Microwave to optical transducer via a Bose-Einstein condensate of atoms

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Stimulated Raman scattering of atoms with Λ-configuration of levels is a promising tool for transducing microwave photons to optical photons. We consider an atomic Bose-Einstein condensate whose large phase-space density highly amplifies the coupling to the microwave field. Due momentum transfer to the condensate, stimulated photon scattering can occur outside of the phase-matched direction, which can be used to separate the converted photons from the strong Raman readout pulse. Conversely, in the phase-matched direction, superradiant scattering due to bosonic enhancement leads to increased efficiency of the microwave to optical conversion. We determine the optimal conditions for the emitted optical photons to be collected into the guided modes of optical fibers.
I. INTRODUCTION

A quantum network consists of a set of quantum processing and storage nodes distributed at different locations and connected by optical fibers that transmit quantum information with photons [1–3]. Various platforms with experimentally verified relevant quantum capabilities have been proposed for the implementation of the nodes. These include nuclear magnetic resonance systems [4], single trapped ions [5], neutral atoms in optical lattices [6], single atoms in optical cavities [7], quantum dots [8, 9], color centers in crystals [10], and perhaps the most promising, superconducting circuits [11, 12]. The latter platform operates in the microwave regime, and quantum-coherent microwave-to-optical converters [13] will play the key role for the realization of a quantum network with the superconducting circuit nodes [14]. This is, however, a daunting task, since for the microwave to optical conversion at a single photon level, strong and coherent coupling is needed between the quantum degrees of freedom which differ by many orders of magnitude in energy. The most promising recent experiments employed hybrid systems to realize transducers, demonstrating remarkable achievements, including bidirectional operation [15], coherent coupling [16, 17], and efficient conversion [18, 19].

Neutral alkali or alkaline-earth atoms have strong optical transitions and also microwave resonances between hyperfine sublevels and thereby provide a natural platform for realizing quantum transducer at a single-photon level [20]. In many situations of interest, such atoms can be modeled by a three-level systems with a $\Lambda$ configuration of the levels. A single microwave photon [21, 22] can be converted into a spin-wave excitation of the atomic hyperfine sublevels, which, in turn, can be transferred to a single optical photon in a stimulated Raman process that is inherently reversible [23, 24]. In Refs. [25, 26], Rydberg transitions of the atoms have been proposed as an alternative route to achieve strong dipole coupling to a microwave field.

In the $\Lambda$ configuration, a hyperfine transition of a single atom does not have sufficient interaction strength with a microwave photon to make a practically useful transducer. The natural mitigation is to use large ensembles of atoms [20, 26]. In this paper we study the microwave to optical conversion realized by ultracold atoms that form a degenerate quantum gas – Bose-Einstein condensate (BEC) – for the increased phase-space density. Trapping an atomic BEC in the vicinity of a superconducting waveguide resonator and coupling the hyperfine atomic states to the microwave resonator field [27, 28] has been already achieved [29].

Already in the early days of Bose-Einstein condensation experiments, two essential features of light-scattering on a BEC was noted: Superradiance due to bosonic enhancement, and the possibility
for the light to excite density waves that extend over the whole atomic sample [30, 31]. We note that phenomena of the same physical origin have more recently been found to lead to dynamical phase transitions and exotic phases when the BEC interacts with a resonator field [32–39] instead of the free-space electromagnetic field, whereas the quantum state of the BEC can be imprinted on that of the light [40–42].

Here we consider a BEC as a transducer scheme based on the stimulated Raman process. Since the converted optical photons are intended for applications in quantum communication, we consider how efficiently such photons can be coupled into guided Gaussian modes focused into an optical fiber by a paraxial optical array [20]. Since the light scattering can create excitations in the BEC that are associated with atomic momentum, the condensate participates in the momentum balance of the stimulated Raman process. In particular, the BEC can take away almost arbitrary momentum from the photonic part of the process, without significantly altering its energetics. This is because the dispersion relation of the BEC excitations is extremely flat on the momentum scale relevant to optical photons, due to the large atomic mass. The photon scattering can then occur in the directions other than the phase-matched one which is determined by the readout light. This side-scattering has an intensity proportional to the number of atoms in the condensate. But for the forward scattering case, with the photon emitted in the phase matched direction, we find a superradiant behavior, with the condensate returning to the ground state, and the scattered intensity being proportional to the square of the atom number in the BEC. Exact momentum conservation can be violated because the BEC state is not an atomic momentum eigenstate, and a Gaussian beam is not a momentum eigenstate for the light. We identify situations where, due to these weak violations of the momentum conservation, the direction of the maximum of even the forward-scattered intensity deviates from the direction of the readout pulse.

