Two Hilbert schemes in computer vision

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Abstract

We study multiview moduli problems that arise in computer vision. We show that these moduli spaces are always smooth and irreducible, in both the calibrated and uncalibrated cases, for any number of views. We also show that these moduli spaces always embed in suitable Hilbert schemes, and that these embeddings are open immersions for more than four views, extending and refining work of Aholt–Sturmfels–Thomason. In follow-up work, we will use the techniques developed here to give a new description of the essential variety that simultaneously recovers seminal work of Demazure and recent results of Kileel–Floystad–Ottaviani.

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1 Introduction

In this paper we study some natural moduli spaces that arise in multiview geometry, a subfield of computer vision. Because this subject is foreign to most algebraic geometers, we spend some time here introducing the basic questions and some of the classical calculations. We will then describe how Hilbert schemes enter into the picture, and finally we will describe our main results, before proving them in the remainder of this paper.

Readers interested in seeing statements of our main results (before or instead of first reading our lightning treatment of multiview geometry) are referred to Sections 1.6 and 1.7.

1.1 Multiview geometry: an outline

A key (vaguely-stated) question in computer vision is the following.

“Given several images of an object in the world, can we reconstruct the relative positions of the cameras that took the images and the shape of the object in the world?”

This holds two questions in one:

1. If we know the positions of the cameras and we have a point we can identify in the images, can we identify the point in the world? (This is also called “intersection” or “triangulation”.)

2. If we know where some points in the world map in several images, can we figure out the relative positions of the cameras? (This is also called “resection.”)
One of the insights of computer vision (dating to the older study of photogrammetry) is that there are algorithms for solving both problems together. This package of camera motion (i.e., a sequence of images) and scene reconstruction is called structure from motion. The underlying mathematical structure (as distinguished from the algorithms, numerical methods, and optimization problems underlying the applications to real data) is known as multiple view geometry or multiview geometry. In this section we briefly review the classical formulation of the problems. Starting in Section 2 we will transform them into functorial algebraic geometry. Everything we describe here is also described in detail in [7] and [14], among many other sources.

For the remainder of this section, we work over the real and complex numbers.

**Definition 1.1.0.1.** A pinhole camera is a perspective transformation \( P : \mathbb{R}^3 \to \mathbb{R}^2 \).

We can describe \( P \) synthetically as follows: given a point \( o \in \mathbb{R}^3 \) and a plane \( I \subset \mathbb{R}^3 \), we define a map

\[
\mathbb{R}^3 \setminus \{o\} \to I
\]

by sending a point \( p \) to the intersection between the line \( \overline{po} \) and the plane \( I \). A beautiful algebraic description of such a \( P \) goes like this: embed \( \mathbb{R}^3 \) into \( \mathbb{P}^3_{\mathbb{R}} \), with plane at infinity \( I_\infty \subset \mathbb{P}^3 \) and \( \mathbb{R}^2 \) into \( \mathbb{P}^2_{\mathbb{R}} \) with line at infinity \( L_\infty \). (We will omit the \( \mathbb{R} \) subscript in what follows for readability.) Then there is a unique linear projection \( \tilde{P} : \mathbb{P}^3 \to \mathbb{P}^2 \) such that the preimage of \( L_\infty \) equals \( I_\infty \), the center of \( \tilde{P} \) (i.e., the point at which \( \tilde{P} \) is undefined) equals \( o \in \mathbb{R}^3 \subset \mathbb{P}^3_{\mathbb{R}} \), and such that the induced map

\[
\tilde{P}|_{\mathbb{R}^3} : \mathbb{R}^3 \to \mathbb{R}^3
\]

equals \( P \). That is, pinhole cameras are just restrictions of linear projections of projective spaces.

**Definition 1.1.0.2.** A point correspondence in \( n \) views is a point \( (p_i) \in (\mathbb{R}^2)^n \).

The structure from motion problem can now be phrased like this:

**Problem 1.1.0.3.** Given \( m \) point correspondences in \( n \) views \( (p_i)_1, \ldots, (p_i)_m \in (\mathbb{R}^2)^n \), find \( n \) cameras \( P_i : \mathbb{R}^3 \to \mathbb{R}^2 \) and \( m \) world points \( \xi_j \in \mathbb{R}^3 \) such that \( P_i(\xi_j) = (p_j)_i \), for all \( i \) and \( j \).

There is a standard way to further convert this problem into pure mathematics. Endowing projective space with homogeneous coordinates, we can represent a camera with a \( 3 \times 4 \)-matrix \( P \in \mathbb{R}^{3\times 4} \), up to scaling. In other words, two matrices \( P \) and \( P' \) represent the same camera if and only if they differ by multiplication by a non-zero real number. A point correspondence in \( n \) views corresponds to a tuple \( (p_i) \) of vectors in \( \mathbb{R}^3 \), and a world point is an element \( \xi \in \mathbb{R}^4 \). The problem can then be rephrased as follows.

**Problem 1.1.0.4** (Rephrased in linear algebra). Given \( m \) tuples \( (p_i)_j \in (\mathbb{R}^3)^n \), find matrices \( P_1, \ldots, P_n \in \mathbb{R}^{3\times 4} \) and vectors \( \xi_1, \ldots, \xi_m \in \mathbb{R}^4 \) such that for all \( i, j \) we have \( P_i \xi_j = \lambda_{ij}(p_j)_i \) for some non-zero scalars \( \lambda_{ij} \in \mathbb{R}^\times \).

If we wish to work purely geometrically, there is still another way to phrase the problem.

**Problem 1.1.0.5** (Rephrased geometrically). Given \( m \) points \( \alpha_1, \ldots, \alpha_m \in (\mathbb{R}^2)^n \), characterize all maps \( \varphi : \mathbb{R}^3 \to (\mathbb{R}^2)^n \) such that
1. the components $\text{pr}_i \circ \varphi$ are pinhole cameras, and
2. each $\alpha_i$ is in the image of $\varphi$.

This will be especially useful in what follows, as we will begin to illustrate in Section 1.2.

Remark 1.1.0.6. There are a few natural questions to ask about this situation. For example, for a given number $n$ of views, how many correspondences will yield only finitely many reconstructions? (This is an example of a minimal problem.) Once we know this, how many solutions do we expect? This is related to the algorithms one might use to solve the problem in applications, since if we know there are only a few solutions, we might hope to solve for them analytically, whereas a large number of expected solutions will force the use of numerical methods.

1.2 The case $n = 2$

It is especially illuminating to consider the structure from motion problem for two views, since the geometry becomes simple. (On the other hand, as we will describe below, the geometry of two views is also misleading in certain ways.) Since we will ultimately prove everything here in greater generality elsewhere in this paper, we freely omit or only sketch proofs.

First consider the geometric formulation of the structure from motion problem. We have $m$ points $\alpha_1, \ldots, \alpha_n$ of $\mathbb{R}^2 \times \mathbb{R}^2$, and we wish to find a pair of cameras $\mathbb{R}^3 \to \mathbb{R}^2 \times \mathbb{R}^2$ that contains each $\alpha_i$ in its image. This is a subtle problem, but we can first try to solve a compactified form of the problem. We can replace the affine spaces with projective spaces and promote the cameras to linear projections $\mathbb{P}^3 \to \mathbb{P}^2$, and then we can replace the image of $\mathbb{P}^3 \to \mathbb{P}^2 \times \mathbb{P}^2$ with its closure, which is now a divisor in $\mathbb{P}^2 \times \mathbb{P}^2$ (assuming that centers of the two cameras are distinct). This divisor is called the joint image or multiview variety associated to the camera pair.

Lemma 1.2.0.1. Given a pair of linear projections $P_1, P_2 : \mathbb{P}^3 \to \mathbb{P}^2$ with distinct centers, the closure of $\text{im}(P_1 \times P_2)$ is a divisor in the linear system $|\mathcal{O}_{\mathbb{P}^2, \mathbb{P}^2}(1, 1)|$. (That is, it is given by a bilinear form in the homogeneous coordinates.)

Proof. This comes from the fact that the components are linear projections. One can then use intersection theory in Chow theory or singular cohomology.

Which divisors arise in this way? One immediately notices two things: (1) the divisor in question must be singular, since the line connecting the camera centers gets contracted by $P_1 \times P_2$; (2) a general member of $|\mathcal{O}(1, 1)|$ is smooth (by Bertini’s theorem). Thus, the divisors that occur from pairs of cameras are a special locus in $\mathbb{P}^8 \cong |\mathcal{O}(1, 1)|$. What locus is this?

Viewing elements of $|\mathcal{O}(1, 1)|$ as divisors associated to bilinear forms in the homogeneous coordinates, one can realize the space of such divisors as $3 \times 3$ matrices modulo scalar multiplication. More precisely, if $X_0, X_1, X_2$ are coordinates on the first copy of $\mathbb{P}^2$ and $Y_0, Y_1, Y_2$ are coordinates on the second copy, we can realize an element of $|\mathcal{O}(1, 1)|$ by choosing a $3 \times 3$-matrix $A$ and considering the equation

$$(X_0, X_1, X_2)A(Y_0, Y_1, Y_2)^T = 0.$$
Proposition 1.2.0.2. The joint images are precisely the divisors in $|\mathcal{O}(1,1)|$ corresponding to matrices of rank 2, and these are precisely the divisors whose singular locus consists of a single closed point.

Proof. Let us give a brief geometric explanation of why $A$ must have rank 2. If $o \in \mathbb{P}^3$ is the center of the first camera $P_1$, then we can resolve the rational map $P_1 \to \mathbb{P}^2$, and the exceptional fiber $E$ surjects into the first image plane. On the other hand, the second camera sends $o$ to some point $b = (b_0 : b_1 : b_2) \in \mathbb{P}^2$. In matrix terms, this says that for every $(a_0, a_1, a_2)$ we have

$$(a_0, a_1, a_2)A(b_0, b_1, b_2)^T = 0,$$

which says precisely that $(b_0, b_1, b_2)^T$ is in the right kernel of $A$. 

A final observation. Historically, the reconstruction problem we are considering is only asked up to projective equivalence, that is, up to an automorphism of $\mathbb{P}^3$. This is simply asking that we perform the reconstruction up to a change of homogeneous coordinates, which have been imposed from without (since there is not intrinsic coordinate frame on the world). It turns out that the notion of joint image interacts with projective equivalence especially nicely.

Lemma 1.2.0.3. Two camera configurations in two views $(P_1, P_2) : \mathbb{P}^3 \to (\mathbb{P}^2)^2$ and $(\tilde{P}_1, \tilde{P}_2) : \mathbb{P}^3 \to (\mathbb{P}^2)^2$ are projectively equivalent if and only if their joint images are equal (as closed subschemes).

Proof. We include a general proof in 2.3.0.11 below. The interesting direction is the implication that equal images yield a projective equivalence, since we only a priori get a birational isomorphism that conjugates the configurations. Regularity of this birational map ultimately follows from the linearity of the projections and a brief analysis of the implications for the equations of the birational map.

Finally, we come to the structure from motion problem. We have the space of all joint images sitting inside the determinantal locus of $\mathbb{P}^8 = \mathbb{P}(\mathcal{O}(1,1))$. What constraints are imposed by a point correspondence?

Lemma 1.2.0.4. A point correspondence $(p_1, p_2) \in (\mathbb{P}^2)^2$ determines a hyperplane in $\mathbb{P}^8 = |\mathcal{O}(1,1)|$.

Proof. This follows from the fact that an element $D$ of $|\mathcal{O}(1,1)|$ contains $(p_1, p_2)$ if and only if a defining equation of $D$ vanishes at $(p_1, p_2)$, which means that the locus of divisors containing $(p_1, p_2)$ is the kernel of an evaluation map, making it a linear subspace of $\mathbb{P}^8$ of codimension 1, as desired.

Since we know that the determinantal locus is a cubic hypersurface (the matrices in question being $3 \times 3$), we immediately see that seven general point correspondences will collectively result in three complex solutions to the structure from motion problem. This has one nice feature: cubics have an analytic solution, so we can actually analytically produce all (complex) candidates for the solution from the list of correspondences. (Algorithm: write down the pencil in coordinates – a linear algebra problem – and then solve the cubic induced by the determinant map. Given the matrix, one can manually produce the transform comparing the two cameras, as in [7 Section 9.5.3].)
Summary 1.2.0.5. Let us briefly summarize the main results in two views.

1. The joint image of a pair of cameras $P_1, P_2 : \mathbb{P}^3 \rightarrow \mathbb{P}^2$ is a closed subscheme of $\mathbb{P}^2 \times \mathbb{P}^2$ that is cut out by a single bilinear form in the homogeneous coordinates on the factors.

2. The joint image of $(P_1, P_2)$ uniquely characterizes the pair up to projective equivalence.

3. The space of all joint images is open in the determinantal locus $(\det(A) = 0) \subset \mathbb{P}^8$.

4. A single point correspondence gives a hyperplane in $\mathbb{P}^8$ and thus seven general point correspondences define a line, and this line will intersect the locus of joint images in three (complex) points, at least one of which must be real. (The situation for non-general point correspondences is quite interesting. See [1] for numerous beautiful examples showing that things can be as badly behaved as one can imagine.)

Remark 1.2.0.6. We can interpret the $\mathbb{P}^8 = |\mathcal{O}(1, 1)|$ as a component of the Hilbert scheme of $\mathbb{P}^2 \times \mathbb{P}^2$, and we see that the space of camera configurations up to projective equivalences is a locally closed subscheme of the Hilbert scheme. In more views this situation gets considerably more interesting. This is one of the beautiful observations of [3]. Since the methods of [ibid.] are deeply computational, it was in our effort to understand them geometrically that we were led to the present effort to recast the classical theory in functorial algebraic geometry. As we discovered, one can exploit the resulting deformation theory of camera configurations to give a geometric proof of a refinement and generalization of this key result of [ibid.], and extend the theory to the calibrated realm.

1.3 Calibrated cameras

There is a nagging problem: we have always been working up to projective equivalence. If one looks at the kinds of distortions of real images one can produce using projective transformations (some examples comparing different kinds of reconstruction can be found in Sections 10.2, 10.3, and 10.4 of [7]), one is tempted to require not simply a projective solution to the structure from motion problem, but rather a Euclidean solution.

