Empirical Optimal Transport between Different Measures Adapts to Lower Complexity

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Abstract
The empirical optimal transport (OT) cost between two probability measures from random data is a fundamental quantity in transport based data analysis. In this work, we derive novel guarantees for its convergence rate when the involved measures are different, possibly supported on different spaces. Our central observation is that the statistical performance of the empirical OT cost is determined by the less complex measure, a phenomenon we refer to as lower complexity adaptation of empirical OT. For instance, under Lipschitz ground costs, we find that the empirical OT cost based on \( n \) observations converges at least with rate \( n^{-1/d} \) to the population quantity if one of the two measures is concentrated on a \( d \)-dimensional manifold, while the other can be arbitrary. For semi-concave ground costs, we show that the upper bound for the rate improves to \( n^{-2/d} \). Similarly, our theory establishes the general convergence rate \( n^{-1/2} \) for semi-discrete OT. All of these results are valid in the two-sample case as well, meaning that the convergence rate is still governed by the simpler of the two measures.

On a conceptual level, our findings therefore suggest that the curse of dimensionality only affects the estimation of the OT cost when both measures exhibit a high intrinsic dimension. Our proofs are based on the dual formulation of OT as a maximization over a suitable function class \( \mathcal{F}_c \) and the observation that the \( c \)-transform of \( \mathcal{F}_c \) under bounded costs has the same uniform metric entropy as \( \mathcal{F}_c \) itself.

Keywords: Wasserstein distance, convergence rate, curse of dimensionality, metric entropy, semi-discrete, manifolds

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1 Introduction

The theory of optimal transport (OT) allows for an effective comparison of probability measures that is faithful to the geometry of the underlying ground space (see Račhev & Rüschendorf 1998a,b; Villani 2003, 2008; Santambrogio 2015 for comprehensive treatments). Origins of OT date back to the seminal work by Monge (1781) and its measure theoretic generalization by Kantorovich (1942, 1958), paving the way for a rich theory and many applications. With recent computational advances (for a survey see Bertsimas & Tsitsiklis 1997; Peyré & Cuturi 2019) OT based methodology is also quickly emerging as a useful tool for data analysis with diverse applications in statistics. This includes bootstrap and resampling (Bickel & Freedman 1981; Sommerfeld et al. 2019; Heinemann et al. 2020), goodness of fit testing (del Barrio et al. 1999; Hallin et al. 2021b), multivariate quantiles and ranks (Chernozhukov et al. 2017; Deb & Sen 2021; Hallin et al. 2021a) and general notions of dependency (Nies et al. 2021; Deb et al. 2021; Mordant & Segers 2022). For a recent survey see Panaretos & Zemel (2019). Further areas of application include machine learning (Arjovsky et al. 2017; Altschuler et al. 2017; Dvurechensky et al. 2018), and computational biology (Evans & Matsen 2012; Schiebinger et al. 2019; Tameling et al. 2021; Wang et al. 2021), among others.

Intuitively, OT aims to transform one probability measure into another one in the most cost-efficient way. For a general formulation, let \( \mu, \nu \in \mathcal{P}(\mathcal{X}) \) be probability measures on Polish spaces \( \mathcal{X} \) and \( \mathcal{Y} \), and consider a measurable cost function \( c: \mathcal{X} \times \mathcal{Y} \to \mathbb{R} \). The optimal transport cost between \( \mu \) and \( \nu \) is defined as

\[
T_c(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \int_{\mathcal{X} \times \mathcal{Y}} c(x, y) \, d\pi(x, y),
\]

where \( \Pi(\mu, \nu) \) represents the set of all couplings between \( \mu \) and \( \nu \), i.e., the probability measures on \( \mathcal{X} \times \mathcal{Y} \) with marginal distributions \( \mu \) and \( \nu \). In statistical problems, the measure \( \mu \) is typically unknown and only i.i.d. observations \( X_1, \ldots, X_n \sim \mu \), defining the empirical measure \( \hat{\mu}_n := \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i} \), are available. A standard approach to estimate \( T(\mu, \nu) \) in this setting is by means of the empirical optimal transport cost \( T_c(\hat{\mu}_n, \nu) \), whose convergence to the population value for increasing \( n \) has been the subject of numerous works. Most research in this context, of which we can only give a selective overview, is devoted to the analysis of the Wasserstein distance (cf. Mallows 1972; Shorack & Wellner 1986; Villani 2008) where \( \mathcal{X} = \mathcal{Y} \) and the cost \( c \) in (1.1) corresponds to the \( p \)-th power of a metric \( d \) on \( \mathcal{X} \). More specifically, for \( \mu, \nu \in \mathcal{P}(\mathcal{X}) \), the \( p \)-Wasserstein distance for \( p \geq 1 \) is defined by

\[
W_p(\mu, \nu) := (T_{d^p}(\mu, \nu))^{1/p},
\]

which is a metric on the space of probability measures on \( (\mathcal{X}, d) \) with finite \( p \)-th moment.

A first fundamental contribution for the analysis of the empirical Wasserstein distance \( W_p(\hat{\mu}_n, \mu) \) in case of \( p = 1 \) was made by Dudley (1969) via metric entropy bounds, asserting

\[
\mathbb{E}[W_1(\hat{\mu}_n, \mu)] \leq n^{-1/d}
\]

for compactly supported probability measures \( \mu \) on \( \mathbb{R}^d \) with \( d \geq 3 \). In particular, if \( \mu \) is absolutely continuous with respect to the Lebesgue measure, this upper bound is tight. Under similar conditions, Dobrić & Yúkichi (1995) derived almost sure limits of \( n^{1/d}W_1(\hat{\mu}_n, \mu) \) through explicit matching arguments for two independent empirical measures \( \hat{\mu}_n \) and \( \mu' \) of a common distribution \( \mu \). Extensions to \( p > 1 \) in Polish metric spaces were obtained by Boissard & Le Gouic (2014) relying on covering arguments of the underlying ground space. For probability measures on Euclidean spaces with possibly

\[1\text{Throughout this work, we write } a_n \leq b_n \text{ for two non-negative real-valued sequences } (a_n)_{n \in \mathbb{N}} \text{ and } (b_n)_{n \in \mathbb{N}} \text{ if there exists a constant } C > 0 \text{ such that } a_n \leq Cb_n \text{ for all } n \in \mathbb{N}. \text{ If } a_n \leq b_n \leq a_n, \text{ we write } a_n \asymp b_n. \]
unbounded support, Dereich et al. (2013) and Fournier & Guillin (2015) derived upper bounds on the \( p \)-th moment \( \mathbb{E}[W_p^p(\hat{\mu}_n, \mu)] \) under certain moment assumptions by explicitly constructing a couplings between \( \hat{\mu}_n \) and \( \mu \). For a compactly supported probability measure \( \mu \) on \( \mathbb{R}^d \), their main result implies for \( n \geq 1 \) that

\[
\mathbb{E}[W_p(\hat{\mu}_n, \mu)] \leq \mathbb{E}[W_p^p(\hat{\mu}_n, \mu)]^{1/p} \leq r_{p,d}(n) := \begin{cases} 
\frac{n^{-1/2}p}{d} & \text{if } d < 2p, \\
\frac{n^{-2/p}p}{d} \log(n)^{1/p} & \text{if } d = 2p, \\
\frac{n^{1/d}}{d} & \text{if } d > 2p.
\end{cases}
\]  

This bound is known to be tight in several settings, e.g., for \( d < 2p \) when \( \mu \) is discretely supported and for \( d > 2p \) when \( \mu = \text{Unif}[0, 1]^d \) is the uniform distribution on the unit cube. For \( d = 2p \), the differences between \( \hat{\mu}_n \) and \( \mu \) at multiple scales culminate in the proof of the upper bound to an additional logarithmic factor, however, it remains open whether it is of correct order. For instance, contributions by Ajtai et al. (1984) and Talagrand (1994) show for \( d = 2 \) and \( p \geq 1 \) that \( \mathbb{E}[W_p(\hat{\mu}_n, \mu)] \leq n^{-1/2} \log(n)^{1/2} \) if \( \mu = \text{Unif}[0, 1]^2 \) (see also Bobkov & Ledoux 2021 for an alternative proof) which improves (1.2) for \( p = 1 \) by an additional \( \log(n)^{1/2} \) factor. Notably, the bounds in (1.2) are known to delimit the accuracy of any estimator \( \hat{\mu}_n \) of \( \mu \) with respect to the Wasserstein distance in the high-dimensional regime. More precisely, without additional assumptions on \( \mu \), the rates in (1.2) are (up to logarithmic factors) minimax optimal (Singh & Póczos 2018), which demonstrates that the estimation of measures in the Wasserstein distance severely suffers from the curse of dimensionality.

To overcome this issue, there has been increased interest in structural properties of \( \mu \) that allow for improved convergence rates. For probability measures on a compact Polish space, Weed & Bach (2019) derived tight bounds in terms of a notion of intrinsic dimension of \( \mu \) (the upper and lower Wasserstein dimension). In particular, if \( \mu \) is compactly supported on \( \mathbb{R}^d \) with upper Wasserstein dimension \( s > 2p \), they established that \( \mathbb{E}[W_p(\hat{\mu}_n, \mu)] \leq n^{-s} \). Moreover, for uniformly distributed \( \mu \) on a compact connected Riemannian manifold of dimension \( d \geq 3 \), Ledoux (2019) derived the bound \( \mathbb{E}[W_p(\hat{\mu}_n, \mu)] \leq n^{-1/d} \), effectively improving upon (1.2) if \( 3 \leq d \leq 2p \). Faster convergence rates can also be obtained under smoothness assumptions on Lebesgue absolutely continuous measures by taking suitable wavelet or kernel density estimators (Weed & Berthet 2019; Deb et al. 2021; Manole et al. 2021), which exploit the smoothness explicitly in contrast to the vanilla empirical OT cost. Under a high degree of smoothness, they approach the population measure in Wasserstein distance nearly with the parametric rate \( n^{-1/2} \) (instead of \( n^{-1/d} \)), but come with additional computational challenges (Vacher et al. 2021).

So far, we only discussed the situation when \( \hat{\mu}_n \) is compared to \( \mu \). From a statistical perspective, however, it is of similar interest to investigate \( W_p(\hat{\mu}_n, \nu) \) for a different measure \( \nu \). We refer to Munk & Czado (1998) and Sommerfeld & Munk (2018) for various applications, such as testing for relevant differences and confidence intervals for \( W_p \). One way to transfer the rates from \( W_p(\hat{\mu}_n, \mu) \) to \( W_p(\hat{\mu}_n, \nu) \) is by means of the triangle inequality,

\[
|W_p(\hat{\mu}_n, \nu) - W_p(\hat{\mu}_n, \nu)| \leq W_p(\hat{\mu}_n, \mu).
\]  

Hence, all of the previous bounds on \( W_p(\hat{\mu}_n, \mu) \) immediately imply the same upper bounds for the convergence rate of \( W_p(\hat{\mu}_n, \nu) \) towards \( W_p(\mu, \nu) \) when \( \mu \) and \( \nu \) are distinct measures on a common metric space. In the two-sample case, when \( \nu \) is additionally estimated by \( \hat{\nu}_n := \frac{1}{n} \sum_{i=1}^n \delta Y_i \), based on an i.i.d. sample \( Y_1, \ldots, Y_n \sim \nu \), the triangle inequality yields

\[
|W_p(\hat{\mu}_n, \hat{\nu}_n) - W_p(\mu, \nu)| \leq W_p(\hat{\mu}_n, \mu) + W_p(\hat{\nu}_n, \nu),
\]  

(1.3b)
which implies the same upper bounds as in (1.2) (as well as all improvements described above) for compactly supported \( \mu, \nu \) on \( \mathbb{R}^d \). Therefore, with \( r_{p,d}(n) \) as in (1.2),
\[
\mathbb{E} \left[ W_p(\hat{\mu}_n, \nu_n) - W_p(\mu, \nu) \right] \leq r_{p,d}(n).
\] (1.4)

These upper bounds match the minimax rates (up to logarithmic factors) among all estimators of \( W_p(\mu, \nu) \) when no additional assumptions are placed on the measures (Liang 2019; Niles-Weed & Rigollet 2019). In particular, this suggests that estimation of the Wasserstein distance between (potentially different) two measures is (without additional assumptions) statistically as difficult as estimation of the underlying measure with respect to Wasserstein loss.

However, crucial to the minimax optimality of (1.4) is the fact that \( \mu \) and \( \nu \) can be chosen to be arbitrarily close. In fact, in case \( \mu \neq \nu \) are sufficiently separated, faster convergence rates may occur. Indeed, for compactly supported \( \mu, \nu \) on \( \mathbb{R}^d \), Chizat et al. (2020) employed the dual formulation of the squared 2-Wasserstein distance (with a similar strategy as for the 1-Wasserstein distance by Sriperumbudur et al. 2012) to derive the bound
\[
\mathbb{E} \left[ W_2^2(\hat{\mu}_n, \hat{\nu}_n) - W_2^2(\mu, \nu) \right] \leq \delta^2 r_{2,d}(n).
\] (1.5a)

If \( \mu \neq \nu \) with \( W_2(\mu, \nu) \geq \delta > 0 \), this implies squared convergence rates
\[
\mathbb{E} \left[ W_2(\hat{\mu}_n, \hat{\nu}_n) - W_2(\mu, \nu) \right] \leq r_{2,d}(n)/\delta
\] (1.5b)
when compared to (1.4). For \( d \geq 5 \), these upper bounds were recently generalized by Manole & Niles-Weed (2021) to arbitrary \( p \geq 1 \), asserting the convergence rate \( n^{-\min(p, 2)/d} \) for the empirical \( p \)-Wasserstein distance. They also provided analogous bounds under convex Hölder smooth costs and proved their sharpness for certain instances as well as minimax rate optimality up to logarithmic factors.

