Discretely self-similar solutions to the Navier-Stokes equations with data in $L^2_{loc}$ satisfying the local energy inequality

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Abstract

Chae and Wolf recently constructed discretely self-similar solutions to the Navier-Stokes equations for any discretely self similar data in $L^2_{loc}$. Their solutions are in the class of local Leray solutions with projected pressure, and satisfy the “local energy inequality with projected pressure”. In this note, for the same class of initial data, we construct discretely self-similar suitable weak solutions to the Navier-Stokes equations that satisfy the classical local energy inequality of Scheffer and Caffarelli-Kohn-Nirenberg. We also obtain an explicit formula for the pressure in terms of the velocity. Our argument involves a new purely local energy estimate for discretely self-similar solutions with data in $L^2_{loc}$ and an approximation of divergence free, discretely self-similar vector fields in $L^2_{loc}$ by divergence free, discretely self-similar elements of $L^3_w$.

1 Introduction

The Navier-Stokes equations describe the evolution of a viscous incompressible fluid’s velocity field $v$ and associated scalar pressure $\pi$. In particular, $v$ and $\pi$ are required to satisfy

\begin{align}
\partial_t v - \Delta v + v \cdot \nabla v + \nabla \pi &= 0, \\
\nabla \cdot v &= 0,
\end{align}

in the sense of distributions. For our purposes, (1.1) is applied on $\mathbb{R}^3 \times (0, \infty)$ and $v$ evolves from a prescribed, divergence free initial data $v_0 : \mathbb{R}^3 \to \mathbb{R}^3$. Solutions to (1.1) exhibit a natural scaling: if $v$ satisfies (1.1), then for any $\lambda > 0$

\begin{equation}
v^\lambda(x, t) = \lambda v(\lambda x, \lambda^2 t),
\end{equation}

is also a solution with pressure

\begin{equation}
\pi^\lambda(x, t) = \lambda^2 \pi(\lambda x, \lambda^2 t),
\end{equation}

in the sense of distributions.
and initial data
\[ v^\lambda_0(x) = \lambda v_0(\lambda x). \] 
A solution is called self-similar (SS) if \( v^\lambda(x,t) = v(x,t) \) for all \( \lambda > 0 \) and is discretely self-similar with factor \( \lambda \) (i.e. \( v \) is \( \lambda \)-DSS) if this scaling invariance holds for a given \( \lambda > 1 \). Similarly, \( v_0 \) is self-similar (a.k.a. \( (-1) \)-homogeneous) if \( v_0(x) = \lambda v_0(\lambda x) \) for all \( \lambda > 0 \) or \( \lambda \)-DSS if this holds for a given \( \lambda > 1 \). These solutions can be either forward or backward if they are defined on \( \mathbb{R}^3 \times (0, \infty) \) or \( \mathbb{R}^3 \times (-\infty, 0) \) respectively. In this note we work exclusively with forward solutions and omit the qualifier “forward”.

Self-similar solutions satisfy an ansatz for \( v \) in terms of a time-independent profile \( u \), namely,
\[ v(x,t) = \frac{1}{\sqrt{t}} u \left( \frac{x}{\sqrt{t}} \right), \] 
where \( u \) solves the Leray equations
\[ \begin{align*}
-\Delta u - \frac{1}{2} u - \frac{1}{2} y \cdot \nabla u + u \cdot \nabla u + \nabla p &= 0 \quad \text{in } \mathbb{R}^3, \\
\nabla \cdot u &= 0
\end{align*} \] 
in the variable \( y = x/\sqrt{t} \). Discretely self-similar solutions are determined by their behavior on the time interval \( 1 \leq t \leq \lambda^2 \) and satisfy the ansatz
\[ v(x,t) = \frac{1}{\sqrt{t}} u(y,s), \] 
where
\[ y = \frac{x}{\sqrt{t}}, \quad s = \log t. \] 
The vector field \( u \) is \( T \)-periodic with period \( T = 2 \log \lambda \) and solves the time-dependent Leray equations
\[ \begin{align*}
\partial_s u - \Delta u - \frac{1}{2} u - \frac{1}{2} y \cdot \nabla u + u \cdot \nabla u + \nabla p &= 0 \quad \text{in } \mathbb{R}^3 \times \mathbb{R}, \\
\nabla \cdot u &= 0
\end{align*} \] 
Note that the similarity transform \((1.8)-(1.9)\) gives a one-to-one correspondence between solutions to \((1.1)\) and \((1.10)\). Moreover, when \( v_0 \) is SS or DSS, the initial condition \( v|_{t=0} = v_0 \) corresponds to a boundary condition for \( u \) at spatial infinity, see \([16, 3, 4]\).

Self-similar solutions are interesting in a variety of contexts as candidates for ill-posedness or finite time blow-up of solutions to the 3D Navier-Stokes equations (see \([10, 11, 12, 19, 21, 22]\) and the discussion in \([3]\). Forward self-similar solutions are compelling candidates for non-uniqueness \([12, 10]\). Until recently, the existence of forward self-similar solutions was only known for small data \([1, 7, 9, 15, 13]\). Such solutions are necessarily unique. In \([11]\), Jia and Šverák constructed forward self-similar solutions for large data where the data is assumed to be Hölder continuous away from the origin. This result has been generalized in a number of directions by a variety of authors \([3, 4, 5, 8, 16, 18, 23]\]. This paper can be understood in the context of \([3, 8, 18]\) and we briefly recall the main results of these papers.
In [3], we generalize [11] in two ways. First, all smoothness assumptions on the initial data are removed; we only require \( v_0 \in L^3_w \) (and \( v_0 \) divergence free and SS or DSS). Second, we allow the data to be DSS for any \( \lambda > 1 \), in which case we obtain DSS solutions as opposed to SS solutions – in contrast, the method of [11] can be adapted to give DSS solutions but only when \( \lambda \) is close to 1 [23]. The method of proof in [3] has since been extended to the half-space in [4] and to initial data in the Besov spaces \( B^{3/p-1}_{p,\infty} \) when \( 3 < p < 6 \) [5]. Solutions which satisfy a rotationally corrected scaling invariance are also constructed in [4].

