$$\Phi_i = \begin{pmatrix} v S^{-1} \Omega \frac{\partial}{\partial \phi^i} \left[ \Omega^{-1} H_0 \right] \\ 2i S^{-1} \bar{\partial} \left[ \Omega \frac{\partial}{\partial \phi^i} \Omega^{-1} \right] S \end{pmatrix}, \quad (4.25)$$

3 The linearized equations of motion in a static BPS background can be summarized into the form of $\Delta^\dagger \Delta \Phi = 0$. Here, the Hermitian conjugate of the operator $\Delta$

$$\Delta^\dagger = \begin{pmatrix} iD^f_z & \frac{g}{2} H^{(0)}_r \\ -\frac{g}{2} H^{(0)\dagger}_r & iD^a_z \end{pmatrix},$$

turns out to have no zero mode [25]: $\Delta^\dagger \Phi = 0 \Rightarrow \Phi = 0$, so that $\{\Phi_i\}$ gives a full set of the zero modes.
By using the solution for $W^{(1)}_\alpha$ given in Eq. (4.18), these zero modes can be summarized as

$$\partial_\alpha \dot{\phi}^i \Phi_i = \left( \frac{D_\alpha H^{(0)}}{g} F^{(1)}_{\alpha \bar{\alpha}} \right).$$

(4.26)

Then, the moduli space metric Eq. (4.21) can be rewritten as inner products of the physical zero modes $\Phi_i$

$$g_{ij} \equiv \langle \Phi_j, \Phi_i \rangle.$$

(4.27)

where we have defined the hermitian inner product for pairs of fields in the fundamental and adjoint representations

$$\langle \Phi, \Phi' \rangle \equiv \int d^2x \text{Tr} \left[ f' f'^\dagger + a^\dagger a' \right], \quad \Phi = \left( \begin{array}{c} f \\ a \end{array} \right), \quad \Phi' = \left( \begin{array}{c} f' \\ a' \end{array} \right).$$

(4.28)

4.2.3 The fourth order effective Lagrangian

Let us now calculate the fourth order effective Lagrangian by solving the linearized equations of motion and eliminating the massive modes. The fourth order Lagrangian takes the form

$$L^{(4)} = \text{Tr} \left[ -\frac{1}{2g^2} (F^{(2)}_{\alpha \beta})^2 + \frac{4}{g^2} D^\alpha W^{(2)}_{\bar{z}} F^{(1)}_{\alpha \bar{\alpha}} + D_\alpha H^{(2)} (D^\alpha H^{(0)})^\dagger + (h.c.) \right]$$

$$- \frac{4}{g^2} \left| iD_{\bar{z}} W^{(2)}_{\bar{z}} + \frac{g^2}{4} H^{(2)} H^{(0)}^\dagger + (h.c.) \right|^2 - \frac{4}{g} \left| D_{\bar{z}} H^{(2)} + iW^{(2)}_{\bar{z}} H^{(0)} \right|^2. \right]$$

(4.29)

Here, terms proportional to $W^{(3)}_{\alpha}, H^{(4)}$ and $W^{(4)}_{\bar{z}}$ automatically vanish due to the same mechanism as Eq. (4.16). It will be convenient to combine $H^{(2)}$ and $W^{(2)}_{\bar{z}}$ into a column vector

$$\Phi^{(2)} \equiv \left( \begin{array}{c} H^{(2)} \\ 2W^{(2)}_{\bar{z}} \end{array} \right).$$

(4.30)

Note that the fluctuations transform under the gauge transformations as

$$H^{(2)} \rightarrow H^{(2)} + i\Lambda H^{(0)}, \quad W^{(2)}_{\bar{z}} \rightarrow W^{(2)}_{\bar{z}} - D_{\bar{z}} \Lambda,$$

(4.31)

where $\Lambda$ is an arbitrary hermitian matrix of order $\partial^2_\alpha$. In order to eliminate this unphysical degrees of freedom, let us impose the following gauge fixing conditions for the the fluctuations

$$iD_{\bar{z}} W^{(2)}_{\bar{z}} + \frac{g^2}{4} H^{(2)} H^{(0)}^\dagger = (h.c.).$$

(4.32)

This constraint is equivalent to the condition that the second order fluctuation $\Phi^{(2)}$ is orthogonal to the unphysical gauge zero modes $\Phi_\Lambda$

$$0 = \langle \Phi_\Lambda, \Phi^{(2)} \rangle + (h.c.), \quad \Phi_\Lambda = \left( \begin{array}{c} i\Lambda H^{(0)} \\ -\frac{2}{g} D_{\bar{z}} \Lambda \end{array} \right).$$

(4.33)
Then, the terms containing \((H^{(2)}, W_z^{(2)})\) in the fourth order effective Lagrangian \(\mathcal{L}_{\text{eff}}^{(4)}\) are summarized as

\[
-4\left\langle \Delta \Phi^{(2)}, \Delta \Phi^{(2)} \right\rangle + \left[ \mathcal{D}_\alpha \Phi^{(2)} \right. \left], \partial^\alpha \phi^i \Phi_i \right\rangle + \lambda^i \left\langle \Phi^{(2)}, \Phi_i \right\rangle + (c.c.) ,
\]

(4.34)

where we have introduced the Lagrange multiplier \(\lambda^i\) to impose the condition that the second order fluctuation \(\Phi^{(2)}\) is orthogonal to the physical zero modes \(\Phi_i\), as we have done in Eq. (3.13) in order to separate the massive modes from the zero modes. The linearized equations of motion for \(H^{(2)}\) and \(W_z^{(2)}\) can be written as (cf. \(H \to \Delta \dagger \Delta\) in Eq. (3.18))

\[
4 \Delta \dagger \Delta \Phi^{(2)} + \mathcal{D}_\alpha (\partial^\alpha \phi^i \Phi_i) = \lambda^i \Phi_i .
\]

(4.35)

Let us first determine the Lagrange multipliers \(\lambda^i\). By taking the inner products of the zero modes \(\Phi_J^\dagger\) and the both hand sides of the linearized equation Eq. (4.35), we find that

\[
\partial_\alpha \partial^\alpha \phi^i + \Gamma^i_jk \partial_\alpha \phi^j \partial^\alpha \phi^k = \lambda^i ,
\]

(4.36)

where we have used

\[
\left\langle \Phi_j, \mathcal{D}_\alpha \Phi_i \right\rangle = \delta_\alpha g_{ij} = g_{ij} \Gamma^l_{ik} \partial_\alpha \phi^l , \quad \left\langle \Phi_j, \Delta \dagger \Delta \Phi^{(2)} \right\rangle = \left\langle \Delta \Phi_j, \Delta \Phi^{(2)} \right\rangle = 0 .
\]

(4.37)

We find that \(\lambda^i = 0\) is nothing but the equation of motion for \(\phi^i\) with the second order Lagrangian Eq. (4.20). Then, the linearized equation Eq. (4.35) becomes

\[
\Delta \dagger \Delta \Phi^{(2)} + \frac{1}{4} \mathbf{P} \left[ \mathcal{D}_\alpha (\partial^\alpha \phi^i \Phi_i) \right] = 0,
\]

(4.38)

where \(\mathbf{P}\) the following projection operator which projects out the zero modes

\[
\mathbf{P} \phi \equiv \phi - \left\langle \Phi_j, \phi \right\rangle g^j_i \Phi_i .
\]

(4.39)