Inspired by these developments, this work is dedicated to a comprehensive understanding of the statistical performance of the empirical OT cost when the underlying probability measures are not only different but may additionally be supported on distinct spaces, for example if \( \mathcal{X} \) and \( \mathcal{Y} \) are submanifolds of \( \mathbb{R}^d \) with (possibly) different dimension. This setting is practically relevant, since the concentration of observations from a high-dimensional ambient space on a low dimensional subspace is a commonly encountered phenomenon, reflected by the popularity of nonlinear dimensionality reduction techniques like manifold learning (see, e.g., Talwalkar et al. 2008; Zhu et al. 2018). Based on the upper bound in (1.3), one is inclined to believe that the convergence rate is determined by the slower rate, i.e., by the measure with higher intrinsic dimension. However, the pivotal (and maybe unexpected) finding of this work is that the convergence rate is actually determined by the measure with lower intrinsic dimension. In this sense, empirical OT naturally adapts to measures with distinct complexity in the most favorable way, and estimating the population value is statistically no harder than estimating the simpler one of the measures \( \mu \) and \( \nu \).

We refer to this phenomenon of OT as lower complexity adaptation (LCA).

**Example.** Consider \( \mathcal{Y} = [0, 1]^{d_2} \) for \( d_2 \geq 1 \) and let \( \mathcal{X} \subset \mathcal{Y} \) be a convex subset with dimension \( d_1 \leq d_2 \). In Section 2.3, we establish that the optimal transportation of any \( \nu \in \mathcal{P}(\mathcal{Y}) \) to any \( \mu \in \mathcal{P}(\mathcal{X}) \) under squared Euclidean costs can be decomposed into two motions (see Figure 1): first an orthogonal projection onto the linear space spanned by \( \mathcal{X} \), and then an OT assignment within that linear space. Since such a projection is statistically negligible when compared to an OT assignment, it follows for \( W_2(\mu, \nu) \geq \delta > 0 \) by (1.5) that
\[
\mathbb{E} \left[ W_2(\hat{\mu}_n, \hat{\nu}_n) - W_2(\mu, \nu) \right] \leq r_{2,d_1}(n)/\delta,
\] (1.6)
which is independent of \( d_2 \), reflecting the LCA principle.
Figure 1: Optimal transport between two- and three-dimensional point clouds (blue and green) for squared Euclidean costs $||x - y||^2$. The optimal assignment in (a) is characterized by first projecting each point to the plane spanned by $X$ before matching the data points, as depicted in (b).

Our core contribution is to show that this phenomenon is a hallmark feature of empirical OT that far exceeds the scope of convex subsets and orthogonal projections. To formalize our main result, let $\mathcal{X}$ and $\mathcal{Y}$ be Polish spaces and consider a continuous bounded cost function $c: \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$. In this setting, the OT cost enjoys a dual formulation (Villani, 2008)

$$
T_c(\mu, \nu) = \max_{f \in \mathcal{F}_c} \int_{\mathcal{X}} f \, d\mu + \int_{\mathcal{Y}} f^c \, d\nu,
$$

where $\mathcal{F}_c$ is a suitable collection of uniformly bounded measurable functions on $\mathcal{X}$ (defined in Section 2.1) and $f^c(y) := \inf_{x \in \mathcal{X}} c(x, y) - f(x)$ denotes the $c$-transform of $f \in \mathcal{F}_c$. To investigate the empirical OT cost, we quantify the complexity of the class $\mathcal{F}_c$ and its $c$-transformed counterpart $\mathcal{F}^c_c = \{ f^c | f \in \mathcal{F}_c \}$ in terms of their uniform metric entropy. The uniform metric entropy of a class $\mathcal{G}$ of real-valued functions on a set $Z$ is defined as the logarithm of the covering number with respect to uniform norm $\| \cdot \|_{\infty}$, which is given for $\varepsilon > 0$ by

$$
\mathcal{N}(\varepsilon, \mathcal{G}, \| \cdot \|_{\infty}) := \inf \left\{ n \in \mathbb{N} \left| \text{there exist } g_1, \ldots, g_n: Z \to \mathbb{R} \text{ with } \sup_{g \in \mathcal{G}} \min_{1 \leq i \leq n} \| g - g_i \|_{\infty} \leq \varepsilon \right. \right\}.
$$

A simple but crucial observation, which lies at the heart of this work, is that $c$-transformation with bounded costs is a Lipschitz operation under the uniform norm. Since $f^c = f$ for all $f \in \mathcal{F}_c$, this in particular implies (Lemma 2.1)

$$
\mathcal{N}(\varepsilon, \mathcal{F}^c_c, \| \cdot \|_{\infty}) = \mathcal{N}(\varepsilon, \mathcal{F}_c, \| \cdot \|_{\infty}). \tag{1.7}
$$

This captures the LCA principle from the dual perspective: we only need to be able to control the complexity of either $\mathcal{F}_c$ or $\mathcal{F}^c_c$. Then, under the growth condition

$$
\log \mathcal{N}(\varepsilon, \mathcal{F}_c, \| \cdot \|_{\infty}) \leq \varepsilon^{-k}
$$

for $\varepsilon > 0$ sufficiently small and a fixed $k > 0$, we prove for arbitrary $\mu \in \mathcal{P}(\mathcal{X})$ and $\nu \in \mathcal{P}(\mathcal{Y})$ that any of the empirical estimators

$$
\hat{T}_{c,n} \in \{ T_c(\hat{\mu}_n, \nu), T_c(\mu, \hat{\nu}_n), T_c(\hat{\mu}_n, \hat{\nu}_n) \}, \tag{1.8}
$$

satisfies the upper bound (Theorem 2.2)

$$
\mathbb{E} \left[ \| \hat{T}_{c,n} - T_c(\mu, \nu) \| \right] \leq \begin{cases} 
n^{-1/2} & \text{if } k < 2, \\
n^{-1/2} \log(n) & \text{if } k = 2, \\
n^{-1/k} & \text{if } k > 2. 
\end{cases}
$$

5
As we discuss in Section 3 suitable bounds for the uniform metric entropy of the function class \( \mathcal{F}_c \) are typically determined by the space \( \mathcal{X} \) and the regularity properties of the cost function. Since \( \mathcal{X} \) can be chosen as the support of \( \mu \), these bounds can often be understood in terms of the intrinsic complexity of \( \mu \) (or the one of \( \nu \), if it turns out to be lower). To explore the consequences of the LCA principle, we put special focus on the setting where the measure \( \mu \) is supported on a space with low intrinsic dimension while \( \nu \) may live on a general Polish space. For example, we obtain convergence rates for semi-discrete OT, where \( \mu \) is supported on finitely many points only. In this setting, we find that the empirical estimator \( \hat{T}_{c,n} \) always enjoys the parametric rate (Theorem 3.2)
\[
\mathbb{E}\left[|\hat{T}_{c,n} - T_c(\mu, \nu)|\right] \leq n^{-1/2}.
\]
This complements distributional limits for the one-sample estimator \( \hat{T}_{c,n} = T_c(\hat{\mu}_n, \nu) \) by del Barrio et al. (2021). We also derive metric entropy bounds for \( \mathcal{F}_c \) if \( \mathcal{X} \) is given in terms of the image of sufficiently regular functions on sufficiently nice domains, where we exploit the Lipschitz continuity, semi-concavity, or Hölder continuity of the cost function to obtain novel theoretical guarantees for the convergence rate of \( T_{c,n} \) (Theorems 3.3, 3.8 and 3.11). For example, a special case of Theorem 3.8 states that the bound (1.6) for the 2-Wasserstein distance on \( \mathbb{R}^d \) remains valid for compactly supported probability measures \( \mu \) and \( \nu \) if \( \mu \) is concentrated on a \( d_1 \)-dimensional \( C^2 \) submanifold (Example 3.9 (iii)).

In Section 4 we gather computational evidence for the LCA principle and present simulation results for various settings where the underlying measures have different intrinsic dimensions. In particular, we observe that the numerical findings are in line with the predictions of our theory. Section 5 concludes our work with a discussion and an outline of some open questions. Appendix A contains bounds on the uniform metric entropy of \( \mathcal{F}_c \).

## 2 Lower Complexity Adaptation (LCA)

Throughout the manuscript, we work with absolutely bounded and continuous cost functions \( c: \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R} \) on Polish spaces \( \mathcal{X} \) and \( \mathcal{Y} \). For convenience, we formulate our theory for costs whose range is restricted to the interval \([0, 1]\). Since \( T_{ac+b} = a \cdot T_c + b \) for any \( a > 0 \) and \( b \in \mathbb{R} \), this is not a genuine restriction, and all of our results can easily be adapted to general costs that are absolutely bounded.

### 2.1 Duality and Complexity

In the following, we consider the dual formulation of the OT problem. This requires the notion of \( c \)-conjugacy under a given cost function \( c: \mathcal{X} \times \mathcal{Y} \rightarrow [0, 1] \) for non-empty sets \( \mathcal{X} \) and \( \mathcal{Y} \). The \( c \)-transforms of \( f: \mathcal{X} \rightarrow \mathbb{R} \) and \( g: \mathcal{Y} \rightarrow \mathbb{R} \) are defined by
\[
 f^c(y) := \inf_{x \in \mathcal{X}} c(x, y) - f(x) \quad \text{and} \quad g^c(x) := \inf_{y \in \mathcal{Y}} c(x, y) - g(y). \tag{2.1}
\]

A function \( f: \mathcal{X} \rightarrow \mathbb{R} \) is called \( c \)-concave if there exists \( g: \mathcal{Y} \rightarrow \mathbb{R} \) such that \( f = g^c \). For Polish spaces \( \mathcal{X} \) and \( \mathcal{Y} \) and a continuous cost function, any \( c \)-transform \( f^c \) or \( g^c \) is upper semi-continuous (as an infimum over continuous functions) and thus (Borel)-measurable. The following existence statement, which is tailored to bounded costs, shows that dual solutions of OT can always be assumed to be bounded and \( c \)-concave.

**Theorem** (Duality). Let \( \mathcal{X} \) and \( \mathcal{Y} \) be Polish spaces and let \( c: \mathcal{X} \times \mathcal{Y} \rightarrow [0, 1] \) be continuous. Denote the class of feasible \( c \)-concave potentials by
\[
\mathcal{F}_c = \left\{ f: \mathcal{X} \rightarrow [-1, 1] \mid f \text{ is } c \text{-concave with } \|f^c\|_{\infty} \leq 1 \right\}. \tag{2.2}
\]
Then, for any $\mu \in \mathcal{P}(\mathcal{X})$ and $\nu \in \mathcal{P}(\mathcal{Y})$, it holds that
\[ T_c(\mu, \nu) = \max_{f \in \mathcal{F}_c} \int f \, d\mu + \int f^c \, d\nu. \tag{2.3} \]

Proof. Strong duality and the existence of maximizers in $\mathcal{F}_c$ follow from Theorem 5.10(iii) in Villani [2008], where the bounds on $f \in \mathcal{F}_c$ and $f^c$ are detailed in step 4 of the proof. \qed

The properties of the feasible $c$-concave potentials $\mathcal{F}_c$ strongly depend on the cost function $c$ and the ground space $\mathcal{X}$. For example, if the family $\{c(\cdot, y) | y \in \mathcal{Y}\}$ of partially evaluated costs has a common modulus of continuity (with respect to a metric that metrizes $\mathcal{X}$), then all $c$-concave functions $f \in \mathcal{F}_c$ are continuous with the same modulus. Hence, if $c(\cdot, y)$ is Lipschitz continuous uniform in $y \in \mathcal{Y}$, then each $f \in \mathcal{F}_c$ is Lipschitz as well (Santambrogio [2015], Section 1.2). We denote the element-wise $c$-transform of the set $\mathcal{F}_c$ by $\mathcal{F}_c^c = \{f^c | f \in \mathcal{F}_c\}$, which is by definition also uniformly bounded by one. A crucial observation is that the uniform metric entropy of the function class $\mathcal{F}_c^c$ is bounded by the one of $\mathcal{F}_c$, no matter how complex the space $\mathcal{Y}$ is or how badly the functions $c(x, \cdot)$ for fixed $x \in \mathcal{X}$ behave.

