ON RATIONAL INJECTIVITY OF KASPAROV’S ASSEMBLY MAP IN DIMENSION $\leq 2$

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To Joachim Cuntz, teacher and friend, on the occasion of his sixtieth birthday, and to Gennadi Kasparov, whose work is a great source of inspiration.

Abstract. In this article the author presents a new proof of injectivity of the composition of the inverse of the rational Chern Character in homology applied to the classifying space $BG$ of a (countable) discrete group $G$ and restricted to dimensions less or equal than two with the rationalized assembly map of Kasparov, (cf. [10])

$$H_{\leq 2}(BG; \mathbb{Q}) \xrightarrow{c^1_{\mathbb{Q}}^{-1}} RK_\ast(BG) \otimes \mathbb{Q} \xrightarrow{A} K_\ast(C^\ast(G)) \otimes \mathbb{Q}.$$

Here the lefthandside is the assemblage of its ordinary homology groups with rational coefficients of dimensions $\leq 2$, whereas the middle term is given by the direct limit of the analytic $K$-homology groups of Kasparov taken with respect to the compact subsets of $BG$ (ordered by inclusion) and tensored with the rationals, and the righthandside denotes the (operator) $K$-theory of the full group $C^\ast$-algebra of $G$, again tensored by the rational numbers.

Foreword. This article was intended to appear in the special edition of the Mathematical Journal of the University of Muenster, Westfalen, in honour of Joachim Cuntz, but at that time the main argument still contained some severe defects which could not be overcome easily, so the paper had to be withdrawn. Some time has elapsed since then and the author hopes that the arguments presented here are correct and understandable to the interested reader. Despite the efforts for clarity that have been made the author must admit, and regrets, that some of the references given are not easily available. This concerns the paper "KK-theory of stable projective limits" of which the first part-dealing with the covariant side of the KK-functor- has appeared in the preprint series of the IML at Luminy, Marseille. The second part dealing with K-homology only exists as a manuscript of the author, but anyway is not relevant for the purposes of this paper which only

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uses results on the covariant side. It is planned to submit the complete paper for publication in the near future. Similarly, the other reference by the author, the treatise on ”Regular algebraic K-theory for groups”, is unpublished up to this date. Only some very minor results of this paper are used – all of them dealing with low dimensions where the algebraic K-theory groups coincide with ordinary group homology – so that a standard textbook on Homological Algebra (cf. [7]) giving the basic definitions of group homology including exact sequences in low dimensions associated with group extensions might be helpful. In addition, the interested reader is encouraged to contact the author for copies of the articles in question.

0. Introduction

There is by now an overwhelming variety of special classes of groups for which the (integral) injectivity of the assembly map as above has been proved in full generality without any restrictions to certain dimensions and in many cases even stronger results are available (Baum-Connes type isomorphism theorems). On the other hand up to this date and knowledge of the author not a single counterexample has been constructed where the full integral injectivity of the assembly map (replacing $BG$ with the classifying space for proper actions in case of a group with torsion) fails. The proofs however all seem to depend on certain geometric properties of the group $G$ and it is questionable whether this method will eventually succeed in covering all (say finitely generated discrete) groups. The Novikov Conjecture on homotopy invariance of higher signatures only depends on rational injectivity of the assembly map which is known for general groups in low dimensions. The first step was taken by Novikov who observed the homotopy invariance of higher signatures associated with onedimensional cohomology classes (the zerodimensional case being trivial), followed by Kasparov who extended the result to arbitrary products of onedimensional classes. The twodimensional case appears in Mathai [11] and a paper of Hanke and Schick [6] where also the injectivity of the rational assembly map restricted to elements which are dual to the subring of cohomology generated by the one- and twodimensional classes is established. The present paper gives a new proof of a somewhat more modest result, namely rational injectivity for twodimensional homology classes (and duals of products of onedimensional cohomology classes). It should be pointed out that this is not sufficient to prove homotopy invariance of the corresponding higher signatures which would however follow from a dual result (rational surjectivity of the dual assembly map) which
is fairly easy to obtain in dimension one as sketched below, thereby confirming Kasparov’s result on such classes.

1. Outline of the argument

We begin by recalling some results of Appendix C of the paper [4] by the author. In this paper there is defined for any normal subgroup $N$ of a discrete group $G$ a sequence of abelian groups $\{K^J_n(N, G)\}_{n \geq 1}$ called the (relative) $K^J$-groups of the pair $(N, G)$ and behaving functorially with respect to (compatible) group homomorphisms. In particular one has for any group $G$ the absolute $K^J$-groups $K^J_n(G) := K^J_n(G, G)$. To any group extension

$$1 \longrightarrow N \longrightarrow G \longrightarrow H \longrightarrow 1$$

there is associated a long sequence of $K^J$-groups of the form

$$\cdots \longrightarrow K^J_{n+1}(H) \xrightarrow{\partial_n} K^J_n(N, G) \longrightarrow K^J_n(G) \longrightarrow K^J_n(H) \xrightarrow{\partial_{n-1}} \cdots$$