The solutions of [3] belong to the class of local Leray solutions. This class was introduced by Lemarié-Rieusset in [17] to provide a local analogue of Leray’s weak solutions [19]. We recall the definition of local Leray solutions in full. For \( q \in [1, \infty) \), we say \( f \in L^q_{uloc} \) if

\[
\|f\|_{L^q_{uloc}} = \sup_{x \in \mathbb{R}^3} \|f\|_{L^q(B(x,1))} < \infty.
\]

**Definition 1.1** (Local Leray solutions). A vector field \( v \in L^2_{loc}(\mathbb{R}^3 \times [0, \infty)) \) is a local Leray solution to (1.1) with divergence free initial data \( v_0 \in L^2_{uloc} \) if:

1. for some \( \pi \in L^{3/2}_{loc}(\mathbb{R}^3 \times [0, \infty)) \), the pair \((v, \pi)\) is a distributional solution to (1.1),

2. for any \( R > 0 \), \( v \) satisfies

\[
\text{ess sup}_{0 \leq t < R} \sup_{x_0 \in \mathbb{R}^3} \int_{B_R(x_0)} \frac{1}{2} |v(x,t)|^2 \, dx + \sup_{x_0 \in \mathbb{R}^3} \int_0^R \int_{B_R(x_0)} |\nabla v(x,t)|^2 \, dx \, dt < \infty,
\]

3. for all compact subsets \( K \) of \( \mathbb{R}^3 \) we have \( v(t) \to v_0 \) in \( L^2(K) \) as \( t \to 0^+ \),

4. \( v \) is suitable in the sense of Caffarelli-Kohn-Nirenberg, i.e., for all cylinders \( Q \) compactly supported in \( \mathbb{R}^3 \times (0, \infty) \) and all non-negative \( \phi \in C_0^{\infty}(Q) \), we have

\[
\int |v(t)|^2 \phi \, dx + 2 \int \int |\nabla v|^2 \phi \, dx \, dt \\
\leq \int \int |v|^2 (\partial_t \phi + \Delta \phi) \, dx \, dt + \int \int (|v|^2 + 2\pi)(v \cdot \nabla \phi) \, dx \, dt,
\]

(1.11)

5. for every \( x_0 \in \mathbb{R}^3 \), there exists \( c_{x_0} \in L^{3/2}(0, T) \) such that

\[
p(x, t) - c_{x_0}(t) = -\frac{1}{3} |v(x, t)|^2 + \frac{1}{4\pi} \int_{B_2(x_0)} K(x - y) : v(y, t) \otimes v(y, t) \, dy \\
+ \frac{1}{4\pi} \int_{\mathbb{R}^3 \setminus B_2(x_0)} (K(x - y) - K(x_0 - y)) : v(y, t) \otimes v(y, t) \, dy,
\]

in \( L^{3/2}(0, T; L^{3/2}(B_1(x_0))) \), where \( K(x) = \nabla^2(1/|x|) \).

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In [17], Lemarié-Rieusset constructed global in time local Leray solutions if \( v_0 \) belongs to \( E^2 \), the closure of \( C_0^\infty \) in the \( L^2_{\text{uloc}}(\mathbb{R}^3) \) norm. See Kikuchi-Seregin [14] for another construction which treats the pressure carefully. Note that [17], [14] and [11, 12] contain alternative definitions of local Leray solutions. On one hand, [14] requires the pressure satisfies a certain formula (we will establish a similar pressure formula for our solutions, see Theorem 1.2). In [11, 12], the explicit pressure formula is replaced by a decay condition imposed on the solution at spatial infinity, namely, for all \( R > 0 \)

\[
\lim_{|x_0| \to \infty} \int_0^{R^2} \int_{B(x_0, R)} |v|^2 \, dx \, dt = 0.
\]

Jia and Šverák claim in [11, 12] that, if \( v \) exhibits this decay, then the pressure formula from [14] is valid. Since the decay property is easier to directly establish for a given solution, this justifies using it in place of the explicit pressure formula in the definition of local Leray solutions. It turns out that these properties are equivalent when \( v_0 \in E^2 \). This can be proved using ideas contained in a recent preprint of Maekawa, Miura, and Prange [20] on the construction of local energy solutions in the half space.

Local Leray solutions are known to satisfy a useful a priori bound. Let \( \mathcal{N}(v_0) \) denote the class of local Leray solutions with initial data \( v_0 \). The following estimate is well known for local Leray solutions (see [11]): for all \( \tilde{v} \in \mathcal{N}(v_0) \) and \( r > 0 \) we have

\[
\text{ess sup} \sup_{0 \leq t \leq r^2} \int_{B_r(x_0)} \frac{|\tilde{v}(x, t)|^2}{2} \, dx + \sup_{x_0 \in \mathbb{R}^3} \int_0^{r^2} \int_{B_r(x_0)} |\nabla \tilde{v}|^2 \, dx \, dt < CA, \quad (1.12)
\]

where

\[
A = \sup_{x_0 \in \mathbb{R}^3} \int_{B_r(x_0)} \frac{|v_0|^2}{2} \, dx, \quad \sigma(r) = c_0 \min\{r^2 A^{-2}, 1\}, \quad (1.13)
\]

for a small universal positive constant \( c_0 \).

Concurrently to the publication of [3], Lemarié-Rieusset published the book [18], which includes a chapter on the self-similar solutions of [11]. Here, Lemarié-Rieusset generalizes the space of initial data to include any \( L^2_{\text{loc}} \), divergence free, self-similar vector field. The main elements of his argument are as follows. He first uses the Leray-Schauder approach of [11] to construct self-similar solutions for initial data \( v_0 \) satisfying \( |v_0(x)| \lesssim |x|^{-1} \). This construction is more general than that in [11] but less general than that in [3]. But, provided \( v_0 \) is self-similar, \( v_0 \in L^2_{\text{loc}} \) if and only if \( v_0 \in L^2_{\text{uloc}} \). And, furthermore, if \( v_0 \) is self-similar and belongs to \( L^2_{\text{uloc}} \), then it can be approximated by a sequence \( v_0^{(k)} \) where each \( |v_0^{(k)}(x)| \lesssim |x|^{-1} \). Then, the first construction gives local Leray solutions for each \( v_0^{(k)} \) and, because local Leray solutions satisfy the a priori bound (1.12) depending only on the \( L^2_{\text{uloc}} \) norm of their initial data, these will converge to a SS local Leray solution with \( L^2_{\text{loc}} \) data. This argument breaks down for DSS solutions since \( L^2_{\text{loc}} \cap DSS \neq L^2_{\text{uloc}} \cap DSS \) (see (1.15) for an example) and, therefore, we cannot get the uniform bound (1.12) on a sequence of approximating solutions for free.

