Analysis of solution trajectories of linear fractional order systems

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Abstract

The behavior of solution trajectories usually changes if we replace the classical derivative in a system by a fractional one. In this article, we throw a light on the relation between two trajectories \( X(t) \) and \( Y(t) \) of such a system, where the initial point \( Y(0) \) is at some point \( X(t_1) \) of trajectory \( X(t) \). In contrast with classical systems, trajectories \( X \) and \( Y \) do not follow the same path. Further, we provide a Frenet apparatus of both trajectories in various cases and discuss their effect.

Keywords: Fractional derivative, Mittag-Leffler functions, Orthogonal transformation, Frenet apparatus.

1 Introduction

In the recent past, fractional differential equations (FDE) became a popular topic among the researchers working in pure as well as applied Mathematics. Applications of FDEs are found in various fields ranging from Physics to Biology. We suggest some selected references [1, 2, 3, 4, 5, 6, 7, 8, 9] on applications of FDEs to the readers.

Mathematical analysis of FDEs is also an interesting and equally important topic of research. Existence and uniqueness [10, 11, 12, 13], stability [14, 15, 16, 17, 18, 19, 20] and positivity [21, 22, 23, 24, 25, 26] of these equations is studied by the researchers in details. Fractional order versions of stable manifold theorem are discussed in [27, 28, 29]. Since FDEs are generalizations of classical differential dynamical systems, we cannot expect the same properties from these models as the classical ones.

In [30], we have shown that the planar linear FDE system \( \frac{d^\alpha}{dt^\alpha} X = AX \) may produce self-intersecting trajectories. Such singular points are not shown by their classical counterparts. We continue our investigations in the present manuscript and discuss the trajectories of FDE systems whose initial point is on a different trajectory of the same system.

2 Preliminaries

This section contains basic definitions and results given in the literature.

Definition 2.1. [31] Let \( \alpha \geq 0 \quad (\alpha \in \mathbb{R}) \). Then Riemann-Liouville (RL) fractional integral of function \( f \in C[0, b], \ t > 0 \) of order \( \alpha \) is defined as,

\[
0_1^I f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau) \, d\tau.
\] (1)

Definition 2.2. [31] The Caputo fractional derivative of order \( \alpha > 0, \ n - 1 < \alpha < n, \ n \in \mathbb{N} \) is defined for \( f \in C^n[0, b], \ t > 0 \) as,

\[
C_0^D f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t - \tau)^{n-\alpha-1} f^{(n)}(\tau) \, d\tau & \text{if } n - 1 < \alpha < n \\ \frac{d^n}{dt^n} f(t) & \text{if } \alpha = n. \end{cases}
\] (2)
Note that $C_0^\alpha D_t^\alpha c = 0$, where $c$ is a constant.

**Definition 2.3.** [31] The one-parameter Mittag-Leffler function is defined as,

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad z \in \mathbb{C}, \ (\alpha > 0).$$ (3)

The two-parameter Mittag-Leffler function is defined as,

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \in \mathbb{C}, \ (\alpha > 0, \ \beta > 0).$$ (4)

**Definition 2.4.** [32] Let $\alpha: I \to \mathbb{R}^n$ be a curve. The speed of $\alpha$ is defined as

$$\nu(t) = \| \alpha'(t) \|.$$ (5)

**Definition 2.5.** [32] An isometry of $\mathbb{R}^n$ is a mapping $F: \mathbb{R}^n \to \mathbb{R}^n$ such that

$$d(F(p), F(q)) = d(p, q)$$ (6)

for all points $p, q$ in $\mathbb{R}^n$. $d(x,y)$ is Euclidean distance.

**Definition 2.6.** [32] Two curves $\alpha, \beta: I \to \mathbb{R}^n$ are congruent provided there exists an isometry $F$ of $\mathbb{R}^n$ such that $\beta = F(\alpha)$; that is, $\beta(t) = F(\alpha(t))$ for all $t$ in $I$.

**Definition 2.7.** [32] A transformation $C: \mathbb{R}^n \to \mathbb{R}^n$ is an Orthogonal transformation if it preserves dot products in the sense that

$$C(p) \cdot C(q) = p \cdot q \quad \text{for all } p, q.$$ (7)

Every orthogonal transformation is an isometry.

**Theorem 2.1.** [33] Solution of homogeneous system of fractional order differential equation

$$C_0^\alpha D_t^\alpha X(t) = AX(t), \quad 0 < \alpha < 1$$ (8)

(where $A$ is $n \times n$ matrix) is given by

$$X(t) = E_{\alpha}(At^\alpha)X(0),$$ (9)

where $E_{\alpha}(At^\alpha)$ is matrix variate Mittag-Leffler function.

**Theorem 2.2.** [32] For planar regular curve $\alpha: I \to \mathbb{R}^2$ given by $\alpha(t) = (x(t), y(t)), \ t \in I$, the Frenet apparatus is given by

$$T = \frac{\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}}{\sqrt{(\dot{x})^2 + (\dot{y})^2}}$$

$$N = \frac{\begin{pmatrix} -\dot{y} \\ \dot{x} \end{pmatrix}}{\sqrt{(\dot{x})^2 + (\dot{y})^2}}$$

$$k = \frac{\det \begin{vmatrix} \dot{x} & \dot{y} \\ \ddot{x} & \ddot{y} \end{vmatrix}}{((\dot{x})^2 + (\dot{y})^2)^{3/2}}.$$ (10)
3 Observations

We have following observations.

