ON COMPUTING THE KRONECKER STRUCTURE OF POLYNOMIAL MATRICES USING Julia

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Abstract. In this paper we discuss the mathematical background and the computational aspects which underly the implementation of a collection of Julia functions in the MatrixPencils package for the determination of structural properties of polynomial matrices. We primarily focus on the computation of the finite and infinite spectral structures (e.g., eigenvalues, zeros, poles) as well as the left and right singular structures (e.g., Kronecker indices), which play a fundamental role in the structure of the solution of many problems involving polynomial matrices. The basic analysis tool is the determination of the Kronecker structure of linear matrix pencils using numerically reliable algorithms, which is used in conjunction with several linearization techniques of polynomial matrices. An example of a polynomial matrix which exhibits all relevant structural features is considered to illustrate the main mathematical concepts and the capabilities of implemented tools.

Key words. Polynomial matrices, rational matrices, matrix pencils, descriptor systems, computational methods.

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1. Introduction. Structural properties such as eigenvalues, zeros, poles, and minimal indices, play a fundamental role in the structure of the solution of many problems involving polynomial matrices. An important application domain is the solution of polynomial eigenvalue problems, where polynomial matrices arise either directly from the mathematical modelling of continuous system dynamics or represent approximations of more general general nonlinear mappings leading to nonlinear eigenvalue problems (see [8] for a fairly complete survey of this subject). Another field of application is control system theory, where polynomial matrix models play a fundamental role in the structural analysis of linear systems [12].

One of the computational approaches to solve polynomial eigenvalue problems is via linearizations, where polynomial matrices of arbitrary degree are replaced by first degree polynomial matrices (also called matrix pencils) which allow to retrieve the structural feature of the original problems. The main appeal of this approach is to allow the use of well established computational techniques for matrix pencil manipulations (e.g., reduction to various Kronecker-like forms in conjunction with the QZ algorithm) to determine the involved structural elements. The most commonly used linearizations are the Frobenius companion forms [13], which can be directly built from the underlying problem data. Alternative linearizations are structured matrix pencils, also called system matrix pencils [12], which share the same pole-zero and singular structures with the original polynomial matrix. Using these latter linearizations usually involves the determination of linearization with special features (e.g., of least dimension).

In this article we present the basic concepts to characterize the structural properties of polynomial matrices such as finite and infinite eigenvalues, minimal indices, zeros and poles, and discuss these concepts also in the particular case of first degree polynomial matrices (i.e., matrix pencils). Numerically reliable matrix pencil reduction techniques play a central role in the determination of these properties and therefore they form the basic numerical ingredients for the investigation of structural features of polynomial matrices via suitable linearizations. Three classes of linearizations are discussed for a given polynomial matrix (companion forms, pencil based system matrix, and descriptor system matrix) and the correspondences between the properties of the original polynomial matrix and its linearizations are described. Additionally, we describe a general linearization technique of structured polynomial system matrices of arbitrary degree. We present succinctly the newly implemented collection of software tools for the Release v1.0 of the Julia package MatrixPencils, which cover the computation of structural elements of polynomial matrices and related computations as described in this paper. The main mathematical concepts and the capabilities of implemented tools are illustrated using a simple polynomial matrix employed in [16], which still exhibits all relevant structural features.

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2. Polynomial matrices. Let $\mathbb{F}(\lambda)$ be the set of rational functions with coefficients in the field $\mathbb{F}$ with indeterminate $\lambda$, and let $\mathbb{F}[\lambda]$ be the set of polynomials with coefficients in the field $\mathbb{F}$. We denote $\overline{\mathbb{F}}$ the algebraic closure of $\mathbb{F}$. The most usual cases are when either $\mathbb{F} = \mathbb{R}$, the set of real numbers, or $\mathbb{F} = \mathbb{C}$, the set of complex numbers. Since polynomials can be assimilated with special rational functions with 1 as denominator, $\mathbb{F}[\lambda] \subset \mathbb{F}(\lambda)$. It is easy to show that $\mathbb{F}(\lambda)$ is closed under the addition and multiplication operations. Both operations are associative and commutative, the multiplication is distributive over addition, and each operation possesses an identity element in $\mathbb{F}(\lambda)$. Finally, there exist inverses for all elements under addition and for all nonzero elements under multiplication. Therefore, the set $\mathbb{F}(\lambda)$ forms a field. The subset of polynomials $\mathbb{F}[\lambda]$ forms only a ring (more exactly, an Euclidean domain with identity), because the only invertible elements in $\mathbb{F}[\lambda]$ are the nonzero elements of $\mathbb{F}$, which are thus the units of the ring.

Let $P(\lambda) \in \mathbb{F}[\lambda]^{m \times n}$ be a $m \times n$ polynomial matrix defined as

\[
P(\lambda) = \begin{bmatrix} p_{11}(\lambda) & \cdots & p_{1n}(\lambda) \\
\vdots & \ddots & \vdots \\
p_{m1}(\lambda) & \cdots & p_{mn}(\lambda) \end{bmatrix},
\]

where each $p_{ij}(\lambda)$ is a polynomial of the form

\[
p_{ij}(\lambda) = a_k \lambda^k + a_{k-1} \lambda^{k-1} + \cdots + a_1 \lambda + a_0
\]

with coefficients in $\mathbb{F}$. Polynomial row vectors, column vectors and even scalar polynomials can be associated to particular polynomial matrices with $m = 1$, $n = 1$ or $m = n = 1$, respectively. The degree $d$ of $P(\lambda)$ is the largest degree of the polynomial entries of $P(\lambda)$

\[
d = \deg P(\lambda) := \max_{i,j} \deg p_{ij}(\lambda).
\]

Polynomials as in (2), with $a_k = 1$, are called monic polynomials.

If $k \geq d$, $P(\lambda)$ can alternatively be written as a grade $k$ matrix polynomial

\[
P(\lambda) = \sum_{i=0}^{k} \lambda^i P_i
\]

with $P_i \in \mathbb{F}^{m \times n}$, for $i = 0, 1, \ldots, k$. For this representation, the degree is simply the largest index $d$ for which $P_d \neq 0$. A polynomial matrix with $k = d = 1$ is called a matrix pencil, or simply a pencil.

Remark. The choice of grade $k$ is a matter of convenience and depends on the intended application. For example, it allows to view a constant polynomial matrix $P(\lambda) = A$ as $A$ (i.e., a constant matrix), or $A + \lambda 0$ (i.e., a pencil), or even as $A + \lambda 0 + \lambda^2 0 + \cdots + \lambda^n 0$ (i.e., a polynomial matrix of grade $k$). For a pertinent discussion of this matter, see [13].

The following definitions are straightforward extensions of familiar notions for constant matrices. $P(\lambda)$ is called regular if $m = n$ and $\det P(\lambda) \neq 0$. Otherwise, $P(\lambda)$ is called singular. Equivalently, $P(\lambda)$, viewed as a rational matrix with entries in the field $\mathbb{F}(\lambda)$, is regular if $P(\lambda)$ is invertible (the inverse is however not a polynomial matrix in general). The normal rank of $P(\lambda)$, denoted $\text{rank} P(\lambda)$, is the size of the largest non-identically-zero minor of $P(\lambda)$. Equivalently, the normal rank of $P(\lambda)$, viewed as a rational matrix, is the number of linearly independent rows or columns of $P(\lambda)$. A regular polynomial matrix $P(\lambda)$ is called unimodular if $\det P(\lambda)$ is a constant (i.e., independent of $\lambda$), or, equivalently, $P(\lambda)$ has an inverse that is also a polynomial matrix.

Two main structural properties of a polynomial matrix $P(\lambda)$ are its eigenvalue structure and its singular structure. In what follows, we address these aspects using both linear algebra results as well as control system theory results.

The eigenvalue structure concerns with the eigenvalues of the polynomial matrix $P(\lambda)$, which are those values of $\lambda$ for which the equation

\[
P(\lambda)x = 0
\]
has nonzero solutions $x$. For example, if $P(\lambda)$ is regular, then the finite eigenvalues are simply the roots of $\det P(\lambda)$. If rank $P_d = n$, this is a polynomial of degree $nd$ and all eigenvalues of $P(\lambda)$ are finite. If rank $P_d < n$, then $\det P(\lambda)$ is a polynomial of degree say $q < nd$ and, therefore $P(\lambda)$ has $q$ finite eigenvalues and $nd - q$ infinite eigenvalues. In what follows, we give the precise definitions of eigenvalues using the Smith canonical form of polynomial matrices.

**Theorem 2.1 (Smith form).** Let $P(\lambda)$ be an $m \times n$ polynomial matrix of rank $r$ with coefficients in $F$. Then, there exist unimodular polynomial matrices $U(\lambda) \in F[\lambda]^{m \times m}$ and $V(\lambda) \in F[\lambda]^{n \times n}$ such that

$$D(\lambda) := U(\lambda)P(\lambda)V(\lambda) = \begin{bmatrix}
    d_1(\lambda) & \cdots & 0_{m-r,r} \\
    \vdots & \ddots & \vdots \\
    0_{m-r,r} & \cdots & d_r(\lambda)
\end{bmatrix},$$

where $d_1(\lambda), \ldots, d_r(\lambda)$ are monic polynomials in $F[\lambda]$ such that $d_i(\lambda)$ divides $d_{i+1}(\lambda)$ for $i = 1, \ldots, r - 1$. Moreover, $D(\lambda)$ is unique and is called the Smith canonical form of $P(\lambda)$.

The monic polynomials $d_1(\lambda), \ldots, d_r(\lambda)$ are called the invariant polynomials of $P(\lambda)$. The (finite) eigenvalues of $P(\lambda)$ are the totality of (finite) zeros (roots) of all invariant polynomials.

For each distinct eigenvalue $\lambda_0 \in F$, we can express each $d_i(\lambda)$ in a factored form as $d_i(\lambda) = (\lambda - \lambda_0)^{\alpha_i}p_i(\lambda)$ with $p_i(\lambda_0) \neq 0$, where $\alpha_i \geq 0$ is called the $i$-th partial multiplicity of $\lambda_0$. If $\alpha_i > 0$ then $(\lambda - \lambda_0)^{\alpha_i}$ is called an elementary divisor at $\lambda_0$. Thus, to each $\lambda_0$, a set of increasingly ordered partial multiplicities $(\alpha_1, \ldots, \alpha_r)$ can be uniquely associated such that $0 \leq \alpha_1 \leq \cdots \leq \alpha_r$ jointly with a collection of elementary divisors $(\lambda - \lambda_0)^{\alpha_i}$ for $\alpha_i > 0$, including repetitions. The sum $\sum_{i=1}^r \alpha_i$ is the algebraic multiplicity of the eigenvalue $\lambda_0$, while the number of nonzero terms in this sum is its geometric multiplicity. An eigenvalue $\lambda_0$ is said to be simple, if its algebraic multiplicity is one.

