The principal eigenfunction of the Dirichlet Laplacian with prescribed numbers of critical points on the upper half of a topological torus ‡

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Abstract

We consider the principal eigenvalue problem for the Laplace-Beltrami operator on the upper half of a topological torus under the Dirichlet boundary condition. We present a construction of the upper half of a topological torus that admits the principal eigenfunction having exact numbers of critical points. Furthermore, we manage to identify the locations of all the critical points of the principal eigenfunction explicitly.

Key words. Principal eigenfunction; the upper half of a topological torus; elliptic equation; Dirichlet Laplacian; critical point.

AMS subject classifications. 35J25; 35J05; 47A75; 58J37

1 Introduction

Let $M^+ = (M^+, g)$ be the upper half of a topological torus equipped with a Riemannian metric $g$ written in local coordinates $x = (x^1, x^2)$. On $M^+$, we consider the principal eigenvalue problem for the Laplace-Beltrami operator under the Dirichlet boundary condition

$$
\begin{cases}
\Delta_g u + \lambda_1 u = 0 & \text{in } M^+, \\
u > 0 & \text{in } M^+, \\
0 & \text{on } \partial M^+, \\
\|u\|_{L^2(M^+)} = 1,
\end{cases}
$$

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where $\lambda_1$ is the principal eigenvalue of the Laplace-Beltrami operator $\Delta_g$ on $M^+$ given by

$$\Delta_g u = \text{div}(\nabla_g u) = \sum_{i=1}^{2} \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} \left( \sqrt{|g|}(\nabla_g u)^i \right).$$

(1.2)

The main objective of this paper is to show the existence of the principal eigenfunction of (1.1) on the upper half of a topological torus with exact numbers of critical points. Furthermore, we aim to disclose the locations of all the critical points explicitly.

In [V], Volkmer considers the eigenvalue problem for the Laplace-Beltrami operator on the standard torus $T^2$ embedded in $\mathbb{R}^3$. In [V, Theorem 1, p. 825], he constructs the system of the eigenfunctions together with the eigenvalues on $T^2$. For the existence of the critical points of eigenfunctions on topological tori, we may refer to the work that have been done by [BLS, JN]. Both studies consider the Liouville metric on the flat torus. In [EP, Theorem 1.2, p. 198], Enciso et al. show that for any compact $d$-dimensional Riemannian manifold, $d \geq 3$, one can find a metric in such a way that the first nontrivial eigenfunction can have as many non-degenerate critical points as one wants. See [JNT, M] for two surveys on geometric properties and critical points of eigenfunctions of the Laplace-Beltrami operator.

In our previous work [KS], we construct topological tori together with the nonconstant stable stationary solutions of reaction-diffusion problems with exactly $4n$ critical points whose locations are explicit. The topological tori are constructed by employing the regular perturbations of the standard torus $T^2$. We replace the radius of the tube by a positive periodic function. We refer to the new topological torus as the perturbed torus $T_\varepsilon^2$ with a small parameter $\varepsilon$.

In this paper, we start with considering (1.1) on the upper half $T^+$ of a standard torus $T^2$. By the symmetry of $T^+$, the principal eigenfunction satisfies an ordinary differential equation, and hence its critical points make one circle on $T^+$. Next, we cut away the lower half of the perturbed torus $T^+_\varepsilon$ in [KS]. We call the remaining surface by the upper half $T^+_\varepsilon$ of the perturbed torus $T^2_\varepsilon$. With the aid of the implicit function theorem, we prove that the principal eigenfunction of (1.1) exists on $T^+_\varepsilon$. The construction of the upper half $T^+_\varepsilon$ of the perturbed torus $T^2_\varepsilon$ provides us an explicit formula for (1.1). Hence, it allows us to identify the locations of all the critical points of the principal eigenfunction of (1.1) on $T^+_\varepsilon$.

In [KS, Proof of Theorem 2.2, p.6] we see that the critical points of stable stationary solutions of reaction-diffusion problems on $T^2$ consist of the two circles $\partial T^+ = T^2 \cap \{x_3 = 0\}$. When we perform the regular perturbation $T^2_\varepsilon$ of $T^2$, we obtain that all the critical
points of the solutions on $T^2_\varepsilon$ lie in $\partial T^+_\varepsilon = T^2_\varepsilon \cap \{x_3 = 0\}$ regardless of the value of $\varepsilon$ (see [KS, Proof of Theorem 1.1, p.14]). On the other hand, the locations of the critical points of the principal eigenfunctions of (1.1) lie in the interior of $T^+_\varepsilon$ because of the Dirichlet boundary condition and they depend on the value of $\varepsilon$. Thus, we need another new argument to examine the locations of the critical points (see Proposition 4.1 in Section 4).

