OPTIMAL NONLINEARITY CONTROL OF SCHRÖDINGER EQUATION

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Abstract. We study the optimal nonlinearity control problem for the nonlinear Schrödinger equation $iu_t = -\Delta u + V(x)u + h(t)|u|^\alpha u$, which is originated from the Feshbach resonance management in Bose-Einstein condensates and the nonlinearity management in nonlinear optics. Based on the global well-posedness of the equation for $0 < \alpha < \frac{4}{N}$, we show the existence of the optimal control. The Fréchet differentiability of the objective functional is proved, and the first order optimality system for $N \leq 3$ is presented.

1. Introduction. In recent years, due to the experimental investigation in Bose-Einstein condensates [7, 18, 21, 22, 24, 26] and nonlinear optics [2, 6, 20, 27], the nonlinear Schrödinger equation with varying nonlinearity has been introduced, which reads

$$iu_t = -\Delta u + V(x)u + h(t)|u|^2 u.$$ 

In Bose-Einstein condensate, this model describes the dynamics of interacting atoms at zero temperature in an external potential $V(x)$, where $h(t)$ is related to the scattering length which can be tuned through an external magnetic field by the technique of Feshbach resonance to produce robust matter-wave, a typical example is to take $h(t) = \gamma_0 + \gamma_1 \cos(\omega t)$ [9]. In optics, it describes nonlinearity management for transverse beam propagation in layered optical media. In the case of $h(t)$ being periodic, the behavior of the solution of this equation have been investigated in [5, 9, 10, 11, 23, 29], et al.

From the mathematical point of view, the management problem mentioned above is essentially a nonlinearity control problem. In comparison, the coherent manipulation of quantum systems via external potentials corresponding to the linear control...
problem which reads
\[
\begin{aligned}
  iu_t &= -\Delta u + f(u) + V(x)u + W(x, t)u, \quad (t, x) \in [0, \infty) \times \mathbb{R}^N, \\
  u(0, x) &= u_0,
\end{aligned}
\]
where \( f(u) \) denotes the nonlinearity term, \( V(x) \) is a fixed external potential, and \( W(x, t) \) corresponds to the controller. A lot of works have been carried out on this problem, we refer to [1, 12, 13, 14, 15, 16, 17, 19, 28] for some related studies. However, for the nonlinearity control problem, although it has been studied in physics literature (see [6, 7, 20, 21, 22, 24] and the references therein), to our knowledge, a mathematical discussion is still lacking.

In this paper, we consider the optimal nonlinearity control problem governed by
\[
\begin{aligned}
  iu_t &= -\Delta u + V(x)u + h(t)|u|^\alpha u, \quad x \in \mathbb{R}^N, \quad t \in [0, T], \\
  u(0, x) &= u_0,
\end{aligned}
\]
where \( T > 0 \) is the final control time given arbitrarily, \( h : [0, T] \to \mathbb{R} \) is a real function which denotes the control parameter.

It is known that in the case \( h(t) = \text{constant} \neq 0 \), equation (1.1) is focusing \((h < 0)\) or defocusing \((h > 0)\), and may have stable solutions of the form \( u(x, t) = v(x)e^{-i\lambda t} \). When \( h(t) \neq \text{constant} \), this is obviously false. Notice that when \( h(t) \) is allowed to be sign-changed, equation (1.1) is of mixed type of focusing and defocusing nonlinearity. In such a situation, the global existence and long-time behavior of solutions become rather complex.

Recall that the energy \( E(t) \) corresponds to (1.1) reads
\[
E(t) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u(t, x)|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x)|u(t, x)|^2 dx + \frac{h(t)}{\alpha + 2} \int_{\mathbb{R}^N} |u(t, x)|^{\alpha + 2} dx,
\]
and its evolution is given by
\[
E'(t) = \frac{h'(t)}{\alpha + 2} \int_{\mathbb{R}^N} |u(t, x)|^{\alpha + 2} dx.
\]

It is easy to see that (1.1) enjoys mass conservation, i.e., \( \|u(t)\|_{L^2_x} = \|u_0\|_{L^2_x} \), for all \( t \in [0, T] \). But unlike the case of \( h(t) = \text{constant} \), the energy \( E(t) \) is not conserved, which will bring some difficulties in our study.

The optimal control problem for Schrödinger equation usually needs to minimize an objective functional and deduce an optimality condition to characterize the minimum of the functional [8]. In general, the objective functional consist of two parts, one is the “distance” between the solution of the state equation and the desired state, the other describes the cost through the control process. There are many possible ways of modeling the cost it takes to reach a certain prescribed expectation value, here we will adopt the framework suggested in [16], and define the objective functional \( J = J(h) \) as
\[
J(h) := (u(T), Au(T))^2_{L^2_x} + \gamma_1 \int_0^T (E'(t))^2 dt + \gamma_2 \int_0^T (h'(t))^2 dt,
\]
where \( h \in H^1(0, T) \), \( \gamma_1 \geq 0 \), \( \gamma_2 > 0 \), \( u(T) \) denotes the final state, and the operator \( A : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N) \) is assumed to be bounded linear and self-adjoint.
Remark 1. A typical choice of $A$ would be $A := I - P_{\psi}$, where $P_{\psi}$ denotes the orthogonal projection onto a given target $\psi \in L^2$. If we choose $\|\psi\|_{L^2} = \|u\|_{L^2} = 1$, then it yields

$$\langle u(T), Au(T) \rangle_{L^2} = 1 - \langle u(T), \psi \rangle_{L^2}.$$  

On the other hand, the distance between $u(T)$ and $\psi$ in $L^2$ reads

$$\int_{RN} |u(T) - \psi|^2 dx = 2 - 2 \langle u(T), \psi \rangle_{L^2},$$

which is the same as that in [19]. So, the first term of the right hand side of (1.4) is used to measure the “distance” between the final state $u(T)$ and the target $\psi$.

Our aim is to find a control function $h^* \in H^1(0, T)$ such that $J(h^*) = \inf_{h \in H^1(0, T)} J(h)$. For this purpose, we set

$$C_h(R) := \{h \in H^1(0, T) : |h_0| \leq R\},$$

where $R > 0$ is a real number given arbitrarily. We will look for a minimizer $h^*_R$ in $C_h(R)$ such that $J(h^*_R) = \inf_{C_h(R)} J(h)$. Because of the complexity of $J(h)$, it is difficult to confirm directly that $\inf_{H^1(0, T)} J(h)$ can be reached on $H^1(0, T)$. However, if this is true, obviously we have $\inf_{H^1(0, T)} J(h) = \inf_{C_h(R)} J(h)$ for $R$ large enough. There then exists $h^*_R \in C_h(R)$, such that $h^* = h^*_R$. In this case, $h^*_R$ is independent of $R$ (that is, $h^*_R = h^*_{R+1} = h^*_{R+2} = \cdots$). Otherwise, if $\inf_{H^1(0, T)} J(h)$ can not be reached on $H^1(0, T)$, then $\inf_{C_h(R)} J(h) > \inf_{H^1(0, T)} J(h)$, and when $R \to \infty$, $h^*_R$ will tends to an idea control, i.e., $J(h^*_R)$ will tends to $\inf_{H^1(0, T)} J(h)$.

In what follows, we will solve the following minimizing problem

$$J_* = \inf_{C_h(R)} J(h).$$  

(1.5)

Our main results can be stated as

Theorem 1.1. Let $0 < \alpha < \frac{4}{N}$, $\gamma_1 \geq 0$, $\gamma_2 > 0$, and $V \in C^\infty(\mathbb{R}^N)$ be subquadratic, then for every $u_0 \in \Sigma$, $R > 0$, the optimal control problem (1.5) has a minimizer $h^*_R \in C_h(R)$.

