Optimal Oracles for Point-to-Set Principles

Don Stull

Northwestern University
Fractal Geometry
- Size of small/irregular sets
- Fractal dimensions - Hausdorff, packing, etc.

Algorithmic Randomness
- Inherent randomness of points (binary sequences)
- Computability theory

We can use computability theory to solve problems in **classical** fractal geometry
In the plane, we parameterize orthogonal projections with the angle the line makes with the $x$-axis, so

$$p_\theta : \mathbb{R}^2 \to \mathbb{R}$$

$$p_\theta(x, y) = x \cos \theta + y \sin \theta.$$
Marstrands Projection Theorem

**Theorem (Marstrand '54)**

Let \( E \subseteq \mathbb{R}^2 \) be an analytic set with \( \dim_H(E) = s \). Then for almost every \( \theta \in (0, 2\pi) \),

\[
\dim_H(p_\theta E) = \min\{s, 1\}.
\]

- Any subset of a line has Hausdorff dimension at most 1.
- Lipschitz functions (like \( p_\theta \)) cannot increase the Hausdorff dimension of a set.
  - MPT shows that for a.e. angle, \( \dim_H(p_\theta E) \) is maximal.
- Testing ground for new techniques.
- Useful in many different problems in fractal geometry.
- Wellspring of generalizations and open problems.
Marstrand’s Projection Theorem

**Theorem (Marstrand ’54)**

Let $E \subseteq \mathbb{R}^2$ be an analytic set with $\dim_H(E) = s$. Then for almost every $\theta \in (0, 2\pi)$,

$$\dim_H(p_\theta E) = \min\{s, 1\}.$$

**Question:** How necessary is the analytic assumption?

- Davies proved that, assuming CH, there are sets for which MPT does not hold.

- T. Slaman and S. (unpublished) proved that, assuming $V = L$, there are $\Pi^1_1$ counterexamples to MPT.

- N. Lutz and S. proved that, if $\dim_H(E) = \dim_P(E)$, then for almost every $\theta \in (0, 2\pi)$,

  $$\dim_H(p_\theta E) = \min\{s, 1\}.$$
Marstrand’s Projection Theorem

**Theorem (Marstrand ’54)**

Let $E \subseteq \mathbb{R}^2$ be an analytic set with $\dim_H(E) = s$. Then for almost every $\theta \in (0, 2\pi)$,

$$\dim_H(p_\theta E) = \min\{s, 1\}.$$  

In this talk, we present the notion of *optimal oracles*.  

- If $E \subseteq \mathbb{R}^n$ has optimal oracles, then MPT holds for $E$.  
- Every analytic set has optimal oracles.  
- Every regular set $E$ ($\dim_H(E) = \dim_P(E)$) has optimal oracles.  
- Assuming CH, there are sets without optimal oracles.  
- Assuming $V = L$, there are $\Pi^1_1$ sets without optimal oracles. (Slaman and S., unpublished)
Let $x \in \mathbb{R}$ and $r \in \mathbb{N}$. The *Kolmogorov complexity of $x$ at precision $r$* is

$$K_r(x) \approx \text{Kolmogorov complexity of the first } r \text{ bits of binary expansion of } x$$

$$\approx \text{length of shortest program outputting the first } r \text{ bits}$$

$$\approx \text{of the binary expansion of } x$$

$$\approx \text{number of bits to specify the closet dyadic to } x \text{ at precision } 2^{-r}$$

- Can generalize this to $\mathbb{R}^n$ in the natural way.

- $0 \leq K_r(x) \leq nr$
Effective Dimensions of Points

Definition (Lutz ’03, Mayordomo ’03)

Let $n \in \mathbb{N}$, and $x \in \mathbb{R}^n$. The (effective Hausdorff) dimension of $x$ is

$$\text{dim}(x) = \lim_{r \to \infty} \inf \frac{K_r(x)}{r}.$$ 

Definition (Athreya et al. ’07, Lutz and Mayordomo ’08)

Let $n \in \mathbb{N}$, and $x \in \mathbb{R}^n$. The (effective) strong dimension of $x$ is

$$\text{Dim}(x) = \lim_{r \to \infty} \sup \frac{K_r(x)}{r}.$$ 

$0 \leq \text{dim}(x) \leq \text{Dim}(x) \leq n$ for every $x \in \mathbb{R}^n$. 
The Point-to-Set Principle

Theorem (J. Lutz and N. Lutz, ’16)

For every set $E \subseteq \mathbb{R}^n$, 

$$\dim_H(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \dim^A(x), \text{ and}$$

$$\dim_P(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \text{Dim}^A(x).$$

- The Hausdorff and packing dimensions of a set are characterized by the corresponding dimension of the points in the set, relative to a Hausdorff and packing oracle.
- Allows us to use algorithmic techniques to answer questions in fractal geometry.
Optimal Oracles

Let \( x \in \mathbb{R}^n \), \( B \subseteq \mathbb{N} \) and \( \epsilon > 0 \). We say that \( B \) is \( \epsilon \)-low for \( x \) if, for almost every \( r \in \mathbb{N} \),

\[
K_r^B(x) \geq K_r(x) - \epsilon r.
\]

**Definition (Stull '21)**

Let \( E \subseteq \mathbb{R}^n \) and \( A \subseteq \mathbb{N} \). We say that \( A \) is Hausdorff optimal for \( E \) if the following conditions are satisfied.

