BOUNDS ON BIPARTITELY SHARED ENTANGLEMENT
REDUCED FROM SUPERPOSED TRIPARTITE QUANTUM STATES

Chang-shui Yu, X. X. Yi, and He-shan Song

Department of Physics, Dalian University of Technology, Dalian 116024, P. R. China

Received: date / Revised version: date

Abstract. For a tripartite pure state superposed by two individual states, the bipartitely shared entanglement can always be achieved by local measurements of the third party. Consider the different aims of the third party, i.e. maximizing or minimizing the bipartitely shared entanglement, we find bounds on both the possible bipartitely shared entanglement of the superposition state in terms of the corresponding entanglement of the two states being superposed. In particular, by choosing the concurrence as bipartite entanglement measure, we obtain calculable bounds for tripartite \( (2 \otimes 2 \otimes n) \)-dimensional cases.

PACS. 03.65.Ud – 03.67.Mn – 03.65.Ta

1 Introduction

How the entanglement of a state depends on the entanglement of individual states in the superposition is the key to well understand the nature of entanglement. It also plays an important role in understanding the entanglement of a mixed state. This question was first addressed by Linden, Popescu and Smolin in Ref. [1], who employed von Neumann entropy of the reduced matrix as entanglement measure (ER) and presented an upper bound on ER of the superposed states in terms of those of the states being superposed. The von Neumann entropy of \( |\Psi\rangle \) was defined [2] by

\[
E(|\Psi\rangle) = S(\rho_x) = -\text{Tr}\rho_x \log_2 \rho_x,
\]

where \( \rho_x = \text{Tr}_x |\Psi\rangle \langle \Psi| \) denotes the reduced density matrix of A or B. Recently this problem becomes active and has been extensively studied. Yu et al have studied the concurrence of superpositions [3] and presented an upper
bound and a lower bound on the concurrence of superpositions. Ou et al [4] give an upper bound on the negativity of superpositions. Niset and Cerf [5] reconsidered the concurrence of superpositions and gave lower and upper bounds simpler than ours. Jitesh R. Bhatt et al [6] addressed the problem by considering two superposed coherent states. Davalvanti et al [7] and Song et al [8] have addressed the entanglement of superpositions for multipartite quantum states by employing different entanglement measures. Most recently, Gour [9] reconsidered the question in Ref. [1] and presented tighter upper and lower bounds.

In this paper, we consider the bounds on entanglement induced by superposition in a new and very interesting case. As we know, one of the important methods to producing bipartite entanglement is the reduction of a multipartite entangled state to an entangled state over fewer parties (e.g. bipartite) via measurements [10]. A natural and very interesting question is that if a tripartite quantum pure state superposed by two individual states is considered, how the entanglement shared by two parties with the third party’s assistance is influenced by the superposition of the two individual states (Here it implies that von Neumann entropy is employed as bipartite entanglement measure). Suppose that tripartite pure states are shared by Alice, Bob and Charlie and we concern bipartite entanglement shared by Alice and Bob. There are two different possibilities considering Charlie’s different aims. One is to minimize the entanglement between Alice and Bob described by bipartite entanglement of their reduced density matrix (say bipartite entanglement in the following for simplification), the other is to maximize the entanglement with assistance of Charlie described by the entanglement of assistance (EoA) [10,11].

We will show how bipartitely shared entanglement on the corresponding entanglement of individual states in the superpositions1. Upper bounds on bipartite entanglement and EoA are derived in terms of the entanglement of the states being superposed. The corresponding lower bounds can be naturally obtained analogously to Ref. [5,8] and the generalization to the case of the superposition of more than two terms is straightforward. Both are briefly stated here. What is more, because bipartite entanglement and EoA are inconvenient to calculate and in particular, EoA

