Stability Analysis of Axial Reflection Symmetric Spacetime

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ABSTRACT

In this paper, we explore instability regions of non-static axial reflection symmetric spacetime with anisotropic source in the interior. We impose linear perturbation on the Einstein field equations and dynamical equations to establish the collapse equation. The effects of different physical factors like energy density and anisotropic stresses on the instability regions are studied under Newtonian and post-Newtonian limits. We conclude that stiffness parameter has a significant role in this analysis while the reflection terms increase instability ranges of non-static axial collapse.

Subject headings: Axial symmetry; Relativistic fluids; Stability

1. Introduction

Self-gravitating objects pass through different intense phases of dynamical activities during the evolution of the star model. Anisotropy cannot be ignored in the study of rotating stars which is closely related with axial symmetry. However, there is no exterior metric which coincides with the sources of such interior. It is noted that in conventional celestial objects radial and tangential pressures exist instead of purely isotropic fluids. The theoretical advances indicate that such objects in which density ranges upto $\mu < 10^{15} \text{gcm}^{-3}$ will be anisotropic (Ruderman 1972; Canuto 1973). Banerjee & Sanyal (1996) classified spatially homogeneous axially symmetric spacetimes for an imperfect fluid configuration by keeping constant ratio between shear and expansion.

Herrera et al. (1979) studied an approach to examine the slow adiabatic contraction of anisotropic spheres. They related the radial and tangential pressures by a quadratic law in

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radial coordinate, defining in this way an anisotropic law (or second equation of state). Rago (1991) explored solutions of the field equations depending upon two arbitrary functions, i.e., anisotropic and generating functions which measure the degree of anisotropy and relevant physical quantities. The solutions are then matched with the Schwarzschild exterior metric. Dev & Gleiser (2002) studied the effects of anisotropic pressure on the properties of gravitationally bound spherically symmetric object. They found that anisotropic pressure can have significant effects on the structure and properties of stellar objects. Particularly, the anisotropy can change the critical mass and surface redshift of the equilibrium configurations. Mak & Harko (2002) presented an exact analytical treatment to the field equations describing spherically symmetric anisotropic matter configuration. They concluded that anisotropic pressure (with radial pressure obeying linear equation of state) increases the maximum radius and mass of the quark star, which in their case is around three solar masses. Dev & Gleiser (2003) have extended the formalism developed by Chandrasekhar to discuss the significance of anisotropic pressure on the stability of spherical objects against radial perturbations in the scenario of Newtonian gravity and general relativity. They have also discussed this formalism to study anisotropic spheres with constant energy density and energy densities of $\frac{1}{r^2}$ profile. Recently, Mahmood et al. (2015) explored spherical collapse and expansion of anisotropic cylindrically self-gravitating systems with charged background.

Chaisi & Maharaj (2005) explored a class of exact solutions for anisotropic spherical stars with a physically reasonable form of energy density. These solutions help to describe anisotropic nature of compact objects under strong gravitational fields. Hossein et al. (2012) discussed the formation of anisotropic compact star with variable cosmological constant and checked all the regularity conditions as well as stability of their model. Sharif & Bhatti (2014a,b) investigated the role of different physical factors including anisotropic pressure on distinct star models.

The stability of self-gravitating stars is an important issue as only stable equilibrium models are viable. A general relativistic treatment is required for a precise evaluation of instability regimes. Stability analysis of self-gravitating stars have been performed by several authors since the pioneering work of Chandrasekhar (1964). The Newtonian (N) and post-Newtonian (pN) approximations have mainly been used to investigate the structure of a rotating star in the framework of general relativity (Arutyunyan et al. 1971). Chandrasekhar & Friedman (1972a) studied general relativistic treatment for the stability of axisymmetric spacetime subject to the radial perturbation along sequence of rotating stars. Vilenkin & Ford (1982) investigated gravitational effects for a specific model and showed that behavior of the system can be drastically changed due to spacetime curvature leading to stability or instability. Barrow & Ottewill (1983) explored the existence and sta-
bility of isotropic homogeneous star subject to perturbations in \( f(R) \) gravity. \cite{Chan1993,Chan1994} discussed stability analysis via perturbation without using equation of state in anisotropic stellar interior whose results with astrophysical relevance has also been studied \cite{Dev2003}. Recently, stability analysis for spherical \cite{Sharif2014a} as well as cylindrical \cite{Sharif2014b} configurations are performed by using radial perturbation in the framework of \( f(R) \) gravity in which they concluded that instability ranges depend only on material variables with zero expansion independent of the fact that how much the fluid is stiff. However, it shows dependence on the stiffness parameter in the presence of expansion scalar.

\cite{Herrera2012} showed that instability range is independent of fluid stiffness with zero expansion which is compatible with the study of Tolman mass. \cite{Babichev2013} studied instability of black holes in massive gravity theory and concluded that linear perturbation around the simplest black hole leads to unstable mode. \cite{Roupaas2013} found a connection between gravity and thermodynamics and discussed stability properties under the influence of cosmological constant. We have explored some instability regions for self-gravitating fluids with and without expansion-free condition by imposing linear perturbation \cite{Sharif2014cd}. \cite{Khamesra2015} used the gauge-invariant perturbation theory to discuss stability of spherically symmetric spacetime with anisotropic fluid under axial perturbation. Recently, \cite{Abbas2014} discussed dynamical properties of commutative black hole and found its total energy.

\cite{Vajk1970} systematically derived three classes of axially symmetric spatially homogeneous spacetime for an ideal fluid using equation of state. \cite{Rao2013} solved the field equations by using anisotropic features of the universe with axially symmetric spacetime and showed that their solution represent evolution of the early universe. The dynamical stability of rotating or axial reflection symmetric stars against linear perturbations as well as the final fate of collapse has not been established definitively. A number of interesting attempts in this direction have already been made with a restricted class in a modified gravity \cite{Sharif2014d,Sharif2015}.

The main idea of this work is to present an analytic treatment in spirit of Chandrasekhar’s work \cite{Chandrasekhar1964} to identify the instability eras of non-static axial geometry with reflection symmetry filled with anisotropic matter. The paper is outlined as follows. In the next section, we present some basic equations for axial spacetime which are used to develop our analysis. Section 3 provides perturbation scheme to all equations obtained in section 2 and consequently derives the collapse equation. In section 4, we identify the instability regions with \( N \) and \( pN \) limits from the collapse equation. In the last section, we summarize our results.
2. Anisotropic Source and Conservation Laws

We take a non-static axial spacetime with reflection symmetry in spherical coordinates given by (Herrera et al. 2014a, b)

\[ ds^2 = -A^2(t, r, \theta)dt^2 + B^2(t, r, \theta)(dr^2 + r^2d\theta^2) + 2G(t, r, \theta)dtd\theta + C^2(t, r, \theta)d\phi^2. \] (1)

It excludes explicitly the term representing rotation in the geometry to avoid the complications in the calculations. We consider anisotropic fluid whose energy-momentum tensor is given as

\[ T^{(m)}_{\alpha\beta} = (\mu + P)V_\alpha V_\beta + Pg_{\alpha\beta} + \Pi_{\alpha\beta}, \] (2)

where

\[ \Pi_{\alpha\beta} = \frac{1}{3}(\Pi_I + 2\Pi_{II})(K_\alpha K_\beta - \frac{1}{3}h_{\alpha\beta}) + \frac{1}{3}(\Pi_I + 2\Pi_{II})(L_\alpha L_\beta - \frac{1}{3}h_{\alpha\beta}) \]

\[ + \Pi_{KL}(K_\alpha L_\beta + K_\beta L_\alpha), \]

which includes

\[ h_{\alpha\beta} = g_{\alpha\beta} + V_\alpha V_\beta, \quad \Pi_{KL} = K_\alpha L_\beta T_{\alpha\beta}, \quad \Pi_I = (2K_\alpha K_\beta - S_\alpha S_\beta - L_\alpha L_\beta)T_{\alpha\beta}, \]

\[ \Pi_{II} = (2L_\alpha L_\beta - K_\alpha K_\beta - S_\alpha S_\beta)T_{\alpha\beta}, \]