The main result of this paper states the following.

**Theorem 1.1.** There exists a number $N \in \mathbb{N}$ such that, for each $n \geq N$, a perturbation $M^+$ of the upper half of a standard torus $T^2$ is constructed in such a way that the principal eigenfunction $u$ of (1.1) has exactly $2n$ critical points.

The paper is organized as follows. In Section 2, we consider problem (1.1) when $M^+$ is replaced by the upper half $T^+$ of a standard torus $T^2$. We show that the set of the critical points of the principal eigenfunction of (1.1) equals a circle in $T^+$. In Section 3, we construct a perturbation $T^+_\varepsilon$ of $T^+$ together with the principal eigenfunction on $T^+_\varepsilon$. In Section 4, we prove Theorem 1.1 along with the locations of all the critical points of the principal eigenfunction on $T^+_\varepsilon$.

## 2 The upper half of a standard torus

The parameterization of the upper half $T^+$ of a standard torus $T^2$ is given by

$$
\begin{align*}
  x_1 &= (R + r \cos \varphi) \cos \theta, \\
  x_2 &= (R + r \cos \varphi) \sin \theta, \\
  x_3 &= r \sin \varphi,
\end{align*}
$$

where $R, r$ are constants with $R > r > 0$ and $S^1$ is the unit circle.

Set $x^1 = \varphi, x^2 = \theta$. Then, the Riemannian metric $g = (g_{ij})$ is given by

$$
(g_{ij})_{i,j=1,2} = \begin{pmatrix}
  r^2 & 0 \\
  0 & (R + r \cos \varphi)^2
\end{pmatrix}.
$$

The Riemannian gradient $\nabla_g u$ of $u$ with respect to $g$ on $T^+$ is given by

$$
\nabla_g u = \begin{pmatrix}
  \frac{1}{r^2} \partial_{\varphi} u \\
  \frac{1}{(R + r \cos \varphi)^2} \partial_{\theta} u
\end{pmatrix}.
$$
The area element $d\sigma$ of $T^+$ is given by

$$d\sigma = \sqrt{|g|}d\varphi d\theta = r(R + r\cos\varphi)d\varphi d\theta. \quad (2.4)$$

Thus, we can express the Laplace-Beltrami operator $\Delta_g$ on $T^+$ as

$$\Delta_g u = \frac{1}{r^2}u_{\varphi\varphi} + \frac{1}{(R + r\cos\varphi)^2}u_{\theta\theta} - \frac{\sin\varphi}{r(R + r\cos\varphi)}u_\varphi. \quad (2.5)$$

Let $U$ be the principal eigenfunction of (1.1) where $M^+$ is replaced by $T^+$. By the symmetry of $T^+$ with respect to the center of the hole of $T^+$, $U$ is a function of one variable $\varphi$.

**Proposition 2.1.** The set of critical points of the principal eigenfunction $U$ of (1.1) on $T^+$ equals a circle in $T^+$ centered at a point belonging to the axis of the symmetry of $T^+$.

**Proof.** Since $U > 0$ in $(0, \pi)$ and $U$ satisfies that for every $\varphi \in (0, \pi)$

$$\left( (R + r\cos\varphi)U' \right)' = -r^2(R + r\cos\varphi)\lambda_1 U < 0, \quad (2.6)$$

we see that $(R + r\cos\varphi)U'(\varphi)$ is strictly decreasing in $(0, \pi)$. By applying Hopf’s boundary point lemma to (1.1) on $T^+$, we have that

$$U'(0) > 0 > U'(\pi).$$

These conditions give rise to the existence of a unique point $\varphi^* \in (0, \pi)$ such that

$$U'(\varphi^*) = 0 \text{ and } U''(\varphi^*) < 0. \quad (2.7)$$

Hence, we conclude that the set of critical points of $U$ of (1.1) on $T^+$ corresponds to

$$(\varphi, \theta) \in \{\varphi^*\} \times S^1.$$ 

This completes the proof. \(\square\)

### 3 The upper half of a perturbed torus

Let us introduce a small perturbation $T^+_{\epsilon}$ of $T^+$ parameterized by

$$\begin{cases}
  x_1 = (R + r_\epsilon(\theta)\cos\varphi)\cos\theta, \\
  x_2 = (R + r_\epsilon(\theta)\cos\varphi)\sin\theta, \quad ((\varphi, \theta) \in I^+) \\
  x_3 = r_\epsilon(\theta)\sin\varphi,
\end{cases} \quad (3.1)$$
where \( n \in \mathbb{N} \), \( r_\varepsilon(\theta) = r + \varepsilon \sin(n\theta) \), and the constants \( R, r, \varepsilon \) satisfy \( R > r + |\varepsilon| = \max_{\theta \in S^1} r_\varepsilon(\theta) \).

