“Extrinsic” and “intrinsic” data in quantum measurements: asymptotic convex decomposition of positive operator valued measures

Andreas Winter

Department of Computer Science, University of Bristol,
Merchant Venturers Building, Woodland Road, Bristol BS8 1UB, United Kingdom
(Dated: 19th November, 2002)

We study the problem of separating the data produced by a given quantum measurement (on states from a memoryless source which is unknown except for its average state), described by a positive operator valued measure (POVM), into a “meaningful” (intrinsic) and a “not meaningful” (extrinsic) part.

We are able to give an asymptotically tight separation of this form, with the “intrinsic” data quantified by the Holevo mutual information of a certain state ensemble associated to the POVM and the source, in a model that can be viewed as the asymptotic version of the convex decomposition of POVMs into extremal ones. This result is applied to a similar separation theorem for quantum instruments and quantum operations, in their Kraus form.

Finally we comment on links to related subjects: we stress the difference between data and information (in particular by pointing out that information typically is strictly less than data), derive the Holevo bound from our main result, and look at its classical case: we show that this includes the solution to the problem of extrinsic/intrinsic data separation with a known source, then compare with the well-known notion of sufficient statistics. The result on decomposition of quantum operations is used to exhibit a new aspect of the concept of entropy exchange of an open dynamics.

An appendix collects several estimates for mixed state fidelity and trace norm distance, that seem to be new, in particular a construction of canonical purification of mixed states that turns out to be valuable to analyze their fidelity.

PACS numbers: 03.65.Ta, 03.67.Hk
Keywords: quantum measurement, data, information, Holevo bound

I. THE PROBLEM

Consider a quantum system, represented by a Hilbert space $\mathcal{H}$ (which we assume to be of dimension $d < \infty$ in the sequel), and a measurement on this system, described by a positive operator valued measure (POVM) $a = (a_1, \ldots, a_m)$, $a_j \in \mathcal{B}(\mathcal{H})$ such that $a_j \geq 0$ and $\sum_j a_j = 1$.

Following [2] and [28] we shall be concerned with the question “How much information is obtained by $a$?”, beginning with a clarification what this question should mean at all. Imagine that a family of states (represented by density operators) $\rho_i$ on $\mathcal{H}$ is given, let us say with a priori probabilities $p_i$, such that the density operator of this source of states is $\rho = \sum_i p_i \rho_i$, then the “information” in question could mean the information in $j$ about $i$, and one way to quantify it would be given by Shannon’s mutual information $I(i \wedge j)$. Note that this is in general less than the amount of raw data, which is operationally quantified by the entropy of the distribution of the $j$: $H(\lambda)$, with $\lambda_j = \text{Tr} (\rho a_j)$, due to Shannon’s source coding theorem [24].

This choice however is rather arbitrary: asking about the identity of the state from a list. Why not allowing a different list, or asking for some property of the state. Also, mutual information is a measure of correct identification; but what if we need only “almost correct” identification, as in quantum statistical detection theory [10]?

It seems hence that specifying the information in measurement results, or even only the amount, in an operationally satisfying way, is problematic, and one reason might be the complementarity of quantum mechanics: qualitatively, accessing some observable property optically entails rather poor performance for others. Nevertheless, it is quite obvious intuitively that in almost any POVM there is “quantum noise”, i.e. redundancy put into the $j$ by the very quantum mechanical probability rule, most simply due to nonorthogonality of the operators $a_j$, for example in an overcomplete system (see e.g. [14]).

Our approach will thus be from the opposite end: instead of attempting the impossible, defining what “useful” means in any circumstances, we adopt a very simple criterion of uselessness: statistical independence form the measured states, because independent randomness can be generated from outside without accessing the quantum system. On the other hand we do not permit a distortion of the measurement itself, so that we are forced to consider a simulation of the original measurement by means of, first, a random choice $\nu$ of a measurement $a^{(\nu)}$ from a list and, second, computation of a result from the outcome of this measurement and the random choice, such that the statistical distribution of these results is indistinguishable from the ones of the original measurements.

*Electronic address: winter@cs.bris.ac.uk
on any prepared state.

Because we can absorb the computation of the results into the labelling of the \( a^{(\nu)} \), this means that we aim at finding such POVMs, whose indices are labelled by the same \( j \) as \( a \) and probabilities \( x_{\nu} \), such that

\[
a = \sum_{\nu} x_{\nu} a^{(\nu)}, \quad \text{i.e. } \forall j \ a_j = \sum_{\nu} x_{\nu} a_j^{(\nu)}. \tag{1}
\]

(The operators must be the same because otherwise there would states that induce distinguishable outcome distributions. Below we will introduce an element of approximation into this scheme).

Why should we want to do such a decomposition, interesting though the structure exhibited (convex set of POVMs) might be mathematically? Observe that each \( a^{(\nu)} \) has its distribution of outcomes, with the probabilities \( \Lambda_j^{(\nu)} = \text{Tr} (a_j^{(\nu)}) \) of \( j \) conditional on \( \nu \). Shannon’s source coding theorem quantifies the amount of data in such a source as the (Shannon) entropy

\[
H(\lambda^{(\nu)}) = \sum_j -\lambda_j^{(\nu)} \log \lambda_j^{(\nu)},
\]

by compression (we note that in this paper all logs and exps are to basis 2). Hence, on average, one needs

\[
H(j|\nu) := \sum_{\nu} x_{\nu} H(\lambda^{(\nu)})
\]

bits to faithfully compress the data (\( j \)), given \( \nu \) as side-information.

This motivates the study of the function

\[
\delta(\rho, a) := \min \left\{ H(j|\nu) : a = \sum_{\nu} x_{\nu} a^{(\nu)} \right\}, \tag{2}
\]

which is the minimum data rate (in Shannon’s sense) for exact reconstruction of the data.

**Example 1** Look at a qubit system, \( \mathbb{C}^2 \), with basis \{\( |0\rangle, |1\rangle \)\}: there let us consider the five “Chrysler” states (in analogy to the “Mercedes” trine states)

\[
|e_t\rangle = \left( \cos \frac{\pi t}{5} |0\rangle + \sin \frac{\pi t}{5} |1\rangle \right), \quad \text{for } t = 0, \ldots, 4.
\]

The collection \( a = (\frac{2}{5} |e_t\rangle \langle e_t|)_{t = 0, \ldots, 4} \) is a POVM, and we can determine its decompositions into extremal ones: these latter are given by putting weights on the \( |e_t\rangle \langle e_t| \), and it is straightforward that for an extremal POVM at most 3 can be nonzero (as the “Chrysler” states form a pentagon on the Bloch sphere equator). In fact, every extremal must be of the form

\[
(\alpha |e_t\rangle |e_t\rangle + \beta |e_t+2\rangle |e_t+2\rangle + \gamma |e_t+3\rangle |e_t+3\rangle), \quad t = 0, \ldots, 4,
\]

indices understood modulo 5. From here one can determine the weights to be

\[
\alpha = 1 - \left( \cot \frac{2\pi}{5} \right)^2 \approx 0.5528
\]

\[
\beta = \frac{1}{2} \left( \sin \frac{2\pi}{5} \right)^{-2} \approx 0.8944
\]

For simplicity now look at the maximally mixed state \( \rho = \frac{1}{2} I \), for which it is unimportant which decomposition into these extremal POVMs is chosen, as all contributions \( \nu \) will give the same Shannon entropy:

\[
\delta(\rho, a) = H(j|\nu) = H(\frac{\alpha + \beta + \beta}{2}, \frac{\beta}{2}, \frac{\beta}{2})
\]

\[
= H\left( 1 - \frac{\beta}{2}, \frac{\beta}{2}, \frac{\beta}{2} \right)
\]

\[
= H(1 - \beta, \beta) + \beta \approx 1.5447
\]

In contrast, the main theorem below will achieve a rate of \( H(\rho) = 1 \), asymptotically.

The computation of \( \delta(\rho, a) \) is an interesting problem in its own right (in particular the question if anything can be gained on \( \delta \) by considering multiple copies, i.e. the additivity problem), however we take a different approach, bearing in mind that the operational content of Shannon’s theorem involves block coding — i.e., a large number \( l \) of independent copies of the simple system described above, and an arbitrarily small yet nonzero error probability:

Thus we are really decomposing the POVM

\[
a^{\otimes l} = (a_{j_1} \otimes \cdots \otimes a_{j_l}), j_l \in \{1, \ldots, m\},
\]

where we have introduced the notation \( j^l = j_1 \ldots j_l \) for a string of symbols, used henceforth. And the error introduced through block compression entails that instead of eq. (1) we will only have

\[
a^{\otimes l} \approx A = \sum_{\nu} x_{\nu} A^{(\nu)}, \tag{3}
\]

where the \( \approx \) sign is made precise to mean “average approximation of outcome statistics”: assuming an ensemble \( \{\sigma_k, q_k\} \) with \( \sum_k q_k \sigma_k = \rho^{\otimes m} \), there is the joint distribution of input \( k \) and output \( j^l \) when applying \( a^{\otimes l} \)

\[
\gamma(k, j^l) = q_k \text{Tr} (\sigma_k a_{j^l}), \tag{4}
\]

and likewise for \( A \):

\[
\Gamma(k, j^l) = q_k \text{Tr} (\sigma_k A_{j^l}). \tag{5}
\]

Then we require that, independent of the particular ensemble,

\[
\frac{1}{2} \| \gamma - \Gamma \|_1 = \sum_{k, j^l} \frac{1}{2} |\gamma(k, j^l) - \Gamma(k, j^l)| \leq \epsilon. \tag{CP}
\]
(It is not difficult to see that eq. (4) raised to the the \( l \)th tensor power, together with Shannon compression of the outcomes of \( a^{(\nu)}_1 \otimes \cdots \otimes a^{(\nu)}_N \) for the probably \( \nu_1 \ldots \nu_l \) yields exactly that). Indeed we can, using the abbreviation \( \omega = \rho^{\otimes l} \), rewrite eq. (4) as

\[
\gamma(k,j') = \text{Tr} \left( \omega^{-1/2} q_k \sigma_k \omega^{-1/2} \sqrt{\omega} a_{j'} \sqrt{\omega} \right),
\]

observing that the \( S_k = \omega^{-1/2} q_k \sigma_k \omega^{-1/2} \) form a POVM on \( \mathcal{H}^{\otimes l} \) (this fact was observed before, and used in \( \| \) to classify all ensembles with a given average state). Similarly

\[
\Gamma(k,j') = \text{Tr} \left( S_k \sqrt{\omega} A_{j'} \sqrt{\omega} \right),
\]

and we can rewrite and estimate the left hand side of (CP) as follows:

\[
\frac{1}{2} \| \gamma - \Gamma \|_1 = \sum_{j'} \sum_k \frac{1}{2} \left| \text{Tr} \left( S_k \sqrt{\omega} (a_{j'} - A_{j'}) \sqrt{\omega} \right) \right| \leq \sum_{j'} \frac{1}{2} \| \sqrt{\omega} (a_{j'} - A_{j'}) \sqrt{\omega} \|_1,
\]

so (CP) is in fact implied by

\[
\sum_{j'} \frac{1}{2} \| \sqrt{\rho^{\otimes l}} (A_{j'} - a_{j'}) \sqrt{\rho^{\otimes l}} \|_1 \leq \epsilon.
\]

(CM)

Notice that the condition can be phrased in a particularly nice way introducing the quantum operations

\[
\varphi^{\otimes l} : \sigma \mapsto \sum_{j'} \text{Tr} (\sigma a_{j'}) (j')^{\otimes l},
\]

\[
\Phi^{\otimes l} : \sigma \mapsto \sum_{j'} \text{Tr} (\sigma A_{j'}) (j')^{\otimes l}.
\]

Namely, for a purification \( \pi \) of \( \rho \), (CM) is easily seen to be equivalent to

\[
\frac{1}{2} \| (\text{id} \otimes \Phi^{\otimes l})(\pi^{\otimes l}) - (\text{id} \otimes \varphi^{\otimes l})(\pi^{\otimes l}) \|_1 \leq \epsilon.
\]

The organization of the paper is as follows: In section \[\text{II}\] we will present our main theorem \[\text{III}\] and its proof, which is much more satisfying than results in previous work \[\text{II}\], that can now be regarded as precursors: they are shown to easily follow from theorem \[\text{III}\] in section \[\text{II}\]. Section \[\text{IV}\] is concerned with the asymptotic optimality of our main theorem, a strong converse result, theorem \[\text{V}\]. After this, in section \[\text{VI}\] we apply our result to a kind of asymptotic normal form of completely positive trace preserving maps (operations as well as instruments), and present an extensive discussion in section \[\text{VII}\]: we restate our observation from \[\text{VI}\] that one ought to distinguish obtained data from information, give a new, conceptually simple proof of the Holevo bound, remark on the classical case of the main theorem (which includes the problem of separating extrinsic and intrinsic data under a known source ensemble), comment on the related concept of sufficient statistics, and discuss the bearing of our results on the concept of entropy exchange of an open dynamics of a system. We close with a challenging open problem. An appendix features several not widely known facts about the mixed state fidelity, in particular introducing canonical purifications of mixed states, a second appendix collects properties of typical sequences and typical subspaces, used in the main text.

