Stochastic Variance Reduced Primal Dual Algorithms for Empirical Composition Optimization

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Abstract

We consider a generic empirical composition optimization problem, where there are empirical averages present both outside and inside nonlinear loss functions. Such a problem is of interest in various machine learning applications, and cannot be directly solved by standard methods such as stochastic gradient descent. We take a novel approach to solving this problem by reformulating the original minimization objective into an equivalent min-max objective, which brings out all the empirical averages that are originally inside the nonlinear loss functions. We exploit the rich structures of the reformulated problem and develop a stochastic primal-dual algorithm, SVRPDA-I, to solve the problem efficiently. We carry out extensive theoretical analysis of the proposed algorithm, obtaining the convergence rate, the computation complexity and the storage complexity. In particular, the algorithm is shown to converge at a linear rate when the problem is strongly convex. Moreover, we also develop an approximate version of the algorithm, named SVRPDA-II, which further reduces the memory requirement. Finally, we evaluate our proposed algorithms on several real-world benchmarks, and experimental results show that the proposed algorithms significantly outperform existing techniques.

1 Introduction

In this paper, we consider the following regularized empirical composition optimization problem:

$$\min_{\theta} \frac{1}{n_X} \sum_{i=0}^{n_X-1} \phi_i\left(\frac{1}{n_Y_i} \sum_{j=0}^{n_Y_i-1} f_\theta(x_i, y_{ij}) \right) + g(\theta),$$

(1)

where \((x_i, y_{ij}) \in \mathbb{R}^{m_x} \times \mathbb{R}^{m_y}\) is the \((i, j)\)-th data sample, \(f_\theta : \mathbb{R}^{m_x} \times \mathbb{R}^{m_y} \to \mathbb{R}^\ell\) is a function parameterized by \(\theta \in \mathbb{R}^d\), \(\phi_i : \mathbb{R}^\ell \to \mathbb{R}^+\) is a convex merit function, which measures a certain loss of the parametric function \(f_\theta\), and \(g(\theta)\) is a \(\mu\)-strongly convex regularization term.

Problems of the form (1) widely appear in many machine learning applications such as reinforcement learning [5, 3, 2, 13], unsupervised sequence classification [12, 22] and risk-averse learning [15, 18, 9, 10, 19] — see our detailed discussion in Section 2. Note that the cost function (1) has an empirical average (over \(x_i\)) outside the (nonlinear) merit function \(\phi_i(\cdot)\) and an empirical average (over \(y_{ij}\)) inside the merit function, which makes it different from the empirical risk minimization problems that are common in machine learning [17]. Problem (1) can be understood as a generalized version of the one considered in [9, 10]. In these prior works, \(y_{ij}\) and \(n_Y_i\) are assumed to be independent of...

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1 and $f_{\theta}$ is only a function of $y_j$ so that problem (1) can be reduced to the following special case:

$$\min_{\theta} \frac{1}{n_X} \sum_{i=0}^{n_X-1} \phi_i \left( \frac{1}{n_Y} \sum_{j=0}^{n_Y-1} f_{\theta}(y_j) \right).$$

(2)

Our more general problem formulation (1) encompasses wider applications (see Section 2). Furthermore, different from [2, 19, 18], we focus on the finite sample setting, where we have empirical averages (instead of expectations) in (1). As we shall see below, the finite-sum structures allows us to develop efficient stochastic gradient methods that converges at linear rate.

While problem (1) is important in many machine learning applications, there are several key challenges in solving it efficiently. First, the number of samples (i.e., $n_X$ and $n_Y$) could be extremely large: they could be larger than one million or even one billion. Therefore, it is unrealistic to use batch gradient descent algorithm to solve the problem, which requires going over all the data samples at each gradient update step. Moreover, since there is an empirical average inside the nonlinear merit function $\phi_i(\cdot)$, it is not possible to directly apply the classical stochastic descent (SGD) algorithm. This is because sampling from both empirical averages outside and inside $\phi_i(\cdot)$ simultaneously would make the stochastic gradients intrinsically biased (see Appendix A for a discussion).

To address these challenges, in this paper, we first reformulate the original problem (1) into an equivalent saddle point problem (i.e., min-max problem), which brings out all the empirical averages inside $\phi_i(\cdot)$ and exhibits useful dual decomposition and finite-sum structures (Section 3.1). To fully exploit these properties, we develop a stochastic primal-dual algorithm that alternates between a dual step of stochastic variance reduced coordinate ascent and a primal step of stochastic variance reduced gradient descent (Section 3.2). It in particular, we develop a novel variance reduced stochastic gradient estimator for the primal step, which achieves better variance reduction with low complexity (Section 3.3). We derive the convergence rate, the finite-time complexity bound, and the storage complexity of our proposed algorithm (Section 4). In particular, it is shown that the proposed algorithms converge at a linear rate when the problem is strongly convex. Moreover, we also develop an approximate version of the algorithm that further reduces the storage complexity without much performance degradation in experiments. We evaluate the performance of our algorithms on several real-world benchmarks, where the experimental results show that they significantly outperform existing methods (Section 5). Finally, we discuss related works in Section 6 and conclude our paper in Section 7.

2 Motivation and Applications

To motivate our composition optimization problem (1), we discuss several important machine learning applications where cost functions of the form (1) arise naturally.

Unsupervised sequence classification: Developing algorithms that can learn classifiers from unlabeled data could benefit many machine learning systems, which could save a huge amount of human labeling costs. In [12, 22], the authors proposed such unsupervised learning algorithms by exploiting the sequential output structures. The developed algorithms are applied to optical character recognition (OCR) problems and automatic speech recognition (ASR) problems. In these works, the learning algorithms seek to learn a sequence classifier by optimizing the empirical output distribution match (Empirical-ODM) cost, which is in the following form (written in our notation):

$$\min_{\theta} \left\{ -\sum_{i=0}^{n_X-1} p_{LM}(x_i) \log \left( \frac{1}{n_Y} \sum_{j=0}^{n_Y-1} f_{\theta}(x_i, y_j) \right) \right\},$$

(3)

where $p_{LM}$ is a known language model (LM) that describes the distribution of output sequence (e.g., $x_i$ represents different $n$-grams), and $f_{\theta}$ is a functional of the sequence classifier to be learned, with $\theta$ being its model parameter vector. The key idea is to learn the classifier so that its predicted output $n$-gram distribution is close to the prior $n$-gram distribution $p_{LM}$ (see [12, 22] for more details). The cost function (3) can be viewed as a special case of (1) by setting $n_Y = 1$, $y_j = y_j$ and $\phi_i(u) = -p_{LM}(x_i) \log(u)$. Note that the formulation (2) cannot be directly used here, because of the dependency of the function $f_{\theta}$ on both $x_i$ and $y_j$.

Risk-averse learning: Another application where (1) arises naturally is the risk-averse learning problem, which is common in finance [15, 18, 9, 10, 19, 21]. Let $x_i \in \mathbb{R}^d$ be a vector consisting of
the rewards from \(d\) assets at the \(i\)-th instance, where \(0 \leq i \leq n - 1\). The objective in risk-averse
learning is to find the optimal weights of the \(d\) assets so that the average returns are maximized while
the risk is minimized. It could be formulated as the following optimization problem:

\[
\min_{\theta} \frac{1}{n} \sum_{i=0}^{n-1} \langle x_i, \theta \rangle + \frac{1}{n} \sum_{i=0}^{n-1} \left( \frac{1}{n} \sum_{j=0}^{n-1} \langle x_j, \theta \rangle \right)^2,
\]

where \(\theta \in \mathbb{R}^d\) denotes the weight vector. The objective function in (4) seeks a tradeoff between
the mean (the first term) and the variance (the second term). It can be understood as a special case of (2)
(which is a further special case of (1)) by making the following identifications:

\[
n_X = n_Y = n, \quad y_i \equiv x_i, \quad f_\theta(y_j) = [\theta^T, -\langle y_j, \theta \rangle]^T, \quad \phi_\theta(u) = \langle x_i, u_{0:d-1} \rangle + u_d - \langle x_i, u_{0:d-1} \rangle,
\]

where \(u_{0:d-1}\) denotes the subvector constructed from the first \(d\) elements of \(u\), and \(u_d\) denotes the
\(d\)-th element. An alternative yet simpler way of dealing with (4) is to treat the second term in (4) as a
special case of (1) by setting \(d^* = 2\) and \(d = 1\) in the first formulation. Therefore, we will adopt this formulation in our experiment section (Section 5).

**Other applications:** Cost functions of the form (1) also appear in reinforcement learning [5, 2, 3] and other applications [13]. In Appendix 3 we demonstrate its applications in policy evaluation.

### 3 Algorithms

#### 3.1 Saddle point formulation

Recall from (1) that there is an empirical average inside each (nonlinear) merit function \(\phi_\theta(\cdot)\), which prevents the direct application of stochastic gradient descent to (1) due to the inherent bias (see Appendix A for more discussions). Nevertheless, we will show that minimizing the original cost function (1) can be transformed into an equivalent saddle point problem, which brings out all the empirical averages inside \(\phi_\theta(\cdot)\). In what follows, we will use the machinery of convex conjugate functions [14]. For a function \(\psi : \mathbb{R}^d \to \mathbb{R}\), its convex conjugate function \(\psi^* : \mathbb{R}^d \to \mathbb{R}\) is defined as:

\[
\psi^*(y) = \sup_{x \in \mathbb{R}^d} (\langle x, y \rangle - \psi(x)).
\]

Under certain mild conditions on \(\psi(x)\) [14], one can also express \(\psi(x)\) as a functional of its conjugate function:

\[
\psi(x) = \sup_{y \in \mathbb{R}} (\langle x, y \rangle - \psi^*(y)).
\]

Let \(\phi_\theta^*(w_i)\) denote the conjugate function of \(\phi_\theta(w_i)\). Then, we can express \(\phi_\theta(w_i)\) as

\[
\phi_\theta(w_i) = \sup_{w_i \in \mathbb{R}^d} (\langle w_i, w_i \rangle - \phi_\theta^*(w_i)),
\]

where \(w_i\) is the corresponding dual variable. Substituting (7) into the original minimization problem (1) into the (2) into the original minimization problem (1), we obtain its equivalent min-max problem as:

\[
\min_{\theta} \max_w \left\{ L(\theta, w) + g(\theta) \right\} \equiv \frac{1}{n_X} \sum_{i=0}^{n_X-1} \left[ \frac{1}{n_Y} \sum_{j=0}^{n_Y-1} f_\theta(x_i, y_{ij}, w_i) - \phi_\theta^*(w_i) \right] + g(\theta),
\]

where \(w \equiv \{w_0, \ldots, w_{n_Y-1}\}\) is a collection of all dual variables. We note that the transformation of the original problem (1) into (3) brings out all the empirical averages that are present inside \(\phi_\theta(\cdot)\). This new formulation allows us to develop stochastic variance reduced algorithms below.

#### 3.2 Stochastic variance reduced primal-dual algorithm

One common solution for the min-max problem (3) is to alternate between the step of minimization (with respect to the primal variable \(\theta\)) and the step of maximization (with respect to the dual variable \(w\)). However, such an approach generally suffers from high computation complexity because each minimization/maximization step requires a summation over many components and requires a full
pass over all the data samples. The complexity of such a batch algorithm would be prohibitively high when the number of data samples (i.e., \( n_X \) and \( n_Y \)) is large (e.g., they could be larger than one million or even one billion in applications like unsupervised speech recognition [22]). On the other hand, problem (8) indeed has rich structures that we can exploit to develop more efficient solutions.

To this end, we make the following observations. First, expression (6) implies that when \( \theta \) is fixed, the maximization over the dual variable \( w \) can be decoupled into a total of \( n_X \) individual maximizations over different \( w_i \)'s. Second, the objective function in each individual maximization (with respect to \( w_i \)) contains a finite-sum structure over \( j \). Third, by (6), for a fixed \( w \), the minimization with respect to the primal variable \( \theta \) is also performed over an objective function with a finite-sum structure. Based on these observations, we will develop an efficient stochastic variance reduced primal-dual algorithm (named SVRPDA-I). It alternates between (i) a dual step of stochastic variance reduced coordinate ascent and (ii) a primal step of stochastic variance reduced gradient descent. The full algorithm is summarized in Algorithm 1 with its key ideas explained below.

**Dual step: stochastic variance reduced coordinate ascent.** To exploit the decoupled dual maximization over \( w \) in (8), we can randomly sample an index \( i \), and update \( w_i \) according to:

\[
\theta^{(k)}_i = \arg \min_{\theta_i} \left\{ \frac{1}{n_Y} \sum_{j=0}^{n_Y-1} f_{\theta^{(k-1)}}(x_i, y_j), w_i \right\} + \phi^*_{\theta_i}(w_i) + \frac{1}{2\alpha_w} \| w_i - w_i^{(k-1)} \|^2 (9),
\]

while keeping all other \( w_j \)'s \((j \neq i)\) unchanged, where \( \alpha_w \) denotes a step-size. Note that each step of recursion (2) still requires a summation over \( n_Y \) components. To further reduce the complexity, we approximate the sum over \( j \) by a variance reduced stochastic estimator defined in (12) (to be discussed in Section 3.3). The dual step in our algorithm is summarized in (13), where we assume that the function \( \phi^*_{\theta_i}(w_i) \) is in a simple form so that the argmin could be solved in closed-form. Note that we flip the sign of the objective function to change maximization to minimization and apply coordinate descent. We will still refer to the dual step as “coordinate ascent” (instead of descent).

**Primal step: stochastic variance reduced gradient descent** We now consider the minimization in (8) with respect to \( \theta \) when \( w \) is fixed. The gradient descent step for minimizing \( L(\theta, w) \) is given by

\[
\theta^{(k)} = \arg \min_{\theta} \left\{ \sum_{i=0}^{n_X-1} \sum_{j=0}^{n_Y-1} \frac{1}{n_X n_Y} f_{\theta^{(k-1)}}(x_i, y_j), w_i^{(k)}(\theta \theta) + \frac{1}{2\alpha_\theta} \| \theta - \theta^{(k-1)} \|^2 \right\} (10),
\]

where \( \alpha_\theta \) denotes a step-size. It is easy to see that the update equation (10) has high complexity, it requires evaluating and averaging the gradient \( f_{\theta}^k(\cdot) \) at every data sample. To reduce the complexity, we use a variance reduced gradient estimator, defined in (15), to approximate the sums in (10) (to be discussed in Section 3.3). The primal step in our algorithm is summarized in (16) in Algorithm 1.

### 3.3 Low-complexity stochastic variance reduced estimators

We now proceed to explain the design of the variance reduced gradient estimators in both the dual and the primal updates. The main idea is inspired by the stochastic variance reduced gradient (SVRG) algorithm [7]. Specifically, for a vector-valued function \( h(\theta) = \frac{1}{n} \sum_{i=0}^{n-1} h_i(\theta) \), we can construct its SVRG estimator \( \delta_k \) at each iteration step \( k \) by using the following expression:

\[
\delta_k = h_i_k(\theta) - h_i_k(\hat{\theta}) + h(\hat{\theta}), (17)
\]

where \( i_k \) is a randomly sampled index from \( \{0, \ldots, n-1\} \), and \( \hat{\theta} \) is a reference variable that is updated periodically (to be explained below). The first term \( h_i(\hat{\theta}) \) in (17) is an unbiased estimator of \( h(\theta) \) and is generally known as the stochastic gradient when \( h(\theta) \) is the gradient of a certain cost function. The last two terms in (17) construct a control variate that has zero mean and is negatively correlated with \( h_i(\theta) \), which keeps \( \delta_k \) unbiased while significantly reducing its variance. The reference variable \( \hat{\theta} \) is usually set to be a delayed version of \( \theta \): for example, after every \( M \) updates of \( \theta \), it can be reset to the most recent iterate of \( \theta \). Note that there is a trade-off in the choice of \( M \): a smaller \( M \) further reduces the variance of \( \delta_k \) since \( \hat{\theta} \) will be closer to \( \theta \) and the first two terms in (17) cancel more with each other; on the other hand, it will also require more frequent evaluations of the costly batch term \( h(\hat{\theta}) \), which has a complexity of \( O(n) \).
Algorithm 1 SVRPDA-I

1: Inputs: data \( \{(x_i, y_i) : 0 \leq i < n_X, 0 \leq j < n_Y\} \); step-sizes \( \alpha_\theta \) and \( \alpha_w \); number of iterations \( M \).
2: Initialization: \( \hat{\theta}_0 \in \mathbb{R}^d \) and \( w_0 \in \mathbb{R}^{n_X} \).
3: for \( s = 1, 2, \ldots \) do
4:   Set \( \hat{\theta} = \hat{\theta}_{s-1} \), \( \theta^{(0)} = \hat{\theta} \), \( \hat{w}_s = \hat{w}_{s-1} \), and compute the batch quantities (for each \( 0 \leq i < n_X \)):
5:     \[ U_0 = \sum_{i=0}^{n_X-1} \sum_{j=0}^{n_Y} f'_\theta(x_i, y_j)w_i^{(0)}; \quad T_s(\hat{\theta}) = \sum_{j=0}^{n_Y-1} f'_\theta(x_i, y_j). \]  
6: for \( k = 1 \) to \( M \) do
7:   Randomly sample \( i_k \in \{0, \ldots, n_X-1\} \) and then \( j_k \in \{0, \ldots, n_Y\} \), independently and then \( j_k \in \{0, \ldots, n_Y - 1\} \) at uniform.
8:   Compute the stochastic variance reduced gradient for dual update:
9:     \[ \delta_k^w = f'(\hat{\theta}) (x_{i_k}, y_{j_k}) - f'(\theta^{(k-1)}) (x_{i_k}, y_{j_k}) + T_{i_k}(\hat{\theta}). \]  
10: Update the dual variables:
11:     \[ w_{i_k}^{(k)} = \arg \min_{w_i} \left[ -\langle \delta_k^w, w_i \rangle + \phi_i^*(w_i) + \frac{1}{2\alpha_w} \|w_i - w_i^{(k-1)}\|^2 \right] \]  
12: Update \( U_k \) (primal batch gradient at \( \hat{\theta} \) and \( w^{(k)} \)) according to the following recursion:
13:     \[ U_k = U_{k-1} + \frac{1}{n_X} T'_s(\hat{\theta})(w_i^{(k)} - w_i^{(k-1)}). \]  
14: Randomly sample \( i'_k \in \{0, \ldots, n_X-1\} \) and then \( j'_k \in \{0, \ldots, n_Y\} \), independently of \( i_k \) and \( j_k \), and compute the stochastic variance reduced gradient for primal update:
15:     \[ \delta_k^\theta = f'_\theta(x_{i'_k}, y_{j'_k})w_{i'_k}^{(k)} - f'_\theta(x_{i'_k}, y_{j'_k})w_{i'_k}^{(k)} + U_k. \]  
16: Update the primal variable:
17:     \[ \theta^{(k)} = \arg \min_{\theta} \left[ \langle \delta_k^\theta, \theta \rangle + g(\theta) + \frac{1}{2\alpha_\theta} \|\theta - \theta^{(k-1)}\|^2 \right]. \]  
18: end for
19: Option I: Set \( \tilde{w}_s = w^{(M)} \) and \( \tilde{\theta}_s = \theta^{(M)} \).
20: Option II: Set \( \tilde{w}_s = w^{(M)} \) and \( \tilde{\theta}_s = \theta^{(t)} \) for randomly sampled \( t \in \{0, \ldots, M-1\} \).
21: end for
22: Output: \( \tilde{\theta}_s \), the last outer-loop iterate.

Based on (17), we develop two stochastic variance reduced estimators, (12) and (15), to approximate the finite-sums in (9) and (10), respectively. The dual gradient estimator \( \delta_k^w \) in (12) is constructed in a standard manner using (17), where the reference variable \( \hat{\theta} \) is a delayed version of \( \theta^{(k)} \). On the other hand, the primal gradient estimator \( \delta_k^\theta \) in (15) is constructed by using reference variables \( (\hat{\theta}, w^{(k)}) \); that is, we use the most recent \( w^{(k)} \) as the dual reference variable, without any delay. As discussed earlier, such a choice leads to a smaller variance in the stochastic estimator \( \delta_k^\theta \) at a potentially higher computation cost (from more frequent evaluation of the batch term). Nevertheless, we are able to show that, with the dual coordinate ascent structure in our algorithm, the batch term \( U_k \) in (15), which is the summation in (10) evaluated at \( (\hat{\theta}, w^{(k)}) \), can be computed efficiently. To see this, note that, after each dual update step in (13), only one term inside this summation in (10), has been changed, i.e., the one associated with \( i = i_k \). Therefore, we can correct \( U_k \) for this term by using recursion (14), which only requires an extra \( O(d\ell) \)-complexity per step (same complexity as (15)).

Note that SVRPDA-I (Algorithm 1) requires to compute and store all the \( T'_s(\hat{\theta}) \) in (11), which is \( O(n_Xd\ell) \)-complexity in storage and could be expensive in some applications. To avoid the cost, we develop a variant of Algorithm 1 named as SVRPDA-II (see Algorithm 2 in the supplementary material), by approximating \( T'_s(\hat{\theta}) \) in (14) with \( f'_\theta(x_{i_k}, y_{i_k}, \tilde{\theta}_s) \), where \( \tilde{\theta}_s \) is another randomly sampled index from \( \{0, \ldots, n_Y - 1\} \), independent of all other indexes. By doing this, we can significantly

4As in [7], we also consider Option II wherein \( \tilde{\theta} \) is randomly chosen from the previous \( M \theta^{(k)} \)’s.
Table 1: The total complexities of different stochastic composition optimization algorithms. For C-SAGA, $\alpha = 2/3$ in the minibatch setting and $\alpha = 1$ when batch-size=1. In the bound for ASCVRG, the dependency on $\kappa$ has been dropped since it was not reported in [10].

| Methods          | SVRPDA-I (Ours) | Comp-SVRG [9] | C-SAGA [23] | MSPBE-SVRG/SAGA [5] | ASCVRG [10] |
|------------------|-----------------|---------------|-------------|--------------------|-------------|
| General: problem | $(n_X n_Y + n_X \kappa n_Y) \ln \frac{1}{\delta}$ | $(n_X n_Y + n_X \kappa n_Y)\ln \frac{1}{\delta}$ | $(n_X n_Y + n_X \kappa n_Y)\ln \frac{1}{\delta}$ | $(n_X n_Y + n_X \kappa n_Y)\ln \frac{1}{\delta}$ | $(n_X n_Y)\ln \frac{1}{\delta}$ |
| Special: problem | $(n_X n_Y + n_X \kappa n_Y)\ln \frac{1}{\delta}$ | $(n_X n_Y + n_X \kappa n_Y)\ln \frac{1}{\delta}$ | $(n_X n_Y + n_X \kappa n_Y)\ln \frac{1}{\delta}$ | $(n_X n_Y + n_X \kappa n_Y)\ln \frac{1}{\delta}$ | $(n_X n_Y)\ln \frac{1}{\delta}$ |

reduce the memory requirement from $O(n_X d\ell)$ in SVRPDA-I to $O(d + n_X \ell)$ in SVRPDA-II (see Section 4.2). In addition, experimental results in Section 5 will show that such an approximation only cause slight performance loss compared to that of SVRPDA-I algorithm.

