OUTER SYNCHRONIZATION OF DELAYED COUPLED SYSTEMS ON NETWORKS WITHOUT STRONG CONNECTEDNESS: A HIERARCHICAL METHOD

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Abstract. We consider the outer synchronization between drive-response systems on networks with time-varying delays, where we focus on the case when the underlying networks are not strongly connected. A hierarchical method is proposed to characterize large-scale networks without strong connectedness. The hierarchical algorithm can be implemented by some programs to overcome the difficulty resulting from the scale of networks. This method allows us to obtain two different kinds of sufficient outer synchronization criteria without the assumption of being strongly connected, by combining the theory of asymptotically autonomous systems with Lyapunov method and Kirchhoff’s Matrix Tree Theorem in graph theory. The theory improves some existing results obtained by graph theory. As illustrations, the theoretic results are applied to delayed coupled oscillators and a numerical example is also given.

1. Introduction. Coupled systems on networks (CSNs) have gained increasing recognition as a mathematical framework in understanding dynamical behaviors of different classes of large-scale artificial and natural systems [25]. From food webs to ecological communities webs, communication networks to social organizations and the World Wide Web to the Internet, a great deal of empirical evidence has revealed the ubiquity of CSNs [7, 12, 20, 21]. Meanwhile, time delays should be taken into account in CSNs to simulate more realistic networks, especially long-distance communication and traffic congestion networks. Thus the exploration of delayed CSNs has been a common task of many basic subjects including physics, biology, sociology and engineering [19, 29, 35, 34], where synchronization, a typical kind of dynamics, has come to play a key role.

In fact, synchronization has attracted rapidly increasing attention in miscellaneous research fields in recent years owing to the extensive applications [2, 3, 14, 36]. Here most of the studies have focused on synchronization among nodes in one network that is regarded as inner synchronization. However, it is well-known that the dynamics among different CSNs also occupy a pivotal and important position.
arising from problems in applied science. Examples includes the infectious disease that spreads among different communities. Another kind of synchronization called outer synchronization was thus put forward in [16] to account for the phenomena that different CSNs could achieve synchronization regardless of synchronization of the inner nodes. Shortly after, outer synchronization has become a new and important topic in the research of CSNs and some results related to this issue have been obtained in [4, 6, 30].

The above literatures reveal that Lyapunov method is a powerful tool to study the outer synchronization between different CSNs. However, challenges always exist in constructing proper Lyapunov functions due to the interplay from the overall topology and the local dynamical properties of the coupled nodes. Fortunately, Li et al. [9, 17] developed a systematic approach to construct global Lyapunov functions for large-scale CSNs via building blocks of individual vertex systems. Following in Li’s footstep, plenty of researchers devoted themselves to the issue of CSNs in virtue of this approach and then some novel and useful results were gained in [1, 10, 24]. We have also put effort into this aspect and achieved some interesting results [15, 18, 26, 33]. In all the works mentioned above, the overarching assumption is that the underlying network is strongly connected. This assumption, however, cannot be satisfied in many real situations as many real-world networks fail to be strongly connected. Further studies are needed for networks without strong connectedness (NWSC) and the following natural questions arise:

(Q1) Is there a systematic approach to characterize large-scale NWSC?
(Q2) How to construct appropriate Lyapunov functions for NWSC?
(Q3) Can the outer synchronization among different NWSC be realized and if so, what conditions are needed?

In this paper we shall answer these questions one by one. Recently, Li et al. [5] pioneered in investigating the impact of network connectivity on the dynamic properties of CSNs by considering a condensed digraph obtained by compressing each strongly connected component into a vertex. Enlightened by this, we tackle question (Q1) by proposing a hierarchical method for the condensed digraph, which divides the condensed digraph into different layers according to the coupling structure. We then present the hierarchical algorithm to overcome the difficulty resulting from the scale of networks. For question (Q2), we set strongly connected components as the vertices of the condensed digraph accordingly and regard strongly connected components as the research objects. Take full advantage of their strongly connected property to construct different Lyapunov functions for different strongly connected components, we can answer question (Q3) with the help of the theory of asymptotically autonomous systems in [22, 28].

Motivated by these, this paper is devoted to provide an analytic study of outer exponential synchronization (OES) between delayed coupled systems on NWSC. A hierarchical method is given to cope with large-scale NWSC. Employing Lyapunov method, graph theory, and the theory of asymptotically autonomous systems [22, 28], we obtain two different kinds of sufficient OES criteria. Compared with some recent results in [1, 10, 15, 18, 24, 33, 26], the main contributions of this paper are as follows,

- To characterize large-scale NWSC, we propose a hierarchical method that can be changed into a program to overcome the difficulty resulting from the scale of networks.
• We first analyze the OES between delayed coupled systems on NWSC by means of graph theory especially removing the assumption of a strongly connected network. The proposed theory presents generality and improves the existing results in above literatures.

The rest of this paper is outlined as follows. In Section 2 we give our model and some necessary preliminaries. Section 3 is introduced by the hierarchical method and the corresponding algorithm. Then an analysis on OES is carried out in Section 4. Also, the theoretical results are applied to delayed coupled oscillators on NWSC in Section 5. Ultimately, a simulation example is presented in Section 6 and an illustrated example to hierarchical algorithm is given in Appendix.

2. Model and preliminaries. This section is devoted to present some necessary preliminaries for this paper. We split it into three subsections: in the first one, we introduce some basic concepts on graph theory; then, in second one the model studied in this paper is given. Finally, the last subsection is devoted to show some necessary definitions. We begin with showing some notations as follows:

Throughout this paper, unless otherwise specified, we let $\mathbb{R}$ and $\mathbb{R}^n$ be the set of real numbers and $n$-dimensional Euclidean space, respectively. Let $\mathbb{R}^1_+ = [0, +\infty)$, $\mathbb{L} = \{1, 2, \ldots, l\}$ and $m = \sum_{i=1}^l m_i$ for $m_i \in \mathbb{Z}^+$. Additionally, we denote the Euclidean norm for vectors by $| \cdot |$ and the superscript “T” is employed to stand for the transpose of a vector. We then set $C^{1,1}(\mathbb{R}^n \times \mathbb{R}^n_+; \mathbb{R}^1_+)$ for the family of all nonnegative functions $V(x,t)$ on $\mathbb{R}^n \times \mathbb{R}^1_+$ that are continuously once differentiable in $x$ and $t$.

2.1. Basic concepts on graph theory. We now conclude some basic concepts and a crucial result concerning graph theory in this subsection; See [31] for details. A digraph $G = (\mathbb{L}, E)$ contains a set $\mathbb{L}$ of vertices and a set $E$ of arcs $(k,h)$ leading from initial vertex $k$ to terminal vertex $h$. A subgraph $\mathcal{H}$ of $G$ is said to be spanning if $\mathcal{H}$ and $G$ have the same vertex set. A digraph $\mathcal{G}$ is weighted if each arc $(h,k)$ is assigned a positive weight $a_{kh}$. Here $a_{kh} > 0$ if and only if there exists an arc from vertex $h$ to vertex $k$ in $G$, and we call $A = (a_{kh})_{l \times l}$ as the weight matrix. The weight $W(G)$ of $G$ is the product of the weights on all its arcs. A directed path $\mathcal{P}$ in $G$ is a subgraph with distinct vertices $\{i_1, i_2, \ldots, i_s\}$ such that its set of arcs is $\{(i_k, i_{k+1}) : k = 1, 2, \ldots, s - 1\}$. If $i_s = i_1$, we call $\mathcal{P}$ a directed cycle. A connected subgraph $\mathcal{T}$ is a tree if it contains no cycles. A tree $\mathcal{T}$ is rooted at vertex $k$, called the root, if $k$ is not a terminal vertex of any arcs, and each of the remaining vertices is a terminal vertex of exactly one arc. A subgraph $\mathcal{Q}$ is unicyclic if it is a disjoint union of rooted trees whose roots form a directed cycle. A digraph $\mathcal{G}$ is strongly connected if, for any pair of distinct vertices, there exists a directed path from one to the other. Denote the digraph with weight matrix $A$ as $(G, A)$ (we also call $(G, A)$ as a network for convenience). The Laplacian matrix of $(G, A)$ is defined as $L = (p_{kh})_{l \times l}$, where $p_{kh} = -a_{kh}$ for $k \neq h$ and $p_{kh} = \sum_{j \neq h} a_{kj}$ for $k = h$. The following lemma about graph theory is crucial in the argument of our main results. However, we here avoid giving any proof or explanation, since this can be easily found in [17].

