Algebraic Bethe Ansatz for $U(1)$ Invariant Integrable Models: 
The Method and General Results

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Abstract

In this work we have developed the essential tools for the algebraic Bethe ansatz solution of integrable vertex models invariant by a unique $U(1)$ charge symmetry. The formulation is valid for arbitrary statistical weights and respective number $N$ of edge states. We show that the fundamental commutation rules between the monodromy matrix elements are derived by solving linear systems of equations. This makes possible the construction of the transfer matrix eigenstates by means of a new recurrence relation depending on $N-1$ distinct types of creation fields. The necessary identities to solve the eigenvalue problem are obtained exploring the unitarity property and the Yang-Baxter equation satisfied by the $R$-matrix. The on-shell and off-shell properties of the algebraic Bethe ansatz are explicitly presented in terms of the arbitrary $R$-matrix elements. This includes the transfer matrix eigenvalues, the Bethe ansatz equations and the structure of the vectors not parallel to the eigenstates.

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1 Introduction

The quantum inverse scattering method has been playing a major role in the development of the theory of two-dimensional integrable models [1, 2, 3]. This method paved the way for the discovery of important generalizations of the six-vertex model [4] as well as it has helped to prompt the notion of quantum group symmetry [5, 6]. This approach also offers us the basic tools to solve exactly quantum integrable systems including the computation of lattice correlation functions [7, 8].

The central object in the quantum inverse scattering method is the monodromy matrix $T_A(\lambda)$ which depends on the continuous spectral parameter $\lambda$. This operator is frequently viewed as a matrix on the auxiliary space $A$ and for an arbitrary $N$-dimensional space one can write,

$$T_A(\lambda) = \sum_{a,b=1}^{N} T_{a,b}(\lambda) e_{a,b},$$

(1)

where $e_{a,b}$ are the standard $N \times N$ Weyl matrices.

The matrix elements $T_{a,b}(\lambda)$ act on the space of states of some quantum physical system and they are the generators of a quadratic algebra denominated Yang-Baxter algebra. The set of relations defining this algebra are,

$$\sum_{f,g=1}^{N} R(\lambda, \mu)^{f,g}_{a,b} T_{f,c}(\lambda) T_{g,d}(\mu) = \sum_{f,g=1}^{N} T_{b,f}(\mu) T_{a,g}(\lambda) R(\lambda, \mu)^{c,d}_{g,f},$$

(2)

In analogy to Lie algebras, functions $R(\lambda, \mu)^{c,d}_{a,b}$ can be interpreted as the structure constants of the Yang-Baxter algebra [2]. They are often viewed as the elements of a $N^2 \times N^2$ $R$-matrix acting on the tensor product of two auxiliary spaces. This matrix can be defined as,

$$R_{12}(\lambda_1, \lambda_2) = \sum_{a,b,c,d=1}^{N} R^{c,d}_{a,b}(\lambda_1, \lambda_2) e_{a,c} \otimes e_{b,d}.$$  

(3)

The associativity of the Yang-Baxter algebra requires that the $R$-matrix satisfies the celebrated Yang-Baxter equation [9],

$$R_{12}(\lambda_1, \lambda_2) R_{13}(\lambda_1, \lambda_3) R_{23}(\lambda_2, \lambda_3) = R_{23}(\lambda_2, \lambda_3) R_{13}(\lambda_1, \lambda_3) R_{12}(\lambda_1, \lambda_2),$$

(4)
where $R_{ab}(\lambda_a, \lambda_b)$ denotes the $R$-matrix acting on the tensor product of the spaces $\mathcal{A}_a \otimes \mathcal{A}_b$.

In this paper we shall consider the solutions of the Yang-Baxter equation (4) that can be normalized in order to satisfy the unitarity property,

$$R_{21}(\lambda, \mu)R_{12}(\mu, \lambda) = I_N \otimes I_N,$$

where $I_N$ is the $N \times N$ identity matrix.

At this point we remark that the unitarity property (5) follows from the Yang-Baxter equation (4) under the extra assumption that the $R$-matrix is regular. This hypothesis is equivalent to say that $R_{ab}(\lambda, \lambda)$ is proportional to the permutator on $C_N^a \otimes C_N^b$. The unitarity assures us that the $R$-matrix is invertible and this property is important to show that the Yang-Baxter algebra leads us to mutually commuting operators. This family of commuting operators is then obtained by taking the trace of the monodromy matrix on the auxiliary space,

$$T(\lambda) = \sum_{a=1}^{N} T_{a,a}(\lambda),$$

In the theory of integrable models the operator $T(\lambda)$ is regarded as the generating function of the corresponding quantum integrals of motion such as the underlying one-dimensional Hamiltonian. The understanding of the physical properties of these quantum systems should therefore include at least the knowledge of the eigenvalues and the eigenvectors of $T(\lambda)$. In principle, the diagonalization of $T(\lambda)$ can be accomplished through the formulation of the algebraic Bethe ansatz. For comprehensive reviews on this subject see for instance the references [3, 10]. The basic idea of this method is to exploit commutation relations between the monodromy matrix elements coming from the Yang-Baxter algebra (2). In particular, the eigenstates of $T(\lambda)$ are constructed by applying appropriate off-diagonal monodromy matrix elements, usually named creation operators, on a previously chosen reference state. The main expected feature of this state is that the action of the monodromy matrix on it gives us as result a triangular matrix. We emphasize, however, that the existence of such reference state does not immediately guarantee the success of an algebraic Bethe ansatz solution. In fact, for general values of $N$, we are not aware of any recipe to implement the algebraic Bethe ansatz even when a possible reference state is the trivial ferromagnetic highest
vector. For this class of models, our current knowledge on algebraic Bethe ansatz formulations remains restricted to very specific integrable systems. A representative category of such models are those directly related to \( R \)-matrices based on the vector representation of some special Lie algebras [11, 12, 13] and \( Z_2 \) graded superalgebras [14, 15, 16].

In order to bring new insights into the algebraic Bethe ansatz approach we have to consider the diagonalization of \( T(\lambda) \) without referring to any particular dependence of the \( R \)-matrix on the spectral parameters. This point of view is rather evident when the \( R \)-matrix commutes with at least one \( U(1) \) charge symmetry. This feature assures us that the ferromagnetic pseudovacuum will play the role of a suitable reference state for the most fundamental integrable model associated to such \( R \)-matrix. The diagonalization of the corresponding \( T(\lambda) \) should then be completed solely on basis of the commutation relations derived from the Yang-Baxter algebra and the constraints imposed by both the Yang-Baxter equation and the unitarity property. Despite of its relevance, this strategy of solving integrable models has so far remained largely ignored in the literature. This is particularly the case when the dimension of the monodromy matrix is \( N \geq 3 \) since we have to consider the presence of different types of creation fields. The basic problem is to unveil the role played by each creation operator on the structure of the eigenvectors. This understanding is certainly more difficult when the \( R \)-matrices elements are not specified. It is expected that identities among the weights will be crucial to solve this problem.

In this paper we report on some progress towards the algebraic Bethe ansatz solution of integrable models with arbitrary \( R \)-matrix. We will consider the simplest possible family of models whose Hilbert space description requires us to consider many independent quasi-particle excitations. This turns out to be the systems that their \( R \)-matrices commute with a single \( U(1) \) symmetry for arbitrary values of \( N \),

\[
[R_{12}(\lambda, \mu), S^z \otimes I_N + I_N \otimes S^z] = 0
\]  

(7)

where \( S^z \) denotes the azimuthal component of an operator with spin \( s = (N - 1)/2 \). Note that this invariance implies that \( R(\lambda, \mu)^{c,d}_{a,b} \neq 0 \) only when the ice rule \( a + b = c + d \) is satisfied.

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1 A small part of our results has been briefly announced in reference [17].
We can use property (7) to express the $R$-matrix in terms of the sectors labeling the $2N - 1$ eigenvalues of the $U(1)$ operator. In the Weyl basis indices these sectors are easily parameterized by the charge $q = a + b - 1$ and the $R$-matrix can be written as,

$$R_{12}(\lambda, \mu) = \sum_{q=1}^{2N-1} \sum_{a,c = M\{1,q+1-N\}}^{m\{q,N\}} R(\lambda, \mu)^{c,q+1-c}_{a,q+1-a} e_{a,c} \otimes e_{q+1-a,q+1-c}.\quad (8)$$

where $M\{x, y\}$ and $m\{x, y\}$ denotes the maximum and minimum integer of the pair $\{x, y\}$, respectively.

These integrable systems can be seen as multistate extensions of the totally asymmetric six-vertex model preserving a single conserved quantum number. The classical example is the model whose $R$-matrix is based on the higher spin representation of the quantum $U_q[SU(2)]$ algebra [18]. Here we stress that a systematic classification of $U(1)$ invariant solutions of the non-linear Yang-Baxter equation is beyond the reach at present. It is therefore rather probable that a variety of $U(1)$ invariant $R$-matrices are still waiting to be discovered. This should include solutions with non-additive $R$-matrices as well as those with infinity number of degrees of freedom typical of non-compact models. An example of the latter system could be the vertex model based on the discrete infinite dimensional representation of the $SL(2, R)$ algebra. Note that such representation has a highest weight which plays the role of possible reference state. The framework developed here has the nice feature of being able to accommodate the algebraic solution of all such different systems in a rather unified way.

We have organized this paper as follows. In next section we describe the basic properties of the vertex model associated to the fundamental representation of the Yang-Baxter algebra based on the $R$-matrices satisfying the ice rule (7). In section 3 we describe the essential tools that are necessary to obtain the appropriate commutation rules between the monodromy matrix elements. It is argued that these relations are obtained by solving coupled systems of linear equations. To fix up the main idea of our method we presented the structure of specific sets of commutation rules. In section 4 we discuss a procedure to obtain suitable identities between the elements of the $R$-matrix from that Yang-Baxter and unitarity relations. These identities are decisive for the solution of the transfer matrix eigenvalue problem. In section 5 we diagonalize the transfer matrix
by means of an algebraic Bethe ansatz. The respective eigenstates are constructed similarly to that of a bosonic Fock space with \( N - 1 \) distinct creation fields. We present the explicit expressions for both the on-shell and off-shell parts of the action of the transfer matrix on the eigenstates in terms of the statistical weights. Our conclusions are presented in section 6. In appendices A and B we summarize the technical details entering the solution of the two and three particle problems, respectively.

## 2 The vertex model representation

Each solution of the Yang-Baxter equation (4) gives rise to representations of the Yang-Baxter algebra (2). The representations depending on the spectral parameter are named Lax operators denoted here by \( \mathcal{L}_{A_i}(\lambda) \). The simplest such representation can be obtained directly from the \( R \)-matrix amplitudes by the expression,

\[
\mathcal{L}_{A_i}(\lambda) = \sum_{a, b, c, d=1}^N R(\lambda, \mu_i)^{c, d}_{a, b} e_{a, c} \otimes e^{(i)}_{b, d},
\]

where \( e_{b, d}^{(i)} \) are \( N \times N \) Weyl matrices acting on the tensor product \( \prod_{i=1}^L \otimes_i^N \) space of an one-dimensional lattice of length \( L \). The variables \( \mu_i \) play the role of free continuous parameters.

The Yang-Baxter algebra (2) has the co-multiplication property that the tensor product of two representations is still another possible representation. Hence, the following ordered product of \( \mathcal{L} \)-operators

\[
T_\Lambda(\lambda) = \mathcal{L}_{A_L}(\lambda, \mu_L)\mathcal{L}_{A_{L-1}}(\lambda, \mu_{L-1}) \cdots \mathcal{L}_{A_1}(\lambda, \mu_1),
\]

is indeed a representation of the quadratic algebra (2).

In the context of classical vertex models of statistical mechanics \( \mathcal{L}_{A_i}(\lambda) \) encoded the structure of the local Boltzmann weights at the \( i \)-th site of a square lattice of size \( L \). The possible states of such statistical system are then associated to the possible bond configurations \( a, b, c, d = 1, \ldots, N \) of each vertex on the \( L \times L \) lattice. The energy of each \( i \)-th vertex configuration is then associated to the statistical weight \( R(\lambda, \mu_i)^{c, d}_{a, b} \). The row-to-row transfer matrix of these vertex models \( T(\lambda) \)
can be written in a compact form with the help of the monodromy matrix \((10)\). It is given by the trace of \(T_A(\lambda)\) over the auxiliary space,

\[
T(\lambda) = Tr_A[\mathcal{L}_{AL}(\lambda, \mu_L)\mathcal{L}_{AL-1}(\lambda, \mu_{L-1}) \ldots \mathcal{L}_{A1}(\lambda, \mu_1)].
\]  

In order to diagonalize the operator \(T(\lambda)\) within the quantum inverse scattering framework one needs to know an exact eigenstate \(|0\rangle\) of \(T(\lambda)\). This vector works a reference state in the construction of the Hilbert space of \(T(\lambda)\) by an algebraic Bethe ansatz. In general, such state is searched by asking that the action of the lower left monodromy matrix elements on \(|0\rangle\) are annihilated for arbitrary \(\lambda\). This means that the monodromy matrix acts as an upper triangular matrix on \(|0\rangle\), namely

\[
\mathcal{T}_{a,b}(\lambda) |0\rangle = \begin{cases} 
  w_a(\lambda) |0\rangle, & \text{for } a = b \\
  0, & \text{for } a > b \\
  |ab\rangle, & \text{for } a < b,
\end{cases}
\]  

where \(|ab\rangle\) denotes a general non-null vector.

The action of the upper and the lower elements \(\mathcal{T}_{a,b}(\lambda)\) on \(|0\rangle\) has therefore distinct meanings. For \(a < b\) they play the role of creation fields while for \(a > b\) they can be thought as annihilators. The presence of the \(U(1)\) invariance \((7)\) makes it possible to build up such reference state in terms of the tensor product of local ferromagnetic vectors,

\[
|0\rangle = \prod_{i=1}^{L} \otimes |s\rangle_i, \quad |s\rangle_i = \begin{pmatrix} 
  1 \\
  0 \\
  \vdots \\
  0
\end{pmatrix}_N.
\]  

The state \(|s\rangle_i\) can be viewed as the highest eigenstate of the spin operators \(S^z_i\) acting on the \(i\)-th site of a chain of length \(L\). The total spin of the reference state is therefore \(\frac{(N-1)L}{2}\) and this is the reason why \(|0\rangle\) is called the ferromagnetic state. From Eqs.\((8, 9, 10)\) it is not difficult to see that property \((12)\) is in fact satisfied and that the expressions for functions \(w_a(\lambda)\) are,

\[
w_a(\lambda) = \prod_{i=1}^{L} R(\lambda, \mu_i)^{a,1}_{a,1}.
\]
The next step would then be the construction of other eigenstates of $T(\lambda)$ besides $|0\rangle$. To this end we shall first consider the action of the total spin operator $\sum_{i=1}^{L} S_i^z$ on the monodromy matrix elements $T_{a,b}(\lambda)$. Considering the structure of the Lax operators (8,9) it is possible to derive the following relation,

$$\left[ T_{a,b}(\lambda), \sum_{i=1}^{L} S_i^z \right] = (b-a)T_{a,b}(\lambda). \quad (15)$$

The above commutation relation is useful to illuminate the physical content of the creation fields $T_{a,b}(\lambda)$ for $a < b$. In fact, by acting Eq.(15) on the reference state $|0\rangle$ one derives the property,

$$\sum_{i=1}^{L} S_i^z T_{a,b}(\lambda) |0\rangle = [s(L-1) + s - b + a] T_{a,b}(\lambda) |0\rangle \quad \text{for} \quad a < b. \quad (16)$$

From Eq.(16) we conclude that the field $T_{a,b}(\lambda)$ for $a < b$ behaves as raising operator of an excitation over $|0\rangle$ whose value of the azimuthal spin component is $s_{a,b} = s - b + a$. This means that all monodromy matrix elements of a given upper diagonal have the same azimuthal spin and therefore describe the same type of excitations. As a consequence of that we end up with only $N-1$ linear independent creations fields which is represented, in the simplest manner, by the first row of the monodromy matrix $T_{1,b}(\lambda)$ for $b = 2, \cdots, N$. The analogy with the Hilbert space of high spin Heisenberg magnets strongly suggests that the eigenstates of the transfer matrix (11) could be build up in an algebraic fashion by using the help of such $N-1$ independent creation fields. The whole construction, however, will depend very much on our ability to recast the Yang-Baxter algebra (2) in the form of convenient commutation rules between the diagonal and the off-diagonal monodromy matrix elements. In next section we will deal with this artistic part of the quantum inverse scattering method.

3 The fundamental commutation rules

The purpose of this section is to discuss the structure of the commutation relations that are relevant in the exact diagonalization of the transfer matrix (11). In principle, these relations between the monodromy matrix elements are derived considering the Yang-Baxter algebra (2).
Let us use the symbol $[\alpha; \beta]$ to represent the $\alpha$-th row and the $\beta$-th column of the $N^2 \times N^2$ matrix defined by Eq. (2). The projection of the Yang-Baxter algebra on the rather general set of entries $[(\bar{a} - 1)N + \bar{b}; (\bar{c} - 1)N + \bar{d}]$ can be written as,

$$
\sum_{\bar{e}=M\{1,\bar{a}+\bar{b}-N\}}^{m\{\bar{a}+\bar{b}-1,N\}} R(\lambda, \mu)_{\bar{b},\bar{a}}^{\bar{e},\bar{a}+\bar{b}-\bar{e}} T_{\bar{e},\bar{c}}(\lambda) T_{\bar{a}+\bar{b}-\bar{e},\bar{d}}(\mu) = \sum_{\bar{e}=M\{1,\bar{e}+\bar{d}-N\}}^{m\{\bar{e}+\bar{d}-1,N\}} T_{\bar{a},\bar{e}}(\mu) T_{\bar{b},\bar{e}+\bar{d}-\bar{e}}(\lambda) R(\lambda, \mu)_{\bar{e}+\bar{d}-\bar{e},\bar{e}}^{\bar{e},\bar{d}} \quad \bar{a}, \bar{b}, \bar{c}, \bar{d} = 1 \ldots N.
$$

(17)

In what follows we will discuss three different families of commutation rules between the diagonal, creation and annihilation operators which can be derived from Eq. (17).

### 3.1 The diagonal and creation fields

The creation operators $T_{1,\bar{b}}(\lambda)$ for $b = 2, \ldots, N$ will provide us a basis to construct the eigenvectors of the transfer matrix $T(\lambda)$. Their commutation rules with the diagonal monodromy matrix elements $T_{\bar{a},\bar{a}}(\lambda)$ for $a = 1, \ldots, N$ are therefore essential in the transfer matrix eigenvalue problem. These relations are obtained from the entries $[a; (a + c - 1)N + b - c]$ upon suitable choices of the extra variable $c$. By substituting $\bar{a} = 1, \bar{b} = a, \bar{c} = a + c$ and $\bar{d} = b - c$ in Eq. (17) we find that the respective equations are,

$$
\sum_{\bar{e}=1}^{a} R(\lambda, \mu)_{\bar{a},\bar{1}}^{\bar{e},\bar{a}-\bar{e}+1} T_{\bar{e},\bar{a}+\bar{c}}(\lambda) T_{\bar{a}+\bar{b}-\bar{c}}(\mu) = \sum_{\bar{e}=M\{1,\bar{a}+\bar{b}+N\}}^{m\{a+b-1,N\}} T_{1,\bar{e}}(\mu) T_{\bar{a},\bar{a}+\bar{b}-\bar{e}+\bar{c}}(\lambda) R(\lambda, \mu)_{\bar{a}+\bar{b}-\bar{e}+\bar{c},\bar{a}+\bar{b}-\bar{c}} (\mu) \quad a \leq b.
$$

(18)

In general, to obtain commutation rules that are useful for the eigenvalue problem, we still have to elaborate on Eq. (18). The additional manipulation consists in making particular combinations of a number of equations derived from Eq. (18) with the help of the variable $c$. In Table 1 we have summarized in detail the linear combination required for each diagonal field.

From Table 1 we see that only the commutation rules for the operators $T_{1,1}(\lambda)$ and $T_{N,N}(\lambda)$ follows directly from Eq. (18). Indeed, by setting $c = b - 1$ for $T_{1,1}(\lambda)$ and $c = 0$ for $T_{N,N}(\lambda)$ one finds that their commutation rules with $T_{1,\bar{b}}(\mu)$ are,

$$
T_{1,1}(\lambda) T_{1,\bar{b}}(\mu) = \frac{R(\mu, \lambda)_{1,1}^{\bar{1},1} T_{1,1}(\lambda)}{\frac{R(\mu, \lambda)_{b,1}^{\bar{1},b}}{R(\mu, \lambda)_{b,1}^{\bar{1},1}} T_{1,\bar{b}}(\mu) T_{1,1}(\lambda) - \sum_{\bar{e}=2}^{b} \frac{R(\mu, \lambda)_{b,1}^{\bar{1},b-\bar{e}+\bar{e}}}{R(\mu, \lambda)_{b,1}^{\bar{1},1}} T_{1,\bar{e}}(\lambda) T_{1,1+b-\bar{e}}(\mu).
$$

(19)
The linear combination made from Eq. (18) to obtain commutation rules between the fields $T_{a,a} (\lambda)$ and $T_{1,b} (\lambda)$. The number of equations entering the linear combination is governed by the index $c$.

\[
T_{N,N}(\lambda) T_{1,b}(\mu) = \frac{R(\lambda, \mu)^{N, b}}{R(\lambda, \mu)^{N, 1}} T_{1,b}(\mu) T_{N,N}(\lambda) + \sum_{\varepsilon = b+1}^{N} \frac{R(\lambda, \mu)^{N, b}}{R(\lambda, \mu)^{N, 1}} T_{\varepsilon,1}(\mu) T_{N,N+b-\varepsilon}(\lambda)
- \sum_{\varepsilon=1}^{N-1} \frac{R(\lambda, \mu)^{\varepsilon, N-\varepsilon+1}}{R(\lambda, \mu)^{N, 1}} T_{\varepsilon,N}(\lambda) T_{N-N+1,b}(\mu)
\]

(20)

The commutation rules for the remaining diagonal operators $T_{a,a} (\lambda)$ for $2 \leq a \leq N - 1$ demand a considerable amount of extra work. In these cases we first have to implement the linear combination according to the indices $c$ exhibited in Table 1. As an example let us consider the linear system associated to the diagonal fields with $2 \leq a \leq N + 1 - b$. We find that the relations coming from Eq. (18) for $c = 0, \ldots, b$ can be arranged in the following form,

\[
A_1^{(a,b)} (\lambda, \mu) \begin{pmatrix} T_{1,1}(\mu) T_{a,a+b-1}(\lambda) \\ T_{1,2}(\mu) T_{a,a+b-2}(\lambda) \\ \vdots \\ T_{1,b}(\mu) T_{a,a}(\lambda) \end{pmatrix} = \sum_{\varepsilon=1}^{a} R(\lambda, \mu)_{a,1}^{\varepsilon,a-\varepsilon+1} \begin{pmatrix} T_{\varepsilon,a}(\lambda) T_{a-\varepsilon+1,b}(\mu) \\ T_{\varepsilon,a+1}(\lambda) T_{a-\varepsilon+1,b-1}(\mu) \\ \vdots \\ T_{\varepsilon,a+b-1}(\lambda) T_{a-\varepsilon+1,1}(\mu) \end{pmatrix}
- \sum_{\varepsilon=b+1}^{a+b-1} v_1^{(a,b)} (\lambda, \mu) T_{1,\varepsilon}(\mu) T_{a,a+b-\varepsilon}(\lambda)
\]

(21)
where the \( b \times b \) matrix \( A_{1}^{(a,b)}(\lambda, \mu) \) is given in terms of the structure constants as

\[
A_{1}^{(a,b)}(\lambda, \mu) = \begin{pmatrix}
R(\lambda, \mu)^{a,b}_{a+b-1,1} & R(\lambda, \mu)^{a,b}_{a+b-2,2} & \cdots & R(\lambda, \mu)^{a,b}_{a,b-1,1} \\
R(\lambda, \mu)^{a+1,b-1}_{a+b-1,1} & R(\lambda, \mu)^{a+1,b-1}_{a+b-2,2} & \cdots & R(\lambda, \mu)^{a+1,b-1}_{a,b-1,1} \\
\vdots & \vdots & \ddots & \vdots \\
R(\lambda, \mu)^{a+b-1,1}_{a+b-1,1} & R(\lambda, \mu)^{a+b-1,1}_{a+b-2,2} & \cdots & R(\lambda, \mu)^{a+b-1,1}_{a,b-1,1}
\end{pmatrix},
\]

(22)

while the \( b \times 1 \) vector \( v_{1}^{(a,b)}(\lambda, \mu) \) is

\[
v_{1}^{(a,b)}(\lambda, \mu) = \begin{pmatrix}
R(\lambda, \mu)^{a,b}_{a+b-\bar{e},\bar{\bar{e}}} \\
R(\lambda, \mu)^{a+1,b-1}_{a+b-\bar{e},\bar{\bar{e}}} \\
\vdots \\
R(\lambda, \mu)^{a+b-1,1}_{a+b-\bar{e},\bar{\bar{e}}}
\end{pmatrix}.
\]

(23)

We have now reached a point in which the properties of linear systems of equations are applicable. In fact, the term \( T_{1,b}(\mu)T_{a,a}(\lambda) \) can be systematically single out from the left-hand side of Eq.(21) with the help of Cramer’s rule. Besides that we note the first non-homogeneous part of Eq.(21) contains the reverse product \( T_{a,a}(\lambda)T_{1,b}(\mu) \) when the sum index is \( \bar{e} = a \). Therefore, by computing determinants of \( b \times b \) matrices we are able to generate a unique linear equation relating the operator products \( T_{a,a}(\lambda)T_{1,b}(\mu) \) and \( T_{1,b}(\mu)T_{a,a}(\lambda) \). This equation is then easily solved in order to establish the commutation rule between \( T_{a,a}(\lambda) \) and \( T_{1,b}(\mu) \) for \( 2 \leq a \leq N + 1 - b \).

Clearly, the same method described above can be used to obtain the commutation rules for the other values \( N + 2 - b \leq a \leq N - 1 \) of the index \( a \). The system of linear equations is however different and for sake of completeness we also present it here. Following Table 1 one finds that the respective Eq.(18) can be written as,

\[
A_{2}^{(a,b)}(\lambda, \mu) = \begin{pmatrix}
T_{1,a+b-N}(\mu)T_{a,N}(\lambda) \\
T_{1,a+b-N+1}(\mu)T_{a,N-1}(\lambda) \\
\vdots \\
T_{1,b}(\mu)T_{a,a}(\lambda)
\end{pmatrix} = \sum_{\bar{e}=1}^{a}R(\lambda, \mu)^{\bar{e},a-\bar{e}+1}_{a,1} \begin{pmatrix}
T_{\bar{e},a}(\lambda)T_{a-\bar{e}+1,b}(\mu) \\
T_{\bar{e},a+1}(\lambda)T_{a-\bar{e}+1,b-1}(\mu) \\
\vdots \\
T_{\bar{e},N}(\lambda)T_{a-\bar{e}+1,a+b-N}(\mu)
\end{pmatrix} - \sum_{\bar{e}=b+1}^{N}v_{2}^{(a,b)}(\lambda, \mu)T_{1,\bar{e}}(\mu)T_{a,a+b-\bar{e}}(\lambda) \quad \text{for} \quad N + 2 - b \leq a \leq N - 1,
\]

(24)
where the \((N + 1 - a) \times (N + 1 - a)\) matrix \(A_{2}^{(a,b)}(\lambda, \mu)\) is given by

\[
A_{2}^{(a,b)}(\lambda, \mu) = \begin{pmatrix}
R(\lambda, \mu)^{a,b}_{N,a+b-N} & R(\lambda, \mu)^{a,b}_{N-1,a+b-N+1} & \cdots & R(\lambda, \mu)^{a,b}_{a,b} \\
R(\lambda, \mu)^{a+1,b-1}_{N,a+b-N} & R(\lambda, \mu)^{a+1,b-1}_{N-1,a+b-N+1} & \cdots & R(\lambda, \mu)^{a+1,b-1}_{a,b} \\
\vdots & \vdots & \ddots & \vdots \\
R(\lambda, \mu)^{N,a+b-N}_{N,a+b-N} & R(\lambda, \mu)^{N,a+b-N}_{N-1,a+b-N+1} & \cdots & R(\lambda, \mu)^{N,a+b-N}_{a,b-N}
\end{pmatrix},
\tag{25}
\]

while the \((N + 1 - a) \times 1\) vector \(v_{2}^{(a,b)}(\lambda, \mu)\) is

\[
v_{2}^{(a,b)}(\lambda, \mu) = \begin{pmatrix}
R(\lambda, \mu)^{a,b}_{a+b-\bar{\varepsilon},\bar{\varepsilon}} \\
R(\lambda, \mu)^{a+1,b-1}_{a+b-\bar{\varepsilon},\bar{\varepsilon}} \\
\vdots \\
R(\lambda, \mu)^{N,a+b-N}_{a+b-\bar{\varepsilon},\bar{\varepsilon}}
\end{pmatrix}.
\tag{26}
\]

As before the desirable commutation relations between \(T_{a,a}(\lambda)\) and \(T_{1,b}(\mu)\) for \(N + 2 - b \leq a \leq N - 1\) are obtained by using Cramer’s rule in the linear system \((24 - 26)\). We emphasize that the commutation rules discussed in this subsection have the property that the corresponding \(T_{1,\bar{\varepsilon}}(\mu)\) fields that appear on the left-hand side of the products satisfy the condition \(\bar{\varepsilon} \geq b\). Here we recall that these type of fields are the ones with clear potential to participate directly on the structure of the eigenstates. The condition \(\bar{\varepsilon} \geq b\) guarantees that such \(T_{1,\bar{\varepsilon}}(\mu)\) operators, generated by taking \(T_{a,a}(\lambda)\) through \(T_{1,b}(\mu)\), will only contribute to eigenvectors made out creation fields with spin lower or equal to \(s_{1,b}\). In this manner we assured that the eigenvectors constructed in terms of the \(N - 1\) \(T_{1,b}(\mu)\) operators can indeed be thought as multiparticle states ordered by their spin values.

Let us now illustrate how the general method explained above works in practice. To this end we shall present explicitly the complete structure of the commutation rules between the first two creation fields with all the diagonal operators. The simplest example concerns to the \(T_{1,2}(\mu)\) operator. Its commutation rules with the diagonal operators play a fundamental role in the transfer matrix eigenvalue problem since they dictate in which way the transfer matrix eigenvalues depend on the arbitrary \(R\)-matrix elements. These relations for \(a = 1\) and \(a = N\) follows directly from Eqs. \((19, 20)\) while those for \(2 \leq a \leq N - 1\) are obtained by solving the \(2 \times 2\) linear system of equations \((21, 23)\). It turns out that the final results for the commutation rules between \(T_{a,a}(\lambda)\) and \(T_{1,2}(\mu)\) are,
\[ T_{1,1}(\lambda) T_{1,2}(\mu) = \frac{R(\mu, \lambda)^{1,1}_{1,2}}{R(\mu, \lambda)^{1,1}_{2,1}} T_{1,2}(\mu) T_{1,1}(\lambda) - \frac{R(\mu, \lambda)^{2,1}_{1,2}}{R(\mu, \lambda)^{2,1}_{2,1}} T_{1,2}(\lambda) T_{1,1}(\mu), \]  

(27)

\[ T_{a,a}(\lambda) T_{1,2}(\mu) = D_2^{(a,0)}(\lambda, \mu) T_{1,2}(\mu) T_{a,a}(\lambda) + \sum_{\bar{e}=3}^{a+1} D_2^{(a,\bar{e}-2)}(\lambda, \mu) T_{1,\bar{e}}(\mu) T_{a,a+2-\bar{e}}(\lambda) \]

\[ + \sum_{\bar{e}=1}^{a} \frac{R(\lambda, \mu)^{a,2}_{a+1,1} R(\lambda, \mu)^{\bar{e},a-\bar{e}+1}_{a,1}}{R(\lambda, \mu)^{a+1,1}_{a,1}} T_{\bar{e},a+1}(\lambda) T_{a-\bar{e}+1,1}(\mu) \]

\[ - \sum_{\bar{e}=1}^{a-1} \frac{R(\lambda, \mu)^{\bar{e},a-\bar{e}+1}_{a,1}}{R(\lambda, \mu)^{a,1}_{a,1}} T_{\bar{e},a}(\lambda) T_{a-\bar{e}+1,2}(\mu) \quad \text{for} \quad 2 \leq a \leq N-1, \]  

(28)

\[ T_{N,N}(\lambda) T_{1,2}(\mu) = \frac{R(\lambda, \mu)^{N,2}_{N,1}}{R(\lambda, \mu)^{N,2}_{N,1}} T_{1,2}(\mu) T_{N,N}(\lambda) + \sum_{\bar{e}=3}^{N} \frac{R(\lambda, \mu)^{N,2}_{N+2-\bar{e},\bar{e}}}{R(\lambda, \mu)^{N,1}_{N,1}} T_{1,\bar{e}}(\mu) T_{N,N+2-\bar{e}}(\lambda) \]

\[ - \sum_{\bar{e}=1}^{N-1} \frac{R(\lambda, \mu)^{\bar{e},N-\bar{e}+1}_{N,1}}{R(\lambda, \mu)^{N,1}_{N,1}} T_{\bar{e},N}(\lambda) T_{N-\bar{e}+1,2}(\mu) \]  

(29)

Function \( D_2^{(a,\bar{e})}(\lambda, \mu) \) is defined by using the determinant of a matrix whose elements combine the first column of \( A_1^{(a,2)}(\lambda, \mu) \) with the vector \( e_1^{(a,2)}(\lambda, \mu) \). Its expression in terms of the \( R \)-matrix elements is,

\[ D_2^{(a,\bar{e})}(\lambda, \mu) = -\frac{\left| \begin{array}{cc} R(\lambda, \mu)^{a,2}_{a+1,1} & R(\lambda, \mu)^{a,2}_{a-\bar{e},\bar{e}+2} \\ R(\lambda, \mu)^{a+1,1}_{a+1,1} & R(\lambda, \mu)^{a+1,1}_{a-\bar{e},\bar{e}+2} \end{array} \right|}{R(\lambda, \mu)^{a,1}_{a,1} R(\lambda, \mu)^{a+1,1}_{a+1,1}}, \]

for \( \bar{e} = 0, \cdots, a-1 \) and \( 2 \leq a \leq N-1 \).  

