HAMILTONICITY OF THE CROSS-JOIN GRAPH OF DE BRUIJN SEQUENCES

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Abstract. A generalized de Bruijn digraph generalizes a de Bruijn digraph to the case where the number of vertices need not be a pure power of an integer. Hamiltonian cycles in these digraphs thus generalize regular de Bruijn cycles, and we will thus refer to them simply as de Bruijn cycles. We define the cross-join to be the graph with all de Bruijn cycles as vertices, there is an edge between two of these vertices if one can be obtained from the other via a cross-join operation. We show that the cross-join graph is connected. This in particular means that any regular de Bruijn cycle can be cross-joined repeatedly to reach any other de Bruijn cycle, generalizing a result about regular binary de Bruijn cycles by Mykkeltveit and Szmidt [15]. Furthermore, we present an algorithm that produces a Hamiltonian path across the cross-join graph, one that we may call a de Bruijn sequence of de Bruijn sequences.

1. Introduction

Consider an alphabet $A$ of size $d$. A de Bruijn cycle of order $k$ over the alphabet $A$ is a periodic sequence of characters of $A$ such that, within one period, every possible string of size $k$ occurs exactly once as a substring. For example 00110 and 0022120110 are de Bruijn sequences of order 2 over the alphabets $\{0, 1\}$ and $\{0, 1, 2\}$ respectively. These sequences were popularized by de Bruijn [11] and Good [10] even though their existence was established much earlier, see Flye-Sainte Marie [7] and Martin [13]. Beyond the mere proof of existence for any $d$ and $k$, de Bruijn [11] and Good [10] established that the number of de Bruijn cycles is $(d!)^{d^k-1}/d^k$.

de Bruijn cycles play a pivotal role in coding theory and cryptography as they are the main building blocks of many stream cyphers. The binary case is especially useful, although there has been much research on sequences with non-binary alphabets. Linear feedback shift register sequences of maximal length are indeed de Bruijn sequences that lack the all zero string, but they are known not to be

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safe and hence not useful for cryptographic applications, see Massey [14], for example. de Bruijn cycles based on nonlinear feedback functions are far more useful and more numerous than linear ones. They are, however, far from being mathematically understood as a class. There are many references in the literature that introduce and study properties of subclasses of de Bruijn cycles that are nonlinearly constructed. Golomb initiated this study in his pioneering work [9], but there are newer methods that have been introduced ever since Golomb first published his book in 1967. Such methods are typically combinatorial or graph-theoretic producing one de Bruijn sequences or a collection of similar sequences. Fredricksen [8] is an excellent review articles that outlines, and often details, many of these known methods and algorithms. Much more research on nonlinear sequences has been done in the past two decades. Some recent publications are [6], [19], [20]. Another breed of nonlinear sequences are those produced by efficient successor rules for both binary and non-binary alphabets as in [17] and [18].

One main method of generating de Bruijn sequences is the cross-join technique, which starts with a de Bruijn sequence and interchanges two appropriately chosen pairs of vertices to obtain another de Bruijn sequence. The aim of this paper is to show that any de Bruijn sequence of any alphabet size can be transformed via a sequence of cross-joins into any other de Bruijn sequence of the same order. In the binary case, this result was recently proven by Mikkeiltveit and Szmidt [15]. Our method applies in the binary case as well. Our main objective is to investigate cross-join connectivity of non-binary de Bruijn sequences. However, it turns out that our method of proof applies without modification to a generalized version of de Bruijn sequences where the number of vertices need not be a pure power of some alphabet size \( d \). We give our proof in the language of the latter, in order to widen the scope of the result as well as to pin-point to the actual assumption that is sufficient for the proof. We explain the set up of generalized de Bruijn digraphs in Section 3. In Section 4, we formulate and prove a fundamental lemma (Lemma 4.2) and use it to prove connectedness of the cross-join graph (Definition 3.2). In Section 5, we present an algorithm that generates a Hamiltonian path of de Bruijn cycles, given an arbitrary de Bruijn cycle. We also use the fundamental lemma to establish the correctness of this algorithm. We round up in Section 6 with some open problems. An appendix gives a working implementation of the algorithm in the open source language R. In the next section we give some background material.

2. Preliminaries

There are several known methods to generate de Bruijn sequences. The one method that is most understood is the algebraic method that we outline next. First we assume that the alphabet \( A \) is residue ring \( \mathbb{Z}_d \). A construction is achieved if we have an initial state \((s_1, \ldots, s_k)\) and a rule that provides the next symbol \( a_{k+1} \) of the alphabet \( A \) when the current state is \((a_1, \ldots, a_k)\). In this case the next state
is \((a_2, \ldots, a_k, a_{k+1})\). Thus, the next state is determined by a recurrence relation of order \(k\) and of the form

\[ a_i = f(a_{i-k}, \ldots, a_{i-1}) \]

where \(i \geq k + 1\) and the feedback function \(f\) maps \(A^k\) to \(A\). Since the number of possible states is finite, iterating the feedback function gives a periodic sequence. de Bruijn sequences correspond to recurrence functions with the maximal period of \(d^k\). The sequence is called linear when the recurrence function is linear and homogeneous. When a linear homogeneous feedback function is applied to an initial string of zeros we obtain the constant zero sequence. Therefore, the maximal possible period of a linear feedback function is \(d^k - 1\). A sequence constructed this way is one that misses the all-zero string of size \(k\). Such a sequence is called a maximal length linear feedback shift register sequence (LFSR), or sometimes a “punctured” linear de Bruijn sequence, because a full de Bruijn sequence can then be obtained by appending a zero next to any of the \(d - 1\) occurrences of \(k - 1\) consecutive zeros.

