Intervortex forces in competing-order superconductors

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Abstract

The standard Ginzburg-Landau model of competing-order superconductors is studied. It is observed that this model possesses two distinct species of vortex, and consequently has two distinct integer valued topological charges. A simple point particle model of long range forces between (anti)vortices of any species is developed and compared with numerical simulations of the full field theory, excellent agreement being found. Some of the results are quite counterintuitive. For example, a parameter regime exists where vortices of one species repel both vortices and antivortices of the other.

1 Introduction

There has been considerable interest recently in systems where superconductivity competes with some other order, for example antiferromagnetic order \[1\] or charge order \[2\]. The latter scenario admits a simple description in the Ginzburg-Landau formalism \[3, 4\]: one has a complex field \(\Delta\) representing the superconducting order parameter and a real field \(\rho\) representing the charge order parameter, subject to the free energy density

\[
\mathcal{E} = \frac{\chi}{2} \left| (\nabla - \frac{2ie}{\hbar} A) \Delta \right|^2 + \frac{1}{8\pi} |\nabla \times A|^2 + \frac{\chi}{2} |\nabla \rho|^2 - |\Delta|^2 - (1 - \delta)\rho^2, \tag{1.1}
\]

where \(A_i\) is the electromagnetic gauge potential and \(\chi, \delta\) are positive constants. Order competition is imposed via the constraint \(|\Delta|^2 + \rho^2 = c^2\), so that \(|\rho|\) is maximal where \(|\Delta|\) vanishes, and vice versa. In mathematical terms, this is an example of a gauged sigma model, objects of strong intrinsic interest.

The purpose of this paper is to develop a theory of the long range interactions between vortices in this model within the point vortex formalism. A key observation is that the model supports two distinct species of vortex which we call North vortices, with \(\rho = c\)
in the vortex core, and South vortices, with $\rho = -c$ in the vortex core, and that each of these has an antivortex counterpart (possessing a quantum of negative magnetic flux). Correspondingly, the model possesses two integer-valued topological charges: the total number $n$ of magnetic flux quanta, and the half-degree $d$ of the map $\mathbb{R}^2 \to S^2$ defined by $(x_1, x_2) \mapsto (\text{Re} \Delta(x_1, x_2), \text{Im} \Delta(x_1, x_2), \rho(x_1, x_2))/c$, or, equivalently the net numbers of North vortices $k_+$ and South antivortices $k_-$. This pair of integers cannot change under any smooth deformation of the fields $\Delta, \rho, A_i$ preserving finite total energy.

We will see that the interaction between (anti)vortices of all types depends crucially on the coupling parameter

$$\mu = \frac{\hbar \delta}{2\sqrt{2\pi e \chi c}}$$

which plays a role analogous to the Ginzburg-Landau parameter in conventional (single component) GL theory. If $\mu > 1$, vortices of any species repel one another, as do antivortices of any species, while vortices always attract antivortices. If $\mu < 1$, the behaviour is more surprising: like vortices attract, as do like antivortices, but unlike vortices repel, as do unlike antivortices, and unlike vortex-antivortex pairs. The regime of critical coupling $\mu = 1$ is particularly subtle with various combinations of vortices and antivortices experiencing no static interactions at all. The situation is summarized in table 1.

|       | $\mu < 1$ | $\mu = 1$ | $\mu > 1$ |
|-------|-----------|-----------|-----------|
| $N$   | attract   | repel     | repel     |
| $\bar{N}$ | attract | repel | repel |
| $S$   | attract   | repel     | repel     |
| $\bar{S}$ | attract | repel | repel |

Table 1: Summary of interactions between (anti)vortex pairs. $N$ denotes North vortex, $S$ denotes South vortex and an overbar denotes the corresponding antivortex. The 0 entries in the $\mu = 1$ table indicate (anti)vortex pairs which experience no interaction: their total energy is independent of their separation.

