NO PRODUCT THEOREM
FOR THE COVERING DIMENSION
OF TOPOLOGICAL GROUPS

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Abstract. Two (strongly) zero-dimensional Lindelöf topological groups whose product has positive covering dimension are constructed. An example of a Lindelöf (strongly) zero-dimensional space whose free and free Abelian topological groups are not strongly zero-dimensional is given.

This paper is concerned with the covering dimension of topological groups. There are two definitions of covering dimension, in the sense of Čech and in the sense of Katetov; following [1], we denote the former by \( \text{dim} \) and the latter by \( \text{dim}_0 \). Recall that, given a topological space \( X \), \( \text{dim}_0 X \) (or \( \text{dim} X \)) is the least integer \( n \geq -1 \) such that any finite open (cozero) cover of \( X \) has a finite open (cozero) refinement of order \( n \), provided that such an integer exists. If it does not exist, then \( \text{dim} X = \infty \) (or \( \text{dim}_0 X = \infty \)). For normal spaces, these dimensions coincide (see [1]). A space \( X \) for which \( \text{dim}_0 (X) = 0 \) is said to be strongly zero-dimensional.

In [2] Shakhmatov asked whether the inequality \( \text{dim}_0 (G \times H) \leq \text{dim}_0 G + \text{dim}_0 H \) holds for arbitrary topological groups \( G \) and \( H \). Various versions of this question can be found in [3]. In the paper, we construct two Lindelöf topological groups \( G \) and \( H \) for which \( \text{dim}_0 (G \times H) > \text{dim}_0 G + \text{dim}_0 H = 0 \) (and \( \text{dim} (G \times H) > \text{dim} G + \text{dim} H = 0 \)), thereby answering (in the negative) Shakhmatov’s question and Questions 6.9 and 6.14 in [3]. A modification of this example also gives a negative answer to Arkhangel’skii’s 1981 question of whether the free (free Abelian) topological group of any strongly zero-dimensional space is strongly zero-dimensional [4] (see also [3, Problem 8.17]).

Our construction of topological groups \( G \) and \( H \) for which \( \text{dim}_0 (G \times H) > \text{dim}_0 G + \text{dim}_0 H \) is based on Przymusiński’s notion of \( n \)-cardinality and \( n \)-Bernstein sets [5] and on his construction of Lindelöf spaces \( X \) and \( Y \) such that \( X \times Y \) is normal and \( \text{dim} X = \text{dim} Y = 0 \) but \( \text{dim} (X \times Y) > 0 \) [6]. Below we recall some details, following the exposition of the construction given in [1].

Let \( X \) be a set, and let \( n \in \mathbb{N} \). The \( n \)-cardinality (with respect to \( X \)) of a set \( A \subset X \), denoted by \( |A|_n \), is the least cardinal \( \kappa \) such that

\[
A \subset \bigcup_{i=1}^{n} (X^{i-1} \times Y \times X^{n-i})
\]

for some \( Y \subset X \) with \( |Y| = \kappa \) (here and in what follows it is assumed that \( X^0 = Y = X^0 \)). Clearly, \( |A|_1 = |A| \) and \( |A|_n \leq |A| \). If \( |A|_n \leq \omega \), then \( A \) is said to be \( n \)-countable; otherwise, \( A \) is said to be \( n \)-uncountable.

Given \( n \in \mathbb{N} \), we say that a set \( B \subset X \) is weakly \( n \)-Bernstein with respect to a topology \( \tau \) on \( X^n \) if \( |A \cap B^n|_n = 2^\omega \) for every \( n \)-uncountable \( \tau \)-closed set \( A \subset X^n \).

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For \( x \in X^n \) and \( i \leq n \), by \( x_i \) we denote the \( i \)th coordinate of \( x \) and by \( \bar{x} = \{x_1, \ldots, x_n\} \).

By abuse of notation, given a topology \( \tau \) on \( X \), we denote the product topology on \( X^n \) by \( \tau^n \).

