EXTRA DIMENSIONS, NONMINIMAL COUPLINGS, HORIZONS AND WORMHOLES

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Received 20 April 1996

Static, spherically symmetric configurations of gravity with nonminimally coupled scalar fields are considered in D-dimensional space-times (D ≥ 4) in the framework of generalized scalar-tensor theories (STT). We seek special cases when the system has no naked singularity but, instead, forms either a black hole, or a wormhole. General conditions when this is possible, are formulated. In particular, some such special cases are indicated for multidimensional Brans-Dicke theory and for linear, massless, nonminimally coupled scalar fields (the coupling $\xi R \phi^2$ where $\xi = \text{const}$ and $R$ is the curvature). It is shown that in the Brans-Dicke theory the only black hole solution corresponds to $D = 4$ and the coupling constant $\omega < -2$, and there is a wormhole solution corresponding to $\omega = 0$. For the $\xi$-coupled linear scalar field it is shown that the only black hole solution is the well-known one, with a conformal scalar field ($\xi = \xi_{\text{conf}} = 1/6$) in 4 dimensions (a black hole with scalar charge), while the known 4-dimensional wormhole solution is generalized to systems with conformal coupling in arbitrary dimension.

1. Introduction

Prof. K.P. Staniukovich believed that general relativity (GR) is not an ultimate theory of gravity even on the classical level and paid much attention to its generalizations. This paper, submitted to an issue dedicated to his memory, touches upon some specific problems on this trend.

As is well-known, in GR all spherically symmetric scalar-vacuum and scalar-electrovacuum configurations possess naked singularities if the scalar field is massless, minimally coupled ($\phi_{\text{min}}$). Their counterparts with a conformally coupled scalar field ($\phi_{\text{conf}}$) provide a wider spectrum of possibilities: in the general case there are naked singularities as well, but more various types of these, and, moreover, in some special cases they describe black holes or wormholes.

A natural question arises: are these new possibilities a distinctive feature of conformal coupling and/or the 4-dimensional nature of the space-time, or they occur for more general nonminimal scalar field couplings in various dimensions?

GR with $\phi_{\text{min}}$ and $\phi_{\text{conf}}$ are special cases of a rather general model called generalized scalar-tensor theory (STT) of gravity (see e.g.), with the Lagrangian

$$L = A(\phi)R + B(\phi)\phi^2 - 2\Lambda(\phi) + L_m$$

where $R$ is the scalar curvature of a Riemannian space-time $V^D$ of arbitrary dimension $D$; $A$, $B$ and $\Lambda$ are functions varying from one specific STT to another and $L_m$ is the nongravitational matter Lagrangian. One frequently considers GR with linear scalar fields, which form a special case of (1), such that

$$A(\phi) = 1 - \xi \phi^2, \quad B \equiv 1,$$

where $\xi$ is the nonminimal coupling constant. In particular, $\xi = 0$ corresponds to minimal coupling and $\xi = \xi_{\text{conf}} = (D - 2)/(4(D - 1))$ to conformal coupling.

There are many reasons for considering nonminimal couplings. Historically, STT were put forward as viable theories other that GR, able to account for the observed effects of relativistic gravity or slightly modify the corresponding predictions of GR. One frequently considers GR with linear scalar fields, which form a special case of (1), such that

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There are many reasons for considering nonminimal couplings. Historically, STT were put forward as viable theories other that GR, able to account for the observed effects of relativistic gravity or slightly modify the corresponding predictions of GR. In modern theory of the early Universe, STT are one of the ways to create successful inflationary models (“extended inflation”, and many others). In quantum field renormalization theory the constant $\xi$ appears as a free parameter to be determined empirically; a closely related issue is the induced gravity concept (and others), where gravity itself essentially results from nonminimal scalar field coupling. Other sources of nonminimal...
couplings are the modern unification theories — (super)string and Kaluza-Klein ones. On the other hand, properties of spherically symmetric configurations are one of the key issues in any theory of gravity.

