RADIATIVE CORRECTIONS AS THE ORIGIN OF SPONTANEOUS SYMMETRY BREAKING

A thesis presented
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to
The Department of Physics
in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the subject of Physics
Harvard University
Cambridge, Massachusetts
May 1973
Abstract

Using a functional formalism, we investigate the effect of radiative corrections on the possibility of spontaneous symmetry breaking. We find that in some models, in particular, massless gauge theories with scalar mesons, the radiative corrections can induce spontaneous symmetry breaking, even though the classical approximation indicates that the vacuum is symmetric. Among the consequences of this phenomenon is a relationship between the masses of the scalar and vector mesons, predicting (for small coupling constants) that the scalar mesons are much lighter. We also apply our analysis to models in which the classical approximation indicates an asymmetric vacuum, including one in which our methods are particularly useful because the classical approximation does not completely specify the nature of the vacuum. It is possible to improve our analysis by the use of renormalization group methods; we do this for several models.
Acknowledgments

I would like to thank my thesis advisor, Professor Sidney Coleman, for his assistance and encouragement. I am grateful to him for showing me that the best way to solve a difficult problem is often to not do the obvious. It has been a privilege to have worked with him.

David Politzer has been of immeasurable assistance in this work; I have had innumerable enjoyable and fruitful conversations with him. In addition, he has helped me learn that two people working together can calculate Feynman diagrams with neither can calculate alone.

I have profitted greatly from discussions with many faculty members, especially Professors Thomas Appelquist, Sheldon Glashow, and Helen Quinn, as well as with Dr. Howard Georgi.

I have learned a great deal from my fellow graduate students. I especially wish to thank Jim Carazzone and Terry Goldman.

Teaching has been an enjoyable and rewarding experience; I thank Professors Paul Bamberg and Arthur Jaffe for the opportunity to have taught with them.

I would like to thank Alan Candiotti for the use of his electric typewriter; without it, this thesis would never have been finished.

Finally, I wish to thank my wife, Carolyn. Her love and understanding have helped me through many of the dark and discouraging moments of graduate study.
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I. Introduction

Spontaneous symmetry breaking has been of great importance in the study of elementary particles during the last decade, first because it explained why pions were almost massless\(^1\), and then because it explained why weak intermediate vector mesons did not have to be massless\(^2\). In all of its applications to particle physics, spontaneous symmetry breaking has been induced by writing the symmetric Lagrangian in a somewhat unnatural manner — with a negative mass-squared term. In this paper we will show that it is possible for a theory to have an asymmetric vacuum even when the Lagrangian does not contain any such unnatural terms; instead, the spontaneous symmetry breaking is induced by the quantum corrections to the classical field theory.\(^3\) In particular, theories which do not contain any mass terms may exhibit this phenomenon; one may consider that the theory takes this form in order to avoid the violent infrared divergences which would otherwise occur.

There are some interesting consequences when theories of this type display spontaneous symmetry breaking. The theory, originally formulated in terms of dimensionless parameters, can be rewritten to show explicit dependence on a dimensional quantity, namely the vacuum expectation value of one of the scalar fields. One of the dimensionless parameters can be eliminated, and one obtains a relationship among other quantities in the theory. In a gauge theory, this is generally a relationship among masses, predicting a scalar meson mass which is much less than the vector meson masses. We begin our discussion in Chapter II by describing a formalism which is particularly suited to our purposes. In Chapter III we apply our methods to the study of a particularly simple model, where we do not find evidence of spontaneous symmetry breaking. In Chapter IV we introduce more complicated theories, including some where the vacuum is asymmetric. All of the models considered in Chapters III and IV are without mass terms. In Chapter V we show how these can be embedded in the more general class of theories with mass terms (of either sign). We also give an example where our methods are particularly useful, even though the classical treatment of the theory already predicts spontaneous symmetry breaking. In Chapter VI we show how the methods of the renormalization group allow us to
improve the analysis of Chapters III and IV. We list our conclusions in Chapter VII. An appendix discusses the relationship between our results and a theorem of Georgi and Glashow.
II. Formalism

In the usual treatment of spontaneous symmetry breaking in quantum field theory, one assumes that by looking at the Lagrangian of a particular theory it is possible to determine whether or not the theory displays spontaneous symmetry breaking. That is, one defines the negative of the non-derivative terms in the Lagrangian to be a potential, assumes that all fields are constant in space-time in the vacuum state, and minimizes the potential as a function of the fields to determine the expectation value of the fields in the vacuum state. If the vacuum expectation value of some field is non-invariant under some symmetry of the Lagrangian, spontaneous symmetry breaking is said to occur.

In a classical field theory, this would be precisely the thing to do. In a quantum field theory, this procedure is approximate at best, since the quantized fields are non-commuting operators and one must be careful in manipulating them. To put things differently, the operator nature of the fields gives rise to internal loops in Feynman diagrams; that is, there are both radiative corrections to the interactions in the Lagrangian and new interactions which are not explicit in the Lagrangian. These effects are clearly relevant in determining the nature of the vacuum; the usual assumption is that their effect is only to make small quantitative changes and that the classical approximation gives a correct qualitative description of the vacuum. Since we are interested in considering cases where this assumption may be false, it is desirable to use a formalism which treats the classical approximation and the quantum corrections on the same basis.

Such a formalism is provided by the functional methods introduced by Schwinger and developed by Jona-Lasinio, whose treatment we follow\(^4\). These methods enable us to define an effective potential which includes all the interactions of the theory, both those explicit in the Lagrangian and those arising from quantum corrections. Furthermore, this effective potential is a function of classical fields, so it can be easily manipulated. Finally, its minima determine the nature of the ground state of the theory.

One begins by adding to the Lagrangian a coupling to external c-number sources,
one source for each field in the theory; that is,

\[ \mathcal{L}(\phi_i, \partial^\mu \phi_i) \rightarrow \mathcal{L}(\phi_i, \partial^\mu \phi_i) + \sum_i \phi_i(x) J_i(x), \]  

(2.1)

where the \( \phi_i \) represent all the fields of the theory, whatever their spin. Next one defines a functional \( W(J) \) in terms of the probability amplitude for the vacuum state in the far past to go into the vacuum state in the far future in the presence of the external sources:

\[ e^{iW(J)} = \langle 0^+ | 0^- \rangle. \]  

(2.2)

The importance of \( W(J) \) lies in the fact that it is the generating functional for the connected Green’s functions; that is, we can write

\[ W(J) = \sum_{n_1, n_2, \ldots, n_k} \frac{1}{n_1! n_2! \ldots n_k!} \int d^4x_1 d^4x_2 \ldots d^4x_{n_1} \ldots d^4w_{n_k} \]

\[ \times G^{(n_1, n_2, \ldots, n_k)}(x_1, x_2, \ldots, w_{n_k}) J_1(x_1) J_1(x_2) \ldots J_1(x_{n_1}) \ldots J_k(w_{n_k}). \]  

(2.3)

Here \( G^{(n_1, n_2, \ldots, n_k)}(x_1, x_2, \ldots, w_{n_k}) \) is the sum of all connected Feynman diagrams with \( n_1 \) external lines of type 1, \( n_2 \) external lines of type 2, etc.

Now one defines classical fields \( \Phi_{ic}(x) \) by

\[ \Phi_{ic}(x) = \frac{\delta W(J)}{\delta J_i(x)} = \frac{\langle 0^+ | \phi_i(x) | 0^- \rangle_J}{\langle 0^+ | 0^- \rangle_J}. \]  

(2.4)

Finally, we define the effective action, \( \Gamma(\Phi_c) \), by a functional Legendre transformation:

\[ \Gamma(\Phi_c) = W(J) - \sum_i \int d^4x J_i(x) \Phi_{ic}(x). \]  

(2.5)

From Eq. (2.5) we can immediately conclude that

\[ \frac{\delta \Gamma(\Phi_c)}{\delta \Phi_{ic}(x)} = -J_i(x). \]  

(2.6)

The effective action is also a generating functional; it can be written

\[ \Gamma(\Phi_c) = \sum_{n_1, n_2, \ldots, n_k} \frac{1}{n_1! n_2! \ldots n_k!} \int d^4x_1 d^4x_2 \ldots d^4x_{n_1} \ldots d^4w_{n_k} \]

\[ \times \Gamma^{(n_1, n_2, \ldots, n_k)}(x_1, x_2, \ldots, w_{n_k}) \Phi_{1c}(x_1) \Phi_{1c}(x_2) \ldots \Phi_{1c}(x_{n_1}) \ldots \Phi_{kc}(w_{n_k}). \]  

(2.7)
The \( \Gamma^{(n_1,n_2,\ldots,n_k)}(x_1,x_2,\ldots,w_{n_k}) \) are the IPI (one-particle-irreducible) Green’s functions, defined as the sum of all connected Feynman diagrams which cannot be disconnected by cutting a single internal line; these are evaluated without propagators on the external lines.

Although Eq. (2.7) is the usual expansion of the effective action, it is not the one best suited for studying the possibility of spontaneous symmetry breaking. A better expansion is obtained by expanding in powers of the external momenta about the point where all external momenta are zero; in other words, a Taylor series expansion about a constant value, \( \varphi_{ic} \), of the classical fields \( \Phi_{ic}(x) \). In this expansion we do not distinguish terms with different numbers of external scalar particles, but we do distinguish terms with different number of external particles with spin. Thus, for a theory containing a scalar field, \( \phi \), and a vector field, \( A^\mu \), this expansion would look like

\[
\Gamma(\Phi_c) = \int d^4x \left\{ -V(\varphi_c) + \frac{1}{2} \partial_\mu \Phi_c(x) \partial^\mu \Phi_c(x) Z(\varphi_c) \\
+ \partial_\mu \Phi_c(x) A^\mu_c(x) \Phi_c(x) G(\varphi_c) + \Phi_c(x)^2 A^\mu_c(x) A^\nu_c K(\varphi_c) + \cdots \right\}.
\]

(2.8)

Note that the coefficients in this expansion are functions, not functionals. The first term, \( V(\varphi_c) \), is called the effective potential; it is equal to the sum of all Feynman diagrams with only external scalar lines and with vanishing external momenta. Those diagrams without loops correspond to the interactions in the Lagrangian, while those with loops correspond to the quantum corrections.

Now we wish to see how spontaneous symmetry breaking is described in this formalism. We assume that the Lagrangian possesses a symmetry which is spontaneously broken if some field acquires a non-zero vacuum expectation value. From Eq. (2.4) we see that if the external sources vanish, the classical field is just the vacuum expectation value of the quantized field; this observation, plus Eq. (2.6), tells us that the condition for spontaneous symmetry breaking is

\[
\frac{\delta \Gamma(\Phi_c(x))}{\delta \Phi_c(x)} = 0, \quad \text{for } \Phi_c(x) \neq 0.
\]

(2.9)

Since spontaneous breaking of Poincaré invariance does not seem to be a physically
significant phenomenon, we will restrict ourselves to theories in which the vacuum
is invariant under translation. If the Lagrangian and the quantization procedure
are also Poincaré invariant, then the vacuum expectation value of the fields will be
constant in space-time, and Eq. (2.9) reduces to

$$\frac{dV(\varphi_c)}{d\varphi_c} = 0, \quad \text{for } \varphi_c \neq 0. \quad (2.10)$$

Thus we have obtained, as promised, a c-number function whose minimum deter-
mines the nature of the ground state. In fact, the effective potential can give us much
more information. If we compare Eqs. (2.7) and (2.8), we see that the $n$th derivative
of $V(\varphi_c)$, evaluated at $\varphi_c = 0$, is just the $n$-point IPI Green’s functions, evaluated
at zero external momenta. Of particular interest is the 2-point IPI Green’s function,
which is the inverse propagator; its value at zero momentum may be taken as the de-
definition of the mass. (It is not exactly the position of the pole in the propagator, but
we usually expect it to be close.) Thus, in a theory without spontaneous symmetry
breaking, we may write the scalar meson mass matrix as

$$m^2_{ij} = \frac{\partial^2 V}{\partial \varphi_i \partial \varphi_j} \bigg|_{\varphi_c=0}. \quad (2.11)$$

If spontaneous symmetry breaking does occur, then Eq. (2.11) no longer holds, since
the mass is defined for an isolated particle; that is, the inverse propagator must be
evaluated in the ground state. To do this, we make the usual redefinition of the scalar
fields; if $\langle \varphi \rangle$ is the vacuum expectation value of $\phi$, we define a new quantum field, $\phi'$,
and a new classical field, $\Phi'$, by

$$\phi'(x) = \phi(x) - \langle \varphi \rangle \quad (2.12)$$

and

$$\Phi'(x) = \Phi(x) - \langle \varphi \rangle. \quad (2.13)$$

The mass matrix is then given by

$$m^2_{ij} = \frac{\partial^2 V}{\partial \varphi'_i \partial \varphi'_j} \bigg|_{\varphi'_c=0} = \frac{\partial^2 V}{\partial \varphi_i \partial \varphi_j} \bigg|_{\varphi_c=\langle \varphi \rangle}. \quad (2.14)$$
There is an additional advantage of the functional formalism, which we will not make use of, but which is worth pointing out: Instead of coupling the external sources in Eq. (2.1) to the elementary fields of the theory, we could have coupled them to more complicated fields. We would then have obtained a set of classical fields which would not have simple relationships to the elementary quantum fields of the theory. This might be desirable in at least two different cases. First, it allows one to study the possibility of spontaneous symmetry breaking in a theory without elementary scalar fields, by having a compound field, rather than an elementary field, acquire a non-zero vacuum expectation value. Second, it has the possibility of describing strong interaction effects in terms of the phenomenological fields, even though these fields quite likely cannot be simply expressed in terms of the fundamental quantum fields.

Thus the study of spontaneous symmetry breaking is reduced to the calculation of the effective potential. Unfortunately, this is not so simply done; even if one accepts the validity of a calculation to all orders of perturbation theory, such a calculation requires summing an infinite number of Feynman diagrams, which is no mean feat. It is therefore necessary to find an approximation for the effective potential which will require the summation of only some subset of the relevant Feynman diagrams.

The first expansion to come to mind is that which is most often used in perturbation theory: expansion in powers of the coupling constants. However, there are two objections to using this method for the problem at hand. First, in theories with spontaneous symmetry breaking one commonly defines shifted fields, with a corresponding redefinition of the coupling constants. This can lead to confusion in comparing the order of diagrams calculated using different shifts. But this can be unraveled if one is careful; it certainly does not invalidate the method. The second objection is more basic: We will often be considering theories which appear to contain massless particles, for example, massless scalar electrodynamics. Two diagrams which contribute to the effective potential in scalar electrodynamics are shown in Fig. 1. The first is of order $e^4$, while the second is of order $e^{24}$. An expansion in powers of coupling constants assumes that the second is negligible in comparison with the first, while in fact it is
much more infrared divergent, and not at all negligible. This is not merely a fluke which occurs because we happen to have considered a theory with massless particles. Spontaneous symmetry breaking, like critical phenomena in many-body theory, is associated with correlations between widely separated points; the presence of massless particles, which can mediated long-range forces, is obviously of extreme importance. In fact, if one tried to study massless scalar electrodynamics and neglected diagrams such as that in Fig. 1b, one would obtain qualitatively different results.

A better approximation is the expansion by the number of loops in a diagram. First, let us show that such an expansion is unaffected by a shift of the fields. We introduce a parameter $b$ into the Lagrangian by writing

\[ \mathcal{L}(\phi_i, \partial^\mu \phi_i; b) = b^{-1} \mathcal{L}(\phi_i, \partial^\mu \phi_i). \]  

(2.15)

The power, $P$, of $b$ associated with a particular diagram will be

\[ P = I - V, \]  

(2.16)

where $I$ is the number of internal lines and $V$ is the number of vertices, since the vertices are obtained from the interaction Lagrangian while the propagators are obtained from the inverse of the free Lagrangian. The number of loops, $L$, is equal to the number of independent internal momenta. This is equal to the number of momenta ($I$), less the number of energy-momentum delta functions ($V$), but not counting the delta function corresponding to overall energy-momentum conservation. In other words

\[ L = I - V + 1 = P + 1. \]  

(2.17)

We see that the number of loops is determined by the power of a quantity which multiplies the whole Lagrangian, and does not depend on the details of how the Lagrangian was written; thus the loop expansion is unaffected by a redefinition of the fields.

