Online Absolute Ranking with Partial Information: A Bipartite Graph Matching Approach

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Abstract

Ever since the introduction of the celebrated secretary problem (described in [1]), the notion of selecting from a pool of candidates before encountering every candidate has been of significant research interest. Motivated by the real-world problem of hiring in a supply-constrained environment, e.g., the tech industry, we extend the secretary problem setting to the case where assessment information is further constrained in the form of (partially) delayed scores. That is, assessment of a part of a candidate’s score will be delayed until after the candidate has left the system, with the complete score of that candidate being revealed only after a certain number of subsequent candidates (characterizing the delay in the partial score) have been encountered. A novel aspect of our study, relevant to this and other settings, is that of predicting each candidate’s absolute rank in an online fashion, i.e. before all of the candidates have been revealed. A key analytical contribution of our study involves the innovative use of weighted bipartite matching to assign absolute ranks by finding the maximum likelihood assignment of candidates to order-statistic distributions. We run extensive experiments on synthetically generated data as well as real-world data obtained from a university course. On synthetic data, we observe an average deviation of at most 1.6 ranks out of 25 from the true ranking, and on real-world partially observed data we observe a median deviation of 5.5 ranks out of 50. On fully observed real-world data, the algorithm successfully identifies the ten best students immediately upon their arrivals into the system, demonstrating that the algorithm has wide applicability for online hiring problems. Furthermore, we generalize the weighted bipartite matching algorithm to match candidates to an arbitrary set of template distributions. This is particularly useful when we are interested in a certain fraction of the candidates only (e.g. the top 10% of all the candidates) and involves significantly fewer distributions than the order statistics. For this case, we introduce a novel sparsified randomized matching algorithm with a better complexity ($O(N^2 \log N)$, where $N$ is the number of candidates) than the naïve algorithm ($O(N^3)$).

1 Introduction

Our problem is motivated by hiring in a supply-constrained environment. The hiring process can be described as a multistage funnel — candidates are screened at the top of the funnel and those
that make it though are assessed further in the subsequent stages. The nature of the assessments in each stage may be quite different – for example, the first stage might be basic domain knowledge and the second stage might test for “soft skills” such as the ability to work in a team. When candidates are likely to be actively sought after, it is important for screening decisions to be made quickly, before the candidate accepts an offer at another company. This allows the company to focus their resources on extensively evaluating top ranked candidates. The online nature of the process makes predicting the absolute rank of a candidate important — it permits the company to gauge them not just relative to those that have been seen, but to future candidates as well. We simplify the multistage funnel to have two steps: a screen that might be a phone interview or a test that can be automatically graded, and a more lengthy step that might involve a site visit or the manual grading of a longer project-based assignment. Our algorithms attempt to predict an absolute candidate ranking based on their performance on the first stage assessment.

Another setting for our work is in the context of generating crowdsourced questions. Candidates take a test in which they answer a set of questions (which are automatically graded), but they are also asked to create one or more new questions for future candidates to answer (which have to be evaluated manually). The test is scored as a combination of their performance on questions they answer and the quality of those that they create, but the candidate is given an estimate of their rank before the quality of their questions is evaluated.

1.1 Related Work and Our Contribution

There has been a series of works on online ranking and learning problems with partial information. Out of them, the most classical method is to assume the delayed data as “missing data” and perform data imputation. If we know the distribution from which the data is generated, the most naïve approach is “mean-imputation”, i.e., replace the missing data by the mean. In [2, 3] the authors explore a generalized version of problem-dependent imputation. Another option to tackle the problem is to infer the missing data from known data and side information ([4, 5]). Note that our situation is a little delicate, where the information is not lost but merely delayed. So, it is natural to ask whether one can leverage the techniques used in delayed online optimization literature. Indeed, [6], [7], and [8] provide a convex optimization formulation for the delayed data as a function of delay. However, these papers do not attempt to provide an absolute ranking. In this paper, we present a weighted bipartite matching approach to deal with missing data. This approach immediately provides a way to obtain the absolute ranking as well. Our contribution can be summarized as follows:

