AN INDEX-TYPE INVARIANT OF KNOT DIAGRAMS GIVING BOUNDS FOR UNKNOTTING FRAMED UNKNOTS

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Abstract. We introduce a new knot diagram invariant called the Self-Crossing Index (SCI). Using SCI, we provide bounds for unknotting two families of framed unknots. For one of these families, unknotting using framed Reidemeister moves is significantly harder than unknotting using regular Reidemeister moves.

We also investigate the relation between SCI and Arnold’s curve invariant St, as well as the relation with Hass and Nowik’s invariant, which generalizes cowrithe. In particular, the change of SCI under Ω3 moves depends only on the forward/backward character of the move, similar to how the change of St or cowrithe depends only on the positive/negative quality of the move.

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Knots in $\mathbb{R}^3$ can be represented using planar diagrams via taking a generic projection onto a plane $\mathbb{R}^2$ and marking each crossing with the information about which strand is an overcrossing and which is an undercrossing. This presentation is not unique, and two diagrams are equivalent if and only if they are connected by a sequence of Reidemeister moves of type $\Omega_1$, $\Omega_2$ and $\Omega_3$ (see Figures 1, 2, 3).

**Figure 1.** Right (a, c) and left (b, d) oriented Reidemeister moves of type I.

An important problem in knot theory is the problem of recognizing the unknot. One way to approach it is through finding upper or lower bounds for the length of a minimal sequence of moves required to untangle a unknot diagram. Recently, Lackenby [7] proved a polynomial upper bound of $(236c)^{11}$ for unknotting, where $c$ is the number of crossings of a diagram. On the other hand, Hass and Nowik presented in [6] a family of diagrams requiring $c^2/25$ moves to unknot, using a diagram invariant introduced in [5]. Another family of unknots with quadratic lower bound for unknotting has been constructed by Hayashi, Hayashi, Sawada and Yamada [4] using curve invariants defined by Arnold [1].

In this paper, we construct a new knot diagram invariant called SCI and prove that it provides bounds for unknotting framed knots. As with usual knots, two knot diagrams represent the same framed knot if and only if they are connected by...
a sequence of framed Reidemeister moves, which include usual $\Omega_2$ and $\Omega_3$ moves, but a different kind of $\Omega_1$ moves, which we call $\Omega_1F$ (see Figure 4).

Figure 4. Unoriented Reidemeister moves of type $\Omega_1F, \Omega_2$ and $\Omega_3$ (top to bottom).
The invariant SCI arises naturally as a version of invariants CI and OCI defined in \[10\], which were used there to distinguish forward and backward Reidemeister moves of type Ω3. Forward moves of type Ω1, Ω2 and Ω3 are presented in Figures 1, 2 and 3 as going from the diagram to the left to the diagram to the right. As we will see, SCI distinguishes these, too. The use of index closely resembles the technique used by Vassiliev to define invariants of ornaments, i.e. sets of curves in a plane \[11\]. Moreover, Shumakovitch \[9\] presented index-type formulas for Arnold’s curve invariant St, while Viro \[12\] proved formulas for Arnold’s curve invariants J⁺ and J⁻. The definition of SCI closely resembles a formula for St given by Shumakovitch, and SCI behaves in a similar manner under Reidemeister moves.

Our main result is the following:

**Theorem 1.1.** To unknot the family of diagrams $D_n$ (Figure 9) as framed unknots one needs to use at least $\frac{1}{2} (3n^2 - n + 2)$ moves of type Ω3.

Since SCI is easy to compute, the above follows easily if we understand how SCI changes under framed Reidemeister moves. The following theorem is the main tool to obtain bounds using SCI.

**Theorem 1.2.** SCI increases by 1 under forward Ω3 moves and does not change under Ω1F moves or Ω2 moves.

Of course, unknotting a framed unknot is not easier than unknotting the same unknot using regular Reidemeister moves. We show, using SCI, that unknotting a framed unknot can be essentially harder:

**Theorem 1.3.** The family of framed unknot diagrams $L_n$ is unknotted in $\Theta(n)$ moves using regular Reidemeister moves and in $\Theta(n^2)$ moves using framed Reidemeister moves.

The article is organized as follows. In Section 2 we describe Arnold’s invariants, including Shumakovitch’s \[9\] and Viro’s \[12\] index-type formulas for the invariants. In particular, we introduce indices of crossings that will be used to define SCI. The main part of the article is Section 3, where we define the invariant SCI. We prove its additivity under connected sum and that it is a Vassiliev diagram invariant of order 1. Then we prove Theorems 1.1, 1.2 and 1.3. Finally, in Section 4, we compare some of the properties of SCI to properties of the Hass-Nowik’s invariant \[5\] (denoted by $I_{lk}$ in their paper). In particular, we explain the relation between different types of Ω3 moves: positive/negative as defined by Arnold for curves \[1\], ascending/descending as defined by Östlund \[8\], and forward/backward as defined by one of the authors in \[10\]. The Appendix summarizes how known diagram invariants change under different types of Reidemeister moves.

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2. Index-type description of Arnold’s curve invariants

In this section, we recall the definition of Arnold’s curve invariants and state Shumakovitch’s \[9\] and Viro’s \[12\] theorems describing these in terms of indices.
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The definitions of indices will prove useful in the definition of the Self-Crossing Index, which is similar to the index-type description of the St curve invariant.