Theorem 1.2. Let $N \leq 3$, $1 \leq \alpha < \frac{4}{N}$, and $V \in C^\infty(\mathbb{R}^N)$ be subquadratic. Assume $h \in H^1(0, T)$, $u_0 \in \Sigma$, and the operator $A$ defined in (1.4) satisfies $A(\Sigma) \subset \Sigma$. Then the functional $J(h)$ is Fréchet-differentiable in $H^1(0, T)$. Moreover, for all directions $\nu \in H^1(0, T)$, it holds

$$J'(h) \nu = \int_0^T \nu(t) \int_{\mathbb{R}^N} \bar{v}(t, x)(|u|^\alpha u)(t, x) dx dt - 2 \int_0^T \nu'(t) h'(t)(\gamma_2 + \frac{\gamma_1}{(\alpha + 2)^2} \omega^2(t)) dt,$$

(1.6)

where

$$\omega(t) = \int_{\mathbb{R}^N} |u(x, t)|^{\alpha + 2} dx,$$

(1.7)

and $v \in C([0, T], \Sigma(\mathbb{R}^N))$ is the solution of the adjoint equation

$$\begin{cases}
iv + \frac{\alpha + 2}{2} h(t)|u|^\alpha v \\
+ \frac{\alpha}{2} h(t)|u|^{\alpha - 2} u^2 \bar{v} - \frac{2\gamma_1}{\alpha + 2} (h'(t))^2 \omega(t)|u|^{\alpha} u, \\
v(T) = 4i \langle u(T), Au(T) \rangle_{L^2} Au(T) .
\end{cases}$$

(1.8)
Remark 2. As an example, we can take $A = id - P_\psi$, where $P_\psi : L^2_x \to \Sigma$ is the orthogonal projection. Then $A$ satisfies the assumption $A(\Sigma) \subset \Sigma$. On the other hand, without the assumption $A(\Sigma) \subset \Sigma$, the higher regularity estimates of the solution for (1.1) is needed. We refer to [16], where the Gâteaux-differentiability of $J(h)$ is obtained by $\Sigma''$-regularity of $u$ with $m > \frac{N}{2}$, and in [15], the Fréchet-differentiability of $J(h)$ is obtained by $\Sigma^2$ regularity of $u$.

Theorem 1.2 yields the first order necessary optimality condition for problem (1.5) as following.

Corollary 1.3. Let $h^*_R \in C_b(R)$ be a solution of (1.5), $u_* = u(h^*_R)$ the solution of (1.1), $v_*$ the solution of the adjoint equation (1.8), and $\omega_*$ the function defined in (1.7) with $u$ replaced by $u_*$. Then for all $h \in C_b(R)$,

$$J'(h^*_R)(h - h^*_R) \geq 0.$$ 

Moreover, if $h^*_R \in C^0_b(R)$, the set of the inner point of $C_b(R)$, then for all $\nu \in H^1(0, T)$,

$$J'(h^*_R)\nu = 0.$$

The rest of the paper is organized as follows. In section 2 we list some lemmas and show the local and global existence of solution of (1.1). In section 3, we give the proof of Theorem 1.1. In section 4, we first show the the well-posedness of the adjoint equation and the Lipschitz continuity of the solution $u(h)$ with respect to $h$, and then prove the Fréchet-differentiability of $J(h)$.

Notations and conventions. We use the abbreviations $L^p_x = L^p(\mathbb{R}^N, \mathbb{C})$, $H^1_x = H^1(\mathbb{R}^N, \mathbb{C})$ and $W^{1,r}_x = W^{1,r}(\mathbb{R}^N, \mathbb{C})$. The scalar product on $L^2_x$ is defined by

$$\langle \zeta, \xi \rangle_{L^2_x} = \mathbb{R} \int_{\mathbb{R}^N} \zeta(x)\bar{\xi}(x)dx,$$

where $\mathbb{R}$ means to take the real part of the complex number $z$.

We also define $\Sigma^{1,r} := \{u \in W^{1,r}_x; xu \in L^r_x\}$ with the norm

$$\|u\|_{\Sigma^{1,r}} = \|u\|_{W^{1,r}_x} + \|xu\|_{L^r_x}.$$ 

$\Sigma^{1,2}$ will be denoted by $\Sigma$, and $\Sigma^*$ is the dual space of $\Sigma$.

The norm of the space $L^q(I, L^s_x)$ is denoted by $\|\cdot\|_{L^q(I, L^s_x)}$. Recall that $(q, r)$ is an admissible pair if $2 \leq r \leq \frac{2N}{N-2}$ ($2 \leq r < \infty$ if $N = 1$, $2 \leq r < \infty$ if $N = 2$) and $\frac{q}{r} = N(\frac{1}{2} - \frac{1}{r})$. The Strichartz space $S(I)$ will also be used in the following, which is defined as the closure of the Schwartz functions under the norm

$$\|u\|_{S(I)} = \sup_{(q,r)\text{-admissible}} \|u\|_{L^q(I, L^s_x)}.$$ 

Similarly, we define the space $S_\Sigma(I)$ with the norm

$$\|u\|_{S_\Sigma(I)} = \sup_{(q,r)\text{-admissible}} \|u\|_{L^q(I, \Sigma^{1,r})}.$$ 

We always assume that $u_0 \in \Sigma$, $h \in H^1(0, T)$, and the external potential $V(x)$ is assumed to be smooth and subquadratic, that is

$$V \in C^\infty(\mathbb{R}^N, \mathbb{R})$$

and $\partial^k V \in L^\infty(\mathbb{R}^N, \mathbb{R})$, $\forall k \in \mathbb{N}^N, |k| \geq 2$.

Throughout this paper, $C > 0$ will stand for a constant that may be different from line to line when it dose not cause any confusion.
2. Local and global existence. In this section, we give some preliminary results. We say \( u \) is a mild solution of (1.1) if it satisfies

\[
\dot{u}(t) = U(t)u_0 - i \int_0^t U(t - s)h(s)|u(s)|^\alpha u(s)ds,
\]

where \( U(t) = e^{-itH}, H = -\triangle + V(x) \). Usually, (2.1) is also called the Duhamel's formulation for (1.1).

The following Strichartz's estimates [3, 4] will be invoked throughout this paper.

**Lemma 2.1 (Strichartz's estimates).** Let \( U(t) \) be defined as above, then there exists an \( \eta > 0 \) such that

1. For any admissible pair \((q,r)\) and every \( \varphi \in L^2 \), there exists a constant \( C(q) > 0 \) such that

\[
\|U(t)\varphi\|_{L^q_t L^\infty_r(0,\eta)} \leq C(q)\|\varphi\|_{L^2}.
\]

2. Let \( s \in \mathbb{R} \). If \((\gamma, \rho)\) is an admissible pair and \( f \in L^\gamma_t L^\rho_r(s, s + \eta) \), let

\[
D_s(f) = \int_s^t U(t - s)f(\tau, x)d\tau,
\]

then for every admissible pair \((q,r)\), there exists a constant \( C(q, \gamma) > 0 \), such that

\[
\|D_s(f)\|_{L^q_t L^\infty_r(s,t)} \leq C(q, \gamma)\|f\|_{L^\gamma_t L^\rho_r(s,t)}
\]

for all \( s < t < s + \eta \) and \( f \in L^\gamma_t L^\rho_r(s, s + \eta) \).

In the next, we establish the local existence for system (1.1).

**Lemma 2.2 (Local-existence).** Assume that \( 0 < \alpha < \frac{4}{N - 2} \) if \( N \geq 3 \) or \( 0 < \alpha < \infty \) if \( N = 1, 2 \). Let \( h \in H^1(0, T) \) and \( u_0 \in \Sigma \), then there exists \( l > 0 \) such that for all admissible pair \((q,r)\), (1.1) admits a unique solution \( u \in C((0, l), \Sigma) \cap L^q((0, l), \Sigma^{1/r}) \). Moreover, \( \|u\|_{L^q((0, l), \Sigma^{1/r})} \leq C\|u_0\|_{\Sigma} \).

**Proof.** The proof of the lemma is similar to that of Theorem 4.11.1 in [4]. The main difficulties are to deal with the term including subquadratic potential \( V(x) \), and give an estimate for \( xu \). We firstly present the proof for \( N \geq 3 \).

