Theoretical analysis of Sinc-collocation methods and Sinc-Nyström methods for initial value problems

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Abstract

A Sinc-collocation method has been proposed by Stenger, and he also gave theoretical analysis of the method in the case of a ‘scalar’ equation. This paper extends the theoretical results to the case of a ‘system’ of equations. Furthermore, this paper proposes more efficient method by replacing the variable transformation employed in Stenger’s method. The efficiency is confirmed by both of theoretical analysis and numerical experiments. In addition to the existing and newly-proposed Sinc-collocation methods, this paper also gives similar theoretical results for Sinc-Nyström methods proposed by Nurmuhammad et al. From a viewpoint of the computational cost, it turns out that the newly-proposed Sinc-collocation method is the most efficient among those methods.

Keywords: Sinc approximation, Sinc indefinite integration, differential equation, Volterra integral equation, tanh transformation, double-exponential transformation

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1. Introduction

The concern of this paper is a system of initial value problems of the form

\[
\begin{cases}
y'(t) = K(t)y(t) + g(t), & a \leq t \leq b, \\
y(a) = r,
\end{cases}
\]

where \( K(t) \) is an \( n \times n \) matrix, and \( y(t), g(t), r \) are column vectors of order \( n \). For Eq. (1), several numerical methods based on the Sinc approximation have been developed so far \([2, 5, 13]\), and in general, those methods converge exponentially. For example, Carlson et al. \([2]\) proposed a Sinc-collocation method for Eq. (1), and they also claimed that its convergence rate is \( O(N^2 \exp(-c \sqrt{N})) \). However, their method is designed not for the finite interval \([a, b]\) but for the infinite interval \((−\infty, \infty)\) or \([0, \infty)\), and users have to know solution’s behavior as \( t \to \infty \) to implement the method. In addition, users also have to know solution’s regularity for implementation. It is not so practical to assume that solution’s behavior and regularity can be known in prior to computation, since the solution is an unknown function.
Instead of solving Eq. (1), Stenger [13] firstly transformed the problem to the Volterra integral equation of the second kind:

\[ y(t) = r + \int_a^t \{ g(s) + K(s)y(s) \} \, ds, \quad a \leq t \leq b, \]  

and derived a Sinc-collocation method for Eq. (2). His method does not require solution’s behavior as \( t \to \infty \) when the given interval \([a, b]\) is finite. In addition, he showed theoretically that solution’s regularity needed for implementation can be found from the known functions. This is an advantage of his method over that of Carlson et al. Moreover, he also showed that the convergence rate of his method is \( O(\sqrt{N} \exp(-c \sqrt{N})) \), where \( c \) is the same constant as the result of Carlson et al. It should be noted that those theoretical results were shown only in the case where Eq. (2) is a scalar equation \((n = 1)\), although the method was proposed for a system of equations. This is because the analysis relies on the explicit form of the solution \( y \) that holds only in the scalar case.

The first objective of this study is to extend Stenger’s theoretical results to a system of equations. That is, this paper shows that even in the case of a system of equations, solution’s regularity actually can be found from the known functions \( K(t) \) and \( g(t) \), and also shows that his method converges with the rate: \( O(\sqrt{N} \exp(-c \sqrt{N})) \).

The second objective, which is more important in this paper, is to improve Stenger’s method. The main idea here is replacement of the variable transformation; the “Single-Exponential transformation” (SE transformation) is employed in the method of Stenger (and also Carlson et al.), but it is replaced with the “Double-Exponential transformation” (DE transformation) in the proposed method. Those two methods are referred to as the SE-Sinc-collocation method and the DE-Sinc-collocation method, respectively. It has been known that the replacement of the variable transformation often accelerates the convergence [3, 15], and in fact, this paper shows by theoretical analysis that the rate is drastically improved to \( O(\exp(-c'N/\log N)) \) by the replacement.

The third objective of this study is to give similar theoretical results for Sinc-Nyström methods for Eq. (2). The methods have been proposed by Nurmuhammad et al. [5], where both the SE transformation and the DE transformation are considered. Those two methods are referred to as the SE-Sinc-Nyström method and the DE-Sinc-Nyström method, respectively. Any convergence analysis has not been given for both Sinc-Nyström methods, and as it stands users have no clue to decide which to choose out of the four methods: SE/DE-Sinc-collocation methods and SE/DE-Sinc-Nyström methods. To improve the situation, this paper analyzes the errors theoretically, and shows that the convergence rate of the SE-Sinc-Nyström method is \( O(\exp(-c'N/\log N)) \), and that of the DE-Sinc-Nyström method is \( O(\frac{\log N}{N} \exp(-c'N/\log N)) \).

From a viewpoint of the convergence rate, the DE-Sinc-Nyström method seems to be the best among the four methods. From a viewpoint of the computational cost, however, the DE-Sinc-collocation method has several advantages (see discussion in Section 4.3). Moreover, according to the theoretical analysis in this study, the difference of the convergence rate between the two is quite small, and in fact we can confirm it in numerical experiments (see Section 5). Therefore, the (proposed) DE-Sinc-collocation method compares favorably with the DE-Sinc-Nyström method.

The remainder of this paper is organized as follows. In Section 2 basic definitions and theorems of Sinc methods are stated. In Section 3 four numerical methods to be considered: SE/DE-Sinc-Nyström methods and SE/DE-Sinc-collocation methods are described. Main theoretical re-
results are stated in Section 4 and their proofs are given in Section 6. Numerical examples are presented in Section 5.

2. Basic definitions and theorems of Sinc methods

In this section, fundamental approximation formulas derived from the Sinc approximation are explained with their convergence theorems.

2.1. Sinc approximation and Sinc indefinite integration over the real axis

The Sinc approximation is expressed as

$$F(x) \approx \sum_{j=-N}^{N} F(jh) S(j, h)(x), \quad x \in \mathbb{R},$$

where the basis function $S(j, h)(x)$ (the so-called Sinc function) is defined by

$$S(j, h)(x) = \frac{\sin(\pi(x/h - j))}{\pi(x/h - j)},$$

and $h$ is a step size appropriately chosen depending on a given positive integer $N$. The Sinc indefinite integration is derived by integrating both sides of Eq. (3) as

$$\int_{-\infty}^{x} F(t) \, dt \approx \sum_{j=-N}^{N} F(jh) \int_{-\infty}^{x} S(j, h)(t) \, dt = \sum_{j=-N}^{N} F(jh) J(j, h)(x), \quad x \in \mathbb{R},$$

where $J(j, h)(x)$ is defined by

$$J(j, h)(x) = h \left\{ \frac{1}{2} + \frac{1}{\pi} \text{Si}\left[\pi(x/h - j)\right] \right\}.$$

Here, Si($x$) is the so-called “sine integral” function, whose routine is available in some numerical libraries (IMSL, NAG, GSL, and so on). The approximation (4) is called the Sinc indefinite integration.

