Strong approximation of continuous local martingales by simple random walks

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Abstract

The aim of this paper is to represent any continuous local martingale as an almost sure limit of a nested sequence of simple, symmetric random walks, time changed by a discrete quadratic variation process. One basis of this is a similar construction of Brownian motion. The other major tool is a representation of continuous local martingales given by Dambis, Dubins and Schwarz (DDS) in terms of Brownian motion time-changed by the quadratic variation. Rates of convergence (which are conjectured to be nearly optimal in the given setting) are also supplied. A necessary and sufficient condition for the independence of the random walks and the discrete time changes or, equivalently, for the independence of the DDS Brownian motion and the quadratic variation is proved to be the symmetry of increments of the martingale given the past, which is a reformulation of an earlier result by Ocone [8].

1 Introduction

The present authors are convinced that both for theoretical and practical reasons, it is useful to search for strong (i.e. pathwise, almost sure) approximations of stochastic processes by simple random walks (RWs). The prototype of such efforts was the construction of Brownian motion (BM) as an almost sure limit of simple RW paths, given by Frank Knight in 1962 [7]. Later this construction was simplified and somewhat improved by Pál Révész [9] and then by one of the authors [11]. It is interesting that this method is asymptotically equivalent to Skorohod embedding of a nested sequence of RWs into BM, [11, Theorem 4]. This elementary approach also led to a strong approximation of Itô integrals of smooth functions of BM [11, Theorem 6]. We mention that this RW construction was extended to fractional BMs as well [12].

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This paper discusses a generalization to continuous local martingales \( M \). Beside the above-mentioned RW construction of BM, the other major tool applied by the paper is a representation of continuous local martingales by Brownian motion, time-changed by the quadratic variation, given by Dambis [11] and Dubins – Schwarz [3] (DDS). Theorem 2 shows that the quadratic variation process \( \langle M, M \rangle \) can be almost surely uniformly approximated by a discrete quadratic variation processes \( N_m \) which are based on stopping times of a Skorohod-type embedding of nested simple RWs into \( M \). This corresponds to an earlier similar result by Karandikar [5]. Theorems 3 and 4 give an approximation of \( M \) by a nested sequence of RWs \( B_m \), time-changed by \( \langle M, M \rangle \) and \( N_m \), respectively. The approximations almost surely uniformly converge on bounded intervals. Rates of convergence (which conjectured to be nearly optimal in the given setting) are also supplied.

It is important to note that the DDS Brownian motion \( W \) and the quadratic variation \( \langle M, M \rangle \) are not independent in general, just like the approximating RW \( B_m \) and the discrete quadratic variation \( N_m \). Since this could be a hindrance both in the theory and applications, a necessary and sufficient condition is given for the independence in Theorem 5. Namely, the approximating RW \( B_m \) and the discrete quadratic variation \( N_m \) (and so \( W \) and \( \langle M, M \rangle \)) are independent if and only if \( M \) has symmetric increments given the past. This is a reformulation of an earlier result by Ocone [8], see [2] and [13] as well.

Possible applications of the discrete approximations discussed in this paper include (a) generating continuous local martingales for which one has a suitable method to generate the discrete quadratic variation process \( N_m \), especially when the independence mentioned in the previous paragraph holds and (b) giving an alternative definition of stochastic integration with almost surely converging sums. (In a forthcoming paper the present authors are going to take up the second topic, extending some results of [11].)

## 2 Random walks and the Wiener process

A main tool of the present paper is an elementary construction of the Wiener process (\( \equiv \) BM). The specific construction we are going to use in the sequel, taken from [11], is based on a nested sequence of simple random walks that uniformly converges to the Wiener process on bounded intervals with probability 1. This will be called RW construction in the sequel. One of our intentions in this paper is to extend the underlying “twist and shrink” algorithm to continuous local martingales.

We summarize the major steps of the RW construction here, see [12] as well. We start with an infinite matrix of i.i.d. random variables \( X_m(k) \), \( P \{X_m(k) = \pm 1\} = 1/2 \) \((m \geq 0, k \geq 1)\), defined on the same underlying probability space \((\Omega, \mathcal{F}, P)\). Each row of this matrix is a basis of an approximation of the Wiener process with a dyadic step size \( \Delta t = 2^{-2m} \) in time and a corresponding step size \( \Delta x = 2^{-m} \) in space, illustrated by the next table.

The second step of the construction is twisting. From the independent random walks we want to create dependent ones so that after shrinking temporal and spatial step sizes, each consecutive RW becomes a refinement of the previous one. Since the spatial unit will be halved at each consecutive row, we define
Table 1: The starting setting for the RW construction of BM

| \(\Delta t\) | \(\Delta x\) | i.i.d. sequence | RW |
|-----------|-----|----------------|-----|
| 1         | 1   | \(X_0(1), X_0(2), X_0(3), \ldots\) | \(S_0(n) = \sum_{k=1}^{n} X_0(k)\) |
| \(2^{-2}\) | \(2^{-1}\) | \(X_1(1), X_1(2), X_1(3), \ldots\) | \(S_1(n) = \sum_{k=1}^{n} X_1(k)\) |
| \(2^{-4}\) | \(2^{-2}\) | \(X_2(1), X_2(2), X_2(3), \ldots\) | \(S_2(n) = \sum_{k=1}^{n} X_2(k)\) |
|          |     |                |     |

stopping times by \(T_m(0) = 0\), and for \(k \geq 0\),

\[
T_m(k + 1) = \min\{n : n > T_m(k), |S_m(n) - S_m(T_m(k))| = 2\} \quad (m \geq 1)
\]

These are the random time instants when a RW visits even integers, different from the previous one. After shrinking the spatial unit by half, a suitable modification of this RW will visit the same integers in the same order as the previous RW. We operate here on each point \(\omega \in \Omega\) of the sample space separately, i.e. we fix a sample path of each RW. We define twisted RWs \(\tilde{S}_m\) recursively for \(k = 1, 2, \ldots\) using \(\tilde{S}_{m-1}\), starting with \(\tilde{S}_0(n) = S_0(n)\) \((n \geq 0)\). With each fixed \(m\) we proceed for \(k = 0, 1, 2, \ldots\) successively, and for every \(n\) in the corresponding bridge, \(T_m(k) < n \leq T_m(k + 1)\). Any bridge is flipped if its sign differs from the desired:

\[
\tilde{X}_m(n) = \begin{cases} X_m(n) & \text{if } S_m(T_m(k + 1)) - S_m(T_m(k)) = 2\tilde{X}_m(k + 1), \\ -X_m(n) & \text{otherwise}, \end{cases}
\]

and then \(\tilde{S}_m(n) = \tilde{S}_m(n - 1) + \tilde{X}_m(n)\). Then \(\tilde{S}_m(n)\) \((n \geq 0)\) is still a simple symmetric random walk \([11\text{ Lemma 1}]\). The twisted RWs have the desired refinement property:

\[
\frac{1}{2} \tilde{S}_m(T_m(k)) = \tilde{S}_{m-1}(k) \quad (m \geq 1, k \geq 0).
\]

The last step of the RW construction is shrinking. The sample paths of \(\tilde{S}_m(n)\) \((n \geq 0)\) can be extended to continuous functions by linear interpolation, this way one gets \(\tilde{S}_m(t)\) \((t \geq 0)\) for real \(t\). Then we define the \(m\)th approximating RW by

\[
\tilde{B}_m(t) = 2^{-m} \tilde{S}_m(t 2^{2m}).
\]

Using the definition of \(T_m\) and \(\tilde{B}_m\) we also get the general refinement property

\[
\tilde{B}_{m+1} \left( T_{m+1}(k) 2^{-2(m+1)} \right) = \tilde{B}_m \left( k 2^{-2m} \right) \quad (m \geq 0, k \geq 0). \quad (1)
\]

Note that a refinement takes the same dyadic values in the same order as the previous shrunked walk, but there is a time lag in general:

\[
T_{m+1}(k) 2^{-2(m+1)} - k 2^{-2m} \neq 0. \quad (2)
\]

Then we quote some important facts from \([11]\) about the above RW construction that will be used in the sequel. These will be stated in somewhat stronger forms but can be read easily from the proofs in the cited reference, cf. Lemmas 2-4 and Theorem 3 there.
Lemma A. Suppose that $X_1, X_2, \ldots, X_N$ is an i.i.d. sequence of random variables, $E(X_k) = 0$, $\text{Var}(X_k) = 1$, and their moment generating function is finite in a neighborhood of 0. Let $S_j = X_1 + \cdots + X_j$, $1 \leq j \leq N$. Then for any $C > 1$ and $N \geq N_0(C)$ one has

$$\Pr \left\{ \sup_{1 \leq j \leq N} |S_j| \geq (2CN \log N)^{\frac{1}{2}} \right\} \leq 2N^{1-C}.$$

We mention that this basic fact, that appears in the above-mentioned reference [11], essentially depends on a large deviation theorem.