(30)

The next simplest case is related to the \( T_{1,3}(\mu) \) operator. In order to get the complete set of commutation relations we have now to solve two different linear systems with sizes 3 x 3 and 2 x 2. According to Table 1 the former is defined by Eqs. (21)(23) while the latter is associated to Eqs. (24)(26). By solving such linear systems of equations we find that the commutation relations between \( T_{a,a}(\lambda) \) and \( T_{1,3}(\mu) \) are given by,

\[ T_{1,1}(\lambda) T_{1,3}(\mu) = \frac{R(\mu, \lambda)^{1,1}_{1,1}}{R(\mu, \lambda)^{3,1}_{3,1}} T_{1,3}(\mu) T_{1,1}(\lambda) - \frac{R(\mu, \lambda)^{3,1}_{2,2}}{R(\mu, \lambda)^{3,1}_{3,1}} T_{1,2}(\lambda) T_{1,2}(\mu) \]

\[ - \frac{R(\mu, \lambda)^{3,1}_{1,3}}{R(\mu, \lambda)^{3,1}_{3,1}} T_{1,3}(\lambda) T_{1,1}(\mu) \]  

(31)
\[ T_{a,a}(\lambda)T_{1,3}(\mu) = D_{3}^{(a,0)}(\lambda, \mu)T_{1,3}(\mu)T_{a,a}(\lambda) + \sum_{\bar{e}=4}^{a+2} D_{3}^{(a,\bar{e}-3)}(\lambda, \mu)T_{1,\bar{e}}(\mu)T_{a,a+3-\bar{e}}(\lambda) \]

\[ - \sum_{\bar{e}=1}^{a-1} \frac{R(\lambda, \mu)_{a,1}^{\bar{e},a-\bar{e}+1}}{R(\lambda, \mu)_{a,1}^{\bar{e},a-\bar{e}+1}} T_{\bar{e},a}(\lambda)T_{a-\bar{e},3}(\mu) \]

\[ + \sum_{\bar{e}=1}^{a} \frac{R(\lambda, \mu)_{a,1}^{\bar{e},a-\bar{e}+1}}{R(\lambda, \mu)_{a,1}^{\bar{e},a-\bar{e}+1}} \begin{cases} R(\lambda, \mu)_{a+2,1}^{a,3} & R(\lambda, \mu)_{a+2,1}^{a,3} \\ R(\lambda, \mu)_{a+2,1}^{a+1,2} & R(\lambda, \mu)_{a+2,1}^{a+1,2} \\ R(\lambda, \mu)_{a+2,1}^{a+2,1} & R(\lambda, \mu)_{a+2,1}^{a+2,1} \end{cases} \]

\[ - \sum_{\bar{e}=1}^{a} \frac{R(\lambda, \mu)_{a,1}^{\bar{e},a-\bar{e}+1}}{R(\lambda, \mu)_{a,1}^{\bar{e},a-\bar{e}+1}} \begin{cases} R(\lambda, \mu)_{a+2,1}^{a,3} & R(\lambda, \mu)_{a+2,1}^{a,3} \\ R(\lambda, \mu)_{a+2,1}^{a+1,2} & R(\lambda, \mu)_{a+2,1}^{a+1,2} \\ R(\lambda, \mu)_{a+2,1}^{a+2,1} & R(\lambda, \mu)_{a+2,1}^{a+2,1} \end{cases} \]

\[ \text{for } 2 \leq a \leq N - 2, \quad (32) \]

\[ T_{N-1,N-1}(\lambda)T_{1,3}(\mu) = -R(\lambda, \mu)_{N,2}^{N-1,3} \frac{T_{1,3}(\mu)T_{N-1,N-1}(\lambda)}{R(\lambda, \mu)_{N,2}^{N-1,3}} - \sum_{\bar{e}=4}^{N} \frac{R(\lambda, \mu)_{N,1}^{N-1,3}}{R(\lambda, \mu)_{N,1}^{N-1,3}} \begin{cases} R(\lambda, \mu)_{N+2,\bar{e}}^{N-1,3} & R(\lambda, \mu)_{N+2,\bar{e}}^{N-1,3} \\ R(\lambda, \mu)_{N+2,\bar{e}}^{N+2-\bar{e},\bar{e}} & R(\lambda, \mu)_{N+2,\bar{e}}^{N+2-\bar{e},\bar{e}} \end{cases} \]

\[ - \sum_{\bar{e}=1}^{N-2} \left[ R(\lambda, \mu)_{N,2}^{N-1,3} \frac{T_{e,N}(\lambda)T_{N-e,2}(\mu) - T_{e,N-1}(\lambda)T_{N-e,3}(\mu)}{R(\lambda, \mu)_{N,2}^{N-1,3}} \right] \]

\[ \times \frac{R(\lambda, \mu)_{N,1}^{N-e}}{R(\lambda, \mu)_{N,1}^{N-e}} + \frac{R(\lambda, \mu)_{N,2}^{N-1,3}}{R(\lambda, \mu)_{N,2}^{N-1,3}} \frac{T_{N-1,N}(\lambda)T_{1,2}(\mu)}{R(\lambda, \mu)_{N,2}^{N-1,3}} \quad (33) \]
themselves but also those involving the operators. It is therefore necessary to disentangle not only the commutation rules among the basis vectors

\[ T_{N,N}^{\lambda,\mu} T_{1,3}^{\mu} = \frac{R(\lambda, \mu)^{N,3}_{N,1}}{R(\lambda, \mu)^{N,1}_{N,1}} T_{1,3}^{\mu} T_{N,N}^{\lambda} + \sum_{\bar{\varepsilon}=4}^{N} \frac{R(\lambda, \mu)^{N,3}_{N+3-\bar{\varepsilon},\bar{\varepsilon}}}{R(\lambda, \mu)^{N,1}_{N,1}} T_{1,\bar{\varepsilon}}^{\mu} T_{N,N+3-\bar{\varepsilon},\bar{\varepsilon}}^{\lambda} \]

\[ - \sum_{\bar{\varepsilon}=1}^{N-1} \frac{R(\lambda, \mu)^{\bar{\varepsilon},N-\bar{\varepsilon}+1}_{N,1}}{R(\lambda, \mu)^{N,1}_{N,1}} T_{\bar{\varepsilon},N}^{\lambda} T_{N-\bar{\varepsilon}+1,3}^{\mu} \]

(34)

where function \( D_3^{(a,\bar{\varepsilon})}(\lambda, \mu) \) is represented by the ratio of 3 x 3 and 2 x 2 determinants, namely

\[ D_3^{(a,\bar{\varepsilon})}(\lambda, \mu) = \begin{vmatrix}
R(\lambda, \mu)^{a,3}_{a+2,1} & R(\lambda, \mu)^{a,3}_{a+1,2} & R(\lambda, \mu)^{a,3}_{a-\bar{\varepsilon},3+\bar{\varepsilon}} \\
R(\lambda, \mu)^{a+1,2}_{a+2,1} & R(\lambda, \mu)^{a+1,2}_{a+1,2} & R(\lambda, \mu)^{a+1,2}_{a-\bar{\varepsilon},3+\bar{\varepsilon}} \\
R(\lambda, \mu)^{a+2,1}_{a+2,1} & R(\lambda, \mu)^{a+2,1}_{a+1,2} & R(\lambda, \mu)^{a+2,1}_{a-\bar{\varepsilon},3+\bar{\varepsilon}}
\end{vmatrix}, \]

\[ R(\lambda, \mu)^{a,1}_{a,1} \begin{vmatrix}
R(\lambda, \mu)^{a+1,2}_{a+2,1} & R(\lambda, \mu)^{a+1,2}_{a+1,2} \\
R(\lambda, \mu)^{a+2,1}_{a+2,1} & R(\lambda, \mu)^{a+2,1}_{a+1,2}
\end{vmatrix}, \]

for \( \bar{\varepsilon} = 0, \ldots, a - 1 \) and \( 2 \leq a \leq N - 2 \).

3.2 The creation fields

The results from previous subsection reveal us that the commutation rules between the operators \( T_{a,a}(\lambda) \) and \( T_{1,b}(\lambda) \) are able to produce additional creation fields other than the basis vectors \( T_{1,b}(\lambda) \) or \( T_{1,b}(\mu) \). Though these extra creation operators do not take a direct part on the multiparticle state basis they are essential for the solution of the transfer matrix eigenvalue problem. It is therefore necessary to disentangle not only the commutation rules among the basis vectors themselves but also those involving the operators \( T_{1,b}(\lambda) \) with the remaining creation fields \( T_{a,b}(\mu) \) for \( b > a = 2, \ldots, N - 1 \). In the course of our analysis we find convenient to describe such commutation relations by using the fields \( T_{1,b_1-d_1}(\lambda) \) and \( T_{a_1-a_1+d_1}(\mu) \) where the underlying indices belong to the following intervals,

\[ 2 \leq a_1 \leq N, \quad 0 \leq d_1 \leq N - a_1, \quad 2 \leq b = b_1 - d_1 \leq N \]

The above arrangement of indices bring us at least two technical advantages. First, we assure that all the mentioned commutations rules between the creation fields will be considered without
unnecessary repetition. Next, Eq. (16) implies that the effective azimuthal spin component associated to the term $T_{1,b} (\lambda) T_{a_1-1,a_1+d_1} (\mu)$ should be indexed by the composed variable $b + d_1$. The parameterization $b = b_1 - d_1$ has therefore the merit of allowing us to keep track of the azimuthal spin of such product of creation fields in terms of a unique index $b_1$. As a consequence of that the number of the distinct commutation relations that are needed in a given multiparticle state sector can now be controlled with the help of the remaining indices $a_1$ and $d_1$. A systematic analysis of the Yang-Baxter relation (17) reveals us that these commutation rules are derived from the entries $[a_1 - 1; (a_1 + c - 1)N + b_1 - c]$ for selected values of the variable $c$. The corresponding relations are obtained from Eq. (17) by choosing $\bar{a} = 1$, $\bar{b} = a_1 - 1$, $\bar{c} = a_1 + c$ e $\bar{d} = b_1 - c$, namely

$$\sum_{\bar{e}=1}^{a_1-1} R(\lambda, \mu)^{\bar{e}_a \bar{e}}_{a_1-1,1} T_{\bar{e}_a, a_1+c} (\lambda) T_{a_1-\bar{e}, b_1-c} (\mu) = \sum_{\bar{e}_1=M \{1,a_1+b_1-N\}}^{m \{a_1+b_1-1,N\}} T_{1,\bar{e}} (\mu) T_{a_1-1,a_1+b_1-\bar{e}} (\lambda) R(\lambda, \mu)^{a_1+c,b_1-c}_{a_1+b_1-\bar{e},\bar{e}}. \quad (37)$$

The majority of the suitable commutation rules between the fields $T_{1,b_1-d_1} (\lambda)$ and $T_{a_1-1,a_1+d_1} (\mu)$ require a considerable amount of manipulations among the relations (37). As before we still have to implement certain linear combinations by using the freedom of the index $c$. This procedure is highly dependent on the variables $a_1, b_1$ and it has been detailed in Table 2.

The simplest type of commutation rules are those involving the creation fields that has direct participation on the eigenvector basis. These relations are sorted out by choosing the index $a_1 = 2$. From Table 2 we see that the commutation rule between $T_{1,b_1-d_1} (\lambda)$ and $T_{1,2+d_1} (\mu)$ for $b_1 \geq N$ follows directly from the entry $[1; (b_1 - d_1 - 1)N + 2 + d_1]$ of Eq. (37), namely

$$T_{1,b_1-d_1} (\lambda) T_{1,2+d_1} (\mu) = \frac{R(\lambda, \mu)^{b_1-d_1,2+d_1}_{b_1-d_1,2+d_1}}{R(\lambda, \mu)^{1,1}_{1,1}} T_{1,2+d_1} (\mu) T_{1,b_1-d_1} (\lambda) + \sum_{\bar{e}=2+b_1-N \atop \bar{e} \neq 2+d_1}^{N} \frac{R(\lambda, \mu)^{b_1-d_1,2+d_1}_{2+b_1-\bar{e},\bar{e}}}{R(\lambda, \mu)^{1,1}_{1,1}} T_{1,\bar{e}} (\mu) T_{1,2+b_1-\bar{e}} (\lambda), \quad \text{for } b_1 \geq N. \quad (38)$$

On the other hand for $b_1 < N$ we have to make a combination of two entries $[1; (b_1 - d_1 - 1)N + 2 + d_1]$ and $[1;b_1N + 1]$ of Eq. (37). These equations can be combined to produce a $2 \times 2$
Table 2: The linear combination derived from Eq. (37) to achieve appropriate suitable commutation rules between the fields $T_{1,b_1-d_1}(\lambda)$ and $T_{a_1-1,a_1+d_1}(\lambda)$. The indices $a_1$, $b_1$ and $d_1$ belong to the intervals defined by Eq. (39).

The desirable commutation relations are derived by solving Eq. (39) for the product $T_{1,2}(\mu)$ $T_{1,b_1}(\lambda)$ with the assistance of Cramer’s rule. As a result we obtain a single linear equation relating the terms $T_{1,b_1-d_1}(\lambda)T_{1,2+d_1}(\mu)$ and $T_{1,2+d_1}(\mu)T_{1,b_1-d_1}(\lambda)$ whose solution gives us the

\[ A_3^{(b_1,d_1)}(\lambda, \mu) = \begin{pmatrix} R(\lambda, \mu)_{b_1+d_1-1} & R(\lambda, \mu)_{b_1+d_1} \\ R(\lambda, \mu)_{b_1+1} & R(\lambda, \mu)_{b_1+1} \end{pmatrix} . \]  

We recall that the product $T_{1,2+d_1}(\mu)T_{1,b_1-d_1}(\lambda)$ appears in the sum of Eq. (39) for $\bar{e} = 2 + d_1 \leq b_1$. 

2
following expression,

\[
\mathcal{T}_{1,b_1-d_1}(\lambda)\mathcal{T}_{1,2+d_1}(\mu) = -\frac{R(\lambda, \mu)^{b_1-d_1,2+d_1}_{b_1+1,1} R(\lambda, \mu)^{b_1-d_1,2+d_1}_{b_1+1,1}}{R(\lambda, \mu)_{1,1} R(\lambda, \mu)_{b_1+1,1}} \mathcal{T}_{1,2+d_1}(\mu) \mathcal{T}_{1,b_1-d_1}(\lambda)
+ \frac{R(\lambda, \mu)^{b_1-d_1,2+d_1}_{b_1+1,1}}{R(\lambda, \mu)_{b_1+1,1}} \mathcal{T}_{1,b_1+1}(\lambda) \mathcal{T}_{1,1}(\mu)
- \sum_{\substack{e=2 \to b_1+1 \\ e \neq 2+d_1}} \frac{R(\lambda, \mu)^{b_1-d_1,2+d_1}_{b_1+1,1} R(\lambda, \mu)^{b_1+d_1,2+d_1}_{b_1+1,1}}{R(\lambda, \mu)_{1,1} R(\lambda, \mu)_{b_1+1,1}} \mathcal{T}_{1,1}(\mu) \mathcal{T}_{1,2+b_1-\bar{e}}(\lambda),
\]

for \( b_1 < N \).

(41)

Let us now discuss the strategy we have used so far in order to obtain the commutation relations for the creation fields. The basic idea is that any possible product of creation operators on the right-hand side of the commutation relations should be equally ordered as far as the rapidities \( \lambda \) and \( \mu \) are concerned. This ordering is certainly the opposite of that which we have started with for the left-hand side product of creation operators. In addition to that, the specific ordering choice is actually dictated by the results established in section 3.1 for the commutation rules among the diagonal fields and the basis vectors. The commutation relations for the creation fields are then constructed to bring a given creation operator with the transfer matrix spectral parameter to the further left position in products of monodromy matrix elements that are not proportional to the eigenvectors.

The above procedure is of special importance when we deal with the commutation rules between the operators \( \mathcal{T}_{1,b_1-d_1}(\mu) \) and \( \mathcal{T}_{a_1-a_1+d_1}(\lambda) \) for \( 3 \leq a_1 \leq N \). In this situation the right-hand side of the respective commutation relations is not allowed to possess terms of the form \( \mathcal{T}_{1,b_1-\bar{e}}(\mu) \mathcal{T}_{a_1-a_1+\bar{e}}(\lambda) \) with \( \bar{e} \geq 0 \). This can be clearly seen by considering in an explicit way the linear systems associated to such commutation rules. From Table 2 we observe they depend very much whether \( b_1 < N \) or \( b_1 \geq N \). The former case is rather similar to the linear combination
discussed in previous subsection and we find that,

\[
A_1^{(a_1,b_1)}(\lambda, \mu) = \begin{pmatrix}
\mathcal{T}_{1,1}^{(a_1,b_1)}(\lambda, \mu) \\
\mathcal{T}_{1,2}^{(a_1,b_1)}(\lambda, \mu) \\
\vdots \\
\mathcal{T}_{1,b_1}(\lambda, \mu)
\end{pmatrix} = \sum_{\ell=1}^{a_1} R^{(a_1,b_1)}(\lambda, \mu)_{a_1-\ell,1} \\
\begin{pmatrix}
\mathcal{T}_{\ell,a_1}(\lambda)_{a_1-\ell,1} \\
\mathcal{T}_{\ell,a_1+1}(\lambda)_{a_1-\ell,1} \\
\vdots \\
\mathcal{T}_{\ell,a_1+b_1-1}(\lambda)_{a_1-\ell,1}
\end{pmatrix}
\]

\[
- \sum_{\ell=b_1+1}^{a_1+b_1-1} v_1^{(a_1,b_1)}(\lambda, \mu) \mathcal{T}_{1,\ell}(\mu) \mathcal{T}_{a_1-\ell,a_1+b_1-\ell}(\lambda)
\]

\[
\text{for } b_1 < N \text{ and } 3 \leq a_1 \leq N + 1 - b_1 \quad (42)
\]

\[
A_2^{(a_1,b_1)}(\lambda, \mu) = \begin{pmatrix}
\mathcal{T}_{1,a_1+b_1-N}(\mu) \mathcal{T}_{a_1-1,N}(\lambda) \\
\mathcal{T}_{1,a_1+b_1-N+1}(\mu) \mathcal{T}_{a_1-1,N-1}(\lambda) \\
\vdots \\
\mathcal{T}_{1,b_1}(\lambda) \mathcal{T}_{a_1-1,a_1}(\lambda)
\end{pmatrix}

= \sum_{\ell=1}^{a_1-1} R^{(a_1,b_1)}(\lambda, \mu)_{a_1-\ell,1} \\
\begin{pmatrix}
\mathcal{T}_{\ell,a_1}(\lambda)_{a_1-\ell,1} \\
\mathcal{T}_{\ell,a_1+1}(\lambda)_{a_1-\ell,1} \\
\vdots \\
\mathcal{T}_{\ell,N}(\lambda)_{a_1-\ell,a_1+b_1-N}(\mu)
\end{pmatrix}
\]

\[
\times \begin{pmatrix}
\mathcal{T}_{\ell,a_1}(\lambda)_{a_1-\ell,b_1}(\mu) \\
\mathcal{T}_{\ell,a_1+1}(\lambda)_{a_1-\ell,b_1-1}(\mu) \\
\vdots \\
\mathcal{T}_{\ell,N}(\lambda)_{a_1-\ell,a_1+b_1-N}(\mu)
\end{pmatrix}

- \sum_{\ell=b_1+1}^{N} v_2^{(a_1,b_1)}(\lambda, \mu) \mathcal{T}_{1,\ell}(\mu) \mathcal{T}_{a_1-\ell,a_1+b_1-\ell}(\lambda)
\]

\[
\text{for } b_1 < N \text{ and } N + 2 - b_1 \leq a_1 \leq N \quad (43)
\]

where the \(b_1 \times b_1\) matrix \(A_1^{(a_1,b_1)}(\lambda, \mu)\), the \(b_1 \times 1\) vector \(v_1^{(a_1,b_1)}(\lambda, \mu)\), the \((N+1-a_1) \times (N+1-a_1)\) matrix \(A_2^{(a_1,b_1)}(\lambda, \mu)\) and the \((N+1-a_1) \times 1\) vector \(v_2^{(a_1,b_1)}(\lambda, \mu)\) are given by Eqs. (22, 23, 25, 26), respectively.

For \(b_1 \geq N\) we have instead a different linear system of equations. In this case the relations
coming from Eq.(37) for \( c = b_1 - N, \cdots, N - a_1 \) are organized as follows,

\[
A_4^{(a_1,b_1)}(\lambda, \mu) = \left( \begin{array}{c} T_{1,a_1+b_1-N}(\mu)T_{a_1-1,N}(\lambda) \\ T_{1,a_1+b_1-N+1}(\mu)T_{a_1-1,N-1}(\lambda) \\ \vdots \\ T_{1,N}(\mu)T_{a_1-1,a_1+b_1-N}(\lambda) \end{array} \right) = \sum_{\bar{e}=1}^{a_1-1} R(\lambda, \mu)_{\bar{e},a_1-\bar{e}}
\]

\times \left( \begin{array}{c} T_{\bar{e},a_1+b_1-N}(\lambda)T_{a_1-\bar{e},N}(\mu) \\ T_{\bar{e},a_1+b_1-N+1}(\lambda)T_{a_1-\bar{e},N-1}(\mu) \\ \vdots \\ T_{\bar{e},N}(\lambda)T_{a_1-\bar{e},a_1+b_1-N}(\mu) \end{array} \right)

for \( b_1 \geq N \) and \( 3 \leq a_1 \leq N \); (44)

where the \((2N+1-a_1-b_1) \times (2N+1-a_1-b_1)\) matrix \(A_4^{(a_1,b_1)}(\lambda, \mu)\) is given by

\[
A_4^{(a_1,b_1)}(\lambda, \mu) = \left( \begin{array}{cccc} R(\lambda, \mu)^a_{1+b_1-N,N} & R(\lambda, \mu)^{a_1+b_1-N,N}_{N-1,a_1+b_1-N+1} & \cdots & R(\lambda, \mu)^{a_1+b_1-N,N}_{a_1+b_1-N} \\ R(\lambda, \mu)^{a_1+b_1-N+1,N-1}_{N,a_1+b_1-N} & R(\lambda, \mu)^{a_1+b_1-N+1,N-1}_{N-1,a_1+b_1-N+1} & \cdots & R(\lambda, \mu)^{a_1+b_1-N+1,N-1}_{a_1+b_1-N,N} \\ \vdots & \vdots & \ddots & \vdots \\ R(\lambda, \mu)^{N,a_1+b_1-N}_{N,a_1+b_1-N} & R(\lambda, \mu)^{N,a_1+b_1-N}_{N-1,a_1+b_1-N+1} & \cdots & R(\lambda, \mu)^{N,a_1+b_1-N}_{a_1+b_1-N,N} \end{array} \right).
\] (45)

Direct inspection of Eqs.(42,43,44) reveal us that the variables of the corresponding system of linear equations are indeed the products \(T_{1,b_1-d_1}(\mu)T_{a_1-1,a_1+d_1}(\lambda)\). The origin of these independent linear systems is directly related to the existence of three distinct intervals for the index \(d_1\) once the variables \(a_1\) and \(b_1\) are fixed. Their structure have been constructed to collect together products of creation fields whose respective values for the index \(d_1\) belong to one of the three possible such intervals. The commutation relations between the operators \(T_{1,b_1-d_1}(\mu)\) and \(T_{a_1-1,a_1+d_1}(\lambda)\) are then determined by means of the systematic application of Cramer’s rule to solve the systems of linear equations \(\{42,43,44\}\). The procedure is similar to that already described in section \(3.1\). For instance, we note that the reversed product terms appear on the first non-homogeneous part of Eqs.(42,43,44) when the sum index is \(\bar{e} = a_1 - 1\). The task is however much more cumbersome since we have to apply Cramer’s rule for each left-hand side product of creation fields entering Eqs.(42,43,44). The number of different commutation rules associated to each linear system for
a given multiparticle state depends strongly on the index $b_1$. This feature and the dependence of the total number of relations for a given $N$ is illustrated in Table 3.

| Linear System | Number of Commutation Rules | \( b_1 \) fixed | \( N \) fixed |
|---------------|-----------------------------|----------------|----------------|
| \( A_1^{(a_1,b_1)}(\lambda, \mu) \) | \( \sum_{a_1=3}^{N+1-b_1} (b_1 - 1) \) | \( \sum_{b_1=2}^{N-2} (N-b_1-1)(b_1-1) \) | \( \frac{(N-1)(N-2)(N-3)}{6} \) |
| \( A_2^{(a_1,b_1)}(\lambda, \mu) \) | \( \sum_{a_1=N+2-b_1}^{N} (N+1-a_1) \) | \( \frac{N-1}{2} \sum_{b_1=2}^{N-1} (b_1-1)b_1 \) | \( \frac{N(N-1)(N-2)}{6} \) |
| \( A_4^{(a_1,b_1)}(\lambda, \mu) \) | \( \sum_{a_1=3}^{2N-b_1} (2N+1-a_1-b_1) \) | \( \sum_{b_1=N}^{2N-2} (2N-b_1-1)(2N-b_1-2) \) | \( \frac{N(N-1)(N-2)}{6} \) |

Table 3: The number of commutations relations between the creation fields for fixed $b_1$ as well as for fixed $N$.

We would like to conclude by presenting an example of a complete set of commutation rules between the creation operators. The simplest non-trivial situation occurs when the respective azimuthal spin component is indexed by $b_1 = 2$. The intervals \([36]\) tell us that such relations are those between the operators $T_{1,2}(\mu)$ and $T_{a_1-1,a_1}(\lambda)$ for $2 \leq a_1 \leq N$. The case $a_1 = 2$ follows from the general expression \([36]\) while for the remaining values we need to solve the linear systems.
Putting the results together we find that,

\[ D_2^{(a_1,0)}(\lambda, \mu)T_{1,2}(\mu)T_{a_1-1,a_1}(\lambda) = \frac{R(\lambda, \mu)^{a_1-1,1}}{R(\lambda, \mu)^{a_1,1}}T_{a_1-1,a_1}(\lambda)T_{1,2}(\mu) - \frac{R(\lambda, \mu)^{a_1-1,1}}{R(\lambda, \mu)^{a_1,1}}R(\lambda, \mu)^{a_1,2}}{R(\lambda, \mu)^{a_1+1,1}} \]

\[ \times T_{a_1-1,a_1+1}(\lambda)T_{1,1}(\mu) - \sum_{\epsilon=1}^{a_1-2} \frac{R(\lambda, \mu)^{\epsilon,a_1-\epsilon}}{R(\lambda, \mu)^{a_1+1,1}}R(\lambda, \mu)^{a_1,2}}{R(\lambda, \mu)^{a_1,1}}T_{\epsilon,a_1+1}(\lambda)T_{a_1-\epsilon,1}(\mu) \]

\[ + \sum_{\epsilon=1}^{a_1-2} \frac{R(\lambda, \mu)^{\epsilon,a_1-\epsilon}}{R(\lambda, \mu)^{a_1,1}}T_{\epsilon,a_1}(\lambda)T_{a_1-\epsilon,2}(\mu) - \sum_{\epsilon=3}^{a_1+1} D_2^{(a_1,\epsilon-2)}(\lambda, \mu)T_{1,\epsilon}(\mu)T_{a_1-1,a_1+2-\epsilon}(\lambda), \quad \text{for} \quad 2 \leq a_1 \leq N - 1 \] (46)

\[ T_{1,2}(\mu)T_{N-1,N}(\lambda) = \frac{R(\lambda, \mu)^{N-1,1}}{R(\lambda, \mu)^{N,2}}T_{N-1,N}(\lambda)T_{1,2}(\mu) + \sum_{\epsilon=1}^{N-2} \frac{R(\lambda, \mu)^{\epsilon,N-\epsilon}}{R(\lambda, \mu)^{N,2}}T_{\epsilon,N}(\lambda)T_{N-\epsilon,2}(\mu) \]

\[ - \sum_{\epsilon=3}^{a_1+1} \frac{R(\lambda, \mu)^{N,2}}{R(\lambda, \mu)^{N,2}}T_{1,\epsilon}(\mu)T_{N-1,N+2-\epsilon}(\lambda), \quad \text{for} \quad 2 \leq a_1 \leq N - 1 \] (47)

where we recall that function \( D_2^{(a_1,\epsilon)}(\lambda, \mu) \) is given by Eq. (30).

### 3.3 The creation and annihilation fields

The third class of commutation rules that still need to be considered are those between the basis vectors \( T_{1,b}(\mu) \) and all possible annihilation operator \( T_{a_1+d_1,a_1-1}(\lambda) \). We stress here that the indices \( a_1, d_1 \) and \( b \) are the same used in subsection (3.2) and therefore they belong to the intervals defined by Eq. (36). In general, the result of taking the operator \( T_{a_1+d_1,a_1-1}(\lambda) \) through the creation fields \( T_{1,b}(\mu) \) is the generation of several distinct annihilation fields besides \( T_{a_1+d_1,a_1-1}(\lambda) \). The basic strategy we shall use to construct these relations is to enforce that all such resulting annihilation fields must be kept in the right-hand side position on the respective commutation rule. A detailed study of the Yang-Baxter algebra reveals us that these commutation rules can be built up from the entries \([a_1+d_1+c_1(N-1); (a_1-2)N+b+c_2(N-1)]\) of Eq. (17). The structure of the corresponding equations are now more complicated since they carry a dependence on two independent indices \( c_1 \) and \( c_2 \) as well as on the particular combination \( a_1 + d_1 \). Taking into account the latter feature we find convenient here to express the expressions by means of the following auxiliary indices,

\[ f_1 = a_1 + d_1 \quad \text{and} \quad f_2 = b - d_1 - 2 \] (48)
The starting point relations are then obtained from Eq. (17) by choosing \( \dd = c_1 + 1, \dd = f_1 - c_1, \dd = a_1 + 2 - 1 \) and \( \dd = b - c_2 \), namely

\[
\sum_{\dd = 1}^{f_1} \sum_{m\{a_1 + b - 2, N\}} R(\lambda, \mu) \dd, f_1 + 1 - \dd \dd, a_1 + c_2 - 1(\lambda) \Delta T_{f_1 + 1 - \dd, b - c_2}(\mu) =
\]

\[
\sum_{\dd = M\{a_1 + b - N - 1\}} T_{c_1 + 1, \dd}(\mu) \Delta T_{f_1, a_1 + b - 1 - \dd}(\lambda) R(\lambda, \mu) a_1 + c_2 - 1, b - c_2.
\]

Eq. (49) turns out to be far more involved than that discussed in the two last subsections. The respective linear combinations have to be performed considering two distinct steps and they will culminate in two coupled linear systems. It is rather illuminating to start this discussion by examining Eq. (49) in the particular situation \( c_1 = c_2 = 0 \),

\[
\sum_{\dd = 1}^{f_1} \sum_{m\{a_1 + b - 2, N\}} R(\lambda, \mu) \dd, f_1 + 1 - \dd \dd, a_1 - 1(\lambda) \Delta T_{f_1 + 1 - \dd, b}(\mu) =
\]

\[
\sum_{\dd = M\{a_1 + b - N - 1\}} T_{1, \dd}(\mu) \Delta T_{f_1, a_1 + b - 1 - \dd}(\lambda) R(\lambda, \mu) a_1 - 1, b.
\]

Let us now concentrate our attention on the left-hand side of Eq. (50). For \( \dd = f_1 \) it gives us the product \( T_{a_1 + d_1, a_1 - 1}(\lambda) T_{1, b}(\mu) \) which is indispensable to build up the desirable commutation rules. Unfortunately, for arbitrary values of \( b \), this term is not the only one involving annihilation and creation operators generated by Eq. (50). From the left-hand side of Eq. (50) we see indeed that \( T_{\dd, a_1 - 1}(\lambda) \) behaves as annihilation or diagonal operator when \( \dd \geq a_1 - 1 \) while \( T_{a_1 + d_1 + 1 - \dd, b}(\mu) \) play the role of creation fields for \( \dd \geq a_1 - f_2 = a_1 - (b - d_1 - 2) \). This product combination clearly spoil the main characteristic one would expect from the commutation rule among \( T_{a_1 + d_1, a_1 - 1}(\lambda) \) and \( T_{1, b}(\mu) \). The first task therefore is to eliminate the left-hand side products \( T_{\dd, a_1 - 1}(\lambda) T_{a_1 + d_1 + 1 - \dd, b}(\mu) \) when the index \( \dd \) takes values on the intersection of the above intervals. Considering the upper value of \( \dd \) in Eq. (50) it is not difficult to see that the mentioned interval is

\[
a_1 - m\{1, f_2 = b - d_1 - 2\} \leq \dd < f_1 = a_1 + d_1.
\]

The constraint (51) tells us that the case \( b = 2 \) is a fortunate exception and therefore there is no need of considering the above mentioned cancellations. This means that the appropriate
commutation rules between $T_{a_1+d_1,a_1-1}(\lambda)$ and $T_{1,2}(\mu)$ can be derived directly from Eq. (50). After few manipulations we find,

$$T_{f_1,a_1-1}(\lambda)T_{1,2}(\mu) = \frac{R(\lambda, \mu)_{a_1-1,2}}{R(\lambda, \mu)_{f_1,1}}T_{1,2}(\mu)T_{f_1,a_1-1}(\lambda) + \sum_{\bar{e}=1}^{a_1} \frac{R(\lambda, \mu)_{a_1-1,\bar{e},\bar{e}}}{R(\lambda, \mu)_{f_1,1}} T_{\bar{e},1}(\mu)T_{f_1,a_1+1-\bar{e}}(\lambda)$$

$$- \sum_{\bar{e}=1}^{f_1-1} \frac{R(\lambda, \mu)_{\bar{e},f_1+1-\bar{e}}}{R(\lambda, \mu)_{f_1,1}} T_{\bar{e},a_1-1}(\lambda)T_{f_1+1-\bar{e},2}(\mu).$$  (52)

On the other hand, for $b \geq 3$ we are forced to get rid of the products $T_{\bar{e},a_1-1}(\lambda)T_{a_1+d_1+1-\bar{e},b}(\mu)$ when the index $\bar{e}$ belongs to the interval $(51)$. This can be done by exploring the linear combination of equations derived from Eq. (49) for suitable values of the index $c_1$ while $c_2$ is fixed at $c_2 = 0$. The structure of the linear combination depends on the sign of the azimuthal spin of the product $T_{f_1,a_1-1}(\lambda)T_{1,b}(\mu)$ as it is illustrated in Table 4.

| Operator eliminated | Creation index $\bar{e}$ | Combination index $c_1$ | Number of Equations |
|---------------------|--------------------------|-------------------------|---------------------|
| $T_{\bar{e},a_1-1}(\lambda)T_{f_1+1-\bar{e},b}(\mu)$ | $b - 2 \leq d_1$ | $c_1 = 0, \ldots, b - 2$ | $b - 1$ |
| $T_{\bar{e},a_1-1}(\lambda)T_{f_1+1-\bar{e},b}(\mu)$ | $b - 2 > d_1$ | $c_1 = 0, \ldots, d_1 + 1$ | $d_1 + 2$ |

Table 4: The linear combination from Eq. (49) for $c_2 = 0$ to cancel the operators $T_{\bar{e},a_1-1}(\lambda)T_{f_1+1-\bar{e},b}(\mu)$ when $\bar{e}$ belongs to the interval $(51)$.

Taking into account Table 4 it is not difficult to start the construction of the first needed linear

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$^3$Recall that from Eq. (15) the corresponding spin value is $f_2 = b - 2 - d_1$. 

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combination for $b \geq 3$. Its main structure is as follows,

$$\begin{aligned}
A_{5}^{(f_1,b)}(\lambda, \mu) &= \left( \begin{array}{c}
T_{f_1-b+2,a_1-1}(\lambda)T_{b-1,b}(\mu) \\
T_{f_1-b+3,a_1-1}(\lambda)T_{b-2,b}(\mu) \\
\vdots \\
T_{f_1,a_1-1}(\lambda)T_{1,b}(\mu)
\end{array} \right) \\
&= \sum_{\bar{e}=1}^{a_1+b-2} R(\lambda, \mu)^{a_1-1, b}_{a_1+b-1-\bar{e}, \bar{e}}
\end{aligned}$$

and

$$\begin{aligned}
A_{6}^{(a_1,d_1)}(\lambda, \mu) &= \left( \begin{array}{c}
T_{a_1-1,a_1-1}(\lambda)T_{d_1+2,b}(\mu) \\
T_{a_1,a_1-1}(\lambda)T_{d_1+1,b}(\mu) \\
\vdots \\
T_{f_1,a_1-1}(\lambda)T_{1,b}(\mu)
\end{array} \right) \\
&= \sum_{\bar{e}=M\{1,a_1+b-N-1\}}^{m\{a_1+b-2,N\}} R(\lambda, \mu)^{a_1-1, b}_{a_1+b-1-\bar{e}, \bar{e}}
\end{aligned}$$

for $b - 2 \leq d_1$,

and

$$\begin{aligned}
A_{6}^{(a_1,d_1)}(\lambda, \mu) &= \left( \begin{array}{c}
T_{a_1-1,a_1-1}(\lambda)T_{d_1+2,b}(\mu) \\
T_{a_1,a_1-1}(\lambda)T_{d_1+1,b}(\mu) \\
\vdots \\
T_{f_1,a_1-1}(\lambda)T_{1,b}(\mu)
\end{array} \right) \\
&= \sum_{\bar{e}=1}^{a_1-2} v_{6}^{(a_1,d_1)}(\lambda, \mu)T_{e,a_1-1}(\lambda)T_{f_1+1-\bar{e},b}(\mu)
\end{aligned}$$

for $b - 2 > d_1$.

The $(b - 1) \times (b - 1)$ matrix $A_{5}^{(f_1,b)}(\lambda, \mu)$ as well as the $(d_1 + 2) \times (d_1 + 2)$ matrix $A_{6}^{(a_1,d_1)}(\lambda, \mu)$ are given by

$$\begin{aligned}
A_{5}^{(f_1,b)}(\lambda, \mu) &= \left( \begin{array}{cccc}
R(\lambda, \mu)^{f_1-b+2,b-1}_{f_1-1,1} & R(\lambda, \mu)^{f_1-b+3,b-2}_{f_1-1,1} & \cdots & R(\lambda, \mu)^{f_1-1}_{f_1-1,1} \\
R(\lambda, \mu)^{f_1-b+2,b-1}_{f_1-1,2} & R(\lambda, \mu)^{f_1-b+3,b-2}_{f_1-1,2} & \cdots & R(\lambda, \mu)^{f_1-1}_{f_1-1,2} \\
\vdots & \vdots & \ddots & \vdots \\
R(\lambda, \mu)^{f_1-b+2,b-1}_{f_1-1,b-1} & R(\lambda, \mu)^{f_1-b+3,b-2}_{f_1-1,b-1} & \cdots & R(\lambda, \mu)^{f_1-1}_{f_1-1,b-1}
\end{array} \right)
\end{aligned}$$

for $b - 2 \leq d_1$,

and

$$\begin{aligned}
A_{6}^{(a_1,d_1)}(\lambda, \mu) &= \left( \begin{array}{cccc}
R(\lambda, \mu)^{f_1-b+2,b-1}_{f_1-1,1} & R(\lambda, \mu)^{f_1-b+3,b-2}_{f_1-1,1} & \cdots & R(\lambda, \mu)^{f_1-1}_{f_1-1,1} \\
R(\lambda, \mu)^{f_1-b+2,b-1}_{f_1-1,2} & R(\lambda, \mu)^{f_1-b+3,b-2}_{f_1-1,2} & \cdots & R(\lambda, \mu)^{f_1-1}_{f_1-1,2} \\
\vdots & \vdots & \ddots & \vdots \\
R(\lambda, \mu)^{f_1-b+2,b-1}_{f_1-1,b-1} & R(\lambda, \mu)^{f_1-b+3,b-2}_{f_1-1,b-1} & \cdots & R(\lambda, \mu)^{f_1-1}_{f_1-1,b-1}
\end{array} \right)
\end{aligned}$$

for $b - 2 > d_1$. 