2.2. (Generalized) SE-Sinc approximation and SE-Sinc indefinite integration

When the target interval $[a, b]$ is finite, the Single-Exponential (SE) transformation

$$t = \psi_{SE}(x) = \frac{b-a}{2} \tanh\left(\frac{x}{2}\right) + \frac{b+a}{2},$$

$$x = \phi_{SE}(t) = (\psi_{SE})^{-1}(t) = \log\left(\frac{1-a}{b-t}\right),$$

is frequently used with the formulas (3) and (4). Since this transformation maps $t \in (a, b)$ onto $x \in \mathbb{R}$, we can use (3) and (4) as

$$f(t) = (f \circ \psi_{SE})(\phi_{SE}(t)) \approx \sum_{j=-N}^{N} f(t_{j}^{SE}) S(j, h)(\phi_{SE}(t)), \quad \int_{a}^{b} f(s) \, ds = \int_{-\infty}^{\phi_{SE}(b)} f(\psi_{SE}(x)) (\psi_{SE})'(x) \, dx \approx \sum_{j=-N}^{N} f(t_{j}^{SE}) (\psi_{SE})'(jh) J(j, h)(\phi_{SE}(t)),$$
where \( t_j^{SE} = \psi^{SE}(jh) \). The approximations (5) and (6) are called the SE-Sinc approximation and the SE-Sinc indefinite integration, respectively.

If \( f \) is non-zero at the endpoints \( t = a \) and \( t = b \), the SE-Sinc approximation does not work accurately near the endpoints, because the right hand side of (5) tends to 0 when \( t \to a \) and \( t \to b \). To remedy the issue, Stenger [13] introduced the auxiliary basis functions \( w_a(t) = (b-t)/(b-a) \) and \( w_b(t) = (t-a)/(b-a) \), and modified the approximation as

\[
f(t) \approx P^\text{SE}_N[f](t) := f(t_N^{SE})w_a(t) + f(t_{-N}^{SE})w_b(t) + \sum_{j=-N}^{N} T^\text{SE}[f](t_j^{SE})S(j,h)(\phi^{DE}(t)),
\]

where \( T^\text{SE} \) is defined by \( T^\text{SE}[f](t) = f(t) - f(t_N^{SE})w_a(t) - f(t_{-N}^{SE})w_b(t) \). Throughout this paper, the formula (7) is called the generalized SE-Sinc approximation.

2.3. (Generalized) DE-Sinc approximation and DE-Sinc indefinite integration

Recently, instead of the SE transformation, the Double-Exponential (DE) transformation

\[
t = \psi^{DE}(x) = \frac{b-a}{2} \tanh \left( \frac{\pi}{2} \sinh(x) \right) + \frac{b+a}{2},
\]

\[
x = \phi^{DE}(t) = \{\psi^{DE}\}^{-1}(t) = \arcsinh \left[ \frac{2}{\pi} \arctanh \left( \frac{2t-b-a}{b-a} \right) \right]
\]

has been employed by several authors [3, 4, 8, 15, 17]. In this case, the formulas (7) and (6) are modified as

\[
f(t) \approx P^\text{DE}_N[f](t) := f(t_{-N}^{DE})w_a(t) + f(t_N^{DE})w_b(t) + \sum_{j=-N}^{N} T^\text{DE}[f](t_j^{DE})S(j,h)(\phi^{DE}(t)),
\]

\[
\int_{a}^{b} f(s) \, ds \approx \sum_{j=-N}^{N} f(t_j^{DE})[\psi^{DE}]'(jh)J(j,h)(\phi^{DE}(t)),
\]

where \( t_j^{DE} = \psi^{DE}(jh) \) and \( T^\text{DE}[f](t) = f(t) - f(t_{-N}^{DE})w_a(t) - f(t_N^{DE})w_b(t) \). Throughout this paper, the formulas (8) and (9) are called the generalized DE-Sinc approximation and the DE-Sinc indefinite integration, respectively.

2.4. Convergence theorems

Here let us introduce function spaces needed to state convergence theorems.

**Definition 1.** Let \( \mathcal{D} \) be a bounded and simply-connected domain (or Riemann surface). Then \( H^\omega(\mathcal{D}) \) denotes the family of functions \( f \) analytic on \( \mathcal{D} \) such that the norm \( \| f \|_{H^\omega(\mathcal{D})} = \sup_{z \in \mathcal{D}} |f(z)| \) is finite.

**Definition 2.** Let \( \alpha \) be a positive constant, and let \( \mathcal{D} \) be a bounded and simply-connected domain (or Riemann surface) which satisfies \( (a, b) \subset \mathcal{D} \). Then \( L_\alpha(\mathcal{D}) \) denotes the family of functions \( f \in H^\omega(\mathcal{D}) \) for which there exists a constant \( L \) such that for all \( z \) in \( \mathcal{D} \)

\[
|f(z)| \leq L|Q(z)|^\alpha,
\]

where the function \( Q \) is defined by \( Q(z) = (z-a)(b-z) \).
Definition 3. Let $\alpha$ be a constant with $0 < \alpha \leq 1$, and let $\mathcal{D}$ be a domain with the same conditions as in Definition 2. Then $M_\alpha(\mathcal{D})$ denotes the family of functions $f \in H^\alpha(\mathcal{D})$ for which there exists a constant $M$ such that for all $z$ in $\mathcal{D}$

$$|f(z) - f(a)| \leq M|z - a|^\alpha,$$
$$|f(b) - f(z)| \leq M|b - z|^\alpha.$$ 

In this paper, $\mathcal{D}$ is either of the following two domains:

$$\psi^{\text{SE}}(\mathcal{D}) = \{ z = \psi^{\text{SE}}(\xi) : \xi \in \mathcal{D} \} \quad \text{or} \quad \psi^{\text{DE}}(\mathcal{D}) = \{ z = \psi^{\text{DE}}(\xi) : \xi \in \mathcal{D} \},$$

where $\mathcal{D}$ is the strip domain defined by $\mathcal{D} = \{ \xi \in \mathbb{C} : |\text{Im} \xi| < d \}$ for a positive constant $d$ (see also Tanaka et al. [16, Figures 1 and 5] for the concrete shape of the domains). Convergence theorems for the generalized SE/DE-Sinc approximations are described as follows.

