HARDY-LITTLEWOOD-SOBOLEV SYSTEMS
AND RELATED LIOUVILLE THEOREMS

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Abstract. We prove some Liouville theorems for systems of integral equations and inequalities related to weighted Hardy-Littlewood-Sobolev inequality type on $\mathbb{R}^N$. Some semilinear singular or degenerate higher order elliptic inequalities associated to polyharmonic operators are considered. Special cases include the Hénon-Lane-Emden system.

1. Introduction. In this paper we shall study some Hardy-Littlewood-Sobolev and related Liouville theorems for solutions to some classes of higher order differential equations and systems on $\mathbb{R}^N$.

These kind of problems have received a great deal of attention in recent years. Tools based on different forms of the maximum principle like the moving hyperplanes method (or moving spheres method), nonlinear capacitary estimates and Pohozaev type identities, energy methods and Harnack inequality type argument, have been proved to be very successful for solving interesting problems related to applications and to the general theory of partial differential equations. We refer to [3, 10, 17, 20, 22] and the references therein for some recent contributions on these topics.

In what follows, our first goal is to study a particular aspect of these kind of problems. In general, when we deal with equations or systems of differential problems on $\mathbb{R}^N$ it is well known that tools based on integral representation formulae for solutions are very useful.

There are of course different possibilities here. If the problem is semilinear, one may try to obtain these formulas by using tools based on classical potential theory [14].

On the other hand, when the problem is quasilinear, the possibility to obtain such formulas are based essentially on Wolff’s potential type arguments and are restricted, as far as we know, to nonnegative solutions of equations that involve essentially the p-Laplacian operator or more generally to operators that share the
same properties of the later. See [13, 2]. In this paper we shall consider only semi-
linear problems either in integral or differential form. We point out that even for
semilinear problems there are interesting issues concerning the existence of represen-
tation formulae of solutions. The first observation is that if we deal with solutions
that change sign, there is no hope to obtain such representation. This is due to the
fact that, roughly speaking, higher oscillations of the possible solutions prevent the
possibility to obtain the expected estimates necessary for the validity of an integral
formula.

However even if we restrict our attention to positive solutions, in general a so-
lution cannot be represented. A trivial example is given by the function
\[ u(x) = e^x \]
which is of course a solution of \( u'' = u \) on \( \mathbb{R} \). It is fairly clear that in general, if we
consider linear problems, possible solutions cannot be represented in integral form.
On the other hand if a problem is semilinear, then the presence of a nonlinearity
may help to obtain the necessary estimates for proving the representation formula.

Consider
\[ (-\Delta)^m u = \mu \quad \text{on} \quad \mathbb{R}^N, \]
(1)
where \( m \geq 1 \) is an integer such that \( N > 2m \) and \( \mu \) denotes a positive Radon
measure. Here, \( u \in L^1_{\text{loc}}(\mathbb{R}^N) \) and (1) is satisfied in the sense of distributions.

A natural and classical problem (see [14] for the case \( m = 1 \)) is to find necessary
and sufficient conditions on the possible solutions \( u \) of (1) such that the following
representation formula holds:
\[ u(x) = l + c(2m) \int_{\mathbb{R}^N} \frac{d\mu(y)}{|x-y|^{N-2m}} \quad \text{a.e. on} \quad \mathbb{R}^N, \]
(2)
where \( l \in \mathbb{R} \) and \( c(2m) \) is an explicit positive constant depending on \( N \) and \( m \).

Equations of the type (1) include several model equations studied in recent years.
For instance, a differential equation or inequality of the form
\[ (-\Delta)^m u \geq f(u) \quad \text{on} \quad \mathbb{R}^N, \]
(3)
where \( u \in L^1_{\text{loc}}(\mathbb{R}^N) \), \( f : \mathbb{R} \to \mathbb{R}^+ \) is a given nonnegative function such that \( f(u) \in L^1_{\text{loc}}(\mathbb{R}^N) \) and (3) is satisfied in the sense of distributions, can be viewed as a
special case of (1).

Indeed, in this case since \( \mu = (-\Delta)^m u \) is a positive distribution greater than
\( f(u) \), by putting \( \mu^* = \mu - f(u)dx \), we can rewrite (3) as
\[ (-\Delta)^m u = \mu^* + f(u) \quad \text{on} \quad \mathbb{R}^N. \]
Applying (2) to the above equation we obtain,
\[ u(x) = l + c(2m) \int_{\mathbb{R}^N} \frac{d\mu^*(y)}{|x-y|^{N-2m}} + c(2m) \int_{\mathbb{R}^N} \frac{f(u(y))}{|x-y|^{N-2m}} dy \]
\[ \geq l + c(2m) \int_{\mathbb{R}^N} \frac{f(u(y))}{|x-y|^{N-2m}} dy \quad \text{a.e. on} \quad \mathbb{R}^N. \]
(4)

A relevant special case of (3) is when \( f(u) = |u|^q \) with \( q > 0 \), that is
\[ (-\Delta)^m u \geq |u|^q \quad \text{on} \quad \mathbb{R}^N. \]
(5)
It is well known that if \( 1 < q \leq \frac{N}{N-2m} \), \( u \in L^q_{\text{loc}}(\mathbb{R}^N) \) and (5) is satisfied in the
sense of distributions, then \( u \equiv 0 \) a.e. on \( \mathbb{R}^N \). The proof of this Liouville theorem
can be obtained by using the method of essential nonlinear capacity as developed in [20]. Observe that no sign condition on \( u \) nor on \((-\Delta)^i u\), for \( i = 1, \ldots, m - 1 \) is required.

On the other hand, if \( 0 < q \leq 1 \), then (5) has non trivial solutions. Indeed, it is easy to see that functions of the form
\[
\begin{cases}
  u(x) := \lambda |x|^{2m/(2m-1)} & \text{if } 0 < q < 1, \\
  u(x) := \mu e^{x_1} & \text{if } q = 1,
\end{cases}
\]
where \( \lambda \) and \( \mu \) are suitable parameters, are solutions of (5).

Now, we recall that \( u \) is called a polysuperharmonic function if
\[(\Delta)^iu \geq 0 \quad \text{a.e. } x \in \mathbb{R}^N \quad \text{for } i = 1, \ldots, m. \tag{6}\]

Going back to (5), it is not difficult to show that if \( 0 < q \leq 1 \) and \( u \) is a nonnegative radial polysuperharmonic solution of (5), then \( u \equiv 0 \) a.e. on \( \mathbb{R}^N \). The proof of this fact readily follows by slightly modifying the proof of Proposition 2.1 of [19].

This suggests that the polysuperharmonicity condition (6) plays an important role for proving Liouville theorems for (3).

An answer to these problems and related inequalities of type (3) has been given in [4].

For easy reference of the interested reader we just mention that if a solution \( u \) of (1) satisfies the following ring condition:
\[
\liminf_{R \to +\infty} \frac{1}{R^N} \int_{R \leq |x-y| \leq 2R} |u(y)| - l \, dy = 0, \tag{7}
\]
then
\[
u(x) = l + c(2m) \int_{\mathbb{R}^N} \frac{d\mu(y)}{|x-y|^{N-2m}} \quad \text{a.e. } x \in \mathbb{R}^N, \tag{8}
\]
and \( u \) satisfies the polysuperharmonicity condition (6) in a weak sense.

We point out that a sufficient condition in order that a function \( u \in L^1_{loc}(\mathbb{R}^N) \) such that \( \text{essinf}(u) = l = 0 \) satisfies (7) is that for any \( R > 0 \), we have
\[
u(x) \geq \frac{1}{|B_R(x)|} \int_{B_R(x)} u(y) \, dy \quad \text{a.e. } x \in \mathbb{R}^N. \tag{9}
\]
In other words \( u \) is superharmonic. Namely, if \( u \) is bounded from below and superharmonic on \( \mathbb{R}^N \), it is not difficult to show that
\[
\lim_{R \to +\infty} \frac{1}{|B_R(x)|} \int_{B_R(x)} u(y) \, dy = \text{essinf}(u) = l,
\]
and (7) follows. It should be noted that if \( u \) satisfies (8) and \( l \geq 0 \), then \( u \geq 0 \) a.e. on \( \mathbb{R}^N \) and (6) holds. As a consequence, proving Liouville theorems for (4) turns into nonexistence of nonnegative solutions for (3).

