SURREAL LIMITS

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Abstract. We note that if a sequence of real numbers converges to some limit, then the sequence of the corresponding strings in the surreal +, − sign expansion representation converges, for a natural notion of string convergence, to the string corresponding to the limit, modulo an infinitesimal. The corresponding statement would be obviously false if we were considering, as strings, decimal or binary representations, instead. The string limit of a possibly transfinite sequence of surreal numbers is always defined and, when considering increasing sequences of ordinals, corresponds to taking the supremum. A transfinite sum can be defined using the string limit and this sum agrees with the representation of a surreal number in Conway normal form.

1. Introduction

Limits of sequences from the surreal point of view. If we compute, say, the number $e$ by means of the usual series expansion $e = \sum_{n=0}^{\omega} \frac{1}{n!}$, we get the following sequence of partial sums:

$$1, 2, 2.5, 2.666\ldots, 2.7083\ldots, 2.7166\ldots, 2.7180\ldots, 2.7182\ldots$$

Since the digits in the above decimal expressions eventually stabilize, we get confident that 2.718… approximates the actual decimal expansion of $e$ (of course, this is not a proof!). Though a similar argument goes well for many limits, it is not always the case that digits eventually stabilize in converging sequences. For example, consider the sequence

$$1.99, 2.01, 1.999, 2.001, 1.9999, 2.0001, \ldots,$$

which obviously converges to 2. In this case, no digit at all stabilizes.

We observe that, instead, if we represent real numbers as sequences of +'s and −’s by means of the surreal sign expansion, and we consider a converging (in the classical sense of real analysis) sequence of real numbers, then the +'s and −’s which stabilize give the surreal sign expansion of the limit, possibly with the difference of an infinitesimal. See Theorem 2.2. This shows that the apparently odd and strange surreal way of representing numbers, in particular, reals, by means of a sequence of +'s and −’s is, in a sense, actually a more natural way, in comparison with the usual decimal or binary expansions. So far, the above result is just slightly more than a mere curiosity, but a few arguments presented below suggest at least some remote eventuality of further developments.

Some history. We refer to Conway [C], Ehrlich [E], Gonshor [G] and Siegel [S] for details and history about surreal numbers, in particular, for details about the sign expansion.

The usual ε-δ definition cannot be applied to sequences of surreals, at least if ε is intended to vary among all surreals, since there are always plenty of surreals between

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the elements of a sequence and any purported limit. Conway himself wrote about this situation as follows:

"For instance, the limit of the sequence 0, 1, 2, 3, ... (ω terms) is not 1, at least in the ordinary sense, because there are plenty of numbers in between. A simpler, but sometimes less convincing, example of the same phenomenon is given by the sequence 0, 1, 2, 3, ... of all finite ordinals, which one would expect to tend to ω, but which obviously can’t since there is a whole Host of numbers greater than every finite integer but less than ω. For the author’s amusement, we recall some of the simplest of them:

ω − 1, ω/2, √ω, ω^{1/ω}, ..."

**Definitions of limit up to now.** There are some existing approaches to overcome with the difficulties described above. Here we unify and extend the approaches from [M, L2], which appeared as archived but not otherwise published manuscripts.

A surreal notion of convergence has been introduced by Mező [M]. The idea from [M] is to consider, as possible limits, only surreals of length at most the superior limit of the lengths of the elements of the sequence. Then convergence is established in the way hinted above, considering sign persistence. In fact, the definition from [M], when taken literally, imports that if \( s \) is a limit of a sequence, then every initial segment of \( s \) is a limit, too. In this sense, for countable sequences, we shall consider here the unique longest Mező’s limit. In [M] countably infinite sums are also considered.

In another direction, Rubinstein-Salzedo and Swaminathan [RS] noticed that the \( ε-δ \) condition makes sense for the surreals, too, provided one considers class \( \text{On-long} \) sequences. For such class-sequences, the definition given in [RS, Definition 19] seems to bear a deep resemblance with the present one (see the beginning of Section 4), though the relationships between the two notions have not been fully analyzed yet.

Then Lipparini [L2] rediscovered the surreal limit and generalized it to (set) ordinal-indexed sequences. His work was motivated by the study of transfinite iterations of the natural sum on the ordinals. See [L1]. Indeed, in the sense of surreal sign expansions, an ordinal is a sequence of +'s. If we have an increasing sequence of ordinals, then every place in the sequence is stable from some point on, hence a construction similar to the above one furnishes the limit, that is, the supremum, of the sequence of ordinals. See Section 3 for more details and for the transfinite sum operation arising from the construction.

The above idea of limit can be carried over in general, dealing with sequences of transfinite strings of symbols, or, even more generally, sequences of labeled linearly ordered sets. In this sense, Lipparini [L2] has introduced a rather general notion; see Section 4 here, in particular, Definition 4.1. However, in the next section we shall be content with the particular case of countable sequences of surreal numbers, mostly, reals. Then in Section 3 we shall deal with transfinite sequences of surreals.

The present version of the paper has been mostly written by Paolo Lipparini. He is the sole responsible for any error, omission or inaccuracy.