Lemma 2.1 (Complexity under $c$-transformation). Let $\mathcal{X}$ and $\mathcal{Y}$ be non-empty sets and $c : \mathcal{X} \times \mathcal{Y} \to [0, 1]$. If $\mathcal{F}$ is a bounded function class on $\mathcal{X}$, it follows for $\varepsilon > 0$ that
\[ \mathcal{N}(\varepsilon, \mathcal{F}^c, \|\cdot\|_\infty) \leq \mathcal{N}(\varepsilon, \mathcal{F}, \|\cdot\|_\infty). \]

Proof. If $N := \mathcal{N}(\varepsilon, \mathcal{F}, \|\cdot\|_\infty) = \infty$, the claim is trivial, so assume $N < \infty$. Let $\{f_1, \ldots, f_N\}$ be an $\varepsilon$-covering for $\mathcal{F}$ with respect to the uniform norm on $\mathcal{X}$. For $f \in \mathcal{F}$, consider $f_i$ such that $\|f - f_i\|_\infty \leq \varepsilon$. Since $f$ and $c$ are both bounded, it follows that the $c$-transform $f^c$ is bounded on $\mathcal{Y}$. For all $y \in \mathcal{Y}$, we obtain
\[
f^c(y) = \inf_{x \in \mathcal{X}} c(x, y) - f(x) = \inf_{x \in \mathcal{X}} c(x, y) - f_i(x) + f_i(x) - f(x) \leq \inf_{x \in \mathcal{X}} c(x, y) - f_i(x) + \sup_{x \in \mathcal{X}} |f_i(x) - f(x)| \leq f_i^c(y) + \varepsilon, \]
which implies $\sup_{y \in \mathcal{Y}} |f^c(y) - f_i^c(y)| \leq \varepsilon$. Thus, $\{f_1^c, \ldots, f_N^c\}$ is an $\varepsilon$-covering of $\mathcal{F}^c$ with respect to the uniform norm. \qed

Since any feasible $c$-concave $f \in \mathcal{F}_c$ fulfills $f = f^{cc}$ (Santambrogio [2015], Proposition 1.34), we conclude that the uniform metric entropies of the function classes $\mathcal{F}_c$ and $\mathcal{F}_c^c$ are identical for any covering radius $\varepsilon > 0$, see [1.7]. Hence, to control the complexity of both function classes simultaneously, it suffices to upper bound only one of them. In particular, in a concrete setting where different bounds for $\mathcal{F}_c$ and $\mathcal{F}_c^c$ are available, their ($\varepsilon$-wise) minimum can be employed to derive an upper bound for the convergence rate in Theorem 2.2 below.

2.2 LCA: Dual Perspective

The observation that $\mathcal{F}_c$ and $\mathcal{F}_c^c$ have identical uniform metric entropies implies the following upper bound on the convergence of the empirical estimators $\hat{T}_{c,n}$ in [1.8], demonstrating the LCA principle. Notably, in the two-sample case $\hat{T}_{c,n} = T_c(\hat{\mu}_n, \hat{\nu}_n)$, the statement holds irrespective of the dependency structure between the empirical measures $\hat{\mu}_n$ and $\hat{\nu}_n$. 7
Theorem 2.2 (General LCA). Let $\mathcal{X}$ and $\mathcal{Y}$ be Polish spaces and let $c: \mathcal{X} \times \mathcal{Y} \to [0, 1]$ be continuous. Consider the function class $\mathcal{F}_c$ from (2.2) and assume that there exist $k > 0$, $K > 0$, and $\varepsilon_0 \in (0, 1)$ such that the uniform metric entropy of $\mathcal{F}_c$ is bounded by
\[
\log \mathcal{N}(\varepsilon, \mathcal{F}_c, \| \cdot \|_\infty) \leq K \varepsilon^{-k} \quad \text{for } 0 < \varepsilon \leq \varepsilon_0.
\]
Then, for any $\mu \in \mathcal{P}(\mathcal{X})$ and $\nu \in \mathcal{P}(\mathcal{Y})$, the empirical estimator $\hat{T}_{c,n}$ from (1.8) satisfies
\[
\mathbb{E} \left[ \left| \hat{T}_{c,n} - T_c(\mu, \nu) \right| \right] \leq \begin{cases} 
\frac{n^{-1/2}}{\sqrt{k}} & \text{if } k < 2, \\
\frac{n^{-1/2} \log(n)}{k} & \text{if } k = 2, \\
\frac{n^{-1/k}}{k} & \text{if } k > 2,
\end{cases}
\]
where the implicit constant only depends on $k$, $K$, and $\varepsilon_0$.

When interpreting this statement, one should keep in mind that it is always possible to assume $\mathcal{X} = \text{supp}(\mu)$ and $\mathcal{Y} = \text{supp}(\nu)$ if this leads to improved bounds for the uniform metric entropy of $\mathcal{F}_c$ (or equivalently $\mathcal{F}_c^c$). In this sense, Theorem 2.2 can be tailored to take advantage of intrinsic properties of (the support of) $\mu$ (or $\nu$). The proof employs arguments from empirical process theory and generalizes the technique of Sriperumbudur et al. (2012) and Chizat et al. (2020), where upper bounds for bounded convex sets $\mathcal{X} = \mathcal{Y} \subset \mathbb{R}^d$ and Euclidean costs $c(x, y) = \|x - y\|^p$ for $p = 1$ and $p = 2$ were derived.

Proof of Theorem 2.2. We first consider $\hat{T}_{c,n} = \hat{T}_c(\hat{\mu}_n, \hat{\nu}_n)$. The definition of $\mathcal{F}_c$ implies
\[
T_c(\hat{\mu}_n, \hat{\nu}_n) - T_c(\mu, \nu) = \max_{f \in \mathcal{F}_c} \left( \int_{\mathcal{X}} f d\hat{\mu}_n + \int_{\mathcal{Y}} f^c d\hat{\nu}_n \right) - \max_{f \in \mathcal{F}_c} \left( \int_{\mathcal{X}} f d\mu + \int_{\mathcal{Y}} f^c d\nu \right) \leq \sup_{f \in \mathcal{F}_c} \int_{\mathcal{X}} f d(\hat{\mu}_n - \mu) + \int_{\mathcal{Y}} f^c d(\hat{\nu}_n - \nu),
\]
and hence
\[
|T_c(\hat{\mu}_n, \hat{\nu}_n) - T_c(\mu, \nu)| \leq \sup_{f \in \mathcal{F}_c} \left| \int_{\mathcal{X}} f d(\hat{\mu}_n - \mu) \right| + \sup_{f \in \mathcal{F}_c} \left| \int_{\mathcal{Y}} f^c d(\hat{\nu}_n - \nu) \right|. \tag{2.6}
\]
Note by Lemma 2.1 that both $\mathcal{F}_c$ and $\mathcal{F}_c^c$ have finite uniform metric entropy for any $\varepsilon > 0$. Hence, both function classes contain subsets of at most countable cardinality that are dense in uniform norm, and the right hand side of (2.6) can thus be considered as a countable supremum and is therefore measurable. Further, recall that all elements in $\mathcal{F}_c$ and $\mathcal{F}_c^c$ are absolutely bounded by one. Taking the expectation and invoking symmetrization techniques (see Wainwright 2019 Proposition 4.11), we obtain
\[
\mathbb{E} \left[ |T_c(\hat{\mu}_n, \hat{\nu}_n) - T_c(\mu, \nu)| \right] \leq 2 \left( \mathcal{R}_n(\mathcal{F}_c) + \mathcal{R}_n(\mathcal{F}_c^c) \right),
\]
where $\mathcal{R}_n(\mathcal{F}_c)$ and $\mathcal{R}_n(\mathcal{F}_c^c)$ denote the Rademacher complexities of the function classes $\mathcal{F}_c$ and $\mathcal{F}_c^c$. The Rademacher complexity of $\mathcal{F}_c$ is defined by
\[
\mathcal{R}_n(\mathcal{F}_c) := \mathbb{E} \left[ \sup_{f \in \mathcal{F}_c} \frac{1}{n} \sum_{i=1}^{n} \sigma_i f(X_i) \right].
\]
for i.i.d. $X_1,\ldots,X_n \sim \mu$ and independent i.i.d. Rademacher variables $\sigma_1,\ldots,\sigma_n \sim \text{Unif}\{-1,1\}$. This quantity is dominated by Dudley’s entropy integral (see \cite{vonLuxburg2004} Theorem 16),

$$\mathcal{R}_n(\mathcal{F}_c) \leq \inf_{\delta \in (0,1]} \left(2\delta + \sqrt{32} \sqrt[4]{n} \int_{\delta/4}^{1} \sqrt{\log N(\varepsilon, \mathcal{F}_c, \| \cdot \|_\infty)} \, d\varepsilon \right).$$

Let $\hat{K} := \sqrt{32} \sqrt{n}$. Since the covering number is a decreasing function in $\varepsilon$, a short calculation shows that the assumption $\log \mathcal{N}(\varepsilon, \mathcal{F}_c, \| \cdot \|_\infty) \leq K \varepsilon^{-k}$ for $\varepsilon \leq \varepsilon_0$ implies that

$$\mathcal{R}_n(\mathcal{F}_c) \leq \inf_{\delta \in (0,1]} \left(2\delta + \hat{K} \frac{n^{-1/2}}{\varepsilon_0^{1/2}} \int_{\delta/4}^{1} \min(\varepsilon, \varepsilon_0)^{-k/2} \, d\varepsilon \right)$$

$$\leq \begin{cases} (\hat{K} \frac{\varepsilon_0^{1-k/2}}{1-k/2} + 1) \frac{n^{-1/2}}{\varepsilon_0^{1/2}} & \text{if } k < 2 \text{ for } \delta = 0, \\ (8 + \hat{K} \log(\varepsilon_0) + \frac{\hat{K}(1-\varepsilon_0) \log(\varepsilon_0)}{\varepsilon_0}) n^{-1/2} + 4 \frac{n^{-1/2}}{\varepsilon_0} \log(n) & \text{if } k = 2 \text{ for } \delta = 4n^{-1/2}, \\ (\hat{K} \frac{\varepsilon_0^{1-k/2}}{1-k/2} + 1) \frac{n^{-1/2}}{\varepsilon_0^{1/2}} + (8 + \frac{\hat{K}}{k-1} n^{-1/k} & \text{if } k > 2 \text{ for } \delta = 4n^{-1/k}, \end{cases}$$

where we assume $n$ large enough such that $\delta \leq \varepsilon_0$ for the respective choices. As $\mathcal{F}_c$ and $\mathcal{F}_c^c$ share the same covering number with respect to uniform norm (Lemma 2.1) and since all functions in $\mathcal{F}_c^c$ are bounded in absolute value by one as well, it follows that $\mathcal{R}_n(\mathcal{F}_c^c)$ can be bounded just like $\mathcal{R}_n(\mathcal{F}_c)$. Finally, for the other estimators $\hat{T}_{c,n}$ in (1.8), we obtain

$$\mathbb{E}[\| T_c(\hat{\mu}_n, \nu) - T_c(\mu, \nu) \|] \leq \mathcal{R}_n(\mathcal{F}_c)$$

and

$$\mathbb{E}[\| T_c(\hat{\nu}_n, \mu) - T_c(\mu, \nu) \|] \leq \mathcal{R}_n(\mathcal{F}_c^c)$$

in analogous fashion, which proves the bounds from (2.5) and finishes the proof. \hfill \Box

### 2.3 LCA: Primal Perspective

The proof of Theorem 2.2 relies on a technical observation about the nature of $c$-transforms and does not convey a geometric interpretation why empirical OT should follow the LCA principle. To provide some additional intuition, we next consider the LCA phenomenon from the primal perspective. Even though this approach yields less general results, its more explicit character has benefits, e.g., for establishing matching lower bounds.

**Proposition 2.3** (Decomposition under additive costs). Let $\mathcal{X}$ be a Polish space and $\mathcal{Y} = \mathcal{Y}_1 \times \mathcal{Y}_2$ be the product of two Polish spaces. Let \( c: \mathcal{X} \times \mathcal{Y} \to [0,1] \) be continuous so that

$$c(x,y) = c_1(x,y_1) + c_2(y_2)$$

for all $x \in \mathcal{X}$ and $y = (y_1,y_2) \in \mathcal{Y}$ with continuous $c_1: \mathcal{X} \times \mathcal{Y}_1 \to [0,1]$ and $c_2: \mathcal{Y}_2 \to [0,1]$, and let $p: \mathcal{Y} \to \mathcal{Y}_1$ be the Cartesian projection to $\mathcal{Y}_1$. Then, for any $\mu \in \mathcal{P}(\mathcal{X})$ and $\nu \in \mathcal{P}(\mathcal{Y})$,

$$T_c(\mu, \nu) = T_{c_1}(\mu, p_{\#} \nu) + R_{c_2}(\nu),$$

where $R_{c_2}(\nu) := \int_\mathcal{Y} c_2(y_2) \, d\nu(y_1,y_2)$.