The above considerations motivate the study of Liouville theorems for solutions of inequalities of the form (4) and its generalizations.

In this respect, as observed in [4], we see that if we consider pseudodifferential problems of the form
\[
(-\Delta)^\frac{q}{2} u = f(u) \quad \text{on } \mathbb{R}^N, \tag{10}
\]
or
\begin{equation}
(-\Delta)^\frac{\alpha}{2} u \geq f(u) \quad \text{on } \mathbb{R}^N,
\end{equation}
where \( N > \alpha > 0 \) and \( f \) is a nonnegative function, then it is natural to expect that solutions of (10) or (11) satisfy respectively
\begin{equation}
u(x) = l_1 + c(\alpha) \int_{\mathbb{R}^N} \frac{f(u(y))}{|x-y|^{N-\alpha}} dy \quad \text{a.e. } x \in \mathbb{R}^N,
\end{equation}
or
\begin{equation}
u(x) \geq l_1 + c(\alpha) \int_{\mathbb{R}^N} \frac{f(u(y))}{|x-y|^{N-\alpha}} dy \quad \text{a.e. } x \in \mathbb{R}^N,
\end{equation}
where \( l_1 \in \mathbb{R} \).

The equivalence between (10) and (12) (or the implication (11) \( \Rightarrow \) (13)) can be made precise under appropriate assumptions [4].

The above considerations can be applied to solutions of systems of the form:
\begin{equation}
\begin{cases}
(-\Delta)^m u \geq f(v) & \text{on } \mathbb{R}^N, \\
(-\Delta)^n v \geq g(u) & \text{on } \mathbb{R}^N,
\end{cases}
\end{equation}
where \( N > \max\{2m, 2n\} \) and \( m, n \geq 1 \) are integers, or more generally to
\begin{equation}
\begin{cases}
(-\Delta)^{\frac{\alpha}{2}} u \geq f(v) & \text{on } \mathbb{R}^N, \\
(-\Delta)^{\frac{\beta}{2}} v \geq g(u) & \text{on } \mathbb{R}^N,
\end{cases}
\end{equation}
where \( N > \alpha, \beta > 0 \) and \( f, g \) are nonnegative functions.

In [4] is studied the possibility to represent the solutions of the system (14) or (15) by the system of integrals
\begin{equation}
\begin{cases}
u(x) \geq l_1 + \int_{\mathbb{R}^N} \frac{f(v(y))}{|x-y|^{N-\alpha}} dy \quad \text{a.e. } x \in \mathbb{R}^N, \\
v(x) \geq l_2 + \int_{\mathbb{R}^N} \frac{g(u(y))}{|x-y|^{N-\beta}} dy \quad \text{a.e. } x \in \mathbb{R}^N,
\end{cases}
\end{equation}
where \( l_1, l_2 \in \mathbb{R} \).

Another motivation for studying systems like (HLS) comes from the so called double weighted Hardy-Littlewood-Sobolev inequality proved by Stein and Weiss in [21]. Indeed, a generalization of the Euler-Lagrange equations associated to the double weighted Hardy-Littlewood-Sobolev, is
\begin{equation}
\begin{cases}
u(x) \geq l_1 + \int_{\mathbb{R}^N} \frac{a_1(y) f(v(y))}{|x-y|^{N-\alpha}} dy \quad \text{a.e. } x \in \mathbb{R}^N, \\
v(x) \geq l_2 + \int_{\mathbb{R}^N} \frac{a_2(y) g(u(y))}{|x-y|^{N-\beta}} dy \quad \text{a.e. } x \in \mathbb{R}^N,
\end{cases}
\end{equation}
where \( l_1, l_2 \in \mathbb{R} \) and \( a_1, a_2 \) are measurable nonnegative functions. See Section 2 for a more detailed discussion.

Actually, the classical (HLS) system corresponds to the case \( \alpha = \beta \) and \( f(v) = v^p \), \( g(u) = u^q \), with \( p, q > 0 \) and equations instead inequalities in (HLS).

The interested reader may refer to very interesting contributions on problems related to (HLS) in [5, 6, 7, 8] and [15, 16].
This paper is organized as follows. In the next section we prove our first nonexistence result of a system of integral equations (or inequalities) related to the generalized version of Hardy-Littlewood-Sobolev (HLS) and additional results for related systems. In Section 3 we apply our techniques to the study of systems of higher order equations/inequalities associated to polyharmonic operators containing singular or degenerate weights. These results apply in particular to the so called Hardy–Hénon-Emden system. See for instance [9, 12] and [18] for the case without singularities or degeneracies.

2. Liouville theorems for double weighted (HLS) systems. In this section we consider some systems of integral containing that can be view as types of (HLS) systems. These problems are connected with the following double weighted Hardy-Littlewood-Sobolev (HLS) inequality proved by Stein and Weiss in [21]

$$\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)g(y)}{|x|^\alpha |x-y|^\lambda |y|^\beta} \, dx \, dy \right| \leq H \|f\|_{L^p(\mathbb{R}^N)} \|g\|_{L^q(\mathbb{R}^N)},$$  \hfill (17)

where \(H := H(\lambda, \alpha, \beta, r, N), 0 < \lambda < N, \alpha + \beta \geq 0, 1 < s, r < \infty,\) and

$$\frac{\alpha}{N} < 1 - \frac{1}{r}, \quad \frac{\beta}{N} < 1 - \frac{1}{s}, \quad \frac{1}{r} + \frac{1}{s} + \frac{\lambda + \alpha + \beta}{N} = 2.$$  

The best constant in (17) is defined by

$$H = \max \left\{ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)g(y)}{|x|^\alpha |x-y|^\lambda |y|^\beta} \, dx \, dy : \|f\|_{L^p(\mathbb{R}^N)} \|g\|_{L^q(\mathbb{R}^N)} = 1 \right\}.$$

The corresponding positive optimizers of (17) are solutions of the following Euler-Lagrange system,

\[
\begin{cases}
\lambda_1 r f(x)^{r-1} = \frac{1}{|x|^\alpha} \int_{\mathbb{R}^N} \frac{g(y)}{|y|^\beta |x-y|^\lambda} \, dy & \text{a.e. } x \in \mathbb{R}^N, \\
\lambda_2 s g(x)^{s-1} = \frac{1}{|x|^\beta} \int_{\mathbb{R}^N} \frac{f(y)}{|y|^\alpha |x-y|^\lambda} \, dy & \text{a.e. } x \in \mathbb{R}^N,
\end{cases}
\hfill (18)
\]

where \(\lambda_1 r = \lambda_2 s = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)g(y)}{|x|^\alpha |x-y|^\lambda |y|^\beta} \, dx \, dy = H.\)

Putting \(U := a_1 f^{r-1}\) and \(V := a_2 g^{s-1}\), where \(a_1, a_2\) are positive normalization constants, and defining \(p := \frac{1}{s-1}, \quad q := \frac{1}{r-1}\) with \(pq \neq 1\), we see that \((U, V)\) is a solution of

\[
\begin{cases}
U(x) = \frac{1}{|x|^\alpha} \int_{\mathbb{R}^N} \frac{V(y)^p}{|y|^\beta |x-y|^\lambda} \, dy & \text{a.e. } x \in \mathbb{R}^N, \\
V(x) = \frac{1}{|x|^\beta} \int_{\mathbb{R}^N} \frac{U(y)^q}{|y|^\alpha |x-y|^\lambda} \, dy & \text{a.e. } x \in \mathbb{R}^N,
\end{cases}
\hfill (19)
\]

where \(U \in L^{p+1}(\mathbb{R}^N), \ V \in L^{p+1}(\mathbb{R}^N), \ 0 < \lambda < N, \alpha + \beta \geq 0, 0 < p, q < \infty,\) and