4 Theoretical Analysis

4.1 Computation complexity

We now perform convergence analysis for the SVRPDA-I algorithm and derive their complexities in computation and storage. To begin with, we first introduce the following assumptions.

Assumption 4.1. The function $g(\theta)$ is $\mu$-strongly convex in $\theta$, and each $\phi_i$ is $1/\gamma$-smooth.

Assumption 4.2. The merit functions $\phi_i(u)$ are Lipschitz with a uniform constant $B_u$:

$$|\phi_i(u) - \phi_i(u')| \leq B_u \|u - u'\|, \quad \forall u, u'; \quad \forall i = 0, \ldots, n_X - 1.$$ 

Assumption 4.3. $f_\theta(x_i, y_{ij})$ is $B_\theta$-smooth in $\theta$, and has bounded gradients with constant $B_f$:

$$\|f'_\theta(x_i, y_{ij})\| \leq B_\theta \|\theta - \theta_i\|, \quad \|f'_\theta(x_i, y_{ij})\| \leq B_f, \quad \forall \theta, \theta_i, \theta_j, \forall i, j.$$ 

Assumption 4.4. For each given $w$ in its domain, the function $L(\theta, w)$ defined in (8) is convex in $\theta$:

$$L(\theta_1, w) - L(\theta_2, w) \geq \langle L'_\theta(\theta_2, w), \theta_1 - \theta_2 \rangle, \quad \forall \theta_1, \theta_2.$$ 

The above assumptions are commonly used in existing compositional optimization works [9], [10], [18], [19], [23]. Based on these assumptions, we establish the non-asymptotic error bounds for SVRPDA-I (using either Option I or Option II in Algorithm 1). The main results are summarized in the following theorems, and their proofs can be found in Appendix E.

Theorem 4.5. Suppose Assumptions 4, 4.3, 4.4 hold. If in Algorithm 1 (with Option I) we choose

$$\alpha_\theta = \frac{1}{n_X \mu (64\kappa + 1)}, \quad \alpha_w = \frac{n_X \mu}{\gamma \alpha_\theta}, \quad M = \left[78.8 n_X \kappa + 1.3 n_X + 1.3\right]$$

where $\lceil x \rceil$ denotes the roundup operation and $\kappa = B^2_\theta/\gamma \mu + B^2_w B^2_\|/\mu^2$, then the Lyapunov function $P_\kappa := E[\|\theta - \theta^*\|^2 + \|
\begin{align*}
\frac{\gamma^2}{\mu \kappa} E[\|\bar{w}_i - w_i\|^2] & \leq (3/4) P_0. \quad \text{Furthermore, the overall computational cost (in number of oracle calls)} \quad \text{for reaching } P_\epsilon \leq \epsilon \text{ is upper bounded by}
\end{align*}

$$O((n_X n_Y + n_X \kappa + n_X) \ln(1/\epsilon)). \quad (18)$$

where, with a slight abuse of notation, $n_Y$ is defined as $n_Y = (n_{Y_0} + \ldots + n_{Y_{n_X-1}})/n_X$.

Theorem 4.6. Suppose Assumptions 4, 4.3, 4.4 hold. If in Algorithm 1 (with Option II) we choose

$$\alpha_\theta = \left(\frac{25 B^2_w}{\gamma} + 10 B_\theta B_w + \frac{80 B^2_w B^2_\|}{\mu}\right)^{-1}, \quad \alpha_w = \frac{\mu}{40 B^2_f}, \quad M = \max\left(\frac{10}{\alpha_\theta \mu \alpha_w \gamma}, \frac{2n_X}{\alpha_\theta \mu}\right),$$

then $P_\kappa := E[\|\theta - \theta^*\|^2 + \frac{\gamma^2}{\mu \kappa} E[\|\bar{w}_i - w_i\|^2] \leq (5/8)^6 P_0$. Furthermore, let $\kappa = \frac{B^2_\theta}{\gamma} + \frac{B^2_w B^2_\|}{\mu^2}$. Then, the overall computational cost (in number of oracle calls) for reaching $P_\epsilon \leq \epsilon$ is upper bounded by

$$O((n_X n_Y + n_X \kappa + n_X) \ln(1/\epsilon)). \quad (19)$$

The above theorems show that the Lyapunov function $P_\kappa$ for SVRPDA-I converges to zero at a linear rate when either Option I or II is used. Since $E[\|\theta - \theta^*\|^2] \leq P_\kappa$, they imply that the computational cost (in number of oracle calls) for reaching $E[\|\theta - \theta^*\|^2] \leq \epsilon$ is also upper bounded by (18) and (19).

\footnote{One oracle call is defined as querying $f_\theta$, $f'_\theta$, or $\phi_i(u)$ for any $0 \leq i < n$ and $u \in \mathbb{R}^\ell$.}
We now briefly discuss and compare the storage complexities of both SVRPDA-I and SVRPDA-II. In Table 2, we report the itemized and total storage complexities for both algorithms, which shows that SVRPDA-II significantly reduces the memory footprint. We also observe that the batch quantities in (11), especially $\bar{f}_i(\hat{\theta})$, dominates the storage complexity in SVRPDA-I. On the other hand, the memory usage in SVRPDA-II is more uniformly distributed over different quantities. Furthermore, although the total complexity of SVRPDA-II, $O(d + n_X \ell)$, grows with the number of samples $n_X$, the $n_X \ell$ term is relatively small because the dimension $\ell$ is small in many practical problems (e.g., $\ell = 1$ in (3) and (4)). This is similar to the storage requirement in SPDC [24] and SAGA [4].

In Appendix D, we also show that our algorithms outperform C-SAGA in experiments.

### Comparison with existing composition optimization algorithms

Table 1 summarizes the complexity bounds for our SVRPDA-I algorithm and compares them with existing stochastic composition optimization algorithms. First, to our best knowledge, none of the existing methods consider the general objective function (1) as we did. Instead, they consider its special case (2), and even in this special case, our algorithm still has better (or comparable) complexity bound than other methods. For example, our bound is better than that of [9] since $\kappa^2 > n_X$ generally holds, and it is better than that of ASCVRG, which does not achieve linear convergence rate (as no strong convexity is assumed). In addition, our method has better complexity than C-SAGA algorithm when $n_X = 1$ (regardless of mini-batch size in C-SAGA), and it is better than C-SAGA for (2) when the mini-batch size is 1\(^6\). However, since we have not derived our bound for mini-batch setting, it is unclear which one is better in this case, and it is an interesting topic for future work. One notable fact from Table 1 is that in this special case (2), the complexity of SVRPDA-I is reduced from $O((n_X + n_Y + n_X \kappa) \ln \frac{1}{\epsilon})$ to $O((n_X + n_Y + n_X \kappa) \ln \frac{1}{\epsilon})$. This is because the complexity for evaluating the batch quantities in (11) (Algorithm 1) can be reduced from $O(n_X n_Y)$ in the general case (1) to $O(n_X n_Y)$ in the special case (2). To see this, note that $f_0$ and $n_Y$ become independent of $i$ in (2) and (11), meaning that we can factor $U_0$ in (11) as $U_0 = \frac{1}{n_X n_Y} \sum_{j=0}^{n_Y-1} f_j'(y_j) \sum_{i=0}^{n_X} w_i^{(0)}$, where the two sums can be evaluated independently with complexity $O(n_Y)$ and $O(n_X)$, respectively. The other two quantities in (11) need only $O(n_Y)$ due to their independence of $i$. Second, we consider the further special case of (2) with $n_X = 1$, which simplifies the objective function (1) so that there is no empirical average outside $\phi_i(\cdot)$. This takes the form of the unsupervised learning objective function that appears in [12]. Note that our results $O((n_Y + \kappa) \log \frac{1}{\epsilon})$ enjoy a linear convergence rate (i.e., log-dependency on $\epsilon$) due to the variance reduction technique. In contrast, stochastic primal-dual gradient (SPDG) method in [12], which does not use variance reduction, can only have sublinear convergence rate (i.e., $O(\frac{1}{\epsilon})$).

### Relation to SPDC [24]

Lastly, we consider the case where $n_Y = 1$ for all $1 \leq i \leq n_X$ and $f_0$ is a linear function in $\theta$. This simplifies (4) to the problem considered in [24], known as the regularized empirical risk minimization of linear predictors. It has applications in support vector machines, regularized logistic regression, and more, depending on how the merit function $\phi$ is defined. In this special case, the overall complexity for SVRPDA-I becomes (see Appendix F):

$$O((n_X + \kappa) \ln(1/\epsilon)),$$

where the condition number $\kappa = B^2 / \mu^2$. In comparison, the authors in [24] propose a stochastic primal dual coordinate (SPDC) algorithm for this special case and prove an overall complexity of $O\left((n_X + \sqrt{n_X \kappa}) \ln \left(\frac{1}{\epsilon}\right)\right)$ to achieve an $\epsilon$-error solution. It is interesting to note that the complexity result in [24] and the complexity result in [24] only differ in their dependency on $\kappa$. This difference is most likely due to the acceleration technique that is employed in the primal update of the SPDC algorithm. We conjecture that the dependency on the condition number of SVRPDA-I can be further improved using a similar acceleration technique.

### 4.2 Storage complexity

We now briefly discuss and compare the storage complexities of both SVRPDA-I and SVRPDA-II. In Table 2, we report the itemized and total storage complexities for both algorithms, which shows that SVRPDA-II significantly reduces the memory footprint. We also observe that the batch quantities in (11), especially $\bar{f}_i(\hat{\theta})$, dominates the storage complexity in SVRPDA-I. On the other hand, the memory usage in SVRPDA-II is more uniformly distributed over different quantities. Furthermore, although the total complexity of SVRPDA-II, $O(d + n_X \ell)$, grows with the number of samples $n_X$, the $n_X \ell$ term is relatively small because the dimension $\ell$ is small in many practical problems (e.g., $\ell = 1$ in (3) and (4)). This is similar to the storage requirement in SPDC [24] and SAGA [4].

| Methods          | $U_0$ | $\langle \bar{f}_i \rangle$ | $\langle \bar{f}_i \rangle$ | $\theta^{(k)}$ | $\hat{\theta}$ | $\{w_i^{(k)}\}$ | $\delta_i^u$ | $\delta_i^w$ | Total          |
|------------------|-------|-----------------------------|-----------------------------|----------------|----------------|----------------|---------------|---------------|----------------|
| SVRPDA-I         | $O(d)$ | $O(n_X \ell)$               | $O(n_X d \ell)$             | $O(d)$         | $O(n_X \ell)$  | $O(d)$         | $O(d)$        | $O(d)$        | $O(n_X d \ell)$ |
| SVRPDA-II        | $O(d)$ | $O(n_X \ell)$               | $O(n_X \ell)$               | $O(d)$         | $O(n_X \ell)$  | $O(d)$         | $O(d)$        | $O(d)$        | $O(d + n_X \ell)$ |
Figure 1: Performance of different algorithms on the risk-averse learning for portfolio management optimization problem. The performance is measured in terms of the number of oracle calls required to achieve a certain objective gap.

5 Experiments

In this section we consider the problem of risk-averse learning for portfolio management optimization \cite{9,10}, introduced in Section 2. Specifically, we want to solve the optimization problem (4) for a given set of reward vectors \( \{x_i \in \mathbb{R}^d : 0 \leq i \leq n - 1 \} \). As we discussed in Section 2, we adopt the alternative formulation (6) for the second term so that it becomes a special case of our general problem (1). Then, we rewrite the cost function into a min-max problem by following the argument in Section 3.1 and apply our SVRPDA-I and SVRPDA-II algorithms (see Appendix C.1 for the details).

We evaluate our algorithms on 18 real-world US Research Returns datasets obtained from the Center for Research in Security Prices (CRSP) website \cite{8}, with the same setup as in \cite{10}. In each of these datasets, we have \( d = 25 \) and \( n = 7240 \). We compare the performance of our proposed SVRPDA-I and SVRPDA-II algorithms with the following state-of-the-art algorithms designed to solve composition optimization problems: (i) Compositional-SVRG-1 (Algorithm 2 of \cite{9}), (ii) Compositional-SVRG-2 (Algorithm 3 of \cite{9}), (iii) Full batch gradient descent, and (iv) ASCVRG algorithm \cite{10}. For the compositional-SVRG algorithms, we follow \cite{9} to formulate it as a special case of the form (2) by using the identification (5). Note that we cannot use the identification (6) for the compositional SVRG algorithms because it will lead to the more general formulation (1) with \( f_{\theta} \) depending on both \( x_i \) and \( y_{ij} \equiv x_j \). For further details, the reader is referred to \cite{9}.

As in previous works, we compare different algorithms based on the number of oracle calls required to achieve a certain objective gap (the difference between the objective function evaluated at the current iterate and at the optimal parameters). One oracle call is defined as accessing the function \( f_{\theta} \), its derivative \( f’_{\theta} \), or \( \phi_i(u) \) for any \( 0 \leq i < n \) and \( u \in \mathbb{R}^\ell \). The results are shown in Figure 1, which shows that our proposed algorithms significantly outperform the baseline methods on all datasets. In addition, we also observe that SVRPDA-II also converges at a linear rate, and the performance loss caused by the approximation is relatively small compared to SVRPDA-I.

\footnote{Additional experiments on the application to policy evaluation in MDPs can be found in Appendix D.}

\footnote{The processed data in the form of .mat file was obtained from https://github.com/tyDLin/SCVRG.}

\footnote{Code: https://github.com/adidevraj/SVRPDA The hyper-parameters can be found in Appendix C.2.}
6 Related Works

Composition optimization have attracted significant attention in optimization literature. The stochastic version of the problem \((2)\), where the empirical averages are replaced by expectations, is studied in \([18]\). The authors propose a two-timescale stochastic approximation algorithm known as SCGD, and establish sublinear convergence rates. In \([19]\), the authors propose the ASC-PG algorithm by using a proximal gradient method to deal with nonsmooth regularizations. The works that are more closely related to our setting are \([9]\) and \([10]\), which consider a finite-sum minimization problem \((2)\) (a special case of our general formulation \((1)\)). In \([9]\), the authors propose the compositional-SVRG methods, which combine SCGD with the SVRG technique from \([7]\) and obtain linear convergence rates. In \([10]\), the authors propose the ASCVRG algorithms that extends to convex but non-smooth objectives. Recently, the authors in \([23]\) propose a C-SAGA algorithm to solve the special case of \((2)\) with \(n_X = 1\), and extend to general \(n_X\). Different from these works, we take an efficient primal-dual approach that fully exploits the dual decomposition and the finite-sum structures.

On the other hand, problems similar to \((1)\) (and its stochastic versions) are also examined in different specific machine learning problems. \([16]\) considers the minimization of the mean square projected Bellman error (MSPBE) for policy evaluation, which has an expectation inside a quadratic loss. The authors propose a two-timescale stochastic approximation algorithm, GTD2, and establish its asymptotic convergence. \([11]\) and \([13]\) independently showed that the GTD2 is a stochastic gradient method for solving an equivalent saddle-point problem. In \([2]\) and \([3]\), the authors derived saddle-point formulations for two other variants of costs (MSBE and MSCBE) in the policy evaluation and the control settings, and develop their stochastic primal-dual algorithms. All these works consider the stochastic version of the composition optimization and the proposed algorithms have sublinear convergence rates. In \([5]\), different variance reduction methods are developed to solve the finite-sum version of MSPBE and achieve linear rate even without strongly convex regularization. Then the authors in \([6]\) extends this linear convergence results to the general convex-concave problem with linear coupling and without strong convexity. Besides, problems of the form \((1)\) was also studied in the context of unsupervised learning \([12, 22]\) in the stochastic setting (with expectations in \((1)\)).

Finally, our work is inspired by the stochastic variance reduction techniques in optimization \([8, 7, 4, 1, 24]\), which considers the minimization of a cost that is a finite-sum of many component functions. Different versions of variance reduced stochastic gradients are constructed in these works to achieve linear convergence rate. In particular, our variance reduced stochastic estimators are constructed based on the idea of SVRG \([7]\) with a novel design of the control variates. Our work is also related to the SPDC algorithm \([24]\), which also integrates dual coordinate ascent with variance reduced primal gradient. However, our work is different from SPDC in the following aspects. First, we consider a more general composition optimization problem \((1)\) while SPDC focuses on regularized empirical risk minimization with linear predictors, i.e., \(n_Y \equiv 1\) and \(f_0\) is linear in \(\theta\). Second, because of the composition structures in the problem, our algorithms also needs SVRG in the dual coordinate ascent update, while SPDC does not. Third, the primal update in SPDC is specifically designed for linear predictors. In contrast, our work is not restricted to that by using a novel variance reduced gradient.

7 Conclusions and Future Work

We developed a stochastic primal-dual algorithms, SVRPDA-I to efficiently solve the empirical composition optimization problem. This is achieved by fully exploiting the rich structures inherent in the reformulated min-max problem, including the dual decomposition and the finite-sum structures. It alternates between (i) a dual step of stochastic variance reduced coordinate ascent and (ii) a primal step of stochastic variance reduced gradient descent. In particular, we proposed a novel variance reduced gradient for the primal update, which achieves better variance reduction with low complexity. We derive a non-asymptotic bound for the error sequence and show that it converges at a linear rate when the problem is strongly convex. Moreover, we also developed an approximate version of the algorithm named SVRPDA-II, which further reduces the storage complexity. Experimental results on several real-world benchmarks showed that both SVRPDA-I and SVRPDA-II significantly outperform existing techniques on all these tasks, and the approximation in SVRPDA-II only caused a slight performance loss. Future extensions of our work include the theoretical analysis of SVRPDA-II, the generalization of our algorithms to Bregman divergences, and applying it to large-scale machine learning problems with non-convex cost functions (e.g., unsupervised sequence classifications).
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Appendix

A Solving (1) in the main paper directly by SGD is biased

Applying the standard chain rule, we obtain the gradient of the cost function in (1) as

$$\frac{1}{n} \sum_{i=0}^{n-1} \phi_i'(\mathcal{F}_i(\theta)) \mathcal{F}_i(\theta)$$

(21)

where:

$$\mathcal{F}_i(\theta) \equiv \frac{1}{n_{y_i}} \sum_{j=0}^{n_{y_i}-1} f_{\theta}(x_i, y_{ij})$$

(22)

and $$f_{\theta}(x, y)$$ denotes the $$d \times \ell$$ matrix, with its $$(i, j)^{th}$$ element defined to be:

$$[f_{\theta}(x, y)]_{i,j} = \frac{\partial}{\partial \theta_i} [(f_{\theta}(x, y)]_j$$

(23)

Note from (21) that there are empirical averages inside and outside $$\phi'_i(\cdot)$$. Therefore, if we sample these empirical averages simultaneously, the stochastic gradient estimator would be biased. In other words, a direct application of stochastic gradient descent to (1) would be intrinsically biased.

B SVRPDA-II algorithm

Algorithm 2 in this supplementary material summarizes the full details of the SVRPDA-II algorithm, which was developed in Section 3.3 of the main paper. Note that it no longer requires the computation or the storage of $$f_i(\tilde{\theta})$$ in (24). Also note that the $$f_i(k)$$ in (27) is replaced with $$f_{\tilde{\theta}}(x_i, y_{i,k})$$ now.

C Experiment details

C.1 Implementation details in risk-averse learning

As we discussed in Section 2, we adopt the alternative formulation (6) for the second term so that it becomes a special case of our general problem (1). Then, using the argument in Section 3.1, the second term in (4) can be rewritten into the objective in (8). Combining it with the first term in (4), the original problem (4) can be reformulated into the following equivalent min-max form:

$$\min_{\theta \in \mathbb{R}^d} \max_w \frac{1}{n} \sum_{i=0}^{n-1} \left( \frac{1}{n} \sum_{j=0}^{n-1} \langle x_i - x_j, \theta \rangle, w \right) - \phi^*(w) - \langle x_i, \theta \rangle$$

(30)

where $$w_i \in \mathbb{R}, \phi^*(w_i) = w_i^2/4$$ and $$w = \{w_0, \ldots, w_{n-1}\}$$. Note that the above min-max problem has an extra $$\langle x_i, \theta \rangle$$ term within the sum. Since it is in a standard empirical average form, we can deal with it in a straightforward manner. Notice that (30) is exactly of the form (1) in the main paper except the last term $$\langle x_i, \theta \rangle$$ within the summation, which as we will show next, can be dealt with in a straightforward manner.

Taking out the $$\langle x_i, \theta \rangle$$ term in (30), based on the discussion in Section 3 of the main paper, the batch gradients used in the algorithm are as follows. Batch gradient of (30) with respect to $$w_i$$, for each $$0 \leq i \leq n - 1$$ can be written as:

$$\overline{f}_i(\theta) = \frac{1}{n} \sum_{j=0}^{n-1} f_{\theta}(x_i, x_j) = \frac{1}{n} \sum_{j=0}^{n-1} \langle x_i - x_j, \theta \rangle$$

(31)
Algorithm 2 SVRPDA-II

1: Inputs: data \{\{x_i, y_{ij}\} : 0 \leq i < n_X, 0 \leq j < n_Y\}; step-sizes \alpha_\theta and \alpha_w; \# inner iterations \(M\).

2: Initialization: \(\tilde{\theta}_0 \in \mathbb{R}^d\) and \(\tilde{w}_0 \in \mathbb{R}^{n_X}\).

3: for \(s = 1, 2, \ldots\) do

4: \hspace{1em} Set \(\tilde{\theta} = \tilde{\theta}_{s-1}, \theta^{(0)}(0) = \tilde{\theta}, \tilde{w}^{(0)}(0) = \tilde{w}_{s-1}\), and compute the batch quantities (for each \(0 \leq i < n_X\)):

\[
U_0 = \sum_{i=0}^{n_X} \sum_{j=0}^{n_Y} \frac{f_\theta(x_i, y_{ij}) w_i^{(0)}}{n_X n_Y}, \quad \tilde{f}_i(\tilde{\theta}) \triangleq \sum_{j=0}^{n_Y} \frac{f_\theta(x_i, y_{ij})}{n_Y}.
\]  

(24)

5: for \(k = 1 \) to \(M\) do

6: Randomly sample \(i_k \in \{0, \ldots, n_X - 1\}\) and then \(j_k \in \{0, \ldots, n_Y - 1\}\) at uniform.

7: Compute the stochastic variance reduced gradient for dual update:

\[
\delta_k^w = f_\theta^{(k-1)}(x_{i_k}, y_{i_k j_k}) - f_\theta(x_{i_k}, y_{i_k j_k}) + \tilde{f}_i(\tilde{\theta}).
\]  

(25)

8: Update the dual variables:

\[
w_i^{(k)} = \arg \min_{w_i} \left[ -\langle \delta_k^w, w_i \rangle + \phi_i^{(w)}(w_i) + \frac{1}{2\alpha_w} \|w_i - w_i^{(k-1)}\|^2 \right]
\]  

if \(i = i_k\).

