Dynamical stabilization of runaway potentials and landscape of vacua at finite density

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Abstract

We study a $SU(2)$ gauge theory with a classical complex modulus. Introducing a chemical potential for a conserved modulus hypercharge causes it to become unstable and start condensing. We show that the modulus condensation in turn generates homogeneous but anisotropic non-abelian field strength condensates. The existence of a stable vacuum at the end point of the condensation process depends on a modulus representation under the gauge group. For a modulus in the fundamental representation, the $global$ vacuum of the theory is a state both with the rotational symmetry and the electromagnetic $U(1)_{em}$ being spontaneously broken. In other words, the system describes an anisotropic superconducting medium. We further explore the landscape of vacua of this theory and identify metastable vacua with an abnormal number of Nambu-Goldstone bosons. The $SO(2)$ symmetry of these vacua corresponds to locking gauge, flavor, and spin degrees of freedom. There are also metastable $SO(3)$ rotationally invariant vacua. For a modulus in the adjoint representation, we show that the theory does not have stable vacua with homogeneous anisotropic non-abelian field strength condensates, although there are metastable vacua. The reason of that is connected with a larger number of the physical components of the modulus in the case of the adjoint representation as compared to the fundamental one.

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1 Introduction

Dynamics in relativistic field theories with chemical potential for bosonic matter is rich and quite sophisticated [1–6]. For example, in these theories the phenomenon of spontaneous symmetry breaking with an abnormal number of Nambu-Goldstone (NG) bosons was revealed [2]. In Ref. [3], a new Higgs-like phase was described with condensates of both gauge and scalar fields, which break gauge, flavor, and rotational symmetries. It is noticeable that both these phenomena have already found applications in studies of dense quark matter [7–10].

In this paper, we consider an essentially soluble (in the weak coupling limit) 3+1 dimensional $SU(2)$ gauge model with a (classically) complex modulus $\Phi$ in some irreducible representation of the gauge group (no fermions are included). We introduce a chemical potential $\mu$ associated with the conserved charge of the flavor (hypercharge) $U(1)_Y$ symmetry. Our goal is to describe the landscape of the ground states in this model.

The Lagrangian density of the model is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^{(a)} F^{\mu\nu (a)} + [(D_\mu - i\mu\delta_{\mu 0})\Phi]^\dagger [(D_\mu - i\mu\delta_{\mu 0})\Phi].$$

(1.1)

The gauge fields are

$$A_\mu = A_\mu^{(a)} T^a,$$

with $T^a$ being the $SU(2)$ Lie algebra generators in the representation of $\Phi$. The field strength tensor and the covariant derivatives are given by

$$F_{\mu\nu}^{(a)} = \partial_\mu A_\nu^{(a)} - \partial_\nu A_\mu^{(a)} + g\epsilon^{abc} A_\mu^{(b)} A_\nu^{(c)}, \quad D_\mu = \partial_\mu - igA_\mu.$$

(1.2)

We consider vacua of the theory (1.1) with homogeneous but generically anisotropic condensates of non-abelian field strength $F_{\mu\nu}^{(a)}$ and with a homogeneous condensate of the complex scalar $\Phi$. The effective potential describing such vacua is obtained by setting all field derivatives to zero. We find

$$V = -\frac{g^2}{2} \left[ \left( A_0^{(a)} A_0^{(a)} \right) \left( A_i^{(a)} A_i^{(b)} \right) - \left( A_0^{(a)} A_i^{(a)} \right) \left( A_0^{(b)} A_i^{(b)} \right) \right] + \frac{g^2}{4} \left( A_i^{(a)} A_i^{(a)} \right) \left( A_j^{(b)} A_j^{(b)} \right)$$

$$- \frac{g^2}{4} \left( A_i^{(a)} A_j^{(a)} \right) \left( A_i^{(b)} A_j^{(b)} \right) - \mu^2 \Phi^\dagger \Phi - 2g\mu\Phi^\dagger A_0^{(a)} T^a \Phi - \frac{g^2}{4} A_\mu^{(a)} A_\mu^{(a)} \Phi^\dagger \Phi.$$

(1.3)

Note that $-\mu^2 \Phi^\dagger \Phi$ has the form of a mass term with the wrong sign. Because of that, the modulus $\Phi$ should start condense. Our aim is to describe the end point in this condensing process.
In section 2, we study classical vacua of the theory with a modulus in the fundamental representation of the gauge group. In this case, condensing the modulus in turn leads to the generation of homogeneous but anisotropic non-abelian field strength condensates. We show that the stable vacuum of the theory is a state both with the rotational symmetry and the electromagnetic $U(1)_{em}$ being spontaneously broken. In other words, the system describes an anisotropic superconducting medium. We emphasize that it is the global vacuum. This result is quite nontrivial. Indeed, there are 10 physical fields in the model, and the problem is equivalent of studying the geometry of a ten dimensional hypersurface corresponding to a physical effective potential. The latter is derived by imposing the Gauss law constraints on potential $V$ (1.3) (see Sec. 2 below). The fact that the global minimum of this hypersurface corresponds to a complicated system where most of the initial symmetries are spontaneously broken is noticeable.\footnote{This solution has been considered in our short note [11]. However, it was only shown there that it is a local minimum. The question about the global vacuum in the model remained unresolved.}

We further explore the landscape of vacua of this theory and identify metastable vacua with an abnormal number of Nambu-Goldstone bosons. The $SO(2)$ symmetry of these vacua corresponds to locking gauge, flavor, and spin degrees of freedom. There are also metastable $SO(3)$ rotationally invariant vacua.

Thus, in this simple but nontrivial model, we show that unstable directions in nonabelian gauge theories induced by a chemical potential for the modulus field can be stabilized by the generation of nonabelian gauge field strength condensates. As will be discussed in Sec. 2, the Gauss law constraints play the crucial role in this stabilization process.

The case with the modulus in the adjoint representation is considered in section 3. We show that, unlike the previous case, this theory does not have stable vacua with homogeneous anisotropic non-abelian field strength condensates, although there are metastable vacua. The physics underlying so dissimilar behaviors of these two models is connected with the different number of physical fields. While in the case of the fundamental representation, there is one physical (Higgs) field connected with the modulus, there are three physical fields relating to the modulus in the adjoint representation. As a result, the effective potential is unbounded from below in the latter.

In section 4, the main results of the paper are summarized. In Appendices
A, B, and C, some useful formulas and relations are derived.

2 \textbf{SU}(2) gauge theory with a fundamental complex modulus at finite density

In this section we discuss the SU(2) gauge theory with a classical complex modulus in the fundamental representation. Thus, we take

\[ T^a = \frac{\tau^a}{2}, \quad (2.1) \]

where \( \tau^a \) are Pauli matrices. Henceforth we will use the unitary gauge with

\[ \Phi^T = (0, \phi), \quad (2.2) \]

where \( \phi \) is real. In this gauge, the only conserved flavor charge is the electric charge \( Q_{\text{em}} = T^3 + Y/2 \).

