EIGENSTATES OF PARAPARTICLE CREATION OPERATORS

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Abstract

Eigenstates of the parabose and parafermi creation operators are constructed. In the Dirac contour representation, the parabose eigenstates correspond to the dual vectors of the parabose coherent states. In order $p = 2$, conserved-charge parabose creation operator eigenstates are also constructed. The contour forms of the associated resolutions of unity are obtained.

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1 Introduction

Eigenstates of the ordinary bose creation operator were constructed in Ref. [1] using Heitler’s contour integral form of the δ-function[2]. These eigenstates correspond to the dual vectors of the coherent states in Dirac’s contour representation of boson systems[3,1,4]. It is natural to investigate whether such creation operator eigenstates can also be constructed for paraparticles[5,6]. Parabose coherent states were proposed in [7], parafermi coherent states in [6], and recently, parabose squeezed states in [8].

In this paper in Sec. 2, the eigenstates for the parabose creation operator are constructed. Heitler’s form of the δ-function is used, so the expansion coefficients for these eigenstates in the parabose number basis are actually distributions. In Sec. 3, paragrassman numbers are used in the construction of the eigenstates of the parafermi creation operator. In the number basis, the expansion coefficients for the $f$ eigenstates and for the $f^\dagger$ eigenstates are paragrassman numbers, and so in this case, there is also an enlargement of the usual Hilbert space description. Lastly, in Sec. 4, the conserved-charge parabose creation operator eigenstates are constructed for the two-mode parabose system in order $p = 2$. In each section, the respective contour forms of the resolution of unity are derived.

2 Eigenstates of the Parabose $a^\dagger$ Operator

For a single-mode parabose system, the number basis is

$$|n> = \frac{(a^\dagger)^n}{\sqrt{n!}}|0>, \quad N_B|n> = n|n>, \quad (1)$$
where $N_B$ is the number operator $N_B = \frac{1}{2} \{a^\dagger, a\} - \frac{p}{2}$ with $p$ the order of the parastatistics. The eigenvalue of the deformed parabose number operator $[N_B]$ is

$$[n] = n + \frac{p-1}{2}(1 - (-)^n),$$

(2)

with $[n]! = [n][n-1] \cdots [1]$, $[0]! \equiv 1$. The parabose number states satisfy

$$a|n> = \sqrt{[n]}|n-1>, \quad a^\dagger|n> = \sqrt{[n+1]}|n+1>$$

(3)

In this basis, the unnormalized coherent states [7] are

$$|z> = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{[n]!}}|n> = E(za^\dagger)|0>, \quad E(x) \equiv \sum_{n=0}^{\infty} \frac{x^n}{[n]!}$$

(4)

with $a|z> = z|z>$. We denote the eigenstate of the creation operator $a^\dagger$ by a “primed” ket $|z>^{'},$

$$a^\dagger|z>^{' = z^*|z>^{'},}$$

(5)

and expand it $|z>^{' = \sum_{n=0}^{\infty} c_n(z^*)|n> in the number basis. By (3), the resulting recursion relations are

$$c_0 z^* = 0, \quad c_1 z^* = \sqrt{[1]} c_0, \cdots; \quad c_n z^* = \sqrt{[n]} c_{n-1},$$

(6)

or $c_n = \frac{\sqrt{[n]!}}{(z^*)^{n+1}} c_0$. By the Cauchy integral formula for an analytic function $f(z^*)$, or alternatively by use of Heitler’s $\delta$-function [4] in the contour integral [7], it follows[1] that

$$c_0 = \frac{1}{z^*}|_{C^*} = \delta(z^*)$$

(7)

$$c_n = \frac{\sqrt{[n]!}}{(z^*)^{n+1}}|_{C^*} = \frac{(-)^n \sqrt{[n]!}}{n!} \delta^{(n)}(z^*).$$

(8)

Note $f(0) = \oint_{C^*} \frac{dz^*}{2\pi i} f(z^*)\delta(z^*)$; the n-th derivative $f^{(n)}(0) = n! \oint_{C^*} \frac{dz^*}{2\pi i} \frac{f(z^*)}{(z^*)^{n+1}} = (-)^n \int_{-\infty}^{\infty} dx f(x) \delta^{(n)}(x).$
The notation $|C^*\rangle$ means that the subsequent integration over $z^*$ must be over the counterclockwise contour $C^*$ enclosing the origin in the complex $z^*$ plane.