Theorem 1 (Okayama [6, Theorem 3], see also Stenger [13]). Let $f \in M_\alpha(\psi^{\text{SE}}(\mathcal{D}))$ for $d > 0 < d < \pi$, let $N$ be a positive integer, and let $h$ be selected by the formula

$$h = \sqrt{\frac{\pi d}{\alpha N}}. \quad (11)$$

Then there exists a constant $C$ which is independent of $N$, such that

$$\max_{a \leq t \leq b} \left| f(t) - \mathcal{P}_N^{\text{SE}}[f](t) \right| \leq C \sqrt{N} \exp \left( -\sqrt{\pi d \alpha N} \right).$$

Theorem 2 (Okayama [6, Theorem 6]). Let $f \in M_\alpha(\psi^{\text{DE}}(\mathcal{D}))$ for $0 < d < \pi/2$, let $N$ be a positive integer, and let $h$ be selected by the formula

$$h = \frac{\log(2N/\alpha)}{N}. \quad (12)$$

Then there exists a constant $C$ which is independent of $N$, such that

$$\max_{a \leq t \leq b} \left| f(t) - \mathcal{P}_N^{\text{DE}}[f](t) \right| \leq C \exp \left\{ -\frac{\pi d N}{\log(2N/\alpha)} \right\}. \quad (12)$$

Convergence theorems for the SE/DE-Sinc indefinite integration have also been given as below.

Theorem 3 (Okayama et al. [10, Theorem 2.9]). Let $(f Q) \in L_\alpha(\psi^{\text{SE}}(\mathcal{D}))$ for $d > 0 < d < \pi$, let $N$ be a positive integer, and let $h$ be selected by the formula (11). Then there exists a constant $C$ which is independent of $N$, such that

$$\max_{a \leq t \leq b} \left| \int_a^t f(s) \, ds - \sum_{j=-N}^N f(t_j^{\text{SE}})(\psi^{\text{SE}} )' (jh) J(j, h)(\psi^{\text{SE}} (t)) \right| \leq C \exp \left( -\sqrt{\pi d \alpha N} \right).$$

Theorem 4 (Okayama et al. [10, Theorem 2.16]). Let $(f Q) \in L_\alpha(\psi^{\text{DE}}(\mathcal{D}))$ for $d > 0 < d < \pi/2$, let $N$ be a positive integer, and let $h$ be selected by the formula (12). Then there exists a constant $C$ which is independent of $N$, such that

$$\max_{a \leq t \leq b} \left| \int_a^t f(s) \, ds - \sum_{j=-N}^N f(t_j^{\text{DE}})(\psi^{\text{DE}} )' (jh) J(j, h)(\psi^{\text{DE}} (t)) \right| \leq C \frac{\log(2N/\alpha)}{N} \exp \left\{ -\frac{\pi d N}{\log(2N/\alpha)} \right\}. \quad (12)
3. Numerical methods

In this section, four numerical methods to be considered in this paper are described. First three methods are existing ones: the SE-Sinc-Nyström method (Section 3.1), the DE-Sinc-Nyström method (Section 3.2), and the SE-Sinc-collocation method (Section 3.3). Fourth one is the newly-proposed method: the DE-Sinc-collocation method (Section 3.4).

3.1. SE-Sinc-Nyström method

As for the functions in Eq. (2), let \( y(\bar{t}) \) and \( g(\bar{t}) \) be each element of the vectors \( y(t) \) and \( g(t) \), respectively, and let \( k_{ij}(t) \) be \((i, j)\)-th element of the matrix \( K(t) \). Assume the following conditions:

- \( (SE1) \) \( k_{ij}(t) \in L_{\infty}(\mathbb{D}_d) \) \((i = 1, \ldots, n, \ j = 1, \ldots, n)\),
- \( (SE2) \) \( g_i(t) \in L_{\infty}(\mathbb{D}_d) \) \((i = 1, \ldots, n, \ n)\),
- \( (SE3) \) \( y_i(t) \in H^m(\mathbb{D}_d) \) \((\bar{t} = 1, \ldots, n)\),

and define \( h \) as Eq. (11). Under those assumptions, the integral in Eq. (2) can be approximated by the SE-Sinc-Nyström method, and we have the new approximated equation:

\[
\begin{align*}
\sum_{j=-N}^{N} \bigg[ \varepsilon \phi^\text{SE}(\bar{t}) + K^\text{SE}(\bar{t})Y^\text{SE} \big( \bar{t} \big) \bigg] \phi^\text{SE}(\bar{t}) \bigg] \phi^\text{SE}(\bar{t}) = h^\text{SE} + (y_0^\text{SE} \phi^\text{SE}(\bar{t})).
\end{align*}
\]

In order to determine the approximated solution \( y^{(N)}(t) \), we have to obtain the unknown coefficients on the right hand side in Eq. (13), i.e.,

\[
\begin{align*}
Y^\text{SE} = [y_1^\text{SE}(t_1^\text{SE}), \ldots, y_1^\text{SE}(t_n^\text{SE}), y_2^\text{SE}(t_1^\text{SE}), \ldots, y_2^\text{SE}(t_n^\text{SE}), \ldots, y^\text{SE}(t_1^\text{SE})]^T,
\end{align*}
\]

which is a column vector of order \((2N + 1) \times n\) (notice that \( n \) is the number of the system of equations, and \( N \) is the number appearing in \( \Sigma \)). To this end, let us discretize Eq. (13) at \((2N + 1)\) sampling points: \( t = t^\text{SE} \) \((i = -N, \ldots, N)\), and derive the system of linear equations. Let us introduce some notation here. Let \( \sigma_k = 1/2 + \text{Si}(\pi k) / \pi \), and let \( I^{(-1)}_N \) be a \((2N + 1) \times (2N + 1)\) matrix defined by

\[
I^{(-1)}_N = [\sigma_{i-j}], \quad i, j = -N, \ldots, N.
\]

Let \( I_N \) and \( I_n \) be identity matrices of order \((2N + 1) \times n\) and \( n \), respectively. Let \( D^\text{SE}_N \) and \( K^\text{SE}_{ij} \) be \((2N + 1) \times (2N + 1)\) diagonal matrices defined by

\[
D^\text{SE}_N = \text{diag}[[\phi^\text{SE}]'(t^\text{SE}), \ldots, [\phi^\text{SE}]'(N^\text{SE})],
\]

and let \( K^\text{SE}_{ij} \) be \( n \times n \) block of the matrices \( K^\text{SE}_{ij} \). Furthermore, let \( R \) and \( G^\text{SE} \) be column vectors of order \((2N + 1) \cdot n\) defined by

\[
R = [r_1, \ldots, r_1, r_2, \ldots, r_2, \ldots, r_n]^T,
\]

\[
G^\text{SE} = [g_1(t_1^\text{SE}), \ldots, g_1(t_1^\text{SE}), g_2(t_1^\text{SE}), \ldots, g_2(t_1^\text{SE}), \ldots, g_n(t_n^\text{SE})]^T.
\]