• We provide a matching algorithm to produce an estimate of the absolute ranks of the candidates, even though they arrive in an online fashion. This can be thought of as a generalized “secretary problem” ([11]). We validate our matching algorithm using both synthetically generated and real-world data. For synthetic data, we observe that among 25 candidates, the average deviation of the rank of a candidate is at most 1.6. Then, on real-world data, we implement a hiring rule based on the absolute rankings given by our algorithm with the goal of hiring the ten best candidates. The hiring rule hires a total of 13 candidates, all of which are ranked within the top fifteen best candidates, and hires the ten best candidates immediately upon their arrivals into the system. If we have $N$ candidates under consideration, the complexity of the algorithm is $\mathcal{O}(N^3)$ (see Subsection 2.2).

• We then move to a more realistic setting, where a fraction of the candidates has incomplete scores, and again using the bipartite matching approach, we obtain the absolute ranking. We conduct extensive experiments with the real-world exam data and the results are quite encouraging: among 50 students, the median difference between the predicted rank of a student’s final exam score (based on midterm exam scores) and the student’s true final exam rank is 5.5. In comparison, the average absolute deviation for a random ranking is 16.67.

• We extend the candidate ranking setup to a more general setting where the goal of the system to choose a certain fraction of the candidates (top 15% say). Here we introduce a novel sparsified bipartite matching algorithm, with complexity of $\mathcal{O}(N^2 \log N)$, in contrast to $\mathcal{O}(N^3)$ given by the naïve matching algorithm.
1.2 Organization of the Paper

The paper is organized as follows. In Section 2 we consider the case where the system has full information of the candidates seen up to now, and no scores are delayed. Generalizing further, in Section 3 we consider the possibility where the candidates may have a delayed score. Both the aforementioned sections are endowed with experimental validations. Furthermore, in Section 4 we extend the bipartite matching algorithm to a sparse matching problem, and provide theoretical guarantees for the new sparse matching algorithm.

2 Online Absolute Ranking with Full Information

Suppose that \( N \) candidates arrive in an online (sequential) fashion, where for \( i = 1, \ldots, N \), the \( i \)th candidate is associated with a real-valued score \( s_i \). Our goal is to maintain an online prediction of the absolute rankings of the candidates according to their scores.

Our problem can be seen as an extension of the classical secretary problem, where the goal is to maximize the probability of hiring the best candidate and the hiring procedure is restricted to making a decision of hiring or rejecting the current candidate based only on the relative ranking of the current candidates among all previously interviewed candidates. For a discussion of the history and other extensions of the secretary problem, we refer readers to [4].

2.1 System Model

To motivate the use of weighted bipartite matching for the ranking problem, suppose that there are two types of candidates: “desirable” candidates and “undesirable” candidates. If the scores of the desirable and undesirable candidates are drawn i.i.d. from known probability distributions \( P_d \) and \( P_u \) respectively, and the number of desirable candidates \( n \) is known in advance, then we can calculate the maximum likelihood matching of candidates to the distributions \( P_d \) and \( P_u \). For a total of \( N \) candidates, instead of computing the likelihood of all the \( \binom{N}{k} \) assignments of the candidates to \( P_d \) and \( P_u \) and choosing the maximizer, we can set up a bipartite graph, where the edge weights are log-likelihoods, and obtain the maximizer by computing the maximum weighted bipartite matching in polynomial time (see Subsection 2.2 for details).

More generally, if there are multiple types of candidates, where each type has a known number of candidates and is associated with a template probability distribution, then we can again compute the maximum likelihood matching of candidates to distributions, in the presence of an oracle which provides the template distributions to us. In the absence of such an oracle, the template distributions can be learned from previous data (see Section 4 for further comments on this approach).

When the template distributions have the interpretation of being ordered (i.e., corresponding to successively more desirable types of candidates) and number of template distributions equals the number of candidates, then a perfect matching of the candidates to the distributions yields a ranking of the candidates.

In our approach, instead of learning the template distributions from data, we use the distributions of the order statistics of the combined scores for the candidate types.

Suppose that the scores are drawn i.i.d. from a density \( p \). For \( k = 1, \ldots, N \), let \( p^{(k)} \) denote the density of the \( k \)th order statistic of the density \( p \).