Lemma 2.1. Assume $l \geq 2$. Let $c_k$ denote the cofactor of the $k$-th diagonal element of Laplacian matrix of $(G, A)$. Then the following formula holds:
Here \( F_{rs}(y_r, y_s), 1 \leq r, s \leq l, \) are arbitrary functions, \( Q \) is the set of all spanning unicyclic graphs of \((G, A), W(Q)\) is the weight of \( Q \) and \( C_Q \) denotes the directed cycle of \( Q \). In particular, if \((G, A)\) is strongly connected, then \( c_k > 0 \) for \( k \in \mathbb{L} \).

2.2. Drive-response systems on networks. In this paper, a network is given by a weighted digraph \((G, A)\). Here \( G \) is a digraph with \( l \) (\( l \geq 2 \)) vertices and \( A = (a_{kh})_{l \times l} \) is weight matrix in which \( a_{kh} \geq 0 \) measures the connection strengths between the coupled nodes. Specially, \( a_{kh} = 0 \) means that there exists no arc from \( h \) to \( k \) in \( G \). The inter-connections or coupling among vertex systems are described by the arcs of \( G \). Consider the inevitability of time delays, towards \( k \)-th vertex, we define a differential system \( x_k'(t) = f_k(x_k(t), x_k(t - \tau_k(t))) + \psi_k(t) \) as the \( k \)-th vertex system. Here, functions \( f_k : \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}^m \) and \( \psi_k : \mathbb{R}_+^l \rightarrow \mathbb{R}^m \) are continuously vector functions. And differentiable function \( \tau_k(t) : \mathbb{R}_+^l \rightarrow [0, \tau] \) stands for the time delay of the \( k \)-th vertex system, whose derivative is bounded by a constant \( \bar{\tau} \in [0, 1) \). Then, the drive system defined on network \((G, A)\) can be written as

\[
x_k'(t) = f_k(x_k(t), x_k(t - \tau_k(t))) + \psi_k(t) + \sum_{h=1}^{l} a_{kh} H_{kh}(x_h(t)), \quad k \in \mathbb{L},
\]

where continuous functions \( H_{kh} : \mathbb{R}^m \rightarrow \mathbb{R}^m \) referred to as the normalized interference functions represent the influence from of vertex \( h \) on vertex \( k \). In order to synchronize this drive system, the response system can be designed as

\[
y_k(t) = f_k(y_k(t), y_k(t - \tau_k(t))) + \psi_k(t) + g_k(y_k(t) - x_k(t)) + \sum_{h=1}^{l} a_{kh} H_{kh}(y_h(t)), \quad k \in \mathbb{L},
\]

where functions \( g_k : \mathbb{R}^m \rightarrow \mathbb{R}^m \) are continuous functions. It is evident that drive system (1) and response system (2) have the identical coupling structure. We here give an example of two networks with identical coupling structure shown in Fig.1 for readability.

We now define \( e_k(t) = y_k(t) - x_k(t) \) as the \( k \)-th synchronization error state vector between drive-response systems (1) and (2). From the drive-response systems, the error system can be obtained as follows,

\[
e_k'(t) = \left[ f_k(y_k(t), y_k(t - \tau_k(t))) - f_k(x_k(t), x_k(t - \tau_k(t))) \right] + g_k(e_k(t))
+ \sum_{h=1}^{l} a_{kh} \left[ H_{kh}(y_h(t)) - H_{kh}(x_h(t)) \right], \quad k \in \mathbb{L}.
\]

Next, we assume that functions \( f_k, g_k \) and \( H_{kh} \) for \( k, h \in \mathbb{L} \) satisfy global Lipschitz condition with Lipschitz constants \( \theta_k \) and \( B_{kh} \), respectively. By Theorem 2.2.3 in [11], error system (3) has a unique solution \( e(t) = [e_1^T(t), e_2^T(t), \ldots, e_l^T(t)]^T \) with initial condition

\[
e(t) = \phi(t), \quad t \in [-\tau, 0],
\]

where \( \phi(t) = [\phi_1^T(t), \phi_2^T(t), \ldots, \phi_l^T(t)]^T \) is a continuous vector function on \([-\tau, 0]\). We further assume that \( g_k(0) = 0 \) for all \( k \in \mathbb{L} \), then error system (3) has a unique zero solution.
2.3. OES of derive-response systems (1) and (2). In the case of outer synchronization, we know that all solutions of error system (3) will tend to zero but we do not know the speed. In practical engineering process, we always hope two networks achieve synchronization in a finite time. Thus, to achieve faster outer synchronization in different systems, OES is introduced in [2, 36] and we do know the two systems will achieve outer synchronization exponentially fast in this case. Before proceeding further, we first give the definition of OES between derive-response systems (1) and (2), which is required throughout this paper.

**Definition 2.2.** Drive-response systems (1) and (2) are said to achieve outer exponential synchronization, if there are positive constants $\beta$ and $M$ such that

$$|e(t; \phi)| \leq M|\phi| e^{-\beta t}, \quad \forall \phi \in C([-\tau, 0]; \mathbb{R}^m), \quad t \geq 0.$$ 

In the following, we assume that the digraph $G$ is connected without loss of generality, since otherwise each connected component of $G$ can be regarded as an independent system and be treated separately.

Analogous to [5], the following definitions with respect to error system (3) is also necessary. For any subgraph $\mathcal{H}$, we define

- **$\mathcal{H}$-subsystem of error system (3):**

  $$e'_k(t) = \left[ f_k(y_k(t), y_k(t-\tau_k(t))) - f_k(x_k(t), x_k(t-\tau_k(t))) \right] + g_k(e_k(t)) + \sum_{h \in V(\mathcal{G})} a_{kh} \left[ H_{kh}(y_h(t)) - H_{kh}(x_h(t)) \right], \quad k \in V(\mathcal{H}).$$

- **Reduced $\mathcal{H}$-subsystem of error system (3):**

  $$e'_k(t) = \left[ f_k(y_k(t), y_k(t-\tau_k(t))) - f_k(x_k(t), x_k(t-\tau_k(t))) \right] + g_k(e_k(t)) + \sum_{h \in V(\mathcal{H})} a_{kh} \left[ H_{kh}(y_h(t)) - H_{kh}(x_h(t)) \right], \quad k \in V(\mathcal{H}).$$

**Remark 1.** These concepts restrict error system (3) into a component of the network. If the component $\mathcal{H}$ is strongly connected, then we are allowed to take advantage of the method developed in [17] to construct Lyapunov functions on the
strongly connected components. In this sense, these concepts are somewhat useful in the argument of our main results.

The framework is a natural setting for investigating complex coupled systems from some characters of the network that coupled systems are established in. Of key interest is the impact of the network structure. To confirm this, we shall analyze the digraph $\mathcal{G}$ on a general network $(\mathcal{G}, A)$ in detail.

3. Hierarchical method for the digraph $\mathcal{G}$ without strong connectedness. As is already mentioned above, in existing results, a key assumption on the dynamics analysis of CSNs utilizing graph theory is that the network $(\mathcal{G}, A)$ is strongly connected. However, the network $(\mathcal{G}, A)$ is not content with this assumption in general. A question now arises naturally: how to cope with a digraph $\mathcal{G}$ without strong connectedness? A positive answer is given in this section by showing a hierarchical method, which is the key argument in this paper.