It seems that in order to solve Eq. (4.38) in terms of \(\Phi^{(2)}\), we need to know the explicit form of the background BPS solution. However, we can find the following formal expression for \(\Delta \Phi^{(2)}\) without solving the BPS equations

\[
\Delta \Phi^{(2)} = \frac{i}{2g} \left( \frac{2}{g^2} \partial_\alpha \phi^i \partial^\alpha \phi^j \right)^{\dagger} \left[ \nabla_i \frac{\partial}{\partial \phi^j} \left( \partial \Omega^{-1} \Omega \right) H^i_{\dagger -1} \right],
\]

(4.40)

where \(\nabla_i\) is the covariant derivative on the moduli space, which acts on \(\frac{\partial}{\partial \phi^j} \left( \partial \Omega^{-1} \Omega \right)\) as

\[
\nabla_i \frac{\partial}{\partial \phi^j} \left( \partial \Omega^{-1} \Omega \right) = \left( \frac{\partial}{\partial \phi^j} \frac{\partial}{\partial \phi^i} - \Gamma^k_{ij} \frac{\partial}{\partial \phi^k} \right) \left( \partial \Omega^{-1} \Omega \right) .
\]

(4.41)
We can confirm that Eq. (4.40) satisfies the linearized equation Eq. (4.38) by checking that the following equivalent equation is satisfied

\[
(\Delta\Delta^\dagger)\Delta\Phi^{(2)} = -\frac{1}{4}\Delta \left[ D_\alpha (\partial^\alpha \phi \Phi_i) \right],
\]

(4.42)

where the operator \( \Delta\Delta^\dagger \) takes the form

\[
\Delta\Delta^\dagger = \begin{pmatrix}
-\mathcal{D}_z^2 \mathcal{D}_z^f + \frac{g^2}{T} (H(0)^t H(0))_r & 0 \\
0 & -\mathcal{D}_a^2 \mathcal{D}_a^a + \frac{g^2}{T} (H(0)^t H(0))_r
\end{pmatrix}.
\]

(4.43)

The solution \( \Delta\Phi^{(2)} \) is unique since the operator \( \Delta\Delta^\dagger \) obviously has no zero mode and hence it is invertible. Although it has apparent singularities due to the factor \( H_0^{-1} \), we can show that the solution Eq. (4.40) is smooth everywhere (see Appendix C for the proof).

It is in general not easy to solve Eq. (4.40) in terms of \( \Phi^{(2)} \). However, the explicit form of \( \Delta\Phi^{(2)} \) is sufficient for the purpose of determining the fourth order Lagrangian since it can be rewritten as

\[
\mathcal{L}_{eff}^{(4)} = \int d^2x \text{Tr} \left[ -\frac{1}{2g^2} (F_{\alpha\beta}^{(2)})^2 \right] + 4 \left\langle \Delta\Phi^{(2)} , \Delta\Phi^{(2)} \right\rangle.
\]

(4.44)

By using the first order solution (4.18), the first term in the fourth order Lagrangian (4.44) can be calculated as

\[
\text{Tr} \left[ -\frac{1}{2g^2} (F_{\alpha\beta}^{(2)})^2 \right] = \frac{2}{g^2} \text{Tr} \left[ \delta_{\alpha}^l (\Omega \delta_{\beta}^l \Omega^{-1}) \delta^{l\alpha}(\Omega \delta_{\beta}^l \Omega^{-1}) \right].
\]

(4.45)

Substituting (4.45) and (4.46) into (4.44), we obtain the following form of the fourth order effective Lagrangian for the moduli fields

\[
\mathcal{L}_{eff}^{(4)} = (A_{ijkl} + B_{ijkl})(\partial_\alpha \phi^i \partial_\beta \phi^j)(\partial_\gamma \phi^k \partial_\delta \phi^l) + 2B_{ijkl}(\partial_\alpha \phi^i \partial_\beta \phi^j)(\partial^\alpha \phi^k \partial^\beta \phi^l),
\]

(4.46)

where the tensors \( A_{ijkl} \) and \( B_{ijkl} \) are given by

\[
A_{ijkl} = \frac{1}{g^2} \int d^2x \text{Tr} \left[ \frac{4}{g^2 v^2} \left( \nabla_k \frac{\partial}{\partial \phi^k} (\partial_\gamma \Omega^{-1}) \right) (H_0 H_0^t)^{-1} \left( \nabla_k \frac{\partial}{\partial \phi^k} (\Omega \partial_\gamma \Omega^{-1}) \right) \Omega \right],
\]

(4.47)

\[
B_{ijkl} = \frac{1}{g^2} \int d^2x \text{Tr} \left[ \frac{\partial}{\partial \phi^k} \left( \Omega \frac{\partial}{\partial \phi^k} \Omega^{-1} \right) \frac{\partial}{\partial \phi^l} \left( \Omega \frac{\partial}{\partial \phi^l} \Omega^{-1} \right) \right].
\]

(4.48)

5 Correction to the single vortex effective Lagrangian

Applying the formula Eq. (4.44) to the single vortex solution Eq. (2.24), we obtain the following fourth order effective Lagrangian for a single non-Abelian vortex

\[
\mathcal{L}_{eff}^{(4)} = \frac{T}{8} |\partial_\alpha Z \partial^\alpha Z|^2 + \frac{4\pi}{g^2} (g_{ij} F_{\alpha \beta}^{FS} \partial_\alpha b^i \partial_\beta \bar{b}^j) \left[ \partial^\alpha Z \partial^\beta Z - \frac{1}{2} \eta^{\alpha\beta} (\partial_\gamma Z \partial^\gamma \bar{Z}) + c(g_{kl} \partial^\alpha b^k \partial^\beta \bar{b}^l) \right].
\]

(5.1)
where $T = 2\pi v^2$ is the vortex tension and $c$ is a constant given by

$$c = \frac{1}{2\pi} \int d^2 x \ (1 - |z - Z|^2 e^{-\psi})^2 \sim 0.830707 \times \frac{1}{g^2 v^2}. \quad (5.2)$$

In summary, we have obtained the following derivative expansion of the effective Lagrangian for a single non-Abelian vortex

$$\mathcal{L}_{\text{eff}} = -T \left[ 1 - \frac{1}{2} \partial_\alpha Z \partial_\alpha \bar{Z} - \frac{1}{8} |\partial_\alpha Z \partial_\alpha \bar{Z}|^2 \right]$$

$$+ \frac{4\pi}{g^2} \left[ \eta^{\alpha\beta} \left( 1 - \frac{1}{2} \partial_\beta Z \partial_\beta \bar{Z} \right) + \partial^{(\alpha} Z \partial^{\beta)} \bar{Z} \right] (g_{ij}^{\text{FS}} \partial_\alpha b^i \partial_\beta \bar{b}^j)$$

$$+ \frac{4\pi c}{g^2} (g_{ij}^{\text{FS}} \partial_\alpha b^i \partial_\beta \bar{b}^j) (g_{kl}^{\text{FS}} \partial_\alpha b^k \partial_\beta \bar{b}^l) + \mathcal{O}(\partial_\alpha^6). \quad (5.3)$$

### 5.1 Comparison with other models

Let us compare our result, Eq. (5.3), of the four derivative terms in the effective action for a single non-Abelian vortex with those in different context. First we compare the translation modes $Z$ with the Nambu-Goto action, and second we compare the orientational modes $b^i$ with the Faddeev-Skyrme model.

