Weak convergence and invariant measure of a full discretization for non-globally Lipschitz parabolic SPDE

Jianbo Cui · Jialin Hong · Liying Sun

Abstract In this article, we consider a parabolic stochastic partial differential equation (SPDE) with non-globally Lipschitz coefficient and its full discretization based on the spatial spectral Galerkin method and the temporal implicit Euler method. By studying both the a priori estimates and regularity estimates of the numerical solution via a variational approach and Malliavin calculus, we give the sharp weak convergence rate of the proposed numerical approximation. To the best of our knowledge, this is the first sharp weak convergence rate result of full discrete numerical approximation for non-globally Lipschitz parabolic SPDE. Moreover, we prove that the invariant measure of non-globally Lipschitz parabolic SPDE can be approximated by the numerical method with the sharp weak convergence rate, if the considered SPDE admits a unique $V$-uniformly ergodic invariant measure. We study this approximate error by using both time-independent weak convergence analysis and regularity estimates of the corresponding Kolmogorov equation. These time-independent regularity estimates are obtained via a decay estimate, the Bismut–Elworthy–Li formula and the $V$-uniform ergodicity. Finally, numerical experiments confirm the theoretical findings.

Keywords non-globally Lipschitz parabolic SPDE · weak convergence rate · $V$-uniform ergodicity · Kolmogorov equation · Bismut–Elworthy–Li formula ·

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1 Introduction

The numerical approximation for stochastic partial differential equations (SPDEs) with local Lipschitz continuous coefficients, as an active area of research, has been widely concerned in the recent years. For the strongly convergent numerical method, we refer to [2,3,4,6,8,15,16,21,25,26,27,29,30] and references therein. However, it is still far from well-understood about the weak approximation of such model, although much progress has been made. We are only aware that there are several results on weak convergent semi-discretizations, such as some temporal splitting methods in [3] and the spatial finite element methods in [14] for parabolic SPDEs with non-globally Lipschitz nonlinearity, and a temporal splitting method in [13] for the stochastic Schrödinger equation with cubic terms. To the best of our knowledge, there have been no essentially sharp weak convergence rate of full discretization for parabolic SPDEs with non-globally Lipschitz coefficient.

One motivation of this present work is to make a contribution in this direction and study numerical approximations for the following parabolic SPDE

\[ dX(t) = (AX(t) + F(X(t)))dt + dW(t), \quad t > 0, \]
\[ X(0) = X_0, \tag{1} \]

where \( A \) is the Laplacian operator on a regular domain \( \Omega \) with homogenous Dirichlet boundary condition, \( F \) is the Nemytskii operator of a real-valued non-globally Lipschitz function \( f \) and \( \{W(t)\}_{t \geq 0} \) is a generalized \( Q \)-Wiener process on a filtered probability space \((\Omega, F, P, \{F_t\}_{t \geq 0})\). After discretizing Eq. (1) by the spectral Galerkin method in space, we temporally propose an implicit Euler method (4). By denoting \( X^N_K \) the proposed full discretization approximating the exact solution \( X(T) \), \( N \) the space dimension and \( \delta t \) the time stepsize, the essentially sharp weak convergence rate of (4) is the following.

Theorem 1 Let Assumptions 1-3 hold with \( \beta \in (0, 1], \gamma \in (0, \beta) \), \( X_0 \in H^{\frac{d}{2} + \epsilon} \) with an arbitrary small positive constant \( \epsilon \), \( T > 0 \) and \( \delta t_0 \in (0, 1 \wedge \frac{1}{\lambda_1 + \lambda_F}) \). Then for any \( \phi \in C^2_b(\mathbb{H}) \), there exists a positive constant \( C(X_0, Q, T) \) such that for any \( \delta t \in (0, \delta t_0) \), \( K \delta t = T, K \in \mathbb{N}^+ \) and \( N \in \mathbb{N}^+ \),

\[ \left| E \left[ \phi(X(T)) - \phi(X^N_K) \right] \right| \leq C(X_0, Q, T) \left( \delta t^\gamma + \lambda^{-\gamma}_N \right). \]

Up to now, there already exist different approaches to studying the weak convergence rate of numerical methods (see e.g. [17-20,21,22,23,31]). For the full discretization of Eq. (1), it is still unclear how to analyze its sharp weak convergence rate. The key points to gain the error estimate in Theorem 1 are applications of the regularity estimates of the regularized Kolmogorov equation by splitting approach, the Malliavin derivative of [4] and the a priori estimate
of numerical solutions in $E := C(O, \mathbb{R})$. We would like to mention that proving this result confronts at two main difficulties, one being the full implicitly of the proposed method and another being to get the a priori estimates independent of both $N$ and $\delta t$ for (4). These a priori estimates are not trivial due to the loss of the maximum principle for the analytic semigroup. To overcome these difficulties, we make use of some Sobolev–Gagliard–Nirenberg inequalities and the equivalence between a random PDE and Eq. (1). Meanwhile, the approach to the weak convergence analysis is also available for other numerical methods in any finite time.

Based on the weak error analysis, we also study whether the proposed method (4) can be applied to approximating the invariant measure of the considered model. In many physical applications, the approximation of the longtime behaviors like the invariant measure is of fundamental importance when the invariant measure of the original system is unknown (see e.g. [20,23]). For the results about approximating the invariant measures of some SPDEs by using numerical approximations, we refer to [5,10,23] and references therein. For instance, the authors in [23] consider the invariant measure of a full discretization and study the error of between the invariant measure of the semi-discretization and that of the full discretization for the stochastic nonlinear Schrödinger equation. The authors in [5] investigate the error between the invariant measure for the temporal semi-implicit method and the original invariant measure for parabolic SPDE with Lipschitz and regular coefficients. However, it is still not well known how to numerically approximate the invariant measure of non-global Lipschitz parabolic SPDEs and to estimate the error of these invariant measures.

To answer these questions, we deduce the time-independent weak convergence analysis of the proposed full discretization (4), which is more involved to get the time-independent weak error estimate than the finite time case. The main difficulties lie on rigorous a priori estimations of numerical solutions, the regularity estimates of Kolmogorov equation with respect to the spectral Galerkin approximation and getting rid of singular terms. Under the strong dissipative condition $\lambda_1 > \lambda_F$, the time-independent regularity estimates of Kolmogorov equation are obtained by using a decay estimate directly. Under the non-degenerate condition, we first study the $V$-uniform ergodicity of the invariant measure of the spectral Galerkin approximation. Then inspired by [5], we prove the time-independent regularity estimates of the corresponding Kolmogorov equation by using the Bismut–Elworthy–Li formula. Finally, we prove the following result.

**Theorem 2** Let Assumptions 1-3 hold with $\beta \in (0, 1], \gamma \in (0, \beta), X_0 \in H^{\frac{d}{2}+\epsilon}$ with an arbitrary small positive constant $\epsilon$ and $\delta t_0 \in (0, 1 \wedge \frac{1}{(-\lambda_1+\lambda_F)\vee 0})$. In addition, under Assumption 4 or 5 for any $\phi \in C^2_b(H)$, there exists a constant $C(X_0, Q)$ such that for $\delta t \in (0, \delta t_0], K \geq 2$ and $N \in \mathbb{N}^+$,

$$|\mathbb{E}[\phi(X(K\delta t, X_0)) - \phi(X^{N}_K(X^N_0))]| \leq C(X_0, Q)(1 + (K\delta t)^{-\gamma})(\delta t^\gamma + \lambda_N^{-\gamma}).$$
Then, as a result of Theorem 2, we obtain the error of approximating the invariant measure through the weak convergence approach by using the proposed full discretization (4) and the exponential ergodicity of Eq. (1).

**Corollary 1** Under the same conditions of Theorem 2, for any \( \phi \in \mathcal{C}_b^2(\mathbb{H}) \), there exist constants \( c > 0, C(X_0, Q) > 0 \) such that for any large \( K, \delta t \in (0, \delta t_0] \) and \( N \in \mathbb{N}^+ \),

\[
\left| \mathbb{E} \left[ \phi(X_N^N(X_0^N)) - \int_{\mathbb{H}} \phi d\mu \right] \right| \leq C(X_0, Q)(\delta t^\gamma + \lambda_N^{-\gamma} + e^{-cK\delta t}),
\]

where \( \mu \) is the unique invariant measure of Eq. (1). Furthermore, if \( \mu^{N,\delta t} \) is an ergodic invariant measure of the numerical solution \( \{X_k^N\}_{k \in \mathbb{N}^+} \), we have

\[
\left| \mathbb{E} \left[ \int_{\mathbb{H}} \phi d\mu^{N,\delta t} - \int_{\mathbb{H}} \phi d\mu \right] \right| \leq C(X_0, Q)(\delta t^\gamma + \lambda_N^{-\gamma}).
\]

To the best of our knowledge, this is the first result on the time-independent weak error analysis and the error between the numerical invariant measure and the original invariant measure for non-globally Lipschitz parabolic SPDEs, especially for Eq. (1).

The outline of this paper is as follows. Section 2 is devoted to listing some notations and assumptions, and giving both the regularity and a priori estimates of numerical solutions, as well as the a priori estimates of semi-discretized stochastic convolution. In Section 3, we use the splitting based regularizing procedure and give an approach to studying the weak convergence rate of full discretization by Malliavin calculus. In Section 4 we show the regularity estimates of Kolmogorov equation by the spectral Galerkin approximation, deduce the time-independent weak error analysis and approximate the invariant measure of Eq. (1) through the weak convergence approach by the proposed method. Finally, numerical tests are shown to verify our theoretical results.

## 2 Preliminaries and full discretization

In this section, we give some basic assumptions and preliminaries, and introduce the spatial spectral Galerkin method and the implicit Euler type full discretization. Furthermore, we show both the strong convergence and some a priori estimates for the proposed method.

### 2.1 Preliminaries and assumptions

Let \((\mathcal{H}, \cdot, \cdot_\mathcal{H})\) and \((\tilde{\mathcal{H}}, \|\cdot\|_{\tilde{\mathcal{H}}})\) be separable Hilbert spaces. We denote \(\mathcal{C}_k^b(\mathcal{H}, \mathbb{R})\), \(k \in \mathbb{N}^+\), the space of \(k\) times continuous differentiable functionals from \(\mathcal{H}\) to \(\mathbb{R}\).
with bounded derivatives up to order $k$, and $B_b(\mathcal{H}, \mathbb{R})$ the space of measurable and bounded functionals. Define

$$
\|\phi\|_0 := \sup_{x \in \mathcal{H}} |\phi(x)|, \quad |\phi|_1 := \sup_{x \in \mathcal{H}} |D \phi(x)|_{\mathcal{H}}, \quad |\phi|_2 := \sup_{x \in \mathcal{H}} |D^2 \phi(x)|_{\mathcal{L}(\mathcal{H})}
$$

with $D^k \phi$, $k = 1, 2$, being the $k$-th derivative of $\phi$, and $\mathcal{L}(\mathcal{H})$ being the space of linear operators from $\mathcal{H}$ into itself. Denote by $\mathcal{L}_2(\mathcal{H}, \tilde{\mathcal{H}})$ the space of Hilbert–Schmidt operators from $\mathcal{H}$ into $\tilde{\mathcal{H}}$, equipped with the usual norm given by \( \| \cdot \|_{\mathcal{L}_2(\mathcal{H}, \tilde{\mathcal{H}})} = (\sum_{k \in \mathbb{N}^+} \| f_k \|_{H}^2)^{1/2} \), where $f_k$, $k \in \mathbb{N}^+$, is any orthonormal basis of $\mathcal{H}$. Given a Banach space $(\mathcal{E}, \| \cdot \|_\mathcal{E})$, we denote by $\gamma(\mathcal{H}, \mathcal{E})$ the space of $\gamma$-radonifying operators endowed with the norm $\| \cdot \|_{\gamma(\mathcal{H}, \mathcal{E})} = (\mathbb{E}[\| \cdot \|_\mathcal{E}]^2)^{1/2}$, where $(\gamma_k)_{k \in \mathbb{N}^+}$ is a sequence of independent $\mathcal{N}(0, 1)$-random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Moreover, we define $\mathcal{H} := L^2(\mathcal{O})$ with the norm $\| \cdot \|$ and the inner product $(\cdot, \cdot)$, and denote $\mathcal{C}^k_b(\mathcal{H}) := \mathcal{L}_2(\mathcal{H}, U_0)$ with $U_0 = Q(\mathcal{H})$, where $Q \in \mathcal{L}(\mathcal{H})$ is self-adjoint and positive. We also use the notation $\mathcal{C}^k_b(\mathcal{H}) := \mathcal{C}^k_b(\mathcal{H}, \mathbb{R})$, $k \in \mathbb{N}^+$. Meanwhile, let $\mathcal{L} := \mathcal{L}(\mathcal{H})$, $E := \mathcal{L}(\mathcal{O}; \mathbb{R})$ $L^q := L^q(\mathcal{O}; \mathbb{R})$, $q \geq 1$ with the norm $\| \cdot \|_{L^q}$, and $H^k$ and $H^k_0, k \in \mathbb{N}^+$ as the usual Sobolev spaces equipped with usual norms.

Let $\mathcal{I} : L^2([0, T]: U_0) \to L^2(\Omega)$ be an isonormal process, i.e., $\mathcal{I}(\psi)$ is the centered Gaussian random variable, for any $\psi \in L^2([0, T]; U_0)$, and $\mathbb{E}[\mathcal{I}(\psi_1)\mathcal{I}(\psi_2)] = (\mathbb{E}[\psi_1\psi_2])_{L^2([0, T]; U_0)}$, for any $\psi_1, \psi_2 \in L^2([0, T]; U_0)$. We denote the family of smooth real-valued cylindrical random variables by

$$
\mathcal{S} = \left\{ \mathcal{X} = g(\mathcal{I}(\psi_1), \cdots, \mathcal{I}(\psi_n)) : g \in C^\infty_p(\mathbb{R}^n), \psi_j \in L^2([0, T]; U_0), j = 1, \cdots, n \right\},
$$

where $C^\infty_p(\mathbb{R}^n)$ is the space of all real-valued $C^\infty$ functions on $\mathbb{R}^n$ with polynomial growth, and the family of smooth cylindrical $\mathcal{H}$-valued random variables by

$$
\mathcal{S}(\mathcal{H}) = \left\{ G = \sum_{i=1}^M \mathcal{X}_i \otimes h_i : \mathcal{X}_i \in \mathcal{S}, h_i \in \mathcal{H}, M \geq 1 \right\}.
$$

For $G = \sum_{i=1}^M g_i(\mathcal{I}(\psi_1), \cdots, \mathcal{I}(\psi_n)) \otimes h_i$, define its Malliavin derivative

$$
\mathcal{D}_s G = \sum_{i=1}^M \sum_{j=1}^n \partial_j g_i(\mathcal{I}(\psi_1), \cdots, \mathcal{I}(\psi_n)) \otimes (h_i \otimes \psi_j(s)).
$$

Let $\mathcal{D}^{1,2}(\mathcal{H})$ be the closure of $\mathcal{S}(\mathcal{H})$ with respect to Malliavin derivative equipped with the norm

$$
\| G \|_{\mathcal{D}^{1,2}(\mathcal{H})} = \left( \mathbb{E}[\| G \|^2] + \mathbb{E} \int_0^T \| \mathcal{D}_s G \|^2 ds \right)^{1/2}.
$$
Then the Malliavin integration by parts formula holds (see, e.g., [18, Section 2]). Namely, for any random variable $G \in \mathcal{D}^{1,2}(\mathbb{H})$ and any predictable process $\Theta \in L^2([0,T]; L^0_2)$, we have
\[
\mathbb{E} \left[ \left\langle \int_0^T \Theta(t) dW(t), G \right\rangle \right] = \mathbb{E} \left[ \int_0^T \left\langle \Theta(t), D_t G \right\rangle_{L^2_0} dt \right].
\] (2)

This property is the key to analyzing the weak convergent rates in Section 3 and 4. Additionally, the Malliavin derivative satisfies the chain rule, that is, for $\sigma(G) \in \mathcal{D}^{1,2}(\mathbb{H})$,
\[
D_y t (\sigma(G)) = D^\sigma(G) D_t G, \quad y \in U_0, \quad G \in \mathcal{D}^{1,2}(\mathbb{H}),
\]
where $\sigma \in C^1_b(\mathbb{H}, \mathbb{H})$.

Furthermore, we use $c, C$ to denote generic constants, independent of $N$ and $\delta t$, which differ from one place to another. Unless otherwise specified, we always assume that $X_0$ is a deterministic function in $\mathbb{H}^{d+\epsilon}$, where $\epsilon > 0$ is a very small positive number. In addition, for the coefficients in (1), we give the following assumptions.

**Assumption 1** Let $\Omega := [0, L]^d, d \leq 3, L > 0$. Let $A : D(A) \subset \mathbb{H} \rightarrow \mathbb{H}$ be the Laplacian operator on $\Omega$ with the homogenous Dirichlet boundary condition, i.e., $A u = \Delta u, u \in D(A)$.

This assumption implies that the operator $A$ generates an analytic and contraction $C_0$-semigroup $S(t), t \geq 0$ in $\mathbb{H}$ and $L^q, q \geq 1$ and that the existence of the eigensystem $\{ \lambda_k, e_k \}_{k \in \mathbb{N}^+}$ of $\mathbb{H}$, such that $\lambda_k > 0$, $-A e_k = \lambda_k e_k$, $\lim_{k \rightarrow \infty} \lambda_k = \infty$ and $\sup_{k \in \mathbb{N}^+} \| e_k \|_E \leq C$. Let $\mathbb{H}^r$ be the Banach space equipped with the norm $\| \cdot \|_r := \| (\cdot)^r \|_E$ for the fractional power $(-A)^r, r \geq 0$. We also remark that Assumption 4 can be extended to the case that $A$ is a second order elliptic operator on a regular domain and a part of $A$ in $E$ generates an analytic semigroup in $E$. This case is more complicated and will be investigated further.

**Assumption 2** Let $W(t)$ be a Wiener process with covariance operator $Q$, where $Q$ is a bounded, linear, self-adjoint and positive definite operator on $\mathbb{H}$ and satisfies $\| A^{\frac{\beta}{2}} \|_2 < \infty$ with $0 < \beta \leq 2$. Assume in addition that $\beta > \frac{d}{2}$ or $A$ commutes with $Q$.

In the case of investigating the strong error estimate, the additional condition that $\beta > \frac{d}{2}$ or $A$ commutes with $Q$ can be weaken. The additional assumption are needed to ensure a priori estimates of exact and numerical solutions when studying the weak convergence rate of numerical methods. In order to get the time-independent error estimate and to approximate the invariant measure, some dissipative condition and non-degenerate condition are proposed in Section 4.
Assumption 3 Let $f$ be a cubic polynomial with $f(\xi) = -a_3\xi^3 + a_2\xi^2 + a_1\xi + a_0$, $a_i \in \mathbb{R}$, $i = 0, 1, 2, 3$, $a_3 > 0$ and let $F : L^0 \rightarrow \mathbb{H}$ be the Nemytskii operator defined by $F(X)(\xi) = f(X(\xi))$.

Denote $\lambda_F = \sup_{\xi \in \mathbb{R}} f'(\xi)$. The above assumption ensures that $F$ satisfies

$$
(F(u) - F(v), u - v) \leq \lambda_F \|u - v\|^2,
$$

$$
\|F(u) - F(v)\| \leq C_f (1 + \|u\|_E^2 + \|v\|_E^2) \|u - v\|,
$$

for $C_f > 0$. In this case, Eq. (1) corresponds to the stochastic Allen–Cahn equation or stochastic Ginzburg–Landau equation. Moreover, based on Assumption 3, the solvability of the proposed method (4) is obtained if the time stepsize $\delta t$ is small. Indeed, if $\delta t_0 < 1 / (\lambda_F - \lambda_1) > 0$, then the proposed method has a unique solution as $\delta t \in (0, \delta t_0]$.

