Yang-Yang generating function and Bergman tau-function

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Abstract

In this paper we consider the symplectic properties of the monodromy map of second order equations
on a Riemann surface whose potential is meromorphic with second order poles. We show that the Poisson
bracket defined by the periods of meromorphic quadratic differential implies the Goldman Poisson structure
on character variety. These results generalize the previous works of the authors [6] where the case of
holomorphic potentials was considered and the paper [17] where the symplectomorphism was proved for
potentials with first order poles. We show that the leading term of the generating function of the monodromy
symplectomorphism (the "Yang-Yang" function) is proportional to the Bergman tau-function on moduli
spaces of meromorphic quadratic differentials with second order poles.

1 Introduction

The goal of this paper is to study symplectic aspects of the monodromy map for second order linear equation on
a Riemann surface of genus $g$ with meromorphic potential having $n$ second order poles. We generalize results
of the recent paper by the authors and C.Norton [6], where the case of holomorphic potential was treated, and
the paper [17] where the potentials with first order poles were considered.

We are going to study the second order equation on a Riemann surface $C$ of genus $g$ in the form

$$ \phi'' + \left( \frac{1}{2} S_0 + Q \right) \phi = 0. $$

(1.1)

where $S_0$ is a fixed holomorphic projective connection on $C$ and depending holomorphically on the moduli of $C$;
$Q$ is a meromorphic quadratic differential on $C$ with $n$ second order poles, which we denote by $z_1, \ldots, z_n$, and
$4g-4+2n$ simple zeros which we denote by $x_j$, $j = 1, \ldots, 4g-4+n$.

The coordinate invariance of the equation (1.1) implies that the solution $\phi$ locally transforms as $-1/2$-
differential [13] under a coordinate change.

Following [6], are going to prove that if, following [6], $S_0$ is chosen to be the Bergman projective connection $S_B$ then the the natural homological Poisson structure on the space of meromorphic quadratic differentials implies the Goldman Poisson structure on the monodromy manifold of equation (1.1). We discuss also the formal WKB expansion of the generating function of the monodromy symplectomorphism (the "Yang-Yang" function [21]) and show that its leading term in $\hbar$ expansion is proportional to the Bergman tau-function on moduli spaces of abelian quadratic differentials [3, 18]:

$$ G_{YY} = 6\pi i \log \tau_B + \hbar^2 G_1 + \hbar^4 G_2 + \ldots $$

(1.2)

where equations for the terms $G_1, G_2 \ldots$ are expressed in terms of periods of the form $v = \sqrt{Q}$ and the
differentials arising in the WKB ansatz.

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2 Moduli space of quadratic differentials with second order poles; homological symplectic structure

We assume that the leading terms of the asymptotics of $Q$ near the pole $z_j$ look as follows ($\xi_j$ is a local coordinate near $z_j$):

$$Q(\xi_j) = \left[-\frac{m_j(m_j+1)}{\xi_j^2} + \ldots\right](d\xi_j)^2$$  \hspace{1cm} (2.1)

for $m_j \neq 0, -1$.

Denote by $Q_{g,n}$ the moduli space of meromorphic quadratic differentials on Riemann surface of genus $g$ with $n$ double poles and $4g-4+2n$ simple zeros. For any differential $Q \in Q_{g,n}$ we define the canonical cover $\hat{C}$ as the locus in $T^*C$ given by by

$$v^2 = Q.$$  \hspace{1cm} (2.2)

The two-sheeted covering $\pi: \hat{C} \to C$ is branched at zeros of $Q$: thus the total number of branch points is $4g-4+2n$ and the genus of $\hat{C}$ equals $\hat{g} = 4g - 3 + n$. Denote the natural involution on $\hat{C}$ by $\mu$ and the projection of $\hat{C}$ to $C$ by $\pi$. The Abelian differential $v$ is an Abelian differential of third kind on $\hat{C}$ with double zeros (on $\hat{C}$) at the branch points $\{x_i\}_{i=1}^{4g-4+2n}$ and is skew-symmetric under the involution $\mu$. The differential $v$ has simple poles at $2n$ points which we denote by $z_i$ and $z_i^\mu$ with residues $\sqrt{-m_i(m_i+1)}$ and $-\sqrt{-m_i(m_i+1)}$, respectively.

Let us decompose the homology group $H_1(\hat{C} \setminus \{z_i, z_i^\mu\}_{i=1}^n)$ into symmetric and skew-symmetric parts under the involution $\mu$:

$$H_1(\hat{C} \setminus \{z_i, z_i^\mu\}_{i=1}^n) = H_+ \oplus H_-$$

where $\dim H_+ = 2g + n - 1$ and $\dim H_- = 6g - 6 + 3n$.