We have a more convenient result in a special case of Hoeffding’s inequality, cf. [4]. Let $X_1, X_2, \ldots$ be a sequence of bounded i.i.d. random variables, such that $b_i \leq X_i \leq a_i$, and let $S_n = \sum_{i=1}^n X_i$. Then by Hoeffding’s inequality, for any $x > 0$ we have

$$\Pr \left\{ |S_n - E(S_n)| \geq x \left( \frac{1}{2} \sum_{i=1}^n (a_i - b_i)^2 \right)^{\frac{1}{2}} \right\} \leq 2 e^{-\frac{x^2}{4}}.$$

If $E(X_i) = 0$ and $b_i = -a_i$ here, then $\frac{1}{2} \sum_{i=1}^n (a_i - b_i)^2 = \sum_{i=1}^n a_i^2 = \text{Var}(S_n)$ if and only if $X_i = a_i X'_i$, where $\Pr \{ X'_i = \pm 1 \} = \frac{1}{2}$, $1 \leq i \leq n$.

Thus if $S = \sum_{i} a_i X'_i$, where not all $a_i$ are zero and $\text{Var}(S) = \sum_{i} a_i^2 < \infty$, we get

$$\Pr \left\{ |S| \geq x (\text{Var}(S))^{\frac{1}{2}} \right\} \leq 2 e^{-\frac{x^2}{4}} \quad (x \geq 0).$$

The summation above may extend either to finitely many or to countably many terms. Let $S_1, S_2, \ldots, S_N$ be arbitrary sums of the above type: $S_k = \sum_{i} a_{kr} X'_i$, $\Pr \{ X'_i = \pm 1 \} = \frac{1}{2}$, $1 \leq k \leq N$, where $X'_k$ and $X'_i$ can be dependent when $k \neq i$. Then by the inequality (3) we obtain the following analog of Lemma A for any $C > 1$ and $N \geq 1$,

$$\Pr \left\{ \sup_{1 \leq k \leq N} |S_k| \geq (2C \log N)^{\frac{1}{2}} \sup_{1 \leq k \leq N} (\text{Var}(S_k))^{\frac{1}{2}} \right\} \leq \sum_{k=1}^N \Pr \left\{ |S_k| \geq (2C \log N \text{Var}(S_k))^{\frac{1}{2}} \right\} \leq 2Ne^{-C \log N} = 2N^{1-C}. \quad (4)$$

Lemma A easily implies that the time lags (2) are uniformly small if $m$ is large enough.

Lemma B. For any $K > 0$, $C > 1$, and for any $m \geq m_0(C)$, we have

$$\Pr \left\{ \sup_{0 \leq k+2m \leq K} |T_{m+1}(k)2^{-2^m(m+1)} - k2^{-2m}| \geq \left( \frac{3}{2} CK \log^2 K \right)^{\frac{1}{2}} m^{\frac{1}{2}} 2^{-m} \right\} \leq 2(K2^{2m})^{1-C},$$

where $\log^* x = \max \{ 1, \log x \}$.

This lemma and the refinement property (1) implies the uniform closeness of two consecutive approximations if $m$ is large enough.
Lemma C. For any $K > 0$, $C > 1$, and for any $m \geq m_1(C)$, we have
\[
P \left\{ \sup_{0 \leq k 2^{-2m} \leq K} |\tilde{B}_{m+1}(k2^{-2m}) - \tilde{B}_m(k2^{-2m})| \geq K^\frac{4}{3} (\log_* K)^{\frac{4}{3}} m 2^{-\frac{m}{3}} \right\} \leq 3(K2^{2m})^{1-C},
\]
where $K_* = \max\{1, K\}$.

Based on this lemma, it is not difficult to show the following convergence result.

Theorem A. The shrunken RWs $\tilde{B}_m(t)$ ($t \geq 0, m = 0, 1, 2, \ldots$) almost surely uniformly converge to a Wiener process $W(t)$ ($t \geq 0$) on any compact interval $[0, K]$, $K > 0$. For any $K > 0$, $C \geq 3/2$, and for any $m \geq m_2(C)$, we have
\[
P \left\{ \sup_{0 \leq t \leq K} |W(t) - \tilde{B}_m(t)| \geq K^\frac{2}{3} (\log_* K)^{\frac{2}{3}} m 2^{-\frac{m}{3}} \right\} \leq 6(K2^{2m})^{1-C}.
\]

Now taking $C = 3$ in Theorem A and using the Borel–Cantelli lemma, we get
\[
\sup_{0 \leq t \leq K} |W(t) - \tilde{B}_m(t)| < O(1)m 2^{-\frac{m}{3}} \quad \text{a.s.} \quad (m \to \infty)
\]
and
\[
\sup_{0 \leq t \leq K} |W(t) - \tilde{B}_m(t)| < K^\frac{2}{3} (\log K)^{\frac{2}{3}} \quad \text{a.s.} \quad (K \to \infty)
\]
for any $m$ large enough, $m \geq m_2(3)$.

Next we are going to study the properties of another nested sequence of random walks, obtained by Skorohod embedding. This sequence is not identical, though asymptotically equivalent to the above RW construction, cf. [11 Theorem 4]. Given a Wiener process $W$, first we define the stopping times which yield the Skorohod embedded process $B_m(k2^{-2m})$ into $W$. For every $m \geq 0$ let $s_m(0) = 0$ and
\[
s_m(k+1) = \inf \{ s : s > s_m(k), |W(s) - W(s_m(k))| = 2^{-m} \} \quad (k \geq 0). \tag{5}
\]
With these stopping times the embedded process by definition is
\[
B_m(k2^{-2m}) = W(s_m(k)) \quad (m \geq 0, k \geq 0). \tag{6}
\]
This definition of $B_m$ can be extended to any real $t \geq 0$ by pathwise linear interpolation. The next lemma describes some useful facts about the relationship between $\tilde{B}_m$ and $B_m$. These follow from [11 Lemmas 5,7 and Theorem 4], with some minor modifications.

In general, roughly saying, $\tilde{B}_m$ is more useful when someone wants to generate stochastic processes from scratch, while $B_m$ is more advantageous when someone needs a discrete approximation of given processes, like in the case of stochastic integration.

Lemma D. For any $C \geq 3/2$, $K > 0$, take the following subset of the sample space:
\[
A_m = \left\{ \sup_{n > m} \sup_{0 \leq k 2^{-2m} \leq K} |2^{-2m}T_{m,n}(k) - k2^{-2m}| < 6(CK_* \log_* K)^{\frac{2}{3}} m^{\frac{2}{3}} 2^{-m} \right\}, \tag{7}
\]
where \( T_{m,n}(k) = T_n \circ T_{n-1} \circ \cdots \circ T_m(k) \) for \( n > m \geq 0 \) and \( k \geq 0 \). Then for any \( m \geq m_3(C) \),
\[
P\{A^c_m\} \leq 4(K2^{2m})^{1-C}.
\]
Moreover, \( \lim_{n \to \infty} 2^{-2n}T_{m,n}(k) = t_m(k) \) exists almost surely and on the set \( A_m \) we have
\[
\tilde{B}_m(k2^{-2m}) = W(t_m(k)) \quad (0 \leq k2^{-2m} \leq K),
\]
cf. (6). Further, on \( A_m \) except for a zero probability subset, \( s_m(k) = t_m(k) \) and
\[
\sup_{0 \leq k2^{-2m} \leq K} |s_m(k) - k2^{-2m}| \leq 6(K_4 \log_+ K)^{\frac{1}{2}}m^{\frac{1}{2}}2^{-m} \quad (m \geq m_3(C)).
\]

If the Wiener process is built by the RW construction described above using a sequence \( \tilde{B}_m \) \((m \geq 0)\) of nested RWs and then one constructs the Skorohod embedded RWs \( B_m \) \((m \geq 0)\), it is natural to ask what the approximating properties of the latter are. The answer described by the next theorem is that they are essentially the same as the ones of \( \tilde{B}_m \), cf. Theorem A.

