Light cone quantization and interactions of a new closed bosonic string inspired to $D1$ string.

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Abstract

We quantize the bosonic part of the $D1$ string with closed boundary conditions on the light cone and we consider the $U(1)$ worldsheet gauge field a dynamical variable. We compute also 3-Reggeon vertex by the overlapping technique. We find that the Fock space is the sum of sectors characterized by the momentum of the $U(1)$ Wilson line and that these sectors do not interact among them. Each sector has exactly the same spectrum of the usual bosonic string when expressed in properly sector dependent rescaled variables. Rescaling is forced by factorization of the string amplitudes. We are also able to determine the relative string coupling constant of the different sectors. It follows a somewhat unexpected picture in which the effective action is always the same independently on the sector but string amplitudes are only the same when expressed in sector dependent rescaled variables.

keywords: Bosonic string, D-brane
1 Introduction

Since the discovery of branes and their role in the non perturbative physics of string theory quite a lot of attention has been devoted to understanding their dynamics and in particular building their actions. Type IIB string enjoys a non perturbative $SL(2, \mathbb{Z})$ invariance in 10 dimensions therefore one of the aims was to construct D-brane actions (see for example [1], [2], [3]) as well as to compute low energy actions which respect this symmetry (see for example [4]). The need of checking these effective actions directly from string computations has led to the development of the instanton calculus in string theory (see for example [5]) on one side and to the study of the action of the S-duality on string amplitudes on the other (see for example [6],[7]).

Among all branes in type IIB $D1(p, q)$ branes play a prominent role since they are close relatives of the fundamental string and therefore they are very relevant in exploring the non perturbative picture of the IIB superstring. One can then wonder whether $D1(p, q)$ brane actions can be taken as fundamental actions and not only as effective ones. There were studies of the quantization of the $D1$ action performed in [8] whose main interest was anyhow to show that it is possible to quantize Green-Schwarz $D1$ string in a covariant gauge. Nevertheless there were also studies more in line with the previous question and they explored the quantization of the $SL(2, \mathbb{Z})$ invariant action for the $D1(p, q)$ [9] in the hope of being able to recover the full $SL(2, \mathbb{Z})$ invariant string amplitudes directly from prime principles.

The aim of this paper is less ambitious and it is to consider the bosonic part of the $D1$ string action as a fundamental action and to quantize it as a first step to understand what the full $\kappa$-symmetric $D1$ action can teach us. Eventually assuming the action of the $D1$ brane as a fundamental action and not an effective one it could happen that studying its interactions one could compute perturbatively in $1/R$ the strong coupled amplitudes in $AdS_5 \times S_5$ and compare them directly with super Yang-Mills results.

In this paper we consider the action with closed string boundary conditions and we quantize it on the light cone because the covariant quantization is more difficult to implement than usual since the constraint algebra involves structure functions and not only structure constants.

Then we compute the spectrum and the critical dimension which turn out to be the same of the normal bosonic string. Our result for the spectrum is at variance from those in the literature [8] since we find that it always contain massless states. We can track down the reason of this difference to the fact that in those papers it was assumed that the worldsheet electric field strength is diagonal in the Fock space while it is actually not. We also find that the Fock space is the direct sum of Fock subspaces characterized by the momentum of the $U(1)$ Wilson line and that each Fock subspace describes particles with the same mass spectrum as the usual bosonic string.

Finally we exam the interactions and find that each sector interacts only with its self and this explains some negative results found in the literature (see for example [10]) where it was found that it is not possible to make $D$
strings and $F$ strings interact perturbatively. We show that all sectors have the same interactions as the bosonic string up to a different string coupling constant whose relative normalization we determine.

2 The action and its quantization

We consider the bosonic action

$$S = - T_0 \int d^2 \xi \left[ \sqrt{\text{det}(\hat{G}_{\alpha\beta} + \kappa F_{\alpha\beta}) + \theta F_{01}} \right]$$

where $\alpha, \beta, \ldots \in \{0, 1\}$ are worldsheet indexes, $\hat{G}_{\alpha\beta} = G_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu$ is the pull back of the spacetime metric, $\mu, \nu, \ldots$ are spacetime indexes and $F_{01} = \partial_0 A_1 - \partial_1 A_0$ is the worldsheet $U(1)$ electric field. $\kappa$ is a constant and $\theta$ is a kind of $\theta$-vacuum constant. We impose closed boundary conditions, i.e. $X(\sigma + l, \tau) = X(\sigma, \tau)$ and $F_{01}(\sigma + l, \tau) = F_{01}(\sigma, \tau)$.

This action can be written in a Brink-Deser-Di Vecchia-Howe-Zumino \[^{11}\] like form with an additional worldsheet cosmological constant $\lambda$ as

$$S = - \frac{1}{2} T_0 \int d\tau \int_0^l d\sigma \left[ \sqrt{\text{det} \gamma_{\alpha\beta}} \gamma^{\alpha\beta}(\hat{G}_{\alpha\beta} + \kappa F_{\alpha\beta}) + \lambda \sqrt{\text{det} g_{\alpha\beta} + 2\theta F_{01}} \right]$$

where $g_{\alpha\beta}$ is the worldsheet metric and the matrix $\gamma_{\alpha\beta} = g_{\alpha\beta} + \epsilon_{\alpha\beta} f$ with $f$ a scalar density is neither symmetric nor antisymmetric. Because of this property the worldsheet supersymmetric action cannot be written immediately.

As usual at the classical level we must set $\lambda = 0$ if we want non trivial solutions of the equation of motion while at the quantum level it must be retained in order to preserve Weyl invariance.