\section{Separating Extrinsic and Intrinsic Data}

We want to represent (up to a small deviation as specified by the (CM) condition) \( a^{\otimes l} \) as a convex combination of POVMs \( \mathbf{A}^{(\nu)} \), with positive weights \( x_\nu \nu = 1, \ldots, N \), each being defined on the set \([m]^l\) and having a small number \( M \) of sequences on which it is supported (i.e. where \( A^{(\nu)}_{j'} \neq 0 \)): this is an even stronger requirement than the entropy condition we had considered in the introduction. Performing \( \mathbf{A} \) amounts to choosing a \( \nu \) (with probability \( x_\nu \)), and performing \( \mathbf{A}^{(\nu)} \), which itself can generate at most \( M \) different outcomes: the \( \nu \)-part of the produced data is obviously independent of the incoming signal, while the measurement outcome (conditional on the \( \nu \) chosen) contains the useful information.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{diagram.png}
\caption{The source represents a number of possible states encountered by the POVM, but there is no way of knowing which is present (apart from the apriori distribution). The data produced by the measurement is then stored in a record. The rates of these processes are represented by the sizes of the different boxes and width of the data flow arrows: originally the rates of the source and of the measurement outcomes are both large.}
\end{figure}

Our central result is:

\textbf{Theorem 2} There exist POVMs \( \mathbf{A}^{(\nu)} \) on \([m]^l\), \( \nu = 1, \ldots, N \), each supported on a set of cardinality at most \( M \), where

\[
M = \exp \left( H(\lambda; \hat{\rho}) + O(\sqrt{l}) \right),
\]

\[
N = \exp \left( I(\lambda; \hat{\rho}) + O(\sqrt{l}) \right),
\]

such that for \( \mathbf{A} = \frac{1}{N} \sum_{\nu} \mathbf{A}^{(\nu)} \) condition (CM) is satisfied. The characteristic constant in the exponent is

\[
I(\lambda; \hat{\rho}) = H(\rho) - \sum_j \lambda_j H(\hat{\rho}_j),
\]
the *entropy defect* of the ensemble (Lebedev and Levitin [19]), or the *quantum mutual information* between a sender producing letter $j$ with probability $\lambda_j$ and a receiver getting the letter state $\hat{\rho}_j$ (see [15, 23]). It is the difference between the von Neumann entropy $H(\rho) = -\text{Tr} \rho \log \rho$ of the ensemble and its *conditional entropy* $H(\hat{\rho} | \lambda) = \sum_j \lambda_j H(\hat{\rho}_j)$.

Observe that not only $\rho$ can be recovered from this ensemble (as its average), but also the POVM $\mathbf{a}$:

$$a_j = \rho^{-1/2} \lambda_j \rho_j \rho^{-1/2}.$$  

This construction is known as the “square root measurement” [3], or “pretty good measurement” [7]. We shall give the proof of theorem [3] in a minute, after a few preparations. A central part of the argument is the following auxiliary result from [2] that we state separately:

**Lemma 3 (Ahlswede, Winter [2], thm. A.19)** Let $X_1, \ldots, X_M$ be independent identically distributed (i.i.d.) random variables with values in the algebra $L(K)$ of linear operators on $K$, which are bounded between 0 and $1$. Assume that the average $\mathbb{E} X_\mu = \sigma \leq s/2$. Then for every $0 < \eta < 1/2$

$$\Pr \left\{ \frac{1}{M} \sum_{\mu=1}^M X_\mu \not\in [(1 \pm \eta)\sigma] \right\} \leq 2 \dim K \exp \left( -M \frac{\eta^2 s}{2 \ln 2} \right),$$

where $[(1 \pm \eta)\sigma] = [(1 - \eta)\sigma; (1 + \eta)\sigma]$ is an interval in the operator order: $[A; B] = \{ X \in B(K) : A \leq X \leq B \}$.

We shall use the concepts of *typical* and *conditionally typical* subspaces in the form of [27], which we collect in appendix [3].

**Proof of theorem 3.** Define the following operators: for $j^l \in T^l_{\lambda, \delta}$ let

$$\xi^l_j = \Pi^l_{\rho, \delta} \Pi^l_{\rho, \delta}(j^l) \hat{\rho}_j \Pi^l_{\rho, \delta}(j^l) \Pi^l_{\rho, \delta}.$$  

We choose $\delta = m \sqrt{\frac{2e}{N}}$, so that

$$S := \lambda^\otimes l(T^l_{\lambda, \delta}) \geq 1 - \epsilon, \quad \text{Tr} \xi^l_j \geq 1 - \epsilon,$$

which is true by Chebyshev’s inequality and eqs. (B2) and (B4), specifying $\epsilon$ later.

Notice that in this way $\text{Tr} \omega' \geq 2 - 2\epsilon$ for

$$\omega' = \sum_{j' \in T^l_{\lambda, \delta}} \lambda_j \xi^l_{j'}. $$

By eq. (B8) we have

$$\Pi^l_{\rho, \delta} \omega \Pi^l_{\rho, \delta} \geq \alpha \Pi^l_{\rho, \delta},$$

with $\alpha = \exp(-lH(\rho) - O(\sqrt{T}))$. Define now $\Pi$ to be the projector onto the subspace spanned by the eigenvectors of $\omega'$ with eigenvalue $\geq \alpha$. By construction we find $\text{Tr} \Omega \geq 1 - 3\epsilon$ for $\Omega = \lambda^\otimes l \Pi \omega / \Pi$.

Now let $\xi^l_j = \Pi \xi^l_j \Pi$ and define i.d. random variables $j^l(\nu) \in T^l_{\lambda, \delta}, \nu = 1, \ldots, N, \mu = 1, \ldots, M$ by

$$\Pr \{ j^l(\nu) = j^l \} = \frac{\lambda^l_j}{S} =: \lambda^l_{j^l},$$

That is, we consider $N$ independent sets of $M$ independent choices each, from $T^l_{\lambda, \delta}$. Observe that $\Omega = \mathbb{E} \xi^{l(\nu)}$, the expected value of the random operators $\xi^{l(\nu)}$.

We shall show that with high probability the following conditions hold:

$$\frac{1}{M} \sum_{\mu=1}^M \xi^{l(\nu)} \in [(1 \pm \epsilon)\Omega], \quad \text{(I)}$$

for all $\nu$, and

$$\frac{1}{NM} \sum_{\nu, \mu=1}^{N, M} \delta^{l(\nu)} \in [(1 \pm \epsilon)L]', \quad \text{(II)}$$

This is most easily seen with the help of lemma [3] according to it

$$\Pr \{ -\text{I} \nu \} \leq 2 \text{Tr} \Pi \exp \left( -M \frac{\epsilon^2 \alpha}{2 \beta \ln 2} \right),$$

$$\Pr \{ -\Pi \} \leq 2 |T^l_{\lambda, \delta}| \exp \left( -NM \frac{\epsilon^2 \gamma}{2 \ln 2} \right) ,$$

FIG. 2: A nice way of picturing the content of theorem [3] is in the form of an elaborate bottleneck between source and outcomes: it is supplied from outside with the extrinsic data $\nu$, and conditional on this and the incoming $k$ produces the intrinsic data $j^l$. Only the intrinsic data are correlated to the signal $k$, while the extrinsic data (though evidently an indispensable part of the whole data) is independent of it. To put it pointedly: while it is difficult and possibly ambiguous to speak of “useful data”, one can clearly identify data of no import in all respects: the unrelated randomness $\nu$. This is put into the focus by theorem [3] and our concept of usefulness is just the remainder after extracting as much uselessness as possible.
with
\[ \gamma = \min \{ \lambda_j : j^1 \in T_{\lambda, \delta} \} \geq \exp \left( -lH(\lambda) - K n \delta \sqrt{I} \right), \]
compare eq. (38). Choosing \( M \) and \( N \) according to the theorem’s statement will force the sum of these probabilities to be less than 1, i.e. with positive probability all the events \((I_v)\) and \((II)\) happen.

Let us assume we fix now values for the \( J_{\mu}^{(\nu)} \) such that all equations \((I_v)\) and \((II)\) are satisfied. Then we may define operators
\[
A_{j^1}^{(\nu)} = \frac{S}{1 + \epsilon} \omega^{-1/2} \left( \frac{1}{M} \sum_{\mu} \xi_{j^1_{\mu}} \right) \omega^{-1/2} = \frac{S}{1 + \epsilon} \frac{1}{M} \sum_{\mu=1}^{M} \xi_{j^1_{\mu}} \omega^{-1/2}.
\]
We check that for each \( \nu \) these form a sub–POVM (i.e., a collection of positive operators with sum upper bounded by \( I \)) using \((I_v)\) and the definitions of \( \Omega \) and \( \omega' \) we find
\[
\sqrt{\omega} \left( \sum_{j^1} A_{j^1}^{(\nu)} \right) \sqrt{\omega'} = \frac{S}{1 + \epsilon} \sum_{\mu=1}^{M} \xi_{j^1_{\mu}} \leq S \Omega = \Pi \omega' \Pi \leq \omega'' \leq \omega.
\]
Finally, we check that condition \((CM)\) holds: it is sufficient to do this for the sub–POVM constructed, because then we can distribute the remaining operator weight to fill up to \( I \) arbitrarily.