It still remains to be seen whether the loop expansion is a good approximation. Certainly the appearance of high powers of $b$ does not make a diagram small, since $b$ must be set equal to one. However, many-loop diagrams must contain many powers...
of the coupling constants, so the loop expansion is at least as good as the coupling constant expansion. Also, diagrams such as that in Fig. 1b are included in the same order as that in Fig. 1a, so the second objection to the coupling constant expansion is avoided. (One may ask why certain two-loop graphs, such as that in Fig. 2, are any less important than that in Fig. 1b; the answer will appear presently.) We shall see as we go on that logarithms will appear in our calculations, and that the validity of the loop expansion will require not only that the coupling constants, but also the logarithms, be small. (Similarly, logarithms appear in the usual applications of the coupling constant expansion, where one expects the results to hold only when the logarithms are small.)

Finally, we note that the zero-loop (tree) approximation is equivalent to treating the theory classically.

At this point it is probably most instructive to consider a specific model in order to see how our methods work.
III. A Simple Example

Let us consider a simple, but illustrative, model: the theory of a single quartically self-coupled scalar field. The Lagrangian for the theory is

\[
L = \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4 + \frac{A}{2} (\partial_\mu \phi)^2 - \frac{B}{2} \phi^2 - \frac{C}{4!} \phi^4 ,
\]  

(3.1)

where the last three terms are counter-terms to be determined, order by order in the loop parameter, by the renormalization conditions. The coupling constant, \(\lambda\), may be either positive or negative. Ordinarily one forbids negative \(\lambda\) on the grounds that it leads to a potential which has no lower bound, but of course this is a statement only about the zero-loop approximation to the effective potential; the loop contributions may (or may not) be such as to put a lower bound on the effective potential, even for negative \(\lambda\).

The theory contains a discrete symmetry, namely \(\phi \rightarrow -\phi\). The conventional wisdom (for positive \(\lambda\)) is that the vacuum is symmetric if \(m^2\) is positive, and that spontaneous symmetry breaking occurs if \(m^2\) is negative. (Goldstone’s theorem does not apply, since there is no continuous symmetry.) However, the conventional wisdom does not say very much about the case where \(m^2\) vanishes, except to warn of the terrible infrared divergences which may arise. In fact, this case is not well defined until the renormalization conditions are specified. Let us choose the mass renormalization so that

\[
\left. \frac{d^2 V}{d\phi_c^2} \right|_{\phi_c = 0} = m^2 = 0 .
\]  

(3.2)

If the vacuum occurs when \(\phi_c = 0\), this means that the inverse propagator at zero external momentum vanishes; in other words, the theory really contains a massless particle. On the other hand, if spontaneous symmetry breaking occurs, this is not a condition on the physical inverse propagator and there is not necessarily a massless particle; one might ask why this should be considered a special case. The reason is that the point \(\phi_c = 0\) is a symmetric one, and the symmetric formulation of a theory is significant; for example, the unitary gauge of a Higgs theory may be useful for calculating amplitudes, but the renormalizable gauge is certainly better for understanding the underlying symmetries of the theory. In any case, let us consider the...
theory with $m^2 = 0$, and see whether spontaneous symmetry breaking occurs. (We still have not completely specified the theory, since the wave function and coupling constant renormalization conditions have not been stated, but these are best left for later, when their motivation will be more apparent.

One last task remains before we can begin calculating the effective potential: we must learn how to count $i$'s, minus signs, and factorials. A term $g\phi^n$ in the Lagrangian leads to a Feynman diagram with $n$ external $\phi$ lines and a value of $ig(n!)$. Remembering that the Lagrangian contains the negative of the potential, we see that the contribution to the effective potential from a diagram with $n$ external lines is

$$\frac{i}{n!}\varphi^n_c \times \text{(diagram)}. \quad (3.3)$$

In the zero-loop approximation only one diagram, that shown in Fig. 3, contributes to the effective potential. Its contribution is, of course,

$$V_{\text{zero-loop}} = \frac{\lambda}{4!}\varphi^4_c, \quad (3.4)$$

In the next order we have the infinite series of diagrams shown in Fig. 4a, as well as the diagrams of Fig. 4b, which arise from the one-loop mass and coupling constant counter-terms. The latter clearly do not contain any loops, but they are to be counted here because we are really expanding in powers of the loop-counting parameter, $b$, and these terms are of order $b^0$.

In calculating the contribution from the diagrams of Fig. 4a, we must keep track of certain combinatoric factors. To understand these, let the external lines have small non-zero momenta, $\epsilon_1, \epsilon_2, \ldots, \epsilon_{2n}$. (Ultimately, we will let these momenta go to zero.) We now count the one-loop diagrams with $2n$ external lines. First, we consider the case where $n \geq 3$. We note the following:

1) There are $(2n)!$ ways of arranging the external momenta.

2) Interchanging the external moment at a vertex does not give a new diagram, so we have a factor of $(\frac{1}{2})^n$.

3) Rotating or reflecting a diagram does not give a new diagram, so we have another factor of $1/(2n)$.
The contributions from one-loop diagrams with \(2n\) external lines \((n \geq 3)\) is thus

\[
\left[ (2n)! \left( \frac{1}{2} \right)^n \right] \left[ \frac{i}{(2n)!} \varphi_c^{2n} \right] (\text{diagram}) = \frac{i}{2n} \left( \frac{\varphi_c^2}{2} \right)^n (\text{diagram}). \tag{3.5}
\]

If \(n\) equals 1 or 2, point (3) must be modified. For \(n = 1\), the diagram cannot be reflected or rotated; for \(n = 2\), reflection and rotation are the same, so there is only a factor of 1/2, rather than 1/4. However, in both of these cases the Feynman rules include an extra factor of 1/2.

We can now write an expression for the one-loop approximation to the effective potential:

\[
V_{\text{one-loop approx}} = \frac{\lambda}{4!} \varphi_c^4 + \sum_{n=1}^{\infty} \left[ \frac{i}{2n} \left( \frac{\lambda \varphi_c^2/2}{k^2 + i\epsilon} \right)^n \right] + \frac{B^{(1)}}{2} \varphi_c^2 + \frac{C^{(1)}}{4!} \varphi_c^4. \tag{3.6}
\]

We notice that the integrals for \(n \geq 2\) are infrared divergent. We also notice that the infinite sum can be done, yielding

\[
V_{\text{one-loop approx}} = \frac{\lambda}{4!} \varphi_c^4 + \frac{i}{2} \int \frac{d^4k}{(2\pi)^4} \log \left( 1 - \frac{\lambda \varphi_c^2/2}{k^2 + i\epsilon} \right) + \frac{B^{(1)}}{2} \varphi_c^2 + \frac{C^{(1)}}{4!} \varphi_c^4. \tag{3.7}
\]

The infrared divergence has disappeared, being replaced by a logarithmic singularity at the origin of classical field space. It is not too surprising that this should occur, since we expect infrared divergences only if the vacuum is at \(\varphi_c = 0\). If spontaneous symmetry breaking occurs, then \(\phi\) does not remain massless, and there is no reason to expect infrared problems.

We do the integral by rotating into Euclidean space and cutting off the integral at \(k^2 = \Lambda^2\). We obtain

\[
V_{\text{one-loop approx}} = \frac{\lambda}{4!} \varphi_c^4 + \frac{1}{64\pi^2} \left\{ \lambda \varphi_c^2 \Lambda^2 + \frac{\lambda \varphi_c^4}{4} \left[ \log \left( \frac{\lambda \varphi_c^2}{2\Lambda^2} - 1 \right) \right] \right\} + \frac{B^{(1)}}{2} \varphi_c^2 + \frac{C^{(1)}}{4!} \varphi_c^4. \tag{3.8}
\]

We must now determine the value of the renormalization counter-terms. The mass renormalization condition was given in Eq. (3.2). From it we deduce that

\[
B^{(1)} = \frac{-\lambda \Lambda^2}{32\pi^2}. \tag{3.9}
\]
Now we come to the coupling constant renormalization condition, which we have left unspecified so far. Two possibilities are

\[
\frac{d^4V}{d\varphi_c^4} \bigg|_{\varphi_c=0} = \lambda \quad (3.10a)
\]

and

\[
\frac{4!}{\varphi_c^4} V \bigg|_{\varphi_c=0} = \lambda. \quad (3.10b)
\]

Unfortunately, neither of these is acceptable, because of the logarithmic singularity at the origin of the classical field space. Instead, we choose an arbitrary value of \(\varphi_c\), which we denote by \(M\), and require either

\[
\frac{d^4V}{d\varphi_c^4} \bigg|_{\varphi_c=M} = \lambda \quad (3.11a)
\]

or

\[
\frac{4!}{\varphi_c^4} V \bigg|_{\varphi_c=M} = \lambda. \quad (3.11b)
\]

It is important to remember that the choice of \(M\) is arbitrary; if a different value is chosen for \(M\), one obtains a different value for \(\lambda\), but the theory remains unchanged. In other words, as \(M\) is varied \(\lambda\) traces out a curve \(\lambda(M)\). Although two parameters appear, there is actually only a one-parameter family of theories, one theory for each curve. We shall return to this point later, when we discuss the use of the renormalization group. (Note that this is analogous to what happens in the usual treatment of theories with massless particles. Because of the infrared divergences, some of the renormalizations cannot be done at zero external momentum; instead, they are done at an arbitrary non-zero momentum.)

For the moment, let us choose Eq. (3.11a) as the renormalization condition, since it is closer to the definition of the physical 4-point IPI Green’s function. (It is, in fact, the definition of the physical Green’s function if \(M\) is equal to the vacuum expectation value of \(\phi\).) We find

\[
C^{(1)} = -\frac{3\lambda^2}{32\pi^2} \left( \log \frac{\lambda M^2}{2\Lambda^2} + \frac{11}{3} \right) \quad (3.12)
\]
and

\[ V_{\text{one-loop approx}} = \frac{\lambda}{4!} \varphi_c^4 + \frac{\lambda^2 \varphi_c^4}{256 \pi^2} \left[ \log \frac{\varphi_c^2}{M^2} - \frac{25}{6} \right]. \quad (3.13) \]

We note several things about our final expression for the effective potential:

1) \( V \) still contains the logarithmic singularity in \( \phi_c \), but the singularity at \( \lambda = 0 \) has disappeared.

2) \( M \) is in fact arbitrary; if we let \( M \) go to \( M' \), we can choose a new coupling constant,

\[ \lambda' = \frac{d^4V}{d\varphi_c^4} \bigg|_{\varphi_c=M'} = \lambda + \frac{3\lambda^2}{32\pi^2} \log \frac{M'^2}{M^2}, \quad (3.14) \]

and find

\[ V = \frac{\lambda'}{4!} \varphi_c^4 + \frac{\lambda'^2 \varphi_c^4}{256 \pi^2} \left[ \log \frac{\varphi_c^2}{M'^2} - \frac{25}{6} \right]. \quad (3.15) \]

This, to the order of approximation to which we are working, describes the same theory as Eq. (3.13)

3) The behavior of \( V \) is shown in Fig. 5. Since the logarithm of a small number is negative, it appears that there is a maximum at the origin and a minimum at a non-zero value of \( \varphi_c \). Differentiating \( V \), we find

\[ \frac{dV}{d\varphi_c} = \frac{\lambda}{6} \varphi_c^3 + \frac{\lambda^2 \varphi_c^3}{64 \pi^2} \left[ \log \frac{\varphi_c^2}{M^2} - \frac{11}{3} \right]. \quad (3.16) \]

Thus the minimum occurs at a value of \( \varphi_c \) determined by

\[ \lambda \log \frac{\varphi_c^2}{M^2} - \frac{11\lambda}{3} = -\frac{32\pi^2}{3}. \quad (3.17) \]

Thus it would appear that this theory exhibits spontaneous symmetry breaking. However, we must consider the effects of many-loop diagrams, and see whether there are solutions of Eq. (3.17) which are consistent with the validity of the one-loop approximation.

It turns out that increasing the number of loops brings in not only additional powers of \( \lambda \), but also additional logarithms; that is, the leading behavior of the \( n \)-loop contribution to the effective potential is of the order of

\[ \lambda \left( \lambda \log \frac{\varphi_c^2}{M^2} \right)^n. \quad (3.18) \]
Thus the one-loop approximation can be valid only if both $|\lambda|$ and $|\lambda \log(\varphi^2/M^2)|$ are small. However, it is clear that Eq. (3.17) cannot be satisfied unless one of these is large. There may be a minimum of the effective potential away from the origin, but if there is, it is in a region where our calculational methods fail. Similarly, we cannot trust the prediction that there is a maximum at the origin. Later we shall see that by the use of the methods of the renormalization group we can improve the situation, but for the present we simply do not know whether spontaneous symmetry breaking occurs.

Of course, we should not be too surprised that our approximation did not give us any reliable evidence of spontaneous symmetry breaking. The zero-loop approximation to the effective potential indicates that spontaneous symmetry breaking does not occur. If we want the one-loop approximation to change this, we should expect that it must be of the same order of magnitude as the zero-loop term; this is possible only if the interaction is strong, in which case a perturbative methods is expected to fail. At this point, we might conclude that our methods will never indicate spontaneous symmetry breaking unless it is already predicted by the classical approximation; however, this is not quite true.

Consider a theory with two interactions, A and B. Suppose that only A contributes to the effective potential in the zero-loop approximation, but that B contributes through loop diagrams. Even if A and B are both weak, it may be possible to make the one-loop B contribution and the zero-loop A contribution be of the same order of magnitude by adjusting the relative strength of A and B. If so, the one-loop contributions might give rise to spontaneous symmetry breaking, not because they are large, but because they involve a new type of interaction. Since both interactions are weak, and there are no other interactions to be introduced, the contributions from diagrams with two or more loops should be small; therefore, the one-loop approximation should be valid.

One might ask whether it is possible to construct a theory in which the loop expansion is valid, but in which the diagrams with two or more loops introduce new interactions which qualitatively change the effective potential; we not found any. To
see the difficulty, consider the electrodynamics of a scalar with charge $e$ and a fermion with charge $g$, with an additional quartic scalar self-coupling with coupling constant $\lambda$. The zero-loop approximation to the effective potential will be of the order of $\lambda$, while the one-loop contributions will be of the order of $\lambda^2$ or $e^4$. The fermion-photon interaction first contributes in the two-loop approximation, through corrections to the photon propagator. These contributions will be of the order of $g^2e^4$; for these to be significant in comparison with the one-loop terms, $g^2$ must be large, in which case the perturbation theory is no longer valid.

Let us now justify Eq. (3.18) by deriving some simple rules which make it possible to sum all the graphs of any particular order as easily as we summed all of the one-loop graphs. Consider, for example, the two-loop graphs of the type shown in Fig. 6a. To obtain the contribution of these graphs to the effective potential, we must sum over all values of $l$, $m$, and $n$ from zero to infinity. In doing the sum, we must remember the following factors:

1) A factor of $(2l + 2m + 2n)!$, which cancels the factorial in Eq. (3.3), just as in the one-loop calculation.

2) A factor of $(1/2)^{l+m+n}$, from interchanging the lines at each of the vertices with two external lines; this also is analogous to the one-loop calculation.

3) A factor of $1/(3!)$, arising from the symmetry of the diagram.