where \( \mu, \Pi_{\alpha\beta} \) and \( h_{\alpha\beta} \) are the energy density, anisotropic stress tensor and projection tensor, respectively. Also, \( P \) is isotropic pressure and \( \Pi_I \neq \Pi_{II} \neq \Pi_{KL} \) indicates anisotropic contribution of stress tensor. Moreover, \( S_\alpha, L_\alpha, K_\alpha \) and \( V_\alpha \) are unit four-vectors and four-velocity, respectively while \( \alpha, \beta \) represent Lorentz indices. In comoving coordinate system, one can assume these four-vectors as

\[ S_\alpha = C_\delta^\alpha, \quad L_\alpha = \frac{\sqrt{r^2A^2B^2 + G^2}}{A} \delta_\alpha^1, \quad K_\alpha = B_\delta_\alpha^1, \quad V_\alpha = -A_\delta_\alpha^0 + \frac{G}{A} \delta_\alpha^2, \] (3)

satisfying

\[ K_\alpha L_\alpha = S_\alpha L_\alpha = K_\alpha S_\alpha = V_\alpha S_\alpha = V_\alpha K_\alpha = V_\alpha L_\alpha = 0, \]

\[ K_\alpha K_\alpha = S_\alpha S_\alpha = -V_\alpha V_\alpha = L_\alpha L_\alpha = 1. \]

Using these unit four-vectors, the non-zero components of Eq.(2) become

\[ T_{00} = \mu A^2, \quad T_{02} = -\mu G, \quad T_{11} = \left(P + \frac{1}{3}\Pi_I\right)B^2, \]

\[ T_{22} = \mu \frac{G^2}{A^2} + \left(\frac{r^2A^2B^2 + G^2}{A^2}\right)\left(P + \frac{1}{3}\Pi_{II}\right), \]
\[ T_{12} = \Pi_{KL} \left( \frac{B}{A}\sqrt{r^2A^2B^2 + G^2} \right), \quad T_{33} = \left[ P - \frac{1}{3} (\Pi_I + \Pi_{II}) \right] C^2. \] (4)

In order to describe dynamical nature of any self-gravitating system, the conservation law, \( T^{\alpha\beta} \_\beta = 0 \), has a crucial role which from Eqs. (2) and (4) for \( \alpha = 0, 1 \) yield the following couple of equations

\[
\dot{\mu} - \mu \left[ \frac{\dot{B}}{B} + \frac{\dot{C}}{C} + \frac{1}{r^2A^2B^2 + G^2} \left\{ \frac{r^2A\dot{A}B^2}{G} + G\dot{G} + \frac{r^2A^2B\dot{B}}{G} \right\} \right] + (\mu + P) \\
\times \left[ \frac{A^2B^2}{r^2A^2B^2 + G^2} \left\{ \frac{2\dot{B}}{B} + \frac{\dot{C}}{C} + \frac{G^2}{A^2B^2} \left( \frac{\dot{B}}{B} + \frac{\dot{C}}{C} - \frac{\dot{A}}{A} + \frac{\dot{C}}{C} \right) \right\} + \frac{\Pi_{II}}{3} \\
\times \left( \frac{\dot{B}}{B} - \frac{\dot{C}}{C} \right) + \frac{\Pi_{III}}{r^2A^2B^2 + G^2} \left[ \frac{r^2A^2B^2}{B - \frac{\dot{C}}{C} + G^2} \left( \frac{\dot{B}}{B} - \frac{\dot{C}}{C} \right) + G^2 \left( \frac{\dot{G}}{G} - \frac{\dot{A}}{A} - \frac{\dot{C}}{C} \right) \right] \right] = 0, \tag{5}
\]

\[
P' + \frac{2}{9} (2\Pi_I' + \Pi_{II}') + \left[ P + \frac{2}{9} (2\Pi_I + \Pi_{II}) \right] \left\{ \frac{C''}{C} + \frac{3GG'}{2} + \frac{r^2A^2B^2}{r^2A^2B^2 + G^2} \right\} \\
\times \left( \frac{A'}{A} + \frac{2B'}{B} + \frac{2}{r} - \frac{(rB')'}{rB} \right) - \frac{r^2AB^5}{(r^2A^2B^2 + G^2)^{\frac{3}{2}}} \Pi_{KL,\theta} - \frac{r^2AB^5}{(r^2A^2B^2 + G^2)^{\frac{3}{2}}} \Pi_{KL} \\
\times \left\{ \frac{A_\theta}{A} + \frac{6B_\theta}{B} + \frac{C_\theta}{C} + \frac{4GG_\theta}{G^2} + \frac{4r^2A^2B^2}{r^2A^2B^2 + G^2} \left( \frac{A_\theta}{A} + \frac{B_\theta}{B} \right) \right\} \Pi_{KL} \\
+ \frac{\mu r^4A^4B^4}{(r^2A^2B^2 + G^2)^2} \left( B\dot{B} + \frac{A'}{A} - \frac{G A_\theta}{r^2AB^2} \right) - \frac{\mu r^2A^2G^2B^2}{(r^2A^2B^2 + G^2)^2} \left( \frac{G'}{2G} + \frac{(rB')'}{rB} \right) \right] = 0, \tag{6}
\]

where dot and prime stand for differentiation with respect to \( t \) and \( r \) while subscript \( \theta \) indicates \( \theta \)-differentiation.

### 3. Perturbation Approach

Here we use perturbation technique to obtain perturbed form of all the previous equations up to first order keeping the perturbation parameter \( \varepsilon \) in the interval \( (0,1) \). The initial configuration of the system is considered to be static while after perturbation it enters into non-static phase with the same time dependence of metric coefficients. Consequently, the metric and matter variables are perturbed as follows [Herrera et al. 2012; Sharif & Yousaf 2014a,b]

\[
A(t, r, \theta) = A_0(r, \theta) + \varepsilon T(t)a(r, \theta), \tag{7}
\]
\[ B(t, r, \theta) = B_0(r, \theta) + \varepsilon T(t)b(r, \theta), \]  
\[ C(t, r, \theta) = C_0(r, \theta) + \varepsilon T(t)c(r, \theta), \]  
\[ G(t, r, \theta) = G_0(r, \theta) + \varepsilon T(t)g(r, \theta), \]  
\[ \mu(t, r, \theta) = \mu_0(r, \theta) + \varepsilon \tilde{\mu}(t, r, \theta), \]  
\[ P(t, r, \theta) = P_0(r, \theta) + \varepsilon \tilde{P}(t, r, \theta), \]  
\[ \Pi_I(t, r, \theta) = \Pi_{I0}(r, \theta) + \varepsilon \tilde{\Pi}_I(t, r, \theta), \]  
\[ \Pi_{II}(t, r, \theta) = \Pi_{II0}(r, \theta) + \varepsilon \tilde{\Pi}_{II}(t, r, \theta), \]  
\[ \Pi_{KL}(t, r, \theta) = \Pi_{KL0}(r, \theta) + \varepsilon \tilde{\Pi}_{KL}(t, r, \theta). \]

Using the above equations, the first conservation law (11) is perturbed as

\[
\dot{\mu} = - \left[ \mu_0 \left\{ \frac{b}{B_0} + \frac{c}{C_0} + \frac{1}{Z_0} \left( r^2 a A_0 B_0^2 + g G_0 + r^2 b B_0 A_0^2 \right) \right\} + (\mu_0 + P_0) \times \frac{A_0^2 B_0^2}{Z_0} \left\{ r^2 \left( \frac{2b}{B_0} \right) + \frac{2c}{C_0} \right\} + \frac{G_0^2}{A_0^2 B_0^2} \left( \frac{b}{B_0} + \frac{g}{G_0} - \frac{a}{A_0} + \frac{c}{C_0} \right) \right\} + \Pi_{I0} \times \left( \frac{b}{B_0} - \frac{c}{C_0} \right) + \frac{\Pi_{I0}}{3Z_0} \left\{ r^2 A_0^2 B_0^2 \left( \frac{b}{B_0} - \frac{c}{C_0} \right) + G_0^2 \left( \frac{g}{G_0} - \frac{a}{A_0} - \frac{c}{C_0} \right) \right\} \right] \dot{T}. 
\]