which is everywhere exact in certain cases (for example if $G$ and hence $H$ are perfect groups and $N$ is full, i.e. equal to the commutator subgroup $[[N, G]]$ generated by commutators $[x, y] = x y x^{-1} y^{-1}$ with $x \in N$, $y \in G$). For a general (normal sub)group the sequence is also everywhere exact except possibly at $K^J_3(N, G)$ and $K^J_3(G)$. There is a natural transformation to group homology which in case of dimensions one and two reduces to a canonical isomorphism. In particular one gets that $K^J_2(G) = H_2(G) = H_2(BG)$ is given by the Schur multiplier of the group $G$. The general theory and possible failure of exactness for general groups explained above is but of no importance to the present argument since the sequence will be exact in any case in dimensions one and two with which we are dealing here. What matters is that one constructs a canonical (mod 2-) degree-preserving map into $RK^*_c(BG)$ for each group $K^J_n(G)$. One easily finds that the image of $K^J_3(G)$ is rationally just the preimage of $H_2(BG)$ for the Chern Character. By construction this map commutes with the boundary maps in the following sense: given a group extension as above one obtains a six-term exact sequence in $RK^*$-homology which relates the corresponding groups for $BG$, $BH$ and the mapping cone $C_\varphi$ of the induced map $\varphi : BG \to BH$ (restricting to compact subsets and passing to the $C^*$-algebras of continuous functions etc.). Then one has a commutative
Composing this canonical map with the (integral) assembly map results in a natural transformation from $K^J_n(N, G)$ to $K^J_*(C^*_\varphi)$ (note that both $BG$ and $C^*_\varphi(G)$ behave completely functorial with respect to group homomorphisms). Again this will result in a natural map from $K^J_n(N, G)$ to the $K$-group of proper dimension of the mapping cone of the induced homomorphism $C^*_\varphi(G) \to C^*_\varphi(H)$ which for the sake of $K$-theory is equivalent to its kernel. Consider a free resolution

$$1 \to R \to F \to G \to 1$$

of $G$ and put $\overline{G} := F/[R, F]$ where $[R, F]$ is the commutator subgroup generated by all single commutators $rxr^{-1}x^{-1}$ with $r \in R$, $x \in F$. Clearly this is a central extension of $G$ whose kernel $K$ contains $K^J_2(G)$ as a direct summand complemented by a free abelian group. (Recall that the Schur multiplier is given by the formula

$$K^J_2(G) := R \cap [F, F]/[R, F].$$

Considering the long sequence of $K^J$-theory applied to the extension

$$1 \to K \to \overline{G} \to G \to 1$$

one gets that $K^J_1(K, \overline{G}) := K/[K, \overline{G}] = K$ because $K$ is central and the boundary map $K^J_2(G) \to K^J_1(K, \overline{G})$ is an injection. Let $C_0^*(G)$ denote the augmentation ideal in $C^*(G)$, i.e. the kernel of the natural map $C^*(G) \to \mathbb{C}$ induced from the trivial representation of $G$ so that the kernel of $C_0^*(\overline{G}) \to C_0^*(G)$ can be naturally identified with the kernel of the quotient map on the full $C^*$-algebras. It is shown in [4] that the image of the maps $K^J_2(G) \to K_*(C^*_\varphi(G))$ actually lie in the splitting subgroup $K_*(C_0^*(G))$ for $n \geq 1$. Let $\mathcal{J}$ denote the kernel of the extension of group $C^*$-algebras as above and consider the commutative diagram

$$K^J_1(K) \to K_1(C^*_0(K))$$

$$\downarrow \quad \downarrow$$

$$K^J_1(K, \overline{G}) \to K_1(\mathcal{J}).$$

The upper horizontal map is known to be rationally injective since $K$ is an abelian group. Thus if we can show that the composition of this
map with the right vertical map is still rationally injective then by a commutative diagram considered above intertwining the boundary maps one gets that also the map $K_2^J(G) \to K_0(C_0^*(G))$ must be rationally injective.

Before entering into the proof let us consider the onedimensional case which is much simpler. In fact by functoriality one gets a commutative diagram

$$
\begin{array}{c}
K_1^J(G) \longrightarrow K_1(C^*(G)) \\
\downarrow \\
K_1^J(G/[G,G]) \longrightarrow K_1(C^*(G/[G,G]))
\end{array}
$$

since $K_1^J(G) = G/[G,G] = K_1^J(G/[G,G])$ and the lower horizontal map is rationally injective as $G/[G,G]$ is abelian. Considering the dual assembly map which in general takes the form

$$K^*(C^*(G)) \longrightarrow \lim_{\leftarrow X} K^*(X)$$

with $X \subseteq BG$ ranging over the compact subsets one rationally has

$$\lim_{\leftarrow X} (K^*(X) \otimes \mathbb{Q}) \simeq \lim_{\leftarrow X} (\oplus_n H^n(X; \mathbb{Q}))$$

and projecting the right-hand-side onto the onedimensional cohomology subgroup gives the dual space of $H_1(BG; \mathbb{Q})$ (at least when $G$ is finitely generated) by the Universal Coefficient Theorem. Since for finitely generated abelian groups the rational dual assembly map is known to be an isomorphism and in general $H^1(G; \mathbb{Q}) = H^1(G/[G,G]; \mathbb{Q})$ one sees that this subgroup is in the image of the dual assembly map. Therefore the pairing between homology and arbitrary products of onedimensional cohomology classes can be pulled back to a pairing of $K_*(C^*(G)) \otimes \mathbb{Q}$ and certain classes in $K^*(C^*(G)) \otimes \mathbb{Q}$ which gives Kasparovs result mentioned in the Introduction.

2. The proof

In view of the discussion above we need to prove the following result

**Proposition.** The natural composition

$$K \longrightarrow K_1(C_0^*(K)) \longrightarrow K_1(J)$$

is rationally injective.

**Remark.** It can be shown that the map $K_1(C_0^*(K)) \longrightarrow K_1(J)$ is not in general rationally injective (for example on the image of $K_3^J(K)$).
Let $H$ be any central extension of $G$ by a (countable) abelian group $K$ so that $C^*(H)$ has the structure of a semicontinuous field of $C^*$-algebras $A$ over the compact dual $\hat{K}$, where semicontinuous field simply means that $A$ is a $C(\hat{K})$-algebra, which is the maximal $C^*$-completion of the algebra of finite sums $\{\sum_{g \in G} a_g s_g\}$ with coefficients in the central subalgebra $C(\hat{K})$ of continuous functions on $\hat{K}$ and $s_g$ is the unitary generator of $C^*(H)$ corresponding to the group element $g \in G$ by some chosen section $s : G \rightarrow H$ to the natural projection. Denote the fibres of the semicontinuous field by $\{A_\omega\}_{\omega \in \hat{K}}$. Then the evaluated elements $\{s_\omega\}$ are unitary generators of $A_\omega$, and for each pair of points $\omega, \omega' \in \hat{K}$ there is a natural densely defined map $A_\omega \rightarrow A_{\omega'}$ given by linear extension of the assignment $s_\omega \mapsto s_{\omega'}$ with domain the finite linear combinations $\sum_{g} a_g s_\omega^g$ which will be called the tautological shift. Also on $A$ there is a $C(\hat{K})$-valued trace $\text{Tr}$ of norm 1 given by evaluating a finite sum $\sum_{g} a_g s_g$ at $a_1$ (and extending by continuity). It is well known that the norm-function $\omega \mapsto \|x\|_\omega$ is upper semicontinuous for any element $x$ of the semicontinuous field(cf. [2], [12]).