3.1. Digraph $\mathcal{G}$ without strong connectedness. As the theoretical basis of hierarchical method, the following concepts have been supplied in [5] and we here conclude them for the convenience of the reader. Given digraph $\mathcal{G}$, we first define a relation among vertices of the digraph $\mathcal{G}$ as partial order $\preceq$. For vertices $k, h \in V(\mathcal{G})$, $k \preceq h$ if there is an oriented path from vertex $k$ to $h$, where $V(\mathcal{G})$ represents the vertex set of digraph $\mathcal{G}$. We say $k \sim h$ if and only if $k \preceq h$ and $h \preceq k$. We can verify readily that relation “$\sim$” is an equivalence relation. We further define strongly connected component $\mathcal{S}_k$ of digraph $\mathcal{G}$ as follows: if the subgraph $\mathcal{S}_k$ is strongly connected and for any vertex $k$, the subgraph consisting of the vertex set $V(\mathcal{S}_k) \cup \{k\}$ is not strongly connected, then $\mathcal{S}_k$ is a strongly connected component. Then we compress each strongly connected component of digraph $\mathcal{G}$ into a single vertex, all of which constitute a new digraph $\mathcal{H}$ treated as condensed graph of $\mathcal{G}$ (see Fig.2). It follows from the equivalence relation “$\sim$” that the equivalence classes form a partition of $V(\mathcal{G})$. The set of equivalence classes is sometimes called the quotient set and is denoted by $V(\mathcal{G})/\sim$, then $V(\mathcal{H}) = V(\mathcal{G})/\sim$ and each vertex $\mathcal{S}_k \in V(\mathcal{H})$ is a strongly connected component of digraph $\mathcal{G}$. In particularly, if digraph $\mathcal{G}$ is strongly connected, then $\mathcal{G}$ has only one strongly connected component which indicates $V(\mathcal{H}) = \{\mathcal{G}\}$ is a singleton. And otherwise, for any $\mathcal{S}_k, \mathcal{S}_h \in V(\mathcal{H})$, a directed edge from $\mathcal{S}_k$ to $\mathcal{S}_h$ exists if there are $k \in \mathcal{S}_k$ and $h \in \mathcal{S}_h$ such that a directed edge from $k$ to $h$ exists in $\mathcal{G}$.

Based on the above discussion, for condensed digraph $\mathcal{H}$, we can also define a canonical partial order $\prec$. In detail, for strongly connected components $\mathcal{S}_k, \mathcal{S}_h \in V(\mathcal{H})$, $\mathcal{S}_k \prec \mathcal{S}_h$ if there are vertex $k \in \mathcal{S}_k$ and vertex $h \in \mathcal{S}_h$ such that $k \preceq h$ in $\mathcal{G}$. The definition of strongly connected component asserts that if $\mathcal{S}_k, \mathcal{S}_h \in V(\mathcal{H})$ satisfy both $\mathcal{S}_k \prec \mathcal{S}_h$ and $\mathcal{S}_h \prec \mathcal{S}_k$, then $\mathcal{S}_k$ and $\mathcal{S}_h$ are the same strongly connected component. Thus relation $\prec$ is a strict partial order. We can further obtain that for a condensed digraph $\mathcal{H}$ with a finite number of vertices, there must be both minimal and maximal elements in $V(\mathcal{H})$ with respect to the strict partial order $\prec$.

As is mentioned above, every digraph $\mathcal{G}$, whether it is strongly connected or not, can be condensed to condensed digraph $\mathcal{H}$ through collapsing each strongly connected component to a single vertex. For the condensed digraph $\mathcal{H}$, we propose a hierarchical method and each vertex will belong to a certain layer. It should be emphasized that each vertex of $\mathcal{H}$ is a strongly connected component of digraph $\mathcal{G}$. Here is the hierarchical method.
Hierarchical method: In the condensed digraph $\mathcal{H}$, all the vertices with zero in-degree belong to the first layer. Vertex $U$ belongs to the $k$-th ($k \geq 2$) layer if and only if the following conditions hold.

- There is a vertex $V$ belonging to the $(k - 1)$-th layer, such that $V \prec U$;
- If there exists a vertex $V$ such that $V \prec U$, then vertex $V$ must belong to the preceding $k - 1$ layers.

The existence of minimal and maximal elements in the condensed digraph $\mathcal{H}$ guarantees the rationality of the hierarchical method. In practice, the vertices in the first layer are supposed to be selected first, since it is rather simple to seek out the vertices with zero in-degree. Then, we can find out those vertices that only the vertices belonging to the first layer point to and they belong to the second layer. Afterwards, the third layer contains the vertices that merely vertices in the first and second layers direct to. Following this way, we can acquire the vertices belonging to the fourth layer and it is the same with every layer. The layered work will be accomplished until each vertex corresponds to a certain layer. To give the reader a better handle on this process, let us now give an example to discuss the digraph $\mathcal{G}$ with 67 vertices showed in Fig.2.
Example 3.1. In Fig.2, the dotted boxes represent both strongly connected components in $G$ and vertex in the corresponding condensed digraph $H$. In accordance with the hierarchical method, the black dotted boxes belong to the first layer on account of their zero in-degree in $H$. The blue dotted boxes influenced only by the black ones are parts of the second layer. Similarly, the red dotted box influenced by the black and blue dotted boxes belongs to the third one. We therefore conclude that $H$ can be layered into three layers.

3.2. Hierarchical algorithm. According to the hierarchical method, we can layer artificially some simple networks with a small number of vertices as shown in Example 3.1. It is somewhat unrealistic to layer some large-scale networks manually, which motivates us to seek for a method that can be transformed into a program implemented by some mathematical software to help us. Hence we propose a hierarchical algorithm by employing the well-known Tarjan’s algorithm [27], an algorithm in graph theory for finding the strongly connected component of a digraph. Moreover, to facilitate the reader to understand this algorithm, we will give an illustrated example in Appendix. Afterwards, one of our innovative work, the hierarchical algorithm, is presented as follows.

1. Give the adjacency matrix $E$ of original digraph $G$;
2. Employ Tarjan’s algorithm [27] to obtain all the strongly connected components of digraph $G$ with order, whose amount is denoted as $m$. Then we can get an array of sequential vertices number defined as $Array1[l]$. Meanwhile, the number of vertices in each strongly connected component can form an array with the corresponding order defined as $Array2[m]$;
3. Adjust the matrix $E$ according to $Array1[l]$. First we alter the position of each row such that the order of each row is in accordance with that of each vertex in $Array1[l]$ and it is the same as each line according to $Array1[l]$. Then we acquire a new matrix defined as $E'$;
4. Divide $E'$ into different blocks according to $Array2[m]$ (for example, we can employ the function $mat2cell$ in Matlab and corresponding parameters are $(E', Array2[m], Array2[m])$);
5. Dispose each block of the adjacency matrix $E'$ in the following way:
   - If all the elements of a block are zero, then replace this block with zero;
   - If not all the elements of a block are zero, then replace this block with one;
   - Replace all diagonal elements with zero.

Afterwards a new matrix $E''$ is obtained, which is the adjacency matrix of the condensed digraph according to the following Theorem 3.2;
6. Layer the condensed digraph as follows: traverse the matrix $E''$ to seek out those vertices whose corresponding rows in the matrix $E''$ are all zero elements. Undoubtedly, these vertices belong to the first layer because of their zero in-degree. Remove these rows and their homologous lines, then we get a new matrix. Again seek out those rows and their homologous lines, then we get a new matrix. Again seek out those rows with all zero elements to acquire the vertices belong to the second layer and remove their rows and homologous lines. Repeat the procedure mentioned above to obtain the vertices belonging to the third layer, fourth layer and all the layers.

Remark 2. We now remark that the adjacency matrix $E$ of original digraph $G$ can be divided into $E' = (E'_{ij})_{m \times m}$ in step (4) after a regulatory ranking in step (3). An obvious fact is that the block matrix $E''_{ii}$ is precisely the adjacency matrix of the
Theorem 3.2. After dividing the matrix $E'$ into $E' = (E'_{ij})_{m \times m}$ according to the step (4), we define $e_{ii} = 0$ and for $i \neq j$,
\[
e_{ij} = \begin{cases} 0, & \text{if all the elements of } E_{ij} \text{ are 0}, \\ 1, & \text{otherwise}, \end{cases}
\]
then $E'' = (e_{ij})_{m \times m}$ is the adjacency matrix of the corresponding condensed digraph.