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and

\[ A^{(a_1,d_1)}_6(\lambda, \mu) = \begin{pmatrix} R(\lambda, \mu)^{a_1-1,d_1+2}_{f_1,1} & R(\lambda, \mu)^{a_1,d_1+1}_{f_1,1} & \cdots & R(\lambda, \mu)^{1,d_1+1}_{f_1,1} \\ R(\lambda, \mu)^{a_1-1,d_1+2}_{f_1,1} & R(\lambda, \mu)^{a_1,d_1+1}_{f_1,1} & \cdots & R(\lambda, \mu)^{1,d_1+1}_{f_1,1} \\ \vdots & \vdots & \ddots & \vdots \\ R(\lambda, \mu)^{a_1-1,d_1+2}_{a_1-1,d_1+2} & R(\lambda, \mu)^{a_1,d_1+1}_{a_1-1,d_1+2} & \cdots & R(\lambda, \mu)^{1,d_1+1}_{a_1-1,d_1+2} \end{pmatrix}, \] (56)

while the \((b-1) \times 1\) vector \(v^{(f_1,b)}_5(\lambda, \mu)\) and the \((d_1+2) \times 1\) vector \(v^{(a_1,d_1)}_6(\lambda, \mu)\) are

\[ v^{(f_1,b)}_5(\lambda, \mu) = \begin{pmatrix} R(\lambda, \mu)^{\bar{\varepsilon},f_1+1-\bar{\varepsilon}}_{f_1,1} \\ R(\lambda, \mu)^{\bar{\varepsilon},f_1+1-\bar{\varepsilon}}_{f_1,1} \\ \vdots \\ R(\lambda, \mu)^{\bar{\varepsilon},f_1+1-\bar{\varepsilon}}_{f_1-b+2,b-1} \end{pmatrix}, \quad v^{(a_1,d_1)}_6(\lambda, \mu) = \begin{pmatrix} R(\lambda, \mu)^{\bar{\varepsilon},f_1+1-\bar{\varepsilon}}_{f_1,1} \\ R(\lambda, \mu)^{\bar{\varepsilon},f_1+1-\bar{\varepsilon}}_{f_1,1} \\ \vdots \\ R(\lambda, \mu)^{\bar{\varepsilon},f_1+1-\bar{\varepsilon}}_{a_1-1,d_1+2} \end{pmatrix}. \] (57)

By construction the linear systems defined by Eqs. (53-54) contain explicitly the product \(T_{f_1,a_1-1}(\lambda)T_{1,b}(\mu)\) which can be calculated by means of Cramer’s rule. From the right-hand side of Eqs. (53-54) we observe that this solution is able to produce products having the general form \(T_{c_1+1,\varepsilon}(\mu)T_{f_1-c_1,a_1+b-1-\varepsilon}(\lambda)\). This type of terms include the desirable reversed product \(T_{1,b}(\mu)T_{f_1,a_1-1}(\lambda)\) but also several other products having rather undesirable properties. The latter feature occurs when \(\bar{\varepsilon} \leq c_1 + f_2 = b + c_1 - d_1 - 2\) because the operator \(T_{f_1-c_1,a_1+b-1-\varepsilon}(\lambda)\) turns out to be a creation field and its presence on the further right position of the right-hand side commutation rule should definitely not be permitted. We stress here that this problem is independent of the character of the companion field \(T_{c_1+1,\varepsilon}(\mu)\). This is clear when \(T_{c_1+1,\varepsilon}(\mu)\) plays the role of either annihilation or diagonal operator since we have exactly the same situation we managed to handle for the left-hand side of the commutation rules. The case when \(T_{c_1+1,\varepsilon}(\mu)\) play the role of creation operators is more subtle because the product \(T_{c_1+1,\varepsilon}(\mu)T_{f_1-c_1,a_1+b-1-\varepsilon}(\lambda)\) could, in principle, contribute to the eigenvector basis. The order of the rapidities in such product is however opposite to that already chosen in section (3.2). For this reason such remaining type of product combination has also to be avoided. Therefore, no matter the role of \(T_{c_1+1,\varepsilon}(\mu)\), a second step is still necessary to provide us the means to compute the product \(T_{c_1+1,\varepsilon}(\mu)T_{f_1-c_1,a_1+b-1-\varepsilon}(\lambda)\).

\[ \text{(56)} \]

\[ \text{(57)} \]
as long as the index $\bar{\epsilon}$ satisfies the relation, 
\[ \bar{\epsilon} \leq c_1 + f_2 = b + c_1 - d_1 - 2. \tag{58} \]

This additional task has to be implemented without spoiling the main construction underlying the first linear system. This can be done by using the freedom of the extra index $c_2$ since so far we have kept it fixed at $c_2 = 0$. In Table 5 we have summarized the second linear combination for a given $c_1$ with the help of index $c_2$. The equations derived from such linear combination are suitable to calculate the products $T_{c_1+1,\bar{\epsilon}}(\mu)T_{f_1-c_1,a_1+b-1-\bar{\epsilon}}(\lambda)$ on the interval $M\{1, a_1 + b - N - 1\} \leq \bar{\epsilon} \leq m\{a_1 + b - 2, N\}$. Recall here that this interval is the total range of the index $\bar{\epsilon}$ in Eqs.\(\text{[53],[54]}\).

| Operators calculated | Creation index | Combination indexes $c_2$ | Number of equations |
|----------------------|----------------|-------------------------|--------------------|
| $T_{c_1+1,\bar{\epsilon}}(\mu)T_{f_1-c_1,a_1+b-1-\bar{\epsilon}}(\lambda)$ | $b - 2 \leq N - a_1$ | $c_2 = d_1 - c_1 + 2, \ldots, b - 1$ | $c_1 + b - d_1 - 2$ |
| $T_{c_1+1,\bar{\epsilon}}(\mu)T_{f_1-c_1,a_1+b-1-\bar{\epsilon}}(\lambda)$ | $b - 2 > N - a_1$ | $c_2 = d_1 - c_1 + 2, \ldots, c_1 + N - a_1 - d_1$ | $N - a_1 + 1$ |

Table 5: The linear combination from Eq.\(\text{[49]}\) to compute products of the form $T_{c_1+1,\bar{\epsilon}}(\mu)T_{f_1-c_1,a_1+b-1-\bar{\epsilon}}(\lambda)$.

By substituting the data of Table 5 in Eq.\(\text{[49]}\) we find that the second system of linear equations are given by,

\[
A_7^{(f_1-c_1,f_2+c_1)}(\lambda, \mu) = \sum_{\bar{\epsilon}=1}^{f_1} R(\lambda, \mu) \bar{\epsilon}^{f_1+1-\bar{\epsilon}} = \sum_{\bar{\epsilon}=b+c_1-d_1-1}^{a_1+b-2} v_7^{(f_1-c_1,f_2+c_1)}(\lambda, \mu) \tag{59}
\]

\[
\times \begin{pmatrix}
  T_{c_1+1,\bar{\epsilon}}(\mu)T_{f_1-c_1,a_1+b-1-\bar{\epsilon}}(\lambda) \\
  T_{c_1+1,\bar{\epsilon}+1}(\mu)T_{f_1-c_1,a_1+b-1-\bar{\epsilon}+1}(\lambda) \\
  \vdots \\
  T_{c_1+1,a_1+b-1}(\mu)T_{f_1-c_1,a_1+b-1}(\lambda)
\end{pmatrix}
\times \begin{pmatrix}
  T_{c_1+1,\bar{\epsilon}}(\mu)T_{f_1-c_1,a_1+b-1-\bar{\epsilon}}(\lambda) \\
  T_{c_1+1,\bar{\epsilon}+1}(\mu)T_{f_1-c_1,a_1+b-1-\bar{\epsilon}+1}(\lambda) \\
  \vdots \\
  T_{c_1+1,a_1+b-1}(\mu)T_{f_1-c_1,a_1+b-1}(\lambda)
\end{pmatrix}
\]

\[
\times \begin{pmatrix}
  T_{c_1+1,\bar{\epsilon}}(\mu)T_{f_1-c_1,a_1+b-1-\bar{\epsilon}}(\lambda) \\
  T_{c_1+1,\bar{\epsilon}+1}(\mu)T_{f_1-c_1,a_1+b-1-\bar{\epsilon}+1}(\lambda) \\
  \vdots \\
  T_{c_1+1,a_1+b-1}(\mu)T_{f_1-c_1,a_1+b-1}(\lambda)
\end{pmatrix}
\]

for \( b \leq N - a_1 + 2, \)
and

\[
A^8_{(f_1-c_1,f_2+c_1)}(\lambda, \mu) = \left( \begin{array}{c}
T_{c_1+1,a_1+b-N-1}(\mu)T_{f_1-c_1,N}(\lambda) \\
T_{c_1+1,a_1+b-N}(\mu)T_{f_1-c_1,N-1}(\lambda) \\
\vdots \\
T_{c_1+1,f_2+c_1}(\mu)T_{f_1-c_1,f_2-c_1+1}(\lambda)
\end{array} \right) = \sum_{\bar{e}=1}^{f_1} R(\lambda, \mu)^{\bar{e}}_{f_1_{-c_1,c_1+1}}
\]

\[
\times \left( \begin{array}{c}
T_{\bar{e},f_1-c_1+1}(\lambda)T_{f_1+1-\bar{e},f_2+c_1}(\mu) \\
T_{\bar{e},f_1-c_1+2}(\lambda)T_{f_1+1-\bar{e},f_2+c_1-1}(\mu) \\
\vdots \\
T_{\bar{e},N}(\lambda)T_{f_1+1-\bar{e},a_1+b-N-1}(\mu)
\end{array} \right) - \sum_{\bar{e}=b+c_1-d_1-1}^{N} v^8_{(f_1-c_1,f_2+c_1)}(\lambda, \mu)
\]

\[
\times T_{c_1+1,\bar{e}}(\mu)T_{f_1-c_1,a_1+b-1-\bar{e}}(\lambda)
\]

for \( b > N - a_1 + 2 \), \( \text{ (60) } \)

where the \( b \times b \) matrix \( A^7_{(a,b)}(\lambda, \mu) \) and the \( N + 1 - a \times N + 1 - a \) matrix \( A^8_{(a,b)}(\lambda, \mu) \) are given in terms of the structure constants as

\[
A^7_{(a,b)}(\lambda, \mu) = \left( \begin{array}{cccc}
R(\lambda, \mu)^{a+1,b}_{a+b+1,1} & R(\lambda, \mu)^{a+1,b}_{a+b+1,2} & \cdots & R(\lambda, \mu)^{a+1,b}_{a+b+1,1} \\
R(\lambda, \mu)^{a+2,b-1}_{a+b+1,1} & R(\lambda, \mu)^{a+2,b-1}_{a+b+1,2} & \cdots & R(\lambda, \mu)^{a+2,b-1}_{a+b+1,1} \\
\vdots & \vdots & \ddots & \vdots \\
R(\lambda, \mu)^{a+b+1}_{a+b+1,1} & R(\lambda, \mu)^{a+b+1}_{a+b+1,2} & \cdots & R(\lambda, \mu)^{a+b+1}_{a+b+1,1}
\end{array} \right)
\]

\( \text{ (61) } \)

and

\[
A^8_{(a,b)}(\lambda, \mu) = \left( \begin{array}{cccc}
R(\lambda, \mu)^{a+1,b}_{N,a+b-N+1} & R(\lambda, \mu)^{a+1,b}_{N-1,a+b-N+2} & \cdots & R(\lambda, \mu)^{a+1,b}_{a+b+1} \\
R(\lambda, \mu)^{a+2,b-1}_{N,a+b-N+1} & R(\lambda, \mu)^{a+2,b-1}_{N-1,a+b-N+2} & \cdots & R(\lambda, \mu)^{a+2,b-1}_{a+b+1} \\
\vdots & \vdots & \ddots & \vdots \\
R(\lambda, \mu)^{N,a+b-N+1}_{N,a+b-N+1} & R(\lambda, \mu)^{N,a+b-N+1}_{N-1,a+b-N+2} & \cdots & R(\lambda, \mu)^{N,a+b-N+1}_{a+b+1}
\end{array} \right)
\]

\( \text{ (62) } \)

while the \( b \times 1 \) vector \( v^7_{(a,b)}(\lambda, \mu) \) and the \( N + 1 - a \times 1 \) vector \( v^8_{(a,b)}(\lambda, \mu) \) are

\[
v^7_{(a,b)}(\lambda, \mu) = \left( \begin{array}{c}
R(\lambda, \mu)^{a+1,b}_{a+b+1-\bar{e},\bar{e}} \\
R(\lambda, \mu)^{a+2,b-1}_{a+b+1-\bar{e},\bar{e}} \\
\vdots \\
R(\lambda, \mu)^{a+b+1}_{a+b+1-\bar{e},\bar{e}}
\end{array} \right), \quad v^8_{(a,b)}(\lambda, \mu) = \left( \begin{array}{c}
R(\lambda, \mu)^{a+1,b}_{a+b+1-\bar{e},\bar{e}} \\
R(\lambda, \mu)^{a+2,b-1}_{a+b+1-\bar{e},\bar{e}} \\
\vdots \\
R(\lambda, \mu)^{N,a+b-N+1}_{a+b+1-\bar{e},\bar{e}}
\end{array} \right)
\]

\( \text{ (63) } \)
Let us now show how the first and the second systems of linear equations work in practice together. In order to see that we shall discuss in detail the commutation rule among the fields $T_{f_1,a_1-1}(\lambda)$ and $T_{1,b}(\mu)$. We start by considering the first linear system of equations (65,66) which for $b = 3$ gives us the following expressions,

$$A^{(f_1,3)}_5(\lambda, \mu) \left( \begin{array}{c} T_{f_1-1,a_1-1}(\lambda)T_{2,3}(\mu) \\ T_{f_1,a_1-1}(\lambda)T_{1,3}(\mu) \end{array} \right) = \sum_{\ell=1}^{a_1+1} R(\lambda, \mu)^{a_1-1,3}_{a_1+2-\ell,\ell} \left( \begin{array}{c} T_{1,\ell}(\mu)T_{f_1,a_1+1-\ell}(\lambda) \\ T_{2,\ell}(\mu)T_{f_1-1,a_1+1-\ell}(\lambda) \end{array} \right)$$

and

$$A^{(a_1,0)}_6(\lambda, \mu) \left( \begin{array}{c} T_{a_1-1,a_1-1}(\lambda)T_{2,3}(\mu) \\ T_{a_1,a_1-1}(\lambda)T_{1,3}(\mu) \end{array} \right) = \sum_{\ell=M\{a_1+2-N\}}^n \left( \begin{array}{c} T_{1,\ell}(\mu)T_{a_1,a_1+1-\ell}(\lambda) \\ T_{2,\ell}(\mu)T_{a_1-1,a_1+1-\ell}(\lambda) \end{array} \right)$$

$$\times R(\lambda, \mu)^{a_1-1,3}_{a_1+2-\ell,\ell} - \sum_{\ell=1}^{a_1-2} v^{(a_1,0)}_6(\lambda, \mu)T_{\ell,a_1-1}(\lambda)T_{a_1+1-\ell,3}(\mu) \quad \text{for} \quad d_1 = 0. \quad (65)$$

The matrices $A^{(f_1,3)}_5(\lambda, \mu)$ and $A^{(a_1,0)}_6(\lambda, \mu)$ are given by

$$A^{(f_1,3)}_5(\lambda, \mu) = \left( \begin{array}{cc} R(\lambda, \mu)^{a_1-1,2}_{a_1+1,1} & R(\lambda, \mu)^{a_1-1,1}_{a_1+1,1} \\ R(\lambda, \mu)^{a_1-1,2}_{a_1+1,1} & R(\lambda, \mu)^{a_1-1,1}_{a_1+1,1} \end{array} \right), \quad A^{(a_1,0)}_6(\lambda, \mu) = \left( \begin{array}{cc} R(\lambda, \mu)^{a_1-1,2}_{a_1+1,1} & R(\lambda, \mu)^{a_1-1,1}_{a_1+1,1} \\ R(\lambda, \mu)^{a_1-1,2}_{a_1+1,1} & R(\lambda, \mu)^{a_1-1,1}_{a_1+1,1} \end{array} \right), \quad (66)$$

while the vectors $v^{(f_1,3)}_5(\lambda, \mu)$ and $v^{(a_1,0)}_6(\lambda, \mu)$ are

$$v^{(f_1,3)}_5(\lambda, \mu) = \left( \begin{array}{c} R(\lambda, \mu)^{\ell,a_1+1-\ell}_{a_1+1,1} \\ R(\lambda, \mu)^{\ell,a_1+1-\ell}_{a_1+1,1} \end{array} \right), \quad v^{(a_1,0)}_6(\lambda, \mu) = \left( \begin{array}{c} R(\lambda, \mu)^{\ell,a_1+1-\ell}_{a_1+1,1} \\ R(\lambda, \mu)^{\ell,a_1+1-\ell}_{a_1+1,1} \end{array} \right). \quad (67)$$
By using Cramer’s rule in Eqs. (61, 65) we can compute the products \( T_{\lambda_1-1}(\lambda)T_{\lambda_3}(\mu) \) and the results are,

\[
T_{\lambda_1-1}(\lambda)T_{\lambda_3}(\mu) = \sum_{\bar{\epsilon}=1}^{\lambda_1+1} R(\lambda, \mu)^{\lambda_1-1, 3}_{a_1+2-\bar{\epsilon}, \bar{\epsilon}} R(\lambda, \mu)^{\lambda_1-1, 2}_{a_1+2-\bar{\epsilon}, \bar{\epsilon}} T_{\lambda_2-\bar{\epsilon}(\mu)}T_{\lambda_1-1, a_1+2-\bar{\epsilon}(\lambda)} - \sum_{\bar{\epsilon}=1}^{\lambda_1+1} R(\lambda, \mu)^{\lambda_1-1, 3}_{a_1+2-\bar{\epsilon}, \bar{\epsilon}} R(\lambda, \mu)^{\lambda_1-1, 2}_{a_1+2-\bar{\epsilon}, \bar{\epsilon}} T_{\lambda_1-\bar{\epsilon}(\mu)}T_{\lambda_1-1, a_1+2-\bar{\epsilon}(\lambda)} - \sum_{\bar{\epsilon}=1}^{\lambda_1-2} R(\lambda, \mu)^{\lambda_1-1, 2}_{a_1+2-\bar{\epsilon}, \bar{\epsilon}} R(\lambda, \mu)^{\lambda_1-1, 2}_{a_1+2-\bar{\epsilon}, \bar{\epsilon}} T_{\lambda_1-\bar{\epsilon}(\mu)}T_{\lambda_1-1, a_1+2-\bar{\epsilon}(\lambda)}
\]

for \( d_1 \geq 1 \), \( \lambda \), \( \mu \), \( a \), \( f \), \( e \), \( T \), \( \bar{\epsilon} \), \( N \), \( R \). (68)

and

\[
T_{\lambda_1, a_1-1}(\lambda)T_{\lambda_3}(\mu) = \sum_{\bar{\epsilon}=M(a_1+2-N)}^{\lambda_1+1} R(\lambda, \mu)^{\lambda_1-1, 3}_{a_1+2-\bar{\epsilon}, \bar{\epsilon}} R(\lambda, \mu)^{\lambda_1-1, 2}_{a_1+2-\bar{\epsilon}, \bar{\epsilon}} T_{\lambda_2-\bar{\epsilon}(\mu)}T_{\lambda_1-1, a_1+2-\bar{\epsilon}(\lambda)} - \sum_{\bar{\epsilon}=M(a_1+2-N)}^{\lambda_1+1} R(\lambda, \mu)^{\lambda_1-1, 3}_{a_1+2-\bar{\epsilon}, \bar{\epsilon}} R(\lambda, \mu)^{\lambda_1-1, 2}_{a_1+2-\bar{\epsilon}, \bar{\epsilon}} T_{\lambda_1-\bar{\epsilon}(\mu)}T_{\lambda_1-1, a_1+2-\bar{\epsilon}(\lambda)} - \sum_{\bar{\epsilon}=1}^{\lambda_1-2} R(\lambda, \mu)^{\lambda_1-1, 2}_{a_1+2-\bar{\epsilon}, \bar{\epsilon}} R(\lambda, \mu)^{\lambda_1-1, 2}_{a_1+2-\bar{\epsilon}, \bar{\epsilon}} T_{\lambda_1-\bar{\epsilon}(\mu)}T_{\lambda_1-1, a_1+2-\bar{\epsilon}(\lambda)}
\]

for \( d_1 = 0 \). \( \lambda \), \( \mu \), \( a \), \( f \), \( e \), \( T \), \( \bar{\epsilon} \), \( N \), \( R \). (69)

At this point we note that the commutation rules (68, 69) between \( T_{\lambda_1=1+4, a_1-1}(\lambda) \) and \( T_{\lambda_3}(\mu) \) possess the appropriate form only when \( d_1 > 1 \). In these cases the right-hand side of Eqs. (68, 69)
does not generate undesirable products of fields and the second system of linear equations is not needed. This is not the situation of the remaining values \( d_1 = 1 \) or \( d_1 = 0 \). For example, when \( d_1 = 1 \) we see that the first term of Eq. (68) for \( \bar{\varepsilon} = 1 \) produces the undesirable product \( T_{2,1}(\mu)T_{a_1,a_1+1}(\lambda) \). This product can, however, be eliminated thanks to the second linear system of equations. Indeed, by considering Eq. (69) for \( c_1 = 1 \) we can compute the product \( T_{2,1}(\mu)T_{a_1,a_1+1}(\lambda) \) as,

\[
T_{2,1}(\mu)T_{a_1,a_1+1}(\lambda) = \sum_{\varepsilon=1}^{a_1+1} \frac{R(\lambda, \mu)^{a_1+2-\varepsilon}}{R(\lambda, \mu)^{a_1+1,1}} T_{\varepsilon,a_1+1}(\lambda)T_{a_1+2-\varepsilon,1}(\mu) - \sum_{\varepsilon=2}^{a_1+1} \frac{R(\lambda, \mu)^{a_1+1,1}}{R(\lambda, \mu)^{a_1+1,1}} T_{2,\varepsilon}(\mu)T_{a_1,a_1+2-\varepsilon}(\lambda) \quad \text{for} \quad a_1 \leq N - 1. \tag{70}
\]

For the value \( d_1 = 1 \) we can now substitute Eq. (70) in Eq. (68). This cancels out the undesirable product, providing us the suitable commutation rule between the operators \( T_{a_1+1,a_1-1}(\lambda) \) and \( T_{1,3}(\mu) \),

\[
T_{a_1+1,a_1-1}(\lambda)T_{1,3}(\mu) = - \sum_{\varepsilon=2}^{a_1+1} \frac{R(\lambda, \mu)^{a_1+1,1}}{R(\lambda, \mu)^{a_1+1,1}} T_{1,\varepsilon}(\mu)T_{a_1+1,a_1+2-\varepsilon}(\lambda) + \sum_{\varepsilon=1}^{a_1+1} \frac{R(\lambda, \mu)^{a_1+2-\varepsilon}}{R(\lambda, \mu)^{a_1+1,1}} \begin{vmatrix}
R(\lambda, \mu)^{a_1-1,3} & R(\lambda, \mu)^{a_1+1,1} \\
R(\lambda, \mu)^{a_1+1,1} & R(\lambda, \mu)^{a_1+1,1}
\end{vmatrix} T_{\varepsilon,a_1+1}(\lambda)T_{a_1+2-\varepsilon,1}(\mu)
\]

\[
- \sum_{\varepsilon=1}^{a_1-1} \frac{R(\lambda, \mu)^{a_1+2-\varepsilon}}{R(\lambda, \mu)^{a_1+1,1}} \begin{vmatrix}
R(\lambda, \mu)^{a_1-1,3} & R(\lambda, \mu)^{a_1+1,1} \\
R(\lambda, \mu)^{a_1+1,1} & R(\lambda, \mu)^{a_1+1,1}
\end{vmatrix} T_{\varepsilon,a_1-1}(\lambda)T_{a_1+2-\varepsilon,3}(\mu) \tag{71}
\]
The case $d_1 = 0$ is more laborious due to the presence of three different types of undesirable products. They come from the first ($\bar{e} = 1, 2$) and the second ($\bar{e} = 1$) terms of Eq. (59) and their respective forms are $T_{2,1}(\mu) T_{a_1-1, a_1+1}(\lambda)$, $T_{2,2}(\mu) T_{a_1-1, a_1}(\lambda)$ and $T_{1,1}(\mu) T_{a_1, a_1+1}(\lambda)$. All these terms can be eliminated with the help of the second linear system of equations as follows. The last undesirable term $T_{1,1}(\mu) T_{a_1, a_1+1}(\lambda)$ is the simplest one to be computed. Its expression comes directly from Eq. (59) by taking the values $c_1 = d_1 = 0$,

$$T_{1,1}(\mu) T_{a_1, a_1+1}(\lambda) = \sum_{\bar{e} = 1}^{a_1} \frac{R(\lambda, \mu)_{a_1+1-\bar{e}}}{R(\lambda, \mu)_{a_1+1, 1}} T_{\bar{e}, a_1+1}(\lambda) T_{a_1+1-\bar{e}, 1}(\mu)$$

$$- \sum_{\bar{e} = 2}^{a_1+1} \frac{R(\lambda, \mu)_{a_1+1, 1}}{R(\lambda, \mu)_{a_1+1, 1}} T_{\bar{e}, \mu}(\mu) T_{a_1, a_1+2-\bar{e}}(\lambda)$$

(72)

The remaining products $T_{2,1}(\mu) T_{a_1-1, a_1+1}(\lambda)$ and $T_{2,2}(\mu) T_{a_1-1, a_1}(\lambda)$ are obtained as a solution of a $2 \times 2$ linear system of equations. It is derived from Eqs. (59) for the values $c_1 = 1$ and $d_1 = 0$ and the final result is,

$$A_7^{(a_1-1, 2)}(\lambda, \mu) \left( \begin{array}{c} T_{2,1}(\mu) T_{a_1-1, a_1+1}(\lambda) \\ T_{2,2}(\mu) T_{a_1-1, a_1}(\lambda) \end{array} \right) = \sum_{\bar{e} = 1}^{a_1} R(\lambda, \mu)_{a_1+1-\bar{e}} \left( \begin{array}{c} T_{\bar{e}, a_1}(\lambda) T_{a_1+1-\bar{e}, 2}(\mu) \\ T_{\bar{e}, a_1+1}(\lambda) T_{a_1+1-\bar{e}, 1}(\mu) \end{array} \right)$$

$$- \sum_{\bar{e} = 3}^{a_1+1} v_7^{(a_1-1, 2)}(\lambda, \mu) T_{\bar{e}, \mu}(\mu) T_{a_1-1, a_1+2-\bar{e}}(\lambda) \quad \text{for} \quad a_1 \leq N - 1,$$

(73)

$$R(\lambda, \mu)^{N, 2} T_{2,2}(\mu) T_{N-1, N}(\lambda) = \sum_{\bar{e} = 1}^{N} R(\lambda, \mu)_{N-1, 2} \bar{e} T_{\bar{e}, N}(\lambda) T_{N+1-\bar{e}, 2}(\mu)$$

$$- \sum_{\bar{e} = 3}^{N} R(\lambda, \mu)_{N+2-\bar{e}, \bar{e}} T_{2, \bar{e}}(\mu) T_{N-1, N+2-\bar{e}}(\lambda),$$

(74)

where

$$A_7^{(a_1-1, 2)}(\lambda, \mu) = \left( \begin{array}{cc} R(\lambda, \mu)_{a_1+1, 1} & R(\lambda, \mu)_{a_1+2, 1} \\ R(\lambda, \mu)_{a_1+1, 1} & R(\lambda, \mu)_{a_1+2, 1} \end{array} \right) \quad \text{and} \quad v_7^{(a_1-1, 2)}(\lambda, \mu) = \left( \begin{array}{c} R(\lambda, \mu)_{a_1+2-\bar{e}, \bar{e}} \\ R(\lambda, \mu)_{a_1+2-\bar{e}, \bar{e}} \end{array} \right).$$

(75)

By applying Cramer’s rule in Eq. (73) we can easily compute the undesirable terms $T_{2,1}(\mu) T_{a_1-1, a_1+1}(\lambda)$ and $T_{2,2}(\mu) T_{a_1-1, a_1}(\lambda)$ for $a_1 \leq N - 1$. We then substitute this result as well
as the expression (72) for $T_{1,1}(\mu)T_{a_1,a_1+1}(\lambda)$ in Eq.(69). In this way we are able to obtain the commutation rule for the fields $T_{a_1,a_1-1}(\lambda)$ and $T_{1,3}(\mu)$ for $a_1 \leq N$. For $a_1 = N$ we just have to eliminate the term $T_{2,2}(\mu)T_{N-1,N}(\lambda)$ of Eq.(69) with the assistance of the relation (74). The final step to obtain suitable commutation relations concerns with the reordering of the products $T_{a_1,a_1+1}(\lambda)T_{1,2}(\mu)$ with the help of Eqs.(28,29).

4 Identities among the weights

In this section we describe a procedure to derive identities between the amplitudes $R(\lambda, \mu)^{c,d}_{a,b}$ from both the unitarity relation and the Yang-Baxter equation. These identities are essential to carry out simplifications on the eigenvalue problem without the necessity of referring to specific weights. We shall start by considering the consequences of the unitarity relation (5).

4.1 Unitarity relation

A systematic study of the unitarity relation (5) revealed us the existence of two key sets of independent weights. They will play a relevant role in the analysis of Eq.(5) and are defined as,

$$a_{j,q}^{j,q_1}_{b,c} = \begin{cases} R(\lambda, \mu)^{c,q_1+1-a}_{q_1+1-b,b}, & \text{for } j = 1, \\ R(\lambda, \mu)^{N-q_1+c,N+1-a}_{N+1-b,N+1-b}, & \text{for } j = 2, \end{cases}$$

(76)

$$\bar{a}_{j,q}^{j,q_1}_{b,c} = \begin{cases} R(\mu, \lambda)^{c,q_1+1-a}_{q_1+1-b,b}, & \text{for } j = 1, \\ R(\mu, \lambda)^{N-q_1+c,N+1-a}_{N+1-b,N+1-b}, & \text{for } j = 2, \end{cases}$$

(77)

where $q_1 = 1, \ldots, N$ and $b, c = 1, \ldots, q_1$.

The above definition explores the block form of the $R$-matrix on the basis of the $U(1)$ operator $S^z \otimes I_N + I_N \otimes S^z$. The index $j = 1, 2$ is used to split the charges with $q = 1, \ldots, N$ ($j = 1$) from those with $q = N + 1, \ldots, 2N - 1$ ($j = 2$). Considering Eqs.(76,77) we are able to write the matrix elements of unitarity relation (5) as,

$$U[b,c]_{q_1}^{q_1} = \sum_{k=1}^{q_1} a_{b,k}^{j,q_1} \bar{a}_{k,c}^{j,q_1} - \delta_{b,c} = 0 \quad \text{for } q_1 = 1, \ldots, N; \quad b, c = 1, \ldots, q_1; \quad j = 1, 2. \quad (78)$$
A non-trivial identity between the weights is already relevant for the analysis of the one-particle problem. It is obtained from the $U[1, 2]_1^2$ component and by substituting the respective indices in Eq. (78) we find,

$$\frac{R(\lambda, \mu)_{1,2}^{1,2}}{R(\lambda, \mu)_{2,1}^{2,1}} = -\frac{R(\mu, \lambda)_{1,2}^{2,1}}{R(\mu, \lambda)_{2,1}^{1,2}}.$$  (79)

The type of relations are necessary because the combined use of the commutation rules, discussed in previous section, forces us to reorder the spectral parameters of some $R$-matrix elements. It is fortunate that we can solve this problem with the assistance of the unitary property. In general, more complex identities are needed and they are derived by analyzing linear systems of equations based on Eq. (78). The first such system is obtained by means of the linear combination

$$U[b; i + 1]_j^{i+2}a_{i+1,i+2}^j - U[b; i + 2]_j^{i+2}a_{i+1,i+1}^j$$

for $b = 1, \ldots, i + 1$. These equations make it possible to write the following linear system for variables $a_{i+1,i+2}^j - \bar{a}_{i+1,i+2}^j = 0$,

$$\begin{pmatrix}
    a_{1,2}^j & a_{1,3}^j & \cdots & a_{1,i+2}^j \\
    a_{2,2}^j & a_{2,3}^j & \cdots & a_{2,i+2}^j \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{i+1,2}^j & a_{i+1,3}^j & \cdots & a_{i+1,i+2}^j
\end{pmatrix}
\begin{pmatrix}
    \bar{a}_{2,1}^j & \bar{a}_{1,1}^j & \cdots & \bar{a}_{i+2,1}^j \\
    \bar{a}_{3,1}^j & \bar{a}_{2,1}^j & \cdots & \bar{a}_{i+2,1}^j \\
    \vdots & \vdots & \ddots & \vdots \\
    \bar{a}_{i+2,1}^j & \bar{a}_{i+1,1}^j & \cdots & \bar{a}_{i+2,1}^j
\end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$  (80)

The solution of this system of equations for three particular components is able to generate classes of important identities. The first family is found by solving the system (80) for the first component. By employing Cramer’s rule we found,

$$-\begin{vmatrix}
    a_{1,1}^j & \cdots & a_{1,i+2}^j \\
    \vdots & \ddots & \vdots \\
    a_{i+1,1}^j & \cdots & a_{i+1,i+2}^j
\end{vmatrix}
\begin{vmatrix}
    \bar{a}_{1,1}^j & \cdots & \bar{a}_{i+2,1}^j \\
    \vdots & \ddots & \vdots \\
    \bar{a}_{i+1,1}^j & \cdots & \bar{a}_{i+2,1}^j
\end{vmatrix} = \begin{vmatrix}
    a_{1,1}^j & \cdots & a_{1,i+2}^j \\
    \vdots & \ddots & \vdots \\
    a_{i+1,1}^j & \cdots & a_{i+1,i+2}^j
\end{vmatrix} \begin{vmatrix}
    \bar{a}_{1,1}^j & \cdots & \bar{a}_{i+2,1}^j \\
    \vdots & \ddots & \vdots \\
    \bar{a}_{i+1,1}^j & \cdots & \bar{a}_{i+2,1}^j
\end{vmatrix}$$  (81)

The remaining families of identities follow from the solution of Eq. (80) for the second and last
components. After making the ratio of these solutions with Eq. (81) we have,

\[
\begin{vmatrix}
\tilde{a}_{1,i+1}^{j,i+2} & \tilde{a}_{1,i+1}^{j,i} \\
\tilde{a}_{3,i+1}^{j,i+2} & \tilde{a}_{3,i+1}^{j,i}
\end{vmatrix} = -\begin{vmatrix}
an_{1,2}^{j,i+2} & an_{1,2}^{j,i+2} & \cdots & an_{1,2}^{j,i+2} \\
\vdots & \vdots & \ddots & \vdots \\
an_{1,2}^{j,i+2} & an_{1,2}^{j,i+2} & \cdots & an_{1,2}^{j,i+2}
\end{vmatrix}. \tag{82}
\]

and

\[
\begin{vmatrix}
\tilde{a}_{1,i+1}^{j,i+2} & \tilde{a}_{1,i+1}^{j,i} \\
\tilde{a}_{3,i+1}^{j,i+2} & \tilde{a}_{3,i+1}^{j,i}
\end{vmatrix} = (-1)^i \begin{vmatrix}
an_{1,2}^{j,i+2} & an_{1,2}^{j,i+2} \\
\vdots & \vdots \\
an_{1,2}^{j,i+2} & an_{1,2}^{j,i+2}
\end{vmatrix}. \tag{83}
\]

We shall now discuss four specific identities that are going to be very useful in the next section. The first two identities are relevant to verify the exchange property of the two-particle state under the respective rapidities. Both of them are derived by selecting the same indices \(i = j = 1\) in Eqs. (81, 82). Considering that \(R(\lambda, \mu)_{1,1}^{1,1} R(\mu, \lambda)_{1,1}^{1,1} = 1\) it is possible to write Eq. (81) with \(i = j = 1\) as,

\[
D_{2}^{(2,0)}(\lambda, \mu) D_{2}^{(2,0)}(\mu, \lambda) = \frac{R(\lambda, \mu)_{1,1}^{1,1} R(\mu, \lambda)_{1,1}^{1,1}}{R(\lambda, \mu)_{2,1}^{2,1} R(\mu, \lambda)_{2,1}^{2,1}}, \tag{84}
\]

while Eq. (82) with \(i = j = 1\) gives us,

\[
D_{2}^{(2,1)}(\lambda, \mu) = -\frac{R(\mu, \lambda)_{3,1}^{2,2} D_{2}^{(2,0)}(\lambda, \mu)}{R(\mu, \lambda)_{3,1}^{3,3}}. \tag{85}
\]

The two remaining examples are going to be used in the three-particle problem. As before they are important to demonstrate the rapidities symmetrization properties of this state. They
are derived from Eqs. (82, 83) by choosing the indices $i = 2$ and $j = 1$, namely

$$
\frac{D_2^{(3,1)}(\lambda_1, \lambda)}{D_2^{(3,0)}(\lambda_1, \lambda)} = \frac{R(\lambda_1, \lambda)^{3,2}_{4,1} R(\lambda_1, \lambda)^{3,2}_{2,3}}{R(\lambda_1, \lambda)^{4,1}_{4,1} R(\lambda_1, \lambda)^{4,1}_{2,3}} = - \begin{vmatrix} R(\lambda, \lambda)^{2,3}_{4,1} & R(\lambda, \lambda)^{2,3}_{3,2} \\ R(\lambda, \lambda)^{4,1}_{4,1} & R(\lambda, \lambda)^{4,1}_{3,2} \end{vmatrix} \quad (86)
$$

$$
\frac{D_2^{(3,2)}(\lambda_1, \lambda)}{D_2^{(3,0)}(\lambda_1, \lambda)} = \frac{R(\lambda_1, \lambda)^{3,2}_{4,1} R(\lambda_1, \lambda)^{3,2}_{2,3}}{R(\lambda_1, \lambda)^{4,1}_{4,1} R(\lambda_1, \lambda)^{4,1}_{2,3}} = - \begin{vmatrix} R(\lambda, \lambda)^{3,2}_{4,1} & R(\lambda, \lambda)^{3,2}_{3,2} \\ R(\lambda, \lambda)^{4,1}_{4,1} & R(\lambda, \lambda)^{4,1}_{3,2} \end{vmatrix} \quad (87)
$$

