The algebraic structure behind the derivative nonlinear Schrödinger equation

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Received 19 February 2013, in final form 17 May 2013
Published 5 July 2013
Online at stacks.iop.org/JPhysA/46/305201

Abstract

The Kaup–Newell (KN) hierarchy contains the derivative nonlinear Schrödinger equation (DNLSE) amongst others interesting and important nonlinear integrable equations. In this paper, a general higher grading affine algebraic construction of integrable hierarchies is proposed and the KN hierarchy is established in terms of a $\hat{sl}_2$ Kac–Moody algebra and principal gradation. In this form, our spectral problem is linear in the spectral parameter. The positive and negative flows are derived, showing that some interesting physical models arise from the same algebraic structure. For instance, the DNLSE is obtained as the second positive, while the Mikhailov model as the first negative flows. The equivalence between the latter and the massive Thirring model is also explicitly demonstrated. The algebraic dressing method is employed to construct soliton solutions in a systematic manner for all members of the hierarchy. Finally, the equivalence of the spectral problem introduced in this paper with the usual one, which is quadratic in the spectral parameter, is achieved by setting a particular automorphism of the affine algebra, which maps the homogeneous into principal gradation.

PACS numbers: 02.30.Ik, 05.45.Yv, 03.50.−z, 11.10.Lm, 02.30.Jr
Mathematics Subject Classification: 35Q51, 37K10, 35Q55

(Some figures may appear in colour only in the online journal)

1. Introduction

The derivative nonlinear Schrödinger equation (DNLSE-I)

$$i\partial_t \psi + \partial_x^2 \psi \pm i\partial_x (|\psi|^2 \psi) = 0$$

is a well-known integrable model with interesting physical applications. In particular, it describes nonlinear Alfvén waves in plasma physics [1–6] and the propagation of ultra-short pulses in nonlinear optics [7–9].
Equation (1), and also other related models that will be mentioned in the following, have been extensively studied for a long time. Its inverse scattering transform (IST) with a vanishing boundary condition (VBC), $\psi \to 0$ as $|x| \to \infty$, was first solved in [10]. Equation (1) is one of the nonlinear evolution equations comprising the Kaup–Newell (KN) hierarchy.

The complete integrability of (1) and the hierarchy of Hamiltonian structures of the KN hierarchy were constructed in [11]. Moreover, the Riemann–Hilbert problem was considered, as well as expansions over the squared solutions [11]. A specific feature of these models is that they have a Lax operator containing a quadratic power on the spectral parameter. More specifically, (1) and its related models can be obtained from a zero-curvature representation where the standard spectral problem is given by

$$ A_x = \left( \begin{array}{cc} \lambda^2 & \lambda q \\ \lambda r & -\lambda^2 \end{array} \right), \quad (\partial_x + A_x) \psi = 0, \quad (2) $$

where $q = q(x, t)$ and $r = r(x, t)$ are the fields and $\lambda$ is the complex spectral parameter. Due to the quadratic power of $\lambda$ the original form of the IST [12] has a divergent Cauchy integral when $|\lambda| \to \infty$. This is the main reason for introducing the revision in the IST [10], through some weight functions that control such divergence. Let us recall that the Zakharov–Shabat approach [12] was initially proposed to solve the Ablowitz–Kaup–Newell–Segur (AKNS) spectral problem

$$ A_x = \left( \begin{array}{cc} \lambda & q \\ r & -\lambda \end{array} \right). \quad (3) $$

Another obvious difference worth mentioning is that in (2) the fields are associated with $\lambda$, unlike (3). This implies in a higher grading construction from the algebraic formalism point of view, in contrast to the AKNS construction where the fields are in a zero grade subspace.

For a nonvanishing boundary condition (NVBC), $\psi \to \text{const.}$ as $|x| \to \infty$, an IST approach was also proposed [13, 14], but it is a difficult procedure due to the appearance of a double-valued function of $\lambda$ and, therefore, the IST had to be developed on its Riemann sheets. A more straightforward method introduces a convenient affine parameter that avoids the construction of the Riemann sheets [15]. Recently, solutions with VBC and also NVBC were constructed through Darboux/Bäcklund transformations yielding explicit and useful formulas [16, 17]. These works are all based on operator (2) or the revised form of the IST [10].

There are also two other known types of DNLSEs, namely, the DNLSE-II [18]

$$ i\partial_t \psi + \partial_x^2 \psi + 4i|\psi|^2 \partial_x \psi = 0 $$

and the DNLSE-III [19]

$$ i\partial_t \psi + \partial_x^2 \psi \mp 4i|\psi|^2 \partial_x \psi \mp 8|\psi|^4 \psi = 0. $$

The gauge equivalence between (1), (4) and (5) was analyzed for the first time in [19] and its Hamiltonian structures have also been extensively studied. (Regarding gauge equivalent models see also [20].) A hierarchy containing (4) within the Sato–Wilson dressing formalism was also considered and it was shown that (4) can be reduced to the fourth Painlevé equation [21].

The well-known massive Thirring model

$$ i\partial_v - mu + 2g|u|^2 v = 0, $$

$$ i\partial_u + mv - 2g|v|^2 u = 0, $$

(6)
was proved to be integrable and solved through the IST for the first time in [22] (see also [23, 24]). The relation between (6) and (1) was also pointed out [11, 19]. We will show this relation precisely through another model, arising naturally from the first negative flow of the KN hierarchy, namely
\[ \partial_t \partial_x \phi - \phi \mp 2i|\phi|^2 \partial_x \phi = 0. \] (7)

This relativistically invariant model has attracted attention only recently, although it was already proposed a long time ago [11, 19] and is known as the Mikhailov model. It is already known that (7) is equivalent to (6) [11, 19]. Equation (7) was also called the Fokas–Lenells equation in recent works and its multi-soliton solutions were obtained through Hirota’s bilinear method [25], where it was pointed out that they have essentially the same form as those of (1) if one introduces the potential \( \psi = \partial_x \phi \) and changes the dispersion relation. We will explain precisely the origin of this relation. Equation (7) was also referred as the modified Pohlmeyer–Lund–Regge model [26] and it can be reduced to the third Painlevé equation.

It is important to mention that there are matrix or multi-field generalizations of DNLS and Thirring-like models [27–29].

The integrable properties of (1), (4), (5) and (6), like their soliton solutions and Hamiltonian hierarchies, have already been thoroughly studied, especially in [10, 11, 19, 22]. Nevertheless, these models, and more recently (7), continue to attract attention as they have not been fully studied through more recent techniques. One of the approaches to obtain soliton solutions is through the dressing method proposed in the pioneer paper [12], which is connected to the Riemann–Hilbert problem. Another approach occurs in connection to \( \tau \)-functions and transformation groups [30], where solitons are obtained through vertex operators. The precise relation between these two methods was established in [31] (see also [32] for a thorough explanation) and it enables one to construct soliton solutions in a purely affine algebraic manner. Thus, we refer to this approach as the algebraic dressing method.

Since then [31], there has been a significant development of affine algebraic techniques [33–36] relying on the algebraic structure underlying the equations of motion, providing general and systematic methods. This algebraic approach is well understood for models that fit into the AKNS construction, where the fields are associated with zero grade operators. Nevertheless, there are some generalizations of this formalism, for instance, the addition of fermionic fields to higher grading operators [37, 38] and a generalization of the dressing method to include NVBC for the AKNS construction [39–41]. Recently, a higher grading construction was proposed that includes the Wadati–Konno–Ichikawa hierarchy, and the algebraic dressing formalism was supplemented with reciprocal transformations [42]. Following this line of thought, it is important to embrace other known integrable models into such an algebraic formalism, to extend these techniques beyond the AKNS scheme. This is accomplished here regarding the KN hierarchy. We introduce its underlying algebraic structure and employ the algebraic dressing method to construct its soliton solutions systematically. Through gauge transformations we also obtain solutions of the other related models like (4)–(6), for instance. An important remark is in order. We construct these models starting from a spectral problem that is not quadratic on the spectral parameter and it simplifies the procedure to construct the solutions. We also prove that our construction is equivalent to the quadratic one (2).