2.1. Arnold’s invariants. When mentioning Reidemeister moves on curves we consider the moves obtained from regular Reidemeister moves by forgetting the information about over- and undercrossings. The distinction between $\Omega_1$, matched $\Omega_2$, unmatched $\Omega_2$, and $\Omega_3$ moves carries over to the case of curves, as well as the notions of left and right $\Omega_1$ moves for oriented curves (see Figures 1, 2 and 3, while we need to choose an orientation to distinguish between matched and unmatched $\Omega_2$ moves, the matched/unmatched type does not depend on the orientation chosen).

By positive (or forward) moves of type $\Omega_1$ or $\Omega_2$ we define moves that create new crossings; their converses are called negative (or backward). In order to define Arnold’s invariant we also need to define what positive and negative moves of type $\Omega_3$ are.

**Definition 2.1** (vanishing triangle). The vanishing triangle of a $\Omega_3$ move is the triangle formed by the three edges contained in the diagram of a $\Omega_3$ move (see 3) which ends are the three crossings involved in a $\Omega_3$ move.

**Definition 2.2** (positive and negative $\Omega_3$ move). Consider an $\Omega_3$ move performed on a closed oriented curve $C$. Consider the vanishing triangle of this move. Assign an orientation to the vanishing triangle corresponding to the order in which its sides appear if we move along $C$ beginning at an arbitrary point. Let $n$ be the number of sides of the vanishing triangle whose orientation agrees with the orientation of the triangle and let $q = (-1)^n$. Then a $\Omega_3$ move is considered positive if it changes $q$ from $-1$ to $+1$ and negative if the reverse occurs.

**Remark 2.3.** The definitions of positive and negative moves carry over to regular Reidemeister moves (i.e. on knot/link diagrams). For moves of type $\Omega_1$ and $\Omega_2$ positive (resp. negative) moves are the same as forward (resp. backward) moves. For moves of type $\Omega_3$, these notions are different, and the relationship between these is clarified in Subsection 4.2.

**Definition 2.4** (Arnold [1]). The Arnold invariants $J^+$, $J^-$, and $St$ are defined by the following rules:

1. Orientation of the curve does not affect the invariants.
2. $J^+$ changes by $+2$ under positive matched $\Omega_2$ moves, and remains unchanged under unmatched $\Omega_2$ moves and $\Omega_3$ moves.
3. $J^-$ changes by $+2$ under positive unmatched $\Omega_2$ moves, and remains unchanged under matched $\Omega_2$ moves and $\Omega_3$ moves.
4. $St$ changes by $+1$ under positive $\Omega_3$ moves, and remains unchanged under $\Omega_2$ moves.
5. For curves $K_0$ and $K_i$, for $i \in \mathbb{N}_0$ (see Figure 5),
   a) $J^+(K_{i+1}) = -2i$, $J^+(K_0) = 0$;
   b) $J^-(K_{i+1}) = -3i$, $J^-(K_0) = -1$;
   c) $St(K_{i+1}) = i$, $St(K_0) = 0$.

Arnold proved that such invariants exist, and their uniqueness follows from the fact that any curve may be obtained from one of the $K_i$’s using $\Omega_2$ and $\Omega_3$ moves.

Note that these also can be used to obtain bounds for unknotting, as did Hayashi, Hayashi, Sawada and Yamada [4].

2.2. Indices with respect to a curve. We now proceed to define indices of points in the plane with respect to a given curve $C$. 

...
Definition 2.5 (the index of a point with respect to a curve). Let $\gamma : S^1 \to \mathbb{R}^2$ represent an oriented curve $C$, and let $p \in \mathbb{R}^2 \setminus \gamma(S^1)$. Then we define the index of $p$ with respect to $C$, denoted as $\text{ind}_C(p)$, to be the degree of the map $\tilde{\gamma}_p : S^1 \to S^1$ defined by

$$\tilde{\gamma}_p(t) = \frac{\gamma(t) - p}{\|\gamma(t) - p\|}.$$ 

We will drop the subscript $C$ from notation and write $\text{ind}(p)$ whenever it causes no confusion. ♦

Since the index with respect to $C$ is equal for all points in a connected component of the complement of $C$, we can define the following:

Definition 2.6 (indices of regions, edges and crossings). Let $C$ be an oriented curve.

Let $r$ be a region of $\mathbb{R}^2$, i.e. a connected component of $\mathbb{R}^2 \setminus C$. Then we define the index of the region $r$ with respect to $C$ to be

$$\text{ind}_C(r) = \text{ind}_C(p)$$

for any $p \in r$. We denote the set of all regions by $\mathcal{R}(C)$.

Denote by $\mathcal{E}(C)$ the set of all crossings of $C$, and by $\mathcal{S}(C)$ the set of all edges of $C$, i.e. connected components of $C \setminus \mathcal{E}(C)$. Let $e$ be an edge of $C$ and define its index with respect to $C$ to be

$$\text{ind}_C(e) = \frac{1}{2} \sum_{r \in \mathcal{R}(e)} \text{ind}_C(r)$$

where $\mathcal{R}(e) \subset \mathcal{R}(C)$ is the set of two regions adjacent to $e$.

Let $c$ be a crossing of $C$, define its index with respect to $C$ to be

$$\text{ind}_C(c) = \frac{1}{4} \sum_{r \in \mathcal{R}(c)} \text{ind}_C(r)$$

where $\mathcal{R}(c) \subset \mathcal{R}(C)$ is the set of four regions adjacent to $c$ (counted with multiplicity). ♦

2.3. Viro’s formulas for $J^+$ and $J^-$. To introduce Viro’s formulas for $J^+$ and $J^-$ we recall the definition of smoothing of a crossing:

Definition 2.7. Let $c$ be a crossing of an oriented curve $C$. Then the smoothing of $C$ consists of two (potentially intersecting) curves created by removing the crossing $c$ and replacing it with two non-intersecting strands that preserve the original orientation (cf. with Figure 4). ♦

Note that if $C$ is not oriented, taking any of the two possible orientations gives the same smoothing. The definition above goes through for crossings between oriented curves as well as for crossings of oriented link diagrams.
Figure 6. Smoothing a crossing of a knot is just smoothing the crossing of the underlying curve and forgetting about the crossing information.