**Theorem 1.** For every \( K > 0, C \geq 3/2 \) and \( m \geq m_3(C) \) we have
\[
P\left\{ \sup_{0 \leq t \leq K} |W(t) - B_m(t)| \geq K^{\frac{1}{2}}(\log_+ K)^{\frac{1}{2}}m^{2^{-\frac{1}{4}}} \right\} \leq 10(K2^{2m})^{1-C}.
\]

**Proof.** By the triangle inequality,
\[
\sup_{0 \leq t \leq K} |W(t) - B_m(t)| \leq \sup_{0 \leq t \leq K} |W(t) - \tilde{B}_m(t)| + \sup_{0 \leq t \leq K} |\tilde{B}_m(t) - B_m(t)|.
\]
By Lemma [D] and equation (6), on the set \( A_m \) defined by (7) we have
\[
\tilde{B}_m(k2^{-2m}) = W(s_m(k)) = B_m(k2^{-2m}),
\]
extcept for a zero probability subset when \( m \geq m_3(C) \). Since both \( \tilde{B}_m(t) \) and \( B_m(t) \) are obtained by pathwise linear interpolation based on the vertices at \( k2^{-2m} \in [0,K] \), they are identical on \( A_m \), except for a zero probability subset of it when \( m \geq m_3(C) \). Thus
\[
P\left\{ \sup_{0 \leq t \leq K} |W(t) - B_m(t)| \geq K^{\frac{1}{2}}(\log_+ K)^{\frac{1}{2}}m^{2^{-\frac{1}{4}}} \right\}
\]
\[
\leq P\{A^c_m\} + P\left\{ \sup_{0 \leq t \leq K} |W(t) - \tilde{B}_m(t)| \geq K^{\frac{1}{2}}(\log_+ K)^{\frac{1}{2}}m^{2^{-\frac{1}{4}}} \right\}
\]
Then by Theorem A and Lemma D we get the statement of the theorem. \( \square \)

### 3 The basic approximation results

Beside the RW construction of standard Brownian motion, the other main tool applied in this paper is a theorem of Dambis (1965) and Dubins–Schwarz (1965) and an extension of it, cf. Theorems [B] and [C] below. Briefly saying, these theorems state that any continuous local martingale \((M(t), t \geq 0)\) can be transformed into a standard Brownian motion by time-change. Then somewhat loosely speaking, the resulting Brownian motion takes on the same values in
the same order as \( M(t) \), only the corresponding time instants may differ. These
and other necessary matters about continuous local martingales will be taken
from and discussed in the style of \([10]\) in the sequel.

Below it is supposed that an increasing family of sub-\(\sigma\)-algebras \( (\mathcal{F}_t, t \geq 0) \) is given in the probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and the given continuous local
martingale \( M \) is adapted to it.

In the case of a continuous local martingale \( M(t) \) vanishing at 0 its quadratic
variation \( \langle M, M \rangle_t \) is a process with almost surely continuous and non-decreasing
sample paths vanishing at 0. This will be one of the two time-changes we are
going to use in the sequel. The other one is a quasi-inverse of the quadratic
variation:

\[
T_s = \inf \{ t : \langle M, M \rangle_t > s \},
\]

where \( \inf(\emptyset) = \infty \) by definition. Then the sample paths of the process \( T_s \) are
almost surely increasing, but only right-continuous, since such a path has a jump
at any value where the quadratic variation has a constant level-stretch. Beside
this, \( T_s \) may be infinite valued. The duality between the two time-changes is
expressed by

\[
\langle M, M \rangle_s = \inf \{ t : T_t > s \}. \tag{9}
\]

Observe that \( T_s \) cannot have constant level-stretches since this would imply jumps for \( \langle M, M \rangle_t \). Also the continuity of \( \langle M, M \rangle_t \)
gives that \( \langle M, M \rangle_t = s \) for \( s \geq 0 \), while we have only \( T_{\langle M, M \rangle_t} \geq t \)
(\( t \geq 0 \)) in the opposite direction. It is clear that

\[
\langle M, M \rangle_t < s \implies t < T_s, \quad \text{but} \quad t < T_s \implies \langle M, M \rangle_t \leq s, \tag{10}
\]

while

\[
\langle M, M \rangle_t \leq s \iff t \leq \tilde{T}_s, \quad \text{t \leq \tilde{T}_s}, \tag{11}
\]

respectively.

**Theorem B.** \([10], V.1.6, p.181\) If \( M \) is a continuous \((\mathcal{F}_t)\)-local martingale
vanishing at 0 and such that \( \langle M, M \rangle_\infty \leq \infty \) a.s., then \( W(s) = M(T_s) \) is an
\((\mathcal{F}_{\tilde{T}_s})\)-Brownian motion and \( M(t) = W(\langle M, M \rangle_t) \).

Similar statement is true when \( \langle M, M \rangle_\infty < \infty \) is possible. Note that on the
set \( \{ \langle M, M \rangle_\infty < \infty \} \) the limit \( M(\infty) = \lim_{t \to \infty} M(t) \) exists with probability 1, 
cf. \([10], IV.2.16, p.131\).

**Theorem C.** \([10], V.1.7, p.182\) If \( M \) is a continuous \((\mathcal{F}_t)\)-local martingale
vanishing at 0 and such that \( \langle M, M \rangle_\infty < \infty \) with positive probability, then there
exists an enlargement \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})\) of \((\Omega, \mathcal{F}_t, \mathbb{P})\) and a Wiener process \( \tilde{\beta} \) on \( \tilde{\Omega} \)
independent of \( M \) such that the process

\[
W(s) = \begin{cases} M(T_s) & \text{if } s < \langle M, M \rangle_\infty, \\ M(\infty) + \tilde{\beta}(s - \langle M, M \rangle_\infty) & \text{if } s \geq \langle M, M \rangle_\infty \end{cases}
\]

is a standard Brownian motion and \( M(t) = W(\langle M, M \rangle_t) \) for \( t \geq 0 \).

From now on, \( W \) will always refer to the Wiener process obtained from \( M \)
by the above time-change, the so-called DDS Wiener process (or DDS Brownian
motion) of \( M \).

Now Skorohod-type stopping times can be defined for \( M \), similarly as for \( W \)
in \([5]\). For \( m \geq 0 \), let \( \tau_m(0) = 0 \) and

\[
\tau_m(k + 1) = \inf \{ t : t > \tau_m(k), |M(t) - M(\tau_m(k))| = 2^{-m} \} \quad (k \geq 0). \tag{12}
\]
The \((m+1)\)st stopping time sequence is a refinement of the \(m\)th in the sense that 
\((\tau_m(k))_{k=0}^{\infty}\) is a subsequence of \((\tau_{m+1}(j))_{j=0}^{\infty}\) so that for any \(k \geq 0\) there exist 
\(j_1\) and \(j_2\), \(\tau_{m+1}(j_1) = \tau_m(k)\) and \(\tau_{m+1}(j_2) = \tau_m(k+1)\), where the difference 
\(j_2 - j_1 \geq 2\), even.

**Lemma 1.** With the stopping times defined by \([\ref{12}]\) from a continuous local 
martingale \(M\) one can directly obtain the sequence of shrunken RWs that almost 
surely converges to the DDS Wiener process \(W\) of \(M\), cf. \([\ref{13}]\):

\[
B_m(k2^{-2m}) = W(s_m(k)) = M(\tau_m(k)), \quad s_m(k) = \langle M, M \rangle_{\tau_m(k)}
\]

but \(\tau_m(k) \leq T_{s_m(k)}\], where for \(m \geq 0\), the non-negative integer \(k\) is taking 
values (depending on \(\omega\)) until \(s_m(k) \leq \langle M, M \rangle_{\infty}\).

**Proof.** By Theorems \([\ref{11}]\) and \([\ref{12}]\) it follows that \(W(\langle M, M \rangle_{\tau_m(k)}) = M(\tau_m(k))\). 
This implies that \(s_m(k) \leq \langle M, M \rangle_{\tau_m(k)}\). Then consider first the case \(k = 1\). If 
\(s_m(1) < \langle M, M \rangle_{\tau_m(1)}\) held, then \(T_{s_m(1)} < \tau_m(1)\) would follow by \([\ref{14}]\), and this would 
lead to a contradiction because \(M(T_{s_m(1)}) = W(s_m(1)) = \pm 2^{-m}\). For 
values \(k > 1\), induction with a similar argument can show the statement of the 
lemma. \(\square\)

Below \(B_m\) will always denote the sequence of shrunken RWs defined by 
Lemma \([\ref{11}]\).

Our next objective is to show that the quadratic variation of \(M\) can be 
obtained as an almost sure limit of a point process related to the above stopping 
times that we will call a discrete quadratic variation process:

\[
N_m(t) = 2^{-2m} \#\{r : r > 0, \tau_m(r) \leq t\} = 2^{-2m} \#\{r : r > 0, s_m(r) \leq \langle M, M \rangle_{t}\} \quad (t \geq 0).
\]

Clearly, the paths of \(N_m\) are non-decreasing pure jump functions, the jumping 
times being exactly the stopping times \(\tau_m(k)\). Moreover, \(N_m(\tau_m(k)) = k2^{-2m}\) 
and the magnitudes of jumps are constant \(2^{−2m}\) when \(m\) is fixed.

**Lemma 2.** Let \(M\) be a continuous local martingale vanishing at 0, let \(\langle M, M \rangle\) 
be the quadratic variation, \(T\) be its quasi-inverse \([\ref{2}]\), and \(N_m\) be the discrete 
quadratic variation defined in \([\ref{13}]\). Fix \(K > 0\) and take a sequence \(a_m = O(m^{−2−2^2}K)\) 
with some \(\epsilon > 0\), where \(a_m \geq K \vee 1\) for any \(m \geq 1\) \((x \vee y = \max(x, y), x \wedge y = \min(x, y))\).