This is a function of the way in which the world (domain) and image plane are coordinatized. In other words, given a plane $I \subset \mathbb{R}^3$ in space and a camera center $o \in \mathbb{R}^3$, we can produce a canonical projection $\mathbb{R}^3 \rightarrow I$ with center $o$ by using the synthetic description after Definition 1.1.0.1. Moreover, we can give $I$ canonical coordinates using the metric structure on $\mathbb{R}^3$: the center of the coordinate system is the point of $I$ closest to $o$ (the so-called principal point), and the $x$- and $y$-axes are chosen to be perpendicular lines spanned by unit vectors in $I$. With these choices, we can write the transformation as $(x, y, z) \mapsto (fx/z, fy/z)$, where $f$ is the distance from $o$ to $I$ (the “focal length”).

If one imagines taking a photograph of Seattle and comparing it to a photograph of a perfect diorama of Seattle, one can see that precomposing any camera with a similarity transformation (a composition of translations, orthogonal transformations, and scaling) will leave the intrinsic geometry of the camera intact. That is, the rays back-projected from two points will have the same angles before and after the similarity transformation, the camera will have the same focal length, and so on.
Definition 1.3.0.1. A pinhole camera $P : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ is calibrated if it differs from a canonical projection $(x, y, z) \mapsto (fx/z, fy/z)$ by a similarity.

It turns out that there is a beautiful way of understanding when the homogeneous coordinates on $\mathbb{P}^3$ and $\mathbb{P}^2$ can be chosen to make a linear projection $\mathbb{P}^3 \rightarrow \mathbb{P}^2$ compatible with the metric structures up to similarity. It requires us to keep track of two key conics determined by the Euclidean metric: the conic $x^2 + y^2 + z^2 = 0$ in the plane at infinity $w = 0$, the so-called “absolute conic”, and something we will call the “Euclidean conic” $x^2 + y^2 + z^2 = 0$ in $\mathbb{P}^2$. (We call it the Euclidean conic because it arises from the Euclidean metric on $\mathbb{R}^3$.)

Lemma 1.3.0.2. A camera $P : \mathbb{P}^3 \rightarrow \mathbb{P}^2$ is calibrated if and only if it takes the absolute conic isomorphically to the Euclidean conic.

Proof. Any linear form on $\mathbb{P}^3$ that vanishes on the absolute conic is a multiple of the equation of the plane at infinity, and thus any such form is uniquely determined by its value at one additional point not on that plane. It follows that any camera $\mathbb{P}^3 \rightarrow \mathbb{P}^2$ (whose center does not lie on the plane at infinity) is uniquely determined by its center and its restriction to the absolute conic. Moreover, any automorphism of the absolute conic extends to an automorphism of $\mathbb{P}^3$, and we can simultaneously act on a conic and swap two points that lie in the complement of the plane spanned by a conic. Thus, any camera that sends the absolute conic to the circle is projectively equivalent to the camera centered at $(0 : 0 : 0 : 1)$ for which the induced map of conics is the identity map. This camera is precisely $(x, y, z) \mapsto (x/z, y/z)$ in affine coordinates. 

Remark 1.3.0.3. The proof of Lemma 1.3.0.2 does not use anything about the base field. If one restricts to the real numbers, one can recover a more classical description of calibrated cameras. A real projective camera $P : \mathbb{P}^3_{\mathbb{R}} \rightarrow \mathbb{P}^2_{\mathbb{R}}$ can be described by a matrix $A \in M_{3 \times 4}(\mathbb{R})$. Assuming that the camera center does not lie on the plane at infinity, we can write the matrix $A$ as

$$A = [M| - MC],$$

where $M$ is an invertible $3 \times 3$-matrix and $C$ is the vector of affine coordinates of the camera center. The RQ-factorization yields an expression

$$A = K[R| - RC],$$

where

$$K = \begin{pmatrix} \alpha_x & s & x_0 \\ 0 & \alpha_y & y_0 \\ 0 & 0 & 1 \end{pmatrix}$$

with $\alpha_x, \alpha_y > 0$ and $R \in O(3)$ is an orthogonal matrix. The matrix $K$ is called the “calibration matrix” and measures the discrepancy between the camera image coordinate system and the orthonormal coordinate system induced by the Euclidean structure of the ambient space (containing the image plane). For example, the camera output may be in pixel coordinates on a CCD, and there is no reason for the pixels to be square, or for the coordinate system to have the principal point as its origin. (This is described beautifully in Chapter 2 of [7].)
Using this matrix formulation, the proof of Lemma 1.3.0.2 for real cameras comes down to the statement that the bilinear form associated to the image of the absolute conic is $(\mathbf{K}\mathbf{K}^T)^{-1}$, and that this also gives the unique Cholesky factorization of that matrix, thus showing that $\mathbf{K}$ uniquely determines the image of the absolute conic. (See [2] Section 8.5.1 for more details.)

Since calibrated cameras are subject to additional constraints, one expects their joint images to lie in a smaller subspace of $\mathbb{P}^8$. And, indeed, this is the case. There is a five-dimensional subvariety of the rank 2 locus, called the essential variety, that corresponds to pairs of calibrated cameras. The matrices that parametrize the corresponding bilinear forms are called essential matrices, and they are charmingly characterized by the property that their two non-zero singular values are equal [2] Section 9.6.1.

1.4 More views, more equations

Consider the situation now with $n$ cameras for some larger number $n$. A camera configuration corresponds to a morphism

$$\Phi : \text{Bl}_{\{o_1,\ldots,o_n\}} \mathbb{P}^3 \to (\mathbb{P}^2)^n.$$ 

**Lemma 1.4.0.1.** When the camera centers $o_1,\ldots,o_n$ are not collinear, $\Phi$ is a closed immersion. Otherwise, it contracts the strict transform of the line containing the $o_i$ and is an immersion on the complement of the line.

**Proof.** The proof in general is described in Corollary 2.2.3.4 below. \hfill \Box

Following the story for $n = 2$, we can try to understand two things.

1. What are the equations for the joint image $\text{im}(\Phi)$?

2. What is the space of all joint images, as $\Phi$ ranges over all configurations of $n$ cameras?

The joint image was studied in [3]. In particular, for general configurations of $n$ cameras, the ideal of the joint image is generated by $\binom{n}{2}$ bilinear and $\binom{n}{3}$ trilinear polynomials (in the various homogeneous coordinates). The number of generators thus grows rapidly with $n$; it is likely that degenerate configurations have even larger ideals.

The space of all images has been studied in two different ways. The more classical approach is to study what are called “$n$-focal tensors”, which are certain multilinear relations on the coordinates of the camera matrices. For $n = 3, 4$ these give embeddings of the moduli space of camera configurations into spaces of tensors; as described in [2] and [16], the ideals of the closures of these loci are extremely complex. In addition, the natural embeddings into spaces of tensors peter out after $n = 4$.

1.5 Hilbert schemes appear

A deep insight into this problem was discovered by Aholt, Thomas, and Sturmfels in [3]: there is a natural place to embed the moduli space for joint images in $n$ views, namely the Hilbert scheme of $(\mathbb{P}^2)^n$, and, moreover, this embedding is generically an open immersion into a single component. That is, the authors of [ibid.] realized that there is a single component of
the Hilbert scheme of \((P^2)^n\) that is a birational model for the moduli space of configurations of \(n\) cameras up to projective equivalence. This is even better than the classical picture for \(n = 2\): rather than sitting in a distinguished closed subspace, the camera configurations make up (most of) an entire irreducible component of the ambient Hilbert scheme. One of our main goals in this paper is to expand upon and refine this insight.

The methods of [ibid.] are very classical and computational: Gröbner bases, Hilbert functions, toric geometry, geometric invariant theory. In particular, the discovery that the camera locus is dense in a single component of the Hilbert scheme arises from a computation of the tangent space to a single degenerate configuration (where all camera centers lie on a line). As we will show below, the functorial approach allows one to directly verify the results at general points using deformation theory, and moreover allows one to give a refined understanding of the locus corresponding to camera configurations.

A key question raised by [ibid.] is what happens for configurations of calibrated cameras, which are necessarily more algebraically complex owing to the presence of additional conditions. We will give the answer below in Section 4.3.

1.6 Our results

The main result is the following. This is proven in Sections 3 and 4. The statements on Hilbert schemes generalize and refine the results of [3] described in Section 1.5.

**Theorem 1.6.0.1.** There are smooth irreducible varieties \(Cam_n\) and \(CalCam_n\) parametrizing \(n\)-view camera configurations and \(n\)-view calibrated camera configurations, respectively. They contain open subspaces \(Cam^{nc}_n\) and \(CalCam^{nc}_n\) parametrizing configurations where the camera centers are not collinear.

1. For all \(n > 1\), sending a configuration to its joint image defines a locally closed embedding

\[
Cam_n \hookrightarrow \text{Hilb}(P^2)^n.
\]

If \(n > 2\) then this morphism is an open immersion on \(Cam^{nc}_n\). If \(n > 4\) then this morphism is an open immersion, so that \(Cam_n\) is identified with an open subscheme of the smooth locus of \(\text{Hilb}(P^2)^n\).

2. For all \(n > 1\), there is a natural locally closed embedding

\[
CalCam_n \hookrightarrow \text{Hilb}_{C_1 \times \cdots \times C_n}(P^2)^n
\]

(where the latter is a diagram Hilbert scheme; see Section 3.3). If \(n > 2\) then this morphism is an open immersion on \(CalCam^{nc}_n\). If \(n > 4\) then this morphism is an open immersion on all of \(CalCam_n\).

3. The natural decalibration morphism \(\nu_n : CalCam_n \rightarrow Cam_n\) is finite, proper and unramified. The morphism \(\nu_2\) is an étale cover with general fiber of order 2. For \(n > 2\) the morphism \(\nu_n\) is generically injective.

1.7 Methodological contributions

There are a few basic principles that set this work apart from other work on multiview geometry.
1. The *functorial method*, common in modern algebraic geometry, gives us insight into the intrinsic geometry of natural moduli problems growing out of the classical constructions.

2. The *geometric view of calibration* via calibration data gives us insight into the structure of the space of calibrated cameras in a way that seems not to have been considered before. In particular, by restricting camera configurations to morphisms between calibrating conics, we get a fibration structure on the moduli space of calibrated camera configurations that is quite useful for studying the moduli space. In Section 3.4, there’s a third Hilbert scheme – the Hilbert scheme of the product of calibrating conics – that is the base of this fibration. This way of thinking about calibration can also be used to understand the essential variety in new ways. In [4], this is used to reproduce results of both [5] and [6] (which used the results of [5]) from first principles, among other things.

3. The use of *diagram Hilbert schemes* puts the calibrated case in a framework very similar to the uncalibrated case and almost transparently recovers the result that the moduli space is open in a Hilbert scheme.

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2 The algebraic geometry of pinhole cameras

In this section we review the basic theory of pinhole cameras, with a geometric emphasis. We include a canonical treatment of calibrated cameras with a greater focus on the geometry of the calibrating conics. For the sake of clarity, we focus in Section 2.1 and Section 2.2 on the geometry over an algebraically closed field. In Section 2.3 we study what happens over a general base, as a preparation for the study of moduli and deformation theory in Section 3.

2.1 Basic Definitions

**Definition 2.1.0.1.** A *pinhole camera* is a surjective rational map $\varphi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ given by three linearly independent sections of $\mathcal{O}_{\mathbb{P}^3}(1)$. The *center* of the camera is the unique point $p \in \mathbb{P}^3$ at which $\varphi$ is undefined.
Note that an equivalent condition on \( \varphi \) is that it is a surjective rational map such that \( \varphi^* \mathcal{O}_{\mathbb{P}^2}(1) \) is isomorphic to \( \mathcal{O}_{\mathbb{P}^3}(1) \). This condition makes sense because \( \varphi \) is regular in codimension 1, by the valuative criterion of properness, hence induces a well-defined pullback map on Picard groups.

**Definition 2.1.0.2.** A calibrated plane is a pair \((\mathbb{P}^2, D)\) with \( D \) a smooth conic.

**Definition 2.1.0.3.** A calibration datum for a pinhole camera \( \varphi \) is a pair of degree 2 curves \( C \subset \mathbb{P}^3 \) and \( D \subset \mathbb{P}^2 \) such that
1. \( D \) is a smooth conic;
2. \( \varphi \) is regular along \( C \);
3. \( \varphi_C \) factors through \( D \).

If \( C \) is smooth, the calibration datum will be called smooth; otherwise it will be called degenerate. If a calibrated plane \((\mathbb{P}^2, D)\) is fixed, a relative calibration datum for a pinhole camera \( \Phi \) is a curve \( C \subset \mathbb{P}^3 \) such that \((C, D)\) is a calibration datum for \( \Phi \).

**Remark 2.1.0.4.** If \( C \) is smooth then it follows from the linearity of the camera projection that \( \Phi \) must map \( C \) isomorphically to \( D \), and that the center of \( \Phi \) is not contained in the plane spanned by \( C \). If \( C \) is degenerate, it must be a divisor-theoretic sum of two lines on the quadric cone in \( \mathbb{P}^3 \) generated by \( D \) under the projection \( \Phi \) (i.e., a union of two distinct rulings or a double ruling). As we will see below, a union of two distinct rulings cannot occur as limit of calibration data.

**Remark 2.1.0.5.** A given camera with calibrated image plane \((\mathbb{P}^2, D)\) has infinitely many relative calibration data: one can take any plane section of the quadric cone in \( \mathbb{P}^3 \) lying over \( D \). Once we look at configurations of two or more cameras, there will be at most two calibration data (smooth or degenerate). This is described at length in Section 4.1.2.

Degenerate calibrations give us closures of natural moduli spaces, including the closure of the classical twisted pair moduli space \( \text{SO}(3) \times \mathbb{P}^2 \) to a finite étale cover of the essential variety described in Section 4.2. Imagining the system of plane sections of the cone over \( D \), one readily sees that degenerate calibration data arise as limits of smooth calibration data.

**Definition 2.1.0.6.** A calibrated camera is a pair \((\varphi, (C, D))\) where \( \varphi \) is a pinhole camera and \((C, D)\) is a calibration datum for \( \varphi \).