This paper is organized as follows. In section II, we introduce the scheme mixing the microwave field, the optical readout pulse, the generated optical photon, and the BEC excitation. We use a second-quantized description in which the equations of motion can be written straightforwardly in the single-excitation subspace. In section III we introduce our theory for emission of the converted optical photon into paraxial guided (Gaussian) modes. Since for a given direction such modes still form a broadband 1D continuum around the frequency of the emitted radiation, the radiated intensity can be written in the Born-Markov approximation, similarly to the free-space spontaneous emission. We distinguish two cases: (A) a side-scattering with the BEC left with a single free-particle excitation, and (B) forward-scattering with the BEC left in its initial state. In section IV, we present our numerical results and discuss the findings.
II. THE FOUR-WAVE MIXING SCHEME

We consider a Bose-Einstein condensate of atoms with the ground state $|g\rangle$, a hyperfine state in the ground-state manifold $|s\rangle$, and an excited state $|e\rangle$, in the $\Lambda$ configuration of levels, see fig. 1. Transition $|g\rangle \rightarrow |s\rangle$ is coupled to a microwave resonator mode $\hat{c}$ of frequency $\omega_c$ with coupling strength $\eta$. The hyperfine magnetic sub-states $|g\rangle$ and $|s\rangle$ are coupled by electric dipole transitions to the excited state $|e\rangle$. An external laser of frequency $\omega_d$ resonantly drives the transition $|s\rangle \rightarrow |e\rangle$ with Rabi frequency $\Omega_d$. The transition $|g\rangle \rightarrow |e\rangle$ is coupled to the free-space modes of the electromagnetic field $\hat{a}_{\kappa,\lambda}$ of frequency $\omega_\kappa$ with strength $g_{\kappa,\lambda}$. Setting $\hbar = 1$, the single-atom Hamiltonian is

$$H = \frac{p^2}{2M} + U(r) (|g\rangle \langle g| + |s\rangle \langle s|) + \omega_{gs} |s\rangle \langle s| + \omega_{ge} |e\rangle \langle e| + \eta \hat{c} e^{-i\omega_c t} |s\rangle \langle g| + \Omega_d e^{-i\omega_d t} e^{i\mathbf{k}_d \mathbf{r}} |e\rangle \langle s| + \sum_{\kappa,\lambda} g_{\kappa,\lambda}(r) \hat{a}_{\kappa,\lambda} e^{-i\omega_\kappa t} |e\rangle \langle g| + \text{h.c.},$$

where $p^2/2M$ is the kinetic energy of the atom that it can acquire from the momentum transfer from the electromagnetic radiation, and $U(r)$ is an optical dipole trapping potential for the atom in states $|g\rangle$ and $|s\rangle$, while the atom in the excited state is assumed free. The diagonal terms of the Hamiltonian proportional to $\omega_{gs}$ and $\omega_{ge}$ can be transformed out in a rotating frame, leading to

$$H = \frac{p^2}{2M} + U(r) (|g\rangle \langle g| + |s\rangle \langle s|) + \Omega_d e^{i\mathbf{k}_d \mathbf{r}} |e\rangle \langle s| + \sum_{\kappa,\lambda} g_{\kappa,\lambda}(r) \hat{a}_{\kappa,\lambda} e^{-i(\omega_\kappa - \omega_{ge}) t} |e\rangle \langle g| + \text{h.c.},$$

where we assumed resonant interactions $\omega_c = \omega_{gs}$ and $\omega_d = \omega_{se}$.

The second-quantized Hamiltonian can be derived from eq. (2) as $\hat{H} = \int dV \hat{\Psi}^\dagger(r) H \hat{\Psi}(r)$ using the field operator ansatz

$$\hat{\Psi}(r) = \sqrt{N} |g\rangle \phi_{\text{BEC}}(r) + \sum_{k \neq 0} |g\rangle \phi_k(r) \hat{b}_{g,k} + \sum_{k} |s\rangle \phi_k(r) \hat{b}_{s,k} + \sum_{k} |e\rangle \phi_k(r) \hat{b}_{e,k},$$

where $N$ is the number of atoms, $\phi_{\text{BEC}}(r)$ is the macroscopically populated BEC wave function normalized as $\int dV \phi_{\text{BEC}}(r) = 1$, $\hat{b}_{g,k}$, $\hat{b}_{s,k}$ and $\hat{b}_{e,k}$ are the annihilation operators corresponding to states with atoms in the internal states $|g\rangle$, $|s\rangle$, $|e\rangle$ and with the external degrees of freedom $k$, while $\phi_k(r) \propto e^{i\mathbf{k} \cdot \mathbf{r}}$ are the corresponding motional wave functions which are the momentum eigenfunctions. The condensate wave function $\phi_{\text{BEC}}(r)$ includes the atom-atom interaction, and it is a broad function filling the trap and having large overlap with the single-particle ground state of the trap and with the zero-momentum eigenfunction $\phi_0(r)$. We can therefore take $k = 0$ out of
FIG. 1: Level scheme depicting the many-body states available to the system starting from \[ |G\rangle \equiv |0, 0, 0\rangle. \] The other states are defined as \[ |S\rangle \equiv |0, 1, 0\rangle, \quad |E_k\rangle \equiv |0, 0, 1_k\rangle, \] and \[ |G_{k'}\rangle \equiv |1_k, 0, 0\rangle. \] Black lines with arrows show the transitions induced by the interaction with the microwave cavity, the drive, and the free electromagnetic fields.