(1) Consider the system

\[ \dot{X}(t) = \begin{bmatrix} -2 & 4 \\ -4 & -2 \end{bmatrix} X(t). \] (11)

Solution of the linear system (11) with initial condition \( X(0) = [1, 1]^T \) is given in the Figure 1 and it is shown by a blue line.

Now, consider the same system (11) with initial condition \( X(0) = [e^{-1}(\cos 2 + \sin 2), e^{-1}(-\sin 2 + \cos 2)]^T \) on the original trajectory, discussed above. Solution of this system is shown in the same figure by a red line.

It can be observed that both the trajectories follow the same path.

![Figure 1: Solution trajectory of system (11) starting at some point on original trajectory.](image)

(2) Consider the non-autonomous system of differential equations

\[ \dot{X}(t) = \begin{bmatrix} 6t \\ 3t^2 - 3 \end{bmatrix}. \] (12)

The solution trajectories of this system with distinct initial points (as above) are shown in Figure 2. It can be observed that (cf. blue curve in Figure 2), the loop in the original trajectory can be eliminated by choosing the initial condition of a new trajectory at a point on the original trajectory after self-intersection.

(3) Consider fractional order system

\[ 0_0^C D_t^{0.7} X(t) = \begin{bmatrix} -1 & 3 \\ -3 & -1 \end{bmatrix} X(t). \] (13)
In Figure 3, we show solutions to this system with different initial conditions. As in the last case, the initial condition of the second system is at some point on the original trajectory. However, the paths followed by these two trajectories are different, unlike in classical model (11).

(4) In paper [30], we have observed self-intersecting trajectories of some linear fractional order systems. Consider the system,

\[ C_0^D t^{0.1} X(t) = \begin{bmatrix} 0.983469 & 0.181075 \\ -0.181075 & 0.983469 \end{bmatrix} X(t). \]  

(14)

If \( X(0) = [1, 1]^T \), the solution trajectory shows self-intersection (see Figure 4(a)). Let us consider this system with initial condition \( X(0) = [\Re(E_{0.1}(0.983469 + 0.181075i)50^{0.1}) + \Im(E_{0.1}(0.983469 + 0.181075i)50^{0.1}), -\Im(E_{0.1}(0.983469 + 0.181075i)50^{0.1}) + \Re(E_{0.1}(0.983469 + 0.181075i)50^{0.1})]^T \) on the original trajectory.

Though we have taken new initial condition on the original trajectory at a point after self-intersection, the singular points cannot be removed unlike in classical case (2). Further, it seems that the new trajectory is some linear transformation of the original one.

(5) Time used to complete the loop:
Consider the system (14) with initial condition \( X(0) = [1, 1]^T \). Consider a node formed by solution trajectory in the time interval (12.35, 34). If we solve the system (14) with initial condition \( X(0) = [\Re(E_{0.1}(0.983469 + 0.181075i)50^{0.1}) + \Im(E_{0.1}(0.983469 + 0.181075i)50^{0.1}), -\Im(E_{0.1}(0.983469 + 0.181075i)50^{0.1}) + \Re(E_{0.1}(0.983469 + 0.181075i)50^{0.1})]^T \)
Our motivation for the present study is to find the linear transformation between original trajectory and the new trajectory of the fractional order system.

4 Analysis

In this section, we consider linear system of integer and fractional differential equations in $\mathbb{R}^2$ and $\mathbb{R}^3$. First we solve the system with initial condition $X(0) = X_0$ to obtain the solution $X(t)$. Then we solve the same system with initial condition at $X(t_1)$ for some $t_1 > 0$ and call the new solution as $Y(t)$. We show that $Y(t) = TX(t)$, where $T$ is some linear transformation.

**Lemma 4.1.** Consider a planar system $\dot{X}(t) = AX(t)$.

New trajectory $Y(t)$ starting at some point $X(t_1)$ on original trajectory $X(t)$ is given by the linear transformation

$$Y(t) = TX(t),$$

(15)
where \( T = e^{At_1} \).

(i) If \( A \) has real-distinct eigenvalues then \( T \) represents scaling (only).

(ii) If \( A \) has complex conjugate eigenvalues \( a \pm ib \) then \( T \) represents both scaling and rotation.

**Proof.** Solution of a system of ODEs \( \dot{X}(t) = AX(t) \), \( X(0) = X_0 \) is given by

\[
X(t) = e^{At}X_0.
\]

Now, let us consider the system \( \dot{Y}(t) = AY(t) \), \( Y(0) = X_1 \), where \( X_1 = X(t_1) = e^{At_1}X_0 \). Then its solution is given by, \( Y(t) = e^{At}X_1 = e^{At_1}e^{At}X_0 = TX(t) \), where \( T = e^{At_1} \).

The qualitative behavior of the system \( X'(t) = AX(t) \) does not change if we replace \( A \) by its canonical form.

(i) If \( A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \), then

\[
T = e^{At_1} = \begin{bmatrix} e^{\lambda_1 t_1} & 0 \\ 0 & e^{\lambda_2 t_1} \end{bmatrix}.
\]

Here \( T \) represents scaling only.