Now we are ready to state

**Theorem 2.8** (Viro [12]). Let \( C \) be a curve with \( n \) double points. Let \( \bar{C} \) be the diagram obtained by smoothing all crossings of \( C \), and let \( \mathcal{R}(\bar{C}) \) be the set of regions in the complement of \( \bar{C} \). Then

\[
J^+(C) = 1 + n - \sum_{r \in \mathcal{R}(\bar{C})} (\chi(r) \text{ind}^2(r)),
\]

\[
J^-(C) = 1 - \sum_{r \in \mathcal{R}(\bar{C})} (\chi(r) \text{ind}^2(r)),
\]

where \( \chi \) is the Euler characteristic and \( n \) the number of crossings of \( C \).

We see that \( J^+ \) and \( J^- \) admit explicit descriptions using indices. This is useful for both calculating the values of \( J^+ \) and \( J^- \) for given diagrams, as well as proving properties of \( J^+ \) and \( J^- \), such as:

**Proposition 2.9.** Choose an orientation of \( C \). Under a left (resp. right) positive \( \Omega_1 \) move, \( J^- \) changes by \(-2 \text{ind}(c) - 1\) (resp. \(2 \text{ind}(c) - 1\)) and \( J^+ \) changes by \(-2 \text{ind}(c)\) (resp. \(2 \text{ind}(c)\)), where \( c \) is the crossing created by the \( \Omega_1 \) move.

*Proof.* The addition of a loop by a left positive \( \Omega_1 \) move affects formulas for \( J^+ \) and \( J^- \) in only two ways: adding a new region to the complement of \( \bar{C} \) and changing the Euler characteristic of the region that the loop is made in. Let \( c \) be the crossing created by the \( \Omega_1 \) move. Then the new region in the complement of \( \bar{C} \) is a disk with Euler characteristic 1 and index \( \text{ind}(c) + 1 \). The Euler characteristic of the region surrounding the loop decreases by 1 and the index remains \( \text{ind}(c) \). Thus, the change to \( J^- \) is

\[
\Delta J^- = -((\text{ind}(c) + 1)^2 - ((\chi_1 - 1) \text{ind}^2(c) - \chi_1 \text{ind}^2(c))) = -2 \text{ind}(c) - 1
\]

under one positive \( \Omega_1 \) move, where \( \chi_1 \) is the original Euler characteristic of the region surrounding the loop.

From \( J^+ = J^- + n \), where \( n \) is the number of crossings, it follows that \( J^+ \) changes by \(-2 \text{ind}(c)\) under a left positive \( \Omega_1 \) move.

Changing the orientation of \( C \), we obtain the desired results for right positive \( \Omega_1 \) moves.

\[\square\]

2.4. **Shumakovitch’s formulas for St.** We proceed to Shumakovitch’s formulas for St. First we need to define *weights*.

Fix an arbitrary point \( p \) on the oriented curve \( C \) which is not one of its \( n \) crossings. Label the edges from 1 to 2\( n \) following the orientation of the curve, with the edge containing \( p \) being labeled by 1.
Definition 2.10 (weight). Consider a crossing $c$. Denote the edges pointing towards $c$ by $e_i$ and $e_j$, where $i$ and $j$ are their respective labels, with $e_i$ crossing $e_j$ from left to right (see Figure 7). Let $\text{sgn}(k)$ be the sign of the integer $k$. Then set

\[ \omega(c) = \text{sgn}(i - j), \]
\[ \omega(e_i) = \text{sgn}(i - j), \]
\[ \omega(e_j) = -\text{sgn}(i - j). \]

Let $r_W$ be the region directly to the left of $e_i$, $r_E$ be the region directly to the right of $e_j$, $r_S$ be the region directly to the right of $e_i$ and left of $e_j$, and $r_N$ be the remaining region surrounding $c$ (see Figure 7). The weight $\omega(r)$ of a region is the sum of the contributions of all adjacent crossings (with multiplicity two if a region is adjacent to a crossing in two ways), denoted $\omega_c(r)$, which are equal to

\[ \omega_c(r_W) = \omega_c(r_E) = \frac{1}{2} \text{sgn}(i - j), \]
\[ \omega_c(r_N) = \omega_c(r_S) = -\frac{1}{2} \text{sgn}(i - j). \]

Remark 2.11. Writhe of a knot diagram $D$ is the sum of signs of all crossings:

\[ w(D) = \sum_{c \in \mathcal{C}(D)} \text{sgn}(c). \]

Using the weights defined above, one can try to define a curve invariant via

\[ \sum_{c \in \mathcal{C}(C)} \omega(c). \]

This is not invariant under the choice of the point $p$, but an easy inspection shows that if we subtract $2 \text{ind}(p)$, one obtains a curve invariant. Checking how it changes under Reidemeister moves and calculating the value on a simple closed curve one obtains the winding number of $C$:

\[ \text{wind}(C) = -2 \text{ind}(p) + \sum_{c \in \mathcal{C}(C)} \omega(c). \]

Figure 7. Calculation of weight for crossing $c$ and the edges $e_i$ and $e_j$ pointing towards $c$, and the contribution to the weight of the surrounding regions.

