Bethe ansatz for the XXX-$S$ chain with non-diagonal open boundaries

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Abstract

We consider the algebraic Bethe ansatz solution of the integrable and isotropic XXX-$S$ Heisenberg chain with non-diagonal open boundaries. We show that the corresponding $K$-matrices are similar to diagonal matrices with the help of suitable transformations independent of the spectral parameter. When the boundary parameters satisfy certain constraints we are able to formulate the diagonalization of the associated double-row transfer matrix by means of the quantum inverse scattering method. This allows us to derive explicit expressions for the eigenvalues and the corresponding Bethe ansatz equations. We also present evidences that the eigenvectors can be build up in terms of multiparticle states for arbitrary $S$.

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1 Introduction

The possibility of constructing $SU(2)$ invariant Heisenberg chain with arbitrary spin-$S$ solvable by Bethe ansatz methods was a remarkable achievement of the representation theory underlying the associative algebra describing the dynamical symmetry of quantum integrable systems \[1\]. It turns out that the Hamiltonian of such spin-$S$ XXX Heisenberg magnet \[2\] commutes with the transfer matrix $T_S(\lambda)$ of a $2S+1$ state vertex on the square $L \times L$ lattice \[1\,1\,3\]. This connection is based on well known relationships between one-dimensional quantum spin chains and two-dimensional statistical mechanics models whose Boltzmann weights satisfy the Yang-Baxter equation \[4\,5\].

The row-to-row transfer matrix $T_S(\lambda)$ of such $2S+1$ state vertex model can be conveniently written as the trace, over an auxiliary space $\mathcal{A} \equiv \mathbb{C}^{2S+1}$, of an ordered product of Boltzmann weights. More specifically,

$$T_S(\lambda) = \text{Tr}_\mathcal{A}[T^{(S)}_\mathcal{A}(\lambda)] = \mathcal{L}_{ab}^{(S)}(\lambda)\mathcal{L}_{ab}^{(S)}(\lambda)\ldots\mathcal{L}_{ab}^{(S)}(\lambda),$$

where $\lambda$ is the spectral parameter and $\mathcal{A}$ represents the horizontal degrees of freedom of the vertex model.

The Boltzmann weight $\mathcal{L}_{ab}^{(S)}(\lambda)$ is solution of the Yang-Baxter equation

$$\mathcal{L}_{ab}^{(S)}(\lambda - \mu)T^{(S)}_a(\lambda)T^{(S)}_b(\mu) = T^{(S)}_b(\mu)T^{(S)}_a(\lambda)\mathcal{L}_{ab}^{(S)}(\lambda - \mu),$$

invariant relative to the $SU(2)$ Lie algebra. It can be viewed as $(2S+1) \times (2S+1)$ matrix on the auxiliary space whose elements are operators acting non-trivially only on the $b$-th factor of the Hilbert space $\prod_{b=1}^{L} \otimes \mathbb{C}^{2S+1}$. Its explicit expression in terms of the spin-$S$ $SU(2)$ generators $\vec{S}_a = (S^x_a, S^y_a, S^z_a)$ is \[1\,2\,3\]

$$\mathcal{L}_{ab}^{(S)}(\lambda) = (\lambda + 2\eta S)^2 \sum_{l=0}^{2S} \prod_{k=l+1}^{2S} \frac{\lambda - \eta k}{\lambda + \eta k} \prod_{n=0}^{2S} \text{sgn}(\vec{S}_a \otimes \vec{S}_b - x_n),$$

where $x_l = \frac{1}{2}l(l+1) - S(S+1)$ and $\eta$ is the so-called quasi-classical parameter.
Besides the Yang-Baxter equation the operator $L^{(S)}_{12}(\lambda)$ satisfies other relevant properties such as

Unitarity: $L^{(S)}_{12}(\lambda)L^{(S)}_{21}(-\lambda) = \zeta_S(\lambda)\text{Id} \otimes \text{Id}$; \hfill (4)

Parity invariance: $P_{12}L^{(S)}_{12}(\lambda)P_{12} = L^{(S)}_{12}(\lambda)$; \hfill (5)

Temporal invariance: $L^{(S)}_{12}(\lambda)t^{12} = L^{(S)}_{12}(\lambda)$; \hfill (6)

Crossing symmetry: $L^{(S)}_{12}(\lambda) = (-1)^{2S} \frac{\zeta_S(\lambda)}{\zeta_S(-\lambda - \eta)} V^{1} L^{(S)}_{12}(-\lambda - \eta)^{12} V^{-1}$; \hfill (7)

where functions $\zeta_S(\lambda) = (2S\eta)^2 - \lambda^2$ and $\zeta_S(\lambda) = \prod_{k=1}^{2S-1} (\lambda + k\eta)$. Here $\text{Id}$ is the $(2S+1) \times (2S+1)$ identity matrix, $P_{12}$ is the permutation operator, $t_\alpha$ denotes transposition on the $\alpha$-th space, $V^1 = V \otimes \text{Id}$ and $V^2 = \text{Id} \otimes V$. The matrix $V$ is anti-diagonal whose non-null elements are $V_{i,j} = -(-1)^i \delta_{i,2S+2-j}$.

This notion of integrability has been extended to include integrable open boundary conditions \cite{6, 7}. In addition to the Yang-Baxter solution $L^{(S)}_{ab}(\lambda)$ determining the dynamics of the bulk one has to introduce $(2S+1) \times (2S+1)$ $K$-matrices $K_S(\lambda)$ whose elements represent the interactions at the left and right ends of the open spin chain. Compatibility with bulk integrability demands that these matrices should satisfy the reflection equation given by \cite{7}

$$L^{(S)}_{12}(\lambda - \mu) \frac{1}{K_S(\lambda)} L^{(S)}_{21}(\lambda + \mu) \frac{2}{K_S(\mu)} L^{(S)}_{12}(\lambda + \mu) \frac{1}{K_S(\lambda)} L^{(S)}_{21}(\lambda - \mu),$$ \hfill (8)

where $\frac{1}{K_S(\lambda)} = K_S(\lambda) \otimes \text{Id}$ and $\frac{2}{K_S(\lambda)} = \text{Id} \otimes K_S(\lambda)$.

In the case of open boundaries the analogue of the transfer matrix is the following double-row operator \cite{7}

$$t_S(\lambda) = \text{Tr}_A \left[ K_S^{(+)}(\lambda) T_A^{(S)}(\lambda) K_S^{(-)}(\lambda) \left[ T_A^{(S)}(-\lambda) \right]^{-1} \right],$$ \hfill (9)

where $K_S^{(-)}(\lambda)$ can be chosen as one of the solutions of the reflection equation \cite{8}. The other matrix $K_S^{(+)}(\lambda)$ can be directly obtained from $K_S^{(-)}(\lambda)$ thanks to the extra relations \cite{4, 7} satisfied by the operator $L^{(S)}_{ab}(\lambda)$. Following a scheme devised in ref.\cite{8} this isomorphism becomes

$$K_S^{(+)}(\lambda) = \left[ K_S^{(-)}(-\lambda - \eta) \right]^\ell.$$ \hfill (10)
The understanding of the physical properties of the XXX-$S$ open chain includes necessarily the exact diagonalization of the double-row operator $\mathcal{D}$. If the $K$-matrices are diagonal this problem can be tackled, for example, by an extension of the quantum inverse scattering method $[7]$ and the use of fusion hierarchy procedures $[9, 10]$. The same does not occur when the $K$-matrices are non-diagonal due to an apparent lack of simple reference states to start Bethe ansatz analysis. In spite of this difficulty, progresses have recently been made for the anisotropic version of the $S = \frac{1}{2}$ Heisenberg magnet usually denominated the XXZ spin chain. These achievements have been made either by a functional Bethe ansatz analysis $[11]$ or by means of the algebraic Bethe ansatz method $[12]$. The latter approach has been based on earlier ideas developed in the context of the eight vertex model $[13]$. In particular, it was argued that the spectrum of the open XXZ chain can be parameterized by Bethe ansatz equation provided certain constraint between the parameters of the Hamiltonian is satisfied. Part of the conclusions were achieved with the help of a numerical study of the spectrum for finite values of $L$ $[14]$. More recently, new results have been obtained in ref.$[15]$ by exploring the description of the open XXZ spin chain in terms of the Temperley-Lieb algebra. The extension of all such analysis for integrable Heisenberg chains with arbitrary spin-$S$ appears to be highly non-trivial and it is indeed an interesting open problem in the field of integrable models.

In this paper we would like to take some steps towards the direction of solving the isotropic higher spin Heisenberg model $[3]$ with non-diagonal open boundaries. We show that the double-row transfer matrix operator associated to the integrable XXX-$S$ Heisenberg chain can be diagonalized by Bethe ansatz at least when the respective $K$-matrices parameters satisfy one out of two possible types of constraints. We find that the roots of the Bethe ansatz equations are fixed by integers $n \leq 2SL$ that play the role of standard particle number sectors. This feature shows that the Hilbert space has a multiparticle structure which should be useful to determine the nature of the ground state and excitations unambiguously.

The outline of this paper is as follows. In section $2$ we argue that the non-diagonal $K$-matrices of the XXX-$S$ Heisenberg model are diagonalizable by spectral independent similarity transformations. In section $3$ suitable quantum space transformations are used to show that
the diagonalization of $t_S(\lambda)$ is similar to an eigenvalue problem with diagonal and triangular $K$-matrices provided that certain constraints are satisfied. In section II we discuss the quantum inverse scattering method for the latter system, presenting the corresponding eigenvalues and Bethe ansatz equations. Explicit expressions for the eigenvectors in terms of similarity transformation acting on creation fields can be written for spin $\frac{1}{2}$ and $1$. In section III our conclusions and further perspectives are discussed. In Appendix A we summarize certain properties of the $K$-matrices. In Appendices B and C we discuss the one and two particle analysis of the eigen-spectrum as well as auxiliary expressions for $S = \frac{3}{2}$, respectively. Finally, in Appendix D we exhibit general relations concerning the one-particle unwanted terms and the two-particle state construction for arbitrary $S$.

## 2 The $K$-matrices properties

The most general reflection $K$-matrix associated to the open XXX-$S$ Heisenberg chain possesses three free parameters. For $S = \frac{1}{2}$ it is given by \[ K^{(-)}_{\frac{1}{2}}(\lambda) = \begin{pmatrix} \xi_- + \frac{\lambda}{\eta} & c_- \frac{\lambda}{\eta} \\ d_- \frac{\lambda}{\eta} & \xi_- - \frac{\lambda}{\eta} \end{pmatrix}, \] (11) while the isomorphism (10) implies that

$$K^{(+)}_{\frac{1}{2}}(\lambda) = \begin{pmatrix} \xi_+ - 1 - \frac{\lambda}{\eta} & -c_+ \left( \frac{\lambda}{\eta} + 1 \right) \\ -d_+ \left( \frac{\lambda}{\eta} + 1 \right) & \xi_+ + 1 + \frac{\lambda}{\eta} \end{pmatrix},$$ (12)

where $\xi_\pm, c_\pm$ and $d_\pm$ are six free parameters.

A remarkable characteristic of these $K$-matrices is that they can be diagonalized by similarity transformations which are independent of the spectral parameter $\lambda$. More precisely, it is possible to rewrite the equations (11,12) as

$$K^{(-)}_{\frac{1}{2}}(\lambda) = \rho^{(-)}_{\frac{1}{2}} G^{(-)}_{\frac{1}{2}} \begin{pmatrix} \bar{\xi}_- + \frac{\lambda}{\eta} & 0 \\ 0 & \bar{\xi}_- - \frac{\lambda}{\eta} \end{pmatrix} \left[ G^{(-)}_{\frac{1}{2}} \right]^{-1},$$ (13)

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and
\[
K^{(+)}_t(\lambda) = \rho^{(+)}_t G^{(+)}_t \begin{pmatrix}
\xi_+ - 1 - \frac{\lambda}{\eta} & 0 \\
0 & \xi_+ + 1 + \frac{\lambda}{\eta}
\end{pmatrix} \left[ G^{(+)}_t \right]^{-1},
\]
(14)
where \( G^{(\pm)}_S \) refer to appropriate \((2S + 1) \times (2S + 1)\) matrices. In what follows we will represent them in terms of the standard Weyl basis \( \hat{e}_{ij} \) by the expression
\[
G^{(\pm)}_S = \sum_{i,j=1}^{2S+1} g^{(\pm)}_{i,j} \hat{e}_{ij}.
\]
(15)

In the specific case of \( S = \frac{1}{2} \) the expressions relating the off-diagonal and the diagonal elements of \( G^{(\pm)}_S \) are
\[
\frac{g^{(\pm)}_{2,1}}{g^{(\pm)}_{1,1}} = \frac{1 + \epsilon_{\pm} \sqrt{1 + c_{\pm} d_{\pm}}}{c_{\pm}}; \quad \frac{g^{(\pm)}_{1,2}}{g^{(\pm)}_{2,2}} = \frac{1 + \epsilon_{\pm} \sqrt{1 + c_{\pm} d_{\pm}}}{d_{\pm}},
\]
(16)
where \( \epsilon_{+} = \epsilon_{-} = \pm 1 \). The other variables \( \xi_{\pm} \) and \( \rho^{(\pm)}_t \) entering in the formulae (13,14) are given by
\[
\bar{\xi}_{\pm} = -\frac{\epsilon_{\pm} \xi_{\pm}}{\sqrt{1 + c_{\pm} d_{\pm}}}; \quad \rho^{(\pm)}_t = -\epsilon_{\pm} \sqrt{1 + c_{\pm} d_{\pm}}.
\]
(17)

The \( K \)-matrices for \( S > \frac{1}{2} \) can be computed either by brute force analysis of the reflection equation [17] or constructed by the so-called fusion procedure [18]. Their matrix elements expressions become very cumbersome as one increases the value of the spin and this fact has been exemplified in Appendix A for spin 1 and \( \frac{3}{2} \) cases. It turns out, however, that we have found out that such \( K \)-matrices can be rewritten in a rather compact and illuminating form with the help of appropriate spectral independent similarity transformations, namely
\[
K^{(\pm)}_S(\lambda) = \rho^{(\pm)}_S G^{(\pm)}_S D^{(\pm)}_S(\lambda) \left[ G^{(\pm)}_S \right]^{-1},
\]
(18)
where the overall normalizing constant is \( \rho^{(\pm)}_S = -\epsilon_{\pm} \frac{2S}{\sqrt{1 + c_{\pm} d_{\pm}}} \left( \frac{\sqrt{1 + c_{\pm} d_{\pm}}}{2S} \right)^{2S} \).

The diagonal matrix \( D^{(\pm)}_S(\lambda) \) is defined by
\[
D^{(\pm)}_S(\lambda) = \sum_{j=1}^{2S+1} f^{(\pm)}_j(S; \lambda, \bar{\xi}_\pm) \hat{e}_{jj},
\]
(19)
where the corresponding diagonal entries are
\[
\begin{align*}
\left. f_\alpha^{(-)} (S; \lambda, \bar{\xi}_-) \right| &= \prod_{\beta=1}^{2S} \xi_- + S + \frac{1}{2} - \beta - \text{sign} \left( \alpha - \frac{1}{2} - \beta \right) \frac{\lambda}{\eta}, \\
\left. f_\alpha^{(+)} (S; \lambda, \bar{\xi}_+) \right| &= \prod_{\beta=1}^{2S} \xi_+ + S + \frac{1}{2} - \beta + \text{sign} \left( \alpha - \frac{1}{2} - \beta \right) \left( \frac{\lambda}{\eta} + 1 \right).
\end{align*}
\]

(20a)

(20b)

Interesting enough, the novel parameters \( \bar{\xi}_\pm \) encode both the dependence on the spin value and on the variables describing the off-diagonal \( K \)-matrices elements. Specifically, we have found that
\[
\bar{\xi}_\pm = - \frac{\epsilon_\pm 2S \xi_\pm}{\sqrt{1 + c_\pm d_\pm}}.
\]

(21)

Finally, the four elements \( g_{1,1}^{(\mp)} \), \( g_{1,2}^{(\mp)} \), \( g_{2,1}^{(\mp)} \) and \( g_{2,2}^{(\mp)} \) of \( G_S^{(\mp)} \) are related by the expressions
\[
\frac{g_{2,1}^{(\pm)}}{g_{1,1}^{(\pm)}} = \frac{(-1)^{2S} \sqrt{2S} \left( 1 + \epsilon_\pm \sqrt{1 + c_\pm d_\pm} \right)}{c_\pm},
\]
\[
\frac{g_{2,2}^{(\pm)}}{g_{1,2}^{(\pm)}} = \frac{(-1)^{2S} \sqrt{2} \left( (S - 1)c_\pm d_\pm + (2S - 1)(1 + \epsilon_\pm \sqrt{1 + c_\pm d_\pm}) \right)}{\sqrt{Sc_\pm} \left( 1 + \epsilon_\pm \sqrt{1 + c_\pm d_\pm} \right)},
\]

(22)

(23)

and the remaining elements are obtained by the following recurrence relations
\[
g_{m,l}^{(\pm)} = \frac{\sqrt{2S(m - 1)(2S + 2 - m)} g_{2,1}^{(\pm)} g_{m-1,l}^{(\mp)} - \sqrt{2S(l - 1)(2S + 2 - l)} g_{1,2}^{(\pm)} g_{m,l-1}^{(\pm)}}{2S(m - l) g_{1,1}^{(\pm)}},
\]

(24)

for \( l \neq m = 1, \ldots, 2S + 1 \) while for \( m = l \) we have
\[
g_{l,l}^{(\pm)} = \frac{g_{2,2}^{(\pm)} g_{l-1,l-1}^{(\mp)} - \sqrt{2(2S - 1)(l - 2)(2S + 3 - l)} g_{3,1}^{(\pm)} g_{l-2,l}^{(\pm)} + 2(l - 2) g_{2,2}^{(\pm)} g_{l-1,l}^{(\pm)}}{\sqrt{2S(2S + 2 - l)(l - 1)} g_{1,1}^{(\pm)}}.
\]

(25)

An important feature of our construction is that the matrices \( G_S^{(\pm)} \) are itself representations, without spectral parameter, of the monodromy matrix associated to the Yang-Baxter algebra \( \mathcal{L}_{ab}^{(S)} (\lambda) \). In fact, the matrix \( \mathcal{L}_{12}^{(S)} (\lambda) \) with four free parameters are the widest possible class of non-diagonal twisted boundary conditions compatible with integrability for the XXX-S spin chain \[19\]. An immediate consequence of this symmetry is the commutation relation
\[
\left[ \mathcal{L}_{12}^{(S)} (\lambda), G_S^{(\pm)} \otimes G_S^{(\pm)} \right] = 0,
\]

(26)

which will be of great use in next section.
3 The eigenvalue problem

The purpose of this section is to show that the eigenvalue problem for the double-row transfer matrix operator $t_S(\lambda)$,

$$ t_S(\lambda) |\psi\rangle = \Lambda_S(\lambda) |\psi\rangle, \quad (27) $$

associated to the XXX-$S$ chain with two general non-diagonal open boundaries can be transformed into a similar problem with only one genuine non-diagonal $K$-matrix.

In order to demonstrate that we use the decomposition property for the $K_S^{(+)}(\lambda)$ matrix described in section 2 and the operator $t_S(\lambda)$ becomes

$$ t_S(\lambda) = \text{Tr}_A \left[ G_S^{(+)} D_S^{(+)}(\lambda) \left[ G_S^{(+)} \right]^{-1} T_A^{(S)}(\lambda) K_S^{(-)}(\lambda) \left[ T_A^{(S)} (-\lambda) \right]^{-1} \right]. \quad (28) $$

We now proceed by inserting identity terms of type $\left[ G_S^{(+)} \right]^{-1} G_S^{(+)}$ in between all the fundamental operators that appear in the trace (28). By using the invariance of the trace under cyclic permutation one can rewrite Eq. (28) as

$$ \frac{t_S(\lambda)}{\rho_S^{(+)}} = \text{Tr}_A \left[ D_S^{(+)}(\lambda) \tilde{T}_A^{(S)}(\lambda) \tilde{K}_S^{(-)}(\lambda) \left[ \tilde{T}_A^{(S)} (-\lambda) \right]^{-1} \right], \quad (29) $$

where $\tilde{T}_A^{(S)}(\lambda) = \tilde{L}_A^{(S)}(\lambda) \tilde{L}_{AL}^{(S)}(\lambda) \cdots \tilde{L}_{A1}^{(S)}(\lambda)$. The new operator $\tilde{L}_A^{(S)}(\lambda)$ and $K$-matrix $\tilde{K}_S^{(-)}(\lambda)$ are given in terms of unitary transformations acting on the auxiliary space by the expressions,

$$ \tilde{L}_A^{(S)}(\lambda) = \left[ G_S^{(+)} \right]^{-1} L_A^{(S)}(\lambda) G_S^{(+)}, \quad (30) $$

and

$$ \tilde{K}_S^{(-)}(\lambda) = \left[ G_S^{(+)} \right]^{-1} K_S^{(-)}(\lambda) G_S^{(+)} \quad (31) $$

It turns out, however, that the gauge transformation (30) on the $L_A^{(S)}(\lambda)$ operators can be reversed with the help of a second transformation on the quantum space [19]. In fact, one can use property (26) to define quantum space matrices $V_j$ acting non-trivially only at the $j$-th site

$$ V_j = \text{Id} \otimes \cdots \otimes \text{Id} \otimes G_S^{(+)} \otimes \text{Id} \otimes \cdots \otimes \text{Id}, \quad (32) $$
such that they are able to undo the transformation \([30]\), namely

\[
V_j^{-1} \tilde{L}_S^{(S)}(\lambda)V_j = L_{A_j}^{(S)}(\lambda).
\]  

(33)

We now can use this property in order to define a new double-row transfer matrix operator \(\bar{t}_S(\lambda)\)

\[
\bar{t}_S(\lambda) = \prod_{j=1}^{L} V_j^{-1} t_S(\lambda) \prod_{j=1}^{L} V_j,
\]  

(34)

having only one non-diagonal \(K\)-matrix

\[
\frac{\bar{t}_S(\lambda)}{\rho_S^{(+)}} = \text{Tr}_A \left[ D_S^{(+)}(\lambda) T_A^{(S)}(\lambda) \tilde{K}_S^{(-)}(\lambda) \left[ T_A^{(S)}(-\lambda) \right]^{-1} \right].
\]  

(35)

Clearly, the operators \(t_S(\lambda)\) and \(\bar{t}_S(\lambda)\) have the same eigenvalues while their eigenstates \(|\psi\rangle\) and \(|\bar{\psi}\rangle\) are related by a similarity transformation,

\[
|\psi\rangle = \prod_{j=1}^{L} V_j |\bar{\psi}\rangle.
\]  

(36)

Though this framework clearly brings a considerable simplification in the original eigenvalue problem, it is not enough to make the diagonalization of the double-row operator \(\bar{t}_S(\lambda)\) with six free boundary parameters amenable to a standard Bethe ansatz analysis. This is because the \(K\)-matrix \(\tilde{K}_S^{(-)}\) is generally non-diagonal which still imposes us the difficulty of finding suitable reference states needed to begin the Bethe ansatz computations. However, a great advantage of this formulation is that one can easily identify the existence of at least three cases of physical interest in which the standard \(SU(2)\) highest weight states could be used as pseudovacuums to build up the whole Hilbert space. The simplest occurs when one of the boundaries is free, say \(K_S^{(-)}(\lambda) = \text{Id}\) while the other is still arbitrary with three free parameters. The next one is when the \(K\)-matrices \(K_S^{(\pm)}(\lambda)\) are diagonalizable in the same basis, i.e. \(G_S^{(\pm)} = G_S^{(-)}\) which implies that we have altogether four distinct couplings say \(c_-, d_-\) and \(\xi_\pm\). This includes the important symmetric situation where the left and right \(K\)-matrices are the same but arbitrarily non-diagonal. As far as the Bethe ansatz technicalities are concerned the most general case in which \(SU(2)\) highest weight vectors can be used as a reference state is when
the effective $\tilde{K}_S(\lambda)^{-1}$ $K$-matrix becomes either upper or lower triangular. This leads us to an open integrable system with five free couplings since such condition imposes certain constraint between the parameters $c_\pm$ and $d_\pm$. Substituting the representation (18) in Eq. (31) and after some algebra we find that there are two possible classes of restrictions satisfying the above mentioned triangularity property. It turns out that these constraints depend only upon the variables $c_\pm$, $d_\pm$ and their expressions are,

\begin{align}
(\text{I}) & \quad \frac{1 + \epsilon_- \sqrt{1 + c_- d_-}}{c_-} = \frac{1 + \epsilon_+ \sqrt{1 + c_+ d_+}}{c_+}, \\
(\text{II}) & \quad \frac{-d_-}{1 + \epsilon_- \sqrt{1 + c_- d_-}} = \frac{1 + \epsilon_+ \sqrt{1 + c_+ d_+}}{c_+}.
\end{align}

Depending on the ratio $\varepsilon = \frac{\epsilon_+}{\epsilon_-}$ the zeros entries of $\tilde{K}_S(\lambda)^{-1}$ are either below or above the principal diagonal. This feature has been summarized in Table 1 for each manifold. Note that the diagonal elements of the triangular matrix $\tilde{K}_S(\lambda)^{-1}$ will necessarily be the eigenvalues of $K_S^-(\lambda)$. By considering decomposition (18) we conclude that such eigenvalues are exactly the entries of the diagonal matrix $\rho_S^-(\lambda) D_S^-(\lambda)$.