Notice that \( T_0^+ = T^+ \).

Set \( x^1 = \varphi \), \( x^2 = \theta \). Then, the Riemannian metric \( g^\varepsilon = (g^\varepsilon_{ij}) \) is given by

\[
(g^\varepsilon_{ij})_{i,j=1,2} = \begin{pmatrix}
\rho^2(\theta) & 0 \\
0 & (R + r_\varepsilon(\theta) \cos \varphi)^2 + (r'_\varepsilon(\theta))^2
\end{pmatrix},
\]

(3.2)

The Riemannian gradient \( \nabla_{g^\varepsilon} u \) of \( u \) with respect to \( g^\varepsilon \) on \( T^+_\varepsilon \) is given by

\[
\nabla_{g^\varepsilon} u = \begin{pmatrix}
\frac{1}{\rho^2(\theta)} \partial_\varphi u \\
\frac{1}{(R + r_\varepsilon(\theta) \cos \varphi)^2 + (r'_\varepsilon(\theta))^2} \partial_\theta u
\end{pmatrix},
\]

(3.3)

The area element \( d\sigma_\varepsilon \) of \( T^+_\varepsilon \) is given by

\[
d\sigma_\varepsilon = \sqrt{|g^\varepsilon|} d\varphi d\theta = r_\varepsilon(\theta) \sqrt{(R + r_\varepsilon(\theta) \cos \varphi)^2 + (r'_\varepsilon(\theta))^2} d\varphi d\theta.
\]

(3.4)

Hence, the Laplace-Beltrami operator \( \Delta_{g^\varepsilon} \) on \( T^+_\varepsilon \) can be expressed as

\[
\Delta_{g^\varepsilon} u = \frac{1}{\rho^2(\theta)} u_{,\varphi\varphi} + \frac{1}{\rho^2(\theta)} u_{,\theta\theta} + \frac{\Phi}{\rho^2(\theta)} u_{,\varphi} + \frac{r'_\varepsilon(\theta) \Phi - r_\varepsilon(\theta) \Phi_{,\theta}}{r_\varepsilon(\theta) \Phi^3} u_{,\theta}.
\]

(3.5)

where we set \( \Phi = \Phi(\varphi, \theta) = \sqrt{(R + r_\varepsilon(\theta) \cos \varphi)^2 + (r'_\varepsilon(\theta))^2} \).

Let \( u^\varepsilon = u^\varepsilon(\varphi, \theta) \) be the principal eigenfunction of (1.1) where \( M^+ \) is replaced by \( T^+_\varepsilon \). Then \( u^\varepsilon \) satisfies

\[
\begin{cases}
\Delta_{g^\varepsilon} u^\varepsilon + \lambda_1^\varepsilon u^\varepsilon = 0 & \text{in } T^+_\varepsilon, \quad (3.6) \\
 u^\varepsilon > 0 & \text{in } T^+_\varepsilon, \quad u^\varepsilon = 0 & \text{on } \partial T^+_\varepsilon, \quad (3.7) \\
 \| u^\varepsilon \|_{L^2(T^+_\varepsilon)} = 1. \quad (3.8)
\end{cases}
\]

Notice that \( u^0 = U \) and \( \lambda_1^0 = \lambda_1 \).

By applying the implicit function theorem to problem (3.6)–(3.8) as in [H, Example 3.2, pp. 32–33], we see that the principal eigenfunction \( u^\varepsilon \) on \( T^+_\varepsilon \) is close to the principal eigenfunction \( U \) in \( T^+ \) in \( C^2 \)-topology for sufficiently small \( |\varepsilon| \).

**Proposition 3.1.** Let \( u^\varepsilon \) be the principal eigenfunction of (1.1) where \( M^+ \) is replaced by \( T^+_\varepsilon \). Then, there exists \( \varepsilon_0 > 0 \) such that for each \( |\varepsilon| \in (0, \varepsilon_0) \), \( \delta_0(\varepsilon) > 0 \) with \( \lim_{\varepsilon \to 0} \delta_0(\varepsilon) = 0 \) exists and satisfies

\[
\| u^\varepsilon - U \|_{C^2(T^+)} < \delta_0(\varepsilon), \quad \text{if } |\varepsilon| \in (0, \varepsilon_0),
\]

where \( U \) is the principal eigenfunction of (1.1) on \( T^+ \) given by Proposition 2.1.
4 Proof of Theorem 1.1