---

1 In other words, we consider the entanglement of reduced density matrix of the superposed tripartite quantum pure states. One should note that it is essentially different from directly considering the entanglement of a given ensemble. The reasons are as follows. 1) The reduced density matrix of two superposed tripartite pure states is obviously different from the classical sum of two corresponding mixed states. 2) For a tripartite pure state, the entanglement shared by two parties can be adjusted by the assistance of the third party; For a given ensemble (or mixed state), its entanglement only denotes the minimal average entanglement of all pure-state realizations, it is no sense in saying the maximal entanglement of a given ensemble. 3) Since EOA (in particular COA) characterizes tripartite entanglement instead of bipartite entanglement [12], what we are considering is tripartite entanglement instead of bipartite entanglement. Our system should include 3 subsystems instead of only bipartite mixed states.
is not an entanglement monotone [12], we also consider that Alice, Bob and Charlie share tripartite \((2 \otimes 2 \otimes n)\)-dimensional pure states and find the bounds on the entanglement bipartitely shared by Alice and Bob by employing concurrence as bipartite entanglement measure. In this case, the minimal and the maximal bipartitely shared entanglement are given, respectively, by Wootters’ concurrence [13] and concurrence of assistance (CoA) [11,14,15] which is shown to be a tripartite entanglement monotone [11].

The paper is organized as follows. In Sec. II, by utilizing von Neumann entropy as bipartite entanglement measure, we derive upper bounds on bipartitely shared entanglement of superposed tripartite quantum states. In Sec. III, calculable bounds on bipartitely shared entanglement are found for \((2 \otimes 2 \otimes n)\)-dimensional quantum states in terms of bipartite concurrence. The conclusion is drawn finally.

2 Bounds in terms of von Neumann Entropy

Given a tripartite quantum pure state \(|\Gamma\rangle_{ABC}\), the entanglement of reduced density matrix can be given by

\[
E(\rho_{AB}) = \min_{i} \sum_{p_i} E(|\Lambda_i\rangle_{AB})
\]

and EoA is defined by [11]

\[
E_a(|\Gamma\rangle_{ABC}) = \max_{i} \sum_{p_i} E(|\Lambda_i\rangle_{AB})
\]

where the minimum and the maximum are taken over all possible decompositions of

\[
\rho_{AB} = \text{Tr}_C(|\Gamma\rangle_{ABC}\langle\Gamma|) = \sum_{i} p_i |\Lambda_i\rangle_{AB}\langle\Lambda_i| .
\]

Note that the subscript \(ABC\) throughout the paper denotes Alice, Bob and Charlie, respectively.

**Theorem 1.** Suppose that \(|\Phi\rangle_{ABC}\) and \(|\Psi\rangle_{ABC}\) are two normalized tripartite pure states, and \(|\Gamma\rangle_{ABC} = \alpha |\Phi\rangle_{ABC} + \beta |\Psi\rangle_{ABC}\) be the superposed state with \(|\alpha|^2 + |\beta|^2 = 1\). Let 
\[
\varrho_{AB} = \text{Tr}_C(|\Gamma\rangle_{ABC}\langle\Gamma|), \quad \varrho_1_{AB} = \text{Tr}_C(|\Phi\rangle_{ABC}\langle\Phi|) \quad \text{and} \quad \varrho_2_{AB} = \text{Tr}_C(|\Psi\rangle_{ABC}\langle\Psi|),
\]

then

\[
||\Gamma\rangle_{ABC}||^2 E(\varrho_{AB}) \leq |\alpha|^2 [E(\varrho_1_{AB}) + E_a(|\Phi\rangle_{ABC})] + |\beta|^2 [E_a(|\Psi\rangle_{ABC}) + E(\varrho_2_{AB})] + 4 |\alpha\beta| , \quad (5)
\]

and

\[
||\Gamma\rangle_{ABC}||^2 E_a(|\Gamma\rangle_{ABC}) \leq 2 \left( |\alpha|^2 E_a(|\Phi\rangle_{ABC}) + |\beta|^2 E_a(|\Psi\rangle_{ABC}) + 2 |\alpha\beta| \right) \quad (6)
\]

where \(|\Gamma\rangle_{ABC}\) denotes \(L_2\) norm of \(|\Gamma\rangle_{ABC}\).