**Lemma 1** (see [1] Lemma 24.1). For a set \( A \subset X^n \) and an infinite cardinal \( \kappa \), the following conditions are equivalent:

(a) \(|A|_n = \kappa\);

(b) \( A \) contains a subset \( B \) of cardinality \( \kappa \) such that \( \bar{p} \cap \bar{q} = \emptyset \) whenever \( p \) and \( q \) are distinct points of \( B \).

The following theorem is based on Theorem 24.3 and Proposition 24.4 in [1].

**Theorem 1.** Let \((X, \tau)\) be a space with separable completely metrizable topology \( \tau \), and let \( \mu \) be a topology on \( X^2 \) with the following properties:

(i) \( \mu \supset \tau^2 \);

(ii) \( X^2 \) contains at most \( 2^\omega \) \( 2 \)-uncountable \( \mu \)-closed sets;

(iii) \(|A|_2 \geq 2^\omega \) for any \( 2 \)-uncountable \( \mu \)-closed set \( A \subset X^2 \).

Then \( X \) contains pairwise disjoint sets \( B_1, B_2, \ldots \) such that every \( B_i \) is weakly \( 2 \)-Bernstein with respect to \( \mu \) and weakly \( n \)-Bernstein with respect to \( \tau^n \) for all \( n \).

**Proof.** First, note that, for each \( n \in \mathbb{N} \), the number of \( n \)-uncountable \( \tau^n \)-closed sets in \( X^n \) does not exceed \( 2^\omega \) and the \( n \)-cardinality of each of them is at least \( 2^\omega \) (see Theorem 24.2 and the beginning of the proof of Theorem 24.3 in [1]). By \( \mathcal{A}_2 \) we denote the family of all \( 2 \)-uncountable \( \mu \)-closed subsets of \( X^2 \) and by \( \mathcal{A}_n \), \( n \neq 2 \), the family of all \( n \)-uncountable \( \tau^n \)-closed subsets of \( X^n \). Note that \( \mathcal{A}_2 \) contains all \( 2 \)-uncountable \( \tau^2 \)-closed subsets of \( X^2 \), because \( \tau^2 \subset \mu \). We set \( \mathcal{A} = \bigcup_{n \in \mathbb{N}} \mathcal{A}_n \) and enumerate all elements of \( \mathcal{A} \) by ordinals in \( 2^\omega \) as \( \mathcal{A} = \{A_\alpha : \alpha \in 2^\omega\} \) so that each element occurs \( 2^\omega \) times in this enumeration. For each \( \alpha < 2^\omega \), we denote the unique \( n \in \mathbb{N} \) for which \( A_\alpha \in \mathcal{A}_n \) by \( n(\alpha) \) and proceed precisely as in the proof of Theorem 24.3 of [1]. Namely, using transfinite induction, we choose points \( \bar{p}(\alpha, i) \in A_\alpha \) for all \( \alpha \in 2^\omega \) and \( i \in \mathbb{N} \) so that \( \bar{p}(\alpha, i) \cap \bar{p}(\beta, j) = \emptyset \) if \( \alpha \neq \beta \) or \( i \neq j \). Let \( \bar{p}(0, i), i \in \mathbb{N} \), be arbitrary points in \( A_0 \) satisfying the condition \( \bar{p}(0, i) \cap \bar{p}(0, j) = \emptyset \) for \( i \neq j \) (they exist by Lemma [1]). Suppose that \( \gamma > 0 \) and points \( \bar{p}(\alpha, i) \) are already chosen for all \( \alpha < \gamma \) and \( i \in \mathbb{N} \). Note that the cardinality of the set

\[
Y = \bigcup_{\alpha < \gamma, i \in \mathbb{N}} \{\bar{p}(\alpha, i) : \alpha < \gamma, i \in \mathbb{N}\}
\]

is less than \( 2^\omega \). By assumption and in view of the remark at the beginning of the proof we have \(|A_\gamma|_{n(\gamma)} = 2^\omega \). By Lemma [1] there exists a \( B \subset A_\gamma \) such that \(|B| = 2^\omega \) and \( \bar{p} \cap \bar{q} = \emptyset \) for any distinct \( p, q \in B \). As \( \bar{p}(\gamma, 1), \bar{p}(\gamma, 2), \ldots \) we choose any different points in

\[
B \setminus \bigcup_{i = n}^{n(\gamma)} (X^{i-1} \times Y \times X^{n(\gamma) - i}).
\]