(26)

if \(i \neq i_k\).

9: Update \(U_k\) according to the following recursion:

\[
U_k = U_{k-1} + \frac{1}{n_X} f_\theta^{(w)}(x_{i_k}, y_{i_k j_k}) (w_i^{(k)} - w_i^{(k-1)}).
\]  

(27)

10: Randomly sample \(i'_k \in \{0, \ldots, n_X - 1\}\) and then \(j'_k \in \{0, \ldots, n_Y - 1\}\), independent of \(i_k\) and \(j_k\), and compute the stochastic variance reduced gradient for primal update:

\[
\delta_k^\theta = f_\theta^{(k-1)}(x_{i'_k}, y_{i'_k j'_k}) w_i^{(k)} - f_\theta(x_{i'_k}, y_{i'_k j'_k}) w_i^{(k)} + U_k.
\]  

11: Update the primal variable:

\[
\theta^{(k)} = \arg \min_{\theta} \left[ \langle \delta_k^\theta, \theta \rangle + g(\theta) + \frac{1}{2\alpha_\theta} \|\theta - \theta^{(k-1)}\|^2 \right].
\]  

(29)

12: end for

13: Option I: Set \(\tilde{w}_s = w^{(k)}\) and \(\tilde{\theta}_s = \theta^{(k)}\).

14: Option II: Set \(\tilde{w}_s = w^{(k)}\) and \(\tilde{\theta}_s = \theta^{(k)}\) for randomly sampled \(t \in \{0, \ldots, M - 1\}\).

15: end for

16: Output: \(\tilde{\theta}_s\) at the last outer-loop iteration.

Batch gradient of (30) with respect to \(\theta\) (without the \langle x_i, \theta \rangle term) is given by:

\[
L_\theta(\theta, w) = \frac{1}{n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} f_\theta(x_i, x_j) w_i
\]  

(32)

\[
= \frac{1}{n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} (x_i - x_j) w_i
\]

For each \(0 \leq i \leq n - 1\), gradient of \(\tilde{f}_{\theta,i}(x_i)\) is given by:

\[
\tilde{f}_{i}(\theta) \triangleq \frac{1}{n} \sum_{j=0}^{n-1} f_\theta(x_i, x_j) = x_i - \frac{1}{n} \sum_{j=0}^{n-1} x_j
\]  

(33)

Based on the above derivation and the expression (17) in the main paper, the stochastic variance reduced gradient for the dual update in both SVRPDA-I and SVRPDA-II is given by

\[
\delta_k^w = \langle x_i - x_j, \theta \rangle + \tilde{f}_i(x_i)
\]  

(34)
and the stochastic variance reduced gradient for the primal update is given by
\[ \delta_k^\theta \triangleq (x_i - x_j)w_i - (x_i - x_j)w_i + \mathcal{L}_\theta(\hat{\theta}, w) = \mathcal{T}_\theta(\hat{\theta}, w) \]  \hfill (35)

Note that, since the function \( f_\theta \) is linear in \( \theta \), the variance reduced gradient for the primal variable is in-fact the full batch gradient.

Next, due to the additional \( \langle x_i, \theta \rangle \) term in (30) (which was omitted in the above definitions), there is an additional term that needs to be added to the variance reduced gradient in (35). Denoting \( g_\theta(x_i) = \langle x_i, \theta \rangle \) and \( \hat{g}_\theta(x_i) = x_i \), the correction batch term is given by:
\[ \bar{g}_\theta = \frac{1}{n} \sum_{i=0}^{n-1} \hat{g}_\theta(x_i) = \frac{1}{n} \sum_{i=0}^{n-1} x_i \]  \hfill (36)

which is independent of \( \hat{\theta} \). In summary, the final variance reduced stochastic gradient for the primal update in both SVRPDA-I and SVRPDA-II is given by:
\[ \delta_k^\theta = \mathcal{T}_\theta(\hat{\theta}, w) - \frac{1}{n} \sum_{i=0}^{n-1} x_i \]  \hfill (37)

\section{C.2 Hyper-parameter choices for algorithms}

In this subsection, we provide the hyper-parameters that are used in our experiments on risk-averse learning (Section 5). We first list the hyper-parameters of our methods below:

- **SVRPDA-I**: \( M = n, \alpha_\theta = 0.0003, \alpha_w = 100. \)
- **SVRPDA-II**: \( M = n, \alpha_\theta = 0.0003, \alpha_w = 100. \)

Then, we provide the hyper-parameters used in the baseline methods:

- Compositional-SVRG-1 (Algorithm 2 of [9]): \( K = n, A = 6, \gamma = 0.0003; \)
- Compositional-SVRG-2 (Algorithm 3 of [9]): \( K = n, A = 3, B = 3, \gamma = 0.0004; \)
- ASCVRG: The results are obtained by using their publicly released code on github: \url{https://github.com/tyDLin/SCVRG} with the same setting and choice of hyper-parameters.
- Batch gradient descent: step size \( \alpha = 0.01; \)

Note that, for the Compositional-SVRGs and batch gradient algorithms, the above choice of the hyper-parameters are obtained by sweeping through a set of hyper-parameters and choosing the ones with the best performance. For ASCVRG, we use the the publicly released code by the authors.

\section{D Additional experiments on MDP policy evaluation}

Consider a Markov decision process (MDP) problem with state space \( S \) and action space \( A \). We assume that both \( S \) and \( A \) are finite, and define \( S = \{1, \ldots, S\} \). For any \( 1 \leq i, j \leq S \), we denote by \( r_{i,j} \) the reward associated with transition from state \( i \) to state \( j \). Given a policy \( \pi : S \to \mathcal{P}(A) \), where \( \mathcal{P}(A) \) denote the probability space over \( A \), we let \( P^\pi \in \mathbb{R}^{S \times S} \) denote the associated state transition probability matrix. The goal in policy evaluation is to estimate the value function \( V^\pi : S \to \mathbb{R} \) associated with the policy \( \pi \), which is a fixed-point solution to the following Bellman equation:
\[ V^\pi(i) = \sum_{j=1}^{S} P_{i,j}^\pi (r_{i,j} + \gamma V^\pi(j)), \quad 1 \leq i \leq S, \]

where \( 0 < \gamma < 1 \) denotes the discount factor. We consider a linear function approximation to the value function: \( V^\pi(i) \approx \langle \Psi_i, \theta \rangle \), where \( \{ \Psi_i \in \mathbb{R}^d : 1 \leq i \leq S \} \) denotes the feature vectors, and \( \theta \in \mathbb{R}^d \) denotes the weight vector to be learned. The problem of finding the optimal weight vector \( \theta^* \) that best approximates \( V^\pi \) can be formulated as the following optimization problem \([19][23]\):
\[ \theta^* = \arg \min_{\theta} \left\{ F(\theta) := \frac{1}{S} \sum_{i=1}^{S} \left( \langle \Psi_i, \theta \rangle - \sum_{j=1}^{S} P_{i,j}^\pi (r_{i,j} + \gamma \langle \Psi_j, \theta \rangle) \right)^2 \right\}. \]  \hfill (38)
Note that the above problem can be expressed as a special case of (1) by using the following identifications: For each $1 \leq i \leq S$, $n_X = n_Y = S$, $\phi_i(u) = u^2$, and

$$f_\theta(x_i, y_j) = \langle \Psi_i, \theta \rangle - S \cdot P_{i, j}^\pi \left( r_{i, j} + \gamma \langle \Psi_j, \theta \rangle \right), \quad 1 \leq i, j \leq S. \quad (39)$$

And it is also possible, although less intuitive, to rewrite (38) as a special case of (2). In existing composition optimization literature such as [23, 19], this is achieved via higher dimensional transformation, with $f_\theta : \mathbb{R}^d \rightarrow \mathbb{R}^{2S}$, $\phi_i : \mathbb{R}^{2S} \rightarrow \mathbb{R}$, $n_Y = S$, and $n_X = S$. Denote by $Q_\theta^i$ the Q-function:

$$Q_\theta^i(i) := \sum_{j=1}^S P_{i,j}^\pi \left( r_{i,j} + \gamma \langle \Psi_j, \theta \rangle \right), \quad 1 \leq i \leq S.$$  

Then, by defining the function $f_\theta$ and $\phi_i$ such that

$$\frac{1}{S} \sum_{j=1}^S f_\theta(y_j) = \left[ \langle \Psi_1, \theta \rangle, Q_\theta^1(1), \ldots, \langle \Psi_S, \theta \rangle, Q_\theta^S(S) \right] \quad (40)$$

and 

$$\frac{1}{S} \sum_{i=1}^S \phi_i \left( \langle \Psi_1, \theta \rangle, Q_\theta^1(1), \ldots, \langle \Psi_S, \theta \rangle, Q_\theta^S(S) \right) = \frac{1}{S} \sum_{i=1}^S \left( \langle \Psi_i, \theta \rangle - Q_\theta^i(i) \right)^2,$$

the problem (38) can be reformulated as (2). The reader is referred to [23, 19] for more details.

We evaluate our algorithms on two experimental settings, one with $S = 10$, $d = 5$, and another with $S = 10^4$ and $d = 10$. In each of these two cases, $\gamma$ was set to be 0.9, and both the transition probability matrix $P^\pi \in \mathbb{R}^{S \times S}$ and the feature vectors $\{\Psi_i \in \mathbb{R}^d : 1 \leq i \leq S\}$ were randomly generated. We compare the performance of algorithms SVRPDA-I and SVRPDA-II with the C-SAGA algorithm of [23], as it is the most recent composition optimization that we are aware of, and is shown to be superior to all existing algorithms on this MDP policy evaluation task [23]. The hyper-parameters for the C-SAGA algorithm were chosen as follows. For $S = 10$ case, we choose all hyper-parameters as in [23]: Mini-batch size $s = 1$ and step-size $\eta = 0.1$. For the $S = 10^4$ case, we choose mini-batch size $s = 100$, and step-size $\eta = 0.0005$ (see [23] for details on what these hyper-parameters mean). The hyper-parameters for the SVRPDA-I and SVRPDA-II are chosen as follows. For the MDP with $S = 10$, we choose $M = 150$, $\alpha_\theta = 0.1$, and $\alpha_w = 0.25$ for SVRPDA-I, and choose $M = 15$, $\alpha_\theta = 0.5$, and $\alpha_w = 1.25$ for SVRPDA-II. For the MDP with $S = 10^4$, we choose $M = 13500$, $\alpha_\theta = 0.01$ and $\alpha_w = 16 \times 10^4$ for SVRPDA-I. For SVRPDA-II, $M = 13500$, $\alpha_\theta = 0.01$ and $\alpha_w = 16 \times 10^4$.

The performance criteria was chosen to be the number of oracle calls required to achieve a certain “objective gap”, defined as $F(\theta) - F(\theta^*)$. Notice that “one function call”, when the function is of the form (39) is not comparable to one function call, when the function is defined according to (40). Due to the fact that $f_\theta$ for the C-SAGA algorithm is of dimension $2S$ (as opposed to 1 in our formulation), we count 2S oracle calls whenever a function of this form is called for the purpose of fair comparison. The results for MDP with $S = 10$ and $S = 10^4$ are reported in Figures 2 and 3, respectively. We observe that despite having much smaller memory requirement, SVRPDA-II has a comparable/better performance than C-SAGA, while SVRPDA-I is clearly better than C-SAGA.
E Convergence and complexity of SVRPDA-I: Proof

In this section, we derive the (non-asymptotic) bound of the SVRPDA-I algorithm and its total computation complexity. For convenience, we first repeat the saddle point formulation and the
Also, recall the definitions of $\tilde{f}_i(\theta)$ and $\tilde{f}_j(\theta)$:

$$\tilde{f}_i(\theta) = \frac{1}{n_{Y_i}} \sum_{j=0}^{n_{Y_i}-1} f_\theta(x_i, y_{ij}), \quad \tilde{f}_j(\theta) = \frac{1}{n_{Y_i}} \sum_{j=0}^{n_{Y_i}-1} f'_\theta(x_i, y_{ij})$$  \hspace{1cm} (42)

Furthermore, we defined $L(\theta, w)$ and its gradient as

$$L(\theta, w) := \frac{1}{n_X} \sum_{i=0}^{n_X-1} \left( \langle f_\theta(x_i, y_i), w_i \rangle - \phi_i^*(w_i) \right)$$  \hspace{1cm} (43)

$$L'_\theta(\theta, w) := \frac{1}{n_X} \sum_{i=0}^{n_X-1} \tilde{f}_i(\theta) w_i$$  \hspace{1cm} (45)

Using the above notations, the saddle point problem (41) can be rewritten as

$$\min_\theta \min_w \frac{1}{n_X} \sum_{i=0}^{n_X-1} \left( \langle \tilde{f}_i(\theta), w_i \rangle - \phi_i^*(w_i) \right) + g(\theta)$$  \hspace{1cm} (46)

### E.1 Compact Notation

Throughout this section, we introduce the following compact notation, to ease exposition of the proof: For any $\theta \in \mathbb{R}^d$, and $0 \leq i \leq n_X - 1$ and $0 \leq j \leq n_{Y_i} - 1$, we denote:

$$f_{ij}(\theta) \equiv f_\theta(x_i, y_{ij})$$  \hspace{1cm} (47)

Therefore, the stochastic variance reduced gradient for dual update defined in (12) of the main paper is rewritten as:

$$\delta^w_k = f_{i_kj_k}(\theta^{(k-1)}) - f_{i_kj_k}(\tilde{\theta}) + \tilde{f}_{i_k}(\tilde{\theta})$$  \hspace{1cm} (48)

Similarly, the stochastic variance reduced gradient for primal update defined in (15) of the main paper is:

$$\delta^\theta_k = f'_{i_kj_k}(\theta^{(k-1)}) v_{i_k}^{(k)} - f'_{i_kj_k}(\tilde{\theta}) v_{i_k}^{(k)} + L'_\theta(\tilde{\theta}, w_k)$$  \hspace{1cm} (49)

where we used the fact that $U_k \equiv L'_\theta(\tilde{\theta}, w_k)$. We now proceed to recall the Algorithm 1 rewritten in a simplified form, using the compact notation.

### E.2 Algorithm

Before we proceed to prove the convergence of the algorithm, we first recall the update equations of the algorithm. The following updates are at stage $s$ of the outerloop; To simplify exposition, we suppress dependency on $s$, and let $\tilde{\theta} \equiv \theta_s$ throughout.

For the dual update, at each iteration $k$, we first randomly pick an index $0 \leq i_k \leq n_X - 1$ at uniform, and then pick another index $0 \leq j_k \leq n_{Y_{i_k}} - 1$ at uniform. For the chosen $(i_k, j_k)$, we first compute the variance reduced stochastic gradient $\delta^w_k$ of $\tilde{f}_i(\theta)$ using (48):

$$\delta^w_k = f_{i_kj_k}(\theta^{(k-1)}) - f_{i_kj_k}(\tilde{\theta}) + \tilde{f}_{i_k}(\tilde{\theta})$$

Then, we update the dual variables according to the recursion (13):

$$w_i^{(k)} = \begin{cases} \arg \min_{w_i} \left[ -\langle \delta^w_k, w_i - w_i^{(k-1)} \rangle + \phi_i^*(w_i) + \frac{1}{2\alpha_w} \|w_i - w_i^{(k-1)}\|^2 \right] & \text{if } i = i_k \\ w_i^{(k-1)} & \text{if } i \neq i_k \end{cases}$$  \hspace{1cm} (50)
We restate the Assumptions in Section 4 here using the notation in (47) to make the reading easier:

**Assumption E.1.** The function \( g(\theta) \) is \( \mu \)-strongly convex in \( \theta \), and each \( \phi_i \) is \( 1/\gamma \)-smooth.

**Assumption E.2.** The merit functions \( \phi_i(u) \) are Lipschitz with a uniform constant \( B_w \):

\[
|\phi_i(u) - \phi_i(u')| \leq B_w \|u - u'\|, \quad \forall u, u' \in \mathbb{R}^d, 0 \leq i \leq n_X - 1.
\]

**Assumption E.3.** \( f_{ij}(\theta) \) is \( B_\theta \)-smooth in \( \theta \), and has bounded gradients with constant \( B_f \): For each \( 0 \leq i \leq n_X - 1 \) and \( 0 \leq j \leq n_{Y_i} - 1 \),

\[
\|f'_{ij}(\theta_1) - f'_{ij}(\theta_2)\| \leq B_\theta \|\theta_1 - \theta_2\|, \quad \|f'_{ij}(\theta)\| \leq B_f, \quad \forall \theta, \theta_1, \theta_2 \in \mathbb{R}^d
\]

**Assumption E.4.** For each given \( w \) in its domain, the function \( L(\theta, w) \) defined in (8) is convex in \( \theta \):

\[
L(\theta_1, w) - L(\theta_2, w) \geq \langle L'_\theta(\theta_2, w), \theta_1 - \theta_2 \rangle.
\]

### E.4 Preliminary results

In this subsection, we introduce lemmas which lay the foundation for the proof of the main convergence result that follows. First, our proof relies on the following important lemma, which is a slightly adjusted version of Lemma 3 in [20] for our problem setting.

**Lemma E.5.** Consider any function of the form \( P(x) = f(x) + g(x) \), with \( x \in \mathbb{R}^d \). Suppose \( f(x) \) is linear in \( x \), and \( g(x) \) is \( \mu_g \)-strongly convex. Then, for \( \alpha > 0 \), the following holds for any vector \( v \in \mathbb{R}^d \) and \( y \in \mathbb{R}^d \):

\[
P(y) \geq P(x^{(+)}) + \frac{1}{\alpha} \langle (x-x^{(+)})\rangle (y-x) + \langle v-f'(x) \rangle, (x^{(+)}) \rangle + \frac{1}{\alpha} \|x-x^{(+)\}}\|^2 + \frac{\mu_g}{2} \|y-x^{(+)\}}\|^2
\]

where:

\[
x^{(+)\}} = \text{prox}_{\alpha g}\{x - \alpha v\}
\]

\[
= \arg\min_w \left\{ \|g(w) + \frac{1}{2\alpha} \|w - x + \alpha v\|^2 \right\}
\]

**Proof.** Based on the definition of \( x^{(+)\}} \), the optimality condition associated with the proximal operator states that there exists a sub-gradient \( \xi \in \partial g(x^{(+)\}} \) such that

\[
\frac{x^{(+)\}} - x + \alpha v}{\alpha} + \xi = 0
\]

(52)
where $\partial g(x^{(+)})$ denotes the sub-differential of $g$ at $x^{(+)}$. Next, by the linearity of $f$ and the strong convexity of $g$, we have, for any $x, y \in \mathbb{R}^d$,

$$P(y) = f(y) + g(y)$$

\[ \begin{align*}
  &\geq f(x^{(+)}) + \langle f'(x), (y-x^{(+)}) \rangle + g(x^{(+)}) + \langle \xi, (y-x^{(+)}) \rangle + \frac{\mu_g}{2} \| y - x^{(+)} \|^2 \\
  &= b \cdot P(x^{(+)}) + \langle f'(x), (y-x^{(+)}) \rangle + \langle \xi, (y-x^{(+)}) \rangle + \frac{\mu_g}{2} \| y - x^{(+)} \|^2 \\
  &= P(x^{(+)}) + \langle f'(x), (y-x^{(+)}) \rangle - \frac{1}{\alpha} \| (x^{(+)}) - x + \alpha v \|, (y-x^{(+)}) \rangle + \frac{\mu_g}{2} \| y - x^{(+)} \|^2 \\
  &\geq P(x^{(+)}) + \langle f'(x) - v, (y-x^{(+)}) \rangle - \frac{1}{\alpha} \| (x^{(+)}) - x \|, (y-x^{(+)}) \rangle + \frac{\mu_g}{2} \| y - x^{(+)} \|^2 \\
  &\geq P(x^{(+)}) + \langle f'(x) - v, (y-x^{(+)}) \rangle + \frac{1}{\alpha} \| x^{(+)} - x \|^2 \\
  &- \frac{1}{\alpha} \| (x^{(+)}) - x \|, (y-x^{(+)}) \rangle + \frac{\mu_g}{2} \| y - x^{(+)} \|^2 \\
\end{align*} \]

where step (a) follows from the linearity of $f$ and the strong convexity of the function $g$, step (b) uses the definition $P(x^{(+)}) = f(x^{(+)}) + g(x^{(+)})$, step (c) substitutes the expression of $\xi$ from (52), step (d) rearrange the second and the third terms, and step (e) completes the proof by adding and subtracting $x$ in the second inner product.

The difference between our Lemma E.5 and Lemma 3 in [20] is that our function $v$ be smaller than a certain positive number in Lemma 3 of [20]). This lemma is useful for deriving a bound when the update recursions are defined by a proximal mapping with an arbitrary update vector $v$. This is particularly helpful for our case as both our primal and dual updates are in proximal mapping form with the update vector $v$ being variance reduced stochastic gradient.

Next, we quote the Lemma 2 of [20] below:

**Lemma E.6.** Let $R$ be a closed convex function on $\mathbb{R}^d$ and let $x, y \in \text{dom}(R)$. Then:

$$\| \text{prox}_R(x) - \text{prox}_R(y) \| \leq \| x - y \|$$

(54)

We next introduce a useful property of the conjugate function:

**Lemma E.7.** When Assumption E.2 holds, the domain of $\phi_i^*(w_i)$, denoted as $\text{dom}(\phi_i)$, satisfies

$$\text{dom}(\phi_i) \subseteq \{ w_i : \| w_i \| \leq B_w \}$$

(55)

That is, for any $w_i$ that satisfies $\| w_i \| > B_w$, we will have $\phi_i^*(w_i) = +\infty$. In consequence, the dual variables $u_i^{(k)}$ obtained from the dual update (50) will always be bounded by $B_w$ throughout the iterations.

