Reminiscence of classical chaos in driven transmons

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Transmon qubits are ubiquitously used in superconducting quantum information processor architectures. Strong drives are required to realize fast, high-fidelity, gates and measurements, including parametrically activated processes. Here, we show that even off-resonant drives, in regimes routinely used in experiments, can cause strong modifications to the structure of the transmon spectrum rendering a large part of it chaotic. Accounting for the full nonlinear dynamics of the transmon in a Floquet-Markov formalism, we find that these chaotic states, often neglected through the hypothesis that the anharmonicity is weak, strongly impact the lifetime of the transmon’s computational states. In particular, we observe that chaos-assisted quantum phase slips greatly enhance band dispersions. In the presence of a measurement resonator, we find that approaching chaotic behavior correlates with strong transmon-resonator hybridization, and an average resonator response centered on the bare resonator frequency. These results lead to a photon number threshold characterizing the appearance of chaos-induced quantum demolition effects during strong-drive operations such as dispersive qubit readout. The phenomena described here are expected to be present in all circuits based on low-impedance Josephson-junctions.

I. INTRODUCTION

Low-impedance Josephson junction circuits, where the Josephson energy dominates over the charging energy, are fundamental building blocks of superconducting quantum processors. Although the most widely used superconducting qubit is the transmon [1], capacitively shunted Josephson junctions appear in other species of qubits, such as the heavy fluxonium [2, 3], the 0 − π qubit [4] and the capacitively shunted flux qubit [5, 6]. Josephson junctions can also be used as simple nonlinear elements for parametrically activated multi-wave mixing [7–10], or as linear inductive elements in a Josephson junction array to realize superinductances [11, 12].

To meet the requirements of quantum information processing with fast and high-fidelity operations, strong driving fields that are off-resonant from the qubit are often used in parametrically activated coupling [7, 13], multi-qubit gates [14, 15] and dispersive readout [16]. However, off-resonant drives of even moderate amplitude are often observed to cause spurious qubit transitions [17–19]. This is the case of the dispersive readout whose quantum-non demolition (QND) character is observed only at very small drive amplitudes, corresponding to a few photons (\( \bar{n} \sim 2 \)) populating the measurement resonator [17]. Models based on perturbative expansion in the qubit-resonator coupling [20] or in qubit anharmonicity and drive amplitude [21, 22] have been explored to understand the origin of these unwanted transitions. At large photon numbers (\( \bar{n} \geq 100 \)), qubit-resonator resonances [23] and structural instabilities [13, 24] have been shown to result in spurious transitions. Recently, a numerical study of the full time dynamics of the transmon has shown that qubit-resonator resonances can lead to leakage of the transmon population to states lying above the Josephson junction cosine potential, something which has been referred to as ionization [25]. In that study, ionization was shown to coincide with the loss of QND-ness experimentally observed at low photon numbers in Ref. [17].

Here, we show that the often-neglected highly excited states of the transmon can play an important role for current experimental parameters and drive amplitudes, even in the absence of ionization [13, 23, 26]. We find that a subset of the highly-excited states in the spectrum of a driven transmon, the so-called chaotic layer, significantly and unexpectedly dresses the charge dispersion of the low-energy transmon spectrum, with detrimental effects on the qubit dephasing time. This phenomenon can be understood as chaos-assisted [27] quantum phase slips [28]. The increased dependence on the offset charge even for the low-energy states suggests that in the presence of strong drives, models which rely on a perturbed harmonic oscillator such as the Kerr nonlinear oscillator [29] cannot give an accurate description of the system, in particular because the selection rules derived from these models are no longer applicable [23]. In addition, we show that the chaotic layer makes steady-state populations deviate significantly from the Boltzmann distribution [24, 30, 31].

We draw upon the Floquet theory [30–32] of nonlinear oscillators [33] to distinguish between the chaotic and regular states [34, 35] of a driven transmon. We intro-
duce a rescaling of the transmon Hamiltonian from which an effective Planck constant emerges, \( h_{\text{eff}} = \sqrt{8E_C/E_J} \) \((h = 1)\). In the transmon regime, where \( h_{\text{eff}} \) is small, the chaotic dynamics of the classical driven transmon becomes more resolved in the quantum spectrum as the number of chaotic states increases. In particular, we show that the spectrum of a single driven transmon is correlated \([36]\). In transmon systems, this type of analysis has been performed in the many-body regime \([37]\). Our study reveals that, within the range of current experimental parameters, the size of the classical chaotic domain is strongly sensitive to the drive frequency, something which can lead to instabilities in the quantum dynamics even at low drive power. Our results suggest ways to avoid experimental realizations of transmons from being plagued by these instabilities.

Moreover, by simulating the full transmon plus resonator circuit QED model, we show that chaos develops along two directions, that of increasing resonator Fock state number and that of increasing drive power. In addition to validating the study of the single driven transmon, we show that the transmon and the resonator strongly hybridize in the chaotic phase. As a result, in the context of the dispersive readout where chaotic effects are present, spurious qubit transitions become possible. In particular, we introduce a critical photon number around which chaos-induced non-QND effects are expected. We also predict that spurious effects below this threshold should be exponentially reduced with \( \sqrt{E_J/8E_C} \).

The remainder of this article is organized as follows. In Sec. II, we introduce a rescaled version of the Hamiltonian along with the key parameters of the dynamics, discuss chaos in the classical driven transmon, and briefly discuss the impact of classical chaos on the quantum system. Sec. III tackles the spectral properties of the system and the dependence of the instability on drive frequency. In Sec. IV, we consider the coupling of the transmon to a bath, and the impact of the chaotic layer on the coherence properties of the low-energy sector is analyzed. Sec. V focuses on the interplay of chaos and the validity of the dispersive approximation in the full circuit QED setup. Sec. VI discusses spurious non-QND effects originating from the interaction with chaotic states.

II. PERIODICALLY DRIVEN TRANSMON

The Hamiltonian of a transmon in a typical circuit QED setup takes the form \([1, 38]\)

\[
H(t) = 4E_C(n - n_g)^2 - E_J \cos(\phi) + \mathbf{nF},
\]  
(1)

where \(n\) and \(\phi\) are respectively the charge and phase operators, \(E_C\) and \(E_J\) are the charging and Josephson energies, and \(n_g\) an offset charge. The phase \(\phi\) is compact and takes its values in the range \((-\pi, \pi]\). The operator \(\mathbf{F}\) represents a classical driving field on the resonator or the coupling to a measurement resonator in a circuit QED setup \([38]\). As such, \(\mathbf{F} = F_c(t) + \mathbf{F}_q\) can be expressed as the sum of a classical part and of a quantum part. The quantum part represents the displaced quadrature of the readout resonator, i.e. \(\mathbf{F}_q = ig(\mathbf{a} - \mathbf{a}^\dagger)\) where \(g\) is the light-matter coupling. On the other hand, the classical part is assumed to take the form \(F_c(t) = \varepsilon_d \cos(\omega_d t)\) and represents either a direct capacitive drive on the transmon or the classical amplitude of the resonator field. In this work, we are concerned with the limit of strong drives, where the classical part dominates over quantum fluctuations, i.e. \(\varepsilon_d > g\). Expressing the drive amplitude \(\varepsilon_d = 2g\sqrt{n}\) in terms of an equivalent number of resonator photons \(\hat{n}\), this strong drive limit corresponds to \(\hat{n} > 1\). In typical cQED setup, one has \(g/2\pi \sim 250\text{MHz}\) and \(\hat{n} \geq 2\), resulting in an effective drive \(\varepsilon_d/2\pi \gtrsim 700\text{MHz}\).

In what follows, we neglect quantum fluctuations \(\mathbf{F}_q\) to study the periodically driven transmon Hamiltonian

\[
H(t) = 4E_C(n - n_g)^2 - E_J \cos(\phi) + \varepsilon_d \cos(\omega_d t)\mathbf{n}.
\]  
(2)

We return to a full circuit QED model accounting for the presence of the resonator in Sec. V.

A. Classical model

We first consider the classical limit of the driven transmon Hamiltonian where we replace the conjugate operators \(\{\phi, n\}\) by the phase-space coordinates \(\{\phi, \mathbf{n}\}\). In doing so, it is useful to rescale energy and time using the relations \(\hat{H} = H/E_J\) and \(\hat{t} = \omega_J t\), where \(\omega_p = \sqrt{8E_J/E_C}/\hbar\) is the plasma frequency of the transmon. Under this transformation, which preserves Hamilton’s equations, the classical Hamiltonian takes the form

\[
\dot{\hat{H}}(\hat{t}) = \left(\hat{n} - \hat{n}_g\right)^2 - \cos(\hat{\phi}) + \varepsilon_d \cos(\tilde{\omega}_d \hat{t})\hat{n}.
\]  
(3)

This corresponds to the Hamiltonian of a driven charged classical pendulum with dimensionless momentum \(\hat{n} = zn\) and position \(\hat{\phi} = \phi\) \([1]\). Here, \(z = \sqrt{8E_C/E_J}\) is the characteristic impedance of the transmon. In this rescaled form, the three relevant parameters of the classical driven transmon are the rescaled drive amplitude \(\hat{\varepsilon}_d = \varepsilon_d/\omega_p\), the rescaled drive frequency \(\tilde{\omega}_d = \omega_d/\omega_p\) and the rescaled offset charge \(\hat{n}_g = zn_g\).

In the absence of a drive, two different types of motion of the system can be distinguished. For \(\hat{H} < 2\) (i.e. \(H < 2E_J\)), the system undergoes small and bounded phase oscillations. On the other hand, for \(\hat{H} > 2\), the system experiences unbounded full \(\pm 2\pi\) rotations of the phase. While manipulations of the transmon qubit are designed such as to only lead to small phase oscillations, the transmon can be promoted to states above \(2E_J\) by strong drives \([13, 23-25]\). In the quantum case, the resulting full rotations correspond to quantum phase slips \([1, 28]\). At the boundary of these two types of motion, defined by the trajectory of energy \(\hat{H} = 2\) also known as the separatrix, small perturbations can have a large
Poincaré sections obtained by plotting the value of the phase space coordinates $\{\phi(t), \tilde{n}(t)\}$ at every period $T = \omega_d/\omega_d$ of the drive for some initial condition [39]. Figure 1(a) first shows this in the absence of a drive. There, the two expected types of motions are clearly visible: the bounded oscillations leading to the closed orbits and the unbounded rotations to the nearly horizontal patterns. In the presence of the drive, see Fig. 1(b), the Poincaré sections break up into regular and chaotic regions. The regular regions consist of weakly perturbed Kolmogorov-Arnold-Moser tori, reminiscent of the motion of the unperturbed system, while the chaotic region develops around the separatrix [33]. The small tori located within the regular unbounded trajectories in Fig. 1(b) are due to two resonances where the drive frequency $\pm \omega_d$ matches the energies of the trajectories which pass in the vicinity of $(\tilde{\phi}, \tilde{n}) = (0, \pm 2.5)$.

B. Quantum model

To compare the quantum dynamics to the classical one, we quantize the rescaled Hamiltonian of Eq. (3). Importantly, because the rescaling does not preserve the phase-space volume, it leads to a renormalization of the Planck constant upon quantization with $\hbar_{\text{eff}} = \hbar \omega_p/\bar{E}_J = z$. As a result, the commutation relation of the rescaled operators is $[\tilde{\phi}, \tilde{n}] = i\hbar_{\text{eff}}$. Consequently, in addition to the three parameters that determine the classical dynamics enumerated above, $\tilde{\varepsilon}_d$, $\tilde{\omega}_d$, and $\tilde{n}_y$, the driven quantum dynamics of the transmon is characterized by a fourth parameter, $\hbar_{\text{eff}}$, characterizing quantum fluctuations.