We will concern ourselves here with a method called the cross-join method, which starts with a de Bruijn cycle of some order and produces another de Bruijn cycle of the same order. To see how this is done, let \(w_k = (x_1, \ldots, x_k)\) be a word of size \(k\). Any word \((\hat{x}_1, x_2, \ldots, x_k)\) with \(\hat{x}_1 \neq x_1\) is a conjugate of \(w_k\). Note that two conjugate words can reach the exact same words \((x_2, \ldots, x_k, a)\) for \(a \in A\). It can be seen now that interchanging the successors of two conjugate words in a de Bruijn cycle splits it into two smaller cycles. These two cycles can be joined back into another de Bruijn cycle by locating a word of one cycle whose conjugate is on the other cycle, and interchanging the successors of these two words. For example, interchanging the successors of 110 and 010 in the sequence 00011100 we get the two cycle 00011100 and 0101 (which can be thought of as the sequences of vertices \((000, 001, 011, 110, 110, 100, 000)\) and \((010, 101, 010)\). Since the conjugate words 001 and 101 are not on the same cycle, interchanging their successors joins the two cycles into the de Bruijn cycle (arranged to start at 000): 0001011100.

Two pairs of vertices that allow to transform a de Bruijn cycle \(u\) to another de Bruijn cycle \(v\) are called cross-join pairs. In the binary case, each vertex has exactly one conjugate, so it is enough to determine one vertex from each pair and the resulting pair is called a cross-join pair. The idea of cross-joining a binary de Bruijin sequence into another has been attractive to many investigators, especially because it is always possible to generate a de Bruijn sequence via LFSR logic and cross-join it one or more times to obtain another nonlinear de Bruijn sequence. Chang et. al. \([2]\) conjectured that any two binary de Bruijn sequences obtained by maximal period LFSR sequences of the same order \(k\) have the same number of cross-join pairs that only depends on \(k\), and therefore the same number of de Bruijn sequences can be made from each LFSR by using a single cross-join operation. This
The common number is \( \frac{(2^{k-1} - 1)(2^{k-1} - 2)}{6} \). The conjecture was proven by Helleseth and Kløve [11]. The author of this paper was not able to find any information in the literature on the number of direct cross-join neighbors of a de Bruijn sequence when it is not based on a maximal period LFSR. Using computation to inspect this number of neighbors for low order binary de Bruijn sequences, we find that many distinct values exist. Indeed, the 16 binary de Bruijn sequences of order 4 are equally divided between sequences with 7 cross-join neighbors and sequences with 10 cross-join neighbors. The case of binary sequences of order 5 is quite different. Table 1 reports the possible number of neighbors along with the frequency of vertices that have this number of neighbors. Notice that the formula of Chang et al [2] is 7 and 35 respectively for \( k = 4 \) and 5, while the corresponding numbers of maximal LFSR sequences are 2 and 15 so that many nonlinear de Bruijn sequences share this number of neighbors with the LFSR sequences.

Recently, Mykkeltveit and Szmidt [15] settled the following question in the affirmative. "Is it possible to obtain any binary de Bruijn sequence by applying a sequence of cross-join pair operations to a given binary de Bruijn sequence?" They mentioned that this is a several-decade-old question that was recently asked at the International Workshop on Coding and Cryptography 2013 in Bergen. They even claim this result as an explanation of the origins of nonlinear Boolean functions that yield de Bruijn cycles. This is based on the fact that once a cross-join pair is identified, the feedback function of the initial de Bruijn sequence can easily be altered to give the feedback function of the new sequence.

As mentioned in the introduction, the main theme of this paper is to examine the main result of [15] in a more general setting. In the binary case, it is well known that for a feedback function \( f \) to produce a pure cycle (including a de Bruijn cycle), it is necessary that \( f(x_1, \ldots, x_k) = x_1 + g(x_2, \ldots, x_k) \). It is the function \( g \) that Mykkeltveit and Szmidt [15] used in their proof. Unfortunately, there is no such representation for non-binary feedback functions, calling for a different method of proof.

We formulate and prove our results for the set of Hamiltonian cycles (the analogues of de Bruijn cycles) in the so-called \( (d, N) \) generalized de Bruijn digraphs, in which the number of vertices is any positive integer \( N \geq 2 \) and which boils down to a regular de Bruijn digraph of order \( k \) when \( N = d^k \).
A de Bruijn sequence is interchangeably called a de Bruijn cycle because it can be seen as a Hamiltonian cycle on the de Bruijn digraph. A de Bruijn digraph with alphabet $A = \{0, 1, \ldots, d - 1\}$ has $d^k$ vertices which can be taken as the set of $d^k$ vectors $(x_1, \ldots, x_k)$. For two vertices $a = (a_1, \ldots, a_k)$ and $b = (b_1, \ldots, b_k)$, there exists an edge connecting $a$ to $b$ if and only if $b_i = a_{i+1}$ for $i = 1, \ldots, k - 1$. When the vertices are regarded as decimal numbers represented in base $d$, the set of edges consists of all pairs $(x, y)$ (or interchangeably $x \rightarrow y$) where $x$ and $y$ are integers in $\{0, \ldots, d^k - 1\}$ and $y = dx + r \mod d^k$, $r = 0, 1, \ldots, d - 1$.