2.1 The classical equations of motion

It is almost immediate to find the general solution of the e.o.m. associated to the previous action in Minkowski spacetime, i.e. when $G_{\mu\nu} = \eta_{\mu\nu}$ as we set in the rest of the paper. In particular the $A_\alpha$ e.o.m. reads

$$\partial_\alpha \left( F^{\alpha\beta}/\sqrt{-\text{det}(\hat{G} + \kappa F)} \right) = 0$$

which implies

$$F_{01}/\sqrt{-\text{det}(\hat{G} + \kappa F)} = c_0$$

with $c_0$ an arbitrary constant which can be identified with $(\Pi^1 + T_0 \theta)/(T_0 \kappa^2)$ where $\Pi^1$ is the $A_1$ conjugate momentum given in eq. \[^{10}\]. It then follows that we can compute $F_{01}$ as

$$F_{01}^2 = - \frac{c_0^2}{1 + \kappa^2 c_0^2} \text{det} \hat{G}$$

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Using this result into the action, even if $A_\alpha$ is dynamical and therefore it is not completely correct, implies that we can rewrite the action up to topological terms as

$$S = -\frac{T_0}{\sqrt{1 + \kappa^2 c_0}} \int d^2 \xi \sqrt{-\det(\hat{G}_{\alpha\beta})}$$

that is we find a usual bosonic string with a rescaled tension. In doing so we are nevertheless forgetting the fact that $A_\alpha$ is dynamical, even if it has only one d.o.f. corresponding to the Wilson line $w = \oint d\sigma A_1$. The consequences of this sloppy treatment is that we are, for example, missing the relative amplitudes normalizations of the different sectors of the theory which are characterized by the $\Pi^1$ eigenvalues and that we can be induced to believe that $F_{01}$ is a constant operator at the quantum level which is not as eq. (37) shows.

In the orthonormal gauge $\dot{X}^2 + X'^2 = \dot{X} \cdot X' = 0$ supplemented by the temporal gauge $A_0 = 0$ the general solution of e.o.m reads

$$X^\mu(\sigma, \tau) = x^\mu + \frac{1}{\pi T} \frac{1}{2} \sum_{n \neq 0} \text{sgn}(n) \left( a_n e^{-i \frac{2\pi n}{\kappa^2} (\tau + \sigma)} + \tilde{a}_n e^{-i \frac{2\pi n}{\kappa^2} (\tau - \sigma)} \right)$$

$$A_1(\sigma, \tau) = a_1 + \frac{c_0}{\sqrt{1 + \kappa^2 c_0}} \int d\tau \ X'^2$$

where we have used the residual gauge invariance to fix the $\tau$ independent integration function appearing in $A_1$ to a constant $a_1$ and set $T = \sqrt{T_0^2 + \frac{1}{\kappa^2} (\Pi^1 + T_0 \theta)^2}$.

### 2.2 Constraints algebra

It is immediate to pass to the Hamiltonian formalism and compute the momenta

$$P_\mu = T_0 \frac{\dot{X}_\mu X'^2 - X'_\mu \dot{X} \cdot X'}{\sqrt{-\dot{X}^2 X'^2 + (\dot{X} \cdot X')^2 - \kappa^2 F_{01}^2}}$$

$$\Pi^1 = T_0 \kappa^2 \frac{F_{01}}{\sqrt{-\dot{X}^2 X'^2 + (\dot{X} \cdot X')^2 - \kappa^2 F_{01}^2}} - T_0 \theta$$

and the Hamiltonian

$$H = \int_0^l d\sigma (-\Pi^1 A'_0)$$

so that the primary and secondary constraints can be written as

$$L_\pm = \left( \frac{P}{T} \pm T X' \right)^2$$

$$\Pi^0 = 0$$

$$\Pi^1' = 0$$
where we defined the a priori momentum dependent tension
\[ T^2(\sigma) = T_0^2 + \frac{1}{\kappa^2} (\Pi^1(\sigma))^2 = T_0^2 S^2(\sigma) \] (15)
and the scaling \( S(\sigma) \) which plays a major role later, see eq. (33). These constraints satisfy an algebra whose non-vanishing elements are the Virasoro algebra
\[ \{L_{\pm}(\sigma_1), L_{\pm}(\sigma_2)\} = \pm 4 \left( \frac{P}{T} + TX' \right) \cdot \left( \frac{P}{T} - TX' \right) \partial_{\sigma_1} \delta(T^{-1}) (\delta(\sigma_1 - \sigma_2) \right) \] (16)
and a further constraint
\[ \{L_{\pm}(\sigma_1), L_{\pm}(\sigma_2)\} = 4 \left( \frac{P}{T} + TX' \right) \cdot \left( \frac{P}{T} - TX' \right) \partial_{\sigma_1} \delta(\sigma_1 - \sigma_2) \] (17)
which involves a structure function which makes the computation of the BRST charge more involved than usual [12]. If we could impose at the constraints algebra level the constraint \( \Pi^1 = 0 \) we would recover the usual algebra but this is not correct. Therefore in order to avoid the issues involved in computing the BRST charge we have chosen to quantize the theory on the light cone.

