ON THE QUATERNIONIC $p$-ADIC $L$-FUNCTIONS ASSOCIATED TO HILBERT MODULAR EIGENFORMS

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ABSTRACT. We construct $p$-adic $L$-functions associated to cuspidal Hilbert modular eigenforms of parallel weight two in certain dihedral or anticyclotomic extensions via the Jacquet-Langlands correspondence, generalizing works of Bertolini-Darmon, Vatsal and others. The construction given here is adelic, which allows us to deduce a precise interpolation formula from a Waldspurger type theorem, as well as a formula for the dihedral $\mu$-invariant. We also make a note of Howard’s nonvanishing criterion for these $p$-adic $L$-functions, which can be used to reduce the associated Iwasawa main conjecture to a certain nontriviality criterion for families of $p$-adic $L$-functions.

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1. INTRODUCTION

Let $F$ be a totally real number field of degree $d$ over $\mathbf{Q}$. Fix a prime ideal $p \subset \mathcal{O}_F$ with underlying rational prime $p$. Fix an integral ideal $\mathfrak{N}_0 \subset \mathcal{O}_F$ with $\text{ord}_p(\mathfrak{N}_0) \leq 1$. Let

\begin{equation}
\mathfrak{N} = \begin{cases} 
\mathfrak{N}_0 & \text{if } p \mid \mathfrak{N}_0 \\
p\mathfrak{N}_0 & \text{if } p \not\mid \mathfrak{N}_0.
\end{cases}
\end{equation}

Hence, $\text{ord}_p(\mathfrak{N}) = 1$. Fix a totally imaginary quadratic extension $K$ of $F$. Assume that the relative discriminant $\mathfrak{D}_{K/F}$ of $K$ over $F$ is prime to $\mathfrak{N}/p$. The choice of $K$ then determines uniquely a factorization

\begin{equation}
\mathfrak{N} = p\mathfrak{N}^+\mathfrak{N}^-
\end{equation}

of $\mathfrak{N}$ in $\mathcal{O}_F$, with $v \mid \mathfrak{N}^+$ if and only if $v$ splits in $K$, and $v \mid \mathfrak{N}^-$ if and only if $v$ is inert in $K$. Let us assume additionally that $\mathfrak{N}^-$ is the squarefree product
of a number of primes congruent to \( d \) mod 2. Fix a Hilbert modular eigenform \( f \in \mathcal{S}_2(\mathfrak{N}) \) of parallel weight two, level \( \mathfrak{N} \), and trivial Nebentypus. Assume that \( f \) is either a newform, or else arises from a newform of level \( \mathfrak{N}/p \) via the process of \( p \)-stabilization. Our hypotheses on \( \mathfrak{N} \) and \( K \) imply that the global root number of the associated Rankin-Selberg \( L \)-function \( L(f, K, s) \) at its central value \( s = 1 \) is equal to 1 (as opposed to \(-1\)). Moreover, the Jacquet-Langlands correspondence allows us to associated to \( f \) an eigenform on a totally definite quaternion algebra over \( F \), which puts us in the setting of Waldspurger’s theorem \([27]\), as refined by Yuan-Zhang-Zhang in \([28]\). Let us view \( f \) as a \( p \)-adic modular form via a fixed embedding \( \mathbb{Q} \to \mathbb{Q}_p \), writing \( \mathcal{O} \) to denote the ring of integers of a finite extension of \( \mathbb{Q}_p \) containing all of the Fourier coefficients of \( f \) under this fixed embedding. Let us assume additionally that \( f \) is either \( p \)-ordinary, by which we mean that its eigenvalue at the Hecke operator \( T_p \) is invertible in \( \mathcal{O} \), or else that \( f \) is \( p \)-supersingular, by which we mean that its eigenvalue at the Hecke operator \( T_p \) is zero. In the case where \( f \) is \( p \)-ordinary, let us write \( \alpha_p = \alpha_p(f) \) to denote the unit root of the Hecke polynomial \( X^2 - a_p(f)X + q \). Here, \( a_p(f) \) denotes the eigenvalue of \( f \) at \( T_p \), and \( q \) denotes the cardinality of the residue field of \( p \). Let \( \delta = [F_p : \mathbb{Q}_p] \). We consider the behaviour of \( f \) in the dihedral \( \mathbb{Z}_\delta \)-extension \( K_{p\infty} \) of \( K \) described by class field theory. Writing the Galois group \( \text{Gal}(K_{p\infty}/K) \cong \mathbb{Z}_\delta \) as \( G_{p\infty} = \lim_{\leftarrow n} G_{p^n} \), we consider the \( \mathcal{O} \)-Iwasawa algebra

\[
\Lambda = \mathcal{O}[[G_{p\infty}]] = \lim_{n} \mathcal{O}[G_{p^n}],
\]

whose elements can be viewed as \( \mathcal{O} \)-valued measures on \( G_{p\infty} \). The purpose of this note is to give a construction of elements \( \mathcal{L}_p(f, K) \in \Lambda \) whose specializations to finite order characters \( \rho \) of \( G_{p\infty} \) interpolate the central values \( L(f, \rho, 1) \) of the twisted Rankin-Selbert \( L \)-functions \( L(f, \rho, s) \). To be more precise, if \( \rho \) is a finite order character of \( G_{p\infty} \), and \( \lambda \) an element of \( \Lambda \) with associated measure \( d\lambda \), let

\[
\rho(\lambda) = \int_{G_{p\infty}} \rho(\sigma) \cdot d\lambda(\sigma)
\]

denote the specialization of \( \lambda \) to \( \rho \). Let \( \mathfrak{P} \) denote the maximal ideal of \( \mathcal{O} \). We define the \( \mu \)-invariant \( \mu = \mu(\lambda) \) associated to an element \( \lambda \in \Lambda \) to be the largest exponent \( c \) such that \( \lambda \in \mathfrak{P}^c \Lambda \). Let \( \pi = \pi_f \) denote the cuspidal automorphic representation of \( \text{GL}_2 \) over \( F \) associated to \( f \), with

\[
L(\pi, ad, s) = \prod_v L(\pi_v, ad, s)
\]

the \( L \)-series of the adjoint representation of \( \pi \). Let

\[
\zeta_F(s) = \prod_v \zeta_v(s)
\]

denote the Dedekind zeta function of \( F \). Let

\[
L(\pi, \rho, s) = \prod_v L(\pi_v, \rho_v, s)
\]
denote the Rankin-Selberg $L$-function of $\pi$ times the theta series associated to $\rho$, with central value at $s = 1/2$. Note that we have an equality of $L$-functions

$$L(\pi, \rho, s - \frac{1}{2}) = \Gamma_C(s)|K/F|L(f, \rho, s),$$

with

$$\Gamma_C(s) = 2(2\pi)^{-s}\Gamma(s),$$

as explained for instance in [10, 0.5]. Let $\pi = JL(\pi')$ denote the Jacquet-Langlands correspondence of $\pi$. Choose a decomposable vector $\Phi = \otimes_v \Phi_v \in \pi'$. Let $\omega = \omega_K/F$ denote the quadratic Hecke character associated to $K/F$, with decomposition $\omega = \otimes_v \omega_v$. Following Yuan-Zhang-Zhang [28], we consider for any prime $v$ of $F$ the local linear functional defined by

$$\alpha(\Phi_v, \rho_v) = \frac{L(\omega_v, 1) \cdot L(\pi_v, \rho_v, 1/2)}{\zeta_v(2) \cdot L(\pi_v, \rho_v, 1/2)} \cdot \int_{K_v^x / F_v^x} \langle \pi_v'(t)\Phi_v, \Phi_v \rangle_v \cdot \rho_v(t) dt.$$ 

Here, $\langle \cdot, \cdot \rangle_v$ denotes a nontrivial hermitian form on the component $\pi_v'$ such that the product $\langle \cdot, \cdot \rangle = \prod_v \langle \cdot, \cdot \rangle_v$ is the Petersson inner product, and $dt$ denotes the Tamagawa measure. We refer the reader to the discussion below for more precise definitions. We show the following result.

**Theorem 1.1** (Theorem [17.7] Corollary [18, 4.1] and Theorem [11.10]). Fix an eigenform $f \in S_2(\mathbb{H})$ subject to the hypotheses above, with $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$ a fixed embedding. Let $\rho$ be a finite order character of the Galois group $G_{p^n}$ that factors though $G_{p^n}$ for some positive integer $n$. Let $| \cdot |$ denote the complex absolute value on $\overline{\mathbb{Q}}_p$, taken with respect to a fixed embedding $\overline{\mathbb{Q}}_p \to \mathbb{C}$.

(i) If $f$ is $p$-ordinary, then there exists a nontrivial element $L_p(f, K_{p^n}) \in \Lambda$ such that the following equality holds in $\overline{\mathbb{Q}}_p$:

$$|\rho(L_p(f, K_{p^n}))| = \frac{\alpha_p^{-2n} \cdot \zeta_F(2)}{2 \cdot L(\pi, \rho, 1/2)} \cdot L(\pi, \rho, 1/2) \cdot L(\pi, \rho^{-1}, 1/2) \cdot \prod_{v | \infty} \alpha(\Phi_v, \rho_v) \cdot \alpha(\Phi_v, \rho_v^{-1}) \bigg|^{\frac{1}{2}}.$$ 

(ii) If $f$ is $p$-supersingular, then there exist two nontrivial elements $L_p(f, K_{p^n})^{\pm} \in \Lambda$ such that the following equalities hold in $\overline{\mathbb{Q}}_p$:

$$|\rho(L_p(f, K_{p^n})^{\pm})| = \frac{\zeta_F(2)}{2 \cdot L(\pi, \rho, 1/2)} \cdot L(\pi, \rho, 1/2) \cdot L(\pi, \rho^{-1}, 1/2) \cdot \prod_{v | \infty} \alpha(\Phi_v, \rho_v) \cdot \alpha(\Phi_v, \rho_v^{-1}) \bigg|^{\frac{1}{2}}.$$ 

(iii) We have that $\mu(L_p(f, K_{p^n})) = \mu(L_p(f, K_{p^n})^{\pm}) = 2\nu$, where $\nu = \nu_\Phi$ denotes the largest integer such that $\Phi$ is congruent to a constant $\Phi^\nu$ mod $\mathfrak{B}^\nu$.

At this point, some remarks are in order. First of all, we note that this construction is the generalization to totally real fields of that given by Bertolini-Darmon [2, § 1.2] and Darmon-Iovita [5], building on the seminal work of Bertolini-Darmon.
It is sketched by Longo in [13], using the language of Gross points. The novelty of the construction given here is that we work adelically rather than $p$-adically, which allows us to use Waldspurger’s theorem directly to deduce the interpolation property. This also allows us to give a simpler proof of the $\mu$-invariant formula than that given by Vatsal in [24]. Finally, this construction allows us to reduce the associated Iwasawa main conjecture to a nonvanishing criterion for these $p$-adic $L$-functions via the theorem of Howard [11, Theorem 3.2.3(c)] (cf. also [23, Theorem 1.3]), as we explain below. In particular, Howard’s criterion (Conjecture 5.1) has the following applications to Iwasawa main conjectures. Let $\text{Sel}_{p^{\infty}}(f, K_{p^{\infty}})$ denote the $p^{\infty}$-Selmer group associated to $f$ in the tower $K_{p^{\infty}}/K$. We refer the reader to [10], [13] or [23] for definitions. Let

$$X(f, K_{p^{\infty}}) = \text{Hom}(\text{Sel}_{p^{\infty}}(f, K_{p^{\infty}}), \mathbb{Q}_p/\mathbb{Z}_p)$$

denote the Pontryagin dual of $\text{Sel}_{p^{\infty}}(f, K_{p^{\infty}})$, which has the structure of a compact $\Lambda$-module. If $X(f, K_{p^{\infty}})$ is $\Lambda$-torsion, then the structure theory of $\Lambda$-modules in [3, § 4.5] assigns to $X(f, K_{p^{\infty}})$ a $\Lambda$-characteristic power series

$$\text{char}_\Lambda(X(f, K_{p^{\infty}})) \in \Lambda.$$  

The associated dihedral main conjecture of Iwasawa theory is then given by

**Conjecture 1.2** (Iwasawa main conjecture). Let $f \in S_2(\mathfrak{N})$ be a Hilbert modular eigenform as defined above, such that the global root number of the complex central value $L(f, K, 1)$ is $+1$. Then, the dual Selmer group $X(f, K_{p^{\infty}})$ is $\Lambda$-torsion, and there is an equality of ideals

$$(L_p(f, K_{p^{\infty}})) = (\text{char}_\Lambda(X(f, K_{p^{\infty}}))) \text{ in } \Lambda. \quad (3)$$

We may now deduce the following result towards this conjecture.