The summation over the free motional states for the ground state \[ |g\rangle. \] The tiny overlap with other free motional state with low momentum \( k \approx 0 \) can safely be neglected as it has no effect on the forward photon scattering amplitude. We can also neglect the influence of the s-wave scattering of free atoms off the condensate. In the following, we will use in place of the zero momentum wave function \( \phi_0(r) \) the BEC wave function \( \phi_{\text{BEC}}(r) \) also for the states \( |s\rangle \) and \( |e\rangle \). We then obtain the second-quantized Hamiltonian

\[
\hat{H} = \sum_k \frac{k^2}{2M} (\hat{b}_{gk}^\dagger \hat{b}_{gk} + \hat{b}_{ek}^\dagger \hat{b}_{ek}) + \left\{ \sqrt{N} \eta \hat{c}^\dagger \hat{c} + \Omega_d \sum_k \hat{b}_{ek}^\dagger \hat{b}_s \int dV \ e^{i\mathbf{k}_s \cdot \mathbf{r}} \phi_{\mathbf{k}}^*(\mathbf{r}) \phi_{\text{BEC}}(\mathbf{r}) \right. \\
+ \sqrt{N} \sum_k \sum_{\kappa, \lambda} \hat{a}_{\kappa, \lambda} e^{-i(\omega_{\kappa} - \omega_{ge})t} \hat{b}_{ek}^\dagger \int dV \ g_{\kappa, \lambda}(\mathbf{r}) \phi_{\mathbf{k}}^*(\mathbf{r}) \phi_{\text{BEC}}(\mathbf{r}) \\
\left. + \sum_k \sum_{\kappa, \lambda} \hat{a}_{\kappa, \lambda} e^{-i(\omega_{\kappa} - \omega_{ge})t} \hat{b}_{ek}^\dagger \hat{b}_{gk} \int dV \ g_{\kappa, \lambda}(\mathbf{r}) \phi_{\mathbf{k}}^*(\mathbf{r}) \phi_{\text{BEC}}(\mathbf{r}) + \text{h.c.} \right\}, \quad (4)
\]

where we used that the motional wave functions are the orthonormal eigenfunctions of the kinetic energy, and neglected the momentum transfer for the microwave transition \[ |g\rangle \rightarrow |s\rangle \] leading to that atoms in state \( |s\rangle \) have non-vanishing wavefunction \( \phi_0(r) \approx \phi_{\text{BEC}}(r) \). In the single-excitation space, the state vector of the system can be expanded as

\[
|\Psi\rangle = \delta_N |0, 0, 0\rangle |1_c\rangle |0\rangle + \varsigma_N |0, 1, 0\rangle |0_c\rangle |0\rangle + \sum_k \epsilon_{N,k} |0, 0, 1_k\rangle |0_c\rangle |0\rangle \\
+ \sum_k \sum_{\kappa, \lambda} \alpha_{N,k,\kappa,\lambda}^k |1_k, 0, 0\rangle |0_c\rangle |1_{\kappa,\lambda}\rangle + \sum_{\kappa, \lambda} \alpha_{N,\kappa,\lambda}^0 |0, 0, 0\rangle |0_c\rangle |1_{\kappa,\lambda}\rangle \quad (5)
\]
where the five terms on the rhs have the following physical meanings:

1. $|0, 0, 0⟩|1_c⟩|0⟩$ denotes the state of the system of ultracold atoms in the pure BEC state, a single photon in the microwave resonator, and no photons the free radiation field;

2. $|0, 1, 0⟩|0_c⟩|0⟩$ denotes the state of the ensemble with a single atom in the internal state $|s⟩$ due to the absorption of the microwave photon;

3. $|0, 0, 1_k⟩|0_c⟩|0⟩$ denotes the state of the ensemble with a single atom in state $|e⟩$ with momentum $k$;

4. $|1_k, 0, 0⟩|0_c⟩|1_{\kappa, \lambda}⟩$ denotes the state of the ensemble with an atom in state $|g⟩$ having the momentum $k$, plus an optical photon in the mode with the wave vector $\kappa$ and the polarization $\lambda$;

5. $|0, 0, 0⟩|0_c⟩|1_{\kappa, \lambda}⟩$ denotes a state with all the atoms in the pure BEC state with no excitations, plus an optical photon in the mode with the wave vector $\kappa$ and the polarization $\lambda$.

Note that the last two states account for the two possible photon emission channels corresponding to the system of ultracold atoms returning to the BEC state $\phi_{\text{BEC}}(r)$ with no excitations ("forward scattering") or to a single motional excitation state $\phi_{k\neq0}(r)$ ("side scattering"), which is illustrated in fig. 1.

