A UNIVERSAL METRIC FOR THE CANONICAL BUNDLE OF A
HOLOMORPHIC FAMILY OF PROJECTIVE ALGEBRAIC MANIFOLDS

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Dedicated to M. Salah Baouendi on the occasion of his 60th birthday.

1. Introduction

In his celebrated work [S-98, S-02], Siu proved that the plurigenera of any algebraic manifold are invariant in families. More precisely, let \( \pi : \mathcal{X} \to \mathbb{D} \) be a holomorphic submersion (i.e., \( d\pi \) is nowhere zero) from a complex manifold \( \mathcal{X} \) to the unit disk \( \mathbb{D} \), and assume that every fiber \( \mathcal{X}_t := \pi^{-1}(t) \) is a compact projective manifold. Then for every \( m \in \mathbb{N} \), the function \( P_m : \mathbb{D} \to \mathbb{N} \) defined by \( P_m(t) := h^0(\mathcal{X}_t, mK_{\mathcal{X}_t}) \) is constant.

Siu’s approach to the problem begins with the observation that the function \( P_m \) is upper semi-continuous. Thus in order to prove that \( P_m \) is continuous (hence constant) it suffices to show that given a global holomorphic section \( s \) of \( mK_{\mathcal{X}_0} \), there is a family of global holomorphic sections \( s_t \) of \( \mathcal{X}_t \) for all \( t \) in a neighborhood of 0, that varies holomorphically with \( t \) and satisfies \( s_0 = s \).

To prove such an extension theorem, Siu establishes a generalization of the Ohsawa-Takegoshi Extension Theorem to the setting of complex submanifolds of a Kahler manifold having codimension 1 and cut out by a single, bounded holomorphic function. This theorem, which we will discuss below, requires the existence of a singular Hermitian metric on the ambient manifold having non-negative curvature current, with respect to which the section to be extended is \( L^2 \). Thus in the presence of the extension theorem, the approach reduces to construction of such a metric.

The case where the fibers \( \mathcal{X}_t \) of our holomorphic family are of general type was treated in [S-98]. In this setting, Siu produced a single singular Hermitian metric \( e^{-\kappa} \) for \( K_X \) so that every \( m \)-canonical section is \( L^2 \) with respect to \( e^{-(m-1)\kappa} \).

However, in the case where the fibers \( \mathcal{X}_t \) of our holomorphic family are assumed only to be algebraic, and not necessarily of general type, Siu’s proof in [S-02] does not construct a single metric as in the case of general type. Instead, Siu constructs for every section \( s \) of \( mK_{\mathcal{X}_0} \), a singular Hermitian metric \( \kappa \) for \( mK_\mathcal{X}_0 \) of non-negative curvature so that \( s \) is \( L^2 \) with respect to this metric.

Definition. Let \( \mathcal{X} \to \Delta \) be a holomorphic family of complex manifolds and \( \mathcal{X}_0 \) the central fiber of \( \mathcal{X} \). A universal canonical metric for the pair \( (\mathcal{X}, \mathcal{X}_0) \) is a singular Hermitian metric \( e^{-\kappa} \) for the canonical bundle \( K_{\mathcal{X}} \) of \( \mathcal{X} \) such that for every global holomorphic section \( s \in H^0(\mathcal{X}_0, mK_{\mathcal{X}_0}) \),

\[
\int_{\mathcal{X}_0} |s|^2 e^{-(m-1)\kappa} < +\infty.
\]

The goal of this paper is to prove that for any holomorphic family \( \mathcal{X} \to \Delta \) of compact complex algebraic manifolds with central fiber \( \mathcal{X}_0 \), the pair \( (\mathcal{X}, \mathcal{X}_0) \) has a universal canonical metric having non-negative curvature current. To this end, our main theorem is the following result.

Theorem 1. Let \( X \) be a complex manifold admitting a positive line bundle \( A \to X \), and \( Z \subset X \) a smooth compact complex submanifold of codimension 1. Assume there is a subvariety \( V \subset X \) not containing \( Z \) such that \( X \setminus V \) is a Stein manifold. Let \( T \in H^0(X, Z) \) be a holomorphic section of

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the line bundle associated to $Z$, thought of as a divisor. Let $E \to X$ be a holomorphic line bundle and denote by $K_X$ the canonical bundle of $X$. Assume we are given singular metrics $e^{-\varphi_E}$ for $E$ and $e^{-\varphi_Z}$ for the line bundle associated to $Z$.

Suppose in addition that the above data satisfy the following assumptions.

(R) The metrics $e^{-\varphi_E}$ and $e^{-\varphi_Z}$ restrict to singular metrics on $Z$.

(B) $\sup_X |T|^2 e^{-\varphi_Z} < +\infty$.