Chae and Wolf, on the other hand, introduced an entirely new method in [8] to construct \( \lambda \)-DSS solutions for any \( \lambda > 1 \) and initial data \( v_0 \in L^2_{\text{loc}}(\mathbb{R}^3) \). These
solutions live in the class of “local Leray solutions with projected pressure,” which means they satisfy a modified local energy inequality instead of the classical local energy inequality (1.11) of [6]. To construct these solutions, Chae and Wolf use a fixed point argument to solve the mollified Navier-Stokes equations (this is the same system studied in [3], but written in physical variables as opposed to the similarity variables, see (3.4) and (3.5)). To apply the fixed point argument, Chae and Wolf first prove existence for the (mollified) linearized equations where the given drift velocity is DSS. They then apply a fixed point theorem (the space for the fixed point argument is a bounded set of the DSS subspace of $L^{18/5}(0,T;L^3(B_1))$ – $B_r$ denotes the ball of radius $r$ centered at the origin – defined below [8, (3.1)]) to prove that there exists a drift velocity which matches the solution. This gives existence of a DSS solution to the mollified Navier-Stokes equations. Note that the approximations satisfy the a priori (energy) bound [8, (2.35)] and the norm of the mollification term can be absorbed for $T$ sufficiently small.

In this paper we give a simple, alternative proof of the result in [8]. The following theorem is our main result.

**Theorem 1.2.** Assume $v_0 \in L^2_{\text{loc}}(\mathbb{R}^3)$ is a divergence free $\lambda$-DSS vector field for some $\lambda > 1$. Then there exists a $\lambda$-DSS distributional solution $v$ to (1.1) and associated pressure $\pi$ so that $v$ is suitable in the sense of [6] and satisfies

$$\lim_{t \to 0^+} \|v(t) - v_0\|_{L^2(K)} = 0,$$

for every compact subset $K$ of $\mathbb{R}^3$. Moreover, for any $T > 0$ and compact subset $K$ of $\mathbb{R}^3$, we have $v \in L^\infty(0,T;L^2(K)) \cap L^2(0,T;H^1(K))$ and $\pi \in L^{3/2}(0,T;L^{3/2}(K))$. Furthermore, for any $(x,t) \in \mathbb{R}^3 \times (0,\infty)$, the pressure satisfies the following formula

$$\pi(x,t) = -\frac{1}{3}|v|^2(x,t)$$

$$+ \lim_{\delta \to 0} \int_{|y|>\delta} K_{ij}(x-y)v_i(y,t)v_j(y,t) dy,$$

in $L^{3/2}_{\text{loc}}(\mathbb{R}^3 \times (0,\infty))$.

**Comments on Theorem 1.2**

1. In [8], the data also belongs to $L^2_{\text{loc}}$, but the solution is not shown to satisfy the local energy inequality of [6]. Instead, it satisfies a “local energy inequality with projected pressure”. Since the solution constructed in Theorem 1.2 satisfies the traditional local energy inequality, this theorem is a slight refinement of the main result of [8]. Furthermore, we are careful to give a precise formulation (1.14) of the pressure and its connection to the velocity. The relationship between $v$ and $\pi$ is less clear in [8].

2. The integral in (1.14) is not a Calderon-Zygmund singular integral because we do not have a global bound of $v$. It is defined in $L^{3/2}_{\text{loc}}$ using the DSS property.
3. Our method of proof is by approximation and is similar to the argument from [18]. The main difference is that we need to construct a sequence of approximating solutions and establish a new \textit{a priori} bound for these solutions for DSS data – in [18] the bound (1.12) is sufficient (and free). Note that an approximation argument using (1.12) was also used by the authors in [3] to construct SS solutions as a limit of DSS solutions where the scaling factors are converging to 1.

4. Generally, the solution \( v \) is not necessarily a local Leray solution because \( v_0 \) may not be in \( L^2_{uloc} \), and we do not assert the uniform bounds (item 2) in Definition 1.1. Consider the DSS function in \( L^2_{loc} \) for \( 0 < a < \frac{3}{2} \),

\[
f_a(x) = \sum_{k \in \mathbb{Z}} \lambda^k f_{a,0}(\lambda^k x), \quad f_{a,0}(x) = |x - x_0|^{-a} \chi(x - x_0),
\]

where \( 1 + r < |x_0| < \lambda - r \) for some \( r > 0 \), and \( \chi \) is the characteristic function of the ball \( B_r(0) \). It is not in \( L^2_{uloc} \) when \( 1 < a < \frac{3}{2} \) for its behavior at infinity. It is in \( L^2_{uloc} \) when \( 0 < a \leq 1 \). The function \( f_1(x) \) for \( a = 1 \) is given in Comment 4 after [4, Theorem 1.2] as an inapplicable example since it is not in \( L^{3,\infty}(\mathbb{R}^3) \).

5. If \( v_0 \in L^2_{uloc} \), then it is not difficult to obtain uniform bounds on \( v \) in the sense of item 2 from Definition 1.1. Furthermore, item 5 from Definition 1.1 can be established whenever \( v_0 \in E^2 \) (see [20]). Thus, our construction yields DSS local Leray solutions whenever the data is DSS, divergence free, and in \( E^2 \).

Our strategy for proving Theorem 1.2 is to approximate a solution with data in \( L^2_{loc} \) using solutions constructed in [3]. There are several steps. First we need to prove that DSS data in \( L^2_{loc} \) can be approximated in \( L^2(B_1) \) by DSS data in \( L^3_w \). This is the subject of §4.1. Then, [3] gives us a sequence of DSS solutions in the local Leray class. To prove that these solutions converge to a solution with \( L^2_{loc} \) data satisfying the desired pressure formula, we need to establish new \textit{a priori} bounds for the solutions from [3] which are independent of the \( L^3_w \) norm of the initial data (this is done in §3) and also prove that they satisfy the pressure formula (see §2). In §4.2 and §4.3, we put these ingredients together to prove Theorem 1.2.

2 A limiting pressure formula for DSS solutions

In this section we will prove that, under certain conditions, the limiting pressure distribution of an approximation scheme for (1.1) inherits the structure of the approximate pressure distributions. This result will be applied in §3 and §4.3.

\textbf{Lemma 2.1.} \textit{Fix } \lambda > 1 \textit{ and } T > 0. \textit{Let } v_0 \in L^2_{loc} \textit{ be a given divergence free, } \lambda\text{-DSS vector field and assume } \{ v_0^{(k)} \} \subset L^2_{loc} \textit{ is a sequence of divergence free, } \lambda\text{-DSS vector fields so that } v_0^{(k)} \rightarrow v_0 \textit{ in } L^2(B_1). \textit{Assume } v_k \textit{ and } \tilde{v}_k \textit{ are divergence free, } \lambda\text{-DSS vector fields and that there exists a distribution } \pi_k \textit{ so that the following conditions are satisfied:}
• $v_k$, $\tilde{v}_k$, and $\pi_k$ solve the system
\[
\partial_t v_k - \Delta v_k + \tilde{v}_k \cdot \nabla v_k + \nabla \pi_k = 0 \quad (x, t) \in \mathbb{R}^3 \times [0, T],
\]
for the initial data $v_0^{(k)}$ and both $v_k$ and $\tilde{v}_k$ converge to $v_0^{(k)}$ in $L^2_{\text{loc}}$.