We can do the sums over $l$, $m$, and $n$ independently. Each sum is of the form

$$\sum_{j=0}^{\infty} \left( -\frac{i\lambda\varphi^2_c}{2} \right)^j \left( \frac{i}{k^2 + i\epsilon} \right)^{j+1} = \frac{i}{k^2 - \frac{1}{2}\lambda\varphi^2_c + i\epsilon}. \quad (3.19)$$

In addition we have a factor of $-i\lambda\varphi_c$ at each of the two remaining vertices and the factor of $1/6$ from point (3). This is just what we would have obtained from the graph of fig. 6b if the $\phi$ had a mass of $\frac{1}{2}\lambda\varphi^2_c$. (The Feynman rules would include the $1/6$, again from the symmetry of the graph.) We shall call those graphs which, like that in Fig. 6b, have no vertices with two external lines prototype graphs. It is clear that the above discussion holds for all prototype graphs. We thus obtain a rule for calculating the $n$-loop contribution to the effective potential: Draw all of the $n$-loop prototype graphs and calculate them using ordinary Feynman rules, except with a
propagator given by Eq. (3.19). The only exception to this rule is the case of one-loop graphs, which alone are invariant under rotation. We note that the prototype graph in this case is a circle, which does not correspond to any Feynman diagram arising from ordinary perturbation theory.

All that remains is to show that there are a finite number of \( n \)-loop prototype graphs. We define a type-\( n \) vertex to be one with \( n \) internal lines, and denote the number of type-\( n \) vertices in a graph by \( V_n \). By definition, an IPI graph does not contain any type-1 vertices, while a prototype graph contains only type-3 and type-4 vertices. For an IPI graph,

\[
V = V_2 + V_3 + V_4. \tag{3.20}
\]

Since every internal line has two ends, we see that

\[
2I = 2V_2 + 3V_3 + 4V_4. \tag{3.21}
\]

Thus,

\[
L = I - V + 1 = \frac{1}{2}V_3 + V_4 + 1. \tag{3.22}
\]

Thus, for a given \( L \), only a finite number of type-3 and type-4 vertices are allowed. Since only a finite number of graphs can be made with a given finite number of vertices, the number of prototype graphs in any order is finite.

A few words of caution:

1) One must not forget diagrams arising from the insertion of counter-terms. These are calculated by drawing prototype graphs with counterterm insertions drawn explicitly, remembering that the order of the counter-term must be included in determining the order of a diagram. The number of such prototype graphs in any order is finite, since all counter-terms are at least one-loop in order.

2) The factor of \( \frac{1}{2} \lambda \phi_c^2 \) in Eq. (3.19) looks like a mass, but it is not; it is not even a constant. However, it is related to the mass if \( \phi_c \) is set equal to the vacuum expectation value of \( \phi \).

3) One must remember that we are summing all IPI graphs, and that the use of prototype graphs is merely a device to perform the sum. Thus we must include the prototype graph in Fig. 7, even though it has no external lines. The reason is
that this prototype graph stands for a sum of graphs which do have external lines. However, it also includes the graph which looks exactly like it and which does not have any external lines; this graph must be subtracted after the prototype graph is calculated. That is, the total contribution of this prototype graph to the effective potential is

\[ i(-i\lambda) \left\{ \left[ \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - \frac{i}{2}\lambda\varphi_c^2 + i\epsilon} \right]^2 - \left[ \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 + i\epsilon} \right]^2 \right\}. \] (3.23)

We can now calculate some many-loop graphs, and verify Eq. (3.18). It is now also possible to specify the wave function renormalization condition, which we have not yet done. This is a condition on the term in the effective action which is of the form

\[ \partial_\mu \Phi_c \partial_\mu \Phi_c \mathcal{Z}(\varphi_c). \] (3.24)

In the zero-loop approximation, \( \mathcal{Z} \) will be 1. The one-loop contribution to \( \mathcal{Z} \) arises from the counter-term and from the one-loop diagrams which have two external \( \phi \)'s with four-momenta \( p \) and \( -p \), and any number of external \( \phi \)'s with vanishing four-momenta. (Actually, it is just the term proportional to \( p^2 \) in the Taylor series expansion of these diagrams.) The sum of these diagrams can be calculated using the methods outlined above: one draws prototype graphs in which the vertices with non-vanishing external momenta are shown explicitly, while the vertices with two zero-momentum external lines are implicit in the propagators. The resulting integrals have ultra-violet divergences and must be cut off; a counter-term is necessary to make \( \mathcal{Z} \) cutoff-independent. The most natural choice for the renormalization condition is

\[ \mathcal{Z}(0) = 1. \] (3.25)

Unfortunately, the same difficulty arises here as arose with the coupling constant renormalization: \( \mathcal{Z} \) contains a logarithmic singularity at \( \varphi_c = 0 \). Thus, we must renormalize at some arbitrary non-zero value of \( \varphi_c \). For simplicity, we choose the value we used for the coupling constant renormalization, and have

\[ \mathcal{Z}(M) = 1. \] (3.26)
Again we see the similarity to the usual method of renormalization in momentum-space. There, the wave function renormalization, like the coupling constant renormalization, would have an infrared divergence, while the mass renormalization would not; in our method, the first two have logarithmic singularities, while the third does not.
IV. Bigger and Better Models

In this chapter we shall extend the methods of the previous chapter to models which include more than one field; we shall continue, however, to restrict ourselves to models in which no mass term appears in the Lagrangian. (As mentioned previously, this does not necessarily imply that the theory contains massless particles.) We shall also restrict ourselves to renormalizable theories, since only for these do we have any assurance that higher order calculations will be finite.

In a theory with many quantized fields, we find it useful to define a classical field corresponding to each quantized field; the effective action is a functional of all these classical fields. However, in expanding \( \Gamma \) in Eq. (2.8), we defined the effective potential to depend only on the spinless classical fields. The reason is that we are only interested in cases where the vacuum is Lorentz invariant; therefore we want the vacuum expectation value of any field with spin to be zero.

Just as with the theory of a single scalar field, it is possible to derive rules which enable us to sum the infinite number of Feynman diagrams which occur in each order in the loop expansion. We begin by considering graphs that contain only spin-zero particles. We arrange the real spinless fields in a vector \( \vec{\phi} \), with components \( \phi_i \). (One should not be misled by the notation to conclude that these fields necessarily belong to a representation of some symmetry group.) Also, we define \( V_0(\vec{\phi}_c) \) to be the zero-loop approximation to the effective potential. (It is, of course, of the same form as the potential terms in the Lagrangian, except that it is a function of the classical, rather than of the quantized, fields.) Finally, we define a matrix \( U(\vec{\phi}_c) \) by

\[
U_{ij}(\vec{\phi}_c) = \frac{\partial V_0(\vec{\phi}_c)}{\partial \phi_i \partial \phi_j}.
\]  

(4.1)

This is exactly the factor which occurs at each scalar vertex with two external lines. We can now calculate the sum of all one-loop diagrams with only spinless particles; by comparison with Eq. (3.8), we see that the result is

\[
V_{\text{spin--0 loop}} = \frac{1}{64\pi^2} \text{Tr} \left\{ 2U(\vec{\phi})\Lambda^2 + U(\vec{\phi})^2 \left[ \log \frac{U(\vec{\phi})}{\Lambda^2} - \frac{1}{2} \right] \right\}. 
\]  

(4.2)
We note that $U$ is a symmetric matrix and can therefore be diagonalized; there is thus no difficulty in interpreting this expression. In order to calculate sums of multi-loop graphs, we determine Feynman rules for prototype graphs. The propagator will now be a matrix; referring to Eq. (3.19), we see that it is given by

$$
\Delta_{ij}(\vec{\varphi}_c) = \left[ \frac{i}{k^2 - U(\vec{\varphi}_c) + i\epsilon} \right]_{ij}.
$$

Next we consider the effect of the spin-$\frac{1}{2}$ fields, which we arrange in a vector $\vec{\psi}$. The Yukawa coupling term in the Lagrangian is of the form

$$
L_{\text{Yuk}} = \bar{\psi}_i F_{ij}(\vec{\varphi}_c) \psi_j = \bar{\psi}_i \left[ A_{ij}(\vec{\varphi}_c)I + B_{ij}(\vec{\varphi}_c)i\gamma^5 \right] \psi_j,
$$

where both $A$ and $B$ are Hermitian matrices. In computing the contribution to the effective potential from diagrams with a single fermion loop, the matrix $F(\vec{\varphi}_c)$ will appear at each vertex. In summing these diagrams one must remember that one-loop diagrams with an odd number of vertices vanish, since the trace of an odd number of Dirac matrices is zero. Thus, it is appropriate to group the vertices in pairs, noting that

$$
\frac{1}{k} F(\vec{\varphi}_c) \frac{1}{k} F(\vec{\varphi}_c) = \frac{1}{k^2} F(\vec{\varphi}_c) F(\vec{\varphi}_c)^*.
$$

The sum also differ from that for the scalar loops in that there is the usual minus sign for a fermion loop and in that, since fermion lines are directed, there is no factor of $1/2$ arising from the reflection symmetry of the diagram. The latter factor is compensated for by another factor of $1/2$ from summing only the even terms in the sum. Thus, the contribution of the fermion one-loop diagrams is

$$
V_{\text{spin-1/2 loop}} = -\frac{1}{64\pi^2} \text{Tr} \left\{ 2\Lambda^2 F(\vec{\varphi}_c) F(\vec{\varphi}_c)^* \\
+ [F(\vec{\varphi}_c) F(\vec{\varphi}_c)^*]^2 \left[ \log \frac{F(\vec{\varphi}_c) F(\vec{\varphi}_c)^*}{\Lambda^2} - \frac{1}{2} \right] \right\},
$$

where the trace is over both Dirac and particle indices.

In determining the fermion propagator to be used in prototype graphs, both even and odd numbers of vertices must be included in the sum, since no trace is being
taken. One obtains

$$S_{ij}(\vec{\varphi}_c) = \left[ \frac{i}{k - F(\vec{\varphi}_c)} \right]_{ij}. \quad (4.7)$$

Lastly we come to the vector particles, which we will arrange in a vector $\vec{A}^\mu$; in the cases we will consider, these will always be associated with gauge symmetries, so that the theory will be renormalizable. Here matters are complicated by the fact that in gauge theories there are two types of scalar-vector vertices, shown in Fig. 8. Because of the vertex of Fig. 8a, there are diagrams, such as that in Fig. 9, which have both a scalar and a vector particle going around the loop. However, these diagrams will not contribute to the effective potential if we work in the Landau gauge, where the vector propagator is transverse. Because the external scalars have zero momentum, the internal scalars and the vectors have the same momentum; thus, when the Landau gauge propagator is multiplied by the momentum factor from the vertex the result is zero. We will do all of our calculations in the Landau gauge.

We define a matrix $G(\vec{\varphi}_c)$ to describe the couplings giving rise to the vertex of Fig. 8b. It appears in the Lagrangian as

$$\mathcal{L} = \cdots + \frac{1}{2} A_i^\mu A_{\mu j} G_{ij}(\vec{\varphi}_c) + \cdots. \quad (4.8)$$

$G$ is also given in terms of the infinitesimal generators, $T_i$, of the gauge group:

$$G_{ij}(\vec{\varphi}_c) = g_i g_j (T_i \vec{\varphi}, T_j \vec{\varphi}). \quad (4.9)$$

Here $g_i$ and $g_j$ are the appropriate gauge coupling constants.

The computation of the sum of one-loop diagrams with a gauge particle going around the loop can now be done. The only new complication is the presence of the numerator factors in the vector propagator. In the Landau gauge these work out quite easily; their only effect is to multiply the result by a factor of 3. Thus, we have

$$V_{\text{gauge loop}} = \frac{3}{64\pi^2} \text{Tr} \left\{ 2\Lambda^2 G(\vec{\varphi}_c) + G(\vec{\varphi}_c)^2 \left[ \log \frac{G(\vec{\varphi}_c)}{\Lambda^2} - \frac{1}{2} \right] \right\}. \quad (4.10)$$

The calculation of the vector meson propagator to be used in prototype graphs is
also straightforward; one obtains

\[ D_{ij}^{\mu \nu} = \left[ \frac{i \left( -g^{\mu \nu} + \frac{k_{\mu} k_{\nu}}{k^2} \right)}{k^2 - G(\vec{\phi}_c) + i\epsilon} \right]_{ij}. \]  

(4.11)

If the vector particles are associated with a non-Abelian gauge group, the theory will also involve fictitious ghost particles\(^7\). If one works in the Landau gauge, the ghosts do not couple to the other scalar particles and therefore need no further consideration at this point. There will be higher-order graphs contributing to the effective potential which have ghosts coupled to the gauge particles, but in these the ghost part of the diagram is calculated as usual. We will see, however, that the ghosts will complicate the renormalization procedure.

We will not discuss the renormalization procedure in detail at this point, since it is similar to that for the theory of Chapter III; when appropriate, we will make some comments in the context of particular theories. Let us now proceed to consider several models, to see if our methods can detect spontaneous symmetry breaking in any of them.

1. A Scalar SO\((n)\) Model

This model is very much like our original simple model; instead of a single scalar fields there are \(n\) real fields, transforming as a vector under the group SO\((n)\). The Lagrangian is

\[ \mathcal{L} = \frac{1}{2} \partial_{\mu} \vec{\phi} \cdot \partial^{\mu} \vec{\phi} - \frac{\lambda}{4!} (\vec{\phi} \cdot \vec{\phi})^2 + \text{counter-terms}. \]  

(4.12)

The effective potential is expected to be a function of the \(n\) classical fields. However, we notice that because of the symmetry of the theory, the effective potential can only be a function of

\[ \varphi_{c}^2 = \sum_{k=1}^{n} \varphi_{kc}^2. \]  

(4.13)

Thus we can calculate the effective potential for the case where only \(\varphi_{1c}\) is non-zero, and then immediately extend the results to the general case. In other words, it is sufficient to calculate only diagrams with external \(\phi_1\)'s. (This is just another way of saying that the vacuum expectation value of \(\vec{\phi}\) must point in a particular direction.)
We expect two different types of one-loop diagrams: those with a $\phi_1$ running around the loop, and those with one of the other $\phi_k$’s running around the loop. Referring to Eqs. (4.1) and (4.2), we see that

$$U(\vec{\phi}_c)|_{\phi_1 \neq 0, \phi_2 = \phi_3 = \ldots = \phi_{nc} = 0} = \begin{pmatrix} \frac{1}{2} \lambda \varphi_{1c}^2 & 0 & \cdots & 0 \\ 0 & \frac{1}{6} \lambda \varphi_{1c}^2 \\ \vdots & \ddots & \ddots \\ 0 & & & \frac{1}{6} \lambda \varphi_{1c}^2 \end{pmatrix}$$

(4.14)

and thus

$$V(\vec{\phi}_c)|_{\phi_1 \neq 0, \phi_2 = \phi_3 = \ldots = \phi_{nc} = 0} = \frac{\lambda}{4!} \varphi_{1c}^4 + \frac{1}{64\pi^2} \left\{ 2\Lambda^2 \left( \frac{\lambda}{2} + \frac{(n-1)\lambda}{6} \right) \varphi_{1c}^2 \\ + \frac{\lambda^2}{4} \varphi_{1c}^2 \left[ \log \frac{\varphi_{1c}^2}{2\Lambda^2} - \frac{1}{2} \right] + \frac{(n-1)\lambda^2}{36} \left[ \log \frac{\varphi_{1c}^2}{6\Lambda^2} - \frac{1}{2} \right] \right\} + \text{counter-terms}.$$  

(4.15)

We see that there are, as expected, two types of one-loop contributions.