For example, this statement can be applied in Euclidean spaces under $l_p^n$ costs if $\mathcal{X} \subset \mathcal{Y}_1$. In this case, we find $c(x,y) = \|x - y_1\|_p + \|y_2\|_p$ (see also Figure 1 in the introduction), which satisfies condition (2.7). If $p = 2$, a relation of the form (2.8) clearly remains valid whenever $\mathcal{X}$ is contained in an affine linear subspace of $\mathcal{Y}$ and $p$ is the corresponding orthogonal projection.
Proof of Proposition 2.3. For any coupling \( \pi \in \Pi(\mu, \nu) \), we consider the decomposition
\[
\int_{X \times Y} c \, d\pi = \int_{X \times Y} c_1 \, d\pi + \int_{X \times Y} c_2 \, d\pi.
\]
The second term on the right is independent of \( \pi \) and equals \( R_{c_2}(\nu) := \int_1 c_2(y_1) \, d\nu(y_1, y_2) \), while the first term can be rewritten in terms of \( \tilde{\pi} = (\text{id}, p) \# \pi \in \Pi(\mu, p \# \nu) \) by a change of variables, such that \( \int_{X \times Y} c_1 \, d\pi = \int_{X \times Y_1} c_1 \, d\tilde{\pi} \). Conversely, the gluing lemma (cf. Villani 2008, Chapter 1) implies that each \( \tilde{\pi} \in \Pi(\mu, p \# \nu) \) gives rise to a \( \pi \in \Pi(\mu, \nu) \) for which \( \int_{X \times Y} c_1 \, d\pi = \int_{X \times Y_1} c_1 \, d\tilde{\pi} \) holds as well. Therefore, we conclude that
\[
T_c(\mu, \nu) = \inf_{\pi \in \Pi(\mu, \nu)} \int_{X \times Y} c_1 \, d\pi + \int_{X \times Y} c_2 \, d\pi
= \inf_{\tilde{\pi} \in \Pi(\mu, p \# \nu)} \int_{X \times Y} c_1 \, d\tilde{\pi} + R_{c_2}(\nu) = T_{c_1}(\mu, p \# \nu) + R_{c_2}(\nu).
\]

Since relation (2.8) in Proposition 2.3 holds for any pair of probability measures \( \mu \in \mathcal{P}(X) \) and \( \nu \in \mathcal{P}(Y) \), we obtain
\[
T_c(\hat{\mu}_n, \hat{\nu}_n) - T_c(\mu, \nu) = T_{c_1}(\hat{\mu}_n, p \# \hat{\nu}_n) - T_{c_1}(\mu, p \# \nu) + R_{c_2}(\hat{\nu}_n - \nu),
\]
where \( \hat{\nu}_n - \nu \) is understood as a signed measure. Thus, the statistical performance when estimating \( T_c(\mu, \nu) \) via the empirical OT cost is governed by the (potentially much simpler) complexity of \( T_{c_1}(\mu, p \# \nu) \). By the reverse triangle inequality, it furthermore follows that
\[
\mathbb{E}[|T_c(\hat{\mu}_n, \hat{\nu}_n) - T_c(\mu, \nu)|] \geq \mathbb{E}[||T_{c_1}(\hat{\mu}_n, p \# \hat{\nu}_n) - T_{c_1}(\mu, p \# \nu)| - |R_{c_2}(\hat{\nu}_n - \nu)||],
\]
where we note \( \mathbb{E}[|R_{c_2}(\hat{\nu}_n - \nu)|] \propto n^{-1/2} \) if \( \sigma_{c_2}^2 := \text{Var}_{Y \sim \nu}[c_2(Y)] > 0 \). Let \( \hat{f}_n \in \mathcal{F}_{c_1} \) denote an optimizer for (2.3) between \( \hat{\mu}_n \) and \( p \# \nu \) under the cost function \( c_1 \). If the i.i.d. random variables \( X_1, \ldots, X_n \sim \mu \) and \( Y_1, \ldots, Y_n \sim \nu \) are independent, we find
\[
\mathbb{E}[T_{c_1}(\hat{\mu}_n, p \# \hat{\nu}_n) - T_{c_1}(\hat{\mu}_n, p \# \nu)]
= \mathbb{E} \left[ \max_{f \in \mathcal{F}_{c_1}} \left( \int_X f \, d\hat{\mu}_n + \int_{Y_1} f^{c_1} \, d\hat{\nu}_n \right) - \max_{f \in \mathcal{F}_{c_1}} \left( \int_X f \, d\hat{\mu}_n + \int_{Y_1} f^{c_1} \, d(p \# \nu) \right) \right]
\geq \mathbb{E} \left[ \mathbb{E}_{X_1, \ldots, X_n} \left[ \mathbb{E}_{Y_1, \ldots, Y_n} \left( \int_X f(Y) \, d\hat{\nu}_n - \int_{Y_1} f^{c_1} \, d(p \# \nu) \right) \right] \right]
= 0,
\]
where independence is crucial for the final equality. In particular, this implies
\[
\mathbb{E}[|T_{c_1}(\hat{\mu}_n, p \# \hat{\nu}_n) - T_{c_1}(\mu, p \# \nu)|] \geq \mathbb{E}[T_{c_1}(\hat{\mu}_n, p \# \hat{\nu}_n) - T_{c_1}(\mu, p \# \nu)]
\geq \mathbb{E}[T_{c_1}(\hat{\mu}_n, p \# \nu) - T_{c_1}(\mu, p \# \nu)],
\]
which means that lower bounds for the two-sample setting can be obtained from lower bounds for the one-sample case. In Examples 3.7 and 3.10 of the subsequent section, we employ relation (2.10) and (2.11) to this end.

Remark 2.4 (Dependent empirical measures). Dependencies between the empirical measures \( \hat{\mu}_n \) and \( \hat{\nu}_n \) can lead to parametric convergence rates of order \( n^{-1/2} \) irregardless of the underlying spaces \( Y_1 \) and \( Y_2 \). For example, if \( X = Y_1 \) and \( \mu = p \# \nu \) with empirical measures related by \( \hat{\mu}_n = p \# \hat{\nu}_n \), it follows from (2.9) that
\[
\mathbb{E}[|T_c(\hat{\mu}_n, \hat{\nu}_n) - T_c(\mu, \nu)|] = \mathbb{E}[|R_{c_2}(\hat{\nu}_n - \nu)|] \times n^{-1/2}
\]
for any non-negative cost function \( c_1 \) with \( c_1(y_1, y_1) = 0 \) for all \( y_1 \in Y_1 \) if \( \sigma_{c_2} > 0 \).
3 Applications and Examples

The LCA principle, as formalized in Theorem 2.2 can readily be employed whenever suitable bounds on the uniform metric entropy of $F_{\epsilon}$ are available. In the following, we consider a number of settings where well-known entropy bounds lead to novel results on the convergence rate of empirical OT. In order to efficiently exploit the properties of the space $X$ in settings of low intrinsic dimensionality, the following observation is useful.

**Lemma 3.1** (Union bound). Let $F$ be a class of real valued functions on a set $X = \bigcup_{i=1}^{I} X_i$ for (not necessarily disjoint) subsets $X_i \subset X$ and $I \in \mathbb{N}$, and let $F_{\epsilon}|_{X_i} := \{ f|_{X_i} : X_i \to \mathbb{R} | f \in F \}$ be the class of functions restricted to $X_i$ for all $i \in \{1, \ldots, I\}$. Then, for each $\epsilon > 0$,

\[ \log N(\epsilon, F, \| \cdot \|_\infty) \leq \sum_{i=1}^{I} \log N(\epsilon, F_{\epsilon}|_{X_i}, \| \cdot \|_\infty). \]

**Proof.** Suppose that the right hand side is finite (otherwise the bound is trivial). Denote by $F_{\epsilon}$ a minimal $\epsilon$-covering of $F_{\epsilon}|_{X_i}$ with respect to the uniform norm. Let $X_1 := X_1$ and define $X_i := X_i \setminus (\bigcup_{j=1}^{i-1} X_j)$ for $i \geq 2$. Then $\{ \sum_{j=1}^{I} f_i \cdot 1_{X_i} | f_i \in F_{\epsilon} \}$ is an $\epsilon$-covering of $F$ under the uniform norm with cardinality at most $\prod_{i=1}^{I} N(\epsilon, F_{\epsilon}|_{X_i}, \| \cdot \|_\infty)$. \(\square\)

### 3.1 Semi-Discrete Optimal Transport

We first address the setting of semi-discrete OT, where $X = \{x_1, \ldots, x_I\}$ is a finite discrete space with $I \in \mathbb{N}$ elements. Structural and computational properties of semi-discrete OT have been investigated extensively, especially in Euclidean contexts (see, e.g., Aurenhäuser et al. 1998; Mérigot 2011; Geiß et al. 2013; Hartmann & Schuhmacher 2020). For our purposes, consider a general Polish space $Y$ and a continuous cost function $c : X \times Y \to [0, 1]$. By definition (2.2), the function class $F_{\epsilon}$ is absolutely bounded by one and we find that

\[ N(\epsilon, F_{\epsilon}|_{\{x_i\}}, \| \cdot \|_\infty) \leq \lceil 1/\epsilon \rceil \]

for any $\epsilon > 0$ and $i \in \{1, \ldots, I\}$. Hence, it follows by Lemma 3.1 that $\log N(\epsilon, F_{\epsilon}, \| \cdot \|_\infty) \leq I \log \lceil 1/\epsilon \rceil \leq I/\epsilon$ and we can apply Theorem 2.2 to derive the following bound.

**Theorem 3.2** (Semi-discrete LCA). Let $X = \{x_1, \ldots, x_I\}$ be a finite discrete space, $Y$ a Polish space, and $c : X \times Y \to [0, 1]$ continuous. Then, for any $\mu \in \mathcal{P}(X)$ and $\nu \in \mathcal{P}(Y)$, the empirical estimator $T_{c,n}$ from (1.8) satisfies

\[ \mathbb{E} \left[ \left| T_{c,n} - T_c(\mu, \nu) \right| \right] \leq n^{-1/2}. \]

This result is in line with recent findings by del Barrio et al. (2021), who derive a central limit theorem for the empirical semi-discrete OT cost. Their result allows for possibly unbounded costs, but it is limited to the one-sample estimator $T_{c,n} = T_c(\tilde{\mu}_n, \nu)$. According to Markov’s inequality, Theorem 3.2 implies that the sequence of random variables $\sqrt{n} \left( T_{c,n} - T_c(\mu, \nu) \right)$ is tight even for $T_{c,n} = T_c(\mu, \tilde{\nu}_n) \text{ or } T_c(\tilde{\mu}_n, \nu)$, which indicates that it might also be possible to derive limit distributions when the measure on the general space $Y$ is estimated empirically. Moreover, Theorem 3.2 asserts novel bounds for the Wasserstein distance when one measure is supported on finitely many points only while the support of the other measure is bounded.
3.2 Optimal Transport under Lipschitz costs

Semi-discrete OT can be regarded as a special OT setting where one probability measure has intrinsic dimension zero. We now broaden this perspective to higher dimensions and consider parameterized spaces and surfaces. The effective dimension is then governed by the (possibly low-dimensional) domain of the parameterization, and not by the (possibly high-dimensional) ambient space. In this section, we work with an additional Lipschitz requirement for the cost function \( c: \mathcal{X} \times \mathcal{Y} \to [0, 1] \) and impose the following condition on the Polish space \( \mathcal{X} \). By rescaling, we may assume that the Lipschitz constant is equal to one.

Assumption (Lip). Suppose \( \mathcal{X} = \bigcup_{i=1}^{I} g_i(\mathcal{U}_i) \) for \( I \in \mathbb{N} \) connected metric spaces \( (\mathcal{U}_i, d_i) \) and maps \( g_i: \mathcal{U}_i \to \mathcal{X} \) so that \( c(g_i(\cdot), y) \) is 1-Lipschitz with respect to \( d_i \) for all \( y \in \mathcal{Y} \).

This setting captures a broad notion of generalized surfaces in an ambient space. Since the mappings \( g_i \) are not required to be injective, self-intersections are possible. Moreover, exploiting the (not necessarily disjoint) decomposition \( \mathcal{X} = \bigcup_{i=1}^{I} \mathcal{X}_i \) with \( \mathcal{X}_i := g_i(\mathcal{U}_i) \) for \( i \in \{1, \ldots, I\} \), it suffices by Lemma 3.1 to control the complexity of \( \mathcal{F}_c|_{\mathcal{X}_i} \) to bound the metric entropy of \( \mathcal{F}_c \). For this purpose, we note that the Lipschitz continuity of \( c(g_i(\cdot), y) \) implies that \( f \circ g_i \) for any \( c \)-concave potential \( f \in \mathcal{F}_c \) is Lipschitz as well. This relates the restricted function class \( \mathcal{F}_c|_{\mathcal{X}_i} \) to the class of bounded Lipschitz functions on \( \mathcal{U}_i \). For the latter, metric entropy bounds in terms of the covering number of \( \mathcal{U}_i \) are available (Kolmogorov & Tikhomirov 1961, Section 9). For \( \varepsilon > 0 \), the covering number of a metric space \( (\mathcal{U}, d) \) is defined by

\[
\mathcal{N}(\varepsilon, \mathcal{U}, d) := \inf \left\{ n \in \mathbb{N} \mid \text{there exist } U_1, \ldots, U_n \subseteq \mathcal{U} \text{ with diam}(U_k) \leq 2\varepsilon \text{ and } \mathcal{U} = \bigcup_{k=1}^{n} U_k \right\},
\]

where diam(\( U \)) := sup_{u,v \in U} d(u, v) denotes the diameter of a subset \( U \subseteq \mathcal{U} \).