Proof of Proposition. Let us first deal with the case where $K$ is finitely generated which is a bit more simple. Assume that the map is not rationally injective. Then there exists a nontorsion element of $K$ which is in the kernel of the composition as above. Without loss of generality we may divide $K$ by its torsion subgroup, and since $K$ is finitely generated we may assume that our chosen element is an element of a basis of the remaining free abelian quotient. Thus completing to a full basis we may further divide by a complementary subgroup obtaining a central extension of $G$ by $\mathbb{Z}$. We continue to denote this extension by $\overline{G}$ and its kernel by $\overline{K}$. If the Proposition is valid for the new extension, it is also valid for the original since the chosen element was completely deliberate. Now $C^*(\overline{K})$ can be identified with continuous functions on the circle $S^1 = \{z \in \mathbb{C} | \|z\| = 1\}$ and $C^*_0(\overline{K})$ corresponds to those functions vanishing at the point 1. The kernel $J$ is a certain $C^*$-completion of the algebra of finite sums $\{\sum_{g \in G} a_g s_g\}$ with coefficients $a_g$ taken from the central subalgebra $C^*_0(\overline{K})$ and $s_g$ is the unitary generator of $C^*(\overline{G})$ corresponding to the group element $g \in G$ by some chosen section $s : G \rightarrow \overline{G}$. Multiplication depends on the particular section, i.e. for each pair $g, h \in G$ there is an element $n(g, h) \in \mathbb{Z}$ such that $s_g \cdot s_h = u^{n(g, h)} s_{gh}$ with $u$ the canonical unitary generator of $C^*(\overline{K})$ acting as a multiplier on $C^*_0(\overline{K})$. One may continuously deform this action with respect
to some parameter \( \theta \in [0, 1] \) by letting \( u_\theta \) denote the multiplier of \( C_0(S^1\setminus \{1\}) \simeq C_0^*(K) \) such that \( u_\theta \equiv 1 \) for all points \( z = e^{2\pi i \varphi} \) with \( \varphi \leq \theta \) and rotating the identical function \( u \) by an angle of \( 2\pi i \theta \) for all other values \( \geq \theta \). One gets a continuous deformation with \( u_0 = u \) and \( u_1 \equiv 1 \). Denote the corresponding semicontinuous field of \( C^* \)-algebras by \( \{ J_\theta \} \). Then \( J_1 \) is just the kernel of \( C^*(K \times G) \to C^*(G) \) and by the splitting projection \( K \times G \to K \) the map \( C_0(K) \to J_1 \) is seen to split, whence the corresponding map in \( K \)-theory is (split) injective.

Now the \( K \)-theory of a semicontinuous field of \( C^* \)-algebras posesses some semicontinuity properties. If an element of \( K_*(C_0^*(K)) \) is in the kernel of the map to \( K_*(J_{\theta_0}) \) for some fixed value \( \theta_0 \), then there exists a neighbourhood of \( \theta_0 \) such that the element is in the kernel of the map to \( K_*(J_\theta) \) for every \( \theta \) in this neighbourhood. This is seen considering the associated semicontinuous field of mapping cones \( C_\theta \) for the inclusions \( C_0^*(K) \to J_\theta \). One gets surjective maps \( C_\theta \to C_0^*(K) \) and the question whether an element is in the kernel of the map on \( K \)-groups as above for some given parameter boils down up to stabilization etc. to the fact whether a given unitary (resp. projection or difference of projections) in the quotient lifts to a unitary (projection) in the corresponding mapping cone extension. But this lifting property will extend to some neighbourhood of \( \theta_0 \) from functional calculus. From Bott Periodicity it suffices to do the case of \( K_1 \). Suppose a unitary \( u \) lifts to a unitary (invertible) element at some given point \( \theta_0 \). Consider an arbitrary lift \( V \) which is invertible at \( \theta_0 \). Then \( V_\theta \) is invertible iff \( |V|_\theta \) and \( |V^*|_\theta \) are both invertible. Suppose that in each neighbourhood of \( \theta_0 \) there exist parameters \( \theta \) where \( |V|_\theta \) (or \( |V^*|_\theta \)) is not invertible. Applying functional calculus to \( |V| \) with respect to a continuous function \( f \) vanishing outside the open unit disc \( \{ z \in \mathbb{C} | |z| < 1 \} \) and such that \( f \equiv 1 \) in some neighbourhood of 0 one gets a contradiction to upper semicontinuity of the norm-function of \( f(|V|) \) at \( \theta_0 \) since this element is zero evaluated at \( \theta_0 \). But then there exists a unitary \( U_\theta \) (using polar decomposition of \( V \) in a given neighbourhood where it is invertible) for each point in the neighbourhood such that its image maps to the image of \( u \) in the \( K_1 \)-group of the quotient evaluated at the given point. Thus one finds that the set of parameters \( \theta \) where the map on \( K \)-groups is injective is closed, since its complement is open.

