Optimizing Content Caching to Maximize the Density of Successful Receptions in Device-to-Device Networking

Derya Malak, Mazin Al-Shalash, and Jeffrey G. Andrews

Abstract

Device-to-device (D2D) communication is a promising approach to optimize the utilization of air interface resources in 5G networks, since it allows decentralized opportunistic short-range communication. For D2D to be useful, mobile nodes must possess content that other mobiles want. Thus, intelligent caching techniques are essential for D2D. In this paper we use results from stochastic geometry to derive the probability of successful content delivery in the presence of interference and noise. We employ a general transmission strategy where multiple files are cached at the users and different files can be transmitted simultaneously throughout the network. We then formulate an optimization problem, and find the caching distribution that maximizes the density of successful receptions (DSR) under a simple transmission strategy where a single file is transmitted at a time throughout the network. We model file requests by a Zipf distribution with exponent $\gamma_r$, which results in an optimal caching distribution that is also a Zipf distribution with exponent $\gamma_c$, which is related to $\gamma_r$ through a simple expression involving the path loss exponent. We solve the optimal content placement problem for more general demand profiles under Rayleigh, Ricean and Nakagami small-scale fading distributions. Our results suggest that it is required to flatten the request distribution to optimize the caching performance. We also develop strategies to optimize content caching for the more general case with multiple files, and bound the DSR for that scenario.

Parts of the manuscript were presented at the 2014 IEEE Globecom Workshops [1] and at the 2015 IEEE ICC Workshops [2].

D. Malak and J. G. Andrews are with the Wireless and Networking Communications Group (WNCG), The University of Texas at Austin, Austin, TX 78701 USA (email: deryamalak@utexas.edu; jandrews@ece.utexas.edu). M. Al-Shalash is with Huawei Technologies, Plano, TX 75075 USA (e-mail: mshalash@huawei.com).

This research has been supported by Huawei.

Manuscript last revised: August 30, 2016.
I. INTRODUCTION

Wireless networks are experiencing a well-known ever-rising demand for enhanced high rate data services, in particular wireless video, which is forecast to consume around 70% of wireless bandwidth by 2019 [3]. Non-real-time video in particular is expected to comprise half of this amount [4], and comprises large files that can be cached in the network. Meanwhile, preliminary D2D techniques have been standardized by 3GPP to allow decentralized file sharing and public safety applications [5]. D2D is intriguing since it allows increased spatial reuse and possibly very high rate communication without increased network infrastructure or new spectrum, but is only viable when the mobile users have content that other nearby users want. Thus, it is clear that smart content caching is essential for D2D.

Caching popular content is a well known technique to reduce resource usage, and increase content access speed and availability [6]. Infrastructure-based caching can reduce delay and when done at the network edge, also reduce the impact on the backhaul network, which in many cases is the bottleneck in wireless networks [7]. However, this type of caching does not reduce the demand on spectral resources. To gain spectral reuse and increase the area spectral efficiency, the content must be cached on wireless devices themselves, which allows short-range communication which is independent of the network infrastructure. D2D communication can enable proximity-based applications involving discovering and communicating with nearby devices [8]. Synchronized distributed network architectures for D2D communications are designed, e.g., FlashLinQ [9] and ITLinQ [10], and caching is shown to provide increased spectral reuse in D2D-enabled networks [11]. Although order optimal solutions for optimal content placement is known under certain channel conditions [12]–[14], it is not known how to best cache content in a D2D network. Intuitively, popular content should be seeded into the users’ limited storage resources in a way that maximizes the probability that a given D2D device can find a desired file within its radio range. Exploring this problem quantitively is the goal of this paper.

A. Related Work

Different aspects of D2D content distribution are studied. Scalability in ad hoc networks is considered [15], where decentralized algorithms for message forwarding are proposed by considering a Zipf product form model for message preferences. Throughput scaling laws with caching have been widely studied [16]–[18]. Optimal collaboration distance, Zipf distribution for content reuse, best achievable
scaling for the expected number of active D2D interference-free collaboration pairs for different Zipf exponents is studied [19]. With a heuristic choice (Zipf) of caching distribution for Zipf distributed requests, the optimal collaboration distance [20] and the Zipf exponent to maximize number of D2D links are determined [17]. However, in general, the caching pmf is not necessarily same as the request pmf. This brings us to the one of the main objectives in this paper, which is to find the best caching pmf that achieves the best density of successful receptions (DSR) in D2D networks.

Under the classical protocol model of ad hoc networks [21], for a grid network model, with fixed cache size $M$, as the number of users $n$ and the number of files $m$ become large with $nM \gg m$, the order optimal caching distribution is studied and the per-node throughput is shown to behave as $\Theta(M/m)$ [12], [22]. The network diameter is shown to scale as $\sqrt{n}$ for a multi-hop scenario [13]. It is shown that local multi-hop yields per-node throughput scaling as $\Theta(\sqrt{M/m})$ [14].

Spatial caching for a client requesting a large file that is stored at the caches with limited storage, is studied [23]. Using Poisson point process (PPP) to model the user locations, optimal geographic content placement and outage in wireless networks are studied [24]. The probability that the typical user finds the content in one of its nearby base stations (BS)s is optimized using the distribution of the number of BSs simultaneously covering a user [25]. Performance of randomized caching in D2D networks from a DSR maximization perspective has not been studied, which we study in this paper.

Although the work conducted in [17], [19] focused on the optimal caching distribution to maximize the average number of connections, the system model was overly simplistic. They assumed a cellular network where each BS serves the users in a square cell. The cell is divided into small clusters. D2D communications are allowed within each cluster. To avoid intra-cluster interference, only one transmitter-receiver pair per cluster is allowed, and it does not introduce interference for other clusters. In this paper, we aim to overcome these serious limitations using a more realistic D2D network model that captures the simultaneous transmissions where there is no restriction in the number of D2D pairs.

B. Contributions

This paper develops optimal content caching strategies that aim to maximize the average density of successful receptions so as to address the demands of D2D receivers. The contributions are as follows.

1The order optimality in [12], [22] is in the sense of a throughput-outage tradeoff due to simple model used.
Physical channel modeling using PPP. We introduce the network model in Sect. III in which the locations of the D2D users are modeled as a homogeneous PPP. Different from the grid-based model in [12], [22], we consider the actual physical channel model. PPP modeling makes our analysis tractable because unlike the cluster-based model in [20], where only a pair of users are allowed to communicate in a square region, we require no constraint on the link distance and allow a random number of simultaneous transmissions. All analysis is for a typical mobile node which is permissible in a homogeneous PPP by Slivnyak’s theorem [26]. The interference due to simultaneously active transmitters, noise and the small-scale Rayleigh fading are incorporated into the analysis. Any transmission is successful as long as the Signal-to-Interference-plus-Noise Ratio (SINR) is above a threshold.

Density of successful receptions (DSR). We propose a new file caching strategy exploiting stochastic geometry and the results of [27], and we introduce the concept of the density of successful receptions (DSR). Although in this paper, we do not investigate the throughput-outage tradeoff as in [12], [22], the DSR is closely related to the outage probability, obtained through the scaling of the coverage, i.e., the complement of the outage probability, with the number of receivers per unit area.

Optimal caching distribution to maximize the DSR for the sequential multi-file model. We study a randomized transmission model for D2D users with storage size 1 in Sect. III. We propose techniques for randomized content caching based on the possible ways of prioritizing different files. In Sect. III, we start with a baseline model with single file to determine the optimal fractions of transmitters $\gamma_1$ and receivers $\gamma_2$ in the D2D network model with PPP distributed user locations that maximizes the DSR. In Sect. IV we consider the more general sequential multi-file transmission scenario, where we investigate the maximum DSR in terms of the optimal fractions of $\gamma_1$ and $\gamma_2$ derived in Sect. III to determine the DSR, and optimize the caching pmf based on the randomized model.

Small-scale fading DSR results. We formulate an optimization problem in Sect. IV-A to find the best caching distribution that maximizes the DSR under a simple transmission strategy where single file is transmitted at a time throughout the network, assuming user demands are modeled by a Zipf distribution with exponent $\gamma_r$. This scheme yields a certain fraction of users to be active at a time based on the distribution of the requests. In Sect. IV-B we optimize the DSR of users for the multi-file setup, where the small-scale fading is Rayleigh distributed. We consider several special cases corresponding to 1) small but non-zero noise, 2) arbitrary noise and 3) an approximation for arbitrary
noise allowing the path loss exponent $\alpha = 4$. For case 1), we show that the optimal caching strategy also has a Zipf distribution but with exponent $\gamma_c = \frac{\gamma_r}{\alpha/2+1}$ where $\alpha > 2$. For case 2), we show that the same result holds based on an approximation of the SINR coverage justified numerically in Sect. [IV-B].

This relation implies that $\gamma_c$ is smaller than $\gamma_r$, i.e., the caching distribution should be more uniform compared to the request distribution, yet more popular files should be cached at a higher number of D2D users. For case 3), we obtain a distribution similar to Benford’s law (detailed in Sect. [IV-B] that optimizes the caching pmf. We also extend our results to the “general request distributions”, and show that cases 1) and 2) are also valid for Ricean and Nakagami fading distributions in Sect. [IV-B].

In general, the optimal DSR and the optimal caching distribution might not be tractable. Therefore, assuming the request and caching probabilities are known a priori, we weight the caching pmf to provide iterative techniques to optimize the DSR under different settings. We propose caching strategies that consider maximizing the DSR of the least desired file and of all files as detailed in Sect. [V-B].

**Optimal caching distribution to maximize the DSR for the simultaneous multi-file model.** In Sect. [VI] we extend our study to the simultaneous transmissions of different files and define popularity-based and global strategies. The popularity-based strategy is in favor of the transmission of popular files and discards unpopular files. On the other hand, the global strategy schedules all the files simultaneously, which leads to lower coverage than the sequential model does. Optimization of the DSR in these cases is very intricate compared to the case of sequential modeling. Therefore, we numerically compare the proposed caching models in Sect. [VI] and observe that the optimal solutions become skewed towards the most popular content in the network. Thus, we infer that under different models, the optimal caching distribution may not be a Zipf distribution as also found in [12]–[14].

**Insights.** Our results show that the optimal caching strategy exhibits less locality of the reference (abbreviated as locality) compared to the input stream of requests, i.e., the demand distribution. We also analyze the special case of $\alpha = 4$ using a tight approximation for standard Gaussian $Q$-function. Using this approach we show that the optimal caching distribution can be approximated by Benford’s law, which is a special bounded case of Zipf’s law [30]. In Sect. [VII] we validate that both Zipf
distribution and Benford’s law have very similar distributional characteristics, further validating the
generality of the results. For the multiple file case, we extend our results by finding lower and upper
bounds for the DSR in Sect. V. Simulations show that the bounds are very accurate approximations
for particular $\gamma_r$ values.

II. SYSTEM MODEL

We consider a mobile network model in which D2D users are spatially distributed as a homogeneous
PPP $\Phi$ of density $\lambda$, where a randomly selected user can transmit or receive information. In the
multiple file scenario, the randomized caching model we propose is shown in Fig. 1. The model can
be summarized as follows. At any time slot, only a fraction of the D2D users scheduled. Any user
transmits with probability $\gamma_1$ and receives with probability $\gamma_2 = 1 - \gamma_1$ independently of other users.
Each user has a cache with storage size 1. If it is selected as a receiver at a time slot, it draws a
sample from the request distribution $p_r(\cdot)$, which is assumed to be Zipf distributed. If it is selected
as transmitter at a time slot, it draws a sample from the caching distribution $p_c(\cdot)$. The selection of
request distribution and the optimization of caching distribution will be detailed in Sect. V. At any
time slot, each receiver is scheduled based on closest transmitter association.

A system model for the D2D content distribution network with multiple files is illustrated in Fig. 2.
For multiple file case, different from the single file case, where the D2D content distribution network
is like a downlink cellular network since nearest transmitter has the content, a farther transmitter is
often the one with the file required by the receiver.
Fig. 2: System model for D2D users with multiple files. Each receiver is associated to its closest transmitter that contains the requested file, where TX(k) and RX(k) denote the set of transmitters and receivers corresponding to file k. For illustration purposes, different types are separated on the plot. However, transmissions of different files can occur simultaneously.