**2. Surreal limits of real numbers**

**The \( s \)-limit.** For notational convenience, we shall identify a surreal number with its sign expansion, that is, its representation as an ordinal-indexed sequence of +'s and −'s. Thus a surreal number is a function \( s : α_s → \{+, −\} \), where \( α_s \) is an ordinal depending on \( s \). The ordinal \( α_s \) will be called the length of \( s \); in the original theory developed by J. Conway, \( α_s \) corresponds to the birthday of \( s \). If \( β < α_s \), \( s(β) \) will be sometimes called place \( β \) (in the representation) of \( s \), and places \( ≥ α_s \) will be dubbed undefined.
**Definition 2.1.** If \((s_n)_{n<\omega}\) is a sequence of surreal numbers, we define the *s-limit of \((s_n)_{n<\omega}\)*, in symbols, \(\text{slim}_{n<\omega} s\), as the surreal \(s\) such that place \(\gamma\) in the sign expansion of \(s\) is defined if and only if there is \(m < \omega\) such that, for every \(n \geq m\), the sign expansions of the \(s_n\)'s are identical (and defined) up to place \(\gamma\) included. If this is the case, place \(\gamma\) of \(s\) is set to be equal to the corresponding place of \(s_m\) (hence also of the \(s_n\)'s which follow). Notice that, by construction, if \(s(\gamma)\) is defined, then \(s(\gamma')\) is defined, too, for every \(\gamma' < \gamma\), hence the definition is well posed.

The s-limit of a sequence is always defined; possibly, it is the empty sequence. Notice also that, as far as we meet a place at which the values of the \(s_n\)'s eventually oscillate, we impose that the sign expansion of the s-limit stops at that point. This is necessary if we want Theorem 2.2 below to become true.. As an additional reason in support of the choice, we shall note at the beginning of Section 4 that the s-limit admits a natural game-theoretical definition in terms of Left and Right options.

**The connection between the classical limit and s-limit.** Since a real number is (or can be considered as) a surreal number, we have also the notion of an s-limit of a sequence of reals. The main observation in the present section is the curious fact that if a sequence of real numbers has a limit in the sense of classical real analysis, then the limit and the s-limit coincide, modulo an infinitesimal.

In the proof of the following theorem we shall freely use the Berlekamp’s Sign-Expansion Rule and the characterization of surreals born at day \(\omega\). See, e.g., [8, VIII, 2] for full details. Recall that, under the sign expansion representation, the surreal order is obtained by comparing the first difference. This is quite similar to the lexicographic order, but formally different; in the surreal sense undefined is considered to be between \(-\) and \(+\), rather than before them. Notice that the following theorem deals only with surreals born at most at day \(\omega\). We shall always consider *addition* \(+\) in the surreal sense (symbols overlapping never causes confusion). Of course, when restricted to real numbers, surreal addition coincides with usual addition.

**Theorem 2.2.** If \((r_n)_{n<\omega}\) is a sequence of real numbers and \(\lim_{n\to\infty} r_n\) exists, possibly equal to \(\infty = \omega\) or \(-\infty = -\omega\), then \(\lim_{n\to\infty} r_n = \text{slim}_{n<\omega} r_n + \varepsilon\), where \(\varepsilon\) is either 0, or \(1/\omega\), or \(-1/\omega\).

**Proof.** Let \(r = \lim_{n\to\infty} r_n\). First, suppose that \(r = 0\). In the sense of sign expansions, this corresponds to the empty sequence. If infinitely many \(r_n\)'s are equal to 0, then \(\text{slim}_{n<\omega} r_n = 0\); indeed, in this case, the first sign cannot eventually stabilize and be defined, hence the definition of the s-limit gives the empty sequence. If all but finitely many \(r_n\)'s are strictly positive, then the first sign in the surreal representation is eventually +. Since the sequence converges in the classical sense, then, for every \(m < \omega\), we have \(r_n \leq 1/2^m\) eventually, that is, the first + sign is eventually followed by at least \(m\) minuses. This means that \(\text{slim}_{n<\omega} r_n = +\underbrace{- - - \cdots}_{m+1} = 1/\omega\). If all but finitely many \(r_n\)'s are strictly negative, the symmetric argument gives \(\text{slim}_{n<\omega} r_n = -1/\omega\). In the remaining case we have infinitely many positive \(r_n\)'s, hence infinitely many occurrences of + at the first place, and infinitely many negative \(r_n\)'s, hence infinitely many occurrences of − at the first place. Thus the first place does not eventually stabilize and, according to Definition 2.1, \(\text{slim}_{n<\omega} r_n = 0\), being the empty sequence.

The case when \(r\) is a dyadic rational is similar. A number \(r'\) greater than \(r\) and sufficiently close to \(r\) has the form \(r + - - - \ldots\), where juxtaposition denotes string concatenation and, as far as \(r'\) approaches \(r\), after the string \(r+\) we get a larger and larger number of minuses, possibly followed by a plus and other signs. Thus if \((r_n)_{n<\omega}\) converges to \(r\) eventually from above, \(\text{slim}_{n<\omega} r_n = r + - - - \cdots = r + 1/\omega\). Symmetrically,
if \((r_n)_{n<\omega}\) converges to \(r\) eventually from below, \(\lim_{n<\omega} r_n = r - 1/\omega\). In the remaining cases, either \(r_n = r\) for infinitely many \(n\)'s, or there are infinitely many \(n\)'s such that \(r_n < r\) and there are infinitely many \(n\)'s such that \(r_n > r\). In each of the above cases \(\lim_{n<\omega} r_n = r\).