**Proof.** Based on HJW theorem [16], any ensemble that represents \(\varrho_{AB}\) can be achieved by the local Positive Operator Value Measurements (POVM) [17] on Charlie’s system. Let \(M_j = I_{AB} \otimes N_j\), \(\sum_j N_j N_j^\dagger = 1\) be the POVM operators on Charlie’s party written in terms of Kraus operators [14], then the corresponding ensemble can be given by \(\{p_{ij}, |\tilde{T}_{ij}\rangle_{AB}\}\) where

\[
\sqrt{p_{ij}} |\tilde{T}_{ij}\rangle_{AB} = \frac{i |M_j |\Gamma\rangle_{ABC}}{||\Gamma\rangle_{ABC}||}
\]

with

\[
p_{ij} = \frac{||i |M_j |\Gamma\rangle_{ABC}||^2}{||\Gamma\rangle_{ABC}||^2}
\]

and \(|i\rangle\) being the computational basis of Charlie’s system.

The average entanglement is \(\sum_{ij} p_{ij} E(|\tilde{T}_{ij}\rangle_{AB})\) by which we have (Note that we have omitted the subscripts for
simplification.)
\[
\sum_{ij} p_{ij} E((\mathcal{T}_{ij})) = \sum_{ij} p_{ij} \times E\left(\alpha \sqrt{\frac{q_{ij}}{p_{ij} \|\Gamma\|^2}} |\Phi^i_j\rangle + \beta \sqrt{\frac{q_{2ij}}{p_{ij} \|\Gamma\|^2}} |\Phi^{i'}_j\rangle\right)
\leq 2 \sum_{ij} p_{ij} \left[\alpha \sqrt{\frac{q_{ij}}{p_{ij} \|\Gamma\|^2}} \ |\Phi^i_j\rangle \right]^2 + \beta \sqrt{\frac{q_{2ij}}{p_{ij} \|\Gamma\|^2}} \ |\Phi^{i'}_j\rangle \right]^2 + \beta \sqrt{\frac{q_{2ij}}{p_{ij} \|\Gamma\|^2}} \ |\Phi^{i'}_j\rangle \right]^2 + \alpha \sqrt{\frac{q_{ij}}{p_{ij} \|\Gamma\|^2}} \ |\Phi^i_j\rangle \right]^2 \right]
\leq 2 \sum_{ij} p_{ij} \left[|\alpha|^2 q_{1ij} E\left(|\Phi^i_j\rangle\right)\right] + |\beta|^2 q_{2ij} E\left(|\Phi^{i'}_j\rangle\right) + 2 |\alpha \beta| \sqrt{q_{1ij} q_{2ij}}, \tag{9}
\]

where
\[
|\Phi^i_j\rangle = \frac{|i\rangle M_j |\Phi\rangle}{\sqrt{q_{ij}}}, \quad |\Phi^{i'}_j\rangle = \frac{|i\rangle M_j |\Psi\rangle}{\sqrt{q_{2ij}}},
\]
\[
q_{1ij} = |\langle i| M_j |\Phi\rangle|, \quad q_{2ij} = |\langle i| M_j |\Psi\rangle|.
\]

In addition, the first inequality in eq. (9) follows from the inequality [11]
\[
h(x) \leq 2 \sqrt{x(1-x)}, \tag{10}
\]
with
\[
h(x) = -x \log_2 x - (1-x) \log_2 (1-x), \tag{11}
\]
and the original bound on entanglement of superposed states given in Ref. [1] by replacing \(\alpha\) and \(\beta\) in Ref. [1] with \(\alpha \sqrt{\frac{q_{ij}}{p_{ij} \|\Gamma\|^2}}\) and \(\beta \sqrt{\frac{q_{2ij}}{p_{ij} \|\Gamma\|^2}}\), respectively. Now, if we suppose \(\{q_{ij}, |\Phi^i_j\rangle\}\) is the optimal decomposition of \(\varrho_{1AB}\) in the sense of minimal average entanglement, it is obvious that
\[
E(\varrho_{1AB}) \leq 2 \left(|\alpha|^2 E(\varrho_{1AB}) + |\beta|^2 E_{\alpha}(|\Psi\rangle_{ABC}) + 2 |\alpha \beta|\right), \tag{12}
\]
where we apply the definitions of \(E(\varrho_{AB})\) and \(E_{\alpha}(|\Psi\rangle_{ABC})\) and the Cauchy-Schwarz inequality
\[
\sum_{ij} \sqrt{q_{1ij} q_{2ij}} \leq 1. \tag{13}
\]