• $v_k$ and $\tilde{v}_k$ are uniformly bounded in $L^\infty(0, T; L^2(B_1)) \cap L^2(0, T; H^1(B_1))$ over all $k \in \mathbb{N}$.

• for all $0 < t \leq T$, $\pi_k$ satisfies the formula
\[
\pi_k(x, t) = -\frac{1}{3} [\tilde{v}_k \cdot v_k](x, t) + \lim_{\delta \to 0} \int_{|y| > \delta} K_{ij}(x - y)(\tilde{v}_k)_i(y, t)(v_k)_j(y, t) dy. \tag{2.1}
\]

• there exists a $\lambda$-DSS solution $v$ in $L^\infty(0, T; L^2(B_1)) \cap L^2(0, T; H^1(B_1))$ with pressure $\pi$ in $L^{3/2}(0, T; L^{3/2})$ so that
\[
\begin{align*}
v_k &\to v \text{ weakly in } L^2(0, T; H^1(B_1)) \\
v_k &\to v \text{ in } L^2(0, T; L^2(B_1)) \\
\pi_k &\to \pi \text{ weakly in } L^{3/2}(0, T; L^{3/2}(B_1)).
\end{align*}
\]

Then, for a.e. $0 < t \leq T$ and $x \in B_{\lambda}$, the pressure $\pi$ satisfies the formula
\[
\pi(x, t) = -\frac{1}{3} |v|^2(x, t) + \lim_{\delta \to 0} \int_{|y| > \delta} K_{ij}(x - y)(v)_i(y, t)(v)_j(y, t) dy, \tag{2.2}
\]
in $L^{3/2}((0, T) \times B_{\lambda})$.

**Remark 2.2.** The purpose of this lemma is to establish the pressure formula (2.2) which, ultimately, will allow us to prove (1.14). It is, however, not needed to establish the other conclusions of Theorem 1.2.

**Proof.** Note that since $v_k$, $\tilde{v}_k$, and $v$ are all uniformly bounded in $L^\infty(0, T; L^2(B_1)) \cap L^2(0, T; H^1(B_1))$, convergence in $L^2(0, T; L^2(B_1))$, Hölder’s inequality, Sobolev embedding, using the equation to get uniform bound of $\partial_t v_k$, and re-scaling the solution, implies that
\[
v_k \text{ and } \tilde{v}_k \to v \text{ in } L^3(0, T; L^3(B_1)).
\]
It also shows that $v_k$, $\tilde{v}_k$, and $v$ are all uniformly bounded in $L^3(0, T; L^3(B_1))$ (at least for $k$ sufficiently large).

Let
\[
\pi_1^k(x, t) = -\frac{1}{3} [\tilde{v}_k \cdot v_k](x, t),
\]

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\[
\pi^2_k(x, t) = \lim_{\delta \to 0} \int_{\lambda^2 < |y| < \delta} K_{ij}(x - y)(\tilde{v}_k)_i(y, t)(v_k)_j(y, t) \, dy,
\]

and
\[
\pi^3_k(x, t) = \int_{y \geq \lambda^2} K_{ij}(x - y)(\tilde{v}_k)_i(y, t)(v_k)_j(y, t) \, dy.
\]

Also let
\[
\pi_1^1(x, t) = -\frac{1}{3} |v|^2(x, t),
\]
\[
\pi_2^1(x, t) = \lim_{\delta \to 0} \int_{\lambda^2 < |y| < \delta} K_{ij}(x - y)v_i(y, t)v_j(y, t) \, dy,
\]

and
\[
\pi_3^1(x, t) = \int_{y \geq \lambda^2} K_{ij}(x - y)v_i(y, t)v_j(y, t) \, dy.
\]

Since \(v_k \text{ and } \tilde{v}_k \to v\) in \(L^3(0, T; L^3(B_\lambda))\), we have \(\pi_k^1 \to \pi^1\) in \(L^{3/2}(0, T; L^{3/2}(B_\lambda))\).

Let
\[
h_{i,j}(y, t) = (\tilde{v}_k)_i(v_k)_j - v_i v_j = \{(\tilde{v}_k)_i[(v_k)_j - v_j] + [(\tilde{v}_k)_i - v_i]v_j\} (y, t).
\]

Using the Calderon-Zygmund theory we clearly have
\[
\int_0^T \int_{B_\lambda} |\pi_k^2(x, t) - \pi^2(x, t)|^{3/2} \, dx \, dt
\]
\[
\leq C \int_0^T \int_{B_{\lambda^2}} |h_{i,j}(x, t)|^{3/2} \, dx \, dt
\]
\[
\leq C \left( \int_0^T \int_{B_{\lambda^2}} \tilde{v}_k^3 \, dx \, dt \right)^{1/2} \left( \int_0^T \int_{B_{\lambda^2}} (v_k - v)^3 \, dx \, dt \right)^{1/2}
\]
\[
+ C \left( \int_0^T \int_{B_{\lambda^2}} v^3 \, dx \, dt \right)^{1/2} \left( \int_0^T \int_{B_{\lambda^2}} (\tilde{v}_k - v)^3 \, dx \, dt \right)^{1/2}. \quad (2.3)
\]

Re-scaling gives
\[
\int_0^T \int_{B_{\lambda^2}} (\tilde{v}_k - v)^3(x, t) \, dx \, dt = \lambda^4 \int_0^{T\lambda^-4} \int_{B_1} (\tilde{v}_k - v)^3(z, \tau) \, dz \, d\tau,
\]

for the obvious choice of \(z\) and \(\tau\). Since the right hand side of the above equation vanishes as \(k \to \infty\), as does the identical term but with \(\tilde{v}_k\) replaced by \(v_k\), we conclude that \(\pi_k^2\) converges to \(\pi^2\) in \(L^{3/2}(0, T; L^{3/2}(B_1))\).