The sum of all partial multiplicities gives the total number of all finite eigenvalues of $P(\lambda)$ and is denoted as $\delta_{\text{fin}}(P)$. This value can be alternatively defined using the degrees of the invariant polynomials as follows

$$\delta_{\text{fin}}(P) = \sum_{i=1}^r \deg d_i(\lambda).$$

For the definition of infinite eigenvalues of $P(\lambda)$ we use the mathematical framework introduced in [6], which we call the GLR framework (using the initials of authors’ names). For $j \geq d$, the $j$-reversal of $P(\lambda)$ is the matrix polynomial $\text{rev}_j P(\lambda) := \lambda^j P(1/\lambda)$. If $j = d$, the $d$-reversal is called simply the reversal of $P(\lambda)$ and denoted $\text{rev} P(\lambda)$. The GLR framework defines, for a grade $k$ polynomial matrix of degree $d$, $\lambda_0 = \infty$ an infinite eigenvalue of $P(\lambda)$ if and only if 0 is an eigenvalue of $\text{rev}_k P(\lambda)$. Using the Smith form of $\text{rev}_k P(\lambda)$, we can define the increasingly ordered partial multiplicities of an infinite eigenvalue as $(\alpha_1^\infty, \ldots, \alpha_r^\infty)$ with $0 \leq \alpha_1^\infty \leq \cdots \leq \alpha_r^\infty$. For each $\alpha_i^\infty > 0$ there exists an infinite elementary divisor of degree $\alpha_i^\infty$ (or and infinite eigenvalue of multiplicity $\alpha^\infty$). The number of infinite eigenvalues of $P(\lambda)$ is given by

$$\delta^\infty(P) = \sum_{i=1}^r \alpha_i^\infty.$$
If $P(\lambda)$ is regarded as a rational matrix, an alternative framework, called the McMillan framework, is widely used in control system theory to characterize the pole-zero structure of $P(\lambda)$ [7], [12]. In a broad sense, a complex value $\lambda_0$ is a pole of $P(\lambda)$ if at least one entry of $P(\lambda_0)$ is infinite, while $\lambda_0$ is a zero if $P(\lambda_0)$ has rank less than $r$ (its normal rank). This interpretation of poles and zeros leads to conceptual difficulties if $\lambda_0$ is both a pole and zero or if $\lambda_0 = \infty$ and therefore we give precise definitions based on the so-called local Smith-McMillan form (see, for example, [7]).

**Theorem 2.3** (Local Smith-McMillan form at $\lambda_0$). Let $P(\lambda)$ be an $m \times n$ rational matrix of rank $r$ with coefficients in $F$ and $\lambda_0$ any finite value in $F$. Then, there exist rational matrices $U_0(\lambda) \in F(\lambda)^{m \times m}$ and $V_0(\lambda) \in F(\lambda)^{n \times n}$, both regular at $\lambda_0$, such that

$$
D_0(\lambda) := U_0(\lambda)P(\lambda)V_0(\lambda) = 
\begin{bmatrix}
(\lambda - \lambda_0)^{\sigma_1} & \cdots & 0_{m-r,r} \\
\vdots & \ddots & \vdots \\
0_{m-r,r} & \cdots & (\lambda - \lambda_0)^{\sigma_r} \\
0_{m-r,n-r} & \cdots & 0_{m-r,n-r}
\end{bmatrix},
$$

where $\sigma_1 \leq \cdots \leq \sigma_r$. Moreover, $D_0(\lambda)$ is unique and is called the local Smith-McMillan form of $P(\lambda)$ at $\lambda_0$.

The values $\sigma_i$, $i = 1, \ldots, r$, are called the finite structural indices at $\lambda_0$ and have the following interpretation. A value $\sigma_i < 0$ defines a finite pole of $P(\lambda)$ at $\lambda_0$ of multiplicity $-\sigma_i$, while a value $\sigma_i > 0$ defines a finite zero of multiplicity $\sigma_i$ of $P(\lambda)$ at $\lambda_0$. $\lambda_0$ is neither pole nor zero if all structural indices are zero. We denote $\delta_{fin}^\sigma(P)$ the number of all finite zeros with their multiplicities, which is the sum of all positive structural indices for $\lambda_0 \in F$ and denote $\delta_{fin}^\sigma(P)$ the number of all finite poles, which is the absolute value of the sum of all negative structural indices for $\lambda_0 \in F$.

For a polynomial matrix $P(\lambda)$ all structural indices are non-negative, and therefore $P(\lambda)$ has no finite poles. It follows that $\delta_{fin}^\sigma(P) = 0$. The following straightforward result states that the finite structural indices of a polynomial matrix $P(\lambda)$ are basically the same as the partial multiplicities of its finite eigenvalues.

**Lemma 2.4.** Let $P(\lambda)$ by a polynomial matrix of rank $r$ and let $\lambda_0$ be a finite eigenvalue of $P(\lambda)$ with $(\alpha_1, \ldots, \alpha_r)$, the associated set of increasingly ordered partial multiplicities. Also, let $(\sigma_1, \ldots, \sigma_r)$ be the set of increasingly ordered structural indices of $P(\lambda)$ at $\lambda_0$. Then, $\alpha_i = \sigma_i$ for $i = 1, \ldots, r$.

A similar result holds for the infinite poles and zeros.

**Theorem 2.5** (Local Smith-McMillan form at $\infty$). Let $P(\lambda)$ be an $m \times n$ rational matrix of rank $r$ with coefficients in $F$. Then, there exist rational matrices $U_\infty(\lambda) \in F(\lambda)^{m \times m}$ and $V_\infty(\lambda) \in F(\lambda)^{n \times n}$, both regular at $\infty$, such that

$$
D_\infty(\lambda) := U_\infty(\lambda)P(\lambda)V_\infty(\lambda) = 
\begin{bmatrix}
(1/\lambda)^{\sigma_1} & \cdots & 0_{r,n-r} \\
\vdots & \ddots & \vdots \\
0_{m-r,r} & \cdots & 0_{m-r,n-r}
\end{bmatrix},
$$

where $\sigma_1 \leq \cdots \leq \sigma_r$. Moreover, $D_\infty(\lambda)$ is unique and is called the local Smith-McMillan form of $P(\lambda)$ at $\infty$.

The values $\sigma_i$, $i = 1, \ldots, r$, are called the infinite structural indices and have a similar interpretation as before. A value $\sigma_i^\infty < 0$ defines an infinite pole of $P(\lambda)$ of multiplicity $-\sigma_i^\infty$, while a value $\sigma_i^\infty > 0$ defines an infinite zero of $P(\lambda)$ of multiplicity $\sigma_i^\infty$. $P(\lambda)$ has neither infinite poles nor infinite zeros if all infinite structural indices are zero. We denote $\delta_{inf}^\sigma(P)$ the number of all infinite zeros with their multiplicities, which is the sum of all positive infinite structural indices, and denote $\delta_{inf}^\sigma(P)$ the number of all infinite poles, which is the absolute value of the sum of all negative infinite structural indices.
For a polynomial matrix \( P(\lambda) \) all its poles are infinite, while its zeros may be both finite and infinite. The McMillan framework interprets infinite zeros as “infinite frequencies” (e.g., as may occur in passive electrical networks), and therefore attaches a physically meaningful interpretation to infinite zeros. The following result shows that the relation between the infinite eigenvalues structure in the GLR framework and infinite zero structure in the McMillan framework can be expressed in term of a simple shift of multiplicities (see [1]).

**Lemma 2.6.** Let \( P(\lambda) \) by a polynomial matrix of rank \( r \), grade \( k \) and let \( (\alpha_1^\infty, \ldots, \alpha_r^\infty) \) be the set of increasingly ordered partial multiplicities associated to the finite eigenvalues of \( P(\lambda) \). Also, let \( (\sigma_1^\infty, \ldots, \sigma_r^\infty) \) be the set of increasingly ordered structural indices of \( P(\lambda) \) at \( \infty \). Then, \( \sigma_i^\infty = \alpha_i^\infty - k \) for \( i = 1, \ldots, r \).

If we know the partial multiplicities of the infinite eigenvalues of \( P(\lambda) \), then we can simply determine the multiplicities of the infinite zeros from the positive structural indices \( \sigma_j^\infty := \alpha_j^\infty - k, j = r-u+1, \ldots, r \), where \( u \) is the number of partial multiplicities \( \alpha_i^\infty \) which satisfy \( \alpha_i^\infty > k \). In a similar way, we can determine the multiplicities of the infinite poles from the negative structural indices \( \sigma_j^\infty := \alpha_j^\infty - k, j = 1, \ldots, l \), where \( l \) is the number of partial multiplicities \( \alpha_i^\infty \) which satisfy \( \alpha_i^\infty < k \). Conversely, if we know the \( \sigma_j^\infty \), \( j = 1, \ldots, l \) and \( \alpha_i^\infty \), \( i = r-u+1, \ldots, r \), then for a grade \( k \) polynomial matrix \( P(\lambda) \), the partial multiplicities of infinite eigenvalues can be reconstructed as

\[
(\sigma_{1}^{\infty} + k, \ldots, \sigma_{l}^{\infty} + k, k, \ldots, k, \sigma_{r-u+1}^{\infty} + k, \ldots, \sigma_{r}^{\infty} + k),
\]

where there are \( r-u-l \) partial multiplicities equal to \( k \). It must be noted that a consequence of Lemma 2.2 is, that, while the partial multiplicities of infinite eigenvalues depends on the chosen grade \( k \) of the polynomial matrix \( P(\lambda) \), the multiplicities of zeros and poles are independent of the choice of \( k \). In particular, for a degree \( d \) polynomial matrix, we always have \( \sigma_{i}^{\infty} = -d \).

The number of finite and infinite poles \( \delta^P(P) := \delta_{fin}^P(P) + \delta_{\infty}^P(P) \) is called the McMillan degree of \( P(\lambda) \) [12] (also called the polar degree). Analogously, the number of finite and infinite zeros is \( \delta^Z(P) := \delta_{fin}^Z(P) + \delta_{\infty}^Z(P) \) (also called the zero degree).