The type of scaling depends on sign of \( \lambda_j \), \( j = 1, 2 \).

(ii) If \( A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \), then

\[
T = e^{At_1} = \begin{bmatrix} e^{at_1} \cos(bt_1) & e^{at_1} \sin(bt_1) \\ -e^{at_1} \sin(bt_1) & e^{at_1} \cos(bt_1) \end{bmatrix}
= \begin{bmatrix} e^{at_1} & 0 \\ 0 & e^{at_1} \end{bmatrix} \begin{bmatrix} \cos(bt_1) & \sin(bt_1) \\ -\sin(bt_1) & \cos(bt_1) \end{bmatrix}
= U \cdot V
\]

where \( U = \begin{bmatrix} e^{at_1} & 0 \\ 0 & e^{at_1} \end{bmatrix} \) is scaling matrix and \( V = \begin{bmatrix} \cos(bt_1) & \sin(bt_1) \\ -\sin(bt_1) & \cos(bt_1) \end{bmatrix} \) is rotation matrix.

**Comment:** Scaling factor depends on real part of eigenvalue whereas imaginary part of eigenvalue represents angle of rotation. The curves \( X(t) \) and \( U^{-1}Y(t) \) are congruent.

**Example 4.1.** Consider the two classical systems

\[
\dot{X}(t) = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix} X(t) \quad (16)
\]

and

\[
\dot{X}(t) = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} X(t). \quad (17)
\]

In Figure 6(a) and 6(b) we sketch the solutions of system (16) and (17) respectively with initial conditions \( X(0) = [1, 1]^T \) (Blue color) and \( Y(0) = X(1) \) (Red color). It can be checked that both the trajectories follow the same path.
Lemma 4.2. Consider a planar system $\frac{C}{0}D_t^\alpha X(t) = AX(t)$, $0 < \alpha < 1$. New trajectory $Y(t)$ starting at some point $X(t_1)$ on original trajectory $X(t)$ is given by the linear transformation

$$Y(t) = TX(t),$$

where $T = E_\alpha(At^\alpha)$.

(i) If $A$ has real-distinct eigenvalues then $T$ represents scaling (only).

(ii) If $A$ has complex conjugate eigenvalues $a \pm ib$ then $T$ represents both scaling and rotation.

Proof. Solution of a system of FDEs $\frac{C}{0}D_t^\alpha X(t) = AX(t)$, $0 < \alpha < 1$, $X(0) = X_0$ is given by

$$X(t) = E_\alpha(At^\alpha)X_0.$$ 

Now, let us consider the system $\frac{C}{0}D_t^\alpha Y(t) = AY(t)$, $Y(0) = X_1$, where $X_1 = X(t_1) = E_\alpha(At^\alpha_1)X_0$. Then its solution is given by, $Y(t) = E_\alpha(At^\alpha)X_1 = E_\alpha(At^\alpha_1)E_\alpha(At^\alpha)X_0 = TX(t)$, where $T = E_\alpha(At^\alpha_1)$. As in Lemma 4.1, we assume that $A$ is in canonical form.

(i) If $A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, then

$$T = E_\alpha(At^\alpha_1) = \begin{bmatrix} E_\alpha(\lambda_1 t^\alpha_1) & 0 \\ 0 & E_\alpha(\lambda_2 t^\alpha_1) \end{bmatrix}.$$
Here $T$ is a scaling matrix.

(ii) If $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ then

$$T = e^{At_1} = \begin{bmatrix} \text{Re}[E_{\alpha}((a + ib)t_1^\alpha)] & \text{Im}[E_{\alpha}((a + ib)t_1^\alpha)] \\ -\text{Im}[E_{\alpha}((a + ib)t_1^\alpha)] & \text{Re}[E_{\alpha}((a + ib)t_1^\alpha)] \end{bmatrix}$$

$$= \begin{bmatrix} |E_{\alpha}((a + ib)t_1^\alpha)| & 0 \\ 0 & |E_{\alpha}((a + ib)t_1^\alpha)| \end{bmatrix}$$

$$= U \cdot V$$

where $U = \begin{bmatrix} |E_{\alpha}((a + ib)t_1^\alpha)| & 0 \\ 0 & |E_{\alpha}((a + ib)t_1^\alpha)| \end{bmatrix}$ is scaling matrix and $V = \begin{bmatrix} \text{Re}[E_{\alpha}((a + ib)t_1^\alpha)] & \text{Im}[E_{\alpha}((a + ib)t_1^\alpha)] \\ \text{Im}[E_{\alpha}((a + ib)t_1^\alpha)] & \text{Re}[E_{\alpha}((a + ib)t_1^\alpha)] \end{bmatrix}$ is rotation matrix.

**Comment:** Unlike in integer order case, the scaling not only depends on $a$ but also on $b$. The curves $X(t)$ and $U^{-1}Y(t)$ are congruent.

**Example 4.2.** General solution of,

$$\frac{d}{dt} X(t) = AX(t), \quad 0 < \alpha < 1,$$ where $A = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix}$ and $X(0) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ (19)

is given by

$$X(t) = c_1 E_{\alpha}(-t^\alpha) + (c_2 - c_1) E_{\alpha}(-2t^\alpha).$$ (20)

Let $Y(t)$ be a solution of $\frac{d}{dt} Y(t) = AY(t)$ with $Y(0) = X(t_1)$, $t_1 > 0$.