2.3 Light cone quantization
We can proceed in fixing all the gauge invariances in the light cone gauge. In particular we fix Weyl invariance by setting
\[ \det \gamma = -1 \] (18)
Notice that we could have used the condition \( \det g = -1 \) as well and we would have obtained the same results but the chosen choice makes computations easier. After this first step we fix almost all worldsheet diffeomorphisms by
\[ X^+(\sigma, \tau) = \chi (\tau - \tau_0) \] (19)
with \( \chi = \pm 1 \). In this section \( \chi = 1 \) but the choice \( \chi = -1 \) is needed when discussing the interactions, see figure [1]. We are then left with residual diffeomorphisms given by \( \tau = \tau', \sigma = \sigma(\sigma', \tau') \) which can be fixed by
\[ \gamma_{11}(\sigma, \tau) = g_{11}(\sigma, \tau) = \hat{\gamma}_{11}(\tau) \] (20)
This can be done because \( \frac{\gamma_{11}(\sigma, \tau)}{\sqrt{-\det \gamma}} d\sigma \) is invariant under these residual diffeomorphisms and can be used to introduce a new worldsheet spacial coordinate as \( \frac{N(\tau)}{\sqrt{-\det \gamma}} d\zeta = \frac{\gamma_{11}(\sigma, \tau)}{\sqrt{-\det \gamma}} d\sigma \). After this step there are still some residual diffeomorphisms given by \( \tau = \tau', \sigma = \sigma' + \sigma_0(\tau') \) which could be used to fix \( \gamma_{01}(\sigma = 0, \tau) = 0 \) but we prefer not to fix it and get the constraint associated with its e.o.m. when we compute the Hamiltonian.
We can now consider the gauge fixing of the $U(1)$ worldsheet symmetry. Since we are working with a worldsheet with a topology of a cylinder we can only set
\[ A_1(\sigma, \tau) = a_1(\tau), \quad a_1(\tau) \equiv a_1(\tau) + \frac{2\pi n}{l}, \quad n \in \mathbb{Z} \] (21)
but we cannot set $a_1 = 0$ since $\exp\left(i \oint d\sigma A_1\right)$ is gauge invariant. As it happened with diffeomorphisms we are left with a residual gauge symmetry with parameter $\epsilon = \epsilon(\tau)$ which can be fixed by
\[ \oint d\sigma A_0 = 0 \leftrightarrow A_0(\sigma, \tau) = A_{0,nzm}(\sigma, \tau) \] (22)
After the gauge fixing the action becomes
\[ S = -\frac{T_0}{2} \int d\tau \left\{ 2\chi \hat{g}_{11}(\tau) \int_0^l d\sigma \dot{X}^- + \dot{a}_1 \int_0^l d\sigma (2\kappa f + 2\theta) \right. \\
+ \int_0^l d\sigma \left[ -(2\kappa f + 2\theta)A_{0,nzm} - \hat{g}_{11}(\tau)\dot{X}_i^- + \frac{1 + f^2 - g_{01}^2}{\hat{g}_{11}} X_i'^2 \\
+ 2g_{01}(-\chi X^- + \dot{X}_i X'_i) + \lambda \sqrt{1 + \hat{f}^2} \right] \} \] (23)
To further simplify it we can solve for the non dynamical fields. From the $A_{0,nzm}$ e.o.m. we get
\[ f(\sigma, \tau) = \hat{f}(\tau) \] (24)
The variation of the conjugate variable $f_{nzm}(\sigma, \tau)$ yields simply
\[ A_{0,nzm}(\sigma, \tau) = \frac{\hat{f}(\tau)}{\kappa \hat{g}_{11}(\tau)} \int d\sigma (X_i')^2 \] (25)
because the $\sigma$ integration of the e.o.m. has no ambiguous integration constant since $A_{0,nzm}$ has no zero modes. Notice however that $\hat{f}(\tau)$ is dynamical and it is proportional to the momentum conjugate to $a_1$ as eq. (30) shows.

The next non dynamical variable we consider is $X_{nzm}^- (\sigma, \tau)$ whose e.o.m. we get
\[ g_{01}(\sigma, \tau) = \hat{g}_{01}(\tau) \] (26)
As before the integration of e.o.m. of the conjugate variable $g_{01,nzm}(\sigma, \tau)$ yields
\[ \chi X_{nzm}^- (\sigma, \tau) = \int d\sigma \left[ \dot{X}_i X'_i - \frac{\hat{g}_{01}(\tau)}{\kappa \hat{g}_{11}(\tau)} (X'_i)^2 \right]. \] (27)
Inserting into the action (23) the previous results we get
\[ S = \int d\tau \left\{ -T_0 l \chi \hat{g}_{11}(\tau) \dot{x}^- - T_0 l (\kappa \hat{f} + \theta) \dot{a}_1 - \frac{T_0 l}{2} \lambda \sqrt{1 + \hat{f}^2} \right. \\
+ \int_0^l d\sigma \left[ \frac{T_0 l}{2} \hat{g}_{11}(\tau) \dot{X}_i^- - \frac{T_0 l}{2} \frac{1 + f^2 - g_{01}^2}{\hat{g}_{11}} X_i'^2 - T_0 \hat{g}_{01} \dot{X}_i X'_i \right] \} \] (28)

\[ 1 \text{ In the following for a generic field } Q Q_{nm} \text{ is the projection of the field } Q \text{ onto the periodic part explicitly } \sigma \text{ dependent, i.e. } \text{given } Q_{zm} = \frac{1}{l} \int_0^l d\sigma Q \text{ we set } Q_{nzm} = Q - Q_{zm}. \]
from which we can read the momenta

\[ p^+ = -p_- = T_0 l \hat{g}_{11}(\tau) \]
\[ p = -T_0 l (\kappa f(\tau) + \theta) \]
\[ \mathcal{P}_i = T_0 (\hat{g}_{11} \dot{X}_i - \hat{g}_{01} X'_i) \]

where \( p \) is the conjugate momentum to \( a_1 \). The Hamiltonian is then easily computed to be

\[ H = \frac{l}{p^+} \int_0^l d\sigma \left[ \frac{1}{2} \mathcal{P}_i^2 + \frac{T_0^2}{2} S^2 X''_i^2 + \frac{T_0 l}{p^+} \hat{g}_{01} \mathcal{P}_i X'_i \right] + \lambda \frac{T_0 l}{2} S \]

where we have introduced the scaling

\[ S(p_w) = \left( 1 + \frac{1}{\kappa^2} \left( \frac{p_w}{T_0} + \theta \right)^2 \right)^{1/2} \]

with \( p_w = p/l \) as defined in the last of eq.s (39).