Form a complete bipartite graph where the \( N \) candidates are the left nodes and the densities \( p^{(1)}, \ldots, p^{(N)} \) are the right nodes. For the edge connecting the \( i \)th candidate with density \( p^{(j)} \), we use the edge weight \( \log p^{(j)}(s_i) \) if candidate \( i \) has been revealed; otherwise, we set the edge value to a large negative constant. The maximum weighted perfect matching in the resulting bipartite graph maximizes the product \( \prod_{i \in R} p^{(\sigma(i))}(s_i) \) over permutations \( \sigma \) of \( \{1, \ldots, N\} \), so the matching has the interpretation of a maximum likelihood estimator for the likelihood function \( \ell(\sigma) = \sum_{i \in R} \log p^{(\sigma(i))}(s_i) \), where \( R \) is the set of indices of the revealed candidates. The assignment corresponding to \( \sigma \) is such the assignment where each candidate \( i \in R \) is assigned to \( \sigma(i) \)th order statistic, i.e. is ranked the \( \sigma(i) \)th lowest.

When all \( N \) candidates have been revealed, the likelihood function is \( \ell(\sigma) = \sum_{i=1}^{N} \log p^{(\sigma(i))}(s_i) \).
If we assume that the scores are absolutely continuous with CDF $F$ and PDF $f$, then
\[
\ell(\sigma) = \sum_{j=1}^{N} \log \left\{ N \left( \frac{N-1}{\sigma(j)-1} \right) F(s_j)^{\sigma(j)-1} [1 - F(s_j)]^{N - \sigma(j)} f(s_j) \right\}
\]
\[
= \text{constant} + \sum_{j=1}^{N} \sigma(j) \log \frac{F(s_j)}{1 - F(s_j)}.
\]
Since the function $x \mapsto \log\{F(x) / [1 - F(x)]\}$ is increasing, a standard exchange argument shows that the maximum likelihood estimator has the interpretation of sorting the candidates according to their scores, i.e., the algorithm exactly recovers the true ranking of the candidates.

### 2.2 Implementation

Given a weighted bipartite graph $G = (V, E, w)$, where $V$ is the disjoint union of $L$ and $R$, and $w_{i,j}$ is the weight of edge $(i, j)$, $i \in L$, $j \in R$, the **maximum weighted bipartite matching** in $G$ is a set of edges $M \subseteq E$ such that $M$ is a *matching* (each vertex is the endpoint of at most one edge in $M$) and $M$ maximizes the sum of weights $\sum_{e \in M} w_e$. The maximum weighted bipartite matching has an linear programming (LP) relaxation

\[
\begin{align*}
\text{maximize} & \quad \sum_{(s_e, e \in E)} x_e w_e \\
\text{subject to} & \quad \sum_{j \in V : (i,j) \in E} x_{i,j} = 1, \quad \forall i \in V \\
& \quad x_e \geq 0, \quad \forall e \in E
\end{align*}
\]

which has a corresponding dual LP

\[
\begin{align*}
\text{minimize} & \quad \sum_{(p_i, i \in V)} p_i \\
\text{subject to} & \quad p_i + p_j \geq w_{i,j}, \quad \forall (i,j) \in E.
\end{align*}
\]

The dual LP problem is the **weighted vertex cover** problem: given a weighted bipartite graph, assign a *price* to each vertex such that the weight of every edge is at most the sum of the prices of the edge’s endpoints, such that the total sum of prices is minimized. The value of any feasible solution to the dual LP is an upper bound to the value of any feasible solution to the primal LP.

According to classical LP duality theory (see [10] §6.3), an optimal primal-dual solution satisfies the *complementary slackness conditions* for LPs; in particular, for all $(i,j) \in E$, if $x^*_{i,j} > 0$, then $p_{i}^* + p_{j}^* = w_{i,j}$ for a pair of optimal solutions $(x^*_{e}, e \in E)$ and $(p^*_{i}, i \in V)$.