$$\frac{\alpha}{N} < \frac{1}{q+1}, \quad \frac{\beta}{N} < \frac{1}{p+1}, \quad \frac{q}{q+1} + \frac{p}{p+1} + \frac{\lambda + \alpha + \beta}{N} = 2.$$
Finally, by defining \( u(x) := |x|^{\alpha} U(x) \) and \( v(x) := |x|^{\beta} V(x) \) it follows that \((u, v)\) is a positive solution of the system
\[
\begin{aligned}
&\left\{
\begin{array}{l}
u(x) = \int_{\mathbb{R}^N} \frac{v(y)^p}{|y|^{\alpha_1}|x-y|^{\lambda}} \, dy \quad \text{a.e. } x \in \mathbb{R}^N, \\
v(x) = \int_{\mathbb{R}^N} \frac{u(y)^q}{|y|^{\alpha_2}|x-y|^{\lambda}} \, dy \quad \text{a.e. } x \in \mathbb{R}^N,
\end{array}
\right.
\tag{20}
\end{aligned}
\]
where \( \alpha_1 := \beta(p+1), \alpha_2 := \alpha(q+1) \) and where \( u^{q+1}|x|^{-\alpha_2}, v^{p+1}|x|^{-\alpha_1} \in L^1(\mathbb{R}^N), 0 < \lambda < N, \frac{\alpha_1}{p+1} + \frac{\alpha_2}{q+1} \geq 0, 0 < p, q < \infty, \) and
\[
\alpha_1 < N, \quad \alpha_2 < N, \quad \frac{N - \alpha_2}{q+1} + \frac{N - \alpha_1}{p+1} = \lambda.
\]

The above description justifies the study of more general systems of the form,
\[
\begin{aligned}
&\left\{
\begin{array}{l}
u(x) \geq \int_{\mathbb{R}^N} \frac{a_1(y) v(y)^p}{|x-y|^{N-\alpha}} \, dy \quad \text{a.e. } x \in \mathbb{R}^N, \\
v(x) \geq \int_{\mathbb{R}^N} \frac{a_2(y) u(y)^q}{|x-y|^{N-\alpha}} \, dy \quad \text{a.e. } x \in \mathbb{R}^N.
\end{array}
\right.
\tag{21}
\end{aligned}
\]

The system (21) has been studied in [4] for \( a_1(y) := \mu |y|^{-\alpha_1}, a_2(y) := \mu |y|^{-\alpha_2} \) \( \mu > 0, \alpha_1 > 0, \alpha_2 > 0, 0 < \alpha, \beta < N \) and \( p, q > 0 \).

In what follows, we shall restrict our study to the following case
\[
\begin{aligned}
&\left\{
\begin{array}{l}
u(x) \geq \mu_1 \int_{\mathbb{R}^N} \frac{v(y)^p}{|y|^{\alpha_1}|x-y|^{N-\alpha}} \, dy \quad \text{a.e. } x \in \mathbb{R}^N, \\
v(x) \geq \mu_2 \int_{\mathbb{R}^N} \frac{u(y)^q}{|y|^{\alpha_2}|x-y|^{N-\alpha}} \, dy \quad \text{a.e. } x \in \mathbb{R}^N,
\end{array}
\right.
\tag{22}
\end{aligned}
\]
with the hypothesis \( 0 < \alpha < N, \mu_1, \mu_2 > 0 \). We shall further assume that
\[
\begin{aligned}
v^p \cdot |\cdot|^{-\alpha_1} &\in L^1_{\text{loc}}(\mathbb{R}^N) \quad \text{and} \quad u^q \cdot |\cdot|^{-\alpha_2} \in L^1_{\text{loc}}(\mathbb{R}^N),
\end{aligned}
\tag{23}
\]
and
\[
\begin{aligned}
-\infty < \alpha_1 < N \quad \text{and} \quad -\infty < \alpha_2 < N.
\end{aligned}
\tag{24}
\]

The above hypotheses (23) and (24) are natural in the sense that if the system (22) has a nontrivial solution then necessarily these assumptions are satisfied.

**Remark 2.1.** The minimal requirement for \((u, v)\) to be solution of (22) is that \(u\) and \(v\) are measurable functions. If we exclude the trivial solutions, namely \((u, v) \equiv (0, 0)\) a.e. and \((u, v) \equiv (\infty, \infty)\) a.e., then the assumptions \(v^p \cdot |\cdot|^{-\alpha_1} \in L^1_{\text{loc}}(\mathbb{R}^N)\) and \(u^q \cdot |\cdot|^{-\alpha_2} \in L^1_{\text{loc}}(\mathbb{R}^N)\) are necessary. Indeed, fix \( R > 0 \) we have
\[
u(x) \geq \mu_1 \int_{|y| < R} \frac{v(y)^p}{|y|^{\alpha_1}|x-y|^{N-\alpha}} \, dy \geq \mu_1 \frac{\|v\|_{L^p_{\text{loc}}}(\mathbb{R}^N)}{(|x|+R)^{N-\alpha}} \int_{|y| < R} \frac{v(y)^p}{|y|^{\alpha_1}} \, dy,
\]
which implies that if \((u, v)\) is not trivial then necessarily \(v^p \cdot |\cdot|^{-\alpha_1} \in L^1_{\text{loc}}(\mathbb{R}^N)\). Analogously, we have \(u^q \cdot |\cdot|^{-\alpha_2} \in L^1_{\text{loc}}(\mathbb{R}^N)\).

**Proposition 2.2.** Let \((u, v)\) be a nontrivial solution of (22). Then (24) holds.
Proof. Arguing by contradiction, we assume that \( \alpha_1 \geq N \) and there exists \((u,v)\) a nontrivial solution of \((22)\). Since \( v^p \mid \cdot \mid^{-\alpha_1} \in L^1_{\text{loc}}(\mathbb{R}^N) \), necessarily we have that
\[
\inf_{B_1} v = 0.
\]
Indeed, if this is not the case, then \( \inf v = c > 0 \) and therefore \( v^p(x) \mid x \mid^{-\alpha_1} \geq c^p \mid x \mid^{-\alpha_1} \notin L^1_{\text{loc}}(\mathbb{R}^N) \). Therefore for any \( \epsilon > 0 \) there exists a set \( A_\epsilon \subset B_1 \) of positive measure such that \( v(x) < \epsilon \) for any \( x \in A_\epsilon \).

Now let \( \epsilon > 0 \) and and let \( x \in A_\epsilon \). From the second inequality in \((22)\), and taking into account that \( |x| < 1 \), we have
\[
\epsilon > v(x) \geq \mu_2 \int_{B^c_1} \frac{u(y)^q}{|y|^\alpha_2 |x-y|^{N-\alpha}} \, dy \geq \mu_2 \int_{B^c_1} \frac{u(y)^q}{|y|^\alpha_2 (1 + |y|)^{N-\alpha}} \, dy.
\]
Since \( \epsilon \) is arbitrary, we have that
\[
\int_{B^c_1} \frac{u(y)^q}{|y|^\alpha_2 (1 + |y|)^{N-\alpha}} \, dy = 0.
\]
That is \( u = 0 \) a.e. on \( \mathbb{R}^N \). This implies, by using the first inequality in \((22)\), that \( v = 0 \) a.e. on \( \mathbb{R}^N \). This contradiction concludes the proof.

By the same argument used in the above proof, one can prove the following.

**Proposition 2.3.** Let \((u,v)\) be a nontrivial solution of \((22)\). Then for any \( R > 0 \) we have
\[
0 < \inf_{B_R} v < +\infty, \quad 0 < \inf_{B_R} u < +\infty.
\]

Therefore, without loss of generality, if not otherwise stated by a solution of \((22)\) we mean a pair of functions \((u,v)\) such that \( v^p \mid \cdot \mid^{-\alpha_1} \in L^1_{\text{loc}}(\mathbb{R}^N) \) and \( u^q \mid \cdot \mid^{-\alpha_2} \in L^1_{\text{loc}}(\mathbb{R}^N) \) and satisfies the system of inequalities \((22)\) for a.e. \( x \in \mathbb{R}^N \).

In order to prove our main results we need the following lemma concerning a priori estimates of the solutions of \((22)\).