**Remark 2.1.0.7.** In the classical literature, a camera is called calibrated when it takes the absolute conic to the Euclidean conic: more precisely, we can endow \( \mathbb{P}^3 \) with coordinates \( x, y, z, w \) and \( \mathbb{P}^2 \) with coordinates \( X, Y, Z \), and then we take the curves \( C \) and \( D \) to be given by the equations \( \{ w = 0, x^2 + y^2 + z^2 = 0 \} \) and \( \{ X^2 + Y^2 + Z^2 = 0 \} \), respectively. Note that any camera as described here with a smooth calibration datum can be transformed to a classically calibrated camera by applying suitable automorphisms to \( \mathbb{P}^3 \) and \( \mathbb{P}^2 \). (This is not unique.) The degenerate calibrations cannot.

There are two reasons to use this more flexible approach:

1. it leads to the “right definition” of the moduli space of calibrated camera configurations (Section 3.4);
by always forcing the absolute conic to map to the Euclidean conic, one makes it impossible to study modular boundary points where the absolute conic is flattened until it collapses (yielding degenerate calibrations). As we will describe below, these degenerate calibrations give geometrically meaningful compactifications of the space of calibrated camera configurations.

2.2 Multiview configurations

In this section, we describe some of the geometry attached to a collection of cameras with distinct centers.

2.2.1 Uncalibrated cameras

**Definition 2.2.1.1.** A multiview configuration is a collection of cameras

\[ \varphi_1, \ldots, \varphi_n : \mathbb{P}^3 \to \mathbb{P}^2. \]

**Notation 2.2.1.2.** We will generally use \( \Phi : \mathbb{P}^3 \to (\mathbb{P}^2)^n \) to denote a multiview configuration, writing \( \Phi_i = \text{pr}_i \circ \Phi \) for its components when necessary. The length of \( \Phi \) is the number of cameras; we will denote it \( \text{len}(\Phi) \). Write \( \text{Center}(\Phi) \subset \mathbb{P}^3 \) for the tuple of camera centers. Write \( \pi : \text{Res}(\Phi) \to \mathbb{P}^3 \) for the blowup of \( \mathbb{P}^3 \) at the reduced closed subscheme supported at the camera centers; if two cameras have the same center we only count it once. Given an index \( i \), let \( E_i \) denote the exceptional divisor over the \( i \)th camera center, with canonical inclusion \( \iota_i : E_i \hookrightarrow \text{Res}(\Phi) \). By the previous convention, this means that there can be \( i \neq j \) for which \( E_i = E_j \).

**Definition 2.2.1.3.** A multiview configuration \( \Phi \) is general if the camera centers are all distinct. It is non-collinear if the camera centers do not all lie on a single line, and collinear otherwise.

**Definition 2.2.1.4.** An isomorphism between multiview configurations \( \Phi^1 \) and \( \Phi^2 \) of common length \( n \) is an automorphism \( \varepsilon : \mathbb{P}^3 \to \mathbb{P}^3 \) fitting into a commutative diagram

\[
\begin{tikzcd}
\mathbb{P}^3 \arrow[r, dashed, \Phi^1] \arrow[d, \varepsilon] & (\mathbb{P}^2)^n \arrow[d, \Phi^2] \\
\mathbb{P}^3
\end{tikzcd}
\]

**Lemma 2.2.1.5.** Let \( Y \) be a scheme, and let \( (\mathcal{L}, s_0, \ldots, s_n) \) be an invertible sheaf with \( n \) sections. If \( Z \) is the zero scheme of \( s_0, \ldots, s_n \) then the rational map induced by this linear series extends uniquely to a morphism \( \text{Bl}_Z Y \to \mathbb{P}^n \).

**Proof.** By definition the sections \( s_0, \ldots, s_n \) define a surjection

\[ \mathcal{O}_Y^{n+1} \to \mathcal{L} \otimes \mathcal{I}_Z, \]
which extends to a surjective map of $\mathcal{O}_Y$-algebras

$$\text{Sym}^n(\mathcal{L}^n) \oplus n^1 \twoheadrightarrow \bigoplus \mathcal{I}^n.$$ 

The induced map on relative $\text{Proj}$ constructions gives the desired morphism.

**Proposition 2.2.1.6.** Given a multiview configuration $\Phi$, there is a unique commutative diagram

$$
\begin{array}{c}
\text{Res}(\Phi) \\
\text{Res}(\Phi) \\
\pi^{-1} \\
\Phi
\end{array}
\xrightarrow{\rho} 
\begin{array}{c}
(\mathbb{P}^2)^{\text{len}(\Phi)} \\
(\mathbb{P}^2)^{\text{len}(\Phi)} \\
\Phi
\end{array}
\xrightarrow{\text{pr}_i} \mathbb{P}^2
$$

The diagram has the property that for each $i$, the composition

$$E_i \xrightarrow{i_i} \text{Res}(\Phi) \xrightarrow{\rho} (\mathbb{P}^2)^{\text{len}(\Phi)} \xrightarrow{\text{pr}_i} \mathbb{P}^2$$

is an isomorphism.

**Proof.** Lemma 2.2.1.5 shows the existence and uniqueness of the desired diagram. To check that the composition is an isomorphism on exceptional divisors one can see that each map is locally isomorphic to the morphism $\text{Bl}_0 \mathbb{A}^3 \to \mathbb{P}^2$ that resolves the canonical presentation $\mathbb{A}^3 \setminus \{0\} \to \mathbb{P}^2$, and here one can simply check that the induced map from the exceptional divisor to the plane is an isomorphism. We omit the details.

There are several equivalent ways to describe a multiview configuration of length $n$.

1. A camera is given by choosing 3 linearly independent global sections of $\mathcal{O}_{\mathbb{P}^3}(1)$. Fixing a basis for the latter space thus makes a single such camera identifiable with a point of the Grassmannian of 3-planes in a fixed 4-dimensional space.

2. A point of the Grassmannian is given by a $3 \times 4$ matrix of full rank.

It is not hard to see that the notion of isomorphism of multiview configuration makes all of these equivalent formulations lead to the same isomorphism classes of objects, even though we have had to choose a basis for $\Gamma(\mathbb{P}^3, \mathcal{O}(1))$ in the latter two.

### 2.2.2 Calibrated cameras

When the cameras are adorned with calibration data, we track these data through the diagrams.

**Definition 2.2.2.1.** Given a multiview configuration $\Phi : \mathbb{P}^3 \to (\mathbb{P}^2)^n$, a *multiview calibration datum* is a pair $(C, (C_1, \ldots, C_n))$ such that for each $i = 1, \ldots, n$ the pair $(C, C_i)$ is a calibration datum for $\Phi_i$. Given a tuple of calibrated planes $(\mathbb{P}^2, C_i)$ for $i = 1, \ldots, n$, a *relative calibration datum* for $\Phi$ is a curve $C \subset \mathbb{P}^3$ such that $(C, (C_1, \ldots, C_n))$ is a calibration datum for $\Phi$. 

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Notation 2.2.2. We will write $C$ for a calibration datum $(C_i, (C_i))$, and then $C_0 = C$ and $C_i = C_i$ for $i = 1, \ldots, n$.

Notation 2.2.3. A calibrated multiview configuration $(\Phi, C)$ will be called non-degenerate if the calibration datum is non-degenerate.

Definition 2.2.4. An isomorphism between multiview configurations with calibration data $(\Phi_1, C_1)$ and $(\Phi_2, C_2)$ of common length $n$ is an isomorphism $\varepsilon : \Phi_1 \to \Phi_2$ of multiview configurations as in Definition 2.2.1.4 such that $\varepsilon(C_1^0) = C_2^0$ and such that for $i = 1, \ldots, n$ we have $C_1^i = C_2^i$.

The equivalent formulations of multiview configurations, etc., in terms of Grassmannians are a bit more baroque, due to the need to track the conics. Perhaps there is a formulation in terms of orthogonal Grassmannians (with respect to a quadratic form), but we will not dwell on that here.

2.2.3 A characterization of isomorphic general configurations

In this section we briefly consider when two multiview configurations $\Phi_1$ and $\Phi_2$ are isomorphic (and similarly when they are endowed with calibration data). This will play a role in studying a particular map from the moduli space to Hilbert schemes in later sections of this paper.

Definition 2.2.3.1. Given a multiview configuration $\Phi$, the associated multiview scheme, also known as the joint image [3, 18], is the scheme-theoretic image of the resolution $\text{Res}(\Phi)$ under the canonical extension $\rho$ of Proposition 2.2.1.6. It is denoted $\text{Sch}(\Phi)$. Working over a field (as we are here), the multiview scheme is a variety, and is called the “multiview variety” in [3].

Definition 2.2.3.2. Given a calibrated multiview configuration $(\Phi, C)$ with calibrated image planes $(P_2, C_i)$, $i = 1, \ldots, n$, the associated multiview flag, denoted $\text{Flag}(\Phi, C)$, is the flag $C \subset \text{Sch}(\Phi)$ contained in $C_1 \times \cdots \times C_n \subset (P_2)^n$.

As we will gradually see, the following lemma is the key result connecting the abstract moduli problems we study here to Hilbert schemes.

Lemma 2.2.3.3. The derived adjunction map $\mathcal{O}_{\text{Sch}(\Phi)} \to R \rho_* \mathcal{O}_{\text{Res}(\Phi)}$ is a quasi-isomorphism.

Proof. This amounts to showing that $\rho^* : \mathcal{O}_{(P_2)^n} \to \rho_* \mathcal{O}_{\text{Res}(\Phi)}$ is surjective and that all higher direct images $R^i \rho_* \mathcal{O}_{\text{Res}(\Phi)}$ (with $i > 0$) vanish.

For the surjectivity statement, note that $\rho_* \mathcal{O}_{\text{Res}(\Phi)}$ is a finite $\mathcal{O}_{(P_2)^n}$-algebra by properness. Moreover, since every non-empty fiber of $\rho$ is geometrically integral (it being an intersection of lines, hence either a point or a line), we see that $\rho^*$ is surjective after base change to any point of $(P_2)^n$. By Nakayama’s lemma, $\rho^*$ is surjective.

Now we show that the higher direct images vanish. By the Theorem on Formal Functions, the completion of $R^j \rho_* \mathcal{O}$ at a point $p$ is isomorphic to $\lim \mathcal{H}^i(X_m, \mathcal{O}_{X_m})$, where $X_m$ is the $m$th infinitesimal neighborhood of the fiber of $\rho$ over $p$. When the fiber is empty or a point, this vanishes. The only interesting case is the unique singular point that is the image of the strict transform of the line through all camera centers, in the collinear case. Note that $\mathcal{O}_{X_m}$
is filtered by subquotients that are symmetric powers of the ideal sheaf \( J_{X_0} \) restricted to \( X_0 \). Given a line \( L \) in \( P^3 \), we have that \( J_L \mid L \cong \mathcal{O}_L(\ell - 1)^{\oplus 2} \). For each point on \( L \) that we blow up, the ideal sheaf gets twisted by \( 1 \) (functions from \( P^3 \) vanish to extra order on the strict transform along the intersection with the exceptional divisor). In fact, if we are blowing up \( n \) points, we have that \( J_{X_0} \mid X_0 \cong \mathcal{O}_{X_0}(n - 1)^{\oplus 2} \). The \( \ell \)th symmetric power will be a sum of copies of \( \mathcal{O}_{X_0}(\ell(n - 1)) \). All such sheaves have vanishing \( H^i \) for all \( i > 0 \).

Write \( J_m \) for the ideal sheaf of \( X_m \) in \( \text{Res}(\Phi) \). Consider the standard exact sequences

\[
0 \to J_{m-1} \to J_m \to \mathcal{O}_{X_m} \to \mathcal{O}_{X_{m-1}} \to 0.
\]

The above calculations show inductively that \( H^i(X_n, \mathcal{O}_{X_n}) = 0 \) for all \( n \geq 0 \) and all \( i > 0 \). This concludes the proof. \( \square \)

**Corollary 2.2.3.4.** If \( \Phi \) is a non-collinear multiview configuration then the map \( \rho : \text{Res}(\Phi) \to (P^2)^n \) is a closed immersion.

**Proof.** By the non-collinearity assumption, the geometric fibers of \( \rho \) all have length at most 1. Thus, \( \rho \) is proper and quasi-finite, hence finite. Applying Lemma 2.2.3.3 then shows that \( \rho \) is a closed immersion. \( \square \)

**Lemma 2.2.3.5.** Suppose \( \varphi_1, \varphi_2 : P^3 \to P^2 \) are cameras and \( \alpha : P^3 \to P^3 \) is a birational automorphism such that \( \varphi_2 = \varphi_1 \circ \alpha \). If \( \alpha \) and \( \varphi_1 \circ \alpha \) are both regular on an open subset \( U \subset P^3 \) whose complement has codimension at least 2 then \( \alpha \) extends to a unique regular automorphism \( P^3 \to P^3 \).

**Proof.** Removing the center of \( \varphi_1 \) if necessary, we may assume that there is an open subscheme \( U \subset P^3 \) on which \( \varphi_1, \varphi_2, \) and \( \alpha \) are all regular and \( \text{codim}(P^3, P^3 \setminus U) \geq 2 \). By assumption, \( \varphi_1^* \mathcal{O}(1) = \mathcal{O}_U(1) \). Thus, \( \alpha^* \mathcal{O}(1) = \mathcal{O}(1) \). Since \( \Gamma(U, \mathcal{O}(1)) = \Gamma(P^3, \mathcal{O}(1)) \), we conclude from the universal property of projective space that the morphism \( \alpha : U \to P^3 \) extends to a unique endomorphism \( \tilde{\alpha} \) of \( P^3 \). Since \( \alpha \) is birational, \( \tilde{\alpha} \) is an isomorphism, as desired. \( \square \)

**Proposition 2.2.3.6.** Two multiview configurations \( \Phi^1 \) and \( \Phi^2 \) of length \( n \) are isomorphic if and only if their associated multiview schemes in \( (P^2)^n \) are equal.

