Infinite-dimensional analyticity in quantum physics

Paul E. Lammert

Department of Physics, 104B Davey Lab
Pennsylvania State University
University Park, PA 16802-6300

(Dated: Aug. 22, 2021)

A study is made, of families of Hamiltonians parameterized over open subsets of Banach spaces in a way which renders many interesting properties of eigenstates and thermal states analytic functions of the parameter. Examples of such properties are charge/current densities. The apparatus can be considered a generalization of Kato’s theory of analytic families of type B insofar as the parameterizing spaces are infinite dimensional. It is based on the general theory of holomorphy in Banach spaces and an identification of suitable classes of sesquilinear forms with operator spaces associated with Hilbert riggings. The conditions of lower-boundedness and reality appropriate to proper Hamiltonians is thus relaxed to sectoriality, so that holomorphy can be used. Convenient criteria are given to show that a parameterization \( x \mapsto h_x \) of sesquilinear forms is of the required sort (regular sectorial families). The key maps \( \mathcal{R}(\zeta, x) = (\zeta - H_x)^{-1} \) and \( \mathcal{E}(\beta, x) = e^{-\beta H_x} \), where \( H_x \) is the closed sectorial operator associated to \( h_x \), are shown to be analytic. These mediate analyticity of the variety of state properties mentioned above. A detailed study is made of nonrelativistic quantum mechanical Hamiltonians parameterized by scalar- and vector-potential fields and two-body interactions.

CONTENTS

1. Introduction
   A. Motivation
   B. An operator prototype
   C. Sketch of the theory
   D. Organization of the paper
   E. Conventions and notations

2. Analyticity in the Banach space setting
   A. Differential calculus
   1. Derivatives
   2. Taylor series and analyticity
   B. Holomorphy
   1. Complex linearity and conjugate linearity
   2. Reduction to 1D domain or range

3. Hilbert riggings
   A. Example: kinetic energy
   B. General construction

4. Families of forms and operators
   A. Sesquilinear forms
   B. Completion and closure
   C. From s-forms to operators
   D. Holomorphy of the \( \mathcal{R} \)-map
   E. Series expansion
   F. Holomorphic families
   G. Operator bounded families
   H. Numerical range and spectrum

5. Magnetic Schrödinger forms
   A. Nonrelativistic \( N \)-particle systems
   B. Scalar potential

1. Bounded potentials
2. Unbounded potentials
3. Modulating the confining potential
4. Removing redundancy

C. Interaction
D. Vector potential
1. bounded \( A \)
2. Sobolev multipliers
3. Removing redundancy again
E. Putting it all together

6. Low-energy Hamiltonians & eigenstate properties
   A. Riesz-Dunford-Taylor integrals
   B. Low-energy Hamiltonians
   C. Schatten classes
   D. Finite rank
   E. Eigenstate perturbation
   1. charge/current density

7. Semigroups and statistical operators
   A. Operator semigroups
   B. The exponential map \( \mathcal{E} \)
   C. Statistical operator and free energy
   D. Thermal expectations
   1. charge/current density

8. Summary

a) Electronic mail: lammert@psu.edu
1. INTRODUCTION

A. Motivation

The mathematical concept of analyticity is ubiquitous in physics. Here is a short list of examples. It is in the background whenever we approximate a function by a few terms of its Taylor series. The question of whether perturbation series converge or not is of interest in many contexts. Kramers-Krönig relations are a manifestation of analyticity in complex half-planes. In thermodynamics, phase transitions are identified with the locus of points in a phase diagram at which free energy fails to be analytic. And, for nonzero temperature: statistical operator (i.e., the thermal state), free energy, thermal expectations, susceptibilities, and so on. The framework developed here can be used to address such analyticity questions with relative ease, as is demonstrated explicitly. The framework is flexible, powerful, and general due to treating Hamiltonians initially as sesquilinear forms, with a relaxation of the physically-grounded requirements of reality and lower-boundedness to sectoriality so that holomorphy (C-differentiability) can be invoked, and complex analysis methods brought to bear.

Kato’s analytic perturbation theory for type B families is concerned with similar questions, but only for families with parameterization domains in the complex numbers C. The move to infinite-dimensional parameterization domains (Banach spaces, specifically) not only increases flexibility, but triggers a conceptual rearrangement, leading to a rephrasing of everything in terms of compositions of holomorphic maps between Banach spaces. It therefore becomes imperative to repackage appropriate classes of unbounded sesquilinear forms as Banach spaces. Section B develops that key part of the apparatus.

B. An operator prototype

It is a familiar and useful fact that the resolvent \( R(\zeta, H) = (H - \zeta)^{-1} \) of operator \( H \) is a holomorphic function of the spectral parameter \( \zeta \). The extension to a holomorphic dependence on \( H \) is worth looking at as a prototype for the theory to be developed.

Definition 1.1. Given: a closed, densely-defined, operator \( T \) on Banach space \( \mathcal{X} \) [denoted \( T \in \mathcal{L}_d(\mathcal{X}) \)]. An operator \( A \) is \( T \)-bounded if dom \( A \supseteq \text{dom } T \) and there are \( a,b \) such that

\[
\forall x \in \text{dom } T, \|Ax\| \leq a\|x\| + b\|Tx\|. \tag{1}
\]

By increasing \( a \), it may be possible to decrease \( b \). The infimum of all \( b \)'s that work is the \( T \)-bound of \( A \).

If \( T \) is closed and invertible, dom \( T \) can be turned into a Banach space with the norm \( \|x\|_T = \|Tx\|_\mathcal{X} \); we understand it as such in the following. The following Lemma brings the notion of analyticity to the surface. (A proof is provided at the end of the subsection, which should probably be skipped until it is called upon.)

Lemma 1.2. Suppose \( T \) is closed with range \( \mathcal{X} \), and \( A \) is \( T \)-bounded. If \( \text{rng}(T + A) = \mathcal{X} \), then

\[
(T + A)^{-1} = T^{-1}(1 + AT^{-1})^{-1} \in \mathcal{L}(\mathcal{X}). \tag{2}
\]

This holds in particular when \( \|AT^{-1}\| < 1 \), which implies convergence of the Neumann series

\[
(T + A)^{-1} = T^{-1}\sum_{n=0}^{\infty}(-AT^{-1})^n. \tag{3}
\]
One would like to hold up series (3) as the demonstration that $(T + A)^{-1}$ is analytic at $A = 0$. That the terms are not simply multiples of powers of $A$, however, shows the need for at least the rudiments of a more general theory of analyticity, a theory which will be reviewed in Section 2. Similarly, a claim that the series converges uniformly on some ball about the origin raises the question of domain and codomain of the map. The codomain is clearly $L(H)$. The domain could be taken the same, but that would be far too timid. Instead, consider $dom\, T$ as a Banach space with norm $\|x\|_T = \|Tx\|$, making $T$ an isomorphism. Then, $A \mapsto (T + A)^{-1}$ can be considered a map from $L(dom\, T; \mathcal{X})$ to $L(\mathcal{X})$. Indeed, the norm of $A$ as an element of $L(dom\, T; \mathcal{X})$ is precisely $\|AT^{-1}\|$, so the series (3) is uniformly convergent on any center-zero ball of radius less than 1.

It might seem very difficult to extend this to families of operators having diverging domains, but we shall find it possible by working with Hamiltonians in the guise not of operators, but of sesquilinear forms. Recall that a sesquilinear form ($s$-form) in Hilbert space $\mathcal{H}$ is a complex-valued function $h[\phi, \psi]$, linear in $\phi$ and conjugate-linear in $\psi$ which range over some subspace of $\mathcal{H}$. Motivations for working with sesquilinear forms are, first, the increased strength. That is needed for DFT applications, for example, where potentials in first, the increased strength. That is needed for DFT applications, for example, where potentials in

$$\begin{align*}
\int (\nabla - iA)\psi^2 \, dx, \\
\psi_A[\psi, \psi] = \int (\nabla + iA)\bar{\phi} \cdot (\nabla - iA)\psi \, dx,
\end{align*}$$

The energy of the state with wavefunction $\psi$ is well-defined as a real-valued and lower-bounded quadratic form defined on a dense subspace of $L^2(\mathbb{R}^3)$. Under a technical condition, $h_A$ is naturally associated to a corresponding self-adjoint operator $H_A$. This well-known theory is recovered as part of the development in Section 4. The resolvent $R(\zeta, H_A) = (H_A - \zeta)^{-1}$ is $\mathbb{C}$-analytic in $\zeta$. Is it also analytic in $A$? Merely posing the question shows that we should take $A$ in some complex space, and therefore allow the field $A(x)$ to be complex-valued. Thus, we generalize (5) to the sesquilinear form ($s$-form)

$$h_A[\phi, \psi] = \int (\nabla + iA)\bar{\phi} \cdot (\nabla - iA)\psi \, dx,$$
part of such an equivalence class $C$ in the right way, and that turns out to be surprisingly easy.

The relation $\preceq$ of relative boundedness induces equivalence classes of $s$-forms on $\mathcal{K}$. Suppose $C$ is one such (technically: containing some closable sectorial form). When $A$ is allowed to be complex, $hA[\psi]$ is no longer real, but appropriate restrictions ensure that it takes values in a right-facing wedge in $C$ as $\psi$ varies among unit vectors in its domain. This property is sectoriality, and the useful generalization of lower-bounded self-adjointness which allows complex analytic methods to be brought into play. The class $C_\preceq$ of all $s$-forms bounded relative to $C$ has a natural Banach space structure, up to norm equivalence. In fact, it can be identified with $\mathcal{L}(\mathcal{K}_+;\mathcal{K}_-)$, where $\mathcal{K}_+ \subset \mathcal{K} \subset \mathcal{K}_-$ is a Hilbert rigging of the ambient Hilbert space $\mathcal{K}$. Such Hilbert riggings play a very important role in our methodology, and are reviewed in Section 3.1.

The sectorial forms in $C$, denoted $C^C$, comprise an open subset of $C_\preceq$, and are the ones of real interest. They induce closed sectorial operators, $H$ corresponding to $h$. A central result is that $h \mapsto H^{-1}$ is holomorphic from an open subset of $C^C$ into $\mathcal{L}(\mathcal{K})$. Note that the same cannot be said of $h \mapsto H$, since the operators have differing domains, hence it is not even clear in what Banach space we can locate them all. This result can be unfolded if one has such a family, that is, what other qualities inherit the holomorphy. Section 4 is concerned with identifying specific holomorphic families of nonrelativistic Hamiltonians — magnetic Schrödinger forms parameterized by both scalar and vector potentials. Sections 5 and 7 on the other hand, are concerned with what can be done if one has such a family, that is, the other qualities inherit the holomorphy. Section 5 studies low-energy Hamiltonians and eigenstate perturbation. Holomorphy of the energy and charge/current densities for isolated nondegenerate eigenstates is derived here, among other things. Section 7 is concerned with holomorphy of the $E$-map and its consequences. Under appropriate conditions, this yields holomorphy of the nonzero-temperature statistical operator in trace-norm, as well of free energy and thermal expectations. Special attention is again given to charge/current density. Section 8 gives a selective summary.

E. Conventions and notations

For convenient reference, some conventions will be listed here. $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ denote generic Banach spaces, $\mathcal{U}$ an open subset of a Banach space, $\mathcal{H}$ denotes a Hilbert space. $\mathcal{L}(\mathcal{X};\mathcal{Y})$ is the space of bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$, with the usual operator norm, $\mathcal{L}(\mathcal{H}) = \mathcal{L}(\mathcal{H};\mathcal{H})$, $\mathcal{L}^1(\mathcal{H})$ and $\mathcal{L}^2(\mathcal{H})$ denote the spaces of trace-class and Hilbert-Schmidt operators, respectively, and $\mathcal{L}_{cl}(\mathcal{X})$ the set of densely-defined closed operators in $\mathcal{X}$, and $\mathcal{L}_{iso}(\mathcal{X};\mathcal{Y})$ that of invertible bounded operators (Banach isomorphisms). Product spaces, e.g., $\mathcal{X} \times \mathcal{Y}$ are usually denoted that way rather than as $\mathcal{X} \oplus \mathcal{Y}$ because the product notion matches the informal interpretation better. $R(z,A) = (A-z)^{-1}$ is the resolvent operator (the notation $\mathcal{R}$ will be overloaded later), $\text{res}A$ the resolvent set, and $\sigma A$ the spectrum of $A$. Topological closure is generally denoted by cl, instead of an overbar. Barred arrows specify functions, while plain arrows display the domain and codomain, e.g., $A \mapsto e^{A}$; $\mathcal{L}(\mathcal{H}) \mapsto \mathcal{L}(\mathcal{H})$ is exponentiation on bounded operators. $(x_n)_{n \in \mathbb{N}}$ or $(x_n)_n$, or even just $(x_n)$ if it is unambiguous, denotes a sequence. Additional notations will be defined as need arises. Definitions and all theorem-like environments share a common counter; the
numbering is merely a navigational aid.

2. ANALYTICITY IN THE BANACH SPACE SETTING

This section reviews the necessary theory of differential calculus and holomorphy in Banach spaces. The material on differential calculus reviewed in Section 2.1 is quite standard and can be found in many places. The theory of holomorphy in Banach spaces discussed in Section 2.2 is much less so. In depth treatments are in monographs of Mujica and Chae. Thm. 2.1 is the reason this section is here at all, and the other results are mostly concerned with making the demonstration of holomorphy as easy as possible.

A. Differential calculus

In this subsection, the base space of Banach spaces \( \mathcal{X}, \mathcal{Y}, \ldots \) may be either \( \mathbb{R} \) or \( \mathbb{C} \). \( \mathcal{U} \) is an open subset of a Banach space (usually \( \mathcal{X} \)).

1. Derivatives

If \( f: \mathcal{U} \rightarrow \mathcal{Y} \) admits a linear approximation near \( a \in \mathcal{U} \) as

\[
f(a + x) = f(a) + Df(a)x + o(\|x\|),
\]

for some continuous linear map \( Df(a): \mathcal{X} \rightarrow \mathcal{Y} \), i.e. \( Df(a) \in \mathcal{L}(\mathcal{X}; \mathcal{Y}) \), then \( Df(a) \) is said to be the Fréchet differential (or derivative) of \( f \) at \( a \).

We are not interested in differentiability at isolated points only, but throughout \( \mathcal{U} \). \( f \) is \( C^1 \) on \( \mathcal{U} \) if \( Df \) is everywhere defined on \( \mathcal{U} \) and continuous. In that case, \( Df: \mathcal{U} \rightarrow \mathcal{L}(\mathcal{X}; \mathcal{Y}) \) is itself a continuous map into a Banach space (with the usual operator norm) and we may ask about differentiability of \( Df \).

If the differential of \( Df \) at \( a \), denoted \( D^2 f(a) \), exists it belongs to \( \mathcal{L}(\mathcal{X} \times \cdots \times \mathcal{X}; \mathcal{Y}) \), by definition. Thus, for \( x, x' \in \mathcal{X} \), \( D^2 f(a)(x)(x') \in \mathcal{L}(\mathcal{X}; \mathcal{Y}) \) (dropping some parentheses and writing simply ' \( D^2 f(a)x' \) is a good idea) and \( D^2 f(a)xx' \in \mathcal{Y} \). Elements of \( \mathcal{L}(\mathcal{X}; \mathcal{L}(\mathcal{X}; \mathcal{Y})) \) are actually \emph{bilinear}, that is, linear in each argument with the other held fixed. Moreover, \( D^2 f(a) \) is symmetric, that is, \( D^2 f(a)x x' = D^2 f(a)x' x \). This symmetry continues to higher orders, as long as differentiability holds, and provides good motivation to think primarily in terms of multilinear mappings rather than nested linear mappings. Eliding the distinction between a nested operator in \( \mathcal{L}(\mathcal{X}; \cdots \mathcal{L}(\mathcal{X}; \mathcal{Y}) \cdots) \), the \( n \)-th differential \( D^n f(a) \in \mathcal{L}(\mathcal{X} \times \cdots \times \mathcal{X}; \mathcal{Y}) \) is a continuous, symmetric, \( n \)-linear map from \( \mathcal{X} \times \cdots \times \mathcal{X} \) into \( \mathcal{Y} \).

2. Taylor series and analyticity

If \( f \) is continuously differentiable, then whenever the line segment from \( a \) to \( a + x \) is in \( \mathcal{U} \), \( f(a + x) = f(a) + \left( \int_0^1 Df(a + tx) \, dx \right) x \). Suspending the question of convergence, one deduces that the Taylor series expansion should be \( \sum_{n=0}^\infty \frac{1}{n!} D^n f(a) x \cdots x \). If, for every point \( a \in \mathcal{U} \), the Taylor series expansion of \( f \) converges to \( f \) uniformly and absolutely on a ball of some nonzero (\( a \)-dependent) radius, \( f \) is said to be \emph{analytic} on \( \mathcal{U} \). This is the favorable situation in which we are interested. The notion of analyticity, and the actual use of a convergent series expansion, is independent of the base field, but it follows from a \emph{prima facie} much weaker condition when the base field is \( \mathbb{C} \), as discussed next.

B. Holomorphy

This subsection is concerned with the equivalence between holomorphy and \( \mathbb{C} \)-analyticity (Thm. 2.1) and ways to make the demonstration of holomorphy easy (nearly everything else).

1. Complex linearity and conjugate linearity

A function of type \( \mathbb{R} \rightarrow \mathbb{R} \) can be differentiable to all orders without being analytic, whereas the situation is remarkably otherwise for those of type \( \mathbb{C} \rightarrow \mathbb{C} \). What is much less well-appreciated is that this contrast persists even in infinite-dimensional Banach spaces. Now we assume that the base field for \( \mathcal{X} \) and \( \mathcal{Y} \) is \( \mathbb{C} \). They can still be regarded as real vector spaces \( \mathcal{X}_\mathbb{R}, \mathcal{Y}_\mathbb{R} \) by restriction of scalars; in that case \( ix \) is considered not a scalar multiple of \( x \), but a vector in an entirely different “direction”. Suppose \( f: \mathcal{X}_\mathbb{R} \rightarrow \mathcal{Y}_\mathbb{R} \) is \( \mathbb{R} \)-differentiable at \( a \), temporarily denote the differential as \( D_{\mathbb{R}} f(a) \), and define \( D_{\mathbb{C}} f(a)(x) = \frac{1}{2} [D_{\mathbb{R}} f(a)(x) - i D_{\mathbb{R}} f(a)(ix)] \), \( \overline{D}_{\mathbb{C}} f(a)(x) = \frac{1}{2} [D_{\mathbb{R}} f(a)(x) + i D_{\mathbb{R}} f(a)(ix)] \). The condition for \( \mathbb{C} \)-differentiability is then \( D_{\mathbb{C}} f(a) = 0 \). This is the analog of the Cauchy-Riemann equation. The function \( f \) is said to be \emph{holomorphic} on \( \mathcal{U} \) if it is \( \mathbb{C} \)-differentiable there. Sometimes (e.g., Cha) \emph{holomorphic} is instead taken synonymous with \( \mathbb{C} \)-analyticity by definition, but it does not really matter as the following remarkable theorem shows.