In a generalized de Bruijn digraph, $d^k$ is replaced by any integer $N > d$. This digraph was introduced to dispense with the restrictive number of vertices in ordinary de Bruijn digraphs. Formally, a generalized de Bruijn digraph $G_B(N, d) = (V, E)$ where the vertex set is $V = \{0, 1, \ldots, N - 1\}$ and $(x, y)$ is contained in the edge set $E$ if and only if $y = dx + r \mod N$ for some $r \in \{0, \ldots, d - 1\}$.

This digraph preserves many of the properties of ordinary de Bruijn digraphs. As shown below, it is a regular digraph. Also, Imase and Itoh [12], Reddy, Pradhan and Kuhl [16] prove that $G_B(N, d)$ has a very short diameter just like de Bruijn digraphs. They also show that $G_B(N, d)$ is strongly connected. Indeed, Du and Hwang [4] show that $G_B(N, d)$ is Hamiltonian when $gcd(d, N) > 1$. Du et al [5] establishes the result when $gcd(d, N) = 1$ and $d > 2$. That is, $G_B(N, d)$ is Hamiltonian except when $d = 2$ and odd $N$. In the rest of the paper the following notation will be used. If $(x, y) \in E$, we say that $y$ is a successor of $x$ and that $x$ is a predecessor of $y$. The set of possible successors of a vertex $x$ is denoted by $\Gamma_x^+$ while the set of predecessors is $\Gamma_x^-$. Two vertices $x_1$ and $x_2$ are said to be conjugate vertices if there exist two vertices $y_1$ and $y_2$ such that $(x_1, y_1), (x_1, y_2), (x_2, y_1)$ and $(x_2, y_2)$ are all edges in $G_B(N, d)$. In this case we also say that $y_1$ and $y_2$ are companion vertices. A path is a sequence of vertices $x_1, \ldots, x_k$ such that $(x_i, x_{i+1})$ is an edge for all $i = 1, \ldots, k - 1$. A path is simple if all of its vertices are distinct. A cycle is a path in which the first and last vertices coincide. A cycle is simple if, except for the first and last, all of its vertices are distinct. A Hamiltonian path (resp. cycle) is a simple path (resp. cycle) that includes all the vertices of the digraph. Since $G_B(N, d)$ generalizes de Bruijn digraphs, we are going to refer to a Hamiltonian cycle of $G_B(N, d)$ simply as a de Bruijn cycle.

A cross-join operation is performed to a de Bruijn cycle $dB$ as follows. First locate a conjugate pair of vertices on $dB$ and interchange their successors, thus splitting $dB$ into two shorter cycles. Next locate another conjugate pair of vertices, each residing on a different cycle, and interchange their successors to obtain a new de Bruijn cycle. We illustrate with two examples. The first one uses the de Bruijn
Figure 1. de Bruijn digraphs $B(3,2)$ (left) and $B(2,3)$ (right).

cycle $0,1,3,7,6,5,2,4,0$ of $G_B(8,2)$ (with a 0 at the end emphasizing the cyclic nature). Interchanging the successors of 1 and 5 (the cross pair) we get the two cycles $0,1,2,4,0$ and $3,7,6,5,3$. Interchanging the successors of the conjugate pair $(2,6)$ (the join pair) we get the de Bruijn sequence $0,1,2,5,3,7,6,4,0$.

The reader is urged to verify that the de Bruijn cycle $0,2,1,5,3,4,0$ of $G_B(6,3)$ can be cross-joined into $0,1,5,3,4,2,0$ using the two pairs $(2,4)$ and $(0,4)$ consecutively.

**Definition 3.1.** Two de Bruijn cycles are called cross-join adjacent, or simply adjacent, if it is possible to cross-join one into another via a single cross-join operation.

This allows for the following definition.

**Definition 3.2.** The cross-join graph $C(N,d)$ of $G_B(N,d)$ has the set of all de Bruijn cycles as the set of vertices. There is an edge between two vertices if they are adjacent in the sense of definition 3.1. In particular, $C(d,d^k)$ is the cross-join graph of a regular de Bruijn digraph of order $k$.

**Lemma 3.3.** $G_B(N,d)$ is $d$-regular. That is, each vertex has $d$ successors and $d$ predecessors.
Proof. We only need to show that each vertex in \( G_B(N, d) \) has exactly \( d \) predecessors. Let \( \gcd(d, N) = \delta \). Given a vertex \( y \) between 0 and \( N - 1 \), we need to count the number of solutions to the equations

\[
y \equiv dx + r \mod N,
\]

where \( r \) can be 0, 1, \ldots, \( d - 1 \). Equivalently, \( dx \equiv (y - r) \mod N \). When \( y - r \) is not a multiple of \( \delta \) there obviously is no solution. Suppose that \( y - r = \delta t \) for some \( t \) in \( \{0, 1, \ldots, N/\delta - 1\} \). The equation implies that \( (d/\delta)x \equiv t \mod N/\delta \). Since \( \gcd(d/\delta, N/\delta) = 1 \), the last equation admits a unique solution modulo \( N/\delta \) and therefore it has \( \delta \) solutions modulo \( N \). The lemma now follows, as there are exactly \( d/\delta \) multiples of \( \delta \) in the \( d \) consecutive integers \( y, y - 1, \ldots, y - d + 1 \) for each fixed \( y \). \( \square \)

Lemma 3.4. Suppose that \( d \) divides \( N \), then for any vertex \( y \in V \), \( \Gamma_y^- = \{\lfloor y/d \rfloor + t \cdot N/d : t = 0, \ldots, d - 1\} \).