Proof. Here we consider the case when $i \neq j$. On one hand, if not all the elements of $E'_{ij}$ are 0, it follows from $i \neq j$ that for some vertex $u \in V(H_j)$, there exists a vertex $v \in V(H_i)$ such that $u \preceq v$. This implies $H_j \prec H_i$ and then $e_{ij} = 1$ in $E''$. On the other hand, as for the case when all the elements of $E'_{ij}$ are 0, it is required to prove $e_{ij} = 0$ in $E''$. Suppose that $e_{ij} = 1$, then we can show $H_j \prec H_i$. It means that there are vertex $u \in V(H_j)$ and vertex $v \in V(H_i)$ satisfying $u \preceq v$, which is in contradiction to the fact that all the elements of $E'_{ij}$ are 0. Thus $e_{ij} = 0$ in this case. This proves Theorem 3.2. \qed

Remark 3. The operation object of the hierarchical algorithm merely the adjacency matrix of the given digraph $\mathcal{G}$. This provides us with great possibility to apply this algorithm to some mathematical software. Actually, we have implemented this algorithm through the corresponding Matlab program. With the help of the program, the hierarchical algorithm presents a rather effective approach to layer a general digraph and can overcome the difficulty resulting from the scale of networks.
There are positive constants \( \sigma_k, \eta_k, \tilde{a}_{kh} \) and function \( P_k(e_k) \) such that

\[
\frac{dV_k(e_k(t), t)}{dt} \leq \eta_k |e_k(t - \tilde{\tau}(t))|^p - \sigma_k |e_k(t)|^p + \sum_{h \in V(\mathcal{H}^i)} \tilde{a}_{kh} \left[ P_h(e_h(t)) - P_k(e_k(t)) \right].
\]

Based on the hierarchical method provided in Section 3, we can now realize OES for network \((G, A)\) failing to be strongly connected by means of combining graph theory, Lyapunov method and the theory of asymptotically autonomous systems [22, 28], and arrive at the following insightful result.

**Theorem 4.2.** Suppose that \( \eta_k < \sigma_k(1 - \tilde{\tau}) \) for all \( k \in \mathbb{L} \). Then OES between drive-response systems (1) and (2) can be realized if error system (3) admits a set of vertex-Lyapunov functions \( \{V_k(e_k, t)\} \) on each strongly connected component \( \mathcal{H}^i \).

**Proof.** As a special component of network \((G, A)\), the strongly connected component \( \mathcal{H}^i \) on first layer of network \((G, A)\) remains unaffected by others in the interactional network. The \( \mathcal{H}^i \)-subsystem of error system (3) is equivalent to the reduced \( \mathcal{H}^i \)-subsystem of error system (3). This property makes the first layer become particularly relevant in OES analysis for network \((G, A)\). Thus we first investigate how to achieve OES between drive-response systems (1) and (2) on the first layer for convenience. And we divide the proof into two steps for ease of exposition.

**Step 1.** OES analysis on the first layer of network \((G, A)\).

As before, we define the synchronization error status of the strongly connected component \( \mathcal{H}^i \) as \( e_{ij} \). Given the fact that system (3) possesses a set of vertex-Lyapunov functions \( \{V_k(e_k, t)\} \) on any \( \mathcal{H}^i \), we set

\[
V_{ij}(e_{ij}, t) = \sum_{k \in V(\mathcal{H}^i)} c^{(ij)}_k V_k(e_k, t),
\]

as the Lyapunov function of the drive-response systems on \( \mathcal{H}^i \). Here \( c^{(ij)}_k \) denotes the cofactor of the \( k \)-th diagonal element of Laplacian matrix of \((\mathcal{H}^i, \mathcal{A}^i)\), where \( \mathcal{A}^i = (\tilde{a}_{kh})_{k, h \in V(\mathcal{H}^i)} \). Noting that \( \mathcal{H}^i \) is strongly connected, it follows from Lemma 2.1 that \( c^{(ij)}_k > 0 \) for all \( k \in V(\mathcal{H}^i) \). Then the definition of the set of vertex-Lyapunov functions \( \{V_k(e_k, t)\} \) for \( \mathcal{H}^i \)-subsystem implies that

\[
V_{ij}(e_{ij}, t) = \sum_{k \in V(\mathcal{H}^i)} c^{(ij)}_k V_k(e_k, t) \leq \sum_{k \in V(\mathcal{H}^i)} c^{(ij)}_k \beta_k |e_k|^p \leq \left[ \sum_{k \in V(\mathcal{H}^i)} c^{(ij)}_k \beta_k \right] |e_{ij}|^p
\]

and

\[
V_{ij}(e_{ij}, t) \geq \sum_{k \in V(\mathcal{H}^i)} c^{(ij)}_k \alpha_k |e_k|^p = \sum_{l \in V(\mathcal{H}^i)} c^{(ij)}_l \alpha_l \sum_{k \in V(\mathcal{H}^i)} \frac{c^{(ij)}_k \alpha_k}{\sum_{h \in V(\mathcal{H}^i)} c^{(ij)}_h \alpha_h} \left( |e_k|^2 \right)^{\frac{p}{2}}
\]
direct application of Lemma 2.1 yields

\[ Q, \beta \] and \( \alpha \)

For simplicity, we denote \( e \), then we have

\[ \alpha_{1j} |e_{1j}| \leq V_{1j} (e_{1j}, t) \leq \beta_{1j} |e_{1j}|. \quad (4) \]

We now make direct calculations as follows,

\[
\frac{dV_{1j}(e_{1j}(t), t)}{dt} \leq - \sum_{k \in V(\mathcal{H}_{1j})} c_k^{(1j)} \sigma_k |e_k(t)|^p + \sum_{k \in V(\mathcal{H}_{1j})} c_k^{(1j)} \eta_k |e_k(t - \tau_k(t))|^p
\]

Then define \( Q_{1j} \) as the set of all spanning unicyclic graphs of \( (\mathcal{H}_{1j}, \hat{A}_{1j}) \), and a direct application of Lemma 2.1 yields

\[
\sum_{k, h \in V(\mathcal{H}_{1j})} c_k^{(1j)} \tilde{a}_{kh} \left[ P_h(e_h) - P_k(e_k) \right] = \sum_{Q \in Q_{1j}} W(Q) \sum_{(s, r) \in E(Q)} \left[ P_r(e_r) - P_s(e_s) \right].
\]

In view of that \( \sum_{(s, r) \in E(Q)} \left[ P_r(e_r) - P_s(e_s) \right] = 0 \), we can show readily that

\[
\frac{dV_{1j}(e_{1j}(t), t)}{dt} \leq - \sum_{k \in V(\mathcal{H}_{1j})} c_k^{(1j)} \sigma_k |e_k(t)|^p + \sum_{k \in V(\mathcal{H}_{1j})} c_k^{(1j)} \eta_k |e_k(t - \tau_k(t))|^p.
\]

On the other hand, in consideration of \( \eta_k < \sigma_k (1 - \tau) \), we can certainly choose a sufficiently small \( \gamma_{1j} > 0 \), such that

\[
\frac{\eta_k e^{\gamma_{1j} t}}{1 - \tau} + \gamma_{1j} \beta_k - \sigma_k < 0.
\]

