TORIC SURFACES OVER AN ARBITRARY FIELD

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Abstract. We study toric varieties over an arbitrary field with an emphasis on toric surfaces in the Merkurjev-Panin motivic category. We explore the decomposition in the motivic category of certain toric varieties into products of finite Azumaya algebras, the geometric and topological information encoded in these Azumaya algebras, and the relationship between the decomposition in the motivic category and the semiorthogonal decomposition of the derived category. We obtain the information mentioned above for toric surfaces by classifying all minimal smooth projective toric surfaces.

1. Introduction

Letting $X$ be a scheme over a field $k$ and letting $K/k$ be a field extension, we say a scheme $Y$ over $k$ is a $K/k$-form of $X$ if base change from $k$ to $K$, the schemes $X_K := X \otimes_k K$ and $Y_K$ are isomorphic as schemes over $K$ [Ser97, Chapter III §1]. For $k^s$ the separable closure of $k$, a $k^s/k$-form is simply called a form or twisted form. The scheme $X_k^s$ has a natural $\Gamma = \text{Gal}(k^s/k)$-action.

We will focus on the study of toric varieties $X$ over $k$. The case of split toric varieties has been extensively studied, for example in [Dan78][Ful93][CLS11]. Since any toric variety $X$ has a torus action over $k$ and is a twisted form of a split toric variety, the study of $X$ is equivalent to the study of the split toric variety $X_k^s$ with $\Gamma$-action on the fan structure as well as the open orbit $U$ of $X$ under the torus action, see §3. This paper is motivated by ideas in [MP97], which studies toric varieties over an arbitrary field in the motivic category $C$, and in particular by the following question in loc. cit.:

Question 1. If $X$ is a smooth projective toric variety over $k$, is $K_0(X_k^s)$ always a permutation $\Gamma$-module?

Definition 1.1. A $\Gamma$-module $M$ is a permutation $\Gamma$-module if there exists a $\Gamma$-invariant $\mathbb{Z}$-basis of $M$. We call such a basis a permutation $\Gamma$-basis or $\Gamma$-basis.

The reason that we care about the $\Gamma$-action on $K_0(X_k^s)$ is that it in some way determines $X$, cf §6. More specifically, if $K_0(X_k^s)$ has a permutation $\Gamma$-basis of line bundles over $X_k^s$, then $X$ has a decomposition into a finite product of finite Azumaya algebras in the motivic category $C$ completely described by this $\Gamma$-basis (Theorem 6.5).

The original motivation for finding the decomposition of a smooth projective variety over $k$ into a product of finite Azumaya algebras in $C$ is to compute higher algebraic $K$-theory of the variety. Quillen [Qui73] computed higher algebraic $K$-theory for Severi-Brauer varieties, cf Example 3.5, and Swan [Swa85] for quadric hypersurfaces. Panin [Pan94] generalized their results by finding the decomposition in $C$ for twisted flag varieties. In fact, these Azumaya algebras also encode arithmetic/geometric information about the variety, and in nice cases, classify all its twisted forms.

Blunk investigated del Pezzo surfaces of degree 6 over arbitrary fields in [Blu10] in this direction, cf Example 3.6. We will investigate the same information for all smooth projective toric surfaces over arbitrary fields, cf §7. Moreover, since Tabuada [Tab14, Theorem 6.10] showed that the motivic category $C$ is a part of the category of noncommutative motives $Hmo$, it follows that certain
semiorthogonal decompositions of the derived category will give a decomposition of the variety in $\mathcal{C}$. We will also briefly discuss the possibility of lifting the decomposition of a smooth projective toric variety with a permutation basis of line bundles to the derived category, cf $\S 8$.

While higher dimensional toric varieties over arbitrary fields are still out of reach, we obtain a rather complete picture of smooth projective toric surfaces in this paper. The main tool is the classification of minimal smooth projective toric surfaces (Theorem 4.9). The classification and its implications are discussed in $\S 4$, $\S 5$, $\S 7$, while results for toric varieties of all dimensions are discussed in $\S 6$ and $\S 8$.

1.1. Organization. The organization of the paper is as follows:

Throughout, the base field $k$ will be arbitrary. Sections 2 and 3 introduce the background on the motivic category $\mathcal{C}$ and toric varieties over $k$, including some basic facts and examples needed for the paper. For more details about $\mathcal{C}$, see [MP97, §1] or [Mer05, §3]. Section 4 classifies minimal smooth projective toric surfaces over $k$ via toric geometry. Section 5 verifies that $K_0(X_{k^s})$ has a permutation $\Gamma$-basis of line bundles for toric surfaces. In section 6, we consider smooth projective toric varieties $X$ of all dimensions where $K_0(X_{k^s})$ has a permutation $\Gamma$-basis of line bundles. We reinterpret the construction for the separable algebra corresponding to a toric variety in [MP97], and deduce that it provides exactly a decomposition of a smooth projective toric variety with the aforementioned property in terms of such a basis. In section 7, we apply the construction in $\S 6$ to toric surfaces. Moreover, we relate the constructed algebras to the open orbit $U$ via Galois cohomology. For details of Galois cohomology, see [Ser97][KMRT98][GS06]. In section 8, we discuss the relationship between the semiorthogonal decomposition of the derived category and the decomposition in the motivic category of toric varieties via noncommutative motives and descent theory for derived categories.

Most of the time, instead of working with $X_{k^s}$ and $\Gamma$-action, we work with $X_l$ and $G = \text{Gal}(l/k)$-action where $l$ is the splitting field of the torus $T$.

1.2. Notation. Fix the base field $k$ and a separable closure $k^s$ of $k$. Let $\Gamma = \text{Gal}(k^s/k)$. $T$ is an algebraic torus over $k$. $l$ is the splitting field of $T$ and $G = \text{Gal}(l/k)$ unless otherwise stated. For any object $Z$ (algebraic groups, varieties, algebras, maps) over $k$ and any extension $K/k$, write $Z \otimes_k K$ as $Z_K$.

For a split toric variety $Y$, we denote $\Sigma$ as the fan structure and $\text{Aut}_\Sigma$ as the group of fan automorphisms. We will freely use the same notation for the ray in the fan, the minimal generator of the ray in the lattice and the Weil divisor corresponding to the ray when the context is clear.

For an algebra $A$, denote $A^{\text{op}}$ as its opposite algebra. Denote $S_n$ as the permutation group of a set of $n$ elements.

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2. Motivic Category $\mathcal{C}$

Definition 2.1. The motivic category $\mathcal{C} = \mathcal{C}_k$ over a field $k$ has:

- Objects: The pair $(X, A)$ where $X$ is a smooth projective variety over $k$, $A$ is a finite separable $k$-algebra
- Morphisms: $\text{Hom}_\mathcal{C}((X, A), (Y, B)) = K_0(X \times Y, A^{\text{op}} \otimes_k B)$

The Grothendieck group $K_0$ of a pair is defined below. A $k$-algebra $A$ is finite separable if $\dim_k(A)$ is finite and for any field extension $K$ of $k$, $A_K$ is semisimple. Equivalently we have:

Definition 2.2. The algebra $A$ is a finite separable $k$-algebra if it is a finite product of Azumaya $l_i$-algebras $A_i$ where $l_i$ is a finite separable field extension of $k$, i.e, $A_i$ is a matrix algebra over a finite dimensional division algebra with center $l_i$. 
Let \( u : (X,A) \to (Y,B) \) and \( v : (Y,B) \to (Z,C) \) be morphisms in \( \mathcal{C} \). Since \( u \in K_0(X \times Y, A^{\text{op}} \otimes_k B) \cong K_0(Y \times X, B \otimes_k A^{\text{op}}) \), the map \( u \) can also be viewed as \( u^{\text{op}} : (Y,B^{\text{op}}) \to (X,A^{\text{op}}) \). The composition \( v \circ u : (X,A) \to (Z,C) \) is given by
\[
\pi_*(q^* v \otimes_B p^* u)
\]
where \( p : X \times Y \times Z \to X \times Y \), \( q : X \times Y \times Z \to Y \times Z \), \( \pi : X \times Y \times Z \to X \times Z \) are projections.

We write \( X \) for \((X,k)\) and \( A \) for \((\text{Spec } k, A)\). Since the morphisms are defined in \( K_0 \), the category is also called \( K\)-correspondences.

### 2.1. Algebraic \( K \)-theory of a pair.

The algebraic \( K \)-theory of a pair \((X,A)\) is defined in the following way and it generalizes the Quillen \( K \)-theory of varieties:

Let \( \mathcal{P}(X,A) \) be the exact category of left \( \mathcal{O}_X \otimes_k A \)-modules which are locally free \( \mathcal{O}_X \)-modules of finite rank and morphisms of \( \mathcal{O}_X \otimes_k A \)-modules. The group \( K_n(X,A) \) of the pair \((X,A)\) is defined as \( K_0^\mathcal{Q}(\mathcal{P}(X,A)) \), the Quillen \( K \)-theory of \( \mathcal{P} \). Let \( \mathcal{M}(X,A) \) be the exact category of left \( \mathcal{O}_X \otimes_k A \)-modules which are coherent \( \mathcal{O}_X \)-modules and morphisms of \( \mathcal{O}_X \otimes_k A \)-modules. The group \( K'_n(X,A) \) of the pair \((X,A)\) is defined as \( K_0^\mathcal{Q}(\mathcal{M}(X,A)) \). The embedding \( \mathcal{P} \subset \mathcal{M} \) induces a map \( K_n(X,A) \to K'_n(X,A) \) and it is an isomorphism if \( X \) is regular (resolution theorem). Note that \( K_n(X,k) \) is the usual \( K_n(X) \) and \( K_n(\text{Spec } k, A) = K_n(\text{Rep}(A)) \) is the \( K \)-theory of representations of \( A \).

In fact, \( K_n \) defines a functor \( K_n : \mathcal{C} \to \text{Ab} \) which sends \((X,A)\) to \( K_n(X,A) \). For \( u : (X,A) \to (Y,B) \), \( x \in K_n(X,A) \), we can define
\[
K_n(u)(x) = q_*(u \otimes_A p^* x)
\]
where \( p : X \times Y \to X \), \( q : X \times Y \to Y \) are projections.

Similarly we can define, for any variety \( V \) over \( k \), a functor \( K^V_n : \mathcal{C} \to \text{Ab} \) where on objects \( K^V_n(X,A) = K'_n(V \times X,A) \).