**Remark.** Following the results of Verghese [19], the pole structure of \( P(\lambda) \) is equivalent to the zero structure of the regular polynomial matrix

\[
\tilde{P}(\lambda) := \begin{bmatrix} P(\lambda) & I_p \\ I_m & 0 \end{bmatrix}.
\]

Thus, we can convert the pole structure determination problem into a zero structure determination problem, which in turn can be solved as an eigenvalue computation problem. \( \square \)

To characterize the singular structure of a polynomial matrix \( P(\lambda) \), the relevant objects are the **right nullspace** and **left nullspace** of \( P(\lambda) \). For this, we regard \( P(\lambda) \) as an \( m \times n \) rational matrix of normal rank \( r \) \(< \min(m,n) \) and consider the sets of left and right annihilators

\[
\mathcal{N}_l(P) := \{ \nu(\lambda) \in \mathbb{F}(\lambda)^{1 \times m} \mid \nu(\lambda) P(\lambda) = 0 \},
\]

\[
\mathcal{N}_r(P) := \{ \nu(\lambda) \in \mathbb{F}(\lambda)^{n \times 1} \mid P(\lambda) \nu(\lambda) = 0 \}.
\]

\( \mathcal{N}_l(P) \) is a linear space of dimension \( m-r \) called the **left nullspace** of \( P(\lambda) \) and \( \mathcal{N}_r(P) \) is a linear space of dimension \( n-r \) called the **right nullspace** of \( P(\lambda) \). It is always possible to choose polynomial bases \( \{ p_1(\lambda), \ldots, p_{m-r}(\lambda) \} \) and \( \{ q_1(\lambda), \ldots, q_{n-r}(\lambda) \} \) for \( \mathcal{N}_l(P) \) and \( \mathcal{N}_r(P) \), respectively. The degree of a polynomial basis is the sum of degrees of the basis polynomial vectors. A **minimal polynomial basis** is one which has the least possible degree. For a minimal polynomial basis \( \{ p_1(\lambda), \ldots, p_{m-r}(\lambda) \} \) of the left nullspace \( \mathcal{N}_l(P) \) the degrees \( \{ q_1(\lambda), \ldots, q_{n-r}(\lambda) \} \) of the polynomial vectors are called the **left minimal indices** (also known as **left Kronecker indices**), while for a minimal polynomial basis \( \{ q_1(\lambda), \ldots, q_{n-r}(\lambda) \} \) of the right nullspace \( \mathcal{N}_r(P) \) the degrees \( \{ e_1, \ldots, e_{n-r} \} \) of the polynomial vectors are called the **right minimal indices** (also known as **right Kronecker indices**). The left and right minimal indices are unique up to permutations and fully characterize the singular structure of a polynomial matrix. The above results have been established in [5] (see also [7] for a textbook presentation).
The degree of the minimal polynomial basis of $\mathcal{N}(P)$ is $\mu(P) := \sum_{i=1}^{n-r} \eta_i$ and, similarly, the degree of the minimal polynomial basis of $\mathcal{N}_r(P)$ is $\mu_r(P) := \sum_{i=1}^{n-r} \epsilon_i$. The sum of all the minimal indices of a given $P(\lambda)$ is
\[
\mu(P) := \mu_r(P) + \mu_r(P).
\]
If $P(\lambda)$ is regular (i.e., square and $\det(P(\lambda)) \neq 0$), then $\mu(P) := 0$. However, $\mu(P) := 0$ may generally occur for a singular polynomial matrix (e.g., $P(\lambda) = A$ for $A$ singular).

There are several fundamental relationships between various structural elements of polynomial matrices. The following result relates the finite and infinite eigenvalues of a regular pencil and is established in [13, Lemma 6.1].

**Lemma 2.7.** Let $P(\lambda)$ be a regular $n \times n$ polynomial matrix of grade $k$, over an arbitrary field. Then
\[
\delta_{fin}(P) + \delta_{\infty}(P) = kn.
\]
The above result is a corollary of the following more general relation involving the infinite eigenvalues, zeros and poles.

**Lemma 2.8.** Let $P(\lambda)$ be an $m \times n$ polynomial matrix of grade $k$, rank $r$, over an arbitrary field. Then
\[
\delta_{fin}(P) + \delta_{\infty}(P) = kr + \delta^*_{\infty}(P) - \delta^p(P).
\]
For the proof of this result we can apply the results of **Lemma 2.4** and **Lemma 2.6**, observing that
\[
\delta_{\infty}(P) = kr + \delta^*_{\infty}(P) - \delta^p_{\infty}(P)
\]
and taking into account that $\delta^p(P) = \delta^p_{\infty}(P)$.

The following result of [21, Theorem 3] relates the number of poles, number of zeros and the singular structure.

**Lemma 2.9.** Let $P(\lambda)$ be an $m \times n$ polynomial matrix over an arbitrary field. Then
\[
\delta^p(P) = \delta^*(P) + \mu(P).
\]
The following result, called in [13] the **Index Sum Theorem**, relates the eigenvalue and singular structures of polynomial matrices.

**Lemma 2.10.** Let $P(\lambda)$ be an $m \times n$ polynomial matrix of grade $k$, rank $r$, over an arbitrary field. Then
\[
\delta_{fin}(P) + \delta_{\infty}(P) + \mu(P) = kr.
\]
This result is Theorem 6.5 in [13] and its proof is given in terms of companion form linearizations of the polynomial matrix $P(\lambda)$. An alternative, much simpler proof is possible by combining the results of **Lemma 2.8** and **Lemma 2.9**.

The handling of the particular case of a constant polynomial matrix $P(\lambda) := P_0$ depends on the choice of grade $k$. For $k = 0$, the polynomial matrix $P(\lambda)$ of rank $r = \text{rank } P_0$ satisfies $P(\lambda) = \text{rev}_0 P(\lambda)$ and therefore both $P(\lambda)$ and $\text{rev}_0 P(\lambda)$ have the trivial Smith-form $\text{diag}(I_r, 0)$. It follows, that $P(\lambda)$ has no finite and infinite eigenvalues, and has $m - r$ right Kronecker indices equal to 0 and $n - r$ left Kronecker indices equal to 0 (both sets may be empty). Regarded as a grade $k \geq 1$ polynomial matrix $P(\lambda) = P_0 + \lambda \eta + \cdots + \lambda^n \eta_0$, $P(\lambda)$ has no finite eigenvalues, but has $kr$ infinite eigenvalues with partial multiplicities $(k, k, \ldots, k)$, and the same left and right Kronecker indices as above.

**3. Matrix pencils.** A matrix pencil $M - \lambda N$ is a grade one polynomial matrix, whose structural properties can be numerically investigated using numerically reliable pencil manipulation algorithms. This allows to determine the structural properties of polynomial matrices via linearization techniques.
In what follows, we assume \( F \) is an algebraically closed field (e.g., \( F = \mathbb{C} \)). The basic mathematical tool for matrix pencils is the Kronecker canonical form (KCF) obtained using strict equivalence transformations, which exhibits both the eigenvalue structure as well as the singular structure of the pencil. Recall that two pencils \( M - \lambda N \) and \( \tilde{M} - \lambda \tilde{N} \) with \( M, N, \tilde{M}, \tilde{N} \in F^{m \times n} \) are strictly equivalent if there exist two invertible matrices \( U \in F^{m \times m} \) and \( V \in F^{n \times n} \) such that

\[
U(M - \lambda N)V = \tilde{M} - \lambda \tilde{N}.
\]

(12)

For a general (singular) pencil, the strict equivalence leads to the KCF.

**Lemma 3.1.** Let \( M - \lambda N \) be an arbitrary pencil with \( M, N \in F^{m \times n} \) and \( F \) is an algebraically closed field. Then, there exist invertible matrices \( U \in F^{m \times m} \) and \( V \in F^{n \times n} \) such that

\[
U(M - \lambda N)V = \begin{bmatrix}
K_r(\lambda) & K_{\text{reg}}(\lambda) \\
K_{\text{reg}}(\lambda) & K_l(\lambda)
\end{bmatrix},
\]

(13)

where:

1) The full row rank pencil \( K_r(\lambda) \) has the form

\[
K_r(\lambda) = \text{diag} \left( L_{e_1}(\lambda), L_{e_2}(\lambda), \ldots, L_{e_{\nu_r}}(\lambda) \right),
\]

with \( L_i(\lambda) \) \((i \geq 0)\) an \( i \times (i+1) \) bidiagonal pencil of form

\[
L_i(\lambda) = \begin{bmatrix}
-\lambda & 1 & & \\
& \ddots & \ddots & \\
& & -\lambda & 1
\end{bmatrix};
\]

(14)

2) The regular pencil \( K_{\text{reg}}(\lambda) \) is in the Weierstrass canonical form

\[
K_{\text{reg}}(\lambda) = \begin{bmatrix}
J_f - \lambda I & \\
I & \lambda J_{\infty}
\end{bmatrix},
\]

(15)

where \( J_f \) is in the Jordan canonical form

\[
J_f = \text{diag} \left( J_{s_1}(\lambda_1), J_{s_2}(\lambda_2), \ldots, J_{s_k}(\lambda_k) \right),
\]

(16)

with \( J_{s_i}(\lambda_i) \) an elementary \( s_i \times s_i \) Jordan block of the form

\[
J_{s_i}(\lambda_i) = \begin{bmatrix}
\lambda_i & 1 & & \\
& \lambda_i & \ddots & \\
& & \ddots & 1 \\
& & & \lambda_i
\end{bmatrix}
\]

and \( J_{\infty} \) is nilpotent and has the (nilpotent) Jordan form

\[
J_{\infty} = \text{diag} \left( J_{s_1}^{(0)}(\lambda_1), J_{s_2}^{(0)}(\lambda_2), \ldots, J_{s_k}^{(0)}(\lambda_k) \right);
\]

(17)

3) The full column rank \( K_l(\lambda) \) has the form

\[
K_l(\lambda) = \text{diag} \left( L^T_{\eta_1}(\lambda), L^T_{\eta_2}(\lambda), \ldots, L^T_{\eta_{\nu_l}}(\lambda) \right).
\]

The Kronecker canonical form (13) exhibits the right and left singular structures of the pencil \( M - \lambda N \) via the full row rank block \( K_r(\lambda) \) and full column rank block \( K_l(\lambda) \), respectively, and the eigenvalue structure via the regular pencil \( K_{\text{reg}}(\lambda) \).
The full row rank pencil $K_r(\lambda)$ is $n_r \times (n_r + \nu_r)$, where $n_r = \sum_{i=1}^{\nu_r} \epsilon_i$, the full column rank pencil $K_f(\lambda)$ is $(n_f + \nu_f) \times n_f$, where $n_f = \sum_{j=1}^{\nu_f} \eta_j$, while the regular pencil $K_{reg}(\lambda)$ is $n_{reg} \times n_{reg}$, with $n_{reg} = n_f + n_\infty$, where $n_f$ is the number of finite eigenvalues of $J_f - \lambda I$ and $n_\infty$ is the number of infinite eigenvalues of $I - \lambda J_\infty$. The $\epsilon_i \times (\epsilon_i + 1)$ blocks $L_\epsilon(\lambda)$ with $\epsilon_i \geq 0$ are the right elementary Kronecker blocks, and $\epsilon_i$, for $i = 1, \ldots, \nu_r$, are called the right Kronecker indices. The $(\eta_j + 1) \times \eta_j$ blocks $L_{\eta}(\lambda)$ with $\eta_j \geq 0$ are the left elementary Kronecker blocks, and $\eta_j$, for $j = 1, \ldots, \nu_f$, are called the left Kronecker indices.