It is well-known that the classical Hungarian algorithm computes the maximum weighted bipartite matching in $\mathcal{O}(|V||E| + |V| \log |V|)$ time ([13]). The Hungarian algorithm is a primal-dual algorithm which maintains a feasible weighted vertex cover while iteratively increasing the cardinality of a matching which obeys the complementary slackness conditions. We next give a brief description of the Hungarian algorithm.

Given a weighted vertex cover $(p_i, i \in V)$, call the edge $(i,j) \in E$ **tight** (with respect to the vertex cover) if $p_i + p_j = w_{i,j}$. Start with a feasible weighted vertex cover by assigning the price $\max_{j \in V : (i,j) \in E} w_{i,j}$ to each $i \in L$, and the price 0 to each $i \in R$. Then, create an auxiliary bipartite graph formed from the original bipartite graph by retaining only the edges which are tight with respect to the weighted vertex cover. Note that if a maximum cardinality matching is found in the auxiliary graph, then the weight of the matching equals the weight of the vertex cover (by the definition of tightness), which provides a certificate of optimality for the primal-dual pair of solutions (the weighted matching and the weighted vertex cover).

To increase the cardinality of a matching in the auxiliary graph, search for an **augmenting path**, that is, a path in the auxiliary graph starting and ending at unmatched vertices such that the edges in the path alternate between unmatched and matched edges (a *matched edge* is an edge present in the current matching). Augmenting paths can be found using breadth-first search, and the existence of an augmenting path implies that the cardinality of the matching can be increased.

If no augmenting paths can be found, then the set of vertices which can be reached (via paths which alternate between unmatched and matched edges) from unmatched left vertices defines a
cut in the graph. Let $\delta$ be the minimum value of $p_i + p_j - w_{i,j}$ for any edge $(i, j)$ across the cut. For each vertex reachable from the unmatched left vertices, decrease the price of the vertex by $\delta$ if the vertex is a left vertex, and increase the price of the vertex by $\delta$ if the vertex is a right vertex. This operation is guaranteed to introduce a new tight edge into the auxiliary graph and decrease the overall sum of prices in the weighted vertex cover, while maintaining the dual feasibility of the weighted vertex cover. The algorithm then continues to find augmenting paths, until the algorithm terminates with an optimal weighted matching and weighted vertex cover whose values coincide.

For further details, we refer readers to [11, §11.2]. For candidate ranking, the bipartite graph is complete, so the runtime is $O(N^3)$.

Although the application of weighted bipartite matching to the candidate ranking problem requires knowledge of the order statistics of the score distribution, these distributions can be approximated via simulation.

### 2.3 Experiments

We simulated 25 candidates with scores which are i.i.d. draws from the standard Gaussian density (Experiment 1) or the uniform distribution on $[0, 1]$ (Experiment 2). The candidates are revealed sequentially, and at each iteration we ran the algorithm on the current list of candidates, producing a prediction of the absolute ranks of the candidates seen so far. We then recorded the difference between the candidate’s true rank among the 25 candidates, and the ranking assigned to the candidate by the algorithm when the candidate first enters the system. The above procedure is then repeated for 1000 trials. The results are summarized in Figure 1.

![Figure 1: The average absolute difference in ranking (AADR) between each candidate’s true rank and the ranking assigned to the candidate by the matching algorithm when the candidate first enters the system is given as a function of time step in which the candidate first enters the system. The results are averaged over 1000 trials.](image)

The results of the experiment demonstrate that the algorithm’s performance slightly improves as more candidates are revealed, which aligns with the intuition that there is less uncertainty in the remaining candidates’ ranking. However, even when only one candidate is observed, the algorithm is quite successful at predicting the absolute rank of the candidate.

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1For the simulation, the distributions are discretized to bins of size 0.01.
The algorithm performs well against both the Gaussian density and the uniform density, but the results for the uniform case are slightly better. Heuristically, this may be because of the greater separation between the order statistics of the uniform distribution.

### 2.3.1 Experiments with Real-World Data

Next, we used anonymized data from a class of 191 students at the University of California, Berkeley. The data consisted of two midterm scores and a final exam score for each student, which we combined into an overall score (with weighting 0.25 for each of the midterms and 0.5 for the final exam). After randomly shuffling the data, we used the empirical mean and variance of the first 141 data points to fit a Gaussian distribution. The scores were quantized by rounding to the nearest multiple of 3.