The rank of the intersection pairing on $H_-$ equals $6g - 6 + 2n$. Introduce a set of generators of $H_-$ denoted by $\{a^-_i, b^-_i\}_{i=1}^{3g-3+n}, \{l_i\}_{i=1}^n$ with the intersection index

$$a^-_i \circ b^-_j = \delta_{ij}/2$$

and such that the intersection index of $l_k$ with all other $(a_j, b_j, l_k)$ generators equals to $0$. The generators $l_k$ satisfying this requirement can be chosen as follows: pick any path between $z_k$ and $z_k^\mu$ which is skew-symmetric under $\mu$ (for example, take a path connecting $z_k$ with one of the branch points by a path and then continue this path to another sheet using the involution $\mu$; then blow up such path into a positively oriented closed loop encircling both $z_k$ and $z_k^\mu$). Then

$$\int_{l_k} v = 4\pi i \sqrt{-m_k(m_k+1)}$$

The integrals of $v$ over cycles $(a_j, b_j)$ will be denoted by

$$A_i = \int_{a^-_i} v \quad B_i = \int_{b^-_i} v \quad i = 1, \ldots, 3g - 3 + n,$$  \hspace{1cm} (2.3)

The subspace of $R_{g,n}$ corresponding to fixed values of $m_1, \ldots, m_n$ we denote by $R_{g,n}[m]$. The periods $\{A_i, B_i\}_{i=1}^{3g-3+n}$ can be used as local homological coordinates on $R_{g,n}[m]$. We define the following symplectic form on $R_{g,n}[m]$:

$$\Omega_{hom} = 2 \sum_{i=1}^{3g-3+n} dA_i \wedge dB_i$$  \hspace{1cm} (2.4)

**Remark 2.1** If all periods of $v$ in $H_-$ are real i.e. $m_i(m_i+1)$ are real and positive and all $\{A_i, B_i\}$ are real then the obtained real slice of $R^R_{g,n}[m]$ coincides with the largest stratum of the combinatorial model based on Strebel differentials [16, 3]. The symplectic form (2.4), being restricted to $R^R_{g,n}[m]$, coincides with the symplectic form used in [16] to introduce the orientation on the largest stratum of the combinatorial model. The real version of the form $\omega$ in notations of [16] equals to $\sum_{i=1}^n p_i^2 \psi_i$ where $\psi_i$ are tautological $\psi$-classes and $p_i^2 = 2\pi m_i(m_i+1)$ [3].

2
3 Admissible projective connections

Here we discuss the holomorphic projective connection \( S_0 \) on \( C \), which is assumed to depend locally holomorphically on the moduli of the Riemann surface \( C \). We start from the following definition:

**Definition 3.1** Two holomorphic projective connections on \( C \) holomorphically depending on moduli, \( S_0 \) and \( S_1 \), are called equivalent if the 1-form, corresponding to family of quadratic differentials \( S_1 - S_0 \) (which is locally defined on the moduli space \( M_g \)), is closed.

The projective connection which plays the main role in our analysis is the Bergman projective connection (in fact, there is an infinite family of Bergman projective connections on each Riemann surface corresponding to different Torelli markings). The Bergman projective connection on \( C \) is defined as follows. Consider a Torelli marking \( \alpha \) of \( C \) defined by a choice of canonical basis \( \{ a_i, b_j \}_{i=1}^g \) in \( H_1(C, \mathbb{Z}) \) with the intersection index \( a_i \circ b_j = \delta_{ij} \). Consider the canonical normalized meromorphic bidifferential \( B^\alpha(x, y) = d_x d_y \log E(x, y) \) where \( E(x, y) \) is the prime-form [8]. This bidifferential depends on the Torelli marking \( \alpha \) of \( C \); it is normalized by the requirement that all of its \( \alpha \)-periods vanish with respect to each variable.

The following asymptotics holds as \( y \to x \):

\[
B^\alpha(x, y) = \left( \frac{1}{(\xi(x) - \xi(y))^2} + \frac{1}{6} S_B^\alpha(\xi(x)) + \ldots \right) d\xi(x) d\xi(y) .
\]

(3.1)

where \( S_B^\alpha \) is the Bergman projective connection which also depends on the Torelli marking.

**Definition 3.2** The projective connection \( S_0 \), holomorphically depending on local moduli of the Riemann surface \( C \), is called admissible (or "integrable", or "Lagrangian") if is equivalent (Def. 3.1) to a Bergman projective connection.

As it was shown in [6], any two Bergman projective connections, corresponding to different Torelli markings, \( \alpha_1 \) and \( \alpha_2 \), are equivalent, and the family of quadratic differentials \( S_B^{\alpha_2} - S_B^{\alpha_1} \) is canonically identified with the 1-form \( d[\det(C\Omega + D)] \) where \( \Omega \) is the period matrix of \( C \) and \( \begin{pmatrix} C & D \\ A & B \end{pmatrix} \) is the \( Sp(2g, \mathbb{Z}) \) matrix transforming the Torelli marking \( \alpha_1 \) to \( \alpha_2 \).