We can now compute the e.o.m in either the Lagrangian formalism or the Hamiltonian one and find that \( p^+, p \) are constant,

\[ \dot{x}^- = -\frac{H}{p^+} \]
\[ \dot{a}_1 = \frac{T_0}{\kappa^2 p^+} \left( \frac{p}{T_0 l} + \theta \right) \int_0^l d\sigma X''_i^2 \]

and at the same time the remaining Lagrange multiplier \( \hat{g}_{01} \) implies the constraint

\[ \int_0^l d\sigma \dot{X}_i X_i' = \int_0^l d\sigma \mathcal{P}_i X_i' = 0 \]

while the \( X^i \) e.o.m reads as usual

\[ \ddot{X}^i - \omega^2(p) X''^i = 0 \]

with \( \omega(p) = \frac{T_0 l}{p^+} S(p) \). It is worth noticing that using the \( a_1 \) e.o.m, we can then compute the on shell expression for the electric field to be the gauge invariant expression

\[ F_{01} = \frac{T l}{p^+ \kappa^2} \left( \frac{p}{T l} + \theta \right) X_i'^2 \]

It is then easy to realize how \( F_{01} \) is not diagonalized on the the mass eigenstates (45) and therefore differently from what asserted in the literature [8] where it is assumed that \( F_{01} \) can be diagonalized the closed D1 string has always massless excitations.
In order to compute the commutation relations between the modes we write the \( X^i \) and \( P_i \) expansions explicitly as

\[
X^i = x^i + \frac{p^i}{p^+} \tau + i \sqrt{\frac{l}{4\pi|p^+|}} \frac{\omega(p)}{\omega(p)} \sum_{n \neq 0} \frac{\text{sgn}(n)}{\sqrt{|n|}} \left( a^i_n e^{-i \frac{2\pi n}{l}(\omega(p)\tau + \sigma)} + \tilde{a}^i_n e^{-i \frac{2\pi n}{l}(\omega(p)\tau - \sigma)} \right)
\]

\[
P_i = \frac{p^i}{l} + \frac{1}{2l} \sqrt{\frac{4\pi|p^+|}{l}} \frac{\omega(p)}{\omega(p)} \sum_{n \neq 0} \sqrt{|n|} \left( a^i_n e^{-i \frac{2\pi n}{l}(\omega(p)\tau + \sigma)} - \tilde{a}^i_n e^{-i \frac{2\pi n}{l}(\omega(p)\tau - \sigma)} \right)
\]

As a consequence of the presence of \( \omega(p) \) which does depend on \( p \) in the previous expansions we find that the operators \( a^i, \tilde{a}^i \) do not commute with \( a_1 \). The proper way to proceed is to perform a canonical transformation at \( \tau = 0 \) for simplicity and introduce the new fields as:

\[
\hat{X}^i(\sigma) = \sqrt{S(p)} X^i(\sigma)
\]

\[
\hat{P}_i(\sigma) = \frac{1}{\sqrt{S(p)}} P_i(\sigma)
\]

\[
\hat{x}^- = \sqrt{S(p)} x^-
\]

\[
\hat{p}^+ = \frac{1}{\sqrt{S(p)}} p^+
\]

\[
w = la_1 + \frac{\partial}{\partial p_w} \ln(\sqrt{S(p)}) \left[ \int_0^l d\sigma \frac{X^i P_i + P_i X^i}{2} - \frac{x^- p^+ + p^+ x^-}{2} \right]
\]

\[
p_w = \frac{p}{l}
\]

with non vanishing commutation relations

\[
[w, p_w] = i
\]

\[
[\hat{x}^-, \hat{p}^+] = -i
\]

\[
[\hat{x}^i, \hat{p}_j] = i \delta^i_j
\]

\[
[a^i_m, \tilde{a}^i_n] = [\tilde{a}^i_m, a^i_n] = \delta_{n,m} \delta^{ij}
\]

The new operator \( w \) corresponds physically to the Wilson line \( \oint d\sigma A_1 \) and it is defined modulo \( 2\pi \) times an integer, i.e. \( w \equiv w + 2\pi \) because of the big gauge transformations generated by \( \epsilon = \exp(i2\pi n^2 \tau) \).

At this stage one could wonder whether to stick with \( x, p \) or use \( \dot{x}, \dot{p} \) since both couples have the same commutation relation and in both cases one could define an operator \( w \) in a way to commute with either of them by using \( \int_0^l d\sigma (X^i P_i X^i P_i + P_i X^i P_i) \) or as done before \( \int_0^l d\sigma (X^i P_i + P_i X^i) \). One of the main results of this paper is that the proper variables are \( \dot{x}, \dot{p} \) since only using them we can factorize amplitudes in the usual way.

\[\footnote{The symmetrization of the product \( \mathcal{P}X \) in the definition of \( w \) is necessary in order to get an Hermitian operator.} \]

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2.4 Fock space and critical dimension