So, the eigenstate of $a^\dagger$ is

$$|z\rangle' = \sum_{n=0}^{\infty} \frac{(-)^n \sqrt{n!}}{n!} \delta^{(n)}(z^*) |n\rangle = \sum_{n=0}^{\infty} \frac{\sqrt{n!}}{(z^*)^{n+1}} |n\rangle > |C^*\rangle = \sum_{n=0}^{\infty} \frac{(a^\dagger)^n}{(z^*)^{n+1}} |0\rangle > |C^*\rangle,$$

and, formally (in the number basis)

$$|z\rangle' = \frac{1}{z^*-a^\dagger} |0\rangle > |C^*\rangle.$$

Note that the action of integer powers of $(a^\dagger)^m$ removes the contribution of the number states $n < m$ in this expansion; e.g.

$$a^\dagger |z\rangle' = a^\dagger \frac{1}{z^*-a^\dagger} |0\rangle > |C^*\rangle = z^\star \frac{1}{z^*-a^\dagger} |0\rangle > |C^*\rangle - |0\rangle > |C^*\rangle = z^\star |z\rangle'$$

since the $|0\rangle > |C^*\rangle$ term gives no contribution because of the subsequent contour integration.

In the Dirac contour representation, the dual vector $<\alpha|$ of the parabose coherent state $|\alpha\rangle$, given above in (4), is [10]

$$<\alpha| = \sum_{n=0}^{\infty} <n| \frac{(\alpha^*)^n}{\sqrt{n!}} \frac{\sqrt{[n]!}}{[n]!} \frac{1}{z^* - \alpha^*}, \quad (|z\rangle > |\alpha\rangle).$$

with $<\alpha|a^\dagger = <\alpha|\alpha^*$. Thus, the eigenstate $|z\rangle'$ of $a^\dagger$ in the number basis corresponds to the parabose coherent state's eigenbra $<\alpha|$ of $a^\dagger$ in the Dirac contour representation.

The inner product of the unnormalized parabose coherent state $|w\rangle$ and the eigenstate $|z\rangle'$ is given by

$$<w|z\rangle' = \sum_{n,m=0}^{\infty} <n| \frac{(w^*)^n}{\sqrt{[n]!}} \frac{\sqrt{[m]!}}{[m]!} \frac{1}{(z^*)^{m+1}} |m\rangle > |C^*\rangle = \sum_{n=0}^{\infty} \frac{(w^*)^n}{(z^*)^{n+1}} |0\rangle > |C^*\rangle$$

$$= \frac{1}{z^* - w^*} |C^*\rangle = \delta(z^* - w^*), \quad (|z\rangle > |w\rangle)$$
and so they satisfy the “contour form” of the resolution of unity, see [3],

\[ \oint_{C^*} \frac{dz^*}{2\pi i} |z| > z = \sum_{n,m=0}^{\infty} \oint_{C^*} \frac{dz^* \sqrt{|n|!}}{(z^*)^{n+1}} |n| < m |(z^*)^m| \sqrt{|m|!} = \sum_{n=0}^{\infty} |n| < n | = I. \] (13)

**Remark:** This resolution of unity can be used to derive a contour integral expressions for the parabose Hermite polynomials[11]: From (13), the parabose coordinate eigenstate \(|x>| can be written as

\[ |x> = \sum_{n=0}^{\infty} |n> \sqrt{|n|!} \oint_{C^*} \frac{dz^*}{2\pi i} < z|x>, \] (14)

where \(<x|z>| is the wave function of the parabose coherent state in the parabose coordinate representation. We consider

\[ <x|z> = \frac{1}{x} <x|x|z> = \frac{1}{x\sqrt{2}} <x|(a + a^\dagger)|z>, \] (15)