Returning to the above question on the occurrence of horizons and wormholes in models more general than 4-dimensional GR with $\varphi_{\text{conf}}$, for a very narrow range of generalizations an answer was given in Ref. [1]: it was shown that static scalar-vacuum solutions of $(d + 2)$-dimensional GR, with $\varphi_{\text{conf}}$ and the space-time structure $U^2 \times S^d$ (where $U^2$ accounts for the radial and temporal variables), can possess horizons only when $d = 2$, i.e., in the conventional 4-dimensional case.

Here we consider the broad class of theories [3] in the case $\Lambda = L_m = 0$, for which a general exact static, spherically symmetric solution is available. The space-time structure and the metric are assumed in the form

$$V^D(g) = U^2 \times S^2 \times V_1 \times \cdots \times V_n,$$

$$\dim M_i = N_i; \quad D = 4 + \sum_{i=1}^n N_i, \tag{3}$$

where $V_i$ ($i = 1, \ldots, n$) are Ricci-flat internal spaces of arbitrary dimensions $N_i$ and signatures:

$$ds_D^2 = e^{2\varphi} dt^2 - e^{2\alpha} du^2 - e^{2\beta} d\Omega^2 + \sum_{i=1}^n e^{2\beta_i(u)} ds_i^2; \tag{4}$$

$$d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$$

and $ds_i^2$ are the linear elements on $S^2$ and $V_i$, respectively.

In Sec. 2 we write out the relevant scalar-vacuum solution. In Sec. 3 we try to find general conditions when this solution can contain horizons or describe a wormhole. Further on the study is specialized to $D$-dimensional Brans-Dicke theory (Sec. 4) and GR with a linear, nonminimally coupled scalar field (Sec. 5). Sec. 6 is a brief conclusion.

## 2. Vacuum solutions of generalized scalar-tensor theories

The system [1] is essentially reduced to that with $\varphi_{\text{min}}$ by a conformal mapping well-known in $4$-dimensional STT [3] and modified for $D$ dimensions as follows [13]:

$$V^D(g) \rightarrow \nabla^D(\overline{g}) : \quad g_{MN} = A^{-2/(D-2)} \overline{g}_{MN}. \tag{5}$$

Indeed, omitting a total divergence, we obtain the following form of the Lagrangian in terms of $\overline{g}$:

$$\mathcal{L} = \overline{\mathcal{R}} + F(\varphi)\overline{g}^{AB}\varphi_A\varphi_B$$

$$+ A^{-D/(D-2)}[-2\Lambda(\varphi) + L_m] \tag{6}$$

where an overbar marks quantities corresponding to $\overline{g}_{AB}$ and

$$F(\varphi) = \frac{1}{A^2} \left[ AB + \frac{D - 1}{D - 2} \left( \frac{dA}{d\varphi} \right)^2 \right]. \tag{7}$$

Putting in [14] $\Lambda = L_m = 0$, it is possible to write down the general static, spherically symmetric scalar-vacuum solution to the field equations in the following form [14, 12]:

$$ds_D^2 = f(u)ds_2^2, \quad f(u) = \left[ A(\varphi) \right]^{-2/(D-2)};$$

$$ds_2^2 = e^{-2b_0u} dt^2 - \frac{e^{2B_0}}{s^2(k, u)} \left[ \frac{du^2}{s^2(k, u)} + d\Omega^2 \right] + \sum_{i=1}^n e^{-2b_i(u)} ds_i^2,$$

$$F(\varphi)(d\varphi/du)^2 = S = \text{const} \tag{8}$$

where $u$ is the radial coordinate, harmonic in $\nabla$ (such that $\nabla^M\nabla_M u = 0$), defined for $u > 0$; the flat-space asymptotic corresponds to $u = 0$; the integration constants $B, b_i, k$ and $S$ are connected by the relations

$$B = b_0 + \sum_{i=1}^n N_i b_i;$$

$$2k^2 \text{sign} k = B^2 + b_0^2 + \sum_{i=1}^n N_i b_i^2 + S; \tag{9}$$

lastly, the function $s(k, u)$ is defined as follows:

$$s(k, u) \equiv \begin{cases} (1/k) \sinh ku, & k > 0, \\ u, & k = 0, \\ (1/k) \sin ku, & k < 0. \end{cases} \tag{10}$$

