Iterative Calculation of Characteristic Modes
Using Arbitrary Full-Wave Solvers

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Abstract—An iterative algorithm is adopted to construct approximate representations of matrices describing the scattering properties of arbitrary objects. The method is based on the implicit evaluation of scattering responses from iteratively generated excitations. The method does not require explicit knowledge of any system matrices (e.g., stiffness or impedance matrices) and is well suited for use with matrix-free and iterative full-wave solvers, such as finite-difference time-domain method, finite-element method, and multilevel fast multipole algorithm. The proposed method allows for significant speed-up compared to the direct construction of a full transition matrix or scattering dyadic. The method is applied to the characteristic mode decomposition of arbitrarily shaped obstacles of arbitrary material distribution. Examples demonstrating the speed-up and complexity of the algorithm are studied with several commercial software packages.

Index Terms—Antenna theory, characteristic modes (CMs), computational electromagnetics, eigenvalues and eigenvectors, scattering.

I. INTRODUCTION

CHARACTERISTIC mode (CM) analysis centers on the decomposition of the scattering properties of arbitrary structures into a basis with convenient properties; see [1], [2], [3], [4], and [5] for a detailed review. While CM analysis is commonly associated with the study of perfectly conducting objects using the method of moments [6], CMs of arbitrary structures can be calculated using any full-wave solver using scattering-based formulations [7], [8], [9]. This approach involves only the forward solution of a series of scattering problems to construct matrices that describe the scattering properties of the system under study.

An object’s transition matrix [10, Sec. 7.8] or its scattering dyadic [10, Sec. 4.3] can be constructed using a series of simulations carried out by any full-wave solver capable of determining scattered fields due to a known excitation. A simple approach to this procedure (discussed in [7] and [9]), involves a number of full-wave solutions coinciding with the selected dimension of the transition matrix or scattering dyadic representation. For large problems necessitating matrix-free methods, e.g., iterative finite-element method (FEM) solvers, fast multipole method (FMM), and finite-difference time-domain method (FDTD), [11, ch. 11], the cost of performing each of these full-wave simulations is typically far higher than the cost of decomposing the final transition matrix or matrix representation of the scattering dyadic.

In this letter, we consider an iterative algorithm for efficiently estimating the complete transition matrix or scattering dyadic of arbitrary scatterers. This method yields CMs without explicit knowledge of the internal structure of a particular numerical method (e.g., impedance or stiffness matrices). The proposed technique represents a matrix-free method for computing CMs in an arbitrary full-wave solver. Additionally, the algorithm readily allows for calculating all CMs with modal significance above a predefined threshold, which may be favorable over algorithms producing a fixed number of modes in decreasing order of modal significance.

II. TRANSITION MATRIX AND SCATTERING DYADIC

Consider an obstacle $\Omega$ illuminated by an incident field $E_i$ and producing a scattered field $E_s$. The transformation between the incident and scattered fields can be described by matrices by projecting both onto wave-specific bases. Utilizing regular (Bessel) and outgoing (Hankel) spherical vector waves to represent the incident and scattered field, the scattering problem is completely defined by the transition matrix $T$ [10]

$$T_a = f$$

where $a$ and $f$ are collections of spherical vector wave expansion coefficients for the incident and scattered field, respectively. Alternatively, the incident field can be described by a spectrum of plane waves

$$E_i(r) = \int_{4\pi} E(\hat{r}') e^{-jk\hat{r}' \cdot r} d\Omega'$$

where $E(\hat{r}')$ represents a plane wave field vector (with units of $\text{Vm}^{-1}\text{sr}^{-1}$, which is orthogonal to propagation direction $\hat{r}'$, $k$ is the background wave number, $j^2 = -1$, $\hat{r} = r/r$, and $r = |r|$. The scattered field in such case is studied in terms of the electric far-field $F$ and the scattering problem is completely described.
by the scattering dyadic $S$ as [10]

$$\frac{4\pi j}{k} \int 4 \pi S(\hat{r}, \hat{r}') \cdot E(\hat{r}') \, d\Omega' = F(\hat{r}).$$

(3)

Applying a quadrature rule to approximate the aforementioned integral transforms the scattering dyadic description of the problem into matrix form; see [9].