#### Translational zero modes and mixed terms

First, by setting the orientational moduli $b^i$ to zero in Eq. (5.3), we have the effective action for the translational modes $Z$ as

$$S_Z = -2\pi v^2 \int d^{d-1} x \left( 1 - \frac{1}{2} \partial_\alpha Z \partial_\alpha \bar{Z} - \frac{1}{8} |\partial_\alpha Z \partial_\alpha \bar{Z}|^2 \right), \quad (5.4)$$

which is precisely the one of an Abelian (ANO) vortex. At this order, this effective Lagrangian coincides with the Nambu-Goto action [42] given by

$$S_{\text{NG}} = -T \int d^{d-1} x \sqrt{-\det(-\gamma_{\alpha\beta})}, \quad (5.5)$$

where $\gamma_{\alpha\beta}$ is the induced metric on the world-volume given by

$$\gamma_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} = \eta_{\alpha\beta} - \frac{1}{2} (\partial_\alpha Z \partial_\beta \bar{Z} + \partial_\alpha \bar{Z} \partial_\beta Z). \quad (5.6)$$

In Eq. (5.3), there are the four derivative terms containing both the translational zero modes $Z$ and the orientational zero modes $b^i$. These are precisely the terms appearing in the derivative expansion of the following action describing the $\mathbb{C}P^{N-1}$ sigma model on the vortex world-volume

$$S_{\text{eff}} = \int d^{d-1} x \sqrt{-\det(-\gamma_{\alpha\beta})} \left[ -T + \frac{4\pi}{g^2} \gamma^{\alpha\beta} (g_{ij}^{\text{FS}} \partial_\alpha b^i \partial_\beta \bar{b}^j) + \mathcal{O}(\partial_\alpha^6) \right]. \quad (5.7)$$
Orientational zero modes

Next, we consider the orientational zero modes in the internal space. For simplicity, let us restrict ourselves to the case of \( N = 2 \) in which the internal moduli space is \( \mathbb{C}P^1 \). By setting \( Z = 0 \) in Eq. (5.3), we obtain the following effective Lagrangian of the orientational zero modes with the higher derivative correction

\[
L_{\mathbb{C}P^1} = \frac{4\pi}{g^2} \left[ \frac{\partial_\alpha b \partial_\alpha \bar{b}}{(1 + |b|^2)^2} + \frac{c |\partial_\alpha b \partial_\alpha b|^2}{(1 + |b|^2)^4} \right].
\] (5.8)

Now let us compare this with the Skyrme-Faddeev model which is a \( \mathbb{C}P^1 \) model with a four-derivative term [47]. To this end, we formulate the \( \mathbb{C}P^1 \) model (at the leading order) by two complex fields \( h = (h^1, h^2) \) charged under \( U(1) \) gauge symmetry. By introducing auxiliary gauge field \( a_\alpha \) and scalar field \( \lambda \) as Lagrange multipliers, it can be written as

\[
L = (\partial_\alpha + ia_\alpha)h (\partial_\alpha - ia_\alpha)h^\dagger - \lambda \left( hh^\dagger - \frac{4\pi}{g^2} \right).
\] (5.9)

The variation of \( \lambda \) gives the constraint \( hh^\dagger = \frac{4\pi}{g^2} \), which can be solved by \( h = \sqrt{\frac{g^2/4\pi}{1 + |b|^2}} (1, b) \). Then the variation of \( a_\alpha \) gives

\[
a_\alpha = \frac{i b \partial_\alpha \bar{b} - b \partial_\alpha \bar{b}}{2(1 + |b|^2)}.
\] (5.10)

Substituting these back into the original Lagrangian (5.9), the \( \mathbb{C}P^1 \) model (at the leading order) is recovered.

The field strength of the gauge field (5.10) is

\[
f_{\alpha\beta} = \partial_\alpha a_\beta - \partial_\beta a_\alpha = -i \frac{\partial_\alpha b \partial_\beta \bar{b} - \partial_\beta b \partial_\alpha \bar{b}}{(1 + |b|^2)^2}.
\] (5.11)

Then the Faddeev-Skyrme term [47] can be written as the field strength squared as

\[
f_{\alpha\beta}f^{\alpha\beta} = 2 \frac{(\partial_\alpha b \partial_\alpha b)^2 - |\partial_\alpha b \partial_\alpha b|^2}{(1 + |b|^2)^4}.
\] (5.12)

This term is quadratic in the time derivative and does not coincide with Eq. (5.8). The other term containing four time-derivatives appears in an \( \mathcal{N} = 1 \) supersymmetric extension of the Skyrme-Faddeev term [51]:

\[
\frac{(\partial_\alpha b \partial_\alpha \bar{b})^2}{(1 + |b|^2)^4}.
\] (5.13)

---

4 This term is also called the baby Skyrme term in \( d = 2 + 1 \) [50].

5 This term also arises when one constructs the Faddeev-Skyrme-like model as the low-energy effective theory of pure \( SU(2) \) Yang-Mills theory [52], and hence is called the Gies term in that context.
As shown in Appendix B, this higher derivative term can be obtained by adding a higher derivative term to the original Lagrangian \((5.9)\). In general the fourth order terms are summarized as

\[
L^{(4)}_{\text{general}} = c_1 f_{\alpha \beta} f^{\alpha \beta} + c_2 \left( \frac{\partial_\alpha b \partial_\bar{\beta} \bar{b}}{1 + |b|^2} \right)^2 = \frac{(2c_1 + c_2)(\partial_\alpha b \partial_\bar{\beta} \bar{b})^2 - 2c_1 |\partial_\alpha b \partial_\bar{\beta} \bar{b}|^2}{(1 + |b|^2)^4}. \tag{5.14}
\]

This reduces to the four derivative term in Eq. \((5.8)\) when

\[
c_2 = -2c_1 = c. \tag{5.15}
\]

The condition \(c_2 = -2c_1\) is precisely the condition for \(\mathcal{N} = 1\) supersymmetry [51]. This must be the case because there remains \(\mathcal{N} = 1\) supersymmetry (four supercharges) in the vortex effective theory, because vortices are 1/2 BPS states in supersymmetric theories with eight supercharges.

### 5.2 Instantons trapped inside a non-Abelian vortex

In this section, we consider \(d = 4 + 1\) dimensions where vortices are membranes with 2+1 dimensional world-volume. Besides the vortices, there also exist Yang-Mills instantons which are particle-like solitons in \(d = 4 + 1\) dimensions. But they cannot exist stably in the Higgs phase which we are considering. Instead, they can stably exist inside the world-volume of a non-Abelian vortex [30, 31]. Instanton-vortex composite configurations are 1/4 BPS states in supersymmetric gauge theories with 8 supercharges. In the supersymmetric \(\mathbb{C}P^{N-1}\) model with 4 supercharges, at the leading order of the 1/2 BPS vortex effective theory, instantons can be regarded as 1/2 BPS sigma model lumps [30] which are point-like solitons in \(d = 2 + 1\) dimensions. Here we show that instantons (= lumps) are not modified even if we include derivative corrections found in the previous sections.

For simplicity, we set \(Z = 0\) and consider the case of \(N = 2\) (the \(\mathbb{C}P^1\) sigma model) with the Lagrangian \((5.8)\) for the orientational modes. In \(d = 4 + 1\) dimensions, the vortex is a membrane which has coordinates \((x^0 = t, x^1, x^2)\). We parametrize the two spatial coordinates by complex variables \(w = x^1 + ix^2\) and \(\bar{w} = x^1 - ix^2\). We discuss only static configurations in this model.

