Stability of the Self-accelerating Universe in Massive Gravity

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I. INTRODUCTION AND SUMMARY

It is a standing question whether the ΛCDM model is the correct description of the recent cosmic acceleration. Modified gravity models, such as massive gravity, may provide an alternative description to the cosmological constant scenario, where the background solution mimics precisely an isotropic and homogeneous background driven by a cosmological constant. Therefore, in order to discriminate between General Relativity (GR) and modified gravity, it is important to understand the evolution of perturbations on these backgrounds.

Recently, a theory of massive gravity has been proposed, characterized by the important property that it propagates at most five degrees of freedom, avoiding the well known Boulware-Deser ghost [1, 2]. In this theory, which we refer as Λ3 massive gravity, several cosmological solutions have been found, with particular attention to self-accelerating vacuum solutions which mimic the ΛCDM background [3–11]. The main goal of this paper is to study in detail the Hamiltonian structure of perturbations around these self-accelerating backgrounds based on the approach developed in [9, 10]. We pay attention to the scalar sector, where the background coordinate choice plays an important role in characterizing the local dynamics.

Our findings suggest that Λ3 massive gravity does act in a frame-dependent way under certain circumstances. For the self-accelerating vacuum backgrounds we consider here, there are two possible behaviours depending on the frame-choice: either the scalar fluctuations propagate, or there is no propagating scalar degree of freedom at the linear order in perturbations. In the first category, we find that the Hamiltonian of the propagating scalar is unbounded from below, signalling instability regardless of the choice of the parameters. For the second category of solutions, we identify the symmetry that eliminates the propagating scalar mode and show that this symmetry exists when the physical metric and the fiducial metric have the same form in the background. Due to the strong coupling behaviour, one should analyse higher order perturbations to determine stability in this case. A particular solution with this strong coupling was, indeed, found to be unstable at a non-linear level [12].

Finally, to make contact with known solutions in the literature, we classify some of these space-times, written in different coordinates, according to these two different behaviours of perturbations. By taking the decoupling limit of these solutions, we then discuss the difference between the decoupling theory and the full theory analysis. It was found that there were regions in the parameter space where the scalar mode was stable in the decoupling theory [13, 14]. On the other hand, vector modes have no dynamics at linear order in perturbations, but instead acquire...
II. EXACT SOLUTIONS IN $\Lambda_3$ MASSIVE GRAVITY

Our starting point is the Lagrangian for the $\Lambda_3$ massive gravity, which has the following form [1]

$$\mathcal{L}_G = \frac{M^2_0}{2} \sqrt{-g} \left[ R - \frac{m^2}{4} U(g_\mu\nu, K_{\mu\nu}) \right], \quad (1)$$

where

$$K_\mu^\nu = \delta_\mu^\nu - \sqrt{\Sigma} \nabla_\mu \nabla_\nu, \quad \Sigma_\mu^\nu = g^\mu\rho \partial_\rho \phi^\alpha \partial_\nu \phi^{\beta} \eta_{\alpha \beta}, \quad (2)$$

and $\phi(x^\mu)$ are the Stückelberg fields, which are introduced to restore the diffeomorphism invariance that was broken by the choice of fiducial metric $\eta_{\alpha \beta}$. The mass term $U$ can be written in terms of $\Sigma$ as

$$U = -m^2 \left[ U_2 + \alpha_3 U_3 + \alpha_4 U_4 \right], \quad (3)$$

with

$$U_2 = (\text{tr}K)^2 - \text{tr}(K^2),$$
$$U_3 = (\text{tr}K)^3 - 3(\text{tr}K)(\text{tr}K^2) + 2\text{tr}K^3,$$
$$U_4 = (\text{tr}K)^4 - 6(\text{tr}K)^2(\text{tr}K^2) + 8(\text{tr}K)(\text{tr}K^3) + 3(\text{tr}K^2)^2 - 6\text{tr}K^4,$$

where $m$ has dimension of a mass, while $\alpha_3$ and $\alpha_4$ are dimensionless parameters.