The advantage of the unitary gauge is in that all auxiliary (gauge dependent) degrees of freedom are removed. In particular, in this gauge the vacuum expectation values (condensates) \( A^{(a)}_\mu \) of gauge fields are well-defined physical quantities. Also, without loss of generality, by SO(3) rotations we can always set

\[ A^{(3)}_1 = A^{(3)}_2 = 0, \quad (2.3) \]

i.e., the \( A^{(3)}_i \) condensate is along the \( z \)-axis.

Although the gauge symmetry is gone in the unitary gauge, the theory still has constraints. In fact, it is a system with second-class constraints, similar to the theory of a free massive vector field \( A_\mu \) described by the Proca Lagrangian (for a thorough discussion of systems with second-class constraints, see Sec. 2.3 in book [12]). In such theories, while the Lagrangian formalism can be used without introducing a gauge, the physical Hamiltonian is obtained by explicitly resolving the constraints. In our case, this implies that to obtain the physical potential \( V^{\text{fund}}_{\text{phys}} \) one has to impose the Gauss law constraints on \( V \) in (1.3).

The Gauss constraints amount to integrating out the time-like components of the gauge potentials \( A^{(a)}_0 \). This could be done by using their equations of motion. We also find to be convenient to parameterize the gauge potential and the modulus expectation values as

\[ A^{(a)}_i = \frac{\mu}{g} a^{(a)}_i, \quad \phi = \frac{\mu}{g} \phi_0. \quad (2.4) \]
Further introducing
\[ X = \left( a_3^{(1)} - a_3^{(1)} a_1^{(2)} \right)^2, \quad Y = \left( a_3^{(2)} a_3^{(1)} - a_3^{(2)} a_2^{(1)} \right)^2, \quad Z = \left( a_1^{(1)} a_2^{(2)} - a_1^{(2)} a_2^{(1)} \right)^2, \]
\[ w_1 = X + Y, \quad w_2 = \sum_{i,j=1}^2 \left( a_i^{(j)} \right)^2, \quad w_3 = \sum_{i=1}^2 \left( a_i^{(i)} \right)^2, \]
after eliminating \( A_0^{(a)} \) from (1.3), the physical potential takes the form
\[ V_{\text{fund}}^{\text{phys}} = \frac{\mu^4}{4g^2} \frac{Q_0^2 (\phi_0^2 + 2(w_2 + w_3) - 8) + Q_n}{Q_d}, \]
where
\[ Q = \phi_0^4 (w_2 + w_3) + 2\phi_0^2 \left( a_3^{(3)} \right)^2 (2w_2 + w_3) + (w_2 + w_3)^2 + 4 \left( a_3^{(3)} \right)^4 w_2 + 4 \left( a_3^{(3)} \right)^2 (w_1 + w_2^2 + w_2 w_3) + 4(w_3 + w_2)(w_1 + Z), \]
\[ Q_n = \phi_0^8 \left( a_3^{(3)} \right)^2 + 2\phi_0^6 \left( 2 \left( a_3^{(3)} \right)^4 + 3 \left( a_3^{(3)} \right)^2 (w_2 + w_3) + w_1 + Z \right) + 4\phi_0^4 \left( \left( a_3^{(3)} \right)^6 \right. \]
\[ + \left( a_3^{(3)} \right)^4 (5w_2 + 3w_3) + \left( a_3^{(3)} \right)^2 (3w_2^2 + 2w_3^2 + 5w_2 w_3 + 2w_1 + 3Z) \]
\[ + 2(w_3 + w_2)(w_1 + Z) + 8\phi_0^2 \left( 2 \left( a_3^{(3)} \right)^6 w_2 + \left( a_3^{(3)} \right)^4 (4w_2^2 + 3w_2 w_3 + 2w_1 + Z) \right. \]
\[ + \left( a_3^{(3)} \right)^2 (w_3^2 + 2w_2^2 w_3 + w_2 w_3^2 + (w_1 + Z) (5w_2 + 3w_3)) + (w_1 + Z) ((w_2 + w_3)^2 \]
\[ + w_1 + Z) + 16 \left( a_3^{(3)} \right)^2 w_2 + w_1 + Z) \left( a_3^{(3)} \right)^4 w_2 + \left( a_3^{(3)} \right)^2 (w_2^2 + w_2 w_3 \]
\[ + w_1) + (w_1 + Z)(w_2 + w_3), \]
\[ (2.5) \]
\[ (2.6) \]
\[ (2.7) \]
\[ (2.8) \]
\[ Q_d = \phi_0^6 + 4\phi_0^4 \left( (a_3^{(3)})^2 + w_2 + w_3 \right) + 4\phi_0^2 \left( (a_3^{(3)})^4 + (a_3^{(3)})^2 (3w_2 + 2w_3) \right) + (w_2 + w_3)^2 + w_1 + Z \right) + 8 \left( (a_3^{(3)})^4 w_2 + (a_3^{(3)})^2 (w_2^2 + w_2w_3 + w_1) \right) + (w_3 + w_2)(w_1 + Z). \] (2.9)

The rest of this section is organized as follows. First, in subsection 2.1 we prove that the physical potential \( V_{\text{phys}} \) is bounded from below and has global minima. In subsection 2.2 we prove that the global minima of \( V_{\text{fund}} \) are vacua where the gauge field condensates spontaneously break both the electromagnetic \( U(1)_{\text{em}} \) and the rotational \( SO(3)_{\text{rot}} \),

\[ U(1)_{\text{em}} \times SO(3)_{\text{rot}} \rightarrow SO(2)_{\text{rot}}. \] (2.10)

There are 3 massless Nambu-Goldstone (NG) modes associated with spontaneous breakdown of the global symmetry. Thus, these vacua describe an anisotropic superconducting medium. In subsection 2.3, we discuss interesting metastable vacua of the theory with an abnormal number of massless NG bosons. Finally, in subsection 2.4, metastable \( SO(3)_{\text{rot}} \) invariant vacua are considered.