If \(r\) is real and not dyadic, then its sign expansion is infinite and neither eventually + nor eventually −. Suppose that \(r > 0\), the case \(r < 0\) being treated symmetrically. Since \(r\) is not dyadic, then, in particular, it is not an integer. Letting \([r]\) denote the integer part of \(r\), we have that \(|r - r_n| < \min\{r - [r], [r] + 1 - r\}\), for sufficiently large \(n\), hence, from some point on, the integer part of \(r_n\) is the same as the integer part of \(r\); moreover, \(r_n\) has a binary point, too, hence these parts of the sign expansion eventually stabilize. What remain are the fractional parts, which are computed like the binary expansion, except possibly for a last sign/digit. Since the sign expansion of \(r\) is neither eventually + nor eventually −, then, for every \(m < \omega\), both \(a +\) and \(a −\) occur after the \(m\)th place of the fractional part of \(r\). Let \(q > m\) be such that both \(a +\) and \(a −\) occur between the \(m + 1\)th place and the \(q\)th place of the fractional part of \(r\). If \(|r - r_n| < 2^q\), then \(r\) and \(r_n\) have the same fractional part up to the \(m\)th place. Since \(m\) is arbitrary, the fractional part of the sequence \((r_n)_{n<\omega}\) eventually stabilizes to the value of the fractional part of \(r\). In conclusion, the whole sign expansions stabilize to the sign expansion of \(r\).

Notice that if \(r\) is not a dyadic rational, the same argument works for the binary expansions, as well. Hence, in the sense of the theorem, the sign expansion representation has some actual advantage over the binary one only for the countable set of dyadic rationals.

Finally, the cases when \(r = \infty\) or \(r = -\infty\) are trivial. \(\square\)

The eventual presence of an infinitesimal in Theorem 2.2 does not seem to be a serious drawback. Anyone interested only in real numbers would surely feel free to ignore it. In a sense, the presence of \(\pm 1/\omega\) has some use, since it tells us when the sequence converges from above or from below. However, notice the asymmetry between the cases of dyadic and nondyadic limits, since the infinitesimal can appear only in the former case.

In the above respect, a more uniform version of Theorem 2.2 holds, with exactly the same proof. A surreal \(s\) is finite if \(-n < s < n\), for some natural number \(n\). Any finite surreal can be expressed uniquely as a sum \(s = R(s) + \varepsilon(s)\), where \(R(s) \in \mathbb{R}\) and \(\varepsilon(s)\) is infinitesimal. Then the proof of Theorem 2.2 together with Berlekamp’s Rule and easy facts about the sign expansion shows the following.

**Corollary 2.3.** If \((s_n)_{n<\omega}\) is a sequence of finite surreal numbers and \(\lim_{n \to \infty} R(s_n)\) exists and is finite, then \(\lim_{n \to \infty} R(s_n) = R(\lim_{n<\omega} s_n)\).

**Comparison of limits.** The notion of limit in the classical sense and the notion of s-limit do not coincide, in general, even modulo infinitesimals. Indeed, the latter limit is always defined, while this is not necessarily the case for the former. As we mentioned during the proof, Theorem 2.2 shows that the sign expansion has some real advantage over the binary expansion only in the case of dyadic numbers. However, independently from Theorem 2.2 the sign expansion has the advantage of representing each real uniquely, while the choice of, say, the binary representation 1.000... in place of 0.111... for the natural number 1 might be perceived as somewhat arbitrary.

We say that the s-limit \(s\) of some sequence \((s_n)_{n<\omega}\) is a full s-limit if the length of \(s\) is the inferior limit of the lengths of the \(s_n\)'s; in other words, if \(s\) is as long as possible, as far as this is compatible with the lengths of the \(s_n\)'s. The relationship between the classical limit and the s-limit reverses, if we take into account only full limits. Indeed, if \((r_n)_{n<\omega}\) is a sequence of real numbers and \(s\) is a full s-limit of \((r_n)_{n<\omega}\), then \(\lim_{n \to \infty} r_n\) exists (possibly \(\omega\) or \(-\omega\)) and is equal to \(s + \varepsilon\), for \(\varepsilon\) either 0, or 1/\(\omega\), or \(-1/\omega\). As apparent from the
not every s-limit is full, even in case the sequence is convergent in the classical sense. E. g., the s-limit of the sequence \( (\frac{-1}{2})^n \) is not full; indeed, the elements of the sequence are represented as +, −+, −++, −+−, +++, +++, . . . Every monotone sequence of real numbers has a full s-limit, but also nonmonotone sequences can have a full s-limit, e. g., +, +−, −−, −+, +++, . . . , converging to \( \frac{2}{3} \).