The constant $S$ has the meaning of a scalar charge; with $S = 0$ we are led to $D$-dimensional GR. We will be naturally interested in the nontrivial case $S \neq 0$.

As follows from the last line of [13], due to $S = \text{const}$ the function $F(\varphi)$ has the same sign in the whole space (or at least in the $u$-chart which includes the asymptotic region). Therefore, applied to our spherically symmetric case, all STT are divided into two large classes: $F(\varphi) > 0$ and $F(\varphi) < 0$, hereafter labelled as normal and anomalous, respectively. In normal STT the gradient term in [13] has its conventional sign and consequently the scalar field energy density is non-negative in this conformal frame.

By [14] we have $k > 0$ for all normal STT. Thus many possible solution behaviours, connected with $k < 0$, are possible only in anomalous STT with $F < 0$.

An alternative form of $ds_2^2$ for $k > 0$ is obtained after the coordinate transformation

$$e^{-2ku} = 1 - 2k/R \equiv P(R), \tag{11}$$

namely,

$$ds_2^2 = P^{a_0} dt^2 - P^{-A} dR^2 - P^{1-A} R^2 d\Omega^2 + \sum_{i=1}^n P^{a_i} ds_i^2 \tag{12}$$
where the constants $k$, $a_i = b_i/k$, $A = B/k$ and $S$ are connected by the relation

$$A^2 + a_0^2 + \sum_{i=1}^n N_i a_i^2 + S/k^2 = 2. \quad (13)$$

3. Search for horizons and wormholes

3.1. Criteria

We will try to find special cases when the metric $ds^2_D$ describes a black hole, i.e., possesses an event horizon. That means, in terms of (4), that at some value of the radial coordinate $(u = u_{\text{hor}})$

A1. $e^\gamma \to 0$, while

A2. $e^\beta$ remains finite and

A3. $e^{\beta_i}$ $(i = 1, \ldots, n)$ remain finite;

A4. No signal can reach $u = u_{\text{hor}}$ for a finite time by a remote observer’s clock, i.e., the integral $\int du e^{\alpha - \gamma}$ diverges as $u \to u_{\text{hor}}$.

In the formulation of Items A1–A4 the radial coordinate is arbitrary, not necessarily harmonic.

We will also look for cases when the solution describes a (static, traversable) wormhole, i.e., there are two flat-space asymptotics connected by a regular bridge. That means that, as well as at $u = 0$, at some other value of the radial coordinate $u = u_\infty$

B1. $e^\gamma$ and $e^{\beta_i}$ $(i = 1, \ldots, n)$ remain finite;

B2. $e^\beta \to \infty$;

B3. There is an infinite path along the radius, i.e., the integral $\int e^{\alpha} du$ diverges;

B4. A correct flat-space circumference-radius ratio for coordinate circles is asymptotically valid, i.e., $e^{\beta_i - \alpha \beta_i} \to 1$.

These criteria are also radial coordinate reparametrization invariant.

3.2. Search for horizons ($k > 0$)

The solution (8) is regular for $u < \infty$ ($R > 2k$), provided the function $f(u)$ is regular (vanishing or blowing-up of $f(u)$ at finite $u$ can lead only to a naked singularity). So a horizon can exist either at the sphere $u = \infty$ ($R = 2k$), or somewhere beyond this sphere if the latter is regular. Consider the first possibility in terms of (13).