Proof. Since \( y/d - 1 < \lfloor y/d \rfloor \leq y/d \), we have

\[
y - d < d \lfloor y/d \rfloor \leq y,
\]

so that \( y = d \lfloor y/d \rfloor + r \) for some \( r \in \{0, \ldots, d - 1\} \). Hence, \( \lfloor y/d \rfloor \) is a predecessor of \( y \), and clearly so is \( \lfloor y/d \rfloor + t \cdot N/d \) for each \( t = 1, \ldots, d - 1 \). By lemma 3.3 these are the only predecessors of \( y \). \( \square \)

If follows that, when \( d \) divides \( N \), for each vertex \( x \in \{0, \ldots, N/d - 1\} \) and each \( t = 1, \ldots, d - 1 \), \( \Gamma_x^+ = \Gamma_{x+(N/d) \cdot t}^+ \). We formulate this in the following form that is more usable below.

Lemma 3.5. Suppose that \( d \) divides \( N \). Let \( y_1 \) and \( y_2 \) be two successors of \( x \). Suppose that \( y_1 \) is the successor of some \( x \neq x_1 \). Then \( y_2 \) is also a successor of \( x_2 \).

The reason we require that \( d \) divides \( N \) in the above two lemmas is precisely to require any two conjugate vertices \( x_1 \) and \( x_2 \) to be completely conjugates, in the sense that \( \Gamma_{x_1}^+ = \Gamma_{x_2}^+ \). In fact, when \( d \) does not divide \( N \), the conjugacy relation is not transitive. That is, it is possible to find vertices \( x_1, x_2, x_3 \) such that \( x_1, x_2 \) is a conjugate pair, \( x_2, x_3 \) is a conjugate pair but \( x_1, x_3 \) is not a conjugate pair. As an illustration, we list in Table 2 all the edges of \( G_B(N, d) \) for \( d = 4 \) and three values of \( N \) with different divisibility conditions. The edges are listed in the form \( x \rightarrow y_1, \ldots, y_{d-1} \), meaning that \( (x, y_i) \) is an edge for each \( i \). In Case (a) note that 0 and 2 are conjugates (as they both have an edge to 0 and 1), 2 and 4 are conjugates but 0 and 4 are not conjugates. Similar intransitive vertices can be found in (b). Only in (c), where \( d|N \), conjugate vertices form an equivalent class and so they are completely conjugate.
4. Connectedness of $C(N,d)$

In this section we formulate and prove our main result. We will first define a metric distance between de Bruijn cycles. To this end let us align de Bruijn cycles as finite sequences that all start with the same vertex. Without loss of generality we choose this initial vertex to be 0. In the following, vertices of the cross-join graph, i.e., de Bruijn sequences, are denoted by $u,v$, etc. while vertices of $G_B(N,d)$ are denoted by $x,y$, etc.

**Definition 4.1.** Let $u$ and $v$ be two vertices of $C(N,d)$ aligned as in the previous paragraph. The distance $D(u,v)$ is defined as $N - L$, where $L$ is the length of the longest initial path that is common to $u$ and $v$. 

![Figure 2. generalized de Bruijn digraphs $G_B(2,6)$ (left), $G_B(2,12)$ (middle), and $G_B(3,10)$ (right).](image)
Table 2.

It is straightforward to verify that $D(u, v)$ satisfies the three axioms of a distance. Also, it is essential to keep in mind that this distance is not related to the distance defined by the graph adjacency on $C(N, d)$, (i.e. the number of vertices in the shortest path across edges of the graph between two vertices). As an example, for $d = 3$ and $N = 10$, the following are two de Bruijn sequences aligned to start at 0, with the last 0 repeated to stress that the sequences cycle back to the initial vertex.

$u = (0, 2, 7, 1, 5, 6, 9, 8, 4, 3, 0)$
$v = (0, 2, 7, 1, 4, 3, 9, 8, 5, 6, 0)$

The maximum common initial path is $(0, 2, 7, 1)$ so $D(u, v) = 10 - 4 = 6$. The following lemma is fundamental for the rest of the paper.

**Lemma 4.2.** Let $u$ and $v$ be two distinct de Bruijn sequences in $G_B(N, d)$ where $d$ divides $N$. Then there exists a de Bruijn sequence $u_1$ which is a neighbor of $u$ in $C(N, d)$ such that $D(u_1, v) < D(u, v)$.

We prove this lemma after we state and prove our main result.

**Theorem 4.3.** When $d$ divides $N$, the cross-join graph $C(N, d)$ is connected.

**Proof.** Let $u$ and $v$ be two distinct vertices in $C(N, d)$. By Lemma 4.2 $u$ has a neighbor $u_1$ on $C(N, d)$ such that $D(u_1, v) < D(u, v)$. If $u_1 = v$ then we are done, otherwise the same argument can be iterated to get a vertex $u_2$, which is a neighbor of $u_1$, with $D(u_2, v) < D(u_1, v)$. Due to the strict inequality, and since the number of vertices of $C(N, d)$ is finite, it is evident that this iterative process must end at $v$ after a finite number $m$, leading to the desired path $u_0 = u, u_1, \ldots, u_m = v$. □
Proof of Lemma 4.2. As a road map, we are going to state and prove three claims within the proof, the main one is Claim 1, while Claim 2 and Claim 3 are stated and proved within the proof of Claim 1.