1. \( A \) is a Hausdorff oracle for \( E \).
2. For every \( B \subseteq \mathbb{N} \) and every \( \epsilon > 0 \) there is a point \( x \in E \) such that \( \dim^{A,B}(x) \geq \dim_H(E) - \epsilon \) and \( B \) is \( \epsilon \)-low for \( x \) relative to \( A \). That is,

\[
K_r^{A,B}(x) \geq K_r^A(x) - \epsilon r.
\]
Definition (Stull ’21)

Let $E \subseteq \mathbb{R}^n$ and $A \subseteq \mathbb{N}$. We say that $A$ is Hausdorff optimal for $E$ if the following conditions are satisfied.

1. $A$ is a Hausdorff oracle for $E$.

2. For every $B \subseteq \mathbb{N}$ and every $\epsilon > 0$ there is a point $x \in E$ such that $\dim^{A,B}(x) \geq \dim_H(E) - \epsilon$ and $B$ is $\epsilon$-low for $x$ relative to $A$.

$A$ is an optimal Hausdorff oracle for $E$ if

- $\dim_H(E) = \sup_{x \in E} \dim^A(x)$.
- Given any oracle $B$, can find a high dimension point in $x \in E$ such that $B$ contains almost no information about $x$. 
Lemma (S. ’21)

If $E$ is regular, i.e. $\dim_H(E) = \dim_P(E)$, then $E$ has optimal oracles.

1. Let $A_1$ and $A_2$ be Hausdorff and packing oracles of $E$. Let $A$ be the join of $A_1, A_2$.

   $\dim_H(E) = \sup_{x \in E} \dim^A(x)$ and $\dim_P(E) = \sup_{x \in E} \text{Dim}^A(x)$

2. Let $\epsilon > 0$ and $B \subseteq \mathbb{N}$. By the point-to-set principle, there is an $x \in E$ such that

   $\dim^{A,B}(x) > \dim_H(E) - \epsilon/4$.

3. Hence, for all sufficiently large $r \in \mathbb{N}$,

   $(\dim_H(E) - \epsilon/2) r \leq K^{A,B}_r(x) 
   \leq K^A_r(x) + O(\log r) 
   \leq (\dim_H(E) + \epsilon/2) r.$
Theorem (S. ’21)

If $E$ is analytic then $E$ has optimal oracles.

1. It suffices to show this for compact $E$.

2. Standard facts from geometric measure theory give the existence of “nice” Borel measure $\mu$ supported on $E$.

3. We encode $\mu$ into an oracle, and join this with a Hausdorff oracle for $E$.

4. Use a generalization of Levin’s coding theorem to prove that, for every $B \subseteq \mathbb{N}$, $B$ is low for $x$, for almost every $x$.
   - Proves a stronger statement: For every $B \subseteq \mathbb{N}$, for $\mu$-a.e. $x \in E$, $K_r^A(x) \geq K_r^{A,B}(x) - O(\log r)$.
Know that a set $E \subseteq \mathbb{R}^n$ has optimal oracles if
- $E$ is regular, or
- $E$ is analytic.

Conditional results:
- (S. ’21) Assuming CH, there are sets without optimal oracles.
- (Slaman and S.) Assuming $V = L$, there is a $\Pi^1_1$ set without optimal oracles.
- Assuming AD, every set has optimal oracles.
Optimal oracles

Theorem

If $E \subseteq \mathbb{R}^2$ has optimal oracles, then for a.e. $\theta \in [0, 2\pi)$

$$\dim_H(p_\theta E) = \min\{\dim_H(E), 1\}.$$
Optimal oracles in other contexts

- Most results in classical geometric measure theory assume that the set is analytic.
- Sometimes this is not necessary
  - N. Lutz theorem on slices and intersections
  - N. Lutz and S. theorem on Furstenberg sets
  - X. Guo and S. theorem on packing dimension of projections (unpublished).
- For certain problems, some assumption on the set is necessary.
  - If $E \subseteq \mathbb{R}$ is a Borel subring, then $\dim_H(E) = 0$ or $E = \mathbb{R}$.
  - If $E \subseteq \mathbb{R}^n$ is analytic and $\dim_H(E) > n/2$, then $\mu(\Delta E) > 0$.
    - Falconer’s distance set problem, open for all $n \geq 2$.
  - (Slaman and S.) Assuming $V = L$, there are $\Pi^1_1$ counterexamples. These counterexamples all very similar to the counterexample for optimal oracles.
The End

Thank you!