Analogously, let \(\{q_{ij}, |\Phi^i_j\rangle\}\) minimize the entanglement of \(\varrho_{2AB}\), then
\[
E(\varrho_{2AB}) \leq 2 \left(|\alpha|^2 E_{\alpha}(|\Phi\rangle_{ABC}) + |\beta|^2 E(\varrho_{2AB}) + 2 |\alpha \beta|\right). \tag{14}
\]

Therefore, inequalities (12) and (14) can also be written in a more symmetric form as inequality (5). If \(\{p_{ij}, |\mathcal{T}_{ij}\rangle_{AB}\}\) is supposed to be the optimal decomposition in the sense of maximal average entanglement, one will obtain
\[
E(\varrho_{1AB}) \leq 2 \left(|\alpha|^2 E_{\alpha}(|\Phi\rangle_{ABC}) + |\beta|^2 E_{\alpha}(|\Psi\rangle_{ABC}) + 2 |\alpha \beta|\right) \tag{15}
\]
The proof is completed. \(\square\)

### 3 Bounds in terms of concurrence

We have employed von Neumann Entropy as bipartite entanglement measure to investigate the minimal and the maximal entanglement shared by Alice and Bob with the assistance of Charlie, provided that they share a tripartite quantum pure state. As we know, in general EoA and bipartite entanglement of mixed states are difficult to calculate. In particular, EoA is not an entanglement monotone. In this sense, EoA of superposition seems not to be a good candidate for the investigation, even though one can not worry about the EoA in the bound of \(E(\varrho_{AB})\) where EoA can be only considered as a function. On the contrary, it
has been shown that \((2 \otimes 2)\)-dimensional concurrence and
CoA of the \((2 \otimes 2 \otimes n)\)-dimensional pure states are
explicitly given and CoA is a good entanglement measure.
Next we restrict tripartite quantum states to \((2 \otimes 2 \otimes n)\)-
dimensional case. CoA of a tripartite pure state \(|\Phi'\rangle_{ABC}\)
can be defined \([10,15]\) by
\[
C_a(|\Phi'\rangle_{ABC}) = \max \sum_i p_i C(|A_i'\rangle_{AB}) = F(\rho'_AB, \tilde{\rho}'AB),
\]
where
\[
\rho'_AB = Tr_C |\Phi'\rangle_{ABC} \langle \Phi'|,
\]
\[
\tilde{\rho}'AB = (\sigma_y \otimes \sigma_y) \rho'_AB (\sigma_y \otimes \sigma_y)
\]
and
\[
F(\sigma', \rho') = Tr\sqrt{\sigma' \rho' \sigma'}.
\]
The concurrence of \(\rho'_AB\) is defined by \([13]\)
\[
C(\rho'_AB) = \min \sum_i p_i C(|A_i'\rangle_{AB})
= \max\{0, \lambda_1 - \sum_{i>1} \lambda_i\},
\]
with \(\lambda_i\) being the square roots of eigenvalues of \(\tilde{\rho}'AB\rho'_AB\)
in decreasing order. The extremums in eq. (16) and eq.
(20) are taken over all possible decompositions of \(\rho'_AB\).

