Alternating group covers of the affine line

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Abstract

For an odd prime $p \equiv 2 \mod 3$, we prove Abhyankar’s Inertia Conjecture for the alternating group $A_{p+2}$, by showing that every possible inertia group occurs for a (wildly ramified) $A_{p+2}$-Galois cover of the projective $k$-line branched only at infinity where $k$ is an algebraically closed field of characteristic $p > 0$. More generally, when $2 \leq s < p$ and $\gcd(p-1, s+1) = 1$, we prove that all but finitely many rational numbers which satisfy the obvious necessary conditions occur as the upper jump in the filtration of higher ramification groups of an $A_{p+s}$-Galois cover of the projective line branched only at infinity.

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1 Introduction

Suppose $\phi : Y \to \mathbb{P}^1_k$ is a $G$-Galois cover of the projective $k$-line branched only at $\infty$ where $G$ is a finite group and $k$ is an algebraically closed field of characteristic $p > 0$. Let $p(G) \subset G$ be the normal subgroup generated by the conjugates of a Sylow $p$-subgroup. Then the $G/p(G)$-Galois quotient cover is a prime-to-$p$ Galois cover of $\mathbb{P}^1_k$ branched only at $\infty$. Since the prime-to-$p$ fundamental group of the affine line $A^1_k$ is trivial, this implies that $p(G) = G$; a group $G$ satisfying this condition is called quasi-$p$. In 1957, Abhyankar conjectured that a finite group $G$ occurs as the Galois group of a cover $\phi : Y \to \mathbb{P}^1_k$ branched only at $\infty$ if and only if $G$ is a quasi-$p$ group [1]. Abhyankar’s conjecture was proved by Raynaud [10] and Harbater [7].

Now suppose $G_0$ is the inertia group at a ramified point of $\phi$. Then $G_0$ is a semi-direct product of the form $G_1 \rtimes \mathbb{Z}/(m)$ where $G_1$ is a $p$-group and $p \nmid m$ [12 IV]. Let $J \subset G$ be the normal subgroup generated by the conjugates
of $G_1$. Then the $G/J$-Galois quotient cover is a tame Galois cover of $\mathbb{P}^1_k$ branched only at $\infty$. Since the tame fundamental group of $\mathbb{A}^1_k$ is trivial, this implies that $J = G$. Based on this, Abhyankar stated the currently unproven Inertia Conjecture.

**Conjecture 1.1** (Inertia Conjecture). [3, Section 16] Let $G$ be a finite quasi-$p$ group. Let $G_0$ be a subgroup of $G$ which is an extension of a cyclic group of order prime-to-$p$ by a $p$-group $G_1$. Then $G_0$ occurs as the inertia group of a ramified point of a $G$-Galois cover $\phi : Y \to \mathbb{P}^1_k$ branched only at $\infty$ if and only if the conjugates of $G_1$ generate $G$.

There is not much evidence to support the converse direction of Conjecture 1.1. For every finite quasi-$p$ group $G$, the Sylow $p$-subgroups of $G$ do occur as the inertia groups of a $G$-Galois cover of $\mathbb{P}^1_k$ branched only at $\infty$ [9]. For $p \geq 5$, Abhyankar’s Inertia Conjecture is true for the quasi-$p$ groups $A_p$ and $\text{PSL}_2(\mathbb{F}_p)$ [3, Thm. 2]. In Theorem 5.2, we prove:

**Theorem 1.2.** If $p \equiv 2 \mod 3$ is an odd prime, then Abhyankar’s Inertia Conjecture is true for the quasi-$p$ group $A_{p+2}$. In other words, every subgroup $G_0 \subset A_{p+2}$ of the form $\mathbb{Z}/(p) \rtimes \mathbb{Z}/(m)$ occurs as the inertia group of an $A_{p+2}$-Galois cover of $\mathbb{P}^1_k$ branched only at $\infty$.

Note that the values of $m$ such that $A_{p+2}$ contains a subgroup $G_0 \simeq \mathbb{Z}/(p) \rtimes \mathbb{Z}/(m)$ are exactly the divisors of $p - 1$. We also give a second proof of Abhyankar’s Inertia Conjecture for the group $A_p$ when $p \geq 5$; this proof uses the original equations of Abhyankar [2] rather than relying on the theory of semi-stable reduction.

More generally, we study the ramification filtrations of $A_n$-Galois covers $\phi : Y \to \mathbb{P}^1_k$ branched only at $\infty$ when $p$ is odd and $p \leq n < 2p$. This condition ensures that the order of $A_n$ is strictly divisible by $p$, and so the ramification filtration is determined by the order of $G_0$ and the upper jump $\sigma$. The upper jump is a rational number that satisfies some necessary conditions, Notation 5.5. One motivation to study the ramification filtration is that it determines the genus of $Y$.

In Theorem 4.9, we compute the order of the inertia group and the upper jump of the ramification filtration of Abhyankar’s $A_{p+s}$-Galois cover $\phi_s : Y_s \to \mathbb{P}^1_k$ branched only at $\infty$ when $2 \leq s < p$. This determines the genus of $Y_s$, which turns out to be quite small. When $\gcd(p - 1, s + 1) = 1$, the inertia group is a maximal subgroup of the form $\mathbb{Z}/(p) \rtimes \mathbb{Z}/(m)$ in $A_{p+s}$. This is
the basis of the proof of Theorem 1.2 when \( s = 2 \). It also leads to another application, Corollary 5.6, where we use the theory of formal patching to prove:

**Corollary 1.3.** Suppose \( 2 \leq s < p \) and \( \gcd(p - 1, s + 1) = 1 \). Then all but finitely many rational numbers \( \sigma \) satisfying the obvious necessary conditions occur as the upper jump of an \( A_{p+s} \)-Galois cover of \( \mathbb{P}^1_k \) branched only at \( \infty \).

In fact, Corollary 1.3 is a strengthening of Theorem 1.2 when \( s = 2 \). When \( s > 2 \), the normalizer of a \( p \)-cycle in \( A_{p+s} \) contains more than one maximal subgroup of the form \( \mathbb{Z}/(p) \rtimes \mathbb{Z}/(m) \). This is because there are many elements of prime-to-\( p \) order that centralize a \( p \)-cycle in \( A_{p+s} \). Thus, when \( s > 2 \), more equations will be needed to verify Abhyankar’s Inertia Conjecture for the group \( A_{p+s} \) using the strategy of this paper.

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## 2 Background

Let \( k \) be an algebraically closed field of characteristic \( p \geq 3 \). A curve in this paper is a smooth connected projective \( k \)-curve. A cover \( \phi \) of the projective line branched only at \( \infty \) will be called a *cover of the affine line* and the inertia group at a ramification point of \( \phi \) above \( \infty \) will be called the *inertia group of \( \phi \)*. A *\( G \)-Galois cover* is a Galois cover \( \phi : Y \to X \) together with an isomorphism \( G \simeq \text{Aut}(Y/X) \); (the choice of isomorphism will not be important in this paper).

### 2.1 Ramification

Let \( K \) be the function field of a \( k \)-curve \( X \). A place \( P \) of \( K/k \) is the maximal ideal of a valuation ring \( O_P \subset K \). Let \( \mathcal{P}_K \) denote the set of all such places. Let \( v_P \) denote the normed discrete valuation on the valuation ring \( O_P \). A *local parameter at \( P \)* is an element \( \alpha \in O_P \) such that \( v_P(\alpha) = 1 \).

Consider a finite separable extension \( F/K \). Let \( \tilde{F} \) be the Galois closure of \( F/K \) and let \( G \) be the Galois group of \( \tilde{F}/K \). A place \( Q \in \mathcal{P}_F \) is said to lie over \( P \in \mathcal{P}_K \) if \( O_P = O_Q \cap \kappa \) and we denote this by \( Q|P \). For any \( Q \in \mathcal{P}_F \) with \( Q|P \), there is a unique integer \( e(Q|P) \) such that \( v_Q(x) = e(Q|P)v_P(x) \) for any \( x \in K \). The integer \( e(Q|P) \) is the *ramification index* of \( Q|P \) in \( F/K \).
The extension $F/K$ is \textit{wildly ramified} at $Q|P$ if $p$ divides $e(Q|P)$. When there exists a ramification point $Q$ such that $p$ divides $e(Q|P)$, we say that the extension is \textit{wildly ramified}.

2.2 Higher Ramification Groups

We will need the following material from \cite[Chapter 3]{13}.

\textbf{Definition 2.1.} For any integer $i \geq -1$ the $i$-th lower ramification group of $Q|P$ is

$$G_i(Q|P) = \{\sigma \in G : \nu_Q(\sigma(z) - z) \geq i + 1 \text{ for all } z \in \mathcal{O}_Q\}.$$ 

We let $G_i$ denote $G_i(Q|P)$ when the places are clear from context.

\textbf{Proposition 2.2.} With the notation above, then:

1. $G_0$ is the inertia group of $Q|P$, and thus $|G_0| = e(Q|P)$, and $G_1$ is a $p$-group.