(G) The line bundles $p(K_X + Z + E) + A$, $0 \leq p \leq m - 1$, are globally generated, in the sense that a finite number of sections of $H^0(X, p(K_X + Z + E) + A)$ generate the sheaf $\mathcal{O}_X(p(K_X + Z + E) + A)$.

(P) $\sqrt{-1} \partial \bar{\partial} \varphi_E \geq 0$ and there exists a constant $\mu$ such that $\mu \sqrt{-1} \partial \bar{\partial} \varphi_E \geq \sqrt{-1} \partial \bar{\partial} \varphi_Z$.

(T) The singular metric $e^{-(\varphi_Z + \varphi_E)}|Z$ has trivial multiplier ideal:

$$\mathcal{I}(Z, e^{-(\varphi_Z + \varphi_E)}|Z) = \mathcal{O}_Z.$$ 

Then there is a metric $e^{-\kappa}$ for $K_X + Z + E$ with the following properties:

(C) $\sqrt{-1} \partial \bar{\partial} \kappa \geq 0$.

(L) For every $m > 0$ and every section $s \in H^0(Z, m(K_Z + E)|Z)$, $|s|^2 e^{-(m-1)\kappa + \varphi_E + \varphi_Z}$ is locally integrable.

(I) For every integer $m > 0$ and every section $s \in H^0(Z, m(K_Z + E))$,

$$\int_Z |s|^2 e^{-(m-1)\kappa + \varphi_E} < +\infty.$$ 

Remarks. (i) For the ambient manifold $X$, we have in mind the following two examples: either $X$ is compact complex projective (in which case the variety $V$ could be taken to be a hyperplane section of some embedding of $X$) or else $X$ is a family of compact complex algebraic manifolds. In the former case, it is well-known that the hypothesis (G) holds for any sufficiently ample $A$, while in the latter case, one might have to shrink $X$ a little to obtain (G). Of course, there are many other examples of such $X$.

(ii) Note that in condition (L), the local functions $|s|^2 e^{-(m-1)\kappa + \varphi_E + \varphi_Z}$ depend on the local trivializations of the line bundles in question. However, the local integrability condition is independent of these choices.

Together with a variant of the Ohsawa-Takegoshi Theorem (Theorem 11 below), Theorem 1 implies a generalization of Siu’s extension theorem to the case where the normal bundle of the submanifold $Z$ is not necessarily trivial. The first extension theorem of this type was established by Takayama [15-05] Theorem 4.1 under some additional hypotheses. The general case was done in [13-06], where Theorem 11 was also established. The argument here is related to that of [14-06], but the focus is on construction of the metric rather than on the extension theorem.

As a result of Theorem 11 we have the following corollary, which is our stated goal.

Corollary 2. For every holomorphic family $\mathcal{X} \to \Delta$ of smooth projective varieties with central fiber $\mathcal{X}_0$, the pair $(\mathcal{X}, \mathcal{X}_0)$ has, perhaps after slightly shrinking the family, a universal canonical metric having non-negative curvature current.

Proof. Let $X$ be a family of compact projective manifolds $\pi : \mathcal{X} \to \mathbb{D}$, and $Z = \mathcal{X}_0$ the central fiber. Take $T = \pi$, $E = \mathcal{O}_{\mathcal{X}}$ and $\varphi_E \equiv 0$. Since $\mathcal{X}_0$ is cut out by a single holomorphic function, the line bundle associated to $\mathcal{X}_0$ is trivial. Take $\varphi_Z \equiv 0$. Then the hypotheses of Theorem 11 are satisfied, perhaps after shrinking the family, and we obtain a metric $e^{-\kappa}$ for $K_{\mathcal{X}}$ such that $\sqrt{-1} \partial \bar{\partial} \kappa \geq 0$ and $|s|^2 e^{-(m-1)\kappa + m}$ is integrable for every integer $m > 0$ and every section $s \in H^0(\mathcal{X}_0, mK_{\mathcal{X}_0})$. \qed
Remark. Note that in the setting of families, the constant $\mu$ is not needed, and the hypotheses (L) and (I) are the same.

Remark. In his paper \cite{Tsuji02}, Tsuji has claimed the existence of a metric with the properties stated in Corollary 2. As in our approach, Tsuji’s proof makes use of an infinite process. It seems that convergence of this process was not checked; in fact, it is demonstrated in \cite{Tsuji02} that Tsuji’s process, as well as any reasonable modification of it, diverges.

