Chaos and hyperchaos in driven interacting quantum systems

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We investigate theoretically the dynamics of one-dimensional coupled quantum systems with energy pumping and dissipation. Our study focusses on coupled Rydberg atoms driven by an external laser field and undergoing spontaneous emission. We show that this class of quantum systems is able to demonstrate a spontaneous onset of chaotic, and even hyperchaotic oscillations, which are characterized by one or several positive Lyapunov exponents. The number of positive Lyapunov exponents grows with the number of coupled elements. Remarkably, hyperchaotic dynamics appears in short chains of only five units, while large chains can demonstrate a highly randomized behavior. We investigate the transition from regular to chaotic behavior and analyze the underlying instabilities.

Recent progress in experimental techniques for the fabrication, control and measurement of quantum systems has allowed the routine creation of moderate-to-large arrays of two-level quantum units, e.g., superconducting qubits or trapped atoms, where a degree of quantum coherence and control over their quantum state can be maintained [1]. While these systems are promising for numerous applications from quantum sensing and communication to universal quantum computing [2], [3], the fundamental importance of their development is in the decisive demonstration that effects such as quantum superposition and entanglement are not limited to the microscopic domain. As a result, such systems provide the experimental tools for direct investigation of quantum-to-classical transition and to evaluate different mechanisms for the emergence of classical behavior.

The well known fundamental impossibility of the direct classical simulation of large quantum systems means that theoretical understanding of their dynamics is limited, hindering experimental progress [4], [5]. Different strategies are emerging for attacking this problem. Of particular interest is the exploitation of a link that is being established between chaotic behavior and ergodicity in quantum and classical systems, entanglement and thermalization [6], [11]. However, there is still little understanding of the associated emergent phenomena, which arise as the size of the system changes.

To address this question, here we theoretically study the dynamics of a chain of interacting two-level quantum systems [Fig. 1(a)]. We find that even a small chain comprising between two and four elements is able to demonstrate chaotic behavior. Remarkably, in systems with five or more elements a phenomenon known as hyperchaos emerges, a dynamical regime characterized by two or more positive Lyapunov exponents. The highly randomized nature of hyperchaos shares some similarities with thermalization even though the system is far from equilibrium. Our results give new insights into the dynamics of large artificial quantum coherent structures, which will be important for the design and control of quantum systems such as Rydberg molecules, quantum processors, simulators and detectors.

To be specific, we theoretically investigate the dynamics of a 1D chain of interacting Rydberg atoms driven by an external electromagnetic field in the presence of decoherence and dissipation [12], [13]. Rydberg systems provide a controllable testbed for studying fundamental quantum dynamics [16], [18] and for developing new quantum chemistry [19], [21]. In particular, arrays of Rydberg atoms can be controlled experimentally with very high precision [22], [23].

The system we consider is an array of N two-level atoms [see Fig. 1(a)] with a ground state |g⟩ and an electronically excited high-energy Rydberg state |e⟩ [Fig. 1(b)]. Rydberg states have large polarizability \( \sim n^2 \) (where n is the principal quantum number), which generates strong and long-range interactions between atoms [25]. To model an electromagnetically driven array of Rydberg atoms, we use the Hamiltonian:

\[
H = \sum_{j=1}^{N} \left[ -\delta_\omega |e⟩⟨e|_j + \frac{\Omega_R}{2} (|e⟩⟨g|_j + |g⟩⟨e|_j) + \sum_{j<k} V_{i,j} |e⟩⟨e|_j \otimes |e⟩⟨e|_k, \right.
\]

where \( \delta_\omega = \omega_j - \omega_0 \) is the detuning between the laser and transition frequencies, \( \Omega_R \) is the Rabi frequency (tuned by the laser field amplitude), and \( V_{i,j} \) characterizes the interaction between \( j \)th and \( j \)th atoms. The dynamics are described by the Liouville-von Neumann equation for the density matrix, \( \rho \),

\[
\dot{\rho} = -i [H, \rho] + L [\rho],
\]

where relaxation and dephasing processes are taken into
account through the appropriate Lindblad operator
\[ \mathcal{L}[\rho] = \gamma \sum_j \left[ \langle g|e_j\rho|g \rangle \langle e_j|g \rangle - \frac{1}{2} \{ \langle e| e_j \rho e_j \rangle - \langle e| e_j \rho \} \right], \tag{2} \]

with \( \gamma \) characterizing the effective rate of both the relaxation and dephasing processes. Note that generally one can introduce separate rates for relaxation and pure dephasing, but this is not essential for our analysis here. Rydberg atom interactions can range over a few tens of micrometers, larger than the physical size of the system. Therefore, to simplify analysis, we first assume that interactions are identical for any pair of atoms, \( V_{i,j} = V \) [see Fig. 1(a,b)].