Then the system of equations can be written as

\[
(I_N \otimes I_n - I_n \otimes [hI^{(-1)}_N D^\text{SE}_N])Y^\text{SE} = I_n \otimes [hI^{(-1)}_N D^\text{SE}_N]G^\text{SE} + R,
\]

where \( \otimes \) denotes the Kronecker product. By solving the system (14), the approximated solution \( y^{(N)}(t) \) is determined by Eq. (13). This procedure is the SE-Sinc-Nyström method.
3.2. DE-Sinc-Nyström method

The important difference from the previous method is the variable transformation; the SE transformation is replaced with the DE transformation here. Assume the following conditions:

(DE1) \( k_{ij} Q \in L_\alpha(\psi^\text{DE}(Q_d)) (i = 1, \ldots, n, \ j = 1, \ldots, n) \),

(DE2) \( g_i Q \in L_\alpha(\psi^\text{DE}(Q_d)) (i = 1, \ldots, n) \),

(DE3) \( y_i \in H_\infty(\psi^\text{DE}(Q_d)) (i = 1, \ldots, n) \),

and define \( h \) as Eq. (12). Under those assumptions, the integral in Eq. (2) can be approximated by the DE-Sinc indefinite integration (9), and we have the new approximated equation:

\[
y^{(N)}(t) = r + \sum_{j=0}^{N} \left( g(t_j^\text{DE}) + K(t_j^\text{DE})y^{(N)}(t_j^\text{DE}) \right) [\psi^\text{DE}]'(jh)J(j, h)(\phi^\text{DE}(t)).
\]  

(15)

In order to determine the approximated solution \( y^{(N)} \), we have to obtain the unknown coefficients:

\[
Y^\text{DE} = [y_1^{(N)}(t_{-N}^\text{DE}), \ldots, y_1^{(N)}(t_N^\text{DE}), y_2^{(N)}(t_{-N}^\text{DE}), \ldots, y_2^{(N)}(t_N^\text{DE}), \ldots, y_n^{(N)}(t_{-N}^\text{DE})]^T,
\]

which is a column vector of order \((2N + 1) \cdot n\). To this end, let us discretize Eq. (15) at \((2N + 1)\) sampling points: \( t = t_i^\text{DE} (i = -N, \ldots, N) \), and derive the system of linear equations. Let \( D_n^\text{DE} \) and \( K_{ij}^\text{DE} \) be \((2N + 1) \times (2N + 1)\) diagonal matrices defined by

\[
D_n^\text{DE} = \text{diag}[\psi^\text{DE}]'(-Nh), \ldots, [\psi^\text{DE}]'(Nh)],
K_{ij}^\text{DE} = \text{diag}[k_{ij}(t_{-N}^\text{DE}), \ldots, k_{ij}(t_N^\text{DE})],
\]

and let \([K_{ij}^\text{DE}]\) be an \( n \times n \) block of the matrices \( K_{ij}^\text{DE} \). Furthermore, let \( G^\text{DE} \) be a column vector of order \((2N + 1) \cdot n\) defined by

\[
G^\text{DE} = [g_1(t_{-N}^\text{DE}), \ldots, g_1(t_N^\text{DE}), g_2(t_{-N}^\text{DE}), \ldots, g_2(t_N^\text{DE}), \ldots, g_n(t_{-N}^\text{DE})]^T.
\]

Then the system of linear equations to be solved is written as

\[
(I_\alpha \otimes I_N - I_n \otimes \{h[I_N^{-(1)}D_N^\text{DE}][K_{ij}^\text{DE}]\})Y^\text{DE} = I_n \otimes \{h[I_N^{-(1)}D_N^\text{DE}]G^\text{DE} + R\}.
\]  

(16)

By solving the system (16), the approximated solution \( y^{(N)} \) is determined by Eq. (15). This procedure is the DE-Sinc-Nyström method.

3.3. SE-Sinc-collocation method

Stenger [13] developed the following SE-Sinc-collocation method independently of Nurmuhammad et al. [5] (actually more than 10 years before), but below we find that it is strongly related to the SE-Sinc-Nyström method described in Section 3.1. Assume the following conditions:

(SE1) \( k_{ij} Q \in L_\alpha(\psi^\text{SE}(Q_d)) (i = 1, \ldots, n, \ j = 1, \ldots, n) \),

(SE2) \( g_i Q \in L_\alpha(\psi^\text{SE}(Q_d)) (i = 1, \ldots, n) \),

(SE4) \( y_i \in M_\alpha(\psi^\text{SE}(Q_d)) (i = 1, \ldots, n) \),

and define \( h \) as Eq. (11). Let \( Y \) be the solution of the system of linear equations (14), and let us write it as

\[
Y = [y_1,-N, y_1,-N+1, \ldots, y_1,N, y_2,-N, y_2,-N+1, \ldots, y_2,N, \ldots, y_n,N]^T.
\]  

(17)
Then the approximated solution \( \tilde{y}^{(N)}(t) = [\tilde{y}_1(t), \ldots, \tilde{y}_n(t)]^T \) is given by

\[
\tilde{y}_i^{(N)}(t) = y_{i,-N}w_a(t) + y_{i,N}w_b(t) + \sum_{j=-N}^{N} [y_{i,j} - y_{j,N}w_a(t_j^{DE}) - y_{i,N}w_b(t_j^{DE})]S(j, h)(\phi^{SE}(t)),
\]

for \( i = 1, \ldots, n \). This procedure is the DE-Sinc-collocation method.

### 3.4. DE-Sinc-collocation method (newly proposed)

In view of Sections 3.1 and 3.2 it is quite natural to replace the SE transformation with the DE transformation in the previous method. Assume the following conditions:

(DE1) \( k_{ij}Q \in L_0(\psi^{DE}(\mathcal{D}_d)) (i = 1, \ldots, n), j = 1, \ldots, n \),

(DE2) \( g_{ij}Q \in L_0(\psi^{DE}(\mathcal{D}_d)) (i = 1, \ldots, n), j = 1, \ldots, n \),

(DE4) \( y_i \in M_0(\psi^{SE}(\mathcal{D}_d)) (i = 1, \ldots, n) \),

and define \( h \) as Eq. (12). Let \( Y \) be the solution of the system of linear equations (16), and let us write it as (17). Then the approximated solution \( \tilde{y}^{(N)}(t) = [\tilde{y}_1(t), \ldots, \tilde{y}_n(t)]^T \) is given by

\[
\tilde{y}_i^{(N)}(t) = y_{i,-N}w_a(t) + y_{i,N}w_b(t) + \sum_{j=-N}^{N} [y_{i,j} - y_{j,N}w_a(t_j^{DE}) - y_{i,N}w_b(t_j^{DE})]S(j, h)(\phi^{DE}(t)),
\]

for \( i = 1, \ldots, n \). This procedure is the DE-Sinc-collocation method.