**Proof.** For any given $w_i$ that satisfies $\| w_i \| > B_w$, define $u_i = u_i + \frac{w_i}{\| w_i \|} t$, where $t$ is an arbitrary real scalar. Then, by the definition of conjugate function, we have

$$\phi_i^*(w_i) = \sup_{u_i} \left[ \langle w_i, u_i \rangle - \phi_i(u_i) \right]$$

\[ \begin{align*}
  &\geq \sup_{t} \left[ \langle w_i, u_i + \frac{w_i}{\| w_i \|} t \rangle - \phi_i \left( u_i + \frac{w_i}{\| w_i \|} t \right) \right] \\
  &= \sup_{t} \left[ \| w_i \| t - \phi_i \left( u_i + \frac{w_i}{\| w_i \|} t \right) + \phi_i(u_i) \right] + \langle w_i, u_i \rangle - \phi_i(u_i) \\
  &= \sup_{t} \left[ \| w_i \| t - B_w \left( \frac{w_i}{\| w_i \|} t \right) \right] + \langle w_i, u_i \rangle - \phi_i(u_i) \\
  &\geq \sup_{t} \left[ \| w_i \| t - B_w \| t \| \right] + \langle w_i, u_i \rangle - \phi_i(u_i) \\
\end{align*} \]
where step (a) uses the fact that the supremum over a subset (line) is smaller, and step (b) uses the following inequality obtained from the Lipschitz property of \( \phi_i(u_i) \):

\[
|\phi_i(u) - \phi_i(u')| \leq B_w \|u - u'\|, \quad \forall u, u' \quad \Rightarrow \quad -\phi_i(u) + \phi_i(u') \geq -B_w \|u - u'\|
\]  

E.5 Dual Bound

In order to derive the bound for the dual update, we first introduce an auxiliary dummy variable \( w'_{ij} \):

\[
w'_{ij} = \arg \min_{w_i} \left[ -\left\langle \delta^w_{ij}, w_i - w_i^{(k-1)} \right\rangle + \phi^*_i(w_i) + \frac{1}{2\alpha_w} \|w_i - w_i^{(k-1)}\|^2 \right]
\]

\[
= \text{prox}_{\alpha_w \phi^*_i} \left[ w_i^{(k-1)} - \alpha_w \delta^w_{ij} \right]
\]

where,

\[
\delta^w_{ij} := f_{ij}(\theta^{(k-1)}) - f_{ij}(\bar{\theta}) + \bar{T}_i(\bar{\theta})
\]

The variable \( w'_{ij} \) can be understood as the updated value of the dual variable if \( i \) and \( j \) is selected.

Our analysis in this section focuses on deriving bounds for the \( \|w_i^{(k)} - w_i^*\|^2 \). We will first examine \( \|w'_i - w_i^*\|^2 \) and then relate it to \( \|w_i^{(k)} - w_i^*\|^2 \). To begin with, for each \( i \) and \( j \), we have

\[
\|w'_i - w_i^*\|^2 = \|w'_i - w_i^{(k-1)} + w_i^{(k-1)} - w_i^*\|^2
\]

\[
= \|w'_i - w_i^{(k-1)}\|^2 + \|w_i^{(k-1)} - w_i^*\|^2 + 2\langle (w'_i - w_i^{(k-1)}), (w_i^{(k-1)} - w_i^*) \rangle
\]

Now, we upper bound the first and the third terms in (60) together. For a given \( \theta^{(k)} \) and \( i \), define

\[
P_{w_i}(x) := -\left\langle \bar{T}_i(\theta^{(k-1)}), x \right\rangle + \phi^*_i(x)
\]

Note that the first part of the function is linear in \( x \) and the second part of the function is \( \gamma \)-strongly convex (since \( \phi_i \) is \( 1/\gamma \)-smooth by Assumption E.1). Furthermore, by (58), the update rule for the dummy variables \( w'_{ij} \) is defined by a proximal operator. Therefore, we can apply Lemma E.5 with \( P(x) \equiv P_{w_i}(x) \) and the following identifications:

\[
f(x) = -\left\langle \bar{T}_i(\theta^{(k-1)}), x \right\rangle \quad g(x) = \phi^*_i(x) \quad v = -\delta^w \quad x = w_i^{(k-1)} \quad x^{(+)} = w'_{ij} \quad y = w_i^* \quad \alpha = \alpha_w
\]

which leads to

\[
-\left\langle \bar{T}_i(\theta^{(k-1)}), w_i^* \right\rangle + \phi^*_i(w_i^*) \geq -\left\langle \bar{T}_i(\theta^{(k-1)}), w'_i \right\rangle + \phi^*_i(w'_i) \]

\[
+ \frac{1}{\alpha_w} \left( (w_i^{(k-1)} - w'_i), (w_i^* - w_i^{(k-1)}) \right)
\]

\[
- \left\langle \delta^w_{ij} - \bar{T}_i(\theta^{(k-1)}), w'_i - w_i^* \right\rangle
\]

\[
+ \frac{1}{\alpha_w} \left\| w_i^{(k-1)} - w'_i \right\|^2 + \frac{\gamma}{2} \left\| w_i^* - w'_i \right\|^2
\]

Furthermore, by definition, since \( w_i^* \) is the optimal solution to the following optimization problem,

\[
w_i^* = \arg \min_{w_i} \left\{ \phi^*_i(w_i) - \left\langle \bar{T}_i(\theta^*), w_i \right\rangle \right\}
\]

and by the fact that the cost function inside the above arg min is \( \gamma \)-strongly convex due to \( \phi^*_i(\cdot) \), we have

\[
-\left\langle \bar{T}_i(\theta^*), w'_i \right\rangle + \phi^*_i(w'_i) \geq -\left\langle \bar{T}_i(\theta^*), w_i^* \right\rangle + \phi^*_i(w_i^*) + \frac{\gamma}{2} \left\| w'_i - w_i^* \right\|^2
\]

Adding (62) and (63) cancels the \( \phi^*_i \) terms and leads to

\[
\left\langle \bar{T}_i(\theta^{(k-1)}) - \bar{T}_i(\theta^*), w'_i - w_i^* \right\rangle \geq \frac{1}{\alpha_w} \left( (w_i^{(k-1)} - w'_i), (w_i^* - w_i^{(k-1)}) \right)
\]

\[
- \left\langle \delta^w_{ij} - \bar{T}_i(\theta^{(k-1)}), w'_i - w_i^* \right\rangle
\]

\[
+ \frac{1}{\alpha_w} \left\| w_i^{(k-1)} - w'_i \right\|^2 + \frac{\gamma}{2} \left\| w_i^* - w'_i \right\|^2
\]
Multiplying both sides by $2\alpha_w$ and rearranging the terms, we obtain

$$2\alpha_w(\mathcal{J}_i(\theta^{(k-1)}) - \mathcal{J}_i(\theta^*) - w_{ij}' - w_{ii}'^*) + 2\alpha_w\langle \delta_{ij}' - \mathcal{J}_i(\theta^{(k-1)}) \rangle, w_{ij}' - w_{ii}'^* \rangle - 2\alpha_w\gamma \|w_{ij}' - w_{ii}'^*\|^2 \geq 2\langle w_{ij}' - w_{ii}'^*(k-1) \rangle_j + 2\|w_{ij}'(k-1) - w_{ij}'\|^2$$ \hspace{1cm} (65)

Now, observe that inequality (65) could be used as an upper bound for the first and third terms on the right hand side of (60). Using this, (60) becomes:

$$\|w_{ij}' - w_{ii}'^*\|^2 \leq \|w_{ij}' - w_{ii}'^*(k-1)\|^2 + 2\|w_{ij}' - w_{ii}'^*(k-1)\|^2 + 2\|w_{ij}' - w_{ii}'^*(k-1)\|^2 + 2\|w_{ij}' - w_{ii}'^*(k-1)\|^2$$ \hspace{1cm} (66)

where step (a) added and subtracted a $\|w_{ij}'(k-1) - w_{ii}'^*(k-1)\|^2$ in order to apply (65) in the following inequality. Dividing both sides by $2\alpha_w$ and combining common terms, we get the following bound:

$$\left(\frac{1}{2\alpha_w} + \gamma\right)\|w_{ij}' - w_{ii}'^*\|^2 \leq \frac{1}{2\alpha_w}\|w_{ij}'(k-1) - w_{ii}'^*\|^2 + \frac{1}{2\alpha_w}\|w_{ij}' - w_{ii}'^*(k-1)\|^2 + \langle \mathcal{J}_i(\theta^{(k-1)}) - \mathcal{J}_i(\theta^*), w_{ij}' - w_{ii}'^* \rangle + \langle \mathcal{J}_i(\theta^{(k-1)}) - \mathcal{J}_i(\theta^*), w_{ij}' - w_{ii}'^* \rangle$$ \hspace{1cm} (67)

Next, we will bound the last term in (67). Consider the full batch dual ascent algorithm. In this case, for each $0 \leq i \leq n_X - 1$, the update rule is given by:

$$\mathcal{W}_i^{(k)} = \arg\min_{w_i} \left[ -\langle \mathcal{J}_i(\theta^{(k-1)}) \rangle_i + \phi_i^*(w_i) + \frac{1}{2\alpha_w}\|w_i - w_{i}^{(k-1)}\|^2 \right] = \text{prox}_{\alpha_w\phi_i^*} \left[ w_i^{(k-1)} - \alpha_w\mathcal{J}_i(\theta^{(k-1)}) \right]$$ \hspace{1cm} (68)

The above update rule will only be used for analysis. Considering the last term in the right hand side of (67), we have:

$$\langle \delta_{ij}' - \mathcal{J}_i(\theta^{(k-1)}) \rangle_i, w_{ij}' - w_{ii}'^* \rangle = \langle \delta_{ij}' - \mathcal{J}_i(\theta^{(k-1)}) \rangle_i, w_{ij}' - \mathcal{W}_i^{(k)} \rangle + \langle \delta_{ij}' - \mathcal{J}_i(\theta^{(k-1)}) \rangle_i, \mathcal{W}_i^{(k)} - w_{ii}'^* \rangle \geq \langle \delta_{ij}' - \mathcal{J}_i(\theta^{(k-1)}) \rangle_i, w_{ij}' - \mathcal{W}_i^{(k)} \rangle + \langle \delta_{ij}' - \mathcal{J}_i(\theta^{(k-1)}) \rangle_i, \mathcal{W}_i^{(k)} - w_{ii}'^* \rangle \geq \alpha_w\|\delta_{ij}' - \mathcal{J}_i(\theta^{(k-1)})\|w_{ij}' - \mathcal{W}_i^{(k)}\|^2 + \langle \delta_{ij}' - \mathcal{J}_i(\theta^{(k-1)}) \rangle_i, \mathcal{W}_i^{(k)} - w_{ii}'^* \rangle$$ \hspace{1cm} (69)

where (a) uses Cauchy-Schwarz inequality, and step (b) substitutes the proximal expressions of $w_{ij}'$ in (68) and $\mathcal{W}_i^{(k)}$ in (69) followed by Lemma E.6. Averaging both sides of the inequality over all $0 \leq j \leq n_Y - 1$ and using the fact that the average of the second term on the right hand side of (69) is zero, we get:

$$\frac{1}{n_Y} \sum_{j=0}^{n_Y-1} \left[ \langle \delta_{ij}' - \mathcal{J}_i(\theta^{(k-1)}) \rangle_i, w_{ij}' - w_{ii}'^* \rangle \right] \leq \frac{\alpha_w}{n_Y} \sum_{j=0}^{n_Y-1} \left[ \|\delta_{ij}' - \mathcal{J}_i(\theta^{(k-1)})\|w_{ij}' - \mathcal{W}_i^{(k)}\|^2 \right] = \frac{\alpha_w}{n_Y} \sum_{j=0}^{n_Y-1} \left[ \|f_{ij}(\theta^{(k-1)}) - f_{ij}(\theta) + \mathcal{J}_i(\theta) - \mathcal{J}_i(\theta^{(k-1)})\|w_{ij}'(k-1)\|^2 \right] = \frac{(a)}{\alpha_w} \frac{\alpha_w}{n_Y} \sum_{j=0}^{n_Y-1} \left[ \left(1 - \frac{1}{n_Y}\right)f_{ij}(\theta^{(k-1)}) \right]$$
\[-\left(1 - \frac{1}{n_{Y_i}}\right) f_{ij}(\bar{\theta}) - \left(\bar{F}_i(\theta^{(k-1)}) - F_i(\bar{\theta}) - \frac{1}{n_{Y_i}} f_{ij}(\theta^{(k-1)}) + \frac{1}{n_{Y_i}} f_{ij}(\bar{\theta})\right)\] 
\leq \frac{2\alpha_w}{n_{Y_i}} \sum_{j=0}^{n_{Y_i} - 1} \left[\left\|\left(1 - \frac{1}{n_{Y_i}}\right) f_{ij}(\theta^{(k-1)}) - \left(1 - \frac{1}{n_{Y_i}}\right) f_{ij}(\bar{\theta})\right\|^2 \right. 
\left. + \left\|\bar{F}_i(\theta^{(k-1)}) - F_i(\bar{\theta}) - \frac{1}{n_{Y_i}} f_{ij}(\theta^{(k-1)}) + \frac{1}{n_{Y_i}} f_{ij}(\bar{\theta})\right\|^2 \right] 
= \frac{2\alpha_w}{n_{Y_i}} \sum_{j=0}^{n_{Y_i} - 1} \left[\left\|\left(1 - \frac{1}{n_{Y_i}}\right) f_{ij}(\theta^{(k-1)}) - \left(1 - \frac{1}{n_{Y_i}}\right) f_{ij}(\bar{\theta})\right\|^2 \right. 
\left. + \left(\frac{n_{Y_i} - 1}{n_{Y_i}}\right)^2 \frac{1}{n_{Y_i} - 1} \left\|\left[f_{ij}(\theta^{(k-1)}) - f_{ij}(\bar{\theta})\right]\right\|^2 \right] 
\leq 2\alpha_w \left[\left(\frac{1}{n_{Y_i}}\right)^2 B_j^2 \|\theta^{(k-1)} - \bar{\theta}\|^2 + \frac{n_{Y_i} - 1}{n_{Y_i}^2} B_j^2 \|\theta^{(k-1)} - \bar{\theta}\|^2 \right] 
= 2\alpha_w B_j^2 \left(1 - \frac{1}{n_{Y_i}}\right) \|\theta^{(k-1)} - \bar{\theta}\|^2 \tag{70}\]

where step (a) follows by adding and subtracting \(\frac{1}{n_{Y_i}} \left(f_{ij}(\theta^{(k-1)}) - f_{ij}(\bar{\theta})\right)\), step (b) uses \(\|a + b\|^2 \leq 2\|a\|^2 + 2\|b\|^2\), step (c) applies Jensen’s inequality to the second term, step (d) applies Assumption E.3 (uniformly bounded gradients implies the uniform Lipschitz continuity of the functions \(f_{ij}\)) to both terms. Averaging both sides of (67) over all \(0 \leq j \leq n_{Y_i} - 1\), and using (70), we obtain:

\[
\left(\frac{1}{2\alpha_w} + \gamma\right) \frac{1}{n_{Y_i}} \sum_{j=0}^{n_{Y_i} - 1} \|w'_{ij} - w_i^*\|^2 \leq \frac{1}{2\alpha_w} \|w_i^{(k-1)} - w_i^*\|^2 - \frac{1}{2\alpha_w} \frac{1}{n_{Y_i}} \sum_{j=0}^{n_{Y_i} - 1} \|w'_{ij} - w_i^{(k-1)}\|^2 \tag{71}\]

Finally, we relate the bound for \(w_{ij}'\) back to the bound for the dual variable \(w_i^{(k)}\). Recall that, for each \(w_i\), there is a probability \(1/n_X\) that it will be selected and updated, and a probability of \((n_X - 1)/n_X\) that it will be kept the same as \(w_i^{(k-1)}\). Furthermore, conditioned on the fact that \(w_i\) is selected, it will be updated to \(w_{ij}'\), with probability \(1/n_{Y_i}\). Therefore, for each \(w_i\), there is a probability \(1/n_X n_{Y_i}\) that it will be updated to \(w_{ij}'\) for \(j = 0, \ldots, n_{Y_i} - 1\), and a probability of \((n_X - 1)/n_X\) that it remains the same. Therefore, letting \(\mathcal{F}_k\) denote the filtration of all events up to the beginning of iteration \(k\)
(before the dual update step), we have:

\[
E\{w_i^{(k)} \mid F_k\} = \frac{1}{n_X n_Y} \sum_{j=0}^{n_Y-1} w_{ij}' + \frac{n_X - 1}{n_X} w_i^{(k-1)}
\]

\[
E\{w_i^{(k)} - w_i^* \mid F_k\} = \frac{1}{n_X n_Y} \sum_{j=0}^{n_Y-1} (w_{ij}' - w_i^*) + \frac{n_X - 1}{n_X} (w_i^{(k-1)} - w_i^*)
\]

\[
E\{||w_i^{(k)} - w_i^*||^2 \mid F_k\} = \frac{1}{n_X n_Y} \sum_{j=0}^{n_Y-1} ||w_{ij}' - w_i^*||^2 + \frac{n_X - 1}{n_X} ||w_i^{(k-1)} - w_i^*||^2
\]

\[
E\{||w_i^{(k)} - w_i^{(k-1)}||^2 \mid F_k\} = \frac{1}{n_X n_Y} \sum_{j=0}^{n_Y-1} ||w_{ij}' - w_i^{(k-1)}||^2
\]

Using (72) in (71), we obtain:

\[
\left(\frac{1}{2\alpha_w} + \gamma\right) \left(n_X E\{||w_i^{(k)} - w_i^*||^2 \mid F_k\} - (n_X - 1)||w_i^{(k-1)} - w_i^*||^2\right)
\]

\[
\leq \frac{1}{2\alpha_w} ||w_i^{(k-1)} - w_i^*||^2 - \frac{n_X}{2\alpha_w} E\{||w_i^{(k)} - w_i^{(k-1)}||^2 \mid F_k\}
\]

\[
+ 2\alpha_w B_f^2 \left(1 - \frac{1}{n_Y}\right) ||\theta^{(k-1)} - \bar{\theta}||^2
\]

\[
+ n_X E\{\langle \mathcal{J}_i(\theta^{(k-1)}) - \mathcal{J}_i(\theta^*), w_i^{(k)} - w_i^* \rangle \mid F_k\}
\]

\[
- (n_X - 1) \left\{\mathcal{J}_i(\theta^{(k-1)}) - \mathcal{J}_i(\theta^*), w_i^{(k-1)} - w_i^*\right\}
\]

Summing both sides of (72) over 0 \leq i \leq n_X - 1, using the fact that \(||w_i^{(k)} - w_i^*||^2 = \sum_{i=0}^{n_X-1} ||w_i^{(k)} - w_i^*||^2\), and then dividing by \(n_X\), we get

\[
\left(\frac{1}{2\alpha_w} + \gamma\right) \left(E\{||w_i^{(k)} - w_i^*||^2 \mid F_k\} - \frac{n_X - 1}{n_X} ||w_i^{(k-1)} - w_i^*||^2\right)
\]

\[
\leq \frac{1}{2\alpha_w n_X} ||w^{(k-1)} - w^*||^2 - \frac{1}{2\alpha_w} E\{||w^{(k)} - w^{(k-1)}||^2 \mid F_k\}
\]

\[
+ 2\alpha_w B_f^2 \left(1 - \frac{1}{n_Y}\right) ||\theta^{(k-1)} - \bar{\theta}||^2
\]

\[
+ n_X E\{L(\theta^{(k-1)}, w^{(k)} - w^*) - L(\theta^*, w^{(k)} - w^*) \mid F_k\}
\]

\[
- (n_X - 1) \left\{L(\theta^{(k-1)}, w^{(k-1)} - w^*) - L(\theta^*, w^{(k-1)} - w^*)\right\}
\]

where, we have used the notation:

\[
\frac{1}{n_Y} := \frac{1}{n_X} \sum_{i=0}^{n_X-1} 1/n_Y_i
\]

Rearranging and combining the common terms, we obtain the final dual bound:

\[
\left(\frac{1}{2\alpha_w} + \gamma\right) E\{||w^{(k)} - w^*||^2 \mid F_k\} + \frac{1}{2\alpha_w} E\{||w^{(k)} - w^{(k-1)}||^2 \mid F_k\}
\]

\[
\leq \left(\frac{1}{2\alpha_w} + \frac{n_X - 1}{n_X}\right) ||w^{(k-1)} - w^*||^2 + 2\alpha_w B_f^2 \left(1 - \frac{1}{n_Y}\right) ||\theta^{(k-1)} - \bar{\theta}||^2
\]

\[
+ E\{L(\theta^{(k-1)}, w^{(k)} - w^*) - L(\theta^*, w^{(k)} - w^*) \mid F_k\}
\]

\[
+ (n_X - 1) E\{L(\theta^{(k-1)}, w^{(k-1)} - w^*) - L(\theta^*, w^{(k-1)} - w^*) \mid F_k\}
\]
We now apply Lemma E.5 with $\theta$ to the following optimization problem:

\[
\text{minimize } \| \theta_{(k)} - \theta^* \|^2 \text{ subject to } \theta_{(k)} \in \Theta
\]

Using (81) to bound the first and the third term in (77), we obtain:

\[
\| \theta_{(k)} - \theta^* \|^2 = \| \theta_{(k)} - \theta_{(k-1)} + \theta_{(k-1)} - \theta^* \|^2 \\
\leq \alpha \| \theta_{(k-1)} - \theta_{(k)} \|^2 + \alpha \| \theta_{(k)} - \theta^* \|^2 + 2 \langle \theta_{(k)} - \theta_{(k-1)}, (\theta_{(k-1)} - \theta^*) \rangle
\]

Similar to the dual bound, we now bound the first term and the third term together. Introduce the following function of $x$ (for fixed $\theta_{(k-1)}$ and $w_{(k)}$):

\[
P_\theta(x) := \langle L_\theta'(\theta_{(k-1)}, w_{(k)}), x \rangle + g(x)
\]

The first part of the function is linear in $x$ (and hence convex), and the second part of the function is $\mu$-strongly convex (Assumption E.1). Recall the primal update rule in (51), which can be written in the following proximal mapping form:

\[
\theta_{(k)} = \arg \min_\theta \left\{ \langle \delta^{\theta}_{k, \theta}, \theta \rangle + g(\theta) + \frac{1}{2\alpha_k} \| \theta - \theta_{(k-1)} \|^2 \right\}
\]

as follows:

\[
\theta_{(k)} = \text{prox}_{\alpha_k \delta^{\theta}_{k}} \left( \theta_{(k-1)} - \alpha_k \delta^{\theta}_{k} \right)
\]

We now apply Lemma E.5 with $P(x) = P_\theta(x)$ and the following identifications:

\[
f(x) = \langle L_\theta'(\theta_{(k-1)}, w_{(k)}), x \rangle, \quad g(x) = g(\theta) \quad v = \delta^{\theta}_{k} \quad x = \theta_{(k-1)} \quad x^{(+)} = \theta_{(k)} \quad w = \theta^* \quad \alpha = \alpha_k
\]

which leads to

\[
\langle L_\theta'(\theta_{(k-1)}, w_{(k)}), \theta^* \rangle + g(\theta^*) \geq \langle L_\theta'(\theta_{(k-1)}, w_{(k)}), \theta_{(k)} \rangle + g(\theta_{(k)}) + \frac{1}{\alpha_k} \langle \delta^{\theta}_{k} - L_\theta'(\theta_{(k-1)}, w_{(k)}), (\theta^* - \theta_{(k-1)}) \rangle \\
- \langle \delta^{\theta}_{k} - L_\theta'(\theta_{(k-1)}, w_{(k)}), (\theta_{(k)} - \theta^*) \rangle
\]