Our work is thus organized as follows. In section 2, we introduce a general higher grading affine algebraic construction of integrable hierarchies. In section 3, when the algebra \( \hat{A}_1 \) with principal gradation is chosen, this construction yields the KN hierarchy as a particular case, where (1) appears as the second positive and (7) as the first negative flows. We introduce the explicit transformations relating equations (1), (4) and (5). Furthermore, we propose a
transformation that takes (7) to the Thirring model (6), demonstrating their equivalence. In section 4, we employ the algebraic dressing method, which along with the representation theory of affine Lie algebras provides a systematic construction of the solutions for all the models within the KN hierarchy. We also obtain the explicit solutions of the other mentioned models. In section 5, we prove the equivalence between our construction, which is linear in $\lambda$, and the usual one (2). Finally, section 6 contains our concluding remarks. We refer the reader to appendix A for the algebraic concepts involved in this paper.

2. A general integrable hierarchy

Let $\hat{G}$ be a semi-simple Kac–Moody algebra with a grading operator $Q$. The algebra is then decomposed into graded subspaces

$$\hat{G} = \sum_{m \in \mathbb{Z}} \hat{G}^{(m)} = \{T_n^a \mid [Q, T_n^a] = m T_n^a\}.$$  \hfill (8)

The parentheses in the superscript of an operator denote its grade according to $Q$ and should not be confused with the affine index $n$ without parentheses, which is the power of the spectral parameter.

Let $E^{(2)}$ be a semi-simple element of grade 2. Define the kernel, $K$, and image, $M$, subspaces as follows:

$$K = \{T_n^a \in \hat{G} \mid [E^{(2)}, T_n^a] = 0\}, \quad \hat{G} = K \oplus M.$$  \hfill (9)

From the Jacobi identity we conclude that

$$[K, K] \subset K, \quad [K, M] \subset M,$$  \hfill (10)

and we assume the symmetric space structure

$$[M, M] \subset K.$$  \hfill (11)

Let $F^{(1)}[\phi] \in M^{(1)}$ be the operator containing the fields of the models, i.e. choose all generators of grade 1 in $M$, $M^{(1)} = \{R_1^{(1)}, \ldots, R_r^{(1)}\}$, and take the linear combination $F^{(1)}[\phi] = \phi_1 R_1^{(1)} + \cdots + \phi_r R_r^{(1)}$ where $\phi = (\phi_1, \ldots, \phi_r)$ and $\phi_i = \phi_i(x, t)$. Now we define the integrable hierarchy of PDEs starting from the zero-curvature condition

$$\partial_x A_t - \partial_t A_x + [A_x, A_t] = 0$$  \hfill (12)

with the potentials defined as follows:

$$A_x \equiv E^{(2)} + F^{(1)}[\phi],$$  \hfill (13a)

$$A_t \equiv \sum_{m=-2M}^{2N} D^{(m)}[\phi]$$  \hfill (13b)

where $N$ and $M$ are arbitrary fixed positive integers that label each model within the hierarchy and $D^{(m)}[\phi] \in \hat{G}^{(m)}$.

Let us now show that with the algebraic structure defined by (13), the zero-curvature equation (12) can be solved nontrivially, yielding a PDE for each choice of $N$ and $M$. Note that (13a) is well defined and independent of $N$ and $M$, therefore, we need to show that each $D^{(m)}$
in (13b) can be uniquely determined in terms of the fields. Projecting (12) into each graded subspace we obtain the following set of equations:

\[ [E^{(2)}, D^{(2N)}] = 0 \quad \text{grade } 2N + 2, \]
\[ [E^{(2)}, D^{(2N-1)}] + [F^{(1)}, D^{(2N)}] = 0 \quad \text{grade } 2N + 1, \]
\[ \partial_x D^{(2N)} + [E^{(2)}, D^{(2N-2)}] + [F^{(1)}, D^{(2N-1)}] = 0 \quad \text{grade } 2N, \]
\[ \vdots \]
\[ \partial_x D^{(2)} + [E^{(2)}, D^{(0)}] + [F^{(1)}, D^{(1)}] = 0 \quad \text{grade } 2, \]
\[ \partial_x D^{(1)} - \partial_t F^{(1)} + [E^{(2)}, D^{(-1)}] + [F^{(1)}, D^{(0)}] = 0 \quad \text{grade } 1, \]
\[ \partial_x D^{(0)} + [E^{(2)}, D^{(-2)}] + [F^{(1)}, D^{(-1)}] = 0 \quad \text{grade } 0, \]
\[ \vdots \]
\[ \partial_x D^{(-2M+2)} + [E^{(2)}, D^{(-2M)}] + [F^{(1)}, D^{(-2M+1)}] = 0 \quad \text{grade } -2M + 2, \]
\[ \partial_x D^{(-2M+1)} + [F^{(1)}, D^{(-2M)}] = 0 \quad \text{grade } -2M + 1, \]
\[ \partial_t D^{(-2M)} = 0 \quad \text{grade } -2M. \]

These equations can be solved recursively, starting from grade \(2N + 2\) until grade 2 and from grade \(-2M\) until grade zero. Each equation still splits into \(\mathcal{K}\) and \(\mathcal{M}\) components according to (10) and (11). We also have the \(\mathcal{K}\) component of the grade 1 equation as a constraint. Thus, each \(D^{(m)}\) is determined in terms of the fields \(\phi\) and, consequently, the equations of motion are obtained from the \(\mathcal{M}\) component of the grade 1 projection

\[ \partial_x D^{(1)} - \partial_t F^{(1)} + [E^{(2)}, D^{(-1)}] + [F^{(1)}, D^{(0)}] = 0. \]

(15)

The potential (13b) generates mixed flows [43]. If one is interested in positive flows only, yielding the positive part of the hierarchy, one must restrict the sum as

\[ A_\ell \equiv \sum_{m=1}^{2N} D^{(m)}[\phi] \]

(16)

and the equations of motion then simplify to

\[ \partial_x D^{(1)}_M - \partial_t F^{(1)} = 0. \]

(17)

For negative flows one must restrict the sum as

\[ A_\ell \equiv \sum_{m=0}^{-2N} D^{(m)}[\phi] \]

(18)

and the equations of motion are then given by

\[ \partial_t F^{(1)} - [E^{(2)}, D^{(-1)}] - [F^{(1)}, D^{(0)}] = 0. \]

(19)

Therefore, we have shown that the hierarchy defined by (13) solves the zero-curvature equation (12). This construction is valid for an arbitrary graded affine Lie algebra \(\tilde{G}\) and the hierarchy is defined through the choice of \(\{\tilde{G}, Q, E^{(2)}\}\), leading to an immediate algebraic classification. Each choice of positive integers \(N\) and \(M\) yields one mixed model. For positive or negative flows the models are labeled by \(N\) only.

3. The Kaup–Newell hierarchy

Let us take the previous construction with the loop-algebra \(\tilde{G} = A_1 = \{\mathcal{H}^n, E^n_o, E^n_{-o}\}\) and principal gradation \(Q = \frac{1}{2}H^0 + 2\tilde{d}\), leading to the decomposition \(\tilde{G}^{(2m)} = \{\mathcal{H}^m\}\) and
\( \hat{G}^{(2m+1)} = \{ E_m^a, E_{-a}^{m+1} \} \). The semi-simple element is chosen as \( E^{(2)} = H^1 \) and then (9) is given by

\[
K^{(2m)} = \{ H^m \}, \quad M^{(2m+1)} = \{ E_m^a, E_{-a}^m \}. \tag{20}
\]

The operator containing the fields must now have the form \( F^{(1)} = q(x, t)E^0_a + r(x, t)E_{-a}^1 \) and (13) then reads

\[
A_t = H^1 + qE^0_a + rE_{-a}^1, \tag{21a}
\]

\[
A_t = \sum_{m=1}^{2N} D^{(m)} \quad \text{or} \quad A_t = -\sum_{m=0}^{-2N} D^{(m)}, \tag{21b}
\]

where \( D^{(2m)} = c_{2m}H^m \) and \( D^{(2m+1)} = a_{2m+1}E^m_a + b_{2m+1}E_{-a}^m \). The coefficients \( a_{2m+1}, b_{2m+1} \) and \( c_{2m} \) will be determined in terms of the fields \( q \) and \( r \) by solving the zero-curvature equation, as explained previously in (14). The first equality in (21b) is valid for the positive flows, while the second one for the negative flows.