Let $M_0$ be the maximal common initial sequence of $u$ and $v$. That is, suppose that the sequence $M_0 : 0 = x_1 \rightarrow x_2 \rightarrow \cdots \rightarrow x_{L_0}$ is common to $u$ and $v$ and $L_0$ is maximal. Since $u \neq v$, $L_0 < N$ and so the successors of $x_{L_0}$ in $u$ and $v$ are both distinct from 0. Let us refer to these successors respectively as $x^{(1)}_{L_0}$ and $x^{(2)}_{L_0+1}$. Since $u$ is a de Bruijn sequence, it contains every vertex of $G_B(N,d)$ so it must contain $x^{(2)}_{L_0+1}$. The latter is evidently one of the vertices of $\bar{M}_0$, the complement of $M_0$ in $u$; that is, the sub-path of $u$ that starts with $x^{(1)}_{L_0}$ and ends with the vertex just before 0. Let $\ast x_0$ be the predecessor of $x^{(2)}_{L_0+1}$ in $u$. Since $x^{(2)}_{L_0+1}$ belongs to $\bar{M}_0$, the vertex $\ast x_0$ is either in $\bar{M}_0$ or it is $x_{L_0}$ itself. But the latter is not possible because otherwise the common initial path of $u$ and $v$ would extend to $x_{L_0+1}$, defying the maximality of $M_0$. Now $x_{L_0}$ and $\ast x_0$ are predecessors of the same vertex. They must be conjugate vertices by Lemma 3.5. Swapping their successors we split $u$ into two cycles, a cycle $C_1$ that includes the vertex 0 and another cycle $\tilde{C}_1$ that includes the edge $\ast x_0 \rightarrow x^{(1)}_{L_0}$.

The cycle $C_1$, aligned to start at 0, and the de Bruijn cycle $v$ have a maximal common initial sequence

$M_1 : 0 = x_1 \rightarrow x_2 \rightarrow \cdots \rightarrow x_{L_0} \rightarrow \cdots \rightarrow x_{L_1}$

where $L_1 \geq L_0 + 1$. The rest of the proof depends on establishing the following

Claim 1: It is possible to join $C_1$ and $\tilde{C}_1$ by using vertices in $\bar{M}_1$, the complement of $M_1$ in $C_1$.

To show this suppose we cannot. Then let the successors of $x_{L_1}$ in $v$ and $C_1$ be $x_{L_1+1}$ and $x^{(2)}_{L_1+1}$ respectively. Obviously, $x_{L_1+1}$ is not on the path $M_1$. Since every possible vertex is either on $C_1$ or on $\tilde{C}_1$, it follows that $x_{L_1+1}$ is on $\bar{M}_1$, the complement of $M_1$ in $C_1$, as it cannot be on $\bar{C}_1$, by our assumption. Let $\ast x_1$ be the predecessor of $x_{L_1+1}$ in $C_1$. Similarly to the previous paragraph, we can argue that $\ast x_1$ is in $\bar{M}_1$.

Interchanging the successors of $x_{L_1}$ and $\ast x_1$ we further split the cycle $C_1$ into two cycles $C_2$ and $\tilde{C}_2$ with $C_2$ being the cycle that includes 0 and shares a larger still initial path with $v$, say,

$M_2 : 0 = x_1 \rightarrow x_2 \rightarrow \cdots \rightarrow x_{L_2}, L_2 > L_1$.

In essence, this process can be iterated, arranging and re-arranging vertices on the initial cycle $C_1$ but without using vertices from $\tilde{C}_1$, only a finite number of times. Let $k$ be the maximal number of iterations and let $C_k$ be the resulting cycle that includes the vertex 0 with maximal initial path

$M_k : 0 = x_1 \rightarrow x_2 \rightarrow \cdots \rightarrow x_{L_k}, L_k > L_{k-1}$.
that is common with the de Bruijn sequence $v$.

Claim 2: The sub-path of the cycle $C_k$ that begins with $x_{L_k}$ and ends with 0 is simply an edge $(x_{L_k}, 0)$. That is, there is no vertex in $C_k$ between $x_{L_k}$ and 0.

To see this, suppose that $x_{(k+1)} \neq 0$ is the successor of $x_{L_k}$ in $C_k$. Let $x_{L_{k+1}}$ be the successor of $x_{L_k}$ in $v$, so that $x_{L_{k+1}}$ and $x_{(k+1)}$ are companion vertices. We then see that $x_{L_{k+1}} \neq 0$ since otherwise the de Bruijn cycle $v$ would be shorter than $C_k$. Since $C_1$ and $\tilde{C}_1$ include all vertices, $x_{L_{k+1}}$ is either in $\tilde{C}_1$ or it is in the part $M_1$ of $C_1$. If the first case is true, swapping the predecessor of $x_{L_{k+1}}$ in $\tilde{C}_1$ with the predecessor of $x_{(k+1)}$ (which is evidently one of the vertices of $M_1$) shows that $C_1$ and $\tilde{C}_1$ can be joined into a de Bruijn sequence using a vertex outside $M_1$, contradicting the original assumption of Claim 1.

If the second case is true, that is, if $x_{L_{k+1}}$ belongs to $M_k$ or any of the cycles made by the previous iteration and that are at most $C_2, \ldots, C_{k-1}, \tilde{C}_2, \ldots, \tilde{C}_{k-1}$ (equivalently, if it is one of the vertices of $M_1$), then we can swap the predecessors of $x_{(k+1)}$ and $x_{L_{k+1}}$ to get yet another cycle $C_{k+1}$ that shares a longer initial segment with $v$, contradicting the maximality of $C_k$.