For a system of \( N \) qubits, \( \rho \) has the dimensionality of its Hilbert space, \( 2^N \). However, quantum correlations, including entanglement, between the quantum states of different qubits are usually more fragile than those of the same qubit mainly for the following reasons. Firstly, dephasing caused by local noise sources suppresses the interqubit correlations stronger. Secondly, the qubit-qubit interactions, which tend to induce quantum correlations between them, decrease with increasing interqubit separation. Consequently, to first approximation, we use a lower-dimension factorized density matrix, i.e., \( \rho \approx \prod_j \rho_j \). This approximation yields the following set of dimensionless equations for the matrix elements
\[ w_j = (\rho_j)_{11} - (\rho_j)_{00} \]
\[ q_j = (\rho_j)_{10} - (\rho_j)_{01} \]
corresponding to the population inversion of the \( j \)th qubit, and \( \rho_j \) corresponding to its coherence [20]:
\[ w_j = -2\Omega \Im q_j - (w_j + 1); \]
\[ \dot{q}_j = i \left[ \Delta - c \sum_{k \neq j} (w_k) \right] q_j - \frac{1}{2} q_j + \frac{i}{2} \Omega w_j. \tag{3} \]

In [3] the dimensionless time \( \tau = \gamma t \) (i.e. \( \dot{x} \equiv dx/d\tau = \gamma^{-1} dx/dt \)), \( \Omega = \Omega_R/\gamma \), \( \Delta = \delta_c/\gamma \), and \( c = V a/\gamma \), where \( a \) is the lattice constant.

Here we study the dynamical regimes of chains of interacting atoms. To characterise the complexity (randomness) of the emergent chaos and hyperchaos we calculate the full spectrum of the Lyapunov exponents [27]. First we analyze two coupled atoms (\( N = 2 \)). Previously, it has been shown that interplay between the energy pumping and dissipation can eventually trigger self-sustained state population oscillations in this system and even lead to the emergence of bistability, when homogeneous and antiferromagnetic states coexist [12]. Our investigation reveals another interesting phenomenon associated with deterministic chaos, which emerges via a cascade of period-doubling bifurcations for a periodic oscillations [27]. In Fig. 1(c), we show a grayscale map illustrating the dependence of the largest Lyapunov exponent \( \Lambda_1 \) on \( \Delta \) and \( \Omega \). A transition from white to black reflects the change from smaller to larger values of \( \Lambda_1 \). The diagram clearly demonstrates the presence of chaotic dynamics, for which \( \Lambda_1 > 0 \), in considerable areas of the parameter plane shaded by dark-gray/black colors. This shows that the discovered chaos is a robust phenomenon existing in our system over wide parameter ranges. Notably, for certain parameter values it coexists with the antiferromagnetic steady state. The detailed descriptions of the bifurcation transitions and analysis of the Lyapunov exponents that characterize the stability of long-term dynamics are presented in the Supplemental materials [26].

For \( N = 3 \) and \( 4 \), the same type of chaotic dynamics exists in the system. However, surprisingly, for \( N \geq 5 \) a new type of chaotic dynamics appears, whose stability is characterized by more than one positive Lyapunov exponent. This type of chaos is known as hyperchaos [28]. A map of different dynamical regimes in the \( (\Delta, \Omega) \) parameter plane is shown in Fig. 2 for \( N = 5 \) and \( c = 5 \). This diagram was built by calculating the spectrum of Lyapunov exponents for the various limit sets that exist in the \( (\Delta, \Omega) \)-plane. When all the Lyapunov exponents are negative, there is a stable equilibrium state (white in the diagram). If the largest Lyapunov exponent equals 0 the oscillations are periodic (light gray/cyan online). If the largest two Lyapunov exponents are both 0, the dynamics are quasiperiodic (dark gray/red online). However,
when one or two Lyapunov exponents are positive, there is chaos (gray/green) or hyperchaos (black), respectively.

