Volume preserving diffeomorphisms
as Poincaré maps for volume preserving flows

D.V. Treschev

All the objects below will be $C^\infty$-smooth. Let $M$ be a compact manifold with $\dim M = m$, and let $\nu$ be a volume form on $M$. Here we mean that $\nu$ is a nowhere vanishing differential $m$-form such that $\int_M \nu > 0$. Let $T = \mathbb{R}/\mathbb{Z} = \{ t \mod 1 \}$. Then $\omega = dt \wedge \nu$ is a volume form on the manifold $T \times M$.

Let $\pi_T: T \times M \to T$ and let $\pi_M: T \times M \to M$ be the natural projections

$$ T \times M \ni (t, x) \mapsto \pi_T(t, x) = t, \quad (t, x) \mapsto \pi_M(t, x) = x. $$

Consider a vector field $v$ on $T \times M$. We assume that

(A) the first component of $v$ is positive: $D\pi_T v = v_T > 0$,

(B) $v$ preserves the form $\omega$: $L_v \omega = 0$, where $L_v$ is the Lie derivative.

Let $g^s_v$ be the flow generated by the vector field $v$ on $T \times M$. Condition (A) implies that the Poincaré map (the first-return map) $P_v: \{0\} \times M \to \{0\} \times M$ is well defined. The map $P_v$ preserves the volume form $\lambda = \iota_v \omega|_{\{t=0\}}$ on $\{0\} \times M$.

**Theorem 1.** Let $P_v$ be the diffeomorphism defined above and let $Q: \{0\} \times M \to \{0\} \times M$ be another diffeomorphism which preserves $\lambda$. Assume that $Q$ is (smoothly) isotopic to $P_v$ in the group of $\lambda$-preserving self-maps of $\{0\} \times M$. Then there exists an $\omega$-preserving vector field $u$ on $T \times M$ such that $D\pi_T u > 0$ and $Q = P_u$.

This result, probably interesting on itself, was obtained as a tool required in the proof of the main result of [1]. Now we sketch the proof of Theorem 1.

(a) Consider an $s$-smooth family of diffeomorphisms $T_M(s): M \to M$ of the manifold $M$ into itself, $s \in \mathbb{R}$. We extend $T_M$ to a family of diffeomorphisms $T^s$ of $\mathbb{R} \times M$ by defining

$$ \mathbb{R} \times M \ni (t, x) \mapsto T^s(t, x) = (t + s, T_M(t + s) \circ T^{-1}_M(t)(x)). \quad (1) $$

A direct computation shows that $T^s$ is a flow, that is, $T^0 = \text{id}$ and $T^{s_2} \circ T^{s_1} = T^{s_1 + s_2}$ for any $s_1, s_2 \in \mathbb{R}$. The flow $T^s$ generates a vector field $\mathcal{U}$ on $\mathbb{R} \times M$:

$$ \mathcal{U} = \left. \frac{d}{ds} \right|_{s=0} T^s \circ T^{-s}, \quad D\pi_\mathbb{R}(\mathcal{U}) = 1, \quad (2) $$

where $\pi_\mathbb{R}: \mathbb{R} \times M \to \mathbb{R}$ is the natural projection.

AMS 2010 Mathematics Subject Classification. Primary 37C10.
(b) We put \( \hat{v} = v/v_T \). Then \( D\pi_T \hat{v} = 1 \). Let \( (t, x) \mapsto g^s_b(t, x), (t, x) \in \mathbb{T} \times M, \) be the flow of the vector field \( \hat{v} \). Then \( g^s_b \) preserves the form \( v_T \omega \). The Poincaré maps \( P_v \) and \( P_{\hat{v}} \) coincide. Hence \( g^s_b(0, x) = (1, P_v(x)) \) for any \( x \in M \).

Let \( G^s \) be the lift of the flow \( g^s_b \) to the covering space \( \mathbb{R} \times M \). We define a family \( \sigma_s: M \rightarrow M \) by \( G^s(0, x) = (s, \sigma_s(x)), s \in \mathbb{R} \).

(c) Let \( \gamma_s \) be an \( s \)-smooth isotopy from the conditions of Theorem 1: for any \( s \in [0, 1] \) the map \( \gamma_s \) is a \( \lambda \)-preserving diffeomorphism of \( M \cong \{0\} \times M \). Then \( \gamma_0 = P_v \) and \( \gamma_1 = Q \). By smoothly changing the parametrization on \( \gamma_s \) we can assume that \( \gamma_s = P_v \) if \( s \) takes values close to 0, and \( \gamma_s = Q \) if \( s \) is close to 1. We extend \( \gamma_s \) to the whole axis \( \mathbb{R} = \{s\} \), for example, by putting \( \gamma_s = P_v \) for \( s < 0 \) and \( \gamma_s = Q \) for \( s > 0 \).