**Example 2.3.** [MP97, Example 1.6(1)] \( M_n(k) \cong k \) in \( \mathcal{C} \).

**Example 2.4.** [MP97, Example 1.6(3)], see also [Tab14, Theorem 9.1]. Let \( A \) and \( B \) be two central simple \( k \)-algebras. Then \( A \cong B \) in \( \mathcal{C} \) if and only if \([A] = [B] \in \text{Br}(k)\).

**Proof.** Previous example indicates that Brauer equivalences give isomorphisms in \( \mathcal{C} \), so \([A] = [B] \in \text{Br}(k)\) implies \( A \cong B \) in \( \mathcal{C} \).

For the opposite direction, since each central simple \( k \)-algebra is Brauer equivalent to a unique division \( k \)-algebra, we can assume \( A,B \) are division algebras. Let \( M : A \to B \) and \( N : B \to A \) be inverse maps in \( \mathcal{C} \). Since \( K_0(A^{\text{op}} \otimes_k B) \cong ZR \) and \( K_0(B^{\text{op}} \otimes_k A) \cong ZR^{\text{op}} \) for \( R \) the unique simple \( A-B \)-bimodule, we have \( M = nR \) and \( N = mR^{\text{op}} \) for some \( m,n \in Z \). \( N \circ M = N \otimes_B M \cong mnR^{\text{op}} \otimes_B R \cong A \), \( M \circ N = M \otimes_A N \cong mnR \otimes_A R^{\text{op}} \cong B \). Since \( A,B \) are simple modules, we have \( mn = 1 \) and we can assume \( M = R, N = R^{\text{op}} \). As a right \( A \)-module and a left \( B \)-module respectively, we have \( M_A \cong A^{\text{op}} \) and \( B_M \cong B^{\text{op}} \). Similarly, \( A N \cong A P \) and \( N_B \cong B^q \). The left \( A \)-module isomorphism \( N \otimes_B M \cong N \otimes_B B^s \cong N^s \cong A^{ps} \cong A \) implies that \( p = s = 1 \). Similarly \( r = q = 1 \). In particular, this implies \( \dim_k A = \dim_k B \).

Finally consider the \( k \)-algebra homomorphism \( f : B \to \text{End}_A(M_A) \cong A \) by sending \( b \) to \( l_b \) left multiplication by \( b \). This is obviously injective, and it is surjective because \( A,B \) have the same dimension, so \( A \cong B \) as \( k \)-algebras.

### 3. Toric Varieties

Let \( T \) be an algebraic torus over \( k \).

**Definition 3.1.** A toric \( T \)-variety \( X \) over \( k \) is a normal geometrically irreducible variety with an action of the torus \( T \) and an open orbit \( U \) which is a principal homogeneous space over \( T \).
By definition, the torus $T_{k^s} \cong \mathbb{G}_{m,k^s}^n$ splits where $n = \dim X$. The torus $T$ corresponds to a cocycle class $[\rho] \in H^1(\Gamma, \text{Aut}_{\mathbb{G}_{m,k^s}}(\mathbb{G}_{m,k^s})) = H^1(\Gamma, \text{GL}(n, \mathbb{Z}))$. Moreover, the torus $T$ splits over a finite Galois extension $l$ of $k$ ($T_l \cong \mathbb{G}_{m,l}^n$), which is called the splitting field of $T$.

Explicitly, tori $T_{k^s}$ and $\mathbb{G}_{m,k^s}^n$ have natural Galois actions with $\Gamma$ acting on the factor $k^s$. This Galois action gives group automorphisms of $T_{k^s}$ over $k$, but not over $k^s$ because $\Gamma$ also acts on the scalar $k^s$. Let $\phi : T_{k^s} \to \mathbb{G}_{m,k^s}^n$ be an isomorphism, then we obtain $\rho : \Gamma \to \text{GL}(n, \mathbb{Z})$ by sending $g$ to $\phi g^{-1} \phi^{-1}$, and we have $\ker(\rho) = \text{Gal}(k^s/l)$ where $l$ is the splitting field. Conversely, the torus $T$ can be constructed from $\rho$ as follows, cf \[V.E83, \S1\]. Let $\rho : G = \text{Gal}(l/k) \to \text{GL}(n, \mathbb{Z})$ be induced by $\rho$ and let $\mu : G \to \text{Aut}_{k}(\mathbb{G}_{m,k}^n)$ act on $\mathbb{G}_{m,k}^n \otimes_k l$ via $\mu(g) = \rho'(g) \otimes g, g \in G$, then $T \cong \mathbb{G}_{m,l}^n / \mu(G)$.

**Definition 3.2.** A toric $T$-variety $X$ over $k$ is called a toric $T$-model if $X(k)$ is nonempty.

By [VA85, \S4 Proposition 4], the set $X(k)$ is nonempty if and only if $U(k)$ is. In this case, the open orbit $U \cong T$ and there is an $T$-equivariant embedding $T \hookrightarrow X$.

**Definition 3.3.** A toric $T$-variety is split if $T$ splits, and is non-split otherwise.

Let $X_{k^s}$ (or $X_1$) be the split toric variety with the fan structure $\Sigma$. Since the $\Gamma$-action on $T_{k^s}$ is compatible with the one on $X_{k^s}$, the image of $\rho$ is contained in $\text{Aut}_\Sigma$, namely

$$\rho(\Gamma) = \text{Gal}(l/k) \subseteq \text{Aut}_\Sigma \subseteq \text{GL}(n, \mathbb{Z}).$$

Let $X_\Sigma$ be the split toric variety over $k$ with fan structure $\Sigma$. If $X$ is a toric $T$-model, then similarly as the torus $T$, the variety $X$ can be recovered from $\rho$ and $\Sigma$ as $(X_\Sigma \otimes k) / \mu(G)$. In general, for each toric $T$-variety $X$, there is a unique (up to $T$-isomorphism) toric $T$-model $X^*$ such that $X_{k^s} \cong (X^*)_k$. We call $X^*$ the associated toric $T$-model of $X$. In detail, the toric $T$-model $X^* = (X \times U)/T$ where $T$ acts on $X \times U$ diagonally, and the toric $T$-variety $X = (X^* \times U)/T$ where $T$ acts on $X^* \times U$ via $t \cdot (x,y) = (tx, yt^{-1})$, cf [VA85, \S4].

In summary, an algebraic torus $T$ is uniquely determined by a 1-cocycle (class) $\rho : \Gamma \to \text{GL}(n, \mathbb{Z})$. A toric $T$-model $X$ is uniquely determined by $\rho$ and fan $\Sigma$ with the restriction $\rho(\Gamma) \subseteq \text{Aut}_\Sigma$. A toric $T$-variety is uniquely determined by its associated $T$-model $X^*$ and a principal homogeneous space $U \in H^1(k, T)$.

**Lemma 3.4.** Let $\phi : X_{\Sigma_1} \to X_{\Sigma_2}$ be a toric morphism of split smooth projective toric varieties over $k^s$, and let $\bar{\phi} : N_1 \to N_2$ be the induced $\mathbb{Z}$-linear map of lattices that is compatible with fans $\Sigma_1$, $\Sigma_2$ for each $i$. Let $\rho_i : \Gamma \to \text{Aut}(N_i)$ be a Galois action on $N_i$ that is compatible with fan $\Sigma_i$ ($\rho_i(\Gamma) \subseteq \text{Aut}_{\Sigma_i}$) such that $\phi$ is $\Gamma$-equivariant. Let $T_i$ be the torus corresponding to $\rho_i$. Then, for any $U_1 \in H^1(k, T_1)$, there exists $U_2 \in H^1(k, T_2)$ such that $\phi$ descends to a map $X_1 \to X_2$ where $X_i$ is the toric variety corresponding to $(\rho_i, \Sigma_i, U_i)$ for $i = 1, 2$.

**Proof.** Restrict $\phi$ to tori $\phi|_{T_{N_i}} : T_{N_1} \to T_{N_2}$. Since $\bar{\phi}$ is $\Gamma$-equivariant, maps $\phi$ and $\phi|_{T_{N_i}}$ descend to $\phi : X^*_1 \to X^*_2$ where $X^*_i$ is the toric $T_i$-model corresponding to $\Sigma_i$ and $\psi : T_1 \to T_2$. $\psi$ induces $H^1(k, T_1) \to H^1(k, T_2)$ and let $U_2$ be the image of $U_1$ under this map. Set $X_i = (X^*_i \times U_i)/T_i$, then $\phi$ descends to a map $X_1 \to X_2$.

**Example 3.5.** Severi-Brauer variety $X(X_{k^s} \cong \mathbb{P}^n)$. Let $A$ be a central simple $k$-algebra of degree $n + 1$, then $X = \text{SB}(A)$ is a toric variety with torus $T = \text{R}_{\text{E}/k}(\mathbb{G}_{m,E})/\mathbb{G}_{m,k}$ where $E$ is a maximal étale $k$-subalgebra of $A$. $X$ has a rational point if and only if $A = M_{n+1}(k)$ if and only if $X \cong \mathbb{P}^n$.

Quillen [Qui73, \S8 Theorem 4.1] showed that $K_m(\text{SB}(A)) \cong K_m(k) \times \prod_{i=1}^n K_m(A^\otimes i)$ for $m \geq 0$, and Panin [Pan94] further showed that $\text{SB}(A) \cong k \times \prod_{i=1}^n A^\otimes i$ in $\mathcal{C}$.

**Example 3.6.** Let $X$ be a del Pezzo surface of degree 6 over $k$ ($K_X$ is anti-ample with $K_X^2 = 6$, $X_{k^s} \cong \text{Bl}_{p_1,p_2,p_3}(\mathbb{P}^2)$ where $p_1, p_2, p_3$ are not collinear). It is a toric $T$-variety where $T$ is the connected component of the identity of $\text{Aut}_k(X)$. 

Blunk [Blu10] showed that $X \cong k \times P \times Q$ in $C$ where $P$ is an Azumaya $K$-algebra of rank $9$ $(\dim_k(P)/\dim_k(K) = 9)$ and $Q$ is an Azumaya $L$-algebra of rank $4$ where $K, L$ are étale $k$-algebras of degree $2$ and $3$ respectively.

**Example 3.7.** Involution surface $X \ (X_{k^s} \cong \mathbb{P}^1 \times \mathbb{P}^1)$. The surface $X$ corresponds to a central simple algebra $(A, \sigma)$ of degree $4$ with a quadratic pair $\sigma$, the even Clifford algebra $C_0(A, \sigma)$ is a quaternion algebra over $K$, the discriminant extension of $X$. Write $B = C_0(A, \sigma)$, then $X$ is the Weil restriction $R_{K/k}SB(B)$, cf [AB15, Example 3.3]. Denote the torus of $SB(B)$ in Example 3.5 as $T$, then $X$ is a toric variety with torus $R_{K/k}T$.