We streamed the remaining 50 students into our algorithm sequentially and used the following hiring rule: if the algorithm ever assigns an absolute rank to a student within the top ten, then the student is immediately hired. With this rule, the algorithm hired 13 students, a superset of the true top ten students. The other three students were also among the top fifteen best students. For all but one student, the algorithm hired the student immediately upon the student’s arrival in to the system (for the exceptional case, the student was hired four time steps after entering the system). Qualitatively, the algorithm is very successful at making timely and accurate hiring decisions.

### 3 Delayed Score Model

To apply the algorithm in practical settings, it is particularly useful to extend the algorithm to the setting in which only partial information is known about the candidates.

Suppose that $N$ candidates arrive in an online (sequential) fashion, where for $i = 1, \ldots, N$, the $i$th candidate has an instantaneous score $s_i$ and a delayed score $d_i$. The instantaneous score is revealed to the decision maker immediately upon the candidate’s arrival into the system, whereas the delayed score is revealed after a delay $\tau$, which is assumed to be a known deterministic positive integer.

The score pairs $(s_1, d_1), \ldots, (s_N, d_N)$ are assumed to be i.i.d. and drawn from a known distribution $P$ on the space $S \times D$. We assume that we are given a function $f : S \times D \to [0, \infty)$ and we call $f(s_i, d_i)$ the combined score of candidate $i$. The goal is to develop a good hiring procedure which selects candidates with large combined scores.

More precisely, assume that the distribution $P$ has a density $p_{s,d}$ with marginal densities $p_s$ and $p_d$. For each $k = 1, \ldots, N$, let $p_{s,d}^{(k)}$ be the conditional distribution of $(s_1, d_1)$ given that $f(s_1, d_1)$ is the $k$th smallest value among $f(s_1, d_1), \ldots, f(s_N, d_N)$; similarly, let $p_{d}^{(k)}$ be the conditional density of $d_1$ given $s_1$ if $f(s_1, d_1)$ is the $k$th order statistic of $f(s_1, d_1), \ldots, f(s_N, d_N)$.

Form a complete bipartite graph where the $N$ candidates are the left nodes and the densities $p_{s,d}^{(1)}, \ldots, p_{s,d}^{(N)}$ are the right nodes. For the edge connecting the $i$th candidate with density $p_{s,d}^{(j)}$, we use the edge weight $\log p_{s,d}^{(j)}(s_i, d_i)$; however, if the delayed score $d_i$ has not yet been revealed, then we use the averaged log-likelihood $\int_D p_{d}^{(j)}(z | s_i) \log p_{s,d}^{(j)}(s_i, z) \, dz = \mathbb{E}[\log p_{s,d}^{(j)}(s_i, d_i) | s_i]$ where $(s^{(j)}, d^{(j)})$ is a pair of random variables with joint distribution $p_{s,d}^{(j)}$.

Therefore, letting $K$ denote the set of encountered candidates with known delayed score, $U$ the set of encountered candidates with unknown delayed score, and $C := K \cup U$, the algorithm computes

\[
\arg \max_{\sigma \in S_N} \sum_{i \in K} \log p^{(\sigma(i))}(s_i, d_i) + \sum_{i \in U} \mathbb{E}[\log p_{s,d}^{(\sigma(i))}(s^{(\sigma(i))}, d^{(\sigma(i))}) | s^{(\sigma(i))} = s_i] \\
= \arg \max_{\sigma \in S_N} \mathbb{E} \left[ \prod_{i \in C} p_{s,d}^{(\sigma(i))}(s^{(\sigma(i))}, d^{(\sigma(i))}) \ \big| \ s^{(\sigma(i))} = s_i \ \forall i \in C, \ d^{(\sigma(i))} = d_i \ \forall i \in K \right],
\]

the maximizer of an expected log-likelihood.
Figure 2: The histograms display the absolute differences in the true ranks of the students and the ranks assigned by the matching algorithm. The combined score functions used are \( f_1 \) (Left) and \( f_2 \) (Right). For \( f_1 \), the average absolute difference in ranks is 8.32 and the median is 5.5; for \( f_2 \), the average absolute difference in ranks is 5.68 and the median is 4.5. For comparison, the average absolute difference in ranks between two permutations of \( \{ 1, \ldots, 50 \} \) chosen independently and uniformly at random is 16.67.