(d) Consider the family of maps \( \mathcal{T}_M(s): M \rightarrow M \) with \( \mathcal{T}_M(s) = \sigma_s \circ P_{\hat{v}}^{-1} \circ \gamma_s \), \( s \in \mathbb{R} \). Then \( \mathcal{T}_M(0) = \text{id} \) and \( \mathcal{T}_M(1) = Q \), hence both \( \mathcal{T}_M(0) \) and \( \mathcal{T}_M(1) \) preserve the form \( \lambda \).

Let \( \mathcal{T}^s \) be the flow \( (1) \) on \( \mathbb{R} \times M \) generated by the family \( \mathcal{T}_M(s) \), and let \( \mathcal{U} \) be the corresponding vector field on \( \mathbb{R} \times M \). Then by \( (1) \)

\[
D\pi_T \mathcal{U} = 1, \quad \mathcal{U}^0 = \text{id}_{\mathbb{T} \times M}, \quad \mathcal{T}^1(0, x) = (1, Q(x)).
\]

By \( (1) \) and \( (2) \)

\[
\mathcal{U} = \left( 1, \frac{d}{ds} \bigg|_{s=0} (\sigma_{t+s} \circ P_{\hat{v}}^{-1} \circ \gamma_{t+s}) \circ (\sigma_t \circ P_{\hat{v}}^{-1} \circ \gamma_t)^{-1} \right) = \hat{v} + \mathcal{W},
\]

\[
\hat{v} = \left( 1, \left( \frac{d}{dt} \sigma_t \right) \circ \sigma_t^{-1} \right), \quad \mathcal{W} = \left( 0, D(\sigma_t \circ P_{\hat{v}}^{-1}) \left( \frac{d}{dt} \gamma_t \right) \circ \gamma_t^{-1} \circ P_{\hat{v}} \circ \sigma_t^{-1} \right).
\]

Near the points \( t = 0 \) and \( t = 1 \) we have: \( d\gamma_t/dt = 0 \), and therefore \( \mathcal{U} = \hat{v} \). Hence \( \mathcal{U}|_{s \in [0, 1]} \) can be extended to an \( s \)-periodic vector field \( \mathcal{W} \) on \( \mathbb{R} \times M \). Let \( \hat{\mathcal{W}}^s \) be the corresponding flow on \( \mathbb{R} \times M \). Since \( \hat{\mathcal{W}} \) is periodic, projections of \( \hat{\mathcal{W}}^s \) and \( \hat{\mathcal{W}} \) on a flow \( \hat{\mathcal{W}}^s \) and a vector field \( U \) on \( \mathbb{T} \times M \) are well defined.

(e) Let \( \mathbf{1} \) be the vector field on \( \mathbb{R} \times M \) defined by the equalities \( D\pi_T \mathbf{1} = 1 \) and \( D\pi_M \mathbf{1} = 0 \). The flow \( \hat{\mathcal{W}}^s \) preserves some volume form \( \Omega \) on \( \mathbb{R} \times M \) which can be chosen so that \( \nu_1 \Omega|_{t=0} = \nu_1 \Omega|_{t=1} = \lambda \).

Any volume form on \( \mathbb{R} \times M \) equals \( \hat{\rho} \omega \), where \( \hat{\rho}: \mathbb{R} \times M \rightarrow \mathbb{R} \) is a positive function. Therefore, \( \Omega = \hat{\rho} \omega \), where \( \hat{\rho}|_{t=0} = \hat{\rho}|_{t=1} = \rho_0 \) and \( \lambda = \rho_0 \nu \).

These equalities and the periodicity of \( \mathcal{U} \) imply that \( \hat{\rho} \) is 1-periodic in \( t \). Hence there exists a function \( \rho: \mathbb{T} \times M \rightarrow \mathbb{R} \) such that \( \hat{\rho} = \rho \circ \pi \), where \( \pi: \mathbb{R} \times M \rightarrow \mathbb{T} \times M \) is the canonical projection. The flow \( \hat{\mathcal{W}}^s \) preserves the volume form \( \rho \omega \).

The vector field \( u = \rho U \) preserves \( \omega \). It remains to note that \( P_u = P_U = Q \).

**Corollary.** The vector field \( \mathbf{w} \) is small if the isotopy \( \gamma_s \) is close to the identity. Moreover, in this case \( U \) is close to \( \hat{v} \), and therefore \( \rho \) is close to \( v_T \) and then \( \mathbf{u} \) is a small perturbation of \( \mathbf{v} \).
Bibliography

[1] B. Khesin, S. Kuksin, and D. Peralta-Salas, *Global, local and dense non-mixing of the 3D Euler equation*, preprint, 2019.

Dmitrii V. Treschev
Steklov Mathematical Institute
of Russian Academy of Sciences, Moscow
E-mail: treschev@mi-ras.ru

Presented by D. O. Orlov
Accepted 25/DEC/19
Translated by THE AUTHOR