We calculate directly from the definitions:
\[
\sum_{j^1} \frac{1}{2} \| \sqrt{\omega} (a_{j^1} - A_{j^1}) \sqrt{\omega} \|_1 \\
= \sum_{j^1} \frac{1}{2} \left\| \lambda_j \hat{\rho}_{j^1} - \frac{S}{1 + \epsilon} \omega^{-1/2} \left( \sum_{\mu=1}^{M} \xi_{j^1_{\mu}} \right) \omega^{-1/2} \right\|_1 \\
\leq \frac{1}{2} (1 - S) + \sum_{j^1 \in T_\lambda} L_{j^1} \left\| \hat{\rho}_{j^1} - \xi_{j^1} \right\|_1 \\
+ \frac{1}{2} \left\| L' - \frac{1}{N M} \sum_{\mu} \delta_{j^1_{\mu}} \right\|_1 \\
\leq \epsilon + \sum_{j^1 \in T_\lambda} L_{j^1} \left( \frac{1}{2} \| \hat{\rho}_{j^1} - \xi_{j^1} \|_1 + \frac{1}{2} \| \xi_{j^1} - \xi_{j^1} \|_1 \right) \leq \epsilon + \sum_{j^1 \in T_\lambda} L_{j^1} \left( \frac{1}{2} \| \hat{\rho}_{j^1} - \xi_{j^1} \|_1 + \frac{1}{2} \| \xi_{j^1} - \xi_{j^1} \|_1 \right).
\]
By the definition of \( \xi_{j^1} \), using eq. (10) and lemma 4 below, we can bound the first of the two terms in brackets by \( \epsilon + \sqrt{2} \epsilon \). It remains to estimate the second: consider
\[
\Omega' = \sum_{j^1 \in T_\lambda} L_{j^1} \xi_{j^1},
\]
and recall that \( \xi_{j^1} = \Pi_{j^1} \Pi \), hence \( \Omega = \Pi \Omega ' \Pi \). By construction we have
\[
\sum_{j^1 \in T_\lambda} L_{j^1} \mathrm{Tr} \xi_{j^1} \geq 1 - 3 \epsilon,
\]
thus, using lemma 4 with each of the \( \xi_{j^1} \) and employing concavity of the square root function, we end up with
\[
\sum_{j^1 \in T_\lambda} L_{j^1} \frac{1}{2} \| \xi_{j^1} - \xi_{j^1} \|_1 \leq \sqrt{6} \epsilon,
\]
which allows us to estimate (11) by \( 2 \epsilon + \sqrt{2} \epsilon + \sqrt{6} \epsilon \). \( \square \)

Here is the lemma that we needed in the proof: it says that a POVM element that is likely to respond to a state acts “gently” on it in the sense of little disturbance.

**Lemma 4 (Lemma V.9 of [27])** For a state \( \rho \) and an operator \( 0 \leq X \leq I, \) if \( \mathrm{Tr} (\rho X) \geq 1 - \lambda \), then
\[
\| \rho - \sqrt{\lambda} \rho \sqrt{\lambda} \|_1 \leq \sqrt{8} \lambda.
\]
The same is true if \( \rho \) is only a subnormalized density operator. \( \square \)

### III. PREVIOUS APPROACHES

The question addressed in the present paper of quantifying the “amount of information obtained by a quantum measurement” has been posed before, in the works [21] and [28], with mathematical modellings different from ours, though there is an evolution leading from the first to the present:

In [21], the POVM \( a \) was assumed to maximize a certain Bayesian gain (there called “fidelity”)
\[
F(a) = \sum_{i,j} p_{ij} \mathrm{Tr} (\rho_{ij} a_{ij}),
\]
to achieve the optimal (i.e. maximal) value \( F_{opt} \). On blocks of length \( l \) the gain (or fidelity) function was extended by defining \( F_{ij} = \frac{1}{l} \sum_{k=1}^l F_{ik,jk} \). This definition has the easily checked property that the gain on blocks of length \( l \),
\[
F(a^{(l)}) = \sum_{i,j} p_{ij} \mathrm{Tr} (\rho_{ij} a_{ij}) F_{ij},
\]
equals the single letter expression \( F(a) \).

Note that in this way the maximum Bayesian gain is still \( F_{opt} \) (which can be seen from eq. (13) below). Then the following theorem was shown:

**Theorem 5 (Massar, Popesecu [21])** For \( \epsilon > 0 \) and \( l \) large enough there exists a POVM \( A \) with fidelity \( F(A) \geq F_{opt} - \epsilon \) and
\[
M \leq \exp(l (H(\rho) + \epsilon))
\]
many outcomes among the \( j^1 \). \( \square \)
This result was interpreted as saying that about any property of the ensemble states, as encoded in the Bayesian gain matrix \(F_{ij}\), one can learn at most one bit per qubit.

In [28] this was extended and clarified as follows: observe that for any POVM \(A = (A^\nu_\mu)_{\mu=1,...,M}\) one has

\[
F(A) = \sum_i p_i \sum_\mu \text{Tr} (\rho_i A^\nu_\mu) \frac{1}{l} \sum_{k=1}^l F_{ikj}\mu k
\]

where (with \([l] = \{1,...,l\}\))

\[
(A|k)_j = \text{Tr} \left( \rho \otimes_k \otimes \mathbb{1}_k \right) \sum_{\mu: \mu_k = j} A^\nu_\mu
\]

\[
= \rho^{-1} \text{Tr} \left( \rho \otimes_k \otimes \mathbb{1}_k \right) \sum_{\mu: \mu_k = j} A^\nu_\mu
\]

\[
= \sqrt{\rho^{-1}} \text{Tr} \left( \rho \otimes_k \otimes \mathbb{1}_k \right) \sqrt{\rho^{-1}}
\]

(13)

For each \(k\), the collection \((A|k)_j = 1,...,m\) obviously is a POVM on \(H\). We may assume (as we shall do in the sequel) that the \(|F_{ij}|\) are bounded by 1: then the fidelity condition of theorem 5, reading

\[
|F(A) - F(a)| \leq \epsilon,
\]

is implied by

\[
\forall k \sum_{ij} |p_i \text{Tr} (\rho_i (A|k)_j) - p_i \text{Tr} (\rho_i a_j)| \leq \epsilon.
\]

(14)

This is itself implied by

\[
\forall k \forall i \sum_j |\text{Tr} (\rho_i (A|k)_j) - \text{Tr} (\rho_i a_j)| \leq \epsilon,
\]

(C0)

C1

which in turn follows from

\[
\forall k \sum_j \|(A|k)_j - a_j\| \leq \epsilon.
\]

(C3)

It was then proved

**Theorem 6 (Winter, Massar [28])** For the state \(\rho\) and the POVM \(a\) define a canonical ensemble \(\{\hat{\rho}_j, \lambda_j\}\), with states

\[
\hat{\rho}_j = \frac{1}{\text{Tr} (\rho a_j)} \sqrt{\rho a_j} \sqrt{\rho}
\]

and probabilities \(\lambda_j = \text{Tr} (\rho a_j)\).

Given \(\epsilon > 0\), there exists a POVM \(A = (A^\nu_\mu)_{\mu=1,...,M}\) with

\[
M \leq \exp \left( l \left( H(\rho) - \sum_j \lambda_j H(\hat{\rho}_j) \right) + C \sqrt{l} \right)
\]

(where \(C\) is a constant depending only on \(\epsilon\), \(d\) and \(m\), and such that (C3) is satisfied.

This theorem is in an asymptotic sense best possible (such an optimality was missing in [21]):

**Theorem 7 (Winter, Massar [28])** Let \(0 < \epsilon \leq (\lambda_0/2)^2\) with \(\lambda_0 = \min_j \lambda_j\). Then for any POVM \(A = (A^\nu_\mu)_{\mu=1,...,M}\) such that (C3) holds, one has

\[
M \geq \exp \left( l \left( H(\rho) - \sum_j \lambda_j H(\hat{\rho}_j) + \frac{3\epsilon}{\lambda_0 d} \log \frac{2e}{\lambda_0 d} \right) \right).
\]

(15)

\[
\]

FIG. 3: In [21] and [28] the original POVM is replaced by an “equivalent” one (as made precise in theorems 5 and 6) with much fewer outcomes. So, POVM and data record need much less rate of processing and storage, respectively. Of course, compared to theorem 5 we lose many potential measurement results in constructing the new POVM.

Here we want to show that the theorems 5 and 6 may be obtained as corollaries of theorem 3.

**Proof of theorem 5** Choose \(x_\nu\) and \(A^{(\nu)}\) according to theorem 3 such that condition (CM) is satisfied for \(A = \sum_\nu x_\nu A^{(\nu)}\), with some \(\epsilon > 0\) (which implies that also (CP) is satisfied with the same \(\epsilon\)). Then, assuming without loss of generality that \(|F_{ij}| \leq 1\), we get immediately out of eq. (13) that

\[
|F(A) - F(a^{(\nu)})| \leq \epsilon.
\]

Since we assume that \(a\) maximizes \(F\) we conclude, using linearity of \(F\) in the POVM:

\[
F_{\text{opt}} - \epsilon = F(a^{(\nu)}) - \epsilon
\]

\[
\leq F(A) = \sum_\nu x_\nu F(A^{(\nu)}).
\]

This finally means that for at least one \(\nu\)

\[
F(A^{(\nu)}) \geq F_{\text{opt}} - \epsilon.
\]
Denoting the smallest nonzero eigenvalue of any of the $E$-operators only), so regardless of $a$, $H(\rho)$ is the rate of intrinsic data of any probing of the ensemble states.

Note further that our derivation does not depend on the particular structure of the block-fidelity: obviously we can as well conclude for any ensemble $\{\sigma_k, q_k\}$ with average $\omega$ and any fidelity matrix $F_{k,j}$ that

$$|F(A) - F(a^{⊕l})| \leq \sum_{k,j} q_k |\text{Tr}(\sigma_k A_{j'}) - \text{Tr}(\sigma_k a_{j'})|F_{k,j}$$

$$\leq \epsilon \|F\|,$$

with $\|F\| := \max_{k,j} |F_{k,j}|$. If now $\|F\| \leq O(F(a^{⊕l}))$ for $l \to \infty$ then we get (for sufficiently large $l$)

$$F(A) \geq (1 - \epsilon) F(a^{⊕l}).$$

Of course, as explained in the introduction, theorem 3 is really a corollary of theorem 2. So, we continue to prove the latter:

**Proof of theorem 2.** Assume that a collection of POVMs $A^{(\nu)}$, $\nu = 1, \ldots, N$ like in theorem 2 is chosen, with probabilities $x_\nu$, such that $A' = \sum_\nu x_\nu A^{(\nu)}$ satisfies (CM). Define i.i.d. random variables $T_1, \ldots, T_Q$, each with $\Pr\{T_q = \nu\} = x_\nu$. We want to study the random POVMs $A^{(T_q)}$, and especially their mean

$$A = \frac{1}{Q} \sum_{q=1}^Q A^{(T_q)}.$$

Observe that $E(A) = E(A^{(T_q)}) = A'$.