Theorem 2.12 (Shumakovich [9]). Let $C$ be an oriented curve. Then

\[ \text{St}(C) = \sum_{c \in \mathcal{C}(C)} (\omega(c) \text{ind}(c)) + \delta^2 - \frac{1}{4}, \]
\[ \text{St}(C) = \frac{1}{2} \sum_{e \in \partial(C)} (\omega(e) \text{ind}^2(e)) + \delta^2 - \frac{1}{4}, \]

\[ \text{St}(C) = \frac{1}{3} \sum_{r \in \partial(C)} (\omega(r) \text{ind}^3(r)) + \delta^2 - \frac{1}{4}, \]

where \( \delta = \text{ind}(e_p) \), where \( e_p \) is is the edge containing \( p \).

Again, these descriptions allow to easily examine some properties of St.

**Proposition 2.13.** Arnold’s invariant \( \text{St} \) changes by \( + \text{ind}(c) \) under a left positive \( \Omega_1 \) move and \( - \text{ind}(c) \) under a right positive \( \Omega_1 \) move, where \( c \) is the new crossing formed by the \( \Omega_1 \) move.

**Proof.** This follows from Equation (1). First, choose a point \( p \) which does not lie on the edge the \( \Omega_1 \) move is applied to. We can do so if the diagram is not a trivial unknot diagram, in which case the proposition is easily checked to be true.

A positive \( \Omega_1 \) move adds a crossing. If we keep the same starting point \( p \), the numbering of edges changes, but weights of any other crossings stay the same, since the labels of adjacent edges all shift by either 0 or 2. Let the three edges connected to the new crossing be numbered \( k, k+1, \) and \( k+2 \). For a left \( \Omega_1 \) move, we get \( \omega(c) = \text{sgn}((k+1)-k) = +1 \), and for a right one we get \( \omega(c) = \text{sgn}(k-(k+1)) = -1 \), proving the proposition. \( \square \)

3. The Self-Crossing Index and bounds for unknotting framed knots

In this section, we introduce a new knot diagram invariant, called the Self-Crossing Index, or SCI. We prove that it is additive under connected sum and that it is Vassiliev of order 1. We finally show how it provides bounds for unknotting framed knots via Theorems 1.1, 1.2 and 1.3.

**3.1. Definition and properties of SCI.** In the previous section, we defined indices of points with respect to a curve, weights of regions, edges and crossings of a closed curve, as well as smoothing of a crossing. All these generalize to the case of a knot diagram by considering the underlying curve of a diagram.

**Definition 3.1** (Self-Crossing Index). Let \( D \) be an oriented knot diagram and let \( \mathcal{C}(D) \) be the set of crossings of \( D \). Then

\[ \text{SCI}(D) = \sum_{c \in \mathcal{C}(D)} \text{sgn}(c) \text{ind}(c) \]

where \( \text{sgn}(c) \) is the \textit{sign} of the crossing \( c \).

We immediately notice similarity with Equation (1). Suppose the knot diagram \( D \) is ascending. Let \( p \) be a lowest point of the diagram \( D \), that is a point such that if we move along \( D \) starting at \( p \), then each crossing is passed through its undercrossing first. If the same point \( p \) is taken to calculate weights as in the previous section, then we obtain that \( \omega(c) = \text{sgn}(c) \) for any crossing \( c \). Thus

\[ \text{St}(C) = \sum_{c \in \mathcal{C}(D)} (\text{sgn}(c) \text{ind}(c)) + \delta^2 - \frac{1}{4}, \]

and therefore
Theorem 3.2. For an ascending knot diagram $D$ and its underlying curve $C$,

$$\text{SCI}(D) = \text{St}(C) - \delta^2 + \frac{1}{4},$$

where $\delta = \text{ind}(p)$, $p$ being a lowest point of $D$.

From this we immediately obtain formulas for SCI similar to Equations (2) and (3), under the assumption that $D$ is ascending. However, we aim to prove such formulas for SCI in full generality. For this, we need to define modified weights depending on the signs of the crossings rather than on the topology of the curve.

Definition 3.3. Using a set-up similar to that used to define weight (Definition 2.10), we define, for a crossing $c$,

$$\tilde{\omega}(e_i) = \text{sgn}(c),$$
$$\tilde{\omega}(e_j) = -\text{sgn}(c),$$
$$\tilde{\omega}_c(r_W) = \tilde{\omega}_c(r_E) = \frac{1}{2} \sum_{c \in \mathcal{C}(r)} \text{sgn}(c),$$
$$\tilde{\omega}_c(r_N) = \tilde{\omega}_c(r_S) = -\frac{1}{2} \sum_{c \in \mathcal{C}(r)} \text{sgn}(c),$$
$$\tilde{\omega}(r) = \sum_{c \in \mathcal{C}(r)} \tilde{\omega}_c(r),$$

where $\mathcal{C}(r)$ is the set of all crossings adjacent to the region (again, counted with multiplicities).

Theorem 3.4. Let $D$ be an oriented knot diagram and let $\mathcal{E}(D)$ and $\mathcal{R}(D)$ be the set of edges and regions of $D$, respectively. Then

(4) $$\text{SCI}(D) = \frac{1}{2} \sum_{e \in \mathcal{E}(D)} \tilde{\omega}(e) \text{ind}^2(e),$$
(5) $$\text{SCI}(D) = \frac{1}{3} \sum_{r \in \mathcal{R}(D)} \tilde{\omega}(r) \text{ind}^3(r).$$

Proof. We follow the argument given in [9].

To show that these three formulas of SCI are equivalent, we will show that the calculations are equivalent in a neighborhood of a crossing $c$. From the definition, the contribution of a crossing $c$ to SCI is $\text{sgn}(c) \text{ind}(c)$.