The rest of this paper is structured as follows. In section 2 we choose length, energy and charge units to reduce the GL model to a standard gauged sigma model, review its topological properties, and construct its (anti)vortices, paying particular attention to their asymptotics at spatial infinity. In section 3 we develop a theory of long range intervortex interactions by modelling vortices as solutions of the linearization of the sigma model about its vacuum, in the presence of appropriate point sources at the vortex centre, chosen to replicate the vortex’s large $r$ behaviour. This models vortices as composite point particles carrying a scalar monopole charge, inducing a real scalar field of mass $\mu$ (roughly, the field $\rho$) and a magnetic dipole moment inducing a vector field of mass 1 (roughly, $A_i$). The interaction energy between pairs of such point particles is easily computed, producing the predictions of table 1 as well as precise asymptotic formulae for the interaction energies valid at large separation. In section 4 we verify these predictions by numerically computing the interaction energy of (anti)vortex
pairs via a gradient descent energy minimization method. Finally, section 3 presents some concluding remarks.

2 Competing-order vortices

We first choose scales to minimize the number of parameters in the free energy \(1.1\). Let

\[
E = \lambda \mathcal{E}^{\text{new}} - c^2, \quad x_i = \lambda x_i^{\text{new}}, \quad A_i = \lambda A_i^{\text{new}},
\]

\[
(u_1 + iu_2, u_3) = (\Delta/c, \rho/c),
\]

\[
D_i u = \frac{\partial u}{\partial x_i^{\text{new}}} - A_i^{\text{new}} e \times u,
\]

(2.1)

where \(e = (0,0,1)\). Then, with the choices

\[
\lambda \mathcal{E} = 4\pi \chi^2 c^4 \left(\frac{2e}{\hbar}\right)^2, \quad \lambda_x^2 = \frac{\chi^2}{\lambda \mathcal{E}}, \quad \lambda_A = \frac{\hbar}{2e\lambda_x},
\]

(2.2)

we find that

\[
\mathcal{E}^{\text{new}} = \frac{1}{2} D_i u \cdot D_i u + \frac{1}{2} (B^{\text{new}})^2 + \frac{\mu^2}{2} (e \cdot u)^2
\]

(2.3)

where \(B^{\text{new}} = \partial_1^{\text{new}} A_2^{\text{new}} - \partial_2^{\text{new}} A_1^{\text{new}}\) and \(\mu\) is defined in equation \(1.2\). We henceforth discard the superscript “new.”

The total energy of a pair of fields \((u, A)\) is the integral

\[
E = \int_{\mathbb{R}^2} \mathcal{E} dx_1 dx_2.
\]

(2.4)

In order for this to be finite, \(u\), at large \(r\) (where \((x_1, x_2) =: r(\cos \theta, \sin \theta)\)), must approach the equator \(u_3 = 0\) on \(S^2\). It need not, however, be constant: it may wind around the equator

\[
u \sim (\cos n\theta, \sin n\theta, 0)
\]

(2.5)

some integer \(n\) times. Then finite energy also implies \(|D u| \sim 0\) as \(r \rightarrow \infty\), so \(A \sim \frac{n}{r} (-\sin \theta, \cos \theta)\), whence, by a standard application of Stokes’s Theorem one finds that the total magnetic flux of any finite energy configuration is quantized,

\[
\int_{\mathbb{R}^2} B dx_1 dx_2 = 2\pi n.
\]

(2.6)

If \(n \neq 0\), there must be points in the plane where \(u_1 + iu_2 = 0\). Note, however, that these come in two distinct species since \(u_3\) may take the value \(+1\) or \(-1\) at each such point. Consider a point \(x^+\) where \(u(x^+) = (0, 0, 1)\). This point itself may be assigned a sign \(\sigma(x^+)\) according to whether the field \(u(x)\) is locally an orientation preserving \((\sigma = +1)\) or orientation reversing \((\sigma = -1)\) map close to \(x^+\). The sum of these signs over all points where \(u = (0, 0, 1)\) is an integer-valued topological invariant of the field \(u\),

\[
k_+ = \sum_{x \in u^{-1}(\epsilon)} \sigma(x)
\]