We sketch the solution trajectories $X(t)$ (Blue color) of the system (19) subject to the initial condition $X(0) = [1, 2]^T$ and $Y(t)$ (Red color) with initial condition $Y(0) = X(1)$ in the Figure 7 (a).

**Example 4.3.** General solution of,

$$\frac{d}{dt} X(t) = AX(t), \quad 0 < \alpha < 1,$$ where $A = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}$ and $X(0) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ (21)

is given by

$$X(t) = \begin{bmatrix} c_1 \text{Re}[E_{\alpha}(2it^\alpha)] + (c_2/2) \text{Im}[E_{\alpha}(2it^\alpha)] \\ -2c_1 \text{Im}[E_{\alpha}(2it^\alpha)] + c_2 \text{Re}[E_{\alpha}(2it^\alpha)] \end{bmatrix}.$$ (22)

Let $Y(t)$ be a solution of $\frac{d}{dt} Y(t) = AY(t)$ with $Y(0) = X(t_1)$, $t_1 > 0$.

We sketch the solution trajectories $X(t)$ (Blue color) of the system (21) with $X(0) = [1, 1]^T$ and $Y(t)$ (Red color) with $Y(0) = X(1)$ in the Figure 7 (b).
Theorem 4.1. Consider a system $\dot{X}(t) = AX(t)$, where $A$ is $3 \times 3$ matrix.

New trajectory $Y(t)$ starting at some point $X(t_1)$ on original trajectory $X(t)$ is given by the linear transformation

$$Y(t) = TX(t),$$

where $T = e^{At_1}$.

(i) If $A$ has real-distinct eigenvalues then $T$ represents scaling (only).

(ii) If $A$ has complex conjugate eigenvalues $a \pm ib$ and a real eigenvalue $\lambda$ then $T$ represents both scaling and rotation.

Proof. (i) If $A$ is in the standard canonical form $A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$, then

$$T = e^{At_1} = \begin{bmatrix} e^{\lambda_1 t_1} & 0 & 0 \\ 0 & e^{\lambda_2 t_1} & 0 \\ 0 & 0 & e^{\lambda_3 t_1} \end{bmatrix}$$

Here $T$ represents scaling only.
(ii) If $A$ is in the standard canonical form $A = \begin{bmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & \lambda \end{bmatrix}$, then $A$ has eigenvalues $a \pm ib$, $\lambda$ and

$$T = e^{At} = \begin{bmatrix} e^{at_1} \cos(bt_1) & e^{at_1} \sin(bt_1) & 0 \\ -e^{at_1} \sin(bt_1) & e^{at_1} \cos(bt_1) & 0 \\ 0 & 0 & e^{\lambda t_1} \end{bmatrix} = \begin{bmatrix} e^{at_1} & 0 & 0 \\ 0 & e^{at_1} & 0 \\ 0 & 0 & e^{\lambda t_1} \end{bmatrix} \begin{bmatrix} \cos(bt_1) & \sin(bt_1) & 0 \\ -\sin(bt_1) & \cos(bt_1) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$ 

where $U = \begin{bmatrix} e^{at_1} & 0 & 0 \\ 0 & e^{at_1} & 0 \\ 0 & 0 & e^{\lambda t_1} \end{bmatrix}$ is scaling matrix (Uniform scaling by factor $e^{at_1}$ of $X$, $Y$-coordinates and scaling of $Z$-coordinate by $e^{\lambda t_1}$) and $V = \begin{bmatrix} \cos(bt_1) & \sin(bt_1) & 0 \\ -\sin(bt_1) & \cos(bt_1) & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is rotation matrix (Rotation about $Z$-axis; angle of rotation is $bt_1$).

The curves $X(t)$ and $U^{-1}Y(t)$ are congruent.

**Example 4.4.** Consider the two classical systems

$$\dot{X}(t) = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & -2 \\ 0 & 2 & -2 \end{bmatrix} X(t)$$ (24)

and

$$\dot{X}(t) = \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix} X(t)$$ (25)

In Figure 8(a) and 8(b) we sketch the solutions of system (24) and (25) respectively with initial conditions $X(0) = [1, 1, 1]^T$ (Blue color) and $Y(0) = X(1)$ (Red color). It can be checked that both the trajectories follow the same path.

**Theorem 4.2.** Consider a system $C_0^1 D_0^\alpha X(t) = AX(t), 0 < \alpha < 1$ where $A$ is $3 \times 3$ matrix.

New trajectory $Y(t)$ starting at some point $X(t_1)$ on original trajectory $X(t)$ is given by the linear transformation

$$Y(t) = TX(t),$$ (26)

where $T = E_\alpha(At_1^\alpha)$.

(i) If $A$ has real-distinct eigenvalues then $T$ represents scaling (only).

(ii) If $A$ has complex conjugate eigenvalues $a \pm ib$ and a real eigenvalue $\lambda$ then $T$ represents both scaling and rotation.

**Proof.** (i) If $A$ is in the standard canonical form $A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$, then

$$T = E_\alpha(At_1^\alpha) = \begin{bmatrix} E_\alpha(\lambda_1 t_1^\alpha) & 0 & 0 \\ 0 & E_\alpha(\lambda_2 t_1^\alpha) & 0 \\ 0 & 0 & E_\alpha(\lambda_3 t_1^\alpha) \end{bmatrix}.$$
Here $T$ represents scaling only.