Since the kinetic energy $\frac{k^2}{2M}$ associated with the atomic motion is negligible\(^1\) in comparison to the optical radiation frequencies $\omega_c$ and $\omega_d$, the diagonal terms of eq. (4) we can be taken out. Then the time evolution of the amplitudes is obtained from the Schrödinger equation

$$i \frac{d |\Psi⟩}{dt} = \hat{H} |\Psi⟩,$$

with the initial conditions $\delta_N(t = 0) = 1$ and $\varsigma_N(0) = \epsilon_{N; k}(0) = \alpha_{N; k; \kappa, \lambda}(0) = \delta_{N; k; \kappa, \lambda}(0) = 0$ for all $k, \kappa, \ldots$

\(^1\) $\frac{k^2}{2M}$ can be expressed as $\frac{\omega^2}{2M}$, so that the ratio $\frac{k^2/2M}{\omega^2}$ is very small because of the large rest mass of the atoms.
and λ, as

\[ i\partial_t \delta_N = \sqrt{N} \eta^{*} \delta_N \]  
\[ i\partial_t \epsilon_N = \sqrt{N} \eta \delta_N + \sum_k \epsilon_{N;k} \Omega_d \int \! dV \, e^{-ik_d r} \phi_k(r) \phi^*_\text{BEC}(r) \]  
\[ i\partial_t \epsilon_{N;k} = \delta_N \Omega_d \int \! dV \, e^{ik_d r} \phi_k^*(r) \phi^*_\text{BEC}(r) \]  
\[ \sum_{k'} \sum_{\kappa,\lambda} \alpha_{N;k',\lambda}^k \epsilon^{-i(\omega_\kappa - \omega_{ge})t} \int \! dV \, g_{\kappa,\lambda}(r) \phi_k^*(r) \phi_{k'}(r) \]  
\[ \int \! dt' \epsilon_{N;k}(t') e^{i(\omega_\kappa - \omega_{ge})t'} \int \! dV \, g_{\kappa,\lambda}^*(r) \phi_k^*(r) \phi_{k'}(r) \]  
\[ \int \! dt' \epsilon_{N;k}(t') e^{i(\omega_\kappa - \omega_{ge})t'} \int \! dV \, g_{\kappa,\lambda}^*(r) \phi_k^*(r) \phi_{k'}^*(r). \]

The first three equations account for the atomic dynamics, while the last two equations describe the generation of the optical radiation field from the atomic source and are integrated to obtain directly the field amplitudes. As seen from eq. (6d), apart from the nontrivial dependence on \( N \) through the amplitude \( \epsilon_{N;k} \), there is a multiplicative factor of \( \sqrt{N} \) due to the bosonic enhancement of photoemission when the condensate returns to the state \( \phi^*_\text{BEC}(r) \).

Our main focus is the spatial profile of the emitted radiation that is collected by Gaussian optics. In the eqs. (6c) and (6d) for the field amplitudes, their spatial profiles are encoded in the overlap integrals which simultaneously depend on time. In order to separate the temporal and spatial dependence in these formulas, we have to separate the temporal and wave-vector dependence of the amplitudes \( \epsilon_{N;k}(t) \). This can be done with a good approximation since the Fourier transform of the BEC wave function has a narrow bandwidth. The evolution of \( \epsilon_{N;k}(t) \) in this narrow bandwidth does not undergo dispersion during the time of interest, and the Rabi oscillation of the amplitudes \( \epsilon_{N;k}(t) \) (first term on the RHS of eq. eq. (6c)) remain synchronized, assuming a weak excitation limit: When the Rabi frequencies contributing to the excitation of each intermediate state \( \{ \epsilon; k \} \) are sufficiently small, the last term in (6b) can be neglected, and thus all the amplitudes \( \epsilon_{N;k} \) increase linearly with time. We can then write

\[ \epsilon_{N;k}(t) \simeq \epsilon_N(t) \int \! dV \, e^{ik_d r} \phi^*_k(r) \phi^*_\text{BEC}(r) = \frac{\epsilon_{N}(t)}{\sqrt{V}} \int \! dV \, e^{ik_d r} e^{-ikr} \phi^*_\text{BEC}(r) = \frac{\epsilon_{N}(t)}{\sqrt{V}} \phi^*_\text{BEC}(k-k_d), \]

(7)
where we took into account the time evolution (6c) of $\epsilon_k(t)$ and used the momentum eigenstates $1/\sqrt{V} e^{i k r}$ for the wavefunction of the intermediate states. Here $\epsilon_N(t)$ denotes the time dependent part of the amplitude (with $k$ dependence factored out) that is driven by the pulsed microwave excitation and the driving laser, while $\tilde{\phi}_{\text{BEC}}$ denotes the Fourier transform of $\phi_{\text{BEC}}$. In this perturbative limit, at the beginning of the process only the first terms of eq. (6c) and eq. (6d) dominate the time evolution of $\zeta_N$ and $\epsilon_N$ and the $N$-dependence of the amplitude of state $|e\rangle$ becomes trivial,

$$\epsilon_N \simeq \sqrt{N} \epsilon_{N=1} \equiv \sqrt{N} \epsilon. \quad (8)$$