**Remark 1.** The assumptions on the solution \( y \), i.e., (SE3), (DE3), (SE4), (DE4) seem to be hard to check, because \( y \) is an unknown function to be determined. In reality, however, those assumptions are unnecessary, because both (SE3) and (SE4) can be shown from the conditions (SE1) and (SE2) and both (DE3) and (DE4) can be shown from the conditions (DE1) and (DE2). To prove the facts is one of the main contributions of this paper, which is explained next (Theorems 6 and 7).

### 4. Theoretical results

In this section, Theoretical results for the four methods in Section 3 are explained. The proofs are given in Section 6.

#### 4.1. Results on the regularity of the solution

As described in Remark 1, the condition on the solution \( y \) is assumed in each scheme. If the given problem (1) is a ‘scalar’ equation \( (n = 1) \), the following result has been known.

**Theorem 5 (Stenger et al. [14, Theorem 2.3]).** Let \( n = 1 \), and let the assumptions (SE1) and (SE2) be fulfilled. Then the initial-value problem (1) has a unique solution \( y_1 \in M_0(\psi^{SE}(\mathcal{D}_d)) \).

This theorem shows the condition (SE4), and since \( M_0(\psi^{SE}(\mathcal{D}_d)) \subset H^\infty(\psi^{SE}(\mathcal{D}_d)) \), the condition (SE3) is also shown. In this paper, the same result is shown in the case of a system of equations (for both SE and DE).

**Theorem 6.** Let the assumptions (SE1) and (SE2) be fulfilled. Then the initial-value problem (1) has a unique solution \( y \) with \( y_i \in M_0(\psi^{SE}(\mathcal{D}_d)) \) for \( i = 1, \ldots, n \).

**Theorem 7.** Let the assumptions (DE1) and (DE2) be fulfilled. Then the initial-value problem (1) has a unique solution \( y \) with \( y_i \in M_0(\psi^{DE}(\mathcal{D}_d)) \) for \( i = 1, \ldots, n \).
4.2. Results on convergence of the numerical solutions

In the case of a ‘scalar’ equation, the convergence of the SE-Sinc-collocation method is analyzed as follows. In what follows, \( C \) denotes a constant independent of \( N \).

**Theorem 8 (Stenger [13, pp. 446–447]).** Let \( n = 1 \), and let the assumptions (SE1) and (SE2) be fulfilled. Then, for all \( N \) sufficiently large, the system (14) is uniquely solvable, and the error of the numerical solution \( \tilde{y}_1 \) of Eq. (18) is estimated as

\[
\max_{a \leq t \leq b} |y_1(t) - \tilde{y}_1(t)| \leq C \sqrt{N} \exp \left( - \sqrt{\pi dN} \right).
\]

This paper extends the result to a system of equations, and to the DE-Sinc-collocation method.

**Theorem 9.** Let the assumptions (SE1) and (SE2) be fulfilled. Then, for all \( N \) sufficiently large, the system (14) is uniquely solvable, and the error of the numerical solution \( \tilde{y}^{(N)} \) of Eq. (18) is estimated as

\[
\max_{i = 1, \ldots, n} \left\{ \max_{a \leq t \leq b} |y_i(t) - \tilde{y}_i^{(N)}(t)| \right\} \leq C \sqrt{N} \exp \left( - \sqrt{\pi dN} \right).
\]

**Theorem 10.** Let the assumptions (DE1) and (DE2) be fulfilled. Then, for all \( N \) sufficiently large, the system (16) is uniquely solvable, and the error of the numerical solution \( \tilde{y}^{(N)} \) of Eq. (19) is estimated as

\[
\max_{i = 1, \ldots, n} \left\{ \max_{a \leq t \leq b} |y_i(t) - \tilde{y}_i^{(N)}(t)| \right\} \leq C \exp \left\{ - \frac{\pi dN}{\log(2dN/\alpha)} \right\}.
\]

Furthermore, this paper also shows the convergence of the SE/DE-Sinc-Nyström methods.

**Theorem 11.** Let the assumptions (SE1) and (SE2) be fulfilled. Then, for all \( N \) sufficiently large, the system (14) is uniquely solvable, and the error of the numerical solution \( y^{(N)} \) of Eq. (13) is estimated as

\[
\max_{i = 1, \ldots, n} \left\{ \max_{a \leq t \leq b} |y_i(t) - y_i^{(N)}(t)| \right\} \leq C \exp \left( - \sqrt{\pi dN} \right).
\]

**Theorem 12.** Let the assumptions (DE1) and (DE2) be fulfilled. Then, for all \( N \) sufficiently large, the system (16) is uniquely solvable, and the error of the numerical solution \( \tilde{y}^{(N)} \) of Eq. (15) is estimated as

\[
\max_{i = 1, \ldots, n} \left\{ \max_{a \leq t \leq b} |y_i(t) - \tilde{y}_i^{(N)}(t)| \right\} \leq C \exp \left\{ - \frac{\pi dN}{\log(2dN/\alpha)} \right\}.
\]

4.3. Discussion about the performance

In view of the convergence rates shown above, the DE-Sinc-Nyström method seems to be the best, and this was then followed by the DE-Sinc-collocation method, the SE-Sinc-Nyström method, and the SE-Sinc-collocation method. However, the DE-Sinc-collocation method (the second one) can be considered as the best, or at least as useful as the DE-Sinc-Nyström method, for the following reasons. Firstly, the difference of convergence between the DE-Sinc-Nyström method and the DE-Sinc-collocation method is quite small, and actually it is almost indistinguishable in the numerical experiments (see Figures 1–2 in Section 5). Secondly, compared to the
the approximate solution of the DE-Sinc-collocation method $\tilde{y}^N$ (Eq. (19)), that of the DE-Sinc-Nyström method $y^N$ (Eq. (15)) has time-consuming terms to evaluate. All of the basis functions in $\tilde{y}^N$ are elementary functions, whereas the basis functions $J(j, h)$ in $y^N$ includes the special function $\text{Si}(x)$. Furthermore, $\tilde{y}^N$ can be computed with $O(nN)$, but $y^N$ needs $O(n^2N)$ because a matrix-vector product is included in $y^N$. Therefore, from the viewpoint of the computational cost, the DE-Sinc-collocation method is better than the DE-Sinc-Nyström method (see also Table 1).