III. SCATTERING-BASED CHARACTERISTIC MODE DECOMPOSITION

CMs are fields or currents excited by characteristic excitations that are solutions to an eigenvalue problem involving either the transition matrix or scattering dyadic [7], [9]. In the case when spherical vector waves are used, characteristic excitations are given by expansion coefficient vectors $a_n$ satisfying [7]

$$T a_n = t_n a_n$$

(4)

while in the case of a plane-wave basis, characteristic excitations are the plane-wave spectra $E_n$ computed from [9]

$$\int 4 \pi S(\hat{r}, \hat{r}') \cdot E_n(\hat{r}') \, d\Omega' = t_n E_n(\hat{r}).$$

(5)

Note that, by the definitions of the transition matrix (1) and scattering dyadic (3), the eigenvectors of either of the aforementioned eigenvalue problems are simultaneous representations of both incident and scattered fields, i.e., both instances of $a_n$ may be replaced by $F_n$ in (4) and both instances of $E_n$ may be replaced by $F_n$ in (5). Additionally, the characteristic excitations $a_n$ or $E_n$ may be applied to reconstruct modal current densities and other modal quantities [8], [9]. The scattering-based CM problem in (5) is equivalent to the impedance-based problem for lossless structures, with explicit algebraic links between the two approaches available for many common integral equation formulations [7]. CM eigenvalues of impedance-based formulations, commonly denoted as $\lambda_n$ [2], are related to those of the scattering-based formulation via $t_n = -(1 + j\lambda_n)^{-1}$.

IV. CONSTRUCTION OF SCATTERING OPERATORS

The transition matrix $T$ and scattering dyadic $S$ are not, in general, known in closed form for arbitrary scattering systems. However, construction of either quantity (or its discrete representation in the case of the scattering dyadic) can be carried out with full-wave simulations capable of mapping arbitrary incident fields onto the resulting scattered fields [8], [9]. The number of incident field configurations (i.e., spherical or plane waves) required to construct accurate representations of $T$ or $S$ is commonly dictated by the electrical size of an obstacle [7], [9] and varies from tens to thousands. However, the resulting matrices are often of low rank and their explicit calculation could be circumvented by a matrix completion technique [12], such as the one introduced in the following section, that constructs approximate representations of $T$ or $S$ with a low number of calls to a full-wave solver.

Throughout this letter, we schematically denote the process of calling a full-wave solver as $L(E^i) = E^o$, where the input (excitation) field $E^i$ and output (scattered) field $E^o$ are assumed to be expressed in bases appropriate for constructing the operator of interest. For example, a solver assembling an incident field out of spherical waves with weights contained in the vector $a$ and returning a scattered field expressed in terms of spherical wave weights $f$ may be written schematically as $L(a) = f$ even if the underlying transition matrix (1) is not explicitly computed during the simulation process. A similar notation is adopted for constructing incident fields with plane-wave weights $E(\hat{r}')$ and calculating the resulting scattered far-fields $F(\hat{r})$. In most cases, the incident fields must be converted from their representation as spherical or plane waves into a form compatible with the selected numerical solver, for example, by projection onto a set of local basis functions.

V. ITERATIVE ALGORITHMS

Procedures for estimating the transition or scattering dyadic matrix of an object using a set of full-wave evaluations are sketched in Algorithms 1 and 2, respectively. Both algorithms follow similar steps, although the implementation details differ in a few small, yet significant, ways. These differences are highlighted in the following description of the procedure, where steps of both algorithms are referenced in parallel. An example of the implementation of the algorithm using the scattering dyadic is given in [13].

In the setup steps 1 and 2, an iteration index $m$ is initialized to zero, while an excitation vector is initialized with random values. For the transition matrix algorithm, the vector $a_0$ is initialized
with random complex numbers, while in the scattering dyadic algorithm, the excitation \( E_0 \) is a random complex function. Utilizing a random initialization increases the likelihood of overlap with multiple dominant modes in the first iteration.

Step 3 represents a condition for terminating the iterating component of the algorithm. Options for this condition are discussed further at the end of this section. Within the iterating loop, excitations are normalized in step 4 using either a vector norm (transition matrix algorithm) or an inner product defined over the unit sphere (scattering dyadic algorithm). The normalized excitation is then used to drive a full-wave simulation in step 5.