First we recall the lump solution in the model without the four derivative correction. The energy density can be written as

\[
E_{\mathbb{C}P^1}^{(2)} = \frac{4\pi i}{g^2} \int d^2 x \sum_{\alpha = 1, 2} \frac{\partial_\alpha b \partial_\bar{\alpha} \bar{b}}{1 + |b|^2} = \frac{8\pi}{g^2} \int d^2 x \frac{2|\partial_\alpha b|^2 + (|\partial_\alpha b|^2 - |\partial_\bar{\alpha} b|^2)}{(1 + |b|^2)^2}. \tag{5.16}
\]

The first term is positive semi-definite \(|\partial_\alpha b|^2 \geq 0\) and the second term can be rewritten as

\[
I = \frac{4\pi i}{g^2} \int \frac{\partial_\alpha b \partial_\bar{\alpha} \bar{b} - \partial_\alpha b \bar{b} \partial_\bar{\alpha} \bar{b}}{(1 + |b|^2)^2} dw \wedge d\bar{w} = \frac{4\pi i}{g^2} \int \frac{db \wedge d\bar{b}}{(1 + |b|^2)^2} = \frac{8\pi^2}{g^2} k, \tag{5.17}
\]

\(22\)
where the integer $k$ is the degree of the map from the vortex world-volume to the target space $\mathbb{C}P^1$. This is the topological charge for lumps: $\pi_2(\mathbb{C}P^1) = \mathbb{Z}$. The energy (5.16) is bound from below by this topological charge $I$, and the Bogomol’nyi bound is saturated by the BPS equation for lumps, given by

$$\partial_{\bar{w}} b = 0.$$  (5.18)

The BPS solutions $b(w)$ are holomorphic in $w$ and satisfy the static equation of motion. If we fix the boundary condition as $b \to \infty$ ($w \to \infty$), the solutions with $I = \frac{8\pi^2}{g^2} k$ are given by $k$-th order rational maps

$$b(w) = \frac{a_0 w^k + a_1 w^{k-1} + \cdots + a_k}{\bar{a}_0 w^{k-1} + \bar{a}_1 w^{k-2} + \cdots + \bar{a}_{k-1}}.$$  (5.19)

This is the general solutions for the 1/2 BPS sigma model lumps.

Next let us consider the effect of the four derivative term

$$E^{(4)}(\mathbb{C}P^1) = -\frac{64\pi c}{g^2} \int d^2 x \frac{|\partial_w b \partial_{\bar{w}} b|^2}{(1 + |b|^2)^4}.$$  (5.20)

The total energy can be rewritten as

$$E^{(2)+(4)}_{\mathbb{C}P^1} = \frac{16\pi}{g^2} \int d^2 x \left[ 1 - 4c \frac{|\partial_w b|^2}{(1 + |b|^2)^2} \right] \frac{|\partial_w b|^2}{(1 + |b|^2)^2} + \frac{8\pi^2}{g^2} k.$$  (5.21)

Again, the first term is positive semi-definite as long as the correction term is sufficiently small and vanishes if the BPS equation (5.18) is satisfied. We thus have found that there is no contribution from the four derivative term to the BPS configurations and consequently the lump solutions are not modified. The energy of lumps $\frac{8\pi^2}{g^2} k$ agrees with that of instantons in the Higgs phase, which can be found by rewriting the energy of the original bulk theory to the Bogomol’nyi form for 1/4 BPS configurations. Therefore, it is natural to conjecture that all higher order terms vanish for 1/2 BPS lump configurations.

By considering 1/2 BPS lumps on the $\mathbb{C}P^{N-1}$ model, we have obtained 1/4 BPS composite states of instantons inside a vortex. On the other hand, there exist another 1/4 BPS composite states of intersecting vortex-membranes [30, 31]. Vortices of one kind have codimensions in $z$-plane and extend to $w$-plane, and those of another kind have codimensions in $w$-plane and extend to $z$-plane. This intersecting vortices can be constructed by considering holomorphic maps of the translational moduli $Z(w)$, instead of the orientational moduli $b^i(w)$ considered in this section. Now let us show that, in general, the fourth derivative terms do not modify the 1/2 BPS states. The BPS equation can be found by rewriting the second order energy as

$$E^{(2)}_{\text{eff}} = \int d^2 x \, 4g_{ij} \partial_w \phi^i \partial_{\bar{w}} \phi^j + \int i g_{ij} d\phi^i \wedge d\phi^j.$$  (5.22)
The second term gives the area of a two cycle of the moduli space on which the map $\phi^i$ wraps. The first term is positive semi-definite and vanishes for any holomorphic maps satisfying

$$\partial w \phi^i = 0. \quad (5.23)$$

On the other hand, it follows from the general form of the higher derivative corrections Eq. (4.46) that the fourth order energy vanishes for the holomorphic maps

$$E^{(4)}_{\text{eff}} \propto \partial_w \phi^i \partial_w \phi^j \partial_{\bar{w}} \phi^k \partial_{\bar{w}} \phi^l = 0. \quad (5.24)$$

Therefore, the higher derivative corrections do not modify the $1/2$ BPS configurations.

There is a possibility that the holomorphic maps are unstable if the fourth order energy is negative. However, this would be an artifact of the truncation at the fourth order. For example, the translational part of the energy can be rewritten as

$$E_{\text{eff}} = T \int d^2 x 2(1 - |\partial_w Z|^2)|\partial_{\bar{w}} Z|^2 + T \int \frac{i}{2} (dw \wedge d\bar{w} + dZ \wedge d\bar{Z}) \quad (5.25)$$

The second term is a topological term and the first term vanishes for a holomorphic map $Z(w)$ but it is not positive semi-definite. Although a holomorphic map $Z(w)$ appears to be an unstable solution if $|\partial_w Z|^2 > 1$, such a case is beyond the validity of our approximation. We can show that all the holomorphic maps are stable BPS solutions if we consider the full order action, i.e. the Nambu-Goto action, whose energy can be rewritten as

$$E_{\text{NG}} = T \int d^2 x \left[ \sqrt{(1 + |\partial_w Z|^2)^2 - |\partial_{\bar{w}} Z|^2} + 4|\partial_{\bar{w}} Z|^2 - (1 + |\partial_w Z|^2 - |\partial_{\bar{w}} Z|^2) \right]$$

$$+ \int \frac{i}{2} (dw \wedge d\bar{w} + dZ \wedge d\bar{Z}) \quad (5.26)$$

The first term is positive semi-definite and can be expanded as

$$T \int d^2 x 2 \left[ 1 - |\partial_w Z|^2 + |\partial_w Z|^2 (|\partial_w Z|^2 + |\partial_{\bar{w}} Z|^2) + \cdots \right]|\partial_{\bar{w}} Z|^2. \quad (5.27)$$

As this example shows, we should take into account the full order corrections to prove the stability of the holomorphic maps.

### 5.3 Higher derivative terms and massive modes

To see the physical meaning of the higher derivative corrections for the internal orientation, let us consider the equation of motion with higher order corrections in the $N = 2$ case. Assuming that the orientational zero mode $b$ is independent of the spatial world-volume coordinates, we
can show that the following configuration satisfies the equation of motion even in the presence of the higher derivative terms:

\[ b = \exp(i\omega t). \] (5.28)

This solution corresponds to an excited state of the non-Abelian vortex whose orientation is rotating along the equator of \( \mathbb{C}P^1 \) (Fig. 2). The rotation of the orientation induces one of three components of the \( SU(2)_{C+F} \) conserved charges, which is given by

\[ \rho_3 = \frac{i}{2} \left( \frac{\partial L}{\partial (\partial_t b)} b - \frac{\partial L}{\partial (\partial_t \bar{b})} \bar{b} \right) = \pi \frac{g^2}{\omega} \left( 1 + \frac{1}{2} c \omega^2 \right) \omega. \] (5.29)

The relation between the angular velocity \( \omega \) and the conserved charge \( \rho_3 \) (shown in Fig. 3-(a)) is modified by the higher derivative term, as in the case of the angular velocity and the angular momentum of the particle (see Eq. (3.32)).

By numerically solving the full equations of motion Eqs. (2.8) and (2.9), we can see that the size of the excited vortex is slightly larger than that of the static configuration (see Fig. 3 (b)). This is analogous to the case of the particle discussed in Sec. 3: the higher derivative term corresponds to the correction from the massive mode (vortex size) which is slightly shifted by the conserved charge.