Let \( u^\varepsilon \) be the principal eigenfunction of (1.1) where \( M^+ \) is replaced by \( T^+_\varepsilon \) for \( |\varepsilon| \in (0, \varepsilon_0) \) as in Proposition 3.1. Then the uniqueness of the principal eigenfunction \( u^\varepsilon \), together with the symmetry of \( T^+_\varepsilon \), provides us the symmetry of \( u^\varepsilon \) with respect to \( T^+_\varepsilon \cap H \) for the following \( n \) planes \( H \):

\[
H = \{-x_1 \sin \theta_k + x_2 \cos \theta_k = 0\} \quad \text{with} \quad k = 0, 1, \ldots, n - 1,
\]

where \( \theta_k = \frac{2k + 1}{2n} \pi \), and hence

\[
\frac{\partial u^\varepsilon}{\partial \theta} = 0 \quad \text{for every} \quad (\varphi, \theta) \in [0, \pi] \times \{\theta_k, \ k = 0, 1, \ldots, 2n - 1\}.
\]

From (2.7) and Proposition 3.1, the following conditions hold: Let \( \delta > 0 \) be sufficiently small. There exists \( \varepsilon_1 \in (0, \varepsilon_0) \) such that if \( |\varepsilon| \leq \varepsilon_1 \), then

\[
\begin{aligned}
\frac{\partial u^\varepsilon}{\partial \varphi} &> 0 \quad \text{for every} \quad \varphi \in [0, \varphi^* - \delta], \\
\frac{\partial u^\varepsilon}{\partial \varphi} &< 0 \quad \text{for every} \quad \varphi \in [\varphi^* + \delta, \pi], \\
\frac{\partial^2 u^\varepsilon}{\partial \varphi^2} &< 0 \quad \text{for every} \quad \varphi \in [\varphi^* - 2\delta, \varphi^* + 2\delta].
\end{aligned}
\]

These conditions imply the existence of a unique point \( \hat{\varphi}(\varepsilon, \theta) \in (\varphi^* - \delta, \varphi^* + \delta) \) with \( \hat{\varphi}(0, \theta) \equiv \varphi^* \) satisfying, if \( |\varepsilon| \leq \varepsilon_1 \) then

\[
\frac{\partial u^\varepsilon}{\partial \varphi}(\hat{\varphi}(\varepsilon, \theta), \theta) = 0 \quad \text{for every} \quad \theta \in S^1.
\]

From (4.3) and (4.4), if \( |\varepsilon| \leq \varepsilon_1 \), we reassure that the set of critical points of the function \( u^\varepsilon \) on \( I^+ (= [0, \pi] \times S^1) \) is contained in \( (\varphi^* - \delta, \varphi^* + \delta) \times S^1 \). To obtain all the critical points of the principal eigenfunction \( u^\varepsilon \) on \( T^+_\varepsilon \), it suffices to examine the derivatives of \( u^\varepsilon \) with respect to \( \theta \) up to the second order only for \( \varphi \in [\varphi^* - \delta, \varphi^* + \delta] \times S^1 \).

We may express the principal eigenfunction \( u^\varepsilon \) as

\[
\begin{aligned}
u^\varepsilon = U + \varepsilon \frac{\partial u^\varepsilon}{\partial \varepsilon} \bigg|_{\varepsilon=0} + o(\varepsilon) \quad \text{as} \ \varepsilon \to 0.
\end{aligned}
\]

Set \( V = \frac{\partial u^\varepsilon}{\partial \varepsilon} \bigg|_{\varepsilon=0} \). By the Dirichlet boundary condition of \( u^\varepsilon \), we have

\[
V(0, \theta) = V(\pi, \theta) = 0 \quad \text{for every} \ \theta \in S^1.
\]
By (4.5), the derivatives of $u^\epsilon$ with respect of $\theta$ up to the second order are given by

$$\frac{\partial u^\epsilon}{\partial \theta} = \epsilon V_\theta + o(\epsilon) \quad \text{and} \quad \frac{\partial^2 u^\epsilon}{\partial \theta^2} = \epsilon V_{\theta\theta} + o(\epsilon) \quad \text{as} \quad \epsilon \to 0. \quad (4.7)$$

Let us differentiate (3.6) with respect to $\epsilon$.