For our purposes it is enough to consider vacuum solutions which mimic GR backgrounds with a positive cosmological constant. In other words, we search for vacuum solutions to the Lagrangian (1) which result in a de Sitter space

$$ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu = -b^2(t,r) dt^2 + a^2(t,r) \left( dr^2 + r^2 d\Omega^2 \right), \quad (4)$$

with the spherically symmetric Stückelberg fields defined as

$$\phi^0 = f(t,r), \quad \phi^i = g(t,r) \frac{x^i}{r}. \quad (5)$$

A change of frame in the background metric is accompanied by a change of the Stückelberg functions $f$, $g$ (see for example the discussion in [2]). Due to the above assumptions, the matrix $\Sigma^\mu_\nu$, defined in (2), takes the form

$$\Sigma = \begin{pmatrix}
\frac{f'^2 - f'^2}{a^2} & \frac{f' - g' - f'}{a^2} & 0 & 0 \\
\frac{f' - g' - f'}{a^2} & \frac{g'^2 - f'^2}{a^2} & 0 & 0 \\
0 & 0 & \frac{a^2}{r^2 a^2} & 0 \\
0 & 0 & 0 & \frac{a^2}{r^2 a^2}
\end{pmatrix} \quad (6)$$

where prime and dot are derivatives with respect to $r$ and $t$, respectively. This metric choice is particularly helpful to calculate the square root needed in the Lagrangian definition (1)–(2). The equations of motion for $f(t,r)$ and $g(t,r)$
In this section, we explore the Hamiltonian structure of scalar linear perturbations, which only depend on time and radius. In the notation of the previous Section, we only consider the following perturbations

\[
\left(\frac{r^2a_3P_1}{\sqrt{X}}\right) - \left(\frac{r^2a_3P_1}{\sqrt{X}}\right) + \mu \left[ \left(\frac{r^2a_2P_1}{\sqrt{X}} + r^2a^2P_2\right) \right] = 0,
\]

(7)

\[
\left(\frac{r^2a_3P_1}{\sqrt{X}}\right) - \left(\frac{r^2a_3P_1}{\sqrt{X}}\right) + \mu \left[ \left(\frac{r^2a_2P_1}{\sqrt{X}} + r^2a^2P_2\right) \right] = ra^2 \left[ P_0 + P_1 \sqrt{X} + P_2 W \right]
\]

where

\[
X = \left(\frac{\dot{f}}{b} + \mu \frac{g'}{a}\right) - \left(\frac{\dot{g}}{b} + \mu \frac{f'}{a}\right),
\]

(8)

\[
W = \mu \left(\frac{\dot{f}g' - \dot{g}f'}{ab}\right).
\]

and \(\mu = \text{sign} \left(\dot{f}g' - \dot{g}f'\right)\). The functions \(P_i\) are defined as

\[
P_0(x) = -12 - 2x(x - 6) - 12(x - 1)(x - 2)a_3 - 24(x - 1)^2a_4,
\]

\[
P_1(x) = 2(3 - 2x) + 6(x - 1)(x - 3)a_3 + 24(x - 1)^2a_4,
\]

\[
P_2(x) = -2 + 12(x - 1)a_1 - 24(x - 1)^2a_4,
\]

and the primes in those functions \(P_i\) represent a derivative with respect to their argument \(x = g/(ra)\). The remaining two equations of motion (with respect to \(a\) and \(b\)) are lengthy and will not be needed for the arguments below, hence we will not show them.

The equation of motion due to \(f\) has a simple solution given by \(g(t, r) = x_0 \cdot a(t, r)\), where \(x_0\) is a constant that satisfies \(P_1(x_0) = 0\). The last equation for \(x_0\) can be solved, resulting in

\[
x_0 = \frac{\alpha + 3\beta \pm \sqrt{\alpha^2 - 3\beta}}{3\beta},
\]

(9)

where \(\alpha = 1 + 3\alpha_3\) and \(\beta = \alpha_3 + 4\alpha_4\). Notice that the special case of \(\alpha_3 = \alpha_4 = 0\) gives \(x_0 = 3/2\). Using this solution for \(g(t, r)\) we can show that the Einstein equation is given by

\[
G^\mu_{\nu} = -\frac{1}{2} m^2 P_0(x_0) \delta^\mu_{\nu}.
\]

(10)

Thus for self-accelerating solutions that satisfy the condition \(g = x_0 r a\), the functions \(a(t, r)\) and \(b(t, r)\) are exactly the same as the scale factor and lapse function in pure GR in presence of a bare cosmological constant. The remaining function, \(f\), can be obtained from the equation (7). The non-linearity of the equation explains why there could be more than one self-accelerating solution in a given coordinate system.