2.2 Full discretization

Now we are in the position to give both the semi-discretization and the full discretization for (1). In the sequel, we let $\delta t \in (0, \delta t_0]$, $\delta t_0 \in (0, 1 / (\lambda_F - \lambda_1))$, $t_k = k\delta t$, $k \in \mathbb{N}^+$ and $N \in \mathbb{N}^+$ for convenience.

Denote the spectral Galerkin projection by $P^N$. Using spectral Galerkin methods in space, we get the following semi-discretization

$$
\begin{align*}
\frac{dX(t)}{dt} &= AX(t) + P^N(F(X(t))) + \delta tP^N dW(t), \\
&= AX^N(t) + \delta tP^N(F(X^N(t))) + \delta tP^N dW(t).
\end{align*}
$$

(3)

For the weak convergence analysis for a finite interval $[0, T]$, we choose $\delta t \in (0, \delta t_0]$, $K \in \mathbb{N}^+$. For the time-independent weak convergence analysis, we fixed the stepsize $\delta t$ and let $K \in \mathbb{N}^+$. Based on the implicit Euler method, we obtain the full discretization

$$
X^N_{k+1} = X^N_k + \delta tAX^N_{k+1} + \delta tP^N(F(X^N_{k+1})) + \delta tP^N(W(t))
$$

where $X^N_0 = P^N X_0$ and $P^N W_k = P^N(W((k+1)\delta t) - W(k\delta t))$. Here, for the sake of simplicity, we omit the dependence on $X_0$ and denote $X^N_k := X^N_k(X^N_0)$, $k \in \mathbb{N}^+$. Denoting $\delta t := (I - A\delta t)^{-1}$, then the full discretization can be rewritten as

$$
X^N_{k+1} = S_{\delta t}X^N_k + \delta tS_{\delta t}P^N(F(X^N_{k+1})) + \delta tP^N \delta W_k,
$$

(4)

which is equivalent to

$$
Y^N_{k+1} = Y^N_k + \delta tAY^N_{k+1} + \delta tP^N(F(Y^N_{k+1}) + Z^N_{k+1}),
$$

(5)

where

$$
Z^N_{k+1} = \sum_{j=0}^{k} S_{\delta t}^{j+1} P^N \delta W_j.
$$

The notation $[t]_{\delta t} := \max\{0, \delta t, \cdots, k\delta t, \cdots\} \cap [0, t]$ is used frequently.
Let Assumptions 1-3 hold with $\beta > \frac{d}{2}$ and $X_0 \in \mathbb{H}^\beta$, or with $A$ commutes with $Q$ and $X_0 \in \mathbb{H}^\beta \cap E$. It can be shown that for any $T > 0$,

$$\sup_{s \in [0,T]} \mathbb{E}\left[\|X(s)\|_E^p\right] + \sup_{s \in [0,T]} \mathbb{E}\left[\|X(s)\|_{H^\beta}^p\right] \leq C(X_0, Q, p)$$

and

$$\mathbb{E}\left[\|X(t) - X(s)\|_E^p\right] \leq C(X_0, Q, p)(t - s)^{\frac{\beta p}{2}},$$

where $p \geq 1$, $0 \leq s < t \leq T$. We have the following strong error estimation, its proof is similar to the proofs of [29, Theorem 4.1] and [14, Theorem 3.1].

**Lemma 1** Let $X_0 \in \mathbb{H}^\beta$, $T = K\delta t$, $K \in \mathbb{N}^+$ and $N \in \mathbb{N}^+$. Under the Assumptions 1-3, the full discretization is strongly convergent and satisfies

$$\sup_{k \leq K} \left\| X_N^k - X(t_k) \right\|_{L^p(\Omega; \mathbb{H})} \leq C(T, X_0, Q) (\delta t^\frac{\beta}{2} + \lambda_N^\frac{-\beta}{2}).$$

**Remark 1** Similar to Lemma 1, for any $T = K\delta t > 0$, $k \leq K$, $k \in \mathbb{N}$ and $N \in \mathbb{N}^+$, we have

$$\sup_{k \leq K} \left\| X_N^k - X_N^k(t_k) \right\|_{L^p(\Omega; \mathbb{H})} \leq C(T, X_0, Q) \delta t^\frac{\beta}{2},$$

and

$$\sup_{t \in [0,T]} \left\| X_N^k(t) - X(t) \right\|_{L^p(\Omega; \mathbb{H})} \leq C(T, X_0, Q) \lambda_N^\frac{-\beta}{2}.$$

Lemma 1 also yields a result on the weak convergence rate. Combining with weak convergence result in Theorem 1, we immediately have the weak convergence rate is $O\left(\delta t^{(\beta-\epsilon_1)\frac{\beta}{2} + \lambda_N^{-(\beta-\epsilon_1)\frac{\beta}{2}}})\right)$, for any small positive number $\epsilon_1$.

Thus in Sections 2-4, we mainly focus on weak convergence rates of numerical methods in the case $\beta \in (0, 1]$.

### 2.3 A priori estimate of the full discretization

In this subsection, our purpose is to give the a time-independent priori estimate of the proposed numerical method. Indeed, it suffices to show a priori estimates of $Y_N^k$ and $Z_N^k$, $k \in \mathbb{N}^+$. We have

**Lemma 2** Under the Assumptions 1-3, for $\gamma \in (0, \beta]$, there exist some positive constants $C(X_0, Q, p)$ and $C(\gamma, X_0, Q, p)$ such that

$$\sup_{t \geq 0} \mathbb{E}\left[\|X_N^k(t, X_0)\|_{H^\gamma}^p\right] \leq C(X_0, Q, p) \quad \text{and} \quad \sup_{t \geq 0} \mathbb{E}\left[\|X_N^k(t, X_0)\|_E^p\right] \leq C(X_0, Q, p).$$

Lemma 2 is about the a priori estimate of the solution $X_N^k$ for the spectral Galerkin method, which is very useful in Section 3. Its proof is similar to that of the numerical solution, see Lemmas and 3.
Lemma 3  Let Assumptions 1-2 hold and \( p \geq 1 \). There exists a constant \( C(Q,p) \) such that the discretized stochastic convolution \( Z_N^k, k \in \mathbb{N}^+ \) satisfies

\[
\sup_{k \in \mathbb{N}^+} \| Z_N^k \|_{L^p(\Omega;E)} \leq C(Q,p).
\]

Proof Under the condition that \( A \) commutes with \( Q \), we apply the Burkholder inequality and get

\[
\| Z_{k+1}^N \|_{L^p(\Omega;E)} = \left\| \sum_{j=0}^k S_{\delta t}^{k+1-j} P^N \delta W_j \right\|_{L^p(\Omega;E)} \\
\leq \left\| S_{\delta t}^{k+1-\lfloor \cdot \rfloor} \right\|_{L^p(\Omega;L^2([0,t_{k+1}];\gamma(H,E)))} \\
\leq C \sqrt{\sum_{i \in \mathbb{N}^+} \sum_{j=0}^k \left( \frac{1}{1 + \lambda_i \delta t} \right)^{2(k+1-j)}} q_i \delta t \\
\leq C \sqrt{\sum_{i \in \mathbb{N}^+} \lambda_i \left( 1 - \left( \frac{1}{1 + \lambda_i \delta t} \right)^{2(k+1)} \right) q_i} \\
\leq C \sqrt{\sum_{i \in \mathbb{N}^+} \lambda_i^{\beta-1} q_i \sup_{i \in \mathbb{N}^+} \frac{1}{\lambda_i^2 (2 + \lambda_i \delta t)} \leq C(Q,p).}
\]

If \( \beta > \frac{d}{2} \), it follows from the Sobolev embedding theorem and the Burkholder–Davis–Gundy inequality that for small \( \epsilon \),

\[
\| Z_{k+1}^N \|_{L^p(\Omega;E)} \leq \left\| \sum_{j=0}^k S_{\delta t}^{k+1-j} P^N \delta W_j \right\|_{L^p(\Omega;\mathcal{H}^{d+1})} \\
\leq C \sqrt{\sum_{j=0}^k \left\| A^{\frac{d+2}{2}} S_{\delta t}^{k+1-j} Q^j \right\|_{L^2}^2 \delta t} \\
\leq C \sqrt{\sum_{j=0}^k \left\| A^{\frac{d+2}{2}+\beta-1} S_{\delta t}^{k+1-j} \right\|_{L^2}^2 \| A^{\frac{d-1}{2}} \|_{L^2}^2 \delta t} \\
\leq C \sqrt{\sum_{j=0}^k \frac{1}{((k+1-j)\delta t)^{\frac{d+1}{2}+\beta-1} (1 + \lambda_i \delta t)^{(k+1-j)(1+\beta-\frac{d-1}{2})-\epsilon} \delta t} \\
\leq C \int_0^\infty t^{-\frac{d+1}{2}+\beta-\epsilon-1} \frac{1}{(1 + \lambda_i \delta t)^{(1+\beta-\frac{d-1}{2}) \delta t}} dt \leq C(Q,p).
\]

Apart from the a priori estimate of \( Z_N^k \), we also need the uniform bound of \( Y_N^k \) to control \( X_N^k, k \in \mathbb{N}^+ \) which is introduced in the following lemma.
Lemma 4 Under the Assumptions 1-3, there exists a positive constant $C(Q, X_0, p)$ such that the solution $Y_{k}^{N}$ of the random PDE (5) satisfies

$$\sup_{k \in \mathbb{N}^+} \| Y_{k}^{N} \|_{L^p(\Omega, E)} \leq C(Q, X_0, p).$$

Proof First, by multiplying $Y_{k+1}^{N}$ on both sides of Eq. (5) and integrating over $O$, we have

$$\frac{1}{2} \| Y_{k+1}^{N} \|^2 \leq \frac{1}{2} \| Y_{k}^{N} \|^2 - \delta t \| \nabla Y_{k+1}^{N} \|^2 - (a_3 - \epsilon) \delta t \| Y_{k+1}^{N} \|_{L^4}^4 + C(\epsilon) \delta t (1 + \| Z_{k+1}^{N} \|_{L^2}^2).$$

The Gronwall inequality, together with the a priori estimate of $Z_{k}^{N}$ in Lemma 3, leads to

$$E \left[ \| Y_{k+1}^{N} \|^2 \right] \leq C(Q, X_0) \sum_{j=0}^{k} \frac{1}{(1 + 2 \lambda_1 \delta t)^{k+1-j}} \delta t \leq C(Q, X_0).$$

Next, we turn to estimate the a priori estimate in $E$ by the mild form of

$$Y_{k+1}^{N} = S_{\delta t} Y_{0}^{N} + \sum_{j=0}^{k} S_{\delta t}^{k+1-j} P^{N} F(Y_{j+1}^{N} + Z_{j+1}^{N}) \delta t.$$ 

The Sobolev embedding theorem and the smooth effect of $S_{\delta t}$ yield that

$$\| Y_{k+1}^{N} \|_{E} \leq \| Y_{0}^{N} \|_{E} + C \sum_{j=0}^{k} \| \nabla Y_{j+1}^{N} \|_{L^6}^6 \leq \| Y_{0}^{N} \|_{E} + C \sum_{j=0}^{k} \frac{1}{((k + 1 - j) \delta t)^{\frac{3}{2}}} \| \nabla F(Y_{j+1}^{N} + Z_{j+1}^{N}) \| \delta t.$$ 

The above estimate can be improved in $d = 1$ by using Gagliard–Nirenberg inequality $\| u \|_{L^6} \leq C \| \nabla u \|_{L^2}^{\frac{3}{2}} \| u \|_{L^2}^{\frac{3}{2}}$ and the estimation of $\sum_{j=0}^{k} \| \nabla Y_{j+1}^{N} \|_{L^2}^2$ (see the proof [13] Proposition 3.1). For higher dimension $d = 2, 3$, case, we need to give the a priori estimate of $\| \nabla Y_{k+1}^{N} \|$. Multiplying the term $-A Y_{k+1}^{N}$ on both
sides of Eq. (3) and integrating over \( \mathcal{O} \), we obtain

\[
\| \nabla Y_{k+1}^N \|^2 \leq \| \nabla Y_k^N \|^2 - 2\delta t \| AY_{k+1}^N \|^2 + 2\delta t \langle \nabla F(Y_{k+1}^N + Z_{k+1}^N), \nabla Y_{k+1}^N \rangle \\
\leq \| \nabla Y_k^N \|^2 - (2 - \epsilon)\delta t \| AY_{k+1}^N \|^2 \\
+ C(\epsilon)\delta t (\| Y_k^N \|^2 + \| Z_{k+1}^N \|^2 + \| Z_{k+1}^N \|^6_{L^6}) \\
- 2\epsilon \delta t (\| Y_k^N \| \nabla Y_k^N + Y_k^N \nabla Y_{k+1}^N) + C\delta t (\| Y_{k+1}^N \|^2 + AY_{k+1}^N) \\
+ C\delta t (\| Y_{k+1}^N \|^2 + AY_{k+1}^N) + C\delta t (\| Y_{k+1}^N + Z_{k+1}^N \|^2, AY_{k+1}^N) \\
\leq \| \nabla Y_k^N \|^2 + C(\epsilon)\delta t (1 + \| Y_{k+1}^N \|^2 + \| Z_{k+1}^N \|^6_{E} + \| Y_{k+1}^N \|^2 \| Z_{k+1}^N \|^2_{L^2}) \\
- (2 - 2\epsilon)\delta t \| AY_{k+1}^N \|^2 + C\delta t \| Z_{k+1}^N \|^2_{E} \| Y_{k+1}^N \|^4_{L^6}.
\]

The Gagliardo–Nirenberg–Sobolev inequality yields that

\[
\| \nabla Y_{k+1}^N \|^2 \leq \| \nabla Y_k^N \|^2 + C(\epsilon)\delta t (1 + \| Y_{k+1}^N \|^2 + \| Z_{k+1}^N \|^6_{E} + \| Y_{k+1}^N \|^2 \| Z_{k+1}^N \|^2_{E}) \\
- (2 - 2\epsilon)\delta t \| AY_{k+1}^N \|^2 + C\delta t \| Z_{k+1}^N \|^2_{E} \| Y_{k+1}^N \|^4_{L^6} \\
\leq \| \nabla Y_k^N \|^2 + C(\epsilon)\delta t (1 + \| Y_{k+1}^N \|^2 + \| Z_{k+1}^N \|^6_{E} + \| Y_{k+1}^N \|^2 \| Z_{k+1}^N \|^2_{E}) \\
- (2 - 3\epsilon)\delta t \| AY_{k+1}^N \|^2 + C(\epsilon)\delta t \| Z_{k+1}^N \|^2_{E} \| Y_{k+1}^N \|^6_{L^6} + 1 - \frac{1 - 2\epsilon}{4}.
\]

Combining the estimations of \( Y_{k+1}^N \) and \( \nabla Y_{k+1}^N \) with the equivalence of the norm in \( H_0^1 \) and \( H^2 \), we obtain

\[
\| Y_{k+1}^N \|^2_{H_0^1} \leq \| Y_k^N \|^2_{H_0^1} + (1 - 3\epsilon)\delta t \| Y_{k+1}^N \|^2_{E} + C(\epsilon)\delta t (1 + \| Y_{k+1}^N \|^2 + \| Z_{k+1}^N \|^6_{E}) \\
+ \| Y_{k+1}^N \|^2 \| Z_{k+1}^N \|^2_{E} + C\delta t \| Z_{k+1}^N \|^2_{E} \| Y_{k+1}^N \|^4_{E} + \| Y_{k+1}^N \|^2 \| Z_{k+1}^N \|^2_{E} + C\delta t \| Z_{k+1}^N \|^2_{E} \| Y_{k+1}^N \|^6_{E} \\
\leq \| Y_k^N \|^2_{H_0^1} + (1 - 3\epsilon)\delta t \| Y_{k+1}^N \|^2_{E} + C(\epsilon)\delta t (\| Y_{k+1}^N \|^2 + \| Z_{k+1}^N \|^2 + \| Z_{k+1}^N \|^6_{E}) \\
+ \| Y_{k+1}^N \|^2 \| Z_{k+1}^N \|^2_{E} + C\delta t \| Z_{k+1}^N \|^2_{E} \| Y_{k+1}^N \|^6_{E}.
\]

By using Gronwall’s inequality and then taking \( p \)-th moment on both sides, combining with the a priori estimate of \( \| Z_{k+1}^N \|^2_{E} \) in Lemma 3 and \( \| Y_{k+1}^N \| \), we complete the proof.

By more refined estimates, one can strengthen the results of Lemmas 3 and 4 and obtain the following result.

**Corollary 2** Under Assumptions 3, 4, for \( p \geq 1 \), there exist a constant \( C(Q, X_0, p) \) such that

\[
\left\| \sup_{k \in \mathbb{N}^+} \| Y_{k+1}^N \|^p_{E} \right\|_{L^p(\Omega; \mathbb{R})} \leq C(Q, X_0, p),
\]

Moreover, it holds that for some constant \( C'(Q, X_0, p) \),

\[
\sup_{k \in \mathbb{N}^+} \| X_{k+1}^N \|^p_{L^p(\Omega; \mathbb{R})} \leq C'(Q, X_0, p).
\]

Beyond the a priori estimate of \( X_{k+1}^N \), we also need the Malliavin regularity of the numerical method to control the stochastic integral error term in the weak convergence analysis in Sections 3 and 4.
Proposition 1. Let Assumptions [14] hold with \( \beta \in (0, 1] \). Then there exists a constant \( C(Q, X_0, p) \) such that for \( s < t_{k+1} \), \( k \in \mathbb{N} \), \( z \in U_0 \),

\[
\| D_s^z X_{k+1}^N \|_{L^p(\Omega, \mathcal{F})} \leq C(Q, X_0, p)(1 + \lambda_F \delta t)^{k+1-|s|}\lambda_s \left( \frac{1}{1 + \lambda_1 \delta t^{(k+1-|s|)N+1}} \right) \| - (A) \|_{A^1}^\beta \|z\|,
\]

and

\[
\| - (A) \|_{A^1}^\beta \|D_s^z X_{k+1}^N\|_{L^p(\Omega, \mathcal{F})} \leq C(Q, X_0, p)(1 + \lambda_F \delta t)^{k+1-|s|}\lambda_s \| - (A) \|_{A^1}^\beta \|z\|.
\]

Proof. For \( s \geq (k+1)\delta t \), \( z \in U_0 \), we have \( D_s^z X_{k+1}^N \) = 0. For \( 0 \leq s < k\delta t \leq T \), \( z \in U_0 \), we obtain

\[
D_s^z X_{k+1}^N = S_{\delta t} D_s^z X_k^N + \delta t S_{\delta t} P^N(DF(X_{k+1}^N) \cdot D_s^z X_{k+1}^N)
= D_s^z X_k^N + \delta t A D_s^z X_{k+1}^N + \delta t P^N(DF(X_{k+1}^N) \cdot D_s^z X_{k+1}^N).
\]

For \( k\delta t \leq s < (k+1)\delta t \), \( z \in U_0 \), we have

\[
D_s^z X_{k+1}^N = P^N S_{\delta t}^k z + \delta t S_{\delta t}^k P^N(DF(X_{k+1}^N) \cdot D_s^z X_{k+1}^N).
\]

From the above calculations, it follows that for \( k \geq [s]_{\delta t} \), \([s]_{\delta t} \delta t \leq s < ([s]_{\delta t} + 1)\delta t \),

\[
D_s^z X_{k+1}^N = P^N S_{\delta t}^k z + \delta t \sum_{j=[s]_{\delta t}}^{k} S_{\delta t}^{k-j} P^N(DF(X_{j+1}^N) \cdot D_s^z X_{j+1}^N).
\]