**Proof.** Since \( \Phi^i \) is birational onto its image for \( i = 1, 2 \), we see that if \( \text{Sch}(\Phi^1) = \text{Sch}(\Phi^2) \) then there is a birational automorphism \( \alpha : P^3 \to P^3 \) such that \( \Phi^2 = \Phi^1 \circ \alpha \). Moreover, \( pr_1 \circ \Phi^1, \alpha, \) and \( pr_1 \circ \Phi^1 \circ \alpha \) are all regular on the open subscheme of \( P^3 \) that is the complement of the line joining the centers of \( \Phi^1 \) (as this maps isomorphically to the smooth locus of \( \text{Sch}(\Phi^1) \)). Applying Lemma 2.2.3.5 we see that \( \alpha \) is regular, as desired. \( \square \)

### 2.3 Relativization

In this section we describe how to generalize the results of Section 2.1 and Section 2.2 to families of cameras over an arbitrary base space.

**Definition 2.3.0.1.** Given a scheme \( S \), a relative pinhole camera over \( S \) is a rational map \( p : P \to P^2_S \) over \( S \) uniquely determined by the following information:
1. the scheme $P$ is a Zariski form of $P^3_S$;
2. there is a map $\sigma : \mathcal{O}_{P^3} \to \mathcal{O}_P(1)$ whose cokernel is an invertible sheaf supported exactly over a section $Z$ of $P \to S$, called the camera center;
3. a representative of $p$ is given by the morphism $P \setminus Z \to P^2_S$ determined by the quotient $\sigma_{P \setminus Z}$ and the universal property of projective space.

Since we assume that there is a section of $P \to S$, it follows from the basic theory of Brauer-Severi schemes that it is in fact a Zariski form of $P^3_S$. For reasons of descent theory, we do not make this an \textit{a priori} assumption.

\textbf{Definition 2.3.0.2.} Given a scheme $S$, a \textit{relative multiview configuration of length $n$} over $S$ is given by a proper $S$-scheme $P \to S$ of finite presentation and a rational map $\Phi : P \to (P^2_S)^n$ over $S$ such that for each $i$ the composition $\text{pr}_i \circ \Phi$ is a relative pinhole camera as in \textbf{Definition 2.3.0.1}.

Two relative multiview configurations

$$\Phi^i : P_i \to P^2_S, \quad i = 1, 2$$

are \textit{isomorphic} if there is an isomorphism $\varepsilon : P_1 \sim P_2$ such that $\Phi^2 = \Phi^1 \circ \varepsilon$.

\textbf{Notation 2.3.0.3.} Given a multiview configuration $\Phi : P \to (P^2)^n$ of length $n$, we will write

1. $S(\Phi)$ for the domain $P$ of $\Phi$;
2. $Z_1(\Phi), \ldots, Z_n(\Phi) \subset P$ for the camera centers;
3. $Z(\Phi)$ for the scheme-theoretic union $Z_1(\Phi) \cup \cdots \cup Z_n(\Phi)$;
4. $\text{Res}(\Phi)$ for the blowup of $S(\Phi)$ in $Z$.

\textbf{Definition 2.3.0.4.} A relative multiview configuration $\Phi$ over $S$ is \textit{general} if the camera centers $Z_1, \ldots, Z_{\text{len}(\Phi)}$ are pairwise disjoint closed subschemes of $P$.

\textbf{Definition 2.3.0.5.} A relative multiview configuration $\Phi : P \to (P^2_S)^n$ over $S$ is \textit{collinear} if there is a closed subscheme $L \subset S(\Phi)$ that is a relative line over $S$ and that contains $Z(\Phi)$. It is \textit{nowhere-collinear} if it is not collinear upon any basechange $S' \to S$.

\textbf{Definition 2.3.0.6.} Given a relative multiview configuration $\Phi$ of length $n$ over $S$, a \textit{(smooth) calibration datum} for $\Phi$ is a pair $(C, (C_1, \ldots, C_n))$ where

1. $C \subset P$ is a (smooth) degree two curve over $S$;
2. $C_i \subset P^2_S$ is a relative smooth conic over $S$ for $i = 1, \ldots, n$;
3. some representative of $\Phi$ is regular along $C$;
4. and the induced morphisms $(\text{pr}_i \circ \Phi)_C$ factors through $C_i$ for $i = 1, \ldots, n$.

If $C$ is smooth, the calibration datum will be called \textit{smooth}; otherwise it will be called \textit{degenerate}.
**Proposition 2.3.0.7.** Given a general relative multiview configuration \( \Phi \) over \( S \), there is a unique commutative diagram

\[
\begin{array}{ccc}
\text{Res}(\Phi) & \xrightarrow{\rho} & (\mathbb{P}^2)^{\text{len}(\Phi)} \\
\pi^{-1} & \searrow & \\
\Phi & \xrightarrow{\text{pr}_i} & \mathbb{P}^2 \\
\end{array}
\]

The diagram has the property that for each \( i \), the composition

\[
E_i \xrightarrow{\iota_i} \text{Res}(\Phi) \xrightarrow{\rho} (\mathbb{P}^2)^{\text{len}(\Phi)} \xrightarrow{\text{pr}_i} \mathbb{P}^2
\]

is an isomorphism. Moreover, this diagram is compatible with arbitrary base change on \( S \).

**Proof.** The arrow \( \rho \) exists again by Lemma 2.2.1.5, and the functoriality follows from the functoriality of Lemma 2.2.1.5 and the flatness of everything over \( S \). Finally, the isomorphism condition can be checked on geometric fibers, which reduces it to Proposition 2.2.1.6.

Write \( \text{MVC}_n(S) \) for the groupoid of general relative multiview configurations of length \( n \) over \( S \). Write \( \text{RVC}_n(S) \) for the groupoid of tuples \( (P, (Z_1, \ldots, Z_n), f) \) where \( \pi : P \to S \) is an fppf form of \( P^3 \to S \), the \( Z_i \in P(S) \) are pairwise non-intersecting sections of \( \pi \), and \( f : P \to (\mathbb{P}^2)^n \) is a morphism from the blowup of \( P \) along \( \cup Z_i \) to \( (\mathbb{P}^2_S)^n \) such that \( \text{pr}_i \circ f \) induces an isomorphism from the \( i \)th exceptional divisor \( E_i \subset \tilde{P} \) to \( \mathbb{P}^2_S \).

**Corollary 2.3.0.8.** Proposition 2.3.0.7 defines a canonical equivalence of categories \( \text{MVC}_n(S) \to \text{RVC}_n(S) \).

**Proof.** The proof is tautological. (This corollary mainly serves to establish notation.)

**Definition 2.3.0.9.** Given a general multiview configuration \( \Phi \) of length \( n \), the image of the morphism \( \rho \) described in Lemma 2.3.0.7 is the multiview scheme of \( \Phi \).

**Notation 2.3.0.10.** The multiview scheme of \( \Phi \) will be denoted \( \text{Sch}(\Phi) \).

**Proposition 2.3.0.11.** Two general multiview configurations \( \Phi^1, \Phi^2 \) of length \( n \) over \( S \) are isomorphic if and only if \( \text{Sch}(\Phi^1) = \text{Sch}(\Phi^2) \) as closed subschemes of \( (\mathbb{P}^2_S)^n \).

The proof of Proposition 2.3.0.11 is a modification of that of Proposition 2.2.3.6. We require a modification of Lemma 2.2.3.5.

**Lemma 2.3.0.12.** Suppose \( A \) is a ring and \( U \subset \mathbb{P}^3_A \) is an open subset such that for every geometric point \( A \to \kappa \) the fiber \( U_\kappa \subset \mathbb{P}^3_\kappa \) has complement of codimension at least 2. Suppose \( \alpha : U \to \mathbb{P}^3_A \) is a morphism such that \( \alpha^*\mathcal{O}(1) = \mathcal{O}_U(1) \). Then \( \alpha \) extends to a unique automorphism of \( \mathbb{P}^3_A \).

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Proof. By the universal property of projective space, it suffices to show that restriction defines an isomorphism
\[ \Gamma(\mathbb{P}^3_A, \mathcal{O}(1)) \cong \Gamma(U, \mathcal{O}(1)). \]
To show this, it suffices to show that the adjunction map \( \nu(1) : \mathcal{O}_{\mathbb{P}^3}(1) \to \iota_* \mathcal{O}_U(1) \) is an isomorphism of sheaves. By the projection formula, it suffices to show that the adjunction map for the structure sheaf
\[ \nu : \mathcal{O}_{\mathbb{P}^3} \to \iota_* \mathcal{O}_U \]
is an isomorphism. But this is precisely Proposition 3.5 of [8]. \( \square \)

**Proposition 2.3.0.13.** If \( \Phi \) is a general multiview configuration over \( S \) then for all base changes \( T \to S \) we have that the natural morphism
\[ \text{Sch}(\Phi) \times_S T \to \text{Sch}(\Phi \times_S T) \]
is an isomorphism. That is, formation of the associated multiview scheme is compatible with base change. Furthermore, \( \text{Sch}(\Phi) \) is flat over the base.

**Proof.** By Lemma 2.2.3.3 the structure morphism \( \mathcal{O}_{(\mathbb{P}^2)^n} \to \rho_* \mathcal{O}_{\text{Res}(\Phi)} \) is surjective. Consider the triangle in the derived category
\[ I \to \mathcal{O}_{(\mathbb{P}^2)^n} \to \mathcal{R} \rho_* \mathcal{O}_{\text{Res}(\Phi)} \]
Let \( i : (\mathbb{P}^2)^n_q \to (\mathbb{P}^2)^n \) be an embedding of a fiber. Pulling back to the fiber and using cohomology and base change we have
\[ L i^* \mathcal{R} \rho_* \mathcal{O}_{\text{Res}(\Phi)} \cong \mathcal{R} \rho_* L i^* \mathcal{O}_{\text{Res}(\Phi)} \mathcal{O}_{\text{Res}(\Phi)} \]
\[ \cong \mathcal{R} \rho_* (\mathcal{O}_{\text{Res}(\Phi)})_q \]
\[ \cong (\mathcal{O}_{\text{Res}(\Phi)})_q \]
Applying [9] Lemma 3.31 to \( \mathcal{R} \rho_* \mathcal{O}_{\text{Res}(\Phi)} \), we see that it is quasi-isomorphic to a sheaf flat over the base. But \( \mathcal{H}^0(\mathcal{R} \rho_* \mathcal{O}_{\text{Res}(\Phi)}) \) is \( \rho_* \mathcal{O}_{\text{Res}(\Phi)} \). Thus, we conclude that the short exact sequence
\[ 0 \to \mathcal{I} \to \mathcal{O}_{(\mathbb{P}^2)^n} \to \rho_* \mathcal{O}_{\text{Res}(\Phi)} \to 0 \]
consists of \( S \)-flat sheaves and is compatible with arbitrary base change. This establishes the result. \( \square \)

# 3 Moduli and deformation theory

## 3.1 Moduli of uncalibrated camera configurations

In this section we describe the basic moduli problem attached to uncalibrated camera configurations. In Section 3.2 we will study the deformation theory of a configuration \( \Phi \), especially as it relates to the deformation theory of the associated scheme \( \text{Sch}(\Phi) \).

**Definition 3.1.0.1.** Given a positive integer \( n \), the *stack of camera configurations of length \( n \)*, denoted \( \text{Cam}_n \), has as objects over a scheme \( S \) the groupoid of general relative multiview configurations of length \( n \).
**Proposition 3.1.0.2.** Let $M^n \subset M^{3 \times 4}$ be the locus of full rank $3 \times 4$ matrices. There is an equivalence of stacks between $[M^n / GL_4]$ and $Cam_n$.

**Proof.** By [19, Theorem 4.46], there is a canonical equivalence between $Cam_n(T)$ and $Cam_{GL_4}^{GL_4}(Q)$, the $GL_4$-equivariant objects of $Cam_n(Q)$. We claim that $[M^n / GL_4](T)$ is equivalent to $Cam_{GL_4}^{GL_4}(Q)$ as well.

An object of $[M^n / GL_4](T)$ is a $GL_4$-torsor, $Q \to T$, with an equivariant map to $M^n$. The map $Q \to M^n$ induces a rational linear map $A : P^3_{GL_4 \times Q} \to (P^2_{GL_4 \times Q})^n \in Cam_n(Q)$. Let $\alpha : GL_4 \times Q \to Q$ be the group action and let $pr_2$ be projection onto $Q$. Equivariance of the map $Q \to M^n$ is precisely the statement that the action of $GL_4$ on $P^3$ induces an isomorphism $\varphi : \alpha^* A \to pr_2^* A$ of objects of $Cam_{GL_4}(GL_4 \times Q)$.

\[
\begin{array}{ccc}
P^3_{GL_4 \times Q} & \xrightarrow{\alpha^* A} & (P^2_{GL_4 \times Q})^n \\
\downarrow & & \downarrow \\
P^3_{GL_4 \times Q} & \xrightarrow{pr_2^* A} & A
\end{array}
\]

Similarly, equivariance gives us a cocycle condition for $\varphi$ satisfying [19, Proposition 3.49], so $A$ is a $GL_4$-equivariant object.

An isomorphism of objects $Q$ and $Q'$ of $[M^n / GL_4](T)$ is an morphism of torsors commuting with their maps to $M^n$. Let the corresponding objects of $Cam_n(Q)$ be $A$ and $A'$. Then

\[
\begin{array}{ccc}
pr_2^* A & \to & A \\
\downarrow & & \downarrow \\
pr_2^* A' & \to & A'
\end{array}
\]

commutes, so by [19, THING 3.48] this is a $GL_4$-equivariant morphism.

This defines a morphism of stacks $[M^n / GL_4] \to Cam_n$. We can check this is an equivalence over strictly Hensalian local rings. In this case, every $P^3$ form is trivial and the description above is an equivalence.

**Corollary 3.1.0.3.** The stack $Cam_n$ is a smooth algebraic space of finite type over $Spec \mathbb{Z}$.

**Proof.** The space of full rank $3 \times 4$ matrices is smooth and has trivial stabilizers.

**Notation 3.1.0.4.** We will write $Cam_n^{nc}$ for the locus of non-collinear configurations. This is an open substack.

### 3.2 Deformations of multiview configurations

In this section, we study the relationship between the infinitesimal deformation theory of a camera configuration and the deformation theory of its associated multiview scheme. We get strong results for non-collinear cameras for arbitrary $n$ and for collinear cameras for $n > 4$. 