**Inferior and superior limits.** The main idea behind the definition of the s-limit is to take into account not only full limits (by the above remark, this would not be sufficient to represent all classical real limits), but to treat also the case in which signs oscillate at a certain place. The choice made in Definition 2.1 is, in a sense, a neutral one: if the sign at a certain place is not eventually constant, we set the sign to be undefined in the limit. On the other hand, we could have chosen either the smallest possible value, or, in the other direction, the largest value. The above considerations suggest the definitions of slim inf and slim sup that we shall formally give in the general case in Section 4. For countable sequences of real numbers, in order to compute slim inf, and starting from the first place, we choose + if this is the eventual sign of the members of the sequence; we choose − if there are infinitely many − ’s in that place in the sequence (notice the asymmetry between the two cases!), and in this latter case we discard all the members of the sequence with a different sign. In the remaining case (infinitely many undefined) we set the place undefined in slim inf and we stop the construction. In case the construction does not stop at a certain place, we proceed in the same way with the next place, and so on. Notice that, when dealing with real numbers or, more generally, surreals of length at most \( \omega \), we never go beyond the \( \omega \)-th step of the construction. The delicate issue of subsequent steps shall be dealt with in Section 4.

A definition symmetric to slim inf gives slim sup.

Notice that slim, slim inf and slim sup do not necessarily strictly agree even for converging sequences of real numbers. Considering, as above, \( r_n = (\frac{-1}{2})^n \), we have \( \text{slim}_{n<\omega} r_n = 0 \), slim inf\( r_n = -\frac{1}{\omega} \), and slim sup\( r_n = \frac{1}{\omega} \). However, for a converging sequence of reals, slim inf and slim sup and the classical real limit lim do agree modulo an infinitesimal. For a general, possibly non converging, sequence of reals, slim inf agrees with the classical lim inf, modulo an infinitesimal, and the same holds for slim sup and lim sup.

Notice that in the definition of slim inf we do need discard certain members of the sequence in the case of infinitely many minus signs, when the sign is not eventually constant. Otherwise, we would have the unwanted result that, for the above sequence, slim inf\( r_n = -\omega \).

**Comparison with the Limit.** Another notion of limit for \( \omega \)-indexed sequences of surreal numbers is known and useful. It is usually called the Limit, denoted by Lim with upper-case L, and is obtained by taking componentwise the limits (in the sense of real analysis) of the coefficients in the Conway normal representation of the elements of the sequence. This Limit is not always defined: the componentwise real limits should always exist and be finite, and the result should give an actual surreal number (if the result contains an ascending sequence of exponents of \( \omega \), it is not a surreal number).

Though Theorem 2.2 shows that the Limit and the s-limit give quite close results when taking the limit of a sequence of reals, on the other hand, for arbitrary surreals, the two limits could turn out to be quite remote. For example, \( \text{Lim}_{n \to \infty} \frac{-1}{\omega} = 0 \), while \( \text{slim}_{n<\omega} \frac{-1}{\omega} = \sqrt{\omega} \). This last s-limit might appear unnatural, but notice that \( \sqrt{\omega} \) is “multiplicatively halfway” between 1 = \( \frac{\omega}{\omega} \) and \( \omega \). However, \( \text{Lim}_{n \to \infty} (\frac{\omega}{\omega} + 1) = 1 \), while \( \text{slim}_{n<\omega} (\frac{\omega}{\omega} + 1) \) is still \( \sqrt{\omega} \). This shows that the s-limit of a finite sum is not always the sum of the s-limits. In another direction, \( \text{Lim}_{n \to \infty} \omega^n = 0 \), while \( \text{slim}_{n<\omega} \omega^n = \omega^\omega \), which seems much closer to intuition and corresponds to a general rule we shall describe in the next
For example, if \( n_\omega \to \infty (\omega - n) \) is undefined, and \( \text{slim}_{n<\omega} (\omega - n) = \frac{\omega}{2} \), again, halfway between \( 0 = \omega - \omega \) and \( \omega \).

Probably, there is not a unique notion of limit for a sequence of surreals which is good for every purpose; in each particular case one should choose the most appropriate notion.

3. Subsequences and sums

Subsequences. Taking \( s \)-limits of surreal numbers is not always a monotone operation. For example, if \( a_n = n \) and \( b_n = \omega - 1 \), for every \( n < \omega \), we have \( a_n < b_n \), for every \( n < \omega \), but \( \text{slim}_{n<\omega} a_n = \omega > \omega - 1 = \text{slim}_{n<\omega} b_n \). Of course, a similar situation should occur for every notion of limit defined on all countable sequences of surreals, just assuming that the limit of a constant sequence gives its constant value. Then if the limit of some sequence is greater than all the elements of the sequence, you can also find a surreal in between the limit and all the members of the sequence, and the same argument as above applies.

It is not always the case that the \( s \)-limit of a subsequence coincides with the \( s \)-limit of the sequence. If \( a_n = n \), for \( n \) even, and \( a_n = \omega - 1 \), for \( n \) odd, then \( \text{slim}_{n<\omega} a_n = \omega \), but \( \text{slim}_{n<\omega}, n \text{ odd} a_n = \omega - 1 \). However, the \( s \)-limit of a nondecreasing sequence gives a result \( \geq \) than all the elements of the sequence; moreover, a cofinal subsequence of a nondecreasing sequence has the same \( s \)-limit.

Since the proofs of the above facts work as well for sequences indexed by ordinals \( > \omega \), we shall present the general results. Notice that Definition 2.1 can be naturally extended to deal with arbitrary ordinal-indexed sequences; just consider those initial segments of sign expansions which are eventually constant. The reader might assume that we are always dealing with sequences indexed by some limit ordinal; otherwise, for a sequence indexed by a successor ordinal, simply set the \( s \)-limit to be the last element of the sequence. See full details in Definition 4.1 below.