\textbf{Theorem 2.1.} For complex Banach spaces \( \mathcal{X} \) and \( \mathcal{Y} \), \( \mathcal{U} \) open in \( \mathcal{X} \), the following properties of \( f: \mathcal{U} \rightarrow \mathcal{Y} \) are equivalent:

(a) holomorphy (\( \mathbb{C} \)-differentiability)
(b) infinite \( \mathbb{C} \)-differentiability
(c) \( \mathbb{C} \)-analyticity

\textbf{Proof.} See §§8 and 14 of Mujica, or Chae, Thm. 14.13.  \qed
Thus, even if we are ultimately interested only in \( \mathbb{R} \)-analyticity in some real subspace \( \mathcal{X} \) of a complex space \( \mathcal{X} \), it can be advantageous to work in \( \mathcal{X} \), establish holomorphy (a comparatively simple property) to get \( \mathbb{C} \)-analyticity in \( \mathcal{X} \) and thence \( \mathbb{R} \)-analyticity in \( \mathcal{X} \) by restriction. This is in the spirit of Jacques Hadamard’s famous dictum, “Le plus court chemin entre deux vérités dans le domaine réel passe par le domaine complexe”.

The following mostly simple permanence properties of holomorphy are important:

- **Composition:** Whenever \( f : \mathcal{X} \supset \mathcal{U} \to \mathcal{Y} \) and \( g : \mathcal{Y} \supset \mathcal{V} \to \mathcal{X} \) are holomorphic, so is \( g \circ f : \mathcal{U} \cap f^{-1}(\mathcal{V}) \to \mathcal{X} \).
- **Inversion:** \( L_{iso}(\mathcal{X}; \mathcal{Y}) \text{ inv} \to L_{iso}(\mathcal{Y}; \mathcal{X}) \) is holomorphic, where \( L_{iso}(\mathcal{X}; \mathcal{Y}) \) denotes the open set of invertible operators in \( L(\mathcal{X}; \mathcal{Y}) \).
- **Products:** If the domain of \( f \) is in a product space \( \mathcal{X}_1 \times \mathcal{X}_2 \), then \( f \) is holomorphic if it is jointly continuous and separately holomorphic.
- **Equivalent norms:** Holomorphy is stable under equivalent renorming of the domain or codomain space.
- **Differentiation:** If \( f \) is holomorphic, so is \( Df \).
- **Sequential limits:** Sequential convergence uniformly on compact sets preserves holomorphy. (See Prop. 2.11)

### 2. Reduction to 1D domain or range

The preceding shows that convergence of series expansions can be deduced from the mere existence of a differential. However, the latter is still complicated by expansions can be deduced from the mere existence of a differential.

**Definition 2.2.** \( f : \mathcal{U} \to \mathcal{Y} \) is \( G \)-holomorphic if for all \( x \in \mathcal{U}, y \in \mathcal{X}, \zeta \mapsto f(x + \zeta y) \) is an ordinary holomorphic function of \( \zeta \) on some neighborhood of zero in \( \mathbb{C} \).

The following fundamental theorem is named after Graves, Taylor, Hille and Zorn.

**Theorem 2.3** (GTHZ). For a map \( f : \mathcal{U} \to \mathcal{Y} \), the following property equivalence holds:

\[ \text{holomorphy} \Leftrightarrow \text{\( G \)-holomorphy and locally boundedness}. \]

**Proof.** See Mujica, Prop. 8.6 and Thm. 8.7; Chad, Thm. 14.9.

**Remark 2.4.** By definition, **locally bounded** means bounded on some neighborhood of each point of the domain. In a Banach space (or even a metric space), this is equivalent to boundedness on compact subsets of the domain.

Maps into spaces of linear operators will be very important in the following and are considered now. In fact, since any Banach space is isometrically embedded in its bidual, this is not really a special case.

**Definition 2.5.** \( f : \mathcal{U} \to \mathcal{Y} \) is **weakly holomorphic** if \( x \mapsto (\lambda, f(x)) \in \mathbb{C} \) is holomorphic for each \( \lambda \in \mathcal{Y}^* \). It is **densely weakly holomorphic** if the condition holds for a set of \( \lambda \)’s dense in \( \mathcal{Y}^* \).

\[ f : \mathcal{U} \to L(\mathcal{Y}; \mathcal{X}) \text{ is strongly holomorphic if } x \mapsto f(x) \in \mathcal{X} \text{ is holomorphic for each } y \in \mathcal{Y} \text{; and weak-operator holomorphic if } x \mapsto (\lambda, f(x)y) \in \mathbb{C} \text{ is holomorphic for each } y \in \mathcal{Y} \text{ and } \lambda \in \mathcal{X}^* \text{.} \]

As for weak holomorphy, these may be modified with dense to indicate that the set of \( \lambda \)'s [resp. pairs \( (y, \lambda) \)] in question is dense in \( \mathcal{Y} \) [resp. \( \mathcal{Y} \times \mathcal{X}^* \)].

The following Lemma is preparation for Propositions 2.9 and 2.10. Some obvious abbreviations (‘st.’ for ‘strong’, ‘loc. bdd.’ for ‘locally bounded’, ‘holo’ for ‘holomorphic’) are used.

**Lemma 2.6.** For \( f : \mathcal{U} \to L(\mathcal{Y}; \mathcal{X}) \), the following property implications hold.

(a) st. \( G \)-holo. \( \Rightarrow \) G-holo.
(b) loc. bdd. \( \in \mathcal{Y} \) dense st. holo. \( \Rightarrow \) st. holo.
(c) st. holo. \( \Rightarrow \) loc. bdd.
(d) loc. bdd. \( \in \mathcal{Y} \) dense st. G-holo. \( \Rightarrow \) holo.
(e) st. holo. \( \Rightarrow \) holo.

**Remark 2.7.** Parts (a), (b), and (c) are really just preparation for (d) and (e).

**Proof.** (a): Since \( G \)-holomorphy concerns affine planes independently, assume that \( \mathcal{U} \subseteq \mathbb{C} \) without loss. Assume (to be justified later) that \( f \) is also continuous. Then, for every \( y \in \mathcal{Y} \) and simple closed contour \( \Gamma \) in \( \mathcal{U} \),

\[ 0 = \oint_{\Gamma} f(\omega) \frac{d\omega}{2\pi i} = \left[ \oint_{\Gamma} f(\omega) \frac{d\omega}{2\pi i} \right] y. \]

Continuity of \( f \) is used here to justify taking \( y \) outside the integral. Since \( y \) ranges over \( \mathcal{Y} \), which is separating for \( L(\mathcal{Y}; \mathcal{X}) \), the integral in square brackets is zero. Finally, Morera’s theorem implies that \( f \) is holomorphic, because \( \Gamma \) is arbitrary.

To complete the proof of (a), we must show that \( f \) is continuous at \( \zeta \in \mathcal{U} \). Suppose not. Then there is a sequence \( \mathcal{U} \ni \zeta_n \to \zeta \) such that \( \|f(\zeta') - f(\zeta)\|/(\zeta_n - \zeta) \to \infty \), and by the uniform boundedness principle, \( y \in \mathcal{Y} \) such that \( (f(\zeta')y - f(\zeta)y)/(\zeta_n - \zeta) \) diverges. However, since \( f \) is strongly holomorphic, the limit of the latter is \( \left\| f(z)y \right\|_{z=\zeta} = \). Contradiction.

(b): We need to show that, for each \( y \in \mathcal{Y}, x \mapsto f(x)y \) (abbreviated here \( f(y) \)) is holomorphic near each point of \( \mathcal{U} \). By the dense strong holomorphy assumption, there is \( D \) dense in \( \mathcal{Y} \) such that, for every \( u \in D, f(y)u \) is holomorphic. Also, for any sequence \( D \ni y_n \to y \), local boundedness implies that the sequence \( f(y_n) \) converges...
not merely pointwise, but locally uniformly, to \( f(y) \), which is therefore holomorphic by Prop. 2.11.(c): Fix compact \( K \subset U \). For every \( y \in \mathcal{Y} \), \( f(y) \) is holomorphic by hypothesis, therefore continuous, therefore bounded on \( K \). The uniform boundedness principle secures boundedness \( f \) on \( K \).

(d): local boundedness & dense strong G-holomorphy implies strong G-holomorphy by the Gâteaux version of (b), which implies G-holomorphy by (a). Finally, holomorphy follows by Thm. 2.3.(e): We have G-holomorphy by (a), and local boundedness by (c). Again, conclude via Thm. 2.3.

### Proposition 2.8

For \( f: U \to \mathcal{Y}^* \), the following are equivalent:

(a) holomorphy
(b) weak-* holomorphy
(c) local boundedness \& dense weak-* G-holomorphy

**Proof.** This follows immediately from Lemma 2.6 for the case \( \mathcal{Y}^* \simeq C(L(\mathcal{Y}; \mathbb{C})) \), realizing that the adjective “strong” there specializes to “weak-*”.

### Proposition 2.9

For \( f: U \to \mathcal{Y} \), the following are equivalent:

(a) holomorphy
(b) weak holomorphy
(c) local boundedness \& dense weak G-holomorphy

Proof. \( \mathcal{Y} \) is isometrically imbedded in its bidual \( \mathcal{Y}^{**} \simeq C(L(\mathcal{Y}; \mathbb{C})) \). Now apply Prop. 2.8.

### Proposition 2.10

For \( f: U \to L(\mathcal{Y}; \mathcal{X}) \), the following are equivalent:

(a) holomorphy
(b) weak-operator holomorphy
(c) loc. bdd. \& dense weak-operator G-holomorphy

**Proof.** Use the same trick as in Prop. 2.9 to write \( f: U \to L(\mathcal{Y}; L(\mathcal{X}; \mathbb{C})) \), apply Lemma 2.6 directly, and then Prop. 2.9.

Although holomorphy for the case \( \mathcal{X} \equiv \mathcal{Y} \equiv \mathbb{C} \) is not usually discussed in terms of linear operators as here, we may note that it fits in perfectly. The operator \( Df(a) \) in that case can be construed simply as multiplication by a complex number, \( \partial f(a) \), so that \( a \mapsto Df(a) \) is identified with the complex function \( a \mapsto \partial f(a) \). Differentiation does not generate objects of a fundamentally different type in that case. For higher-dimensional Banach spaces, however, it does so, and part (b) of Thm. 2.1 thereby gains in importance. The \( D^n f \), as \( n \) varies, all have distinct codomains, yet they are all holomorphic if \( f \) is so.

We close this Section with a proof of the sequential permanence property mentioned earlier, which is also found as Prop. 9.13 of Mujica.

**Proposition 2.11.** If \( f_n: U \to \mathcal{Y} \) is a sequence of holomorphic mappings converging to \( f \) uniformly on compact subsets of \( U \), then \( f \) is holomorphic.

### Proof.

Use Thm. 2.3 (G-holomorphic and locally bounded \( \Leftrightarrow \) holomorphic). For any compact subset \( K \) of \( U \), the \( f_n \)’s are bounded, and converge uniformly to \( f \), hence \( f \) is bounded. By Prop. 2.9 G-holomorphy of \( f \) reduces to the case \( U \subset \mathcal{Y} = \mathbb{C} \), which is a well-known result of classical complex analysis.

### 3. HILBERT RIGGINGS

This section is also primarily background, although Prop. 3.2 is not standard and will play an important rôle. Section 3A is a concrete illustration of Hilbert rigging intended primarily for those unfamiliar with the idea. A Hilbert rigging of a Hilbert space \( \mathcal{H} \) is a sandwiching \( \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- \) by two other Hilbert spaces such that \( \mathcal{H}_- \) is the dual space of \( \mathcal{H}_+ \) with respect to the original inner product on \( \mathcal{H} \). They will be used through the identification of a family \( \mathcal{C}_S \) of \( S \)-forms in \( \mathcal{H} \) with \( L(\mathcal{H}_+; \mathcal{H}_-) \) for an appropriate \( \mathcal{H}_+ \). Prop. 3.2 concerns the identification of isomorphisms from \( \mathcal{H}_+ \) to \( \mathcal{H}_- \) with closed operators on \( \mathcal{H} \).

#### A. Example: kinetic energy

Before presenting the abstract construction of Hilbert rigging, we illustrate briefly with the concrete and pertinent example of kinetic energy. The reader unfamiliar with Hilbert riggings may find it helpful to keep this example in mind in Section 3B.

Thus, take \( \mathcal{H} \) to be \( L^2(\mathbb{R}^n) \); the inner product is

\[
\langle u|v \rangle = \int u(x)^* v(x) \, dx = \int \overline{u(p)} \overline{v(p)} \, dp.
\]

(9)

Fourier transform will be indicated (in this subsection only) by an over-tilde, as above.

Now, a sesquilinear form corresponding to kinetic energy is

\[
\langle \phi|\psi \rangle_+ := \langle \phi|\psi \rangle + \sum_{i=1}^n \langle \partial_i \phi|\partial_i \psi \rangle.
\]

(10)

To be precise, \( \|\psi\|_2^2 = \langle \psi|\psi \rangle_+ \) is the kinetic energy of vector state \( \psi \), up to the addition of \( \|\psi\|^2 \). The notation suggests, as indeed is the case, that this sesquilinear form is a legitimate inner product. Moreover, it corresponds to a Hilbert space \( \mathcal{H}_+ \) based on a dense subspace of \( \mathcal{H} \). That this is so is best seen in momentum space, a move which also alleviates the technical complication that we must be careful to *a priori* interpret the derivatives in (10) in a weak or distributional sense. The momentum space expression is

\[
\langle \phi|\psi \rangle_+ = \int \tilde{\phi}(p)^* \tilde{\psi}(p) (1 + |p|^2) dp.
\]

(11)
This clarifies both that there really is a subspace of \( \mathcal{H} \)
which is complete for the new inner product \( \langle \cdot | \cdot \rangle_+ \) and
why we included the term \( \langle \phi | \psi \rangle \) in \( \text{(11)} \).

Authorized by the Riesz-Fréchet theorem, we could identify \( \mathcal{H}_+ \) with its dual space as usual, associating
\( \phi \in \mathcal{H}_+ \) with the functional \( \psi \mapsto \langle \phi | \psi \rangle_+ \). However, we want to identify the dual with respect to \( \langle \cdot | \cdot \rangle_+ \),
but with respect to \( \langle \cdot | \cdot \rangle \). The momentum-space expression \( \text{(11)} \) makes clear how to do this: Define \( J \) by
\[
J\phi(p) = (1 + |p|^2)^{-1/2}\phi(p),
\]
so that \( \langle \phi | \psi \rangle_+ = \langle J\phi | \psi \rangle \), where the last represents some extension of the inner product on \( \mathcal{H} \). With the inner product
\[
\langle \phi | \psi \rangle_- = \int \overline{\phi(p)} \psi(p) \, dp,
\]
we get another Hilbert space \( \mathcal{H}_- \) such that
\( \mathcal{H}_+ \subset \mathcal{H}_- \subset \mathcal{H}_+ \), and \( J : \mathcal{H}_+ \to \mathcal{H}_- \) is unitary.
All three of these spaces consist of functions in momentum space, but elements of \( \mathcal{H}_+ \) are actually tempered distributions, in general. For instance, if \( n = 1 \), \( \mathcal{H}_- \) contains delta-functions. Now we can clarify the meaning of \( J\phi \): The map \( \mathcal{H}_+ \times \mathcal{H}_- \ni (\phi, \psi) \mapsto \langle \phi | \psi \rangle \) admits an extension by continuity to either \( \mathcal{H} \) in both factors (yielding the ordinary inner product), or to \( \mathcal{H}_- \) in one factor.

\section*{B. General construction}

We now review the abstract idea of a Hilbert rigging as summarized in the (not commutative!) diagram
\[
\begin{array}{ccc}
\mathcal{H}_+ & \xrightarrow{\iota_+} & \mathcal{H} \\
\downarrow J & & \downarrow J \\
\mathcal{H}_- & \xrightarrow{\iota_-} & \mathcal{H}
\end{array}
\]
\( \text{(13)} \)

Expositions of this technology can be found in \S II.2 of Simon\textsuperscript{19}, \S VIII.6 of Reed & Simon\textsuperscript{17}, Ch. 4 of de Oliveira\textsuperscript{18}, or \S 14.1 of Berezansky\textsuperscript{19}.

Start with a Hilbert space \( \mathcal{H} \) with inner produce \( \langle \cdot | \cdot \rangle \), and a dense subspace equipped with stronger inner product \( \langle \cdot | \cdot \rangle_+ \), which makes it into a Hilbert space \( \mathcal{H}_+ \), so that the inclusion of one underlying vector space \( \{ \mathcal{H}_+ \} \) into the other \( \{ \mathcal{H} \} \) induces a continuous injection \( \iota_+ : \mathcal{H}_+ \hookrightarrow \mathcal{H} \).

The adjoint of \( \iota_+ \), defined by
\[
\langle \iota_+^* u | \psi \rangle_+ = \langle u | \iota_+ \psi \rangle
\]
is also injective with dense image, since taking adjoints swaps those properties. Use \( \iota_+^* \) to define a new inner product on \( \{ \mathcal{H} \} \) via
\[
\langle u | v \rangle_- := \langle \iota_+^* u | \iota_+^* v \rangle_+,
\]
equipped with which it becomes the preHilbert space \( \{ \mathcal{H} \}_- \), with a completion denoted \( \mathcal{H}_- \). The inclusion of \( \{ \mathcal{H} \} \) into \( \mathcal{H}_- \) is \( \iota_0 \). By construction, \( \iota_+^* \) extends by continuity to a unitary mapping
\[
J^{-1} : \mathcal{H}_- \to \mathcal{H}_+.
\]
Thus, suppressing the injection \( \iota_+ \) of \( \mathcal{H}_+ \) into \( \mathcal{H} \), we may rewrite \( \text{(14)} \) as
\[
\langle \psi | \phi \rangle = \langle J^{-1} \psi | \phi \rangle_+.
\]
Furthermore, according to the preceding, the right-hand side extends by continuity to a continuous sesquilinear map on \( \mathcal{H}_- \times \mathcal{H}_+ \) with \( J^{-1} \mathcal{H}_+ = \mathcal{H}_+ \). Using \( \text{(17)} \), then to define an extension of the \( \mathcal{H} \) inner product \( \langle \cdot | \cdot \rangle \) to \( \mathcal{H}_- \times \mathcal{H}_+ \), we say that \( \mathcal{H}_- \) realizes the dual space of \( \mathcal{H}_+ \) relative to the original inner product.