Our renormalization conditions are similar to those of the theory with a single scalar particle, namely

$$\frac{\partial^2 V}{\partial \varphi_{1c}^2} |_{\varphi_1 = \ldots = \varphi_{nc} = 0} = 0,$$  

(4.16a)

$$\frac{\partial^4 V}{\partial \varphi_{1c}^4} |_{\varphi_1 = \ldots = \varphi_{nc} = 0} = \lambda,$$  

(4.16b)

and

$$Z(\varphi_{1c} = M, \varphi_{2c} = \ldots = \varphi_{nc} = 0) = 1.$$  

(4.16c)

Performing the renormalization, we obtain the final expression for the one-loop approximation to the effective potential:

$$V(\vec{\varphi}) = \frac{\lambda}{4!} \varphi_{1c}^4 + \frac{1}{64\pi^2} \left( \frac{\lambda^2}{4} + \frac{(n-1)\lambda^2}{36} \right) \varphi_{1c}^2 \left[ \log \frac{\varphi_{1c}^2}{M^2} - \frac{25}{6} \right].$$  

(4.17)

This effective potential is of the same form as that of our previous model, but it differs from the previous one in having the factor of $n - 1$ in the logarithmic term. Thus, we might hope that for sufficiently large $n$ this potential will exhibit a minimum.
within the range of validity of the one-loop approximation. (In terms of the comments of the previous chapter, we can attribute this to the presence in this theory of two types of interactions: the $\phi_i^4$ interaction and the $\phi_i^2\phi_j^2$ interaction ($i \neq j$). The latter first contributes to the effective potential in the one-loop approximation.) If we differentiate, we obtain, instead of Eq. (3.17),

$$\lambda \left(1 + \frac{n - 1}{9}\right) \left(\log \frac{\varphi^2}{M^2} - \frac{11}{3}\right) = -\frac{32\pi^2}{3}. \quad (4.18)$$

Certainly this equation can be satisfied with both $|\lambda|$ and $|\log(\varphi^2/M^2)|$ being kept small if $n$ is made sufficiently large; we can obtain a minimum by having many types of loops, rather than by making a single loop contribution large. Unfortunately, letting $n$ be large also invalidates the one-loop approximation. For example, consider the two-loop prototype graphs of the form shown in Fig. 7. Each of the $n\phi$'s can be allowed to run around the left loop and and each of the $n\phi$'s can be allowed to run around the right loop; thus there are $n^2$ prototype graphs of this type. Each higher order will bring in another factor of $n$, so the $L$-loop contribution will be of the order of

$$\lambda \left(n\log \frac{\varphi^2}{M^2}\right)^L. \quad (4.19)$$

Thus, if $n$ is large enough to satisfy Eq. (4.18), it is large enough to invalidate the one-loop approximation. (We see that we were mistaken in identifying two types of interactions; a better way of describing the theory is to say that it contains a single many-component field and only one interaction.) This model is no improvement over our first one.

2. A Yukawa Model

Next we consider a theory with a single scalar field and a single fermion field, with the Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + i \bar{\psi} \gamma^5 \psi + ig \bar{\psi} \gamma^5 \phi \psi + \frac{\lambda}{4!} \phi^4 + \text{counter-terms}. \quad (4.20)$$

(The quartic scalar self-coupling is required for renormalizability.) Because this theory contains two genuinely distinct interactions, it may be possible to adjust matters so that the one-loop corrections qualitatively change the nature of the vacuum.
Because there is only one scalar field and one fermion field, the matrices $U$ and $F$ are trivial:

$$U = \frac{1}{2} \lambda \varphi_c^2$$  \hspace{1cm} (4.21a)

and

$$F = ig\varphi_c \gamma^5.$$  \hspace{1cm} (4.21b)

The calculation of the one-loop approximation to the effective potential is straightforward; after renormalization one obtains

$$V = \frac{\lambda}{4!} \varphi_c^4 + \frac{1}{64\pi^2} \left( \frac{\lambda^2}{4} - 4g^2 \right) \varphi_c^4 \left[ \log \frac{\varphi_c^2}{M^2} - \frac{25}{6} \right].$$  \hspace{1cm} (4.22)

Indeed, the Yukawa coupling, which first contributes to the effective potential in the one-loop approximation, does have an important qualitative effect: The effective potential has the shape of the previous ones (Fig. 5) only if $g^4$ is less than $\lambda^2/16$; if $g^4$ is larger than $\lambda^2/16$, the effective potential has the shape shown in Fig. 10. The latter case is clearly unacceptable; the effective potential has no lower bound as the classical field approaches infinity, so the vacuum does not exist (unless higher-order contributions make the effective potential turn upward, but in that case the one-loop approximation is still useless). And the former case has the same difficulties as our earlier models, but to a greater degree: the minimum is even further from the region of validity of the one-loop approximation. Thus, we have found a theory where the one-loop contribution to the effective potential has an important qualitative effect; unfortunately, it isn’t the sort of effect we were looking for.

3. Massless Scalar Electrodynamics

Now we consider the theory of two real (or one complex) scalar fields coupled to a gauge vector meson, with the Lagrangian

$$\mathcal{L} = \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} \left( \partial_\mu \phi_1 - e A_\mu \phi_2 \right)^2 + \frac{1}{2} \left( \partial_\mu \phi_2 + e A_\mu \phi_1 \right)^2 - \frac{\lambda}{4!} \left( \phi_1^2 + \phi_2^2 \right)^2$$  \hspace{1cm} (4.23)$$

+ \text{counter-terms},$$

where

$$F^{\mu\nu} = \partial^\nu A^\mu - \partial^\mu A^\nu.$$  \hspace{1cm} (4.24)
If this theory has a symmetric vacuum, it describes the electrodynamics of a massless charged scalar meson. If spontaneous symmetry breaking occurs, then by the usual Higgs mechanism we obtain the theory of a single massive scalar meson coupled to a massive vector meson. To determine which is the case is to determine whether the infrared divergences associated with massless charged scalar particles are strong enough to prevent such particles from existing.

The calculation of the one-loop approximation to the effective potential is straightforward, as is the renormalization; one obtains

\[
V = \frac{\lambda}{4!} \varphi_c^4 + \frac{1}{64\pi^2} \left( \frac{\lambda^2}{4} + \frac{\lambda^2}{36} + 3e^4 \right) \varphi_c^4 \log \left( \frac{\varphi_c^2}{M^2} - \frac{25}{6} \right).
\]

(4.25)

where

\[
\varphi_c^2 = \varphi_1^2 + \varphi_2^2.
\]

(4.26)

This effective potential also has the shape shown in Fig. 5; it appears to have a minimum at a non-zero value of \( \varphi_c \). We again must determine whether the minimum occurs within the range of validity of the one-loop approximation. We can simplify our equations if we recall that \( M \) is an arbitrary parameter; we are certainly allowed to choose \( M \) to be the location of the minimum of the effective potential, \( \langle \phi \rangle \). We find

\[
\left. \frac{dV}{d\varphi_c} \right|_{\varphi_c = M = \langle \phi \rangle} = \frac{\lambda}{6} (\langle \phi \rangle)^3 + \frac{1}{64\pi^2} \left( \frac{\lambda^2}{4} + \frac{\lambda^2}{36} + 3e^4 \right) \left( -\frac{44}{3} (\langle \phi \rangle)^2 \right).
\]

(4.27)

Thus, we must see if we can satisfy

\[
\lambda = \frac{11}{8\pi^2} \left( 3e^4 + \frac{\lambda^2}{4} + \frac{\lambda^2}{36} \right)
\]

(4.28)

while keeping both \( e \) and \( \lambda \) small enough that our approximation is valid. Clearly we can do this if we choose \( \lambda \) to be of the order of \( e^4 \). If we do this, we should ignore the \( \lambda^2 \) terms, since they are of the same order of magnitude as the \( e^8 \) terms we expect from two-loop diagrams. Thus, in the one-loop approximation we find that if

\[
\lambda = \frac{33e^4}{8\pi^2},
\]

(4.29)
there will be a minimum in the effective potential which is within the region where the approximation is valid. The gauge coupling, which only contributes to the effective potential through loop diagrams, has caused spontaneous symmetry breaking.

If we substitute Eq. (4.29) back into Eq. (4.25), we obtain

\[ V = \frac{3e^4}{64\pi^2} \varphi_c^4 \left[ \log \frac{\varphi_c^2}{\langle \phi \rangle^2} - \frac{1}{2} \right]. \]  

(4.30)

All reference to the parameter \( \lambda \) has disappeared from the effective potential. Note that even if we had chosen to define \( \lambda \) differently (for example, by using Eq. (3.11b)), we would have obtained different results for Eqs. (4.25-29), but the same result for Eq. (4.30).

At first glance, a very strange thing seems to have occurred: Starting with two dimensionless parameters, \( e \) and \( \lambda \), we have obtained a theory with one dimensionless parameter, \( e \), and one dimensional parameter, \( \langle \phi \rangle \). Furthermore, the dependence on \( \langle \phi \rangle \) is trivial, being determined solely by dimensional considerations. A little thought will make this seem less strange. Originally, the theory was really characterized by three parameters: \( e \), \( \lambda \), and \( M \); however, one two of these were independent and the dependence on \( M \) was not explicitly shown. The final description of the theory is also characterized by three parameters: \( e \), \( \langle \phi \rangle \), and \( \langle \phi \rangle / M \); again only two are truly independent, and we have not shown the dependence on the third parameter, which we have set equal to one. Of course, we could have renormalized at a point other than the minimum of the effective potential (that is, with \( \langle \phi \rangle / M \neq 1 \)), and would have obtained a different expression for the effective potential and a different relationship between \( \lambda \) and \( e \), although the physics would have been the same. In other words, both descriptions contain two dimensionless and one dimensional parameters, with only two being truly independent. The difference between the two descriptions is that in the second one dimensional analysis tells us a great deal; there is a two-parameter family of spontaneously broken theories, but changing one of the parameters, \( \langle \phi \rangle \), is completely equivalent to changing the scale in which masses are measured, so that effectively there is only a one-parameter family.

Another important point is the singularity in the effective potential (and thus
in the effective action) at \( \varphi_c = 0 \). Because of this branch point, we cannot expand the effective action in a Taylor series about \( \varphi_c = 0 \); the expansion in terms of the \( n \)-point IPI Green’s functions is not valid. If it were, we could approximate the effective potential near \( \varphi_c = 0 \) (although not everywhere) by the sum of the first few Green’s functions. It isn’t, and therefore we must do an infinite sum even to get an approximation to the effective potential.

Even if we had taken the theory at face value, as massless scalar electrodynamics, there would not have been a simple interpretation of the Green’s functions in terms of S-matrix elements, because of the presence of massless particles. To calculate any physical process we would have had to sum over sets of degenerate states with various numbers of particles at infinitesimal momenta.

At this point, we can transform the fields as is usually done in Higgs models to obtain a massive scalar field and a massive vector field. If we define these masses in terms of the values of the inverse propagators at zero momentum, we obtain

\[
m^2(S) = \left. \frac{d^2V}{d\varphi_c^2} \right|_{\varphi_c = \langle \phi \rangle} = \frac{3e^4}{8\pi^2} \langle \phi \rangle^2
\]

and

\[
m^2(V) = e^2 \langle \phi \rangle^2
\]

and thus

\[
\frac{m^2(S)}{m^2(V)} = \frac{3e^2}{8\pi^2} = \frac{3}{2\pi} \alpha.
\]

It is important to remember that the expression we have obtained for the effective potential is only valid in the region of classical field space where \( \log(\varphi_c^2/M^2) \) is small; in other words, it is not valid for very large or very small \( \varphi_c \). Thus, there might be deeper minima than the one we have found, which lie in regions inaccessible to our computational methods. However, the value of the effective potential at the origin will remain zero, even if higher-order effects cause this to be a minimum rather than the maximum which the one-loop approximation indicates. Since the effective potential is negative at the minimum which we have found, and since this minimum is in a region where higher-order effects are expected to be small, we see that the
absolute minimum cannot occur at the origin; spontaneous symmetry breaking occurs for small values of \( e \), at least when \( \lambda \) is of the order of \( e^4 \).

4. Other Gauge Theories

The results for a more complicated (possibly non-Abelian) gauge theory containing many spin-0 and spin-1/2 particles are, in general, qualitatively similar to those for scalar electrodynamics. First, we note that in these theories the matrices \( U(\varphi_c) \), \( F(\varphi_c) \) and \( G(\varphi_c) \) have a simple physical interpretation: when evaluated at the vacuum expectation value of \( \phi \), they are the zero-loop approximations to the mass matrices of the spin-0, spin-1/2, and spin-1 particles, respectively. If the vector meson masses are sufficiently larger than those of the scalar mesons and fermions, the effects of the scalar and fermion loops will be negligible compared to those of the vector loops. (Remember that the contribution of the loop diagrams is roughly proportional to the fourth power of the relevant mass.) In this case, we obtain

\[
V = V_0(\varphi_c) + \frac{3e^4}{64\pi^2} \text{Tr} \left\{ G(\varphi_c)^2 \left[ \log \frac{G(\varphi_c)}{M^2} - \frac{25}{6} \right] \right\}
\]

as the one-loop approximation to the effective potential. If there is a single multiplet of scalar particles whose self-interactions are described by a single quartic self-coupling, we can absorb \( V_0 \) into the vector loop by replacing \( M \) by \( \mu \), where \( \mu \), unlike \( M \), is not arbitrary, since its value determines the strength of the scalar self-interaction. We can then write

\[
V = \frac{3e^4}{64\pi^2} \text{Tr} \left\{ G(\varphi_c)^2 \left[ \log \frac{G(\varphi_c)}{\mu^2} - \frac{1}{2} \right] \right\}.
\]

This equation is of the same form as Eq. (4.30); we expect it to have the same consequences: a minimum in the effective potential, which gives rise to spontaneous symmetry breaking and a single relationship among masses. Otherwise, we expect theories with our mass renormalization condition to be similar to the more usual ones with a negative mass term; by adding one restriction, we have obtained one new result.

To illustrate all this, we consider the Weinberg-Salam model of leptons,\(^8\) modified by requiring that the scalar mass term in the Lagrangian vanish. This model is
based on an SU(2) × U(1) gauge symmetry; the gauge fields are denoted by $\vec{W}_\mu$ and $B_\mu$, with coupling constants $g$ and $g'$, respectively. There is a single complex scalar doublet; with no loss of generality, we can assume that only the real part of one of its components has a non-zero vacuum expectation value. Because the leptons are light, their effects on the effective potential can be neglected. We assume that the final massive scalar meson is light enough that the scalar loop contributions can also be neglected. The matrix $G(\varphi_c)$ is

$$
G(\varphi_c) = \begin{pmatrix}
\frac{1}{4}g^2\varphi_c^2 & 0 & 0 & 0 \\
0 & \frac{1}{4}g^2\varphi_c^2 & 0 & 0 \\
0 & 0 & \frac{1}{4}g^2\varphi_c^2 & \frac{1}{4}gg'\varphi_c^2 \\
0 & 0 & \frac{1}{4}gg'\varphi_c^2 & \frac{1}{4}g'^2\varphi_c^2
\end{pmatrix} .
$$

(4.36)

Its eigenvalues and the corresponding eigenvectors are

$$
\frac{1}{4}g^2\varphi_c^2 : W_1^\mu \\
\frac{1}{4}g^2\varphi_c^2 : W_2^\mu \\
0 : A^\mu \equiv \frac{1}{\sqrt{g^2 + g'^2}} (gB^\mu - g'W_3^\mu) \\
\left(\frac{g^2 + g'^2}{4}\right)\varphi_c^2 : Z^\mu \equiv \frac{1}{\sqrt{g^2 + g'^2}} (g'B^\mu + g'W_3^\mu) .
$$

(4.37)

The $W_1^\mu$ and $W_2^\mu$ correspond to the charge weak intermediate vector boson, The $A^\mu$ to the photon, and the $Z^\mu$ to a massive neutral vector boson. Identifying the coupling constant of the photon as the electric charge, we find

$$
e^2 = \frac{g^2g'^2}{g^2 + g'^2} .
$$

(4.38)

Knowing the eigenvalues of $G(\varphi_c)$, we can easily calculate the one-loop approximation to the effective potential, and see that the minimum occurs when $\varphi_c$ is equal to $\mu$. 
Thus, the massive vector mesons have masses

\[ m^2(W) = \frac{1}{4} g^2 \mu^2 \]

and

\[ m^2(Z) = \frac{1}{4} (g^2 + g'^2) \mu^2 . \]

The mass of the scalar meson is

\[ m^2(\phi) = \frac{d^2 V}{d\phi^2} \bigg|_{\phi = \mu} = \frac{3}{128\pi^2} \left[ 2g^4 + (g^2 + g'^2)^2 \right] \mu^2 . \]

We thus obtain the relation

\[ m^2(\phi) = \frac{3}{32\pi^2} \left( 2g^2 m^2(W^\pm) + (g^2 + g'^2) m^2(Z) \right) . \]

A similar analysis can be applied to any of the many gauge theories of weak and electromagnetic interactions which have been recently proposed. One must, however, be careful in treating models with heavy leptons; if the leptons are made massive enough, the fermion loop contributions will dominate the effective potential and, as we have seen, will destroy not only the spontaneous symmetry breaking, but also the vacuum.
V. Massive Theories

So far, we have restricted ourselves to the consideration of theories in which the Lagrangian does not contain a mass term; we will now remove that restriction and consider theories in which mass terms (either positive or negative) are present. The models we have considered previously are special cases of this large class of theories — special not because they contain massless particles (they don’t necessarily), but because their only dimensional parameter is the renormalization point, which is arbitrary and which can be changed without affecting the physics. While this is significant, it does not seem to be the sort of property that should characterize the transition point between normal theories and spontaneously broken ones. Therefore, we might expect quantum corrections to invalidate the conventional wisdom — perhaps there are theories which are spontaneously broken even though the Lagrangian contains a positive mass-squared, or theories which have a negative mass-squared in the Lagrangian and yet have a symmetric vacuum. We shall study a few models to see if we can find any.