**Theorem 2:** Let \(|\Phi'\rangle_{ABC}\) and \(|\Psi'\rangle_{ABC}\) be two \((2 \otimes 2 \otimes n)\)-
dimensional pure states. The superposition state is
given by \(|\Gamma'\rangle_{ABC} = \alpha |\Phi'\rangle_{ABC} + \beta |\Psi'\rangle_{ABC}, |\alpha|^2 + |\beta|^2 = 1\).
Let \(\varrho'_{AB} = Tr_C (|\Gamma'\rangle_{ABC} \langle \Gamma'|)\), \(\varrho'_1AB = Tr_C (|\Phi'\rangle_{ABC} \langle \Phi'|)\)
and \(\varrho'_2AB = Tr_C (|\Psi'\rangle_{ABC} \langle \Psi'|)\), then
\[
||\Gamma'\rangle_{ABC}||^2 C(\varrho'_{AB}) \leq \frac{|\alpha|^2}{2} [C(\varrho'_1AB) + C(|\Phi'\rangle_{ABC})] + \frac{|\beta|^2}{2} [C(|\Psi'\rangle_{ABC}) + C(\varrho'_2AB)] + 2 |\alpha| |\beta|,
\]
and
\[
||\Gamma'\rangle_{ABC}||^2 C_a(|\Gamma'\rangle_{ABC})
\leq |\alpha|^2 C_a(|\Phi'\rangle_{ABC}) + |\beta|^2 C_a(|\Psi'\rangle_{ABC}) + 2 |\alpha| |\beta|.
\]
where \(||\Gamma'\rangle_{ABC}||\) denotes \(l_2\) norm of \(|\Gamma'\rangle_{ABC}\).

**Proof.** Let \(\{p_{ij}'\}, |\varrho'_{ij}\rangle_{AB}\) be any ensemble that represents \(\varrho'_AB\) corresponding to a POVM operation \(M_j\) on
Charlie’s system analogous to that in **Theorem 1**. The
average concurrence is given by \(\sum_{ij} p_{ij}' C(|\varrho'_{ij}\rangle_{AB})\) where
\[
\sqrt{p_{ij}' |\varrho'_{ij}\rangle_{AB}} = \frac{|\langle i| M_j |\Gamma'\rangle_{ABC}||^2}{||\Gamma'\rangle_{ABC}||}
\]
with
\[
p_{ij}' = |\langle i| M_j |\Phi'\rangle_{ABC}||^2
\]
\[
|\varrho'_{ij}\rangle = \sqrt{q_{ij}'}, |\varrho'_{ij}\rangle = \sqrt{q_{ij}''},
\]
\[
q_{ij}' = |\langle i| M_j |\Phi'\rangle|, q_{ij}'' = |\langle i| M_j |\Psi'\rangle|.
\]
Note that we have omitted the subscripts for simplification
again. Thus,
\[
\sum_{ij} p_{ij}' C(|\varrho'_{ij}\rangle_{AB})
\leq \sum_{ij} p_{ij}' \left( \alpha \sqrt{\frac{q_{ij}'}{p_{ij}' ||\Gamma'||}} |\Phi''_{ij}\rangle + \beta \sqrt{\frac{q_{ij}''}{p_{ij}' ||\Gamma'||}} |\Psi''_{ij}\rangle \right)
\]
\[
\leq \sum_{ij} p_{ij}' \left[ \alpha \sqrt{\frac{q_{ij}'}{p_{ij}' ||\Gamma'||}} \frac{||\Phi''_{ij}||^2}{||\Gamma'||^2} |\Phi''_{ij}\rangle + \beta \sqrt{\frac{q_{ij}''}{p_{ij}' ||\Gamma'||}} \frac{||\Psi''_{ij}||^2}{||\Gamma'||^2} |\Psi''_{ij}\rangle \right]
\]
Here the first inequality follows from the definition of concurrence for pure state and the inequality $\sum_k |z_k| \geq |\sum_k z_k|$. The second inequality is derived from the inequality (13) and

$$
|\langle \Psi''_{ij} | \sigma_y \otimes \sigma_y | \Phi''_{ij} \rangle| \leq |\langle \Psi''_{ij} \rangle| \cdot |\langle \Phi''_{ij} \rangle| = 1. \quad (29)
$$