General models for the multi-cell SINR using stochastic geometry were developed in [27], where the downlink coverage probability was derived as:

$$ p_{\text{cov}}(T, \lambda, \alpha) \triangleq \mathbb{P}[\text{SINR} > T] = \pi \lambda \int_{0}^{\infty} e^{-\pi \lambda \beta(T, \alpha) - \mu T \sigma^2} r^{\alpha/2} dr, $$

where $\beta(T, \alpha) = \frac{2(\mu T)^{2/\alpha}}{\alpha} \mathbb{E}[g^{2/\alpha} (\Gamma(-2/\alpha, \mu T g) - \Gamma(-2/\alpha))]$. The expectation is with respect to the interference power distribution $g$, the transmit power is $1/\mu$, and Signal-to-Noise Ratio (SNR) is defined at a distance of $r = 1$ and is $\text{SNR} = 1/(\mu \sigma^2)$. A summary of the symbol definitions and important network parameters are given in Table I.

**Definition 1. Density of successful receptions (DSR).** *The performance of a randomly chosen receiver is determined by its SINR coverage. For the homogeneous PPP $\Phi$ with density $\lambda$, let $\gamma_1$ fraction of all users be the transmitter process $\Phi_t$, and $\gamma_2$ fraction of users be the receiver process $\Phi_r$, where $0 < \gamma_1, \gamma_2 < 1$. The coverage probability of a randomly chosen receiver is $p_{\text{cov}}(T, \lambda \gamma_1, \alpha)$, which is the same for all receivers, and the total average number of receivers is proportional to the density $\lambda \gamma_2$. Hence, the DSR, which denotes the mean number of successful receptions per unit area, equals*

$$ \text{DSR} = \lambda \gamma_2 p_{\text{cov}}(T, \lambda \gamma_1, \alpha) = \lambda \gamma_2 \left( \pi \lambda \gamma_1 \int_{0}^{\infty} e^{-\pi \lambda \gamma_1 r \beta(T, \alpha) - \mu T \sigma^2 r^{\alpha/2}} dr \right), $$

where $p_{\text{cov}}(T, \lambda \gamma_1, \alpha)$ is obtained by combining (1) with the thinning property of the PPP, i.e., $\Phi_t$, which is obtained through the thinning of $\Phi$, is a homogeneous PPP with density $\lambda \gamma_1$ [31, Ch. 1].

We consider the generalized file caching problem in PPP networks where every user randomly requests or caches some files based on the availabilities. Our goal is to maximize the DSR in (2) for
single file and multiple files. We discuss the details of our optimization problem in Sects. III and IV.

III. DSR FOR A SINGLE FILE

We first assume that there is a single file in the network. The single file case is the baseline model for the more general multi-file model presented in Sect. IV. Sampled uniformly at random from the PPP $\Phi$, a fraction $\gamma_1$ of the users form the process $\Phi_t$ of the users possessing the file, and a fraction $\gamma_2$ of the users form the process $\Phi_r$ of the users who want the same file. The receivers communicate with the nearest transmitter while all other transmitters act as interferers, and each transmitter can serve multiple receivers. A receiver is in coverage when its SINR from its nearest transmitter is larger than some threshold $T$. Given the total density of receivers is given by $\lambda \gamma_2$, and each receiver is successfully covered with probability $p_{\text{cov}}(T, \lambda \gamma_1, \alpha)$, the DSR, i.e., DSR, is given by their product. In the single file scenario, since there is only 1 file being transmitted in the network, there is no caching pmf. Our objective in this section is to determine the optimal fractions of transmitters $\gamma_1$ and receivers $\gamma_2$ in the PPP network that maximizes the DSR. In Sect. IV we consider the multiple file transmission scenario, where we use the optimal fractions of transmitters and receivers $\gamma_1$ and $\gamma_2$, respectively, derived in this section, to determine the DSR, and optimize the caching pmf based on the randomized model outlined in Sect. III. We formulate the following optimization problem to determine $\gamma_1$ and $\gamma_2$:

$$\begin{align*}
\text{DSR}^* &= \max_{\gamma_1 > 0, \gamma_2 > 0} \lambda \gamma_2 p_{\text{cov}}(T, \lambda \gamma_1, \alpha) \\
\text{s.t.} \quad &\gamma_1 + \gamma_2 = a, \quad 0 < a \leq 1,
\end{align*}$$

where $p_{\text{cov}}(T, \lambda \gamma_1, \alpha)$ is the coverage probability of a typical user, and $a \leq 1$ is the total fraction of transmitting and receiving users in a PPP network $\Phi$ with density $\lambda$.

Lemma 1. The fraction of transmitters should be less than that of receivers, i.e., the solution of (3) satisfies the following relation: $\gamma_1 < a/2 < \gamma_2 < a \leq 1$.

Proof: See Appendix A.

Lemma 2. The maximum DSR for arbitrary noise and $\alpha = 4$ is given by

$$\begin{equation}
\text{DSR}^* = \lambda (a - \gamma_1) \left/ \left( \frac{1}{\gamma_1} \left[ \frac{1}{\gamma_1} - \frac{1}{a - \gamma_1} \right] \frac{2\mu T \sigma^2}{(\pi \lambda)^2 \beta(T, 4)} + \beta(T, 4) \right) \right.
\end{equation}$$
TABLE I: Important network parameters.

| Symbol | Definition |
|--------|------------|
| $T$: $\alpha > 2$ | SINR threshold; Path loss exponent |
| $\gamma_1$: $\gamma_2$ | Fraction of transmitting users; fraction of receiving users |
| $\Phi$: $\Phi_t$: $\Phi_r$ | Homogeneous PPP of all D2D users; transmitter process; receiver process |
| $\lambda$: $\lambda_t$ | Intensity of $\Phi$; intensity of $\Phi_t$ |
| $\mu^{-1}$: $\sigma^2$ | The constant transmit power; Noise variance |
| $g \sim \exp(\mu)$ | Interference power distribution |
| $\gamma_r$: $\gamma_c$ | Zipf request parameter; Zipf caching parameter |
| $M$: $1$ | Size of the file catalog; storage size of any user |
| $p_r(\cdot)$; $p_c(\cdot)$ | Popularity pmf; caching pmf |
| $p_{cov}(T, \lambda, \alpha)$ | Coverage probability for the sequential transmission model |
| $\beta(T, \alpha)$ | A function of interference in the exponent of $p_{cov}$ |
| $F_B(\cdot)$ | The pmf of the Benford’s distribution |
| $\Theta(\cdot)$; $o(\cdot)$ | Big O notation; Little-o notation |

Proof: See Appendix B.

Corollary 1. Low SNR case, $\alpha = 4$. As $\sigma^2 \to \infty$, the coverage can be approximated as $p_{cov}(T, \lambda, \alpha) = \mathbb{P}[\text{SNR} > T] = \mathbb{P}[\text{SINR} > T] \approx \pi \lambda \int_0^{\infty} e^{-\pi \lambda x - \mu T \sigma^2 x^{\alpha/2}} \, dx$. Hence, the maximum DSR is given as

$$DSR^* = \lambda (a - \gamma_1) \left/ \left( \frac{1}{\gamma_1} \left[ \frac{1}{\gamma_1} - \frac{1}{a - \gamma_1} \right] \right) \right. \frac{2\mu T \sigma^2}{(\pi \lambda)^2 + 1},$$

where optimal $\gamma_1$ satisfies

$$a - 3 \sigma_1 + 3 \gamma_1^2 = \frac{(\pi \lambda)^2}{4 \mu T \sigma^2}.$$  

Corollary 2. No noise (degenerative) case. For no noise, $p_{cov}(T, \lambda, \alpha) = \beta(T, \alpha)^{-1}$. Maximum DSR for single file for $0 < a \leq 1$, Rayleigh fading, no noise, and $\alpha > 2$ is $DSR^* = \max_{\gamma_1 > 0} \lambda (a - \gamma_1) \left/ \beta(T, \alpha) \right. = \frac{\lambda (a - \gamma_1^*)}{\beta(T, \alpha)}$, obtained for the optimal value of $\gamma_1$, i.e., $\gamma_1^* = \varepsilon > 0$ so that there is one transmitter.

Next, we consider the low noise approximation of the success probability that is more easily computable than the constant noise power expression and more accurate than the no noise approximation for $\sigma^2 = 0$. Using the expansion $\exp(-x) = 1 - x + o(x)$ for $\sigma^2 \neq 0$ as $x \to 0$, the term $p_{cov}(T, \lambda, \alpha)$

1In the no noise case the single file result is trivial. In multiple file case, there will be interference due to the simultaneous transmissions of multiple files, which will be discussed in Sect. IV.
for small but non-zero noise case can be calculated after an integration by parts of (1) as follows

\[ p_{cov}(T, \lambda, \alpha) = \frac{1}{\beta(T, \alpha)} - \frac{\mu T \sigma^2 (\lambda \pi)^{-\frac{\alpha}{2}}}{\beta(T, \alpha)^{\frac{\alpha}{2}+1}} \Gamma \left( 1 + \frac{\alpha}{2} \right) + o(\sigma^2). \]

**Lemma 3.** The maximum DSR for a single file for \( a = 1 \), Rayleigh fading, small noise is equal to

\[ DSR^* = \frac{\lambda \alpha}{\beta(T, \alpha)} \left[ \frac{1}{\alpha} - \frac{(\gamma_1^* - 1)}{\alpha + \gamma_1^*(2 - \alpha)} o(\sigma^2) \right]. \]

*Proof:* See Appendix C.

For \( \alpha = 4 \), there is a closed form expression for \( \beta(T, 4) \) as follows: \( \beta(T, 4) = 1 + \sqrt{T} \arctan(\sqrt{T}) \), which we use for the derivation of Lemma 4.

**Lemma 4.** The maximum DSR for small but non-zero noise and \( \alpha = 4 \) is

\[ DSR^* = \frac{2\lambda(a - \gamma_1)}{(1 + \sqrt{T} \arctan(\sqrt{T}))} \left[ 1 - \frac{\mu T \sigma^2 a}{\mu T \sigma^2 (2a - \gamma_1) + o(\sigma^2)} \right] + o(\sigma^2). \]  \((5)\)

*Proof:* See Appendix D.

**Discussion.** In Fig. 3(a), we illustrate the relation between DSR* and SNR for \( T = SNR/2, \lambda=0.1 \). To simplify the notation, we assume that \( \gamma_1 + \gamma_2 = 1 \) and let \( \gamma = \gamma_1 \) and \( \gamma_1^* = \gamma_{opt} \). As SNR increases for \( T = SNR/2 \), the DSR decreases and \( \gamma_{opt} \) decreases. Note that the solid lines denote the simulation results for the PPP model. In Fig. 3(b), the variation of DSR* with respect to \( T \) for \( SNR = 10, \lambda=0.1 \) is shown. The coverage \( p_{cov}(T, \lambda \gamma_1, \alpha) \) is monotonically decreasing in \( T \) and a concave increasing function of \( \gamma_1 \). For increasing \( T \), the value of DSR becomes very small, and to maximize the DSR, a
higher fraction of the users should be transmitters (i.e., higher $\gamma_1$) to compensate the outage. For low $T$, to maximize the DSR, the fraction of the receivers $\gamma_2$ should be higher. Therefore, as $T$ decreases, the DSR increases and becomes right-skewed, but $\gamma_{opt}$ decreases only slightly, which is negligible. Thus, we conclude that $\gamma_{opt}$ is largely invariant to $T$ and mainly determined by SNR. In Fig. 3 (c), we show the variation of $DSR^*$ with $\lambda$. The DSR increases with $\lambda$. On the other hand, $\gamma_{opt}$ decreases as the density of users increases and transmissions from increased number of users cause high interference.