The solutions to the time-dependent Schrödinger equation associated to the Hamiltonian in Eq. (2) are the time-dependent Floquet states $|\psi_k(t)\rangle = \exp(-i\varepsilon_k t)|\phi_k(t)\rangle$, characterized by the rescaled quasienergies $\varepsilon_k$ and the time-periodic Floquet modes $|\phi_k(t)\rangle$ [32]. The quasienergies of the Floquet modes are defined up to integer multiples of the drive frequencies, and hence are not indicative of the amount of energy stored in the system. However, the mean energy per cycle for mode $|\phi_k(t)\rangle$ can be defined as [40]

$$\langle H \rangle = \frac{1}{T} \int_0^T dt \langle \phi_k(t) | \mathbf{H}(t) | \phi_k(t) \rangle ,$$

where $T$ is the period of the drive. This mean energy per cycle is plotted in Fig. 1(c) as a function of the drive amplitude $\tilde{\varepsilon}_d$ and different Floquet states $k$ for a very weakly nonlinear transmon qubit with $\hbar_{\text{eff}}^{-1} = 7.91$ ($\bar{E}_J/\bar{E}_C = 500$). As the drive amplitude increases, the perturbation hybridized the states around the energy $\langle H \rangle/\bar{E}_J = 2$, corresponding to the energy of the separatrix in the classical system, and does so in an increasingly large bandwidth around that energy. Remarkably the main features subsist in the more experimentally relevant case of $\hbar_{\text{eff}}^{-1} = 3$ ($\bar{E}_J/\bar{E}_C = 72$), although the hybridization is visually less pronounced because the level separations are larger, see Fig. 1(d).
The diffusion of classical trajectories through the chaotic domain translates to delocalized Husimi functions of the Floquet modes for the driven quantum system [31, 41]. This can be intuitively understood by the fact that the Floquet modes are eigenstates of the propagator over one period of the drive, which is the quantum analog of the stroboscopic Poincaré map defined above [42]. In the insets of Fig. 1(c) and Fig. 1(d), we plot the Husimi functions at time $T/8$ of the Floquet modes indicated by the red markers on the spectra at $\tilde{\varepsilon}_d = 0.5$. Because the phase is defined on $(-\pi, \pi]$, we use the definition of Ref. [43] for a coherent state on a circle. The insets indicated by the red triangle correspond to the Husimi functions of the Floquet modes of the first excited state. Because it is outside of the region where the states are strongly mixed (i.e. outside of the chaotic layer in the classical system), its wavefunction remains globally regular and is reminiscent of the first Fock state. On the other hand, the insets marked by a red dot correspond to a state located close to the separatrix. The corresponding Husimi functions are irregular and delocalized over the region corresponding to the chaotic layer in the classical case, see Fig. 1(b). The black squares in the upper right corner of the insets indicate the phase space area $h_{\text{eff}}$ occupied by one state. A smaller $h_{\text{eff}}$ results in a smaller amplitude of quantum fluctuations, and therefore in more resolved features in the Husimi functions.

III. QUANTUM SIGNATURES OF CHAOS IN THE DRIVEN SPECTRUM

A. Level-spacing statistics

The chaotic nature of the states near the separatrix can be confirmed through the correlated nature of the Floquet spectrum, which manifests itself in the distribution of level spacings $P(\Delta)$. Indeed, the strong level hybridization observed near the separatrix results in level repulsion and, in turn, to a Floquet spectrum with strong correlations in the distribution of the spacing between levels. In that situation, $P(\Delta)$ is expected to follow the Wigner-Dyson distribution (dashed line in Fig. 2) [36]. In contrast, in the regular regime levels are uncorrelated and their distribution is Poissonian, as is characteristic of random uncorrelated events (dotted line in Fig. 2) [36].

To display this distribution, we consider a set of $N$ states with sorted quasienergies $\varepsilon_1 \leq \cdots \leq \varepsilon_N$ lying within the first Brillouin zone, i.e. $|\varepsilon_n| \leq \omega_d/2$ for all $n$. The spacing between adjacent quasienergies is defined by $\Delta_n = (\varepsilon_{n+1} - \varepsilon_n)/\Delta$ for $n = 1, \ldots, N - 1$, where $\Delta = \omega_d/N$ is the mean level spacing. We also define $\Delta_N = (\varepsilon_0 - \varepsilon_N + \omega_d)/\Delta$ at the boundary of the Brillouin zone. In the limit of large $N$, the distribution $P(\Delta)$ is expected to follow the Wigner-Dyson statistics for correlated spectrum [44], whereas it is expected to follow a Poisson distribution for an uncorrelated spectrum.

![Figure 2](image.png)

In analogy with the study of classical chaos, we define the chaotic domain using the mean energy per cycle $\langle H \rangle$. As can first be seen in Fig. 1(c) for $h_{\text{eff}} = 7.91$, the mean energies of the chaotic states are concentrated around $2E_J$, and are separated from the regular states by a gap. The relevant energy bandwidth of the chaotic zone depends on the drive amplitude and, for $\tilde{\varepsilon}_d = 0.5$, we take the $N$ states whose energies satisfy $1.6 < \langle H \rangle / E_J < 2.5$. The statistics is generated using the Floquet spectra corresponding to 200 values of $n_g$ uniformly distributed over the interval $[0, 0.5]$. The driven spectrum distribution follows the Wigner-Dyson distribution (dashed line), while the distribution of energies for the undriven system follows the Poisson distribution (dotted line). The inset shows the integrated distribution $I(\Delta) = \int_0^\Delta ds P(s)$.

In the absence of a drive, the system is regular and $P(\Delta)$ approaches the Wigner-Dyson distribution. The number of states $N$ in the selected energy bandwidth has increased in the chaotic case. This is due to the hybridization of states with mean energy close to the separatrix region $\langle H \rangle \sim 2E_J$, as mentioned above. More generally, the extent of the chaotic layer is seen
to increase with the drive amplitude, something which is particularly clear in Fig. 1(c). The inset of Fig. 2 shows the integrated distribution \( I(\Delta) = \int_{0}^{\Delta} ds P(s) \) for which the statistical variations are reduced because of the integration, thereby allowing for a clearer distinction between uncorrelated and correlated spectra [36].

### B. Drive frequency dependence

In addition to spreading over an increasingly large energy bandwidth with increasing drive amplitude, the size of the chaotic region also strongly depends on the drive frequency. This is illustrated in Fig. 3(a)-(d) which shows the Poincaré sections for \( \hat{\omega}_d = 0.75, 1.25, 2.25 \) and 4.25. The mean energy spectra of the corresponding quantum systems are shown in Fig. 3(e)-(h) as a function of \( \epsilon_d/\hat{\omega}_d \) for \( h^{\text{eff}} = 2.45 \) and \( n_0 = 0.25 \). To compare the effect of different drive frequencies, the ac-Stark shift computed from the quasenergies of the ground and first excited states is fixed to 100 MHz in panels (a)-(d) (see Appendix A). The corresponding drive amplitudes are represented by a vertical black dashed line in Fig. 3(e)-(h). In panels (a)-(d), the width of the chaotic layer is observed to be maximal for intermediate drive frequencies \( \hat{\omega}_d \sim 1 - 2 \). This observation is in qualitative agreement with the approximate expression for the width of the chaotic layer around the separatrix

\[
W_{\epsilon}/E_i = \frac{\hat{\epsilon}_d}{\hat{\omega}_d} \text{sech}\left(\frac{\pi \hat{\omega}_d}{2}\right), \quad (5)
\]

a result which is valid for \( \hat{\omega}_d > 1 \) [39, 45]. Following this expression, the chaotic layer is expected to have a maximal width for \( 1 < \hat{\omega}_d < 2 \), and to exponentially decrease in width with increasing \( \hat{\omega}_d \) for \( \hat{\omega}_d > 2 \).

The approximate expression for the width, however, does not account for resonances that result in the additional tori observed in Fig. 3(a)-(d). In particular, in the range \( 0.6 \lesssim \hat{\omega}_d \lesssim 3 \), which includes the regime of operation of current experiments, we find that resonances play an essential role in drive-induced instabilities. The origin of these resonances can be qualitatively understood by expressing the classical Hamiltonian of Eq. (3) in the equivalent form

\[
\hat{H}(\hat{t}) = \frac{(\hat{n} - \bar{n}_g)^2}{2} - \cos \left[ \hat{\phi} + \frac{\hat{\epsilon}_d}{\hat{\omega}_d} \sin(\hat{\omega}_d \hat{t}) \right]. \quad (6)
\]

Using the Jacobi-Anger expansion to first order in \( \hat{\epsilon}_d/\hat{\omega}_d \), this can be approximated as

\[
\hat{H}(\hat{t}) \approx \frac{(\hat{n} - \bar{n}_g)^2}{2} - J_0(\frac{\hat{\epsilon}_d}{\hat{\omega}_d}) \cos \hat{\phi} + J_1(\frac{\hat{\epsilon}_d}{\hat{\omega}_d}) \left[ \cos(\hat{\phi} - \hat{\omega}_d \hat{t}) - \cos(\hat{\phi} + \hat{\omega}_d \hat{t}) \right], \quad (7)
\]

where \( J_k(z) \) is the \( k \)-th Bessel functions of the first kind. The first line of Eq. (7), \( \hat{H}_{\hat{\omega}_d} = (\hat{n} - \bar{n}_g)^2/2 - J_0(\hat{\epsilon}_d/\hat{\omega}_d) \cos \hat{\phi} \), describes an undriven pendulum with a potential energy reduced by the factor \( J_0(\hat{\epsilon}_d/\hat{\omega}_d) \). The second line is of smaller amplitudes and describes the first harmonic of the time-dependent perturbation. Higher-order harmonics of the drive are neglected because their amplitude is suppressed by the factor \( J_k(\hat{\epsilon}_d/\hat{\omega}_d) \) and because the corresponding resonances occur at higher frequency, \( \hat{\omega} > 2\hat{\omega}_d \). The effects of the perturbation can be understood by inserting in the second line of Eq. (7) the solution \( (\hat{\phi}(t), \hat{n}(t)) \) of the system under the time-independent Hamiltonian \( \hat{H}_{\hat{\omega}_d} \) with initial conditions \( (\hat{\phi}_0, \hat{n}_0) \). Depending on the drive frequency, resonances can occur either within the bounded states or the unbounded states.

The trajectories representing the bounded states of the pendulum can be generated in the Poincaré sections with the initial conditions \( (\hat{\phi}_0 < \pi, \hat{n}_0 = 0) \). For small oscillations, the pendulum behaves as a slightly anharmonic oscillator, and its oscillation frequency decreases as the oscillation amplitude \( \hat{\phi}_0 \) increases. For \( |\hat{\phi}_0| < 0.8\pi \), the nonlinearity does not play an important role and the pendulum frequency varies smoothly. This corresponds to a frequency range \( 0.65 \lesssim \hat{\omega} < 1 \) for the pendulum oscillation. In this case, under \( \hat{H}_{\hat{\omega}_d} \) the trajectory that passes through \( (\hat{\phi}_0, \hat{n}_0) \) takes the standard form \( \hat{\phi}(t) = \hat{\phi}_0 \sin(\hat{\omega}_d t) \) where we have neglected the higher harmonics of the motion. Inserting this expression for \( \hat{\phi}(t) \) in the second line of Eq. (7) leads to the slowly rotating terms \( J_1(\hat{\epsilon}_d/\hat{\omega}_d) J_{2k+1}(\hat{\phi}_0) \cos(\hat{\omega}_d(2k+1)t) \).

The case \( k = 0 \) corresponds to a 1:1 resonance, i.e. \( \hat{\omega}_d = \hat{\omega} \in [0.65, 1] \), which strongly impacts the low-energy states of the system. At \( \hat{\omega}_d = 0.75 \) this resonance appears as a second set of tori located close to the central tori, see the red-colored region in Fig. 3(a). Large instability results from the overlap between these two tori, with a chaotic layer arising at their separatrices [33]. The resulting increased width of the chaotic layer is not captured by the approximate expression of Eq. (5). The case \( k = 1 \) leads to a 3:1 resonance for \( \hat{\omega}_d = 3\hat{\omega} \in [2, 3] \). For \( \hat{\omega}_d = 2.25 \) this resonance results in 3 sets of tori surrounding the central regular island, see the three green-colored regions in Fig. 3(c). Although of smaller amplitudes than the 1:1 resonance, the overlap of this resonance with the main set of tori is likely to produce unstable motion at the boundary of these. Note also the appearance of a weak 5:1 resonance at \( \hat{\omega}_d = 4.25 \) resulting in 5 small tori within the regular island, see the green-colored region in Fig. 3(d).

In contrast, no direct resonance occurs for the case \( 1 < \hat{\omega}_d < 1.5 \) which is common for the dispersive readout, see Fig. 3(b) for \( \hat{\omega}_d = 1.25 \). Nevertheless, as expected from Eq. (5), the width of the chaotic layer is large at this drive frequency.

In the quantum case, the resulting large chaotic layer translates into a large hybridization of the states even at low drive amplitude, see the mean energy spectra Fig. 3(e-g). This hybridization can be further observed from the modification of the rate matrices (see Ap-
Figure 3. (a)-(d): Poincaré sections at time $T/8$ for the drive frequencies (a) $\tilde{\omega}_d = 0.75$, (b) $\tilde{\omega}_d = 1.25$, (c) $\tilde{\omega}_d = 2.25$ and (d) $\tilde{\omega}_d = 4.25$. The drive amplitudes are chosen such that the frequency of the $0 \rightarrow 1$ transition in the quantum system is ac-Stark shifted by 100 MHz for all $\tilde{\omega}_d$. (e)-(f): Mean energy spectra as a function of $\tilde{\varepsilon}_d/\tilde{\omega}_d$ at the same respective drive frequencies, with $\tilde{\eta}_{\text{eff}} = 2.45$ and $n_g = 0.25$. In each case, the drive amplitude yielding an ac-Stark shift of 100 MHz is indicated by the vertical black dashed line – the above Poincaré sections have been computed for these amplitudes. For the parameters of panels (a), (c) and (d), a 1:1 resonance occurs between the drive and the system. In (a), this resonance results in a second set of tori within the bounded states of the system (red region). The overlap of the small oscillation and this resonance results in a large chaotic layer. In the quantum system (e), this leads to strong level hybridization. In panels (c) and (d), the drive comes in resonance with the unbounded states, resulting in two additional out-of-phase sets of tori rather than one (red regions). From (c) to (d), these tori move away from the center as the drive frequency increases. In (c) and (d), 3:1 and 5:1 resonances emerge (green region). In (c), the proximity of the 1:1 (red) and 3:1 (green) resonances to the regular island also causes large instabilities, both in the classical and quantum systems. In (d), the resonances are far and the width of the chaotic layer is smaller, as expected from Eq. (5). In (b), the resonance is absent.