Establishing the convergence of \(\pi_k^3\) to \(\pi^3\) is more difficult. Let
\[
p_k(x, t) = \pi_k^3(x, t) - \pi^3(x, t) = \int_{|y| \geq \lambda^2} K_{ij}(x - y)h_{i,j}(y, t) \, dy.
\]
Fix \( x \in B_{\lambda} \). Then
\[
|p_k(x, t)|^{3/2} \leq C \left| \int_{|y|\geq\lambda^2} \frac{1}{|y|^3} |h_{i,j}(y, t)| \, dy \right|^{3/2} \\
\leq C \left( \int_{|y|\geq\lambda^2} \frac{1}{|y|^4} \, dy \right)^{1/2} \int_{|y|\geq\lambda^2} \frac{1}{|y|^{5/2}} |h_{i,j}(y, t)|^{3/2} \, dy \\
= C \int_{|y|\geq\lambda^2} \frac{1}{|y|^{5/2}} |h_{i,j}(y, t)|^{3/2} \, dy.
\]

Let \( A_k = \{ x : \lambda^{k-1} \leq |x| < \lambda^k \} \) for \( k \in \mathbb{Z} \). Then, using the scaling properties of \( h \),
\[
\int_{|y|\geq\lambda^2} \frac{1}{|y|^{5/2}} |h_{i,j}(y, t)|^{3/2} \, dy = \sum_{k=3}^{\infty} \int_{A_k} \frac{1}{|y|^{5/2}} |h_{i,j}(y, t)|^{3/2} \, dy \\
\leq C(\lambda) \sum_{k=3}^{\infty} \frac{1}{\lambda^{5k/2}} \int_{A_k} |h_{i,j}(y, t)|^{3/2} \, dy \\
\leq C(\lambda) \sum_{k=3}^{\infty} \frac{1}{\lambda^{5k/2}} \int_{B_1} |h_{i,j}(z, t\lambda^{-2k})|^{3/2} \, dz.
\]

Thus,
\[
\int_0^T \int_{B_{\lambda}} |p_k(x, t)|^{3/2} \, dt \leq \lambda^3 C(\lambda) \int_0^T \sum_{k=3}^{\infty} \frac{1}{\lambda^{5k/2}} \int_{B_1} |h_{i,j}(z, t\lambda^{-2k})|^{3/2} \, dz \, dt \\
\leq C(\lambda) \sum_{k=3}^{\infty} \frac{1}{\lambda^{k/2}} \int_0^T \int_{B_1} |h_{i,j}(z, \tau)|^{3/2} \, dz \, d\tau \\
\leq C(\lambda) \int_0^T \int_{B_1} |h_{i,j}(z, \tau)|^{3/2} \, dz \, d\tau.
\]

Thus this term is bounded as (2.3).

We have now shown that \( \pi_k(x, t) \) converges weakly to both \( \pi^1(x, t) + \pi^2(x, t) + \pi^3(x, t) \) and \( \pi(x, t) \) in \( L^{3/2}(0, T; L^{3/2}(B_\lambda)) \), implying that \( \pi(x, t) = \pi^1(x, t) + \pi^2(x, t) + \pi^3(x, t) \) as distributions. In other words, \( \pi(x, t) \) satisfies (2.2) in \( L^{3/2}((0, T) \times B_\lambda) \). \( \Box \)

### 3 Properties of DSS solutions with data in \( L^3_w \)

The goal of this section is to obtain a bound on the local evolution of DSS solutions \( v \) constructed in [3] that is independent of both the \( L^3_w \) and \( L^2_{uloc} \) norms of \( v \) and to establish an explicit representation formula for the pressure.

Assume \( v_0 \in L^3_w(\mathbb{R}^3) \) and \( v \) is a DSS solution evolving from \( v_0 \) as constructed in [3]. For a generic solution to (1.1), we cannot close energy estimates for \( \phi v \) solely in terms of \( v_0|_{B_\lambda} \) – there is always some spillover. Proposition 3.1 states that this is possible for DSS solutions as a result of their scaling properties. In our argument, we must
work with a quantity that is continuous in time. This is not known for \( \int_{B_1} |v(t)|^2 \, dx \) when \( v \) is a local Leray solution. Hence, we need to work at the level of a mollified approximation scheme \([3, (2.24)]\), see (3.4) below. Note that in \([3]\), the mollified scheme is used to approximate a solution to the time-periodic Leray equations and the mollification is time-independent. Undoing the similarity transformation results in a time-dependent mollification of the drift component of the nonlinear term of the solution in the physical variables, see (3.5) below; this matches the mollification used in \([8]\).

**Proposition 3.1.** Fix \( \lambda > 1 \). Assume \( v_0 \in L^3_w(\mathbb{R}^3) \) is \( \lambda \)-DSS and divergence-free, and \( v \) is a \( \lambda \)-DSS local Leray solution evolving from \( v_0 \) constructed in \([3]\) (in particular, it is the limit of the mollified approximation scheme \([3, (2.24)]\)) and \( \pi \) is its associated pressure. Let \( \alpha_0 = \|v_0\|_{L^2(B_1)}^{\lambda} \). Then, there exist positive \( T = T(\alpha_0, \lambda) \) and \( C(\alpha_0, \lambda) \) independent of \( \|v_0\|_{L^2_{uloc}} \) and \( \|v_0\|_{L^3_w} \) so that

\[
\text{ess sup}_{0 \leq t \leq T} \int_{B_1} |v(x, t)|^2 \, dx + \int_0^T \int_{B_1} |\nabla v|^2 \, dx \, dt < C(\alpha_0, \lambda),
\]

and

\[
\int_0^T \int_{B_1} |\pi(x, t)|^{3/2} \, dx \, dt < C(\alpha_0, \lambda).
\]

Moreover, for \( x \in B_1 \) and \( t \in (0, T) \), the pressure satisfies the formula

\[
\pi(x, t) = -\frac{1}{3} |v|^2(x, t) + \lim_{\delta \to 0} \int_{|y| < \delta} K_{ij}(x - y)v_i(y, t)v_j(y, t) \, dy,
\]

in \( L^{3/2}(B_1 \times (0, T)) \).

Typically, the best pressure decompositions we have for local Leray solutions depend on a particular ball containing the spatial point at which the pressure is being computed. The resulting formula consists of a local Calderon-Zygmund part and a far-field part with a singular kernel that is decaying faster than \( K \) kernel. The formula (3.3) does not involve such a decomposition, and, as is evident in the proof, the integral in (3.3) is defined using the DSS property.

The proof of \([3]\) shows that the left sides of (3.1) and (3.2) are bounded by constants depending on \( v_0 \), in particular its \( L^3_w(\mathbb{R}^3) \)-norm. For this application, we need a bound depending only on \( \|v_0\|_{L^2(B_1)} \) and \( \lambda \).