(2.7)
which we may interpret as the net excess of North vortices over North antivortices in the field configuration. We may similarly assign a sign \( \sigma(x^-) \) to each point \( x^- \) in the plane at which \( u(x^-) = (0, 0, -1) \). Again, \( \sigma(x^-) = +1 \) if \( u(x) \) is locally orientation preserving and \( u(x^-) = -1 \) if it is locally orientation reversing. One should note, however, that, while \((u_1, u_2)\) is a good oriented local coordinate system for \( S^2 \) in a neighbourhood of \((0, 0, 1)\), it is anti-oriented in a neighbourhood of \((0, 0, -1)\), so each point with \( \sigma(x^-) = +1 \) contributes negatively to the winding of the field \( u \) about the equator in \( S^2 \). Hence, the integer-valued topological invariant associated with the South (anti)vortex positions

\[
k_- = \sum_{x \in u^{-1}(-e)} \sigma(x)
\]

represents the net excess of South antivortices over South vortices in the field configuration. One sees that the winding number at spatial infinity, which determines the total magnetic flux, is determined by \( k_+, k_- \) as

\[
n = k_+ - k_-
\]

Furthermore, the total signed area in \( S^2 \) covered by the mapping \( u(x) \) is \( 2\pi(k_+ + k_-) \), so we may identify \( k_+ + k_- \) has the half-degree of the map \( u : \mathbb{R}^2 \rightarrow S^2 \). The four types of (anti)vortex supported by this model are summarized pictorially in Figure 1.

To understand the (anti)vortices in more detail, we must numerically solve the Euler-Lagrange equations for the functional \( E \),

\[
P_u(-D_iD_i u + \mu^2 (e \cdot u)e) = 0,
\]

\[
-\partial_i \partial_j A_j + \partial_j \partial_i A_i - e \cdot (u \times D_i u) = 0,
\]

where \( P_u \) denotes projection orthogonal to \( u \), that is, \( P_u(v) := v - (u \cdot v)u \). These are consistent with the ansatz

\[
u^N = (\sin f(r) \cos \theta, \sin f(r) \sin \theta, \cos f(r))
\]

\[
A^N = \frac{a(r)}{r}(-\sin \theta, \cos \theta)
\]

where the profile functions \( f, a \), satisfy the coupled ODE system

\[
f'' + \frac{1}{r} f' - \frac{(1 - a)^2}{r^2} \sin f \cos f + \mu^2 \sin f \cos f = 0
\]

\[
a'' - \frac{1}{r} a' + \sin^2 f(1 - a) = 0
\]

subject to the boundary conditions \( f(0) = a(0) = 0, f(\infty) = \pi/2, a(\infty) = 1 \). Having found \( f \) and \( a \), we may easily construct the other three species of (anti)vortex,

\[
u^S = (\sin f(r) \cos \theta, \sin f(r) \sin \theta, \cos f(r))\]

\[
A^S = \frac{a(r)}{r}(-\sin \theta, \cos \theta)
\]

\[
u^\hat{N} = (\sin f(r) \cos \theta, -\sin f(r) \sin \theta, \cos f(r))
\]

\[
A^\hat{N} = \frac{a(r)}{r}(\sin \theta, -\cos \theta)
\]

\[
u^\hat{S} = (\sin f(r) \cos \theta, -\sin f(r) \sin \theta, -\cos f(r))
\]

\[
A^\hat{S} = \frac{a(r)}{r}(\sin \theta, -\cos \theta).
\]
Figure 1: The field values attained by the four species of (anti)vortex. The field \( u(\mathbf{x}) \) wraps the circle at spatial infinity once around the equator in the direction indicated, anticlockwise for vortices, clockwise for antivortices (viewed from above the North pole). The (anti)vortex interior then covers either the Northern or the Southern hemisphere once. The topological charges \( k_+, k_- \) measure the number of times the field assumes the value \( (0, 0, 1) \) and \( (0, 0, -1) \) respectively, counted with orientation.
The system (2.14), (2.15) does not appear to be integrable, so we resort to numerical integration to find $f, a$. Regularity at the origin requires $f(r) \sim \alpha_1 r$ and $a(r) \sim \alpha_2 r^2$ for some constants $\alpha_1, \alpha_2$. For large $r$, $\hat{f}(r) := f(r) - \pi/2$ and $\hat{a}(r) := a(r) - 1$, being small, should be asymptotic to decaying solutions of the linearization of the system about $(f, a) = (\pi/2, 1)$,