Considering the above discussions, we find that the formulation of a Bethe ansatz solution for the eigenspectrum of $\bar{t}_S(\lambda)$ on the parameters manifolds (I) and (II) is certainly worthwhile to pursue. It will leads us to benefit from the knowledge of the exact spectrum with five out of six possible boundary couplings, a considerable number of free parameters at our disposal. A fundamental ingredient in the algebraic Bethe ansatz is the quadratic relations satisfied by the matrix elements of the double-row monodromy matrix defined by [7]

\begin{equation}
\mathcal{T}_A^{(s)}(\lambda) = \mathcal{T}_A^{(s)}(\lambda) \tilde{K}_S^{-}(\lambda) \left[ \mathcal{T}_A^{(s)}(-\lambda) \right]^{-1},
\end{equation}

and consequently the double-row operator $\bar{t}_S(\lambda)$ can be written in the form

\begin{equation}
\frac{\bar{t}_S(\lambda)}{\rho_S^+(\lambda)} = \text{Tr}_A \left[ D_S^{(+)}(\lambda) \mathcal{T}_A^{(s)}(\lambda) \right].
\end{equation}

Taking into account the property [26] we see that the effective $\tilde{K}_S(\lambda)^{-1}$ matrix satisfies the same reflection equation [8] as the original $K$-matrix $K_S^-(\lambda)$. As a consequence of that and
the fact the entries of $\tilde{K}_S^{(-)}(\lambda)$ are $c$-numbers it follows that $\tilde{T}_A^{(S)}(\lambda)$ is also a solution of the reflection equation, namely

$$L_{12}^{(S)}(\lambda - \mu) \tilde{T}_A^{(S)}(\lambda) L_{21}^{(S)}(\lambda + \mu) \tilde{T}_A^{(S)}(\mu) = \tilde{T}_A^{(S)}(\mu) L_{12}^{(S)}(\lambda + \mu) \tilde{T}_A^{(S)}(\lambda).$$

(41)

In the next section we will explore such quadratic algebra together with the existence of a pseudovacuum on which $\tilde{T}_A^{(S)}(\lambda)$ acts triangularly to present the expressions for eigenvalues of $\tilde{t}_S(\lambda)$ as well as the corresponding Bethe ansatz equations.

4 Algebraic Bethe ansatz

In the next subsections we will consider the diagonalization of the operator $\tilde{t}_S(\lambda)$ in the most general restrictive condition (I) or (II) by an algebraic formulation of the Bethe ansatz. The other two situations mentioned in section 3 are special cases and the corresponding eigenvalues and Bethe ansatz results can be derived from the results, for example, obtained for manifold (I). This is obvious when $G_{S}^{(+)} = G_{S}^{(-)}$ and in the case $K_{S}^{(-)}(\lambda) = \text{Id}$ one needs to consider the limit $\bar{\xi} - \to \infty$ with fixed $\rho_{S}^{(-)} = 1$ in the results to be given bellow.

4.1 The spin-$\frac{1}{2}$ solution

Here we shall consider the diagonalization of the double-row transfer matrix $\tilde{t}_{\frac{1}{2}}(\lambda)$ by means of the quantum inverse scattering method [5, 7]. The corresponding bulk Boltzmann weights (3) are those of the isotropic six-vertex model,

$$L_{12}^{(\frac{1}{2})} (\lambda) = \begin{pmatrix} \lambda + \eta & 0 & 0 & 0 \\ 0 & \lambda & \eta & 0 \\ 0 & \eta & \lambda & 0 \\ 0 & 0 & 0 & \lambda + \eta \end{pmatrix}.$$  

(42)

Following the remarks of section 3 we are assuming that the boundary couplings $c_{\pm}$ and $d_{\pm}$ satisfy one of the two possible constraints described by Eqs. [37-38]. In this situation
the effective $\tilde{K}_{S}^{(-)}(\lambda)$ $K$-matrix is triangular and its diagonal entries are proportional to the eigenvalues $f_{j}^{(-)}(\frac{1}{2}; \lambda, \bar{c}_{-})$. Without losing generality one can clearly consider the case in which $\tilde{K}_{\frac{1}{2}}^{(-)}(\lambda)$ is upper triangular, and after some simplifications in Eq.(31) we find that

$$\tilde{K}_{\frac{1}{2}}^{(-)}(\lambda) = \varepsilon \rho_{\frac{1}{2}}^{(-)} \begin{pmatrix}
  f_{1}^{(-)}(\frac{1}{2}; \lambda, \bar{c}_{-}) & \frac{\sigma_{12}^{g_{22}^{(+)}}}{g_{11}^{(+)}} \frac{\lambda}{\lambda - \mu} \\
  0 & f_{2}^{(-)}(\frac{1}{2}; \lambda, \bar{c}_{-})
\end{pmatrix}. \tag{43}
$$

The off-diagonal term in Eq.(43) is not expected to affect the eigenvalues of $\tilde{t}_{\frac{1}{2}}(\lambda)$ but it will certainly be relevant in the structure of the eigenvectors. The explicit expression for $\sigma_{12}$ has been presented in Appendix A. As discussed in section 3 a direct consequence of the upper triangular property of $\tilde{K}_{S}(\lambda)$ is that the following $SU(2)$ highest state vector

$$|\bar{0}_{S}\rangle = \prod_{j=1}^{L} |S, S\rangle_{j}, \quad |S, S\rangle = \begin{pmatrix}
  1 \\
  0 \\
  \vdots \\
  0
\end{pmatrix}_{2S+1}, \tag{44}
$$

is an exact eigenvector of the double transfer matrix $\tilde{t}_{S}(\lambda)$.

This means that the state $|\bar{0}_{\frac{1}{2}}\rangle$ can be used as pseudovacuum to build up the other eigenvectors of $\tilde{t}_{\frac{1}{2}}(\lambda)$ following the strategy of the algebraic Bethe ansatz approach [5, 7]. A main step in this method involves writing the double-row monodromy matrix $T_{A}^{(\frac{1}{2})}(\lambda)$ in the $2 \times 2$ form

$$T_{A}^{(\frac{1}{2})} = \begin{pmatrix}
  A(\lambda) & B(\lambda) \\
  C(\lambda) & D(\lambda)
\end{pmatrix}. \tag{45}
$$

By using the intertwining relation (41) and following the procedure devised first by Sklyanin [7] one can derive the commutation rules

$$A(\lambda)B(\mu) = \frac{(\mu - \lambda + \eta)(\mu + \lambda)(\mu - \lambda)}{(\mu + \lambda + \eta)(\mu - \lambda)}B(\mu)A(\lambda) - \frac{2\mu}{(2\mu + \eta)(\mu - \lambda)} B(\lambda)A(\mu) - \frac{\eta}{(\mu + \lambda + \eta)} B(\lambda)\tilde{D}(\mu), \tag{46}
$$

$$\tilde{D}(\lambda)B(\mu) = \frac{(\lambda + \mu + 2\eta)(\lambda - \mu + \eta)(\lambda - \mu)}{(\lambda - \mu)(\lambda + \mu + \eta)}B(\mu)\tilde{D}(\lambda) - \frac{2\eta(\lambda + \eta)}{(2\lambda + \eta)(\lambda - \mu)}B(\lambda)\tilde{D}(\mu) + \frac{4\mu(\lambda + \eta)}{(\lambda + \mu + \eta)(2\lambda + \eta)(2\mu + \eta)} B(\lambda)A(\mu), \tag{47}$$
where the new field $\tilde{D}(\lambda)$ is introduced in order to simplify the commutation relations. It is given by the following combination between the operators $A(\lambda)$ and $D(\lambda)$,

$$\tilde{D}(\lambda) = D(\lambda) - \frac{\eta}{2\lambda + \eta}A(\lambda).$$

(48)

In terms of the operators $A(\lambda)$ and $\tilde{D}(\lambda)$ the double-row transfer matrix eigenvalue problem can now be written as

$$\left[ f_1^{(+)}\left(\frac{1}{2}; \lambda, \xi_+ \right) + f_2^{(+)}\left(\frac{1}{2}; \lambda, \xi_+ \right)\frac{\eta}{2\lambda + \eta} \right] A(\lambda) |\tilde{\psi}\rangle + f_2^{(+)}\left(\frac{1}{2}; \lambda, \xi_+ \right)\tilde{D}(\lambda) |\tilde{\psi}\rangle = \frac{\Lambda_{\frac{1}{2}}(\lambda)}{\rho_{\frac{1}{2}}^{(+)} \rho_{\frac{1}{2}}^{(-)}} |\tilde{\psi}\rangle,$$

(49)

while the action of the fields $A(\lambda)$, $\tilde{D}(\lambda)$ and $C(\lambda)$ on the reference state $|\tilde{0}_{\frac{1}{2}}\rangle$ are given by

$$A(\lambda) |\tilde{0}_{\frac{1}{2}}\rangle = \epsilon \rho_{\frac{1}{2}}^{(-)} f_1^{(-)}\left(\frac{1}{2}; \lambda, \epsilon \xi_- \right) \left[ \frac{(\lambda + \eta)^2}{\zeta_{\frac{1}{2}}(\lambda)} \right] L |\tilde{0}_{\frac{1}{2}}\rangle,$$

(50a)

$$\tilde{D}(\lambda) |\tilde{0}_{\frac{1}{2}}\rangle = \epsilon \rho_{\frac{1}{2}}^{(-)} \left[ f_2^{(-)}\left(\frac{1}{2}; \lambda, \epsilon \xi_- \right) - f_1^{(-)}\left(\frac{1}{2}; \lambda, \epsilon \xi_- \right) \frac{\eta}{2\lambda + \eta} \right] \left[ \frac{\lambda^2}{\zeta_{\frac{1}{2}}(\lambda)} \right] L |\tilde{0}_{\frac{1}{2}}\rangle,$$

(50b)

$$C(\lambda) |\tilde{0}_{\frac{1}{2}}\rangle = 0.$$

(50c)

The fields $B(\lambda)$ are interpreted as a kind of creation operators over the pseudovacuum $|\tilde{0}_{\frac{1}{2}}\rangle$ and the multiparticle Bethe states $|\tilde{\psi}_n(\lambda_1, \ldots, \lambda_n)\rangle$ are supposed to be given by

$$|\tilde{\psi}_n(\lambda_1, \ldots, \lambda_n)\rangle = B(\lambda_1) \ldots B(\lambda_n) |\tilde{0}_{\frac{1}{2}}\rangle.$$

(51)

The rapidities $\lambda_j$ will be determined by solving the eigenvalue problem with the above ansatz for the eigenvectors. This is done with the help of the commutation relations (46-47) to move $A(\lambda)$ and $\tilde{D}(\lambda)$ in Eq. (49) over the creation fields until they reach the reference state $|\tilde{0}_{\frac{1}{2}}\rangle$. The terms proportional to the eigenvectors (51) are easily collected by keeping only the first pieces of the commutation rules. After using expressions (50a, 50b) and some simplifications we find that the final result for the eigenvalues are

$$\frac{\Lambda_{\frac{1}{2}}(\lambda)}{\rho_{\frac{1}{2}}^{(+)} \rho_{\frac{1}{2}}^{(-)}} = \left[ \frac{(\lambda + \eta)^2}{\zeta_{\frac{1}{2}}(\lambda)} \right] L \left[ \frac{\epsilon^2(\lambda + \eta)(\lambda + \epsilon \bar{\xi}_-)(-\lambda + \eta \xi_+)}{(2\lambda + \eta)^2} \right] \prod_{j=1}^{n} \left( \frac{\lambda_j - \lambda + \frac{i\eta}{2}}{(\lambda_j - \lambda - \frac{i\eta}{2})(\lambda_j + \lambda + \frac{i\eta}{2})} \right) + \left[ \frac{\lambda^2}{\zeta_{\frac{1}{2}}(\lambda)} \right] L \left[ \frac{\epsilon^2(\lambda - \lambda + \epsilon \bar{\xi}_-)(\lambda + \eta \xi_+ + \eta)}{(2\lambda + \eta)^2} \right] \prod_{j=1}^{n} \left( \frac{\lambda_j - \lambda - \frac{3i\eta}{2}}{(\lambda_j - \lambda - \frac{3i\eta}{2})(\lambda_j + \lambda + \frac{3i\eta}{2})} \right).$$

(52)
where we have used the values of \( f_j^{(\pm)}(\frac{1}{2}, \lambda, \xi_{\pm}) \) taken from Eqs. (20a, 20b) and performed the displacements \( \lambda_i \rightarrow \lambda_i - \frac{\eta}{2} \) on the rapidities.

The remaining terms that are not proportional to \( |\tilde{\psi}(\lambda_1, \ldots, \lambda_n)\rangle \) can be canceled out by imposing further restrictions on the rapidities \( \lambda_j \). These are known as the Bethe ansatz equations which in our case are given by

\[
\left[\frac{\lambda_j + \frac{\eta}{2}}{\lambda_j - \frac{\eta}{2}}\right]^{2L} = \left(\frac{\lambda_j - \varepsilon \eta \xi_- + \frac{\eta}{2}}{\lambda_j + \varepsilon \eta \xi_- - \frac{\eta}{2}}\right) \left(\frac{\lambda_j + \frac{\eta}{2}}{\lambda_j - \frac{\eta}{2}}\right) \prod_{i=1}^{n} \left(\frac{\lambda_j - \lambda_i + \eta}{\lambda_j + \lambda_i + \eta}\right) \left(\frac{\lambda_j - \lambda_i - \eta}{\lambda_j + \lambda_i - \eta}\right)
\]

(53)

We now can derive similar results for the open spin-\( \frac{1}{2} \) chain that commutes with the double-row transfer matrix \( t_{\frac{1}{2}}(\lambda) \). The corresponding Hamiltonian is proportional to the first-order expansion of \( t_{\frac{1}{2}}(\lambda) \) in the spectral parameter \( t_{\frac{1}{2}} \)

\[
H_{\frac{1}{2}} = \frac{1}{\eta} \sum_{i=1}^{L-1} \left[ \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \sigma_i^z \sigma_{i+1}^z \right] + \frac{1}{\eta \xi_-} \left[ \sigma_1^z + c_- \sigma_1^+ + d_- \sigma^- \right] - \frac{1}{\eta \xi_+} \left[ \sigma_L^z + c_+ \sigma_L^+ + d_+ \sigma^- \right],
\]

(54)

where \( \sigma_i^x, \sigma_i^y \) and \( \sigma_i^z \) are the Pauli matrices with \( \sigma_{\alpha}^\pm = \frac{1}{2} (\sigma_{\alpha}^x \pm i \sigma_{\alpha}^y) \). Its eigenvalues \( E_n(\lambda) \) are obtained in terms of the rapidities \( \lambda_j \) that satisfy the Bethe ansatz equation (53) by the following expression

\[
E_n(\lambda) = 2\eta \sum_{k=1}^{n} \frac{1}{\lambda_k^2 - \frac{\eta^2}{4}} + \frac{L}{\eta} - \frac{1}{\eta} \left[ 1 + \frac{\rho_+^{(+)}}{\xi_+} - \varepsilon \frac{\rho_-^{(-)}}{\xi_-} \right].
\]

(55)

We would like to close this section with the following remark. The ferromagnetic \( \eta < 0 \) Hamiltonian (53) is known to describe the stochastic dynamics of symmetric hopping of particles in one dimension provided that certain relations are satisfied by the boundary parameters (20). More specifically, letting \( \alpha(\gamma) \) be the rate of injection (ejection) of particles at the left boundary and \( \delta(\beta) \) the corresponding rate at the right boundary we have (20, 21)

\[
\alpha - \gamma = \frac{1}{\eta \xi_-}, \quad \alpha = \frac{d_-}{2\eta \xi_-}, \quad \gamma = \frac{c_-}{2\eta \xi_-},
\]

(56)

and

\[
\beta - \delta = \frac{1}{\eta \xi_+}, \quad \beta = -\frac{c_+}{2\eta \xi_+}, \quad \delta = -\frac{d_+}{2\eta \xi_+}.
\]

(57)
The above particular parameterization of the boundary parameters $c_\pm$ and $d_\pm$ satisfies the constraints (I) or (II) for arbitrary values of the particle injection and ejection rates. Though the spectrum at this special case have been determined before [20, 21] not much is known about the behavior of the wave functions. This information can now be in principle extracted by combining the unitary transformation (32, 36) with the multiparticle state structure (51). This knowledge of the eigenvectors can be used to calculate correlation functions, thanks to recent developments made in the quantum inverse scattering method [22, 23] which allows us to reconstruct local spin operators in terms of the monodromy matrix fields. We hope to return to this problem since this could provide us with new insights on the physics of stochastic dynamics of interacting particle systems.

4.2 The spin-1 solution

The statistical system associated to the integrable XXX-Heisenberg model with spin-1 is a three-state vertex model with nineteen non-null Boltzmann weights given by

$$\mathcal{L}_{12}^{(1)}(\lambda) = \begin{pmatrix}
\tilde{a}(\lambda) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \tilde{b}(\lambda) & 0 & \tilde{h}(\lambda) & 0 & 0 & 0 & 0 \\
0 & 0 & \tilde{c}(\lambda) & 0 & \tilde{d}(\lambda) & 0 & \tilde{c}(\lambda) & 0 \\
0 & 0 & \tilde{h}(\lambda) & 0 & \tilde{b}(\lambda) & 0 & 0 & 0 \\
0 & 0 & \tilde{d}(\lambda) & 0 & \tilde{g}(\lambda) & 0 & \tilde{d}(\lambda) & 0 \\
0 & 0 & 0 & 0 & 0 & \tilde{b}(\lambda) & 0 & \tilde{h}(\lambda) \\
0 & 0 & \tilde{c}(\lambda) & 0 & \tilde{d}(\lambda) & 0 & \tilde{c}(\lambda) & 0 \\
0 & 0 & 0 & 0 & 0 & \tilde{h}(\lambda) & 0 & \tilde{b}(\lambda) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \tilde{a}(\lambda)
\end{pmatrix}, \quad (58)$$

with

$$\tilde{a}(\lambda) = \lambda + 2\eta, \quad \tilde{b}(\lambda) = \lambda, \quad \tilde{c}(\lambda) = \frac{2\eta^2}{\lambda + \eta}, \quad \tilde{d}(\lambda) = \frac{2\eta^2}{\lambda + \eta}, \quad (59)$$

$$\tilde{c}(\lambda) = \frac{\lambda(\lambda - \eta)}{\lambda + \eta}, \quad \tilde{g}(\lambda) = \tilde{b}(\lambda) + \tilde{c}(\lambda), \quad \tilde{h}(\lambda) = 2\eta. \quad (60)$$
As before we can consider the situation when the effective \( \tilde{K}_1(\lambda) \) matrix is upper triangular. In this case, carrying on few algebraic simplifications in Eq. (31) we find that

\[
\tilde{K}_1(\lambda) = \rho_1 (-) \begin{pmatrix}
  f_1(1; \lambda, \varepsilon \xi_-) & 2g^{(+)}_{11} \frac{\lambda}{\eta} \left[ \kappa_{12} \left( \frac{1}{2} - \frac{\lambda}{\eta} \right) - \kappa_{23} \right] & 2\kappa_{13} \left( \frac{g^{(+)}_{11}}{g_{11}} \right)^2 \left( \frac{1}{2} - \frac{\lambda}{\eta} \right) \\
  0 & f_2(1; \lambda, \varepsilon \xi_-) & -2g^{(+)}_{11} \frac{\lambda}{\eta} \left[ \kappa_{12} \left( \frac{1}{2} - \frac{\lambda}{\eta} \right) + \kappa_{23} \right] \\
  0 & 0 & f_3(1; \lambda, \varepsilon \xi_-) 
\end{pmatrix},
\]

where the off-diagonal coefficients \( \kappa_{12}, \kappa_{13} \) and \( \kappa_{23} \) have been collected in Appendix A.

At this point we need to start introducing suitable notation for the double monodromy operator \( T_A^{(S)}(\lambda) \). Here we shall use a representation which can be easily extended to accommodate arbitrary spin-\( S \) case,

\[
T_A^{(1)}(\lambda) = \begin{pmatrix}
  A_1(\lambda) & B_{12}(\lambda) & B_{13}(\lambda) \\
  C_{21}(\lambda) & A_2(\lambda) & B_{23}(\lambda) \\
  C_{31}(\lambda) & C_{32}(\lambda) & A_3(\lambda) 
\end{pmatrix},
\]

The next step is to rewrite the eigenvalue problem in terms of the double monodromy matrix elements. To perform this task is convenient to define new diagonal operators \( \tilde{A}_i(\lambda) \) in terms of appropriate linear combinations of the fields \( A_i(\lambda) \). This is done in such way that the action of the new fields on the state \( |\bar{0}_1\rangle \) will be proportional to a single bulk term. Keeping in mind possible extension to general values of the spin we define,

\[
A_1(\lambda) = \tilde{A}_1(\lambda),
\]

\[
A_2(\lambda) = \tilde{A}_2(\lambda) + \frac{\bar{h}(2\lambda)}{\bar{a}(2\lambda)} \tilde{A}_1(\lambda),
\]

\[
A_3(\lambda) = \tilde{A}_3(\lambda) + \frac{\bar{c}(2\lambda)}{\bar{a}(2\lambda)} \tilde{A}_1(\lambda) + \frac{\bar{h}_1(2\lambda)}{\bar{h}_2(2\lambda)} \tilde{A}_2(\lambda),
\]

where the functions \( \bar{h}_1(\lambda) \) and \( \bar{h}_2(\lambda) \) are the following determinants

\[
\bar{h}_1(\lambda) = \begin{vmatrix}
  \bar{a}(\lambda) & \bar{c}(\lambda) \\
  \bar{h}(\lambda) & \bar{h}(\lambda) 
\end{vmatrix},
\quad
\bar{h}_2(\lambda) = \begin{vmatrix}
  \bar{a}(\lambda) & \bar{h}(\lambda) \\
  \bar{h}(\lambda) & \bar{g}(\lambda) 
\end{vmatrix}.
\]
Taking into account the representation (62) and the above redefinitions of the diagonal fields, the diagonalization of the doubled transfer matrix \( \tilde{t}_1(\lambda) \) becomes equivalent to the problem
\[
\sum_{i=1}^{3} \omega_i^{(+))(\lambda)} \tilde{A}_i(\lambda) |\tilde{\psi}_n(\lambda_1, \ldots, \lambda_n)\rangle = \frac{\Lambda_i(\lambda)}{\rho_i^{(+)})} |\tilde{\psi}_n(\lambda_1, \ldots, \lambda_n)\rangle, \tag{67}
\]
with
\[
\omega_1^{(+)}(\lambda) = f_1^{(+)}(1; \lambda, \xi_+) + \frac{\tilde{h}(2\lambda)}{a(2\lambda)} f_2^{(+)}(1; \lambda, \xi_+) + \frac{\tilde{c}(2\lambda)}{a(2\lambda)} f_3^{(+)}(1; \lambda, \xi_+), \tag{68}
\]
\[
\omega_2^{(+)}(\lambda) = f_2^{(+)}(1; \lambda, \xi_+) + \frac{\tilde{h}_1(2\lambda)}{h_2(2\lambda)} f_3^{(+)}(1; \lambda, \xi_+), \tag{69}
\]
\[
\omega_3^{(+)}(\lambda) = f_3^{(+)}(1; \lambda, \xi_+). \tag{70}
\]