### 2.1 \( V_{\text{fund}} \) is bounded from below and has global minima

Notice that \( \{ X, Y, Z, Q, Q_n, Q_d, w_1, w_2, w_3 \} \) in (2.5), (2.7)-(2.9), are explicitly non-negative. Additionally, in the physical potential (2.6) the only negative contribution comes from a term in the numerator multiplied by \( Q \). Thus, run-away directions of the potential (2.6) (if they were to exist) would have come only from the large values of the condensates that keep \( \{ \phi_0^2, w_2, w_3 \} \) finite. Recalling definitions of \( w_2 \) and \( w_3 \) from (2.5), we conclude that the only run-away direction could come from \( a_3^{(3)} \) condensate, and only assuming that \( \phi_0 \neq 0 \). However, rescaling

\[ a_3^{(3)} \rightarrow \lambda a_3^{(3)}, \quad \lambda \rightarrow \infty, \] (2.11)

we find

\[ V_{\text{fund}} \bigg|_{a_3^{(3)} \rightarrow \lambda a_3^{(3)}} \rightarrow \lambda^2 \frac{\mu^4}{4g^2} \left( \phi_0^2 + 2w_2 \right) \left( a_3^{(3)} \right)^2, \] (2.12)

which becomes infinitely large, since \( \phi_0 \neq 0 \). Thus we conclude that \( V_{\text{fund}} \) given by (2.6) is in fact bounded from below.
2.2 Global minima of $V_{\text{phys}}^{\text{fund}}$

By using the inequality
\[ w_1 \leq w_2w_3, \quad (2.13) \]
we will first prove that all the possible minima are either in the class
\[ (1) : \ a_3^{(3)} = 0, \quad (2.14) \]
or
\[ (2) : \ \phi_0 = a_1^{(1,2)} = 0, \quad a_3^{(3)} \neq 0. \quad (2.15) \]
Then it will be shown that the global minima of potential (2.6) are in the first class (2.14), with gauge field condensates spontaneously breaking both the $U(1)_{em}$ and the $SO(3)_{rot}$ rotational symmetry down to the $SO(2)_{rot}$.

The inequality (2.13) could be proved by using definitions (2.5) and setting
\begin{align*}
a_1^{(1)} &= U \sin(\alpha) \cos(\theta_1), \quad a_1^{(2)} = U \sin(\alpha) \sin(\theta_1), \\
a_2^{(1)} &= U \cos(\alpha) \cos(\theta_2), \quad a_2^{(2)} = U \cos(\alpha) \sin(\theta_2), \\
a_3^{(1)} &= V \cos(\theta_3), \quad a_3^{(2)} = V \sin(\theta_3), \quad U, V \geq 0. \quad (2.16)
\end{align*}
Clearly, we will have
\[ w_1 = U^2V^2 \left( \sin^2 \alpha \sin^2(\theta_3 - \theta_1) + \cos^2 \alpha \sin^2(\theta_3 - \theta_2) \right) \leq U^2V^2 = w_2w_3. \quad (2.17) \]
To find the minima, we analyze $\frac{\partial V_{\text{phys}}^{\text{fund}}}{\partial a_3^{(3)}}$. Any minimum of the potential necessarily satisfies $\frac{\partial V_{\text{phys}}^{\text{fund}}}{\partial a_3^{(3)}} = 0$. Using (2.6), we have then
\[ 0 = a_3^{(3)} P \left( w_1, w_2, w_3, a_3^{(3)}, \phi_0 \right), \quad (2.18) \]
where $P$ is a polynomial of $\{ w_1, w_2, w_3, a_3^{(3)}, \phi_0 \}$:
\[ P = 32 \left( w_2w_3 - w_1 \right) \phi_0^4 \left( \phi_0^4 + 2\phi_0^2 \left( w_2 + w_3 \right) + 4(w_1 + Z) \right) + \hat{P}, \quad (2.19) \]
with polynomial $\hat{P} \equiv \hat{P}(w_1, w_2, w_3, a_3^{(3)}, \phi_0)$ being manifestly non-negative. From (2.13), we can see that $P$ is semi-positive definite. Moreover, we find that $P = 0$ (with $a_3^{(3)} \neq 0$) implies
\[ \phi_0 = w_2 = 0. \quad (2.20) \]
Notice that (2.20) further implies that \( w_1 = Z = 0 \). Therefore the solution to (2.18) is indeed given either by (2.14) or (2.15).

Solution (2.15) is a trivial (unstable) vacuum with zero vacuum energy. With (2.14), the physical potential (2.6) simplifies

\[
V_{\text{phys}} = \frac{\mu^4}{g^2} \left[ \frac{\phi_0^2 (w_2 + w_3) (\phi_0^2 + 2(w_2 + w_3) - 8)}{4(\phi_0^2 + 2(w_2 + w_3))} + \frac{1}{2} (w_1 + Z) \right].
\]  

(2.21)

It is straightforward to analyze (2.21). It has a global minimum

\[
\min \left\{ V_{\text{fund phys}} \bigg|_{a_3^{(3)} = 0} \right\} = -\frac{\mu^4}{2g^2} \equiv V_{\text{phys}}^{\text{min}},
\]  

(2.22)

attained with

\[
\phi_0^2 = 2, \quad w_2 + w_3 = 1, \quad w_1 = Z = 0.
\]  

(2.23)

We emphasize that for large values of the chemical potential — \( \mu \) being much larger than the strong coupling scale of the gauge theory — the gauge coupling \( g \) is self-consistently weak and quantum correction to (2.22) are suppressed. \(^2\)

Let us show that in vacua (2.23) the rotational symmetry is broken to the \( SO(2)_{\text{rot}} \) subgroup. Recalling the definitions (2.5), the most general parameterization of the vector potential condensates takes form

\[
a_1^{(1)} = \sin \theta_1 \cos \psi, \quad a_2^{(1)} = \cos \theta_1 \sin \theta_2 \cos \psi, \quad a_3^{(1)} = \cos \theta_1 \cos \theta_2 \cos \psi,
\]

\[
a_1^{(2)} = \sin \theta_1 \sin \psi, \quad a_2^{(2)} = \cos \theta_1 \sin \theta_2 \sin \psi, \quad a_3^{(2)} = \cos \theta_1 \cos \theta_2 \sin \psi,
\]  

(2.24)

with arbitrary \( \{\theta_1, \theta_2, \psi\} \). But from (2.24) vector condensates \( A_i^{(1)} \) and \( A_i^{(2)} \) are collinear,

\[
A_i^{(2)} = \tan \psi A_i^{(1)},
\]  

(2.25)

\(^2\)A comment concerning the possibility of the one-loop Coleman-Weinberg (CW) effect [13] in this model is in order. A one-loop Coleman-Weinberg (CW) potential would generate an effective quartic coupling of order \( g^4 \). However, the important difference between the CW model (zero chemical potential) and the present one (finite chemical potential) is that while there are no tree level contributions from gauge fields in the former, the dominant effect in the present model is based on the tree level contributions of gauge bosons (vector condensates). Therefore, while in CW model the one-loop contribution of gauge fields is dominant, it is subleading in our case. In fact, one can show that a one-loop CW effective quartic potential for a complex fundamental modulus will modify our classical analysis at order \( O(g^2 \ln g^2) \). This correction is small at weak coupling.
while \( A_i^{(3)} = 0 \). Thus, the rotational symmetry is broken to the \( SO(2)_{\text{rot}} \) subgroup indeed.