III. RADIATION INTO GAUSSIAN MODES

Our aim is to collect the emitted radiation into an optical fiber. The modes of a single-mode fiber form a one-dimensional manifold, which can be parametrized by a scalar wavenumber $k$. The important parameters are the size of the fiber core and the orientation of the fiber with respect to the axis of the driving field that determines the phase-matched emission direction. To model the incoupling process, we assume a generic paraxial optical array, which transforms a set of free, nearly paraxial beams to the strongly confined guided mode, see fig. 2. The fiber modes confined in the core of size $d_{\text{core}}$ are transformed in free space to Gaussian modes with waist $w_0$ determined by the parameters of the optical array at the position of the ensemble:\footnote{According to the ABCD rule of paraxial optics [43], this transformation depends on the wavenumber, but in our case the relevant wavenumbers are very strongly confined to the vicinity of the two-photon resonance, hence we neglect this dependence.}

$$d_{\text{core},k} \! @ \! \text{fiber entrance} \iff w_0,k \! @ \! \text{position of the BEC.}$$

On the basis of the foregoing considerations, from the full set of free-space modes $\alpha_{k,\lambda}$, we can separate a set of guided modes $\beta_{k,\lambda}$\footnote{All the possible wavenumbers $k$ of the amplitude $\beta_{k,\lambda}$ are parallel to the axis of the fiber.} that are coupled by the optical array into the fiber. These modes can be well approximated by Gaussian modes with wavenumber $k$ and waist $w_0$,

$$f_k(r) = \frac{z_R}{q^*(z)} \exp \left( ik \left[ z + \frac{\rho^2}{2 q^*(z)} \right] \right)$$

$$= i \frac{w_0}{w(z)} \exp \left[ ikz - \frac{\rho^2}{w(z)^2} + i \frac{k \rho^2}{2 R(z)} - i \phi_{\text{Gouy}} \right] \equiv \varphi_G(r) e^{ikz}, \quad (9)$$

where $z$ is the position along the direction of the fiber axis, $\rho = \sqrt{x^2 + y^2}$ is the radial coordinate, $q(z) = z + iz_R$ is the complex beam parameter with the Rayleigh range $z_R = kw_0^2/2$, $w(z) = w_0\sqrt{1 + z^2/z_R^2}$ is the transverse beam size, $R(z) = z + z_R^2/z$ is the radius of curvature of the wave.
FIG. 2: The geometry of the system of atoms, paraxial optics and the fibre.

front, and $\phi_{\text{Gouy}} = \arctan(z/z_R)$ is the Gouy phase. The electric field that is coupled into the fiber is

$$E_G(r) = i \sum_{k,\lambda} \sqrt{\frac{\omega}{2\varepsilon_0 V}} \beta_{k,\lambda}(t) \mathbf{e}_\lambda \mathbf{f}_k(r), \quad (10)$$

where $V = \mathcal{A}L$ is the mode volume with $\mathcal{A} = \pi w_0^2/4$ being its cross-sectional area and $L$ the fictitious quantization length along the propagation axis. The fibre-coupled modes form a broadband continuum and the field $E_G$ can be treated similarly to that of the three-dimensional electromagnetic vacuum surrounding the atom [44]. Irreversible photon emission from an excited-state atom into the fibre takes place, and the photon emission rate is determined by the local coupling between the atom and the modes,

$$g_{k,\lambda}(r) = \sqrt{\frac{\omega}{2\varepsilon_0 V}} d_{k,\lambda} \mathbf{f}_k(r) = g_{k,\lambda} f_k(r). \quad (11)$$

A. Free final state of the atom

Let us first consider the case when the ensemble of ultracold atoms excited into the state $|e\rangle$, on emitting a photon, does not return to the BEC state $\phi_{\text{BEC}}(r)$ but ends up in an arbitrary free
state with momentum $p'$. The total photon number $\Delta n(t)$ generated during a time $\Delta t$ and coupled into the fibre can be expressed as the sum of the photon numbers in the corresponding modes,

$$\Delta n(t) = \sum_{p'} \sum_{k,\lambda} |\Delta \beta_{k,\lambda}^{p'}(t)|^2$$

$$= N \sum_{p'} \sum_{\pi_1, \pi_2} \sum_{k, \lambda} |g_{k, \lambda}|^2 \int_{t-\Delta t}^{t} dt_1 \int_{t-\Delta t}^{t} dt_2 \epsilon_{\pi_1}^*(t_1) \epsilon_{\pi_2}(t_2) e^{i(\omega_k - \omega_{ge})(t_2-t_1)} \times \int dV \int dV' f_k(r) \phi_{\pi_1}(r) \phi_{\pi_2}(r') \phi_{p'}^*(r'). \quad (12)$$