We set

\[
B_i = \bigcup_{\alpha < 2^\omega} \{\bar{p}(\alpha, i) : \alpha < 2^\omega\}
\]

for each \( i \in \mathbb{N} \). Clearly, \( B_i \cap B_j = \emptyset \) if \( i \neq j \). Any \( n \)-uncountable \( \tau^n \)-closed subset \( A \) of \( X^n \) equals \( A_n \) for \( 2^\omega \) indices \( \alpha \in 2^\omega \), and we have \( \bar{p}(\alpha, i) \in A \cap B^{n(\alpha)}_i \) and \( n(\alpha) = n \) for each of these \( \alpha \) and all \( i \in \mathbb{N} \). Since \( \bar{p}(\alpha, i) \cap \bar{p}(\beta, j) = \emptyset \) for \( \alpha \neq \beta \), it follows that \(|A \cap B^{n(\alpha)}_i| \geq 2^\omega \) by Lemma [1]. Similarly, we have \(|A \cap B^{n(\beta)}_i| \geq 2^\omega \) for any \( 2 \)-uncountable \( \mu \)-closed subset \( A \subset X^2 \). \( \square \)
Let $C$ be the Cantor set in $[0,1] \subset \mathbb{R}$, and let $\tau$ be the usual (Euclidean) topology on $C$. In [1] proof of Theorem 27.5] (with a reference to [2]) a topology $\mu$ on $C^2$ satisfying conditions (i)--(iii) in Theorem 3.1 for $X = C$ was defined and, given an arbitrary decomposition $\{S, S_1, S_2\}$ of $C$ into pairwise disjoint sets weakly 2-Bernstein with respect to $\mu$, topologies $\tau_1$ and $\tau_2$ on $C$ were constructed, which have, in particular, the following properties: for $i = 1, 2$,

(1) $\tau_i \supset \tau$;
(2) any $\tau_i$-neighborhood of any point of $S_i$ is a $\tau$-neighborhood;
(3) $\tau_i$ has a base consisting of $\tau$-closed sets;
(4) $\dim(C, \tau_i) = \dim_0(C, \tau_i) = 0$;
(5) $(C, \tau_1) \times (C, \tau_2)$ is normal and $\dim((C, \tau_1) \times (C, \tau_2)) = \dim_0((C, \tau_1) \times (C, \tau_2)) = 1$.

Using Theorem 3.1 we can choose $S_1$ and $S_2$ in the above construction to be $n$-weakly Bernstein with respect to $\tau^n$ for all $n \in \mathbb{N}$. We fix the corresponding topologies $\tau_1$ and $\tau_2$ and set $C_i = (C, \tau_i)$ for $i = 1, 2$.

**Lemma 2.** The spaces $C^n_i$ are Lindelöf for all $n \in \mathbb{N}$.

**Proof.** We argue by induction on $n$.

Let $\gamma$ be a $\tau_i$-open cover of $C_i$. In view of (2), each point $s \in S_i$ has a $\tau$-open neighborhood $U_s$ contained in an element $V_s$ of $\gamma$. Let $U = \bigcup_{s \in S_i} U_s$. Since $S_i$ is weakly 1-Bernstein with respect to $\tau$ and $C \setminus U$ is a $\tau$-closed set disjoint from $S_i$, it follows that $C \setminus U$ is 1-countable, that is, countable. For each $x \in C \setminus U$, choose an element $V_x$ of $\gamma$ containing $x$. Let $\{U_{s_k} : k \in \mathbb{N}\}$ be a countable subcover of the $\tau$-open cover $\{U_s : s \in S_i\}$ of $S_i$. Then $\{V_{s_k} : k \in \mathbb{N}\} \cup \{V_x : x \in C \setminus U\}$ is a countable subcover of $\gamma$.