From (3), c.f. eq.(14) for the parabose deformed derivative \(D/Dz\) in [10],

\[ a^\dagger|z> = \frac{\partial}{\partial z} |z> + (\frac{p-1}{2z})|z> - (\frac{p-1}{2z})|z> - z>, \] (16)

so (15) gives

\[ \frac{\partial}{\partial z} <x|z> = (-z + x\sqrt{2} - \frac{p-1}{2z}) <x|z> + (\frac{p-1}{2z}) <x|z> - z>. \] (17)

This has the solution

\[ <x|z> = N_0 e^{-\frac{x^2}{2} - \frac{p^2}{2} E(\sqrt{2}xz)} \] (18)

with \(N_0\) a normalization constant. Substituting this into (14) gives

\[ |x> = N_0 e^{-\frac{x^2}{2}} \sum_{n=0}^{\infty} |n> \sqrt{|n|!} \oint_{C^*} \frac{dz^*}{2\pi i} e^{-\frac{(z^*)^2}{2}} E(\sqrt{2}xz^*) \] (19)

But from [11], in the parabose coordinate representation

\[ |x> = \sum_{n=0}^{\infty} |n>|x> = N_0 e^{-\frac{x^2}{2}} \sum_{n=0}^{\infty} |n> \frac{H_n^p(x)}{\sqrt{n!}}, \] (20)
\[ H_n^{(p)}(x) = [n]! \sum_{k=0}^{[\frac{n}{2}]} \frac{(-)^k (2x)^{n-2k}}{k! [n-2k]!}, \]  
where \([k]\)' denotes the largest integer less than or equal to \(k\). So since the \(|n>\) are complete,

\[ H_n^{(p)}(x) = \frac{[n]!}{2\pi i} \oint_C \frac{dz}{(z)^{n+1}} e^{-\frac{z^2}{4}} E(\sqrt{2xz}). \]  

3 Eigenstates of the Parafermi \(f^\dagger\) Operator

In the finite dimensional Hilbert space of a single-mode parafermi system, the number states can be written as

\[ |n> = (f^\dagger)^n |0>, \quad N_f |n> = n |n> \]  
where \(N_f = \frac{1}{2} [f^\dagger, f] + \frac{p}{2}\) is the parafermi number operator. Here

\[ \{n\} = n(p+1-n), \quad \{n\}! = \{n\} \{n-1\} \cdots \{1\}, \quad \{0\}! \equiv 1. \]  
with \(n\) an integer, \(0 \leq n \leq p\). In this basis, \(f^\dagger |n> = \sqrt{\{n+1\}} |n>, \quad f |n> = \sqrt{\{n\}} |n-1>\). Since \(f^\dagger |p> = 0\), there is the useful fact that

\[ |n> = \sqrt{\frac{\{n\}!}{\{p\}!}} f^{p-n} |p>. \]  

To describe the parafermi eigenstates of \(f\) (and of \(f^\dagger\)) in this number basis, we use [6] paragrassman numbers \(\xi\) obeying \(\xi^{p+1} = 0\). The unnormalized eigenstate of the parafermi annihilation operator \(f\)

\[ |\xi> = \sum_{n=0}^{p} |n> \frac{\xi^n}{\sqrt{\{n\}!}} \]  
satisfies the eigenequation \(f |\xi> = |\xi> \cdot \xi\). In this formulation, \(|\xi>\) is expandable in the number basis, c.f. [6]. Note that \(\xi\) stands to the right of \(|\xi>\). The overlap of two eigenstates \(|\xi>\) and
\[ |\zeta > \text{ is} \]
\[ < \xi |\zeta > = \sum_{n=0}^{p} \frac{(\xi^*)^n \zeta^n}{\{n\}!} \]  
(27)

where \( \xi^* \) is the conjugate of \( \xi \). By the paragrassmann integral formula (see appendix)
\[ \int \xi^n \, d\mu(\xi, \xi^*) \, (\xi^*)^m = \delta_{n,m} \{n\}! \]  
(28)

and (26), there is the resolution of unity
\[ \int |\xi > d\mu(\xi, \xi^*) < \xi| = I, \]
\[ d\mu(\xi, \xi^*) = d^p\xi \, d^p\xi^* \, e^{-\frac{1}{2}[\xi^* \xi]} \]  
(29)