Another important ingredient is to determine the action of the double monodromy matrix elements on the pseudovacuum \(|\bar{0}_1\rangle\). This can be done with the help of the Yang-Baxter algebra [24] and the triangularity properties of both \( \mathcal{L}_{ab}^{(1)}(\lambda) \) and \( \tilde{K}_1^{(-)}(\lambda) \) operators upon \(|\bar{0}_1\rangle\). Following ref.[24] and taking into account Eq.(61) we have
\[
\tilde{A}_1(\lambda) |\bar{0}_1\rangle = \rho_1^{(-)} \omega_1^{(-)}(\lambda) \left[ \frac{a(\lambda)^2}{\zeta_1(\lambda)} \right]^{L} |\bar{0}_1\rangle, \tag{71}
\]
\[
\tilde{A}_2(\lambda) |\bar{0}_1\rangle = \rho_1^{(-)} \omega_2^{(-)}(\lambda) \left[ \frac{\tilde{b}(\lambda)^2}{\zeta_1(\lambda)} \right]^{L} |\bar{0}_1\rangle, \tag{72}
\]
\[
\tilde{A}_3(\lambda) |\bar{0}_1\rangle = \rho_1^{(-)} \omega_3^{(-)}(\lambda) \left[ \frac{\tilde{c}(\lambda)^2}{\zeta_1(\lambda)} \right]^{L} |\bar{0}_1\rangle, \tag{73}
\]
\[
C_{21}(\lambda) |\bar{0}_1\rangle = C_{31}(\lambda) |\bar{0}_1\rangle = C_{32}(\lambda) |\bar{0}_1\rangle = 0,
\]
with
\[
\omega_1^{(-)}(\lambda) = f_1^{(-)}(1; \lambda, \varepsilon \xi_-), \tag{74}
\]
\[
\omega_2^{(-)}(\lambda) = f_2^{(-)}(1; \lambda, \varepsilon \xi_-) - \frac{\tilde{h}(2\lambda)}{a(2\lambda)} f_1^{(-)}(1; \lambda, \varepsilon \xi_-), \tag{75}
\]
\[
\omega_3^{(-)}(\lambda) = f_3^{(-)}(1; \lambda, \varepsilon \xi_-) - \frac{\tilde{h}_1(2\lambda)}{h_2(2\lambda)} f_2^{(-)}(1; \lambda, \varepsilon \xi_-) - \frac{\tilde{h}_3(2\lambda)}{h_2(2\lambda)} f_1^{(-)}(1; \lambda, \varepsilon \xi_-), \tag{76}
\]
where the new function \( \tilde{h}_3(\lambda) = \begin{vmatrix} \tilde{c}(\lambda) & \tilde{h}(\lambda) \\ \tilde{h}(\lambda) & \tilde{g}(\lambda) \end{vmatrix} \).
Also one expects that the operators $B_{12}(\lambda)$, $B_{13}(\lambda)$ and $B_{23}(\lambda)$ play the role of creation operators over the reference state $|\bar{0}_1\rangle$. Therefore it is natural to seek for other eigenvectors of $\hat{t}_1(\lambda)$ as linear combinations of products of these creation fields acting on $|\bar{0}_1\rangle$. This is done by exploring the commutation rules between the diagonal $\tilde{A}_i(\lambda)$ and the creation fields which can be derived from the boundary Yang-Baxter algebra [11]. A careful analysis of these relations reveals us that the construction of the eigenvectors can be based on either $B_{12}(\lambda)$ and $B_{13}(\lambda)$ or $B_{23}(\lambda)$ and $B_{13}(\lambda)$ pair of fields rather than on arbitrary combination of the three possible creation operators. We remark that this redundancy is not particular of this system, but it is a general feature of the algebraic Bethe ansatz framework developed in ref.[25, 26] for a large family of integrable vertex models with periodic boundary. This formalism has been generalized by Li et al ref.[27] to include vertex models with open boundaries based on ideas first envisaged by Fan [24] and later extended for systems solvable by nested Bethe ansatz [28]. We also recall that in the context of three state vertex models this approach was recently reviewed in ref.[29]. Considering that such algebraic framework has already been described in these references, we shall not repeat the details here, and in what follows we will present only the main results for the eigenvectors and the eigenvalues. Here we consider that the eigenvectors will be constructed in terms of a linear combination of products of the creation fields $B_{12}(\lambda)$ and $B_{13}(\lambda)$ acting on the vector $|\bar{0}_1\rangle$. It turns out that the eigenstates $|\bar{\psi}_n(\lambda_1, \ldots, \lambda_n)\rangle$ form a multiparticle structure and they can be constructed as

$$|\bar{\psi}_n(\lambda_1, \ldots, \lambda_n)\rangle = \varphi_n(\lambda_1, \ldots, \lambda_n)|\bar{0}_1\rangle$$

(77)

such that the vector $\varphi_n(\lambda_1, \ldots, \lambda_n)$ satisfy a second order recursion relation of the form

$$\varphi_n(\lambda_1, \ldots, \lambda_n) = B_{12}(\lambda_1)\varphi_{n-1}(\lambda_2, \ldots, \lambda_n) + B_{13}(\lambda_1)\sum_{i=2}^{n}\varphi_{n-2}(\lambda_2, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_n) \times \left(\Gamma_1^{(i)}(\lambda_1, \ldots, \lambda_n)\tilde{A}_1(\lambda_i) + \Gamma_2^{(i)}(\lambda_1, \ldots, \lambda_n)\tilde{A}_2(\lambda_i)\right),$$

(78)

Here we are assuming the identification $|\bar{\psi}_0\rangle \equiv |\bar{0}_1\rangle$. The functions $\Gamma_1^{(i)}(\lambda_1, \ldots, \lambda_n)$ and
\[ \Gamma_2^{(i)}(\lambda_1, \ldots, \lambda_n) \] are given by

\[
\Gamma_1^{(i)}(\lambda_1, \ldots, \lambda_n) = \frac{\bar{d}(\lambda_1 - \lambda_i)}{b(\lambda_1 + \lambda_i)} \bar{p}(\lambda_1, \lambda_i) \prod_{j=2}^{i-1} \frac{\bar{h}_4(\lambda_j - \lambda_i)}{\bar{a}(\lambda_j - \lambda_i) \bar{e}(\lambda_j - \lambda_i)} \prod_{k=2}^{n} \frac{\bar{a}(\lambda_k - \lambda_i) \bar{b}(\lambda_k + \lambda_i)}{b(\lambda_k - \lambda_i) \bar{a}(\lambda_k + \lambda_i)},
\]

and

\[
\Gamma_2^{(i)}(\lambda_1, \ldots, \lambda_n) = \frac{\bar{d}(\lambda_1 + \lambda_i)}{b(\lambda_1 + \lambda_i)} \prod_{j=2}^{i-1} \frac{\bar{h}_4(\lambda_j - \lambda_i)}{\bar{a}(\lambda_j - \lambda_i) \bar{e}(\lambda_j - \lambda_i)} \prod_{k=2}^{n} \frac{\bar{h}_4(\lambda_i - \lambda_k) \bar{h}_2(\lambda_k + \lambda_i)}{b(\lambda_i - \lambda_k) \bar{e}(\lambda_i - \lambda_k) \bar{a}(\lambda_k + \lambda_i) b(\lambda_k + \lambda_i)},
\]

with \( \bar{p}(x, y) = \frac{\bar{e}(x + y)}{\bar{e}(x - y)} - \frac{\bar{h}(2y)}{\bar{a}(2y)} \bar{d}(x - y) \) and \( \bar{h}_4(\lambda) = \begin{vmatrix} \bar{g}(\lambda) & \bar{d}(\lambda) \\ \bar{d}(\lambda) & \bar{e}(\lambda) \end{vmatrix} \).

The action of the doubled transfer matrix \( \bar{t}_1(\lambda) \) on the state \( |\bar{\psi}(\lambda_1, \ldots, \lambda_n)\rangle \) is performed relying on similar data for the \((n-1)\) and \((n-2)\) particle states and with help of mathematical induction. Adapting the discussion of refs. \[27, 29\] to our case we can infer that the eigenvalue expression is

\[
\frac{\Lambda_1(\lambda)}{\rho_1^{(+)}(\lambda_1)} = \omega_1^{(+)}(\lambda) \omega_1^{(-)}(\lambda) \begin{vmatrix} \bar{a}(\lambda) & \bar{a}(\lambda - \lambda) & \bar{b}(\lambda_j + \lambda) \\ \bar{h}_4(\lambda) & \bar{h}_4(\lambda - \lambda) & \bar{h}_2(\lambda_j + \lambda) \\ \lambda \end{vmatrix} \prod_{j=1}^{n} \bar{e}(\lambda_j - \lambda_j) \bar{b}(\lambda + \lambda_j) \bar{b}(2\lambda_j)
\]

\[
+ \omega_2^{(+)}(\lambda) \omega_2^{(-)}(\lambda) \begin{vmatrix} \bar{b}(\lambda) & \bar{b}(\lambda - \lambda) & \bar{a}(\lambda_j + \lambda) \\ \bar{e}(\lambda - \lambda) & \bar{e}(\lambda - \lambda) & \bar{e}(\lambda + \lambda_j) \bar{a}(\lambda + \lambda_j) \bar{b}(\lambda + \lambda_j) \\ \lambda \end{vmatrix} \prod_{j=1}^{n} \bar{e}(\lambda - \lambda_j) \bar{e}(\lambda + \lambda_j) \bar{b}(\lambda + \lambda_j)
\]

\[
+ \omega_3^{(+)}(\lambda) \omega_3^{(-)}(\lambda) \begin{vmatrix} \bar{e}(\lambda) & \bar{e}(\lambda - \lambda) & \bar{b}(\lambda + \lambda_j) \\ \bar{e}(\lambda - \lambda) & \bar{e}(\lambda - \lambda) & \bar{e}(\lambda + \lambda_j) \bar{b}(\lambda + \lambda_j) \\ \lambda \end{vmatrix} \prod_{j=1}^{n} \bar{e}(\lambda - \lambda_j) \bar{e}(\lambda + \lambda_j) \bar{b}(\lambda + \lambda_j),
\]

\[
\text{where } \bar{h}_5(\lambda) = \begin{vmatrix} \bar{b}(\lambda) & \bar{d}(\lambda) \\ \bar{d}(\lambda) & \bar{b}(\lambda) \end{vmatrix} \text{ and provided that the rapidities } \lambda_j \text{ satisfy the following Bethe ansatz equations}
\]

\[
\begin{vmatrix} \bar{a}(\lambda_j) \\ \bar{b}(\lambda_j) \end{vmatrix} \frac{2L}{\bar{h}_2(2\lambda_j)} \begin{vmatrix} \omega_1^{(+)}(\lambda_j) \omega_1^{(-)}(\lambda_j) & \bar{b}(2\lambda_j) \\ \omega_2^{(+)}(\lambda_j) \omega_2^{(-)}(\lambda_j) & \bar{h}_2(2\lambda_j) \end{vmatrix}^2 = \prod_{i=1}^{n} \frac{\bar{b}(\lambda_j - \lambda_i) \bar{h}_2(\lambda_j + \lambda_i)}{\bar{e}(\lambda_j - \lambda_i) \bar{b}(\lambda_j + \lambda_i)}.\]

Now we are almost ready to get standard expressions for the eigenvalues and Bethe ansatz equations. By introducing the new set of variables \( \lambda_i = \lambda_i - \eta \) and performing many simplifi-
cations in the functions entering Eqs.\((81, 82)\) we conclude that the eigenvalues are

\[
\frac{\Lambda_1(\lambda)}{\rho_1^{(+)}\rho_1^{(-)}} = \left[ \frac{(2\lambda + 3\eta)(\lambda + \varepsilon\eta\xi - \frac{\eta}{2})(\lambda + \varepsilon\eta\xi + \frac{\eta}{2})(\lambda - \eta\xi + \frac{\eta}{2})}{(2\lambda + \eta)\eta^4} \right] \\
\times \left[ \frac{(\lambda + 2\eta)^2}{\zeta_1(\lambda)} \right] \prod_{j=1}^{n} \frac{(\lambda_j - \lambda + \eta)(\lambda_j + \lambda - \eta)}{(\lambda_j - \lambda - \eta)(\lambda_j + \lambda + \eta)} \\
+ \left[ \frac{(\lambda + \varepsilon\eta\xi - \frac{\eta}{2})(\lambda - \varepsilon\eta\xi + \frac{3\eta}{2})(\lambda - \eta\xi + \frac{\eta}{2})}{\eta^4} \right] \\
\times \left[ \frac{\lambda^2}{\zeta_1(\lambda)} \right] \prod_{j=1}^{n} \frac{(\lambda_j - \lambda + \eta)(\lambda + \lambda_j - \eta)(\lambda - \lambda_j + 2\eta)(\lambda + \lambda_j + 2\eta)}{(\lambda_j - \lambda - \eta)(\lambda + \lambda_j + \eta)(\lambda - \lambda_j)(\lambda + \lambda_j)} \\
+ \left[ \frac{(2\lambda - \eta)(\lambda - \varepsilon\eta\xi + \frac{\eta}{2})(\lambda - \varepsilon\eta\xi + \frac{3\eta}{2})(\lambda + \eta\xi + \frac{\eta}{2})}{(2\lambda + \eta)\eta^4} \right] \\
\times \left[ \frac{\lambda(\lambda - \eta)}{\zeta_1(\lambda)} \right] \prod_{j=1}^{n} \frac{(\lambda - \lambda_j + 2\eta)(\lambda + \lambda_j + 2\eta)}{(\lambda - \lambda_j)(\lambda + \lambda_j)}, \quad (83)
\]

while the Bethe ansatz equations for the new rapidities \(\lambda_i\) becomes

\[
\left[ \frac{\lambda_i + \eta}{\lambda_i - \eta} \right]^{2L} = \left( \frac{\lambda_j - \varepsilon\eta\xi - \frac{\eta}{2}}{\lambda_j + \varepsilon\eta\xi - \frac{\eta}{2}} \right) \left( \frac{\lambda_j + \varepsilon\eta\xi + \frac{\eta}{2}}{\lambda_j - \eta\xi + \frac{\eta}{2}} \right) \prod_{i=1}^{n} \frac{(\lambda_j - \lambda_i + \eta)(\lambda_j + \lambda_i + \eta)}{(\lambda_j - \lambda_i - \eta)(\lambda_j + \lambda_i - \eta)}. \quad (84)
\]

where \(\xi_{\pm}\) are taken from Eq.\((21)\) with \(S = 1\).

We finally remark that the results of this subsection offers us in principle the basis to solve the \(O(3)\) non-linear sigma model with non-diagonal open boundaries. Due to the isomorphism \(O(3) \sim SU(2)\) the elements of the operator \(58\) can indeed be interpreted as the scattering amplitudes of the \(S\)-matrix associated to the \(O(3)\) field theory \(30\). One expects that similar relation is also valid for the boundary scattering matrices \(31\). In this case, we need to adapt our results to include the solution of the eigenspectrum of an open transfer matrix in the presence of inhomogeneities, following for example the lines of ref.\(32\). It would be interesting to exploit this possibility to determine the effects of the boundaries in the physics of the \(O(3)\) model.
4.3 The spin-$S$ solution

The classical analogue of the solvable spin-$S$ XXX model is the $2S + 1$-state vertex model \cite{3} having the total number of $\frac{1}{2}(2S + 1) [2(2S + 1)^2 + 1]$ non-null Boltzmann weights. The transformed upper triangular $\tilde{K}_S^{(-)}(\lambda)$ matrix corresponding to the left boundary in $\bar{t}_S(\lambda)$ is

$$
\tilde{K}_S^{(-)}(\lambda) = (\varepsilon)^{2S} \rho_s^{(-)} \left( \begin{array}{cccccc}
 f_1^{(-)}(S; \lambda, \varepsilon \xi_-) & * & * & \cdots & * \\
 0 & f_2^{(-)}(S; \lambda, \varepsilon \xi_-) & * & \cdots & * \\
 0 & 0 & f_3^{(-)}(S; \lambda, \varepsilon \xi_-) & \vdots & \\
 \vdots & \vdots & \ddots & * \\
 0 & 0 & \cdots & 0 & f_{2S+1}^{(-)}(S; \lambda, \varepsilon \xi_-) \\
\end{array} \right),
$$

where $*$ denotes non-vanishing values that can be directly determined from Eq. (31).

To implement the quantum inverse scattering framework we will represent the doubled monodromy matrix by the following structure

$$
\mathcal{T}_A^{(S)}(\lambda) = \left( \begin{array}{cccccc}
 A_1(\lambda) & B_{12}(\lambda) & \cdots & B_{1(2S+1)}(\lambda) \\
 C_{21}(\lambda) & A_2(\lambda) & \cdots & B_{2(2S+1)}(\lambda) \\
 \vdots & \vdots & \ddots & \vdots \\
 C_{(2S+1)1}(\lambda) & C_{(2S+1)2}(\lambda) & \cdots & C_{(2S+1)2S}(\lambda) & A_{2S+1}(\lambda) \\
\end{array} \right). \quad (86)
$$

The next step in the algebraic formulation consists in determining the action of the $\mathcal{T}_A^{(S)}(\lambda)$ elements on the reference state $|\bar{0}_S\rangle$ which helps us to distinguish creation and annihilation fields as well as to reformulate the eigenvalues problem in terms of appropriate linear combinations of diagonal fields. To perform that we need to know certain commutation relations between the operators $\mathcal{T}_A^{(S)}(\lambda)$ and $\left[\mathcal{T}_A^{(S)}(\lambda)\right]^{-1}$. This can be obtained by using Eq. (2) with $\mu = -\lambda$ \cite{24} to get the following general matrix relation

$$
\left[\mathcal{T}_2^{(S)}(-\lambda)\right]^{-1} \mathcal{L}_{12}^{(S)}(2\lambda) \mathcal{T}_1^{(S)}(\lambda) = \mathcal{T}_1^{(S)}(\lambda) \mathcal{L}_{12}^{(S)}(2\lambda) \left[\mathcal{T}_2^{(S)}(-\lambda)\right]^{-1}. \quad (87)
$$

By applying both sides of relation \cite{87} on the pseudovacuum $|\bar{0}_S\rangle$ and by taking into account the upper triangular property of both $\mathcal{L}_{12}^{(S)}(2\lambda)$ and $\tilde{K}_S^{(-)}(\lambda)$ when acting on the state $|\bar{0}_S\rangle$ we
conclude that all the fields $C_{\alpha\beta}(\lambda)$ are annihilators,

$$C_{\alpha\beta}(\lambda) \langle 0_S \rangle = 0$$  \hspace{1cm} (88)

while $B_{\alpha\beta}(\lambda)$ acts as creation fields upon $|0_S\rangle$.

We also see that $A_i(\lambda) \langle 0_S \rangle$ for $i = 2, \ldots, 2S + 1$ turns out to be proportional to many distinct bulk terms of the form $[t_i(\lambda)]^{2L}$ since it involves the action of upper elements of the operator $\left[T_A(S)(-\lambda) \right]^{-1}$ on $|0_S\rangle$. In the specific case of the $XXX - S$ model the functions $t_i(\lambda)$ are

$$t_i(\lambda) = (\lambda + 2\eta S) \prod_{k=s-i+2}^{s} \frac{\lambda + \eta k - \eta S}{\lambda + \eta k + \eta S}. \hspace{1cm} (89)$$

As remarked in previous sections this difficulty can be circumvented by writing the fields $A_i(\lambda)$ as linear combinations of new operators $\tilde{A}_i(\lambda)$ such that their action on $|0_S\rangle$ is proportional only to $[t_i(\lambda)]^{2L}$ term. The solution of this problem involves a considerable amount of algebraic work but the final answer can fortunately be given in terms of the determinants of certain $j \times j$ matrices that shall be denoted by $M_{j,i}^{(+)\lambda}$. Its elements are determined in terms of the entries of the $L_{12}^{(S)}(\lambda)$ operator. More precisely, by writing $L_{12}^{(S)}(\lambda) = \sum_{abcd=1}^{2S+1} R_{a,b}^{c,d}(\lambda) \hat{e}_{cb} \otimes \hat{e}_{ad}$ we find that such linear combination is

$$A_i(\lambda) = \sum_{j=1}^{i} \frac{M_{j,i}^{(+)\lambda}(2\lambda)}{M_{j,j}^{(+)\lambda}(2\lambda)} \tilde{A}_j(\lambda) \hspace{1cm} (90)$$

where the $j \times j$ matrix $M_{j,i}^{(+)\lambda}(\lambda)$ is given by

$$M_{j,i}^{(+)\lambda}(\lambda) = \begin{pmatrix}
R_{1,1}^{1,1}(\lambda) & R_{1,1}^{2,2}(\lambda) & \cdots & R_{1,1}^{j-i-1,1}(\lambda) & R_{1,1}^{i,i}(\lambda) \\
R_{2,1}^{1,1}(\lambda) & R_{2,2}^{2,2}(\lambda) & \cdots & R_{2,2}^{j-i-1,2}(\lambda) & R_{2,2}^{i,i}(\lambda) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
R_{j,1}^{1,1}(\lambda) & R_{j,2}^{2,2}(\lambda) & \cdots & R_{j,j}^{j-i-1,j}(\lambda) & R_{j,j}^{i,i}(\lambda)
\end{pmatrix}_{j \times j}. \hspace{1cm} (91)$$

By using the relation (90) and the action of all $A_i(\lambda)$ on the reference state we find that

$$\tilde{A}_i(\lambda) \langle 0_S \rangle = \rho_S^{(-)} \omega_i^{(-)}(\lambda) \left[ \frac{t_2^L(\lambda)}{\zeta_S(\lambda)} \right] \langle 0_S \rangle, \hspace{1cm} (92)$$

21
where
\[ \omega_i^{(-)}(\lambda) = (\varepsilon)^{2S} \left[ f_i^{(-)}(S; \lambda, \varepsilon \xi_-) - \sum_{k=1}^{i-1} \left| \frac{M_{i-1,k}^{(-)}(2\lambda)}{M_{i-1,i-1}^{(+)}(2\lambda)} \right| f_k^{(-)}(S; \lambda, \varepsilon \xi_-) \right], \quad (93) \]

while the entries of a second \( j \times j \) auxiliary matrix \( M^{(-)}_{j,i}(\lambda) \) are given by
\[ M^{(-)}_{j,i}(\lambda) = \begin{pmatrix} R^{1,1}_{1,1}(\lambda) & R^{2,2}_{1,1}(\lambda) & \cdots & R^{i-1,i-1}_{1,1}(\lambda) & R^{i+1,i+1}_{1,1}(\lambda) & \cdots & R^{i,j}_{1,1}(\lambda) \\ R^{1,1}_{2,2}(\lambda) & R^{2,2}_{2,2}(\lambda) & \cdots & R^{i-1,i-1}_{2,2}(\lambda) & R^{i+1,i+1}_{2,2}(\lambda) & \cdots & R^{i,j}_{2,2}(\lambda) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ R^{1,1}_{j,j}(\lambda) & R^{2,2}_{j,j}(\lambda) & \cdots & R^{i-1,i-1}_{j,j}(\lambda) & R^{i+1,i+1}_{j,j}(\lambda) & \cdots & R^{i,j}_{j,j}(\lambda) \end{pmatrix}_{j \times j}. \quad (94) \]

Equipped with Eq. (90) one can now write the eigenvalue problem in terms of the new diagonal fields \( \tilde{A}_i(\lambda) \), namely
\[ \sum_{k=1}^{2S+1} \omega_k^{(+)}(\lambda) \tilde{A}_k(\lambda) \left| \tilde{\psi} \right\rangle = \frac{\Lambda_S(\lambda)}{\rho^{(+)}_S} \left| \tilde{\psi} \right\rangle, \quad (95) \]

where
\[ \omega_k^{(+)}(\lambda) = \sum_{i=k}^{2S+1} \left| \frac{M_{k,i}^{(+)}(2\lambda)}{M_{k,k}^{(+)}(2\lambda)} \right| f_i(S; \lambda, \xi_+). \quad (96) \]

At this stage we would like to emphasize that the role construction presented above is applicable to any multistate vertex model whose Boltzmann weights are invariant by one \( U(1) \) charge conservation symmetry. In order to get manageable expressions for the eigenvalues, however, one still needs to carry on cumbersome simplifications on the general formulae given in Eqs. (93 and 96). In the case of the \( XXX-S \) model we are able to show that all the contributions to \( \omega_i^{(+)}(\lambda) \) miraculously factorized in the following product forms
\[ \omega_i^{(\pm)}(\lambda) = \tau_i^{(\pm)}(\lambda) \chi_i^{(\pm)}(\lambda, \xi_\pm) \quad (97) \]

where
\[ \tau_i^{(+)}(\lambda) = \prod_{k=1}^{i} \frac{2\lambda + [2S + 3 - i - k] \eta}{2\lambda + [2 - k] \eta}, \quad (98) \]
\[ \tau_i^{(-)}(\lambda) = (\varepsilon)^{2S} \prod_{k=i}^{2S+1} \frac{2\lambda + [2 - 2i + k] \eta}{2\lambda + [1 + k - i] \eta}. \quad (99) \]
and
\[
\chi_i^{(+)}(\lambda, \bar{\xi}_+)(\lambda) = \prod_{j=1}^{2S+1-i} \left[ \bar{\xi}_+ + S + \frac{1}{2} - j - \frac{\lambda}{\eta} \right] \prod_{j=1}^{i-1} \left[ \bar{\xi}_+ + S + \frac{3}{2} - j + \frac{\lambda}{\eta} \right],
\]
(100)

\[
\chi_i^{(-)}(\lambda, \bar{\xi}_-)(\lambda) = \prod_{j=i}^{2S} \left[ \bar{\xi}_- + S + \frac{1}{2} - j + \frac{\lambda}{\eta} \right] \prod_{j=2S+2-i}^{2S} \left[ \bar{\xi}_- + S - \frac{1}{2} - j - \frac{\lambda}{\eta} \right].
\]
(101)