Comment. If instead of the principal gradation one considers the homogeneous gradation \( Q = \hat{d} \), yielding \( \hat{G}^{(m)} = \{ E^m_a, E_{-a}^m, H^m \} \), and the semi-simple element \( E^{(2)} = H^2 \), we obtain the following Lax pair for the KN hierarchy:

\[
A_t = H^2 + qE^0_a + rE_{-a}^1, \tag{22a}
\]

\[
A_t = \sum_{m=1}^{2N} D^{(m)} \quad \text{or} \quad A_t = -\sum_{m=0}^{-2N} D^{(m)}, \tag{22b}
\]

where \( D^{(m)} = a_mE^m_a + b_mE_{-a}^m + c_mH^m \). The operator (22a) is exactly the standard one found in the literature (2) [10, 11, 19, 22] (see the matrix representation in appendix A). With the construction (22) we obtain precisely the same equations of motion as the construction (21), which will be derived in the sequel. Moreover, we will demonstrate the equivalence between both constructions in section 5. Note, however, that (21a) does not have a quadratic power on the spectral parameter, in contrast to (22a). The convenience of using (21a) appears clearly when constructing the solutions of the models within the hierarchy.

### 3.1. Positive flows

The models within the positive part of the hierarchy are obtained from the zero-curvature equation

\[
\left[ \partial_t + H^1 + qE^0_a + rE_{-a}^1, \partial_t + D^{(2N)} + D^{(2N-1)} + \cdots + D^{(1)} \right] = 0. \tag{23}
\]

For \( N = 1 \) we have the trivial equations \( \partial_t q = \partial_t q \) and \( \partial_t r = \partial_t r \). For \( N = 2 \), after solving each equation in (14), we get the following equations of motion:

\[
2\partial_t q + \partial_t^2 q + \partial_t (q^2 r) = 0, \tag{24a}
\]

\[
2\partial_t r - \partial_t^2 r + \partial_t (q^2 r) = 0, \tag{24b}
\]

whose explicit Lax pair is given by

\[
A_t = H^1 + qE^0_a + rE_{-a}^1, \tag{25a}
\]

\[
A_t = H^2 + qE^0_a + rE_{-a}^2 - \frac{1}{2}qrH^1 - \frac{1}{2}(q^2 r + \partial_t q)E^0_a - \frac{1}{2}(q^2 r - \partial_t r)E_{-a}^1. \tag{25b}
\]
Taking (24) under the transformations $x \rightarrow ix$, $t \rightarrow 2it$, $q = \psi$ and $r = \pm \psi^*$, we obtain precisely equation (1),

$$i \partial_t \psi + \partial_t^2 \psi \pm i \partial_x \left( |\psi|^2 \psi \right) = 0. \quad (26)$$

Considering $N = 3$, and after solving (14), we obtain the model

$$4 \partial_t q - \partial_t^2 q - 3 \partial_q (qr \partial_q q) - \frac{1}{2} \partial_q (q^2 r^2) = 0,$$

$$4 \partial_t r - \partial_t^2 r + 3 \partial_q (qr \partial_q r) - \frac{1}{2} \partial_q (q^2 r^2) = 0, \quad (27a)$$

$$4 \partial_t q - 3 \partial_q (qr \partial_q q) + \frac{1}{2} \partial_q (q^2 r^2) = 0,$$

$$4 \partial_t r - 3 \partial_q (qr \partial_q r) - \frac{1}{2} \partial_q (q^2 r^2) = 0, \quad (27b)$$

together with its Lax pair

$$A_s = H^1 + qE_a^0 + rE_{-a}, \quad (28a)$$

$$A_t = H^3 + qE_a^2 + rE_{-a}^2 - \frac{1}{2} qr H^2 - \frac{1}{2} (q^2 r + \partial_t q \partial_t r)E_a^1$$

$$- \frac{1}{2} (q^2 - \partial_t r)E_{-a}^1 + \frac{1}{2} (\partial_t q - q \partial_q r + \frac{1}{2} q^2 r^2)H^1$$

$$+ \frac{1}{2} (\partial_t^2 q + 3 qr \partial_q q + \frac{1}{2} q^3 r^2)E_a^0 + \frac{1}{3} (\partial_t^2 r - 3 qr \partial_t r + \frac{1}{2} q^2 r^3)E_{-a}^1. \quad (28b)$$

The system (27) under $x \rightarrow ix$, $t \rightarrow -4it$, $q = \psi$ and $r = \pm \psi^*$ becomes

$$\partial_t \psi - \partial_t^2 \psi \mp 3 \partial_q (|\psi|^2 \partial_t \psi) + \frac{1}{2} \partial_x \left( |\psi|^4 \psi \right) = 0. \quad (29)$$

Continuing in this way for $N = 4, 5, \ldots$ one can obtain higher order nonlinear PDEs.

### 3.2. Negative flows

The negative flows of the KN hierarchy are obtained from the zero-curvature equation

$$\left[ \partial_t + H^1 + qE_a^0 + rE_{-a}, \partial_t + D^{(-2N)} + D^{(-2N+1)} + \cdots + D^{(0)} \right] = 0. \quad (30)$$

Taking $N = 1$, after solving (14), we obtain the following nonlocal field equations:

$$\frac{1}{4} \partial_t q - \int_{-\infty}^{\infty} g \, dx' + g \int_{-\infty}^{x} q \, dx' \int_{-\infty}^{x} r \, dx' = 0, \quad (31a)$$

$$\frac{1}{4} \partial_t r - \int_{-\infty}^{x} r \, dx' - r \int_{-\infty}^{x} q \, dx' \int_{-\infty}^{x} r \, dx' = 0. \quad (31b)$$

These equations can be cast in a local form if we introduce new fields defined by

$$q \equiv \partial_t \phi, \quad r \equiv \partial_t \rho, \quad (32)$$

and then we obtain the relativistically invariant Mikhailov model [11, 19]

$$\frac{1}{4} \partial_t \partial_t \phi - \phi + \phi \rho \partial_t \phi = 0, \quad (33a)$$

$$\frac{1}{4} \partial_t \partial_t \rho - \rho - \rho \phi \partial_t \rho = 0. \quad (33b)$$

Its explicit Lax pair is given by

$$A_s = H^1 + \partial_t \phi E_a^0 + \partial_t \rho E_{-a}^1, \quad (34a)$$

$$A_t = H^{(-1)} + 2 \phi E_{-a}^2 - 2 \rho E_{-a}^0 + 2 \phi H^0. \quad (34b)$$

Taking (33) with $x \rightarrow \frac{1}{2} x$, $t \rightarrow -\frac{1}{2} t$, $\phi = \psi$ and $\rho = \pm \psi^*$ we obtain exactly (7),

$$\partial_t \partial_t \psi \mp 2i |\psi|^2 \partial_t \psi = 0. \quad (35)$$

This model can also be derived from the following Lagrangian:

$$\mathcal{L} = \frac{1}{2} (\partial_t \phi \partial_t \phi^* + \partial_t \phi^* \partial_t \phi) + |\psi|^2 \mp \frac{1}{2} |\psi|^2 (\phi \partial_t \phi^* - \phi^* \partial_t \phi). \quad (36)$$

If one consider $N = 2, 3, \ldots$ it is possible to obtain higher order integro-differential equations like (31).
3.3. Relations between DNLS type equations

Let us now introduce the transformations connecting the three types of DNLSes. From (24) the continuity equation immediately follows:

\[ \partial_t (qr) + \partial_j j = 0, \quad j = \frac{1}{2} (r \partial_j q - q \partial_j r) + \frac{3}{4} (qr)^2. \]

(37)

Let us define new fields through the following gauge transformation:

\[ \tilde{q} \equiv -\frac{1}{2} q e^{-\mathcal{J}}, \quad \tilde{r} \equiv \frac{1}{2} r e^{-\mathcal{J}}, \quad \mathcal{J} \equiv \int_{-\infty}^{\infty} qr \, dx', \]

(38)

where \( c \) is a constant. Upon using (37) the equations of motion (24) can written in terms of these new fields, yielding

\[ 2 \partial_t \tilde{q} + \tilde{q}^2 \tilde{q} - Aq^2 \tilde{r}^2 + Bq^2 \partial_\tau \tilde{r} - C \partial_\tau (q^2 \tilde{r}) = 0, \]

(39a)

\[ 2 \partial_t \tilde{r} - \tilde{q}^2 \tilde{r} + Aq^2 \tilde{r}^3 + Bq^2 \partial_\tau \tilde{q} - C \partial_\tau (q^2 \tilde{r}) = 0, \]

(39b)

where \( A = 8c (2c - 1) \), \( B = 4c \) and \( C = 4 (1 - c) \). Note that in (39) we have a fifth-order nonlinearity and two types of derivative nonlinear terms. Let us use the same transformation leading to equation (26), i.e. \( x \rightarrow ix, \, t \rightarrow 2it, \) \( \tilde{q} = \tilde{\psi} \) and \( \tilde{r} = \pm \psi^* \). If besides this we set \( c = \frac{1}{2} \), we obtain equation (4).

\[ i \partial_t \tilde{\psi} + \tilde{q}^2 \tilde{\psi} \pm 4i \tilde{\psi}^2 \partial_\tau \tilde{\psi} = 0. \]

(40)

On the other hand if we set \( c = 1 \), we obtain equation (5).

\[ i \partial_t \tilde{\psi} + \tilde{q}^2 \tilde{\psi} \pm 4i \tilde{\psi}^2 \partial_\tau \tilde{\psi} + 8|\tilde{\psi}|^4 \tilde{\psi} = 0. \]

(41)

Therefore, the transformation (38) connects explicitly the three types of DNLSes. If one knows a solution of (24), it is possible to obtain a solution of (39), which in particular yields the solutions of (40) and (41).