(a) \(\)Then for any \(C \geq 3/2\) and \(m \geq m_4(C)\) we have

\[
P \left\{ \sup_{0 \leq t \leq K} |\langle M, M \rangle_t \wedge a_m - N_m(t \wedge T_{a_m})| \geq 12(Ca_m \log a_m)\frac{2^{m+2}}{m+2} \right\} \leq 3(a_m2^{2m+1})^{1-C}.
\]

(b) \(\)Suppose that the quadratic variation satisfies the following tail-condition: a sequence \((a_m)\) fulfilling the above assumptions can be chosen so that

\[
P \left\{ \langle M, M \rangle_t \wedge a_m \right\} \leq D(t)m^{-1-\epsilon},
\]

where \(D(t)\) is some finite valued function of \(t \in \mathbb{R}_+\). Then for any \(C \geq 3/2\) and 
\(m \geq m_4(C)\) it follows that

\[
P \left\{ \sup_{0 \leq t \leq K} |\langle M, M \rangle_t - N_m(t)| \geq 12(Ca_m \log a_m)\frac{2^{m+2}}{m+2} \right\} \leq 3(a_m2^{2m+1})^{1-C} + D(K)m^{-1-\epsilon}.
\]
Strong approximation of continuous martingales

Proof. The basic idea of the proof is that the Skorohod stopping times of a Wiener process are asymptotically uniformly distributed as shown by [8], while the case of a continuous local martingale can be reduced to the former by the DDS representation, cf. Lemma 1.

Introduce the abbreviation $h_{a,m} = 11.1 \left(Ca_m \log a_m\right)^{\frac{1}{2}} m^{\frac{1}{2}} 2^{-m}$. Then $h_{a,m} = O(m^{-r}) \to 0$ as $m \to \infty$. We need a truncation here using the sequence $a_m$, since the quadratic variation $(M, M)_t$ is not a bounded random variable in general. By (11) and (13),

$$N_m(t \wedge T_{a_m}) = 2^{-2m} \# \{r: r > 0, \tau_m(r) \leq t \wedge T_{a_m}\} = 2^{-2m} \# \{r: r > 0, s_m(r) \leq (M, M)_t \wedge a_m\}.$$

On the event

$$A_{a,m} = \left\{ \sup_{0 \leq r 2^{-2m} \leq 2a_m} \left| s_m(r) - r 2^{-2m} \right| \leq h_{a,m} \right\},$$

if $r = \lfloor (M, M)_t \wedge a_m + h_{a,m} \rfloor 2^{2m} \rfloor + 1$, then $s_m(r) > (M, M)_t \wedge a_m$, so $s_m(r)$ is not included in $N_m(t \wedge T_{a_m})$. Observe here that $a_m + h_{a,m} + 2^{-2m} \leq 2a_m$ if $m$ is large enough, $m \geq m_4(C)$, where we also suppose that $m_4(C) \geq m_3(C)$ and $m_3(C)$ is defined by Lemma 15. This explains why the sup is taken for $r 2^{-2m} \leq 2a_m$ in the definition of $A_{a,m}$. Similarly on $A_{a,m}$, if $r = \lfloor (M, M)_t \wedge a_m - h_{a,m} \rfloor 2^{2m} \rfloor$, then $s_m(r) \leq (M, M)_t \wedge a_m$, so $s_m(r)$ must be included in $N_m(t \wedge T_{a_m})$. Hence

$$(M, M)_t \wedge a_m - h_{a,m} - 2^{-2m} \leq N_m(t \wedge T_{a_m}) \leq (M, M)_t \wedge a_m + h_{a,m} + 2^{-2m},$$

(15)

for any $t \in [0, K]$ on $A_{a,m}$.

Now 6 $(C 2a_m \log (2a_m))^{\frac{1}{2}} m^{\frac{1}{2}} 2^{-m} \leq 11.1 \left(Ca_m \log a_m\right)^{\frac{1}{2}} m^{\frac{1}{2}} 2^{-m} = h_{a,m}$, since $\log (2a_m) \leq (1 + \log 2) \log a_m$. Hence it follows by Lemma 15 that

$$\mathbb{P} \left\{ A_{a,m}^c \right\} \leq 4(2a_m 2^{2m})^{1-C} \leq 3(a_m 2^{2m})^{1-C},$$

when $C \geq 3/2$ and $m \geq m_4(C)$. Noticing that $0.9(Ca_m \log a_m)^{\frac{1}{2}} m^{\frac{1}{2}} 2^{-m} > 2^{-2m}$ for any $m \geq 1$, this and (15) prove (a).

Part (b) follows from (a), the inequality

$$| (M, M)_t - N_m(t) | \leq |(M, M)_t - (M, M)_t \wedge a_m| + |(M, M)_t \wedge a_m - N_m(t \wedge T_{a_m})| + |N_m(t \wedge T_{a_m}) - N_m(t)|,$$

(16)

and from the following simple relationships between events:

$$\{ (M, M)_t \wedge a_m \neq (M, M)_t \} = \{ (M, M)_t \geq a_m \}$$

and

$$\{ N_m(t \wedge T_{a_m}) \neq N_m(t) \} = \{ t > T_{a_m} \} = \{ (M, M)_t > a_m \},$$

cf. (11).
Theorem 2. Using the same notations as in Lemma \( \mathcal{2} \) and taking a sequence \( (c_m) \) increasing to \( \infty \) arbitrary slowly, we have

\[
\sup_{0 \leq t \leq K} |\langle M, M \rangle_t - N_m(t)| < c_m m^{\frac{4}{5}} 2^{-m} \quad \text{a.s.} \quad (m \to \infty).
\]

Under the condition of Remark \( \mathcal{1} \) we also have

\[
\sup_{0 \leq t \leq K} |\langle M, M \rangle_t - N_m(t)| < g(K)^{\frac{4}{5}} (\log g(K))^{\frac{1}{2}} \quad \text{a.s.} \quad (K \to \infty)
\]

for any \( K > 0 \), \( C \geq 3/2 \) and \( m \geq m_4(C) \).

Proof. To show the first statement take e.g. \( C = 3/2 \) and \( a_m = c_m \) in Lemma \( \mathcal{2} \) (a). Consider the inequality \( \mathcal{10} \). Since \( \langle M, M \rangle \) is finite-valued and \( c_m \to \infty \), if \( m \) is large enough, depending on \( \omega \), \( \langle M, M \rangle < a_m \) holds and then \( t < T_{a_m} \) holds as well by \( \mathcal{10} \). These remarks show that the first and the third terms on the right hand side of inequality \( \mathcal{10} \) are zero if \( m \) is large enough. Further, statement of Lemma \( \mathcal{2} \) (a) can be applied to the second term. This, with the Borel–Cantelli lemma, proves the first statement of the theorem.

The second statement of theorem follows similarly from Lemma \( \mathcal{2} \) (a) by the Borel–Cantelli lemma, taking \( C = 3 \) and \( a_m = g(K) \). \( \square \)

Lemma 3. Let \( M \) be a continuous local martingale vanishing at \( 0 \), let \( \langle M, M \rangle \) be the quadratic variation and \( T \) be its quasi-inverse \( \mathcal{3} \). Denote by \( B_m \) the sequence of shrunken RWs embedded into \( M \) by Lemma \( \mathcal{3} \). Fix \( K > 0 \) and take a sequence \( a_m = O(m^{-\frac{7}{4} - \epsilon} 2^{m})K \) with some \( \epsilon > 0 \), where \( a_m \geq K \vee 1 \) for any \( m \geq 1 \).

(a) Then for any \( C \geq 3/2 \) and \( m \geq m_3(C) \) we have

\[
P \left\{ \sup_{0 \leq t \leq K} |M(t \wedge T_{a_m}) - B_m(\langle M, M \rangle_t \wedge a_m)| \geq a_m^{\frac{7}{4}} (\log a_m)^{\frac{1}{2}} m 2^{-m} \right\} \leq 10(a_m 2^{m})^{1-C}.
\]

(b) Under the tail-condition \( \mathcal{12} \), for any \( C \geq 3/2 \) and \( m \geq m_3(C) \) it follows that

\[
P \left\{ \sup_{0 \leq t \leq K} |M(t) - B_m(\langle M, M \rangle_t)| \geq a_m^{\frac{7}{4}} (\log a_m)^{\frac{1}{2}} m 2^{-m} \right\} \leq 10(a_m 2^{m})^{1-C} + D(K)m^{1-C}.
\]
Proof. First, take the DDS Wiener process $W(s)$ obtained from $M(t)$ by the time-change $T_s$ as described by Theorems [3] and [C]. Since below we are going to use $W(s)$ and also the time change $T_s$ only for arguments $s \leq (M, M)_\infty$, we can always assume that $W(s) = M(T_s)$ and $M(t) = W((M, M)_t)$, irrespective of the fact whether $(M, M)_\infty = \infty$ or not. Second, define the nested sequence of shrunken RWs $B_m(s)$ by Lemma [I]. Then a quasi-inverse time-change $(M, M)_t$ is applied to $B_m(s)$ that gives $B_m((M, M)_t)$ which will be the sequence of time-changed shrunken RWs approximating $M(t)$.