Although the single file case is trivial in the sense that it boils down to the optimization of the fractions of the transmitters and receivers that maximizes the DSR, it is the baseline model for the multiple file case where the main objective is to determine the optimal caching distribution over the set of files. We discuss the multiple file setup next.

IV. Optimizing the DSR of the Sequential Serving Model with Multiple Files

We determine the optimal caching distribution for the transmitters to maximize the DSR for the sequential serving-based strategy, in which one type of file is transmitted at a time. Later, in Sect. VI, we study the general case, where the transmissions of different files can take place simultaneously.

File Popularity Distribution. To model the file popularity in a general PPP network, we use Zipf distribution for $p_r$, which is commonly used in the literature [19]. Then, the popularity of file $i$ is given by $p_r(i) = \frac{1}{i^{\gamma_r}} / \sum_{j=1}^{M} \frac{1}{j^{\gamma_r}}$, for $i = 1, \ldots, M$, where $\gamma_r$ is the Zipf exponent and there are $M$ files in total. The demand distribution $p_r \sim Zipf(\gamma_r)$ is the same for all receivers of the PPP model.

A. Sequential Serving-based Model

In this model, only the set of transmitters having a specific file transmits simultaneously. Hence, this is the special case where only one file is transmitted at a time network-wide. This is illustrated in Fig. 1 in Sect. II. If a user is selected as a receiver at a time slot, it draws a sample from the request distribution $p_r(\cdot)$, which is known. If any user is randomly selected as the transmitter at a time slot with probability $\gamma_1$, it draws a sample from the caching distribution $p_c(\cdot)$, which is not known yet. At any time slot, each receiver is scheduled based on closest transmitter association. According to this

---

4This follows from the separability assumption of $p_{cov}(T, \lambda \gamma_1, \alpha)$ in $\lambda \gamma_1$ and $T$, thus insensitivity of the DSR maximization problem to the value of $T$, which is further detailed in Assumption 1 of Sect. IV.B and verified in Appendix F.
model, since file \( i \) is available at each transmitter with \( p_c(i) \), using the thinning property of the PPP [31, Ch. 1], the probability of coverage for file \( i \) is

\[
p_{\text{cov}}(T, \lambda t p_c(i), \alpha) = \pi \lambda t p_c(i) \int_{0}^{\infty} e^{-\pi \lambda t p_c(i)r^{\beta(T,\alpha)} - \mu T r^{\gamma/2}} \, dr,
\]

where \( \lambda_t = \lambda \gamma_1 \) is the total density of the transmitting users.

Given that the requests are modeled by the Zipf distribution, our objective is to maximize the DSR of users for the sequential serving-based model, denoted by \( \text{DSR}_S \) for a PPP model with density \( \lambda \):

\[
\max_{p_c} \text{DSR}_S \\
\text{s.t.} \quad \sum_{i=1}^{M} p_c(i) = 1; \quad p_r(i) = \frac{1}{i^{\gamma_r}} / \sum_{j=1}^{M} \frac{1}{j^{\gamma_r}}, \quad i = 1, \ldots, M,
\]

where \( \text{DSR}_S = \lambda \gamma_2 \sum_{i=1}^{M} p_r(i) p_{\text{cov}}(T, \lambda \gamma_1 p_c(i), \alpha) \), the first constraint is the total probability law for the caching distribution, and the second constraint is the demand distribution modeled as Zipf with exponent \( \gamma_r \), and \( \gamma_2 = 1 - \gamma_1 \), and \( M \) is the number of files.

Note that \( p_{\text{cov}}(T, \lambda \gamma_1 p_c(i), \alpha) \) in (7) is obtained for a sequential transmission or scheduling model and it is same as the formulation given in (1) which follows from Theorem 1 of [27]. This model can be generalized to different scheduling schemes. For example, in Sect. VI we introduce a more general model where multiple files are simultaneously transmitted, and obtain a coverage expression \( \mathcal{P}_{\text{cov}}(T, \cdot, \alpha) \) that is different from \( p_{\text{cov}}(T, \cdot, \alpha) \) in (7), which is detailed in Theorem 2 of Sect. VI.

Similar to the optimal fractions of the transmitter and receiver processes calculated in Sect. III for the single file case, optimal values of \( \gamma_1 \) and \( \gamma_2 = 1 - \gamma_1 \) for multi-file case can be found by taking the derivative of (7) with respect to \( \gamma_1 \), which yields the following expression:

\[
\sum_{i=1}^{M} \lambda p_r(i) p_c(i) \left\{ \int_{0}^{\infty} \left[ \frac{1}{\gamma_1} - \frac{1}{1 - \gamma_1} - \pi \lambda p_c(i) \beta(T, \alpha) \right] e^{-\pi \lambda \gamma_1 p_c(i)r^{\beta(T,\alpha)} - \mu T r^{\gamma/2}} \, dr \right\} = 0,
\]

where optimal value of \( \gamma_1 \) and the pmf \( p_c(\cdot) \) are coupled. Therefore, we first solve (7) by optimizing the pmf \( p_c(\cdot) \) and then, determine the \( \gamma_1 \) value that satisfies (8).

We now investigate different special network scenarios where significant simplification is possible.
B. Rayleigh Fading DSR Results

We optimize the DSR of users for the multi-file setup, where interference fading power follows an exponential distribution with \( g \sim \exp(\mu) \). We consider several special cases corresponding to 1) small but non-zero noise, 2) arbitrary noise and 3) an approximation for arbitrary noise allowing the path loss exponent \( \alpha = 4 \). We find the optimal caching distribution corresponding to each scenario.

**Lemma 5. Small but non-zero noise, \( \alpha > 2 \).** The optimal caching distribution is

\[
p_c(i) = \frac{1}{\gamma_c} \sum_{j=1}^{M} \frac{1}{j^{\gamma_c}}, \quad i = 1, \ldots, M,
\]

which is also Zipf distributed, where \( \gamma_c = \frac{\gamma_r}{\alpha/2+1} \) is the Zipf exponent for the caching pmf.

**Proof:** See Appendix E.

Assuming \( \alpha > 2 \), the caching pmf exponent satisfies \( \gamma_c < \frac{\gamma_r}{2} \), which implies that the optimal caching pmf that maximizes the DSR has a more uniform distribution exhibiting less locality of reference compared to the request distribution that is more skewed towards the most popular files.

**Assumption 1. Separability of coverage distribution.** For Rayleigh, Ricean and Nakagami small-scale fading distributions, the function \( \beta(T, \alpha)^{\alpha/2} \) can be approximated as a linear function of \( T \) as shown in Fig. 4. This relation greatly simplifies the analysis of the optimization problem given in (7).

**Lemma 6. Arbitrary Noise, \( \alpha > 2 \).** For arbitrary noise, from Assumption 1, the optimal caching distribution \( p_c(\cdot) \) can be approximated as a Zipf distribution given by

\[
p_c(i) \approx \frac{1}{i^{\gamma_c}} \sum_{j=1}^{M} \frac{1}{j^{\gamma_c}}, \quad i = 1, \ldots, M,
\]

where \( \gamma_c = \frac{\gamma_r}{\alpha/2+1} < \frac{\gamma_r}{2} \) is the Zipf exponent for the caching pmf assuming \( \alpha > 2 \).

**Proof:** See Appendix F.

Interestingly, this result is the same as Rayleigh fading with small but non-zero noise model developed in Sect. [V-B] which follows from the monotonic transformation [32] caused by increasing the noise power \( \sigma^2 \) in (6). According to the pmf given in (9), the optimal caching strategy exhibits less locality of reference than the input stream of requests. Therefore, it is a good caching strategy, which

---

5 Although the expression \( \beta(T, \alpha)^{\alpha/2} / T \) is not analytically tractable, we can approximate \( \beta(T, \alpha)^{\alpha/2} \) as a linear function of \( T \) because the lower incomplete Gamma function has light-tailed characteristics. Since the channel power distribution -which is exponential due to Rayleigh fading- is also light tailed, we can expect to observe such a linear approximation in our numerical results.
will be further verified in Sect. VII. Lemma 6 suggests that files with higher popularity should be cached less frequently than the demand for this file, and unpopular files should be cached more frequently than the demand for the file. However, high popularity files should be still cached at more locations compared to the low popularity files. The path loss evens out the file popularities and the caching distribution should be more uniform compared to the request distribution. The sequential transmission model shows that for a Zipf request distribution with exponent $\gamma_r$, which is skewed towards the most popular files, the optimal caching pmf should be also Zipf distributed with the relation $\gamma_c < \frac{\gamma_r}{2}$ for $\alpha > 2$, implying that the caching pmf is more uniform than the request pmf.

The next result generalizes Lemma 6 to any request distribution $p_r(\cdot)$ rather than the Zipf distribution, and is derived solving (29) in Appendix F using the separability of coverage from Assumption 1.

**Theorem 1.** For arbitrary noise, if the small-scale fading is Rayleigh, Nakagami or Ricean distributed, from Assumption 1 for a general request pmf, $p_r(\cdot)$, the optimal caching pmf is approximated as

$$p_c(i) \approx p_r(i)^{\frac{1}{\alpha/2+1}}/\sum_{j=1}^{M} p_r(j)^{\frac{1}{\alpha/2+1}}, \quad i = 1, \ldots, M.$$  \hspace{1cm} (10)

From (10), it is required to flatten the request pmf to optimize the caching performance. Examples include the case of uniform demands, where the optimal caching distribution should be also uniform, and Geometric($p$) request distribution, for which the caching distribution satisfies Geometric($q$), where $q = 1 - (1 - p)^{\frac{1}{\alpha/2+1}}$. In the case of Zipf demands, we can derive the same result as in Lemma 6.

**Lemma 7.** An Approximation for Arbitrary Noise with $\alpha = 4$. For a total number of files $M$ and
arbitrary noise with $\alpha = 4$, the optimal caching pmf is

$$p_c(i) = a_i + b \log \left( \frac{i + 1}{i} \right), \quad i = 1, \ldots, M,$$

(11)

where $b = \sqrt{\frac{\mu T + \gamma r}{\pi \lambda T}}$, $a_i = \frac{1}{M} + \frac{b}{M} \sum_{j=1}^{M} \log \left( \frac{j}{i+1} \right)$, and the pmf is valid only if $b \leq [M \log(M) - \log(M!)]^{-1}$.

Proof: See Appendix G.

The distribution $p_c(\cdot)$ in (11) of Lemma 7 is a variety of Benford’s law [30], which is a special bounded case of Zipf’s law. Benford’s law refers to the frequency distribution of digits in many real-life sources of data and is characterized by the pmf

$$F_B(i) = \log_{10} \left( \frac{i+1}{i} \right), \quad i \in \{1, \ldots, 9\}.$$  

In distributed caching problems, the number of files, $M$, is generally much greater than 9. Therefore, we generalize the law as

$$F_B(i) = \log_{M+1} \left( \frac{i+1}{i} \right), \quad i \in \{1, \ldots, M\}.$$  

The result in (11) has a very similar form as the Benford law with shift parameter $a_i$ for file $i$ and a scaling parameter $b$, as determined in Lemma 7.

V. A LOWER AND UPPER BOUND ON THE DSR AND DIFFERENT CACHING STRATEGIES

The analysis of the DSR becomes intractable for the multiple file case when the caching pdf does not have a simple form. Therefore, we derive a lower and upper bound to characterize the DSR for the sequential serving model and provide two different caching strategies to maximize DSR$_S$.

A. Bounds on DSR$_S$

We provide a lower and upper bound for DSR$_S$, the DSR of the sequential serving-based transmission model with multiple files. We discussed the optimal file caching problem for multiple file scenarios in [1]. Here, we compare our solution to the several bounds and other caching strategies.

1) Upper Bound (UB): Using the concavity of $p_{cov}(T, \lambda TP_c(i), \alpha)$ in $p_c(i)$, a UB is found as

$$\sum_{i=1}^{M} p_r(i)p_{cov}(T, \lambda TP_c(i), \alpha) \leq p_{cov}(T, \lambda T \sum_{i=1}^{M} p_r(i)p_c(i), \alpha) \quad (a) \leq p_{cov}(T, \lambda TP_r(1), \alpha),$$

(12)

where $(a)$ follows from Jensen’s inequality, and $(b)$ follows from the assumption $p_r(1) > p_r(i)$ for $1 < i \leq M$ that yields $\sum_{i=1}^{M} p_r(i)p_c(i) < p_r(1) \sum_{i=1}^{M} p_c(i) = p_r(1)$, where $p_r(1) = \left( \sum_{j=1}^{M} j^{-\gamma} \right)^{-1}$.