Rearranging the terms in the above inequality, we obtain

\[
\| \theta_{(k-1)} - \theta_{(k)} \|^2 + \langle \theta_{(k)} - \theta_{(k-1)} \rangle, (\theta_{(k-1)} - \theta^*) \rangle \\
\leq \alpha_k \langle L_\theta'(\theta_{(k-1)}, w_{(k)}), \theta^* \rangle + \alpha_k g(\theta^*) - \alpha_k g(\theta_{(k)}) - \alpha_k \langle L_\theta'(\theta_{(k-1)}, w_{(k)}), \theta_{(k)} \rangle \\
+ \alpha_k \langle \delta^{\theta}_{k} - L_\theta'(\theta_{(k-1)}, w_{(k)}), \theta_{(k)} - \theta^* \rangle - \frac{\alpha_k \mu}{2} \| \theta^* - \theta_{(k)} \|^2
\]

Using (81) to bound the first and the third term in (77), we obtain:

\[
\| \theta_{(k)} - \theta^* \|^2 = \| \theta_{(k)} - \theta_{(k-1)} \|^2 + \| \theta_{(k-1)} - \theta^* \|^2 + 2 \langle \theta_{(k)} - \theta_{(k-1)}, (\theta_{(k-1)} - \theta^*) \rangle \\
\leq \langle \delta^{\theta}_{k}, \theta_{(k)} \rangle + \| \theta_{(k-1)} - \theta^* \|^2 + \| \theta_{(k-1)} - \theta^* \|^2 + 2 \| \theta^* - \theta_{(k-1)} \|^2 \\
+ 2 \langle \theta_{(k)} - \theta_{(k-1)} \rangle, (\theta_{(k-1)} - \theta^*) \rangle \\
\leq \| \theta_{(k)} - \theta^* \|^2 - \| \theta_{(k)} - \theta_{(k-1)} \|^2 - 2 \| \theta^* \|^2
\]

where step (a) subtracts and adds the second term. Furthermore, note that $\theta^*$ is the optimal solution to the following optimization problem:

\[
\theta^* = \arg \min_\theta \left[ \langle L(\theta, w^*), x \rangle + g(\theta) \right]
\]
We will now upper bound the right hand side of (87). To this end, we have:

\[
\begin{align*}
L(\theta^{(k)}, w^*) + g(\bar{\theta}^{(k)}) &\geq L(\theta^*, w^*) + g(\bar{\theta}^*) + \frac{\mu}{2}||\theta^{(k)} - \theta^*||^2 \\
\end{align*}
\]  

(83)

Multiplying both sides of the above inequality by \(2\alpha_0\) and then adding it to (82), we obtain:

\[
\begin{align*}
(1 + 2\alpha_0\mu)||\theta^{(k)} - \theta^*||^2 \leq ||\theta^{(k-1)} - \theta^*||^2 - ||\theta^{(k)} - \theta^{(k-1)}||^2 \\
+ 2\alpha_0\left[L(\theta^{(k)}, w^*) - L(\theta^*, w^*)\right] \\
+ 2\alpha_0\left[\left\langle L_0'(\theta^{(k-1)}, w^{(k)}), \theta^* - \theta^{(k)}\right\rangle - \left\langle L_0'(\theta^{(k-1)}, w^{(k)}), \theta^{(k-1)}\right\rangle\right] \\
+ 2\alpha_0\left(\delta^0_k - L_0'(\theta^{(k-1)}, w^{(k)}), (\theta^{(k)} - \theta^*)\right)
\end{align*}
\]  

(84)

Next, we bound the last term in (84). To this end, we first introduce the following auxiliary variable \(\theta_k\), which is the updated primal variable if full batch gradient were used:

\[
\theta_k = \arg\min_{\theta} \left[\left\langle L_0'(\theta^{(k-1)}, w^{(k)}), \theta\right\rangle + g(\theta) + \frac{1}{2\alpha_0}||\theta - \theta^{(k-1)}||^2\right]
\]

(85)

Note that both (79) and (85) are written in proximal mapping form. We now bound the last term (84):

\[
\begin{align*}
\delta^0_k - L_0'(\theta^{(k-1)}, w^{(k)}) , \theta^{(k)} - \theta^* \\
\leq \alpha_0(\delta^0_k - L_0'(\theta^{(k-1)}, w^{(k)}), (\theta^{(k)} - \theta^*) \\
+ \left\langle L_0'(\theta^{(k-1)}, w^{(k)}), (\theta^{(k)} - \theta^*)\right\rangle
\end{align*}
\]  

(86)

where step (a) adds and subtracts \(\theta^{(k)}\), step (b) uses Cauchy-Schwartz inequality, and step (c) substitutes (79) and (85) and then applies Lemma E.6. Let \(\mathcal{F}_k^{(+)}\) denote the filtration of all events up to and including the dual update in the \(k\)-th iteration. Applying expectation to both sides of (86) conditioned on \(\mathcal{F}_k^{(+)}\), we have:

\[
\begin{align*}
E\left\{\left\langle \delta^0_k - L_0'(\theta^{(k-1)}, w^{(k)}), (\theta^{(k)} - \theta^*)\right\rangle | \mathcal{F}_k^{(+)}\right\} \\
\leq \alpha_0E\left\{\left\langle \delta^0_k - L_0'(\theta^{(k-1)}, w^{(k)}), (\theta^{(k)} - \theta^*)\right\rangle | \mathcal{F}_k^{(+)}\right\} \\
\leq \alpha_0E\left\{\left\langle \delta^0_k - L_0'(\theta^{(k-1)}, w^{(k)}), (\theta^{(k)} - \theta^*)\right\rangle | \mathcal{F}_k^{(+)}\right\} \\
\leq \alpha_0E\left\{\left\langle \delta^0_k - L_0'(\theta^{(k-1)}, w^{(k)}), (\theta^{(k)} - \theta^*)\right\rangle | \mathcal{F}_k^{(+)}\right\}
\end{align*}
\]  

(87)

where step (a) uses the fact that \(\theta^{(k)}\), \(\theta^{(k-1)}\) and \(w^{(k)}\) are deterministic conditioned on \(\mathcal{F}_k^{(+)}\), and step (b) uses the fact that the conditional expectation of \(\delta^0_k\) is the batch gradient.

We will now upper bound the right hand side of (87). To this end, we have:

\[
\begin{align*}
E\left\{\left\langle \delta^0_k - L_0'(\theta^{(k-1)}, w^{(k)}), (\theta^{(k)} - \theta^*)\right\rangle | \mathcal{F}_k^{(+)}\right\} \\
\leq \alpha_0E\left\{\left\langle \delta^0_k - L_0'(\theta^{(k-1)}, w^{(k)}), (\theta^{(k)} - \theta^*)\right\rangle | \mathcal{F}_k^{(+)}\right\} \\
\leq \alpha_0E\left\{\left\langle \delta^0_k - L_0'(\theta^{(k-1)}, w^{(k)}), (\theta^{(k)} - \theta^*)\right\rangle | \mathcal{F}_k^{(+)}\right\}
\end{align*}
\]  

(88)

(\(d\))
where step (a) uses the definition of $\delta_k^\theta$, step (b) uses $E[X - E[X]]^2 \leq E[X^2]$, step (c) uses Lemma E.7, and step (d) uses the Lipschitz continuity of the gradients (Assumption E.3). Substituting (88) into (87), we get:

$$E\left\{ (\delta_k^\theta - L_\theta(\theta(k-1), w(k))), (\theta(k) - \theta^*) \right\} \leq \alpha_\theta B_w^2 B_\theta^2 \|\theta(k-1) - \bar{\theta}\|^2$$

Finally, substituting (89) into (84) and then further applying expectation conditioned on $F_k$, we obtain:

$$(1 + 2\alpha_\theta \mu) E\left\{ \|\theta(k) - \theta^*\|^2 \mid F_k \right\} \leq \|\theta(k-1) - \theta^*\|^2 - E\left\{ \|\theta(k) - \theta(k-1)\|^2 \mid F_k \right\}$$

$$+ 2\alpha_\theta E\left\{ \left[ L(\theta(k), w^*) - L(\theta^*, w^*) \right] \mid F_k \right\}$$

$$+ 2\alpha_\theta E\left\{ \left[ L_\theta(\theta(k-1), w(k)), \theta^* \right] - \left[ L_\theta(\theta(k-1), w(k)), \theta(k) \right] \right\} \mid F_k \right\}$$

$$+ 2\alpha_\theta^2 B_w^2 B_\theta^2 \|\theta(k-1) - \bar{\theta}\|^2$$

Dividing both sides by $2\alpha_\theta$ and combining common terms, we obtain the final bound for the primal variable:

$$\left( \frac{1}{2\alpha_\theta} + \mu \right) E\left\{ \|\theta(k) - \theta^*\|^2 \mid F_k \right\} + \frac{1}{2\alpha_\theta} E\left\{ \|\theta(k) - \theta(k-1)\|^2 \mid F_k \right\}$$

$$\leq \frac{1}{2\alpha_\theta} \|\theta(k-1) - \theta^*\|^2 + \alpha_\theta B_w^2 B_\theta^2 \|\theta(k-1) - \bar{\theta}\|^2$$

$$+ E\left\{ \left[ L_\theta(\theta(k-1), w(k)), \theta^* \right] - \left[ L_\theta(\theta(k-1), w(k)), \theta(k) \right] \right\} \mid F_k \right\}$$

$$+ E\left\{ \left[ L(\theta(k), w^*) - L(\theta^*, w^*) \right] \mid F_k \right\}$$

E.7 Convergence for Option I

Based on the derived primal and dual bounds above, we now proceed to prove the convergence of SVRPDA-I with Option I: updating $\bar{\theta}$ using the most recent $\theta(k)$ (see Algorithm 1).

Adding (76) and (91) we obtain the total bound for the primal and dual variable updates:

$$\left( \frac{1}{2\alpha_w} + \gamma \right) E\left\{ \|w(k) - w^*\|^2 \mid F_k \right\} + \frac{1}{2\alpha_w} E\left\{ \|w(k) - w(k-1)\|^2 \mid F_k \right\}$$

$$\left( \frac{1}{2\alpha_\theta} + \mu \right) E\left\{ \|\theta(k) - \theta^*\|^2 \mid F_k \right\} + \frac{1}{2\alpha_\theta} E\left\{ \|\theta(k) - \theta(k-1)\|^2 \mid F_k \right\}$$

$$\leq \left( \frac{1}{2\alpha_w} + \frac{\gamma(n - 1)}{n} \right) \|w(k-1) - w^*\|^2 + \frac{1}{2\alpha_w} \|\theta(k-1) - \theta^*\|^2$$

$$+ \left( \frac{1}{2\alpha_w} B_w^2 \left( 1 - 1/n \right) + \alpha_\theta B_w^2 B_\theta^2 \right) \|\theta(k-1) - \bar{\theta}\|^2$$

$$+ E\left\{ L(\theta(k-1), w(k)) - L(\theta^*, w^*) \mid F_k \right\}$$

$$+ E\left\{ \left[ L_\theta(\theta(k-1), w(k)), \theta^* \right] - \left[ L_\theta(\theta(k-1), w(k)), \theta(k) \right] \right\} \mid F_k \right\}$$

$$+ (n - 1) E\left\{ L(\theta(k-1), w(k) - w(k-1)) - L(\theta^*, w(k) - w(k-1)) \mid F_k \right\}$$

Next, we need to upper bound the $L$ terms on the right-hand side of the above inequality. To this end, we first show the following inequality:

$$- L(\theta^*, w(k) - w^*) + L(\theta(k), w^*) - L(\theta^*, w^*) + \left[ L_\theta(\theta(k-1), w(k)), \theta^* \right]$$

$$- \left[ L_\theta(\theta(k-1), w(k)), \theta(k) \right] + L(\theta(k), w(k) - w^*)$$

$$= - L(\theta^*, w(k)) + L(\theta(k), w(k)) + \left[ L_\theta(\theta(k-1), w(k)), \theta^* - \theta(k) \right]$$

26
\begin{align}
\leq & -\langle L_0^{(k)}(\theta^{(k)}, w^{(k)}), \theta^{*} - \theta^{(k)} \rangle + \langle L_0^{(k)}(\theta^{(k-1)}, w^{(k)}), \theta^{*} - \theta^{(k)} \rangle \\
= & \langle L_0^{(k)}(\theta^{(k)}, w^{(k)}) - L_0^{(k)}(\theta^{(k-1)}, w^{(k)}), \theta^{(k)} - \theta^{*} \rangle \\
\leq & \frac{1}{n_X} \sum_{i=0}^{n_X-1} \left\langle \left( \mathbf{T}_i^{\prime}(\theta^{(k)}) - \mathbf{T}_i^{\prime}(\theta^{(k-1)}) \right) w_i^{(k)}, \theta^{(k)} - \theta^{*} \right\rangle \\
\leq & \frac{1}{n_X} \sum_{i=0}^{n_X-1} \left\langle \left( \mathbf{T}_i^{\prime}(\theta^{(k)}) - \mathbf{T}_i^{\prime}(\theta^{(k-1)}) \right) w_i^{(k)}, \theta^{(k)} - \theta^{*} \right\rangle \\
\leq & \frac{1}{n_X} \sum_{i=0}^{n_X-1} \left\| \mathbf{T}_i^{\prime}(\theta^{(k)}) - \mathbf{T}_i^{\prime}(\theta^{(k-1)}) \right\| w_i^{(k)} \cdot \| \theta^{(k)} - \theta^{*} \| \\
\leq & \frac{1}{n_X} \sum_{i=0}^{n_X-1} \left\| \mathbf{T}_i^{\prime}(\theta^{(k)}) - \mathbf{T}_i^{\prime}(\theta^{(k-1)}) \right\| \cdot \| w_i^{(k)} \| \cdot \| \theta^{(k)} - \theta^{*} \| \\
\leq & \frac{1}{n_X} \sum_{i=0}^{n_X-1} \left\| \mathbf{T}_i^{\prime}(\theta^{(k)}) - \mathbf{T}_i^{\prime}(\theta^{(k-1)}) \right\| \cdot \| w_i^{(k)} \| \cdot \| \theta^{(k)} - \theta^{*} \| \\
\leq & \frac{1}{n_X} \sum_{i=0}^{n_X-1} \left\| \mathbf{T}_i^{\prime}(\theta^{(k)}) - \mathbf{T}_i^{\prime}(\theta^{(k-1)}) \right\| \cdot \| w_i^{(k)} \| \cdot \| \theta^{(k)} - \theta^{*} \| \\
\leq & \frac{1}{n_X} \sum_{i=0}^{n_X-1} \left\| \mathbf{T}_i^{\prime}(\theta^{(k)}) - \mathbf{T}_i^{\prime}(\theta^{(k-1)}) \right\| \cdot \| w_i^{(k)} \| \cdot \| \theta^{(k)} - \theta^{*} \| \\
= & B_\theta B_w \| \theta^{(k)} - \theta^{(k-1)} \| \cdot \| \theta^{(k)} - \theta^{*} \| \\
\leq & \frac{B_\theta B_w}{\beta_0} \| \theta^{(k)} - \theta^{(k-1)} \|^2 + \frac{\beta_0 B_\theta B_w}{\beta_0} \| \theta^{(k)} - \theta^{*} \|^2 \\
\leq & \frac{B_\theta B_w}{\beta_0} \| \theta^{(k)} - \theta^{(k-1)} \|^2 + \frac{\beta_0 B_\theta B_w}{\beta_0} \| \theta^{(k)} - \theta^{*} \|^2
\end{align}

where step (a) uses the fact that $L(\theta, w)$ is convex with respect to the $\theta$, step (b) substitutes the expression of $L_0^{(k)}$, step (c) uses Jensen’s inequality, step (d) uses Cauchy-Schwartz inequality, step (e) substitutes the expression for $\mathbf{T}_i$, step (f) uses Jensen’s inequality, step (g) uses the Lipschitz gradient property of $f_\theta$ and Lemma 4.3, step (h) uses $ab \leq \frac{1}{\beta_0} a^2 + \frac{\beta_0}{\beta_0} b^2$. In consequence, the above inequality implies that

\begin{align}
- L(\theta^{*}, w^{(k)} - w^{*}) + L(\theta^{(k)}, w^{(k)}) - L(\theta^{*}, w^{*}) + \langle L_0^{(k)}(\theta^{(k-1)}, w^{(k)}), \theta^{*} \rangle - \langle L_0^{(k)}(\theta^{(k-1)}, w^{(k)}), \theta^{(k)} \rangle \\
\leq - L(\theta^{(k)}, w^{(k)} - w^{*}) + \frac{B_\theta B_w}{\beta_0} \| \theta^{(k)} - \theta^{(k-1)} \|^2 + \frac{\beta_0 B_\theta B_w}{\beta_0} \| \theta^{(k)} - \theta^{*} \|^2
\end{align}

Using (94), the $L$ terms in (92) becomes (notice that we keep the first and last $L$ terms in (92) intact)

\begin{align}
L(\theta^{(k-1)}, w^{(k)} - w^{*}) - L(\theta^{*}, w^{(k)} - w^{*}) + L(\theta^{(k)}, w^{(k)}) - L(\theta^{*}, w^{*}) + \langle L_0^{(k)}(\theta^{(k-1)}, w^{(k)}), \theta^{*} \rangle - \langle L_0^{(k)}(\theta^{(k-1)}, w^{(k)}), \theta^{(k)} \rangle \\
\leq L(\theta^{(k-1)}, w^{(k)} - w^{*}) - L(\theta^{(k)}, w^{(k)} - w^{*}) + (n_X - 1) \left[ L(\theta^{(k-1)}, w^{(k)} - w^{(k-1)}) - L(\theta^{*}, w^{(k)} - w^{(k-1)}) \right] \\
+ \frac{B_\theta B_w}{\beta_0} \| \theta^{(k)} - \theta^{(k-1)} \|^2 + \frac{\beta_0 B_\theta B_w}{\beta_0} \| \theta^{(k)} - \theta^{*} \|^2 \\
= L(\theta^{(k-1)}, w^{(k)} - w^{*} + (n_X - 1)(w^{(k)} - w^{(k-1)})) - L(\theta^{(k)}, w^{(k)} - w^{*} + (n_X - 1)(w^{(k)} - w^{(k-1)})) \\
+ \frac{B_\theta B_w}{\beta_0} \| \theta^{(k)} - \theta^{(k-1)} \|^2 + \frac{\beta_0 B_\theta B_w}{\beta_0} \| \theta^{(k)} - \theta^{*} \|^2
\end{align}
\[
\begin{aligned}
&\leq \left( L'_0 \beta^{k-1}, w^{(k)} - w^* + (n_X - 1)(w_i^{(k)} - w_i^{(k-1)}) \right), \theta^{(k-1)} - \theta^{(k)} \right) \\
&+ \frac{B_0 B_w}{\beta_0} \|\theta^{(k)} - \theta^{(k-1)}\|^2 + \beta_0 B_0 B_w \|\theta^{(k)} - \theta^*\|^2 \\
&= \left( \frac{1}{n_X} \sum_{i=0}^{n_X-1} \mathcal{T}_i(\theta^{(k-1)})(w_i^{(k)} - w_i^* + (n_X - 1)(w_i^{(k)} - w_i^{(k-1)})), \theta^{(k-1)} - \theta^{(k)} \right) \\
&+ \frac{B_0 B_w}{\beta_0} \|\theta^{(k)} - \theta^{(k-1)}\|^2 + \beta_0 B_0 B_w \|\theta^{(k)} - \theta^*\|^2 \\
&= \left( \frac{1}{n_X} \sum_{i=0}^{n_X-1} \mathcal{T}_i(\theta^{(k-1)})(w_i^{(k-1)} + n_X(w_i^{(k)} - w_i^{(k-1)}) - w_i^*), \theta^{(k-1)} - \theta^{(k)} \right) \\
&+ \frac{B_0 B_w}{\beta_0} \|\theta^{(k)} - \theta^{(k-1)}\|^2 + \beta_0 B_0 B_w \|\theta^{(k)} - \theta^*\|^2 \\
&\leq 2\beta_1 \left( \frac{1}{n_X} \sum_{i=0}^{n_X-1} \mathcal{T}_i(\theta^{(k-1)})(w_i^{(k-1)} - w_i^*) \right)^2 + 2\beta_1 \left( \sum_{i=0}^{n_X-1} \mathcal{T}_i(\theta^{(k-1)})(w_i^{(k)} - w_i^{(k-1)}) \right)^2 \\
&+ \frac{1}{\beta_1} \|\theta^{(k-1)} - \theta^{(k)}\|^2 + \frac{B_0 B_w}{\beta_0} \|\theta^{(k)} - \theta^{(k-1)}\|^2 + \beta_0 B_0 B_w \|\theta^{(k)} - \theta^*\|^2 \\
&\leq 2\beta_1 \frac{B_0 B_w}{\beta_0} \sum_{i=0}^{n_X-1} \|w_i^{(k-1)} - w_i^*\|^2 + 2\beta_1 \left( \sum_{i=0}^{n_X-1} \mathcal{T}_i(\theta^{(k-1)})(w_i^{(k)} - w_i^{(k-1)}) \right)^2 \\
&+ \left( \frac{1}{\beta_1} + \frac{B_0 B_w}{\beta_0} \right) \|\theta^{(k-1)} - \theta^{(k)}\|^2 + \beta_0 B_0 B_w \|\theta^{(k)} - \theta^*\|^2
\end{aligned}
\]

where step (a) applies \((94), (b)\) uses convexity of \(L\) in \(\theta\), step (c) uses \(\langle a, b \rangle \leq \beta_1 \|a\|^2 + \frac{1}{\beta_1} \|b\|^2\), for some \(\beta_1 > 0\) to be chosen later, step (d) uses \(\|a + b\|^2 \leq 2\|a\|^2 + 2\|b\|^2\), and step (e) applies Jensen’s inequality and bounded gradient assumption to the first term. Before we proceed, we note that, by taking expectation of the second term conditioned on \(\mathcal{F}_k\), we get