The maps in \( \text{(13)} \) naturally induce two bounded linear mappings
\[
T \mapsto \iota_0 T \iota_+ : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}_+; \mathcal{H}_-),
\]
\[
T \mapsto \iota_+ \iota_0 T : \mathcal{L}(\mathcal{H}_-; \mathcal{H}_+) \to \mathcal{L}(\mathcal{H}).
\]
These will be useful below. More interesting, though, is a map that takes arbitrary \( \hat{T} \in \mathcal{L}(\mathcal{H}_+; \mathcal{H}_-) \) into a (generally unbounded) linear operator \( T \) on \( \mathcal{H} \) according to the following notational convention.

\textbf{Convention 3.1.} For \( \hat{T} \in \mathcal{L}(\mathcal{H}_+; \mathcal{H}_-) \), \( T \) denotes the restriction of \( \hat{T} \) to \( \text{dom} \, \hat{T} = \{ \psi \in \mathcal{H}_+ | \hat{T} \psi \in \mathcal{H} \} \), considered simply as an operator in \( \mathcal{H} \).

Not every linear operator in \( \mathcal{H} \) comes from an operator in \( \mathcal{L}(\mathcal{H}_+; \mathcal{H}_-) \) in this way, so one should not think of the hat as a map or transform of some sort; the map actually goes the other way.

The following Proposition can be viewed as an analog of Lemma \ref{lem:1.2}. It plays an important rôle in the theory.

\textbf{Proposition 3.2.} Given \( \hat{T} \in \mathcal{L}_{\text{iso}}(\mathcal{H}_+; \mathcal{H}_-) \),
\begin{enumerate}[a)]
\item \( \hat{T} \in \mathcal{L}(\mathcal{H}; \mathcal{H}_-) \), i.e., it is closed with dense domain.
\item \( \hat{T} \mapsto \hat{T} : \mathcal{L}_{\text{iso}}(\mathcal{H}_+; \mathcal{H}_-) \to \mathcal{L}(\mathcal{H}) \) is holomorphic.
\end{enumerate}

\textbf{Proof.} \( T^{-1} = \iota_+ \hat{T}^{-1} \iota_0 \) is bounded with domain \( \mathcal{H} \), hence closed, hence so is \( T \). Since \( \iota_0 \) and \( \iota_+ \) have dense image, \( \text{dom} \, T \) is dense in \( \mathcal{H} \). Finally, \( T \to T^{-1} \) is holomorphic since it is explicitly a composite of inversion and composition with a linear map, which are holomorphic operations.

\rule{\textwidth}{0.5pt}

\section*{4. FAMILIES OF FORMS AND OPERATORS}

This section is the technical core of the paper, preparing for applications in Sections \ref{sec:2} and \ref{sec:2} of Section \ref{sec:2}. 

recalls some basic ideas and definitions connected with sesquilinear forms (s-forms). That is preparation for consideration of sectorial forms parameterized over an open set $U$ of some Banach space. We want these parameterizations to be holomorphic, hence the generalization of the $\mathbb{R}$-centered notion of lower-bounded hermitian to sectorial. However, this can make sense only if relevant classes of s-forms have a Banach space structure themselves. Thm. 4.3 solves this problem, showing that the class $C_\phi^{\mathcal{C}}$ of s-forms relatively bounded with respect to an equivalence class $C$ of closable forms is naturally identified with $L(\mathcal{H}_a;\mathcal{H}_c)$, where $\mathcal{H}_c \subset \mathcal{H} \subset \mathcal{H}_a$ is an Hilbert rigging. Attention is then turned to the closed operators associated with the sectorial forms $C_\phi^{\mathcal{C}}$ in $\mathcal{C}$. Thm. 4.10 is the second main result, showing that the operator $h$ associated with $h \in C_\phi^{\mathcal{C}}$ is invertible iff $h$ viewed as an element of $L(\mathcal{H}_a;\mathcal{H}_c)$ is so. This gives holomorphy of the $\mathbb{R}$-map $(\zeta,h) \mapsto (H-\zeta)^{-1}$ on its natural domain in $\mathbb{C} \times C_\phi^{\mathcal{C}}$, which will be a basic tool in Sections VI.1,2 of Kato’s treatise.

A. Sesquilinear forms

This section consists mostly of definitions and notational conventions, as well as some notational conventions. A standard source for this material is §§ VI.1,2 of Kato’s treatise.

(1) A sesquilinear form (s-form henceforth) $h$ on complex vector space $\mathcal{H}$ is a map $\langle \phi, \psi \rangle \mapsto h[\phi, \psi] : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ linear in the second variable and conjugate-linear in the first. (Conjugate-linearity distinguishes these from bilinear forms.) Dirac-style notation will also be used: $\langle \phi | h | \psi \rangle \equiv h[\phi, \psi]$. To a sesquilinear form is associated a quadratic form $h[\psi] := h[\psi, \psi]$. The sesquilinear form can be recovered by polarization, so we will always use the term s-form for economy.

We write $|t|$ for the map $\psi \mapsto |t[\psi]|$. This is not an s-form, unless $|t| = t$.

(2) The adjoint of the s-form $h$ is $h^* [\phi, \psi] := h[\psi, \varphi]$. If $h = h^*$, $h$ is hermitian. $h$ is split into real and imaginary hermitian parts as $h = h^r + ih^i$ with $h^r = \frac{1}{2}(h + h^*)$, $h^i = \frac{1}{2i}(h - h^*)$. Hermitian quadratic forms are partially ordered similarly to self-adjoint operators: $h \leq h'$ means $\forall \psi \in \mathcal{H}, h[\psi] \leq h'[\psi]$. The inner product of the ambient Hilbert space provides the special s-form $1[\phi, \psi] := \langle \phi | \psi \rangle$.

(3) The numerical range of $h$ is the set
$$\text{num } h := \{ h[\psi] \mid \psi \in \text{dom } h, \| \psi \| = 1 \}. \quad (18)$$

The role of numerical range for s-forms somewhat analogous to that of spectrum for operators.

**Lemma 4.1.** \textbf{num } $h$ is a convex set.

**Proof.** We need to show that the line segment in $\mathbb{C}$ from $h[\psi]$ to $h[\varphi]$ is in $\text{num } h$, for unit vectors $\psi, \varphi \in \text{dom } h$. By suitable scaling and translation (replace $h$ by $a h + b I$), we may assume that $h[\psi] = 0$ and $h[\varphi] = 1$.

Define $\varphi(s) = (1-s)\psi + se^{i\theta}\varphi$ for $0 \leq s \leq 1$, with $\theta$ to be chosen.

$$h[\varphi(s)] = s^2 + s(1-s)\{ e^{i\theta}h[\psi, \varphi] + e^{-i\theta}h[\varphi, \psi] \}.$$ 

For suitable choice of $\theta$, the quantity in braces, thus $h[\varphi(s)]$ is real. $h[\varphi(s)]$ goes continuously from 0 to 1 as $s$ increases from 0 to 1, and therefore covers at least the segment $[0, 1]$. Since $\varphi(0)$ and $\varphi(1)$ are already normalized, normalizing $\varphi(s)$ will not alter this conclusion.

(4) An open sector is a right-facing wedge,
$$\text{Sectr } (c, \theta) := \{ c + re^{i\varphi} \mid r > 0, |\varphi| < \theta \},$$
in $\mathbb{C}$ for some vertex $c \in \mathbb{C}$ and half-angle $\theta < \pi/2$, and the closed sector $\text{Sectr } (c, \theta)$ is its closure. If sector $\Sigma$ is contained in the interior of $\Sigma'$ and $\Sigma'$ has a strictly larger half-angle than does $\Sigma$, then $\Sigma'$ is a dilation of $\Sigma$.

(5) $h$ is sectorial if its numerical range is contained in some sector, and any such will be said to be a sector for $h$. $\Sigma$ is an ample sector for $h$ if it is a dilation of some sector for $h$.

For any sectorial form $h$, $h^+$ will denote an arbitrary translate $m 1 + h^r$ such that $1 \leq h^r$. (Of course, the choice of $m$ can be standardized, but for our purposes there is no need.)

(6) Any operator $T$ in $\mathcal{H}$ naturally induces an s-form on $\text{dom } T$ by $\langle \phi, \psi \rangle \mapsto \langle \phi | T \psi \rangle$. The numerical range of $T$ is simply the numerical range of this s-form. Caution: a closed operator is called sectorial if its spectrum lies in a sector. This is not the same thing as the associated s-form being sectorial; the latter is a stronger condition. The relation between numerical range and spectrum is taken up in Section IV.1.

(7) The vector space of s-forms on a dense subspace $\mathcal{K}$ of $\mathcal{H}$ will be denoted $\text{SF}^{\mathcal{K}}(\mathcal{H})$. The set of sectorial s-forms on $\mathcal{K}$, denoted $\text{SF}_{\text{sect}}^{\mathcal{K}}(\mathcal{H})$, is a cone in $\text{SF}^{\mathcal{K}}(\mathcal{H})$. Generally, superscript ‘sect’ indicates the sectorial members of any class of s-forms.

(8) s-form $t$ is bounded relative to s-form $h$, denoted $t \leq h$, if dom $t \supseteq$ dom $h$ and there exist $a, b > 0$ such that $|t[\psi]| \leq a h[\psi] + b |h[\psi]|$ for every $\psi \in \text{dom } h$. The relation $\leq$ is reflexive and transitive.
If \( t \preceq h \) and \( h \preceq t \), then \( t \) and \( h \) are equivalent, denoted \( t \sim h \). Equivalent \( s \)-forms have the same domain.

Sectoriality of \( h \) can be expressed as: \( h^+ \) is bounded below and \( h^- \preceq h^+ \).

\( \preceq \) has a modest but useful calculus. For instance,

\[
B \in \mathcal{L}(\mathcal{H}) \implies B \preceq h, \\
c \in \mathbb{C} \setminus \{0\} \implies h \sim ch, \\
t \preceq h \implies h + t \preceq h, \\
t, h \text{ sectorial} \implies h \preceq h + t.
\]

(9) A sequence \( (\psi_n) \) in \( \text{dom } h \) is \( h\)-Cauchy if \( \langle (|h| + 1)(\psi_n - \psi_n) \rangle \rightarrow 0 \). It \( h \)-converges to \( \psi \) if \( \langle (|h| + 1)(\psi_n - \psi) \rangle \rightarrow 0 \). \( h \) is closed if all \( h \)-Cauchy sequences \( h \)-converge, closable if it has a closed extension.

Note that \( t \preceq h \) is equivalent to every \( h \)-Cauchy sequence is \( t \)-Cauchy.

### B. Completion and closure

The notion of Cauchy-ness in item (9) above is common across an equivalence (\( \sim \)) class of \( s \)-forms. This is an important fact, as it points the way to a "completion" of an entire equivalence class on a common domain. Therefore, we consider an equivalence class \( \mathcal{C} \) of \( s \)-forms defined on a dense subspace \( \mathcal{H} \subseteq \mathcal{H} \), containing a sectorial \( s \)-form \( h \), and therefore a hermitian \( s \)-form \( h^+ \geq 1 \). The class of all forms on \( \mathcal{H} \) which are bounded relative to those in \( \mathcal{C} \) is denoted \( \mathcal{C}_s \). The various sets of \( s \)-forms involved here are related as

\[
\mathcal{C}^d = \mathcal{C} \cap \text{SF}^d(\mathcal{H}) \subset \mathcal{C} \subset \mathcal{C}_s \subset \text{SF}(\mathcal{H}).
\]

The set \( \mathcal{C}^d \), the sectorial forms among \( \mathcal{C} \), is a cone, while \( \mathcal{C}_s \) is a vector space. It will emerge that it has a natural Banach space structure, up to norm-equivalence.

Two \( \mathcal{C} \)-Cauchy sequences \( (x_n) \) and \( (y_n) \) are equivalent if \( (x_n - y_n) \) is \( \mathcal{C} \)-Cauchy. This is written as \( x \sim y \), and the equivalence class of \( (x_n) \) is denoted \( x^- \). Vectors in \( \mathcal{H} \) are identified with the classes of constant sequences. The completion of \( \mathcal{C} \) is constructed on the vector space

\[
\mathcal{H} := \{ \sim \text{-classes of } \mathcal{C} \text{-Cauchy sequences in } \mathcal{H} \},
\]

and \( s \)-forms in \( \mathcal{C} \) are extended to \( \mathcal{H} \) according to

\[
\langle x^\sim | y^\sim \rangle := \lim_{n \to \infty} \langle x_n | y_n \rangle,
\]

as we now discuss.

The term completion suggests that we are dealing with the ordinary completion of a relevant preHilbert space structure on \( \mathcal{H} \). That is correct, and the inner product represented by any \( h^+ \geq 1 \) in \( \mathcal{C} \) will do. Let \( h \in \mathcal{C} \) be sectorial, \( h^+ \) as in item [5] above, and \( (\mathcal{H}, h^+) \) be the preHilbert space structure consisting of the space \( \mathcal{H} \) with inner product \( \langle \phi | \psi \rangle_h := \langle \phi | h^+ | \psi \rangle \). \( C \)-Cauchy is the same thing as \( (\mathcal{H}, h^+) \)-Cauchy in the usual sense, and the usual Hilbert space completion of \( (\mathcal{H}, h^+) \) can be viewed as being carried on \( \mathcal{H} \). In order to see that \( s \)-forms in \( \mathcal{C} \) can be extended to \( \mathcal{H} \), we need to know that they satisfy a Cauchy-Schwarz-like inequality.

**Lemma 4.2.** Suppose \( 1 \leq h^+ \) and \( t \preceq h \). Then, there is some \( M > 0 \) such that for every \( x, y \in \text{dom } h^+ \),

\[
| \langle x | t | y \rangle |^2 \leq M h^+ \| x \| h^+ \| y \|
\]

**Proof.** Only the case \( t \) hermitian, \( \| t \| \leq h^+ \), \( \| x | y \| \) real, \( h^+ \| x \| = h^+ \| y \| = 1 \) need be checked, since the general case follows by rescaling, multiplying \( x \) by a phase \( e^{it} \), and \( \| t | x | y \| \leq \| t^* | x | y \| + \| t | x | y \| \). Here is the verification of the special case:

\[
4 \| t | x | y \| = t | x + y | - t | x - y | \\
\leq \| t | x + y | \| + \| t | x - y | \| \\
\leq h^+ \| x + y \| + h^+ \| x - y \| = 4
\]

This lemma asserts that every \( t \in \mathcal{C}_s \) is a bounded sesquilinear form on the dense subspace \( \mathcal{H} \) of \( (\mathcal{H}, h^+) \), hence extends by continuity to the full space so as to satisfy (20). Each such extended \( s \)-form is represented by a bounded operator on \( (\mathcal{H}, h^+) \); for instance, \( h^+ \) itself is represented by the identity.

However, we also desire to identify \( \mathcal{H} \) with a subspace of the ambient Hilbert space \( \mathcal{H} \). Certainly, the inclusion \( \iota: \mathcal{H} \hookrightarrow \mathcal{H} \) extends by continuity to a bounded operator \( \tilde{\iota}: (\mathcal{H}, h^+) \rightarrow \mathcal{H} \). The only question is whether it is injective. It fails to be so only if there are two inequivalent \( C \)-Cauchy sequences in \( \mathcal{H} \), which converge as sequences in \( \mathcal{H} \) to the same vector. By linearity, only the case \( x_n \not\to 0 \) in \( \mathcal{H} \) need be considered: \( x \not\sim 0 \) fails if and only if \( t \langle x_n \rangle \not\to 0 \), for any \( t \in \mathcal{C} \). The test may therefore be performed for any member of the class \( \mathcal{C} \). If \( \iota \) is injective, we simply identify \( \mathcal{H} \) with its image, and thereby obtain a closed \( s \)-form in \( \mathcal{H} \) for every \( t \) in \( \mathcal{C} \).

In that case, \( \mathcal{C} \) is said to be closable. Although it must be checked, only closable classes are of interest to us, so closability is assumed henceforth.

### C. From \( s \)-forms to operators

With \( (\mathcal{H}, h^+) \) in the role of \( \mathcal{H}_+ \), we obtain a Hilbert rigging as in Section 3.B, from which we now take over various notations.

Any bounded sesquilinear form \( t \) on \( \mathcal{H}_+ \), (in particular, one in \( \mathcal{C} \)) is represented by a unique operator \( [t]_+ \in \mathcal{L}(\mathcal{H}_+) \) satisfying

\[
\langle \phi | [t]_+^* | \psi \rangle = \langle \phi | t | \psi \rangle
\]
for all $\phi, \psi \in \mathcal{H}$. Using the unitary isomorphism $J: \mathcal{H}_+ \to \mathcal{H}_-$, we get another “representation” $[t]_0^- := J[t]_0^+ \in \mathcal{L}(\mathcal{H}_+; \mathcal{H}_-)$ of $t$ satisfying
\[
\langle \phi | [t]_0^- \psi \rangle = \langle \phi | t \psi \rangle .
\] (23)

[Recall that the inner product on $\mathcal{H}$ extends to $\mathcal{H}_+ \times \mathcal{H}_+ \cup \mathcal{H}_+ \times \mathcal{H}_-$ as in (17).] The notation $[t]_0^\pm$ indicates the domain and range spaces in the subscript and superscript, respectively. It is unambiguous, but cumbersome. Fortunately, it will not be needed much. Restricting $[t]_0^+$ to those $\psi$ such that $[t]_0^+ \psi$ is in $\mathcal{H}_u$ yields yet a third operator, $[t]_0^\dagger \in \mathcal{L}_0(\mathcal{H})$. The domain and range of this operator are subspaces of $\mathcal{H}$.

D. Holomorphy of the $\mathcal{R}$-map

The operator guise of $t$ which is ultimately of most interest is $\lfloor t \rfloor_0^\pm$. However, the $\lfloor t \rfloor_0^-$ and $\lfloor t \rfloor_0^+$ forms have some especially nice properties, collectively:

**Theorem 4.3.** the map $t \mapsto [t]_0^- \in \mathcal{B}$ is a bijection between $\mathcal{C}_\#$ and $\mathcal{L}(\mathcal{H}_+; \mathcal{H}_-)$. The image $\mathcal{C}_\#$ of $\mathcal{C}^\delta$ under this map is an open subset of $\mathcal{L}_0(\mathcal{H}_+; \mathcal{H}_-) \cap \mathbb{R}_+$. 

**Proof of Thm. 4.3** part 1. We already know from Section 3.1 that there is a natural bijection between $\mathcal{C}_\#$ and $\mathcal{L}(\mathcal{H}_+; \mathcal{H}_-)$. By means of the unitary $J$, this is mapped into $\mathcal{L}(\mathcal{H}_+; \mathcal{H}_-)$. □

**Convention 4.4.** From now on, we consider $\mathcal{C}_\#$ to be equipped with this Banach space structure — up to norm-equivalence. This structure is independent of the choice of $h^+$ used to construct $\mathcal{H}_+$, and therefore intrinsic.