Recall the definition of marginal POVMs. Obviously, by linearity of this definition, we have

$$(A|k) = \frac{1}{Q} \sum_{q=1}^Q (A^{(T_q)}|k)$$

and

$$E(A|k) = E(A^{(T_q)}|k) = (A'|k).$$

From condition (CM) and the monotonicity of the trace norm under partial trace we get now, for every $k$,

$$\sum_{j} \frac{1}{2} \|\sqrt{\rho}(A'|k)_j - a_j\sqrt{\rho}\|_1 \leq \epsilon.$$  \hfill (15)

Denoting the smallest nonzero eigenvalue of any of the $\sqrt{\rho}a_j\sqrt{\rho}$ by $u$, and choosing $\epsilon$ small enough, this assures that $\sqrt{\rho}(A'|k)_j \sqrt{\rho}$ restricted to the support of $\sqrt{\rho}a_j\sqrt{\rho}$ is lower bounded by $u/2$. Then we can apply lemma 4 and obtain

$$\Pr\{\sqrt{\rho}(A|k)_j \sqrt{\rho} \notin [(1 \pm \epsilon)\sqrt{\rho}(A'|k)_j \sqrt{\rho}] \text{ on supp } \hat{\rho}_j\}$$

$$\leq 2d \exp\left(-Q \frac{c^2 u}{4 \ln 2}\right).$$

Thus we can estimate the sum of these probabilities over all $k = 1, \ldots, l$ and $j = 1, \ldots, m$ to less than 1 if

$$Q \geq 1 + \frac{4 \ln 2}{c^2 u} \log(2dlm).$$

This implies that there exist actual values of the $T_q$ such that for all $k$

$$\sum_{j} \frac{1}{2} \|\sqrt{\rho}(A|k)_j \sqrt{\rho}_{|\text{supp } \hat{\rho}_j} - \sqrt{\rho}a_j\sqrt{\rho}\|_1 \leq 2\epsilon,$$  \hfill (16)

where we observed that the $\sqrt{\rho}(A'|k)_j \sqrt{\rho}$ all have trace at most 1, and have used eq. (15). Hence we get (with $A_{kj} = \text{Tr}(\rho(A|k)_j)$ and $A_{kj}^2 = \text{Tr}(\sqrt{\rho}(A|k)_j \sqrt{\rho})$)

$$\sum_{j} A_{kj} \text{Tr} \left(\hat{\rho}_{kj|\text{supp } \hat{\rho}_j}\right) \geq 1 \mp 2\epsilon,$$

and using lemma 4, this gives

$$\sum_{j} \frac{1}{2} \|\sqrt{\rho}(A|k)_j \sqrt{\rho}_{|\text{supp } \hat{\rho}_j} - \sqrt{\rho}(A|k)_j \sqrt{\rho}\|_1 \leq 2\sqrt{\epsilon}.$$

(17)

Now (10) and (17) yield

$$\sum_{j} \frac{1}{2} \|\sqrt{\rho}(A|k)_j - a_j\sqrt{\rho}\|_1 \leq 2\epsilon + 2\sqrt{\epsilon},$$

Denoting the minimal eigenvalue of $\rho$ by $r$ (which we assumed to be positive) this readily implies

$$\sum_{j} \frac{1}{2} \|(A|k)_j - a_j\| \leq \frac{2\epsilon + 2\sqrt{\epsilon}}{rd},$$

and we are done, since $A$ has only $MQ$ many possible outcomes. \hfill \Box

**IV. STRONG CONVERSE**

In this section we prove the asymptotic optimality of the separation of the measurement from theorem 2. To be precise, it is

**Theorem 8** Whenever there are POVMs $A^{(\nu)}$ on $[m]^l$, $\nu = 1, \ldots, N$, each supported on at most $M$ elements, and probability weights $x_\nu > 0$, such that $A = \sum_\nu x_\nu A^{(\nu)}$ satisfies condition (CM), for some $\epsilon < 1$, then

$$M \geq \exp \left(H(\lambda; \hat{\rho}) - O(\sqrt{\lambda})\right),$$

$$MN \geq \exp \left(H(\lambda) - O(\sqrt{\lambda})\right),$$

where the constants depend only on $\epsilon$.

**Proof.** Let us begin with the second inequality: by construction the set $\mathcal{R} \subset [m]^l$ of possible outcomes of $A$ has
for $j$ we consider

By a well known trick [30] the lower bound now follows:

inequality and eq. (B2)

A

which in turn implies

\begin{equation}
\lambda_{\otimes l} (\mathcal{R}) \geq \Lambda (\mathcal{R}) - \epsilon = 1 - \epsilon.
\end{equation}

which in turn implies

By a well known trick [30] the lower bound now follows:

we consider $\mathcal{R} = \mathcal{R} \cap \mathcal{T}_{\lambda, \delta}$, with $\delta = \sqrt{\frac{1}{2m}}$, whence we have, using Chebyshev’s inequality

\begin{equation}
\lambda_{\otimes l} (\mathcal{R}') \geq \frac{1 - \epsilon}{2}.
\end{equation}

Using the fact (compare eq. (B8))

\[ \forall j, l \in \mathcal{T}_{\lambda, \delta} \quad \lambda_{jl} \leq \exp \left( -lH(\lambda) + K m \delta \sqrt{l} \right), \]

we conclude

\[ MN \geq |\mathcal{R}| \geq |\mathcal{R}'| \geq \frac{1 - \epsilon}{2} \exp \left( lH(\lambda) - K m \delta \sqrt{l} \right). \]

Now for the first inequality: introduce the ensembles $\{ \hat{\rho}_{jl}, \lambda^{(l)}_{jl} \}_{jl}$ with

\[ \lambda^{(l)}_{jl} = \sqrt{\omega} \rho_{jl} \sqrt{\omega}, \]

all of which have average $\omega$. Then we define the (subnormalized) density operators

\[ \hat{\rho}_{jl} = \Pi_{j, \delta} (j') \rho_{j' \delta} \Pi_{j, \delta} (j'), \]

\[ \hat{P}_{jl}^{(l)} = \Pi_{j, \delta} (j') \rho_{j' \delta} \Pi_{j, \delta} (j'), \]

for $j, l \in \mathcal{T}_{\lambda, \delta}$, with $\delta = \sqrt{\frac{4 m n}{1 - \epsilon}}$. Then by Chebyshev inequality and eq. (B2)

\[ \frac{1}{2} \left\| \omega - \sum_{jl \in \mathcal{T}_{\lambda, \delta}} \lambda_{jl} \hat{\rho}_{jl} \right\|_1 \leq \frac{1 - \epsilon}{2}, \]

while from (CM) we get

\[ \frac{1}{2} \left\| \sum_{jl \in \mathcal{T}_{\lambda, \delta}} \lambda_{jl} \hat{\rho}_{jl} - \sum_{\nu} x_{\nu} \sum_{jl \in \mathcal{T}_{\lambda, \delta}} \lambda^{(l)}_{jl} \hat{P}_{jl}^{(l)} \right\|_1 \leq \frac{1 + \epsilon}{2} =: \epsilon'. \]

These immediately imply

\[ \sum_{\nu} x_{\nu} \sum_{jl \in \mathcal{T}_{\lambda, \delta}} \lambda^{(l)}_{jl} \Tr \hat{P}_{jl}^{(l)} \geq 1 - \epsilon', \]

so there exists at least one $\nu$ such that

\[ \sum_{jl \in \mathcal{T}_{\lambda, \delta}} \lambda^{(l)}_{jl} \Tr \hat{P}_{jl}^{(l)} \geq 1 - \epsilon'. \]

Now consider the (subnormalized) density operators

\[ \Theta^{(l)}_{jl} = \sqrt{\hat{P}_{jl}^{(l)}} \Omega_{j, \delta} \sqrt{\hat{P}_{jl}^{(l)}}, \]

which evidently satisfy

\[ \theta := \sum_{jl \in \mathcal{T}_{\lambda, \delta}} \lambda^{(l)}_{jl} \Theta^{(l)}_{jl} \leq \sum_{jl} \lambda^{(l)}_{jl} \hat{P}_{jl}^{(l)} = \omega. \]

Denoting with $\Pi$ the projection onto the support of $\theta$ and inserting $\Tr \Theta^{(l)}_{jl} = \Tr \hat{P}_{jl}^{(l)}$, we arrive at

\[ \Tr (\omega \Pi) \geq 1 - \epsilon', \]

from where we conclude

\[ \rank \Pi = \Tr \Pi \geq \exp \left( lH (\rho) - O (\sqrt{l}) \right). \]

This follows by a standard reasoning (which we take from [27]); for $F = \Pi_{j, \delta} \Pi_{j', \delta}$, choosing $\delta$ large enough, we get

\[ \Tr (\Pi_{j, \delta} \omega \Pi_{j', \delta}) = \Tr (\omega F) \geq \frac{1 - \epsilon'}{2}. \]

By eq. (B8) the inequality follows.

We may relax a bit the condition of the theorem regarding the parameter $M$: if we allow the different POVMs $\Lambda^{(l)}_{\nu}$ to have different numbers $M_{\nu}$ of possible outcomes, then we can prove the slightly stronger estimate

\[ M := \sum_{\nu} M_{\nu} \geq \exp \left( lH (\hat{\rho}) - O (\sqrt{l}) \right). \]

(while the second inequality obviously holds for $\sum_{\nu} M_{\nu}$). To see this go back to eq. (24) and observe that by a Markov inequality argument

\[ \Pr \left\{ \nu: \sum_{\nu} \lambda^{(l)}_{jl} \Tr \hat{P}_{jl}^{(l)} \geq 1 - \sqrt{\epsilon'} \right\} \geq 1 - \sqrt{\epsilon'}, \]

whence the claim directly follows.

**Remark 9** While in the above proof we assumed the property (CM) for $\epsilon < 1$, we conjecture that (CP) for all sources with average $\omega$, with $\epsilon < 1$, is sufficient to arrive at its conclusion.

Let us inspect this possibility along the lines of the proof: crucial were the estimates [14] and [13], the former being an immediate consequence of (CP), so we
would have to show this only for the latter. However, this demonstration has escaped us so far.

Finally, a comment on why this converse is strong: optimality of theorem \(3\) is proved already by our observation in the previous section that it implies theorem \(4\) and the lower bound of theorem \(3\). However, closer inspection of this lower bound reveals that it coincides with the upper bound only in the limit \(\epsilon \to 0\). For positive \(\epsilon\) it leaves room for a tradeoff between compression and error (not untypical for the type of error concept we had used). This is known in information theory as a weak converse \(2\). The strong converse in contrast shows optimality of the upper bound in the asymptotic limit \(l \to \infty\), with any \(\epsilon\) bounded away from 1.

V. ASYMPTOTIC DECOMPOSITION OF INSTRUMENTS AND OPERATIONS

An interesting generalization of our main theorem arises from the point of view that POVMs are just a special case of general open dynamics: the most general form of evolution is a completely positive, trace-preserving linear map \(\varphi\) from states on \(H\) to states on \(K\). Such a map can (non-uniquely) be represented in the Kraus form

\[
\varphi_\ast : \pi \mapsto \sum_{j=1}^{m} V_j \pi V_j^*,
\]

where \(V_j : H \to K\) are \(\mathbb{C}\)-linear and \(\sum_j V_j^* V_j = \mathbb{1}\). The representation can be made unique by considering it as a partial measurement, and including the outcome \(j\) extend the output system to \(K \otimes J\), and modify the map \(\varphi_\ast\) to

\[
\tilde{\varphi}_\ast : \pi \mapsto \sum_{j=1}^{m} V_j \pi V_j^* \otimes |j\rangle\langle j|.
\]

(Quite this will amount to a change of the Kraus operators, too, but we will not need the details here.) This is the notion of an instrument (Davies and Lewis \(6\)). One can see that it is representable in Kraus form, too, so we will in the sequel always look at a particular Kraus representation.

In analogy to the question about POVMs of this work we would like to approximate \(\varphi_\ast^{\otimes l}\) by the average of some \(\Phi^{(\nu)}_\ast\), \(\nu = 1, \ldots, N\), each of which should have a Kraus representation with a small number of contributing operators. As is well known this number is the dimension of the ancillary system (environment) sufficient to emulate the effect of the operation by a unitary interaction and subsequent partial trace. Its logarithm is an upper bound on the “information leakage” from the system to the environment.

Note that (apart from looking at approximation) we are considering here the problem of convex decomposition of completely positive maps, like we did before for POVMs. Of course, every completely positive map has a decomposition into extremal such ones, with possibly fewer terms in the Kraus representation. For this one can employ a theorem of Choi \(6\), saying that \(\varphi_\ast\) from eq. (21) is extremal if and only if the family of operators \(V_j^* V_k\) is linearly independent (in particular, then \(m \leq d\)).