It follows that $x_{L_k}$ is a predecessor of 0. We next prove

Claim 3: $C_k$ includes all predecessors of 0.

We prove this claim in a way similar to the proof of the previous claim. In effect, suppose that $y$ is a predecessor of 0 that is not on $C_k$. If $y$ belongs to $\tilde{C}_1$ we get a contradiction because we could have joined $C_1$ and $\tilde{C}_1$ by swapping the successors of $y$ and $x_{L_k}$ (which is on $M_1$). Likewise, the presence of $y$ on any of the intermediate cycles $C_2, \ldots, C_{k-1}, \tilde{C}_2, \ldots, \tilde{C}_{k-1}$ contradicts the maximality of $k$.

The validity of this last claim means that the sequence $M_k$ cannot be continued into a de Bruijn sequence as it cannot cycle back to 0 without using one of the predecessors of 0 for a second time. This of course is not true because $M_k$ is already an initial path of the de Bruijn sequence $v$.

We have thus proven that $C_1$ and $\tilde{C}_1$ can be joined by swapping the successor of a vertex in $M_1$ with that of a conjugate vertex in $\tilde{C}_1$. This makes a new de Bruijn sequence $u_1$ which is a neighbor of $u$ on $C(N, d)$. Since $L_0 < L_1$, $N - L_1 < N - L_0$ and $u_1$ satisfies the inequality

$$D(u_1, v) < D(u, v)$$

as desired. \hfill \Box

5. Hamiltonicity of $C(N, d)$

The next natural question is whether or not the cross-join graph admits a Hamiltonian path or cycle. There is no simple answer as we don’t have information about the number of cross-join neighbors of each vertex/sequence, so none of the known
sufficient conditions for Hamiltonicity can be applied. It turns out that the principle upon which Lemma 4.2 is based can be used again to show the existence of a Hamiltonian path by actually generating one. Namely, that we can cross-join a sequence $u$ into another sequence by keeping the longest possible initial subsequence of $u$ unchanged. The following algorithm describes the steps.

**ALGORITHM H.**

Input: A de Bruijn sequence $u_1 = (x_1, x_2, \ldots, x_N)$
Output: A Hamiltonian path along the cross-join graph $C(N, d)$

1. For the current sequence $u_k$, $k \geq 1$ form the set
   
   $$A_k = \{(i, i') : i < i' \text{ and } (x_i, x_{i'}) \text{ is a conjugate pair}\}$$

2. If $A_k$ is empty, output $u_k$ and halt
3. Locate the lexicographically largest pair $(i, i')$ in $A_k$
4. Form the set
   
   $$B_k(i, i') = \{(j, j') : i + 1 \leq j' \leq i' < j \text{ and } (x_j, x_{j'}) \text{ is a conjugate pair}\}$$

5. If $B_k(i, i')$ is empty, delete $(i, i')$ from $A_k$ and go to Step (2)
6. Locate the lexicographically largest pair $(j, j')$ in $B_k(i, i')$
7. Form the cross-joined sequence $u'$ based on the cross-join pairs $(x_i, x_{i'})$ and $(x_j, x_{j'})$:
   
   $$s' = (x_1, \ldots, x_i, x_{i'}+1, \ldots, x_j, x_{j'}+1, \ldots, x_{i'}, x_{i'+1}, \ldots, x_{j'}, x_{j'+1}, \ldots, x_N)$$

8. If $u'$ occurred earlier as an output sequence, delete $(j, j')$ from $B_k(i, i')$ and go to Step (5)
9. Output $u_k$
10. Let $k = k + 1$
11. Let the current sequence $u_k$ be $u'$ and go to Step (1)

**Remark 5.1.** The sets $A_k$ and $B_k(i, i')$ need not be evaluated a priori, but they simply make the presentation easier. Also, the choice of $(j, j')$ as the largest is not strictly needed for Hamiltonicity. It is included to provide a definitive way of choosing the next neighbor. In fact, as will be seen in the proof of correctness next, the essential part is that the cross-join neighbor of a current state $u_k$ that shares the highest initial subsequence with $u_k$ will be tried first. This neighbor is accepted if it is a new output, otherwise the next largest $(i, i')$ is tried, noting that (for $d > 2$) $i$ may remain the same as before while $i'$ is lowered. That is, the next lower pair in lexicographical order is tried.

**Proof of Correctness of Algorithm H.** Let $u_1, \ldots, u_K$ be the total output sequence. For the sake of getting a contradiction, suppose that $v$ is a de Bruijin sequence that is not in the output. Let $k_0$ be the maximal index between 1 and $K$ such that
$u_{k_0}$ shares the longest possible initial subsequence with $v$. Let $l$ be the length of this common largest initial subsequence. By Lemma 4.2 there exists a de Bruijn sequence $\hat{v}$ that is a cross-join neighbor of $u_{k_0}$ such that $D(\hat{v}, v) < D(u_{k_0}, v)$, where $D$ is as given in Definition 4.1. Recalling the proof of Lemma 4.2 this implies that

(a) $\hat{v}$ shares with $u_{k_0}$ the same initial subsequence of length $l$, and

(b) it shares an initial subsequence with $v$ that is longer than $l$.