In order to construct further relevant identities we found the necessity of combining the results of two systems of linear equations. The first of them is obtained by means of the linear combination $U[b; k+2]_j a_{1,i+2}^{j,i+2} - U[b; i+2]_j a_{1,k+2}^{j,i+2}$ for $b = 1, 2, 3$ and $k = 1, \ldots, i - 1$. From these equations we are able to write a $3 \times 3$ system of equations for variables $a_{1,i+2}^{j,i+2}, a_{1,k+2}^{j,i+2}$ for $\bar{e} = i, i+1, i+2$, namely

$$
\begin{vmatrix} a_{1,i+2}^{j,i+2} & a_{1,i+1}^{j,i+2} & a_{1,i+2}^{j,i+2} \\ a_{2,i}^{j,i+2} & a_{2,i+1}^{j,i+2} & a_{2,i+2}^{j,i+2} \\ a_{3,i}^{j,i+2} & a_{3,i+1}^{j,i+2} & a_{3,i+2}^{j,i+2} \end{vmatrix} \begin{vmatrix} a_{1,i+2}^{j,i+2} & a_{1,i+2}^{j,i+2} & a_{1,i+2}^{j,i+2} \\ a_{1,k+2}^{j,i+2} & a_{1,k+2}^{j,i+2} & a_{1,k+2}^{j,i+2} \\ a_{1,i+2}^{j,i+2} & a_{1,i+2}^{j,i+2} & a_{1,i+2}^{j,i+2} \end{vmatrix} = - \sum_{\bar{e}=2}^{i-1} \begin{vmatrix} a_{2,1+\bar{e}}^{j,\bar{e}+2} \\ a_{2,1+\bar{e}}^{j,\bar{e}+2} \\ a_{2,1+\bar{e}}^{j,\bar{e}+2} \end{vmatrix} + \delta_{k,1} \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix} \quad (88)
$$

The next system of equations is generated by solving Eq. (88) for the first variable $a_{1,i+2}^{j,i+2} a_{1,k+2}^{j,i+2} - a_{1,i+2}^{j,i+2} a_{1,k+2}^{j,i+2}$. By using Cramer’s rule we find,

$$
\sum_{\bar{e}=1}^{i-1} \begin{vmatrix} a_{1,i+2}^{j,i+2} & a_{1,i+2}^{j,i+2} & a_{1,i+2}^{j,i+2} \\ a_{2,1+\bar{e}}^{j,\bar{e}+2} & a_{2,1+\bar{e}}^{j,\bar{e}+2} & a_{2,1+\bar{e}}^{j,\bar{e}+2} \\ a_{3,1+\bar{e}}^{j,\bar{e}+2} & a_{3,1+\bar{e}}^{j,\bar{e}+2} & a_{3,1+\bar{e}}^{j,\bar{e}+2} \end{vmatrix} = \delta_{k,1} \begin{vmatrix} a_{1,i+2}^{j,i+2} \\ a_{1,i+2}^{j,i+2} \\ a_{1,i+2}^{j,i+2} \end{vmatrix} \begin{vmatrix} a_{1,i+2}^{j,i+2} & a_{1,i+2}^{j,i+2} & a_{1,i+2}^{j,i+2} \\ a_{2,1+\bar{e}}^{j,\bar{e}+2} & a_{2,1+\bar{e}}^{j,\bar{e}+2} & a_{2,1+\bar{e}}^{j,\bar{e}+2} \\ a_{3,1+\bar{e}}^{j,\bar{e}+2} & a_{3,1+\bar{e}}^{j,\bar{e}+2} & a_{3,1+\bar{e}}^{j,\bar{e}+2} \end{vmatrix},
$$

for $k = 1, \ldots, i - 1$, (89)
Note that Eqs. (89) can be interpreted as a system of linear equations where the variables are the $3 \times 3$ determinants. In order to obtain our final identity we first solve Eqs. (89) for the $3 \times 3$ determinant indexed by $\bar{e} = 1$. Next we make the ratio of this solution with Eq. (81) considering the replacements $i \rightarrow i + 1$ and $\lambda \leftrightarrow \mu$. After some cumbersome simplifications we find the following identity,

$$
\begin{vmatrix}
    a_{1,i}^{j+1} & a_{1,i+1}^{j+2} & a_{1,i+1}^{j+1} \\
    a_{2,i}^{j+2} & a_{2,i+1}^{j+2} & a_{2,i+1}^{j+1} \\
    a_{3,i}^{j+2} & a_{3,i+1}^{j+2} & a_{3,i+1}^{j+1}
\end{vmatrix}
\begin{vmatrix}
    a_{1,i}^{j+1} & a_{1,i+1}^{j+2} & a_{1,i+1}^{j+1} \\
    a_{2,i}^{j+1} & a_{2,i+1}^{j+1} & a_{2,i+1}^{j+2} \\
    a_{3,i}^{j+1} & a_{3,i+1}^{j+1} & a_{3,i+1}^{j+2}
\end{vmatrix}
= (-1)^i

\begin{vmatrix}
    a_{1,i}^{j+1} & a_{1,i+1}^{j+2} & a_{1,i+1}^{j+1} \\
    a_{2,i}^{j+1} & a_{2,i+1}^{j+1} & a_{2,i+1}^{j+2} \\
    a_{3,i}^{j+1} & a_{3,i+1}^{j+1} & a_{3,i+1}^{j+2}
\end{vmatrix}
\begin{vmatrix}
    a_{1,i}^{j+2} & a_{1,i+1}^{j+2} & a_{1,i+1}^{j+2} \\
    a_{2,i}^{j+2} & a_{2,i+1}^{j+2} & a_{2,i+1}^{j+2} \\
    a_{3,i}^{j+2} & a_{3,i+1}^{j+2} & a_{3,i+1}^{j+2}
\end{vmatrix}

(90)

We close this section by presenting the explicit expressions of two other identities in terms of the $R$-matrix elements. These relations are going to work together with those coming from the Yang-Baxter equation in order to simplify the eigenvalue problem. In what follows we shall present them in a notation that will be useful for next subsection. The first one follows from Eq. (90) choosing $j = 1$,

$$
\frac{D_3^{(i,0)}(\lambda, \mu)}{D_2^{(i,0)}(\lambda, \mu)} = \frac{D_4^{(i+1,2)}(\lambda, \mu) D_4^{(i+2,4)}(\lambda, \mu)}{D_4^{(i+1,3)}(\lambda, \mu) D_4^{(i+2,3)}(\lambda, \mu)},

(91)

$$

where the determinant $D_4^{(i,b)}(\lambda, \mu)$ is given by,

$$
D_4^{(i,b)}(\lambda, \mu) = \begin{vmatrix}
    R(\mu, \lambda)^{b,i+1-b}_{i,i+1} & \ldots & R(\mu, \lambda)^{b,1}_{i,1} \\
    \vdots & \ddots & \vdots \\
    R(\mu, \lambda)^{b,1}_{i+1-b,i+1} & \ldots & R(\mu, \lambda)^{b,1}_{i+1,1}
\end{vmatrix}

\text{for } i \leq N

(92)

The second one follows from Eq. (82) also with $j = 1$ and by performing the rapidity exchange $\lambda \leftrightarrow \mu$ we find,

$$
\frac{D_2^{(i+1,1)}(\lambda, \mu)}{D_2^{(i+1,0)}(\lambda, \mu)} = \frac{D_5^{(i+2,2)}(\lambda, \mu)}{D_4^{(i+2,3)}(\lambda, \mu)},

(93)

36
such that the determinant $D_{5}^{(i,2)}(\lambda, \mu)$ is,

$$D_{5}^{(i+2,2)}(\lambda, \mu) = \begin{vmatrix} R(\mu, \lambda)^{2,i+1}_{i+2,1} & R(\mu, \lambda)^{4,i-1}_{i+2,1} & \cdots & R(\mu, \lambda)^{i+2,1}_{i+2,1} \\ \vdots & \vdots & \ddots & \vdots \\ R(\mu, \lambda)^{2,i+1}_{3,i} & R(\mu, \lambda)^{4,i-1}_{3,i} & \cdots & R(\mu, \lambda)^{i+2,1}_{3,i} \end{vmatrix}$$

for $i + 2 \leq N$. (94)

4.2 Yang-Baxter equation

In practice to extract informations from the Yang-Baxter equation one has to project out the $N^3 \times N^3$ matrix \[4\] on the Weyl basis. Considering the projection on row $[(a_1 - 1)N^2 + (a_2 - 1)N + a_3]$ and column $[(c_1 - 1)N^2 + (c_2 - 1)N + c_3]$ we find,

$$R(\lambda_1, \lambda_2)^{b_1, b_2}_{a_1, a_2} R(\lambda_1, \lambda_3)^{c_1, b_3}_{b_1, a_3} R(\lambda_2, \lambda_3)^{c_2, c_3}_{b_2, b_3} = R(\lambda_2, \lambda_3)^{b_2, b_3}_{a_2, a_3} R(\lambda_1, \lambda_3)^{b_1, c_3}_{a_1, b_3} R(\lambda_1, \lambda_2)^{c_1, c_2}_{b_1, b_2},$$

where sum on the repeated indices $b_i$ is assumed.

In order to describe our approach we shall denote Eq.(95) for a given set of indices $a_i$ and $c_i$ by the symbol $Y B_{a_1, a_2, a_3}^{c_1, c_2, c_3}(\lambda_1, \lambda_2, \lambda_3)$. In general, we are required to elaborate on Eq.(95) to obtain suitable weights identities that are directly useful in the eigenvalue problem. In fact, there exists only two exceptions that follow from Eq.(95) without further manipulations. These identities come from the entries $Y B_{3,1,1}^{2,2,1}(\lambda_1, \lambda_2, \lambda)$ and $Y B_{N-1,3,1}^{N-2,2}(\lambda_1, \lambda_2)$. After appropriate normalizations they are given by,

$$R(\lambda_2, \lambda_1)^{1,1}_{1,1} R(\lambda_1, \lambda_2)^{2,2}_{2,1} R(\lambda_1, \lambda_2)^{3,3}_{3,1} = R(\lambda_2, \lambda_1)^{2,1}_{2,1} R(\lambda_1, \lambda_2)^{2,2}_{2,1} R(\lambda_1, \lambda_2)^{3,3}_{3,1} + R(\lambda_1, \lambda_2)^{2,2}_{2,1} R(\lambda_1, \lambda_2)^{3,3}_{3,1}$$

(96)

and

$$R(\lambda, \lambda_2)^{N,2}_{N,2} R(\lambda_1, \lambda_2)^{2,2}_{3,1} \frac{R(\lambda_1, \lambda_1)^{N,3}_{N,3}}{R(\lambda_1, \lambda_2)^{3,1}_{3,1} R(\lambda_1, \lambda_1)^{N,2}_{N,2}} = R(\lambda, \lambda_2)^{N,2}_{N,2} R(\lambda, \lambda_1)^{N,3}_{N,3} \frac{R(\lambda, \lambda_1)^{N,2}_{N,2}}{R(\lambda, \lambda_2)^{N,1}_{N,1}}.$$ (97)

The main feature of Eqs.(96,97) is that they factorized two different types of weights products into a single product. Identities of this sort are essential to carry out the necessary simplifications on the transfer matrix eigenvalue problem. This task, however, will require more complex classes of products factorization which involve at least three distinct terms. In order to derive such complicated relations we have to make combinations of a special set of equations which follows
Two families of linear combinations used to derive factorization identities of three weights products. 

Table 7: Two families of linear combinations used to derive factorization identities of three weights products.

from Eq. (95). In table (7) we have summarized two families of linear combinations that will lead to three terms factorization identities.

Let us now explain how to use the linear combination described in the first row of table (7). We start by first considering the situation when row index is $k = 0$. In this case we are left with equations $Y B_{3,a,1}^{2+j,a-j,2}(\lambda_1, \lambda_2) \ (k = 0)$ for $0 \leq j \leq a - 1$ which can be combined as follows,

$$
\begin{pmatrix}
R(\lambda_1, \lambda)^{a+1}_{a+1,1} & \ldots & R(\lambda_1, \lambda)^{2}_{a,2}
\end{pmatrix}
\begin{pmatrix}
R(\lambda, \lambda_2)^{1}_{a,1}R(\lambda_1, \lambda_2)^{a+1,2}_{3,a}
\end{pmatrix}
= 
\begin{pmatrix}
v_{2,1}^{0,0}
\end{pmatrix}
$$
for $2 \leq a \leq N - 2$, (98)

The right-hand side components of the nonhomogenous system (98) are particular cases of more general elements $v_{2,1}^{k,j}$, 

$$
v_{2,1}^{k,j} = \sum_{b_2=1}^{a-j+1} R(\lambda_1, \lambda)^{a+3-b_2, b_2}_{3+k,a-k} R(\lambda_1, \lambda_2)^{2+j,a-j+2-b_2}_{a+3-b_2,1} R(\lambda, \lambda_2)^{a-j,2}_{b_2, a-j+2-b_2}, \quad \text{for } k = 0, \ldots, a - 1 \quad (99)
$$

The main purpose of the linear combination (98) is that it provides us the means to get rid of the right-hand side of equation $Y B_{3,a,1}^{2+j,a-j,2}(\lambda_1, \lambda_2)$. For the remaining values $k = 1, \ldots, a - 1$ we are therefore asked to eliminate the following terms,

$$
\sum_{b_2=1}^{a-k} R(\lambda, \lambda_2)^{b_2,a-k+1-b_2}_{a-k,1} R(\lambda_1, \lambda_2)^{a+2-b_2,2}_{3+k,a-k+1-b_2} R(\lambda_1, \lambda)^{2+j,a-j}_{a+2-b_2, b_2}, \quad (100)
$$
In order to do that we first rewrite the sum (100) in an equivalent matrix form, namely

\[
\sum_{b_2=1}^{a-k} R(\lambda, \lambda_2)^{b_2,a-k+1-b_2,1} R(\lambda_1, \lambda_2)^{a+2-b_2,2} \begin{pmatrix} R(\lambda_1, \lambda)^{2,a}_{a+2-b_2,2} \\ \vdots \\ R(\lambda_1, \lambda)^{a+1,1}_{a+2-b_2,2} \end{pmatrix} \quad \text{for } 1 \leq k \leq a-1. \tag{101}
\]

We next note that the right-hand side column of Eq. (101) is exactly the \( b_2 \) column of the matrix defining the linear system (98). If now we substitute the last column of the \( a \times a \) matrix of Eq. (98) by the sum (101) we will end up with a new matrix whose determinant is null since we are computing the sum of \( a - k \) determinants with two equal columns. Therefore, to eliminate the terms (100) for \( k = 1, \ldots, a-1 \) we just have to replace the last column of the matrix defining the linear system (98) by the left-hand side of equations \( Y B_{3+k,a-k,1}^{2+j,a-j,2}(\lambda_1, \lambda, \lambda_2) \). This leads us to the following condition,

\[
\begin{vmatrix}
R(\lambda_1, \lambda)^{2,a}_{a+1,1} & \cdots & R(\lambda_1, \lambda)^{2,a}_{3,a-1} & v^{k,0}_{2,1} \\
R(\lambda_1, \lambda)^{3,a-1}_{a+1,1} & \cdots & R(\lambda_1, \lambda)^{3,a-1}_{3,a-1} & v^{k,1}_{2,1} \\
\vdots & \ddots & \vdots & \vdots \\
R(\lambda_1, \lambda)^{a+1,1}_{a+1,1} & \cdots & R(\lambda_1, \lambda)^{a+1,1}_{3,a-1} & v^{k,a-1}_{2,1}
\end{vmatrix} = 0 \quad \text{for } 1 \leq k \leq a - 1, \tag{102}
\]

In order to have a single relation for all values of \( k \) we proceed as follows. For \( k = 0 \) we solve the linear system (98) for the last component \( R(\lambda, \lambda_2)^{a+1}_{a+2,2} \) while for \( 1 \leq k \leq a - 1 \) we single out the element \( v^{k,0}_{2,1} \) from Eq. (102). After considering these steps we find,

\[
\sum_{b_2=1}^{a+1} R(\lambda_1, \lambda)^{a+3-b_2,2}_{3+k,a-k} R(\lambda_1, \lambda_2)^{2,a+2-b_2}_{a+3-b_2,1} R(\lambda_1, \lambda_2)^{a,2}_{b_2,a+2-b_2} = \delta_{k,0} S^0_{2,0} + S^k_{2,1},
\]

for \( k = 0, \ldots, a-1 \) and \( 2 \leq a \leq N-2 \). \tag{103}

The dependence of functions \( S^0_{2,0} \) and \( S^k_{2,1} \) on the \( R \)-matrix amplitudes are

\[
S^0_{2,0} = (-1)^{a+1} R(\lambda, \lambda_2)^{a+1}_{a+1,1} R(\lambda_1, \lambda_2)^{2,2}_{3,1} \frac{D^{(a+1,2)}(\lambda, \lambda_1)}{D^{(a+1,3)}(\lambda, \lambda_1)} \tag{104}
\]
and

\[
S_{2,1}^k = (-1)^a \begin{vmatrix}
R(\lambda_1, \lambda)^{2,a}_{a+1,1} & \ldots & R(\lambda_1, \lambda)^{2,a}_{3,a-1} & 0 \\
R(\lambda_1, \lambda)^{3,a-1}_{a+1,1} & \ldots & R(\lambda_1, \lambda)^{3,a-1}_{3,a-1} & v_{2,1}^k \\
\vdots & \ddots & \vdots & \vdots \\
R(\lambda_1, \lambda)^{a+1,1}_{a+1,1} & \ldots & R(\lambda_1, \lambda)^{a+1,1}_{3,a-1} & v_{2,1}^{k,a-1}
\end{vmatrix}
D_{4}^{(a+1,3)}(\lambda, \lambda_1),
\tag{105}
\]

such that the determinant \( D_{4}^{(i,b)}(\lambda, \lambda_1) \) is defined by Eq. (92).

We now turn our attention to the linear combination associated to the second row of table (7). As before we have to make a distinction between the cases \( k = 0 \) and that involving the remaining values \( k = 1, \ldots, a - 1 \). In the former situation we have to combine equations \( YB_{3,a,1}^{4+j,a-1-j,1}(\lambda_1, \lambda, \lambda_2) \) \( (k = 0) \) with \( j = -2 \) and \( 0 \leq j \leq m\{a - 2, N - 4\} \) which can be arranged as,

\[
\begin{pmatrix}
R(\lambda_1, \lambda)^{2,a+1}_{a+2,1} & \ldots & R(\lambda_1, \lambda)^{2,a+1}_{3,a} \\
R(\lambda_1, \lambda)^{3,a-1}_{a+2,1} & \ldots & R(\lambda_1, \lambda)^{3,a-1}_{3,a} \\
\vdots & \ddots & \vdots \\
R(\lambda_1, \lambda)^{a+2,1}_{a+2,1} & \ldots & R(\lambda_1, \lambda)^{a+2,1}_{3,a}
\end{pmatrix}
\begin{pmatrix}
R(\lambda, \lambda_2)^{1,a}_{a+1}R(\lambda_1, \lambda_2)^{a+1,1}_{3,a} \\
R(\lambda, \lambda_2)^{2,a-1}_{a+1}R(\lambda_1, \lambda_2)^{a+1,1}_{3,a-1} \\
\vdots \\
R(\lambda, \lambda_2)^{a+1,1}_{a+1}R(\lambda_1, \lambda_2)^{a+1,1}_{3,a-1}
\end{pmatrix}
= \begin{pmatrix}
v_{1,1}^{0,-2} \\
v_{1,1}^{0,0} \\
\vdots \\
v_{1,1}^{0,a-2}
\end{pmatrix}
\tag{106}
\]

where the components \( v_{1,1}^{0,j} \) are obtained from the following more general elements,

\[
v_{1,1}^{k,j} = \sum_{b_2=1}^{a-j-1} R(\lambda_1, \lambda)^{a+3-b_2-b_2}_{3+k,a-k} R(\lambda_1, \lambda_2)^{4+j,a-j-b_2}_{a+3-b_2,1} R(\lambda, \lambda_2)^{a-j-1,1}_{b_2,a-j-b_2} \quad \text{for} \quad k = 0, \ldots, a - 1. \tag{107}
\]

By the same token, the corresponding algebraic manipulations for the values \( k = 1, \ldots, a - 1 \) consist in the substitution of the last column of the matrix (106) by left-hand side of equations \( YB_{3,k,a-k,1}^{4+j,a-1-j,1}(\lambda_1, \lambda, \lambda_2) \) with \( j = -2 \) and \( 0 \leq j \leq a - 2 \). This leads us to the condition,

\[
\begin{vmatrix}
R(\lambda_1, \lambda)^{2,a+1}_{a+2,1} & \ldots & R(\lambda_1, \lambda)^{2,a+1}_{4,a-1} & v_{1,1}^{k,-2} \\
R(\lambda_1, \lambda)^{3,a-1}_{a+2,1} & \ldots & R(\lambda_1, \lambda)^{3,a-1}_{4,a-1} & v_{1,1}^{k,0} \\
\vdots & \ddots & \vdots & \vdots \\
R(\lambda_1, \lambda)^{a+2,1}_{a+2,1} & \ldots & R(\lambda_1, \lambda)^{a+2,1}_{4,a-1} & v_{1,1}^{k,a-2}
\end{vmatrix}
= 0 \quad \text{for} \quad 1 \leq k \leq a - 1, \tag{108}
\]

40
Now by solving Eq. (106) for the component \( R(\lambda, \lambda)^{a,1}_2 R(\lambda_1, \lambda_2)^{3,1}_3 \) and by calculating \( v_{1,1}^k \) from Eq. (108) we find,

\[
\sum_{b_2=1}^{a+1} R(\lambda_1, \lambda)_{3+k,a-k}^{a+3-b_2,b_2} R(\lambda_1, \lambda_2)_{a+3-b_2,1}^{2,a+2-b_2} R(\lambda, \lambda_2)_{b_2,a+2-b_2}^{a+1,1} = \delta_k,0 \begin{bmatrix} S_0^0 & S_k^k \end{bmatrix},
\]

for \( k = 0, \ldots, a-1 \) and \( 2 \leq a \leq N-2 \),

\[
(109)
\]

where functions \( S_{1,0}^0 \) and \( S_{1,1}^k \) are given by

\[
S_{1,0}^0 = (-1)^a \frac{R(\lambda, \lambda)^{a,1}_2 R(\lambda_1, \lambda_2)^{3,1}_3 D_5^{(a+2,2)}(\lambda, \lambda_1)}{D_4^{(a+2,4)}(\lambda, \lambda_1)}
\]

and

\[
S_{1,1}^k = (-1)^a \frac{\begin{vmatrix}
R(\lambda_1, \lambda)^{2,a+1}_4 R(\lambda_1, \lambda_2)^{2,a+1}_2 R(\lambda, \lambda_2)^{4,a-1}_4 R(\lambda_1, \lambda_2)^{4,a-1}_2 & 0 \\
\vdots & \ddots & \vdots \\
R(\lambda_1, \lambda)^{a+2,1}_4 R(\lambda_1, \lambda_2)^{a+2,1}_2 R(\lambda, \lambda_2)^{a+2,1}_4 R(\lambda_1, \lambda_2)^{a+2,1}_2 & v_{1,1}^k \\
\end{vmatrix}}{D_4^{(a+2,4)}(\lambda, \lambda_1)},
\]

\[
(111)
\]

such that the determinant \( D_5^{(i+2,2)}(\lambda, \lambda_1) \) is defined by Eq. (94).

At this point we have at hand the basic ingredients that are necessary to obtain useful identities for the eigenvalue problem. We just have to construct the total number of ‘a’ \( 2 \times 2 \) linear systems of equations by considering the expressions (103-109). These \( a \) systems are composed by Eqs. (103-109) for \( 0 \leq k \leq a-1 \), namely

\[
\begin{bmatrix}
R(\lambda, \lambda_2)^{a,2}_2 R(\lambda_1, \lambda_2)^{a,2}_2 \\
R(\lambda, \lambda_2)^{a+1,1}_2 R(\lambda_1, \lambda_2)^{a+1,1}_2 \\
\end{bmatrix}
\times
\begin{bmatrix}
\begin{vmatrix}
R(\lambda_1, \lambda)^{2,1}_2 R(\lambda_1, \lambda)^{2,a+1}_2 & R(\lambda_1, \lambda)^{2,1}_2 R(\lambda_1, \lambda_2)^{2,a+1}_2 \\
R(\lambda_1, \lambda)^{a+1,1}_2 R(\lambda_1, \lambda_2)^{a+1,1}_2 & R(\lambda_1, \lambda)^{a+1,1}_2 R(\lambda_1, \lambda_2)^{a+1,1}_2 \\
\end{vmatrix} \\
R(\lambda_1, \lambda_2)^{a+2,2}_2 R(\lambda_1, \lambda_2)^{a+2,2}_2 \\
R(\lambda_1, \lambda_2)^{a+1,1}_2 R(\lambda_1, \lambda_2)^{a+1,1}_2 \\
\end{bmatrix} R(\lambda_1, \lambda)^{a+3-b_2,b_2}_3 R(\lambda_1, \lambda_2)^{3,a}_3 + \sum_{b_2=1}^{a-1} R(\lambda_1, \lambda_2)^{2,a+2-b_2}_2 R(\lambda_1, \lambda_2)^{3,a}_3 R(\lambda_1, \lambda_2)^{3,a}_3 + \delta_k,0 \begin{bmatrix} S_{2,1}^k & S_{1,1}^k \end{bmatrix}
= \delta_k,0 \begin{bmatrix} S_{2,0}^0 & S_{1,0}^0 \end{bmatrix},
\]

for \( 0 \leq k \leq a-1 \) and \( 2 \leq a \leq N-2 \).

\[
(112)
\]

To obtain the desirable identity we proceed as follows. We first compute the products

\[
R(\lambda_1, \lambda_2)^{2,2}_2 R(\lambda_1, \lambda_2)^{3,a}_3
\]

by applying Cramer’s rule in the system (112). This will lead us to the
following relations

\[
\sum_{b_2=1}^{a} \begin{vmatrix}
R(\lambda, \lambda_2)^{a,2}_{a+1,b_2} & R(\lambda, \lambda_2)^{a,2}_{b_2,a+2-b_2} \\
R(\lambda, \lambda_2)^{a+1,1}_{a+1,1} & R(\lambda, \lambda_2)^{a+1,2}_{a+1,a+2-b_2}
\end{vmatrix} R(\lambda_1, \lambda_2)^{2,a+2-b_2}_{a+3-b_2} R(\lambda_1, \lambda)^{a+3-b_2}_{3+k,a-k} \\
= \begin{vmatrix}
R(\lambda, \lambda_2)^{a,2}_{a+1,1} S^k_{2,1} & \delta_{k,0} \\
R(\lambda, \lambda_2)^{a+1,1}_{a+1,1} S^k_{1,1} & R(\lambda, \lambda_2)^{a+1,1}_{a+1,1} S^0_{0,0}
\end{vmatrix},
\]

for \(0 \leq k \leq a-1\) and \(2 \leq a \leq N-2\). \hspace{1cm} (113)

The next step is to solve the linear system (113) for the variable \(b_2 = a\) which allows us to eliminate function \(D^{(a,0)}_2(\lambda, \lambda_2) R(\lambda_1, \lambda_2)^{2,2}_{3,1}\). By normalizing the result by \(D^{(a+2,3)}_4(\lambda, \lambda_1) R(\lambda, \lambda_2)^{a+1,1}_{a,1} R(\lambda, \lambda_2)^{3,1}_{a+1,1} R(\lambda_1, \lambda_2)^{3,1}_{3,1}\) and after few manipulations we find that,

\[
D^{(a,0)}_2(\lambda, \lambda_2) \frac{R(\lambda_1, \lambda_2)^{2,2}_{3,1}}{R(\lambda_1, \lambda_2)^{3,1}_{3,1}} = -\frac{R(\lambda, \lambda_2)^{a,2}_{a+1,1} D^{(a+2,2)}_5(\lambda, \lambda_1)}{R(\lambda, \lambda_2)^{a+1,1}_{a+1,1} D^{(a+2,3)}_4(\lambda, \lambda_1)} + \frac{D^{(a+1,2)}_5(\lambda, \lambda_1) R(\lambda, \lambda_2)^{a-1,2}_{a,1}}{D^{(a+1,3)}_4(\lambda, \lambda_1)} + \frac{R(\lambda_1, \lambda_2)^{2,2}_{3,1} D^{(a+1,2)}_4(\lambda, \lambda_1)}{R(\lambda_1, \lambda_2)^{3,1}_{3,1} D^{(a+1,3)}_4(\lambda, \lambda_1)} \frac{D^{(a+2,4)}_4(\lambda, \lambda_1)}{D^{(a+2,3)}_4(\lambda, \lambda_1)}
\]

for \(2 \leq a \leq N-2\). \hspace{1cm} (114)

We finally discuss the special situation when \(a = N-1\). It turns out that the steps in this case are fairly parallel to those already described for \(2 \leq a \leq N-2\). The respective identity is obtained from Eq. (114) by choosing \(a = N-1\) except for the terms involving the determinants \(D^{(N+1,b)}_4(\lambda, \lambda_1)\) and \(D^{(N+1,2)}_5(\lambda, \lambda_1)\). These determinants are then replaced by the following analytic continuations,

\[
D^{(N+1,b)}_4(\lambda, \lambda_1) = \lim_{a \to N-1} \frac{D^{(a+2,b)}_4(\lambda, \lambda_1)}{R(\lambda_1, \lambda)^{a+2,1}_{a+2,1}} = (-1)^{N+1-b} \begin{vmatrix}
R(\lambda_1, \lambda)^{b,N+2-b}_{N,2} & \ldots & R(\lambda_1, \lambda)^{N,2}_{N,2} \\
\vdots & \ddots & \vdots \\
R(\lambda_1, \lambda)^{b,N+2-b}_{b,N+2-b} & \ldots & R(\lambda_1, \lambda)^{N,2}_{b,N+2-b}
\end{vmatrix}
\]

(115)
and
\[ D_{5}^{(N+1,2)}(\lambda, \lambda_{1}) \equiv \lim_{a \to N-1} \frac{D_{5}^{a+2,2}(\lambda, \lambda_{1})}{R(\lambda_{1}, \lambda)^{a+2,1}}. \]

\[ = (-1)^{N} \begin{vmatrix} R(\lambda_{1}, \lambda)^{2,N}_{N,2} & R(\lambda_{1}, \lambda)^{4,N}_{N,2} & \ldots & R(\lambda_{1}, \lambda)^{N,2}_{N,2} \\ \vdots & \vdots & \ddots & \vdots \\ R(\lambda_{1}, \lambda)^{2,N}_{3,N-1} & R(\lambda_{1}, \lambda)^{4,N-2}_{3,N-1} & \ldots & R(\lambda_{1}, \lambda)^{N,2}_{3,N-1} \end{vmatrix}. \quad (116) \]

Though the element \( R(\lambda_{1}, \lambda)^{N+1,1}_{N+1,1} \) in Eqs. (115,116) are not defined we observe that in the identity (114) such determinants always appear as ratios \( \frac{D_{4}^{(N+1,4)}(\lambda, \lambda_{1})}{D_{4}^{(N+1,3)}(\lambda, \lambda_{1})} \) or \( \frac{D_{4}^{(N+1,2)}(\lambda, \lambda_{1})}{D_{4}^{(N+1,3)}(\lambda, \lambda_{1})} \). This means that the common element \( R(\lambda_{1}, \lambda)^{N+1,1}_{N+1,1} \) will be canceled out in such ratios. We end by mentioning that the very particular case \( a = 2, e N = 3 \) is obtained by setting \( D_{4}^{(4,4)}(\lambda, \lambda_{1}) = 1. \)

5 The eigenvalue problem

In this section we shall consider the diagonalization of the transfer matrix (111) related to the inhomogeneous \( U(1) \) vertex model defined in section 2. The \( U(1) \) invariance of the respective statistical weights implies that the transfer matrix (111) commutes with the total spin operator,

\[ [T(\lambda), \sum_{i=1}^{L} S_{z}^{i}] = 0. \quad (117) \]

A direct consequence of Eq.(117) is that the Hilbert space can be separated in disjoint sectors labeled by the eigenvalues of the spin operator, namely

\[ \sum_{i=1}^{L} S_{z}^{i} |\Phi_{n}\rangle = [Ls - n] |\Phi_{n}\rangle, \quad (118) \]

where \( |\Phi_{n}\rangle \) denotes the eigenvector on the \( n \)-th sector.

In terms of the monodromy matrix elements the transfer matrix eigenvalue problem can then be defined by,

\[ \sum_{a=1}^{N} T_{a,a}(\lambda) |\Phi_{n}\rangle = \Lambda_{n}(\lambda) |\Phi_{n}\rangle. \quad (119) \]
Here we are going to solve the eigenvalue problem (119) by solely relying on the three different families of commutation rules of section 3, the identities between the $R$-matrix elements of section 4 as well as on the pseudovacuum state properties (12). To this end we will seek the eigenvectors $|\Phi_n\rangle$ in the form of multiparticle states which can formally be represented by the following expression,

$$|\Phi_n\rangle = \phi_n(\lambda_1, \ldots, \lambda_n)|0\rangle,$$

where the rapidities $\lambda_1, \ldots, \lambda_n$ parameterize the momenta of the particles.

For consistency the vector with null number of particles $\phi_0 \equiv 1$ is directly identified to the reference state $|0\rangle$. The property (16) concerning the spin of the monodromy matrix elements suggests us that the mathematical structure of the vectors $\phi_n(\lambda_1, \ldots, \lambda_n)$ for general $n$ can be searched with the help of linear combination of the product of the creation operators. In what follows we are going to argue that such vectors are indeed given in terms of a recurrence relation involving the $N - 1$ creation fields $T_{1,\ell}(\lambda)$.

### 5.1 The one-particle problem

The one-particle state $n = 1$ corresponds to an excitation of spin $s - 1$ over the reference state $|0\rangle$. From property (16) we see that among the basis vectors $T_{1,\ell}(\lambda)$ the operator responsible for this excitation is $T_{1,2}(\lambda)$. The one-particle vector $\phi_1(\lambda_1)$ is then given by,

$$\phi_1(\lambda_1) = T_{1,2}(\lambda_1).$$

The solution of the eigenvalue problem (119) for the one-particle state is achieved by using the commutation rules between the operators $T_{a,a}(\lambda)$ and $T_{1,2}(\lambda)$. Taking into account the form
of Eqs. (27-29) as well as the pseudovacuum properties (12) we find,

$$ T_{1,1}(\lambda) |\Phi_1\rangle = w_1(\lambda) \frac{R(\lambda_1, \lambda)_{1,1}^{1,1}}{R(\lambda_1, \lambda)_{2,1}^{1,1}} |\Phi_1\rangle - w_1(\lambda_1) \frac{R(\lambda_1, \lambda)_{1,1}^{2,1}}{R(\lambda_1, \lambda)_{2,1}^{2,1}} T_{1,2}(\lambda) |0\rangle , \quad (122) $$

$$ T_{a,a}(\lambda) |\Phi_1\rangle = w_a(\lambda) D_2^{(a,0)}(\lambda, \lambda_1) |\Phi_1\rangle + w_1(\lambda_1) \frac{R(\lambda, \lambda_1)_{a+1,1}^{a,1}}{R(\lambda, \lambda_1)_{a+1,1}^{a+1,1}} T_{a,a+1}(\lambda) |0\rangle $$

$$ - w_2(\lambda_1) \frac{R(\lambda, \lambda_1)_{a+1,1}^{a-1,2}}{R(\lambda, \lambda_1)_{a,1}^{a-1,2}} T_{a-1,a}(\lambda) |0\rangle , \quad \text{for} \quad 2 \leq a \leq N - 1, \quad (123) $$

$$ T_{N,N}(\lambda) |\Phi_1\rangle = w_N(\lambda) \frac{R(\lambda, \lambda_1)_{N,2}^{N,2}}{R(\lambda, \lambda_1)_{N,1}^{N,1}} |\Phi_1\rangle - w_2(\lambda_1) \frac{R(\lambda, \lambda_1)_{N-1,1}^{N-1,2}}{R(\lambda, \lambda_1)_{N,1}^{N-1,2}} T_{N-1,N}(\lambda) |0\rangle . \quad (124) $$

It is possible to gather Eqs. (122,124) into a single expression for all values of the diagonal index $a$. This requires first the definition of a discrete function that is able to project out a set of indices. Keeping in mind the extension to multi-particle states we shall define this function by,

$$ \delta_{i,j_1,\ldots,j_n}^j = 1 - \sum_{k=1}^n \delta_{i,j_k} , \quad (125) $$

By using identity (79) to reorder the rapidities of the last term in Eq. (122) we find that Eqs. (122,124) can be recast as,

$$ T_{a,a}(\lambda) |\Phi_1\rangle = w_a(\lambda) P_a(\lambda, \lambda_1) |\Phi_1\rangle - \delta_{a,a+1}^N T_{a,a+1}(\lambda) w_1(\lambda_1) \mathcal{F}_1^{(a)}(\lambda, \lambda_1) |0\rangle $$

$$ - \delta_{a,a-1}^1 T_{a-1,a}(\lambda) w_2(\lambda_1) \bar{\mathcal{F}}_1^{(a-1)}(\lambda, \lambda_1) |0\rangle , \quad \text{for} \quad 1 \leq a \leq N, \quad (126) $$

The polynomials $P_a(\lambda, \mu)$ are the terms proportional to the eigenstate $|\Phi_1\rangle$ in Eqs. (122,124), namely

$$ P_a(\lambda, \mu) = \begin{cases} R(\mu, \lambda)_{1,1}^{1,1}, & \text{for} \quad a = 1 \\ \frac{R(\mu, \lambda)_{2,1}^{2,1}}{R(\mu, \lambda)_{2,1}^{2,1}} & \text{for} \quad 2 \leq a \leq N - 1 \\ D_2^{(a,0)}(\lambda, \mu), & \text{for} \quad 2 \leq a \leq N - 1 \\ R(\lambda, \mu)_{N,2}^{N,2}, & \text{for} \quad a = N. \end{cases} \quad (127) $$

The auxiliary functions $\mathcal{F}_1^{(a)}(\lambda, \mu)$ and $\bar{\mathcal{F}}_1^{(a)}(\lambda, \mu)$ are proportional to the terms not parallel to the one-particle state. This notation for these off-shell amplitudes have been defined for later
convenience. They are given by,

\[ 0 \mathcal{F}^{(a)}_1(\lambda, \mu) = -1 \mathcal{F}^{(a)}_1(\lambda, \mu) = \frac{R(\lambda, \mu_{a+1,1})^2}{R(\lambda, \mu_{a+1,1})} \]  

(128)

The solution of the eigenvalue problem is obtained by summing Eq.(128) over the index \( a \). This gives us the action of the transfer matrix on the one-particle state,

\[ T(\lambda) |\Phi_1\rangle = \sum_{a=1}^{N} w_a(\lambda) P_a(\lambda, \lambda_1) |\Phi_1\rangle + \left[w_1(\lambda_1) - w_2(\lambda_1)\right] \sum_{a=1}^{N-1} 0 \mathcal{F}^{(a)}_1(\lambda, \lambda_1) T_{a,a+1}(\lambda) |0\rangle . \]  

(129)

We see that the terms \( T_{a,a+1}(\lambda) |0\rangle \) for \( a = 1, \cdots, N \) are undesirable states because they are not proportional to the one-particle eigenstate \( |\Phi_1\rangle \). Note that these states are built up from the many possible creation fields possessing azimuthal spin \( s = 1 \). All of them are eliminated by imposing that the rapidity \( \lambda_1 \) satisfies the one-particle Bethe equation,

\[ \frac{w_1(\lambda_1)}{w_2(\lambda_1)} = 1. \]  

(130)

which implies that the corresponding one-particle eigenvalue is,

\[ \Lambda_1(\lambda) = \sum_{a=1}^{N} w_a(\lambda) P_a(\lambda, \lambda_1) , \]  

(131)

We would like to close this section by commenting on the meaning of the notation we have introduced for the off-shell function \( \mathcal{F}^{(a)}_{b-a}(\lambda, \mu) \). The two right-hand side indices \( a \) and \( b-a \) are directly related to the companion unwanted operator \( T_{a,b}(\lambda) \), making explicit its azimuthal spin value \( s_{a,b} = b-a \). The remaining index \( c \) accounts for the number of weight \( w_1(\lambda_i) \) that is present in the respective undesirable term of form \( \prod_{i=1}^{c} w_1(\lambda_i) T_{a,b}(\lambda) |0\rangle \). We shall see that this is indeed a suitable notation to accommodate similar results for general multiparticle states.