Before proceeding with further results we stress that the above explicit expressions for \(\omega_i^{(+)}(\lambda)\) with arbitrary \(S\) are novel in the literature since they were unknown even in the case of diagonal boundaries [10]. Now we reached a point in which we have gathered the basic ingredients to start an algebraic Bethe ansatz analysis of the eigenspectrum of \(\bar{t}_S(\lambda)\). In particular the vector \(\bar{0}_S\rangle\) is itself an eigenstate of \(\bar{t}_S(\lambda)\) with the eigenvalue
\[
\Lambda_{S}^{(0)}(\lambda) = \sum_{i=1}^{2S+1} \omega_i^{(+)}(\lambda)\omega_i^{(-)}(\lambda) \left[ \frac{\eta_i^{2}(\lambda)}{\zeta S(\lambda)} \right]^L.
\]
(102)

The other eigenvectors of \(\bar{t}_S(\lambda)\) are looked as states created by the action of the fields \(B_{\alpha\beta}(\lambda)\) on the reference state \(\bar{0}_S\rangle\). A single particle excitation is made by lowering the value of the azimuthal spin component by an unity on the ferromagnetic pseudovacuum \(\bar{0}_S\rangle\). From the point of view of the algebraic Bethe ansatz framework this excitation can be represented by \(B_{jj+1}(\lambda_1) \bar{0}_S\rangle\) for any \(j = 1, \ldots, 2S\). As far as commutation relations are concerned we find that it is simpler to choose the one-particle state as
\[
\bar{\psi}_1^j(\lambda_1)\rangle = B_{12}(\lambda_1) \bar{0}_S\rangle.
\]
(103)

The action of the double transfer matrix \(\bar{t}_S(\lambda)\) on this state can be computed with the aid of the commutation relations between the fields \(\bar{A}_i(\lambda)\) and \(B_{12}(\lambda)\) that can be obtained from the boundary Yang-Baxter algebra [11]. In Appendix B we present details of our analysis of the one-particle eigenvalue problem for \(S = \frac{3}{2}\). This study together with the previous results for \(S = 1\) [27, 29] and the help of mathematical induction lead us to the following general
expression

\[
\frac{\tilde{t}_S(\lambda)}{\rho_S^{(+)}, (\lambda)} |\tilde{\psi}_1(\lambda_1)\rangle = \sum_{i=1}^{2S+1} \left[ \frac{t^2(\lambda)}{\xi_S(\lambda)} \right]^L \omega^{(+)}_i(\lambda) \omega^{(-)}_i(\lambda) Q_i(\lambda, \lambda_1) |\tilde{\psi}_1(\lambda_1)\rangle \\
+ \sum_{i=1}^{2S} B_{i+1}(\lambda) \left[ q^{(1)}_i(\lambda, \lambda_1) \bar{A}_1(\lambda_1) + q^{(2)}_i(\lambda, \lambda_1) \bar{A}_2(\lambda_1) \right] |\tilde{0}_S\rangle,
\]

(104)

where function \(Q_i(\lambda, \lambda_j)\) is given by

\[
Q_i(\lambda, \lambda_j) = \begin{cases} 
R^{1,1}_{1,1}(\lambda_j - \lambda) R^{2,1}_{1,2}(\lambda_j + \lambda) & i = 1 \\
R^{1,1}_{1,1}(\lambda_j - \lambda) R^{2,1}_{1,1}(\lambda_j + \lambda) & \\
R^{1,1}_{2,1}(\lambda - \lambda_j) R^{1,2}_{2,1}(\lambda - \lambda_j) & i = 2, \ldots, 2S \\
R^{1,1}_{2,1}(\lambda - \lambda_j) R^{1,1}_{2,1}(\lambda - \lambda_j) & \\
R^{2,1}_{2,2}S+1(\lambda - \lambda_j) R^{2,1}_{2,2}S+1(\lambda - \lambda_j) & i = 2S + 1 \\
R^{2,1}_{2,2}S+1(\lambda - \lambda_j) R^{2,1}_{2,2}S+1(\lambda - \lambda_j) & \\
R^{2,1}_{2,2}S+1(\lambda - \lambda_j) R^{2,1}_{2,2}S+1(\lambda - \lambda_j) & \\
R^{2,1}_{2,2}S+1(\lambda - \lambda_j) R^{2,1}_{2,2}S+1(\lambda - \lambda_j) & \\
R^{2,1}_{2,2}S+1(\lambda - \lambda_j) R^{2,1}_{2,2}S+1(\lambda - \lambda_j) & \\
R^{2,1}_{2,2}S+1(\lambda - \lambda_j) R^{2,1}_{2,2}S+1(\lambda - \lambda_j) & \\
R^{2,1}_{2,2}S+1(\lambda - \lambda_j) R^{2,1}_{2,2}S+1(\lambda - \lambda_j) & \\
\end{cases}
\]

(105)

From (104) we see that the unwanted terms proportional to \(B_{i+1}(\lambda)\) can be eliminated by imposing that the rapidity \(\lambda_1\) satisfies the following one-particle Bethe ansatz equation,

\[
\left[ \frac{t_1(\lambda_1)}{t_2(\lambda_1)} \right]^{2L} \frac{\omega^{(-)}_i(\lambda_1)}{\omega^{(-)}_i(\lambda_1)} = -\frac{q^{(2)}_i(\lambda, \lambda_1)}{q^{(1)}_i(\lambda, \lambda_1)}, \quad i = 1, \ldots, 2S.
\]

(106)

We note that though the expressions for \(q^{(1)}_i(\lambda, \lambda_1)\) and \(q^{(2)}_i(\lambda, \lambda_1)\) have in general a very involved dependence on the i-th index, see for instance Appendix B, we have found out that the ratio \(\frac{q^{(2)}_i(\lambda, \lambda_1)}{q^{(1)}_i(\lambda, \lambda_1)}\) does not depend of such index. Its expression for arbitrary \(S\), coming directly from the commutation rules, involves many complicated terms and it has been collected in Appendix D. It turns out, however, that it is possible to carry out further simplifications in equations (D.1-D.2) thanks to several identities between the Boltzmann weights \(R^{c,d}_{a,b}(\lambda)\). This also leads us to conclude that the ratio \(\frac{q^{(2)}_i(\lambda, \lambda_1)}{q^{(1)}_i(\lambda, \lambda_1)}\) does not depend on the spectral parameter \(\lambda\). This is consistent to what one would expect from a standard Bethe ansatz analysis and the
simplified expression for such ratio reads
\[
\frac{q_i^{(2)}(\lambda, \lambda_1)}{q_i^{(1)}(\lambda, \lambda_1)} = \frac{\omega_i^{(+)}(\lambda_1)}{\omega_i^{(+)}(\lambda_1)} \Theta(\lambda_1)
\]  
(107)
where for later convenience we define function \(\Theta(\lambda)\) separately, namely
\[
\Theta(\lambda) = \frac{R_{1,1}(\lambda)R_{2,2}(\lambda) - R_{1,2}(\lambda)R_{2,1}(\lambda)}{[R_{1,2}(\lambda)]^2}.
\]  
(108)

Putting all these results together we find that \(|\tilde{\psi}_1(\lambda_1)\rangle\) is an eigenvector of \(\tilde{t}_S(\lambda)\) with eigenvalue \(\Lambda_1(\lambda, \lambda_1)\) given by
\[
\frac{\Lambda_S^{(1)}(\lambda, \lambda_1)}{\rho_S^{(+)}/\rho_S^{(-)}} = \sum_{i=1}^{2S+1} \left[ \frac{t_i^{(2)}(\lambda)}{\zeta_S(\lambda)} \right]^L \omega_i^{(+)}(\lambda)\omega_i^{(-)}(\lambda)Q_i(\lambda, \lambda_1)
\]  
(109)
provided that the variable \(\lambda_1\) satisfies the restriction
\[
\left[ \frac{t_1^{(2)}(\lambda_1)}{t_2^{(2)}(\lambda_1)} \right]^{2L} \frac{\omega_1^{(+)}(\lambda_1)\omega_1^{(-)}(\lambda_1)}{\omega_2^{(+)}(\lambda_1)\omega_2^{(-)}(\lambda_1)} = \Theta(\lambda_1).
\]  
(110)

Here we remark that the equation (110) is equivalent to the condition of analyticity of \(\Lambda_1(\lambda, \lambda_1)\) as a function of the rapidity \(\lambda_1\). This fact is indeed an extra verification of the validity of our Bethe ansatz analysis.

We now turn to the analysis of the two-particle state. In this case one expects that this state should be given in terms of two linearly independent vectors \(B_{12}(\lambda_1)B_{12}(\lambda_2) |0_S\rangle\) and \(B_{13}(\lambda_1) |0_S\rangle\). Previous experience in determining two-particle states [24, 25, 26] suggests us to look first for the commutation rule between the fields \(B_{12}(\lambda_1)\) and \(B_{12}(\lambda_2)\). To avoid overcrowding this section with more heavier formulae we have exhibited this relation for \(S \geq 1\) in Appendix D. From equations (90) and the observations made in Appendix D we clearly see that the state
\[
|\tilde{\psi}_2(\lambda_1, \lambda_2)\rangle = B_{12}(\lambda_1)B_{12}(\lambda_2) + B_{13}(\lambda_1) \left[ \frac{R_{1,2}(\lambda_1 + \lambda_2)}{R_{1,2}(\lambda_1 + \lambda_2)} \tilde{A}_2(\lambda_2) + \left( \frac{R_{1,2}(\lambda_1 + \lambda_2)}{R_{1,2}(\lambda_1 + \lambda_2)} \right) \frac{M_{1,2}^{(2)}(\lambda_1, \lambda_2)}{M_{1,1}^{(2)}(\lambda_2)} \tilde{M}_{1,2}^{(2)}(\lambda_1, \lambda_2) - \left( \frac{R_{1,2}(\lambda_1 + \lambda_2)}{R_{1,2}(\lambda_1 + \lambda_2)} \right) \tilde{A}_1(\lambda_2) \right] |0_S\rangle,
\]  
(111)
is symmetric under the exchange of the rapidities $\lambda_1$ and $\lambda_2$. In other words we have that

$$\ket{\tilde{\psi}_2(\lambda_1, \lambda_2)} = Z_{S}(\lambda_1, \lambda_2) \ket{\tilde{\psi}_2(\lambda_2, \lambda_1)}$$

where $Z_S(\lambda_1, \lambda_2)$ is the following function,

$$Z_S(\lambda_1, \lambda_2) = -\frac{R_{2,1}(\lambda + \lambda_1)}{R_{1,2}(\lambda + \lambda_1)} \begin{vmatrix} R_{1,2}^3(\lambda - \lambda_1) & R_{1,3}^3(\lambda - \lambda_1) \\ R_{2,2}^3(\lambda - \lambda_1) & R_{2,3}^3(\lambda - \lambda_1) \\ R_{1,1}^1(\lambda - \lambda_1) & R_{1,3}^1(\lambda - \lambda_1) \end{vmatrix}$$

This state is therefore an educated ansatz for the two-particle vector for general $S \geq 1$. Note that it reproduces the previous state for $S = 1$ \[221, 27, 29\] and in Appendix B we have presented all the needed evidences that it is indeed a suitable eigenvector for $S = \frac{3}{2}$. The corresponding eigenvalue can be calculated by keeping only the terms proportional to the vector $B_{12}(\lambda_1)B_{12}(\lambda_2)$ coming from the first part of the commutation relations between the fields $\tilde{A}_i(\lambda)$ and $B_{12}(\lambda_i)$. Taking into account our previous experience with the one-particle state and the structure of the commutation rules discussed in Appendices B and D we find that

$$\Lambda^{(2)}_S(\lambda, \{\lambda_1, \lambda_2\}) = \frac{2^{S+1}}{\rho^+_S \rho^-_S} \sum_{i=1}^{2S+1} \left[ \frac{t_2^2(\lambda)}{t_1(\lambda)} \right]^L \omega^{(+)}_i(\lambda)\omega^{(-)}_i(\lambda) \prod_{j=1}^{n=2} Q_i(\lambda, \lambda_j)$$

The associated Bethe ansatz equations are expected to be the condition on the rapidities such that the residues at the simple poles $\lambda = \lambda_1, \lambda_2$ present in functions $Q_i(\lambda, \lambda_j)$ vanish. This condition is equivalent to the following system of equations

$$\left[ \frac{t_j(\lambda_j)}{t_2(\lambda_j)} \right]^{2L} \frac{\omega^{(+)}_j(\lambda_j)\omega^{(-)}_j(\lambda_j)}{\omega^{(+)}_2(\lambda_j)\omega^{(-)}_2(\lambda_j)} = \Theta(\lambda_j) \prod_{i=1}^{n=2} \frac{Q_2(\lambda_j, \lambda_i)}{Q_1(\lambda_j, \lambda_i)} \quad j = 1, \ldots, n = 2.$$

By the same token, one expects that general multiparticle states can in principle be constructed in terms of a recurrence relation of order $2S$ that involves the creation fields $B_{1j}(\lambda) \quad j = 2, \ldots, 2S + 1$. The precise structure of such relation for arbitrary $S$ has however eluded us so far. This by no means prevents us to propose general expressions for the corresponding eigenvalues and Bethe ansatz equations. In any factorizable theory, it is believed that the two-particle results already contain the main flavour about the content of the spectrum.
This means that the expressions (114) and (115) are expected to be valid for general values of \( n \leq 2LS \). Considering these observations and after working out the explicit expressions for functions \( Q_i(\lambda, \lambda_j) \) we find that the \( n \)-particle eigenvalue \( \Lambda^{(n)}_s(\lambda, \{ \lambda_i \}) \) is given by

\[
\frac{\Lambda^{(n)}_s(\lambda, \{ \lambda_i \})}{\rho^{(+)}_{\rho}(-)} = \sum_{i=1}^{2S+1} \left[ \frac{t_i^2(\lambda)}{\zeta^{(+)}_s(\lambda)} \right]^L \omega^{(+)}_i(\lambda) \omega^{(-)}_i(\lambda)
\]

while the Bethe ansatz equations are given by

\[
\left[ \lambda_j + \eta S \right]^{2L} \left[ \lambda_j - \eta S \right] = \left( \frac{\lambda_j - \varepsilon \eta \bar{\xi}_- + \frac{\eta}{2}}{\lambda_j + \varepsilon \eta \bar{\xi}_- - \frac{\eta}{2}} \right) \left( \frac{\lambda_j + \eta \bar{\xi}_+ + \frac{\eta}{2}}{\lambda_j - \eta \bar{\xi}_+ - \frac{\eta}{2}} \right) \prod_{i=1}^{n} \left( \frac{\lambda_j - \lambda_i + \eta}{\lambda_j - \lambda_i - \eta} \right) \left( \frac{\lambda_j + \lambda_i + \eta}{\lambda_j + \lambda_i - \eta} \right),
\]

where we have performed the displacement \( \lambda_i \rightarrow \lambda_i - \eta S \) in order to bring the Bethe ansatz equations in a more symmetrical form.

At this point it should be emphasized that the right hand side of the Bethe ansatz equations (117) depend on both the spin \( S \) and the off-diagonal elements \( c_{\pm}, d_{\pm} \) through the renormalized variable \( \bar{\xi}_\pm \) defined in Eq.(21). We also mention that we have verified numerically for several values of \( L \) and \( S \) that the equations (116,117) indeed reproduces the ground state and few low-lying excitations of the double-row transfer matrix \( \bar{t}_S(\lambda) \). In particular, we have been able to check the completeness of the Bethe ansatz solution for \( L = 2 \) up to \( S = \frac{3}{2} \). Finally, we remark that the final results for the eigenvalues (114) and Bethe ansatz equations (113) are expected to be valid for any integrable vertex model whose underlying \( R \)-matrix possesses an unique \( U(1) \) charge symmetry and the invariance (5,6).

5 Conclusions

The purpose of this paper was to solve the integrable XXX-\( S \) Heisenberg model with open boundary conditions by means of the quantum inverse scattering approach. We first argued that the corresponding \( K \)-matrices are diagonalizable by special similarity transformations...
without a dependence on the spectral parameter. This fact together with the property of reversing gauge transformed Boltzmann weights leads us to an eigenvalue problem with only one non-diagonal effective $K$-matrix. In the cases when such $K$-matrix are either upper or lower triangular we managed to present explicit expressions for the eigenvalues of the doubled transfer matrix operator $t_S(\lambda)$ as well as the associated Bethe ansatz equations for arbitrary values of the spin-$S$. This condition was shown to be equivalent to two possible constraints between the four off-diagonal boundary parameters, leading us with five free parameters out of six possible ones.

We hope that the ideas developed in this paper will be also suitable to solve a broad class of isotropic integrable systems with non-diagonal open boundaries. In fact, the method devised here has been first applied to the fundamental $SU(N)$ isotropic vertex model under more restrictive open boundary conditions [33]. We expect that the nested Bethe ansatz approach could be further generalized to tackle effective triangular $K$-matrices which will provide us the solution of the associated doubled transfer matrix operator with fewer constrained boundary parameters as compared to that presented in ref. [33]. We also hope that other vertex models based on higher rank symmetries such as $O(N)$ and $sp(2N)$ Lie algebras could be dealt by the framework discussed in this work. This assumes that certain classes of non-diagonal $K$-matrices of these vertex models can be classified in terms of similarity transformations that are itself symmetries of the corresponding $\mathcal{L}$-operator, acting on spectral dependent diagonal solutions for the reflection equation. This would means that our observation of section (2) for $SU(2)$ could be generalized to other Lie algebras as well. We plan to investigate such rather interesting possibility in a future work.

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Appendix A: The $K$-matrix properties

In this Appendix we briefly summarize the explicit expressions of the $K$-matrix elements satisfying the reflection equation (8) for $S = 1$ and $\frac{3}{2}$. For $S = 1$ the corresponding matrix is given by

$$K_1(\lambda) = \begin{pmatrix}
    k_{11}(\lambda) & k_{12}(\lambda) & k_{13}(\lambda) \\
    k_{21}(\lambda) & k_{22}(\lambda) & k_{23}(\lambda) \\
    k_{31}(\lambda) & k_{32}(\lambda) & k_{33}(\lambda)
\end{pmatrix},$$

(A.1)

where the elements $k_{ij}(\lambda)$ are given by

$$k_{11}(\lambda) = -\frac{1}{4} \left( 2\xi - \frac{1}{2} + \frac{\lambda}{\eta} \right) \left( 2\xi + \frac{1}{2} + \frac{\lambda}{\eta} \right) + \frac{cd}{8} \left( \frac{1}{2} - \frac{\lambda}{\eta} \right),$$

(A.2)

$$k_{12}(\lambda) = \frac{c}{2\sqrt{2}} \left( 2\xi - \frac{1}{2} + \frac{\lambda}{\eta} \right) \frac{\lambda}{\eta},$$

(A.3)

$$k_{13}(\lambda) = \frac{c^2}{4} \left( \frac{1}{2} - \frac{\lambda}{\eta} \right) \frac{\lambda}{\eta},$$

(A.4)

$$k_{21}(\lambda) = \frac{d}{2\sqrt{2}} \left( 2\xi - \frac{1}{2} + \frac{\lambda}{\eta} \right) \frac{\lambda}{\eta},$$

(A.5)

$$k_{22}(\lambda) = -\frac{1}{4} \left( 2\xi + \frac{1}{2} - \frac{\lambda}{\eta} \right) \left( 2\xi - \frac{1}{2} + \frac{\lambda}{\eta} \right) + \frac{cd}{4} \left( \frac{1}{2} - \frac{\lambda}{\eta} \right) \left( \frac{1}{2} + \frac{\lambda}{\eta} \right),$$

(A.6)

$$k_{23}(\lambda) = \frac{c}{2\sqrt{2}} \left( 2\xi + \frac{1}{2} - \frac{\lambda}{\eta} \right) \frac{\lambda}{\eta},$$

(A.7)

$$k_{31}(\lambda) = \frac{d^2}{4} \left( \frac{1}{2} - \frac{\lambda}{\eta} \right) \frac{\lambda}{\eta},$$

(A.8)

$$k_{32}(\lambda) = \frac{d}{2\sqrt{2}} \left( 2\xi + \frac{1}{2} - \frac{\lambda}{\eta} \right) \frac{\lambda}{\eta},$$

(A.9)

$$k_{33}(\lambda) = -\frac{1}{4} \left( 2\xi - \frac{1}{2} - \frac{\lambda}{\eta} \right) \left( 2\xi + \frac{1}{2} - \frac{\lambda}{\eta} \right) + \frac{cd}{8} \left( \frac{1}{2} - \frac{\lambda}{\eta} \right).$$

(A.10)

On the other hand for $S = \frac{3}{2}$ we have

$$K_2(\lambda) = \begin{pmatrix}
    \bar{k}_{11}(\lambda) & \bar{k}_{12}(\lambda) & \bar{k}_{13}(\lambda) & \bar{k}_{14}(\lambda) \\
    \bar{k}_{21}(\lambda) & \bar{k}_{22}(\lambda) & \bar{k}_{23}(\lambda) & \bar{k}_{24}(\lambda) \\
    \bar{k}_{31}(\lambda) & \bar{k}_{32}(\lambda) & \bar{k}_{33}(\lambda) & \bar{k}_{34}(\lambda) \\
    \bar{k}_{41}(\lambda) & \bar{k}_{42}(\lambda) & \bar{k}_{43}(\lambda) & \bar{k}_{44}(\lambda)
\end{pmatrix},$$

(A.11)
where the elements $\bar{k}_{ij}(\lambda)$ are given by

$$\bar{k}_{11}(\lambda) = \frac{cd}{18} \left[ \xi - \left( 3\xi + \frac{\lambda}{\eta} \right) \left( 1 - \frac{\lambda}{\eta} \right) \right] + \frac{1}{27} \left( 3\xi + \frac{\lambda}{\eta} \right) \left( 3\xi - 1 + \frac{\lambda}{\eta} \right) \left( 3\xi + 1 + \frac{\lambda}{\eta} \right),$$

(A.12)

$$\bar{k}_{12}(\lambda) = -\frac{c}{18\sqrt{3}} \left[ cd \left( 1 - \frac{\lambda}{\eta} \right) - 2 \left( 3\xi - 1 + \frac{\lambda}{\eta} \right) \left( 3\xi + \frac{\lambda}{\eta} \right) \right] \frac{\lambda}{\eta},$$

(A.13)

$$\bar{k}_{13}(\lambda) = -\frac{c^2}{9\sqrt{3}} \left( \frac{1}{2} - \frac{\lambda}{\eta} \right) \left( 3\xi - 1 + \frac{\lambda}{\eta} \right) \frac{\lambda}{\eta},$$

(A.14)

$$\bar{k}_{14}(\lambda) = \frac{c^2}{27} \left( \frac{1}{2} - \frac{\lambda}{\eta} \right) \left( 1 - \frac{\lambda}{\eta} \right) \frac{\lambda}{\eta},$$

(A.15)

$$\bar{k}_{21}(\lambda) = -\frac{d}{18\sqrt{3}} \left[ cd \left( 1 - \frac{\lambda}{\eta} \right) - 2 \left( 3\xi - 1 + \frac{\lambda}{\eta} \right) \left( 3\xi + \frac{\lambda}{\eta} \right) \right] \frac{\lambda}{\eta},$$

(A.16)

$$\bar{k}_{22}(\lambda) = \frac{2cd}{27} \left[ \left( \frac{\lambda}{\eta} \right)^3 + \left( 3\xi - \frac{5}{4} \right) \left( \frac{\lambda}{\eta} \right)^2 + \left( \frac{1}{4} + \frac{3\xi}{4} \right) \frac{\lambda}{\eta} - \frac{3\xi}{2} \right]$$

$$+ \frac{1}{27} \left( 3\xi + 1 - \frac{\lambda}{\eta} \right) \left( 3\xi + \frac{\lambda}{\eta} \right) \left( 3\xi - 1 + \frac{\lambda}{\eta} \right),$$