We are interested in injectivity for the parameter \( \theta = 0 \), so it suffices to show that the map is injective for all positive parameters. Obviously for every positive parameter \( \theta \) there is an interval on the circle where the action of the multiplier \( u_\theta \) reduces to the identity, moreover we can easily contract the original embedding \( C_0^*(K) \to J_\theta \) by homotopy to an embedding which is trivial outside the interval in question. However
we cannot just cut off the complementary interval so as to reduce to the case of the parameter $\theta = 1$. Put $B = C^*_\theta(K)$. We will now show the converse, namely that if the map $K_1(B) \hookrightarrow K_1(J_{\theta_0})$ is injective for some parameter $\theta_0 \in [0, 1]$ then there exists a neighbourhood $U_{\theta_0}$ of $\theta_0$ where the map $K_1(B) \hookrightarrow K_1(J_\theta)$ is injective for all $\theta \in U_{\theta_0}$.

To this end assume given $\theta_0$ such that the first map as above is injective. Clearly this implies that the map $K_1(B) \hookrightarrow K_1(J_{\theta})$ is also injective for each $\theta \geq \theta_0$ because the first map will factor over the second one by the embedding of $J_\theta$ as an ideal of $J_{\theta_0}$ using some convenient reparametrization. Now assume that injectivity fails for every $\theta < \theta_0$. Choose a monotonously increasing sequence $\{\theta_k\}_{k \geq 1}$ of points converging up to $\theta_0$. The assumption implies on embedding $J_{\theta_0}$ as an ideal in $J_\theta$ and using the six-term exact sequence of $K$-theory, that the generator of the kernel of the composition

$$K_1(B) \hookrightarrow K_1(J_{\theta_0}) \rightarrow K_1(J_\theta)$$

has a lift to the group $K_0(J_{\theta}/J_{\theta_0})$ for any $\theta < \theta_0$. This quotient algebra is again a semicontinuous field over the halfopen interval $(\theta_0, \theta]$. On tensoring with $K$ we can assume that all algebras are stable so that the set of all $A_n = K \otimes (J_{\theta}/J_{\theta_n})$ defines a stable projective system of $C^*$-algebras in the sense of [5] and there is a natural map from $A_\theta = K \otimes (J_{\theta}/J_{\theta_0})$ into its stable projective limit $A_\infty = \lim_{\leftarrow} A_n$. Put $S_n = J_{\theta_n}$ and $C_n = S_n/S_0$ which can be viewed as semicontinuous fields over the open interval $(\theta_n, 1 + \theta_n)$ resp. the halfopen interval $(\theta_n, \theta_0]$ (the parametrization is adjusted to the multiplier action of $u$ on the fibres of $J$, i.e. $u$ corresponds to multiplication by the complex number $\exp(2\pi i(1 - \theta))$ for a parameter $0 \leq \theta < 1$ and to the identity for $\theta \geq 1$). If $A_\theta$ denotes the fibre of $J$ at the point $1 - \theta$ there is, for each $n$, a surjective evaluation map from $C_n$ onto $A_{\theta_0}$ commuting with the natural inclusion maps. Let $O_n$ be the kernel of this map.

To clarify the situation consider the following three inverse systems of short exact sequences of $C^*$-algebras

$$1 \rightarrow K \otimes S_n \rightarrow K \otimes S_1 \rightarrow A_n \rightarrow 1$$

$$1 \rightarrow S_0 \rightarrow S_n \rightarrow C_n \rightarrow 1$$

$$1 \rightarrow O_n \rightarrow C_n \rightarrow A_{\theta_0} \rightarrow 1.$$
injective inverse system as given above by the \( \{ S_n \} \) or \( \{ C_n \} \) such that
the smaller algebras sit as ideals of the larger ones one constructs an
equivalent stable projective system considering the associated system
of mapping cones \( \{ C_{\varphi_n} \} \) with \( \varphi_n : K \otimes S_1 \to K \otimes S_1 / K \otimes S_n \) etc.. Here
equivalent means that there is a natural compatible \( E \)-equivalence of
the factors. When alluding to the \( K \)-theory of the stable projective
limit of an injective system of ideals \( \{ I_n \} \) we tacitly assume that the
projective limit of the corresponding system of mapping cones is un-
derstood which is then well defined up to \( E \)-equivalence and denoted
\( I_\infty \). From Theorem 2 of [5] there is for any stable projective system
\( \{ A_n, \rho_n \} \) a Milnor lim\(^1\)-sequence of the form
\[
1 \longrightarrow \lim^1 K_{*+1}(A_n) \longrightarrow K_*(\lim_{\leftarrow} A_n) \longrightarrow \lim_{\leftarrow} K_*(A_n) \longrightarrow 1.
\]
The \( \lim \)- resp lim\(^1\)-groups appear as kernel resp. cokernel of the map
\[
\prod_{n=1}^\infty K_*(A_n) \xrightarrow{\Pi(id-\rho_\cdot)^*} \prod_{n=1}^\infty K_*(A_n).
\]
Recall that given a sequence of (stable) \( C^* \)-algebras \( \{ A_n \} \) the \( C^* \)-algebra
\( \prod_{n=1}^\infty A_n \) consists of norm-bounded sequences \( x = \{ x_n \} \) with
\( x_n \in A_n \) and product type involution, multiplication and \( \| x \| = \sup_{n \in \mathbb{N}} \{ \| x_n \| \} \). For this particular case the Milnor lim\(^1\)-sequence yields
a natural identification \( K_*(\Pi A_n) = \Pi K_*(A_n) \). The \( C^* \)-algebra \( \Pi A_n \)
contains the direct sum \( \oplus A_n \) as a proper ideal and the quotient will
be denoted by \( \Pi / \oplus A_n \), i.e. one has an exact sequence of the form
\[
1 \longrightarrow \oplus A_n \longrightarrow \prod A_n \longrightarrow \prod / \oplus A_n \longrightarrow 1.
\]
Our first objective is to show that the natural map \( K_*(\Pi / \oplus C_n) \to K_*(\Pi / \oplus A_{\theta_0}) \) obtained from the evaluation map at \( \theta_0 \) is trivial. Since
the map
\[
K_*(\oplus C_n) \xrightarrow{\oplus(id-\rho_n)^*} K_*(\oplus C_n)
\]
is an isomorphism, the kernel and cokernel of
\[
K_*(\Pi C_n) \xrightarrow{\Pi(id-\rho_n)^*} K_*(\Pi C_n)
\]
are the same as kernel and cokernel of
\[
K_*(\Pi / \oplus C_n) \xrightarrow{\Pi / \oplus(id-\rho_n)^*} K_*(\Pi / \oplus C_n).
\]
Then our assertion implies that also the evaluation map \( K_*(C_\infty) \to K_*(A_{\theta_0}) \) is trivial since it factors over \( K_*(\Pi / \oplus C_n) \) by the embedding
of \( K_*(A_{\theta_0}) \) into \( \Pi / \oplus K_*(A_{\theta_0}) \) as the image of the constant sequences.
One uses the following two facts. First, there is to each pair of indices
(θ, σ) a *-homomorphism δ_σ : A_θ → A_σ ⊗ A_{1−σ+θ} (say maximal tensor product on the right) defined by sending the unitary generators s^g to the diagonal elementary tensors s^g ⊗ s^{1+θ−σ}, extending linearly and continuously. Second, putting θ' = 1 − θ there is for each index θ an antilinear isomorphism