3.4. The massive Thirring model

The massive Thirring model is obtained from the Lagrangian

\[ \mathcal{L} = \tilde{\Phi} (i \gamma^\mu \partial_\mu - m) \Phi + \frac{g}{2} J_\mu J^\mu, \quad J^\mu = \tilde{\Phi} \gamma^\mu \Phi, \]

(42)

where \( m \) is the mass, \( g \) is the coupling constant and

\[ \Phi = \begin{pmatrix} u \\ v \end{pmatrix}, \quad \gamma^0 = \gamma_0 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = -\gamma_1 = i \sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \]

(43)

The \( \sigma_i \) are the usual Pauli matrices and \( \Phi \equiv \Phi^\dagger \gamma^0 \). The equations of motion are then given by

\[ i \gamma^\mu \partial_\mu \Phi - m \Phi + g J_\mu \gamma^\mu \Phi = 0, \]

(44)

or written in component form and in the light cone coordinates \( x = \frac{1}{2} (x^1 + x^0) \) and \( t = \frac{1}{2} (x^1 - x^0) \); we thus have

\[ i \partial_t v - mu + 2g |u|^2 v = 0, \]

(45a)

\[ i \partial_t u + mv - 2g |v|^2 u = 0. \]

(45b)

An important conclusion pointed out in [24] is that the solutions of (45) cannot, in general, correspond to solutions of the sine-Gordon model. However, as will be shown below, its solutions can be obtained, in general, from the solutions of the model (35).

\[ ^4 \text{It will be shown that, in fact, } \mathcal{J} \text{ is a local function.} \]
Consider (33) under the transformations $x \to -\frac{1}{2} x$, $t \to \frac{m^2}{4} t$ and $\phi = \varphi = \rho^*$. Thus, we have the equation of motion given by

$$\partial_t \partial_x \varphi = m^2 \varphi - 2i m^2 |\varphi|^2 \partial_x \varphi,$$

from which the continuity equation follows:

$$\partial_t |\partial_x \varphi|^2 = m^2 |\partial_x \varphi|^2.$$  \hfill (47)

Let us define new fields through the following relations:

$$u \equiv \frac{1}{\sqrt{2g}} (\partial_x \varphi) e^{i \mathcal{J}}, \quad v \equiv -\frac{i m}{\sqrt{2g}} \varphi e^{i \mathcal{J}}, \quad \mathcal{J} \equiv \int_{-\infty}^{x} |\partial_x \varphi|^2 \, dx'.$$

Then, calculating $\partial_t v$ and also $\partial_t u$, making use of (46) and (47), after writing the result in terms of the fields $u$ and $v$ we get precisely equations (45). Therefore, if $\varphi$ is a solution of (46), then (48) yields a solution of the Thirring model (45). \hfill 5

4. Dressing approach to the KN hierarchy

We now employ the algebraic dressing method [31] to construct soliton solutions for the KN hierarchy, assuming $q \to 0$ and $r \to 0$ when $|x| \to \infty$. To extract the fields in the dressing formalism it is necessary to employ a highest weight representation of the algebra, so we need the full Kac–Moody algebra $A_1$ including the central term. Thus, consider the Lax pair (21) but with $A_1$ in the following slightly different form:

$$\tilde{A}_x = H^1 + q E_0^0 + r E_{-a}^1 - (\partial_t v - 2i \delta_{N+1}^0) \hat{c},$$

where $v$ is a function to be determined. Note that the central term does not change the equations of motion, since it commutes with all other generators. The vacuum solution, obtained by setting $q \to 0$, $r \to 0$ and $v \to 0$, is then given by

$$\tilde{A}_x = H^1 + 2i \delta_{N+1}^0 \hat{c},$$

$$\tilde{A}_t = H^N,$$  \hfill (50b)

where $N = 2, 3, \ldots$ for the positive flows and $N = -1, -2, \ldots$ for the negative flows. Note that (50a) remains the same for every model, unless for the central term, while (50b) changes according to each model. The potentials (50) can still be written in the pure gauge form $\tilde{A}_\mu = -\partial_\mu \tilde{\Psi} \tilde{\Psi}^{-1}$ with

$$\tilde{\Psi} = e^{-H^1 x} e^{-H^1 x'} e^{-2 \delta_{N+1}^0 \rho \hat{c}}.$$  \hfill (51)

The idea of the dressing method is to obtain the general potentials $A_\mu$, with a nontrivial field configuration, out from the vacuum $\tilde{A}_\mu$, through the gauge transformations

$$A_\mu = \Theta \pm \tilde{A}_\mu \Theta^{-1} - \partial_\mu \Theta \Theta^{-1}.$$  \hfill (52)

Furthermore, we have the gauge freedom $\tilde{\Psi} \to \Psi' = \Psi g$ where $g$ is a constant group element. Hence, the dressing operators must satisfy $\Theta \pm \tilde{\Psi} = \Theta \pm \Psi g$, which is the Riemann–Hilbert problem

$$\Theta^{-1} \Theta = \tilde{\Psi} g \tilde{\Psi}^{-1}.$$  \hfill (53)

\hfill 5 If we consider the same kind of transformation for both equations (33), without requiring $\rho = \phi^*$, and define $\chi_1 = \frac{m^2}{8} \partial_x \phi \phi^* e^{i \mathcal{J}}$, $\chi_2 = \frac{m^2}{8} \partial_x \rho \rho^* e^{i \mathcal{J}}$, $\chi_3 = \frac{m^2}{8} \rho e^{-i \mathcal{J}}$ and $\chi_4 = -\frac{m^2}{8} \phi e^{i \mathcal{J}}$, where $\mathcal{J} = \int_{-\infty}^{x} \partial_x \phi \partial_x \rho \, dx'$, we obtain the four component Thirring-like model considered in [29], equation (2.31).
Assuming a Gauss decomposition, the dressing operators can be further factorized as
\[ \Theta_+ = e^{A^{(0)}} e^{B^{(1)}} e^{B^{(2)}} \ldots \]  
\[ \Theta_- = e^{B^{(0)}} e^{B^{(-1)}} e^{B^{(-2)}} \ldots \]  
where \( A^{(0)} \) and \( B^{(m)} \) are graded elements. According to the principal gradation these operators must have the following form:
\[ B^{(2m)} = \chi_{2m+1} E_{m} + \psi_{2m+1} E_{-m} + \phi_{2m} H_{m}, \]  
where the fields \( \chi_{2m+1}, \psi_{2m+1} \) and \( \phi_{2m} \) are now our unknowns.