Appendix A). This state hybridization can lead to loss of the QND character of the dispersive readout. As a concrete example, the value of $h_{\text{eff}}^{-1}$ and $\tilde{\omega}_d = 0.75$ of Fig. 3(f) was chosen to match the experimental parameters of Walter et al. [26] where the dispersive readout fidelity was observed to degrade for measurement photon numbers $\tilde{n} > 2.5$, something which was attributed to measurement-induced mixing of unknown origin. Using the light-matter coupling of $g/2\pi = 208$ MHz reported in Ref. [26], $\tilde{n} = 2.5$ can be converted to an effective drive $\tilde{\varepsilon}_d = 0.105$ on the transmon. As can be observed in Fig. 3(e), the first excited state is “absorbed” in the chaotic layer for $\varepsilon_d \gtrsim 0.1$. Since chaotic states often lead to strong hybridization between the transmon and the readout resonator (see Sec. V), this hints at chaos-induced state mixing and non quantum-demolition effects in the readout beyond that drive amplitude. Although the above mean-field analysis can qualitatively predict unstable behavior of the transmon, the agreement cannot be expected to be quantitative at low resonator photon numbers since vacuum fluctuations of the resonator field and qubit-resonator parametric processes occurring in the presence of a readout tone can play an important role. We address some of these mechanisms in Sec. V.

The drive can also come in resonance with unbounded states of the pendulum with energy $\tilde{H}_{\tilde{\varepsilon}_d} > 2J_0(\tilde{\varepsilon}_d/\tilde{\omega}_d) \sim 2$. The trajectories at those energies take the approximate form $\tilde{\phi}(\tilde{t}) = \pm \tilde{\omega}_d t + F(\tilde{t})$, where $F$
is a function of period $2\pi/\bar{\omega}$ [33]. Using this expression in Hamilton’s equation of motion for the phase, we find that the oscillation frequency satisfies $\bar{\omega} = |\bar{\eta} - \bar{n}_g| \geq 1.5$, where $|\bar{\eta} - \bar{n}_g|$ is the averaged momentum of the trajectory over one period. This lower bound can vary with the drive amplitude through the effect of the reduction factor $J_0(\bar{\eta}/\bar{\omega})$. To leading order, substituting this expression for $\phi(t)$ in the second line of Eq. (7) results in the term $A_0 J_1(\bar{\eta}/\bar{\omega}) \cos(\pm(\bar{\omega} - \bar{\omega}(t))t)$, where $A_0$ is the zeroth Fourier component of $\cos(F(t))$. Hence, for $\bar{\omega}_d = \bar{\omega} \geq 1.5$, this resonance appears in the form of two tori moving clockwise and anti-clockwise, corresponding to $\pm \bar{\omega}_d$, as depicted by the red-coloured regions in Fig. 3(d) for $\bar{\omega}_d = 2.25$ and Fig. 3(d) for $\bar{\omega}_d = 4.25$. At $\bar{\omega}_d = 2.25$, the proximity of the resonance with the regular island results in a large chaotic domain which translates in strong state hybridization in the quantum case, see Fig. 3(g). Because the average momentum associated to this resonance is $|\bar{\eta} - \bar{n}_g| = \bar{\omega}_d$, the pair of tori move further away from the center of phase space with increasing drive frequency. For this reason, at $\bar{\omega}_d = 4.25$, this resonance is far from the separatrix of the undriven system and does not affect the chaotic layer. As a result, for $\bar{\omega}_d = 4.25$ and at larger drive frequencies, the width of the chaotic layer is exponentially suppressed as expected from Eq. (5). In this situation, the absence of instability yields a mean energy spectrum with little state hybridization, see Fig. 3(h). This fact is further illustrated by the regular structure of the charge matrix elements in Appendix A.

To summarize the above, the frequency ranges $0.65 < \bar{\omega}_d < 1$ and $1.5 \leq \bar{\omega}_d \leq 3$ lead to strong instabilities, in particular for the former. The parameter regime $1.1 < \bar{\omega}_d < 1.5$ avoids resonances but has a large chaotic layer around the separatrix. For $\bar{\omega}_d \geq 3.5$, resonances do not affect the bounded states (or very weakly through resonances of order greater than 5), and the width of the chaotic layer is suppressed. As shown in the next sections, the presence of the chaotic layer impacts the coherence times of the transmon qubit, and it should therefore be minimized when operating the transmon with strong drive, e.g. in dispersive qubit readout. The frequency ranges $1.1 \leq \bar{\omega}_d \leq 1.5$ and $\bar{\omega}_d \geq 3.5$ seem to be more benign.

**IV. IMPACT ON COHERENCE PROPERTIES**

Having established that the driven capacitively shunted Josephson junction exhibits signatures of chaos even at the large $\hbar_{\text{eff}}$ corresponding to the transmon regime, we now turn to the impacts of this observation on coherence properties of the transmon in the presence of a bosonic bath. In this situation, the total Hamiltonian now takes the form

$$H(t) = 4E_C(n - n_g)^2 - E_J \cos(\phi) + \epsilon_d \cos(\omega_d t)n + \sum_k g_k (b_k^\dagger b_k + \sum_k \Delta_{jk} b_k^\dagger b_k, $$

where $b_k$ and $b_k^\dagger$ are the annihilation and creation operators of the bath modes. Because of hybridization with states in the chaotic layer which have a strong charge dispersion, it is important to keep the gate charge $n_g$ in Eq. (8) even when interested in the coherence properties of the low-lying eigenstates of the system.

**A. Rate matrix and steady-state population**

Within the Floquet-Markov description of the system-bath coupling, the bath-induced transition rate from Floquet state $j$ to $i$ is given by [32]

$$\Gamma_{ij} = \sum_k |n_{ijk}|^2 \{\Theta(\Delta_{ijk}) + n_B(|\Delta_{ijk}|)\} J(|\Delta_{ijk}|),$$

Figure 4. Top panels : Rate matrices in units of the coupling strength to the bath, at $\bar{\nu}_{\text{drive}} = 0.4$, $h_{\text{eff}} = 3$, $\bar{\omega}_d = 1.34$ and $T = 10\text{ mK}$ for (a) $n_g = 0.5$ and (b) $n_g = 0.25$. The states are sorted by their mean energy ($\langle H \rangle$). The blue (red) squares correspond to the sum of rates involving an even (odd) number of drive photons $k$. The insets show the rate matrices of the undriven systems. At finite drive, the instability develops around the separatrix leading to an irregular block in the rate matrix also allowing for upward excitation rates. At the symmetric point $n_g = 0.5$, the red and blue squares do not overlap, while at $n_g = 0.25$, hybridization of parity sectors can be observed through the presence of purple squares even in the low-energy sector. The purple squares in (a) are due to numerical errors (see text). Bottom panels : Steady-state population as derived from the above rate matrices for (c) $n_g = 0.5$ and (d) $n_g = 0.25$. In the driven system, a plateau forms over the states that are part of the chaotic block of the rate matrices, instead of the regular exponential distribution in the undriven case, as shown in the insets.
where $\Theta(x)$ denotes the Heaviside function, $n_B(x)$ is the thermal occupation number of the bath, and $\Delta_{ijk} = \epsilon_j - \epsilon_i - k\omega_d$. The spectral function $J(x)$ is assumed to be that of an ohmic bath, i.e. $J(x) \propto x \exp(-|x|/\omega_c)$, where $\omega_c$ is a high-frequency cut-off. In the expression for $\Gamma_{ij}$, $k$ can be interpreted as the number of drive photons participating positively or negatively to the transition.

We have also introduced the charge operator matrix elements

$$n_{ijk} = \frac{\omega_d}{2\pi} \int_0^{2\pi/\omega_d} dt \langle \phi_i(t) | n | \phi_j(t) \rangle \exp(i\Delta_{ijk}t), \quad (10)$$

where $|\phi_j(t)\rangle$ are the Floquet modes of the driven system.

The rate matrices of the driven transmon at $\tilde{\epsilon}_d = 0.4$ and a temperature of $T = 10$ mK are shown for $n_g = 0.5$ in Fig. 4(a) and for $n_g = 0.25$ in Fig. 4(b). As a comparison, the insets show the rate matrices in the undriven case. For a given transition $\Gamma_{ij}$, the blue (red) squares sum the contributions from even (odd) values of $k$, with purple squares indicating contributions from both even and odd values. At zero drive, the upper triangular sector of the rate matrices (corresponding to upward transitions) contains negligible but non-zero elements due to the finite temperature (not visible in the insets). At finite drive, the instability develops around the states located on the separatrix (typically around the 5th excited state), forming an irregular block in the rate matrix. In particular, states within the chaotic layer are all coupled to one another through the charge operator. In addition, because of the drive photons, upward transitions are now apparent. The appearance of an irregular block in the rate matrix directly relates to the repulsive statistics of the quasienergies. In fact, chaotic systems can be accurately described by random matrices in the limit of a large number of chaotic states [36]. In Appendix E, we leverage this property to estimate the average charge matrix element between two chaotic states when all the low-energy states are chaotic.

In the transmon regime, the low-energy sector is almost independent of the offset charge and, if one neglects its influence, the Hamiltonian becomes effectively symmetric under the parity transformation $\phi \rightarrow -\phi$ and $n \rightarrow -n$ (see Appendix B). Because it neglects the gate charge, this symmetry is implicit in the Kerr nonlinear oscillator model of the transmon. Although this symmetry is exact only at $n_g = 0$ and $n_g = 0.5$, at zero drive it results in a suppression of the matrix elements of the charge operator $n_{i,i+2}$ in the low-energy sectors for all values of $n_g$, and forbids the transition $i \rightarrow i + 2$, see insets of Fig. 4(a,b). The fact that this is only an approximate symmetry at $n_g = 0.25$ is apparent for the states in the separatrix region and above.

In the presence of a drive term $\tilde{\epsilon}_d \cos(\omega_d t) n$, the inversion symmetry only holds together with the time translation $t \rightarrow t + \pi/\omega_d$ [32]. This symmetry of the driven Hamiltonian defines even and odd parity sectors among the time-dependent Floquet states (see Appendix B). Because the charge operator is anti-symmetric, under this generalized parity symmetry, transitions through the charge operator between two states of the same (opposite) parity can only involve an odd (even) number of drive photons $k$. As can be seen in Fig. 4(a) for $n_g = 0.5$, blue and red squares do not mix (i.e. there are no purple squares) indicating that the symmetry is respected for that gate charge. The purple squares appearing for states 18 and 19 only result from numerical precision errors. The situation is very different at $n_g = 0.25$ where purple squares appear in the chaotic layer but also for transitions involving the low-energy states. This results in transitions that are otherwise forbidden at the symmetric points $n_g = 0$ and $n_g = 0.5$. As discussed in further details in the next section, the breaking of this effective symmetry in the low-energy sector is a consequence of a strong increase of the band dispersion in the presence of drive. Interestingly, transitions forbidden by the apparent inversion symmetry of the transmon were experimentally observed under strong drives, but remained unexplained [23].

The rate matrix can also be used to compute the system’s steady-state density matrix $\rho_{ss}$. Under the assumption of weak system-bath coupling, the steady state is diagonal in the Floquet basis

$$\rho_{ss}(t) = \sum_j p_j |\phi_j(t)\rangle \langle \phi_j(t)|, \quad (11)$$

where the populations $p_j$ satisfy the rate equations $\dot{p}_j = \sum_i \Gamma_{ij} p_i - \sum_i \Gamma_{ji} p_j$. The insets of Fig. 4(c) and Fig. 4(d) show these steady-state populations for the undriven systems for $h_{\text{eff}}^{-1} = 3$ which is typical of the transmon regime. As expected, the populations follow the thermal distribution with populations quickly dropping below $10^{-10}$. In contrast, in the driven systems (here with $\tilde{\epsilon}_d = 0.4$), the steady-state populations form a plateau corresponding to the chaotic layer. This behavior is typical of chaotic systems [30, 31]. Increasing the drive amplitude, the chaotic layer and therefore the plateau grow until all the low-energy states are part of the plateau, including the ground state. This results in a dramatic decrease of the purity of the transmon’s steady state, an observation which is in agreement with numerical [24] and experimental results [13].