Then we show the regularity estimate of the Malliavin derivative \( D_s^z X_{k+1}^N \) by using similar arguments in [14] Proposition 4.2. Since in each step, \( D_s^z X_{k+1}^N \) can be viewed as

\[
D_s^z X_{k+1}^N = D_s^z X_k^N + \delta t A D_s^z X_{k+1}^N + \delta t P^N(DF(X_{k+1}^N) \cdot D_s^z X_{k+1}^N),
\]

it follows that

\[
\| D_s^z X_{k+1}^N \|^2 \leq \| D_s^z X_k^N \|^2 - \delta t \| \nabla D_s^z X_{k+1}^N \|^2 + \delta t (DF(X_{k+1}^N) \cdot D_s^z X_{k+1}^N, D_s^z X_{k+1}^N) \leq \| D_s^z X_k^N \|^2 + \delta t (-\lambda_1 + \lambda_F) \| D_s^z X_{k+1}^N \|^2,
\]

which implies that

\[
\| D_s^z X_{k+1}^N \|^2 \leq \frac{1 + \lambda_F \delta t}{1 + \lambda_1 \delta t} \| D_s^z X_k^N \|^2.
\]

Next we aim to estimate the regularity of \( D_s^z X_{k+1}^N \). By defining \( V_{z}^{N}(k+1, s) := D_s^z X_{k+1}^N - P^N S_{\delta t}^{k+1-|s|} z \), it follows that

\[
V_{z}^{N}(k+1, s) = V_{z}^{N}(k, s) - \delta t A V_{z}^{N}(k+1, s) + \delta t P^N(DF(X_{k+1}^N) \cdot V_{z}^{N}(k+1, s)) + \delta t P^N(DF(X_{k+1}^N) \cdot P^N S_{\delta t}^{k+1-|s|} z).
\]
By similar arguments in [14, Proposition 4.1], we obtain

\[
\|V_N^z(k + 1, s)\| \leq \delta t \sum_{j=[s]_\delta t}^k \left( \frac{1 + \lambda_F \delta t}{1 + \lambda_1 \delta t} \right)^{k+1-j} \|DF(X_{j+1}) \cdot S_{j+1} + (j + 1 - [s]_\delta t) \delta t \|^{-\alpha} \\
\times \frac{1}{(1 + \lambda_1 \delta t)^{(1-\alpha)(j+1-[s]_\delta t)}} \|(-A)^{-\alpha} z\|
\]

Taking expectation, combined with the smoothy effect of \(S_\delta t\) leads that

\[
\|\mathcal{D}_s X_N^{k+1}\| \leq \|V_N^z(k + 1, s)\|_{L^p(\Omega, \mathcal{H})} + \|P^N S_{\delta t}^{k+1-[s]_\delta t} z\| \\
\leq C(Q, p) \left( 1 + \sup_{j \in \mathbb{N}^+} \|X_j\|^2_{L^2(\Omega, \mathcal{H})} \right) \left( 1 + \lambda_F \delta t \right)^{k+1-[s]_\delta t} \\
\times \left( 1 + \frac{1}{t^{\alpha}_{k+1-[s]_\delta t}} \frac{1}{(1 + \lambda_1 \delta t)^{(k+1-[s]_\delta t)(1-\alpha)}} \right) \|(-A)^{-\alpha} z\|.
\]

Based on the above estimate, taking expectation and taking \(\alpha = \frac{1-\beta}{2}\), we finish the proof of the first desired estimate. Similar arguments in the proof of [14, Proposition 4.2] lead the second desired estimate. \(\square\)

**Remark 2** If in addition \(\lambda_1 \geq \lambda_F\), we have the following time-independent estimate

\[
\|\mathcal{D}_s X_N^{k+1}\|_{L^p(\Omega, L^2_\alpha)} \leq C(Q, X_0, p) \left( 1 + t^{\frac{\alpha}{1-\alpha}}_{k+1-[s]_\delta t} \frac{1}{(1 + \lambda_1 \delta t)^{(k+1-[s]_\delta t)(1-\alpha)}} \right) \|A^{\frac{\alpha}{1-\alpha}}\|_{L^2_\alpha},
\]

and

\[
\|A^{\frac{\alpha}{1-\alpha}} \mathcal{D}_s X_N^{k+1}\|_{L^p(\Omega, L^2)} \leq C(Q, X_0, p) \|A^{\frac{\alpha}{1-\alpha}}\|_{L^2},
\]

for some constant \(C(Q, X_0, p)\).

Based on the strong convergence, the a priori estimate and the Malliavin regularity of [4], we are able to deal with the weak convergence of the proposed method in the next section.
3 Weak convergence analysis of the full discretization

In this section, we aim to give the weak error analysis for the considered numerical method approximating Eq. (4). By the idea of [9,14], we also introduce the auxiliary regularized stochastic PDE and its corresponding Kolmogorov equation.

We introduce the auxiliary problem proposed in [9,14]
\[
\frac{dX(t)}{dt} = -AX(t)dt + \Psi_\delta(X(t))dt + dW(t), \quad X(0) = X_0,
\]
(6)
where \( \tau \) is regularizing parameter of this splitting approach, \( \Psi_\delta(x) := \frac{2\Phi(x) - x}{\delta}, \) \( t > 0 \) and \( \Psi_0(x) = F(x) \), \( \Phi_t \) is the phase flow of the differential equation \( dx(t) = f(x(t))dt, \quad x(0) = \xi \in \mathbb{R} \).

Next, we give the regularity estimate of Kolmogorov equation with respect to Eq. (6) shown in [14],
\[
\frac{\partial U(t,x)}{\partial t} = \langle Ax + \Psi_\delta(x), DU(t,x) \rangle + \frac{1}{2} tr[Q^T(t)D^2 U(t,x)Q(t)].
\]
(7)

**Lemma 5** For every \( \alpha, \theta, \gamma \in (0,1) \), \( \theta + \gamma < 1 \), \( 0 < \tau \leq \tau_0(f) \), there exist \( C(T,\tau_0) \) and \( C(T,\tau_0,\alpha) \) such that for \( \tau \in (0,\tau_0) \), \( x \in E, y,z \in H \) and \( t \in (0,\tau] \),
\[
|DU(t,x).y| \leq C(T,\tau_0,\alpha)(1 + |x|_E^\gamma)\|A^{-\alpha}y\|,
\]
(8)
\[
|D^2U(t,x).(y,z)| \leq C(T,\tau_0,\theta,\gamma)(1 + |x|_E^{\theta+\gamma})\|A^{-\theta}y\||A^{-\gamma}z||.\]
(9)

**Proof of Theorem 1** Based on the above estimates, now we give the weak error estimate of (4). The main idea of deducing the sharp weak convergence rate lies on the decomposition of \( \mathbb{E} \left[ \phi(X(T)) - \phi(X_N^\tau) \right] \) into \( \mathbb{E} \left[ \phi(X(T)) - \phi(X^\tau(T)) \right] \) and \( \mathbb{E} \left[ \phi(X(T)) - \phi(X_N^\tau) \right] \). The first term is estimated by Lemma 6 and possesses the strong convergence order 1 with respect to the parameter \( \tau \). The second term is controlled by Theorem 3. Combining these estimations together, we complete the proof of Theorem 1. \( \square \)

**Lemma 6** Let Assumptions 1,2 hold. Then the solution \( X^\tau \) of Eq. (6) is strongly convergent to the solution \( X \) of Eq. (1) and satisfies
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \|X(t)\|_E^p \right] \leq C(T,Q,p)(1 + \|X_0\|_E^p),
\]
\[
\left\| \sup_{t \in [0,T]} \|X(t) - X(t)\|_{L^p(\Omega)} \right\| \leq C(T,Q,X_0,p)\tau,
\]
for any \( p \geq 2 \).
Thus, we mainly focus on the estimate of $E\left[ \phi(X^\tau(T)) - \phi(X_N^\tau) \right]$. For convenience, we introduce the continuous interpolation of the implicit full discretization. Similar to [8], we define for $k \in \mathbb{N}^+$, $t \in [t_k, t_{k+1}]$, $\hat{X}^N(t_k) = X_k^N$,

$$d\hat{X}^N = (AS_{st}X_k^N + S_{st}P^NF(X_{k+1}^N))dt + S_{st}P^NdW(t).$$

**Theorem 3** Let Assumptions 1-3 hold, $\beta \in (0, 1]$, $\gamma \in (0, \beta)$ and $T = K\delta t$.

Then for any $\phi \in C_0^2(\mathbb{R})$, there exist $\tau$ and $C(X_0, T, Q)$ such that

$$\left| E\left[ \phi(X^\tau(T)) - \phi(X_N^\tau) \right] \right| \leq C(X_0, T, Q)\left( \delta t^\gamma + \lambda N^{-\gamma} \right).$$

**Proof** We decompose the error $E\left[ \phi(X^\tau(T)) - \phi(X_N^\tau) \right]$ into

$$E\left[ U^\tau(T, X_0) \right] - E\left[ U^\tau(0, X_N^\tau) \right] = \left( E\left[ U^\tau(T, X_0) \right] - E\left[ U^\tau(T, X_0^N) \right] \right) + \left( E\left[ U^\tau(T, X_0^N) \right] - E\left[ U^\tau(0, X_N^\tau) \right] \right).$$

The first term is controlled, by the regularity of $U^\tau$ in Lemma 5 as

$$\left| E\left[ U^\tau(T, X_0) \right] - E\left[ U^\tau(T, X_0^N) \right] \right| \leq \int_0^1 \left| E\left[ DU^\tau(T, \theta X_0 + (1 - \theta)X_0^N) \cdot (I - P^N)X_0 \right] \right| d\theta \leq C(1 + \|X_0\|_\mathcal{L}_2^2 + \|X_0^N\|_\mathcal{L}_2^2) \min(T^{-\alpha}(\lambda_N)^{-\alpha}\|X_0\|, (\lambda_N)^{-\frac{\alpha}{2}}\|X_0\|_\mathcal{H}^\alpha).$$

By using the Itô formula for Skorohod integrals (see e.g. [28, Chapter 3]), the Kolmogorov equation (7) and Malliavin integration by parts, the second term
is split into

\[ \mathbb{E}[U^*(T, X_0^N)] - \mathbb{E}[U^*(0, X_0^N)] \]
\[ = \sum_{k=0}^{K-1} \mathbb{E}[U^*(T - t_k, X_k^N)] - \mathbb{E}[U^*(T - t_{k+1}, X_{k+1}^N)] \]
\[ = \mathbb{E}[U^*(T, X_0^N)] - \mathbb{E}[U^*(T - \delta t, X_1^N)] \]
\[ - \sum_{k=1}^{K-1} \mathbb{E}\left[ \int_{t_k}^{t_{k+1}} \sum_{l \in \mathbb{N}^+} D^2U^*(T - t, \tilde{X}_N(t)) \cdot (\mathcal{D}_t \tilde{X}_N(t)Q^l_{\tau}e_l, S_{\delta t}Q^l_{\tau}e_l) \right] dt \]
\[ + \sum_{k=1}^{K-1} \left( \int_{t_k}^{t_{k+1}} \mathbb{E}\left[ (DU^*(T - t, \tilde{X}_N(t)), A\tilde{X}_N(t) - AS_{\delta t}X_k^N) \right] dt \right. \]
\[ + \int_{t_k}^{t_{k+1}} \sum_{j \in \mathbb{N}^+} \mathbb{E}\left[ D^2U^*(T - t, \tilde{X}_N(t)) \cdot \langle Q^j_{\tau}e_j, Q^j_{\tau}e_j \rangle - (S_{\delta t}P^NQ^{j+1}_{\tau}e_j, S_{\delta t}P^NQ^{j+1}_{\tau}e_j) \right] dt \right) \]
\[ := \mathbb{E}[U^*(T, X_0^N)] - \mathbb{E}[U^*(T - \delta t, X_1^N)] + \sum_{k=1}^{K-1} I^k_1 + I^k_2 + I^k_3 + I^k_4. \]

The Markov property of \( X_k^N \), the regularity estimate (S) of \( U^* \) in Lemma \( \Box \) and the a priori estimates of \( X^N \) in Lemma \( \Box \) and \( X_k^N \) in Corollary \( \Box \) lead that for \( 0 < \alpha < 1 \)

\[ \left| \mathbb{E}[U^*(T, X_0^N)] - \mathbb{E}[U^*(T - \delta t, X_1^N)] \right| \]
\[ = \left| \mathbb{E}[U^*(T - \delta t, X_N(\delta t)) - U^*(T - \delta t, X_N)] \right| \]
\[ \leq C(1 + \mathbb{E}[\|X_N(\delta t)\|^2_2] + \mathbb{E}[\|X_N\|^2_2])(1 + (T - \delta t)^{-\alpha})\delta t^\alpha \]
\[ \leq C(Q, X_0)(1 + (T - \delta t)^{-\alpha})\delta t^\alpha. \]

For the term \( I^k_1 \), the regularity of \( U^* \) and the a priori estimate of \( \tilde{X}_N \) yield that

\[ \left| \sum_{k=1}^{K-1} I^k_1 \right| \leq C \sum_{k=1}^{K-1} \int_{t_k}^{t_{k+1}} (T - t) \frac{\delta t}{(t - t_k)} \mathbb{E}\left[ (1 + \|\tilde{X}_N(t)\|^2_2)\|D_t \tilde{X}_N(t)\|^2_2\|S_{\delta t}\|_c(-A)^{\frac{\alpha - 1}{2}}\|_c \right] dt \]
\[ \leq C(T, Q, X_0)\delta t \sum_{k=1}^{K-1} \int_{t_k}^{t_{k+1}} (T - t) \frac{\delta t}{(t - t_k)} \frac{\delta t}{(t - t_k)} dt, \]

where we use the fact that for \( t_k \leq t \leq t_{k+1}, \)

\[ \mathcal{D}_s \tilde{X}_N(t) = S_{\delta t}D_sX_N(t_k) + (t - t_k)P^NS_{\delta t}DF(\tilde{X}_N(t_{k+1}))D_s\tilde{X}_N(t_k) + D_s \int_{t_k}^{t} S_{\delta t}dW(s) \]
\[ = (t - t_k)P^NS_{\delta t}DF(\tilde{X}_N(t_{k+1}))D_s\tilde{X}_N(t_k). \]
We estimate $I_2^k$, $I_3^k$ and $I_4^k$, $k \geq 1$ separately. The definition of $\hat{X}$ leads to

$$
I_2^k = \int_{t_k}^{t_{k+1}} E \left[ \langle DU^\tau(T - t, \hat{X}^N(t)), A(X_k^N - S_{\delta t} X_k^N) \rangle \right] dt 
+ \int_{t_k}^{t_{k+1}} E \left[ \langle DU^\tau(T - t, \hat{X}^N(t)), (t - t_k)(-A)^2 S_{\delta t} X_k^N \rangle \right] dt 
+ \int_{t_k}^{t_{k+1}} E \left[ \langle DU^\tau(T - t, \hat{X}^N(t)), (t - t_k)A S_{\delta t} P^N F(X_{k+1}^N) \rangle \right] dt 
+ \int_{t_k}^{t_{k+1}} E \left[ \langle DU^\tau(T - t, \hat{X}^N(t)), A \int_{t_k}^{t} S_{\delta t} dW(s) \rangle \right] dt 
:= I_{21}^k + I_{22}^k + I_{23}^k + I_{24}^k.
$$

It follows from the property $I - S_{\delta t} = -A \delta t (I - A \delta t)^{-1}$, the mild form of $X_k^N$, the a priori estimate of $\hat{X}$, and the regularity of $U^\tau$ and $S_{\delta t}$ that for $k \geq 1$, any small $\epsilon_1 > 0$,

$$
|I_{21}^k| \leq \left| \int_{t_k}^{t_{k+1}} E \left[ \langle DU^\tau(T - t, \hat{X}^N(t)), (-A)^2 \delta t S_{\delta t}^{k+1} X_0^N \rangle \right] dt \right| 
+ \left| \int_{t_k}^{t_{k+1}} \sum_{j=0}^{k-1} E \left[ \langle DU^\tau(T - t, \hat{X}^N(t)), (-A)^2 \delta t S_{\delta t}^{k+1-j} P^N F(X_{j+1}^N) \rangle \right] dt \right| 
+ \left| \int_{t_k}^{t_{k+1}} E \left[ \langle DU^\tau(T - t, \hat{X}^N(t)), (-A)^2 \delta t \sum_{j=0}^{k-1} S_{\delta t}^{k+1-j} P^N \delta W_j \rangle \right] dt \right| 
\leq C \delta t \int_{t_k}^{t_{k+1}} (T - t)^{-\alpha} E \left[ (1 + \|\hat{X}(t)\|_2^2)(-A)^{1-\epsilon_1} S_{\delta t}^k \|(-A)^{1-\alpha+\epsilon_1} S_{\delta t} \| X_0^N \| \right] dt 
\leq C T, X_0, Q(\delta t)^{\alpha-\epsilon_1} \int_{t_k}^{t_{k+1}} (T - t)^{-\alpha} (t_k)^{1+\epsilon_1} dt 
+ \left| \int_{t_k}^{t_{k+1}} E \left[ \langle DU^\tau(T - t, \hat{X}^N(t)), (-A)^2 \delta t \sum_{j=0}^{k-1} S_{\delta t}^{k+1-j} P^N \delta W_j \rangle \right] dt \right| 
\leq C T, X_0, Q(\delta t)^{\alpha-\epsilon_1} \int_{t_k}^{t_{k+1}} (T - t)^{-\alpha} (t_k)^{1+\epsilon_1} dt 
+ \left| \int_{t_k}^{t_{k+1}} E \left[ \langle DU^\tau(T - t, \hat{X}^N(t)), (-A)^2 \delta t \sum_{j=0}^{k-1} S_{\delta t}^{k+1-j} P^N \delta W_j \rangle \right] dt \right|. 
$$
By using Malliavin calculus integration by parts and Malliavin differentiability of \( \hat{X}^N \), we have

\[
\left| \int_{t_k}^{t_{k+1}} \mathbb{E}\left[ (DU^T(T - t, \hat{X}^N(t)), (-A)^2 \delta t \sum_{j=0}^{k-1} S_{jt}^{k+1-j} P^N \delta W_j) \right] dt \right|
\]

\[
\leq C \delta t \int_{t_k}^{t_{k+1}} \left\| D^{2T} (T - t, \hat{X}^N(t)) \cdot (D_s^{Q^\delta \epsilon_1} \hat{X}^N(t), (-A)^2 S_{jt}^{k+1-j} P^N Q^\delta \epsilon_1) \right\| ds dt
\]

\[
\leq C \delta t \int_{t_k}^{t_{k+1}} (T - t)^{-1+\epsilon} \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} \mathbb{E}\left[ (1 + \| \hat{X}^N(t) \|_{L_D^2}) (-A)^{\frac{k-j}{2}} D_s \hat{X}^N(t) \right] \left\| A^{1-\epsilon} S_{jt}^{k-j} \right\| A^{1+\epsilon} S_{jt} \left\| (-A)^{\frac{k-j}{2}} \right\| L_{2}^2 ds dt
\]

\[
\leq C(T, X_0, Q)(\delta t)^{1-\epsilon} \int_{t_k}^{t_{k+1}} (T - t)^{-1+\epsilon} \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} (t_k - [s]_\delta t)^{-1+\epsilon} ds dt.
\]

The above analysis leads to

\[
|I_{22}^n| \leq C(T, X_0, Q)(\delta t)^{\alpha-\epsilon} \int_{t_k}^{t_{k+1}} (T - t)^{-\alpha}(t_k)^{-1+\epsilon} dt
\]

\[
+ C(T, X_0, Q)(\delta t)^{1-2\epsilon} \int_{t_k}^{t_{k+1}} (T - t)^{1-\epsilon} \int_{t_k}^{t_{k+1}} (t_k - [s]_\delta t)^{-1+\epsilon} ds dt,
\]

for \( k \geq 1 \). Since the estimation for \( I_{22}^n \) for \( k \geq 1 \) is similar, we omit the procedures. For \( I_{23}^n \), by the regularity of \( DU^T \), we have

\[
|I_{23}^n| \leq C \delta t \int_{t_k}^{t_{k+1}} (T - t)^{-1+\epsilon} \mathbb{E}\left[ (1 + \| \hat{X}^N(t) \|_{L_D^2}) (A^{1+\epsilon} S_{\delta t} \| F(X^N_{k+1}) \|) dt
\]

\[
\leq C(T, Q, X_0)(\delta t)^{1-\epsilon} \int_{t_k}^{t_{k+1}} (T - t)^{-1+\epsilon} dt.
\]