Theorem 3.3 (Lipschitz LCA). Let \( \mathcal{X} \) and \( \mathcal{Y} \) be Polish spaces and let \( c: \mathcal{X} \times \mathcal{Y} \to [0, 1] \) be continuous. If Assumption (Lip) holds and there exists \( k > 0 \) so that for all \( i \in \{1, \ldots, I\} \)

\[
\mathcal{N}(\varepsilon, \mathcal{U}_i, d_i) \leq \varepsilon^{-k} \quad \text{for } \varepsilon > 0 \text{ sufficiently small},
\]

then, for any \( \mu \in \mathcal{P}(\mathcal{X}) \) and \( \nu \in \mathcal{P}(\mathcal{Y}) \), the empirical estimator \( \hat{T}_{c,n} \) from (1.8) satisfies

\[
\mathbb{E} \left[ |\hat{T}_{c,n} - T_c(\mu, \nu)| \right] \leq \begin{cases} n^{-1/2} & \text{if } k < 2, \\
^{-1/2} \log(n) & \text{if } k = 2, \\
^{-1/k} & \text{if } k > 2. \end{cases}
\]

Proof. Lemma A.2 in Appendix A shows that the uniform metric entropy in this setting is bounded by \( \log \mathcal{N}(\varepsilon, \mathcal{F}_c, ||x||_{\infty}) \leq \varepsilon^{-k} \). Applying Theorem 2.2 then yields bound (3.3). \( \square \)

Remark 3.4 (Disconnected domains). If the metric spaces \( \mathcal{U}_1, \ldots, \mathcal{U}_I \) in Assumption (Lip) consist of finitely many connected components, Theorem 3.3 remains valid at the price of a possibly larger constant. Moreover, if some \( \mathcal{U}_i \) has infinitely many components, Lemma A.2 ensures

\[
\log \mathcal{N}(\varepsilon, \mathcal{F}_c, ||x||_{\infty}) \leq \varepsilon^{-k} \log(\varepsilon^{-1}) \leq \varepsilon^{-k-\delta}
\]

for any \( \delta > 0 \) (where the implicit constant depends on \( \delta \)). Hence, Theorem 2.2 shows that bound (3.3) still holds when \( k \) is replaced by \( k + \delta \).
We emphasize once more that no additional assumptions on the complexity of the Polish space $\mathcal{Y}$ are necessary. To highlight applications and noteworthy consequences of Theorem 3.3 we consider a number of examples.

**Example 3.5** (Metric spaces with Lipschitz costs). Let $\mathcal{X}$ and $\mathcal{Y}$ be closed subsets of a Polish metric space $(\mathcal{Z}, d)$ and consider costs $c: \mathcal{Z}^2 \to \mathbb{R}$ that are continuous and absolutely bounded on $\mathcal{X} \times \mathcal{Y}$. Furthermore, assume that $c(\cdot, y)$ is Lipschitz on $\mathcal{X}$ uniformly in $y \in \mathcal{Y}$, which, for example, holds for $c(x, y) = d^p(x, y)$ and $p \geq 1$ if $\mathcal{Y}$ is bounded. Then, Theorem 3.3 provides convergence rates whenever $\mathcal{N}(\varepsilon, \mathcal{X}, d) \leq \varepsilon^{-k}$. This condition holds if the upper Minkowski-Bouligand dimension of $\mathcal{X}$ (Mattila, 1995, Section 5.3), defined by

$$\dim_{\text{M}}(\mathcal{X}) := \limsup_{\varepsilon \to 0} \frac{\log \mathcal{N}(\varepsilon, \mathcal{X}, d)}{\log(1/\varepsilon)},$$

is strictly dominated by $k$. Note that non-integral values of $\dim_{\text{M}}(\mathcal{X})$ are possible and that $\mathcal{X}$ does not necessarily have to be connected (Remark 3.4).

**Example 3.6** (Euclidean spaces with locally Lipschitz costs). Consider a locally Lipschitz cost function $c: \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ for $d \in \mathbb{N}$. This setting entails the choice $c(x, y) = |x - y|^p$ for the Euclidean norm $|\cdot|$, or the $\ell_p$-costs $c(x, y) = \sum_{i=1}^d |x_i - y_i|^p$, both of which are popular options for the Wasserstein distance of order $p \geq 1$. In the following examples, we implicitly assume that $\mathcal{X}$ and $\mathcal{Y}$ are Polish subsets of $\mathbb{R}^d$.

(i) **Bounded sets**: If $\mathcal{X}, \mathcal{Y} \subset \mathbb{R}^d$ are bounded sets, then $c(\cdot, y)$ is Lipschitz continuous on $\mathcal{X}$ uniformly in $y \in \mathcal{Y}$ and $\mathcal{N}(\varepsilon, \mathcal{X}, |\cdot|) \leq \varepsilon^{-d}$. Since one can always enlarge $\mathcal{X}$ and $\mathcal{Y}$ to connected compact sets, Theorem 3.3 asserts that the convergence rate of the empirical OT cost is dominated by (3.3) with $k = d$.

(ii) **Surfaces**: Improved bounds can be derived if $\mathcal{X} = \bigcup_{i=1}^I g_i(\mathcal{U}_i)$ for Lipschitz maps $g_i: \mathcal{U}_i \to \mathbb{R}^d$ on bounded and connected sets $\mathcal{U}_i \subset \mathbb{R}^s$ with $d > s \in \mathbb{N}$. In this setting, the partially evaluated cost $c(g_i(\cdot), y)$ is Lipschitz on $\mathcal{X}$ uniformly in $y \in \mathcal{Y}$ for bounded sets $\mathcal{Y}$, and it holds that $\mathcal{N}(\varepsilon, \mathcal{U}_i, |\cdot|) \leq \varepsilon^{-s}$. Hence, bound (3.3) is valid for $k = s$.

(iii) **Manifolds**: The examples outlined in (ii) include compact $s$-dimensional immersed $\mathcal{C}^1$ submanifolds of $\mathbb{R}^d$, possibly with boundary (Lee, 2013). Due to compactness, a finite atlas of charts with connected and bounded co-domains $\mathcal{U}_i$ can always be found.

We next show that the upper bounds in Theorem 3.3 can be complemented by matching lower bounds in settings where the primal approach to the LCA principle (see Proposition 2.3) is available.

**Example 3.7** (Lower bounds under Lipschitz costs). Let $\mathcal{Y} = \mathcal{Y}_1 \times \mathcal{Y}_2$ be the product of two connected Polish spaces and let $\mathcal{X} = \mathcal{Y}_1$. Consider the costs $c(x, y) = d_1(x, y_1) + c_2(y_2)$, where $d_1$ is a Polish metric on $\mathcal{Y}_1$ such that $0 < \text{diam}(\mathcal{Y}_1) \leq 1$ and $c_2: \mathcal{Y}_2 \to [0, 1]$ is continuous. Then $c(\cdot, y)$ is Lipschitz with respect to $d_1$ uniformly in $y \in \mathcal{Y}$, so Assumption (Lip) is fulfilled. If $\mathcal{N}(\varepsilon, \mathcal{X}, d_1) \gtrsim \varepsilon^{-k}$ for some $k > 0$ as $\varepsilon \searrow 0$ and if the empirical measures $\hat{\mu}_n$ and $\hat{\nu}_n$ are independent, one can show that

$$\sup_{\mu, \nu} \mathbb{E} \left[ |T_c(\hat{\mu}_n, \hat{\nu}_n) - T_c(\mu, \nu)| \right] \gtrsim \begin{cases} n^{-1/2} & \text{if } k \leq 2, \\ n^{-1/k} & \text{if } k > 2, \end{cases} \quad (3.4)$$

where the supremum is taken over $\mu \in \mathcal{P}(\mathcal{X})$ and $\nu \in \mathcal{P}(\mathcal{Y})$. For $k \leq 2$, inequality (3.4) follows by selecting suitable discrete measures $\mu$ and $\nu$ (see Sommerfeld et al, 2019).
Section 5). For $k > 2$, inequality (3.4) follows by (2.10) and (2.11) in conjunction with the minimax lower bounds for the $1$-Wasserstein distance $W_1(\mu, \nu) = T_{d_1}(\mu, \nu)$ by Singh & Póczos (2018, Theorem 2)
\[
\sup_{\mu, \nu} \mathbb{E}[|T_{d_1}(\hat{\mu}_n, \hat{\nu}) - T_{d_1}(\mu, \nu)|] \geq \sup_{\mu} \mathbb{E}[|T_{d_1}(\hat{\mu}_n, \mu)|] \geq n^{-1/k}.
\]
Hence, the upper bounds from (3.3) are sharp for $k \neq 2$ and sharp up to logarithmic terms in case of $k = 2$.

### 3.3 Optimal Transport under Semi-Concave Costs

It is known that better convergence rates than those offered by Theorem 3.3 can be obtained on Euclidean spaces for more regular cost functions (Manole & Niles-Weed 2021). In this section, we consider improvements for semi-concave costs (before we turn to Hölder smooth costs in Section 3.4). A function $f : U \to \mathbb{R}$ on a bounded, convex domain $U \subset \mathbb{R}^d$ for $d \in \mathbb{N}$ is called $\Lambda$-semi-concave with modulus $\Lambda \geq 0$ if the map
\[
u \mapsto f(u) - \Lambda \|u\|^2
\]
is concave, where $\|\cdot\|$ denotes the Euclidean norm (cf. Gangbo & McCann 1996). The uniform metric entropy of the class of bounded, Lipschitz, and semi-concave functions on $U$ is of order $\varepsilon^{-d/2}$ (Bronshtein 1976; Guntuboyina & Sen 2013), compared to $\varepsilon^{-d}$ without semi-concavity. Since boundedness, Lipschitz continuity, and semi-concavity are all inherited to $\mathcal{F}_c$ by the cost function, we impose the following conditions. For convenience, we assume the Lipschitz constant and the modulus of semi-concavity to be equal to one, since other constants can be accommodated by scaling the cost function.

**Assumption (SC).** Suppose $\mathcal{X} = \bigcup_{i=1}^I g_i(\mathcal{U}_i)$ for $I \in \mathbb{N}$ bounded, convex subsets $\mathcal{U}_i \subset \mathbb{R}^d$ and maps $g_i : \mathcal{U}_i \to \mathcal{X}$ so that $c(g_i(\cdot), y)$ is 1-Lipschitz and 1-semi-concave for all $y \in \mathcal{Y}$.

Similar to Assumption (Lip) in the previous section, Assumption (SC) enables the application of Lemma 3.1 to the union $\mathcal{X} = \bigcup_{i=1}^I \mathcal{X}_i$ with $\mathcal{X}_i = g_i(\mathcal{U}_i)$ for all $i \in \{1, \ldots, I\}$. Combined with the uniform metric entropy bounds by Bronshtein (1976) and Guntuboyina & Sen (2013), this leads to the following result.

**Theorem 3.8** (Semi-concave LCA). Let $\mathcal{X}$ and $\mathcal{Y}$ be Polish spaces and $c : \mathcal{X} \times \mathcal{Y} \to [0, 1]$ be continuous. If Assumption (SC) holds, then, for any $\mu \in \mathcal{P}(\mathcal{X})$ and $\nu \in \mathcal{P}(\mathcal{Y})$, the empirical estimator $\hat{T}_{c,n}$ from (1.8) satisfies
\[
\mathbb{E}[|\hat{T}_{c,n} - T_c(\mu, \nu)|] \leq \begin{cases} 
 n^{-1/2} & \text{if } d \leq 3, \\
 n^{-1/2} \log(n) & \text{if } d = 4, \\
 n^{-2/d} & \text{if } d \geq 5.
\end{cases} \tag{3.5}
\]

**Proof.** Lemma A.3 in Appendix A shows that the uniform metric entropy in this setting is bounded by $\log \mathcal{N}(\varepsilon, \mathcal{F}_c, \|\cdot\|_\infty) \leq \varepsilon^{-d/2}$. This implies bound (3.5) via Theorem 2.2.

Compared to Theorem 3.3, which would guarantee a convergence rate of $n^{-1/d}$ for $d \geq 5$ under Assumption (SC), Theorem 3.8 provides the considerably faster rate $n^{-2/d}$. To explore applications, we revisit the settings discussed in Example 3.6 and show how they fare under the stronger Assumption (SC). For this purpose, note that semi-concavity of a $C^2$ function $f$ on a convex domain is implied by the boundedness of the Eigenvalues of its Hessian. Indeed, if the Eigenvalues are bounded from above by $2\Lambda > 0$, then $u \mapsto f(u) - \Lambda \|u\|^2$ has a negative semi-definite Hessian and is therefore concave (see Luenberger 2003, Section 6.4, Proposition 5).
Example 3.9 (Euclidean spaces with $C^2$ costs). Extending Example 3.6, consider a twice continuously differentiable cost function $c: \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ for $d \in \mathbb{N}$. This setting includes $c(x, y) = \| x - y \|^p$ as well as $c(x, y) = \sum_{i=1}^d |x_i - y_i|^p$ for $p \geq 2$. We implicitly assume that $\mathcal{X}$ and $\mathcal{Y}$ are Polish subsets of $\mathbb{R}^d$ in the following.

(i) **Bounded sets:** Let $\mathcal{X}, \mathcal{Y} \subset \mathbb{R}^d$ be bounded sets. Since $\mathcal{X}$ and $\mathcal{Y}$ can always be enlarged to convex compact sets, the functions $c(\cdot, y)$ are Lipschitz continuous and semi-concave on $\mathcal{X}$ uniformly in $y \in \mathcal{Y}$. Hence, Theorem 3.8 provides the upper bounds (3.5).