This discussion sheds light on the distinction between the time-dynamics of the transmon ionization numerically observed in [25], and the ionization in the steady state [13, 24]. In the former, one captures the dynamics of the dispersive readout of a transmon qubit. As the cavity rings up on a timescale $\kappa^{-1}$, where $\kappa$ is the photon loss rate, the effective field amplitude $\tilde{\epsilon}_d$ on the transmon increases. Starting in the ground or first excited states, the system follows the corresponding first or second line in the mean energy spectrum of Fig. 1(c). Ionization occurs when one of those lines crosses a large resonance with a chaotic state, i.e. when the mean energy suddenly increases in Fig. 1(c), something which happens at $\tilde{\epsilon}_d \approx 1.1$ for the ground state and at $\tilde{\epsilon}_d \approx 0.75$ for the first excited state. Once a chaotic state is populated, it decays either through the transmon-bath cou-
In the previous section, we have seen that the Hamiltonian of the driven transmon can be approximately divided into regular blocks of states: phase-like states at the bottom of the cosine potential well; charge-like states above that potential, on which the drive acts perturbatively; and one chaotic block with a strongly correlated spectrum. From perturbation theory performed on the regular block, one would expect a drive to lead to a slow hybridization amongst regular low-lying states. In this situation, the ground and first excited states would therefore weakly inherit an offset-charge sensitivity from weak dressing with higher energy states, leading to an overall small increase of the band dispersion with the drive amplitude. In practice, we find that the presence of the chaotic layer results in the energy dispersion of these states to be significantly modified when the system is driven. This is illustrated by the solid lines in Fig. 5(a) for the first excited state of the transmon with values of $h_{\text{eff}}$ in the range 1 to 7 as labeled by the different colors. As a comparison, the dashed lines correspond to the undriven transmon and for which the exponential suppression of the charge dispersion is clearly observed. In contrast, in the driven transmon the energy bands are disrupted by peaks of multiple orders of magnitude in addition to being, on average, substantially larger than in the undriven case. The sharpness of these peaks is indicative of a weak resonance between the first excited state and strongly $n_g$-dependent states located in the chaotic layer. These results are obtained by identifying the first excited state in the undriven case, then tracking the corresponding Floquet mode $|\phi_1(t)\rangle$ as a function of drive amplitude.

This phenomenon is closely related to chaos-assisted tunneling (CAT) [27]. In CAT, tunneling between two sets of disjoint regular states is facilitated by their coupling to delocalized states in the chaotic layer. Moreover, because of the participation of states within the chaotic layer, the tunneling rates are expected to vary widely with the control parameters [27]. For the transmon, chaos can assist tunneling between different wells of the cosine potential. Because the phase of the transmon is compact, tunneling between wells distant by $\delta\phi=2\pi n$ translates to $n$ $2\pi$-swings of the transmon phase or, equivalently, to quantum phase-slips of order $n$. The transmon states acquire their $n_g$-dependence through these full phase rotations [1]. In the undriven transmon, the rate of these events decreases exponentially with $n$ for large $h_{\text{eff}}^{-1}$.

To evaluate the phase slip rate in the presence of a drive, we compute the Fourier transform of the energy bands of the system [46]. Indeed, the Fourier components $t_n = \int_{0.5}^{0.5} d\epsilon \delta \epsilon \langle \epsilon (\epsilon(n)) e^{i\phi_1(t)} \rangle$ of the energy $\epsilon (n_g)$ of the first excited state corresponds to the rate of phase slips of order $n$ when the system is in the first excited state [47]. The components $t_n$ are plotted as a function of the index $n$ in Fig. 5(b). The inset shows the exponential suppression of $t_n$ expected for the undriven case. In the driven case (main panel), $t_n$ is no longer exponen-
tially suppressed with \( n \). Instead, it shows a long tail, indicating that the sharp features seen in Fig. 5(a) results from long-range hopping between the wells. Note that the tunnelling rates \( t_n \) remain globally suppressed with \( h_{\text{eff}}^{-1} \), see Fig. 5(b). This is explained by an exponential suppression of the matrix elements \( n_{ijk} \) between the regular states and the chaotic states with \( h_{\text{eff}}^{-1} \). We discuss this point in Sec. VI.

In cold atoms trapped in a driven optical lattice, the signatures of CAT have been observed by measuring coherent oscillations between states localized in distinct wells [48, 49]. Because the phase coordinate of the transmon is compact, tunneling oscillation between wells cannot be measured. However, phase slips (due to CAT or not) lead to a phase accumulation which depends on the gate charge \( n_g \) due to the Aharonov-Casher effect [12]. Because the gate charge is a fluctuating function of time, this results in enhanced dephasing of the transmon. The increased phase slip rate due to CAT can thus in principle be witnessed through sharp variations of the pure dephasing rate \( \gamma_\phi \) associated to the transmon's 0-1 transition, as a functions of \( n_g \) and of the drive amplitude \( \tilde{\varepsilon}_d \).

Within the Floquet-Markov theory, the dephasing rates takes the form [50]

\[
\gamma_\phi = A_e|g_{0,\phi}|\sqrt{\log \omega_1\omega_m}} + \sum_{k=0}^{n} 2S(\omega_d)|g_{k,\phi}|^2. \tag{12}
\]

The first term represents \( 1/f \) charge noise, while the second term comes from the possible conversion of a photon loss to a dephasing event due to the hybridization of the logical states \( |0 \rangle \) and \( |1 \rangle \). In the first term, \( A_e \) is the amplitude of charge noise, \( g_{k,\phi} = n_{11k} - n_{00k} \) where \( n_{ijk} \) is a matrix element of the charge operator defined in Eq. (10), \( \omega_1 \) is the infrared cut-off, and \( \omega_m \) is the characteristic measurement time. Typical values are \( \sqrt{\log \omega_1\omega_m}} \sim 4 \) and \( A_e = 10^{-4}e \) [51]. In the second term, \( S(\omega) \) is the spectral function of the relevant bath, here assumed to be associated to dielectric losses.

The dephasing rate obtained from Eq. (12) and the numerically obtained Floquet modes \( |\phi_0(t)\rangle \) and \( |\phi_1(t)\rangle \) corresponding to the logical states in the presence of a drive is shown in Fig. 6(b) as a function of \( \tilde{\varepsilon}_d \) for 50 values of \( n_g \) uniformly spaced in the range \( [0, 0.5] \) (red lines). All curves show a slow quadratic increase of \( \gamma_\phi \) with the drive amplitude due to dielectric losses through the hybridization of \( |0 \rangle \) and \( |1 \rangle \) by the drive. This quadratic increase in the rate with the amplitude of the drive is a signature of a perturbative effect. More interestingly, the dephasing rate also displays sharp peaks whose position strongly depends on \( n_g \) and \( \tilde{\varepsilon}_d \), as is expected for CAT. To better understand the origin of these structures, the mean energy of the different Floquet mode is plotted in Fig. 6(a) for \( n_g = 0.13 \) as a representative example. The value of \( \gamma_\phi \) obtained for the same gate charge is highlighted in Fig. 6(b) (blue line). By comparing the two plots (see the vertical dashed lines), it becomes clear that the sharp increases in \( \gamma_\phi \) correspond to resonances between regular states (ground or first excited state) and chaotic states. As mentioned above, the resulting hybridization of the computational states with states that have a strong charge dispersion leads to a sharp increase in the dephasing rate. The black line is an average over all realizations of the gate charges. Depending on the time scale of the charge fluctuations [52, 53] and of time needed to measure \( \gamma_\phi \), this average may be more representative of potential experimental observations.

V.CIRCUIT QED: TRANSMON COUPLED TO A RESONATOR

We have so far treated the drive seen by the qubit as a purely classical field. To account for vacuum fluctuations and the richer structure of the energy levels in the pres-
In this situation, the Hamiltonian takes the form

\[ \mathbf{H}(t) = 4E_C(n - n_d)^2 - E_d \cos(\phi) + \varepsilon_d \cos(\omega_d)\mathbf{n} + \omega_a a^\dagger a - i\gamma n(a - a^\dagger) - (a - a^\dagger) \sum_k g_k(b_k^\dagger b_k) + \sum_k \omega_k b_k^\dagger b_k. \]  

The first line of this expression is the Hamiltonian of the driven transmon qubit, as in Eq. (2). The second line contains the free cavity Hamiltonian defined by a frequency \( \omega_a \), together with the transmon-cavity capacitive coupling \( g \). We take \( g/2\pi = 250 \text{ MHz} \) and \( \omega_a/2\pi = 8 \text{ GHz} \), with the drive detuned from the cavity at \( \omega_d/2\pi = 7.5 \text{ GHz} \), while keeping the same parameters for the transmon qubit as in the previous sections. Finally, the last line represents the capacitive coupling of the cavity to a bosonic bath. The drive term on the qubit can either be seen as directly acting on the qubit in the laboratory frame, or as resulting from a drive on the resonator. In that latter situation, the Hamiltonian of Eq. (13) is to be understood as expressed in a displaced frame where the drive on the cavity has been removed, i.e. \( \varepsilon_d = 2g\sqrt{\bar{n}} \) where \( \bar{n} \) is the cavity steady-state photon number in the laboratory frame. In this section, we find the Floquet spectrum of Eq. (13) in the absence of coupling to the bath, and then calculate transition rates in linear response theory [24]. Details on the Floquet simulations are provided in Appendix C.

### A. Structure of the Floquet spectrum

To characterize the structure of the Floquet spectrum, for each Floquet mode we compute expectation values of a pair of operators that are good quantum numbers in the undriven and decoupled Hamiltonian: the transmon excitation number \( N_t = \sum_{i,n} i|n\rangle \langle n| \) and the resonator excitation number \( N_r = \sum_{i,n} n|n\rangle \langle n| \), where \( |n\rangle \) is a bare state of the joint transmon-cavity system. In Fig. 7 we show these quantities on a two-dimensional grid in the \((\langle N_t \rangle, \langle N_r \rangle)\) plane, the panels (a)-(d) representing different drive amplitudes \( \varepsilon_d \). As in Eq. (4), the double angle brackets represent the time-averaged expectation value of the operator in a given Floquet mode over a period of the drive. In the absence of coupling and drive, this grid is expected to be rectangular. In the coupled case, for high enough resonator photon number \( \langle N_r \rangle \) for strong enough drive power, some Floquet modes deviate strongly from the regular rectangular grid. These strong deviations are associated with state dressing, and a significant hybridization of the two subsystems. More precisely, horizontal deviations from the rectangular grid can be interpreted as hybridization of transmon states, while vertical deviations originate from hybridization of Fock states. The hybridization of Fock states usually occurs through entanglement of the two systems, implying also a hybridization of the transmon states.

The degree of hybridization between the two subsystems is measured by the purity of the transmon reduced density matrix. Encoding this purity into the color of the grid points in Fig. 7, deviations from the rectangular grid appear to correlate with drops in purity, i.e. an increase of entanglement entropy between the transmon and the resonator. In particular, purity drops are drastic for states whose transmon excitation number corresponds to the chaotic layer identified in the previous sections for similar drive powers.
We can gain further understanding of the purity drop by first diagonalizing the time-dependent Hamiltonian of the driven transmon [first line of Eq. (13)], and then expressing the transmon-resonator coupling $-i g n (a - a^\dagger)$ in the joint basis of the transmon Floquet states $\{|\tilde{j}\rangle\}$ and cavity Fock states. In a frame rotating at the drive frequency for the resonator and at the quasienergies for the transmon, the Hamiltonian reduces to

$$H(t) = \Delta a^\dagger a - i g \sum_{i,j,k} n_{ijk} |\tilde{j}\rangle \langle \tilde{j}| (e^{i \Delta_{ijk,k-1} t} a - e^{i \Delta_{ijk,k-1} t} a^\dagger),$$  

(14)

where $\Delta = \omega_a - \omega_d$. The charge matrix elements $n_{ijk}$ and the energy differences $\Delta_{ijk}$ are defined in Sec. IV, and the Floquet modes of the transmon are evaluated at $t = 0$. Large hybridization occurs between the states $|\tilde{j}, n\rangle$ and $|\tilde{j}, n + 1\rangle$ if the coupling strength is of the order of the transition frequency, i.e. $2g \sqrt{n} + I n_{ijk} \sim \Delta_{ijk,k-1}$. Note that, for small detuning $\Delta$, the transition frequencies $\Delta_{ijk,k-1} - \Delta$ are small for $k = 0, 1$, and consequently hybridization of the states is mainly due to the matrix elements $n_{ijk,k0}$ and $n_{ijk,k1}$. For chaotic states, we derive in Appendix E an estimate of the matrix elements $n_{ijk}$ and the frequencies $\Delta_{ijk,k-1}$. Using these estimates and the previous expression allows us to obtain a threshold value on the photon number $n$ for strong transmon-resonator hybridization (see details in Appendix F). For the parameters of Fig. 7, we find that the transmon states become strongly coupled even at the lowest Fock states. This strong hybridization can result in large decay rates for chaotic states, as explained in the paragraph below.