To understand better the physics in this vacua, it will be convenient to introduce the charged fields \( W_\mu^{(\pm)} = \frac{1}{\sqrt{2}} (A_\mu^{(1)} \pm i A_\mu^{(2)}) \) with \( Q_{em} = \mp 1 \). Let us choose the direction of the collinear condensates \( A_i^{(1)} \) and \( A_i^{(2)} \) along the \( z \)-axis. Then, from Eqs. (2.23) and (2.25), we find that the \( W \) condensate is

\[
W_3^{(\pm)} = \frac{\mu}{\sqrt{2} g} e^{\pm i \psi}.
\]

Therefore the electromagnetic \( U(1)_{em} \) is spontaneously broken and these vacua are anisotropic superconducting media.

It is noticeable that solution (2.26) describes a nonzero field strength \( F_{\mu \nu} \) which corresponds to the presence of \textit{non-abelian} constant “chromoelectric”-like condensates in the ground state. In order to see this, note that, as follows from the Gauss constraint, \( A_0^{(3)} = \mu/g \) for this solution (see Appendix A). Then, choosing for simplicity the vacuum with the angle \( \psi = 0 \), one finds

\[
E_3^{(2)} = F_{03}^{(2)} = g A_3^{(1)} A_0^{(3)} = \frac{\mu^2}{g}.
\]

We emphasize that while an abelian constant electric field in different media always leads to an instability, \(^3\) non-abelian constant chromoelectric fields do not in many cases. For a discussion of the stability problem for constant non-abelian fields, see Refs. [8] and [14]. On a technical side, this difference is connected with that while a vector potential corresponding to a constant abelian electric field depends on spatial and/or time coordinates, a constant non-abelian chromoelectric field is expressed through constant vector potentials, as takes place in our case, and therefore momentum and energy are good quantum numbers in the latter.

The mass spectrum of excitations in this vacuum can be found from Lagrangian density (1.1) by evaluating zeroes of the determinant of the quadratic form of small fluctuations around the vacuum solution,

\[
A_\mu^{(a)}(x) = \frac{\mu}{g} a_\mu^{(a)} + \delta A_\mu^{(a)}, \quad \Phi^T(x) = \left( 0, \mu/g \phi_0 + \delta \phi(x)/\sqrt{2} \right).
\]

\(^3\)In metallic and superconducting media, such an instability is classical in its origin. In semiconductors and insulators, this instability is manifested in creation of electron-hole pairs through a quantum tunneling process.
One finds 10 physical states, 7 massive ones and 3 massless NG bosons associated with spontaneous breakdown of the global symmetries:

\[ m^2 = 0, \ [\times 3], \]
\[ m^2 = 2\mu^2, \ [\times 2], \]
\[ m^2 = 5\mu^2, \ [\times 2], \]
\[ m^2 = 4\mu^2, \ [\times 1], \]
\[ m^2 = \left(\frac{5}{2} + \frac{\sqrt{13}}{2}\right)\mu^2, \ [\times 1], \]
\[ m^2 = \left(\frac{5}{2} - \frac{\sqrt{13}}{2}\right)\mu^2, \ [\times 1], \]

where the square bracket denotes the multiplicity of states. Notice that in this vacuum the number of the massless NG bosons coincides with the number of moduli parameterizing the global minima (2.24).

2.3 Metastable vacuum with abnormal number of massless NG bosons

Besides the 3 dimensional moduli space of the global minima with the rotational \( SO(3)_{\text{rot}} \) broken down to \( SO(2)_{\text{rot}} \), discussed in the previous subsection, this model has an intricate structure of metastable vacua, some of which were discussed in our note [11]. In this subsection, we identify interesting metastable vacua of this theory with an abnormal number of massless NG bosons.

Consider gauge field condensates such that \( \vec{a}^{(3)} \equiv \{a_i^{(3)}\} = \vec{0} \), and with vectors \( \vec{a}^{(1)} \equiv \{a_i^{(1)}\} \) and \( \vec{a}^{(2)} \equiv \{a_i^{(2)}\} \) satisfying\(^4\)

\[ |\vec{a}^{(1)}| = |\vec{a}^{(2)}| = A, \quad \vec{a}^{(1)} \cdot \vec{a}^{(2)} = 0, \]

where \( A \) is a constant. Such vacua indeed exist:

\[ A = \frac{\sqrt{2}(3 - \sqrt{3})}{3}, \quad \phi_0 = \frac{2}{3} \sqrt{6(\sqrt{3} - 1)}, \]

\(^4\)An obvious generalization would be to consider vacua such that \( \vec{a}^{(a)} \cdot \vec{a}^{(b)} = A^2 \delta^{ab} \), for \( \{a, b\} = \{1, 2, 3\} \). However, such vacua do not exist.
with vacuum energy necessarily higher than that of the global minimum \(2.22\)
\[
V_{\text{fund}}^\text{phys} = -\frac{\mu^4}{g^2} \frac{8(2\sqrt{3} - 3)}{9} > V_{\text{fund}}^\text{min} = -\frac{\mu^4}{2g^2}.
\]
(2.32)

We show below that vacua (2.30) are perturbatively stable, thus they are metastable.

Clearly, vacua (2.30) completely break the \(SO(3)\text{rot}\) symmetry and form a 3 parameter family. Naively, the full pattern of the global symmetry breaking takes the form
\[
U(1)_{\text{em}} \times SO(3)_{\text{rot}} \rightarrow 1,
\]
and thus one would expect 4 massless NG bosons. In fact, there is a \(SO(2)\) symmetry in the vacua (2.30) whose generator is a linear combination of the gauge \(SU(2)\) generator \(\hat{T}_3\), the hypercharge \(\hat{Y}\), and the generator \(\hat{J}_3\) of the \(SO(3)\text{rot}\): it will be appropriate to call it a gauge-flavor-spin locked (GFSL) symmetry. Thus the correct pattern of the symmetry breaking takes the form
\[
SU(2)_{\text{gauge}} \times U(1)_Y \times SO(3)_{\text{rot}} \rightarrow SO(2)_{\text{GFSL}}.
\]
(2.34)

To describe explicitly the \(SO(2)_{\text{GFSL}}\) symmetry, let’s choose, without loss of generality, a representative vacuum of (2.30) as
\[
a_1^{(1)} = a_2^{(2)} = \mathcal{A}.
\]
(2.35)

Notice that the condensates \(a_i^{(j)}\) with \(\{i, j = 1, 2\}\),
\[
\hat{a} \equiv \begin{pmatrix}
    a_1^{(1)} \\
    a_1^{(2)} \\
    a_2^{(1)} \\
    a_2^{(2)}
\end{pmatrix},
\]
(2.36)
are closed under the action of the generators \(\hat{T}_3\) and \(\hat{J}_3\). In parameterization (2.36), our representative vacuum (2.35) is given by
\[
\hat{a}_{\text{meta}} \equiv \mathcal{A} \begin{pmatrix}
    1 & 0 \\
    0 & 1
\end{pmatrix}.
\]
(2.37)