2) Lower Bound (LB): Using the fact that given $p_r(\cdot)$ is Zipf distributed, the optimal $p_c(\cdot)$ also has Zipf distribution as proven in Lemma 6 as a solution of the DSR$_S$ maximization problem in (7).
As a result, any distribution that is not skewed towards the most popular files will yield a suboptimal DSR. Hence a uniform caching distribution performs worse than the Zipf law, and a LB is found as
\[
\sum_{i=1}^{M} p_r(i)p_{\text{cov}}(T, \frac{\lambda_t}{M}, \alpha) > \sum_{i=1}^{M} p_r(i)p_{\text{cov}}\left(T, \frac{\lambda_t}{M}, \alpha\right) = p_{\text{cov}}\left(T, \frac{\lambda_t}{M}, \alpha\right). \tag{13}
\]

**B. Caching Strategies for the Sequential Serving Model with Multiple Files**

We propose two optimization formulations to maximize DSR in the presence of multiple files, where the request and caching probabilities are known a priori because in general the optimal DSR and the optimal caching distribution is not tractable. The first strategy, where we maximize the DSR for the least popular file, favors the least desired file, i.e., the file with the lowest popularity, to prevent from fading away in the network. Therefore, we introduce the variables \(0 \leq \rho_i \leq 1\) for files \(i \in \{1, \ldots, M\}\) to weight the caching pmf \(p_c(\cdot)\). The second strategy aims to maximize the DSR of all files by optimizing the fraction \(\rho_i\)'s of the users for each file type. We assume the caching distribution is given. Then, we provide iterative techniques to solve the problems presented in this section.

1) **Maximum DSR of the Least Desired File:** Our motivation behind maximizing the DSR of the least desired file is to prevent the files with low popularity from fading away in the network.

**Lemma 8.** The caching probability of each file is weighted by \(\rho_i < 1\) so that the total fraction of transmissions for all files, denoted by \(\xi\) satisfies \(\xi = \sum_{i=1}^{M} \rho_i p_c(i) \leq 1\). Given \(\eta = \max_{i, \rho_i=1} p_r(i)p_c(i) = p_r(j)p_c(j)\) for some \(j\), the optimal solution is given by \(\rho_i = 1_{\{i \geq j\}} + \frac{\eta}{p_r(i)p_c(i)} 1_{\{1 \leq i < j\}}\).

**Proof:** See Appendix [H]

2) **Maximum DSR of All Files:** We maximize the DSR for all files without any prioritization.

**Lemma 9.** The optimal solution to maximize the DSR for all files is given by \(\rho_i = 1\) for all \(i\).

**Proof:** See Appendix [I]

As well as maximizing the DSR for the sequential model, one might wish to select a file with a particular request probability, and use D2D to distribute this file and all files with higher probability or simultaneously cache all files using D2D as detailed in Sect. [VI]. In the next section, we describe the simultaneous transmission of multiple files, and derive expressions for SINR distribution and DSR.
VI. Simultaneous Transmissions of Different Files with Arbitrary Noise

We consider the multiple file case, where a typical receiver requires a specific set of files, and the set of its transmitter candidates are the ones that contain any of the requested files. Each receiver gets the file from the closest transmitter candidate. The rest of the active transmitters that do not have the files requested are the interferers. We provide a detailed analysis for the SINR coverage next.

Assume that each receiver has a state, determined by the set of files it requests. For a receiver in state \( j \), the set of requested files is \( f_r(j) \). Let the tagged receiver be \( y \in \Phi_r \) and in state \( j \), and \( \Phi_t(j) \) be the set of transmitters that a receiver in state \( j \) can get data from. Hence, the set of transmitter candidates for user in state \( j \) is the superposition given by \( \Phi_t(j) = \sum_{i \in f_r(j)} \Phi_{t,i} \), where \( \Phi_{t,i} \) is the set of transmitters containing file \( i \). Let \( \lambda_j \) be the density of \( \Phi_t(j) \), where \( \lambda_j = \lambda t p_j = \lambda \gamma_1 p_j \). The rest of the transmitters, i.e., \( \sum_{i \notin f_r(j)} \Phi_{t,i} \), is an independent process with density \( \lambda_t - \lambda_j = \lambda_t (1 - p_j) = \lambda \gamma_1 (1 - p_j) \).

The sum \( p_j = \sum_{i \in f_r(j)} p_c(i) \) gives the probability that the user has at least one of the files requested by any receiver in state \( j \). Hence, the density of the transmitter candidates \( \lambda_j \) for a receiver in state \( j \) are given by the product of \( \lambda \gamma_1 \) and \( \sum_{i \in f_r(j)} p_c(i) \), i.e., \( \lambda_j = \lambda t p_j = \lambda \gamma_1 \sum_{i \in f_r(j)} p_c(i) \). Hence, using the nearest neighbor distribution of the typical receiver in state \( j \), the distance to its nearest transmitter is distributed as Rayleigh(\( \sigma_j \)) \( \sim \frac{r}{\sigma_j} \exp \left( -\frac{r^2}{2\sigma_j^2} \right) \), for \( \sigma_j = 1/\sqrt{2\pi\lambda_j} \) and \( r \geq 0 \).

We assume that all users experience Rayleigh fading with mean 1, and constant transmit power of \( 1/\mu \). Assuming user \( y \) is at \( o \), in state \( j \) and is a receiver, and \( x \) is the tagged transmitter denoted by \( b_o \), and the distance between them is \( r \), then the SINR at user \( y \) is \( \text{SINR}_y = \frac{hr^{-\alpha}}{\sigma^2 + I_r(j)} \), where \( h \) is the channel gain parameter between \( x \) and \( y \), \( \sigma^2 \) is the white Gaussian noise, and \( I_r(j) \) is the total interference at node \( y \) in state \( j \), and given by the following expression: \( I_r(j) = \sum_{x \in \Phi_t(j) \setminus b_o} g_x r_x^{-\alpha} = \sum_{x \in \Phi_t(j) \setminus b_o} g_x r_x^{-\alpha} + \sum_{x \in \Phi_t(j) \setminus f_r(j)} g_x r_x^{-\alpha} \), where \( g_x \) is the channel gain from the interferer \( z \) and the receiver \( y \), \( r_z \) is the interferer \( z \) to receiver distance, on RHS, the first term is the interference due to the set of transmitters that has the files requested by the receiver, and the second term is the interference due to the rest of the transmitters that do not have any of the desired files by the receiver. The total interference depends on the transmission scheme. Compared to the nearest user association [27], it is hard to characterize the interference in dynamic caching models with different association techniques.
Theorem 2. The probability of coverage of a typical user conditioned on being at state \( j \) is given by:

\[
\mathcal{P}_{\text{cov}}(T, \lambda_j, \alpha) = \pi \lambda_j \int_0^\infty e^{-\pi \lambda_j v(1-\rho_2(T,\alpha) - \pi \lambda_j v(\rho_1(T,\alpha)+\rho_2(T,\alpha)) - T \sigma^2 v^2/2)} \, dv,
\]

where \( \rho_1(T, \alpha) = T^{2/\alpha} \int_{-\infty}^{T^{-2/\alpha}} \frac{1}{1+u^{\alpha/2}} \, du \) and \( \rho_2(T, \alpha) = T^{2/\alpha} \int_{0}^{T^{-2/\alpha}} \frac{1}{1+u^{\alpha/2}} \, du \).

Proof: See Appendix J.

We now consider the special case of the path loss exponent \( \alpha = 4 \), which is more tractable.

Corollary 3. Letting \( H(T, \lambda_t, p_j) = \left( \frac{p_j}{\sqrt{T}} - p_j \tan^{-1} \left( \frac{1}{\sqrt{T}} \right) + \frac{\pi}{2} \right) \frac{\pi \lambda_t}{\sqrt{2\sigma^2}} \), the probability of coverage of a typical user conditioned on being at state \( j \) for the special case of \( \alpha = 4 \) and \( \mu = 1 \) is given by

\[
\mathcal{P}_{\text{cov}}(T, \lambda_j, 4) = \pi \lambda_t p_j \sqrt{\frac{\pi}{T \sigma^2}} e^{rac{H(T, \lambda_t, p_j)^2}{2}} Q(H(T, \lambda_t, p_j)).
\]

Proof: See Appendix K.

Since the term \( \sqrt{T} \tan^{-1} \left( \frac{1}{\sqrt{T}} \right) \) is increasing in \( T \) and converges to 1 in the limit as \( T \) goes to infinity, \( H(T, \lambda_t, \cdot) \) is increasing in \( p_j \), and positive. Furthermore, \( \mathcal{P}_{\text{cov}}(T, \lambda_j, \alpha) \) is monotonically increasing in \( p_j \). This observation is essential in the characterization of the DSR under different user criteria.

We consider two different strategies for the simultaneous transmission of multiple files, namely popularity-based and global models, which differ mainly in the set of files cached at the transmitters.

A. Popularity-based DSR

In this approach, a set of files corresponding to the most popular ones in the network is cached simultaneously at all transmitters. We define DSR\(_P\), which stands for the DSR of the popularity-based approach, and is calculated over the set of most popular files as

\[
\text{DSR}_P = \lambda \gamma_2 \sum_{k \in \mathcal{K}} p_r(k) \mathcal{P}_{\text{cov}}(T, \xi_l, \alpha),
\]

where \( \mathcal{K} \) is the set of the \( K \) most popular files, and \( \xi_l = \lambda \gamma_1 \sum_{i \in \mathcal{L}} p_c(i) \), where \( \mathcal{L} \) is a set corresponding to the most popular \( K \) files cached at the transmitters among the set of available files in the caches.

\(^6\)The definition of \( \mathcal{P}_{\text{cov}}(T, \lambda_j, \alpha) \) here is different from the definition of the classical downlink coverage probability \( p_{\text{cov}}(T, \lambda, \alpha) \) given in [1] due to the possibility of simultaneous transmissions of different file types.
Consider the special case of (16), where only the most popular file in the network is cached at all the transmitters if available, i.e., $|K| = 1$, which modifies (16) as

$$DSR_p = \lambda\gamma_2 p_r(k) P_{cov}(T, \lambda\gamma_1 p_c(k), \alpha)$$

$^{(a)}$ = $\lambda\gamma_2 p_r(k) p_{cov}(T, \lambda\gamma_1 p_c(k), \alpha)$, where $(a)$ follows from the fact that for $|K| = 1$, the coverage probability becomes same as the sequential serving-based model in Sect. IV and the most popular file index $k$ can be found from the demand distribution and is given by $k = \arg\max_{i \in \{1, \ldots, M\}} p_r(i)$, and hence the corresponding density of the transmitters is $\lambda\gamma_1 p_c(k)$, where $p_r(k) \geq p_r(l)$ for all $l = 1, \ldots, M$.

B. Global DSR

Global DSR is defined as the average performance of all users in the network, which is determined by the spatial characteristics of file distributions and the coverage of a typical user. The DSR function in our model is state dependent since the coverage probability of a user is determined according to the files requested by the user. The expected global DSR is given as follows:

$$DSR_G = \lambda\gamma_2 \sum_{i=1}^{M} p_r(i) P_{cov}(T, \gamma_1 \lambda p_c(i), \alpha).$$

(17)

A Discussion on the Various Transmission Models. Popularity-based transmission and global model in this section do not depend on the cache states. Instead, they both depend on the global file popularity distributions, and have similar characteristics as given in (16) and (17). It is intuitive to observe that the optimal caching distributions in both models follow similar trends as the request distribution. Sequential serving-based model in Sect. IV-A boils down to the scenario characterized in [27] where only a subset of transmitters and their candidate receivers are active simultaneously. Hence, this model mitigates interference and provides higher coverage than the other models. However, since the DSR is a weighted function of the file transmit pmf $p_c(\cdot)$, the DSR of the model is reduced.