3.1 Experiments with Partial Information

We return to the student data set introduced in Subsection 2.3. After randomly shuffling the data, we used the empirical mean and covariance of the first 141 data points to fit a multivariate Gaussian distribution. We model the pair of midterm grades as the instantaneous score and the final exam grade as the delayed score. All scores were quantized by rounding to the nearest multiple of 5.

We consider two combined score functions. The combined score function \( f_1 \) considers only the final exam grade (Experiment 3), and the combined score function \( f_2 \) uses a weighted sum of the midterm grades and final exam grade, which is typical in a classroom setting (Experiment 4).

\[
f_1(s, d) = d, \quad f_2(s, d) = 0.25s_1 + 0.25s_2 + 0.5d.
\]

We used our algorithm to predict the absolute ranks of the remaining 50 students using only their midterm grades. For each of the two combined score functions, we plotted a histogram of the absolute difference in rank between the true ranking of the student and the ranking given by our algorithm. The results are displayed in Figure 2.

The quality of the ranking produced by the algorithm is better for the second combined score function \( f_2 \) since it places less weight on the unknown delayed score. For both choices of combined score function, the algorithm easily outperforms choosing a random ranking of the students.

Notably, there are a few outliers in the data. In Figure 2 (Right), there are two students for which the algorithm’s assigned rankings differ from the student’s true ranking by more than 30. These students have scores (normalized to be out of 100 total) of (70.8, 72, 15.4) and (58.3, 72, 15.4), which are very unusual and not predictable from the data.

It is also of interest to ask whether the algorithm is successful at identifying the top five scorers. In both Experiments 3 and 4, four of the overall top five scorers were among the top ten candidates in the ranking given by the algorithm.

4 Sparse Hungarian Algorithm for Matching Template Distributions

Another practically relevant goal is the identification of the top fraction of candidates rather than outputting a ranking of all of the candidates. For example, suppose that our goal is to hire the top 10% of the candidates and we have access to past candidates’ data. In this case, a reasonable approach is to divide the data into 10 quantiles and use an unsupervised approach to learn a template distribution for each quantile. Then, applying a matching approach, we hire the candidates that are matched to the template distribution for the top quantile.

This motivates us to study the problem of constrained matching in the setting where each distribution is associated with a large number of observations. For this problem, we give a ran-
domized algorithm which improves the $O(N^3)$ runtime of naively applying the Hungarian algorithm to $O(N^2 \log N)$. The runtime improvement is based on a random sparsification of the bipartite graph.

4.1 Problem Statement

Formally, let $k$ and $n$ be positive integers, where $k$ is the number of distributions and $n$ is the number of candidates to be matched to each distribution, so that $N = nk$ (we assume for now that the numbers of candidates coming from each distribution are equal). Thus, our candidate list is indexed by $L = \{1, \ldots, kn\}$, our distribution list is indexed by $R = \{1, \ldots, k\}$ and we wish to find the maximum likelihood assignment of $n$ candidates to each distribution.

This is solved by the general problem where one is given a bipartite graph $G = (L, R, E)$ and a weight function $w : E \to \mathbb{R}$ such that $|L| = N = nk$ and $|R| = k$. A subset of edges $M$ is called an $N : k$ matching if each left node is incident to exactly one edge in $M$ and each right node is incident to exactly $n$ edges in $M$. The value of the matching $M$ is $\sum_{e \in M} w(e)$, and the goal is to find an $N : k$ matching of maximum value.

Our initial solution is to form the bipartite graph $G' = (L, R', E')$, where $L$ is the original set of left nodes and $R'$ and $E'$ consist of each node in $R$ along with its edges in the original graph duplicated $n$ times, resulting in a bipartite graph with $kn$ left nodes and $kn$ right nodes. By duplicating all of the edges, a maximum weighted perfect matching in the augmented graph corresponds to an optimal $N : k$ matching for the original bipartite graph. However, application of the Hungarian algorithm to the augmented graph (a complete bipartite graph with $kn$ left nodes and $kn$ right nodes) takes $\Omega(|L||E'|) = \Omega(k^3 n^3)$ time.