2. $G_{-1} \supseteq G_0 \supseteq \cdots$ and $G_h = \{\text{Id}\}$ for sufficiently large $h$.

\textbf{Theorem 2.3} (Hilbert’s Different Formula). The different exponent of $F/K$ at $Q|P$ is

$$d(Q|P) = \sum_{i=0}^{\infty} (|G_i(Q|P)| - 1).$$

Here is the Riemann-Hurwitz formula for wildly ramified extensions.

\textbf{Theorem 2.4} (Riemann-Hurwitz Formula). Let $g$ (resp. $g'$) be the genus of the function field $K/k$ (resp. $F/k$). Then

$$2g' - 2 = [F : K](2g - 2) + \sum_{P \in \mathcal{P}_K} \sum_{Q|P} d(Q|P).$$

2.3 Properties of Ramification Groups

Suppose that the order of $G$ is strictly divisible by $p$. Suppose that $F/K$ is wildly ramified at $Q$. The following material about the structure of the inertia group and the higher ramification groups can be found in \cite[IV]{12}. 

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Lemma 2.5. [12, IV, Cor. 4] If \( F/K \) is wildly ramified at \( Q \in \mathcal{P}_F \) with inertia group \( G_0 \) such that \( p^2 \nmid |G_0| \), then \( G_0 \) is a semidirect product of the form \( \mathbb{Z}/(p) \rtimes \mathbb{Z}/(m) \) for some prime-to-\( p \) integer \( m \).

The lower numbering on the filtration from Definition 2.1 is invariant under sub-extensions. There is a different indexing system on the filtration, whose virtue is that it is invariant under quotient extensions.

Definition 2.6. [12, IV, Section 3] The lower jump of \( F/K \) of \( Q \mid P \) is the largest integer \( h \) such that \( G_h \neq \{1\} \). Let \( \varphi(i) = |G_0|^{-1} \sum_{j=1}^i |G_j| \). Define \( G^{\varphi(i)} = G_i \). Then \( \varphi(h) = h/m \). The rational number \( \sigma = h/m \) is the upper jump; it is the jump in the filtration of the higher ramification groups in the upper numbering.

Let \( \tau \in G_0 \) have order \( p \) and \( \beta \in G_0 \) have order \( m \), so that \( G_0 \cong \langle \tau \rangle \rtimes \langle \beta \rangle \).

Lemma 2.7. [12, IV, Prop. 9] With notation as above:

1. If \( \beta \in G_0 \) has order \( m \) and \( h \) is the lower jump, then \( \beta \tau \beta^{-1} = \beta^h \tau \).

2. \( G_0 \) is contained in the normalizer \( N_G(\langle \tau \rangle) \).

2.4 Alternating groups

Suppose that \( G \) is an alternating group \( A_n \). Let \( p \leq n < 2p \) so that \( p^2 \nmid |G| \). The following lemmas give an upper bound for the size of the inertia group.

Lemma 2.8. Let \( \tau = (12\ldots p) \). Then \( N_{A_p}(\langle \tau \rangle) = \langle \tau \rangle \rtimes \langle \beta_o \rangle \) for some \( \beta_o \in A_p \) with \( |\beta_o| = (p-1)/2 \).

Proof. Let \( n_p \) be the number of Sylow \( p \)-subgroups of \( A_p \); then \( n_p = [A_p : N_{A_p}(\langle \tau \rangle)] \). There are \( (p-1)! \) different \( p \)-cycles in \( A_p \), each generating a group with \( p-1 \) non-trivial elements. It follows that \( n_p = (p-2)! \). Therefore, \( |N_{A_p}(\langle \tau \rangle)| = p(p-1)/2 \).

Clearly, \( \langle \tau \rangle \subseteq N_{A_p}(\langle \tau \rangle) \); we show the existence of \( \beta_o \). Let \( a \in \mathbb{F}_p^* \) with \( |a| = p-1 \). There exists \( \theta \in S_p \) such that \( \theta \tau \theta^{-1} = \tau^a \). The permutation \( \theta \) exists since all \( p \)-cycles in \( S_p \) are in the same conjugacy class. Let \( \beta_o = \theta^2 \). Then \( \beta_o \in A_p \) and \( \beta_o \in N_{A_p}(\langle \tau \rangle) \). Also, for any \( r \),

\[
\beta_o^r \tau \beta_o^{-r} = \theta^{2r} \tau \theta^{-2r} = \tau^{a^{2r}}.
\]
Choosing \( r = (p - 1)/2 \) shows that \( \beta_0^{(p-1)/2} \) is contained in the centralizer \( C_{A_p}(\langle \tau \rangle) = \langle \tau \rangle \), and it follows that \( \beta_0^{(p-1)/2} = 1 \). If \( 1 \leq r < (p - 1)/2 \), then \( \beta_0^r \not\in C_{A_p}(\langle \tau \rangle) \) and thus \( \beta_0^r \neq 1 \). It follows that \( \beta_0 \) normalizes \( \langle \tau \rangle \) in \( A_p \) and \( \beta_0 \) has order \( (p - 1)/2 \). \( \square \)

Recall that \( C_{S_n}(\langle \tau \rangle) = \langle \tau \rangle \times H \) where \( H = \{ \omega \in S_n : \omega \text{ is disjoint from } \tau \} \).

**Lemma 2.9.** Let \( 2 \leq s < p \) and let \( \tau = (12...p) \). Let \( H_s \subset S_{p+s} \) be the subgroup of permutations of the set \( \{p+1, p+2, \ldots, p+s\} \). Then there exists \( \theta \in S_p \) such that \( \vert \theta \vert = p - 1 \) and \( N_{A_{p+s}}(\langle \tau \rangle) \) is the intersection of \( A_{p+s} \) with \( (\langle \tau \rangle \times \langle \theta \rangle) \times H_s \).

**Proof.** The permutation \( \theta \) in the proof of Lemma 2.8 has order \( p - 1 \) and normalizes \( \tau \). The elements of \( H_s \) commute with \( \tau \) and \( \theta \). Thus \( (\langle \tau \rangle \times \langle \theta \rangle) \times H_s \subset N_{S_{p+s}}(\langle \tau \rangle) \). Performing a similar count as for Lemma 2.8 we find that the number of Sylow \( p \)-subgroups in \( S_{p+s} \) is \( (p + s)!/(s!p(p - 1)) \). Therefore \( \vert N_{S_{p+s}}(\langle \tau \rangle) \vert = s!p(p - 1) \). Thus \( (\langle \tau \rangle \times \langle \theta \rangle) \times H_s = N_{S_{p+s}}(\langle \tau \rangle) \). The result follows by taking the intersection with \( A_{p+s} \). \( \square \)

Note that the order of \( N_{A_p}(\langle \tau \rangle) \) forces \( \theta \) to be an odd permutation. Suppose \( G_0 = \langle \tau \rangle \times \langle \beta \rangle \) is a subgroup of \( A_{p+s} \). Then \( \beta = \theta^i \omega \) where \( \omega \in H_s \) and \( \omega \) is an even permutation if and only if \( i \) is even.

Recall that for an inertia group \( G_0 \) with \( p^2 \nmid \vert G_0 \vert \), there is a unique lower jump \( \ell \) which encodes information about the filtration of higher ramification groups. The following two lemmas relate the congruence class of \( h \) modulo \( m \) to the order of the centralizer \( C_{G_0}(\langle \tau \rangle) \).

**Lemma 2.10.** Let \( \pi : X \to \mathbb{P}^1_k \) be an \( A_p \)-Galois cover which is wildly ramified at a point \( Q \) above \( \infty \) with inertia group \( G_0 \). If \( \vert G_0 \vert = pm \) and \( \pi \) has lower jump \( \ell \) at \( Q \), then \( \gcd(h, m) = 1 \).

**Proof.** Let \( \beta \in A_p \) be such that \( G_0 = \langle \tau \rangle \times \langle \beta \rangle \). Notice that \( C_{G_0}(\langle \tau \rangle) = \langle \tau \rangle \) since there are no elements of \( A_p \) disjoint from \( \tau \). Then \( \beta^i \not\in C_{G_0}(\langle \tau \rangle) \) for all \( 1 \leq i < m \). By Lemma 2.7(1), if \( 1 \leq i < m \), then \( \tau \neq \beta^i \tau \beta^{-i} = \beta^i \tau \). Notice that \( \beta^ih \neq 1 \) which implies that \( m \nmid \ell \) for each \( 1 \leq i < m \). Hence \( \gcd(h, m) = 1 \). \( \square \)

**Lemma 2.11.** Let \( 2 \leq s < p \), and let \( \phi : Y \to \mathbb{P}^1_k \) be an \( A_{p+s} \)-Galois cover which is wildly ramified at a point \( Q \) above \( \infty \) with inertia group \( G_0 \). If \( \vert G_0 \vert = pm \) and \( \phi \) has lower jump \( \ell \) at \( Q \), then \( C_{G_0}(\langle \tau \rangle) \cong \mathbb{Z}/(p) \times \mathbb{Z}/(m') \) where \( m' = \gcd(h, m) \).
Proof. Let $\beta \in A_{p+s}$ be such that $G_0 = \langle \tau \rangle \times \langle \beta \rangle$. Let $m' = \gcd(h, m)$. Then Lemma 2.7(1) implies $\beta^{m/m'} \tau^{m/m'} = \beta^{m-h/m'} \tau = \tau$. The last equality is true because $|\beta| = m$ and $h/m' \in \mathbb{Z}$. It follows that $\beta^{m/m'} \in C_{G_0}(\langle \tau \rangle)$, that is $\langle \tau \rangle \times \langle \beta^{m/m'} \rangle \subset C_{G_0}(\langle \tau \rangle)$.