Transition rates, defined within linear response theory for the charge operator of the resonator $-i (a - a^\dagger)/\sqrt{2}$ in a similar way as in Eq. (9), are shown as gray arrows connecting the grid points in Fig. 7. At low drive power, rates are predominantly local, consisting of single-photon relaxation (vertical downward-pointing arrows connecting neighboring grid points), and qubit Purcell decay (horizontal left-pointing arrows also connecting neighboring grid points). The existence of one dominant rate allows to identify states corresponding to definite transmon excitation number $|\tilde{j}\rangle$. On the contrary, in the chaotic layer where the purity is low, at finite drive amplitude the rate matrix becomes non-local connecting non-adjacent points. Note that here, dissipation on the resonator is treated in linear response theory. In Appendix G, we verify that typical loss rates of readout resonators ($k / 2 \pi \approx 10$ MHz) do not alter the spectrum of the undriven system. In particular, hybridization is not weakened by the linewidth of the Fock states considered in Fig. 7.

Using the transitions rates, we can now find the steady state of the Floquet Markov master equation [24]. In order to avoid the saturation of the resonator Hilbert space (see Appendix C), we impose a cutoff by setting all rates involving states with $\langle N_r \rangle \geq 15$ to vanish. We find that this only causes a small quantitative change in the steady-state density matrix. The transmon and resonator excitation numbers in this steady state are represented on Fig. 7 as a red cross for each drive power. Moreover, the occupation $p_i$ of each Floquet mode in the steady state is encoded in the area of a circle centered at each grid point. For sufficiently low drive power, the steady state is dominated by the state with $\langle N_r, t \rangle \sim 0$, i.e. the dressed-state closest to the vacuum state. Beyond a threshold drive amplitude, the steady state of the Floquet-Markov Lindblad master equation has significant weights on the chaotic states (see Fig. 14 which shows similar results as Appendix C but for larger drive amplitudes). This threshold corresponds to transmon ionization where states above the cosine potential of the transmon become populated [13, 24, 25]. Notably, strong hybridization leads to nonzero resonator photon number in the steady state, that is $\langle N_r \rangle \geq 2$ at $\tilde{\epsilon}_d = 0.95$, see Fig. 13.

Comparing to the mean-field study of a driven transmon of the previous sections, we find that the qualitative features of the spectrum of the off-resonantly driven circuit QED Hamiltonian match those of the driven transmon qubit. To illustrate this, we plot in Fig. 8(a) the transmon excitation number $\langle N_j \rangle$ for the full circuit QED model of Eq. (13) (colored dots) versus drive amplitude and compare it to the corresponding observable in the Floquet spectrum of Eq. (2) with the same parameters, but corresponding to a transmon driven by a classical field (black crosses). As in Fig. 7, for the former, color encodes the purity of the transmon reduced density matrix. While the agreement between the full circuit QED simulation and the mean field driven transmon is excellent for regular states, deviations appear within the chaotic layer where the purity drops. Differences within the chaotic layer between the two models are expected due to strong hybridization between the transmon and the resonator (not accounted in the mean field model), and due to fine sensitivity of chaotic spectra on external parameters [36]. Nevertheless, in the two models, the threshold values of $\tilde{\epsilon}_d$ for the onset of chaos agree.

### B. Frequency response of the resonator

Performing the analog of a numerical two-tone spectroscopy experiment, in Fig. 8(b) we plot the pulled frequency response of the cavity versus drive amplitude $\tilde{\epsilon}_d$. To obtain these results, we first identify the cavity vacuum states corresponding to each transmon occupation number. Then, for each of these vacuum states, labeled $i$, we identify the transition frequency corresponding to the largest matrix element $|n_{ijk}|$. The corresponding transition rate is encoded in the symbol sizes. As in Fig. 8(a), the symbol color encodes the purity of the corresponding vacuum state. Only states with purity above 0.85 are plotted, for which the cavity vacuum can be well identified as the states remain close to a tensor product state.

Figure 8(b) shows that, as expected, low-energy regu-
The purity drops and the connecting arrows in Fig. 7 between the full circuit QED simulation (color encodes purity of transmon reduced density matrix; only states satisfying $\langle N_t \rangle \leq 0.73$ are shown) of Eq. (13) and the driven transmon model of Eq. (2) (black crosses). Pulled resonator frequency as obtained from two-tone spectroscopy, with colors representing purities of Floquet modes corresponding to the resonator vacuum (only purities above 0.85 were retained, thus excluding cavity frequency pulls due to chaotic states). Pulls from perturbation theory Eq. (15) are shown as black crosses. (c) Resonator frequency response in the steady state. Only states where the steady-state population times the square of the transition matrix element exceeded 0.001$\kappa$.

Figure 8. (a) Agreement of transmon population $\langle N_t \rangle$ between the full circuit QED simulation (color encodes purity of transmon reduced density matrix; only states satisfying $\langle N_t \rangle \leq 0.73$ are shown) of Eq. (13) and the driven transmon model of Eq. (2) (black crosses). Pulled resonator frequency as obtained from two-tone spectroscopy, with colors representing purities of Floquet modes corresponding to the resonator vacuum (only purities above 0.85 were retained, thus excluding cavity frequency pulls due to chaotic states). Pulls from perturbation theory Eq. (15) are shown as black crosses. (b) Pulled resonator frequency as obtained from two-tone spectroscopy, with colors representing purities of Floquet modes corresponding to the resonator vacuum (only purities above 0.85 were retained, thus excluding cavity frequency pulls due to chaotic states). Pulls from perturbation theory Eq. (15) are shown as black crosses. (c) Resonator frequency response in the steady state. Only states where the steady-state population times the square of the transition matrix element exceeded 0.001$\kappa$.

V. NON-QND EFFECTS ORIGINATING FROM THE CHAOTIC STATES

The purity drops and the connecting arrows in Fig. 7 can be used to quantify the quantum demolition character of the light-matter interaction in Eq. (13). For exam-
ple, through its coupling with the resonator, the transmon can relax through the resonator bath. This effect, known as the Purcell decay, results in the left-pointing arrows connecting the low-energy states of the transmon in Fig. 7(a). Moreover, approximating the transmon as a weakly nonlinear oscillator, recent works have shown that the small anharmonicity of the transmon can result in non-QNDness via correlated photon emission of the transmon-resonator systems [21, 22, 57]. However, the magnitude of these corrections, controlled by the relative transmon anharmonicity and the relative detuning of the transmon with the resonator, remains small compared to Purcell effect at sufficiently large detuning as considered in this paper. In addition, these correlated effects occur primarily at the qubit frequency, which could be efficiently suppressed by a Purcell filter. Here, we show that the presence of chaotic states in the transmon spectrum lead to non-QND effects that cannot be captured by a perturbative treatment.

In the previous sections, we have shown that part of the transmon spectrum is rendered chaotic when the system is driven and/or coupled to a resonator. Through the coupling with the resonator, see Eq. (14), regular states can decay to chaotic states through multi-photon transitions. This is further illustrated by the large non-local arrows in Fig. 7 connecting distant dots in the \( (\langle N_i \rangle, \langle N_r \rangle) \) plane. The inherently strong nonlinearity of these states cannot be accounted for by perturbative models taking the relative anharmonicity as a small parameter. Moreover, as shown in Sec. IV, chaotic states strongly depend on the gate charge \( n_g \), while this parameter can be gauged away in oscillator models for the transmon where the phase becomes non-compact. We stress that, as compared to perturbative treatments where correlated effects occur at a few dominant frequencies, the transition frequencies involved in this decay are numerous and highly depend on the system parameters.

Spurious bath-assisted decay from the regular to the chaotic states can therefore result in a source of non-QNDness during strong-drive operations such as dispersive readout. In this section, in order to characterize such an effect, we introduce a ‘chaos-assisted’ critical photon number \( n_{ca} \), based on the onset of large hybridization between low-lying regular states and higher-energy chaotic states. Due to finite matrix elements between these states, transitions to the chaotic layer arise below \( n_{ca} \). We propose to exponentially suppress this effect by increasing the ratio \( E/J/E_C \).

### A. Chaos-assisted critical photon number

Going back to our analysis of a driven transmon in the absence of quantum fluctuations from the resonator of Sec. II, an ‘ionization threshold’ between regular and chaotic levels occurs at a critical drive amplitude, and can be graphically identified from the plots of the mean energy of the transmon, see Fig. 1(c,d). For each regular state, the critical drive corresponds to the drive amplitude \( \varepsilon_{d,ca} \) for which the level gets ‘absorbed’ into the chaotic layer. As can be observed in Fig. 1(c) and (d), as the drive amplitude increases, the chaotic layer grows into a cone-like shape whose tip corresponds to the separatrix energy, and thus the boundary between chaotic layer and regular states depends on drive amplitude. The closer a state is to the separatrix at zero drive, the smaller the critical drive amplitude necessary to have it completely hybridize with the chaotic layer. This critical drive amplitude then matches the so-called ‘ionization threshold’ of the respective regular state [25].

Having associated the critical drive amplitude to the boundary between regular and chaotic states, we can observe this boundary in Fig. 1(c) and (d) for two values of \( h_{eff} \). From these figures, we see that the dependence of the mean energy \( \langle H/E_j \rangle \) on \( \varepsilon_d \) is roughly independent of \( h_{eff} \), as this is ultimately tied to the energies of orbits at the regular-to-chaotic boundaries in the classical phase space in Fig. 1(b). Remarkably, this implies that the mean energy \( \langle H/E_j \rangle \) at the boundary solely depends on the parameters of the classical driven pendulum, i.e. \( \varepsilon_d = \varepsilon_d/\omega_p \) and \( \omega_d = \omega_d/\omega_p \).

As emphasized in Sec. II, what does depend on \( h_{eff} \) is the number of regular states below the chaotic layer. This number controls which mean energies \( \langle H/E_j \rangle \) are accessible by the quantum states, and ultimately the critical drive amplitudes at which regular states get absorbed into the chaotic layer. As discussed in Sec. IIIB, the size of the regular central island in the Poincaré sections, e.g. in Fig. 3(a)-(d), depends on the drive frequency and decreases with drive amplitude. When quantizing the system, in the semiclassical limit, we expect the number of states contained in the low-energy regular subspace to be directly proportional to the inverse of the effective Planck constant \( h_{eff} \) and to the area of the regular island [34, 35]. Hence, the critical drive amplitude for, say, the excited state, is defined as the amplitude at which the area of the regular island is equal to \( 2h_{eff} \), i.e. it contains exactly two states, the ground and the excited. The critical drive amplitude therefore increases with \( h_{eff}^{-1} \) for any given state: the larger the ratio \( E_j/E_C \), the larger the critical drive amplitude needed to absorb the level into the chaotic layer.

Similarly, for the case of an undriven transmon coupled to a resonator, one can define a critical photon number \( n_{ca} = (\varepsilon_{d,ca}/2g)^2 \), where \( \varepsilon_{d,ca} \) is the critical drive amplitude of the driven transmon without coupling to a resonator. We justify this definition by noting that the two systems, i.e. the driven transmon and the undriven transmon coupled to a resonator, have similar spectra (see Appendix D). This hints towards the fact that one could replace the driving field by a quantum field without affecting the ionization threshold. For photon numbers \( n > n_{ca} \), we expect the first excited state to be absorbed in the chaotic layer.

Note that this definition of \( n_{ca} \) differs from the critical photon number \( n_{crit} \) defined in [1, 20]. In these works,
of the transmon, i.e. larger ratio \( \frac{E_j}{E_C} \), is a favorable regime in the presence of drive. Indeed, another benefit of working in the large \( \frac{E_j}{E_C} \) regime for the transmon is that the magnitude of the spurious interactions between the low-energy states and the chaotic states is exponentially suppressed with \( \frac{1}{\sqrt{E_j/E_C}} \). This suppression mechanism is related to that of charge dispersion with \( \sqrt{E_j/E_C} \) \cite{1}. Charge dispersion manifests itself through phase-slip events, which are in turn exponentially suppressed with \( \sqrt{E_j/E_C} \) in an undriven transmon. In a driven transmon, chaos-assisted phase slips can occur through the interaction of the low-energy states with the chaotic states (see Sec. IV). Since phase slips involve the full nonlinearity of the cosine potential, they result in multiphoton transitions \cite{58}. Nonetheless, as explained below, these chaos-assisted events are also exponentially suppressed with \( \frac{1}{\sqrt{E_j/8E_C}} \).

In a quantum system, the regular island and the chaotic layer are not dynamically separated as is the case in the classical system. Spurious transitions from the computational states to the chaotic states might occur well below the critical drive amplitude. The regular states have a finite overlap with the chaotic layer in phase space, resulting in finite coupling between regular and chaotic states. Note that CAT, studied in Sec. IV, is an example of such possible regular-to-chaos transition. When a bath or a resonator is coupled to the driven transmon through the charge operator, this results in a finite decay from the regular states to the chaotic layer. Being concerned with high-fidelity operations such as qubit readout, one wished to minimize such effects.