Now, we present some numerical results on the general transmission models discussed and present results related to the popularity-based DSR, global DSR and sequential DSR.

State dependent coverage probability. We illustrate the SINR coverage probability for varying $p_j$ for a fixed fraction of transmitters ($\gamma_1 = 0.4$) in Fig. 5. The coverage probability is state dependent$^7$ and for the receiver in state $j$, the density of transmitters is given by $\lambda_j = \lambda p_j$ where $p_j = \gamma_1 \sum_{i \in f_r(j)} p_c(i)$.

$^7$The receiver’s state refers to the collection of files it requests.
If the requested files are available in the set of transmitters, then the receiver has higher coverage. Therefore, for higher fraction of transmitters $\gamma_1$, the coverage probability is higher.

**Caching performance of the proposed transmission models.** The optimal caching strategies that maximize the caching problems of Sect. [VI] given in (16) and (17) are not necessarily Zipf distributed. However, without the Zipf distribution assumption, the optimization formulations become intractable since $p_{\text{cov}}(T, \lambda_j, \alpha)$ in (14) is nonlinear in the density of the users. Therefore, for simulation purposes, we find the optimal Zipf caching exponents that maximize the proposed functions.

**DSR comparison.** We investigate the variation of the sequential model $\text{DSR}_S$ with respect to the caching parameter $\gamma_c$. From Fig. 6 we observe that $\gamma_c$ increases with the request distribution parameter $\gamma_r$, assuming both distributions are Zipf. In Figs. 7 and 8 we illustrate the variation of the popularity-based
based model $\text{DSR}_P$ and the global model $\text{DSR}_G$ with $\gamma_c$. In both figures, it is clearly seen that as the requests become more skewed (higher $\gamma_r$), the DSR increases. It also increases with $\gamma_c$, which implies that the optimal caching distribution should also be skewed towards the highly popular files.

VII. Numerical Results and Discussion

We evaluate the optimal caching distributions that maximize the DSR. The simulation results are based on Sects. IV and V. We consider a general PPP network model with Rayleigh fading distribution with $\mu = 1$ and $\alpha = 4$ for small and general noise solutions. The requests are modeled by Zipf($\gamma_r$).

**Benford versus Zipf distributions.** In Figs. 9 and 10 we illustrate the trend of optimal Zipf caching distribution and the Benford law developed in Sect. IV for different numbers of total files. As seen from Fig. 9, these two distributions have similar characteristics. However, as $\gamma_r$ increases, the range of $M$ for which Benford caching distribution in (11) and Zipf laws are comparable becomes narrower. For $\gamma_r > 0.3$, it is not practical to approximate the Benford law with a Zipf distribution. In fact, as described in Sect. IV as the noise level decreases, i.e., $b = \sqrt{\mu T \sigma^2 \gamma_r / (\pi \lambda_t \beta(T, 4))}$ drops, the optimal caching strategy converges to Zipf distribution. As seen in Fig. 10 for small noise, i.e., high SNR, these laws behave similarly for relatively high $\gamma_r$ values compared to the general noise case.

We now compare the DSR of the sequential serving model for various $\gamma_r$ based on the optimal solutions that are also Zipf distributed, as derived in Sect. IV and the lower and upper bounds obtained in Sect. V. The numerical solutions are obtained by calculating the DSR of various (random) caching distributions and picking the best one that achieves the highest DSR.
Zipf caching with $\gamma_c = \frac{\gamma_r}{(\alpha/2+1)}$ is a good approximation to maximize the DSR. In Fig. 11, we compare the performances of different caching strategies for a Zipf request distribution with parameter $\gamma_r = 0.5$ and SNR = 1. The Zipf caching distribution with parameter $\gamma_r/3$ is very close to the optimal solution evaluated numerically that is also very close to the simple lower bound derived in (13). Furthermore, Benford distribution has very similar characteristics as the optimal caching distribution solution. There is a huge gap between the UB and the no noise in terms of the DSR, and the DSR for the no noise case is the highest among all for all SNR or T values.

LB and UB get closer together as the SNR increases. In Fig. 12, we compare the performance of the caching distributions for a Zipf request pmf with parameter $\gamma_r = 0.5$ and SNR = 10. At high SNR, the UB and LB are closer. Still, the numerical solution and the Zipf caching pmf with parameter $\gamma_r/3$ give similar densities of successful communication, which is very close to the lower bound because...
for that choice of $\gamma_r$, the request distribution converges to a uniform distribution. Benford caching distribution does not perform as well as the Zipf caching distribution, and is even worse than the LB. In Fig. 13 where $\gamma_r = 2$ and SNR = 1, the Zipf caching pmf with parameter $\gamma_r/3$ does not have the same performance as the optimal solution evaluated numerically. Benford distribution has also similar performance as the Zipf caching pmf. In Fig. 14 we also show that Zipf caching pmf and Benford distributions have similar performance as the numerical solution for $\gamma_r = 2$ and SNR = 10.

**Transmit Diversity.** In the sequential serving model, where only one file is transmitted at a time network-wide, as discussed in Sect. IV, using a transmitter diversity scheme will improve the DSR. For the second scenario presented in Sect. VII, in which different files are transmitted simultaneously, a similar diversity scheme can be applied instead of treating the other transmitters as interferers. Diversity combining techniques include the maximal-ratio combining (e.g., of the $k$ closest transmitters [25]), where the received signals are weighted with respect to their SINR and then summed, equal-gain combining, where all the received signals are summed coherently, i.e., the shot-noise model [31, Ch. 2], and the selection combining, which is based on the strongest D2D user association, in which the received signal power (e.g., from the $k$ strongest users [25]) is considered.

Although diversity can decrease the outage probability, how to achieve this in practice is a critical issue. Diversity would seem to require synchronization of all transmitting devices at the physical layer unless higher layer coding is used, which might not be very practical for content distribution. Assuming full synchronization provides an upper bound on what could be achieved. Due to space constraints, we leave such analysis to future work.

**VIII. Conclusions**

Content distribution using direct D2D communications is a promising approach for optimizing the utilization of air-interface resources in 5G network. This work is the first attempt to derive closed form expressions for the optimal content caching distribution and the optimal caching strategies providing maximum DSR in terms of the optimal fractions of transmitters and receivers in a D2D network by using a homogeneous PPP model with realistic noise, interference and Rayleigh fading. We derive the SINR coverage for different transmission strategies in D2D networks with some idealized modeling aspects, i.e., simultaneous scheduling of the users containing the same type of files and Zipf distributed
content caching assumption for the general multi-file transmissions. Our results for the sequential transmission model show that the optimal caching pmf can also be modeled using the Zipf law and its exponent $\gamma_c$ is related to $\gamma_r$ through a simple expression involving the path loss exponent: $\gamma_c = \frac{\gamma_r}{(\alpha/2+1)}$. The optimal content placement for more general demand profiles under Rayleigh, Ricean and Nakagami fading distributions suggests to flatten the request distribution to optimize the caching performance.

The limitations of the model can be overcome by investigating the optimal caching distributions that maximize the DSR for the more general transmission settings incorporating the transmit diversity, and developing intelligent scheduling techniques, which are left as future work. The dynamic settings capturing the changes in the file popularities over time and the interference caused by simultaneous transmissions should also be considered. Future issues include the minimization of backhaul transmissions and BS overhead to optimize resource utilization through D2D collaboration. Future work could also include the design of distributed caching strategies to maximize the hit probability for users by using an SINR coverage model or a distance-based coverage process given the limited range of D2D.

**APPENDIX**

**A. Proof of Lemma 1**

The problem in (3) is equivalent to $\text{DSR}^* = \max_{\gamma_1<\alpha \leq 1} \lambda \left(\frac{a-\gamma_1}{1-\gamma_1}\right) \text{DSR}_T(\gamma_1)$, where $\text{DSR}_T(\gamma_1) = \gamma_2 \rho_{cov}(T,\lambda,\gamma_1,\alpha)$ is the DSR for $a = 1$, and $\text{DSR}_T(\gamma_1)$ can be rewritten as $\text{DSR}_T(\gamma_1) = (1 - \gamma_1) \pi \lambda \gamma_1 \int_{0}^{\infty} e^{-\pi \lambda \gamma_1 r^\beta(T,\alpha) - \mu T \sigma^2 r^2} dr$. Taking the derivative with respect to $\gamma_1$, we obtain

$$\frac{\partial \text{DSR}_T(\gamma_1)}{\partial \gamma_1} + \frac{(a-1)}{(1-\gamma_1)(a-\gamma_1)} \text{DSR}_T(\gamma_1) = 0.$$

(18)

Using (11), the solution of (18) yields the following expression: $0 < \frac{\left(\frac{a-1}{1-\gamma_1}\right)}{\pi \lambda \beta(T,\alpha)} = \frac{\int_{0}^{\infty} e^{-\pi \lambda \gamma_1 r^\beta(T,\alpha) - \mu T \sigma^2 r^2} r \, dr}{\int_{0}^{\infty} e^{-\pi \lambda \gamma_1 r^\beta(T,\alpha) - \mu T \sigma^2 r^2} r \, dr}$. Thus, the positivity condition yields $a - \gamma_1 > \gamma_1$, implying $\gamma_1 < a/2$ and $\gamma_2 > a/2$.

**B. Proof of Lemma 2**

We use the relation $\int_{0}^{\infty} e^{-(x-b)^2} x \, dx = \frac{1}{2} e^{-b^2} + \sqrt{\pi} b Q(-\sqrt{2}b)$, which follows from a change of variables $u = x - b$ and separating into two integrals, where $Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-y^2/2} \, dy$ is the standard
Gaussian tail probability. Using this relation, \( p_{\text{cov}}(T, \lambda \gamma_1, 4) \) is given as

\[
p_{\text{cov}}(T, \lambda \gamma_1, 4) = \pi \lambda \gamma_1 \int_0^\infty e^{-\pi \lambda \gamma_1 r \beta(T, 4) - \mu T \sigma^2 r^2} \, dr = \frac{\pi^2 \lambda \gamma_1}{\sqrt{\mu T \sigma^2}} e^{\frac{(\lambda \gamma_1 \gamma_1 \pi \beta(T, 4))^2}{2 \mu T \sigma^2}} Q\left(\frac{\lambda \gamma_1 \pi \beta(T, 4)}{\sqrt{2 \mu T \sigma^2}}\right). \tag{19}
\]

The solution of (18) in Appendix A for \( \alpha = 4 \) yields the following expression:

\[
\frac{1}{\gamma_1} \left[ 1 - \frac{1}{a - \gamma_1} \right] = \frac{e^{\mu T \sigma^2 u_0^2} \int_0^\infty e^{-\mu T \sigma^2 (r+u_0^2)} \, dr}{e^{\mu T \sigma^2 u_0^2} \int_0^\infty e^{-\mu T \sigma^2 (r+u_0^2)} \, dr} \tag{a}\]

\[
= \frac{1}{2\mu T \sigma^2} \int_{\mu T \sigma^2 u_0^2}^{\infty} e^{-v_1} \, dv_1 - \frac{\pi \lambda \gamma_1 \beta(T, 4)}{2\mu T \sigma^2} \Gamma(1) + \frac{\pi \lambda \gamma_1 \beta(T, 4)}{2\mu T \sigma^2} \int_{\mu T \sigma^2 u_0^2}^{\infty} e^{-v_2} \, dv_2 \tag{b}\]

\[
= \frac{\pi \lambda \gamma_1}{2\mu T \sigma^2} \left[p_{\text{cov}}(T, \lambda \gamma_1, 4)^{-1} - \beta(T, 4)\right]. \tag{20}
\]

where \( u_0 = \frac{\pi \lambda \gamma_1 \beta(T, 4)}{2\mu T \sigma^2} \), \( a \) follows from employing a change of variables \( u = r + \frac{\pi \lambda \gamma_1 \beta(T, 4)}{2\mu T \sigma^2} \), \( b \) also follows from employing change of variables \( v_1 = \mu T \sigma^2 u_2 \) and \( v_2 = \sqrt{\mu T \sigma^2 u} \), and \( c \) follows from employing the definition of standard Gaussian tail probability, and employing the definition of the coverage probability for Rayleigh fading, noise and \( \alpha = 4 \) [27].