First, we derive some computational rules analogous to Eq. (4.1-11). We begin by considering the effects of a scalar mass term, \(-\frac{1}{2}m^2\phi^2\), where \(m^2\) in general, is a matrix (with either positive or negative eigenvalues). For the moment we consider \(U(\phi_c)\) to not contain any contribution from the mass term. We see immediately that the contribution to the effective potential from the diagrams with one scalar loop is

\[
V_{\text{spin-0 loop}} = \text{Tr} \frac{i}{2} \int \frac{d^4k}{(2\pi)^4} \log \left( 1 - \frac{U(\phi_c)}{k^2 - m^2 + i\epsilon} \right)
= \text{Tr} \frac{i}{2} \int \frac{d^4k}{(2\pi)^4} \left[ \log \left( 1 - \frac{m^2 + U(\phi_c)}{k^2 + i\epsilon} \right) - \log \left( 1 - \frac{m^2}{k^2 + i\epsilon} \right) \right].
\]

(5.1)

If we now consider \(U(\phi_c)\) to include the contribution from the mass term, we obtain

\[
V_{\text{spin-0 loop}} = \frac{1}{64\pi^2} \text{Tr} \left\{ 2U(\phi_c)\Lambda^2 + U(\phi_c)^2 \left( \log \frac{U(\phi_c)}{\Lambda^2} - \frac{1}{2} \right) \right\} - \left[ 2U(0)\Lambda^2 + U(0)^2 \left( \log \frac{U(0)}{\Lambda^2} - \frac{1}{2} \right) \right].
\]

(5.2)
The second term is constant in classical field space; its only role it to assure that the effective potential vanishes when \( \varphi_c = 0 \). The scalar propagator to be used in prototype graphs is even simpler to calculate; clearly, Eq. (4.3) remains true if \( U(\varphi_c) \) is interpreted to include the mass term contributions. The rules for spin-1/2 and spin-1 fields are just as simple. \( F(\varphi_c) \) and \( G(\varphi_c) \) are now understood to include contributions from the appropriate mass terms. The formulas for the propagators remain unchanged, while the rules for calculating the one-loop contributions are altered only by the subtraction of the value of the loop at \( \varphi_c = 0 \).

After renormalization, however, the expression for the effective potential becomes more complicated than previously; this is a consequence of the fact that \( U(\varphi_c), F(\varphi_c), \) and \( G(\varphi_c) \) are more complicated functions of \( \varphi_c \) than before, which makes the derivatives of \( V(\varphi_c) \) more complicated also.

1. Massive \( \lambda \phi^4 \) Theory

We return to our original model, adding a mass term. The Lagrangian is now

\[
\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4 + \text{counter-terms}
\]  

(5.3)

The renormalization conditions we impose are

\[
\frac{d^2 V}{d \phi^2} \bigg|_{\varphi = 0} = m^2,
\]

(5.4)

\[
\frac{d^4 V}{d \phi^4} \bigg|_{\varphi = M} = \lambda,
\]

(5.5)

and

\[
Z(M) = 1.
\]

(5.6)

Notice that we have retained the arbitrary non-zero renormalization point, even though there is no singularity at \( \varphi_c = 0 \) when \( m^2 \neq 0 \). The reason should be clear; if we want to consider our previous model as a special case of a more general class, we should impose the same renormalization conditions throughout. (Although \( M \) is still arbitrary, the dependence of the effective potential on \( M \) is no longer determined by dimensional analysis, since the theory now contains a second dimensional parameter.)
First, we consider the case where both $m^2$ and $\lambda$ are positive. A straightforward (but agonizing) calculation yields

$$V = \frac{m^2}{2} \varphi^2_c + \frac{\lambda}{4!} \varphi_4^4 + \frac{1}{64\pi^2} \left( m^2 + \frac{\lambda \varphi^2_c}{2} \right)^2 \log \left( 1 + \frac{\lambda \varphi^2_c}{m^2} \right)$$

$$- \frac{1}{64\pi^2} \left[ \frac{1}{2} m^2 \lambda \varphi^2_c + \frac{\lambda^2 \varphi^4_c}{m^2 + \frac{1}{2} \lambda M^2} \left( \frac{3}{8} m^4 + \frac{7}{8} m^2 \lambda M^2 + \frac{25}{96} \lambda^2 M^4 \right) \right]$$

$$- \frac{\lambda^2 \varphi^4_c}{256\pi^2} \log \left( \frac{m^2 + \frac{1}{2} \lambda M^2}{m^2} \right).$$

(5.7)

The effective potential vanishes at the origin, as it should. We note that the first three terms are always non-negative, while the terms in square brackets, which are negative, are negligible unless $\lambda$ is large (of the order of $64\pi^2$). Only the last term can make the effective potential turn negative, which evidently will happen if $\frac{1}{2} \lambda M^2$ is much greater than $m^2$. Of course, this will put the minimum at a small value of $\varphi_c/M$ (but with $\lambda \varphi^2_c > m^2$), which is outside the region of validity of the one-loop approximation, but we do see our previous results, for $m^2 = 0$, emerging as the limit of the massive ones.

Now we let $m^2$ be negative, and write $m^2 = -\mu^2$. We obtain

$$V = -\frac{\mu^2}{2} \varphi^2_c + \frac{\lambda}{4!} \varphi_4^4 + \frac{1}{64\pi^2} \left( -\mu^2 + \frac{\lambda \varphi^2_c}{2} \right)^2 \log \left( \frac{|\mu^2 - \frac{1}{2} \lambda M^2|}{\mu^2} \right)$$

$$+ \frac{1}{64\pi^2} \left[ \frac{1}{2} \mu^2 \lambda \varphi^2_c + \frac{\lambda^2 \varphi^4_c}{(\mu^2 - \frac{1}{2} \lambda M^2)^2} \left( \frac{3}{8} \mu^4 + \frac{7}{8} \lambda \mu^2 M^2 - \frac{25}{96} \lambda^2 M^4 \right) \right]$$

$$- \frac{\lambda^2 \varphi^4_c}{256\pi^2} \log \left( \frac{|\mu^2 - \frac{1}{2} \lambda M^2|}{\mu^2} \right) + \frac{i}{64\pi} \left[ \left( \frac{\lambda \varphi^2_c}{2} - \mu^2 \right)^2 \theta \left( \mu^2 - \frac{\lambda \varphi^2_c}{2} \right) - \mu^4 \right].$$

(5.8)

Postponing for a moment the discussion of the imaginary part, let us investigate the behavior of the real part of the effective potential as $\mu^2$ is varied while $\frac{1}{2} \lambda M^2$ is held fixed. There is a singularity at $\mu^2 = \frac{1}{2} \lambda M^2$, but the one-loop approximation clearly fails near this point, since additional loops will bring in additional factors of
log \left( \frac{\mu^2 - \frac{1}{2} \lambda \varphi_c^2}{\mu^2 - \frac{1}{2} \lambda M^2} \right); we will therefore assume that this represents a failure of computational method rather than a real physical effect. Away from this region, the terms in square brackets are small (for small $\lambda$), and may be neglected. Let us now consider the two limiting cases:

1) $\mu^2 > \frac{1}{2} \lambda M^2$: In this region, both of the logarithmic terms are small; the effective potential is dominated by the zero-loop terms, and has a minimum at $\varphi_c \approx \sqrt{\frac{6\mu^2}{\lambda}}$, in agreement with ancient lore.

2) $\mu^2 \ll \frac{1}{2} \lambda M^2$: In this region, the logarithmic factors dominate, and the position of the minimum is given roughly by Eq. (3.17).

We now see how the transition from the normal mode to the spontaneously broken mode occurs: With $\frac{1}{2} \lambda M^2$ held fixed, we let $m^2$ decrease from a large positive value. The theory is normal unit $m^2$ becomes much smaller than $\frac{1}{2} \lambda M^2$ (roughly $\frac{1}{2} \lambda M^2 \exp\left(-\frac{8\pi^2}{3\lambda}\right)$). At this point a minimum develops, but it is in a region of small $\varphi_c$, where the one-loop approximation is not reliable. As $m^2$ decreases through zero and becomes negative, the minimum begins to move outward toward the region where the approximation is valid. As $-m^2$ become larger than $\frac{1}{2} \lambda M^2$, the minimum reaches its limiting position, $\sqrt{-\frac{6m^2}{\lambda}}$. Of course, we cannot be confident about these predictions for the regions of large or small $\varphi_c/M$, but we do obtain one important result in which we can have confidence: If $|m^2|$ is greater than $\frac{1}{2} \lambda M^2$, the loop diagrams give a small correction, and the classical approximation predicts the correct qualitative behavior.

Now we must return to the imaginary part of the effective potential, which arises because the argument of the logarithms in the loop integral becomes negative for certain values of $\varphi_c$. At first one may be inclined to object that the action, and thus the potential, must be real, and that we must have made some grave error. It is true that the quantized action, which is a functional of the quantized fields, must be a Hermitian operator, but the effective action, which is a functional of the classical fields, clearly must be complex. This becomes obvious if one realizes that the effective action is the generating functional of the IPI Green’s functions, which must be complex for at least some values of the external momenta. Still, the effective
potential will normally remain real, since the imaginary part corresponds to on-shell
intermediate states, which normally cannot be produced if all the external momenta
vanish. However, if the theory contains a particle with imaginary mass, on-shell
intermediate states can be produced, even though the external momenta vanish. Of
course, the theory doesn’t really contain any particles with imaginary mass, but
we are doing perturbation theory as if it did, and the Green’s functions only have
meaning in the context of a particular perturbation theory; if we redefine the fields
so as to eliminate the appearance of a negative mass-squared, the Green’s functions
are correspondingly redefined, and have no simple relation to the previously defined
ones. (Since we are assuming that the potential is unchanged by a redefinition of
the fields, we must conclude that the imaginary part vanishes when calculated to all
orders; this does not forbid its presence to any finite order.)

Having convinced ourselves that the imaginary part of the effective potential is not
nonsense, we must see how matters are altered by its presence. Since the quantized
action, and thus the counter-terms, must be real, we can only renormalize the real
part of the effective action; that is, we require that

\[ \frac{d^2(\text{Re } V)}{d\phi_c^2} \bigg|_{\phi_c=0} = -\mu^2 \quad \text{(5.9)} \]

and

\[ \frac{d^4(\text{Re } V)}{d\phi_c^4} \bigg|_{\phi_c=M} = \lambda. \quad \text{(5.10)} \]

(This was done in obtaining Eq. (5.8).) We note that the imaginary part is finite, even
without being renormalized. Also, we looked for spontaneous symmetry breaking by
searching for the minimum of the real part of the effective potential, but this does
not matter; for the imaginary part has two terms, of which one is independent of \( \phi_c \),
and thus physically meaningless, while the other vanishes unless \( \phi_c < \sqrt{\frac{2\mu^2}{\lambda}} \), which is
always below the neighborhood of the minimum.

So far, we have been assuming that \( \lambda \) is positive; what happens when \( \lambda \) is negative? In
this case, the zero-loop approximation to the effective potential has no lower bound.
It is possible that the contributions from diagrams with one or more loops might
make the effective potential turn upward at large $\varphi_c$ (as the one-loop terms appear to do), but our approximation cannot determine this. Later we shall see how to improve our approximation, and will find that this does not happen. In any case, there is no minimum within the region of validity of our approximation, no matter what the sign of $m^2$.

2. Massive Scalar Electrodynamics

Next we turn to massive scalar electrodynamics, described by the Lagrangian

$$\mathcal{L} = \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} (\partial_{\mu} \phi_1 - e A_{\mu} \phi_2)^2 + \frac{1}{2} (\partial_{\mu} \phi_2 + e A_{\mu} \phi_1)^2$$

$$- \frac{1}{2} m^2 (\phi_1^2 + \phi_2^2) - \frac{\lambda}{4!} (\phi_1^2 + \phi_2^2)^2 + \text{counter-terms}. \quad (5.11)$$

If $\lambda$ is much greater than $e^4$, we expect that the contributions to the effective potential from scalar loop diagrams will dominate those from photon loop diagrams and that the effective potential will behave qualitatively like that of the $\lambda \phi^4$ theory. Therefore, let us assume that $\lambda$ is of the order of $e^4$, or smaller. In the one-loop approximation, we should then neglect the scalar loop diagrams, since they are of the same order of magnitude as the diagrams with two photon loops. We obtain for the effective potential

$$V = \frac{1}{2} m^2 \varphi_c^2 + \frac{\lambda}{4!} \varphi_c^4 + \frac{3 e^4}{64 \pi^2} \varphi_c^4 \left( \log \frac{\varphi_c^2}{M^2} - \frac{25}{6} \right). \quad (5.12)$$

Here $M$ is, as usual, the value of $\varphi_c$ at which the renormalizations are done. There is no restriction on the sign of either $m^2$ or $\lambda$.

Now we define a quantity $\mu$, with the dimensions of mass, by

$$\frac{\lambda}{4!} = \frac{3 e^4}{64 \pi^2} \left( \log \frac{M^2}{\mu^2} + \frac{11}{3} \right). \quad (5.13)$$

We then obtain

$$V = \frac{1}{2} m^2 \varphi_c^2 + \frac{3 e^4}{64 \pi^2} \varphi_c^4 \left( \log \frac{\varphi_c^2}{\mu^2} - \frac{1}{2} \right). \quad (5.14)$$

Note that $\mu$, unlike $M$, is not arbitrary; Eq. (5.14) does not contain any redundant parameters. We see that $\mu$ is the position of the minimum if $m^2$ vanishes.
To study the behavior of the effective potential, we will need the following equations:

\[
\frac{dV}{d\phi_c} = \phi_c \left( m^2 + \frac{3 e^4 \mu^2 \phi_c^2}{16\pi^2 \mu^2} \log \frac{\phi_c^2}{\mu^2} \right), \tag{5.15}
\]

\[
\frac{d^4V}{d\phi_c^4} = \frac{9 e^4}{8\pi^2} \left( \log \frac{\phi_c^2}{\mu^2} + \frac{11}{3} \right). \tag{5.16}
\]

It will also be useful to consider the behavior of the function \( f(x) = x \log(x) \), shown in Fig. 11.