5.2 The two-particle problem

The two-particle state lies on the \( n = 2 \) sector and is constructed by considering the possible excitations with spin \( s = 2 \). From Eq.(129) we conclude that all products of form \( T_{1,k_1}(\lambda_{m_1}) T_{1,k_2}(\lambda_{m_2}) \)
where indices is constrained by \( k_1 + k_2 = 4 \) can in principle contribute to such state. Therefore, the most general ansatz for vector \( \varphi_2(\lambda_1, \lambda_2) \) one could start with is,

\[
\varphi_2(\lambda_1, \lambda_2) = \sum_{m_1, m_2} \sum_{k_1, k_2} \bar{c}^{(m_1, m_2)}_{k_1, k_2}(\lambda_{m_1}, \lambda_{m_2}) T_{1, k_1}(\lambda_{m_1}) T_{1, k_2}(\lambda_{m_2}), \tag{132}
\]

where \( \bar{c}^{(m_1, m_2)}_{k_1, k_2}(\lambda_{m_1}, \lambda_{m_2}) \) are the coefficients of an arbitrary linear combination.

However, the commutation rules of the operators \( T_{1,1}(\lambda) \) and \( T_{1,3}(\mu) \) as well as between \( T_{1,2}(\lambda) \) and \( T_{1,2}(\mu) \) shows us that not all the terms entering in Eq.\( \varphi_2(\lambda_1, \lambda_2) \) are linearly independent. In fact, by using the commutation relations (31,46) we note that the products \( T_{1,1}(\lambda_1) T_{1,3}(\lambda_2), T_{1,1}(\lambda_2) T_{1,3}(\lambda_1) \) and \( T_{1,2}(\lambda_2) T_{1,2}(\lambda_1) \) can be eliminated from the linear combination (132). Taking this observation into account we conclude that an educated ansatz for the two-particle vector is,

\[
\varphi_2(\lambda_1, \lambda_2) = T_{1,2}(\lambda_1) T_{1,2}(\lambda_2) + g_{1}^{(1)}(\lambda_1, \lambda_2) T_{1,3}(\lambda_1) T_{1,1}(\lambda_2) + g_{2}^{(2)}(\lambda_1, \lambda_2) T_{1,3}(\lambda_2) T_{1,1}(\lambda_1), \tag{133}
\]

where \( g_{i}^{(i)}(\lambda_1, \lambda_2) \) are arbitrary functions to be determined.

The next task is to investigate under which conditions the ansatz (133) is able to provide us the two-particle transfer matrix eigenstates. In other to do that we have to commute the matrix elements \( T_{a,a}(\lambda) \) with the creation operators present in the two-particle ansatz \( \varphi_2(\lambda_1, \lambda_2) \). Let us start by describing the steps necessary to disentangle the action of \( T_{a,a}(\lambda) \) on the first term of the two-particle ansatz (133). In this case we have to use all the three classes of commutation rules presented in section 3. The first step is to move the diagonal operators through the creation fields \( T_{1,2}(\lambda_1) \) and \( T_{1,2}(\lambda_2) \) with the help of relations (27-29). This generates products between creation fields \( T_{1,2}(\lambda) \) and \( T_{a-1,a}(\mu) \) that are not ordered in the unique desirable form \( T_{a-1,a}(\lambda) T_{1,2}(\lambda_1) \) or \( T_{a-1,a}(\lambda) T_{1,2}(\lambda_2) \) for \( a = 2, \ldots, N \). This common rapidity ordering are then established by using the commutation rules (38,41) among the fields \( T_{1,2}(\lambda_1) \) and \( T_{a-1,a}(\lambda) \). The final step concerns with the presence of annihilation fields on the middle of products involving three monodromy matrix elements that are needed to be carried out to the further right in order to act on the reference state \( |0\rangle \). This is sorted out by employing the commutation rules (52) between the annihilation fields with the creation operator \( T_{1,2}(\lambda_2) \).
the two-particle ansatz (133) is however much simpler. In this case we should not worry about
the right-hand side diagonal field $T_{1,1}(\lambda_i)$ since its action on the reference state $|0\rangle$ is known to
be given by Eq.(12). Therefore, the respective computation involves only the set of commutation
rules among the operators $T_{a,a}(\lambda)$ and $T_{1,3}(\lambda_i)$ for $i = 1, 2$. This task is easily accomplished with
the help of Eqs.(31132).

A direct consequence of the above described commutations is the generation of several types
of terms that are not proportional to none of the terms present in the two-particle ansatz (133).
Among many different types of such undesirable terms there exists a very special family that need
to be single out first. This family of unwanted terms are always produced by the same diagonal
operator $T_{a,a}(\lambda)$ for $a = 2, \ldots, N$ and carries a dependence either on the couplings $g_2^{(i)}(\lambda_1, \lambda_2)$ or on
the specific amplitude ratio $R_{3,1}^{2,2}(\lambda_1, \lambda_2)/R_{3,1}^{3,1}(\lambda_1, \lambda_2)$. Therefore, the elimination of such unwanted
terms are only possible after an appropriate choice of the coefficients $g_2^{(i)}(\lambda_1, \lambda_2)$. From now on we
shall refer to the class of undesirable terms that are responsible for fixing the linear combination
as easy unwanted terms. In the two-particle problem there are four different terms of this sort
and their structures are,

\begin{align}
\bullet w_1(\lambda_2) & \frac{R(\lambda, \lambda_1 \lambda_1)_{a,1}^{a-1,2}}{R(\lambda, \lambda_1)_{a,1}^{\alpha,1}} \left[ \frac{R(\lambda_1, \lambda_2 \lambda_2)_{3,1}^{2,2}}{R(\lambda_1, \lambda_2)_{3,1}^{\alpha,1}} + g_2^{(1)}(\lambda_1, \lambda_2) \right] T_{a-1,a}(\lambda) T_{2,3}(\lambda_1) \\
\bullet w_1(\lambda_2) w_3(\lambda_1) & \frac{R(\lambda, \lambda_1 \lambda_1)_{a,1}^{a-2,3}}{R(\lambda, \lambda_1)_{a,1}^{\alpha,1}} \left[ \frac{R(\lambda_1, \lambda_2 \lambda_2)_{3,1}^{2,2}}{R(\lambda_1, \lambda_2)_{3,1}^{\alpha,1}} + g_2^{(1)}(\lambda_1, \lambda_2) \right] T_{a-2,a}(\lambda) \\
\bullet w_1(\lambda_1) & \frac{R(\lambda, \lambda_2 \lambda_2)_{a,1}^{a-1,2}}{R(\lambda, \lambda_2)_{a,1}^{\alpha,1}} g_2^{(2)}(\lambda_1, \lambda_2) T_{a-1,a}(\lambda) T_{2,3}(\lambda_2) \\
\bullet w_1(\lambda_1) w_3(\lambda_2) & \frac{R(\lambda, \lambda_2 \lambda_2)_{a,1}^{a-2,3}}{R(\lambda, \lambda_2)_{a,1}^{\alpha,1}} g_2^{(2)}(\lambda_1, \lambda_2) T_{a-2,a}(\lambda) 
\end{align}

(134)

From Eq.(134) we conclude that all the above easy unwanted terms are canceled out provided
we choose functions $g_2^{(i)}(\lambda_1, \lambda_2)$ as

\begin{equation}
g^{(1)}(\lambda_1, \lambda_2) = - \frac{R(\lambda_1, \lambda_2)_{3,1}^{2,2}}{R(\lambda_1, \lambda_2)_{3,1}^{\alpha,1}}, \quad g^{(2)}(\lambda_1, \lambda_2) = 0.
\end{equation}

(135)

Now, by comparing Eq.(128) with Eq.(135) we see that the coupling $g^{(1)}(\lambda_1, \lambda_2)$ is exactly the
amplitude $1\mathbf{F}_1^{(2)}(\lambda_1, \lambda_2)$ introduced in the one-particle problem. This means that the two-particle
vector can be written as,

$$\phi_2(\lambda_1, \lambda_2) = T_{1,2}(\lambda_1)T_{1,2}(\lambda_2) + _1F_{1}^{(2)}(\lambda_1, \lambda_2)T_{1,3}(\lambda_1)T_{1,1}(\lambda_2). \quad (136)$$

The two-particle vector (136) has the nice feature of being ordered as far as the rapidity $\lambda_1$ is concerned. Another important property is its symmetry under the permutation of the rapidities $\lambda_1$ and $\lambda_2$. This property is derived by substituting the commutation rule (46) for the operators $T_{1,2}(\lambda_1)$ and $T_{1,2}(\lambda_2)$ in Eq. (136) and by considering the definition of $_1F_{1}^{(2)}(\lambda_1, \lambda_2)$ given in Eq. (128). As a result we find,

$$\phi_2(\lambda_1, \lambda_2) = \theta(\lambda_1, \lambda_2) \left[ T_{1,2}(\lambda_2)T_{1,2}(\lambda_1) + \frac{D_2^{(2,1)}(\lambda_1, \lambda_2)}{D_2^{(2,0)}(\lambda_1, \lambda_2)}T_{1,3}(\lambda_2)T_{1,1}(\lambda_1) \right], \quad (137)$$

where the exchange function $\theta(\lambda, \mu)$ is,

$$\theta(\lambda, \mu) = D_2^{(2,0)}(\lambda, \mu) \frac{R(\lambda, \mu)^{2,1}_{1,1}}{R(\lambda, \mu)^{1,1}_{1,1}} = \begin{vmatrix} R(\lambda, \mu)^{2,2}_{2,2} & R(\lambda, \mu)^{2,2}_{3,1} \\ R(\lambda, \mu)^{3,1}_{2,2} & R(\lambda, \mu)^{3,1}_{3,1} \\ R(\lambda, \mu)^{1,1}_{2,2} & R(\lambda, \mu)^{1,1}_{3,1} \end{vmatrix}. \quad (138)$$

The proof of the permutation symmetry is completed by using two identities coming from the unitarity property derived in section 4. In fact, if we substitute the identity (85) in Eq. (137) we find that $\phi_2(\lambda_1, \lambda_2)$ satisfies the exchange property,

$$\phi_2(\lambda_1, \lambda_2) = \theta(\lambda_1, \lambda_2)\phi_2(\lambda_2, \lambda_1), \quad (139)$$

while the second identity (84) assures us the consistency condition,

$$\theta(\lambda_1, \lambda_2)\theta(\lambda_2, \lambda_1) = 1. \quad (140)$$

The exchange symmetry is very helpful to simplify the expressions for the action of the diagonal operators $T_{a,a}(\lambda)$ on the two-particle state $|\Phi_2\rangle$. It makes possible to represent the various distinct contributions to the unwanted terms, generated by the commutation between $T_{a,a}(\lambda)$ and $\phi_2(\lambda_1, \lambda_2)$, in a compact and illuminating form. The details concerning such simplifications are rather cumbersome and for this reason they have been summarized in Appendix A. In particular,
we show how the identities derived in section 4 are very useful to write a recursive formula for the
two-particle off-shell amplitudes. In what follows we shall present the main conclusions that can
be reached from the results of Appendix A. Considering the discrete function (125) we find that
\( T_{a,a}(\lambda) |\Phi_2\rangle \) can be written as,
\[
T_{a,a}(\lambda) |\Phi_2\rangle = w_a(\lambda) \prod_{i=1}^{2} P_a(\lambda, \lambda_i) |\Phi_2\rangle \\
- \bar{\delta}^N a_{a+1}(\lambda) \sum_{i,j=1 \atop j \neq i}^{2} \phi_1(\lambda_j) w_{1}(\lambda_i) F_1^{(a)}(\lambda, \lambda_i) R(\lambda_j, \lambda_i) \frac{1}{R(\lambda_j, \lambda_i) \theta_<(\lambda_j, \lambda_i) |0\rangle}
- \bar{\delta}^1 a_{1,a}(\lambda) \sum_{i,j=1 \atop j \neq i}^{2} \phi_1(\lambda_j) w_{2}(\lambda_i) F_1^{(a-1)}(\lambda, \lambda_i) R(\lambda_j, \lambda_i) \frac{1}{R(\lambda_j, \lambda_i) \theta_<(\lambda_j, \lambda_i) |0\rangle}
- \bar{\delta}^{N-1} a_{a+2}(\lambda) w_{1}(\lambda_1) w_{1}(\lambda_2) F_2^{(a)}(\lambda, \lambda_1, \lambda_2) |0\rangle
- \bar{\delta}^{N-1} a_{1,a}(\lambda) \sum_{i,j=1 \atop j \neq i}^{2} w_{1}(\lambda_i) w_{2}(\lambda_j) F_2^{(a-1)}(\lambda, \lambda_i, \lambda_j) \theta_<(\lambda_j, \lambda_i) |0\rangle
- \bar{\delta}^{1,2} a_{a-2,a}(\lambda) w_{2}(\lambda_1) w_{2}(\lambda_2) F_2^{(a-2)}(\lambda, \lambda_1, \lambda_2) |0\rangle, \quad 1 \leq a \leq N. \tag{141}
\]
where function \( \theta_<(\lambda_i, \lambda_j) \) is defined by,
\[
\theta_<(\lambda_i, \lambda_j) = \begin{cases} 
\theta(\lambda_i, \lambda_j), & \text{for } i < j \\
1, & \text{for } i \geq j.
\end{cases} \tag{142}
\]

The off-shell functions \( F_2^{(a)}(\lambda, \lambda_1, \lambda_2) \) for \( c = 0, 1, 2 \) represent the three distinct contributions
of unwanted operators carrying azimuthal spin \( s = 2 \). In appendix A we have shown that they
are given in terms of the following recursive relations,
\[
0 F_2^{(a)}(\lambda, \lambda_1, \lambda_2) = \frac{R(\lambda, \lambda_1)^{a,2}}{R(\lambda, \lambda_1)^{a+2,1}} F_1^{(a+1)}(\lambda, \lambda_2) + \frac{R(\lambda, \lambda_1)^{a,3}}{R(\lambda, \lambda_1)^{a+2,1}} F_1^{(2)}(\lambda, \lambda_2) \tag{143}
\]
\[
1 F_2^{(a)}(\lambda, \lambda_1, \lambda_2) = 0 F_1^{(a)}(\lambda, \lambda_2) F_1^{(a+1)}(\lambda, \lambda) \frac{R(\lambda, \lambda_1)^{1,1}}{R(\lambda_2, \lambda_1)^{2,1}} \tag{144}
\]
\[
2 F_2^{(a)}(\lambda, \lambda_1, \lambda_2) = - F_2^{(a)}(\lambda, \lambda_1, \lambda_2) - \sum_{i,j=1 \atop j \neq i}^{2} F_2^{(a)}(\lambda, \lambda_i, \lambda_j) \frac{R(\lambda, \lambda_i)^{2,1}}{R(\lambda, \lambda_i)^{1,1}} \frac{R(\lambda, \lambda_j)^{1,1}}{R(\lambda_2, \lambda_1)^{2,1}} \theta_<(\lambda_i, \lambda_j). \tag{145}
\]
The symmetry of the two-particle vector under the exchange of rapidities is expected to be reflected in functions $\mathcal{F}_2^{(a)}(\lambda, \lambda_1, \lambda_2)$ and $\mathcal{F}_2^{(a)}(\lambda, \lambda_1, \lambda_2)$. In fact, from Eq.(141) we observe that such functions are proportional to products that do not contain function $\theta(\lambda_1, \lambda_2)$. Therefore, the invariance of the two-particle state $\vert \Phi_2 \rangle$ by the permutation $\lambda_1 \leftrightarrow \lambda_2$ implies that

$$c \mathcal{F}_2^{(a)}(\lambda, \lambda_1, \lambda_2) = \theta(\lambda_1, \lambda_2)c \mathcal{F}_2^{(a)}(\lambda, \lambda_2, \lambda_1) \quad \text{for} \quad c = 0, 2. \quad (146)$$

We now pause to comment on the results we have so far obtained for the two-particle problem. The alert reader will observe that the role of the discrete functions $\delta_{a}^{i}$ and $\delta_{a}^{j_1,j_2}$ are merely to project out undefined monodromy matrix elements. These are operators having either the form $\mathcal{T}_{a-i,a}(\lambda)$ with $a-i < 1$ or $\mathcal{T}_{a,a+i}(\lambda)$ with $a+i > N$ such that $i = 1, 2$. This remark suggests Eq.(141) can still be written in a more compact manner. Indeed, we find that Eq.(141) is equivalent to the following expression,

$$\mathcal{T}_{a,a}(\lambda) \vert \Phi_2 \rangle = w_a(\lambda) \prod_{i=1}^{2} P_a(\lambda, \lambda_i) \vert \Phi_2 \rangle - \sum_{t=1}^{2} \sum_{p=M\{0, a-t-N\}}^{2} \mathcal{T}_{a-p,a+t-p}(\lambda)$$

$$\times \sum_{1 \leq j_1 < \cdots < j_{t-p} \leq 2}^{t-p} \phi_{2-t}(\{\lambda_i\}_{i=1, j_1, \ldots, j_{t-p}}) \mathcal{T}_{a-p,a+t-p}(\lambda)$$

$$\times \left( \prod_{k=1}^{t-p} w_1(\lambda_{j_k}) \prod_{i=1}^{2} R(\lambda_i, \lambda_{j_k})^{1,1}_{1,1} \theta_{<(\lambda_i, \lambda_{j_k})} \right)$$

$$\times \left( \prod_{l=t-p+1}^{t} w_2(\lambda_{j_l}) \prod_{i=1}^{2} R(\lambda_l, \lambda_{j_l})^{1,1}_{2,1} \theta_{<(\lambda_j, \lambda_{j_l})} \right)$$

$$\prod_{k=1}^{t-p} \prod_{l=t-p+1}^{t} \theta_{<(\lambda_j, \lambda_{j_l})} \vert 0 \rangle, \quad (147)$$

where $\ast$ in the sum means that any terms with $j_k = j_l$ for $l \in \{t - p + 1, \ldots, t\}$ and $k \in \{1, \ldots, t - p\}$ are different. Moreover, the notation $\{\lambda_i\}_{i=1, \ldots, n}$ represents that out of the many possible rapidities $\lambda_1, \ldots, \lambda_n$ those indexed by $\lambda_{j_1}, \ldots, \lambda_{j_p}$ are not allowed in the set.

The latter results contains all the basic ingredients we need to solve the two-particle problem. The solution of the transfer matrix eigenvalue problem consists basically in taking the sum of
either Eq.(141) or Eq.(147) over the diagonal index $a$. By performing few rearrangements in this sum we find that,

\[
T(\lambda) |\Phi_2\rangle = w_a(\lambda) \prod_{i=1}^{2} P_a(\lambda, \lambda_i) |\Phi_2\rangle \\
- \sum_{a=1}^{N-1} \sum_{i,j=1}^{2} \left[ w_1(\lambda_i)_{1} \mathcal{F}_1^{(a)}(\lambda, \lambda_i) R(\lambda_j, \lambda_i)_{1,1}^{1,1} + w_2(\lambda_i)_{0} \mathcal{F}_1^{(a)}(\lambda, \lambda_i) R(\lambda_i, \lambda_j)_{2,1}^{1,1} \theta(\lambda_i, \lambda_j) \right] \\
\times \theta_<(\lambda_j, \lambda_i) T_{a,a+1}(\lambda) \phi_1(\lambda_j) |0\rangle \\
- \sum_{a=1}^{N-2} \left[ w_1(\lambda_1) w_1(\lambda_2)_{2} \mathcal{F}_2^{(a)}(\lambda, \lambda_1, \lambda_2) + \sum_{i,j=1}^{2} w_1(\lambda_i) w_2(\lambda_j)_{1} \mathcal{F}_2^{(a)}(\lambda, \lambda_i, \lambda_j) \theta_<(\lambda_j, \lambda_i) \right] \\
+ w_2(\lambda_1) w_2(\lambda_2)_{0} \mathcal{F}_2^{(a)}(\lambda, \lambda_1, \lambda_2) \right] T_{a,a+2}(\lambda) |0\rangle ,
\]

(148)

From Eq.(148) we can see the possible classes of unwanted terms that should be canceled out.

The first term come as direct extension of the structure present in the off-shell one-particle state. The second term is inherent to the two-particle state and corresponds to the three possible manners of constructing unwanted terms with azimuthal spin $s = 2$ of type $T_{a,a+2}(\lambda) w_i(\lambda_1) w_j(\lambda_2) |0\rangle$ for $i, j = 1, 2$. In order to cancel all such unwanted terms we have to use the properties of the auxiliary functions $\mathcal{F}_1^{(a)}(\lambda, \lambda_1)$ and $\mathcal{F}_2^{(a)}(\lambda, \lambda_1, \lambda_2)$ given by Eqs.(128, 145). More specifically, considering that $\mathcal{F}_1^{(a)}(\lambda, \lambda_1) = -\theta \mathcal{F}_1^{(a)}(\lambda, \lambda_1)$ as well as by substituting the amplitude $\mathcal{F}_2^{(a)}(\lambda, \lambda_1, \lambda_2)$ in
Eq. (148) we find,

\[
T(\lambda) |\Phi_2\rangle = w_a(\lambda) \prod_{i=1}^{2} P_a(\lambda, \lambda_i) |\Phi_2\rangle \\
+ \sum_{a=1}^{N-1} \sum_{i,j=1}^{2} \left[ w_1(\lambda_j) \frac{R(\lambda_i, \lambda_j)_1^1}{R(\lambda_i, \lambda_j)_2^2} - w_2(\lambda_j) \frac{R(\lambda_j, \lambda_i)_1^1}{R(\lambda_j, \lambda_i)_2^2} \theta(\lambda_j, \lambda_i) \right] \mathcal{F}_1^{(a)}(\lambda, \lambda_j) \\
\times \theta(\lambda_i, \lambda_j) T_{a,a+1}(\lambda) \phi_1(\lambda_i) |0\rangle \\
+ \sum_{a=1}^{N-2} \left\{ [w_1(\lambda_1)w_1(\lambda_2) - w_2(\lambda_1)w_2(\lambda_2)] \mathcal{F}_2^{(a)}(\lambda, \lambda_1, \lambda_2) \\
+ \sum_{i,j=1}^{2} w_1(\lambda_i) \left[ w_1(\lambda_j) \frac{R(\lambda_i, \lambda_j)_1^1}{R(\lambda_i, \lambda_j)_2^2} - w_2(\lambda_j) \frac{R(\lambda_j, \lambda_i)_1^1}{R(\lambda_j, \lambda_i)_2^2} \theta(\lambda_j, \lambda_i) \right] \frac{R(\lambda_j, \lambda_i)_1^1}{R(\lambda_j, \lambda_i)_2^2} \\
\times \mathcal{F}_2^{(a)}(\lambda, \lambda_i, \lambda_j) \theta(\lambda_i, \lambda_j) \right\} T_{a,a+2}(\lambda) |0\rangle .
\]

From Eq. (149) we see that the unwanted terms either proportional to \( T_{a,a+1}(\lambda) \phi_1(\lambda_j) |0\rangle \) or to \( T_{a,a+2}(\lambda) |0\rangle \) vanish provided that the rapidities \( \lambda_1 \) and \( \lambda_2 \) satisfy the following Bethe ansatz equation:

\[
\frac{w_1(\lambda_j)}{w_2(\lambda_j)} = \prod_{i=1}^{2} \theta(\lambda_j, \lambda_i) \frac{R(\lambda_j, \lambda_i)_1^1}{R(\lambda_j, \lambda_i)_2^2} \frac{R(\lambda_j, \lambda_i)_2^2}{R(\lambda_j, \lambda_i)_1^1} \quad \text{for} \quad j = 1, 2.
\]

and consequently the two-particle eigenvalue is,

\[
\lambda_2(\lambda) = \sum_{a=1}^{N} \omega(\lambda) \prod_{i=1}^{2} P_a(\lambda, \lambda_i).
\]

The typical feature of integrable theories is that the two-particle results already contain the main flavour about the structure of the spectrum. This means that expressions (151) and (150) are believed to be valid for general \( n \)-particle states. To benefit from the knowledge of the eigenvectors one has, however, to perform the complete analysis for few other multiparticle states with \( n > 2 \).

\[\text{Note that Eq. (150) implies the relation } \prod_{i=1}^{2} \frac{\omega_1(\lambda_i)}{\omega_2(\lambda_i)} = 1 \text{ thanks to the property } \theta(\lambda_1, \lambda_2)\theta(\lambda_2, \lambda_1) = 1.\]
5.3 The three-particle problem

The three-particle state is constructed in terms of a linear combination of products of creation fields with spin \( s - 3 \). Thus, the most general ansatz for the three-particle vector is,

\[
\phi_3(\lambda_1, \lambda_2, \lambda_3) = \sum_{m_1,m_2,m_3=1}^{3} \sum_{k_1,k_2,k_3=1}^{N} c_{k_1,k_2,k_3}^{(m_1,m_2,m_3)}(\lambda_{m_1}, \lambda_{m_2}, \lambda_{m_3}) T_{1,k_1}(\lambda_{m_1}) T_{1,k_2}(\lambda_{m_2}) T_{1,k_3}(\lambda_{m_3}),
\]

(152)

where \( c_{k_1,k_2,k_3}^{(m_1,m_2,m_3)}(\lambda_{m_1}, \lambda_{m_2}, \lambda_{m_3}) \) are arbitrary coefficients.

This general ansatz can be further simplified by considering the following two-step procedure. First we use the commutation rules between the fields \( T_{i,1}(\lambda) \) and \( T_{i,b}(\mu) \) for \( b = 2, \ldots, 4 \) to eliminate from Eq. (152) the linear dependent products \( T_{i,1}(\lambda_{m_1}) T_{i,1}(\lambda_{m_2}) T_{i,4}(\lambda_{m_3}), \)

\( T_{i,1}(\lambda_{m_1}) T_{i,4}(\lambda_{m_2}) T_{i,1}(\lambda_{m_3}), \)

\( T_{i,1}(\lambda_{m_1}) T_{i,2}(\lambda_{m_2}) T_{i,3}(\lambda_{m_3}), \)

\( T_{i,1}(\lambda_{m_1}) T_{i,3}(\lambda_{m_2}) T_{i,2}(\lambda_{m_3}), \)

\( T_{i,2}(\lambda_{m_1}) \times T_{i,1}(\lambda_{m_2}) T_{i,3}(\lambda_{m_3}) \) and \( T_{i,3}(\lambda_{m_1}) T_{i,1}(\lambda_{m_2}) T_{i,2}(\lambda_{m_3}) \). The next step is to reorder products of creation operators associated to the combinations \( T_{i,2}(\lambda) \) with \( T_{i,2}(\mu) \), \( T_{i,2}(\lambda) \) with \( T_{i,3}(\mu) \) and \( T_{i,3}(\lambda) \) with \( T_{i,2}(\mu) \). This task is performed with the help of the commutation rules given by Eq. (111). Taking into account these steps we find that the ansatz for the three-particle vector (152) becomes,

\[
\phi_3(\lambda_1, \lambda_2, \lambda_3) = T_{i,2}(\lambda_1) \phi_2(\lambda_2, \lambda_3) + g_3^{(1)}(\lambda_1, \lambda_2, \lambda_3) T_{i,3}(\lambda_1) \phi_1(\lambda_2) T_{i,1}(\lambda_3)
\]

\[+ g_3^{(2)}(\lambda_1, \lambda_2, \lambda_3) T_{i,3}(\lambda_1) \phi_1(\lambda_2) T_{i,1}(\lambda_2) + g_3^{(3)}(\lambda_1, \lambda_2, \lambda_3) T_{i,3}(\lambda_2) \phi_1(\lambda_3) T_{i,1}(\lambda_1)
\]

\[+ g_3^{(4)}(\lambda_1, \lambda_2, \lambda_3) T_{i,4}(\lambda_1) T_{i,1}(\lambda_2) T_{i,1}(\lambda_3) + g_3^{(5)}(\lambda_1, \lambda_2, \lambda_3) T_{i,4}(\lambda_2) T_{i,1}(\lambda_1) T_{i,1}(\lambda_3)
\]

\[+ g_3^{(6)}(\lambda_1, \lambda_2, \lambda_3) T_{i,4}(\lambda_3) T_{i,1}(\lambda_1) T_{i,1}(\lambda_2) + g_3^{(7)}(\lambda_1, \lambda_2, \lambda_3) T_{i,2}(\lambda_1) T_{i,3}(\lambda_2) T_{i,1}(\lambda_3)
\]

\[+ g_3^{(8)}(\lambda_1, \lambda_2, \lambda_3) T_{i,2}(\lambda_1) T_{i,3}(\lambda_3) T_{i,1}(\lambda_2) + g_3^{(9)}(\lambda_1, \lambda_2, \lambda_3) T_{i,2}(\lambda_2) T_{i,3}(\lambda_3) T_{i,1}(\lambda_1),
\]

(153)

where \( g_3^{(i)}(\lambda_1, \lambda_2, \lambda_3) \) are going to be fixed below.

Note that the three-particle ansatz has been written in terms of the one-particle and two-particle vectors. In this manner it will be able to use the previous results already established in sections 5.1 and 5.2. Further progress is made by investigating the result of the commutation of the diagonal operators with the four distinct types of product of operators defining the three-particle
As in the two-particle problem there is no need to worry about the right-hand side diagonal fields $T_{1,1}(\lambda)$. In what follows we describe the main steps one has to perform in order to solve this problem.

Let us start by considering the commutation of $T_{a,a}(\lambda)$ with the most involved term $T_{1,2}(\lambda_1) \times \phi_2(\lambda_2, \lambda_3)$. The commutation of the diagonal fields with $T_{1,2}(\lambda_1)$ is once again done by using Eqs. (27-29). This step generates besides the diagonal fields several annihilation operators acting on the two-particle vector $\phi_2(\lambda_2, \lambda_3)$. More specifically, we have carried on operators of type $T_{d+a,a}(\lambda)$ or $T_{d+a,a}(\lambda_1)$ for $d \geq 3$ but it requires a number of computations for $d = 0, 1, 2$. In particular, we have to reorder, as far as the rapidity $\lambda$ is concerned, the following products of operators $T_{1,2}(\lambda_1)T_{a_1,a_1+1}(\lambda)T_{2,3}(\lambda_j)$ with $j = 2, 3$ and $T_{1,2}(\lambda_1)T_{a_2,a_2+2}(\lambda)$ as well as $T_{1,3}(\lambda_1)T_{a_1,a_1+1}(\lambda)$ where $a_i = a, \ldots, a - i$. These computations are rather cumbersome and have been summarized in Appendix B.

We now discuss the commutation of the diagonal fields with the second term $T_{1,3}(\lambda_i)\phi_1(\lambda_j)$. In this case we start by commuting the operators $T_{a,a}(\lambda)$ with $T_{1,3}(\lambda_1)$ by means of Eqs. (31-34) and as before we produce diagonal and annihilation fields. These operators are then carried out to the right with the help of Eqs. (52-126). The final step consists of reordering products of creation fields such as $T_{1,3}(\lambda_1)T_{a_1,a_1+1}(\lambda)T_{1,2}(\lambda_j)$ and $T_{1,3}(\lambda_1)T_{a_1,a_1+1}(\lambda)$ having the rapidity $\lambda$ in the further right. This problem is resolved by using the commutation rules given by Eq. (41) as well as by the linear system of equations (42,43) with $d_1 = 0, b_1 = 3$. The commutations between $T_{a,a}(\lambda)$ and the third operator $T_{1,4}(\lambda_i)$ are obtained more directly. In the cases $a = 1$ and $a = N$ they follow from Eqs. (19,20) with $b = 4$, respectively. For $2 \leq a \leq N - 1$ we have to solve the linear system of equations (21,24) with $b = 4$ by applying the standard Cramer’s rule.

We finally discuss the commutation of the diagonal fields with the last term $T_{1,2}(\lambda_i)T_{1,3}(\lambda_j)$ for $i, j = 1, 2, 3$ such that $i \neq j$ This turns out to be the simplest case in our analysis due to the following reason. The commutation of $T_{a,a}(\lambda)$ with $T_{1,2}(\lambda_i)T_{1,3}(\lambda_j)$, by employing Eqs. (27-29,31-34), produces a special class of easy unwanted products that are not generated by any of the previous three terms entering the three-particle ansatz. The form of such undesirable products is
which are canceled out provided we choose $g$ given by ,

\begin{equation}
    g_3^{(4+i+j)}(\lambda_1, \lambda_2, \lambda_3)T_{a-3,a-1}(\lambda)T_{3,3}(\lambda_j) \quad \text{for } a > 3 \tag{154}
\end{equation}

\begin{equation}
    g_3^{(4+i+j)}(\lambda_1, \lambda_2, \lambda_3)T_{a-4,a-1}(\lambda)T_{3,3}(\lambda_i)T_{3,3}(\lambda_j) \quad \text{for } a > 4. \tag{155}
\end{equation}

Therefore we must project out the last term $T_{1,2}(\lambda_i)T_{1,3}(\lambda_j)$ from the linear combination (153) which is achieved by setting $g_3^{(7)}(\lambda_1, \lambda_2, \lambda_3) = g_3^{(8)}(\lambda_1, \lambda_2, \lambda_3) = g_3^{(9)}(\lambda_1, \lambda_2, \lambda_3) = 0$. This choice prompts a sequence of similar cancellations of other terms in the linear combination (153). The first one concerns with the product $T_{1,3}(\lambda_2)\phi_1(\lambda_3)$ which generates the following easy unwanted terms,

\begin{itemize}
    \item $g_3^{(3)}(\lambda_1, \lambda_2, \lambda_3)T_{a-1,a}(\lambda)T_{2,3}(\lambda_2)\phi_1(\lambda_3) \quad \text{for } a > 1 \tag{156}$
    \item $g_3^{(3)}(\lambda_1, \lambda_2, \lambda_3)T_{a-2,a}(\lambda)T_{2,3}(\lambda_2)T_{2,2}(\lambda_3) \quad \text{for } a > 2 \tag{157}$
    \item $g_3^{(3)}(\lambda_1, \lambda_2, \lambda_3)T_{a-3,a}(\lambda)T_{3,3}(\lambda_2)T_{2,2}(\lambda_3) \quad \text{for } a > 3. \tag{158}$
\end{itemize}

which are eliminated provided that $g_3^{(3)}(\lambda_1, \lambda_2, \lambda_3) = 0$

The next case is associated to the operator $T_{1,4}(\lambda_j)$ for $j = 2, 3$. Considering the above constraints, the commutation of $T_{a,a}(\lambda_j)$ with $T_{1,4}(\lambda_j)$ generates the following types of easy unwanted terms,

\begin{itemize}
    \item $g_3^{(3+j)}(\lambda_1, \lambda_2, \lambda_3)T_{a-1,a}(\lambda)T_{2,4}(\lambda_j) \quad \text{for } a > 1 \tag{159}$
    \item $g_3^{(3+j)}(\lambda_1, \lambda_2, \lambda_3)T_{a-2,a}(\lambda)T_{3,4}(\lambda_j) \quad \text{for } a > 2 \tag{160}$
    \item $g_3^{(3+j)}(\lambda_1, \lambda_2, \lambda_3)T_{a-3,a}(\lambda)T_{4,4}(\lambda_j) \quad \text{for } a > 3, \tag{161}$
    \item $g_3^{(3+j)}(\lambda_1, \lambda_2, \lambda_3)T_{a-1,a+1}(\lambda)T_{2,3}(\lambda_j) \quad \text{for } 1 < a < N \tag{162}$
    \item $g_3^{(3+j)}(\lambda_1, \lambda_2, \lambda_3)T_{a-2,a+1}(\lambda)T_{3,3}(\lambda_j) \quad \text{for } 2 < a < N. \tag{163}$
\end{itemize}

which are canceled out provided we choose $g_3^{(5)}(\lambda_1, \lambda_2, \lambda_3) = g_3^{(6)}(\lambda_1, \lambda_2, \lambda_3) = 0$.