**Theorem 1.3.** Suppose that $F = \mathbb{Q}$. Let $f \in S_2(\mathfrak{N})$ be a Hilbert modular eigenform as defined above. Assume that the residual Galois representation associated to $f$ is surjective, as well as ramified at each prime $q | \mathfrak{N}$ such that $q \equiv \pm 1 \mod p$. Then, the dual Selmer group $X(f, K_{p^{\infty}})$ is $\Lambda$-torsion, and there is an inclusion of ideals

$$(L_p(f, K_{p^{\infty}})) \subseteq (\text{char}_\Lambda(X(f, K_{p^{\infty}}))) \text{ in } \Lambda.$$  

Moreover, if Conjecture 5.1 below holds, then there is an equality of ideals

$$(L_p(f, K_{p^{\infty}})) = (\text{char}_\Lambda(X(f, K_{p^{\infty}}))) \text{ in } \Lambda.$$  

**Proof.** The result follows from the refinement of the Euler system argument of Bertolini-Darmon [2] given by Pollack-Weston [18], which satisfies the hypotheses of Howard [10, Theorem 3.2.3], since it removes the $p$-isolatedness condition from the work of [2]. □

We can state the criterion of Conjecture 5.1 in the following more explicit way. Let us now assume for simplicity that $\mathfrak{N}$ is prime to the relative discriminant of $K$ over $F$. Fix a positive integer $k$. Let us define a set of admissible primes $\mathfrak{S}_k$ of $F$, all of which are inert in $K$, with the condition that for any ideal $\mathfrak{n}$ in the set $\mathfrak{S}_k$ of squarefree products of primes in $\mathfrak{S}_k$, there exists a nontrivial eigenform $f^{(\mathfrak{n})}$ of level $\mathfrak{n}\mathfrak{N}$ such that we have the following congruence on Hecke eigenvalues:

$$f^{(\mathfrak{n})} \equiv f \mod \mathfrak{p}^k.$$
Let $\mathfrak{L}_k^+ \subset \mathfrak{L}_k$ denote the subset of primes $v$ for which $\omega(v\mathfrak{N}) = -1$, equivalently for which the root number of the Rankin-Selberg $L$-function $L(f^{(n)}, K, s)$ equals $1$. Let $\mathfrak{S}_k^+$ denote the set of squarefree products of primes in $\mathfrak{L}_k^+$, including the so-called empty product corresponding to $1$. Now, for each vertex $n \in \mathfrak{S}_k^+$, we have a $p$-adic $L$-function $L_p(f^{(n)}, K_{p^\infty})$ or $L_p(f^{(n)}, K_{p^\infty})^\pm$ in $\Lambda$ by the construction given above. As we explain below, each of these $p$-adic $L$-functions is given by a product of completed group ring elements $\theta_{\sigma}, \theta_{\sigma^*}$, where $\theta_{\sigma^*} \in \Lambda$ is constructed in a natural way from the eigenform $f^n$ via the Jacquet-Langlands correspondence and strong approximation at $p$, and $\theta_{\sigma^*}$ is the image of $\theta_{\sigma^*}$ under the involution of $\Lambda$ sending $\sigma$ to $\sigma^{-1}$ in $G_{p^\infty}$. Let us write $\lambda_n$ to denote the completed group ring element $\theta_{\sigma^*}$, which is only well defined up to automorphism of $G_{p^\infty}$. We then obtain from Theorem 1.3 the following result, following the discussion in [11, Theorem 3.2.3 (c)] (cf. also [23, Theorem 1.3]).

**Corollary 1.4.** Keep the setup of Theorem 1.3. Suppose that for any height one prime $\Omega$ of $\Lambda$, there exists an integer $k_0$ such that for all integers $j \geq k_0$ the set

$$\{ \lambda_n \in \Lambda/(\Omega^j) : n \in \mathfrak{S}_j^+ \}$$

contains at least one completed group ring element $\lambda_n$ with nontrivial image in $\Lambda/(\Omega, \mathfrak{P}^{k_0})$. Then, there is an equality of ideals

$$(L_p(f, K_{p^\infty})) = (\text{char}_\Lambda (X(f, K_{p^\infty}))) \text{ in } \Lambda.$$  

The results of Theorem 1.3 and Corollary 1.4 extend to the general setting of totally real fields, as explained in Theorem 1.3 of the sequel paper [23]. We omit the statements here for simplicity of exposition. Finally, let us note that while we have not treated the construction for higher weights, the issue of Jochnowitz congruences (following Vatsal [24] with Rajaei [19]), or Howard’s criterion itself in this note, these problems in fact motivate this work.

**Notations.** Let $A_F$ denote the adeles of $F$, with $A = A_Q$. Let $A_f$ denote the finite adeles of $Q$. We shall sometimes write $T = \text{Res}_{F/Q}(K^\times)$ to denote the algebraic group associated to $K^\times$, and $Z = \text{Res}_{F/Q}(F^\times)$ the algebraic group associated o $F^\times$. Given a finite prime $v$ of $F$, fix a uniformizer $\varpi_v$ of $F_v$, and let $\kappa_v$ denote the residue field of $F_v$ at $v$, with $q = q_v$ its cardinality.

2. Some preliminaries

**Ring class towers.** Given an ideal $\mathfrak{c} \subset \mathcal{O}_F$, let $\mathcal{O}_\mathfrak{c} = \mathcal{O}_F + \mathfrak{c}\mathcal{O}_K$ denote the $\mathcal{O}_F$-order of conductor $\mathfrak{c}$ in $K$. The ring class field of conductor $\mathfrak{c}$ of $K$ is the Galois extension $K[\mathfrak{c}]$ of $K$ characterized by class field theory via the identification

$$\mathcal{K}\times/\mathcal{O}_\mathfrak{c}\times \xrightarrow{\text{rec}_K} \text{Gal}(K[\mathfrak{c}]/K).$$

Here, rec$_K$ denotes reciprocity map, normalized to send uniformizers to their corresponding geometric Frobenius endomorphisms. Let $G[\mathfrak{c}]$ denote the Galois group $\text{Gal}(K[\mathfrak{c}]/K)$. Let $K[p^\infty] = \bigcup_{n \geq 0} K[p^n]$ denote the union of all ring class extensions of $p$-power conductor over $K$. Let us write $G[p^\infty]$ to denote the Galois group $\text{Gal}(K[p^\infty]/K)$, which has the structure of a profinite group $G[p^\infty] = \lim_{\leftarrow n} G[p^n]$.

**Lemma 2.1.** The Galois group $G[p^\infty]$ has the following properties.
(i) The reciprocity map $\text{rec}_K$ induces an isomorphism of topological groups

$$\hat{K}^\times/K^\times U \xrightarrow{\text{rec}_K} G[\mathbb{p}^\infty],$$

where

$$U = \bigcap_{n \geq 0} \hat{O}_p^\times = \{ x \in \hat{O}_K^\times : x_p \in \mathcal{O} F_p \}.$$ 

(ii) The torsion subgroup $G[\mathbb{p}^\infty]_{\text{tors}}$ of $G[\mathbb{p}^\infty]$ is finite. Moreover, there is an isomorphism of topological groups

$$G[\mathbb{p}^\infty]/G[\mathbb{p}^\infty]_{\text{tors}} \rightarrow \mathbb{Z}_p^\delta,$$

where $\delta = [F_p : \mathbb{Q}_p].$

Proof. See [4, § Lemma 2.1 and Corollary 2.2].

We shall use the following notations throughout. Let $G_{p^\infty} = G[\mathbb{p}^\infty]/G[\mathbb{p}^\infty]_{\text{tors}}$ denote the $\mathbb{Z}_p^\delta$-quotient of $G[\mathbb{p}^\infty].$ Let $K_{p^\infty}$ denote the dihedral or anticyclotomic $\mathbb{Z}_p^\delta$-extension of $K,$ so that

$$G_{p^\infty} = \text{Gal}(K_{p^\infty}/K) \cong \mathbb{Z}_p^\delta.$$ 

Given a positive integer $n,$ we then let $K_{p^n}$ denote the extension of $K$ for which

$$G_{p^n} = \text{Gal}(K_{p^n}/K) \cong (\mathbb{Z}/p^n\mathbb{Z})^\delta,$$

so that $G_{p^\infty} = \lim_{\leftarrow n} G_{p^n}.$

**Central values.** Here, we record the refinement of Waldspurger’s theorem [27] given by Yuan-Zhang-Zhang [28], as well as the nonvanishing theorem given by Cornut-Vatsal [4]. Let $B$ be a totally definite quaternion algebra defined over $F.$ We view the group of invertible elements $B^\times$ as an algebraic group with centre $\mathbb{Z}.$ We then view the group $K^\times$ as a maximal torus $T$ of $B^\times$ via a fixed embedding $K \rightarrow B.$ Fix an idele class character

$$\rho = \otimes_v \rho_v : A_F^\times \rightarrow \mathbb{C}^\times.$$

Let $\pi = \otimes_v \pi_v$ be a cuspidal automorphic representation of $\text{GL}_2(A_F),$ assumed to have trivial central character. Let $L(\pi, \rho, s) = \prod_v L(\pi_v, \rho_v, s)$ denote the Rankin-Selberg $L$-function associated to $\pi$ and $\rho,$ with central value at $s = 1/2.$ Let $\epsilon(\pi_v, \rho_v, 1/2) \in \{ \pm 1 \}$ denote the local root number of $L(\pi, \rho, 1/2)$ at a prime $v$ of $F.$ Let $\omega = \omega_{K/F}$ denote the quadratic character associated to $K/F.$ The set of places $v$ of $F$ given by

$$\Sigma = \{ v : \epsilon(\pi_v, \rho_v, 1/2) \neq \rho_v \cdot \omega_v(-1) \}$$

is known to have finite cardinality, and the global root number to be given by the product formula

$$\epsilon(\pi, \rho, 1/2) = \prod_v \epsilon(\pi_v, \rho_v, 1/2) = (-1)^{|\Sigma|}.$$

A theorem of Tunnell and Saito gives a criterion to determine whether or not a given prime of $F$ belongs to this set $\Sigma.$ To state this theorem, we must first say a word or two about the Jacquet-Langlands correspondence. That is, the theorem of Jacquet and Langlands [12] establishes an injection $\Pi \rightarrow JL(\Pi)$ from the set of automorphic representations of $(B \otimes A_F)^\times$ of dimension greater than 1 to the set of cuspidal automorphic representations of $\text{GL}_2(A_F).$ Moreover, it characterizes the
Let \( \langle \cdot \rangle \) denote the Petersson inner product on \( (\pi_v \otimes \rho_v, C) \) treated as zero if \( \pi_v \) is not discrete.

**Theorem 2.3** (Waldspurger, Yuan-Zhang-Zhang). Assume that \( \Phi \in \pi' \) is nonzero and decomposable. Then for any prime \( v \) of \( F \), the local functional defined by

\[
\alpha(\Phi, \rho_v) = \frac{L(\omega, 1) \cdot L(\pi_v, \text{ad}, 1)}{\zeta_v(2) \cdot L(\pi_v, \rho_v, 1/2)} \cdot \int_{K_v^\times / F_v^\times} \langle \pi_v'(t) \Phi_v, \Phi_v \rangle_v \cdot \rho_v(t) dt
\]

does not vanish, and equals 1 for all but finitely many primes \( v \) of \( F \). Moreover, we have the identity

\[
|l(\Phi, \rho)|^2 = \frac{\zeta_F(2) \cdot L(\pi, \rho, 1/2)}{2 \cdot L(\pi, \text{ad}, 1)} \cdot \prod_v \alpha(\Phi, \rho_v),
\]

where the value \( \zeta_F(s) \) is algebraic. Here, \( \zeta_F(s) = \prod_v \zeta_{F,v}(s) \) is the Dedekind zeta function of \( F \), and \( L(\pi, \text{ad}, s) = \prod_v L(\pi_v, \text{ad}, s) \) the \( L \)-series of the adjoint representation.

**Proof.** See \[27\] or \[28\] Proposition 1.2.1. In this setting, the value \( \zeta_F(s) \) is known to lie in the field \( \mathbb{Q}(\pi, \rho) \) generated by values of \( \pi \) and \( \rho \) as a consequence of the fact that \( B \) is totally definite (and hence compact).

Let us now record the following nonvanishing theorem for the central values \( L(\pi, \rho, 1/2) \), whose proof relies on the results of Tunnell-Saito and Waldspurger, as explained in the introduction of \[4\]. Given an integer \( n \geq 1 \), let us call a character that factors through \( G[p^n] \) primitive of conductor \( n \) if it does not factor
though \(G[p^{n-1}]\). Let \(P(n, \rho_0)\) denote the set of primitive characters on \(G[p^n]\) with associated (tamely ramified) character \(\rho_0\) on \(G[p^\infty]_{\text{tors}}\).

**Theorem 2.4** (Cornut-Vatsal). Assume that the root number \(\epsilon(\pi, \rho, 1/2)\) is 1, and that the prime to \(p\)-part of the level of \(\pi\) is prime to the relative discriminant \(\mathcal{D}_{K/F}\). Then, for all \(n\) sufficiently large, there exists a primitive character \(\rho \in P(n, \rho_0)\) such that \(L(\pi, \rho, 1/2) \neq 0\).

**Proof.** See [4] Theorem 1.4, as well as the main result of [25] for \(F = \mathbb{Q}\). \(\square\)

We may then deduce from the algebraicity theorem of Shimura [21], which applies to the values \(L(\pi, \rho, 1/2)\), the following strengthening of this result.

**Corollary 2.5.** Assume that the root number \(\epsilon(\pi, \rho, 1/2)\) is 1, and that the prime to \(p\)-part of the level of \(\pi\) is prime to the relative discriminant \(\mathcal{D}_{K/F}\). Let \(Y\) denote the set of all finite order characters of the Galois group \(G[p^\infty]\), viewed as idele class characters of \(K\) via the reciprocity map \(\text{rec}_K\). Then, for all but finitely many characters \(\rho\) in \(Y\), the central value \(L(\pi, \rho, 1/2)\) does not vanish.

3. Modular forms on totally definite quaternion algebras

We follow the exposition given in Mok [14] §2. Fix an ideal \(\mathfrak{N} \subset \mathcal{O}_F\) as defined in (1) above. Fix a totally imaginary quadratic extension \(K/F\) of relative discriminant \(\mathcal{D}_{K/F}\) prime to \(\mathfrak{N}\), so that we have the factorization (2) of \(\mathfrak{N}\) in \(\mathcal{O}_F\). Let \(B\) denote the totally definite quaternion algebra over \(F\) of discriminant \(\mathfrak{N}^-\). Let \(R\) also fix isomorphisms \(\iota_v : B_v \cong M_2(F_v)\) for all primes \(v \notin \text{Ram}(B)\) of \(F\). Note that \(B\) splits at \(p\) by our hypotheses on the level \(\mathfrak{N}\).

**Basic definition.** Let \(\mathcal{O}\) be any ring. An \(\mathcal{O}\)-valued automorphic form of weight 2, level \(H\), and trivial central character on \(B^\times\) is a function

\[
\Phi : B^\times \backslash \widehat{B}^\times / H \rightarrow \mathcal{O}
\]

such that for all \(g \in B^\times, b \in \widehat{B}^\times, h \in H, \text{ and } z \in \widehat{F}^\times,\)

\[
\Phi(gbhz) = \Phi(b).
\]

Let \(S^B_2(H; \mathcal{O})\) denote the space of such functions, modulo those which factor through the reduced norm homomorphism \(\text{nr}d\). A function in \(S^B_2(H; \mathcal{O})\) is called an \(\mathcal{O}\)-valued modular form of weight 2, level \(H\), and trivial central character on \(B^\times\).