The proof of the second part of Thm. 4.3 relies on the following three Lemmas.

**Lemma 4.5.** If $t \in \mathcal{C}^\delta$, then $[t]_0^\pm = ([t]_0^*)^\dagger$ for all choices of $a$ and $b$. (N.B., the two $*$’s mean slightly different things.)

**Proof.** For $[t]_0^+$ and $[t]_0^-$, this is a simple matter of checking definitions. $[t]_0^\dagger$ involves some consideration of domains. $\psi \in \mathcal{H}_+$ is in dom $[t]_0^\dagger$ if $\phi \mapsto \langle \psi | t | \phi \rangle$ extends to a bounded functional on $\mathcal{H}_+$, whereas $\psi \in \mathcal{H}_-$ is in dom $([t]_0^*)^\dagger$ if $\phi \mapsto \langle \psi | t | \phi \rangle$ does so. Hence $[t]_0^\dagger \subseteq ([t]_0^*)^\dagger$ is clear. To see the opposite, recognize that these are both closed sectorial operators, and without loss we may suppose that they are both surjective. □

**Lemma 4.6.** If $t \in \mathcal{C}^\delta$ satisfies $1 \leq [t]_0^+ \in \mathcal{L}_0(\mathcal{H}_+; \mathcal{H}_-)$. 

**Proof.** For notational simplicity, set $T := [t]_0^-$. Also, we may assume that $t^*$ dominates $\| \cdot \|_2^+$ without loss, since some multiple does so.

$\ker T = \{0\}$ and $\text{rng } T$ closed: $\| \psi \|_2 \leq \| T \psi \|_2$ follows from $\| T \psi \|_2 \leq \| T \| \| \psi \|_2$. By Lemma 4.5 $\ker T^* = \{0\}$ follows just as $\ker T = \{0\}$ above.

$\text{rng } T = \mathcal{H}_-$; $\text{rng } T$ is both closed and dense in $\mathcal{H}_-$. □

**Lemma 4.7.** Suppose $\Sigma$ is an ample sector for $t$. Then, $\Sigma$ is an ample sector for all $s$ in some neighborhood of $t$.

**Proof.** Without loss of generality, we may add a constant to $t$ so that $1 \leq t$, and choose the form used to turn $\mathcal{H}$ into a Hilbert space such that $[t]_0^+ = 1 + iK$, with $K$ hermitian operator in $\mathcal{L}(\mathcal{H}_+)$. Then, with $[s]_0^+ = (1 + A) + i(K + B),

\begin{align*}
[t \psi] - s[\psi] &= \| (\psi(A + iB) \psi) \|_2^2
\end{align*}

\begin{align*}
&\leq (\| A \| + \| B \|) \| \psi \|_2^2
\leq (\| A \| + \| B \|) \| t \psi \|
\end{align*}

□

**Proof of Thm. 4.3** part 2. If $t \in \mathcal{C}^\delta$, then for some $m > 0$, $t + m1$ satisfies the hypotheses of Lemma 4.6. It follows that $[t]_0^+ \in \mathcal{L}_0(\mathcal{H}_+; \mathcal{H}_-) \cap \mathbb{R}_+$. It only remains to show that some neighborhood of $[t]_0^+$ in $\mathcal{L}_0(\mathcal{H}_+; \mathcal{H}_-)$ corresponds to sectorial forms. This follows from Lemma 4.7. □

The pieces are now in place for a holomorphy-of-resolvent type result.

**Convention 4.8.** If $H \in \mathcal{L}_3(\mathcal{H})$, then $\mathcal{R}(\zeta, H)$ is the resolvent of $H$ at $\zeta$. This is thought of as a function of $\zeta$, in a context specified by $H$. We will overload this notation, writing $\mathcal{R}(\zeta, h)$ for $\mathcal{R}(\zeta, [h]_0^+)$, or in the case of an explicit parameterization, $\mathcal{R}(\zeta, x)$ for $\mathcal{R}(\zeta, H_3)$. In the latter two cases, we use the name $\mathcal{R}$-map for $\mathcal{R}$ (even though that’s redundant), rather than resolvent. The $\mathcal{R}$-map has two arguments; the context is specified by a regular sectorial family.

We show now that the $\mathcal{R}$-map is holomorphic on
\[
\Omega := \{ (\zeta, h) \in C \times \mathcal{C}^\delta | \zeta \in \text{res } [h]_0^+ \} .
\] (24)

Since $(\zeta, h) \mapsto [h]_0^+ - \zeta \in \mathcal{L}(\mathcal{H}_+; \mathcal{H}_-) \cap \mathbb{R}_+$ is linear, this reduces to the question (recall Convention 3.1) whether $T \mapsto T^{-1}$ is holomorphic on the subset of $\mathcal{L}_0(\mathcal{H}_+; \mathcal{H}_-) \cap \mathbb{R}_+$ where it is well-defined. Prop. 3.2 addressed the case of $\mathcal{L}_0(\mathcal{H}_+; \mathcal{H}_-)$, and it is now a simple matter to extend it:

**Proposition 4.9.** Given $\hat{T} \in \mathcal{L}_0(\mathcal{H}_+; \mathcal{H}_-) \cap \mathcal{L}(\mathcal{H})$.

(a) $\hat{T} \in \mathcal{L}_0(\mathcal{H}_+)$. 
(b) $\hat{T}$ is injective iff $\hat{T}$ is injective.
(c) If $T : \text{dom } T \to \mathcal{H}$ is bijective, $\hat{T} \in \mathcal{L}_0(\mathcal{H}_+; \mathcal{H}_-)$. 

**Proof.** (a) This follows immediately from Prop. 3.2 and Lemma 1.3.
(b) If $\hat{T} \phi = 0$, then $\phi \in \text{dom } T$. 

11
(c) By assumption, \( \hat{T} + B \in \mathcal{L}_{\text{iso}}(\mathcal{H}^+;\mathcal{H}^-) \) for some \( B \in \mathcal{L}(\mathcal{H}) \). Hence, given \( \xi \in \mathcal{H} \), there is \( \phi \in \mathcal{H} \) such that \( \xi = (\hat{T} + B)\phi = \hat{T}\phi + B\phi \). But \( B\phi \in \mathcal{H} \), so the equation \( B\phi = \hat{T}\psi \) can be solved for \( \psi \in \mathcal{H}^+ \), yielding \( \xi = \hat{T}(\phi + \psi) \). That is, \( \hat{T} \) is not only bounded, but bijective as well, so \( \hat{T} \in \mathcal{L}_{\text{iso}}(\mathcal{H}^+;\mathcal{H}^-) \) (Open Mapping Theorem).

Therefore, the supposed extension from \( \mathcal{L}_{\text{iso}}(\mathcal{H}^+;\mathcal{H}^-) \) to \( \mathcal{L}_{\text{iso}}(\mathcal{H}^+;\mathcal{H}^-) - \mathcal{C} \) is illusory; all the operators we are interested in here are actually already in the former set. The following main result now follows immediately from the preceding work.

**Theorem 4.10.** For \( h \in \mathcal{C}^q \), the closed operator \( H = [h]^0 \) has an inverse in \( \mathcal{L}(\mathcal{H}) \) iff \( \hat{H} = [h]^- \) has an inverse in \( \mathcal{L}(\mathcal{H}^+;\mathcal{H}^-) \), and \( \mathcal{R} \) is holomorphic on \( \Omega \) [see (24)].

### E. Series expansion

We can reframe some of the main result in terms of the simplest ideas about series expansions. Suppose \( \hat{H} \) is in \( \text{Lin}(\mathcal{H}^+;\mathcal{H}^-) \) and \( \hat{T} \) is in \( \mathcal{L}(\mathcal{H}^+;\mathcal{H}^-) \). Then, \( H^{-1} \) exists in \( \mathcal{L}_{\text{iso}}(\mathcal{H}^+;\mathcal{H}^-) \).

In case \( \|\hat{T}\|_{\mathcal{L}(\mathcal{H}^+;\mathcal{H}^-)} < (\|H^{-1}\|_{\mathcal{L}(\mathcal{H}^+;\mathcal{H}^-)})^{-1} \), both \( \hat{T}H^{-1} \in \mathcal{L}(\mathcal{H}^-) \) and \( H^{-1}\hat{T} \in \mathcal{L}(\mathcal{H}^+) \) are operators of norm less than one, and

\[
(\hat{H} + \hat{T})^{-1} = \sum_{n=0}^{\infty} (\hat{H}^{-1}\hat{T})^n \hat{H}^{-1} = \hat{H}^{-1} \sum_{n=0}^{\infty} (\hat{T}\hat{H}^{-1})^n.
\]

Therefore, if \( \hat{H} = [h]^- \) and \( \hat{T} = [t]^+ \), we have a more or less explicit formula for \( ([h] + [t])^{-1} \), which we write \( (H + T)^{-1} \) (recognizing that ‘+’ here must be interpreted indirectly): Merely sandwich the expansions in between \( t_0 \) and \( t_+ \). This exhibits holomorphy in a very direct way. However, it does not itself show that \( H + T \) (or even \( H \)) is closed, nor does it show that invertibility of \( H \) implies invertibility of \( \hat{H} \).

### F. Holomorphic families

We now return to the idea of parameterizing families of sectorial forms by an open set in a Banach space.

**Definition 4.11.** Let \( \mathcal{X} \) be a dense subset of Hilbert space \( \mathcal{H} \), and \( \mathcal{U} \) a connected open subset of a Banach space. The map \( h: \mathcal{U} \to \mathcal{SF}(\mathcal{X}) \)

(a) \( [G\text{-}] \text{holomorphic family} \) in \( \mathcal{SF}(\mathcal{X}) \) parameterized over \( \mathcal{U} \) iff \( h_x[\psi]: \mathcal{U} \to \mathbb{C} \) is \([G\text{-}] \text{holomorphic}\) for each \( \psi \in \mathcal{X} \).

(b) \( \text{regular sectorial family} \) in \( \mathcal{C}^q \) parameterized over \( \mathcal{U} \) iff \( h: \mathcal{U} \to \mathcal{C}^q \) is holomorphic with range in \( \mathcal{C}^q \), where \( \mathcal{C} \) is an equivalence class of closable s-forms on \( \mathcal{X} \) and \( \mathcal{C}_\infty \) has the Banach space structure of Convention [14].

Various adjectives (“in \( \mathcal{SF}(\mathcal{X})/\mathcal{C}^q \), “parameterized over \( \mathcal{U} \)” may be omitted when context disambiguates.

**Remark 4.12.** By polarization, holomorphy of \( h \) immediately implies that \( h_x[\phi, \psi] \) is holomorphic in \( x \) for every \( \phi, \psi \in \mathcal{H} \).

If \( h: \mathcal{U} \to \mathcal{C}^q \) is a regular sectorial family, then the composition of \( h \) with any holomorphic function on \( \mathcal{C}^q \), such as \( R \) or (as shown in Section 7) \( E \), is automatically holomorphic:

**Corollary 4.13.** If \( h \) defined on \( \mathcal{U} \) is a regular sectorial family, then \( \mathcal{R} \) is holomorphic from its open domain in \( \mathbb{C} \times \mathcal{U} \) into \( \mathcal{L}(\mathcal{H}) \).

On the other hand, the requirement to be merely a holomorphic family is weak and easily checkable in application. Hence to get the abstract machinery appropriately hooked up to specific parameterized families, the only real question is when a holomorphic or \( G \)-holomorphic family is actually regular sectorial.

**Proposition 4.14.** A \( G \)-holomorphic family \( h: \mathcal{U} \to \mathcal{SF}(\mathcal{X}) \) is regular sectorial if any of the following criteria holds.

(a) \( h(\mathcal{U}) \) consists of equivalent, closed s-forms.

(b) \( h: \mathcal{U} \to \mathcal{C}^q \subset \mathcal{C}_\infty \) is locally bounded for some closable class \( \mathcal{C} \subset \mathcal{SF}(\mathcal{X}) \).

(c) \( h(\mathcal{U}) \) consists of equivalent, closable s-forms, and for every \( x \), \( h_x \) is uniformly bounded with respect to \( h_y \) for \( y \) in some neighborhood of \( x \).

**Proof.** For (a), note that the assumption is that the forms \( h_x \) are already closed on \( \mathcal{X} \). Hence, holomorphy amounts to weak-operator holomorphy on \( \mathcal{H} \). Conclude with Prop. (2.10(b)). For (b), appeal to Prop. 2.10(c). Criterion (c) is a rephrasing of criterion (b) in light of the preceding theory.

### G. Operator bounded families

This subsection discusses an important kind of holomorphic family constructed on the basis of a given lower-bounded self-adjoint operator \( H \). It is not used until Section 7 and can safely be skipped until then.

Take \( H \) to be a lower-bounded self-adjoint operator in \( \mathcal{H} \), and assume \( 1 \leq H \), which can be arranged without loss by adding a constant. Choose \( \mathcal{X} = \text{dom} H \). On \( \mathcal{X} \), \( H \) defines an s-form \( h \) by

\[
h[\psi] := \langle \psi | H \psi \rangle.
\]

We denote the equivalence class of s-forms to which \( h \) belongs by \( \mathcal{C}(H) \), or simply by \( \mathcal{C} \) in this subsection, when
there is no ambiguity. Once it is known that \( C \) is closable, the theory developed in this section shows that 
\[ C_\infty \simeq L(\mathcal{H}_\tau; \mathcal{H}_\tau), \]
where \( \mathcal{H}_\tau \) is the completion of \( \text{dom} \ H \) under the inner product \( \langle \phi | H \psi \rangle \).

**Lemma 4.15.** \( C(H) \) is closable.

*Proof.* \( \mathcal{H}_\tau \) exists at least as the abstract completion of \( \text{dom} \ H \). Let \( (\psi_n) \subset \text{dom} \ H \) be an \( \mathcal{H}_\tau \)-Cauchy sequence, such that \( \| \psi_n \| \to 0 \), \( \| \psi_n - \psi_m \| \to 0 \). It needs to be shown that \( \psi = 0 \). Taking the limit of \( \| \psi_m \| = \| \psi_m \|_2^2 + \| \psi_m \|_2^2 - 2 \text{Re} \langle \psi_n | H \psi_m \rangle \) as \( n \to \infty \), yields \( 0 = \| \psi \|_2^2 + \lim_{m \to \infty} \| \psi_m \|_2^2 \), showing that \( \psi = 0 \), as required.

Define the real subspace \( \mathcal{X}^r(H) \) of \( \mathcal{SF}(\text{dom} \ H) \) to consist of hermitian \( s \)-forms such that the norm
\[
\| t \|_H = \sup \left\{ \frac{| \langle \phi | t \psi \rangle |}{\| \phi \|_H \| \psi \|_H} : 0 \neq \phi, \psi \in \text{dom} \ H \right\}.
\] (27)

\( \mathcal{X}^r(H) \) corresponds precisely to the set of symmetric operators on \( \text{dom} \ H \) which are operator bounded with respect to \( H \). Now, let
\[
\mathcal{B}(H) := \mathcal{X}^r(H) \oplus i \mathcal{X}^r(H)
\]
be the complexification of \( \mathcal{X}^r(H) \), with the norm extended according to
\[
\| t \|_H := \| t^* \|_H + | t^t \|_H.
\] (29)

We aim to show that \( \mathcal{B}(H) \) is continuously embedded in \( \mathcal{C}(H) \). The following Lemma is the key step.

**Lemma 4.16.** For \( t \in \mathcal{X}^r(H) \), \( | t^t \|_H \leq 1 \).

*Proof.* Assume \( \| t \|_H = 1 \); the general case follows by homogeneity.

If for some \( \psi \), \( | t^t \psi \| \geq | t \| \psi \|_H \), the numerical range of at least one of \( h + t \) and \( h - t \) contains a negative number. Therefore, it suffices to show that \( 0 \notin \text{dom} \ (h + t) \), and even, by Prop. 4.12 below, that \( (-\infty, 0] \in \text{res}(H + T) \), where \( T \) is the operator on \( \text{dom} \ H \) induced by \( t \). That will be the case if \( \| T \mathcal{R}(x, H) \| < 1 \) for \( x \leq 0 \) (Lemma 1.2). But this follows immediately from the definition of the norm \( \| \cdot \|_H \):
\[
\| T \mathcal{R}(x, H) \| < \| H \mathcal{R}(x, H) \| \leq 1.
\]

The desired result follow immediately.

**Proposition 4.17.** Given lower-bounded self-adjoint operator \( H \), \( \mathcal{B}(H) \) is a Banach space continuously embedded in \( \mathcal{C}(H) \). Moreover, if \( \| t - h \|_H < 1 \), then \( \text{Sctr} (0, \frac{1}{2}) \) is a sector for \( t \).

**H. Numerical range and spectrum**

This subsection collects somewhat auxiliary results relating the numerical ranges and spectra of operators. In general, the relationship is subtle. Prop. 4.19 shows that the spectrum of a closed sectorial operator is contained in the closure of its numerical range, but in general, the spectrum could be much smaller: consider the matrix \( \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right) \), with spectrum \( \{0\} \) and numerical range a disk of radius \( 1/2 \).

**Lemma 4.18.** Let \( S \) be a symmetric operator with numerical range in \([0, c]\) with \( c < \infty \), and suppose that \( (\eta_n) \) is a sequence of vectors such that \( \langle \eta_n | S \phi_n \rangle \to 0 \). Then, \( S \phi_n \to 0 \).

*Proof.* Assume for a contradiction that there is a subsequence \( n(k) \) such that \( \| S \phi_{n(k)} \| > \epsilon > 0 \). Without any loss, we may assume that the subsequence is the entire sequence. Hence, there exists a sequence of unit vectors \( (\eta_n) \) such that
\[
\epsilon \leq \| \eta_n \| S \phi_n \rangle \langle \eta_n \| \phi_n \| S \phi_n \rangle \leq c \langle \phi_n | S \phi_n \rangle \to 0.
\]
The second inequality here is Cauchy-Schwarz, and the contradiction finishes the proof.

**Proposition 4.19.** For \( T \) a symmetric operator, \( \inf \text{num} T \in \text{spec} T \).

*Proof.* Assume that \( \inf \text{num} T = 0 \), since that can be arranged by adding a constant, unless \( \text{num} T \) is unbounded below, in which case the Lemma is vacuous anyway. Then, there is a sequence \( (\psi_n) \) of unit vectors in \( \text{dom} T \) such that
\[
\langle \psi_n | T \psi_n \rangle \to 0.
\] (30)

Assume, for a contradiction, that \( 0 \in \text{res} T \), i.e., \( T^{-1} \in \mathcal{L}(\mathcal{H}) \). We will show this implies \( \psi_n \to 0 \). Multiplying \( T \) by a constant if necessary, we may assume \( \| T^{-1} \| = 1 \). Since
\[
\langle \psi | T \psi \rangle = \langle T^{-1} T \psi | T \psi \rangle \in \| T \psi \| (\| T^{-1} \|),
\] (31)
on-negative of \( \text{num} T \) implies the same for \( \text{num} T^{-1} \), so that Lemma 4.18 will apply to \( T^{-1} \).