**Proof.** Since \( v \) is a solution from \([3]\), its image under the similarity transform \((1.9)\) solves the time-periodic Leray equations and is the limit of a mollified approximation scheme \([3, (2.24)]\). In particular, for each \( \epsilon > 0 \), there exists a time periodic solution \( u_\epsilon \) to the problem

\[
\left( \partial_s u_\epsilon - \Delta u_\epsilon - \frac{1}{2} u_\epsilon - \frac{1}{2} y \cdot \nabla u_\epsilon + (\eta_\epsilon * u_\epsilon) \cdot \nabla u_\epsilon + \nabla p_\epsilon \right)(y, s) = 0,
\]

(3.4)
where \( \eta_\epsilon(y) = \frac{1}{\epsilon^3} \eta(y/\epsilon) \) and \( \eta \in C_0^\infty(\mathbb{R}^3) \), is non-negative, and satisfies \( \int \eta(y) \, dy = 1 \). Applying (1.8)-(1.9) we obtain a \( \lambda \)-DSS vector field \( v_\epsilon \) satisfying
\[
\partial_t v_\epsilon(x, t) - \Delta v_\epsilon(x, t) + (\eta_{\epsilon\sqrt{\tau}} \ast v_\epsilon) \cdot \nabla v_\epsilon(x, t) + \nabla \pi_\epsilon(x, t) = 0. \tag{3.5}
\]
Note the time dependence of the convolution kernel \( \eta_{\epsilon\sqrt{\tau}} \) in (3.5).

By the convergence properties of \( u_\epsilon(y, s) \) to \( u(y, s) = \sqrt{\tau} v(x, t) \) [3, p. 1108] and discretely self-similar scaling (to extend the estimates down to \( t = 0 \)), it follows that for all \( T > 0 \) and all compact sets \( K \subset \mathbb{R}^3 \),
\[
\begin{align*}
v_\epsilon &\to v \text{ weakly in } L^2(0, T; H^1(K)), \\
v_\epsilon &\to v \text{ strongly in } L^2(0, T; L^2(K)), \\
v_\epsilon(s) &\to v(s) \text{ weakly in } L^2(K) \text{ for all } s \in [0, T].
\end{align*}
\]

Note also that \( v_\epsilon(t) \to v_0 \) in \( L^2_{\text{loc}} \), i.e. the mollification does not affect the initial data. Furthermore, because each \( v_\epsilon \) is smooth on \( \mathbb{R}^3 \times (0, \infty) \) and right continuous in \( L^2_{\text{loc}} \) at \( t = 0 \), it follows that
\[
\alpha_\epsilon(t) = \int_{B_1} |v_\epsilon(x, t)|^2 \, dx,
\]
and
\[
\tilde{\alpha}_\epsilon(t) = \sup_{0 \leq \tau \leq t} \alpha_\epsilon(\tau)
\]
are continuous as functions of \( t \). This is not clearly true for \( \int_{B_1} |v(x, t)|^2 \, dx \).

Note that, for any \( k \in \mathbb{Z} \) and \( q \in [1, \infty) \), since \( v_\epsilon(x, t) = \lambda^{-k} v_\epsilon(\lambda^{-k} x, \lambda^{-2k} t) \),
\[
\int_{B_{\lambda k}} |v_\epsilon(x, t)|^q \, dx = \lambda^{(3-q)k} \int_{B_1} |v_\epsilon(\bar{x}, \lambda^{-2k} t)|^q \, d\bar{x}. \tag{3.6}
\]

Our goal is to establish local in time a priori bounds for \( \alpha_\epsilon(t) \) that are independent of \( \epsilon \). Note that \( v_\epsilon \) satisfies the local energy equality, i.e.,
\[
\int |v_\epsilon|^2 \phi(t) \, dx + 2 \int_0^t \int |\nabla v_\epsilon|^2 \phi \, dx \, ds
\]
\[
= \int |v_0|^2 \phi \, dx + \int_0^t \int |v_\epsilon|^2 (\partial_s \phi + \Delta \phi) \, dx \, ds \tag{3.7}
\]
\[
+ \int_0^t \int |v_\epsilon|^2 ((\eta_{\epsilon\sqrt{\tau}} \ast v_\epsilon) \cdot \nabla \phi) \, dx \, ds + \int_0^t \int 2\pi_\epsilon(v_\epsilon \cdot \nabla \phi) \, dx \, ds,
\]
for any non-negative \( \phi \in C_0^\infty(\mathbb{R}^3 \times [0, \infty)) \). Fix \( \chi \in C^\infty(\mathbb{R}) \) with \( \chi(t) = 1 \) if \( t \leq 1 \) and \( \chi(t) = 0 \) if \( t \geq \lambda \). We now fix \( \phi \) in (3.7) as
\[
\phi(x, t) = \chi^2(|x|) \cdot \chi(t).
\]

We will estimate the terms on the right hand side of (3.7) for \( 0 < t \leq 1 \), and we can treat \( \phi \) as \( t \)-independent from now on. The first term is bounded by \( \alpha_0 \). For the
second, using the scaling properties (3.6) of \( v_\epsilon \), we have
\[
\int_0^t \int_{B_\lambda} |v_\epsilon|^2 (\partial_s \phi + \Delta \phi) \, dx \, ds \leq C \int_0^t \int_{B_\lambda} |v_\epsilon|^2 \, dx \, ds \\
\leq C \lambda^3 \int_0^{t/\lambda^2} \int_{B_1} |v_\epsilon|^2 \, dx \, ds \\
\leq C(\lambda) \int_0^t \tilde{\alpha}_\epsilon(s) \, ds.
\]

For the cubic term, we begin by using Young’s inequality to obtain
\[
\int_0^t \int_{B_\lambda} |v_\epsilon|^2 ((\eta_\epsilon \sqrt{s} * v_\epsilon) \cdot \nabla \phi) \, dx \, ds \leq C \int_0^t \int_{B_\lambda} |v_\epsilon|^3 \, dx \, ds \\
+ C \int_0^t \int_{B_\lambda} |(\eta_\epsilon \sqrt{s} * v_\epsilon)|^3 \, dx \, ds.
\]

Re-scaling the non-mollified term and making the obvious change of variables results in the estimate
\[
\int_0^t \int_{B_\lambda} |v_\epsilon|^3 \, dx \, ds \leq C(\lambda) \int_0^{t/\lambda^2} \int_{B_1} |v_\epsilon|^3 \, dy \, d\tau \leq C(\lambda) \int_0^t \int |v_\epsilon|^3 \phi^{3/2} \, dx \, ds.
\]
By the Gagliardo-Nirenberg inequality and re-scaling (3.6), we have, for any $s > 0$, that

$$
\| \phi^{1/2} v_\epsilon(s) \|_{L^3} \leq C \| \nabla \otimes (\phi^{1/2} v_\epsilon) \|_{L^2}^{1/2} \| \phi^{1/2} v_\epsilon \|_{L^2}^{1/2}(s)
$$

$$
\leq C(\lambda) (\tilde{\alpha}_\epsilon(s))^{1/2} + \| \phi^{1/2} \nabla v_\epsilon(s) \|_{L^2}^{1/2} (\tilde{\alpha}_\epsilon(s))^{1/4}.
$$