The Weierstrass canonical form (15) exhibits the finite and infinite eigenvalues of the pencil $M - \lambda N$. Each $s_i \times s_i$ Jordan block $J_{s_i}(\lambda_i)$ corresponds to a finite elementary divisor $(\lambda - \lambda_i)^{s_i}$ and, by including all multiplicities, there are $n_f = \sum_{i=1}^{\nu_f} s_i$ finite eigenvalues. Each $s_i^\infty \times s_i^\infty$ nilpotent Jordan block $J_{s_i^\infty}(0)$ corresponds to an infinite elementary divisor of order $s_i^\infty$ and there are $n_\infty = \sum_{i=1}^{\nu_f} s_i^\infty$ infinite eigenvalues. Infinite eigenvalues with $s_i^\infty = 1$ are called simple infinite eigenvalues. If $M - \lambda N$ is regular, then there are no left- and right-Kronecker structures and the Kronecker canonical form is simply the Weierstrass canonical form.

The normal rank $r$ of the pencil $M - \lambda N$ results as

$$r := \text{rank}(M - \lambda N) = n_r + n_f + n_\infty + n_1.$$

We can also express the rank $\ell$ of $N$ as

$$\ell := \text{rank} N = n_r + n_f + \text{rank} J_\infty + n_1 = n_r + n_f + \sum_{i=1}^{\nu_f} (s_i^\infty - 1) + n_1 = r - h.$$

Assuming $s_1^\infty \leq s_2^\infty \leq \cdots \leq s_{\nu_f}^\infty$, then the $r$ partial multiplicities of the infinite eigenvalues are

$$\left(\alpha_1^\infty, \alpha_2^\infty, \ldots, \alpha_r^\infty\right) = (0, \ldots, 0, s_1^\infty, \ldots, s_{\nu_f}^\infty),$$

(18)

where the first $\ell = r - h$ partial multiplicities are equal to zero.

The pole-zero structure at $\infty$ of the pencil $M - \lambda N$ can be retrieved from the KCF using the result of [16, Theorem 2].

**Lemma 3.2.** Let $M - \lambda N$ be an $m \times n$ linear matrix pencil of normal rank $r$ and let $\ell = \text{rank} N$. Then, assuming $0 < s_1^\infty \leq s_2^\infty \leq \cdots \leq s_\nu_f^\infty$ are the ordered sizes of the nilpotent Jordan blocks of $J_\infty$, then the structural indices at $\infty$ of the pencil $M - \lambda N$ are determined by the KCF (13) as follows:

$$\left(\sigma_1^\infty, \sigma_2^\infty, \ldots, \sigma_r^\infty\right) = (-1, \ldots, -1, s_1^\infty - 1, \ldots, s_{\nu_f}^\infty - 1),$$

where there are $\ell$ structural indices equal to $-1$.

It follows that $M - \lambda N$ has $\ell$ poles at $\infty$, all of multiplicities equal to one, while the number of infinite zeros is $\sum_{i=1}^{\nu_f} (s_i^\infty - 1) = n_\infty - h$.

The computation of the Kronecker-canonical form may involve the use of ill-conditioned transformations and, therefore, is potentially numerically unstable. Fortunately, alternative so-called Kronecker-like forms (KLFs), allow to obtain basically the same (or only a part of) structural information on the pencil $M - \lambda N$ by employing exclusively unitary transformations if $F = \mathbb{C}$ (i.e., $U^* U = I$ and $V^* V = I$) or orthogonal transformations if $F = \mathbb{R}$ (i.e., $U^T U = I$ and $V^T V = I$).

An arbitrary pencil $M - \lambda N$ can be reduced using orthogonal or unitary transformations $U$ and $V$ to the block-upper triangular form [14]

$$U(M - \lambda N)V = \begin{bmatrix}
M_r - \lambda N_r & * & * & * \\
0 & M_\infty - \lambda N_\infty & * & * \\
0 & 0 & M_f - \lambda N_f & * \\
0 & 0 & 0 & M_l - \lambda N_l
\end{bmatrix},$$

(19)

where
1) $M_r - \lambda N_r$ has full row rank for all $\lambda \in F$, has only a right nullspace, and contains information on the right Kronecker indices;

2) $M_\infty - \lambda N_\infty$ is regular and contains information on the infinite elementary divisors (i.e., the multiplicities of infinite eigenvalues);

3) $M_f - \lambda N_f$ is regular with $N_f$ invertible and contains the finite elementary divisors (i.e., the finite eigenvalues);

4) $M_l - \lambda N_l$ has full column rank for all $\lambda \in F$, has only a left nullspace, and contains information on the left Kronecker indices.

The KLF (19) can be obtained using numerically stable pencil reduction algorithms as proposed in [14, 2, 3, 11], which at the same time determine the left and right Kronecker indices and the infinite elementary divisors of $M - \lambda N$ from the fine block structure of subpencils $M_r - \lambda N_r$, $M_\infty - \lambda N_\infty$, and $M_l - \lambda N_l$. The finite eigenvalues can be computed using the QZ algorithm to compute the generalized eigenvalues of the pair $(M_f, N_f)$ [10].

Remark. The KLF (19) separates the finite and infinite eigenvalues of $M - \lambda N$ as the eigenvalues of the regular subpencils $M_f - \lambda N_f$ and $M_\infty - \lambda N_\infty$, respectively, provides the information on the multiplicities of infinite eigenvalues (i.e., on the infinite elementary divisors of $M_\infty - \lambda N_\infty$), but does not provide further information on the multiplicities of the finite eigenvalues (i.e., on the finite elementary divisors of $M_f - \lambda N_f$). For the determination of the partial multiplicities associated to a known finite eigenvalue $\lambda_0$ (e.g., computed using the QZ algorithm), the following approach, suggested in [14], can be employed. The pencil reduction algorithm is applied to the shifted pencil $N_f - \tilde{\lambda}(M_f - \lambda_0 N_f)$ to determine its infinite elementary divisors. This corresponds to a transformation of the indeterminate as $\lambda = 1/(\tilde{\lambda} - \lambda_0)$, which maps all finite eigenvalues at $\lambda_0$ of $M_f - \lambda N_f$ into infinite eigenvalues of $N_f - \tilde{\lambda}(M_f - \lambda_0 N_f)$ for which the pencil reduction algorithm determines the partial multiplicities.

The algorithms for the computation of Kronecker-like forms of linear pencils perform repeatedly column and row compressions of matrices using orthonal or unitary transformations. These operations involve rank determinations, for which rank revealing decompositions as the QR-decomposition with column pivoting or the more reliable (but also computationally more involved) singular value decomposition (SVD) can be used. The use of SVD-based rank determinations is the basis of the algorithms proposed in [14, 3]. Albeit numerically reliable, these algorithms have a computational complexity $O(n^4)$, where $n$ is the minimum of row or column dimensions of the pencil. More efficient algorithms of complexity $O(n^3)$ have been proposed in [2, 11], which rely on using QR decompositions with column pivoting for rank determinations. An enhanced version of algorithm of [11] can be devised by combining QR-decompositions (without column pivoting) and SVD-based rank determinations. Both compression techniques have been employed in the implementations of the basic tools to compute various KLFs in the MatrixPencils package along the lines of procedures described in [17; see Procedure PRE-duce, Section 10.1.6]. Functions are also available for several applications of Kronecker-like forms as the computation of Kronecker indices, finite and infinite eigenvalues and zeros, normal rank. These functions served as building blocks for the implemented software for handling polynomial matrices.

4. Linearizations. The standard way to address eigenvalue and structural analysis problems of matrix polynomials is via a linearization, which replaces a given polynomial matrix $P(\lambda)$ by a matrix pencil $L(\lambda) = M - \lambda N$, which (ideally) preserves the eigenvalue and singular structures of $P(\lambda)$. The structural analysis problems for $L(\lambda)$ are then solved using pencil reduction techniques in conjunction with the QZ-algorithm, as described in the previous section. Depending on the employed linearization, the structural properties of $P(\lambda)$ are retrieved from those of $L(\lambda)$.

Assume $P(\lambda)$ is a $p \times m$ polynomial matrix of grade $k$. A pencil $L(\lambda)$ is called a linearization of $P(\lambda)$ if there exist unimodular matrices $U(\lambda)$ and $V(\lambda)$ and $s \geq 0$ such that

$$U(\lambda)L(\lambda)V(\lambda) = \text{diag}\{P(\lambda), I_s\}.$$  

Thus, a linearization $L(\lambda)$ preserves the finite elementary divisors and thus the finite eigenvalues of $P(\lambda)$. It also preserves the dimensions of the right and left nullspaces of $P(\lambda)$. If in addition,
rev. \( L(\lambda) \) is a linearization of rev. \( P(\lambda) \), then \( L(\lambda) \) is said to be a strong linearization of \( P(\lambda) \). For a strong linearization the infinite elementary divisors are also preserved. Therefore, the key property of a strong linearization is that \( L(\lambda) \) and \( P(\lambda) \) have the same finite and infinite elementary divisors. However, for a singular \( P(\lambda) \) other structural features are also desirable to be preserved by \( L(\lambda) \).

Among many existing strong linearizations, the Frobenius companion form linearizations are widely used in solving eigenvalue problems of polynomial matrices. A main appeal of these linearization is that they can be directly constructed from the coefficient matrices of \( P(\lambda) \). Besides the preservation of finite and infinite eigenvalue structures, these linearizations allow to easily retrieve information on the minimal indices. A potential drawback of these linearizations is that they usually do not reflect any structural feature which may be present in \( P(\lambda) \) (e.g., symmetry).

A second category of linearizations is suitable for the investigation of pole-zero and singular structures using the McMillan framework. These linearizations are built as least order system matrix pencils \( L(\lambda) \), corresponding to particular type of realizations of the original polynomial matrix \( P(\lambda) \). The notion of strong linearization can be extended to this framework, by requiring the preservation of the complete pole-zero structure and singular structure of \( P(\lambda) \). Two linearizations in this category are the pencil based linearization and the descriptor system based linearization. Important computational ingredients to determine these linearizations are minimal realization algorithms specific to each type of realization.