We show now how to solve this problem as a consequence of theorem \(3\) with an additional reasoning mainly directed to quantum state fidelities:

Formally, we are looking for a family of maps

\[
\Phi^{(\nu)}_\ast : \mathcal{B}(H^{\otimes l}) \to \mathcal{B}(K^{\otimes l}),
\]

\[
\sigma \mapsto \sum_{\mu=1}^{M} W^{(\nu)}_{\mu} \sigma W^{(\nu)\ast}_{\mu}
\]

and probabilities \(x_\nu\) such that for \(\Phi_\ast = \sum_\nu x_\nu \Phi^{(\nu)}_\ast\) and any ensemble \(\{\sigma_k, q_k\}\) with average \(\omega = \rho^{\otimes l}\) the following condition holds:

\[
\sum_k q_k \frac{1}{2} \left\| \varphi^{\otimes l}_\ast (\sigma_k) - \Phi_\ast (\sigma_k) \right\|_1 \leq \epsilon. \quad (CO)
\]

In fact, there is an appealing way to state them all together, and strengthen the content at the same time: for a purification \(\pi\) of \(\rho\) on an extended system \(H \otimes H'\) we ask for

\[
\frac{1}{2} \left\| (\varphi_\ast \otimes \mathbb{I})^l (\pi^\otimes l) - (\Phi_\ast \otimes \mathbb{I})(\pi^\otimes l) \right\|_1 \leq \epsilon. \quad (CO^*)
\]

Indeed, this implies \((CO)\): just observe that by choosing a POVM \(\{T_k\}_{k=1}^N\) on \(H^{\otimes l}\) one can “induce” any ensemble \(\{\sigma_k, q_k\}\) on \(H^{\otimes l}\) for \(\omega\), in the following sense:

\[
q_k \sigma_k = \text{Tr}_{H'^{\otimes l}} (\pi^\otimes l (\mathbb{1} \otimes T_k)).
\]

How to do this is explained in detail in \(10\) (or see appendix A below). Note that this generalizes the implication of \((CP)\) from \((CM)\), discussed earlier, when we view the POVMs as the quantum operations eqs. \(6\) and \(7\).

Conversely, assuming \((CO)\) for all ensembles for \(\omega\) does unfortunately not imply \((CO^*)\) with a comparable error parameter. (Examples are not hard to construct for which \((CO)\) holds with a small \(\epsilon\) while the bound in \((CO^*)\) is close to 1.)

With \(\varphi_\ast\) there is associated the POVM

\[
a = (a_j = V_j^* V_j : i = j, \ldots, m),
\]

and with this goes the ensemble \(\{\hat{\rho}_j, \lambda_j\}\), as before.

**Theorem 10** With the above notation and \(\epsilon > 0\) there exist quantum operations in the form of eq. (22), with

\[
M \leq \exp \left( H(\lambda; \hat{\rho}) + O(\sqrt{l}) \right),
\]

\[
N \leq \exp \left( l (H(\lambda) - I(\lambda; \hat{\rho})) + O(\sqrt{l}) \right),
\]

and such that \(\Phi_\ast = \frac{1}{M} \sum_\nu \Phi^{(\nu)}_\ast\) satisfies \((CO^*)\).

These bounds are asymptotically best possible if \(\varphi_\ast\) is an instrument.
Proof. Let $A^{(\nu)}$ and $x_\nu$ be the POVMs and probabilities constructed in theorem 3 from $a^{\otimes l}$ and $\omega = \rho^{\otimes l}$, and let $A = \sum_\nu x_\nu A^{(\nu)}$. We use the notation from the proof of this theorem and from section IV

$$\sqrt{\mu} \sqrt{\mu} = \lambda_j \hat{\rho}_j,$$

$$\sqrt{\omega} A_j^{(\nu)} \sqrt{\omega} = \Lambda_j^{(\nu)} \hat{\rho}_j^{(\nu)},$$

$$\Lambda_j = \sum_\nu x_\nu \Lambda_j^{(\nu)}.$$

Note that by the proof of theorem 3 the $\hat{\rho}_j^{(\nu)}$ either are 0 or equal to $\hat{\rho}_j := \frac{S}{1+\epsilon} \hat{\xi}_j$.

Introduce the unitaries $U_j$ by the polar decomposition

$$V_j \sqrt{\rho} = U_j \sqrt{\rho V_j} V_j \sqrt{\rho} = U_j \sqrt{\lambda_j} \hat{\rho}_j,$$

and let $U_j = U_j^* \cdots U_j$. Now define $W_j^{(\nu)}$ by letting

$$W_j^{(\nu)} \sqrt{\omega} = U_j^* \sqrt{\lambda_j^{(\nu)} \hat{\rho}_j^{(\nu)}},$$

and observe that for fixed $\nu$ only $M$ of them are nonzero, and that for fixed $j$ these are all multiples of each other. Hence these operators define a quantum operation $\Phi_\nu$ according to the theorem, and $\Phi_\nu = \sum_\nu x_\nu \Phi_\nu^{(\nu)}$.

With these definitions we check that $(CO^*)$ is satisfied: using $\pi^{\otimes l} = (\sqrt{\omega} \otimes \mathbb{I}^{\otimes l}) |I\rangle \langle I| (\sqrt{\omega} \otimes \mathbb{I}^{\otimes l})$ (see lemma 14 in the appendix) and eqs. (23) and (24) we calculate

$$\left\| \left( \pi^{\otimes l} \otimes \mathbb{I}^{\otimes l} \right) - \left( \Phi_\nu \otimes \mathbb{I}^{\otimes l} \right) \right\|_1 \leq \sum_j \left\| \frac{1}{\sqrt{\lambda_j}} \left( V_j^* \otimes \mathbb{I}^{\otimes l} \right) \pi^{\otimes l} \left( V_j^* \otimes \mathbb{I}^{\otimes l} \right) \right. - \sum_\nu x_\nu \left( W_j^{(\nu)} \otimes \mathbb{I}^{\otimes l} \right) \pi^{\otimes l} \left( W_j^{(\nu)*} \otimes \mathbb{I}^{\otimes l} \right) \left\|_1

= \sum_j \left. \lambda_j \left( \sqrt{\rho} \otimes \mathbb{I}^{\otimes l} \right) |I\rangle \langle I| \left( \sqrt{\rho} \otimes \mathbb{I}^{\otimes l} \right) - \Lambda_j \left( \sqrt{\rho} \otimes \mathbb{I}^{\otimes l} \right) |I\rangle \langle I| \left( \sqrt{\rho} \otimes \mathbb{I}^{\otimes l} \right) \right\|_1

\leq \left\| \lambda^{\otimes l} - \Lambda \right\|_1

+ \sum_j \lambda_j \left\| \frac{1}{\sqrt{\lambda_j}} \left( \sqrt{\rho} \otimes \mathbb{I}^{\otimes l} \right) |I\rangle \langle I| \left( \sqrt{\rho} \otimes \mathbb{I}^{\otimes l} \right) - \left( \sqrt{\rho} \otimes \mathbb{I}^{\otimes l} \right) |I\rangle \langle I| \left( \sqrt{\rho} \otimes \mathbb{I}^{\otimes l} \right) \right\|_1,$$

(25)

The last line here is estimated as follows: the first term is bounded by $2\epsilon$ (see the proof of theorem 3), and for the other we use lemma 14, observe that for each $j$ the two terms inside the trace norm are the canonical purifications of $\hat{\rho}_j$ and $\hat{P}_j$, respectively. Thus we get

$$\left\| \left( \sqrt{\rho} \otimes \mathbb{I}^{\otimes l} \right) |I\rangle \langle I| \left( \sqrt{\rho} \otimes \mathbb{I}^{\otimes l} \right) - \left( \sqrt{\rho} \otimes \mathbb{I}^{\otimes l} \right) |I\rangle \langle I| \left( \sqrt{\rho} \otimes \mathbb{I}^{\otimes l} \right) \right\|_1 \leq 2\sqrt{2} \left\| \hat{\rho}_j - \hat{P}_j \right\|_1,$$

and using concavity of the root function and the estimate of eq. (24) we can upper bound the last line of eq. (25) by $O(e^{-\epsilon^2})$.

If $\varphi_\star$ is an instrument any approximate convex decomposition of $\varphi^{\otimes l}$ implies a similar decomposition for the POVM $a^{\otimes l}$. Hence theorem 8 gives the optimality of the bounds for $M$ and $N$. \hfill \Box

Interestingly, the bounds of theorem 10 depend on the Kraus representation (24) of the map $\varphi_\star$: all other such representations are related by unitary transforms, i.e.

$$\varphi_\star(\sigma) = \sum_j V_j^* \sigma V_j^*$$

if and only if

$$V_j = \sum_j U_j^* V_j,$$

with a unitary matrix $(U_j)_j$ of complex numbers. (This is essentially a consequence of the uniqueness up to unitaries of the Stinespring dilation (25) of $\varphi$, which implies the Kraus representation. This fact is also discussed in detail in (22)).

This motivates the introduction of

$$\Sigma(\rho; \varphi_\star) := \min_{\text{Kraus repr. of } \varphi_\star} I(\lambda; \hat{\rho}),$$

(26)

i.e. the minimum rate of the parameter $M$ in decompositions of $\varphi_\star$ according to theorem 10.

Note that, according to (26), the minimum of $H(\lambda)$ over all Kraus representations is exactly $S_\epsilon$, the entropy exchange of the map $\varphi_\star$ (with respect to $\rho$). For a discussion see subsection VI E below, and the forthcoming (29).

VI. DISCUSSION

We have introduced a separation into extrinsic and intrinsic data of a quantum measurement. It was shown to have definite minimal rates for either of these, and that it encompasses all previously known results on “meaningful” data in quantum measurements. A particular advantage of theorem 2 before theorems 5 and 6 is that it not even requires a new POVM (which might be experimentally difficult to realize). Instead, it can be understood as a mere re-interpretation of the data delivered by $a^{\otimes l}$: in fact, by our construction in the proof of theorem 3 for
all $\nu$ and $j^i$ either $A_{j^i}^{(\nu)}$ is 0 or very close to a multiple of $a_{j^i}$, in the sense of (CM). Hence the random variable $N$, defined as a function of $j^i$:

$$\Pr\{N = \nu|j^i\} = \frac{x_{\nu}}{A_{j^i}^{(\nu)}} \text{Tr} \left( \omega A_{j^i}^{(\nu)} \right) = x_{\nu} \frac{\text{Tr} \left( \omega A_{j^i}^{(\nu)} \right)}{\text{Tr} \left( \omega a_{j^i} \right)},$$

(27)

(up to a scaling factor, close to 1 for typical $j^i$), is almost independent from the source ensemble $\{\sigma_k, q_k\}$ in (CP). More precisely,

$$\sum_{k\nu j^i} \left| q_k x_{\nu} \text{Tr} (\sigma_k A_{j^i}^{(\nu)}) - q_k \text{Tr} (\sigma_k a_{j^i}) \Pr\{N = \nu|j^i\} \right| \leq \epsilon,$$

and in fact, we even have

$$\sum_{\nu j^i} \left| x_{\nu} \sqrt{a_{j^i}} - \Pr\{N = \nu|j^i\} \sqrt{a_{j^i}} \right| \leq \epsilon.$$

This means that one can reproduce the statistics of the whole diagram in figure 2 from the outcomes of $a^{\otimes t}$, by inventing the $\nu$ distributed according to eq. (27). This gives a new view on the extrinsic/intrinsic separation: rather than replacing the original POVM by a fancy construction, one can from the original data $j^i$ compute the extrinsic data $\nu$, and conditionally on that the intrinsic part. Then one can successfully pretend that this separation was delivered by the mixture of the POVMs $A^{(\nu)}$.