Define \( \phi_n = T \psi_n \). Then, \( \| \phi_n \| \geq \| \psi_n \| = 1 \) because \( \| T^{-1} \| = 1 \), and (30) is rewritten as
\[
\langle \phi_n | T^{-1} \phi_n \rangle \to 0.
\]
By Lemma 4.18, \( \psi_n = T^{-1} \phi_n \to 0 \). Contradiction.

**Proposition 4.20.** For an operator \( T \),
(a) Each connected component of the open set \( \mathcal{C} \setminus \text{cl} \text{num} T \) is either disjoint from \( \text{res} T \), or contained in it.
(b) In components contained in \( \text{res} T \), the resolvent is bounded as
\[
\| \mathcal{R}(\zeta, T) \| \leq \frac{1}{\text{dist}(\zeta, \text{num} T)}.
\] (32)

(c) If \( T \) is closed sectorial, \( \text{spec} T \subseteq \text{cl num} T \).

**Proof.** For brevity, write \( G \) for the open set \( \mathbb{C} \setminus \text{cl num} T \).

We first demonstrate the bound \([33]\) for arbitrary \( \zeta \in G \cap \text{res} T \), and use that to show that both \( G \cap \text{res} T \) and \( G \cap \text{spec} T \) are open. That is equivalent to (a), and proves the remaining part of (b).

Thus, check for any unit vector \( \psi \in \text{dom} T \):
\[
\| (T - \zeta)\psi \| \geq | \langle \psi | (T - \zeta)\psi \rangle | \geq | \langle \psi | T\psi \rangle - \zeta | \\
\geq \text{dist}(\zeta, \text{num} T).
\] (33)

This establishes \([32]\).

\( G \cap \text{res} T \) is open: The open disk with center \( \zeta \) and radius \( \| \mathcal{R}(\zeta, T) \|^{-1} \geq \text{dist}(\zeta, \text{num} T) \) is contained in \( \text{res} T \). (See Lemma 1.2)

\( G \cap \text{spec} T \) is open: For \( \omega \) in \( G \cap \text{spec} T \), if there is \( \zeta \) in \( \text{res} T \) with \( \omega - \zeta < \frac{1}{2} \text{dist}(\omega, \text{num} T) \), then \( \omega - \zeta < \text{dist}(\zeta, \text{num} T) \), contradicting the previous paragraph.

For part (c), Since \( \text{cl num} T \) is convex, it is geometrically more-or-less obvious that it is either bounded, a closed sector, or bounded by two parallel lines. The last is impossible since \( T \) is sectorial, and \( G \) has exactly one component in either of the other two cases. The conclusion follows from \( \text{res} T \neq \emptyset \) (\( T \) is closed).

5. MAGNETIC SCHRODINGER FORMS

The core theory of the previous section is inert on its own. To use it, we need some interesting regular sectorial families, and some associated quantities and objects which are holomorphic. The following two sections will take up the latter issue. This section is concerned with regular sectorial families of nonrelativistic Hamiltonians which are parameterized by scalar and vector potential fields and a two-body interaction. Though intended to be more illustrative than exhaustive, the results are nevertheless nontrivial. See Section 5.6 for the summary conclusion.

A. Nonrelativistic \( N \)-particle systems

We consider a system of \( N \) identical particles moving in three-dimensional euclidean space. Hence, the ambient Hilbert space is \( \mathcal{H} \equiv L^2(\mathbb{R}^3)^N \). As s-forms, the Hamiltonians we wish to consider are sums
\[
h_{A,u,v} = k_A + u_{u_0 + u} + v_{v_0 + v},
\] (34)

where
\[
k_A[\psi] = \int_{\mathbb{R}^3} \sum_{\alpha=1}^N \| \nabla_{\alpha} - i A(x_\alpha) \| \psi |^2 \, dx,
\] (35)
is kinetic energy with magnetic vector potential \( A \);
\[
u_u[\psi] = \int_{\mathbb{R}^3} \sum_{\alpha} u(x_\alpha) |\psi(x)|^2 \, dx \quad \text{(36)}
is a one-body potential energy for scalar potential \( u \); and
\[
v_v[\psi] = \int_{\mathbb{R}^3} \frac{1}{2} \sum_{\alpha \neq \beta} v(x_\alpha - x_\beta) |\psi(x)|^2 \, dx \quad \text{(37)}
is a two-body interaction. These are taken to be defined on the space \( \mathcal{H} \equiv C^\infty(\mathbb{R}^3)^N \) of compactly supported, infinitely differentiable functions. In \([34]\), \( A_0, u_0 \) and \( v_0 \) are fixed background or unperturbed fields, while \( A, u \) and \( v \) are variable, drawn from appropriate Banach spaces (to be determined) so that \( h_{A,u,v} \) is a regular sectorial family.

We do not say anything here about statistics because all the s-forms/operators to be considered are invariant under particle permutations; thus, one can simply restrict attention to the subspace carrying the desired representation of the permutation group. Taking spin explicitly into account is similarly unnecessary since we are concerned with spin-independent Hamiltonians.

The perturbed scalar and interaction potentials are taken to be locally integrable, non-negative functions:
\[
u_0, v_0 \in L_{\text{loc}}(\mathbb{R}^3)^+. \quad \text{(38)}
\]

An interesting and natural choice for the unperturbed scalar potential \( v_0 \) is some kind of confining potential, e.g., \( |x|^2 \) or \( |x|^4 \). It is actually somewhat artificial to consider an interaction potential which could not be treated as a perturbation, since the Coulomb interaction \( v(x) = |x|^{-1} \) can be. Since we are not aiming for an exhaustive treatment, \( A_0 \) is dropped (or taken identically zero). Other choices complicate the analysis considerably.

Local integrability of \( u_0 \) and \( v_0 \) ensures that \( \mathcal{H} \) is in the domains of \( u_{u_0} \) and \( v_{v_0} \), while positivity then implies closability on \( \mathcal{H} \). Indeed, \( u_{u_0} \) is closed on the space of functions square integrable with respect to \( |1 + \sum_{\alpha} u(x_{\alpha})| \, dx \), and similarly for \( v_0 \). Closability of \( k_0 \) was already considered in Section 3A Closability of the sum \( k_0 + u_{u_0} + v_{v_0} \) is a new problem (the pieces are not equivalent), which, however, is easily solved as follows (see paragraph VI.1.6 of Kato[10]): if \( t,s \) are sectorial s-forms closable on a common domain \( \mathcal{H} \) (with closures \( \overline{t}, \overline{s} \)), then \( t+s \) is also closable on \( \mathcal{H} \). This is true because a Cauchy sequence with respect to \( t^+ + s^+ \) is Cauchy with respect to each of \( t^+ \) and \( s^+ \) separately, hence has a limit in \( \text{dom} t \cap \text{dom} s \). By induction, this extends to any finite number of closable s-forms.
In summary, the unperturbed form $h_{0,0,0}$ is sectorial and closable on $\mathcal{H} \equiv C^\infty_c((\mathbb{R}^3)^n)$, being a member of an equivalence class $\mathcal{C}$ of forms on $\mathcal{H}$. We are interested in perturbations $h_{A,u,v}$ lying in $\mathcal{C}^5$, and which, moreover, vary holomorphically with the parameters $(A, u, v)$. Note that the identification of an unperturbed form is arbitrary. Any form in $\mathcal{C}^5$ would do. Even on a practical level, this can be so to some extent. For example, one may prefer $u_0(x) = R^1$ if $|x| \leq R$, otherwise $|x|^k$ to the “simpler” $|x|^k$, and if there is a background magnetic field, different gauge choices may recommend themselves.

The following new concept, an extreme sort of relative boundedness, is now going to be very important.

**Definition 5.1.** For $s$-forms $t$ and $h$: $t$ is *(Kato) tiny* with respect to $h$ iff, for any $b > 0$, $|t| \leq a1 + b|h|$, for some $a \in \mathbb{R}$. This is denoted $t \ll h$.

Here are some simple yet useful properties of $\ll$.

1. $t \sim 0 \Rightarrow t \ll h$ for any $h$
2. $\{t \mid t \ll h\}$ is a vector space.
3. Given $t \ll h$:
   - (a) $h + t \sim h$
   - (b) $h' \sim h \Rightarrow t \ll h'$
   - (c) $h$ sectorial $\Rightarrow h + t$ sectorial
   - (d) $h$, $h'$ sectorial $\Rightarrow t \ll h + h'$
   - (e) $t' \prec t$, $h \prec h' \Rightarrow t' \ll h'$

The main point here is that we can accumulate tiny perturbations indefinitely without danger of moving out of $\mathcal{C}^\infty$. Because of item 3(b), it makes sense to write $t \ll C$. But, beware: a set of forms tiny with respect to $C$ might still be unbounded in $\mathcal{C}_{\ll}$.

1. **Bounded potentials**

These are *complex* functions, $u \in L^\infty(\mathbb{R}^3)$, even though ultimately we are (probably) only interested in the real subspace $L^\infty(\mathbb{R}^3, \mathbb{R})$. This expansion is for the sake of holomorphy, as usual; we need to work in the complex space to use the theory of Section 1.

This simple case is a good illustration of the basic method: check that the perturbation does not move $h_{A,u,v}$ out of $\mathcal{C}^5$; check G-holomorphy; check local boundedness.

Now,

$$|u_0[\psi]| \leq \|u\|_{L^\infty(\mathbb{R}^3)} 1[\psi]. \quad (39)$$

This immediately demonstrates local boundedness of $h_{0,u,0}$ due to the factor $\|u\|_{L^\infty(\mathbb{R}^3)}$, as well as that $u_3 \sim 0 \ll h_{0,u,0}$. G-holomorphy of $u_0[\psi]$ in $u$ is trivial because it is linear. In complete detail:

$$u_{u+zu}[\psi] = u_0[\psi] + zu_u[\psi],$$

so the issue reduces to holomorphy of the right-hand side in $z$, which only requires that $u_0[\psi]$ and $u_u[\psi]$ be well-defined. N.B. This argument has nothing to do with the topology of $L^\infty$ and will hold for any vector space for which $u_u \in \mathcal{S}(\mathcal{H})$. Conclusion: $L^\infty(\mathbb{R}^3) \ni u \mapsto h_{0,u,0}$ is a regular sectorial family.

2. **Unbounded potentials**

Now we move on to a space of unbounded potentials, namely, $u \in L^{3/2}(\mathbb{R}^3)$. The following Lemma provides a bound playing the same role as (39). The Sobolev inequality used can be found in books on Sobolev spaces, \cite{20}, partial differential equations\cite{21}, and general analysis\cite{22}.

**Lemma 5.2.** If $u \in L^{3/2}(\mathbb{R}^3)$, then

$$|u_0| \leq c''\|u\|_{L^{3/2}(\mathbb{R}^3)} k_0. \quad (40)$$

**Proof.** For fixed $y \equiv (x_2, \ldots, x_N)$, the Hölder inequality gives

$$\int u(x_1)|\psi(x_1, y)|^2 \, dx_1 \leq c\|u\|_{L^{3/2}} \left\{ \int |\psi(x_1, y)|^6 \, dx \right\}^{1/3} = c\|u\|_{L^{3/2}} \|\psi(\cdot, y)\|_{L^{3/2}(\mathbb{R}^3)}^6.$$

For the integral here, use the Sobolev inequality

$$\|f\|_{L^q(\mathbb{R}^d)} \leq c'\|f\|_{W_k^p(\mathbb{R}^d)}, \quad p \leq q \leq \frac{pd}{d - kp} \quad (41)$$

with the values $d = 3$, $k = 1$, $p = 2$, $q = 6$ to obtain

$$\int \|\psi(\cdot, y)\|^2_{L^{3/2}(\mathbb{R}^3)} \, dy \leq c' \int |\nabla_1 \psi(x_1, y)|^2 \, dy.$$

Adding up the inequalities with each of $x_2, \ldots, x_N$ in place of $x_1$ yields

$$u_0[\psi] \leq cc'\|u\|_{L^{3/2}} k_0[\psi].$$

This demonstrates local boundedness of $L^{3/2}(\mathbb{R}^3) \ni u \mapsto h_{0,u,0}$, but would allow us to conclude that $h_{0,u,0} \in \mathcal{C}^5$ only for $\|u\|_{L^{3/2}}$ sufficiently small (depending on $c''$). Fortunately, it can be improved by using density of $L^\infty$ in $L^{3/2}$.
Lemma 5.3. For \( u \in L^{3/2}(\mathbb{R}^3) \), \( u_n \sim k_0 \).

Proof. Split \( u \) as \( u = u' + u'' \), with \( u' \in L^\infty(\mathbb{R}^3) \) and \( u'' \in L^{3/2}(\mathbb{R}^3) \). \( \|u''\|_{L^{3/2}(\mathbb{R}^3)} \) can be made as small as desired by choosing \( u' \) appropriately (e.g. \( u' = u \{ |u| \leq M \} \) for large \( M \)).

Conclusion: \( L^{3/2}(\mathbb{R}^3) \ni u \mapsto h_{0,u,0} \) is a regular sectorial family.

Remark 5.4. This result is very important to Lieb’s framework for DFT.

3. Modulating the confining potential

The final kind of scalar potential to be considered is modulation of the background (confining) potential: \( L^\infty(\mathbb{R}^3) \ni f \mapsto uf_{u_0} \). Evidently,

\[
\|uf_{u_0}\| \leq \|f\|_{L^\infty(\mathbb{R}^3)} u_{u_0}. \tag{42}
\]

Local boundedness is thus secure, but \( h_{0,fu_0,0} \) will generally fail to be sectorial if \( u_0 \) is anything like what we have in mind. Thus, we need to restrict \( f \) to the open unit ball \( B(L^\infty(\mathbb{R}^3)) \). With that restriction, another regular sectorial family is obtained.

4. Removing redundancy

Combine the preceding three kinds of scalar potential perturbation yields a holomorphic map

\[
L^\infty(\mathbb{R}^3) \oplus L^{3/2}(\mathbb{R}^3) \oplus L^\infty(\mathbb{R}^3) \to \mathcal{C}_x
\]
given by

\[
(u', u'', f) \mapsto u_{u'} + u_{u''} + fu_{u_0} = u_{u'+u''+fu_0}. \tag{43}
\]

However, this should be restricted to the open set

\[
\mathcal{U} := \left\{ (u', u'', f) \in L^\infty \oplus L^{3/2} \oplus L^\infty \mid \|f\| < 1 \right\} \tag{44}
\]
to ensure that \( h_{0,u'+u''+fu_0,0} \) is in \( \mathcal{C}_x \). Thus, we have a regular sectorial family in \( \mathcal{C}_x \) parameterized over \( \mathcal{U} \) above.

However, this is not entirely satisfactory because there is redundancy: many distinct triples \( (u', u'', f) \) may give the same total potential \( u' + u'' + fu_0 \). To cure this infelicity, we pass to a quotient. Recall that the quotient \( \mathcal{X}/\mathcal{M} \) of a Banach space \( \mathcal{X} \) by a closed subspace \( \mathcal{M} \) is a Banach space with norm

\[
\|\pi x\|_{\mathcal{X}/\mathcal{M}} := \inf \left\{ \|x + m\|_{\mathcal{X}} \mid m \in \mathcal{M} \right\},
\]

where \( \pi : \mathcal{X} \to \mathcal{X}/\mathcal{M} \) is the canonical projection. A continuous linear map \( f : \mathcal{X} \to \mathcal{Y} \) naturally induces a linear map on the quotient \( \mathcal{X}/\ker f \), eliminating directions along which \( f \) is constant.

This simple picture is complicated in situations which interest us for two reasons. \( f \) is not sure to be either linear or defined on the entire space \( \mathcal{X} \), hence a slightly generalized notion of kernel is needed, and taking a quotient by a subspace is not an immediately sensible thing to do.

Lemma 5.5. Given \( \mathcal{U} \subseteq \mathcal{X} \) open and convex, and \( \mathcal{U} \xrightarrow{\cdot \mathcal{M}} \mathcal{Y} \), holomorphic, let

\[
\mathcal{M} = \cap_{x \in \mathcal{U}} \ker Df(x).
\]

Then, \( f \) has a unique holomorphic extension to \( \mathcal{U} + \mathcal{M} \), given by \( f(x + m) = f(x) \) for \( m \in \mathcal{M} \). In turn, a holomorphic map \( \tilde{f} : (\mathcal{U} + \mathcal{M})/\mathcal{M} \to \mathcal{Y} \) is induced on the quotient, given by \( \tilde{f}(\pi x) = f(x) \).

Proof. First, note that \( \ker Df(x) \) is a closed subspace of \( \mathcal{X} \) for each \( x \in \mathcal{U} \), so \( \mathcal{M} \) is indeed a closed subspace. To see that the asserted extension is well-defined, suppose that \( y = x + m = x' + m' \), for \( x, x' \in \mathcal{U} \), \( m, m' \in \mathcal{M} \). Denote the affine (two-C-dimensional) subspace containing \( x, x', y \) by \( A \), and consider the restriction of \( f \) to \( A \cap \mathcal{U} \), which is convex. The restriction of \( Df \) is everywhere zero, hence \( f \) is constant on \( A \cap \mathcal{U} \), i.e., \( f(x) = f(x') \) and the extended \( f \) is well-defined. That the extension is holomorphic follows immediately from \( Df(x + m) = Df(x) \), and unicity from \( \mathcal{U} + \mathcal{M} \) being connected and \( f \) given on an open set, namely \( \mathcal{U} \).

Therefore, \( f \) is well-defined on \( \mathcal{U}/\mathcal{M} \) according to the given formula, and it remains only to show that it is holomorphic. As usual, we use the equivalence with \( \mathcal{G} \)-holomorphy plus local boundedness (Thm. 2.3). For \( \mathcal{G} \)-holomorphy, note that \( \tilde{f}(\pi x + \zeta \pi y) = f(x + \zeta y) \), so the question reduces to \( \mathcal{G} \)-holomorphy of \( f \) itself. For local boundedness, note that \( \|\pi x \prec \prec \tilde{y}\| < \epsilon \) implies that \( x \) is within distance \( \epsilon \) of \( \pi^{-1}\tilde{y} \).

To apply this, one only need check that \( \mathcal{U} \) in

\[
\inf \left\{ \|u'\|_{L^\infty} + \|u''\|_{L^{3/2}} + \|f\|_{L^\infty} \mid u = u' + u'' + fu_0 \right\}
\]

is finite. This is a norm \( \|u\| \) making \( L^\infty + L^{3/2} + u_0 L^\infty \) a Banach space, and the subset \( \mathcal{U}_0 \) consisting of \( u \) with some decomposition obeying the constraint \( \|f\|_{L^\infty} < 1 \) is open. Conclusion: the map \( u \mapsto h_{0,u,0} \) is a regular sectorial family parameterized over \( \mathcal{U}_0 \).