4.1. Companion forms based linearizations. These linearizations are widely used in the numerical linear algebra community, where the GLR framework is mostly employed. Assume the \( p \times m \) polynomial matrix \( P(\lambda) \) is given as a grade \( k \) matrix polynomial of the form

\[
P(\lambda) = P_0 + P_1\lambda + \ldots + P_k\lambda^k.
\]

The degree \( d \) of \( P(\lambda) \) is the maximum value of \( i = 0, 1, \ldots, k \) for which \( P_i \neq 0 \) and \( d \leq k \).

The first Frobenius companion form linearization of \( P(\lambda) \) is the linear pencil \( C_1(\lambda) := M_1 - \lambda N_1 \), with

\[
M_1 = \begin{bmatrix} -P_{k-1} & -P_{k-2} & \cdots & -P_0 \\ I_m & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & I_m & 0 \end{bmatrix}, \quad N_1 = \begin{bmatrix} P_k \\ I_m \\ \vdots \\ I_m \end{bmatrix},
\]

where \( M_1 \) and \( N_1 \) are \((p + (k-1)m) \times km\) matrices. If \( P(\lambda) \) is regular then \( C_1(\lambda) \) is regular as well. This linearization can be employed to recover the eigenvalue structure, zero structure, and singular structures of \( P(\lambda) \) from the Kronecker structure of \( C_1(\lambda) \) using the following results [13]:

**Proposition 4.1.** Let \( P(\lambda) = \sum_{i=0}^{k} P_i \lambda^i \) be a \( p \times m \) matrix polynomial with grade \( k \geq 2 \), and let \( C_1(\lambda) \) be its first Frobenius companion form linearization. Then:

(a) the finite and infinite elementary divisors of \( P(\lambda) \) and \( C_1(\lambda) \) are the same, thus \( \delta_{\text{fin}}(P) = \delta_{\text{fin}}(C_1) \) and \( \delta_{\text{inf}}(P) = \delta_{\text{inf}}(C_1) \);

(b) if \( r = \text{rank} \ P(\lambda) \) and \( r_1 = \text{rank} \ C_1(\lambda) \), then the structural indices of \( P(\lambda) \) at \( \infty \) \((\sigma_{\infty}^{1}, \ldots, \sigma_{\infty}^{r})\) and the partial multiplicities \((\tilde{\alpha}_{\infty}^{1}, \ldots, \tilde{\alpha}_{\infty}^{r})\) of the infinite eigenvalues of \( C_1(\lambda) \) are related as

\[
\tilde{\alpha}_{\infty}^{i} = 0, \quad i = 1, \ldots, r_1 - r, \quad \tilde{\alpha}_{\infty}^{i} = \sigma_{\infty}^{i} + k, \quad i = 1, \ldots, r;
\]

(c) the right minimal indices \((\epsilon_{1}, \ldots, \epsilon_{r})\) of \( P(\lambda) \) and the right minimal indices \((\tilde{\epsilon}_{1}, \ldots, \tilde{\epsilon}_{r})\) of \( C_1(\lambda) \) are related as

\[
\epsilon_{i} = \tilde{\epsilon}_{i} - (k - 1), \quad i = 1, \ldots, r;
\]

(d) the left minimal indices of \( P(\lambda) \) and \( C_1(\lambda) \) are the same, and hence

\[
\mu(P) = \mu(C_1) - (k - 1)\nu;
\]
ON COMPUTING THE KRONECKER STRUCTURE OF POLYNOMIAL MATRICES USING JULIA

\[ P \text{ matrix } S \]

In the control system literature, the matrix pencil

\[ -P \text{ polynomial matrix} \]

(22)

\[ A - \lambda E \]

\[ B - \lambda F \]

\[ C - \lambda G \]

\[ D - \lambda H \]

where \( P \) is the linear pencil \( C_2(\lambda) := M_2 - \lambda N_2 \), with

\[
M_2 = \begin{bmatrix}
-\lambda_{k-1} & I_p & 0 \\
-\lambda_{k-2} & 0 & \cdots \\
\vdots & \vdots & \ddots & I_p \\
-\lambda_0 & 0 & \cdots & 0
\end{bmatrix}, \quad N_2 = \begin{bmatrix}
\lambda P_k & I_p \\
I_p & \cdots & I_p
\end{bmatrix},
\]

where \( M_2 \) and \( N_2 \) are \( pk \times (m + (k - 1)p) \) matrices. This linearization can be employed to recover the eigenvalue structure, zero structure, and singular structures of \( P(\lambda) \) from the Kronecker structure of \( C_2(\lambda) \) using the following results [13]:

**Proposition 4.2.** Let \( P(\lambda) = \sum_{i=0}^{k} P_i \lambda^i \) be a \( p \times m \) matrix polynomial with grade \( k \geq 2 \), and let \( C_2(\lambda) \) be its second Frobenius companion form linearization. Then:

(a) the finite and infinite elementary divisors of \( P(\lambda) \) and \( C_2(\lambda) \) are the same, thus \( \delta_{\text{fin}}(P) = \delta_{\text{fin}}(C_2) \) and \( \delta_{\infty}(P) = \delta_{\infty}(C_2) \);

(b) if \( r = \text{rank}(P) \) and \( r_2 = \text{rank}(C_2) \), then the structural indices of \( P(\lambda) \) at \( \infty \) (\( \sigma^f_1, \ldots, \sigma^f_{r_2} \)) and the partial multiplicities (\( \tilde{\alpha}^\infty_1, \ldots, \tilde{\alpha}^\infty_{r_2} \)) of the infinite eigenvalues of \( C_2(\lambda) \) are related as

\[
\tilde{\alpha}^\infty_i = 0, \quad i = 1, \ldots, r_2 - r, \quad \tilde{\alpha}^\infty_{r_2+i} = \sigma^f_i + k, \quad i = 1, \ldots, r;
\]

(c) the right minimal indices of \( P(\lambda) \) and \( C_2(\lambda) \) are the same;

(d) the left minimal indices \( (\eta_1, \ldots, \eta_{\nu}) \) of \( P(\lambda) \) and the left minimal indices \( (\tilde{\eta}_1, \ldots, \tilde{\eta}_{\nu}) \) of \( C_2(\lambda) \) are related as

\[
\eta_i = \tilde{\eta}_i - (k - 1), \quad i = 1, \ldots, \nu;
\]

and hence

\[
\mu(P) = \mu(C_2) - (k - 1)\nu;
\]

(e) the normal ranks of \( P(\lambda) \) and \( C_2(\lambda) \) are related as

\[
\text{rank}(P) = \text{rank}(C_2) - p(k - 1).
\]

**4.2. Pencils based linearization.** For a rational matrix \( P(\lambda) \) (and therefore also for a polynomial matrix \( P(\lambda) \)) we can use a linearization of the form

\[
S(\lambda) = \begin{bmatrix}
A - \lambda E & B - \lambda F \\
C - \lambda G & D - \lambda H
\end{bmatrix},
\]

where \( A - \lambda E \) is an \( n \times n \) regular pencil and the quadruple of linear pencils \( (A - \lambda E, B - \lambda F, C - \lambda G, D - \lambda H) \) is a pencils based realization of \( P(\lambda) \) which satisfies

\[
P(\lambda) = (C - \lambda G)(\lambda E - A)^{-1}(B - \lambda F) + D - \lambda H.
\]

In the control system literature, the matrix pencil \( S(\lambda) \) is called the Rosenbrock’s system matrix [12] of the pencils based realization of \( P(\lambda) \).

Of particular interest are realizations which allow to retrieve the structural elements of \( P(\lambda) \) from those of \( S(\lambda) \). A realization \( (A - \lambda E, B - \lambda F, C - \lambda G, D - \lambda H) \) is called strongly irreducible [19] if it is strongly controllable and strongly observable, for which the equivalent conditions are that the pencils

\[
\begin{bmatrix}
A - \lambda E & B - \lambda F & 0 \\
C - \lambda G & D - \lambda H & I_p
\end{bmatrix}, \quad \begin{bmatrix}
A - \lambda E & B - \lambda F & 0 \\
C - \lambda G & D - \lambda H & I_p
\end{bmatrix}
\]

are strongly controllable and strongly observable.
have no finite and infinite zeros. These conditions are fulfilled if the pair \((A - \lambda E, B - \lambda F)\) is \(E\)-strongly controllable and the pair \((A - \lambda E, C - \lambda G)\) is \(E\)-strongly observable for which the equivalent conditions are that the pencils

\[
\begin{bmatrix}
A - \lambda E & B - \lambda F
\end{bmatrix}
\begin{bmatrix}
A - \lambda E \\
C - \lambda G
\end{bmatrix}
\]

have no finite and infinite eigenvalues \([4]\). In this case, the realization is called strongly minimal and \(n\) is the least achievable value such that (23) holds. This value is called in [12] the least order and denoted with \(\nu(P)\). The main importance of strongly minimal realizations is that they can be computed in a relatively simply way from a non-minimal realization (e.g., using a procedure proposed in [4]) using standard pencil manipulation algorithms.

A linear pencil based linearization can be easily derived (by inspection) for a grade \(k \geq 2\) polynomial matrix \(P(\lambda)\) in (3) as follows. A strongly-controllable realization of order \(n = m(k - 1)\) is given by

\[
(A - \lambda E, B - \lambda F, C - \lambda G, D - \lambda H) = \begin{bmatrix}
I_m & -\lambda I_m & 0 \\
I_m & -\lambda I_m & 0 \\
\vdots & \ddots & \vdots \\
\vdots & \ddots & \vdots \\
I_m & -\lambda I_m & 0 \\
0 & \cdots & -\lambda I_m \\
0 & \cdots & -\lambda I_m \\
0 & \cdots & -\lambda I_m \\
0 & \cdots & -\lambda I_m \\
0 & \cdots & -\lambda I_m \\
0 & \cdots & -\lambda I_m \\
P_k - \lambda P_k & P_{k-2} & \cdots & P_1 & P_0
\end{bmatrix}
\]

A strongly-observable realization of order \(n = p(k - 1)\) is given by

\[
(A - \lambda E, B - \lambda F, C - \lambda G, D - \lambda H) = \begin{bmatrix}
I_p & -\lambda I_p & 0 \\
I_p & -\lambda I_p & 0 \\
\vdots & \ddots & \vdots \\
\vdots & \ddots & \vdots \\
I_p & -\lambda I_p & 0 \\
0 & \cdots & -\lambda I_p \\
0 & \cdots & -\lambda I_p \\
0 & \cdots & -\lambda I_p \\
0 & \cdots & -\lambda I_p \\
0 & \cdots & -\lambda I_p \\
0 & \cdots & -\lambda I_p \\
0 & \cdots & -\lambda I_p \\
0 & \cdots & -\lambda I_p \\
0 & \cdots & -\lambda I_p \\
-P_{k-1} + \lambda P_k & P_{k-2} & \cdots & P_1 & P_0
\end{bmatrix}
\]

For \(k = 1\), a realization of order \(n = 0\) is given by \(D - \lambda H := P_0 + \lambda P_1\), while for a constant polynomial matrix (i.e., \(k = 0\)), we take \(D := P_0\) and \(H\) an empty matrix.