5. Numerical results

In this section, numerical examples of the SE/DE-Sinc-Nyström methods and the SE/DE-Sinc-collocation methods are presented. The computation was done on Mac OS X 10.6, Mac Pro two 2.93 GHz 6-Core Intel Xeon with 32 GB DDR3 ECC SDRAM. The computation programs were implemented in C++ with double-precision floating-point arithmetic, and compiled by GCC 4.0.1 with no optimization. The linear systems (14) and (16) are solved by using the LU decomposition. In what follows, $\pi_-$ denotes an arbitrary positive number less than $\pi$, and it was set as $\pi_- = 3.14$ in actual computation. Firstly, let us consider the following two examples.

**Example 1.** Consider the following initial value problem (the Halm equation [12]) over the interval $[0, 1]$:

$$(1 + x^2)^2y''(t) - 2y = 0, \quad y(0) = 0, \quad y'(0) = 1,$$

which is equivalent to the system

$$y'_1(t) = y_2(t), \quad y_1(0) = 0,$$

$$y'_2(t) = \frac{2}{(1 + x^2)^2}y_1(t), \quad y_2(0) = 1,$$

whose solution is $y_1(t) = \sqrt{1 + x^2}\sinh(\arctan x)$, $y_2(t) = y'_1(t)$.

**Example 2.** Consider the following initial value problem over the interval $[0, 2]$:

$$y'_1(t) = -y_1(t) + \frac{1}{2 \sqrt{t}}y_2(t), \quad y_1(0) = 0,$$

$$y'_2(t) = -\frac{1}{\sqrt{t}}y_1(t), \quad y_2(0) = 1,$$

whose solution is $y_1(t) = \sqrt{t} \exp(-t)$, $y_2(t) = \exp(-t)$.

As for Example 1, the conditions (SE1) and (SE2) are satisfied with $\alpha = 1$ and $d = 3\pi_-/4$. In the DE case, let us set $p = \pi_-/(2 \log 2)$ and

$$q = \sqrt{\left[1 + 7p^2 + \sqrt{(1 + 7p^2)^2 + (6p)^2}\right]/2},$$

and furthermore set $x = -(1 - q)/(4p)$, $y = 3(1 - (1/q))/4$, and $d_-=\arcsin(y/\sqrt{x^2 + y^2})$. Then, the conditions (DE1) and (DE2) are satisfied with $\alpha = 1$ and $d = d_-$. As for Example 2 which is a harder example because of the singularity at the origin, (SE1) and (SE2) are satisfied with $\alpha = 1/2$ and $d = \pi_-$, and (DE1) and (DE2) are satisfied with $\alpha = 1/2$ and $d = \pi_-/2$. The numerical errors
The errors in Example 1 are plotted in Figure 1, and the errors in Example 2 are plotted in Figure 2. In the graphs, “maximum error” denotes the maximum absolute error at 999 equally-spaced points (say \( t_l \)) on the interval \([a, b] \), i.e.,

\[
\text{maximum error} = \max_{i=1,\ldots,999} \max_{t_l=1,\ldots,999} |y_i(t_l) - \hat{y}_i(t_l)|,
\]

where \( \hat{y}_i \) means each numerical solution. From both figures, we can confirm the results of Theorems 9–12. More precisely, as for the (newly-proposed) DE-Sinc-collocation method, its convergence rate is actually much higher than that of the SE-Sinc-collocation method. As described in Section 4.3, although the theoretical rate of the DE-Sinc-Nyström method is a bit higher than that of the DE-Sinc-collocation method, both rates are almost indistinguishable in the numerical results. Moreover, as seen in Table 1, the DE-Sinc-Nyström methods needs times twice as much as the DE-Sinc-collocation method to obtain \(10^{-8} \) accuracy (the same applies in the SE case). At least from the result, we can conclude that the DE-Sinc-collocation method is the most efficient.

In the examples above, all the assumptions (SE1), (SE2), (DE1), and (DE2) are satisfied with some \( \alpha \) and \( d \). Let us have a look at another case here.

**Example 3.** Set a function \( F \) as \( F(t) = \sqrt{\cos(4 \arctanh t) + \cosh(\pi)} \), and consider the following initial value problem over the interval \([-1, 1] \):

\[
\begin{align*}
y'_1(t) &= -\frac{2[tF^2(t) + \sin(4 \arctanh t)]}{F(t)}y_2(t), & y_1(-1) = 0, \\
y'_2(t) &= \frac{2[tF^2(t) + \sin(4 \arctanh t)]}{F(t)}y_1(t), & y_2(-1) = 1,
\end{align*}
\]

whose solution is \( y_1(t) = \sin[(1 - t^2)F(t)] \), \( y_2(t) = \cos[(1 - t^2)F(t)] \).
This is a quite hard example to solve numerically, due to the bad behavior of $F$ at $t = \pm 1$ (non-regular points are densely distributed around the endpoints). Fortunately, the assumptions (SE1) and (SE2) are satisfied with $\alpha = 1$ and $d = \pi/2$, but (DE1) and (DE2) are not satisfied with any $d > 0$ (we easily see $\alpha = 1$, though). Therefore, Theorems [10] and [12] cannot be used in this case. However, according to the recent result [11], even in such a case, DE’s methods may achieve the same convergence rate with that of SE, by setting $d = \arcsin(d_{SE}/\pi)$, where $d_{SE}$ denotes SE’s $d$. We can in fact observe it in Figure 3. DE’s methods seem to converge with the similar rate to that of SE. Since the computational cost is the same as that of the previous examples, we can consider that the DE-Sinc collocation method still keeps the lead even in this case.

6. Proofs

6.1. Proofs on the regularity of the solution

The idea here is to apply the standard contraction mapping theorem, which holds not only in the scalar case but also in the case of a system of equations. Set $X = \{H^{\infty}(\mathbb{D})\}^n$ and $Y = \{M_{\alpha}(\mathbb{D})\}^n$, and define $\|f\|_X = \max_{i=1,...,n}\|f_i\|_{H^{\infty}(\mathbb{D})}$. The goal is to show $y \in Y$, but it is not easy because $Y$ is not a Banach space. For this reason, firstly $y \in X$ is shown ($X$ is a Banach space), and by using the result, $y \in Y$ is shown. Let us introduce the integral operator $J : X \to X$ as $J[f](t) = \int_a^t f(s) \, ds$, and $V : X \to X$ as

$$V[f](t) = \int_a^t K(s)f(s) \, ds,$$

where $K$ satisfies the assumption (SE1) or (DE1). If the operator is multiplied repeatedly, it becomes a contraction map.