\[
\begin{aligned}
&\mathbb{E}\left\{ \left\| \sum_{i=0}^{n_X-1} \mathcal{T}_i(\theta^{(k-1)})(w_i^{(k)} - w_i^{(k-1)}) \right\|^2 \mid \mathcal{F}_k \right\} \\
&= \sum_{i=0}^{n_X-1} \sum_{j=0}^{n_Y-1} \frac{1}{n_X n_Y} \left\| \mathcal{T}_i(\theta^{(k-1)})(w_i^{(k)} - w_i^{(k-1)}) \right\|^2 \\
&\leq B^2 \sum_{i=0}^{n_X-1} \sum_{j=0}^{n_Y-1} \frac{1}{n_X n_Y} \left\| w_i^{(k)} - w_i^{(k-1)} \right\|^2 \\
&= B^2 \sum_{i=0}^{n_X-1} \mathbb{E}\left\{ \left\| w_i^{(k)} - w_i^{(k-1)} \right\|^2 \mid \mathcal{F}_k \right\}
\end{aligned}
\]
Therefore, the conditional expectation of all the $L$ terms are bounded by
\[
\frac{2\beta_1 B_f^2}{n_X} \sum_{i=0}^{n_X-1} \left\| \theta_i^{(k-1)} - w_i^* \right\|^2 + 2\beta_1 B_f^2 \sum_{i=0}^{n_X-1} E\left\{ \left\| \theta_i^{(k-1)} - w_i^{(k)} \right\|^2 | F_k \right\}
\]
\[+ \left( \frac{1}{\beta_1} + \frac{B_0 B_w}{\beta} \right) E\left\{ \left\| \theta^{(k)} - \theta^{(k-1)} \right\|^2 | F_k \right\} + \beta_0 B_0 B_w E\left\{ \left\| \theta^{(k)} - \theta^* \right\|^2 | F_k \right\}
\]
\[= \frac{2\beta_1 B_f^2}{n_X} \left\| w^{(k)} - w^* \right\|^2 + 2\beta_1 B_f^2 E\left\{ \left\| w^{(k)} - w^{(k-1)} \right\|^2 | F_k \right\}
\]
\[+ \left( \frac{1}{\beta_1} + \frac{B_0 B_w}{\beta_0} \right) E\left\{ \left\| \theta^{(k)} - \theta^{(k-1)} \right\|^2 | F_k \right\} + \beta_0 B_0 B_w E\left\{ \left\| \theta^{(k)} - \theta^* \right\|^2 | F_k \right\} \quad (97)
\]
Therefore, the total bound (92) becomes
\[
\left( \frac{1}{2\alpha_w} + \gamma \right) E\left\{ \left\| \theta^{(k)} - \theta^* \right\|^2 | F_k \right\} + \frac{1}{2\alpha_w} E\left\{ \left\| w^{(k)} - w^{(k-1)} \right\|^2 | F_k \right\}
\]
\[+ \left( \frac{1}{2\alpha_w} + \mu \right) E\left\{ \left\| \theta^{(k)} - \theta^* \right\|^2 | F_k \right\} + \frac{1}{2\alpha_w} E\left\{ \left\| \theta^{(k)} - \theta^{(k-1)} \right\|^2 | F_k \right\}
\]
\[\leq \left( \frac{1}{2\alpha_w} + \frac{\gamma (n_X - 1)}{n_X} \right) \left\| w^{(k)} - w^* \right\|^2 + \frac{1}{2\alpha_w} \left\| \theta^{(k)} - \theta^* \right\|^2
\]
\[+ \left( 2\alpha_w B_f^2 \left( 1 - \frac{1}{n_Y} \right) + \alpha_\theta B_0^2 B_w^2 \right) \left\| \theta^{(k-1)} - \theta^* \right\|^2
\]
\[+ \frac{2\beta_1 B_f^2}{n_X} \left\| w^{(k-1)} - w^* \right\|^2 + 2\beta_1 B_f^2 E\left\{ \left\| w^{(k)} - w^{(k-1)} \right\|^2 | F_k \right\}
\]
\[+ \left( \frac{1}{\beta_1} + \frac{B_0 B_w}{\beta_0} \right) E\left\{ \left\| \theta^{(k-1)} - \theta^* \right\|^2 | F_k \right\} + \beta_0 B_0 B_w E\left\{ \left\| \theta^{(k)} - \theta^* \right\|^2 | F_k \right\} \quad (98)
\]
By combining the common terms, we obtain
\[
\left( \frac{1}{2\alpha_w} + \mu - \beta_0 B_0 B_w \right) E\left\{ \left\| \theta^{(k)} - \theta^* \right\|^2 | F_k \right\} + \left( \frac{1}{2\alpha_w} + \gamma \right) E\left\{ \left\| w^{(k)} - w^* \right\|^2 | F_k \right\}
\]
\[+ \left( \frac{1}{2\alpha_w} - \frac{1}{\beta_1} - \frac{B_0 B_w}{\beta_0} \right) E\left\{ \left\| \theta^{(k)} - \theta^{(k-1)} \right\|^2 | F_k \right\} + \left( \frac{1}{2\alpha_w} - 2\beta_1 B_f^2 \right) E\left\{ \left\| w^{(k)} - w^{(k-1)} \right\|^2 | F_k \right\}
\]
\[\leq \frac{1}{2\alpha_w} \left\| \theta^{(k-1)} - \theta^* \right\|^2 + \left( \frac{1}{2\alpha_w} + \gamma - \frac{\gamma}{n_X} + \frac{2\beta_1 B_f^2}{n_X} \right) \left\| w^{(k-1)} - w^* \right\|^2
\]
\[+ \left( 2\alpha_w B_f^2 \left( 1 - \frac{1}{n_Y} \right) + \alpha_\theta B_0^2 B_w^2 \right) \left\| \theta^{(k-1)} - \theta^* \right\|^2 \quad (99)
\]
Applying inequality $\|x + y\|^2 \leq 2\|x\|^2 + 2\|y\|^2$ to the last term in (99), we obtain
\[
\left( \frac{1}{2\alpha_w} + \mu - \beta_0 B_0 B_w \right) E\left\{ \left\| \theta^{(k)} - \theta^* \right\|^2 | F_k \right\} + \left( \frac{1}{2\alpha_w} + \gamma \right) E\left\{ \left\| w^{(k)} - w^* \right\|^2 | F_k \right\}
\]
\[+ \left( \frac{1}{2\alpha_w} - \frac{1}{\beta_1} - \frac{B_0 B_w}{\beta_0} \right) E\left\{ \left\| \theta^{(k)} - \theta^{(k-1)} \right\|^2 | F_k \right\} + \left( \frac{1}{2\alpha_w} - 2\beta_1 B_f^2 \right) E\left\{ \left\| w^{(k)} - w^{(k-1)} \right\|^2 | F_k \right\}
\]
\[\leq \left( \frac{1}{2\alpha_w} + 4\alpha_w B_f^2 \left( 1 - \frac{1}{n_Y} \right) + 2\alpha_\theta B_0^2 B_w^2 \right) \left\| \theta^{(k-1)} - \theta^* \right\|^2
\]
\[+ \left( \frac{1}{2\alpha_w} + \gamma - \frac{\gamma}{n_X} + \frac{2\beta_1 B_f^2}{n_X} \right) \left\| w^{(k-1)} - w^* \right\|^2 + \left( 4\alpha_w B_f^2 \left( 1 - \frac{1}{n_Y} \right) + 2\alpha_\theta B_0^2 B_w^2 \right) \left\| \theta - \theta^* \right\|^2 \quad (100)
\]
Taking full expectation of the above inequality, we obtain:
\[
\left( \frac{1}{2\alpha_w} + \mu - \beta_0 B_0 B_w \right) E\left\{ \left\| \theta^{(k)} - \theta^* \right\|^2 \right\} + \left( \frac{1}{2\alpha_w} + \gamma \right) E\left\{ \left\| w^{(k)} - w^* \right\|^2 \right\}
\]
We choose $\beta$ where $r$ which simplifies the recursion to be
following conditions:

\[ \text{Inequality (103)} \text{ can also be further written as} \]
\[ E\|\theta(k) - \theta^*\|^2 + \frac{1}{\alpha_w} \|w(k) - w^*\|^2 \]
\[ \leq r_P E\|\theta(k-1) - \theta^*\|^2 + \frac{1}{\alpha_w} + \mu - \beta_0 B_\theta B_w \]
\[ + \frac{4 \alpha_w B_f^2 (1 - \frac{1}{n_Y}) + 2 \alpha_0 B_w^2 B_\theta^2}{2 \alpha_w + \mu - \beta_0 B_\theta B_w} E\|\tilde{\theta} - \theta^*\|^2 \]

We choose $\beta_0$, $\beta_1$, and the primal and the dual step-sizes to be

\[ \beta_0 = \frac{\mu}{2 B_\theta B_w}, \quad \beta_1 = \frac{\gamma}{4 B_f^2} \]
\[ \alpha_w = \frac{1}{64 n_X \left( \frac{B_f^2}{\mu^2} + \frac{B_w^2 B_\theta^2}{\mu^2} \right) + n_X} = \frac{1}{\mu} \cdot \frac{1}{64 n_X \kappa + n_X} \]

(102)
where $\kappa$ is the condition number defined as

$$\kappa = \frac{B_f^2}{\mu\gamma} + \frac{B^2_\theta B^2_w}{\mu^2}$$

(105)

It can be verified that the above choice of step-sizes satisfies the condition (102). With our choice of the parameters, we also have

$$\frac{1}{2\alpha_w} + \frac{\gamma}{\mu - \beta_0 B_\theta B_w} = \frac{1}{2\alpha_w} + \frac{\gamma}{\mu} = \frac{64\kappa + 3}{64n_X \kappa + n_X + 1}$$

(106)

and

$$4\alpha_w B_f^2 \left(1 - \frac{1}{n_Y}\right) + 2\alpha_\theta B^2_w B^2_\theta = \frac{4\alpha_w B_f^2 \left(1 - \frac{1}{n_Y}\right) + 2\alpha_\theta B^2_w B^2_\theta}{\frac{1}{2\alpha_w} + \frac{\gamma}{\mu}}$$

$$= \frac{8B_f^2 \left(1 - \frac{1}{n_Y}\right) + 4B^2_\theta B^2_w \cdot \frac{1}{n_X}}{(64\kappa + 1)(64n_X \kappa + n_X + 1)}$$

(107)

Substituting (106–107) into (104), we obtain

$$E\|\theta^{(k)} - \theta^*\|^2 + \frac{\gamma}{\mu} \cdot \frac{64\kappa + 3}{64n_X \kappa + n_X + 1} E\|w^{(k)} - w^*\|^2$$

$$\leq r_P E\|\theta^{(k-1)} - \theta^*\|^2 + r_D \cdot \frac{\gamma}{\mu} \cdot \frac{64\kappa + 3}{64n_X \kappa + n_X + 1} E\|w^{(k-1)} - w^*\|^2$$

$$+ \frac{8B_f^2 \left(1 - \frac{1}{n_Y}\right) + 4B^2_\theta B^2_w \cdot \frac{1}{n_X}}{(64\kappa + 1)(64n_X \kappa + n_X + 1)} E\|\bar{\theta} - \theta^*\|^2$$

(108)

Furthermore, the primal and the dual ratios can be upper bounded as

$$r_P = \frac{1 + 8\alpha_\theta \alpha_w B_f^2 \left(1 - \frac{1}{n_Y}\right) + 4\alpha_\theta^2 B^2_\theta B^2_w}{1 + 2\alpha_\theta \mu - 2\alpha_\theta \beta_0 B_w B_w}$$

$$= \frac{1 + 8\alpha_\theta \alpha_w B_f^2 \left(1 - \frac{1}{n_Y}\right) + 4\alpha_\theta^2 B^2_\theta B^2_w}{1 + \alpha_\theta \mu}$$

$$= 1 - \frac{\alpha_\theta \mu - 8\alpha_\theta \alpha_w B_f^2 \left(1 - \frac{1}{n_Y}\right) - 4\alpha_\theta^2 B^2_\theta B^2_w}{1 + \alpha_\theta \mu}$$

$$\leq 1 - \frac{1}{1024} \frac{n_X \kappa + 16n_X}{16n_X + 16}$$

$$\leq 1 - \frac{1}{78.8n_X \kappa + 1.3n_X + 1.3}$$

(109)

$$r_D = 1 - \frac{1}{\alpha_w \gamma} \frac{\alpha_w B_f^2 \left(1 - \frac{1}{n_Y}\right)}{1 + 2\alpha_\theta \gamma}$$

$$= 1 - \frac{1}{\alpha_w \gamma} \frac{\alpha_w \gamma}{n_X 1 + 2\alpha_\theta \gamma}$$

$$= 1 - \frac{1}{64n_X \kappa + 3n_X} < r_P$$

(110)

Therefore, inequality (108) can be further upper bounded as

$$E\|\theta^{(k)} - \theta^*\|^2 + \frac{\gamma}{\mu} \cdot \frac{64\kappa + 3}{64n_X \kappa + n_X + 1} E\|w^{(k)} - w^*\|^2$$

31
\[ \begin{align*}
\leq & \ r_P \left( E\|\theta^{(k-1)} - \theta^*\|^2 + \frac{\gamma}{\mu} \cdot \frac{64\kappa + 3}{64n_X + n_X + 1} E\|w^{(k-1)} - w^*\|^2 \right) \\
& + \frac{8B^2_1 \left( 1 - \frac{1}{n_Y} \right)}{\mu \gamma} + \frac{4B^2_wB^2_\theta}{\mu^2} \cdot \frac{1}{n_X} E\|\tilde{\theta} - \theta^*\|^2 \\
\leq & \ r_P \left( E\|\theta^{(k-1)} - \theta^*\|^2 + \frac{\gamma}{\mu} \cdot \frac{64\kappa + 3}{64n_X + n_X + 1} E\|w^{(k-1)} - w^*\|^2 \right) \\
& + \frac{8B^2_1 \left( 1 - \frac{1}{n_Y} \right)}{\mu \gamma} + \frac{4B^2_wB^2_\theta}{\mu^2} \cdot \frac{1}{n_X} \left( E\|\tilde{\theta} - \theta^*\|^2 + \frac{\gamma}{\mu} \cdot \frac{64\kappa + 3}{64n_X + n_X + 1} E\|\tilde{w} - w^*\|^2 \right) \\
\overset{(a)}{=} & \ r_P \left( E\|\theta^{(k-1)} - \theta^*\|^2 + \frac{\gamma}{\mu} \cdot \frac{64\kappa + 3}{64n_X + n_X + 1} E\|\tilde{w} - w^*\|^2 \right) \\
& + \frac{8B^2_1 \left( 1 - \frac{1}{n_Y} \right)}{\mu \gamma} + \frac{4B^2_wB^2_\theta}{\mu^2} \cdot \frac{1}{n_X} \left( E\|\tilde{\theta}_{s-1} - \theta^*\|^2 + \frac{\gamma}{\mu} \cdot \frac{64\kappa + 3}{64n_X + n_X + 1} E\|\tilde{w}_{s-1} - w^*\|^2 \right)
\end{align*} \]

where step (a) uses the fact that the $\tilde{\theta} = \tilde{\theta}_{s-1}$ and $\tilde{w} = \tilde{w}_{s-1}$ when we are considering the $s$-th stage/outer-loop (see Algorithm I in the main paper). Define the following Lyapunov functions:

\[ P_s, k = E\|\theta^{(k)} - \theta^*\|^2 + \frac{\gamma}{\mu} \cdot \frac{64\kappa + 3}{64n_X + n_X + 1} E\|w^{(k)} - w^*\|^2 \]

\[ P_s = E\|\tilde{\theta} - \theta^*\|^2 + \frac{\gamma}{\mu} \cdot \frac{64\kappa + 3}{64n_X + n_X + 1} E\|\tilde{w} - w^*\|^2 \]

As a result, we can rewrite inequality (111) as

\[ P_{s,k} \leq r_P \cdot P_{s,k-1} + \frac{8B^2_1 \left( 1 - \frac{1}{n_Y} \right)}{\mu \gamma} + \frac{4B^2_wB^2_\theta}{\mu^2} \cdot \frac{1}{n_X} P_{s-1} \]

\[ \leq \left( 1 - \frac{1}{78.8n_X + 1.3n_X + 1.3} \right) \cdot P_{s,k-1} + \frac{8B^2_1 \left( 1 - \frac{1}{n_Y} \right)}{\mu \gamma} + \frac{4B^2_wB^2_\theta}{\mu^2} \cdot \frac{1}{n_X} P_{s-1} \]

(112)

Furthermore, at the $s$-th stage (outer loop iteration), when Option I is used in Algorithm I in the main paper, we have $\tilde{\theta}_s = \theta^{(M)}$ and $\tilde{w}_s = w^{(M)}$. Therefore, it holds that

\[ P_s = P_{s,M} \]

\[ \leq \left( 1 - \frac{1}{78.8n_X + 1.3n_X + 1.3} \right)^M P_{s,0} \]

\[ + \sum_{k=0}^{M-1} \left( 1 - \frac{1}{78.8n_X + 1.3n_X + 1.3} \right)^k \times \frac{8B^2_1 \left( 1 - \frac{1}{n_Y} \right)}{\mu \gamma} + \frac{4B^2_wB^2_\theta}{\mu^2} \cdot \frac{1}{n_X} P_{s-1} \]

\[ \leq \left( 1 - \frac{1}{78.8n_X + 1.3n_X + 1.3} \right)^M P_{s,0} \]

\[ + \sum_{k=0}^{+\infty} \left( 1 - \frac{1}{78.8n_X + 1.3n_X + 1.3} \right)^k \times \frac{8B^2_1 \left( 1 - \frac{1}{n_Y} \right)}{\mu \gamma} + \frac{4B^2_wB^2_\theta}{\mu^2} \cdot \frac{1}{n_X} P_{s-1} \]

\[ = \left( 1 - \frac{1}{78.8n_X + 1.3n_X + 1.3} \right)^M P_{s,0} \]

\[ + \left( \frac{8B^2_1 \left( 1 - \frac{1}{n_Y} \right)}{\mu \gamma} + \frac{4B^2_wB^2_\theta}{\mu^2} \cdot \frac{1}{n_X} \right) \left( 1 + 3 \right) P_{s-1} \]

\[ \leq \left( 1 - \frac{1}{78.8n_X + 1.3n_X + 1.3} \right)^M P_{s,0} + \left( \frac{8B^2_1 \left( 1 - \frac{1}{n_Y} \right)}{\mu \gamma} + \frac{4B^2_wB^2_\theta}{\mu^2} \cdot \frac{1}{n_X} \right) \frac{1.3}{64} P_{s-1} \]
We will now bound the $L$ terms in (116). First consider the second $L$ term in (116):

$$
(n_X - 1) \left[ L(\theta^{(k-1)}, w^{(k)} - w^{(k-1)}) - L(\theta^*, w^{(k)} - w^{(k-1)}) \right]
$$
We further lower bound the first term in (117). Ignoring the scaling factors, this can be rewritten as
\[
\|L'(\theta^{(k-1)}), w(k) - w(k-1)\|_2^2 = \frac{1}{n_X} \sum_{i=0}^{n_X-1} J'_i(\theta)(w_i'(k) - w_i'(k-1))^2
\]
\[
\leq \beta_2 (n_X - 1)\|L'(\theta^{(k-1)}, w(k) - w(k-1))\|_2^2 + \frac{1}{\beta_2} \|\theta(k-1) - \theta^*\|_2^2
\]
(117)
for some \(\beta_2 > 0\) to be determined later, where step (a) uses convexity of the function \(L\) in its first variable, and (b) uses \(a^2 + b^2 \geq 2ab \geq ab\).

We further lower bound the first term in (117). Ignoring the scaling factors, this can be rewritten as follows:
\[
\|L'(\theta^{(k-1)}, w(k) - w(k-1))\|_2^2 \leq \frac{1}{n_X} \sum_{i=0}^{n_X-1} J'_i(\theta)(w_i'(k) - w_i'(k-1))^2
\]
\[
= \frac{1}{n_X} \sum_{i=0}^{n_X-1} J'_i(\theta)(w_i'(k) - w_i'(k-1))^2
\]
(118)
where (a) follows from the definition (45), (b) is direct, by removing the \(1/n_X\) outside the \(\| \cdot \|_2^2\) operator, and (c) uses the bounded gradients assumption (Assumption E.3).

Notice that, conditioned on \(\mathcal{F}_k\), \(w(k)\) is the only random variable in (118). Furthermore, for each \(i\) and \(j\), using (58) we have,
\[
\sum_{i=0}^{n_X-1} w_i'(k) - w_i'(k-1) = w_i' - w_i'(k-1) \quad \text{with probability} \quad 1/n_X n_Y, \quad (119)
\]
Taking expectation, conditioned on \(\mathcal{F}_k\) on both sides of (118),
\[
E \left\{ \|L'(\theta^{(k-1)}, w(k) - w(k-1))\|_2^2 \mid \mathcal{F}_k \right\} \leq \frac{B_f^2}{n_X} E \left\{ \left\| \sum_{i=0}^{n_X-1} w_i'(k) - w_i'(k-1) \right\|_2^2 \mid \mathcal{F}_k \right\}
\]
\[
= \frac{B_f^2}{n_X} \frac{1}{n_X n_Y} \sum_{i=0}^{n_X-1} \sum_{j=0}^{n_Y-1} \| w_i' \|_2^2 \quad (120)
\]
where (a) follows from (119), (b) follows from the last identity in (72), and (c) is just definition. It follows from (117) and (120) that:
\[
(n_X - 1) E \left\{ \left\| L(\theta^{(k-1)}, w(k) - w(k-1)) - L(\theta^*, w(k) - w(k-1)) \right\|_2^2 \mid \mathcal{F}_k \right\}
\]
\[
\leq \beta_2 B_f^2 \frac{(n_X - 1)^2}{n_X} E \left\{ \| w(k) - w(k-1) \|_2^2 \mid \mathcal{F}_k \right\} + \frac{1}{\beta_2} \|\theta(k-1) - \theta^*\|_2^2
\]
(121)
Next consider the remaining \(L\) terms (lines 4 and 5 of (116)). We have:
\[
L(\theta^{(k-1)}, w(k) - w^*) - L(\theta^*, w(k) - w^*) + L(\theta, w^*) - L(\theta^*, w^*)
\]
\[
= L(\theta^{(k-1)}, w(k)) - L(\theta, w(k)) + L(\theta, w^*) - L(\theta, w^*) + L(\theta^*, w^*) - L(\theta^*, w^*)
\]
\[
= L(\theta^{(k-1)}, w(k)) - L(\theta, w(k)) + L(\theta, w^*) - L(\theta^*, w^*)
\]
(122)
Therefore, the remaining \(L\) terms of (116) can be bounded as:
\[
L(\theta, w^*) - L(\theta^*, w^*) + L(\theta^{(k-1)}, w(k) - w^*) - L(\theta^*, w(k) - w^*) - \langle L'_{\theta}(\theta^{(k-1)}, w(k)), \theta(k) - \theta^* \rangle
\]
\[
L(\theta^{(k-1)}, w^{(k)}) - L(\theta^*, w^*) + L(\theta^{(k)}, w^*) - L(\theta^{(k-1)}, w^*) - \langle L'_\theta(\theta^{(k-1)}, w^{(k)}), \theta^{(k)} - \theta^* \rangle \\
\leq \langle L'_\theta(\theta^{(k-1)}, w^{(k)}), \theta^{(k)} - \theta^* \rangle + \langle L'_\theta(\theta^{(k)}, w^*), \theta^{(k)} - \theta^* \rangle - \langle L'_\theta(\theta^{(k-1)}, w^{(k)}), \theta^{(k)} - \theta^* \rangle \\
= -\langle L'_\theta(\theta^{(k-1)}, w^{(k)}), \theta^{(k)} - \theta^* \rangle + \langle L'_\theta(\theta^{(k)}, w^*), \theta^{(k)} - \theta^* \rangle \\
= \langle L'_\theta(\theta^{(k-1)}, w^*), \theta^{(k)} - \theta^* \rangle - \langle L'_\theta(\theta^{(k-1)}, w^{(k)}), \theta^{(k)} - \theta^* \rangle \\
= \langle L'_\theta(\theta^{(k-1)}, w^*), \theta^{(k)} - \theta^* \rangle - \langle L'_\theta(\theta^{(k-1)}, w^{(k)}), \theta^{(k)} - \theta^* \rangle \\
= \langle L'_\theta(\theta^{(k-1)}, w^*), \theta^{(k)} - \theta^* \rangle - \langle L'_\theta(\theta^{(k-1)}, w^{(k)}), \theta^{(k)} - \theta^* \rangle \\
\leq \langle L'_\theta(\theta^{(k-1)}, w^*), \theta^{(k)} - \theta^* \rangle - \langle L'_\theta(\theta^{(k-1)}, w^{(k)}), \theta^{(k)} - \theta^* \rangle \\
= \langle L'_\theta(\theta^{(k-1)}, w^*), \theta^{(k)} - \theta^* \rangle - \langle L'_\theta(\theta^{(k-1)}, w^{(k)}), \theta^{(k)} - \theta^* \rangle \tag{123}
\]