The relation in (20) can be rearranged to obtain the following equality:

\[
\frac{1}{\gamma_1} \left[ 1 - \frac{1}{a - \gamma_1} \right] = \frac{(\pi \lambda)^2 \beta(T, 4)^2}{2\mu T \sigma^2} \left[ \frac{\beta(T, 4)^{-1}}{p_{\text{cov}}(T, \lambda \gamma_1, 4)} - 1 \right] \tag{21}\]

where \( a \) follows from the fact that \( p_{\text{cov}}(T, \lambda \gamma_1, 4) \leq \beta(T, 4)^{-1} \), where \( \beta(T, 4)^{-1} \) is the no noise coverage. Given that \( \gamma_1 \) is optimal, i.e., it satisfies (21), the maximum of the DSR \( p_{\text{cov}}(T, \lambda \gamma_1, 4) \) is

\[
p_{\text{cov}}(T, \lambda \gamma_1, 4) = \left( \frac{1}{\gamma_1} \left[ 1 - \frac{1}{a - \gamma_1} \right] \frac{2\mu T \sigma^2}{(\pi \lambda)^2 \beta(T, 4)} + \beta(T, 4) \right)^{-1}, \tag{22}
\]

using which DSR* can be obtained. Combining (22) with (19), the optimal value of \( \gamma_1 \) is found.

C. Proof of Lemma [3]

The problem is equivalent to the formulation

\[
\text{DSR}^* = \max_{\gamma_1} \lambda(1-\gamma_1) \left[ \frac{1}{\beta(T, \alpha)} - \frac{e^{\mu T \sigma^2 (\lambda \gamma_1)^{\frac{\alpha}{2} - 1}}}{\beta(T, \alpha)^{\frac{\alpha}{2} + 1}} \Gamma \left( 1 + \frac{\alpha}{2} \right) + o(\sigma^2) \right],
\]

which is concave in \( \gamma_1 \). Hence, taking its derivative with respect to \( \gamma_1 \), we obtain

\[
\frac{1}{\beta(T, \alpha)} = \frac{e^{\mu T \sigma^2 (\lambda \gamma_1)^{\frac{\alpha}{2} - 1}}}{\beta(T, \alpha)^{\frac{\alpha}{2} + 1}} \Gamma \left( 1 + \frac{\alpha}{2} \right) \gamma_1 \frac{\alpha}{2} \gamma_1^{-\frac{\alpha}{2}}. \]

The solution \( \gamma_1^* \) satisfies the polynomial equation

\[
\left[ \frac{2\beta(T, \alpha)}{2} \frac{(\lambda \gamma_1^*)^{\frac{\alpha}{2} - 1}}{\mu T \sigma^2 \Gamma \left( 1 + \frac{\alpha}{2} \right)} \right] (\gamma_1^*)^{\frac{\alpha}{2} + 1} o(\sigma^2) - \gamma_1^* = \frac{\alpha}{2 - \alpha},
\]

using which DSR* is found.
D. Proof of Lemma 4

Using (18), and letting $\alpha = 4$, we have

\[
\frac{1}{\gamma_1} - \frac{1}{\alpha - \gamma_1} = \int_0^\infty e^{-\pi \gamma_1 r (T, 4) - \mu T \sigma^2 r^2} \, dr = \frac{1}{\pi \lambda (T, 4)} \int_0^\infty e^{-\pi \lambda \gamma_1 r (T, 4) - \mu T \sigma^2 r^2} \, dr \]

\[
= \frac{\pi \lambda \gamma_1 (T, 4)}{\pi \lambda \gamma_1 (T, 4)^3 - 2(\mu T \sigma^2)(\pi \lambda \gamma_1 (T, 4)) + o(\sigma^2)},
\]

where (a) follows from $\exp(-x) = 1 - x + o(x)$ for $x \to 0$. Then, using (23), we obtain

\[
\frac{2a}{\gamma_1} = \frac{(\pi \lambda (T, 4))^3 \gamma_1^2 + (\mu T \sigma^2)(\pi \lambda (T, 4))^2 + o(\sigma^2)}{(\mu T \sigma^2)(\pi \lambda (T, 4))^2 + o(\sigma^2)} = \frac{(\pi \lambda \gamma_1 (T, 4))^2 + 2\mu T \sigma^2 + o(\sigma^2)}{2\mu T \sigma^2 + o(\sigma^2)}.
\]

Given that $\gamma_1$ is optimal, i.e., it satisfies (24), $p_{\text{cov}}(T, \lambda \gamma_1, 4)$ is given by

\[
p_{\text{cov}}(T, \lambda \gamma_1, 4) = \frac{1}{\beta(T, 4)} - \frac{2\mu T \sigma^2 (\lambda \gamma_1)^{-2}}{\beta(T, 4)^3} + o(\sigma^2) = \frac{2}{\beta(T, 4)} \left[ \frac{\mu T \sigma^2 (a - \gamma_1) + o(\sigma^2)}{\mu T \sigma^2 (2a - \gamma_1) + o(\sigma^2)} \right] + o(\sigma^2),
\]

using which the final result can be obtained. As $\sigma^2 \to 0$, \( \lim_{\sigma^2 \to 0} p_{\text{cov}}(T, \lambda \gamma_1, 4) = \frac{2}{\beta(T, 4)} \left[ \frac{\alpha - \gamma_1}{2a - \gamma_1} \right].
\]

E. Proof of Lemma 5

For small but non-zero noise case, and given that \( \sum_{i=1}^M p_c(i) = 1 \), (7) can be rewritten as

\[
\max_{p_c} \lambda \left[ \frac{1}{\beta(T, \alpha)} + o(\sigma^2) \right] = \Gamma \left( 1 + \frac{\alpha}{2} \right) \left[ \frac{\mu T \sigma^2 (\lambda \pi)^{-\alpha/2}}{\beta(T, \alpha)^{\alpha/2+1}} \right] \sum_{i=1}^M \lambda p_r(i) p_c(i)^{-\alpha/2},
\]

equivalent to minimizing \( \sum_{i=1}^M p_r(i) p_c(i)^{-\alpha/2} = (\sum_{j=1}^M \frac{1}{\gamma_j})^{-1} \sum_{i=1}^M \frac{p_c(i)^{-\frac{\alpha}{2}}}{\gamma_j} \) subject to \( \sum_{i=1}^M p_c(i) = 1 \).

Using the Lagrange multiplier method \[33\], we define, \( \Lambda(p_c, \eta) = \sum_{i=1}^M \frac{1}{\gamma_j} p_c(i)^{-\alpha/2} + \eta \left( \sum_{i=1}^M p_c(i) - 1 \right) \), where \( p_c = [p_c(1) \ldots p_c(M)] \), and \( \eta \) is the Lagrange multiplier. To maximize \( \Lambda(p_c, \eta) \), we take its partial derivatives with respect to \( p_c \). The partial derivative \( \frac{\partial \Lambda(p_c, \eta)}{\partial \gamma_j} \) reduces to the constraint equation.

Partial derivative of \( \Lambda(p_c, \eta) \) with respect to \( p_c(i) \) gives \( (-\alpha/2) \frac{1}{\gamma_j} p_c(i)^{-\alpha/2-1} + \eta = 0, \quad i = 1, \ldots, M \). Hence, for any \( i \neq j \) pair of file indices, we require \( (-\alpha/2) \frac{1}{\gamma_j} p_c(i)^{-\alpha/2-1} = (-\alpha/2) \frac{1}{\gamma_j} p_c(j)^{-\alpha/2-1} \), implying that \( p_c(j)/p_c(i) = (i/j)^{\alpha/2+1} \). Then, \( p_c(i) \) also has Zipf distribution with exponent \( \gamma_c = \frac{\alpha}{\alpha + 1} \).
F. Proof of Lemma 6

We investigate the general solution of (7). Using the Lagrange multiplier method \[33\], we define \( \Lambda(p_c, \eta) = \sum_{i=1}^{M} \lambda_ip_r(i)p_{cov}(T, \lambda_ip_c(i), \alpha) + \eta \left( \sum_{i=1}^{M} p_c(i) - 1 \right) \). The partial derivatives of \( \Lambda(p_c, \eta) \) with respect to \( p_c(i) \) for \( i = 1, \ldots, M \) give \( M \) equations.

\[
\frac{\partial \Lambda(p_c, \eta)}{\partial p_c(i)} = \lambda_ip_r(i)\frac{\partial p_{cov}(T, \lambda_ip_c(i), \alpha)}{\partial p_c(i)} + \eta = \lambda_ip_r(i) \pi \lambda_t \int_0^\infty e^{-\pi \lambda_ip_r(i)r\beta(T, \alpha) - \mu T\sigma^2 r^{\alpha/2}} dr - (\pi \lambda_t)^2 \beta(T, \alpha)p_c(i) \int_0^\infty e^{-\pi \lambda_ip_r(i)r\beta(T, \alpha) - \mu T\sigma^2 r^{\alpha/2}} r dr + \eta.
\]

To maximize \( \Lambda(p_c, \eta) \), we equate the RHS of (25) to 0 and obtain

\[
\int_0^\infty \left[ 1 - \pi \lambda_t \beta(T, \alpha)p_c(i)r \right] e^{-\pi \lambda_t p_r(i)r\beta(T, \alpha) - \mu T\sigma^2 r^{\alpha/2}} dr = -\frac{\eta}{p_r(i)\pi \lambda_t^2}.
\]

The partial derivative of \( p_{cov}(T, \lambda_ip_c(i), \alpha) \) with respect to \( \lambda_t \) is given as

\[
\frac{\partial p_{cov}(T, \lambda_ip_c(i), \alpha)}{\partial \lambda_t} = \pi p_c(i) \int_0^\infty e^{-\pi \lambda_ip_r(i)r\beta(T, \alpha) - \mu T\sigma^2 r^{\alpha/2}} dr - (\pi \lambda_t)^2 \beta(T, \alpha)p_c(i) \int_0^\infty e^{-\pi \lambda_ip_r(i)r\beta(T, \alpha) - \mu T\sigma^2 r^{\alpha/2}} r dr \frac{p_c(i)}{\lambda_t} \frac{\partial p_{cov}(T, \lambda_ip_c(i), \alpha)}{\partial p_c(i)}.
\]

Combining the relations (26) and (27) results in \( \frac{\partial p_{cov}(T, \lambda_ip_c(i), \alpha)}{\partial \lambda_t} = -\eta \frac{p_c(i)}{\lambda_t^2 p_r(i)} \). Using the definition of \( p_{cov}(T, \lambda_ip_c(i), \alpha) \), we can easily note that \( p_{cov}(T, \lambda_ip_c(i), \alpha) = p_{cov}(T(p_c(j)/p_c(i))^{\alpha/2}, \lambda_ip_c(j), \alpha) \).