It is straightforward to compute
\[
e^{i\alpha (\hat{T}_3 + \frac{1}{2} \hat{Y}) + i\beta \hat{J}_3} \hat{a}_{\text{meta}} = U(\alpha) \hat{a}_{\text{meta}} U^T(\beta)
\]
\[
= \mathcal{A} \begin{pmatrix}
    \cos(\alpha - \beta) & \sin(\alpha - \beta) \\
    -\sin(\alpha - \beta) & \cos(\alpha - \beta)
\end{pmatrix},
\]
where $\alpha, \beta$ are arbitrary angles and
\[
U(\alpha) = \begin{pmatrix}
\cos \alpha & \sin \alpha \\
-\sin \alpha & \cos \alpha
\end{pmatrix}.
\] (2.39)

Recall that while gauge bosons carry no hypercharge, the hypercharge of the scalar $\Phi$ equals +1. Then,
\[
e^{i \alpha \hat{T}_3 + \frac{1}{2} \hat{Y}} + i \beta \hat{J}_3 \begin{pmatrix} 0 \\ \phi_0 \end{pmatrix} = \begin{pmatrix} 0 \\ \phi_0 \end{pmatrix}.
\] (2.40)

From Eqs. (2.38) and (2.40) we see that the vacuum $\{\tilde{a}_{\text{meta}}, \phi_0\}$ is invariant under transformations (2.38) with $\alpha = \beta$, i.e., under the $SO(2)$ transformations with the generator
\[
\hat{G}_{SO(2)_{\text{GFL}}} = \hat{T}_3 + \frac{1}{2} \hat{Y} + \hat{J}_3,
\] (2.41)

which corresponds to locking gauge, flavor, and spin degrees of freedom (compare with the color-flavor locking phase in dense QCD [15]).

The mass spectrum of excitations in vacuum (2.37) (including their degeneracies) is
\[
m^2 = 0, \quad [\times 2],
\]
\[
m^2 = \frac{4}{3} \mu^2, \quad [\times 1],
\]
\[
m^2 = 4 \mu^2, \quad [\times 1],
\]
\[
m^2 = \frac{16}{3} \mu^2, \quad [\times 1],
\]
\[
m^2 = \left(4 - \frac{4}{\sqrt{3}}\right) \mu^2, \quad [\times 1],
\]
\[
m^2 = \left(8 - \frac{8}{\sqrt{3}} + \frac{4 \sqrt{6(\sqrt{3} - 1)}}{3}\right) \mu^2, \quad [\times 1],
\] (2.42)
\[
m^2 = \left(8 - \frac{8}{\sqrt{3}} - \frac{4 \sqrt{6(\sqrt{3} - 1)}}{3}\right) \mu^2, \quad [\times 1],
\]
\[
m^2 = \frac{8}{3} \left(\sqrt{3} - 1 + \sqrt{2\sqrt{3} - 3}\right) \mu^2, \quad [\times 1],
\]
\[
m^2 = \frac{8}{3} \left(\sqrt{3} - 1 - \sqrt{2\sqrt{3} - 3}\right) \mu^2, \quad [\times 1].
\]
Thus we conclude that vacuum (2.37) is perturbatively stable. It is metastable since its energy is above the global minimum of the physical potential $V_{\text{fund}}^{\text{phys}}$ (2.22). Quite unexpectedly, this vacuum has only 2 massless NG bosons, rather than 3 as expected from the true pattern of the symmetry breaking (2.33). It is the same phenomenon as that found in a non-gauge relativistic field model at finite density in Ref. [2]. As we show in Appendix B, only one of the massless states in (2.42) has a linear dispersion relation in the infrared, $\omega \sim k$, while the other has quadratic dispersion relation $\omega \sim k^2$. So, after all, the number of massless excitation in vacuum (2.37) is in accordance with the Nielsen-Chadha counting rule [16]. Finally, from Appendix B, the infrared dispersion relations exhibit $SO(2)_{\text{GFSL}}$ symmetry (2.41).

An interesting feature of this phase is that besides a chromoelectric field strength condensate, there exists also a chromomagnetic one:

$$H_3^{(3)} = F_{12}^{(3)} = g A_1^{(1)} A_2^{(2)} = \frac{\mu^2}{g} A^2.$$  

(2.43)

It is also interesting that a possibility of the existence of a similar, color-spin locking, phase in dense QCD has been recently discussed in Ref. [10]. While the question concerning the existence of the latter is open, it is quite noticeable that a phase with locking gauge and spin degrees of freedom does exist in the present model.

### 2.4 $SO(3)_{\text{rot}}$ invariant metastable vacuum of $V_{\text{fund}}^{\text{phys}}$

For completeness, in this subsection, we briefly describe $SO(3)_{\text{rot}}$ invariant metastable vacua considered in our note [11] (their energy density is larger than that of the GFSL vacuum).

It is easy to show that these $SO(3)_{\text{rot}}$-invariant solutions are

$$A_0^{(3)} = \frac{2\mu}{g}, \quad \phi_0 = \text{arbitrary},$$  

(2.44)

and no other condensates. This solution is also invariant with respect to the electromagnetic $U(1)_{\text{em}}$.

These $SO(3)$ invariant vacua (a line of moduli) with $V_{\text{fund}}^{\text{phys}} = 0$ are either metastable or unstable. In order to show this, we calculated the spectrum of excitations in this vacuum. The spectrum includes a single massless $\phi_0$-modulus excitation and 9 massive modes assigned to three $SO(3)_{\text{rot}}$ triplets (vector modes). Their masses are $m_0 = g\phi_0$ and $m_{\pm} = (g\phi_0 \pm 2\sqrt{2}\mu)$. While
the mass $m_0$ is connected with the neutral $A^{(3)}$ vector boson ($Q_{em} = 0$), $m_{\pm}$ are the masses of charged $W^{(\pm)}$ vector bosons ($Q_{em} = \mp 1$). For a large scalar condensate $\phi_0 > 2\sqrt{2}\mu/g$, all the masses are positive and therefore the vacua are not unstable, although metastable: their energy density $V_{\text{fund}} = 0$ is larger than both $V_{\text{phys}}^{\text{fund-min}}$ in the ground state (see Eq. (2.22)) and the energy density in the GFSL vacuum.

On the other hand, at the values of $\phi_0$ less than $2\sqrt{2}\mu/g$, the mass $m_-$ becomes negative and therefore the process of a crossover of particle-antiparticle levels takes place. The latter is a signature of the Bose-Einstein instability: at these values of $\phi_0$, the condensate of charged $W^{(+)\text{}}$ vector bosons is dynamically generated.

Thus, in this simple but nontrivial model, we showed that unstable directions in nonabelian gauge theories induced by a chemical potential for the modulus field can be stabilized by the generation of nonabelian gauge field strength condensates. Such condensates are homogeneous but anisotropic.