Criterion A3 implies that all $a_i$ $(i \geq 1)$ are equal, and with no loss of generality we will assume that there is only one internal space $V_1$, with $\dim V_1 = N_1 = N$; as also follows from A3,

$$f(u) \sim P^{-a_1} \quad \text{as } R \to 2k. \quad (14)$$

From A1 and A2 it follows

$$a_0 > a_1, \quad (15)$$

$$a_0 + (N + 1)a_1 = 1, \quad (16)$$

respectively. Finally, A4 leads to $A + a_0 \geq 2$, whence

$$a_1(N + 2) \leq 0. \quad (17)$$

On the other hand, Eq. (13) with (14) gives

$$S/k^2 = a_1(N + 2)[2 - (N + 1)a_1]. \quad (18)$$

By (17), $a_1 \leq 0$; but $a_1 = 0$ with (18) leads to $S = 0$, i.e., the trivial case $a_0 = A = 1$, $S = 0$, that is, $\varphi = \text{const}$ and the Schwarzschild metric with “frozen” extra dimensions (the only black hole solution in the minimal coupling case $F(\varphi) = \text{const} \, [16]$).

The other option, $a_1 < 0$, leads, by (18), to $S < 0$ and consequently to $F < 0$. We arrive at

**Proposition 1.** Nontrivial (non-Schwarzschild) black holes with horizons at $u = \infty$ can exist only in anomalous STT.

Moreover, in this case

$$f(u) \equiv |A(\varphi)|^{-2/(D-2)} \sim P^{-a_1} \Rightarrow 0$$

$$A(\varphi) \to \infty \quad \text{as } u \to \infty, \quad (19)$$

i.e., the effective gravitational coupling ($\sim 1/A$) vanishes at the horizon.

If $u = \infty$ is a regular sphere, the solution behaviour depends on $A(\varphi)$ and cannot be determined in a general manner; however, the known example of black holes with a conformal scalar field in 4 dimensions makes sure that horizons beyond such a sphere are, in principle, possible.

This sphere is regular if all the metric coefficients in (8) are finite, i.e., all $a_i$ are equal and, in addition,

$$a_i = 1 - A \Rightarrow a_i = 1/(N + 2). \quad (20)$$

Then Eq. (13) implies

$$S/k^2 = 1 + 1/(N + 2) = 1/(4\xi_{\text{conf}}). \quad (21)$$

The result $S > O$ means that $F(\varphi) > 0$. So we have proved

**Proposition 2.** A continuation beyond $u = \infty$ is possible only in normal STT.

Moreover, $f(u) \sim P^{-a_0} \to \infty$ as $u \to \infty$, i.e., the effective gravitational coupling tends to infinity. The experience of dealing with such behaviour in 4 dimensions indicates that a strong gravitational instability can probably develop near such a sphere.
We also notice that in the present case the functions $g_{\alpha\alpha}(u)$ ($a > 0$) and $g_{00}(u)$ coincide up to a constant scale factor. This coincidence will be naturally preserved beyond $u = \infty$ as well. Therefore Criteria A1 and A3 cannot be fulfilled together. We can conclude the following:

**Proposition 3.** Event horizons beyond the sphere $u = \infty$ are possible only in the case $D = 4$.

### 3.3. Search for horizons ($k \leq 0$)

In the case $k = 0$ the solution is regular at $u < \infty$ provided $f(u)$ is regular. As $u \to \infty$, $e^\beta$ behaves like $e^{const \cdot u}$, while $e^{\omega}$ behaves like just $e^{const \cdot u}$. Therefore $g_{22}$ and $g_{\alpha\alpha}$ ($a > 3$) cannot simultaneously tend to constants and for $D > 4$ the surface $u = \infty$ can be neither a regular sphere, nor a horizon.

If $k = 0$, $D = 4$, a horizon at $u = \infty$ is possible if

$$b_0 > 0, \quad f(u) \sim u^2 e^{-2b_0 u}. \quad (22)$$

Then $g_{00} \sim u^2 e^{-4b_0 u} \to 0$ and A4 is also valid. So this is a possible black hole case.