### A. Data vs. Information

One (as it turns out, rather careless) interpretation of our result could be that the “useful” information produced by the POVM $a$ amounts to $I(\lambda; \hat{\rho})$. This in itself is not yet precise, so let’s fix “information” to mean “communicable information” in the sense of Shannon [24]: for any source $\{\sigma_i, \mu_i\}$ with average $\sum_i \mu_i \sigma_i = \rho$ the source and measurement outcome are random variables $X$ and $Y$ with a joint distribution

$$\Pr\{X = i, Y = j\} = \mu_i \text{Tr} (\sigma_i a_j),$$

and the mutual information of these is

$$I(X \wedge Y) = H(X) + H(Y) - H(XY).$$

We repeat here the discussion of [28] regarding the relation between this quantity and $I(\lambda; \hat{\rho})$:

Observe first that the joint distribution of $X$ and $Y$ can be rewritten as

$$\Pr\{X = i, Y = j\} = \text{Tr} \left( \rho^{-1/2} \mu_i \sigma_i \rho^{-1/2} \sqrt{a_j} \sqrt{\rho} \right) = \lambda_j \text{Tr} (\hat{\rho} a_j S_i),$$

where the $S_i = \rho^{-1/2} \mu_i \sigma_i \rho^{-1/2}$ form a POVM (compare [10] where this correspondence between POVMs and ensembles was used to classify the latter with given density matrix). But here the Holevo bound [12] applies, with the ensemble $\{\hat{\rho}_j, \lambda_j\}$, and thus we have proved:

**Theorem 11** Let $\{\sigma_i, \mu_i\}$ be any ensemble whose average state $\sum_i \mu_i \sigma_i$ equals $\rho$. Define random variables $X, Y$ with joint distribution

$$\Pr\{X = i, Y = j\} = \mu_i \text{Tr} (\sigma_i a_j)$$

(this is the probability for $\sigma_i$ to occur and that $j$ is observed on this state). Then

$$I(X \wedge Y) \leq I(\lambda; \hat{\rho}).$$

Note that in general maximization over the ensemble $\{\sigma_i, \mu_i\}$ (yielding the accessible information

$$J_\rho (a) = I_{\text{acc}}(\lambda; \hat{\rho}),$$

because in the above proof it corresponds to an information maximization over the POVM $S_i$) does not achieve the upper bound: see [12], where it is shown that it does if and only if all the $\hat{\rho}_j$ commute.

Furthermore, by a result from [11]

$$J_{\rho^{\otimes t}} (a^{\otimes t}) = I_\rho (a),$$

hence the gap remains even asymptotically! For further discussion of this point we refer the reader to [28], section VII C. We record here only the consequence that one ought to distinguish between data (collected by measurement) and information (about a property of the states): the latter is never larger than the former, and typically in quantum situations it is strictly less. However, this seems nothing to worry about: after all, this is an observation quite familiar from our experience, though it is worth stressing that in the present context it is a purely quantum phenomenon.

Peter Shor has remarked the notable fact that in the presence of entanglement, however, this distinction disappears: the entanglement-assisted capacity [9] for the quantum–classical channel that is represented by our POVM, i.e. $\varphi$, from eq. (4), with the average of the sent symbols required to be $\rho$ (this means that in the formula for the entanglement-assisted capacity one has to put a purification of $\rho$) coincides with our $I(\lambda; \hat{\rho})$! In fact, our result can be understood as a weak version of the conjectured “Quantum Reverse Shannon Theorem” [9] for quantum–classical channels.

To end this part of the discussion note that the bound of theorem 10 in the case of a maximally refined measurement is simply the von Neumann entropy $H(\rho)$ of the source, and this regardless of the nature of the POVM and of the source. In this sense, there is “democracy among measurements”, at least the maximally refined ones.

It is thus appealing to view our result as a dual to the creation of a density operator by mixing pure states: it is well known that in any representation $\rho = \sum \rho_i \sigma_i$, with pure states $\sigma_i$, $H(\rho) \geq H(\rho_i)$, with equality if the $\sigma_i$ are mutually orthogonal eigenstates of $\rho$: hence, $H(\rho)$ is the minimum entropy needed to generate $\rho$. In the present work we identify $H(\rho)$ as the maximum entropy of measurement data correlated to $\rho$. 
B. Holevo bound

Here we show how to turn around the previous argument to actually prove the Holevo information bound. The statement is as follows:

**Theorem 12 (Holevo [12])** Let \( \{ \hat{\rho}_j, \lambda_j \}_{j=1,...,m} \) be an ensemble of states with average \( \rho = \sum_j \lambda_j \hat{\rho}_j \), and \( \{ S_i \}_{i=1,...,n} \) a POVM. Define the joint distribution of random variables \( Y, X \) to be

\[
\Pr\{ Y = j, X = i \} = \lambda_j \Tr (\hat{\rho}_j S_i).
\]

Then the inequality

\[
I(Y \land X) \leq I(\lambda; \hat{\rho}) = H(\rho) - \sum_j \lambda_j H(\hat{\rho}_j)
\]

holds.

**Proof.** To begin with, observe that eq. (28) may be rewritten as

\[
\Pr\{ Y = j, X = i \} = p_i \Tr (\sigma_i a_j),
\]

with \( a_j = \rho^{-1/2} \lambda_j \hat{\rho}_j \rho^{-1/2} \) and the ensemble \( \{ \sigma_i, p_i \} \), where \( p_i \sigma_i = \sqrt{p_i} \sqrt{\rho_i} \). Now consider i.i.d. realizations \( X_1, Y_1, \ldots, X_l, Y_l \) of the pair \( X, Y \). We shall apply theorem 2 to \( a^{\otimes l} \) and \( \rho^{\otimes l} \), with parameter \( 0 < \epsilon < 1 \). Hence, for \( A = \sum_\nu x_\nu A^{(\nu)} \) and the ensemble \( \{ \sigma^{i}_i, p^{\nu}_i \} \) the condition (CP) holds. Let us define random variables \( \xi, \nu \)

\[
\Pr\{ \nu = j', \xi = i' \} = p_i \Tr (\sigma^{i}_i A^{j'}).
\]

Then we may calculate (with \( f(\epsilon) := \epsilon (\log m + 2 \log n) \))

\[
I(Y \land X) = I(X' \land Y') \leq I(\xi \land \nu) + I(\xi \land \nu i) + I(\xi \land \nu i') + I(\xi \land \nu i') + I(\xi \land \nu') + I(\xi \land \nu') + I(\xi \land \nu i) + I(\xi \land \nu i') + I(\xi \land \nu i') + I(\xi \land \nu') + I(\xi \land \nu') + I(\xi \land \nu i) + I(\xi \land \nu i') + I(\xi \land \nu i') + I(\xi \land \nu') + I(\xi \land \nu').
\]

Only classical entropy relations have been used: line 2 is by lemma [3] stated below, line 3 is by data processing, as \( \nu \) is a function of \( \nu \) and \( \mu \), line 4 is a standard identity, and line 5 by independence of \( \nu \) and \( \xi \) and the standard inequality \( I(\xi \land \mu) \leq H(\mu) \).

Now divide by \( l \) and let \( l \to \infty \):

\[
I(Y \land X) \leq I(\lambda; \hat{\rho}) + \epsilon (\log m + 2 \log n).
\]

As \( \epsilon > 0 \) was arbitrary, the theorem follows. \( \square \)

**Lemma 13 (Fano [4])** Let \( P \) and \( Q \) be probability distributions on a set with finite cardinality \( a \), such that \( \frac{1}{a} \| P - Q \|_1 \leq \lambda \). Then

\[
|H(P) - H(Q)| \leq \lambda \log a + 2H(\lambda, 1 - \lambda).
\]

\( \square \)

The reader may want to compare this proof to our earlier one in [23]: despite similarities they are conceptually completely different! In fact, there we introduced the Holevo mutual information as a certain fidelity measure (which may seem slightly artificial) and applied theorem 1 while here we directly exploit the “bottleneck” nature of our main result (compare again fig. 2), thus providing a much more natural approach.

C. Fixed source ensemble and classical case

Our approach has concentrated on universal properties of the POVM, leaving the source as free as possible. What happens if we fix the source \( \{ p_i, \sigma_i \} \)? Note firstly that the whole situation is fully classical now, as we only have to regard the correlation between source issues \( X = i \) and measurement results \( Y = j \). Thus it is modelled by the classical case of the initial problem: the source is \( \{ |i\rangle \langle i|, p_i \} \), and the POVM \( b \) consists of operators

\[
b_j = \sum_i \Tr (\rho_i a_j)|i\rangle\langle i|.
\]

This model has the same joint statistics of \( i \) and \( j \) as the above described one (most generally, \( b_j \) can be any operator with eigenbasis \( \{ |i\rangle \} \)).

Now observe the following: as long as the POVMs \( A^{(\nu)} \) are diagonal in the basis \( \{ |i\rangle \} \), too (this is the classicality condition for the POVMs), the validity of (CP) for all ensembles with average

\[
P = \sum_i p_i |i\rangle\langle i|
\]

is implied by its validity for the ensemble \( \{ |i\rangle\langle i|, p_i \} \). This is because source states \( \rho^{\nu} \) and \( \sum_i |i\rangle\langle i| \rho^{\nu} |i\rangle\langle i| \) produce the same statistics, so only sources consisting of mixtures of the \( |i\rangle\langle i| \) have to be considered. The condition (CP) for them clearly is implied by its validity for \( \{ |i\rangle\langle i|, p_i \} \).

At this point theorems 2 and 8 can be applied: because the induced ensemble for source state \( P \) and POVM \( b \) is \( \{ \sigma_j, \lambda_j \} \), with

\[
\lambda_j = \sum_i p_i \Tr (\rho_i a_j) = \Tr (\rho a_j),
\]

\[
\sigma_j = \sum_i \frac{1}{\lambda_j} p_i \Tr (\rho_i a_j)|i\rangle\langle i|,
\]

we obtain \( I(X \land Y) \), that is the Shannon mutual information between the source and the measurement, as the rate of intrinsic data. More precisely, we can perform a data separation by postprocessing, according to the prescription of the beginning of this section, eq. (27), into extrinsic \( \nu \), almost independent of \( i' \), and intrinsic \( j' \) depending on \( i' \) and \( \nu \).

However, this is not exactly what we set out to initially: theorem 8 allows us to decompose the \( b_j \) into convex
combinations of operators

\[ B_j^{(v)} = \sum_i \beta_j^{(v)}_{i|i} |j^i\rangle \langle i^j|, \]

but it is not clear that these can be obtained from POVMs \( A^{(v)} \), in the sense that

\[ \forall \nu \forall j^i \forall i^j \beta_j^{(v)}_{i|i} = \text{Tr} \left( \rho_i A_j^{(v)} \right). \]

For this to hold the vectors \( (\beta_j^{(v)}_{i|i})_i^j \) (for all \( j^i \)) must belong to the cone spanned by the vectors \( (\langle \psi | \rho_i | \psi \rangle)_i^j \). It is conceivable that under this condition the obtainable intrinsic data rate increases. We have to leave this interesting question for the moment.

For classical sources and measurements we thus obtain that intrinsic data equals mutual information. On the other hand, we can come back to their being distinct in truly quantum situations: we pointed out in subsection VI F that intrinsic data equals mutual information. On this account, we conclude (1), we conclude

\[ \forall \nu \forall j^i \forall i^j \beta_j^{(v)}_{i|i} = \text{Tr} \left( \rho_i A_j^{(v)} \right). \]

This should be viewed especially in the light of the conjecture implied in subsection VI F.

D. Sufficient statistics

The reader familiar with classical statistical theories may have been reminded by our above discussion of the concept of sufficient statistics, at least when the quantum source and the observation are essentially classical, i.e. when all the \( p_i \) and \( a_j \) commute: the former are then just probability distributions and the latter form a statistical decision rule, with distribution of \( j \) conditional on \( i \) denoted \( q(j|i) \). As there is also a distribution \( p_i \) on the \( ri \) we have here a statistical model in the sense of estimation theory (we refer the reader to [18] for detailed explanations).