The following result has been stated in [19] for strongly irreducible realizations (and thus also valid for strongly minimal realizations):

**Proposition 4.3.** Let \(P(\lambda)\) be a \(p \times m\) rational matrix and let \((A - \lambda E, B - \lambda F, C - \lambda G, D - \lambda H)\) be a strongly irreducible linearization satisfying (23). Then:
(a) the finite and infinite zero structures and the singular Kronecker structures of \(P(\lambda)\) and \(S(\lambda)\) are the same;
(b) the finite and infinite pole structures of \(P(\lambda)\) and the finite and infinite zero structures of the pole pencil

\[
S_p(\lambda) := \begin{bmatrix}
A & B & 0 \\
C & D & I_p \\
0 & I_m & 0
\end{bmatrix} - \lambda \begin{bmatrix}
E & F & 0 \\
G & H & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

are the same.

**Remark.** For computational purposes, the reduced pole pencil

\[
\tilde{S}_p(\lambda) = \begin{bmatrix}
A & 0 & 0 \\
0 & 0 & I_p \\
0 & I_m & 0
\end{bmatrix} - \lambda \begin{bmatrix}
E & F & 0 \\
G & H & 0 \\
0 & 0 & 0
\end{bmatrix}
\]
can be employed instead of (27) to compute both the finite and infinite poles of $P(\lambda)$, as the zeros of $\tilde{S}_p(\lambda)$ using pencil manipulation techniques. The finite poles can alternatively be determined as the finite eigenvalues of $A - \lambda E$. If $P(\lambda)$ is a polynomial matrix, it has no finite poles. In this case, $E$ is nilpotent and $A - \lambda E$ is unimodular.

Using (11) it is straightforward to relate the number of infinite eigenvalues of a polynomial matrix $P(\lambda)$ and the system matrix $S(\lambda)$ as

$$\delta_{\infty}(P) - \delta_{\infty}(S) = dr - \text{rank}(S).$$

Note that for an $n$-th order realization, $\text{rank}(S) = r + n$ and we have

$$\delta_{\infty}(P) - \delta_{\infty}(S) = (d - 1)r - n. \tag{28}$$

It follows that, knowing $\delta_{\infty}(S)$, the number of infinite eigenvalues $\delta_{\infty}(P)$ can be recovered using (28). Alternatively, knowing the pole-zero structure (i.e., the multiplicities of infinite poles $\sigma_p^\infty, j = 1, \ldots, l$ and the multiplicities of infinite zeros $\sigma_r^\infty, j = r - u + 1, \ldots, r$), then for a grade $k$ polynomial matrix $P(\lambda)$, the partial multiplicities of infinite eigenvalues can be reconstructed from (7).

Building linearizations of the form (22) for a polynomial matrix $P(\lambda)$ based on a strongly irreducible realization $(A - \lambda E, B - \lambda F, C - \lambda G, D - \lambda H)$ usually involves two steps. First, build a strongly controllable realization as in (25), which however may not be strongly observable, because the pencil $[A - \lambda E, C - \lambda G]$ may have infinite eigenvalues. These infinite eigenvalues can be removed using the procedure proposed in [4]. A completely similar approach can be devised by starting at the first step with a strongly observable realization as in (26) and then removing the infinite eigenvalues of the pencil $[A - \lambda E, B - \lambda F]$ using the procedure of [4]. The decision on which of these approaches to be used can be guided by the goal to minimize the computational effort in the second step, by choosing the initial realization of lower order. Therefore, if $p > m$, the realization (25) of order $m(k - 1)$ is to be preferred, while if $p < m$ the realization (26) of order $p(k - 1)$ may be preferable.

### 4.3. Descriptor system based linearization

For a rational matrix $P(\lambda)$ (and therefore also for a polynomial matrix $P(\lambda)$) we can alternatively use a linearization with a system matrix of the form

$$S(\lambda) = \begin{bmatrix} A - \lambda E & B \\ C & D \end{bmatrix}, \tag{29}$$

where $(A - \lambda E, B, C, D)$ is called a *descriptor system realization* of $P(\lambda)$ and satisfies

$$P(\lambda) = C(\lambda E - A)^{-1}B + D. \tag{30}$$

The descriptor realization is called *irreducible* if it is *controllable* and *observable*, for which equivalent conditions are that the pencils

$$[A - \lambda E, B], \quad [A - \lambda E, C]$$

have no finite and infinite zeros [20]. If additionally the pencil $A - \lambda E$ has no first order infinite elementary divisors (also called *no non-dynamic modes*), then the descriptor realization is called *minimal* and $n$, the order of $A$, is the least achievable dimension.

Any descriptor realization of $P(\lambda)$ is a particular pencil realization, which is strongly irreducible if the descriptor realization is irreducible. Therefore, the results of Proposition 4.3 apply also to an irreducible descriptor system realization with the system matrix $S(\lambda)$ in (29).

An advantage of using descriptor system realizations for pole computations is that the finite and infinite poles of $P(\lambda)$ can be determined as the finite and infinite zeros of the reduced pole pencil

$$\tilde{S}_p(\lambda) := A - \lambda E. \tag{31}$$
Using (28), the number of infinite eigenvalues \( \delta_\infty(P) \) can be recovered from those of the system matrix \( S(\lambda) \) in (29), while the partial multiplicities of infinite eigenvalues of \( P(\lambda) \) can be retrieved from the infinite zero structures of \( S(\lambda) \) and \( \tilde{S}_p(\lambda) \) in (31).

For a polynomial matrix \( P(\lambda) \) it is always possible to build a strongly irreducible realization with the system matrix (29) with \( A = I \) and \( E \) nilpotent (and therefore \( A - \lambda E \) unimodular). Such a realization can be determined following the suggestions from [21] by building a (standard) minimal realization \( (E - \lambda I, B, C, P_0) \) of the strictly proper rational matrix \( \lambda^{-1}(P(\lambda^{-1}) - P_0) \) satisfying \( \lambda^{-1}(P(\lambda^{-1}) - P_0) = -C(\lambda I - E)^{-1}B \). For this purpose, minimal realization procedures as suggested in [15] can be used. Then, the realization of \( P(\lambda) \) is simply \((I - \lambda E, B, C, P_0)\).

Building linearizations of the form (29) for a polynomial matrix \( P(\lambda) \) based on an irreducible realization of the form \((I - \lambda E, B, C, P_0)\) can be done in two steps. First, we build a controllable realization of \( P(\lambda) \) in the form

\[
\begin{bmatrix}
I - \lambda E & B \\
C & D
\end{bmatrix}
\begin{bmatrix}
I_m & -\lambda I_m & 0 \\
I_m & -\lambda I_m & 0 \\
\vdots & \vdots & \vdots \\
I_m & -\lambda I_m & 0 \\
I_m & -\lambda I_m & 0 \\
P_k & P_{k-1} & \cdots & P_1 & 0 & P_0
\end{bmatrix}
\]

of order \( m(k + 1) \), which however may not be observable at infinity, because the pair \([I - \lambda E]\) may have infinite (decoupling) zeros, or equivalently, the standard pair \((E, C)\) may have unobservable null eigenvalues. These unobservable eigenvalues can be removed by reducing the pair \((E, C)\) to the observability staircase form [15] from which an observable realization can be obtained. A suitable algorithm for this purpose is described, for example, in [15].

A completely similar approach can be devised by starting at the first step with an observable realization of \( P(\lambda) \) in the form

\[
\begin{bmatrix}
I - \lambda E & B \\
C & D
\end{bmatrix}
\begin{bmatrix}
I_p & -\lambda I_p & 0 \\
I_p & -\lambda I_p & 0 \\
\vdots & \vdots & \vdots \\
I_p & -\lambda I_p & 0 \\
I_p & -\lambda I_p & 0 \\
-I_p & 0 & \cdots & 0 & 0 & P_{k-1} & 0 & P_0
\end{bmatrix}
\]

of order \( p(k + 1) \), and then removing the uncontrollable infinite eigenvalues of the pencil \([I - \lambda E B]\) using the procedure of [15]. The decision on which of these approaches to be used can be guided by the goal to minimize the computational effort in the second step, by choosing the initial realization of lower order. Therefore, if \( p > m \), the realization (32) of order \( m(k + 1) \) is to be preferred, while if \( p < m \) the realization (33) of order \( p(k + 1) \) may be preferable.

4.4. Linearization of polynomial system matrices. A polynomial system matrix has the form

\[
S(\lambda) = \begin{bmatrix}
-T(\lambda) & U(\lambda) \\
V(\lambda) & W(\lambda)
\end{bmatrix},
\]

where \( T(\lambda), U(\lambda), V(\lambda) \) and \( W(\lambda) \) are polynomial matrices of sizes \( n \times n, n \times m, p \times n \) and \( p \times m \), respectively, and \( T(\lambda) \) is regular. This polynomial system matrix is associated to the rational transfer function matrix

\[
R(\lambda) = V(\lambda)T^{-1}(\lambda)U(\lambda) + W(\lambda)
\]

and it was used in the works of Rosenbrock [12] and later of Verghese [21, 19] to study the pole-zero and singular structures of \( R(\lambda) \). Particular system matrices with first order polynomial
matrices (i.e., pencils) have already been considered in (22) and (29), and are well-suited for computational purposes. Therefore, the linearization of a general polynomial matrix to obtain a first order polynomial form is often necessary.

The linearization of the polynomial system matrix \( S(\lambda) \) in (34) to a first order polynomial matrix of the form (22) having the same transfer function matrix \( R(\lambda) \) can be performed using the pencil based linearizations of \( S(\lambda) \) as in (25) or (26). Assume that \( S(\lambda) \) has the following pencil based linearization

\[
\tilde{C}_1(\lambda) = \begin{bmatrix}
\tilde{A} - \lambda \tilde{E} & \tilde{B}_1 - \lambda \tilde{F}_1 \\
\tilde{C}_1 - \lambda \tilde{G}_1 & \tilde{D}_{11} - \lambda \tilde{H}_{11} & \tilde{D}_{12} - \lambda \tilde{H}_{12} \\
\tilde{C}_2 - \lambda \tilde{G}_2 & \tilde{D}_{21} - \lambda \tilde{H}_{21} & \tilde{D}_{22} - \lambda \tilde{H}_{22}
\end{bmatrix}
\]

where \( \tilde{A} - \lambda \tilde{E} \) is a regular \( \tilde{n} \times \tilde{n} \) pencil with \( \tilde{n} \) depending on the chosen linearization (25) or (26), \( \tilde{D}_{11} - \lambda \tilde{H}_{11} \) is an \( n \times n \) pencil, and \( \tilde{D}_{22} - \lambda \tilde{H}_{22} \) is a \( p \times m \) pencil (the dimensions of the rest of subpencils implicitly result). The resulting \( (n + \tilde{n}) \times (n + \tilde{n}) \) pencil \( A - \lambda E \) is regular and it can be shown that

\[
R(\lambda) = (C - \lambda G)(\lambda E - A)^{-1} (B - \lambda F) + D - \lambda H.
\]

This realization is usually not strongly minimal and the reduction to a least order linearization can be achieved using the techniques described in [4]. A similar approach can be devised to arrive to a descriptor system based linearization of the form (29).