It is enough to consider (52) for the Lax operator \( A_x \). Taking it first with the operator \( \Theta_+ \), the projection into the grade zero subspace yields
\[ A^{(0)} = \nu \hat{c}. \]  
The projection into the grade 1 subspace yields
\[ qE_{m}^{0} + rE_{-m}^{1} = -\partial_x B^{(1)}, \]  
therefore
\[ q = -\partial_x \chi_1, \quad r = -\partial_x \psi_1. \]  
In this way, by considering higher grade projections we can determine all operators appearing in \( \Theta_+ \), but just relations (57) are enough for our purposes. Now, let us consider (52) with the operator \( \Theta_- \). Taking the grade 2 projection we have
\[ B^{(0)} = \phi_0 H^{0}. \]  
Note that we already have a central term in (56) so we do not need to include another one in \( B^{(0)} \). The field \( \phi_0 \) will be determined by the next lower grades. Considering the grade 1 projection we obtain
\[ q = -2 \chi_{-1} e^{2\phi_0}, \quad r = 2 \psi_{-1} e^{-2\phi_0}. \]  
The grade zero projection gives one equation for \( \phi_{-2} \), which we do not need, and also an equation for \( \phi_0 \) which is then given by
\[ \phi_0 = -\frac{1}{2} \int_{-\infty}^{\infty} q r \, dx'. \]  
Let us point out some important facts that came out naturally from this approach. Comparing (57) with (32) we see that the solutions of the first negative flow (33) are given by
\[ \phi = -\chi_1, \quad \rho = -\psi_1, \]  
while the solutions of (24) contain an extra derivative (57). This explains why the solutions of (26) are connected to those of (35) through the potential variable \( \psi = \partial_x \phi \) [25, 29]. Comparing (59) and (60) with the gauge transformation (38), we note that the form of the transformation connecting the three types of DNLSEs is already contained in the dressing operators. The same also applies to relations (48). Moreover, we see from (60) that \( \mathcal{J} = -2\phi_0 \). Precisely for the case \( c = 1 \), corresponding to equation (41), we have from (59) that \( \tilde{q} = \chi_{-1} \) and \( \tilde{r} = \psi_{-1} \). For equations (39) and (45) we need to include the arbitrary constants. The term \( e^{\pm i\phi_0} \) is precisely the weight function introduced in the revised form of the IST [10].

4.1. Tau functions

We now introduce an important class of functions that contain the explicit space-time dependence of the solutions. They are directly related to the fields \( \chi_{2m+1}, \psi_{2m+1} \) and \( \phi_{2m} \) of
and, consequently, to the physical fields in the equations of motion through relations (57)–(61). Consider the highest weight states $\{|\lambda_0\rangle, |\lambda_1\rangle\}$ and let us also introduce the following notation for convenience:

$$|\lambda_2\rangle \equiv E^{a^*_a}|\lambda_0\rangle, \quad |\lambda_3\rangle \equiv E^{b^*_b}|\lambda_1\rangle. \quad (62)$$

The procedure to extract the fields in the dressing approach is to project the left-hand side of (53) between appropriate states. Thus, we have

$$e^v = \langle \lambda_0|\Theta^{-1}\Theta_+|\lambda_0\rangle, \quad \psi_1 e^v = \langle \lambda_0|\Theta^{-1}\Theta_+|\lambda_2\rangle, \quad \chi_1 e^v = \langle \lambda_1|\Theta^{-1}\Theta_+|\lambda_3\rangle, \quad (63)$$

$$\chi_{-1} e^v = -\langle \lambda_2|\Theta^{-1}\Theta_+|\lambda_0\rangle, \quad \psi_{-1} e^v = -\langle \lambda_3|\Theta^{-1}\Theta_+|\lambda_1\rangle,$$

Note that the right-hand side of (53) contains the explicit space-time dependence through (51), so we define the $\tau$-functions as

$$\tau_{ab} \equiv \langle \lambda_a|\hat{\Psi}\hat{\Psi}^{-1}|\lambda_b\rangle, \quad (64)$$

where $a, b = 0, 1, 2, 3$. The $\tau$-functions are classified according to the arbitrary group element $g$. Combining the results of (63) and (64) we have

$$\phi_0 = \ln\left(\frac{\tau_{00}}{\tau_{11}}\right), \quad \psi_1 = \frac{\tau_{02}}{\tau_{00}}, \quad \chi_1 = \frac{\tau_{13}}{\tau_{11}}, \quad \psi_{-1} = -\frac{\tau_{31}}{\tau_{11}}, \quad \chi_{-1} = -\frac{\tau_{20}}{\tau_{00}}. \quad (65)$$

From these relations we can express the solutions of all previous models in terms of $\tau$-functions, which can be algebraically calculated if we have an appropriate form for the group element $g$. For instance, from (61) the solutions of (33) are expressed as

$$\phi = -\frac{\tau_{13}}{\tau_{11}}, \quad \rho = -\frac{\tau_{02}}{\tau_{00}}, \quad (66)$$

while from (57) we have the solutions for the positive flows, like (24) and (27), given by

$$q = -\partial_1 \left(\frac{\tau_{13}}{\tau_{11}}\right), \quad r = -\partial_2 \left(\frac{\tau_{02}}{\tau_{00}}\right). \quad (67)$$

The integral appearing in the gauge transformations (38) can be obtained from (60) yielding

$$\mathcal{J} = 2\ln\frac{\tau_{11}}{\tau_{00}}, \quad (68)$$

showing that it is indeed a local function. Then, the solutions of (39) are given by

$$\tilde{q} = \frac{1}{2} \left(\frac{\tau_{11}}{\tau_{00}}\right)^2 \partial_1 \left(\frac{\tau_{13}}{\tau_{11}}\right), \quad \tilde{r} = -\frac{1}{2} \left(\frac{\tau_{00}}{\tau_{11}}\right)^2 \partial_2 \left(\frac{\tau_{02}}{\tau_{00}}\right). \quad (69)$$

In particular, for equation (41) ($c = 1$) it is easier to use directly (59); thus,

$$\tilde{q} = -\frac{\tau_{20}}{\tau_{00}}, \quad \tilde{r} = -\frac{\tau_{31}}{\tau_{11}}. \quad (70)$$

Recall that we must further impose the transformations $x \rightarrow ix, t \rightarrow 2it, \tau_{31}/\tau_{11} = \pm (\tau_{20}/\tau_{00})^*$. For the massive Thirring model (45) we obtain from (48) the following solution:

$$u = -\frac{1}{\sqrt{2g}} \frac{\tau_{11}}{\tau_{00}} \partial_1 \left(\frac{\tau_{13}}{\tau_{11}}\right), \quad v = \frac{im}{\sqrt{2g}} \frac{\tau_{13}}{\tau_{00}}. \quad (71)$$

We must further impose the transformations $x \rightarrow -\frac{i}{2}x$ and $t \rightarrow \frac{2m}{2}t$ in the space-time dependence of the $\tau$-functions and also $\tau_{02}/\tau_{00} = (\tau_{13}/\tau_{11})^*$. 

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6 See appendix A.

7 These are not highest weight states of the algebra.
4.2. Vertex operators

In order to be able to evaluate (64) explicitly let us assume that \( g \) is given in the following form [31, 33]:

\[
g \equiv \prod_{j=1}^{n} \exp \left( \Gamma_j \right), \quad \Gamma_j = \Gamma \left( \kappa_j \right)
\]  

(72)

where \( \Gamma_j \) is a vertex operator depending on a complex parameter \( \kappa_j \). Furthermore, let us assume that the vertex operator satisfies an eigenvalue commutation relation with (50),

\[
\left[ \Gamma_j, \hat{A}_x + \hat{A}_t \right] = \eta_j(\alpha, \tau) \Gamma_j.
\]  

(73)

The function \( \eta_j \) encodes the dispersion relation. From (51) we then have

\[
\hat{\Psi} g \hat{\Psi}^{-1} = \prod_{j=1}^{n} \exp(\Gamma_j) \hat{\Psi}^{-1} = \prod_{j=1}^{n} \exp(\hat{\Psi} \Gamma_j) \hat{\Psi}^{-1} = \prod_{j=1}^{n} \exp(e^{\eta_j} \Gamma_j).
\]  

(74)

The vertex operators satisfy the nilpotency property between the states, which eliminates terms containing its powers, \( (\Gamma_j)^m \) for \( m \geq 2 \), and truncates the exponential series in (74). Therefore, the \( \tau \)-functions (64) assume the following form:

\[
\tau_{ab} = \langle \lambda_a | \prod_{j=1}^{n} (1 + e^{\eta_j(\alpha, \tau) \Gamma_j}) | \lambda_b \rangle.
\]  

(75)

Let us now apply these general concepts to the KN hierarchy. Consider the following two vertex operators:

\[
\Gamma_j = \sum_{n=-\infty}^{\infty} \kappa_j^{-n} E_{\alpha+n}, \quad \Gamma'_j = \sum_{n=-\infty}^{\infty} \kappa_j^{n} E_{\alpha+n}^t,
\]  

(76)

which satisfy the following eigenvalue commutation relations with the vacuum (50):

\[
[\Gamma_j, H^m] = -2\kappa_j^m \Gamma_j, \quad [\Gamma'_j, H^m] = 2\kappa_j^m \Gamma'_j.
\]  

(77)

The dispersion relations of the KN hierarchy are, therefore, given by

\[
\eta_j = -2\kappa_j x - 2\kappa_j^2 t, \quad \eta'_j = 2\kappa_j x + 2\kappa_j^2 t,
\]  

(78)

where \( \alpha \) is the integer labeling the respective flow, i.e. \( N = 2, 3, \ldots \) for the positive flows or \( N = -1, -2, \ldots \) for the negative flows. Thus, (78) explains the change in the dispersion relation between models (35) and (26) [25].