Examples of admissible projective connections are extensively discussed in [6]; known examples include Schottky, Quasi-Fuchsian and Bergman projective connections.

The following proposition gives an alternative characterization of admissible projective connections which does not refer to the Bergman projective connection:

**Proposition 3.1** The projective connection \( S_0 \) is admissible if the following locally defined 1-form on \( \mathcal{R}_{g,n}[m] \)

\[
\Theta = \sum_{j=1}^{3g-3+n} \left[ \left( \int_{b_j} S_0 - S_v \right) dA_j - \left( \int_{a_j} \frac{S_0 - S_v}{v} \right) dB_j \right]
\]

(3.2)

is closed, \( d\Theta = 0 \). In (3.2) \( S_v = \{ \int^x v, \cdot \} \) is the meromorphic projective connection with poles at zeros and poles of \( v \); \( (S_0 - S_v)/v \) is a meromorphic abelian differential on \( C \).

If \( S_0 \) is a Bergman projective connection \( S_B^\alpha \) then \( \Theta = d\log \tau_B \) where \( \tau_B \) is the Bergman tau-function on the space of meromorphic quadratic differentials with second order poles [14, 3, 18].

4 Variational formulas and Poisson brackets for potential

Let us now choose in (1.1) \( S_0 \) to be the Bergman projective connection corresponding to a Torelli marking \( \alpha \). Then (1.1) takes the form

\[
\phi'' + \left( \frac{1}{2} S_B^\alpha + Q \right) \phi = 0 .
\]

(4.1)
Denote two linearly independent solutions of (4.1) by \( \phi_1 \) and \( \phi_2 \) and introduce functions \( \psi_{1,2} = \phi_{1,2}\sqrt{v} \) where \( \phi_{1,2} \) are two linearly independent solutions of (1.1). Denote by \( \Psi \) the Wronskian matrix of \( \psi_1 \) and \( \psi_2 \). The matrix \( \Psi \) satisfies the first order matrix equation
\[
d\Psi = \begin{pmatrix} 0 & v \\ uw & 0 \end{pmatrix} \Psi ,
\]
where the meromorphic function \( u \) on \( \mathcal{C} \) is given by
\[
u = -\frac{S_B - S_u}{2Q} - 1
\]
and \( S_v(\xi) = S(z(\xi), \xi) \) is the Schwarzian derivative of the coordinate \( z(x) = \int_{x_1}^x v \) with respect to a local coordinate \( \xi \). The coefficients \( v \) and \( uv \) in (4.2) are meromorphic differentials on \( \mathcal{C} \), which are skew-symmetric under the involution \( \mu \).

The homological symplectic structure (2.4) on the space \( \mathcal{R}_{g,n}[m] \) induces the Poisson structure on the space of coefficients \( u \) of the equation (4.2). The Poisson bracket between \( u(z) \) and \( u(\zeta) \) (for constant \( z \) and \( \zeta \)) can be computed using the variational formula (4.6) below:
\[
\{ u(z), u(\zeta) \} = \frac{1}{2} \sum_{i=1}^{3g-3+n} \left( \frac{\partial u(z)}{\partial A_i} \frac{\partial u(\zeta)}{\partial B_i} - \frac{\partial u(z)}{\partial B_i} \frac{\partial u(\zeta)}{\partial A_i} \right),
\]
which gives, in analogy to the proof of Proposition 4.4 of [6]:
\[
\frac{4\pi i}{3} \{ u(z), u(\zeta) \} = \mathcal{L}_z \left[ \int_\gamma h(z, \zeta) dz \right] - \mathcal{L}_\zeta \left[ \int_\gamma h(z, \zeta) d\zeta \right],
\]
where
\[
\mathcal{L}_z = \frac{1}{2} \partial_z^3 - 2u(z) \partial_z - u_z(z)
\]
is called the “Lenard’s operator” in the theory of Korteveg de Vries equation. In the computation of the bracket (4.5) it is assumed that the arguments \( z \) and \( \zeta \) of \( u \) are independent of moduli.

The proof of (4.5) is identical to the proof of the same formula in the case of holomorphic potentials (see Prop.4.4 of [6]). It uses variational formulas for the potential \( u (4.3) \) on the space \( \mathcal{R}_{g,n}[m] \) which take the form (see (3.35) of [6] and Section 11 of [18]):
\[
\left. \frac{\partial u(x)}{\partial A_j} \right|_{z(x)=\text{const}} = \frac{3}{2\pi i} \int_{y_j} \frac{B^2(x, y)}{v^2(x)v(y)} \, , \quad \left. \frac{\partial u(x)}{\partial B_j} \right|_{z(x)=\text{const}} = -\frac{3}{2\pi i} \int_{y_j} \frac{B^2(x, y)}{v^2(x)v(y)} .
\]
Substituting these formulas into (4.4), using the Riemann bilinear identities and computing the resulting contour integral by residues leads to (4.5).