Suppose that $\alpha \in \langle \beta \rangle \cap C_{G_0}(\langle \tau \rangle)$. Lemma 2.7(1) implies $\tau = \alpha \tau \alpha^{-1} = \alpha^h \tau$. It follows that $|\alpha|$ divides $h$ and $m$, so $|\alpha|$ divides $m'$ and $\alpha \in \langle \beta^{m/m'} \rangle$. Hence $C_{G_0}(\langle \tau \rangle) = \langle \tau \rangle \times \langle \beta^{m/m'} \rangle$.

3 Newton Polygons

Suppose $f$ defines a degree $n$ extension $F$ of $k(x)$ that is ramified above the place $(x)$. Let $\tilde{F}$ be the splitting field of $f$ over $k(x)$. Let $Q$ be a ramified place in $\tilde{F}$ above $(x)$. Let $G_0$ be the inertia group of $\tilde{F}/k(x)$ at $Q$. Let $\epsilon$ be a local parameter of the valuation ring $\mathcal{O}_Q$. Let $v_Q$ denote the valuation at $Q$.

The Galois extension $\tilde{F}/k(x)$ yields a totally ramified Galois extension of complete local rings $k[[\epsilon]]/k[[x]]$. Let $f_2 \in k[[x]][y]$ be the minimal polynomial for $\epsilon$. Let $e = e(Q|0)$ be the degree of $f_2$. Define a polynomial $N(z) \in \hat{\mathcal{O}}_Q[z]$ such that

$$
\epsilon^{-e} N(z) := \epsilon^{-e} f_2(\epsilon(z + 1)) = \prod_{\omega \in G_0} \left(z - \left(\frac{\omega(\epsilon) - \epsilon}{\epsilon}\right)\right) .
$$

(3.1)

Define coefficients $b_i \in \hat{\mathcal{O}}_Q$ such that $N(z) = \sum_{i=1}^{e} b_i z^i$. The Newton polygon $\Delta$ of $N(z)$ is obtained by taking the lower convex hull of the set of points $\{(i, v_Q(b_i))\}_{i=1}^{e}$. Since $f_2$ is monic, the polygon is a sequence of line segments with increasing negative slopes.

The next proposition shows that the higher ramification groups of $\tilde{F}/k(x)$ at $Q$ are determined by the slopes of $\Delta$. This is not surprising because, as in [8, Chapter 2], the Newton polygon of $N(z)$ relates the valuations of the coefficients and roots of $N(z)$ and the higher ramification groups are determined by studying the valuation of the roots of $N(z)$.

Proposition 3.1. ([11, Thm. 1]) Let $\{V_0, V_1, \ldots, V_r\}$ be the vertices of $\Delta$ and $-h_j$ the slope of the edge joining $V_{j-1}$ and $V_j$. The slopes are integral and the lower jumps in the sequence of higher ramification groups are $h_r < h_{r-1} < \cdots < h_1$. 

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Lemma 3.2. For $1 < t < p - 2$, let $f_{1,t}(y) = y^p - xy^{p-t} + x \in k(x)[y]$. Let $F_t/k(x)$ be the corresponding extension of function fields and $\tilde{F}_t/k(x)$ its Galois closure. Let $Q$ be a place of $\tilde{F}_t$ lying over 0. Then $e(Q) = pm$ for some integer $m$ such that $p \nmid m$. Then the Newton polygon $\Delta_t$ of $\tilde{F}_t/k(x)$ has two line segments, one having integral slope $-m(p-t)/(p-1)$ and the other having slope 0.

Proof. Let $G$ be the Galois group of the extension $\tilde{F}_t/k(x)$. Notice that $G$ is contained in $S_p$; therefore the order of $G$ is strictly divisible by $p$. The extension is branched over $x = 0$. Let $P$ and $Q$ be places lying above 0 in $F_t$ and $\tilde{F}_t$ respectively. The format of the equation $f_{1,t}$ implies that $e(P) = p$; let $m$ be the integer such that $e(Q) = pm$. Then $p \nmid m$ since $p^2 \nmid |G|$. Let $G_0$ be the inertia group at $Q$. Let $x, \eta$, and $\epsilon$ be local parameters of $\tilde{O}_x, \tilde{O}_P$, and $\tilde{O}_Q$ respectively. The extension $\tilde{O}_Q/k[[x]]$ is totally ramified with Galois group $G_0$ of order $pm$.

| Field | Complete Local Ring | Local Parameter |
|-------|---------------------|-----------------|
| $\tilde{F}_t$ | $\tilde{O}_Q$ | $\epsilon$ |
| $F_t$ | $\tilde{O}_P$ | $\eta$ |
| $k(x)$ | $k[[x]]$ | $x$ |

Notice that any root of $f_{1,t}$ is a local parameter at $P$ since

$$p = v_P(x) = v_P\left(\frac{y^p}{y^{p-t}+1}\right) = pv_P(y).$$

Thus we can assume that $\eta$ is a root of $f_{1,t}$. Now consider $\eta$ as an element of $\tilde{O}_Q$. Then $\eta$ can be expressed as a power series in the local parameter $\epsilon$ with coefficients in $k$, that is $\eta = u \cdot \epsilon^m$ where $u$ is a unit of $\tilde{O}_Q$. Also $u$ is an $m$-th power in the complete local ring $\tilde{O}_Q$ so by changing the local parameter $\epsilon$ we can suppose $\eta = \epsilon^m$. It follows that $\epsilon$ satisfies the equation

$$f_{2,t}(\epsilon) = \epsilon^{pm} - x\epsilon^{m(p-t)} + x = 0. \quad (3.2)$$

The polynomial $f_{2,t}(\epsilon)$ is Eisenstein at the prime $(x)$. Now we consider

$$N(z) = f_{2,t}(\epsilon(z + 1)) = \epsilon^{pm}(z + 1)^{pm} - x\epsilon^{m(p-t)}(z + 1)^{m(p-t)} + x. \quad (3.3)$$
Dividing both sides of Equation 3.3 by \( \varepsilon \) produces a vertical shift by \(-\varepsilon \) to the Newton polygon \( \Delta_t \). Vertical and horizontal shifts do not affect the slopes of the line segments of \( \Delta_t \). Substituting \( x = \varepsilon^{\varepsilon m}/(\varepsilon^{m(p-t)} - 1) \) and letting \( d = 1/(\varepsilon^{m(p-t)} - 1) \), then

\[
\frac{N(z)\varepsilon}{\varepsilon_{pm}} = (z + 1)^{pm} - d\varepsilon^{m(p-t)}(z + 1)^{m(p-t)} + d.
\]

Notice that \( N(0) = 0 \) so we can factor a power of \( z \) from \( N(z) \). The effect on \( \Delta_t \) is a shift in the horizontal direction by \(-1\). This results in

\[
\frac{N(z)}{z\varepsilon^{pm}} = \sum_{i=0}^{m-1} \left( m \atop i \right) z^{p(m-i)-1} - d\varepsilon^{m(p-t)} \sum_{i=0}^{m(p-t)-1} \left( m(p-t) \atop i \right) z^{m(p-t)-i-1}.
\]

Let \( z^{-1}\varepsilon^{-pm}N(z) = \sum_{j=0}^{pm-1} b_j z^j \). The valuation of each \( b_j \) is greater than or equal to zero. The ramification polygon \( \Delta_t \) is determined by calculating the valuations of the specific coefficients that determine the lower convex hull of \( \Delta_t \):

1. \( \nu_Q(b_0) = \nu_Q(dmte^{m(p-t)}) = m(p-t) \).
2. For \( 1 \leq j < p - 1 \), let \( i_j = m(p-t) - j - 1 \), then
   \[ \nu_Q(b_j) = \nu_Q\left( -d\varepsilon^{m(p-t)} \left( m(p-t) \atop i_j \right) \right) \geq m(p-t). \]
3. \( \nu_Q(b_{p-1}) = \nu_Q\left( m - d\varepsilon^{m(p-t)} \left( m(p-t) \atop m(p-t) - p \right) \right) = 0. \)
4. \( \nu_Q(b_{pm-1}) = \nu_Q(1) = 0. \)

The vertices of \( \Delta_t \) are thus \((0, m(p-t))\), \((p - 1, 0)\), and \((pm - 1, 0)\).