In the following section we consider perturbations around these self-accelerating solutions in a general framework, without assuming any particular choice of coordinates, or any particular profile for \(f\). In Section VI we present some particular solutions.

### III. Hamiltonian Analysis of Perturbations

In this section, we explore the Hamiltonian structure of scalar linear perturbations, which only depend on time and radius. In the notation of the previous Section, we only consider the following perturbations

\[
a(t, r) = a_0(t, r) + \Delta a(t, r), \quad b(t, r) = b_0(t, r) + \Delta b(t, r),
\]

(11)

\[
f(t, r) = f_0(t, r) + \Delta f(t, r), \quad g(t, r) = g_0(t, r) + \Delta g(t, r),
\]

where the fields with sub-index 0 refer to the background solution. Actually, the expressions are simplified if one uses the self-accelerating direction coordinate \(\delta\Gamma\), which is defined as

\[
\delta\Gamma = \delta g - x_0 r \delta a.
\]

(12)

The Lagrangian (11), to second order in perturbations, reduces to

\[
\mathcal{L} = \delta f \left( A_1 \delta \Gamma + A_2 \delta \Gamma + A_3 \delta \Gamma \right) + \delta \Gamma \left( B_1 \delta \Gamma + B_2 \delta a + B_3 \delta a + B_4 \delta a + B_5 \delta b \right)
\]

(13)

\[
+ \delta a (D_1 \delta b + D_2 \delta a) + \delta b \left( D_3 \delta a + D_4 \delta b + D_5 \delta a \right) + D_0 \delta a^2 + \delta a \left( E_1 \delta b + E_2 \delta a \right),
\]
where all the capital letters represent functions of \((t, r)\), fixed by the background solution. We used the background solution for \(g = x_0 r_0\), which defines the self-accelerating solutions. The functions \(A_1, B_4\) and \(E_1\) are associated with the mass term, thus have an overall factor of \(M_P^2 m^2\), while the \(D_i\) arise from the Hilbert-Einstein piece, hence containing a factor of \(M_P^2\) only. In what follows we do not need the explicit form of these functions \[12\], except for the relation

\[ D_2^2 = 4 D_4 D_6, \tag{14} \]

which ensures the lapse function is a Lagrange multiplier. Note that there is a special choice of parameters characterised by \(\alpha^2 - 3 \beta = 0\). In this case, \(A_i = B_j = 0\) and there is no propagating scalar mode. In the rest of this paper, we will not consider this special case.

In order to construct the Hamiltonian, we need the momentum conjugates of \(\delta a, \delta b, \delta f\) and \(\delta \Gamma\), which read

\[
\begin{align*}
P_a &= B_3 \delta \Gamma + D_5 \delta b + 2 D_6 \delta a, \\
P_b &= 0, \\
P_f &= 0, \\
P_\Gamma &= A_2 \delta f.
\end{align*}
\[
\tag{15}
\]

Before constructing the Hamiltonian in detail, let us explain which term is the crucial one for the following analysis. It turns out that \(A_2\) is the term that sets the two different behaviours that we mentioned earlier, and it is related to the fact that the fiducial metric \(\Sigma_{\mu\nu}\) has the same form as the physical metric \(g_{\mu\nu}\): this condition is essentially a choice of frame. We will come back to this choice of \(\Sigma_{\mu\nu}\) later on, but for now and to explain the different behaviours of the scalar perturbations, let us consider the Hamiltonian for each case separately, first for \(A_2 = 0\) and then for \(A_2 \neq 0\).

IV. CASE \(A_2 = 0\): NO SCALAR DEGREES OF FREEDOM

In this case \(P_f = 0\), which results in a constraint, and the Hamiltonian reads

\[
\begin{align*}
\mathcal{H} &= \frac{1}{4 D_6} (P_a - B_3 \delta \Gamma)^2 - \delta \Gamma (B_1 \delta \Gamma + B_2 \delta a - B_2 \delta a') - D_2 \delta a'^2 - E_2 \delta a^2 \\
&\quad - \delta f (A_4 \delta \Gamma + A_3 \delta \Gamma') - \delta b \left( \frac{D_5}{2 D_6} (P_a - B_3 \delta \Gamma) + (E_1 \delta a + B_2 \delta \Gamma + D_1 \delta a' + D_3 \delta \Gamma') \right),
\end{align*}
\]