**Lemma 2.4.** Let \( \mu_1, \mu_2 > 0 \), \( \alpha_1, \alpha_2 < N \), \( 0 < \alpha < N \) and \( p,q > 0 \). Let \((u,v)\) be a nontrivial solution of \((22)\). Then for any \( R > 0 \) we have
\[
\left( \int_{B_R} \frac{u(x)^q}{|x|^\alpha_2} \, dx \right)^{1-pq} \geq \mu_1^{pq} \mu_2 C_1^p C_2 R^{(N-\alpha)(1-pq)+(\alpha-\alpha_1)+\alpha-\alpha_2}, \quad (25)
\]
\[
\left( \int_{B_R} \frac{v(x)^p}{|x|^\alpha_2} \, dx \right)^{1-pq} \geq \mu_1^{pq} \mu_2 C_1^p C_2 R^{(N-\alpha)(1-pq)+(\alpha-\alpha_2)+\alpha-\alpha_1}, \quad (26)
\]
where
\[
C_1 := \frac{\omega_N \, 2F_1((N-\alpha)p, N-\alpha_1; N-\alpha_1 + 1; -1)}{N-\alpha_1},
\]
\[
C_2 := \frac{\omega_N \, 2F_1((N-\alpha)q, N-\alpha_2; N-\alpha_2 + 1; -1)}{N-\alpha_2},
\]
\( \omega_N \) is the surface area of \( S^{N-1} \) and \( 2F_1 \) is the hypergeometric function.

**Proof.** Raising the first inequality of \((22)\) to the power \( q \), multiplying by \( \mid \cdot \mid^{-\alpha_2} \) and integrating on \( B_R \) we get:
\[
\int_{B_R} \frac{u(x)^q}{|x|^\alpha_2} \, dx \geq \int_{B_R} \frac{\mu_1^q}{|x|^\alpha_2 (R + |x|)^{(N-\alpha)q}} \, dx \left( \int_{B_R} \frac{v(y)^p}{|y|^\alpha_2} \, dy \right)^q. \quad (27)
\]
Analogously from the last inequality in (22), we get
\[
\int_{B_R} \frac{v(x)^p}{|x|^\alpha} \, dy \geq \int_{B_R} \frac{\mu_2^p}{|x|^\alpha (R + |x|)^{(N-\alpha)p}} \, dx \left( \int_{B_R} \frac{u(y)^q}{|y|^\alpha} \, dy \right)^q. 
\] (28)

Setting
\[ A := \int_{B_R} \frac{u(y)^q}{|y|^\alpha} \, dy, \quad B := \int_{B_R} \frac{v(y)^p}{|y|^\alpha} \, dy, \]
and using standard change of variables and polar coordinates we find,
\[
\int_{B_R} \frac{1}{|x|^{\alpha_2}(R + |x|)^{(N-\alpha)q}} \, dx = R^{N-\alpha_2-(N-\alpha)q} \int_{B_1} \frac{1}{|y|^{\alpha_2}(1 + |y|)^{(N-\alpha)q}} \, dx \\
= R^{N-\alpha_2-(N-\alpha)q} \omega_N \int_0^1 t^{N-1-\alpha_2} \left( \frac{1}{1 + t} \right)^{(N-\alpha)q} \, dt.
\]

Now from the integral representation of the hypergeometric series
\[
_{2}F_{1}(a; b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-zt)^a} \, dt, \quad \text{for } c > b > 0, \ |z| < 1,
\] (29)
due to Euler (see [11]), by letting \( z \to -1 \), and choosing \( a = (N-\alpha)q, b = N-\alpha_2, c = b + 1 \) in (29), we have
\[
\int_{B_R} \frac{1}{|x|^{\alpha_2}(R + |x|)^{(N-\alpha)q}} \, dx = C_2 R^{N-\alpha_2-(N-\alpha)q}.
\]

Analogously we have
\[
\int_{B_R} \frac{1}{|x|^{\alpha_1}(R + |x|)^{(N-\alpha)p}} \, dx = C_1 R^{N-\alpha_1-(N-\alpha)p}.
\]

From (27) and (28) we obtain
\[
A \geq \mu_1^p C_2 R^{N-\alpha_2-(N-\alpha)q} B^q, \quad B \geq \mu_2^p C_1 R^{N-\alpha_1-(N-\alpha)p} A^p,
\]
which imply our claim. \(\square\)

**Lemma 2.5.** Let \( \alpha_1, \alpha_2 < N, \ 0 < \alpha < N, \ p, q > 0 \). Let \((u, v)\) be a solution of (22). Let \( \delta > 0 \) and \( K \geq 0 \) be constants such that \( v(x) \geq K \) for \( |x| < \delta \). Then for any \( |x| < \delta \) we have the following estimates
\[
u(x) \geq \mu_1 K^p \frac{\omega_N}{\delta} \frac{1}{\gamma_1}, \quad \text{(30)}\]
\[
u(x) \geq \mu_1 K^p \frac{\omega_N}{\delta} \frac{1}{\gamma_1}, \quad \text{(31)}\]
\[
u(x) \geq \mu_1 K^p \frac{\omega_N}{\delta} \frac{1}{\gamma_1}, \quad \text{(32)}\]

where \( \gamma_i := \omega_N \frac{_{2}F_{1}(N - \alpha_i, N - \alpha; N - \alpha_i + 1; -1)}{N - \alpha_i} \quad (i = 1, 2) \).

**Proof.** Let \( |x| < \delta \). Then, arguing as in the proof of Lemma 2.4
\[ u(x) \geq \mu_1 \int_{\mathbb{R}^N} \frac{v(y)^p}{|y|^{\alpha_1} |x - y|^{N-\alpha}} \, dy \geq \mu_1 \int_{|y| < \delta} \frac{K^p}{|y|^{\alpha_1} |x - y|^{N-\alpha}} \, dy \geq \mu_1 K^p \frac{\omega_N}{\delta} \int_0^1 \frac{t^{N-\alpha_1-1}}{(1 + t)^{N-\alpha}} \, dt = \mu_1 K^p \frac{\omega_N}{\delta} \frac{1}{\gamma_1}. \]
Now arguing as in the proof of Lemma 2.4 one recognizes that the constant \( \gamma_1 \) is the constant written in the statement. This proves (30).

Next, arguing as above, integrating on \(|y| < |x|\) instead of \(|y| < \delta\), we have

\[
u(x) \geq \mu_2 \int_{|y|<\delta} \frac{u(y)^q}{|y|^{\alpha_2}} \frac{K_{pq} \gamma_1}{\delta^{(\alpha_1-\alpha)q}} \int_{|y|<\delta} \frac{dy}{|y|^{\alpha_2} (|x| + |y|)^n-\alpha} \geq \mu_2 \mu_1 K_{pq} \gamma_1 \gamma_2 \frac{1}{\delta^{(\alpha_1-\alpha)q+\alpha_2-\alpha}},
\]

which is (31).

Now, using (30), for \(|x| < \delta\), we have

\[
u(x) \geq \mu_2 \int_{|y|<\delta} \frac{u(y)^q}{|y|^{\alpha_2}} \frac{K_{pq} \gamma_1}{\delta^{(\alpha_1-\alpha)q}} \int_{|y|<\delta} \frac{dy}{|y|^{\alpha_2} (|x| + |y|)^n-\alpha} \geq \mu_2 \mu_1 K_{pq} \gamma_1 \gamma_2 \frac{1}{\delta^{(\alpha_1-\alpha)q+\alpha_2-\alpha}}.
\]

This completes the proof.

2.1. The superlinear case \( pq > 1 \).

**Theorem 2.6.** Let \( \mu_1, \mu_2 > 0, \alpha_1, \alpha_2 < N, 0 < \alpha < N \) and \( pq > 1 \). If

\[
\max \left\{ \frac{q(\alpha-\alpha_1) + \alpha - \alpha_2}{pq-1}, \frac{p(\alpha-\alpha_2) + \alpha - \alpha_1}{pq-1} \right\} \geq N - \alpha
\]

then the problem (22) has no nontrivial solution.

The case \( \alpha_1, \alpha_2 \geq 0 \) has been studied in [4].