**Choice of levels.** Fix \(\mathfrak{M} \subset \mathcal{O}_F\) an integral ideal prime to the discriminant \(\mathfrak{N}^-\) (we shall often just take \(\mathfrak{M} = p\mathfrak{N}^+\)). Given a finite prime \(v \subset F\), let \(R_v\) be a local order of \(B_v\) such that

\[
R_v \text{ is maximal if } v \nmid \mathfrak{N}^-, \text{ or Eichler of level } \text{ord}_v(\mathfrak{M}) \text{ if } v \nmid \mathfrak{N}^-.
\]

Write \(\widehat{R} = \prod_v R_v\). Let \(R = B \cap \widehat{R}\) denote the corresponding Eichler order of \(B\). Let \(H_v = R_v^\times\), so that \(H = \prod_v H_v \subset \widehat{B}^\times\) is the corresponding compact open subgroup.

We shall assume throughout that a compact open subgroup \(H \subset \widehat{B}^\times\) takes this form, in which case we refer to it as \(\mathfrak{M}\)-level structure on \(B\).

**Hecke operators.** We define Hecke operators acting on the space \(S^B_2(H; \mathcal{O})\).
The operators $T_v$. Given any finite prime $v$ of $F$ that splits $B$ and does not divide the level of $H$, we define Hecke operators $T_v$ as follows. Note that since $R_v \subset B_v$ is maximal by [3], we have the identification $\iota_v : R_v^x \cong \text{GL}_2(O_{F_v})$. Suppose now that we have any double coset decomposition

$$ GL_2(O_{F_v}) \left( \begin{array}{cc} \varpi_v & 0 \\ 0 & 1 \end{array} \right) GL_2(O_{F_v}) = \prod_{a \in \mathbb{P}^1(\kappa_v)} \sigma_a GL_2(O_{F_v}). $$

The Hecke operator $T_v$ is then defined by the rule

$$ (T_v \Phi)(b) = \sum_{a \in \mathbb{P}^1(\kappa_v)} \Phi \left( b \cdot \iota_v^{-1}(\sigma_a) \right). $$

An easy check with the transformation property (7) shows that these definitions do not depend on choice of representatives $\{\sigma_a(v)\}_{a \in \mathbb{P}^1(\kappa_v)}$.

The set of representatives $\{\sigma_a\}_{a \in \kappa_p}$ in (9) has the following lattice description. Let $q = q_v$ denote the cardinality of $\kappa_v$. Let $\{L(a)\}_{a \in \kappa_v}$ denote the set of $q + 1$ sublattices of $O_{F_v} \oplus O_{F_v}$ of index equal to $q$. Arrange the matrix representatives $\sigma_a$ so that $\sigma_a(O_{F_v} \oplus O_{F_v}) = L(a)$. The set of lattices $\{L(a)\}_{a \in \mathbb{P}^1(\kappa_v)}$ then describes the set of representatives.

The operator $U_p$. In general, given any finite prime $v$ of $F$ and any integer $m \geq 1$, we let $I_{vm}$ denote the Iwahori subgroup of level $v^m$ of $GL_2(O_{F_v})$,

$$ I_{vm} = \{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in GL_2(O_{F_v}) : c \equiv 0 \mod \varpi_v^m \}. $$

Let us now suppose that $v$ is a finite prime of $F$ not dividing $\mathfrak{N}^{-}$ where the level is not maximal, though we shall only be interested in the special case of $v = p$. Hence, taking $v = p$, let us assume that the level $H = \prod_{a} R_a^x$ is chosen so that $\iota_p : R_p^x \cong I_p$. Suppose we have any double coset decomposition

$$ I_p \left( \begin{array}{cc} 1 & 0 \\ 0 & \varpi_p \end{array} \right) I_p = \prod_{a \in \kappa_p} \sigma_a I_p. $$

The operator $U_p$ is then defined by the rule

$$ (U_p \Phi)(b) = \sum_{a \in \kappa_p} \Phi \left( b \cdot \iota_p^{-1}(\sigma_a) \right). $$

Another easy check with the transformation property (7) shows that this definition does not depend on choice of representatives $\{\sigma_a\}_{a \in \kappa_p}$.

The set of representatives $\{\sigma_a\}_{a \in \kappa_p}$ in (10) has the following lattice description. Recall that we fix a uniformizer $\varpi_p$ of $p$, and let $q = q_p$ denote the cardinality of the residue field $\kappa_p$. Let $\{L(a)\}_{a \in \kappa_p}$ denote the set of sublattices of index $q$ of $O_{F_p} \oplus \varpi_p O_{F_p}$, minus the sublattice $\varpi_p (O_{F_p} \oplus O_{F_p})$. (Note that there are exactly $q$ lattices in this set, as there are exactly $q + 1$ sublattices of “distance 1” away from $O_{F_p} \oplus \varpi_p O_{F_p}$ – see [2] § II.2 or [10] below). We consider translates of the pair

$$ (O_{F_p} \oplus O_{F_p}, O_{F_p} \oplus \varpi_p O_{F_p}) $$

which is stabilized by the Iwahori subgroup $I_p$. Let us now arrange the representatives $\{a_n\}_{n \in \mathcal{K}_p}$ so that

$$\sigma_a (\mathcal{O}_{F_p} + \mathcal{O}_{F_p}) = \mathcal{O}_{F_p} + \varpi_p \mathcal{O}_{F_p} = L(a)$$

$$\sigma_a (\mathcal{O}_{F_p} + \varpi_p \mathcal{O}_{F_p}) = L'(a).$$

We shall then consider the pairs of lattice translates given by $(L(a), L'(a))$.

**Strong approximation.** Let $F_p$ denote the totally positive elements of $F$, i.e. the elements of $F$ whose image under any real embedding of $F$ is greater than 0. Let $\{x_i\}_{i=1}^h$ be a set of representatives for the narrow class group

$$\mathcal{Cl}_F = F_p^\times \backslash \hat{F}_p^\times / \hat{O}_F^\times$$

of $F$, such that $(x_i)_p = 1$ for each $i = 1, \ldots, h$.

**Lemma 3.1.** Given $H \subset \hat{B}^\times$ any compact open subgroup, there is a bijection

$$\prod_{i=1}^h B^\times x_i B_p^\times H \cong \hat{B}^\times.$$

Here, each $x_i$ is an element of $\hat{B}^\times$ such that $(x_i)_p = 1$ and nrd$(x_i) = x_i$.

**Proof.** This is a standard result, it can be deduced from the strong approximation theorem ([26, § III.4, Théorème 4.3]) applied to the norm theorem ([26, § III.4, Théorème 4.1]).

Let us now fix a set of representatives $(x_i)$ for the modified class group $\mathcal{Cl}_F / F_p^\times$ such that $(x_i)_p = 1$ for each $i = 1, \ldots, h$. Let us then choose $x_i \in \hat{B}^\times$ such that $(x_i)_p = 1$ and nrd$(x_i) = x_i$ for each $i$. Now, for each $i$, let us define a subgroup

$$\Gamma_i = \{ b \in B^\times : b_v \in (x_i,v) H_v (x_i,v)^{-1} \text{ for all } v \not\mid p \} \subset B^\times.$$

Observe that each $\Gamma_i$ embeds discretely into $B_p^\times$. (That is, each $\Gamma_i$ can be described as the intersection of $B^\times$ with some conjugate of $H$. This intersection can then be embedded discretely into $B_p^\times = \prod_{\nu \in S} B_v^\times$, where $S$ denotes the set of archimedean places $\Sigma_F$ of $F$ along with the prime $p$. On the other hand, recall that $B$ is ramified at all of the archimedean places of $F$, and hence that $B_p^\times$ is compact. It is then follows from a standard fact in the theory of topological groups that the intersection defining each $\Gamma_i$ embeds discretely into $B_p^\times$ modulo the compact subgroup $B_{\Sigma_F}^\times$. Hence, each $\Gamma_i$ embeds discretely into $B_p^\times$. We may therefore view each $\Gamma_i$ as a discrete subgroup of $\GL_2(F_p)$ via our fixed isomorphism $\iota_p : B_p^\times \cong \GL_2(F_p)$.

**Corollary 3.2.** We have a bijection

$$\prod_{i=1}^h \Gamma_i \backslash B_p^\times / H_p \cong B^\times \backslash \hat{B}^\times / H$$

via the map given on each component by $[g] \mapsto [\xi_i \cdot g]$. That is, for $g \in B_p^\times$, the class of $g$ in each component $\Gamma_i \backslash B_p^\times / H_p$ goes to the class of $\xi_i \cdot g$ in $B^\times \backslash \hat{B}^\times / H$.

Fix $\Phi \in S^B_p(H; O)$. Via a fixed bijection (13), we may view $\Phi$ as an $h$-tuple of functions $\{\phi^i\}_{i=1}^h$ on $\GL_2(F_p)$ that satisfy the relation

$$\phi^i(\gamma b h z) = \phi^i(b)$$
Let us write \( \phi \) for all \( \gamma \in \Gamma_i \), \( b \in B \), \( h \in H_p \), and \( z \in \hat{F}_p^\infty \). The identification (13) then allows us to describe \( \Phi \in S^B_2(H; F) \) as a vector of functions on homothety classes of full rank lattices of \( F_p \oplus F_p \) in the following way. Let \( L(F_p \oplus F_p) \) denote the set of homothety classes of full rank lattices of \( F_p \oplus F_p \).

**Case I: \( p \nmid \mathfrak{M} \).** Fix \( \Phi \in S^B_2(H; F) \) as above, associated to a vector of functions \( (\phi^i)^{h}_{i=1} \). For each \( \phi^i \), we can define a corresponding function \( c_{\phi^i} \) on \( L(F_p \oplus F_p) \) as follows. Given a class \([L] \in L(F_p \oplus F_p)\) with fixed representative \( L \), let \( g_L \in GL_2(F_p) \) be any matrix such that \( g_L(O_{F_p} \otimes O_{F_p}) = L \). Let

\[
c_{\phi^i}([L]) = \phi^i(g_L).
\]

This definition does not depend on choice of matrix representative \( g_L \), as a simple check using relation (14) reveals. It also follows from (14) that

\[
c_{\phi^i}([\gamma L]) = c_{\phi^i}([L])
\]

for all \( \gamma \in \Gamma_i \), for instance by taking \( g_{\gamma L} = \gamma g_L \), then seeing that \( \phi^i(g_{\gamma L}) = \phi^i(\gamma g_L) = \phi^i(g_L) \).

**Case II: \( p \mid \mathfrak{M} \).** Consider \( \Phi \in S^B_2(H; F) \) as above, associated to a vector of functions \( (\phi^i)^{h}_{i=1} \). For each \( \phi^i \), we can define a corresponding function \( c_{\phi^i} \) on pairs \(([L_1], [L_2])\) of classes in \( L(F_p \oplus F_p) \) (with a fixed pair of representatives \((L_1, L_2)\)) satisfying the property that \( L_2 \subset L_1 \) with index \( q \). That is, fix such a pair of classes \((L_1, L_2)\). Let \( g_L \in GL_2(F_p) \) be any matrix such that

\[
\begin{align*}
g_L(O_{F_p} \otimes O_{F_p}) &= L_1, \\
g_L(O_{F_p} \oplus \varpi_p O_{F_p}) &= L_2.
\end{align*}
\]

Let

\[
c_{\phi^i}([L_1], [L_2]) = \phi^i(g_L).
\]

As before, a simple check using relation (14) reveals that \( c_{\phi^i}([L_1], [L_2]) \) does not depend on choice of matrix representative \( g_L \). It also follows from (14) that

\[
c_{\phi^i}([\gamma L_1], [\gamma L_2]) = c_{\phi^i}([L_1], [L_2])
\]

for all \( \gamma \in \Gamma_i \).

Let us write \( c_{\Phi} \) to denote the vector of lattice class functions \( (c_{\phi^i})^{h}_{i=1} \) associated to \( (\phi^i)^{h}_{i=1} \) in either case on the level \( \mathfrak{M} \). We then have the following description of Hecke operators for these functions:

**Case I: \( p \nmid \mathfrak{M} \).** The Hecke operator \( T_p \) on \( c_{\Phi} \) is given by

\[
c_{T_p \Phi}([L]) = \sum_{L' \subset L} c_{\Phi}([L'])
\]

Here, fixing representatives, the sum runs over the \( q + 1 \) sublattices \( L' \) of \( L \) having index equal to \( q \).

**Case II: \( p \mid \mathfrak{M} \).** The Hecke operator \( U_p \) on \( c_{\Phi} \) is given by

\[
c_{U_p \Phi}([L_1], [L_2]) = \sum_{L' \subset L_2} c_{\Phi}([L_2], [L'])
\]

Here, fixing representatives, the sum runs over the sublattices \( L' \subset L_2 \) of index \( q \), minus the sublattice \( L' \) corresponding to \( \varpi_p L_2 \).
The Bruhat-Tits tree of \( \text{PGL}_2(F_p) \). The description above of modular forms \( \Phi \in S^g_2(H;O) \) as functions \( c_\Phi \) on the set of homothety classes \( \mathcal{L}(F_p \oplus F_p) \) has the following combinatorial interpretation. To fix ideas, let \( \mathcal{M}(M_2(F_p)) \) denote the set of maximal orders of \( M_2(F_p) \). The group \( \text{PGL}_2(F_p) \cong B^\times_p/F^\times_p \) acts on by conjugation on \( \mathcal{M}(M_2(F_p)) \).

**Lemma 3.3.** The conjugation action of \( \text{PGL}_2(F_p) \) on \( \mathcal{M}(M_2(F_p)) \) is transitive. Moreover, there is an identification of \( \text{PGL}_2(F_p)/\text{PGL}_2(O_{F_p}) \) with \( \mathcal{M}(M_2(F_p)) \).

**Proof.** This is a standard result, see for instance [26 § II.2]. □

Let \( T_p = (V_p, E_p) \) denote the Bruhat-Tits tree of \( B^\times_p/O^\times_p \cong \text{PGL}_2(F_p) \), by which we mean the tree of maximal orders of \( B_p \cong M_2(F_p) \), such that

(i) The vertex set \( V_p \) is indexed by maximal orders of \( M_2(F_p) \).

(ii) The edgset \( E_p \) is indexed by Eichler orders of level \( p \) of \( M_2(F_p) \).

(iii) The edgset \( E_p \) has an orientation, i.e. a pair of maps

\[
\begin{align*}
  s, t : E_p &\to V_p, \\
  e &\mapsto (s(e), t(e))
\end{align*}
\]

that assigns to each edge \( e \in E_p \) a source \( s(e) \) and a target \( t(e) \). Once such a choice of orientation is fixed, let us write \( E^*_p \) to denote the “directed” edgset of \( T_p \).

