A symmetric scalar constraint for loop quantum gravity

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In the framework of loop quantum gravity, we define a new Hilbert space of states which are solutions of a large number of components of the diffeomorphism constraint. On this Hilbert space, using the methods of Thiemann, we obtain a family of gravitational scalar constraints. They preserve the Hilbert space for every choice of lapse function. Thus adjointness and commutator properties of the constraint can be investigated in a straightforward manner. We show how the space of solutions of the symmetrized constraint can be defined by spectral decomposition, and the Hilbert space of physical states by subsequently fully implementing the diffeomorphism constraint.

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I. INTRODUCTION

In any known canonical formulation of general relativity, the general covariance of the theory is encoded in a number of constraints imposed on phase space. These constraints generate the hypersurface-deformation algebra under Poisson brackets, which is universal for generally covariant theories.

For a canonical quantization of general relativity, it is thus vital that the constrains are implemented in the quantum theory. For the case of loop quantum gravity (LQG, see \cite{1,2} for a review), the diffeomorphism constraints have been dealt with successfully, resulting in a Hilbert space \(\mathcal{H}_{\text{diff}}\) of quantum states that are invariant under spatial diffeomorphisms \cite{3}. The scalar constraints are technically much more demanding, as they have a more complicated action on the canonical variables employed in LQG.

Thiemann \cite{4,5}, based in part on ideas by Rovelli and Smolin \cite{6,7} and other researchers \cite{8}, and using the quantum volume operator \cite{9}, has devised a way to get a well defined action of the quantum scalar constraint \(\hat{C}(N)\) on \(\mathcal{H}_{\text{diff}}\). The invariance under spatial diffeomorphisms is very important, technically, as it allows to deal with the quantization of curvature terms in the constraint. Due to the presence of the lapse function \(N\), the operator \(\hat{C}(N)\) is not invariant under spatial diffeomorphisms, and hence does not preserve the Hilbert space of diffeomorphism invariant states. In fact, no Hilbert space which is invariant under \(\hat{C}(N)\) is known. This has turned out to be a substantial obstacle, as it precludes the straightforward discussion of adjointness relations, spectral resolutions, commutator algebra etc. of the \(\hat{C}(N)\)’s. Partial workarounds have been suggested, see for example \cite{10,11}, but the issues of self-adjointness and anomaly-freeness have never been settled in a straightforward way.

The solution we present has \(\hat{C}(N)\) act on states that are almost diffeomorphism invariant states. These new states can be thought of as being obtained from the spin network states of LQG by averaging over their images under diffeomorphisms which leave fixed sets of vertices in the spatial manifold invariant, schematically,

\[
\sqrt{\gamma} \propto \sum_{f \in \text{Diff}(\Sigma)_{v_1 \ldots v_n}} U_f \Psi_\gamma \quad (1.1)
\]

where \(\gamma\) is a graph with vertices \(v_1 \ldots v_n\), the sum is over elements of the stabilizer of the vertex set, and \(U\) is the unitary action of the diffeomorphisms. Thiemann \cite{4} defines a regulated operator \(\hat{C}_R(N)\) and then shows that the limit \(R \to 0\) is well defined in a relatively weak sense, namely on diffeomorphism invariant distributions. Technically, the regulator is similar to that in lattice gauge theory – curvature is approximated by traces of holonomies. When acting with the regulated operator

\[
on the new Hilbert space \( \mathcal{H}_{\text{fix}} \), the partial group averaging \([\overline{\cdot}]\) is enough to obtain a well defined operator in the limit of vanishing regulator. On the other hand, mostly due to the nature of spatial volume in loop quantum gravity, the resulting state will still belong to \( \mathcal{H}_{\text{fix}} \).

In the new Hilbert space, adjointness and commutator properties of the constraint can be investigated, and a physical Hilbert space defined by using spectral decomposition. We should note that there is a very interesting different line of thought, \([11–14]\), which also suggests that one should use a different Hilbert space to represent the (diffeomorphism and scalar) constraints. Those methods carry the additional benefit that they address the question of anomalies in a direct fashion.

The present article is organized as follows. In section II, we briefly recall some basic notions of loop quantum gravity. Section III A introduces the new Hilbert space, which is used in sections III C and III D, respectively, for the quantization of Euclidean and Lorentzian scalar constraints. The space of solutions to all constraints is discussed in section III E. A summary of anomalies in a direct fashion.

The main concern of the present work is the implementation in the quantum theory of the scalar constraint

\[ \delta F = \frac{\beta}{8\pi G} \int_{\Sigma} d^3x \left( \delta G \frac{\delta E^a}{\delta A^a_i(x)} - \delta A^a_i(x) \right) \delta E^a_i(x) \left( \delta F \right). \]

is the scalar constraint of vacuum gravity. \( F \) is the curvature of \( A \) and \( K \) is the extrinsic curvature of \( \Sigma \), which is a function of \( A \) and \( E \). For the Lorentzian gravity \( \sigma = -1 \). The Euclidean model of gravity is defined by \( \sigma = 1 \).

B. Kinematic Hilbert space

In the present section, we will quantize the kinematic phase space \( \Gamma \), resulting in a Hilbert space \( \mathcal{H} \). The quantum states in LQG are cylindrical functions of the variable \( A \), i.e., they depend on \( A \) only through finitely many holonomies

\[ h_\varepsilon[A] = \text{Pexp} \left( - \int_\varepsilon A \right) \]

where \( \varepsilon \) ranges over finite curves – we will also refer to them as \textit{edges} – in \( \Sigma \).

To spell out the definition we need to be precise about the meaning of “embedded graph” used in the definition of the cylindrical function. A graph \( \gamma \) embedded in \( \Sigma \) is a set of \( e \) edges (un-oriented) embedded in \( \Sigma \), \( \gamma = \{ e_1, \ldots, e_n \} \), of three types:

1. embedded closed interval (two end points),
2. immersed interval, such that the endpoints of the image coincide, and there is no more self-intersections (one endpoint),
3. embedded circle (no endpoints).