In order to estimate the nonlinearity \( h|u|^\alpha u \), we choose an admissible pair \((q_0, r_0)\) which satisfies

\[
1 - \frac{2}{r_0} = \frac{\alpha}{2} \quad \text{and} \quad \frac{1}{q_0} = \frac{1}{\theta} + \frac{1}{q_0},
\]

an easy calculation shows that

\[
\theta = \frac{4}{4 - (N - 2)\alpha} > 1, \quad \text{for} \ N \geq 3.
\]

Since \( H^1(0, T) \) can be embedded continuously into \( L^p(0, T) \) for all \( 1 \leq p \leq \infty \), from \( h \in H^1(0, T) \) we know that \( h \in L^p(0, T) \).

For every \( u_0 \in \Sigma \), from Lemma 2.1, there exist \( \eta \) and \( K \) such that

\[
\|U(\cdot)\varphi\|_{L^\infty((0, \eta), \Sigma)} + \|U(\cdot)u_0\|_{L^{q_0}(0, \eta), \Sigma^{1/r_0}} \leq K\|u_0\|_{\Sigma}.
\]

Let \( l \leq \min\{T, \eta\} \) and set

\[
E := \{ u \in L^\infty((0, l), \Sigma) \cap L^{q_0}((0, l), \Sigma^{1/r_0}) \mid \|u\|_{L^{\infty}(0, l)} \leq (K + 1)\|u_0\|_{\Sigma}\}.
\]

Endow \( E \) with the metric \( d(u, v) = \|u - v\|_{S((0, l), \Sigma)} \), then \( (E, d) \) is a complete metric space. Denote the right hand side of the (2.1) by \( \Phi(u)(t) \), in order to prove Lemma 2.2, it suffices to show that \( \Phi \) is a contraction mapping from \( E \) into itself.
Choose $l$ small enough, by Lemma 2.1, Hölder’s inequality and Sobolev’s inequality, we have
\[
\|\Phi(u)\|_{L_t^\infty L_x^p(0,l)} \leq K\|u_0\|_{L_x^2} + K\|h\|_{L_x^{\infty} L_t^{\infty}(0,l)} \|u\|_{L_t^\infty L_x^p(0,l)} \|u\|_{L_t^\infty L_x^p(0,l)} + C_1 K \|h\|_{L_t^\infty L_x^p(0,l)} \|u\|_{L_t^\infty L_x^p(0,l)} + C_1 K \|h\|_{L_t^\infty L_x^p(0,l)} (K + 1)^{\alpha + 1} \|u_0\|_{L_x^{\infty}} \|u_0\|_{L_x^{\infty}}.
\] (2.2)

On the other hand

\[
i\partial_t [\nabla, U(t)] = H[\nabla, U(t)] + H[\nabla, U(t)] = H[\nabla, U(t)] + \nabla V U(t),
\]
which implies
\[
[\nabla, U(t)] = -i \int_0^t U(t - s) \nabla V U(s) ds.
\]
Similarly, it holds that
\[
[x, U(t)] = -i \int_0^t U(t - s) \nabla U(s) ds.
\]
Hence, back to (2.1), we obtain
\[
\nabla \Phi(u)(t) = U(t) \nabla u_0 - i \int_0^t U(t - s) h(s) \nabla (|u(s)|^\alpha u(s)) ds
\]
\[- i \int_0^t U(t - s) \Phi(u)(s) \nabla V ds.
\]
and
\[
x \Phi(u)(t) = U(t) x u_0 - i \int_0^t U(t - s) h(s) |u(s)|^\alpha (x u(s)) ds
\]
\[- i \int_0^t U(t - s) \nabla \Phi(u)(s) ds.
\]
Since $\nabla V$ is sublinear by the assumption of $V(x)$, there exists a $C > 0$ which depends only on $V$ such that
\[
\|\nabla V \Phi(u)\|_{L_x^2} \leq C\|x \Phi(u)\|_{L_x^2} + C\|\Phi(u)\|_{L_x^2}.
\]
Therefore, we have
\[
\|\nabla \Phi(u)\|_{L_t^\infty L_x^p(0,l)} \leq K\|\nabla u_0\|_{L_x^2} + C_1 K \|h\|_{L_t^{\infty} L_x^p(0,l)} \|\Phi(u)\|_{L_t^\infty L_x^p(0,l)}
\]
\[+ C_2 K \|h\|_{L_t^{\infty} L_x^p(0,l)} \|\Phi(u)\|_{L_t^\infty L_x^p(0,l)} + C_1 K \|h\|_{L_t^{\infty} L_x^p(0,l)} \|u_0\|_{L_x^{\infty}} \|u_0\|_{L_x^{\infty}} \] (2.5)

and
\[
\|x \Phi(u)\|_{L_t^\infty L_x^p(0,l)} \leq K\|x u_0\|_{L_x^2} + C_1 K \|h\|_{L_t^{\infty} L_x^p(0,l)} \|u\|_{L_t^\infty L_x^p(0,l)} \|u\|_{L_t^\infty L_x^p(0,l)} + C_2 K \|h\|_{L_t^{\infty} L_x^p(0,l)} \|\Phi(u)\|_{L_t^\infty L_x^p(0,l)}
\]
\[+ C_1 K \|h\|_{L_t^{\infty} L_x^p(0,l)} \|u_0\|_{L_x^{\infty}} \|u_0\|_{L_x^{\infty}} \] (2.6)

Collecting (2.2), (2.5) and (2.6), we have
\[
\|\Phi(u)\|_{L_t^\infty L_x^p(0,l)} \leq K\|u_0\|_{L_x^{\infty}} + C_1 K \|h\|_{L_t^{\infty} L_x^p(0,l)} (K + 1)^{\alpha + 1} \|u_0\|_{L_x^{\infty}} + C_2 K \|\Phi(u)\|_{L_t^\infty L_x^p(0,l)}.
\] (2.7)
Since \((\gamma, \rho)\) in (2.7) is an arbitrarily admissible pair, we infer that 
\[
\|\Phi(u)\|_{S^2(0,t)} \leq (1 - C_2 Kl)^{-1}[K\|u_0\|_\Sigma + C_1 K\|h\|_{H^1(0,t)}(K + 1)^\alpha + 1\|u_0\|_\Sigma^{\alpha + 1}],
\]
for \(0 < l < (C_2 K)^{-1}\).

Choosing \(l\) small enough such that 
\[
(1 - C_2 Kl)^{-1} K \leq K + \frac{1}{2},
\]
and 
\[
C_1 \|h\|_{H^1(0,t)}(K + 1)^\alpha + 1\|u_0\|_\Sigma < \frac{1}{2},
\]
it is easy to see that \(\Phi\) maps \(E\) into itself and is a contraction. And then the contraction mapping theorem gives the existence and uniqueness of the solution for (1.1) on \([0, t]\).

If \(N = 2\), the proof is the same as the case \(N \geq 3\), except we set \(\theta > 1\), \((2 - \alpha)\theta \leq 1\), and let \(r_0 = 2\theta\).

If \(N = 1\), the lemma holds by setting \(\theta = 1\), \((q_0, r_0) = (\infty, 2)\), and 
\[
E := \{u \in L^\infty((0, t), \Sigma); \|u\|_{S^2(0,t)} \leq (K + 1)\|u_0\|_\Sigma\}.
\]
This completes the proof.

Because we need the global existence of solutions of equation (1.1), hereafter we mainly deal with the case \(0 < \alpha < \frac{4}{N}\), because for the case of \(\frac{4}{N} \leq \alpha \leq \frac{4}{N-2}\), the solution of (1.1) may blow up in finite time [5, 11, 22]. Comparing with [4] and [5], our method is slightly different.

The following Gronwall-type estimate is important in our research, for more details we refer to [5, 15].