The restrictions we have found so far bring a considerable simplification to the structure of

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the three-particle ansatz (153). We are now left with only four independent terms,

\[
\phi_3(\lambda_1, \lambda_2, \lambda_3) = \mathcal{T}_{1,2}(\lambda_1)\phi_2(\lambda_2, \lambda_3) + g_3^{(1)}(\lambda_1, \lambda_2, \lambda_3)\mathcal{T}_{1,3}(\lambda_1)\phi_1(\lambda_2)\mathcal{T}_{1,1}(\lambda_3) + g_3^{(2)}(\lambda_1, \lambda_2, \lambda_3)\mathcal{T}_{1,3}(\lambda_1)\phi_1(\lambda_3)\mathcal{T}_{1,1}(\lambda_2) + g_3^{(4)}(\lambda_1, \lambda_2, \lambda_3)\mathcal{T}_{1,4}(\lambda_1)\mathcal{T}_{1,1}(\lambda_2)\mathcal{T}_{1,1}(\lambda_3)
\]

(164)

The structure of the three-particle vector (164) shows some remarkable characteristics. We first note that the left-hand side of the linear combination is always commanded by the creation fields \(\mathcal{T}_{1,b}(\lambda_1)\) where the index \(b\) is limited by the maximum possible spin value. Next, each term of the linear combination is multiplied by the previous admissible (4-b)-particle state with rapidity \(\lambda_2\) and \(\lambda_3\). Finally, diagonal fields \(\mathcal{T}_{1,1}(\lambda_i)\) are added to the further right to complete the total number of three possible rapidities. Altogether, these are rather illuminating features that will be of great help to set up the ansatz for arbitrary multiparticle states. To determine completely the linear combination (164) we have to return to the analysis of the easy unwanted terms. Now the same form of a given remaining easy unwanted term is generated by two distinct types of operators present in the three-particle vector (164). In particular, the operators \(\mathcal{T}_{1,2}(\lambda_1)\phi_2(\lambda_2, \lambda_3)\) and \(\mathcal{T}_{1,3}(\lambda_1)\phi_1(\lambda_3)\mathcal{T}_{1,1}(\lambda_2)\) generate the following class of easy unwanted terms,

\[
R(\lambda, \lambda_1)\phi_2(\lambda, \lambda_3) = \mathcal{T}_{1,2}(\lambda_1)\phi_2(\lambda_2, \lambda_3) + g_3^{(1)}(\lambda_1, \lambda_2, \lambda_3)\mathcal{T}_{1,3}(\lambda_1)\phi_1(\lambda_2)\mathcal{T}_{1,1}(\lambda_3) + g_3^{(2)}(\lambda_1, \lambda_2, \lambda_3)\mathcal{T}_{1,3}(\lambda_1)\phi_1(\lambda_3)\mathcal{T}_{1,1}(\lambda_2) + g_3^{(4)}(\lambda_1, \lambda_2, \lambda_3)\mathcal{T}_{1,4}(\lambda_1)\mathcal{T}_{1,1}(\lambda_2)\mathcal{T}_{1,1}(\lambda_3)
\]

(165)

On the other hand operators \(\mathcal{T}_{1,2}(\lambda_1)\phi_2(\lambda_2, \lambda_3)\) and \(\mathcal{T}_{1,4}(\lambda_1)\) are able to produce yet another family of easy unwanted terms,

\[
R(\lambda, \lambda_1)\phi_2(\lambda, \lambda_3) = \mathcal{T}_{1,2}(\lambda_1)\phi_2(\lambda_2, \lambda_3) + g_3^{(1)}(\lambda_1, \lambda_2, \lambda_3)\mathcal{T}_{1,3}(\lambda_1)\phi_1(\lambda_2)\mathcal{T}_{1,1}(\lambda_3) + g_3^{(2)}(\lambda_1, \lambda_2, \lambda_3)\mathcal{T}_{1,3}(\lambda_1)\phi_1(\lambda_3)\mathcal{T}_{1,1}(\lambda_2) + g_3^{(4)}(\lambda_1, \lambda_2, \lambda_3)\mathcal{T}_{1,4}(\lambda_1)\mathcal{T}_{1,1}(\lambda_2)\mathcal{T}_{1,1}(\lambda_3)
\]

(166)

The cancellation of these two families of easy unwanted terms is obtained by choosing functions
The permutation for the rapidities $\lambda_2 \leftrightarrow \lambda_3$ and $\lambda_1 \leftrightarrow \lambda_2$. In what follows we shall show that this vector satisfies the relations,

$$\phi_3(\lambda_1, \lambda_2, \lambda_3) = \theta(\lambda_2, \lambda_3)\phi_3(\lambda_1, \lambda_3, \lambda_2) = \theta(\lambda_1, \lambda_2)\phi_3(\lambda_2, \lambda_1, \lambda_3)$$

The permutation for the rapidities $\lambda_2$ and $\lambda_3$ follows immediately from the two-particle exchange symmetry. In this case we just have to use Eqs. (139, 146). The technical steps to show the permutation for the rapidities $\lambda_1$ and $\lambda_2$ are more intricate. We start by commuting the operators $T_{1,2}(\lambda_1)$ and $T_{2,1}(\lambda_2)$ in the product $T_{1,2}(\lambda_1)\phi_2(\lambda_2, \lambda_3)$ by using the two-particle expression (136) and its symmetry property (140). We next carry on the operators $T_{1,1}(\lambda_1)$ and $T_{1,1}(\lambda_2)$ through the operator $T_{1,2}(\lambda_3)$ by using Eq. (27). By performing these two steps procedure we find the
expression,

\[ T_{1,2}(\lambda_1)\phi_2(\lambda_2, \lambda_3) = \theta(\lambda_1, \lambda_2)T_{1,2}(\lambda_2)\phi_2(\lambda_1, \lambda_3) + \theta(\lambda_1, \lambda_2)1F_1^{(2)}(\lambda_2, \lambda_1)T_{1,3}(\lambda_2) \]

\[ \times \left[ P_1(\lambda_1, \lambda_3)T_{1,2}(\lambda_3)T_{1,1}(\lambda_1) - 1F_1^{(1)}(\lambda_1, \lambda_3)T_{1,2}(\lambda_1)T_{1,1}(\lambda_3) \right] \]

\[ - 1F_1^{(2)}(\lambda_1, \lambda_2)T_{1,3}(\lambda_1) \left[ P_1(\lambda_2, \lambda_3)T_{1,2}(\lambda_3)T_{1,1}(\lambda_2) - 1F_1^{(1)}(\lambda_2, \lambda_3)T_{1,2}(\lambda_2)T_{1,1}(\lambda_3) \right] \]

\[ + 1F_1^{(2)}(\lambda_2, \lambda_3)T_{1,2}(\lambda_1)T_{1,3}(\lambda_2)T_{1,1}(\lambda_3) \]

\[ - \theta(\lambda_1, \lambda_2)1F_1^{(2)}(\lambda_1, \lambda_3)T_{1,2}(\lambda_2)T_{1,3}(\lambda_1)T_{1,1}(\lambda_3). \] (172)

The next task is concerned with the elimination of products \( T_{1,2}(\lambda_2)T_{1,3}(\lambda_1)T_{1,1}(\lambda_3) \) and \( T_{1,2}(\lambda_1) \times T_{1,3}(\lambda_2)T_{1,1}(\lambda_3) \) from Eq. (172). This is accomplished by using the commutation rule between \( T_{1,3}(\lambda) \) and \( T_{1,2}(\mu) \) given by Eq. (111) with \( d_1 = 0 \) and \( b_1 = 3 \). After commuting these operators in Eq. (172) we find,

\[ T_{1,2}(\lambda_1)\phi_2(\lambda_2, \lambda_3) = \theta(\lambda_1, \lambda_2)\left[ T_{1,2}(\lambda_2)\phi_2(\lambda_1, \lambda_3) + 1F_1^{(2)}(\lambda_2, \lambda_1)P_1(\lambda_1, \lambda_3)T_{1,3}(\lambda_2)T_{1,2}(\lambda_3) \right] \]

\[ \times \left[ T_{1,1}(\lambda_1) + 1\tilde{H}_1^{(2)}(\lambda_2, \lambda_1, \lambda_3|2)T_{1,3}(\lambda_2)T_{1,2}(\lambda_1)T_{1,1}(\lambda_3) + 2\tilde{F}_2^{(2)}(\lambda_2, \lambda_1, \lambda_3)T_{1,4}(\lambda_2) \right] \]

\[ \times \left[ T_{1,1}(\lambda_1)T_{1,1}(\lambda_3) \right] - \left[ 1F_1^{(2)}(\lambda_1, \lambda_2)P_1(\lambda_2, \lambda_3)T_{1,3}(\lambda_1)T_{1,2}(\lambda_3)T_{1,1}(\lambda_2) \right] \]

\[ + 1\tilde{H}_1^{(2)}(\lambda_1, \lambda_2, \lambda_3|2)T_{1,3}(\lambda_1)T_{1,2}(\lambda_2)T_{1,1}(\lambda_3) + 2\tilde{F}_2^{(2)}(\lambda_1, \lambda_2, \lambda_3)T_{1,4}(\lambda_1)T_{1,1}(\lambda_2)T_{1,1}(\lambda_3) \] (173)

where functions \( 1\tilde{H}_1^{(2)}(\lambda, \lambda_1, \lambda_2|2) \) and \( 2\tilde{F}_2^{(2)}(\lambda, \lambda_1, \lambda_2) \) are given by

\[ 1\tilde{H}_1^{(2)}(\lambda, \lambda_1, \lambda_2|2) = \frac{-D_2^{(2,0)}(\lambda, \lambda_1) R(\lambda, \lambda_2)_{3,1}^{2,2} R(\lambda, \lambda_1)_{3,1}^{2,1}}{D_2^{(3,0)}(\lambda, \lambda_1) R(\lambda, \lambda_2)_{3,1}^{3,1} R(\lambda, \lambda_1)_{3,1}^{3,1}} - \frac{R(\lambda, \lambda_1)_{3,1}^{2,2} R(\lambda, \lambda_2)_{3,1}^{2,1}}{R(\lambda, \lambda_1)_{3,1}^{3,1} R(\lambda, \lambda_2)_{3,1}^{3,1}} \]

\[ - \frac{R(\lambda_1, \lambda_2)_{3,1}^{2,2} D_2^{(3,1)}(\lambda_1, \lambda)}{R(\lambda_1, \lambda_2)_{3,1}^{3,1} D_2^{(3,0)}(\lambda_1, \lambda)} \] (174)

\[ 2\tilde{F}_2^{(2)}(\lambda, \lambda_1, \lambda_2) = \frac{-D_2^{(2,0)}(\lambda, \lambda_1) R(\lambda, \lambda_2)_{3,1}^{2,2} R(\lambda, \lambda_1)_{3,1}^{2,1}}{D_2^{(3,0)}(\lambda, \lambda_1) R(\lambda, \lambda_2)_{3,1}^{3,1} R(\lambda, \lambda_1)_{3,1}^{3,1}} - \frac{R(\lambda, \lambda_1)_{3,1}^{2,2} R(\lambda, \lambda_2)_{3,1}^{2,1}}{R(\lambda, \lambda_1)_{3,1}^{3,1} R(\lambda, \lambda_2)_{3,1}^{3,1}} \]

\[ - \frac{R(\lambda, \lambda_2)_{3,1}^{2,2} D_2^{(3,1)}(\lambda_1, \lambda)}{R(\lambda, \lambda_2)_{3,1}^{3,1} D_2^{(3,0)}(\lambda_1, \lambda)} \] (175)

We now reached a point in which we have to implement simplifications on the expressions (174, 175). This is done by taking into account identities coming from the unitarity relation. We
first use Eq. (79) to change the order of the rapidities $\lambda_1$ and $\lambda_2$ on the second term of Eq. (174).

We then consider the results (86, 87) to reorder the rapidities $\lambda$ and $\lambda_1$ on the determinant ratios $\frac{D_2^{(3,1)}(\lambda_1, \lambda)}{D_2^{(3,0)}(\lambda_1, \lambda)}$ and $\frac{D_2^{(3,2)}(\lambda, \lambda)}{D_2^{(3,0)}(\lambda, \lambda)}$ of Eqs. (174, 175). After implementing these steps we find the following identities for functions $1\mathcal{H}_1^{(2)}(\lambda, \lambda_1, \lambda_2|2)$ and $2\mathcal{F}_2^{(2)}(\lambda, \lambda, \lambda_2)$,

$$1\mathcal{H}_1^{(2)}(\lambda, \lambda_1, \lambda_2|2) = 1\mathcal{H}_1^{(2)}(\lambda, \lambda_1, \lambda_2|2) = \theta(\lambda_1, \lambda_2)\frac{R(\lambda_1, \lambda_2, \lambda)}{R(\lambda_1, \lambda_2, \lambda_2)} 1\mathcal{F}_1^{(2)}(\lambda, \lambda_2)$$

$$2\mathcal{F}_2^{(2)}(\lambda, \lambda_1, \lambda_2) = 2\mathcal{F}_2^{(2)}(\lambda, \lambda_1, \lambda_2)$$

(176)

(177)

where $1\mathcal{H}_1^{(2)}(\lambda, \lambda_1, \lambda_2|2)$ has been defined in Eqs. (A.12).

Now if we substitute the above expressions in Eq. (173) we find that the product $T_{1,2}(\lambda_1)\phi_2(\lambda_2, \lambda_3)$ can be finally be written as,

$$T_{1,2}(\lambda_1)\phi_2(\lambda_2, \lambda_3) = \theta(\lambda_1, \lambda_2) \left[ T_{1,2}(\lambda_2)\phi_2(\lambda_1, \lambda_3) + 1\mathcal{F}_1^{(2)}(\lambda_2, \lambda_1)P_1(\lambda_1, \lambda_3)T_{1,3}(\lambda_2)T_{1,2}(\lambda_3) \right.$$

$$\times T_{1,1}(\lambda_1) + \theta(\lambda_1, \lambda_3) 1\mathcal{F}_1^{(2)}(\lambda_2, \lambda_3)P_1(\lambda_3, \lambda_1)T_{1,3}(\lambda_2)T_{1,2}(\lambda_1)T_{1,1}(\lambda_3) + 2\mathcal{F}_2^{(2)}(\lambda_2, \lambda_1, \lambda_3)$$

$$\times T_{1,4}(\lambda_2)T_{1,1}(\lambda_1)T_{1,1}(\lambda_3)] - \left[ 1\mathcal{F}_1^{(2)}(\lambda_1, \lambda_2)P_1(\lambda_2, \lambda_3)T_{1,3}(\lambda_1)T_{1,2}(\lambda_3)T_{1,1}(\lambda_2) \right.$$

$$+ \theta(\lambda_2, \lambda_3) 1\mathcal{F}_1^{(2)}(\lambda_1, \lambda_3)P_1(\lambda_3, \lambda_2)T_{1,3}(\lambda_1)T_{1,2}(\lambda_2)T_{1,1}(\lambda_3) + 2\mathcal{F}_2^{(2)}(\lambda_1, \lambda_2, \lambda_3)$$

$$\times T_{1,4}(\lambda_1)T_{1,1}(\lambda_2)T_{1,1}(\lambda_3) \right]$$

(178)

which is indeed equivalent to exchange property (171).

We now turn our attention to the action of diagonal fields $T_{a,a}(\lambda)$ on the three-particle state $|\Phi_3\rangle$. The permutation property (171) is once again of fundamental help to gather the contributions of $T_{a,a}(\lambda)|\Phi_3\rangle$ in closed forms. In Appendix B we have described main technical details of these computations and our final results are given by Eq. (B.16). It turns out that this expression can
also be rewritten in a neat compact form, namely

\[
T_{a,a}(\lambda) \Phi_3 = w_a(\lambda) \prod_{i=1}^3 P_a(\lambda, \lambda_i) \Phi_3 - \sum_{t=1}^3 \sum_{p=M(0,a+t-N)}^{m(a-t)} T_{a-p,a+t-p}(\lambda) \times \sum_{1 \leq j_1 < \cdots < j_{t-p} \leq 3}^{*} \phi_{3-t}(\{\lambda_i\}_{i \neq j_1, \ldots, j_t}) t-p F_t^{(a-p)}(\lambda, \lambda_{j_1}, \ldots, \lambda_{j_t}) \times \left( \prod_{k=1}^{t-p} w_1(\lambda_{j_k}) \prod_{i=1, i \neq j_1, \ldots, j_t}^3 \frac{R(\lambda_i, \lambda_{j_k})_{1,1}^{1,1}}{R(\lambda_i, \lambda_{j_k})_{1,2}^{2,1}} \theta_<(\lambda_i, \lambda_{j_k}) \right) \times \left( \prod_{l=t-p+1}^t w_2(\lambda_{j_l}) \prod_{i=1, i \neq j_1, \ldots, j_t}^3 \frac{R(\lambda_{j_l}, \lambda_{j_l})_{1,1}^{1,1}}{R(\lambda_{j_l}, \lambda_{j_l})_{2,1}^{2,1}} \theta_<(\lambda_{j_l}, \lambda_{j_l}) |0) \right)
\]

(179)

As expected the three-particle state generates new unwanted terms proportional to the creation operators \(T_{a,a+3}(\lambda)\) carrying spin \(s = 3\). The corresponding off-shell amplitudes \(c F_3^{(a)}(\lambda, \lambda_1, \lambda_2, \lambda_3)\) for \(c = 0, 1, 2, 3\) are obtained from the following relations,

\[
_0 F_3^{(a)}(\lambda, \lambda_1, \lambda_2, \lambda_3) = \frac{R(\lambda, \lambda_1)_{a+2,1}^{2,1}}{R(\lambda, \lambda_1)_{a+3,1}^{3,1}} F_2^{(a+1)}(\lambda, \lambda_2, \lambda_3) + \sum_{i,j=2, j \neq i}^{n=3} \frac{R(\lambda, \lambda_{1})_{a+2,1}^{a,3}}{R(\lambda, \lambda_{1})_{a+3,1}^{a,3}} F_1^{(a+2)}(\lambda, \lambda_{i}) \times \frac{F_1^{(2)}(\lambda_1, \lambda_j) R(\lambda_i, \lambda_j)_{1,1}^{1,1}}{R(\lambda_i, \lambda_j)_{2,1}^{2,1}} \theta_<(\lambda_i, \lambda_j) + \frac{R(\lambda, \lambda_{1})_{a+4,1}^{a,3}}{R(\lambda, \lambda_{1})_{a+3,1}^{a,3}} F_2^{(2)}(\lambda_1, \lambda_2, \lambda_3)
\]

(180)

\[
_1 F_3^{(a)}(\lambda, \lambda_1, \lambda_2, \lambda_3) = _0 F_2^{(a)}(\lambda, \lambda_2, \lambda_3) _1 F_1^{(a+2)}(\lambda, \lambda_1) R(\lambda_2, \lambda_1)_{1,1}^{1,1} R(\lambda_3, \lambda_1)_{1,1}^{1,1} \frac{R(\lambda_2, \lambda_1)_{2,1}^{2,1}}{R(\lambda_2, \lambda_1)_{2,1}^{2,1}} R(\lambda_3, \lambda_1)_{2,1}^{2,1}
\]

(181)

\[
_2 F_3^{(a)}(\lambda, \lambda_1, \lambda_2, \lambda_3) = _0 F_1^{(a)}(\lambda, \lambda_3) _2 F_2^{(a+1)}(\lambda, \lambda_1, \lambda_2) R(\lambda_3, \lambda_1)_{1,1}^{1,1} R(\lambda_3, \lambda_2)_{1,1}^{1,1} \frac{R(\lambda_3, \lambda_1)_{2,1}^{2,1}}{R(\lambda_3, \lambda_1)_{2,1}^{2,1}} R(\lambda_3, \lambda_2)_{2,1}^{2,1}
\]

(182)
The next step consists of collecting together the contributions proportional to the same type of unwanted terms \( T_{a,a+\varepsilon}(\lambda) \). This task is performed by plugging in Eq. (185) the off-shell amplitudes
After some simplifications we find the following expression,

\[
T(\lambda) | \Phi_3 \rangle = \sum_{a=1}^{N} w_a(\lambda) \prod_{i=1}^{3} P_a(\lambda, \lambda_i) | \Phi_3 \rangle - \sum_{t=1}^{3-N} \sum_{a=1}^{N-t} T_{a,a+t}(\lambda) \sum_{p=0}^{t-1} \sum_{1 \leq i_1 < \cdots < i_p \leq t} \sum_{1 \leq j_{(p+1)} < \cdots < j_r \leq 3}^{*} \phi_{3-t}(\{\lambda_i\}_{i \neq j_1, \ldots, j_r})
\]

\[\times p \mathcal{F}_t^{(a)}(\lambda, \lambda_{j_1}, \ldots, \lambda_{j_t}) \left( \prod_{r=p+1}^{t} \prod_{s=1}^{p} \theta(\lambda_{j_r}, \lambda_{j_s}) \prod_{i=1}^{3} \theta(\lambda_i, \lambda_{j_r}) \right) \left[ \prod_{r=p+1}^{t} w_1(\lambda_{j_r}) \prod_{i=1}^{3} \frac{R(\lambda_i, \lambda_{j_r})_{1,1}^{1,1}}{R(\lambda_i, \lambda_{j_r})_{2,1}^{2,1}} \theta(\lambda_{j_r}, \lambda_i) \right]
\]

\[\times \prod_{s=1}^{p} \theta(\lambda_{j_r}, \lambda_{j_s}) - \prod_{r=p+1}^{t} w_1(\lambda_{j_r}) \prod_{i=1}^{3} \frac{R(\lambda_i, \lambda_{j_r})_{1,1}^{1,1}}{R(\lambda_i, \lambda_{j_r})_{2,1}^{2,1}} \prod_{s=1}^{p} \frac{R(\lambda_{j_r}, \lambda_{j_s})_{1,1}^{1,1}}{R(\lambda_{j_r}, \lambda_{j_s})_{2,1}^{2,1}} \prod_{s=1}^{p} \frac{R(\lambda_{j_r}, \lambda_{j_s})_{1,1}^{1,1}}{R(\lambda_{j_r}, \lambda_{j_s})_{2,1}^{2,1}} \right]. \tag{186}
\]

From Eq. (186) we conclude that all unwanted terms proportional to \( T_{a,a+t}(\lambda) \) can be eliminated by imposing that functions inside the brackets are null, namely

\[
\prod_{r=p+1}^{t} \frac{w_1(\lambda_{j_r})}{w_2(\lambda_{j_r})} = \prod_{r=p+1}^{t} \prod_{i=1}^{3} \theta(\lambda_{j_r}, \lambda_i) \frac{R(\lambda_{j_r}, \lambda_i)_{1,1}^{1,1}}{R(\lambda_{j_r}, \lambda_i)_{2,1}^{2,1}} \prod_{s=1}^{p} \frac{R(\lambda_{j_r}, \lambda_{j_s})_{1,1}^{1,1}}{R(\lambda_{j_r}, \lambda_{j_s})_{2,1}^{2,1}} \prod_{s=1}^{p} \frac{R(\lambda_{j_r}, \lambda_{j_s})_{1,1}^{1,1}}{R(\lambda_{j_r}, \lambda_{j_s})_{2,1}^{2,1}} \text{ for } t = 1, 2, 3 \text{ and } p = 0, \ldots, t-1. \tag{187}
\]

It turns out that this condition is fulfilled provided that the rapidities \( \lambda_i \) satisfy the following Bethe ansatz equations,

\[
\frac{w_1(\lambda_j)}{w_2(\lambda_j)} = \prod_{i=1}^{3} \theta(\lambda_j, \lambda_i) \frac{R(\lambda_j, \lambda_i)_{1,1}^{1,1}}{R(\lambda_j, \lambda_i)_{2,1}^{2,1}} \text{ for } j = 1, 2, 3. \tag{188}
\]

We observe that Eq. (188) follows directly from Eq. (187) for \( t = 1 \). To show that the other unwanted terms proportional to \( T_{a,a+2}(\lambda) \) and \( T_{a,a+3}(\lambda) \) are also canceled out we just have to
substitute Eq.\( \text{(188)} \) in the left-hand side of Eq.\( \text{(187)} \). As a result we obtain,

\[
\prod_{r=p+1}^{t} \prod_{i=1}^{3} \frac{\theta(\lambda_{jr}, \lambda_{ri})}{R(\lambda_{jr}, \lambda_{ri})} \frac{R(\lambda_{i}, \lambda_{jr})_{2,1}^{1,1}}{R(\lambda_{i}, \lambda_{jr})_{2,1}^{1,1}} = \prod_{r=p+1}^{t} \prod_{i=1}^{3} \frac{\theta(\lambda_{jr}, \lambda_{i})}{R(\lambda_{jr}, \lambda_{i})} \frac{R(\lambda_{i}, \lambda_{jr})_{2,1}^{2,1}}{R(\lambda_{i}, \lambda_{jr})_{2,1}^{2,1}} \times \prod_{s=1}^{p} \theta(\lambda_{jr}, \lambda_{js}) \frac{R(\lambda_{js}, \lambda_{ri})_{1,1}^{1,1}}{R(\lambda_{js}, \lambda_{ri})_{1,1}^{1,1}}.
\] (189)

By carrying out few simplifications on Eq.\( \text{(189)} \) one is able to show that it is equivalent to the expression,

\[
\prod_{i,r=p+1}^{t} \theta(\lambda_{jr}, \lambda_{ri}) \theta(\lambda_{jr}, \lambda_{ri}) = 1 \quad \text{for} \quad t = 2, 3.
\] (190)

which is satisfied thanks to the property \( \text{(140)} \) of function \( \theta(\lambda, \mu) \). As a consequence of the cancellation of all unwanted terms we find that the three-particle eigenvalue is given by,

\[
\Lambda_{3}(\lambda) = \sum_{a=1}^{N} w_{a}(\lambda) \prod_{i=1}^{3} P_{a}(\lambda, \lambda_{i}).
\] (191)

We conclude by observing that our results for the two-particle and three-particle states are already capable to unveil us a common pattern not only for the on-shell properties but also to the structure of the off-shell Bethe vectors. It should be emphasized that this fact has become possible only because we have been working without using any specific \( U(1) \) \( R \)-matrix.

### 5.4 The multi-particle state

The previous solution of the two-particle and three-particle problems provides us basic guidelines that are useful to construct multi-particle states. The first lesson we learn is that among the possible product of creation operators entering the linear combination there exists a large subset that in practice do not contribute to the respective state. These terms produce unique easy unwanted terms that are canceled only by setting the corresponding linear combination coefficient to zero. Our results indicate that the surviving product of creation operators possess the following characteristics. From Eqs.\( \text{(136, 170)} \) we observe that the left-hand side of such products is always governed by the creation fields \( T_{1,b}(\lambda_{1}) \) with \( 2 \leq b \leq m\{n+1, N\} \) where \( n \) denotes \( n \)-particle sector.
In addition, each operator $T_{1,b}(\lambda_1)$ is then multiplied by the previously determined $(n + 1 - b)$-particle vector such that the spin of the $n$-particle state is reproduced. Finally, a number of $(b−2)$ diagonal fields $T_{1,1}(\lambda_i)$ are added to the further right to complete the total number of rapidities $\lambda_1, \ldots, \lambda_n$ present in the $n$-particle vector. Combining these features together we find that such building blocks can be written as,

$$T_{1,1+\bar e}(\lambda_1)\phi_{n-\bar e}(\{\lambda_i\}_{i=2}^{\bar e}) \prod_{k_1=1}^{\bar e-1} T_{1,1}(\lambda_{j_{k_1}}) \text{ for } \bar e = 1, \ldots, m\{n, N-1\}. \quad (192)$$

where the indices $j_l$ for $l = 1, \ldots, \bar e−1$ take values on the interval $2 \leq j_l \leq n$ subjected to the condition $2 \leq j_1 < j_2 < \cdots < j_{\bar e−1} \leq n$.

Considering the above discussion $\phi_n(\lambda_1, \ldots, \lambda_n)$ is obtained by taking the most general linear combination of the terms (192), namely

$$\phi_n(\lambda_1, \ldots, \lambda_n) = \sum_{\bar e=1}^{m(n,N−1)} \phi_n^{(\bar e)}(\lambda_1, \ldots, \lambda_n), \quad (193)$$

where

$$\phi_n^{(\bar e)}(\lambda_1, \ldots, \lambda_n) = T_{1,1+\bar e}(\lambda_1) \sum_{\bar e}^{\star} \phi_{n-\bar e}(\lambda_{j_{\bar e+1}}, \ldots, \lambda_{j_n}) \times \prod_{k_1=2}^{\bar e} T_{1,1}(\lambda_{j_{k_1}}) g_{\bar e}^{(j_2, \ldots, j_{\bar e})}(\lambda_1, \ldots, \lambda_n), \quad (194)$$

where $g_{\bar e}^{(j_2, \ldots, j_{\bar e})}(\lambda_1, \ldots, \lambda_n)$ represent the coefficients of an arbitrary linear combination.

At the present stage the number of unknown coefficients for given $n$ and $\bar e$ is $\frac{(n - 1)!}{(n - \bar e)!(\bar e - 1)!}$. Here we will choose to normalize the $n$-particle state by the coefficient proportional to the first term $T_{1,2}(\lambda_1)\phi_{n-1}(\lambda_2, \ldots, \lambda_n)$ of the linear combination (193). Therefore, the total number of coefficients we have to determine is $2^{n−1} − 1$ which at first sight appears as an impracticable task. This number, however, can be truly reduced by invoking the exchange property of the $n$-particle vector under the permutation of the rapidities $\lambda_1, \ldots, \lambda_n$. As emphasized in the previous sections this is a rather important symmetry present in the explicit construction of the two-particle and three-particle states. Inspired by this result we demand that the $n$-particle ansatz (193) should
satisfy similar exchange property. For general \( n \) the permutation \( \lambda_j \leftrightarrow \lambda_{j+1} \) for \( j = 1, \ldots, n-1 \) reads,

\[
\phi_n(\lambda_1, \ldots, \lambda_j, \lambda_{j+1}, \ldots, \lambda_n) = \theta(\lambda_j, \lambda_{j+1})\phi_n(\lambda_1, \ldots, \lambda_{j-1}, \lambda_{j+1}, \lambda_j, \lambda_{j+2}, \ldots, \lambda_n). \tag{195}
\]

In our analysis of the three-particle vector \( \phi_3(\lambda_1, \lambda_2, \lambda_3) \) we noticed that the corresponding permutation symmetry \( \lambda_2 \leftrightarrow \lambda_3 \) depends only on the exchange property of the two-particle vector. It is expected that this feature extends to general \( n \)-particle states providing us a recursive manner to solve the problem. From now on we assume that the permutation property (195) of the \( n \)-particle vector for \( j \geq 2 \) follows as a consequence of the exchange symmetry of the previous constructed \((n-1)\)-particle vector. We stress that this assertion has been explicitly verified up to the four-particle state. This implies that relation (195) should then be valid for each term of the linear combination, namely

\[
\phi_n^{(\bar{e})}(\lambda_1, \ldots, \lambda_n) = \theta(\lambda_j, \lambda_{j+1})\phi_n^{(\bar{e})}(\lambda_1, \ldots, \lambda_{j-1}, \lambda_{j+1}, \lambda_j, \lambda_{j+2}, \ldots, \lambda_n),
\]

for \( j \geq 2 \) and \( 1 \leq \bar{e} \leq n \). \tag{196}

We now start to explore the consequences of the building blocks permutation symmetry (196). The first non-trivial case is \( \bar{e} = 2 \) and by substituting \( \phi_n^{(2)}(\lambda_1, \ldots, \lambda_n) \) in Eq.(196) we find,

\[
g_2^{(j+1)}(\lambda_1, \ldots, \lambda_n) = \theta(\lambda_j, \lambda_{j+1})g_2^{(j)}(\lambda_1, \ldots, \lambda_{j-1}, \lambda_{j+1}, \lambda_j, \lambda_{j+2}, \ldots, \lambda_n). \tag{197}
\]

which can recursively be solved and as result we obtain,

\[
g_2^{(j+1)}(\lambda_1, \ldots, \lambda_n) = \prod_{i=2}^{n} \theta_\prec(\lambda_i, \lambda_{j+1})g_2^{(j)}(\lambda_1, \lambda_{j+1}, \{\lambda_i\}_{i \neq j+1}). \tag{198}
\]

In order to make further progress one needs to find the expression for the amplitude \( g_2^{(2)}(\lambda_1, \ldots, \lambda_n) \). At this point the expressions for the two-particle and three-particle states obtained in sections 5.2 and 5.3 are of great utility. Direct comparison between Eqs.(193,194) and Eqs.(136,170) for \( n = 2, 3 \) reveals us that,

\[
g_2^{(2)}(\lambda_1, \lambda_2) = \mathcal{F}_1^{(2)}(\lambda_1, \lambda_2), \quad g_2^{(2)}(\lambda_1, \lambda_2, \lambda_3) = \frac{R(\lambda_3, \lambda_2)}{R(\lambda_3, \lambda_2)^2} \mathcal{F}_1^{(2)}(\lambda_1, \lambda_2). \tag{199}
\]

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We next note that the \( R \)-matrices prefactor in Eq. (199) is directly related to the wanted term originated from the commutation rule among the field \( T_{1,1}(\lambda_2) \) and the \((n-2)\)-particle vector \( \phi_{n-2}(\lambda_3, \ldots, \lambda_n) \). Considering this information we are able to infer that the expression for \( g_2^{(2)}(\lambda_1, \ldots, \lambda_n) \) is,

\[
g_2^{(2)}(\lambda_1, \ldots, \lambda_n) = \prod_{i=3}^{n} \frac{R(\lambda_i, \lambda_2)_{11}^{11}}{R(\lambda_i, \lambda_j)_{21}^{21}} F_1^{(2)}(\lambda_1, \lambda_2).
\]  

(200)

and by substituting Eq. (200) in Eq. (198) we finally find,

\[
g_2^{(j)}(\lambda_1, \ldots, \lambda_n) = \prod_{i=3}^{n} \frac{R(\lambda_i, \lambda_j)_{11}^{11}}{R(\lambda_i, \lambda_j)_{21}^{21}} \theta_< (\lambda_i, \lambda_j) F_1^{(2)}(\lambda_1, \lambda_j).
\]  

(201)

To emphasize the main points of our procedure we shall also consider explicitly the next case \( \tilde{e} = 3 \). By substituting \( \phi_3^{(3)}(\lambda_1, \ldots, \lambda_n) \) in Eq. (196) we found that functions \( g_3^{(j_2,j_3)}(\lambda_1, \ldots, \lambda_n) \) satisfy the following constraints,

\[
g_3^{(j_2,j_1)}(\lambda_1, \ldots, \lambda_n) = \theta(\lambda_j, \lambda_{j+1}) g_3^{(j_2,j_3)}(\lambda_1, \ldots, \lambda_{j-1}, \lambda_j, \lambda_{j+1}, \lambda_j, \lambda_{j+2}, \ldots, \lambda_n),
\]

for \( j_2 + 1 < j_3 \),

(202)

\[
g_3^{(j_2,j+1)}(\lambda_1, \ldots, \lambda_n) = \theta(\lambda_j, \lambda_{j+1}) g_3^{(j_2,j_3)}(\lambda_1, \ldots, \lambda_{j-1}, \lambda_j, \lambda_{j+1}, \lambda_j, \lambda_{j+2}, \ldots, \lambda_n),
\]

for \( j_2 + 1 < j_3 \),

(203)

\[
g_3^{(j,j+1)}(\lambda_1, \ldots, \lambda_n) = \theta(\lambda_j, \lambda_{j+1}) g_3^{(j_2,j_3)}(\lambda_1, \ldots, \lambda_{j-1}, \lambda_j, \lambda_{j+1}, \lambda_j, \lambda_{j+2}, \ldots, \lambda_n),
\]

for \( j_2 + 1 = j_3 \).