$$0 = \frac{1}{r_\epsilon^2} u^\epsilon_{\varphi\varphi} + \frac{\partial}{\partial \epsilon} \left( \frac{1}{r_\epsilon^2} u^\epsilon_{\varphi\varphi} + \frac{1}{\Phi^2} u^\epsilon_{\varphi\theta\epsilon} + \frac{\partial}{\partial \epsilon} \left( \frac{1}{\Phi^2} \right) u^\epsilon_{\theta\theta} \right) + \Phi_{\varphi} \frac{r_\epsilon}{r_\epsilon^2 \Phi_{\varphi}} u^\epsilon_{\varphi\theta} + \frac{\partial^2}{\partial \epsilon^2} \left( \frac{r_\epsilon^2}{r_\epsilon^3} \Phi_{\varphi} \Phi_{\varphi} \right) u^\epsilon_{\varphi} + \frac{\partial}{\partial \epsilon} \left( \frac{r_\epsilon^2 \Phi_{\varphi} - r_\epsilon \Phi_{\varphi}}{r_\epsilon^3 \Phi_{\varphi}^3} \right) u^\epsilon_{\theta\epsilon} + \frac{\partial}{\partial \epsilon} \left( \frac{r_\epsilon^2 \Phi_{\varphi}}{r_\epsilon^3 \Phi_{\varphi}^3} \right) u^\epsilon_{\varphi} + \frac{\partial}{\partial \epsilon} \left( \frac{r_\epsilon^2 \Phi_{\varphi}}{r_\epsilon^3 \Phi_{\varphi}^3} \right) u^\epsilon_{\theta\theta} + \frac{\partial}{\partial \epsilon} \frac{\partial}{\partial \epsilon} \frac{\partial}{\partial \epsilon} \frac{\partial}{\partial \epsilon}. \quad (4.8)$$

Then, we set $\epsilon = 0$ to obtain

$$\Delta_g V + \lambda_1 V + \frac{\partial \lambda_1}{\partial \epsilon} \bigg|_{\epsilon=0} U = \frac{2}{r} \sin(n \theta) \left[ -\lambda_1 U + \frac{R \sin \varphi}{2r (R + r \cos \varphi)^2} U \right] \quad (4.9)$$

for every $(\varphi, \theta) \in I^+$. Multiply both sides of (4.8) by $U$ and integrate them on $I^+$ to obtain

$$\int_{I^+} \left\{ \left( \Delta_g U + \lambda_1 U \right) V + \frac{\partial \lambda_1}{\partial \epsilon} \right|_{\epsilon=0} U^2 \right\} d\sigma = 0. \quad (4.9)$$

By using the integration by parts and the fact that $\int_{I^+} U^2 d\sigma = 1$, we obtain

$$\int_{I^+} \left( \Delta_g U + \lambda_1 U \right) V d\sigma + \frac{\partial \lambda_1}{\partial \epsilon} \bigg|_{\epsilon=0} = 0. \quad (4.10)$$

Since $\Delta_g U + \lambda_1 U = 0$, $\frac{\partial \lambda_1}{\partial \epsilon} \bigg|_{\epsilon=0} = 0$. Hence, for every $(\varphi, \theta) \in I^+$

$$\Delta_g V + \lambda_1 V = \frac{2}{r} \sin(n \theta) \left[ -\lambda_1 U + \frac{R \sin \varphi}{2r (R + r \cos \varphi)^2} U \right]. \quad (4.10)$$

The right-hand side of (4.10) can be regarded as a product of a function of one variable $\varphi$ and a function of one variable $\theta$ and all the coefficients of the left-hand side of (4.10) are independent of $\theta$. Moreover, the right-hand side of (4.10) is infinitely differentiable. Then, by the standard regularity theory for elliptic partial differential equations (see [GT]), we may differentiate both sides of (4.10) with respect to $\theta$ twice to obtain

$$\Delta_g \left( \frac{V_{\theta\theta}}{n^2} + V \right) + \lambda_1 \left( \frac{V_{\theta\theta}}{n^2} + V \right) = 0 \quad \text{in} \quad T^+. \quad (4.11)$$

In addition, by (4.6)

$$\frac{V_{\theta\theta}}{n^2} + V = 0 \quad \text{on} \quad \partial T^+. \quad (4.12)$$
Since the principal eigenfunction \( U \) of (1.1) on \( T^+ \) is unique, (4.11) together with (4.12) yields that

\[
\frac{V_{\theta\theta}}{n^2} + V = cU
\]  

(4.13)

for some constant \( c \in \mathbb{R} \). Hence, the function \( V \) can be expressed as

\[
V(\varphi, \theta) = C_1(\varphi) \cos(n\theta) + C_2(\varphi) \sin(n\theta) + cU \quad \text{for every } (\varphi, \theta) \in I^+,
\]  