We proceed as usual to define the vacuum as

\[ a_n^i |0⟩ = \tilde{a}_n^i |0⟩ = p^i |0⟩ = p^+_n |0⟩ = p_w |0⟩ = 0, \quad n > 0 \] (44)

and then construct the Fock space by

\[ |\{N, \tilde{N}, k^i, k^+, n_w\}⟩ = \prod_{i=1}^{D-2} \prod_{n_i=1}^{∞} \frac{a_{-n_i}^i a_{-n_i}^i}{\sqrt{N_{n_i}}!} \langle 0 | e^{i k^i \hat{x}^i + i k^+ \hat{x}^- + i n_w w} \] (45)

Now we have well defined normal ordering prescription we can compute the Hamiltonian

\[ H = \sqrt{S} \left[ (\hat{p}_i)^2 + 4πT_0 \left( \sum_{n=1}^{∞} n (a_{-n}^i a_{n}^i + \tilde{a}_{-n}^i \tilde{a}_{n}^i) - \frac{D-2}{12} \right) \right] \] (46)

As usual we have reabsorbed the divergence from the regularized normal ordering constant

\[ \frac{S}{2p^r} \cdot 4πT_0 \cdot 2(D-2) \sum_{n=1}^{∞} n e^{-\epsilon 2πn / (l^2 p^r)} = -\frac{D-2}{12} \frac{p^+ \omega}{p^r} - \frac{D-2}{12} \frac{p^+ \omega}{p^r} + O(\epsilon) \] into a shift of the two dimensional cosmological constant \( \lambda \) in eq. (32) while the constant part, regularization independent is the zero point energy.

It would therefore seem that the spectrum of the theory is sector dependent but it is not. There are two reasons why we need rescaling \( P^- \). The first is that if we can hope to have a Poincaré invariant theory we cannot rescale all translator generators but \( P^- \). The second is that in order to factorize amplitudes with more than 4 legs we need to use hatted zero modes as discussed in the next section. Therefore we need the hatted Hamiltonian

\[ \hat{P}^-_{(2)} = \hat{H}_{(2)} = \frac{1}{\sqrt{S}} H \] (47)

which commutes with \( w \) differently from \( H = P^- \) which does not commute with \( w \), i.e. \([H, w] \neq 0\). As a consequence the mass spectrum in term of the hatted operators is independent on the sector \( p_w \):

\[ M^2 = 2\hat{P}^-_{(2)} \hat{p}^+ - (\hat{p}_i)^2 = 4πT_0 \left( \sum_{n=1}^{∞} n (a_{-n}^i a_{n}^i + \tilde{a}_{-n}^i \tilde{a}_{n}^i) - 2 \right) \] (48)

Let us now discuss the Lorentz invariance and the critical dimension of the theory. The non dynamical Lorentz generators are exactly the same as in the usual theory when expressed using hatted operators. The dynamical generators break by definition the gauge condition (19) which must be restored by a diffeomorphism. On the other side \( w = \oint dσ A_1 \) is diffeomorphism invariant therefore the dynamical generators have the same expressions as in the usual theory when written using the hatted operators and hence the critical dimension is left unchanged.
3 Interactions

We now determine the three strings interaction vertex by generalizing the overlapping conditions used by Cremmer, Gervais, Kaku and Kikkawa [13] to the case where there are the worldsheet metric and a gauge field on the world sheet. The same vertex, but only for on shell states, can be obtained by writing the vertex operators associated to the physical states obtained from DDF operators [14] and then computing the string amplitudes as explicitly done in [15].

We consider the case where string no. 3 with width \( l_3 \) is incoming and splits into outgoing string no. 1 and no. 2 with width \( l_1 \) and \( l_2 \) respectively. The local worldsheet coordinates on the three string are related at the interaction point \((\sigma_i, \tau_i)\) as

\[
\sigma^{(3)}_i = \begin{cases} 
\frac{l_1 - \sigma^{(1)}_i}{l_1} & 0 \leq \sigma^{(1)}_i \leq l_1 \quad [0 \leq \sigma^{(3)}_i \leq \sigma^{(3)}] \\
\frac{l_2 - \sigma^{(2)}_i}{l_2} & 0 \leq \sigma^{(2)}_i \leq l_2 \quad [\sigma^{(3)}_3 \leq \sigma^{(3)} \leq l^{(3)}_3] 
\end{cases}
\] (49)

and

\[
\tau^{(3)}_i - \tau^{(3)} = -(\tau^{(1)}_i - \tau^{(1)}_0) = -(\tau^{(2)}_i - \tau^{(2)}_0) \quad (50)
\]

as shown in figure (1).

![Figure 1: Interaction among three strings with their local worldsheet coordinates.](image)

Let us consider the overlapping condition at \( \tau = \tau_i = 0 \) for \( \det \gamma \), it reads

\[
\det \gamma^{(3)} = \begin{cases} 
\det \gamma^{(1)} \left( -\frac{d\sigma^{(1)}}{d\tau^{(3)}} \right)^2 & 0 \leq \sigma^{(1)} \leq l_1 \\
\det \gamma^{(2)} \left( -\frac{d\sigma^{(2)}}{d\tau^{(3)}} \right)^2 & 0 \leq \sigma^{(2)} \leq l_2 
\end{cases}
\] (51)

Since the previous condition must be consistent with the Weyl gauge fixing we deduce

\[
\frac{\sigma_i}{l_1} = \frac{l_3 - \sigma_i}{l_2} = 1, \quad (52)
\]

and with the help of the overlap condition for the worldsheet metric component \( g_{11} \)

\[
g_{11}^{(3)} = \begin{cases} 
g_{11}^{(1)} & 0 \leq \sigma^{(1)} \leq l_1 \\
g_{11}^{(2)} & 0 \leq \sigma^{(2)} \leq l_2 
\end{cases}
\] (53)
we obtain Mandelstam parametrization since $p^+ = T_0 l \chi_{11}$.