We therefore estimate \( e^{\gamma_{1j} t} V_{1j}(e(t), t) \) for any \( t \geq 0 \) as follows,

\[
e^{\gamma_{1j} t} V_{1j}(e_{1j}(t), t)
\]

\[
\leq V_{1j}(\phi_{1j}(0), 0) + \int_0^t e^{\gamma_{1j} s} \left[ \sum_{k \in V(\mathcal{H}_{1j})} c_k^{(1j)} \eta_k |e_k(s - \tau_k(s))|^p \right] ds
\]

\[
+ \int_0^t e^{\gamma_{1j} s} \left[ - \sum_{k \in V(\mathcal{H}_{1j})} c_k^{(1j)} \sigma_k |e_k(s)|^p + \gamma_{1j} \sum_{k \in V(\mathcal{H}_{1j})} c_k^{(1j)} V_k(e_k(s), s) \right] ds
\]
where \( \phi \) is the initial value of the strongly connected component \( \mathcal{S}_{1j} \). Applying inequality (4) yields the desired assertion,
\[
e^{\gamma t} \alpha |e_{1j}(t)|^p \leq e^{\gamma t} V_j(e_{1j}, t)
\]
\[
\leq \left[ e^{\gamma t} \frac{-1}{\gamma_{1j}(1 - \bar{\tau})} \sum_{k \in \mathcal{V}(\mathcal{S}_{1j})} c_k^{(1)} \eta_k + \beta_{1j} \right] |\phi_{1j}|^p,
\]
where \( L := \max_{i,j} \left\{ \frac{e^{\gamma t} - 1}{\gamma_{1j}(1 - \bar{\tau})} \sum_{k \in \mathcal{V}(\mathcal{S}_{1j})} c_k^{(1)} \eta_k + \beta_{1j} \right\}, \gamma := \min_{i,j} \gamma_{ij} > 0 \) and \( \alpha := \min_{i,j} \alpha_{ij} > 0 \). After a simple manipulation, we have
\[
|e_{1j}(t)| \leq (L/\alpha)^{\frac{1}{p}} |\phi_{1j}| e^{-\frac{\gamma t}{\bar{\tau}}}.
\]
Recalling Definition 2.2, we can conclude that systems (1) and (2) on \( \mathcal{S}_{1j} \) have achieved OES.

**Step 2.** OES analysis on other layers of network \((G, A)\).

We shall show that other layers of network \((G, A)\) are also outer exponentially synchronized after OES on the first layer has been realized. To see this, it is illuminating to discuss OES on any strongly connected component \( \mathcal{S}_{2j} \). Since OES on the first layer of network \((G, A)\) has been realized, \( \mathcal{S}_{1j} \)-subsystem has an equilibrium \( e_{1j}^* = 0 \) that attracts all solutions. Allowing for \( \mathcal{S}_{2j} \) is affected only by strongly connected components in the first layer, we can write \( \mathcal{S}_{2j} \)-subsystem as
\[
e'_k(t) = \left\{ f_k \left[ y_k(t), y_k(t - \tau_k(t)) \right] - f_k \left[ x_k(t), x_k(t - \tau_k(t)) \right] \right\} + g_k(e_k(t))
\]
\[
+ \sum_{h \in \mathcal{V}(\mathcal{S}_{2j})} a_{kh} \left[ H_{kh}(y_h(t) - H_{kh}(x_h(t)) \right]
\]
\[
+ \sum_{h \in \mathcal{V}(\mathcal{S}_{1j})} a_{kh} \left[ H_{kh}(y_h(t) - H_{kh}(x_h(t)) \right], k \in \mathcal{V}(\mathcal{S}_{2j}).
\]

The theory of asymptotically autonomous systems [22, 28] asserts that the asymptotic behaviors of \( \mathcal{S}_{2j} \)-subsystem are the same as those of the corresponding limit
system. The continuity of $H_{kh}$ implies that the limit system is precisely equivalent to the reduced $H_{2j}$-subsystem. We now turn to consider the reduced $H_{2j}$-subsystem for realizing OES on $H_{2j}$. Same with $H_{1j}$-subsystem, the reduced $H_{2j}$-subsystem is also immune to the dynamic behaviors of other layers. Thus we are allowed to realize OES on $H_{2j}$ analogous to Step 1, and then $H_{2j}$-subsystem has an equilibrium $e_{2j}^* = 0$ attracting all solutions.

When we turn to consider OES on $H_{3j}$, the same argument shows that it suffices to research the reduced $H_{3j}$-subsystem of error system (3) and thus OES on $H_{3j}$ can be realized similarly. We can just keep repeating the above process to achieve OES on any strongly connected component $H_{ij}$, which completes this proof.

**Remark 4.** As mentioned in Section 1, the difficulty to investigate the dynamics of CSNs results from their complex structure. It remains poorly understood how the dynamical behavior depends on the network architecture since the technical difficulties in the study of CSNs. In the previous research results [8, 9, 15, 17, 18, 26, 33], we assumed that the underlying network is strongly connected when we tried to use graph theory and Lyapunov method to explore the dynamics of CSNs, but we were helpless in the face of general NWSC. This paper just makes up for the inadequacy of this aspect by providing a hierarchical method to study the dynamics of coupled systems on general networks. Theorem 4.2 suggests that the assumption of a strongly connected network is dispensable. This is an attempt to analyze the dynamics of coupled systems on NWSC and perform the characteristics of the network by the hierarchical method. Moreover, we can treat a strongly connected network as a special case of a general network, where this network has only one strongly connected component and the corresponding condensed digraph is a singleton. In this sense, the obtained result in Theorem 4.2 presents generality and improves that in above literatures.

**Remark 5.** It’s well-known that constructing appropriate Lyapunov functions is a quite difficult task. In this paper, in term of digraph $G$, we treat every strongly connected component $H_{ij}$ as a vertex of the corresponding condensed digraph $H$. And then the strongly connected property of $H_{ij}$ allows us to construct a Lyapunov function $V_{ij}$ for $H_{ij}$-subsystem of error system (3). More precisely, The results in [17] suggests that $V_{ij}$ can be constructed with the help of the sets of vertex-Lyapunov functions $\{V_k(e_k, t)\}$ on $H_{ij}$ and the topology structure of networks. That is, $V_{ij}(e, t) = \sum_{k \in V(H_{ij})} c_k^{(ij)} V_k(e_k, t)$, where $c_k^{(ij)}$ is the cofactor of the $k$-th diagonal element of Laplacian matrix of $\tilde{A}_{ij}$. By means of every Lyapunov function $V_{ij}$ and the theory of asymptotically autonomous systems [22, 28], we can synchronize drive-response systems (1) and (2).

**Remark 6.** We can now answer the proposed question as to what role the first layer of network plays in synchronizing systems (1) and (2). Indeed it is shown from the proof of Theorem 4.2 that the drive-response systems on the first layer can realize OES if the conditions in Theorem 4.2 are satisfied, regardless of strongly connected components in other layers. The interesting fact is determined by the property that strongly connected components in the first layer remain unaffected by others in networks. However, the dynamics of the other layers will be affected by the first one seriously. Naturally, one can ask whether the derive-response systems on other layers can be synchronized even if the first one fails to achieve OES? In fact, we can synchronize some parts of the network in this case and an exhaustive
explanation is shown in the following corollary. This proof is similar to Step 2 of the proof in Theorem 4.2, and therefore is omitted.

**Corollary 1.** Define the set $\Pi$ as all strongly connected components $\mathcal{H}_{kh}$ on which derive-response systems (1) and (2) can achieve OES. Then the derive-response systems on $\mathcal{H}_{ij}$ can arrive at OES if they satisfy the conditions in Theorem 4.2 and

$$\{\mathcal{H}_{rs}|\mathcal{H}_{rs} \prec \mathcal{H}_{ij}, r, s \in \mathbb{L}\} \subseteq \Pi.$$ 

Theorem 4.2 has provided a criterion to guarantee OES between systems (1) and (2) but is somewhat inconvenient in applications. This is not only because the conditions in Theorem 4.2 are not related to the coefficients $f_k$ and $H_{kh}$ explicitly but also because it appears to be very difficult to construct proper vertex Lyapunov functions sets for error system (3). It is in this spirit that we would like to explore the possibility of using the coefficients of error system (3) to determine sufficient conditions for OES between the derive-response systems.