Again using Malliavin calculus integration by parts yields that

\[
|I_{24}^n| = \left| \int_{t_k}^{t_{k+1}} \mathbb{E}\left[ (DU^T(T - t, \hat{X}^N(t)), A \int_{t_k}^{t} P^N S_{\delta t} dW(s)) \right] dt \right|
\]

\[
= \left| \int_{t_k}^{t_{k+1}} \int_{t_k}^{t} \mathbb{E}\left[ (D^{2U^T}(T - t, \hat{X}^N(t))D_s \hat{X}^N(t), P^N S_{\delta t} A^{1+\epsilon} A^{1+\epsilon} D_s \hat{X}^N(t), A^{1+\epsilon} P^N S_{\delta t} A^{1+\epsilon} L_{2}^2) \right] ds dt \right|
\]

\[
\leq C(T, Q, X_0)(\delta t)^{1-\epsilon} \int_{t_k}^{t_{k+1}} (T - t)^{-1+\epsilon} dt.
\]
Thus we have

\[ |I_3^k| \leq C(T,Q,X_0)(\delta t)^{2-2\epsilon_1} \left( \int_{t_k}^{t_{k+1}} (T - t)^{-1+\epsilon_1} \left( 1 + \int_0^t (s_{3t} - [s_{3t}])^{-1+\epsilon_1} ds \right) dt \right). \]

Now, we are in the position to control \( I_3^k \). It follows from the continuity of \( \Psi \) in [14, Lemma 4.2] and the regularity of \( DU^\tau \) that

\[
|I_3^k| \leq \left| \int_{t_k}^{t_{k+1}} \mathbb{E} \left[ \langle DU^\tau(T - t, \hat{X}^N(t)), \Psi_\epsilon(\hat{X}^N(t)) - F(\hat{X}^N(t)) \rangle \right] dt \right| \\
+ \left| \int_{t_k}^{t_{k+1}} \mathbb{E} \left[ \langle DU^\tau(T - t, \hat{X}^N(t)), (I - P^N)F(\hat{X}^N(t)) \rangle \right] dt \right| \\
+ \left| \int_{t_k}^{t_{k+1}} \mathbb{E} \left[ \langle DU^\tau(T - t, \hat{X}^N(t)), (I - S_{3t})P^N F(X_{k+1}^N) \rangle \right] dt \right| \\
+ \left| \int_{t_k}^{t_{k+1}} \mathbb{E} \left[ \langle DU^\tau(T - t, \hat{X}^N(t)), P^N(F(\hat{X}^N(t)) - F(X_{k+1}^N)) \rangle \right] dt \right| \\
\leq C \tau \int_{t_k}^{t_{k+1}} \mathbb{E} \left[ 1 + \|\hat{X}^N(t)\|_E^2 \right] dt + C((\lambda_N)^{-\alpha} + (\delta t)^\alpha) \int_{t_k}^{t_{k+1}} (T - t)^{-\alpha} \\
\mathbb{E} \left[ (1 + \|\hat{X}^N(t)\|_E^2)(1 + \|\hat{X}^N(t)\|_{L^8}^3 + \|X_{k+1}^N\|_{L^8}^3) \right] dt \\
+ \left| \int_{t_k}^{t_{k+1}} \mathbb{E} \left[ \langle DU^\tau(T - t, \hat{X}^N(t)), P^N(F(\hat{X}^N(t)) - F(X_{k+1}^N)) \rangle \right] dt \right|.
\]

Thus it suffices to estimate the last term in the above inequality. It follows from the Taylor expansion of \( F \), the regularity of \( DU^\tau \) and the a priori estimate of \( \hat{X}^N \) that

\[
\int_{t_k}^{t_{k+1}} \mathbb{E} \left[ \langle DU^\tau(T - t, \hat{X}^N(t)), P^N(F(\hat{X}^N(t)) - F(X_{k+1}^N)) \rangle \right] dt \\
\leq \int_{t_k}^{t_{k+1}} (T - t_{k+1}) \mathbb{E} \left[ \langle DU^\tau(T - t, \hat{X}^N(t)), P^N(DF(\hat{X}^N(t)) \cdot (AS_{3t}X_{k+1}^N)) \rangle \right] dt \\
+ \int_{t_k}^{t_{k+1}} (T - t_{k+1}) \mathbb{E} \left[ \langle DU^\tau(T - t, \hat{X}^N(t)), P^N(DF(\hat{X}^N(t)) \cdot (S_{3t}P^N F(X_{k+1}^N)) \rangle \right] dt \\
+ \int_{t_k}^{t_{k+1}} \mathbb{E} \left[ \langle DU^\tau(T - t, \hat{X}^N(t)), P^N(DF(\hat{X}^N(t)) \cdot (\int_0^t P^N S_{3t} dW(s))) \rangle \right] dt \\
+ \int_{t_k}^{t_{k+1}} \mathbb{E} \left[ \langle DU^\tau(T - t, \hat{X}^N(t)), P^N(\int_0^1 (1 - \theta) D^2 F(\theta \hat{X}^N(t) + (1 - \theta) X_{k+1}^N) \cdot (X^N(t) - X_{k+1}^N, X^N(t) - X_{k+1}^N)(d\theta)) \rangle \right] dt := I_{31}^k + I_{32}^k + I_{33}^k + I_{34}^k.
\]
The mild form of $X^N_{k+1}$ and Malliavin calculus integration by parts yield that

$$
|I_{31}^k| = \left| \int_{t_k}^{t_{k+1}} (t - t_{k+1}) \left( \mathbb{E} \left[ (DU^*(T - t, X^N(t)), P^N(DF(X^N(t)) \cdot (A_{S\delta t}^{k+1} X^N_0)) \right] \right) dt \\
+ \delta t \mathbb{E} \left[ (DU^*(T - t, \tilde{X}^N(t)), P^N(DF(\tilde{X}^N(t)) \cdot \left( \sum_{j=0}^{k-1} A_{S\delta t}^{k+1-j} P^N(X^N_{j+1})) \right) \right] \\
+ \mathbb{E} \left[ (DU^*(T - t, \tilde{X}^N(t)), P^N(DF(\tilde{X}^N(t)) \cdot \left( \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} A_{S\delta t}^{k+1-j} dW(s)) \right) \right) dt \right| \\
\leq C\delta t^2 \sup_{t \in [0, T]} \mathbb{E} \left[ (1 + \|\tilde{X}^N(t)\|^2_{L^2}) \|(\tilde{A})^{1-c_1} S_{\delta t}^k \|A_{S\delta t}^1 \|X_0\| \right] \\
+ C\delta t^2 \int_{t_k}^{t_{k+1}} \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} \sum_{l \in \mathbb{N}^+} \mathbb{E} \left[ D^2U^*(T - t, \tilde{X}^N(t) \cdot P^N((DF(\tilde{X}(t)) \cdot (DU^*(T - t, \tilde{X}^N(t)) \cdot P^N(D^2F(\tilde{X}(t)) \cdot (DU^*(T - t, \tilde{X}^N(t)) \cdot P^N(D^2F(\tilde{X}(t))).

By the a priori estimate of $\tilde{X}^N$ and Sobolev embedding theorem $E \hookrightarrow H^{k+\epsilon}$, we have

$$
|I_{31}^k| \leq C(T, Q, X_0)\delta t^{1-c_1} (t_k^{-1+c_1} \delta t + \delta t \sum_{j=0}^{k-1} t_k^{-1+c_1}) \\
+ C\delta t \int_{t_k}^{t_{k+1}} \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} \sum_{l \in \mathbb{N}^+} \mathbb{E} \left[ (1 + \|\tilde{X}(t)\|^2_{L^2}) \|A_{S\delta t}^{k+1-j} Q^2_{S\delta t} \|D^2_{S\delta t}^{c_1} \tilde{X}(t)\| \right] ds dt \\
+ C\delta t \int_{t_k}^{t_{k+1}} \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} \sum_{l \in \mathbb{N}^+} \mathbb{E} \left[ \|(\tilde{A})^{1-\epsilon} D^2U^*(T - t, \tilde{X}(t))\| \|(\tilde{A})^{-\epsilon} D^2F(\tilde{X}(t)).
$$
for $\eta > \frac{d}{2} + \frac{d}{2}\epsilon$. And by using the smoothy properties of $S_{st}$, the Malliavin regularity and the a priori estimate of $\tilde{X}(t)$, we have

$$|I_{31}^{k}| \leq C(T, Q, X_{0})\delta t^{1-\epsilon_{1}}(t_{k}^{-1+\epsilon_{1}}\delta t + \delta t^{k+1})$$

$$+ C(T, Q, X_{0})\delta t \int_{t_{k}}^{t_{k+1}} \sum_{j=0}^{k-1} \int_{t_{j}}^{t_{j+1}} E\left[\left\|(-A)^{\frac{3-\alpha}{2}}S_{st}^{k+1-j}(-A)^{\frac{3-\alpha}{2}}\right\|_{\mathbb{L}^{2}}\right] ds dt$$

$$+ C\delta t \int_{t_{k}}^{t_{k+1}} \sum_{j=0}^{k-1} \int_{t_{j}}^{t_{j+1}} (T-t)^{-\eta} E\left[\left(1 + \left\|\tilde{X}(t)\right\|_{E}^{\frac{3}{2}}\right)\left\|(-A)^{\frac{3-\alpha}{2}}S_{st}^{k+1-j}A^{-\frac{3-\alpha}{2}}\right\|_{\mathbb{L}^{2}}\right] ds dt$$

$$\leq C(T, Q, X_{0})\delta t^{1-\epsilon_{1}}(t_{k}^{-1+\epsilon_{1}}\delta t) + C(T, Q, X_{0})(\delta t)^{\beta-\epsilon_{1}} \int_{t_{k}}^{t_{k+1}} \int_{t_{0}}^{t_{k}} (t_{k} - [s]_{\delta t})^{-\frac{\alpha+1}{\epsilon_{1}}} ds dt$$

Similarly, we get

$$|I_{32}^{k}| \leq C\delta t \int_{t_{k}}^{t_{k+1}} E\left[(1 + \left\|\tilde{X}^{N}(t)\right\|_{E}) (1 + \left\|\tilde{X}^{N}(t)\right\|_{L^{0}})^{3}\right] dt \leq C(\delta t)^{2},$$

and

$$|I_{33}^{k}| \leq \left|\int_{t_{k}}^{t_{k+1}} \sum_{l \in \mathbb{N}^{+}} \left[ D^{2}U^{T}(T-t, \tilde{X}^{N}(t)) \cdot \left(P^{N}(DF(\tilde{X}^{N}(t))) \cdot (P^{N}S_{st}^{\beta}e_{1}) \cdot D_{2}^{N}q_{t}^{2}\hat{X}^{N}(t))\right) ds dt\right|$$

$$+ \left|\int_{t_{k}}^{t_{k+1}} \sum_{l \in \mathbb{N}^{+}} \left[ (D^{T}U^{T}(T-t, \tilde{X}^{N}(t)), P^{N}(D^{2}F(\tilde{X}^{N}(t))) \cdot (D_{2}^{N}q_{t}^{2}\hat{X}^{N}(t), P^{N}S_{st}^{\beta}e_{1}))\right] ds dt\right|$$

$$\leq C(T, Q, X_{0})(\delta t)^{2-\beta_{1}}.$$
It is concluded that

\[ |I_1^k| \leq C(T, Q, X_0)(\delta t)^{\beta - \epsilon_1} (1 + t_k^{1+\epsilon_1} \delta t + \int_{t_k}^{t_{k+1}} (T - t)^{-\eta} \int_0^t (t_k - [s]_\delta)^{-1 + \epsilon_1} dt). \]

For \( I_2^k \), by applying the regularity of \( D^2U^\tau, \) we obtain

\[ |I_2^k| \leq \left| \int_{t_k}^{t_{k+1}} \sum_{j \in \mathbb{N}^+} \mathbb{E} \left[ D^2U^\tau(T - t, \hat{X}^N(t)) \cdot \left( (I - P^N)Q^\frac{1}{2} e_j, (I + P^N)Q^\frac{1}{2} e_j \right) \right] dt \right| 
+ \left| \int_{t_k}^{t_{k+1}} \sum_{j \in \mathbb{N}^+} \mathbb{E} \left[ D^2U^\tau(T - t, \hat{X}^N(t)) \cdot \left( P^N(I - S_{\delta t})Q^\frac{1}{2} e_j, P^N(I + S_{\delta t})Q^\frac{1}{2} e_j \right) \right] dt \right| 
\leq C \int_{t_k}^{t_{k+1}} (T - t)^{-1+\epsilon_1} \mathbb{E} \left[ (1 + \|\hat{X}^N(t)\|_E^2) \|A^{\frac{1}{2}}\|_E^2 \|A^{-\frac{1}{2}} + \epsilon_1 (I - P^N)A^{-\frac{1}{2}}\|_E^2 \right] dt
+ C \int_{t_k}^{t_{k+1}} (T - t)^{-1+\epsilon_1} \mathbb{E} \left[ (1 + \|\hat{X}^N(t)\|_E^2) \|A^{\frac{1}{2}}\|_E^2 \|A^{-\frac{1}{2}} + \epsilon_1 (I - S_{\delta t})A^{-\frac{1}{2}}\|_E^2 \right] dt
\leq C(T, Q, X_0)((\delta t)^{\beta - \epsilon_1} + (\lambda_N)^{-\beta + \epsilon_1}) \int_{t_k}^{t_{k+1}} (T - t)^{-1+\epsilon_1} dt. \]

Combining all the estimations of \( I_1^k - I_2^k \) and summing up over \( k \), taking \( \tau = O(\delta t^2) \) or \( O(\lambda_N^{-\beta}) \), we finish the proof. \( \square \)

4 Time-independent weak convergence analysis and approximation of invariant measures

In this section, we consider whether the proposed method can approximate the invariant measure \( \mu \). Different from analyzing weak error in Section 2, we need to give the time-independent regularity estimates of the Kolmogorov equation, which are more complicated.

4.1 V-uniform ergodicity for the semi-discretization

To study the ergodic invariant measure numerically, we give more assumptions as following, that is, the dissipative condition in Assumption 4 and the non-degenerate condition in Assumption 5.

Assumption 4 Let \( \lambda_F < \lambda_1 \) and \( \|(-A)^{\frac{1}{2}}\|_{L^2_{\mathbb{R}}} < \infty, \beta \leq 1. \)

The above Assumption 4 immediately implies the following result on exponential convergence to equilibrium.

Proposition 2 Under Assumptions 4 and 5, there exist \( c > 0, C > 0 \) such that for any \( \phi \in C^1_b(\mathbb{R}), t \geq 0 \) and \( x_1, x_2 \in \mathbb{R}, \)

\[ |\mathbb{E}[\phi(X(t, x_1)) - \phi(X(t, x_2))]| \leq C|\phi|_1 e^{-\epsilon t} (1 + \|x_1\|^2 + \|x_2\|^2). \]
Remark 3 Based on the proof of Lemma 1 and Corollary 3, together with the strict dissipative condition $\lambda_F < \lambda_1$ in Assumption 4, the full discretization is strongly convergent and satisfies

$$\sup_{k \in \mathbb{N}^+} \left\| X_k^N - X(t_k) \right\|_{L^p(\mathbb{H})} \leq C(X_0, Q)(\delta t^\frac{\lambda}{2} + \lambda_N^\frac{\beta}{2}).$$

In some situations, it may occur that $\lambda_1 \leq \lambda_F$, which leads that Assumption 4 does not hold. In this case, we give the following non-degenerate condition.

Assumption 5 Let the covariance operator $Q$ be invertible and commute with $A$, $\|Q^{-\frac{1}{2}}(-A)^{-\frac{1}{2}}\| < \infty$ and $\|(-A)^{\frac{\beta}{2}}\|_{L_2^\infty} < \infty$, $\beta \leq 1$.

Under Assumption 5, the existence of the unique invariant measure $\mu$ for Eq. 1, as well as the invariant measure $\mu^N$ for the spatial Galerkin method, will be obtained according to Doob theorem for general $\lambda_F \in \mathbb{R}$. Besides the ergodicity of the invariant measure, we also need the following exponential convergence result in Proposition 3. Its proof lies heavily on the strong Feller property and $V$-uniform ergodicity of the Markov semi-group $P_t$ generated by the solution of Eq. 11 and Eq. 13 (see e.g. 10, 22). In fact, we first follow the proof of 22 to show the a priori estimate of a Lyapunov functional $V$ and to obtain the existence of the invariant measure. Then we prove that the Markov semigroup of the solution is strong Feller and irreducible, which implies the existence of the unique and ergodic invariant measure. By using again a priori estimate of a Lyapunov functional $V$, one can obtain the $V$-uniform ergodicity. In particular, we choose $\phi \in B_b(\mathbb{H})$ to get the exponential ergodicity of the invariant measure, which immediately implies Proposition 3.

Proposition 3 Under Assumptions 4, 5 and Assumptions 5, there exist $c > 0$, $C > 0$ such that for any $\phi \in B_b(\mathbb{H})$ and for $t \geq 0$, any $x_1, x_2 \in \mathbb{H}$ and $y_1^N, y_2^N \in P^N(\mathbb{H})$, we have

$$|E[\phi(X(t, x_1)) - E[\phi(X(t, x_2))]| \leq C\|\phi\|_0 e^{-ct}(1 + \|x_1\|^2 + \|x_2\|^2), \quad (10)$$

$$|E[\phi(X^N(t, y_1^N)) - E[\phi(X^N(t, y_2^N))]| \leq C\|\phi\|_0 e^{-ct}(1 + \|y_1^N\|^2 + \|y_2^N\|^2). \quad (11)$$

Proof For the exponential convergence to equilibrium of the original equation, we refer to 22. Thus we focus on the semi-discretization. First, we can define

$$P_t^N \phi(x) = E_{\phi}(X^N(t, x)), \quad \phi \in B_b(P^N(\mathbb{H})), \quad t \geq 0.$$

For the sake of simplicity, we omit the index $N$ of $P_t^N$ for convenience. The Markov property and Feller property of $P_t$ can be obtained by the similar arguments in 17, Chapter 4]. The left proof will be divided into three steps.

Step 1: $P_t$ is strong Feller. To get the strong Feller property, $P_t(B_b(P^N(\mathbb{H}))) \subseteq C_b(P^N(\mathbb{H}))$ for $t > 0$, it suffices to show that for any $\phi \in C_b(P^N(\mathbb{H}))$ and $T > t > 0$, there exists $C_T > 0$ such that $\sup_{x \in P^N(\mathbb{H})} \|DP_t\phi(x)\| \leq C_T\|\phi\|_0$. In
Indeed, the strong Feller property follows from $|P_t \phi(x) - P_t \phi(y)| \leq C_T \| \phi \|_0 \| x - y \|, x, y \in P^N(\mathbb{H})$ and the density of $C_b(P^N(\mathbb{H}))$ in $B_b(P^N(\mathbb{H}))$.