(A.17)

$$\bar{k}_{23}(\lambda) = -\frac{c}{54} \left[ cd \left( 1 - \frac{\lambda}{\eta} \right) \left( 1 + 2\frac{\lambda}{\eta} \right) + 4 \left( \left( 1 - \frac{\lambda}{\eta} \right)^2 - (3\xi)^2 \right) \right] \frac{\lambda}{\eta},$$

(A.19)

$$\bar{k}_{24}(\lambda) = -\frac{c^2}{9\sqrt{3}} \left( \frac{1}{2} - \frac{\lambda}{\eta} \right) \left( 3\xi + 1 - \frac{\lambda}{\eta} \right) \frac{\lambda}{\eta},$$

(A.20)

$$\bar{k}_{31}(\lambda) = -\frac{d^2}{9\sqrt{3}} \left( \frac{1}{2} - \frac{\lambda}{\eta} \right) \left( 3\xi - 1 + \frac{\lambda}{\eta} \right) \frac{\lambda}{\eta},$$

(A.21)

$$\bar{k}_{32}(\lambda) = -\frac{d}{54} \left[ cd \left( 1 - \frac{\lambda}{\eta} \right) \left( 1 + 2\frac{\lambda}{\eta} \right) + 4 \left( \left( 1 - \frac{\lambda}{\eta} \right)^2 - (3\xi)^2 \right) \right] \frac{\lambda}{\eta},$$

(A.22)

$$\bar{k}_{33}(\lambda) = -\frac{2cd}{27} \left[ \left( \frac{\lambda}{\eta} \right)^3 - \left( 3\xi + \frac{5}{4} \right) \left( \frac{\lambda}{\eta} \right)^2 + \left( \frac{1}{4} - \frac{3\xi}{4} \right) \frac{\lambda}{\eta} + \frac{3\xi}{2} \right]$$

$$+ \frac{1}{27} \left( 3\xi - \frac{\lambda}{\eta} \right) \left( 3\xi + 1 - \frac{\lambda}{\eta} \right) \left( 3\xi - 1 + \frac{\lambda}{\eta} \right),$$

(A.23)

$$\bar{k}_{34}(\lambda) = -\frac{c}{18\sqrt{3}} \left[ cd \left( 1 - \frac{\lambda}{\eta} \right) - 2 \left( 3\xi - \frac{\lambda}{\eta} \right) \left( 3\xi + 1 - \frac{\lambda}{\eta} \right) \right] \frac{\lambda}{\eta},$$

(A.25)

$$\bar{k}_{41}(\lambda) = \frac{d^3}{27} \left( \frac{1}{2} - \frac{\lambda}{\eta} \right) \left( 1 - \frac{\lambda}{\eta} \right) \frac{\lambda}{\eta},$$

(A.26)

$$\bar{k}_{42}(\lambda) = -\frac{d^2}{9\sqrt{3}} \left( \frac{1}{2} - \frac{\lambda}{\eta} \right) \left( 3\xi + 1 - \frac{\lambda}{\eta} \right) \frac{\lambda}{\eta},$$

(A.27)
\[ k_{43}(\lambda) = -\frac{d}{18\sqrt{3}} \left[ cd \left( 1 - \frac{\lambda}{\eta} \right) - 2 \left( 3\xi - \frac{\lambda}{\eta} \right) \left( 3\xi + 1 - \frac{\lambda}{\eta} \right) \right] \frac{\lambda}{\eta}, \quad (A.28) \]
\[ k_{44}(\lambda) = \frac{cd}{18} \left[ \xi - \left( 3\xi - \frac{\lambda}{\eta} \right) \left( 1 - \frac{\lambda}{\eta} \right) \right] + \frac{1}{27} \left( 3\xi - \frac{\lambda}{\eta} \right) \left( 3\xi - 1 - \frac{\lambda}{\eta} \right) \left( 3\xi + 1 - \frac{\lambda}{\eta} \right), \quad (A.29) \]

Next we list the dependence of the off-diagonal coefficients of the transformed \( K \)-matrix \( \tilde{K}^{(\text{S})}_S(\lambda) \) on the parameters \( c_\pm \) and \( d_\pm \). For \( S = \frac{1}{2} \) we find
\[ \sigma_{12} = -\frac{\epsilon_+ [c_+d_- + c_-d_+ - 2c_+d_+] + (c_-d_+ - d_-c_+) \sqrt{1 + c_+d_+}}{2d_+ \sqrt{1 + c_+d_+}}, \quad (A.30) \]
while for \( S = 1 \) we have
\[ \kappa_{12} = \frac{(2 + c_-d_+ + c_+d_+)}{32\sqrt{2}(1 + c_+d_+)} \left( -\epsilon_+ 2(c_+ - c_-) \sqrt{1 + c_+d_+} - \Delta \right), \quad (A.31) \]
\[ \kappa_{13} = -\epsilon_+ \Theta^0 + 2c_+^4 (d_+ - d_-)^2 + \frac{\epsilon_+^2}{32} \left[ d_+^2 \Theta^1 - 2d_+d_- \Theta^2 + d_-^2 \Theta^3 \right], \quad (A.32) \]
\[ \kappa_{23} = \frac{\xi-}{8\sqrt{2}} \left( 2(c_+ - c_-) \sqrt{1 + c_+d_+} + \epsilon_+ \Delta \right), \quad (A.33) \]
where \( \Delta \) and \( \Theta^i \) are given by
\[ \Delta = 2c_- - c_+(2 - c_-d_+ + c_+d_-), \quad (A.34) \]
\[ \Theta^0 = c_+^4(d_+ - d_-) \sqrt{1 + c_+d_+} [d_- (2 + c_+d_+) - d_+ (2 + c_-d_+)], \quad (A.35) \]
\[ \Theta^1 = 4c_+d_+ + c_-d_+ (4 + c_-d_+), \quad (A.36) \]
\[ \Theta^2 = 2c_-d_+ + c_+d_+ (6 + c_-d_+), \quad (A.37) \]
\[ \Theta^3 = c_+d_+ (8 + c_+d_+). \quad (A.38) \]

**Appendix B: One and Two particle states for \( S = \frac{3}{2} \)**

The purpose of this Appendix is to present some of the technical details entering the analysis of the one and two particle states for \( S = \frac{3}{2} \). In order to do that it is convenient to work with
a new matrix \( \tilde{R}_{ab}^{(S)}(\lambda) = P_{ab} L_{ab}^{(S)}(\lambda) \) where \( P_{ab} \) is the permutator. This matrix plays a direct role in the quantum inverse scattering method and Eq. (B.1) is rewritten in terms of \( \tilde{R}_{ab}^{(S)}(\lambda) \) as

\[
\tilde{R}_{12}^{(S)}(u - v) \frac{1}{T_A} A_{12}^{(S)}(u) \tilde{R}_{12}^{(S)}(u + v) \frac{1}{T_A} A_{12}^{(S)}(v) = \frac{1}{T_A} A_{12}^{(S)}(v) \tilde{R}_{12}^{(S)}(u + v) \frac{1}{T_A} A_{12}^{(S)}(u) \tilde{R}_{12}^{(S)}(u - v),
\]

(B.1)

In the specific case of a 44 vertex model, the \( \tilde{R}_{ab}^{(S)}(\lambda) \) operator can be expressed in terms of the following matrix:

\[
\begin{pmatrix}
    a(\lambda) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & b(\lambda) & 0 & 0 & c(\lambda) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & e(\lambda) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & f(\lambda) & 0 & 0 & h(\lambda) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & g(\lambda) & 0 & 0 & i(\lambda) & 0 & 0 & j(\lambda) & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & b(\lambda) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & r(\lambda) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & r(\lambda) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q(\lambda) & 0 & 0 & i_1(\lambda) & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & n(\lambda) & 0 & 0 & r_1(\lambda) & 0 & 0 & h_1(\lambda) & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f_1(\lambda) & 0 & 0 & t_1(\lambda) & 0 & 0 & f_1(\lambda) & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_1(\lambda) & 0 & 0 & c_1(\lambda) & 0 & 0 & e_1(\lambda) & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d(\lambda) & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & n(\lambda) & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e_1(\lambda) & 0 & 0 & b_1(\lambda) & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_1(\lambda)
\end{pmatrix}
\]

(B.2)

In order to solve the one-particle problem one first needs to obtain the appropriate commutation rules between the fields \( A_i(u) \) and \( B_{12}(v) \) coming from the boundary Yang-Baxter equation (B.1). Using the symbol \([i,j]\) to represent the i-th row and the j-th column of Eq. (B.1) we conclude that such suitable commutation rules are derivable from the entries \([1,2],[2,3],[3,4],[2,6],[3,7] \) and \([4,8]\). Further progress are made replacing the fields \( A_i(u) \) by \( \tilde{A}_i(u) \) in these equations with the help of the relations (90). After several algebraic manipulations we obtain
the following structure

\[
\tilde{A}_1(u)B_{12}(v) = a_1^1(u,v)B_{12}(v)\tilde{A}_1(u) + a_2^1(u,v)B_{12}(u)\tilde{A}_1(v) + a_3^1(u,v)B_{12}(u)\tilde{A}_2(v) \\
+ a_4^1(u,v)B_{13}(v)C_{21}(u) + a_5^1(u,v)B_{13}(u)C_{21}(v) + a_6^1(u,v)B_{13}(u)C_{32}(v) \\
+ a_7^1(u,v)B_{14}(v)C_{31}(u) + a_8^1(u,v)B_{14}(u)C_{31}(v) + a_9^1(u,v)B_{14}(u)C_{42}(v) \\
\]  

(B.3)

\[
\tilde{A}_2(u)B_{12}(v) = a_1^2(u,v)B_{12}(v)\tilde{A}_2(u) + a_2^2(u,v)B_{12}(u)\tilde{A}_1(v) + a_3^2(u,v)B_{12}(u)\tilde{A}_2(v) \\
+ a_4^2(u,v)B_{23}(u)\tilde{A}_1(v) + a_5^2(u,v)B_{23}(u)\tilde{A}_2(v) + a_6^2(u,v)B_{13}(v)C_{21}(u) \\
+ a_7^2(u,v)B_{13}(v)C_{32}(u) + a_8^2(u,v)B_{13}(u)C_{21}(v) + a_9^2(u,v)B_{13}(u)C_{32}(v) \\
+ a_{10}^2(u,v)B_{24}(u)C_{21}(v) + a_{11}^2(u,v)B_{24}(u)C_{32}(v) + a_{12}^2(u,v)B_{14}(v)C_{31}(u) \\
+ a_{13}^2(u,v)B_{14}(v)C_{42}(u) + a_{14}^2(u,v)B_{14}(u)C_{31}(v) + a_{15}^2(u,v)B_{14}(u)C_{42}(v) \\
\]  

(B.4)

\[
\tilde{A}_3(u)B_{12}(v) = a_1^3(u,v)B_{12}(v)\tilde{A}_3(u) + a_2^3(u,v)B_{12}(u)\tilde{A}_1(v) + a_3^3(u,v)B_{12}(u)\tilde{A}_2(v) \\
+ a_4^3(u,v)B_{23}(u)\tilde{A}_1(v) + a_5^3(u,v)B_{23}(u)\tilde{A}_2(v) + a_6^3(u,v)B_{34}(u)\tilde{A}_1(v) \\
+ a_7^3(u,v)B_{34}(u)\tilde{A}_2(v) + a_8^3(u,v)B_{13}(v)C_{21}(u) + a_9^3(u,v)B_{13}(v)C_{32}(u) \\
+ a_{10}^3(u,v)B_{13}(v)C_{43}(u) + a_{11}^3(u,v)B_{13}(u)C_{21}(v) + a_{12}^3(u,v)B_{13}(u)C_{32}(v) \\
+ a_{13}^3(u,v)B_{24}(u)C_{21}(v) + a_{14}^3(u,v)B_{24}(u)C_{32}(v) + a_{15}^3(u,v)B_{14}(v)C_{31}(u) \\
+ a_{16}^3(u,v)B_{14}(v)C_{42}(u) + a_{17}^3(u,v)B_{14}(u)C_{31}(v) + a_{18}^3(u,v)B_{14}(u)C_{42}(v) \\
\]  

(B.5)
\[ \tilde{A}_4(u)B_{12}(v) = a_1^4(u, v)B_{12}(v)\tilde{A}_4(u) + a_2^4(u, v)B_{12}(u)\tilde{A}_1(v) + a_3^4(u, v)B_{12}(u)\tilde{A}_2(v) \\
+ a_4^4(u, v)B_{23}(u)\tilde{A}_1(v) + a_5^4(u, v)B_{23}(u)\tilde{A}_2(v) + a_6^4(u, v)B_{34}(u)\tilde{A}_1(v) \\
+ a_7^4(u, v)B_{34}(u)\tilde{A}_2(v) + a_8^4(u, v)B_{13}(v)C_{21}(u) + a_9^4(u, v)B_{13}(v)C_{32}(u) \\
+ a_{10}^4(u, v)B_{13}(v)C_{43}(u) + a_{11}^4(u, v)B_{13}(u)C_{21}(v) + a_{12}^4(u, v)B_{13}(u)C_{32}(v) \\
+ a_{13}^4(u, v)B_{24}(u)C_{21}(v) + a_{14}^4(u, v)B_{24}(u)C_{32}(v) + a_{15}^4(u, v)B_{14}(v)C_{31}(u) \\
+ a_{16}^4(u, v)B_{14}(v)C_{42}(u) + a_{17}^4(u, v)B_{14}(u)C_{31}(v) + a_{18}^4(u, v)B_{14}(u)C_{42}(v) \tag{B.6} \]

Before proceeding we would like to remark that several identities between the Boltzmann weights have been used in order to obtain relations (B.3–B.6). We also note that many of the coefficients \(a_i^j(u, v)\) are proportional to annihilation operators and not all of them are relevant in the calculations. In Appendix C we have listed only those that indeed play an important role in our analysis since in general they are sufficiently cumbersome. By applying Eqs. (B.3–B.6) on the pseudovacuum \(|\tilde{0}_2\rangle\) we see that one can rearrange the action of the double transfer matrix \(\tilde{t}_2(\lambda)\) on the one-particle state \(B_{12}(\lambda_1)|\tilde{0}_2\rangle\) as in Eq. (107). Furthermore, it turns out that the functions \(Q_i(\lambda, \lambda_1), q_i^{(1)}(\lambda, \lambda_1)\) and \(q_i^{(2)}(\lambda, \lambda_1)\) can therefore be explicitly read off, namely

\[
Q_i(\lambda, \lambda_1) = a_i^1(\lambda, \lambda_1) \tag{B.7}
\]

\[
q_i^{(1)}(\lambda, \lambda_1) = \sum_{j=i}^{4} \omega_j^{(+)}(\lambda)a_{2j}^i(\lambda, \lambda_1) \tag{B.8}
\]

\[
q_i^{(2)}(\lambda, \lambda_1) = \sum_{j=i}^{4} \omega_j^{(+)}(\lambda)a_{2j+1}^i(\lambda, \lambda_1) \tag{B.9}
\]

where \(i = 1, \ldots, 4\) and function \(\omega_i^{(+)}(\lambda)\) has been defined in Eq. (98).

As mentioned in the main text the ratio \(\frac{q_i^{(2)}(\lambda, \lambda_1)}{q_i^{(1)}(\lambda, \lambda_1)}\) is independent of the \(i\)-th index and of the spectral parameter \(\lambda\). In our case this ratio is given by

\[
\frac{q_i^{(2)}(\lambda, \lambda_1)}{q_i^{(1)}(\lambda, \lambda_1)} = -\frac{\omega_i^{(+)}(\lambda_1)}{\omega_i^{(+)}(\lambda)} \left( \frac{a(\lambda_1)l(\lambda_1) - b(\lambda_1)b(\lambda_1)}{[e(\lambda_1)]^2} \right) \tag{B.10}
\]

Next we turn to the two-particle state. In order to obtain an ansatz to this vector we have considered the commutation rules [1,3] and [1,6] coming from Eq. (B.1). Acting these relations
on $|\tilde{0}_\frac{3}{2}\rangle$ leads us to the following expression

$$\left[ B_{12}(u)B_{12}(v) + B_{13}(u) \left( \alpha_2(u,v)\tilde{A}_2(v) + \alpha_1(u,v)\tilde{A}_1(v) \right) \right] |\tilde{0}_\frac{3}{2}\rangle = Z_{\frac{3}{2}}(u,v) \left[ B_{12}(v)B_{12}(u) + B_{13}(v) \left( \alpha_3(u,v)\tilde{A}_2(u) + \alpha_4(u,v)\tilde{A}_1(u) \right) \right] |\tilde{0}_\frac{3}{2}\rangle,$$  \hspace{1cm} (B.11)

where functions $\alpha_i(u,v)$ for $i=1,\ldots,4$ and $Z_{\frac{3}{2}}(u,v)$ are given by

$$\alpha_1(u,v) = \frac{f(u+v) b(2v)}{e(u+v) a(2v)} - \frac{f(u-v) h(u+v)}{h(u-v) e(u+v)},$$ \hspace{1cm} (B.12)

$$\alpha_2(u,v) = \frac{f(u+v)}{e(u+v)}, \quad \alpha_3(u,v) = \frac{f(u+v)}{e(u+v)},$$ \hspace{1cm} (B.13)

$$\alpha_4(u,v) = \frac{f(u+v) b(2u)}{e(u+v) a(2u)} + \left( \frac{h(u-v)f(u-v) - f(u-v)c(u-v)}{h(u-v)l(u-v) - f(u-v)r(u-v)} \right) \frac{h(u+v)}{e(u+v)},$$ \hspace{1cm} (B.14)

$$Z_{\frac{3}{2}}(u,v) = \left( \frac{h(u-v)l(u-v) - f(u-v)r(u-v)}{a(u-v)h(u-v)} \right) \frac{e(u+v)}{e(u+v)}.$$ \hspace{1cm} (B.15)

From Eq. (B.11) it follows that an appropriate two-particle state should be

$$|\tilde{\psi}_2(\lambda_1,\lambda_2)\rangle = \left[ B_{12}(\lambda_1)\tilde{B}_{12}(\lambda_2) + B_{13}(\lambda_1) \left( \alpha_2(\lambda_1,\lambda_2)\tilde{A}_2(\lambda_2) + \alpha_1(\lambda_1,\lambda_2)\tilde{A}_1(\lambda_2) \right) \right] |\tilde{0}_\frac{3}{2}\rangle,$$ \hspace{1cm} (B.16)

since it is symmetric $|\tilde{\psi}_2(\lambda_1,\lambda_2)\rangle = Z_{\frac{3}{2}}(\lambda_1,\lambda_2) |\tilde{\psi}_1(\lambda_2,\lambda_1)\rangle$ under the exchange of rapidities.

The next step is to solve the eigenvalue problem for the two-particle state $|\tilde{\psi}_2(\lambda_1,\lambda_2)\rangle$. In order to do that we need extra commutations rules between the fields $\tilde{A}_i(u)$ and $B_{13}(v)$, $B_{12}(u)$ and $B_{j+1}(v)$, $C_{j+1j}(u)$ and $B_{12}(v)$. In the case of the fields $C_{j+1j}(u)$ and $B_{12}(v)$ the rules comes from the entries $[2,5],[3,6]$ and $[4,7]$ of Eq. (B.11) and the ones for the other operators are obtained from $[1,3],[2,4],[1,6],[2,7],[3,8],[2,10],[3,11]$ and $[4,12]$ entries. After long algebraic manipulations we are able to obtain the following expressions

$$\tilde{A}_i(u)B_{13}(v) |\tilde{0}_\frac{3}{2}\rangle = b_i^+(u,v)B_{13}(v)\tilde{A}_i(u) |\tilde{0}_\frac{3}{2}\rangle + \text{unwanted terms} \quad i = 1,\ldots,4$$ \hspace{1cm} (B.17)
where by “unwanted terms” we mean those that do not give contributions proportional to (B.17)-(B.23) and after some algebra we conclude that the two-particle wanted terms have the particle state \(\psi_{\bar{3} 2}\).

We have now the main ingredients to study the action of the operators \(\tilde{A}_i(\lambda)\) on the two-particle state \(\left|\tilde{\psi}_2(\lambda_1, \lambda_2)\right\rangle\). The functions \(b_i^k(u, v), c_{ij}^k(u, v)\) and \(d_i^k(u, v)\) are once again very involved and have been presented in Appendix C.

We have now the main ingredients to study the action of the operators \(\tilde{A}_i(\lambda)\) on the two-particle state \(\left|\tilde{\psi}_2(\lambda_1, \lambda_2)\right\rangle\). Taking into account the commutation rules Eqs. (B.3)-(B.6) and (B.17)-(B.23) and after some algebra we conclude that the two-particle wanted terms have the
following structure

\[
\frac{\tilde{f}_\lambda(\lambda)}{\rho_{\lambda}^{(+)}} |\tilde{\psi}_2(\lambda_1, \lambda_2)\rangle = B_{12}(\lambda_1)B_{12}(\lambda_2) \sum_{i=1}^{4} \omega_i^{(+)}(\lambda) \left( \prod_{j=1}^{n} a_i^{(\lambda, \lambda)} \right) \tilde{A}_i(\lambda) \left| 0_{\lambda} \right\rangle 
\]

\[
+ B_{13}(\lambda_1)\tilde{A}_4(\lambda) \left( \Lambda_2^{42}(\lambda, \{\lambda_i\})\tilde{A}_2(\lambda_2) + \Lambda_2^{41}(\lambda, \{\lambda_i\})\tilde{A}_1(\lambda_2) \right) \left| 0_{\lambda} \right\rangle 
\]

\[
+ B_{13}(\lambda_1)\tilde{A}_3(\lambda) \left( \Lambda_2^{32}(\lambda, \{\lambda_i\})\tilde{A}_2(\lambda_2) + \Lambda_2^{31}(\lambda, \{\lambda_i\})\tilde{A}_1(\lambda_2) \right) \left| 0_{\lambda} \right\rangle 
\]

\[
+ B_{13}(\lambda_1)\tilde{A}_2(\lambda) \left( \Lambda_2^{22}(\lambda, \{\lambda_i\})\tilde{A}_2(\lambda_2) + \Lambda_2^{21}(\lambda, \{\lambda_i\})\tilde{A}_1(\lambda_2) \right) \left| 0_{\lambda} \right\rangle 
\]

\[
+ B_{13}(\lambda_1)\tilde{A}_1(\lambda) \left( \Lambda_2^{12}(\lambda, \{\lambda_i\})\tilde{A}_2(\lambda_2) + \Lambda_2^{11}(\lambda, \{\lambda_i\})\tilde{A}_1(\lambda_2) \right) \left| 0_{\lambda} \right\rangle 
\]