Hence, for any $\gamma > 0$,

$$
\| \phi^{1/2} v_\epsilon(s) \|_{L^3}^2 \leq C(\lambda) (\gamma^{-3} \tilde{\alpha}_\epsilon(s)^3 + \gamma \tilde{\alpha}_\epsilon(s))^1 + \gamma \| \phi^{1/2} \nabla v_\epsilon(s) \|_{L^2}^2.
$$

Thus,

$$
\int_0^t \int |v_\epsilon|^2 ((\eta_\sqrt{\tau} * v_\epsilon) \cdot \nabla \phi) \, dx \, ds \leq C(\lambda, \gamma, \eta) \int_0^t (\tilde{\alpha}_\epsilon(s)^3 + \tilde{\alpha}_\epsilon(s)) \, ds
$$

$$
+ C(\lambda) \gamma \int_0^t \int |\nabla v_\epsilon|^2 \phi \, dx \, ds.
$$

Provided $\gamma$ is small enough, the gradient term can be absorbed into the left hand side of (3.7).

We next estimate the pressure term. For this we need a formula for the pressure which we presently justify. Let $w_\epsilon = v_\epsilon - V_0$ where $V_0(x, t) = e^{t\Delta} v_0$. We have

$$
\partial_t w_\epsilon - \Delta w_\epsilon + \nabla \pi_\epsilon = g, \quad \text{div } w_\epsilon = 0,
$$

where $g_i = -\partial_j G_{ji}$ with

$$
G = (\eta_\sqrt{\tau} * v_\epsilon) \otimes v_\epsilon
$$

$$
= (\eta_\sqrt{\tau} * w_\epsilon + \eta_\sqrt{\tau} * V_0) \otimes (w_\epsilon + V_0).
$$

For $0 < t_1 < t_2 < \infty$, we have

$$
V_0 \in C([t_1, t_2]; L^4(\mathbb{R}^3) \cap L^6(\mathbb{R}^3)),
$$

$$
w_\epsilon \in L^\infty(t_1, t_2; L^2(\mathbb{R}^3)) \cap L^2(t_1, t_2; L^6(\mathbb{R}^3)) \subset L^4(t_1, t_2; L^3(\mathbb{R}^3)).
$$

By Young’s convolution inequality,

$$
\| G \|_{L^2(t_1, t_2; L^2)} \lesssim \| \eta_\sqrt{\tau} \|_{L^\infty(t_1, t_2; L^6(\mathbb{R}^3) \cap L^1)} (\| w_\epsilon \|_{L^4(t_1, t_2; L^3(\mathbb{R}^3))} + \| V_0 \|_{L^4(\mathbb{R}^3 \times [t_1, t_2])})^2.
$$

Since $g \in L^2([t_1, t_2]; H^{-1})$, [6, Lemma A.2] implies $w_\epsilon \in C([t_1, t_2]; L^2)$ (after modification on a set of time of measure zero; since the modified vector field still satisfies the above system distributionally, this does not affect our argument).

Consider the following non-stationary Stokes system with forcing $g$

$$
\partial_t V - \Delta V + \nabla P = g, \quad \text{div } V = 0,
$$

with initial data $V_0 = w_\epsilon(t_1) \in L^2(\mathbb{R}^3)$. It is well known that if $g \in L^\infty(t_1, t_2; H^{-1})$ and $V_0 \in L^2$, then there exists a unique $V \in C_w([t_1, t_2]; L^2(\mathbb{R}^3)) \cap L^2([t_1, t_2]; H^1(\mathbb{R}^3))$
and unique $\nabla P$ solving the above non-stationary stokes system (see [3, p. 1107-1108]). Letting $V = w_\epsilon$ and $P = \pi_\epsilon$, this implies that $w_\epsilon$ and $\nabla \pi_\epsilon$ are unique. Up to a function $\pi_\ast(t)$ independent of $x$,

$$\pi_\epsilon(x, t) - \pi_\ast(t) = -\frac{1}{3}[(\eta_{\sqrt{\tau}} * v_\epsilon) \cdot v_\epsilon](x, t) + \lim_{\delta \to 0} \int_{|y| > \delta} K_{ij}(x - y)(\eta_{\sqrt{\tau}} * v_\epsilon)_j(y, t)(v_\epsilon)_j(y, t) dy,$$

where $K_{ij}(x) = \partial_i \partial_j \frac{1}{4\pi|x|}$. The right side is defined in $L^2([t_1, t_2]; L^2(\mathbb{R}^3))$. Since the only appearance of $\pi_\epsilon$ in (3.5) is $\nabla \pi_\epsilon$, we can re-define $\pi_\epsilon$ to equal $\pi_\epsilon - \pi_\ast(t)$ and, therefore, can drop $\pi_\ast(t)$ from (3.11).

The pressure $\pi_\epsilon$ given by (3.11) is already bounded in $L^2([t_1, t_2]; L^2(\mathbb{R}^3))$ for any $0 < t_1 < t_2 < \infty$ but the bound depends on $t_1$, $t_2$ and $\epsilon$. We now bound it in $L^{3/2}(0, T; L^{3/2}(B_\lambda))$. Bounding the first term from (3.11) is simple given Hölder’s inequality, (3.8), and (3.9). In particular, we have for any $\gamma > 0$

$$\int_0^t \| \frac{1}{3}[(\eta_{\sqrt{\tau}} * v_\epsilon)(\cdot, s)]v_\epsilon(\cdot, s)] \|_{L^{3/2}(B_\lambda)}^{3/2} ds$$

$$\leq C(\lambda, \gamma, \eta) \int_0^t (\tilde{\alpha}_\epsilon(s)^3 + \tilde{\alpha}_\epsilon(s)^1) ds + \gamma \int_0^t \int \|\nabla v_\epsilon\|^2 \phi dx ds.$$