**Lemma 2.3.** Let \(T > 0\), \(1 \leq p_1 < q_1 \leq \infty\), \(1 \leq p_2 < q_2 \leq \infty\) and \(A, B_1, B_2 > 0\). Then there exists \(\Gamma = \Gamma(B_1, B_2, p, q, T)\) such that if \(f_1 \in L^{q_1}(0, T), f_2 \in L^{q_2}(0, T)\) satisfies 
\[
\|f_1\|_{L^{q_1}(0,t)} + \|f_2\|_{L^{q_2}(0,t)} \leq A + B_1 \|f_1\|_{L^{p_1}(0,t)} + B_2 \|f_2\|_{L^{p_2}(0,t)},
\]
for all \(0 < t < T\), then 
\[
\|f_1\|_{L^{q_1}(0,t)} + \|f_2\|_{L^{q_2}(0,t)} \leq \Gamma.
\]

**Lemma 2.4 (Global-existence).** Assume \(0 < \alpha < \frac{4}{N}\), \(h \in H^1(0, T)\) and \(u_0 \in \Sigma\), then the solution of (1.1) is global. Moreover, for all \(t \in [0, T]\), it holds
\[
\|xu(t)\|_{L^4_x}^2 + \|\nabla u(t)\|_{L^4_x}^2 \leq M_1 e^{M_2 t},
\]
where \(M_1 = M_1(T, u_0, V, \|h\|_{H^1(0, T)})\) and \(M_2 = M_2(T, V, \|h\|_{H^1(0, T)})\) are two constants.

**Proof.** We only need to show \(u \in C([0, T], \Sigma)\), and then \(u \in L^\gamma([0, T], \Sigma^{1, \nu})\) can be obtained from Strichartz’ s estimates and Lemma 2.3.

To prove \(u \in C([0, T], \Sigma)\), it suffices to show (2.9). Since \(V\) is subquadratic by the assumption, we deduce from (1.2) and (1.3) that, for all \(t \in [0, T]\),
\[
\frac{1}{2}\|\nabla u(t)\|_{L^4_x}^2 \leq E(t) + \frac{1}{2} \int_{\mathbb{R}^N} |V(x)||u(t, x)|^2 dx + \frac{|h(t)|}{\alpha + 2}\|u(t)\|_{L^\nu_x}^{\alpha + 2}
\]
\[
\leq |E(0)| + C_1 \|u(t)\|_{L^4_x}^2 + C_2 \|xu(t)\|_{L^4_x}^2
\]
\[
+ \int_0^t \frac{|h'(s)|}{\alpha + 2}\|u(s)\|_{L^\nu_x}^{\alpha + 2} ds + \frac{\|h\|_{L^\infty(0, T)}}{\alpha + 2}\|u(t)\|_{L^\nu_x}^{\alpha + 2},
\]
(2.10)
where $C_1$ and $C_2$ only depend on $V$.

Using Gagliardo-Nirenberg’s inequality, we obtain

$$\|u(t)\|_{L^{n+2}}^{n+2} \leq C \|\nabla u(t)\|_{L^2}^{\frac{n^2}{n+2}} \|u(t)\|_{L^2}^{n+2-\frac{n^2}{n+2}}.$$  

Since $0 < \alpha < \frac{4}{n}$, we have $\frac{n^2}{n+2} < 2$, by virtue of Young’s inequality with $\epsilon$ and the conservation of mass, we obtain

$$\|u(t)\|_{L^{n+2}}^{n+2} \leq \delta \|\nabla u\|_{L^2}^2 + C_\delta,$$  

(2.11)

where $\delta > 0$ satisfies $\frac{4\delta}{\delta+2} \|h\|_{L^\infty(0,T)} < \frac{1}{4}$, and

$$C_\delta = \frac{4\delta}{N\alpha} - \frac{N\alpha}{\epsilon-\alpha} \left(1 - \frac{N\alpha}{4}\right) \|u_0\|_{L^2}^\frac{4\delta}{\delta+2}.$$  

Substituting (2.11) into (2.10), we have

$$\left(\frac{1}{2} - \frac{\delta \|h\|_{L^\infty(0,T)}}{\alpha+2}\right) \|\nabla u(t)\|_{L^2}^2 \leq |E(0)| + C_1 \|u_0\|_{L^2}^2 + \frac{C_\delta}{\alpha+2} (\|h\|_{L^\infty(0,T)} + T^\frac{\alpha}{2} \|h'\|_{L^2(0,T)})$$

$$+ C_2 \|xu(t)\|_{L^2}^2 + \int_0^t \frac{\delta |h'(s)|}{\alpha+2} \|\nabla u(t)\|_{L^2}^2 ds.$$  

As $\left(\frac{1}{2} - \frac{\delta \|h\|_{L^\infty(0,T)}}{\alpha+2}\right)^{-1} \leq 4$, it follows that

$$\|\nabla u(t)\|_{L^2}^2 \leq B_1 + 4C_2 \|xu(t)\|_{L^2}^2 + \frac{4\delta}{\alpha+2} \int_0^t |h'(s)| \|\nabla u(t)\|_{L^2}^2 ds,$$  

(2.12)

where $B_1 = 4|E(0)| + C_1 \|u_0\|_{L^2}^2 + \frac{C_\delta}{\alpha+2} (\|h\|_{L^\infty(0,T)} + T^\frac{\alpha}{2} \|h'\|_{L^2(0,T)})$.

On the other hand

$$\left| \frac{d}{dt} \int_{\mathbb{R}^N} |x|^2 |u(t,x)|^2 dx \right| = \left| 43 \int_{\mathbb{R}^N} x\bar{u}(t,x) \nabla u(t,x) dx \right|$$

$$\leq 2 \|\nabla u(t)\|_{L^2}^2 + 2 \|xu(t)\|_{L^2}^2,$$  

(2.13)

where $\Im z$ means to take the imaginary part of the complex number $z$.

Using Gronwall’s inequality, (2.12) yields

$$\|\nabla u(t)\|_{L^2}^2 \leq (B_1 + 4C_2 \|xu(t)\|_{L^2}^2) \exp\left(\frac{4\delta T^\frac{\alpha}{2}}{\alpha+2} \|h'\|_{L^2(0,T)}\right)$$

$$+ 4C_2 \int_0^t \exp\left(\frac{4\delta}{\alpha+2} \int_s^t |h'(\tau)| \frac{d}{d\tau} ds\right) \|xu(s)\|_{L^2}^2 ds$$

$$\leq B_1B_2 + 4B_2C_2 \|xu(t)\|_{L^2}^2$$

$$+ 8B_2C_2 \int_0^t (\|\nabla u(s)\|_{L^2}^2 + \|xu(s)\|_{L^2}^2) ds,$$  

(2.14)

where $B_2 = \exp\left(\frac{4\delta T^\frac{\alpha}{2}}{\alpha+2} \|h'\|_{L^2(0,T)}\right)$. 


Combining (2.13) with (2.14), we have
\[
\left| \frac{d}{dt} \| xu(t) \|_{L^2_x}^2 \right| + 3 \| xu(t) \|_{L^2_x}^2 + 3 \| \nabla u(t) \|_{L^2_x}^2 \\
\leq 5 (\| xu(t) \|_{L^2_x}^2 + \| \nabla u(t) \|_{L^2_x}^2) \\
\leq 5B_1B_2 + (20B_2C_2 + 5) \| xu(t) \|_{L^2_x}^2 \\
+ 40B_2C_2 \int_0^t (\| \nabla u(s) \|_{L^2_x}^2 + \| xu(s) \|_{L^2_x}^2) ds.
\]
Set \( B_3 = \max\{20B_2C_2 + 5, \frac{40}{3}B_2C_2\} \), using Gronwall’s inequality once again, we get
\[
\| xu(t) \|_{L^2_x}^2 + \| \nabla u(t) \|_{L^2_x}^2 \leq (5B_1B_2 + B_3 \| xu_0 \|_{L^2_x}^2) e^{B_3 t},
\]
which implies
\[
\| xu(t) \|_{L^2_x}^2 + \| \nabla u(t) \|_{L^2_x}^2 \leq (5B_1B_2 + B_3 \| xu_0 \|_{L^2_x}^2) e^{B_3 t} \text{ for all } t \in [0, T].
\]
This proves (2.9). With the help of Lemma 2.2, we conclude that \( u \in C([0, T], \Sigma) \).