+ unwanted terms

\[
(B.24)
\]

where functions \( \Lambda_2^{lk}(\lambda, \{\lambda_i\}) \) are given by

\[
\Lambda_2^{42}(\lambda, \{\lambda_i\}) = \omega_4^{(+)}(\lambda) \left( b_1^{(4)}(\lambda, \lambda_1)\alpha_2(\lambda_1, \lambda_2) + a_1^{(4)}(\lambda, \lambda_2)\lambda_2 c_2^{(4)}(\lambda, \lambda_2) + a_1^{(4)}(\lambda, \lambda_1)\lambda_1 a_2^{(4)}(\lambda, \lambda_2)\lambda_2 d_2^{(4)}(\lambda, \lambda_1) \right) 
\]

\[
+ \omega_3^{(+)}(\lambda) \left( a_1^{(2)}(\lambda, \lambda_1)\lambda_1 a_2^{(2)}(\lambda, \lambda_2)\lambda_2 d_2^{(2)}(\lambda, \lambda_1) \right) 
\]

\[
(B.25)
\]

\[
\Lambda_2^{41}(\lambda, \{\lambda_i\}) = \omega_4^{(+)}(\lambda) \left( b_1^{(4)}(\lambda, \lambda_1)\alpha_1(\lambda_1, \lambda_2) + a_1^{(4)}(\lambda, \lambda_2)\lambda_2 c_1^{(4)}(\lambda, \lambda_2) + a_1^{(4)}(\lambda, \lambda_1)\lambda_1 a_2^{(4)}(\lambda, \lambda_2)\lambda_2 d_2^{(4)}(\lambda, \lambda_1) \right) 
\]

\[
+ \omega_3^{(+)}(\lambda) \left( a_1^{(2)}(\lambda, \lambda_1)\lambda_1 a_2^{(2)}(\lambda, \lambda_2)\lambda_2 d_2^{(2)}(\lambda, \lambda_1) \right) 
\]

\[
(B.26)
\]

\[
\Lambda_2^{32}(\lambda, \{\lambda_i\}) = \omega_4^{(+)}(\lambda) \left( a_1^{(4)}(\lambda, \lambda_1)\lambda_1 c_2^{(4)}(\lambda, \lambda_2) + a_0^{(4)}(\lambda, \lambda_1)\lambda_1 a_0^{(4)}(\lambda, \lambda_2)\lambda_2 d_2^{(4)}(\lambda, \lambda_1) \right) 
\]

\[
+ a_1^{(4)}(\lambda, \lambda_1)\lambda_1 a_2^{(4)}(\lambda, \lambda_2)\lambda_2 d_2^{(4)}(\lambda, \lambda_1) \right) + \omega_3^{(+)}(\lambda) \left( b_1^{(4)}(\lambda, \lambda_1)\alpha_2(\lambda_1, \lambda_2) + a_1^{(4)}(\lambda, \lambda_1)c_2^{(4)}(\lambda, \lambda_2) \right) 
\]

\[
+ a_0^{(4)}(\lambda, \lambda_1)c_2^{(4)}(\lambda, \lambda_2) + a_0^{(4)}(\lambda, \lambda_1)\lambda_2 a_2^{(4)}(\lambda, \lambda_2)\lambda_2 d_2^{(4)}(\lambda, \lambda_1) \right) + \omega_2^{(+)}(\lambda) \left( a_2^{(4)}(\lambda, \lambda_1)c_2^{(4)}(\lambda, \lambda_2) + a_1^{(4)}(\lambda, \lambda_1)a_2^{(4)}(\lambda, \lambda_2)\lambda_2 d_2^{(4)}(\lambda, \lambda_1) \right) 
\]

\[
(B.27)
\]
\[ \Lambda_{21}^{31}(\lambda, \{\lambda_i\}) = \omega_4^{(+)}(\lambda) \left( a_{10}^4(\lambda, \lambda_1)c_{12}^3(\lambda, \lambda_2) + a_{10}^4(\lambda, \lambda_1)c_{12}^3(\lambda, \lambda_2) + a_{10}^4(\lambda, \lambda_1)a_{12}^4(\lambda, \lambda_2)d_{23}^2(\lambda, \lambda_1) 
\]
\[ \qquad + a_{10}^4(\lambda, \lambda_1)a_{12}^4(\lambda, \lambda_2)d_{23}^2(\lambda, \lambda_1) \right) + \omega_3^{(+)}(\lambda) \left( b_{10}^4(\lambda, \lambda_1)a_{12}^4(\lambda, \lambda_2)d_{23}^2(\lambda, \lambda_1) + a_{10}^4(\lambda, \lambda_1)c_{12}^3(\lambda, \lambda_2) 
\]
\[ \quad + a_{10}^4(\lambda, \lambda_1)c_{12}^3(\lambda, \lambda_2) + a_{10}^4(\lambda, \lambda_1)a_{12}^4(\lambda, \lambda_2)d_{23}^2(\lambda, \lambda_1) + a_{10}^4(\lambda, \lambda_1)a_{12}^4(\lambda, \lambda_2)d_{23}^2(\lambda, \lambda_1) \right) 
\]
\[ \quad + \omega_2^{(+)}(\lambda) \left( a_{12}^4(\lambda, \lambda_1)c_{12}^3(\lambda, \lambda_2) + a_{12}^4(\lambda, \lambda_1)a_{12}^4(\lambda, \lambda_2)d_{23}^2(\lambda, \lambda_1) \right) \]  
(B.28)

\[ \Lambda_{21}^{22}(\lambda, \{\lambda_i\}) = \omega_4^{(+)}(\lambda) \left( a_{10}^4(\lambda, \lambda_1)c_{22}^3(\lambda, \lambda_2) + a_{10}^4(\lambda, \lambda_1)c_{22}^3(\lambda, \lambda_2) + a_{10}^4(\lambda, \lambda_1)c_{22}^3(\lambda, \lambda_2) 
\]
\[ \quad + a_{10}^4(\lambda, \lambda_1)a_{12}^4(\lambda, \lambda_2)d_{23}^2(\lambda, \lambda_1) + a_{10}^4(\lambda, \lambda_1)a_{12}^4(\lambda, \lambda_2)d_{23}^2(\lambda, \lambda_1) + a_{10}^4(\lambda, \lambda_1)a_{12}^4(\lambda, \lambda_2)d_{23}^2(\lambda, \lambda_1) \right) 
\]
\[ \quad + \omega_3^{(+)}(\lambda) \left( a_{10}^4(\lambda, \lambda_1)c_{22}^3(\lambda, \lambda_2) + a_{10}^4(\lambda, \lambda_1)c_{22}^3(\lambda, \lambda_2) + a_{10}^4(\lambda, \lambda_1)c_{22}^3(\lambda, \lambda_2) 
\]
\[ \quad + a_{10}^4(\lambda, \lambda_1)c_{22}^3(\lambda, \lambda_2) + a_{10}^4(\lambda, \lambda_1)c_{22}^3(\lambda, \lambda_2) + a_{10}^4(\lambda, \lambda_1)c_{22}^3(\lambda, \lambda_2) \right) 
\]
\[ \quad + a_{10}^4(\lambda, \lambda_1)a_{12}^4(\lambda, \lambda_2)d_{23}^2(\lambda, \lambda_1) + a_{10}^4(\lambda, \lambda_1)a_{12}^4(\lambda, \lambda_2)d_{23}^2(\lambda, \lambda_1) + a_{10}^4(\lambda, \lambda_1)a_{12}^4(\lambda, \lambda_2)d_{23}^2(\lambda, \lambda_1) \right) 
\]
\[ \quad + \omega_2^{(+)}(\lambda) \left( a_{12}^4(\lambda, \lambda_1)c_{22}^3(\lambda, \lambda_2) + a_{12}^4(\lambda, \lambda_1)c_{22}^3(\lambda, \lambda_2) + a_{12}^4(\lambda, \lambda_1)c_{22}^3(\lambda, \lambda_2) \right) \]  
(B.29)

\[ \Lambda_{21}^{23}(\lambda, \{\lambda_i\}) = \omega_4^{(+)}(\lambda) \left( a_{10}^4(\lambda, \lambda_1)c_{12}^3(\lambda, \lambda_2) + a_{10}^4(\lambda, \lambda_1)c_{12}^3(\lambda, \lambda_2) + a_{10}^4(\lambda, \lambda_1)c_{12}^3(\lambda, \lambda_2) 
\]
\[ \quad + a_{10}^4(\lambda, \lambda_1)a_{12}^4(\lambda, \lambda_2)d_{23}^2(\lambda, \lambda_1) + a_{10}^4(\lambda, \lambda_1)a_{12}^4(\lambda, \lambda_2)d_{23}^2(\lambda, \lambda_1) + a_{10}^4(\lambda, \lambda_1)a_{12}^4(\lambda, \lambda_2)d_{23}^2(\lambda, \lambda_1) \right) 
\]
\[ \quad + \omega_3^{(+)}(\lambda) \left( a_{10}^4(\lambda, \lambda_1)c_{12}^3(\lambda, \lambda_2) + a_{10}^4(\lambda, \lambda_1)c_{12}^3(\lambda, \lambda_2) + a_{10}^4(\lambda, \lambda_1)c_{12}^3(\lambda, \lambda_2) \right) 
\]
\[ \quad + a_{10}^4(\lambda, \lambda_1)c_{12}^3(\lambda, \lambda_2) + a_{10}^4(\lambda, \lambda_1)c_{12}^3(\lambda, \lambda_2) + a_{10}^4(\lambda, \lambda_1)c_{12}^3(\lambda, \lambda_2) \right) 
\]
\[ \quad + a_{10}^4(\lambda, \lambda_1)a_{12}^4(\lambda, \lambda_2)d_{23}^2(\lambda, \lambda_1) + a_{10}^4(\lambda, \lambda_1)a_{12}^4(\lambda, \lambda_2)d_{23}^2(\lambda, \lambda_1) + a_{10}^4(\lambda, \lambda_1)a_{12}^4(\lambda, \lambda_2)d_{23}^2(\lambda, \lambda_1) \right) 
\]
\[ \quad + \omega_2^{(+)}(\lambda) \left( a_{12}^4(\lambda, \lambda_1)c_{12}^3(\lambda, \lambda_2) + a_{12}^4(\lambda, \lambda_1)c_{12}^3(\lambda, \lambda_2) + a_{12}^4(\lambda, \lambda_1)c_{12}^3(\lambda, \lambda_2) \right) \]  
(B.30)
\[
\Lambda_{2}^{12}(\lambda, \{\lambda_i\}) = \omega^{(+)}_4(\lambda)\left(4a_{10}^4(\lambda, \lambda_1)c_{21}^3(\lambda, \lambda_2) + a_{6}^4(\lambda, \lambda_1)c_{21}^2(\lambda, \lambda_2) + a_{8}^4(\lambda, \lambda_1)c_{21}^1(\lambda, \lambda_2) + a_{1}^4(\lambda, \lambda_1)a_{2}^2(\lambda, \lambda_2)d_{1}^1(\lambda, \lambda_1)\right) +
\]
\[
\omega^{(+)}_3(\lambda)\left(3a_{10}^3(\lambda, \lambda_1)c_{21}^2(\lambda, \lambda_2) + a_{6}^3(\lambda, \lambda_1)c_{21}^1(\lambda, \lambda_2) + a_{8}^3(\lambda, \lambda_1)c_{21}^0(\lambda, \lambda_2) + a_{1}^3(\lambda, \lambda_1)a_{2}^1(\lambda, \lambda_2)d_{1}^1(\lambda, \lambda_1)\right) \times
\]
\[
a_{2}^2(\lambda, \lambda_2)d_{1}^2(\lambda, \lambda_1) + a_{3}^2(\lambda, \lambda_1)a_{3}^2(\lambda, \lambda_2)d_{1}^2(\lambda, \lambda_1) + a_{1}^2(\lambda, \lambda_1)a_{3}^2(\lambda, \lambda_2)d_{1}^1(\lambda, \lambda_1)\right) +
\]
\[
\omega^{(+)}_2(\lambda)\left(2a_{1}^2(\lambda, \lambda_1)c_{21}^2(\lambda, \lambda_2) + a_{6}^2(\lambda, \lambda_1)c_{21}^1(\lambda, \lambda_2) + a_{1}^2(\lambda, \lambda_1)a_{2}^1(\lambda, \lambda_2)d_{1}^1(\lambda, \lambda_1)\right) \times
\]
\[
a_{2}^1(\lambda, \lambda_2)d_{1}^1(\lambda, \lambda_1) + a_{3}^1(\lambda, \lambda_1)a_{3}^1(\lambda, \lambda_2)d_{1}^1(\lambda, \lambda_1) + a_{1}^1(\lambda, \lambda_1)a_{3}^1(\lambda, \lambda_2)d_{1}^1(\lambda, \lambda_1)\right) +
\]
\[
\omega^{(+)}_1(\lambda)\left(b_{1}^1(\lambda, \lambda_1)a_{1}(\lambda, \lambda_2) + a_{4}^1(\lambda, \lambda_1)c_{21}^1(\lambda, \lambda_2) + a_{1}^1(\lambda, \lambda_1)a_{3}^1(\lambda, \lambda_2)d_{1}^1(\lambda, \lambda_1)\right) \quad (B.31)
\]

\[
\Lambda_{2}^{11}(\lambda, \{\lambda_i\}) = \omega^{(+)}_4(\lambda)\left(4a_{10}^4(\lambda, \lambda_1)c_{11}^3(\lambda, \lambda_2) + a_{6}^4(\lambda, \lambda_1)c_{11}^2(\lambda, \lambda_2) + a_{8}^4(\lambda, \lambda_1)c_{11}^1(\lambda, \lambda_2) + a_{1}^4(\lambda, \lambda_1)a_{2}^2(\lambda, \lambda_2)d_{1}^1(\lambda, \lambda_1)\right) +
\]
\[
\omega^{(+)}_3(\lambda)\left(3a_{10}^3(\lambda, \lambda_1)c_{11}^2(\lambda, \lambda_2) + a_{6}^3(\lambda, \lambda_1)c_{11}^1(\lambda, \lambda_2) + a_{8}^3(\lambda, \lambda_1)c_{11}^0(\lambda, \lambda_2) + a_{1}^3(\lambda, \lambda_1)a_{2}^1(\lambda, \lambda_2)d_{1}^1(\lambda, \lambda_1)\right) \times
\]
\[
a_{2}^2(\lambda, \lambda_2)d_{1}^2(\lambda, \lambda_1) + a_{3}^2(\lambda, \lambda_1)a_{3}^2(\lambda, \lambda_2)d_{1}^2(\lambda, \lambda_1) + a_{1}^2(\lambda, \lambda_1)a_{3}^2(\lambda, \lambda_2)d_{1}^1(\lambda, \lambda_1)\right) +
\]
\[
\omega^{(+)}_2(\lambda)\left(2a_{1}^2(\lambda, \lambda_1)c_{11}^2(\lambda, \lambda_2) + a_{6}^2(\lambda, \lambda_1)c_{11}^1(\lambda, \lambda_2) + a_{1}^2(\lambda, \lambda_1)a_{2}^1(\lambda, \lambda_2)d_{1}^1(\lambda, \lambda_1)\right) \times
\]
\[
a_{2}^1(\lambda, \lambda_2)d_{1}^1(\lambda, \lambda_1) + a_{3}^1(\lambda, \lambda_1)a_{3}^1(\lambda, \lambda_2)d_{1}^1(\lambda, \lambda_1) + a_{1}^1(\lambda, \lambda_1)a_{3}^1(\lambda, \lambda_2)d_{1}^1(\lambda, \lambda_1)\right) +
\]
\[
\omega^{(+)}_1(\lambda)\left(b_{1}^1(\lambda, \lambda_1)a_{1}(\lambda, \lambda_2) + a_{4}^1(\lambda, \lambda_1)c_{11}^1(\lambda, \lambda_2) + a_{1}^1(\lambda, \lambda_1)a_{3}^1(\lambda, \lambda_2)d_{1}^1(\lambda, \lambda_1)\right) \quad (B.32)
\]

It turns out that many identities between the Boltzmann weights can be used in order to show the following remarkable property

\[
\Lambda_{2}^{3k}(\lambda, \{\lambda_i\}) = \alpha_{k}(\lambda_1, \lambda_2)\omega^{(+)}_1(\lambda)\prod_{i=1}^{n=2} a_{1}^i(\lambda, \lambda_i) \quad (B.33)
\]

Considering Eq. (B.24), Eq. (B.33) and Eq. (92) together it is not difficult to derive the
Appendix C: Auxiliary functions for \( S = \frac{3}{2} \)

The purpose of this Appendix is to list the expressions of the functions \( a_i^j(u, v), b_i^j(u, v), c_i^j(u, v) \) and \( d_i^j(u, v) \) used in the previous Appendix. To sort notation we shall used the symbol \( u_\pm = u \pm v \) and we emphasize that the most complicated functions \( a_2^1(u, v) \) and \( a_3^4(u, v) \) have been collected at the end of this Appendix:

\[
a_1^2(u, v) = \frac{a(-u_-) e(u_+)}{a(u_+) e(-u_-)} \\
a_2^7(u, v) = -\left( \frac{e(u_+) b(-u_-) + \frac{b(2v)e(-u_-) b(u_+)}{a(2v)}}{a(u_+) e(-u_-)} \right) \\
a_3^4(u, v) = -\left( \frac{b(u_+)}{a(u_+)} \right) \\
a_4^1(u, v) = \frac{a(-u_-) f(u_+)}{a(u_+)} \\
a_1^3(u, v) = \frac{(-b(u_+) b(u_+)) + a(u_+) l(u_+) (h(u_-) l(u_-) - f(u_-) r(u_-))}{a(u_+) e(-u_-) h(u_-)} \\
a_2^2(u, v) = \frac{e(u_+) b(-u_-) \left( \frac{b(2v) a(u_+)}{a(2u)} + \frac{b(u_-) b(u_+)}{e(u_-) e(u_+)} \right) + b(2v) \left( \frac{b(2v) b(u_+)}{a(2v) a(u_+)} + \frac{b(u_-) (b(u_+)^2 - a(u_+) l(u_-))}{a(u_+) e(u_+) e(u_+)} \right)}{a(u_+) e(-u_-) + \frac{a(u_-) b(u_+) (h(u_-) l(u_-) - f(u_-) r(u_-))}{a(u_+) e(u_-)^2 h(u_-)} + b(u_-) f(u_-) r(u_+)} + \frac{a(2v)}{a(u_+) e(u_+)} \\
\]

As a final comment we would like to stress that we have also performed extensive checks verifying that in fact the unwanted terms are canceled out provided the rapidities \( \lambda_i \) satisfy the restriction \( |\Pi| \) [34].
\[ a_3^2(u, v) = \frac{b(2u) b(u_+)}{a(2u) a(u_+)} + \frac{b(u_-) (b(u_+)^2 - a(u_+)^2)}{a(u_+) \epsilon(u_-) \epsilon(u_+)} \]  
(C.7)

\[ a_4^2(u, v) = -\left( \frac{b(2v)f(u_+)}{a(2v) \epsilon(u_+)} \right) + \frac{f(u_-) h(u_+)}{\epsilon(u_+) h(u_-)} \]  
(C.8)

\[ a_5^2(u, v) = -\left( \frac{f(u_+)}{\epsilon(u_+)} \right) \]  
(C.9)

\[ a_6^2(u, v) = -\frac{b(u_-) b(u_+) (h(u_-) l(u_-) - f(u_-) r(u_-)) f(u_+)}{a(u_+) \epsilon(u_-) \epsilon(u_+) h(u_-)} \]  
(C.10)

\[ a_7^2(u, v) = \frac{(a(u_+) m(u_-) - b(u_+) c(u_+)) (h(u_-) l(u_-) - f(u_-) r(u_-))}{a(u_+) \epsilon(u_-) \epsilon(u_+) h(u_-)} \]  
(C.11)

\[ a_1^3(u, v) = \frac{(- (i(u_-) i_1(u_-)) + j(u_-) q(u_-)) (e(u_+) q(u_+))}{e(u_+) h(u_-) h(u_+) j(u_-)} \]  
(C.12)

\[ a_2^3(u, v) = \frac{c(u_-) g_1(u_+) i(u_-)}{h(u_-) h(u_+) j(u_-)} + a(u_-) c(u_+) \epsilon(u_+) (- (g_1(u_-) i(u_-)) + j(u_-) m(u_-))}{a(u_+) \epsilon(u_-) h(u_-) h(u_+) j(u_-)} + \frac{f(u_+) (j(u_-) q(u_-) - i(u_-) i_1(u_-)) (a(u_+) b(u_-) \epsilon(u_+) r(u_+) - a(u_-) b(u_+) \epsilon(u_+) r(u_-))}{a(u_+) \epsilon(u_+) \epsilon(u_+) h(u_-) \epsilon(u_+) h(u_+) j(u_-)} \]  
(C.13)
\[
a^3_3(u, v) = -\left( \frac{c(u_-) m(u_+)}{h(u_-) h(u_+)} \right) - \left( \frac{b(2u) b(u_+) a(u_+)}{a(2u) a(u_+)} + \frac{b(u_-) (b(u_+)^2 - a(u_+) l(u_+))}{a(u_+)} \right) \\
\times \left( \frac{-(c(2u) b(2u)) + a(2u) m(2u)}{-b(2u) b(2u) + a(2u) l(2u)} + \frac{f(u_-) r(u_+)}{h(u_-) h(u_+)} \right) \\
+ \frac{b(u_+)}{a(u_+)} \left( \frac{c(2u) + c(u_-) c(u_+) + b(2u) f(u_-) r(u_+)}{a(2u) h(u_-) h(u_+)} \right) \\
\frac{a(u_+)}{h(u_-) h(u_+)} (C.14)
\]

\[
a^3_4(u, v) = \frac{f(u_-) i(u_-) i_1(u_+)}{h(u_-) h(u_+)} + \frac{e(u_-) f(u_+) \left( -(i(u_-) i_1(u_+)) + j(u_-) q(u_-) \right)}{e(u_+)} h(u_-)^2 j(u_-) \\
- \left( \frac{-(c(2u) b(2u)) + a(2u) m(2u)}{-b(2u) b(2u) + a(2u) l(2u)} + \frac{f(u_-) r(u_+)}{h(u_-) h(u_+)} \right) \\
\frac{b(2v) f(u_+)}{a(2v) h(u_-) h(u_+)} (C.15)
\]

\[
a^3_5(u, v) = -\left( \frac{f(u_-) q(u_+)}{h(u_-) h(u_+)} \right) + \frac{f(u_+)}{e(u_+)} \left( \frac{-(c(2u) b(2u)) + a(2u) m(2u)}{-b(2u) b(2u) + a(2u) l(2u)} + \frac{f(u_-) r(u_+)}{h(u_-) h(u_+)} \right) \\
(C.16)
\]

\[
a^3_6(u, v) = -\left( \frac{b(2v) i(u_+)}{a(2v) h(u_+)} \right) + \frac{i(u_-) j(u_+)}{h(u_+)} j(u_-) (C.17)
\]

\[
a^3_7(u, v) = -\left( \frac{i(u_+)}{h(u_+)} \right) (C.18)
\]

\[
a^3_8(u, v) = -\left( \frac{a(-u_-) f(u_+)}{a(u_+)} \right) \frac{c(2u) + c(u_-) c(u_+) + b(2u) f(u_-) r(u_+)}{a(2u) h(u_-) h(u_+)} \\
+ \frac{g(u_-) j(u_+)}{h(u_-) h(u_+)} j(u_-) f_1(u_+) - \frac{c(u_+)}{a(u_+)} \frac{b(u_-) (j(u_-) m(u_-) - g_1(u_-) i(u_-)) f(u_+)}{h(u_-) h(u_+)} j(u_-) \\
+ \frac{f(u_-) (j(u_-) q(u_-) - i(u_-) i_1(u_-)) (b(u_+) b(u_-) r(u_-) f(u_+) - a(u_+) e(u_-) c(u_-) q(u_+))}{a(u_+) e(u_-) e(u_+) h(u_+)^2 h(u_+)} j(u_-) \\
- \left( \frac{-(c(2u) b(2u)) + a(2u) m(2u)}{-b(2u) b(2u) + a(2u) l(2u)} + \frac{f(u_-) r(u_+)}{h(u_-) h(u_+)} \right) \\
\left( \frac{a(-u_-) f(u_+)}{a(u_+)} \right) \frac{b(2u) + b(u_-) h(u_+)}{e(u_-) e(u_+)} (C.19)
\]
\[ a_9^3(u, v) = \frac{(-g_1(u_+ i(u_+)) + j(u_-) m(u_-))}{a(u_+ h(u_-) h(u_+ j(u_-))} (a(u_+) l_1(u_+) - c(u_+ c(u_+)) + f(u_+ (j(u_-) q(u_-) - i(u_-) i_1(u_-)) (b(u_+) e(u_-) c(u_+) r(u_-) - a(u_+) e(u_-) m(u_+) r(u_-))}{a(u_+) e(u_-) e(u_+) h(u_-)^2 h(u_+) j(u_-)} \]

\[-(a(u_+) m(u_+) - b(u_+) c(u_+)) (h(u_-) l(u_-) - f(u_-) r(u_-))}{a(u_+) e(u_-) e(u_+) h(u_-)} \times \left( -\frac{(c(2 u) b(2 u)) + a(2 u) m(2 u)}{(b(2 u) b(2 u)) + a(2 u) l(2 u)} + \frac{f(u_-) r(u_+)}{h(u_-) h(u_+)} \right) \] (C.20)

\[ a_{10}^3(u, v) = \frac{(e(u_+) f_1(u_+) - f(u_+) g_1(u_+))}{e(u_+) h(u_-) h(u_+)} \left( - (i(u_-) i_1(u_-)) + j(u_-) q(u_-) \right) \] (C.21)

\[ a_4^4(u, v) = \frac{h_1(u_-) (h(u_+) h_1(u_+) - i(u_+ i_1(u_+))}{h(u_+)} \] (C.22)

\[ a_3^3(u, v) = \frac{b(u_+)}{a(u_+)} + \left( \frac{b(2 u) b(u_+)}{a(2 u) a(u_+)} \right) + \frac{b(u_-) (b(u_+)^2 - a(u_+ l(u_+))}{a(u_+) e(u_-) e(u_+)} \]

\[-\left( \frac{b(2 u) b(u_+)}{a(2 u) a(u_+)} \right) + \frac{b(u_-) (b(u_+)^2 - a(u_+ l(u_+))}{a(u_+) e(u_-) e(u_+)} \times \left( \frac{g(u_-) g_1(u_+)}{j(u_-) j(u_+)} + \frac{i(u_-) i_1(u_+)}{j(u_-) j(u_+)} \left| M_{2,3}^{(2 u)} \right| + \left| M_{2,4}^{(2 u)} \right| \right) \]

\[-\left( \frac{i(u_-) i_1(u_+)}{j(u_-) j(u_+)} \left| M_{3,4}^{(2 u)} \right| \right) \left( - \left( c(u_-) m(u_+)) \right) h(u_-) h(u_+) \right) \]