The fibre-coupled modes form a broadband continuum around the transition frequency $\omega_{ge}$ and the bandwidth of the modes is much larger than the inverse of the short time interval $\Delta t$, which is analogous to spontaneous emission in the three-dimensional free-space modes. We also assume weak coupling and neglect the reabsorption of photons within the cloud. Hence we can adopt the Born-Markov approximation to evaluate the sum in Eq. (12). Formally, because of the broadband summation over $k$ of the exponential terms $e^{i(\omega_k - \omega_{ge})(t_2-t_1)}$, only the time instant $t_1 \approx t$ contributes to the combined summation over $k$ and the time integrals over $t_{1,2}$. Therefore, the atomic amplitudes $\epsilon(t_{1,2})$ can be replaced by $\epsilon(t)$ and taken out of the time integral. The remaining time integrals can be carried out, and the summation over $k$ can be evaluated in the continuous limit as an integral, as follows:

$$\Delta n(t) = N \sum_{p'} \sum_{\pi_1, \pi_2} \sum_{k, \lambda} |g_{k, \lambda}|^2 \int_{t-\Delta t}^{t} dt_1 \int_{t-\Delta t}^{t} dt_2 \epsilon_{\pi_1}^*(t_1) \epsilon_{\pi_2}(t_2) e^{i(\omega_k - \omega_{ge})(t_2-t_1)} \times \int dV \int dV' \ldots$$

$$= N \sum_{p'} \sum_{\pi_1, \pi_2} \frac{L}{2\pi} \int d\omega \int_{t-\Delta t}^{t} dt_1 \int_{t-\Delta t}^{t} dt_2 \epsilon_{\pi_1}^*(t_1) \epsilon_{\pi_2}(t_2) e^{i(\omega - \omega_{ge})(t_2-t_1)} \times \int dV \int dV' \ldots$$

$$= \frac{N\omega_{ge} d_{fe}^2}{4\pi \varepsilon_0 c A} \sum_{p'} \sum_{\pi_1, \pi_2} \int dt_1 \int_{t-\Delta t}^{t} dt_2 \epsilon_{\pi_1}^*(t_1) \epsilon_{\pi_2}(t_2) 2\pi \delta(t_2 - t_1) \times \int dV \int dV' \ldots$$

$$= \frac{N\omega_{ge} d_{fe}^2}{2\varepsilon_0 c A} \Delta t \sum_{p'} \sum_{\pi_1, \pi_2} \epsilon_{\pi_1}(t) \epsilon_{\pi_2}(t) \int dV f_{kge}(r) \phi_{\pi_1}(r) \phi_{p'}^*(r) \int dV' f_{kge}^*(r') \phi_{\pi_2}(r') \phi_{p'}^*(r')$$

$$= N \Gamma \frac{\sigma_n}{A} \Delta t \sum_p \int dV f_{kge}(r) \phi_{\pi}(r) \phi_{p'}^*(r)^2, \quad (13)$$

where for brevity we omitted the expressions under the volume integrals (last two terms on the RHS of eq. (12)), with the understanding that from the third line inclusive, $k$ must be replaced by...
also in those expressions. The sum over the polarization $\lambda$ could be simplified by assuming the optimum configuration with the driving laser linearly polarized perpendicular to the plane of fig. 2, i.e., the plane spanned by the propagation of the driving laser beam $k_d$ and the fibre-coupled modes. The induced atomic dipole moment $d_{ge}$ has then the same out-of-plane direction. One of the two possible polarizations of the fibre-coupled modes is also perpendicular to this plane, hence parallel to the dipole moment $d_{ge}$. The other one is in-plane, being orthogonal and thus decoupled from the atomic dipole. The remaining derivation follow the standard Born-Markov (Wigner-Weisskopf) procedures. We then identify the free-space atomic spontaneous emission rate $\Gamma = \omega_{ge}^3 d_{ge}^2 / (6\pi\varepsilon_0 c^3)$ and the radiative cross section of the atomic dipole $\sigma_A = 3 \lambda^2 / (2\pi)$ in order to put the final result in a compact form.

Using the approximation (7) for the amplitudes $\epsilon_k(t)$ and specifying the wave functions $\phi_k(r)$ and $\phi_{p'}(r)$ as plane waves, the rate of photons $I = \Delta n / \Delta t$ coupled into the fibre is

$$I(t) = N \Gamma |\epsilon(t)|^2 \frac{\sigma_A}{A} \sum_{p'} \left| \frac{1}{V\sqrt{V}} \sum_k \tilde{\phi}_0(k - k_d) \int dV \phi_G^*(r) e^{i(k - k_{ge} - p')r} \right|^2$$

$$= N \Gamma |\epsilon(t)|^2 \frac{\sigma_A}{A} \sum_{p'} \left| \int dV \frac{1}{\sqrt{V}} \phi_{BEC}(r) \phi_G^*(r) e^{i(k_d - k_{ge} - p')r} \right|^2$$

$$= N \Gamma |\epsilon(t)|^2 \frac{\sigma_A}{A} \int dV |\phi_{BEC}(r) \phi_G(r)|^2 \equiv N \Gamma |\epsilon(t)|^2 \frac{\sigma_A}{A} \xi,$$  \hspace{1cm} (14)

where we replaced the summations over the discrete variables $k$ and $p'$ by integrals over continuous variables. We thus find that the photon scattering rate into the fibre is proportional to (i) the number of atoms $N$, (ii) the single-atom photon scattering rate $\Gamma |\epsilon(t)|^2$ in the full solid angle, (iii) the ratio of the atomic radiative cross section and the cross section of the fibre coupled beam, and (iv) a geometrical factor $\xi$ which includes the non-trivial condition on the spatial matching of light beams and the atom cloud.