Denote $\alpha = \text{ind}(c)$. Then $\text{ind}(e_i) = \alpha + \frac{1}{2}$, $\text{ind}(e_j) = \alpha - \frac{1}{2}$. Therefore the contribution of the edges $e_i, e_j$ to the sum (4) is

$$\frac{1}{2} \left( \text{sgn}(c) \left( \alpha + \frac{1}{2} \right)^2 - \text{sgn}(c) \left( \alpha - \frac{1}{2} \right)^2 \right) = \alpha \text{sgn}(c) = \text{sgn}(c) \text{ind}(c).$$

This proves the identity (4).

Similarly, we can consider the regions around $c$, and consider the contribution to $\tilde{\omega}(r)$ that $c$ makes (i.e. $\pm \text{sgn}(c)/2$). Since $\tilde{\omega}(r)$ is just the sum of contributions of adjacent crossings, we can rewrite (5) as

$$\text{SCI}(D) = \frac{1}{3} \sum_{c \in \mathcal{C}(r)} \sum_{r \in \mathcal{R}(c)} \pm \text{sgn}(c) \text{ind}^3(r)/2,$$
where $\mathcal{C}(D)$ is the set of crossings of $D$, $\mathcal{R}(c)$ is the set of four regions surrounding a crossing $c$ and the sign $\pm$ depends on whether $r$ is $r_E, r_W, r_N$ or $r_S$ for that crossing. However, since $\text{ind}(r_W) = \alpha + 1$, $\text{ind}(r_E) = \alpha - 1$, and $\text{ind}(r_N) = \text{ind}(r_S) = \alpha$, we have

$$\frac{1}{3} \sum_{r \in \mathcal{R}(c)} \pm \text{sgn}(c) \text{ind}^3(r)/2 = \frac{1}{3} \left( \frac{\text{sgn}(c)}{2} \left( (\alpha + 1)^3 + (\alpha - 1)^3 \right) - \frac{\text{sgn}(c)}{2} \left( \alpha^3 + \alpha^3 \right) \right) = \alpha \text{sgn}(c),$$

which proves (5). \qed

Another remarkable property of SCI is its additivity under connected sums.

**Theorem 3.5.** Let $D$ and $E$ be two knot diagrams and $D \# E$ denote their connected sum. Then

$$(6) \quad \text{SCI}(D \# E) = \text{SCI}(D) + \text{SCI}(E).$$

**Proof.** The connected sum of $D$ and $E$ leaves the signs of the crossings unchanged. In addition, the indices of the regions do not change, as the operation simply merges together two regions with the same index. Thus the indices and signs of the crossings stay the same and summing along all the crossings of $D$ and $E$ gives the identity in the proposition. \qed

We also note that SCI is a **Vassiliev (diagram) invariant** of order 1.

**Definition 3.6** (finite type/Vassiliev diagram invariant [2]). Let $D$ be a knot diagram. Let $S$ be a subset of crossings of $D$, $S \subset \mathcal{C}(D)$. For a knot diagram invariant $I$ we define, inductively,

$$I_S(D) = I_{(S \setminus \{c\})}(D) - I_{(S \setminus \{c\})}(D_c),$$

where $c$ is an arbitrary crossing in $S$ and $D_c$ is the diagram $D$ with the crossing $c$ changed. Equivalently,

$$I_S(D) = \sum_{X \subseteq S} (-1)^{|X|} I(D_X),$$

where $D_X$ is the diagram $D$ with all the crossings from $X$ changed.

We define $I$ to be a **Vassiliev invariant of order at most** $m \geq 0$ (or **finite type invariant**) if $I_S(D) = 0$ for any diagram $D$ and any subset $S \subset \mathcal{C}(D)$ such that $|S| = m + 1$. We say that $I$ is exactly **of order** $m$ if it is of order at most $m$ and there is a diagram $D$ and set $S \subset \mathcal{C}(D)$ such that $|S| = m$ and $I_S(D) \neq 0$. \diamond

**Remark 3.7.** Diagram invariants arising from curve invariants (e.g. $\text{St}$, $J^+$, $J^-$) are Vassiliev of order 0.

**Theorem 3.8.** SCI is a **Vassiliev invariant of order 1**.

**Proof.** Take any knot diagram $D$ with at least two crossings. Let $a \neq b$ be two crossings of $D$. Clearly, changing a crossing does not change any indices of crossings and it changes the sign of one crossing. Thus we have

$$\text{SCI}_{\{a\}}(D) = \text{SCI}(D) - \text{SCI}(D_a) = 2 \text{sgn}(a) \text{ind}(a).$$

But similarly we have $\text{SCI}_{\{a\}}(D_b) = 2 \text{sgn}(a) \text{ind}(a)$. Thus $\text{SCI}_{\{a,b\}}(D) = 0$, so SCI is Vassiliev of order at most 1.
Finally, taking any diagram $D$ which has a crossing $a$ of index $\text{ind}(a) \neq 0$ we get that SCI is of order 1.

3.2. Bounds for unknotting via SCI. We now prove Theorem 1.2 which is the key tool in establishing bounds for unknotting framed knots using SCI.

Proof of Theorem 1.2. For $\Omega 1 F$ or $\Omega 2$ moves, each move creates or removes two crossings of opposite sign and of the same index, thus preserving SCI.

For $\Omega 3$ moves, the signs of the crossings remain unchanged, so it is enough to consider changes of indices of the crossings. There are eight cases that need to be considered (cf. Figure 3). We consider the case of an $\Omega 3 a$ move since all the other cases are similar.