(ii) If $A$ is in the standard canonical form $A = \begin{bmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & \lambda \end{bmatrix}$ then

$$T = e^{At} = \begin{bmatrix} \text{Re}[E_\alpha((a+ib)t_1^\alpha)] & \text{Im}[E_\alpha((a+ib)t_1^\alpha)] & 0 \\ -\text{Im}[E_\alpha((a+ib)t_1^\alpha)] & \text{Re}[E_\alpha((a+ib)t_1^\alpha)] & 0 \\ 0 & 0 & E_\alpha(\lambda t_1^\alpha) \end{bmatrix}$$

$$= \begin{bmatrix} |E_\alpha((a+ib)t_1^\alpha)| & 0 & 0 \\ 0 & |E_\alpha((a+ib)t_1^\alpha)| & 0 \\ 0 & 0 & E_\alpha(\lambda t_1^\alpha) \end{bmatrix}$$

$$U \cdot V$$

where $U = \begin{bmatrix} |E_\alpha((a+ib)t_1^\alpha)| & 0 & 0 \\ 0 & |E_\alpha((a+ib)t_1^\alpha)| & 0 \\ 0 & 0 & E_\alpha(\lambda t_1^\alpha) \end{bmatrix}$ is scaling matrix (Uniform scaling by factor $|E_\alpha((a+ib)t_1^\alpha)|$ of $X$, $Y$-coordinates and scaling of $Z$-coordinate by $E_\alpha(\lambda t_1^\alpha)$)

and $V = \begin{bmatrix} \text{Re}[E_\alpha((a+ib)t_1^\alpha)] & \text{Im}[E_\alpha((a+ib)t_1^\alpha)] & 0 \\ -\text{Im}[E_\alpha((a+ib)t_1^\alpha)] & \text{Re}[E_\alpha((a+ib)t_1^\alpha)] & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is rotation matrix (Rotation about $Z$-axis; angle of rotation is $\theta = \cos^{-1} \left[ \frac{\text{Re}[E_\alpha((a+ib)t_1^\alpha)]}{|E_\alpha((a+ib)t_1^\alpha)|} \right]$).

The curves $X(t)$ and $U^{-1}Y(t)$ are congruent.

Comments:-

New trajectories are transformed versions of original trajectories. In integer order case, both trajectories follow same path because $e^{At_1(t_1+t_2)} = e^{At_1}e^{At_2}$.

This is not the case with fractional order systems because $E_\alpha(A(t_1+2)t_2^\alpha) \neq E_\alpha(At_1^\alpha)E_\alpha(At_2^\alpha)$, in general.
Example 4.5. General solution of,

\[ C_0^\alpha \text{D}_t^\alpha X(t) = AX(t), \quad 0 < \alpha < 1, \text{ where } A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & -2 \\ 0 & 2 & -2 \end{bmatrix} \text{ and } X(0) = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \] (27)

is given by

\[ X(t) = \begin{bmatrix} c_1 E_\alpha(t^\alpha) + 2c_2(E_\alpha(2t^\alpha) - E_\alpha(t^\alpha)) + c_3(E_\alpha(t^\alpha) - E_\alpha(2t^\alpha)) \\ (c_2/3)(4E_\alpha(2t^\alpha) - E_\alpha(-t^\alpha)) + (2c_3/3)(E_\alpha(-t^\alpha) - E_\alpha(2t^\alpha)) \\ (2c_2/3)(E_\alpha(2t^\alpha) - E_\alpha(-t^\alpha)) + (c_3/3)(4E_\alpha(-t^\alpha) - E_\alpha(2t^\alpha)) \end{bmatrix}. \] (28)

\[ Y(t) \text{ be a solution of } C_0^\alpha \text{D}_t^\alpha Y(t) = AY(t) \text{ with } Y(0) = X(t_1), \quad t_1 > 0. \]

In the Figure 9 (a), we sketch the solution trajectories \( X(t) \) (Blue color) of the system (27) subject to the initial condition \( X(0) = [1, 1, 1]^T \) and \( Y(t) \) (Red color) with initial condition \( Y(0) = X(1) \).

Example 4.6. Repeating the same exercise as in Example 4.5 with

\[ A = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & 1 & 1 \end{bmatrix} \text{ and } X(0) = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \] (29)

we get

\[ X(t) = \begin{bmatrix} c_1 E_\alpha(-3t^\alpha) \\ c_2 Re[E_\alpha((2 + i)t^\alpha)] + (c_2 - 2c_3)Im[E_\alpha((2 + i)t^\alpha)] \\ c_3 Re[E_\alpha((2 + i)t^\alpha)] + (c_2 - c_3)Im[E_\alpha((2 + i)t^\alpha)] \end{bmatrix}. \] (30)

In the Figure 9 (b), we sketch the solution trajectories \( X(t) \) (Blue color) of the system (29) with initial condition \( X(0) = [1, 1, 1]^T \) and \( Y(t) \) (Red color) subject to the initial condition \( Y(0) = X(1) \).
Figure 9: Trajectories of fractional order systems (27) and (29).