We begin by considering the case of positive \( m^2 \). We see that there is a minimum at the origin, but that if \( m^2 \) is sufficiently small, the effective potential will have a second minimum arising from the logarithmic term; the situation will be as in either Fig. 12a or Fig. 12b. To determine matters more quantitatively, we consider Eq. (5.15), and see that the condition for an extremum away from the origin is

\[
m^2 = -\frac{3 e^4 \mu^2}{16\pi^2} f \left( \frac{\phi_c^2}{\mu^2} \right). \tag{5.17}
\]

If \( m^2 > \frac{3 e^4 \mu^2}{16\pi^2} e^{-1} \), Eq. (5.17) cannot be satisfied; the effective potential behaves as in Fig. 12c. If \( 0 < m^2 < \frac{3 e^4 \mu^2}{16\pi^2} e^{-1} \), there will be two solutions, one for \( \phi_c^2/\mu^2 < e^{-1} \) and one for \( \phi_c^2/\mu^2 > e^{-1} \). Clearly the former corresponds to a maximum and the latter to a minimum. We must now determine whether this is an absolute minimum. The effective potential vanishes at the origin; From Eqs. (5.14) and (5.17) we see that at \( \phi_c^2/\mu^2 = \beta \) it is

\[
V(\beta \mu^2) = \frac{\beta \mu^4}{4} \left( m^2 - \frac{3 e^4 \mu^2}{32\pi^2} \beta \right). \tag{5.18}
\]

The condition for an absolute minimum away from the origin is therefore

\[
\beta > \frac{32\pi^2 m^2}{3 e^4 \mu^2}, \tag{5.19}
\]

where \( \beta \) satisfies

\[
\beta \log \beta = -\frac{16\pi^2 m^2}{3 e^4 \mu^2}. \tag{5.20}
\]

After a little algebra, we see that this condition is satisfied if \( \beta > e^{-1/2} \). Thus, there three regions:
1) \(0 < m^2 < \frac{3e^4\mu^2}{32\pi^2}e^{-1/2}\) : The effective potential behaves as in Fig. 12a; spontaneous symmetry breaking occurs.

2) \(\frac{3e^4\mu^2}{32\pi^2}e^{-1/2} < m^2 < \frac{3e^4\mu^2}{16\pi^2}e^{-1}\) : The effective potential behaves as in Fig. 12b; the vacuum is symmetric.

3) \(m^2 > \frac{3e^4\mu^2}{16\pi^2}e^{-1}\) : The effective potential behaves as in Fig. 12c; again there is no spontaneous symmetry breaking.

The results for region (1) seem in contradiction with the conventional wisdom — they predict spontaneous symmetry breaking for a theory with a positive mass-squared. To show that there is no contradiction, we calculate \(\lambda\), using Eq. (5.16):

\[
\lambda = \frac{dV}{d\varphi^4}\bigg|_{\varphi_c=M} = \frac{9e^4}{8\pi^2} \left( \log \frac{M^2}{\mu^2} + \frac{11}{3} \right).
\]

(5.21)

Normally, we would choose \(M\) to be of the order of magnitude of \(m\), since \(m\) is the only dimensional parameter in the initial formulation of the theory. (It is true that there is another dimensional parameter, the position of the minimum of the effective potential, which is much greater than \(m\), but this is not originally evident.) If we choose \(M \approx m\), and let \(m^2\) be in region (1), we find that \(\lambda\) is negative (unless \(e^4\) is absurdly large). This is not the theory of which the conventional wisdom speaks; instead, it is one which is usually discarded as not having a lower bound on the effective potential. However, we see that the effect of the quantum corrections is to make the effective potential turn upward at large \(\varphi_c\), and thus remedy the situation. To obtain the more usual theory, with a positive \(\lambda\), \(m^2\) must be in either region (2) or region (3), where the vacuum is, in fact, symmetric.

Now we turn to the case of negative \(m^2\). There is a maximum at the origin, and a minimum determined by

\[
f\left(\frac{\varphi_c^2}{\mu^2}\right) = \frac{\varphi_c^2}{\mu^2} \log \frac{\varphi_c^2}{\mu^2} = \frac{16\pi^2 m^4}{3e^4 \mu^2} = \frac{16\pi^2 |m^2|}{3e^4 \mu^2}\left|m^2\right|.
\]

(5.22)

We see from Fig. 11 that this equation has only one solution. For \(|m^2| \ll \mu^2\), this solution is at \(\varphi_c \approx \mu\), which is in agreement with our results for \(m^2 = 0\). In general, the minimum occurs when

\[
|m^2| = \frac{3e^4}{16\pi^2 \varphi_c^2} \log \frac{\varphi_c^2}{\mu^2}.
\]

(5.23)
To determine the mass of the scalar particle, we calculate

\[
m^2(\phi) = \left. \frac{d^2V}{d\phi^2} \right|_{\phi=\langle \phi \rangle} = -m^2 + \frac{3e^4}{16\pi^2} \langle \phi \rangle^2 \left( 3 \log \frac{\langle \phi \rangle^2}{\mu^2} + 2 \right) = 2|m^2| + \frac{3e^4}{8\pi^2} \langle \phi \rangle^2.
\]

(5.24)

The first term is the usual (zero-loop) result for modes of this type, while the second term is just our result for the case of \( m^2 = 0 \). As \( \lambda \) (defined at the minimum of the effective potential) is increased above \( e^4 \), we find that

\[
\langle \phi \rangle^2 \approx \frac{6|m^2|}{\lambda}
\]

and

\[
m^2(\phi) = 2|m^2| \left( 1 + \frac{9e^4}{8\pi^2\lambda} \right).
\]

(5.25)

(5.26)

For \( \lambda \) much greater than \( e^4 \), the second term on the right hand side of Eq. (5.26) is negligible, and the classical result holds.

3. Electrodynamics with Massive Photons

A model which is useful for understanding the physical principals behind our results is the theory of a massive charged scalar meson coupled to a massive photon.\(^9\) If we assume that the scalar quartic coupling constant, \( \lambda \), is of the order of \( e^4 \), we need only consider diagrams with photon loops in the one-loop approximation. The expression we obtain for the effective potential is

\[
V = \frac{\lambda}{4!} \phi^4 c + \frac{3}{64\pi^2} \left\{ (m^2 + e^2\phi^2 c) \log \left( 1 + \frac{e^2\phi^2 c}{m^2} \right) - e^2\phi^2 c m^2 \right. \\
- \frac{e^4\phi^4 c}{(m^2 + e^2M^2)^2} \left( \frac{3}{2} m^4 + 7e^2m^2M^2 + \frac{25}{6} e^4M^4 \right) \\
\left. + e^4\phi^4 c \log \left( \frac{m^2}{m^2 + e^2M^2} \right) \right\},
\]

(5.27)

where \( M \) is the value of \( \phi_c \) at which \( \lambda \) is defined and \( m \) is the mass of the photon. We see that if \( m^2 \) is much larger than \( e^2M^2 \), the effective potential does not possess
a minimum away from the origin. In other words, the coupling to the photon can produce spontaneous symmetry breaking only if the photon is sufficiently light. But this is equivalent to saying that spontaneous symmetry breaking occurs only if the range of the electromagnetic interaction is sufficiently long. This agrees with physical intuition, since spontaneous symmetry breaking is essentially a correlation of the field through all of space, which should require long-range forces. It is true that long-range forces are not evident in the usual examples of the Goldstone phenomenon, where spontaneous symmetry breaking is induced by a negative mass-squared term in the Lagrangian, but this is such an unnatural term that we really don’t have any physical interpretation for it until we redefine the fields. (And when we do redefine the fields, we obtain massless particles, and thus long-range forces.)

4. A Model with Pseudo-Goldstone Bosons

Finally we consider a model in which the quantum corrections must be considered in order to completely determine the character of the vacuum, even though the Lagrangian contains a negative mass-squared term. The model is based on an SU(3) gauge symmetry; in addition to the gauge vector mesons, it has an octet of spinless mesons, \( \phi \), which we write as a \( 3 \times 3 \) traceless Hermitian matrix. If we impose the requirement that the Lagrangian be invariant under the transformation \( \phi \rightarrow -\phi \), the most general form for the non-derivative scalar meson terms in the Lagrangian is

\[
\mathcal{L}_{\text{scalar}} = -\mu^2 \text{Tr} \phi^2 + a \left( \text{Tr} \phi^2 \right)^2 + b \text{Tr} \phi^4.
\]

(5.28)

However, we note that for three-dimensional traceless Hermitian matrices

\[
\text{Tr} \phi^4 = \frac{1}{2} \left( \text{Tr} \phi^2 \right)^2.
\]

(5.29)

Thus the zero-loop approximation so the effective potential is of the form

\[
V_0 = -\mu^2 \text{Tr} \phi^2 + \lambda \left( \text{Tr} \phi^2 \right)^2.
\]

(5.30)

We note that this expression involves only \( \text{Tr}(\phi^2) \), which is invariant under the transformations of SO(8); minimizing it will determine only \( \text{Tr}(\phi^2) \), but not \( \text{Det}(\phi) \), which
is an invariant under SU(3) transformations but not under SO(8) transformations. Yet, because of the presence of the vector mesons and their couplings to the scalar mesons, the Lagrangian is invariant only under SU(3), and not under SO(8). Therefore, we must calculate the one-loop contributions to the effective potential, where the effects of the vector mesons first appear, in order to completely determine the nature of the spontaneous symmetry breaking.

The SO(8) invariance of the zero-loop effective potential has another important consequence: The spontaneous breaking of SO(8) symmetry gives rise to a single massive scalar and seven massless Goldstone bosons; we will obtain the same result in the zero-loop approximation to our model, since to that order our model is the same as an SO(8) symmetric one. However, when SU(3) is spontaneously broken, either SU(2) × U(1) or U(1) × U(1) remains as an unbroken subgroup; only four or six of the Goldstone bosons can be eaten by the vector meson through the Higgs-Kibble mechanism. The remaining massless bosons acquire a mass when loop effects are included; they are only pseudo-Goldstone bosons.\(^\text{10}\)

The most general form for the vacuum expectation value of \(\phi\) can be written as

\[
\langle \phi \rangle = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, \quad a + b + c = 0.
\] (5.31)

If \(a = b\) (or \(a = c\), etc.), there will be an unbroken SU(2) × U(1) subgroup; otherwise the unbroken subgroup will be U(1) × U(1). We assume that the vector meson masses are enough larger than those of the scalar mesons that only the contribution to the effective potential from the vector meson loops is important. The matrix \(G(\langle \phi \rangle)\) is
calculated to be

\[
G(\langle \phi \rangle) = 2g^2 \begin{pmatrix}
(b - a)^2 & 0 & 0 & \cdots & 0 \\
0 & (b - a)^2 & 0 & \cdots & 0 \\
0 & 0 & (a - c)^2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & (b - c)^2
\end{pmatrix}.
\]

(5.32)

(Note that there will be either four or two massless vector mesons depending on whether or not \(a = b\) (or \(a = c\), etc.), in agreement with our above remarks.) We can write the one-loop contribution to the potential as

\[
V_{\text{loop}} = \frac{3g^4}{8\pi^2} \left\{ (a - b)^4 \left( \log \left( \frac{(a - b)^2}{\mu^2} \right) - \frac{1}{2} \right) + (b - c)^4 \left( \log \left( \frac{(b - c)^2}{\mu^2} \right) - \frac{1}{2} \right) \\
+ (c - a)^4 \left( \log \left( \frac{(c - a)^2}{\mu^2} \right) - \frac{1}{2} \right) \right\},
\]

(5.33)

where \(\mu\) is a mass determined by the vacuum expectation value of \(\text{Tr}(\phi^2)\). Since we are interested in the relative magnitude of \(a\), \(b\), and \(c\), but not in their absolute magnitude, we minimize the one-loop contributions, leaving \(\mu\) fixed. (We ignore the zero-loop terms, since they do not depend on the relative magnitude of \(a\), \(b\), and \(c\).) One can readily believe, and somewhat less readily prove, that there is a minimum when

\[
a = b = \frac{\mu}{3}, \quad c = -\frac{2\mu}{3},
\]

(5.34)

and that this minimum is unique (aside from permutations of \(a\), \(b\), and \(c\)). Thus, there will be an unbroken SU(2) \(\times\) U(1) subgroup.

Next we turn our attention to the spinless mesons. We identify the physical meson fields with the perturbations about the minimum; we denote these by the usual
notation for the pseudoscalar octet. These are related to diagonal perturbations by

\[ a = \frac{\mu}{3} + \frac{1}{\sqrt{2}} \pi^0 + \frac{1}{\sqrt{6}} \eta, \]
\[ b = \frac{\mu}{3} - \frac{1}{\sqrt{2}} \pi^0 + \frac{1}{\sqrt{6}} \eta, \]
\[ c = \frac{2\mu}{3} - \frac{2}{\sqrt{6}} \eta. \]  

(5.35)

Substituting this into Eq. (5.33), we find

\[ \Delta m^2(\pi^0) = \frac{3g^4\mu^2}{\pi^2} \]  

(5.36)

and

\[ \Delta m^2(\eta) = \frac{9g^4\mu^2}{\pi^2}, \]  

(5.37)

where \( \Delta m^2 \) indicates the contribution of the one-loop terms to the scalar meson masses. Since isospin is an unbroken symmetry, the results for the \( \pi^+ \) and \( \pi^- \) must be the same as that for the \( \pi^0 \). Furthermore, since there are only four massive scalars, the \( K \)’s must all be Goldstone bosons, which are eaten by the four vector mesons which become massive. Finally, we note that the \( \eta \) is clearly the meson which acquired a mass in zeroth order; the pions are the pseudo-Goldstone bosons, and their entire mass is given by Eq. (5.36).

Now we use Eqs. (5.32) and (5.34) to obtain the vector meson masses; we find that the four massive vector mesons have a mass given by

\[ m^2(V) = 2g^2\mu^2. \]  

(5.38)

We thus have the relationship

\[ \frac{m^2(\pi)}{m^2(V)} = \frac{3g^2}{2\pi^2}. \]  

(5.39)
VI. The Renormalization Group

In this chapter we show that for the class of theories in which the Lagrangian does not contain any masses (or any other dimensional parameters) it is possible to improve the one-loop approximation to the effective potential by the use of renormalization group methods.\(^{11}\) Recall that the validity of the one-loop approximation required that both the coupling constants and the logarithms (e.g., \(\log(\varphi_c^2/M^2)\)) be small. However, these are not two completely independent conditions; \(M\) is arbitrary in that a variation of \(M\) can be compensated for by a suitable variation of other quantities in the theory, leaving the physics unchanged. For this to be so, it is clear that the dependence of the effective action on the coupling constants and on \(M\) must be intertwined. The nature of the interdependence is most easily determined in theories which contain no dimensional parameter other than \(M\), for in these the dependence on \(M\) is determined by dimensional analysis.

To formulate these considerations more exactly, let us consider a theory with \(n\) coupling constants \(\lambda_i\) and \(m\) fields \(\psi_j\). The \(\psi_j\) include all fields, regardless of their spin. The statement that a variation of \(M\) can be compensated for by a variation of the \(\lambda_i\) and of the normalization of the \(\psi_j\) is

\[
0 = \left[ M \frac{\partial}{\partial M} + \sum_i \tilde{\beta}_i(\lambda) \frac{\partial}{\partial \lambda_i} + \sum_j \tilde{\gamma}_j(\lambda) \int d^4x \psi_{cj}(x) \frac{\delta}{\delta \psi_{cj}(x)} \right] \Gamma, 
\]

(6.1)

where \(\Gamma\) is the effective action. The \(\tilde{\beta}_i\) and the \(\tilde{\gamma}_j\) can be functions of only the \(\lambda_i\), since there are no other dimensionless parameters available. If we apply this equation to the expansion of \(\Gamma\) in terms of the IPI Green’s functions, we obtain the usual equation of the renormalization group. However, as we have said before, this is not the best expansion for our purposes. Instead, we work with the Taylor series expansion of \(\Gamma\). Since each term in this expansion is identified by the power of momentum and the number of fields with spin, the action of the differential operator of Eq. (6.1) does not mix terms, and so Eq. (6.1) must hold term by term. Let us specialize to the case where there is only a single spinless field \(\phi(x)\), which we have now expanded about \(\Phi_c(x) = \varphi_c\), and any number of fields with spin. (The results are unchanged
if we replace the single spinless field by an SO($n$) vector of spinless fields.) A typical term in our expansion of $\Gamma$ will contain $n_j$ fields of type $j$ (possibly including $\Phi_c(x)$), some power of momentum, and a dimensionless function $F(\varphi_c, \lambda_i, M)$. Analogous to Eq. (6.1), we will obtain

$$\left[ M \frac{\partial}{\partial M} + \sum_i \beta_i(\lambda) \frac{\partial}{\partial \lambda_i} + \sum_j (n_j \gamma_j(\lambda)) + \varphi_c \frac{\partial}{\partial \varphi_c} \right] F(\varphi_c, \lambda_i, M)$$

(6.2)

Now we use the fact that $M$ is the only dimensional parameter in the theory; dimensional analysis tells us that $F$ can be a function of only $\varphi_c/M$ and the $\lambda_i$. If we define

$$t = \log \frac{\varphi_c}{M}, \quad (6.3)$$

$$\beta_i = \frac{\beta_i}{1 - \bar{\gamma}_\varphi}, \quad (6.4)$$

and

$$\gamma_j = \frac{\bar{\gamma}_j}{1 - \bar{\gamma}_\varphi}, \quad (6.5)$$

we can rewrite Eq. 6.2) as

$$\left[ -\frac{\partial}{\partial t} + \sum_i \beta_i(\lambda) \frac{\partial}{\partial \lambda_i} + \sum_j n_j \gamma_j(\lambda) \right] F(t, \lambda). \quad (6.6)$$

Assuming that we know the $\beta_i$ and the $\gamma_j$, we can write down the general solution to Eq. (6.6). It is

$$F(t, \lambda) = f(\lambda'(t, \lambda)) \exp \int_0^t dt' \sum_j n_j \gamma_j(\lambda'(t', \lambda)), \quad (6.7)$$

where $\lambda'_i(t, \lambda)$ is determined by

$$\frac{\partial \lambda'_i(t, \lambda)}{\partial t} = \beta_i(\lambda') \quad (6.8)$$

and

$$\lambda'_i(0, \lambda) = \lambda_i \quad (6.9)$$
and $f$ is an arbitrary function of $\lambda'$.