where step (a) substitutes \((122)\), step (b) uses the convexity of \(L(\theta, w)\) in \(\theta\) by applying \(f(x) - f(y) \leq (f'(x), x - y)\), step (c) merges the first and the third terms in line (b), step (d) adds and subtracts the same term (i.e., the second and the third terms), step (e) merges the first term with the second term and also merges the third term and the fourth term, step (f) uses the linearity of \(L(\theta, w)\) in \(w\). We now proceed to bound the two terms in \((123)\). For a \(\beta_1 > 0\) (to be determined later), the first term in \((123)\) can be upper bounded as

\[
\left| \langle L'_\theta(\theta^{(k-1)}, w^{(k)} - w^*), \theta^{(k)} - \theta^{(k-1)} \rangle \right| \\
\leq \frac{1}{\beta_1} \| L'_\theta(\theta^{(k-1)}, w^{(k)} - w^*) \|^2 + \beta_1 \| \theta^{(k)} - \theta^{(k-1)} \|^2 \\
\leq \frac{B_1^2}{\beta_1 n_X} \| w^{(k)} - w^* \|^2 + \beta_1 \| \theta^{(k)} - \theta^{(k-1)} \|^2 \tag{124}
\]

where (a) uses Cauchy-Schwartz inequality and the fact that \(a^2 + b^2 \geq 2ab \geq ab\), and step (b) uses the definition of \(L'_\theta\) in \((125)\) and Jensen’s inequality for \(\| \cdot \|^2\). Next, the second term in \((123)\) can be upper bounded as

\[
\left| \langle L'_\theta(\theta^{(k)}, w^*), \theta^{(k)} - \theta^{(k-1)} \rangle \right| \\
\leq \| L'_\theta(\theta^{(k)}, w^*) - L'_\theta(\theta^{(k-1)}, w^*) \| \cdot \| \theta^{(k)} - \theta^{(k-1)} \| \\
\leq B_0 B_w \| \theta^{(k)} - \theta^{(k-1)} \|^2 \tag{125}
\]

where step (a) uses Cauchy-Schwartz inequality and step (b) uses Lipschitz condition of the gradient \(f'_\theta\) together with the boundedness of \(w^*\) (Lemma \(E.7\)). Substituting \((124) - (125)\) into \((123)\), we obtain

\[
L(\theta^{(k)}, w^*) - L(\theta^*, w^*) + L(\theta^{(k-1)}, w^{(k)} - w^*) - L(\theta^*, w^{(k)} - w^*) - \langle L'_\theta(\theta^{(k-1)}, w^{(k)}), \theta^{(k)} - \theta^* \rangle \\
\leq \frac{B_1^2}{\beta_1 n_X} \| w^{(k)} - w^* \|^2 + \beta_1 \| \theta^{(k)} - \theta^{(k-1)} \|^2 + B_0 B_w \| \theta^{(k)} - \theta^{(k-1)} \|^2 \\
= \frac{B_1^2}{\beta_1 n_X} \| w^{(k)} - w^* \|^2 + (\beta_1 + B_0 B_w) \| \theta^{(k)} - \theta^{(k-1)} \|^2 \tag{126}
\]

We have now bounded all the \(L\) terms in \((116)\).
Substituting both (121) and (126) in (116), we get the final bound, without the $L$ terms as follows:

\[
\left(\frac{1}{2\alpha_w} + \gamma\right) E\{\|w^{(k)} - w^*\|^2 \mid \mathcal{F}_k\} + \frac{1}{2\alpha_w} E\{\|w^{(k)} - w^{(k-1)}\|^2 \mid \mathcal{F}_k\} = \\
\left(\frac{1}{2\alpha_\theta} + \mu\right) E\{\|\theta^{(k)} - \theta^*\|^2 \mid \mathcal{F}_k\} + \frac{1}{2\alpha_\theta} E\{\|\theta^{(k)} - \theta^{(k-1)}\|^2 \mid \mathcal{F}_k\} \\n\leq \left(\frac{1}{2\alpha_w} + \gamma \frac{(n_X - 1)}{n_X}\right) \|w^{(k-1)} - w^*\|^2 + \frac{1}{2\alpha_\theta} \|\theta^{(k-1)} - \theta^*\|^2 \\
+ \left(2\alpha_w B_f^2 \left(1 - \frac{1}{n_Y}\right) + \alpha_\theta B_w^2 B_\theta^2\right) \|\theta^{(k-1)} - \tilde{\theta}\|^2 \\
+ \frac{\beta_1 B_w\sqrt{(n_X - 1)^2}}{n_X^2} E\{\|w^{(k)} - w^{(k-1)}\|^2 \mid \mathcal{F}_k\} + \frac{1}{\beta_2} \|\theta^{(k-1)} - \theta^*\|^2 \\
+ \frac{B^3_w}{\beta_1 n_X} E\{\|w^{(k)} - w^*\|^2 \mid \mathcal{F}_k\} + (\beta_1 + B_\theta B_w) E\{\|\theta^{(k)} - \theta^{(k-1)}\|^2 \mid \mathcal{F}_k\} \\
\tag{127}
\]

Combining common terms and rearranging,

\[
\left(\frac{1}{2\alpha_\theta} + \mu\right) E\{\|\theta^{(k)} - \theta^*\|^2 \mid \mathcal{F}_k\} + \left(\frac{1}{2\alpha_w} + \gamma - \frac{B_f^2}{\beta_1 n_X}\right) E\{\|w^{(k)} - w^*\|^2 \mid \mathcal{F}_k\} \\
\leq \left(\frac{1}{2\alpha_w} + \gamma \frac{(n_X - 1)}{n_X}\right) \|w^{(k-1)} - w^*\|^2 \\
+ \left(\frac{1}{2\alpha_\theta} + \frac{1}{\beta_2} + 4\alpha_w B_f^2 \left(1 - \frac{1}{n_Y}\right) + 2\alpha_\theta B_w^2 B_\theta^2\right) \|\theta^{(k-1)} - \theta^*\|^2 \\
+ \left(\beta_1 + B_\theta B_w - \frac{1}{2\alpha_\theta}\right) E\{\|\theta^{(k)} - \theta^{(k-1)}\|^2 \mid \mathcal{F}_k\} \\
+ \left(\frac{\beta_2 B_f^2\sqrt{n_X - 1}}{n_X^2} - \frac{1}{2\alpha_w}\right) E\{\|w^{(k)} - w^{(k-1)}\|^2 \mid \mathcal{F}_k\} \\
+ \left(4\alpha_w B_f^2 \left(1 - \frac{1}{n_Y}\right) + 2\alpha_\theta B_w^2 B_\theta^2\right) \|\theta - \theta^*\|^2 \\
\tag{128}
\]

where we have also used the inequality $(a + b)^2 \leq 2a^2 + 2b^2$ for the $\|\theta^{(k-1)} - \tilde{\theta}\|^2$ term.

We now choose the step-sizes $\alpha_\theta$, $\alpha_w$ and $M$ as in Theorem 4.6

\[
\alpha_\theta = \left(\frac{25B_f^2}{\gamma} + 10B_\theta B_w + \frac{80B_w^2 B_\theta^2}{\mu}\right)^{-1} \\
\alpha_w = \frac{\mu}{40B_f^2} \\
M = \max\left(\frac{10}{\alpha_\theta \mu}, \frac{2n_X}{\alpha_\theta \gamma}, 4n_X\right) \\
\tag{129-131}
\]

The choice of $\alpha_\theta$ in (129) ensures the following three bounds:

\[
\alpha_\theta \leq \frac{\gamma}{25B_f^2} \quad \text{or} \quad \alpha_\theta \leq \frac{1}{10B_\theta B_w} \quad \text{or} \quad \alpha_\theta \leq \frac{\mu}{80B_w^2 B_\theta^2} \\
\tag{132}
\]

Furthermore, we choose $\beta_1$ and $\beta_2$ as follows:

\[
\beta_1 = -B_\theta B_w + \frac{1}{2\alpha_\theta} \\
\tag{133}
\]

\[36\]
where we have also used (1

The second inequality in (132) will ensure positivity of \( \beta_1 \).

Applying the above choice of hyper-parameters: Based on (133) we have:

\[
\beta_1 + B_\theta B_w - \frac{1}{2\alpha_\theta} = 0
\]  

(135)

and using (134):

\[
\frac{\beta_2 B_f^2(n_X - 1)^2}{n_X^2} - \frac{1}{2\alpha_w} = \frac{1}{2\alpha_w} B_f^2(n_X - 1)^2 - \frac{1}{2\alpha_w} = \frac{1}{2\alpha_w} (n_X - 1)^2 - \frac{1}{2\alpha_w} < 0
\]  

(136)

Equations (135) and (136) will ensure that the third and fourth terms on the right hand side of (128) are either 0 or negative, and therefore can be ignored, reducing the bound in (128) to:

\[
\left( \frac{1}{2\alpha_\theta} + \mu \right) \mathbb{E}\{ \| \theta^{(k)} - \theta^* \|^2 | F_k \} + \left( \frac{1}{2\alpha_w} + \gamma - \frac{B_f^2}{\beta_1 n_X} \right) \mathbb{E}\{ \| w^{(k)} - w^* \|^2 | F_k \}
\]

\[
\leq \left( \frac{1}{2\alpha_w} + \frac{\gamma (n_X - 1)}{n_X} \right) \| w^{(k-1)} - w^* \|^2
\]

\[
+ \left( \frac{1}{\beta_2} + \frac{1}{2\alpha_w} + 4\alpha_w B_f^2 + 2\alpha_\theta B_w^2 B_\theta^2 \right) \| \theta^{(k-1)} - \theta^* \|^2
\]

\[
+ \left( 4\alpha_w B_f^2 + 2\alpha_\theta B_w^2 B_\theta^2 \right) \| \bar{\theta} - \theta^* \|^2
\]

(137)

where we have also used \((1 - 1/n_Y) \leq 1\).

The \( \bar{\theta} \) in the (137) is \( \bar{\theta}_{s-1} \), the fixed primal variable at the beginning of stage \( s \). Denote \( \bar{\theta}_s \) to be the primal variable randomly chosen among \( \theta^{(k)} \) for \( 1 \leq k \leq M \) at the end of stage \( s \). We define \( \bar{w}_{s-1} \) and \( \bar{w}_s \) in a similar manner (though neither of them are used in the algorithm), and also note that \( \theta^{(0)} \) and \( w^{(0)} \) are initialized to \( \theta_{s-1} \) and \( \bar{w}_{s-1} \) at the beginning of stage \( s \). Summing both sides of (137) over all \( 1 \leq k \leq M \),

\[
\left( \frac{1}{2\alpha_\theta} + \mu \right) \mathbb{E}\{ \| \theta^{(M)} - \theta^* \|^2 | F_s \} + \left( \frac{1}{2\alpha_w} + \gamma - \frac{B_f^2}{\beta_1 n_X} \right) \mathbb{E}\{ \| w^{(M)} - w^* \|^2 | F_s \}
\]

\[
+ \left( \mu - \frac{1}{\beta_2} - 4\alpha_w B_f^2 - 2\alpha_\theta B_w^2 B_\theta^2 \right) \sum_{k=1}^{M-1} \mathbb{E}\{ \| \theta^{(k)} - \theta^* \|^2 | F_s \}
\]

\[
+ \left( \gamma - \frac{B_f^2}{\beta_1 n_X} \right) \sum_{k=1}^{M-1} \mathbb{E}\{ \| w^{(k)} - w^* \|^2 | F_s \}
\]

\[
\leq \left( \frac{1}{2\alpha_w} + \frac{\gamma (n_X - 1)}{n_X} \right) \| w^{(0)} - w^* \|^2 + \left( \frac{1}{\beta_2} + \frac{1}{2\alpha_\theta} \right) \| \theta^{(0)} - \theta^* \|^2
\]

\[
+ M \left( 4\alpha_w B_f^2 + 2\alpha_\theta B_w^2 B_\theta^2 \right) \| \bar{\theta} - \theta^* \|^2
\]

(138)

Substituting \( w^{(0)} = \bar{w}_{s-1} \) and \( \theta^{(0)} = \bar{\theta}_{s-1} \), and also noting that the first two terms on the left hand side of (138) can be combined with the second two terms (note that the difference in coefficients are
positive, and positive terms on the left hand side of the inequality can be ignored):

\[
\left( \mu - \frac{1}{\beta_2} - 4\alpha_w B_f^2 - 2\alpha_\theta B_w^2 B_\theta^2 \right) \sum_{k=1}^{M} \mathbb{E}\{ \| \theta^{(k)} - \theta^* \|^2 \mid \mathcal{F}_s \} \\
+ \left( \frac{\gamma}{n_X} - \frac{B_f^2}{\beta_1 n_X} \right) \sum_{k=1}^{M} \mathbb{E}\{ \| w^{(k)} - w^* \|^2 \mid \mathcal{F}_s \} \\
\leq \left( \frac{1}{2\alpha_w} + \frac{\gamma(n_X - 1)}{n_X} \right) \| \bar{w}_{s-1} - w^* \|^2 \\
+ \left( \frac{1}{\beta_2} + \frac{1}{2\alpha_\theta} + M \left( 4\alpha_w B_f^2 + 2\alpha_\theta B_w^2 B_\theta^2 \right) \right) \| \bar{\theta}_{s-1} - \theta^* \|^2
\]

(139)

Dividing both sides of (139) by \( M \) and applying Jensen’s inequality on the left hand side, we obtain:

\[
\left( \mu - \frac{1}{\beta_2} - 4\alpha_w B_f^2 - 2\alpha_\theta B_w^2 B_\theta^2 \right) \mathbb{E}\{ \| \bar{\theta}_s - \theta^* \|^2 \mid \mathcal{F}_s \} \\
+ \left( \frac{\gamma}{n_X} - \frac{B_f^2}{\beta_1 n_X} \right) \mathbb{E}\{ \| \bar{w}_s - w^* \|^2 \mid \mathcal{F}_s \} \\
\leq \left( \frac{1}{2M\alpha_w} + \frac{\gamma(n_X - 1)}{M n_X} \right) \| \bar{w}_{s-1} - w^* \|^2 \\
+ \left( \frac{1}{M\beta_2} + \frac{1}{2M\alpha_\theta} + 4\alpha_w B_f^2 + 2\alpha_\theta B_w^2 B_\theta^2 \right) \| \bar{\theta}_{s-1} - \theta^* \|^2
\]

(140)

We now substitute the hyper-parameter values in (129) (130) (131) (133) (134) in (140) to obtain a linear rate.

Substituting these values in the coefficient of the first term on the left hand side of (140) (we’re using the third bound for \( \alpha_\theta \) in (132) here):

\[
\mu - \frac{1}{\beta_2} - 4\alpha_w B_f^2 - 2\alpha_\theta B_w^2 B_\theta^2 = \mu - 2\alpha_w B_f^2 - 4\alpha_w B_f^2 - 2\alpha_\theta B_w^2 B_\theta^2 \\
\geq \mu - 2\frac{\mu}{40 B_f^2} B_f^2 - 4\frac{\mu}{40 B_f^2} B_f^2 - 2\frac{\mu}{80 B_w^2 B_\theta^2} B_w^2 B_\theta^2 \\
= \mu - \frac{\mu}{20} - \frac{\mu}{10} - \frac{\mu}{40} \\
= \mu - \frac{7\mu}{40} \\
= \frac{33\mu}{40} \\
\geq \frac{4\mu}{5}
\]

(141)
Next, substituting for the coefficient of the second term on the left hand side of (140) (here we use the first and second bounds for \( \alpha \) in (132), both in the inequality in the fourth line):

\[
\frac{\gamma}{n_X} - \frac{B_j^2}{\beta_2 n_X} = \frac{\gamma}{n_X} - \frac{B_j^2}{n_X} \left( -B_\theta B_w + \frac{1}{2\alpha_\theta} \right)^{-1}
\]

\[
= \frac{\gamma}{n_X} - \frac{B_j^2}{n_X} \left( \frac{1 - 2\alpha_\theta B_\theta B_w}{2\alpha_\theta} \right)
\]

\[
= \frac{\gamma}{n_X} - \frac{B_j^2}{n_X} \left( \frac{2\alpha_\theta}{1 - 2\alpha_\theta B_\theta B_w} \right)
\]

\[
\geq \frac{\gamma}{n_X} - \frac{B_j^2}{n_X} \times \frac{2\gamma}{25B_j^2} \times \frac{1}{1 - \frac{2}{10B_\theta B_w} B_\theta B_w}
\]

\[
= \frac{\gamma}{n_X} - \frac{2\gamma}{25n_X} \times \frac{5}{4}
\]

\[
= \frac{\gamma}{n_X} - \frac{\gamma}{10n_X}
\]

\[
\geq \frac{4\gamma}{5n_X}
\]

(142)

Next consider the coefficient of the first term on the right hand side of (140) (we use the second and third values of \( M \) in (131), both in the second line):

\[
\frac{1}{2M\alpha_\theta} + \frac{\gamma(n_X - 1)}{Mn_X} \leq \frac{1}{2M\alpha_w} + \frac{\gamma}{M}
\]

\[
\leq \frac{\gamma}{4n_X} + \frac{\gamma}{4n_X}
\]

\[
= \frac{\gamma}{2n_X}
\]

(143)

Finally, we consider the coefficient of the second term on the right hand side of (140) (we use the third bound for \( \alpha_\theta \) in (132), and the first and third definitions of \( M \) in (131)):

\[
\frac{1}{M\beta_2} + \frac{1}{2M\alpha_\theta} + 4\alpha_w B_j^2 + 2\alpha_\theta B_w B_\theta^2 \leq \frac{2\alpha_\theta B_j^2}{M} + \frac{\mu}{20} + \frac{4}{40} \frac{\mu}{B_j^2} + \frac{\mu}{80B_w B_\theta^2} 2B_w B_\theta^2
\]

\[
\leq \frac{\mu}{40B_j^2} 2B_j^2 + \frac{\mu}{20} + \frac{\mu}{10} + \frac{\mu}{40}
\]

\[
= \frac{9\mu}{40}
\]

\[
= \frac{\mu}{4}
\]

(144)

Using (141), (142), (143), and (144) in (140), we have:

\[
\frac{4\mu}{5} \mathbb{E} \left\{ \| \tilde{\theta}_s - \theta^* \|^2 \right\} + \frac{4\gamma}{5n_X} \mathbb{E} \left\{ \| \tilde{w}_s - w^* \|^2 \right\} \]

\[
\leq \frac{\gamma}{2n_X} \| \tilde{w}_{s-1} - w^* \|^2 + \frac{\mu}{4} |\tilde{\theta}_{s-1} - \theta^*|^2
\]

(145)

Applying full expectation to both sides of the above inequality, we obtain

\[
\frac{4\mu}{5} \mathbb{E} \| \tilde{\theta}_s - \theta^* \|^2 + \frac{4\gamma}{5n_X} \mathbb{E} \| \tilde{w}_s - w^* \|^2
\]

\[
\leq \frac{\gamma}{2n_X} \mathbb{E} \| \tilde{w}_{s-1} - w^* \|^2 + \frac{\mu}{4} \mathbb{E} |\tilde{\theta}_{s-1} - \theta^*|^2
\]

(146)

Dividing both sides by \( 4\mu/5 \), we have

\[
\mathbb{E} \| \tilde{\theta}_s - \theta^* \|^2 + \frac{\gamma}{n_X \mu} \mathbb{E} \| \tilde{w}_s - w^* \|^2
\]
where, by (129) and (130), the step-sizes which can be further bounded as
\[ E\|\tilde{\theta}_s - \theta^*\|^2 + \frac{\gamma}{n_X \mu} E\|\tilde{w}_s - w^*\|^2 \]
\leq \frac{5}{8} \left( E\|\tilde{w}_{s-1} - w^*\|^2 + \frac{\gamma}{\mu n_X} E\|\tilde{\theta}_{s-1} - \theta^*\|^2 \right). \tag{148}

Define the Lyapunov function \( P_s \) to be
\[ P_s = E\|\tilde{\theta}_s - \theta^*\|^2 + \frac{\gamma}{n_X \mu} E\|\tilde{w}_s - w^*\|^2. \]

Then, inequality \( [146] \) can be expressed as
\[ P_s \leq \frac{5}{8} P_{s-1} \leq \left( \frac{5}{8} \right)^s P_0. \tag{149} \]

Therefore, \( P_s \) converges to zero at a linear rate of \( 5/8 \). In order to achieve \( \epsilon \)-precision solution (i.e., \( P_s \leq \epsilon \)), it requires a total of \( O(\ln \frac{1}{\epsilon}) \) outer-loop iterations (stages). And for each outer loop iteration, it requires \( M \) steps of inner-loop primal-dual updates, which is \( O(1) \) per step (in number of oracle calls), and \( O(n_X n_Y) \) for evaluating the batch gradients for the control variates, where \( n_Y = (n_{Y_1} + \cdots + n_{Y_{n_X-1}})/n_X \). Therefore, the complexity per outer loop iteration is \( O(n_X n_Y + M) \) so that the total complexity can be written as:
\[ O \left( (n_X n_Y + M) \ln \left( \frac{P_0}{\epsilon} \right) \right). \tag{150} \]

Recall from \([131]\) that \( M \) is given by
\[ M = \frac{10}{\mu \alpha_\theta} + \frac{2n_X}{\alpha_w \gamma} + 4n_X. \]

where, by \([129]\) and \([130]\), the step-sizes \( \alpha_\theta \) and \( \alpha_w \) are given by
\[ \alpha_\theta = \left( \frac{25B_1^2}{\gamma} + 10B_0 B_w + \frac{80B_2^2 B_0^2}{\mu} \right)^{-1}, \quad \alpha_w = \frac{\mu}{40B_2^2}. \]

This implies that \( M = O(B_1^2/\mu \gamma + B_2^2 B_0^2/\mu^2 + (B_3^2/\mu \gamma)n_X + n_X) \). In consequence, the total complexity is
\[ O \left( (n_X n_Y + n_X \kappa + n_X) \ln \frac{1}{\epsilon} \right), \tag{151} \]

where
\[ \kappa = B_3^2/\mu \gamma + B_2^2 B_0^2/\mu^2. \tag{152} \]

Noting that \( E\|\tilde{\theta}_s - \theta^*\|^2 \leq P_s \), the bound \( [149] \) implies that \( E\|\tilde{\theta}_s - \theta^*\|^2 \) also converges to zero at a linear rate of \( 5/8 \) and the total complexity to reach \( E\|\tilde{\theta}_s - \theta^*\|^2 \leq \epsilon \) is also given by \([151]\).