Let us now address the case $k < 0$, so that in (8) $s(k, u) = k^{-1} \sin ku$ and the solution is defined for $0 < u < u_{max} = \pi/|k|$. All the exponential functions are finite. So, in the general case, when $f(u)$ is regular for $0 \leq u \leq u_{max}$, the solution, as is easily verified, describes a wormhole $[3, 14]$. If, however, $f(u) \to 0$ as $u \to 0$, so that $g_{22}$ be finite, then at the same time $g_{00} \to 0$ and $g_{\alpha\alpha} \to 0$ ($a > 3$), in contrast to Criterion A3. Thus, in addition to Proposition 3, we have

**Proposition 4.** Event horizons for $k < 0$ can exist only if $D = 4$.

Criterion A4 is then valid as well and, in addition, $\mathcal{A} \to \infty$ as $u \to u_{max}$, so that again the effective gravitational coupling blows up.

### 3.4. Search for wormholes

We have seen that for $k < 0$ (that is, only in anomalous STT) wormholes appear in the general case. Let us find out when they are possible in normal STT.

They are evidently possible when the space-time is continued beyond $u = \infty$, as shown by an explicit example of this sort $[3]$ ($D = 4$, GR with $\omega_{\text{conf}}$).

Another possibility is that the second flat asymptotic occurs at $u = \infty$ ($R = 2k$). In this case one must have in $[4]$

$$a_0 = a_1 = 1/(D-2); \quad S/k^2 = (D-1)/(D-2) \quad (29)$$

and the function $f(u)$ takes the form

$$f(u) = e^{-2s u/(D-2)} \equiv P^c, \quad c = \pm \sqrt{\frac{D-1}{(D-2)^3} \omega} \quad (30)$$

Then the regularity condition $f(u)P^{a_0} \to \text{const}$ as $u \to \infty$ implies that in $[4]$ the minus sign must be chosen and

$$\omega = (D-1)/(D-2) \Rightarrow \omega = 0. \quad (31)$$

Thus the continuation is possible only in the special case of the STT $[5]$ with $\omega = 0$. Under the above conditions the metric takes the simple form

$$ds_D^2 = dt^2 - \frac{dR^2}{1 - 2k/R} - R^2 d\Omega^2 + ds^2. \quad (32)$$

On the other hand, Criteria B2 and B3 lead to

$$a_0 \geq 2/(N + 2), \quad (25)$$

whence by (24)

$$S/k^2 \leq -2N, \Rightarrow F < 0, \quad (26)$$

i.e., this situation is possible only for anomalous STT. Moreover, the asymptotic circumference-to-radius ratio is $\sim P^{-1/2} \to \infty$, i.e., Criterion B4 is violated: this configuration is not a wormhole. A conclusion is:

**Proposition 5.** In normal STT wormholes are possible only with a continuation beyond $u = \infty$.

### 4. Example: the Brans-Dicke theory

A multidimensional generalization of the Brans-Dicke STT is specified by the functions

$$\mathcal{A}(\phi) = \varphi, \quad \mathcal{B}(\phi) = \omega/\varphi, \quad \omega = \text{const.} \quad (27)$$

Consequently,

$$F(\varphi) = \varphi^2, \quad \varpi = \omega + \frac{D-1}{D-2},$$

$$f(u) = \varphi^{-2/(D-2)}, \quad F\varphi^2 = S \Rightarrow \varphi = \varphi_0 e^{su} \quad (28)$$

where $s = \text{const} = \pm \sqrt{S/\varpi}$. In what follows we omit the unessential constant $\varphi_0$. Recall that the metric is given in [3] or, for $k > 0$, [14].