To construct the eigenstates of the parafermi creation operator \( f^\dagger \), we recall (25) and consider
\[ |\xi >' = \sum_{n=0}^{p} |n > (-\xi^*)^{p-n} \sqrt{\{n\}! / \{p\}!} \]  
(30)

These are the desired eigenstates since
\[ f^\dagger |\xi >' = \sum_{n=0}^{p-1} |n + 1 > (-\xi^*)^{p-n} \sqrt{\{n+1\}! / \{p\}!} = \sum_{n=1}^{p} |n > (-\xi^*)^{p-n+1} \sqrt{\{n\}! / \{p\}!} \]
\[ = -|\xi >' \xi^*. \]  
(31)

The overlap of these eigenstates is
\[ < \xi |\zeta >' = \sum_{n=0}^{p} \frac{\{n\}! \xi^p \zeta^n}{\{p\}!} = \sum_{n=0}^{p} \frac{\{p-n\}! \xi^n \zeta^p}{\{p\}!} \]  
(32)

As for the \( f \) eigenstates in (29), the eigenstates \( |\xi >' \) obey a resolution of unity
\[ \int |\xi >' d\mu(\xi^*, \xi) < \xi| = I \]  
(33)

where
\[ d\mu(\xi^*, \xi) = d^p\xi \, d^p\xi^* \, e^{-\frac{1}{2}[\xi^* \xi]} \]  
(34)
This follows from (26) and (30) by

\[
\int |\xi >' d\mu(\xi',\xi) < \xi | \\
= \sum_{n,m=0}^p |n > \int (-\xi^*)^{p-n} d\mu(\xi',\xi)(-\xi)^{p-m} < m| \sqrt{n!m!} \{p\}!
= \sum_{n,m=0}^p \frac{\{n\}!}{\{p\}!} (p-n)! |n >|n| = I
\]

(35)

where we have used the fact that \( \{n\}! = \frac{n!}{(p-n)!} \).

Furthermore, with the aid of the differentiation formula (see appendix)

\[
\frac{\partial}{\partial \xi} \xi^n = \{n\} \xi^{n-1} = \xi^n \frac{\partial}{\partial \xi}, \quad (0 \leq n \leq p),
\]

(36)

we have

\[
f^\dagger |\xi > = |\xi > \frac{\xi}{\partial \xi}, \quad f |\xi >' = - |\xi > \frac{\xi}{\partial \xi}.
\]

(37)

which give the matrix elements of \( f^\dagger, f \),

\[
< \xi| f^\dagger |\xi >' = \xi^* < \xi|\xi >' = - < \xi|\xi >' \xi^*;
\]

\[
< \xi| f |\xi >' = \frac{\partial}{\partial \xi^*} < \xi|\xi >' = - < \xi|\xi >' \frac{\partial}{\partial \xi^*}.
\]

(38)

Alternatively, these equations follow from (26) and (30) since

\[
< \xi|\xi >' = \frac{(-)^p}{\sqrt{\{p\}!}} \sum_{n=0}^p (-)^n (\xi^*)^n (\xi^*)^{p-n}
\]

(39)

Lastly, as in the parabose case (13), there is a contour-like-form resolution of unity for the \( f^\dagger \) and \( f \) eigenstates:

\[
\int |\xi >' dp \xi^* < \xi | = \sum_{n,m=0}^p \sqrt{\frac{\{n\}!}{\{p\}!}} m > \int (-\xi^*)^{p-n} dp \xi^* (\xi^*)^m < m| \frac{1}{\sqrt{\{m\}!}}
\]
\[ = \sum_{n=0}^{p} |n><n| = I \]  

(40)

where we have used

\[
\int d^p \xi^* (\xi^*)^p = p!, \quad \int d^p \xi^* (\xi^*)^n = 0 \quad (0 \leq n \leq p),
\]

(41)

and \( \xi^* d^p \xi^* = -d^p \xi^* \xi^* \). Note that in (40), as in the parabose case (13), the integration is only over a single variable in the contour form of the resolution of unity, whereas in (29) and (35) it is over two variables as for the usual parabose coherent states.