(16)

It is interesting to mention here that only the non-static part of first conservation law exists while the static part vanishes for this case. Similarly, the static part of second conservation law leads to

\[
P_0 + \frac{2}{9} (2\Pi_{I0} + \Pi_{II0}) + \left[ P_0 + \frac{2}{9} (2\Pi_{I0} + \Pi_{II0}) \right] \left\{ \frac{C_0'}{C_0} + \frac{3G_0G_0'}{2} + \frac{r^2 A_0^2 B_0^2}{Z_0^2} \right\} = 0. 
\]

The non-static part of the second conservation law after perturbation turns out to be

\[
\frac{1}{B_0^2} \left\{ \overline{P}' + \frac{2}{9} (2\overline{\Pi}' + \overline{\Pi}'_{II}) \right\} + \frac{1}{B_0^2} \left\{ \overline{P} + \frac{2}{9} (2\overline{\Pi} + \overline{\Pi}_{II}) \right\} \left\{ \frac{C_0'}{C_0} + \frac{3G_0G_0'}{2} \right\} + \frac{r^2 A_0^2 B_0^2}{Z_0^2} \left( \frac{A_0'}{A_0} + \frac{B_0'}{B_0} + \frac{2}{r} - \frac{1}{r} - \frac{B_0'}{B_0} \right) \right\} \left\{ \frac{G_0'}{2G_0} + \frac{1}{r} + \frac{B_0'}{B_0} \right\} \right\} = 0. 
\]
The non-static part of 02-component of the field equation as obtained in Eq. (A3) can be written as

\[ l\dddot{T} + m\ddot{T} + n\dot{T} = 0, \]

here \( l, m \) and \( n \) are functions of \( r \) and \( \theta \). The solutions of this equation involve stable and unstable configurations. To describe the instability range, we concentrate over the unstable part which has the following form (Herrera et al. 2012; Sharif & Yousaf 2014a,b; Sharif & Bhatti 2014a,b).

\[
T(t) = -\exp(\sqrt{\alpha}t), \quad \text{where} \quad \alpha = \frac{-m + \sqrt{m^2 - 4ln}}{2l}.
\]

For the solution to be real, we take \( \alpha > 0 \) with certain constraints that \( m < 0 \) while \( l, n > 0 \). Such a solution describes static system which undergoes collapse with large past time.
Now, we calculate non-static part of anisotropic stresses in terms of static matter profiles to evaluate the instability regimes. We take the equation of state introduced by Harrison et al. (1965) which relates pressure with energy density using adiabatic index $\Gamma$ as follows

$$
P = \Gamma \frac{P_0}{\mu_0 + P_0} \bar{\mu},$$

(20)

where $\Gamma$ represents rigidity or stiffness in the fluid which is taken to be constant in our stability analysis. The value of $\bar{\mu}$ from the non-static part of first conservation law after perturbation can be obtained by integrating Eq. (19) with respect to $t$ as follows

$$
\bar{\mu} = - \left[ \mu_0 \left\{ \frac{b}{B_0} + \frac{c}{C_0} + \frac{1}{Z_0} \left( r^2 a A_0 B_0^2 + g G_0 + r^2 b B_0 A_0^2 \right) \right\} + \left( \mu_0 + P_0 \right) \right.
\times \frac{A_0^2 B_0^2}{Z_0} \left\{ r^2 \left( \frac{2b}{B_0} + \frac{2c}{C_0} \right) + \frac{G_0^2}{A_0^2 B_0^2} \left( \frac{b}{B_0} + \frac{g}{G_0} - \frac{a}{A_0} + \frac{c}{C_0} \right) \right\} + \frac{\Pi_{I0}}{3}
\times \left( \frac{b}{B_0} - \frac{c}{C_0} \right) + \frac{\Pi_{IiO}}{3Z_0} \left\{ r^2 A_0^2 B_0^2 \left( \frac{b}{B_0} - \frac{c}{C_0} \right) + G_0^2 \left( \frac{g}{G_0} - \frac{a}{A_0} - \frac{c}{C_0} \right) \right\} \right] T.
$$

Using this value of $\bar{\mu}$ in Eq. (20), we obtain

$$
\bar{\Pi}_I = -\Gamma \frac{\Pi_{I0}}{\mu_0 + \Pi_{I0}} \chi T, \quad \bar{\Pi}_{II} = -\Gamma \frac{\Pi_{II0}}{\mu_0 + \Pi_{II0}} \chi T,
\bar{\Pi}_{KL} = -\Gamma \frac{\Pi_{KL0}}{\mu_0 + \Pi_{KL0}} \chi T, \quad \bar{P} = -\Gamma \frac{P_0}{\mu_0 + P_0} \chi T,
$$

(21)

here

$$
\chi = \mu_0 \left\{ \frac{b}{B_0} + \frac{c}{C_0} + \frac{1}{Z_0} \left( r^2 a A_0 B_0^2 + g G_0 + r^2 b B_0 A_0^2 \right) \right\} + \left( \mu_0 + P_0 \right) \right.
\times \frac{A_0^2 B_0^2}{Z_0} \left\{ r^2 \left( \frac{2b}{B_0} + \frac{2c}{C_0} \right) + \frac{G_0^2}{A_0^2 B_0^2} \left( \frac{b}{B_0} + \frac{g}{G_0} - \frac{a}{A_0} + \frac{c}{C_0} \right) \right\} + \frac{\Pi_{I0}}{3}
\times \left( \frac{b}{B_0} - \frac{c}{C_0} \right) + \frac{\Pi_{IiO}}{3Z_0} \left\{ r^2 A_0^2 B_0^2 \left( \frac{b}{B_0} - \frac{c}{C_0} \right) + G_0^2 \left( \frac{g}{G_0} - \frac{a}{A_0} - \frac{c}{C_0} \right) \right\} .
$$