The end points of the edges of a given graph \( \gamma \) form the set \( \{ v_1, \ldots, v_m \} \) of the vertices of \( \gamma \). Intersection of two different edges is either empty or consists of vertices of \( \gamma \),

\[ e_I \cap e_J \subset \{ v_1, \ldots, v_m \}, \quad \text{whenever} \quad I \neq J. \]

In particular, each edge of the type 3 (circle) does not intersect any other edge of \( \gamma \).
Definition II.1. A function \( \Psi : A \to \Psi[A] \) is called cylindrical if there is a graph \( \gamma \) such that
\[
\Psi[A] = \psi(h_{e_1}[A], \ldots, h_{e_n}[A]) \quad (\mathrm{II.5})
\]
with a function function \( \psi : \text{SU}(2)^n \to \mathbb{C} \). Here, for every edge \( e_i \) we choose an orientation to define the parallel transport \( h_{e_i}[A] \). For each edge \( e_I \) of the type \( S \), we also choose arbitrary beginning-end point, and assume that
\[
\psi(h_1, \ldots, h_J, \ldots) = \psi(h_1, \ldots, g^{-1}h_Jg, \ldots) \quad \forall g \in \text{SU}(2).
\]

Some remarks about this definition are in order: Firstly, we understand \( \text{II.5} \) to include the case of \( n \) where \( \rho \) with a function function transport functions \( \Psi \) and \( \Psi' \)ial one.

To calculate the scalar product between two cylindrical functions \( \Psi \) and \( \Psi' \) defined by using graphs \( \gamma \) and \( \gamma' \), respectively, we find a refined graph \( \gamma'' \) such that both the functions can be written as
\[
\Psi[A] = \psi(h_{e_1}'[A], \ldots, h_{e_n}'[A]), \\
\Psi'[A] = \psi'(h_{e_1}''[A], \ldots, h_{e_n}''[A]).
\]
The scalar product is
\[
\langle \Psi | \Psi' \rangle = \int dg_1 \ldots dg_n \psi(g_1, \ldots, g_n)\psi'(g_1, \ldots, g_n). \quad (\text{II.6})
\]
We denote the space of all the cylindrical functions defined as above with a graph \( \gamma \) by \( \text{Cyl}_\gamma \) and, respectively, the space of all the cylindrical functions by \( \text{Cyl} \). The Hilbert space \( \mathcal{H}_{\text{kin}} \) is the completion
\[
\mathcal{H}_{\text{kin}} = \overline{\text{Cyl}}
\]
with respect to the Hilbert norm defined by \( \text{II.6} \).

Every cylindrical function \( \Psi \) is also a quantum operator
\[
(\Psi(\tilde{A})\Psi')[A] = \Psi[A]\Psi'[A]. \quad (\text{II.7})
\]
A connection operator \( \tilde{A} \) by itself is not defined.

The field \( E \) is naturally quantized as
\[
\tilde{E}_i^a \Psi[A] = \frac{\hbar}{i} \{ \Psi[A], E_i^a(x) \} = \frac{8\pi\beta}{i} \delta_{\text{kin}}^a \Psi[A]. \quad (\text{II.8})
\]

1 The existence is ensured by assuming suitable differentiability class of the edges. A save assumption is analyticity of the edges. Since analytic diffeomorphism are not local enough, in \( \text{II.6} \) we introduced a new category of manifolds we called \( \text{semi} \)-analytic. Briefly, \( \text{semi} \)-analytic means differentiability of a given finite order, and suitably defined piecewise analyticity. Then, all the edges and diffeomorphisms are assumed to be \( \text{semi} \)-analytic.

It turns into well defined operators in \( \mathcal{H}_{\text{kin}} \) after smearing along a 2-surface \( S \subset \Sigma \)
\[
\int \frac{1}{2} \xi^i \tilde{E}_i^a \epsilon_{abc} dx^b \wedge dx^c \quad f : S \to \text{su}(2)
\]
where \( f \) may involve parallel transports \( \xi \):
\[
f(x) = \tilde{f}(x)(h_{pxx_0}h_{px_0})^{-1}
\]
where \( S \ni x \mapsto p_{xx_0} \) assigns to each point \( x \in S \) a path \( p_{xx_0} \) connecting a fixed point \( x_0 \) to \( x \), \( \xi \in \text{su}(2) \), and \( f : S \to \mathbb{R} \).

There is an orthogonal decomposition of \( \mathcal{H}_{\text{kin}} \) with respect to subspaces labelled by the graphs defined above. Given a graph \( \gamma \), denote by \( \tilde{\mathcal{H}}_{\gamma} \) the subspace of \( \mathcal{H}_{\text{kin}} \) defined by the cylindrical functions \( \text{II.5} \) corresponding to \( \gamma \). Whenever a graph \( \gamma \) can be obtained from a graph \( \gamma' \) by a sequence of the following steps:

- cutting an edge \( e_I' \) into two: \( e_I' = e_J \circ e_K \)
- adding a new edge: \( \gamma = \{ e_1', \ldots, e_{n-1}', e_n \} \), \( \gamma' = \{ e_1', \ldots, e_{n-1}' \} \)

then
\[
\tilde{\mathcal{H}}_{\gamma'} < \tilde{\mathcal{H}}_{\gamma},
\]
that is, \( \tilde{\mathcal{H}}_{\gamma'} \) is a proper subset of \( \tilde{\mathcal{H}}_{\gamma} \). Hence,
\[
\mathcal{H}_{\text{kin}} = \bigcup_\gamma \tilde{\mathcal{H}}_{\gamma},
\]
but this is not an orthogonal decomposition.