Proposition 3. For each integer $m > 0$, fix a basis $s_1^{(m)}, \ldots, s_N^{(m)}$ of $H^0(X, m(K_Z + E|Z))$. Choose constants $\varepsilon_m$ such that the metric

$$\kappa_0 := \log \left( \sum_{m=1}^{\infty} \varepsilon_m \left( \sum_{\ell=1}^{N_m} |s_{\ell}^{(m)}|^2 \right)^{1/m} \right)$$

is convergent. Suppose $e^{-\varphi E}$ is locally integrable. Then for each $m > 0$ and every $s \in H^0(X, m(K_Z + E|Z))$,

$$\int_Z |s|^2 e^{-(m-1)\kappa_0 + \varphi E} < +\infty.$$

Proof. Fix $s \in H^0(X, m(K_Z + E|Z))$, and let $\kappa_{0,m} = \log \left( \sum_{\ell=1}^{N_m} |s_{\ell}^{(m)}|^2 \right)^{1/m}$. Note that $e^{-\kappa_0} \lesssim e^{-\kappa_{0,m}}$, and thus we have

$$\int_Z |s|^2 e^{-(m-1)\kappa_0 + \varphi E} \lesssim \int_Z |s|^2/m e^{-(m-1)\kappa_{0,m} + \varphi E}$$

$$= \int_Z |s|^2/m \left( \frac{|s|^2}{|s_1^{(m)}|^2 + \ldots + |s_N^{(m)}|^2} \right)^{(m-1)/m} e^{\gamma E - \varphi E} e^{-\gamma E}$$

$$\lesssim \int_Z |s|^2/m e^{\gamma E - \varphi E} e^{-\gamma E}$$

$$\lesssim \left( \int_Z |s|^2 e^{\gamma E - \varphi E} e^{-\gamma E} e^{-n(m-1)} \right)^{1/m} \left( \int_Z e^{\gamma E - \varphi E} e^{n-1} \right)^{(m-1)/m},$$

where $\omega$ is a fixed Kähler form for $Z$ and $e^{-\gamma E}$ is a smooth metric for $E|Z$. The last inequality is a consequence of Hölder’s Inequality. Since $e^{-\varphi E}$ is locally integrable, we are done. \hfill \square

A calculation similar to the proof of Proposition 3 shows that $|s|^2 e^{-(m-1)\kappa_0 + \varphi E}$ is locally integrable on $Z$. Thus in view of Proposition 3, Theorem 4 follows if we construct a metric $e^{-\kappa}$ with non-negative curvature current such that $e^{-\kappa}|Z = e^{-\kappa_0}$. This is precisely what we do. We employ a technical simplification, due to Paun \cite{Paun05}, of Siu’s original idea of extending metrics using an Ohsawa-Takegoshi-type extension theorem for sections.

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2. The Ohsawa-Takegoshi Extension Theorem

Let $Y$ be a Kähler manifold of complex dimension $n$. Assume there exists an analytic hypersurface $V \subset Y$ such that $Y - V$ is Stein. Examples of such manifolds are Stein manifolds (where $V$ is empty) and projective algebraic manifolds (where one can take $V$ to be the intersection of $Y$ with a projective hyperplane in some projective space in which $Y$ is embedded).

Fix a smooth hypersurface $Z \subset Y$ such that $Z \not\subset V$. In [V-06] we proved the following generalization of the Ohsawa-Takegoshi Extension Theorem.

**Theorem 4.** Suppose given a holomorphic line bundle $H \to Y$ with a singular Hermitian metric $e^{-\psi}$, and a singular Hermitian metric $e^{-\varphi_Z}$ for the line bundle associated to the divisor $Z$, such that the following properties hold.

(i) The restrictions $e^{-\psi}|Z$ and $e^{-\varphi_Z}|Z$ are singular metrics.

(ii) There is a global holomorphic section $T \in H^0(Y, Z)$ such that

$$Z = \{T = 0\} \quad \text{and} \quad \sup_Y |T|^2 e^{-\varphi_Z} = 1.$$

(iii) $\sqrt{-1} \partial \bar{\partial} \psi \geq 0$ and there is an integer $\mu > 0$ such that $\mu \sqrt{-1} \partial \bar{\partial} \psi \geq \sqrt{-1} \partial \bar{\partial} \varphi_Z$.

Then for every $s \in H^0(Z, K_Z + H)$ such that

$$\int_Z |s|^2 e^{-\psi} < +\infty \quad \text{and} \quad s \wedge dT \in \mathcal{A}(e^{-(\varphi_Z + \psi)}|Z),$$

there exists a section $S \in H^0(Y, K_Y + Z + H)$ such that

$$S|Z = s \wedge dT \quad \text{and} \quad \int_Y |S|^2 e^{-(\varphi_Z + \psi)} \leq 40 \pi \mu \int_Z |s|^2 e^{-\psi}.$$

3. Inductive Construction of Certain Sections by Extension

Fix a holomorphic line bundle $A \to X$ such that the property (G) in Theorem 1 holds.