These equations are very pretty, but they involve the $\beta_i$ and the $\gamma_j$, which we don’t know until we have solved the theory exactly, at which point we don’t need the equations. However, our perturbation method allows us to obtain approximations for the $\beta_i(\lambda')$ and the $\gamma_j(\lambda')$ which are valid for small $\lambda'$: We use the one-loop calculations to obtain $\beta_i(\lambda'(0, \lambda))$ and $\gamma_j(\lambda'(0, \lambda))$, and then use Eq. (6.8) to obtain an expression for $\lambda'_i(t, \lambda)$ which is valid in that range of $t$ where $\lambda'$ remains small. What we have gained in comparison with our previous method is that $\lambda'$ may remain small even when $t$ is large, so that our new results will be valid in this region, even though our old ones were not. Let us now demonstrate the application of these methods to some models.

1. $\lambda \phi^4$ Theory

Because of its simplicity, we return to our original model. In this section we modify it slightly by choosing Eq. (3.11b) as the coupling constant renormalization condition. We can then define a function $U(t, \lambda)$ by

$$V(\varphi_c) = \frac{\varphi_c^4}{4!} U(t, \lambda).$$  \hspace{1cm} (6.10)

Our coupling constant and wave function renormalization conditions are then

$$U(0, \lambda) = \lambda$$ \hspace{1cm} (6.11)

and

$$Z(\lambda) = 1.$$ \hspace{1cm} (6.12)

If we substitute $Z(t, \lambda)$ and $U(t, \lambda)$ for $F(t, \lambda)$ in Eq. (6.6), and evaluate at $t = 0$, we obtain

$$- \left. \frac{\partial Z}{\partial t} \right|_{t=0} + 2\gamma(\lambda) = 0$$ \hspace{1cm} (6.13)

and

$$- \left. \frac{\partial U}{\partial t} \right|_{t=0} + \beta(\lambda) + 4\lambda \gamma(\lambda) = 0.$$ \hspace{1cm} (6.14)
Furthermore, we can substitute $Z(t, \lambda)$ and $U(t, \lambda)$ for $F(t, \lambda)$ in Eq. (6.7), again evaluate at $t = 0$, and obtain

$$Z(0, \lambda) = 1 = f_Z(\lambda'(0, \lambda)) = f_Z(\lambda)$$  \hspace{1cm} (6.15)

and

$$U(0, \lambda) = \lambda = f_U \lambda(\lambda'(0, \lambda)) = f_U(\lambda).$$ \hspace{1cm} (6.16)

Knowing the form of $f_Z$ and $f_U$, we can now write

$$Z(t, \lambda) = \exp \left[ 2 \int_0^t dt' \gamma(\lambda'(t', \lambda)) \right]$$  \hspace{1cm} (6.17)

and

$$U(t, \lambda) = \lambda'(t, \lambda) [Z(t, \lambda)]^2.$$  \hspace{1cm} (6.18)

Using Eqs. (6.13), (6.14), (6.17), and (6.18), and one-loop calculations for $Z(t, \lambda)$ and $U(t, \lambda)$, we obtain an improved approximation for the effective potential.

Eq. (6.18) enables us to gain further understanding of the physical significance of $\lambda'(t, \lambda)$. Suppose we change the renormalization point from $M$ to $M^\ast$. We obtain a rescaled classical field $\Phi^\ast_c$, given by

$$\Phi^\ast_c = \left[ Z \left( \log \frac{M^\ast}{M}, \lambda \right) \right]^{1/2} \Phi_c,$$  \hspace{1cm} (6.19)

and a new coupling constant $\lambda^\ast$, given by

$$\lambda^\ast = 4! \frac{V|_{\varphi_c=M^\ast}}{(\varphi^\ast_c)^4}$$

$$= \left[ Z \left( \log \frac{M^\ast}{M}, \lambda \right) \right]^{-2} U \left( \log \frac{M^\ast}{M}, \lambda \right)$$  \hspace{1cm} (6.20)

$$= \lambda' \left( \log \frac{M^\ast}{M}, \lambda \right).$$

We now proceed to our calculations. From our results in Chapter III, we obtain the one-loop approximation for $U(t, \lambda)$; it is

$$U(t, \lambda) = \lambda + \frac{3\lambda^2}{32\pi^2} \log \frac{\varphi_c^2}{M^2} = \lambda + \frac{3\lambda^2 t}{16\pi^2}.$$  \hspace{1cm} (6.21)
The one-loop approximation to $Z(t, \lambda)$ is obtained from the sum of all one-loop graphs which have two external lines carrying momenta $p$ and $-p$, respectively, and all other external momenta vanishing; we call the sum of these graphs $\Sigma(p^2)$. $Z(t, \lambda)$ is given by

$$Z(t, \lambda) = \frac{d}{dp^2} \Sigma(p^2) \bigg|_{p^2=0}.$$  

(6.22)

Dimensional analysis simplifies matters considerably; before renormalization $Z$ must be a dimensionless function of $\varphi_c^2/\Lambda^2$, so only the logarithmically divergent terms in $Z$ will survive after renormalization. Thus we need consider only the logarithmically divergent part of $Z$, which arises from the quadratically divergent graphs in $\Sigma(p^2)$. For the theory we are considering, there is only one such graph, shown in Fig. 13. Furthermore, the value of this diagram is clearly independent of $p^2$, so the one-loop approximation to $Z(t, \lambda)$ vanishes, and in the one-loop approximation we have

$$Z(t, \lambda) = 1.$$  

(6.23)

If we substitute Eqs. (6.21) and (6.23) into Eqs. (6.13) and (6.14), we obtain

$$\gamma(\lambda) = 0$$  

(6.24)

and

$$\beta(\lambda) = \frac{3\lambda^2}{16\pi^2}.$$  

(6.25)

Thus we determine $\lambda'$ from

$$\frac{d\lambda'}{dt} = \frac{3\lambda'^2}{16\pi^2}$$  

(6.26)

together with the boundary condition, Eq. (6.9), and obtain

$$\lambda'(\lambda, t) = \frac{\lambda}{1 - \frac{3\lambda}{16\pi^2}}.$$  

(6.27)

Therefore,

$$V(\varphi_c) = \frac{\varphi_c^4}{4!} U(t, \lambda) = \frac{\varphi_c^4}{4!} \lambda'(\lambda, t)$$

$$= \frac{1}{4!} \frac{\lambda \varphi_c^4}{1 - \frac{3\lambda}{32\pi^2} \log \frac{\varphi_c^2}{M^2}}.$$  

(6.28)
How does this compare with our former expression for the effective potential, Eq. (3.13) (allowing for the redefinition of $\lambda$)? The two expressions agree in the region where $|\lambda| \ll 1$ and $|\lambda \log(\varphi_c^2/M^2)| \ll 1$, which is the region where the earlier expression was valid. However, our new result is valid as long as we remain in the region where $\lambda'$ is small. First, suppose that $\lambda$ is negative. In this case, $\lambda'$ remains small as $t$ becomes large and positive, so Eq. (6.28) is valid in the region of large $\varphi_c$; we see that the effective potential has no lower bound and the theory does not exist. Thus, the only physically meaningful case is that of positive $\lambda$. In this case, $\lambda'$ remains small as $t$ becomes large and negative, but as $t$ becomes large and positive $\lambda'$ has a pole; we cannot continue past the pole. That is, Eq. (6.28) is valid for arbitrarily small $\varphi_c$, but fails for large $\varphi_c$. It tells us that the effective potential has a minimum at the origin; Eq.(3.13) predicted a maximum, but was not a reliable approximation in that region. Furthermore, Eq. (6.28) does not predict any other minima in its range of validity; this theory does not appear to display spontaneous symmetry breaking.

2. The Scalar SO($n$) Theory

If we repeat this analysis for the SO($n$) model of Chapter IV, we obtain similar results. We find that

$$\beta = \frac{3\lambda^2}{16\pi^2} \left(1 + \frac{n-1}{9}\right)$$  \hspace{1cm} (6.29)

and

$$V(\varphi_c) = \frac{\frac{1}{3!} \lambda \varphi_c^4}{1 - \frac{3\lambda}{32\pi^2} \left(1 + \frac{n-1}{9}\right) \log \frac{\varphi_c^2}{M^2}}.$$  \hspace{1cm} (6.30)

Particularly interesting is the case of very large $n$, where our result for the effective potential can be approximated by

$$V(\varphi_c) = \frac{\frac{1}{3!} \lambda \varphi_c^4}{1 - \frac{n\lambda}{96\pi^2} \log \frac{\varphi_c^2}{M^2}}.$$  \hspace{1cm} (6.31)

There is an alternative method of improving our approximation when $n$ is very large. We note that the dominant contribution in each order will be from the diagrams shown in Fig. 14, both because these are most numerous for large $n$ and because these have the most logarithms for a given power of $\lambda$. Summing these diagrams
is somewhat complicated by the fact that the coupling constant at a vertex can be either $\lambda/3$ or $\lambda$. However, if $n$ is large, almost all vertices will have a factor of $\lambda/3$; if we use this factor for all vertices (this is in the same spirit as that which motivated Eq. (6.31)), we obtain the renormalization group result, Eq. (6.31). In other words, the renormalization group approach has enabled us to sum the “leading logarithms”.

3. Massless Scalar Electrodynamics

Next we apply these methods to massless scalar electrodynamics. Our expansion of the effective action is

$$\Gamma = \int d^4x \left\{ -V(\varphi_c) - \frac{1}{4} (\partial^\mu A_c^\mu - \partial^\nu A_c^\nu)^2 H(\varphi_c) + \frac{1}{2} \left[ (\partial_\mu \Phi_1c)^2 + (\partial_\mu \Phi_2c)^2 \right] Z(\varphi_c) + e \left[ -\partial_\mu \Phi_1c A_\mu^c \Phi_2c + \partial_\mu \Phi_2c A_\mu^c \Phi_1c \right] F(\varphi_c) + e^2 \left[ \Phi_1c A_{\mu c} A_\mu^c + \Phi_2c A_{\mu c} A_\mu^c \right] G(\varphi_c) + \cdots \right\},$$  

(6.32)

where the dots represent terms which do not concern us here because they are not involved with the renormalization procedure. Our renormalization conditions are

$$H(M) = 1,$$  

(6.33)

$$Z(M) = 1,$$  

(6.34)

$$F(M) = 1,$$  

(6.35)

and

$$U(M) = \frac{4!}{\varphi_c^4} V(\varphi_c)|_{\varphi_c=M} = \lambda.$$  

(6.36)

(Note that we have modified the definition of the scalar quartic coupling constant, as in the previous sections.) We will also find that

$$G(M) = 1.$$  

(6.37)

If we apply Eq. (6.6) to $H$, $Z$, $eF$, and $U$ and evaluate at $\varphi_c = M$, we obtain

$$-\frac{\partial H}{\partial t} + 2\gamma_A = 0,$$  

(6.38)
\[ \frac{\partial Z}{\partial t} + 2\gamma \varphi = 0, \quad (6.39) \]

\[ -\frac{\partial (eF)}{\partial t} + \beta_e + e\gamma_A + 2e\gamma \varphi = 0, \quad (6.40) \]

and

\[ -\frac{\partial U}{\partial t} + \beta_\lambda + 4\lambda\gamma \varphi = 0. \quad (6.41) \]

Once we have calculated \( H, Z, F, \) and \( U, \) we can use these equations to find the \( \beta \)'s and the \( \gamma \)'s, and then determine \( \lambda' \) and \( e' \).

Before we calculate the functions \( H, Z, F, \) and \( G, \) we should consider the consequences of electromagnetic gauge invariance. First of all, there is the usual result that \( Z_1 = Z_2, \) where \( Z_1 \) and \( Z_2 \) are the usual rescaling factors in the quantized action; they are not the same as any of the functions which appear in our expansion of the effective action. Because \( Z_1 = Z_2, \) there is a relationship among the counter-terms for the scalar wave function renormalization and the scalar-photon vertices. The same relationship must hold among those terms in \( Z, F, \) and \( G \) which are logarithmic in \( \varphi_c, \) since these terms were made finite by the counter-terms. That is, \( Z, F, \) and \( G \) are equal, up to terms which are finite and independent of \( \varphi_c. \) The reason that finite differences might exist is that these functions are not the same as the functions one considers in the usual renormalization procedure; for example, \( Z \) involves a sum of diagrams with various numbers of external lines, while the usual scalar self-energy involves only diagrams with two external lines.

It is also a usual consequence of the Ward identities that the photon self-energy vanishes at zero four-momentum and is transverse for non-zero four-momentum. This is still true, but not of immediate use to us, since we consider the self-energy (which has no external scalar lines) in conjunction with diagrams with many external scalar lines. However, the counter-terms are restricted by this result, so we should not expect any divergent non-transverse term in the photon self-energy.

As we showed previously, we need consider only those diagrams which give logarithmically divergent contributions to the functions we are calculating. These diagrams are shown in Figs. 15–18. (Diagrams 18b–d are included because there are
implicit external scalar lines, even though they are not shown explicitly on the prototype graph.) Several of these diagrams can be neglected: Diagrams 15b, 15c, 16b, 16c, and 16d are independent of $p^2$, so they cannot contribute to the wave function renormalization, while Diagram 17c must vanish if either of the external scalar momenta is zero and thus must be at least of order $p^2$ and finite. The remaining diagrams must be calculated using a regulator, such as that of 't Hooft and Veltman, which preserves the gauge invariance. Doing this, one obtains

$$H = 1 - \frac{e^2}{24\pi^2}t$$

(6.42)

and

$$Z = F = G + 1 + \frac{3e^2}{8\pi^2}t.$$  

(6.43)

(It is also interesting to note that the contributions from Diagrams 18b–d cancel, as might be expected from a naive application of the Ward identities.) We obtain $U$ from our previous calculation; it is

$$U = \lambda + \left(\frac{5\lambda^2}{24\pi^2} + \frac{9e^4}{4\pi^2}\right)t.$$  

(6.44)

We now use Eqs. (6.38–41) to obtain

$$\gamma_A = -\frac{e^2}{48\pi^2},$$  

(6.45)

$$\gamma_\phi = \frac{3e^2}{16\pi^2},$$  

(6.46)

$$\beta_e = \frac{e^3}{48\pi^2},$$  

(6.47)

and

$$\beta_\lambda = \frac{1}{4\pi^2} \left(\frac{5}{6}\lambda^2 - 3e^2\lambda + 9e^4\right).$$  

(6.48)

We must then solve

$$\frac{de'}{dt} = \frac{3e'^3}{48\pi^2}$$  

(6.49)

and

$$\frac{d\lambda'}{dt} = \frac{1}{4\pi^2} \left(\frac{5}{6}\lambda'^2 - 3e'^2\lambda' + 9e'^4\right).$$  

(6.50)
The former equation is easily solved, yielding

\[ e'^2 = \frac{e^2}{1 - \frac{e^2}{24\pi^2} t}. \]  

(6.51)

To solve the latter equation, we define \( R = \lambda'/e'^2 \), and obtain the equivalent equation

\[ e'^2 \frac{dR}{d(e'^2)} = 5R^2 - 19R + 54. \]  

(6.52)

This can be easily solved, and we finally obtain

\[ \lambda' = \frac{e'^2}{10} \left[ \sqrt{719} \tan \left( \frac{\sqrt{719}}{2} \log e'^2 + \theta \right) + 19 \right], \]  

(6.53)

where \( \theta \) is chosen so that \( \lambda' = \lambda \) when \( e' = e \).