\[-\left( \frac{b(2 u) b(u_+)}{a(2 u) a(u_+)} \right) + \frac{b(u_-) (b(u_+)^2 - a(u_+ l(u_+))}{a(u_+) e(u_-) e(u_+)} \times \left( \frac{M_{2,3}^{(2 u)}}{M_{2,4}^{(2 u)}} \right) + \frac{f(u_-) r(u_+)}{h(u_-) h(u_+)} \]

\[ b(u_+) \left( \frac{c(2 u) e(u_+)}{a(u_+ h(u_-) h(u_+))} + \frac{b(2 u) f(u_-) r(u_+)}{a(2 u) h(u_-) h(u_+)} \right) \] (C.23)
\[
\begin{align*}
\quad a_4^i(u, v) &= \frac{h_1(u_-) i(u_+)}{J(u_-) j(u_+)} \left( -\frac{e(u_-) f(u_+) h(u_+) i(u_-)}{e(u_+) h(u_+)} + f(u_-) i_1(u_+) \right) - \frac{b(2 v) f_1(u_+) g(u_-)}{a(2 v) j(u_-) j(u_+)} \\
+ &\frac{e(u_-) f_1(u_-) g(u_+) h(u_+)}{e(u_+) h(u_-) j(u_+)} - \left( \frac{f(u_-) h(u_+)}{e(u_+) h(u_-)} - \frac{b(2 v) f(u_+)}{a(2 v) e(u_+)} \right) \\
\times &\left( \frac{g(u_-) g_1(u_+)}{j(u_-) f(u_+)} + \frac{i(u_-) i_1(u_+)}{j(u_-) j(u_+)} \left( M_{23}^{(+)}(2 u) \right) + \frac{M_{23}^{(+)}(2 u)}{M_{23}^{(+)}(2 u)} \right) \\
- &\left( \frac{i(u_-) i_1(u_+)}{j(u_-) j(u_+)} + \frac{M_{34}^{(+)}(2 u)}{M_{33}^{(+)}(2 u)} \right) \left( \frac{f(u_-) i(u_+)}{h(u_-) h(u_+)} + \frac{M_{34}^{(+)}(2 u)}{M_{33}^{(+)}(2 u)} \right)
\end{align*}
\]

\[
\begin{align*}
\quad a_5^i(u, v) &= -\left( \frac{f_1(u_+)}{e(u_+)} + \frac{f(u_+)}{e(u_+)} \left( \frac{g(u_-) g_1(u_+)}{j(u_-) j(u_+)} + \frac{i(u_-) i_1(u_+)}{j(u_-) j(u_+)} \left( M_{23}^{(+)}(2 u) \right) + \frac{M_{23}^{(+)}(2 u)}{M_{23}^{(+)}(2 u)} \right) \right) \\
- &\left( \frac{i(u_-) i_1(u_+)}{j(u_-) j(u_+)} + \frac{M_{34}^{(+)}(2 u)}{M_{33}^{(+)}(2 u)} \right) \left( -\left( \frac{f(u_-) q(u_+)}{h(u_-) h(u_+)} + \frac{f(u_+)}{e(u_+)} \left( \frac{M_{34}^{(+)}(2 u)}{M_{33}^{(+)}(2 u)} + \frac{f(u_-) r(u_+)}{h(u_-) h(u_+)} \right) \right) \right)
\end{align*}
\]

\[
\begin{align*}
\quad a_6^i(u, v) &= \frac{h(u_-) h_1(u_-) i(u_+)}{h(u_-) j(u_-)^2} - \frac{b(2 v) h_1(u_+) i(u_-)}{a(2 v) j(u_-) j(u_+)} - \left( \frac{b(2 v) i(u_+)}{a(2 v) h(u_+)} + \frac{i(u_-) j(u_+)}{h(u_+) j(u_-)} \right) \\
\times &\left( \frac{i(u_-) i_1(u_+)}{j(u_-) j(u_+)} + \frac{M_{34}^{(+)}(2 u)}{M_{33}^{(+)}(2 u)} \right)
\end{align*}
\]

\[
\begin{align*}
\quad a_7^i(u, v) &= -\left( \frac{h_1(u_+) i(u_-)}{j(u_-) j(u_+)} + \frac{i(u_+)}{h(u_+)} \left( \frac{i(u_-) i_1(u_+)}{j(u_-) j(u_+)} + \frac{M_{34}^{(+)}(2 u)}{M_{33}^{(+)}(2 u)} \right) \right)
\end{align*}
\]
\[
\begin{align*}
a_0^4(u,v) &= \frac{\n(u_\pm) (a(u_+)+b_1(u_+)+d(u_+)c(u_+))}{a(u_+) j(u_+)} \\
&\quad - \left( \frac{g(u_-)g_1(u_+) + i(u_-) i_1(u_+)}{j(u_-) j(u_+)} \right) \left( \begin{array}{c} M_{2,3}^{(+)}(2u) \\ M_{2,2}^{(+)}(2u) \\ M_{2,2}^{(+)}(2u) \end{array} \right) \\
&\quad \times (a(u_+) m(u_+)-b(u_+) c(u_+)) \left( h(u_-) l(u_-) - f(u_-) r(u_-) \right) \\
&\quad + \frac{f_1(u_-) g(u_+)}{a(u_+) e(u_-) e(u_+)} \left( \frac{h(u_-) l(u_-) - c(u_+) e(u_+)}{j(u_-) j(u_+)} \right) \\
&\quad - \left( \frac{i(u_-) i_1(u_+)}{j(u_-) j(u_+)} + \left| \frac{M_{2,3}^{(+)}(2u)}{M_{3,3}^{(+)}(2u)} \right| \right) \\
&\quad \times \left( \frac{-g_1(u_-)}{a(u_+) h(u_-)} j(u_-) \right) \left( a(u_+) l_1(u_+) - c(u_+) c(u_+) \right) \\
&\quad + \frac{f(u_+) (j(u_-) q(u_-) - i(u_-) i_1(u_-))}{a(u_+) e(u_-) e(u_+)} \left( b(u_+) e(u_-) c(u_+) r(u_-) - a(u_+) e(u_-) m(u_+) r(u_-) \right) \\
&\quad + \frac{f_1(u_-) g_1(u_+)}{a(u_+) e(u_-) e(u_+)} h(u_-) \left( \frac{M_{3,3}^{(+)}(2u)+f(u_-) r(u_-)}{h(u_-) h(u_+)} \right) \\
&\quad - \frac{a(u_+) m(u_+)-b(u_+) c(u_+)}{a(u_+) e(u_-) e(u_+)} h(u_-) \left( \begin{array}{c} M_{2,3}^{(+)}(2u) \\ M_{2,2}^{(+)}(2u) \end{array} \right) \\
&\quad \left( C.28 \right)
\end{align*}
\]

\[
\begin{align*}
a_{10}^4(u,v) &= - \left( \frac{f_1(u_+)}{h(u_+)} - \frac{f_1(u_+)}{e(u_+)} \right) \left( \frac{h(u_-) l(u_-) - c(u_+) e(u_+)}{j(u_-) j(u_+)} \right) \\
&\quad + \frac{e_1(u_+) f_1(u_-)}{j(u_-) j(u_+)} \\
&\quad - \frac{f_1(u_-) g(u_+)}{e(u_+)} \left( \frac{i(u_-) i_1(u_+)}{j(u_-) j(u_+)} + \left| \frac{M_{2,3}^{(+)}(2u)}{M_{3,3}^{(+)}(2u)} \right| \right) \\
&\quad \times \left( \frac{e(u_+)}{e(u_+)} f_1(u_+)-f(u_+)) g_1(u_+) \right) \left( j(u_-) q(u_-) - i(u_-) i_1(u_-) \right) \\
&\quad \left( C.29 \right)
\end{align*}
\]

\[
b_1^4(u,v) = \frac{a(\pm u_+) h(u_+)}{a(u_+) h(\pm u_-)} \left( C.30 \right)
\]
\[ b_1^2(u, v) = \left( e(u_+) q(u_+) - r(u_+) f(u_+) \right) \times \left( \frac{((-g_1(u_+) i(u_-)) + j(u_-) m(u_-)) \left( (-i(u_-) m(u_-) - g(u_-) q(u_-)) \right)}{e(u_-) i(u_-) \left( -(-i(u_-) m(u_-) - g(u_-) q(u_-)) \right)} + \frac{g(u_-) m(u_-) + i(u_-) q(u_-)}{e(u_-) i(u_-)} \right) \]  
(C.31)

\[ b_1^3(u, v) = \left( h_1(u_+) l_1(u_-) - f_1(u_-) r_1(u_-) \right) \times \left( \frac{l_2(u_+)}{h(u_+)(-b(u_+))} \left( \frac{m(u_+)(a(u_+)) - b(u_+)}{h(u_+)} \right) \right) \frac{c(u_+) (-b(u_+)) + f_1(u_-) r_1(u_-)}{h(u_+)(-b(u_+)) + a(u_+)}(u_+) \right) \right) \]  
(C.32)

\[ b_1^4(u, v) = \left( e_1(u_-) \right) \left( \frac{c_1(u_+)}{j(u_+)} \right) \frac{g(u_-) (-g_1(u_+) q(u_-)) + f_1(u_) r(u_-)}{j(u_+)(-e(u_-) q(u_-)) + f_1(u_-) r(u_-)} \right) \frac{e(u_-) f_1(u_-) - f(u_-) g_1(u_+) r_1(u_-)}{j(u_+)(e(u_+)) q(u_-) - f(u_-) r(u_-)} \right) \]  
(C.33)

\[ c_{22}^1(u, v) = -\left( \frac{b(u_+)}{a(u_+)} \right) \]  
(C.34)

\[ c_{21}^1(u, v) = -\left( \frac{b(2 u) b(u_+)}{a(2 u) a(u_+)} \right) - \frac{b(u_-) e(u_+)}{a(u_+) e(u_-)} \]  
(C.35)

\[ c_{12}^1(u, v) = -\left( \frac{b(2 v) b(u_+)}{a(2 v) a(u_+)} \right) + \frac{b(u_-) e(u_+)}{a(u_+) e(u_-)} \]  
(C.36)

\[ c_{11}^1(u, v) = \frac{b(2 u)}{a(2 u)} \left( \frac{-\frac{b(u_+)}{a(u_+)} \left( \frac{b(u_-) e(u_+)}{a(u_+) e(u_-)} \right)}{a(2 u)} + \frac{b(u_+)}{a(u_+) e(u_-)} - \frac{b(u_-) b(2 v)}{a(2 v) a(u_+)} \right) \]  
(C.37)

\[ c_{23}^2(u, v) = -\left( \frac{f(u_+)}{e(u_+)} \right) \]  
(C.38)

\[ c_{22}^2(u, v) = \frac{b(u_+)}{a(u_+)} - f(u_-) \left( \frac{h(u_+)}{e(u_+)} \right) \]  
(C.39)

\[ c_{21}^2(u, v) = \left( -\frac{c(2 u) f(u_+) h(u_-)}{a(2 u)} + \frac{b(2 u) b(u_+)}{a(2 u) a(u_+)} + \frac{b(u_-) e(u_+)}{a(u_+) e(u_-)} \right) f(u_-) b(u_+) \]  
(C.40)
\[ c_{13}^2(u, v) = - \left( \frac{b(2v) f(u_+)}{a(2v) e(u_+)} \right) + \frac{f(u_-) h(u_+)}{e(u_+)^2 h(u_-)} \]  
\[ (C.41) \]

\[ c_{12}^2(u, v) = - \left( \frac{\left( - \frac{b(2v) b(u_+)}{a(2v) a(u_+)} + \frac{b(u_-) e(u_+)}{a(u_+)^2 e(u_-)} \right) f(u_+) b(u_+)}{e(u_+)^2 h(u_-)} \right) + \frac{f(u_+) l(u_-)}{e(u_+)^2 h(u_-)} \]
\[ + \frac{f(u_-) h(u_+)}{e(u_+)^2 h(u_-)} \left| M_{2,3}^{(+)} (2u) \right| \frac{b(2v) \left( f(u_-) l(u_+) + \frac{f(u_+) h(u_-)}{e(u_+)^2 h(u_-)} \left| M_{2,3}^{(+)} (2u) \right| \right)}{a(2v) e(u_+)^2 h(u_-)} \]  
\[ (C.42) \]

\[ c_{11}^2(u, v) = \left( \frac{c(2u) f(u_-) h(u_+)}{a(2u)} \right) - \left( \frac{b(2u) \left( - \frac{b(2v) b(u_+)}{a(2v) a(u_+)} + \frac{b(u_-) e(u_+)}{a(u_+)^2 e(u_-)} \right) f(u_+) b(u_+)}{a(2u)} \right) + \frac{b(u_+) e(u_-)}{a(u_+)^2 e(u_-)} \]
\[ \frac{b(u_-) b(2v) e(u_+)}{a(2v) a(u_+)^2 e(u_-)} \right) f(u_-) b(u_+) + \frac{b(2u) f(u_-) l(u_-)}{a(2u)} + c(u_+) f(u_-) \]
\[ - \frac{b(2v) \left( \frac{c(2u) f(u_-) h(u_-)}{a(2u)} \right) + \frac{b(2u) f(u_-) l(u_-)}{a(2u)} + c(u_-) f(u_+) \right)}{a(2v)} \]  
\[ (C.43) \]

\[ c_{24}^2(u, v) = - \left( \frac{i(u_+)}{h(u_+)} \right) \]  
\[ (C.44) \]

\[ c_{23}^2(u, v) = - \frac{i(u_+)}{h(u_+)} \left| M_{3,4}^{(+)} (2u) \right| - \frac{i(u_-) q(u_+)}{h(u_+)^2 j(u_-)} + \frac{f(u_+) i(u_-) r(u_+)}{e(u_+)^2 h(u_-) j(u_-)} \]  
\[ (C.45) \]

\[ c_{22}^3(u, v) = \frac{b(u_+) g(u_-) c(u_+)}{a(u_+)^2 h(u_+) j(u_-)} - \frac{g(u_-) m(u_+)}{h(u_+)^2 j(u_-)} + \frac{i(u_+) j(u_-) \left| M_{2,4}^{(+)} (2u) \right|}{h(u_+)^2 j(u_-)} + \frac{i(u_-) \left| M_{2,3}^{(+)} (2u) \right| q(u_+)}{h(u_+)^2 j(u_-)} \]
\[ \frac{b(u_-) f(u_-) b(u_+)}{a(u_+)^2} - f(u_-) l(u_+) - f(u_+)^2 h(u_-) + \frac{f(u_+) h(u_-) \left( - \frac{(c(2u) b(2u)+a(2u) m(2u))}{(b(2u) b(2u)+a(2u) l(2u))} \right) r(u_+)}{e(u_+)^2 h(u_-) j(u_-)} \]  
\[ (C.46) \]
\[c_{21}^3(u, v) = \frac{b(2u) b(u_+)}{a(2u) a(u_+)} + \frac{b(u_-) e(u_+)}{a(u_+) e(u_-)} \frac{g(u_-) c(u_+)}{h(u_+)} \]
\[d(u_-) g(u_+) + \frac{d(2u) i(u_-) j(u_-)}{a(2u)} + \frac{b(2u) g(u_-) m(u_+)}{a(2u)} + c(2u) i(u_-) q(u_+) \frac{h(u_+)}{a(2u)} \]
\[- \left( i(u_-) r(u_+) \left( - \left( \frac{c(2u) f(u_-) h(u_-)}{a(2u)} \right) + \left( \frac{b(2u) b(u_+)}{a(2u) a(u_+)} + \frac{b(u_-) e(u_+)}{a(u_+) e(u_-)} \right) f(u_-) b(u_+) \right) \frac{b(2u) f(u_-) l(u_+)}{a(2u)} - c(u_-) f(u_+) \right) / \left( e(u_+) h(u_-) h(u_+) j(u_-) \right) \]
\[c_{14}^3(u, v) = - \left( \frac{b(2v) i(u_+)}{a(2v) h(u_+)} \right) + \frac{i(u_-) j(u_+)}{h(u_+)} \frac{h(u_+)}{j(u_-)} \]
\[c_{13}^3(u, v) = \frac{i(u_-) j(u_+)}{h(u_+)} \left( M_{3,4}^{(+)}(2u) \right) + \frac{i(u_-) q(u_+)}{h(u_+)} - \frac{b(2v) \left( \frac{i(u_+)}{h(u_+)} \right) M_{3,4}^{(+)}(2u)}{a(2v)} + \frac{i(u_-) q(u_+)}{h(u_+)} \frac{h(u_+)}{j(u_-)} \]
\[c_{12}^3(u, v) = \frac{g(u_+)}{h(u_+)} m(u_-) + \frac{i(u_-) j(u_+)}{h(u_+)} \left( M_{2,4}^{(+)}(2u) \right) \]
\[- \left( - \left( \frac{b(2v) b(u_+)}{a(2v) a(u_+)} \right) + \frac{b(u_-) e(u_+)}{a(u_+) e(u_-)} \right) g(u_-) c(u_+) \frac{i(u_+)}{h(u_+)} \frac{h(u_+)}{j(u_-)} \]
\[b(2v) \left( \frac{g(u_-) m(u_+)}{h(u_+)} + \frac{i(u_+)}{h(u_+)} \right) M_{3,2}^{(+)}(2u) + \frac{i(u_-)}{h(u_+)} q(u_+ \frac{h(u_+)}{j(u_-)} \right) \]
\[b(2v) \left( \frac{g(u_-) m(u_+)}{h(u_+)} + \frac{i(u_+)}{h(u_+)} \right) M_{3,2}^{(+)}(2u) + \frac{i(u_-)}{h(u_+)} q(u_+) \frac{h(u_+)}{j(u_-)} \right) \]
\[b(2v) \left( \frac{g(u_-) m(u_+)}{h(u_+)} + \frac{i(u_+)}{h(u_+)} \right) M_{2,2}^{(+)}(2u) + \frac{i(u_-)}{h(u_+)} q(u_+) \frac{h(u_+)}{j(u_-)} \right) \]
\[b(2v) \left( \frac{g(u_-) m(u_+)}{h(u_+)} + \frac{i(u_+)}{h(u_+)} \right) M_{2,2}^{(+)}(2u) + \frac{i(u_-)}{h(u_+)} q(u_+) \frac{h(u_+)}{j(u_-)} \right) \]
\[- \left( \frac{i(u_-)}{e(u_+)} f(u_-) b(u_+ \frac{h(u_+)}{e(u_+) h(u_-)} \right) + f(u_+) L(u_-) \frac{h(u_+)}{e(u_+) h(u_-)} \right) + f(u_+) L(u_-) \frac{h(u_+)}{e(u_+) h(u_-)} \right) \]
\[b(2v) \left( \frac{f(u_-) L(u_+)}{a(2v) e(u_+)} \right) / \left( h(u_+) j(u_-) \right) \]
\[(C.50)\]
\[ c_{11}(u, v) = \frac{d(u_+)}{h(u_+)} \frac{g(u_-)}{j(u_-)} + \frac{d(2u)}{a(2u)} \frac{i(u_-)}{h(u_+)} \frac{j(u_+)}{j(u_-)} + \frac{b(2u)}{a(2u)} \frac{g(u_+)}{h(u_+)} \frac{m(u_-)}{j(u_-)} + \frac{b(u_+)}{a(2u)} \frac{e(u_-)}{e(u_-)} \frac{c(u_+)}{a(2u)} \frac{g(u_-)}{c(u_+)} \]

\[ - \frac{b(2u) \left( - \frac{b(u_+)}{a(2u)} \frac{h(u_+)}{a(u_+)} e(u_-) + \frac{b(u_+)}{a(2u)} \frac{c(u_+)}{a(u_+)} e(u_-) \right)}{a(2u)} \frac{g(u_-)}{c(u_+)} \]

\[ + \frac{c(2u) i(u_+)}{a(2u)} \frac{q(u_-)}{j(u_-)} - \frac{b(2v) \left( \frac{d(2u)}{a(2u)} \frac{i(u_+)}{h(u_+)} + \frac{d(u_-)}{a(2u)} \frac{g(u_+)}{h(u_-)} + \frac{b(2u)}{a(2u)} \frac{g(u_+)}{h(u_+)} \frac{m(u_-)}{j(u_-)} + \frac{c(2u) i(u_-)}{a(2u)} \frac{q(u_+)}{q(u_-)} \right)}{a(2v)} \frac{h(u_+)}{j(u_-)} \]

\[ - \left( i(u_-) r(u_+) \right) \left( \frac{c(2u) f(u_-)}{h(u_+)} - \frac{b(2u) \left( \frac{b(u_+)}{a(2u)} \frac{b(u_+)}{a(u_+)} \frac{e(u_-)}{e(u_-)} \right)}{a(2v)} \right) f(u_-) b(u_+) + \frac{b(2u) \frac{f(u_+)}{a(2u)}}{a(2u)} + c(u_+) f(u_-) \]

\[ \frac{b(2v) \left( \frac{c(2u) f(u_+)}{a(2u)} \frac{h(u_-)}{a(u_+)} + \frac{b(2u) \frac{f(u_-)}{a(2u)} \frac{l(u_+)}{l(u_-)}}{a(2v)} + c(u_-) f(u_+) \right)}{a(2v)} \right) \right) / \left( e(u_+) \frac{h(u_-) h(u_+)}{j(u_-)} j(u_-) \right) \]

\[ (C.51) \]

\[ d_1^1(u, v) = - \frac{f(u_+)}{e(u_+)} \]

\[ (C.52) \]

\[ d_1^1(u, v) = - \left( \frac{h(u_+)}{e(u_+)} \frac{- (f(u_-) c(u_-)) + h(u_-) f(u_-)}{h(u_-) l(u_-) - f(u_-) r(u_-)} \right) - \frac{b(2u) f(u_+)}{a(2u)} \]

\[ (C.53) \]

\[ d_3^2(u, v) = - \left( \frac{a(u_+) m(u_+) - b(u_+) c(u_+)}{-(b(u_+) b(u_+)) + a(u_+) l(u_+)} \right) \]

\[ (C.54) \]

\[ d_2^2(u, v) = \left[ - \left( M_{2,3}^{(+)} (2u) \right) \left( a(u_+) m(u_+) - b(u_+) c(u_+) \right) \right] \]

\[ - \frac{a(u_+) \left( - (g_1(u_-) i(u_-)) + j(u_-) m(u_-) \right) q(u_-)}{- (i(u_-) i_1(u_-)) + j(u_-) q(u_-)} + \frac{b(u_+)}{a(2u)} \frac{r(u_-) f(u_-)}{h(u_-)} \]

\[ + \left( \frac{a(u_+)}{e(u_+)} \frac{- (g_1(u_-) i(u_-)) + j(u_-) m(u_-)}{- (i(u_-) i_1(u_-)) + j(u_-) q(u_-)} - \frac{b(u_+)}{a(2u)} \frac{e(u_+)}{e(u_-)} \right) f(u_-) \]

\[ / \left( a(u_+) l(u_+) - b(u_+) b(u_+) \right) \]

\[ (C.55) \]
\[ d_1^2(u, v) = \left( \frac{c(2u)}{a(2u)} \left( - (a(u_+) m(u_+)) + b(u_+) c(u_+) \right) \right. \\
- a(u_+) i_1(u_+) \left( - (d(u_-) i(u_-)) + g(u_-) j(u_-) \right) \\
- \left. \frac{a(u_+) b(2u) \left( - (g_1(u_-) i(u_-)) + j(u_-) m(u_-) \right) q(u_+)}{a(2u)} \right) \\
+ \frac{b(u_+) h(u_+) \left( - (l(u_-) c(u_-)) + r(u_-) f(u_-) \right)}{- (h(u_-) l(u_-)) + f(u_-) r(u_-)} + j(u_-) m(u_-) - g_1(u_-) i(u_-)) \right) \\
/ \left( - (b(u_+) b(u_+)) + a(u_+) l(u_+) \right) \) \) (C.56)

\[ d_3^2(u, v) = - \left( \frac{e(u_+ f_1(u_+) - f(u_+) g_1(u_+))}{e(u_+) q(u_+) - f(u_+) r(u_+)} \right) \) \) (C.57)

\[ d_3^2(u, v) = \left( \begin{array}{c}
- (e(u_+) f_1(u_+) + f(u_+) g_1(u_+)) \left| M_{3,4}^{(+)} (2u) \right|
\end{array} \right) \\
+ \left( \begin{array}{c}
\frac{e(u_+) f_1(u_-) h(u_+)}{h(u_+)} \left( \frac{l_1(u_+)}{h(u_+)} - \frac{m(u_+)(a(u_+ m(u_+)) - b(u_+) c(u_+))}{h(u_+)} \right.
\left. - \frac{c(u_+)(- (b(u_+) m(u_+)) + l(u_+) c(u_+))}{h(u_+)} \right)
\frac{- h(u_+)(- (b(u_+ b(u_+)) + a(u_+ l(u_+))) (h_1(u_-) l_1(u_-) - f_1(u_-) r_1(u_-))}{h_1(u_-)(f_1(u_-) r_1(u_-) - h_1(u_-) l_1(u_-))} \right) \\
/ \left( h_1(u_-)(f_1(u_-) r_1(u_-) - h_1(u_-) l_1(u_-)) \right) / \left( e(u_+) q(u_+) - f(u_+) r(u_+) \right) \) \) (C.58)
\[ d_2^3(u, v) = - \left( \frac{e(u_+) f_1(u_-) l_1(u_+)}{h_1(u_-) M_{2,3}^{(+)}(2u)} \right) + \left( \frac{(e(u_+) f_1(u_-) - f(u_+) g_1(u_+)) M_{2,4}^{(+)}(2u)}{h_1(u_-) M_{2,2}^{(+)}(2u)} \right) \]

\[ - \frac{f(u_+) (i_1(u_-) m(u_-) - g_1(u_-) q(u_-)) q(u_+)}{h_1(u_-) M_{2,2}^{(+)}(2u)} + \frac{n(u_-) e(u_+) f_1(u_+)}{h_1(u_-)} \]

\[ + \left( a(u_+) \frac{(a(u_+) m(u_-) - b(u_+) c(u_+))}{M_{2,3}^{(+)}(2u)} + b(u_+) \frac{r(u_-) f(u_+)}{h(u_-)} \right) \]

\[ - \left( \frac{a(u_+) - (g_1(u_-) i(u_-)) + j(u_-) m(u_-)) q(q(u_-))}{h(u_-)} - b(u_+) \frac{e(u_+) r(u_+)}{h(u_-)} \right) \]

\[ e(u_+) \]

\[ \text{/ (} h_1(u_-) \left( a(u_+) l(u_-) - b(u_+) b(u_+) \right) \right) + c(u_+) e(u_+) f_1(u_-) \]

\[ \times \left( \frac{M_{2,3}^{(+)}(2u)}{h_1(u_-) M_{2,2}^{(+)}(2u)} \right) \]

\[ b(u_+) \frac{(a(u_+) m(u_-) - l(u_+)) c(u_+)}{h(u_-)} \]

\[ + \frac{b(u_+) - (g_1(u_-) i(u_-)) + j(u_-) m(u_-)) q(q(u_-))}{h(u_-)} - l(u_+) \frac{r(u_-) f(u_+)}{h(u_-)} \]

\[ - \left( \frac{b(u_+) - (g_1(u_-) i(u_-)) + j(u_-) m(u_-))}{h(u_-)} - b(u_+) \frac{e(u_+) r(u_+)}{h(u_-)} \right) \]

\[ e(u_+) \]

\[ \text{/ (} h_1(u_-) \left( a(u_+) l(u_-) - b(u_+) b(u_+) \right) \right) \]

\[ / (e(u_+) q(u_-) - f(u_+) r(u_+)) \]