![Figure 8. Changes to indices of regions adjacent to the crossings involved in a $\Omega 3 a$ move.](image)

For a forward $\Omega 3 a$ move (left to right in Figure 8), all three indices of crossings increase by 1. The signs of the crossings are $+1$, $+1$ and $-1$, so the overall change to SCI equals $+1$, as claimed.

![Figure 9. The family of unknots $D_n$.](image)

Corollary 3.9. For an unknot diagram $D$, the number of Reidemeister moves of type $\Omega 3$ needed to unknot $D$ is greater or equal to $|\text{SCI}(D)|$.

Proof. Since SCI is zero for any trivial knot diagram, the result follows from Theorem 1.2. □
Proof of Theorem 1.1. Through computation, we find that
\[ \text{SCI}(D_n) = \frac{1}{2} (3n^2 - n + 2). \]
The desired result follows from Corollary 3.9. □

Hass and Nowik [6] found a lower bound of \(2n^2 + 3n - 2\) using HN (see Definition 4.1) for unknotting a similar family of unknots using regular Reidemeister moves. Using the same procedure as Hass and Nowik [6], one obtains a quadratic lower bound of \(2n^2 + 2n - 1\) for unknotting \(D_n\), even in the framed setting. Indeed, we have
\[ \text{HN}(D_n) = nX_n + nX_{-n} + (2n-1)X_{-1} + (4n-1)Y_0. \]
Let \(g: \mathbb{Z} \rightarrow \mathbb{Z}\) be the homomorphism defined by \(g(X_k) = 1 + |k|\) and \(g(Y_k) = -1 - |k|\). Then
\[ g(\text{HN}(D_n)) = 2n^2 + 2n - 1. \]
Let \(R\) be the set of \(\pm(X_k + Y_k), \pm(X_k + Y_{k+1}), \pm(X_{k+1} - X_k)\) and \(\pm(Y_{k+1} - Y_k)\), for all integers \(k\), which represent all possible changes of HN under framed Reidemeister moves (see [5] for discussion on changes of HN under Reidemeister moves). Since \(|g(r)| \leq 1\) for all \(r \in R\), the lower bound for the number of framed Reidemeister moves to unknot \(D_n\) obtained from HN is \(2n^2 + 2n - 1\).

This lower bound is higher than that found by SCI, which has \(\frac{3}{2}\) as the quadratic coefficient. However, SCI is still useful, as it provides bounds on the minimal number of \(\Omega^3\) moves. \(g(\text{HN})\) does not provide such a bound, as it changes under unmatched \(\Omega^2\) moves. In fact, the change of HN under a \(\Omega^3\) move can be expressed as a sum of changes under a matched and unmatched \(\Omega^2\) move. Therefore, the value of HN is not sufficient to distinguish \(\Omega^3\) moves from combinations of \(\Omega^2\) moves.

Finally, we show that the minimal number of framed Reidemeister moves for unknotting can be degrees higher than the number of regular Reidemeister moves needed. One example is the following family of unknot diagrams \(L_n\).

\[\begin{array}{c}
\text{n positive loops} \\
\text{...} \\
\text{n negative crossings} \\
\end{array}\]

Figure 10. A family of unknots \(L_n\).
Proof of Theorem [1.3]. Clearly, $L_n$ may be unknotted using $2n$ Reidemeister moves. Moreover, $L_n$ has $2n$ crossings, and thus needs at least $n$ Reidemeister moves to unknot. Therefore the minimal unknotting sequence has length $\Theta(n)$.

As a framed unknot, $L_n$ may be unknotted inductively in the following way. “Push” a loop from outside onto the loop in the middle of the diagram using $2n$ moves of type $\Omega_2$ and $n$ moves of type $\Omega_3$. Then, use a $\Omega_2$ move in the middle of the diagram to obtain $L_{n-1}$.

On the other hand, calculating SCI for $L_n$ gives that $SCI(L_n) = \frac{n(n + 1)}{2}$, so we need at least this number of $\Omega_3$ moves to unknot $L_n$ as a framed unknot. This proves $L_n$ is optimally unknotted in $\Theta(n^2)$ moves. □

Remark 3.10. Since SCI gives bounds for the number of $\Omega_3$ moves, we may strengthen the bound in the proof above by considering other invariants (e.g. the number of crossings) and bounds on $\Omega_1$ and $\Omega_2$ moves that these provide.

4. Comparison with the Hass-Nowik invariant

In this section, we recall the definition of a knot diagram invariant $HN$ given by Hass and Nowik in [5]. The invariant was used by Hass and Nowik to prove quadratic bounds for unknotting a family of diagrams almost identical to the family $D_n$. We prove it is additive under connected sum and is not a Vassiliev invariant. We end this section with a discussion of the relationship between forward/backward and positive/negative character of $\Omega_3$ moves, which is established in Proposition 4.8.

4.1. $HN$ and its properties.

**Definition 4.1** (Hass-Nowik diagram invariant [5]). Let $D$ be an oriented knot diagram. Denote by $lk$ the linking number of a two-component link. For such $D$, we define

$$HN(D) = \sum_{c \in \mathcal{C}_+(D)} X_{lk(D^c)} + \sum_{c \in \mathcal{C}_-(D)} Y_{lk(D^c)}$$

where $\mathcal{C}_+(D)$ be the set of positive crossings and $\mathcal{C}_-(D)$ be the set of negative crossings of $D$, and $D^c$ denotes the two-component link obtained by smoothing $D$ at $c$. This invariant takes values in $\mathbb{Z}$, the free abelian group with basis $\{X_k, Y_k\}_{k \in \mathbb{Z}}$.