(4.14)

for some functions \( C_1(\varphi), C_2(\varphi) \) of class \( C^2 \). By (4.6), we have

\[
C_1(0) = C_2(0) = C_1(\pi) = C_2(\pi) = 0.
\]  

(4.15)

Let us substitute (4.14) into (4.10). Then, \( C_1, C_2 \) satisfy the following two ordinary differential equations:

\[
C''_1(\varphi) - \frac{r \sin \varphi}{R + r \cos \varphi} C'_1(\varphi) - B_n(\varphi) C_1(\varphi) = 0,
\]  

(4.16)

\[
C''_2(\varphi) - \frac{r \sin \varphi}{R + r \cos \varphi} C'_2(\varphi) - B_n(\varphi) C_2(\varphi) = A(\varphi),
\]  

(4.17)

where we set

\[
A(\varphi) = 2r \left[ -\lambda_1 U + \frac{R \sin \varphi}{2r(R + r \cos \varphi)^2} U_{\varphi} \right] \quad \text{and} \quad B_n(\varphi) = r^2 \left[ \frac{n^2}{(R + r \cos \varphi)^2} - \lambda_1 \right].
\]

Choose \( N \in \mathbb{N} \) in Theorem 1.1 as \( N > \sqrt{\lambda_1}(R + r) \). Then, for every \( n \geq N \), \( B_n(\varphi) \) is positive in \((0, \pi)\). Let \( n \geq N \). Hence, by applying the maximum principle to (4.16) we have that \( C_1 \equiv 0 \). Therefore, we have

\[
V(\varphi, \theta) = C_2(\varphi) \sin(n\theta) + cU \quad \text{for every } (\varphi, \theta) \in I^+.
\]  

(4.18)

To evaluate \( V(\varphi, \theta) \) for \( \varphi \in [\varphi^* - \delta, \varphi^* + \delta] \), we need to evaluate \( C_2(\varphi) \) for \( \varphi \in [\varphi^* - \delta, \varphi^* + \delta] \). In Proposition 4.1 below, we will show that \( C_2(\varphi^*) \) is positive. Then, the continuity of \( C_2 \) gives us \( C_2(\varphi) > 0 \) for every \( \varphi \in [\varphi^* - \delta, \varphi^* + \delta] \), provided that \( \delta > 0 \) is chosen sufficiently small.

An elementary calculation of the derivatives of (4.18) with respect to \( \theta \) up to the second order shows that, for every \( \varphi \in [\varphi^* - \delta, \varphi^* + \delta] \)

\[
\begin{align*}
V_\theta(\varphi, \theta) \neq 0 & \quad \text{if} \quad \theta \notin \{\theta_k | k = 0, 1, 2, \ldots, 2n - 1\}, \\
V_{\theta\theta}(\varphi, \theta) \neq 0 & \quad \text{if} \quad \theta \in \{\theta_k | k = 0, 1, 2, \ldots, 2n - 1\}.
\end{align*}
\]  

(4.19)
Then, it follows from (4.2) and (4.7) that there exists $\epsilon_2 \in (0, \epsilon_1)$ such that if $|\epsilon| < \epsilon_2$, for every $\varphi \in [\varphi^* - \delta, \varphi^* + \delta]$

$$
\begin{align*}
\frac{\partial u^\epsilon}{\partial \theta}(\varphi, \theta) \neq 0 & \quad \text{if } \theta \notin \{\theta_k | k = 0, 1, 2, \ldots, 2n - 1\}, \\
\frac{\partial^2 u^\epsilon}{\partial \theta^2}(\varphi, \theta) > 0 & \quad \text{if } \theta \in N_k, k \text{ is an odd number in } \{0, 1, \ldots, 2n - 1\}, \\
\frac{\partial^2 u^\epsilon}{\partial \theta^2}(\varphi, \theta) < 0 & \quad \text{if } \theta \in N_k, k \text{ is an even number in } \{0, 1, \ldots, 2n - 1\},
\end{align*}
$$

where $N_k = [\theta_k - \delta, \theta_k + \delta]$.

We conclude that the set of critical points of the principal eigenfunction $u^\epsilon$ on $T^+_\epsilon$ corresponds to:

$$(\varphi, \theta) \in \{ (\hat{\varphi}(\epsilon, \theta_k), \theta_k) | k = 0, 1, \ldots, 2n - 1 \},$$

which consists of exactly $2n$ critical points in $T^+_\epsilon$. This completes the proof.