By (b), it follows that $\hat{v}$ could not have appeared earlier in the output, as this contradicts the definition of $k_0$. This in turn implies that $u_{k_0}$ is not the last output sequence (i.e. $k_0 < K$), for otherwise it has at least one neighbor that did not previously appear ($\hat{v}$ constitutes one such neighbor). Thus let $i$ be the length of the initial subsequence that is common between $u_{k_0}$ and $u_{k_0} + 1$ and let us consider two cases:

Case 1: $l \leq i$. Then $u_{k_0} + 1$ shares with $v$ a common initial subsequence of length $l$. But this contradicts the maximality of $k_0$.

Case 2: $l > i$. Then $\hat{v}$ is a neighbor of $u_{k_0}$ that leaves the initial subsequence of length $l$ unchanged. This says that there must be an index $l'$ such that the entries of $u_{k_0}$ at $l$ and $l'$ are conjugate a pair that leads to a cross-join neighbor that was not the previous output. This contradicts the choice of $(i, i')$ as the lexicographically largest such pair.

The two contradictions show that the output is a Hamiltonian path. \hfill $\square$

Table 5 displays two Hamiltonian paths for the case when $N = 16$ and $d = 2$. The one on the left is the result of Algorithm H, while the one on the right uses a version of Algorithm H that uses the lexicographically smallest $(j, j')$. As a first guess, we used the prefer-one sequence, see [8], as a starting sequence but none of the paths, for $N = 16$, come up to be a Hamiltonian cycle. However, we do get a Hamiltonian cycle when we apply Algorithm H with any of the sequences (2), (3), (5), (6), (9), (10), (11), (12), where the labels are as in Table 5. By an extensive search we also find that the de Bruijn sequence

0, 1, 3, 7, 15, 31, 30, 28, 24, 17, 2, 5, 10, 21, 11, 23, 14, 29, 26, 20, 9, 19, 6, 13, 27, 22, 12, 25, 18, 4, 8, 16

serves as an initial sequence of Hamiltonian cycle for $N = 32$ and $d = 2$.

For $d = 2$ and $N = 4, 6, 8, 10, 12, 14, 18, 20, 22, 24, 26, 28, 30$, the algorithm can easily locate an initial de Bruijn sequence that initiates a Hamiltonian cycle. An implementation in Rstudio is given at the end of this paper.

6. Conclusion

When $d$ divides $N$, we have proven that the cross-join graph that corresponds to the generalized de Bruijn digraph $G_B(N, d)$ with an arbitrary number of vertices $N$ is connected, and indeed Hamiltonian, in the sense that it contains Hamiltonian paths. This of course includes the class of all regular de Bruijn sequences of
arbitrary alphabet size \( d \). Usefulness of this result includes in the fact that a linearly generated sequence can be cross-joined a number of times to obtain virtually any nonlinear sequence. Even though the Hamiltonian path can be altered into a Hamiltonian cycle in all cases considered in computation, the existence of these cycles is formally still open. Of particular interest is a condition for an initial de Bruijn sequence that makes Algorithm \( H \) a Hamiltonian cycle.

As the proof of the fundamental Lemma 4.2 depends on the fact that the conjugacy of vertices of \( G_B(N, d) \) is an equivalence relation, the results of this paper are still open when \( d \) does not divide \( N \).

**APPENDIX A. IMPLEMENTATION IN RSTUDIO**

```r
largest_i<-function(i,dB,d){
  #input: a de Bruijn sequence dB and a pair of locations \( i[1]<i[2] \)
  #output: The largest conjugate pair \( (i[1],i[2]) \) with \( i[1]<i[2] \) that
  #is less than or equal, lexicographically, to the input i
  N=length(dB);
  i1=i[1];
  i2=i[2];
  M=N/d;
  while(i1>0){
    while(i2>i1){
      if(dB[i1]%/%M==dB[i2]%/%M) break
      else i2=i2-1
    }
  }
  if(i2==i1){
    #remaining cases
    
    # Check if i2 is a multiple of M
    if(i2==M*round(i2/M)){
      # If yes, then i2 is the largest conjugate pair
      return(i2)
    }
  }
  # If no conjugate pair is found, return 0
  return(0)
}
```

**TABLE 3.** \( i \) and \( j \) are overlaid while \( j' \) and \( j'' \) are underlined. The labels on the right refer to the sequences on the left.
i1 = i1 - 1;
i2 = N - 1
}
else break #avoid infinite loop
}
return(c(i1, i2))

largest_j <- function(i, j, dB, d)
{
  # INPUT: a de Bruijn sequence dB, a conjugate pair (i[1], i[2])
  # in dB with i[1] < i[2], and another pair of locations (j[1], j[2])
  # with j[1] > i[2], i[1] < j[2] <= i[2]
  # OUTPUT: The largest conjugate pair (j[1], j[2]) with j[1] > i[2]
  # and i[1] + 1 < j[2] <= i[2] that is less than or equal, lexicographically,
  # to the input j and that makes a cross-join pair with (i[1], i[2])
  # it returns j[1] = i[2] if no such pair j is possible for pair i
  N <- length(dB);
  j1 <- j[1];
  j2 <- j[2];
  M <- N / d;
  while (j1 > i[2]){
    while (j2 > i[1]){
      if (dB[j1] %% M == dB[j2] %% M) break # found a conj. pair at j1, j2
      else j2 <- j2 - 1
    }
    if (j2 == i[1]){
      j1 <- j1 - 1;
      j2 <- i[2] # begin with (j1, j2) as lex-largest with new reduction in j1
    }
    else break # avoid infinite loop: continue inner break
  }
  return(c(j1, j2))
}
crossjoin_HP <- function(dB, d = 2)
{
  # INPUT: generalized deBruijn cycle dB that starts with 0, d is multiplier
  # OUTPUT: a list X that includes a Hamiltonian path across
  # the cross-join graph starting with dB as the first vertex
  X <- list(dB);
  N <- length(dB);
  L <- 1; # will store current length of Hamil path
  largest_j <- function(i, j, dB, d)
  {...
  