Let us first assume that the STT is normal, i.e., $\varpi > 0$. As follows from the above considerations, black holes or wormholes are then possible only beyond a regular sphere $u = \infty$ ($R = 2k$). One easily finds that this sphere can be regular if

$$a_0 = a_1 = 1/(D-2); \quad S/k^2 = (D-1)/(D-2) \quad (29)$$

and the function $f(u)$ takes the form

$$f(u) = e^{-2s u/(D-2)} \equiv P^c, \quad c = \pm \sqrt{\frac{D-1}{(D-2)^3} \omega} \quad (30)$$

Then the regularity condition $f(u)P^{a_0} \to \text{const}$ as $u \to \infty$ implies that in [4] the minus sign must be chosen and

$$\omega = (D-1)/(D-2) \Rightarrow \omega = 0. \quad (31)$$

Thus the continuation is possible only in the special case of the STT [5] with $\omega = 0$. Under the above conditions the metric takes the simple form

$$ds_D^2 = dt^2 - \frac{dR^2}{1 - 2k/R} - R^2 d\Omega^2 + ds^2. \quad (32)$$
The continuation beyond $R = 2k$ is realized, for example, by putting $x = \sqrt{R - 2k}$ and allowing $x$ to take all real values. Then

$$ds_D^2 = dt^2 - 4(x^2 + 2k)dx^2 - (x^2 + 2k)^2 d\Omega_k^2 + ds_1^2,$$

(33)

i.e., a wormhole, manifestly symmetric under the substitution $x \rightarrow -x$, with the neck radius $\sqrt{2k}$, trivial extra dimensions and zero mass (since $g_{00} = \text{const}$). However, the scalar field is nontrivial: $\varphi = x/\sqrt{x^2 + 2k}$.

In the anomalous case $\omega < 0$ there is no continuation beyond $u = \infty$ and the only nontrivial possibility is that of a horizon at $u = \infty$ for $k > 0$. Indeed, for $k < 0$, when $u$ is defined on a finite segment, the exponential conformal factor $f(u)$ cannot change the metric qualitatively; for $k = 0$, $g_{22}$ behaves like $e^{c \text{const} u}/u$ and cannot tend to a finite limit as $u \rightarrow \infty$.

For $k > 0$ the requirements A1–A4 with $a_1 \neq 0$ are fulfilled if and only if $\omega < -2$ and

$$a_0 = 1 - (N + 1)a_1; \quad a_1 = \frac{2}{(N + 2)(\omega + 2)} < 0,$$

(34)

so that the only remaining free integration constant is $k$, connected with the black hole mass. The “scalar charge” $S$ is also expressed in terms of $k$ and $\omega$:

$$S = \frac{4}{\omega + 2} \left[ 1 + \frac{N + 1}{(N + 2)(\omega + 2)} \right].$$

(35)

The assumption $a_1 = 0$ leads to a Schwarzschild black hole with trivial extra dimensions.

However, the case $D = 4$, when the condition A3 is cancelled, must be considered separately. It turns out that a non-Schwarzschild horizon at $u = \infty$ indeed takes place if and only if $\omega < -2$ and

$$a_0 = \frac{1}{|\omega|} - \frac{1}{|\omega| - 2} > 1, \quad S/k^2 = 2(1 - a_0^2).$$

(36)

There is again only one free integration constant $k$.

5. Linear, nonminimally coupled scalar fields

5.1. The general case

We have seen that there are indeed some cases when the conformal factor $f(u)$ induces new qualitative features of the solution behaviour, the existence of black hole and wormhole configurations. Let us now consider probably the most interesting special case of STT, namely, that described by Eq. (3). In this case, in Eq. (3)

$$F(\varphi) = \frac{1 - \eta \varphi^2}{(1 - \varphi^2)^2}, \quad \eta = \xi(1 - \xi/\xi^{\text{conf}}).$$

(37)

We assume $\xi \varphi^2 < 1$, i.e., the values of $\varphi$ when the STT is normal at the asymptotic (in the opposite case, the effective gravitational coupling would be negative, which seems nonphysical). Then the possibility of black holes or wormholes may be connected only with a continuation beyond $u = \infty$.