Let us fix bases

$$\{\tilde{\sigma}_j^{(m,0,p)} ; 1 \leq j \leq M_p\}$$

of $H^0(X, p(K_X + Z + E) + A)$. We let $\sigma_j^{(m,0,p)} \in H^0(Z, p(K_Z + E|Z) + A|Z)$ be such that

$$\tilde{\sigma}_j^{(m,0,p)}|Z = \sigma_j^{(m,0,p)} \wedge (dT)^{\otimes p}.$$

We also fix smooth metrics

$$e^{-\gamma_Z} \text{ and } e^{-\gamma_E} \text{ for } Z \to X, \text{ and } E \to X$$

respectively. Finally, let us fix bases

$$s_1^{(m)}, \ldots, s_N^{(m)} \text{ for } H^0(X, m(K_Z + E|Z)), \quad m = 1, 2, \ldots,$$

orthonormal with respect to the singular metric $(\omega^{-(n-1)e^{-\gamma_E}})^{m-1} e^{-\varphi_E}$ for $(m-1)K_Z + mE|Z$.

(Since $e^{-\varphi_E}$ is locally integrable, every holomorphic section is integrable with respect to this metric.)

**Proposition 5.** For each $m = 1, 2, \ldots$ there exist a constant $C_m < +\infty$ and sections

$$\tilde{\sigma}_{j,\ell}^{(m,k,p)} \in H^0(X, (km + p)(K_X + Z + E) + A)$$

where $p = 1, 2, \ldots, m - 1$, $1 \leq j \leq M_p$, $1 \leq \ell \leq N_m$ and $k = 1, 2, \ldots$, with the following properties.

(a) $\tilde{\sigma}_{j,\ell}^{(m,k,p)}|Z = (s_\ell^{(m)})^{\otimes k} \otimes \sigma_j^{(m,0,p)} \wedge (dT)^{(km+p)}$
(b) If \( k \geq 1 \),
\[
\int_X \frac{\sum_{j=1}^{M_0} |\tilde{\sigma}_{j,\ell}^{(m,k,0)}|^2 e^{-(\gamma_Z + \gamma_E)}}{\sum_{j=1}^{M_{m-1}} |\tilde{\sigma}_{j,\ell}^{(m,k-1,m-1)}|^2} \leq C_m.
\]

(c) For \( 1 \leq p \leq m - 1 \),
\[
\int_X \frac{\sum_{j=1}^{M_p} |\tilde{\sigma}_{j,\ell}^{(m,k,p)}|^2 e^{-(\gamma_Z + \gamma_E)}}{\sum_{j=1}^{M_{m-1}} |\tilde{\sigma}_{j,\ell}^{(m,k,p-1)}|^2} \leq C_m.
\]

Proof. (Double induction on \( k \) and \( p \).) Fix a constant \( \hat{C}_m \) such that the
\[
\sup_X \frac{\sum_{j=1}^{M_0} |\tilde{\sigma}_{j}^{(m,0,0)}|^2 \omega^n e^{m-1}(\gamma_Z + \gamma_E)}{\sum_{j=1}^{M_{m-1}} |\tilde{\sigma}_{j}^{(m,0,m-1)}|^2} \leq \hat{C}_m
\]
and
\[
\sup_Z \frac{\sum_{j=1}^{M_0} |\sigma_j^{(m,0,0)}|^2 \omega^{n-1} m e^{m-1}(\gamma_Z + \gamma_E)}{\sum_{j=1}^{M_{m-1}} |\sigma_j^{(m,0,m-1)}|^2} \leq \hat{C}_m,
\]
and for all \( 0 \leq p \leq m - 2 \),
\[
\sup_X \frac{\sum_{j=1}^{N_{p+1}} |\tilde{\sigma}_{j}^{(m,0,p+1)}|^2 \omega^p e^{-(\gamma_Z + \gamma_E)}}{\sum_{j=1}^{M_p} |\tilde{\sigma}_{j}^{(m,0,p)}|^2} \leq \hat{C}_m,
\]
and
\[
\sup_Z \frac{\sum_{j=1}^{N_{p+1}} |\sigma_j^{(m,0,p+1)}|^2 \omega^{-1} e^{-(\gamma_Z + \gamma_E)}}{\sum_{j=1}^{M_p} |\sigma_j^{(m,0,p)}|^2} \leq \hat{C}_m.
\]

\((k = 0)\) We set \( \tilde{\sigma}_{j,\ell}^{(m,0,p)} := \tilde{\sigma}_{j}^{(m,0,p)} \) and simply observe that
\[
\int_X \frac{\sum_{j=1}^{M_p} |\tilde{\sigma}_{j,\ell}^{(m,0,p)}|^2 e^{-(\gamma_Z + \gamma_E)}}{\sum_{j=1}^{M_{m-1}} |\tilde{\sigma}_{j,\ell}^{(m,0,p-1)}|^2} \leq \hat{C}_m \int_X \omega^n.
\]