Let us now turn to prove \( u \in L^\gamma_t(L^\rho_x(I_t)) \). Take \( M > 0 \) such that for all \( t \in [0, T] \), \( \| u(t) \|_{L^\rho_x} \leq M \). Set \( I_t := [t, t + \tau] \cap [0, T] \) for \( 0 < \tau \leq \eta \), it then follows from Lemma 2.1 and Hölder’s inequality that
\[
\| u \|_{L^\gamma_t(L^\rho_x(I_t))} \leq C M + CM^\alpha \| h \|_{H^\infty(I_t)} \| u \|_{L^\gamma_t(L^\rho_x(I_t))}, \tag{2.15}
\]
where \((q_0, r_0)\) is defined in the proof of Lemma 2.2.

Moreover, let \((\gamma, \rho) = (q_0, r_0)\), we conclude from Lemma 2.3 that
\[
\| u \|_{L^\gamma_t(L^\rho_x(I_t))} \leq C < +\infty,
\]
where \( C \) is independent of \( I_t \).

By Hölder’s inequality, we have
\[
\| u \|_{L^\gamma_t(L^\rho_x(I_t))} \leq \frac{q_0 - q_0}{\alpha \rho} \| u \|_{L^\gamma_t(L^\rho_x(I_t))} \leq C \eta^{\frac{q_0 - q_0}{\alpha \rho} q_0}.
\]
Substituting it into (2.15), we infer that for any admissible pair \((\gamma, \rho)\),
\[
\| u \|_{L^\gamma_t(L^\rho_x(I_t))} < +\infty.
\]
Through a finite summation over intervals \( I_j := [j\tau, (j + 1)\tau] \cap [0, T] \), \( j = 0, 1, \ldots \), we get
\[
\| u \|_{L^\gamma_t(L^\rho_x(0, T))} = \sum_j \int_{I_j} \| u(t) \|_{L^\rho_x}^\gamma dt = \sum_j \| u \|_{L^\gamma_t(L^\rho_x(I_j))} < +\infty.
\]
Similarly, we deduce from (2.3) and (2.4) that
\[
\| \nabla u \|_{L^\gamma_t(L^\rho_x(0, T))} < +\infty \text{ and } \| xu \|_{L^\gamma_t(L^\rho_x(0, T))} < +\infty.
\]
In summary,
\[
\| u \|_{L^\gamma_t(L^\rho_x(0, T))} < +\infty.
\]
This completes the proof. □
3. **Proof of Theorem 1.1.** The proof of Theorem 1.1 proceeds in three steps. In step 1, we investigate the convergence of the minimizing sequence \( \{ h_n \} \) as well as the corresponding solution sequence \( \{ u_n \} \). In step 2, we show that the limit of \( u_n \) is the solution for (1.1) with respect to the limit of \( \{ h_n \} \). In step 3, by studying the weak lower semi-continuity of the objective functional, we conclude that the limit of \( \{ h_n \} \) is indeed a minimizer of optimal control problem (1.5).

**Step 1.** For any given \( h \in C_b(R) \), Lemma 2.4 implies that \( 0 \leq J(h) < +\infty \). Thus \( J \) is bounded from below. For a minimizing sequence \( \{ h_n \} \subset C_b(R) \), the corresponding sequence \( J(h_n)(n \in \mathbb{N}) \) of the objective function is bounded on \( \mathbb{R} \), hence
\[
J(h_n) \leq C < +\infty \quad \text{for all } n \in \mathbb{N}.
\]
Since \( \gamma_2 > 0 \), we have
\[
\int_0^T |h'_n(t)|^2 \, dt \leq C < +\infty.
\]
For \( |h_n(0)| \leq R \) and \( \{ h_n \} \subset C[0,T] \), by the embedding, we have
\[
h_n(t) = h_n(0) + \int_0^t h'_n(s) \, ds \leq R + T^{1 \over 2} \left( \int_0^T |h'_n(s)|^2 \, ds \right)^{1 \over 2} < +\infty,
\]
therefore \( \{ h_n \} \) is uniformly bounded in \( L^\infty(0,T) \), and thus in \( H^1(0,T) \). Then there exists a subsequence (still denoted by \( \{ h_n \} \)) and \( h^*_R \in C_b(R) \) such that
\[
h_n \to h^*_R \quad \text{in } H^1(0,T).
\]
Moreover, there exists a subsequence such that \( h_n \to h^*_R \) in \( L^2(0,T) \cap L^p(0,T) \), and \( h_n(t) \to h^*_R(t) \), for a.e. \( t \in [0,T] \).

On the other hand, from Lemma 2.4, we have
\[
\|\nabla u_n(t)\|_{L^2}^2 + \|x u_n(t)\|_{L^2}^2 \leq M_{1,n} e^{M_{2,n}T} \quad \text{for all } t \in [0,T].
\]
In the proof of Lemma 2.4, we know that \( \{ M_{1,n} \} \) and \( \{ M_{2,n} \} \) are uniformly bounded. Therefore, there is a constant \( C \) independent of \( n \), such that
\[
\|u_n(t)\|_{\Sigma} \leq C \quad \text{for all } t \in [0,T],
\]
that is
\[
\|u_n\|_{L^\infty((0,T),\Sigma)} \leq C,
\]
so \( u_n \) is uniformly bounded in \( L^\infty((0,T),\Sigma) \), and particularly in \( L^2((0,T),\Sigma) \). By reflexivity of \( L^2((0,T),\Sigma) \), there exist a subsequence (we still denote by \( \{ u_n \} \)) and \( u_* \in L^2((0,T),\Sigma) \), such that
\[
u_n \to u_* \quad \text{in } L^2((0,T),\Sigma).
\]
On the other hand, since \( u_n \) is a solution of (1.1), we get
\[
\|u_n\|_{L^\infty((0,T),\Sigma^\ast)} \leq C.
\]
Note that the embedding \( \Sigma \hookrightarrow L^p \) is compact for \( 2 \leq p < 2^* \), by the Aubin-Lions Lemma [25], the set \( \{ u \in L^2((0,T),\Sigma) | \partial_t u \in L^\infty((0,T),\Sigma^\ast) \} \) can be embedded into \( \Sigma \) compactly for \( 2 \leq p < 2^* \). Therefore
\[
u_n \to u_\ast \quad \text{in } L^2((0,T),L^p) \quad \text{for } 2 \leq p < 2^*.
\]
By passing to a subsequence, it holds
\[
u_n(t) \to u_\ast(t) \quad \text{in } L^2_x \quad \text{and} \quad u_n(t) \to u_\ast(t) \quad \text{in } L^{2+2}_x \quad \text{for a.e. } t \in [0,T].
\]
Moreover, we can confirm that $u_n(t) \to u_*(t)$ in $\Sigma$ for a.e. $t \in [0, T]$. Indeed, fix $t \in [0, T]$ such that $u_n(t) \to u_*(t)$ in $L^2_x$. In view of (3.1), up to a subsequence, $u_n(t)$ converge weakly in $\Sigma$, this weak limit must be $u_*(t)$, since $u_n(t) \to u_*(t)$ in $L^2_x$. Furthermore, (3.1) yields that $u_* \in L^\infty((0, T), \Sigma)$.