Next we can exam the overlap conditions for the coordinates $X^\mu$ which are then expressed as

$$X_{(3)}^i (\sigma_{(3)}, \tau_{i}^-) = \begin{cases} 0 \leq \sigma_{(1)} \leq l_1 \\ 0 \leq \sigma_{(2)} \leq l_2 \end{cases}$$

$$X_{(3)}^- = X_{(1)}^- = X_{(2)}^- \quad (54)$$

to which one has to add eq. (50) which expresses the $X^+$ overlap after the gauge fixing. It is the $X^+ = \chi (\tau - \tau_0)$ overlap along with the symmetric choice of worldsheet coordinates pictured in fig. [1] which forces the $X^+$ gauge fixing with $\chi_{(1,2)} = -1$.

It is also worth writing the overlap condition for the worldsheet 1-form current $J_\mu = 2 * \left( \frac{\delta S}{\delta \partial X^\mu} d\xi^\alpha \right) = P_\mu d\sigma + \ldots$ which can then expressed as

$$P_{i(3)}(\sigma_{(3)}, \tau_{i}^-) = \begin{cases} -\frac{l_{1i}}{\sigma_i} P_{i(1)}(\sigma_{(1)}, \tau_{i}^+) & 0 \leq \sigma_{(1)} \leq l_1 \\ -\frac{l_{2i}}{\sigma_i} P_{i(2)}(\sigma_{(2)}, \tau_{i}^+) & 0 \leq \sigma_{(2)} \leq l_2 \end{cases}$$

$$p_{i(3)}^+ + p_{i(1)}^+ + p_{i(2)}^+ = 0 \quad (55)$$

The condition for $p^+$ is also consistent with the overlap condition for the worldsheet metric component $g_{11}$, the $X^+$ gauge fixing $\chi_{(3)} = -\chi_{(1,2)} = 1$, the constraint [52] on the parametrization and the explicit expression [29] which relates $p^+$ and $g_{11}$.

To these usual conditions we must add the conditions that follow from the $A_\alpha d\xi^\alpha$ overlap conditions

$$A_{1(3)}(\sigma_{(3)}, \tau_{i}^-) d\sigma_{(3)} = \begin{cases} 0 \leq \sigma_{(1)} \leq l_1 \\ 0 \leq \sigma_{(2)} \leq l_2 \end{cases}$$

$$A_{0(3)}(\sigma_{(3)}, \tau_{i}^-) = \begin{cases} 0 \leq \sigma_{(1)} \leq l_1 \\ 0 \leq \sigma_{(2)} \leq l_2 \end{cases} \quad (56)$$

The previous conditions are the natural ones when we work in Hamiltonian formalism where we have only the freedom of performing $\sigma$ redefinitions. After the gauge fixing the previous conditions become

$$e^{i(l_3 a_{1(3)} + l_1 a_{1(2)} + l_2 a_{1(2)})} = 1 \quad (57)$$

$$\frac{p_{(3)}}{l_3} = \frac{p_{(1)}}{l_1} = \frac{p_{(2)}}{l_2} \quad (58)$$

We use the exponential version for the $la_1$ condition because the Wilson line $la_1$ is defined modulo $2\pi$ times an integer. Finally the condition on $p$ is the only one which is compatible with the $la_1$ condition and the commutation relations, moreover it can be directly derived from the expression [30] which relates $p$ and $f$ and the continuity condition for $f$. 

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We want now proceed in computing the interaction 3-vertex $|V_3\rangle$ by imposing the previous conditions. Because of the $p_w = p/l$ overlap condition
\[ (p_w(r) - p_w(s)) |V_3\rangle = 0 \quad r, s = 1, 2, 3 \] (59)
almost all the previous overlap conditions can be written using hatted operators simply by substituting the unhatted operators. Explicitly we can write
\[ (\hat{x}_c^\nu(r) - \hat{x}_c^\nu(s)) |V_3\rangle = 0 \] (60)
\[ \sum_{r=1}^3 \hat{p}_c^\nu(r) |V_3\rangle = 0 \] (61)
\[ (\hat{X}_{(3)}^{\nu} - \theta_1 \hat{X}_{(1)}^{\nu}) (\sigma_{(1)}) - \theta_2 \hat{X}_{(2)}^{\nu} (\sigma_{(2)}) |V_3\rangle = 0 \] (62)
\[ (\hat{P}_{(3)}^{\nu} + \theta_1 \hat{P}_{(1)}^{\nu} (\sigma_{(1)}) + \theta_2 \hat{P}_{(2)}^{\nu} (\sigma_{(2)}) |V_3\rangle = 0 \] (63)
where we have defined $\theta_2 = \theta(\sigma_1 - \sigma_{(2)})$ and similarly for $\theta_3 = \theta(\sigma_{(2)} - \sigma_1)$. Care must be nevertheless used in rewriting the overlap condition for $l a_1$ in term of $w$ since $w$ is obtained from $l a_1$ with a shift by quantities which are not normal ordered and a function of its momentum $p_w$. In particular we start writing
\[ e^{il a_1} = e^{i w (p_w + 1)} \left( \frac{S(p_w + 1)}{S(p_w)} \right)^{1 \cdot \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{8} \cdot \frac{1}{16}} \] (64)
then using eq. [59] and the reflection properties for $\hat{X}_{(3)}$ and $\hat{P}_{(3)}$ in $e^{i \sum_{r=1}^3 w(r) |V_3\rangle = 1}$ we see that all the terms involving $\hat{X}_{(1,2,3)}$ and $\hat{P}_{(1,2,3)}$ (or $X_{(1,2,3)}$ and $P_{(1,2,3)}$) cancel and we are left with the contributions from the lightcone zero modes $\hat{x}^\nu$ and $\hat{p}^\nu$. Finally with the help of the reflection properties for $\hat{x}^\nu$ and $\hat{p}^\nu$ we get
\[ e^{i \sum_{r=1}^N w(r) |V_3\rangle = \left( \frac{S(p_w(1))}{S(p_w(1) - 1)} \right)^{\frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{8} \cdot \frac{1}{16} |V_3\rangle} \] (65)
At first sight it seems curious that the only non trivial contribution comes from the light cone zero modes $\hat{x}^\nu$ and $\hat{p}^\nu$ and one would expect that also the transverse zero modes contribute but, as discussed in appendix [A], the regularized contribution from the transverse non zero modes cancel the one from the transverse zero modes. There is also an intuitive explanation why the only contribution comes from $\hat{x}^\nu$ and $\hat{p}^\nu$ and it is related to the fact that only these d.o.f. are treated symmetrically in outgoing and incoming strings (as overlap eqs. [60], [61] and eq. [75] show) and we want the relative normalization of the different $p_w$ sectors to be independent on whether we consider a $1 \rightarrow 2$ or $2 \rightarrow 1$ interaction.