\((p = 0)\): Consider the sections \((s_{\ell}^{(m)})^{k} \otimes \sigma_j^{(m,0,0)},\) and define the semi-positively curved metric
\[
\psi_{k,\ell,0} := \log \sum_{j=1}^{M_{m-1}} |\sigma_{j,\ell}^{(m,k-1,m-1)}|^2
\]
for the line bundle \((mk - 1)(K_X + Z + E) + A\). Observe that locally on \( Z \),
\[
|s_{\ell}^{(m)} \wedge dT^m|^k \otimes \sigma_j^{(m,0,0)}|^2 e^{-(\varphi_Z + \psi_{k,\ell,0} + \varphi_E)} = |s_{\ell}^{(m)} \wedge dT^m|^2 |\sigma_j^{(m,0,0)}|^2 e^{-(\varphi_Z + \varphi_E)} \sum_{j=1}^{M_{m-1}} |\sigma_j^{(m,0,m-1)}|^2
\]
\[
\lesssim |s_{\ell}^{(m)}|^2 e^{-(\varphi_Z + \varphi_E)}.
\]

Moreover, we have
\[
\sqrt{-1} \partial \bar{\partial} (\psi_{k,\ell,0} + \varphi_E) \geq 0 \quad \text{and} \quad \mu \sqrt{-1} \partial \bar{\partial} (\psi_{k,\ell,0} + \varphi_E) \geq \sqrt{-1} \partial \bar{\partial} \varphi_Z.
\]
Finally,

\[
\int_Z \left| (s^{(m)}_{\ell})^k \otimes \sigma_j^{(m,0,0)} \right|^2 e^{-(\psi_{k,\ell,0} + \varphi_E)} \leq 40\pi \mu \int_Z \left| s^{(m)}_{\ell} \right|^2 \frac{\left| \sigma_j^{(m,0,0)} \right|^2 e^{-(\varphi_E + \varphi_E)}}{\sum_{j=1}^{N_{m-1}} \left| \sigma_j^{(m-1)} \right|^2} < +\infty.
\]

We may thus apply Theorem 4 to obtain sections

\[
\tilde{\sigma}_{j,\ell}^{(m,k,0)} \in H^0(X, mk(K_X + Z + E) + A), \quad 1 \leq j \leq M_0, \quad 1 \leq \ell \leq N_m,
\]
such that

\[
\tilde{\sigma}_{j,\ell}^{(m,k,0)}|_Z = (s^{(m)}_{\ell})^k \otimes \sigma_j^{(m,0,0)} \wedge (dT)^{\otimes km}, \quad 1 \leq j \leq M_0, \quad 1 \leq \ell \leq N_m,
\]

and

\[
\int_X \left| \tilde{\sigma}_{j,\ell}^{(m,k,0)} \right|^2 e^{-(\psi_{k,\ell,0} + \varphi_Z + \varphi_E)} \leq 40\pi \mu \int_Z \left| s^{(m)}_{\ell} \right|^2 \frac{\left| \sigma_j^{(0)} \right|^2 e^{-(\varphi_E + \varphi_E)}}{\sum_{j=1}^{N_{m-1}} \left| \sigma_j^{(m-1)} \right|^2}.
\]

Summing over \( j \), we obtain

\[
\int_X \sum_{j=1}^{M_0} \frac{\left| \tilde{\sigma}_{j,\ell}^{(m,k,0)} \right|^2 e^{-(\gamma_Z + \gamma_E)}}{\sum_{j=1}^{N_{m-1}} \left| \sigma_j^{(m-1)} \right|^2} \leq \sup_X e^{\varphi_Z + \varphi_E - \gamma_Z - \gamma_E} \int_X \sum_{j=1}^{M_0} \frac{\left| \tilde{\sigma}_{j,\ell}^{(m,k,0)} \right|^2 e^{-(\varphi_Z + \varphi_E)}}{\sum_{j=1}^{N_{m-1}} \left| \sigma_j^{(m-1)} \right|^2} \leq 40\pi \sup_X e^{\varphi_Z + \varphi_E - \gamma_Z - \gamma_E} \int_X \left| s^{(m)}_{\ell} \right|^2 \frac{\sum_{j=1}^{M_0} \left| \sigma_j^{(m,0,0)} \right|^2 e^{-\varphi_E}}{\sum_{j=1}^{N_{m-1}} \left| \sigma_j^{(m,0,0)} \right|^2} e^{-\kappa} \leq 40\pi \tilde{C}_m \sup_X e^{\varphi_Z + \varphi_E - \gamma_Z - \gamma_E} \int_X \left| s^{(m)}_{\ell} \right|^2 \frac{\omega^{-(n-1)(m-1)} e^{-(m-1)(\gamma_E + \varphi_E)}}{\sum_{j=1}^{N_{m-1}} \left| \sigma_j^{(m-1)} \right|^2} = 40\pi \tilde{C}_m \sup_X e^{\varphi_Z + \varphi_E - \gamma_Z - \gamma_E} \cdot
\]