Now, we are in the position to deduce the regularity estimate of $P_t$, $\| DP_t \phi \|_0 \leq C_T \| \phi \|_0$. Recall that $\eta^h(t, x) = D\mathbb{E}[X^N(t, x)] \cdot h$ satisfies

$$\frac{1}{2} \| \eta^h(t, x) \|^2 + \int_0^t \| \nabla \eta^h(t, x) \|^2 \, ds \leq \frac{1}{2} \| h \|^2 + \int_0^t \lambda_F \| \eta^h(s, x) \|^2 \, ds.$$ 

This, combined with the equivalence of Sobolev spaces $H^1$ and $H \cap H^1_0$, implies that

$$\int_0^T \| (-A)^{\frac{1}{2}} \eta^h(t, x) \|^2 \, ds \leq C(T) \| h \|^2.$$ 

The Bismut-Elworthy-Li formula

$$(DP_t \phi(x), h) = \frac{1}{t} \mathbb{E}[\phi(X^N(t, x))] \int_0^t \langle Q^{-\frac{1}{2}} \eta^h(s, x), P^N dW(s) \rangle,$$ 

together with the Hölder inequality, leads to

$$\| DP_t \phi(x) \|^2 \leq \frac{1}{T^2} \| \phi \|^2 \mathbb{E} \left[ \int_0^T \| P^N(Q^{-\frac{1}{2}} \eta^h(s, x)) \|^2 \, ds \right] \leq C(T) \| \phi \|^2 \| Q^{-\frac{1}{2}} (-A)^{-\frac{1}{2}} \|^2,$$

for $T > 0$, which implies the strong Feller property of $P_t$.

Step 2: $P_t$ is irreducible. A basic tool for proving the irreducibility is using the approximate controllability of the following system

$$d \tilde{X}^N(t) = A \tilde{X}^N(t) \, dt + P^N(\tilde{X}^N(t)) \, dt + P^N(Q_{\tilde{u}}(t)) \, dt, \ t > 0 \quad (12)$$

$$\tilde{X}^N = x,$$

where $x \in P^N(\mathbb{H})$ and $u \in L^2([0, T]; P^N(\mathbb{H}))$. Denoting by $\tilde{X}^N(t, x, u)$ the mild solution of the above system, it follows that

$$\tilde{X}^N(t) = e^{tA} x + \int_0^t e^{(t-s)A} P^N(F(\tilde{X}^N(s))) \, ds + \int_0^t e^{(t-s)A} P^N(Q_{\tilde{u}}(s)) \, ds.$$ 

Thus it needs to show that for any fixed time $T > 0$, for any $\epsilon > 0$, $x_0, x_1 \in P^N(\mathbb{H})$, there exists $u \in L^2([0, T]; P^N(\mathbb{H}))$ such that $\| \tilde{X}^N(T, x_0, u) - x_1 \| \leq \epsilon$. Now, we denote $\alpha_{x_0, x_1}(t) = \frac{t}{2} x_0 + \frac{1}{2} x_1$ and $\beta_{x_0, x_1}(t) = \frac{t}{2} \alpha_{x_0, x_1}(t) - A \alpha_{x_0, x_1}(t) - P^N F(\alpha_{x_0, x_1}(t)), t \in [0, T]$.

Since $x_0, x_1 \in P^N(\mathbb{H}) \subset D(A), Q$ is invertible, we choose $u \in C([0, T]; P^N(\mathbb{H}))$ such that $\| \beta_{x_0, x_1}(t) - Q_{\tilde{u}}(t) \| \leq C \epsilon, t \in [0, T]$. Denote $z(t) = X^N(t, x_0, u)$ ...
\[ \frac{1}{2} \| z(t) \|^2 \leq \int_0^t -\| \nabla z(t) \|^2 ds + \int_0^t \langle F(X^N(s, x_0, u)) - F(\alpha_{x_0, x_1}(s)), z(s) \rangle ds \]
\[ + \int_0^t \langle Q^\perp u(s) - \beta_{x_0, x_1}(s), z(s) \rangle ds \]
\[ \leq \int_0^t \left( \frac{1}{2} + \lambda_F - \lambda_1 \right) \| z(s) \|^2 ds + \frac{T}{2} C^2 \epsilon^2. \]

Then the Gronwall inequality implies that \( \| z(T) \| \leq C \sqrt{T} e^{\frac{1}{2}(\frac{1}{2} + \lambda_F - \lambda_1)T \epsilon}. \)

Choosing a proper \( C \) completes the proof of approximate controllability. By applying the approximate controllability of the skeleton equation (12), we deduce that for \( x_0, x_1 \in P^N(H) \) and \( T > 0, P(\| X(T, x_0) - x_1 \| < \epsilon) > 0. \)

Indeed, the approximate controllability leads to the existence of a control \( u \in L^2([0, T]; P^N(H)) \) such that \( \| X^N(T, x_0, u) - x_1 \| \leq \frac{\epsilon}{2}. \)

Then we have
\[ P(\| X^N(T, x_0) - x_1 \| \geq \epsilon) \leq P \left( \| X^N(T, x_0) - \tilde{X}^N(T, x_0, u) \| \geq \frac{\epsilon}{2} \right). \]

Similar arguments in the proof of the priori estimate of \( X^N \) lead to
\[ \| X^N(t, x_0) - \tilde{X}^N(t, x_0, u) \| \leq \| Y^N(t, x_0) - \tilde{Y}^N(t, x_0, u) \| + \| Z^N(t) - \tilde{Z}^N(t, u) \|, \]

where \( \tilde{X}^N = \tilde{Y}^N + \tilde{Z}^N, \tilde{Y}^N \) and \( \tilde{Z}^N \) satisfy
\[ \frac{d}{dt} \tilde{Z}^N = A\tilde{Z}^N + Q^\perp u(t), \tilde{Z}^N(0) = 0, \]
\[ \frac{d}{dt} \tilde{Y}^N = A\tilde{Y}^N + P^N \tilde{F} \tilde{Y}^N + \tilde{Z}^N, \tilde{Y}^N = x_0. \]

By the monotonicity of \( F \) and dissipativity of \( A \), we have
\[ \frac{1}{2} \| Y^N(t, x_0) - \tilde{Y}^N(t, x_0, u) \|^2 \]
\[ \leq \int_0^t -\lambda_1 \| Y^N(s, x_0) - \tilde{Y}^N(s, x_0, u) \|^2 ds \]
\[ + \int_0^t \langle F(Y^N + Z^N) - F(\tilde{Y}^N + \tilde{Z}^N), Y^N - \tilde{Y}^N \rangle ds \]
\[ \leq \int_0^t (-\lambda_1 + \lambda_F) \| Y^N(s, x_0) - \tilde{Y}^N(s, x_0, u) \|^2 ds \]
\[ + \int_0^t \langle F(\tilde{Y}^N + Z^N) - F(\tilde{Y}^N + \tilde{Z}^N), Y^N - \tilde{Y}^N \rangle ds \]
\[ \leq \int_0^t C \| Y^N(s, x_0) - \tilde{Y}^N(s, x_0, u) \|^2 ds \]
\[ + \int_0^t C(1 + \| \tilde{Y}^N \|^4_E + \| \tilde{Z}^N \|^4_E + \| Z^N \|^4_E) \| Z^N - \tilde{Z}^N \|^2 ds. \]
Then the Gronwall inequality leads to
\[
\|Y^N(T, x_0) - \tilde{Y}^N(T, x_0, u)\| \leq C e^{CT} \sqrt{T} \|Z^N - \tilde{Z}^N\|_{C([0,T], P^N(H))} (1 + \|\tilde{Y}^N\|^2_{C([0,T], E)} + \|\tilde{Z}^N\|^2_{C([0,T], E)}).
\]

The Sobolev embedding theorem \(H^2 \hookrightarrow E\), the inverse inequality \(\|x\|_{H^2} \leq \lambda_N \|x\|\), and the uniform boundedness of \(\tilde{Y}^N, \tilde{Z}^N\) and \(Z^N\) imply that
\[
\|Y^N(T, x_0) - \tilde{Y}^N(T, x_0, u)\| \leq C(\lambda_N) e^{CT} \sqrt{T} \|Z^N - \tilde{Z}^N\|_{C([0,T], P^N(H))} (1 + \|\tilde{Y}^N\|^2_{C([0,T], P^N(H))} + \|\tilde{Z}^N\|^2_{C([0,T], P^N(H))} + \|Z^N\|^2_{C([0,T], P^N(H))}).
\]

It is concluded that
\[
P\left(\|X^N(T, x_0) - \tilde{X}^N(T, x_0, u)\| \geq \frac{\epsilon}{2}\right)
\leq P\left(\|Y^N(T, x_0) - \tilde{Y}^N(T, x_0, u)\| + \|Z^N(T) - \tilde{Z}^N(T, u)\| \geq \frac{\epsilon}{2}\right)
\leq P\left(C(\lambda_N) e^{CT} \sqrt{T} \|Z^N - \tilde{Z}^N\|_{C([0,T], P^N(H))} (1 + \|\tilde{Y}^N\|^2_{C([0,T], P^N(H))} + \|\tilde{Z}^N\|^2_{C([0,T], P^N(H))} \geq \frac{\epsilon}{2}\right).
\]

Since \(Z^N\) is full in \(C([0,T]; P^N(H))\) and \(C(\lambda_N)\) is polynomially dependent on \(\lambda_N\), we have that there exists \(R = R(\lambda_N, \epsilon, T)\) such that
\[
P\left(\|X^N(T, x_0) - \tilde{X}^N(T, x_0, u)\| \geq \frac{\epsilon}{2}\right)
\leq P\left(\|Z^N - \tilde{Z}^N\|_{C([0,T], P^N(H))} \geq R(\lambda_N, \epsilon, T)\right) < 1,
\]
which completes the proof of the irreducibility.

Step 3: Existence of the invariant measure and \(V\)-uniformly ergodicity. Similar arguments in the proof of Lemma 2 imply the uniform estimate of \(X^N\) in \(H^\beta\). The existence of the invariant measure \(\mu^N\) of Eq. (4) is ensured by the uniform estimate of \(X^N\) in \(H^\beta\) and the Sobolev compact embedding theorem. To show the exponential ergodicity of the invariant measure, by Theorem 12.1, it suffices to show that the \(p\)-th moment of \(X^N(t, x)\) is ultimately bounded, i.e., \(E[|X^N(t, x)|^p] \leq k|x|^p e^{-\omega t} + c\), \(t \geq 0, x \in P^N(H)\), for some positive constants \(k, \omega, c,\) and \(p\).

For convenience, we only prove the case \(p = 2\). Due to the fact that \(X^N(t) = Y^N(t) + Z^N(t)\), we estimate the \(\mathbb{H}\)-norm of \(Z^N\) and \(Y^N\), respectively. The mild form of \(Z^N\) yields that
\[
E[\|Z^N(t)\|^2] \leq E[\|Z^N(t)\|_E^2] \leq C(Q).
\]
It follows from the variational approach, Poincare inequality, Young and Hölder inequalities that

\[
\mathbb{E}[|Y^N(t)|^2] = \mathbb{E}[|Y^N(0)|^2] - 2 \int_0^t \|\nabla Y^N(s)\| ds + 2 \int_0^t \langle F(Y^N(s) + Z^N(s)), Y^N(s) \rangle ds
\]

\[
\leq \mathbb{E}[|Y^N(0)|^2] - 2\lambda_1 \int_0^t \|Y^N(s)\|^2 ds + C(\epsilon) \int_0^t (1 + \|Z^N\|^2_L) ds
\]

\[
+ \int_0^t (-2a_3 - c)\|Y^N(s)\|^4_L + C\|Y^N(s)\|^2 ds
\]

\[
\leq \mathbb{E}[|Y^N(0)|^2] - 2\lambda_1 \int_0^t \|Y^N(s)\|^2 ds + C(\epsilon) \int_0^t (1 + \|Z^N\|^2_L) ds.
\]

Then the Gronwall inequality implies that

\[
\mathbb{E}[|Y^N(0)|^2] \leq e^{-2\lambda_1 t}\|x\|^2 + \int_0^t e^{-2\lambda_1 (t-s)} C(Q) ds \leq e^{-2\lambda_1 t}\|x\|^2 + C(Q, \lambda_1).
\]

From [22, Theorem 12.1], it follows that \(\{P_t^N\}_{t \geq 0}\) is \(V\)-uniformly ergodic with \(V = 1 + \|x\|^2\), i.e.,

\[
\sup_{\|\phi\|_V \leq 1} \left| P_t^N \phi(x) - \int_{P^N(H)} \phi d\mu^N \right| \leq CV(x) e^{-\alpha t}, \quad x \in P^N(H),
\]

where \(C, \alpha\) depend on \(\lambda_1\) and \(Q\), and \(\phi \in B^V_0(P^N(H))\), i.e, \(\phi\) is Borel-measurable and \(\|\phi\|_V := \sup_{x \in P^N(H)} \frac{\|\phi(x)\|}{\|\chi(x)\|} < \infty\). Now taking any \(\phi \in B_b(P^N(H))\), we have \(\|\phi\|_V \leq \|\phi\|_0\). Combining with \(V\)-uniformly ergodicity of \(P_t\), it is deduced that

\[
\left| P_t^N \phi(x) - \int_{P^N(H)} \phi d\mu^N \right| \leq C(\|\phi\|_0) V(x) e^{-\alpha t}, \quad x \in P^N(H).
\]

By the fact that \(B_b(H) \subset B_b(P^N(H))\), we obtain the exponential ergodicity of the unique invariant measure. Taking two different initial data \(y_1^N, y_2^N\), combining with the exponential ergodicity of \(\mu^N\), we complete the proof. \(\square\)

**Remark 4** Under the same conditions of Proposition 3, one can obtain more stronger ergodicity result, that is, the uniformly exponentially ergodicity (\(V = 1\)) of \(P_t^N, t \geq 0\) (see e.g. [22]),

\[
\left| P_t^N \phi(x) - \int_{P^N(H)} \phi d\mu^N \right| \leq C e^{-\alpha t},
\]

which can be used to improve the bound of regularity estimates in Lemma 3. The condition \(\|Q^{R}(A)^{-\frac{1}{2}}\| < \infty\) in Assumption 3 is necessary for the strong Feller property of \(P_t\). However, from the proof of the strong Feller property of \(P_t^N\), the estimate \([11]\) holds even for the case \(\|Q^{R}(A)^{-\frac{1}{2}}\| = \infty\).
4.2 Time-independent regularity estimate of Kolmogorov equation

In order to obtain the time-independent optimal weak error estimate, we need more stronger regularity estimates of the Kolmogorov equation. However, the regularizing approach by the splitting strategy is not a proper way to get such time-independent regularity estimates. To overcome this difficulty, we investigate time-independent regularity estimates of the Kolmogorov equation by means of a finite dimensional approximation. Recall the Kolmogorov equation of the Galerkin approximation

$$\frac{\partial U^M(t, x)}{\partial t} = (Ax + PMF(x), DU^M(t, x)) + \frac{1}{2} tr[P^M Q^\frac{1}{2} D^2 U^M(t, x) P^M Q^\frac{1}{2}].$$

(13)

On the regularity estimates in the following lemmas, the conditional expectations are needed since we can’t prove $\sup_{s \in [0, t]} \mathbb{E}||X^M(s, x)||_E^p \leq C(1 + ||x||_E^2)$ for some positive numbers $p$ and $q$.

**Lemma 7** Let Assumptions [14] hold. For $\eta > \frac{2}{3}$, $\alpha, \theta, \gamma \in [0, 1]$, $0 < \gamma < 1$, there exist $c > 0$, $C(Q, \alpha)$ and $C(Q, \theta, \gamma)$ such that for $x, h, k \in P^M(\mathbb{H})$, $M \in \mathbb{N}^+$ and $t > 0$,

$$|DU^M(t, x) \cdot h| \leq C(Q, \alpha)(1 + \sup_{s \in [0, t]} \mathbb{E}[||X^M(s, x)||_E^2])(1 + t^{-\alpha})e^{-ct\|(-A)^{-\alpha}h\|},$$

(14)

$$|D^2 U^M(t, x) \cdot (h, k)| \leq C(Q, \theta, \gamma)(1 + \sup_{s \in [0, t]} \mathbb{E}[||X^M(s, x)||_E^7])(1 + t^{-\eta} + t^{-\beta - \gamma})e^{-ct\|(-A)^{-\theta}h\| \|(-A)^{-\gamma}k\|}.$$  

(15)

**Proof** Similar to the proof of [14] Proposition 4.1, we have

$$DU^M(t, x) \cdot h = \mathbb{E}[D\phi(X^M(t, x)) \cdot \eta^h(t, x)],$$

$$DU^M(t, x) \cdot (h, k) = \mathbb{E}[D\phi(X^M(t, x)) \cdot \zeta^{h,k}(t, x)]$$

$$+ \mathbb{E}[D^2 \phi(X^M(t, x)) \cdot (\eta^h(t, x), \eta^k(t, x))],$$

for $h, k, x \in P^M(\mathbb{H})$, $t \geq 0$, where $\eta^h$ and $\zeta^{h,k}$ satisfy

$$\frac{\partial \eta^h(t, x)}{\partial t} = A\eta^h(t, x) + P^M(DF(X^M(t, x))\eta^h(t, x)), \eta^h(0, x) = h,$$

$$\frac{\partial \zeta^{h,k}(t, x)}{\partial t} = A\zeta^{h,k}(t, x) + P^M(DF(X^M(t, x))\zeta^{h,k}(t, x))$$

$$+ P^M(D^2 F(X^M(t, x))\eta^h(t, x)\eta^k(t, x)), \zeta^{h,k}(0, x) = 0.$$

For convenience, the parameter $M$ is omitted in the notations of $\eta^h$ and $\zeta^{h,k}$. Consider the following auxiliary equation

$$\frac{\partial V(t, s)h}{\partial t} = (A + P^M DF(X^M(t, x)))V(t, s)h, \ V(s, s)h = h.$$
The straightforward argument leads to \( \|V(t, s)h\|^2 \leq e^{-2(\lambda_1 + \lambda_F)(t-s)}\|h\|^2 \).