Define \( \Psi \in \tilde{\mathcal{H}}_{\gamma} \) to be a proper element of \( \tilde{\mathcal{H}}_{\gamma} \) if this is true that
\[
\Psi \perp \tilde{\mathcal{H}}_{\gamma'} \iff \tilde{\mathcal{H}}_{\gamma} < \tilde{\mathcal{H}}_{\gamma'}.
\]

Given \( \gamma \), the proper states form a subspace \( \mathcal{H}_\gamma \subset \tilde{\mathcal{H}}_{\gamma} \). The family \( (\mathcal{H}_\gamma)_\gamma \) does provide an orthogonal decomposition
\[
\mathcal{H}_{\text{kin}} = \bigoplus_\gamma \mathcal{H}_\gamma. \quad (\text{II.9})
\]

This decomposition can be also applied directly to the cylindrical functions
\[
\text{Cyl}_\gamma := \text{Cyl} \cap \mathcal{H}_\gamma \subset \text{Cyl}, \quad (\text{II.10})
\]
\[
\text{Cyl} = \bigoplus \text{Cyl}_\gamma \quad (\text{II.11})
\]

III. QUANTUM SCALAR CONSTRAINT

The scalar constraint
\[
C(N) = \int d^3x N(x)C(x)
\]
has not been successfully quantized in the kinematical Hilbert space \( \mathcal{H}_{\text{kin}} \) of the previous section. We will introduce now a new Hilbert space which admits quantum operators \( \tilde{C}(N) \).
A. A new Hilbert space

The idea of the new Hilbert space we will introduce now is to average each of the subspaces \( \mathcal{H}_\gamma \) with respect to all the diffeomorphisms \( \text{Diff}(\Sigma) \) which act trivially on the set \( v_1, \ldots, v_m \) of the vertices of \( \gamma \) \(^2\)

Every \( f \in \text{Diff}(\Sigma) \) defines a unitary operator \( U_f : \mathcal{H}_{\text{kin}} \to \mathcal{H}_{\text{kin}} \),

\[
U_f \Psi[A] = \Psi[f^*A].
\]

Given a graph \( \gamma \) consisting of edges and vertices

\[
E(\gamma) = \{e_1, \ldots, e_n\}, \quad \text{Vert}(\gamma) = \{v_1, \ldots, v_m\},
\]

the action of \( U_f \) on a cylindrical function (II.5) reads

\[
U_f \Psi[A] = \psi(h_{f(e_1)}(A), \ldots, h_{f(e_n)}(A)), \quad \text{(III.1)}
\]

where for the parallel transport along each edge \( f(e_1) \) we choose the orientation induced by the map \( f \) and by the orientation of \( e_1 \) chosen in (II.5). Denote by \( \text{TDiff}(\Sigma)_\gamma \) the subset of \( \text{Diff}(\Sigma) \) which consists of all the diffeomorphisms acting trivially in \( \mathcal{H}_\gamma \). On the other hand, for a general \( f \in \text{Diff}(\Sigma) \), we have a unitary isomorphism

\[
U_f : \mathcal{H}_\gamma \to \mathcal{H}_{f(\gamma)}.
\]

The maps \( \mathcal{H}_\gamma \to \mathcal{H}_{\text{kin}} \) obtained by the diffeomorphisms \( \text{Diff}(\Sigma) \) are in the 1-1 correspondence with the elements of the quotient

\[
D_\gamma := \text{Diff}(\Sigma)_{\text{Vert}(\gamma)}/\text{TDiff}(\Sigma)_\gamma. \quad \text{(III.2)}
\]

Still, \( D_\gamma \) is a non-compact set and we do not know any probability measure on it. Therefore it is not surprising, that given \( \Psi \in \mathcal{H}_\gamma \), the result of the averaging will not, in general, be an element of \( \mathcal{H}_{\text{kin}} \). However, it will be well defined as an element of the space \( \text{Cyl}^* \) dual to \( \text{Cyl} \). Given \( \Psi \in \mathcal{H}_\gamma \), we turn it into \( (\Psi | e) \in \text{Cyl}^* \),

\[
(\Psi | : \Psi' \mapsto (\Psi | \Psi'),
\]

and average in \( \text{Cyl}^* \),

\[
\eta(\Psi) = \frac{1}{N_\gamma} \sum_{\{f\} \in D_\gamma} \left| U_f \Psi \right|, \quad \text{(III.3)}
\]

\( N_\gamma \) is a normalization factor that will be determined in a moment.

Lemma III.1. \( \eta(\Psi) \) is a well defined linear functional

\[
\eta(\Psi) : \text{Cyl} \to \mathbb{C}
\]

which is invariant under \( \text{Diff}(\Sigma)_{\text{Vert}(\gamma)} \).

\(^2\) The general idea of the averaging with respect to the diffeomorphisms has been already used in LQG, see [2]. We apply it now for our purposes.

Proof. Each term in the sum (III.3) is independent of the choice of a representative \( f \in \{f\} \) because the action of \( \text{TDiff}(\Sigma) \) on \( \mathcal{H}_\gamma \) is trivial. Given \( \Psi' \in \text{Cyl} \), any finite set of terms in the sum is not zero. Hence, the sum is finite. The sum is invariant, because if \( \{f_i\} \) is a set of representatives for the classes \( D_\gamma \), then so is \( \{f_0 f_i\}, f_0 \in \text{Diff}(\Sigma)_{\text{Vert}(\gamma)} \).

We define the map

\[
\mathcal{H}_\gamma \ni \Psi \mapsto \eta(\Psi) \in \text{Cyl}^*
\]

every for embedded graph \( \gamma \), and extend it by linearity to the algebraic orthogonal sum

\[
\eta : \bigoplus_{\gamma} \mathcal{H}_\gamma \to \text{Cyl}^*. \quad \text{(III.4)}
\]

Notice, that \( \text{Cyl} \subset \bigoplus_{\gamma} \mathcal{H}_\gamma \), therefore \( \text{Cyl} \) is in the domain of the averaging map \( \eta \).

Definition III.1. The new Hilbert space \( \mathcal{H}_{\text{ext}} \) is defined as the completion

\[
\mathcal{H}_{\text{ext}} = \overline{\eta(\text{Cyl})} \quad \text{(III.5)}
\]

under the norm induced by the scalar product

\[
(\eta(\Psi) | \eta(\Psi')) := \eta(\Psi)(\Psi'). \quad \text{(III.6)}
\]

One can check [3] that (III.6) has indeed all the properties of a scalar product, and hence \( \mathcal{H}_{\text{ext}} \) really is a Hilbert space. It has an orthogonal decomposition that is reminiscent of (II.9):

Lemma III.2. Let \( \text{FS}(\Sigma) \) be the set of finite subsets of \( \Sigma \). Then

\[
\mathcal{H}_{\text{ext}} = \bigoplus_{V \in \text{FS}(\Sigma)} \mathcal{H}_V \quad \text{(III.7)}
\]

\[
\mathcal{H}_V = \bigoplus_{\gamma(V)} \mathcal{H}_{[\gamma(V)]} \quad \text{(III.8)}
\]

\[
\mathcal{H}_{[\gamma(V)]} = \eta(\mathcal{H}_{[\gamma(V)]}), \quad \text{(III.9)}
\]

where the second sum is over the diffeomorphism \( \gamma \) with vertex set \( V = \text{Vert}(\gamma) \).