### 6 Mass deformations

#### 6.1 Higher order corrections from mass terms

In this section, we consider higher order corrections from supersymmetric mass deformations of the original theory. Without breaking the supersymmetry, we can deform the original model by
Fig. 3: (a) The relation between the angular velocity $\omega$ and the conserved charge $\rho_3$ for $g = v = 1$. Compared to the second order relation $\rho_3 = \frac{1}{g} \omega$ (dashed line), the fourth order relation $\rho_3 = \frac{1}{g} (1 + \frac{c}{2} \omega^2) \omega$ (solid line) shows the better agreement with the numerical result (dots) obtained by solving the full equations of motion Eqs. (2.8) and (2.9). (b) The energy density distributions in the transverse plane ($z$-plane) for $g = v = 1$. The size of the excited vortex with $\omega = 0.5$ (solid line) is slightly larger than that of the static vortex with $\omega = 0$ (dashed line).

Adding the following terms to the Lagrangian Eq. (2.1)

$$
\mathcal{L}_{\text{adjoint+mass}} = \text{Tr} \left[ \frac{1}{g^2} D_\mu \Sigma_I D^\mu \Sigma_I + \frac{1}{2g^2} [\Sigma_I, \Sigma_J]^2 - |\Sigma_I H - H M_I|^2 \right],
$$

where $\Sigma_I$ ($I = 1, \ldots, n$) are the real adjoint scalar fields in the vector multiplets and $M_I$ are mass matrices in the Cartan subalgebra of $SU(N)_F$

$$
M_I \equiv m_I \cdot H = \text{diag}(m_{I,0}, m_{I,1}, \ldots, m_{I,N-1}).
$$

If all the masses are non-degenerate, the $SU(N)_F$ flavor symmetry is maximally broken to the Cartan subalgebra of $SU(N)_F$. These mass deformations can be obtained by dimensional reductions from the six dimensional model on $\mathbb{R}^{6-n} \times (S^1)^n$ with the following twisted boundary conditions around periodic dimensions

$$
H(x^\mu, \theta_I + 2\pi R_I) = H(x^\mu, \theta_I) e^{2\pi i R_I M_I}.
$$

Ignoring the infinite tower of the Kaluza-Klein modes, we obtain the mass deformation Eq. (6.1) with the following identification

$$
\Sigma_I(x^\mu) = -W_{\theta_I}(x^\mu), \quad H(x^\mu) e^{i\theta_I M_I} = H(x^\mu, \theta_I).
$$

The mass terms do not change the VEV of $H$ (given in Eq. (2.3)), while they induce those of the adjoint scalars

$$
\langle \Sigma_I \rangle = M_I.
$$
In this mass deformed model, the color-flavor global symmetry $SU(N)_{\text{C+F}}$ is explicitly broken to the Cartan subgroup $U(1)^{N-1}$. Hence the orientational moduli, which were the Nambu-Goldstone zero modes of $SU(N)_{\text{C+F}}$, are lifted by a potential induced by the mass terms. The potential on the moduli space can be calculated by finding the minimum energy configuration and evaluating the energy for each values of the moduli parameters. This can be done perturbatively with respect to the masses $m_{I,A}$. As a zeroth order configuration, we consider the vortex solution Eq. (2.10) satisfying the BPS equations without the mass deformations. Then we can determine the corrections to $\Sigma_I$ by solving their equations of motion

$$
\frac{2}{g^2} \left( D_\mu D^\mu \Sigma_I + [\Sigma_J [\Sigma_J, \Sigma_I]] \right) = (\Sigma_I H - H M_I) H^\dagger + H (H^\dagger \Sigma_I - M_I H^\dagger).
$$

(6.6)

Due to the fact that $\Sigma_I$ originate from the higher dimensional gauge fields $W_{\theta_I}$, we can solve the equations of motion in the similar way to the gauge fields $W_\alpha$

$$
\Sigma_I^{(1)} = M + i(\delta_I S^\dagger S^{\dagger -1} - S^{-1} \delta_I^\dagger S) + O(m^3, m \partial^2),
$$

(6.7)

where we have assumed that the derivatives $\partial_\alpha$ and the masses $m$ are of the same order

$$
m \sim \partial_\alpha.
$$

(6.8)

The differential operators $\delta_I$ and $\delta_I^\dagger$ are defined by

$$
\delta_I = k_i^I \frac{\partial}{\partial \phi^i}, \quad \delta_I^\dagger = \bar{k}_i^I \frac{\partial}{\partial \bar{\phi}^i},
$$

(6.9)

where $k_i^I$ are holomorphic Killing vectors on the moduli space which are the following linear combinations of the Killing vectors of the unbroken symmetry $U(1)^{N-1}$

$$
k_i^I \equiv m_I \cdot \xi^i.
$$

(6.10)

For example, the holomorphic Killing vectors on the moduli space of the single vortex solutions are given by

$$
k_i^I = i(m_{I,i} - m_{I,0})b^i \quad \text{(no sum over $i$)}.
$$

(6.11)

Inserting the solution $\Sigma_I^{(1)}$ into the deformation terms Eq. (6.1), we obtain the effective potential of the form

$$
V_{\text{eff}}^{(2)} = g_{ij} k_i^I k_j^I.
$$

(6.12)

This is the sum of the squared norm of the Killing vectors $k_i^I$ evaluated with respect to the moduli space metric $g_{ij}$ given in Eq. (4.21).
The potential of this form can also be obtained by dimensional reductions from the four
dimensional vortex world-volume action (the effective theory of the vortex in six spacetime di-
mensions)

\[ S_{\text{eff}}^{4d} = \int d^4 x g_{ij} \left[ \partial_\alpha \phi^i \partial^\alpha \phi^j - \partial_\theta \phi^i \partial_\theta \phi^j \right]. \]  
\hspace{1cm} (6.13)

As in the case of the bulk theory, we impose the twisted boundary condition for the moduli fields

\[ \phi^i(x^\alpha, \theta_I + 2\pi R_I) = \phi^i(x^\alpha, \theta_I)e^{2\pi R_I k^i_I}. \]  
\hspace{1cm} (6.14)

Keeping only the lowest modes and evaluating the kinetic terms in the effective action, we obtain
the effective action with the potential Eq. (6.12)

\[ S_{\text{eff}}^{(4-n)d} = \int d^{4-n} x g_{ij} \left[ \partial_\alpha \phi^i \partial^\alpha \phi^j - k_I^j k^i_I \right]. \]  
\hspace{1cm} (6.15)

Therefore, the procedure of the dimensional reductions and the calculation of the effective action
are commutative.