\((1 \leq p \leq m - 1)): \) Assume that we have obtained the sections \( \tilde{\sigma}_{j,\ell}^{(m,k,p-1)}, \) \( 1 \leq j \leq M_{p-1}, \) \( 1 \leq \ell \leq N_m. \) Consider the non-negatively curved singular metric

\[
\psi_{k,\ell,p} := \log \sum_{j=1}^{M_{p-1}} \left| \tilde{\sigma}_{j,\ell}^{(m,k,p-1)} \right|^2
\]

for \((km + p - 1)(K_X + Z + E) + A\). We have

\[
\left| (s^{(m)}_{\ell})^k \otimes \sigma_j^{(m,0,0)} \right|^2 e^{-(\varphi_Z + \psi_{k,\ell,p} + \varphi_E)} = \frac{\left| \sigma_j^{(m,0,p)} \right|^2 e^{-(\varphi_Z + \varphi_E)}}{\sum_{j=1}^{M_{p-1}} \left| \sigma_j^{(m,0,p-1)} \right|^2} \leq e^{-(\varphi_Z + \varphi_E)},
\]

which is locally integrable on \( Z \) by the hypothesis (T). Next,

\[
\int_Z \left| (s^{(m)}_{\ell})^k \otimes \sigma_j^{(m,0,0)} \right|^2 e^{-(\psi_{k,\ell,p} + \varphi_E)} = \int_Z \frac{\left| \sigma_j^{(m,0,p)} \right|^2 e^{-\varphi_E}}{\sum_{j=1}^{M_{p-1}} \left| \sigma_j^{(m,0,p-1)} \right|^2} \leq C^* \int_Z e^{\gamma_Z} \frac{\left| \sigma_j^{(m,0,p)} \right|^2 e^{-(\varphi_Z + \varphi_E)}}{\left| \sigma_j^{(m,0,p-1)} \right|^2} < +\infty,
\]

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where
\[ C^* := \sup_{z} e^{\varphi z - \gamma z}. \]
Moreover,
\[ \sqrt{-1} \partial \bar{\partial} (\psi_{k,\ell,p} + \varphi_E) \geq 0 \quad \text{and} \quad \sqrt{-1} \partial \bar{\partial} (\psi_{k,\ell,p} + \varphi_E) \geq \sqrt{-1} \partial \bar{\partial} \varphi_Z. \]
By Theorem 4 there exist sections
\[ \tilde{\sigma}_{j,\ell}^{(m,k,p)} \in H^0(X, (mk + p)(K_X + Z + E) + A), \quad 1 \leq j \leq M_0 \]
such that
\[ \tilde{\sigma}_{j,\ell}^{(m,k,p)}|Z = (\sigma_{\ell}^{(m)}) \otimes k \otimes \sigma_{j,\ell}^{(m,0,p)} \wedge (dT)^{\otimes km + p}, \quad 1 \leq j \leq M_p, \]
and
\[ \int_X |\tilde{\sigma}_{j,\ell}^{(m,k,p)}|^2 e^{-(\psi_{k,\ell,p} + \varphi_E)} \leq 40\pi \mu \int_Z |\sigma_j^{(m,0,p)}|^2 e^{-\varphi_E}. \]
Summing over \( j \), we obtain
\[ \int_X \sum_{j=1}^{M_p} |\tilde{\sigma}_{j,\ell}^{(m,k,p)}|^2 e^{-(\gamma_Z + \gamma_E)} \leq 40\pi \mu \sup_X e^\varphi Z + \varphi E - \gamma Z - \gamma E \tilde{C}_m \int_Z e^{-\varphi E} \omega^{n-1}. \]
Letting
\[ C_m := 40\pi \mu \tilde{C}_m \max \left( \int_X \omega^n, \sup_X e^{\varphi Z + \varphi E - \gamma Z - \gamma E} \right) \]
completes the proof.