In step 6, the transition matrix or scattering dyadic is estimated by a set of outer products between all previous excitations and scattered fields. This construction ensures that incident fields map to the observed scattered fields when the incident fields are orthogonal in an appropriate inner product. The estimate of the scattering dyadic or transition matrix is then used to generate an estimate of the set of eigenvalues \( \{ t_n \} \) in step 7 using the appropriate choice of eigenvalue problem in (4) or (5).

In steps 8 and 9, the Gram–Schmidt procedure (GSP) [14] is used to generate a new excitation based on the previous scattered field with zero projection onto the set of previous excitations. The index \( n \) is then incremented and the procedure repeats from step 4 until one or more stopping criteria are met. Note that the classical GSP described in step 8 is unstable and the excitations lose orthogonality after a few iterations. The modified GSP stabilizes the procedure [14] and should be used in step 8. This modification also highlights the similarity with Arnoldi iteration [14], but for conceptual simplicity, the modified GSP is not described in Algorithms 1 and 2. Additionally, the algorithms may be reinitialized with a random excitation in the orthogonal complement to \( P_m \) or \( P^\perp_m \) to improve robustness.

An example of a stopping criterion is detecting an updated excitation norm less than a predefined value. Alternatively, the routine may be terminated when all modes with modal significance above a threshold \( t_{\text{th}} \) are sufficiently converged.

VI. EXTENSION TO TRANSIENT SIMULATIONS

The presented iterative algorithms consider transition matrix and scattering dyadic representations at a single frequency. However, in multifrequency cases, such as FDTD simulations, where a wide frequency response follows each numerical evaluation, the algorithm could be altered to consider the entire frequency interval. Provided multifrequency simulations in which all frequency points are excited with the same weighted incident field, steps 5–7 are evaluated for each frequency point. This information is compiled and processed to provide an updated incident field in steps 8–9. One could consider specific points in the interval; the highest frequency point, as without any prior knowledge of the object, this is the region with the most potential modes; or possibly the frequency point at which the sum of the estimated modal significances is the highest. Use of the latter option during every iteration of the algorithm provided the best results in the examples considered in Section VII.

VII. NUMERICAL EXAMPLES

To verify the proposed procedure, we analyze CMs of an example structure previously studied in the literature. To this end, we consider the cylindrical dielectric resonator antenna on a finite ground plane documented in [15]. The cylinder has a diameter of 10.5 mm, a height of 2.3 mm, and a relative permittivity of \( \varepsilon_r = 38 \). It is placed on top of the center of a perfectly electrically conducting square plate of side length 42 mm; see Fig. 1.

To calculate the CMs of this structure, a realization of Algorithm 2 was implemented using an iterative FEM-MoM (Method of moments) hybrid solver within Altair FEKO [16]. The discrete representation of the scattering dyadic is set up with Lebedev quadrature [9] of a degree 146 and a dimension of \( N = 292 \).

Convergence of CM eigenvalues is demonstrated in Fig. 1, where the number of correctly evaluated digits is plotted as a function of mode index and iteration number. Dark colors indicate convergence, while light colors correspond to modes that are not yet converged. For all three studied frequencies, the number of converged modes is approximately the number of iterations (see diagonal \( n = n \) line), though an offset of approximately ten iterations is required to obtain any converged eigenvalues in the highest frequency case.

Horizontal lines indicate the number of modes with modal significances above \( 10^{-2} \) and \( 10^{-5} \), while vertical lines indicate the number of iterations to resolve all modes above these thresholds with at least one correct digit. In practical applications of CMs, only modes above the \( |t_n| \geq 10^{-2} \) line are typically considered. For the lowest frequency (\( f = 1 \) GHz), the data in Fig. 1 show that only six simulations are required to accurately compute these dominant modes, much fewer than the 292 solutions required to construct the entire matrix representing the scattering dyadic. This trend also holds for the two higher frequencies, where 14 and 30 iterations are required, yielding approximate speed-up factors \( \{ N/6, N/14, N/30 \} \approx \{ 49, 21, 10 \} \) for \( f = \{ 1, 4, 7 \} \) GHz.