It is remarkable that, in the plane of the driving laser and the fibre-coupled mode, the intensity $I(t)$ is independent of the angle between the wave vector $k_d$ and the fibre axis. The reason is that momentum conservation in the photon scattering process is not restrictive in the case of a BEC initial state of the atomic ensemble. Due to its extremely flat dispersion relation compared to that of the photons, the BEC acts as a momentum reservoir that can absorb away any momentum mismatch without significantly altering the energetics of the photon scattering process. A broad angular distribution has a significant consequence in practical applications: the generated photon can be easily separated from the driving laser beam, without the need for narrowband spectral filters, which necessarily degrade the coupling efficiency into the fibre. We note that this isotropic
distribution (or, dipole pattern in the case of a fixed direction of the atomic dipole moments) holds only for the first photon; already the second photon, encountering the Bragg grating created by the excitation left by the first photon, will preferentially scatter into the same direction [45].

B. BEC final state of the atom

A case of special interest is that of the final state of the atoms after the photon emission being the same BEC state as in the beginning. Although the final state is now a single motional state $\phi_{\text{BEC}}$, and not a broad continuum, the scattered intensity is sizable because of the bosonic enhancement. The probability of this transition is strongly enhanced by the macroscopic part of the atomic population in the BEC state. The intensity due to this photon scattering channel is

$$ I_0(t) = N^2 \Gamma |\epsilon(t)|^2 \frac{\varphi^*_{\text{BEC}}(r)}{A} \int dV |\varphi_{\text{BEC}}(r)|^2 \varphi_G(r)e^{i(k_d - k_{ge})r} \bigg| \equiv N^2 \Gamma |\epsilon(t)|^2 \frac{\sigma_A}{A} |\xi_0|^2. \quad (15) $$

Note that the intensity scales now quadratically with the number of atoms, while the geometric factor involves the BEC wavefunction in a different power, as compared to Eq. (14). Moreover, the geometric factor $\xi_0$ is longer isotropic: there is no momentum transfer to the final BEC state that would compensate the momentum mismatch of the readout and emitted photon, so the outgoing photon has to take the momentum of the absorbed photon from the driving laser, resulting in predominantly forward scattering.

C. Analytical approximations for the scattered intensity

In order to proceed with analytic expressions, we approximate the BEC wavefunction by a Gaussian corresponding to the ground state of a non-interacting gas in a harmonic trap with cylindrical symmetry,

$$ \phi_{\text{BEC}}(r) = \frac{1}{(2\pi)^{3/4} \sigma_{\text{osc}}^{1/2}} \exp\left( -\frac{x^2 + y^2}{4\sigma_x^2} - \frac{z^2}{4\sigma_z^2} \right), \quad (16) $$

where $\sigma_x = \sigma_y = \sigma$ and $\sigma_z$ are the oscillator lengths of the trap. We consider the configuration in which the axis of the optical fibre coincides with the long axis of the condensate (cf. fig. 2), characterized by the oscillator length $\sigma_z$. In the case of a free atom in the final state, the geometric
factor $\xi$ is

$$
\xi = \int dV |\phi_{\text{BEC}}(r)\varphi_G(r)|^2 \\
= \frac{w_0^2}{(2\pi)^{3/2}\sigma^2 z} \int dV \frac{1}{w(z)^2} \exp \left[ -\frac{x^2 + y^2}{2} \left( \frac{1}{\sigma^2} + \frac{1}{w(z)^2} \right) - \frac{z^2}{2\sigma_z^2} \right] \\
= \frac{w_0^2}{\sqrt{2\pi} \sigma_z} \int dz \frac{\exp \left( -\frac{z^2}{2\sigma_z^2} \right)}{\sigma^2 + w(z)^2} \approx \frac{w_0^2}{\sigma^2 + w_0^2}, \tag{17}
$$

where the last approximation is valid in the typical case when the longitudinal length $\sigma_z$ of the condensate is much smaller than the Rayleigh range, so that the transverse beam size $w(z)$ can be approximated by the beam waist $w_0$. The result shows that the geometrical factor $\xi \lesssim 1$. The larger the beam waist, the better the geometrical coupling factor. This is reasonable as the optics that couples into the fibre collects light from larger solid angle.

If the ensemble of ultracold atoms returns to the BEC state, the geometric factor, within the same approximations, is

$$
\xi_0(\theta) = \int dV |\phi_{\text{BEC}}(r)|^2 \varphi_G^*(r)e^{i(k_d - k_{ge})r} \\
= \frac{z_R}{(2\pi)^{3/2}\sigma^2 z} \int dV \frac{1}{q(z)} \exp \left[ -\frac{x^2 + y^2}{2} \left( \frac{1}{\sigma^2} + \frac{i k_d}{q(z)} \right) - \frac{z^2}{2\sigma_z^2} + i k_d (\cos \theta - 1) z + i k_d \sin \theta x \right] \\
= \frac{z_R}{\sqrt{2\pi} \sigma_z} \int dz \frac{1}{q(z) + i k_d \sigma^2} \exp \left[ -\frac{i k_d^2 \sigma^2 q(z)}{2(q(z) + i k_d \sigma^2)} \sin^2 \theta + i k_d (\cos \theta - 1) z - \frac{z^2}{2\sigma_z^2} \right], \tag{18}
$$