Panin [Pan94] showed that $X \cong k \times B \times A$ in $C$.

3.1. $K_0$ of split toric varieties. Let $Y$ be a split smooth proper toric $T$-variety with fan $\Sigma$.

For $\sigma \in \Sigma$, denote $O_{\sigma}$ the closure of the $T$-orbit corresponding to $\sigma$ and $J_{\sigma}$ the sheaf of ideals defining $O_{\sigma}$. Write $\sigma(1)$ as the set of rays span $\sigma$. For $\sigma, \tau \in \Sigma$, if $\sigma(1) \cap \tau(1) = \emptyset$ and $\sigma(1) \cup \tau(1)$ span a cone in $\Sigma$, denote the cone by $\langle \sigma, \tau \rangle$, otherwise $\langle \sigma, \tau \rangle = 0$.

From [A.A92], we have

**Theorem 3.8 (Klyachko, Demazure).** As an abelian group, $K_0(Y)$ is generated by $O_{\sigma} = 1 - J_{\sigma}$ with relations

(1) $O_{\sigma} \cdot O_{\tau} = \begin{cases} O_{\langle \sigma, \tau \rangle}, & \langle \sigma, \tau \rangle \neq 0 \\ 0, & \text{otherwise} \end{cases}$

(2) $\prod_{e \in \Sigma(1)} J_f(e) = 1, f \in \text{Hom}(N, \mathbb{Z}) = M = T^*$

where $T^*$ is the group of characters of $T$.

**Theorem 3.9 (Klyachko).** The abelian group $K_0(Y)$ is free with rank equal to the number of the maximal cones. In addition, sheaves $O_y$ and $O_{y'}$ coincide in $K_0(Y)$ for any rational closed points $y, y' \in Y$.

4. Minimal Toric Surfaces

Let $X$ be a smooth projective toric surface over $k$. We say $X$ is minimal if any birational morphism $f : X \to X'$ from $X$ to another smooth projective surface $X'$ defined over $k$ is an isomorphism. In this section, we will classify minimal smooth projective toric surfaces.

Let $Y$ be a split smooth projective toric surface with fan $\Sigma$ over a field $K$ and let $G$ be a finite subgroup of $\text{GL}(2, \mathbb{Z})$. Define $T_{G, K} = \{Y | G \text{ can be embedded into } \text{Aut}_\Sigma(Y)\}/\text{isomorphisms over } K$ (we will simply write $T_G$ if the base field $K$ is not important). Note that $T_{G, K}$ only depends on the conjugacy classes of $G$ in $\text{GL}(2, \mathbb{Z})$.

**Definition 4.1.** The surface $Y$ is minimal with respect to $G$ over $K$ if $Y \in T_{G, K}$ and any birational morphism $f : Y \to Y'$, $Y' \in T_{G, K}$, defined over $K$ is an isomorphism.

For $X$ a smooth projective toric $T$-surface over $k$, let $\rho : \Gamma \to \text{GL}(2, \mathbb{Z})$ be the map corresponding to $T$ as described before. We can redefine minimal toric surfaces as

**Definition 4.2.** $X$ is a minimal toric surface if $X_{k^s}$ is minimal with respect to $G = \rho(\Gamma)$ over $k^s$.

In general, there is a finite chain of blow-ups of toric $T$-surfaces

$$X = X_0 \to X_1 \to \cdots \to X_n = X'$$

where $(X')_{k^s}$ is minimal with respect to $G$.

Now, to classify all minimal smooth projective toric surfaces over $k$ is the same as classifying, for each finite subgroup $G$ of $\text{GL}(2, \mathbb{Z})$, minimal surfaces with respect to $G$ over $k^s$. It is well
known that, when $G$ is trivial, the minimal (toric) surfaces are $\mathbb{P}^2$ and Hirzebruch surfaces $F_a = \text{Proj}(O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(a))$ for $a \geq 0, a \neq 1$.

There are 13 non-conjugate classes of finite subgroups of $\text{GL}(2, \mathbb{Z})$ and they can only be either cyclic or dihedral groups. See Table 1 below.

**Table 1. Non-conjugate classes of finite subgroups of $\text{GL}(2, \mathbb{Z})$ and their generators**

| Cyclic | Dihedral | Generators |
|--------|----------|------------|
| $C_1 = \langle I \rangle$ | $D_3 = \langle C \rangle$ | $A = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ |
| $D_2 = \langle C' \rangle$ | $D_4 = \langle -I, C \rangle$ | $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ |
| $D_4 = \langle -I, C' \rangle$ | $D_6 = \langle A^2, C \rangle$ | $C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ |
| $D_6 = \langle A, C \rangle$ | $D_12 = \langle A, C' \rangle$ | $C' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ |

We will make use of the following simple fact from toric geometry [OM78, Proposition 6.7]:

**Proposition 4.3.** Let $Y$ be a split smooth projective toric surface with the fan structure $\Sigma$. Counterclockwise label (the minimal generators of) the rays of $\Sigma$ as $y_1, \ldots, y_n$ and denote $D_i$ the divisor corresponding to $y_i$, then $y_{i-1} + y_{i+1} + a_iy_i = 0$ where $a_i = D_i^2$ ($y_{n+1} = y_1$).

**Definition 4.4.** Let $Y$ be a split smooth projective toric surface. We can assign a sequence $a = (a_1, \ldots, a_n)$ to $Y$ where $a_i$ comes from Proposition 4.3. We refer this sequence as the self-intersection sequence of $Y$.

There is an induced action of $\text{Aut}_\Sigma(Y)$ on $a = (a_1, \ldots, a_n)$ that fixes this sequence: Let $\alpha \in \text{Aut}_\Sigma(Y)$ and define $\alpha(i)$ so that $\alpha(y_i) = y_{\alpha(i)}$, then $\alpha(a) = (a_{\alpha(1)}, \ldots, a_{\alpha(n)})$. Applying $\alpha$ to the relation $y_{i-1} + y_{i+1} + a_iy_i = 0$, we get $a_i = a_{\alpha(i)}$ and thus $\alpha(a) = a$.

More specifically, consider the case where $\text{Aut}_\Sigma(Y) \cap \text{SL}(2, \mathbb{Z}) = C_t$ is nontrivial and let us look at the action of $C_t$ on $a$. As indicated in Table 1, the cyclic group $C_t$ is generated by powers of $A$ or $B$ where $B$ is the rotation by $\pi/4$ and $A$ is conjugate in $\text{GL}(2, \mathbb{R})$ to the rotation by $\pi/3$. In particular, the action of $C_t$ on the fan $\Sigma$ is free which implies $t \mid n$. Let $n = tm$ and let $\sigma$ be the generator of $C_t$ that rotates counterclockwise, then $\sigma(a) = (a_{m+1}, \ldots, a_m) = a$.

**Lemma 4.5.** Let $\text{Aut}_\Sigma(Y) \cap \text{SL}(2, \mathbb{Z}) = C_t$ be nontrivial (i.e., $t = 2, 3, 4, 6$). If the number of rays of the fan $\geq \max\{4, t\}$, then $Y$ is not minimal with respect to $C_t$, that is, there exists a split smooth projective toric surface $Y'$ such that $Y \rightarrow Y'$ is a blow-up along torus invariant points and the group of fan automorphisms of $Y'$ contains $C_t$.

Therefore, minimal surfaces with respect to $C_t$ have the number of rays $\leq \max\{4, t\}$.

**Proof.** Denote counterclockwise $y_1, \ldots, y_n$ as rays of $\Sigma$ and let $a = (a_1, \ldots, a_n)$ be its self-intersection sequence. If $n > 4$, $Y$ is not $\mathbb{P}^2$ or $F_a$, then there exists $i$ such that $a_i = -1$. Let $\sigma$ be a generator of $C_t$ and as discussed above, $\sigma$ rotates the rays. If $n > t$, then the ray $\sigma(y_i)$ is not adjacent to $y_i$, and thus $\sigma(y_i)$ stays a (-1)-curve after $Y$ blowing down $y_i$. Hence $Y'$ can be obtained by successively blowing down $y_i, \sigma(y_i), \ldots, \sigma^{t-1}(y_i)$. \hfill $\square$

**Lemma 4.6.** $D_2$ fixes rays generated by $\pm(1, 1)$ or maximal cones generated by $(1, 0)$ and $(0, 1)$ or by $(-1, 0)$ and $(0, -1)$; $D_4$ fixes rays generated by $(1, 0)$.

Now we are ready to classify split minimal toric surfaces with respect to a finite subgroup $G$ of $\text{GL}(2, \mathbb{Z})$. 
Proposition 4.7. Let $Y$ be a split smooth projective toric surface. If $Y$ is minimal with respect to a finite subgroup $G$ of $\mathrm{GL}(2, \mathbb{Z})$, then $Y$ must be one of the following varieties:

| $G$     | $Y$                          |
|---------|------------------------------|
| $D_2$  | $\mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1, F_{2a+1}, a \geq 1$ |
| $D'_2$ | $F_{2a}, a \geq 0$          |
| $C_2$  | $\mathbb{P}^1 \times \mathbb{P}^1$ |
| $D_4, D'_4$ | $\mathbb{P}^1 \times \mathbb{P}^1$ |
| $C_3$  | $\mathbb{P}^2$              |
| $D_6$  | $\mathbb{P}^2$              |
| $D'_6$ | $S$                         |
| $C_4$  | $\mathbb{P}^1 \times \mathbb{P}^1$ |
| $D_8$  | $\mathbb{P}^1 \times \mathbb{P}^1$ |
| $C_6$  | $S$                         |
| $D_{12}$ | $S$                        |

where $F_a = \text{Proj}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a))$ is the Hirzebruch surface and $S$ is the blow-up $\text{Bl}_{p_1,p_2,p_3}(\mathbb{P}^2)$ of $\mathbb{P}^2$ along three torus invariant points.

Proof. Let $\Sigma$ be the fan structure of $Y$.

$G = D_2$: If $D_2$ fixes at least one maximal cone, then $Y = \mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1 = F_0$. Otherwise $\Sigma$ has rays $\pm(1, 1)$, and the rays counterclockwise before and after $(1, 1)$ must be $(a+1, a)$ and $(a, a+1)$ respectively. $Y$ is isomorphic to $F_{2a+1}$ in this case. For $F_1$, it has a $G$-invariant $(-1)$-curve so it is the blow-up of a rational point on $\mathbb{P}^2$, not minimal. So we have $a \geq 1$.