First, to avoid these effects, one can maximize the size of the regular central island by driving at a suitable frequency, as discussed in Sec. III B. Second, one can maximize the number of states contained in the regular island. Previous works \cite{41,59} have shown that for a state of a given excitation number, the tunneling rate to the chaotic layer is exponentially suppressed with \( \frac{1}{\sqrt{E_j}} \). Intuitively, this follows from an exponential suppression, as a function of \( \frac{1}{\sqrt{E_j}} \), of the phase-space overlap of a regular state with the chaotic layer decreases exponentially. In other words, the coupling of a regular state to the chaotic layer is mediated by the regular states in between, resulting in an exponential suppression with the number of intermediate states, thus with \( \frac{1}{\sqrt{E_j}} \). Note that at large values of \( \frac{1}{\sqrt{E_j}} \), other processes such as resonance-assisted tunneling can alter this exponential law \cite{59}.

In practice, this translates to an exponential suppression of the matrix elements of the charge operator between regular and chaotic Floquet modes. As mentioned in Sec. V, the matrix elements \( n_{ijk} \) with \( k = -1,0 \) between Floquet modes determine the ability of the driven transmon to hybridize with the resonator and potentially decay into the resonator bath similarly to a Purcell effect. For a Floquet state \( |\ell\rangle \) of the driven transmon, we define its unitless coupling to the subspace of ‘high-energy’ Floquet states above a threshold index \( j > i \) as

\[
N_{ij,\ell \in (-1,0)} = \sqrt{\sum_{lk \in (-1,0)} |n_{ijk}|^2},
\]

where the matrix elements \( n_{ijk} \) are defined in Eq. (10), and where the Floquet states are sorted through their mean energies defined in Eq. (4). By its definition in Eq. (16), the dimensionless coupling \( N_{ij,\ell \in (-1,0)} \) is a monotonically decreasing function. Relevant information about the coupling to the chaotic layer is encompassed in how fast this decrease is as a function of the threshold state index \( j \).

In Fig. 9(a) (respectively (b)), for three values of \( \frac{1}{\sqrt{E_j}} \), and the same parameters as in Fig. 5, we plot the coupling \( N_{ij,\ell \in (-1,0)} \) as a function of the index \( j \) for the ground state \( i = 0 \) (respectively the first excited state \( i = 1 \)), at zero drive (dotted line) and \( \tilde{\xi}_d = 0.1 \) (solid line). As the matrix elements fluctuate as a function of the offset charge, they are averaged over 50 values of \( n_g \). At zero drive, the matrix elements are exponentially suppressed with the state index \( j \) for all three values of \( \frac{1}{\sqrt{E_j}} \). At finite drive, after a rapid decay, all three curves stabilize on a plateau marked by comparatively slower decrease in both Fig. 9(a) and (b). This plateau is due to the near-equality of the matrix elements \( n_{ijk} \) when \( l \) belongs to the chaotic layer. While the width of the plateau should grow with \( \frac{1}{\sqrt{E_j}} \), as the chaotic states are more numerous, its value is exponentially reduced with \( \frac{1}{\sqrt{E_j}} \). Due to its proximity to the chaotic states, the first excited state is more strongly coupled to the chaotic states than the ground state, which translates to larger values of the dimensionless coupling \( N_{ij,\ell \in (-1,0)} \) for the excited state than for the ground state, both within and away from the plateaus.

Note that at this relatively small drive amplitude \( \tilde{\xi}_d = 0.1 \) and for a light-matter coupling strength \( g/2\pi = 250 \text{ MHz} \), the transmon-resonator interaction term involves transitions with the chaotic states at a coupling strength of approximately 20 MHz in the case \( \frac{1}{\sqrt{E_j}} = 3 \) \((E_j/E_C = 72)\). The associated frequencies \( \Delta_{ij,k-1} - \Delta \) strongly depend on the drive amplitude and the offset charge. With the fluctuations of the offset charge and the ac-Stark shift leading to a sweep across the spectrum during readout \cite{23,25}, these transitions might result in non-QND effects during transmon readout.
This important population of the chaotic states also mean that, in order for numerical studies to capture chaos-induced effects on the low-energy states of a single driven transmon, it is necessary to use a Hilbert space size which contains the entire chaotic layer. That is, in some instances it may be necessary to revisit the conventional wisdom that using a few transmon states is sufficient for accurate simulations of the driven transmon. Moreover, based on the study of instabilities in the classical system, it may be possible to identify favorable frequency placement for which the width of the chaotic layer is minimal, results which can be used to find optimal parameters for operations with strong drives such as dispersive qubit readout. The identification of the chaotic layer as a function of the classical parameters $\omega_d/\omega_p$ and $\epsilon_d/\omega_p$ leads to the definition of a chaos-induced critical photon number in the quantum system, which increases as a function of $E_J/E_C$. In particular, spurious transitions during strong-drive operations are expected to be minimized in the regime of large $E_J/E_C$.

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Appendix A: Suppression of chaotic behavior at high-frequency driving

In Sec. III B, we study the dependence of the chaotic layer on the drive frequency. To have a fair comparison between the effects at different frequencies, we fixed the absolute value of the ac-Stark shift of the 0 - 1 transition to 100 MHz, and correspondingly chose the drive amplitude. This is made possible by tracking the ground and first excited states as a function of drive amplitude to obtain the difference of the quasienergies, $\epsilon_1 - \epsilon_0$. The negative absolute value of the ac-Stark shift is plotted in Fig. 10 as a function of $\xi_d$ for various frequencies. Disruptions of the lines indicate that tracking is no longer possible.

As a complementary study to Sec. III B, in Fig. 11(a)-(d), we show the rate matrices at an ac-Stark shift of 100 MHz for the same drive frequencies. The size of the chaotic block follows that of the chaotic layer in the Poincaré sections of Fig. 3(a)-(d). In particular, the rate matrix corresponding to Fig. 3(d) remains close to that of the undriven system [see inset of Fig. 4(b)], although the 0-1 transition frequency is shifted by 100 MHz.
amplitude which is well above that needed to work at a fixed drive. In Fig. 3 are plotted. Tracking is lost beyond a certain drive amplitude which is well above that needed to work at a fixed shift of \(-100\) MHz.

Appendix B: Inversion symmetry sector of the driven system

In this section, we provide further explanation on the inversion symmetry discussed in Sec. IV. First, let us consider the eigenvalue equation \(H|\psi\rangle = E|\psi\rangle\) with \(H\) the Hamiltonian Eq. (2) of the undriven transmon \((\varepsilon_d = 0)\). The eigenstates of \(H\) are eigenstates of the parity operator \(P\), defined as \(P : |n\rightarrow -n, \phi \rightarrow -\phi\), if and only if \(n_g \equiv 0, 0.5 \bmod 1\). To prove this, we define the boost operator along charge \(U = \exp(-in_g\phi)\). After transformation under this boost, the Hamiltonian \(H' = UHU^\dagger\) does not depend on \(n_g\) and is therefore symmetric under \(P\). Thus parity acting onto an eigenstate must yield the same eigenstate if the spectrum is nondegenerate. That is, \(P|\psi\rangle = e^{i\theta}|\psi\rangle\), or \((-\phi)|\psi\rangle = e^{i\theta}\langle\phi|\psi\rangle\), for all \(\phi\), and some phase \(\theta\) independent of \(\phi\). Additionally, in the boosted frame, the eigenstates obey the ‘twisted-periodic’ boundary condition \((\phi + 2\pi)|\psi\rangle = e^{-2\pi n_g}\langle\phi|\psi\rangle\).

Writing the two conditions above at \(\phi = \pi\) gives \((-\pi)|\psi\rangle = e^{i\theta}\langle\pi|\psi\rangle = e^{-i2\pi n_g}\langle\pi|\psi\rangle\), which implies that \(e^{i\theta} = e^{i2\pi n_g}\). The wavefunctions are eigenstates of parity iff \(\theta = 0, \pi\), that is for \(n_g = 0, 0.5 \bmod 1\).

As explained in the main text, in the transmon regime, the low-energy sector is almost independent of the offset charge and the Hamiltonian becomes effectively parity symmetric. When parity is a symmetry of \(H\), matrix elements of the charge operator \(\langle j|n|j\rangle \neq 0\) iff \(i\) and \(j\) belong to different parity sectors. Therefore if \(\langle j|n|j\rangle \neq 0\) then parity symmetry is broken. By the Hellmann-Feynman theorem, \(\langle j|n|j\rangle \propto dE_j/dn_g\), so whenever a band has nonzero group velocity, parity symmetry is broken. This is why sweet spots (zero group velocity) occur at \(n_g = 0, 0.5 \bmod 1\). Exponentially small group velocity, as in the transmon limit \(E_J \gg E_C\), translates to exponentially weak breaking of parity. Thus, in the inset of Fig. 4 (b) computed for \(n_g = 0.25\) and therefore nonzero group velocity, purple squares appear, but they are exponentially dim. By extension, in the low-energy sectors, transitions that are forbidden by parity selection rules at \(n_g = 0, 0.5 \bmod 1\) are exponentially suppressed when parity symmetry is broken, leading to an effective parity symmetry. Note that it is this effective symmetry in the transmon limit \(E_J \gg E_C\) that allows us to ignore charge dispersion effects and express the transmon \(H\) as a Kerr nonlinear oscillator.

At zero drive, this effective symmetry results in a suppression of the matrix elements of the charge operator \(n_{i,i+2}\), and forbids the transition \(i \rightarrow i + 2\) in the low-energy sectors for both \(n_g = 0.5\) and \(n_g = 0.25\). This can be seen in the rate matrices of the undriven systems – insets of Fig. 4 (a, b) – where the squares corresponding to these transitions remain empty. Importantly, this symmetry is not respected at \(n_g = 0.25\) for states in the separatrix region where charge dispersion become non-negligible.

In the presence of a drive \(\varepsilon_d \cos(\omega_d t)\mathbf{n}\), the inversion symmetry holds if one additionally applies the operation \(t \rightarrow t + \pi/\omega_d\) [32]. This symmetry of the driven Hamiltonian defines even and odd parity sectors among the time-dependent Floquet states. Note that the parity of a Floquet state depends on the Brillouin zone it belongs to in the undriven case. For example, in the undriven system, the eigenenergy of the first excited state decomposes as \(E_1 = \varepsilon_1 + k\omega_d\), where the quasienergy satisfies \(|\varepsilon_1| \leq \omega_d/2\), and \(k = 1\) for the parameters used in Fig. 4. Within the Floquet formalism, the Floquet state, corresponding to the eigenstate \(\exp(-iE_1 t)|1\rangle\) of the time evolution operator, reads \(|\psi_1(t)\rangle = \exp(-i\varepsilon_1 t)|\phi_1(t)\rangle\), with the Floquet mode \(|\phi_1(t)\rangle = \exp(-i\omega_d t)|1\rangle\). Since the Floquet mode \(|\phi_1(t)\rangle\) is invariant under the transformation \(t \rightarrow t + \pi/\omega_d\) and \(\mathbf{n} \rightarrow -\mathbf{n}\), it belongs to the even sector. When increasing the drive, the state \(|\phi_1(t)\rangle\) will remain in the even sector, and expands only on even states \(\exp(-i(2k+1)\omega_d t)|2n + 1\rangle\), \(\exp(-i2k\omega_d t)|2n\rangle\), with \(k \in Z\) and \(n \in N\).

Under this symmetry, transitions through the charge operator between two states of the same (opposite) parity can only involve an odd (even) number of drive photons \(k\), as the charge operator already changes the parity. This results in superposition of blue and red squares in at \(n_g = 0.5\), see Fig. 4 (a). On the contrary, at \(n_g = 0.25\), purple squares, indicative of red and blue contributions, light up not only in the chaotic block, but also for transitions involving the low-energy states.

Transitions which do not respect the apparent symmetry of the transmon were experimentally observed in Sank et al. [23] but remained unexplained. There, the situation is slightly more complicated as a harmonic mode is capacitively coupled to the transmon, making this effectively a 3-mode problem. Nonetheless, the situation is similar, and the same conclusions can be drawn. Indeed, the inversion symmetry for the full Hamiltonian holds upon adding the transformation \(\mathbf{n} \rightarrow -\mathbf{n}\), where
Figure 11. Rate matrices and steady-state populations at for the same parameters as in Fig. 3, with the drive amplitude corresponding to an ac-Stark shift of $-100$ MHz. The rate matrices (b) and (d) are more regular, resulting in a smaller plateau in the steady-state distribution.

$$n_r = -\sqrt{2}(a - a^\dagger)$$ is the charge operator of the resonator. Because the resonator is low-Q, transitions between states are most likely to be induced by the bath coupled to the resonator through the resonator charge operator $n_r$. Since $n_r$ maps one parity sector to another similarly to $n$, the single driven transmon analysis remains valid for the full driven circuit QED setup.