5 Monodromy representation and Goldman bracket

The ratio \( f = \phi_1/\phi_2 \) of two linearly independent solutions of (1.1) solves the Schwarzian equation
\[
S(f, \xi) = S_B(\xi) + Q(\xi) ,
\]
where \( \xi \) is an arbitrary local parameter on \( \mathcal{C} \) and \( S \) denotes Schwarzian derivative. The Schwarzian equation determines a \( \text{PSL}(2, \mathbb{C}) \) monodromy representation of the fundamental group \( \pi_1(\mathbb{C} \setminus \{ z_i \}^n_{i=1}) \) which turns out to be liftable to an \( \text{SL}(2, \mathbb{C}) \) representation for the case \( n = 0 \) [9].

Denote the standard generators of the fundamental group by \( \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g, \kappa_1, \ldots, \kappa_n \); these generators satisfy the relation
\[
(\kappa_n \ldots \kappa_1) \prod_{i=g}^1 \alpha_i^{-1} b_i^{-1} \alpha_i \beta_i = \text{id} .
\]
The monodromy matrices satisfy the inverse relation

$$\prod_{i=1}^{g} M_{\alpha_i} M_{\beta_i} M_{\alpha_i}^{-1} M_{\beta_i}^{-1} (M_{\kappa_1} \cdots M_{\kappa_n}) = I.$$ \hfill (5.2)

The diagonal form of the matrices $M_{\kappa_j}$ is given by

$$\Lambda_j = \begin{pmatrix} e^{2\pi i m_j} & 0 \\ 0 & e^{-2\pi i m_j} \end{pmatrix}.$$ \hfill (5.3)

A rather technical computation in [6] allows to find the Poisson bracket between traces of monodromy matrices of the equation (4.2) along two arbitrary loops $\gamma$ and $\tilde{\gamma}$ using (4.5). The result is the Goldman's bracket [10]:

$$\{ \text{tr} M_\gamma, \text{tr} M_{\tilde{\gamma}} \} = \frac{1}{2} \sum_{p \in \gamma \cap \tilde{\gamma}} \nu(p) \left( \text{tr} M_{\gamma p \tilde{\gamma}} - \text{tr} M_{\gamma p \tilde{\gamma}^{-1}} \right)$$ \hfill (5.4)

where the monodromy matrices $M_\gamma, M_{\tilde{\gamma}} \in \text{PSL}(2, \mathbb{C})$; $\gamma p \tilde{\gamma}$ and $\gamma p \tilde{\gamma}^{-1}$ are paths obtained by resolving the intersection point $p$ in two different ways (see [10]); $\nu(p) = \pm 1$ is the contribution of the point $p$ to the intersection index of $\gamma$ and $\tilde{\gamma}$.

The symplectic form which inverts the Goldman bracket on a symplectic leaf $m_j = \text{const}$ can be written in terms of Fock-Goncharov (complex shear) coordinates as follows: consider a triangulation of the surface $C$ with $n$ vertices at poles of $Q$. In principle, such triangulation can be chosen arbitrarily, but keeping in mind the WKB analysis (see [2, 7]) it is convenient to choose this triangulation in a special way that we presently explain (see Fig. 1). First, consider the critical bi-partite graph $\Gamma_Q$ of the differential $Q$ defined by its critical trajectories $\text{Im} \int_{x_1}^{x} \nu = 0$, where $x_1$ is an arbitrarily chosen zero of $Q$ and assume that $Q$ does not have any "saddle connections" i.e. none of the critical trajectories connect two zeros of $Q$. Such differentials are generic and the standard face of $\Gamma_Q$ is 4-valent with one pair of opposite vertices being formed by two zeros (white vertices) and another pair by two poles (black vertices) of $Q$. All white vertices of $\Gamma_Q$ are three-valent since all zeros of $Q$ are simple. From the critical graph $\Gamma_Q$ one can construct two other graphs: the "black" graph $\Sigma_Q$
which gives a triangulation of $\mathcal{C}$ with $N$ vertices, and the "white" graph which is dual to $\Sigma_Q$ and denoted by $\Sigma^*_Q$. All vertices of $\Sigma^*_Q$ are three-valent.