where we have used \[14\] to simplify the expression. By looking at the above Hamiltonian, it is obvious that \(\delta b\) and \(\delta f\) appear linearly, hence their equations of motion are constraints. Therefore, we end up with the following five primary constraints

\[
\begin{align*}
C_1 &= P_b, \\
C_2 &= P_f, \\
C_3 &= \frac{\partial \mathcal{H}}{\partial \delta f} = A_1 \delta \Gamma + A_3 \delta \Gamma', \\
C_4 &= \frac{\partial \mathcal{H}}{\partial \delta b} = -\frac{D_5}{2 D_6} (P_a - B_3 \delta \Gamma) - (E_1 \delta a + B_2 \delta \Gamma + D_1 \delta a' + D_3 \delta \Gamma'), \\
C_5 &= P_\Gamma.
\end{align*}
\]

In addition, consistency conditions on these primary constraints lead to an additional secondary constraint, \(C_6\), corresponding to the time evolution of \(C_4\). The Poisson algebra of all six constraints results in

\[
\begin{align*}
\{C_j, C_i\} &= 0 & j = 1, 2 & \text{ and } i \text{ arbitrary} \\
\{C_j, C_i\} &\neq 0 & i, j \neq 1, 2.
\end{align*}
\]

Therefore, there are two first class constraints, \(C_1\) and \(C_2\), and four second class constraints \(C_3, C_4, C_5\) and \(C_6\), which in total remove 8 coordinates of the phase space \[18\]. Therefore, in the case of \(A_2 = 0\), the algebra of constraints removes all dynamical variables, leaving no propagating scalar degrees of freedom in the Hamiltonian expanded at quadratic order in perturbations. Scalar degrees of freedom may acquire non-trivial dynamics at higher order in perturbations. Indeed it was found that non-linear perturbations lead to instability \[12\].
A. Gauge symmetry and the same structure of $\Sigma_{\mu\nu}$ and $g_{\mu\nu}$

The absence of a propagating degree of freedom for $A_2 = 0$ can also be understood in terms of a new gauge symmetry due to the first class constraint $C_2$, i.e. $P_f = 0$. To see this explicitly, consider the transformation $\delta f \to \delta f + \lambda(t, r)$, which induces a change in the Lagrangian \[ \Delta L = A_1 \lambda(t, r) \delta \Gamma + A_2 \lambda(t, r) \delta \dot{\Gamma} + A_3 \lambda(t, r) \delta \Gamma' = A_2 \lambda(t, r) \delta \dot{\Gamma}, \] (18)

where we have used the constraint $C_3$ in the last equality. So for vanishing $A_2$ we obtain $\Delta L = 0$.

Furthermore, a vanishing $A_2$ implies another interesting symmetry for the fiducial metric $\Sigma_{\mu\nu}$; it presents a same structure as the physical metric $g_{\mu\nu}$. In order to probe this statement, let us begin by using equation of motion for $g_{00}$, given in (7), which explicitly reads

$$\left[(r^2 a_0^2) f'_0 - (r^2 a_0^2) f_0\right] - 2\mu r a_0^2 b_0 \left[x_0 - \sqrt{x_0}\right] = 0,$$

(19)

where we have used $g_0 = x_0 r a_0$ to restrict ourselves to the self-accelerating backgrounds. Moreover, using again the self-accelerating condition, $g_0 = x_0 r a_0$, one may write $A_2 = 0$ as

$$a_0^2 f_0 = (ra_0)'\left[(ra_0)' f_0 - (ra_0) f'_0\right].$$

(20)

Now by plugging (20) into (19), and using the definition of $X_0$ from equation (8), we arrive at the following equation

$$\frac{1}{b_0^2} f_0 - x_0^2 ((ra_0)'^2)^2 - \frac{a_0^2 f_0^2}{((ra_0)'^2)^2} + 2\mu x_0 b_0 \left[f_0 (ra_0)' a_0 - f'_0 (ra_0)' a_0 - a_0 f_0 (ra_0)'\right] + \left[x_0^2 ((ra_0)'^2)^2 - f_0^2 a_0^2 - x_0^2\right] = 0.$$

(21)

Since the lapse function $b_0$ represents the gauge freedom and it can be arbitrary, all three brackets in the above equation should vanish simultaneously. From these conditions, one can show that the fiducial metric $\Sigma_{\mu\nu}$ takes the following form

$$\Sigma_{\mu\nu} dx^\mu dx^\nu = -\left(\frac{a_0 f_0}{(ra_0)'}\right)^2 dt^2 + x_0^2 a_0^2 (dr^2 + r^2 d\Omega^2),$$

(22)

which has exactly the same form as the physical metric (4). Note that $a_0$ and $f_0$ are functions of $(t, r)$.