4 Conserved-Charge Parabose Creation Operator

Eigenstates for Order \( p = 2 \)

The parabose creation and annihilation operators for the two-mode system satisfy the trilinear commutation relations

\[
[a_k, \{a_l^\dagger, a_m\}] = 2\delta_{kl} a_m, \quad [a_k, \{a_l^\dagger, a_m^\dagger\}] = 2\delta_{kl} a_m^\dagger + 2\delta_{km} a_l^\dagger, \]

\[
[a_k, \{a_l, a_m\}] = 0, \quad (k, l, m = 1, 2)
\]

(42)

where \( a_1 = a, \ a_2 = b \). Since \( ab \neq ba \) for \( p \geq 2 \), there is a degeneracy in the states with \( n \) parabosons \( a \) and \( m \) parabosons \( b \). For such states, we find [9] that the degree of degeneracy is “\( \min(n, m) + 1 \)”. The complete set of state vectors is:

\[
|n, m; i > = \frac{1}{\sqrt{N_{i,n,m}^i}} (a^\dagger)^{n-i+S(b^\dagger))^{m-2\frac{(1-\varepsilon)}{2}} (a^\dagger b^\dagger)^2 (a^\dagger)^i - S - 2\frac{(1-\varepsilon)}{2}) |0 >
\]

(43)

where \( N_{i,n,m}^i \) is the normalization constant, and \( S = \frac{1}{2}(1 - (-)^m) \), and \( i \) is the degeneracy index \( 1 \leq i \leq \min(n, m) + 1 \). For parastatistics of order \( p = 2 \), the \( \{|n, m; i > \} \) are an orthonormal set.

\[^4\text{Note that here in Sec. 4, but not in Sec. 2, } [x] \text{ denotes the integer part of } x \text{ for } x \geq 0.\]
basis vectors with normalization constant

\[(N_{n,m}^i)^2 = 2^{n+m} \frac{(n+i)!}{2}! \frac{(n+1-i)!}{2}! \frac{(m+i)!}{2}! \frac{(m+1-i)!}{2}!\]  

(44)

In this basis, \(a^\dagger, b^\dagger, a, b\) also act as raising and lowering operators (the explicit formulas are given in eqs.(15-18) in [9]).

If we consider \(a\) and \(b\) to be two types of parabose quanta possessing abelian charges “+1” and “-1”, then the charge operator is

\[Q \equiv N_a - N_b\]  

(45)

with \(N_a = \frac{1}{2}\{a^\dagger, a\} - 1, N_b = \frac{1}{2}\{b^\dagger, b\} - 1\). This charge operator \(Q\) commutes with the operators \(a^\dagger b^\dagger\) and \(b^\dagger a^\dagger\), so their common eigenstate should satisfy the eigenequations

\[Q | q, z, w > = q | q, z, w >,\]

\[a^\dagger b^\dagger | q, z, w > = w^* | q, z, w >, \quad b^\dagger a^\dagger | q, z, w > = z^* | q, z, w >.\]  

(46)

Expanding \(| q, z, w >\) in terms of the complete set of orthonormal basis vectors \(|n, m; i >\) for the two-mode parabose system, for \(q \geq 0\) we have from the \(Q\) eigenequation (46)

\[| q, z, w > = \sum_{m=0}^{\infty} \sum_{i=1}^{m+1} c_{q+m,m}^i |q + m, m; i >\]  

(47)

From the remaining two eigenequations, we obtain the coefficients

\[c_{q+m,m}^i = \frac{(-)^m 2^m \sqrt{\frac{q+m+i}{2}! \frac{q+m+1-i}{2}!}}{\sqrt{\frac{q+i}{2}! \frac{q+1+i}{2}! \frac{m+i}{2}! \frac{m+1-i}{2}!}} \delta(s(w^*)) \delta(r(z^*))\]

\[= \frac{1}{\sqrt{\frac{q+i}{2}! \frac{q+1+i}{2}!}} \frac{2^m \sqrt{\frac{q+m+i}{2}! \frac{q+m+1-i}{2}! \frac{m+i}{2}! \frac{m+1-i}{2}!}}{\delta(w^*)^{1+s} \delta(z^*)^{1+r}} |\Gamma^*, \Delta^*\]  

(48)

where the integers

\[r \equiv \left[ \frac{m-(-)^q+m+i}{2} + \frac{1-(-)^q}{4} \right],\]

\[s \equiv \left[ \frac{m+(-)^q+m+i}{2} + \frac{1+(-)^q}{4} \right].\]
The counterclockwise contours $C^*$ and $B^*$ enclose respectively the origins in the complex $z^*$ and $w^*$ planes. Since for a specific $q$-sector the overall $\frac{1}{\sqrt{[\frac{1}{2}]!\left[\frac{q+1}{2}\right]!}}$ is constant, we omit it in the following analysis.