5. Implemented software. In what follows, we succinctly describe the newly implemented software tools for the Release v1.0 of the Julia package MatrixPencils [18]. These functions cover the computation of structural elements of polynomial matrices and related computations as described in this paper. The required basic computational tools, as for example, tools for the computation of the Kronecker structure of matrix pencils or computation of least order linearizations have been already implemented for the (previous) Release v0.5 and will be described elsewhere. The implemented functions focus only on the computation of structural elements such as eigenvalues, zeros, Kronecker indices, but do not address the computation of vectors associated with them, as eigenvectors, zero directions, or bases vectors of certain nullspaces.

A polynomial matrix \( P(\lambda) \) can be entered in the Julia language in two formats. The first possibility is to enter it as a matrix with Polynomial type elements as defined in the Polynomials package (https://github.com/JuliaMath/Polynomials.jl). This input format is mainly intended as a convenient way to enter polynomial matrices in a quasi-symbolic form using matrices or vectors with polynomial entries (or even scalar polynomials) as input data. The second input format relies on the monomial basis representation \( P(\lambda) = P_0 + P_1 \lambda + \ldots + P_k \lambda^k \) by storing the coefficient matrices \( P_0, P_1, \ldots, P_k \) of the successive powers \( \lambda^0, \lambda^1, \ldots, \lambda^k \) in an 3-dimensional array \( \mathcal{P} \), where \( \mathcal{P}[;,:,1] \) contains \( P_{i+1} \). This format is internally used in all computational routines and is also suited for alternative representations of \( P(\lambda) \) (e.g., in other polynomial bases).

The following mnemonics have been used in the naming of functions:

| Mnemonic | Denotation |
|----------|------------|
| 1p       | linear pencil |
| 1s       | linear system in descriptor form |
| 1ps      | linear pencil system |
| pm       | polynomial matrix |
| spm      | structured polynomial matrix; also polynomial system matrix |
| poly     | polynomial matrix, polynomial vector or scalar polynomial$^1$ |
| 2        | place holder for “conversion to” |

$^1$Based on the Polynomial type provided by the Polynomials package https://github.com/JuliaMath/Polynomials.jl
The following table lists the main functions available in Release v1.0 of the MatrixPencils package for polynomial matrices:

| Function          | Description                                                                 |
|-------------------|-----------------------------------------------------------------------------|
| poly2pm           | Conversion of a polynomial matrix used in Polynomials package to a polynomial matrix represented as a 3-dimensional matrix |
| pm2poly           | Conversion of a polynomial matrix represented as a 3-dimensional matrix to a polynomial matrix used in Polynomials package |
| pmdeg             | Determination of the degree of a polynomial matrix                           |
| pemeval           | Evaluation of a polynomial matrix for a given value of its argument          |
| pmreverse         | Building the reversal of a polynomial matrix                                 |
| pm2lpCF1          | Building a linearization in the first Frobenius companion form               |
| pm2lpCF2          | Building a linearization in the second Frobenius companion form              |
| pm2ls             | Building a structured linearization \( [A-\lambda E \ B \ C \ D] \) of a polynomial matrix |
| ls2pm             | Computation of the polynomial matrix from its structured linearization       |
| pm2lps            | Building a pencil based structured linearization \( [A-\lambda E \ B-\lambda F \ C \ D] \) of a polynomial matrix |
| lps2pm            | Computation of the polynomial matrix from its pencil based structured linearization |
| spm2ls            | Building a structured linearization \( [A-\lambda E \ B \ C \ D] \) of a structured polynomial matrix \( \begin{bmatrix} T(\lambda) & U(\lambda) \\ V(\lambda) & W(\lambda) \end{bmatrix} \) |
| spm2lps           | Building a pencil based structured linearization \( [A-\lambda E \ B-\lambda F \ C \ D] \) of a structured polynomial matrix \( \begin{bmatrix} T(\lambda) & U(\lambda) \\ V(\lambda) & W(\lambda) \end{bmatrix} \) |
| pmkstruct         | Determination of the Kronecker structure and the multiplicities of infinite poles and zeros using companion form based linearizations |
| pmeigvals         | Computation of the finite and infinite eigenvalues using companion form based linearizations |
| pmzeros           | Computation of the finite and infinite zeros using companion form based linearizations |
| pmzeros1          | Computation of the finite and infinite zeros using pencil based structured linearizations |
| pmzeros2          | Computation of the finite and infinite zeros using structured linearizations |
| pmroots           | Computation of the roots of the determinant of a regular polynomial matrix (i.e., finite zeros) |
| pmpoles           | Computation of the finite and infinite poles using companion form based linearizations |
| pmpoles1          | Computation of the finite and infinite poles using pencil based structured linearizations |
| pmpoles2          | Computation of the finite and infinite poles using structured linearizations |
| pmrank            | Determination of the normal rank of a polynomial matrix                      |
| ispmregular       | Checking the regularity of a polynomial matrix                               |
| ispmunimodular    | Checking the unimodularity of a polynomial matrix                            |

6. Example. To illustrate the main concepts related to polynomial matrices, we present an example which possesses all discussed essential structural features and can be handled both
analytically and numerically. This example, taken from [16], is a \( p \times m \) matrix with \( p = m = 3 \)

\[
P(\lambda) = \begin{pmatrix}
\lambda^2 + \lambda + 1 & 4\lambda^2 + 3\lambda + 2 & 2\lambda^2 - 2 \\
\lambda & 4\lambda - 1 & 2\lambda - 2 \\
\lambda^2 & 4\lambda^2 - \lambda & 2\lambda^2 - 2
\end{pmatrix}
\]

of degree \( d = 2 \) and rank \( r = 2 \). \( P(\lambda) \) can be alternatively expressed as the matrix polynomial \( P(\lambda) = P_0 + P_1\lambda + P_2\lambda^2 \) in the standard monomial basis, with

\[
P_0 = \begin{pmatrix}
1 & 2 & -2 \\
0 & -1 & -2 \\
0 & 0 & 0
\end{pmatrix}, \quad P_1 = \begin{pmatrix}
1 & 3 & 0 \\
1 & 4 & 2 \\
0 & -1 & -2
\end{pmatrix}, \quad P_2 = \begin{pmatrix}
1 & 4 & 2 \\
0 & 0 & 0 \\
1 & 4 & 2
\end{pmatrix}.
\]

To study the finite eigenvalue structure we use the unimodular matrices from [16]

\[
U(\lambda) = \begin{pmatrix}
1 & -1 & -1 \\
-\lambda & \lambda + 1 & \lambda \\
0 & -\lambda & 1
\end{pmatrix}, \quad V(\lambda) = \begin{pmatrix}
1 & -3 & 6 \\
0 & 1 & -2 \\
0 & 0 & 1
\end{pmatrix},
\]

(35)

to obtain the Smith-form of \( P(\lambda) \) as

\[
D(\lambda) = U(\lambda)P(\lambda)V(\lambda) = \begin{pmatrix}
1 & 0 & 0 \\
0 & \lambda - 1 & 0 \\
0 & 0 & 0
\end{pmatrix},
\]

which exhibits the finite eigenvalue \( \lambda = 1 \) with partial multiplicities \((0, 1)\), and the (normal) rank \( r = 2 \) of \( P(\lambda) \).

To study the structure at infinity of \( P(\lambda) \), we determine the multiplicities of the infinite eigenvalues as the multiplicities of the null eigenvalues of the reversal \( P_{rev}(\lambda) := \lambda^2 P(1/\lambda) \). We used the following two unimodular matrices

\[
\tilde{U}(\lambda) = \begin{pmatrix}
-\lambda & 0 & \lambda^2 + 2\lambda + 1 \\
\lambda^2 - \lambda + 1 & 0 & -\lambda^3 - \lambda^2 - 1 \\
0 & -1 & \lambda
\end{pmatrix}, \quad \tilde{V}(\lambda) = \begin{pmatrix}
3\lambda + 1 & 3\lambda^3 + \lambda^2 - 3\lambda - 4 & 6 \\
-\lambda & -\lambda^3 + \lambda + 1 & -2 \\
0 & 0 & 1
\end{pmatrix},
\]

to obtain the Smith-form of \( P_{rev}(\lambda) \) as

\[
D_{rev}(\lambda) = \tilde{U}(\lambda)P_{rev}(\lambda)\tilde{V}(\lambda) = \begin{pmatrix}
1 & 0 & 0 \\
0 & \lambda^2(\lambda - 1) & 0 \\
0 & 0 & 0
\end{pmatrix},
\]

which exhibits the finite eigenvalue \( \lambda = 0 \) with partial multiplicities \((0, 2)\), and therefore two infinite eigenvalues with the same partial multiplicities, and, additionally, the finite zero \( \lambda = 1 \) with partial multiplicities \((0, 1)\). This nonzero finite zero of \( P_{rev}(\lambda) \) is the reciprocal of the finite zero of \( P(\lambda) \), and has evidently the same structural indices. As expected, the (normal) rank \( P_{rev}(\lambda) \) is 2. It follows that the spectrum of \( P(\lambda) \), formed of the finite and infinite eigenvalues, is \( \mathcal{E} = \{1, \infty\} \), with partial multiplicities \((0, 1)\) and \((0, 2)\), respectively. The finite zero structure (according to McMillan [12]) and the finite eigenvalue structure at \( \lambda = 1 \) coincide, with the finite structural indices \((0, 1)\). The infinite pole-zero structure is given by the infinite structural indices \((-2, 0)\) and indicates an infinite pole of multiplicity two and no infinite zero (recall that the partial multiplicities of infinite eigenvalues are in excess with \( d = 2 \)).