3 SU(2) gauge theory with an adjoint complex modulus at finite density

In this section, we discuss the $SU(2)$ gauge theory with a classical complex modulus in the adjoint representation. Thus, we take

$$T^1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad T^2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad T^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

for the $SU(2)$ generators in adjoint (vector) representation, canonically normalized as

$$\text{Tr} \left( T^a T^b \right) = 2 \delta^{ab}, \quad \left[ T^a, T^b \right] = i \epsilon^{abc} T^c.$$

Henceforth we will use the unitary gauge with

$$\Phi^T = \frac{\mu}{g} \left( \phi_1 + i\phi_0, 0, \phi_3 + i\phi_0 \right),$$

where $\{\phi_0, \phi_1, \phi_3\}$ are all real. Notice that our unitary gauge removes three degrees of freedom from the complex scalar, which is expected given that

\footnote{One can verify that for the vacua of interest (3.3) provides a good gauge choice, see Appendix C.}
the gauge group is completely broken. As in section 2, by $SO(3)$ rotations we can always set (2.3). Again, the physical potential $V_{phys}^{adj}$ is obtained by integrating out the time-like components $a_0^{(a)}$ of the gauge potentials.

The rest of the section is organized as follows. First, in subsection 3.1 we explicitly exhibit runaway directions of the physical potential $V_{phys}^{adj}$. Thus, unlike the gauge theory with fundamental matter, here, the runaway directions can not be stabilized by homogeneous (though generically anisotropic) gauge field strength condensates. The reason for that is discussed in the next subsection. Necessarily, in this theory any vacuum is at most metastable. In subsection 3.2, we identify metastable $SO(3)_{rot}$ invariant vacua with zero energy density. In subsection (3.3), we identify $SO(2)_{rot}$ invariant vacua with negative $V_{phys}^{adj}$. Unfortunately, these vacua are perturbatively unstable.

### 3.1 Runaway directions of $V_{phys}^{adj}$

We will not present here the general expression for $V_{phys}^{adj}$. We find though

$$V_{phys}^{adj} \bigg|_{a_1^{(1,2)},a_2^{(1,2)}=a_3^{(1,2)}=0} = \frac{\mu^4}{g^2} \left[ \left( \left( a_3^{(3)} \right)^2 - 1 \right) \left( \phi_1^2 + \phi_3^2 + 2\phi_0^2 \right) + \frac{\left( \phi_1^2 - \phi_3^2 \right)^2}{\phi_1^2 + \phi_3^2 + 2\phi_0^2} \right]$$

for arbitrary $\{a_3^{(3)}, \phi_0, \phi_1, \phi_3\}$. The runaway directions are explicit in (3.4).

For example,

$$\phi_1^2 + \phi_3^2 < \text{const} , \quad \left| a_3^{(3)} \right| < 1 , \quad \phi_0 \to \infty$$

and

$$(\phi_1^2 - \phi_3^2)^2 < \text{const} , \quad \left| a_3^{(3)} \right| < 1 , \quad \phi_1 \to \infty.$$ (3.5)

In fact, at $|a_3^{(3)}| < 1$, any direction with the second term in the square brackets in (3.4) being bounded, and at least one of the fields $\phi_0, \phi_1, \phi_3$ going to infinity, is a runaway one.

So, indeed the potential $V_{phys}^{adj}$ is unbounded from below for homogeneous (generically anisotropic) adjoint complex scalar and gauge field strength condensates. It seems that the reason of this feature is connected with a larger number (three) of physical fields connected with the modulus in this case as compared to the fundamental one. For example, if one removes the field $\phi_0$ (coming from the imaginary part of the modulus) and one of the fields $\phi_1, \phi_3$ in Eq. (3.4), this expression becomes bounded from below.
3.2 $SO(3)_{\text{rot}}$ invariant metastable vacuum of $V_{\text{phys}}^{\text{adj}}$

There are simple $SO(3)_{\text{rot}}$ invariant vacua of $V_{\text{phys}}^{\text{adj}}$ with zero energy density:

$$a_{1,2,3}^{(1,2,3)} = 0, \quad \phi_1 = \phi_0 = 0, \quad \phi_3 = \text{arbitrary}. \quad (3.6)$$

These vacua are also invariant under the electromagnetic $U(1)_{\text{em}}$ with the generator $Q_{\text{em}} = T^3 + Y$.

The mass spectrum of fluctuations, including degeneracies, in vacua (3.6) is

$$m^2 = 0, \quad [\times 1],$$
$$m^2 = 4\mu^2, \quad [\times 2],$$
$$m^2 = 2\mu^2 \phi_3^2, \quad [\times 3],$$
$$m^2 = \mu^2 (\phi_3 + 1)^2, \quad [\times 3],$$
$$m^2 = \mu^2 (\phi_3 - 1)^2, \quad [\times 3]. \quad (3.7)$$

For generic $\phi_3$, there is a single massless $\phi_3$-modulus excitation. Vacua (3.6) are perturbatively stable for $\phi_3 > 1$. They are metastable since the physical potential $V_{\text{phys}}^{\text{adj}}$ is unbounded from below. When $\phi_3 < 1$ the mass $m_- \equiv \mu(\phi_3 - 1)$ is negative and therefore the process of a crossover of particle-antiparticle levels takes place, making vacuum (3.6) unstable. The same phenomenon occurs in $SO(3)_{\text{rot}}$ invariant vacua of the $SU(2)$ gauge theory with a complex fundamental modulus considered in subsection 2.4 above.

3.3 $SO(2)_{\text{rot}}$ invariant unstable vacuum of $V_{\text{phys}}^{\text{adj}}$

In this subsection we consider a one parameter family of the vacua of $V_{\text{phys}}^{\text{adj}}$ which have the following pattern of the spontaneous symmetry breaking

$$U(1)_Y \times SO(3)_{\text{rot}} \rightarrow SO(2)_{\text{rot}}, \quad (3.8)$$

which is similar to that considered in subsection 2.2.