Remark 4.2. In [5], $HN$ is a part of a larger family $I_\phi$ defined for any 2-component link invariant $\phi$. Precisely, $HN = I_{lk}$.

It turns out that $HN$ is additive under connected sum, similarly to SCI.

**Theorem 4.3.** For any two knot diagrams $D$ and $E$,

$$HN(D\#E) = HN(D) + HN(E).$$

**Proof.** Let $\mathcal{C}(D)$ and $\mathcal{C}(E)$ be the sets of crossings of $D\#E$ that come from $D$ and $E$ respectively. Let $\hat{D}$ and $\hat{E}$ be the parts of the diagram $D\#E$ that come from $D$ and $E$, respectively. The linking number of a two-component link is equal to the half of the sum of signs of crossings between the components. After smoothing a crossing $a \in \mathcal{C}(D)$, $\hat{E}$ is contained entirely within one of the two components.
Thus, none of the crossings in $C(E)$ contribute to the linking number of the two-component link, meaning that the link $(D\#E)^a$ has the same linking number as $D^a$.

The same reasoning shows that $\text{lk}(D\#E)^b = \text{lk}(E^b)$ for any $b \in C(E)$. Thus, $\text{HN}(D\#E) = \text{HN}(D) + \text{HN}(E)$. □

Unlike SCI, HN is not a Vassiliev invariant. We show this using the standard diagrams of $(2,p)$-torus knots (for $p$ odd), which we denote $T(2,p)$. These diagrams are characterized by the property that they have $p$ positive crossings and are alternating (cf. Figure 11).

Figure 11. Diagram of the $(2,5)$-torus knot

Lemma 4.4. Let $S$ be the set of all crossings of $T(2,p)$. Let $S_k$ be any subset of $S$ with cardinality $k$, and $T(2,p)_{S_k}$ be the knot diagram of $T(2,p)$ with the crossings of $S_k$ changed. Then

$$\text{HN}(T(2,p)_{S_k}) = (p - k)X_{\frac{p-2k-1}{2}} + kY_{\frac{p-2k+1}{2}}.$$  

Proof. First, notice that any crossing created by smoothing a crossing in $T(2,p)$ is a crossing between different components of the link obtained. Since $T(2,p)$ differs from $T(2,p)_{S_k}$ by just crossing changes, the same is true for $T(2,p)_{S_k}$.

Therefore, smoothing a positive crossing of $T(2,p)_{S_k}$ leaves $p - k - 1$ positive crossings and $k$ negative crossings, so each positive crossing contributes $X_{\frac{p-2k-1}{2}}$ to HN. Similarly, smoothing a negative crossing contributes $Y_{\frac{p-2k+1}{2}}$ to HN since there are $p - k$ positive and $k - 1$ negative crossings left. □

Theorem 4.5. HN is not a Vassiliev diagram invariant.

Proof. Let $C_p$ be the set of all crossings of $T(2,p)$. From Lemma 4.4 we obtain that

$$\text{HN}_{C_p}(T(2,p)) = \sum_{S \subset C_p} (p - |S|)X_{\frac{p-2|S|-1}{2}} + |S|Y_{\frac{p-2|S|+1}{2}}$$

and since there is only one subset $S \subset C_p$ such that $|S| = 0$, therefore the coefficient of $X_{\frac{p-1}{2}}$ in the sum above is equal to $p \neq 0$, so $\text{HN}_{C_p}(T(2,p)) \neq 0$. This finishes the proof, since if HN was a Vassiliev invariant of order $n$, then for any $p > n$ we would have $\text{HN}_{C_p}(T(2,p)) = 0$. □

4.2. Forward/backward, positive/negative and ascending/descending $\Omega 3$ moves. The relationship of forward/backward to positive/negative $\Omega 3$ moves is best understood using the notions of ascending and descending $\Omega 3$ moves introduced by Östlund [8]:
Definition 4.6 (ascending and descending Ω3 moves). Follow the orientation of the knot diagram. An Ω3 move is ascending if the three segments involved are passed in the order bottom-middle-top, and descending if the three segments involved are passed in the order top-middle-bottom.

Remark 4.7. The ascending/descending classification of a move does not change when reversing the move. For instance, if we consider diagrams in Figure 12, it does not matter if we go from left to right or from right to left. On the contrary, reversing a move changes its forward/backward or positive/negative classification.

Proposition 4.8. An ascending Ω3 move is forward if and only if it is positive. A descending Ω3 move is forward if and only if it is negative.

Proof. The eight cases of Ω3 moves can be placed into two groups based on the bottom-middle-top orientation of the vanishing triangle. In cases a, d, e, and g it is clockwise, and in cases b, c, f, and h it is counterclockwise. We will prove Proposition 4.8 for Ω3a and Ω3b moves, as others are dealt with similarly.

For an ascending Ω3a move, of the three strands involved in the move, the bottom strand connects to the middle strand, the middle to the top, and the top to the bottom (see Figure 12). The order-of-appearance orientation of the vanishing triangle, introduced in Definition 2.2, is clockwise, which agrees with orientations of all three strands. This makes $q' = -1$ for this diagram. Once the Ω3 move is made, the order-of-appearance orientation remains clockwise, but it now disagrees with orientations of all three strands, so $q' = +1$. It follows that a forward ascending Ω3a move is positive.

For a descending Ω3a move, the order-of-appearance orientation is opposite (as it was in the case of Ω3a move), so a forward descending Ω3a move is negative.