To each edge $e$ of the graph $\Sigma_Q$ is we assign a coordinate $\zeta_e$ which is logarithm of the corresponding Fock-Goncharov (or "shear") coordinate. Then the symplectic form

$$\Omega_{\text{mon}} = 2 \sum_{v \in V(\Sigma_Q)} \sum_{e,e' \perp v, e < e'} \, d\zeta_e \wedge d\zeta_{e'}$$

is the inverse of the Poisson bracket (5.4) on each symplectic leaf. The graph is considered to be "ciliated" i.e. the edges $e$ are ordered counter-clockwise at each vertex [5]. The Casimirs $m_j$ are expressed in terms of coordinates $\zeta_e$ as follows:

$$m_j = \exp \left( \sum_{e \perp v} \zeta_e \right) \quad j = 1, \ldots, n$$

The symplectic potential $\theta_G$ for the Goldman’s symplectic form, satisfying $d\theta_{\text{mon}} = \Omega_{\text{mon}}$, can then be chosen in the following form:

$$\theta_{\text{mon}} = \sum_{v \in V(\Sigma_Q)} \sum_{e,e' \perp v} (\zeta_e d\zeta_{e'} - \zeta_{e'} d\zeta_e) \quad \text{(5.6)}$$

On the other hand, the symplectic form (2.4) can be equivalently rewritten using the graph $\Sigma^*_Q$ dual to $\Sigma_Q$; the vertices of $\Sigma^*_Q$ are the zeros of the differential $Q$. We can write the form (2.4) as follows (see Theorem 2.11 of [3]):

$$\Omega_{\text{hom}} = 2 \sum_{f \in F(\Sigma_Q)} \sum_{l,l' \in \partial f, l \prec l'} \, dP_l \wedge dP_{l'} \quad \text{(5.7)}$$

where for integrals of $v$ over edges of $\Sigma^*_Q$ we use the notation

$$P_l = \int_l v \quad \text{(5.8)}$$

We assume that the integral of $v$ in (5.8) is taken inside of the corresponding face of the graph $\Sigma^*_Q$ i.e. implicitly the branch cuts for $v$ are chosen along the edges of $\Sigma^*_Q$. Since the faces of $\Sigma^*_Q$ are in one to one correspondence to the vertices of the graph $\Sigma_Q$ and the edges $l$ of $\Sigma^*_Q$ are in one to one correspondence with the edges $e$ of $\Sigma_Q$, we shall assign an index $e$ to $l$ and rewrite the formula (5.7) as follows:

$$\Omega_{\text{hom}} = 2 \sum_{v \in V(\Sigma_Q)} \sum_{e,e' \perp v} \, dP_{l_e} \wedge dP_{l_{e'}} \quad \text{(5.9)}$$

where the edges incident at $v$ are ordered counterclockwise starting from a chosen cilium; the notation $l_e$ denotes the edge of $\Sigma^*_Q$ dual the edge $e$ of $\Sigma_Q$, and with the induced orientation.

Then the symplectic potential of the homological symplectic form written as (5.7) can be chosen as

$$\theta_{\text{hom}} = \sum_{v \in V(\Sigma_Q)} \sum_{e,e' \perp v} \left( P_{l_e} dP_{l_{e'}} - P_{l_{e'}} dP_{l_e} \right) \quad \text{(5.10)}$$

such that

$$\Omega_{\text{hom}} = d\theta_{\text{hom}} \quad \text{(5.11)}$$

In the next section we use the symplectic potentials $\theta_{\text{hom}}$ and $\theta_{\text{mon}}$ to introduce the generating function of the monodromy symplectomorphism.

6 Generating function of monodromy symplectomorphism and Bergman tau-function

We start from the following definition.
Definition 6.1 The generating function of the monodromy symplectomorphism (the "Yang-Yang" function of [21]) is defined by

$$dG_{YY} = \theta_{\text{mon}} - \theta_{\text{hom}} \quad (6.1)$$

We notice that one can associate many different generating functions to a given symplectic map; the difference is in the choice of symplectic potentials on both sides. Our choice of symplectic potentials (5.6) for Goldman symplectic form and (5.10) for the homological symplectic form is made for technical convenience and symmetry.

The following proposition is a corollary of Proposition 5.1 of [6]

Proposition 6.1 Let $G^\varphi_{YY}$ and $G'^\varphi_{YY}$ be Yang-Yang generating functions corresponding to two Torelli markings, $\alpha$ and $\alpha'$, defining the Bergman projective connections $S_\alpha^\varphi$ and $S'_{\alpha'}^\varphi$. Let the Torelli markings $\alpha'$ and $\alpha$ be related by an $Sp(2g, \mathbb{Z})$ matrix

$$\sigma = \begin{pmatrix} C & D \\ A & B \end{pmatrix}; \quad \begin{pmatrix} b \\ a \end{pmatrix}^\sigma = \begin{pmatrix} b \\ a \end{pmatrix} \quad (6.2)$$

Then

$$G'^\varphi_{YY} = G^\varphi_{YY} + 6\pi i \log \det(C\Omega + D) \quad (6.3)$$

where $\Omega$ is the period matrix of $\mathcal{C}$ corresponding to the Torelli marking $\alpha$.