**Lemma 13.** Let the assumption (SE1) be fulfilled. Then it holds for all positive integers $m$ and $z \in \psi^{\infty}(\mathbb{D}_d)$ that

$$|V^m[f](z)| \leq \frac{nL(b-a)^{2\alpha-1}c_1 B(\psi_1(x), \alpha, \alpha)^m}{m!} \|f\|_X [1, 1, \ldots, 1]^T,$$

12
where \( x = \text{Re}[\psi^{SE}(z)] \), \( \psi_1(x) = (\tanh(x/2) + 1)/2 \), \( B(x, \alpha, \beta) \) is the incomplete beta function, \( L \) is the constant in Eq. (10), and \( c_1 \) is a constant depending only on \( d \).

**Lemma 14.** Let the assumption (DE1) be fulfilled. Then it holds for all positive integers \( m \) and \( z \in \psi^{SE}(\mathcal{D}_d) \) that
\[
|V^m[f](z)| \leq \frac{(nL(b-a)^{2\alpha-1}c_2B(\psi_2(x), \alpha, \alpha))^m}{m!} \|f\|_X \|1, 1, \ldots, 1\|_X^T,
\]
where \( x = \text{Re}[\psi^{SE}(z)] \), \( \psi_2(x) = (\tanh(\pi \sinh(x)/2) + 1)/2 \), \( L \) is the constant in Eq. (10), and \( c_2 \) is a constant depending only on \( d \).

These lemmas are straightforward extension from the existing ones [9, Lemmas 5.4 and 5.6], and the proofs are omitted. Then in both cases it holds that
\[
\|V^m f\|_X \leq \frac{(nL(b-a)^{2\alpha-1}c_2B(\psi_2(x), \alpha, \alpha))^m}{m!} \|f\|_X,
\]
and thus for sufficiently large \( m \), \( V^m \) is a contraction map, from which we have the next theorem.

**Theorem 15.** Let the assumptions (SE1) and (SE2) be fulfilled. Then Eq. (2) has a unique solution \( y \in X \), i.e., \( y_i \in H^\infty(\psi^{SE}(\mathcal{D}_d)) \) for \( i = 1, \ldots, n \).

**Theorem 16.** Let the assumptions (DE1) and (DE2) be fulfilled. Then Eq. (2) has a unique solution \( y \in X \), i.e., \( y_i \in H^\infty(\psi^{DE}(\mathcal{D}_d)) \) for \( i = 1, \ldots, n \).

**Proof.** Before applying the contraction mapping theorem, the only thing we have to show is (SE2)/(DE2) \( \Rightarrow J g \in X \), which is done by Lemmas [17] and [19] (since \( M_\alpha(\mathcal{D}) \subset H^\infty(\mathcal{D}) \)).

The next lemma is a result for SE, which completes the proof of Theorem [15].

**Lemma 17 (Stenger [13, Theorem 4.1.3]).** Let \( gQ \in L_\alpha(\psi^{SE}(\mathcal{D}_d)) \), and set \( q(t) = \int_a^t g(s) \, ds \). Then \( q \in M_\alpha(\psi^{SE}(\mathcal{D}_d)) \).

In the case of DE (for Theorem [16]), we need the next lemma.

**Lemma 18 (Okayama et al. [7, Lemma A.4]).** For \( x \in \mathbb{R} \) and \( y \in (-\pi/2, \pi/2) \), it holds that
\[
\psi_2(x) := \frac{1}{2} \tanh \left( \frac{\pi \cos y}{2} \sinh x \right) + \frac{1}{2} \leq \left| \frac{1}{2} \tanh \left( \frac{\pi}{2} \sinh(x + iy) \right) + \frac{1}{2} \right|.
\]

Using Lemma [18] we show the following lemma (DE version of Lemma [17]).

**Lemma 19.** Let \( gQ \in L_\alpha(\psi^{DE}(\mathcal{D}_d)) \), and set \( q(t) = \int_a^t g(s) \, ds \). Then \( q \in M_\alpha(\psi^{DE}(\mathcal{D}_d)) \).
Proof. Clearly \( q \in H^\omega(\mathcal{D}_d) \) holds. Let us show the Hölder continuity at the endpoint \( a \) (showing it at \( b \) is omitted, because it is quite similar). Putting \( n = 1, f \equiv 1, K = g \) in Lemma 14 we have
\[
\left| \int_a^z g(w) \, dw \right| \leq L(b - a)^{2\alpha - 1} c_2 B(\psi_2(x), \alpha, \alpha).
\]
Then it holds that
\[
|q(z) - q(a)| = \left| \int_a^z g(w) \, dw \right| \leq L(b - a)^{2\alpha - 1} c_2 B(\psi_2(x), \alpha, \alpha) \leq L(b - a)^{2\alpha - 1} c_2 B(1, \alpha, \alpha)\psi_2(x)^\alpha.
\]
Furthermore, from Lemma 18 it holds that
\[
(b - a)\psi_2(x) \leq \left| \frac{b - a}{2} \tan \left( \frac{\pi}{2} \sinh(x + iy) \right) + \frac{b - a}{2} \right| = |\psi^{\text{DE}}(x + iy) - a| = |z - a|.
\]
Thus there exists a constant \( \tilde{L} \) such that \(|q(z) - q(a)| \leq \tilde{L}|z - a|^\alpha\) for all \( z \in \psi^{\text{DE}}(\mathcal{D}_d) \).

Now showing \( y \in X \) is finished. For \( y \in Y \), what is left is to show the Hölder continuity. In view of the right hand side of Eq. (2), clearly \( y \) (on the left hand side) is Hölder continuous of \( \alpha \)-order. Hence Theorems 6 and 7 are established.