Proof. Both decompositions follow from definition (no spurious vertices) and the orthogonality of the subspaces \( \mathcal{H}_\gamma \), together with (III.3). \( \square \)

To understand the structure of each of the subspaces \( \mathcal{H}_\gamma \), decompose the space \( \mathcal{H}_\gamma \) into the kernel of \( \eta \), and the orthogonal completion

\[
\mathcal{H}_\gamma = \text{Ker}(\eta) \cap \mathcal{H}_\gamma \oplus S_\gamma. \quad \text{(III.10)}
\]

The orthogonal completion \( S_\gamma \) consists of all the \( \Psi \) such that for every \( f \in \text{Diff}(\Sigma)_{\text{Vert}(\gamma)} \)

\[
f(\gamma) = \gamma \Rightarrow U_f \Psi = \Psi. \quad \text{(III.11)}
\]
In other words, elements of $S_\gamma$ are invariant with respect to the symmetry group

$$\text{Sym}_\gamma = \{ f \in \text{Diff}(\Sigma)_{(x_1, \ldots, x_m)} : f(\gamma) = \gamma \} / \text{TDiff}(\Sigma)_\gamma$$

(III.12)
of the graph $\gamma$. In fact, it is straightforward to show the following

**Lemma III.3.** The map

$$\eta : S_\gamma \to \eta(\mathcal{H}_\gamma)$$

is a unitary embedding modulo the an overall factor $|\text{Sym}_\gamma|/N_\gamma$, where $N_\gamma$ is the free constant in the definition (III.3) of $\eta$.

In the following, we set

$$\frac{|\text{Sym}_\gamma|}{N_\gamma} = 1.$$ 

Finally, we point out that $\mathcal{H}_{\text{vtx}}$ carries a natural action of $\text{Diff}(\Sigma)$, which we will also denote by $U$. It is defined by

$$U_f \eta(\Psi) := \eta(U_f \Psi), \quad f \in \text{Diff}(\Sigma) \quad (\text{III.13})$$

and extension by continuity. A short calculation shows

**Lemma III.4.** $U_f$ as in (III.13) is unitary and maps $\mathcal{H}_V$ to $\mathcal{H}_{f(V)}$ in the decomposition (III.7).

@B. Lifting operators to $\mathcal{H}_{\text{vtx}}$

In the kinematical Hilbert space $\mathcal{H}$ one often considers quantum operators defined on the domain $\text{Cyl}$, and such that

$$\bar{\mathcal{O}} : \text{Cyl} \to \text{Cyl}.$$ 

(III.14)

Each of them passes, by duality, to an operator $\bar{\mathcal{O}}^*$ defined in $\text{Cyl}^*$. In particular, it is defined on $\eta(\text{Cyl}) \subset \mathcal{H}_{\text{vtx}}$. However, while $\bar{\mathcal{O}}^*$ maps $\eta(\text{Cyl})$ into $\text{Cyl}^*$, the image is not necessarily in $\mathcal{H}_{\text{vtx}}$. Importantly, sometimes the domain is actually mapped back into $\mathcal{H}_{\text{vtx}}$. Then $\bar{\mathcal{O}}^*$ becomes an operator in $\mathcal{H}_{\text{vtx}}$.

**Lemma III.5.** Suppose, an operator as in (III.14) is the form

$$\bar{\mathcal{O}}(N) = \sum_{x \in \Sigma} N(x) \bar{\mathcal{O}}_x,$$

where the $\bar{\mathcal{O}}_x$ are operators that have a local action for all $x \in \Sigma$

$$U_f \bar{\mathcal{O}}_x = \bar{\mathcal{O}}_x U_f \quad \text{for } f \in \text{Diff}_x$$

(III.15)

and $\bar{\mathcal{O}}_x|_{\mathcal{H}_x} = 0$ for $x \notin \text{Vert}(\gamma)$.

Then $\bar{\mathcal{O}}^*$ is an operator on $\mathcal{H}_{\text{vtx}}$.

**Proof.** Using the conditions of locality (III.15, III.16), one can pull the action of $\bar{\mathcal{O}}$ through the average (III.3), resulting in an element of $\text{Cyl}^*$ of the same form.

There are several important operators which have this property: the quantum volume element smeared against an arbitrary function $N \in C(\Sigma)$,

$$\tilde{V}(N) = \int d^3 x N(x) \sqrt{|\text{det} E|} = \sum_{x \in \Sigma} N(x) \tilde{V}_x$$

the Gauss constraint operator (for $\Lambda \in C(\Sigma, \text{su}(2))$)

$$\int d^3 x \Lambda^i D_a E^a_i = \sum_{x \in \Sigma} \Lambda^i(x) \tilde{\sigma}_{tx},$$

and also the integral of the Ricci scalar operator

$$\tilde{R}(N) = \int d^3 x N(x) \sqrt{|\text{det} E|} R(x) = \sum_{x \in \Sigma} N(x) \tilde{R}_x.$$}

which has recently been introduced [13]. Our quantum scalar constraint operator will take a similar form in $\mathcal{H}_{\text{vtx}}$, although it will not be well defined in $\text{Cyl}$ itself. It will be defined directly in $\mathcal{H}_{\text{vtx}}$.

**C. Scalar constraint operator for Euclidean gravity**

The Euclidean scalar constraint in the absence of matter can be obtained from (II.3) by setting the metric signature $\sigma$ to 1. For the choice of $\beta = 1$, the expression simplifies because the second term drops out. The remaining term, in Thiemann form, is proportional to

$$C_{\text{Euc}}(N) = \frac{-2}{(8\pi G)^2 \beta^2} \int d^3 x \epsilon^{abc} \text{Tr} F_{ab}(x) \{ A_b(x), V(N) \}$$

(III.17)

where

$$V(N) = \int d^3 x N(x) \sqrt{\epsilon}$$

and $N : \Sigma \to \mathbb{R}$ is an arbitrary lapse function.