4. CONSTRUCTION OF THE METRIC

4.1. A metric associated to \( m(K_X + Z + E) \). Fix a smooth metric \( e^{-\psi} \) for \( A \to X \). Consider the functions
\[ \lambda_{\ell,N}^{(m)} := \log \sum_{j=1}^{M_p} |\tilde{\sigma}_{j,\ell}^{(m,k,p)}|^2 \omega^{-n(mk + p)} e^{-(km(\gamma_Z + \gamma_E) + \psi)}, \]
where \( N = mk + p \). Set
\[ \lambda^{(m)}_{N} := \log \sum_{\ell=1}^{N_m} e^{\lambda_{\ell,N}^{(m)}}. \]

LEMMA 6. For any non-empty open subset \( V \subset X \) and any smooth function \( f : V \to \mathbb{R}_+ \),
\[ \frac{1}{\int_V f \omega^n} \int_V (\lambda_{N}^{(m)} - \lambda_{N-1}^{(m)}) f \omega^n \leq \log \left( \frac{N_m C_m \sup_V f}{\int_V f \omega^n} \right). \]

Proof. Observe that by Proposition 5 there exists a constant \( C_m \) such that for any open subset \( V \subset X \),
\[ \int_V (e^{\lambda_{\ell,N}^{(m)} - \lambda_{\ell,N-1}^{(m)}} - 1) f \omega^n \leq C_m \sup_V f, \]
and thus
\[ \int_V (e^{\lambda_{N-1}^{(m)} - \lambda_{N-2}^{(m)}} - 1) f \omega^n = \sum_{\ell=1}^{N_m} \int_V (e^{\lambda_{\ell,N}^{(m)} - \lambda_{\ell,N-1}^{(m)}} - 1) f \omega^n \leq N_m C_m \sup_V f. \]
An application of (the concave version of) Jensen’s inequality to the concave function \( \log \) then gives
\[ \frac{1}{\int_V f \omega^n} \int_V (\lambda_{N}^{(m)} - \lambda_{N-1}^{(m)}) f \omega^n \leq \log \left( \frac{N_m C_m \sup_V f}{\int_V f \omega^n} \right). \]
The proof is complete.

Consider the function
\[ \Lambda_k^{(m)} = \frac{1}{k} \lambda_{mk}. \]

Note that \( \Lambda_k^{(m)} \) is locally the sum of a plurisubharmonic function and a smooth function. By applying Lemma \( \square \) and using the telescoping property, we see that for any open set \( V \subset X \) and any smooth function \( f : V \to \mathbb{R}_+ \),

\[ \frac{1}{\int_V f \omega^n} \int_V \Lambda_k^{(m)} f \omega^n \leq m \log \left( \frac{N_mC_m \sup_V f}{\int_V f \omega^n} \right). \]

**Proposition 7.** There exists a constant \( C^{(m)}_0 \) such that
\[ \Lambda_k^{(m)}(x) \leq C^{(m)}_0, \quad x \in X. \]

**Proof.** Let us cover \( X \) by coordinate charts \( V_1, \ldots, V_N \) such that for each \( j \) there is a biholomorphic map \( F_j \) from \( V_j \) to the ball \( B(0,2) \) of radius 2 centered at the origin in \( C^n \), and such that if \( U_j = F_j^{-1}(B(0,1)) \), then \( U_1, \ldots, U_N \) is also an open cover. Let \( W_j = V_j \setminus F_j^{-1}(B(0,3/2)) \).

Now, on each \( V_j \), \( \Lambda_k^{(m)} \) is the sum of a plurisubharmonic function and a smooth function. Say \( \Lambda_k^{(m)} = h + g \) on \( V_j \), where \( h \) is plurisubharmonic and \( g \) is smooth. Then for constant \( A_j \) we have
\[ \sup_{U_j} \Lambda_k^{(m)} \leq \sup_{U_j} g + \sup_{U_j} h \]
\[ \leq \sup_{U_j} g + A_j \int_{W_j} h \cdot F_j \ast dV \]
\[ \leq \sup_{U_j} g - A_j \int_{W_j} g \cdot F_j \ast dV + A_j \int_{W_j} \Lambda_k^{(m)} \cdot F_j \ast dV \]

Let
\[ C^{(m)}_j := \sup_{U_j} g - A_j \int_{W_j} g \cdot F_j \ast dV \]
and define the smooth function \( f_j \) by
\[ f_j \omega^n = F_j \ast dV. \]

Then by (1) applied with \( V = W_j \) and \( f = f_j \), we have
\[ \sup_{U_j} \Lambda_k^{(m)} \leq C^{(m)}_j + m A_j \log \left( \frac{N_mC_m \sup_{W_j} f_j}{\int_{W_j} f_j \omega^n} \right) \int_{W_j} f_j \omega^n. \]

Letting
\[ C^{(m)}_0 := \max_{1 \leq j \leq N} \left\{ C^{(m)}_j + m A_j \log \left( \frac{N_mC_m \sup_{W_j} f_j}{\int_{W_j} f_j \omega^n} \right) \int_{W_j} f_j \omega^n \right\} \]
completes the proof. \( \square \)

Since the upper regularization of the lim sup of a uniformly bounded sequence of plurisubharmonic functions is plurisubharmonic (see, e.g., [H-90 Theorem 1.6.2]), we essentially have the following corollary.
Corollary 8. The function

\[ \Lambda^{(m)}(x) := \limsup_{y \to x} \limsup_{k \to \infty} \Lambda^{(m)}_k(y) \]

is locally the sum of a plurisubharmonic function and a smooth function.