Suppose that $n > 1$ and we have already proved that $C^n_k$ is Lindelöf for all $k < n$. Let $\gamma$ be a $\tau^n_i$-open cover of $C^n_i$. In view of (2), each point $s \in S^n_i$ has a $\tau^n_i$-open neighborhood $U_s$ contained in an element $V_s$ of $\gamma$. Let $U = \bigcup_{s \in S^n_i} U_s$. Since $S_i$ is weakly $n$-Bernstein with respect to $\tau^n$ and $C^n \setminus U$ is a $\tau$-closed set disjoint from $S^n_i$, it follows that $C^n \setminus U$ is $n$-countable, that is, there exists a countable set $Y \subset C$ such that $C^n \setminus U$ is contained in a countable union of spaces of the form $C^{k-1}_1 \times \{x\} \times C^{n-k}_1$, each of which is Lindelöf by the induction hypothesis.

It remains to choose a countable subfamily of $\gamma$ covering $C^n \setminus U$ and a countable subfamily of $\{V_s : s \in S^n_i\}$ covering $U$, which exists because $\{V_s : s \in S^n_i\}$ has the $\tau^n$-open refinement $\{U_s : s \in S^n_i\}$. \hfill \Box

For a completely regular Hausdorff space $X$, let $F(X)$ and $A(X)$ denote, respectively, its free and free Abelian topological groups (see, e.g., [3]).

**Theorem 2.** For the spaces $C_1$ and $C_2$ defined above, $\dim A(C_1) = \dim_0 A(C_1) = \dim A(C_2) = \dim_0 A(C_2) = 0$, while $\dim(a(C_1) \times A(C_2)) > 0$ and $\dim_0(A(C_1) \times A(C_2)) > 0$.

**Proof.** It is well known (see, e.g., [3] p. 417) that, for any $X$, the group $A(X)$ is the union of its closed subspaces $A_n(X) = \{\epsilon_1 x_1 + \cdots + \epsilon_n x_n : \epsilon_i = \pm 1, x_i \in X\}$, each of which is the continuous image under the natural addition map of the $n$th power of the disjoint union $X \sqcup \{0\} \sqcup X^{-1}$, where $X^{-1}$ is a homeomorphic copy of $X$ and $\{0\}$ is a singleton (0 is the zero element of $A(X)$). Thus, $A(C_i)$ is Lindelöf for $i = 1, 2$. Since $\dim C_1 = 0$, we have $\ind A(C_1) = 0$ [3] and hence $\dim A(C_1) = \dim_0 A(C_1) = 0$, because the covering dimension of a Lindelöf space does not exceed its small inductive dimension and $\dim = \dim_0$ for normal spaces (see [1]).

According to [3] Theorem 7 (version 2)], $C_i$ is a retract of $A(C_i)$. Hence $C_1 \times C_2$ is a retract of $A(C_1) \times A(C_2)$. Thus, $C_1 \times C_2$ is $C$-embedded in $A(C_1) \times A(C_2)$, which,
together with (5) and Theorem 11.22 of [1], implies \( \dim_0(A(C_1) \times A(C_2)) > 0 \). Clearly, any space \( X \) with \( \dim X = 0 \) is strongly zero-dimensional (because any disjoint open cover is cozero), whence \( \dim(A(C_1) \times A(C_2)) > 0 \). \( \square \)

**Theorem 3.** The space \( X = C_1 \oplus C_2 \) has the following properties:

(i) \( \dim X = \dim_0 X = 0 \);

(ii) \( \dim A(X) > 0 \) and \( \dim_0 A(X) > 0 \);

(iii) \( \dim F(X) > 0 \) and \( \dim_0 F(X) > 0 \).