In the Lorentzian case, the second term in (II.3) can not be made to vanish for real Ashtekar variables, thus $C_{\text{Euc}}(N)$ is only one part of $C(N)$. But even in this case, $C_{\text{Euc}}(N)$ plays a vital part in the quantization of the whole constraint [4].

To quantize the Euclidean scalar constraint, we express $F$ by the parallel transports along suitable loops $\alpha_\sigma^e$ and we express $A$ in terms of parallel transport along suitable curves $s_\sigma^e$,

$$C_{\text{Euc}}(N)(A, E) = \sum_{\sigma} B_\sigma \left( \rho_\sigma(h_{s_\sigma^e}) - \rho_\sigma(h_{(s_\sigma^e)^{-1}}) \right) \cdot \text{Tr} \left( \rho_\sigma(h_{(s_\sigma^e)^{-1}}) \{ \rho_\sigma(h_{s_\sigma^e}), V(N) \} \right)$$

(III.18)

The $B_\sigma$ are $\epsilon$-independent constants and the $\rho_\sigma$ representations of $\text{SU}(2)$. The loops $\alpha$ and curves $s$ approach
points in the limit \( \epsilon \to 0 \). Moreover, \( B_\sigma, \rho_\sigma, \alpha_\sigma \) and \( s_\sigma \) are chosen such that, the entire expression converges to

\[
\lim_{\epsilon \to 0} \mathcal{C}_\text{Euc}^\epsilon(N)(A, E) = \mathcal{C}_\text{Euc}(N)(A, E) \tag{III.19}
\]

for every smooth \((A, E)\).

For every fixed value of \( \epsilon \), the operator

\[
\mathcal{C}_\text{Euc}^\epsilon(N) = \frac{1}{i\hbar} \sum_{\sigma} B_\sigma \text{Tr}
\left( \left( \rho_\sigma \left( \overline{h_{\alpha_\sigma}} \right) \right) - \rho_\sigma \left( \overline{h_{\alpha_\sigma}}^{-1} \right) \cdot \rho_\sigma \left( \overline{h_{\alpha_\sigma}}^{-1} \right) \right), \tag{III.20}
\]

is well defined in the kinematic Hilbert space \( \mathcal{H}_\text{kin} \) in the domain \( \text{Cyl} \). However, the limit \( \epsilon \to 0 \) does not exist. Also, before taking the limit, for a constant \( \epsilon \), the operator is not diffeomorphism covariant. The finite loops break the covariance. Remarkably, there is a way out. First, we improve the regularization. For that we apply the decomposition [(II.9)], and adapt the regulated expression to each subspace \( \mathcal{H}_\sigma \) independently. We will do it below in such a way, that for \( (\Psi_1 | \in \eta(\text{Cyl}) \in \mathcal{H}_{\text{vtx}} \) and \( \Psi_2 \in \text{Cyl}_v \subset \mathcal{H}_\gamma \), the number

\[
(\Psi_1 | (\mathcal{C}^\epsilon_{\text{Euc}}(N) \Psi_2) \tag{III.21}
\]

will be \( \epsilon \)-independent – either because it vanishes, or due to the symmetries of \( (\Psi_1 | \in \mathcal{H}_{\text{vtx}} \). In this way, we will define the limit

\[
\mathcal{C}^\epsilon_{\text{Euc}}(N) := \lim_{\epsilon \to 0} \left( \mathcal{C}^\epsilon_{\text{Euc}}(N) \right)^* \tag{III.22}
\]
as an operator on \( \text{Cyl}^\epsilon \), by setting

\[
(\mathcal{C}^\epsilon_{\text{Euc}}(N)(\Psi_1))(\Psi_2) := \lim_{\epsilon \to 0} (\Psi_1 | (\mathcal{C}^\epsilon_{\text{Euc}}(N) \Psi_2). \tag{III.23}
\]

Note that this involves some abuse of notation, as \( \mathcal{C}^\epsilon_{\text{Euc}} \) is not the dual of any operator defined in \( \text{Cyl} \).

We explain now, in what way we achieve the \( \epsilon \)-independence of (III.21). We are making the same assumptions about the loop-path assignment

\[
(\gamma, v) \mapsto \{\alpha_\sigma, s_\sigma | \sigma = 1, 2, \ldots\} \tag{III.24}
\]
as in Sec. VI.C of [1]. For each \( \sigma \), the pair \( s_\sigma \) and \( \alpha_\sigma \) is based at a point \( v \in \Sigma \). If \( v \) is not a vertex of \( \gamma \), then the corresponding term of the operator vanishes. Here, \( v \in \text{Vert}(\gamma) \), and \( \sigma \) labels the pairs of loops and segments based at \( v \). As a result, the action of the regulator-dependent operator \( \mathcal{C}^\epsilon_{\text{Euc}}(N) \) defined on \( \mathcal{H}_\gamma \), takes the following form,

\[
\mathcal{C}^\epsilon_{\text{Euc}}(N) = \sum_{v \in \text{Vert}(\gamma)} N(v) \sum_{\sigma} \mathcal{C}^\epsilon_{\gamma v \sigma}, \tag{III.25}
\]

where

\[
\mathcal{C}^\epsilon_{\gamma v \sigma} : \text{Cyl}_\gamma \to \text{Cyl}_{\gamma \cup (\alpha_\sigma)} \subset \mathcal{H}_{\gamma \cup (\alpha_\sigma)}. \tag{III.26}
\]