C. Interaction

The message of this subsection is that two-body interactions can be treated in much the same way as one-body potentials, an observation that goes back centuries. Indeed, instead of coordinatizing configuration space \( (\mathbb{R}^3)^N \) with \( x_1, x_2, \ldots, x_N \), we may use \( \frac{x_1-x_2}{\sqrt{2}}, \frac{x_1+x_2}{\sqrt{2}}, x_3, \ldots, x_N \), and thereby control an interaction between particles 1 and 2 by the kinetic energy just as an external potential.
for particle 1. As long as we use only Kato tiny perturbations, as was done in Section 5.1, then, owing to property 2 of Section 5.1, it is not possible that each perturbation alone is controllable, while the combination is not. We have, for example, an regular sectorial family of pair interactions parameterized over \( U_\epsilon = L^\infty(\mathbb{R}^3) + L^{3/2}(\mathbb{R}^3) \).

D. Vector potential

For our purposes, the form of \( k_A \) given in (35) is not good for complex vector potentials. In order that \( k_A \) be holomorphic in \( A \), it should not appear complex-conjugated. The correct definition is

\[
k_A[\psi] = \sum_{\alpha=1}^N \int_{\mathbb{R}^{3N}} (\nabla_\alpha + i A(x_\alpha)) \overline{\psi} \cdot (\nabla_\alpha - i A(x_\alpha)) \psi \, dx
\]

\[
= \langle (\nabla - i \overline{A})\psi | (\nabla - i A)\psi \rangle
\]

We take a somewhat different approach with this than for scalar potentials. \( \nabla_\alpha \) is a bounded operator from \( W_2^2(\mathbb{R}^{3N}) \) into \( \tilde{L}^2(\mathbb{R}^{3N}) \) (we use an over-arrow to indicate ordinary, complex, three-dimensional vectors). The integral in (45) will be a legitimate \( \tilde{L}^2 \) inner product if multiplication by \( A \) (or \( \overline{A} \)) has the same property. This is very natural train of thought, but before pursuing it, we consider bounded vector potentials.

1. bounded \( A \)

Lemma 5.6. If \( A \) is bounded, then \( k_A \) is a tiny perturbation of \( k_0 \).

Proof. For an arbitrary \( \psi \in W_2^2(\mathbb{R}^{3N}) \),

\[
|k_A[\psi] - k_0[\psi]| = |\langle (\nabla - i \overline{A})\psi | (\nabla - i A)\psi \rangle - \|\nabla\psi\|^2|
\]

\[
\leq \|A\|_{L^\infty} \|\psi\|^2 + 2 \|A\|_{L^\infty} \|\nabla\psi\| \|\psi\|.
\]

Control the final term with the inequality

\[
2 \|\nabla\psi\| \|\psi\| \leq \epsilon \|\nabla\psi\|^2 + \frac{1}{\epsilon} \|\psi\|^2, \quad \epsilon > 0.
\]

Since \( \epsilon \) can be taken as small as desired here,

\[
k_A - k_0 \prec k_0.
\]

Just as \( \overline{G} \)-holomorphy of \( u_\epsilon \) followed from holomorphy of \( C \ni z \mapsto z \), \( \overline{G} \)-holomorphy of \( k_A \) follows from holomorphy of \( z \mapsto z^2 \). Local boundedness of \( k_A[\psi] \) as a function of \( A \in \tilde{L}^\infty(\mathbb{R}^3) \) follows from an estimate like that in (46).

Thus, \( \tilde{L}^\infty(\mathbb{R}^3) \ni A \mapsto k_A \) is a regular sectorial family.

2. Sobolev multipliers

Now we return to the idea mentioned at the beginning of this section. Multiplication of elements of \( W_2^2(\mathbb{R}^d) \) by a fixed function \( f \) is a linear operation. If it is actually a bounded linear operator into \( L^2(\mathbb{R}^d) \), then \( f \) is a member of the space \( M(W_2^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)) \) of Sobolev multipliers. This space is nontrivial (it contains \( L^\infty \)) and is a Banach space with the norm it inherits from \( L(W_2^2(\mathbb{R}^d); L^2(\mathbb{R}^d)) \):

\[
\|f\|_{M(W_2^2 \rightarrow L^2)} := \sup \left\{ \|f\psi\|_{L^2} \mid \|\psi\|_{W_2^2} = 1 \right\}
\]

Therefore, we consider \( A \in \tilde{M}(W_2^2(\mathbb{R}^d); L^2(\mathbb{R}^d)) \). One needs to check that this lifts from 3-dimensional space properly, but that is simple: abbreviating \( y \equiv (x_2, \ldots, x_N) \),

\[
\int |A(x_1)\psi(x_1, y)|^2 \, dx_1
\]

\[
\leq \|A\|_{\tilde{M}(W_2^2 \rightarrow L^2)} \int |\nabla_1\psi(x_1, y)|^2 \, dx_1.
\]

Integration over \( y \) shows that the norm is independent of \( N \).

G-holomorphy has nothing to do with the topology of the space over which \( A \) ranges, so it follows for \( \tilde{M}(W_1^1(\mathbb{R}^{3N}); L^2(\mathbb{R}^{3N})) \) just as for bounded vector potentials. Local boundedness follows from a calculation much like (46):

\[
\left| k_{A+a}[\psi] - k_A[\psi] \right| \leq \|\nabla - i A\| \|\psi\| + \|A\| \|\nabla\psi\| + \|\psi\|^2.
\]

This establishes that, for \( A \in \tilde{M}(W_2^2(\mathbb{R}^{3N}); L^2(\mathbb{R}^{3N})) \), \( k_A \preccurlyeq k_0 \). However, the opposite, \( k_0 \preccurlyeq k_A \), is problematic in general, although it does hold if \( \|A\|_{M(W_2^2 \rightarrow L^2)} < 1 \). The situation looks at first like what we faced with \( u \in L^{3/2} \) for \( u_\epsilon \). However, \( L^\infty \) is not dense in \( M(W_2^2 \rightarrow L^2) \). The norm is an operator norm and we face the familiar problem that strong convergence does not imply norm convergence. Thus, we settle for what is clear, \( k_0 \sim k_A \) for \( A \) in the unit ball \( B(M(W_2^2 \rightarrow L^2)) \).

On one level the preceding is entirely satisfactory. The Sobolev-multiplier norm is natural. However, one might prefer something more familiar and easier to work with, such as given in the following Lemma.

Lemma 5.7. For \( A \in \tilde{L}^3(\mathbb{R}^3) \), \( k_A \sim k_0 \).

Proof. Use a Hölder inequality and the Sobolev inequality cited in Lemma 5.2 to obtain

\[
\|A\psi\|_{L^2} \leq \|A\|_{L^3} \|\psi\|_{L^6} \leq \epsilon \|A\|_{L^3} \|\psi\|_{W_2^2}.
\]

Again, just as in Lemma 5.3, a bounded vector field can be subtracted from \( A \) so that the \( L^3 \) norm of the residual is as small as desired.
3. Removing redundancy again

As for scalar potentials, there is also redundancy here, since $\bar{L}^\infty$ intersects $\bar{L}^3$, and is contained in $\bar{H}(W^1_2 \to L^2)$. It can be solved in exactly the same way to obtain a regular sectorial family parameterized over an open set $U_A$ in $L^\infty + \bar{M}(W^1_2 \to L^2)$ or all of $L^\infty + \bar{L}^3$.

E. Putting it all together

Here is the summary of preceding investigation. With lower-bounded locally integrable background potential and interaction, $h_{A,m,v}$ is a regular sectorial family on all of $\bar{L}^3 \times (L^{3/2} + L^\infty) \times (L^{3/2} + L^\infty)$. Alternatively, the $L^3$ summand for $A$ can be replaced by $\bar{M}(W^1_2(\mathbb{R}^3))$ and summands $u_0 L^\infty(\mathbb{R}^3)$ and $v_0 L^\infty(\mathbb{R}^3)$ added to the potential and interaction factors with restriction to an open neighborhood $U$ of the origin. The condition to be in $U$ does not factorize.

6. LOW-ENERGY HAMILTONIANS & EIGENSTATE PROPERTIES

The previous section was concerned with one component of application, namely the construction of regular sectorial families useful for nonrelativistic quantum mechanics. This section and the next tackle the question: given a regular sectorial family $h$ defined on $U$, what interesting functions/quantities are holomorphic? To a considerable extent, this can be fruitfully discussed without reference to any concrete regular sectorial family. This section uses Riesz-Dunford-Taylor integral methods to discuss “low-energy Hamiltonians” in case there is a gap in the spectrum, i.e., a curve $\Gamma$ in the resolvent set of $H_x$ running top-to-bottom in $\mathbb{C}$ (recall, we deal in “Hamiltonians” which are sectorial but not necessarily self-adjoint). The part of the spectrum to the left of $\Gamma$ then corresponds to a bounded Hamiltonian which is holomorphic on some neighborhood of $x$. Properties of nondegenerate eigenstates associated with isolated eigenvalues are considered in section 6E. The holomorphicity itself and expectations of all ordinary observables, as well as of generalized observables such as charge-density and current-density (when they make sense) are holomorphic. Some of the material here, primarily Section 6A and Prop. 6.3 are appealed to in section 7.

A. Riesz-Dunford-Taylor integrals

Recall that one of the main conclusions of Section 4 was holomorphy of the map $(\zeta,x) \mapsto \mathcal{R}(\zeta,H_x)$. As a function of the single complex variable $\zeta$, it is natural to integrate this around contours. The Riesz-Dunford-Taylor calculus constructs a holomorphic function $f(A)$ of an arbitrary bounded operator $A$ by integrating $f(\zeta)\mathcal{R}(\zeta,A)$ around a contour enclosing the entire spectrum $\text{spec } A$, where $f$ is an ordinary holomorphic function. Some basic references for this technology are §III.6 of Kato, Chap. 6 of Hislop & Sigal or §3.3 of Kadison & Ringrose. Since we deal with unbounded operators, we cannot do that, but the idea can be modified for some interesting purposes.

The first basic idea is that, if $H$ is a closed operator, $E$ is an isolated eigenvalue, and $\Gamma$ is a simple anticlockwise closed contour in $\text{res } H$, surrounding $E$ but no other part of $\text{spec } H$, then

$$P(\Gamma) = -\oint_{\Gamma} \mathcal{R}(\zeta,H) \frac{d\zeta}{2\pi i}$$  \hspace{1cm} (50)

is a projection onto the corresponding eigenspace. For normal (in particular, self-adjoint) operators this is straightforward as the relevant part of the resolvent looks like $(E - \zeta)^{-1} P$, where $P$ is an orthogonal projector onto the eigenspace, so that $P(\Gamma) = P$. The restriction of a non-normal operator to an eigenspace is generally not simply a multiple of the identity if the algebraic multiplicity exceeds one. Consequently, the resolvent generally has higher-order poles. If the eigenvalue is nondegenerate however, that cannot happen and its value can be extracted as

$$E = -\text{Tr} \left[ \oint_{\Gamma} \mathcal{R}(\zeta,H) \frac{d\zeta}{2\pi i} \right] - \frac{1}{2\pi i}$$  \hspace{1cm} (51)

We can profitably generalize somewhat. First, we have the basic result

**Proposition 6.1.** Given: $A \in \mathcal{L}_d(\mathcal{H})$ and $\Gamma$ a simple anticlockwise contour in $\text{res } A$, surrounding the part $\sigma$ of $\text{spec } A$. Then,

(a) $P(\Gamma) := -\oint_{\Gamma} \mathcal{R}(\zeta,A) \frac{d\zeta}{2\pi i}$  \hspace{1cm} (52)

is a projection with $\text{rng } P(\Gamma) \subseteq \text{dom } A$.

(b) $A(\Gamma) := -\oint_{\Gamma} \zeta \mathcal{R}(\zeta,A) \frac{d\zeta}{2\pi i}$  \hspace{1cm} (53)

satisfies $A(\Gamma) = AP(\Gamma) = P(\Gamma)A$ (hence maps $\text{rng } P(\Gamma)$ into itself), annihilates $\ker P(\Gamma)$, and its spectrum as an operator on $\text{rng } P(\Gamma)$ is $\sigma$.

(c) More generally, for open $U$ containing $\Gamma$ and the region it surrounds, and $f$ an arbitrary holomorphic function on $U$,

$$f(A|\Gamma) := -\oint_{\Gamma} f(\zeta)\mathcal{R}(\zeta,A) \frac{d\zeta}{2\pi i}$$  \hspace{1cm} (54)

maps $\text{rng } P(\Gamma)$ into itself and annihilates $\ker P(\Gamma)$. $P(\Gamma) = 1(A|\Gamma)$ and $A(\Gamma) = \text{id}(A|\Gamma)$ are special cases. $f \mapsto f(A|\Gamma)$ is a Banach algebra morphism from the space of holomorphic functions on $U$ (with uniform norm) into $\mathcal{L}(\text{rng } P(\Gamma))$. 

18
Proof. For the parts concerning $P(\Gamma)$ and $A(\Gamma)$, see Hislop & Sigal\textsuperscript{26} Prop. 6.9. For the Banach algebra aspects, see Kadison and Ringrose.\hfill $\square$

We are not nearly so interested in varying $f$ in (54), however, as in varying $A$ for a few simple cases of $f$, principally 1 and id.

Theorem 6.2. Given regular sectorial family $h$ and simple closed contour $\Gamma \subset \text{res } H_x$, there is a neighborhood $W$ of $x$ such that $y \in W \Rightarrow \Gamma \subset \text{res } H_y$ and for each $f$ holomorphic on and inside $\Gamma$, $y \mapsto f(H_y|\Gamma) \colon W \to \mathcal{L}(\mathcal{H})$ is holomorphic.

Proof. By compactness of $\Gamma$ and holomorphicity of $(\zeta, y) \mapsto R(\zeta, H_y)$.

$\square$

B. Low-energy Hamiltonians

No contour can be drawn around the entire spectrum of an unbounded operator $H_x$. However, since $H_x$ is bounded below, it might be possible to surround the part of spec $H_x$ in some left-half-plane, if there is a gap. That such a contour will continue to surround the “low energy” part of the spectrum when $x$ is perturbed is not immediately evident: Each $H_y$ is bounded below, but is it possible that spec $H_y$ has a part that drifts off to $-\infty$ as $y \to x^2$? Fortunately, such pathology is ruled out by Lemma 4.7 which says that a slight enlargement of a sector for one member of a regular sectorial family is a sector for all sufficiently close members.

More generally than a vertical line, we may start with a continuous curve $\Gamma \subset \mathbb{C}$ such that each horizontal line $\text{Im } z = \text{constant}$ intersects $\Gamma$ in exactly one point. In other words, $\Gamma$ goes from bottom to top of the plane without overhangs, as illustrated in Fig. 1. Such a curve, with upward orientation, will be called a right-boundary. Suppose $\Gamma$ is a right-boundary contained in $\text{res } H_x$, and let $\Sigma$ be a sector for $h_x$ with vertex to the left of $\Gamma$ (Fig. 1). Then we may form a closed contour by running along the lower edge of $\Sigma$ away from the vertex until meeting $\Gamma$, then running upward along $\Gamma$ until meeting the upper edge of $\Sigma$, and then back to the vertex. This contour, called $\Gamma$, encircles all the numerical range of $h_x$ to the left of $\Gamma$, hence the part of spec $H_x$ in that region. And, therefore, according to the preceding paragraph, $\Gamma$ also encloses all of spec $H_y$ lying to the left of $\Gamma$, for $y$ in some neighborhood of $x$. Now we extend the notation in (52), (53), and (54) (as long as $f$ is holomorphic on the region to the left of $\Gamma$), writing for instance $f(H_y|\Gamma)$ for the integral taken around $\Gamma$. The point is that it does not matter how $\Gamma$ is completed to a closed contour as long as all the spectrum to the left of $\Gamma$ is enclosed. Since that can always be done (assuming $\Gamma \subset \text{res } H_x$), the notation is justified.

C. Schatten classes

The preceding part of this Section showed how we get a variety of holomorphic maps $f \colon U \to \mathcal{L}(\mathcal{H})$. What if the image of $f$ happens to be in some restricted class of operators which has its own Banach space structure, for instance, the trace-class operators $\mathcal{L}^1(\mathcal{H})$, the Hilbert-Schmidt operators $\mathcal{L}^2(\mathcal{H})$, or more generally a Schatten $p$-class $\mathcal{L}^p(\mathcal{H})$? Nearly automatic holomorphy in these situations is shown in Prop. 6.5 below. Only the trace-class $\mathcal{L}^1(\mathcal{H})$ is used in this Section, but other Schatten classes $\mathcal{L}^p(\mathcal{H})$ will be put to work in Section 7.

First, we recall some basic facts about the Schatten $p$-classes\textsuperscript{28–30} that we will use.

Definition 6.3. For $1 \leq p < \infty$, $\mathcal{L}^p(\mathcal{H})$ is the set of compact operators $T$ such that $|T|^p \in \mathcal{L}^1(\mathcal{H})$, where $|T| = (T^*T)^{1/2}$.

Proposition 6.4. The classes $\mathcal{L}^p(\mathcal{H})$ have the following properties.

1. Equipped with the norm $\|T\|_p = (\text{Tr}|T|^p)^{1/p}$, $\mathcal{L}^p(\mathcal{H})$ is a Banach space.

2. $\mathcal{L}^p(\mathcal{H})$ is also a two-sided $*$-ideal: $\|ACB\|_p \leq \|A\|\|C\|_p\|B\|$ and it contains $C^*$ whenever it contains $C$.

3. $\mathcal{L}^1(\mathcal{H})$ is the dual space of the compact operators $\mathcal{L}_0(\mathcal{H})$ with the usual operator norm, while for $1 < p < \infty$, $\mathcal{L}^p(\mathcal{H})$ realizes the dual of $\mathcal{L}^q(\mathcal{H})$, where $1/p + 1/q = 1$, via the pairing $(S,T) \mapsto \text{Tr}ST$. On the other hand, the finite-rank operators are dense in $\mathcal{L}_0(\mathcal{H})$ as well as $\mathcal{L}^p(\mathcal{H})$ for $1 < p < \infty$. Thus, every $\mathcal{L}^p(\mathcal{H})$ ($1 \leq p < \infty$) is the dual space of a Banach space in which the finite-rank operators are dense.
Proposition 6.5. Given: $f: \mathcal{U} \to \mathcal{L}(\mathcal{H})$ holomorphic. If $f$ is a locally bounded map into $\mathcal{L}^p(\mathcal{H})$ ($1 \leq p < \infty$), then $f$ is holomorphic into $\mathcal{L}^p(\mathcal{H})$.