If we suppose $\{q''_{ij}, |\Phi''_{ij}\rangle\}$ is the optimal decomposition of $\rho'_{1AB}$ in the sense of minimal average concurrence, we have

$$
||\Gamma''||^2 C(\rho'_{1AB}) \leq |\alpha|^2 C(\rho'_{1AB}) + |\beta|^2 C(|\Psi''_{ABC}\rangle) + 2|\alpha| |\beta|. \quad (30)
$$

Let $\{q''_{2ij}, |\Phi''_{ij}\rangle\}$ minimize the average concurrence of $\rho'_{2AB}$, then

$$
||\Gamma''||^2 C(\rho'_{1AB}) \leq |\alpha|^2 C(|\Phi''_{ABC}\rangle) + |\beta|^2 C(\rho'_2AB) + 2|\alpha| |\beta|. \quad (31)
$$

Inequalities (30) and (31) can be rewritten in the more symmetric form as inequality (22). If $\{q'_{ij}, |\Phi'_{ij}\rangle\}$ is supposed to be the optimal decomposition maximizing the average concurrence, one will get

$$
||\Gamma'||^2 C(\rho'_{1AB}) \leq |\alpha|^2 C(|\Phi'_{ABC}\rangle) + |\beta|^2 C(\rho'_{2AB}) + 2|\alpha| |\beta|. \quad (32)
$$

The proof is completed.

As an application, by Matlab 6.5 we generate two random $(2 \otimes 2 \otimes 4)$-dimensional pure states

$$
|\phi\rangle_{ABC} = \begin{bmatrix} 0.4061, 0.1119, 0.1321, 0.4155, \\
0.2188, 0.3618, 0.0422, 0.3351, \\
0.2407, 0.1541, 0.1120, 0.0759, \\
0.2656, 0.2659, 0.2019, 0.2402 \end{bmatrix}^T, \quad (33)
$$

and

$$
|\psi\rangle_{ABC} = \begin{bmatrix} 0.3868, 0.0250, 0.4408, 0.0716, \\
0.1171, 0.1588, 0.1093, 0.0930, \\
0.0581, 0.2613, 0.1253, 0.0290, \\
0.2439, 0.4571, 0.3642, 0.3189 \end{bmatrix}^T. \quad (34)
$$
The relation between CoA of the superposed state $|\gamma\rangle_{ABC} = \alpha |\phi\rangle_{ABC} + \beta |\psi\rangle_{ABC}$, $|\alpha|^2 + |\beta|^2 = 1$, and its upper bound with different $|\alpha|$ and the relation between bipartite concurrence and its upper bound are shown in Fig. 1 and Fig. 2, respectively. In fact, we have random chosen $10^6$ pairs of $(2 \otimes 2 \otimes 4)$- dimensional $|\phi\rangle_{ABC}$ and $|\psi\rangle_{ABC}$, all the numerical results show that Theorem 2 can provide good upper bounds. From Fig. 2 one can find that the upper bound is as large as twice the actual value of concurrence, however, the ratio of the upper bound and the actual concurrence is between 1 and 2. The minimal value corresponds to the superposition including a W-type state [17] with the probability trending to 1 and the maximal one corresponds to the superposition of two GHZ-type [17] state with equal probabilities. In fact, the state-dependent tightness exists in all the relevant works [1,3-9]. The tighter bound still needs further efforts.

4 Discussion and Conclusion

Before the end, we would like to taking eq. (5) in Theorem 1 as an example (the others are analogous.) to briefly discuss the lower bounds and the generalization to the case of the superposition of more than two terms. At first, we briefly state the introduction of lower bound. The state $|\Gamma\rangle_{ABC} = \alpha |\Phi\rangle_{ABC} + \beta |\Psi\rangle_{ABC}$ can always be rewritten by

$$|\Phi\rangle_{ABC} = \frac{||\Gamma\rangle_{ABC}|}{\alpha} |\Gamma\rangle_{ABC} - \frac{\beta}{\alpha} |\Psi\rangle_{ABC}.$$  \hspace{1cm} (35)