6.1 Bergman tau-function over spaces of quadratic differentials

The Bergman tau-function on moduli spaces appear in various context - from isomonodromy deformations to spectral geometry, Frobenius manifolds and random matrices, see the review [18]. In the context of moduli spaces of quadratic differentials with second order poles the Bergman tau-function was discussed in detail in [3], Section 4.1, where it is denoted by $\tau_+$. In our present context we only need the main properties of $\tau_B$: the defining equations, the transformation under a change of Torelli marking and transformation under rescaling of $Q$ by a constant. To write the defining equations we choose one zero $x_{f_j}$ at the boundary of each face $f_j$ of the graph $\Sigma_Q$ and connect it by a path $r_j$ to the pole $z_j$ lying inside the face $f_j$. The choice of the zero $x_{f_j}$ is dictated by the choice of ciliation of the graph $\Sigma_Q$ used to define the ordering of edges $e$ at each pole $z_j$. Then one chooses the paths $a_j^-$ and $b_j^+$ in $H_-$ such that they don't intersect the paths $r_j$. Then the defining equations for $\tau_B$ can be written in one of the following forms. With respect to periods of $v$ over cycles $(a_j^-, b_j^+)$ one has:

$$\frac{\partial \log \tau_B}{\partial \mathcal{P}_{a^-_j}} = \frac{3}{2\pi i} \int_{b^-_j} \frac{S_B - S_v}{v} \quad \frac{\partial \log \tau_B}{\partial \mathcal{P}_{b^+_j}} = \frac{3}{2\pi i} \int_{a^-_j} \frac{S_B - S_v}{v} \quad (6.4)$$

such that pippo

$$d \log \tau_B = \frac{3}{2\pi i} \sum_{j=1}^{3g-3+n} \left[ \left( \int_{a^-_j} \frac{S_B - S_v}{v} \right) d\mathcal{P}_{b^-_j} - \left( \int_{b^-_j} \frac{S_B - S_v}{v} \right) d\mathcal{P}_{a^-_j} \right] \quad (6.5)$$

Alternatively, the same equations can be written in terms of integrals of $v$ over edges of $\Sigma_Q^*$:

$$\frac{\partial \log \tau_B}{\partial \mathcal{P}_l} = -\frac{3}{2\pi i} \sum_{l', \ell \in f, l < l'} \int_{v} \frac{S_B - S_v}{v} \quad (6.6)$$

where, since the differential $(S_B - S_v)/v$ is singular (but residue-free) at the endpoints of $l'$, the integral in the r.h.s. is understood as 1/2 of the integral over the corresponding cycle in $H_-$. Then the differential (6.5) can be alternatively written as

$$d \log \tau_B = \frac{3}{2\pi i} \sum_{f \in \mathcal{F}(\Sigma_Q^*)} \sum_{l, \ell' \in f, l < l'} \left( \int_{v} \frac{S_B - S_v}{v} \right) d\mathcal{P}_l \quad (6.7)$$

where $l, l'$ here denote edges of the dual graph $\Sigma_Q^*$ on the boundary of the face $f$ and ordered counterclockwise starting from the ciliation.
The following two properties of $\tau_B$ are important in the present context (see [3] for details). First, if the Torelli marking of $C$ transforms according to (6.2), then $\tau_B$ transforms as follows:

$$\tau_B^\alpha = \tau_B^0 \det(C\Omega + D).$$

(6.8)

Second, if we don’t change the Torelli marking but rescale $Q$ by a coefficient $\kappa$ then

$$\tau_B(\kappa Q) = \kappa^{5(2g-2+n)/72} \tau_B(Q).$$

(6.9)

7  WKB expansion and Yang-Yang generating function

The problem of understanding of the generating function of the monodromy symplectomorphism (the Yang-Yang function) for equation (1.1) was posed in [21]. Here we briefly discuss the WKB expansion of the Yang-Yang function in the framework of recent results of [2] where the relationship between Voros symbols and the complex shear coordinates (the $SL(2)$ case of the Fock-Goncharov coordinates) on the character variety was established.

We shall follow the conventions of [6] and write the main equation as follows:

$$\phi'' + \frac{1}{2} S_B + h^{-2} Q \phi = 0$$

(7.1)

Define $v = \sqrt{Q}$. Then in terms of coordinate $z(x) = \int_{x_0}^x v$ and the function $\psi(x) = \phi \sqrt{v(x)}$ we have

$$\psi_{zz} + (q(z) + h^{-2})\psi = 0$$

(7.2)

where

$$q = \frac{S_B - S_v}{2v^2}$$

To study the limit $h \to 0$ we introduce the function $S$ via

$$\phi = v^{-1/2} \exp \left\{ \int_{x_0}^x (h^{-1} s_{-1} + s_0 + h S_1 + \ldots) v \right\}$$

(7.3)

where $s_k$ are meromorphic functions on $\hat{C}$.