(204)

Once again it is possible to solve Eqs. (202,204) in a recursive manner and the result is,

\[
g_3^{(j_2,j_1)}(\lambda_1, \ldots, \lambda_n) = \prod_{i=2}^{n} \theta_<(\lambda_i, \lambda_{j_2}) \theta_<(\lambda_i, \lambda_{j_3}) g_3^{(2,3)}(\lambda_1, \lambda_{j_2}, \lambda_{j_3}, \{\lambda_i\}_{i=2,\ldots,n}).
\]  

(205)

provided that function \( g_3^{(2,3)}(\lambda_1, \ldots, \lambda_n) \) satisfies the following exchange property,

\[
g_3^{(2,3)}(\lambda_1, \ldots, \lambda_n) = \theta(\lambda_2, \lambda_3) g_3^{(2,3)}(\lambda_1, \lambda_3, \lambda_2, \lambda_4, \ldots, \lambda_n).
\]  

(206)

The remaining task consists in the determination of the fundamental amplitude \( g_3^{(2,3)}(\lambda_1, \ldots, \lambda_n) \) which is done by considering the same kind of arguments employed in the case
\( e = 2 \). By comparing Eqs. (193,194) for \( n = 3 \) with Eq. (170) tell us that function \( g_3^{(2,3)}(\lambda_1, \lambda_2, \lambda_3) = 2F_2^{(2)}(\lambda_1, \lambda_2, \lambda_3) \). The prefactors for general values of \( n \) are now governed by the first term of the commutation rule between the product \( T_{1,1}(\lambda_2)T_{1,1}(\lambda_3) \) and the creation fields \( T_{1,2}(\lambda_i) \) present in the leading part of vector \( \phi_{n-3}(\lambda_1, \ldots, \lambda_n) \). Collecting all these informations together we find,

\[
g_3^{(2,3)}(\lambda_1, \ldots, \lambda_n) = \prod_{i=4}^{n} R(\lambda_i, \lambda_2) \frac{R(\lambda_i, \lambda_3)}{R(\lambda_i, \lambda_j)} \frac{2F_2^{(2)}}{R(\lambda_i, \lambda_j)} (\lambda_1, \lambda_2, \lambda_3). \tag{207}
\]

We clearly see that Eq. (207) satisfies the permutation relation (206) thanks to the exchange symmetry of the two-particle function \( 2F_2^{(2)}(\lambda_1, \lambda_2, \lambda_3) \). The coefficients \( g_3^{(j_2,j_3)}(\lambda_1, \ldots, \lambda_n) \) are obtained by substituting Eq. (207) in Eq. (205) which gives us,

\[
g_3^{(j_2,j_3)}(\lambda_1, \ldots, \lambda_n) = \prod_{i=2, j \neq j_2}^{n} \frac{R(\lambda_i, \lambda_{j_2})}{R(\lambda_i, \lambda_{j_3})} \frac{2F_2^{(2)}}{R(\lambda_i, \lambda_{j_3})} (\lambda_1, \lambda_{j_2}, \lambda_{j_3}). \tag{208}
\]

Let us now generalize the above results for arbitrary index \( e \). The analysis of the consequences of the permutation symmetry (196) is made by using mathematical induction, leading us to the following general constraint,

\[
g_{e}^{(j_2,\ldots,j_e)}(\lambda_1, \ldots, \lambda_n) = \prod_{i=2, j \neq j_2}^{n} \prod_{l=2}^{e} \theta(<\lambda_i, \lambda_{j_l}) \frac{g_{e}^{(2,\ldots,e)}(\lambda_1, j_2, \ldots, j_e)}{g_{e}^{(2,\ldots,e)}(\lambda_1, \lambda_{j_2}, \ldots, \lambda_{j_e})}. \tag{209}
\]

We are again left to determine the expression of the basic function \( g_{e}^{(2,\ldots,e)}(\lambda_1, \ldots, \lambda_n) \). To make progress in this direction we first pause to discuss the results we have so far obtained. From Eqs. (200,207) we note that functions \( g_2^{(2)}(\lambda_1, \ldots, \lambda_n) \) and \( g_3^{(2,3)}(\lambda_1, \ldots, \lambda_n) \) are given in terms of product among universal prefactors with the off-shell amplitudes \( 1F_1^{(2)}(\lambda_1, \lambda_2) \) and \( 2F_2^{(2)}(\lambda_1, \lambda_2, \lambda_3) \). The former amplitude is a central element of the eigenvalue problem for the one-particle state while the latter play analogous role for the two-particle state. It is natural to assume that the structure of functions \( g_{e}^{(2,\ldots,e)}(\lambda_1, \ldots, \lambda_n) \) for arbitrary \( e \) will follow the same pattern noted above. In other words, that these elementary functions can be fixed after the solution of the eigenvalue problem for the previous \( (\bar{e} - 1) \)-particle state is completed. In analogy to sections 5.1,5.3 such solution will provides the expression for the respective amplitude \( \bar{e}-1F_{\bar{e}-1}^{(2)}(\lambda_1, \lambda_{j_2}, \ldots, \lambda_{j_e}) \).
The expression for \( g_{\bar{e},...\bar{e}}^{(2,..,\bar{e})}(\lambda_1, \ldots, \lambda_n) \) is then obtained by multiplying prefactors involving the ratios \( R(\lambda_i, \lambda_j)^{11}_{11} / R(\lambda_i, \lambda_j)^{21}_{21} \) with the amplitude \( e_{-1}^{(2)} F^{(2)}_{e-1} (\lambda_1, \lambda_j, \ldots, \lambda_{j_e}) \). As before the prefactors are easily accounted through the commutation of the product \( T_{1,1}(\lambda_2) \ldots T_{1,1}(\lambda_\bar{e}) \) with operator \( \phi_{n-e}(\lambda_{\bar{e}+1}, \ldots, \lambda_n) \). Considering this reasoning we obtain,

\[
g_{\bar{e},...\bar{e}}^{(2,..,\bar{e})}(\lambda_1, \ldots, \lambda_n) = \prod_{i=\bar{e}+1}^{n} \prod_{j=2}^{\bar{e}} \frac{R(\lambda_i, \lambda_j)^{11}_{11}}{R(\lambda_i, \lambda_j)^{21}_{21}} e_{-1}^{(2)} F^{(2)}_{e-1} (\lambda_1, \ldots, \lambda_\bar{e}). \tag{210}
\]

and by substituting Eq. (210) in Eq. (209) we found,

\[
g_{\bar{e},...\bar{e}}^{(j_2,...,j_{\bar{e}})}(\lambda_1, \ldots, \lambda_n) = \prod_{k_1=2}^{\bar{e}} \prod_{k_2=2}^{j} \frac{R(\lambda_{k_2}, \lambda_{j_{k_1}})^{11}_{11}}{R(\lambda_{k_2}, \lambda_{j_{k_1}})^{21}_{21}} \theta_<(\lambda_{k_2}, \lambda_{j_{k_1}}) e_{-1}^{(2)} F^{(2)}_{e-1} (\lambda_1, \lambda_j, \ldots, \lambda_{j_{\bar{e}}}). \tag{211}
\]

We have reduced the determination of an arbitrary \( n \)-particle vector to the computation of only \( N - 1 \) coefficients \( e_{-1}^{(2)} F^{(2)}_{e-1} (\lambda_1, \lambda_j, \ldots, \lambda_{j_{\bar{e}}}) \) for \( \bar{e} = 2, \ldots, N \). Collecting together Eqs. (193, 194) and Eq. (211) we are able to propose that an educated ansatz for the \( n \)-particle vector should be,

\[
\phi_n(\lambda_1, \ldots, \lambda_n) = \sum_{\bar{e}=1}^{m(n,N-1)} T_{1,1+\bar{e}}(\lambda_1) \sum_{2\leq j_2<\ldots<j_{\bar{e}}\leq n}^* \phi_{n-\bar{e}}(\lambda_{j_{\bar{e}+1}}, \ldots, \lambda_{j_{n}}) e_{-1}^{(2)} F^{(2)}_{e-1} (\lambda_1, \lambda_j, \ldots, \lambda_{j_{\bar{e}}})
\]

\[
\times \prod_{k_1=2}^{\bar{e}} T_{1,1}(\lambda_{j_{k_1}}) \prod_{k_2=\bar{e}+1}^{n} \frac{R(\lambda_{j_{k_2}}, \lambda_{j_{k_1}})^{11}_{11}}{R(\lambda_{j_{k_2}}, \lambda_{j_{k_1}})^{21}_{21}} \theta_<(\lambda_{j_{k_2}}, \lambda_{j_{k_1}}). \tag{212}
\]

where the normalization \( \theta F^{(2)}_0 (\lambda) = 1 \) is assumed.

Having at hand a proposal for the \( n \)-particle vector we can turn our attention to the action of the diagonal operators \( T_{a,a}(\lambda) \) on the multi-particle state \( |\Phi_n\rangle = \phi_n(\lambda_1, \ldots, \lambda_n) |0\rangle \). This study is fundamental to find the expressions of the off-shell functions \( e_{-1}^{(2)} F^{(2)}_{e-1} (\lambda_1, \lambda_j, \ldots, \lambda_{j_{\bar{e}}}) \) in terms the \( R \)-matrix elements. It should be stressed that the approach put forward here is clearly self-consistent. After the solution of the eigenvalue problem for the sector with \( (n-1) \) particles we have the basic ingredients to pursue the next sector having one more particle and so forth. Considering our previous results for one-particle, two-particle and three-particle together with the help of mathematical induction we are able to determine the action of the diagonal fields on \( |\Phi_n\rangle \).
The final results are,

$$\mathcal{T}_{a,b}(\lambda | \Phi_n) = w_a(\lambda) \prod_{i=1}^{n} P_a(\lambda, \lambda_i) | \Phi_n \rangle - \sum_{t=1}^{n} \sum_{P=M(0,a+t-N)} m(a-1,t) \mathcal{T}_{a-p,a+t-p}(\lambda)$$

$$\times \sum_{1 \leq j_1 < \ldots < j_t \leq n} \phi_{n-1}(\lambda_{j_1}, \ldots, \lambda_{j_t}) \, t_p \mathcal{F}_{t}^{(a-p)}(\lambda, \lambda_{j_1}, \ldots, \lambda_{j_t})$$

$$\times \left( \prod_{k=1}^{t-p} w_1(\lambda_{j_k}) \prod_{i=1}^{n} \frac{R(\lambda_i, \lambda_{j_k})_{1,1}^{1,1}}{R(\lambda_i, \lambda_{j_k})_{2,1}^{2,1}} \right) \left( \prod_{l=t-p+1}^{t} \prod_{i=1}^{n} \frac{R(\lambda_{j_l}, \lambda_i)_{1,1}^{1,1}}{R(\lambda_{j_l}, \lambda_i)_{2,1}^{2,1}} \right) \theta_{<}(\lambda_{j_l}, \lambda_i)$$

$$\sum_{1 \leq j_1 < \ldots < j_t \leq n} \phi_{n-1}(\lambda_{j_1}, \ldots, \lambda_{j_t}) \, t_p \mathcal{F}_{t}^{(a-p)}(\lambda, \lambda_{j_1}, \ldots, \lambda_{j_t})$$

$$\sum_{1 \leq j_1 < \ldots < j_t \leq n} \phi_{n-1}(\lambda_{j_1}, \ldots, \lambda_{j_t}) \, t_p \mathcal{F}_{t}^{(a-p)}(\lambda, \lambda_{j_1}, \ldots, \lambda_{j_t})$$

$$\times \left( \prod_{k=1}^{t-p} w_1(\lambda_{j_k}) \prod_{i=1}^{n} \frac{R(\lambda_i, \lambda_{j_k})_{1,1}^{1,1}}{R(\lambda_i, \lambda_{j_k})_{2,1}^{2,1}} \right) \left( \prod_{l=t-p+1}^{t} \prod_{i=1}^{n} \frac{R(\lambda_{j_l}, \lambda_i)_{1,1}^{1,1}}{R(\lambda_{j_l}, \lambda_i)_{2,1}^{2,1}} \right) \theta_{<}(\lambda_{j_l}, \lambda_i)$$

$$\sum_{1 \leq j_1 < \ldots < j_t \leq n} \phi_{n-1}(\lambda_{j_1}, \ldots, \lambda_{j_t}) \, t_p \mathcal{F}_{t}^{(a-p)}(\lambda, \lambda_{j_1}, \ldots, \lambda_{j_t})$$

The structure of the recurrence relations for the off-shell amplitudes $c \mathcal{F}_{b}^{(a)}(\lambda, \lambda_1, \ldots, \lambda_b)$ is as follows. For $c \neq 0$ and $c \neq b$ our findings for the two-particle and three-particle states given in Eqs. (144, 181, 182) are sufficient to guide us to the following general structure,

$$c \mathcal{F}_{b}^{(a)}(\lambda, \lambda_1, \ldots, \lambda_b) = \sum_{i=a+1}^{b} R(\lambda_i, \lambda_{a+1}) \frac{\mathcal{F}_{b-1}(\lambda, \lambda_{a+1}, \ldots, \lambda_{b-1})}{\prod_{j=1}^{b-1} \frac{R(\lambda_{i-j}, \lambda_{a+1})_{1,1}^{1,1}}{R(\lambda_{i-j}, \lambda_{a+1})_{2,1}^{2,1}}}$$

for $b = 2, \ldots, N-1$; $a = 1, \ldots, N-b$; $c = 1, \ldots, b-1$. (214)

The expressions for $c = 0, b$ are, however, more complicated and their multi-particle extension required us to perform explicit computations up to the four-particle state. Considering the help of mathematical induction as well as the expected properties of these functions under exchange of their rapidities, see for instance Eqs. (146, 184), we are able to infer that,

$$0 \mathcal{F}_{b}^{(a)}(\lambda, \lambda_1, \ldots, \lambda_b) = \sum_{\varepsilon=1}^{b} R(\lambda, \lambda_1) \frac{\mathcal{F}_{b-\varepsilon}^{(a+\varepsilon)}(\lambda, \lambda_{b-\varepsilon+1}, \ldots, \lambda_{b-1})}{\prod_{l=1}^{b-\varepsilon} \frac{R(\lambda_{l-1}, \lambda_{b-\varepsilon+1})_{1,1}^{1,1}}{R(\lambda_{l-1}, \lambda_{b-\varepsilon+1})_{2,1}^{2,1}}}$$

for $b = 1, \ldots, N-1$; $a = 1, \ldots, N-b$ (215)
and

\[ bF_b^{(a)}(\lambda, \lambda_1, \ldots, \lambda_b) = -\sum_{f=0}^{b-1} \sum_{1 \leq f_1 < \cdots < f_{b-f} \leq b} fF_b^{(a)}(\lambda, \{\lambda_i\}_{i \neq f_1, \ldots, f_{b-f}}) \lambda_{f_1}, \ldots, \lambda_{f_{b-f}} \]

\[ \times \prod_{s=1}^{b-f} \prod_{i=1 \atop i \neq f_1, \ldots, f_{b-f}}^{b} \theta_<(\lambda_i, \lambda_{f_s}) R(\lambda_i, \lambda_{f_s}^{1,1}) R(\lambda_i, \lambda_{f_s}^{2,1})/R(\lambda_i, \lambda_{f_s}^{1,1}) \]

for \( b = 1, \ldots, N - 1; \ a = 1, \ldots, N - b. \) \( (216) \)

Here we stress that the recurrence relations \((214, 215, 216)\) have as initial condition the overall normalization \(F_0^{(a)}(\lambda) = 1\) as well as the expressions for the one-particle off-shell amplitudes \(F_1^{(a)}(\lambda, \mu)\) and \(1F_1^{(a)}(\lambda, \mu)\) defined by Eq. \((128)\).

The solution of the eigenvalue problem for the multiparticle state is completed by summing over Eq. \((213)\) for \(a = 1, \ldots, N\). By substituting the expression \((216)\) for the off-shell amplitudes \(bF_b^{(a)}(\lambda, \lambda_1, \ldots, \lambda_b)\) we find,

\[ T(\lambda) |\Phi_n\rangle = \sum_{a=1}^{N} w_a(\lambda) \prod_{i=1}^{n} P_a(\lambda, \lambda_i) |\Phi_n\rangle - \sum_{t=1}^{N-t} \sum_{a=1}^{N-t} T_{a,a+t}(\lambda) \sum_{p=0}^{t-1} \sum_{1 \leq j_{(p+1)} < \cdots < j_{t} \leq n}^{\ast} \]

\[ \times \phi_{n-t}(\{\lambda_i\}_{i \neq j_{1}, \ldots, j_{t}}) \prod_{s=1}^{p} w_1(\lambda_{j_s}) \prod_{i=1 \atop i \neq j_{1}, \ldots, j_{t}}^{n} R(\lambda_i, \lambda_{j_s}^{1,1}) R(\lambda_i, \lambda_{j_s}^{2,1}) \theta_<(\lambda_i, \lambda_{j_s}) \]

\[ \times \left( \prod_{r=p+1}^{t} \prod_{s=1}^{p} \theta_<(\lambda_{j_s}, \lambda_{j_r}) \prod_{i=1 \atop i \neq j_{1}, \ldots, j_{t}}^{n} \theta_<(\lambda_i, \lambda_{j_s}) \right) \left[ \prod_{r=p+1}^{t} w_2(\lambda_{j_r}) \prod_{i=1 \atop i \neq j_{1}, \ldots, j_{t}}^{n} R(\lambda_r, \lambda_{j_s}^{1,1}) R(\lambda_r, \lambda_{j_s}^{2,1}) \theta(\lambda_{j_r}, \lambda_i) \right] \]

\[ \times \prod_{s=1}^{p} \theta(\lambda_{j_s}, \lambda_{j_s}) - \prod_{r=p+1}^{t} w_1(\lambda_{j_r}) \prod_{i=1 \atop i \neq j_{1}, \ldots, j_{t}}^{n} R(\lambda_i, \lambda_{j_s}^{1,1}) R(\lambda_i, \lambda_{j_s}^{2,1}) \prod_{s=1}^{p} R(\lambda_{j_s}, \lambda_{j_s}^{1,1}) R(\lambda_{j_s}, \lambda_{j_s}^{2,1}) \right] . \] \( (217) \)

The unwanted terms are canceled out by requiring that functions inside the bracket of Eq. \((217)\) are null for \( t = 1, \ldots, m(n, N - 1) \). The situation here is exactly the same found already for the two-particle and three-particle state. It is sufficient to consider the constraint coming from the
case \( t = 1 \) which leads us to the following Bethe ansatz equations,

\[
\frac{w_1(\lambda_{j_1})}{w_2(\lambda_{j_1})} = \prod_{i=1}^{n} \theta(\lambda_{j_1}, \lambda_i) \frac{R(\lambda_{j_1}, \lambda_i)^{1,1}_{1,1} R(\lambda_i, \lambda_{j_1})^{2,1}_{2,1}}{R(\lambda_{j_1}, \lambda_i)^{2,1}_{2,1} R(\lambda_i, \lambda_{j_1})^{1,1}_{1,1}} \quad \text{for } j_1 = 1, \ldots, n. \quad (218)
\]

To verify the cancellations for the remaining values of index \( t \) one just has to substitute Eq.\( (218) \) on the bracket of Eq.\( (217) \) and to consider the exchange property \( (140) \) of functions \( \theta(\lambda_i, \lambda_j) \). The elimination of all unwanted terms by Eq.\( (218) \) implies that the \( n \)-particle eigenstate is,

\[
\Lambda_n(\lambda) = \sum_{a=1}^{N} \Lambda^a(\lambda) \prod_{i=1}^{n} P_a(\lambda, \lambda_i). \quad (219)
\]

We close by mentioning that all formulae of this section pass by the following tests. For \( N = 2 \) we recover the well known algebraic Bethe ansatz solution of the six-vertex model \( [10] \). In this special case one has to note that both numerator and denominator of Eq.\( (138) \) vanish and the respective limit gives us \( \theta(\lambda, \mu) = R(\lambda, \mu)^{2,2}_{2,2}/R(\lambda, \mu)^{1,1}_{1,1} \). For \( N = 3 \) we reproduce the algebraic Bethe ansatz construction proposed by Tarasov \( [12] \) for the nineteen-vertex models. Strictly speaking, our results should be seen as extensions of the above mentioned works since no assumption of spectral parameter dependence for the \( R \)-matrix has been made. We also verified that the expressions \( (218, 219) \) reproduce the Bethe ansatz equations and eigenvalues of the higher spin generalization of the six-vertex model \( [11] \). We have also computed explicitly the most complicated off-shell amplitudes of the four-particle state such as \( 0\mathcal{F}^{(a)}_4(\lambda, \lambda_1, \ldots, \lambda_4) \) and \( 4\mathcal{F}^{(a)}_4(\lambda, \lambda_1, \ldots, \lambda_4) \). These computations have been checked to be in accordance with the recurrence relations \( (215, 216) \).

### 6 Conclusion

In this paper we have shown how the algebraic Bethe ansatz method works for arbitrary vertex models whose \( R \)-matrix commutes with one \( U(1) \) symmetry. This invariance guarantees the existence of a reference state in which the respective monodromy matrix acts triangularly. We recall that this is necessary requirement to start performing an algebraic Bethe ansatz analysis.
The transfer matrix eigenvalue problem is viewed under the general perspective that its solution should not depend on a specific functional form of the $R$-matrix. We argued that the algebraic formulation of the Bethe states of the transfer matrix can be done solely on basis of the Yang-Baxter algebra, the Yang-Baxter equation and the unitarity property satisfied by the $R$-matrix. In fact, we have presented a method to obtain the appropriate commutation rules between the monodromy matrix elements for arbitrary number $N$ of edge states. Moreover, the necessary identities among the $R$-matrix elements to solve the eigenvalue problem are derived from the respective Yang-Baxter and unitary relations.

This approach provided us the expressions for the on-shell properties such as the transfer matrix eigenvalues and the Bethe ansatz equations as well as the structure of the off-shell Bethe vectors in terms of the arbitrary $R$-matrix weights. The respective off-shell amplitudes are determined by means of a recurrence formula whose input are special entries of the $R$-matrix. Note that the previous understanding of off-shell structure of $U(1)$ integrable models was restricted to the six-vertex [10] and nineteen-vertex models [12]. By contrast, this paper offers us the basic ingredients to compute such properties for any $U(1)$ vertex model. In a forthcoming paper [20] we shall indeed exhibit the explicit expressions of the off-shell amplitudes of some $U(1)$ invariant vertex models such as the classic higher spin $XXZ - S$ Heisenberg system and the non-compact vertex model based on the $SL(2, R)$ algebra. Interesting enough, we found that all the off-shell amplitudes $c F^{(a)}(\lambda, \lambda_1, \ldots, \lambda_b)$ factorize in terms of product of elementary functions.

We hope that the framework developed in this paper will also be relevant to solve other families of integrable models by the algebraic Bethe ansatz. In particular, that the universal structure of our expressions for the eigenvectors and the off-shell amplitudes should play the role of cornerstones in nested Bethe ansatz solutions of integrable models with $R$-matrices commuting with more than one $U(1)$ symmetry. This idea has indeed worked for the Hubbard model whose algebraic Bethe ansatz solution [19] encoded the main features found in the algebraic Bethe ansatz formulation of the $N = 3$ $U(1)$ vertex model. In general, we expect that the algebraic solution of a $N$-state integrable model with $m$ distinct $U(1)$ conserved charges will be described in terms of a $(N - m + 1) \times (N - m + 1)$ monodromy matrix. The partition of the monodromy matrix from
$N \times N$ to $(N - m + 1) \times (N - m + 1)$ are going to be dictated by the form of the $U(1)$ operators that commute with the $R$-matrix. This means that the respective length of the recurrence relations for the eigenvectors and off-shell amplitudes will be governed by the effective number $\tilde{N} = N - m + 1$. Now the off-shell amplitudes $c \mathcal{F}_b^{(a)}(\lambda, \lambda_1, \ldots, \lambda_b)$ will be seen as vectors with $m^2$ components while $\theta(\lambda, \mu)$ will behave as a $m \times m$ auxiliary factorized $\bar{R}(\lambda, \mu)$ $R$-matrix satisfying the unitary property. We recall that this scenario is known to work for specific vertex models with $N - 2$ $U(1)$ symmetries such as those based on the non-exceptional Lie algebras [13]. This has been also verified for the vertex model related to the fundamental representation of the $U_q[G_2]$ algebra [21].

This system is rather involved since the size of its $R$-matrix is $7 \times 7$ while its algebraic Bethe ansatz solution is formulated by a $5 \times 5$ monodromy matrix. In all these cases the corresponding auxiliary $\bar{R}$-matrices $\bar{R}(\lambda, \mu)$ is derived from the analysis of the commutation relations following the general method explained in section 3. Their factorization and that they fulfill the unitarity relation were verified using the respective specific weights. The results of this paper suggest that such properties can be derived as a consequence of the Yang-Baxter equation and the unitary relation of the original $R$-matrix we have started with. Hopefully, this could be shown by generalizing the method discussed in section 4.

Finally, we think that this paper highlights the expectation that the Bethe ansatz properties of integrable models should be exhibited in terms of universal formulae depending only on the $R$-matrix amplitudes. We hope that our results will inspire the search of a general recipe to solve integrable systems invariant by many $U(1)$ symmetries through the algebraic Bethe ansatz method.

Appendix A: The two-particle state

In this appendix we provide the technical details entering the solution of the two-particle eigenvalue problem. Here we shall assume that the easy unwanted terms mentioned in section 5.2 have been already canceled out. Therefore, the results to be described in what follows concern with the
symmetric two-particle ansatz,

\[ |\Phi_2\rangle = \mathcal{T}_{1,2}(\lambda_1)T_{1,2}(\lambda_2) |0\rangle - \frac{R(\lambda_1, \lambda_2)^{2,2}}{R(\lambda_1, \lambda_2)^{3,1}} w_1(\lambda_2) \mathcal{T}_{1,3}(\lambda_1) |0\rangle \quad (A.1) \]

The elimination of the easy unwanted terms brings considerable simplifications on the action of the diagonal operators \( \mathcal{T}_{a,a}(\lambda) \) on \( |\Phi_2\rangle \). Considering all the steps mentioned in section 5.2, we find that \( \mathcal{T}_{a,a}(\lambda) |\Phi_2\rangle \) has to be divided in five different parts. We find that they are given by,

\[
\mathcal{T}_{a,a}(\lambda) |\Phi_2\rangle = w_a(\lambda) P_a(\lambda, \lambda_1) \left[ P_a(\lambda, \lambda_2) \left| \Phi_2^{(1)} \right\rangle + P_a(\lambda, \lambda_1, \lambda_2) \left| \Phi_2^{(2)} \right\rangle \right] 
- \delta_a^{1N} \sum_{i,j=1 \atop i \neq j}^2 w_1(\lambda_i) \mathcal{H}^{(a)}_1(\lambda, \lambda_1, \lambda_2 | i) \mathcal{T}_{a,a+1}(\lambda) \mathcal{T}_{1,2}(\lambda_j) |0\rangle 
- \delta_a \sum_{i,j=1 \atop i \neq j}^2 w_2(\lambda_i) \mathcal{H}^{(a-1)}_1(\lambda, \lambda_1, \lambda_2 | i) \mathcal{T}_{a-1,a}(\lambda) \mathcal{T}_{1,2}(\lambda_j) |0\rangle 
- \delta_a^{1N} w_1(\lambda_1) w_1(\lambda_2) 2 \mathcal{F}_2^{(a)}(\lambda, \lambda_1, \lambda_2) \mathcal{T}_{a,a+2}(\lambda) |0\rangle 
- \delta_a \sum_{i,j=1 \atop i \neq j}^2 w_1(\lambda_i) w_2(\lambda_j) \mathcal{H}^{(a-1)}_2(\lambda, \lambda_1, \lambda_2 | i) \mathcal{T}_{a-1,a+1}(\lambda) |0\rangle 
- \delta_a^{1,2} w_1(\lambda_1) w_2(\lambda_2) 0 \mathcal{F}_2^{(a-2)}(\lambda, \lambda_1, \lambda_2) \mathcal{T}_{a-2,a}(\lambda) |0\rangle 
\]

\[
\text{for } 1 \leq a \leq N, \quad (A.2)
\]

Here we recall that functions \( P_a(\lambda, \mu) \) have been defined in the one-particle problem, see Eq. (137). The dependence of the extra functions \( \tilde{P}_a(\lambda, \lambda_1, \lambda_2) \) on the \( R \)-matrix elements are given by,

\[
\tilde{P}_1(\lambda, \lambda_1, \lambda_2) = \frac{R(\lambda_1, \lambda_2)^{3,1}}{R(\lambda_1, \lambda_2)^{2,2}} \left[ \frac{R(\lambda_2, \lambda)^{2,1}}{R(\lambda_1, \lambda)^{3,1}} + \frac{R(\lambda_1, \lambda)^{2,2}}{R(\lambda_1, \lambda)^{3,1}} \right] \quad (A.3)
\]

\[
P_a(\lambda, \lambda_1, \lambda_2) = \frac{R(\lambda_1, \lambda_2)^{3,1}}{R(\lambda_1, \lambda_2)^{2,2}} \left[ \frac{R(\lambda_2, \lambda)^{a,2}}{R(\lambda_1, \lambda)^{a+1,1}} D_2^{(a+1,1)}(\lambda, \lambda_1) + \frac{R(\lambda_1, \lambda)^{a,2}}{R(\lambda_1, \lambda)^{a+1,1}} D_2^{(a+1,0)}(\lambda, \lambda_1) \right] 
- \frac{D_2^{(a,1)}(\lambda, \lambda_1)}{D_2^{(a,0)}(\lambda, \lambda_1)} R(\lambda_1, \lambda_2)^{a-1,2} \quad \text{for } 2 \leq a \leq N - 2 \quad (A.4)
\]
\[ \bar{P}_{N-1}(\lambda, \lambda_1, \lambda_2) = \frac{R(\lambda, \lambda_2)^{3,1}_{2,1} R(\lambda, \lambda_1)^{N,1}_{N,1} R(\lambda, \lambda_1)^{N,2}_{N,2}}{R(\lambda, \lambda_2)^{2,2}_{3,1} R(\lambda, \lambda_1)^{N,2}_{N,1} R(\lambda, \lambda_1)^{N,2}_{N,2}} + \frac{R(\lambda, \lambda_2)^{2,2}_{3,1} R(\lambda, \lambda_1)^{N,1}_{N,1}}{R(\lambda, \lambda_2)^{3,1}_{3,1} R(\lambda, \lambda_1)^{N,2}_{N,2}} \]

\[ \bar{P}_N(\lambda, \lambda_1, \lambda_2) = \frac{R(\lambda, \lambda_2)^{3,1}_{3,1} R(\lambda, \lambda_1)^{N,3}_{N,2}}{R(\lambda, \lambda_2)^{2,2}_{3,1} R(\lambda, \lambda_1)^{N,2}_{N,2} R(\lambda, \lambda_1)^{N,1}_{N,2}} - \frac{D^{(N-1,1)}_2(\lambda, \lambda_1) R(\lambda, \lambda_2)^{N-2,2}_{N-1,1}}{D^{(N-1,0)}_2(\lambda, \lambda_1) R(\lambda, \lambda_2)^{N-1,1}_{N-1,1}} \] (A.5)

For sake of clarity we have divided the functions proportional to the unwanted terms in two distinct categories. The first class consists of those that carry an internal discrete index dependence \( \mathcal{H}^{(a)}_b(\lambda, \lambda_1, \lambda_2|1) \) or \( \mathcal{H}^{(a)}_b(\lambda, \lambda_1, \lambda_2|2) \) where \( 1 \leq b \leq 2, \quad 1 \leq a \leq N - b, \quad b - 1 \leq c \leq 1 \). The expressions for functions \( \mathcal{H}^{(a)}_b(\lambda, \lambda_1, \lambda_2|1) \) are

\[ \mathcal{H}^{(a)}_1(\lambda, \lambda_1, \lambda_2|1) = \frac{R(\lambda, \lambda_1)^{1,1}_{1,1}}{R(\lambda, \lambda_2)^{1,1}_{2,1}} \mathcal{F}^{(a)}_1(\lambda, \lambda_1) \quad \text{for} \quad a = 1, \ldots, N - 1 \] (A.7)

\[ \mathcal{H}^{(a)}_0(\lambda, \lambda_1, \lambda_2|1) = P_2(\lambda_1, \lambda_2) \mathcal{F}^{(a)}_1(\lambda, \lambda_1) \quad \text{for} \quad a = 1, \ldots, N - 1 \] (A.8)

\[ \mathcal{H}^{(a-1)}_2(\lambda, \lambda_1, \lambda_2|1) = \mathcal{F}^{(a)}_1(\lambda, \lambda_1) \mathcal{H}^{(a-1)}_2(\lambda, \lambda_1, \lambda_2|2) \quad \text{for} \quad a = 2, \ldots, N - 1 \] (A.9)

The explicit expressions for functions \( \mathcal{H}^{(a)}_b(\lambda, \lambda_1, \lambda_2|2) \) are in general very cumbersome. Fortunately, thanks to the exchange symmetry of the two-particle vector \( \mathcal{E}_{139} \) there exists a direct relationship between functions \( \mathcal{H}^{(a)}_b(\lambda, \lambda_1, \lambda_2|2) \) and \( \mathcal{H}^{(a)}_b(\lambda, \lambda_1, \lambda_2|1) \). In fact, by considering the permutation \( \lambda_1 \leftrightarrow \lambda_2 \) on the two-particle state result \( \mathcal{E}_{12} \) we are able to derive the following consistency relation,

\[ \mathcal{H}^{(a)}_b(\lambda, \lambda_1, \lambda_2|2) = \theta(\lambda_1, \lambda_2) \mathcal{H}^{(a)}_b(\lambda, \lambda_2, \lambda_1|1) \] (A.10)

However, in the course of our analysis we find that explicit expressions for the following special families \( \mathcal{H}^{(a)}_1(\lambda, \lambda_1, \lambda_2|2) \) and \( \mathcal{H}^{(a+1)}_1(\lambda, \lambda_1, \lambda_2|2) \) for \( a = 1, \cdots, N - 1 \) are indeed very relevant. For this reason it is necessary to quote their expressions, namely

\[ \mathcal{H}^{(1)}_1(\lambda, \lambda_1, \lambda_2|2) = \frac{R(\lambda_2, \lambda_1)^{2,1}_{1,2}}{R(\lambda_2, \lambda_1)^{2,1}_{2,1}} P_2(\lambda_1, \lambda) - \frac{R(\lambda_1, \lambda_2)^{2,1}_{1,2}}{R(\lambda_1, \lambda_2)^{2,1}_{2,1}} \mathcal{H}^{(1)}_1(\lambda, \lambda_1, \lambda_2|2) \] (A.11)
The remaining functions to be defined are \( 0\mathcal{F}_2^{(a)}(\lambda, \lambda_1, \lambda_2) \) and \( 2\mathcal{F}_2^{(a)}(\lambda, \lambda_1, \lambda_2) \). Their explicit expression in terms of the weights are,

\[
0\mathcal{F}_2^{(a)}(\lambda, \lambda_1, \lambda_2) = \frac{R(\lambda, \lambda_1)^{a \lambda_1 + 1} R(\lambda, \lambda_2)^{a \lambda_2 + 1}}{R(\lambda, \lambda_1)^{a \lambda_1 + 1} R(\lambda, \lambda_2)^{a \lambda_2 + 1}} - \frac{R(\lambda, \lambda_1)^{a \lambda_1 + 1} R(\lambda, \lambda_2)^{a \lambda_2 + 1}}{R(\lambda, \lambda_1)^{a \lambda_1 + 1} R(\lambda, \lambda_2)^{a \lambda_2 + 1}}
\]

for \( 1 \leq a \leq N - 2 \) (A.13)

\[
2\mathcal{F}_2^{(a)}(\lambda, \lambda_1, \lambda_2) = \frac{R(\lambda, \lambda_1)^{a \lambda_1 + 1} R(\lambda, \lambda_2)^{a \lambda_2 + 1}}{R(\lambda, \lambda_1)^{a \lambda_1 + 1} R(\lambda, \lambda_2)^{a \lambda_2 + 1}} - \frac{R(\lambda, \lambda_1)^{a \lambda_1 + 1} R(\lambda, \lambda_2)^{a \lambda_2 + 1}}{R(\lambda, \lambda_1)^{a \lambda_1 + 1} R(\lambda, \lambda_2)^{a \lambda_2 + 1}}
\]

for \( 2 \leq a \leq N - 2 \) (A.14)

To make progress on the structure of the two-particle problem we first investigate the wanted terms. It is essential to assure that the functional form of the products proportional to either \( \Phi_2^{(1)} \) or \( \Phi_2^{(2)} \) are exactly the same. This means that we have to demonstrate the following
relation,
\[ \tilde{P}_a(\lambda, \lambda_1, \lambda_2) = P_a(\lambda, \lambda_2) \quad \text{for} \quad a = 1, \ldots, N. \] (A.17)

The factorization property (A.17) follows from a series of identities we have derived in section (4.2). The particular cases \(a = 1\) and \(a = N\) are an immediate consequence of the Yang-Baxter equation. This is easily seen by comparing the definitions of the corresponding \(\bar{a} = (114)\) by using the relations (93,91) derived from the unitarity property. In order to show property (A.17) for the remaining cases \(2 \leq a \leq N - 1\) we have to combine together the identities derived from both the unitarity relation and the Yang-Baxter equation. In order to show Eq.(A.17) for \(2 \leq a \leq N - 2\) we just have to replace the determinants \(D_4^{i,b}(\lambda, \lambda_1)\) and \(D_5^{i,2}(\lambda, \lambda_1)\) in the Yang-Baxter identity (114) by using the relations (93,91) derived from the unitarity property. The proof for the case \(a = N - 1\) requires to consider the extension of Eqs.(93,91) when the index \(i\) goes to the value \(N - 1\). As a result of such analytical continuation we found,

\[
\lim_{i \to N-1} \frac{D_2^{(i+1,1)}(\lambda, \lambda_1)}{D_2^{(i+1,0)}(\lambda, \lambda_1)} = \frac{R(\lambda, \lambda_1)_{N-1,3}^{N,2}}{R(\lambda, \lambda_1)_{N,2}^{N,2}} = -\frac{D_5^{(N+1,2)}(\lambda, \lambda_1)}{D_4^{(N+1,3)}(\lambda, \lambda_1)} \tag{A.18}
\]

and

\[
\lim_{i \to N-1} \frac{D_3^{(i,0)}(\lambda, \lambda_1)}{D_2^{(i,0)}(\lambda, \lambda_1)} = \begin{vmatrix}
R(\lambda, \lambda_1)_{N-1,3}^{N,2} & R(\lambda, \lambda_1)_{N,2}^{N,2} & R(\lambda, \lambda_1)_{N,1}^{N,1} \\
R(\lambda, \lambda_1)_{N-1,3}^{N-1,3} & R(\lambda, \lambda_1)_{N-1,3}^{N,2} & R(\lambda, \lambda_1)_{N-1,3}^{N,1} \\
R(\lambda, \lambda_1)_{N-1,2}^{N,1} & R(\lambda, \lambda_1)_{N-1,2}^{N,1} & R(\lambda, \lambda_1)_{N-1,2}^{N,2}
\end{vmatrix} = \frac{D_4^{(N,2)}(\lambda, \lambda_1)_{N-1,2}^{N+1,4}(\lambda, \lambda_1)}{D_4^{(N,3)}(\lambda, \lambda_1)_{N-1,2}^{N+1,3}(\lambda, \lambda_1)}. \tag{A.19}
\]

To complete the proof we have to consider the analytical continuation of Eq.(114) for \(a = N - 1\) by means of relations (115-116). The replacement of the determinants \(D_4^{i,b}(\lambda, \lambda_1)\) and \(D_5^{i,2}(\lambda, \lambda_1)\) in this analytical extension of Eq.(114) with the help of Eqs.(93,A.18,A.19) leads us to relation (A.17) for \(a = N - 1\).