$$\hat{f}'' + \frac{1}{r} \hat{f}' - \mu^2 \hat{f} = 0, \quad (2.17)$$

$$\hat{a}'' - \frac{1}{r} \hat{a}' - \hat{a} = 0. \quad (2.18)$$

Hence, at large $r$,

$$f(r) \sim \frac{\pi}{2} + \frac{q}{2\pi} K_0(\mu r), \quad a(r) \sim 1 + \frac{m}{2\pi} r K_1(r), \quad (2.19)$$

where $K_0, K_1$ are modified Bessel’s functions of the second kind, and $q, m$ are unknown constants. The factors of $2\pi$ are included for later convenience. Our numerical strategy is to solve (2.14), (2.15) on $[r_0, R]$, with $r_0 > 0$ small and $R$ large by shooting rightwards from $r_0$, using $(\alpha_1, \alpha_2)$ as shooting parameters, leftwards from $R$ using $(q, m)$ as shooting parameters, and imposing that $f, a$ and their derivatives match at some interior point $r_1$ of order 1. The results of this scheme for various values of the coupling $\mu$ are depicted in Figure 2. Of particular interest are the values of the constants $(q, m)$ as functions of $\mu$, depicted in Figure 3. Note that $q \equiv m$ when $\mu = 1$. This is not a coincidence: the system (2.14), (2.15) reduces to a first order system at this critical value of the coupling,

$$f' = \frac{1 - a}{r} \sin f, \quad a' = r \cos f, \quad (2.20)$$

from which it follows immediately that $q \equiv m$. This is a symptom of the self duality (or BPS property) enjoyed by the model at $\mu = 1$, whose full consequences are both deep and far ranging [5, 6]. In this paper we will concentrate on the case $\mu \neq 1$, however.

### 3 The point vortex model

It is convenient to think of (anti)vortices as static solutions of the Lorentz invariant model on $(2 + 1)$-dimensional Minkowski space whose static energy is $E$, that is, the model with Lagrangian density

$$\mathcal{L} = \frac{1}{2} D_\mu u \cdot D^\mu u - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\mu^2}{2} (e \cdot u)^2, \quad (3.1)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, spacetime indices $\mu, \nu$ run over 0, 1, 2, and the Minkowski metric has signature $+--$. We emphasize that this is a mathematical device. It allows us to access some techniques and results familiar in the study of topological solitons in high energy physics. We certainly do not assert that the time dynamics defined by this relativistic extension is relevant to competing order superconductors.

The key observation is that static vortices, far from their core, are indistinguishable from solutions of the linearization of the model (3.1) about the vacuum (meaning $A_\mu = 0$, $u = (1, 0, 0)$)
Figure 2: The profiles functions $f(r)$ (blue curves) and $a(r)$ (red curves) of a North vortex at couplings $\mu = 2$ (top), $\mu = 1$ (middle) and $\mu = 0.5$ (bottom).

in the presence of appropriate point sources placed at the vortex centre. Since physics is model independent, the forces between well-separated vortices should coincide with those between the corresponding point sources interacting via the fields they induce in the linear theory. These are easily computed, yielding an asymptotic formula for the interaction energy between well-separated vortices. This underlying idea was introduced by Manton to study long-range forces between magnetic monopoles [7], and subsequently applied to nuclear Skyrmions by Schroers [8]. It was adapted to vortices in the conventional Ginzburg-Landau model in [9], then multicomponent vortices in [10].

Our first task is to identify the point sources that replicate the vortex asymptotics, and to do this we must first re-write it in the gauge in which, as $r \to \infty$, $\mathbf{u} \to (1, 0, 0)$ in every direction, that is, the gauge where $u_2 = 0$ and $u_1 \geq 0$. This is accomplished by applying the singular (at $r = 0$) gauge transformation $(u_1 + i u_2, u_3) \mapsto (e^{-i\theta}(u_1 + i u_2), u_3)$. The order parameter takes the form $\mathbf{u} = (\cos \Theta, 0, \sin \Theta)$ in this gauge, the vacuum is $\Theta = 0$ and the North vortex has

$$\Theta(r) = f(r) - \frac{\pi}{2} \sim \frac{q}{2\pi} K_0(\mu r),$$

$$\left( A_0, A_1, A_2 \right) = \frac{a(r) - 1}{r} (0, -\sin \theta, \cos \theta) \sim \frac{m}{2\pi} (0, \partial_2, -\partial_1) K_0(r). \quad (3.2)$$