4.3. Two-vertex solution

Using a single vertex operator in (75) we get a non-interesting simple exponential solution with one of the fields vanishing, which corresponds to a linearization of the equations of motion. The first nontrivial solution is obtained with two vertices in the form

\[
g = \exp \left( \Gamma_j \right) \exp \left( \Gamma'_j \right).
\]  

(79)

Then, from (75) we obtain

\[
\tau_{ab} = \langle \lambda_a | 1 + \Gamma_j e^{\eta_j} + \Gamma'_j e^{\eta'_j} + \Gamma_j \Gamma'_j e^{\eta_j+\eta'_j} | \lambda_b \rangle.
\]  

(80)

The matrix elements are calculated in appendix B, leading to the following \( \tau \)-functions:

\[
\tau_{00} = 1 + \frac{\kappa_2}{\kappa_1 - \kappa_2} e^{\eta_1+\eta'_2}, \quad \tau_{02} = \frac{1}{\kappa_2} e^{\eta'_2}, \quad \tau_{13} = \frac{1}{\kappa_1} e^{\eta_1}, \quad \tau_{11} = 1 + \frac{\kappa_1}{\kappa_1 - \kappa_2} e^{\eta_1+\eta'_1}, \quad \tau_{20} = e^{\eta_1}, \quad \tau_{31} = e^{\eta'_1}.
\]  

(81)
Replacing (81) in relations (66)–(71) we can obtain explicit one-soliton solutions for all the previous models considered in this paper. Let us also recall that we must pick the right dispersion relation (78) for the corresponding flow $N$. For instance, from (66) we have a solution of (33), that under the reduction $x \to \frac{1}{2} x$, $t \to -\frac{1}{2} t$ and $\kappa_2 = \kappa^*_1 = \kappa$, which is compatible with the choice $\phi = \rho^* = \varphi$, yields a solution of (35) with the minus sign. Still writing $\kappa = \kappa_R + i \kappa_I$ we have found

$$|\varphi|^2 = \frac{\left(\kappa_R^2 + \kappa_I^2\right)^{-1} e^{\theta}}{1 - \frac{\kappa_R^2}{2\kappa_I^2} e^{\theta} + \frac{\kappa_R^4 + \kappa_I^4}{16\kappa_I^2} e^{2\theta}}, \quad \theta = -2\kappa_I \left(x - \frac{1}{\kappa_R^2 + \kappa_I^2} t\right).$$

Using (67) we have a solution of (24), that with the appropriate reduction yields a solution of (26) with the plus sign, whose square modulus reads

$$|\psi|^2 = \frac{4e^{\theta}}{1 - \frac{\kappa_R^2}{2\kappa_I^2} e^{\theta} + \frac{\kappa_R^4 + \kappa_I^4}{16\kappa_I^2} e^{2\theta}}, \quad \theta = -4\kappa_I \left(x + 4\kappa_R t\right).$$

Both functions (82) and (83) are plotted in figure 1. Solution (82) has an unusual behavior, as can be seen from its graph. The soliton gets wider as its height and velocity increases. This behavior does not occur for (83) that shows the usual solitonic profile.

From (71) we have the following one-soliton solution for the Thirring model (45):

$$u = \frac{1}{\sqrt{2g}} \frac{e^{\eta}}{1 + \frac{|\eta|^2}{(\kappa x - m^2 \kappa^{-1} t)} e^{\eta + \eta'}}, \quad v = \frac{im}{\sqrt{2g}} \frac{(\kappa^*_1)^{-1} e^{\eta}}{1 + \frac{|\eta|^2}{(\kappa x - m^2 \kappa^{-1} t)} e^{\eta + \eta'}},$$

where $\eta = -i \left(\kappa x - m^2 \kappa^{-1} t\right)$. Taking the square modulus of these solutions we obtain exactly the behavior of (82) for $v$ and the behavior of (83) for $u$, which are sketched in figure 1. The formulas are almost identical.

4.4. Four-vertex solution

Let us consider a more complex solution by choosing the group element with four vertices in the form

$$g = \exp \left(\Gamma_1\right) \exp \left(\Gamma_2\right) \exp \left(\Gamma_3\right) \exp \left(\Gamma_4^*\right).$$

13
We then calculate the $\tau$-functions analogously to (80). After calculating the matrix elements, which are presented in appendix B, we obtain

$$
\tau_{00} = \langle \lambda_0 | 1 + \Gamma_1 \gamma_1 e^{i \eta_1 + \eta_2} + \Gamma_2 \gamma_2 e^{i \eta_2 + \eta_3} + \Gamma_3 \gamma_3 e^{i \eta_3 + \eta_4} + \Gamma_4 \gamma_4 e^{i \eta_4 + \eta_1} + \gamma_1 \gamma_2 \gamma_3 \gamma_4 e^{i \eta_1 + \eta_2 + \eta_3 + \eta_4} | \lambda_0 \rangle,
$$

$$
\tau_{11} = \langle \lambda_1 | 1 + \Gamma_1 \gamma_1 e^{i \eta_1 + \eta_2} + \Gamma_2 \gamma_2 e^{i \eta_2 + \eta_3} + \Gamma_3 \gamma_3 e^{i \eta_3 + \eta_4} + \Gamma_4 \gamma_4 e^{i \eta_4 + \eta_1} + \gamma_1 \gamma_2 \gamma_3 \gamma_4 e^{i \eta_1 + \eta_2 + \eta_3 + \eta_4} | \lambda_1 \rangle,
$$

$$
\tau_{02} = \langle \lambda_0 | \Gamma_2 e^{i \eta_2} + \Gamma_4 e^{i \eta_4} + \gamma_2 \gamma_4 e^{i \eta_2 + \eta_4} + \gamma_1 \gamma_2 \gamma_3 \gamma_4 e^{i \eta_1 + \eta_2 + \eta_3 + \eta_4} | \lambda_2 \rangle,
$$

$$
\tau_{13} = \langle \lambda_1 | \Gamma_1 e^{i \eta_1} + \Gamma_3 e^{i \eta_3} + \gamma_1 \gamma_3 e^{i \eta_1 + \eta_3} + \gamma_1 \gamma_2 \gamma_3 \gamma_4 e^{i \eta_1 + \eta_2 + \eta_3 + \eta_4} | \lambda_3 \rangle,
$$

$$
\tau_{20} = \langle \lambda_2 | \Gamma_1 e^{i \eta_1} + \Gamma_3 e^{i \eta_3} + \Gamma_1 \gamma_2 \gamma_3 \gamma_4 e^{i \eta_1 + \eta_2 + \eta_3 + \eta_4} | \lambda_0 \rangle,
$$

$$
\tau_{31} = \langle \lambda_3 | \Gamma_2 e^{i \eta_2} + \Gamma_4 e^{i \eta_4} + \gamma_2 \gamma_4 e^{i \eta_2 + \eta_4} + \gamma_1 \gamma_2 \gamma_3 \gamma_4 e^{i \eta_1 + \eta_2 + \eta_3 + \eta_4} | \lambda_1 \rangle.
$$

Considering special transformations, for instance, $\kappa_1 = \kappa_2 = \kappa$ and $\kappa_3 = \kappa_4 = \zeta$ one obtains a two-soliton solution in the same way as (82)–(84). We will not write down further explicit formulas but in figure 2 we show a graph of the two-soliton solution of the Thirring model (45) obtained in this way from (71).

The one- and two-soliton solutions, given by (81) and (86), respectively, were explicitly checked against all the previous models mentioned in this paper, with the aid of symbolic computation. We also want to stress that in spite of the technical difficulty, the procedure to compute the $\tau$-functions with $n$ vertices is well defined and systematic.