Substituting these values in Eq. (18), it follows that

$$
- \frac{T \Gamma}{B_0^2} \left( \frac{P_0}{\mu_0 + P_0} + \frac{4\Pi_{I0} \chi}{9(\mu_0 + \Pi_{I0})} + \frac{2\Pi_{II0} \chi}{9(\mu_0 + \Pi_{II0})} \right) \left( \frac{C}{C_0} + \frac{3G_0 G_0'}{2Z_0} \right) + \frac{r^2 A_0^2 B_0^2}{Z_0} \left( \frac{A_0'}{A_0} + \frac{1}{r} \right) \right\} \right] + \frac{T r^2 A_0 B_0^3 \Gamma}{Z_0^2}
\times \left( \frac{\Pi_{KL0} \chi}{\mu_0 + \Pi_{KL0}} + \Gamma \frac{\Pi_{II0} \chi}{\mu_0 + \Pi_{II0}} \frac{T r^2 A_0 B_0^3}{Z_0^2} \right) \left\{ \frac{A_0}{A_0} + \frac{6B_0}{B_0} + \frac{C_0}{C_0} + \frac{4G_0 G_0'}{Z_0} \right\}.
$$
\[
\begin{align*}
+ \frac{4r^2 A_0 B_0^2}{Z_0} \left( \frac{A_{0\theta} + B_{0\theta}}{A_0} \right) &= T \left\{ \frac{\chi r^4 A_0^4}{Z_0^2} \left( \frac{A_0'}{A_0} \right) - \frac{r^2 A_0 B_0^2}{Z_0} \left( \frac{a}{A_0} + \frac{b}{B_0} \right) \right\} \\
\times \left\{ P_0' + \frac{2}{9} (2\Pi_{I0} + \Pi_{II0}) \right\} - \left\{ \left( \frac{c}{C_0} \right)' + \frac{3G_0G_0'}{2Z_0} \left( \frac{g}{G_0} + \frac{g'}{G_0} - \frac{Z}{Z_0} \right) \right\} \\
+ \frac{r^2 A_0 B_0^2}{Z_0} \left( \frac{2a}{A_0} + \frac{2b}{B_0} - \frac{Z}{Z_0} \right) \left( \frac{A_0'}{A_0} + \frac{1}{r} \right) + \frac{r^2 A_0 B_0^2}{Z_0} \left( \frac{a}{A_0} + \frac{b}{B_0} \right) \right\} \\
\times \left\{ P_0 + \frac{2}{9} (2\Pi_{I0} + \Pi_{II0}) \right\} - \frac{1}{B_0^2} \left\{ P_0 + \frac{2}{9} (2\Pi_{I0} + \Pi_{II0}) \right\} \left\{ \frac{C_0'}{C_0} + \frac{3G_0G_0'}{2Z_0} \right\} \\
+ \frac{r^2 A_0 B_0^2}{Z_0} \left( \frac{a'}{A_0} + \frac{1}{r} \right) + \frac{r^2 A_0 B_0^3}{Z_0^2} \Pi_{KL,0} \left( \frac{a}{A_0} + \frac{3b}{B_0} \frac{3Z}{Z_0} \right) + \frac{r^2 A_0 B_0^3 \Pi_{KL,0}}{Z_0^2} \\
\times \left\{ \left( \frac{A_0}{A_0} + \frac{3b}{B_0} \frac{3Z}{Z_0} \right) \right\} + \frac{r^2 A_0 B_0^3}{Z_0^2} \Pi_{KL,0} \left[ \frac{6B_{0\theta}}{B_0} \left( \frac{b_{0\theta}}{B_{0\theta}} + \frac{b}{B_0} \right) \left( \frac{a}{A_0} + \frac{c}{C_0} \right) + \frac{4G_0G_{0\theta} \Pi_{KL,0}}{Z_0} \right] \\
\times \left\{ \left( \frac{g_{0\theta} + g_{\theta}}{G_0} - \frac{Z}{Z_0} \right) + \frac{4r^2 A_0 B_0^2}{Z_0} \left( \frac{2a}{A_0} + \frac{2b}{B_0} - \frac{Z}{Z_0} \right) \left( \frac{a}{A_0} + \frac{b}{B_0} \right) \right\} \\
- \frac{\mu_0 r^4 A_0^4}{Z_0^2} \left\{ \left( \frac{a}{A_0} \right)' - \frac{G_0 A_{0\theta}}{r^2 A_0 B_0^2} \left( \frac{g}{G_0} + \frac{a_{0\theta}}{A_0} - \frac{a}{A_0} \right) \right\} + \frac{\mu_0 G_0^2 A_0^2 r^2}{Z_0^2} \\
\times \left\{ \frac{2g}{G_0} + \frac{2a}{A_0} - \frac{2Z}{Z_0} \right\} + \frac{\mu_0 G_0^2 A_0^2 r^2}{Z_0^2} \left( \frac{g}{G_0} + \frac{b}{B_0} \right)' - \frac{G_0^2 A_0^2}{Z_0^2} \\
\times \left\{ \frac{C_0'}{2G_0} + \frac{1}{r} + \frac{B_{0\theta}}{B_0} \right\}.
\end{align*}
\]

where

\[
Z_0 = r^2 A_0^2 B_0^2 + G_0^2,
\]
\[
\bar{Z} = 2r^2 A_0^2 B_0^2 \left( \frac{a}{A_0} + \frac{b}{B_0} \right) + 2gG_0.
\]

This is the required collapse equation with the constraints \(P_0', \Pi_{I0}', \Pi_{II0} < 0\) which is very useful to investigate the instability regions for our systematic analysis.

\[\text{4. Vorticity Tensor and Instability Regions}\]

This section investigates vorticity tensor and dynamical instability ranges for non-static axial spacetime with the help of equations obtained in the previous section particularly the collapse equation subject to \(N\) and \(pN\) limits. The role of stiffness parameter and its dependence on physical factors are also analyzed in this scenario.
The kinematical variable responsible for producing local spinning action of anisotropic fluid configurations is the vorticity tensor. For reflection axisymmetric spacetime, this tensor in terms of four vectors, $K_\alpha$ and $L_\alpha$, can be expressed as

$$\Omega_{\alpha\beta} = \Omega (K_\beta L_\alpha - L_\beta K_\alpha),$$

where

$$\Omega = \frac{G}{2B\sqrt{Z}} \left( \frac{G'}{G} - \frac{2A'}{A} \right).$$

For the vanishing of vorticity scalar, either $G = 0$ or $\frac{G'}{G} - \frac{2A'}{A} = 0$. If we take $G = 0$ then it leads to the vanishing of vorticity scalar. On the other hand, if we take $\frac{G'}{G} - \frac{2A'}{A} = 0$, then it vanishes the metric coefficient $A(t, r, \theta)$ describing the temporal component of the spacetime as follows

$$\frac{G'}{G} - \frac{2A'}{A} = 0,$$

$$\ln \left( \frac{G\tilde{C}}{A^2} \right) = 0,$$

where $\tilde{C} = \tilde{C}(t, \theta)$ is an arbitrary function of integration. Consequently, $G\tilde{C} = A^2$, which implies that for $G = 0$, we have $A = 0$ disturbing the existence of our non-static axial spacetime. Hence,

$$\frac{G'}{G} - \frac{2A'}{A} \neq 0.$$

Thus, we take $G = 0$ with regularity condition at the center indicating that vorticity of axisymmetric spacetime exists if and only if its reflection degrees of freedom exist or more precisely $\Omega = 0 \Leftrightarrow G = 0$. Consequently the assumption $\Omega = 0$ in the dynamical evolution of non-static axisymmetric anisotropic metric gives zero value to non-diagonal scale factor, $G$, whose dynamics has already been discussed in GR (Sharif & Bhatti 2014b) as well as in modified gravity theory (Sharif & Yousaf 2014c).

4.1. Newtonian Limit

For N limit, we take $A_0 = 1$, $B_0 = 1$, $C_0 = r$, $G_0 = r$ so that $Z_0$ turns out to be $r^2$ for the instability analysis. We also discard the terms of order $\frac{m_0}{r}$, where $m_0$ is the static profile of the mass function. The physical requirement of the collapsing matter, i.e., $P_0$, $\Pi_{I0}$, $\Pi_{II0} < 0$, is also imposed in N approximation. By making use of the above mentioned constraints, the
collapse equation yields

\[
T \Gamma \left[ \left( 3b + \frac{2c}{r} + \frac{g}{r} \right) \left\{ P_0 + \frac{2}{9} (2 \Pi_{I0} + \Pi_{II0}) \right\} \right]' + \frac{11}{4r} \left( P_0 + \frac{2}{9} (2 \Pi_{I0} + \Pi_{II0}) \right) \\
- \left[ \frac{\Pi_{KL0}}{2 \sqrt{2} r} \left( 3b + \frac{2c}{r} + \frac{g}{r} \right) \right]' = -T \left[ \left\{ P_0 + \frac{2}{9} (2 \Pi_{I0} + \Pi_{II0}) \right\} \left\{ \frac{1}{2} \left( a + b \right)' \right\} \\
+ \left( \frac{c}{r} \right)' + \frac{1}{2r} \left( 2a + 11b - \frac{Z}{2r^2} \right) + \frac{\Pi_{KL0}}{2 \sqrt{2}} \left[ 2(a + b)' \left( 2a + 2b - \frac{Z}{2r^2} \right) \right] \\
\frac{3}{4r} \left( g' + \frac{g}{r} - \frac{Z}{2r^2} \right) \left\{ P_0 + \frac{2}{9} (2 \Pi_{I0} + \Pi_{II0}) \right\} \right] + \frac{\mu_0}{4} \left( 2b' - a' + \frac{7}{2r} + \frac{6c}{r} - \frac{Z}{r^2} \right). 
\]

The system will be unstable until it satisfies the following relation

\[
\Gamma < \frac{\mu_0}{\eta'} \frac{\left( 2b' + \frac{6c}{r} + \frac{7}{2r} - a' \right) + G_2 + A_1}{\eta + \frac{11}{4r} \left( G_1 - \frac{\Pi_{KL0}}{2 \sqrt{2} r} \left( 3b + \frac{2c}{r} \right) \right)_0}, \tag{24}
\]

where

\[
\eta = \left( 3b + \frac{2c}{r} + \frac{g}{r} \right) \xi_1,
\]

while remaining quantities are defined in Appendix A. It is well-known that instability will emerge as long as all the terms given in the above inequality are positive. For this purpose, we need to take \(|A_1|, |G_1| \text{ and } |G_2|\) instead of \(A_1, G_1 \text{ and } G_2\). We find that the adiabatic index depends on the static profile of matter variables of axial geometry.