$G = D'_2$: $\Sigma$ has rays $\pm(1, 0)$, and the rays counterclockwise before and after $(1, 0)$ are $(a, -1)$ and $(a, 1)$ respectively. $Y$ is isomorphic to $F_{2a}, a \geq 0$.

$G = C_2$: $\Sigma$ should have rays $x, y, -x, -y$ and $x, y$ form a basis of the lattice, thus $Y \cong \mathbb{P}^1 \times \mathbb{P}^1$.

$G = D_4, D'_4$: $Y \cong \mathbb{P}^1 \times \mathbb{P}^1$ which follows from the case $G = C_2$.

For cases $G \supseteq C_t, t > 2$, let $n$ be the number of rays of $\Sigma$. Recall that $t \mid n$ and by Lemma 4.5, $n \leq \max\{4, t\}$.

$G = C_3$: $3 \mid n, n \leq 4$, so $n = 3$ and $Y \cong \mathbb{P}^2$.

$G = C_4$: $4 \mid n, n \leq 4$, so $n = 4$ and $Y \cong \mathbb{P}^1 \times \mathbb{P}^1$.

$G = C_6$: $6 \mid n, n \leq 6$, so $n = 6$. Following from the case $G = C_3, Y$ is the blow up of $\mathbb{P}^2$ along three torus invariant closed points.

$G = D_6$: Following from the case $G = C_3, Y \cong \mathbb{P}^2$.

$G = D'_6$: Following from the case $G = C_3, Y$ is either $\mathbb{P}^2$ or the blow-ups of $\mathbb{P}^2$ along three torus invariant points. As $\text{Aut}_\Sigma(\mathbb{P}^2)$ is isomorphic to the conjugacy class of $D_6$, $Y$ can not be $\mathbb{P}^2$.

$G = D_8$: Following from the case $G = C_4, Y \cong \mathbb{P}^1 \times \mathbb{P}^1$.

$G = D_{12}$: Following from the case $G = C_6, Y$ is the blow up of $\mathbb{P}^2$ along three torus invariant closed points.

□

Lemma 4.8. Let $X$ be a toric surface that is a form of $F_a, a \geq 1$, then $X$ is a $\mathbb{P}^1$-bundle over a smooth conic curve. If $X$ has a rational point, then $X \cong F_a$.

Proof. Let $X$ correspond to $(\rho_1, \Sigma_1, U_1)$ and let $\Sigma_1$ be the fan of $F_a$ with rays $(1, 0), (0, 1), (-1, a), (0, -1)$. Let $\phi : \mathbb{Z}^2 \to \mathbb{Z}$ be the projection to the first factor, which corresponds to $\phi : F_a \to \mathbb{P}^1$. Let $\rho_2 = \det \circ \rho_1 : \Gamma \to \text{GL}(1, \mathbb{Z})$, then $\phi$ is Galois equivariant with respect to $\rho_1$ and $\rho_2$. By Lemma 3.4, $\phi$ descends to $X \to C$. As a form of $\mathbb{P}^1$, $C$ is a smooth plane conic curve ([GS06, Corollary 5.4.8] for characteristic not 2 and [EKM08, §45A] for any characteristic).

Either $X \cong F_a$ or $\rho_1$ permutes rays $(1, 0), (-1, a)$. Let $D'$ be the Cartier divisor corresponding to the ray $(0, -1)$, then it is Galois invariant in both cases. Thus, $D'$ descends to a Cartier divisor
$D$ on $X$, and $X \cong \text{Proj}(O_C \oplus O_C(D))$ is a $\mathbb{P}^1$-bundle over $C$. If $X$ has a rational point, so does $C$. Therefore, $C \cong \mathbb{P}^1, X \cong F_n$. □

By Proposition 4.7, a minimal smooth projective toric surface $X$ is a form of (i) $F_a, a \geq 2$; (ii) $\mathbb{P}^2$; (iii) $\mathbb{P}^1 \times \mathbb{P}^1$; (iv) $\text{Bl}_{p_1,p_2,p_3}(\mathbb{P}^2)$ where $p_1, p_2, p_3$ are not collinear. Furthermore, we have

**Theorem 4.9.** $X$ is a minimal smooth projective toric surface if and only if $X$ is (i) a $\mathbb{P}^1$-bundle over a smooth conic curve; (ii) the Severi-Brauer surface; (iii) an involution surface; (iv) the del Pezzo surface of degree 6 with Picard rank 1.

*Proof.* It follows from Lemma 4.8, Example 3.5, 3.6, 3.7 and the fact that a minimal del Pezzo surface of degree not equal to 8 has Picard rank 1 [CTKM08, Theorem 2.4]. □

5. $K_0$ of Toric Surfaces

In this section, we will show that $K_0(X_{k^p})$ is a permutation $\Gamma$-module for $X$ smooth projective toric surface over $k$. First recall how $K_0$ behaves under blow-ups:

**Theorem 5.1.** [GIT71, VII 3.7] Let $X$ be a noetherian scheme and let $i : Y \to X$ be a regular closed immersion of pure codimension $d$. Let $p : X' \to X$ be the blow up of $X$ along $Y$ and $Y' = p^{-1}Y$. There is a split short exact sequence

$$0 \to K_0(Y) \overset{w}{\to} K_0(Y') \oplus K_0(X) \overset{v}{\to} K_0(X') \to 0$$

and the splitting map $w$ for $u$ is given by $w(y', x) = p|_{Y'}(y'), y' \in K(Y'), x \in K(X)$.

This gives us an isomorphism $K_0(X') \cong \text{ker}(w) \cong K_0(X) \oplus \bigoplus_{d-1} K_0(Y')$ which fits into the split short exact sequence

$$0 \to K_0(X) \overset{p^*}{\to} K_0(X') \to \bigoplus K_0(Y) \to 0.$$ 

Now let $X$ be a smooth projective toric $T$-surface over $k$ that splits over $l$. Let $Y$ be a $T$-invariant reduced subscheme of $X$ of dimension 0, then $Y_i$ is a disjoint union of $T_l$-invariant points fixed by $G = \text{Gal}(l/k)$. Set $X' = \text{Bl}_Y X$. We have

$$0 \to K_0(X_i) \overset{p^*}{\to} K_0(X'_i) \to K_0(Y_i) = \bigoplus \mathbb{Z} \to 0$$

where $p^*$ is a $G$-homomorphism. Each $\mathbb{Z}$ is generated by exceptional divisors $E_i$ corresponding to the points in $Y_i$ and $G$ permutes $E_i$ the same way as $G$ permutes the points in $Y_i$.

If we know $K_0(X_i)$ has a permutation $G$-basis $\gamma$, then $K(X'_i)$ has a permutation $G$-basis consisting of $p^*\gamma$ (total transforms of $\gamma$) and $E_i$.

**Theorem 5.2.** Let $X$ be a smooth projective toric $T$-surface over $k$ that splits over $l$, $G = \text{Gal}(l/k)$. Then $K_0(X_i)$ has a permutation $G$-basis of line bundles on $X_l$.

*Proof.* By previous discussion and the fact that $G \subseteq \text{Aut}_X$, it suffices to prove that $K_0(X_i)$ has a permutation $\text{Aut}_X$-basis of line bundles for $X$ minimal. By Theorem 4.9, we only need to consider the following cases for $X_i$:

(i): $F_a, a \geq 2$, $\text{Aut}_X = S_2$.

(ii): $\mathbb{P}^2$, $\text{Aut}_X = D_6$.

(iii): $\mathbb{P}^1 \times \mathbb{P}^1$, $\text{Aut}_X = D_8$.

(iv): del Pezzo surface of degree 6, $\text{Aut}_X = D_{12}$.

We will use Equation (2) in Theorem 3.8 with $f = (1, 0)$ and $(0, 1)$ in producing relations and finding a permutation basis. We will write $x_i$ for rays in the fan and $J_i = O(-D_i)$ where $D_i$ is the divisor corresponding to $x_i$. 

(i): Rays \( x_1 = (1, 0), \) \( x_2 = (0, 1), \) \( x_3 = (-1, a), \) \( x_4 = (0, -1). \) \( S_2 \) fixes \( x_2, x_4 \) and permutes \( x_1, x_3. \) Relations are:

\[
\begin{aligned}
J_3 &= J_1 \\
J_4 &= J_2 J_3^a
\end{aligned}
\]

Let \( x \) be a rational point of \( X_i, \) then the sheaf \( \mathcal{O}_x = (1 - J_1)(1 - J_2) \) in \( K_0. \) A permutation basis is \( 1, J_1, J_2, J_1 J_2. \)

(ii): Rays \( x_1 = (1, 0), \) \( x_2 = (0, 1), \) \( x_3 = (-1, -1). \) \( D_6 \) rotates \( x_i \) and reflects along lines in \( x_1, x_2, x_3. \) Relations are \( J_1 = J_2 = J_3. \) A permutation basis is \( 1, J_1, J_1^2. \)

(iii): Rays \( x_1 = (1, 0), \) \( x_2 = (0, 1), \) \( x_3 = (-1, 0), \) \( x_4 = (0, -1). \) \( D_8 \) rotates \( x_i \) and reflects along lines in \( x_1, x_2, (1, 1), (-1, 1). \) Relations are:

\[
\begin{aligned}
J_3 &= J_1 \\
J_4 &= J_2
\end{aligned}
\]

A permutation basis is \( 1, J_1, J_2, J_1 J_2. \)

(iv): Rays \( x_1 = (1, 0), \) \( x_2 = (0, 1), \) \( x_3 = (-1, -1), \) \( y_1 = (-1, 0), \) \( y_2 = (0, -1), \) \( y_3 = (1, 1). \) \( D_{12} \cong S_2 \times S_3 (S_2, S_3 \text{ permutation groups}). \) \( S_2 \) switches between \( x_i \) and \( y_i. \) \( S_3 \) permutes the pair of rays \( (x_i, y_i). \) Let \( J_i' \) correspond to \( y_i. \) Relations are

\[
\frac{J_1}{J_1'} = \frac{J_2}{J_2'} = \frac{J_3}{J_3'}
\]

As proved in [Blu10, Theorem 4.2], we have a permutation basis \( 1, P_1, P_2, P_3, Q_1, Q_2 \) where

\[
\begin{aligned}
P_1 &= J_1 J_2' \\
P_2 &= J_2 J_3' \\
P_3 &= J_3 J_1' \\
Q_1 &= J_1 J_2 J_3' \\
Q_2 &= J_1' J_2' J_3
\end{aligned}
\]

\[\square\]

Remark 5.3. The difficulties to generalize Theorem 5.2 to higher dimensions (at least using the approach of this paper) are:

(1) The classification of non-conjugacy classes of finite subgroups of \( \text{GL}(n, \mathbb{Z}) \) is difficult and not complete which often only provides algorithms and requires the help of computer even for small \( n. \) Also, the number of those finite subgroups grows very fast as \( n. \) For example, there are total of 73 for \( \text{GL}(3, \mathbb{Z}) \) and 710 for \( \text{GL}(4, \mathbb{Z}) \).