In other words, the operator adds the loops \( \alpha_\sigma \) to \( \gamma \), while the paths \( s_\sigma \), possibly new elements of the graph \( \gamma \), do not change the graph, regarded as a subset in \( \Sigma \). Every loop \( \alpha_\sigma \) appearing in (III.26) begins and ends at a vertex of \( \gamma \), does not intersect \( \gamma \) in any other point, and does not have self intersections. Hence \( \alpha_\sigma \) becomes an edge (of type 2) of the new graph \( \gamma \cup \{\alpha_\sigma\} \). One of the two key properties we ask of the loop assignment (III.24) is that for every \( \epsilon_1 \) and \( \epsilon_2 \) there is \( f \in \text{Diff}_{\text{Vert}(\gamma)} \) such that

\[
\mathcal{C}^\epsilon_{\gamma v \sigma} = U_f \mathcal{C}^\epsilon_{\gamma v \sigma}. \tag{III.27}
\]

This is the property that ensures the independence of (III.21) of \( \epsilon \):

\[
(\Psi_1 | (\mathcal{C}^\epsilon_{\gamma v \sigma} \Psi_2) = (\Psi_1 | (U_f \mathcal{C}^\epsilon_{\gamma v \sigma} \Psi_2). \tag{III.28}
\]

Consequently, the limit (III.23) on \( \eta(\mathcal{H}_\gamma) \) and, by linearity over the \( \gamma \)-sectors, on \( \eta(\text{Cyl}) \subset \mathcal{H}_{\text{vtx}} \), can be taken. However, the result is not necessarily an element of \( \mathcal{H}_{\text{vtx}} \). For example, in general it may not be \( \text{Diff}(\Sigma) )_{\text{Vert}(\gamma)} \)-invariant. To ensure the invariance, we need to coordinate the assignments (III.24) for graphs that are equivalent under \( \text{Diff}(\Sigma) )_{\text{Vert}(\gamma)} \). Also, we want the resulting operator valued distribution to be invariant with respect to all the \( \text{Diff}(\Sigma) \). Hence, as our second key property, we ask the following: Given a graph \( \gamma \), and \( f_1 \in \text{Diff}(\Sigma) \) then there exists \( f_2 \in \text{Diff}(\Sigma) \) such that

\[
\mathcal{C}^\epsilon_{f_1(\gamma)}(v_1) \Psi_1 = U_{f_2} \mathcal{C}^\epsilon_{\gamma v \sigma}. \tag{III.29}
\]

A simple calculation then completes the proof following:

**Proposition III.6.** Let \( \mathcal{C}^\epsilon_{\text{Euc}}(N) \) be an operator on \( \text{Cyl}^\epsilon \) obtained as a limit

\[
\mathcal{C}^\epsilon_{\text{Euc}}(N) := \lim_{\epsilon \to 0} (\mathcal{C}^\epsilon_{\text{Euc}}(N))^*,
\]

where \( \mathcal{C}^\epsilon_{\text{Euc}}(N) \) is of the form (III.25) satisfying

(a) covariance (III.27) under changes of \( \epsilon \),

(b) covariance (III.29) under \( \text{Diff}_\gamma \).

Then \( \mathcal{C}^\epsilon_{\text{Euc}}(N) \) preserves \( \mathcal{H}_{\text{vtx}} \), i.e.,

\[
\mathcal{C}^\epsilon_{\text{Euc}}(N)(\Psi | \in \mathcal{H}_{\text{vtx}}, \tag{III.30}
\]

and it is diffeomorphism covariant, i.e.,

\[
\mathcal{C}^\epsilon_{\text{Euc}}(N)U_f = U_f \mathcal{C}^\epsilon_{\text{Euc}}(f^{-1} N). \tag{III.31}
\]

Above, the assumption made in [1] and adopted here, that in (III.24) the assigned loops \( \alpha_\sigma \) do not overlap \( \gamma \) is relevant for (III.30). Otherwise, the operator could produce non-normalizable elements of \( \text{Cyl}^\epsilon \). We proceed to discuss some further properties of \( \mathcal{C}^\epsilon_{\text{Euc}}(N) \) under the
assumptions of the previous proposition. Firstly, note that we can write

$$\tilde{C}_{\text{Euc}}(N) = \sum_{x \in \Sigma} N(x) \tilde{C}_{\text{Euc},x}$$  \hspace{1cm} (III.32)

where $\tilde{C}_{\text{Euc},x}$ has the following properties: It preserves the spaces $\eta(\text{Cyl})_V := \eta(\text{Cyl}) \cap H_V$ for $V \in \text{FS}(\Sigma)$, 

$$\tilde{C}_{\text{Euc},x} \eta(\text{Cyl})_V \subseteq \eta(\text{Cyl})_V.$$  

This makes clear that the operator $\tilde{C}_{\text{Euc}}(N)$ preserves the decomposition (III.7) of $H_{\text{kin}}$ into sectors labeled by finite subsets $V$ of $\Sigma$. Moreover, 

$$\tilde{C}_{\text{Euc},x} |_{\eta(\text{Cyl})_V} = 0,$$  

unless $x \notin V$.

Also, $\tilde{C}_{\text{Euc},x}$ is covariant, 

$$U_f \tilde{C}_{\text{Euc},x} U_{f^{-1}} = \tilde{C}_{\text{Euc},f(x)},$$

for every $f \in \text{Diff}(\Sigma)$.

Finally, $\tilde{C}_{\text{Euc}}(N)$ does not preserve the decomposition (III.8). Rather, by the duality to (III.26), the operator annihilates the loops created by each $\tilde{C}_{\gamma \nu \sigma}$.

The operator $\tilde{C}_{\text{Euc}}(N)$ is not symmetric. But the Hermitian adjoint

$$(\tilde{C}_{\text{Euc}}(N))^\dagger$$

is well defined. A typical proposal for a symmetric quantum scalar constraint operator is

$$\tilde{C}_{\text{Euc}}(N) := \frac{1}{2} \left( \tilde{C}_{\text{Euc}}(N) + (\tilde{C}_{\text{Euc}}(N))^\dagger \right).$$  \hspace{1cm} (III.33)

The (essential) self-adjointness is an open issue.