*Figure 2.* Graphs of solutions (71) involving four vertices. In the left column we have plotted the time evolution of $|\psi|^2$ and in the right column the time evolution of $|\psi|^2$. We have set the parameters as $m = 1$, $g = \xi$, $\kappa_2 = \kappa_1 = \kappa = 1 + i$ and $\kappa_4 = \kappa_3 = \zeta = 2 + i$. The waves travel from right to left. Note that for $|\psi|^2$ the small soliton is faster than the largest one, while for $|\psi|^2$ the higher soliton is faster.
5. Equivalent construction

The standard construction of the KN hierarchy is given by the following Lax operator, with the homogeneous gradation [10, 11, 19, 22]:

\[
A_\alpha = H^2 + qE^1_{\alpha} + rE^{-1}_{\alpha} = \begin{pmatrix} \lambda^2 & \lambda q \\ \lambda r & -\lambda^2 \end{pmatrix}, \quad Q = h = \lambda \frac{d}{d\lambda}.
\] (87)

Given an affine Lie algebra \( \hat{G} \) with generators \( T_n^a \), the conjugation by a group element \( h \), i.e. \( T_n^a \mapsto h T_n^a h^{-1} \), is an automorphism since it preserves the commutator (A.1). Consider the algebra \( \hat{A}_1 \) and let us define the following group element

\[
h = \exp\left\{ -\frac{1}{2} \ln(\lambda)H^0 \right\},
\] (88)

which yields the following mapping:

\[
hHnh^{-1} = H^n, \quad hE^a_{\alpha}h^{-1} = E^{a-1}_{\alpha}, \quad hE^{-a}_{\alpha}h^{-1} = E^{a+1}_{\alpha}.
\] (89)

Thus, (87) is mapped into

\[
A_\alpha \mapsto hA_\alpha h^{-1} = H^2 + qE^1_{\alpha} + rE^{-1}_{\alpha} = \begin{pmatrix} \lambda^2 & \lambda q \\ \lambda r & -\lambda^2 \end{pmatrix}, \quad Q \mapsto hQh^{-1} = \frac{1}{2}H^0 + \hat{d}.
\] (90)

If we now introduce a new spectral parameter \( \zeta \equiv \lambda^2 \), then \( \frac{d}{d\lambda} \mapsto \frac{d}{d\zeta} \) and by redefining the operators in (90) with respect to the spectral parameter \( \zeta \), we map (87) into another algebraic construction with principal gradation

\[
\begin{pmatrix}
A_\alpha = H^2 + qE^1_{\alpha} + rE^{-1}_{\alpha} \\
Q = h = \lambda \frac{d}{d\lambda} \\
\end{pmatrix} \quad \xrightarrow{\lambda = \sqrt{\zeta}} \quad \begin{pmatrix}
hA_\alpha h^{-1} = H^1 + qE^0_{\alpha} + rE^{-1}_{\alpha} \\
hQh^{-1} = \frac{1}{2}H^0 + 2\hat{d} \\
\end{pmatrix}
\] (91)

Therefore, we have demonstrated the equivalence of the standard construction (22) and the one proposed in this paper (21). We have introduced the construction (21) because it is much simpler and more natural to be treated under the dressing method than (87), and it eliminates the spurious quadratic power of the spectral parameter that is responsible for divergences in some complex integrals appearing in the IST.

Comment. It is possible to apply the method used in this paper to the homogeneous construction (22) and the same relations (56)–(61) arises, but the vacuum (51) now stays in the form

\[
\Psi = e^{-H^2} e^{-H^{01}} e^{-2\lambda \beta_{b+1,0}i}.
\] (92)

The vertex operators (76) still satisfy the eigenvalue relations with this vacuum but are not uniformly graded according to the homogeneous gradation. Moreover, to be able to obtain the correct solution we must redefine the spectral parameter in (92) \( \zeta \equiv \lambda^2 \). Therefore, the procedure does not occur in a natural way as in the case of principal gradation and it is necessary to introduce some ingredients by hand.

6. Concluding remarks

The KN hierarchy was obtained from a higher grading affine algebraic construction with the algebra \( \hat{A}_1 \) and principal gradation. In fact, we have proposed a general construction that can generates novel integrable models if different affine Lie algebras are employed. The results of this paper should extend naturally to these cases.
The main models within the KN hierarchy were derived. The DNLSE-I (26) arises from the second positive flow, while the Mikhailov model (35) is obtained from the first negative flow. The gauge transformation (38) connects the system (24) to (39), which in particular yields relations between the three kinds of DNLSEs, namely, (26), (40) and (41). Furthermore, we have demonstrated a general relation between the model (35) and the massive Thirring model (45) through (48).

We developed the algebraic dressing method for the KN hierarchy and several relations found previously in the literature emerge naturally. For instance, the form of the gauge transformations linking the three DNLSEs and the precise connection between the solutions of (35) and those of (26). Moreover, the weight function introduced in the revised IST [10] also arises from the algebraic dressing method. We stress that this method is general and systematic, and relies only on the algebraic structure of the hierarchy. The solutions of all models considered in this paper were expressed in terms of \( \tau \)-functions, which can be systematically calculated through a vertex representation theory of the algebra. We considered explicitly one- and two-soliton solutions. The solitons of the model (35), given by (82), possess an unusual behavior where its width increases with its height, as shown in figure 1.

Finally, we demonstrated that our construction (21) is conjugate related to the usual construction found in the literature (22). However, the dressing procedure applied to our Lax pair (21) is greatly simplified compared to (22). Several works on VBC and also NVBC are based on the standard Lax pair and the revised IST. We conclude that these problems can be simplified if one considers our construction instead. We plan to illustrate this fact more precisely regarding NVBC within the dressing formalism in a future opportunity.

Acknowledgments

We thank CAPES, CNPq and Fapesp for financial support. GSF thanks the support from CNPq under the ‘Ciência sem fronteiras’ program.

Appendix A. Algebraic concepts

Let \( \mathcal{G} \) be a finite-dimensional Lie algebra, with commutator \([T_a, T_b]\) for \( T_a, T_b \in \mathcal{G} \) and with a symmetric bilinear Killing form \((T_a|T_b)\). The infinite-dimensional loop-algebra is defined by \( \hat{\mathcal{G}} = \mathcal{G} \otimes \mathbb{C}(\lambda, \lambda^{-1}) \), i.e. \( T_a \mapsto \hat{T}_a = T_a \otimes \lambda^n \in \hat{\mathcal{G}} \) for \( n \in \mathbb{Z} \) and \( \lambda \) is the so-called complex spectral parameter. Let us introduce the central term \( \hat{c} \), which commutes with every other generator, and also the derivative operator \( \hat{d} = \partial_{\hat{c}} \). The Kac–Moody algebra is then defined by \( \hat{\mathcal{G}} = \mathcal{G} \oplus \mathbb{C}\hat{c} \oplus \mathbb{C}\hat{d} \) with commutator

\[
[T_a^n, T_b^m] = [T_a, T_b] \otimes \lambda^{n+m} + \hat{c} n \delta_{n+m,0} (T_a|T_b).
\]  

If we set \( \hat{c} = 0 \), we have the commutator for the loop-algebra \( \hat{\mathcal{G}} \). For example, considering the algebra \( \hat{A}_1 = \{ E_{\pm 1}, H \} \), where \( [E_\pm 1, E_{\mp 1}] = H \) and \( [H, E_{\pm 1}] = \pm 2E_{\pm 1} \), we have the Kac–Moody algebra \( \hat{A}_1 \) with generators \( \{ E_{\pm 1}^n, E_{\mp 1}^n, H^n, \hat{c}, \hat{d} \} \) and commutation relations

\[
[H^n, H^m] = 2n \delta_{n+m,0} \hat{c}, \quad [E_\pm 1^m, E_{\mp 1}^n] = H^{n+m} + n \delta_{n+m,0} \hat{c}, \quad [\hat{d}, T^n] = nT^n, \quad [\hat{c}, T^n] = 0,
\]  

where \( T^n \in \{ H^n, E_{\pm 1}^n \} \).