**Proof of Theorem 2.6.** Let \((u, v)\) be a non trivial solution of (22). Assume that

\[
\frac{q(\alpha-\alpha_1) + \alpha - \alpha_2}{pq-1} \geq N - \alpha
\]

or, since \( pq > 1 \), equivalently

\[
\theta := (N - \alpha)(1 - pq) + q(\alpha-\alpha_1) + \alpha - \alpha_2 \geq 0.
\]

From (25), we get

\[
\int_{B_R} \frac{u(y)^q}{|y|^{\alpha_2}} dy \leq cR^{-\frac{n}{n-\alpha}}.
\]

If \( \theta > 0 \), from (34) we deduce that \( u \equiv 0 \).

If \( \theta = 0 \), (34) implies that \( u(y)^q |y|^{-\alpha_2} \in L^1(\mathbb{R}^N) \). Now repeating the argument used in the proof of Lemma 2.4, that is, raising the first inequality of (22) to the power \( q \), multiplying by \(|x|^{-\alpha_2}\) integrating on the annulus \( A_R = B_{2R} \setminus B_R \), and using (28) we have

\[
\int_{A_R} \frac{u^q(y)}{|x|^{\alpha_2}} dy \geq c \left( \int_{B_R} \frac{u^q(y)}{|x|^{\alpha_2}} dy \right)^{pq}.
\]

Next, since \( u^q |y|^{-\alpha_2} \in L^1(\mathbb{R}^N) \) we have

\[
\lim_{R \to +\infty} \int_{A_R} \frac{u^q(y)}{|x|^{\alpha_2}} dy = 0,
\]

and this fact combined with (35), gives the claim.

We proceed similarly if

\[
\frac{p(\alpha-\alpha_2) + \alpha - \alpha_1}{pq-1} \geq N - \alpha
\]

holds.
Theorem 2.9. Let existence and argument holds for \( v \) defined in (39). Then the problem (22) has no nontrivial solution.

Proof of Theorem 2.9. Let \( u,v \) be a solution of (22). It is clear that if \( v \), or equivalently \( u \) has a local infimum equal to 0 then \( u \equiv 0 \) and \( v \equiv 0 \) (see Proposition 2.3).

In what follows for a given \( \delta \) we shall put \( K_v(\delta) := \inf_{|x|<\delta} v(x) \). Arguing by absurd assume that \( 0 < K_v(\delta) \) and \( \alpha_1 + \alpha_2 < N \leq N \).

Let \( 1 > \delta > 0 \). From (32) it follows that for any \( |x| < \delta \) we have,

\[
v(x) \geq \mu_1^2 \mu_2 (K_v(\delta))^{p \alpha q} \left( \frac{1}{\delta^{(n-\alpha_1)\alpha+\alpha_2}} \right)^{\frac{1}{\sigma}}.
\]
therefore,
\[ K_v(\delta) \geq \mu_1^q \mu_2^p (K_v(\delta))^{pq} \frac{1}{\delta(\alpha_1 - \alpha)q + \alpha_2 - \alpha}, \]
which, in turn, yields
\[ K_v(\delta) \leq \delta^{(\alpha_1 - \alpha)q + \alpha_2 - \alpha} (\mu_1^q \mu_2^p)^{\frac{1}{1 - pq}} \leq (\mu_1^q \mu_2^p)^{\frac{1}{1 - pq}}, \quad (41) \]
that is a uniform bound on \( K_v(\delta) \) for \( 0 < \delta < 1 \).

Next, set \( \epsilon := q(\alpha_1 - \alpha) + \alpha_2 - \alpha \geq 0 \). Since \( u \) satisfies the estimates (31) with \( K = K_v(\delta) \), we get
\[
v(x) \geq \mu_2 \int_{|y| < \delta} \frac{u(y)^q}{|x - y|^{N - \alpha}} dy
\geq \mu_2 \int_{|y| < \delta} \frac{K_v(\delta)^{pq}}{|y|^{(\alpha_1 - \alpha)q} |x|^{\alpha_2} |y|^{N - \alpha}}
\geq \mu_2 \int_{|y| < \delta} \frac{K_v(\delta)^{pq}}{|y|^{N + \epsilon} |x| + |y|^{N - \alpha}}.
\]
By Fatou Lemma we have
\[
\liminf_{x \to 0} v(x) \geq \mu_2 \int_{|y| < \delta} \liminf_{x \to 0} \frac{dy}{|y|^{\alpha + \epsilon} (|x| + |y|)^{N - \alpha}} = K_v(\delta)^{pq} \int_{|y| < \delta} \frac{dy}{|y|^{N + \epsilon}} = +\infty.
\]
This implies that \( \lim_{x \to 0} v(x) = +\infty \), hence a contradiction with the upper bound (41).

\[ \square \]

2.2. The sublinear case \( pq < 1 \). For systems which satisfy the natural “sublinear” assumption \( pq < 1 \) we have,

**Theorem 2.10.** Let \( \mu_1, \mu_2 > 0, \alpha_1, \alpha_2 < N, 0 < \alpha < N, p, q > 0 \) and assume that \( pq < 1 \). If
\[
\min \{ \alpha - \alpha_1 + p(\alpha - \alpha_2), \alpha - \alpha_2 + q(\alpha - \alpha_1) \} \leq (N - \alpha)(pq - 1), \quad (42)
\]
or
\[
\max \{ \alpha - \alpha_1 + p(\alpha - \alpha_2), \alpha - \alpha_2 + q(\alpha - \alpha_1) \} \geq 0, \quad (43)
\]
then the problem (22) has no nontrivial solution.

**Proof.** Assume that (42) holds. Without loss of generality we may suppose that
\[
\min \{ \alpha - \alpha_1 + p(\alpha - \alpha_2), \alpha - \alpha_2 + q(\alpha - \alpha_1) \} = \alpha - \alpha_2 + q(\alpha - \alpha_1) \leq (N - \alpha)(pq - 1), \quad (44)
\]
that is \( \theta := \alpha - \alpha_2 + q(\alpha - \alpha_1) + (N - \alpha)(1 - pq) \leq 0 \). Let \((u, v)\) be a non trivial solution of (22). From the estimate (25) of Lemma 2.4, we deduce that for any \( R \in (0, 1) \) we have,
\[
\left( \int_{B_R} \frac{u(x)^q}{|x|^{\alpha_2}} dx \right)^{1 - pq} \geq \mu_1^q \mu_2^p C_1^q C_2 R^\theta \geq \mu_1^q \mu_2^p C_1^q C_2.
\]
By letting \( R \to 0 \) in the above inequality we obtain,
\[
0 \geq \mu_1^q \mu_2^p C_1^q C_2.
\]
This completes the proof.
Next suppose that (43) holds and that without loss of generality
\[ \max\{\alpha - \alpha_1 + p(\alpha - \alpha_2), \alpha - \alpha_2 + q(\alpha - \alpha_1)\} = \alpha - \alpha_1 + p(\alpha - \alpha_2) \geq 0. \]

Let \((u, v)\) be a non trivial solution of (22). From the first inequality in (22) we know that
\[ u(x) \geq c (R + |x|)^{\alpha - N} \int_{B_R} \frac{v(y)^p}{|y|^q} \, dy, \]
and from (26) and \(R \geq |x|\),
\[ u(x) \geq c(R + |x|)^{\alpha - N} \frac{R^{(N - \alpha)(1 - pq) + p(\alpha - \alpha_2) + \alpha - \alpha_1}}{1 - pq} \geq c R^\theta, \]
where
\[ \theta := \frac{\alpha - \alpha_1 + p(\alpha - \alpha_2)}{1 - pq} \geq 0. \]

If \(\theta > 0\) we are done. In the remaining case \(\theta = 0\) we proceed as follows.