Apply eq. (5) in Theorem 1 to eq. (35), one can obtain the analogous bound on $E(\phi_{AB})$. The upper bound includes $E(\phi_{AB})$ and $E_a(|\Gamma\rangle_{ABC})$ which is further limited by eq. (6). On the contrary, a lower bound on $E(\phi_{AB})$ can be derived from the upper bound. An analogous lower bound based on the analogous expression to eq. (35) for $|\Psi\rangle_{ABC}$ can also be obtained. The minimal lower bound serves as a potential lower bound. Our theorems can be straightforwardly generalized to the case of superposition of more than two terms. Without loss of generality, suppose the superposition of three terms as $|\Pi\rangle_{ABC} = a |\Phi\rangle_{ABC} + b |\Psi\rangle_{ABC} + c |\Theta\rangle_{ABC}$ which can be rewritten by

$$|\Pi\rangle_{ABC} = \sqrt{|a|^2 + |b|^2} |\Gamma_p\rangle_{ABC} + c |\Theta\rangle_{ABC}$$ \hspace{1cm} (36)

with

$$|\Gamma_p\rangle_{ABC} = \frac{a |\Phi\rangle_{ABC} + b |\Psi\rangle_{ABC}}{\sqrt{|a|^2 + |b|^2}}.$$ \hspace{1cm} (37)

then we can apply eq. (5) to eq. (36) and obtain the upper bound on $E(\sigma_{AB})$ in terms of $E(\sigma_{AB})$, $E_a(\sigma_{AB1})$ and $E_a(\sigma_{AB2})$ where $\sigma_{AB} = Tr_C (|\Pi\rangle_{ABC} \langle \Pi|)$, $\sigma_{AB1} = Tr_C (|\Gamma_p\rangle_{ABC} \langle \Gamma_p|)$ and $\sigma_{AB2} = Tr_C (|\Theta\rangle_{ABC} \langle \Theta|)$. Employ eq. (5) and eq. (6) again on eq. (37), one can obtain upper bounds on $E(\sigma_{AB1})$ and $E_a(\sigma_{AB1})$, respectively, which will lead to the final upper bound on $E(\sigma_{AB})$ in terms of the bipartitely shared entanglement of the three superposed quantum states.

In summary, for a tripartite quantum pure state shared by Alice, Bob and Charlie and superposed by two individual states, we have presented upper bounds on the entanglement shared by Alice and Bob with assistance of Charlie in terms of his different aims. We consider the bounds by employing von Neumann entropy and concurrence respectively. The latter provides calculable bounds.
for \((2 \otimes 2 \otimes n)\)-dimensional quantum states. In particular, although CoA maximizes the concurrence shared by Alice and Bob, CoA is a tripartite entanglement monotone instead of bipartite entanglement monotone [12]. It is worthy of being noted that the lower bounds can be naturally obtained similarly to Ref. [5,8], which is only a simple algebra and briefly stated here. What is more, the generalization of the presented bounds to the case where there are more than two terms in the superposition is straightforward. We believe the tighter bounds on the entanglement of superpositions is still important and interesting for further work.

5 Acknowledgement

Yu and Song thank the support by the National Natural Science Foundation of China (NNSFC), under Grant No. 10747112, No. 10575017 and No.60703100. Yi thanks the support by NNSFC, under Grant No. 60578014.

References

1. Noah Linden, Sandu Popescu and John A. Smolin, Phys. Rev. Lett. 97, 100502 (2006).
2. C. H. Bennett, H. J. Bernstein, S. Popescu, and B. Schumacher, Phys. Rev. A 53, 2046 (1996).
3. Chang-shui Yu, X. X. Yi and He-shan Song, Phys. Rev. A 75, 022332 (2007).
4. Yong-Cheng Ou, Heng Fan, quant-ph/0704.0757.
5. J. Niset and N. J. Cerf, quant-ph/0705.4650.
6. Jitesh R. Bhatt and Prasanta K. Panigrahi, quant-ph/0708.0470.