Hence, we obtain from Lemma 3.3 the following immediate

**Corollary 3.4.** The induced conjugation action of \( \text{PGL}_2(F_p) \) on \( T_p \) is transitive. Moreover, there is an identification of \( \text{PGL}_2(F_p)/\text{PGL}_2(O_{F_p}) \) with \( V_p \).

Now, recall that we let \( \mathcal{L}(F_p \oplus F_p) \) denote the set of homothety classes of full rank lattices of \( F_p \oplus F_p \).

**Lemma 3.5.** There is a bijection \( \mathcal{L}(F_p \oplus F_p) \cong \mathcal{M}(M_2(F_p)) \).

**Proof.** See [26 II §2]. Let \( V = F_p \oplus F_p \), viewed as a 2-dimensional \( F_p \)-vector space. Fixing a basis \( \{z_1, z_2\} \) of \( V \), we obtain the standard identification

\[
\begin{array}{l}
M_2(F_p) \to \text{End}_{F_p}(V) \\
\gamma \mapsto (v \mapsto \gamma \cdot v)
\end{array}
\]

Here, \( v = xz_1 + yz_2 \) denotes any element of \( V \), and \( \gamma \cdot v \) the usual matrix operation of \( \gamma \) on \((x, y)\). Hence, we obtain a bijective correspondence between maximal orders of \( M_2(F_p) \) and maximal orders of \( \text{End}_{F_p}(V) \). Now, it is well known that the maximal orders of \( \text{End}_{F_p}(V) \) take the form of \( \text{End}_{F_p}(L) \), with \( L \) any full rank lattice of \( F_p \oplus F_p \), as shown for instance in [26 II § 2, Lemme 2.1(1)]. Since the rings \( \text{End}_{F_p}(L) \) correspond in a natural way to classes \( [L] \in \mathcal{L}(F_p \oplus F_p) \), the claim follows. □

Hence, fixing an isomorphism \( \mathcal{L}(F_p \oplus F_p) \cong \mathcal{M}(M_2(F_p)) \), we can associate to each class \( [L] \in \mathcal{L}(F_p \oplus F_p) \) a unique vertex \( v_{[L]} \in V_p \). Let us also for future reference make the following definition (cf. [26 § II.2]). Let \( [L], [L'] \in \mathcal{L}(F_p \oplus F_p) \) be a pair of classes with fixed representatives \((L, L')\) such that \( L \supset L' \). Fix bases \((z_1, z_2)\) and \((z_1', z_2')\) for \( L \) and \( L' \) respectively. We define the distance \( d(v_{[L]}, v_{[L']}) \) between the associated vertices \( v_{[L]} \) and \( v_{[L']} \) by

\[
d(v_{[L]}, v_{[L']}) = |b - a|.
\]
This definition does not depend on choice of representatives or bases. Let \( v_0 \) denote the vertex of \( V_p \) corresponding to the maximal order \( M_2(O_{F_p}) \). The length of any vertex \( v \in V_p \) is then given by the distance \( d(v, v_0) \).

Now, the subgroups \( \Gamma_i \subset B^\times_p \) defined in (12) above act naturally by conjugation on \( T_p \), and the quotient graphs \( \Gamma_i \backslash T_p \) are finite. We may therefore consider the disjoint union of finite quotient graphs

\[
\coprod_{i=1}^h \Gamma_i \backslash T_p = \left( \coprod_{i=1}^h \Gamma_i \backslash V_p, \coprod_{i=1}^h \Gamma_i \backslash E_p^* \right).
\]

Moreover, we may consider the following spaces of modular forms defined on these disjoint unions of finite quotient graphs.

**Definition** Given a ring \( O \), let \( S_2 \left( \coprod_{i=1}^h \Gamma_i \backslash T_p; O \right) \) denote the space of vectors \( (\phi^i)^h_{i=1} \) of \( O \)-valued, \( (\Gamma_i)^h_{i=1} \)-invariant functions on \( T_p = (V_p, E_p^*) \). Here, it is understood that \( \Phi \in S_2 \left( \coprod_{i=1}^h \Gamma_i \backslash V_p; O \right) \) is a function on \( \coprod_{i=1}^h \Gamma_i \backslash V_p \) if \( p \nmid M \), or a function on \( \coprod_{i=1}^h \Gamma_i \backslash E_p^* \) if \( p \mid M \).

**Proposition 3.6.** Let \( O \) be any ring. We have a bijection of spaces

\[
S_2 \left( \coprod_{i=1}^h \Gamma_i \backslash T_p; O \right) \cong S_2^B(H; O).
\]

**Proof.** Fix a function \( \Phi \in S_2^B(H; O) \). Recall that by (13), we have a bijection

\[
\prod_{i=1}^h \Gamma_i \backslash B_p^\times / H_p \rightarrow B^\times / \hat{B}^\times / H,
\]

and moreover that we can associate to \( \Phi \) a vector of functions \( c_\Phi = (c_{\phi^i}) \) on the set of homothety classes \( L(F_p \oplus F_p) \). The vector \( c_\Phi \) is then clearly determined uniquely by the transformation law for \( \Phi \) in view of this bijection. Hence, by Lemma 3.5, we may view \( c_\Phi \) as a vector of functions on the set of maximal orders \( \mathcal{M}(M_2(F_p)) \). Since we saw above that each of the functions \( c_{\phi^i} \) is \( \Gamma_i \)-invariant, the claim follows.

Let us for ease of notation write \( \Phi \) to denote both a function in the space of modular forms \( S_2^B(H; O) \), as well as its corresponding vector of functions \( c_\Phi \) on maximal orders of \( M_2(F_p) \) in \( S_2 \left( \coprod_{i=1}^h \Gamma_i \backslash T_p; O \right) \). We can then write down the following combinatorial description of the Hecke operators \( T_p \) and \( U_p \) defined above, dividing into cases on the level structure.

**Case I:** \( p \nmid M \). We obtain the following description of the operator \( T_p \). Let \( \Phi(v) \) denote the \( h \)-tuple of functions \( (c_{\phi^i}(v))^h_{i=1} \) evaluated at a fixed vertex \( v \) of \( V_p \). By (17), we obtain the description

\[
(T_p \Phi)(v) = \sum_{w \rightarrow v} c_\Phi(w).
\]

Here, the sum ranges over all \( q + 1 \) vertices \( w \) adjacent to \( v \).
Case II: $p | \mathfrak{M}$. We obtain the following description of the operator $U_p$. Let $\Phi(\epsilon)$ denote the $h$-tuple of functions $\{c_{\phi}(\epsilon)\}_{i=1}^h$ evaluated at a fixed edge $\epsilon$. By (18), we obtain the description

$$(U_p \Phi)(\epsilon) = \sum_{s(\epsilon') = t(\epsilon)} c_{\phi}(\epsilon').$$

(22)

Here, the sum runs over the $q$ edges $\epsilon' \in \mathcal{E}_p^*\epsilon$ such that $s(\epsilon') = t(\epsilon)$, minus the edge $\epsilon$ obtained by reversing orientation of $\epsilon$.

The Jacquet-Langlands correspondence. We now give a combinatorial version of the theorem of Jacquet and Langlands [12] under the bijection (20). Let us first review some background from the theory of Hilbert modular forms. We refer the reader to [7], [8] or [9] for basic definitions and background.

Hilbert modular forms. Let $f \in \mathcal{S}_2(\mathfrak{M})$ be a cuspidal Hilbert modular form of parallel weight two, level $\mathfrak{M} \subset \mathcal{O}_F$, and trivial Nebentypus with associated vector of functions $\{f_i\}_{i=1}^h$ on the $d$-fold product $H_d$ of the complex upper-half plane. We write $\mathcal{S}_2(\mathfrak{M})$ to denote this space of functions, which comes equipped with an action by classically or adelically defined Hecke operators for each finite prime $v$ of $F$, which we denote by $T_v$ in an abuse of notation. Let us also write $U_v$ to denote these operators when $v$ divides $\mathfrak{M}$. Fix a set of representatives $\{(t_i)_{i=1}^h\}$ for the narrow class group $\text{Cl}_F$. By the weak approximation theorem, we may choose these representatives in such way that $(t_i)_\infty = 1$ for each $i$. Given a vector $z = (z_1, \ldots, z_d) \in F \otimes \mathbb{R}$, let us also define operations

$$\text{Tr}(z) = \sum_{i=1}^d z_i, \quad \mathcal{N}(z) = \prod_{i=1}^d z_i.$$ 

Let us then define $e_F(z) = \exp(2\pi i \text{Tr}(z))$. It can be deduced from the transformation law satisfied by $f$ that each $f_i$ admits a Fourier series expansion

$$f_i(z) = \sum_{\mu \in \mathfrak{t}_i} a_i(\mu) e_F(\mu z),$$

where the sum over $t_i$ means the sum over ideals generated by the idèles $t_i$, and $\mu \gg 0$ means that $\mu$ is strictly positive. Now, any fractional ideal $\mathfrak{m}$ of $F$ can be written uniquely as some $(\mu)t_i^{-1}$ with $\mu \in t_i$ totally positive. We use this to define a normalized Fourier coefficient $a_{\mathfrak{m}}(f)$ of $f$ at $\mathfrak{m}$ in the following way. That is, let

$$a_{\mathfrak{m}}(f) = \begin{cases} a_i(\mu) \cdot \mathcal{N}(t_i)^{-1} & \text{if } \mathfrak{m} \text{ is integral} \\ 0 & \text{else.} \end{cases}$$

Equivalently, we can define the normalized Fourier coefficient $a_{\mathfrak{m}}(f)$ using the adelic Fourier expansion of $f$. That is, let $| \cdot |_{\mathbb{A}_F}$ denote the adelic norm, and let $\chi_F : \mathbb{A}_F/F \rightarrow \mathbb{C}^\times$ denote the standard additive character whose restriction to archimedean components agrees with the restriction to archimedean components of the character $e_F$. We then have the following adelic Fourier series expansion at infinity:

$$f \left( \begin{bmatrix} y & x \\ 0 & 1 \end{bmatrix} \right) = |y|_{\mathbb{A}_F} \cdot \sum_{\xi \in \mathcal{O}_F} a_{\xi y \mathcal{O}_F}(f) \cdot e_F(\xi y_{\infty}) \cdot \chi_F(\xi x).$$
Here, \( i \) denotes the \( d \)-tuple \((i, \ldots, i)\), and \( y_\infty \) denotes the archimedean component of \( y \). Hence, we could also take this to be our definition of normalized Fourier coefficients. A Hilbert modular form \( f \in S_2(\frak{n}) \) is said to be a \textit{normalized eigenform} if it is a simultaneous eigenvector for all of the Hecke operators \( T_v \) and \( U_v \) with \( a_{O_p}(f) = 1 \). In this case, we write

- \( T_v f = a_v(f) \cdot f \) for all \( v \nmid \frak{n} \).
- \( U_v f = \alpha_v(f) \cdot f \) for all \( v \mid \frak{n} \).

A normalized eigenform \( f \in S_2(\frak{n}) \) is said to be a \textit{newform} if there does not exist any other form \( g \in S_2(\frak{n}) \) with \( \frak{n} \mid \frak{m} \) and \( \frak{m} \supseteq \frak{n} \) such that \( a_n(f) = a_n(g) \) for all integral ideals \( n \) of \( F \) prime to \( \frak{n} \). An eigenform \( f \in S_2(\frak{n}) \) is \( p \)-\textit{ordinary} if its \( T_p \)-eigenvalue \( a_p(f) \) is a \( p \)-adic unit with respect to any fixed embedding \( \overline{Q} \to \overline{Q}_p \).

In this case, there exists a \( p \)-adic unit root \( \alpha_p(f) \) to the Hecke polynomial

\[(23) \quad x^2 - a_p(f)x + q,\]

where \( q \) denotes the cardinality of the residue field at \( p \). Let us now fix an integral ideal \( \frak{n} \subset \mathcal{O}_F \) as in \([1]\), with underlying integral ideal \( \frak{n}_0 \subset \mathcal{O}_F \). Fix a Hilbert modular eigenform \( f_0 \in S_2(\frak{n}_0) \) that is new at all primes dividing the level \( \frak{n}_0 \). Let \( \frak{n}_0 \subset \mathcal{O}_F \) be an integral ideal that is not divisible by \( p \). The \textit{\( p \)-stabilization} \( f \in S_2(\frak{n}) \) of \( f_0 \in S_2(\frak{n}_0) \) is the eigenform given by

\[(24) \quad f = f_0 - \beta_p(f_0) \cdot (T_p f_0),\]

where \( \beta_p(f_0) \) denotes the non-unit root to \((23)\). This is a \( p \)-ordinary eigenform in \( S_2(\frak{n}) \) with \( U_p \)-eigenvalue \( \alpha_p(f_0) \).

Let us now consider an eigenform \( f \in S_2(\frak{n}) \) given by \( f_0 \) if \( p \) divides \( \frak{n}_0 \), or given by the \( p \)-stabilization of \( f_0 \) if \( p \) does not divide \( \frak{n}_0 \). We have the following quaternionic description of \( f \) in either case. Let \( B \) denote the totally definite quaternion algebra over \( F \) of discriminant \( \frak{n}^- \). To be consistent with the notations above, let us also write \( U_v \) for the Hecke operators \( T_v \) on \( S^B_2(H; \mathcal{O}) \) when \( v \mid \frak{n}^+ \).

**Proposition 3.7 (Jacquet-Langlands).** Given an eigenform \( f \in S_2(\frak{n}) \) as above, there exists a function \( \Phi \in S_2 \left( \prod_{i=1}^h \Gamma_i \backslash T_p; C \right) \) such that

- \( T_v \Phi = a_v(f) \cdot \Phi \) for all \( v \nmid \frak{n} \).
- \( U_v \Phi = \alpha_v(f) \cdot \Phi \) for all \( v \mid \frak{n}^+ \).
- \( U_p \Phi = \alpha_p(f) \cdot \Phi \).

This function is unique up to multiplication by non-zero complex numbers. Conversely, given an eigenform \( \Phi \in S_2 \left( \prod_{i=1}^h \Gamma_i \backslash T_p; C \right) \), there exists an eigenform \( f \in S_2(\frak{n}) \) such that

- \( T_v f = a_v(\Phi) \cdot f \) for all \( v \nmid \frak{n} \).
- \( U_v f = \alpha_v(\Phi) \cdot f \) for all \( v \mid \frak{n}^+ \).
- \( U_p f = \alpha_p(\Phi) \cdot f \).