First we observe that from the above inequality it follows \(u(x) \geq c > 0\) for a.e. \(x \in \mathbb{R}^N\). Using this information in the second inequality in (22) we get
\[ v(x) \geq c \int_{\mathbb{R}^N} \frac{1}{|y|^{\alpha_2}|x - y|^{N - \alpha}} =: cw(x) \]
and the function \(w\) is defined finite for a.e. \(x \in \mathbb{R}^N\). By standard argument one recognizes that \(w\) is radially symmetric, and by an homogeneity argument we get \(w(x) = c|x|^{\alpha - \alpha_2}\) where \(c > 0\) is a suitable constant. This implies that \(v(x) \geq c|x|^{\alpha - \alpha_2}\). Inserting this estimate in the right hand side of the first inequality in (22), and taking into account that \(\theta = 0\), we see that
\[ u(x) \geq \mu_1 \int_{\mathbb{R}^N} \frac{c^p|y|^{(\alpha - \alpha_2)p}}{|y|^{\alpha_1}|x - y|^{N - \alpha}} \, dy = \mu_1 c^p \int_{\mathbb{R}^N} \frac{1}{|y|^{\alpha}|x - y|^{N - \alpha}} = +\infty. \]
This completes the proof.

The remaining case \(\max\{\alpha - \alpha_1 + p(\beta - \alpha_2), \beta - \alpha_2 + q(\alpha - \alpha_1)\} = \beta - \alpha_2 + q(\alpha - \alpha_1) \geq 0\), can be handled in a similar way. We omit the details. \(\square\)

**Theorem 2.11.** Let \(\mu_1, \mu_2 > 0, \alpha_1, \alpha_2 < N, 0 < \alpha < N, p, q > 0\) such that \(pq < 1\). If (42) and (44) are not satisfied then (36) has nontrivial solution. In particular the functions defined in (39) are solution of (36).

**Proof.** The proof follows the same pattern of the proof of Theorem 2.7. \(\square\)

The above Theorems 2.6, 2.7, 2.9, 2.10, 2.11, can be summarized in the following,

**Theorem 2.12.** Let \(\mu_1, \mu_2 > 0, \alpha_1, \alpha_2 < N, 0 < \alpha < N, p, q > 0\) such that \(pq \neq 1\). The problem (22) has nontrivial solution if and only if
\[ N - \alpha > \max \left\{ \frac{\alpha - \alpha_1 + p(\alpha - \alpha_2)}{pq - 1}, \frac{\alpha - \alpha_2 + q(\alpha - \alpha_1)}{pq - 1} \right\} \geq \min \left\{ \frac{\alpha - \alpha_1 + p(\alpha - \alpha_2)}{pq - 1}, \frac{\alpha - \alpha_2 + q(\alpha - \alpha_1)}{pq - 1} \right\} > 0. \quad (45) \]
Moreover if (45) holds then even the system of equations (36) has nontrivial solution.

In the scalar case the above theorem reads as
Corollary 2.13. Let $\mu > 0$, $\alpha_1 < N$, $0 < \alpha < N$, $p > 0$ such that $p \neq 1$. The problem
\[
 u(x) \geq \mu \int_{\mathbb{R}^N} \frac{u(y)^p}{|y|^{\alpha_1}|x-y|^{N-\alpha}} \, dy \quad \text{a.e. } x \in \mathbb{R}^N, \tag{46}
\]
has nontrivial solution if and only if
\[
 N - \alpha > \frac{\alpha - \alpha_1}{p-1} > 0 \quad \text{(47)}
\]
Moreover if (47) holds then even the equation
\[
 u(x) = \mu \int_{\mathbb{R}^N} \frac{u(y)^p}{|y|^{\alpha_1}|x-y|^{N-\alpha}} \, dy \quad \text{a.e. } x \in \mathbb{R}^N, \tag{48}
\]
has nontrivial solution.

For a general result concerning equations of the form (48) in the case $\alpha = 2$ see [1].

2.3. The linear case $pq = 1$.

Theorem 2.14. Let $\mu_1, \mu_2 > 0$, $\alpha_1, \alpha_2 < N$, $0 < \alpha < N$, $p, q > 0$ and assume that $pq = 1$.

Necessary conditions for nontrivial nonnegative solutions of (22) to exist are given by
\begin{enumerate}
  \item[(i)] $\alpha - \alpha_1 + p(\alpha - \alpha_2) = 0$,
  \item[(ii)] $\alpha - \alpha_2 + q(\alpha - \alpha_1) = 0$.
\end{enumerate}

Moreover, if (i) and (ii) hold then there exist two positive constants $\lambda^* \geq \lambda_* > 0$ such that
\begin{enumerate}
  \item[1.] if $\mu_1 \mu_2 > \lambda^*$ then the system of inequalities (22) does not have nontrivial solution.
  \item[2.] if $\mu_1 \mu_2 < \lambda_*$ then the system of equations (36) has nontrivial solution.
\end{enumerate}

Moreover the following estimate on $\lambda^*$ and $\lambda_*$ hold.
\[
 \lambda^* \leq \min \left\{ \frac{1}{C_1 C_2 p}, \frac{1}{\gamma_1 \gamma_2} \right\} \tag{49}
\]
where the constants $C_1, C_2, \gamma_1, \gamma_2$ are defined in Lemma 2.4;
\[
 \lambda_* \geq \sup \left\{ f(x)f(qx + \alpha_2 - \alpha), \, x \in [a, b] \right\} \tag{50}
\]
with $a := \max\{0, (\alpha - \alpha_2)/q\}$, $b := \min\{N - \alpha, (N - \alpha_2)/q\}$ and $f$ is defined as
\[
 f(x) := \frac{1}{\omega_N B(\alpha/2, N/2)} \frac{B(\frac{N-x}{2}, \frac{\alpha+x}{2})}{B(\frac{\alpha}{2}, \frac{N-\alpha-x}{2})} \tag{51}
\]
where $B$ denotes the Beta function.

Proof. Let $(u, v)$ be a nontrivial solution of (22). From (25) we have
\[
 1 \geq \mu_1^q \mu_2^p C_1 C_2 R^{\alpha - \alpha_2 + q(\alpha - \alpha_1)}. \tag{52}
\]
Taking the limit as $R \to +\infty$ and $R \to 0^+$ in (52) we get that necessarily (ii) holds. Similarly, from (26) we obtain (i).
In order to prove the claim in 1. we argue as follow. Let \((u,v)\) be a nontrivial solution of \((22)\). Since \((52)\) and (ii) hold, we obtain \(\mu_1^p \mu_2 \leq (C_1^p C_2)^{-1}\), which, by raise to power \(p\), is equivalent to
\[
\mu_1^p \mu_2 \leq (C_1^p C_2)^{-1}.
\]
(53)
For \(R > 0\), set \(K(R) := \inf_{B_R} v\). From (32) we have
\[
K(R) \geq \mu_1^p \mu_2 K(R)^{pq} \frac{1}{\delta(\alpha_1 - \alpha)q + \alpha_2 - \alpha},
\]
which can be written as
\[
\mu_1^p \mu_2 \leq (\gamma_1^q \gamma_2)^{-1},
\]
(54)
since \(pq = 1\) and (ii) holds. Raising to power \(p\) the inequality \((54)\) and from \((53)\) we have that necessarily
\[
\mu_1^p \mu_2 \leq \min \left\{(C_1^p C_2)^{-1}, (\gamma_1^p \gamma_2)^{-1}\right\}.
\]
This concludes the proof of point 1. and of the estimate \((49)\).