F Special case: SVRPDA-I with \( n_{Y_1} \equiv 1, f_\theta \) linear in \( \theta \) and no outer loop

First, we observe that in this special case, our SVRPDA-I algorithm will become a single-loop algorithm, and that the outer-loop in Algorithm \([1]\) is no longer needed. To see this, first note that when \( n_{Y_1} \equiv 1, \delta_{\theta_k}^w \) is independent of \( \theta \) because the last two terms in \([12]\) would cancel each other. Second, when \( f_\theta \) is linear in \( \theta \), the term \( \hat{T}_k(\bar{\theta}) \) in \([14]\) and \( U_0 \) in \([11]\) are independent of \( \bar{\theta} \), which further implies that \( U_k \) (that is recursively defined in \([14]\)) is also independent of \( \bar{\theta} \). Finally, we also note that, with linear \( f_\theta \), the first two terms in \([15]\) cancel with each other, so that \( \delta_k^w \equiv U_k \) is independent of \( \bar{\theta} \). As a result, the inner loop in Algorithm \([1]\) does not require an outer-loop to update the reference variable \( \bar{\theta} \).

The following theorem establishes the complexity bound for the SVRPDA-I algorithm in this special case.

\[ \leq \frac{5}{16} E\|\tilde{w}_{s-1} - w^*\|^2 + \frac{5\gamma}{8\mu n_X} E\|\tilde{\theta}_{s-1} - \theta^*\|^2, \tag{147} \]
Adding (155) and (156), we obtain the combined primal-dual bound:

**Theorem F.1.** Suppose Assumptions 4.2, 4.4 hold. Furthermore, suppose \( n_{Y_i} = 1, 1 \leq i \leq n_X \), and \( f_\theta \) is a linear function of \( \theta \). Consider just the the inner loop of Algorithm 7 with \( s = 1 \) fixed, and

\[
\alpha_\theta = \frac{\gamma}{16B_2^2 + 4n_X \mu \gamma}, \quad \text{and} \quad \alpha_w = \frac{3n_X + \kappa + 1}{2\kappa \mu n_X} 
\]

where \( \kappa = B_2^2 / \gamma \mu \) is the condition number. Then, the Lyapunov function

\[
\Delta^{(k)} := \left( \frac{1}{2\alpha_\theta} + \mu \right) \mathbb{E} \left\{ \| \theta^{(k)} - \theta^* \|^2 \mid \mathcal{F}_k \right\} + \left( \frac{1}{2\alpha_w} + \gamma \right) \mathbb{E} \left\{ \| w^{(k)} - w^* \|^2 \mid \mathcal{F}_k \right\}
\]

satisfies \( \Delta^{(k)} \leq (1 - 1/(1 + 2\kappa + 2n_X))^k \Delta^{(0)} \). Furthermore, the overall computational cost (in number of oracle calls) for reaching \( \Delta^{(k)} \leq \epsilon \) is upper bounded by

\[
O \left( \left( n_X + \kappa \right) \ln \left( \frac{1}{\epsilon} \right) \right).
\]

In comparison, the authors in [24] propose a stochastic primal dual coordinate (SPDC) algorithm for this special case and prove an overall complexity of \( O \left( (n_X + \sqrt{n_X \kappa}) \ln \left( \frac{1}{\epsilon} \right) \right) \) to achieve an \( \epsilon \)-error solution, where the condition number \( \kappa = B_2^2 / \mu \gamma \). This is by far the best complexity for this class of problems. It is interesting to note that the complexity result in (153) and the complexity result in [24] only differ in their dependency on \( \kappa \). This difference is most likely due to the acceleration technique that is employed in the primal update of the SPDC algorithm. We conjecture that the dependency on the condition number of SVRPDA-I can be further improved using a similar acceleration technique.

**F.1 Proof of Theorem F.1**

It is useful to first discuss the main implications of choosing \( f_\theta \) to be linear in \( \theta \) and \( n_{Y_i} = 1 \) for all \( i \). First, based on Assumption 4.2 (or equivalently Assumption 4.3), we have \( B_\theta = 0 \), since \( f_\theta \) is independent of \( \theta \). This also implies that \( L_\theta' \) is independent of \( \theta \), and therefore, Assumption 4.4 (or equivalently Assumption 4.4) holds with equality:

\[
L(\theta_1, w) - L(\theta_2, w) = (L_\theta'(\theta_2, w), \theta_1 - \theta_2).
\]

In particular, for any \( \theta \in \mathbb{R}^d \) and \( w \in \mathbb{R}^e \), \( L(\theta, w) = (L_\theta'(\theta, w), \theta) \). Finally, \( n_{Y_i} = 1 \) implies \( 1/n_{Y_i} = 1/n_{Y} = \frac{1}{n_X} \sum_{i=0}^{n_X-1} 1/n_{Y_i} = 1 \).

Using the above implications in the primal bound (91) (in particular, letting \( B_\theta = 0 \), and using linearity of \( L(\theta, w) \)), we obtain the primal bound for the special case as follows:

\[
\left( \frac{1}{2\alpha_\theta} + \mu \right) \mathbb{E} \left\{ \| \theta^{(k)} - \theta^* \|^2 \mid \mathcal{F}_k \right\} + \left( \frac{1}{2\alpha_w} + \gamma \right) \mathbb{E} \left\{ \| w^{(k)} - w^* \|^2 \mid \mathcal{F}_k \right\} 
\leq \frac{1}{2\alpha_\theta} \| \theta^{(k-1)} - \theta^* \|^2 - \mathbb{E} \left\{ \left[ L(\theta^{(k)}), w^{(k)} - w^* \right] - L(\theta^*, w^{(k-1)} - w^*) \right\} \mid \mathcal{F}_k \right\} 
\]

(155)

Similarly, using the above implications in the dual bound (76) (in particular, letting \( (1 - 1/n_{Y}) = 0 \), the dual bound for the special case becomes:

\[
\left( \frac{1}{2\alpha_w} + \gamma \right) \mathbb{E} \left\{ \| w^{(k)} - w^* \|^2 \mid \mathcal{F}_k \right\} + \frac{1}{2\alpha_w} \mathbb{E} \left\{ \| w^{(k)} - w^{(k-1)} \|^2 \mid \mathcal{F}_k \right\} 
\leq \left( \frac{1}{2\alpha_w} + \gamma \left( \frac{n_X - 1}{n_X} \right) \right) \| w^{(k-1)} - w^* \|^2 
+ \mathbb{E} \left\{ L(\theta^{(k-1)}, w^{(k-1)} - w^*) - L(\theta^*, w^{(k-1)} - w^*) \mid \mathcal{F}_k \right\} 
+ n_X \mathbb{E} \left\{ L(\theta^{(k-1)}, w^{(k-1)} - w^{(k-1)}) - L(\theta^*, w^{(k-1)} - w^{(k-1)}) \mid \mathcal{F}_k \right\} 
\]

(156)

Adding (155) and (156), we obtain the combined primal-dual bound:

\[
\left( \frac{1}{2\alpha_\theta} + \mu \right) \mathbb{E} \left\{ \| \theta^{(k)} - \theta^* \|^2 \mid \mathcal{F}_k \right\} + \left( \frac{1}{2\alpha_w} + \gamma \right) \mathbb{E} \left\{ \| w^{(k)} - w^* \|^2 \mid \mathcal{F}_k \right\} 
\]
As done in the previous proofs, we will first consider all the \( \theta \) terms that appear on the right hand side of (157). Following exactly the same steps as (93), (94), (95) and (96), we obtain the final bound for the \( \theta \) terms as given in (177), but with \( B_{\theta} = 0 \). We still write out the whole simplification details here for completeness, and also to show how this special case is much simpler than the more general case.

Considering all the \( \theta \) terms in (157), we have:

\[
\begin{align*}
- \left[ L(\theta^{(k)}, w^{(k)} - w^*) - L(\theta^*, w^{(k)} - w^*) \right] + \left[ L(\theta^{(k-1)}, w^{(k-1)} - w^*) - L(\theta^*, w^{(k-1)} - w^*) \right] \\
+ n_X \left[ L(\theta^{(k-1)}, w^{(k)} - w^{(k-1)}) - L(\theta^*, w^{(k)} - w^{(k-1)}) \right]
\end{align*}
\]

\[
\begin{align*}
&= L(\theta^{(k-1)}, w^{(k-1)} - w^*) - L(\theta^{(k)}, w^{(k-1)} - w^*) \\
&+ n_X \left[ L(\theta^{(k-1)}, w^{(k)} - w^{(k-1)}) - L(\theta^*, w^{(k)} - w^{(k-1)}) \right]
\end{align*}
\]

\[
\begin{align*}
&= L(\theta^{(k-1)}, w^{(k-1)} - w^*) - L(\theta^{(k)}, w^{(k-1)} - w^*) \\
&+ \left( n_X - 1 \right) \left[ L(\theta^{(k-1)}, w^{(k)} - w^{(k-1)}) - L(\theta^*, w^{(k)} - w^{(k-1)}) \right]
\end{align*}
\]

\[
\begin{align*}
&= L(\theta^{(k-1)}, w^{(k-1)} - w^*) - L(\theta^{(k)}, w^{(k-1)} - w^*) \\
&+ \left( n_X - 1 \right) \left[ L(\theta^{(k-1)}, w^{(k)} - w^{(k-1)}) - L(\theta^*, w^{(k)} - w^{(k-1)}) \right]
\end{align*}
\]

\[
\begin{align*}
&\leq \left< L_{\theta} \left( \theta^{(k-1)} - \theta^* + (n_X - 1)(w^{(k)} - w^{(k-1)}) \right), \theta^{(k-1)} - \theta^{(k)} \right>
\end{align*}
\]

\[
\begin{align*}
&= \left( \frac{1}{n_X} \sum_{i=0}^{n_X-1} T_i(\theta^{(k-1)})(w_i^{(k)} - w_i^*) \right) \left( \theta^{(k-1)} - \theta^{(k)} \right)
\end{align*}
\]

\[
\begin{align*}
&= \left( \frac{1}{n_X} \sum_{i=0}^{n_X-1} T_i(\theta^{(k-1)})(w_i^{(k-1)} + n_X(w_i^{(k)} - w_i^{(k-1)})) \right) \left( \theta^{(k-1)} - \theta^{(k)} \right)
\end{align*}
\]

\[
\begin{align*}
&\leq \beta_1 \left< \frac{1}{n_X} \sum_{i=0}^{n_X-1} T_i(\theta^{(k-1)})(w_i^{(k-1)} + n_X(w_i^{(k)} - w_i^{(k-1)})) \right>, \theta^{(k-1)} - \theta^{(k)} \right>
\end{align*}
\]

\[
\begin{align*}
&= \frac{1}{n_X} \sum_{i=0}^{n_X-1} T_i(\theta^{(k-1)})(w_i^{(k-1)} + n_X(w_i^{(k)} - w_i^{(k-1)})), \theta^{(k-1)} - \theta^{(k)} \right>
\end{align*}
\]

\[
\begin{align*}
&\leq 2\beta_1 \left< \frac{1}{n_X} \sum_{i=0}^{n_X-1} T_i(\theta^{(k-1)})(w_i^{(k-1)} - w_i^*)), \theta^{(k-1)} - \theta^{(k)} \right>
\end{align*}
\]

\[
\begin{align*}
&= \beta_1 \left< \frac{1}{n_X} \sum_{i=0}^{n_X-1} T_i(\theta^{(k-1)})(w_i^{(k-1)} - w_i^*), \theta^{(k-1)} - \theta^{(k)} \right>
\end{align*}
\]

\[
\begin{align*}
&\leq 2\beta_1 \left< \frac{1}{n_X} \sum_{i=0}^{n_X-1} T_i(\theta^{(k-1)})(w_i^{(k-1)} - w_i^*)), \theta^{(k-1)} - \theta^{(k)} \right>
\end{align*}
\]
\[ + \frac{1}{\beta_1} \| \theta^{(k-1)} - \theta^{(k)} \|^2 \]

\[
\leq \frac{2\beta_1 B_f^2}{n_X} \sum_{i=0}^{n_X-1} \| w_i^{(k-1)} - w_i^* \|^2 + 2\beta_1 \left\| \sum_{i=0}^{n_X-1} f'_i(\theta^{(k-1)})(w_i^{(k)} - w_i^{(k-1)}) \right\|^2 + \frac{1}{\beta_1} \| \theta^{(k-1)} - \theta^{(k)} \|^2
\]

(158)

where step (a) uses convexity of \( L \) in \( \theta \), step (b) uses \( \langle a, b \rangle \leq \beta_1 \| a \|^2 + \frac{1}{\beta_1} \| b \|^2 \), step (c) uses \( \| a + b \|^2 \leq 2 \| a \|^2 + 2 \| b \|^2 \), and step (d) applies Jensen’s inequality and bounded gradient assumption to the first term. The second term in (158) can be simplified as follows:

\[
E \left\{ \left\| \sum_{i=0}^{n_X-1} f'_i(\theta^{(k-1)})(w_i^{(k)} - w_i^{(k-1)}) \right\|^2 \right\}_{\mathcal{F}_k} = \frac{2\beta_1 B_f^2}{n_X} \sum_{i=0}^{n_X-1} \sum_{j=0}^{n_X n_Y - 1} \| w_i^{(k)} - w_i^{(k-1)} \|^2 \leq B_f^2 \sum_{i=0}^{n_X-1} \sum_{j=0}^{n_X n_Y - 1} \| w_i^{(k)} - w_i^{(k-1)} \|^2 = B_f^2 \sum_{i=0}^{n_X-1} E \{ \| w_i^{(k)} - w_i^{(k-1)} \|^2 \} \right\}_{\mathcal{F}_k}
\]

(159)

Therefore, using (158) and (159), the conditional expectation of all the \( L \) terms in (157) are bounded by:

\[
\frac{2\beta_1 B_f^2}{n_X} \sum_{i=0}^{n_X-1} \| w_i^{(k-1)} - w_i^* \|^2 + 2\beta_1 B_f^2 \sum_{i=0}^{n_X-1} E \{ \| w_i^{(k)} - w_i^{(k-1)} \|^2 \mid \mathcal{F}_k \} + \frac{1}{\beta_1} E \{ \| \theta^{(k-1)} - \theta^{(k)} \|^2 \mid \mathcal{F}_k \}
\]

(160)

Substituting the above upper bound for the \( L \) terms in (157) leads to

\[
\left( \frac{1}{2\alpha_\theta} + \mu \right) E \{ \| \theta^{(k)} - \theta^* \|^2 \mid \mathcal{F}_k \} + \left( \frac{1}{2\alpha_w} + \gamma \right) E \{ \| w^{(k)} - w^* \|^2 \mid \mathcal{F}_k \} + \left( \frac{1}{2\alpha_w} - \frac{1}{\beta_1} \right) E \{ \| \theta^{(k-1)} - \theta^{(k)} \|^2 \mid \mathcal{F}_k \} + \left( \frac{1}{2\alpha_w} - 2\beta_1 B_f^2 \right) E \{ \| w^{(k)} - w^{(k-1)} \|^2 \mid \mathcal{F}_k \} \leq \frac{1}{2\alpha_\theta} \| \theta^{(k-1)} - \theta^* \|^2 + \left( \frac{1}{2\alpha_w} + \gamma - \frac{2\beta_1 B_f^2}{n_X} \right) \| w^{(k-1)} - w^* \|^2
\]

(161)

which will be the final bound we will analyze.

**Substituting hyper-parameter choices**

Recall the choice of step-sizes \( \alpha_\theta \) and \( \alpha_w \) defined in Theorem [E.3]

\[
\alpha_\theta = \frac{\gamma}{16B_f^2 + 4n_X \mu \gamma}
\]

\[
\alpha_w = \frac{3n_X + \kappa + 1}{2\gamma n_X + \kappa + 1}
\]

(162)

where the condition number \( \kappa \) is also defined in Theorem [E.1]

\[
\kappa = B_f^2 / \gamma \mu
\]

Furthermore, choosing \( \beta_1 \) such that

\[
\frac{1}{2\alpha_\theta} = \frac{1}{\beta_1}
\]

(163)

43
we have:
\[ \alpha_\theta \alpha_w = \frac{\gamma}{16B_f^2 + 4n_X \mu \gamma} \times \frac{1}{2} \frac{3n_X + \kappa + 1}{2\gamma n_X + \kappa + 1} \]
\[ = \frac{1}{32B_f^2 + 8n_X \mu \gamma} \times \frac{3n_X + \kappa + 1}{n_X + \kappa + 1} < \frac{1}{32B_f^2 + 8n_X \mu \gamma} \times \frac{3n_X + 3\kappa + 3}{n_X + \kappa + 1} = \frac{3}{32B_f^2 + 8n_X \mu \gamma} < \frac{1}{8B_f^2} \]

so that,
\[ \frac{1}{2\alpha_w} > 4\alpha_\theta B_f^2 \]
\[ \implies \frac{1}{2\alpha_w} > 2\beta_1 B_f^2 \] (165)

Identities (163) and (165) make sure that the third and the fourth terms on the left hand side of (161) are either zero or positive, so that they can be ignored, resulting in:
\[ \left( \frac{1}{2\alpha_\theta} + \mu \right) E\left\{ \| \theta^{(k)} - \theta^* \|^2 \big| F_k \right\} + \left( \frac{1}{2\alpha_w} + \gamma \right) E\left\{ \| w^{(k)} - w^* \|^2 \big| F_k \right\} \]
\[ \leq \frac{1}{2\alpha_\theta} \| \theta^{(k-1)} - \theta^* \|^2 + \left( \frac{1}{2\alpha_w} + \gamma - \frac{\gamma}{n_X} + \frac{2\beta_1 B_f^2}{n_X} \right) \| w^{(k-1)} - w^* \|^2 \] (166)

We further look at the coefficients of other error terms in (166). To this end, we have:
\[ \frac{1}{2\alpha_\theta \mu} = \frac{16B_f^2 + 4n_X \mu \gamma}{2\mu \gamma} \]
\[ = \frac{8B_f^2}{\mu \gamma} + 2n_X \]
\[ = 2\kappa + 2n_X \] (167)

and therefore, letting \( r_P \) to denote the ratio of the coefficients of \( \| \theta^{(k-1)} - \theta^* \|^2 \) and \( E\left\{ \| \theta^{(k)} - \theta^* \|^2 \big| F_k \right\} \), we have:
\[ r_P := \left( \frac{1}{2\alpha_\theta} \right) / \left( \frac{1}{2\alpha_\theta} + \mu \right) \]
\[ = 1 - \frac{2\alpha_\theta \mu}{1 + 2\alpha_\theta \mu} \]
\[ = 1 - \frac{1}{1 + \frac{2\alpha_\theta \mu}{1 + 2\kappa + 2n_X}} \] (168)
Also, for the dual terms we have \( r_D \) denoting the ratio of the coefficients of \( \|w^{(k-1)} - w^*\|^2 \) and \( \mathbb{E}\{\|w^{(k)} - w^*\|^2 \mid \mathcal{F}_k\} \), so that:

\[
    r_D := \left( \frac{1}{2\alpha_w} + \gamma - \frac{\gamma}{n_X} + \frac{2\beta_1 B_f^2}{\gamma} \right) / \left( \frac{1}{2\alpha_w} + \gamma \right)
    = 1 - \frac{2\alpha_w \gamma - 8\alpha_w \alpha_B B_f^2}{n_X (1 + 2\alpha_w \gamma)}
\]

where step (a) follows from (164), and step (b) follows from (170) below:

\[
    \frac{2\alpha_w \gamma - 1}{2\alpha_w \gamma + 1} = \frac{3n_X + \kappa + 1}{3n_X + \kappa + 1} - 1
    = \frac{3n_X + \kappa + 1 - n_X - \kappa - 1}{3n_X + \kappa + 1 + n_X + \kappa + 1}
    = \frac{2n_X}{4n_X + 2\kappa + 2}
\]  

From (168), the above bound (169) on \( r_D \) implies \( r_D \leq r_P \).

Recalling the definition of \( \Delta^{(k)} \), the Lyapunov function defined in Theorem 6.1. Substituting for the step-size values in (162), we have:

\[
    \Delta^{(k)} = \left( \frac{1}{2\alpha_w} + \mu \right) \mathbb{E}\left\{ \|\theta^{(k)} - \theta^*\|^2 \right\} + \left( \frac{1}{2\alpha_w} + \gamma \right) \mathbb{E}\left\{ \|w^{(k)} - w^*\|^2 \mid \mathcal{F}_k \right\}
    = \left( \mu (2\kappa + 2n_X + 1) \right) \mathbb{E}\left\{ \|\theta^{(k)} - \theta^*\|^2 \right\} + \left( \gamma \left( \frac{n_X + \kappa + 1}{3n_X + \kappa + 1} + 1 \right) \right) \mathbb{E}\left\{ \|w^{(k)} - w^*\|^2 \mid \mathcal{F}_k \right\}
    = \left( \mu (2\kappa + 2n_X + 1) \right) \mathbb{E}\left\{ \|\theta^{(k)} - \theta^*\|^2 \right\} + \left( \gamma \left( \frac{4n_X + 2\kappa + 2}{3n_X + \kappa + 1} \right) \right) \mathbb{E}\left\{ \|w^{(k)} - w^*\|^2 \mid \mathcal{F}_k \right\}
\]

Based on (168) and (169), and the fact that \( r_D \leq r_P \), the inequality (161) then implies:

\[
    \Delta^{(k)} \leq r \Delta^{(k-1)}
\]

where,

\[
    r = r_P = 1 - 1/(1 + 2\kappa + 2n_X).
\]

The bound (171) implies that after \( k \) iterations, the error \( \Delta^{(k)} \) satisfies:

\[
    \Delta^{(k)} \leq r^k \Delta^{(0)}
\]

Therefore, for \( \Delta^{(k)} < \epsilon \), it suffices to have \( r^k \Delta^{(0)} < \epsilon \), so that the number of iterations \( k \) satisfies:

\[
    k \ln r \leq \ln \left( \frac{\epsilon}{\Delta^{(0)}} \right)
    \implies k \ln \left( 1 - 1/(1 + 2\kappa + 2n_X) \right) \leq \ln \left( \frac{\epsilon}{\Delta^{(0)}} \right)
    \implies -k/(1 + 2\kappa + 2n_X) \leq \ln \left( \frac{\epsilon}{\Delta^{(0)}} \right)
    \implies k \geq \left( 1 + 2\kappa + 2n_X \right) \ln \left( \frac{\Delta^{(0)}}{\epsilon} \right)
\]

where we have used \(-\ln(1-x) \geq x\). (172) implies the final complexity result of Theorem 6.1. \( \square \)