### 4.2. Coefficients-type criterion

The main goal of this subsection is to form another sufficient OES criterion for systems (1) and (2) using to coefficients of error system (3). For this purpose, we first impose the following additional hypothesis in terms of $f_k$ in error system (3).

**Assumption 1.** For any $k, h \in \mathbb{L}$, there are positive constants $\delta_k$ and $\mu_k$ such that

$$e_k^T [f_k(y_k, \hat{y}_k) - f_k(x_k, \hat{x}_k)] \leq -\delta_k |e_k|^2 + \mu_k |e_k| |\hat{y}_k - \hat{x}_k|.$$ 

Under this standing assumption, another OES criterion is stated as follows.

**Theorem 4.3.** Under Assumption 1, OES between drive-response systems (1) and (2) on the strongly connected component $\mathcal{H}_{ij}$ can be achieved if

$$\mu_k < \sigma_k (1 - \tau),$$ 

where $\sigma_k = p(\delta_k - \theta_k) - (p - 1) \mu_k - p \sum_{h \in V(\mathcal{H}_{ij})} a_{kh} B_{kh}$ with some constant $p \geq 2$ and the Lipschitz constants $\theta_k$, $B_{kh}$ corresponding to functions $g_k$ and $H_{kh}$, respectively.

**Proof.** Choose the vertex Lyapunov functions set $\{V_k(e_k(t)|k \in V(\mathcal{H}_{ij})\}$ on $\mathcal{H}_{ij}$ described by

$$V_k(e_k(t)) = |e_k|^p.$$ 

Obviously, the condition (A1) in Definition 4.1 is valid. Taking the time derivative of $V_k(e_k(t), t)$ along the trajectories of error system (3) and applying the Lipschitz continuity of $g_k$ and $H_{kh}$ as well as Assumption 1 show that

$$\frac{dV_k(e_k(t), t)}{dt} = p|e_k(t)|^{p-2} e_k^T(t) [f_k(y_k(t), y_k(t - \tau_k(t))) - f_k(x_k(t), x_k(t - \tau_k(t))) + p|e_k(t)|^{p-2} e_k^T(t) \sum_{h \in V(\mathcal{H}_{ij})} a_{kh} [H_{kh}(y_h) - H_{kh}(x_h)]$$

$$\leq - p(\delta_k - \theta_k)|e_k(t)|^p + p\mu_k|e_k(t)|^{p-1}|e_k(t - \tau_k(t))|$$

$$+ p \sum_{h \in V(\mathcal{H}_{ij})} a_{kh} B_{kh}|e_k(t)|^{p-1}|e_h(t)|.$$ 

(5)
Young’s inequality yields that
\[ |e_k(t)|^{p-1} |\xi(t)| \leq \frac{p-1}{p} |e_k(t)|^p + \frac{1}{p} |\xi(t)|^p, \] (6)
Choosing \( \xi(t) = e_k(t - \tau_k(t)) \) or \( \xi(t) = e_k(t) \) in (6) and substituting them into inequality (5) produces the following inequality chain,
\[
\frac{dV_k(e_k(t), t)}{dt} \leq \left[-p(\delta_k - \theta_k) + (p-1)\mu_k \right]|e_k(t)|^p + \mu_k|e_k(t - \tau_k(t))|^p + \sum_{h \in V(\Omega_{ij})} a_{kh} B_{kh} |e_h(t)|^p \\
+ (p-1) \sum_{h \in V(\Omega_{ij})} a_{kh} B_{kh} |e_k(t)|^p \\
= \left[-p(\delta_k - \theta_k) + (p-1)\mu_k + (p-1) \sum_{h \in V(\Omega_{ij})} a_{kh} B_{kh} \right]|e_k(t)|^p + \mu_k|e_k(t - \tau_k(t))|^p \\
+ \sum_{h \in V(\Omega_{ij})} a_{kh} B_{kh} |e_k(t)|^p + \sum_{h \in V(\Omega_{ij})} a_{kh} B_{kh} \left[|e_h(t)|^p - |e_k(t)|^p \right] \\
= - \left[p(\delta_k - \theta_k) - (p-1)\mu_k - p \sum_{h \in V(\Omega_{ij})} \tilde{a}_{kh} \right]|e_k(t)|^p + \mu_k|e_k(t - \tau_k(t))|^p \\
+ \sum_{h \in V(\Omega_{ij})} \tilde{a}_{kh} \left[P_h(e_h(t)) - P_k(e_k(t)) \right],
\]
where \( \tilde{a}_{kh} = a_{kh} B_{kh} \geq 0 \) and \( P_k(e_k) = |e_k|^p \). We write \( \sigma_k = [p(\delta_k - \theta_k) - (p-1)\mu_k - p \sum_{h \in V(\Omega_{ij})} a_{kh} B_{kh}] \) and \( \eta_k = \mu_k \), which are all positive constants. This implies the conditions (A1) and (A2) in Definition 4.1 are fulfilled. Thus error system (3) possesses a vertex Lyapunov functions set on \( \Omega_{ij} \). It follows from the given conditions in Theorem 4.3 that \( \eta_k < \sigma_k(1 - \tau) \) and all conditions in Theorem 4.2 are satisfied. Therefore, OES between systems (1) and (2) can be achieved according to Theorem 4.2. The proof is completed. \( \square \)

5. An application to delayed coupled oscillators on NWSC. Arising from problems in applied science, coupled oscillators with time-varying delays have been extensively studied, with consequent emphasis on their widespread applications describing a wide variety of physical and biological phenomena [32, 13, 23]. To embody the validity of our results above, in this section, we consider OES of delayed coupled oscillators on a general network. Specially, this coupled system is modeled in a digraph \( G \) with \( l \) (\( l \geq 2 \)) vertices, without strongly connectedness. For the \( k \)-th vertex it is assigned a time-varying delay oscillator described by
\[
x_k'(t) + \epsilon_k x_k^2(t) + x_k(t) + \varepsilon_k x_k(t - \tau_k(t)) = Q(t),
\]
where \( \epsilon_k \geq 0 \) is damping coefficient and \( \varepsilon_k \geq 0 \). We assume that the intensity and form of influence from vertex \( h \) to vertex \( k \) are described by \( b_{kh} \) and \( N_h : \mathbb{R} \to \mathbb{R} \), where \( b_{kh} \geq 0 \), and \( b_{kh} = 0 \) if and only if there exists no arc from \( h \) to \( k \) in \( G \). By introducing \( \tilde{x}_k(t) = x_k'(t) + \gamma_k x_k(t), \gamma_k > 0 \), we can get the \( k \)-th system:
Assume that Assumption 2 holds and Theorem 5.1. and (8), an easily verifiable OES result is now stated as follows.

Similarly, the corresponding response system can be written as

\[
\begin{align*}
\dot{y}_k(t) &= \gamma_k y_k(t) + \varphi_k(y_k(t) - x_k(t)), \\
\dot{\gamma}_k(t) &= (-\epsilon_k + \gamma_k)\gamma_k(t) + (\epsilon_k\gamma_k - \gamma^2_k - 1)y_k(t) + Q(t) - \epsilon_k y_k(t - \tau_k(t)) \\
&- \sum_{h=1}^l b_{kh} N_h(y_h(t)).
\end{align*}
\]

An additional assumption to ensure the existence and uniqueness of zero solution to drive-response systems is shown as

**Assumption 2.** For any \(k, h \in \mathbb{L}\), functions \(\varphi_k\) and \(N_h\) satisfy global Lipschitz condition and \(\varphi_k(0) = 0\). That is, there are positive constants \(\hat{B}_h\) and \(\hat{\theta}_k\), such that

\[
|\varphi_k(x_k)| \leq \hat{\theta}_k |x_k|, \quad |N_h(y_h) - N_h(x_h)| \leq \hat{B}_h |y_h - x_h|.
\]