Proposition 3.1. Suppose that \((s_\beta)_{\beta<\alpha}\) is a nondecreasing sequence of surreals with \( s \)-limit \( s \). Then

(a) \( s_\delta \leq s \), for every \( \delta < \alpha \).

\(\text{If (} t_\zeta)_{\zeta<\eta} \text{ is another nondecreasing sequence, with } s \)-limit } t, \text{ then} (b) \( \text{If } (t_\zeta)_{\zeta<\eta} \text{ is (an order-preserving rearrangement of) a cofinal subsequence of } (s_\beta)_{\beta<\alpha}, \text{ then } t = s. \)

(c) Suppose that \((s_\beta)_{\beta<\alpha}\) and \((t_\zeta)_{\zeta<\eta}\) are chained, in the sense that, for every \( \beta < \alpha \), there is \( \zeta < \eta \) such that \( s_\beta \leq t_\zeta \) and conversely. Then \( s = t \).

Proof. Let us say that two surreals \( \gamma \)-agree if their sign expansions agree up to place \( \gamma \) included (allowing the possibility of corresponding places to be both undefined).

(a) Let \( \delta < \alpha \), we want to show that \( s_\delta \leq s \). If, for every ordinal \( \gamma \), place \( \gamma \) in the subsequence \((s_\beta)_{\beta \leq \delta < \alpha}\) is constant (allowing undefined values), then the subsequence itself is constant, the definition of the \( s \)-limit gives \( s_\delta = s \) and we are done. Otherwise, let \( \gamma \) be the first ordinal such that place \( \gamma \) in the subsequence \((s_\beta)_{\delta \leq \beta < \alpha}\) is not constant. By the definition of \( \gamma \), all the \( s_\beta \)'s \( \gamma \)-agree, for every \( \gamma' < \gamma \) and \( \beta \geq \delta \), hence, again, by the definition of the \( s \)-limit, \( s \) and \( s_\delta \) \( \gamma \)-agree, for every \( \gamma' < \gamma \). Since the \( s_\beta \)'s \( \gamma \)-agree, for every \( \gamma' < \gamma \) and \( \beta \geq \delta \), and the sequence is nondecreasing, the only possible transitions at place \( \gamma \) are from \( - \) to undefined or \( + \) and from undefined to \( + \). By the definition of \( \gamma \), at least one transition occurs and, since there is a finite number of possible transitions and no cycle is possible, place \( \gamma \) eventually stabilizes to some value, which will be the value in \( s \). Thus \( s_\delta < s \).
(b) Suppose by contradiction that \( t \neq s \) and let \( \gamma \) be the first place at which they disagree. Hence at least one between \( t(\gamma) \) and \( s(\gamma) \) is defined. Suppose that \( t(\gamma) \) is defined; the other case is similar and easier. By the definition of the \( s \)-limit, there is some \( \zeta \) such that \( t_\zeta \)'s \( \gamma \)-agree with \( t \), for every \( \zeta \geq \zeta \). If \( t_\zeta \leq u < t_\zeta \) and \( t_\zeta \gamma \)-agree, then they \( \gamma \)-agree with \( u \). Since \( t_\zeta \) is cofinal in \( (s_\beta)_{\beta < \alpha} \) and \( (s_\beta)_{\beta < \alpha} \) is nondecreasing, then the \( s_\beta \)'s eventually \( \gamma \)-agree with \( t_\zeta \); hence with \( t \). Then the definition of \( s \)-limit gives that \( s \gamma \)-agrees with \( t \), a contradiction.

(c) Consider a new sequence made by all the elements of both sequences, ordered in such a way that the new big sequence is still nondecreasing. Since the original sequences are nondecreasing, this can be accomplished in such a way that they actually become subsequences of the big sequence. Since the original sequences are chained, they are both cofinal in the big sequence (except perhaps for the trivial case in which all the sequences are eventually constant). If \( u \) is the \( s \)-limit of the big sequence, then, by (b), \( u = s \) and \( u = t \), thus \( s = t \). □

**s-limits of ordinals.** As we hinted to in the introduction, the \( s \)-limit of a non decreasing ordinal-indexed sequence of ordinals is their supremum. In general, the \( s \)-limit of an ordinal-indexed sequence of ordinals is their inferior limit, i. e., the supremum of the set of those ordinals \( \alpha \) such that the members of the sequence are eventually \( \geq \alpha \).

**Surreal series.** Since an addition operation is defined among surreal numbers, any notion of limit entails the definition of a series. If \( (s_n)_{n<\omega} \) is a sequence of surreals, let \( \sum_{n<\omega} s_n \) be \( \text{lim}_{n<\omega} S_n \), where \( S_n \) denotes the partial sum \( s_0 + s_1 + \cdots + s_{n-1} \). By Proposition 3.1(c), if all the \( s_n \)'s are nonnegative, then \( \sum_{n<\omega} s_n \) is invariant under permutations, since the corresponding partial sums are chained in the sense of \( \text{Corollary 3.1(c)} \). Here we are using the general commutative-associative property of + and the fact that the nonnegative surreals form an ordered monoid.