Proof. One need only observe that the function \( \Lambda_k \) is obtained from a singular metric on the line bundle \( m(K_X + Z + E) \) (this singular metric \( e^{-\kappa^{(m)}} \) will be described shortly) by multiplying by a fixed smooth metric of the dual line bundle. \( \square \)

Consider the singular Hermitian metric \( e^{-\kappa^{(m)}} \) for \( m(K_X + Z + E) \) defined by

\[ e^{-\kappa^{(m)}} = e^{-\Lambda^{(m)}_k} \omega^{-nm} e^{-m(\gamma_Z + \gamma_E)}. \]

This singular metric is given by the formula

\[ e^{-\kappa^{(m)}}(x) = \exp\left( -\limsup_{y \to x} \limsup_{k \to \infty} \kappa^{(m)}_k(y) \right), \]

where

\[ e^{-\kappa^{(m)}_k} = e^{-\Lambda^{(m)}_k} \omega^{-nm} e^{-m(\gamma_Z + \gamma_E)}. \]

The curvature of \( e^{-\kappa^{(m)}} \) is thus

\[ \sqrt{-1} \partial \bar{\partial} \kappa^{(m)}_k = \frac{\sqrt{-1}}{k} \partial \bar{\partial} \log \sum_{\ell=1}^{N_m} \sum_{j=1}^{N_0} |a^{(m,k,0)}_{\ell j}|^2 - \frac{1}{k} \sqrt{-1} \partial \bar{\partial} \psi \]

\[ \geq -\frac{1}{k} \sqrt{-1} \partial \bar{\partial} \psi \]

We claim next that the curvature of \( e^{-\kappa} \) is non-negative. To see this, it suffices to work locally. Then we have that the functions

\[ \kappa^{(m)}_k + \frac{1}{k} \psi \]

are plurisubharmonic. But

\[ \limsup_{y \to x} \limsup_{k \to \infty} \kappa^{(m)}_k + \frac{1}{k} \psi = \limsup_{y \to x} \limsup_{k \to \infty} \kappa^{(m)}_k = \kappa^{(m)}. \]

It follows that \( \kappa^{(m)} \) is plurisubharmonic, as desired.

4.2. The metric for \( K_X + Z + E \); Proof of Theorem \( \square \) Let \( \varepsilon_m \) be constants, chosen so \( \varepsilon_m \searrow 0 \) sufficiently rapidly that the sum

\[ e^\kappa := \sum_{m=1}^{\infty} \varepsilon_m e^{\frac{1}{m} \kappa^{(m)}} = \sum_{m=1}^{\infty} \exp\left( \frac{1}{m} \kappa^{(m)} + \log \varepsilon_m \right) \]

converges everywhere on \( X \) (to a metric for \( -(K_X + Z + E) \)). It is possible to find such constants since, by Proposition \( \square \) each \( \kappa^{(m)} \) is locally uniformly bounded from above. (The lower bound \( e^{\kappa^{(m)}} \geq 0 \) is trivial.) Moreover, by elementary properties of plurisubharmonic functions, \( \kappa \) is plurisubharmonic. Indeed, for any \( r \in \mathbb{N}, \) the function

\[ \psi_r := \log \sum_{m=1}^{r} \exp\left( \frac{1}{m} \kappa^{(m)} + \log \varepsilon_m \right) \]

is plurisubharmonic, and \( \psi_r \searrow \kappa. \) It follows that \( \kappa = \sup_r \psi_r \) is plurisubharmonic. (Again, see \( \square \) Theorem 1.6.2.) Thus \( e^{-\kappa} \) is a singular Hermitian metric for \( K_X + Z + E \) with non-negative curvature current.
Observe that, after identifying $K_Z$ with $(K_X + Z)|Z$ by dividing by $dT$,

$$\kappa_k^{(m)}|Z = \log \left( \sum_{\ell=1}^{N_m} |s^{(m)}_\ell|^2 \right) + \frac{1}{k} \log \sum_{j=1}^{M_0} |\sigma^{(m,0,0)}_j|^2.$$  

Thus we obtain $e^{-\kappa^{(m)}}|Z = \left( \sum_{\ell=1}^{N_m} |s^{(m)}_\ell|^2 \right)^{-1}$. It follows that

$$e^{-\kappa}|Z = \frac{1}{\sum_{m=1}^{\infty} \varepsilon_m \left( \sum_{\ell=1}^{N_m} |s^{(m)}_\ell|^2 \right)^{2/m}}.$$  

In view of the short discussion following the proof of Proposition 3, the metric $e^{-\kappa}$ satisfies the conclusions of Theorem 1. The proof of Theorem 1 is thus complete.  

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