\[ j_θ : A_θ → A_{θ'} \]

from antilinear extension of the assignment \( s^g_0 \mapsto s^{θ'}_0 \) which gives a K-equivalence of the underlying real Banach algebras and a KK-equivalence (\(*\)-isomorphism) of the complex \( C^*\)-algebra \( A_θ \otimes_\mathbb{R} \mathbb{C} \) with the complex \( C^*\)-algebra \( A_{θ'} \otimes_\mathbb{R} \mathbb{C} \). One notes that in case of a group \( C^*\)-algebra one has an isomorphism \( C^*(G) \otimes_\mathbb{R} \mathbb{C} ≃ C^*(G) \oplus C^*(G) \) which is compatible with the projections induced by \( \overline{G} \to G \) so it passes to the kernel \( J \otimes_\mathbb{R} \mathbb{C} ≃ J \oplus J \). If \( τ \) denotes the flip of the two copies of the original group \( C^*\)-algebra in \( C^*(\overline{G}) \otimes_\mathbb{R} \mathbb{C} \) then \( (C^*(\overline{G}) \otimes_\mathbb{R} \mathbb{C}) \rtimes \mathbb{Z}_2 ≃ M_2(C^*(\overline{G})) \) and similarly with \( J \). One has to keep in mind however that the flip automorphism is not in general compatible with evaluation at the fibres, only if one considers intervals which are symmetric around the origin. To obtain the analogue of \( δ_σ \) in this context one uses the identity \( C^*(\overline{G}) \otimes_\mathbb{R} \mathbb{C} ≃ C^*(\overline{G}) \oplus C^*(\overline{G}) \).

The comultiplication \( δ \) on the group \( C^*\)-algebra gives a map

\[ C^*(\overline{G}) \otimes_\mathbb{R} \mathbb{C} \xrightarrow{δ \oplus δ} \left( C^*(\overline{G}) \otimes C^*(\overline{G}) \right) \oplus \left( C^*(\overline{G}) \otimes C^*(\overline{G}) \right) \]

Composing with the evaluation map at a point \( σ \) in the left factor gives

\[ C^*(\overline{G}) \otimes_\mathbb{R} \mathbb{C} \xrightarrow{δ_σ} \left( A_σ \otimes \mathbb{C} \right) \otimes \left( C^*(\overline{G}) \otimes_\mathbb{R} \mathbb{C} \right). \]

If \( J_θ \) is an ideal of \( C^*(\overline{G}) \) consisting of functions vanishing at a given point \( θ \) then restriction of \( δ_σ \) to \( J_θ \otimes_\mathbb{R} \mathbb{C} \) results in a homomorphism

\[ J_θ \otimes_\mathbb{R} \mathbb{C} \xrightarrow{δ_σ} \left( A_σ \otimes \mathbb{C} \right) \otimes \left( J_θ \otimes_\mathbb{R} \mathbb{C} \right) \]

where the second factor consists of functions vanishing at the point \( θ − σ \) (or \( 1 + θ − σ \) for that matter), so \( δ_σ \) passes to the quotient algebras

\[ A_θ \otimes_\mathbb{R} \mathbb{C} \xrightarrow{δ_σ} \left( A_σ \otimes \mathbb{C} \right) \otimes \left( A_{1+θ−σ} \otimes_\mathbb{R} \mathbb{C} \right). \]

Without loss of generality we may replace every \( C^*\)-algebra \( A \) considered above by \( A \otimes_\mathbb{R} \mathbb{C} \) which we will do without changing the notation.

If we wish to refer to the original algebra \( A \) we use the notation \( A^0 \).