**Step 2.** We show that $u_*$ is a mild solution of (1.1) with control $h^*_R$. Indeed, for $v_1, v_2 \in L_x^{\alpha+2}$, we have

$$
\|v_1^\alpha v_1 - |v_2|^{\alpha} v_2\|_{L_x^{\frac{\alpha+2}{\alpha}}} \leq C(\|v_1\|_{L_x^{\alpha+2}}^\alpha + \|v_2\|_{L_x^{\alpha+2}}^\alpha)\|v_1 - v_2\|_{L_x^{\alpha+2}},
$$

therefore, for a.e. $t \in [0, T]$,

$$
|u_n|^\alpha u_n(t) \to |u_*|^\alpha u_*(t) \text{ in } L_x^{\frac{\alpha+2}{\alpha}}.
$$

On the other hand

$$
u_n(t) = U(t)u_0 - i \int_0^t U(t-s)h_n(s)|u_n(s)|^\alpha u_n(s)ds \text{ for all } t \in [0, T].
$$

Let $\chi \in C^\infty_0(\mathbb{R}^N)$, then

$$
\langle u_n(t), \chi \rangle_{L_x^2} = \langle U(t)u_0, \chi \rangle_{L_x^2} - i \int_0^t \langle U(t-s)h_n(s)|u_n(s)|^\alpha u_n(s), \chi \rangle_{L_x^2}ds.
$$

By the aid of Lemma 2.4 in [16], it follows from the Lebesgue dominated convergence theorem that

$$
u_*(t) = U(t)u_0 - i \int_0^t U(t-s)h^*_R(s)|u_*(s)|^\alpha u_*(s)ds \text{ for a.e. } t \in [0, T],
$$

so $u_* \in L^\infty((0, T), \Sigma) \cap W^{1, \infty}((0, T), \Sigma^*)$ and satisfies (3.4). Based on Strichartz’s estimates, the uniqueness of the weak $\Sigma-$solution $u_*$ can be deduced by the standard argument. Using the same discussion as in the proof of Theorem 3.3.9 in [4], we conclude that $u_*$ is indeed a mild solution of (1.1), i.e., $u_* = u(h^*_R)$ satisfies $u_* \in C((0, T), \Sigma) \cap C^1((0, T), \Sigma^*)$. Furthermore

$$
u_n \to \nu_*, \text{ in } C([0, T], L_x^2) \cap C([0, T], L_x^{\alpha+2}).
$$

**Step 3.** In order to prove that $h^*_R \in C_h(R)$ is indeed a minimizer of the control problem, we need only to show

$$
J_* = \liminf_{n \to \infty} J(h_n) \geq J(h^*_R).
$$

Since $A : \Sigma \to L_x^2$ is a bounded linear operator by assumption, the sequence $(Au_n(T))_{n \in \mathbb{N}}$ converges weakly to $Au_*(T)$ in $L_x^2$, combining (3.3) with (3.5), we have

$$
|\langle u_n(T), Au_n(T) \rangle_{L_x^2} - \langle u_*(T), Au_*(T) \rangle_{L_x^2}| \to 0.
$$

As $h_n \to h^*_R$ in $H^1(0, T)$, we have

$$
\liminf_{n \to \infty} \int_0^T |h'_n(t)|^2 dt \geq \int_0^T |(h^*_R)'(t)|^2 dt.
$$

To deal with the term $\int_0^T |E'(t)|^2 dt$, in view of (1.3), we define

$$
\omega_n(t) := \int_{\mathbb{R}^N} |u_n(x, t)|^{\alpha+2} dx, \quad \omega_*(t) := \int_{\mathbb{R}^N} |u_*(x, t)|^{\alpha+2} dx.
$$
Notice that $0 \leq \omega_n(t) \leq \|u_n\|^{\alpha+2}$, then (3.2) implies that $\omega_n, \omega_s \in L^\infty(0, T)$. By the convexity of the function $|z|^{\alpha+2}$ for $z \in \mathbb{C}$, it follows from (3.3) that
\[
|\omega_n(t) - \omega_s(t)| \leq \int_{\mathbb{R}^N} \|u_n(x, t)\|^{\alpha+2} - |u_s(x, t)|^{\alpha+2} dx
\]
\[
\leq \int_{\mathbb{R}^N} |u_n(x, t)| - |u_s(x, t)|^{\alpha+2} dx
\]
\[
\to 0,
\]
which implies that $\omega_n \to \omega_s$ in $L^\infty(0, T)$. Therefore
\[
\liminf_{n \to \infty} \int_0^T |h_n'(t)|^2 |\omega_n(t)|^2 dt
\]
\[
\geq \liminf_{n \to \infty} \int_0^T |h_n'(t)|^2 |\omega_s(t)|^2 dt
\]
\[
+ \liminf_{n \to \infty} \int_0^T |h_n'(t)|^2 (|\omega_n(t)|^2 - |\omega_s(t)|^2) dt
\]
\[
\geq \int_0^T |(h_n')^2(t)|^2 |\omega_s(t)|^2 dt.
\]
Collecting (3.7), (3.8) and (3.9), we derive (3.6). This completes the proof of Theorem 1.1.

4. The Fréchet differentiability. The main purpose in this section is to prove Theorem 1.2. To this aim, we first analyze the adjoint problem corresponding to the control problem of (1.1).

Equation (1.1) can be rewritten in an abstract form
\[
P(u, h) := iu_t + \Delta v - V(x)u - h(t)|u|^\alpha u = 0.
\]
Differentiating with respect to $u$ formally, we have
\[
\partial_u P(u, h)v = iv_t + \Delta v - V(x)v - \frac{\alpha + 2}{2} h(t)|u|^\alpha v - \frac{\alpha}{2} h(t)|u|^\alpha - 2u^2 \bar{v},
\]
where $v \in L^2(\mathbb{R}^N) \subset \Sigma^\ast$. By a similar argument as in [16], we can get the following adjoint equation
\[
\begin{aligned}
iv_t + \Delta v - V(x)v - \frac{\alpha + 2}{2} h(t)|u|^\alpha v - \frac{\alpha}{2} h(t)|u|^\alpha - 2u^2 \bar{v} &= \frac{\delta J}{\delta u(t)}, \\
v(T, x) &= i \frac{\delta J}{\delta u(T)},
\end{aligned}
\]
where $u$ is the solution of (1.1) with respect to the control $h$, $\frac{\delta J}{\delta u(t)}$ and $\frac{\delta J}{\delta u(T)}$ denote the first variations of $J$ with respect to $u(t)$ and $u(T)$ respectively. Explicitly
\[
\frac{\delta J}{\delta u(t)} = \frac{2\gamma_1}{\alpha + 2} |h'(t)|^2 \left( \int_{\mathbb{R}^N} |u(t, x)|^{\alpha+2} dx \right) |u(t, x)|^\alpha u(t, x)
\]
\[
= \frac{2\gamma_1}{\alpha + 2} |h'(t)|^2 \omega(t)|u(t, x)|^\alpha u(t, x),
\]
and
\[
\frac{\delta J}{\delta u(T)} = 4(u(T, \cdot), Au(T, \cdot))_{L^2} Au(T, x).
\]
Then, (4.2) defines a Cauchy problem for $v$, and one can solve it backwards in time. Now, Let us consider the well-posedness of the Cauchy problem (4.2) as follows.
Proposition 4.1. Let \( T > 0 \), \( 0 < \alpha < \frac{4}{N} \), \( V \in C^\infty(\mathbb{R}^N) \) be subquadratic, \( h \in H^1(0,T) \) and \( u_0 \in \Sigma \). In addition, assume that the operator \( A \) satisfies \( A(\Sigma) \subset \Sigma \). Then, equation (4.2) admits a unique mild solution \( v \in C([0,T],\Sigma) \cap L^\gamma((0,T),\Sigma) \) for all admissible pairs \((\gamma,\rho)\).

Proof. By Lemma 2.4, we have \( u \in C([0,T],\Sigma) \cap L^\gamma((0,T),\Sigma) \). Then, it is easily to infer that \( \frac{\delta J(\tilde{u},h)}{\delta u} \in L^1((0,T),L^\frac{4}{\alpha+2}) \). According to the assumption on \( A \), we have \( \frac{\delta J(u,h)}{\delta u} \in \Sigma \).

Due to the inequality
\[
\| |u|^{\alpha}v_1 - |u|^{\alpha}v_2 \|_{L^\frac{4}{\alpha+2}} \leq C\|v_1 - v_2\|_{L^{\alpha+2}},
\]
and the fact that \( h \in L^\infty(0,T) \), we can get the local well-posedness by virtue of Theorem 4.6.1 in [4]. Since equation (4.2) is linear with respect to \( v \), by a classical argument for Schrödinger equation, we can get the global existence of the solution for (4.2).

Now we turn to the Fréchet differentiability of the functional \( J(h) = J(u(h),h) \).

Firstly we study the local Lipschitz continuity of \( u \) with respect to the control \( h \).