**Lemma 3.3.** For \( 2 \leq s < p \), let \( g_s(y) = y^{p+s} - xy^s + 1 \in k(x)[y] \). Let \( L_s/k(x) \) be the corresponding extension of function fields and \( \tilde{L}_s/k(x) \) its Galois closure. Let \( Q \) be a place of \( \tilde{L}_s \) lying over \( \infty \). Then \( e(Q|\infty) = pm \) for some integer \( m \) such that \( p \nmid m \) and the Newton polygon \( \Delta_s \) of \( \tilde{L}_s/k(x) \) has two line segments, one having integral slope \(-m(p + s)/(p - 1)\) and the other having slope 0.
Proof. Let $G$ be the Galois group of the extension $\tilde{L}_s/k(x)$. Notice that $G$ is contained in $S_{p+s}$, therefore the order of $G$ is strictly divisible by $p$. The extension is branched over $\infty$. Let $P_{(\infty,0)}$ and $P_{(\infty,\infty)}$ be the two places of $L_s$ lying above $\infty$. The format of the equation $g_s$ implies that $P_{(\infty,0)}$ and $P_{(\infty,\infty)}$ have ramification indices $p$ and $s$ respectively, see e.g., Lemma 4.8. Let $Q$ be a place of $\tilde{L}_s$ lying above $P_{(\infty,0)}$. Let $m$ be the integer $e(Q|\infty)/p$. Let $G_0$ be the inertia group at $Q$. Let $x^{-1}, \eta,$ and $\epsilon$ be local parameters of $\mathcal{O}_{x^{-1}}, \mathcal{O}_{P_{(\infty,0)}}$, and $\mathcal{O}_Q$ respectively.

| Field | Complete Local Ring | Local Parameter |
|-------|---------------------|-----------------|
| $\tilde{L}_s$ | $Q_m$ | $\hat{\mathcal{O}}_Q$ | $m$ |
| $L_s$ | $P_{(\infty,0)}_p$ | $\hat{\mathcal{O}}_{P_{(\infty,0)}}$ | $m$ |
| $k(x)$ | $\infty$ | $k[[x^{-1}]]$ | $x^{-1}$ |

Then $\hat{\mathcal{O}}_Q/k[[x^{-1}]]$ is a totally ramified Galois extension with Galois group $G_0$ of order $pm$. By the same reasoning as for Lemma 3.2, there exists a local parameter $\epsilon$ of $\hat{\mathcal{O}}_Q$ that satisfies $\epsilon^m = \eta$. Therefore $\epsilon$ satisfies the irreducible equation

$$g_{2,s}(\epsilon) = \epsilon^{m(p+s)} - x\epsilon^{ms} + 1 = 0. \quad (3.4)$$

We calculate the ramification polygon $\Delta'_s$ by considering

$$N(z) = g_{2,s}(\epsilon(z + 1)) = \epsilon^{m(p+s)}(z + 1)^{m(p+s)} - x\epsilon^{ms}(z + 1)^{ms} + 1. \quad (3.5)$$

Since $N(0) = 0$, it follows that

$$\frac{N(z)}{z^{m(p+s)}} = (z + 1)^{ms} \sum_{i=0}^{m-1} \binom{m}{i} z^{p(m-i)-1} - (1 + \epsilon^{-m(p-s)}) \sum_{i=0}^{ms-1} \binom{ms}{i} z^{ms-1-i}.$$ 

Let $z^{-1}\epsilon^{-m(p+s)}N(z) = \sum_{j=1}^{m(p+s)-1} b_j z^j$. The valuation of each $b_j$ is non-negative. The ramification polygon $\Delta'_s$ is determined when we calculate the valuations of the specific coefficients that determine the lower convex hull of $\Delta'_s$.

1. $\nu_Q(b_0) = m(p+s)$.
2. $\nu_Q(b_j) \geq m(p+s)$ for $1 \leq j < p - 1$. 

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3. $v_Q(b_{p-1}) = 0$.

4. $v_Q(b_{m(p+s)-1}) = 0$.

The vertices of $\Delta'_s$ are thus $(0, m(p-s))$, $(p-1, 0)$, and $(m(p+s)-1, 0)$.

4  $A_n$-Galois covers of the affine line

Suppose $\pi : X \to \mathbb{P}^1_k$ is an $A_n$-Galois cover branched only at $\infty$. The cover is wildly ramified at each point $Q \in X$ above $\infty$. The complexity of the wild ramification is directly related to the power of $p$ that divides the ramification index $e(Q|\infty)$. For this reason, we concentrate on Galois groups $A_n$ such that the order of $A_n$ is strictly divisible by $p$. We use some equations of Abhyankar to study $A_n$-Galois covers when $p$ is odd and $p \leq n < 2p$. The goal is to determine the inertia groups and upper jumps that occur for $A_n$-Galois covers $\pi : X \to \mathbb{P}^1_k$ branched only at $\infty$. This ramification data also determines the genus of the curve $X$.

4.1 Two useful lemmas

The following is a version of Abhyankar’s Lemma which will be needed to construct a $G$-Galois cover of $\mathbb{P}^1_k$ branched only at $\infty$ from a $G$-Galois cover of $\mathbb{P}^1_k$ branched at 0 and $\infty$.

**Lemma 4.1** (Refined Abhyankar’s Lemma). Let $m$, $r_1$, and $r_2$ be prime-to-$p$ integers. Suppose $\pi : X \to \mathbb{P}^1_k$ is a $G$-Galois cover with branch locus $\{0, \infty\}$. Suppose $\pi$ has ramification index $r_1$ above 0 and inertia group $G_0 \cong \mathbb{Z}/(p) \times \mathbb{Z}/(m)$ above $\infty$ with lower jump $h$. Let $\psi : \mathbb{P}^1_k \to \mathbb{P}^1_k$ be an $r_2$-cyclic cover with branch locus $\{0, \infty\}$. Assume that $\pi$ and $\psi$ are linearly disjoint.

Then the pullback $\pi' = \psi^* \pi$ is a $G$-Galois cover $\pi' : X' \to \mathbb{P}^1_k$ with branch locus contained in $\{0, \infty\}$, with ramification index $r_1/gcd(r_1, r_2)$ above 0, with inertia group $G'_0 \subset G_0$ of order $pm/gcd(m, r_2)$ above $\infty$ and with lower jump $hr_2/gcd(m, r_2)$. If $\sigma$ and $\sigma'$ are the upper jumps of $\pi$ and $\pi'$ respectively, then $\sigma' = r_2 \sigma$.

**Proof.** Consider the fibre product:

\[
\begin{array}{ccc}
X & \leftarrow & X' \\
\pi & \downarrow & \pi' \\
\mathbb{P}^1_k & \leftarrow & \mathbb{P}^1_k \\
\psi & & \\
\end{array}
\]
All the claims follow from the classical version of Abhyankar’s Lemma [7, 
Lemma X.3.6] except for the information about the lower and upper jumps of 
$\pi'$. Consider the composition $\psi \pi'$ which has ramification index $pmr_2/gcd(m, r_2)$ 
above $\infty$. Since upper jumps are invariant under quotients, the upper jump 
of $\psi \pi'$ equals $\sigma$. Thus the lower jump of $\psi \pi'$ equals $\sigma mr_2/gcd(m, r_2) = 
h r_2/gcd(m, r_2)$ by Definition 2.6. This equals the lower jump of $\pi'$ since 
lower jumps are invariant for subcovers and the claim about the upper jump 
of $\pi'$ follows from Definition 2.6.

The following lemma is useful to compare ramification information about 
a cover and its Galois closure. Let $S_n^1 := Stab_{S_n}(1)$.

Lemma 4.2. If $\rho : Z \to W$ is a cover with Galois closure $\pi : X \to W$, then 
the branch locus of $\rho$ and of $\pi$ are the same.

Proof. The branch locus of $\rho$ is contained in the branch locus of $\pi$ since 
ramification indices are multiplicative. Assume that $b$ is in the branch locus 
of $\pi$ but not in the branch locus of $\rho$. We will show that this is impossible. 
The Galois group $H$ of $\pi$ is a transitive subgroup of $S_n$, where $n$ is the degree 
of $\rho$. The Galois group $H'$ of $X \to Z$ is a subgroup of $H$ with index $n$. After 
identifying $H$ with a subgroup of $S_n$, we can assume without loss of generality 
that $H' \subset S_n^1$. Let $Q \in X$ be a ramification point lying above $b$ with inertia 
group $G_0$. Conjugating $G_0$ by an element $\omega \in H$ results in an inertia group 
at some point of $X$ above $b$. Since $b$ is not a branch point of $\rho$, we have that 
$\omega G_0 \omega^{-1} \subset S_n^1$ for all $\omega \in H$. This is impossible since $H$ is transitive on the 
set $\{1, 2, \ldots, n\}$. Therefore the branch loci must be the same. \hfill \square

4.2 $A_p$-Galois covers of the affine line

Let $p \geq 5$. In this section, we find $A_p$-Galois covers $\pi : X \to \mathbb{P}^1_k$ branched 
only at $\infty$ with a small upper jump.