Here, we present a randomized algorithm that improves the complexity to $O(k^2 n^2 \log n)$.

4.2 The Proposed Algorithm

We form the augmented weighted bipartite graph $G^* = (L, R', E^*)$ where $R'$ consists of $k$ copies each of the $n$ right nodes in $R$. We use the term group to refer to a set of copies of a right node in $R$; thus, $R'$ consists of $k$ groups, each containing $n$ vertices. An edge $(i, j')$ in $E^*$, where $j'$ is a copy of the vertex $j \in R$, inherits the weight $w(i, j)$ from $G$.

The idea of the algorithm is very simple. We construct $E^*$ in the following way: for each vertex $i \in L$ and each group in $R'$ independently, add $c \log n$ edges chosen without replacement, where $c$ is a constant to be chosen later. This produces a graph with $O(kn \log n)$ edges, and we run the Hungarian algorithm on the augmented graph, so the time complexity $O(k^2 n^2 \log n)$ as advertised. See Figure 3 for a visual depiction of the sparsified graph.

4.3 Analysis of the Algorithm

The correctness of the algorithm hinges on the following:

**Observation:** If all of the edges of any optimal $N : k$ matching in the original bipartite graph are included in the augmented bipartite graph, then the perfect matching recovered by the Hungarian algorithm will correspond to an optimal assignment in the original bipartite graph.

In light of the observation, it suffices to bound the probability that no optimal $N : k$ matchings in the original bipartite graph are present in the augmented bipartite graph. The key is the following:

**Theorem 1** (Hall’s Marriage Theorem, [12 Theorem 5.1]). Let $G = (L, R, E)$ be a bipartite graph on $n$ left nodes and $n$ right nodes, and for any subset $A \subseteq L$, let $\Gamma(A)$ denote the set of neighbors of vertices in $A$. A necessary and sufficient condition for $G$ to have a perfect matching is that for every subset $A \subseteq L$, $|A| \leq |\Gamma(A)|$.

Fix an optimal $N : k$ matching $M$ in $G$. We say that $M$ survives in $G^*$ if there exists a perfect matching $M'$ in $G^*$ such that for every $(i, j) \in M$, $i$ is matched to a copy of $j$ in $M'$. In order for $M$ to survive in $G^*$, it is necessary and sufficient that for each group $U \subseteq R'$, if $\{v_1, \ldots, v_n\} \subseteq L$ is the set of left nodes which are matched (in $M$) to the right node corresponding to the group $U$, then the subgraph of $G^*$ induced by $\{v_1, \ldots, v_n\}$ and $U$ has a perfect matching. Therefore, our goal is to prove the following:
Theorem 2. Let $G = (L, R, E)$ be a bipartite graph with $|L| = |R| = n$, such that each vertex in $L$ is connected independently to a uniformly random subset of $c \ln n$ vertices in $R$. Then, for a universal constant $c$, $G$ has a perfect matching with high probability.

Proof. We can bound the probability that a matching $M$ does not survive in $G^*$ by taking a union bound over the groups $U$ in $R'$ on the event that there does not exist a perfect matching between $U$ and the vertices in $L$ which are matched by $M$ to the vertex corresponding to $U$. The probability of the latter is just the probability that a perfect matching exists in a bipartite graph of $n$ vertices where each left vertex has edges to $c \ln n$ random right vertices. By Hall’s Marriage Theorem (Theorem 1),

$$p_n := P(\exists A \subseteq L, |A| > |\Gamma(A)|) \leq \sum_{k=c \ln n + 1}^{n} P(\exists A \subseteq L, k = |A| > |\Gamma(A)|)$$

$$\leq \sum_{k=c \ln n + 1}^{n-1} \binom{n}{k}^2 \left( \frac{c \ln n}{n} \right)^k + n \left( \sum_{k=c \ln n + 1}^{n-1} \binom{n}{k} \right) \frac{c \ln n - 1}{n}.$$  