5 Differential geometry of trajectories of fractional order systems

Frenet apparatus is a tool which is very useful to describe the shape of a curve. In this section we find Frenet apparatus for solution trajectories of FDEs

\[ t^\alpha X(t) = AX(t), \quad 0 < \alpha < 1, \quad X(0) = X_0 = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \]

where \( A \) is in canonical form.

(1) Let, \( A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \), where \( \lambda_1 \neq \lambda_2 \) are real numbers.

The Frenet apparatus of solution trajectory of \( t^\alpha X(t) = AX(t), \quad 0 < \alpha < 1, \quad X(0) = (c_1, c_2)^T \) (respectively \( t^\alpha Y(t) = AY(t), \quad 0 < \alpha < 1, \quad Y(0) = X(t_1), \quad t_1 > 0 \)) is \( T_1, N_1, \kappa_1 \) (respectively \( T_2, N_2, \kappa_2 \)).

If \( \nu_1(t) = \sqrt{(\dot{x})^2 + (\dot{y})^2} \), is speed of \( X = (x, y)^T \) then

\[ \nu_1(t) = \sqrt{c_1^2 \left( \lambda_1^{t^\alpha} E_{\alpha,\alpha} \left( \lambda_1^{t^\alpha} \right) \right)^2 + c_2^2 \left( \lambda_2^{t^\alpha} E_{\alpha,\alpha} \left( \lambda_2^{t^\alpha} \right) \right)^2} \]

Similarly, speed of \( Y \) is given by

\[ \nu_2(t) = \sqrt{c_1^2 \left( E_{\alpha} \left( \lambda_1^{t_1^\alpha} \right) \right)^2 \left( \lambda_1^{t_1^\alpha} E_{\alpha,\alpha} \left( \lambda_1^{t_1^\alpha} \right) \right)^2 + c_2^2 \left( E_{\alpha} \left( \lambda_2^{t_1^\alpha} \right) \right)^2 \left( \lambda_2^{t_1^\alpha} E_{\alpha,\alpha} \left( \lambda_2^{t_1^\alpha} \right) \right)^2}. \]
If \( u_1 = \lambda_1 t^{\alpha-1} E_{\alpha,\alpha}(\lambda_1 t^\alpha) \) then we have
\[
\kappa_1 = \left| \frac{c_1 c_2 u_1}{\nu^3} \right| \quad \text{and} \quad \kappa_2 = \left| \frac{c_1 c_2 E_{\alpha}(\lambda_1 t^\alpha) E_{\alpha}(\lambda_2 t^\alpha) u_1}{\nu^3} \right|. 
\]

Therefore,
\[
\kappa_2 = \left| \frac{E_{\alpha}(\lambda_1 t^\alpha) E_{\alpha}(\lambda_2 t^\alpha) \nu^3}{\nu^3} \right| \kappa_1. 
\]

The unit tangent vectors
\[
T_1 = \left( \frac{c_1 \lambda_1 t^{\alpha-1} E_{\alpha,\alpha}(\lambda_1 t^\alpha), c_2 \lambda_2 t^{\alpha-1} E_{\alpha,\alpha}(\lambda_2 t^\alpha)}{\nu_1(t)} \right)
\]
and
\[
T_2 = \left( \frac{c_1 E_{\alpha}(\lambda_1 t^\alpha) \lambda_1 t^{\alpha-1} E_{\alpha,\alpha}(\lambda_1 t^\alpha), c_2 E_{\alpha}(\lambda_2 t^\alpha) \lambda_2 t^{\alpha-1} E_{\alpha,\alpha}(\lambda_2 t^\alpha)}{\nu_2(t)} \right).
\]
\[
\therefore T_2 = \frac{\nu_1(t)}{\nu_2(t)} \begin{bmatrix} E_{\alpha}(\lambda_1 t^\alpha) & 0 \\ 0 & E_{\alpha}(\lambda_2 t^\alpha) \end{bmatrix} T_1.
\]

Similarly, the unit normal vectors
\[
N_1 = \left( \frac{-c_2 \lambda_2 t^{\alpha-1} E_{\alpha,\alpha}(\lambda_2 t^\alpha), c_1 \lambda_1 t^{\alpha-1} E_{\alpha,\alpha}(\lambda_1 t^\alpha)}{\nu_1(t)} \right)
\]
and
\[
N_2 = \left( \frac{-c_2 E_{\alpha}(\lambda_2 t^\alpha) \lambda_2 t^{\alpha-1} E_{\alpha,\alpha}(\lambda_2 t^\alpha), c_1 E_{\alpha}(\lambda_1 t^\alpha) \lambda_1 t^{\alpha-1} E_{\alpha,\alpha}(\lambda_1 t^\alpha)}{\nu_2(t)} \right).
\]
\[
\therefore N_2 = \frac{\nu_1(t)}{\nu_2(t)} \begin{bmatrix} E_{\alpha}(\lambda_2 t^\alpha) & 0 \\ 0 & E_{\alpha}(\lambda_1 t^\alpha) \end{bmatrix} N_1.
\]

Note that, if \( \lambda_1 = \lambda_2 = \lambda \) then \( \nu_2(t) = E_\alpha(\lambda t^\alpha) \nu_1(t) \), \( T_2 = T_1 \), \( N_2 = N_1 \), \( \kappa_2 = \kappa_1 = 0 \).