We list results for only the $q \geq 0$ sector: in it the unnormalized dual vectors are

$$|q, z, w> = \sum_{m=0}^{\infty} \sum_{i=1}^{m+1} \frac{2^m \sqrt{[\frac{m+i}{2}]!\left[\frac{m+1-i}{2}\right]!\left[\frac{q+m+i}{2}\right]!\left[\frac{q+m+1-i}{2}\right]!}}{(w^*)^{1+s}(z^*)^{1+r}} |q + m, m; i > |_{C^*, B^*},$$

whereas the unnormalized parabose conserved-charge coherent states themselves are

$$|q, v, u> = \sum_{m=0}^{\infty} \sum_{i=1}^{m+1} \frac{v^r u^s}{2^m \sqrt{[\frac{m+i}{2}]!\left[\frac{m+1-i}{2}\right]!\left[\frac{q+m+i}{2}\right]!\left[\frac{q+m+1-i}{2}\right]!}} |q + m, m; i >.$$

The inner product of $|q, z, w>^*$ and $|q, v, u>$ is

$$<q, v, u| q, z, w>^* = \sum_{m=0}^{\infty} \sum_{i=1}^{m+1} \left(\frac{v^*}{z^*}\right)^r \left(\frac{u^*}{w^*}\right)^s \left(\frac{1}{z^*w^*}\right) |q + m, m; i > |_{C^*, B^*} = \frac{1}{z^* - v^*} |_{C^*} \frac{1}{w^* - u^*} |_{B^*} = \delta(z^* - v^*) \delta(w^* - u^*),$$

$$<|v^*|, |w^*| > |u^*| > 0.$$

These satisfy the contour form of the resolution of unity

$$\oint_{C^*} \oint_{B^*} \frac{d z^* d w^*}{2\pi i 2\pi i} |q, z, w>^* <q, z, w| = \sum_{m=0}^{\infty} \sum_{i=1}^{m+1} |q + m, m; i > <q + m, m; i| = I_q.$$

where $I_q$ is the unity operator in the $q \geq 0$ sector.

In summary, working in the number basis, in this paper we construct the creation operator eigenvectors for single-mode parabosons and parafermions, and for the two-mode conserved-charge parabosons. The contour forms of the associated resolutions of unity are obtained.

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Appendix

A Proof of paragrassman integration formula:

We write $\xi = \sum_{i=1}^{p} \xi_i$, where the Green components $\xi_i$ satisfy the relations

$$\{\xi_i, \xi_i\} = 0, \quad [\xi_i, \xi_j] = 0 \ (i \neq j).$$

(53)

Also,

$$\frac{1}{2}[\xi^*, \xi] = \sum_{i=1}^{p} \xi_i^* \xi_i, \quad (\frac{1}{2}[\xi^*, \xi])^2 = 2! \sum_{i<j} \xi_i^* \xi_j^* \xi_i \xi_j, \ldots,$$

$$\frac{1}{2}[\xi^*, \xi]^n = n! \sum_{i_1<\cdots<i_n} \xi_i^* \cdots \xi_i^* \xi_{i_1} \cdots \xi_{i_n}, \ldots, \quad (\frac{1}{2}[\xi^*, \xi])^p = p! \xi_1^* \cdots \xi_p^* \xi_1 \cdots \xi_p,$$

(54)

so

$$e^{-\frac{1}{2}[\xi^*, \xi]} = 1 + \sum_i \xi_i \xi_i^*$$

$$+ \sum_{i<j} \xi_i \xi_j^* \xi_j + \cdots + \sum_{i_1<\cdots<i_n} \xi_i \cdots \xi_i \xi_{i_1}^* \cdots \xi_{i_n}^* + \cdots + \xi_1 \cdots \xi_p \xi_1^* \cdots \xi_p^*$$

(55)

For paragrassman integration, we adopt

$$\int d^p \xi \xi_1 \cdots \xi_p = 1, \quad \int d^p \xi \xi_1 \cdots \xi_{i_n} = 0 \ (0 \leq n < p)$$

(56)

where $d^p \xi \equiv d\xi_1 \cdots d\xi_p$. These give (41).