**Proof.** Property (i) obviously follows from property (4) of the topologies \( \tau_i \). Property (ii) follows from Theorem 2 and the fact that the group \( A(C_1) \times A(C_2) \) is topologically isomorphic to \( A(C_1 \oplus C_2) \) [10], Proposition 4. The isomorphism \( i: A(C_1 \oplus C_2) \to A(C_1) \times A(C_2) \) takes each point \( x \in C_1 \oplus C_2 \) to \( (x, 0_2) \) if \( x \in C_1 \) and to \( (0_1, x) \) if \( x \in C_2 \) (by \( 0 \), we denote the zero element of \( A(C_1) \)).

Let us prove (iii). The space \( X^2 \) is topologically embedded in \( F(X) \) as a closed subspace consisting of two-letter words of the form \( xy \), where \( x, y \in X \) (see [7], Theorem 7.1.13]). Therefore, \( C_1 \times C_2 \) is topologically embedded in \( F(X) \) as a closed subspace \( Y \) consisting of two-letter words of the form \( xy \), where \( x \in C_1 \) and \( y \in C_2 \). Let us show that \( Y \) is \( C \)-embedded in \( F(X) \).

Recall that \( A(X) \) is a topological quotient of \( F(X) \). Let \( h: F(X) \to A(X) \) be the natural quotient homomorphism. It takes \( Y \) to the set \( h(Y) \) of all elements of \( A(X) \) of the form \( x + y \), where \( x \in C_1 \) and \( y \in C_2 \). Note that \( i(x + y) = (x, y) \in A(C_1) \times A(C_2) \) for any such \( x \) and \( y \), so that the restriction \( h|Y: Y \to h(Y) \) is one-to-one. Moreover, \( h|Y \) is a homeomorphism, because its inverse is the composition of the homeomorphism \( i|_{h(Y)}: h(Y) \to C_1 \times C_2 \subset A(C_1) \times A(C_2) \) and the natural multiplication map \( C_1 \times C_2 \to F(C_1 \oplus C_2) \).

As mentioned above, \( C_i \) is a retract of \( A(C_i) \) for \( i = 1, 2 \), and hence \( C_1 \times C_2 \) is a retract of \( A(C_1) \times A(C_2) \). Let \( r: A(C_1) \times A(C_2) \to C_1 \times C_2 \) be a retraction. Then \( r \circ i: A(C_1 \oplus C_2) \to C_1 \times C_2 \) is a continuous map whose restriction to \( h(Y) \) is a homeomorphism \( h(Y) \to C_1 \times C_2 \).

Take any continuous function \( f: Y \to \mathbb{R} \). Since \( C_1 \times C_2 \) is a retract of \( A(C_1) \times A(C_2) \), it follows that \( i(h(Y)) = C_1 \times C_2 \) is \( C \)-embedded in \( A(C_1) \times A(C_2) \). Let \( \hat{f} \) be a continuous extension of \( f \circ (h|Y)^{-1} \circ i^{-1}: C_1 \times C_2 \to \mathbb{R} \) to \( A(C_1) \times A(C_2) \). Then \( f \circ \hat{h}: F(X) \to \mathbb{R} \) is a continuous extension of \( f \) to \( F(X) \).

Thus, \( Y \) is \( C \)-embedded in \( F(X) \). Since \( Y \) is homeomorphic to \( C_1 \times C_2 \), we have \( \dim_0 Y > 0 \) (see property (5) of the topologies \( \tau_i \)). Therefore, \( \dim_0 F(X) > 0 \) by Theorem 11.22 of [1], and hence \( \dim F(X) > 0 \). \( \square \)

**Remark 1.** the group \( A(C_2) \) in our example can be made to have countable network weight. Indeed, setting \( C_2 = (S_2, \tau_2|_{S_2}) \) instead of \( C_2 = (C, \tau_2) \), we obtain an example of two strongly zero-dimensional spaces \( C_1 \) and \( C_2 \), one Lindelöf to any finite power and the other separable and metrizable, for which \( \dim_0(C_1 \times C_2) > 0 \) [1][11][11].

**Remark 2.** The space \( X \) in Theorem [8] is Lindelöf.

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