The general consideration of Sec. 3 implies that, for naked-singularity-free solutions, a regular sphere $u = \infty$ and a continuation beyond it must be provided. For such a sphere $A \rightarrow 0$, therefore let us consider the asymptotic of $f(u)$ as $A \rightarrow 0$.

Evidently, in this case $\xi > 0$, $\varphi^2 \rightarrow 1/\xi$ and $1 - \eta \varphi^2 \rightarrow \xi/\xi^{\text{conf}}$. Hence the last line of (3) gives:

$$\sqrt{S}du \sim \frac{\xi}{\xi^{\text{conf}}} \frac{d\varphi}{1 - \xi \varphi^2}$$

(38)

which yields after integration

$$\sqrt{\xi} \varphi = \tanh h(u + u_0), \quad h = \sqrt{S/\xi} \xi^{\text{conf}}$$

(39)

where $u_0$ is an integration constant and “$\sim$” means the greatest term in a possible series decomposition kept in mind. From (39) it follows

$$f(u) \sim e^{4hu/(D - 2)} \quad (u \rightarrow \infty).$$

(40)

On the other hand, a regular sphere at $u = \infty$ can exist only if (see (11) and (21))

$$f(u) \sim e^{2hu/(D - 2)}.$$  

(41)

Comparing the two expressions for $f(u)$ and excluding $h$, one obtains

$$S/k^2 = \xi/(2\xi^{\text{conf}})^2.$$  

(42)

Meanwhile, as follows from (21), $S/k^2 = 1/(4\xi^{\text{conf}})$, which is compatible with (42) only when $\xi = \xi^{\text{conf}}$.

We conclude that the continuation is possible only with a conformally coupled field, $\varphi^{\text{conf}}$. Therefore, although it is straightforward, we will not determine $f(u)$ explicitly in the general case.

5.2. Conformal coupling

Let us now give an explicit form of the solution for $\xi = \xi^{\text{conf}}$. One easily finds:

$$\sqrt{\xi} \varphi = \tan [\sqrt{\xi S}(u + u_0)];$$

$$f(u) = \left\{ \cosh [\sqrt{\xi S}(u + u_0)] \right\}^{4/(D - 2)}.$$  

(43)

Under the above regularity conditions for the sphere $u = \infty$, $\sqrt{\xi S} = k/2$ and the corresponding special solution can be transformed from (3) into

$$ds_D^2 = \frac{c}{2} \left( 1 + \frac{x}{c^2} \right)^{4/(D - 2)} \times \left\{ dt^2 - \frac{4k^2}{(1 - x^2)^2} \left[ \frac{4dx^2}{(1 - x^2)^2} + d\Omega_k^2 \right] + ds_1^2 \right\},$$

$$\varphi(x) = \frac{1}{\sqrt{\xi} c^2} - \frac{x}{c^2}.$$  

(44)
where we have introduced
\[ x = e^{-ku}, \quad c = e^{ku_0}/2. \] (45)

So \( x = 1 \) corresponds to spatial infinity, \( x = 0 \) to the regular sphere \( u = \infty \) and \( x < 0 \) is the new domain uncovered by the \( u \)-chart. The solution behaviour turns out to strongly depend on the value of \( c \), the constant determining the asymptotic value of the \( \varphi \) field:

1. \( c < 1 \). At \( x = -c^2 > -1 \) the metric in (44) has a naked singularity due to conformal factor vanishing.

2. \( c = 1 \). The coordinate \( x \) is defined up to \( x = -1 \).
   For \( D = 4 \), after the further reparametrization
   \[ k/(1-x) = r, \] (46)

   we obtain the old solution describing a black hole with a scalar charge [4, 3].

   \[ ds^2 = \left(1 - \frac{k}{2r}\right)^2 dt^2 - \left(1 - \frac{k}{2r}\right)^{-2} dr^2 - r^2 d\Omega^2, \]

   \[ \varphi(r) = \frac{k}{\sqrt{2} r} \] (47)

For \( D > 4 \), as \( x \to -1 \), the spherical radius \( r = \sqrt{-g_{22}} \) tends to infinity. This is a kind of horizon, displaced infinitely far beyond a neck (a minimum of \( r \)), since both the proper length along the radial direction and the light travel time diverge as \( x \to -1 \).