- In the above instability constraint, the quantities \(|G_1| \text{ and } |G_2|\) incorporate meridional effects that arise due to non-zero vorticity vector of the collapsing system. It is well-known from the work of Herrera et al. (2014a) that invoking of reflection effects in axially symmetric anisotropic stellar object causes the emission of gravitational radiations. These radiations induce the loss of both energy and angular momentum, which consequently boosts up the instability of the reflectional axisymmetric body.

- The quantity \(A_1\) includes anisotropic contribution of axial geometry. It is seen from the expression (24) that anisotropy tends to produce complications in understanding its role in the stability of axial systems. However, if one considers positivity of all terms in denominator and numerator of the above expression then it is seen from (24) that anisotropic pressure tends to increase instability regions. This result is well-consistent with Chan et al. (1993).
4.2. Post-Newtonian Limit

For the instability era in the pN approximation, we assume

\[ A_0 = 1 - \frac{m_0}{r_1}, \quad B_0 = 1 + \frac{m_0}{r_1}, \]

and the terms of the order \( \frac{m_0}{r} \) while discarding the terms containing higher orders of \( \frac{m_0}{r} \). The system will be unstable in the pN region if it satisfies the inequality

\[ \Gamma < \frac{\mu_0 \sigma \zeta + G_3 + A_2 + X_1}{A_3}, \]

where static profile terms \( X_1 \) and \( G_3 \) are non-diagonal and diagonal components of scale factors at pN epoch, respectively, while \( A_2 \) and \( A_3 \) incorporate anisotropic effects in the evolutionary phases of collapsing self-gravitating axial stellar object. These terms are given in Appendix A.

4.2.1. Restricted Class of Anisotropic Axial Spacetime

On assuming \( G_3 = 0 \), our instability constraint at pN approximation of axisymmetric object with reflection symmetry reduces to

\[ \Gamma < \frac{\mu_0 \sigma \zeta}{A_3}. \]

This describes instability range of the restricted class of non-static axial geometry since it excludes explicitly rotations around the symmetry axis, i.e., \( dt d\phi \) as well as the reflection terms. This result coincides and supports already calculated solution (Sharif & Bhatti 2014b).

4.2.2. Reflection and Restricted Class of Isotropic Axial Spacetime

On taking equal all principal stresses as well as zero value to \( G_3 \), one can find instability regions of restricted class of isotropic axisymmetric spacetime from expression (26). However, apart from that by assuming only first of above limits, one can get dynamical instability constraint of reflection axisymmetric compatible with perfect fluid. All possible stellar models of reflection axial symmetric system coupled with perfect (isotropic) matter configurations have have been explored in detail by Herrera et al. (2015).

We see that the adiabatic index \( \Gamma \) plays a central role to investigate dynamical instability of the relativistic system. It is worth mentioning that for \( \Gamma < \frac{4}{3} \) and \( \Gamma < 1 \), the spherical and
cylindrical relativistic objects become unstable respectively thereby enforces the importance of index $\Gamma$. Infact, the adiabatic index also known as stiffness parameter demonstrates how much relativistic fluid is stiff. We have established the relevance of such index in the dynamical instability as seen from expressions (24) and (26) depending upon the static profile of the structural properties of the system. The system would be in complete hydrostatic equilibrium, if (during evolution) adiabatic index is able to attain value equal to the right hand side of the expressions given in (24) and (26). However, if stiffness parameter attains a value greater than the right hand side of expressions (24) and (26), then the relativistic system begins to move in the stable window, thereby ceasing the collapsing mechanism.

5. Conclusions

This paper is devoted to investigate dynamical instability of non-static axially symmetric spacetime by choosing reflection term in the geometry. Since rotating stars are more stable than non-rotating, so for the instability regions, we have neglected the term representing rotation in the general non-static axial spacetime. It is worth mentioning that for axially symmetric sources perfect fluid distribution seems to be inflexible restriction, even in the static case. On the other hand, Bondi coordinates are known to be very useful for the treatment of gravitational radiation in vacuum, but are not particularly suitable within the source. An analytical approach, which shares some similarities with ours, although restricted to the perfect fluid case, can be found in the literature. Therefore, here, we have considered a source which includes all nonvanishing stresses compatible with the symmetry of the problem to carry out our systematic analysis.

We have explored the field equations and corresponding conservation laws in this scenario. We have found three independent components from the conservation law while there exist only two components in the case of spherical and cylindrical spacetimes (Herrera et al. 2012; Sharif & Bhatti 2014a,b,c,d). The radial perturbation is used for metric as well as material variables to obtain perturbed form of these dynamical equations. We have explored static and non-static parts of independent components of the conservation law. It is found that only non-static part for the first conservation law exists and static part vanishes while the remaining equations have both static as well as non-static components. Using 02-component of the field equations with perturbation technique, we have found a solution which corresponds to both stable and unstable configurations and start collapsing at large past time diminishing its areal radius (Sharif & Yousal 2014a,b,c, 2015).

We have developed a general collapse equation to examine the instability regions using non-static parts of anisotropic stresses and the solution (19). We have explored two insta-
bility ranges under \(N\) as well as \(pN\) limits and found that instability range is defined by the adiabatic index (Sharif & Yousaf 2014c, 2015) unlike expansion-free case [where it has no role (Herrera et al. 2012)]. The adiabatic index depends upon static profile of the energy density, anisotropic pressure and the reflection term in the spacetime. We conclude that reflection symmetry increases the unstable range of the axial geometry. The system will remain unstable until it satisfies the relations (24) and (26) while their violation will lead to stable configuration of the model. It would be interesting to examine the role of dissipative terms like heat flux on the stability of non-static axial geometry.

**Appendix A**

The 02-component of the Einstein tensor corresponding to our line element in Eq. (11) takes the form

\[
G_{02} = -\frac{1}{4(r^2 A^2 B^2 + G^2)^2} \left[ 4G^4 \left\{ \frac{\dot{C}_\theta}{C} + \frac{\dot{B}_\theta}{B} + \frac{\dot{B}C_\theta}{BC} + \frac{\dot{C}B_\theta}{CB} \right\} + 4r^4 A^4 B^2 G \left\{ \frac{A'C'}{AC} - \frac{B'^2}{B^2} - 2A'B' \frac{A'}{AB} + \frac{1}{r} \left( \frac{C'}{C} + \frac{G'}{G} - \frac{A'}{A} \right) - \frac{G''}{2G} + \frac{A'G'}{2AG} + \frac{G'B'}{GB} - \frac{G'C'}{GC} + \frac{A''}{A} + \frac{B''}{B} + \frac{C''}{C} + \frac{1}{r^2} \left( \frac{B_{\theta\theta}}{B} + \frac{C_\theta}{C} \right) \right\} + 4r^2 A^2 G^3 \left\{ \frac{A'C'}{AC} - \frac{3G' B'}{2GB} + \frac{3C'^2}{4G^2} - \frac{B'C'}{BC} + \frac{A'^2}{A} - \frac{3A' G'}{2AG} + \frac{A''}{A} + \frac{B''}{B} + \frac{2C''}{C} + \frac{1}{r^2} \left( \frac{A_{\theta\theta} B_\theta}{AB} - \frac{B_{\theta\theta} G_\theta}{BG} \right) \right\} + 4r^4 A^2 B^4 G \left\{ \frac{\dot{A}B}{AB} + \frac{\dot{A}C}{AC} - \frac{\dot{B}C}{BC} \right\} + 4r^2 A^2 B^2 G \left\{ \frac{B_\theta G_\theta}{BG} - \frac{\dot{A} B_\theta}{AB} - \frac{\dot{B} B_\theta}{B^2} + \frac{C_\theta G_\theta}{CG} + \frac{\dot{C} G_\theta}{GC} - \frac{\dot{A} C_\theta}{AC} + \frac{\dot{B} C_\theta}{BC} \right\} + 4r^4 A^4 B^4 \left\{ \frac{\dot{B} A_\theta}{BA} - \frac{\dot{B} B_\theta}{B^2} + \frac{A_\theta \dot{C}}{AC} + \frac{\dot{B} C_\theta}{BC} - \frac{\dot{C} \theta}{C} \right\} + 4r^2 B^2 G^3 \left\{ \frac{\dot{G} C}{GC} - \frac{\dot{B} B'}{B^2} + \frac{4B \dot{G} C}{BG} - \frac{2B \dot{B} C}{BC} - \frac{\dot{C}}{C} - \frac{\dot{B}}{B} \right\} + 4G^5 \left\{ \frac{G''}{G} - \frac{G'^2}{2G^2} - \frac{2B'C'}{BC} + \frac{G'' C'}{2GC} + \frac{C''}{C} - \frac{G'B'}{GB} \right\} \right].
\]
For $\alpha = 3$, the conservation law, $T^{\alpha\beta} = 0$, leads to the following equation