These are precisely [9] the fields induced in the linearized model

$$\mathcal{L}_{\text{lin}} = \frac{1}{2} \partial_\mu \Theta \partial^\mu \Theta - \frac{\mu^2}{2} \Theta^2 + \rho \Theta - \frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \frac{1}{2} A_\mu A^\mu + j_\mu A^\mu \quad (3.3)$$
Figure 3: The large \( r \) shooting parameters \( q, m \) of the North vortex solution as functions of the coupling \( \mu \). These may be interpreted as the scalar monopole charge \((q)\) and magnetic dipole moment \((m)\) of the corresponding point vortex.
by the static sources
\[ \rho = q \delta(x), \quad (j_0, j_1, j_2) = m(0, \partial_2, -\partial_1)\delta(x), \quad (3.4) \]
so our linearized model of a North vortex is a composite point source consisting of a scalar monopole of charge \( q^N = q \) inducing a real scalar field \( \Theta \) of mass \( \mu \), and a magnetic dipole of moment \( m^N = m \) inducing a Proca field \( A_\mu \) of mass 1. The corresponding sources for the other species of (anti)vortex follow immediately by unwinding (2.16). All are scalar monopole/magnetic dipole composites, with charges \( (q^N, m^N) = (q, m), \ (q^S, m^S) = (-q, m), \ (q^{\bar{N}}, m^{\bar{N}}) = (q, -m), \ (q^{\bar{S}}, m^{\bar{S}}) = (-q, -m). \) (3.5)

The interaction Lagrangian for a pair of sources \( (\rho^{(1)}, j_\mu^{(1)}), (\rho^{(2)}, j_\mu^{(2)}) \) is
\[ L_{\text{int}} = \int_{\mathbb{R}^2} (\rho^{(1)} \Theta^{(2)} + j_\mu^{(1)} A_\mu^{(2)}) dx_1 dx_2 \quad (3.6) \]
where \( (\Theta^{(2)}, A_\mu^{(2)}) \) are the fields induced by the second source. We apply this in the case where the sources are static scalar monopole/magnetic dipole composites of charges \( (q_1, m_1), \ (q_2, m_2) \) located at \( y \) and \( z \) respectively. The result is a function of \( s := |y - z| \), the vortex separation. It may be interpreted as minus the interaction energy of the source pair, so
\[ E_{\text{int}}(s) = -L_{\text{int}} = \frac{1}{2\pi} [m_1 m_2 K_0(s) - q_1 q_2 K_0(\mu s)]. \quad (3.7) \]
If \( \mu > 1 \), the first term, representing magnetic interactions, dominates at large \( s \), whereas if \( \mu < 1 \), the second term, representing scalar interactions dominates. By choosing \( (q_1, m_1), \ (q_2, m_2) \) from the list (3.5), we obtain long range interaction energies between (anti)vortices of any species. The nature of these interactions is summarized in Table 1. The zero entries for critical coupling, \( \mu = 1 \), follow from the observation that \( q = m \) here. Our calculation establishes that the leading order interactions for \( NN, SS, N\bar{S} \) and \( S\bar{N} \) pairs vanish in this case. In fact, the self-duality structure can be used to prove that the interaction vanishes exactly for these pairs [5]: static solutions exist with the individual vortices placed at any points in the plane when \( \mu = 1 \).

Of course, these predicted interaction potentials are based on a leap of faith – that physics is model independent. This particular faith allows, indeed encourages, scepticism in its acolytes. Luckily it also admits a definitive test: we can compute the energy between vortices held at a fixed separation by numerical simulation of the original nonlinear model. This is the subject of the next section.