**Appendix C: Floquet simulations for circuit QED Hamiltonian**

In this Appendix, we provide more details for the numerical simulations presented in Sec. V. To capture the entirety of the chaotic layer even at strong drives $\tilde{\varepsilon}_d \geq 1$, we have used a local Hilbert space size of 35 for the transmon. For the resonator, we use 20 states, which is pertinent in the regime of off-resonant drives (here, the drive frequency is 500 MHz below the bare cavity frequency at 8 GHz). We characterize truncation errors by plotting the error in the bosonic commutation relation in Fig. 12, and observe significant errors of the commutator only for states with $\langle N_r \rangle \geq 12$. We also observe a ‘reflection’ at the Hilbert space boundary [63], as indicated by monotonically decreasing values of $\langle N_r \rangle$ versus $\langle N_t \rangle$ (upper regions of the four panels of Fig. 12).

Boundary effects in the Hilbert space become important at strong drives, where states with high $\langle N_r \rangle$ become populated, see Fig. 13. In particular, averaging the commutator error in the steady state of the driven-dissipative evolution, we find that it correlates well with the number of occupied Floquet modes as determined from the entanglement entropy of the steady-state density matrix [61] [Fig. 13(b,c)], which occurs at about the same threshold power $\tilde{\varepsilon}$ as ionization, defined as a significant increase in $\langle N_i \rangle$, see Fig. 13(a).

When computing the steady state, a minimal bound for the required truncation of the resonator Hilbert space can be estimated based on energy considerations. As discussed in Sec. V, one can first diagonalize the driven transmon Hamiltonian, and then write the transmon-resonator coupling in the new basis (Floquet basis for the

Figure 12. Commutator error, encoded in symbol color, for each Floquet mode, expressed as $|1 - \langle[a, a^\dagger]\rangle|$. 

### Figure 12

(a) $\tilde{\varepsilon}_d = 0.0$

(b) $\tilde{\varepsilon}_d = 0.5$

(c) $\tilde{\varepsilon}_d = 1.0$

(d) $\tilde{\varepsilon}_d = 1.89$
transmon and Fock basis for the resonator). At strong drives, we need to account for the full chaotic layer of the driven transmon and its corresponding energy span. Although the mean energies of these states concentrate around $\langle H \rangle \sim 2E_J$, their Fourier decompositions extend over multiple Brillouin zones of width $\omega_d$. More precisely, the highest populated Brillouin zone is approximately given by the energy of the charge state that lies on the external boundary of the chaotic layer. For instance, on Fig. 1(c) the border of the regular/chaotic domain is located around $\langle H \rangle_{\max} = 3.5E_J$ at $\tilde{\epsilon}_d = 0.5$, and around $\langle H \rangle_{\max} = 5.5E_J$ at $\tilde{\epsilon}_d = 1$.

Through the transmon-resonator coupling, the energy stored in the Floquet modes can be converted into resonator photons. In fact, the chaotic behaviour of the system with increasing Fock state number (strong hybridization) shows the ability for the resonator to absorb energy. The maximum energy $\langle H \rangle_{\max}$ at $\tilde{\epsilon}_d = 0.5$ (resp. $\tilde{\epsilon}_d = 1$), can be converted into $\langle H \rangle_{\max}/\omega_n \approx 7$ (resp. 11) resonator photons. Interestingly, this is in agreement with the range of $\langle |N_r| \rangle$ of populated states in Fig. 7 (highlighted with circles). Therefore, the dimension of the resonator Hilbert space has to be chosen larger than the estimated highest populated Fock state.

Figure 13. (a) Steady state population of the transmon and resonator as a function of drive strength. Both expectation values become non-monotonic beyond a threshold at $\tilde{\epsilon}_d \approx 1$. (b) Commutator error (see also Fig. 12) in the steady state. (c) Number of occupied Floquet modes in the steady state [61] as calculated from the entanglement entropy $N_{occ} = \exp(-\sum_i p_i \log p_i)$.

Figure 14. Same as Fig. 7, but for higher drive strengths. Steady-state expectation values of resonator photon number and transmon excitation number deviate significantly from zero. The steady state has significant weight over a large number of chaotic states. An arrow is plotted if $\Gamma_{ij} > 10^{-3}\kappa$ for (a)-(b), or $10^{-2}\kappa$ for (c)-(d).

Appendix D: Chaos in the undriven cQED system

We consider an undriven cQED system composed of a transmon coupled to a resonator with Hamiltonian

$$H = 4E_C(n - n_g)^2 - E_J \cos(\phi) + \omega_n a^\dagger a - i\gamma(n a^\dagger a).$$

(D1)

This system is analogous to a single driven transmon where the drive is replaced by a quantum field. We show that correlations develop in the spectrum with increasing Fock state number. By comparing the spectrum of this system with the Floquet spectrum of a driven transmon where an effective drive amplitude is varied, we find that the spectra agree at low photon number and for regular states, but show significant deviations for chaotic states.

We bring the corresponding eigenenergies $E_i$ to the first Brillouin zone delimited by the resonator frequency $\omega_n$, by defining $\epsilon_i = E_i[\omega_n] \in [-\omega_n/2, \omega_n/2]$. In Fig. 15 (black dots), for each eigenstate $|i\rangle$ satisfying $\langle N_r \rangle_i < 20$, we show its energy $\epsilon_i$ and its mean photon number $\langle N_r \rangle_i$, in the $(\epsilon, \langle N_r \rangle)$ plane. We note the appearance of seemingly continuous lines as a function of $\langle N_r \rangle$ (note that these are made of unlabeled dots), that can be associated to transmon states [25]. We refer to these lines as branches.

For eigenstates with a given mean photon number
Figure 15. Black dots: eigenenergies of the undriven cQED Hamiltonian in Eq. (D1). For each eigenstate, the energy $\varepsilon_i$ and its mean photon number $\langle N_r \rangle_i$ are plotted in the $(\varepsilon_i, \langle N_r \rangle_i)$ plane. The energies $\varepsilon_i$ are the eigenenergies folded back into the first Brillouin zone defined by the resonator frequency. Only states satisfying $\langle N_r \rangle < 20$ are shown. The parameters are the same as in Fig. 7, with $\omega_d/2\pi = 8$ GHz and $g/2\pi = 0.250$ GHz. Red dots: quasienergies of the Floquet Hamiltonian Eq. (2) with $\omega_d/2\pi = 8$ GHz as a function of $n = (\varepsilon_d/2g)^2$, for states satisfying $\langle N_r \rangle < 20$. The behaviour of the two spectra is qualitatively similar, but significant deviations appear at large photon number.

$\bar{n} = \langle N_r \rangle$, we can compare the spectrum of the undriven system with the quasispectrum of a single transmon driven (no resonator) at an amplitude $\varepsilon_d = 2g\sqrt{\bar{n}}$ (red dots in Fig. 15). Note that large anti-crossings appear in both spectra, which is the signature of a correlated spectrum. Although the two spectra qualitatively agree at low photon number, significant deviations appear at large photon number.

As recently observed numerically in Ref. [25], in the undriven system, the frequency pull on the resonator goes to zero as $\langle N_r \rangle$ increases. To see this, let us consider the effect of a probe on the system in a given state $|i\rangle$, with the probe frequency being close to that of the resonator, at $\omega_a + \delta$, where $\delta$ is small. A weak enough probe can only cause transitions to states with neighbouring mean photon number $\langle N_r \rangle_i \pm 1$. In addition, since the probe frequency is $\omega_a + \delta$, it can only connect $|i\rangle$ with states of energy $\varepsilon_i \pm \delta$, as all the energies are plotted modulo the resonator frequency. Hence, the probe causes transitions to neighbouring states in the $(\varepsilon_i, \langle N_r \rangle_i)$ plane, which correspond to neighbouring states on the same branch. The frequency pull is then $\varepsilon_j - \varepsilon_i$, where $|j\rangle$ is the state on the same branch with $\langle N_r \rangle_j \approx \langle N_r \rangle_i \pm 1$. Hence, the frequency pulls correspond to the slopes of the apparent branches, which decrease with increasing photon number $\langle N_r \rangle$.

To explain this, let us consider an eigenstate state $|i\rangle$ on one of the energy branches, at $\langle N_r \rangle_i$. We pick the state $|j\rangle$ that has maximum overlap with $a_i^\dagger |i\rangle$, so that $i \rightarrow j$ is the brightest transition upon driving the resonator when the system is in state $|i\rangle$. By using

$$\langle j| H a_i^\dagger |i\rangle = \langle j| [a_i^\dagger H + [H, a_i^\dagger]] |i\rangle,$$

we obtain the general relation

$$\langle E_j - E_i - \omega_a \rangle = g \frac{\langle j| n |i\rangle}{\langle j| a_i^\dagger |i\rangle}.$$  

While $|\langle j| n |i\rangle|$ grows sublinearly with the square root of the photon number $\sqrt{\langle N_r \rangle_i}$ (see Appendix E), $\langle j| a_i^\dagger |i\rangle$ scales as $\sqrt{\langle N_r \rangle_i}$ by definition of $|j\rangle$. Hence, the frequency difference satisfies

$$|\varepsilon_j - \varepsilon_i| = |E_j - E_i - \omega_a| \propto g/\sqrt{\langle N_r \rangle_i} \rightarrow 0.$$  

Appendix E: Standard deviation of the transmon dipole moment in the chaotic phase

We consider a single transmon driven at high enough power to render all of the phase-like states chaotic. The Hilbert space is then simply composed of a finite number $N$ of chaotic states and an infinity of regular charge states. We define $U$ the unitary that diagonalizes the Floquet propagator $U_T = \mathcal{T} \exp[-i \int_0^T H(t) dt]$, where $\mathcal{T}$ is the time-ordering operator. Because the system is chaotic, the matrix elements $U_{ij}$ of the unitary $U$ projected on the chaotic subspace can be modeled as random matrices, distributed with variance $1/N$ and zero mean [36]. The charge matrix element $n_{ij}$ in the Floquet basis (at a given time) are related to the charge matrix element $\tilde{n}_{ij}$ in the Floquet basis at zero drive via the unitary $U$, through $n = U \tilde{n} U^\dagger$. We can therefore write

$$n_{ij} = \sum_{r,s} U_{ir} \tilde{n}_{rs} U_{js}^\dagger.$$  

The $n_{ij}$’s can thus be modeled as random variables. Taking the average of these random variables of the modulus squared of the above equality, we obtain

$$\mathbb{E}(|n_{ij}|^2) = \sum_{r,s} \mathbb{E}(U_{ir} U_{js}^\dagger U_{it}^\dagger U_{jt}) \tilde{n}_{rt} \tilde{n}_{ts}^\dagger = \sum_{r,s} \mathbb{E}(|U_{ir}|^2) |U_{js}|^2 |\tilde{n}_{rs}|^2 \quad \mathcal{E}(U_{js}^2) |\tilde{n}_{rs}|^2 = \sum_{r,s} |\tilde{n}_{rs}|^2 / N_{ch}^2.$$  

The second line is obtained by observing that only squared matrix elements $|U_{ir}|^2$ survive when taking the average value. In going from the second to third line, we obtain an approximation by neglecting the correlations between $|U_{ir}|^2$ and $|U_{js}|^2$. Finite correlations exist, since the $U_{ir}$ obey orthonormality conditions. However, using the probability distributions derived in Ref. [36] for random matrices, we checked numerically that these second-order correlations are small if the number of chaotic
states is large enough, i.e. $N_{ch} > 8$, and vanish in the large $N_{ch}$ limit. The last line is obtained using the equality $\mathbb{E}(|U_{ii}|^2) = 1/N_{ch}$.

The term $\sum_{r,s} |\hat{n}_{rs}|^2$ can be easily estimated. Indeed, one can write $\sum_{r,s} |\hat{n}_{rs}|^2 = |\mathbf{P}_{ch}\mathbf{n}\mathbf{P}_{ch}|^2$, where $\mathbf{P}_{ch}$ is the projector on the chaotic subspace. As the regular states are charge states, a basis of the chaotic subspace are the $2N_{ch}$ charge states $|n_{charge}\rangle$ with $n_{charge} < N_{ch}/2$. We can therefore calculate $|\mathbf{P}_{ch}\mathbf{n}\mathbf{P}_{ch}|^2$ in the charge basis, giving

$$|\mathbf{P}_{ch}\mathbf{n}\mathbf{P}_{ch}|^2 = \frac{N_{ch}/2}{n_{charge} = -N_{ch}/2} n_{charge}^2$$

$$= \frac{(N_{ch}/2)(N_{ch}/2 + 1)(N_{ch} + 1)}{3}.$$ 

This leads to

$$\sqrt{\mathbb{E}(|n_{ij}|^2)} \approx \sqrt{\frac{N_{ch}}{12}}.$$ 

In other words, the standard deviation of the dipole moment for a given transition increases with the size of the chaotic layer. Loosely speaking, the action of the unitary $U$ is to “randomize” the matrix elements of the charge operator. As a consequence, the regular structure of the matrix elements of the undriven system, e.g. the harmonic oscillator-like structure $n_{ij} \propto \delta_{j,i+1} + \delta_{j,i-1}$ of the low-energy states, is not preserved.