(2) The \( K \)-group \( K_0(X_i) \) in question may not stay a permutation module after blow-ups if \( X \) is not a surface.

6. Construction of Separable Algebras

Let \( X \) be a smooth projective toric \( T \)-variety over \( k \) that splits over \( l, \) and \( X^* \) be its associated toric model, cf §3. [MP97, Theorem 5.7] states that there is a split monomorphism \( u : X^* \to A \) in the motivic category \( \mathcal{C} \) from \( X^* \) to an \( \text{étale} \) \( k \)-algebra \( A \) and \( u \) is represented by an element \( Q \) in \( \text{Pic}(X^* \otimes_k A). \) We can construct \( u' : X \to B \) out of \( u. \) [MP97, Theorem 7.6] states that \( u' \) is also a split monomorphism in \( \mathcal{C}. \) In this section, we will recall the construction of \( u' \) and consider the case when \( u \) is an isomorphism.
Write $X_A = X \otimes_k A$ and we have $f : X_t \to X_t^*$, a $T_i$-isomorphism. Consider the diagram:

$$
\begin{array}{ccc}
X_{A\otimes_k l} & \xrightarrow{f_A} & X_{A\otimes_k l}^* \\
\downarrow^{\pi_X} & & \downarrow^{\pi_X^*} \\
X_A & \xrightarrow{\pi_X} & X_A^*
\end{array}
$$

(3)

Let $P' = f^*(\pi^*_X(Q))$, then $B = \text{End}_{X_A}(\pi^*_{X_A}(P')) \in \text{Br}(A)$ and $u' : X \to B$ is represented by $\pi_{X_A}(P')$, namely, $u' = \phi_*(P') \in K_0(X, B)$ where $\phi$ is the projection $X_A \otimes_k l \to X$.

The following criterion, which is [MP97, Proposition 4.5], checks when a toric model is isomorphic to an étale algebra in $\mathcal{C}$.

**Proposition 6.1.** Let $X^*$ be a smooth projective toric model over $k$ that splits over $l$ and $G = \text{Gal}(l/k)$. Assume that $K_0(X_t^*)$ is a permutation $G$-module, then $X^* \cong \text{Hom}_G(P, l)$ in the motivic category $\mathcal{C}$ for any permutation $G$-basis $P$ of $K_0(X_t^*)$.

**Remark 6.2.** In particular, this implies that for any split smooth projective toric variety $Y$ over $k$, $Y \cong \mathbb{k}^n$ in $\mathcal{C}$ where $n$ equals to the rank of $K_0(Y)$ (also equals to the number of maximal cones of the fan). Note that a smooth projective toric variety $Y$ over $k$ where the fan of $Y_t$ has no symmetry is automatically split.

**Lemma 6.3.** Let $X^*, G$ be the same as before, then there is an isomorphism $u : X^* \to A$ in $\mathcal{C}$ where $A$ is an étale $k$-algebra and $u$ is represented by an element $Q \in \text{Pic}(X_A^*)$ if and only if $K_0(X_t^*)$ has a permutation $G$-basis of line bundles on $X_t^*$.

**Proof.** $\Rightarrow$ : decompose $A$ as $\prod_{i=1}^t k_i$ where $k_i$ is a finite separable field extension of $k$, then $X_A = \coprod_{i=1}^t X_A^*$ is the disjoint union of $X_A^*$ and $Q = \coprod_{i=1}^t Q_i$ where $Q_i$ is a line bundle on $X_A^i$. Let $g_i : X_A^i \to X^*$ be the projections, then $u = \bigoplus_{i=1}^t q_is_i$ be the projections and $G_i = \text{Gal}(k_i/k)$, then $u_{k_i} = \bigoplus_{i=1}^t p_i^*q_i^*q_is_i(Q_i) = \bigoplus_{i=1}^t \bigoplus_{g \in G_i}p_i^*(gQ_i)$ and $A_{k_i} \cong (k^*)^n$ where $n = \sum_{i=1}^t |G_i|$. View $u$ as $u^{op} : A^{op} = A \to X^*$, then $u^{op}_k$ induces an isomorphism $K_0((k^*)^n) \to K_0(X^*_{k_i})$ where the canonical basis of the former sends to $\{p_i^*(gQ_i) \mid g \in G_i, 1 \leq i \leq t\}$ and this set gives a permutation $G$-basis of $K_0(X^*_{k_i})$ of line bundles. As $\text{Gal}(k^*/l)$ acts trivially on $K_0(X^*_{k_i})$, this basis descends to $X^*_{k_i}$.

$\Leftarrow$ : Assume $P$ is a permutation $G$-basis of $K_0(X_t^*)$ of line bundles on $X_t^*$ and $P$ divides into $t$ $G$-orbits. Let $\{P_i \mid 1 \leq i \leq t\}$ be the set of representatives of $G$-orbits, and $\text{Gal}(l/k_i)$ be the stabilizer of $P_i$. Set $A = \text{Hom}_G(P, l)$, then $A \cong \prod_{i=1}^t k_i$. As $X^*$ has a rational point, by [CTKM08, Proposition 5.1], $P_i \in \text{Pic}(X_t^*)^{\text{Gal}(l/k_i)} \cong \text{Pic}(X^*_{k_i})$, namely $P_i \cong p_i^*(Q_i)$ for some $Q_i \in \text{Pic}(X^*_{k_i})$ where $p_i : X_t^* \to X^*_{k_i}$ is the projection. There is a morphism $u : X^* \to A$ which is represented by $\prod_{i=1}^t P_i \in \text{Pic}(X_A^*)$, and by construction, $u_t$ induces an isomorphism $K_0(X^*_t) \cong K_0(A_t)$.

Using the following lemma, we have $u$ is an isomorphism.

**Lemma 6.4.** Let $X^*$ be the same as before and $A$ is an étale $k$-algebra. If $u : X^* \to A$ is a morphism in $\mathcal{C}$ such that $K_0(u_{k^*}) : K_0(X^*_{k^*}) \to K_0(A_{k^*})$ is an isomorphism, then so is $u$.

**Proof.** There is a commutative diagram:

$$
\begin{array}{ccc}
K_0(X^*) & \xrightarrow{K_0(u)} & K_0(A) \\
\downarrow & & \downarrow \\
K_0(X^*_{k^*}) & \xrightarrow{K_0(u_{k^*})} & K_0(A_{k^*})
\end{array}
$$

The right vertical map is an isomorphism as $A$ is étale and so is $K_0(u_{k^*})$ by assumption. The left vertical map is an isomorphism by [MP97, Corollary 5.8]. Thus, $K_0(u)$ is also an isomorphism.
Write \( w = u^{\text{op}} : A \to X^* \), then by the splitting principle [MP97, Proposition 6.1 and the proof], \( K_0^\ast(w): K_0(X^* \times A) \to K_0(X^* \times X^*) \) is surjective. Thus, there exists \( v \in K_0(X^* \times A): X^* \to A \) such that \( w \circ v = K_0^\ast(w)(v) = 1_{X^*} \), and then \( K_0(w \circ v) = K_0(w)K_0(v) = 1_{K_0(X^*)} \). Since \( K_0(w) = \phi \) is an isomorphism, \( K_0(v) = \phi^{-1} \) and \( K_0(v \circ w) = K_0(v)K_0(w) = 1_{K_0(A)} \). This implies \( v \circ w = 1_A \) and thus \( v \) is a two sided inverse of \( w \) in \( \mathcal{C} \).

The proof of (3) \( \iff \) (4) in [MP97, Proposition 7.9] shows that the \( T_l \)-isomorphism \( f : X_l \to X_l^i \) induces a \( G = \text{Gal}(l/k) \)-module isomorphism \( f^\ast : K_0(X_l^i) \to K_0(X_l) \). Thus, \( K_0(X_l^i) \) has a permutation \( G \)-basis of line bundles on \( X_l^i \) if and only if \( K_0(X_l) \) has such a basis. Note that the proof (1) \( \iff \) (2) (an isomorphism \( u : X^* \to A \) gives an isomorphism \( u' : X \to B \), which uses the construction (3) recalled at the beginning of the section, works only when \( u \) is represented by an element \( Q \in \text{Pic}(X^i_\ast) \). Thus, we have the following instead:

**Theorem 6.5.** Let \( X \) be a smooth projective toric \( T \)-variety over \( k \) that splits over \( l \) and \( G = \text{Gal}(l/k) \). Assume \( K_0(Y_l) \) has a permutation \( G \)-basis \( P \) of line bundles on \( X_l \). Let \( \{ P_i | 1 \leq i \leq t \} \) be \( G \)-orbits of \( P \), and \( \pi : X_l \to X \) be the projection. For any \( S_i \in P_l, \) set \( B_i = \text{End}_{\mathcal{O}_X}(\pi_\ast(S_i)) \) and \( B = \prod_{i=1}^t B_i \), then the map \( u = \bigoplus_{i=1}^t \pi_\ast(S_i) : X \to B \) gives an isomorphism in the motivic category \( \mathcal{C} \).

**Proof.** By Lemma 6.3, we have an isomorphism \( u : X^* \to A \) represented by \( Q \in \text{Pic}(X^i_\ast) \). Here \( A \cong \prod_{i=1}^t k_i \) where \( \text{Gal}(l/k_i) \) is the stabilizer of \( S_i \) under \( G \)-action. \( Q = \prod_{i=1}^t Q_i \) and \( Q_i \in \text{Pic}(X^i_{k_i}) \) descends from \( (f^\ast)^{-1}(S_i) \in \text{Pic}(X^i_{k_i}) \). Now we run the construction (3) for \( Q_i \):

\[
\begin{array}{ccc}
X_{k_i \otimes k_l} & \xrightarrow{f_i} & X^i_{k_i \otimes k_l} \\
\downarrow \pi_X & & \downarrow \pi_X^* \\
X_{k_i} & & X^i_{k_i}
\end{array}
\]

Let \( p : X_l \to X_{k_i} \) and \( q : X_{k_i} \to X \) be the projections, then \( \pi_X^if_i^\ast\pi_X^\ast(Q_i) \cong p_\ast(S_i) \otimes k_i \) where its \( \mathcal{O}_{X_{k_i}} \)-module structure comes from the one on \( p_\ast(S_i) \). Thus, \( \text{End}_{\mathcal{O}_{X_{k_i}}}(p_\ast(S_i)) \otimes \text{End}_{k}(k_i) \) is Brauer equivalent to \( B_i' = \text{End}_{\mathcal{O}_{X_{k_i}}}(p_\ast(S_i)) \). It remains to prove that \( B_i \cong B_i' \). There is a \( G \)-isomorphism: \( B_i \otimes_k l \cong \text{End}_{\mathcal{O}_{X_{k_i}}}(\pi_\ast\pi_\ast(S_i)) \cong \text{End}_{\mathcal{O}_{X_{k_i}}}(p^\ast q_\ast p_\ast(S_i)) \cong \text{End}_{\mathcal{O}_{X_{k_i}}}(p^\ast(S_i) \otimes_k k_i) \cong (B_i' \otimes_k l) \otimes_k k_i \cong B_i' \otimes_k l \). The fourth isomorphism follows from Lemma 6.6. Take \( G \)-invariants on both sides, we have \( B_i \cong B_i' \).