By (56), the integral $\int \xi^n \text{d}\mu(\xi, \xi^*) \ (\xi^*)^m$ with $d\mu(\xi, \xi^*) = d^p \xi \ d^p \xi^* \ e^{-\frac{1}{2}[\xi^*, \xi]}$ is non-zero only when $n = m$. So in this integration, we identify

$$\xi^n(\xi^*)^n \sim (n!)^2 \sum_{i_1<\cdots<i_n} \xi_{i_1} \cdots \xi_{i_n} \xi_{i_1}^* \cdots \xi_{i_n}^*,$$

$$\xi^p(\xi^*)^p \sim (p!)^2 \xi_1 \cdots \xi_p \xi_1^* \cdots \xi_p^*.$$

(57)

(58)
In the sum in (57) there are a total of \( \binom{p}{n} \) terms and each such term contributes only once in the paragrassman integration; thus,

\[
\int \xi^n \, d^n \xi \, d^n \xi^* \, e^{-\frac{1}{2}[\xi^* \xi]} \, (\xi^*)^n = (n!)^2 \binom{p}{n} = \frac{n! p!}{(p - n)!} = \{n\}!.
\]

(59)

B Proof of paragrassman differentiation formula:

The left-differentiation with respect to \( \xi \) is defined by

\[
\frac{\partial}{\partial \xi} = \sum_{i=1}^{p} \frac{\partial}{\partial \xi_i},
\]

(60)

where

\[
\{ \frac{\partial}{\partial \xi_i}, \frac{\partial}{\partial \xi_j} \} = 0, \quad [ \frac{\partial}{\partial \xi_i}, \frac{\partial}{\partial \xi_j} ] = 0 \quad (i \neq j),
\]

\[
\{ \frac{\partial}{\partial \xi_i}, \xi_i \} = 1, \quad [ \frac{\partial}{\partial \xi_i}, \xi_j ] = 0 \quad (i \neq j).
\]

(61)

In terms of Green components, \( \xi^n \) can be expressed as

\[
\xi^n = n! \sum_{i_1 < \cdots < i_n} \xi_{i_1} \cdots \xi_{i_n}
\]

(62)

where there are \( \binom{p}{n} \) terms in the sum. For each \( j \), in

\[
\frac{\partial}{\partial \xi} \xi^n = n! \sum_{j=1}^{p} \frac{\partial}{\partial \xi_j} \left( \sum_{i_1 < \cdots < i_n} \xi_{i_1} \cdots \xi_{i_n} \right),
\]

(63)

there are \( \binom{p-1}{n-1} \) terms in the inner summation which involve \( \xi_j \) and which survive after \( \frac{\partial}{\partial \xi_j} \). So there are a total of \( n! \, p \, \binom{p-1}{n-1} \) terms on the “rhs” of (63) and they are of the form \( \xi_{i_1} \cdots \xi_{i_{n-1}} \)

with \( i_1 < \cdots < i_{n-1} \). These are precisely the terms appearing in the Green component expansion

\footnote{\( \binom{p}{n} \) denotes the ordinary binomial coefficient.}
of $\xi^{n-1}$. By considering the symmetry of the Green components, we obtain the proportionality factor

$$n! p\binom{p-1}{n-1} \left(\frac{p}{n-1}\right)^{(n-1)} = n(p + 1 - n) = \{n\}$$

so

$$\frac{\partial}{\partial \xi} \xi^n = \{n\} \xi^{n-1} \quad (0 \leq n \leq p). \quad (65)$$

Right-differentiation can be dealt with in a similar manner.

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