V. CASE $A_2 \neq 0$: A SINGLE SCALAR DEGREE OF FREEDOM

The fact that $A_2 \neq 0$ implies $P_T \neq 0$, and since $\delta \Gamma$ appears linearly in the Lagrangian we need to define $\delta f = P_T/A_2$ to have a well-defined Hamiltonian. By plugging $\delta f$ in terms of $P_T$ into the Hamiltonian, we obtain

$$H = -\frac{1}{A_2} P_T (A_1 \delta \Gamma + A_3 \delta \Gamma') + \frac{1}{4D_0} (P_a - B_3 \delta \Gamma') - \delta \Gamma (B_1 \delta \Gamma + B_2 \delta a + B_4 \delta a')$$

$$- D_2 \delta a'^2 - E_2 \delta a^2 - \delta b \left(\frac{D_5}{2D_0} (P_a - B_3 \delta \Gamma) + (E_1 \delta a + B_5 \delta \Gamma + D_1 \delta a' + D_3 \delta a'')\right).$$

(23)

We get the four following primary constraints

$$C_1 = P_T,$$

$$C_2 = P_f,$$

$$C_3 = \frac{\partial H}{\partial b} = -\frac{D_5}{2D_0} (P_a - B_3 \delta \Gamma) - (E_1 \delta a + B_5 \delta \Gamma + D_1 \delta a' + D_3 \delta a''),$$

$$C_4 = P_T - A_2 \delta f.$$

(24)

Again, consistency conditions on these primary constraints result in one additional secondary constraint, $C_5$, which corresponds to the time evolution of $C_3$. The Poisson algebra of the constraints is then

$$\{C_j, C_i\} = 0 \quad j = 1 \quad \text{and} \quad i \text{ arbitrary}$$

$$\{C_j, C_i\} \neq 0 \quad i, j \neq 1$$

In this case, we have one first class constraint only, $C_1$, and four second class constraints. Hence we have 2 coordinates in phase space, corresponding to a single propagating degree of freedom in the system. It is worth mentioning that in this case $C_2 = P_f$ is not a first class constraint, thus we do not expect the associated gauge symmetry we had in the previous case.
A. Instability of scalar degree of freedom

In the case $A_2 \neq 0$, it is interesting to analyse the stability of the remaining scalar degree of freedom. One can remove the metric perturbations and their canonical momenta (i.e. $\delta a, \delta b$ and $P_a$) using the constraints $C_3$ and $C_5$, and obtain the following Lagrangian
\[ \mathcal{L} = A_2 \delta f \delta \Gamma + A_1 \delta f \delta \Gamma + A_3 \delta f \delta \Gamma' + T(B_i, D_i, E_i) \delta \Omega^2. \]

The function $T(B_i, D_i, E_i)$ is a complicated expression of the coefficients $B_i, D_i$ and $E_i$, which appears as a consequence of integrating out $P_a, \delta a$. The Hamiltonian derived from the Lagrangian \((25)\) is given by
\[ \mathcal{H}_\Gamma = -\frac{A_1}{A_2} P_1 \delta \Gamma - \frac{A_3}{A_2} P_1 \delta \Gamma' - T(B_i, D_i, E_i) \delta \Omega^2. \]

Notice that $P_1$ appears linearly, implying that this Hamiltonian is unbounded from below for generic values of the $A_i$, or equivalently, for arbitrary choices of the self-accelerating backgrounds solutions. This “linear” instability is similar to the instability that appears in higher derivative theories known as Ostrogradski instability \([16]\). This instability on its own is not a bad thing at least classically but this can lead to a catastrophic instability when this mode couples to healthy degrees of freedom whose Hamiltonian is bounded from below.