D. The quantum Lorentzian scalar constraint of matter free gravity

To define the quantum scalar constraint operator of the Lorentzian gravity and with a general value of the Barbero-Immirzi parameter $\beta$, we go back to the classical theory. The gravitational part of the scalar constraint is

$$C(N) = \sqrt{\beta} C_{\text{Euc}}(N) - \frac{1}{2} \left( 1 + \beta^2 \right) \left( 8 \pi G \right)^4 \beta^6 T(N)$$

where $T$ is written in a way compatible with the LQG as follows

$$T(N) = -2 \int d^3 x e^{A_{\alpha}} \text{Tr}(\{ A_{\alpha}(x), \{ C_{\text{Euc}}(1), V(1) \}) \cdot \{ A_{\alpha}(x), \{ C_{\text{Euc}}(1), V(1) \} \{ A_{\alpha}(x), V(N) \}))$$  \hspace{1cm} (III.34)

As before, for every subspace $H_\gamma$ in the decomposition (III.9) we use the family of paths $s^\gamma_\sigma$ introduced above, and a regulated classical expression

$$T^\epsilon(N) = \sum_{\sigma, \sigma', \sigma''} e^{\sigma \sigma' \sigma''} \text{Tr}(\{ H^{-1}_\sigma \{ H_\sigma, K \} \cdot H^{-1}_\sigma \{ H_\sigma, K \} H_\sigma \{ H^{-1}_\sigma, V(N) \}))$$  \hspace{1cm} (III.35)

with

$$K := \{ C_{\text{Euc}}(1), V(1) \}, \quad H_\sigma := \rho(h_{\sigma \sigma'})$$

such that as $\epsilon \to 0$, the paths are shrunk, the constants $e^{\sigma \sigma' \sigma''}$ are independent of $\epsilon$, and

$$\lim_{\epsilon \to 0} T^\epsilon(N)[A, E] = T(N)[A, E].$$  \hspace{1cm} (III.36)

We introduce that regulation for every graph $\gamma$. Next, in the kinematical $H_{\text{kin}}$ we define a quantum operator\footnote{The path assignment $\sigma \mapsto s^\gamma_\sigma$ obtained by ignoring the loops $\alpha$ in the assignments $\sigma \mapsto \alpha$, $s^\gamma_\sigma$ used previously may assign the same segment to two different $\sigma \neq \sigma'$. That may be compensated by choosing suitable values for the constants $e^{\sigma \sigma' \sigma''}$.} $\tilde{T}^\epsilon(N) : \text{Cyl} \to \text{Cyl}^\ast$ via

$$\tilde{T}^\epsilon(N) = \frac{1}{i\hbar} \sum_{A, E} e^{\sigma \sigma' \sigma''} \text{Tr}(\{ H^{-1}_\sigma \{ H_\sigma, \tilde{R}^\ast \} \cdot H^{-1}_\sigma \{ H_\sigma, \tilde{R}\} H_\sigma \{ H^{-1}_\sigma, V(N) \}))$$  \hspace{1cm} (III.37)

with

$$\tilde{R}^\ast := \frac{1}{i\hbar} [\tilde{C}_{\text{Euc}}(1), \tilde{V}(1)].$$

As in the case of the Euclidean quantum gravitational constraint,

$$\eta(\Psi) (\tilde{T}^{\epsilon - 1}(N) \Psi') = \eta(\Psi) (\tilde{T}^{\epsilon - 1}(N) \Psi')$$

for $\Psi, \Psi', \in \text{Cyl}$, hence the limit is well defined as an operator

$$\tilde{T}^\ast(N) : \eta(\text{Cyl}) \to \text{Cyl}^\ast.$$  

If the constants $e^{\sigma \sigma' \sigma''}$ are assigned to each graph in a $\text{Diff}(\Sigma)_{\text{Vert}(\gamma)}$ invariant way, then analogously to the Euclidean case,

$$\tilde{T}^\ast(N) \eta(\text{Cyl}) \subseteq \eta(\text{Cyl}),$$

\footnote{Notice, that in the classical regulated expression $T^\epsilon(N)$ we have $C_{\text{Euc}}$ whereas in the quantum regulated expression we use $\tilde{C}_{\text{Euc}}$. If in the classical expression we replaced $C_{\text{Euc}}$ by $\tilde{C}_{\text{Euc}}$, then (III.36) would not be true. On the other hand, we can not use $\tilde{C}_{\text{Euc}}$ in the quantum $\tilde{T}^\ast(N)$, because the expression would not make sense. This is a drawback of the regularization procedure of the Lorentzian part of the scalar constraint.}
hence \( \hat{T}^*(N) \) becomes an operator in \( \mathcal{H}_{\text{vtx}} \) with domain \( \eta(\text{Cyl}) \). The operator has a similar structure as \( \bar{C}_{\text{Euc}}^*(N) \):

\[
\hat{T}^*(N) = \sum_{x \in \Sigma} N(x) \hat{T}_x^*,
\]

where, for any \( V \in \text{FS}(\Sigma) \),

\[
\hat{T}_x^* : \mathcal{H}_V \rightarrow \mathcal{H}_V,
\]

and

\[
\hat{T}_x^*|_{\mathcal{H}_V} = 0, \quad \text{unless} \quad x \in V.
\]

If the constants \( e^{\sigma \sigma'} \) are assigned to each graph in a \( \text{Diff}(\Sigma) \)-invariant way, then

\[
U_f \hat{T}_x^* U_f^{-1} = \hat{T}_{f(x)}^*,
\]

that is the distribution \( x \mapsto \hat{T}_x^* \) is \( \text{Diff}(\Sigma) \)-invariant.