Since we can define the \( s \)-limit of every ordinal-indexed sequence of surreals, the above \( s \)-sum of length \( \omega \) can be extended to the transfinite. Details go as follows. The \( s \)-sum of the empty sequence is 0. If \( \sum_{0<\beta} s_\alpha \) has been already constructed, let \( \sum_{0<\beta+1} s_\alpha = s_\beta + \sum_{0<\beta} s_\alpha \). Finally, if \( \beta \) is limit, let \( \sum_{0<\beta} s_\alpha = \text{lim}_{\beta<\beta} \sum_{0<\beta} s_\alpha \). In the case when all the \( s_\alpha \)'s are ordinals, the above iterated natural sum has been studied in Lipparini [L1]. It will be probably interesting to see which results from [L1] extend to the surreal framework. Notice that, when restricted to ordinals, \( \sum s \) is different from the usual transfinite ordinal sum \( \sum \). Though the limiting process is the same, the successor steps in defining \( \sum s \) correspond to taking the natural ordinal sum, while in \( \sum \) the usual noncommutative ordinal sum is used.

Invariance of \( \sum s \) under permutations does not extend beyond \( \omega \); actually, invariance fails already at stage \( \omega + 1 \). Just take \( s_0 = 0 \) and all the other \( s_\alpha \)'s to be 1. Then \( \sum_{0<\omega+1} s_\alpha = \omega + 1 \), but if we permute \( s_0 \) with \( s_\omega \), we get \( \omega \) instead. It will be probably interesting to consider transfinite \( s \)-sums in the case when all the \( s_\alpha \)'s are equal. Cf. Altman [A] and references there for the ordinal case.

The \( s \)-sum equals the surreal sum in many cases, notably, the following corollary is immediate from Conway [C, Chapter 3] or Gonshor [G, Theorem 5.12].

**Corollary 3.2.** For every sequence \( (s_\alpha)_{0<\beta} \) of surreals and every sequence \( (r_\alpha)_{0<\beta} \) of reals, the following identity holds

\[
\sum_{0<\beta} s_\alpha \omega r_\alpha = \sum_{0<\beta} s_\alpha \omega r_\alpha,
\]

provided the latter sum represents the Conway normal form of some surreal.
Comparison with classical series. One could try to extend the classical series expansions of real analysis to infinite surreal numbers by using the s-sum, for example, by considering $f(s) = \sum_{n=0}^{\omega} s^n$. Though $f(\omega)$ gives the expected value $\omega^2$, which is equal to $\exp(\omega)$ in the sense of the surreal exponentiation, on the other hand, $f(\omega + 1) = \omega^2$, too, hence series expansions through the s-sum generally give unwanted results. Actually, as follows from the proof of Theorem 2.2, if $e^r = d$ is dyadic $> 1$, then $f(r) = d - 1/\omega \neq d$, hence series obtained by using the s-sum do not always assume the exact wanted value even for real numbers. However, there is perhaps the possibility of modifying the s-limit and hence the s-sum in order to make things work better, but this is still to be developed. See the last sentence in Remark 4.2.

4. Further remarks and generalizations

The s-limit and the canonical representation of surreals. According to [G] Theorem 2.8], the canonical representation of a surreal $s$ is $\{F | G\}$, where $F$, $G$, respectively, are the sets of those surreals which are initial segments of $s$ and are $< s$, respectively, $> s$. If $(s_\beta)_{\beta < \alpha}$ is a sequence of surreals and $\{F_\beta | G_\beta\}_{\beta < \alpha}$ are their respective canonical representations, then a representation of $\text{slim}_{\beta < \alpha} s_\beta$ is $\{F | G\}$, where $F = \bigcup_{\beta < \alpha} \bigcap_{\gamma \geq \beta} F_\gamma$ and $G = \bigcup_{\beta < \alpha} \bigcap_{\gamma \geq \beta} G_\gamma$, in words, we take as representatives only those elements which are eventually in the lower, respectively, upper sets.

The same remark holds if we start considering, as another representation, those $F$ and $G$ which are the sets of those surreals born strictly before $s$ and are $< s$, respectively, $> s$.

However, the above considerations do not always hold for arbitrary representations of the $s_\beta$’s. For example, for every $n \in \omega$, we have $n + 1 = \{n | \}$, but then the above formulas would give $\text{slim}_{n<\omega} n = \{\} = 0 \neq \omega$.

Surreal inferior and superior limits. If $(s_\beta)_{\beta < \alpha}$ is a sequence of surreal numbers, we define $s = \text{slim inf}_{\beta < \alpha} s_\beta$ by defining $s(\gamma)$ by transfinite induction on $\gamma$, simultaneously constructing an auxiliary set $A(\gamma)$ cofinal in $\alpha$. Suppose that $\gamma$ is an ordinal and that both $s(\delta)$ and $A(\delta)$ have been defined, for every $\delta < \gamma$. Let $B = \alpha$, if $\gamma = 0$; $B = A(\delta)$, if $\gamma = \delta + 1$; and $B = \bigcap_{\delta < \gamma} A(\delta)$, if $\gamma$ is limit. We are now ready to define $s(\gamma)$ and $A(\gamma)$. If $B$ is not cofinal in $\alpha$, we set $s(\gamma)$ to be undefined and the construction stops. Henceforth, suppose that $B$ is cofinal in $\alpha$. If there is some $\tilde{\beta} < \alpha$ such that $s_\beta(\gamma) = +$, for every $\beta \geq \tilde{\beta}$, $\beta \in B$, we set $s(\gamma) = +$ and $A(\gamma) = \{\beta \in B \mid s_\beta(\gamma) = +\}$. If the set of those $\beta \in B$ such that $s_\beta(\gamma) = -$ is cofinal in $B$, we set $s(\gamma) = -$ and $A(\gamma) = \{\beta \in B \mid s_\beta(\gamma) = -\}$. In all the other cases, we let $s(\gamma)$ be undefined and the construction stops.