Next, we denote that \(X_k = [x_k, \tilde{x}_k]^T\), \(\bar{X}_k = [\tilde{x}_k, \tilde{x}_k]^T\), \(G_k(X_k) = [0, -N_h(x_k)]^T\), \(\Psi_k(t) = [0, Q(t)]^T\) and \(F_k(X_k, \bar{X}_k) = H_k(X_k) - I_k(\bar{X}_k)\), where \(H_k(X_k) = [\tilde{x}_k - \gamma_k x_k, (-\epsilon_k + \gamma_k)\tilde{x}_k + (\epsilon_k\gamma_k - \gamma^2_k - 1)x_k]^T\), \(I_k(\bar{X}_k) = [0, \varphi_k(\tilde{x}_k)]^T\). Then we rewrite the derive system as

\[
X_k(t) = F_k\left(X_k(t), X_k(t - \tau_k(t))\right) + \Psi_k(t) + \sum_{h=1}^l b_{kh} G_h\left(X_h(t)\right), \quad k \in \mathbb{L}. \tag{7}
\]

Similarly, the corresponding response system can be written as

\[
Y'_k(t) = F_k\left(Y_k(t), Y_k(t - \tau_k(t))\right) + \Psi_k(t) + \Phi_k\left(Y_k(t) - X_k(t)\right) + \sum_{h=1}^l b_{kh} G_h\left(Y_h(t)\right), \quad k \in \mathbb{L}, \tag{8}
\]

where \(Y_k = [y_k, \tilde{y}_k]^T\) and \(\Phi_k(X_k) = [\varphi_k(x_k), 0]^T\). For drive-response systems (7) and (8), an easily verifiable OES result is now stated as follows.

**Theorem 5.1.** Assume that Assumption 2 holds and \(\gamma_k(\epsilon_k - \gamma_k) \leq 1, \epsilon_k \leq 2\gamma_k < 2\epsilon_k\) for all \(k \in \mathbb{L}\). Then the drive-response systems (7) and (8) can achieve OES on \(\mathcal{S}_{ij}\) if there exists a constant \(p \geq 2\) such that

\[
\epsilon_k < \sigma_k(1 - \tau).
\]

where \(\sigma_k = p\left(\frac{2}{\tau} - \hat{\theta}_k\right) - (p - 1)\epsilon_k - p \sum_{h \in \mathcal{V}(\mathcal{S}_{ij})} b_{kh}\hat{B}_h\).

**Proof.** Let \(E_k = Y_k - X_k = [y_k - x_k, \tilde{y}_k - \tilde{x}_k]^T := [\epsilon_k, \tilde{e}_k]^T\). Then the error system of the drive-response systems (7) and (8) can be written as

\[
E'_k(t) = F_k\left(Y_k(t), Y_k(t - \tau_k(t))\right) - F_k\left(X_k(t), X_k(t - \tau_k(t))\right) + \Phi_k\left(E_k(t)\right) + \sum_{h=1}^l b_{kh} (G_h(Y_h(t)) - G_h(X_h(t))).
\]
Assumption 2 implies that the error system possesses a unique zero solution. For arbitrary \( \bar{X}_k = (\bar{x}_k, \bar{x}_k)^T \) and \( \bar{Y}_k = (\bar{y}_k, \bar{y}_k)^T \), we can get from \( \gamma_k(\epsilon_k - \gamma_k) \leq 1 \) and \( \epsilon_k \leq 2\gamma_k < 2\epsilon_k \) that

\[
E_k^T \left[ F_k(Y_k, \bar{Y}_k) - F_k(X_k, \bar{X}_k) \right] \\
= E_k^T \left[ H_k(Y_k) - H_k(X_k) \right] - E_k^T \left[ I_k(\bar{Y}_k) - I_k(\bar{X}_k) \right] \\
\leq \epsilon_k \bar{e}_k + (2+\epsilon_k)\epsilon_k \bar{e}_k^2 + (\epsilon_k \gamma_k - \gamma_k^2) \bar{e}_k + E_k^T \left[ I_k(\bar{Y}_k) - I_k(\bar{X}_k) \right] \\
\leq \frac{\gamma_k^2 \epsilon_k - \epsilon_k \gamma_k - \gamma_k^2}{2} \bar{e}_k^2 + \epsilon_k + \epsilon_k E_k |\bar{y}_k - \bar{x}_k| \\
= \frac{\gamma_k}{2} |E_k|^2 + \epsilon_k |\bar{y}_k - \bar{x}_k|,
\]

where \( \epsilon^2 = \epsilon_k - \gamma_k > 0 \). Thus Assumption 1 is satisfied. In view of Assumption 2, it can be verified straightforwardly that

\[
|G_h(Y_h(t)) - G_h(\hat{x}_h(t))| = |N_h(yh) - N_h(xh)| \leq \hat{B}_h |yh - xh| \leq \hat{B}_h |E_h|,
\]

and

\[
|\Phi_k(X_h)| = |\varphi_k(xh)| \leq \hat{\theta}_h |xh| \leq \hat{\theta}_h |X_h|.
\]

Recalling the definition of \( \sigma_k \) in Theorem 5.1, all conditions in Theorem 4.3 are fulfilled in view of \( \epsilon_k < \sigma_k(1 - \bar{\tau}) \). Theorem 4.3 thus yields that drive-response systems (7) and (8) can achieve OES.

**Remark 7.** The stability and periodicity of the coupled oscillators have been investigated in [17] and [8], where a necessary assumption is that the network that the coupled oscillators are established on is strongly connected. This assumption is removed in Theorem 5.1 by means of the hierarchical method introduced in Section 3. Aware of this issue, the method seems to be practicable to study their stability and periodicity on NWSC, which is our further research orientation.

6. **Numerical simulations.** In this section, we present a numerical example to confirm the main results proposed in Section 4. In the simulation, the Euler numerical scheme is used to simulate by Matlab. We proceed to consider delayed coupled oscillators system (7) as a drive system, where we assume that \( Q(t) = 100 \cos(t) \) and \( \varphi_k(xh) = 0.1xh \) for \( k \in \{1, 2, \ldots, 20\} \).

Specially, we choose \( p = 2 \), \( \tau = 1 \), \( \bar{\tau} = 0.5 \) and \( \hat{B}_h = 1 \) for all \( h \in \{1, 2, \ldots, 20\} \), then we set the forms of time delays \( \tau_h(t) \) and coupling terms \( N_h(xh) \) as

| Forms | \( k \in \{1, \ldots, 5\} \) | \( k \in \{6, \ldots, 10\} \) | \( k \in \{11, \ldots, 15\} \) | \( k \in \{16, \ldots, 20\} \) |
|-------|-----------------|-----------------|-----------------|-----------------|
| \( \tau_h(t) \) | \( 0.5(\sin(t) + 1) \) | \( 0.5(\cos(t) + 1) \) | \( 0.4(\sin(t) + 1) \) | \( 0.4(\cos(t) + 1) \) |
| \( N_h(xh) \) | \( \sin(xh) \) | \( \cos(xh) \) | \( 0.5 \sin(2xh) \) | \( 0.5 \cos(2xh) \) |

Naturally, it can be verified that \( \tau_h(t) \in [0, \tau] \) and \( |N_h(yh) - N_h(xh)| \leq \hat{B}_h |yh - xh| \) for all \( h \in \{1, 2, \ldots, 20\} \), \( \bar{\tau}_h(t) \leq 0.5 \) for \( k \in \{1, 2, \ldots, 10\} \) and \( \bar{\tau}_h(t) \leq 0.4 \) for \( k \in \{11, 12, \ldots, 20\} \). Thus the derivatives of \( \tau_h(t) \) are bounded by a constant \( \bar{\tau} \in (0, 1) \) and Assumption 2 is satisfied. We further define \( \gamma_k = 1.6 \) for all \( k \in \).
\{1,2,\ldots,20\} \text{ and write the parameters } \epsilon_k, \varepsilon_k \text{ as}

| Parameters \ k | \epsilon_k | \varepsilon_k |
|---------------|-----------|-----------|
| \{1,\ldots,5\} | 1.8      | 0.05      |
| \{6,\ldots,10\}    | 1.9      | 0.1       |
| \{11,\ldots,15\}   | 2.0      | 0.15      |
| \{16,\ldots,20\}   | 2.1      | 0.2       |