If the operator \( (\bar{C}_{\text{Euc}}^*)^I \) exists, then so does \( (\hat{T}^*(N))^I \). In that case we can define a symmetric operator

\[
\hat{T}(N) = \frac{1}{2} \left( \hat{T}^*(N) + (\hat{T}^*(N))^I \right).
\]

(III.38)

The final result is a quantum gravitational scalar constraint operator

\[
\bar{C}(N) = \sqrt{\beta} \bar{C}_{\text{Euc}}(N) - 2 \left( \frac{1 + \beta^2}{8 \pi G} \right) \hat{T}(N) \quad \text{(III.39)}
\]

defined in \( \mathcal{H}_{\text{vtx}} \) in the domain \( \eta(\text{Cyl}) \). As a consequence of the properties of \( \bar{C}_{\text{Euc}}(N) \) and \( T(N) \), it is again local and covariant,

\[
\bar{C}(N) = \sum_{x \in \Sigma} N(x) \bar{C}_x,
\]

\[
\bar{C}_x \mathcal{H}_V \subseteq \mathcal{H}_V
\]

\[
\bar{C}_x|_{\mathcal{H}_V} = 0, \quad \text{unless} \quad x \in V,
\]

\[
U_f \bar{C}_x U_f^{-1} = \bar{C}_{f(x)}.
\]

### E. Solutions to the quantum constraints.

Suppose the quantum constraint operators \( \bar{C}_x, \ x \in \Sigma \), are essentially self adjoint. Since

\[
[\bar{C}_x, \bar{C}_{x'}] = 0,
\]

every subspace \( \mathcal{H}_{\{x_1, \ldots, x_m\}} \) can be decomposed using the spectral decomposition of the operators \( \bar{C}_{x_I}, \ I = 1, \ldots, m, \)

\[
\mathcal{H}_{\{x_1, \ldots, x_m\}} = \int_0^\infty d\mu(c_1) \ldots d\mu(c_m) \mathcal{H}_{\{x_1, \ldots, x_m\}}^{c_1, \ldots, c_m}.
\]

The elements of the subspace

\[
\mathcal{H}_{\{x_1, \ldots, x_m\}}^{0, \ldots, 0}
\]

are solutions to the quantum scalar constraint. If \( (c_1, \ldots, c_m) = (0, \ldots, 0) \) is a point of the measure zero, then some continuity in the map

\[
(c_1, \ldots, c_m) \mapsto \mathcal{H}_{\{x_1, \ldots, x_m\}}^{c_1, \ldots, c_m}
\]

is used to determine individual spaces \( \mathcal{H}_{\{x_1, \ldots, x_m\}}^{0, \ldots, 0} \). In the general case,

\[
\mathcal{H}_{\{x_1, \ldots, x_m\}}^{0, \ldots, 0} \in (\eta(\text{Cyl})_{x_1, \ldots, x_m})^*.
\]

The elements are (finite or formal infinite) linear combinations

\[
\mathcal{H}_{\{x_1, \ldots, x_m\}}^{0, \ldots, 0} \ni \Psi = \sum_{[\gamma(V)]} \eta(\Psi^\gamma) \equiv
\]

where \([\gamma(V)]\) ranges over the set of \( \text{Diff}(\Sigma)_V \) equivalence classes of the graphs with vertices \( V \). In fact, there is a natural embedding

\[
\mathcal{H}_{\{x_1, \ldots, x_m\}}^{0, \ldots, 0} \rightarrow \text{Cyl}^*.
\]

To turn elements of \( \mathcal{H}_{\{x_1, \ldots, x_m\}}^{0, \ldots, 0} \) into solutions to the quantum Diffeomorphism constraint we average them with respect to the remaining diffeomorphisms

\[
\Psi = \sum_{[\gamma(V)]} \eta(\Psi^\gamma) = \sum_{[\gamma(V)]} \sum_{[f]} \eta(U_f \Psi^\gamma)
\]

where

\[
[f] \in \text{Diff}(\Sigma)/\text{Diff}(\Sigma)_{\text{Vert}(\gamma)}.
\]

The result is the subspace

\[
\tilde{\eta}(\mathcal{H}_{\{x_1, \ldots, x_m\}}^{0, \ldots, 0}) \subseteq \text{Cyl}^*,
\]

and its elements are \( \text{Diff}(\Sigma) \)-invariant. On the other hand, the operator \( \bar{C}(N) \) we have defined can be applied directly on each \( \text{Diff}(\Sigma) \) invariant element of \( \text{Cyl}^* \). In fact,

\[
\bar{C}(N)\tilde{\eta}(\Psi) = 0, \quad \text{for} \quad \Psi \in \mathcal{H}_{\{x_1, \ldots, x_m\}}^{0, \ldots, 0}.
\]

Solving the Gauss constraint is ensured either by restricting \( \mathcal{H} \) to the Yang-Mills gauge invariant elements, or by introducing a third rigging map, integration with respect to the \( \text{SU}(2) \) transformations in \( V \) for each space \( \mathcal{H}_{\{x_1, \ldots, x_m\}}^{0, \ldots, 0} \).

### IV. SUMMARY AND OUTLOOK

In this article, we have introduced a new Hilbert space \( \mathcal{H}_{\text{vtx}} \) of quantum states for the gravitational field. It can be decomposed into sectors

\[
\mathcal{H}_{\text{vtx}} = \bigoplus_{V \in \text{FS}(\Sigma)} \mathcal{H}_V
\]
where the states in $\mathcal{H}_V$ are invariant under all the spatial diffeomorphisms that leave invariant the finite set $V$.

Using the ideas of [4, 5], together with the class of regularizations introduced in [1], we were able to find quantizations of the scalar constraint of pure gravity $\tilde{C}(N)$ as operators leaving $\mathcal{H}_{\text{vac}}$ invariant. This removes a longstanding technical problem, as previous quantizations were defined on fully diffeomorphism invariant states $\mathcal{H}_{\text{dir}}$, but mapped out of that space.

Consequently, it is straightforward to symmetrize the operator, see (III.33,III.38). Moreover, one can immediately work out the commutation relations. Since

$$\tilde{C}(N) = \sum_{x\in V} N(x) \tilde{C}_x, \quad (IV.1)$$

$$[\tilde{C}_x, \tilde{C}_{x'}] = 0 \quad (IV.2)$$

we find

$$[\tilde{C}(M), \tilde{C}(N)] = 0. \quad (IV.3)$$

To discuss the question of anomalies of this quantization, one would thus have to investigate the quantization of the diffeomorphism generator which would classically result from the Poisson bracket of two scalar constraints, as has been done for Thiemann’s quantization, [6, 10, 19]. It is interesting to note that (IV.3) immediately results for any quantization of the form (IV.1) under the reasonable condition (IV.2).

There is a very interesting different line of thought, [11, 14], which also suggests that one should use a different Hilbert space to represent the (diffeomorphism and scalar) constraints. Those methods carry the additional benefit that they address the question of anomalies in a direct fashion. What connection, if any, they have to the constructions of the present article, remains to be seen.

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