Here, \( a_v(\Phi) \) denotes the eigenvalue for \( T_v \) of \( \Phi \) if \( v \nmid \frak{n} \), and \( \alpha_v(\Phi) \) the eigenvalue for \( U_v \) of \( \Phi \) if \( v \mid \frak{n} \).

**Proof.** We generalize the argument given for \( F = Q \) in \([2]\) Proposition 1.3).

**Case I:** Suppose first that \( p \) divides \( \frak{n}_0 = \frak{n} \), hence that \( f \in S_2(\frak{n}) \) is new at \( p \). Let \( R_0 \subset B \) be an Eichler order of level \( p \frak{n}^+ \). The theorem of Jacquet-Langlands \([12]\) then associates to \( f \) an eigenform \( \Phi \in S^B_2(\hat{R}_0^\times; C) \) such that

- ...
On the other hand, observe that $\phi$ Let us now define $\Phi$ to be an Eichler order of level $\mathfrak{N}$. The theorem of Jacquet-Langlands [12] then associates to $f$ an eigenform $\Phi_0 \in S_2^0(\mathcal{R}_0; \mathbb{C})$ such that

\begin{align*}
T_v\Phi &= a_v(f) \cdot \Phi \quad \text{for all } v \nmid \mathfrak{N} \\
U_v\Phi &= \alpha_v(f) \cdot \Phi \quad \text{for all } v \mid \mathfrak{N}^+ \\
U_p\Phi &= \alpha_p \cdot \Phi.
\end{align*}

This function $\Phi$ is unique up to multiplication by nonzero elements of $\mathbb{C}$. It can be identified with a function in $S_2 \left( \prod_{i=1}^h \Gamma_i \backslash \mathbb{H} \right)$ by Proposition 3.6.

**Case II:** Suppose that $p$ does not divide $\mathfrak{m}_0$, hence that $f \in S_2(\mathfrak{N})$ is not new at $p$. Let $\mathfrak{R}_0 \subset B$ be an Eichler order of level $\mathfrak{N}$. Then $\Phi$ does not divide $\phi$. It follows that $\phi$ is an eigenvector for the Hecke operator $U_p$ with eigenvalue $\alpha_p$. Let us then define $\Phi = (\phi^h_{i=1})$. Thus, $\Phi$ is a function in the space $S_2 \left( \prod_{i=1}^h \Gamma_i \backslash \mathcal{E}_p^*; \mathbb{C} \right)$.

Now we can construct from this function $\Phi_0 = (\phi^h_{i=1})$ the following functions of $\phi^h_{i=1}$ of $S_2 \left( \prod_{i=1}^h \Gamma_i \backslash \mathcal{E}_p^*; \mathbb{C} \right)$. For each component $\phi^h_{i=1}$, define a pair of functions $\phi^h_{i=1} : \mathcal{E}_p^* \rightarrow \mathbb{C}$ by the rules

\begin{equation}
\phi^h_{i=1}(e) = \phi^h_{i=1}(s(e)), \quad \phi^h_{i=1}(t(e)) = \phi^h_{i=1}(t(e)).
\end{equation}

We have by construction that

\begin{align*}
T_v\phi^h_{i=1} &= a_v(f) \cdot \phi^h_{i=1} \quad \text{for all } v \nmid \mathfrak{m}_0, \\
U_v\phi^h_{i=1} &= \alpha_v(f) \cdot \phi^h_{i=1} \quad \text{for all } v \mid \mathfrak{N}_0.
\end{align*}

Now, observe that

\begin{equation}
(U_p\phi^h_{i=1})(e) = \sum_{s(e') = t(e)} \phi^h_{i=1}(e') = \sum_{s(e') = t(e)} \phi^h_{i=1}(t(e)) = q \cdot \phi^h_{i=1}(e).
\end{equation}

On the other hand, observe that

\begin{equation}
(U_p\phi^h_{i=1})(e) = \sum_{s(e') = t(e)} \phi^h_{i=1}(e) = (T_p\phi^h_{i=1})(t(e)) - \phi^h_{i=1}(s(e)) = \alpha_p(f_0) \cdot \phi^h_{i=1}(t(e)) - \phi^h_{i=1}(s(e)).
\end{equation}

Let us now define $\phi^h = \phi^h_{i=1} - \alpha_p \cdot \phi^h_{i=1}$. Using (26) and (27), we find that

\begin{align*}
(U_p\phi^h)(e) &= \sum_{s(e') = t(e)} \phi^h_{i=1}(e') - \alpha_p \cdot \phi^h_{i=1}(e') \\
&= \sum_{s(e') = t(e)} \phi^h_{i=1}(s(e')) - \alpha_p \cdot \sum_{s(e') = t(e)} \phi^h_{i=1}(t(e')) \\
&= q \cdot \phi^h_{i=1}(t(e)) - \alpha_p \cdot (\alpha_p \cdot \phi^h_{i=1}(t(e)) - \phi^h_{i=1}(s(e))) \\
&= \alpha_p \cdot (\phi^h_{i=1}(s(e)) - \alpha_p \cdot \phi^h_{i=1}(t(e))) \\
&= \alpha_p \cdot \phi^h(e),
\end{align*}

i.e. since $\alpha_p$ being a root of (24) implies that $\alpha_p^2 = a_p(f_0) \cdot \alpha_p - q$. It follows that $\phi^h$ is an eigenvector for the Hecke operator $U_p$ with eigenvalue $\alpha_p$. Let us then define $\Phi = (\phi^h_{i=1})$. Thus, $\Phi$ is a function in the space $S_2 \left( \prod_{i=1}^h \Gamma_i \backslash \mathcal{E}_p^*; \mathbb{C} \right)$. It is
an simultaneous eigenvector for all of the operators $T_v$ with $v \nmid \mathfrak{M}$, $U_v$ with $v \mid \mathfrak{M}$, and $U_p$ having the prescribed eigenvalues. Moreover, it is the unique such function up to multiplication by non-zero elements of $C$.

The converse in either case can be established as follows. Given such a function $\Phi$ in $S_2 \left( \prod_{i=1}^{h} \Gamma_i \backslash T_p ; C \right)$, consider its image under the bijection (20). The theorem of Jacquet and Langlands then associates to this image an eigenform $f \in S_2(\mathfrak{M})$ having the proscribed eigenvalues. □

4. Construction of measures

Let $f \in S_2(\mathfrak{M})$ be an eigenform as defined for Proposition 3.7, with $\Phi$ the associated quaternionic eigenform. Recall that we fixed an embedding $\mathbb{Q} \to \mathbb{Q}_p$, and take $\mathcal{O}$ to be the ring of integers of a finite extension of $\mathbb{Q}_p$ containing all of the Fourier coefficients of $f$. Recall as well that we let $\Lambda$ denote the $\mathcal{O}$-Iwasawa algebra $\mathcal{O}[G_{p^\infty}]$. We construct in this section elements of $\Lambda$, equivalently $\mathcal{O}$-valued measures on $G_p^\infty$, that interpolate the central values $L(\Phi, \rho, 1/2)$. Here, $\rho$ is any finite order character of $G_p^\infty$. The construction below generalizes those of Bertolini-Darmon ([1], [2]) in the ordinary case, as well as constructions of Darmon-Iovita [5] and Pollack [17] in the supersingular case. These constructions are also sketched in the ordinary case by Longo [13], using the language of Gross points. We use the Yuan-Zhang-Zhang generalization Waldpurger’s theorem, as described in Theorem 2.3 above, to deduce an interpolation formula for these measures (Theorem 4.7). We then give a formula for the associated $\mu$-invariant (Theorem 4.10), generalizing the work of Vatsal [24].

Fix an integral ideal $\mathfrak{M} \subset \mathcal{O}_F$ having the factorization in $F$ defined in (2). Let $B$ denote the totally definite quaternion algebra over $F$ of discriminant $\mathfrak{M}^-$. Let $Z$ denote the maximal $\mathcal{O}_F[\frac{1}{p}]$-order in $K$, and let $R \subset B$ be an Eichler $\mathcal{O}_F[\frac{1}{p}]$-order of level $\mathfrak{M}^+$. We fix an optimal embedding $\Psi$ of $\mathcal{O}$ into $R$, i.e. an injective $F$-algebra homomorphism $\Psi : K \to B$ such that

$$\Psi(K) \cap R = \Psi(Z).$$

Such an embedding exists if and only if all of the primes dividing the level of $R$ are split in $K$ (see [26, § II.3]), so our choice of factorization (2) ensures that we may choose such an embedding.

**Galois action on the Bruhat-Tits tree.** The reciprocity map $\text{rec}_K$ induces an isomorphism

$$\tilde{K}^\times / \left( K^\times \prod_{v \mid p} Z_v^\times \right) \xrightarrow{\text{rec}_K} G[p^\infty],$$

as implied for instance from Lemma 2.1(i). Passing to the adelization, the optimal embedding $\Psi$ then induces an embedding

$$\tilde{K}^\times / \left( K^\times \prod_{v \mid p} Z_v^\times \right) \xrightarrow{\Psi} B^\times \backslash \widehat{B}^\times / \prod_{v \mid p} R_v^\times.$$ 

Taking the subset of $\widehat{B}^\times$ defined by $\prod_{v \mid p} R_v^\times$, with associated subgroups $\Gamma_i$ as defined in (12), strong approximation (14) gives an isomorphism

$$\prod_{i=1}^{h} \Gamma_i \backslash B^\times / F_p^\times \xrightarrow{\eta_p} B^\times \backslash \widehat{B}^\times / \prod_{v \mid p} R_v^\times.$$
We may therefore consider the composition of maps given by

\[
G[p^\infty] \xrightarrow{r_{K}^{-1}} \hat{K}^\times / \left( K^\times \prod_{v \mid p} Z_v^\times \right) \xrightarrow{\hat{\Psi}} B^\times \backslash \hat{B}^\times / \prod_{v \mid p} R_v^\times \xrightarrow{\eta_p^{-1}} \prod_{i=1}^h \Gamma_i \backslash B_p^\times / F_p^\times.
\]

The composition

\[(28) \eta_p^{-1} \circ \hat{\Psi} \circ r_{K}^{-1}\]

induces an action \(\ast\) of the Galois group \(G[p^\infty]\) on the Bruhat-Tits tree \(T_p = (\mathcal{V}_p, E_p^\times)\). This action factors through that of the local optimal embedding \(\Psi_p : K_p \rightarrow B_p\). We can give a more precise description, following [5, § 2.2]. That is, the optimal embedding \(\Psi : K \rightarrow B\) induces a local optimal embedding \(\Psi_p : K_p^\times \rightarrow B_p^\times\), which in turn induces an action (by conjugation) of \(K_p^\times / F_p^\times\) on \(B_p^\times / F_p^\times\). The dynamics of this action depend on the decomposition of \(p\) in \(K\). Hence, we divide into cases. Let us write \(K_p = K \otimes F_p\) to denote the localization of \(K\) at \(p\).

**Case I:** \(p\) splits in \(K\). In this case, \(K_p^\times\) is a split torus, and so the action of \(K_p^\times / F_p^\times\) on \(T_p\) does not fix any vertex. Fix a prime \(\mathfrak{P}\) above \(p\) in \(K\) (not to be confused with the maximal ideal \(\mathfrak{P} \subset \mathcal{O}\) defined above). Define a homomorphism

\[
|| ||_{\mathfrak{P}} : K_p^\times / F_p^\times \longrightarrow \mathbb{Z}
\]

\[
x \longmapsto \text{ord}_{\mathfrak{P}} \left( \frac{x}{p} \right).
\]

Note that the choice of \(\mathfrak{P}\) only changes the homomorphism above by a sign. For later applications, we shall choose \(\mathfrak{P}\) in accordance with our fixed embedding \(\bar{Q} \rightarrow \bar{\mathcal{O}}_p\).

Consider the maximal compact subgroup of \(K_p^\times / F_p^\times\) defined by

\[
U_0 = \ker (|| ||_{\mathfrak{P}}).
\]

Consider the natural decreasing filtration of compact subgroups

\[(29) \ldots \subset U_j \subset \ldots U_1 \subset U_0\]

satisfying the condition that \([U_0 : U_j] = q^j - 1(q + 1)\) for each index \(j \geq 1\). The action of \(U_0\) on \(T_p\) fixes a geodesic \(J = J_{\mathfrak{P}}\) of \(T_p\), i.e. an infinite sequence of consecutive vertices. Now, the quotient \(K_p^\times / F_p^\times / U_0\) acts by translation on \(J\). Let us define the distance between any vertex \(v \in \mathcal{V}_p\) and the geodesic \(J\) to be

\[
d(v, J) := \min_{w \in J} d(v, w).
\]

Here, \(w \in J\) runs over all of the vertices of \(J\). If \(d(v, J) = j\), then it is simple to see from the definitions of distance above that \(\text{Stab}_{K_p^\times / F_p^\times}(v) = U_j\). Moreover, we see that the quotient \(K_p^\times / F_p^\times / U_j\) acts simply transitively on the set of vertices of distance \(j\) from \(J\). In this case, let us fix a sequence of consecutive vertices \(\{v_j\}_{j \geq 0}\) with \(d(v_j, J) = j\) such that \(\text{Stab}_{K_p^\times / F_p^\times}(v_j) = U_j\).

**Case II:** \(p\) is inert in \(K\). In this case, the quotient \(K_p^\times / F_p^\times\) is compact, and so the action of \(K_p^\times / F_p^\times\) on \(T_p\) fixes a distinguished vertex \(v_0\). Hence, we can take
\( \mathcal{U}_0 = K_p^\times / F_p^\times \) to be the maximal compact subgroup in the construction above, with associated natural decreasing filtration of compact subgroups

\[
\ldots \subset \mathcal{U}_j \subset \ldots \mathcal{U}_1 \subset \mathcal{U}_0 = K_p^\times / F_p^\times
\]

satisfying the condition that \( [\mathcal{U}_0 : \mathcal{U}_j] = q^{j-1}(q + 1) \) for each index \( j \geq 1 \). If \( d(v_0, v) = j \) for some vertex \( v \in \mathcal{V}_p \), then we have that \( \text{Stab}_{K_p^\times / F_p^\times}(v) = \mathcal{U}_j \). In this case, let us fix a sequence of consecutive vertices \( \{v_j\}_{j \geq 0} \) with \( d(v_0, v_j) = j \) such that \( \text{Stab}_{K_p^\times / F_p^\times}(v_j) = \mathcal{U}_j \).