(ii) **Surfaces:** Improved bounds can be obtained if $\mathcal{X} = \bigcup_{i=1}^l \mathcal{U}_i$ with bounded second derivatives on open, bounded, and convex sets $\mathcal{U}_i \subset \mathbb{R}^s$ for $s < d$. In this setting, the partially evaluated cost $c_{\mathcal{U}_i}(\cdot, y)$ is Lipschitz and semi-concave on $\mathcal{U}_i$ uniformly in $y \in \mathcal{Y}$ for bounded $\mathcal{Y}$. Hence, bound (3.5) holds with $d$ replaced by $s$.

(iii) **Manifolds:** The setting described in (ii) includes compact $s$-dimensional immersed $C^2$ submanifolds of $\mathbb{R}^d$ (Lee 2013). Compactness ensures the existence of an atlas with finitely many charts such that all involved maps have bounded second derivative and all co-domains of charts are convex and bounded.

We continue with a setting in which lower bounds for the empirical OT cost that match the upper bounds in Theorem 3.8 can be derived via Proposition 2.3.

Example 3.10 (Lower bounds under semi-concave costs). Let $1 \leq d_1 \leq d_2$ and consider the unit cubes $\mathcal{X} = [0,1]^{d_1}$ and $\mathcal{Y} = [0,1]^{d_2}$. If $\mathcal{X}$ is embedded into $\mathcal{Y}$ along the first $d_1$ coordinates, the squared Euclidean cost function amounts to

$$c(x, y) = \| x - y_1 \|^2 + \| y_2 \|^2 =: c_1(x, y_1) + c_2(y_2),$$

where $y = (y_1, y_2) \in [0,1]^{d_1 + (d_2 - d_1)}$. Up to scaling of the cost function, this setting satisfies Assumption (SC). For independent empirical measures $\hat{\mu}_n$ and $\hat{\nu}_n$, one can show that

$$\sup_{\mu, \nu} \mathbb{E}[|T_c(\hat{\mu}_n, \hat{\nu}_n) - T_c(\mu, \nu)|] \geq \begin{cases} n^{-1/2} & \text{if } d_1 \leq 4, \\ n^{-2/d_1} & \text{if } d_1 \geq 5, \end{cases}$$

(3.6)

where $\mu \in \mathcal{P}(\mathcal{X})$ and $\nu \in \mathcal{P}(\mathcal{Y})$ in the supremum. For $d_1 \leq 4$, this lower bound follows by selecting suitable discrete measures (see Sommerfeld et al. [2019], Section 5), and for $d_1 \geq 5$, it follows from inequality (2.10) in conjunction with Proposition 21 in Manole & Niles-Weed [2021] (which is also applicable for more general strictly convex cost functions). Overall, the upper bounds in (3.5) match the lower bounds (3.6) in case $d_1 \neq 4$ and are sharp up to logarithmic factors for $d_1 = 4$.

We want to highlight that the fast rates of Theorem 3.8 (compared to Theorem 3.3) can often be expected in the settings of Example 3.9 even if the cost function is not $C^2$ on all of $\mathbb{R}^{2d}$. For example, if the set

$$\mathcal{D} = \{(x, y) \mid c \text{ is not } C^2 \text{ at } (x, y)\} \subset \mathbb{R}^{2d}$$

is strictly separated from $\Sigma \coloneqq \text{supp}(\mu) \times \text{supp}(\nu) \subset \mathbb{R}^{2d}$, one can extend the restriction $c|\Sigma$ to a $C^2$ cost function on all of $\mathbb{R}^{2d}$ (by the extension theorem of Whitney [1934]) without altering $T_{c,n}$ or $T_c(\mu, \nu)$. If $c(x, y) = \| x - y \|^p$ for any $0 < p < 2$, we find $\mathcal{D} = \{(x, y) \mid x \in \mathbb{R}^{d}\}$, implying the fast convergence rates in (3.5) whenever the supports of $\mu$ and $\nu$ are strictly separated. Similar observations were pointed out by Manole & Niles-Weed [2021, Corollary 3(ii)].
under additional convexity assumptions. In contrast, considering the \( l_p \)-cost function \( c(x, y) = \sum_{i=1}^{d} |x_i - y_i|^p \) for \( 0 < p < 2 \), the set
\[
\mathcal{D} = \{(x, y) \mid x_i = y_i \text{ for some } i \in \{1, \ldots, d\}\} \subset \mathbb{R}^{2d}
\]
is notably larger. Therefore, the \( p \)-Wasserstein distance based on the Euclidean norm may exhibit faster convergence rates than its \( l_p \) counterpart, e.g., if \( \mu \) is a translation of \( \nu \) along a coordinate axis.

We conjecture that the faster rates implied by Theorem 3.8 will occur under even more general circumstances. For example, in some settings it might suffice that \( \mathcal{D} \) is negligible under the actual optimal transport, meaning that \( \pi(\mathcal{D}) = 0 \) for an OT plan between \( \mu \) and \( \nu \). A proof of this claim, however, would likely require quantitative statements on the regularity of the cost function along the support of empirical OT plans and lies beyond the scope of this work. Still, we pick up on this hypothesis and observe some numerical evidence in Section 4.

### 3.4 Optimal Transport under Hölder Costs

In Section 3.2 and 3.3, we have shown that the rate of convergence of the empirical OT cost in \( \mathbb{R}^d \) is bounded by \( n^{-1/d} \) for Lipschitz and \( n^{-2/d} \) for \( C^2 \) costs (if \( d \geq 5 \)). The recent work of Manole & Niles-Weed (2021) demonstrated that these results can be generalized to \( \alpha \)-Hölder smooth costs for \( 0 < \alpha \leq 2 \), deriving the rates \( n^{-\alpha/d} \). In this section, we employ similar arguments to bound the uniform metric entropy of the class \( \mathcal{F}_c \) by \( \varepsilon^{-d/\alpha} \) (for any \( d \in \mathbb{N} \) and \( 0 < \alpha \leq 2 \)) in settings resembling the ones of Assumption (Lip) and (SC).

We say that a function \( f: U \to \mathbb{R} \) on a convex domain \( U \subset \mathbb{R}^d \) is \((\alpha, \Lambda)\)-Hölder smooth for \( 0 < \alpha \leq 1 \) and \( \Lambda > 0 \) if \( \|f\|_{\infty} < \Lambda \) and
\[
|f(x) - f(y)| \leq \Lambda \cdot \|x - y\|^\alpha.
\]
Moreover, we say that \( f \) is \((\alpha, \Lambda)\)-Hölder smooth for \( 1 < \alpha \leq 2 \) if \( \|f\|_{\infty} < \Lambda \) and \( f \) is differentiable with \((\alpha - 1, \Lambda)\)-Hölder smooth partial derivatives. If the convex domain \( U \) is not open, we assume the existence of a Hölder smooth function on an open subset of \( \mathbb{R}^d \) containing \( U \) that coincides with \( f \) on \( U \).

**Assumption (Hol).** Let \( \alpha \in (0, 2] \) and suppose that \( \mathcal{X} = \bigcup_{i=1}^{I} \mathcal{U}_i \) for \( I \in \mathbb{N} \) compact, convex subsets \( \mathcal{U}_i \subset \mathbb{R}^d \) and maps \( g_i: \mathcal{U}_i \to \mathcal{X} \) so that \( c(g_i(\cdot), y) \) is \((\alpha, 1)\)-Hölder for all \( y \in \mathcal{Y} \).

**Theorem 3.11** (Hölder LCA). Let \( \mathcal{X} \) and \( \mathcal{Y} \) be Polish spaces and \( c: \mathcal{X} \times \mathcal{Y} \to [0, 1] \) be continuous. If Assumption (Hol) holds, then, for any \( \mu \in \mathcal{P}(\mathcal{X}) \) and \( \nu \in \mathcal{P}(\mathcal{Y}) \), the empirical estimator \( \hat{T}_{c,n} \) from (1.8) satisfies
\[
\mathbb{E}[\|\hat{T}_{c,n} - T_c(\mu, \nu)\|] \leq \begin{cases} n^{-1/2} & \text{if } d < 2\alpha, \\ n^{-1/2} \log(n) & \text{if } d = 2\alpha, \\ n^{-\alpha/d} & \text{if } d > 2\alpha. \end{cases}
\]

**Proof.** Lemma A.4 in Appendix A shows that the uniform metric entropy in this setting is bounded by \( \log \mathcal{N}(\varepsilon, \mathcal{F}_c, \|\cdot\|_{\infty}) \leq \varepsilon^{-d/\alpha} \). An application of Theorem 2.2 yields the claim. \( \square \)

Theorem 3.11 can be used to derive upper bounds for Hölder smooth cost functions on Euclidean spaces analogous to Example 3.9. Moreover, it is again possible to derive lower bounds in certain situations. For instance, in the setting of Example 3.10, we can
consider $\alpha$-Hölder costs of the form $c(x, y) = \sum_{i=1}^{d_1} |x_i - y_i|^{\alpha} + \sum_{i=d_1+1}^{d_2} |y_2|^{\alpha}$ for $\alpha \in (0, 2]$. Then, Assumption (Hol) is fulfilled (for $d = d_1$) and one can show that

$$\sup_{\mu, \nu} \mathbb{E} \left[ |T_c(\hat{\mu}_n, \hat{\nu}_n) - T_c(\mu, \nu)| \right] \geq \begin{cases} n^{-1/2} & \text{if } d_1 \leq 2\alpha, \\ n^{-\alpha/d_1} & \text{if } d_1 > 2\alpha, \end{cases}$$

where the supremum is taken over $\mu \in \mathcal{P}(\mathcal{X})$ and $\nu \in \mathcal{P}(\mathcal{Y})$. Herein, the lower bound for $d_1 \leq 2\alpha$ follows by selecting discretely supported measures, whereas the regime $d_1 > 2\alpha$ is covered by Proposition 21 in Manole & Niles-Weed (2021). In particular, in case of $d_1 \neq 2\alpha$, the upper bound from Theorem 3.11 matches the lower bound, whereas for $d_1 = 2\alpha$ it is sharp only up to a logarithmic factor.

### 4 Simulations

In the previous sections, we investigated the convergence rate for the empirical OT cost in various settings, stressing than an intrinsic adaptation to the less complex measure governs asymptotic statistical properties. We now turn to the question if these asymptotic properties can already be observed in the finite sample regime accessible to numerical analysis. For this purpose, we fix probability measures $\mu$ and $\nu$ and approximate the mean absolute deviation

$$\Delta_n = \mathbb{E} \left[ |\hat{T}_{c,n} - T_c(\mu, \nu)| \right]$$

for various values of $n$ via Monte-Carlo simulations with 2000 independent repetitions.\footnote{We employed the network-simplex based C++ solver by Bonneel et al. (2011) for the computation of the empirical OT cost. The full source code used to produce the data in this section can be found under 
https://gitlab.gwdg.de/staudt1/lca} Since the value of $T_c(\mu, \nu)$ has to be known with high accuracy for conclusive results, we are restricted to relatively simple settings where analytical approaches are feasible and the optimal transport cost or map can be computed explicitly. The spaces $\mathcal{X}$ and $\mathcal{Y}$ are considered to be subsets of $\mathbb{R}^d$, with either $l_1$ and $l_2$ cost functions of the form

$$c_1(x, y) = \sum_{i=1}^{d} |x_i - y_i| \quad \text{or} \quad c_2(x, y) = \sum_{i=1}^{d} |x_i - y_i|^2.$$

In total, we look at the following choices for $\mu$ and $\nu$. The intrinsic dimension of the former is denoted by $d_1$, and the one of the latter by $d_2$, where $d_1 \leq d_2 \leq d$. The $r$-dimensional unit-sphere is denoted by $S^r \subset \mathbb{R}^{r+1}$.

(i) **Cube**: We choose $\mu = \text{Unif}(\mathcal{X})$ for $\mathcal{X} = [0, 1]^{d_1} \times \{0\}^{d_2-d_1}$ and $\nu = \text{Unif}(\mathcal{Y})$ for $\mathcal{Y} = [0, 1]^{d_2}$. As the one-sample estimates $T_c(\hat{\mu}_n, \nu)$ and $T_c(\mu, \hat{\nu}_n)$ are computationally infeasible, we employ the two-sample estimates $\hat{T}_{c,n} = T_c(\hat{\mu}_n, \hat{\nu}_n)$ for up to $n = 2^{11} = 2048$. The value of $T_c(\mu, \nu)$ is calculated analytically.

(ii) **Sphere**: We choose $\mu = \text{Unif}(\mathcal{X})$ for $\mathcal{X} = S^{d_1} \times \{0\}^{d_2-d_1}$ and $\nu = \text{Unif}(\mathcal{Y})$ for $\mathcal{Y} = S^{d_2}$. The two-sample estimate $\hat{T}_{c,n} = T_c(\hat{\mu}_n, \hat{\nu}_n)$ is used for up to $n = 2^{11} = 2048$ and $T_c(\mu, \nu)$ is approximated numerically (the optimal transport map between $\nu$ to $\mu$ can be established due to the symmetry of the setting).