4 Numerical results

How can we compute the interaction energy \( E_{\text{int}}^{NN}(s) \) between two North vortices held distance \( s \) apart? Note that no such static solution exists (unless \( s = 0 \), or \( \mu = 1 \)), precisely because vortices exert forces on one another. The answer is that we solve a constrained minimization
problem for the energy functional $E$: we minimize among all fields having $k_+ = 2$ and $k_- = 0$ subject to the constraint that $u(s/2, 0) = u(-s/2, 0) = e$. In practice, we discretize space, replacing spatial derivatives by difference operators on a regular $n_1 \times n_2$ lattice of spacing $h$ (we used $n_1 = n_2 = 251$ and $h = 0.1$). This replaces the continuum energy functional $E(u, A)$ by a discrete approximant $E_{\text{dis}} : C_{\text{dis}} \to \mathbb{R}$ where $C_{\text{dis}} = (S^2)^{n_1 n_2} \times (\mathbb{R}^2)^{n_1 n_2}$ is the discretized configuration space. We then construct an appropriate initial guess $u_{i,j}, A_{i,j}$ with, around the boundary of the lattice, $u_{i,j} \cdot e = 0$ and winding 2, and

$$u_{\pm i_0,0} = e,$$

where $s = 2i_0 h$. We then minimize $E_{\text{dis}}$ among all points in $C_{\text{dis}}$ satisfying the constraint (4.1) using arrested Newton flow $[11]$ for the function $E_{\text{dis}}$, but never updating $u_{\pm i_0,0}$ (or $u, A$ on the boundary of the lattice). Having computed the lowest energy among all $(k_+, k_-) = (2, 0)$ field configurations with $u(\pm s/2, 0) = e$, we then subtract twice the energy of a single North vortex to obtain $E_{\text{ind}}^{NN}(s)$.

Interaction energies for any other vortex combination can be computed similarly by modifying the constraint (4.1) and boundary behaviour of the field configuration appropriately. By symmetry, $NN \equiv N\bar{N} \equiv SS \equiv S\bar{S}, N\bar{N} \equiv SS, NS \equiv N\bar{S}$ and $N\bar{S} \equiv N\bar{S}$, so only 4 of the 10 distinct (anti)vortex pairs need be considered, and we can, without loss of generality, assume that the left vortex is $N$. The results are depicted in Figure 4. They match perfectly the predictions of our simple point vortex model.

5 Concluding remarks

In this paper we have developed a simple point vortex model of long range interactions between (anti)vortices in the usual Ginzburg-Landau model of competing-order superconductors. The model supports two distinct species of vortex, each with a matching antivortex, and hence there are 10 different (anti)vortex pairs possible. Symmetries reduce this to 4 energetically distinct pairs: $NN, N\bar{N}, NS$ and $N\bar{S}$. The point vortex model predicts asymptotic formulae for the interaction of energy of each of these pairs, as a function of separation, with considerable success. The qualitative nature of the interactions depends on a single parameter $\mu$, and if $\mu < 1$ the interactions display some counterintuitive features. For example, the interaction between vortices of one species and antivortices of the other is repulsive.

It would be interesting to study vortex lattices in this model in an applied magnetic field. Although, for $\mu > 1$, pure $N$ (or pure $S$) arrays are energetically favoured over $NS$ mixtures, if the state emerges from disorder, presumably some species mixing is inevitable.

It would also be interesting to consider in detail the effect of adding a small term linear in $\rho$ to the original Ginzburg-Landau theory, the upshot of which is that (after rescaling) the energy density becomes

$$\mathcal{E} = \frac{1}{2} D_i u \cdot D_i u + \frac{1}{2} B^2 + \frac{\mu^2}{2} (\tau - e \cdot u)^2,$$

where $\tau$ is an extra small parameter. This term breaks the symmetry between $N$ and $S$ vortices: if $\tau > 0$ then $S$ vortices are slightly more energetically costly than $N$ vortices.
Figure 4: Plot of the interaction energies for different vortex pairs and separations $E_{int} = E - 2E^1$. The dashed lines are the point vortex approximations given by (3.7). Note that the interactions agree with table [1].
(and *vice versa* if $\tau < 0$). Remarkably, when $\mu = 1$, the model still enjoys a self-duality structure, and $N$ vortices exert no net force on $S$ antivortices. The basic point-vortex model of intervortex forces is similar to the one developed here, in that a point vortex still consists of a scalar monopole of some charge $q$ and a magnetic dipole of some moment $m$, but these sources induce fields of mass $\sqrt{1 - \tau^2} \mu$ and $\sqrt{1 - \tau^2}$, and there is no symmetry relating $q^N$ with $q^S$ or $m^N$ with $m^S$. Introducing a linear term has the effect of increasing the range of intervortex forces, therefore, as well as breaking the degeneracy of $N$ and $S$ vortices.

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