Let us note that in either case of the decomposition of \( p \) in \( K \), the filtration subgroup \( \mathcal{U}_j \) is simply the standard compact subgroup of \( K_p^\times / F_p^\times \) of the form

\[
\mathcal{U}_j = (1 + p^j \mathcal{O}_K \otimes \mathcal{O}_{F_p})^\times / (1 + p^j \mathcal{O}_{F_p})^\times.
\]

In either case, we obtain from the filtration \( (29) \) or \( (30) \) an infinite sequence of consecutive edges \( \{e_j\}_{j \geq 1} \), with each edge \( e_j \) joining two vertices \( v_{j-1} \leftrightarrow v_j \), and satisfying the property that

\[
\text{Stab}_{K_p^\times / F_p^\times}(e_j) = \mathcal{U}_j.
\]

We refer the reader to [5, § 1] for some more details. Let us for simplicity write \( \{w_j\} = \{v_j\} \) to denote either the sequence of consecutive vertices \( \{v_j\}_{j \geq 0} \) or the induced sequence of consecutive edges \( \{e_j\}_{j \geq 1} \).

A pairing. Fix an eigenform \( \Phi = (\phi^i)_{i=1}^h \in \mathcal{S}_2(\prod_{i=1}^h \Gamma_i \backslash \mathcal{T}_p \otimes \mathcal{O}) \). Fix a sequence of consecutive edges or vertices \( \{w_j\} \). We define for each index \( j \) a function

\[
\Phi_{K,j} : K_p^\times / F_p^\times / \mathcal{U}_j \to \mathcal{O},
\]

\[
\gamma \mapsto \Phi (\gamma \ast w_j) = (\phi^i (\gamma \ast w_j))^h_{i=1}.
\]

Let us now simplify notation by writing

\[
\mathcal{H}_\infty = \text{rec}_K^{-1}(G[p^\infty]) = \tilde{K}^\times / \left( K^\times \prod_{v \in \mathcal{P}} Z_v^\times \right).
\]

Let us commit an abuse of notation in writing \( \mathcal{U}_j \) to also denote the image of the filtration subgroup defined in \( (31) \) above in \( \mathcal{H}_\infty \). We then have the relation

\[
\mathcal{H}_\infty = \lim_{\mathcal{U}_j} \mathcal{H}_\infty / \mathcal{U}_j.
\]

To be more precise, \( \mathcal{H}_\infty \) is profinite, hence compact. The open subgroups \( \mathcal{U}_j \) then have finite index in \( \mathcal{H}_\infty \). Since \( \mathcal{H}_\infty \) must also be locally compact, its open subgroups form a base of neighbourhoods of the identity. We claim that the collection \( \{\mathcal{U}_j\}_{j \geq 0} \) in fact forms a base of neighbourhoods of the identity, in which case the natural map \( \mathcal{H}_\infty \to \lim_{\mathcal{U}_j} \mathcal{H}_\infty / \mathcal{U}_j \) is seen to be both continuous and injective. Since its image is dense, a standard compactness argument then implies that the map must be an isomorphism. We now claim that the functions \( \Phi_{K,j} \) defined above in fact descend to functions on the quotients \( \mathcal{H}_j = \mathcal{H}_\infty / \mathcal{U}_j \). Indeed, this we claim is clear from the composition of maps \( (28) \), as the part of the image of \( \mathcal{H}_\infty \) that does not land in \( K_p^\times / F_p^\times \) must lie in one of the subgroups \( \Gamma_i \). Since each eigenform \( \phi^i \) is
\( \Gamma_r \)-invariant, the claim follows. The functions \( \Phi_{K,j} \) are then seen to give rise to a natural pairing

\[
[ , ]_\Phi : H_\infty \times T_p \longrightarrow \mathcal{O}
\]

\[
(t, w_j) \mapsto \Phi(\eta^{-1}_p \circ \psi(t) \ast w_j),
\]

and under the reciprocity map \( \text{rec}_K \) a natural pairing

\[
[ , ]_\Phi : G[p^\infty] \times T_p \longrightarrow \mathcal{O}
\]

\[
(\sigma, w_j) \mapsto \Phi(\eta^{-1}_p \circ \psi \circ \text{rec}_K^{-1}(\sigma) \ast w_j).
\]

Let us write \([ , ]_\Phi\) to denote either pairing, though it is a minor abuse of notation.

\begin{lemma}
We have the following distribution relations with respect to the eigenform \( f_0 \) associated to \( \Phi \) in the setting of Proposition \[ \text{3.7} \].
\end{lemma}

\begin{enumerate}
\item If \( \{ w_j \} = \{ v_j \} \), then

\[
\pi_{j+1,j} \Phi(t) \cdot t \in \mathcal{O}[H_j].
\]

\item If \( \{ w_j \} = \{ e_j \} \), then

\[
\pi_{j+1,j} \Phi(t) \cdot t \in \mathcal{O}[H_j].
\]
\end{enumerate}

\[ \text{Proof.} \] See \[ \text{5, Lemma 2.6} \]. The same approach applied to each component \( \phi^i \) of \( \Phi \) works here. That is, we have by direct calculation on each \( \phi^i \) that

\[
\pi_{j+1,j} \Phi(t) \cdot t \in \mathcal{O}[H_j].
\]
On the other hand, we have by definition that

\[
\sum_{x \in \mathcal{H}_{j+1}} \Phi_{K,j+1}(xt_{j+1}) = \sum_{x \in \mathcal{H}_{j+1}} \Phi((xt_{j+1}) \ast w_{j+1}).
\]  

Suppose that \( \{w_j\} = \{v_j\}_{j \geq 0} \) is a sequence of consecutive vertices. Then, the sum on the right hand side of (36) corresponds on each component \( \phi^j \) to the sum over the \( q + 1 \) vertices adjacent to \( t_j \ast v_j \), minus the vertex \( t_j \ast v_{j-1} \). We refer the reader to \cite[Figure 3, p. 12]{5} for a visual aid, as it also depicts the situation here. In particular, we find that the inner sum of (35) is given by

\[
T_p(\Phi)(t_j \ast v_j) - \Phi(t_j \ast v_{j-1}).
\]

We may then deduce from Theorem 3.7 applied to \( \Phi \) that

\[
\sum_{x \in \mathcal{H}_{j+1}} \Phi((xt_{j+1}) \ast v_{j+1}) = a_p(f_0) \cdot \Phi(t_j \ast v_j) - \Phi((t_j \ast v_j) - \Phi(t_{j+1} \ast v_{j-1})
\]

\[
= a_p(f_0) \cdot \Phi_{K,j}(t_j) - \Phi_{K,j-1}(t_j).
\]

The first part of the claim then follows from (35) and (36), using the definition of \( \phi_{K,j} \). Suppose now that \( \{v_j\} = \{e_j\}_{j \geq 1} \) is a consecutive sequence of edges. Then, the sum on the right hand side of (36) corresponds on each component \( \phi^j \) to the sum over the \( q + 1 \) edges emanating from \( t_j \ast e_j \), minus the edge obtained by reversing orientation. In particular, we find that the inner sum of (35) is given by

\[
U_p(\Phi)(t_j \ast e_j).
\]

We may then deduce from Proposition 3.7 that

\[
\sum_{x \in \mathcal{H}_{j+1}} \Phi((xt_{j+1}) \ast e_{j+1}) = a_p(f_0) \cdot \Phi((t_j \ast e_j)).
\]

The second part of the claim then follows as before from (35) and (36), using the definition of \( \phi_{K,j} \).

\[\square\]

**The ordinary case.** Let us assume now that the eigenform \( f_0 \) is \( p \)-ordinary, i.e. that the image of the eigenvalue \( a_p(f_0) \) under our fixed embedding \( \overline{Q} \to \overline{Q}_p \) is a \( p \)-adic unit. Recall that we let \( \alpha_p = a_p(f_0) \) denote unit root of the Hecke polynomial \( Q_j \). Fix a sequence of consecutive directed edges \( \{w_j\} = \{e_j\}_{j \geq 1} \). Let us consider the system of maps

\[
\phi_{K,j} : \mathcal{H}_j \to \mathcal{O}
\]

defined for each index \( j \geq 1 \) by the assignment of an element \( t \in \mathcal{H}_j \) to the value

\[
\phi_{K,j}(t) = \alpha_p^{-j} \cdot \Phi_{K,j}(t).
\]

For each \( j \geq 1 \), let us also define a group ring element

\[
\theta_{K,j}(\mathcal{H}_j) = \alpha_p^{-j} \cdot \sum_{t \in \mathcal{H}_j} \Phi_{K,j}(t) \cdot t
\]

\[
= \alpha_p^{-j} \cdot \theta_{K,j} \in \mathcal{O}([\mathcal{H}_j]).
\]

**Lemma 4.2.** The system of maps \( \{\phi_{K,j}\}_{j \geq 1} \) defined in (37) determines an \( \mathcal{O} \)-valued measure on the group \( \mathcal{H}_\infty = \text{rec}^{-1}_K(G[p^\infty]) \).
Proof. Lemma [14](ii) implies that for each \( j \geq 1 \),
\[
\pi_{j+1, j} (\theta_{\Phi,j+1}) = \pi_{j+1, j} \left( \alpha_p^{-(j+1)} \cdot \theta_{\Phi,j+1} \right) = \alpha_p^{-(j+1)} \cdot \pi_{j+1, j} (\theta_{\Phi,j+1})
\]
\[
= \alpha_p^{-(j+1)} \cdot \alpha_p \cdot \theta_{\Phi,j}
\]
\[
= \alpha_p^{-j} \cdot \theta_{\Phi,j}
\]
\[
= \theta_{\Phi,j}.
\]
Hence, the system of maps \( \{ \varphi_{\Phi,j} \}_{j \geq 1} \) defines a bounded \( \mathcal{O} \)-valued distribution on \( H_\infty \), as required. \( \Box \)

**Corollary 4.3.** The system of maps \( \{ \varphi_{\Phi,j} \}_{j \geq 1} \), under composition with the reciprocity map \( \text{rec}_K \) followed by projection to the Iwasawa algebra \( \Lambda = \mathcal{O}[G_p^\infty] \), defines an \( \mathcal{O} \)-valued measure \( d\theta_\Phi \) on the Galois group \( G_p^\infty \).

Let us now consider the associated completed group ring element
\[
(38) \quad \theta_\Phi = \lim_{j} \alpha_p^{-j} \cdot \sum_{\sigma \in \Omega_{\Phi,j}} [\sigma, e_j]_\Phi \cdot \sigma \in \Lambda
\]
Observe that a different choice of sequence of directed edges \( \{ e_j \}_{j \geq 1} \) has the effect of multiplying \( \theta_\Phi \) by an automorphism of \( G_p^\infty \). To correct this, we let \( \mathcal{L}_\Phi \) denote the image of \( \mathcal{L}_\Phi \) under the involution of \( \mathcal{O}[G_p^\infty] \) that sends \( \sigma \mapsto \sigma^{-1} \in G_p^\infty \).

**Definition** Let \( \mathcal{L}_p(\Phi, K) = \theta_\Phi \theta_\Phi^* \).

Hence, \( \mathcal{L}_p(\Phi, K) \) is a well-defined element of \( \Lambda \).

**The supersingular case.** Assume now that \( a_p(f_0) = 0 \). Fix a sequence of consecutive vertices \( \{ w_j \} = \{ v_j \}_{j \geq 0} \). Here, we give a construction of the \( p \)-adic \( L \)-function of the quaternionic eigenform \( \Phi \) associated to \( f_0 \) by Proposition [7.1] following [3], building on techniques of Pollack [17]. Recall that by Lemma [2.1] (ii), we have an isomorphism of topological groups \( G_p^\infty \cong \mathbb{Z}_p^\delta \), with \( \delta = [F_p : \mathbb{Q}_p] \). Fixing \( \delta \) topological generators \( \gamma_1, \ldots, \gamma_\delta \) of \( G_p^\infty \), we can then define an isomorphism
\[
(39) \quad \mathcal{O}[G_p^\infty] \rightarrow \mathcal{O}[T_1, \ldots, T_\delta]/\left( (T_1 + 1)^{p^n} - 1, \ldots, (T_\delta + 1)^{p^n} - 1 \right)
\]
via the map that sends each \( \gamma_i \mod G_p^\infty \) to the class \( T_i + 1 \mod \left( (T_i + 1)^{p^n} - 1 \right) \).

Granted this identification \( (39) \), we claim to have the following power series description of the group ring operator \( \xi_n \) defined in \( (32) \):
\[
(T_1, \ldots, T_\delta) \xrightarrow{\xi_n} \left( \Sigma_{p^n}(T_1 + 1), \ldots, \Sigma_{p^n}(T_\delta + 1) \right).
\]
Here, \( \Sigma_{p^n} \) denotes the cyclotomic polynomial of degree \( p^n \). Let \( \Omega_n \) denote the power series operation that sends
\[
(T_1, \ldots, T_\delta) \rightarrow \left( (T_1 + 1)^{p^n} - 1, \ldots, (T_\delta + 1)^{p^n} - 1 \right)
\]
\[
= \left( T_1 \cdot \prod_{j=1}^{n} \Sigma_{p^n}(T_1 + 1), \ldots, T_\delta \cdot \prod_{j=1}^{n} \Sigma_{p^n}(T_\delta + 1) \right).
\]
Here, the last equality follows from the fact that

\[(T + 1)^{p^n} = T \cdot \prod_{j=1}^{n} \Sigma_{p^j}(T + 1).\]

Let us also define power series operations

\[\tilde{\Omega}^+_n = \tilde{\Omega}^+_n(T_1, \ldots, T_3) = \prod_{j=2}^{n} \xi_j(T_1, \ldots, T_3)\]

\[\tilde{\Omega}^-_n = \tilde{\Omega}^-_n(T_1, \ldots, T_3) = \prod_{j=1}^{n} \xi_j(T_1, \ldots, T_3)\]

\[\Omega^+_n = \Omega^+_n(T_1, \ldots, T_3) = (T_1, \ldots, T_3) \ast \tilde{\Omega}^+_n.\]

Here, we write \((T_1, \ldots, T_3) \ast \) to denote the dot product, i.e. the multiplication operation that sends \((X_1, \ldots, X_3)\) to \((T_1X_1, \ldots, T_3X_3)\), and \(\xi_j\) is the group ring operation defined above in \([\text{2}2]\). Let us for simplicity of notation make the identification \(\Lambda \cong \mathbb{O}[[T_1, \ldots, T_3]]\) implicitly in the construction that follows.