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As we will see below, this gives strong results on the relationship between \( \text{Cam}_n \) and \( \text{Hilb}_n(\mathbb{P}^2)^n \), clarifying and improving the groundbreaking results of \cite{[3]}. In particular, our infinitesimal analysis will apply at all points, showing density in the Hilbert scheme; special points are handled well for \( n > 4 \) by a straightforward argument using the cotangent complex, giving the enhancement in those cases. These methods are very different from the ideal-theoretic methods of \cite{[3]}. It would be especially interesting to understand how the cotangent complex argument of Section 3.2.3 relate to the Gröbner basis calculations in \cite{[3]}.

**Definition 3.2.0.1.** Fix a ring \( A \) containing an ideal \( I \) such that \( I^2 = 0 \) and let \( A_0 = A/I \). Suppose \( \Phi^0 \) is a relative multiview configuration of length \( n \) over \( A_0 \). An infinitesimal deformation of \( \Phi^0 \) to \( A \) is a pair \((\Phi, \varepsilon)\), where \( \Phi \) is a multiview configuration of length \( n \) over \( A \) and \( \varepsilon : \Phi \otimes_A A_0 \to \Phi^0 \) is an isomorphism of relative multiview configurations.

An isomorphism between infinitesimal deformations \((\Phi, \varepsilon)\) and \((\Phi', \varepsilon')\) of \( \Phi^0 \) is an isomorphism \( \alpha : \Phi \to \Phi' \) of relative multiview configurations such that \( \varepsilon' \circ \alpha \otimes_A A_0 = \varepsilon \).

Our goal in this section is to prove the following.

**Theorem 3.2.0.2.** If \( \Phi \) is a general multiview configuration of length \( n \) with associated multiview variety \( V \subset (\mathbb{P}^2)^n \) then, assuming either that \( \Phi \) is non-collinear or that \( n > 4 \), we have that the infinitesimal deformations of \( \Phi \) are in bijection with the infinitesimal deformations of \( V \) as a closed subscheme of \( (\mathbb{P}^2)^n \).

The proof will work roughly as follows.

1. First, we will study the abstract deformations of \( V \) as a scheme. As we will see, \( V \) has a property that we will call essential rigidity.
2. Using this essential rigidity, we will show that any deformation of \( V \) as a closed subscheme of \( (\mathbb{P}^2)^n \) arises from a deformation of \( \Phi \). In the collinear case this is non-trivial, because \( \text{Res}(\Phi) \to (\mathbb{P}^2)^n \) contracts a line. As we will explain, this can be worked around by a mild study of the cotangent complex of the contraction map in the collinear case, as long as \( n > 4 \).
3. Using Proposition 2.3.0.11 we have that two deformations of \( \Phi \) give rise to the same deformation of \( V \) if and only if they are isomorphic, completing the proof.

It is worth noting (as hinted at in this outline) that the proof we give here is almost purely geometric. We do not rely on dimension estimates, ideal-theoretic calculations, masses of cohomology, etc. The arguments are simple variants of classical Italian geometric arguments, first used to study the geometry of projective surfaces.

### 3.2.1 Essential rigidity of blowups of \( \mathbb{P}^3 \)

In this section we fix a commutative ring \( A_0 \), a square-zero extension \( I \subset A \to A_0 \), and a collection of pairwise everywhere-disjoint sections \( \sigma_i : \text{Spec} A_0 \to \mathbb{P}^3_{A_0} \).

We write \( P_0 \) for the blowup \( \text{Bl}_{Z_0} \mathbb{P}^3_{A_0} \), where \( Z_0 \) is the reduced closed subscheme of \( \mathbb{P}^3_{A_0} \) supported on the union of the images of the \( \sigma_i \).
Proposition 3.2.1.1. Given a deformation $P$ of $P_0$ over $A$, there is a unique morphism

$$\beta : P \to P^3_A$$

deforming the canonical blow-down map

$$\beta_0 : P_0 \to P^3_{A_0},$$

up to infinitesimal automorphism of $P^3_A$. Moreover, $\beta$ realizes $P$ as the blowup of $P^3_A$ at a closed subscheme $Z$ that deforms $Z_0$ (and $Z$ is a union of $n$ sections of $P^3_A$).

**Proof.** Via the universal property of projective space, the morphism $\beta_0$ is given by the natural map

$$\mathcal{O}_{P_0}^\oplus 4 \to \beta_0^* \mathcal{O}(1)$$

arising from pulling back the natural map of sheaves

$$\mathcal{O}^\oplus 4 \to \mathcal{O}(1)$$

on $P^3_{A_0}$. By the Theorem on Formal Functions, the adjunction map

$$\alpha : \mathcal{O}_{P^3_{A_0}} \to R(\beta_0)_* \mathcal{O}_{P_0}$$

is an isomorphism. The deformation theory of $\beta_0^* \mathcal{O}(1)$ to $P$ is governed by the cohomology groups $H^2(P, \mathcal{O})$ (where obstructions live) and $H^1(P, \mathcal{O})$ (acting simply transitively on deformations). Since $\alpha$ is an isomorphism, the projection formula shows that these groups are naturally isomorphic to $H^i(P^3_{A_0}, \mathcal{O}) \otimes_{A_0} I$, $i = 1, 2$, which vanish by cohomology and base change and the calculation of the cohomology of projective space.

This shows two things: first, that $\beta_0^* \mathcal{O}(1)$ admits a unique deformation $L$ to $P$, and second that all sections of $\beta_0^* \mathcal{O}(1)$ admit lifts to sections of $L$ over $P$. There is thus a commutative diagram

$$\begin{array}{ccc}
\mathcal{O}^\oplus 4 & \longrightarrow & L \\
\downarrow & & \downarrow \\
\mathcal{O}_{P_0} & \longrightarrow & \beta_0^* \mathcal{O}(1)
\end{array}$$

of coherent sheaves on $P$ (where the bottom row is the pushforward of the displayed sheaves from $P_0$), yielding a commutative diagram

$$\begin{array}{ccc}
P & \xrightarrow{\beta} & P^3_A \\
\uparrow & & \uparrow \\
P_0 & \xrightarrow{\beta_0} & P^3_{A_0}.
\end{array}$$

We claim that this deformed morphism is also a blow-down. One way to see this is the following. For each exceptional divisor $E \subset P_0$, the normal sheaf is $\mathcal{O}_E(-1)$. Cohomology and base change tells us that

$$H^0(E, \mathcal{O}_E(-1)) = 0 = H^1(E, \mathcal{O}_E(-1)).$$
which shows that each $E_i$ has a unique deformation to an $A$-flat divisor in $P$. The invertible sheaf $L$ is trivial on each $E_i$, so they are all collapsed under $\beta$. More concretely, by Nakayama’s Lemma the Stein factorization of

$$\sqcup E_i \to P$$

produces a union of sections $Z \subset P^3_A$ deforming $Z_0 \subset A$. The pullback of the ideal sheaf of $Z$ to $P$ is precisely the ideal sheaf of $\sqcup E_i$, showing that there is a unique factorization

$$\begin{array}{ccc}
P & \longrightarrow & P^3_A \\
\gamma \downarrow & & \downarrow \\
\operatorname{Bl}_Z P^3_A.
\end{array}$$

The morphism $\gamma$ becomes an isomorphism over $A_0$, whence it must be an isomorphism over $A$ by Nakayama’s Lemma and the $A$-flatness of $P$ and $\operatorname{Bl}_Z P^3_A$. 

### 3.2.2 Lifting deformation for non-collinear configurations

In this section, we explain how any deformation of a non-collinear multiview scheme lifts to a deformation of the associated multiview configuration. Fix a deformation situation

$$I \subset A \to A_0$$

and a non-collinear multiview configuration $\Phi^0$ of length $n$ over $A_0$ with scheme $\operatorname{Sch}(\Phi^0)$.

**Proposition 3.2.2.1.** If $X \subset (\mathbb{P}^2)_n^A$ is an $A$-flat deformation of $\operatorname{Sch}(\Phi^0)$ then there is a deformation $\Phi$ of $\Phi^0$ such that $\operatorname{Sch}(\Phi) = X$ as closed subschemes of $(\mathbb{P}^2)^n$. Moreover, $\Phi$ is unique up to unique isomorphism of deformations of $\Phi^0$ over $A$.

**Proof.** Since $\Phi^0$ is non-collinear, the natural morphism

$$\operatorname{Res}(\Phi^0) \to \operatorname{Sch}(\Phi^0) \subset (\mathbb{P}^2)^n$$

is an isomorphism. By Proposition 3.2.1, any deformation of $\operatorname{Sch}(\Phi^0)$ is a blowup $P$ of $P^3_A$ at $n$ disjoint sections over $\operatorname{Spec} A$. The deformation thus results in a rational map

$$\Phi : \mathbb{P}^3_A \dashrightarrow (\mathbb{P}^2_A)^n$$

extending $\Phi^0$. We wish to show that $\Phi$ is a relative multiview configuration in the sense of Definition 2.3.0.4. To do this, it suffices to check that composition with each projection is a relative pinhole camera. Write $p : \mathbb{P}^3_A \dashrightarrow \mathbb{P}^2_A$ for one such projection; we will abuse notation and also write $p$ for the corresponding map $P \to \mathbb{P}^2_A$ from the blowup. We will write $E$ for the exceptional divisor associated to $p$ and $Z$ for the section blown up to make $E$. That is, we assume that $p$ is the $i$th projection of $\Phi$ and that $E$ is the preimage of the $i$th section in $\mathbb{P}^3_A$, which we call $Z$, uniformly omitting $i$ from the notation. By the pinhole camera assumptions on $\Phi^0$, $p|_{E_A}$ maps $E$ isomorphically to $\mathbb{P}^2_{A_0}$. It follows from Nakayama’s lemma that $p|_E$ maps $E$ isomorphically to $\mathbb{P}^2_A$. 22
Write \( U \subset \mathbb{P}_A^3 \) for the complement of the sections that are blown up to resolve \( \Phi \). By the previous paragraph, we see that \( U_{A_0} \subset \mathbb{P}_A^3 \) is precisely the complement of the camera centers of \( \Phi^0 \). By the universal property of projective space, the morphism \( p \) is given by a surjective morphism

\[
\lambda : \mathcal{O}_P^{\oplus 3} \rightarrow \mathcal{L}
\]

for some \( \mathcal{L} \) in \( \text{Pic}(P) \). Write \( \pi : P \rightarrow \mathbb{P}_A^3 \) for the blow-down map. We know from the definition of pinhole cameras, the rigidity of invertible sheaves on \( P \), and the canonical way to extend morphisms generically across blowups that \( \mathcal{L} \cong \pi^*(\mathcal{O}(1))(-E) \). Moreover, the resulting arrow

\[
f : \pi_* \mathcal{O}^{\oplus 3} \rightarrow \mathcal{O}_{\mathbb{P}_A^3}(1)
\]

has the property that its image is precisely \( \mathcal{O}_{\mathbb{P}_A^3}(1) \otimes \mathcal{I}_Z \), where \( \mathcal{I}_Z \) is the ideal sheaf of \( Z \). (This follows from the universal property of blowing up.) This shows that the cokernel of \( f \) is an invertible sheaf supported on \( Z \), showing that \( p \) is a relative pinhole camera, as desired.

It remains to show that any two such realizations \( \Phi_1 \) and \( \Phi_2 \) are conjugate by an infinitesimal automorphism of \( \mathbb{P}_A^3 \). But this follows immediately from Proposition 2.3.0.11.

### 3.2.3 Lifting deformations for collinear configurations

For the sake of computational ease, in this section we consider a deformation situation \( I \subset A \rightarrow A_0 \) in which \( A \) is an Artinian local ring with maximal ideal \( m \) and \( mI = 0 \). Write \( k = A/m \).

We start with a multiview configuration \( \Phi : \mathbb{P}_{A_0}^3 \rightarrow (\mathbb{P}^2)^n \) whose special fiber \( \Phi_k \) is collinear. Thus, the morphism

\[
\text{Res}(\Phi_k) \rightarrow \text{Sch}(\Phi_k) \subset (\mathbb{P}^2)^n
\]

contracts a line \( \ell \subset \text{Res}(\Phi_k) \). To make things easier to read, write \( R = \text{Res}(\Phi_k) \) and \( B = \text{Sch}(\Phi_k) \). Write \( L_{R/B} \) for the cotangent complex of the morphism \( R \rightarrow B \). In addition, write \( E_1, \ldots, E_n \subset R \) for the exceptional divisors. The usual calculations show that \( K_R = \pi^* K_{\mathbb{P}_A^3} + E_1 + \cdots + E_n \).

**Lemma 3.2.3.1.** If \( n > 4 \) then \( \text{Ext}^2_R(L_{R/B}, \mathcal{O}_R) = 0 \).

**Proof.** Consider the standard spectral sequence

\[
E_2^{pq} = \text{Ext}^p(R, \mathcal{H}^{-q}(L_{R/B}, \mathcal{O}_R)) \Rightarrow \text{Ext}^{p+q}(L_{R/B}, \mathcal{O}_R).
\]

We know that \( \mathcal{H}^0(L_{R/B}) = \Omega^1_{R/B} \), and that \( \mathcal{H}^{-j}(L_{R/B}) \) is supported on \( \ell \) for all \( j \geq 0 \). By Serre duality, we can compute the terms in the spectral sequence as

\[
\text{Ext}^p(R, \mathcal{H}^{-q}(L_{R/B}, \mathcal{O}_R)) = H^{3-p}(R, \mathcal{H}^{-q}(L_{R/B})(K_R))^\vee.
\]

Since the cohomology sheaves of \( L_{R/B} \) are all supported on \( \ell \), all columns of the \( E_2^{pq} \) page vanish except (possibly) for \( p = 2, 3 \). It follows that

\[
\text{Ext}^2_R(L_{R/B}, \mathcal{O}_R) \cong H^1(R, \Omega^1_{R/B}(K_R))^\vee.
\]

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A local calculation shows that $\Omega_{R/B}^1$ is annihilated by the ideal of $\ell$, so that $\Omega_{R/B}^1 = \Omega_{\ell/\text{Spec} \ k}^1$, and thus

$$H^1(R, \Omega_{R/B}^1(K_R))^\vee \cong H^1(\ell, \mathcal{O}_\ell(K_\ell + K_R))^\vee \cong H^0(\ell, \mathcal{O}_\ell(-K_\ell)) = H^0(\ell, \mathcal{O}(4-n)) = 0,$$

as desired. 