In a symmetric way we define slim sup.

Extensions and variations on slim. As we mentioned in the introduction, Definition 2.1 can be naturally extended in order to deal with sequences of strings of arbitrary symbols, not just $+$ and $-$. Actually, we shall present a generalization in which we take into account linear orders, not only well-orders.

Definition 4.1. Let $A$ be any set and $L$, $M$ be linearly ordered sets. We shall consider sequences of the form $(a_\ell)_{\ell \in L}$, where each $a_\ell$ is a function from some initial segment of $M$ to $A$. Here both the empty set and $M$ itself are considered to be initial segments.

It is useful to visualize the $a_\ell$’s as rows in an infinite $L \times M$ matrix with possibly empty entries. In this sense, $a_\ell$ is the $\ell^{th}$ row of the matrix, and $a_\ell(m)$ is the element in the $m^{th}$ column of the $\ell^{th}$ row. (Warning: in the case when $L$ or $M$ is an ordinal, the above terminology might be misleading, since, say, 0 is the $1^{st}$ ordinal.)

We define the s-limit of $(a_\ell)_{\ell \in L}$, in symbols, slim$_{\ell \in L} a_\ell$ as follows.
If $L$ has a maximum $\ell$, then we set $\text{slim}_{\ell \in L} a_\ell = a_\ell$.

If $L$ has no maximum, then $\text{slim}_{\ell \in L} a_\ell$ is the function $a$ given by the following prescriptions. If $m \in M$, we declare $a(m)$ to be defined in case there is some $\ell(m) \in L$ such that, for every $m' \leq m$ and every $\ell, \ell' \geq \ell(m)$, we have $a_\ell(m') = a_{\ell'}(m')$. If this is the case, we let $a(m) = a_{\ell(m)}(m)$. It is immediate from the definition that the domain $\text{dom}(a)$ of $a$ is an initial (possibly empty) segment of $M$. In particular, the $s$-limit is always and uniquely defined.

Under the above matrix visualization, $a(m)$ is defined if there is some $\ell(m)$ such that all the columns before (and including) the $m$th column are eventually constant from the $\ell(m)$th row on. Notice that we could have declared $a(m)$ to be defined just in case the $m$th column is eventually constant. This would give a different definition of a limit; this latter definition has the drawback that it does not imply that $\text{dom}(a)$ is an initial segment of $M$. To avoid the trouble, we can define another notion of limits of strings, call it $\text{slim}^\circ$, by declaring $a(m)$ to be defined if the $m$th column is eventually constant and all the preceding columns are eventually constant, too (the difference with slim is that here we make no assumption about the points from which the columns become constant).

We believe slim to be more natural than slim$^\circ$. For sure, though the version of Proposition 3.1(a) holds for slim$^\circ$ with the same proof, the analogues of Proposition 3.1(b)(c) do not hold for slim$^\circ$. Consider the following increasing sequence of strings of length $\omega + 1$:

\[
a_0 = - - - - \cdots , \quad a_1 = + - - - \cdots + , \quad a_2 = + + - - \cdots + , \quad \ldots \quad (\text{the main point is that signs in the last place alternate}).
\]

If $b_{n<\omega}$ is the subsequence consisting of the strings with odd index, then slim$^\circ_{n<\omega} a_n = \text{slim}^\circ_{n<\omega} b_n = \omega$, but slim$^\circ_{n<\omega} b_n = \omega - 1 \neq \text{slim}^\circ_{n<\omega} a_n$, in particular, slim$^\circ$ and slim differ on $(b_n)_{n<\omega}$. The counterexample to 3.1(c) is obtained by considering also the subsequence consisting of the strings with even index. However, for many arguments in this note the two definitions would turn out to be essentially equivalent.

Notice that in the above definitions we simply discard those columns for which the entries are not eventually constant; actually, we discard all further columns which follow a column as above. In the case when we do not have to discard columns, we speak of a full limit. Formally, we say that $a$ is the full limit of $(a_\ell)_{\ell \in L}$ if slim$_{\ell \in L} a_\ell = a$ and, in addition, for every $m \in M$, if there is some $\ell(m) \in L$ such that $a_\ell(m)$ is defined, for every $\ell \geq \ell(m)$, then $a(m)$ is defined, too. In other words, the $s$-limit $a$ of a sequence is the full limit of the sequences truncated at dom$(a)$, where dom$(a)$ is the largest possible initial segment of $M$ such that the truncated sequences do admit a full limit. In the above example, slim$_{n<\omega} b_n$ is not a full limit, though, in the sense of slim$^\circ$, slim$^\circ_{n<\omega} b_n$ would actually be a full limit.