If we regard \( P(\lambda) \) as a rational matrix, then we can alternatively use for our analysis the local Smith-McMillan form of \( P(\lambda) \) (as in [16]). For the finite eigenvalue structure, the analysis based on the Smith form is satisfactory. For the analysis of the infinite structure, we employ two matrices \( U_\infty(\lambda) \) and \( V_\infty(\lambda) \), which are regular at \( \lambda = \infty \), to determine the structure of \( P(\lambda) \) at infinity. For reference purposes we give the expressions of these matrices

\[
U_\infty(\lambda) = \begin{pmatrix}
\frac{1}{\lambda} & 0 & \frac{(\lambda + 1)^2}{\lambda} \\
\frac{\lambda^2 - \lambda + 1}{\lambda} & 0 & \frac{\lambda^3 + \lambda + 1}{\lambda} \\
0 & -1 & \lambda
\end{pmatrix}, \quad V_\infty(\lambda) = \begin{pmatrix}
\frac{3}{\lambda} + 1 & -4\lambda^2 - 3\lambda^3 + \lambda^2 + \frac{3}{\lambda} \\
-\frac{1}{\lambda} & \frac{\lambda^3 + \lambda^2 - 1}{\lambda} & -2 \\
0 & 0 & 1
\end{pmatrix}
\]
and the resulting local Smith-McMillan form at $\infty$ 

$$D_\infty(\lambda) = U_\infty(\lambda)P(\lambda)V_\infty(\lambda) = \begin{bmatrix} (1/\lambda)^{-2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. $$

The above form shows that $\infty$ is indeed a pole of multiplicity two of $P(\lambda)$ and $P(\lambda)$ has no zeros at infinity.

The last column of $V(\lambda)$ in (35) is a right annihilator of $P(\lambda)$ of degree 0 and, represents a minimal polynomial basis of the right nullspace $N_r(P)$ of $P(\lambda)$. Similarly, the last row of $U(\lambda)$ in (35) is a left annihilator of $P(\lambda)$ of degree 1 and, represents a minimal polynomial basis of the left nullspace $N_l(P)$ of $P(\lambda)$. It follows, that the singularity of $P(\lambda)$ is characterized by the right Kronecker index $\tilde{\epsilon}_1 = 1$ and the left Kronecker index $\eta_1 = 0$.

In what follows, we determine the structural properties of $P(\lambda)$ by employing the three type of discussed linearizations.

6.1. Using a companion form linearization. Using the first Frobenius companion form linearization of $P(\lambda)$ of grade $k = d$, we obtain the pencil $C_1(\lambda) = M_1 - \lambda N_1$, with

$$M_1 = \begin{bmatrix} -1 & -3 & 0 & -1 & -2 & 2 \\ -1 & -4 & -2 & 0 & 1 & 2 \\ 0 & 1 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad N_1 = \begin{bmatrix} 1 & 4 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. $$

The computation of the Kronecker structure of $C_1(\lambda)$ reveals the following: a finite eigenvalue 1 with the partial multiplicities $(0, 1)$ (not explicitly determined) and hence $\delta_{fin}(C_1) = 1$; two infinite eigenvalues and the corresponding partial multiplicities $(0, 2)$, hence $\delta_{\infty}(C_1) = 2$; the right Kronecker index $\tilde{\epsilon}_1 = 1$ and the left Kronecker index $\eta_1 = 1$, and hence $\mu(C_1) = 2$. From this information, we can recover the right Kronecker index $\epsilon_1$ of $P(\lambda)$ (see Proposition 4.1) as $\epsilon_1 = \tilde{\epsilon}_1 - (d-1) = 0$. The normal rank of $P(\lambda)$ results as rank $P(\lambda) = \text{rank} C_1(\lambda) - m(d-1) = 2$, where rank $C_1(\lambda) = \delta_{fin}(C_1) + \delta_{\infty}(C_1) + \mu(C_1) = 5$. The finite zero structure of $P(\lambda)$ is the same as the finite eigenvalue structure of $C_1(\lambda)$, while the infinite pole-zero structure of $P(\lambda)$ results from the resulted partial multiplicities of infinite eigenvalues (i.e., $(0,2)$), which exceed with $d = 2$ the infinite structural indices $(-2,0)$, thus indicating an infinite pole of multiplicity 2 and no infinite zeros.

Similar results can be obtained using the second Frobenius companion form linearization of $P(\lambda)$.

6.2. Using pencil based linearization. A strongly minimal realization of $P(\lambda)$ can be determined by inspection, observing that the coefficient matrix $P_2$ of $\lambda^2$ can be expressed in a full rank factorized form $P_2 = LR$, with

$$L = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T, \quad R = \begin{bmatrix} 1 & 4 & 2 \end{bmatrix},$$

which immediately leads to the strongly minimal realization of order 1 with

$$A = -1, \quad E = 0, \quad B = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}, \quad F = R, \quad C = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T, \quad G = L, \quad D = P_0, \quad H = -P_1.$$

The computation of the Kronecker structure of the system matrix pencil $S(\lambda)$ in (22) reveals the following: a finite eigenvalue 1 with the partial multiplicities $(0, 1)$ (not explicitly determined) and hence $\delta_{fin}(S) = 1$; an infinite eigenvalue and the corresponding partial multiplicities $(0, 1)$, hence $\delta_{\infty}(S) = 1$; the right Kronecker index $\epsilon_1 = 0$ and the left Kronecker index $\eta_1 = 1$, and hence $\mu(S) = 1$. The system matrix $S(\lambda)$ has a finite zero at 1 and no infinite zeros, and therefore the zero and singular structures of $P(\lambda)$ and $S(\lambda)$ coincide and $\delta^z(P) = 1$.

The computation of the Kronecker structure of the pole pencil $S_p(\lambda)$ in (27) reveals the following: no finite eigenvalues and hence $\delta_{fin}(S_p) = 0$; seven infinite eigenvalues and the
corresponding partial multiplicities \((1, 1, 1, 3)\), hence \(\delta_\infty(S_p) = 7\); no right and left Kronecker indices, and hence \(\mu(S_p) = 0\). The pole pencil \(S_p(\lambda)\) has no finite zeros and has two infinite zeros of multiplicity two, and therefore the pole and singular structures of \(S_p(\lambda)\) coincide and \(\delta^p(\lambda) = \delta^s(S_p) = 2\). The condition \(\delta^p(\lambda) = \delta^s(\lambda) + \mu(\lambda)\) is fulfilled, because \(\delta^p(\lambda) = \delta^s(S_p) = \delta^s(S) + \mu(S)\).

From the knowledge of the infinite pole-zero structure with the infinite structural indices \((-2, 0)\) (i.e., two infinite poles and no infinite zero), we can determine the infinite eigenvalue structure by shifting these values with \(d = 2\) (the degree of \(P(\lambda)\)). We obtain the expected partial multiplicities of infinite eigenvalues \((0, 2)\).

6.3. Using descriptor system realization based linearizations. A third possibility to determine the pole-zero and the singular Kronecker structures is to use a descriptor system realization based linearization of \(P(\lambda)\) of the form (29) where \((A - \lambda E, B, C, D)\) satisfies (30). Recall that, if the descriptor realization is irreducible (i.e., controllable and observable), then the zero and singular structures of \(S(\lambda)\) in (29) and \(P(\lambda)\) coincide.

Consider the irreducible realization of order \(n = 4\) (also used in [9]) with

\[
A = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & -1 & 0
\end{bmatrix}, \quad E = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 4 & 2 \\
0 & -1 & -2
\end{bmatrix},
\]

\[
C = \begin{bmatrix}
0 & 0 & -1 & -1 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix}, \quad D = \begin{bmatrix}
1 & 2 & -2 \\
0 & -1 & -2 \\
0 & 0 & 0
\end{bmatrix}.
\]

The computation of the Kronecker structure of the system matrix pencil \(S(\lambda)\) in (29) reveals the following: a finite eigenvalue 1 with the partial multiplicities \((0, 1)\) (not explicitly determined) and hence \(\delta_{\text{fin}}(S) = 1\); four infinite eigenvalues and the corresponding partial multiplicities \((1, 1, 1, 1)\), hence \(\delta_\infty(S) = 4\); the right Kronecker index \(\epsilon_1 = 0\) and the left Kronecker index \(\eta_1 = 1\), and hence \(\mu(S) = 1\). The system matrix \(S(\lambda)\) has a finite zero at 1 and no infinite zeros, and therefore the zero and singular structures of \(P(\lambda)\) and \(S(\lambda)\) coincide.

The computation of the Kronecker structure of the pole pencil \(S_p(\lambda) = A - \lambda E\) in (27) reveals the following: no finite eigenvalues and hence \(\delta_{\text{fin}}(S_p) = 0\); four infinite eigenvalues and the corresponding partial multiplicities \((1, 3)\), hence \(\delta_\infty(S_p) = 4\); no right and left Kronecker indices, and hence \(\mu(S_p) = 0\). The pole pencil \(S_p(\lambda)\) has no finite zeros and has two infinite zeros of multiplicity two, and therefore the pole and singular structures of \(P(\lambda)\) and the zeros and singular structures of \(S_p(\lambda)\) coincide and the condition \(\delta^p(\lambda) = \delta^s(P) + \mu(P)\) is fulfilled, because \(\delta^p(\lambda) = \delta^s(S_p) = \delta^s(S) + \mu(S) = 2\).

Similar results have been obtained using a minimal descriptor realization (i.e., without non-dynamic modes) of order \(n = 3\). For reference purposes we give the matrices of employed realization

\[
A = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{bmatrix}, \quad E = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
0 & -1 & -2 \\
0 & 0 & 0 \\
1 & 4 & 2
\end{bmatrix},
\]

\[
C = \begin{bmatrix}
0 & -1 & -1 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{bmatrix}, \quad D = \begin{bmatrix}
1 & 3 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

It is worth mentioning, that for the computation of zeros and poles, the use of a minimal realization instead of an irreducible one has no practical advantages. This is because the determination of a minimal realization usually involves, besides the determination of an irreducible realization using orthogonal similarity transformations, the additional step of eliminating the non-dynamic modes, which however involves matrix inversions and thus cannot be performed using only orthogonal reductions.
7. Conclusions. In this article we presented the main theoretical results which are relevant for the determination of the Kronecker and pole-zero structures of polynomial matrices using linearization based computational techniques. The companion form based linearizations served as basis to implement the basic structural analysis functions of the MatrixPencils package to compute eigenvalues, singular structures and pole-zero structures of polynomial matrices. Alternatively, linearizations based on pencil and descriptor system representations are used to implement functions for the determination of the pole-zero and singular structures. Future extensions of these tools will serve as basis for the implementation of similar functions for handling rational matrices.

Some useful links for the MatrixPencils package are listed below:
- download site of the latest release https://github.com/andreasvarga/MatrixPencils.jl;
- alternative download site https://zenodo.org/record/3837409;
- latest version of the documentation https://andreasvarga.github.io/MatrixPencils.jl/dev/;
- complete list of available functions https://sites.google.com/site/andreasvarga/contact/home/software/matrix-pencils-in-julia.

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