At first sight, this result does not seem to agree with the decoupling limit analysis which shows that there is a parameter space where the Hamiltonian is bounded from below for some self-accelerating solutions. We will discuss in section VII this issue; but in order to compare with the decoupling limit result, we need to know the explicit form of the coefficients that appear in the Hamiltonian. In the next section we will discuss explicit solutions for the background functions.

VI. EXAMPLES OF BACKGROUND SOLUTIONS

In this section we will consider three kinds of solutions for the special case of $\alpha_3 = \alpha_4 = 0$ (a generalisation to any $\alpha_3$ and $\alpha_4$ is straightforward). These solutions include those that are previously found in \([3, 4, 6]\) (see \([11]\) for a recent review), as well as a new solution. The solutions are presented in different coordinates and we show the existence of a scalar degree of freedom in each particular frame choice.

As we have seen, the condition for self-acceleration is $g_0 = 3a_0 r/2$. This form of $g_0$ leaves no unique solution for $f_0$, implying that there could be several branches of solutions. In the literature it has been argued that one branch is defined when $\Sigma_{\mu \nu}$ has the same symmetries as the physical metric. However, this property does not hold in all the reference systems as we will see in what follows. In order to keep the discussion closed and show enough examples of this coordinate dependence of the background, it is enough to consider the following backgrounds:

- An open-FRWL, with a physical metric given by
\[ b_0(t, r) = 1, \quad a_0(t, r) = \frac{\sinh(HT)}{4 - H^2 r^2}, \]
where $H = m/2$. As mentioned before, the self-accelerating backgrounds condition is $g_0 = 3a_0 r/2$. We show three different solutions for $f_0$. The first solution, found in \([3, 4]\) is given by
\[ f_0^I = \frac{3}{2H} \left[ \text{arctanh} \left( \frac{4Hr}{4 - H^2 r^2} \sinh(HT) \right) + \text{arctanh} \left( \frac{4 + H^2 r^2}{4 - H^2 r^2} \tanh(HT) \right) - \frac{4Hr}{4 - H^2 r^2} \sinh(HT) \right]. \]

The second solution, found in \([4]\) but now written in the form of \([5]\), is given by
\[ f_0^{II} = \frac{3}{2H} \frac{4 + H^2 r^2}{4 - H^2 r^2} \sinh(HT). \]

Finally, the third and new solution is
\[ f_0^{III} = -\frac{3}{H} \frac{1}{4 - H^2 r^2} \cosh \left( \frac{HT}{2} \right) \left( -16 - H^4 r^4 + 8 H^2 r^2 \cosh(HT) \right) \frac{1}{2}. \]
• A flat-FRWL, with a physical metric given by

\begin{equation}
 b_0(t, r) = 1 \quad a_0(t, r) = \frac{1}{2} e^{Ht},
\end{equation}

where again \( H = m/2 \). As mentioned before, the self-accelerating backgrounds have \( g_0 = 3a_0r/2 \) and the three solutions equivalent to those shown above are as follows

\begin{equation}
 f_0^I = \frac{3}{2H} \left[ \text{arctanh} \left( \frac{1}{2} H r e^{Ht} \right) + \text{arctanh} \left( \frac{(4 + H^2 r^2) e^{2Ht} - 4}{(4 - H^2 r^2) e^{2Ht} + 4} \right) - \frac{1}{2} H r e^{Ht} \right],
\end{equation}

\begin{equation}
 f_0^{II} = \frac{3}{16H} e^{-Ht} \left( (4 + H^2 r^2) e^{2Ht} - 4 \right),
\end{equation}

\begin{equation}
 f_0^{III} = \frac{3}{4H} \sqrt{[1 + e^{-Ht}] \times [H^2 r^2 e^{2Ht} - 4(1 + e^{Ht})]}.
\end{equation}

• Conformally flat, with a physical metric given by

\begin{equation}
 b_0(t, r) = a_0(t, r) = \frac{4}{4 + H^2(r^2 - t^2)},
\end{equation}

where again \( H = m/2 \). Once again, the spatial part of the St"uckelberg fields is \( g_0 = 3a_0r/2 \), while the three solutions become

\begin{equation}
 f_0^I = \frac{3}{2H} \left[ \text{arctanh} \left( \frac{4Hr}{4 + H^2(r^2 - t^2)} \right) + \text{arctanh} \left( \frac{4Ht}{4 - H^2(r^2 - t^2)} \right) - \frac{4Hr}{4 + H^2(r^2 - t^2)} \right],
\end{equation}

\begin{equation}
 f_0^{II} = \frac{6t}{4 + H^2(r^2 - t^2)},
\end{equation}

\begin{equation}
 f_0^{III} = \frac{6\sqrt{H^2t^2 - 4}}{H(4 + H^2(r^2 - t^2))}.
\end{equation}

From the last expression, we see that solution III is valid for times larger than the Hubble scale, i.e. \( t \geq 1/H \).