  return(c(j1, j2))
}
crossjoin_HP(dB, d = 2)

i=c(N-2,N-1);
j=c(N,N-1);
while(i[1]>0){
    i=largest_i(i,dB,d); #get lex largest conjugate pair i that is less or equal to input i
    j=c(N,i[2]);
    if (i[1]<=0) break;
    while(j[1]>i[2]){ 
        j=largest_j(i,j,dB,d); 
        if((j[1]>i[2]) && (j[2]>i[1])){#just located cross-join partner j for i
            B=c(dB[1:(i[1])],dB[(i[2]+1):(j[1])]);
            if ((i[2]-i[1])==1) B=c(B,dB[i[2]]); else{
                if (j[2]<i[2]) B=c(B,dB[(j[2]+1):(i[2])]);# else B=c(B,dB[i[2]])
                if (i[1]<j[2]) B=c(B,dB[(i[1]+1):(j[2])]);# else B=c(B,dB[j[2]])
            }
            if (j[1]<N) B=c(B,dB[(j[1]+1):N]);
            if (is_new(B,X)){
                #L=L+1;
                L=length(X)+1;
                X[[L]]=B;
                dB=B;
                #reset i and j for new iteration
                i=c(N-2,N-1); j=c(N,N-1);
                break
            }
        }
        else {#adjust j
            if (j[2]>i[1]+1) j[2]=j[2]-1
            else {j[1]=j[1]-1;j[2]=i[2]}
        }
    }
}
if(i[2]>i[1]+1) i[2]=i[2]-1
else {i[1]=i[1]-1;i[2]=N-1}
return(X)

References
[1] N. G. de Bruijn, A Combinatorial Problem, Koninklijke Nederlandse Akademie v. Wetenschappen 49, pp. 758764, (1946).
[2] T. Chang, I. Song, and S. H. Cho, Some properties of cross-join pairs in maximum length linear sequences. Proc. ISZTA 90, Honolulu, Hawaii, pp. 1077- 1079, (1990).
[3] P. Dabrowski, G. Labuzek, T. Rachwalik, J. Szymt, Searching for nonlinear feedback shift registers with parallel computing. IACR Cryptology ePrint Archive 2013/542. 2013 Military Communications and Information Systems Conference. MCC 2013, Malto, France.

[4] D.Z. Du, F.K. Hwang, Generalized de Bruijn Digraphs, Networks, vol. 18, pp. 27-38, (1988).

[5] D.Z. Du, D.F. Hsu, F.K. Hwang, X. M Zhang, The Hamiltonian property of generalized de Bruijn digraphs, Journal of Combinatorial Theory, Series B, vol. 52, pp.1-8, (1991).

[6] E. Dubrova, A scalable method for constructing Galois NLFSRs with period $2n-1$ using cross-join pairs. IEEE Trans. on Inform. Theory, 59(1), pp. 703-709, (2013).

[7] C. Flye-Sainte Marie, Solution to problem number 58, l’Intermediare des Mathmaticiens, vol. 1, pp. 107-110, (1894).

[8] H. Fredricksen, A Survey of Full Length Nonlinear Shift Register Cycle Algorithms. SIAM Review, Vol. 24, No. 2, pp. 195-221, (1982).

[9] S. Golomb, Shift register sequences. San Fransisco, Holden-Day, (1967), revised edition, Laguna Hills, CA, Aegean Park Press, 1982.

[10] I. J. Good, Normal recurring decimals. Journal of the London Mathematical Society 21 (3) pp. 167-169, (1946).

[11] T. Helleseth, T. Kløve, The Number of Cross-join pairs in maximum length linear sequences. IEEE Trans. on Inform. theory, 31, pp. 1731-1733, (1991).

[12] M. Imase, M. Itoh, A Design for Directed Graphs with Minimum Diameter. IEEE Transactions on Computers. Vol. C-32, No. 8, August 1983.

[13] M. H. Martin, A Problem in Arrangements. Bulletin of the American Mathematical Society, 40, 859-864, (1934).

[14] J. L. Massey, Shift-register synthesis and BCH decoding. IEEE Trans. Information Theory, IT-15 (1): 122-127 (1969).

[15] J. Mykkelteit, J. Szymt, On Cross Joining de Bruijn Sequences. Contemporary Mathematics, 63, pp.335-346, (2015).

[16] S. M. Reddy, D. K. Pradhan and J. G. Kuhl, Direct Graphs with Minimum Diameter and Maximal Connectivity. School of Engineering, Oakland University Tech. Rep., July 1980.

[17] J. Sawada, A. Williams, and D. Wong. A Surprisingly Simple de Bruijn Sequence Construction. Discrete Math., 339, pp 127-131, (2016).

[18] J. Sawada, A. Williams, and D. Wong. A Simple Shift Rule for k-ary de Bruijn sequences. Discrete Math., 340, pp 524-531, (2017).

[19] J. Szymt, Nonlinear feedback shift registers and Zech logarithms. arXiv:1710.09556v2, (2017).

[20] M.S. Turan, On the nonlinearity of maximum-length NFSR feedbacks. Cryptography and Communications, 4(3-4), pp. 233-243, (2012).