Now there is a complex isomorphism \( j_θ : A_θ → A_{θ'} \). All the algebras we consider may be indexed by subsets of the unit circle, the
fibrés $A_\theta$ correspondant à des points. Si $\mathcal{A}$ est tout autre algèbre, la notation $\mathcal{A}^{-\theta}$ désigne l’algèbre correspondante avec l’ensemble d’index repoussé de $-\theta$. Soit $\{A_\lambda\}_{\lambda \in \Lambda}$ une suite cofinale d’algèbres $C^*$-separables de $\Pi/\oplus C_n$. Pour chaque $A_\lambda$, nous construirons une homotopie de quasihomomorphismes, en représentant un élément de $E(A_\lambda, \Pi/\oplus A_{\theta_0})$ (en principe, on peut utiliser $KK$-théorie mais avec nos limites projectives stables, seules les limites projectives bien définies définissent une limite stable). Connectant la carte d’évaluation $A_\lambda \rightarrow \Pi/\oplus A_{\theta_0}$ avec la carte triviale du tracé, et de telle sorte que ces homotopies constituent un système cohérent d’éléments de $E$-théorie avec rapport à l’inclusion. La carte $A_\lambda$ est le préimage de la carte $A_\lambda$ dans $\Pi C_n$ et choisissons une suite dense $\{x^\lambda_k\}$ de l’unité de la sphère de $A_\lambda$. Pour chaque $n$, considérons l’image de $A_\lambda$ dans $C_n^{-\theta} \otimes A_\theta$ par la projection sur $C_n$ composée avec $\delta_\theta$ où $\theta_n \leq \theta \leq \theta_0$. Les fibres de $C_n^{-\theta}$ sont indexées par l’intervalle demi-ouvert $(\theta_n - \theta, \theta_0 - \theta)$ et il existe un intervalle $I_n^\theta$ à l’origine de l’origine de l’origine telle que remplaçant la multiplication dans les fibres correspondantes par multiplication de l’algèbre $C^*$-algèbre $C^*(G)$ utilisant la projection triviale $s^\theta \mapsto s^\theta_0$, l’erreur de la projection triviale est uniformément petite de l’ordre de $1/n$ lorsque restreint aux images des finies nombres d’éléments $\{x^\lambda_1, \ldots, x^\lambda_n\}$ pour tout $\theta$ dans l’intervalle compact $[\theta_n, \theta_0]$. Laissez $C_{n,+}^{-\theta}$ être l’image de $C_n^{-\theta}$ correspondant à l’évaluation sur la partie positive $[0, \theta_0 - \theta]$ de l’ensemble d’index et de même $C_{n,-}^{-\theta}$ est l’image correspondant à l’évaluation sur la partie négative $(\theta_n - \theta, 0]$. Le principe est de découper la carte $x_n$ par $j_\sigma$ sur $C_{n,-}^{-\theta}$ et $C_{n,+}^{-\theta}$, puis prendre la différence de $x_+$ avec $x_-$ définissant un quasihomomorphisme en $C_{n,+}^{-\theta} \otimes A_\theta$. L’idée est de remettre ce processus en utilisant des petits intervalles autour de l’origine. L’idée est de s’embêter et de projeter $C_n^{-\theta}$ sur $\overline{D}_n^{-\theta}$, où le dernier est l’algèbre semi-continue de la fibre correspondant à l’intervalle fermé $[\theta - \theta_0, \theta_0 - \theta]$ qui est symétrique autour de l’origine. Ensuite, embête $\overline{D}_n^{-\theta}$ sur $\overline{D}_n^{-\theta} \rtimes \mathbb{Z}_2$ dont la fibre à l’origine est égale à $M_2(C^*(G))$, donc le flip devient interne. En supposant stabilité, on peut choisir une trajectoire continue strictement stable de unitaires $\{U_t\}$ dans l’algèbre de multiplicateurs stable indexée par l’intervalle $[\theta - \theta_0, 0]$, tel que $U_t$ agisse sur la fibre correspondante à l’origine $t$, de sorte que $U_0$ inverse le flip et telle que $U_t$ égale l’unité à l’antépôle négatif de l’intervalle $I_n^\theta$ et de toutes les parties au-delà. Conjuguant l’image de $x$ dans $\overline{D}_n^{-\theta}$ avec l’unitaire défini par la trajectoire $\{U_t\}$, puis prenant la
difference of the original image with the conjugated and flipped image defines a quasihomomorphism which evaluated at the origin is trivial. Tensoring with $A_\theta$ and evaluating at the point $\theta_0 - \theta$ (and its conjugate point) yields for each fixed $\theta_n \leq \theta \leq \theta_0$ a quasihomomorphism from $C_n$ to a certain subalgebra of $\mathcal{C}_n \times \mathbb{C}^2 \times \mathbb{Z}_2 \times A_\theta$. One notes that in case that $\theta$ is not contained in $I_n$ the image lies in the subalgebra $A_{\theta_n - \theta} \otimes A_\theta$. On the other hand, for $\theta \in I_n$ the first factor is approximately equal $M_2(G)$ (or rather $M_2(C^*(G)) \oplus M_2(C^*(G))$) on symmetric fibres different from the origin) as far as the finitely many elements $\{x_1^n, \cdots, x_n^n\}$ are concerned. Also the image will be contained in the subalgebra of this tensor product generated by diagonal elementary tensors of the form $\{j^* s_{\theta_0 - \theta} \otimes j^* s_\theta\}$ (resp. also $\{j^* s_{\theta_0} \otimes j^* s_\theta\}$ and $\tau$ if $\theta \in I_n$) where $j^*$ is the element corresponding to the complex number $i$ in the underlying real Banach algebra of $A^0$ and $\varepsilon$ can take the values 0 or 1. Now for each pair of indices $(\theta_0, \theta)$ the $C^*$-algebra $\hat{A}^0_{\theta_0 - \theta} \otimes \mathbb{C}$ contains (commuting) copies of $A_{\theta_0 - \theta}$ and $A_\theta$ from which follows by the universal property of the maximal tensor product that one has a surjective homomorphism

$$A_{\theta_0 - \theta} \otimes A_\theta \to (A^0_{\theta_0 - \theta} \otimes A^0_\theta) \otimes \mathbb{C}.$$ 