Since $T_s$ may have jumps, we get that

\[
\sup_{0 \leq t \leq K} |M(t) - B_m((M, M)_t)| \geq \sup_{0 \leq s \leq (M, M)_K} |M(T_s) - B_m((M, M)_{T_s})| = \sup_{0 \leq s \leq (M, M)_K} |W(s) - B_m(s)|. \tag{17}
\]

Recalling however that the intervals of constancy are the same for $M(t)$ and for $(M, M)_t$, [10] IV (1.13), p.125, there is in fact equality in (17). To go on, we need a truncation using the sequence $a_m$, since the quadratic variation $(M, M)_t$ is not a bounded random variable in general. Then (17) (with equality as explained above) and (11) imply

\[
\sup_{0 \leq t \leq K} |M(t \wedge T_{a_m}) - B_m((M, M)_{t \wedge a_m})| = \sup_{0 \leq s \leq (M, M)_K} |M(T_s \wedge T_{a_m}) - B_m((M, M)_{T_s \wedge a_m})| = \sup_{0 \leq s \leq a_m \wedge (M, M)_K} |W(s) - B_m(s)| \leq \sup_{0 \leq s \leq a_m} |W(s) - B_m(s)|.
\]

Hence by Theorem [1] with $m \geq m_2(C)$,

\[
P \left\{ \sup_{0 \leq t \leq K} |M(t \wedge T_{a_m}) - B_m((M, M)_{t \wedge a_m})| \geq a_m^4 (\log a_m)^{3/2} m 2^{-\frac{s}{2}} \right\} = \sup_{0 \leq s \leq a_m} |W(s) - B_m(s)| \geq a_m^4 (\log a_m)^{3/2} m 2^{-\frac{s}{2}} \leq 10(a_m 2^{2m})^{1-C}.
\]

This proves (a).

To show (b) it is enough to consider the inequality

\[
\sup_{0 \leq t \leq K} |M(t) - B_m((M, M)_t)| \leq \sup_{0 \leq t \leq K} |M(t) - M(t \wedge T_{a_m})| + \sup_{0 \leq t \leq K} |M(t \wedge T_{a_m}) - B_m((M, M)_{t \wedge a_m})| + \sup_{0 \leq t \leq K} |B_m((M, M)_{t \wedge a_m}) - B_m((M, M)_t)|.
\]

From this point the proof is similar to the proof of Lemma [2] (b).

Kiefer [5] proved in the Brownian case $M = W$ that using Skorohod embedding one cannot embed a standardized RW into $W$ with convergence rate better than $O(1)n^{-\frac{s}{3}}(\log n)^{s}(\log \log n)^{s}$, where $n$ is the number of points used.
in the approximation. Since the next theorem gives a rate of convergence \( O(1)n^{-\frac{7}{14}}\log n \) (the number of points used is \( n = K2^{2m} \)), this rate is close to the best we can have with a Skorohod-type embedding. The same remark is valid for Theorem 4 below.

**Theorem 3.** Applying the same notations as in Lemma 3 and taking a sequence \((c_m)\) increasing to \( \infty \) arbitrary slowly, we have

\[
\sup_{0 \leq t \leq K} |M(t) - B_m(\langle M, M \rangle_t)| < c_m m^{2 - \frac{14}{13}} \quad \text{a.s.} \quad (m \to \infty).
\]

Under the condition of Remark 1, we also have

\[
\sup_{0 \leq t \leq K} |M(t) - B_m(\langle M, M \rangle_t)| < g(K)^\frac{1}{2} (\log g(K))^{\frac{3}{2}} \quad \text{a.s.} \quad (K \to \infty)
\]

for any \( m \) large enough, \( m \geq m_3(3) \).

**Proof.** The statements follow from Lemma 3 in a similar way as Theorem 2 followed from Lemma 2.

We mention that when \( M \) is a continuous local martingale vanishing at 0 and there is a deterministic function \( f \) on \( \mathbb{R}_+ \) such that \( \langle M, M \rangle_t = f(t) \) a.s., then it follows that \( M \) is Gaussian and has independent increments, see [10, V (1.14), p.186]. Conversely, if \( M \) is a continuous Gaussian martingale, then \( \langle M, M \rangle_t = f(t) \) a.s., see [10, IV (1.35), p.133]. In this case the “twist and shrink” construction of Brownian motion described in Section 2 can be extended to a construction of \( M(t) \) (or a simulation algorithm in practice). Namely, we have

\[
|\langle M(t) - \bar{B}_m(f(t)) \rangle| \leq O(1) m^{2 - \frac{14}{13}} \quad \text{a.s.} \quad (m \to \infty).
\]

Here \( \bar{B}_m(t) = 2^{-m} \tilde{S}_m(t2^{2m}) \) (\( m \geq 0 \)) denotes the nested sequence of the RW construction described in Section 2.

Combining the previous results one can replace \( \langle M, M \rangle_t \) by the discrete quadratic variation \( N_m(t) \) when approximating \( M(t) \) by time-changed shrunken RWs.

**Lemma 4.** Let \( M \) be a continuous local martingale vanishing at 0, let \( \langle M, M \rangle \) be the quadratic variation, \( T \) be its quasi-inverse, \( \tilde{T} \) and \( N_m \) be the discrete quadratic variation defined by (13). Denote by \( B_m \) the sequence of shrunken RWs embedded into \( M \) by Lemma 4. Fix \( K > 0 \) and take a sequence \( a_m = O(m^{-7-\epsilon}2^{2m})K \) with some \( \epsilon > 0 \), where \( a_m \geq K \lor 1 \) for any \( m \geq 1 \).

(a) Then for any \( C \geq 3/2 \) and \( m \geq m_5(C) \) we have

\[
P \left\{ \sup_{0 \leq s \leq K} |M(t \wedge T_{a_m}) - B_m(N_m(t \wedge T_{a_m})| \geq 2a_m (\log a_m)^\frac{\epsilon}{2} m^{2 - \frac{14}{13}} \right\} \leq 14(a_m 2^{2m})^{1-C}.
\]

(b) Under the tail-condition (14), for any \( C \geq 3/2 \) and \( m \geq m_5(C) \) it follows that

\[
P \left\{ \sup_{0 \leq t \leq K} |M(t) - B_m(N_m(t))| \geq 2a_m (\log a_m)^\frac{\epsilon}{2} m^{2 - \frac{14}{13}} \right\} \leq 14(a_m 2^{2m})^{1-C} + D(K)m^{-1-\epsilon}.
\]
**Proof.** For proving (a) we use the triangle inequality
\[
\sup_{0 \leq t \leq K} |M(t \wedge T_{a_m}) - B_m(N_m(t \wedge T_{a_m}))| \\
\leq \sup_{0 \leq t \leq K} |M(t \wedge T_{a_m}) - B_m((M, M)_t \wedge a_m)| \\
+ \sup_{0 \leq t \leq K} |B_m((M, M)_t \wedge a_m) - B_m(N_m(t \wedge T_{a_m}))|.
\]

Since the first term on the right hand side can be estimated by Theorem 3, we have to consider the second term. For \(m \geq 1\) introduce the abbreviation
\[
L_{a,m} = 13(C(a_m \log a_m)^{\frac{1}{2}}2^{-m} \geq 12(Ca_m \log a_m)^{\frac{1}{2}}m^{\frac{1}{2}}2^{-m} + 2^{-2m}).
\]

By Theorem 2 (a), with \(C \geq 3/2\) and \(m \geq m_4(C)\) it follows that
\[
\sup_{0 \leq t \leq K} |B_m((M, M)_t \wedge a_m) - B_m(N_m(t \wedge T_{a_m}))| \\
\leq \sup_{0 \leq t \leq K} |B_m((M, M)_t \wedge a_m)2^{2m}2^{-2m}) - B_m(N_m(t \wedge T_{a_m}))| + 2^{-m} \\
\leq \sup_{0 \leq k \leq 2^{2m}2^{-2m} \leq L_{a,m} \wedge a_m} \sup_{|r-k|2^{-2m} \leq L_{a,m}} |B_m(k2^{-2m}) - B_m(r2^{-2m})| + 2^{-m} \\
\leq \sup_{0 \leq k \leq 2^{-2m} \leq a_m \wedge a_m} \sup_{0 \leq r \leq 2^{-2m} \leq L_{a,m}} |B_{m}^{(k)}(r2^{-2m})| + 2^{-m},
\]
except for an event of probability \(\leq 3(a_m2^{2m})^{1-C}\), since the difference of a shrunken RW at two dyadic points equals the value of some shrunken RW \(B_{m}^{(k)}\) at a dyadic point.