**Lemma 6.6.** Take a proper variety over \( k \) and assume that there is a finite group \( G \) acting on Cartier divisors \( \text{CDiv}(X) \). Let \( D \in \text{CDiv}(X) \) and \( g \in G \) such that \( D \) and \( gD \) are not linearly equivalent, then \( \text{Hom}_{\mathcal{O}_X}(\mathcal{O}_X(D), \mathcal{O}_X(gD)) = 0 \).

**Proof.** Assume that \( \text{Hom}_{\mathcal{O}_X}(\mathcal{O}_X(D), \mathcal{O}_X(gD)) \neq 0 \), which is equivalent to \( \mathcal{O}_X(gD - D) \) has a nonzero global section \( s \). Since \( G \) is a finite group, \( g^n = 1 \) for some \( n \). Therefore, \( \mathcal{O}_X(D - gD) = (g^{-n-1} \otimes \cdots \otimes g \otimes 1)\mathcal{O}_X(gD - D) \) has a nonzero global section \( t = g^{-n-1}s \otimes \cdots \otimes s \). View \( s, t \) as \( s : \mathcal{O}_X(D) \to \mathcal{O}_X(gD) \) and \( t : \mathcal{O}_X(gD) \to \mathcal{O}_X(D) \). Since \( st, ts \in \Gamma(X, \mathcal{O}_X) = k \) are nonzero, \( \mathcal{O}(gD - D) \cong \mathcal{O}_X \), contradiction.

**Remark 6.7.** There is a more “economical” description of an algebra isomorphic to \( X \) in \( \mathcal{C} \):

Write \( S_i = \mathcal{O}(-D_i) \) where \( D_i \) is torus invariant. Let \( \text{Gal}(l/l_i) \) be the stabilizer of \( D_i \) under \( G \)-action and let \( \pi_i : Y_{l_i} \to Y \) be the projections. \( D_i \) and thus \( S_i \) descend to \( Y_{l_i} \), and we use the same notation. Then \( Y \cong \prod_{i=1}^t \text{End}_{\mathcal{O}_X}(\pi_\ast(S_i)) \). In effect, it replaces all \( M_n(k) \) in \( B \) constructed in the theorem by \( k \) which is an isomorphism in \( \mathcal{C} \).

**Remark 6.8.** A question remains: If \( K_0(X_l) \) is a permutation \( G \)-module, can we always find a permutation \( G \)-basis of line bundles?
Recall that for $n \geq 0$, $K_n$ defines a functor $K_n : \mathcal{C} \to Ab$, hence we have

**Corollary 6.9.** $K_n(X) \cong \prod_{i=1}^{t} K_n(B_i)$.

### 7. Separable Algebras for Toric Surfaces

#### 7.1. Separable algebras for minimal toric surfaces.

Recall the families of minimal toric surfaces described in Theorem 5.2: Let $X$ be a minimal smooth projective toric $T$-surface over $k$ that splits over $l$, and $X^*$ be its associated toric model. Let $\pi : X_l \to X$ be the projection. All isomorphisms below are taken in the motivic category $\mathcal{C}$.

(i): $X_l \cong F_a, a \geq 2$. $X^* \cong k^4$ and $X \cong k \times Q \times k \times Q$ where $Q \cong \text{End}_{O_X}(\pi_*J_1)$ is a quaternion $k$-algebra.

(ii): More generally, let $X = \text{SB}(A)$ be a Severi-Brauer variety of dimension $n$ and $J = \mathcal{O}_{X_l}(-1)$. $X^* \cong k^{n+1}$ and $X \cong k \times \prod_{i=1}^{n} A^{\otimes i}$ where $A^{\otimes i} \cong \text{End}_{O_X}(\pi_*J^i)$, cf Example 3.5.

(iii): $X_l \cong \mathbb{P}^1 \times \mathbb{P}^1$. $X^* \cong k \times \mathbb{K} \times k$ where $K$ is a quadratic étale algebra and the discriminant extension of $X$. $X \cong k \times B \times A$ where $B \cong \text{End}_{O_X}(\pi_*J_1)$ is an Azumaya $K$-algebra of rank 4 and $A \cong \text{End}_{O_X}(\pi_*J_2)$ is a central simple $k$-algebra of degree 4, cf Example 3.7.

(iv): See Example 3.6 where $X^* \cong k \times \mathbb{K} \times l$ and $P \cong \text{End}_{O_X}(\pi_*P_1)$ and $Q \cong \text{End}_{O_X}(\pi_*Q_1)$.

Now let $X$ be a smooth projective toric $T$-variety over $k$ that splits over $l$ and $G = \text{Gal}(l/k)$. Recall that $X$ is uniquely determined by the associated toric model $X^*$, which corresponds to $\rho : \Gamma \to \text{GL}(n, \mathbb{Z})$ and fan $\Sigma$ such that $\rho(\Gamma) \subseteq \text{Aut}_\Sigma$, and a principal homogeneous space $U \subseteq H^1(k, T)$. Every variety within a family above has the same fan. Let $\rho' : G \hookrightarrow \text{Aut}_\Sigma(X_l)$ be the inclusion induced by $\rho$. We want to see how the separable algebras described above relate to $\rho'$ and $U$.

Let $\dim X = n$ and $N$ be the number of rays in the fan $\Sigma$, then the Picard rank of $X_l$ is $m = N - n$. Write $M$ for the group of characters of $T_l$ and $\text{CDiv}_{T_l}$ for $T_l$-invariant Cartier divisors. There is a natural action of $\text{Aut}_\Sigma(X_l)$ on $M$ and $\text{CDiv}_{T_l}(X_l)$ and an induced action on $\text{Pic}(X_l)$ via the canonical morphism $\text{CDiv}_{T_l}(X_l) \to \text{Pic}(X_l)(D \to \mathcal{O}_{X_l}(D))$.

We have a short exact sequence of $\text{Aut}_\Sigma(X_l)$-modules and therefore of $G$-modules via $\rho'$:

\[
0 \to M \to \text{CDiv}_{T_l}(X_l) \to \text{Pic}(X_l) \to 0,
\]

or simply

\[
0 \to \mathbb{Z}^n \to \mathbb{Z}^N \to \mathbb{Z}^m \to 0.
\]

It corresponds to the short exact sequence of tori over $l$:

\[
1 \to \mathbb{G}_m, l \to \mathbb{G}_m, N \to \mathbb{G}_m, m \to 1
\]

and the sequence descends to

\[
1 \to S \to V \to T \to 1.
\]

Let $i : \text{Aut}_\Sigma(X_l) \hookrightarrow S_N$ where $S_N$ is the group of permutations of the canonical $\mathbb{Z}$-basis of the lattice $\mathbb{Z}^N$ and it induces $i_* : H^1(G, \text{Aut}_\Sigma) \to H^1(G, S_N)$. Let $[\alpha] = i_*[\rho']$ and $E$ be the corresponding étale $k$-algebra of degree $N$, then $V = R_{E/k}(\mathbb{G}_{m,E})$. Let $j : \text{Aut}_\Sigma(X_l) \to \text{GL}(m, \mathbb{Z})$ be the map induced by the action of $\text{Aut}_\Sigma(X_l)$ on $\text{Pic}(X_l)$ and it induces $j_* : H^1(G, \text{Aut}_\Sigma) \to H^1(G, \text{GL}(m, \mathbb{Z}))$. Let $[\beta] = j_*[\rho']$, then $S$ is the torus corresponding to $[\beta]$. The short exact sequence of tori over $k$ gives

\[
0 \to H^1(G, T) \to H^2(G, S) \to \text{Br}(E).
\]

Here $H^1(G, V) = H^1(G, R_{E/k}(\mathbb{G}_{m,E})) = \prod H^1(\text{Gal}(E_l/k), \mathbb{G}_{m,E_l}) = 0$ by Hilbert 90 Theorem where $E = \prod E_l$ and the $E_l$ are finite separable field extensions of $k$.

Let $S^* = \text{Hom}(S_l, \mathbb{G}_{m,l})$ be the group of characters over $l$. Then sequence (4) can be rewritten as

\[
0 \to T^* \to V^* \to S^* \to 0
\]
Lemma 7.1. The following diagram is commutative:

\[
H^1(G,T) \otimes H^0(G,S^*) \xrightarrow{\delta \otimes 1} H^1(G,T) \otimes H^1(G,T^*)
\]

\[
\delta \otimes 1 \downarrow \quad \quad \quad \quad \quad \delta \otimes 1
\]

\[
H^2(G,S) \otimes H^0(G,S^*) \xrightarrow{\cup} \text{Br}(k)
\]

Proof. Let \( a \in H^1(G,T), \varphi \in H^0(G,S^*) \). For each \( a_g \in T(l), g \in G \), pick \( b_g \in V(l) \) that maps to \( a_g \), then \((\delta a)_{g,h} = b_{g,h}^{-1}b_g b_h \cdot g,h \in G \). Pick \( \phi \in V^* \) that maps to \( \varphi \), then \((\partial \varphi)_g = \phi^{-1} \phi \). Let \( \alpha = a \cup (\partial \varphi) \) and \( \beta = (\delta a) \cup \varphi \), then \( \alpha_{g,h} = \phi(\partial^{-1} \phi)(a_g) = \phi(\delta^{-1} \phi)(b_g)(b_h)(\beta_{g,h} = (\partial \varphi)_{g,h}) = (\delta^{-1} \phi)(b_g)(b_h) \cdot (\delta^{-1} \phi)(b_h) \cdot (\delta^{-1} \phi)(b_h) \cdot (\delta^{-1} \phi)(b_h) \). Set \( \theta_g = (\delta^{-1} \phi)(b_g) \), then \( \beta_{g,h} = \theta^{-1} \theta_\gamma \theta \delta_\gamma \theta \delta \Theta \). Thus, \( \alpha \) and \( \beta \) give the same cycle class in \( \text{Br}(k) \).