(C.59)
$$d^3_{1}(u, v) = \left( \frac{d(2u)}{a(2u)} \left(- (c(u_+) f_1(u_+) + f(u_+) g_1(u_+)) \right) + \frac{f(u_+) i_1(u_+) (g(u_-) i_1(u_-) - d(u_-) q(u_-))}{i(u_-) i_1(u_-) - j(u_-) q(u_-)} \right)$$

$$+ \frac{a(-u_-) e(u_+) f_1(u_-) h(u_+)}{a(2u)} \left( \frac{e(2u) h(u_-) h(u_+)}{a(2u)} + c(u_-) c(u_+) + \frac{b(2u) f(u_-) r(u_+)}{a(2u)} \right)$$

$$+ \frac{b(2u) \left( \frac{f(u_+) (i_1(u_-) m(u_-) - g_1(u_-) q(u_-)) q(u_+)}{i(u_-) i_1(u_-) - j(u_-) q(u_-)} - \frac{e(u_+) (f_1(u_-) r_1(u_-) - n(u_-) l_1(u_-)) f_1(u_+)}{- (h_1(u_-) l_1(u_-)) + f_1(u_-) r_1(u_-)} \right)}{a(2u)}$$

$$= \left( \frac{f(u_+) r(u_+)}{i(u_-) i_1(u_-) - j(u_-) q(u_-)} \right) \left( \frac{e(u_+) g_1(u_+) (f_1(u_-) r_1(u_-) - n(u_-) l_1(u_-))}{e(u_+) (h(u_-) l_1(u_-) - f(u_-) r(u_-)) + \frac{b(2u) f(u_+)}{a(2u) e(u_+)} \right)$$

$$/ (e(u_+) q(u_+) - f(u_+) r(u_+)) \right) \right) \quad \text{(C.60)}$$
\[ a_2^4(u, v) = - \left( \frac{b(2v)n(u_+)d(u_-)}{a(2v)j(u_-)j(u_+)} \right) + \frac{a(u_-)n(u_-)d(u_+)e(u_+)}{a(u_+)e(u_-)j(u_-)j(u_+)} \\
+ \left( \frac{d(2u)}{a(2u)} + \frac{d(u_-)d(u_+)}{j(u_-)j(u_+)} + \frac{b(2u)g(u_-)g_1(u_+)}{a(2u)j(u_-)j(u_+)} + \frac{c(2u)i(u_-)i_1(u_+)}{a(2u)j(u_-)j(u_+)} \right) \times \left( \frac{e(u_+)b(-u_-)}{a(u_+)e(-u_-)} + \frac{b(2v)b(u_+)}{a(u_+)} \right) \\
+ f_1(u_-)g(u_+) \left( a(u_+)b(u_-)e(u_-)r(u_+) - a(u_-)b(u_+)e(u_+)r(u_-) \right) \\
+ \frac{f(2u)}{a(u_+)e(u_-)h(u_-)j(u_+)} \left( g(u_-)g_1(u_+) + i(u_-)i_1(u_+) \right) \left( \frac{M_{3,3}^{(+)}(2u)}{M_{2,3}^{(+)}(2u)} \right) + \frac{M_{2,4}^{(+)}(2u)}{M_{2,2}^{(+)}(2u)} \right) \times \left( \frac{e(u_+)b(-u_-)}{a(u_+)e(-u_-)} + \frac{b(2v)}{a(u_+)} \left( \frac{h(2u)b(u_+)}{a(2u)a(u_+)} + \frac{b(u_-)(b(u_+)^2 - a(u_+)l(u_+))}{a(u_+)e(u_-)e(u_+)} \right) \right) \\
+ \frac{a(u_-)b(u_+) \left( h(u_-)l(u_-) - f(u_-)r(u_-) \right)}{a(u_+)^2 h(u_-)} + \frac{b(u_-)f(u_-)r(u_+)}{e(u_-)e(u_+)h(u_-)} + h_1(u_-)i(u_+)} \\
\times \frac{c(u_-)g_1(u_+)}{a(u_+)e(u_-)h(u_-)j(u_+)} \left( \frac{M_{3,3}^{(+)}(2u)}{M_{2,3}^{(+)}(2u)} \right) \left( \frac{c(u_-)g_1(u_+)}{h(u_-)h(u_+)} \right) - \frac{b(2v)c(u_-)m(u_+)}{a(2v)h(u_-)h(u_+)} \\
+ \frac{a(u_-)c(u_+)e(u_+) \left( j(u_-)m(u_-) - g_1(u_-)i(u_+) \right)}{a(u_+)^2 h(u_-)h(u_+)j(u_+)} + f(u_+) \left( j(u_-)q(u_-) - i(u_-)i_1(u_-) \right) \left( a(u_+)b(u_-)e(u_-)r(u_+) - a(u_-)b(u_+)e(u_+)r(u_-) \right) \\
\times \left( \frac{e(u_+)b(-u_-)}{a(u_+)e(-u_-)} + \frac{b(2v)}{a(u_+)} \left( \frac{h(2u)b(u_+)}{a(2u)a(u_+)} + \frac{b(u_-)(b(u_+)^2 - a(u_+)l(u_+))}{a(u_+)e(u_-)e(u_+)} \right) \right) \\
+ \frac{a(u_-)b(u_+) \left( h(u_-)l(u_-) - f(u_-)r(u_-) \right)}{a(u_+)^2 h(u_-)} + \frac{b(u_-)f(u_-)r(u_+)}{e(u_-)e(u_+)h(u_-)} + h_1(u_-)i(u_+)} \\
\times \left( - \frac{c(2u)b(2u)}{a(2u)} + \frac{b(2u)m(2u)}{- (b(2u)b(2u)) + a(2u)l(2u)} + \frac{f(u_-)r(u_+)}{h(u_-)h(u_+)} \right) \left( \frac{c(u_-)c(u_+)}{h(u_-)h(u_+)} + \frac{a(u_-)c(u_+)e(u_-)e(u_+)}{a(2u)h(u_-)h(u_+)} \right) \\
\times \left( \frac{e(u_+)}{a(u_+)} \frac{b(-u_-)}{a(2v)} + \frac{b(2v)e(-u_-)b(u_+)}{a(2u)} \left( \frac{c(2u)}{a(2u)} + \frac{c(u_-)c(u_+)}{h(u_-)h(u_+)} + \frac{b(2u)f(u_-)r(u_+)}{a(2u)h(u_-)h(u_+)} \right) \right) \left( C.61 \right)
\[ a_8^4(u,v) = - \left( \frac{a(-u_-) f(u_+) \left( \frac{d(2u_-)}{a(2u_-)} + \frac{d(u_-) d(u_+)}{j(u_-) j(u_+)} + \frac{b(2u_-) g(u_-) g_1(u_+)}{a(2u_-) j(u_-) j(u_+)} + \frac{c(2u_-) i(u_-) i_1(u_+)}{a(2u_-) j(u_-) j(u_+)} \right)}{a(u_+) e(-u_-)} \right) \]

\[ + \frac{f_1(u_-) g(u_+) \left( b(u_+) b(u_-) r(u_-) f(u_+) - a(u_+) e(u_-) c(u_-) q(u_+) \right)}{a(u_+) e(u_-) e(u_+) h(u_-) j(u_-) j(u_+)} \]

\[ - \frac{n(u_-) d(u_+) b(u_-) f(u_+)}{a(u_+) e(u_-) j(u_-) j(u_+)} - \left( \frac{g(u_-) g_1(u_+)}{j(u_-) j(u_+)} + \frac{i(u_-) i_1(u_+)}{M_{2,3}^{(+)}(2u_-)} \right) \]

\[ \times - \frac{M_{2,4}^{(+)}(2u_-)}{M_{2,2}^{(+)}(2u_-)} \]

\[ \times - \frac{h(u_-) f_1(u_+)}{j(u_-) j(u_+)} \]

\[ - \left( \frac{i(u_-) i_1(u_+)}{j(u_-) j(u_+)} + \frac{M_{3,4}^{(+)}(2u_-)}{M_{3,3}^{(+)}(2u_-)} \right) \]

\[ \left( - \frac{(d(-u_-) i(u_-)) + g(u_-) j(u_-)}{h(u_-) h(u_+) j(u_+)} \right) \]

\[ - \frac{f_1(u_+)}{h(u_-) h(u_+) j(u_+)} \]

\[ \left( \frac{a(-u_-) f(u_+) \left( \frac{c(2u_-)}{a(2u_-)} + \frac{c(u_-) c(u_+)}{h(u_-) h(u_+)} + \frac{b(2u_-) f(u_-) r(u_+)}{a(2u_-) h(u_-) h(u_+)} \right)}{a(u_+) e(-u_-)} \right) \]

\[ - \frac{c(u_+) b(u_-) \left( - (g_1(u_-) i(u_-)) + j(u_-) m(u_-) \right) f(u_+)}{a(u_+) e(u_-) h(u_-) h(u_+) j(u_-)} \]

\[ \times \frac{f(u_+)) (j(u_-) q(u_-) - i(u_-) i_1(u_-)) (b(u_+) b(u_-) r(u_-) f(u_+) - a(u_+) e(u_-) c(u_-) q(u_+))}{a(u_+) e(u_-) h(u_-)^2 h(u_+)^2 j(u_-)^2} \]

\[ - \frac{f_1(u_-) g(u_+)}{a(u_+) e(u_-) e(u_+) h(u_-) h(u_+) j(u_-)} \]

\[ \times \left( \frac{- \frac{(c(2u_-) b(2u_-)) + a(2u_-) m(2u_-)}{a(2u_-)} + f(u_-) r(u_+)}{h(u_-) h(u_+)} \right) \]

\[ \times \left( \frac{- \frac{(a(-u_-) f(u_+) \left( \frac{b(2u_-)}{a(2u_-)} + \frac{b(u_-) b(u_+)}{e(u_-) e(u_+)} \right)}{a(u_+) e(-u_-)} \right) + \frac{g(u_+)) (h(u_-) f(u_+) - f(u_-) c(u_-))}{e(u_-) e(u_+) h(u_-)} \]

\[ - \frac{b(u_+) b(u_-) \left( h(u_-) l(u_-) - f(u_-) r(u_-) \right) f(u_+)}{a(u_+) e(u_-)^2 e(u_+) h(u_-)} \)} \]

(C.62)
Appendix D: Relations for arbitrary $S$

In this Appendix we present certain expressions concerning the unwanted terms of the one-particle problem as well as the construction of the two-particle vector for arbitrary $S$.

The commutation rules used in the solution of one-particle eigenvalue problem come from the entries $[1,2]$, $[2,3]$, $[2S,2S+1]$, $[2,2+2S+1]$, $[3,3+2S+1]$, $[2S+1,2(2S+1)]$ of the boundary Yang-Baxter equation (B.1). To cancel the unwanted terms we need to know how to compute the ratio $\frac{q^{(2)}_{2S}(\lambda,\lambda_1)}{q^{(1)}_{1}(\lambda,\lambda_1)}$ which is not expected to have a dependence on the $i$-th index. This means that this ratio can be calculated collecting the simplest unwanted contributions which turns out to be those coming from the commutation rules between the fields $\tilde{A}_{2S}(\lambda)$, $\tilde{A}_{2S+1}(\lambda)$ and $B_{12}(\lambda_1)$. Considering the help of mathematical induction we find that the function $q^{(2)}_{2S}(\lambda,\lambda_1)$ is

$$q^{(2)}_{2S}(\lambda,\lambda_1) = -\omega^{(+)}_{2S}(\lambda)\frac{R^{2S+1,2}_{1,2S}(\lambda + \lambda_1)}{R^{2S-1}_{1,2S}(\lambda + \lambda_1)} + \omega^{(+)}_{2S+1}(\lambda)$$

$$\times \begin{pmatrix}
R^{2S+1,2}_{1,2S}(\lambda + \lambda_1) & M^{(+)}_{2S,2S+1}(2\lambda) \\
R^{2S-1}_{1,2S}(\lambda + \lambda_1) & M^{(+)}_{2S,2S}(2\lambda)
\end{pmatrix} - \begin{pmatrix}
R^{2S+1,2}_{1,2S}(\lambda - \lambda_1) \\
R^{2S+1,2}_{2,2S+1}(\lambda + \lambda_1)
\end{pmatrix}$$

$$\begin{pmatrix}
R^{2S+1,2}_{1,2S}(\lambda + \lambda_1) \\
R^{2S+1,2}_{2,2S+1}(\lambda + \lambda_1)
\end{pmatrix}$$

$$\begin{pmatrix}
R^{2S+1,2}_{1,2S}(\lambda - \lambda_1) \\
R^{2S+1,2}_{2,2S+1}(\lambda + \lambda_1)
\end{pmatrix}$$

(D.1)
After some algebra we find that commutation relation between the creation operators is derived by combining the entries \([1,3]\) and \([1,3+2S]\) of Eq.(B.1).

We close this Appendix discussing the construction of the two-particle state. The appropriate commutation relation is derived by combining the entries \([1,3]\) and \([1,3+2S]\) of Eq.(B.1). After some algebra we find that commutation relation between the creation operators \(B_{12}(u)\) while \(q_{2S}^{(1)}(\lambda, \lambda_1)\) is given by

\[
q_{2S}^{(1)}(\lambda, \lambda_1) = \omega_{2S}^{(+)}(\lambda) \left( \frac{R_{1,2S}^{2S+1,2}(\lambda - \lambda_1)}{R_{1,2S+1}^{2S+1,1}(\lambda - \lambda_1)} \frac{R_{1,2S}^{2S+1,1}(\lambda + \lambda_1)}{R_{1,2S+1}^{2S+1,1}(\lambda + \lambda_1)} - \frac{R_{1,2S}^{2S+1,2}(\lambda + \lambda_1)}{R_{1,2S+1}^{2S+1,1}(\lambda + \lambda_1)} \right) \\
+ \omega_{2S+1}^{(+)}(\lambda) \left[ - \frac{M_{1,2}^{(+)}(2\lambda_1)}{M_{2S,2S+1}^{(+)}(2\lambda)} \right] \left( \frac{R_{1,2S}^{2S+1,2}(\lambda - \lambda_1)}{R_{1,2S+1}^{2S+1,1}(\lambda - \lambda_1)} \frac{R_{1,2S}^{2S+1,1}(\lambda + \lambda_1)}{R_{1,2S+1}^{2S+1,1}(\lambda + \lambda_1)} - \frac{R_{1,2S}^{2S+1,2}(\lambda + \lambda_1)}{R_{1,2S+1}^{2S+1,1}(\lambda + \lambda_1)} \right) \\
- \frac{M_{1,1}^{(+)}(2\lambda_1)}{M_{1,1}^{(+)}(2\lambda)} \frac{R_{1,2S}^{2S+1,1}(\lambda - \lambda_1)}{R_{1,2S+1}^{2S+1,1}(\lambda - \lambda_1)} \frac{R_{1,2S}^{2S+1,1}(\lambda + \lambda_1)}{R_{1,2S+1}^{2S+1,1}(\lambda + \lambda_1)} \\
+ \frac{R_{1,2S}^{2S+1,1}(\lambda - \lambda_1)}{R_{1,2S+1}^{2S+1,1}(\lambda - \lambda_1)} \frac{R_{1,2S}^{2S+1,1}(\lambda + \lambda_1)}{R_{1,2S+1}^{2S+1,1}(\lambda + \lambda_1)} \frac{R_{1,2S}^{2S+1,2}(\lambda - \lambda_1)}{R_{1,2S+1}^{2S+1,1}(\lambda - \lambda_1)} \frac{R_{1,2S}^{2S+1,2}(\lambda + \lambda_1)}{R_{1,2S+1}^{2S+1,1}(\lambda + \lambda_1)} \\
\frac{R_{1,2S}^{2S+1,1}(\lambda - \lambda_1)}{R_{1,2S+1}^{2S+1,1}(\lambda - \lambda_1)} \frac{R_{1,2S}^{2S+1,1}(\lambda + \lambda_1)}{R_{1,2S+1}^{2S+1,1}(\lambda + \lambda_1)} \frac{R_{1,2S}^{2S+1,2}(\lambda - \lambda_1)}{R_{1,2S+1}^{2S+1,1}(\lambda - \lambda_1)} \frac{R_{1,2S}^{2S+1,2}(\lambda + \lambda_1)}{R_{1,2S+1}^{2S+1,1}(\lambda + \lambda_1)} \\
\right] \\
(\text{D.2})

Taking into account the explicit expressions for the Boltzmann weights and for the functions \(\omega_j^{\pm}(\lambda)\) one can verify that the ratio \(\frac{q_{2S}^{(1)}(\lambda, \lambda_1)}{q_{2S}^{(1)}(\lambda, \lambda_1)}\) satisfy Eq.(107).

We close this Appendix discussing the construction of the two-particle state. The appropriate commutation relation is derived by combining the entries \([1,3]\) and \([1,3+2S]\) of Eq.(B.1). After some algebra we find that commutation relation between the creation operators \(B_{12}(u)\)
and \(B_{12}(v)\) is

\[
B_{12}(u)B_{12}(v) + \frac{R^3_{1,2}(u_+)}{R^1_{1,2}(u_+)} B_{13}(u)A_2(v) + \sum_{j=4}^{2S+1} \frac{R^{j-1}_{1,2}(u_+)}{R^1_{1,2}(u_+)} B_{1j}(u)C_{j-12}(v) \\
\left( - \frac{R^3_{1,2}(u_-)}{R^1_{1,2}(u_-)} \right) \left( \frac{R^3_{1,3}(u_+)}{R^2_{1,2}(u_+)} B_{13}(u)A_1(v) + \sum_{j=4}^{2S+1} \frac{R^{j-2}_{1,3}(u_+)}{R^2_{1,2}(u_+)} B_{1j}(u)C_{j-21}(v) \right) \\
= Z_S(u, v) \left[ B_{12}(v)B_{12}(u) + \frac{R^1_{3,2}(u_+)}{R^1_{2,1}(u_+)} B_{13}(v)A_2(u) + \sum_{j=4}^{2S+1} \frac{R^{1,j-1}_{2,1}(u_+)}{R^1_{2,1}(u_+)} B_{1j}(v)C_{j-12}(u) \right] \\
+ \begin{vmatrix}
R^3_{1,2}(u_-) & R^3_{1,3}(u_-) \\
R^3_{1,2}(u_-) & R^1_{3,3}(u_-) \\
R^3_{1,2}(u_-) & R^1_{3,3}(u_-) \\
R^2_{1,2}(u_-) & R^2_{1,3}(u_-)
\end{vmatrix} \left[ \frac{R^1_{3,3}(u_+)}{R^1_{2,1}(u_+)} B_{13}(v)A_1(u) + \sum_{j=4}^{2S+1} \frac{R^{1,j-2}_{2,1}(u_+)}{R^1_{2,1}(u_+)} B_{1j}(v)C_{j-21}(u) \right] \
\tag{D.3}
\]

where function \(Z_S(u, v)\) has been defined in Eq. (D.3).

The above relation allows us to define the following vector

\[
\phi(u, v) = B_{12}(u)B_{12}(v) + \frac{R^3_{1,2}(u_+)}{R^1_{1,2}(u_+)} B_{13}(u)A_2(v) + \sum_{j=4}^{2S+1} \frac{R^{j-1}_{1,2}(u_+)}{R^1_{1,2}(u_+)} B_{1j}(u)C_{j-12}(v) \\
- \left( \frac{R^3_{1,2}(u_-)}{R^1_{1,2}(u_-)} \right) \left( \frac{R^3_{1,3}(u_+)}{R^2_{1,2}(u_+)} B_{13}(u)A_1(v) + \sum_{j=4}^{2S+1} \frac{R^{j-2}_{1,3}(u_+)}{R^2_{1,2}(u_+)} B_{1j}(u)C_{j-21}(v) \right) \
\tag{D.4}
\]

which is symmetric under the exchange of the variables \(u\) and \(v\), thanks to certain identities between the Boltzmann weights. More precisely, we have

\[
\phi(u, v) = Z_S(u, v)\phi(v, u) \
\tag{D.5}
\]

The two-particle state is now obtained by acting the vector (D.4) on the pseudovacuum \(|\bar{0}_S\rangle\) leading us to

\[
|\bar{\psi}_2(\lambda_1, \lambda_2)\rangle = \left( B_{12}(\lambda_1)B_{12}(\lambda_2) + \frac{R^3_{1,2}(\lambda_1 + \lambda_2)}{R^1_{1,2}(\lambda_1 + \lambda_2)} B_{13}(\lambda_1)A_2(\lambda_2) \right. \\
- \left. \frac{R^3_{1,2}(-\lambda_1 - \lambda_2)}{R^1_{1,3}(\lambda_1 - \lambda_2)} \frac{R^3_{1,3}(\lambda_1 + \lambda_2)}{R^1_{1,2}(\lambda_1 + \lambda_2)} B_{13}(\lambda_1)A_1(\lambda_2) \right) |\bar{0}_S\rangle \
\tag{D.6}
\]

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Finally, taking into account Eq. we then recover the expression exhibited in section 4.3.

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\[ \varepsilon = \varepsilon_+/\varepsilon_- = \varepsilon + \frac{1}{\varepsilon} - \tilde{K}_S(\lambda) \]

| Manifold | $\varepsilon = \varepsilon_+/\varepsilon_-$ | $\tilde{K}_S^{(-)}(\lambda)$ |
|----------|---------------------------------------------|---------------------------------|
| I        | +                                           | Upper                           |
| II       | −                                           | Triangular                      |
| I        | −                                           | Lower                           |
| II       | +                                           | Triangular                      |

Table 1: The triangular property dependence of $\tilde{K}_S^{(-)}(\lambda)$ on the ratio $\varepsilon = \varepsilon_+/\varepsilon_-$. 