**Proposition 4.1.** $C_2(\varphi^*)$ is strictly positive.

**Proof.** Suppose on the contrary that $C_2(\varphi^*) \leq 0$. We will show that $C_2'(\varphi^*) > 0$.

Suppose that $C_2'(\varphi^*) = 0$. Then, from (4.17) we have

$$
C_2''(\varphi^*) = B_n(\varphi^*)C_2(\varphi^*) + A(\varphi^*) \leq A(\varphi^*) = -2r\lambda_1U(\varphi^*) < 0.
$$

Since $C_2(\pi) = 0$, there exists a point $\eta \in (\varphi^*, \pi)$ that satisfies

$$
\begin{cases}
C_2(\eta) = \min_{[\varphi^*, \pi]} C_2 < 0, \\
C_2'(\eta) = 0, \quad C_2''(\eta) \geq 0.
\end{cases}
$$

(4.20)

Substituting $\eta$ into the left-hand side of (4.17) yields

$$
C_2''(\eta) - \frac{r \sin \eta}{R + r \cos \eta}C_2'(\eta) - B_n(\eta)C_2(\eta) > 0.
$$

(4.21)

On the other hand,

$$
A(\eta) = 2r \left[ -\lambda_1U(\eta) + \frac{R \sin \eta}{2r(R + r \cos \eta)^2}U_{\eta}(\eta) \right] < 0,
$$

(4.22)

which is a contradiction. Hence, $C_2'(\varphi^*) \neq 0$.

Next, we suppose that $C_2''(\varphi^*) < 0$. Since $C_2(\varphi^*) \leq 0$ and $C_2(\pi) = 0$, there exists $\eta \in (\varphi^*, \pi)$ satisfying (4.20). Exactly in the same way as in the previous case, we get
a contradiction. Eventually, we conclude that \( C_2'(\varphi^*) > 0 \). Under this circumstance, we shall prove that \( C_2(\varphi) < 0 \) for every \( \varphi \in [0, \varphi^*) \), which contradicts \( C_2(0) = 0 \). 

Recall that \( A(\varphi^*) < 0 \). Set

\[
\gamma = \inf \left\{ \Lambda \in (0, \varphi^*) \left| A(\varphi) < 0 \text{ for every } \varphi \in (\Lambda, \varphi^*) \right. \right\}. \tag{4.23}
\]

Then, \( 0 \leq \gamma < \varphi^* \). Note that \( \gamma \) is independent of \( n \). Since \( A(0) = 0 \), it follows that \( A(\gamma) = 0 \). Equation (4.17) is represented as

\[
\left( (R + r \cos \varphi)C_2(\varphi) \right)' = (R + r \cos \varphi) \left[ B_n(\varphi)C_2(\varphi) + A(\varphi) \right]. \tag{4.24}
\]

Since \( C_2(\varphi^*) \leq 0 \) and \( C_2'(\varphi^*) > 0 \), there exists \( \beta \in (0, \varphi^* - \gamma) \) such that

\( C_2(\varphi) < 0 \) for every \( \varphi \in (\varphi^* - \beta, \varphi^*) \). \( \tag{4.25} \)

If \( \varphi \in (\varphi^* - \beta, \varphi^*) \), then

\[
\int_{\varphi}^{\varphi^*} \left( (R + r \cos h)C_2(h) \right)' \, dh = \int_{\varphi}^{\varphi^*} (R + r \cos h) \left[ B_n(h)C_2(h) + A(h) \right] \, dh < 0,
\]

with the result that

\( (R + r \cos \varphi)C_2'(\varphi) > (R + r \cos \varphi^*)C_2'(\varphi^*) > 0 \) for every \( \varphi \in (\varphi^* - \beta, \varphi^*) \).

Hence,

\[
C_2'(\varphi) > 0 \text{ and } C_2(\varphi) < C_2(\varphi^*) \leq 0 \text{ for every } \varphi \in (\varphi^* - \beta, \varphi^*). \]

Set

\[
H = \inf \left\{ \Lambda \in (\gamma, \varphi^*) \left| C_2'(\varphi) > 0 \text{ for every } \varphi \in (\Lambda, \varphi^*) \right. \right\}. \tag{4.26}
\]

Then \( \gamma \leq H < \varphi^* \). Let us show that \( H = \gamma \). For this purpose, we suppose that \( H > \gamma \). From the definition of \( H \),

\[
C_2'(\varphi) > 0 \text{ and } C_2(\varphi) < C_2(\varphi^*) \leq 0 \text{ for every } \varphi \in (H, \varphi^*). \tag{4.27}
\]

Integrating (4.24) in \( \varphi \) from \( H \) to \( \varphi^* \) yields that

\[
\int_{H}^{\varphi^*} \left( (R + r \cos h)C_2'(h) \right) \, dh = \int_{H}^{\varphi^*} (R + r \cos h) \left[ B_n(h)C_2(h) + A(h) \right] \, dh < 0,
\]

and hence

\( (R + r \cos H)C_2'(H) > (R + r \cos \varphi^*)C_2'(\varphi^*) > 0 \).
Then $C_2'(H) > 0$. This contradicts the definition of $H$. Therefore $H = \gamma$.