To control the summation, we split the summation into two parts.

$$\sum_{k=c \ln n + 1}^{\lfloor n/2 \rfloor} \left( \frac{n}{k} \right)^2 \left( \frac{c \ln n}{n} \right)^k \leq \sum_{k=c \ln n + 1}^{\lfloor n/2 \rfloor} \left( \frac{c \ln n - 2}{n} \right)^k \leq \sum_{k=c \ln n + 1}^{\lfloor n/2 \rfloor} \left( \frac{c \ln n - 2}{n} \right)^k e^{2k}$$

$$\leq \sum_{k=c \ln n + 1}^{\lfloor n/2 \rfloor} \left( \frac{1}{2} \right)^k \frac{c \ln n - 2}{n} e^{2k} \leq \sum_{k=c \ln n + 1}^{\infty} \left( \frac{4e^2}{n^2} \right)^k = O(n^{-c})$$
for \( c, n \) sufficiently large. For the second part of the summation, for \( c > 4 \),

\[
\sum_{k=\lceil n/2 \rceil}^{n-1} \binom{n}{k} \frac{ck \ln n}{n} = \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n}{k} \frac{(n-k) \ln n}{n} \\
\leq \sum_{k=1}^{\lfloor n/2 \rfloor} \left(\frac{en}{k}\right)^{2k} \exp\left\{ -\frac{ck(n-k) \ln n}{n} \right\} \leq \sum_{k=1}^{\lfloor n/2 \rfloor} \left(\frac{en}{2k}\right)^{2k} \exp\left\{ -\frac{ck \ln n}{2} \right\} \\
\leq \sum_{k=1}^{\lfloor n/2 \rfloor} \left(\frac{en}{2k}\right)^{2k} n^{-ck/2} \leq \sum_{k=1}^{\lfloor n/2 \rfloor} \left(\frac{e^2}{n^c/2-2}\right)^{k} = O(n^{-(c/2-3)}).
\]

Finally, the last term is

\[
n\left(\frac{n-1}{n}\right)^{cn \ln n} \leq n \exp\{-c \ln n\} = O(n^{-(c-1)}).
\]

Thus, we get \( p_n \to 0 \) as \( n \to \infty \). In fact, if we take \( c = 10 \), then \( p_n = O(n^{-2}) \).

The previous result shows that for each left node, if we attach \( 10 \ln n \) edges to each group in \( G^* \), then the probability that any particular group will fail to have a maximum weight perfect matching surviving from \( G \) is \( O(n^{-2}) \), and taking a union bound over the \( k \) groups shows that the algorithm recovers the optimal assignment with probability \( 1 - O(n^{-1}) \) (and by using \( 2(r+4) \ln n \) edges to each group for each vertex, we can achieve a probability of \( 1 - \mathcal{O}(n^{-r}) \) for any positive integer \( r \)).

The intuition of the algorithm is clearly brought out by the following calculation. In the setting of Theorem 2, the number of possible perfect matchings is \( n! \), and the probability that a particular perfect matching appears in the random graph is \([c \ln n]/n^n\). Thus, the expected number of perfect matchings in the graph is \( \approx (c e^{-1} \ln n)^n \), which grows exponentially with \( n \). In fact, even if we only connect a constant \( c \) edges to each group for each vertex, then the expected number of perfect matchings still goes to \( \infty \) exponentially fast. The choice of connectivity \( c \ln n \) is used in the proof to make the probability of error go to 0, but it is not wasteful: even with a constant number of edges per vertex per group, the complexity of the Hungarian algorithm is still \( \mathcal{O}(k^2 n^2 \log n) \).

## 5 Conclusion and Future Work

We addressed the problem of absolute ranking in an online setting using bipartite matching algorithms. A shortcoming of this approach is that the complexity of the algorithm is cubic in the number of candidates, and hence it is prohibitive when the number of candidates is large. One immediate goal is to come up with a matching algorithm with better complexity. Also, instead of formulating a matching problem, we would like to model this as a delayed convex optimization problem. We keep these as our future endeavors.

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