Denote \( \eta^h(t, x) := \tilde{\eta}^h(t, x) + e^{tA}h = \eta^h(t, x) - e^{tA}h + e^{tA}h \). It follows from the smooth effect of \( e^{tA} \) and the estimate of \( V(t, s) \), 0 \( \leq s \leq t \), that for some \( \lambda_1 > c > 0 \),

\[
|\mathbb{E}[D\phi(X^M(t, x)) \cdot e^{tA}h]| \leq |\phi|_1 C_t t^{-\alpha} e^{-ct} \|(-A)^{-\alpha}h\|,
\]

and

\[
\|\tilde{\eta}^h(t, x)\| = \left\| \int_0^t V(t, s)(P^M(DF(X^M(t, x))e^{sA}h)ds \right\|
\leq \int_0^t e^{-(\lambda_1 + \lambda_F)(t-s)} \|DF(X^M(s, x))\|_E \|e^{sA}h\| ds
\leq C|\phi|_1 \int_0^t (1 + \|X^M(s, x)\|_E^2) e^{-(\lambda_1 + \lambda_F)(t-s)} s^{-\alpha} e^{-cs} ds \|(-A)^{-\alpha}h\|.
\]

Taking \( c > \lambda_1 - \lambda_F \), we obtain

\[
|\mathbb{E}[D\phi(X^M(t, x)) \cdot \tilde{\eta}^h(t, x)]| \leq C|\phi|_1 \int_0^t (1 + \mathbb{E}[\|X^M(s, x)\|_E^2]) e^{-(\lambda_1 + \lambda_F)(t-s)} s^{-\alpha} e^{-cs} ds \|(-A)^{-\alpha}h\|
\leq C e^{-(\lambda_1 - \lambda_2)\beta} \int_0^t (1 + \mathbb{E}[\|X^M(s, x)\|_E^2]) s^{-\alpha} e^{-(c-\lambda_1 + \lambda_F)s} ds \|(-A)^{-\alpha}h\|
\leq C (1 + \sup_{s \in [0, t]} \mathbb{E}[\|X^M(s, x)\|_E^2]) e^{-(\lambda_1 - \lambda_2)\beta} \|(-A)^{-\alpha}h\|.
\]

The above two estimates imply that

\[
|\mathbb{E}[D\phi(X^M(t, x)) \cdot \eta^h(t, x)]| \leq C_\alpha (1 + \sup_{s \in [0, t]} \mathbb{E}[\|X^M(s, x)\|_E^2]) (1 + t^{-\alpha}) e^{-ct} \|(-A)^{-\alpha}h\|.
\]

Similarly, we have

\[
|\mathbb{E}[D^2\phi(X^M(t, x)) \cdot (\eta^h(t, x), \eta^k(t, x))]| \leq C_{\beta, \gamma} |\phi|_2 (1 + \sup_{s \in [0, t]} \mathbb{E}[\|X^M(s, x)\|_E^2]) (1 + t^{-(\beta+\gamma)}) e^{-ct} \|(-A)^{-\beta}h\| \|(-A)^{-\gamma}h\|.
\]

Now similar arguments, combined with the decomposition \( V(t, s)h = \tilde{V}(t, s)h + e^{-(t-s)A}h \), imply that for \( t > s \geq 0 \), \( 0 \leq \alpha < 1 \),

\[
\|V(t, s)h\| \leq C e^{-(\lambda_1 + \lambda_F)(t-s)}(t-s)^{-\alpha} + \int_s^t (r-s)^{-\alpha} e^{-(c-\lambda_1 + \lambda_F)(r-s)} \|DF(X^M(r, x))\|_E dr \|A^{-\alpha}h\|,
\]

\[(16)\]
Based on the representation of $\zeta^{h,k}$ and (16), we obtain
\[
\|\zeta^{h,k}(t, x)\| = \left\| \int_0^t V(t, s) P^M \left( D^2 F(X^M(s, x)) \eta^h(s, x) \eta^G(s, x) \right) ds \right\|
\leq C_\eta \int_0^t e^{-c(t-s)}(t-s)^{-\eta} \left( 1 + \int_s^t e^{-c(t-r)}(r-s)^{-\eta} \| D^2 F(X^M(r, x)) \|_{E^{d\tau}} \right) \|(-A)^{-\gamma} P^M ((D^2 F(X^M(s, x)) \eta^h(s, x) \eta^G(s, x)) \| ds,
\]
for $\eta > \frac{4}{3}$. Thus we have
\[
E[D\phi(X^M(t, x)) \cdot \zeta^{h,k}(t, x)]
\leq C_\eta e^{-ct} (1 + \sup_{r \in [0, t]} E[\|X^M(r, x)\|_E^2]) \int_0^t e^{-cs}(t-s)^{-\eta} \left( 1 + s^{-\theta-\gamma} \right)(1 + \sup_{r \in [0, t]} E[\|X^M(r, x)\|_E^2]) \|(-A)^{-\gamma} k\| ds
\leq C_\eta e^{-ct}(1 + t^{-\eta})(1 + \sup_{r \in [0, t]} E[\|X^M(r, x)\|_E^2]) \|(-A)^{-\gamma} k\|,
\]
which completes the proof. □

**Lemma 8** Let Assumptions (38) and Assumption (39) hold. For $\eta > \frac{4}{3}$, $\alpha, \theta, \gamma \in [0, 1]$, $\theta + \gamma < 1$, there exist $c > 0$, $C(Q, \alpha)$ and $C(Q, \theta, \gamma)$ such that for $x, y, z \in P^M(\mathbb{H})$ and $t \in (0, T]$,
\[
|D^2 U^M(t, x) \cdot y| \leq C(Q, \alpha)(1 + \sup_{s \in [0, t]} E[\|X^M(s, x)\|_E^2]) (1 + t^{-\alpha} e^{-ct} \|A^{-\alpha} y\|),
\]
\[
|D^2 U^M(t, x) \cdot (y, z)| \leq C(Q, \theta, \gamma)(1 + \sup_{s \in [0, t]} E[\|X^M(s, x)\|_E^2])(1 + t^{-\eta + t^{-\theta-\gamma}} e^{-ct} \|A^{-\gamma} y\|) \|A^{-\gamma} z\|.
\]

**Proof** By the similar arguments in Lemma 7, we can obtain the regularity estimate for $0 < t \leq T$. For convenience, we take $T = 1$ and get for $0 < t \leq 1$
\[
\eta^h(t, x) \leq C_\alpha (1 + \sup_{s \in [0, t]} \|X^M(s, x)\|_E^2)(1 + t^{-\alpha}) \|(-A)^{-\alpha} h\|,
\]
\[
\zeta^{h,k}(t, x) \leq C_{\theta, \gamma}(1 + \sup_{s \in [0, t]} \|X^M(s, x)\|_E^2)(1 + t^{-\eta + t^{-\theta-\gamma}}) \|(-A)^{-\theta} h\| \|(-A)^{-\gamma} h\|.
\]
To get the exponential decay of the regularity estimate, we need another a priori estimate of $X^M$ independent of $x$. According to the fact $X^M = Y^M + Z^M$ and the evolution of $\|Y^M\|^2$, we have
\[
\frac{\partial}{\partial t} \|Y^M(t)\|^2 = -2\|\nabla Y^M(t)\|^2 + 2\langle F(Y^M(t) + Z^M(t)), Y^N(t) \rangle
\leq -c\|Y^M(t)\|^2 + C(1 + \|Z^M(t)\|_E^2),
\]
Thus, we get 
\[ \sup_{x \in \mathbb{R}} \|Y^M(t)\|^p \leq C(p, t)(t \wedge 1)^{-\frac{p}{2}}, \]

(20)

for \( p \geq 1 \), where \( C(p, t) \) has finite moments of any order. Combining with the equivalence of norms in finite dimension, we have

\[ \|\eta^h(t, x)\| \leq C(M, t, |\phi|_1)\|h\|, \]

\[ \|\zeta^{h,k}(t, x)\| \leq C(M, t, |\phi|_2)\|h\|\|k\|, \]

for \( t > 0 \). Indeed, by the chain rule, we have

\[ \|\eta^h(t, x)\|^2 + \int_0^t \|\nabla \eta^h(s, x)\|^2 ds \leq \|h\|^2 + \int_0^t \langle DF(X^M(s))\eta^h(s, x), \eta^h(s, x)\rangle ds. \]

Therefore, the Gronwall inequality leads that for \( t > 0 \)

\[ \|\eta^h(t, x)\|^2 + \int_0^t \|\nabla \eta^h(s, x)\|^2 ds \leq \|h\|^2. \]

The same arguments, together with the Sobolev embedding theorem and inverse inequality, yield that

\[ \|\zeta^{h,k}(t, x)\| \leq \| \int_0^t V(t - s)D^2 F(X^M(s))\eta^h(s, x)\eta^k(s, x)ds \|
\leq e^{\lambda t} \int_0^t \|X^M(s)\|_{L^6}\|\eta^h(s, x)\|_{L^6}\|\eta^k(s, x)\|_{L^6} ds
\leq e^{\lambda t} \lambda_M \int_0^t \|X^M(s)\|_{H_1}\|\eta^h(s, x)\|_{H_1}\|\eta^k(s, x)\|_{H_1} ds
\leq e^{\lambda t} \lambda_M^2 \int_0^t \|X^M(s)\|_{H^1}\|\eta^h(s, x)\|_{H^1}\|\eta^k(s, x)\|_{H^1} ds
\leq e^{\lambda t} \lambda_M^2 \int_0^t \|X^M(s)\|_{H^{1,2}}\|\eta^h(s, x)\|_{H^{1,2}}\|\eta^k(s, x)\|_{H^{1,2}} ds.
\]

Thus, we get

\[ |DU^M(t, x) \cdot h| \leq e^{\lambda t} \|h\|, \]

\[ |D^2U^M(t, x) \cdot (h, k)| \leq C(p, t)e^{\lambda t} \lambda_M^2 \sqrt{t} \|h\|\|k\|, \]

which implies that for any \( \phi \in C^2_b(\mathbb{R}) \), \( U^M(t) = P_t \phi \in C^2_b(\mathbb{R}) \), \( t > 0 \).

The Bismut–Elworthy–Li formula states that if \( \Phi: \mathbb{P} \to \mathbb{R} \) belongs to \( C^2_b(\mathbb{P}) \) and \( |\Phi(x)| \leq M(\Phi)(1 + |x|^q) \), \( q \geq 1 \), then we can calculate the first and second order derivatives of \( U^M(t, x) := \mathbb{E}[\Phi(X^M(t, x))] \) with respect to \( x \). Indeed, we have

\[ DU^M(t, x) \cdot h = \frac{1}{h} \mathbb{E} \left[ \int_0^t \langle Q \frac{1}{h} \eta^h(s, x), d\tilde{W}(s) \rangle \Phi(X^M(t, x)) \right],\]

\[ D^2U^M(t, x) \cdot (h, k) = \frac{1}{h} \mathbb{E} \left[ \int_0^t \langle Q \frac{1}{h} \eta^h(s, x), d\tilde{W}(s) \rangle \Phi(X^M(t, x)) \right],\]
for any \( x \in P^M(\mathbb{H}) \), \( h \in P^M(\mathbb{H}) \). The Markov property of \( P_t \) implies that

\[
DU^M_{\phi}(t, x) \cdot h = \frac{2}{t} \mathbb{E} \left[ \int_0^t \langle Q^{-\frac{1}{2}} \eta^h(s, x), d\tilde{W}(s) \rangle U^N_{\phi} \left( t, X^M \left( \frac{t}{2}, x \right) \right) \right],
\]

where \( \tilde{W} \) is the cylindrical Wiener process. By applying again the Bismut–Elworthy–Li formula, we get a formula of the second derivative

\[
D^2U^M_{\phi}(t, x) \cdot (h, k) = \frac{2}{t} \mathbb{E} \left[ \int_0^t \langle Q^{-\frac{1}{2}} \zeta^{h,k}(t, x), d\tilde{W}(s) \rangle U^M_{\phi} \left( t, X^M \left( \frac{t}{2}, x \right) \right) \right]
\]

\[
+ \frac{2}{t} \mathbb{E} \left[ \int_0^t \langle Q^{-\frac{1}{2}} \eta^h(t, x), d\tilde{W}(s) \rangle DU^M_{\phi} \left( t, X^M \left( \frac{t}{2}, x \right) \right) \cdot \eta^k \left( t, x \right) \right],
\]

for \( x \in P^M(\mathbb{H}) \), \( h, k \in P^M(\mathbb{H}) \). By using a priori estimates of \( \eta^h \) and \( \zeta^{h,k} \), we obtain

\[
|DU^M_{\phi}(t, x) \cdot h| \leq \frac{1}{t} \sqrt{\mathbb{E}[\Phi(X^M(t, x))]^2} \sqrt{\mathbb{E} \left[ \int_0^t \| Q^{-\frac{1}{2}} \eta^h(s, x) \|^2 ds \right]}
\]

\[
\leq \frac{1}{t} C(t) M(\Phi) (1 + \sqrt{\mathbb{E}[\| X^M(t, x) \|^2 q]}) \| h \|
\]

and

\[
|D^2U^M_{\phi}(t, x) \cdot (h, k)| \leq \frac{2}{t} \sqrt{\mathbb{E}[U^M_{\phi} \left( t, X^M \left( \frac{t}{2}, x \right) \right)]^2} \sqrt{\mathbb{E} \left[ \int_0^t \| Q^{-\frac{1}{2}} \zeta^{h,k}(s, x) \|^2 ds \right]}
\]

\[
+ \frac{1}{t} \sqrt{\mathbb{E} \left[ \int_0^t \| Q^{-\frac{1}{2}} \eta^h(s, x) \|^2 ds \right] \sqrt{\mathbb{E}[\| DU^M_{\phi} \left( t, X^M \left( \frac{t}{2}, x \right) \right) \cdot \eta^k \left( t, x \right) \|^2]}}
\]

\[
\leq \frac{1}{t} C(t) M(\Phi) (1 + \sqrt{\mathbb{E}[\| X^M(t, x) \|^2 q]}) \sqrt{\mathbb{E} \left[ \int_0^t \| Q^{-\frac{1}{2}} \zeta^{h,k}(s, x) \|^2 ds \right]}
\]

\[
+ \frac{1}{t^2} C(t, Q) M(\Phi) (1 + \sqrt{\mathbb{E}[\| X^M \left( \frac{t}{2}, x \right) \|^2 q]}) \| h \| \| k \|
\]

for \( 0 < t \leq 1 \). To estimate \( \mathbb{E} \left[ \int_0^t \| Q^{-\frac{1}{2}} \zeta^{h,k}(s, x) \|^2 ds \right] \), we consider the \( M \)-independent estimation of \( \zeta^{h,k} \) and have

\[
\| \zeta^{h,k}(t, x) \| \leq \| \int_0^t V(t - s) D^2F(X^M(s)) \eta^h(s, x) \eta^k(s, x) ds \|
\]

\[
\leq C(t) \left( 1 + \sup_{s \in [0, t]} \| X^M(s) \|_{L^4} \| \eta^h(s, x) \|_{L^4} \right) \| \eta^k(s, x) \|_{L^4} ds
\]

\[
\leq C(t) \left( 1 + \sup_{s \in [0, t]} \| X^M(s) \|_{L^4} \right) \| \eta^h(s, x) \|_{L^4} \| \eta^k(s, x) \|_{L^4} ds
\]

\[
\leq C(t) \left( 1 + \sup_{s \in [0, t]} \| X^M(s) \|_{L^4} \right) \| h \| \| k \|
\]
which implies that

\[
\int_0^t \|Q^{-\frac{1}{2}}\zeta^{h,k}(s,x)\|^2 ds \leq C(t)\|\zeta^{h,k}(t,x)\|^2 + \int_0^t \langle D^2 F(X^M(s))\eta^h(s,x)\eta^k(s,x),\zeta^{h,k}(s,x) \rangle ds \\
\leq e^{Ct}C(t)(1 + \sup_{s\in[0,t]}\|X^M(s)\|_E^2)\|h\|^2\|k\|^2 + C\sup_{s\in[0,t]}\|\zeta^{h,k}(s,x)\|
\]

\[
(1 + \sup_{s\in[0,t]}\|X^M(s)\|_E)\int_0^t \|(-A)^{\frac{1}{2}}\eta^h(s,x)\|\|(-A)^{\frac{1}{2}}\eta^k(s,x)\| ds \\
\leq e^{Ct}C(t)(1 + \sup_{s\in[0,t]}\|X^M(s)\|_E^2)\|h\|^2\|k\|^2.
\]

It is concluded that

\[
|D^2U^M_{\Phi}(t,x) \cdot (h,k)| \leq \left(\frac{1}{t} + \frac{1}{t^2}\right)C(t)M(\Phi)(1 + \sqrt{\mathbb{E}[\|X^M(t/2,x)\|^2]})
\]

\[
(1 + \sup_{s\in[0,t]}\sqrt{\mathbb{E}[\|X^M(s,x)\|^2]})\|h\|\|k\|.
\]

For any \( t \geq 1 \), we have \( U^M(t,x) = \mathbb{E}[U^M(t-1, X^M(1,x))] \). The exponential convergence estimate (11) yields that

\[
|U^M(t-1,x) - \int P_{\mu}M| \leq Ce^{-c(t-1)}(1 + \|x\|^2).
\]

Inspired by [5], we choose \( \Phi(x) = U^M(t-1,x) - \int P_{\mu}M \Phi M \), we have that \( D\Phi \) and \( D^2\Phi \) are uniformly bounded by (21). The above estimate (11) in \( 0 < t \leq 1 \) and the fact that \( U^M(t,x) = \mathbb{E}[\Phi(X^M(1,x))] + \int P_{\mu}M \Phi M \) lead to

\[
|DU^M(t,x) \cdot h| \leq Ce^{-c(t-1)}(1 + \sqrt{\mathbb{E}[\|X^M(1,x)\|^4]})\|h\|
\]

\[
|D^2U^M(t,x) \cdot (h,k)| \leq Ce^{-c(t-1)}(1 + \sqrt{\mathbb{E}[\|X^M(t/2,x)\|^4]})
\]

\[
(1 + \sup_{s\in[0,1]}\sqrt{\mathbb{E}[\|X^M(s)\|^2]})\|h\|\|k\|,
\]

for \( t \geq 1 \). Thus we conclude that

\[
\|DU^M(t,x)\| \leq Ce^{-ct}(1 + \sup_{s\in[0,1]}\sqrt{\mathbb{E}[\|X^M(s,x)\|^4]})
\]

\[
\|D^2U^M(t,x)\|_{\mathcal{L}(\mathbb{H})} \leq Ce^{-ct}(1 + \sup_{s\in[0,1]}\sqrt{\mathbb{E}[\|X^M(s,x)\|^4]})(1 + \sup_{s\in[0,1]}\sqrt{\mathbb{E}[\|X^M(s,x)\|^2]}).
\]
Combining with the Markov property of $X^M$, we have
\[
|DU^M(t, x) \cdot h| \leq Ce^{-c(t-1)} \left( 1 + \sup_{s \in [0, t]} E[\|X^N(s, x)\|^4] \right)^{\frac{1}{4}} \left( E[\|\eta^N(1, 1)\|^2] \right)^{\frac{1}{2}}
\leq Ce^{-c(t-1)} \left( 1 + \sup_{s \in [0, t]} E[\|X^M(s, x)\|^4] \right)^{\frac{1}{4}} \|h\|_{H^{-\sigma}}
\]
\[
|D^2U^M(t, x) \cdot (h, k)| = |E[D^2(U^M(t-1, X^M(1, x))) \cdot (h, k)]|
\leq |E[D^2U^M(t-1, X^M(1, x)) \cdot (\eta^N(1, 1), \eta^N(1, 1))]|
+ |E[DU^M(t-1, X^M(1, x)) \cdot \zeta^{h,k}(1, x)]|
\leq Ce^{-c(t-1)} \left( 1 + \sup_{s \in [0, t]} E[\|X^M(s, x)\|^4] \right)^{\frac{1}{4}} \|h\|_{H^{-\sigma}} \|k\|_{H^{-\gamma}}.
\]
The above estimate, together with (29) completes the proof. □

4.3 Time-independent weak error estimate and approximation of the invariant measure

To get weak convergence analysis, we introduce the another solution $X^M$, $M \gg N$, of spectral Galerkin method. The regularity estimates in Lemmas 7 and 8 of $U^M$ are crucial. The proof is similar to that of Theorem 1 and needs a preciser procedure. Before that, we first give a useful estimate to deal with the conditional expectation appeared in the regularity estimate of $U^M$ in Lemmas 7 and 8. For convenience, we use the notation $E_x$ as the conditional expectation at $x \in E$.