\[
\frac{\mu r^2 A^2 B^2 G}{(r^2 A^2 B^2 + G^2)^2} \left[ \frac{\dot{\mu}}{\mu} + \frac{\dot{A}}{A} + \frac{3\dot{B}}{B} + \frac{\dot{G}}{G} + \frac{\dot{C}}{C} + \frac{1}{r^2 B^2} \left( \frac{\mu_0}{\mu} + \frac{2G_0}{G} + \frac{2A_0}{A} \right) \right] \\
+ \frac{1}{r^2 A^2 B^2 + G^2} \left\{ 4r^2 A^2 \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} \right) - \frac{4\dot{G}}{G} - GA^2 \left( \frac{5A_0}{A} + \frac{2B_0}{B} \right) \right\} \\
+ r^2 A^2 B^2 \left( \frac{\dot{G}}{G} + \frac{\dot{B}}{B} \right) + \frac{r^2 A^2 B^2 A_0}{G} \right\} - \frac{4G^2 G_0 (r^2 A^2 B^2 + G^2)}{r^2 B^2} \\
+ \frac{\mu A^2 G^2}{(r^2 A^2 B^2 + G^2)^2} \left( \frac{B_0}{B} + \frac{C_0}{C} - \frac{r^2 B G \dot{B}}{r^2 A^2 B^2 + G^2} \right) - \frac{r^2 A B^3 \Pi_{KL}}{(r^2 A^2 B^2 + G^2)^2} \\
\times \left[ \left( \frac{3}{r} + \frac{4B'}{B} \right) \left( \frac{A'}{A} + \frac{C'}{C} + \frac{3}{r^2 A^2 B^2 + G^2} \right) \right] \left\{ (2A^2 + A) \left( \frac{A_0}{A} + \frac{B_0}{B} \right) - G \left( \frac{B}{B} \right) \right\} \\
+ \frac{2AB_0}{B} \right\} + 2AA_0 + \frac{A^2C_0}{C} - \frac{r^2 B G \dot{B}}{r^2 A^2 B^2 + G^2} - \frac{2A^2 G G_0}{r^2 A^2 B^2 + G^2} - \frac{G B}{B} \\
- \frac{P}{C(r^2 A^2 B^2 + G^2)} \left( \frac{G \dot{C} + A^2 C_0}{G} \right) \right\} = 0. \\
\text{(A2)}
\]

Using Eqs. (4), (A1) and (7-11), the non-static part of the 02-component of the field equations, $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$, becomes

\[
- \frac{2r^4 G_0 A_0 B_0 A_1 B_1}{Z_0^4} \left( \frac{g}{G_0} + \frac{3a}{A_0} + \frac{b}{B_0} + \frac{a'}{A_0} - \frac{2\dot{G}}{Z_0} \right) T + \frac{3r^2 A_0^2 G_0^2 G_0 B_0'}{B_0 Z_0^2} \\
\times \left( \frac{g'}{G_0} + \frac{b'}{B_0} + \frac{2a}{A_0} + \frac{2g}{G_0} - \frac{2\dot{Z}}{Z_0} - \frac{b}{B_0} \right) T + \frac{G_0 B_0 A_0^2 r^2 B_{06} \dot{g} T}{Z_0^2} \\
- \frac{r^2 a G_0^2 A_0^2 Z_0^2 T}{Z_0^4} - \frac{2g}{G_0} - \frac{2\dot{Z}}{Z_0} \right) T + \left( \frac{C_0 G_0^2}{C_0 Z_0^2} + \frac{3r^2 G_0 G_0^2 A_0^2}{Z_0^2} \left( \frac{g'}{G_0} + \frac{2a'}{A_0} - \frac{2\dot{Z}}{Z_0} \right) \right) T \\
+ \frac{3r A_0^2 B_0 G_0}{B_0 Z_0^2} \left( \frac{2a}{A_0} + \frac{b}{B_0} - \frac{3g}{Z_0} \right) T - \frac{A_0^2 G_0^2 B_{00} G_{00} T}{B_0 Z_0^2} \left( \frac{2a}{A_0} + \frac{b}{B_0} - \frac{3g}{Z_0} \right) T
\]
\[\begin{align*}
&+ \frac{2g}{G_0} + \frac{b_0}{B_{00}} + \frac{g_0}{G_{00}} - \frac{g^2 Z}{B_0 Z_0} - r^2 B_0' C_0' A_0^2 G_0^3 \left( \frac{b'}{B_0'} + \frac{c'}{C_0'} + \frac{2a}{A_0} + \frac{3g}{G_0} \right) \left( - \frac{b}{B_0} - \frac{c}{C_0} - \frac{2Z}{Z_0} \right) T - \frac{G_0 A_0^3 r^4 B_0^2}{Z_0^4} \left( \frac{g}{G_0} + \frac{4a}{A_0} + \frac{2b}{B_0} - \frac{2Z}{Z_0} \right) T \\
&+ \frac{r^4 G_0 B_0^2 A_0^3 A_0' C_0'}{C_0 Z_0^4} \left( \frac{g}{G_0} + \frac{2b}{B_0} + \frac{3a}{A_0} + \frac{a'}{A_0'} + \frac{c'}{C_0'} - \frac{c}{C_0} - \frac{2Z}{Z_0} \right) T \\
&+ \frac{4G_0 B_0 r^2 C_0 B_0' g T}{C_0 Z_0^2} + \frac{G_0 A_0^2 B_0^2 r^2 G_{00} c T}{C_0 A_0^2} - \frac{r^2 B_0^2 a A_0 C_{00} T}{C_0 Z_0^2} + \frac{r^2 A_0 A_0' C_0' G_0^3}{C_0 Z_0^2} \\
&\times \left( \frac{3g}{G_0} + \frac{a'}{A_0'} - \frac{c}{C_0} - \frac{2Z}{Z_0} \right) T + \frac{G_0 A_0^3 B_0 r^2 G_{00} b' T - B_0^3 b_0 A_0^2 r^4 T}{Z_0^4} \\
&- \frac{G_0^3 B_0 r^2 b T}{Z_0^3} + \frac{r^2 A_0^2 G_0^3}{Z_0^2} \left( \frac{2a'}{A_0'} + \frac{3g}{G_0} - \frac{2Z}{Z_0} \right) T + \frac{A_0^3 B_0 G_{00} C_0^3 T}{Z_0^2} + \frac{2a}{A_0} + \frac{2g}{G_0} \\
&+ \frac{B_{00} c T}{Z_0^2} + \frac{B_0 C_0 Z_0^2}{Z_0} - \frac{2Z}{Z_0} \right) T + \frac{4A_0^2 G_0^3 G_{00} T}{C_0 Z_0^2} - \frac{2a}{A_0} + \frac{2g}{G_0} \\
&+ \frac{B_{00} c T}{Z_0^2} + \frac{B_0 C_0 Z_0^2}{Z_0} - \frac{2Z}{Z_0} \right) T + \frac{2a}{A_0} + \frac{2g}{G_0} \\
&- \frac{r^4 A_0^3 B_0^5 c_0 T}{B_0^3 Z_0^4} - \frac{br^2 B_0 A_0 G_{00} T}{C_0 Z_0^2} + \frac{br^4 A_0^2 B_0^2 B_{00} T}{Z_0^2} + \frac{br^4 A_0^3 B_0^3 A_{00} T}{Z_0^2} \\
&+ \frac{r^4 A_0^3 B_0^2 A_0' G_0' T}{2G_0^2} \left( \frac{2b}{B_0} + \frac{3a}{A_0} + \frac{a'}{A_0'} + \frac{b}{B_0} + \frac{g}{G_0'} - \frac{2Z}{Z_0} \right) T + \frac{4r^4 B_0 A_0' B_0' G_0' T}{Z_0^2} \\
&\times \left( \frac{3g}{G_0} + \frac{a}{A_0} + \frac{a_0}{A_{00}} + \frac{b_0}{B_{00}} - \frac{b}{B_0} \right) \left( \frac{3r A_0^3 G_0^3}{Z_0^2} + \frac{2g'}{G_0'} - \frac{2Z}{Z_0} \right) T - \frac{r^4 A_0 B_0^2 G_0' c_0 T}{Z_0^3} + \frac{G_0^3 A_0 A_{00} c_{00}}{C_0 Z_0^2} \\
&+ \frac{2r^3 A_0^4 B_0^3 G_0' T}{Z_0^2} \left( \frac{4a}{A_0} + \frac{2b}{B_0} + \frac{g'}{G_0} - \frac{2Z}{Z_0} \right) T + \frac{r^4 A_0 B_0^2 G_0' c_0 T}{Z_0^3} + \frac{G_0^3 A_0 A_{00} c_{00}}{C_0 Z_0^2} \\
&- \frac{4a}{A_0} + \frac{2b}{B_0} - \frac{c}{C_0} - \frac{2Z}{Z_0} \right) T + \frac{r^4 A_0 B_0^2 G_0' c_0 T}{Z_0^3} - \frac{TC_0 B_0^4 G_0^4}{2B_0^3 Z_0^2} \left( \frac{g'}{G_0'} + \frac{b'}{B_0'} + \frac{4g}{G_0} \\
&- \frac{3b}{B_0} - \frac{2Z}{Z_0} \right) + \frac{2G_0 c_{00} G_0^4}{B_0^3 C_0 Z_0^2} \left( \frac{g'}{G_0'} + \frac{c'}{C_0'} + \frac{4g}{G_0} - \frac{2b}{B_0} - \frac{c}{C_0} - \frac{2Z}{Z_0} \right) T
\end{align*}\]
Using pN approximation, Eq. (22) takes the form