The function $s = \sum_{k=-1}^\infty h^k s_k$ satisfies the Riccati equation

$$ds + vs^2 = -qv - h^{-2} v$$

(7.4)

(notice that on both sides of this equation we have 1-forms) or

$$d \left( \sum_{k=-1}^{\infty} h^k s_k \right) + v \left( \sum_{k=-1}^{\infty} h^k s_k \right)^2 = -qv - h^{-2} v.$$  

The coefficients of $h^{-2}$, $h^{-1}$ and $h^0$ give

$$s_{-1} = \pm i, \quad s_0 = 0, \quad s_1 = \pm i q/2.$$  

(7.5)

For all other coefficients we have

$$ds_k + v \sum_{j,l \geq -1} s_j s_l = 0, \quad k > 0.$$  

(7.6)

Thus

$$s_{k+1} = \pm i \left( \frac{d s_k}{v} + \sum_{j+l=k, j,l \geq 0} s_j s_l \right)$$

(7.7)

In particular, for $k = 1, 2$ we get

$$s_2 = \frac{i d s_1}{2v}; \quad s_3 = \frac{s_1^2}{2} - \frac{1}{4v} d \left( \frac{d s_1}{v} \right).$$

(7.8)
Lemma 7.1 Functions $s_{2k+1}$ are symmetric under $\mu$, while $s_{2k}$ are anti-symmetric. Therefore, 1-forms $s_2v$ are meromorphic 1-forms on $\mathbb{C}$, while $s_{2k+1}v$ are meromorphic 1-forms on $\hat{\mathbb{C}}$ anti-symmetric under $\mu$.

Proof. We proceed by induction. The induction step yields

$$s_{2k+1} = \pm \frac{i}{2} \left( \frac{ds_{2k}}{v} + \sum_{j+l=2k+1, j,l \geq 0} s_j s_l \right) .$$  \hfill (7.9)

By induction hypothesis all products in the sum are even, as well as the first term; thus $s_{2k+1}$ is even under the involution $\mu$. Similarly,

$$s_{2k} = \pm \frac{i}{2} \left( \frac{ds_{2k-1}}{v} + \sum_{j+l=2k-1, j,l \geq 0} s_j s_l \right)$$  \hfill (7.10)

and all the terms in the right-hand side are odd by induction hypothesis.

Consider now the corresponding even and odd parts:

$$s_{\text{even}} = \sum_{l=0}^{\infty} s_{2l} \hbar^{2l} \quad s_{\text{odd}} = \sum_{s=0}^{\infty} s_{2s+1} \hbar^{2s+1}$$  \hfill (7.11)

Notice that $s_{\text{even}}(x^\mu) = -s_{\text{even}}(x)$ and $s_{\text{odd}}(x^\mu) = s_{\text{odd}}(x)$

Lemma 7.2 The following equation holds:

$$ds_{\text{odd}} = -2v s_{\text{even}} s_{\text{odd}} .$$  \hfill (7.12)

Proof. This directly follows from equations (7.6). Namely, the statement of the lemma is written as

$$\sum_{k=-1}^{\infty} \hbar^{2k+1} ds_{2k+1} = -2v \left( \sum_{l=0}^{\infty} s_{2l} \hbar^{2l} \right) \left( \sum_{j=-1}^{\infty} s_{2j+1} \hbar^{2j+1} \right)$$  \hfill (7.13)

which is equivalent to (7.6) (the factor of 2 appears since the sum in (7.6) goes twice over all pairs of indices).

According to Theorem 7.17 of [2], the sum of Vörös symbols over a path between two zeros of $v$ lying in some rectangle bounded by critical trajectories of $v$ (understood in the sense of the Borel resummation scheme), tends to the Fock-Goncharov coordinate corresponding to the opposite diagonal of the same rectangle in the limit $\hbar \to 0$.