**Lemma 4.4.** Given an integer \(n \geq 0\), let \(\varepsilon\) denote the sign of \((-1)^n\). Multiplication by \(\tilde{\Omega}^-_n\) induces a bijection \(\Lambda/\langle \Omega^+_n \rangle \longrightarrow \tilde{\Omega}^-_n \Lambda/\langle \Omega^+_n \rangle\).

**Proof.** Cf. \([\text{5}]\) Lemma 2.7, where the result is given for \(\delta = 1\). A similar argument works here. That is, let \(g\) be any polynomial in \(\Lambda\). We consider the map that sends \(g \mapsto \tilde{\Omega}^-_n g\). Observe that \(\tilde{\Omega}^-_n \tilde{\Omega}^+_n = \Omega_n\). Hence if \(\Omega^+_n \mid g\), then \(\tilde{\Omega}^-_n g \equiv 0 \mod \Omega_n\). It follows that the map is injective. Since \(\Lambda\) is a unique factorization domain, the map is also seen to be surjective. \(\square\)

**Proposition 4.5.** Given a positive integer \(n\), let \(\varepsilon\) denote the sign of \((-1)^n\).

(i) We have that \(\Omega^+_n \vartheta_{\Phi,n} = 0\).

(ii) There exists a unique element \(\Theta_{\Phi,n} \in \Lambda/\Omega^+_n \Lambda\) such that \(\vartheta_{\Phi,n} = \tilde{\Omega}^-_n \Theta_{\Phi,n}\).

**Proof.** Cf. \([\text{5}]\) Proposition 2.8. Let us first suppose that \(n > 2\) is even. We then have that

\[\Omega^+_n \vartheta_{\Phi,n} = \Omega^+_n \vartheta_{\Phi,n} \xi_n \vartheta_{\Phi,n} = \Omega^+_n \xi_n \vartheta_{\Phi,n}.\]

Since \(a_p(f_0) = 0\), we obtain from Lemma \([\text{1}]\) (i) that

\[\Omega^+_n \vartheta_{\Phi,n} = -\Omega^+_n \xi_n \vartheta_{\Phi,n}.\]

This allows us to reduce to the case of \(n = 2\) by induction. Now, we find that

\[\Omega^+_2 \vartheta_{\Phi,2} = (T_1 \cdots T_3) \ast \xi_2 \vartheta_{\Phi,2} = (T_1 \cdots T_3) \ast \xi_2 \vartheta_{\Phi,2} = (T_1 \cdots T_3) \ast \xi_2 \vartheta_{\Phi,2} = (T_1 \cdots T_3).\]

Observe that

\[(T_1 \cdots T_3) \ast \xi_1 \xi_2 = (T_1 \cdots T_3) \ast (\Sigma_p(T_1 + 1) + \Sigma_p(T_3 + 1)) = \Omega_2(T_1, \ldots, T_3).\]
which is 0 in the group ring $O[G_p^2]$ by (39). This proves claim (i) for $n$ even. The case of $n$ odd can be shown in the same way. To see (ii), observe that $\Omega_n = \Omega_n^- \Omega_n^-$. Using Lemma 4.4, deduce that any element in $\Lambda$ annihilated by $\Omega_n$ must be divisible by $\Omega_n^-$. We know that $\Omega_n^- \partial \Phi, n = 0$. Thus, we find that $\partial \Phi, n$ must be divisible by $\Omega_n^-$. Since $\Lambda$ is a unique factorization domain, this concludes the proof. □

Using Proposition 4.5(ii), we may define elements

\[ \partial^\pm \Phi, n = (-1)^n \cdot \Theta^\pm \Phi, n \quad \text{if } n \equiv 0 \mod 2, \]
\[ \partial^- \Phi, n = (-1)^{n+1} \cdot \Theta^- \Phi, n \quad \text{if } n \equiv 1 \mod 2. \]

**Lemma 4.6.** The sequence $\{\partial^\pm \Phi, n\}_{n \equiv (-1)^n \mod (2)}$ is compatible with respect to the natural projection maps $\Lambda/\Omega_n \rightarrow \Lambda/\Omega_n^{-1}$.

**Proof.** Cf. [5, Lemma 2.9]. Let us choose lifts to $\Lambda$ of the group ring elements $\partial \Phi, n$ and $\Theta \Phi, n$ for all $n \geq 0$. We denote these lifts by the same symbols. Let us first suppose that $n$ is even. Lemma 4.1(i) implies that $\partial \Phi, n = -\xi_n \partial \Phi, n - 2 \mod \Omega_n$.

Using Proposition 4.5(ii), it follows that there exists a polynomial $f \in \Lambda$ such that

\[ \tilde{\Omega}_n - \partial^+ \Phi, n = -\xi_n \tilde{\Omega}_n - 2 \tilde{\Omega}^+ \Phi, n + \Omega_n f. \]  
(41)

Observe that we have the identity

\[ \Omega_n = \Omega_n^+ \tilde{\Omega}_n^- \]  
(42)

Observe that we also have the identity

\[ \tilde{\Omega}^-_n = \xi_n \tilde{\Omega}^- \]  
(43)

Using (42), we may cancel out by $\tilde{\Omega}^-_n$ on either side of (41) to obtain that

\[ \Theta^+ \Phi, n = -\Theta^+ \Phi, n - 2 + \Omega_n^{-1} f \]

by (43). This proves the result for $n$ even. The case of $n$ odd can be shown in the same way. □

Using Lemma 4.6, we may define elements

\[ \vartheta^\pm = \lim_{\overline{n}} \vartheta^\pm \Phi, n \in \lim_{\overline{n}} \Lambda/\Omega_n \]  
(44)

Observe again however that a different choice of sequence of consecutive vertices $\{v_j\}_{j \geq 0}$ has the effect of multiplying $\vartheta^\pm$ by some element $\sigma \in G_{p^\infty}$. As in the ordinary case, we correct this by making the following

**Definition** Let $L_p(\Phi, K)^\pm = \vartheta^\pm \Phi \cdot (\vartheta^\pm)^\ast$.

Note that $L_p(\Phi, K)^\pm$ is then a well-defined element of $O[[G_{p^\infty}]]$.

**p-adic L-functions.** In both cases on $f_0$, we refer to the associated element $L_p(\Phi, K)$ or $L_p(\Phi, K)^\pm \in \Lambda$, with $\Phi$ the eigenform associated to $f_0$ by Proposition 3.7, the (quaternionic) $p$-adic L-function associated to $f_0$ and $G_{p^\infty} = \text{Gal}(K_{p^\infty}/K)$. 

Interpolation properties. Recall that we let $\Lambda$ denote the $O$-Iwasawa algebra $O[[G_p]]$. Let $\rho$ be any finite order character of the Galois group $G_p$. Let $\rho(L_p(\Phi, K))$ denote the specialization of $L_p(\Phi, K)$ to $\rho$. To be more precise, a continuous homomorphism $\rho : G_p \rightarrow C_p$ extends to an algebra homomorphism $\Lambda \rightarrow C_p$ by the rule

$$\rho(\lambda) = \int_{G_p} \rho(x)d\lambda(x),$$

with $d\lambda$ the $O$-valued measure of $G_p$ associated to an element $\lambda \in \Lambda$.

**Remark** Note that a product of elements $\lambda_1 \lambda_2 \in \Lambda$ corresponds to convolution of measures $d(\lambda_1 \boxtimes \lambda_2)$ under specialization, i.e.

$$\rho(\lambda_1 \lambda_2) = \int_{G_p} \rho(x)d(\lambda_1 \boxtimes \lambda_2) = \int_{G_p} \left( \int_{G_p} \rho(x+y)d\lambda_1(x) \right) d\lambda_2(y).$$

We now state the following consequence of Theorem 2.3. Let $L_p(\Phi, K)^*$ denote any of the $p$-adic $L$-functions $L_p(\Phi, K)$, $L_p(\Phi, K)^+$, or $L_p(\Phi, K)^-$.  

**Theorem 4.7.** Fix embeddings $\overline{Q} \rightarrow \overline{Q}_p$ and $\overline{Q}_p \rightarrow C$. Let $\rho$ be any finite order character of $G_p$ such that factors through $G_{p^m}$ for some integer $m \geq 1$. Let us view the values of $\rho$ and $dL_p(\Phi, K)^*$ as complex values via $\overline{Q}_p \rightarrow C$, in which case we let $|\rho(L_p(\Phi, K)^*)|$ denote the complex absolute value of the specialization $\rho(L_p(\Phi, K)^*)$. We have the following interpolation formulae in the notations of Theorem 2.3 above.

(i) If $\Phi$ is $p$-ordinary, then

$$|\rho(L_p(\Phi, K))| = \frac{\zeta_p(2)^m}{2 \cdot L(\pi, \text{ad}, 1)} \left[ L(\pi, \rho, 1/2) \cdot L(\pi, \rho^{-1}, 1/2) \cdot \prod_{v|\infty} \alpha(\Phi_v, \rho_v) \cdot \alpha(\Phi_v, \rho_v^{-1}) \right]^{1/2}.$$

(ii) If $\Phi$ is $p$-supersingular, then

$$|\rho(L_p(\Phi, K)^\pm)| = \frac{\zeta_p(2)}{2 \cdot L(\pi, \text{ad}, 1)} \left[ L(\pi, \rho, 1/2) \cdot L(\pi, \rho^{-1}, 1/2) \cdot \prod_{v|\infty} \alpha(\Phi_v, \rho_v) \cdot \alpha(\Phi_v, \rho_v^{-1}) \right]^{1/2}.$$

**Note** that the values on the right hand sides of (i) and (ii) are both algebraic as a consequence of Theorem 2.6 and hence can be viewed as values in $\overline{Q}_p$ via our fixed embedding $\overline{Q} \rightarrow \overline{Q}_p$.

Observe in particular that the specialization $\rho(L_p(\Phi, K)^*)$ vanishes if and only if the complex central value $L(\pi, \rho, 1/2)$ vanishes. Hence, we obtain from Theorem 2.4 (or the stronger result deduced in Corollary 2.5 above) the following important

**Corollary 4.8.** The element $L_p(\Phi, K)^*$ does not vanish identically in $O[[G_p]]$.

To prove Theorem 4.7 let us first consider the following basic result. Recall that given an element $\lambda \in \Lambda$, we let $\lambda^*$ denote the image of $\lambda$ under the involution sending $\sigma \mapsto \sigma^{-1} \in G_p$.  

(45)
Lemma 4.9. We have that $\rho(\lambda^*) = \rho^{-1}(\lambda)$ for any $\lambda \in \Lambda$.

Proof. Since $\rho : G_p^\infty \to \mathbb{C}^\times$ is a homomorphism of groups, we have that $\rho(\sigma^{-1}) = \rho(\sigma)^{-1}$ for any $\sigma \in G_p^\infty$, as a consequence of the basic identities

$$\rho(\sigma)\rho(\sigma^{-1}) = \rho(\sigma)^{-1}\rho(\sigma) = 1.$$  

Using the definition of $\lambda^*$, we then find that

$$\rho(\lambda^*) = \int_{G_p^\infty} \rho(\sigma) d\lambda^*(\sigma) = \int_{G_p^\infty} \rho(\sigma^{-1}) d\lambda(\sigma) = \int_{G_p^\infty} \rho(\sigma^{-1}) d\lambda(\sigma) = \rho^{-1}(\lambda).$$

We now prove Theorem 4.7.

Proof. Suppose first that the eigenform $\Phi$ is $p$-ordinary, hence that the $T_p$-eigenvalue of $\Phi$ is a $p$-adic unit with respect to our fixed embedding $\mathbb{Q} \to \overline{\mathbb{Q}}_p$. Recall that in this case, we define $L_p(\Phi, K) = \theta_p^\ast \theta_p^\ast$ as in (38). Let $\rho$ be a finite order character of $G_p^\infty$ that factors through $G_{p^m}$. We have by definition that

$$\rho(\Phi) = \int_{G_p^\infty} \rho(\sigma) \cdot d\Phi(\sigma)$$

$$= \alpha_p^{-m} \cdot \int_{G_p^\infty} \rho(\sigma) \cdot \Phi_K, m(\sigma)$$

$$= \alpha_p^{-m} \cdot \int_{G_p^\infty} \rho(\sigma) \cdot \Phi \left( (\eta_p \circ \Psi) \circ \text{rec}_K^{-1}(\sigma) \right).$$

Here, $\epsilon_m$ denotes the $m$-th directed edge in the fixed sequence $\{\epsilon_j\}_{j \geq 1}$ defined above. Let $\epsilon_m^0$ to denote the directed edge defined by $\eta_p^{-1} \circ \Psi \circ \text{rec}_K^{-1}(\sigma) \ast \epsilon_m$, where $\ast$ denotes the induced conjugation action. We argue that the value $\Phi(\epsilon_m^0)$ can be identified with the value $\Phi(t)$, where $t = \text{rec}_K^{-1}(\sigma)$, and $\Phi(t)$ denotes the evaluation at $t$ of the corresponding eigenform $\Phi \in S(J_H^2(H; O))$. That is, recall from the discussion above that the action of the Galois group $G_{p^m}$ on the Bruhat-Tits tree $T_p$ factors through the induced conjugation action by the quotient $K_p^\infty F_p^\infty / U_m$. In particular, the quotient $(K_p^\infty:F_p^\infty) / U_m$ acts simply transitively on the set of vertices of distance $m$ away from the geodesic $J$ of vertices fixed by the maximal compact subgroup $U_0 \subseteq K_p^\infty / F_p^\infty$. Now, $\epsilon_m$ is given by the intersection of 2 maximal orders corresponding to vertices $(\eta, m_1, \eta, m)$ say, where $d(\eta, m, J) = j$. Using that $G_{p^m}$ acts simply transitively, we deduce that $\epsilon_m^\ast$ is given by the intersection of 2 maximal orders corresponding to vertices $(\eta, m_1, \eta, m)$, where $\epsilon_m^\ast = \eta_p^{-1} \circ \Psi \circ \text{rec}_K^{-1}(\sigma) \ast \epsilon_m$ is the vertex obtained from the action of Galois, having $d(\eta, m, J) = m$. Now, recall that by Lemma 3.5, we have a bijection between the set of maximal orders of $M_2(F_p)$ and the set of homothety classes of full rank lattices of $F_p \oplus F_p$. The origin vertex $0$ corresponds to the class of the lattice $O_{F_p} \oplus O_{F_p}$ under any such bijection. Since $d(\eta, m, J) = m - 1$ by construction, we may take

$$L_1 = O_{F_p} \oplus \pi_p^{m-1} O_{F_p}$$

as a lattice representative for the class corresponding to the vertex $\eta, m$. Here, $\pi_p$ is a fixed uniformizer of $F_p$. Similarly, we may take

$$L_2 = O_{F_p} \oplus \pi_p^m O_{F_p}$$
Recall that we let $\Lambda$ denote the $O$-ated to any of the $p$-completed group ring $O[\Gamma_p]$. Recall that by definition, $c_\Phi([L_1],[L_2]) = \Phi(g_L)$. Here, we have fixed a pair of representatives $(L_1, L_2)$ for the pair of classes $([L_1],[L_2])$, and $g_L$ is any matrix in $GL_2(F_p)$ such that

$$g_L \left( O_{F_p} \oplus O_{F_p} \right) = L_1$$
$$g_L \left( O_{F_p} \oplus \pi_p^{m-1} O_{F_p} \right) = L_2.$$