In order to prove the claim in 2. and the estimate \((50)\), we shall use the Selberg relation \((40)\) and we shall look for solutions of \((36)\) of the form
\[
u(x) := c_1 |x|^{-\gamma}, \quad v(x) := c_2 |x|^{-\delta}.
\]
(55)
Let \(F(x) := f(x)f(qx + \alpha_2 - \alpha)p, S := \sup \{F(x), x \in [a,b]\}\) and \(\mu_1 \mu_2 < S\). It is easy to recognize that \(a < b\) and the function \(F\) is positive and continuous on \([a,b]\).
Therefore the set \(I := F([a,b])\) is an interval and we claim that \([0,S[ \subset I \subset [0,S]\).
Indeed, since \(f(x) \to 0\) as \(x \to 0\) or \(x \to N - \alpha\), we get that \(F(x) \to 0\) as \(x \to a\) or \(x \to b\).
Therefore there exists \(\gamma \in [a,b]\) such that \(F(\gamma) = \mu_1 \mu_2^p\). Now, fixed \(c_2 > 0\), we set
\[
\delta := \gamma q + \alpha_2 - \alpha, \quad c_1 := \mu_1 c_2^p \frac{c(N - \gamma)}{c(N - \gamma - \alpha)c(\alpha)}.
\]
The functions defined in \((55)\) with these choices are solutions of \((36)\). Indeed plugging \(u\) and \(v\) in \((36)\) one obtains
\[
\left\{
\begin{array}{l}
\frac{c_1}{|x|^{-\gamma}} = \frac{c_2^p}{|x|^{-\delta p + \alpha_2}} \frac{1}{|x|^{-\alpha}}(x) = \mu_1 c_2^p \frac{c(N - \delta p + \alpha_1 + \alpha)}{c(N - \delta p - \alpha_1 + \alpha)c(\alpha)} \frac{1}{|x|^{-\delta p + \alpha_2 + \alpha}} \\
\frac{c_2}{|x|^{-\delta}} = \frac{c_1}{|x|^{-\gamma q + \alpha_2}} \frac{1}{|x|^{-\alpha}}(x) = \mu_2 c_2^q \frac{c(N - \gamma q + \alpha_2 + \alpha)}{c(N - \gamma q - \alpha_2 + \alpha)c(\alpha)} \frac{1}{|x|^{-\gamma q + \alpha_2 - \alpha}}
\end{array}
\right.
\]
(56)
By (i) and (ii), it is easy to see that the first equation is satisfied. In order to check that the second equation in \((56)\) is fulfilled it suffices to take into account the definition of \(c(\alpha)\) and the relation \(B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\).

In the scalar case the above results reads as follows.

**Corollary 2.15.** Let \(\mu > 0, \alpha_1 < N, 0 < \alpha < N\).

If there exists a nontrivial nonnegative solutions of \((46)\) with \(p = 1\). Then \(\alpha_1 = \alpha\).
Moreover, if \(\alpha_1 = \alpha\), then there exist two positive constants \(\lambda^* \geq \lambda_* > 0\) such that

1. if \(\mu > \lambda^*\) then the inequalities \((46)\) with \(p = 1\) does not have nontrivial solution.
2. if $\mu < \lambda_*$ then the equation (48) with $p = 1$ has nontrivial solution. Moreover the following estimates on $\lambda^*$ and $\lambda_*$ hold

$$\lambda^* \leq \frac{N - \alpha}{\omega_N \ 2F_1(N - \alpha, N - \alpha; N - \alpha + 1; -1)};$$

$$\lambda_* \geq \sup \left\{ f(x), \ x \in [0, N - \alpha[ \right\},$$

with $f$ defined in (51).

3. Applications to systems of differential inequalities. Finally, consider the following (possibly) singular system.

$$\begin{cases} (-\Delta)^m u \geq \lambda_1 \frac{|v|^p}{|x|^{\alpha_1}} & \text{on } \mathbb{R}^N, \\ (-\Delta)^m v \geq \lambda_2 \frac{|u|^q}{|x|^{\alpha_2}} & \text{on } \mathbb{R}^N, \end{cases}$$

(59)

where $\lambda_1, \lambda_2 > 0$, $N > \alpha_1, \alpha_2$, $N > 2m$ and $p, q > 0$.

Here, by a solution of (59) we mean a pair of functions $(u, v) \in L^1_{\text{loc}}(\mathbb{R}^N) \times L^1_{\text{loc}}(\mathbb{R}^N)$ such that $v^p \frac{\cdot}{|x|^{\alpha_1}} \in L^1_{\text{loc}}(\mathbb{R}^N)$ and $u^q \frac{\cdot}{|x|^{\alpha_2}} \in L^1_{\text{loc}}(\mathbb{R}^N)$ and satisfies the system of inequalities (59) in the distributional sense.

If the solutions of (59) are also solutions of the system (22) then, by applying the results of [4] we obtain Liouville’s type theorems of (22). In [4] it is proved that if (here $w$ denotes a component of the possible solution of (59)),

$$\liminf_{R \to +\infty} \int_{R < |y-x| < 2R} |w(y)| \ dy = 0 \quad \text{for a.e. } x \in \mathbb{R}^N$$

(60)

then the solutions of (59) solve the system of integral inequalities (22).

More precisely we have the following.

**Theorem 3.1.** Let $(u, v)$ be solution of (59). If one of the following assumptions holds,

1. $u$ and $v$ satisfy the ring condition (60);
2. $u$ and $v$ are weakly superharmonic and nonnegative;
3. $p, q \geq 1$, $pq \geq 1$, $2m - \alpha_1 + (2m - \alpha_2)p > 0$, $2m - \alpha_2 + (2m - \alpha_1)q > 0$;
4. $p, q \geq 1$, $pq > 1$, $2m > \alpha_1, 2m > \alpha_2$;
5. $v$ satisfies (60), $q \geq 1$ and $2m \geq \alpha_2$;
6. $u$ satisfies (60), $p \geq 1$ and $2m \geq \alpha_1$;

then $(u, v)$ solves (22) with $\alpha := 2m$, $\mu_i := C(2m)\lambda_i$ ($i = 1, 2$) and $C(2m) := \Gamma\left(\frac{N - 2m}{2}\right)(2^{2m} \pi^{N/2} \Gamma(m))^{-1}$.

Moreover if $(u, v)$ solves the system of equations in (59), that is

$$\begin{cases} (-\Delta)^m u = \lambda_1 \frac{|v|^p}{|x|^{\alpha_1}} & \text{on } \mathbb{R}^N, \\ (-\Delta)^m v = \lambda_2 \frac{|u|^q}{|x|^{\alpha_2}} & \text{on } \mathbb{R}^N, \end{cases}$$

(61)

and one of the assumptions 1., 3., 4., 5., or 6. holds then $(u, v)$ solves even the system of integral equations (36).
Proof. 1. The claim is a direct consequence of Theorem 2.4 of [4].

2. Let \( l := \inf u \). Since \( l \geq 0 \) the representation of \( u \) follows from Theorem 2.4 and Remark 2.7 in [4]. Analogously for \( v \).

3. We shall prove that the case 1. occurs. Fix \( x_0 \in \mathbb{R}^N \) and for \( R > 0 \) we set \( B_R := B_R(x_0) \setminus B_R(x_0) \) and \( \phi_R(x) := \phi_1(\|x - x_0\|/R) \) with \( \phi_1 \) a standard \( C_0^\infty \) cut off function \( (0 \leq \phi_1 \leq 1, \phi_1(t) = 1 \) for \( t < 1 \) and \( \phi_1(t) = 0 \) for \( t > 2 \)).

From now on we choose \( R > 2 \|x_0\| \). With this choice it follows that if \( x \in A_R \) then \( R/2 < \|x\| < 3R \).

We claim that
\[
\int \frac{|u|^q}{|x|^q} \phi_R \leq cR^{q + \frac{N}{p} - 2m} \left( \int_{A_R} \frac{|v|^p}{|x|^p} \phi_R \right)^{1/p}, \tag{62}
\]
where if \( p > 1 \), then \( p' := p/(p - 1) \) denotes the conjugate exponent of \( p \); while if \( p = 1 \) we agree that \( 1/p' = 0 \). Indeed, by definition of weak solution, we have
\[
\int \frac{|u|^q}{|x|^q} \phi_R \leq \int v(\Delta)^m \phi_R \leq cR^{q + \frac{N}{p' - 2m}} \int_{A_R} \frac{|v|^p}{|x|^p} \phi_R^{1/p} (\Delta)^m \phi_R. \tag{63}
\]

By standard argument (i.e. by using Hölder inequality if \( p > 1 \), and eventually by replacing \( \phi_1 \) with \( \phi_1^2 \) and \( \gamma \) large enough; see for instance [20] for more details) we get (62).