The higher order corrections to \( H \) and \( W_z \) can also be determined from the equations of
motion with the mass deformations. They can also be solved in a similar way to the case without
mass deformation as

\[ \Delta \Phi^{(2)} = \left( \frac{i}{2g} \right) \frac{2}{g^v} \left[ \partial_\alpha \phi^i \partial^\alpha \phi^k - k_I^j k^i_I \right] S^+ \left[ \nabla J \frac{\partial}{\partial \phi^k} \left( \bar{\phi} \Omega^{-1} \right) H_0^{+1} \right] S \]. \]  
\hspace{1cm} (6.16)

Then the higher order corrections are obtained by substituting the solution into

\[ \mathcal{L}_{\text{eff}}^{(4)} = \int d^2 x \text{Tr} \left[ -\frac{1}{2g^2}(F^{(2)}_{\alpha\beta})^2 + \frac{1}{g^2}(\mathcal{D}_\alpha \Sigma_I^{(1)})^2 + \frac{1}{2g^2}[\Sigma_I^{(1)}, \Sigma_J^{(1)}]^2 \right] + 4 \left( \Delta \Phi^{(2)}, \Delta \Phi^{(2)} \right). \]  
\hspace{1cm} (6.17)

In the case of the effective Lagrangian of a single vortex, the induced terms in take the form

\[ \mathcal{L}_{\text{eff mass}}^{(2)+(4)} = \frac{4\pi c}{g^2} g_{ij}^{FS} k_I^j \bar{b}^i \left( 1 - \frac{1}{2} \partial_\alpha Z \partial^\alpha \bar{Z} \right) \]
\[ + \frac{4\pi c}{g^2} \left[ (g_{ij}^{FS} k_I^j \partial_\alpha \bar{b}^i)(g_{kl}^{FS} k_I^k \partial^\alpha \bar{b}^l) + \text{c.c.} \right] \]
\[ + \frac{4\pi c}{g^2} (g_{ij}^{FS} k_I^j k_I^j)(g_{kl}^{FS} k_I^k k_I^l). \]  
\hspace{1cm} (6.18)

These corrections terms can also be obtained by using the dimensional reductions from the four-
dimensional world-volume with the higher derivative corrections.
6.2 Kink monopoles

In this section we consider a single 1/2 BPS non-Abelian vortex-string in four spacetime dimensions and its low-energy effective theory in two spacetime dimensions. Besides the vortex, another important topological soliton, 1/2 BPS monopole, arises in four dimensions. However, since the gauge symmetry is completely broken in the Higgs phase, the monopole cannot stay alone and must be accompanied with the vortex-string. If the vortex-strings are attached to the monopoles from the both left and right hand sides, the composite vortex-monopole state is indeed a stable BPS state preserving 1/4 supersymmetry (two supercharges) \[29\]. Such confined monopoles can also be viewed as kinks in the two-dimensional vortex effective theory. For example, let us consider $N = 2$ case with only one mass term

$$M = \frac{1}{2}(m, -m). \quad (6.19)$$

The orientational part of the second order effective Lagrangian of a single vortex-string in the mass deformed $U(2)$ theory takes the form

$$\mathcal{L}^{(2)}_{\text{eff}} = \frac{4\pi}{g^2} \frac{\partial_\alpha b \partial^\alpha \bar{b} - m^2 |b|^2}{(1 + |b|^2)^2}, \quad (\alpha = 0, 1). \quad (6.20)$$

In this sigma model, the orientational moduli are lifted by the potential and there are discrete vacua at $b = 0, \infty$. In the $(1 + 1)$-dimensional vortex world-sheet, we can consider a BPS kink interpolating between the discrete vacua

$$E^{(2)}_{\text{eff}} = \int dx \left[ \frac{4\pi}{g^2} \frac{\partial_x b - mb}{(1 + |b|^2)^2} + \partial_x \sigma \right], \quad (6.21)$$

where $\sigma$ is the moment map of the unbroken $U(1)$ symmetry defined by

$$\sigma = -\frac{2\pi m}{g^2} \frac{1 - |b|^2}{1 + |b|^2}. \quad (6.22)$$

The BPS equation can be easily solved as

$$\partial_x b = mb, \quad \rightarrow \quad b(x) = \exp \left[ m(x - x_0) + i\theta \right], \quad (6.23)$$

where $x_0$ and $\theta$ are kink position and phase moduli, respectively. The mass of the kink is

$$M_{\text{BPS}} = \sigma(x \to \infty) - \sigma(x \to -\infty) = \frac{4\pi m}{g^2}. \quad (6.24)$$

As expected, this BPS mass of the kink coincides with that of the monopole in $(3+1)$ dimensions.

To see the effect of the higher order corrections to the BPS kinks, let us take into account the forth order corrections in the effective theory

$$\mathcal{L}^{(2+4)}_{\text{eff}} = \frac{4\pi}{g^2} \left[ \frac{\partial_\alpha b \partial^\alpha \bar{b} - m^2 |b|^2}{(1 + |b|^2)^2} + c \frac{\partial_\alpha b \partial^\alpha b + m^2 b^2 |b|^2}{(1 + |b|^2)^4} \right]. \quad (6.25)$$
The energy of static configurations can be rewritten as
\[ E^{(2+4)}_{\text{eff}} = \int dx \left[ \frac{4\pi}{g^2} \left( 1 - c \frac{\partial_x b + mb}{1 + |b|^2} \right) + \partial_x \sigma \right]. \] (6.26)

Therefore, the BPS kink solution and its mass Eqs. (6.23) and (6.24) are not modified by the higher order corrections. Thus, we again encountered the case where the topological soliton in the vortex effective theory is exact at the second order. This is consistent with the observation that the kink on the vortex corresponds to the monopoles attached by the vortices in the four dimensional full theory.

### 6.3 Q-solitons

Let us again consider the effective theory of the single non-Abelian vortex in 4 + 1 dimensions. In the case of \( N = 2 \), the effective theory is (2 + 1)-dimensional massive \( \mathbb{C}P^1 \) sigma model. The effective Lagrangian at the second order is the same as Eq. (6.20) with \( \alpha = 0, 1, 2 \). As is well known, from the Derrick’s theorem, there are no stable lump solitons: the potential makes them collapse. However, one can still construct stable solitons by adding Noether charges to the lumps. It is the so-called Q-lumps \([54, 55]\). The Q-lump solutions have the same form as the \( \mathbb{C}P^1 \) but their phases are time-dependent. To see this, let us see the energy

\[ E^{(2)} = \frac{4\pi}{g^2} \int d^2x \left( |\dot{b}|^2 + |\partial_x b|^2 - m^2|b|^2 \right) \frac{1}{(1 + |b|^2)^2} + I + Q, \] (6.27)

with \( I \) and \( Q \) being the topological charge and the Noether charge associated with the \( U(1) \) global symmetry

\[ I = \frac{8\pi i}{g^2} \int \frac{db \wedge \bar{d}b}{(1 + |b|^2)^2} = \frac{8\pi^2 k}{g^2}, \] (6.28)

\[ Q = \pm \frac{4\pi m}{g^2} \int d^2x \frac{i(b\bar{b}^* - \bar{b}b^*)}{(1 + |b|^2)^2}. \] (6.29)

The BPS equations are of the form

\[ \dot{b} = \pm imb, \quad \partial_w b = 0. \] (6.30)

Thus the solution is given by

\[ b(t, x^1, x^2) = e^{\pm int} \tilde{b}(w), \] (6.31)

where \( \tilde{b}(w) \) can be any holomorphic function of \( w \) as in the case of the static configuration Eq. (5.19). Note that the Q-lump solution with the minimal winding number \( \tilde{b} = \frac{\lambda}{w} \) has infinite
energy because its Noether charge density has an asymptotic tail $\sim 1/|\lambda^2 w|$. The minimal configuration with finite energy, for example, is given by $\tilde{b} = \frac{\lambda}{w-w_1} - \frac{\lambda}{w-w_2}$.

Let us next see the effects from the higher derivative corrections: Do the Q-lumps receive corrections from the higher order terms? Indeed, as one can easily check, the solution given in Eq. (6.30) solves the full equations of motion including the fourth order corrections. In the presence of the higher order corrections, the energy density can be rewritten as follows

$$E^{(4)} = \frac{4\pi c}{g^2} \int d^2x \frac{2(|\dot{b}|^4 - |(\partial_t b)^2 - m^2 b|^2) + |\dot{b}^2 - (\partial_t b)^2 + m^2 b|^2}{(1 + |b|^2)^4}$$

$$= \frac{32\pi c}{g^2} \int d^2x \frac{m^2 (\tilde{b}^\alpha \partial_\alpha \tilde{b} \partial_\alpha \tilde{b} + \tilde{b}^\alpha \partial_\alpha \tilde{b} \partial_\alpha \tilde{b}) - 2|\partial_\alpha \tilde{b} \partial_\alpha \tilde{b}|^2}{(1 + |\tilde{b}|^2)^4}.$$  (6.32)

Since $\tilde{b}$ is a holomorphic function in $w$, we can immediately conclude that $E^{(4)} = 0$ for the Q-lump solutions $b = e^{i\omega t}b(w)$. Therefore, the BPS Q-lump solutions and their masses are not modified by the higher order corrections.