We now turn our attention to the structure of the unwanted terms. In order to obtain the result (141) presented in section 5.2 we first identify functions \(1H_2^{(a)}(\lambda, \lambda_1, \lambda_2|1)\) and \(1F_2^{(a)}(\lambda, \lambda_1, \lambda_2),\)

\[1H_2^{(a)}(\lambda, \lambda_1, \lambda_2|1) = 1F_2^{(a)}(\lambda, \lambda_1, \lambda_2) \tag{A.20}\]

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The next step is to carry out some simplifications on functions \( c\mathcal{F}_2^{(a)}(\lambda, \lambda_1, \lambda_2) \) for \( c = 0, 1, 2 \) proportional to undesirable terms with spin \( s = 2 \). It is possible to express these functions in terms of recurrence relations involving the one-particle weights \( a\mathcal{F}_1^{(a)}(\lambda, \lambda_1, \lambda_2), 1\mathcal{F}_1^{(a)}(\lambda, \lambda_1, \lambda_2) \) and certain \( R \)-matrix amplitudes. For function \( 0\mathcal{F}_2^{(a)}(\lambda, \lambda_1, \lambda_2) \) this follows directly by using the definition (128) in Eq.(A.14), namely

\[
0\mathcal{F}_2^{(a)}(\lambda, \lambda_1, \lambda_2) = \frac{R(\lambda, \lambda_1)_{a+1,1}^{a+2,1}}{R(\lambda, \lambda_1)_{a+2,1}} 0\mathcal{F}_1^{(a+1)}(\lambda, \lambda_2) + \frac{R(\lambda, \lambda_1)_{a+2,1}^{a+3,1}}{R(\lambda, \lambda_1)_{a+2,1}} 1\mathcal{F}_1^{(2)}(\lambda_1, \lambda_2). \tag{A.21}
\]

In the case of \( 1\mathcal{F}_2^{(a)}(\lambda, \lambda_1, \lambda_2) \) we have to use the identification (A.20) together with the definitions (A.8, A.9) as well as the property (A.10). After few manipulations we find

\[
1\mathcal{F}_2^{(a)}(\lambda, \lambda_1, \lambda_2) = 0\mathcal{F}_1^{(a)}(\lambda, \lambda_2) 1\mathcal{F}_1^{(a+1)}(\lambda, \lambda_1) R(\lambda_2, \lambda_1)_{1,1}^{1,1} \frac{1}{R(\lambda_2, \lambda_1)_{2,1}^{2,1}}. \tag{A.22}
\]

where we have used the relation \( R(\lambda_2, \lambda_1)_{2,1}^{2,1} P(\lambda_2, \lambda_1) \theta(\lambda_1, \lambda_2) = R(\lambda_2, \lambda_1)_{1,1}^{1,1} \) which can be derived considering Eqs.(127,138,140).

The last simplification concerns to show that the definitions (A.15, A.16) for functions \( 2\mathcal{F}_2^{(a)}(\lambda, \lambda_1, \lambda_2) \) are equivalent to the following expression,

\[
2\mathcal{F}_2^{(a)}(\lambda, \lambda_1, \lambda_2) = -0\mathcal{F}_2^{(a)}(\lambda, \lambda_1, \lambda_2) - \sum_{i,j=1, j \neq i}^{2} 1\mathcal{F}_2^{(a)}(\lambda, \lambda_i, \lambda_j) \frac{R(\lambda_j, \lambda_i)_{2,1}^{2,1}}{R(\lambda_2, \lambda_1)_{1,1}^{1,1}} \frac{R(\lambda_i, \lambda_j)_{2,1}^{2,1}}{R(\lambda_i, \lambda_j)_{2,1}^{2,1}} \prod_{k=1}^{j-1} \theta(\lambda_k, \lambda_j). \tag{A.23}
\]

In order to show the equivalence between Eqs.(A.15, A.16) and (A.23) it is necessary to use particular relations coming from the symmetry relation Eq.(A.10), namely

\[
0\mathcal{H}_1^{(a+1)}(\lambda, \lambda_1, \lambda_2|2) = \theta(\lambda_1, \lambda_2) 0\mathcal{H}_1^{(a+1)}(\lambda, \lambda_2, \lambda_1|1) \quad \text{for} \quad 1 \leq a \leq N - 2 \tag{A.24}
\]

\[
1\mathcal{H}_1^{(a)}(\lambda, \lambda_1, \lambda_2|2) = \theta(\lambda_1, \lambda_2) 1\mathcal{H}_1^{(a)}(\lambda, \lambda_2, \lambda_1|1) \quad \text{for} \quad 1 \leq a \leq N - 2. \tag{A.25}
\]

We first substitute the expressions for these functions given by Eqs.(A.7, A.11, A.12) in the relation (A.25). We then reorder the corresponding amplitude ratios of type \( \frac{R(\lambda, \mu)_{1,1}^{1,1}}{R(\lambda, \mu)_{2,1}^{2,1}} \) for \( (\lambda, \mu) = ... \)
\((\lambda_2, \lambda_1)\) and \((\lambda, \mu) = (\lambda_1, \lambda)\) with the help of identity (29). As a result we obtain,

\[
\frac{R(\lambda_2, \lambda_1)^{2,1}_{1,2}}{R(\lambda_2, \lambda_1)^{2,1}_{2,1}} P_2(\lambda_1, \lambda) - \frac{R(\lambda, \lambda_1)^{2,1}_{1,2}}{R(\lambda, \lambda_1)^{2,1}_{2,1}} R(\lambda_1, \lambda_2)^{1,2}_{2,1} - \frac{R(\lambda_1, \lambda_2)^{2,2}_{3,1}}{R(\lambda, \lambda_1)^{2,1}_{3,1}} R(\lambda_1, \lambda)^{3,1}_{3,1}
\]

\[
= \theta(\lambda_1, \lambda_2) R(\lambda_1, \lambda_2)^{1,1}_{1,2} F_1^{(a)}(\lambda, \lambda_2) \quad \text{for } a = 1
\]

(A.26)

\[
- \frac{D_2^{(a,0)}(\lambda, \lambda_1)}{D_2^{(a+1,0)}(\lambda, \lambda_1)} \left[ \frac{R(\lambda, \lambda_1)^{a,1}_{a+1,1}}{R(\lambda, \lambda_1)^{a+1,1}_{a+1,1}} \right] - \theta(\lambda_1, \lambda_2) R(\lambda_1, \lambda_2)^{1,1}_{2,1} F_1^{(a)}(\lambda, \lambda_2)
\]

(A.27)

We next multiply Eq. (A.27) by function \(0 F_1^{(a+1)}(\lambda, \lambda_1)\) and Eq. (A.24) by function \(0 F_1^{(a)}(\lambda, \lambda_1)\). By subtracting the former equation from the latter equation and by considering the explicit expressions for \(0 H_1^{(a+1)}(\lambda, \lambda_1, \lambda_2|2)\) given in Eq. (A.13) we found,

\[
0 F_1^{(a)}(\lambda, \lambda_1) 0 F_1^{(a+1)}(\lambda, \lambda_2) \frac{R(\lambda, \lambda_1)^{a+1,1}_{a+1,1}}{R(\lambda, \lambda_1)^{a+1,1}_{a+1,1}} + \frac{D_2^{(a,0)}(\lambda, \lambda_1)}{D_2^{(a+1,0)}(\lambda, \lambda_1)} 0 F_1^{(a)}(\lambda, \lambda_2) 0 F_1^{(a+1)}(\lambda, \lambda_1)
\]

\[
\times \frac{R(\lambda, \lambda_1)^{a,1}_{a+1,1}}{R(\lambda, \lambda_1)^{a+1,1}_{a+1,1}} - \frac{R(\lambda_1, \lambda_2)^{2,2}_{3,1}}{R(\lambda_1, \lambda_2)^{3,1}_{3,1}} F_1^{(a+1)}(\lambda, \lambda_1)
\]

\[
= \theta(\lambda_1, \lambda_2) R(\lambda_1, \lambda_2)^{1,1}_{1,2} F_1^{(a)}(\lambda, \lambda_2) 0 F_1^{(a+1)}(\lambda, \lambda_1).
\]

(A.28)
After few manipulations in Eq. (A.28) we are able to write it as,

\[ \frac{D_2^{(a,0)}(\lambda, \lambda_1)}{D_2^{(a+1,0)}(\lambda, \lambda_1)} R(\lambda, \lambda_1)^{a+1,1} \left( \frac{R(\lambda, \lambda_1)^a_1}{R(\lambda, \lambda_1)^{a+1,1}} - \right) \begin{array}{c|c|c}
R(\lambda, \lambda_1)^{a,3}_0 & R(\lambda, \lambda_1)^{a,3}_1 & R(\lambda, \lambda_1)^{a,3}_2 \\
R(\lambda, \lambda_1)^{a+1,2}_0 & R(\lambda, \lambda_1)^{a+1,2}_1 & R(\lambda, \lambda_1)^{a+1,2}_2 \\
R(\lambda, \lambda_1)^{a+1,2}_0 & R(\lambda, \lambda_1)^{a+1,2}_1 & R(\lambda, \lambda_1)^{a+1,2}_2
\end{array} \]

\[ \times \frac{R(\lambda_1, \lambda_2)^{2,2}_3}{R(\lambda_1, \lambda_2)^{2,1}_3} = \frac{R(\lambda_1, \lambda_2)^{2,2}_3}{R(\lambda_1, \lambda_2)^{2,1}_3} R(\lambda, \lambda_1)^{a+1,1} - 0 \mathcal{F}_1^{(a+1)}(\lambda, \lambda_2) R(\lambda, \lambda_1)^{a,2}_{a+1,1} - 0 \mathcal{F}_1^{(a)}(\lambda, \lambda_1) \]

\[ \times_1 \mathcal{F}_1^{(a+1)}(\lambda, \lambda_2) \frac{R(\lambda_2, \lambda_1)^{1,1}_2}{R(\lambda_2, \lambda_1)^{2,1}_2} - \theta(\lambda_1, \lambda_2) \frac{R(\lambda_1, \lambda_2)^{1,1}_2}{R(\lambda_1, \lambda_2)^{2,1}_2} \mathcal{F}_1^{(a)}(\lambda, \lambda_2) \mathcal{F}_1^{(a+1)}(\lambda, \lambda_1). \quad (A.29) \]

which in fact shows that Eqs. (A.16)(A.23) are equivalent once we consider Eq. (A.22).

Finally, it remains to show that Eq. (A.23) for \( a = 1 \) is equivalent to Eq. (A.15). In this case, we multiply Eq. (A.24) for \( a = 1 \) by \( 0 \mathcal{F}_1^{(1)}(\lambda, \lambda_1) \) and Eq. (A.26) by \( \mathcal{F}_1^{(2)}(\lambda, \lambda_1) \). By subtracting these equations we found,

\[ 0 \mathcal{F}_1^{(2)}(\lambda, \lambda_2) \mathcal{F}_1^{(1)}(\lambda, \lambda_1) \frac{R(\lambda_1, \lambda_1)^{2,1}_2}{R(\lambda_1, \lambda_1)^{3,1}_3} - \frac{R(\lambda_2, \lambda_1)^{2,1}_2}{R(\lambda_2, \lambda_1)^{3,1}_3} P_2(\lambda_1, \lambda) 0 \mathcal{F}_1^{(2)}(\lambda, \lambda_1) \]

\[ + 0 \mathcal{F}_1^{(2)}(\lambda_1, \lambda_2) 0 \mathcal{F}_1^{(2)}(\lambda, \lambda_1) \frac{R(\lambda_1, \lambda_1)^{3,1}_2}{R(\lambda_1, \lambda_1)^{3,1}_3} = \frac{R(\lambda_2, \lambda_1)^{1,1}_2}{R(\lambda_2, \lambda_1)^{2,1}_2} 0 \mathcal{F}_1^{(2)}(\lambda, \lambda_2) 0 \mathcal{F}_1^{(1)}(\lambda, \lambda_1) \]

\[ - \theta(\lambda_1, \lambda_2) \frac{R(\lambda_1, \lambda_2)}{R(\lambda_1, \lambda_2)^{2,1}_2} \mathcal{F}_1^{(1)}(\lambda, \lambda_2) 0 \mathcal{F}_1^{(2)}(\lambda, \lambda_1). \quad (A.30) \]

We have now to eliminate the terms \( P_2(\lambda_1, \lambda) \) and \( 0 \mathcal{F}_1^{(2)}(\lambda_1, \lambda_1) \) \( \frac{R(\lambda_1, \lambda)^{3,1}_2}{R(\lambda_1, \lambda)^{3,1}_3} \) from Eq. (A.30). This step is implemented with the help of two identities derived from the unitarity relation. More specifically, these are the identity (85) and that coming from \( U[1, 3]^3 \) of Eq. (78), namely

\[ R(\lambda, \mu)^{1,3}_3 R(\mu, \lambda)^{3,1}_3 + R(\lambda, \mu)^{2,2}_3 R(\mu, \lambda)^{3,1}_2 + R(\lambda, \mu)^{3,1}_3 R(\mu, \lambda)^{3,1}_1 = 0. \quad (A.31) \]
and as result we obtain,

\[
\frac{R(\lambda_2, \lambda_1)}{R(\lambda_2, \lambda_1)} \cdot \frac{D_2(2,1)}{D_2(2,1)}(\lambda_1, \lambda) + 1F_1^{(2)}(\lambda_1, \lambda_2) \cdot \frac{R(\lambda_1, \lambda)}{R(\lambda_1, \lambda)} = \frac{R(\lambda_2, \lambda_1)}{R(\lambda_2, \lambda_1)} \cdot \frac{D_2(2,1)}{D_2(2,1)}(\lambda_1, \lambda) + 1F_1^{(2)}(\lambda_1, \lambda_1) \cdot \frac{R(\lambda_1, \lambda)}{R(\lambda_1, \lambda)}
\]

which shows the equivalence of Eqs. (A.15, A.23) for \(a = 1\).

Putting all these results together we will find the final results presented in section 5.2.

Appendix B: The three-particle state

In this appendix we describe some of the technical details used to solve the three-particle problem. The steps necessary for the last two terms of the three-particle vector have already been discussed in the text of section 5.3. Therefore we shall concentrate our attention on the most intricate part of the calculations which is associated to the first term \(T_{1,2}(\lambda_1) \phi_2(\lambda_2, \lambda_3)\). In this case we have to carry the operators \(T_{1,a}(\lambda)\) through the state \(\Phi_3^{(1)} = T_{1,2}(\lambda_1) \phi_2(\lambda_2, \lambda_3) |0\rangle\). The first step is done with the help of Eqs. (27, 29) and as result we obtain,

\[
T_{1,1}(\lambda) \Phi_3^{(1)} = \left[ \frac{R(\lambda_1, \lambda)}{R(\lambda_1, \lambda)} \cdot \frac{D_2(2,1)}{D_2(2,1)}(\lambda_1, \lambda) \cdot T_{1,2}(\lambda_1) \cdot T_{1,1}(\lambda) \right] \Phi_2(\lambda_2, \lambda_3) |0\rangle,
\]

(B.1)

\[
T_{a,a}(\lambda) \Phi_3^{(1)} = \left[ D_2^{(a,0)}(\lambda, \lambda_1) \cdot T_{1,2}(\lambda_1) \cdot T_{a,a}(\lambda) + \sum_{\tilde{a} = 3}^{a+1} D_2^{(a,\tilde{a}-2)}(\lambda, \lambda_1) \cdot T_{1,\tilde{a}}(\lambda_1) \cdot T_{a,a+2-\tilde{a}}(\lambda) \right] \Phi_2(\lambda_2, \lambda_3) |0\rangle, \quad \text{for} \quad 2 \leq a \leq N - 1,
\]

(B.2)
\[ T_{N,N}(\lambda) \left| \phi_3^{(1)} \right\rangle = \left[ \frac{R(\lambda, \lambda_1)_{N,1}^{N,2}}{R(\lambda, \lambda_1)_{N,1}^{N,1}} T_{1,2}(\lambda_1) T_{N,N}(\lambda) + \sum_{\varepsilon=3}^{N} \frac{R(\lambda, \lambda_1)_{N,1}^{N,2}}{R(\lambda, \lambda_1)_{N,1}^{N,1}} T_{1,\varepsilon}(\lambda_1) T_{N,N+2-\varepsilon}(\lambda) - \sum_{\varepsilon=1}^{N-1} \frac{R(\lambda, \lambda_1)_{N,1}^{\varepsilon,N-\varepsilon+1}}{R(\lambda, \lambda_1)_{N,1}^{N,1}} T_{\varepsilon,N}(\lambda) T_{N-\varepsilon+1,2}(\lambda_1) \right] \phi_2(\lambda_2, \lambda_3) \left| 0 \right\rangle. \] (B.3)

By examining Eqs. (B.1-B.3) we conclude that our next problem is to compute the action of operators \( T_{d+a,a}(\lambda) \) with \( d = 0, \ldots, N - a \) on the two-particle state \( \phi_2(\lambda_2, \lambda_3) \left| 0 \right\rangle \). The main stages to disentangle this problem is as follows. For the diagonal fields \( d = 0 \) this task is implemented by using the results (141) for the two-particle state. This operation, however, is able to produce products of creation operators of the form \( T_{1,2}(\lambda_1) T_{a_1,a_1+1}(\lambda) T_{1,2}(\lambda_j) \left| 0 \right\rangle \) with \( j = 2, 3 \) and \( T_{1,2}(\lambda_1) T_{a_2,a_2+2}(\lambda) \left| 0 \right\rangle \) for \( a_i = a - i, \ldots, a \) that need to be reordered as far as the rapidity \( \lambda \) is concerned. The product of the creation fields is sorted out by commuting the operators \( T_{1,2}(\lambda_1) \) and \( T_{a_1,a_1+1}(\lambda) \) by using Eqs. (46,47). After that we carry on the diagonal and annihilation operators through the field \( T_{1,2}(\lambda_j) \) with the help of Eq. (52,126). These two steps together are able to generate the following product of creation fields \( T_{1,3}(\lambda_1) T_{a_1,a_1+1}(\lambda) \) which also has a wrong order on the rapidities \( \lambda \) and \( \lambda_1 \). This term as far as \( T_{1,2}(\lambda_1) T_{a_2,a_2+2}(\lambda) \left| 0 \right\rangle \) are fortunately fixed by another set of commutation rules defined by Eq. (41) as well as by the linear system of equations (42,43) with \( d_1 = 0 \) and \( b_1 = 3 \). Putting together all the above steps we are finally able to reorder the products of creation fields in a suitable manner.

We now turn to the computations involving \( T_{d+a,a}(\lambda) \phi_2(\lambda_2, \lambda_3) \left| 0 \right\rangle \) for \( d = 1, \ldots, N - a \). To this end we shall first discuss the cases where \( d \geq 3 \). In this situation the computations are somehow simpler since the azimuthal spin of the operator \( T_{d+a,a}(\lambda) \) exceeds that associated to the two-particle state \( \phi_2(\lambda_2, \lambda_3) \left| 0 \right\rangle \). In fact, by using the commutation rules (52,68,72,126) we can reduce the corresponding calculations to the action of several annihilators on the reference state \( \left| 0 \right\rangle \) and consequently we find,

\[ T_{d+a,a}(\lambda) \phi_2(\lambda_2, \lambda_3) \left| 0 \right\rangle = 0 \quad \text{for} \quad d \geq 3, \] (B.4)

The results for \( d = 1 \) and \( d = 2 \) are obtained after a number of extra steps are performed. First we have to commute the operators \( T_{a+1,a}(\lambda) \) and \( T_{a+2,a}(\lambda) \) with the fields \( T_{1,2}(\lambda_j) \) for
$j = 2, 3$ present on the first part of the state $\phi_2(\lambda_2, \lambda_3) |0\rangle$. This task is accomplished with the help of Eqs. (62)-(65) which generates the following type of products $T_{1,1}(\lambda_2)T_{a,a+1}(\lambda) |0\rangle$, $T_{1,1}(\lambda_2)T_{a+1,a+2}(\lambda) |0\rangle$, $T_{a,a}(\lambda)T_{2,3}(\lambda_2) |0\rangle$ and $T_{a+1,a}(\lambda)T_{2,3}(\lambda_2) |0\rangle$. We note that these terms possess either a diagonal or an annihilation operator that still need to be carried out to the right-hand side. This operation for the first two terms is done by using Eq. (72). For the last two terms we have to employ Cramer’s rule in Eq. (65) to compute the commutation rules among the fields $T_{a,a}(\lambda)$ and $T_{a+1,a}(\lambda)$ with $T_{2,3}(\mu)$. The last step concerns with the commutations among the operators $T_{a+1,a}(\lambda)$, $T_{a+1,a}(\lambda)$ and $T_{1,3}(\lambda_2)$. This is easily performed with the help of the commutation rules described at end of section 5.3 by Eqs. (68)-(74). Collecting together all the above mentioned steps we find that,

$$T_{a+2,a}(\lambda)\phi_2(\lambda_2, \lambda_3) |0\rangle = w_{a+2}(\lambda)w_1(\lambda_2)w_1(\lambda_3)T_{2,2}(\lambda_2, \lambda_3) |0\rangle$$

$$+ w_{a+1}(\lambda)w_1(\lambda_2)w_1(\lambda_3)T_{2,2}(\lambda_2, \lambda_3)T_{2,2}(\lambda_2, \lambda_3) |0\rangle$$

$$- w_{a+1}(\lambda)w_1(\lambda_2)w_1(\lambda_3)T_{2,2}(\lambda_2, \lambda_3)H_{1}(a+1)(\lambda, \lambda_2, \lambda_3) |0\rangle$$

$$+ w_a(\lambda)w_2(\lambda_2)w_2(\lambda_3)T_{2,2}(\lambda_2, \lambda_3) |0\rangle, \text{ for } a \leq N - 2$$

(B.5)

$$T_{a+1,a}(\lambda)\phi_2(\lambda_2, \lambda_3) |0\rangle = T_{1,2}(\lambda_2) T_{2,2}(\lambda_2, \lambda_3) T_{2,2}(\lambda_2, \lambda_3) T_{2,2}(\lambda_2, \lambda_3) |0\rangle$$

$$- w_a(\lambda)w_2(\lambda_2)P_{a,2}(\lambda, \lambda_2, \lambda_3) |0\rangle$$

$$+ T_{1,2}(\lambda_2) [w_{a+1}(\lambda)w_1(\lambda_3)T_{1,2}(\lambda, \lambda_2, \lambda_3)] |0\rangle$$

$$- w_a(\lambda)w_2(\lambda_3)P_{2,a}(\lambda, \lambda_2, \lambda_3) |0\rangle$$

$$+ T_{a+1,a+2}(\lambda)w_1(\lambda_2)w_1(\lambda_3)T_{2,2}(\lambda_2, \lambda_3) |0\rangle$$

$$- T_{a,a+1}(\lambda)w_1(\lambda_2)w_2(\lambda_2)H_{1}(a+1)(\lambda, \lambda_2, \lambda_3) |0\rangle$$

$$+ w_2(\lambda_2)w_1(\lambda_3)T_{2,2}(\lambda_2, \lambda_3)T_{2,2}(\lambda_2, \lambda_3) |0\rangle$$

$$+ \delta_{a}^{1}T_{a-1,a}(\lambda)w_2(\lambda_2)w_2(\lambda_3)T_{2,2}(\lambda_2, \lambda_3) |0\rangle, \text{ for } a \leq N - 1$$

(B.6)
where functions $2\mathcal{F}_2^{(a)}(\lambda, \lambda_2, \lambda_3|1)$, $P_{1,a+1}(\lambda, \lambda_2, \lambda_3)$ and $P_{2,a+2}(\lambda, \lambda_2, \lambda_3)$ are given by,

$$2\mathcal{F}_2^{(a)}(\lambda, \lambda_2, \lambda_3|1) = \frac{R(\lambda, \lambda_2)_{a+1,2}^{a+1,1} \mathcal{F}_1^{(2)}(\lambda_2, \lambda_3) + P_{2}(\lambda, \lambda_2, \lambda_3) \mathcal{F}_1^{(a+1)}(\lambda, \lambda_2) \mathcal{F}_2^{(a)}(\lambda, \lambda_3)}{R(\lambda, \lambda_2)_{a+1,1}^{a+1,1} \mathcal{F}_1^{(2)}(\lambda, \lambda_3) - \mathcal{F}_1^{(a+1)}(\lambda, \lambda_2) \mathcal{F}_2^{(a)}(\lambda, \lambda_3)} + \mathcal{F}_2^{(a)}(\lambda, \lambda_2) \mathcal{F}_1^{(a+1)}(\lambda, \lambda_3)$$

$$- \frac{R(\lambda, \lambda_2)_{a+1,1}^{a+1,1} \mathcal{F}_1^{(a)}(\lambda, \lambda_2) \mathcal{F}_1^{(a)}(\lambda, \lambda_3)}{R(\lambda, \lambda_2)_{a+1,1}^{a+1,1} \mathcal{F}_1^{(2)}(\lambda, \lambda_3) - \mathcal{F}_1^{(a+1)}(\lambda, \lambda_2) \mathcal{F}_2^{(a)}(\lambda, \lambda_3)} \frac{R(\lambda, \lambda_2)^{a+2,1}}{R(\lambda, \lambda_2)^{a+1,1}} \frac{R(\lambda, \lambda_2)^{a+2,1}}{R(\lambda, \lambda_2)^{a+1,1}}$$

$$P_{1,a+1}(\lambda, \lambda_2, \lambda_3) = P_{a+1}(\lambda, \lambda_3) \mathcal{F}_1^{(a)}(\lambda, \lambda_2) \mathcal{F}_1^{(1)}(\lambda, \lambda_2) \mathcal{F}_2^{(a)}(\lambda, \lambda_3) - \mathcal{F}_1^{(a+1)}(\lambda, \lambda_3) \mathcal{F}_1^{(a)}(\lambda, \lambda_2)$$

$$P_{2,a}(\lambda, \lambda_2, \lambda_3) = P_{a}(\lambda, \lambda_2) \mathcal{F}_1^{(a)}(\lambda, \lambda_2) \mathcal{F}_1^{(1)}(\lambda, \lambda_2) + \mathcal{F}_1^{(a)}(\lambda, \lambda_2) \mathcal{F}_1^{(a)}(\lambda, \lambda_3) \frac{R(\lambda, \lambda_2)^{a+1,1}}{R(\lambda, \lambda_2)^{a+1,1}}$$

It turns out that the above functions can be further simplified as follows. Considering Eqs. [A.13, A.14, A.22] we are able to rewrite $2\mathcal{F}_2^{(a)}(\lambda, \lambda_2, \lambda_3|1)$ as,

$$2\mathcal{F}_2^{(a)}(\lambda, \lambda_2, \lambda_3|1) = -\mathcal{F}_2^{(a)}(\lambda, \lambda_2, \lambda_3) - \mathcal{F}_2^{(a)}(\lambda, \lambda_2, \lambda_3) \frac{R(\lambda, \lambda_2)^{1,1} \mathcal{F}_1^{(1)}(\lambda_2, \lambda_3) + \mathcal{F}_1^{(a)}(\lambda, \lambda_2) \mathcal{F}_1^{(a)}(\lambda, \lambda_3)}{R(\lambda, \lambda_2)^{2,1} \mathcal{F}_1^{(2)}(\lambda, \lambda_3) - \mathcal{F}_1^{(a+1)}(\lambda, \lambda_2) \mathcal{F}_2^{(a)}(\lambda, \lambda_3)}$$

$$+ \mathcal{F}_1^{(a)}(\lambda, \lambda_2) \mathcal{H}_1^{(a+1)}(\lambda, \lambda_2, \lambda_3|2).$$

By using the Eq. [A.8, A.10] and then the direct comparison between Eq. [A.23] and Eq. [B.10] leads us to the identity,

$$2\mathcal{F}_2^{(a)}(\lambda, \lambda_2, \lambda_3|1) = 2\mathcal{F}_2^{(a)}(\lambda, \lambda_2, \lambda_3).$$

Next, by exploring the permutation property of the two-particle vector $\phi_2(\lambda_2, \lambda_3)$ it is not difficult to conclude that the right-hand side of Eqs. [B.5, B.6] must be symmetrical under the rapidity exchange $\lambda_2 \leftrightarrow \lambda_3$ and therefore we find that,

$$P_{1,a+1}(\lambda, \lambda_2, \lambda_3) = \theta(\lambda_2, \lambda_3) P_1(\lambda_3, \lambda_2) P_{a+1}(\lambda, \lambda_2) \mathcal{F}_1^{(a)}(\lambda, \lambda_3)$$

$$P_{2,a}(\lambda, \lambda_2, \lambda_3) = \theta(\lambda_2, \lambda_3) P_2(\lambda_3, \lambda_2) P_{a}(\lambda, \lambda_2) \mathcal{F}_1^{(a)}(\lambda, \lambda_3).$$
Taking into account the identities (B.11, B.12, B.13) we are able to bring Eqs. (B.5, B.6) into their more symmetrical form, namely

\[
T_{a+2}\phi_2(\lambda_2, \lambda_3) |0\rangle = w_{a+2}(\lambda) w_1(\lambda_2) w_1(\lambda_3) F_2^{(a)}(\lambda, \lambda_2, \lambda_3) |0\rangle \\
+ w_{a+1}(\lambda) w_2(\lambda_2) w_1(\lambda_3) F_2^{(a)}(\lambda, \lambda_2, \lambda_3) \left( \frac{R(\lambda_2, \lambda_3)_{1,1}^{1,1}}{R(\lambda_2, \lambda_3)_{2,1}^{2,1}} \right) \theta(\lambda_2, \lambda_3) |0\rangle \\
+ w_{a+1}(\lambda) w_1(\lambda_2) w_2(\lambda_3) F_2^{(a)}(\lambda, \lambda_2, \lambda_3) \left( \frac{R(\lambda, \lambda_3)_{1,1}^{1,1}}{R(\lambda, \lambda_2)_{2,1}^{2,1}} \right) \theta(\lambda, \lambda_3) |0\rangle \\
+ w_a(\lambda) w_2(\lambda_2) w_2(\lambda_3) F_2^{(a)}(\lambda, \lambda_2, \lambda_3) |0\rangle, \text{ for } a \leq N-2
\]

(B.14)

\[
T_{a+1, a}\phi_2(\lambda_2, \lambda_3) |0\rangle = T_{1,2}(\lambda_3) \phi_1(\lambda_2) \left( w_{a+1}(\lambda) w_1(\lambda_2) \frac{R(\lambda, \lambda_2)_{1,1}^{1,1}}{R(\lambda, \lambda_2)_{2,1}^{2,1}} P_{a+1}(\lambda, \lambda_3) \\
- w_a(\lambda) w_2(\lambda_2) \theta(\lambda_2, \lambda_3) \frac{R(\lambda, \lambda_3)_{1,1}^{1,1}}{R(\lambda, \lambda_2)_{2,1}^{2,1}} P_{a+1}(\lambda, \lambda_2) \\
+ T_{1,2}(\lambda_3) \phi_1(\lambda_3) \left( w_{a+1}(\lambda) w_1(\lambda_3) \theta(\lambda, \lambda_3) \frac{R(\lambda, \lambda_3)_{1,1}^{1,1}}{R(\lambda, \lambda_2)_{2,1}^{2,1}} P_{a+1}(\lambda, \lambda_2) \\
- w_a(\lambda) w_2(\lambda_3) \frac{R(\lambda, \lambda_2)_{1,1}^{1,1}}{R(\lambda, \lambda_2)_{2,1}^{2,1}} P_{a+1}(\lambda, \lambda_2) \\
+ T_{a+1, a+2}(\lambda) w_1(\lambda_2) w_1(\lambda_3) F_2^{(a)}(\lambda, \lambda_2, \lambda_3) |0\rangle \\
- T_{a, a+1}(\lambda) F_1^{(a)}(\lambda, \lambda_2) \phi_1(\lambda_3) \left( w_1(\lambda_2) w_2(\lambda_3) \frac{R(\lambda, \lambda_2)_{1,1}^{1,1}}{R(\lambda, \lambda_2)_{2,1}^{2,1}} P_{a+1}(\lambda, \lambda_2) \\
+ w_2(\lambda_2) w_1(\lambda_3) \frac{R(\lambda, \lambda_3)_{1,1}^{1,1}}{R(\lambda, \lambda_2)_{2,1}^{2,1}} \theta(\lambda, \lambda_3) \right) |0\rangle \\
+ \delta_a T_{a-1, a}(\lambda) w_2(\lambda_2) w_2(\lambda_3) F_2^{(a-1)}(\lambda, \lambda_2, \lambda_3) |0\rangle, \text{ for } a \leq N-1
\]

(B.15)

The commutation of the diagonal field with the last two terms $T_{1,3}(\lambda_i) \phi_1(\lambda_j)$ and $T_{1,4}(\lambda_1)$ have been already discussed in the main text of subsection 5.3 Therefore there is no need to repeat the procedure used for such two terms once again. Considering the above results together with the steps explained for the operators $T_{1,3}(\lambda_i) \phi_1(\lambda_j)$ and $T_{1,4}(\lambda_1)$ we find out the action of the operator
\( \mathcal{T}_{a,a}(\lambda) \) on the three particle ansatz \(|\Phi_3\rangle\) is given by

\[
\mathcal{T}_{a,a}(\lambda) |\Phi_3\rangle = w_a(\lambda) \prod_{i=1}^{3} P_a(\lambda, \lambda_i) |\Phi_3\rangle
\]

\[
-\tilde{\delta}_{a}^{N} \mathcal{T}_{a,a+1}(\lambda) \sum_{1 \leq i_1 < i_2 \leq 3} \sum_{j_1=1}^{3} \phi_2(\lambda_{i_1}, \lambda_{i_2}) w_1(\lambda_{j_1}) \prod_{k=1}^{2} \frac{R(\lambda_{i_k}, \lambda_{j_1})}{R(\lambda_{i_k}, \lambda_{j_1})}^{(a)}(\lambda_{j_1}, \lambda_{j_1}) |0\rangle
\]

\[
-\tilde{\delta}_{a}^{1} \mathcal{T}_{a-1,a}(\lambda) \sum_{1 \leq i_1 < i_2 \leq 3} \sum_{i_1=1}^{3} \phi_2(\lambda_{i_1}, \lambda_{i_2}) w_2(\lambda_{i_1}) \prod_{k=1}^{2} \frac{R(\lambda_{i_1}, \lambda_{i_2})}{R(\lambda_{i_1}, \lambda_{i_2})}^{(a)}(\lambda_{i_1}, \lambda_{i_1}) |0\rangle
\]

\[
-\tilde{\delta}_{a}^{N-1,N} \mathcal{T}_{a,a+2}(\lambda) \sum_{i_1=1}^{3} \sum_{j_1=1}^{3} \sum_{i_1=1}^{3} \phi_1(\lambda_{i_1}) w_1(\lambda_{j_1}) w_2(\lambda_{j_2}) \prod_{k=1}^{2} \frac{R(\lambda_{i_1}, \lambda_{j_1})}{R(\lambda_{i_1}, \lambda_{j_1})}^{(a)}(\lambda_{j_1}, \lambda_{j_1}) |0\rangle
\]

\[
\times \theta(\lambda_{i_1}, \lambda_{j_1}) \frac{R(\lambda_{i_1}, \lambda_{j_1})^{(a)}}{R(\lambda_{i_1}, \lambda_{j_1})^{(a)}} |0\rangle - \tilde{\delta}_{a}^{1,2} \mathcal{T}_{a-2,a}(\lambda)
\]

\[
\times \sum_{i_1=1}^{3} \sum_{1 \leq i_1 < i_2 \leq 3} \sum_{j_1=1}^{3} \phi_1(\lambda_{i_1}) w_2(\lambda_{i_2}) \prod_{k=1}^{2} \frac{R(\lambda_{i_1}, \lambda_{i_2})}{R(\lambda_{i_1}, \lambda_{i_2})}^{(a)}(\lambda_{i_1}, \lambda_{i_1}) |0\rangle
\]

\[
-\tilde{\delta}_{a}^{N-2,N-1,N} \mathcal{T}_{a,a+3}(\lambda) w_1(\lambda_1) w_1(\lambda_2) w_2(\lambda_{i_3}) \prod_{k=1}^{2} \frac{R(\lambda_{i_k}, \lambda_{j_1})}{R(\lambda_{i_k}, \lambda_{j_1})}^{(a)}(\lambda_{i_1}, \lambda_{j_1}) |0\rangle - \tilde{\delta}_{a}^{1,N-1,N} \mathcal{T}_{a-1,a+2}(\lambda)
\]

\[
\times \sum_{1 \leq j_1 < j_2 \leq 3} \sum_{i_1=1}^{3} w_1(\lambda_{j_1}) w_1(\lambda_{j_2}) \prod_{k=1}^{2} \theta(\lambda_{j_1}, \lambda_{j_2}) \prod_{i_1=1}^{2} \frac{R(\lambda_{i_k}, \lambda_{j_1})}{R(\lambda_{i_k}, \lambda_{j_1})}^{(a)}(\lambda_{j_1}, \lambda_{j_1}) |0\rangle
\]

where functions \(\mathcal{F}_3^{(a)}(\lambda, \lambda_1, \lambda_2, \lambda_3)\) for \(c = 0, 1, 2, 3\) have been summarized in Eqs(180-183).

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