Let \( P \in \text{Pic}(X_l) \) be a line bundle on \( X_l \) with stabilizer group \( \text{Gal}(l/k) \) under \( G \)-action. Since \( P \) is in \( \text{Pic}(X_l)^{\text{Gal}(l/k)} \cong \text{Pic}(X^*) \), \( P \) corresponds to a character \( \chi : \Sigma_k \rightarrow \mathbb{G}_m \) over \( k \), or equivalently \( \chi : \Sigma_k \rightarrow \mathbb{G}_m \). Let \( \pi : X_l \rightarrow X \) be the projection.

Proposition 7.2. Consider the composition of maps \( \delta_P : H^1(G,T) \xrightarrow{\delta} H^2(G,S) \xrightarrow{\chi} \text{Br}(k) \), then \( \delta_P[U] = \left[ \text{End}_{\mathcal{O}_X}(\pi_* P) \right] \in \text{Br}(k) \).

Proof. First we prove the case when \( k = k \). In this case, the line bundle \( P \in \text{Pic}(X_l)^G \cong \text{Pic}(X^*) \). Thus, there is \( Q \in \text{Pic}(X^*) \) such that \( P \cong f^* \pi_*^* Q \), where \( \pi_* : X_l^* \rightarrow X^* \) is the projection and \( f : X_l \rightarrow X^* \) is the \( T \)-isomorphism. [MP97, Lemma 7.3] shows that \( \cup \cup [Q] = \left[ \text{End}_{\mathcal{O}_X}(\pi_* P) \right] \in \text{Br}(k) \). On the other hand, \( \delta_P([U]) = \delta([U]) \cup [\chi] = \delta([U]) \cup [\chi] \). By Lemma 7.1, \( \delta_P([U]) = [U] \cup [\partial Q] = [U] \cup [Q_T]. \)

In general, let \( H = \text{Gal}(l/k) \) and consider the restriction map \( \text{Res} : H^1(G,T) \rightarrow H^1(H,T_k) \) which sends \([U] \) to \([U_k] \). There is a commutative diagram:

\[
\begin{array}{ccc}
H^1(G,T) & \xrightarrow{\delta} & H^2(G,S) & \xrightarrow{\chi} & \text{Br}(k) \\
\text{Res} \downarrow & & \text{Res} \downarrow & & \\
H^1(H,T_k) & \xrightarrow{\delta} & H^2(H,S_k) & \xrightarrow{\chi} & \text{Br}(k)
\end{array}
\]

Thus, \( \delta_P[U] = \left[ \text{End}_{\mathcal{O}_{X_k}}(\pi_* P) \right] \) where \( \pi_k : X_l \rightarrow X_k \) is the projection. By the proof of Lemma 6.3, \( \text{End}_{\mathcal{O}_{X_k}}(\pi_* P) \cong \text{End}_{\mathcal{O}_X}(\pi_* P) \).

Corollary 7.3. Let \( X \) be a smooth projective toric variety over \( k \) that splits over \( l \) and \( G = \text{Gal}(l/k) \). Assume \( \text{Pic}(X_l) \) is a permutation \( G \)-module, i.e, \( S \) is quasi-trivial, hence \( S \) has the form \( \prod_{i=1}^t \text{R}_{k_i/k} \mathbb{G}_{m,k_i} \) where \( k_i \) is a finite separable field extension of \( k \). Then the principal homogeneous space \( U \) is uniquely determined by \( (B_i \in \text{Br}(k_i))_{1 \leq i \leq t} \) where \( B_i \) splits over \( E \). Let \( \{P_i | 1 \leq i \leq t\} \) be the set of representatives for \( G \)-orbits of \( \text{Pic}(X_l) \), then \( B_i \) comes from \( \text{End}_{\mathcal{O}_X}(\pi_* P_i) \).

Proof. The result follows from the exact sequence \( 0 \rightarrow H^1(k,T) \rightarrow \prod_{i=1}^t \text{Br}(k_i) \rightarrow \text{Br}(E) \) and Proposition 7.2.

Remark 7.4. Families (i)(ii)(iii) and their blow-ups have permutation Picard groups.
(ii): $X = SB(A)$ is a Severi-Brauer variety of dimension $n$, $\text{Aut}_\Sigma(X_l) = S_{n+1}$. We have
$$1 \to \mathbb{G}_{m,k} \to R_{E/k}(\mathbb{G}_{m,E}) \to T \to 1$$
which induces
$$0 \to H^1(G,T) \xrightarrow{\delta} \text{Br}(k) \to \text{Br}(E).$$
Then $\delta(U) = [A]$ and $A$ splits over $E$, cf [MP97, Example 8.5].

(i): $X_l = F_{a}, a \geq 2, \text{Aut}_\Sigma = S_2$ and $E$ factors as $k \times F \times k$ where $F$ is the quadratic étale $k$-algebra corresponding to $[p'] \in H^1(G, S_2)$. We have
$$1 \to \mathbb{G}_{m,k} \to \mathbb{G}_{m,k} \times R_{F/k}(\mathbb{G}_{m,F}) \to T \to 1$$
where $\mathbb{G}_{m,k} \to \mathbb{G}_{m,k}$ is the $a$-th power homomorphism. It induces
$$0 \to H^1(G,T) \xrightarrow{\delta} \text{Br}(k) \to \text{Br}(k) \times \text{Br}(F)$$
where $[U] \mapsto [Q] \mapsto ([Q^{\otimes a}],[Q_F])$. By Lemma 4.8, the toric surface $X$ is a $\mathbb{P}^1$-bundle over some conic curve $Z$. We have the torus of $Z$ is $T' = R_{F/k}(\mathbb{G}_{m,F})/\mathbb{G}_{m,k}$. There is a commutative diagram with exact rows:

$\begin{array}{cccccc}
1 & \to & \mathbb{G}_{m,k} & \to & \mathbb{G}_{m,k} \times R_{F/k}(\mathbb{G}_{m,F}) & \to & T & \to & 1 \\
\text{surj.} & & \downarrow & & \downarrow h & & \downarrow & & \\
1 & \to & \mathbb{G}_{m,k} & \to & R_{F/k}(\mathbb{G}_{m,F}) & \to & T' & \to & 1
\end{array}$

Hence, the image of $[U]$ under $\delta \circ h_*$: $H^1(G,T) \to H^1(G,T') \to \text{Br}(k)$ is $[Q]$, and then $Z = SB(Q)$. Since a quaternion algebra has a period at most 2 in the Brauer group, $(Q^{\otimes a}) \in \text{Br}(k)$ is trivial implies that $Q = M_2(k)$ if $a$ is odd. Thus we have

**Proposition 7.5.** Let $X$ be a toric surface that is a form of $F_{2a+1}$, then $X \cong F_{2a+1}$.

**Remark 7.6.** Iskovskih showed that any form of $F_{2a+1}$ is trivial [Isk79, Theorem 3(2)]. The above proposition reprove this result in the case of toric surfaces.

(iii): $X_l = \mathbb{P}^1 \times \mathbb{P}^1, \text{Aut}_\Sigma = D_8$. In this case, $\beta : G \to \text{GL}(2,Z)$ factors through $\gamma : G \to S_2$ where $S_2$ permutes $\mathcal{O}(1,0)$ and $\mathcal{O}(0,1)$. Then the quadratic étale algebra $K$ corresponds to $\gamma$. We have
$$1 \to R_{K/k}(\mathbb{G}_{m,K}) \to R_{E/k}(\mathbb{G}_{m,E}) \to T \to 1$$
which induces
$$0 \to H^1(G,T) \xrightarrow{\delta} \text{Br}(K) \to \text{Br}(E)$$
Then $\delta(U) = [B]$ and $B$ splits over $E$. Let $N_{K/k} : R_{K/k}(\mathbb{G}_{m,K}) \to \mathbb{G}_{m,k}$ be the norm map which induces $\text{cor}_{K/k} : \text{Br}(K) \to \text{Br}(k)$, then $[A] = \text{cor}_{K/k}[B]$.

### 7.2. Separable algebras for toric surfaces.

Let $X$ be a smooth projective toric $T$-surface over $k$ that splits over $l$ and $G = \text{Gal}(l/k)$. Recall that we have a finite chain of blow-ups of toric $T$-surfaces
$$X = X_0 \to X_1 \to \cdots \to X_n = X'$$
where $X'$ is minimal. For $1 \leq i \leq n$, let $f_i : (X_{i-1})_l \to (X_i)_l$ and this is a blow-up of a $G$-set of $T_l$-invariant points. Let $E_i$ be the $G$-set of the exceptional divisors of $f_i$ and $X' \cong B$ in $\mathcal{C}$.

**Proposition 7.7.** $X \cong B \times \prod_{i=1}^n \text{Hom}_G(E_i,l)$ in $\mathcal{C}$.

**Proof.** We only need to consider the following simple case:

$f : Y \to Z$ is a blow-up of toric $T$-surfaces and $E = \{P_j\}$ is the $G$-set of the exceptional divisors of $g = f_l$. We assume further that the $G$-action on $E$ is transitive.

Let $p : Y_l \to Y$ and $q : Z_l \to Z$ be the projections, then we have a commutative diagram:
Recall that if $K_0(Z_i)$ has a $G$-basis $\gamma$, then $g^*(\gamma) \cup E$ is a $G$-basis of $K_0(Y_i)$. As $Z$ is a toric surface, we can assume $\gamma$ consists of line bundles over $Z_i$. Let $P \in \gamma$, then

$$\text{End}_{O_Y}(p_*g^*P) \cong \text{End}_{O_Y}(f^*q_*P) \cong \text{Hom}_{O_Z}(q_*P, f_*f^*(q_*P)) \cong \text{End}_{O_Z}(q_*P)$$

where $f_*f^*$ is identity as $f$ is flat proper and $f_*O_Y = O_Z$.

As for $G$-orbit $E$, we have $\bigoplus_j P_j = p^*Q$ for some locally free sheaf $Q$ on $Y$. By Lemma 6.6 and the assumption that $G$ acts transitively on $E$, $\text{End}_{O_Y}(Q) \cong \text{Hom}_G(E, l)$. It is Brauer equivalent to $\text{End}_{O_Y}(p_*P_j)$ for any $P_j \in E$. Thus the result follows from Theorem 6.5. \qed

8. Derived categories of toric varieties

Let $X$ be a smooth projective variety over $k$ and $D^b(X)$ be the bounded derived category of coherent sheaves on $X$. We will define exceptional objects and collections in a generalized way.