These vacua are characterized by the condensates:

$$a_{1,2}^{(1,2,3)} = a_3^{(3)} = 0,$$
$$a_3^{(1)} = -\frac{\sqrt{5\sqrt{17} - 19}}{2} \cos \alpha, \quad a_3^{(2)} = -\frac{\sqrt{5\sqrt{17} - 19}}{2} \sin \alpha. \quad (3.9)$$
\[
\phi_0 = \frac{\sqrt{2} (7\sqrt{17} - 33) \sqrt{17} - 3}{4 (1 + \sqrt{17}) \sqrt{\cos^2 \alpha (7\sqrt{17} - 33) + 9 + \sqrt{17}}} \cos \alpha \sin \alpha,
\]

\[
\phi_1 = \frac{\sqrt{2} (28 - 4\sqrt{17} - \sqrt{17} - 3 \cos^2 \alpha (7\sqrt{17} - 33) + 9 + \sqrt{17})}{4 (1 + \sqrt{17}) \sqrt{\cos^2 \alpha (7\sqrt{17} - 33) + 9 + \sqrt{17}}} ,
\]

\[
\phi_3 = \frac{\sqrt{2} (28 - 4\sqrt{17} + \sqrt{17} - 3 \cos^2 \alpha (7\sqrt{17} - 33) + 9 + \sqrt{17})}{4 (1 + \sqrt{17}) \sqrt{\cos^2 \alpha (7\sqrt{17} - 33) + 9 + \sqrt{17}}},
\]

(3.10)

where \( \alpha \) is an arbitrary parameter. The vacuum energy corresponding to (3.9), (3.10) is

\[
V_{\text{adj,phys}}^{\text{adj}} = \frac{\mu^4}{g^2} \frac{17 \sqrt{17} - 71}{16} < 0.
\]

(3.11)

Note also that because the time component \( a_0^{(3)} \), determined from the Gauss law constraint, is nonzero for this solution, \( a_0^{(3)} = \frac{1}{2} \sqrt{17} - 3 \) (see Appendix A), there are chromomagnetic field strengths condensates \( F_{03}^{(1)} \) and/or \( F_{03}^{(2)} \) in these vacua.

The mass spectrum of fluctuations in vacua (3.9), (3.10) is

\[
m^2 = 0, \quad [\times 3],
\]

\[
m^2 = (5 - \sqrt{17}) \mu^2, \quad [\times 2],
\]

\[
m^2 = \left( \frac{7}{2} - \frac{\sqrt{17}}{2} \right) \mu^2, \quad [\times 2],
\]

\[
m^2 = \left( \frac{15}{4} - \frac{\sqrt{17}}{4} \right) \mu^2, \quad [\times 1],
\]

(3.12)

and the remaining 4 states have mass

\[
m^2 = \lambda \mu^2,
\]

(3.13)

where \( \lambda \) is a root of the following forth order polynomial

\[
4\lambda^4 + (1 - 7\sqrt{17}) \lambda^3 + (58\sqrt{17} - 182) \lambda^2 + (993\sqrt{17} - 4151) \lambda - 8382\sqrt{17} + 34578 = 0.
\]

(3.14)
Notice from (3.12) that there are 3 massless NG bosons, as expected from the pattern of the spontaneous symmetry breaking (3.8). Also, 2 of the roots of (3.14) are imaginary. Indeed, solving numerically (3.14) we find
\[ \lambda = \{0.5457; 4.397; 1.011 \pm 0.9307 i\} . \] (3.15)
We conclude that vacua (3.9), (3.10), albeit having lower vacuum energy than the metastable vacua (3.6), are unstable.

4 Conclusion

It is rather surprising that dynamics of a relatively simple model, such as the present one, leads to such a rich landscape of stable and metastable vacua. In particular, it is noticeable that the anisotropic superconducting vacuum and the gauge-flavor-spin locking one discussed in Sec. 2 have their analogues in dense quark matter (see Refs. [8–10, 15]), where their studies are limited by complexities in infrared dynamics in QCD. One can say that the present consideration yields a proof that the dynamics with vector condensates of gauge fields is a real thing.

The model with the field \( \Phi \) in the fundamental representation of the gauge group considered here is a very special case of that in Ref. [3], with the quartic coupling constant \( \lambda \) and the mass of the scalar field \( \Phi \) chosen to be zero. This special limit has two important advantages. First, it yields the dynamics with moduli, and this led us to revealing the phenomenon of the dynamical stabilization of runaway potentials at finite density. Secondly, this limit, retaining richness of the dynamics, simplifies the analysis of the structure of the vacuum manifold. This allowed us to establish the structure of the global vacuum and find some interesting metastable ones in the model.

We also found that the stability properties of the gauge theory with classical complex modulus and chemical potential are intrinsically different for theories in the fundamental and adjoint representations: while the one in the fundamental representation is stabilized by a condensation of gauge fields, the theory in the adjoint one is always unstable. The reason for that is connected with different numbers of physical components in the moduli in these two cases, being larger in the latter. A special role of the fundamental representation could be welcome, taking into account that, after all, the electroweak Higgs field is assigned to this representation.

We would not be surprised if the dynamics revealed here find applications in such different areas as cosmology and condensed matter.
Acknowledgments

A.B. would like to thank Maxim Pospelov for valuable discussions. A.B. would like to thank the Aspen Center for Physics for hospitality where part of this work was done. A.B. research at Perimeter Institute is supported in part by the Government of Canada through NSERC and by the Province of Ontario through MEDT. A.B. gratefully acknowledges further support by an NSERC Discovery grant. The work of J.J. and V.A.M. was supported by the Natural Sciences and Engineering Research Council of Canada. V.A.M. is grateful to Prof. Taichiro Kugo and Prof. Teiji Kunihiro for their warm hospitality during his stay at Yukawa Institute for Theoretical Physics, Kyoto University.

A Gauss Law constraints

In this section we collect expressions for the expectation values of the time components of non-abelian gauge potential condensates constrained by the Gauss Law.

- For the model with a complex modulus in a fundamental representation, the values of the gauge potential time components in global minimum (2.22) are
  \[ A_0^{(1)} = 0, \quad A_0^{(2)} = 0, \quad A_0^{(3)} = \frac{\mu}{g}. \]  
  \hfill (A.1)

- For the model with a complex modulus in a fundamental representation, the values of the gauge potential time components in metastable vacua in subsection 2.3 are
  \[ A_0^{(1)} = 0, \quad A_0^{(2)} = 0, \quad A_0^{(3)} = \frac{2\sqrt{3} \mu}{3g}. \]  
  \hfill (A.2)

- For the model with a complex modulus in a fundamental representation, the values of the gauge potential time components in the \( SO(3)_{rot} \) invariant vacua in subsection 2.4 are
  \[ A_0^{(1)} = 0, \quad A_0^{(2)} = 0, \quad A_0^{(3)} = \frac{2\mu}{g}. \]  
  \hfill (A.3)

- For the model with a complex modulus in an adjoint representation, the
values of the gauge potential time components in $SO(3)_{\text{rot}}$ invariant vacua in subsection 3.2 are

$$A_0^{(1)} = 0, \quad A_0^{(2)} = 0, \quad A_0^{(3)} = \frac{\mu}{g}.$$ (A.4)

For the model with a complex modulus in an adjoint representation, the values of the gauge potential time components in the $SO(2)_{\text{rot}}$ invariant vacua in subsection 3.3 are

$$A_0^{(1)} = 0, \quad A_0^{(2)} = 0, \quad A_0^{(3)} = \frac{\sqrt{17} - 3\mu}{2g}.$$ (A.5)