### 7 Summary and Discussion

We have proposed a systematic method to obtain higher derivative terms in the low-energy effective theories on solitons. We have applied our method to a single non-Abelian vortex and have obtained four-derivative terms in the $\mathbb{C} \times \mathbb{C}P^{N-1}$ model on the vortex world-volume. We have compared our four-derivative terms with the Nambu-Goto action and the Faddeev-Skyrme model. The action for the translational moduli $Z$ coincides with the Nambu-Goto action and the terms for the orientational moduli $b^i$ coincide with those for the supersymmetric extension of the Faddeev-Skyrme term. We have also shown that the contribution from the four-derivative terms disappears for 1/4 BPS states of instantons trapped inside a non-Abelian vortex and consequently the solutions are not modified in the presence of the four derivative terms.

In this paper, we have derived the four derivative terms. In principle we can go on to any order in our formalism. The sixth order is considered to be prominently important. We have confirmed that the effective action of the translational zero modes $Z$ at the fourth order is consistent with the Nambu-Goto action. On the other hand, in the width expansion from the Nambu-Goto action, the first correction term written as the extrinsic curvature squared starts from the sixth derivatives in the derivative expansion [44]. In field theory calculation, it seems that there is no agreement on the signature of that term [46]. Therefore, we can in principle determine that term for the BPS case.

In this paper we have studied local non-Abelian vortices which exist in the theory with the
number $N_F$ of flavors equals to the number $N_C$ of color. When the theory has more flavors, $N_F > N_C$, vortices are called semi-local [59]. It is known in this case that the orientational zero modes of a single vortex is non-normalizable, i.e., the integration over the codimensions diverges [57], unless the size modulus is zero and the vortex shrinks to a local vortex [58]. For two vortices, the relative orientational zero modes are normalizable even with a non-zero size moduli, while the overall orientational zero modes are non-normalizable [58]. Although we can formally extend our method to semi-local vortices, we should check if there exists a divergence in four derivative terms even for normalizable moduli. Remember that in the derivative expansion, derivatives are assumed to be less than the lowest mass $m$ of the mass spectrum in the vacuum. The existence of the vacuum moduli for $N_F > N_C$ implies that $m = 0$ and hence the convergence radius of the derivative expansion seems to be zero. Therefore, it is interesting to see if there exists a special mechanism which justifies the derivative expansion for semi-local vortices. Non-Abelian vortices were extended to arbitrary gauge groups $G$ in the form of $(U(1) \times G)/C(G)$ with the center $C(G)$ of $G$ [59]. Especially the cases of $G = SO(N), USp(2N)$ have been studied in detail [60, 61]. We can straightforwardly extend our analysis to the cases of arbitrary gauge groups but we should be careful to the normalizability since they are semi-local vortices in general.

Although we have studied BPS vortices in supersymmetric gauge theories, our method to obtain higher derivative corrections by solving equations of motion for massive fields is robust and can be extended to non-supersymmetric theories. For instance, we can apply it to non-Abelian vortices in non-supersymmetric theory [62]. In reality, non-Abelian vortices exist in high-density QCD which may be realized in the core of neutron stars [63]. In this case, the low-energy world-sheet theory on the vortex is described by the bosonic $CP^2$ model at the leading order [64]. The four-derivative correction to it should be important especially for the fate of confined monopoles which have been recently shown to exist as kinks on the vortex [65]. This is because they are non-BPS and higher derivative terms do not vanish automatically, unlike BPS instantons discussed in Sec.5.2.

Finally our method is general so that we can apply it to other BPS solitons such as domain walls, monopoles and instantons, or non-BPS solitons such as Skyrmions. In the same spirit, four derivative term in the form of the Skyrme term was obtained in the effective theory of non-Abelian domain walls [23], and four derivative terms for collective coordinates of a rotating Skyrmion were calculated [24].

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A O(3) model

From the isomorphism $\mathbb{C}P^1 \simeq S^2 \simeq O(3)/O(2)$, the $\mathbb{C}P^1$ model is equivalent to the $O(3)$ model. In order to see this equivalence, let us introduce a three vector $n = (n_1, n_2, n_3)$ by

$$n = \left( \frac{1}{\sqrt{1+|\beta|^2}}, \frac{\beta}{\sqrt{1+|\beta|^2}} \right) \vec{\sigma} \left( \frac{1}{\sqrt{1+|\beta|^2}} \beta \frac{1}{\sqrt{1+|\beta|^2}} \right)$$

$$\quad = \left( \frac{\beta + \bar{\beta}}{1+|\beta|^2}, -i \frac{\beta - \bar{\beta}}{1+|\beta|^2}, \frac{1-|\beta|^2}{1+|\beta|^2} \right) \quad (A.1)$$

which satisfies the constraint

$$n^2 = 1. \quad (A.2)$$

Conversely, $\beta$ is the stereographic coordinate, given by

$$\beta = \frac{n_1 + i n_2}{1+n_3} = \frac{1-n_3}{n_1-ia_2}. \quad (A.3)$$

The kinetic term becomes

$$\frac{\partial_\mu \beta \partial_\nu \bar{\beta}}{(1+|\beta|^2)^2} = \frac{1}{2} \partial_\mu n \cdot \partial_\mu n, \quad (A.4)$$

and the field strength can be rewritten as

$$f_{\mu\nu} = n \cdot (\partial_\mu n \times \partial_\nu n). \quad (A.5)$$

Therefore the Skyrme-Faddeev term and the other four derivative term become

$$f_{\mu\nu} f^{\mu\nu} = (\partial_\mu n \times \partial_\nu n)^2, \quad (A.6)$$

$$\frac{(\partial_\mu \beta \partial_\nu \bar{\beta})^2}{(1+|\beta|^2)^4} = \frac{1}{4} (\partial_\mu n \cdot \partial_\mu n)^2, \quad (A.7)$$