Proposition 4.2. Let \( 0 < \alpha < \frac{4}{N} \), \( V \in C^\infty(\mathbb{R}^N) \) be subquadratic, \( u_0 \in \Sigma, h,\tilde{h} \in H^1(0,T), \) and \( u,\tilde{u} \in C([0,T],\Sigma) \) be the correspond solutions of (1.1) with the controls \( h \) and \( \tilde{h} \) respectively. Then for every \( h \in H^1(0,T) \), there exists \( \epsilon > 0 \) such that if \( \|\tilde{h} - h\|_{H^1(0,T)} < \epsilon \) for every admissible pair \((\gamma,\rho)\), it holds
\[
\|\tilde{u} - u\|_{L^\gamma((0,T),\Sigma)} \leq C\|\tilde{h} - h\|_{H^1(0,T)},
\]
where \( C \) only depends on \( u_0, T, \gamma \) and \( \|h\|_{H^1(0,T)} \).

Proof. The Duhamel’s formulation (2.1) yields
\[
\tilde{u}(t) - u(t) = i \int_0^t U(t-s)[\tilde{h}(s)|u(s)|^\alpha u(s) - \tilde{h}(s)|\tilde{u}(s)|^\alpha \tilde{u}(s)]ds.
\]
Applying Strichartz’s estimates and Hölder’s inequality,
\[
\begin{align*}
\|\tilde{u} - u\|_{L^r_t L^q_x(0,t)} &\leq C\|\tilde{h}u\|_r - \tilde{h}|u|_r \tilde{u}|u|_r L^\alpha_t L^{\alpha'}_r(0,t) \\
&\leq C\|\tilde{h} - h\|_{L^r_t L^q_x(0,t)} \|u\|_{L^\infty_t L^{\infty}_x(0,t)} \|u\|_{L^\infty_t L^{\infty}_x(0,t)} + C\|\tilde{h}\|_{L^\infty_t L^{\infty}_x(0,t)} \||\tilde{u} - u\|_{L^\alpha_t L^{\alpha'}_x(0,t)} \|u\|_{L^\infty_t L^{\infty}_x(0,t)} \
&\leq C\|\tilde{h} - h\|_{H^1(0,T)} + C\|\tilde{u} - u\|_{L^\alpha_t L^{\alpha'}_x(0,t)},
\end{align*}
\]
where \((q_0,r_0)\) and \( \theta \) are defined in the proof of Lemma 2.2. Together with Lemma 2.3, we get
\[
\|\tilde{u} - u\|_{L^\infty_t L^{\infty}_x(0,T)} \leq C\|\tilde{h} - h\|_{H^1(0,T)},
\]
where \( C \) depends only on \( \|u\|_{L^\infty((0,T),\Sigma)}, \|\tilde{u}\|_{L^\infty((0,T),\Sigma)}, T, \gamma \).

Analogously, from (2.3) and (2.4), we deduce
\[
\|\nabla \tilde{u} - \nabla u\|_{L^r_t L^q_x(0,t)} \leq C\|\tilde{h} - h\|_{L^r_t L^q_x(0,t)} \|u\|_{L^\infty_t L^{\infty}_x(0,t)} \|\nabla u\|_{L^\infty_t L^{\infty}_x(0,t)}
\]
By repeating the argument above (4.1), we recall a fact below.

Let \( u \in \Sigma^+ \) and \( \chi_u \in \Sigma \) be the solution of the equation

\[-\Delta \chi_u + |x|^2 \chi_u + \chi_u = u,\]

where \( C \) depends only on \( \|u\|_{L^\infty((0,T),\Sigma)}, \|\tilde{u}\|_{L^\infty((0,T),\Sigma)}, T, \gamma. \)

And

\[
\|x\tilde{u} - xu\|_{L^1_0 L^\infty_2(t,0)} \\
\leq C\|h - \tilde{h}\|_{L^\infty((0,T))} \|u\|_{L^\infty_0 L^\infty_2(t,0)} \|\tilde{u}\|_{L^\infty_0 L^\infty_2(t,0)} \\
+ C\|\tilde{h}\|_{L^\infty((0,T))} \|u\|_{L^\infty_0 L^\infty_2(t,0)} \|x\tilde{u} - xu\|_{L^\infty_0 L^\infty_2(t,0)} \\
+ C\|\tilde{u}\|_{L^\infty_0 L^\infty_2(t,0)} + C\|\tilde{u}\|_{L^\infty_0 L^\infty_2(t,0)} + C\|\tilde{u}\|_{L^\infty_0 L^\infty_2(t,0)}
\]

(4.5)

where \( C \) depends only on \( \|u\|_{L^\infty((0,T),\Sigma)}, \|\tilde{u}\|_{L^\infty((0,T),\Sigma)}, T, \gamma. \)

Collecting (4.4), (4.5) and (4.6), using Lemma 2.3, we derive

\[
\|\tilde{u}\|_{L^\gamma((0,T),\Sigma)} \leq C\|h - \tilde{h}\|_{H^1(0,T)},
\]

where \( C \) depends only on \( \|u\|_{L^\infty((0,T),\Sigma)}, \|\tilde{u}\|_{L^\infty((0,T),\Sigma)}, T, \gamma. \)

In order to prove (4.3), we need to show that there exists \( \epsilon > 0 \), such that if

\[
\|\tilde{h} - h\|_{H^1(0,T)} < \epsilon,
\]

then there exists an \( \epsilon > 0 \) such that when

\[
\|\tilde{h} - h\|_{H^1(0,T)} < \epsilon,
\]

\[
C_1(K + 1)M^n \|\tilde{h}\|_{L^\gamma(s,s+l)} < 1.
\]

Following the proof of Lemma 2.2, we get

\[
\|\tilde{u}\|_{L^\infty((0,T),\Sigma)} \leq (K + 1)\|u_0\|_{\Sigma},
\]

and thus \( \|\tilde{u}(l)\|_{\Sigma} \leq M \). Since \( T > 0 \) is finite, we have

\[
[0, T] \subset \bigcup_{k=1}^{[\frac{T}{l}]+1} [(k - 1)l, kl] .
\]

By repeating the argument above \([\frac{T}{l}] + 1 \) times, we can derive

\[
\|\tilde{u}\|_{L^\infty((0,T),\Sigma)} \leq M\frac{l}{T}.
\]

This completes the proof. \( \square \)

To get the Fréchet-differentiability of the unconstrained functional \( J \) under \( \Sigma \)-regularity of \( u \), we recall a fact below.
In view of Proposition 4.2, we can deduce easily that

then for any $v \in \Sigma$,

$$
\langle u, v \rangle_{\Sigma^*, \Sigma} = \Re \int_{\mathbb{R}^3} (\nabla \chi_u \nabla \bar{v} + |x|^2 \chi_u \bar{v} + \chi_u \bar{v}) dx
= \Re \int_{\mathbb{R}^3} u \bar{v} dx,
$$

(4.7)

where $\langle u, v \rangle_{\Sigma^*, \Sigma}$ is the dual product.

**Proof of Theorem 1.2.** Assume that $h$ and $\tilde{h} = h + \phi$ satisfy the assumption in Proposition 4.2, $u$ and $\tilde{u}$ are the solutions of (1.1) corresponding to the control $h$ and $\tilde{h}$ respectively. By the definition of Fréchet differentiability, we will show that

$$J(\tilde{h}) - J(h) = \text{the linear terms of } \phi + \mathcal{O}(\|\phi\|^2_{H^1(0,T)}).$$

then as $\|\phi\|_{H^1(0,T)} \to 0$, the desired result can be obtained.