### B Infrared dispersion relation for fluctuations (2.42)

Introduce

$$k_\perp^2 \equiv k_1^2 + k_2^2, \quad k^2 \equiv k_\perp^2 + k_3^2.$$ (B.1)

Dispersion relations for the fluctuations (2.42) in the infrared region $k^2 \ll \mu^2$ are

$$\omega^2 = \frac{1}{3} k^2 + \mathcal{O}(k^4),$$ (B.2)

$$\omega^2 = \frac{7 - 4\sqrt{3}}{16} k^2 \left( k_3^2 + (2\sqrt{3} + 3)k_\perp^2 \right) + \mathcal{O}(k^6),$$ (B.3)

$$\omega^2 = \frac{4}{3} \mu^2 + \left( \frac{5}{3} - \frac{\sqrt{3}}{3} \right) k_3^2 + \left( \frac{9707}{6623} - \frac{916\sqrt{3}}{6623} \right) k_\perp^2 + \mathcal{O}(k^4),$$ (B.4)

$$\omega^2 = 4 \mu^2 - \left( 1 + \sqrt{3} \right) k_3^2 + \left( \frac{71\sqrt{3}}{9} + \frac{118}{9} \right) k_\perp^2 + \mathcal{O}(k^4),$$ (B.5)

$$\omega^2 = \frac{16}{3} \mu^2 + \left( \frac{4\sqrt{3}}{3} + \frac{10}{3} \right) k_3^2 + \left( 6\sqrt{3} - 8 \right) k_\perp^2 + \mathcal{O}(k^4),$$ (B.6)

$$\omega^2 = \left( 4 - \frac{4}{\sqrt{3}} \right) \mu^2 + \left( \frac{2}{3} + \frac{\sqrt{3}}{3} \right) k_3^2 + \left( 11 - 6\sqrt{3} \right) k_\perp^2 + \mathcal{O}(k^4).$$ (B.7)
\[
\omega^2 = \left( 8 - \frac{8}{\sqrt{3}} + \frac{4\sqrt{6(\sqrt{3} - 1)}}{3} \right) \mu^2 + \left( \frac{3}{2} - \frac{\sqrt{3}}{6} + \left( \frac{\sqrt{3}}{9} + \frac{1}{6} \right) \sqrt{6(\sqrt{3} - 1)} \right) k_3^2
\]
\[+ \left( \frac{6763}{3222} - \frac{469\sqrt{3}}{3222} + \left( \frac{617}{3222} + \frac{3583\sqrt{3}}{19332} \right) \sqrt{6(\sqrt{3} - 1)} \right) k_1^2 + O(k^4),
\]
(B.8)

\[
\omega^2 = \left( 8 - \frac{8}{\sqrt{3}} - \frac{4\sqrt{6(\sqrt{3} - 1)}}{3} \right) \mu^2 + \left( \frac{3}{2} - \frac{\sqrt{3}}{6} - \left( \frac{\sqrt{3}}{9} + \frac{1}{6} \right) \sqrt{6(\sqrt{3} - 1)} \right) k_3^2
\]
\[+ \left( \frac{6763}{3222} - \frac{469\sqrt{3}}{3222} - \left( \frac{617}{3222} + \frac{3583\sqrt{3}}{19332} \right) \sqrt{6(\sqrt{3} - 1)} \right) k_1^2 + O(k^4),
\]
(B.9)

\[
\omega^2 = \frac{8}{3} \left( \sqrt{3} - 1 + \sqrt{2\sqrt{3} - 3} \right) \mu^2 + \left( 1 + \left( 1 + \frac{2\sqrt{3}}{3} \right) \sqrt{2\sqrt{3} - 3} \right) k_3^2
\]
\[\quad - \left( \frac{224}{37} + \frac{138\sqrt{3}}{37} + \left( \frac{190\sqrt{3}}{37} + \frac{332}{37} \right) \sqrt{2\sqrt{3} - 3} \right) k_1^2 + O(k^4),
\]
(B.10)

\[
\omega^2 = \frac{8}{3} \left( \sqrt{3} - 1 - \sqrt{2\sqrt{3} - 3} \right) \mu^2 + \left( 1 - \left( 1 + \frac{2\sqrt{3}}{3} \right) \sqrt{2\sqrt{3} - 3} \right) k_3^2
\]
\[\quad - \left( \frac{224}{37} + \frac{138\sqrt{3}}{37} - \left( \frac{190\sqrt{3}}{37} + \frac{332}{37} \right) \sqrt{2\sqrt{3} - 3} \right) k_1^2 + O(k^4).
\]
(B.11)

### C Unitary gauge

We show here that the unitary vacuum choice (3.3) is the most general. Indeed, in the conventional (purely imaginary) choice for the \(SU(2)\) generators in adjoint representation

\[
T^1_c = -i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad T^2_c = -i \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad T^3_c = -i \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]
(C.1)
vacuum (3.3) is represented by

\[
\Phi_{c,0} = \text{Re} \left( \Phi_{c,0} \right) + i \text{Im} \left( \Phi_{c,0} \right) = \begin{bmatrix}
0 \\
-\frac{\sqrt{2}}{2} (\phi_1 + \phi_3) \\
0
\end{bmatrix} + i \begin{bmatrix}
\frac{\sqrt{2}}{2} (\phi_1 - \phi_3) \\
-\sqrt{2} \phi_0 \\
0
\end{bmatrix}.
\]  

(C.2)

Notice that the most general vacuum in canonical representation (C.1) can be parameterized by two real vectors \{\text{Re}(\Phi_c), \text{Im}(\Phi_c)\} in three dimensional space with gauge transformations acting as rotations separately on \{\text{Re}(\Phi_c)\} and \{\text{Im}(\Phi_c)\}. Thus, to demonstrate the validity of (3.3), all we need to do is to show that in the canonical representation an arbitrary vacuum can be parameterized as in (C.2).

But it is easy to show this. Let \( \vec{v}_1 \equiv \text{Re}(\Phi_c) \) and \( \vec{v}_2 \equiv \text{Im}(\Phi_c) \) are two vectors representing the generic vacuum. First, without any loss of generality we can choose coordinate axes in such a way that the vector \( \vec{v}_1 \) is aligned along the \( y \)-axis. Then, obviously we can identify \( \vec{v}_1 \) with \( \text{Re}(\Phi_{c,0}) \). At this stage, the vector \( \vec{v}_2 \) is arbitrary, but with our choice of coordinate axes we can still do rotations about \( y \)-axis. Any rotation of this type does not change \( \vec{v}_1 = \text{Re}(\Phi_{c,0}) \). Clearly, no matter what \( \vec{v}_2 \) is, by a \( y \)-rotation we can make it to have \( z \)-component to vanish — but then the rotated \( \vec{v}_2 \) would be identified with \( \text{Im}(\Phi_{c,0}) \).

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