Notation 4.3. Let $t$ be an integer with $1 < t < p-2$ and let $f_t = y^p - y^t + x$. 
Consider the curve $Z_t$ with function field $F_t := k(x)[y]/(f_t)$. Let $\pi_t : X_t \to \mathbb{P}^1_k$ 
be the Galois closure of $\rho_t : Z_t \to \mathbb{P}^1_k$; the function field of $X_t$ is the Galois 
closure $\tilde{F}_t$ of $F_t/k(x)$. Let $\zeta$ be a $(p-t)$th root of unity.

Abhyankar proved that the Galois group of $\pi_t$ is $A_p$ when $t$ is odd and 
$S_p$ when $t$ is even [2, Section 20]. For the proof, he showed that the Galois 
group is doubly transitive on the set $\{1, 2, \ldots, p\}$ and contains a certain cycle
type. We now study the ramification of the cover \( \pi_t \). The following result can be found in [2, Section 20].

**Lemma 4.4.** The cover \( \pi_t : X_t \to \mathbb{P}_k^1 \) has one ramified point above \( x = 0 \) with ramification index \( t \) and is unramified above all other points of \( \mathbb{A}_k^1 \).

**Proof.** When \( x = 0 \) in the equation \( f_t \), then \( y = 0 \) or \( y = \zeta^i \) for some \( 1 \leq i \leq p - t \). There are \( p - t + 1 \) points in the fibre of \( \rho \) above the point \( x = 0 \) which we denote by \( P_{(0,0)} \) and \( P_{(0,\zeta)}, \ldots, P_{(0,1)} \). Since \( p - t + 1 < p \), then \( x = 0 \) is a branch point of \( \rho \).

The value \( y = 0 \) is the only solution to \( \partial f/\partial y = 0 \). Therefore \( P_{(0,0)} \) is the only ramification point above \( \mathbb{A}_k^1 \). The Galois group \( H \) of \( \pi_t \) is either \( S_p \) or \( A_p \). Thus Lemma 4.2 implies that \( \pi_t \) is unramified above all points of \( \mathbb{A}_k^1 \) except \( x = 0 \).

Because \( p = \sum_{P \mid 0} e(P \mid 0) \), it follows that \( P_{(0,0)} \) has ramification index \( t \).

Let \( Q \in X_t \) be a point lying above \( P_{(0,0)} \). It remains to show that \( e(Q \mid P_{(0,0)}) = 1 \).

Let \( H' \) be the Galois group of \( X_t \to Z_t \). Without loss of generality we can suppose that \( H' \subset S_p^1 \). Since \( p \nmid |H'| \), Lemma 2.5 implies that \( G_0(Q \mid 0) \) is a cyclic group of order \( t \cdot c \) for some prime-to-\( p \) integer \( c \).

Assume \( c \neq 1 \). If \( \omega \) is a generator for \( G_0(Q \mid 0) \); then \( \omega \notin S_p^1 \) since \( P_{(0,0)} \) is ramified over 0. Then \( G_0(Q \mid P_{(0,0)}) = \langle \omega^i \rangle \subset S_p^1 \). By the assumption on \( c \), the automorphism \( \omega^i \) is not the identity. Since \( H \) is transitive on \( \{1, 2, \ldots, p\} \), there exists \( \gamma \in H \) such that \( \gamma \phi^i \gamma^{-1} \notin S_p^1 \).

There exists a point \( \tilde{Q} \) in the fiber of \( X_t \) above 0 such that \( G_0(\tilde{Q} \mid 0) = \langle \gamma^{-1} \phi \gamma \rangle \). Since \( \gamma \notin S_p^1 \), the point \( \tilde{Q} \) is in the fibre of \( \pi_t \) over 0 but not in the fibre above \( P_{(0,0)} \). Furthermore, \( \gamma \phi^i \gamma^{-1} = (\gamma \phi \gamma^{-1})^i \in G_0(\tilde{Q} \mid 0) \). Hence \( G_0(\tilde{Q} \mid 0) \notin S_p^1 \). Therefore, for some \( i \), the extension \( P_{(0,\zeta)} \mid 0 \) is ramified. This gives a contradiction so the assumption that \( c \neq 1 \) is false. \( \square \)

Lemma 2.5 implies that the inertia group \( G_0 \) at a point of \( X_t \) over \( \infty \) is of the form \( \mathbb{Z}/(p) \rtimes \mathbb{Z}/(m) \) where \( p \nmid m \). To determine the upper jump \( \sigma \) of \( \pi_t \) over \( \infty \), we use the equation \( f_t \) to understand the ramification that occurs in the quotient map \( \rho_t : Z_t \to \mathbb{P}_k^1 \).

**Lemma 4.5.** The cover \( \pi_t : X_t \to \mathbb{P}_k^1 \) has ramification index \( p(p - 1)/\gcd(p - 1, t - 1) \) and upper jump \( \sigma = (p - t)/(p - 1) \) above \( \infty \).
Proof. Let \( P_\infty \) be a point of \( Z_t \) that lies above \( \infty \). Then

\[-e(P_\infty|\infty) = v_{P_\infty}(x) = v_{P_\infty}(y^p - y^t) = pv_{P_\infty}(y).\]

Therefore \( p|e(P_\infty|\infty) \) and it follows that \( \rho \) is totally ramified at \( P_\infty \).

Consider the change of variables \( x \mapsto 1/x \) and \( y \mapsto 1/y \). Understanding the ramification over \( \infty \) is equivalent to understanding the ramification over \( x = 0 \) of the cover with equation \( f_{1,t} = y^p - xy^{p-t} + x \). By Lemma 3.2 the absolute value of the non-zero slope of the ramification polygon \( \Delta_t \) of \( \tilde{F}_t/k(x) \) is \( m(p-t)/(p-1) \); this is the lower jump by Lemma 3.1. By Definition 2.6, the upper jump is \( \sigma = (p-t)/(p-1) \).

By Lemma 2.10, \( h \) and \( m \) are co-prime, therefore \( h = (p-t)/\gcd(p-1, t-1) \) and \( m = (p-1)/\gcd(p-1, t-1) \) and \( |G_0| = pm \). \( \square \)

**Theorem 4.6.** For \( 1 < t < (p-2) \), let \( m_t = (p-1)/\gcd(p-1, t(t-1)) \). Then there exists an \( A_p \)-Galois cover \( \pi'_t : X'_t \to \mathbb{P}^1_k \) branched only at \( \infty \) with ramification index \( pm_t \) and upper jump \( \sigma'_t = t(p-t)/(p-1) \). The genus of \( X'_t \) is \( 1 + |A_p|(t(p-t) - p - 1/m_t)/2p \).

**Proof.** Let \( d_1 = \gcd(p-1, t-1) \) and let \( m = (p-1)/d_1 \). Consider the Galois cover \( \pi : X_t \to \mathbb{P}^1_k \) from Notation 4.3. Lemma 4.4 states that \( \pi_t \) has ramification index \( t \) above \( 0 \) and is unramified above \( A^1_k - \{0\} \). Lemma 4.3 states that the inertia group \( G_0 \) above \( \infty \) has order \( pm \) and upper jump \( \sigma = (p-t)/(p-1) \).

If \( t \) is odd, then \( \pi_t \) has Galois group \( A_p \). Let \( m^* = \gcd(m, t) \). Since \( A_p \) is simple, the cover \( \pi_t \) is linearly disjoint from the \( t \)-cyclic cover \( \psi : \mathbb{P}^1_k \to \mathbb{P}^1_k \) with equation \( z^t = x \). Applying Lemma 4.4 the pullback \( \pi'_t = \psi^*\pi_t \) is a cover \( \pi'_t : X'_t \to \mathbb{P}^1_k \) with Galois group \( A_p \). The map \( \pi'_t \) is branched only at \( \infty \) with inertia group \( G'_0 \) of order \( pm/m^* \) and upper jump \( \sigma'_t = t(p-t)/(p-1) \). Notice that \( d_1 m^* = \gcd(p-1, t(t-1)) \), so the inertia group has order \( pm/m^* = p(p-1)/\gcd(p-1, t(t-1)) \).

If \( t \) is even, then \( \pi_t \) has Galois group \( S_p \). Let \( Y_t \) be the smooth projective curve corresponding to the fixed field \( \tilde{F}^{A_p}_t \). Let \( \mu_t : X_t \to Y_t \) be the subcover with Galois group \( A_p \).