Proof. For $B$ finite-rank, $\text{Tr} f(x)B$ is a finite sum of terms of the form $\langle \phi_\alpha, f(x)\psi_\alpha \rangle$, each of which is holomorphic by hypothesis. Hence, the result follows from the remark about density of such operators in the pre-dual which precedes the Proposition together with Prop. 2.8. □

D. Finite rank

Prop. 6.5 does not quite give holomorphy due to the hypothesis of local boundedness. However, if we specialize to Riesz-Dunford-Taylor integrals and ask that $P_x(\Gamma)$ have finite rank, holomorphy into $\mathcal{L}^1(\mathcal{H})$ follows without an explicit local boundedness assumption. The next two well-known Lemmas encapsulate the simple key observations.

Lemma 6.6. If $P$ and $Q$ are projections (not necessarily orthogonal), $\|P - Q\| < 1$ implies that rank $P = \text{rank} Q$.

Proof. If $\text{rng} Q \ni \phi \mapsto P\phi$ is injective, then $\text{rank} P \geq \text{rank} Q$, which suffices by symmetry of the situation. However, for $\phi \in \text{rng} Q$,

$$\|P\phi\| = \|Q\phi + (P - Q)\phi\| \geq \|\phi\| - \|P - Q\|\|\phi\| > 0.$$ □

Lemma 6.7. A continuous function into $\mathcal{L}(\mathcal{H})$ with range in operators of rank $\leq N < \infty$ is actually continuous into $\mathcal{L}^1(\mathcal{H})$.

Proof. $\|A - B\|_1 \leq (\text{rank} A + \text{rank} B)\|A - B\|$. □

Proposition 6.8. rank $P_x(\Gamma) = N < \infty$ implies that $x$ has a neighborhood $W$ such that $\text{rank} P_y(\Gamma) = N$ for every $y \in W$, and $y \mapsto f(\mathcal{H}_y(\Gamma)):\mathcal{W} \to \mathcal{L}^1(\mathcal{H})$ is holomorphic.

Proof. Lemma 6.6 ensures existence of $W$ such that rank $P_y(\Gamma) = N$ for $y \in W$. Therefore $f(\mathcal{H}_y(\Gamma))$ also has rank $N$ since it maps rng $P_y(\Gamma)$ into itself while annihilating ker $P_y(\Gamma)$. Lemma 6.7 then completes the proof. □

E. Eigenstate perturbation

The extreme case is rank $P_x(\Gamma) = 1$. Then we are in the venerable context of eigenstate perturbation. A general rank-1 projection can be written as

$$|\phi\rangle\langle\eta|, \text{ with } \langle\eta|\phi\rangle = 1 \text{ and } \|\phi\| = 1,$$ (55)

where $\phi$ and $\eta$ are determined up to a common phase factor $e^{i\theta}$. Suppose, now, that $h$ is a regular sectorial family that $H_x$ has an isolated nondegenerate eigenvalue at $E_x$, and let $\Gamma$ be a contour which separates $E_x$ from the rest of spec $H_x$. Then, Prop. 6.8 shows that as $y$ varies in some neighborhood of $x$,

$$P_y(\Gamma) = |\phi_y\rangle\langle\eta_y|, \langle\eta_y|\phi_y\rangle = 1, \|\phi_y\| = 1,$$ (56)

and

$$H_y(\Gamma) = E_y|\phi_y\rangle\langle\eta_y|,$$ (57)

with $P_y(\Gamma)$ and $H_y(\Gamma)$ holomorphic. A fortiori, $E_y$ moves continuously with $y$ as long as it remains separated from the rest of spec $H_y$ — the isolation condition, for short. As $y$ moves along any continuous curve in $\mathcal{H}$ beginning at $x$ and respecting the isolation condition $E_y$ can be continuously tracked, but if the path returns to $x$, we may not return to $E_x$ unless the path can be contracted to a point without violating the isolation condition. Therefore, we consider $W$, a maximal simply connected open set containing $x$ and with the isolation condition satisfied everywhere in $W$. For $y \in W$, we can simply write $P_y$ and $E_y$, since the particular choice of $\Gamma$ is immaterial.

Now, $E_y$ is holomorphic as a $\mathbb{C}$-valued function and $P_y$ and $H_y$ as $\mathcal{L}^1(\mathcal{H})$-valued functions, for $y \in W$. Therefore, for any bounded observable $B \in \mathcal{L}(\mathcal{H})$, its “expectation”

$$y \mapsto \text{Tr} B|\phi_y\rangle\langle\eta_y| = \langle\eta_y|B\phi_y\rangle$$ (58)

is holomorphic on $W$. The quotation marks are because this coincides with the usual notion of expectation only when $\eta_y = \phi_y$, e.g., when $H_y$ is self-adjoint.

There are other interesting holomorphic quantities which do not fall into this category, however. $E_y$ itself,

$$E_y = \langle\eta_y|H_y\phi_y\rangle,$$ (59)

is one such. The charge and current density are others when our parameter space includes scalar and vector potentials. This is because these quantities are the derivatives of $E_y$ with respect to scalar and vector potential, respectively. At a heuristic level, this claim is straightforward, but there are delicate details, which we will now check.

Lemma 6.9 (Hellmann-Feynman). Suppose finite-rank projections $P_y$ and bounded operators $A_y$ depend differentiably on parameter $y$, and that $[P_y, A_y] = 0$. Then $D_y \text{Tr} P_y A_y = \text{Tr} P_y D_y A_y$.

Proof. ($y$ subscripts will be suppressed for notational simplicity) Differentiating $P(1 - P) = 0$, deduce that $D P$ maps rng $P$ into rng$(1 - P)$ and vice versa. Since both rng $P$ and $\text{rng}(1 - P)$ are invariant under $A$, it immediately follows that $\text{Tr}(D P)A = 0$ (put $(P + 1 - P)$ on each side and use cyclicity of trace). □

Since $H_y(\Gamma)$ is analytic, the preceding Lemma gives

$$D_y E_y|_{y=x} = \langle\eta_x|DH_y(\Gamma)|x\phi_x\rangle$$

$$= \oint_{\Gamma} \langle\eta|D_y \mathcal{R}(\zeta, H_y)\phi\rangle \frac{d\zeta}{2\pi i}.$$ (60)
This shows that to continue, we need

equalities of Schrödinger forms as in Section 5. The differences of charge/current density (Lemma 4.5).

\[ \langle \eta | \mathcal{R}(\zeta, H_y) \rangle = \langle \eta | \hat{H}_y \rangle \]

To do much with this requires explicit knowledge of \( \eta, \phi \) and \( J \) or \( \delta \).

Proposition 6.11.

\[ D_y \langle \eta | \mathcal{R}(\zeta, H_y) \rangle \phi = \langle \eta | (\hat{H}_y - \zeta)^{-1} D_y \hat{H}_y (\hat{H}_y - \zeta)^{-1} \phi \rangle. \]

(61)

To continue, we need

Lemma 6.10. \( H_y^* \eta_y = \overline{E_y \eta_y} \) and \( \eta_y \in \mathcal{H}_+. \)

Proof. First, note that \( P_y^* \eta_y = \eta_y \). Now (Prop. 6.1), \( H_y = P_y H_y + (1 - P_y) H_y (1 - P_y) \), and \( P_y \) commutes with \( H_y \) on \( \eta_y \) in \( \mathcal{H} \). Therefore,

\[ \psi \in \text{dom } H_y \Rightarrow \]

\[ \langle \eta_y | H_y \psi \rangle = \langle \eta_y | P_y H_y \psi \rangle = \langle \eta_y | H_y P_y \psi \rangle = E_y \langle \eta_y | \phi_y \rangle. \]

(62)

This shows that \( H_y^* \eta = \overline{E_y \eta} \). Also, \( \eta_y \in \mathcal{H}_+ \), because (Lemma 4.5) \( H_y^* = [h_n]_0^\oplus \) and \( h_y \in \mathcal{C}^4 \) even if not in our parameterization.

Using this Lemma, the previous display is rewritten as

\[ - (\zeta - E_y)^{-2} (\eta | D \hat{H}_y \phi \rangle \]

which, inserted into the contour integral allows an easy evaluation. In conclusion,

Proposition 6.11.

\[ D_y E_y \bigg|_x = D_y \langle \eta_x | h_y | \phi_x \rangle \bigg|_x. \]

(63)

To do much with this requires explicit knowledge of \( h \).

1. charge/current density

For a concrete case, consider a regular sectorial family of Schrödinger forms as in Section 5. The differentials of \( E \) with respect to \( u \) and \( A \) are linear forms on a perturbation \( \delta u \) or \( \delta A \) (the ‘\( \delta \)’ doesn’t actually have any independent meaning from our perspective), given by

\[ D_u E \cdot \delta u = \sum \langle \eta | \delta u(x_\alpha) \phi \rangle \]

\[ =: \int \delta u \rho \, dx, \]

(64)

and

\[ D_A E \cdot \delta A = \langle \delta A | \phi \rangle \]

\[ =: - \int \delta A \cdot J \, dx \]

(65)

using the abbreviated notation of (63). These define the charge density \( \rho \) and current density \( J \) of the state in question. In classical notation, one writes \( \rho = \delta E / \delta u \) and \( J = - \delta E / \delta A \). More explicitly,

\[ \rho(x) = \sum \int \eta \phi(x_\alpha = x) \, dx \]

(66)

and

\[ J(x) = 2A(x) \rho(x) + \sum \int i \eta \phi(x_\alpha = x) \, dx \]

(67)

where the notation means that integration is over all positions except those of particle \( \alpha \), which is set equal to \( x \).

Of course, when \( h \) is not hermitian, the physical interpretation of these as charge/current densities is rather unclear, but the identifications are natural generalizations, indeed analytic continuations.

Restricted to hermitian \( h, \rho \) and \( J \) are \( \mathbb{R} \)-analytic, but as maps into what Banach spaces? Simplifying very slightly what we had in Section 5 we take \( u \) and \( A \) in \( \mathcal{F}_u = L^{3/2} (\mathbb{R}^3) + L^\infty (\mathbb{R}^3) \) and \( \mathcal{F}_A = L^1 (\mathbb{R}^3) + L^\infty (\mathbb{R}^3) \), respectively. As differentials of a scalar function on \( \mathcal{F}_u \times \mathcal{F}_A \), then, \( (\rho, J) \) is in \( \mathcal{F}_u^* \times \mathcal{F}_A^* \), a priori. This is highly inconvenient due to the presence of the \( L^\infty \) summands. Fortunately, we can show that \( \rho \in \mathcal{F}_u = L^3 \cap L^1 \), and \( J \in \mathcal{F}_A = L^{3/2} \cap L^1 \). Then, if characteristic \( (u, A) \) analytic into \( \mathcal{F}_u \times \mathcal{F}_A \) because (61) \( \mathcal{F}_u = \mathcal{F}_u^* \), which implies that \( \mathcal{F}_u \) is embedded into \( \mathcal{F}_u^* \) as a closed subspace, and similarly \( \mathcal{F}_A \) into \( \mathcal{F}_A^* \). Here, we understand \( L^p \cap L^q \) to be equipped with the max norm \( || f || = \max (|| f ||_p, || f ||_q) \).

It suffices to show that \( \rho \) and \( J \) are integrable, since the integral forms (64, 65), and the fact that they induce linear functionals on \( L^{3/2} \) and \( L^3 \), respectively, then shows that \( \rho \in L^3 \) and \( J \in L^{3/2} \). Here are the required bounds: First, from (66), \( || \rho || \leq N \| \eta \|^2 = N \| \rho^* P \| \leq N \| P \|^2 \| \rho \|^2 \), \( P \) being the state projector [see (60)]. Then, from (67), what was just shown establishes that \( \rho A \) is integrable, and the Cauchy-Bunyakovsky-Schwarz inequality shows that the second term is also, since \( \eta, \phi \in \mathcal{H}_+ \). As discussed in the Introduction, these conclusions are relevant to density functional theory (DFT), current-density functional theory (CDFT), and magnetic-field density functional theory.

7. SEMIGROUPS AND STATISTICAL OPERATORS

Whereas the ideas of the previous section trace their lineage back to the primitive notion of inversion, the progenitor of this section is exponentiation. We will study the operator family \( e^{-\beta H} \) as \( \beta \) ranges over a vertex-zero sector and \( H \) over operators associated with a regular sectorial family. In quantum statistical mechanics, \( e^{-\beta H} \), assuming it is trace-class, is the unnormalized statistical operator of a system with Hamiltonian \( H \) at temperature \( T = \beta^{-1} \). The trace, \( Z_{\beta, H} = \text{Tr} e^{-\beta H} \), is the partition function, and \( F_{\beta, H} = - \beta^{-1} \ln Z_{\beta, H} \) is interpreted as
thermodynamic free energy. At nonzero temperature, the statistical operator and free energy play roles analogous to those played by the ground state and ground state energy at zero temperature. Temperature, however, is not the only thermodynamic control parameter. For a system with variable particle number(s), for instance, there are chemical potentials \( \mu_i \) for the various species, i.e. \( \beta H \) should be replaced by \( \beta (H - \sum \mu_i N_i) \), where \( N_i \) is the number of particles of species \( i \). This can be treated as a Hamiltonian on a Fock space with variable particle number. Another thermodynamic parameter, volume can be incorporated in the form of a confining potential. In this way, we naturally move in the direction of considering the Hamiltonian as being a highly variable object and studying the dependence of the statistical operator and free energy on it.

This statistical interpretation ceases to be viable if the trace-class requirement is dropped, but this more relaxed setting also has physical interest, especially in connection with ideas around “imaginary time” evolution. Here, the semigroup aspects come to the fore. \([0, \infty) \ni \beta \mapsto T(\beta) \equiv e^{-\beta H} \) should be the operator semigroup generated by \(-H\). As Cor. 4.13 showed that the \( \mathcal{R} \)-map \( (\zeta, x) \mapsto \mathcal{R}(\zeta, x) = (\zeta - Hx)^{-1} \) is holomorphic on its natural domain in \( \mathbb{C} \times \mathcal{U} \), Cor. 7.8 shows that the \( \mathcal{E} \)-map \( (\beta, x) \mapsto \mathcal{E}(\beta, x) = e^{-\beta H x} \) is holomorphic, where \( \beta \) in the right half-plane \( \mathbb{C}_+ \) is restricted only by the requirement of sectoriality. Section 7 C considers a case where the statistical interpretation is viable. With \( H_0 \) a lower-bounded self-adjoint operator with resolvent in some Schatten class, and an regular sectorial family in \( \mathcal{B}(H_0) \), \( F_{\alpha x} \) is holomorphic for \( \beta \) in some neighborhood of \( \mathbb{R}_+ \) and \( x \) in some neighborhood of zero. Similarly to the case of nondegenerate eigenstates considered in section 6 E, this implies analyticity of (generalized) observables. Charge-density and current-density are again examined in detail.

A. Operator semigroups

We begin with a recollection of some relevant definitions. A map \( U : [0, \infty) \to \mathcal{L}(\mathcal{X}) \) is a strongly continuous operator semigroup if

1. It respects the semigroup structure of \([0, \infty)\): \( U(0) = \text{id} \) and \( U(s + t) = U(s)U(t) \).
2. For each \( x \in \mathcal{X} \), the orbit map \( t \mapsto U(t)x \) is continuous.

The generator \( A \) of the semigroup is defined by

\[
Ax = \lim_{t \downarrow 0} \frac{Ax - x}{t},
\]

(68)

dom \( A \) being the subspace on which the limit exists. \( A \) is a closed operator with dense domain and for \( x \in \text{dom} \( A \) \), \( \frac{d}{dt} U(t)x = U(t)A x \) (e.g., Engel & Nagel, Thm II.1.4 and Lemma II.1.1). The semigroup \( U(t) \) is often denoted \( e^{tA} \), which can be understood in a very straightforward (power series) sense when \( A \) is bounded. A strongly continuous semigroup is necessarily locally bounded in operator norm.

If we leave everything above the same, except to expand the domain from \([0, \infty) \) to \( \text{Sctr}(0, \theta) \cup \{0\} \) (also a semigroup), \( U \) is a holomorphic semigroup. That the appellation is deserved follows from denseness of \( \text{dom} \( A \) \) and local boundedness, which implies that \( U \) is strongly holomorphic, and therefore [Lemma 2.6(e)] holomorphic \( \text{Sctr}(0, \theta) \to \mathcal{L}(\mathcal{X}) \).

Now, if \( H \) were bounded, \( e^{-\beta H} \) could be obtained with a Riesz-Dunford-Taylor integral of the function \( e^{-\beta \zeta} \) along a contour surrounding the entire spectrum. If \( H \) is sectorial, though, its spectrum is unbounded only toward the right in \( \mathbb{C} \), where \( e^{-\beta \zeta} \) is rapidly decreasing, assuming \( |\arg \beta| \) is not too large. This suggests that a contour such as \( \Gamma \) in Fig. 2 might work. That it does so is the content of the following theorem, for the proof of which we refer to the secondary literature.

**Definition 7.1.** The contour \( \Gamma \) in \( \mathbb{C} \) parameterized by arc-length \( s \) is adapted to sector \( \Sigma \) if \( \text{Re} \Theta(s) \to +\infty \) as \( s \to \pm \infty \), and \( \Gamma \) is exterior to some dilation of \( \Sigma \) item (4) Sec. 4A).

**Theorem 7.2.** Let \( A \) be a densely-defined operator with \( \text{spec} \( A \) \) contained in a sector \( \Sigma \) of half-angle \( \theta \), such that

\[
\zeta \not\in \Sigma' \Rightarrow \|R(\zeta, A)\| \leq \frac{M(\Sigma')}{|\zeta| + 1}.
\]

(69)

for every dilation \( \Sigma' \) of \( \Sigma \). Then, with \( \Gamma \) a contour adapted to \( \Sigma \), a holomorphic semigroup \( \text{Sctr}(0, \pi - \theta) \to \mathcal{L}(\mathcal{X}) \) with generator \( A \) is defined by

\[
\beta \mapsto e^{-\beta A} = \int_{\Gamma} R(\zeta, A)e^{-\beta \zeta} \frac{d\zeta}{2\pi i}.
\]

(70)
Proof. See §II.4 of Engel & Nagel, §IX.1.6 of Katz, or §X.8 of Reed & Simon.

Because $e^{-\beta A}$ is holomorphic into bounded operators, it has a strong regularizing property not enjoyed by the generic operator semigroup:

Corollary 7.3. $\beta \mapsto e^{-\beta A}$ is a continuous linear map of $\mathcal{H}$ into $\text{dom } A$ (with the $A$-norm).

B. The exponential map $\mathcal{E}$

Just as we earlier expanded the usual holomorphy of the resolvent $\mathcal{R}(\zeta, H)$ with respect to the spectral parameter to find that it was holomorphic in a parameterization of $H$ via a regular sectorial family, we will in this subsection (Thm. 7.6) expand the holomorphy of $\beta \mapsto e^{-\beta H}$ just discussed to include $H$. If we imagine varying $A$ in $\mathcal{E}$, we see that we should restrict to $A$ with spectrum in a sector to which $\Gamma$ is adapted. Since we deal with operators coming from $s$-forms, we want to consider sectors for the numerical ranges, not the spectra.