All the previous conditions can be satisfied if we write
\[ |V_3\rangle = c_0 |V_3\rangle \otimes \sum_{n_w \in \mathbb{Z}} S(n_w) n_w^{\frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{8} \cdot \frac{1}{16} |V_3\rangle (p_w(1) = p_w(3) = p_w(3) = n_w) \] (66)
where $c_0 S(0)$ is the string coupling constant in the $n_w = 0$ sector, $|V_3\rangle_{\hat{X}}$ is the usual bosonic 3-string vertex operator written in term of the hatted operators. Would we not use hatted quantities we could not have written the 3-vertex in a factorized form since the overlapping matrices would explicitly depend on $p_w(r) = n_w$: this is clearly shown by eq. (72) in appendix. Therefore if we use hatted operators we can completely separate the “stringy” operators from the worldsheet electric variables. Since interactions take place only among strings with fixed and equal $p_w(r) = n_w$ the computations of interacting lightcone Hamiltonian (67) and amplitudes factorization work in the same way as usual in each sector only when we use hatted operators, in particular we need also to rescale the Hamiltonian to obtain a hatted Hamiltonian which is also the $P_{(2)}^-$ generator, is independent on the sector $n_w$ and commutes with $w$ differently from the unhatted Hamiltonian.

### 3.1 The effective action

The key point is that vertexes and consequently amplitudes are completely factorized in a part which depends on $\hat{X}$ times a part which depends on $w$. Therefore the full interacting lightcone Hamiltonian can be written as

$$\hat{P}^- = \hat{P}_{(2)}^- + g(p_w)\hat{P}_{(3)}^- + g^2(p_w)\hat{P}_{(4)}^- + \ldots$$

(67)

with $\hat{P}_{(N)}^-$ the $N$ particle contact Hamiltonian which does not depend on $w$ and $p_w$. Hence every sector $n_w$ has the same lightcone Hamiltonian and the same effective action when written in term of the hatted coordinates. The only difference is that each sector has a different, rescaled string coupling constant given by

$$g(n_w) = c_0 S(n_w)^{1/4} = c_0 \left(1 + \frac{1}{\kappa^2} \left(\frac{n_w}{T_0} + \theta\right)^2\right)^{1/4}$$

(68)

which can be read from (66). It is also possible to verify directly and easily the dependence on $n_w$ of the string coupling constant for the cases with $N \geq 4$ interacting strings using the overlapping conditions for special configurations where all interactions happen at the same time as pictured in figure (2) for the case $N = 4$. In all these case one gets immediately eq. (65) with the appropriate $N$.

### 4 Conclusions

In this paper we have quantized the bosonic part of the closed $D1$ string action. We have found that it is viable as a fundamental action and describes an infinite number of sectors. The sectors are characterized by the eigenvalue of the momentum $p_w$ of the $U(1)$ Wilson line $w = \oint A_1 d\sigma$. Each sector interacts only with strings of the same sector and has the same spectrum and
same interactions as the bosonic string. The only difference in the interactions is that each sector has a different string coupling constant.

Actually we have the same interactions as the bosonic string when we use sector dependent rescaled momenta (and positions). This rescaling is not at all arbitrary but it is dictated by the factorization of multipoint string amplitudes. It follows that all sectors have the same effective action but string amplitudes in different sectors are not equal when written using the natural momenta which are derived from the action but become equal when kinematical variables are rescaled. This very same picture continues to hold when the full $D1$ action in Green-Schwarz formalism is considered [16].

This suggests a solution of the puzzle which seems to arise when one considers the action of $SL(2,\mathbb{Z})$ symmetry on string amplitudes [6]. In fact if one acts with a $SL(2,\mathbb{Z})$ transformation on a string amplitude and takes the resulting amplitudes as they stand it seems that for any element of the $SL(2,\mathbb{Z})$ symmetry we have a different spectrum. The solution suggested from this paper is that the kinematical factors of amplitudes obtained by the action of a $SL(2,\mathbb{Z})$ element should be rescaled in such a way to have the same poles as the usual amplitudes.