**Definition 8.1.** Let $A$ be a finite simple $k$-algebra. An object $V$ in $D = D^b(X)$ is called $A$-exceptional if $\text{Hom}_D(V, V) = A$ and $\text{Ext}^i_D(V, V) = 0$ for $i \neq 0$.

**Definition 8.2.** A set of objects $\{V_1, \ldots, V_n\}$ in $D = D^b(X)$ is called an exceptional collection if for each $1 \leq i \leq n$, $V_i$ is $A_i$-exceptional for some finite simple $k$-algebra $A_i$, and $\text{Ext}^r_D(V_i, V_j) = 0$ for any integer $r$ and $i > j$. The collection is full if the thick triangulated subcategory $\langle V_1, \ldots, V_n \rangle$ generated by the $V_i$ is equivalent to $D^b(X)$.

**Definition 8.3.** A set of objects $\{V_1, \ldots, V_n\}$ in $D \in D^b(X)$ is called an exceptional block if it is an exceptional collection and $\text{Ext}^r_D(V_i, V_j) = 0$ for any integer $r$ and $i \neq j$. Note that the ordering of the $V_i$ in this case does not matter.

Assume $\{V_1, \ldots, V_n\}$ is a full exceptional collection as above. Since $(V_i)$ is equivalent to $D^b(A_i)$, the bounded derived category of right $A_i$-modules, we have semiorthogonal decompositions $D^b(X) = \langle V_1, \ldots, V_n \rangle = \langle D^b(A_1), \ldots, D^b(A_n) \rangle$.

The semiorthogonal decomposition of $D^b(X)$ can be lifted to the world of dg categories. For details about dg categories, see [Ke106]. There is a dg enhancement of $D^b(X)$, denoted as $D^{dg}_g(X)$ where $D^{dg}_g(X)$ is the dg category with same objects as $D^b(X)$ and whose morphism has a dg $k$-module structure such that $H^0(\text{Hom}_{D^{dg}_g(X)}(x, y)) = \text{Hom}_{D^b(X)}(x, y)$. $\text{perf}_{dg}(X)$ is the dg subcategory of perfect complexes. Since $X$ is smooth projective, $\text{perf}_{dg}(X)$ is quasi-equivalent to $D^{dg}_g(X)$. For an $A$-exceptional object $V$, the pretriangulated dg subcategory $\langle V \rangle_{dg}$ generated by $V$ is quasi-equivalent to $D^{dg}_g(A)$. Therefore, there is a dg enhancement of the semiorthogonal decomposition $D^{dg}_g(X) = \langle V_1, \ldots, V_n \rangle_{dg}$, which is quasi-equivalent to $\langle D^{dg}_g(A_1), \ldots, D^{dg}_g(A_n) \rangle_{dg}$.

Let $dgcat$ be the category of all small dg categories, there is a universal additive functor $U : dgcat \to \text{Hmoq}$ where $\text{Hmoq}$ is the category of noncommutative motives, see [Tab15, §2.1-2.4]. We have $U(\text{perf}_{dg}(X)) \simeq \bigoplus_{i=1}^n U(D^{dg}_g(A_i)) \simeq \bigoplus_{i=1}^n U(A_i)$. On the other hand, the motivic category $\mathcal{C}$ is a full subcategory of $\text{Hmoq}$ by sending a pair $(X, A)$ to $\text{perf}_{dg}(X, A)$, the dg category of complexes of right $O_X \otimes_k A$-modules which are also perfect complexes of $O_X$-modules [Tab14, Theorem 6.10] or [Tab15, Theorem 4.17]. The above discussion gives the following well-known fact:

**Theorem 8.4.** Let $X$ be a smooth projective variety over $k$. Assume $D^b(X)$ has a full exceptional collection of objects $\{V_1, \ldots, V_n\}$ where $V_i$ is $A_i$-exceptional, then $X \cong \prod_{i=1}^n A_i$ in the motivic category $\mathcal{C}$. 

Now we explore the existence of a full exceptional collection for a smooth projective toric variety of higher dimension.

**Lemma 8.5.** Let \( Y \) be a split smooth projective toric variety over \( k \) of dimension \( n \geq 3 \) and let \( D \) be a Cartier divisor on \( Y \), then \( H^i(Y, \mathcal{O}(D)) = 0 \) for \( 0 < i < n \).

**Proof.** Let \( \Sigma \) be the fan structure of \( Y \), then the support \( |\Sigma| = \mathbb{R}^n \). Let \( \varphi \) be the support function corresponding to \( D \). [Dan78, Theorem 7.2] states that \( H^i(Y, \mathcal{O}(D)) = \bigoplus_{u \in M} H^i_{Z(u)}(Y, \mathcal{O}(D)) \) where \( M \) is the dual lattice, \( H^i_{Z(u)}(Y, \mathcal{O}(D)) = H^i(|\Sigma|, |\Sigma| \setminus Z(u)) \) and \( Z(u) = \{ v \in |\Sigma| \mid \langle u, v \rangle \geq \varphi(v) \} \).

Therefore, we have short exact sequences:

\[
0 \to H^0_{Z(u)}(Y, \mathcal{O}(D)) \to H^0(|\Sigma|) \to H^0(|\Sigma| \setminus Z(u)) \to H^1_{Z(u)}(Y, \mathcal{O}(D)) \to 0
\]

and \( H^i_{Z(u)}(Y, \mathcal{O}(D)) \cong H^{i-1}(|\Sigma| \setminus Z(u)) \) for \( i \geq 2 \). Since \( Z(u) \) is a union of cones, \( |\Sigma| \setminus Z(u) = \emptyset, \mathbb{R}^n \setminus \{0\} \) or is homotopy equivalent to \( \mathbb{R}^{n-1} \setminus \{p_1, \ldots, p_m\} \) for \( m \geq 0 \). From topology, we have

\[
H^i(\mathbb{R}^{n-1} \setminus \{p_1, \ldots, p_m\}) = \begin{cases} 
  k, & i = 0 \\
  k^m, & i = n - 2 \\
  0, & \text{otherwise}
\end{cases}
\]

Therefore, \( H^i(Y, \mathcal{O}(D)) = 0 \) for any \( D \) and \( 1 \leq i < n-1 \). By Serre duality, \( H^{n-1}(Y, \mathcal{O}(D)) = 0 \). \( \square \)

**Lemma 8.6.** Let \( Y, D \) be the same as before. Let \( G \) be a finite group that acts on Cartier divisors \( \text{CDiv}(Y) \) and fixes \( K_Y \). Assume that \( D \) is not linear equivalent to \( gD \) for any \( g \in G \), then \( \{ \mathcal{O}(gD) \mid g \in G \} \) is an exceptional block.

**Proof.** It suffices to show that \( H^i(Y, \mathcal{O}(gD - D)) = 0 \) for any \( g \in G \) and \( i = 0, n \). The case \( i = 0 \) is proved in Lemma 6.6. By Serre duality, \( H^i(Y, \mathcal{O}(gD - D)) \) is the dual of \( H^0(Y, \mathcal{O}(K_Y + D - gD)) \). Write \( E = K_Y + D - gD \) and there exists \( m \geq 1 \) such that \( g^m = 1 \). Assume there is a nonzero global section \( s \in \Gamma(Y, \mathcal{O}(E)) \), then there is a nonzero global section \( \otimes_{i=0}^{m-1} g^i s \) of \( \mathcal{O}(E + gE + \cdots + g^{m-1}E) = \mathcal{O}(mK_Y) \). But since \( Y \) is rational, \( H^0(Y, \mathcal{O}(mK_Y)) = 0 \), contradiction. \( \square \)

Assume \( X \) satisfies the conditions of Theorem 6.5, i.e. \( X \) is a smooth projective toric variety over \( k \) that splits over \( l \) where \( K_0(X_l) \) has a permutation \( G = \text{Gal}(l/k) \)-basis \( P \) of line bundles over \( X_l \). By Lemma 8.6, each \( G \)-orbit of \( P \) is an exceptional block. Let \( \pi : X_l \to X \) be the projection.

**Theorem 8.7.** If there is an ordering for \( G \)-orbits \( \{P_i\}_{i=1}^l \) of \( P \) such that \( \{P_1, \ldots, P_l\} \) forms a full exceptional collection of \( D^b(X_l) \), then for any \( S_i \in P_l \), \( \{\pi_* S_1, \ldots, \pi_* S_l\} \) is a full exceptional collection of \( D^b(X) \).

**Proof.** First we show that \( \{\pi_* S_1, \ldots, \pi_* S_l\} \) is an exceptional collection. Since \( \pi \) is flat and finite, both \( \pi^* : D^b(X) \to D^b(X_l) \) and \( \pi_* : D^b(X_l) \to D^b(X) \) are exact functors. The result follows from \( \text{Ext}^r_{D^b(X_l)}(\pi_* S_i, \pi_* S_j) \otimes_k l \cong \text{Ext}^r_{D^b(X_l)}(\pi^* \pi_* S_i, \pi^* \pi_* S_j) \cong \bigoplus_{g, g' \in G} \text{Ext}^r_{D^b(X_l)}(g S_i, g' S_j) \). In particular, \( \pi_* S_i \) is an exceptional object, thus \( \{\pi_* S_l\} \) is an admissible subcategory of \( D^b(X) \). Since \( \langle \pi_* S_i \otimes_k l \rangle = \langle P_i \rangle \) and \( D^b(X_l) = \langle P_1, \ldots, P_l \rangle \), by [AB15, Lemma 2.3], \( D^b(X) = \langle \pi_* S_1, \ldots, \pi_* S_l \rangle \). \( \square \)

**Remark 8.8.** In the cases of smooth projective toric surfaces \( X \), using the known results of semiorthogonal decompositions for projective spaces, projective bundles, del Pezzo surfaces and blow-ups, cf [Kuz14][BSS11][AB15], there exists such an ordering for the permutation \( G \)-basis of line bundles of \( K_0(X_l) \) constructed in Theorem 5.2. Therefore, the decomposition of \( X \) in the motivic category \( C \) constructed in §7 can be lifted to the semiorthogonal decomposition of \( D^b(X) \).

**Question 2.** For smooth projective toric varieties \( X \) satisfying the conditions of Theorem 6.5, is there always a choice of ordering for \( G \)-orbits \( \{P_i\}_{i=1}^l \) of \( P \) such that \( \{P_1, \ldots, P_l\} \) forms a full exceptional collection of \( D^b(X_l) \)?
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