The branch locus of the degree 2 quotient cover \( Y_t \to \mathbb{P}^1_k \) is contained in \{0, \infty\}. The ramification index must be 2 over both 0 and \( \infty \). By the Riemann-Hurwitz formula, \( Y_t \) has genus 0. Therefore \( \mu_t : X_t \to \mathbb{P}^1_k \) is an \( A_p \)-Galois cover of the projective line.
Let $P_0 \text{ (resp. } P_\infty \text{)}$ be the point of $Y_t$ above 0 (resp. $\infty$). Since ramification indices are multiplicative, $\mu_t$ has ramification index $t/2$ over $P_0$ and $|G_0|/2$ over $P_\infty$. It can be seen that $|G_0|/2 = pm/2$ is an integer from Lemma 4.6 since $t$ is even. The lower jump of $\mu_t$ is the same as the lower jump of $\pi_t$ since lower jumps are invariant under subextensions. Therefore the upper jump of $\mu_t$ is $2\sigma$.

The cover $\mu_t$ is linearly disjoint from the $t/2$-cyclic cover $\psi : P^1_k \to P^1_k$ with equation $z^{t/2} = x$. Let $m = \gcd(m/2, t/2)$. Applying Lemma 4.1, the pullback $\pi'_t = \psi^* \mu_t$ is an $A_p$-Galois cover $\pi'_t : X'_t \to P^1_k$. The map $\pi'_t$ is branched only at $\infty$ where it has inertia group $G'_0$ of order $pm/(2m)$ and upper jump $\sigma'_t = t(p - t)/(p - 1)$. Notice that $pm/2m = pm/m^*$, so $|G'_0| = p(p - 1)/\gcd(p - 1, t(t - 1))$.

The genus calculation is immediate from the Riemann-Hurwitz formula, Theorem 2.4.

The smallest genus for an $A_p$-Galois cover obtained using the method of Theorem 4.6 is

$$g = 1 + |A_p|(p^2 - 5p + 2)/2p(p - 1).$$

This occurs when $t = 2$ or $t = p - 2$ and the upper jump is $\sigma = 2(p - 2)/(p - 1)$. To see this, consider the derivative $d\sigma/dt = (p - 2t)(p - 1)$. Since this value of $\sigma$ is less than 2, it is possible that this is the smallest genus that occurs among all $A_p$-Galois covers of the affine line. We find $A_p$-Galois covers with slightly larger upper jumps in Section 5.4.

### 4.3 $A_{p+s}$-Galois covers of the affine line

In this section, we find $A_n$-Galois covers of the projective line branched only at $\infty$ with small upper jump when $p$ is odd and $p < n < 2p$.

**Notation 4.7.** Let $s$ be an integer with $2 \leq s < p$. Consider the group $A_{p+s}$ of even permutations on $p+s$ elements and the subgroup $H_s \subset S_{p+s}$ of permutations on $\{p+1, p+2, \ldots, p+s\}$. Let $g_s = y^{p+s} - xy^s + 1$. Consider the curve $Z'_s$ with function field $L_s := k(x)[y]/(g_s)$. Let $\phi_s : Y_s \to P^1_k$ be the Galois closure of $\rho'_s : Z'_s \to P^1_k$; the function field of $Y_s$ is the Galois closure $\tilde{L}_s$ of $L_s/k(x)$.

Abhyankar proved that the Galois group of $\phi_s$ is $A_{p+s}$ except when $p = 7$ and $s = 2$ [2 Section 11]. The following result can be found in [2 Section 21].
Lemma 4.8. The cover $\rho_s$ is branched only at $\infty$. The fibre over $\infty$ consists of two points $P_{(\infty,0)}$ and $P_{(\infty,\infty)}$ which have ramification indices $p$ and $s$ respectively.

Proof. There are no simultaneous solutions to the equations $g_s = 0$ and $\partial g_s/\partial y = 0$. Therefore the cover $\rho_s'$ is not branched over any points of $A^1_k$. Since the tame fundamental group of $A^1_k$ is trivial, $\rho_s'$ must be wildly ramified above $\infty$. The fibre of $Z_1'$ over $\infty$ consists of two points $P_{(\infty,0)}$ and $P_{(\infty,\infty)}$. The first point can be seen by applying the change of variables $x \mapsto 1/x$ to $g_s$. This produces the equation $x y^{p+s} - y^s + x$. Taking the partial derivative with respect to $y$ yields the point $P_{(\infty,0)}$. The second point can be seen by applying the change of variables $y \mapsto 1/y$ to $x y^{p+s} - y^s + x$ resulting in the equation $x - y^p + x y^{p+s}$. Taking the partial derivative with respect to $y$ yields the point $P_{(\infty,\infty)}$. To show that $e(P_{(\infty,0)}|\infty) = p$ and $e(P_{(\infty,\infty)}|\infty) = s$, let $P$ be either $P_{(\infty,0)}$ or $P_{(\infty,\infty)}$ and consider the valuation $v_P$. The result follows since

$$-e(P|\infty) = v_P(x) = v(y^p + y^{-s}) = \min\{pv_P(y), -sv_P(y)\}.$$

\[\square\]

Theorem 4.9. Let $2 \leq s < p$. If $p = 7$, assume $s \neq 2$. Let $m_s = (p-1)s/gcd(p-1, s(s+1))$. Then there exists an $A_{p+s}$-Galois cover $\phi_s : Y_s \to \mathbb{P}^1_k$ branched only at $\infty$ with inertia group $G_0$ of order $pm_s$ and upper jump $\sigma_s = (p+s)/(p-1)$. The genus of $Y_s$ is $1 + |A_{p+s}|(s-1/m_s)/2p$.

In [41 Cor. 2.2], the author proves that the genus of $Y_s$ in Theorem 4.9 is the smallest genus that occurs among all $A_{p+s}$-Galois covers of the affine line.

Proof. Consider the cover $\phi_s : Y_s \to \mathbb{P}^1_k$ defined in Notation 4.7. Abhyankar proved that $\phi_s$ has Galois group $A_{p+s}$ [2 Section 11]. By Lemmas 4.2 and 4.8, $\infty$ is the only branch point of $\phi_s$. Let $Q$ be a point of $Y_s$ lying above $\infty$. The cover $\phi_s$ is wildly ramified at $Q$ with $p^3 \nmid e(Q|\infty)$. By Lemma 2.5, the inertia group $G_0$ at $Q$ is of the form $\mathbb{Z}/(p) \times \mathbb{Z}/(m)$ for some prime-to-$p$ integer $m$.

Let $h$ be the lower jump of $\phi_s$ at $Q$. The Newton polygon of $\phi_s$ is the same as the Newton polygon $\Delta'_s$ calculated in Lemma 3.3. Therefore, $h = m(p+s)/(p-1)$, because this is the negative of the slope of the line segment of $\Delta'_s$. By Definition 2.6, the upper jump is $\sigma_s = (p+s)/(p-1)$.
Write $m = m'm''$ where $m'$ is the order of the prime-to-$p$ center of $G_0$. Lemma 2.7(1) implies that $m' = \gcd(h, m)$. Since $h/m = \sigma_s = (p+s)/(p-1)$, it follows that $m'' = (p-1)/\gcd(p-1, s+1)$.

Without loss of generality, we can suppose that $\tau = (12\ldots p) \in G_0$. By Lemma 2.9, $G_0 = \langle \tau \rangle \rtimes \langle \beta \rangle$ for some $\beta$ of the form $\beta = \theta^i \omega$. Recall that $\theta \in S_p$ acts faithfully by conjugation on $\tau$ and $\omega \in H_s$ commutes with $\tau$. The inertia group $G_0$ acts transitively on $\{p+1, p+2, \ldots, p+s\}$ by Lemma 4.8. Thus $\omega$ is a cycle of length $s$.

The order of $\beta$ is $m$, the order of $\theta^i$ is $m''$, and the order of $\beta$ is $s$. Thus $m = \text{lcm}(m'', s)$. It follows that $m' = s/\gcd(p-1, s)$ and $m = (p-1)s/\gcd(p-1, s(s+1))$. The genus calculation is immediate from the Riemann-Hurwitz formula, Theorem 2.4.

## 5 Applications

### 5.1 Support for the Inertia Conjecture

In this section, we first give a new proof of Abhyankar’s Inertia Conjecture for the group $A_p$; this proof does not use the theory of semi-stable reduction. Then we prove Abhyankar’s Inertia Conjecture for the group $A_p+2$ for an odd prime $p \equiv 2 \mod 3$.

**Corollary 5.1.** [5, Cor. 3.1.5] Let $p \geq 5$. Abhyankar’s Inertia Conjecture is true for the alternating group $A_p$. In other words, every subgroup $G_0 \subset A_p$ of the form $\mathbb{Z}/(p) \rtimes \mathbb{Z}/(m)$ can be realized as the inertia group of an $A_p$-Galois cover of $\mathbb{P}^1_k$ branched only at $\infty$.

**Proof.** Suppose $G_0 \subset A_p$ satisfies the conditions of Conjecture 1.1. Since $p^2 \nmid |A_p|$, then $G_0 \simeq \mathbb{Z}/(p) \rtimes \mathbb{Z}/(m)$ for some prime-to-$p$ integer $m$. Thus the second claim implies the first.