Then we can apply estimate (11) with some \(C' > 1\) for the last expression:
\[
P \left\{ \sup_{0 \leq k \leq 2^{-2m} \leq a_m \wedge a_m} \sup_{0 \leq r \leq 2^{-2m} \leq L_{a,m}} |B_{m}^{(k)}(r2^{-2m})| \right\} \leq 2N^{1-C'},
\]
where \(N = [a_m2^{2m}][L_{a,m}2^{2m}]\) and \(\sup_{k,r} \text{Var} \left( B_{m}^{(k)}(r2^{-2m}) \right) \leq L_{a,m}\). Choose here \(C'\) so that \(1 - C' = \frac{2}{3}(1 - C)\). Then a simple computation shows that
\[
2N^{1-C'} \leq (a_m2^{2m})^{1-C}, \quad \text{also log} N \leq 8m \log, C \log a_m, \text{and}
\]
\[
\left(2C' \log N \sup_{k,r} \text{Var} \left( B_{m}^{(k)}(r2^{-2m}) \right) \right)^{\frac{1}{2}} + 2^{-m} \leq a_m^{\frac{1}{2}}(\log, a_m)^{\frac{1}{2}}m2^{-\frac{m}{2}}
\]
if \(m \geq m_5(C) \geq m_4(C)\). This argument and Theorem 3(a) applied to (18) give (a).

Statement (b) again follows from (a) in a similar way as in Lemma 2. \(\square\)

**Theorem 4.** Using the same notations as in Lemma 4 and taking a sequence \((c_m)\) increasing to \(\infty\) arbitrary slowly, we have
\[
\sup_{0 \leq t \leq K} |M(t) - B_m(N_m(t))| < c_mm^{\frac{1}{2}} \quad \text{a.s.} \quad (m \to \infty).
\]
Under the condition of Remark 4 we also have

\[
\sup_{0 \leq t \leq K} |M(t) - B_m(N_m(t))| < g(K)^4 (\log g(K))^2 \quad \text{a.s.,} \quad (K \to \infty)
\]

for any \( m \) large enough, \( m \geq m_5(3) \).

Proof. The statements follow from Lemma 4 in a similar way again as Theorem 2 followed from Lemma 2 \( \square \)

4 Independence of the DDS BM and \( \langle M, M \rangle \)

It is important both from theoretical and practical (e.g. simulation) points of view that the shrunken RW \( B_m \) and the corresponding discrete quadratic variation process \( N_m \) be independent when approximating \( M \) as in Theorem 4.

This leads to the question of independence of the DDS Brownian motion \( W \) and quadratic variation \( \langle M, M \rangle \) in the case of a continuous local martingale \( M \).

For, by Lemma 1 \( B_m \) depends only on \( W \) and, by (13), \( N_m \) is determined by \( \langle M, M \rangle \) alone. Conversely, if the processes \( B_m \) and \( N_m \) are independent for any \( m \) large enough, then so are \( W \) and \( \langle M, M \rangle \) too by Theorems 1 and 2. It will turn out from the next theorem that the basic notion in this respect is the symmetry of the increments of \( M \) given the past. Thus we will say that a stochastic process \( M(t) \) (\( t \geq 0 \)) is symmetrically evolving (or has symmetric increments given the past) if for any positive integer \( n \), reals \( 0 \leq s < t_1 < \cdots < t_n \) and Borel sets of the line \( U_1, \ldots, U_n \) we have

\[
P \{ \Gamma \mid \mathcal{F}_s^0 \} = P \{ \Gamma^- \mid \mathcal{F}_s^0 \},
\]  

where \( \Gamma = \{ M(t_1) - M(s) \in U_1, \ldots, M(t_n) - M(s) \in U_n \} \), \( \Gamma^- \) is the same, but each \( U_j \) replaced by \( -U_j \), and \( \mathcal{F}_s^0 = \sigma(M(u), 0 \leq u \leq s) \) is the filtration generated by the past of \( M \). If \( M(t) \) has finite expectation for any \( t \geq 0 \), then this condition expresses a very strong martingale property.

Condition (19) is clearly equivalent to the following one: for arbitrary positive integers \( n, j \), reals \( 0 \leq s_j < \cdots < s_1 \leq s < t_1 < \cdots < t_n \) and Borel sets \( V_1, \ldots, V_j, U_1, \ldots, U_n \) one has

\[
P \{ \Gamma \cap \Lambda \} = P \{ \Gamma^- \cap \Lambda \},
\]  

where \( \Gamma \) and \( \Gamma^- \) are defined above and \( \Lambda = \{ M(s_1) \in V_1, \ldots, M(s_j) \in V_j \} \).

Our Theorem 5 below is basically a reformulation of Ocone’s Theorem A of [8]. There it is shown that a continuous local martingale \( M \) is conditionally (w.r.t. to the sigma algebra generated by \( \langle M, M \rangle \)) Gaussian martingale if and only if it is \( J \)-invariant. Here \( J \)-invariance means that \( M \) and \( \int_0^t \alpha \, dM \) have the same law for any predictable process \( \alpha \) with range in \( \{-1, 1\} \). In fact, it is proved there too that \( J \)-invariance is equivalent to \( H \)-invariance which means that it is enough to consider deterministic integrands of the form \( \alpha^{(r)}(t) = I_{[0,r]}(t) - I_{(r,\infty)}(t) \). Moreover, Theorem B there extends the above result to càdlàg local martingales with symmetric jumps.

Dubins, Emery and Yor in [2] and Vostrikova and Yor in [13] gave shorter proofs with additional equivalent conditions in the case when \( M \) is a continuous martingale. In these references the equivalent condition of the independence
of the DDS BM and \( \langle M, M \rangle \) explicitly appears. Besides, in \cite{2}, the conjecture
that a continuous martingale \( M \) has the same law as its Lévy transform \( M = \int \text{sgn}(M) \, dM \) if and only if its DDS BM and \( \langle M, M \rangle \) are independent is proved to be equivalent to the conjecture that the Lévy transform is ergodic. Below we give a new, long, but elementary proof for any continuous local martingale \( M \) that the DDS BM and \( \langle M, M \rangle \) are independent if and only if \( M \) is symmetrically evolving.

**Theorem 5.** (a) If the Wiener process \( W(t) \) (\( t \geq 0 \)) and the non-decreasing, vanishing at 0, continuous stochastic process \( C(t) \) (\( t \geq 0 \)) are independent, then \( M(t) = W(C(t)) \) is a symmetrically evolving continuous local martingale vanishing at 0, with quadratic variation \( C \).

(b) Conversely, if \( M \) is a symmetrically evolving continuous local martingale, then its DDS Brownian motion \( W \) and its quadratic variation \( \langle M, M \rangle \) are independent processes.

**Proof.** To prove (a) suppose that \( W \) and \( C \) are independent. By \cite{10} V (1.5), p. 181], \( M(t) = W(C(t)) \) is a continuous local martingale. For simplicity, we will use only three sets in showing that \( M \) is symmetrically evolving, i.e. equation (20) holds, the generalization being straightforward:

\[
P \{ M(s_1) \in V_1, M(t_1) - M(s) \in U_1, M(t_2) - M(s) \in U_2 \} = \int \mathbb{P} \{ W(x_1) \in V_1 \}
\]

\[
\int_{U_1} \mathbb{P} \{ W(y_2) - W(y_1) \in U_2 - u \} \mathbb{P} \{ W(y_1) - W(y_0) \in du \}
\]

\[
\times \mathbb{P} \{ C(s_1) \in dx_1, C(s) \in dy_0, C(t_1) \in dy_1, C(t_2) \in dy_2 \}
\]

\[
= \int \mathbb{P} \{ W(x_1) \in V_1 \}
\]

\[
\int_{U_1} \mathbb{P} \{ W(y_1) - W(y_2) \in U_2 - u \} \mathbb{P} \{ W(y_0) - W(y_1) \in du \}
\]

\[
\times \mathbb{P} \{ C(s_1) \in dx_1, C(s) \in dy_0, C(t_1) \in dy_1, C(t_2) \in dy_2 \}
\]

\[
= \mathbb{P} \{ M(s_1) \in V_1, M(s) - M(t_1) \in U_1, M(s) - M(t_2) \in U_2 \} ,
\]

using the independence of \( B \) and \( C \) on one hand and the symmetry and independence of the increments of Brownian motion on the other hand.

