Abstract We study Saccheri’s three hypotheses on a two right-angled isosceles quadrilateral, with a rectilinear summit side. We claim that in the Hilbert’s foundation of geometry the euclidean parallelism is a theorem and as that it can be used, in the hyperbolic geometry.

Keywords rectilinear quadrilaterals · euclidean parallelism · rectilinear bisector

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1 Introduction

We study Saccheri’s three hypotheses on a two right-angled isosceles quadrilateral, with a rectilinear summit side. We claim that in the Hilbert’s foundation of geometry the euclidean parallelism is a theorem and as that it can be used, in the hyperbolic geometry. The paper is organized in the following way, at section 2 we present the used Definitions, Postulates, and theorems. At section 3 we prove that the rectilinear bisector of the vertex angle of a rectilinear isosceles triangle and the perpendicular bisector of the congruent sides of this triangle are intersecting. At section 4 we prove that, if for Saccheri’s rectilinear quadrilaterals the hypothesis of the acute angle, or the hypothesis of the obtuse angle is true. Then a rectilinear isosceles triangle there exists, in which the perpendicular bisector to the congruent sides and the rectilinear bisector of the vertex angle are not intersecting.

2 Used definitions, postulates and theorems

In this paper we consider a Hilbert plane, satisfying the axioms ( I1 )-( I3 ), ( B1 )-( B4 ), ( C1 )-( C6 ), with the definitions and theorems deduced by them.
Fig. 1

[1]. Also: To prove the theorem of exterior angle we double the median BD of any triangle ABC by drawing the circle ( D, DB ). To prove the theorem that the perpendicular to a line through a point not on the line is unique, we use the theorem of exterior angle and the Hilbert’s theorem that all right angles are congruent to one another.

3 Theorem 1

The rectilinear bisector of the vertex angle of a rectilinear isosceles triangle and the perpendicular bisector of the congruent sides of this triangle are intersecting.

Case A: In the rectilinear isosceles triangle ABΓ (figure 1) we are given that sides AB=BΓ and the bisector BΔ=ΔΔ=ΔΓ/2. In this case the bisector BΔ, and the perpendicular bisector to the side AB are intersecting at the midpoint Δ of the base AΓ of the above triangle. Proof: In the isosceles triangle AΔB the line segment which joins the midpoint E of the base AB with the vertex Δ is perpendicular bisector to the side AB.

Case B: In the isosceles triangle ABΓ (figure 2) we are given that sides AB=BΓ (1) and the bisector BΔ>ΔΔ=ΔΓ/2 (2) this implies that angle BAΔ>angle ABΔ (3). In this case the bisector BΔ and the perpendicular bisectors to the congruent sides AB and BΓ are intersecting. Proof: By (3), given the Pasch’s postulate, is ensured that the perpendicular bisector IT, to the side AB of the isosceles triangle ABΓ will intersect the bisector BΔ, of the vertex angle ABΓ. Since in the opposite assumption, which states that the perpendicular bisector IT, to the side AB will intersect the side ΔΔ (in the triangle ABΔ) at a point M, implies that the angle BAΔ<angle ABΔ which is a contradiction. Given that in the isosceles triangle ABΓ two perpendicular bisectors are intersecting at a point, it easily proved that the third perpendicular bisector passes through this point too. This last implies that all the triangles of the case B are inscribed in circles.

Case Γ: In the isosceles triangle AMΓ (figure 3) we are given that sides AM=MG and MΔ<ΔΔ=ΔΓ/2, then the bisector MΔ, of the vertex angle AMΓ and the perpendicular bisector of the side AM are intersecting. Proof: For any inscribed in a circle γ isosceles triangle ΔΣΓ (figure 3) of the case B, corresponds one and only one isosceles triangle ANΓ of the case Γ, which has the same base ΔΓ.
inscribed in the circle $\gamma$. For all these pairs of triangles that have the same base, but are inscribed in different circles there exists an inverse relation between their altitudes. This last means that given a base $A\Gamma$ and its perpendicular bisector $\Sigma Z$ (figure 3) the inscribed isosceles triangle $A\Sigma \Gamma$, of the case $B$, with the maximum altitude $\Sigma \Delta$, determines the corresponding inscribed isosceles triangle $AN\Gamma$, of the case $\Gamma$, with the minimum altitude $N\Delta$, this is also the isosceles triangle of the case $\Gamma$ with the shortest altitude that can exist, as it follows. The maximum altitude $\Sigma \Delta$ for the inscribed isosceles triangle $A\Sigma \Gamma$, or the same the circle $\gamma$ with the maximum radius, which can pass through the given base $A\Gamma$, implies that the arc $A\Gamma$ nearly coincides with its chord, since the curvature $k$ of a circle of radius $r$ defined by $k = 1/r$. Any isosceles triangle $AM\Gamma$ of the case $\Gamma$ with an altitude $M\Delta$, greater in length than the minimum altitude $N\Delta$ is also an inscribed triangle, as it follows. Because in the triangle $A\Sigma M$, the strict inequality, the chord $A\Sigma < \Sigma N < \Sigma M$ implies that the angle $\Sigma AM > \angle \Sigma MA$, this last, as in the case $B$, ensures that the perpendicular bisector to the side $AM$ of the isosceles triangle $AM\Gamma$ and the line segment $\Sigma Z$, on which the bisector or the altitude $M\Delta$ belongs are intersecting.

4 Theorem 2

If for Saccheri’s rectilinear quadrilaterals the hypothesis of the acute angle, or the hypothesis of the obtuse angle is true. Then a rectilinear isosceles triangle there exists, in which the perpendicular bisector to the congruent sides and the rectilinear bisector of the vertex angle are not intersecting. Proof: For the Saccheri’s quadrilateral $AEZ\Delta$ and its common perpendicular $\Gamma M$ to the base $\Delta Z$ and to the summit $AE$, is valid (a) $A\Delta=EZ > \Gamma M$ if the hypothesis of the acute angle is true (figure 4) and (b) $A\Delta=EZ < \Gamma M$ if the hypothesis of the obtuse angle is true (figure 5). Also, on the straight line $M\Gamma$ there exists a point $B$, so that $A\Delta=EZ=BG\Gamma$, to the right of the point $M$ or to the left of the point $M$, if the case (a) or the case (b) is respectively valid. The point $B$ together with the side $AE$ of the quadrilateral $AEZ\Delta$ forms the rectilinear isosceles triangle $ABE$. Since
the congruent sides $AB$ and $BE$ of the triangle $ABE$ are also the summits of the acute-angled (figure 4) or the obtuse-angled (figure 5) isosceles quadrilaterals $AB\Gamma\Delta$ and $BEZ\Gamma$ it implies that the perpendicular bisectors $II\Pi$ and $T\Sigma$ to the congruent sides of the isosceles triangle $ABE$, as well the bisector $B\Gamma$ of the vertex angle $ABE$ are perpendiculars to the base $\Delta Z$ of the acute-angled or the obtuse-angled quadrilateral $AEZ\Delta$, and since the perpendicular to a line through a point not on the line is unique, we can conclude that they are not intersecting. But this
conclusion contradicts to the results of the theorem 1 and we have to reject the above two hypotheses.

5 Conclusions

Since we reject the hypotheses of the acute angles and the obtuse angles, the acceptance that the rectilinear Saccheri’s quadrilaterals are only rectangle quadrilaterals, implies that the Euclidean parallelism is proved as a theorem in the Hilbert’s foundation of geometry, and as that it can be used, in the hyperbolic geometry.

References

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