Furthermore, a multifrequency implementation of Algorithm 2 was constructed using the scattering matrix obtained through an FDTD solver, finite integration technique in CST Studio Suite [17]. Similar to the aforementioned FEM calculations, the scattering matrix has dimension \( N = 292 \) at each frequency point. The 20 highest modal significances from the complete scattering matrix are displayed in the frequency interval 1–7 GHz in the three panels of Fig. 2. In dashed lines and serve as a visual reference for the results of the discretized Algorithm 2 extended to transient simulations displayed in solid lines. Due to limitations in software, the full matrices were used to generate the responses of simultaneous excitations with arbitrary complex weights. With seven iterative excitations, top panel, the emergence of the modes can be seen with an only partial agreement. Using an intermediate of 14 excitations, most significant modes are present but not fully reconstructed. Finally, in this numerical example, 28 excitations were sufficient to produce visually indistinguishable modes compared to the full matrix, a reduction of 90%.
Fig. 1. Number of correctly evaluated digits of eigenvalues $t_n$ depending on the number of iterations $m$. The number of digits is evaluated from the comparison with the decomposition of fully constructed scattering dyadic as described in [9], 292 plane waves being used. The full-wave solver used to gather the data is FEM-MoM hybrid implemented in Altair FEKO [16]. The horizontal solid and dashed lines indicate CMs reaching depicted thresholds of modal significance. The corresponding vertical solid and dashed line shows the number of iterations $m$ required to acquire these modes.

Fig. 2. Twenty highest modal significances of a dielectric cylinder on a finite ground plane in the frequency interval 1–7 GHz obtained from a scattering matrix obtained using 292 excitations (dashed) and the estimates from $m \in \{7, 14, 28\}$ iterative evaluations (solid) with a discretized version of Algorithm 2.

VIII. DISCUSSION

To naively construct the transition matrix or a discrete representation of the scattering dyadic of dimension $N$, one requires $N$ full-wave evaluations. Estimates for the minimum dimension $N_{\text{min}}$ required to resolve CMs to a prescribed precision are available in the literature [7], [9], although these estimates tend to be quite conservative. A dominant trend, however, is that the dimension $N_{\text{min}}$ generally increases with the electrical size of the object under consideration. In the iterative approach presented here, the dominant $K$ eigenvalues of the scattering dyadic or transition matrix are estimated accurately using approximately $K_0 + K$ iterations, i.e., in linear time with some offset $K_0$. While the values of the parameters $N_{\text{min}}$ and $K_0$ are difficult to quantitatively predict due to their dependence on both geometric complexity and electrical size, it holds that if the scattering dyadic or transition matrix has dimension $N$ sufficient to resolve $K$ CMs, then the iterative cost $K_0 + K$ is at worst equal to $N$. This indicates that the iterative algorithm cannot perform worse, in terms of computational time, than full, explicit calculation. Nevertheless, the exact quantitative speed-up afforded by the iterative algorithm depends on the dimension $N$ of the matrices being approximated, which can be skewed by a priori knowledge of the modal structure of the system being studied. For instance, incorrectly setting $N$ unnecessarily high will lead to artificially high speed-up, while prior knowledge of the number of significant modes might allow for the naïve approach to realize accurate results with small $N$, thus greatly reducing the observed speed-up afforded by the iterative approach.

While based on a fundamentally different formulation of CMs, the proposed iterative approach shares some motivational aspects with matrix-free methods for computing CMs of large structures [18], where matrix-vector multiplications are implemented using MLMFA within iterative eigenvalue algorithms to avoid the high cost of computing, storing, and inverting large impedance matrices. In contrast to that work, however, the approach taken here utilizes an iterative approach to reduce the high computational burden of computing the scattering matrix, rather than circumventing its computation altogether.

Finally, the proposed method is general for use with any full-wave solver capable of solving scattering problems. Hence, given an appropriate solver, the algorithm can be utilized to efficiently calculate CMs for objects for which impedance-based approaches are not readily available or computationally viable, e.g., structures containing anisotropic, inhomogeneous, and even nonreciprocal media. In this way, the proposed method represents a significant addition to the existing body of methods for computing CMs in practical antennas and propagation problems.
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