where $\theta$ is the angle between the direction of the incoming driving field $k_d$ and the direction of the optical fiber $k$. For the forward scattering direction, $\theta = 0$, which was already calculated in [20], we have

$$
\xi_0 = \frac{z_R}{(2\pi)^{3/2}\sigma^2 z} \int dV \frac{1}{q(z)} \exp \left[ -\frac{x^2 + y^2}{2} \left( \frac{1}{\sigma^2} + \frac{i k_d}{q(z)} \right) - \frac{z^2}{2\sigma_z^2} \right] \\
= -i \sqrt{\frac{\pi}{2}} \frac{z_R}{\sigma_z} e^{-\frac{(z_R + k_d \sigma_z^2)^2}{2\sigma_z^2}} \text{erfc} \left( \frac{z_R + k_d \sigma_z^2}{\sqrt{2} \sigma_z} \right) \approx \frac{w_0^2}{2\sigma^2 + w_0^2}, \tag{19}
$$

where in last line we again used the approximation that the longitudinal length $\sigma_z$ of the condensate is much smaller than the Rayleigh range $z_R$.

So far we have seen that for every $(k_d, k_{ge})$ the collected by the Gaussian optics radiation has two components: one which is independent of the angle $\theta$, and is proportional to $N$, and the other which is confined into a small solid angle around $\theta = 0$ and is proportional to $N^2$. Certainly, this latter dominates the photon emission in the forward direction for sufficiently large atom number.
IV. NUMERICAL RESULTS AND DISCUSSION

We consider a cylindrically symmetric harmonic trap and an optical fiber orientated in the $z$ direction, and address the question as to which part of the incoupled intensity, section IIIA or eq. (15), gives larger contribution to the collected radiation for a given trap geometry $\sigma, \sigma_z$ and beam waist $w_0$, at different angles $\theta$. It was already shown in [20] that the intensity of the forward scattering is optimized if the beam waist takes the value $\sqrt{2}\sigma$. So in the following we are going to fix the $w_0 = \sqrt{2}\sigma$. In the limit of small longitudinal size relative to the Rayleigh range, the geometrical factors of eq. (17) and eq. (19) become $\xi = 2/3$ and $|\xi_0|^2 = 1/4$, respectively. To compare the two channels of the radiation, we investigate the distinct parts of the intensities, section IIIA and eq. (15), namely the geometrical factors $|\xi_0(\theta)|^2$ and $\xi/N$ (see fig. 3).

If the BEC has an oblate or a nearly spherical shape, the superradiant part of the geometrical factor takes a maximum value of $1/4$ at the forward direction $\theta = 0$, while the isotropic channel results in $\xi = 2/3$. In this case $\sigma^2 > \sigma_z$ which is equivalent to the setup in which the Rayleigh range is much larger than the longitudinal size $\sigma_z$. As the spheroid becomes prolate and $\sigma_z$ becomes comparable to $z_R$, the geometrical factors of both channels decrease. At a certain point, where the shape of the BEC becomes very elongated, the maximum of the anisotropic radiation is displaced to a nontrivial angle. The reason for this is that the curvature and the Gouy phase term of the Gaussian beam become significant, which can be compensated for in eq. (18) only by a nonzero $\theta$. 

FIG. 3: Angular dependence of the geometrical factor $|\xi_0(\theta)|^2$ (continuous lines) corresponding to the superradiance and the value of the geometrical factor $\xi/N$ (horizontal, dashed lines) for the isotropic radiation for a fixed number of atoms $N = 10$, for various values of the transverse size $\sigma_z$ (a) and the longitudinal size $\bar{\sigma}$ (b)
Around $\theta = 0$ the dominant contribution comes from superradiant emission from the BEC, but above a particular angle $\theta^*$, which depends on the trap geometry, the isotropic radiation exceeds the superradiant one. This critical angle increases if the geometrical parameters $\sigma$ and $\sigma_z$ decrease.

For small number of atoms, as in fig. 4, only the isotropic part is significant for nearly all angles $\theta$. But as the number of atoms increases, the superradiance becomes important in an ever-widening range around the forward direction, as it is amplifies by a factor of $N$ compared to the isotropic radiation.

To summarize, we have considered the photon emission by a Raman process with three level atoms in the $\Lambda$ configuration of levels. In addition to the internal atomic state, we have taken into account the external, motional degree of freedom of the atoms in order to describe the momentum transfer in the photon recoil. We performed the analysis in the perturbative, weak excitation limit, in which we were able to separate the spatial dependence of the emitted radiation from its temporal dynamics. Under these approximations, we have shown that there are two channels for the converted radiation: the supperadiant one, which corresponds to the phase-matched photon scattering in the forward direction; and the isotropic one, in which the Bose-Einstein condensate takes away arbitrary momentum mismatch between the incoming and emitted radiation. The contribution of the latter one is more significant in the side scattering directions, while the width of this region varies with the number of atoms in the condensate and with the dimensions of the harmonic trap.
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