We write

$$J(\tilde{h}) - J(h) = J_1 + J_2 + J_3,$$

where

$$
J_1 := \langle \tilde{u}(T), A\tilde{u}(T) \rangle_{L^2}^2 - \langle u(T), Au(T) \rangle_{L^2}^2,
$$

$$J_2 := \gamma_2 \int_0^T |(\tilde{h}'(t))^2 - (h'(t))^2| dt,$$

and

$$J_3 := \frac{\gamma_1}{(\alpha + 2)^2} \int_0^T (\tilde{h}'(t))^2 (\int_{\mathbb{R}^N} |\tilde{u}|^{\alpha+2} dx)^2 - (h'(t))^2 (\int_{\mathbb{R}^N} |u|^{\alpha+2} dx)^2) dt.$$

In view of Proposition 4.2, we can deduce easily that

$$
J_1 = 4 \langle u(T), Au(T) \rangle_{L^2} \langle \tilde{u}(T) - u(T), Au(T) \rangle_{L^2} + \mathcal{O}(\|\phi\|^2_{H^1(0,T)}),
$$

(4.8)

$$J_2 = 2\gamma_2 \int_0^T h'(t) \phi'(t) dt + \mathcal{O}(\|\phi\|^2_{H^1(0,T)}).
$$

(4.9)

Next, rewrite $J_3$ as

$$J_3 = \frac{\gamma_1}{(\alpha + 2)^2} \int_0^T (\tilde{h}'(t))^2 (\int_{\mathbb{R}^N} |\tilde{u}|^{\alpha+2} dx)^2 - (h'(t))^2 (\int_{\mathbb{R}^N} |u|^{\alpha+2} dx)^2) dt
+ \frac{\gamma_1}{(\alpha + 2)^2} \int_0^T (\tilde{h}'(t))^2 (\tilde{\omega}(t)^2 - \omega^2(t)) dt,$$

where $\tilde{\omega}(t)$ and $\omega(t)$ are defined as in (1.7). Then, we can expand these terms using quadratic expansions in both $\tilde{\omega}(t)$ and $\tilde{h}'(t)$. Firstly, we have

$$(\tilde{h}'(t))^2 = (h'(t) + \phi'(t))^2 = (h'(t))^2 + 2h'(t)\phi'(t) + (\phi'(t))^2.$$

By the Lipschitz property of $u(h)$, any term with higher order than the quadratic order error of $\|\tilde{u} - u\|_{L^\infty((0,T),\Sigma)}$ is bounded by $\mathcal{O}(\|\phi\|^2_{H^1(0,T)})$, so

$$\tilde{\omega}(t) = \int_{\mathbb{R}^N} |\tilde{u}(t, x)|^{\alpha+2} dx
= \omega(t) + (\alpha + 2) \Re \int_{\mathbb{R}^N} |u|^\alpha \tilde{u} - u(t, x) dx + \mathcal{O}(\|\tilde{u} - u\|^2_{L^2}).$$
Therefore

\[
J_3 = \frac{2\gamma_1}{(\alpha + 2)} \int_0^T (h'(t))^2 \omega(t) \Re \int_{\mathbb{R}^N} |u|^\alpha u(\tilde{u} - u)(t, x) dx dt
\]

\[
+ \frac{2\gamma_1}{(\alpha + 2)} \int_0^T \phi(t) h'(t) \omega^2(t) dt + O(\|\phi\|^2_{H^1(0,T)}).
\]

(4.10)

Notice that \(\frac{2\gamma_1}{(\alpha + 2)} (h'(t))^2 \omega(t)|u|^\alpha u\) is actually the right hand side of the adjoint equation (1.8), thus the first term of the right hand side of (4.10) can be rewritten as

\[
\int_0^T \left( \frac{2\gamma_1}{(\alpha + 2)} (h'(t))^2 \omega(t) \right) |u|^\alpha (\tilde{u} - u) \Sigma \Sigma^* dt.
\]

Let \(f_j(t) \in \Sigma\) and \(f'_j(t) \in \Sigma^*, j = 1, 2,\) we have

\[
\frac{d}{dt} \Re \int_{\mathbb{R}^N} f_1 f_2 dx = \langle f_1', f_2 \rangle_{\Sigma^*, \Sigma} + \langle f_1, f'_2 \rangle_{\Sigma, \Sigma^*}.
\]

Then, by \(\bar{u}(0) = u(0),\) it follows from (4.2) and (4.7) that

\[
\int_0^T \langle v(t, \cdot), \partial_u P(u, h)(\tilde{u} - u)(t, \cdot) \rangle_{\Sigma, \Sigma^*} dt
\]

\[
- \Re \int_{\mathbb{R}^N} \bar{i} \bar{v}(T, x) (\tilde{u}(T, x) - u(T, x)) dx.
\]

(4.11)

Since \(v \in C([0, T], \Sigma)\) by Proposition 4.1 and \(\bar{u}, u \in C([0, T], \Sigma) \cap W^{1, \infty}((0, T), \Sigma^*),\) the right hand side of (4.11) is well-defined. Combining with (4.1), we have

\[
\partial_u P(u, h)(\tilde{u} - u) = i\partial_t (\tilde{u} - u) + \Delta(\tilde{u} - u) - V(\tilde{u} - u) + h(t)|u|^\alpha u
\]

\[
- \tilde{h}(t)|\tilde{u}|^\alpha + \phi(t)|\bar{u}|^\alpha \tilde{u} + h(t)R(\tilde{u}, u)
\]

where the remainder \(R(\tilde{u}, u)\) is given by

\[
R(\tilde{u}, u) = |\tilde{u}|^\alpha \tilde{u} - |u|^\alpha u - \frac{\alpha + 2}{2} |u|^\alpha (\tilde{u} - u) - \frac{\alpha}{2} |u|^\alpha - 2|u|^2 (\tilde{u} - \tilde{u}).
\]

For the complex-valued function \(|u|^\alpha u\), Taylor’s formula gives

\[
|\tilde{u}|^\alpha \tilde{u} = |u|^\alpha u - \frac{\alpha + 2}{2} |u|^\alpha (\tilde{u} - u) - \frac{\alpha}{2} |u|^\alpha - 2|u|^2 (\tilde{u} - \tilde{u})
\]

\[
+ \frac{\alpha}{2} |u|^\alpha + 2 |u|^\alpha (\tilde{u} - u)^2 + u^* |u - \tilde{u}|^2
\]

\[
+ \frac{\alpha - 2}{2} |u|^\alpha - 2|u|^2 (\tilde{u} - u)^2 + u^* |\tilde{u} - u|^2
\]

\[
+ 2|u^*|^\alpha (\tilde{u} - u)^2,
\]

where \(u^* = u + \sigma(\tilde{u} - u),\) for some \(\sigma \in [0, 1].\) So we have

\[
|\Re(\tilde{u}, u)| \leq C|u^*|^\alpha - 1|\tilde{u} - u|^2.
\]
By the assumption $\alpha \geq 1$ (this is the reason why we need $N \leq 3$), we have
\[
\int_0^T \int_{\mathbb{R}^N} |v(t, x)h(t)||\mathcal{R}(\tilde{u}, u)|dxdt \\
\leq C(T)\|h\|_{L^\infty(0, T)}\|v\|_4^{(q+2)} L^{q+2}_{L^2} \|u^*\|_{L^\infty L^{q+2}(0, T)}^{-1} \|\tilde{u} - u\|_2^2 \|L^\infty L^{q+2}_0(0, T)} \tag{4.12}
\]
\[
\leq O(\|\phi\|^2_{H^1(0, T)}).
\]
On the other hand
\[
\int_0^T \int_{\mathbb{R}^N} \bar{v}(t, x)\phi(t)|\bar{u}|^\alpha \bar{u}dxdt \\
= \int_0^T \int_{\mathbb{R}^N} \bar{v}(t, x)\phi(t)udxdt + O(\|\phi\|^2_{H^1(0, T)}).
\tag{4.13}
\]
and
\[
v(T, X) = 4i\langle u(T, \cdot), Au(T, \cdot)\rangle_{L^2} Au(T, x). \tag{4.14}
\]
Combining (4.8), (4.12), (4.13) with (4.14), we see that (4.11) is equal to
\[
-J_1 + \int_0^T \phi(t)\Re \int_{\mathbb{R}^N} \bar{v}(t, x)|u|^\alpha u\partial_t udxdt + O(\|\phi\|^2_{H^1(0, T)}). \tag{4.15}
\]
Collecting (4.8), (4.9), (4.10) and (4.15), taking $\|\phi\|_{H^1(0, T)} \to 0$, we obtain (1.6). This completes the proof. $\square$

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