3. \( c > 1 \). The conformal factor before the curly bracket in (44) does not qualitatively affect the metric behaviour. One can easily verify that in this case \( x \to -1 \) is another flat-space asymptotic and all the wormhole criteria B1–B4 are fulfilled. This wormhole solution of arbitrary dimension generalizes that found in [3] for \( D = 4 \).

### 6. Conclusion

We have studied static, spherically symmetric scalar-vacuum configurations with both minimal and nonminimal scalar-metric couplings in space-times of arbitrary dimension. It has turned out that the introduction of nonminimal couplings can indeed widen the spectrum of solution behaviours, but for “normal” STT (roughly, when the scalar field energy is positive-definite) something different from naked singularities, namely, black holes or wormholes, takes place in very rare special cases. For instance, black holes can appear only in 4 dimensions. For anomalous STT, apart from the general wormhole case \( (k < 0 \) in the solution), some special black hole configurations may exist.

The situation was discussed in detail for the Brans-Dicke theory and the coupling \( \xi R \varphi^2 \). For this \( \xi \)-coupling it has been shown that black holes and wormholes appear only in the conformal case, \( \xi = \xi^{\text{conf}} \) (see [3]); the only black hole solution is the well-known one \( (D = 4, \varphi \equiv 0) \), a black hole with scalar charge [3, 4], while the known 4-dimensional wormhole solution [3] is generalized to arbitrary \( D \).

### Acknowledgement

This work was supported in part by the Russian Ministry of Science.

### References

[1] K.P. Staniukovich, “Gravitational Field and Elementary Particles”, Nauka, Moscow, 1965 (in Russian).
[2] K.P. Staniukovich and V.N.Melnikov, “Hydrodynamics, Fields and Constants in Gravitation Theory”. Energoatomizdat, Moscow 1983 (in Russian).
[3] K.A. Bronnikov, Acta Phys. Polon. B4, 251 (1973).
[4] N.M. Bocharova, K.A. Bronnikov and V.N. Melnikov, Vestnik Mosk. Univ., Fiz., Astron., No. 6 (1970), 706 (in Russian).
[5] J.D. Bekenstein, Ann. Phys. N.Y. 82, 535 (1974); 91, 72 (1975).
[6] R. Wagoner, Phys. Rev. D 1, 3209 (1970).
[7] P. Jordan, Nature 164, 637 (1949).
[8] C. Brans and R.H. Dicke, Phys. Rev. 124, 925 (1961).
[9] D. La and P. Steinhardt, Phys. Rev. Lett. 62, 376 (1989).
[10] A. Zee, Phys. Rev. Lett. 42, 417 (1979).
[11] B.C. Xanthopoulos and T.E. Dialynas, J. Math. Phys. 33, 1463 (1992).
[12] K.A. Bronnikov and V.N. Melnikov, in “Results of Science and Technology. Gravitation and Cosmology” (V.N. Melnikov, Ed.), Vol.4, p.67, VINITI Publ., Moscow, 1992 (in Russian).
[13] K.A. Bronnikov and Yu.N. Kireyev, Phys. Lett. 67A, 95 (1978).
[14] H.G. Ellis, J. Math. Phys. 14, 104 (1973).
[15] K.A. Bronnikov and V.N. Melnikov, preprint RGA-CSVR-003/94, gr-qc/9403064 Gen. Relat. & Gravit. 27, 465 (1995).
[16] K.A. Bronnikov and V.D. Ivashchuk, in: Materials of the 7th Russian Gravitational Conference, Yerevan, 1988, p. 156.