\[ - \frac{\dot{T}_r^2 c B_0^2 C_0^3}{C_0 Z_0^2} - \frac{\dot{T}_r^4 c A_0^4 B_0^2 G_0}{C_0 Z_0^2} - \frac{r^4 A_0^4 B_0^2 C_0'' T}{2 Z_0^2} \left( \frac{c''}{C_0''} + \frac{g''}{G_0''} + \frac{2b}{B_0} + \frac{4a}{A_0} - \frac{2\dot{Z}}{Z_0} \right) \\
+ \frac{TC_0^4 G_0^4}{2 Z_0^2 B_0^2} \left( \frac{g''}{C_0''} + \frac{4g}{G_0} - \frac{2\dot{Z}}{Z_0} - \frac{2b}{B_0} \right) + \frac{TC_0'' G_0^5}{Z_0^2 B_0^2 C_0} \left( \frac{c''}{C_0''} + \frac{5g}{G_0} - \frac{2b}{B_0} - \frac{c}{C_0} - \frac{2\dot{Z}}{Z_0} \right) \\
+ A_0^2 G_0^3 C_{\theta\theta\theta} \left( \frac{2a}{A_0} + \frac{c_{\theta\theta}}{C_{\theta\theta\theta}} + \frac{3g}{G_0} - \frac{2\dot{Z}}{Z_0} - \frac{c}{C_0} \right) T + \frac{r^2 A_0 A_0'' G_0^3 T}{Z_0^2} \left( \frac{a}{A_0} \right) \\
+ A_0'' \left( \frac{3g}{G_0} - \frac{\dot{Z}}{Z_0} \right) + r^2 A_0 B_0'' G_0^3 T \left( \frac{2a}{A_0} + \frac{b''}{B_0''} + \frac{3g}{G_0} - \frac{b}{B_0} - \frac{\dot{Z}}{Z_0} \right) \\
+ \frac{r^4 A_0^2 B_0^2 G_0 A_0^4}{Z_0^2} \left( \frac{g}{G_0} + \frac{2b}{B_0} + \frac{3a}{A_0} + \frac{a''}{A_0''} - \frac{\dot{Z}}{Z_0} \right) T + \frac{r^2 A_0^2 B_0 B_0''}{Z_0^2} \left( \frac{g}{G_0} \right) \\
+ \frac{4a}{A_0} + \frac{b}{B_0} + \frac{b_{\theta\theta}}{B_{\theta\theta}} \left( \frac{\dot{Z}}{Z_0} \right) T + \frac{G_0 A_0^4 B_0 r^2 A_0''}{Z_0^2} \left( \frac{g}{G_0} + \frac{4a}{A_0} - \frac{b}{B_0} + \frac{b''}{B_0''} - \frac{\dot{Z}}{Z_0} \right) T \\
+ \frac{G_0^3 C_{\theta\theta\theta}}{B_0 Z_0^2} \left( \frac{2a}{A_0} + \frac{b_{\theta\theta}}{B_{\theta\theta}} + \frac{3g}{G_0} - \frac{b}{B_0} - \frac{\dot{Z}}{Z_0} \right) T + \frac{2C_{\theta\theta}'' A_0^3 G_0^3}{C_0 Z_0^2} \left( \frac{c''}{C_0''} \right) \\
- \frac{Z}{Z_0} - \frac{c}{C_0} + \frac{2a}{A_0} + \frac{3g}{G_0} - \frac{c}{C_0} \left( \frac{g}{G_0} \right) + \frac{c''}{C_0''} + \frac{2b}{B_0} + \frac{4a}{A_0} \\
- \frac{c}{C_0} - \frac{2\dot{Z}}{Z_0} \right) T + 4 G_0 A_0^2 B_0^2 r^2 C_{\theta\theta\theta} \left( \frac{g}{G_0} + \frac{4a}{A_0} + \frac{2b}{B_0} + \frac{c_{\theta\theta}}{C_{\theta\theta\theta}} - \frac{c}{C_0} \right) T \\
\left( - \mu_0 \left\{ \frac{b}{B_0} + \frac{c}{C_0} + \frac{1}{Z_0} \left( r^2 a A_0^2 B_0^2 + g G_0 + r^2 b B_0 G_0 \right) \right\} + \left( \mu_0 + P_0 \right) \right) \\
\times \frac{A_0^2 B_0^2}{Z_0^2} \left\{ r^2 \left( \frac{b}{B_0} + \frac{2c}{C_0} \right) + \frac{A_0^2 B_0^2}{A_0^2 B_0^2} \left( \frac{b}{B_0} + \frac{g}{G_0} - \frac{a}{A_0} + \frac{c}{C_0} \right) \right\} + \frac{\Pi_{10}}{3} \\
\times \left( \frac{b}{B_0} - \frac{c}{C_0} \right) \left( \frac{r^2 A_0^2 B_0^2}{B_0^2} - \frac{c}{C_0} \right) + G_0^2 \left( \frac{g}{G_0} - \frac{a}{A_0} - \frac{c}{C_0} \right) \right] T. \quad (A3)
\[
+ \frac{1}{2r^2} \left( 1 - \frac{4m_0^2}{r_1^2} \right) \left( 2a \left( 1 + \frac{m_0}{r_1} \right) + 2b \left( 1 - \frac{m_0}{r_1} \right) - \frac{Z}{2r^2} \right) \left( 1 - \frac{m_0}{r_1} \right) \]
\times \left( 1 + \frac{m_0}{r_1} \right) + \frac{1}{2} \left( a \left( 1 + \frac{m_0}{r_1} \right) + b \left( 1 - \frac{m_0}{r_1} \right) \right) \left( P_0 + \frac{2}{9} (2\Pi_{f_0} + \Pi_{I_0}) \right) \left( 1 - \frac{m_0}{r_1} \right) \left( 1 - \frac{m_0}{r_1} \right)
\times \left( \left( 1 - \frac{m_0}{r_1} \right) \left( 1 - \frac{m_0}{r_1} \right) \right) + \frac{1}{2\sqrt{2r}} \Pi_{KLO,\theta} \left( 1 - \frac{m_0}{r_1} \right) \left( 1 + \frac{3m_0}{r_1} \right)
\times \left( a \left( 1 + \frac{m_0}{r_1} \right) + b \left( 1 - \frac{m_0}{r_1} \right) \right) \left( 1 - \frac{m_0}{r_1} \right) \left( 1 - \frac{m_0}{r_1} \right)
\times \left( 1 + \frac{m_0}{r_1} \right) \left( 1 - \frac{2m_0}{r_1} \right) \left( \frac{g}{r} + a_\theta \left( 1 + \frac{m_0}{r_1} \right) - a \left( 1 + \frac{m_0}{r_1} \right) \right)
\times \left( 1 + \frac{m_0}{r_1} \right) \left( 1 - \frac{2m_0}{r_1} \right) \left( \frac{g}{r} + 2a \left( 1 + \frac{m_0}{r_1} \right) - \frac{Z}{2r^2} \right) \left( 1 - \frac{m_0}{r_1} \right)
\times \left( 1 + \frac{m_0}{r_1} \right) \left( 1 - \frac{m_0}{r_1} \right) \left[ \frac{3}{2r} + \left( 1 + \frac{m_0}{r_1} \right) \left( 1 - \frac{m_0}{r_1} \right) \right].
\]