Notice that if $L = \omega$, then slim$_{n<\omega} a_n$ is invariant under permutations of the $a_n$'s. For arbitrary $L$, slim satisfies a nice continuity property. If $L$ is partitioned into convex subsets as $(L_i)_{i \in I}$, then $I$ inherits from $L$ the structure of a linearly ordered set, and then slim$_{i \in L} a_\ell = \text{slim}_{i \in I} \text{slim}_{\ell \in L} a_\ell$, for every sequence $a_\ell$. The $s$-limit behaves only partially well with respect to string concatenation, that we shall denote by juxtaposition. Though slim$_{i \in L} b \ell = b \text{slim}_{i \in L} a_\ell$ for all strings, it is not necessarily always the case that slim$_{i \in L} a_\ell b = (\text{slim}_{i \in L} a_\ell) b$: just take $L = \omega$, $a_n$ of length $n$, for $n \in \omega$, and $b$ of length 1. However, slim$_{i \in L} a_\ell b \ell = (\text{slim}_{i \in L} a_\ell) \text{slim}_{i \in L} b_\ell$ holds when all the $a_\ell$'s have the same length and slim$_{i \in L} a_\ell$ is a full limit.

If $a = \text{slim}_{i \in L} a_\ell$ then either the $a_\ell$'s restricted to dom$(a)$ are eventually constant, or $\text{cf} \text{dom}(a) = \text{cf} \ L$. The $s$-limit of a sequence is equal to the $s$-limit of some subsequence cofinal in $L$, but, in general, as we showed before Proposition 3.1, it is not necessarily the case that every cofinal subsequence has the same $s$-limit. However, if slim$_{i \in L} a_\ell$ is a full limit, then every cofinal subsequence has the same limit.
Remark 4.2. One can introduce a shorthand for the surreal sign expansion. We can consider a surreal number as a sequence of nonzero signed ordinals. As in the standard case, 0 is represented by the empty sequence. If the first sign in the expansion of the surreal $s$ is $+$, and we have exactly $\alpha$ consecutive $+$’s at the beginning, the first element of the shorthand is $\alpha$; then if we have a certain number $\beta$ of consecutive $-$’s, the second element of the shorthand is $-\beta$, and so on. In such a shorthand, ordinals and negated ordinals alternate. Let $\text{sh}(s)$ denote the shorthand of $s$ in the above sense. Taking Definition 4.1 literally would give us strange results, such as $\text{slim}_{n<\omega} \text{sh}(n) = 0$. However, we can adapt the definition by taking place by place the inferior limit for places which consist eventually of positive ordinals, taking the superior limit for places which consist eventually of negative ordinals, and considering a place undefined in the limit if it is not of the above kind, with the usual convention that also all the places which follow should be considered undefined. Let us denote by $\text{slim}^*$ this modified limit acting on shorthands. It gives results different from slim. For example, the $s$-limit of the sequence $++, ++-, +++, ---, \ldots$, is $++\cdot\ldots=\omega$, while the $s$-limit* is $++\cdot\ldots-\ldots=\omega/2$.

Of course, in the general sense of Definition 4.1, $\text{slim}^*$ can be defined when $A$ is a complete lattice. Maybe there are useful variations on the above limit, say, using still different representations of surreal numbers. This has still to be investigated.

Remark 4.3. As is the case for most notions of convergence, Definition 4.1 can be extended to the situation when we work modulo some filter. Under the notations in Definition 4.1 and if $F$ is a filter over $L$, we let the $F$-limit $F\text{-lim} a_\ell$ be the function $a$ such that $a(m)$ is defined in case there is $X \in F$ such that, for every $m' \leq m$ and every $\ell, \ell' \in X$, we have $a_\ell(m') = a_{\ell'}(m')$. If this is the case, we let $a(m) = a_\ell(m)$, for some $\ell \in X$. Notice that the definition of the $s$-limit in Definition 4.1 is the particular case of the above $F$-limit when $F$ is the unbounded filter over $L$. The definition in the present remark looks particularly promising, since if $A$ is finite, $F$ is an ultrafilter and, for every $m < \omega$, the $a_\ell$’s are eventually of length $\geq m$, then $F\text{-lim} a_\ell$ has length $\geq \omega$. We could have introduced a variant of the above definition (in the same spirit as of $\text{slim}^*$) by saying that $a(m)$ is defined if, for every $m' \leq m$, there is some $X_{m'} \in F$ such that $a_\ell(m') = a_{\ell'}(m')$, for every $\ell, \ell' \in X_{m'}$.

Conclusions. In conclusion, at least from some point of view, the $s$-limit appears to be quite unnatural. For example, the $s$-limit of a sum does not always equal the sum of the $s$-limits; moreover, the $s$-limit of any countable sequence with infinitely many positive numbers and infinitely many negative numbers, no matter their size, is always 0. This is however essentially a consequence of the pleasant fact that the $s$-limit is always defined. On the other hand, the $s$-limit has a very natural order/string-theoretical definition, an interpretation in the game theoretical sense, as explained at the beginning of this section, and is well-behaved with respect to nondecreasing sequences, see Proposition 3.1. Moreover, the $s$-limit coincides with—or, better, incorporates— classical notions of limits in some significant cases, as shown in Theorem 2.2 and Section 3, in particular Corollary 3.2.

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