**Conclusions:**
1. Let, \( A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \).
2. In this case, the general solution of the system is, \( X(t) = \begin{bmatrix} c_1 E_{\alpha}(\lambda t^\alpha) + c_2 \frac{t^\alpha}{\alpha} E_{\alpha,\alpha}(\lambda t^\alpha) \\ c_2 E_{\alpha}(\lambda t^\alpha) \end{bmatrix} \).

\[
\nu_1(t) = \left( c_1^2 + c_2^2 \right) \left( \sum_{n=1}^{\infty} \frac{\lambda^n t^{\alpha n-1}}{\Gamma(an)} \right)^2 + c_2^2 \left( \sum_{n=1}^{\infty} \frac{\lambda^n (an + \alpha) t^{\alpha n + \alpha - 1}}{\Gamma(an + \alpha)} \right)^2 + \frac{2c_1 c_2}{\alpha} \sum_{n=1}^{\infty} \frac{\lambda^n t^{\alpha n-1}}{\Gamma(an)} \sum_{n=1}^{\infty} \frac{\lambda^n (an + \alpha) t^{\alpha n + \alpha - 1}}{\Gamma(an + \alpha)} \right)^{1/2}.
\]

If
\[
u_2(t) = \left( \sum_{n=1}^{\infty} \frac{\lambda^n (an + \alpha) t^{\alpha n + \alpha - 1}}{\Gamma(an + \alpha)} \right) \sum_{n=1}^{\infty} \frac{\lambda^n t^{\alpha n - 2}}{\Gamma(an - 1)} - \sum_{n=1}^{\infty} \frac{\lambda^n (an + \alpha) t^{\alpha n + \alpha - 2}}{\Gamma(an + \alpha - 1)} \sum_{n=1}^{\infty} \frac{\lambda^n t^{\alpha n - 1}}{\Gamma(an)},
\]
then
\[
\kappa_1 = \left| \frac{c_2^2 \nu_2(t)}{(\nu_1(t))^3} \right|.
\]
In this case, the general solution of the system is,

\[ u, \nu \]

Let,

\[ \sum_{n=1}^{\infty} \frac{\lambda^\nu n^{\alpha-1}}{\Gamma(\alpha)} + \frac{c_2}{\alpha} \sum_{n=1}^{\infty} \frac{\lambda^\nu n^{\alpha-1}}{\Gamma(\alpha)} = 0, \]

and

\[ \sum_{n=1}^{\infty} \frac{\lambda^\nu n^{\alpha-1}}{\Gamma(\alpha)} = 0. \]

Similarly, \( Y(t) = \left[ (c_1 E_\alpha(\lambda t^\alpha) + \frac{c_2}{\alpha} E_{\alpha,\alpha}(\lambda t^\alpha))E_\alpha(\lambda t^\alpha) + \frac{c_2}{\alpha} E_{\alpha,\alpha}(\lambda t^\alpha) \right] \]

where \( E_{\alpha,\alpha}(\lambda t^\alpha) \) is the generalized hypergeometric function.

\[ \nu_2(t) = \left( \frac{(E_\alpha(\lambda t^\alpha))^2}{\nu_2(t)} \right) \left( \sum_{n=1}^{\infty} \frac{\lambda^\nu n^{\alpha-1}}{\Gamma(\alpha)} \right)^2 + \frac{c_2}{\alpha^2} \left( \sum_{n=1}^{\infty} \frac{\lambda^\nu n^{\alpha-1}}{\Gamma(\alpha)} \right)^2 \]

where \( \alpha = 1 \) and \( \nu = 2 \).

\[ \therefore \kappa_2 = \left| \frac{(\nu_1(t))^3}{(\nu_2(t))^3} E_\alpha(\lambda t^\alpha) \right| \kappa_1, \]

\[ T_2 = \frac{\nu_1(t)E_\alpha(\lambda t^\alpha)}{\nu_2(t)} T_1 + \frac{t_1^\alpha E_{\alpha,\alpha}(\lambda t^\alpha)}{\nu_2(t)} V, \]

where \( V = \left( \frac{c_2}{\alpha} \sum_{n=1}^{\infty} \frac{\lambda^\nu n^{\alpha-1}}{\Gamma(\alpha)} \right) \) and

\[ N_2 = \frac{\nu_1(t)E_\alpha(\lambda t^\alpha)}{\nu_2(t)} N_1 + \frac{t_1^\alpha E_{\alpha,\alpha}(\lambda t^\alpha)}{\nu_2(t)} W, \]

where \( W = \left( \frac{c_2}{\alpha} \sum_{n=1}^{\infty} \frac{\lambda^\nu n^{\alpha-1}}{\Gamma(\alpha)} \right). \)

(4) Let, \( A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}. \)

(i) In this case, the general solution of the system is, \( X(t) = \left[ c_1 \text{Re}[E_\alpha((a + ib)t^\alpha)] + c_2 \text{Im}[E_\alpha((a + ib)t^\alpha)] \right]. \)