**Proposition 3.2.3.2.** Suppose $n > 4$. If $X \subset (\mathbb{P}^2)_A^n$ is an $A$-flat deformation of $\text{Sch}(\Phi^0)$ then there is a deformation $\Phi$ of $\Phi^0$ such that $\text{Sch}(\Phi) = X$ as closed subschemes of $(\mathbb{P}^2)^n$. Moreover, $\Phi$ is unique up to unique isomorphism of deformations of $\Phi^0$ over $A$.

**Proof.** By Lemma 3.2.3.1 and [10, Proposition III.2.2.4], the obstruction to deforming $\text{Res}(\Phi^0) \to \text{Sch}(\Phi^0)$ over $A$ vanishes, resulting in a deformation $R \to X$. Applying the results of Section 3.2.1, we see that this arises from a deformation $\Phi$, as desired. The uniqueness of $\Phi$ up to isomorphism is an immediate consequence of Proposition 2.3.0.11.

### 3.2.4 Proof of Theorem 3.2.0.2

In this section we complete the proof of Theorem 3.2.0.2. In the present terminology, this is equivalent to the following Proposition.

**Proposition 3.2.4.1.** If $\Phi$ is a non-collinear general multiview configuration or $\text{len}(\Phi) > 4$ then the morphism

$$\text{Sch} : \text{Def}_{\Phi^0} \to \text{Def}_{\text{Sch}(\Phi^0)}$$

is an isomorphism of deformation functors.

**Proof.** The injectivity of $\text{Sch}$ follows from Proposition 2.3.0.11. Surjectivity follows from Proposition 3.2.2.1 in the non-collinear case and Proposition 3.2.3.2 in the case $\text{len}(\Phi) > 4$.

### 3.3 Diagram Hilbert schemes

In this section, we briefly explain a basic idea that is hard to find in the literature: diagram Hom-schemes and diagram Hilbert schemes. They are a mild elaboration of the idea of a flag Hilbert scheme, and we will see that they play a key role in the moduli theory of calibrated camera configurations.

#### 3.3.1 Definition and examples

Fix a base scheme $S$, a category $I$, and a functor $\underline{X} : I \to \text{AlgSp}_S$.

**Definition 3.3.1.1.** The **diagram Hilbert functor**

$$\text{Hilb}_{\underline{X}} : \text{Sch}_S \to \text{Sets}$$

is the functor whose value on an $S$-scheme $T$ is the set of isomorphism classes of natural transformations $Y \to \underline{X} \times_S T$ of functors $I \to \text{Sch}_T$ where for each $i \in I$ the associated arrow $Y(i) \to \underline{X}(i) \times_S T$ is a $T$-flat family of proper closed subschemes of $\underline{X}(i)$ of finite presentation over $T$. 24
Example 3.3.1.2. The usual Hilbert scheme is an example: just take $I$ to be the singleton category. So is the flag Hilbert scheme of length $n$: in this case the category $I$ is the category $\{1, \ldots, n\}$, and the functor $X$ is the constant functor $X : \mathcal{X} \to \mathcal{X}$. A natural transformation $Y \to X$ defines a nested sequence of closed subschemes of $X$. This is the flag Hilbert scheme (of length 2 flags).

There is also a stricter kind of flag scheme: suppose $X_1 \subset X_2$ is a closed immersion and one wants to parameterize pairs $Y_i \subset X_i$ such that $Y_1 \subset Y_2$. That is precisely the diagram Hilbert functor associated to the poset-category $\mathcal{P} = \{0 < 1\}$ with the functor $\mathcal{P} \to \text{Sch}_S$ sending $i$ to $X_i$. This last example is the one that will arise naturally for us in the context of calibrated cameras. (We record more general results here in case someone in the future needs this general idea of diagram Hilbert scheme.)

Notation 3.3.1.3. If the diagram in question is a single morphism $X \to Y$, we will write $\text{Hilb}_{X \to Y}$ for the associated Hilbert functor.

3.3.2 Representability

The main result about diagram Hilbert functors is that they are representable.

Proposition 3.3.2.1. Let $I$ be a finite category and $X : I \to \text{AlgSp}_S$ a functor whose components are separated algebraic spaces. Then the diagram Hilbert functor $\text{Hilb}_X$ is representable by an algebraic space locally of finite presentation over $S$. If the $X(i)$ are locally quasi-projective schemes then $\text{Hilb}_X$ is represented by a locally quasi-projective $S$-scheme.

Proof. There is a natural functor

$$F : \text{Hilb}_X \to \prod_{i \in I} \text{Hilb}_{X(i)},$$

and we know that the latter is representable by algebraic spaces (resp. schemes) satisfying the desired conditions. It thus suffices to show the same for $F$, i.e., that $F$ is representable by spaces of the required type.

For each $i \in I$, let

$$Z_i \subset X(i) \times \prod_i \text{Hilb}_{X(i)}$$

denote the universal closed subscheme (pulled back over the product). Let $A$ denote the set of arrows in $I$; for an arrow $a \in A$, let $s(a)$ and $t(a)$ denote the source and target of $a$. Consider the scheme

$$H := \prod_{a \in A} \text{Hom}_{\text{Hilb}_X(i)}(Z(s(a)), Z(t(a))),$$

which naturally fibers over $\prod_i \text{Hilb}_{X(i)}$. The standard theory of Hom-schemes shows that $H \to \prod_i \text{Hilb}_{X(i)}$ is representable by spaces of the desired type.

The final observation to make is that composition of two arrows gives equations $b \circ a = c$ in $A$, and these translate into closed conditions on $H$ because all of the subschemes $Z(i)$ are separated. Since the conditions desired are stable under taking closed subspaces, we have proven the result. □
3.4 Moduli of calibrated camera configurations

Let \( \mathcal{C} \) denote the space of smooth conics in \( \mathbb{P}^2_{\text{Spec } \mathbb{Z}} \), and let \( C_{\text{univ}} \subset \mathbb{P}^2_{\text{Spec } \mathbb{Z}} \) denote the universal smooth conic. (The space \( \mathcal{C} \) is an open subscheme of the bundle of sections of \( \mathcal{O}_{\mathbb{P}^2_{\text{Spec } \mathbb{Z}}} (2) \).) The tuple of conics \( (C_{\text{univ}}, \ldots, C_{\text{univ}}) \) inside \( (\mathbb{P}^2)^n \) will be called the universal calibration.

**Definition 3.4.0.1.** Given a positive integer \( n \), the stack of calibrated camera configurations of length \( n \), denoted \( \text{CalCam}_n \), is the stack over \( \mathcal{C}^n \) whose value over a point \( t : S \to \mathcal{C}^n \) consists of the groupoid of general relative calibrated multiview configurations of length \( n \) with calibration datum of the form \( (C, t^*(C_{\text{univ}}, \ldots, C_{\text{univ}})) \).

In down-to-earth terms, we are just describing the stack of \( n \)-tuples of calibrated cameras with pairwise non-intersecting centers, together with arbitrary but specified calibration data. In the existing literature, the word “calibrated” usually means that one has fixed the calibrating conics to be the canonical absolute conic in space (attached to the Euclidean distance form on \( \mathbb{P}^3 \)) and the circle in the plane. Since any two smooth conics are conjugate under a homography, this seems harmless. As we hope to describe in this section, thinking more geometrically and tracking the conics as data instead of normalizing them gives us a great deal of insight into the underlying moduli problem. The point of the universal conic in \( \mathbb{P}^2 \) is that we only want to allow the conic in \( \mathbb{P}^3 \) to vary; that is, we fix calibration data on the image planes when we define the moduli problem. By working with the universal conic, we allow those fixed planar data to be arbitrary.

**Notation 3.4.0.2.** Since we are fixing the calibration data on the image planes to be the universal conic, we will omit them from the notation for a calibration datum. Thus, we will write \( (\Phi, C) \) for a calibrated configuration. When we need to refer to the image plane calibrating curves, we will use \( C_i \) for the curve in the \( i \)th plane, It is key to remember that while \( C_i \) can vary as the base varies (depending upon how it maps to \( \mathcal{C}^n \), this is determined solely by the base and not by the object of \( \text{CalCam}_n \) over that point of the base.

The main result of this section is the following.

**Proposition 3.4.0.3.** The stack \( \text{CalCam}_n \) is a smooth algebraic space of finite type over \( \mathcal{C}^n \).

Let \( \tau_n : \text{CalCam}_n \to \text{CalCam}_{n-1} \times_{\mathcal{C}^{n-1}} \mathcal{C}^n \) be the morphism given by forgetting the last camera (and retaining the last calibrating plane conic).

**Lemma 3.4.0.4.** The morphism \( \tau_n \) is representable by separated schemes of finite presentation.

**Proof.** Let \( ((\varphi_1, \ldots, \varphi_{n-1}, C), C_n) \) by a \( T \)-valued point of \( \text{CalCam}_{n-1} \times_{\mathcal{C}^{n-1}} \mathcal{C}^n \). The fiber of \( \tau_n \) is given by the set of cameras \( \varphi_n \) with the same domain \( \mathbf{P} \to T \) as the first \( n-1 \) cameras, with the following additional properties.

1. The center of \( \varphi_n \) avoids the centers of \( \varphi_i \) for \( i = 1, \ldots, n-1 \).
2. The center of \( \varphi_n \) avoids \( C \).
3. The restriction \( \varphi_n|C \) factors through the closed subscheme \( C_n \subset \mathbf{P} \).
The space of camera centers satisfying the first two conditions is an open subscheme \( P^o \subset P \), and taking the center gives a natural map
\[
\text{CalCam}_n \to P^o \times \text{CalCam}_{n-1} \times \mathcal{C}^{n-1} \times \mathcal{C}^n.
\]

It suffices to show that this map is representable, and thus we may assume that the center is a given section \( \sigma : T \to P \). Blowing up along \( \sigma(T) \) to yield \( \tilde{P} \), with exceptional divisor \( E \), we can then realize the cameras inside the open locus of the \( \text{Hom} \)-scheme \( \text{Hom}(\tilde{P}, P^2) \) parametrizing maps \( f : \tilde{P} \to P^2 \) for which \( f^* \mathcal{O}_{P^2}(1) \) is isomorphic to \( \mathcal{O}(1)(-E) \) on each geometric fiber over \( T \). This locus is of finite type. Finally, the condition that \( C \) lands in \( C_n \) is closed (and of finite presentation), completing the proof. \( \square \)

**Proposition 3.4.0.5.** The morphism \( \tau_n \) is smooth.

**Proof.** By Lemma 3.4.0.4 and [17, Tag 02H6], it suffices to show that \( \tau_n \) is formally smooth. Let \( A \to A_0 \) be a square-zero extension of rings, and suppose that \( (\varphi_1, \ldots, \varphi_n, C) \in \text{CalCam}_n(A_0) \) is fixed. To show formal smoothness we can work Zariski-locally and thus assume that the domains of \( \varphi_1, \ldots, \varphi_n \) are \( P^3_{A_0} \). Now suppose that we fix a deformation \((\varphi'_1, \ldots, \varphi'_{n-1}, C_{A_0}) \in \text{CalCam}_{n-1}(A_0) \times \mathcal{C}^{n-1}(A_0) \).

(Because we are working over the universal conic in each image plane, we have to specify the deformation of the conic that we will use in attempting to deform the \( n \)th calibrated camera.) To show formal smoothness is suffices to extend \( \varphi_n \) to a morphism \( \varphi'_n \) that maps \( C_{A_0} \) to \( C_n \).

The choice of deformation of \( C \) to \( C_A \) induces a lift of \( C \to P^2_{A_0} \) to \( C_A \to P^2_A \). This is because embeddings of degree two curves are given by choosing sections of \( \mathcal{O}_{P^1}(2) \), so any collection of sections embedding \( C \) can be extended to an embedding of \( C_A \).

We are thus reduced to the following: we are given a tuple of three sections \( \sigma_0, \sigma_1, \sigma_2 \in \Gamma(P^3_{A_0}, \mathcal{O}(1)), \) a degree two curve \( C_A \subset P^3_A \), and lifts of the \( \sigma_j|_C \) to \( \Gamma(C_A, \mathcal{O}(1)) \). We wish to lift these extensions to sections \( \tilde{\sigma}_j \in \Gamma(P^3_A, \mathcal{O}(1)) \). We can do this one section at a time. Since \( C_A \) is a degree two curve, it is contained in a canonically defined plane in \( P^3_A \); we will write \( C_A \subset P^2_A \subset P^3_A \) and similarly for \( A_0 \). (If the plane is not trivial, we can further shrink \( A \) to make it so; this is immaterial for the calculations and is only a notational device.)

Consider the diagrams
\[
\begin{align*}
0 \to & \Gamma(P^3_{A_0}, \mathcal{O}) \otimes_{A_0} I \to \Gamma(P^3_A, \mathcal{O}) \to \Gamma(P^3_{A_0}, \mathcal{O}) \to 0 \\
0 \to & \Gamma(P^3_{A_0}, \mathcal{O}(1)) \otimes_{A_0} I \to \Gamma(P^3_A, \mathcal{O}(1)) \to \Gamma(P^3_{A_0}, \mathcal{O}(1)) \to 0 \\
0 \to & \Gamma(P^2_{A_0}, \mathcal{O}(1)) \otimes_{A_0} I \to \Gamma(P^2_A, \mathcal{O}(1)) \to \Gamma(P^2_{A_0}, \mathcal{O}(1)) \to 0
\end{align*}
\]
and

\[
\begin{array}{cccc}
0 & \longrightarrow & \Gamma(\mathbb{P}^2_{A_0}, \mathcal{O}(-1)) \otimes_{A_0} I & \longrightarrow & \Gamma(\mathbb{P}^2_{A}, \mathcal{O}(-1)) & \longrightarrow & \Gamma(\mathbb{P}^2_{A_0}, \mathcal{O}(-1)) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \Gamma(\mathbb{P}^2_{A_0}, \mathcal{O}(1)) \otimes_{A_0} I & \longrightarrow & \Gamma(\mathbb{P}^2_{A}, \mathcal{O}(1)) & \longrightarrow & \Gamma(\mathbb{P}^2_{A_0}, \mathcal{O}(1)) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \Gamma(C, \mathcal{O}(1)) \otimes_{A_0} I & \longrightarrow & \Gamma(C_A, \mathcal{O}(1)) & \longrightarrow & \Gamma(C, \mathcal{O}(1)) & \longrightarrow & 0.
\end{array}
\]

By the usual calculations of the cohomology of projective space, these two diagrams have exact columns. A simple diagram chase then shows that we can lift sections to \(\mathbb{P}^3\) given values on \(\mathbb{P}^2_{A_0}\) and \(C_A\), completing the proof.