The quantities introduced in the unstable range in pN limit are given as

\[
\xi_1 = P_0 + \frac{2}{9} (2\Pi_{f_0} + \Pi_{I_0}), \quad \psi = \left( \frac{P_0}{\mu_0 + P_0} + \frac{4\Pi_{f_0}}{9(\mu_0 + \Pi_{f_0})} + \frac{2\Pi_{I_0}}{9(\mu_0 + \Pi_{I_0})} \right),
\]
\[
\psi_1 = \frac{7}{4r} + \frac{1}{2} \sigma \left( 1 - \frac{m_0^2}{r_1^2} + \frac{1}{r} \right), \quad \psi_2 = \left( 1 - \frac{m_0}{r_1} \right) \left( 1 + \frac{3m_0}{r_1} \right),
\]
\[
\phi_1 = \left( 1 - \frac{4m_0}{r_1} \right) \left( 1 + \frac{m_0}{r_1} \right) \left( 1 - \frac{3m_0}{r_1} \right) \left( 1 - \frac{m_0}{r_1} \right), \quad \sigma = 1 - \frac{4m_0^2}{r_1^2},
\]
\[
\eta_1 = 2a \left( 1 + \frac{m_0}{r_1} \right) + 2b \left( 1 - \frac{m_0}{r_1} \right), \quad \phi_4 = \left( 1 - \frac{m_0}{r_1} \right) \left( 1 + \frac{m_0}{r_1} \right),
\]
\[
- \frac{Z}{r^2}, \quad \eta_2 = \left( 1 - \frac{m_0}{r_1} \right) \left( 1 + \frac{m_0}{r_1} \right), \quad \phi = \left( 1 - \frac{m_0}{r_1} \right) \left( 1 + \frac{m_0}{r_1} \right).
\]
\[ \phi_2 = \left(1 - \frac{m_0}{r_1}\right)' \left(1 + \frac{m_0}{r_1}\right), \quad \phi_3 = \left(1 - \frac{m_0}{r_1}\right) \left(1 + \frac{m_0}{r_1}\right), \]

\[ \zeta = \left(a + \frac{am_0}{r_1}\right)' - \frac{\omega \phi_4}{r} \left(\frac{g}{r} + a_\theta \left(1 + \frac{m_0}{r_1}\right) + \frac{3\bar{Z}}{2r^2 - \eta}\right), \]

\[ X_1 = -\frac{\mu_0}{4} \left(1 - \frac{4m_0}{r_1}\right) \left\{ \left(a \left(1 + \frac{m_0}{r_1}\right)\right)' - \frac{1}{r} \left(1 - \frac{m_0}{r_1}\right) \left(1 + \frac{m_0}{r_1}\right) \right\} \]

\[ \times \left\{ \left(1 - \frac{2m_0}{r_1}\right) \left(a_\theta \left(1 + \frac{m_0}{r_1}\right) - a \left(1 + \frac{m_0}{r_1}\right) - 2b \left(1 - \frac{m_0}{r_1}\right)\right) \right\} \]

\[ + \frac{\mu_0}{2\sqrt{2}r} \left(1 - \frac{2m_0}{r_1}\right) \left\{ 2a \left(1 + \frac{m_0}{r_1}\right) + \left\{ b \left(1 - \frac{m_0}{r_1}\right) \right\}' \right\} - \frac{\chi}{2\sqrt{2}} (1 \]

\[ - \frac{2m_0}{r_1} \left\{ \frac{3}{2r} + \left(1 + \frac{m_0}{r_1}\right)' \left(1 - \frac{m_0}{r_1}\right) \right\}, \]

\[ \mathcal{A}_2 = \xi_1 \omega_1 \left[ \frac{3}{4r} \left( g' + \frac{g}{r} - \frac{\bar{Z}}{2r^2} \right) + \frac{\sigma \eta}{2r^2} \left( \phi_2 + \frac{1}{r} \right) + \frac{1}{4} \left( \eta_1 - \frac{\bar{Z}}{r^2} \right) \right] - 2b \omega_1 \xi_1 \psi_1 \]

\[ + \frac{\Pi_{KL_0}}{2\sqrt{2}r} \left[ 6\phi_3 \left\{ b_\theta \left(1 - \frac{m_0}{r_1}\right) + b \left(1 + \frac{m_0}{r_1}\right) \right\} - a \left(1 + \frac{m_0}{r_1}\right) + \frac{c}{r} \right] \]

\[ + 2\sigma_1 \left( \eta_1 - \frac{\bar{Z}}{r^2} \right)' \]

\[ + b \chi_1 \omega_1 \left( \frac{\psi}{r} \right)' \]

\[ + \frac{1}{2\sqrt{2}} \psi_2 \eta \Pi_{KL_0,\theta}, \]

\[ \mathcal{G}_3 = -\frac{\mu_0}{4} \left(1 - \frac{4m_0}{r_1}\right) \left\{ \frac{-1}{r} \left(1 - \frac{m_0}{r_1}\right) \left(1 + \frac{m_0}{r_1}\right) \left(1 - \frac{2m_0}{r_1}\right) \left(\frac{g}{r}\right)' \right\} + \frac{\mu_0}{2} \]

\[ \times \left(1 - \frac{2m_0}{r_1}\right) \left(\frac{2g}{r} - \frac{\bar{Z}}{r^2} \right) + \frac{\mu_0}{2\sqrt{2}} \left(1 - \frac{2m_0}{r_1}\right) \left(\frac{g}{r}\right)' \]

\[ \mathcal{A}_3 = \omega_1 (\psi \chi)' - \psi \psi_1 \omega_1 + \psi_2 \left( \frac{\Pi_{KL_0} \chi}{\mu_0 + \Pi_{KL_0}} \right) \theta. \]

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