(iii) **Semi-discrete**: We choose $\nu = \text{Unif}(\mathcal{Y})$ for $\mathcal{Y} = [0, 1]^d$ and set $\mu = \text{p}_\# \nu$, where $\text{p}(y) = \arg\min_{x \in \mathcal{X}} c(x, y)$ denotes the $c$-projection onto the finite set $\mathcal{X} = \{x_i\}_{i=1}^I \subset [0, 1]^d$ with $I \in \mathbb{N}$. Consequently, $\mu(\{x_i\})$ equals the fraction of the volume of $[0, 1]^d$ that lies
Figure 2: Simulations of the mean absolute deviation $\Delta_n$ in the cube and sphere settings. The different curves correspond to $1 \leq d_1 \leq 10$. Green lines mark the dimensions $d_1$ for which the upper bounds in Theorem 3.3 and Theorem 3.8 suggest $\sqrt{n}\Delta_n$ to be bounded, while yellow and blue lines enjoy no such guarantee.

closest to $x_i$. The positions $x_i$ have been fixed once for each pair $(I, d)$ by drawing them uniformly in $[0, 1]^d$. The one-sample estimator $\hat{T}_{c,n} = T_c(\mu, \hat{\nu}_n)$ is used for up to $n = 2^{15} = 32768$ and $T_c(\mu, \nu)$ is approximated numerically (based on the observation that $p$ is the optimal transport map between $\nu$ and $\mu$).

A first set of simulation results in the cube and sphere settings with smooth $l^2_2$ and non-smooth $l^1_1$ costs for fixed $d_2 = 10$ can be seen in Figure 2. As anticipated by the LCA principle, the smaller dimension $d_1$ appears to dictate the convergence rate of $\Delta_n$ towards zero as $n$ becomes large. For smooth costs, $\Delta_n$ seems to converge with the rate $1/\sqrt{n}$ for $d_1 \leq 3$ (and even in the critical case $d_1 = 4$, which is in line with results by Ledoux [2019]), while the convergence for $d_1 \geq 5$ is perceivably slower in both settings. This is in good agreement with the upper bounds (3.5) established by Theorem 3.8 for $C^2$ cost functions. For non-smooth $l^1_1$ costs, in contrast, the cube setting again exhibits the behavior to be expected if the bounds (3.3) of Theorem 3.3 were sharp (i.e., only $d_1 = 1$ leads to a clear $n^{-1/2}$ convergence), but the results for the sphere setting somewhat resemble the ones for smooth costs. This salient difference might be explained along the lines discussed in the context of equation (3.7). In fact, if $\pi$ denotes an optimal transport plan for $T_c(\mu, \nu)$, then it is straightforward to see that the cost function $c_1$ is not differentiable $\pi$-almost surely in the cube setting (since all optimal movement of mass leaves the first $d_1$ coordinates unchanged), while it is differentiable $\pi$-almost surely in the sphere setting (almost all mass is moved such that each coordinate changes).

To understand the influence of a different choice of $d_2$, we also conducted analogous simulations with $d_2 = 100$ and $d_2 = 1000$. While the basic conclusions remained unchanged and the LCA principle could be confirmed (see Figure 3b), the statistical fluctuation of the Monte-Carlo estimate of $\Delta_n$ increased with increasing $d_2$ and larger sample sizes $n$ were typically needed to observe linearity of $\Delta_n$ in the presented log-log plots. In this
regard, Figure 3b indicates for $d_2 = 10$ in the cube setting that maximal sample sizes of $n = 2^{11} = 2048$ might not suffice to confidently discern the actual asymptotic convergence rates $n^{-2/d_1}$ for $d_1 \geq 5$ (see Examples 3.9 and 3.10).

Finally, we turn to the results obtained in the semidiscrete setting. According to upper bound (3.2) established in Theorem 3.2, we anticipate asymptotically a convergence rate of order $n^{-1/d_2}$ for $\Delta_n$ independent of the choice of the cost $c$, the cardinality $I$ of $\mathcal{X}$, and the dimension $d_2$ of the ambient space. Figure 4 confirms this expectation under smooth $l_2^2$ cost. Indeed, in simulations with $I = 50$ (and higher), the convergence of $\Delta_n$ seems to be quicker than $n^{-1/2}$ at first, but eventually slows down for sufficiently large $n$. Comparable results were observed for $l_1$ costs as well.

5 Discussion

In this work, we have established novel statistical guarantees for the empirical OT cost between different probability measures, showing that the mean convergence rate is governed by the less complex measure. In a broader sense, the LCA phenomenon suggests that the curse of dimensionality only affects the estimation of the OT cost when both probability measure exhibit high intrinsic dimensions – an observation with possibly significant repercussions for OT based data analysis applications, as the empirical OT functional.
automatically adapts to the complexity of the simpler measure and not to the ambient space. In particular, our theory can also be applied for the popular Wasserstein distance $W_p(\mu, \nu) = T_{dp}(\mu, \nu)^{1/p}$ with $p \geq 1$, since

$$E[|\hat{W}_{p,n} - W_p(\mu, \nu)|] \geq E[|\hat{T}_{dp,n} - T_{dp}(\mu, \nu)|]$$

for fixed measures $\mu \neq \nu$ on compact metric spaces, where $\hat{W}_{p,n} = \hat{T}_{dp,n}$.

Several extensions of our theory seem to be natural targets for future research. First, our arguments crucially rely on uniform metric entropy bounds, which essentially restricts our results to bounded costs and spaces. A generalization to measure-dependent notions of metric entropy, or a technique to properly exploit the concentration of measures, might serve to overcome these limitations. In a similar vein, it might be possible to adapt the LCA principle to more general notions of dimensionality. While our theory already includes general compact Lipschitz and $C^2$ surfaces in $\mathbb{R}^d$, and even metric spaces with finite Minkowski-Bouligand dimension, it is as of yet unclear if general extensions to the Hausdorff dimension (Mattila, 1995, Chapter 4) or the (concentration-dependent) Wasserstein dimension (Weed & Bach, 2019) are viable.

Another interesting problem is to find non-trivial settings where the upper bounds in Theorem 2.2 fail to provide sharp rates. While it is easy to find simple examples where the rates suggested by Theorem 3.3 and 3.8 are not sharp, the bottleneck for these examples is typically a suboptimal bound for the uniform metric entropy of $\mathcal{F}_c$ when applying Theorem 2.2. For instance, additive costs of the form $c(x, y) = c_1(x) + c_2(y)$ always lead to the parametric convergence rate $n^{-1/2}$, which is not necessarily captured by Theorem 3.3 and 3.8. Adapting Theorem 2.2 to these specific costs, however, yields the correct rate for non-constant costs.

We finally stress that our proof technique explicitly relies on empirical measure based estimators under i.i.d. observations. It would be interesting to analyze whether the same (or even faster) convergence rates can be verified for other estimators or for dependent observations. In particular, it remains an open question to what extent estimators leveraging smoothness properties of the underlying measures, e.g., when $\mu$ and $\nu$ are measures on Riemannian manifolds with sufficiently regular densities with respect to the canonical volume forms, also obey the LCA principle.

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A Bounds on the Uniform Metric Entropy

In this appendix, we establish various upper bounds for the uniform metric entropy of the function class $F_c$ defined by equation (2.2). To cover the settings introduced in Section 3, the following observation about uniform covering numbers under composition is useful.

**Lemma A.1** (Composition bound). Let $g: \mathcal{U} \to \mathcal{V}$ be a surjective map between sets $\mathcal{U}$ and $\mathcal{V}$, and let $F$ be a real-valued function class on $\mathcal{V}$. For any $\varepsilon > 0$, the class $F \circ g := \{ f \circ g | f \in F \}$ satisfies

$$N(\varepsilon, F \circ g, \cdot |_{\mathcal{U}}) \leq N(\varepsilon, F, \cdot |_{\mathcal{V}}).$$

**Proof.** Assume that $N := N(\varepsilon, F \circ g, \cdot |_{\mathcal{U}})$ is finite, otherwise the inequality is trivial. Let $\{f_1, \ldots, f_N\}$ be an $\varepsilon$-covering of $F \circ g$ and let

$$f_i(v) := \sup_{u \in g^{-1}(v)} \tilde{f}_i(u).$$

For any $f \in F$, there is an $\tilde{f}_i$ such that $|f \circ g - \tilde{f}_i| \leq \varepsilon$ on $\mathcal{U}$. By definition of $f_i$, this implies $f - f_i \leq \varepsilon$ and $f - f_i \geq -\varepsilon$ on $\mathcal{V}$, which shows that $\{f_1, \ldots, f_N\}$ is an $\varepsilon$-covering of $F$. \qed

We now provide upper bounds on the metric entropy of $F_c$ under the respective assumptions [Lip], [SC], and [Hol]. Due to the union bound (Lemma 3.1) in conjunction with the composition bound (Lemma A.1), it is in all three cases sufficient to bound

$$\log N(\varepsilon, F_c \circ g_i, \cdot |_{\mathcal{U}_i})$$

for all $i \in \{1, \ldots, I\}$ and sufficiently small $\varepsilon > 0$. For convenience, we suppress the index $i$ in the following proofs and work with generic $g := g_i$ and $\mathcal{U} := \mathcal{U}_i$, as well as $d := d_i$ for Assumption [Lip].
Lemma A.2 (Metric entropy under Lipschitz costs). Let \( \mathcal{X} \) and \( \mathcal{Y} \) be Polish spaces and let \( c: \mathcal{X} \times \mathcal{Y} \to [0,1] \) be continuous so that Assumption (Lip) is fulfilled. Then, for any \( \varepsilon > 0 \),

\[
\log \mathcal{N}(\varepsilon, \mathcal{F}_c, \cdot \mid \infty) \leq \sum_{i=1}^{I} \left( \mathcal{N}(\varepsilon/4, \mathcal{U}_i, d_i) + \log(\varepsilon^{-1}) \right) \tag{A.1a}
\]

Moreover, without the connectedness assumption on \( \mathcal{U} \) in (Lip) it holds that

\[
\log \mathcal{N}(\varepsilon, \mathcal{F}_c, \cdot \mid \infty) \leq \sum_{i=1}^{I} \mathcal{N}(\varepsilon/4, \mathcal{U}_i, d_i) \log(\varepsilon^{-1}). \tag{A.1b}
\]

The implicit constants in (A.1) are universal.

Proof. By Assumption (Lip) it holds that \( c(g(\cdot), y) \) is 1-Lipschitz for each \( y \in \mathcal{Y} \). Hence, the class \( \mathcal{F}_c \circ g \) is contained in \( \text{BL}_1(\mathcal{U}, d) \), which denotes the 1-Lipschitz functions on \( (\mathcal{U}, d) \) that are absolutely bounded by one (Santambrogio, 2015, Section 1.2). For connected \( \mathcal{U} \), their uniform metric entropy is bounded by (Kolmogorov & Tikhomirov, 1961, Section 9)

\[
\mathcal{N}(\varepsilon, \text{BL}_1(\mathcal{U}, d), \cdot \mid \infty, \mathcal{U}) \leq (2[2/\varepsilon] + 1) 2^{\mathcal{N}(\varepsilon/4, \mathcal{U}, d)}, \tag{A.2a}
\]

while general metric spaces only permit the bound

\[
\mathcal{N}(\varepsilon, \text{BL}_1(\mathcal{U}, d), \cdot \mid \infty, \mathcal{U}) \leq (2[2/\varepsilon] + 1)^{\mathcal{N}(\varepsilon/4, \mathcal{U}, d)}. \tag{A.2b}
\]

This implies claim (A.1). Note that (A.2a) is a variation of equation (238) in Kolmogorov & Tikhomirov (1961), which only proves the stated bound for connected subsets of a centrable metric space (with some improvements, e.g., \( \varepsilon/4 \) can be relaxed to \( s\varepsilon/(s+1) \) for \( s \in \mathbb{N} \) at the cost of a possibly worse constant). However, with minor adaptations, the proof also works without requiring centrability for \( \varepsilon/4 \).

Lemma A.3 (Metric entropy under semi-concave costs). Let \( \mathcal{X} \) and \( \mathcal{Y} \) be Polish spaces and let \( c: \mathcal{X} \times \mathcal{Y} \to [0,1] \) be continuous so that Assumption (SC) is fulfilled. Then, for \( \varepsilon > 0 \) sufficiently small,

\[
\log \mathcal{N}(\varepsilon, \mathcal{F}_c, \cdot \mid \infty) \leq I \varepsilon^{-d/2}, \tag{A.3}
\]

where the implicit constant depends on the sets \( \mathcal{U}_1, \ldots, \mathcal{U}_I \subset \mathbb{R}^d \).