For an ascending Ω3b move, the order-of-appearance orientation is opposite (as it was in the case of Ω3a move), so a forward descending Ω3b move is negative. □
With the relationships between different kinds of $\Omega_3$ moves sorted out, we can precisely describe the behavior of $HN$ under Reidemeister moves:

**Proposition 4.9.** Under forward Reidemeister moves, $HN$ changes by:
- $X_0$ (resp. $Y_0$) under $\Omega_1$ moves creating positive (resp. negative) crossing;
- $X_k + Y_{k+1}$ (for some $k \in \mathbb{Z}$) under $\Omega_2m$ moves;
- $X_k + Y_k$ under $\Omega_2u$ moves;
- $X_k - X_{k+1}$ (resp. $Y_{k+1} - Y_k$) under ascending (i.e. positive) $\Omega_3$ moves with positive (resp. negative) crossing between top and bottom strands;
- $X_{k+1} - X_k$ (resp. $Y_k - Y_{k+1}$) under descending (i.e. negative) $\Omega_3$ moves with positive (resp. negative) crossing between top and bottom strands.

**Proof.** Changes of $HN$ under Reidemeister moves are described section 2 of [5]. The only detail not determined there is the sign of changes under moves of type $\Omega_3$, i.e. whether the change is equal to $X_k - X_{k+1}$ or $-X_k + X_{k+1}$ (resp. $Y_k - Y_{k+1}$ or $-Y_k + Y_{k+1}$). This can be easily checked through casework as demonstrated in Proposition 3.5 in [10] for ascending $\Omega_3a$ moves. \qed

From $HN$ we may also obtain *cowrithe*. While SCI seems to closely resemble $St$, the changes of SCI under $\Omega_3$ moves depend on the forward/backward character of the $\Omega_3$ move, and the changes of $St$ depend on the positive/negative quality of the move. Cowrithe resembles the behavior of $St$ in that its change under $\Omega_3$ moves depends only the positive/negative type of the move (since it is the case with $HN$).

**Definition 4.10** (cowrithe, [3, 5]). Let $D$ be an oriented knot diagram. Let $f: \mathbb{Z} \rightarrow \mathbb{Z}$ be the homomorphism defined by $f(X_n) = -n$ and $f(Y_n) = n$. Then the cowrithe of $D$ is $f(HN_{lk})$.

From Proposition 4.9 we obtain a description of changes of cowrithe.

**Corollary 4.11.** Cowrithe does not change under $\Omega_1$ and unmatched $\Omega_2$ moves. It increases by 1 under forward matched $\Omega_2$ and positive $\Omega_3$ moves.
5. Appendix

In Table 1 we summarize how some of the known knot diagram invariants change under various types of forward Reidemeister moves.

| Invariant | $\Omega_1$ | $\Omega_1 F$ | $\Omega_2 m$ | $\Omega_2 u$ | $\Omega_3^{asc}$ | $\Omega_3^{desc}$ |
|-----------|------------|-------------|--------------|--------------|-----------------|-----------------|
| $w =$writhe | $\text{sgn}(c) = \pm 1$ | 0 | 0 | 0 | 0 | 0 |
| $n =$number of crossings | +1 | +2 | +2 | +2 | 0 | 0 |
| winding number | $w(c) = \pm 1$ | $2w(c)$ | 0 | 0 | 0 | 0 |
| SCI | $\text{sgn}(c) \text{ ind}(c)$ | 0 | 0 | 0 | +1 | +1 |
| HN | $X_0$ if $\text{sgn}(c) = +1$, $Y_0$ if $\text{sgn}(c) = -1$ | $X_0 + Y_0$ | $X_k + Y_{k+1}$ | $X_k + Y_k$ | $X_k - X_{k+1}$, $Y_{k+1} - Y_k$ | $X_{k+1} - X_k$, $Y_k - Y_{k+1}$ |
| $x =$cowrithe | 0 | 0 | 0 | +1 | 0 | +1 |
| $J^+$ | $-2w(c) \text{ ind}(c)$ | $-4w(c) \text{ ind}(c)$ | +2 | 0 | 0 | 0 |
| $J^+/2 + \text{St}$ | 0 | 0 | 0 | +1 | 0 | +1 |
| $A_n$ | 0 | 0 | 0 | 0 | ? | 0 |
| $D_n$ | 0 | 0 | 0 | 0 | 0 | ? |
| $W_n$ | 0 | 0 | 0 | 0 | ? | ? |

Table 1. Knot diagram invariants and their changes under forward Reidemeister moves for knots and framed knots. $c$ denotes a crossing created by the $\Omega_1$ (or $\Omega_1 F$) move, and "?" denotes lack of a combinatorial formula for the change.

Note that a crossing $c$ created by a move of type $\Omega_1$ (or $\Omega_1 F$) has $w(c) = 1$ if it is created on the left side of the strand and $w(c) = -1$ if it is created on the right side of it ($w(c)$ is the weight of the crossing, see Definition 2.10).

As observed by Hass and Nowik [5], the difference between cowrithe and $J^+/2 + \text{St}$ is equal to $4c_2$, where $c_2$ is the second coefficient of the Conway polynomial of a knot. The fact that this difference is a knot invariant is reflected in Table 1.

The precise description of the entries for HN in the table above belongs to Proposition 4.9 and mostly follows [5], and from these one computes the changes for cowrithe. Results for St and $J^+$ follow directly from Shumakovich’s (Theorem 2.12) and Viro’s (Theorem 2.8) formulas. The invariants $A_n, D_n$ for $n = 4, 5, 6, \ldots$ and $W_n$ for $n = 3, 5, 7, \ldots$ are described in [8]. It is worth to note that there is a combinatorial formula for the change of $W_3$ under moves of type $\Omega_3$ (see [8]). While the change may be large, it is bounded by the number of the crossings of a diagram.
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