We will consider the values of \( i \) and \( j \) as random variables: then a sufficient statistics is a random variable \( k \) which is a function of \( j \) (whose distribution conditional on \( i \) we denote \( q(k|i) \)), such that the distribution of \( j \) conditional on \( k \) is independent of \( i \):

\[ \text{Pr}\{j|k\} = \text{Pr}\{j|k,i\} \quad \forall i. \]

Let us denote these conditional probabilities by \( r(j|k) \).

This implies that we can simulate the distribution of \( j \) conditional on \( i \) from \( k \):

\[ q(j|i) = \sum_k r(j|k) q(k|i). \]

In words, to each entry \( k \) of the new data record there exists a distribution on the \( j \) of the original data record such that the latter’s distribution is recovered as a convolution; in terms of stochastic maps it is factorized into \( \tilde{q} \) and \( r \):

\[ i \overset{\tilde{q}}{\rightarrow} k \overset{r}{\rightarrow} j. \]

On the other hand, our theorem [8] provides something appearing to be dual to this (apart from holding only approximately and in an asymptotic setting; these things are easily introduced in sufficient statistics, too): a random variable \( \nu \) with distribution \( \bar{\nu} \) independent of \( i \) and \( j \), and conditional on it a stochastic map \( a(\nu,j|i) \) such that

\[ q(j|i) = \sum_{\nu} x_{\nu} a_{\nu}(j|i). \]

In a diagram:

\[ \nu \downarrow \]

\[ i \overset{\tilde{q}}{\rightarrow} \overset{r}{\rightarrow} j. \]

Like \( k \) in the case of sufficient statistics, the pair \( \nu \mu \) is a function of \( j \), but unlike there, where \( \tilde{q} \) and \( r \) were stochastic maps with independent sources of randomness (when stochastic maps are viewed as set function valued random variables, this is expressed by the independence of \( \tilde{q} \) and \( r \)), the maps \( \tilde{Q} \) and \( \tilde{R} \) draw their randomness from the same source \( \nu \).

In summary, there is no direct isomorphism between our concept of data reduction and sufficient statistics (which, too, can be used to reduce the entropy of data sets): the latter appears as a special case where the maps \( \tilde{Q} \) and \( \tilde{R} \) are independent.

E. Entropy exchange

We want to discuss an application of theorem [8] to the entropy exchange of quantum operations, introduced by Schumacher [22] (and previously by Lindblad [20]): for a quantum operation \( \varphi_\ast \) in the form (21) it is defined as

\[ S_\ast(\rho; \varphi_\ast) = H(W), \quad \text{with } W_{jk} = \text{Tr}(V_j \rho V_k^\ast). \]

It can be shown to be independent of the Kraus representation, by identifying it with the entropy increase in an initially pure environment of the system by a Stinespring dilation of \( \varphi_\ast \), see [22]. In the latter work a number of interesting relations between \( S_\ast \) and other entropic quantities are shown.

In particular, returning to the notation of section III, it is shown that there is a (in this sense, minimal) Kraus representation of \( \varphi_\ast \) such that \( H(\lambda) = S_\ast(\rho; \varphi) \). Because of \( I(\lambda; \rho) \leq H(\lambda) \) (this is simply data processing inequality [4]), we conclude

\[ \Sigma(\rho; \varphi_\ast) \leq S_\ast(\rho; \varphi_\ast). \]
By the derivation this quantity may be dubbed genuinely quantum entropy exchange of a channel, as it is part of the noise that cannot be accounted for classically.

From a different point of view, in fact also the maximum of $I(\Lambda; \rho)$ over all Kraus representations of $\varphi_*$ (compare eq. (26)) is interesting: in a cryptographic setting, where $\varphi_*$ connects users $A$ and $B$, and is controlled by an eavesdropper $E$, it is the amount of data collected by $E$ about $A$’s messages in the worst case.

A deeper investigation of these concepts is relegated to another occasion [29].

F. An open problem

An interesting and challenging question is about the amount of data collected by $a$ under the hypothesis of an arbitrarily varying source (AVS), instead of the i.i.d. model considered here:

An AVS is a collection of source ensembles $\{\rho_s, p_s\}$ (with average state $\rho_s$, labelled by $s \in S$), which we make into a discrete memoryless source by considering the ensembles (labelled by $s' \in S'$)

$$\{\rho_{iv}, p_{iv} \}_{iv}.$$  

The idea is that at each position $k = 1, \ldots, l$ the source may be arbitrarily in one of the internal states $s \in S$. We have no — not even statistical information — about $s$, so our data separation must work for all $s' \in S'$: formally the condition on $A = \sum_s x_s A^{(s)}$ is

$$\forall s' \sum_{j'} \frac{1}{2} \left\| \sqrt{\omega(s')} (A_{j'} - A_{j'}) \sqrt{\omega(s')} \right\|_1 \leq \varepsilon,$$

(AVCM)

where $\omega(s') = \rho_{s_1} \otimes \cdots \otimes \rho_{s_l}$ is the average state of the source when in internal state $s'$.  

A natural candidate for the minimum data rate of the $A^{(s)}$ seems to be

$$\max \{ I(\Lambda; \tilde{\rho}) : \rho \in \text{conv} \{ \rho_s : s \in S \} \},$$

with $\lambda_0 \tilde{\rho}_j = \sqrt{\rho_{a_j} \tilde{\rho}}$, and $\text{conv}$ denoting the closed convex hull.

If this is true, then in particular the quantity

$$\Delta(a) = \max_\rho I(\lambda; \tilde{\rho})$$

is the amount of data collected by $a$, regardless of any source ensemble.

Acknowledgments

I am indebted to Serge Massar for his introducing me to the problem addressed in this paper and for interesting discussions, and to Hiroshi Nagaoka for pointing me at the possible relation between the present approach and sufficient statistics. Thanks to Peter Shor who supplied the insight that the difference between data and information disappears in the presence of entanglement. I thank Masanaka Ozawa for pointing out to me that theorem 10, initially only formulated for operations, is in fact valid for instruments. Part of this work was done during my stay at the ERATO project “Quantum Computation and Information”, Tokyo (August/September 2001). I thank the members of the project for their hospitality, and especially Keiji Matsumoto for discussions on the content of the appendix, on which I also enjoyed conversation with Richard Jozsa and Masahide Sasaki. Last but not least, special thanks are due to Marco P. Carota for constant encouragement during the course of this work.

APPENDIX A: CANONICAL PURIFICATIONS

In this appendix we collect a few facts about mixed state fidelity and a certain kind of purification of mixed states, which we call canonical, that seem not to be widely known. These are used in the main text, but seem to be of interest in their own right.

For the state $\omega$ on $H_1$ consider a purification $|\psi\rangle = \sum_i \sqrt{\lambda_i} |i \rangle \otimes |i \rangle$ on a bipartite system $H_1 \otimes H_2$, that we already have put in Schmidt polar form. Then on both systems there exist ($\mathbb{R}$–linear) complex conjugation maps with respect to the basis $\{ |i \rangle \}$:

$$|\phi\rangle = \sum_i \alpha_i |i \rangle \longrightarrow \sum_i \overline{\alpha_i} |i \rangle =: |\overline{\phi}\rangle.$$  

Then, with $|I\rangle = \sum_i |i \rangle \otimes |i \rangle$, it can be checked that

$$|\psi\rangle|\psi\rangle = (\sqrt{\omega} \otimes \mathbb{1})|I\rangle|I\rangle (\sqrt{\omega} \otimes \mathbb{1})$$

$$= (\mathbb{1} \otimes \sqrt{\omega})|I\rangle|I\rangle (\mathbb{1} \otimes \sqrt{\omega}),$$

see also the following lemma [14]. Then

$$\left( \mathbb{1} \otimes \sqrt{S_k} \right) |\psi\rangle|\psi\rangle = \left( \mathbb{1} \otimes \sqrt{S_k \sqrt{\omega}} \right) I|I\rangle$$

$$= \left( \mathbb{1} \otimes \sqrt{S_k \sqrt{\omega}} \right) I|I\rangle (\mathbb{1} \otimes \sqrt{S_k \sqrt{\omega}})$$

$$= q_k (\mathbb{1} \otimes U_k) \left[ \left( \mathbb{1} \otimes \sqrt{S_k \sqrt{\omega}} \right) |I\rangle|I\rangle (\mathbb{1} \otimes \sqrt{S_k \sqrt{\omega}}) \right] (\mathbb{1} \otimes U_k)$$

$$= q_k (\mathbb{1} \otimes U_k) |t_k\rangle|t_k\rangle (\mathbb{1} \otimes U_k),$$

the third line introducing $q_k = \sqrt{\omega S_k \sqrt{\omega}}$ on $H_2$, and the polar decomposition $\sqrt{S_k \sqrt{\omega}} = U_k \sqrt{q_k \tau_k}$, the fourth the canonical purification $|t_k\rangle$ on $H_1 \otimes H_2$ of $\tau_k$ (with respect to $|I\rangle|I\rangle$), see lemma [14] below. By this lemma we can infer

$$\text{Tr}_{H_2} |\psi\rangle|\psi\rangle (\mathbb{1} \otimes S_k) = q_k \text{Tr}_{H_2} |t_k\rangle|t_k\rangle$$

$$= q_k \tau_k,$$

with the complex conjugated operator $\tau_k$, which is defined as

$$\tau_k = \sum_i \overline{|\phi_i\rangle} \langle \phi_i|, \text{ if } \tau_k = \sum_i |\phi_i\rangle \langle \phi_i|.$$
Note that this is uniquely defined, regardless of the convex decomposition chosen, and in particular independent of the phases of the $|\phi_i\rangle$.

The ensemble \(\{\kappa_k, q_k\}\) has average $\omega = \omega$, and conversely, the above formulas show how to induce any ensemble \(\{\kappa_k, q_k\}\) for $\omega$ on $\mathcal{H}_2$: let $S_k = \omega^{-1/2} q_k \varphi_k \omega^{-1/2}$ (this was noted before in [34] in the context of classifying ensembles with a given density operator).

**Lemma 14 ("Pretty good purifications")**

Consider orthonormal bases of spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, both denoted \(\{|i\}\), and introduce \(|I\rangle = \sum_i |i\rangle \otimes |i\rangle\). As before, we denote the complex conjugation with respect to this basis by $\overline{n}$. Then for a state $\rho = \sum_i \alpha_i |\psi_i\rangle \langle \psi_i| \) (in diagonalized form),

$$|r\rangle |r\rangle = (\sqrt{\rho} \otimes I) |I\rangle |I\rangle (\sqrt{\rho} \otimes I),$$

with

$$|r\rangle = \sum_i \sqrt{\alpha_i} |\psi_i\rangle \otimes \overline{\psi_i},$$

is a purification of $\rho$. We call it the canonical purification with respect to $|I\rangle$. (Note that this definition makes sense as it is independent of phases in the $|\psi_i\rangle$).

If $|s\rangle |s\rangle$ is the canonical purification of another state $\sigma$ then for the fidelity between these:

$$F(|r\rangle |s\rangle) = \langle r | s \rangle = \langle r | s \rangle = \frac{1}{2}\|\sqrt{\rho} - \sqrt{\sigma}\|_1. \quad (A1)$$

Furthermore

$$\text{Tr} \sqrt{\rho} \sqrt{\sigma} \geq 1 - \|\sqrt{\rho} - \sqrt{\sigma}\|_1, \quad (A2)$$

$$\frac{1}{2}\|\sqrt{\rho} - \sqrt{\sigma}\|_1 \leq \sqrt{4} \|\rho - \sigma\|_1. \quad (A3)$$

**Proof.** The formula for the canonical purification is a straightforward calculation. With its help, it is also straightforward to check the fidelity identity, eq. \(\text{(A1)}\).