Simple calculations implies that \(\gamma_k(\epsilon_k - \gamma_k) \leq 1\) and \(\epsilon_k \leq 2\gamma_k < 2\epsilon_k\) for all \(k \in \{1,2,\ldots,20\}\). Corresponding to this drive system, we define system (8) with the same parameters as a response system. Analogous to the process in Section 5, the \(k\)-th error system can be calculated as

\[
\begin{align*}
\hat{e}'_k(t) &= \hat{e}_k - \gamma_k \hat{e}_k(t) + \varphi_k(e_k(t)), \\
\hat{e}''_k(t) &= (-\epsilon_k + \gamma_k)\hat{e}_k(t) + (\epsilon_k \gamma_k - \gamma_k^2 - 1)e_k(t) - \varepsilon_k e_k(t - \tau_k(t)) \\
&\quad - \sum_{h=1}^{20} b_{kh} \left[ N_h(y_h(t)) - N_h(x_h(t)) \right].
\end{align*}
\]

Then we model the drive-response systems on two identical networks \((G, B)\) described in Fig.3, where \(B = (b_{ij})_{20 \times 20}\) and \(b_{ij} = 0\) except for the following values:

\[
\begin{align*}
b_{21} &= b_{56} = b_{9,10} = b_{14,13} = b_{18,17} = b_{3,9} = b_{4,14} = 0.001 \\
b_{32} &= b_{6,7} = b_{10,11} = b_{15,14} = b_{19,18} = b_{4,12} = b_{7,18} = 0.002 \\
b_{43} &= b_{7,8} = b_{11,12} = b_{16,15} = b_{20,19} = b_{8,10} = b_{12,13} = 0.003 \\
b_{14} &= b_{85} = b_{12,9} = b_{13,16} = b_{17,20} = b_{7,11} = b_{11,17} = 0.004.
\end{align*}
\]

Figure 3. Digraph \((G, B)\) with 20 vertices and a not strongly connected network.

As shown in Fig.3, the network \((G, B)\) is not strongly connected. Allowing for the hierarchical method, we can conclude that strongly connected components enclosed by the green boxes belong to the first layer, those enclosed by the black box belong to the second one and by the red box belong to the third one. Digraph \(G\) is divided into three layers. They are \(\delta_1 = \{\delta_{11}, \delta_{12}\}\), \(\delta_2 = \{\delta_{21}\}\) and \(\delta_3 = \{\delta_{31}, \delta_{32}\}\), where \(\delta_i \ (i = 1,2,3)\) represents the \(i\)-th layer and \(\delta_{11}, \delta_{12}, \delta_{21}, \delta_{31}, \delta_{32}\) are all strongly...
connected components of digraph $\mathcal{G}$. Afterwards, if we define $\sigma_k = p(\gamma_k^2 - 0.1) - (p - 1) \varepsilon_k - p \sum_{h \in V(\mathcal{G}_{ij})} b_{kh} \tilde{B}_h$ when $k \in V(\mathcal{G}_{ij})$, then it follows that $\varepsilon_k < \sigma_k(1 - \bar{\tau})$, which signifies that all conditions in Theorem 5.1 are satisfied. And thus the drive-response systems can realize OES. The corresponding simulation results for drive system, response system and synchronization error system are shown in Figs. 4-6, respectively. Fig. 6 suggests that the drive-response systems can achieve OES even if network $(\mathcal{G}, B)$ fails to be strongly connected. The computer simulations of the drive-response systems agree well with the proposed theory.

**Figure 4.** The pathes of the derive system (7) with initial value $(x_{1+5i} = 50, \tilde{x}_{1+5i} = -50, x_{2+5i} = 60, \tilde{x}_{2+5i} = -60, x_{3+5i} = 0, \tilde{x}_{3+5i} = 0, x_{4+5i} = 60, \tilde{x}_{4+5i} = -60, x_{5+5i} = 70, \tilde{x}_{5+5i} = -70, i = 0, 1, 2, 3)$.

**Figure 5.** The pathes of response system (8) with initial value $(y_{1+5i} = 0, \tilde{y}_{1+5i} = -170, y_{2+5i} = -70, \tilde{y}_{2+5i} = 30, y_{3+5i} = 0, \tilde{y}_{3+5i} = 0, y_{4+5i} = -30, \tilde{y}_{4+5i} = 30, y_{5+5i} = 40, \tilde{y}_{5+5i} = 20, i = 0, 1, 2, 3)$.

**Appendix.** In this appendix, in order to visually display the hierarchical algorithm presented in Subsection 3.2 for the reader, the digraph $\mathcal{G}$ with ten vertices in Fig. 7 is taken for example. From Fig. 7(a), one can see that the vertices enclosed by a box belong to the same strong connected component. Then we collapse each strong connected component into a single vertex and obtain the corresponding condensed digraph $H$ shown in Fig. 7(b). Recalling the hierarchical method presented in Section 3, we can find that the strong connected component $\mathcal{G}_i$ belongs to the $i$-th layer. For better comprehension, we shall display the algorithm step by step.
Figure 6. The paths of the synchronization error system with initial value \((e_{1+5i} = -50, \tilde{e}_{1+5i} = -120, e_{2+5i} = -130, \tilde{e}_{2+5i} = 90, e_{3+5i} = 0, \tilde{e}_{3+5i} = 0, e_{4+5i} = -100, \tilde{e}_{4+5i} = 100, e_{5+5i} = -40, \tilde{e}_{5+5i} = 110, i = 0, 1, 2, 3)\).

Figure 7. Digraph \((G, A)\) and its strongly connected components \(H_i\) are shown in (a). The corresponding condensed digraph \(H\) is shown in (b).

Step (1). According to the hierarchical algorithm, we first write the adjacent matrix \(E\) of the original digraph \(G\) as

\[
E = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]
Step(2). With the help of the Tarjan’s algorithm [27], all strongly connected components can be obtained. Then $Array_{1}[10]$ and $Array_{2}[4]$ are both given here,

\[Array_{1}[10] = [3, 5, 4, 7, 6, 9, 10, 8], \quad Array_{2}[4] = [3, 2, 2, 3].\]

Step(3). Alter the matrix $E$ according to $Array_{1}[10]$. Change the position of each row such that the order of each row is in accordance with that of each vertex in $Array_{1}[10]$ and so did each column. We then acquire a new matrix $E'$ as

\[
E = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{pmatrix}.
\]

Step(4). Divide the matrix $E'$ according to $Array_{2}[4]$. We can use the function `mat2cell` in Matlab for instance to achieve it and corresponding parameters are $(E', Array_{2}[4], Array_{2}[4])$. Then we have $E' = (E'_{ij})_{4 \times 4}$, where

\[
E'_{11} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad E'_{22} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad E'_{33} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad E'_{44} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},
\]

represent the adjacency matrix of strong connected components $\mathcal{H}_{1}$, $\mathcal{H}_{2}$, $\mathcal{H}_{3}$ and $\mathcal{H}_{4}$, respectively.

Step(5). Each matrix block is dealt through the rule described in Step (5) of hierarchical algorithm. We can acquire a new matrix $E''$,

\[
E'' = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\end{pmatrix},
\]

which is exactly the adjacent matrix of the condensed digraph according $\mathcal{H}$ to the Theorem 3.2.

Step(6). Layer the condensed digraph $\mathcal{H}$. The elements in the 1-th row are all zero and thus $\mathcal{H}_{1}$ belongs to the first layer. Remove the 1-th row and 1-th column from matrix $E''$ to seek out the 2-th row whose elements are all zero, which implies $\mathcal{H}_{2}$ belongs to the second layer. Proceeding similarly to the rest vertices finds that the third layer contains $\mathcal{H}_{3}$ and the fourth one contains $\mathcal{H}_{4}$. Namely, the strong connected component $\mathcal{H}_{i}$ belongs to the $i$-th layer ($i = 1, 2, 3, 4$), which asserts the correctness of hierarchical algorithm.

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