In order to exhibit the different behaviours of scalar perturbations, it is useful to write the explicit form of \( A_2 \), which is given by

\begin{equation}
 A_2 = -4 \frac{3f_0}{2b_0W_0} - \frac{\mu (r_0)^{1/2}}{a_0},
\end{equation}

where as mentioned before index 0 shows the background variables. From this coefficient, one can determine if there is a propagating d.o.f. using the analysis of the previous Sections. Table I summarises the three solutions (I, II and III) in the three different frames we have written above (open-FRWL, flat-FRWL and conformally flat). It is interesting to notice that solution II, found in \[6\], only has strong coupling in scalar sector in the open-FRWL frame, in agreement with \[8\]. Moreover, solution I, found in \[3, 4\], does propagate a scalar d.o.f. in all three frames given here. Finally, the new solution (III) in the conformal frame does not propagate a scalar mode at linear order in perturbations.

| Background solution | I     | II   | III  |
|---------------------|-------|------|------|
| open-FRWL           | A_2 \neq 0 | A_2 = 0 | A_2 \neq 0 |
| flat-FRWL           | A_2 \neq 0 | A_2 \neq 0 | A_2 \neq 0 |
| conformally flat    | A_2 \neq 0 | A_2 \neq 0 | A_2 = 0 |

TABLE I: Three self-accelerating solutions with the corresponding \( A_2 = 0 \) condition in three different background coordinate choices. Solutions which satisfy \( A_2 = 0 \) have no propagating scalar d.o.f. at linear order in perturbations, whereas solutions with \( A_2 \neq 0 \) propagate a single scalar mode.
VII. DECOUPLING LIMIT

In this section, we discuss the decoupling limit case and clarify the difference between the decoupling limit theory and the full theory analysis. The decoupling limit is defined as \( m \rightarrow 0, M_{pl} \rightarrow \infty \) with \( \Lambda_3 \equiv M_{pl}m^2 \) fixed. In order to take this limit we need to normalise the fields in the following way:

\[
\delta a \rightarrow M_P^{-1} \delta a, \quad \delta b \rightarrow M_P^{-1} \delta b, \quad \delta f \rightarrow \Lambda_3^{-1} \delta f \quad \text{and} \quad \delta g \rightarrow \Lambda_3^{-1} \delta g.
\]

Under this rescaling, the Lagrangian \([13]\) reads

\[
\mathcal{L} = D_1 \delta b \delta a' + \tilde{D}_1 \delta b \delta a'' + \tilde{D}_2 \delta b \delta a' + \tilde{D}_4 \delta b \delta a + \tilde{D}_a \delta a^2 + m^2 \left[ \tilde{E}_1 \delta b \delta a + \tilde{E}_2 \delta a^2 \right] + \frac{1}{m^2} \left[ \tilde{A}_3 \delta f \delta \Gamma + \tilde{A}_4 \delta f \delta \Gamma' + \tilde{A}_5 \delta f \delta \Gamma'' + \tilde{B}_1 \delta \Gamma' \right] + \left[ \tilde{B}_2 \delta \Gamma \delta a + \tilde{B}_3 \delta \Gamma \delta a + \tilde{B}_4 \delta \Gamma \delta a' + \tilde{B}_5 \delta \Gamma \delta b \right],
\]

where we have pulled out all the \( m \) and \( M_{pl} \) dependence from the capital functions \( A_i, B_i, D_i \) and \( E_i \) (leaving expressions with a tilde) and also used \( M_{pl} = \Lambda_3/m^2 \) to write everything in terms of \( m \) and \( \Lambda_3 \). The decoupling limit is then obtained by the \( m \rightarrow 0 \), with \( \Lambda_3 \) fixed. It is worth mentioning that the first line comes from pure Einstein Hilbert action and the three other lines come from the mass term.