7.1 WKB expansion of the Yang-Yang function

According to [2] the Fock-Goncharov coordinates $\zeta_e$ have the following $\hbar$-expansion in terms of integrals of differentials $S_{2k+1}$ over the dual edge $l_e$ of the graph $\Sigma_Q^*$:

$$\zeta_e \sim \frac{1}{\hbar} P_{l_e} + \hbar \int_{l_e} s_1 + \hbar^3 \int_{l_e} s_3 + \ldots = \sum_{k=-1}^{\infty} \hbar^{2k+1} \int_{l_e} s_{2k+1}$$  \hfill (7.14)

Let us introduce the integrals of differentials $s_{2k+1}$ over the edges of the graph $\Sigma_Q^*$; since $s_{2k+1}, k \geq 0$ are singular but residue-less at the endpoints of the edges $l_e$, one defines these periods as $1/2$ of the period of $s_{2k+1}$ over the corresponding cycle in $H_-$. We denote

$$P_{l_e}^{(2k+1)} = \int_{l_e} s_{2k+1} .$$  \hfill (7.15)
Notice that $P_{l_e}^{(-1)} = P_{l_e}$ and coincide with the homological coordinate. Therefore we have now

$$\theta_{\text{hom}} = \frac{1}{\hbar} \sum_{v \in V(\Sigma)} \sum_{e \in e' \perp v} (P_{l_e} \delta P_{l_{e'}} - P_{l_{e'}} \delta P_{l_e})$$

(7.16)

and the $\hbar$-expansion of the difference between the potential $\theta_{\text{mon}}$ (5.6) and $\theta_{\text{hom}}$, using (7.14) can be written as follows:

$$\theta_{\text{mon}} - \theta_{\text{hom}} = \sum_{v \in V(\Sigma)} \sum_{e \in e' \perp v} \left( \sum_{d=-1}^{\infty} P_{l_e}^{(2d+1)} \hbar^{2d+1} \right) d \left( \sum_{k=-1}^{\infty} P_{l_{e'}}^{(2k+1)} \hbar^{2k+1} \right) + (e \leftrightarrow e')$$

$$= \sum_{v \in V(\Sigma)} \sum_{e \in e' \perp v} \sum_{r=-2}^{\infty} \hbar^{2r+2} \sum_{l+k=r} \left[ \left( P_{l_e}^{(2l+1)} \delta P_{l_{e'}}^{(2k+1)} - P_{l_{e'}}^{(2l+1)} \delta P_{l_e}^{(2k+1)} \right) + \left( P_{l_{e'}}^{(2l+1)} \delta P_{l_e}^{(2k+1)} - P_{l_e}^{(2l+1)} \delta P_{l_{e'}}^{(2k+1)} \right) \right] + (e \leftrightarrow e')$$

$$= \sum_{j=1}^{3g-3+n} \sum_{r=-2}^{\infty} \hbar^{2r+2} \sum_{l+k=r} \left[ \left( P_{a_j}^{(1)} \delta P_{b_j}^{(1)} - P_{b_j}^{(1)} \delta P_{a_j}^{(1)} \right) + \left( P_{a_j}^{(1)} \delta P_{b_j}^{(1)} - P_{b_j}^{(1)} \delta P_{a_j}^{(1)} \right) \right] + O(\hbar^2)$$

$$= d \left[ \sum_{j=1}^{3g-3+n} \left( P_{a_j}^{(1)} \delta P_{b_j}^{(1)} - P_{b_j}^{(1)} \delta P_{a_j}^{(1)} \right) \right] + 2 \sum_{j=1}^{3g-3+n} \left( P_{a_j}^{(1)} \delta P_{b_j}^{(1)} - P_{b_j}^{(1)} \delta P_{a_j}^{(1)} \right) + O(\hbar^2).$$

(7.17)

The second sum in (7.17) is, up to a multiplicative constant, the differential of $\log \tau_B$; indeed the equations for the tau-function (6.7) can be written as

$$3\pi i \, d \log \tau_B = \sum_{j=1}^{3g-3+n} \left( P_{a_j}^{(1)} \delta P_{b_j}^{(1)} - P_{b_j}^{(1)} \delta P_{a_j}^{(1)} \right).$$

(7.18)

The first sum can be computed using equations for the tau-function as follows

$$\sum_{j=1}^{3g-3+n} \left( P_{a_j}^{(1)} \delta P_{b_j}^{(1)} - P_{b_j}^{(1)} \delta P_{a_j}^{(1)} \right) = -\frac{3}{2\pi i} \sum_{j=1}^{3g-3+n} \left( P_{a_j} \frac{\partial}{\partial P_{a_j}} + P_{b_j} \frac{\partial}{\partial P_{b_j}} \right) \log \tau_B = -\frac{5(2g-2+n)}{48\pi i},$$

due to homogeneity property of the Bergman tau-function (see (4.7) of [3]); thus the first term in (7.17) vanishes. This implies the following

Theorem 7.1 The generating function $G_{YY}$ has the following asymptotics as $\hbar \to 0$

$$G_{YY} = 6\pi i \log \tau_B + \hbar^2 G_1 + \hbar^4 G_2 + \ldots$$

(7.19)

The remaining terms $G_1$, $G_2$, $\ldots$ of the asymptotic expansion in (7.19) satisfy a system of equations involving periods of the other differentials $s_{2k+2}v$; an explicit evaluation of these coefficients in terms of theta-functions presents an interesting problem.

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