respectively. The total four derivative terms in Eq. (5.14) can be rewritten as

$$\mathcal{L}_4 = c_1 (\partial_\mu n \times \partial_\nu n)^2 + \frac{c_2}{4} (\partial_\mu n \cdot \partial_\mu n)^2. \quad (A.8)$$
In this section, we consider a gauged linear sigma model with a higher derivative term which reproduces the orientational part of the higher derivative terms in Eq. (5.3). The model is described by \( N \) charged scalar fields \( h = (h_1, \ldots, h_N) \) coupled to a \( U(1) \) auxiliary gauge field \( a_\alpha \). The Lagrangian is given by
\[
L = D_\alpha h (D^\alpha h)^\dagger + d D_\alpha h (D^\alpha h)^\dagger D^\alpha h (D^\beta h)^\dagger + \lambda (hh^\dagger - v^2),
\]
where \( d \) is a constant and \( \lambda \) is a Lagrange multiplier for the constraint \( hh^\dagger = v^2 \), which is solved by
\[
h = \frac{v}{\sqrt{1 + |b_i|^2}} (b_1, \ldots, b_{N-1}, 1).
\]
The covariant derivative is defined by \( D_\alpha h = (\partial_\alpha + ia_\alpha) h \). The equation of motion for the auxiliary gauge field \( a_\alpha \) can be solved as
\[
a_\alpha = -\frac{i}{2} \frac{b_\alpha \partial_\alpha b_i - \partial_\alpha b_i \bar{b}_i}{1 + |b_i|^2}.
\]
This solution is independent of the parameter \( d \). In other words, the higher derivative term in Eq. (B.1) does not change the solution for the auxiliary gauge field. We can easily show that Eq. (B.3) is the solution by using the following relations
\[
D_\alpha hh^\dagger = h (D_\alpha h)^\dagger = 0.
\]
Substituting Eq. (B.3) back into the Lagrangian Eq. (B.1), we obtain
\[
L = v^2 g^{FS}_{ij} \partial_\alpha b^i \partial^\alpha \bar{b}^j + v^4 d (g^{FS}_{ij} \partial_\alpha b^i \partial_\beta \bar{b}^j) (g^{FS}_{kl} \partial^\alpha b^k \partial^\beta \bar{b}^l).
\]
Therefore, if \( v^2 = \frac{4\pi}{g^2} \) and \( d = \frac{\pi^2}{4\pi^2} c \), this Lagrangian coincides with the orientational part of the vortex effective action Eq. (5.3).

C Regularity of the second order solution

In this section, we show that there is no singularity in the first component of the solution \( \Delta \Phi^{(2)} \) given in Eq. (4.40). Here we assume that \( Z_I \neq Z_J (I \neq J) \), so that the constant matrix \( \Psi^I \equiv [(z - Z_I) H_0^{-1}]_{z=Z_I} \) is well-defined. Then the singular behavior around the \( I \)-th vortex position \( z = Z_I \) can be written as
\[
\frac{1}{\bar{z} - Z_I} \left[ \Psi_I \nabla_{\bar{z}} \frac{\partial}{\partial \phi^j} (\Omega \partial_2 \Omega^{-1}) \right] \bigg|_{z=Z_I} + O(1).
\]
We can show that
\[
\left[ \nabla_i \frac{\partial}{\partial \phi^j} \left( \Omega \partial_z \Omega^{-1} \right) \right]_{z=Z_I} = \nabla_i \frac{\partial}{\partial \phi^j} \left[ \Omega \partial_z \Omega^{-1} \right]_{z=Z_I} - \nabla_i \left[ \frac{\partial Z_I}{\partial \phi^j} \partial_z (\Omega \partial_z \Omega^{-1}) \right]_{z=Z_I},
\]
(C.2)
where we have used the following identity for any function of the form \( f(z, \bar{z}, \phi^i, \bar{\phi}^j) \)
\[
\left[ \frac{\partial}{\partial \phi^j} f \right]_{z=Z_I} = \frac{\partial}{\partial \phi^j} [f]_{z=Z_I} - \frac{\partial Z_I}{\partial \phi^j} [\partial_z f]_{z=Z_I}. \tag{C.3}
\]

Then, the singular part can be rewritten into the following form
\[
- \frac{g^2 v^2}{4} \frac{1}{\bar{z} - Z_I} \left[ \Psi_I \nabla_i \frac{\partial}{\partial \phi^j} \left( B_I - \bar{Z}_I 1_N \right) \right]^\dagger, \quad \text{with} \quad B_I \equiv - \frac{4}{g^2 v^2} [\Omega \partial_z \Omega^{-1}]_{z=Z_I},
\]
(C.4)
where we used the following relations which can be derived from the master equation Eq. (2.12):
\[
\Psi_I \partial_z \left[ \Omega \partial_z \Omega^{-1} \right]_{z=Z_I} = - \frac{g^2 v^2}{4} \Psi_I, \quad \Psi_I \partial_z^2 \left[ \Omega \partial_z \Omega^{-1} \right]_{z=Z_I} = 0.
\]
(C.5)

Now let us use the explicit form of the generic moduli matrix Eq. (2.30). Then, \( \Psi_I \) is given by
\[
\Psi_I = \prod_{I \neq J} \frac{1}{Z_I - Z_J} \begin{pmatrix} 1 & 0 \\ -\bar{b} & 0 \end{pmatrix}.
\]
(C.6)

By assuming that the matrix \( B_I \) takes the form
\[
B_I \equiv \begin{pmatrix} p_I & (\bar{q}_I)^T \\ \bar{r}_I & s_I \end{pmatrix},
\]
(C.7)
we can rewrite the singular part as
\[
- \frac{g^2 v^2}{4} \frac{1}{\bar{z} - Z_I} \left[ \prod_{J \neq I} \frac{1}{Z_I - Z_J} \left( \begin{array}{cc} 1 & 0 \\ -\bar{b} & 0 \end{array} \right) \nabla_i \frac{\partial}{\partial \phi^j} \left( \begin{array}{c} p_I - \bar{Z}_I (\bar{q}_I)^T \\ \bar{r}_I \\ s_I - Z_I \end{array} \right) \right]^\dagger.
\]
(C.8)

We can show that \( \nabla_i \left[ \frac{\partial}{\partial \phi^j} \left( \bar{Z}_I - p_I \right) \right] = \nabla_i \frac{\partial}{\partial \phi^j} \bar{q}_I = 0 \) as follows. It has been shown that the matrix \( B_I \) is related to the moduli space metric as [10]
\[
g_{ij} = \pi v^2 \sum_{l=1}^k \left( \frac{\partial Z_I}{\partial \phi^j} \frac{\partial Z_I}{\partial \phi^l} + \text{Tr} \left[ \frac{\partial H_0}{\partial \phi^j} \Psi_I \frac{\partial B_I^T}{\partial \phi^l} \right]_{z=Z_I} \right),
\]
(C.9)

\[
= \pi v^2 \sum_{l=1}^k \left( \frac{\partial Z_I}{\partial \phi^j} \frac{\partial}{\partial \phi^l} (\bar{Z}_I - p_I) + \prod_{J \neq I} \frac{1}{Z_I - Z_J} \left[ \frac{\partial \bar{b}_I}{\partial \phi^j} - \sum_{l=1}^k \frac{\partial Z_I}{\partial \phi^l} D_{lJ} \bar{b}_{I,j} \right] \cdot \frac{\partial \bar{q}_I}{\partial \phi^l} \right).
\]

where \( D_{lJ} \equiv [\partial_x e_{lJ}(z)]_{z=Z_I} \). Since the vortex moduli space is a Kähler manifold, there exist a Kähler potential \( K \) such that
\[
g_{ij} = \frac{\partial K}{\partial \phi^j \partial \phi^l}.
\]
(C.10)
By comparing this equation with Eq. (C.9), we can read $p_I$ and $\vec{q}_I$ as

$$\frac{\partial}{\partial \phi^j}(Z_I - p_I) = \frac{1}{\pi v^2} \left[ \frac{\partial K}{\partial \phi^j \partial Z_I} - \sum_{j=1}^{k} D_{IJ} \vec{b}_I \cdot \frac{\partial K}{\partial \phi^j \partial \vec{b}_J} \right], \quad (C.11)$$

$$\frac{\partial \vec{q}_I}{\partial \phi^j} = \frac{1}{\pi v^2} \prod_{J \neq I} (Z_I - Z_J) \frac{\partial K}{\partial \phi^j \partial \vec{b}_I}, \quad (C.12)$$

Since the moduli space metric is covariantly constant, it follows that

$$\nabla_i \frac{\partial}{\partial \phi^j}(Z_I - p_I) = \nabla_i \frac{\partial \vec{q}_I}{\partial \phi^j} = 0. \quad (C.13)$$

This shows that there is no singularity in the solution Eq. (4.40).
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