It is then clear that we can take

$$g_L = \begin{pmatrix} 1 & 0 \\ 0 & \pi_p^{m-1} \end{pmatrix}$$

for this matrix representative. We claim it is also clear that this matrix $g_L$ is contained in the image of the local optimal embedding

$$\left( \mathbb{K}_p^\infty / F_p^\infty \right) / \mathcal{U}_m \xrightarrow{\Psi_p} B_p^\infty \cong GL_2(F_p).$$

Granted this claim, we see that the matrix $g_L$ factors through the action of the Galois group $G_{\mathbb{P}}$ on the directed edge set. In particular, we deduce by transitivity of the action that

$$\{ \Phi(c'_m) \}_{\Phi \in \mathbb{G}_p^m} = \{ \Phi(t) \}_{t \in \text{rec}_K^{-1}(\mathbb{G}_p^m)},$$

where $\Phi(t)$ denotes the evaluation of the global eigenform $\Phi \in S^D_2(H;O)$ on a (torus) class $t$. In particular, we deduce that $\Phi(c'_m) = \Phi(\text{rec}_K^{-1}(\sigma)) = \Phi(t)$. Granted this identification, the specialization $\rho(\theta_\Phi)$ is then given by

$$\alpha^{-m} \cdot \int_{\text{rec}_K^{-1}(\mathbb{G}_p^\infty)} \rho(t) \cdot \Phi(t)\ dt,$$

with $t = \text{rec}_K^{-1}(\sigma)$. Here, $dt$ denotes the counting Haar measure, which coincides with the Tamagawa measure. We are now in a position to invoke the special value formula of Theorem 2.3 above directly. That is, since $\rho$ extends to an algebra homomorphism $\Lambda \rightarrow \mathbb{C}_p$, it follows from Lemma 4.9 that $\rho(\theta_\Phi\theta_\Phi^*) = \rho(\theta_\Phi) \cdot \rho^{-1}(\theta_\Phi)$. Hence, we find that

$$|\rho(\mathcal{L}_p(\Phi,K))| = |\rho(\theta_\Phi)| \cdot |\rho^{-1}(\theta_\Phi)| = |l(\Phi,\rho)| \cdot |l(\Phi,\rho^{-1})|.$$

Here, $l(\Phi,\rho)$ denotes the period integral defined in [20]. The result then follows directly from Theorem 2.3. In the case that $\Phi$ is $p$-supersingular, hence that the $T_p$-eigenvalue of $\Phi$ is zero, the same argument gives the analogous interpolation formula for $|\rho(\mathcal{L}_p(\Phi,K))|$. □

The invariant $\mu$. We now give an expression for the Iwasawa $\mu$-invariant associated to any of the $p$-adic $L$-functions $\mathcal{L}_p(\Phi,K)^*$, following the method of Vatsal [24]. Recall that we let $\Lambda$ denote the $O$-Iwasawa algebra of $G_{\mathbb{P}}$, which is the completed group ring $O[[G_{\mathbb{P}}]]$. Recall as well that we define the $\mu$-invariant $\mu(Q)$ of an element $Q \in \Lambda$ to be the largest exponent $c$ such that $Q \in \mathfrak{p}^c \Lambda$.

**Definition** Given an eigenform $\Phi \in S^D_2 \left( \prod_{i=1}^n \Gamma_i \backslash \mathbb{T}_p;O \right)$, let $\nu = \nu_\Phi$ denote the largest integer such that $\Phi$ is congruent to a constant modulo $\mathfrak{p}^\nu$.

**Theorem 4.10.** The $\mu$-invariant $\mu(\mathcal{L}_p(\Phi,K)^*)$ is given by $2\nu$. 

Proof. See Vatsal [24, Proposition 4.1, § 4.6], which proves the analogous result for $F = \mathbb{Q}$. Let us assume first that $\Phi$ is $p$-ordinary, hence that the image of its $T_p$-eigenvalue under our fixed embedding $\overline{\mathbb{Q}} \to \mathbb{Q}_p$ is a $p$-adic unit. Recall that in this case, we define $L_p(\Phi, K) = \theta_\Phi \theta_\Phi^*$ by the formula (35). Let $\rho$ be any ring class character of $K$ that factors through $G_{p^m}$ for some integer $m \geq 1$. Observe that by definition, we have the congruence $\rho(\theta_\Phi, m) \equiv 0 \mod \mathfrak{p}^\nu$. Hence, we find that $\mu(\theta_\Phi) \geq \nu$. Our approach is now to find a coefficient in the power series expansion for $\theta_\Phi$ having $\mathfrak{p}$-adic valuation at most $\nu$. Let us then write the completed group ring element $\theta_\Phi$ as

$$\theta_\Phi = \lim_{\nu} \left( \sum_{\sigma \in G_{p^j}} c_j(\sigma) \cdot \sigma \right) \in \Lambda.$$ 

Writing 1 for the identity in $G_{p^\infty}$, we then obtain from (35) the expression

$$c_\infty(1) := \lim_{\nu} c_j(1) = \lim_{\nu} \left( a_p^{-1} \cdot [1, \varepsilon_j]_{\Phi} \right)$$

for the constant term in the power series expansion of $\theta_\Phi$. We claim that $c_\infty(1)$ has $\mathfrak{p}$-adic valuation at most $\nu$. Equivalently, we claim that there exists a sequence of directed edges $\{\varepsilon_j\}_{j \geq 1}$ such that

$$\lim_{\nu} ([1, \varepsilon_j]_{\Phi}) \neq \lim_{\nu} ([1, \varepsilon_j]_{\Phi}) \mod \mathfrak{p}^{\nu+1}.$$ 

Indeed, suppose otherwise. Then, for any sequence of directed edges $\{\varepsilon_j\}_{j \geq 1}$, we would have that

$$\Phi(\varepsilon_j) \equiv \Phi(\varepsilon'_j) \mod \mathfrak{p}^{\nu+1}.$$ 

In particular, it would follow that $\Phi$ were congruent to a constant modulo $\mathfrak{p}^{\nu+1}$, giving the desired contradiction. Using the same argument for the element $\theta^*_\Phi$, we find that $\mu(\mathcal{L}_p(\Phi, K)) = 2\nu$. Assume now that $\Phi$ is $p$-supersingular, hence that its $T_p$-eigenvalue is 0. We claim that for each of the $p$-adic $L$-functions $\mathcal{L}_p(\Phi, K)$, we have that $\mu(\mathcal{L}_p(\Phi, K)) = \mu(\vartheta_\Phi \vartheta_\Phi^*)$, as the contribution of trivial zeroes from $\Omega_n$ will not affect the $\mathfrak{p}$-adic valuation. The same argument given above then shows that $\mu(\vartheta_\Phi \vartheta_\Phi^*) = 2\nu$, which concludes the proof.

5. HOWARD’S CRITERION

We conclude with the nonvanishing criterion Howard, [11 Theorem 3.2.3(c)]. This criterion, if satisfied, has important consequences for the associated Iwasawa main conjecture by the combined works of Howard [11 Theorem 3.2.3], and Pollack-Weston [18], as explained in Theorem 1.3 above for the case of $F = \mathbb{Q}$. If also has applications to the analogous Iwasawa main conjectures for general totally real fields, as explained in Theorem 1.3 of the sequel work [23].

Fix a Hilbert modular eigenform $f \in S_j(M)$, with $M \subset O_F$ an integral ideal having the factorization (2). Let us for simplicity assume that $M$ is prime to the relative discriminant of $K$ over $F$. Given a positive integer $k$, let us define a set of admissible primes $\mathfrak{L}_k$ of $O_F$, all of which are inert in $K$, with the condition that for any ideal $n$ in the set $\mathfrak{S}_k$ of squarefree products of primes in $\mathfrak{L}_k$, there exists a nontrivial eigenform $f^{(n)}$ of level $nM$ such that

$$(46) \quad f^{(n)} \equiv f \mod \mathfrak{p}^k.$$
Here, \( (\ref{16}) \) denotes a congruence of Hecke eigenvalues. Let \( \mathfrak{S}_k^+ \subset \mathfrak{S}_k \) denote the subset of ideals \( n \) for which \( \omega_{K/F}(n\mathfrak{O}) = -1 \), where recall \( \omega_{K/F} \) denotes the quadratic Hecke character associated to \( K/F \). Equivalently, we can let \( \mathfrak{S}_k^+ \subset \mathfrak{S}_k \) denote the subset of ideals \( n \) for which the root number of the complex \( L \)-function \( L(f, K, s) \) is \(+1\). Note that this set \( \mathfrak{S}_k^+ \) includes the so-called “empty product” corresponding to \( 1 \). Given an ideal \( n \in \mathfrak{S}_k^+ \), we have an associated \( p \)-adic \( L \)-function \( L_p(f^{(n)}, K)^* = (\theta_{f^{(n)}}, \theta_{f^{(n)}}^*)^* \). Let us then for simplicity write \( \lambda_n \) to denote the associated completed group ring element \( \theta_{f^{(n)}} \), with \( \lambda_1 \) the base element \( \theta_f \). Let \( \mathfrak{Q} \) be any height one prime ideal of \( \Lambda = \mathcal{O}[[G_{p^\infty}]] \). We say that \textit{Howard’s criterion for} \( f \) \textit{and} \( K \) \textit{holds at} \( \mathfrak{Q} \) if there exists an integer \( k_0 \) such that for each integer \( j \geq k_0 \), the set

\[
(\ref{47}) \quad \{ \lambda_n \in \Lambda/(\mathfrak{Q}^j) : n \in \mathfrak{S}_j^+ \}
\]

contains at least one element \( \lambda_n \) with nontrivial image in \( \Lambda/(\mathfrak{Q}, \mathfrak{Q}^{k_0}) \). Following the result of Howard \( \text{[11, Theorem 3.2.3 (c)]} \), as well as the generalization given in \( \text{[23, Theorem 1.3]} \), we make the following

\textbf{Conjecture 5.1.} \textit{Howard’s criterion for} \( f \) \textit{and} \( K \) \textit{holds at any height one prime ideal} \( \mathfrak{Q} \) \textit{of} \( \Lambda \).

**Remark** Note that Conjecture \( \text{5.1} \) holds trivially if \( \text{ord}_\mathfrak{Q}(\lambda_n) = 0 \) for some \( \lambda_n \) in the set \( (\ref{16}) \). It is also easy to see that Conjecture \( \text{5.1} \) holds at the height one prime defined by \( \mathfrak{Q} = (\mathfrak{P}) \), using the characterization of the \( \mu \)-invariant given in Theorem \( \text{4.10} \) above. Observe moreover that Conjecture \( \text{5.1} \) holds trivially for all height one primes of \( \Lambda \) if one of the elements \( \lambda_n \) in the set \( (\ref{47}) \) is a unit in \( \Lambda \). Hence, as explained in Theorem \( \text{1.3} \) above, or more generally in Theorem 1.3 of the sequel paper \( \text{[24]} \), this condition would often be strong enough to imply the full associated Iwasawa main conjecture, i.e. that the equality of ideals \( (\ref{3}) \) indeed holds in Conjecture \( \text{1.2} \) above.

We conclude this discussion with a reformulation Conjecture \( \text{5.1} \) at the height one prime of \( \Lambda \) defined by \( \mathfrak{Q} = (\gamma_1 - 1, \ldots, \gamma_\delta - 1) \) into a conjecture about the nonvanishing of central values of complex Rankin-Selberg \( L \)-functions. Let \( 1 \) denote the trivial character of the Galois group \( G_{p^\infty} \). Recall that we write \( L(f, K, s) \) to denote the Rankin-Selberg \( L \)-function of \( f \) times the theta series associated to \( K \), normalized to have central value at \( s = 1 \). Consider the following easy result.

\textbf{Lemma 5.2.} \textit{Howard’s criterion holds at} \( \mathfrak{Q} = (\gamma_1 - 1, \ldots, \gamma_\delta - 1) \) \textit{if and only if there exists an integer} \( k_0 \) \textit{such that for all integers} \( j \geq k_0 \), the set \( \mathfrak{S}_j^+ \) \textit{contains an ideal} \( n \) \textit{such that the associated central value} \( L(f^{(n)}, K, 1) \) \textit{does not vanish}.

**Proof.** We claim that \( \mathfrak{Q} \) divides an element \( \lambda \in \Lambda \) if and only if the specialization

\[
1(\lambda) = \int_{G_{p^\infty}} 1(\sigma)d\lambda(\sigma)
\]

does not vanish. This can be seen by translating to the power series description of \( \Lambda \). The claim then follows immediately from our interpolation formula for these \( p \)-adic \( L \)-functions (Theorem \( \text{4.7} \)), using the central value formula of Theorem \( \text{2.3} \). \( \square \)

We therefore conclude this note with the following intriguing
Conjecture 5.3. Let $f \in S_2(\mathfrak{A})$ be a cuspidal Hilbert modular eigenform, and $K/F$ a totally imaginary quadratic extension for which the root number of $L(f, K, s)$ equals $+1$. Then, there exists a positive integer $k_0$ such that the following property is satisfied: for each integer $j \geq k_0$, there exists an ideal $n \in \mathcal{E}_j^+$ such that the central value $L(f^{(n)}, K, 1)$ does not vanish.

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