Since $C_2(\varphi) < 0$ for every $\varphi \in (\gamma, \varphi^*)$, if $\varphi \in (\gamma, \varphi^*)$ then

$$
\int_{\varphi}^{\varphi^*} \left( (R + r \cos h)C_2'(h) \right) dh = \int_{\varphi}^{\varphi^*} \left( (R + r \cos h) \left[ B_n(h)C_2(h) + A(h) \right] \right) dh
$$

$$
< \int_{\varphi}^{\varphi^*} (R + r \cos h)A(h) \, dh.
$$

Let us set

$$K(\varphi) = \int_{\varphi}^{\varphi^*} (R + r \cos h)A(h) \, dh \text{ for every } \varphi \in (\gamma, \varphi^*).$$

Since $A(\varphi) < 0$ for every $\varphi \in (\gamma, \varphi^*)$, we have

$$(R + r \cos \varphi^*)C_2'(\varphi^*) - (R + r \cos \varphi)C_2'(\varphi) < K(\varphi) < 0 \text{ for every } \varphi \in (\gamma, \varphi^*).$$

Hence,

$$(R + r \cos \varphi)C_2'(\varphi) > (R + r \cos \varphi^*)C_2'(\varphi^*) - K(\varphi) > -K(\varphi) \text{ for every } \varphi \in (\gamma, \varphi^*).$$

Thus,

$$C_2'(\varphi) > -\frac{K(\varphi)}{R + r \cos \varphi} \text{ for every } \varphi \in (\gamma, \varphi^*). \quad (4.28)$$

Therefore, it follows that

$$C_2(\gamma) = C_2(\varphi^*) - \int_{\gamma}^{\varphi^*} C_2'(h) \, dh \leq - \int_{\gamma}^{\varphi^*} C_2'(h) \, dh < \int_{\gamma}^{\varphi^*} \frac{K(h)}{R + r \cos h} \, dh.$$  

If we set $\tau = -\int_{\gamma}^{\varphi^*} \frac{K(h)}{R + r \cos h} \, dh (> 0)$, $\tau$ is independent of $n$ and $C_2(\gamma) < -\tau$. Moreover, letting $\varphi \to \gamma^+$ in (4.28) yields that

$$C_2'(\gamma) \geq -\frac{K(\gamma)}{R + r \cos \gamma} > 0. \quad (4.29)$$

Next we set

$$H^* = \inf \left\{ \Lambda \in (0, \gamma) \mid C_2'(\varphi) > 0 \text{ for every } \varphi \in (\Lambda, \gamma) \right\}.$$

Then $0 \leq H^* < \gamma$ and hence

$$C_2'(\varphi) > 0 \text{ and } C_2(\varphi) < C_2(\gamma) < -\tau \text{ for every } \varphi \in (H^*, \gamma).$$
Let us show that $H^* = 0$. For this purpose, we suppose that $H^* > 0$. Recall that $n \geq N > \sqrt{\lambda_1}(R + r)$. Since $\frac{A(\varphi)}{\tau}$ is independent of $n$, we may update $N \in \mathbb{N}$ with a large number such that if $n \geq N$ we have

$$B_n(\varphi) > \frac{A(\varphi)}{\tau} \quad \text{for every } \varphi \in (0, \pi).$$

Let $n \geq N$. As a consequence,

$$\int_{H^*} \left( (R + r \cos h)C_2(h) \right)' dh = \int_{H^*} (R + r \cos h) \left[ B_n(h)C_2(h) + A(h) \right] dh < \int_{H^*} (R + r \cos h) \left[ -B_n(h)\tau + A(h) \right] dh < 0.$$

Then, we have from (4.29) that

$$0 < (R + r \cos \gamma)C_2'(\gamma) < (R + r \cos H^*)C_2'(H^*).$$

Thus $C_2'(H^*) > 0$, which contradicts the definition of $H^*$. Therefore $H^* = 0$ and hence

$$C_2'(\varphi) > 0 \quad \text{and} \quad C_2(\varphi) < -\tau \quad \text{for every } \varphi \in (0, \gamma).$$

In particular, we have $C_2(0) < 0$ which contradicts the fact that $C_2(0) = 0$. Eventually, $C_2(\varphi^*) > 0$. \[\Box\]

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