To go further we need to know the behaviour of coefficients in the \( m \rightarrow 0 \) limit. For this purpose we use the decoupling limit of the background solutions given in the previous section. For the self-accelerating solutions, the Hubble parameter \( H \) is proportional to \( m \). Thus in the decoupling limit we take the limit \( H t, H r \ll 1 \). In order to have a Minkowski spacetime in this limit, we use the conformal metric frame when taking this limit. We should note that the decoupling limit of the solution III is not well defined, because \( f^{III}_0 \) becomes imaginary in this limit. This is a special solution where there is no propagating degree of freedom, thus it does not contradict the decoupling limit analysis of \([13, 14]\), which showed that the self-accelerating solution in the decoupling limit propagates a single scalar mode unless \( \alpha^2 - 3 \beta = 0 \). On the other hand solutions I and II have the same decoupling limit solutions \([14]\). Note that solution II has a propagating scalar mode in the conformally flat frame, in contrast to the same solution in the open-FRWL frame where the full theory has no propagating scalar degree of freedom. Again this is not a contradiction, as the decoupling limit is not well defined in the open-FRW frame. In the decoupling limit, the background solutions are given by

\[
a_0 = b_0 = 1 - \frac{H^2}{2} (r^2 - t^2), \quad f_0 = x_0^3 t, \quad g_0 = x_0^3 r, \quad H^2 = \frac{m^2}{3} \left( \frac{2 + \sqrt{\alpha^2 - 3 \beta}}{3 \alpha + \sqrt{\alpha^2 - 3 \beta}} \right)^2 (40),
\]

with \( x_0, \alpha \) and \( \beta \) defined in and below \([9]\). It is possible to show that in \( m \rightarrow 0 \) limit the relevant terms come from the first and third line in \([40]\). If one then describes the scalar mode in the usual way in the decoupling theory (i.e. \( \phi^\mu = x^\mu - \partial^\mu \pi \), where \( \pi \) is the scalar mode, and is equivalent to \( \delta f = -\pi \) and \( \delta \Gamma = \pi \)) then the scalar Lagrangian in the decoupling limit becomes \([13, 14]\)

\[
\mathcal{L}_{\text{kin.}} = \pm 3 \sqrt{\alpha^2 - 3 \beta} \Lambda_3^2 \left( \frac{H}{m} \right)^2 \pi \partial \pi. \quad (42)
\]

The associate Hamiltonian is

\[
\mathcal{H}_\pi = \pm \left( \frac{1}{\sqrt{\alpha^2 - 3 \beta}} \right) \frac{12}{\Lambda_3^2} \left( \frac{m}{H} \right)^2 \left( \frac{H}{m} \right)^2 P_\pi^2 + 3 \sqrt{\alpha^2 - 3 \beta} \Lambda_3^2 \left( \frac{H}{m} \right)^2 \pi^2, \quad (43)
\]

which implies that the scalar perturbations are stable (unstable) for the \( + \) (\( - \)) branch \([13, 14]\). For the special case \( \beta = 0 \), which includes \( \alpha_1 = \alpha_2 = 0 \), the \( + \) branch of solutions disappears and there is always a ghost.

At first sight, this result seems inconsistent with our previous full theory analysis, where we found that the Hamiltonian is unbounded from below for all the self-accelerating solutions if \( A_2 \neq 0 \). However, one should remember that the decoupling limit is not an expansion in field perturbations, but instead a suitable expansion on the graviton mass \( m \) (keeping only the leading terms to a finite scale \( \Lambda_3 \)). Therefore, some of the features, such as the instability, in the full theory at linear order in perturbations may not be captured by the decoupling theory at linear order. However,
they may emerge at higher order in perturbations in the decoupling limit. This interpretation is supported by previous findings on the dynamics of vector degrees of freedom in the decoupling limit of massive gravity [14]. In these papers, it was shown that vector modes have no dynamics at linear order in perturbations, but instead acquire dynamics at higher order in fluctuations, which in turn, lead to a Hamiltonian that is unbounded from below – exactly as we find in the full theory analysis. Hence, we conclude that, physically, our results on the behaviour of perturbations in the two regimes, the decoupling limit and the full theory, do agree with each other.

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[17] For their explicit form one can see the Appendix in [11]. Note that we used some integration by parts.
[18] Each first class constraint removes two coordinates of the phase space, while each second class constraint removes a single coordinate.