Analogously we have
\[
\int \frac{|v|^p}{|x|^p} \phi_R \leq cR^{q + \frac{N}{p'} - 2m} \left( \int_{A_R} \frac{|u|^q}{|x|^q} \phi_R \right)^{1/q}. \tag{64}
\]

Gluing together (62) and (64) we obtain
\[
\left( \int \frac{|u|^q}{|x|^q} \phi_R \right)^{1 - \frac{q}{q'}} \leq cR^a, \text{ with } a := \frac{\alpha_1}{p} + \frac{N}{p'} - 2m + \frac{1}{p} \left( \frac{\alpha_2}{q} + \frac{N}{q'} - 2m \right). \tag{65}
\]

In order to prove (60) we have
\[
\int_{A_R} |u| \leq \left( \int_{A_R} |u|^q \right)^{1/q} \leq c \left( \frac{R^{\alpha_2}}{R^N} \int_{A_R} \frac{|u|^q}{|x|^q} \right)^{1/q} \leq cR^{\frac{\alpha_2 - N}{q} - \frac{pq}{pq - 1}} a \leq cR^b,
\]
where
\[
b := \frac{\alpha_2 - N}{q} + \frac{1}{q} \frac{pq}{pq - 1} a = \frac{\alpha_1 - 2m + (\alpha_2 - 2m)p}{pq - 1} < 0,
\]
and then \( u \) satisfies (60). Similarly one obtains that also \( v \) satisfies (60).

4. The claim follows directly from 3.

5. Arguing as in the proof of point 3., from (63) we have
\[
\int_{A_{R/2}} |u|^q \leq \int_{B_R} |u|^q |x|^{q} \phi_R \leq \int_{A_R} v(\Delta)^m \phi_R \leq cR^{-2m} \int_{A_R} |v|
\]
that is
\[
\int_{A_R} |u|^q \leq cR^{\alpha_2 - 2m} \int_{A_{2R}} |v| \leq c \int_{A_{2R}} |v|
\]
which in turn implies the claim since \( v \) satisfies (60) and \( q \geq 1 \).

6. This claim is the dual of 5. and it follows similarly.

Looking at the proof of point 3 of the above theorem, we realize that the following holds.

**Lemma 3.2.** Let \((u, v)\) be non negative solution of (59). Assume that \( p, q \geq 1, \ pq > 1, 2m - \alpha_1 + (2m - \alpha_2)p > 0 \).

Then \( u \) solves the first inequality in (22) with \( \alpha := 2m, \mu_1 := C(2m)\lambda_1 \). Moreover if \( u \) solves also the first equation in (61) then \( u \) solves the first equation in (36) as well.

**Remark 3.3.** The assumptions \( pq > 1 \) and \( 2m - \alpha_1 + (2m - \alpha_2)p > 0, 2m - \alpha_2 + (2m - \alpha_1)q > 0 \) in condition 3. of the above theorem are sharp. Indeed, if they are not satisfied, then there exist solutions of (59) that cannot be represented in the integral form. Moreover these solutions show also the presence of two different phenomena. If \( pq = 1 \), then Theorem 2.14 holds true. While, for the differential case (59) there exist solutions even if (i) and (ii) are not satisfied. Furthermore, there exist solutions of (59) for large value of \( \lambda_1 \lambda_2^p \).

A simple computation shows that the functions \( u(x) = v(x) = e^{|x|^\alpha} \) are components of a solution of the system,

\[
\begin{align*}
\Delta^2 u &= \lambda_1 \frac{r^p}{|x|^{\alpha + \alpha}} \quad \text{on } \mathbb{R}^N, \\
\Delta^2 v &= \lambda_2 \frac{u^q}{|x|^{\alpha + \alpha}} \quad \text{on } \mathbb{R}^N,
\end{align*}
\]

for \( \lambda_1 = \lambda_2 = 1, \alpha_1 = \alpha_2 = 0 \) and \( p = q = 1 \).

Clearly this solution does not solves the associated integral system.

On the other hand assume that \( pq = 1 \), (i) and (ii) of Theorem 2.14 hold and that \( \lambda_1 \lambda_2^p \) is large enough so that the problem

\[ r(\gamma) r^p(\gamma + \alpha_1 - 4) = \lambda_1 \lambda_2^p, \quad \gamma > \max\{2, 2/q + 4 - \alpha_1\}, \]

admits a solution \( \gamma \), where \( r(t) := t(t-2)(t+N-2)(t+N-4) \). Next, by choosing \( u(x) := c_1 |x|^{\gamma} \) and \( v(x) := c_2 |x|^{q(\gamma + \alpha_1 - 4)} \), where \( c_1 \) and \( c_2 \) are suitable positive constants, we see that \((u, v)\) is a solution of the system (66). We leave the details to the interested reader.

Also in this case this function does not solves the associated integral system. The reason relies on the fact that the necessary condition of Proposition 2.3 is violated.

Next suppose that \( pq \neq 1 \).

We set \( \gamma := -\frac{4 - \alpha_1 + p(4 - \alpha_2)}{pq - 1}, \delta := \frac{4 - \alpha_2 + q(4 - \alpha_1)}{pq - 1} \) and assume that \( \gamma > 2 \) and \( \delta > 2 \). The functions \( u(x) := c_1 |x|^\gamma, v(x) := c_2 |x|^\delta \) are the nonnegative components of a solution of (66). Again, the pair \((u, v)\) is not a solution of the associated integral system (see Proposition 2.3 or Theorem 2.12).

A converse of Theorem 3.1 is the following.

**Theorem 3.4.** Let \((u, v)\) be a solution of the system of integral equations (36). If \( u, v \in L^1_{loc}(\mathbb{R}^N) \), then \((u, v)\) is a distributional solution and solves the system of differential equations (61). Moreover
1. $u$ and $v$ are weakly polysuperharmonic that is
\[ (-\Delta)^k u \geq 0, \quad (-\Delta)^k v \geq 0 \] in distributional sense for $k = 0, 1, \ldots, m - 1$;
2. $\inf u = \inf v = 0$,
3. $u$ and $v$ satisfy (60).

As a consequence of the above results we have,

**Theorem 3.5.** Let $(u, v)$ be a nontrivial distributional solution of (59) and assume that one of the conditions 1. - 6. of Theorem 3.1 is fulfilled.
If $pq \neq 1$, then necessarily (45) holds with $\alpha = 2m$.
If $pq = 1$, then
\[ q(2m - \alpha_1) + 2m - \alpha_2 = 0, \quad p(2m - \alpha_2) + 2m - \alpha_1 = 0, \]
\[ \lambda_1 \lambda_2^p < \lambda_*, \]
where $\lambda_* = \lambda_*(N, p, q, 2m, \alpha_1, \alpha_2) > 0$ is a universal constant independent on $u$ and $v$.

**Corollary 3.6.** (Hénon-Lane-Emden System). Let $\lambda_1, \lambda_2 > 0$, $N > \alpha_1, \alpha_2$, $N > 2m$, $p, q > 0$ and consider the problem
\[
\begin{cases}
(-\Delta)^m u = \lambda_1 \frac{|v|^p}{|x|^{\alpha_1}}, & -\Delta u \geq 0, \quad \text{on } \mathbb{R}^N, \\
(-\Delta)^m v = \lambda_2 \frac{|u|^q}{|x|^{\alpha_2}}, & -\Delta v \geq 0, \quad \text{on } \mathbb{R}^N, \\
\inf u = 0, & \inf v = 0.
\end{cases}
\]

If $pq \neq 1$ then problem (69) has nontrivial distributional solution if and only if (45) holds with $\alpha = 2m$.
If $pq = 1$ and (69) has nontrivial distributional solution then necessarily (67) holds.
If $pq = 1$ and (67) holds then there exist two positive constants $\lambda^* \geq \lambda_* > 0$ such that
1. If $\lambda_1 \lambda_2^p > \lambda^*$ then (69) does not have nontrivial distributional solution.
2. If $\lambda_1 \lambda_2^p < \lambda^*$ then (69) has nontrivial distributional solution.

The proof of the Liouville theorems are a direct consequences of the representation Theorem 3.1. The existence of distributional solutions is a consequence of Theorems 3.5, 2.12, 2.14 and the observation that the solutions provided in those theorems for the integral equations are actually in $L^1_{\text{loc}}(\mathbb{R}^N)$. We leave the details to the interested reader.

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