At this point in our consideration of the \( \lambda \phi^4 \) model we used Eq. (6.18) to obtain an expression for the effective potential which was valid in a larger region than the one we had originally obtained; can we do this in the present case? If \( t \) becomes large and positive (large \( \varphi_c \)), \( e' \) becomes large; it \( t \) becomes large and negative (small \( \varphi_c \)), \( e' \) becomes small but \( \lambda' \) becomes large. Thus, our new expression for the effective potential is valid only in the region of small \( t \), where our previous expression was valid.

If there are minima hiding in regions of very large or very small \( \varphi_c \), we have not found them; we still know only of the minimum we found originally. Although we have not found any more minima, we have not wasted our time in doing these calculations. Previously, we required that \( \lambda \) be of the order of \( e^4 \) in order that the minimum occur; we can now escape this restriction. Looking at Eq. (6.53), we see that as \( e' \) is varied over a relatively small range, the argument of the tangent changes by \( 2\pi \), so that \( \lambda' \) takes on all possible values. (Of course, we can only trust this statement for that part of the variation in which \( \lambda' \) remains small.) Thus, if \( \lambda \) is not of the order of \( e^4 \), but is still small, we can change the renormalization point so that \( \lambda \), defined at the new renormalization point, is of the order of the new \( e^4 \). Our original calculations, using the new \( \lambda, e, \) and \( M \), are then valid for small \( t \) (that is,
$\varphi_c$ near $M$) and we find a minimum in the effective potential. That is, for any small values of $\lambda$ and $\epsilon$, massless scalar electrodynamics does not exist.

4. A Massless Yang-Mills Theory

Finally, we consider a non-Abelian gauge theory: the theory of an SU(2) gauge particle coupled to a scalar triplet, with the Lagrangian

$$\mathcal{L} = -\frac{1}{4} \left( \partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu - g \vec{A}_\mu \times \vec{A}_\nu \right)^2 + \frac{1}{2} \left( \partial_\mu \vec{\phi} - g \vec{A}_\mu \times \vec{\phi} \right)^2 - \frac{\lambda}{4!} \left( \vec{\phi} \cdot \vec{\phi} \right)^2$$

(6.54)

+ counter-terms.

In calculating Feynman diagrams, we must include terms from the so-called “ghost particles”. In the Landau gauge the couplings of the fictitious ghost particle are given by the effective Lagrangian

$$\mathcal{L}_{\text{ghost}} = \partial_\mu \bar{\chi} \partial^\mu \chi - g \bar{\chi} \vec{A}_\mu \cdot \partial^\mu \chi.$$  

(6.55)

We note that in this gauge the ghosts do not couple to the scalar mesons; as a result, the ghost propagator to be used in prototype graphs remains

$$\frac{i}{k^2}$$

(6.56)

Thus, there will be infrared divergences arising from ghost loops, even when $\varphi_c$ is non-vanishing; these infrared divergences complicate the renormalization procedure considerably.

The usual way to renormalize a theory such as this would be to choose some arbitrary point in momentum-space and to do all renormalizations there, rather than at zero four-momentum. Furthermore, by seeing how the coupling constants and the wave function normalizations changed when this point was varied, we could hope to obtain information about the high or low momentum behavior of the Green’s functions. However, this is not quite what we want; we want to know about the behavior of the effective potential at high or low field strength, which means that we must vary a renormalization point in field strength-space. (In the theories we considered previously, both methods would have yielded the same $\beta$’s and $\gamma$’s, since
the same diagrams contribute in either method. For the theory at hand this is no longer true; the contribution from ghost loop diagrams will change if a momentum-space renormalization point is varied, but not if a field strength-space renormalization point is changed.

Thus, we must introduce two arbitrary dimensional parameters, one in field strength space \( (M) \) and one in momentum space \( (\mu) \). It would seem that we would have to replace Eq.(6.1) by a pair of more complicated equations, corresponding to the presence of two arbitrary parameters. Fortunately, we can avoid this complication, at least for some regions of field strength-space. We note that there are two types of one-loop diagrams: those without ghost loops, which are infrared convergent, and those with ghost loops, which are infrared divergent but independent of \( \varphi_c \). Now suppose that we choose \( gM \) to be much larger than \( \mu \), and that we consider field strengths such that \( g\varphi_c \) is much larger than \( \mu \). Since the contribution of the first type of diagram is analytic at \( \mu = 0 \), we can approximate it by its value at that point. Since the contribution of the second type is independent of \( \varphi_c \), it disappears from the effective action once we subtract at \( \varphi_c = M \).

Actually, the constraints imposed by the Ward identities, which must be preserved in order that the theory be renormalizable, add a further complication. To illustrate this, consider the wave function renormalization of the vector meson. We denote by \( \Pi^{\mu\nu} \) the sum of all graphs with two external vector mesons, with momenta \( p \) and \( -p \), and any number of scalars with vanishing momenta. We can write

\[
\Pi^{\mu\nu} = \left( g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) A(p^2, \varphi_c) + \frac{p^\mu p^\nu}{p^2} B(p^2, \varphi_c). \tag{6.57}
\]

It is tempting to assume that the Ward identities require \( B \) to vanish, but this is not so because we are considering graphs with external scalars, and not just vacuum polarization graphs. However, the Ward identities do require that the counter-terms contribute to \( A \) only; \( B \) must be cutoff independent without renormalization. It is, being of the order of

\[
g^2 \log \left( \frac{g^2 \varphi_c^2}{\mu^2} \right). \tag{6.58}
\]
\( B \) is non-vanishing because the ghost and the non-ghost diagrams are not separately transverse, although their logarithmically divergent longitudinal parts cancel. A similar effect will arise in the renormalization of the \( A^\mu A^\nu A^\lambda \) and \( A^\mu A^\nu A^\lambda A^\tau \) vertices; because of the restrictions on the counter-terms, there will be terms in the effective action containing \( \log(g^2M^2/\mu^2) \). Although we can avoid these terms in our one-loop renormalization group analysis, they will arise in diagrams with two or more loops. Since we want to be able to neglect two-loop graphs, we must require that

\[
g^2 \log \left( \frac{g^2 M^2}{\mu^2} \right) \ll 1 \tag{6.59}
\]

and we must stay in a region of field strength-space where

\[
g^2 \log \left( \frac{g^2 \varphi_c^2}{\mu^2} \right) \ll 1. \tag{6.60}
\]

If we remain within these restrictions, \( \mu \) will have a negligible effect on the renormalization group analysis.

However, we now find that there is an ambiguity in the definition of \( g \). If we had followed the usual procedure of renormalizing in momentum-space, we would have obtained the same results whether we defined \( g \) in terms of the \( A^\mu A^\nu A^\lambda \) vertex or in terms of the \( \phi \phi A^\mu \) vertex. Using our procedure, the two definitions give different results, since a ghost loop contributes to the former but not to the latter. The difference is due to having pushed different amounts of \( \log(g^2M^2/\mu^2) \) into the higher loop contributions; therefore, if we are to be able to consistently neglect two-loop graphs, the ambiguity must be small. We will find that it is, and that different choices of the renormalization conditions (even renormalizing in momentum-space) lead to only small quantitative, but not qualitative, changes.

Let us choose to define \( g \) in terms of the \( \phi \phi A^\mu \) vertex. We then have the same renormalization conditions as in scalar electrodynamics, namely Eqs. (6.33–36), where the functions are defined by the obvious generalization of Eq. (6.32). The one-loop diagrams contributing to \( H, Z, \) and \( F \) are shown in Figs. 19–21. Ignoring finite terms of the order of \( \mu^2/g^2 M^2 \), we obtain

\[
H = 1 + \frac{23}{6} \frac{g^2}{16\pi^2} \log \frac{\varphi_c^2}{M^2}, \tag{6.61}
\]
\begin{align*}
Z &= 1 + 6 \frac{g^2}{16\pi^2} \log \frac{\varphi_c^2}{M^2}, \quad (6.62) \\
\text{and} \\
F &= 1 + \frac{9}{2} \frac{g^2}{16\pi^2} \log \frac{\varphi_c^2}{M^2}.
\end{align*}

From a calculation of the effective potential, we obtain

\[ U = \lambda + \left( 36g^4 + \frac{11}{6} \lambda^2 \right) \frac{1}{16\pi^2} \log \frac{\varphi_c^2}{M^2}. \quad (6.64) \]

Following our previous analysis, we obtain from Eqs. (6.62) and (6.64)

\[ \gamma_\varphi = 6 \frac{g^2}{16\pi^2} \quad (6.65) \]

and

\[ \beta_\lambda = \frac{1}{16\pi^2} \left( 72g^4 - 24\lambda g^2 + \frac{11}{3} \lambda^2 \right). \quad (6.66) \]

These two results are unaffected by the ambiguity in the definition of \( g \). We now use Eqs. (6.61), (6.63), and (6.65) to obtain

\[ \gamma_A = \frac{23}{6} \frac{g^2}{16\pi^2} \quad (6.67) \]

and

\[ \beta_g = -\frac{41}{6} \frac{g^2}{16\pi^2}. \quad (6.68) \]

These results are affected by a change in the renormalization conditions; however, the effect is slight (e.g., the \(-\frac{41}{6}\) in \( \beta_g \) is replaced by \(-7\) if renormalization is done in momentum-space).

Proceeding as before, we obtain

\[ g'^2 = \frac{g^2}{1 + \frac{41}{3} \frac{g^2}{16\pi^2}} \quad (6.69) \]

and

\[ \lambda' = -\frac{41}{22} g^2 \left\{ 41 \sqrt{8543} \tan \left( \frac{\sqrt{8543}}{82} \log g^2 + \theta \right) - \frac{31}{41} \right\}. \quad (6.70) \]

These results are reminiscent of the results for scalar electrodynamics, Eqs. (6.51) and (6.53). However, we see that \( g' \) is small when \( t \) is large and negative, which is
just the opposite of the manner in which $e'$ behaves. As before, $\lambda'$ becomes large in either case, so the range of validity of the one-loop approximation to the effective potential cannot be extended; however, we can remove any restriction on the relative magnitude of $\lambda$ and $g$. 
VII. Conclusions

Our most important result is the discovery that spontaneous symmetry breaking can be induced by radiative corrections in a theory which at first glance appears to have a symmetric vacuum; in particular, this occurs in massless gauge theories. It is true that we have based this conclusion on perturbation theory calculations, but there does not seem to be any obvious reason why it should be any less plausible than any other conclusion based on perturbative calculations in quantum field theory, at least for small coupling constants.

Although we have obtained interesting results in some of the theories we have considered, it does not appear that they will enable us to formulate a renormalizable theory of the weak and electromagnetic interactions that will be any less ugly than the plethora of existing models. Rather, it seems profitable to abstract a physical principle from our results: When the infrared divergences in a massless theory become unbearable, nature avoids them by the introduction of an asymmetric vacuum. If we are willing to accept this speculation, we may further speculate that this principle remains true even when the theory does not contain any fundamental scalar fields, that is, when some compound field develops the vacuum expectation value that induces the spontaneous symmetry breaking. We may then hope that when a satisfactory renormalizable model of the weak and electromagnetic interactions is found, it will be possible to show that it is equivalent to a theory containing neither negative mass-squared terms nor Higgs scalars. The latter theory, like the ones we have considered, would have relationships among its coupling constants; pushing our optimism to a probably absurd level, we might hope that it contained only one parameter, the fine structure constant, which would then be determined.

Thus it seems clear that it is desirable to apply our methods to theories without scalar particles to see if any can be found in which radiative corrections induce spontaneous symmetry breaking. As mentioned in Chapter II, our formalism can be readily applied to the study of such theories. Unfortunately, this search has so far been fruitless.
Appendix

Connection with a Theorem of Georgi and Glashow

There is a theorem due to Georgi and Glashow\textsuperscript{14} which states that if the Lagrangian contains an unbroken subgroup of its symmetry group, the relationships arising from the presence of the unbroken symmetry remain true even when the parameters in the Lagrangian are changed by a small amount (for example, by radiative corrections), unless the theory contains a massless particle which is not a Goldstone boson. Our results for massive scalar electrodynamics appear to contradict this result; we found that, for a sufficiently small mass, the one-loop corrections to the effective potential could induce spontaneous symmetry breaking, even though the Lagrangian contained an unbroken symmetry. The resolution of this conflict lies in the requirement that the variation of the parameters be small; the one-loop corrections are not. If we make them arbitrarily small by letting $e$ approach zero, we see from the discussion following Eq. (5.20) that $m^2$ does not remain in region (1), the region where spontaneous symmetry breaking occurs; instead, it moves into region (2) or (3), and the symmetry of the theory remains unbroken.
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Figure 1: Two diagrams which contribute to the effective potential in scalar electrodynamics. The wiggly lines represent photons, while the straight lines represent scalar mesons.

Figure 2: A two-loop diagram which contributes to the effective potential in scalar electrodynamics.
Figure 3: The only zero-loop diagram in the $\lambda\phi^4$ model.

(a) + + + . . . (a)

(b) $B^{(1)}$

Figure 4: The one-loop diagrams which contribute to the effective potential in the $\lambda\phi^4$ model.

(b) $C^{(1)}$
Figure 5: The behavior of the effective potential in many models.

Figure 6: a) A two-loop graph occurring in the $\lambda\phi^4$ model.  
b) The corresponding prototype graph.
Figure 7: A prototype graph which contributes to the effective potential in the $\lambda\phi^4$ model.

Figure 8: The two types of scalar-vector vertices in gauge theories. The wiggly lines represent vector mesons, the straight lines represent scalar mesons.
Figure 9: A diagram which does not contribute to the effective potential in Landau gauge.

Figure 10: The behavior of the effective potential in the Yukawa model when \( g^4 > \lambda^2/16 \).
Figure 11: The function $f(x) = x \log(x)$. 
Figure 12: The behavior of the effective potential in massive scalar electrodynamics.
Figure 13: The only diagram contributing to $Z(\varphi_c)$ in the $\lambda\phi^4$ model in the one-loop approximation.

Figure 14: The most important prototype graphs for the calculation of the effective potential in the scalar $SO(n)$ model.
Figure 15: The graphs which contribute to $H(\varphi_c)$ in scalar electrodynamics in the one-loop approximation.
Figure 16: The graphs which contribute to $Z(\varphi_c)$ in scalar electrodynamics in the one-loop approximation.
Figure 17: The graphs which contribute to $F(\varphi_c)$ in scalar electrodynamics in the one-loop approximation.
Figure 18: The graphs which contribute to $G(\varphi_c)$ in scalar electrodynamics in the one-loop approximation.
Figure 19: The one-loop graphs which give a non-vanishing contribution to $H(\varphi_c)$ in the massless Yang-Mills model.

Figure 20: The one-loop graphs which give a non-vanishing contribution to $Z(\varphi_c)$ in the massless Yang-Mills model.
Figure 21: The one-loop graphs which give a non-vanishing contribution to $F(\varphi_c)$ in the massless Yang-Mills model.