**Lemma 9** Under the Assumptions 4-8 for any $T > t \geq 0$, there exists a constant $C(X_0, Q, p)$ such that for any $p \geq 2$,

\[
E\left[ \sup_{s \in [0, T-t]} E_{X^N(t)} \left[ \|X^M(s, \hat{X}^N(t))\|^p \right] \right] \leq C(Q, X_0, p).
\]

**Proof** Without loss of generality, we assume that $t \in [t_k, t_{k+1}]$, $k \leq K-1$. By the procedure in proving Corollary 2 we let, for $0 \leq s \leq T-t$,

\[
X^M(s, \hat{X}^N(t)) = Y^M(s, \hat{Y}^N(t)) + Z^M(s, \hat{Z}^N(t)).
\]

Here $Y^M$ and $Z^M$ satisfy
\[
dY^M = AY^M ds + P^MF(Y^M + Z^M) ds,
\]
\[
dZ^M = AZ^M ds + P^M d\hat{W}(s),
\]
where $\hat{W}(s) = W(t + s) - W(t)$ and initial data $Y^M(0) = \hat{Y}^N(t)$, and $Z^M(0) = \hat{Z}^N(t)$, where
\[
\hat{Y}^N(t) = Y^N(t) = Y^N = \gamma^N + \frac{\delta t}{\delta s} Y^N_k(t - \delta t) + \frac{\delta t}{\delta s} P^N(F(Y^N_k + Z^N_k)(t - \delta t))
\]
\[
\hat{Z}^N(t) = Z^N(t) = Z^N = \gamma^N + \frac{\delta t}{\delta s} Z^N_k(t - \delta t) + \frac{\delta t}{\delta s} P^N(W(t) - W(t_k)).
\]
Now we show that \( \| \hat{Y}^N(t) \|_{\mathbb{H}^1} \) and \( \| \hat{Z}^N(t) \|_E \) have any finite \( q \)-th moment, \( q \geq 2 \). Indeed, we have

\[
\| \hat{Y}^N(t) \|_{\mathbb{H}^1} \leq C \| Y_k^N \|_{\mathbb{H}^1} + C(1 + \| Y_{k+1}^N \|_{\mathbb{H}^1}^\alpha + \| Z_{k+1}^N \|_E^\beta)
\]

\[
\| \hat{Z}^N(t) \|_E \leq C \| Z_k^N \|_E + \| S_{dt} P^N(W(t) - W(t_0)) \|_E,
\]

which, together with the estimations in Lemmas 3 and 4 implies the boundedness of any \( q \)-th moment of \( \| \hat{Y}^N(t) \|_{\mathbb{H}^1} \) and \( \| \hat{Z}^N(t) \|_E \). Similar arguments in Lemmas 3 and 4 yield that for \( s \in [0, T - t] \)

\[
\| Z^M(s, \hat{Z}^N(t)) \|_E \leq \| \hat{Z}^N(t) \|_E + \| \int_t^{t+s} S(s - r)dW(r) \|_E,
\]

\[
\| Y^M(s, \hat{Y}^N(t)) \|_E \leq \| \hat{Y}^N(t) \|_E + C_d(\| \hat{Y}^N(t) \|_{\mathbb{H}^1}, Q, \sup_{s \in [T-t]} \| Z^M(s, \hat{Z}^N(t)) \|_E),
\]

where \( C_d(\| \hat{Y}^N(t) \|_{\mathbb{H}^1}, Q, \sup_{s \in [T-t]} \| Z^M(s, \hat{Z}^N(t)) \|_E) \) is a random variable and polynomially depends on \( \| \hat{Y}^N(t) \|_{\mathbb{H}^1} \) and \( \| \hat{Z}^N(t) \|_E \). Similar to the proof of Lemma 4 in \( d = 1 \), we do not need the a priori estimate of \( \| Y^M \|_{\mathbb{H}^1} \). Combining with the a priori estimate of stochastic convolution of \( Z^M \), we have that \( C_d \) has finite \( q \)-th moments, \( q \geq 2 \), which leads to the desired result. \( \square \)

**Proof of Theorem 2** Let \( K \delta t = T > 0 \). We transform the error estimate from \( \mathbb{H} \) into \( P^M(\mathbb{H}) \),

\[
|\mathbb{E}[\phi(X(K\delta t, X_0)) - \phi(X^M_k)]| \leq |\mathbb{E}[\phi(X(K\delta t, X_0)) - \phi(X^M(K\delta t, X^M_0))]| + |\mathbb{E}[\phi(X^M(K\delta t, X^M_0)) - \phi(X^N_k)]|.
\]

From the strong convergence analysis in Lemma 1 and Remark 1 it follows that for \( M \in \mathbb{N}^+ \),

\[
|\mathbb{E}[\phi(X(K\delta t, X_0)) - \phi(X^M(K\delta t, X^M_0))]| \leq C(K\delta t, X_0)\lambda_M^{-\frac{2}{\alpha}}.
\]

Then, after taking \( M \to \infty \), it suffices to estimate \( |\mathbb{E}[\phi(X^M(K\delta t, X^M_0)) - \phi(X^N_k)]| \). We decompose \( \mathbb{E}[\phi(X^M(K\delta t, X^M_0)) - \phi(X^N_k)] \) as

\[
\mathbb{E}
\left[
U^M(K\delta t, X^M_0) - U^M(0, X^N_k)
\right] = \left( \mathbb{E}
\left[
U^M(K\delta t, X^M_0)
\right] - \mathbb{E}
\left[
U^M(K\delta t, X^N_0)
\right] \right)
\]

\[
+ \left( \mathbb{E}
\left[
U^M(K\delta t, X^N_0)
\right] - \mathbb{E}
\left[
U^M(0, X^N_0)
\right] \right).
\]

The first term is controlled, by the regularity of \( U^M \) in Lemma 2 as

\[
\left| \mathbb{E}
\left[
U^M(K\delta t, X^M_0)
\right] - \mathbb{E}
\left[
U^M(K\delta t, X^N_0)
\right] \right| \\
\leq \int_0^1 \left| \mathbb{E}
\left[
D U^M(K\delta t, \theta X^M_0 + (1 - \theta) X^N_0) \cdot (I - P^N) X^M_0
\right] \right| d\theta
\]

\[
\leq C(1 + \| X^M_0 \|^2_E + \| X^N_0 \|^2_E) \min((1 + (K\delta t)^{-\alpha})e^{-\alpha \lambda} \| X_0 \|, (\lambda N)^{-\frac{2}{\alpha}} \| X_0 \|_H).
\]
Using the Itô formula for Skorohod integrals, the Kolmogorov equation and Malliavin integration by parts, the second term is split into

$$
\mathbb{E}\left[U^M(K\delta t, X^N_{0})\right] - \mathbb{E}\left[U^M(0, X^N_{k})\right] \\
= \sum_{k=0}^{K-1} \mathbb{E}\left[U^M(K\delta t - k\delta t, X^N_{k})\right] - \mathbb{E}\left[U^M(K\delta t - (k+1)\delta t, X^N_{k+1})\right] \\
= \sum_{k=0}^{K-1} \int_{t_k}^{t_{k+1}} \sum_{l \in \mathbb{N}^+} \mathbb{E}\left[D^2 U^M(T - t, \tilde{X}^N(t)) \cdot (\mathcal{D}_t \tilde{X}^N(t) P^N Q^2 c_t, S_{\delta l} P^N Q^2 e_t)\right] dt \\
+ \sum_{k=0}^{K-1} \left( \int_{t_k}^{t_{k+1}} \mathbb{E}\left[DF_{t,K\delta t,X} - AS_{\delta t} X^N_k\right]\right) dt \\
+ \left( \int_{t_k}^{t_{k+1}} \sum_{l \in \mathbb{N}^+} \mathbb{E}\left[D^2 U^M(T - t, \tilde{X}^N(t)) \cdot \left(P^N Q^2 c_t, P^M Q^2 e_t\right) - (S_{\delta l} P^N Q^2 c_t, S_{\delta l} P^N Q^2 e_t)\right]\right) dt \\
:= \mathbb{E}\left[U^M(K\delta t, X^N_{0})\right] - \mathbb{E}\left[U^M(K\delta t - \delta t, X^N_{1})\right] + \sum_{k=1}^{K-1} II^k_1 + II^k_2 + II^k_3 + II^k_4.
$$

The estimation for the first term can be easily obtained by the similar arguments in the proof of Theorem $3$ and thus we focus on the estimations on $II^k_1-II^k_4$, $k \geq 1$. By the regularity of $U^M$ in Lemmas 7 and 8 and the a priori estimate of $\tilde{X}^N$, we have

$$
| \sum_{k=1}^{K-1} II^k_1 | \leq \left| \sum_{k=0}^{K-1} \int_{t_k}^{t_{k+1}} \mathbb{E}\left[D^2 U^M(T - t, \tilde{X}^N(t)) \cdot (\mathcal{D}_t \tilde{X}^N(t) P^N Q^2 c_t, S_{\delta l} P^N Q^2 e_t)\right] dt \right| \\
\leq C \sum_{k=0}^{K-1} \int_{t_k}^{t_{k+1}} \left[1 + (T - t)^{\frac{\alpha}{\beta - 1}}\right] e^{-c(T-t)} \mathbb{E}\left[1 + \sup_{s \in [0,T-t]} \|X^M(s, \tilde{X}^N(t))\|_E^2\right] \left\|\mathcal{D}_t \tilde{X}^N(t)\right\|_{L_2^\beta}^2 \left\|\|(-A)^{\frac{\alpha}{\beta - 1}} S_{\delta t}\right\|_{L_2^\beta}^2 \left\|(-A)^{\frac{\alpha}{\beta - 1}}\right\|_{L_2^\beta}^2 \right] dt \\
\leq C(Q, X_0) \delta t^{\frac{\alpha}{\beta - 1}} \sum_{k=1}^{K-1} \left(1 + (T - t)^{\frac{\alpha}{\beta - 1}}\right) e^{-c(T-t)(t_{k+1} - t)} \leq C(Q, X_0) \delta t^{\frac{\alpha}{\beta - 1}},
$$

where we use the a priori estimate in Proposition 1 and the fact that for $t_k \leq t \leq s \leq t_{k+1}$,

$$
\mathcal{D}_s \tilde{X}^N(t) = S_{\delta t} \mathcal{D}_s X^N(t_k) + (t - t_k) P^N S_{\delta t} DF(\tilde{X}^N(t_{k+1})) \mathcal{D}_s \tilde{X}^N(t_{k+1}) + \mathcal{D}_s \int_{t_k}^{t} S_{\delta t} P^N dW(s) \\
\quad = (t - t_k) P^N S_{\delta t} DF(\tilde{X}^N(t_{k+1})) \mathcal{D}_s \tilde{X}^N(t_{k+1}).
$$
Then we estimate $I_{21}^k, I_{22}^k, I_{23}^k,$ and $I_{24}^k$, $k \geq 1$ separately. The definition of $\hat{X}$ leads to

$$I_{22}^k = \int_{t_k}^{t_{k+1}} \mathbb{E}\left[(DU^M(T-t, \hat{X}^N(t)), A(X_k^N - S_{\delta t}X_k^N))\right] dt + \int_{t_k}^{t_{k+1}} \mathbb{E}\left[(DU^M(T-t, \hat{X}^N(t)), (t-t_k)A^2S_{\delta t}X_k^N)\right] dt + \int_{t_k}^{t_{k+1}} \mathbb{E}\left[(DU^M(T-t, \hat{X}^N(t)), (t-t_k)AS_{\delta t}D^NF(X_{k+1}^N))\right] dt + \int_{t_k}^{t_{k+1}} \mathbb{E}\left[(DU^M(T-t, \hat{X}^N(t)), A\left(\int_{t_k}^{t} S_{\delta t}dW(s)\right)\right] dt$$

$$:= I_{21}^k + I_{22}^k + I_{23}^k + I_{24}^k.$$ 

From $I - S_{\delta t} = -\delta t(I - As)^{-1}$, the mild form of $X_k^N$, the a priori estimate of $\hat{X}$ in Lemma 3 and the regularity of $U^M$ in Lemmas 7 and 8 and Lemma 9 it follows for $k \geq 1$ and any small $\epsilon_1 > 0$

$$|I_{21}^k| \leq C\delta t \int_{t_k}^{t_{k+1}} (1 + (T-t)^{-\alpha})e^{-c(T-t)}\mathbb{E}\left[1 + \sup_{s \in [0,T-t]} \mathbb{E}X(s, \hat{X}^N(t))^4\right] dt$$

$$\|A^{1-\epsilon_1}S_{\delta t}^k\|\|A^{1-\alpha+\epsilon_1}S_{\delta t}\|\|X_k^N\|dt$$

$$+ C\delta t \int_{t_k}^{t_{k+1}} (1 + (T-t)^{-\alpha})e^{-c(T-t)}\sum_{j=0}^{k-1} \mathbb{E}\left(1 + \sup_{s \in [0,T-t]} \mathbb{E}X(s, \hat{X}^N(t))^4\right)$$

$$\|A^{1-\epsilon_1}S_{\delta t}^{k-j}\|\|A^{1-\alpha+\epsilon_1}S_{\delta t}\|\|F(X_{j+1}^N)\|dt$$

$$+ \left|\int_{t_k}^{t_{k+1}} \mathbb{E}\left[(DU^M(T-t, \hat{X}^N(t)), A^2\delta t \sum_{j=0}^{k-1} S_{\delta t}^{k+1-j}P^N\delta W_j)\right] dt\right|$$

$$\leq C(X_0, Q)(\delta t)^{\alpha-\epsilon_1} \int_{t_k}^{t_{k+1}} (1 + (T-t)^{-\alpha})e^{-c(T-t)}(1 + (t_k)^{-1+\epsilon_1}e^{-\epsilon_1t_k}) dt + I_{211}^k.$$ 

To deal with the last term, we use the idea in 3. that the lack of regularity and bad time behavior do not happen at the same time. In deed, we split the last term as

$$|I_{211}^k| \leq \left|\int_{t_k}^{t_{k+1}} \mathbb{E}\left[(DU^M(T-t, \hat{X}^N(t)), A^2\delta t \int_{0}^{\max(t_k-1,0)} P^N S_{\delta t}^{k+1-s}dW(s))\right] dt\right|$$

$$+ \left|\int_{t_k}^{t_{k+1}} \mathbb{E}\left[(DU^M(T-t, \hat{X}^N(t)), A^2\delta t \int_{\max(t_k-1,0)}^{t_k} P^N S_{\delta t}^{k+1-s}dW(s))\right] dt\right|.$$ 

For the first part, it follows from the Cauchy–Schwarz inequality, the regularity estimate of $U^M$, a priori estimate of $X_N$ and the smoothy effect of $S_{\delta t}$ and
Lemma 9 that

\[
\left| \int_{t_k}^{t_{k+1}} \mathbb{E} \left[ (DU^M(T - t, \hat{X}^N(t)), A^2 \delta t \int_0^{\max(t_k-1,0)} P^N S_{\delta t}^{k+1-[s]\delta t} dW(s) ) \right] dt \right|
\]

\[
\leq C \delta t \int_{t_k}^{t_{k+1}} \sqrt{\mathbb{E} \left[ \| \int_0^{\max(t_k-1,0)} A^2 P^N S_{\delta t}^{k+1-[s]\delta t} dW(s) \|^2 \right] \mathbb{E} \left[ \| DU^M(T - t, \hat{X}^N(t)) \|^2 \right] dt}
\]

\[
\leq C(Q, X_0) \delta t \int_{t_k}^{t_{k+1}} \sqrt{\mathbb{E} \left[ \| \int_0^{\max(t_k-1,0)} A^2 P^N S_{\delta t}^{k+1-[s]\delta t} dW(s) \|^2 \right] e^{-c(T-t)} dt}
\]

\[
\leq C(Q, X_0) \delta t \int_{t_k}^{t_{k+1}} \sqrt{\mathbb{E} \left[ \| \int_0^{\max(t_k-1,0)} (-A)^{\frac{1}{2}} P^N S_{\delta t}^{k+1-[s]\delta t} \|_{L^2(E)}^2 \right] e^{-c(T-t)} dt}
\]

Applying Malliavin calculus integration by parts, Malliavin differentiability of \( \hat{X}^N \) and the regularity estimate of \( U^M \) and Lemma 9 we have

\[
\left| \int_{t_k}^{t_{k+1}} \mathbb{E} \left[ (DU^M(T - t, \hat{X}^N(t)), A^2 \delta t \int_0^{\max(t_k-1,0)} P^N S_{\delta t}^{k+1-[s]\delta t} dW(s) ) \right] dt \right|
\]

\[
= \delta t \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} \sum_{t \in \mathbb{N}^+} \mathbb{E} \left[ \left| D^2 U^M(T - t, \hat{X}^N(t)) \cdot (D_Q^2 \hat{X}^N(t), A^2 S_{\delta t}^{k+1-[s]\delta t} P^N Q^2 \hat{e}_1) \right| \right] dt ds dt
\]

\[
\leq C \delta t \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} \sum_{t \in \mathbb{N}^+} \mathbb{E} \left[ \left| \left( -A \right)^{\frac{1}{2}} \int_0^{t} e^{-c(T-t)} (1 + (T - t) - 1 + \epsilon_1) dt \int_{t_k}^{t_{k+1}} \mathbb{E} \left[ \left( 1 + \sup_{r \in [0, T-t]} \left\| \hat{X}^N(r) \right\|_{L^2(E)} \right) \left\| X^M(r, \hat{X}^N(t)) \right\|_{L^2(E)}^4 \right] \right| \right] ds dt
\]

\[
\leq C(Q, X_0)(\delta t)^{\beta -2\epsilon_1} \int_{t_k}^{t_{k+1}} e^{-c(T-t)} (1 + (T - t) - 1 + \epsilon_1) \int_{t_k}^{t_{k+1}} (t_k - [s]_{\delta t})^{-1+\epsilon_1} e^{-c(t_k-[s]_{\delta t})} \left\| (A - A)^{\frac{1}{2}} \int_0^{t_k} D_s \hat{X}^N(t) \|_{L^2(\Omega, L^2)}^2 \right\| ds dt
\]

\[
\leq C(Q, X_0)(\delta t)^{\beta -2\epsilon_1} \int_{t_k}^{t_{k+1}} e^{-c(T-t)} (1 + (T - t) - 1 + \epsilon_1) dt.
\]
The above analysis leads to

\[ |II_{21}^k| \leq C(X_0, Q)(\delta t)^{\alpha - \epsilon_1} \int_{t_k}^{t_{k+1}} (1 + (T - t)^{-\alpha}) e^{-c(T-t)} dt
\]

\[ + C(X_0, Q)(\delta t)^{3 - 2\epsilon_1} \int_{t_k}^{t_{k+1}} (1 + (T - t)^{-1 + \epsilon_1}) e^{-c(T-t)} dt, \]

for \( k \geq 1 \). Since the estimation for \( II_{22}^k \) and \( II_{23}^k \) for \( k \geq 1 \) is similar, we omit the procedures. Again using Malliavin integration by parts yields that

\[ |II_{24}^k| = \left| \int_{t_k}^{t_{k+1}} \mathbb{E} \left[ \langle DU^M(T - t, \tilde{X}^N(t)), A \int_t^t S\delta_t dW(s) \rangle \right] dt \right|
\]

\[ = \left| \int_{t_k}^{t_{k+1}} \int_{t_k}^t \mathbb{E} \left[ \langle D^2U^M(T - t, \tilde{X}^N(t))D_\tilde{X} \tilde{X}^N(t), AS\tilde{X} \rangle \right] ds dt \right|
\]

\[ \leq \left| \int_{t_k}^{t_{k+1}} \int_{t_k}^t \mathbb{E} \left[ \langle A^{k+\frac{1}{2}} e^{-\epsilon_1} D^2U^M(T - t, \tilde{X}^N(t))A^{\frac{1}{2}}_s D_\tilde{X} \tilde{X}^N(t), A^{\frac{1}{2}}_s S\tilde{X}^{X^N} \rangle \right] ds dt \right|
\]

\[ \leq C(Q, X_0)(\delta t)^{3 - \epsilon_1} \int_{t_k}^{t_{k+1}} (1 + (T - t)^{-1 + \epsilon_1}) e^{-c(T-t)} dt. \]

It follows that

\[ |II_{21}^k| \leq C(X_0, Q)(\delta t)^{3 - 2\epsilon_1} \int_{t_k}^{t_{k+1}} e^{-c(T-t)} (1 + (T - t)^{-1 + \epsilon_1}) dt
\]

\[ + C(X_0, Q)(\delta t)^{\alpha - \epsilon_1} \int_{t_k}^{t_{k+1}} (1 + (T - t)^{-\alpha}) e^{-c(T-t)} (1 + (t_k)^{-1 + \epsilon_1} e^{-c(t_k)}) dt. \]