The subalgebra of the right side generated by diagonal elementary tensors $\{s_{\theta_0 - \theta} \otimes s_\theta\}$ is isomorphic with $A_{\theta_0}$. To see this note that the tensor product algebras $A^0_{\theta_0} \otimes A^0_\theta$ arise as fibres of the semicontinuous field $C^*(\mathcal{C}_\times \mathcal{C}) \simeq C^*(\mathcal{C}) \otimes C^*(\mathcal{C})$ over $\hat{K} \times \hat{K}$. Restricting the semicontinuous field to the $C^*$-subalgebra $C^*(\delta(G) \times K)$ where $\delta(G)$ denotes the diagonal in $\mathcal{C}_\times \mathcal{C}$ and the other factor $K$ is identified with the central subgroup $K$ in the second copy of $\mathcal{C}$, one finds that (the injective image in $C^*(\mathcal{C}_\times \mathcal{C})$ of) $C^*(\delta(G) \times K)$ is isomorphic with the tensor product $C^*(\mathcal{C}) \otimes C^*(\mathcal{C})$ so that each of its fibres corresponds to some fibre $A^0_{\theta_0}$ of $C^*(\mathcal{C})$, and on the other hand it is sent to the diagonal in $A^0_{\theta_0 - \theta} \otimes A^0_\theta$ modulo the ideal of functions vanishing at the point $(\theta_0 - \theta, \theta)$, so that both $C^*$-algebras must be isomorphic. For supposing that the map on fibres is not injective consider an element in its kernel and, choosing some small interval around the given point where the tautological shift is approximately contractive for the given element (resp. a suitable approximation from the dense subalgebra of finite sums in the generators $\{s_\theta\}$), also the images in the tensor product of fibres corresponding to points in the interval must become uniformly small for the shifted elements by upper semicontinuity if the interval is small enough. Then multiplying by some $\delta$-shaped
positive realvalued function of norm one whose domain is contained inside the interval and equal to 1 at $\theta_0$ one gets a contradiction because the norm of the image of the constructed extension is arbitrarily small, while on the other hand the injection $C^*(\delta(G) \times K) \hookrightarrow C^*(G \times G)$ is an isometry. Returning to the general argument one combines the quasihomomorphisms constructed above for all values of $\theta \in [\theta_n, \theta_0]$ and for all $n \in \mathbb{N}$. This yields a quasihomomorphism from $\overline{A}_\lambda$ to the outer direct product $\Pi/ \oplus (M_2(A_{\theta_0}) \otimes C[\theta_n, \theta_0])$. Restriction of this quasihomomorphism to the direct sum $\oplus C_n$ is trivial, so that by standard techniques the construction yields an element of

$$E(A_\lambda, \Pi/ \oplus (A_{\theta_0} \otimes C[\theta_n, \theta_0]))$$

which when evaluated at the end points $\theta_0$ gives the trivial element while when evaluated at the sequence of points $\{\theta_n\}$ it gives the element induced by the evaluation map $A_\lambda \to \Pi/ \oplus A_{\theta_0}$. But $\Pi/ \oplus (A_{\theta_0} \otimes C[\theta_n, \theta_0])$ is $E$-equivalent to $\Pi/ \oplus A_{\theta_0}$ by either projection, which follows from the corresponding result for the direct sum and the direct product using the six-term exact sequence of $E$-theory. Then the evaluation map $\Pi/ \oplus C_n \to \Pi/ \oplus A_{\theta_0}$ is trivial in $K$-theory as asserted. The rest of the argument is a simple diagram chase. Suppose that for some $x \in K_1(C_0(\mathbb{R}))$ the composition

$$K_1(C_0(\mathbb{R})) \to K_1(S_0) \to K_1(S_n)$$

is trivial for every $n > 0$. Then the image $y$ in $K_1(S_\infty)$ is contained in the subgroup $\lim^1 K_0(S_n)$ which can be identified with the cokernel of

$$\Pi/ \oplus K_1(S_n) \xrightarrow{\Pi/ \oplus (id - \rho_n)_*} \Pi/ \oplus K_1(S_n).$$

Choose a representative $\xi$ for this element in $K_1(\Pi/ \oplus S_n)$. Its image in $K_1(\Pi/ \oplus C_n)$ has a preimage for $\Pi/ \oplus (id - \rho_n)_*$ which maps to zero in $K_1(\Pi/ \oplus A_{\theta_0})$ by evaluation at $\theta_0$, hence it lifts to an element of $K_1(\Pi/ \oplus C_n)$ which again maps to an element $\zeta$ of $K_1(\Pi/ \oplus S_n)$. The difference $\xi - \Pi/ \oplus (id - \rho_n)_*(\zeta)$ represents the same element of the cokernel as $\xi$, and lifts to $K_1(\Pi/ \oplus S_0)$. But the latter group has trivial cokernel for $\Pi/ \oplus (id - \rho_n)_*$ so that also $y$ must be trivial. Then the image of $x$ in $K_1(S_0)$ lifts to an element of $K_0(C_\infty)$ which maps to the zero element of $K_0(A_{\theta_0})$ under evaluation, so it lifts again to an element of $K_0(C_\infty)$ whose image in $K_1(S_0)$ is trivial by exactness, giving a contradiction.

In the general case (of nonfinitely generated $K$) one proceeds similarly. Consider the push out of the central extension $\overline{G}$ by the map $K \to K \otimes \mathbb{Q}$ and check that it is well defined from the fact that $K$ is
central. If the map of the proposition is not rationally injective there exists a nontorsion element of $K$ in its kernel which may be completed to a vector space basis of $K \otimes \mathbb{Q}$. As above one now divides by the complementary subspace as to obtain a central extension of $G$ by $\mathbb{Q}$ which we continue to denote by $\overline{G}$, and its kernel by $K$. The kernel $\mathcal{J}$ of the surjection $C^*(\overline{G}) \to C^*(G)$ is again a module over the central subalgebra $C_0^*(K)$ isomorphic to the algebra of continuous functions on the compact dual $\hat{K}$ of $K = \mathbb{Q}$ vanishing at the identity element $\{1\}$, which is the maximal $C^*$-completion of the algebra of finite sums as above with coefficients in $C_0^*(K)$. The algebra $C^*(K)$ can be realized as continuous functions on the real line which are periodic with integer period $q$ for some $q \geq 1$. $C_0^*(K)$ contains those functions vanishing at the point 0. Instead of a single multiplier $u$ defining multiplication on the algebra of finite sums $\{\sum a_g s_g\}$ with respect to some given section $G \rightarrow \overline{G}$ one now has to consider the family of functions $\{u_q | q \in \mathbb{N}\}$ defined by $u_q(t) = e^{2\pi it/q}$, i.e. for each $g, h \in G$ there exist natural numbers $p = p(g, h)$, $q = q(g, h)$ such that the relation $s_g s_h = u_q^p s_{gh}$ holds in $C^*(\overline{G})$. Let $C_{0, q}^*(K)$ denote the ideal in $C_0^*(K)$ of functions vanishing at all integral multiples of $q$. Then $C_0^*(K)$ is the inductive limit of the net $\{C_{0, q}^*(K)\}$ directed by the natural inclusions. Let $\{\mathcal{J}_q\}$ be the corresponding increasing net of ideals with limit $\mathcal{J}$. In order to show injectivity of the restriction $K_1(C_{0, q}^*(K)) \rightarrow K_1(\mathcal{J}_q)$ one proceeds exactly as above, deforming the action of $u_q$ by some parameter $\theta \in [0, 1]$ such that $u_{q, \theta}$ is periodic of period $q$ and equals the identity function on the interval $[0, q\theta]$ whereas on the interval $[q\theta, q]$ one has $u_{q, \theta}(t) = u(t - q\theta)$. Also this deformation can be extended in a compatible way to any multiplier $u_{mq}$ on taking the $m$-th root of the deformation of $u_q$. Although $u_{mq, \theta}$ is not continuous it does define a multiplier of $C_{0, q}^*(K)$, and $u_{q, 1} \equiv 1$. The argument is now the same as for finitely generated $K$, considering the corresponding continuous field of $C^*$-algebras $\{\mathcal{J}_{q, \theta}\}_{\theta \in [0, 1]}$. Passing to inductive limits gives the general result. This completes the proof.