Taking the derivative of this expression with respect to \( \lambda_t \), we have

\[
\frac{\partial p_{cov}(T, \lambda_ip_c(j), \alpha)}{\partial \lambda_t} = -\eta \frac{p_c(j)}{\lambda_t^2 p_r(j)} = \frac{\partial p_{cov}(T, \lambda_ip_c(i), \alpha)}{\partial \lambda_t} \frac{p_r(i)/p_c(i)}{p_r(j)/p_c(j)}.
\]

We can rewrite (28) using the expression for \( p_{cov}(T, \lambda_ip_c(i), \alpha) \) as follows

\[
\frac{\partial p_{cov}(T, \lambda_ip_c(j), \alpha)}{\partial \lambda_t} = \frac{\partial p_{cov}(T(p_c(j)/p_c(i))^{\alpha/2}, \lambda_ip_c(j), \alpha)}{\partial \lambda_t} \frac{p_r(i)/p_c(i)}{p_r(j)/p_c(j)}.
\]

Next, by employing a change of variables \( v = r\beta(T, \alpha) \), we can rewrite (11) in Definition 1 as

\[
p_{cov}(T, \lambda_t, \alpha) = \frac{\pi \lambda_t}{\beta(T, \alpha)} \int_0^\infty e^{-\pi \lambda_tv - \mu \left( \frac{v}{\beta(T, \alpha)^{\alpha/2}} \right) \sigma^2 v^{\alpha/2}} dv.
\]

We investigate the relation between \( \beta(T, \alpha)^{\alpha/2} \) and \( T \) in Fig. 4 for practical \( \alpha \) and \( \mu \) values, and observe
the linear dependence, where the slope is mainly determined by \( \alpha \), and changes only slightly by varying \( \mu \). Based on these simulations, since \( \beta(T, \alpha)^{\alpha/2}/T \) is invariant to \( T \) and using the relation in (30), it is reasonable to write \( p_{\text{cov}}(T, \lambda_T p_c(j), \alpha) \) as a separable function which is the form \( f(\lambda_T p_c(j), \alpha) g(T) \). By taking its derivative with respect to \( \lambda_c \), we can then rewrite (29) as \( g(T) = \left( T \left( \frac{p_c(j)}{p_c(i)} \right)^{\alpha/2} \right) \frac{Q_p(i)}{p_c(i)} (\lambda_T)^{\gamma_r} \). Taking the derivative of both sides with respect to \( T \), we obtain \( \frac{dg(T)}{dT} = \left( \frac{p_c(j)}{p_c(i)} \right)^{\alpha/2} \frac{dg(T)}{dT} \frac{Q_p(i)}{p_c(i)} (\lambda_T)^{\gamma_r} \), implying that \( p_c(j)/p_c(i) = (i/j)^{\gamma_r/2+1} \). Then, \( p_c(\cdot) \) is also Zipf(\( \gamma_c \)) distributed with \( \gamma_c = \frac{\gamma_r}{\alpha/2+1} \).

G. Proof of Lemma 7

The coverage probability \( p_{\text{cov}}(T, \lambda_T p_c(i), 4) \) for Rayleigh fading and general noise with \( \alpha = 4 \) is \( p_{\text{cov}}(T, \lambda_T p_c(i), 4) = \frac{e^{1/2 \lambda_T T^2} x}{\beta(T, 4)} Q(x) \exp(x^2/2) \), where \( x = \frac{\pi \lambda_T p_c(i) \beta(T, 4)}{\sqrt{2 \mu T \sigma^2}} \). The details of the derivation follow from [27]. We approximate \( p_{\text{cov}}(T, \lambda_T p_c(i), 4) \) by using the following tight approximation for Q-function in [34] as \( Q(x) \approx \frac{1 - \exp(-1.135x)}{1.135x} \exp(-x^2/2) \), \( x > 0 \). Hence, \( p_{\text{cov}}(T, \lambda_T p_c(i), 4) \approx \frac{1}{\sqrt{2 \mu T \sigma^2}} (1 - \exp(-\frac{\pi \lambda_T p_c(i) \beta(T, 4)}{\sqrt{2 \mu T \sigma^2}})) \)). Using the Lagrange multiplier method [33] to find the solution of the maximum DSR problem defined in (7), for the file indices \( i \) and \( j \), we obtain the relation \( \frac{1}{\sqrt{2 \mu T \sigma^2}} \exp(-\frac{\pi \lambda_T p_c(i) \beta(T, 4)}{\sqrt{2 \mu T \sigma^2}}) = \frac{1}{\sqrt{2 \mu T \sigma^2}} \exp(-\frac{\pi \lambda_T p_c(j) \beta(T, 4)}{\sqrt{2 \mu T \sigma^2}}) \) that yields the following difference between the file caching probabilities as a function of the network parameters, Zipf exponent \( \gamma_r \) and the file indices, which is given as \( p_c(i) - p_c(j) = -\frac{\sqrt{2 \mu T \sigma^2 \gamma_r}}{\pi \lambda_T \beta(T, 4)} \log\left( \frac{i}{j} \right) \). Using the relation \( \sum_{i=1}^M p_c(i) = 1 \), the caching distribution is found as \( p_c(i) = \frac{1}{M} + \frac{\sqrt{2 \mu T \sigma^2 \gamma_r}}{M \pi \lambda_T \beta(T, 4)} \sum_{j=1}^M \log\left( \frac{j}{i} \right), i = 1, \ldots, M \), and the final result can be obtained by rearranging the terms. The required condition for the pmf \( p_c(\cdot) \) to be valid is \( p_c(i) > 0 \), for \( i = 1, \ldots, M \). Since \( p_c(i) \) is decreasing in \( i \), a sufficient condition is \( p_c(M) = \frac{1}{M} + \frac{\sqrt{2 \mu T \sigma^2 \gamma_r}}{M \pi \lambda_T \beta(T, 4)} \sum_{j=1}^M \log\left( \frac{j}{M} \right) \geq 0 \). Thus, for a total number of files \( M \), we require that \( \sqrt{2 \mu T \sigma^2 \gamma_r} \leq \pi \lambda_T \beta(T, 4)[M \log(M) - \log(M!)]^{-1} \).

H. Proof of Lemma 8

The optimization formulation to maximize the DSR of the least popular file is given as

\[
\max_{\rho_i} \min_i \quad \xi^{-1} p_r(i) p_i p_c(i) p_{\text{cov}}(T, \lambda_T \xi, \alpha),
\]

where \( \xi = \sum_{i=1}^M \rho_i p_c(i), 0 \leq \rho_i \leq 1 \). We can calculate \( p_{\text{cov}}(T, \lambda_T \xi, \alpha) \) if \( p_r(i), p_c(i) \) and \( \xi \) are known priorly. The caching distribution can be modeled by \( p_c(i) \sim \text{Zipf}(\gamma_c) \), where \( \gamma_c = \gamma_r/(\alpha/2+1) \) based
The optimal solution of (32) is \( \xi = p_c(i) \) if \( \xi = 1 \) since \( \xi \) cannot be larger than \( p_c(i) \). Defining \( \xi = \sum_{i=1}^{M} \xi_i \), if \( \xi < 1 \), then \( \xi_i \leq p_c(i) \). The optimal solution can be found by equating \( p_r(i)\xi_i \) for all \( i \in \{1, \ldots, M\} \) so that the least desired file with \( p_r(m) \) is multiplied by the highest \( \xi_m \). Then, the required condition is \( p_r(i)\xi_i = p_r(j)\xi_j \) for \( i \neq j \), which is equivalent to \( \frac{\rho_i}{\rho_j} = \frac{p_r(i)p_c(j)}{p_r(j)p_c(i)} \) for \( i \neq j \). Using the constraint equation of (32), we obtain \( \xi = \sum_{j=1}^{M} \rho_j p_c(j) = \rho_i p_c(i) + \rho_i p_c(i) \sum_{j=1, j \neq i}^{M} \frac{p_r(i)}{p_r(j)} \), and solving for \( \rho_i \), for all \( i \in \{1, \ldots, M\} \), we get \( \rho_i = \frac{\xi}{\text{cov}(T, \lambda \xi, \alpha)} \left( 1 + \sum_{j=1, j \neq i}^{M} \frac{p_r(i)}{p_r(j)} \right)^{-1} = \frac{\xi}{\sum_{j=1}^{M} \frac{p_r(i)p_c(i)}{p_r(j)}}. \)

Hence, the optimal \( \rho_i \)'s for \( i \in \{1, \ldots, M\} \) should satisfy \( p_r(i)p_c(i)\rho_i = \eta \), if \( \rho_i < 1 \), and \( p_r(i)p_c(i) < \eta \), if \( \rho_i = 1 \), implying that for any \( \rho_i < 1 \), \( \eta < p_r(i)p_c(i) \), yielding \( \eta < \min_{i, \rho_i < 1} \rho_i p_c(i) \), and for any \( \rho_i = 1 \), \( \eta > \max_{i, \rho_i = 1} \rho_i p_c(i) \) for some constant \( \eta \). Hence, the objective of (31) can be rewritten as

\[
\xi^{-1}p_r(i)p_c(i)p_{\text{cov}}(T, \lambda \xi, \alpha) = \frac{p_{\text{cov}}(T, \lambda \xi, \alpha)}{\sum_{j=1}^{M} \frac{1}{p_r(j)}} < \frac{\eta}{\xi}p_{\text{cov}}(T, \lambda \xi, \alpha), \quad \rho_i = 1, \quad i = 1, \ldots, M, (33)
\]

which is increasing in \( \eta \). The solution of (31) can be found by letting \( \eta = \max_{i, \rho_i = 1} \rho_i p_c(i) \), implying that \( \rho_i = 1 \) for \( i \geq j \) for some \( j \), and for \( 1 \leq i \leq j - 1 \), we have \( \rho_i = \frac{\eta}{p_r(i)p_c(i)} = \frac{p_r(j)p_c(i)}{p_r(i)p_c(i)} < 1. \)

I. Proof of Lemma 9

The formulation to maximize the DSR for all files is given by

\[
\text{DSR}^*_{\text{tot}} = \max_{\rho_i} \xi^{-1}p_{\text{cov}}(T, \lambda \xi, \alpha) \sum_{i=1}^{M} p_r(i)p_c(i), \quad (34)
\]

where \( \xi = \sum_{i=1}^{M} \rho_i p_c(i), \quad 0 \leq \rho_i \leq 1. \) For \( \xi = 1 \), as \( \xi_i \leq p_c(i) \) and \( \sum_{i=1}^{M} p_r(i)\xi_i \leq \sum_{i=1}^{M} p_r(i)p_c(i) \), the optimal solution is \( \xi_i = p_c(i) \), and \( \rho_i = 1 \) for all \( i \). For \( \xi < 1 \), the solution of the maximum DSR problem can be found using water-filling, where \( 1 \geq \rho_1 \geq \rho_2 \geq \ldots \geq \rho_M \) as \( p_r(i)p_c(i) > p_r(j)p_c(j) \) for \( i < j \). Using the constraint equation \( \xi = \sum_{i=1}^{M} \rho_i p_c(i) \), we have \( \xi - \rho_1 p_c(1) = \sum_{i=2}^{M} \rho_i p_c(i) \).

To simplify the notation, we let \( \tilde{\xi}_i = \sum_{j=1}^{M} \rho_j p_c(j) \). We propose the following update mechanism to
determine the optimal $\rho_i$’s to find the optimal solution of (34), where $\rho_i^*$ and $\tilde{\xi}_i$ are updated as

$$\rho_i^* = \arg \min_{\rho_i \leq \rho_i^*} \left( \tilde{\xi}_i - \rho_i p_r(i) \right)^+,$$

where $y^+ = \max\{y, 0\}$ and $\rho_i^* = 1$. Since $\xi^{-1} p_{\text{cov}}(T, \lambda_t \xi, \alpha)$ is decreasing in $\xi$, and $\sum_{i=1}^{M} p_r(i) \rho_i p_c(i)$ is an increasing function of $\xi$, (34) has an optimal solution. Now, consider the following formulation:

$$\max_{\rho_i} \sum_{i=1}^{M} p_r(i) \rho_i p_c(i)$$

subject to $\xi = \sum_{i=1}^{M} \rho_i p_c(i)$, where $0 \leq \rho_i \leq 1$. Through a duality argument [35], it is trivial to show that the solution of (35) is $\rho_i = \xi_i, i \in \{1, \ldots, M\}$. Using which the formulation (34) is upper bounded by $\text{DSR}_t \leq \max_{\xi} p_{\text{cov}}(T, \lambda_t \xi, \alpha) \sum_{i=1}^{M} p_r(i) p_c(i)$, and the upper bound is achieved for $\xi = 1$.