Consider a subgroup $G_0 \subset A_p$ of the form $\mathbb{Z}/(p) \rtimes \mathbb{Z}/(m)$. The goal is to show that $G_0$ is the inertia group of an $A_p$-Galois cover of the affine line. Without loss of generality, we can suppose that $\tau = (12\ldots p) \in G_0$. By Lemma 2.7(2), $G_0 \subset N_{A_p}(\langle \tau \rangle)$. Lemma 2.8 implies that $N_{A_p} \simeq \mathbb{Z}/(p) \rtimes \mathbb{Z}/((p-1)/2)$.

It thus suffices to prove, for every $m \mid (p-1)/2$, that there exists an $A_p$-Galois cover of the affine line, with an inertia group of order $pm$. Letting $t = 2$, Theorem 4.10 shows the existence of such a cover $\pi_2$ with an inertia
group of order $p(p - 1)/2$. Since $A_p$ is simple, $\pi_2$ is linearly disjoint from the degree $r_2$ cyclic cover of $\mathbb{P}^1_k$ which is branched at 0 and $\infty$. The proof then follows by Lemma 4.1, taking $r_2 = (p - 1)/2m$. \qed

**Corollary 5.2.** If $p \equiv 2 \mod 3$ is an odd prime, then Abhyankar’s Inertia Conjecture is true for $G = A_{p+2}$. In other words, every subgroup $G_0 \subset A_{p+2}$ of the form $\mathbb{Z}/(p) \rtimes \mathbb{Z}/(m)$ can be realized as the inertia group of an $A_{p+2}$-Galois cover of $\mathbb{P}^1_k$ branched only at $\infty$.

**Proof.** Suppose $G_0 \subset A_{p+2}$ satisfies the conditions of Conjecture 1.1. Since $p^2 \nmid |A_{p+2}|$, then $G_0 \simeq \mathbb{Z}/(p) \rtimes \mathbb{Z}/(m)$ for some prime-to-$p$ integer $m$. Thus the second claim implies the first.

Consider a subgroup $G_0 \subset A_{p+2}$ of the form $\mathbb{Z}/(p) \rtimes \mathbb{Z}/(m)$. The goal is to show that $G_0$ is the inertia group of an $A_{p+2}$-Galois cover of the affine line. Without loss of generality, we can suppose that $\tau = (12\ldots p) \in G_0$. By Lemma 2.7(2), $G_0 \subset N_{A_{p+2}}(\langle \tau \rangle)$. By Lemma 2.9, $N_{A_{p+2}}(\langle \tau \rangle) = \langle \tau \rangle \rtimes \langle \beta \rangle$ where $\beta = \theta(p + 1 p + 2)$. Recall that $\theta$ is an odd permutation of order $p - 1$ defined in the proof of Lemma 2.8.

It thus suffices to prove, for every $m \mid (p - 1)$, that there exists an $A_{p+2}$-Galois cover of the affine line, with an inertia group of order $pm$. Letting $s = 2$, Theorem 4.9 shows the existence of such a cover $\phi_2$ with an upper jump $\sigma_2 = (p+2)/(p-1)$. Since $p \equiv 2 \mod 3$, the upper jump $\sigma_2$ is written in lowest terms and thus $m = p - 1$. Since $A_{p+2}$ is simple, $\phi_2$ is linearly disjoint from a degree $r_2$ cyclic cover of $\mathbb{P}^1_k$ which is branched at 0 and $\infty$. The proof then follows by Lemma 4.1, taking $r_2 = (p - 1)/m$. \qed

When $s > 2$, more equations are needed to prove Abhyankar’s Conjecture for $A_{p+s}$ because the normalizer $N_{A_{p+s}}(\langle \tau \rangle)$ contains more that one maximal subgroup of the form $\mathbb{Z}/(p) \rtimes \mathbb{Z}/(m)$.

### 5.2 Formal Patching Results

Suppose $\pi : X \to \mathbb{P}^1_k$ is a $G$-Galois cover which is wildly ramified above $\infty$ with last upper jump $\sigma$. Using the theory of formal patching, it is possible to produce a different $G$-Galois cover with the same branch locus, but with a larger upper jump above $\infty$. The formal patching proof is non-constructive and we do not describe it in this paper. Here are the results that we will use: the first allows us to change the congruence value of the lower jump modulo $m$ and the second allows us to increase the lower jump by a multiple of $m$. 

"..."
Lemma 5.3. [5, Prop. 3.1.1] Suppose $\pi : X \to \mathbb{P}_k^1$ is a $G$-Galois cover branched only at $\infty$ with inertia group $G_0 \cong \mathbb{Z}/(p) \times \mathbb{Z}/(m)$ with $p \nmid m$ and with lower jump $h$. For each $d \in \mathbb{N}$ such that $1 \leq d \leq m$, let $m_d = m/gcd(m,d)$ and $h_d = dh/gcd(m,d)$. Let $G_0^d \subset G_0$ be the subgroup of order $pm_d$. Then there exists a $G$-Galois cover $\pi' : X' \to \mathbb{P}_k^1$ branched only at $\infty$ with inertia group $G_0^d$ and lower jump $h_d$. If $\sigma$ and $\sigma'$ are the upper jumps of $\pi$ and $\pi'$ respectively, then $\sigma' = d\sigma$.

Theorem 5.4. [9, Special case of Theorem 2.3.1] Let $\pi : X \to \mathbb{P}_k^1$ be a $G$-Galois cover branched only at $\infty$ with inertia group $\mathbb{Z}/(p) \times \mathbb{Z}/(m)$ and upper jump $\sigma = h/m$. Then for $i \in \mathbb{N}$ with $gcd(h + im, p) = 1$, there exists a $G$-Galois cover branched only at $\infty$ with the same inertia group and upper jump $\sigma' = \sigma + i$.

5.3 Realizing almost all upper jumps for $A_{p+s}$-Galois covers

Here are the necessary conditions on the upper jump of an $A_{p+s}$-Galois cover of the affine line.

Notation 5.5. Let $2 \leq s < p$. Suppose $\phi : Y \to \mathbb{P}_k^1$ is an $A_{p+s}$-Galois cover branched only at $\infty$ where it has upper jump $\sigma = h'/m''$ written in lowest terms. Then $\sigma$ satisfies these necessary conditions: $\sigma > 1$; $p \nmid h'$; and $m'' | (p-1)$.

Corollary 5.6. Suppose $2 \leq s < p$ and $gcd(p - 1, s + 1) = 1$. Then all but finitely many rational numbers $\sigma$ satisfying the necessary conditions of Notation 5.5 occur as the upper jump of an $A_{p+s}$-Galois cover of $\mathbb{P}_k^1$ branched only at $\infty$.

Proof. Theorem 4.9 implies that there exists an $A_{p+s}$-Galois cover of $\mathbb{P}_k^1$ branched only at $\infty$ with upper jump $\sigma_s = (p + s)/(p - 1)$. The condition on $s$ implies that $\sigma$ is written in lowest terms and thus $m'' = p - 1$. The corollary then follows from Lemma 5.3 and Theorem 5.4.

5.4 Realizing lower jumps for $A_n$-Galois covers with inertia $\mathbb{Z}/(p)$

Question 5.7. Suppose $G$ is a quasi-$p$ group whose order is strictly divisible by $p$. For which prime-to-$p$ integers $h$ does there exist a $G$-Galois cover
\( \pi : X \to \mathbb{P}^1_k \) branched only at \( \infty \) with inertia group \( \mathbb{Z}/(p) \) and lower jump \( h \)?

By Theorem [5.4], all sufficiently large prime-to-\( p \) integers \( h \) occur as the lower jump of a \( G \)-Galois cover of the affine line with inertia \( \mathbb{Z}/(p) \). The question is thus how large \( h \) needs to be to guarantee that it occurs as the lower jump of such a cover. In [5, Thm. 3.1.4], the authors prove that every prime-to-\( p \) integer \( h \geq p - 2 \) occurs as the lower jump of an \( A_p \)-Galois cover of the affine line with inertia group \( \mathbb{Z}/(p) \). The next corollary improves on that result.

**Corollary 5.8.** Let \( p \geq 5 \). Let \( h_0 = (p + 1)/\gcd(p + 1, 4) \). There exists an \( A_p \)-Galois cover of \( \mathbb{P}^1_k \) branched only at \( \infty \) with inertia group \( \mathbb{Z}/(p) \) and lower jump \( h \) for every prime-to-\( p \) integer \( h \geq h_0 \).

**Proof.** It suffices to prove that there exists an \( A_p \)-Galois cover of the affine line with inertia group \( \mathbb{Z}/(p) \) and lower jump \( h_0 \); once this small value is realized for the lower jump of such a cover, then all larger prime-to-\( p \) integers occur as the lower jump of such a cover by Theorem [5.4]. Note that the upper and lower jumps are equal when the inertia group has order \( p \).