Notation 7.4. For a sector $\Sigma$, $\text{Op}(\Sigma)$ denotes the set of closed, densely defined operators on $\mathcal{H}$ with numerical range in $\Sigma$.

A key ingredient of the theorem is the following lemma, which shows that the resolvent bound in Thm. 7.2 is respected.

Lemma 7.5. Given sector $\Sigma$, and $\Sigma'$, a dilation of $\Sigma$, there is a constant $M(\Sigma, \Sigma')$ such that

$$\zeta \not\in \Sigma' \Rightarrow \|\mathcal{R}(\zeta, H)\| < \frac{M(\Sigma, \Sigma')}{|\zeta| + 1} \quad (71)$$

for every $H \in \text{Op}(\Sigma)$.

Proof. This is an immediate consequence of Prop. 1.20.

Theorem 7.6. Let $h$ be a regular sectorial family. With the notation $H_y = [h_y]^0$ as in Sec. 4.B,

$$y \mapsto e^{-H_y} : \mathcal{U} \to \mathcal{L}(\mathcal{H}) \quad (72)$$

is holomorphic.

Proof. Let $\Sigma$ be an ample sector for $h_x$. Thus $\text{cl num } H_x$ and, a fortiori, $\text{spec } H_x$ is contained in $\Sigma$. Furthermore, by Lemma 1.17 there is a neighborhood $\mathcal{V}$ of $x$ such that for $y \in \mathcal{V}$, the same holds for spec $H_y$.

Now, let $\Gamma$ be a contour adapted to $\Sigma$ (Fig. 2) parameterized by arc length $s$, and $\Gamma_n$ the restriction to $-n \leq s \leq n$, for $n \in \mathbb{N}$. The integrals

$$\mathcal{I}_n(y) := \int_{\Gamma_n} \mathcal{R}(\zeta, H_y)e^{-\zeta \frac{d\zeta}{2\pi i}}, \quad (73)$$

and $\mathcal{I}(y)$, the integral over the entire contour, are well-defined on $\mathcal{V}$. Since $\Gamma_n$ is compact, Thm. 4.10 guarantees that $y \mapsto \mathcal{I}_n(y)$ is holomorphic.

Finally, holomorphy of $\mathcal{I}$ will be secured by uniform convergence $\mathcal{I}_n \to \mathcal{I}$ on $\mathcal{V}$, according to Prop. 2.11. Such convergence holds due to the damping factor $e^{-\Re \zeta}$ in the definition of $\mathcal{I}(y)$ combined with the resolvent bound in Lemma 7.5, which holds uniformly on $\mathcal{V}$.

Definition 7.7. For a regular sectorial family $h$, the $\mathcal{E}$-map is defined by

$$\mathcal{E}(\beta, x) = e^{-\beta H_x} \quad (74)$$

on the domain

$$\Omega := \{ (\beta, x) \in \mathcal{C}_1 \times \mathcal{U} \mid \beta H_x \text{ is sectorial} \}, \quad (75)$$

where $\mathcal{C}_1$ is the open right half-plane $\Re \beta > 0$.

As with the $\mathcal{R}$-map, we may also write $\mathcal{E}(\beta, t)$ for a particular $s$-form $t$, thinking of $C^\sigma$ as a regular sectorial family parameterized over itself.

Corollary 7.8. Let $h$ be a regular sectorial family defined on $\mathcal{U}$, with associated family $x \mapsto H_x$ of closed operators. Then $\mathcal{E} : \Omega \to \mathcal{L}(\mathcal{H})$ is holomorphic.

C. Statistical operator and free energy

In quantum statistical mechanics, $e^{-\beta H_x}$ is used in the following way: with real $\beta$ interpreted as inverse temperature, the partition function is $Z_{\beta, x} = \text{Tr } e^{-\beta H_x}$, the free energy is $F_{\beta, x} = -\beta^{-1}\ln Z_{\beta, x}$, and the statistical operator is $\rho_{\beta, x} = Z_{\beta, x}^{-1}e^{-\beta H_x}$. The latter describes the (mixed) thermal state at inverse temperature $\beta$ under Hamiltonian $H_x$, so that the thermal expectation of (bounded, at least) observable $B$ in this state is

$$(B)_{\beta, x} = \text{Tr } \rho_{\beta, x} B. \quad (76)$$

The basic condition for this to make mathematical sense is that $e^{-\beta H_x}$ be trace-class. When we generalize to allow non-real $\beta$ and non-self-adjoint $H_x$, the additional condition that $Z_{\beta, x} \neq 0$ is required.

Phase transitions are generally identified with points of non-analyticity of the free energy density in the thermodynamic limit (quantity of matter tends to infinity at fixed temperature and pressure, or whatever parameters are appropriate). For simple lattice models in particular, it is easy to see that free energy density is analytic for finite systems, while (not so easy to see) singularities can occur in the thermodynamic limit. This is strongly connected with the dogma that phase transitions are phenomena purely of the thermodynamic limit. One may well ask, however, to what extent we may rule out non-analyticity with more realistic Hamiltonians and a greater, possibly infinite, number of parameters, without any thermodynamic limit. This question is addressed here. Thm. 7.6 is an important stepping stone, but the
conclusion of holomorphy into \( L(\mathcal{H}) \) must be strengthened.

A very useful frame in which to think is that of an “unperturbed” Hamiltonian with a polynomial-bounded energy density of states. This seems to be about the right assumption, since it will allow a good perturbation theory as we shall see, while being satisfied in the usual models. For instance, for \( N \) distinguishable particles moving in \( d \) dimensions, the density of states for a harmonic oscillator hamiltonian is \( O(E^{3N-1}) \), and for the usual kinetic energy in a box with periodic boundary conditions, \( O(E^{3N-2}/2) \). Nontrivial quantum statistics or repulsive interactions only improve matters, by the min-max principle. The main theorem (7.9) is framed in the context of the space \( B(H_0) \) of Sec. 1G, where \( H_0 \) is a lower-bounded self-adjoint operator with resolvent in \( L^p(\mathcal{H}) \) for some \( p \), and says that \( e^{-\beta H_0} \) is holomorphic for \( \beta \) in a nontrivial sector and \( x \) in some neighborhood of 0. The key ideas involved are Prop. 6.5 bounding the integral (70) simply by bounding the \( R(\zeta,H_0) \), and using elementary semigroup properties to get an \( L^1(\mathcal{H}) \) bound from an \( L^p(\mathcal{H}) \) bound.

Here is the main result of this subsection.

**Theorem 7.9.** If self-adjoint \( H_0 \) is such that \( R(\zeta, H_0) \) is in \( L^1(\mathcal{H}) \) for one (hence every) resolvent point and some 1 \( \leq p < \infty \), then

\[
E : \text{Sctr} \left( 0, \frac{\pi}{4} \right) \times B_1(B(H_0)) \rightarrow L^1(\mathcal{H})
\]

is holomorphic.

Proof. The theorem proceeds through four lemmas. The first reduces the context from \( L^1(\mathcal{H}) \) to \( L^p(\mathcal{H}) \).

**Lemma 7.10.** Suppose

\[
E : \text{Sctr} \left( 0, \theta \right) \times \mathcal{U} \rightarrow L^p(\mathcal{H})
\]

is locally bounded. Then, \( E \) is holomorphic into \( L^1(\mathcal{H}) \).

Proof. According to Thm. 7.6 and Prop. 6.5, what needs to be shown is that \( E \) is a locally bounded map into \( L^1(\mathcal{H}) \). Given the hypotheses, though, that follows from the generalized Hölder inequality

\[
\|e^{-\beta H}\|_1 \leq \|e^{-(\beta/p)H}\|_p. \tag{77}
\]

To make use of this Lemma, we need conditions which will ensure the hypothesized local \( L^p \)-boundedness. In the next two Lemmas, sector \( \Sigma \), and \( \Sigma' \) a dilation of \( \Sigma \), and a point \( \zeta_0 \notin \Sigma' \) are understood as given, while \( H \) is arbitrary in \( \text{Op}(\Sigma) \). They reduce the problem to one of bounding \( \|R(\zeta_0, H)\|_p \).

**Lemma 7.11.**

\[
\|R(\zeta, H)\|_p \leq C(\Sigma, \Sigma', \zeta_0)\|R(\zeta_0, H)\|_p \tag{78}
\]

Proof. Lemma 7.5 ensures that the factor in square brackets in the resolvent identity

\[
R(\zeta, H) = [1 + R(\zeta, H)(\zeta - \zeta_0)]R(\zeta_0, H), \tag{79}
\]

is bounded uniformly for \( \zeta \notin \Sigma' \).

**Lemma 7.12.**

\[
\|e^{-\beta H}\|_p \leq M(\Sigma, \Sigma', \zeta_0, \beta)\|R(\zeta_0, H)\|_p, \tag{80}
\]

with \( M(\Sigma, \Sigma', \zeta_0, \beta) \) locally bounded in \( \beta \in \text{Sctr} (0, \frac{\pi}{2} - \theta) \), where \( \theta \) is the half-angle of \( \Sigma' \).

Proof. Let contour \( \Gamma \) satisfy \( \zeta_0 \in \Gamma \subset \Sigma' \) (hence, \( \Gamma \) is adapted to \( \Sigma \)). Then,

\[
\|e^{-\beta H}\|_p = \left\| \int_\Gamma R(\zeta, H)e^{-\beta \zeta} \frac{d\zeta}{2\pi i} \right\|_p \leq \int_\Gamma \|R(\zeta, H)\|_p e^{-Re\beta \zeta} \frac{d|\zeta|}{2\pi} \leq C(\Gamma, \beta) \sup_{\zeta \in \Gamma} \|R(\zeta, H)\|_p \leq M(\Sigma, \Sigma', \zeta_0, \beta)\|R(\zeta_0, H)\|_p. \tag{81}
\]

The third line follows since \( \int |e^{-Re\beta \zeta}| d|\zeta| < \infty \), and the fourth line is by Lemma 7.11.

**Lemma 7.13.** If \( \text{dom} H \subseteq \text{dom} A \) and \( \|A R(\zeta_0, H_0)\| < 1 \), then

\[
\|R(\zeta_0, H + A)\|_p \leq (1 + \|A R(\zeta_0, H)\|)^{-1}\|R(\zeta_0, H)\|_p \tag{82}
\]

Proof. Immediate.

**Completion of Proof of Thm. 7.9.** Now it is merely a matter of stringing the pieces together. Prop. 6.5 asserts that local boundedess of \( e^{-\beta H_0} \) in \( L^1(\mathcal{H}) \) suffices to establish holomorphy, Lemma 7.10 shows that \( L^1(\mathcal{H}) \) can be replaced by \( L^p(\mathcal{H}) \); Lemma 7.12 that we only need a local bound on \( R(\zeta_0, H_0) \); and Lemma 7.13 shows how big the perturbation can be. According to the definition of \( B(H_0) \) (see Section 4G, especially Prop. 4.17), it suffices that \( |t - h| \|H_0 \| < 1 \). The restriction on \( \beta \) is needed to insure that \( \beta H_0 \) is sectorial for all \( x \) in \( B_1(B(H_0)) \).

**D. Thermal expectations**

This subsection is concerned with consequences of Thm. 7.9. In other words, what do we do with the holomorphic statistical operator? We Suppose given a regular sectorial family in \( B_1(B(H_0)) \), and adopt the notational convention that \( h_x \) corresponds to the operator \( H_0 + T_x \) on \( \text{dom} H_0 \) (i.e., this is \( [h_x]_0 \)). We can be fairly explicit about the Taylor series expansion of \( e^{-\beta (H_0 + T_x)} \). By appeal to Cor. 7.3

\[
e^{-\beta (H_0 + T_x)} = \int_0^1 e^{-s\beta (H_0 + T_x)}(-\beta T_x)e^{-(1-s)\beta H_0} \, ds \tag{82}\]
for any $h_x \in B(H_0)$. Iteration shows that the $n$-th term of the Taylor series has the familiar form
\[
(-\beta)^n \int_{s_0 \geq 0} e^{-s_0+\beta H} T_x e^{-s_0+\beta H} \ldots T_x e^{-s_0+\beta H} ds_0.
\]
Thm. 7.9 implies that this actually converges for small enough $x$.

In the following, we will be concerned only with the first term, however. For $(\beta, x)$ in some neighborhood of $\mathbb{R}_+ \times \{0\}$, $Z_{\beta, x}$ is nonzero and therefore the free energy $F_{\beta, x}$ is well-defined and holomorphic. According to (82), the derivative (also holomorphic) $-\beta D_x F_{\beta, x}$ is the expectation value $\text{Tr} \rho_{\beta, x} D_x T_x$.

1. charge/current density

Parallel to the treatment of properties of energetically-isolated eigenstates in Section 6.1, we will consider charge and current-density in the thermal context for an system of $N$ nonrelativistic particles in a three-dimensional box, under a periodic boundary condition Hamiltonian consisting of a kinetic energy operator $K_A = \sum_{n=1}^{N} |\nabla_n + A(x_n)|^2$, a one-body potential operator $U_u = \sum_{n} u(x_n)$, and a repulsive two-body interaction $V_{\nu} = \frac{1}{2} \sum_{\neq \beta} v(x_\beta - x_\beta)$. The variables here are $u$ and $A$. $H_0 = K_A + V_{\nu}$ has a polynomial-bounded density of states, hence Thm. 7.9 applies and the free energy is holomorphic. Charge and current-density are obtained by differentiating the free energy with respect to $u$ and $A$, respectively, hence also holomorphic if the perturbed Hamiltonians comprise a regular sectoral family in $B(H_0)$.

Now, apply the result of Kato (§ 5.5.3 of Kat) Thm. 6.2.2 of de Oliveira [13] or Example 13.4 of Hislop & Sigal [23] that a potential in $L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$ is relatively bounded with respect to the $A = 0$ kinetic energy operator $-\Delta$ with relative bound zero (Def. 1.1). Since the system is confined to a box, a bounded potential is automatically square-integrable.

For the kinetic energy operator
\[
|\nabla + A|^2 = -\Delta + 2i A \cdot \nabla + i \text{div} A + |A|^2,
\]
$A$ must be restricted so that each of the last three terms is adequately tame. It will suffice that $A \in \tilde{L}^4(\text{Box})$ if we work in Coulomb gauge, i.e., $\text{div} A = 0$, or in Fourier components, $q \cdot A(q) = 0$. We will denote this subspace of “transverse” vector fields by $\tilde{L}^4(\text{Box})_{\text{trans}}$. That restriction obviously takes care of the divergent term.

\[
\|A\|^2_{L^2} \leq c \|A\|^2_{L^4},
\]
by Hölder’s inequality. Finally, since the box is bounded, $\tilde{L}^4(\text{Box})$ is continuously embedded in $\tilde{L}^4(\text{Box})$, so $|A \cdot \nabla \psi|_{L^2} \leq c$ when $\psi \in \text{dom} (-\Delta)$. For the scalar potential, no trickery is required to apply the result cited above. Simply assume $u, v \in L^2(\text{Box})$.

Thus, we obtain a regular sectorial family in $B(H_0)$ defined for $x \equiv (u, A)$ on some neighborhood of the origin in $L^2(\text{Box}) \times \tilde{L}^4(\text{Box})_{\text{trans}}$. The charge/current density $\rho(A) = -\beta D_x F_{\beta, x}$ is then an analytic function of $x$ valued in $L^2(\text{Box}) \times \tilde{L}^{4/3}(\text{Box})$.

8. SUMMARY

Here is a summary of the apparatus developed here, from an application-oriented perspective. The starting point is a family $h: \mathcal{H} \supseteq \mathcal{U} \rightarrow \mathbf{SF}^3(\mathcal{H})$ of closable, mutually relatively bounded, sectoral s-forms parameterized over $\mathcal{U}$. Thinking of these as generalized Hamiltonians, sectoriality is an appropriate generalization of lower-bounded and hermitian, which allows use of holomorphy. If quantities related to these forms $h_x$ or their associated operators $H_x$ are holomorphic in the parameter $x \in \mathcal{U}$, then real analyticity results for proper Hamiltonians by restriction. If $x \rightarrow h$ is a regular sectoral family, then holomorphic of $\langle \zeta, x \rangle \mapsto R(\zeta, x) = (\zeta - H_x)^{-1}$ and $(\beta, x) \mapsto \mathcal{E}(\beta, x) = e^{-\beta H_x}$, as maps into $\mathcal{L}(\mathcal{H})$, is secured on natural domains. This is the content ofCors. 7.8 and 11.13 respectively. Prop. 11.14 provides a few sets of convenient criteria for $x \rightarrow h$ to be a regular sectoral family. One of these is, (a) G-holomorphy: for each $x \in \mathcal{U}$, $w \in \mathcal{H}$, and $\psi \in \mathcal{H}$, $\zeta \mapsto h_{x + \varepsilon w}[\psi]$ is holomorphic on some neighborhood of the origin in $\mathcal{C}$; and (b) local boundedness: each $x \in \mathcal{U}$ has a neighborhood such that $h_x$ is bounded uniformly relative to $h_x$ for $y$ in that neighborhood. The practicality of these criteria is demonstrated in Section 5, where a regular sectorial family of multi-particle Schrödinger forms is constructed.

The $R$-map and $E$-map are themselves mostly means to an end. An important tool in using them is Prop. 6.5 which says that either is actually holomorphic into the Schatten class $\mathcal{L}^b(\mathcal{H})$ (not just into $\mathcal{L}(\mathcal{H})$) if it is merely locally bounded into $\mathcal{L}^b(\mathcal{H})$. Using this, we can effectively deal with properties of isolated eigenstates, or of thermal states when the $E$-map is verified to be locally bounded into trace-class operators (Thm. 7.9). Particularly interesting are derivatives of the energy or free energy with respect to scalar potential $u$ or vector potential $A$, which give (expectation of) charge-density and current-density, respectively. As differentials of holomorphic functions, these are automatically holomorphic themselves. In the case of isolated eigenstates (Section 6.1), $(\rho, J)$ is analytic in $(L^1(\mathbb{R}^3) \cap L^1(\mathbb{R}^3) \times (L^{3/2}(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)))$, as function of $(u, A)$ in $(L^{3/2} + L^\infty) \times (\tilde{L}^2 + \tilde{L}^\infty)$. For thermal states, additional restrictions are required on the potentials to ensure existence of the free energy. For a system in a box (Section 7.1), $(\rho, J)$ is analytic in $L^2(\text{Box}) \times \tilde{L}^{4/3}(\text{Box})$ as function of $(u, A)$ in $L^2 \times \tilde{L}^4$, with $A$ in Coulomb gauge.