A Another way of performing the main computation

In order to elucidate why only the lightcone zero modes contribute to the $w$ overlapping conditions we proceed as done by Kato and Kikkawa which followed Goto and Naka [17] and we define

$$|V_{3(0)}\rangle = e^{i \int_{0}^{\sigma_{(3)}} d\sigma_{(3)} \mathcal{P}_{(3)}(\sigma_{(3)}) \cdot X_{(3)}(\sigma_{(1)}) + i \int_{\tau_{(3)}}^{\tau_{(2)}} d\tau_{(3)} \mathcal{P}_{(3)}(\sigma_{(3)}) \cdot X_{(2)}(\sigma_{(2)})} |V_{3}\rangle$$ (69)

Using this new object the overlapping conditions for $X$ and $\mathcal{P}$ read

$$X_{(3)}^{\dagger}(\sigma_{(3)}, \tau_{3})|V_{3(0)}\rangle = \mathcal{P}_{i(1)}(\sigma_{(1)}, \tau_{1})|V_{3(0)}\rangle = \mathcal{P}_{i(2)}(\sigma_{(2)}, \tau_{2})|V_{3(0)}\rangle = 0$$ (70)
which imply the very simple solution

\[
|V_{3}(0)\rangle = \prod_{i=1}^{D-2} \left[ e^{\sum_{n=1}^{\infty} \left( \tilde{x}_{n(3)}^i \tilde{a}_{n(3)}^i - \tilde{a}_{n(1)}^i \tilde{x}_{n(1)}^i - \tilde{a}_{n(2)}^i \tilde{x}_{n(2)}^i \right) } \delta (\tilde{e}_{(3)}^i) |\tilde{p}_{(1)}^i = \tilde{p}_{(2)}^i = \tilde{p}_{(3)}^i = 0 \right] \]

\[
\otimes \delta (p_{(3)}^+ + p_{(1)}^+ + p_{(2)}^+)[x_{(1)}^- = x_{(2)}^- = x_{(3)}^- = 0]
\]

\[
\otimes \sum_{n_w \in \mathbb{Z}} c_{n_w} |p_{w(1)} = p_{w(2)} = p_{w(3)} = n_w\rangle
\]

with arbitrary \( c_{n} \). If we compute \( |V_{3}\rangle \) from this explicit expression for \( |V_{3}(0)\rangle \) by inverting eq. (69) we realize immediately that if we do not use hatted zero modes we get overlapping matrices explicitly dependent on \( p_{w(r)} = n_w \) and we cannot separate the stringy d.o.f. from the electric one, for example using the obvious reflection property for \( a \) modes we get overlapping matrices explicitly dependent on \( \alpha \in (0, 1) \).

Now eq. (64) can be written in an explicit way using oscillators as

\[
\int d\sigma_3 \, \hat{P}_3(\sigma_3) \cdot \hat{X}_3(1)(\sigma_1) \supset \frac{\sigma_3}{2\alpha_3} \sqrt{4\pi T_0 S(p_{w(3)})} \sum_{n} \sqrt{|n|} (a_{n(3)}^i + \tilde{a}_{-n(3)}^i) x_{(1)}^i A_{n0}^{(31)}
\]

(71)

Now eq. (64) can be written in an explicit way using oscillators as

\[
e^{iw} = \left( \frac{S(p_w)}{S(p_w - 1)} \right)^{\frac{1}{2}} \left[ \sum_{n=1}^{\infty} (a_n^i \tilde{a}_{-n}^i - a_{-n}^i \tilde{a}_n^i) - i \frac{1}{2} (\hat{a}_r^+ \hat{a}_r^- + \hat{\tilde{a}}_r^+ \hat{\tilde{a}}_r^-) \right] e^{ia_1}
\]

(73)

Using the obvious reflection property for \( a \) we read from (71) we get

\[
e^{i(w(3)+w(1)+w(2))} |V_{3}(0)\rangle = \left( \frac{S(p_{w(1)})}{S(p_{w(1)} - 1)} \right)^{\frac{1}{2}[-(D-2) \sum_{n=1}^{\infty} 1 + \frac{D-2}{2}]} \prod_{r=2}^{3} \left( \frac{S(p_{w(r)})}{S(p_{w(r)} - 1)} \right)^{\frac{1}{2}[-(D-2) \sum_{n=1}^{\infty} 1 - \frac{D-2}{2}]} \prod_{r=1}^{3} \left( \frac{S(p_{w(r)})}{S(p_{w(r)} - 1)} \right)^{\frac{1}{2} \left( \hat{\tilde{a}}_r^+ \hat{a}_r^- + \hat{a}_r^+ \hat{\tilde{a}}_r^- \right)} |V_{3}(0)\rangle
\]

(74)

The lightcone zero modes can be treated using

\[
e^{i\alpha \sum_{r=1}^{N} (\hat{\tilde{x}}_r^- \hat{p}_r^+ + \hat{p}_r^- \hat{\tilde{x}}_r^-) \delta (\sum_{r=1}^{N} p_r^+)[x^- = 0] = e^{i(N-2)\alpha} \delta (\sum_{r=1}^{N} p_r^+)[x^- = 0]
\]

(75)

while regularizing \( \sum_{n=1}^{\infty} 1 \) either with \( \zeta \) function or as \( \sum_{n=1}^{\infty} e^{-r1\pi n/(\sqrt{r} \pi)} \) as we did before we get that the finite part is \( -\frac{1}{2} \) hence

\[
e^{i(w(3)+w(1)+w(2))} |V_{3}(0)\rangle = \left( \frac{S(p_{w(3)})}{S(p_{w(3)} - 1)} \right)^{\frac{1}{2}[(N-2)|N=3]} |V_{3}(0)\rangle
\]

(76)

as we derived in the main text.
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