Proof. Let \( s := \dim(\text{span}(\mathcal{U} - u)) \leq d \) for an arbitrary \( u \in \mathcal{U} \). If \( s = 0 \), the metric entropy of \( \mathcal{F}_c \circ g \) is bounded as in (5.1), so we consider \( s \geq 1 \). By translation and rotation, we may w.l.o.g. assume that \( \mathcal{U} \) is a bounded convex subset of \( \mathbb{R}^n \) that contains the origin. By Assumption (SC) and the properties of the \( c \)-transform, any \( f \in \mathcal{F}_c \) is absolutely bounded by one, and the composition \( f \circ g \) is 1-Lipschitz and 1-semi-concave on \( \mathcal{U} \). Thus, the function \( u \mapsto f \circ g(u) - \|u\|^2 \) is concave, \( L \)-Lipschitz with \( L := 1 + 2 \text{diam}(\mathcal{U}) \), and absolutely bounded by \( 1 + \text{diam}(\mathcal{U})^2 \). According to Dragoniulescu & Ivan (1992, Theorem 1 and Remark 2(ii)) there exists a concave extension \( f \) of this function to \( \mathbb{R}^n \) with identical Lipschitz-modulus. If \( \mathcal{D} \subset \mathbb{R}^d \) denotes a bounded closed cube that contains \( \mathcal{U} \), then \( f \) is absolutely bounded on \( \mathcal{D} \) by \( B := 1 + \text{diam}(\mathcal{U})^2 + L \text{diam}(\mathcal{D}) \). Denoting the class of concave functions on \( \mathcal{D} \) that are absolutely bounded by \( B \) and \( L \)-Lipschitz by \( C_{B,L}(\mathcal{D}) \), we conclude for small \( \varepsilon > 0 \)

\[
\mathcal{N}(\varepsilon, \mathcal{F}_c \circ g, \cdot \mid \infty, \mathcal{U}) = \mathcal{N}(\varepsilon, \mathcal{F}_c \circ g - \| \cdot \|^2, \cdot \mid \infty, \mathcal{U}) \leq \mathcal{N}(\varepsilon, C_{B,L}(\mathcal{D}), \cdot \mid \infty, \mathcal{D}) \leq B^{-d/2} \leq \varepsilon^{-d/2},
\]

where we used uniform metric entropy bounds for the class \( C_{B,L}(\mathcal{D}) \) provided in (Bronshtein 1976) and Guntuboyina & Sen (2013). The implicit constants depend on \( B, L, \) and \( \mathcal{D} \), which in turn depend on \( \mathcal{U} \).
Lemma A.4 (Metric entropy under Hölder costs). Let $\mathcal{X}$ and $\mathcal{Y}$ be Polish spaces and let $c: \mathcal{X} \times \mathcal{Y} \to [0, 1]$ be continuous so that Assumption (Hol) is fulfilled for some $\alpha \in (0, 2]$. Then, for $\varepsilon > 0$ sufficiently small,

$$\log N(\varepsilon, F_c, \| \cdot \|_\infty) \leq I \varepsilon^{-d/\alpha},$$

where the implicit constant depends on $\alpha$ and the sets $U_1, \ldots, U_l \subseteq \mathbb{R}^d$.

Proof. We consider $\alpha \in (0, 1)$ first. An $(\alpha, 1)$-Hölder function with respect to the Euclidean norm $\| \cdot \|$ is a 1-Lipschitz function with respect to the metric induced by $\| \cdot \|^{\alpha}$. It follows by Assumption (Hol) that $F_c \circ g \subseteq BL_1(\mathcal{U}, \| \cdot \|^{\alpha})$ (see the proof of Lemma A.2). Furthermore, each function in $BL_1(\mathcal{U}, \| \cdot \|^{\alpha})$ can be extended to an element in $BL_1(D, \| \cdot \|^{\alpha})$, where $D \subseteq \mathbb{R}^d$ is bounded, connected, and contains $\mathcal{U}$ (McShane, 1934, Corollary 2). Thus, noting that $N(\varepsilon, D, \| \cdot \|^{\alpha}) = N(\varepsilon^{1/\alpha}, D, \| \cdot \|)$ $\leq \varepsilon^{-d/\alpha}$ and employing (A.2a), we find

$$\log N(\varepsilon, F_c \circ g, \| \cdot \|^{1/\alpha}) \leq I \log N(\varepsilon, BL_1(D, \| \cdot \|^{\alpha}), \| \cdot \|_{\infty, D}) \leq \varepsilon^{-d/\alpha}.$$  

For $\alpha \in (1, 2]$, we apply Lemma A.5 to $c(g(\cdot), y)$ for each $y \in \mathcal{Y}$ separately to define a collection of smoothed, approximated cost functions $c_{\sigma} : D \times \mathcal{Y} \to \mathbb{R}$ for $\sigma \in (0, 1)$, where $D \subseteq \mathbb{R}^d$ contains $\mathcal{U}$ and is convex, open, and bounded. Furthermore, there is $K > 0$ so that the functions $c_{\sigma}$ satisfy, for all $y \in \mathcal{Y}$,

$$\|c(g(\cdot), y) - c_{\sigma}(\cdot, y)\|_{\infty, D} \leq K\sigma^\alpha$$
and

where the $C^2(D)$-norm of a twice continuously differentiable function $f : D \to \mathbb{R}$ is

$$\|f\|_{C^2(D)} := \max_{|\beta| \leq 2} \|D^\beta f\|_{\infty, D},$$

for $\beta \in \mathbb{N}_0^d$. Note that a function with $\|f\|_{C^2(D)} \leq I$ for $I > 0$ is absolutely bounded by $I$, $I$-Lipschitz, and $dI$-semi-concave (since the Eigenvalues of its Hessian are bounded by $d \cdot I$). For each $f \in F_c$, we define $f_{\sigma} : D \to \mathbb{R}$, $u \mapsto \inf_{y \in \mathcal{Y}} c_{\sigma}(u, y) - f^*(y)$. Due to $f = f^{cc}$ (Santambrogio, 2015, Proposition 1.34) combined with the first inequality in (A.5), we conclude $|f \circ g - f_{\sigma}| \leq K\sigma^\alpha$ on $\mathcal{U}$. For $\sigma(\varepsilon) := (\varepsilon/2K)^{1/\alpha}$, this implies $|f \circ g - f_{\sigma(\varepsilon)}| \leq \varepsilon/2$ on $\mathcal{U}$. Consequently, defining $F_{c,\sigma} := \{f_{\sigma} \mid f \in F_c\}$,

$$N(\varepsilon, F \circ g, \| \cdot \|^{1/\alpha}) \leq N(\varepsilon/2, F_{c,\sigma(\varepsilon)}, \| \cdot \|_{\infty, D}) \leq N(\varepsilon/2, F_{c,\sigma(\varepsilon)}, \| \cdot \|_{\infty, D}).$$

Since the functions $c_{\sigma}(\cdot, y)/dI_{\sigma}$ are bounded by one, 1-Lipschitz, and 1-semi-concave, we can apply the metric entropy bounds derived in the proof of Lemma A.3 to conclude

$$\log N\left(\varepsilon/2, F_{c,\sigma(\varepsilon)}, \| \cdot \|_{\infty, D}\right) = \log N\left(\frac{\varepsilon}{2dI_{\sigma}}, \frac{F_{c,\sigma(\varepsilon)}}{dI_{\sigma}}, \| \cdot \|_{\infty, D}\right) \leq \left(\frac{\varepsilon}{I_{\sigma(\varepsilon)}}\right)^{-d/2} \times \varepsilon^{-d/\alpha},$$

where the constants depend on $\alpha$ and $D$, which in turn depends on $\mathcal{U}$.

Lemma A.5. Let $D \subseteq \mathbb{R}^d$ be bounded, convex, and open, and let $\mathcal{U} \subset D$ be a compact and convex subset. Then, there exists $K > 0$ such that for any $(\alpha, 1)$-Hölder function $h$ on $\mathcal{U}$ with $1 < \alpha \leq 2$ there is a collection of smooth functions $h_{\sigma} : D \to \mathbb{R}$ such that

$$\|h - h_{\sigma}\|_{\infty, \mathcal{U}} \leq K\sigma^\alpha$$
and

$$\|h_{\sigma}\|_{C^2(D)} \leq K\sigma^{\alpha-2} \quad \text{for } \sigma \in (0, 1].$$

(A.6)
Proof. Recall the definition of \((\alpha,A)\)-Hölder smooth functions for \(1 < \alpha \leq 2\) and \(A > 0\) from Section 3.4, and let \(u,u_0 \in \mathcal{U}\). If we denote \(z := u - u_0\), then the mean value theorem asserts the existence of \(t \in [0,1]\) such that \(h(u) = h(u_0) + \langle \nabla h(u_0 + tz), z \rangle\). This implies

\[
h(u) = h(u_0) + \langle \nabla h(u_0), z \rangle + R_{u_0}(u), \quad \text{where} \quad R_{u_0}(u) = \langle \nabla h(u_0 + tz) - \nabla h(u_0), z \rangle.
\]

Due to the \((\alpha,1)\)-Hölder smoothness of the partial derivatives of \(h\), we find

\[
|R_{u_0}(u)| \leq |\nabla h(u_0 + tz) - \nabla h(u_0)||z| \leq \sqrt{d} |u - u_0|^\alpha.
\]

(A.7)

This shows that the function \(h\) is an element of the class \(\text{Lip}(\alpha,\mathcal{U})\) defined in \cite{Stein} Chapter VI, Section 3. By Theorem 4 in the same reference, the function \(\tilde{h}\) admits an extension \(\tilde{h}\) to \(\mathbb{R}^d\) that is \((\alpha,K')\)-Hölder on \(\mathbb{R}^d\) for some \(K' > 0\) (which is independent of \(\mathcal{U}\) and \(h\)). For an even and smooth mollifier \(M: \mathbb{R}^d \to [0,\infty)\) supported on the unit ball \(B_1\), we define \(M_\sigma := \sigma^{-d}M(\cdot/\sigma)\), which is supported on the ball with radius \(\sigma \in (0,1]\), and set

\[
h_\sigma : \mathcal{D} \rightarrow \mathbb{R}, \quad u \mapsto (\tilde{h} * M_\sigma)(u) = \int_{\mathbb{R}^d} \tilde{h}(u - z)M_\sigma(z) \, dz,
\]

where integration is over \(\mathbb{R}^d\) (i.e., effectively over the support of \(M_\sigma\)). The desired properties (A.6) now follow analogously to the proof of Lemma 8 of \cite{ManoleNiles-Weed}. For completeness, we include the arguments here. We first observe

\[
\tilde{h}(u) = \tilde{h}(u_0) + \langle \nabla \tilde{h}(u_0), u - u_0 \rangle + \tilde{R}_{u_0}(u), \quad \text{where} \quad |\tilde{R}_{u_0}(u)| \leq \sqrt{d}K'|u - u_0|^\alpha,
\]

for any \(u,u_0 \in \mathcal{D}\), which can be derived analogously to (A.7). For the first bound in (A.6), we note that \(M\) is even, implying \(\int z_i M_\sigma(z) \, dz = 0\) for all \(1 \leq i \leq d\). By expanding \(\tilde{h}(u - z)\) around \(u \in \mathcal{U}\) for \(|z| \leq \sigma\), it follows that

\[
|h_\sigma(u) - h(u)| = \left| \int \left( \tilde{h}(u - z) - \tilde{h}(u) \right) M_\sigma(z) \, dz \right| \\
\leq \int |\tilde{R}_u(u - z)| M_\sigma(z) \, dz \\
\leq \sqrt{d}K'\sigma^\alpha.
\]

For the second inequality in (A.6), we fix some \(u_0 \in \mathcal{D}\) and observe for any \(u \in \mathcal{D}\) that

\[
h_\sigma(u) = \int \tilde{h}(u - z)M_\sigma(z) \, dz \\
= \int \left( \tilde{h}(u_0) + \langle \nabla \tilde{h}(u_0), u - z - u_0 \rangle + \tilde{R}_{u_0}(u - z) \right) M_\sigma(z) \, dz \\
=: A_1 + \langle A_2, u \rangle + A_3(u),
\]

where \(A_1 \in \mathbb{R}\), \(A_2 \in \mathbb{R}^d\), and \(A_3(u) = \int \tilde{R}_{u_0}(z)M_\sigma(u - z) \, dz\) (after a change of variables). For \(\beta \in \mathbb{N}_0^d\) with \(|\beta| = 2\), we evaluate \(D^\beta h_{\sigma}\) at \(u_0\). Exchanging differentiation and integration in the first inequality, and employing substitution in the final one, we observe

\[
|D^\beta h_{\sigma}(u_0)| = |D^\beta A_3(u_0)| \leq \sigma^{-d-2} \int |\tilde{R}_{u_0}(z)| \left| D^\beta M \left( \frac{u_0 - z}{\sigma} \right) \right| \, dz \\
\leq \sqrt{d}K'\sigma^{-d-2} \int |u_0 - z|^\alpha \left| D^\beta M \left( \frac{u_0 - z}{\sigma} \right) \right| \, dz \\
\leq \sqrt{d}K'\sigma^{\alpha-2} \int |z|^\alpha |D^\beta M(z)| \, dz \\
= K''\sigma^{\alpha-2}
\]

for some \(0 < K'' < \infty\). Since this holds for any \(u_0 \in \mathcal{D}\) with \(K''\) independent of \(u_0\) and \(\sigma\), we conclude \(|D^\beta h_{\sigma}|_{\infty,\mathcal{D}} \leq K''\sigma^{\alpha-2}\). Analogous inequalities for \(|\beta| < 2\) follow from the fact that \(\mathcal{D}\) is convex and bounded, so \(D^\beta h_{\sigma}\) can be bounded in terms of the second derivatives of \(h_{\sigma}\) and the diameter of \(\mathcal{D}\).