Figure 3(a) illustrates the bifurcation transitions between the different dynamical regimes when $\Omega = 2.5$ and $\Delta$ changes along the dashed line in Fig. 2. The gray (green online) curves represent the evolution of the four largest Lyapunov exponents $\Lambda_1 > \Lambda_2 > \Lambda_3 > \Lambda_4$. A number of distinctive phases emerge as we increase $\Delta$ from 1 to 9. The stable equilibrium state, which exists for small $\Delta$, switches to a periodic solution as a result of an Andronov-Hopf bifurcation at $\Delta \approx 1.7$, where $\Lambda_1$ becomes zero. At $\Delta \approx 3.55$ the periodic oscillations lose their stability via a Neimark-Sacker bifurcation, resulting in the onset of quasiperiodic oscillations ($\Lambda_1 = \Lambda_2 = 0$). Increasing $\Delta$ further leads to chaotic dynamics.

To better illustrate the transition to the chaotic regime, in Fig. 3(b) we present a zoom of the region of Fig. 3(a) framed by the black rectangle. The corresponding bifurcation diagram, shown in Fig. 3(c), is constructed by plotting the points corresponding to the local maxima $w_{1,\text{max}}$ in the time-evolution of $w_1(t)$, calculated for given $\Delta$. For a particular value of $\Delta$, periodic solutions are represented by one or few single dots on the graph, while the complex sets of many points for a specific $\Delta$ reflect quasiperiodic or chaotic dynamics. As $\Delta$ increases, the quasiperiodic oscillations where $\Lambda_1 = \Lambda_2 = 0$ [Fig. 3(b)] are replaced (at $\Delta \approx 3.728$) by complex periodic oscillations due to saddle-node bifurcation. These periodic oscillations are characterized by $\Lambda_1 = 0$ and $\Lambda_{2,3} < 0$ [Fig. 3(b)] and represented in Fig. 3(c) by few isolated dots for a fixed $\Delta$. For $\Delta \geq 3.739$, the periodic solutions undergo a cascade of period-doubling bifurcations giving rise to chaos with one positive Lyapunov exponent at $\Delta \approx 3.745$ [Fig. 3(b)]. The perioddoubling bifurcations do not affect the spectrum of Lyapunov exponents of the stable solution, since the latter remains periodic. However, each period-doubling causes additional Lyapunov exponent to approach zero [Fig. 3(b)], which manifests itself via doubling of the number of dots in Fig. 3(c). Thus for $N = 5$, the bifurcation mechanism leading to the onset of chaos stays the same as for the case of $N = 2$ [20]. Further investigation shows that this mechanism is also present at larger $N$.

Increasing $\Delta$ makes the chaotic oscillations more complicated and leads to the gradual development of hyperchaos, which emerges at $\Delta \approx 4.4$ [see Fig. 3(a)]. This transition is linked to further instability of already unstable periodic orbits, which form the skeleton of the chaotic attractor. Accumulation of the corresponding bifurcations causes the second Lyapunov exponent to become positive.

The dynamics of the $N = 5$ coupled atoms in different dynamical regimes is exemplified in Fig. 4. Typical periodic solution is presented in Fig. 4(a). Here, $n = 1, \ldots, 5$ denotes the atom number in the chain, $t$ is time, and the grayscale indicates the value of $w_n$. All atoms oscillate with the same frequency, but with different phases. However, the phase shift is constant for each pair of atoms, meaning that they are synchronized. The latter is also indicated by the clearly periodic pattern in Fig. 4(a). The quasiperiodic regime when $\Delta = 3.72$ is shown in Fig. 4(b). Now the phase shifts between oscillations of different atoms are no longer constant, and the periodic pattern of the spatio-temporal dynamics is lost, reflect-
In conclusion, we have shown that interplay between dissipation and energy pumping in quantum systems comprising chains of qubits can produce highly nontrivial emergent phenomena associated with the onset of complex chaotic and even hyperchaotic oscillations. The complexity of the hyperchaos increases with the number of elements in the chains. Our results demonstrate a mechanism for randomizing the evolution of coupled atoms, which arises due to dynamical phenomena far from equilibrium and, thus, is unrelated to thermalization processes. The model we have used is generic and can be applied and implemented directly in, e.g., chains of Rydberg atoms or electromagnetically driven superconducting qubits. Our results will be useful for the development of quantum systems with large numbers of unit quantum elements. In particular, they suggest a controllable way of switching between different dynamical regimes via regular to chaos transitions. The latter could be of interest for the emerging fields of quantum random generators and quantum chaotic cryptography. A natural extension of this research will consider the effect of interqubit entanglement on the instabilities introduced here, and whether it can be utilised...
for controlling this far-from-equilibrium behavior.

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