For proving (b) we want to show that the sequences \( \tau_m(k) \) (\( k = 1, 2, \ldots \)) and \( M(\tau_m(j)) - M(\tau_m(j - 1)) \) (\( j = 1, 2, \ldots \)) are independent. Since \( N_m(t) \) depends only on the number of stopping times \( \tau_m(k) \leq t \), cf. \cite{13}, while the shrunk random walk \( B_m \) is determined by the steps \( 2^{-m}X_m(j) = M(\tau_m(j)) - M(\tau_m(j - 1)) \), cf. Lemma \cite{11} this would imply their independence and so the independence of \( W \) and \( \langle M, M \rangle \) too by Theorems \cite{4} and \cite{2}. For this it is enough to show that with arbitrary integers \( m \geq 0, n \geq 1, 0 \leq k < n \), reals \( t_1, \ldots , t_n \) and \( \delta_1 = \pm 2^{-m}, \ldots , \delta_n = \pm 2^{-m} \) (we fix these parameters for the remaining part of the proof) one has

\[
P \{ A \cap B_{\leq k} \cap B_{> k} \} = P \{ A \cap B_{\leq k} \cap B_{> k} \} , \quad (21)
\]

where \( A_{\leq k} = \bigcap_{r=1}^k \{ \tau_m(r) \leq t_r \} \), \( A_{> k} \) is similar, but with \( r = k + 1, \ldots , n \), \( A = A_{\leq k} \cap A_{> k} \), \( B_{\leq k} = \bigcap_{r=1}^k \{ M(\tau_m(r)) - M(\tau_m(r - 1)) = \delta_r \} \), \( B_{> k} \) is similar,
but with \( r = k + 1, \ldots, n \), \( B = B_{<k} \cap B_{>k} \), and finally \( B_{>k} \) is the same as \( B_{>k} \), but each \( \delta_j \) is replaced by \(-\delta_j\). For, if one can reflect all \( \delta_j \)s for \( k < j \leq n \) without changing the probability, then one has the same probability with arbitrary changed signs of \( \delta_j \)s too, since any such change can be reduced to a finite sequence of reflections of the above type. Let \( B^* \) be similar to \( B \), but with arbitrarily changed signs of \( \delta_j \)s. Then, as we said, (21) implies that \( \mathbb{P} \{ A \cap B \} = \mathbb{P} \{ A \cap B^* \} \). Since \( \mathbb{P} \{ B \} = \mathbb{P} \{ B^* \} \) by Lemma 4, the desired independence follows.

We will prove (21) in several steps.

Step 1. In condition (19) one can replace \( s \) by an arbitrary stopping time \( \sigma \) adapted to the filtration \( (\mathcal{F}_t^0) \): for any \( u_j \geq 0 \) (\( 1 \leq j \leq N \)),

\[
\mathbb{P} \{ F \mid \mathcal{F}_\sigma \} = \mathbb{P} \{ F^- \mid \mathcal{F}_\sigma \},
\]

(22)

where

\[
F = \bigcap_{j=1}^N \{ M(u_j + \sigma) - M(\sigma) \in U_j \},
\]

(23)

and \( F^- \) is the same, but each \( U_j \) replaced by \(-U_j\). This is somewhat similar to the optional stopping theorem, see [10, II (3.2), p.69]. Indeed, for discrete valued stopping times \( \sigma \) the statement is obvious, since then

\[
\mathbb{P} \{ F \mid \mathcal{F}_\sigma \} = \sum_{s_r} \mathbb{I} \{ \sigma = s_r \} \mathbb{P} \{ F \mid \mathcal{F}_{s_r} \},
\]

where \( \{s_r\} \) denotes the range of \( \sigma \), including possibly \( \infty \), and \( \mathbb{I} \{ S \} \) denotes the indicator of the set \( S \). For every stopping time \( \sigma \) there exists a decreasing sequence of discrete valued stopping times \( \sigma_i \) almost surely converging to \( \sigma \). Let us denote the events defined according to (23) for \( \sigma_i \) by \( F_i \) and \( F_i^- \), respectively. Further, denote the operators projecting \( L^2(\Omega) \) onto its subspace of random variables measurable w.r.t. \( \mathcal{F}_{\sigma_i}^0 \) and \( \mathcal{F}_{\sigma_i}^0 \) by \( P_i \) and \( P \), respectively. Then

\[
\| \mathbb{P} \{ F_i \mid \mathcal{F}_{\sigma_i}^0 \} - \mathbb{P} \{ F \mid \mathcal{F}_{\sigma_i}^0 \} \|_2 = \| P_i \mathbb{I} \{ F_i \} - P \mathbb{I} \{ F \} \|_2 \\
\leq \| P_i \mathbb{I} \{ F_i \} - P \mathbb{I} \{ F \} \|_2 + \| P \mathbb{I} \{ F \} - P \mathbb{I} \{ F \} \|_2 \\
\leq \| \mathbb{I} \{ F_i \} - \mathbb{I} \{ F \} \|_2 + \| \mathbb{E} (\mathbb{I} \{ F \} \mid \mathcal{F}_{\sigma_i}^0) - \mathbb{E} (\mathbb{I} \{ F \} \mid \mathcal{F}_{\sigma_i}^0) \|_2,
\]

which goes to \( 0 \) as \( i \to \infty \). Here we used that \( \mathbb{E} (\mathbb{I} \{ F \} \mid \mathcal{F}_{\sigma_i}^0) \) is a bounded, reversed-time martingale converging to \( \mathbb{E} (\mathbb{I} \{ F \} \mid \mathcal{F}_{\sigma_i}^0) \). Hence for any \( \epsilon > 0 \),

\[
\| \mathbb{P} \{ F \mid \mathcal{F}_{\sigma_i}^0 \} - \mathbb{P} \{ F^- \mid \mathcal{F}_{\sigma_i}^0 \} \|_2 \leq \| \mathbb{P} \{ F \mid \mathcal{F}_{\sigma_i}^0 \} - \mathbb{P} \{ F_i \mid \mathcal{F}_{\sigma_i}^0 \} \|_2 \\
+ \| \mathbb{P} \{ F^- \mid \mathcal{F}_{\sigma_i}^0 \} - \mathbb{P} \{ F_i^- \mid \mathcal{F}_{\sigma_i}^0 \} \|_2 < \epsilon,
\]

if \( i \) is large enough. This shows that the left hand side of the inequality is zero, so (22) holds.

Step 2. Then for arbitrary reals \( 0 \leq u_j < v_j \) and Borel sets \( U_j \) (\( 1 \leq j \leq N \)) we have

\[
\mathbb{P} \{ G \mid \mathcal{F}_{\sigma_i}^0 \} = \mathbb{P} \{ G^- \mid \mathcal{F}_{\sigma_i}^0 \},
\]

where

\[
G = \bigcap_{j=1}^N \{ M(v_j + \sigma) - M(u_j + \sigma) \in U_j \},
\]

(24)
and \(G^-\) is the same, but each \(U_j\) is replaced by \(-U_j\). For simplicity we prove this only for two factors, the general case being similar:

\[
\mathbb{P}\left\{M(v_1 + \sigma) - M(u_1 + \sigma) \in U_1, M(v_2 + \sigma) - M(u_2 + \sigma) \in U_2 \mid \mathcal{F}_\sigma^0\right\} = \int 1_{\{x_1 - x_2 \in U_1\}} 1_{\{x_3 - x_4 \in U_2\}} \times \mathbb{P}\left\{M(v_1 + \sigma) - M(u_1 + \sigma) - M(\sigma) \in dx_1, M(u_1 + \sigma) - M(\sigma) \in dx_2, M(v_2 + \sigma) - M(u_2 + \sigma) - M(\sigma) \in dx_3, M(u_2 + \sigma) - M(\sigma) \in dx_4 \mid \mathcal{F}_\sigma^0\right\} = \mathbb{P}\left\{M(u_1 + \sigma) - M(v_1 + \sigma) \in U_1, M(u_2 + \sigma) - M(v_2 + \sigma) \in U_2 \mid \mathcal{F}_\sigma^0\right\}.
\]

**Step 3.** Let \(\Delta \tau_m(i) = \tau_m(i) - \tau_m(i-1)\) and \(a \in [0, \infty)\). Consider the event

\[
S(a) = \{\Delta \tau_m(k + 1) \geq a\} = \left\{\inf\{u > 0 : |M(u + \tau_m(k)) - M(\tau_m(k))| \geq 2^{-m}\} > a\right\} = \left\{\sup_{0 < u \leq a} \{|M(u + \tau_m(k)) - M(\tau_m(k))| < 2^{-m}\} \right\} = \bigcap_{0 < u \leq a} \{|M(u + \tau_m(k)) - M(\tau_m(k))| < 2^{-m}\}. \quad (25)
\]

Introduce the set of dyadic numbers \(D_l = \{r2^{-l} : r \in \mathbb{Z}\}\) \((l \geq 0)\), and the events

\[
S_{j,l}(a) = \bigcap_{q \in D_l, 0 < q \leq a} \{|M(q + \tau_m(k)) - M(\tau_m(k))| \leq 2^{-m} - 2^{-j}\}. \quad (26)
\]

when \(j \geq m\). For \(j\) fixed, \((S_{j,l})_{l=0}^\infty\) is a decreasing sequence of events. Take

\[
S_j(a) = \bigcap_{l=0}^\infty S_{j,l}(a).
\]

Since \(S_{j,l}(a)\) is increasing with growing \(j\), so is \(S_j(a)\). Put

\[
S^*(a) = \bigcup_{j=m}^\infty S_j(a) = \bigcup_{j=m}^\infty \bigcap_{l=0}^\infty S_{j,l}(a).
\]

We want to show that \(S^*(a) = S(a)\), where the latter is defined by (25).