**Proof of Proposition 3.4.0.3.** It remains to show smoothness. We use Proposition 3.4.0.5 and induction on \(n\). For \(n = 1\), we see that \text{CalCam}_1\ is smooth over \(\mathcal{C}\), which is itself open in a projective space, hence smooth.

### 3.5 Deformation theory of calibrated camera configurations

In this section we prove the following analogue of Theorem 3.2.0.2.

**Theorem 3.5.0.1.** If \((\Phi, C)\) is a non-degenerate calibrated general multiview configuration of length \(n\) with associated multiview flag

\[(C \subset V) \hookrightarrow (C_1 \times \cdots \times C_n \subset (\mathbb{P}^2)^n)\]

then, assuming either that \(\Phi\) is non-collinear or that \(n > 4\), we have that the infinitesimal deformations of \((\Phi, C)\) are in bijection with the infinitesimal deformations of \(C \subset V\) as a closed subscheme diagram of \(C_1 \times \cdots \times C_n \subset (\mathbb{P}^2)^n\).

**Proof.** The proof leverages the proof of Theorem 3.2.0.2. In particular, we can forget the calibrations and apply Theorem 3.2.0.2 to see that under the given hypotheses any deformation of \(\text{Flag}(\Phi, C)\) induces a deformation of \(\text{Sch}(\Phi)\) that is the image of a deformation \(\tilde{\Phi}\) of \(\Phi\). The assumption that the deformation of \(\text{Sch}(\Phi)\) arises from a deformation of \(\text{Flag}(\Phi, C)\) means that there is also an associated deformation of \(C\). Since \(\Phi\) is an isomorphism onto its image in a neighborhood of \(C\), this deformation of \(C\) canonically lifts to give a calibration of \(\tilde{\Phi}\).

### 4 Comparison morphisms

In this section we compare \(\text{Cam}_n\) and \(\text{CalCam}_n\) by the natural forgetful morphism and we study the natural maps from the two camera moduli problems to appropriate Hilbert schemes.
4.1 The decalibration morphism \( \nu_n : \text{CalCam}_n \to \text{Cam}_n \times \mathcal{E}^n \)

In this section, we study a natural morphism
\[
\text{CalCam}_n \to \text{Cam}_n \times \mathcal{E}^n
\]
given by forgetting the camera calibration datum.

**Definition 4.1.0.1.** The *decalibration morphism* is the morphism
\[
\nu_n : \text{CalCam}_n \to \text{Cam}_n \times \mathcal{E}^n
\]
given by sending \((\Phi, C)\) to \(\Phi\).

Our main result is that \(\nu_n\) is unramified and non-injective. Thus, while \(\text{CalCam}_n\) is smooth over \(\mathcal{E}^n\), its image in \(\text{Cam}_n \times \mathcal{E}^n\) need not be smooth. And this happens in practice: for \(n = 2\), if we take the standard circle in each image plane as calibration datum, the morphism \(\nu_n\) becomes (in the fiber over the “circles” calibration datum) a map into the variety of fundamental matrices whose image is the subvariety of essential matrices. The latter is singular. It would be interesting to understand precisely how its singularities arise from the point of view we take here. (Perhaps this singular locus is precisely the locus where there is only a single calibrating conic in \(\mathbb{P}^3\).)

4.1.1 Intersections of conic cones

Before we delve into the geometry of \(\nu_n\), we need a few preliminaries about intersections of conic cones in \(\mathbb{P}^3\). We thank Bianca Viray for pointing out an omission in the previous version of this Proposition and helping us think through the correct list of possibilities.

**Proposition 4.1.1.** Let \(X_1\) and \(X_2\) be two conic cones in \(\mathbb{P}^3\) with distinct cone points. The intersection \(X_1 \cap X_2\) is one of the following.

1. An irreducible curve of degree 4.
2. A union of a twisted cubic and a line.
3. A union of two smooth conics.
4. A union of one smooth conic and a doubled line.

It can never be a doubled conic, a quadrupled line, or contain two distinct lines.

*Proof.* Sections of \(\mathcal{O}_{\mathbb{P}^3}(2)\) correspond to symmetric \(4 \times 4\)-matrices (at least if 2 is invertible on the base scheme). The conic cones correspond to the rank 3 matrices. Thus, they form a dense open in a hypersurface in \(|\mathcal{O}_{\mathbb{P}^3}(2)|\) of degree 4.

The intersection \(X_1 \cap X_2\) is an effective Cartier divisor on \(X_1\) with class \(\mathcal{O}_{X_1}(2)\). Since a general pencil of sections of \(\mathcal{O}_{\mathbb{P}^3}(2)\) will intersect the rank 3 locus in four points, a general pair of cones span a general pencil, and thus they will have smooth intersection by Bertini’s theorem. Thus, the intersection \(X_1 \cap X_2\) can be (in fact, usually is) a smooth curve of degree 4.

Given a twisted cubic \(C \subset \mathbb{P}^3\), choose two distinct points \(x_1, x_2 \in C\). Consider linear projections \(\pi_i : \mathbb{P}^3 \to \mathbb{P}^2\) centered at \(x_i\). The image of \(C\) under a general such projection
will be a smooth conic curve $D_i \subset \mathbb{P}^2$. Intersecting the cones $X_1$ and $X_2$ over $D_1$ and $D_2$ yields $C \cup L$, where $L$ is a line. (Indeed, the total curve must have degree 4, so the residual curve has degree 1 and must be a line.) Note that $X_1 \cap X_2$ cannot contain a singular cubic space curve. Indeed, any reduced singular cubic must be a plane curve (since otherwise a general projection from a smooth point on the curve will map it to an irreducible singular conic in the plane, which is impossible), and a tripled line on $X_1 \cap X_2$ cannot be Cartier at the cone points, hence $X_1 \cap X_2$ must contain another line, which forces the cone points to coincide.

If the intersection contains a conic (which can be arranged by fixing a conic $C$ and noting that the kernel of the restriction map

$$\Gamma(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) \rightarrow \Gamma(C, \mathcal{O}_C(2)) = \Gamma(\mathbb{P}^1, \mathcal{O}(4))$$

has dimension 5, thus producing many pencils through $C$), then the residual curve is either another conic, a doubled line, a pair of intersecting lines, or another copy of the same conic (i.e., the conic is doubled). We analyze these separately.

First, one can have two smooth conics. It suffices to find a single example to see that this is general behavior within the locus of pencils with non-smooth intersection. A simple example is furnished by the conic cones given in homogeneous coordinates by $X^2 + Y^2 + Z^2 = 0$ and $Y^2 + Z^2 + W^2 = 0$.

What if the residual curve is two distinct lines meeting at a point? Since this must be in both cones, and any such pair of lines must meet at the cone point, we would conclude that $X_1$ and $X_2$ have the same cone point, contrary to our assumption. Thus, this cannot happen.

The residual curve can be a doubled line. An example is given by the pair of cones $X^2 - YZ$ and $(X - \alpha W)^2 - (Y - \alpha W)(Z - \alpha W)$ for any non-zero $\alpha$. (The first cone is being translated along one of its rulings. The resulting cones are tangent along this ruling, leading to a double line of intersection.)

The intersection cannot be a quadrupled line. A quadrupled line is the intersection of two doubled planes. So we can take a pencil generated by two doubled planes and show that it cannot contain any rank 3 forms. After a change of coordinates, we may assume that the doubled planes are given by $X^2 = 0$ and $Y^2 = 0$. The pencil they span cannot contain any form of rank greater than 2.

It remains to rule out a doubled conic. Note that a doubled conic is the intersection of $X_1$ with a doubled plane $2P \in \mathcal{O}_{\mathbb{P}^3}(2)$. We can rule out this case if we can show that the pencil spanned by $X_1$ and a doubled plane not containing its cone point does not contain any more conic cones. This readily reduces to the following matrix calculation. We can represent the cone $X_1$ by the matrix

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
$$

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and an arbitrary doubled plane missing the cone point by the matrix
\[
\begin{pmatrix}
  a^2 & ab & ac & a \\
  ab & b^2 & bc & b \\
  ac & bc & c^2 & c \\
  a & b & c & 1
\end{pmatrix}
\]
for \( a, b, c \in k \). Searching for a conic cone in the pencil corresponds to finding \( \lambda \) such that the matrix
\[
\begin{pmatrix}
  a^2 + \lambda & ab & ac & a \\
  ab & b^2 + \lambda & bc & b \\
  ac & bc & c^2 + \lambda & c \\
  a & b & c & 1
\end{pmatrix}
\]
has rank 3. But row-reducing that matrix yields
\[
\begin{pmatrix}
  \lambda & 0 & 0 & 0 \\
  0 & \lambda & 0 & 0 \\
  0 & 0 & \lambda & 0 \\
  a & b & c & 1
\end{pmatrix},
\]
and this matrix can never have rank 3.

\[\boxed{}\]

**Lemma 4.1.1.2.** Fix a smooth conic \( C \subset \mathbb{P}^3 \). The space \( \Xi \subset \mathbb{P}^9 \) of conic cones in the linear system \( |\mathcal{O}_{\mathbb{P}^3}(2)| \) that contain \( C \) has dimension 3.

**Proof.** The restriction map
\[
\Gamma(\mathbb{P}^3, \mathcal{O}(2)) \to \Gamma(\mathbb{P}^2, \mathcal{O}(2))
\]
induces a surjective linear projection of linear systems
\[
\rho : \mathbb{P}^9 \longrightarrow \mathbb{P}^5.
\]
Moreover, since any pair of smooth conics in \( \mathbb{P}^3 \) are conjugate under an automorphism of \( \mathbb{P}^3 \), we have that the fibers over any two smooth conics are isomorphic. Since smooth conics are general points of \( \mathbb{P}^5 \), we see that the fiber over a smooth conic has dimension 4 (i.e., 9 – 5).

On the other hand, the locus \( S \subset \mathbb{P}^9 \) of conic cones is open in the hypersurface of singular members of \( |\mathcal{O}_{\mathbb{P}^3}(2)| \). Since the cone over \( C \) contains \( C \), we have that \( S \cap \rho^{-1}([C]) \) is non-empty, hence is a threefold in \( \rho^{-1}([C]) \).

**Lemma 4.1.1.3.** Suppose \( A, B \in \Gamma(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) \) is a regular sequence of elements (i.e., they intersect properly everywhere). Let \( I = A \cap B \) the scheme-theoretic intersection. Then the kernel of
\[
\Gamma(\mathbb{P}^3, \mathcal{O}(2)) \to \Gamma(I, \mathcal{O}(2))
\]
is the subspace spanned by \( A \) and \( B \).

**Proof.** By the regular sequence assumption, we have a resolution
\[
0 \to \mathcal{O}(-4) \to \mathcal{O}(-2)^{\oplus 2} \to \mathcal{O} \to \mathcal{O}_I \to 0.
\]
Twisting by 2 and using the vanishing of \( H^1(\mathbb{P}^3, \mathcal{O}(a)) \) (for all \( a \)) gives the result. \( \square \)
4.1.2 The geometry of $\nu_n$

Fix a point $\xi$ of $\Cam_n \times \mathcal{C}^n$. That is, fix conics $C_1, \ldots, C_n$ in $\mathbb{P}^2$ and a multiview configuration $\Phi$. In this section we compute the fiber of $\nu_n$ over $\xi$.

**Proposition 4.1.2.1.** The scheme-theoretic fiber $\nu_n^{-1}(\xi)$ is a reduced $\kappa(\xi)$-scheme of length at most 2.

*Proof.* The fiber $\nu_n^{-1}(\xi)$ is precisely the scheme of smooth conics in the intersection of the cones over the image conics $C_i$ inside the ambient $\mathbb{P}^3$. The result is thus immediate from Proposition 4.1.1.1. (In particular, the lack of doubled conic means that the fibers are discrete.) \[\square\]

**Corollary 4.1.2.2.** The morphism $\nu_n$ is unramified.

*Proof.* This is an immediate consequence of Proposition 4.1.2.1. \[\square\]

**Proposition 4.1.2.3.** The morphism $\nu_n$ is proper.

*Proof.* Suppose we have a multiview configuration $\Phi$ of length 2 over a complete dvr $R$ with fraction field $K$, degree two curves $C_1, \ldots, C_n \subset \mathbb{P}^2_R$ and a degree two curve $C_K \subset \mathbb{P}^3_K$ such that $\Phi_K$ maps $C_K$ isomorphically to the generic fiber of each $C_i$. By the valuative criterion for properness it suffices to extend $C_K$ to a degree two curve $C_R$.

Assume we have a multiview configuration $\Phi$ of length 2 over a complete dvr $R$ with fraction field $K$, and suppose we have conics $C_1, \ldots, C_n \subset \mathbb{P}^2_R$ in each image plane. Write $\overline{C}_i \subset \mathbb{P}^3$ for the cone over $C_i$ under $\text{pr}_i \circ \Phi$ and $I = \overline{C}_1 \cap \cdots \cap \overline{C}_n$. Finally, assume that there is a conic $C_K \subset \mathbb{P}^3_K$ such that $\Phi_K$ maps $C_K$ isomorphically to the generic fiber of each $C_i$; that is, $C_K \subset I_K$. Let $C_R$ be the specialization of $C_K$ in the closed fiber $C_0$. The curve $C_R$ is degree 2, giving us a calibrated configuration over $R$. \[\square\]

Note that even if $C_k$ is a non-degenerate conic, $C_0$ need not be. This is why we need to add degenerate conics.