For a numerical illustration of the above, we define the mean dipole moment for a Floquet state $i$ by

$$\langle n_i \rangle = \sqrt{\frac{1}{M} \sum_{j=0}^{M} |n_{ij}|^2}, \quad (E3)$$

where $M$ is an integer larger than the number of chaotic states $N_{ch}$. The matrix elements $n_{ij}$ are taken between Floquet modes at time $t = 0$. We also define the mean dipole moment

$$\langle n \rangle = \sqrt{\frac{1}{M} \sum_{i=0}^{M} \langle n_i \rangle^2}.$$ 

Using the same argument as above, we have $\langle n \rangle = |\mathbf{P}_{M}\mathbf{n}\mathbf{P}_{M}| \approx \sqrt{M/12}$, where $\mathbf{P}_{M}$ is the projector on the manifold spanned by the charge states $|m\rangle$ such that $|m| \geq M/2$.

In Fig. 16(a), the mean dipole moment per Floquet state $\langle n_i \rangle$ is plotted for the first 25 Floquet states (sorted by mean energy) as a function of the drive amplitude. For clarity, we highlight in red states that are starting close to the separatrix, in blue the low-energy states, and in black the charge-like states. In green is plotted the mean dipole moment $\langle n \rangle$. Its constant value for $\tilde{\varepsilon}_d < 2.5$ is a good indication that the first $M$ states do not hybridize with higher charge-like states for this range of drive amplitude. Note that $M = 25$ gives $\langle n \rangle = \sqrt{M/12} \approx 1.44$, which agrees with the green curve. For $\tilde{\varepsilon}_d > 2.5$, we note that $\langle n \rangle$ increases, indicating the increasing hybridization of higher-energy states with the 25 states represented here. This is also an indication that at $\tilde{\varepsilon} = 2.5$, the first 25 states are hybridized. As the drive amplitude increases, the mean dipole moments per Floquet states $\langle n_i \rangle$ converge slowly towards $\langle n \rangle$.

However, in the definition Eq. (E3) of the mean dipole, the diagonal matrix elements $n_{ii}$ also contribute to the average, yielding large dipole moments for charge-like states. As $n_{ii}$ does not result in transitions to other states, we define the more meaningful dipole moment for
the state $i$,

$$\langle n^+_i \rangle = \sqrt{\frac{1}{M-1} \sum_{j=0, j \neq i}^{M} |n_{ij}|^2},$$  \hspace{1cm} (E4)$$

and the mean dipole moment

$$\langle n^* \rangle = \sqrt{\frac{1}{M} \sum_{i=0}^{M} \langle n^+_i \rangle^2}.$$  

These quantities are plotted in Fig. 16(b), with the same color code used in panel (a). If the values of $\langle n^+_i \rangle$ are qualitatively the same as in Fig. 16(a) for the blue and red states, the behaviour is dramatically different for the charge-like states, for which $\langle n^+_i \rangle$ is small at low drive power. This is a consequence of the fact that the charge operator is mostly diagonal for charge-like states. Crucially, we note that as the drive strength increases, the mean dipole moments $\langle n^+_i \rangle$ converge toward $\langle n^* \rangle$, which increases (green curve). This is due to the increasing participation of charge-like states in the chaotic layer. Note that at $\tilde{\varepsilon}_d = 2.5$, all 25 states are hybridized, and $\langle n^* \rangle$ has come close to $\langle n \rangle$ (green dashed line in Fig. 16(b)).

Finally, note that the mean dipole moment of the ground state, corresponding to the lowest blue line in Fig. 16(b), undergoes a 4-fold increase upon entering the chaotic layer for $\tilde{\varepsilon}_d > 1$.

**Appendix F: Onset of strong hybridization between the transmon and the resonator in the presence of drive**

In this section, we study in detail the onset of the hybridization of the transmon and resonator when the (driven) transmon is in a chaotic state. We start with the Hamiltonian of the driven transmon coupled to a resonator, Eq. (13) in the main text, reproduced here in the absence of the coupling to the bath

$$\mathbf{H}(t) = 4E_C(n - n_g)^2 - E_J \cos(\phi) + \varepsilon_d \cos(\omega_d t) \mathbf{n} + \omega_a \mathbf{a}^\dagger - ig \mathbf{n} (\mathbf{a} - \mathbf{a}^\dagger).$$

Moving to the Floquet basis $\{|i\rangle\}$ for the driven transmon, and in the frame rotating at the drive frequency, this Hamiltonian reduces to

$$\mathbf{H}(t) = \sum_i \varepsilon_i |i\rangle \langle i| + \Delta \mathbf{a}^\dagger \mathbf{a} - ig \sum_{i,j,k} n_{ijk} |i\rangle \langle j| (e^{i(k-1)\omega_d t} \mathbf{a} - e^{i(k+1)\omega_d t} \mathbf{a}^\dagger),$$  \hspace{1cm} (F1)$$

where $\Delta = \omega_a - \omega_d$, and $\varepsilon_i$ are the quasienergies of the driven transmon. Assuming that $|\Delta| \ll \omega_d$ and as $|\varepsilon_i| < \omega_d/2$, the interaction term (second line) can come close to resonance for $k = 0, 1$ (term in $\mathbf{a}$) and $k = -1, 0$ (term in $\mathbf{a}^\dagger$). In order to study the impact of chaotic transmon states on the resonator, for simplicity we assume that the drive strength is large enough for all the phase-like states to be ionized, as in Appendix E.

We consider the tensor product state $|\tilde{i}\rangle \otimes |n\rangle$, composed of a chaotic transmon Floquet mode $|\tilde{i}\rangle$ and the n-th Fock state of the resonator. The term $-i gn_{ij,-1} \mathbf{a}^\dagger$ couples this state to the state $|\tilde{j}\rangle \otimes |n + 1\rangle$ with a coupling strength $gn_{ij,-1}\sqrt{n}$. Let us first give an intuitive picture of the physics by projecting the Hamiltonian in Eq. (F1) on the vacuum and first few Fock states of the resonator. The corresponding situation is depicted in Fig. 17, where we plot the quasispectra of the uncoupled system corresponding to the Hamiltonian in Eq. (F1) with $g = 0$, for the states $|\tilde{i}\rangle \otimes |0\rangle$ (black dots) and $|\tilde{i}\rangle \otimes |1\rangle$ (blue dots). Only the first 13 states $|\tilde{i}\rangle$ (sorted through mean energies) are represented, which are all part of the chaotic layer of
the driven transmon at $\epsilon_d \geq 1$. The two quasispectra are detuned by $\Delta$, due to the term $\Delta a^\dagger a$ in Eq. (F1). Here, we chose $\Delta/2\pi = 200$ MHz for visual clarity.

At $\epsilon_d = 0.05$, the possible transitions, due to the term $-i g n_{ij,k-1} a^\dagger + h.c.$, involving the state $|1\rangle\otimes|0\rangle$ and the states $|j\rangle\otimes|1\rangle$, are shown as green vertical lines, where the thickness of the lines is proportional to the coupling strength. These lines couple one black dot and several blue dots. As expected from the harmonic oscillator-like structure of the matrix elements, one transition is dominant, corresponding to the transition with the state $|0\rangle\otimes|1\rangle$. The coupling with the latter is the main contribution to the energy shift of the state $|1\rangle \otimes |0\rangle$. Other green lines, representing coupling to other states, do not appear here as their coupling strength is more than a thousand times smaller than that of the dominant transition.

At $\epsilon_d = 1.15$, almost all states represented are strongly hybridized. We pick one state of the form $|j\rangle \otimes |0\rangle$ (represented as a black dot), and similarly, the possible transitions and coupling strength are represented by red lines. To be able to visually distinguish between the numerous lines, we slightly offset these lines from each other on the x-axis. However, they represent the couplings of one black dot (corresponding to $|j\rangle \otimes |0\rangle$) to multiple blue dots at the same drive amplitude. Contrary to the situation at low drive power, the transitions are numerous and of similar coupling strength, along with positive and negative detuning. The resulting energy shift on $|j\rangle \otimes |0\rangle$ is null on average (in the sense of random matrix model, see Appendix E).

Strong hybridization between the resonator and the driven transmon can occur through the terms $-i g n_{ij,k-1} a^\dagger + h.c.$, depending on the ratio between the coupling strength $g_{\text{eff}} = 2 g n_{ij,k-1} \sqrt{n + 1}$ and the effective detuning $\delta_{\text{eff}}$ of the transition $i \leftrightarrow j$ [38]. Below, we propose a possible estimate of the minimal Fock state number $n$ for which this happens [38]. Recall from Appendix E that for chaotic states, we have $\sqrt{\mathbb{E}[|n_{ij,k-1}|^2]} = \sqrt{N_{\text{ch}}/12}$, where $N_{\text{ch}}$ is the number of chaotic states (typically 10 at the ionization point of the ground state for the parameters in Sec. V). This yields an effective coupling strength $g_{\text{eff}} = 2 g \sqrt{n + 1} N_{\text{ch}}/12$. Note that this is an upper bound, as the contributions from the matrix elements $n_{ij,k-1}$ are usually smaller. Besides, the mean level-spacing of the driven transmon is $\omega_d/N_{\text{ch}}$. Accounting for the detuning $\Delta$ of the resonator with respect to the drive, the effective detuning between the set of states $|i\rangle \otimes |n\rangle$ and the states $|j\rangle \otimes |n+1\rangle$, can be as low as $\delta_{\text{eff}} = \min(\omega_d/N_{\text{ch}}, |\omega_d/N_{\text{ch}} - |\Delta||)$. Noting that due to the repulsive statistics, the energy levels do not bunch, and the resulting variance of the effective detuning is rather small. On average, one has $\delta_{\text{eff}} = \omega_d/N_{\text{ch}}$.

The condition $g_{\text{eff}} \sim \delta_{\text{eff}}$ reads $2 g \sqrt{n + 1} N_{\text{ch}}/12 \sim \omega_d/N_{\text{ch}}$, leading to the definition of a critical photon number,

$$n_{\text{crit}} \sim \frac{3\omega_d^2}{g^2 N_{\text{ch}}^3}$$

above which strong transmon-resonator hybridization occurs. Note that this relation is valid for chaotic transmon states. This criterion is sensitive to the fluctuations of the energy levels and of the charge matrix elements as a function of the system parameters, and therefore can only give a rough estimate. The strong hybridization between the systems occurs in the steady state if this condition is satisfied for the lowest Fock states as well.

As an example, for the parameters of Sec. V, at a drive amplitude such that $N_{\text{ch}} \sim 12$ ($N_{\text{ch}}$ can be estimated by how many grid points of the first row come together in Fig. 7), we find $g_{\text{eff}}/2\pi \sim 0.5$ GHz and $\delta_{\text{eff}}/2\pi \sim 0.6$ GHz, leading to $n_{\text{crit}} \sim 0.5$. A large drop in purity is expected to occur even for the lowest Fock states (see Fig. 7(d)). As a second example, we consider the parameters of Sec. V system with $g = 0.250/(\sqrt{6}$ GHz instead of $g = 0.250$ GHz in Fig. 18. This reduction of the coupling makes the value of $n_{\text{crit}}$ an order of magnitude higher.

**Appendix G: Effect of the resonator dissipation on the spectrum**

In Sec. V, we have not accounted for the effect of the resonator dissipation on the spectrum. In the limit of very large $\kappa$, the two systems decouple and chaos disap-

![Figure 18](image-url)

Figure 18. Same as Fig. 7, for $g/2\pi$ reduced by a factor $\sqrt{6}$. 
For typical experimental parameters, $\kappa/2\pi = 10$ MHz and $g/2\pi = 250$ MHz, dissipation will affect Fock states with $n \geq (g/\kappa)^2 = 625$. As a numerical check, we diagonalize the non-hermitian undriven Hamiltonian

$$H = 4E_c(n - n_g)^2 - E_J \cos(\phi) + (\omega_a - i\kappa/2) a^\dagger a - i gn(a - a^\dagger).$$

In Fig. 19, the real part of the energy $\epsilon_i$ and the photon number $\langle N_r \rangle_i$ of each state are plotted in the $(\epsilon_i, \langle N_r \rangle_i)$ plane for $\kappa = 0$ (blue dots) and $\kappa/2\pi = 10$ MHz (red dots). Qualitatively, the spectrum is the same in both cases. In particular, anti-crossings, which are signatures of resonances of the coupled system, do not seem to be attenuated.

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