Now, we are in the position to estimate \( II_3^k \). By the regularity of \( DU^M \) and an a priori estimate of \( \tilde{X}^N \), we have

\[ |II_{31}^k| \leq \left| \int_{t_k}^{t_{k+1}} \mathbb{E} \left[ \langle DU^M(T - t, \tilde{X}^N(t)), P^M(I - P^N)F(\tilde{X}^N(t)) \rangle \right] dt \right|
\]

\[ + \left| \int_{t_k}^{t_{k+1}} \mathbb{E} \left[ \langle DU^M(T - t, \tilde{X}^N(t)), (I - S\delta_t)P^N F(\tilde{X}^N(t + 1)) \rangle \right] dt \right|
\]

\[ + \left| \int_{t_k}^{t_{k+1}} \mathbb{E} \left[ \langle DU^M(T - t, \tilde{X}^N(t)), P^N(F(\tilde{X}^N(t)) - F(\tilde{X}^N(t + 1))) \rangle \right] dt \right|
\]

\[ \leq C((\lambda N)^{-\alpha} + (\delta t)^{\alpha}) \int_{t_k}^{t_{k+1}} (1 + (T - t)^{-\alpha}) e^{-c(T-t)}
\]

\[ \mathbb{E} \left[ (1 + \sup_{r \in [0, T - t]} \mathbb{E}\tilde{X}^N(t)[\|X^M(r, \tilde{X}^N(t))[L^2(\tilde{X})])(1 + \|\tilde{X}(t)[L^2(\tilde{X})] dt
\]

\[ + \left| \int_{t_k}^{t_{k+1}} \mathbb{E} \left[ \langle DU^M(T - t, \tilde{X}^N(t)), P^N(F(\tilde{X}^N(t)) - F(\tilde{X}^N(t + 1))) \rangle \right] dt \right|. \]
Thus it sufficient to estimate the last term in the above inequality. It follows from Taylor expansion of \( F \), the regularity of \( \hat{X}^N \) and \( DU^M \), and the a priori estimate of \( \hat{X}^N \) that

\[
\int_{t_k}^{t_{k+1}} \mathbb{E} \left[ (DU^M(T-t, \hat{X}^N(t)), P^N(F(\hat{X}^N(t)) - F(X^N_{k+1})) \right] dt \\
\leq \int_{t_k}^{t_{k+1}} (t-t_{k+1}) \mathbb{E} \left[ (DU^M(T-t, \hat{X}^N(t)), P^N(DF(\hat{X}^N(t)) \cdot (AS_{Dt}X^N_k)) \right] dt \\
+ \int_{t_k}^{t_{k+1}} (t-t_{k+1}) \mathbb{E} \left[ (DU^M(T-t, \hat{X}^N(t)), P^N(DF(\hat{X}^N(t)) \cdot (S_{Dt}P^NF(X^N_{k+1})) \right] dt \\
+ \int_{t_k}^{t_{k+1}} \mathbb{E} \left[ (DU^M(T-t, \hat{X}^N(t)), P^N(DF(\hat{X}^N(t)) \cdot \left( \int_{t}^{t_{k+1}} S_{Dt}dW(s) \right) \right] dt \\
+ \int_{t_k}^{t_{k+1}} \mathbb{E} \left[ (DU^M(T-t, \hat{X}^N(t)), P^N(\int_{0}^{1} D^2 F(\theta \hat{X}^N(t) + (1 - \theta X^N_{k+1})) \\
\cdot (X(t) - X^N_{k+1}, X(t) - X^N_{k+1})(1 - \theta) d\theta) \right] dt := I_{31}^k + I_{32}^k + I_{33}^k + I_{34}^k.
\]

The estimation of \( I_{31}^k \) is similar to the estimation of \( I_{32}^k \) and we need to use a proper decomposition of the stochastic integral. The mild form of \( X^N_{k+1} \), Malliavin integration by parts and Lemma 9 yield that

\[
|I_{31}^k| := \left| \int_{t_k}^{t_{k+1}} (t-t_{k+1}) \mathbb{E} \left[ (DU^M(T-t, \hat{X}^N(t)), P^N(DF(\hat{X}^N(t)) \cdot (AS_{Dt}^{k+1}X_0^N)) \right] dt \\
+ \int_{t_k}^{t_{k+1}} (t-t_{k+1}) \mathbb{E} \left[ (DU^M(T-t, \hat{X}^N(t)), P^N(DF(\hat{X}^N(t)) \cdot \sum_{j=0}^{k-1} A \\
S_{Dt}^{k+1-j} P^N(F(X^N_{j+1})) \right] dt + |I_{311}^k| \\
\leq C \delta t \int_{t_k}^{t_{k+1}} e^{-c(T-t)} \mathbb{E} \left[ (1 + \sup_{r \in [0,T-t]} \mathbb{E} \hat{X}^N(t) || X^M(r, \hat{X}^N(t)) ||_E) (1 + || \hat{X}^N(t) ||_E) \\
\| A^{1-c_1} S_{Dt}^{k+1-j} \| A^{c_1} S_{Dt} \| X_0^N \| \right] dt \\
+ C \delta^2 t \int_{t_k}^{t_{k+1}} e^{-c(T-t)} \sum_{j=0}^{k-1} \mathbb{E} \left[ (1 + \sup_{r \in [0,T-t]} \mathbb{E} \hat{X}^N(t) || X^M(r, \hat{X}^N(t)) ||_E) (1 + || \hat{X}^N(t) ||_E) \\
\| A^{1-c_1} S_{Dt}^{k+1-j} \| A^{c_1} S_{Dt} \| X_0^N \| \right] dt + |I_{311}^k| \\
\leq C(Q, X_0) \int_{t_k}^{t_{k+1}} e^{-c(T-t)} dt \delta t^{1-c_1} (t_k^{-1+c_1} e^{-c_1 t_k} + \delta t \sum_{j=0}^{k-1} t_k^{-1+c_1} e^{-c_1 (t_k-t_j)}) + |I_{311}^k|.
\]

The estimation of \( I_{311}^k \) is similar to the estimation of \( I_{211}^k \) and we have

\[
|I_{311}^k| \leq \left| \int_{t_k}^{t_{k+1}} (t_{k+1} - t) \mathbb{E} \left[ (DU^M(t - t, \tilde{X}^N(t)), A \int_0^{\max(t_k - 1, 0)} P_N S_{\delta t}^{k+1 - |s|st} dW(s) \right] dt \right|
+ \left| \int_{t_k}^{t_{k+1}} (t_{k+1} - t) \mathbb{E} \left[ (DU^M(t - t, \tilde{X}^N(t)), A \int_{\max(t_k - 1, 0)}^{t_k} P_N S_{\delta t}^{k+1 - |s|st} dW(s) \right] dt \right|.
\]

Similar arguments in estimating \( I_{211}^k \) lead to

\[
\left| \int_{t_k}^{t_{k+1}} (t_{k+1} - t) \mathbb{E} \left[ (DU^M(t - t, \tilde{X}^N(t)), A \int_0^{\max(t_k - 1, 0)} P_N S_{\delta t}^{k+1 - |s|st} dW(s) \right] dt \right|
\leq C_0 \delta t \int_{t_k}^{t_{k+1}} \sqrt{\mathbb{E} \left[ (DU^M(t - t, \tilde{X}^N(t)), A \int_0^{\max(t_k - 1, 0)} P_N S_{\delta t}^{k+1 - |s|st} dW(s) \right]^2} dt
\leq C_0 (Q_0, Q) \delta t \int_{t_k}^{t_{k+1}} e^{-c(T - t)} dt.
\]

It follows from the smooth property of \( S_{\delta t} \), the Malliavin regularity, the a priori estimate of \( \tilde{X}(t) \) and Sobolev embedding theorem \( E \hookrightarrow \mathbb{H}^{\frac{n}{4} + \eta} \) that

\[
\left| \int_{t_k}^{t_{k+1}} (t_{k+1} - t) \mathbb{E} \left[ (DU^M(t - t, \tilde{X}^N(t)), A \int_0^{\max(t_k - 1, 0)} P_N S_{\delta t}^{k+1 - |s|st} dW(s) \right] dt \right|
\leq C(Q, Q_0) \delta t \int_{t_k}^{t_{k+1}} e^{-c(T - t)} \int_{\max(t_k - 1, 0)}^{t_k} \sum_{l \in \mathbb{N}^+} \| A^l S_{\delta t}^{k+1 - |s|st} Q^l_0 \|_F \| \mathbb{D}_0^l \tilde{X}(t) \|_{L^2(\Omega)} ds dt
+ C(Q, Q_0) \delta t \int_{t_k}^{t_{k+1}} e^{-c(T - t)} \int_{\max(t_k - 1, 0)}^{t_k} \sum_{l \in \mathbb{N}^+} \| A^l DU^M(t - t, \tilde{X}(t)) \|_{L^2(\Omega, \mathbb{R}^d)} \| A^{-\eta} D^2 F(\tilde{X}(t)) \|_{L^2(\Omega, \mathbb{R}^d)} ds dt
\leq C(Q, Q_0) \delta t \int_{t_k}^{t_{k+1}} e^{-c(T - t)} \int_{\max(t_k - 1, 0)}^{t_k} \| A^{\frac{3}{2} - \eta} S_{\delta t}^{k+1 - |s|st} A^{\frac{3}{2} - \eta} \|_{L^2(\Omega, \mathbb{R}^d)} ds dt
+ C(Q, Q_0) \delta t \int_{t_k}^{t_{k+1}} e^{-c(T - t)} (1 + (T - t)^{-\eta}) \int_{\max(t_k - 1, 0)}^{t_k} \| A^{\frac{3}{2} - \eta} S_{\delta t}^{k+1 - |s|st} A^{\frac{3}{2} - \eta} \|_{L^2(\Omega, \mathbb{R}^d)} ds dt
\leq C(Q, Q_0)(\delta t)^{3 - \eta} \int_{t_k}^{t_{k+1}} e^{-c(T - t)} (1 + (T - t)^{-\eta}) \int_{0}^{(t_k - s)^{-1 + \epsilon} e^{-c_1(t_k - s)}} ds dt,
\]

for \( \eta > \frac{3}{4} + \frac{\epsilon}{4} \). Similarly, we have

\[
|I_{32}^k| \leq C_0 \delta t \int_{t_k}^{t_{k+1}} e^{-c(T - t)} \mathbb{E} \left[ \left( 1 + \sup_{s \in [0, T - t]} \mathbb{E} \tilde{X}^N(t) \| X^M(s, \tilde{X}^N(t)) \|_{L^2} \right) (1 + \| \tilde{X}^N(t) \|_{L^2}) \right] dt,
\]
and

\[ \left| I_{34}^{k} \right| \leq \left| \int_{t_{k}}^{t_{k+1}} \int_{t_{N}}^{t_{N+1}} \mathbb{E} \left[ D^{2} U^{M}(T-t, \tilde{X}^{N}(t)) \cdot (P^{N}(DF(\tilde{X}^{N}(t)) \cdot (P^{N} S_{\delta t} \tilde{\xi}(t_{k+1}), D \tilde{X}^{N}(t)) \right] dsdt \right| \\
+ \left| \int_{t_{k}}^{t_{k+1}} \int_{t_{N}}^{t_{N+1}} \mathbb{E} \left[ (DU^{M}(T-t, \tilde{X}^{N}(t)), P^{N}(D^{2} F(\tilde{X}^{N}(t)) \cdot (D^{2} \tilde{\xi}(t_{k+1}), P^{N} S_{\delta t} \tilde{\xi}(t_{k})) \right] dsdt \right| \\
\leq C(Q, X_{0}) \delta t \int_{t_{k}}^{t_{k+1}} (1 + (T-t)^{-\eta}) e^{-c(T-t)} dt, \]

where we utilize the fact that for \( t_{k} \leq t \leq s \leq t_{k+1} \),

\[ \mathcal{D}_{s} \tilde{X}^{N}(t) = S_{\delta t} \mathcal{D}_{s} X^{N}(t_{k}) + (t - t_{k})P^{N} S_{\delta t} F(\tilde{X}^{N}(t_{k+1})) + \mathcal{D}_{s} X^{N}(t_{k+1}) + \mathcal{D}_{s} \int_{t_{k}}^{t_{k+1}} P^{N} S_{\delta t} dW(s) \]

Combining with the continuity of \( \tilde{X}^{N} \), for \( t \in [t_{k}, t_{k+1}] \),

\[ \| \tilde{X}^{N}(t) - \tilde{X}^{N}(t_{k+1}) \|_{L^{p}(\Omega, \mathbb{H})} \leq (t_{k+1} - t) \| A_{1}^{-1} \tilde{\xi} \|_{L^{p}(\Omega, \mathbb{H})} + C \| X(t_{k}) - X^{N}(t_{k}) \|_{L^{p}(\Omega, \mathbb{H})} \]

\[ + C \| F(\tilde{X}^{N}(t_{k+1})) \|_{L^{p}(\Omega, \mathbb{H})}(t_{k+1} - t) + \| \int_{t_{k}}^{t_{k+1}} P^{N} S_{\delta t} dW(s) \|_{L^{p}(\Omega, \mathbb{H})} \]

\[ \leq C(X_{0}, Q)(t_{k+1} - t)^{\frac{3}{4}}, \]

we deduce that

\[ \left| I_{34}^{k} \right| \leq C \int_{t_{k}}^{t_{k+1}} \mathbb{E} \left[ (DU^{M}(T-t, \tilde{X}^{N}(t)), P^{N}(t_{k}^{1} D^{2} F(\theta \tilde{X}^{N}(t) + (1 - \theta \tilde{X}^{N}(t_{k+1}))) \right] \]

\[ \leq C(Q, X_{0}) \int_{t_{k}}^{t_{k+1}} e^{-c(T-t)} (T-t)^{-\eta} \| \tilde{X}^{N}(t) - \tilde{X}^{N}(t_{k+1}) \|_{L^{2}(\Omega, \mathbb{H})} dt \]

\[ \leq C(Q, X_{0})(\delta t)^{\frac{3}{2}} \int_{t_{k}}^{t_{k+1}} e^{-c(T-t)} (T-t)^{-\eta} dt. \]

It follows that

\[ \left| I_{34}^{k} \right| \leq C(Q, X_{0}) \int_{t_{k}}^{t_{k+1}} e^{-c(T-t)} dt \delta t^{1-\varepsilon_{1}}(t_{k}^{1+\varepsilon} - c_{1} t_{k}) + \delta t \sum_{j=0}^{k-1} t_{k-j}^{1+\varepsilon} e^{-c_{1}(t_{k}-t_{j})} \]

\[ + C(Q, X_{0})(\delta t)^{\frac{3}{2}} \int_{t_{k}}^{t_{k+1}} e^{-c(T-t)} (1 + (T-t)^{-\eta}) \int_{0}^{t_{k}} (t_{k} - s)^{1-\varepsilon_{1}} e^{-c_{1}(t_{k}-s)} ds dt. \]
Corollary 3

Let Assumptions hold with \( d = 1 \) and \( \beta \in (0, 1] \), \( \gamma \in (0, \beta) \), \( X_0 \in E \), \( T > 0 \), \( \delta t_0 < (0, 1, \frac{1 - \gamma}{1 + \gamma}) \). Then for any \( \phi \in C^2_b(\mathbb{H}) \), there exists a positive constant \( C(X_0, Q, T) \) such that for any \( \delta t \in (0, \delta t_0) \), \( K\delta t = T \), \( K \in \mathbb{N}^+ \) and \( h \in (0, 1] \),

\[
\mathbb{E} \left[ \phi(X(T)) - \phi(X_k^h) \right] \leq C(X_0, T, Q) \left( \delta t^\gamma + h^{2\gamma} \right).
\]

In addition, under Assumption or for any \( \phi \in C^2_b(\mathbb{H}) \), there exist constants \( c > 0 \), \( C(X_0, Q) > 0 \) such that for any large \( K \), \( \delta t \in (0, \delta t_0) \) and \( h \in (0, 1] \),

\[
\mathbb{E} \left[ \phi(X_k^h(X_0^h)) - \int_\mathbb{E} \phi d\mu \right] \leq C(X_0, Q) (\delta t^\gamma + h^{2\gamma} + e^{-cK\delta t}).
\]
Furthermore, if $\mu^{h,\delta t}$ is an ergodic invariant measure of the numerical solution \( \{X_k^h\}_{k \in \mathbb{N}^+} \), we have
\[
\left| \mathbb{E} \left[ \int_{V_h} \phi d\mu^{h,\delta t} - \int_{\Omega} \phi d\mu \right] \right| \leq C(\delta t^\gamma + h^{2\gamma}).
\]

**Remark 5** The weak convergence analysis can be extended to the functional space $C^2_p(\mathbb{R})$, i.e., for $\phi \in C^2_p(\mathbb{R})$, the first and second derivatives of $\phi$ grow polynomially. For instance, under Assumption 5, one can first use the arguments in the proof of Lemma 2 to get the regularity estimate of Kolmogorov equation in a finite time $T$. Then similar arguments in Lemma 5 yield the exponential decay estimate for $t \geq T$ by using the $x$-independent uniform boundedness of $X^N$ (see the estimate (20)) and the Bismut-Elworthy-Li formula. Combining with the proof of Theorem 8 we can obtain the similar convergence rate of the proposed method for $C^2_p(\mathbb{R})$.

5 Numerical experiments

In this section, we present several numerical tests to verify the weak convergence rates of (26) in temporal direction and ergodicity. Apply (26) to Eq. (1) where $f(\xi) = -\xi^3 + \lambda_F \xi$, $\lambda_F \in \mathbb{R}$ and $W(t, \xi) = \sum_{j=1}^{\infty} \frac{1}{\sqrt{j!}} \sqrt{2} \sin(j\pi\xi) \beta_j(t)$ with $C$ characterizing the regularity of noise. In our numerical tests, we truncate the series by the first $M$ terms, $M \in \mathbb{N}^+$. We first investigate the weak convergence order in temporal direction of the proposed method (26) in this experiment. In order to show the rate of weak convergence, we fix $N = 2^6$ and take the method (26) with small time stepsize $\delta t^{ref} = 2^{-11}$ as the reference solution. Moreover, we choose four different kinds of functions (a) $\phi(u) = \cos(||u||^2)$, (b) $\phi(u) = \exp(-||u||^2)$, (c) $\phi(u) = \sin(||u||)$ and (d) $\phi(u) = ||u||^4$ as the test functions for weak convergence. Fig 1 plots the value $\ln(\mathbb{E}\phi(X^N(T)) - \mathbb{E}\phi(X^{N}_K))$, against $\ln(\delta t)$ for five different step sizes $\delta t = [2^{-5}, 2^{-6}, 2^{-7}, 2^{-8}, 2^{-9}]$ at $T = 1$, where $X^N(T)$ and $X^K_N$ represent the exact and numerical solutions at time $T$, respectively. Here, the expectation
$\mathbb{E}$ is approximated by taking average over 2000 realizations. It can be seen that $\mathbb{E}$ is of weak order 0.5 for cylindrical Wiener process, i.e. $C = 0$, and of weak order 1 for Q-Wiener process with $C = 0.5, 2$, which are indicated by the reference lines. These coincide with the theoretical analysis.

Then we consider the longtime behaviors of $\mathbb{E}$. Based on the definition of ergodicity, if numerical solution $\mathbb{E}$ is strong mixing, the average $\mathbb{E}[\phi(X^N_k)]$, $k > 0$, started from different initial values will converge to the spatial average for almost every path. To verify this property and to make clear how the average value changes when time $t$ goes, Fig. 2 shows the average of the fully discrete method started from five different initial values with the terminal time being 5 and $C = 0$. It can be seen that $\mathbb{E}[\phi(X^N_k(X_0))]$ started from different initial values converge to the same value in a short time for three different kinds of continuous and bounded functions $\phi$. Although the terminal time chosen here is not very large, in fact, this phenomenon still holds when time $t$ goes into infinity. Moreover, aiming at verifying that the mixed ergodicity does not need the condition $\lambda_F < \lambda_1$, we also show the case $\lambda_F = 12$ which implies $\lambda_F > \lambda_1$. From Fig. 3 it can be seen that for different test functions, the averages will converge to the same value. Numerical tests confirm theor-
ical findings. In fact, the averages started from different initial values will also converge for the Q-Wiener process case. For simplicity, we do not show those figures here.

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