Let,

\[ c_\alpha(a, b, t) = \sum_{n=1}^{\infty} \frac{(\text{Re}(a + ib)^n)\lambda^\nu n^{\alpha-1}}{\Gamma(\alpha)}, \quad s_\alpha(a, b, t) = \sum_{n=1}^{\infty} \frac{(\text{Im}(a + ib)^n)\lambda^\nu n^{\alpha-1}}{\Gamma(\alpha)} \]

and

\[ u_3(t) = (c_1^2 + c_2^2) \left( -c_\alpha(a, b, t) \sum_{n=1}^{\infty} \frac{(\text{Im}(a + ib)^n)(\alpha n - 1)\lambda^\nu n^{\alpha-2}}{\Gamma(\alpha)} + s_\alpha(a, b, t) \sum_{n=1}^{\infty} \frac{(\text{Re}(a + ib)^n)(\alpha n - 1)\lambda^\nu n^{\alpha-2}}{\Gamma(\alpha)} \right). \]

\( \therefore \nu_1(t) = \left( c_\alpha(a, b, t)^2 + s_\alpha(a, b, t)^2 \right)^{1/2}, \)

\[ \kappa_1 = \left| \frac{u_3(t)}{\nu_1(t)} \right|^2, \]

\[ T_1 = \frac{1}{\nu_1(t)} \left( c_1 c_\alpha(a, b, t) + c_2 s_\alpha(a, b, t), -c_1 s_\alpha(a, b, t) + c_2 c_\alpha(a, b, t) \right) \]

and

\[ N_1 = \frac{1}{\nu_1(t)} \left( c_1 s_\alpha(a, b, t) - c_2 c_\alpha(a, b, t), c_1 c_\alpha(a, b, t) + c_2 s_\alpha(a, b, t) \right). \]
Similarly,
\[ Y(t) = \left[ p\text{Re}[E_\alpha((a+ib)t^\alpha)] + q\text{Im}[E_\alpha((a+ib)t^\alpha)] \right] - p\text{Im}[E_\alpha((a+ib)t^\alpha)] + q\text{Re}[E_\alpha((a+ib)t^\alpha)], \]
where \( p = c_1\text{Re}[E_\alpha((a+ib)t^\alpha)] + c_2\text{Im}[E_\alpha((a+ib)t^\alpha)] \) and \( q = -c_1\text{Im}[E_\alpha((a+ib)t^\alpha)] + c_2\text{Re}[E_\alpha((a+ib)t^\alpha)]. \)

\[ \therefore \nu_2(t) = \sqrt{c_1^2 + c_2^2} |E_\alpha((a+ib)t^\alpha)| \left((c_\alpha(a,b,t))^2 + (s_\alpha(a,b,t))^2\right)^{1/2}, \]

\[ \kappa_2 = \frac{1}{|E_\alpha((a+ib)t^\alpha)|^{\kappa_1}}, \]

\[ T_2 = \frac{\text{Re}[E_\alpha((a+ib)t^\alpha)]\nu_1(t)}{\nu_2(t)} T_1 - \frac{\text{Im}[E_\alpha((a+ib)t^\alpha)]\nu_1(t)}{\nu_2(t)} N_1 \text{ and } \]

\[ N_2 = \frac{\text{Re}[E_\alpha((a+ib)t^\alpha)]\nu_1(t)}{\nu_2(t)} N_1 + \frac{\text{Re}[E_\alpha((a+ib)t^\alpha)]\nu_1(t)}{\nu_2(t)} T_1. \]

**Comment:** Speed of the new curve is affected by scaling factor.

### 6 Bifurcation Analysis

Since the new trajectory \( Y(t) \) starting at some point \( X(t_1) \) on original trajectory is a transformation of solution \( X(t) \) of \( C_0\text{D}_t^\alpha X(t) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} X(t), \) \( 0 < \alpha < 1, \) it is worth studying the effect of fractional order on such transformations.

(i) Fix \( t_1 = 1, \) \( a = 0.983469 \) and \( b = 0.181075. \)

In the graph of \( \theta, \) there is local maximum at \( \alpha = 0.06144 \) as shown in the Figure 10.

![Figure 10: Angle of rotation \( \theta \) versus \( \alpha. \)](#)

(ii) In Figure 11 we fix \( a = 1 \) and sketch the surface \( (\alpha, b, \theta). \)

It is observed that angle of rotation \( \theta \) is having maxima at some values of parameters \( b \) and \( \alpha. \)

In the Figure 12 we sketch a parametric curve of maximum \( (\theta) \) for different values of \( \alpha \) and \( b. \)

It can be checked that most of the points of maximum \( (\theta) \) lie on a straight line

\[ b = -0.0066 + 2.9128\alpha. \] (32)
Figure 11: surface $(\alpha, b, \theta)$ for $0 < \alpha < 1$.

Figure 12: $\max(\theta)$ shown by dotted curve and it is approximated by a straight line $[32]$ (red color).

7 Conclusion

The systems of fractional differential equations are not the dynamical systems in a classical sense. The solution $\phi_t(X_0)$ of fractional order initial value problem $\frac{\partial}{\partial t} C_0^{\alpha} X = f(X), \ X(0) = X_0$ does not satisfy the property $\phi_t \circ \phi_s = \phi_{t+s}$ of flow of classical differential equation. However, the two trajectories $\phi_t(X_0)$ and $\phi_{t+s}(X_0)$ are closely related if we take $f(X) = AX$, a linear function.

In this article, we have shown that the new trajectory is a linear transformation of original one. Further, we provided analysis of such trajectories with the help of Frenet apparatus.

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