**Proposition 4.1.2.4.** The morphism $\nu_2$ has smooth image and general fiber of length 2. For any $n > 2$ the morphism $\nu_n$ is generically injective.

*Proof.* The projective closure of the image of a fiber of $\CalCam_2$ over $\mathcal{C}^2$ under $\nu_2$ is known as the “essential variety”, and its singularities are well-known (see [6, Proposition 2.1]); none of its singular points lies in the image of $\nu_2$. To study the general fiber, it suffices by the irreducibility of all spaces involved to produce a single example of a camera configuration of length two such that the fiber of $\nu_n$ has length 2. (Indeed, the locus of quartic curves in $\mathbb{P}^3$ containing a conic has a union of two conics as its generic point.) To do this, it further suffices to find a single example of two conic cones $C_1, C_2 \subset \mathbb{P}^3$ whose intersection is a pair of smooth conics. (Indeed, general projections from the two cone points give image planes together with calibrating conics that give rise to fibers of $\nu_2$ of length 2.) But this has already been written down in the proof of Proposition 4.1.1.1.

We now show that $\nu_n$ is generically injective for $n > 2$. By Lemma 4.1.1.2, given a smooth conic $C$ in $\mathbb{P}^3$, the locus in $|\mathcal{O}_{\mathbb{P}^3}(2)|$ consisting of conic cones containing $C$ is 3-dimensional. Thus, we can find three non-collinear conic cones that contain any given smooth conic $C$. 

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On the other hand, by Lemma 4.1.3, given two conic cones $C_1, C_2$, the set of conic cones that vanish on their entire intersection $C_1 \cap C_2$ is contained in the pencil spanned by the $C_i$. We conclude that if $C_1 \cap C_2$ is reducible, then we can choose general cones $C_3, \ldots, C_n$ containing a smooth conic in $C_1 \cap C_2$ such that $C_i$ is not in the pencil spanned by $C_1$ and $C_2$ for each $i > 2$. The joint vanishing locus $C_1 \cap C_2 \cap C_3 \cap \cdots \cap C_n$ is a smooth conic.

Since this is generic behavior, this shows that $\nu_n$ is generically injective for all $n > 2$.

It is a potentially interesting problem to characterize the locus over which $\nu_n$ is not injective, and the singular locus of its image (the “variety of calibrated $n$-focal tensors”, which is studied for $n = 3$ in coordinatized form in [11]).

**Corollary 4.1.2.5.** The morphism $\nu_n$ is finite.

*Proof.* We have shown that $\nu_n$ quasi-finite and proper and thus, finite.

### 4.2 Twisted pairs and moduli

In this section we study the morphism $\nu_2$ in more detail, showing how the Hilbert scheme gives a natural compactification of the classical “twisted pair” construction. To explicitly compare this new treatment with the literature, in this section we will fix the calibrating conics to be $v(x_0^2 + x_1^2 + x_2^2) \subset \mathbb{P}^2$. Also, we will often think of an essential matrix as the corresponding pair of calibrated cameras in normalized coordinates. In these coordinates we can fix notation $P_1 = [I|0]$ and $P_2 = R[I|t]$ where $t = (a, b, c)$.

#### 4.2.1 Twisted pairs

As shown in Section 5.2 of [14], the locus $M$ of essential matrices is smooth (over $\mathbb{C}$) and admits an étale surjection $SO(3) \times \mathbb{P}^2 \rightarrow M$. We can understand this surjection as follows. Given a calibrated camera $P : \mathbb{P}^3 \rightarrow \mathbb{P}^2$, we can make a new camera $Q$ by composing with a rotation (an element of $SO(3)$) and a translation (and element of $\mathbb{A}^3 \setminus \{0\}$). Since there is always a scaling ambiguity in reconstruction, we may assume we are translating by a unit vector, so the translation is really an element of $\mathbb{P}^2$. (This scaling ambiguity also allows us to replace any arbitrary orthogonal coordinate transformation with an element of $SO(3)$, by scaling by $-1$.) This gives a map $\pi : SO(3) \times \mathbb{P}^2 \rightarrow M$. In terms of matrices we send $(R, t)$ to the camera pair $P = [I|0], Q = [R|t]$ which has essential matrix $[t] \times R$. Since coordinates can be normalized to write any pair of calibrated cameras in this form the morphism $\pi$ is surjective on geometric points. Moreover, one can check in local coordinates that the map is étale [5, Proposition 3.2].

For any real essential matrix $M \in M(\mathbb{R})$, the fiber of $\pi$ over $M$ contains two points: one can take a pair of cameras $P_1, P_2$ and replace it with the pair $P_1, \tilde{P}_2$ where $\tilde{P}_2$ results from rotating $P_2$ by 180 degrees around the axis connecting the centers of $P_1$ and $P_2$. In normalized coordinates, the matrix

$$
R_t = \begin{pmatrix}
2a^2 - 1 & 2ab & 2ac & 0 \\
2ab & 2b^2 - 1 & 2bc & 0 \\
2ac & 2bc & 2c^2 - 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
$$
When $a^2 + b^2 + c^2 = 1$. The pair $(P_1, P_2)$ is called a twisted pair, what we have described is a well-known construction in computer vision [2] Result 9.19. The key thing to note is that the rotation construction described above preserves calibrations for real cameras. For complex cameras, things get more complicated, and for displacements $(a, b, c)$ such that $a^2 + b^2 + c^2 = 0$, the corresponding transformation produces a new camera pair $(\tilde{P}_1, \tilde{P}_2)$ for which $\tilde{P}_2$ is no longer calibrated.

### 4.2.2 Compactification of the twisted pair construction

The morphism $\nu_2 : \text{CalCam}_2 \to \text{Cam}_2 \times \mathbb{C}^n$ gives a double covering of a closed subscheme that generalizes the twisted pair covering of the essential variety. A point of $\text{CalCam}_2$ is the datum $(P_1, P_2, C)$ where $P_1$ and $P_2$ are cameras and $C$ is a conic contained in the intersection of the cones defined by the preimage of $C_{\text{univ}}$ via $P_1$ and $P_2$. Proposition 4.1.1.1 tells us that this intersection must contain either another non-degenerate conic or a doubled line. In either case denote this other degree two curve by $\tilde{C}$. The general fibers of $\nu_2$ are the triples $(P_1, P_2, C)$ and $(P_1, P_2, \tilde{C})$.

This double covering agrees with the twisted pairs covering on real points. In normalized coordinates $\tilde{C}$ is defined by the simultaneous vanishing of

$$x^2 + y^2 + z^2 = 0 \text{ and } (a^2 + b^2 + c^2)w - 2(ax + by + cz) = 0.$$

When $a^2 + b^2 + c^2 = 1$, as it must over $\mathbb{R}$ (up to scaling), one can check that changing coordinates on $\mathbb{P}^3$ via the automorphism

$$H = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-2a & -2b & -2c & 1
\end{pmatrix}$$

sends the triple $(P_1, \tilde{P}_2, C)$ to the triple $(P_1, P_2, \tilde{C})$.

However, over the complex numbers there exist essential matrices such that $a^2 + b^2 + c^2 = 0$. This is exactly the condition that $\tilde{C}$ is a doubled line. In this situation the twisted pair construction fails because the camera $\tilde{P}$ no longer has a trivial calibration. Mathematically speaking, we are really discussing the fact that the twisted pairs morphism $\pi$, while always étale, is not finite. Allowing degenerate calibrations (doubled lines) extends the twisted pair morphism $\pi$ to $\nu_2$.

**Proposition 4.2.2.1.** There exists a fixed-point free involution, $\chi : \text{CalCam}_2 \to \text{CalCam}_2$ over $\text{Cam}_2$ given by fixing the cameras and swapping calibrating conics. More precisely, $\nu_2 \circ \chi = \nu_2$.

**Proof.** Given a pair of cameras $\Phi : \mathbb{P}^2 \to \mathbb{P}^2 \times \mathbb{P}^2$ and smooth conics $D_1, D_2 \subset \mathbb{P}^2$, we can pull back to get two cones $X_1, X_2 \subset \mathbb{P}^3$. Let $F = X_1 \cap X_2$. Blowing up the camera centers, the strict transform of these cones, $\tilde{X}_1, \tilde{X}_2 \subset \text{Bl}_{Z_1, Z_2} \mathbb{P}^3$, are smooth surfaces in $\mathbb{P}^3$. The intersection is a relative effective Cartier divisor and $\tilde{X}_1 \cap \tilde{X}_2 \simeq F$ since the cone centers are distinct.

A point in $\text{CalCam}_2$ is a pair $(\Phi, C)$ where $C$ is a relative effective Cartier divisor contained in $F$. By [17] Tag 0B8V there exists another relative effective Cartier divisor $C'$
such that $C' + C = F$. Checking at a geometric point, Proposition 4.1.1.1 shows that $C'$ is a degree 2 curve, and that no geometric point of $\text{CalCam}_2$ is fixed by $\chi$. This argument is functorial and so induces the desired involution. Since $\chi$ only changes the calibrating conic we have $\nu_2 \circ \chi = \nu_2$.

**Theorem 4.2.2.2.** The morphism $\nu_2$ factors as a finite étale morphism followed by a closed immersion.

**Proof.** By Corollary 4.1.2.5 $\nu_2$ is a finite morphism, hence closed. This yields a factorization $\text{CalCam}_2 \to Z \to \text{Cam}_2$ with the second arrow a closed immersion and the first scheme-theoretically surjective. Let $A$ be a strictly Henselian local ring and $\text{Spec} \ A \to Z$ a morphism. The finiteness of $\nu_2$ yields a diagram

$$\begin{array}{ccc}
\text{Spec} \ B & \longrightarrow & \text{CalCam}_2 \\
\downarrow \psi & & \downarrow \nu_2 \\
\text{Spec} \ A & \longrightarrow & \text{Cam}_2
\end{array}$$

By [17, Tag 04GH], $B$ is the product of local Henselian rings. By Proposition 4.1.2.4 the general fibers of $\psi$ are length 2, corresponding to the two possible calibrating conics, so $\text{Spec} \ B \simeq \text{Spec} \ B_1 \sqcup \text{Spec} \ B_2$. By Corollary 4.1.2.2 $\psi$ is unramified, and thus (by [17, Tag 04GL]) restricts to a closed embedding on each $\text{Spec} \ B_i$.

$$\begin{array}{ccc}
\text{Spec} \ B_i & \longrightarrow & \text{Spec} \ B_1 \sqcup \text{Spec} \ B_2 \\
\downarrow \psi & & \downarrow \\
\text{Spec} \ A & \longrightarrow & \text{Cam}_2
\end{array}$$

The involution described in Proposition 4.2.2.1 induces an isomorphism $f : \text{Spec} \ B_1 \to \text{Spec} \ B_2$. In other words both components map isomorphically to the image, so $\nu_2$ is étale over $Z$, as claimed.

**4.3 Morphisms to Hilbert schemes**

The following describes the main result relating the moduli problems $\text{Cam}_n$ and $\text{CalCam}_n$ to Hilbert schemes. Because the statements and proofs are so similar, we combine everything into a single omnibus Proposition. This gives the generalization of the results of [3, Section 6], leveraging the novel methods of this paper to give more information about the uncalibrated case and the appropriate result in the calibrated case.

**Proposition 4.3.0.1.** The associations $\Phi \mapsto \text{Sch}(\Phi)$ and $(\Phi, C) \mapsto \text{Flag}(\Phi, C)$ define monomorphisms

$$\text{Sch} : \text{Cam}_n \to \text{Hilb}(\mathbb{P}^2)^n / \text{Spec} \ Z$$

and

$$\text{Flag} : \text{CalCam}_n \to \text{Hilb}^{\text{univ}}_{\text{Cam}}(\mathbb{P}^2)^n / \mathbb{C}$$

such that

1. the restriction of $\text{Sch}$ (resp. $\text{Flag}$) to the non-collinear locus $\text{Cam}^{\text{nc}}_n$ (resp. $\text{CalCam}^{\text{nc}}_n$) is an open immersion into the smooth locus $\text{Hilb}^{\text{sm}}(\mathbb{P}^2)^n / \text{Spec} \ Z$ (resp. $\text{Hilb}^{\text{sm}}_{\text{Cam}}(\mathbb{P}^2)^n / \mathbb{C}^n)$.
2. when $n > 4$, the morphism $\text{Sch}$ (respectively, $\text{Flag}$) itself is an open immersion into $\text{Hilb}^{\text{sm}}_{(\mathbb{P}^2)^n/\text{Spec} \mathbb{Z}}$ (respectively, $\text{Hilb}^{\text{sm}}_{\text{C}_{\text{univ}}^n/\mathbb{C}^n}$);

3. the arrows $\text{Sch}$ and $\text{Flag}$ together with the forgetful maps give a commutative diagram

$$
\begin{array}{ccc}
\text{CalCam}_n & \xrightarrow{\nu_n} & \text{Hilb}_{\text{C}_{\text{univ}}^n/\mathbb{C}^n} \\
\downarrow & & \downarrow \\
\text{Cam}_n \times_{\text{Spec} \mathbb{C}^n} \mathbb{C}^n & \rightarrow & \text{Hilb}_{(\mathbb{P}^2)^n/\mathbb{C}^n}.
\end{array}
$$

In particular, every geometric fiber of $\text{Sch}$ over $\text{Spec} \mathbb{Z}$ is a dominant monomorphism of $\text{Cam}_n$ into a single irreducible component of the Hilbert scheme, and when $n > 4$ the fiber of $\text{Sch}$ itself is open in the smooth locus of the Hilbert scheme, and similarly for geometric fibers of $\text{Flag}$ and components of the diagram Hilbert scheme.

**Proof.** Proposition 2.3.0.13 and Proposition 2.3.0.11 show that $\text{Flag}$ is well-defined and a monomorphism. Since $\text{CalCam}_n$ is smooth over $\mathbb{C}^n$, we have that $\text{Flag}$ is an open immersion in a neighborhood of any point where it induces an isomorphism of deformation functors. Theorem 3.5.0.1 then applies to give the two desired statements. \hfill \square

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