Remark. If $G = \mathbb{Z} \oplus \mathbb{Z}$ the fibres $A_\theta$ of the group algebra of the universal central extension as above are just the irrational rotation algebras which have been studied extensively by various authors (cf. [13], [14], [3]). It is well known that in this particular case the fibres are all $KK$-equivalent so the cone-like extensions corresponding to the $C_n$ are (globally) contractible which is a much stronger statement than the local property derived above. For a general group $G$ there is an interesting characterization of the stable fibres $\mathcal{K} \otimes A_\theta$ as a crossed
product of the algebra of compact operators on $L^2(G)$ by an outer group action of $G$ which goes as follows. Consider the left regular representation of $G$ on $L^2(G)$. If $\{\epsilon_k | k \in G\}$ denotes the standard orthonormal basis, then the left regular representation of $A_\theta$ (compare [12]) is defined by

$$\lambda(s^\theta_g)(\epsilon_k) = \exp(2\pi i(1-\theta)n(g,k)) \cdot \epsilon_{gk}.$$ 

The enveloping $C^*$-algebra $\lambda(A_\theta)$ acts on the commutative $C^*$-algebra $C_0(G)$ embedded in the usual way by diagonal matrices in $\mathcal{K}(L^2(G))$ by $\lambda(s^\theta_g)p_h = p_{gh}$ where $p_h$ denotes the onedimensional orthogonal projection onto the subspace spanned by $\epsilon_h$. The corresponding "crossed product" $C_0(G) \rtimes \lambda(A_\theta)$ is contained in $\mathcal{K}(L^2(G))$. Since the elements $\{\lambda(s^\theta_g)\}$ differ from the generators $\{\lambda(s^1_g)\}$ of $\lambda(G)$ in the left regular representation only modulo diagonal invertible matrices which are multipliers of $C_0(G)$, the twisted crossed product is in fact equal to $\mathcal{K}(L^2(G))$. The integer-valued function $n(g, h)$ is defined by the chosen section $G \curvearrowright \widehat{G}$. Changing from one section to another amounts to multiplying each generator $s_g, g \neq 1$ with a complex number $e^{2\pi im(g)}$ where $m : G \setminus \{1\} \to \mathbb{Z}$ is some integer valued (or more generally real valued) function. This changes the 2-cocycle $n$ to $	ilde{n}(g, h) = n(g, h) + m(h) - m(gh).$. It is clear that one can always find a selfadjoint section satisfying $n(g, g^{-1}) = 0$ for all $g \in G$ (if $G$ has 2-torsion one has to consider halfinteger-valued modifications $m$). This implies that $s_g^{-1} = s_g^*$ and hence $n(g, h) = -n(h^{-1}, g^{-1})$. For $0 < \theta \leq 1$ put $\theta' = 1 - \theta$, $\omega = \exp(2\pi i\theta')$, $\omega' = \exp(2\pi i\theta)$. Put $u^\theta_g = \lambda(s^\theta_g)$ and let $G$ act on the $C^*$-algebra $C_0(G) \rtimes \lambda(A_\theta)$ by the formula $a^\theta_g(p_h) = p_{gh}$ plus the following twisted adjoint operation on the image of $A_\theta$

$$a^\theta_g(u^\theta_h) = \omega^{n(g,h^{-1}) - n(g,hg^{-1})} \cdot u^\theta_{ghg^{-1}}$$

and check that this defines an action of $G$ compatible with the actions of $G$ and $A_\theta$ on $C_0(G)$ so that the action extends to an action $a^\theta$ on $\mathcal{K}(L^2(G))$. The corresponding (maximal) crossed product is not (in an obvious way) isomorphic to $\mathcal{K}(L^2(G)) \otimes C^*(G)$ since, although any automorphism of $\mathcal{K}$ is inner, the unitaries implementing the action do not define a (unitary) representation of $G$. Let $\{v^\theta_g | g \in G\}$ denote the unitaries corresponding to group elements in the multipliers of the crossed product algebra. Putting $u^\theta_g = v^\theta_{g^{-1}} \cdot v^\theta_g$ defines a representation of $A_\theta$ mapping $s^\theta_g$ to the unitary $v^\theta_g$. Also the $\{w^\theta_g\}$ commute with $\mathcal{K}(L^2(G))$. Thus one gets an (isomorphic) map $\mathcal{K}(L^2(G)) \otimes A_\theta \longrightarrow \mathcal{K}(L^2(G)) \rtimes a^\theta G$ for each $\theta$. This result should
remains valid on replacing the maximal group algebras and maximal crossed products by the corresponding reduced ones throughout.

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