Now for the last two estimates: begin with

$$1 - \text{Tr} \sqrt{\rho} \sqrt{\sigma} = \text{Tr} (\sqrt{\rho} (\sqrt{\rho} - \sqrt{\sigma}))$$

$$\leq \|\sqrt{\rho} (\sqrt{\rho} - \sqrt{\sigma})\|_1$$

$$\leq \|\sqrt{\rho}\|_2 \|\sqrt{\rho} - \sqrt{\sigma}\|_2$$

$$\leq \|\sqrt{\rho} - \sqrt{\sigma}\|_2$$

$$= \|\rho - \sigma\|_1,$$

invoking two nontrivial inequalities: in the third line we use Cor. IV.2.6 of [15] (which is a kind of Hölder or Cauchy–Schwarz inequality), in the fourth line Thm. X.1.3 from the same book.

Finally, use the well known identity

$$\frac{1}{2}\|\sqrt{\rho} - \sqrt{\sigma}\|_1 = \sqrt{1 - F(|r\rangle |s\rangle)}$$

to obtain

$$\frac{1}{2}\|\sqrt{\rho} - \sqrt{\sigma}\|_1 = \sqrt{1 - \langle r | s \rangle}$$

which we wanted to show. \(\Box\)

**Remark 15** Observe $\text{Tr} \sqrt{\rho} \sqrt{\sigma} \leq \|\sqrt{\rho} \sqrt{\sigma}\|_1$, the square of this latter quantity being known as the (mixed state) fidelity [14]. By theorems by Uhlmann [24] and Jozsa [17] the mixed state fidelity $F(\rho, \sigma) = \|\sqrt{\rho} \sqrt{\sigma}\|_1$ equals the maximum over the pure state fidelities of all possible purifications of $\rho$ and $\sigma$. Because of well known relations between mixed state fidelity and trace norm distance (see [3]), more precisely

$$1 - \sqrt{F(\rho, \sigma)} \leq \frac{1}{2}\|\rho - \sigma\|_1 \leq \sqrt{1 - F(\rho, \sigma)}, \quad (A4)$$

the lemma tells us that at least for (mixed state) fidelity close to 1 the canonical purifications are not too far off the optimum with respect to (pure state) fidelity.

**APPENDIX B: TYPICAL SEQUENCES AND SUBSPACES**

For a probability distribution $P$ on the finite set $\mathcal{X}$ define set of typical sequences (with $\delta > 0$)

$$\mathcal{T}_{P,\delta} = \left\{ x^l : \forall x^l |N(x^l)|^2 - 1P_x^l |\leq \delta \sqrt{l}\sqrt{P_x^l(1 - P_x^l)} \right\},$$

where $N(x^l)$ counts the number of occurrences of $x$ in the word $x^l = x_1 \ldots x_n$.

For a state $\rho$ fix eigenstates $e_1, \ldots, e_d$ (with eigenvalues $R_1, \ldots, R_d$) and define for $\delta > 0$ the typical projector as

$$\Pi^l_{\rho, \delta} = \sum_{t^l \in \mathcal{T}_{P,\delta}} e_{t_1} \otimes \cdots \otimes e_{t_l},$$

For a collection of states $\hat{\rho}_j$, $j = 1, \ldots, m$, and $j^l \in [m]^l$ define the conditional typical projector as

$$\Pi^l_{\rho, \delta}(j^l) = \bigotimes_j \Pi^l_{\rho_j, \delta},$$

where $I_j = \{k : j_k = j\}$ and $\Pi^l_{\rho_j, \delta}$ is meant to denote the typical projector of the state $\rho_j$ on the subsystem composed of the tensor factors $I_j$ in the tensor product of $l$ factors. From [27] we cite the following properties of these projectors:

$$\text{Tr} (\rho^2 \Pi^l_{\rho, \delta}) \geq 1 - \frac{d}{\delta^2}, \quad (B1)$$

$$\text{Tr} (\hat{\rho}_{j^l} \Pi^l_{\rho, \delta}(j^l)) \geq 1 - \frac{md}{\delta^2}, \quad (B2)$$

$$\text{Tr} (\hat{\rho}_j \Pi^l_{\rho, \delta}) \geq 1 - \frac{m^2d}{\delta^2}, \quad (B3)$$

$$\text{Tr} \Pi^l_{\rho, \delta} \leq \exp (\frac{1}{l} H(\rho) + Kd\delta \sqrt{l}), \quad (B4)$$

$$\text{Tr} \Pi^l_{\rho, \delta} \geq \left(1 - \frac{d}{\delta^2}\right) \exp (\frac{1}{l} H(\rho) - Kd\delta \sqrt{l}), \quad (B5)$$
\[
\text{Tr} \Pi_{\tilde{\rho}, \delta}(j^l) \leq \exp \left( lH(\tilde{\rho}) + Kmd\delta \sqrt{l} \right), \quad (B6)
\]
\[
\text{Tr} \Pi_{\tilde{\rho}, \delta}(j^l) \geq \left(1 - \frac{md}{\delta^2}\right) \exp \left( lH(\tilde{\rho}) + Kmd\delta \sqrt{l} \right), \quad (B7)
\]
for an absolute constant \(K > 0\), and the empirical distribution \(P_{j'}\) of letters \(j\) in the word \(j^l\):
\[
P_{j'}(j) = \frac{1}{l} N(j|j^l).
\]
Finally, with
\[
\alpha = \exp \left( -lH(\tilde{\rho}) - Kd\delta \sqrt{l} \right),
\]
\[
\alpha' = \exp \left( -lH(\tilde{\rho}) + Kd\delta \sqrt{l} \right),
\]
we have
\[
\alpha' \Pi_{\tilde{\rho}, \delta} \geq \Pi_{\tilde{\rho}, \delta} \tilde{\rho} \odot \Pi_{\tilde{\rho}, \delta} \geq \alpha \Pi_{\tilde{\rho}, \delta}, \quad (B8)
\]
\[
\beta = \exp \left( -lH(\tilde{\rho}) + Kmd\delta \sqrt{l} \right),
\]
\[
\beta' = \exp \left( -lH(\tilde{\rho}) - Kmd\delta \sqrt{l} \right),
\]
\[
\beta' \Pi_{\tilde{\rho}, \delta}(j^l) \leq \Pi_{\tilde{\rho}, \delta}(j^l) \tilde{\rho}_j \Pi_{\tilde{\rho}, \delta}(j^l) \leq \beta \Pi_{\tilde{\rho}, \delta}(j^l). \quad (B9)
\]

[1] R. Ahlswede, P. L"ober, “Quantum Data Processing”, IEEE Trans. Inf. Theory, vol. 47, no. 1, pp. 474–478, 2001.
[2] R. Ahlswede, A. Winter, “Strong converse for identification via quantum channels”, IEEE Trans. Inf. Theory, vol. 48, no. 3, pp. 569–579, 2002.
[3] C. H. Bennett, P. W. Shor, J. A. Smolin, A. V. Thapliyal, “Entanglement–assisted capacity of a quantum channel and the reverse Shannon theorem”, e–print quant–ph/0106052, 2001.
[4] R. Bhattacharyya, Matrix Analysis, Graduate Texts in Mathematics 169, Springer Verlag, Berlin New York, 1997.
[5] M.–D. Choi, “Completely positive linear maps on complex matrices”, Linear Algebra and Appl., vol. 10, pp. 285–290, 1975.
[6] E. B. Davies, J. T. Lewis, “An operational approach to quantum probability”, Comm. Math. Phys., vol. 17, pp. 239–260, 1970.
[7] R. M. Fano, Class Notes for Transmission of Information, Course 6.574, MIT, Cambridge MA, 1952. See also R. M. Fano, Transmission of Information, Wiley and Sons, New York, 1961.
[8] C. A. Fuchs, J. van de Graaf, “Cryptographic Distinguishability Measures for Quantum–Mechanical States”, IEEE Trans. Inf. Theory, vol. 45, no. 4, pp. 1216–1227, 1999.
[9] P. Hausladen, W. K. Wootters, “A ‘pretty good’ measurement for distinguishing quantum states”, J. Modern Opt., vol. 41, no. 12, pp. 2385–2390, 1994.
[10] C. W. Helstrom, Quantum Detection and Estimation Theory, Academic Press, New York, 1976.
[11] A. S. Holevo, “Information–theoretical aspects of quantum measurement”, Probl. Inf. Transm., vol. 9, no. 2, pp. 110–118, 1973.
[12] A. S. Holevo, “Bounds for the quantity of information transmitted by a quantum channel”, Probl. Inf. Transm., vol. 9, no. 3, pp. 177–183, 1973.
[13] A. S. Holevo, “Asymptotically optimal hypotheses testing in quantum statistics”, Theor. Probability Appl., vol. 23, no. 2, pp. 411–415, 1979.
[14] A. S. Holevo, Probabilistic and Statistical Aspects of Quantum Theory, North Holland, Amsterdam, 1982.
[15] A. S. Holevo, “The Capacity of the Quantum Channel with General Signal States”, IEEE Trans. Inf. Theory, vol. 44, no. 1, pp. 269–273, 1998.
[16] L. P. Hughston, R. Jozsa, W. K. Wootters, “A complete classification of quantum ensembles having a given density matrix”, Phys. Lett. A, vol. 183, no. 1, pp. 14–18, 1993.
[17] R. Jozsa, “Fidelity for mixed quantum states”, J. Mod. Optics, vol. 41, pp. 2315–2323, 1994.
[18] E. L. Lehmann, G. Casella, Theory of Point Estimation, 2nd edition, Springer Texts in Statistics, Springer, Berlin New York, 1998.
[19] D. S. Lebedev, L. B. Levitin, “The maximum amount of information transmissible by an electromagnetic field”, Dokl. Akad. Nauk (SSSR), vol. 149, no. 6, pp. 1299–1302, 1963 (Russian). [English translation: Soviet Physics Dokl., vol. 8, pp. 377–379, 1963.]
[20] G. Lindblad, “Quantum entropy and quantum measurements”, in: C. Bendjaballah, O. Hirota, S. Reynaud (eds.), Quantum Aspects of Optical Communications, Lecture Notes in Physics, vol. 378, pp. 71–80, Springer Verlag, Berlin, 1991.
[21] S. Massar and S. Popescu, “Amount of information obtained by a quantum measurement”, Phys. Rev. A, vol. 61, 062303, 2000.
[22] B. Schumacher, “Sending entanglement through noisy quantum channels”, Phys. Rev. A, vol. 54, no. 4, pp. 2614–2628, 1996.
[23] B. Schumacher, M. D. Westmoreland, “Sending classical information via noisy quantum channels”, Phys. Rev. A, vol. 56, no. 1, pp. 131–138, 1997.
[24] C. E. Shannon, “A mathematical theory of communication”, Bell Syst. Tech. Journal, vol. 27, pp. 379–423, 623–656, 1948.
[25] W. F. Stinespring, “Positive functions on C∗–algebras”, Proc. Amer. Math. Soc., vol. 6, pp. 211–216, 1955.
[26] A. Uhlmann, “The ‘transition probability’ in the state space of a *–algebra”, Rep. Math. Physics, vol. 9, pp. 273–279, 1976.
[27] A. Winter, “Coding Theorem and Strong Converse for
Quantum Channels”, IEEE Trans. Inf. Theory, vol. 45, no. 7, pp. 2481–2485, 1999.

[28] A. Winter, S. Massar, “Compression of quantum–measurement operations”, Phys. Rev. A., vol. 64, 012311, 2001.

[29] A. Winter, S. Massar, “Convex decompositions of completely positive maps and quantum information”, in preparation, 2002.

[30] J. Wolfowitz, Coding Theorems of Information Theory, 2nd edition, Springer Verlag, Berlin, 1964.