Let \( t = (p - 1)/2 \). Then \( \gcd(p - 1, t(t - 1)) = (p - 1)/2 \) if \( p \equiv 1 \mod 4 \) and equals \( p - 1 \) if \( p \equiv 3 \mod 4 \). Consider the \( A_p \)-Galois cover \( \pi_t : X_t \to \mathbb{P}^1_k \) in Theorem [4.6] which is branched only at \( \infty \). If \( p \equiv 3 \mod 4 \), then \( \pi_t \) has inertia group of order \( p \) and upper jump \( (p + 1)/4 \). When \( p \equiv 1 \mod 4 \), then \( \pi_t \) has inertia group of order \( 2p \) and upper jump \( (p + 1)/4 \). In the latter case, taking \( d = 2 \) in Lemma [5.3] yields an \( A_p \)-Galois cover of the affine line with inertia group of order \( p \) and upper jump \( (p + 1)/2 \).

We now provide a partial answer to Question [5.7] for all other alternating groups whose order is strictly divisible by \( p \).

**Corollary 5.9.** Let \( 2 \leq s < p \). If \( p = 7 \), assume \( s \neq 2 \). Let \( h_s = s(p + s)/\gcd(p - 1, s(s + 1)) \). There exists an \( A_{p+s} \)-Galois cover of \( \mathbb{P}^1_k \) branched only at \( \infty \) with inertia group \( \mathbb{Z}/(p) \) and lower jump \( h \) for every prime-to-\( p \) integer \( h \geq h_s \).

**Proof.** By Theorem [4.9] there exists an \( A_{p+s} \)-Galois cover \( \phi_s : Y_s \to \mathbb{P}^1_k \) branched only at \( \infty \) with inertia group \( G_0 \) of order \( pm_s \) and upper jump \( \sigma_s = (p + s)/(p - 1) \) where \( m_s = (p - 1)s/\gcd(p - 1, s(s + 1)) \). Applying Lemma [4.1] with \( r_2 = m_s \) produces an \( A_{p+s} \)-Galois cover of the affine line with inertia group \( \mathbb{Z}/(p) \) and lower jump \( h_s \). This completes the proof by Theorem [5.4].

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Corollary 5.10. Let $p \neq 7$ be an odd prime. Let $h_1 = 2(p+2)/\gcd(p-1,3)$. There exists an $A_{p+1}$-Galois cover of $\mathbb{P}^1_k$ branched only at $\infty$ with inertia group $\mathbb{Z}/(p)$ and lower jump $h$ for every prime-to-$p$ integer $h \geq h_1$.

Proof. By Theorem 4.9, letting $s = 2$, there exists an $A_{p+2}$-Galois cover $\phi_2 : Y_2 \to \mathbb{P}^1_k$ branched only at $\infty$ with inertia group $G_0$ of order $pm_2$ and upper jump $\sigma_2 = (p+2)/(p-1)$ where $m_2 = (p-1)/\gcd(p-1,3)$. The lower jump $h$ of $\phi_2$ equals $(p+2)/\gcd(p-1,3)$.

Consider the $A_{p+1}$-Galois subcover $\tilde{\phi} : Y_2 \to Z'_2$ of $\phi_2$. It is branched above $P(\infty,0)$ where it has ramification index $m_2$ and above $P(\infty,\infty)$ where it has ramification index $pm_2/2$. The lower jump of $\tilde{\phi}$ above $P(\infty,\infty)$ equals the lower jump $h$ of $\phi_2$. The upper jump of $\tilde{\phi}$ is thus $\tilde{\sigma} = 2(p+2)/(p-1)$. Applying the Riemann-Hurwitz formula to $\phi_2$ and $\tilde{\phi}$, we note that $Z'_2$ has genus 0. Another way to see this is that the equation $g_2$ yields that $x = (y^{p+s} + 1)/y$ and so the function field of $Z'_2$ is $L_2 \simeq k(y)$.

Thus $\tilde{\phi}$ is an $A_{p+1}$-Galois cover of the projective line branched at two points. Note that $\tilde{\phi}$ is disjoint from an $m_2$-cyclic cover of the projective line branched at $\{0, \infty\}$. Applying Lemma 4.1 with $r_2 = m_2$ removes the tamely ramified branch point. In particular, it yields a Galois cover $\tilde{\phi}' : Y'_2 \to \mathbb{P}_k^1$ branched only at $\infty$, with ramification index $p$. The upper (and lower) jump of $\tilde{\phi}'$ is $\sigma' = m_2\tilde{\sigma}$ which equals $2(p+2)/\gcd(p-1,3)$. This completes the proof by Theorem 5.4.

5.5 Realizing small upper jumps for $A_p$-Galois covers

The upper jump $\sigma = h/m$ of an $A_p$-Galois cover of the affine line satisfies the necessary conditions $\sigma > 1$, $\gcd(h, m) = 1$, $m \mid (p-1)/2$, and $p \nmid h$. As a generalization of Question 5.7, we can ask which $\sigma$ satisfying the necessary conditions occur as the upper jump of an $A_p$-Galois cover of the affine line.

In [5, Thm. 2], the authors prove that all but finitely many $\sigma$ which satisfy the necessary conditions occur as the upper jump of an $A_p$-Galois cover of the affine line. That result generalizes both Corollary 5.8 (where $m = 1$) and Corollary 5.1 (which can be rephrased as stating that all divisors of $(p-1)/2$ occur as the denominator of $\sigma$ for such a cover). Specifically, given a divisor $m$ of $(p-1)/2$ and a congruence value of $h$ modulo $m$, [5, Thm. 3.1.4] provides a lower bound on $h$ above which all $\sigma = h/m$ (satisfying the necessary conditions) are guaranteed to occur. The bound is $a(p-2)$ where $a$ is such that $1 \leq a \leq m$ and $a \equiv -h \mod m$. 21
Theorem 4.6 improves on [5, Thm. 3.1.4] by providing some new values of \( \sigma \) which were not previously known to occur as the upper jump of an \( A_p \)-Galois cover of the affine line. Corollary 5.8 is an example of that improvement; here are two more examples.

**Example 5.11.** Small primes: The first column of the table shows the values of \( \sigma \) that are achieved in [5, Thm. 3.1.4]. The second column contains rational numbers satisfying the necessary conditions whose status was not known from [5]. The final column contains new values of \( \sigma \) which are guaranteed to occur in Theorem 4.6.

| \( p \) | \( \sigma \) obtained from [5, Thm. 3.1.4] | \( \sigma \) unknown from [5] | Theorem 4.6 |
|-------|---------------------------------|-----------------|-----------|
| 5     | 3, 4, 6, \ldots                | 2               | None      |
|       | 3/2, 7/2, 9/2, \ldots          | None            |           |
| 7     | 5, 6, 8, \ldots               | 2, 3, 4         | 2, 3, 4   |
|       | 5/3, 8/3, 10/3, \ldots         | 4/3             |           |
| 11    | 9, 10, 12, \ldots             | 2, 3, 4, 5, 6, 7, 8 | 3, 4, 5, 6, 7, 8 |
|       | 9/5, 14/5, 19/5, \ldots        | 6/5, 7/5, 8/5, 12/5 | 12/5     |
| 13    | 11, 12, 14, 15, \ldots        | 2, 3, \ldots, 10 | 3, 4, 5, 6, 7, 8, 9, 10 |
|       | 11/2, 15/2, 17/2, \ldots      | 3/2, 5/2, 7/2, 9/2 | 5/2, 7/2, 9/2 |
|       | 11/3, 14/3, 17/3, \ldots      | 4/3, 5/3, 7/3, 8/3, 10/3 | 10/3 |
|       | 11/6, 17/6, 23/6, \ldots      | 7/6             |           |

**Example 5.12.** Suppose \( p \equiv 1 \mod 3 \) and \( m = (p - 1)/6 \) and \( h \equiv -1 \mod m \). Then the lower bound on \( h \) to guarantee that \( h/m \) occurs as the upper jump of an \( A_p \)-Galois cover of the affine line from [5, Thm. 3.1.4] is \( p - 2 \) and from Theorem 4.6 is \( (p - 3)/2 \). Suppose \( p \equiv 2 \mod 3 \) and \( m = (p - 1)/2 \) and \( h \equiv -3 \mod m \). Then the lower bound on \( h \) to guarantee that \( h/m \) occurs as the upper jump of an \( A_p \)-Galois cover of the affine line from [5, Thm. 3.1.4] is \( 3(p - 2) \) and from Theorem 4.6 is \( 3(p - 3)/2 \).

**Proof.** The previous lower bounds are a direct application of [5, Thm. 3.1.4]. For the new lower bounds, when \( t = 3 \), then Theorem 4.6 states that \( \sigma_3 = 3(p - 3)/(p - 1) \) occurs as an upper jump of an \( A_p \)-Galois cover of the affine line. If \( p \equiv 1 \mod 3 \), then \( m = (p - 1)/6 \) and \( h = (p - 3)/2 \); (note that \( h \equiv -1 \mod m \)). If \( p \equiv 2 \mod 3 \), then \( m = (p - 1)/2 \) and \( h = 3(p - 3)/2 \); (note that \( h \equiv -3 \mod m \)).
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