We can introduce a grading operator \( Q \), that splits the algebra into graded subspaces in the following way. For \( T_{a|m}^n \in \hat{\mathcal{G}} \), if \( [Q, T_{a|m}^n] = mT_{a|m}^n \) for \( m \in \mathbb{Z} \), then \( \hat{\mathcal{G}} = \bigoplus_{m \in \mathbb{Z}} \hat{\mathcal{G}}^{(m)} \), where \( \hat{\mathcal{G}}^{(0)} = \{ T_{a|0}^n | [Q, T_{a|0}^m] = mT_{a|0}^n \} \). In the case of \( \hat{\mathcal{G}} = \hat{A}_1 \) the grading operator \( Q = \hat{d} \).
(homogeneous) induces a natural gradation, \( \hat{G}^{(m)} = \{H^m, E^m, \hat{E}^m\} \). For \( Q = \frac{1}{2}H^0 + 2\hat{d} \) (principal) we have \( \hat{G}^{(2m+1)} = \{E^m, \hat{E}^{m+1}\} \) and \( \hat{G}^{(2m)} = \{H^m\} \). The operators \( \hat{c} \) and \( \hat{d} \) have zero grade.

The highest weight states of the algebra is a set of states satisfying \( T_n^a|\lambda_n\rangle = 0 \), if \( T_n^a \) have grade higher than zero. Precisely for the case of \( \hat{A}_1 \), the highest weight states are \( |\lambda_0\rangle, |\lambda_1\rangle \) and obey the following actions:

\[
\begin{align*}
E^0_{\alpha\beta}|\lambda_n\rangle &= 0 \quad (n > 0), \\
H^0|\lambda_n\rangle &= 0, \\
E^0\hat{c}|\lambda_n\rangle &= 0, \\
H^n|\lambda_0\rangle &= 0 \quad (n > 0), \\
H^0|\lambda_1\rangle &= |\lambda_1\rangle, \\
\hat{c}|\lambda_0\rangle &= |\lambda_0\rangle,
\end{align*}
\]  

(A.3)

where \( a = 0, 1 \). The adjoint relations are \( (H^n)^\dagger = H^{-n}, (E^n)^\dagger = E^{-n} \) and \( \hat{c}^\dagger = \hat{c}^* \).

A \( 2 \times 2 \) matrix representation of the \( \hat{A}_1 \) generators can be given as follows:

\[
H^n = \begin{pmatrix} \lambda^n & 0 \\ 0 & -\lambda^n \end{pmatrix}, \quad E^a\hat{c} = \begin{pmatrix} 0 & \lambda^n \\ 0 & 0 \end{pmatrix}, \quad E^a = \begin{pmatrix} 0 & 0 \\ \lambda^n & 0 \end{pmatrix}, \quad \hat{c} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]  

(A.4)

Appendix B. Matrix elements

The relevant states are \( |\lambda_0\rangle, |\lambda_1\rangle, |\lambda_2\rangle = E_{\alpha}^{-1}|\lambda_0\rangle \) and \( |\lambda_3\rangle = E^{0}_{\alpha}|\lambda_1\rangle \). The vertices for the KN hierarchy are

\[
\Gamma_j = \sum_{n=-\infty}^{\infty} \kappa_j^{-n}E_{\alpha}^{-1}, \quad \Gamma_j' = \sum_{n=-\infty}^{\infty} \kappa_j^{-n}E_{\alpha},
\]  

(B.1)

and satisfy the eigenvalue equations (77). We will write only the nonvanishing matrix elements used in (80) and (86). The nilpotency property of the vertices reads \( \langle \lambda_n| (\Gamma_j)^n |\lambda_h\rangle = \langle \lambda_n| (\Gamma_j')^n |\lambda_h\rangle = 0 \) for \( n \geq 2 \). In addition, any matrix element having a power of a vertex vanishes, e.g. \( \langle \lambda_n| (\Gamma_j)^2 |\lambda_h\rangle = 0 \). Another useful result is that the number of \( E^a_{\alpha} \) and \( E^{-a}_{\alpha} \) in a matrix element must be balanced in pairs, e.g. \( \langle \lambda_n| \Gamma_j |\lambda_h\rangle \neq 0 \), while \( \langle \lambda_0| \Gamma_j |\lambda_2\rangle = \langle \lambda_0| \Gamma_j |\lambda_0\rangle = 0 \). Thus, the nonvanishing matrix elements relevant for the solution with two vertices are

\[
\langle \lambda_2| \Gamma_j |\lambda_0\rangle = 1,
\]  

(B.2)

\[
\langle \lambda_1| \Gamma_1 |\lambda_0\rangle = \frac{1}{\kappa_1},
\]  

(B.3)

\[
\langle \lambda_0| \Gamma_2 |\lambda_0\rangle = \frac{1}{\kappa_2},
\]  

(B.4)

\[
\langle \lambda_3| \Gamma_2 |\lambda_1\rangle = 1,
\]  

(B.5)

\[
\langle \lambda_0| \Gamma_3 |\lambda_0\rangle = \frac{\kappa_2}{(\kappa_1 - \kappa_2)^2},
\]  

(B.6)

\[
\langle \lambda_1| \Gamma_3 |\lambda_1\rangle = \frac{\kappa_1}{(\kappa_1 - \kappa_2)^2}.
\]  

(B.7)

Note also that \( \langle \lambda_0| \Gamma_j^3 |\lambda_0\rangle = \langle \lambda_0| \Gamma_j^3 |\lambda_0\rangle \). Besides these elements, the nonvanishing elements for the solution with four vertices are

\[
\begin{align*}
\langle \lambda_0| \Gamma_0^2 |\lambda_2\rangle &= \frac{\kappa_2^2 (\kappa_2 - \kappa_4)^2}{\kappa_2 \kappa_4 \kappa_1 (\kappa_2 - \kappa_3)^2 (\kappa_3 - \kappa_4)^2}, \\
\langle \lambda_0| \Gamma_1^2 |\lambda_2\rangle &= \frac{\kappa_1^2 (\kappa_2 - \kappa_4)^2}{\kappa_2 \kappa_4 \kappa_1^2 (\kappa_1 - \kappa_2)^2 (\kappa_1 - \kappa_4)^2}.
\end{align*}
\]  

(B.8)
\begin{equation}
\langle \lambda_1 | \Gamma_1 \Gamma_2 \Gamma_3 | \lambda_2 \rangle = \frac{k_2^2 (k_1 - k_3)^2}{k_1 k_3 (k_1 - k_2)^2 (k_2 - k_3)^2}, \tag{B.10}
\end{equation}

\begin{equation}
\langle \lambda_1 | \Gamma_1 \Gamma_3 \Gamma_4 | \lambda_3 \rangle = \frac{k_2^2 (k_1 - k_3)^2}{k_1 k_3 (k_1 - k_4)^2 (k_3 - k_4)^2}, \tag{B.11}
\end{equation}

\begin{equation}
\langle \lambda_2 | \Gamma_1 \Gamma_2 \Gamma_3 | \lambda_0 \rangle = \frac{k_2^2 (k_1 - k_3)^2}{(k_1 - k_2)^2 (k_2 - k_3)^2}, \tag{B.12}
\end{equation}

\begin{equation}
\langle \lambda_2 | \Gamma_1 \Gamma_3 \Gamma_4 | \lambda_0 \rangle = \frac{k_4 (k_1 - k_3)^2}{(k_1 - k_4)^2 (k_3 - k_4)^2}, \tag{B.13}
\end{equation}

\begin{equation}
\langle \lambda_3 | \Gamma_2 \Gamma_3 \Gamma_4 | \lambda_1 \rangle = \frac{k_3 (k_2 - k_4)^2}{(k_2 - k_3)^2 (k_3 - k_4)^2}, \tag{B.14}
\end{equation}

\begin{equation}
\langle \lambda_3 | \Gamma_2 \Gamma_3 \Gamma_4 | \lambda_1 \rangle = \frac{k_1 (k_2 - k_4)^2}{(k_1 - k_2)^2 (k_1 - k_4)^2}, \tag{B.15}
\end{equation}

\begin{equation}
\langle \lambda_0 | \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4 | \lambda_0 \rangle = \frac{k_2 k_4 (k_1 - k_3)^2 (k_2 - k_4)^2}{(k_1 - k_2)^2 (k_1 - k_4)^2 (k_2 - k_3)^2 (k_3 - k_4)^2}, \tag{B.16}
\end{equation}

\begin{equation}
\langle \lambda_1 | \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4 | \lambda_1 \rangle = \frac{k_1 k_3 (k_1 - k_3)^2 (k_2 - k_4)^2}{(k_1 - k_2)^2 (k_1 - k_4)^2 (k_2 - k_3)^2 (k_3 - k_4)^2}. \tag{B.17}
\end{equation}

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