6.2. Proofs on convergence of the numerical solutions

6.2.1. SE/DE-Sinc-Nyström method

Firstly, the SE-Sinc-Nyström method is considered. Notice that in this subsection, set \( C = \{C([a, b])\}^n \), and all operators here are discussed on this function space. Let us introduce the operator \( \mathcal{J}_N^{\text{SE}} \), which is an approximation of \( \mathcal{J} \), as
\[
\mathcal{J}_N^{\text{SE}}[f](t) = \sum_{j=-N}^N f(t_j^{\text{SE}})(\psi^{\text{SE}})'\phi^\alpha J(j, h)\phi^{\text{SE}}(t),
\]
and \( \mathcal{V}_N^{\text{SE}} \) as \( \mathcal{V}_N^{\text{SE}}[f](t) = \mathcal{J}_N^{\text{SE}}[Kf](t) \). Then consider the following three equations:
\[
(I - \mathcal{V}_N^{\text{SE}})[y] = r + \mathcal{J}_N^{\text{SE}} g \quad \text{(Eq. (2))},
\]
\[
(I - \mathcal{V}_N^{\text{SE}})[y]^{(N)} = r + \mathcal{J}_N^{\text{SE}} g \quad \text{(Eq. (13))},
\]
\[
(I_n \otimes I_N - I_n \otimes \{h_1^{(-1)} D_N^{\text{SE}}[K^{\text{SE}}]\}) Y^{\text{SE}} = I_n \otimes \{h_1^{(-1)} D_N^{\text{SE}}\} G^{\text{SE}} + R \quad \text{(Eq. (14))}.
\]
Using the standard arguments (e.g., see [9, Lemma 6.1]), we can see the unique solvability of Eq. (14) is equivalent to that of Eq. (13). If the unique solvability of Eq. (13) is shown, i.e., \((I - \mathcal{V}_N^{\text{SE}})^{-1}\) exists, we have
\[
y - y^{(N)} = (I - \mathcal{V}_N^{\text{SE}})^{-1} \{ (I - \mathcal{V}_N^{\text{SE}})[y] - (r + \mathcal{J}_N^{\text{SE}} g) \}
\]
\[
= (I - \mathcal{V}_N^{\text{SE}})^{-1} \{ (r + \mathcal{J}_N^{\text{SE}} g + \mathcal{V}_N^{\text{SE}} y) - \mathcal{V}_N^{\text{SE}} y - \mathcal{V}_N^{\text{SE}} y - \mathcal{J}_N^{\text{SE}} g \}
\]
\[
= (I - \mathcal{V}_N^{\text{SE}})^{-1} \{ (\mathcal{J}_N^{\text{SE}} g - \mathcal{J}_N^{\text{SE}} g) + (\mathcal{V}_N^{\text{SE}} y - \mathcal{V}_N^{\text{SE}} y) \},
\]
and finally using Theorem 8 the desired error estimate (Theorem 11) is obtained. Therefore, what is left is to show the existence and boundedness of \((I - \mathcal{V}_N^{\text{SE}})^{-1}\). For the purpose, the next theorem is useful.

**Theorem 20** (Atkinson [1, Theorem 4.1.1]). Assume the following four conditions:
1. Operators $X$ and $X_n$ are bounded operators on $C$ to $C$.
2. The operator $(I - X) : C \rightarrow C$ has a bounded inverse $(I - X)^{-1} : C \rightarrow C$.
3. The operator $X_n$ is compact on $C$.
4. The following inequality holds:
\[
\|(X - X_n)X_n\|_{L(C,C)} < \frac{1}{\|(I - X)^{-1}\|_{L(C,C)}}.
\]

Then $(I - X_n)^{-1}$ exists as a bounded operator on $C$ to $C$, with
\[
\|(I - X_n)^{-1}\|_{L(C,C)} \leq \frac{1 + \|(I - X)^{-1}\|_{L(C,C)}\|(X - X_n)X_n\|_{L(C,C)}}{1 - \|(I - X)^{-1}\|_{L(C,C)}\|(X - X_n)X_n\|_{L(C,C)}}.
\]

We need to show the four conditions in this theorem as $X = V$ and $X_n = V^{SE}_n$. The first condition clearly holds, and the second condition is known as a classical result. The third condition immediately follows from the Arzelá–Ascoli theorem. The fourth condition is shown by the next lemma, which is straightforward extension from the existing one [9, Lemma 6.5].

**Lemma 21.** Let the assumption (SE1) be fulfilled. Then there exists a constant $C$ independent of $N$ such that
\[
\|(V - V^{SE}_N)\|_{L(C,C)} \leq \frac{C}{\sqrt{N}}.
\]
Furthermore, $\|V^{SE}_N\|_{L(C,C)}$ is uniformly bounded, since $V^{SE}_Nf$ converges to $Vf$ for any $f \in C$. Thus, from Eq. (20), we obtain the desired result: $(I - V^{SE}_N)^{-1}$ exists and uniformly bounded for all sufficiently large $N$. This completes the proof of Theorem [11] (the SE-Sinc-Nyström method).

The proof for the DE-Sinc-Nyström method goes on in exactly the same way. Let us introduce the operator $J^{DE}_N$ as
\[
J^{DE}_N[f](t) = \sum_{j=-N}^{N} f(t_j^{DE}) (\psi^{DE}_j)(j)hJ(j,h)(\phi^{DE}(t)),
\]
and $V^{DE}_N$ as $V^{DE}_N[f](t) = J^{DE}_N[Kf](t)$. The difference from the SE is the next lemma, which is also straightforward extension from the existing one [9, Lemma 6.9].

**Lemma 22.** Let the assumption (DE1) be fulfilled. Then there exists a constant $C$ independent of $N$ such that
\[
\|(V - V^{SE}_N)\|_{L(C,C)} \leq C \left( \frac{\log(2dN/\alpha)}{N} \right)^2.
\]
This completes the proof of Theorem [12] (the DE-Sinc-Nyström method).

### 6.2.2. SE/DE-Sinc-collocation method

Let us consider the SE-Sinc-collocation method first. Notice the relation $\tilde{y}_i^{(N)}(t) = P^{SE}_N[y_i^{(N)}](t)$, where $y_i^{(N)}$ is the solution of the SE-Sinc-Nyström method (see Eq. (13)), and $\tilde{y}_i^{(N)}$ is the solution of the SE-Sinc-collocation method (see Eq. (13)). Then we have
\[
\|y_i - \tilde{y}_i^{(N)}\|_C \leq \|y_i - P^{SE}_N[y_i^{(N)}]\|_C + \|P^{SE}_N\|_{L(C,C)} \|y_i^{(N)} - \tilde{y}_i^{(N)}\|_C.
\]
The first term on the right hand side can be estimated by Theorem 1. On the second term, use Theorem 11 for \(|y_i - y_i^N|\), and use the next lemma to obtain \(\|P_{SE}\|_{L^2(C,C)} \leq C \log(N + 1)\).

**Lemma 23 (Stenger [13, p. 142]).** Let \(h > 0\). Then it holds that
\[
\sup_{\xi \in \mathbb{R}} \sum_{j=-N}^{N} |S(j, h)(\xi)| \leq \frac{2}{\pi} (3 + \log N).
\]

This completes the proof of Theorem 9 (the SE-Sinc-collocation method).

The proof for the DE-Sinc-collocation method goes on in exactly the same way. By using the relation \(y_i^N(t) = P_{DE}^N[y_i^N](t)\), we have the similar inequality as Eq. (21) (just replace SE with DE).

By estimating each term via Theorems 2 and 12 and Lemma 23, Theorem 10 is established.

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