### J. Proof of Theorem 2

The probability of coverage of a typical randomly located user $j$ is given by

$$P_{\text{cov}}(T, \lambda_j, \alpha) = \mathbb{P}(\text{SINR}_j > T) = \int_{r > 0} f_{R_j}(r) \mathbb{P}[h > Tr^\alpha (\sigma^2 + I_{r(j)})] \, dr,$$

where $\mathbb{P}[h > Tr^\alpha (\sigma^2 + I_{r(j)})] = e^{-\mu Tr^\alpha \sigma^2} \mathcal{L}_{I_{r(j)}}(\mu Tr^\alpha) \overset{(a)}{=} e^{-\mu Tr^\alpha \sigma^2} \mathcal{L}_{I_{r(j)}}^c(\mu Tr^\alpha) \mathcal{L}_{I_{r(j)}}(\mu Tr^\alpha)$, where $(a)$ follows from independence of $I_{r(j)}^c$ and $I_{r(j)}^u$. The Laplace transform of $I_{r(j)}^c$ is given as follows:

$$\mathcal{L}_{I_{r(j)}^c}(s) = \mathbb{E} \left[ \prod_{l \in \Phi_{r(j)}^e} \exp \left( -sgl R_{l(t)} - \alpha \right) \right] \overset{(a)}{=} e^{-2\pi \lambda_j \int_{0}^{\infty} (1 - \mathbb{E}[\exp (-sgu^{-\alpha})]) u \, du},$$

where $(a)$ follows from the iid distribution of $g_l$, and its independence from $\Phi$, and the probability generating functional (PGFL) of the PPP [26]. Similarly, the Laplace transform of $I_{r(j)}^u$ is

$$\mathcal{L}_{I_{r(j)}^u}(s) = \mathbb{E} \left[ \prod_{l \in \Phi_{r(j)}^u} \mathbb{E}[\exp \left( -sgl R_{l(t)}^{-\alpha} \right)] \right] = e^{-2\pi (\lambda_t - \lambda_j) \int_{0}^{\infty} (1 - \mathbb{E}[\exp (-sgu^{-\alpha})]) u \, du}.$$

If the fading is Rayleigh with parameter $\mu$, then the Laplace transforms of $I_{r(j)}^c$ and $I_{r(j)}^u$ equal

$$\mathcal{L}_{I_{r(j)}^c}(s) = e^{-2\pi \lambda_j \int_{0}^{\infty} \left( \frac{1}{1 + u^{-\alpha}} \right) u \, du}, \quad \mathcal{L}_{I_{r(j)}^u}(s) = e^{-2\pi (\lambda_t - \lambda_j) \int_{0}^{\infty} \left( \frac{1}{1 + u^{-\alpha}} \right) u \, du}.$$
Thus, the probability of coverage of a typical randomly located user is

\[
P_{\text{cov}}(T, \lambda_j, \alpha) = \int_{r>0} e^{-\lambda_j r^2} e^{-\mu T r^\alpha} \left( \mu T r^\alpha \right)^2 2\pi \lambda_j r \, dr, \tag{38}
\]

which is obtained using (36), (37), a change of variables \( v = s - \frac{2}{\pi} u^2 \), and letting \( s = \mu T r^\alpha \), and the final result is obtained by a change of variables \( v = r^2 \) and the definitions of \( \rho_1(T, \alpha) \) and \( \rho_2(T, \alpha) \).

K. Proof of Corollary 3

For Rayleigh fading with \( \mu = 1 \), using (36) and (38), the coverage of a typical user for \( \alpha = 4 \) is

\[
P_{\text{cov}}(T, \lambda_j, 4) = \int_{r>0} e^{-\lambda_j r^2} e^{-T r^4} \left( \mu T r^4 \right)^2 2\pi \lambda_j r \, dr
= \int_{r>0} e^{-\left( \gamma_j + \frac{2}{\pi} \lambda_\gamma T - \lambda_j \sqrt{T} \tan^{-1}\left( \frac{1}{\sqrt{T}} \right) \right) r^2} e^{-T r^4} 2\pi \lambda_j r \, dr,
\]

where the final result stems from \( \lambda_j = \gamma_1 \sum_{i \in F_{\gamma}(j)} p_\gamma(i) = \lambda_t p_j \), and a change of variables \( u = r^2 \), and the identity \( \int_0^\infty e^{-ax} e^{-bx^2} \, dx = \sqrt{\pi} b \exp \left( \frac{a^2}{4b} \right) Q \left( \frac{a}{\sqrt{2b}} \right) \), and letting \( H(T, \lambda_t, p_j) = \left( \frac{p_j}{\sqrt{T}} - p_j \tan^{-1}\left( \frac{1}{\sqrt{T}} \right) \right) + \frac{\pi \lambda_t}{\sqrt{2\pi}} \). The relation \( P_{\text{cov}}(T, \lambda_j, 4) \) depends on the receiver state \( j \) through \( \lambda_j = \lambda_t p_j \).

REFERENCES

[1] D. Malak and M. Al-Shalash, “Optimal caching for Device-to-Device content distribution in 5G networks,” in Proc., IEEE Globecom Workshops, Austin, TX, Dec. 2014.
[2] ——, “Device-to-Device content distribution: Optimal caching strategies and performance bounds,” in Proc., IEEE ICC Workshops, London, UK, Jun. 2015.
[3] “Cisco visual networking index: Global mobile data traffic forecast update, 2014-2019,” white paper, Cisco, Feb. 2015.
[4] H. Sarkissian, “The business case for caching in 4G LTE networks,” 2012.
[5] X. Lin, J. G. Andrews, A. Ghosh, and R. Ratasuk, “An overview of 3GPP device-to-device proximity services,” IEEE Commun. Mag., vol. 52, no. 4, pp. 40–48, Apr. 2014.
[6] X. Wang, M. Chen, T. Taleb, A. Ksentini, and V. C. M. Leung, “Cache in the air: Exploiting content caching and delivery techniques for 5G systems,” IEEE Commun. Mag., pp. 131–139, Feb. 2014.
[7] K. Shanmugam, N. Golrezaei, A. G. Dimakis, A. F. Molisch, and G. Caire, “FemtoCaching: Wireless content delivery through distributed caching helpers,” IEEE Trans. Information Theory, vol. 59, no. 12, pp. 8402–8413, Dec. 2013.
[8] X. Lin, J. G. Andrews, and A. Ghosh, “Spectrum sharing for Device-to-Device communication in cellular networks,” IEEE Trans. Wireless Comm., vol. 13, no. 12, pp. 6727–6740, Dec. 2014.
[9] X. Wu, S. Tavildar, S. Shakkottai, T. Richardson, R. L. J. Li, and A. Jovicic, “FlashLinQ: A synchronous distributed scheduler for Peer-to-Peer ad hoc networks,” IEEE/ACM Trans. Netw., vol. 21, no. 4, pp. 1215–1228, 2013.
[10] N. Naderializadeh and A. S. Avestimehr, “ITLinQ: A new approach for spectrum sharing in Device-to-Device communication systems,” IEEE Journ. on Sel. Areas in Comm., vol. 32, no. 6, pp. 1139–1151, 2014.
[11] N. Naderializadeh, D. T. Kao, and A. S. Avestimehr, “How to utilize caching to improve spectral efficiency in Device-to-Device wireless networks,” in Proc., Annual Allerton Conference Allerton House, UIUC, Illinois, USA, October 1 - 3 2014.
[12] M. Ji, G. Caire, and A. F. Molisch, “The throughput-outage tradeoff of wireless one-hop caching networks,” *IEEE Trans. Inf. Theory*, vol. 61, no. 12, pp. 6833–6859, 2015.

[13] S. Gitzenis, G. Paschos, and L. Tassiulas, “Asymptotic laws for joint content replication and delivery in wireless networks,” *IEEE Trans. Inf. Theory*, vol. 59, no. 5, pp. 2760–2776, 2013.

[14] S. W. Jeon, S. N. Hong, M. Ji, and G. Caire, “Caching in wireless multihop device-to-device networks,” in *Proc., IEEE ICC*, London, UK, Jun. 2015, pp. 6732–6737.

[15] R. L. Cruz, “Ad-hoc networks at global scale,” in *Proc., IEEE ICNC*, Jan. 2013, pp. 813–817.

[16] M. A. Maddah-Ali and U. Niesen, “Fundamental limits of caching,” *IEEE Trans. Inf. Theory*, vol. 60, no. 5, pp. 2856–2867, 2014.

[17] S. Golrezaei, M. Alexandros G. Dimakis, and A. F. Molisch, “Device-to-Device collaboration through distributed storage,” in *Proc., IEEE GlobeCom*, Dec. 2012, pp. 2397–2402.

[18] M. Ji, G. Caire, and A. F. Molisch, “Fundamental limits of caching in wireless D2D networks,” *IEEE Trans. Inf. Theory*, vol. 62, no. 2, pp. 849–869, Feb. 2016.

[19] N. Golrezaei, A. G. Dimakis, and A. F. Molisch, “Scaling behavior for Device-to-Device communications with distributed caching,” *IEEE Trans. Inf. Theory*, vol. 60, no. 7, pp. 4286–4298, Jul. 2014.

[20] N. Golrezaei, P. Mansourifard, A. F. Molisch, and A. G. Dimakis, “Base-station assisted device-to-device communications for high-throughput wireless video networks,” *IEEE Trans. Wireless Comm.*, vol. 13, no. 7, pp. 3665–3676, 2014.

[21] P. Gupta and P. R. Kumar, “The capacity of wireless networks,” *IEEE Trans. Inf. Theory*, vol. 46, no. 2, pp. 388–404, Mar. 2000.

[22] M. Ji, G. Caire, and A. F. Molisch, “Wireless Device-to-Device caching networks: Basic principles and system performance,” *IEEE Journ. on Sel. Areas in Comm.*, vol. 34, no. 1, pp. 176–189, 2016.

[23] E. Altman, K. Avrachenkov, and J. Goseling, “Coding for caches in the plane,” *arXiv preprint arXiv:1309.0604*, 2013.

[24] B. Blaszczyszyn and A. Giovanidis, “Optimal geographic caching in cellular networks,” in *Proc., IEEE ICC*, London, UK, 2015.

[25] H. Keeler, B. Blaszczyszyn, and M. Karray, “SINR-based k-coverage probability in cellular networks with arbitrary shadowing,” in *Proc., IEEE ISIT*, Istanbul, July 2013, pp. 1167 – 1171.

[26] D. Stoyan, W. Kendall, and J. Mecke, *Stochastic Geometry and Its Applications*, 2nd ed. John Wiley and Sons, 1996.

[27] J. G. Andrews, F. Baccelli, and R. K. Ganti, “A tractable approach to coverage and rate in cellular networks,” *IEEE Trans. Comm.*, vol. 59, no. 11, pp. 3122–3134, Nov. 2011.

[28] S. Vanichpun, “Comparing strength of locality of reference: Popularity, temporal correlations, and some folk theorems for the miss rates and outputs of caches,” Ph.D. dissertation, University of Maryland, 2005.

[29] A. M. Makowski and S. Vanichpun, *Comparing Locality of Reference- Some Folk Theorems for the Miss Rates and the Output of Caches*. Springer US, 2005, ch. 13, pp. 333–365.

[30] L. Pietronero, E. Tosatti, V. Tosatti, and A. Vespignani, “Explaining the uneven distribution of numbers in nature: The laws of Benford and Zipf,” *Physica A: Statistical Mechanics and its Applications*, vol. 293, no. 1-2, pp. 297–304, Apr. 2001.

[31] F. Baccelli and B. Blaszczyszyn, *Stochastic Geometry and Wireless Networks, Volume I — Theory*, ser. Foundations and Trends in Networking. Now Publishers, 2009, vol. 3, no. 3–4.

[32] M. Carter, *Foundations of mathematical economics*. MIT Press, 2001.

[33] D. P. Bertsekas, *Nonlinear programming*. Cambridge, MA: Athena Scientific, 1999.

[34] G. K. Karagiannidis and A. S. Lioumpas, “An improved approximation for the Gaussian Q-function,” *IEEE Commun. Lett.*, vol. 11, no. 8, pp. 644–646, 2007.

[35] S. Boyd and L. Vandenberghe, *Convex optimization*. Cambridge university press, 2009.