First fix an \(\omega \in S(a)\). (We suppress \(\omega\) in the notations below.) Then with this \(\omega\),

\[
\sup_{0 < u \leq a} \{|M(u + \tau_m(k)) - M(\tau_m(k))| < 2^{-m}\} =: s < 2^{-m}.
\]

If \(j > m\) is such that \(2^{-j} < 2^{-m}-s\), then \(\omega \in S_{j,l}(a)\) for any \(l \geq 0\). So \(\omega \in S_j(a)\) for each \(j\) large enough, consequently, \(\omega \in S^*(a)\).
Second, fix an \( \omega \notin S(a) \). Then there exists a real \( u_0 \leq a \) (depending on \( \omega \)) so that \( |M(u_0 + \tau_m(k)) - M(\tau_m(k))| = 2^{-m} \). Since the path of \( M \) is a continuous function, for any \( j \geq m \) there exists an \( \ell \geq 0 \) and \( q \in D_\ell, 0 < q \leq a \) such that

\[
|M(q + \tau_m(k)) - M(\tau_m(k))| > 2^{-m} - 2^{-j}.
\]

That is, \( \omega \notin S_j(a) \) if \( j \geq m \), thus \( \omega \notin S^*(a) \).

In other words, we have proved above that

\[
\{\Delta \tau_m(k + 1) > a\} = S(a) = \lim_{j \to \infty} \lim_{l \to \infty} S_{j,l}(a)
\]

\[
= \lim_{j \to \infty} \lim_{l \to \infty} \bigcap_{q \in D_\ell, 0 < q \leq a} \{ |M(q + \tau_m(k)) - M(\tau_m(k))| \leq 2^{-m} - 2^{-j} \}.
\]

Consequently, any event \( \{\Delta \tau_m(k + 1) > a\} = S(a) \) can be written in terms of monotonic sequences of intersections of finitely many events of the form

\[
\{|M(q + \tau_m(k)) - M(\tau_m(k))| \leq c\} \quad (c \geq 0).
\]

Moreover, such an approximation can be applied to \( \{\Delta \tau_m(k + 1) \in (a, b)\} = S(a) \setminus S(b) \) as well, with any \( 0 \leq a < b \).

Step 4. First, Steps 2 and 3 imply that for any \( a \geq 0 \),

\[
P \left\{ G \cap S_{j,l}(a) \mid \mathcal{F}_{\tau_m(k)}^0 \right\} = P \left\{ G^- \cap S_{j,l}(a) \mid \mathcal{F}_{\tau_m(k)}^0 \right\},
\]

because of the absolute value in definition \( (25) \) of \( S_{j,l}(a) \). Throughout Step 4 \( G \) and \( G^- \) are defined according to \( (24) \) with \( \sigma = \tau_m(k) \), but otherwise with arbitrary parameters, possibly different from case to case. Then taking limit as \( j \to \infty \) and \( l \to \infty \) it follows from Steps 2 and 3 that

\[
P \left\{ G \cap \{\Delta \tau_m(k + 1) > a_{k+1}\} \mid \mathcal{F}_{\tau_m(k)}^0 \right\} = P \left\{ G^- \cap \{\Delta \tau_m(k + 1) > a_{k+1}\} \mid \mathcal{F}_{\tau_m(k)}^0 \right\}.
\]

We want to extend this symmetry property by induction over \( i = k + 1, \ldots, n + 1 \). Taking arbitrary reals \( a_i \geq 0 \) and integers \( l_i \geq 0 \), \( r_i > 0 \), \( (k + 1 \leq i \leq n + 1) \) define the events

\[
B_i = \bigcap_{p=k+1}^i \{M(\tau_m(p)) - M(\tau_m(p - 1)) = \delta_p\},
\]

\[
H_i = \bigcap_{p=k+1}^i \{\Delta \tau_m(p) > a_p\},
\]

\[
K_{i,l}(r) = \bigcap_{p=k+1}^i \{\Delta \tau_m(p) \in [(r_p - 1)2^{-l} + r_p2^{-l}] \}
\]

\[
L_{i,l}(r) = \{ \delta_i \left( M(r_i2^{-l} + \cdots + r_{k+1}2^{-l} + \tau_m(k)) - M(r_i12^{-l} + \cdots + r_{k+1}2^{-l} + \tau_m(k)) \right) > 0 \},
\]

and \( B_i^-, L_i^- \) similarly, but multiplying each \( \delta_p \) by \( (-1) \). Suppose that we have already proved that

\[
P \left\{ G \cap B_{i-1}^- \cap H_i \mid \mathcal{F}_{\tau_m(k)}^0 \right\} = P \left\{ G^- \cap B_{i-1}^- \cap H_i \mid \mathcal{F}_{\tau_m(k)}^0 \right\}.
\]
where $B_k = \Omega$. Define the following event as a generalization of (26):

\[
S_{i,j,l}(a,r) = \bigcap_{q \in D_i, 0 < q \leq a} \left\{ \left| M(q + r_i2^{-i} + \cdots + r_{k+1}2^{-i} + \tau_m(k)) - M(r_i2^{-i} + \cdots + r_{k+1}2^{-i} + \tau_m(k)) \right| \leq 2^{-m} - 2^{-j} \right\},
\]

where $j \geq m$. Then by the induction hypothesis we get that

\[
P \left\{ G \cap B_{i-1} \cap H_i \cap \bigcup_{r_{k+1}, \ldots, r_i=1}^{2^{2i+1}} K_{i,l}(r) \cap L_{i,l}(r) \cap S_{i,j,l}(a+1,r) \mid \mathcal{F}_m^{0} \right\} = P \left\{ G^{-} \cap B_{i}^{-} \cap H_{i+1} \mid \mathcal{F}_m^{0} \right\},
\]

where we agree that when $r_p = 2^{2i} + 1$, the interval $[(r_p - 1)2^{-i}, r_p2^{-i}]$ in the definition of $K_{i,l}(r)$ is replaced by $[(r_p - 1)2^{-i}, \infty) = (2^{2i}, \infty]$. Notice here that the events in $K_{i,l}$ can be written in terms of a difference of events appearing in $H_i$, while the events in $L_{i,l}$ and $S_{i,j,l}$ are both of the type appearing in $G$, though $S_{i,j,l}$ is not affected by reflections because of the absolute values in its definition. Then taking limit as $j \to \infty$ and $l \to \infty$ it follows that

\[
P \left\{ G \cap B_{i} \cap H_{i+1} \mid \mathcal{F}_m^{0} \right\} = P \left\{ G^{-} \cap B_{i}^{-} \cap H_{i+1} \mid \mathcal{F}_m^{0} \right\}.
\]

This completes the induction.

Comparing the notations introduced in this step with the ones introduced above, observe that $B_{>k} = B_n$ and $H_{>k} = H_n = H_{n+1}$, if $a_{n+1} = 0$. Thus one obtains that

\[
P \left\{ B_{>k} \cap H_{>k} \mid \mathcal{F}_m^{0} \right\} = P \left\{ B_{>k}^{-} \cap H_{>k} \mid \mathcal{F}_m^{0} \right\}.
\]

\textbf{Step 5.} The result of Step 4 implies that

\[
P \left\{ A_{>k} \cap B_{>k} \mid \mathcal{F}_m^{0} \right\} = \int_{I} \left\{ \tau_m(k) + x_{k+1} \leq l_{k+1}, \ldots, \tau_m(k) + x_{k+1} + \cdots + x_n \leq t_n \right\} \times P \left\{ B_{>k} \cap \{ \Delta \tau_m(k + 1) \in dx_{k+1}, \ldots, \Delta \tau_m(n) \in dx_n \} \mid \mathcal{F}_m^{0} \right\}
= \int_{I} \left\{ \tau_m(k) + x_{k+1} \leq l_{k+1}, \ldots, \tau_m(k) + x_{k+1} + \cdots + x_n \leq t_n \right\} \times P \left\{ B_{>k}^{-} \cap \{ \Delta \tau_m(k + 1) \in dx_{k+1}, \ldots, \Delta \tau_m(n) \in dx_n \} \mid \mathcal{F}_m^{0} \right\}
= P \left\{ A_{>k} \cap B_{>k}^{-} \mid \mathcal{F}_m^{0} \right\}.
\]
Step 6. Finally, it follows from Step 5 that
\[
\mathbb{P} \{ A \cap B \} = \mathbb{E} \left( \mathbb{E} \left( I \{ A \leq k \cap B \leq k \} | \mathcal{F}^{0}_{\tau_{m}(k)} \right) \right).
\]
This proves (21), and so completes the proof of the theorem. □

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