To bound the principal value integral in (3.11), we need to split the integral into local and non-local parts as follows,

$$\lim_{\delta \to 0} \int_{|y| > \delta} K(x - y)(\eta_{\sqrt{\tau}} * v_\epsilon)(y, t)v_\epsilon(y, t) dy$$

$$= \lim_{\delta \to 0} \int_{B_{\lambda^2} \setminus B_\delta} K(x - y)(\eta_{\sqrt{\tau}} * v_\epsilon)(y, t)v_\epsilon(y, t) \chi_{B_{\lambda^2}}(y) dy$$

$$+ \int_{|y| > \lambda^2} K(x - y)(\eta_{\sqrt{\tau}} * v_\epsilon)(y, t)v_\epsilon(y, t) dy$$

$$=: \pi_{\text{near}}(x, t) + \pi_{\text{far}}(x, t).$$

To bound $\pi_{\text{near}}$ note that, by the Calderon-Zygmund theory,

$$\|\pi_{\text{near}}(\cdot, t)\|_{L^{3/2}(B_\lambda)} \leq \|\eta_{\sqrt{\tau}} * v_\epsilon)(\cdot, t)v_\epsilon(\cdot, t)\|_{L^{3/2}(B_{\lambda^2})},$$

and, arguing as above using (3.8) but with $B_{\lambda^2}$ in place of $B_\lambda$ (see the note following (3.8)), it follows that

$$\int_0^t \|\pi_{\text{near}}(\cdot, s)\|_{L^{3/2}(B_\lambda)}^{3/2} ds \leq C(\lambda, \gamma, \eta) \int_0^t (\tilde{\alpha}_\epsilon(s)^3 + \tilde{\alpha}_\epsilon(s)^1) ds + \gamma \int_0^t \int \|\nabla v_\epsilon\|^2 \phi dx ds.$$

Bounding the term $\pi_{\text{far}}$ is more complicated. Let $A_k = \{x : \lambda^{k-1} \leq |x| < \lambda^k\}$. We
start with the following pointwise estimate which is valid whenever \( x \in B_\lambda, \)

\[
|\pi_{\text{far}}(x, t)| \leq C \sum_{k=3}^{\infty} \int_{A_k} \frac{1}{|x - y|^3} |(\eta_{\epsilon \sqrt{t}} * v_\epsilon)(y, t)| |v_\epsilon(y, t)| \, dy
\]

\[
\leq C(\lambda) \sum_{k=3}^{\infty} \frac{1}{\lambda^{2k}} \int_{A_0} |(\eta_{\epsilon \sqrt{t}} * v_\epsilon)(z, t \lambda^{-2k})| |v_\epsilon(z, t \lambda^{-2k})| \, dz
\]

\[
\leq C(\lambda) \sum_{k=3}^{\infty} \frac{1}{\lambda^{2k}} \|(\eta_{\epsilon \sqrt{t}} * v_\epsilon)(t \lambda^{-2k})\|_{L^2(B_1)} \|v_\epsilon(t \lambda^{-2k})\|_{L^2(B_1)}
\]

\[
\leq C(\lambda) \sum_{k=3}^{\infty} \frac{1}{\lambda^{2k}} \|v_\epsilon(t \lambda^{-2k})\|_{L^2(B_{\lambda^2})}^2
\]

\[
\leq C(\lambda) \tilde{\alpha}_\epsilon(t),
\]

where we have used (3.6), (3.8) and re-scaled the solution. Therefore,

\[
\int_0^t \|\pi_{\text{far}}(\cdot, s)\|_{L^{3/2}(B_\lambda)}^{3/2} \, ds \leq C(\lambda) \int_0^t \tilde{\alpha}_\epsilon(s)^{3/2} \, ds.
\]

After using Hölder’s inequality, (3.9), the above bounds, and \( \alpha^{3/2} \leq \alpha + \alpha^3 \) for \( \alpha > 0, \) it is clear that

\[
\int_0^t \|\pi_\epsilon(\cdot, s)\|_{L^{3/2}(B_\lambda)}^{3/2} \, ds + \int_0^t \int 2\pi_\epsilon(v_\epsilon \cdot \nabla \phi) \, dx \, ds
\]

\[
\leq C(\lambda, \gamma, \eta) \int_0^t (\tilde{\alpha}_\epsilon(s)^3 + \tilde{\alpha}_\epsilon(s)^1) \, ds + \gamma \int_0^t \int |\nabla v_\epsilon|^2 \phi \, dx \, ds.
\]

Combining the above estimates (and taking \( \gamma \) sufficiently small to absorb the gradient terms on the right hand side), we obtain

\[
\alpha_\epsilon(t) + \int_0^t \int_{B_1} |\nabla v_\epsilon|^2 \, dx \, ds \leq \alpha_0 + C(\lambda, \eta, \gamma) \int_0^t (\tilde{\alpha}_\epsilon(s)^3 + \tilde{\alpha}_\epsilon(s)^1) \, ds. \tag{3.12}
\]

Therefore,

\[
\tilde{\alpha}_\epsilon(t) \leq \alpha_0 + C \int_0^t (\tilde{\alpha}_\epsilon(s)^3 + \tilde{\alpha}_\epsilon(s)^1) \, ds. \tag{3.13}
\]

By continuity of \( \alpha_\epsilon(t), \) we have

\[
\tilde{\alpha}_\epsilon(t) \leq 2\alpha_0, \quad \forall t < T \tag{3.14}
\]

for some \( T > 0. \) By a continuity argument, we may take \( T = (C(2 + 8\alpha_0^2))^{-1}. \)
Letting $\epsilon \to 0$ yields

$$(v, \chi_{B_1}v)(t) \leq \liminf_{\epsilon \to 0} (v_\epsilon, \chi_{B_1}v_\epsilon)_{L^2}(t) \leq 2\alpha_0,$$

for all $t \leq T$. Note that (3.12) gives uniform (in $\epsilon$) control of

$$\int_0^T \int_{B_1} |\nabla v_\epsilon|^2 \, dx \, dt \leq C(\alpha_0, \lambda)$$

for some constant $C(\alpha_0, \lambda)$. From [3] we have that $v_\epsilon$ converges weakly to $v$ in $L^2(\frac{1}{k}; T; H^1(B_1))$ for every $k \in \mathbb{N}$. Hence,

$$\int_{1/k}^T \int_{B_1} |\nabla v|^2 \, dx \, dt \leq \sup_{\epsilon > 0} \int_0^T \int_{B_1} |\nabla v_\epsilon|^2 \, dx \, dt,$$

and, letting $k \to \infty$, it follows that

$$\int_0^T \int_{B_1} |\nabla v|^2 \, dx \, dt \leq C(\alpha_0, \lambda).$$

Similarly, since $\pi_\epsilon \in L^{3/2}(0, T; L^{3/2}(B_1))$ with uniformly bounded norms, it follows that $\pi \in L^{3/2}(0, T; L^{3/2}(B_1))$. Applying Lemma 2.1 yields the desired pressure representation in $L^{3/2}(0, T; L^{3/2}(B_1))$ and concludes the proof.

4 DSS solutions with data in $L^2_{\text{loc}}(\mathbb{R}^3)$

In this section we prove Theorem 1.2. To do this, we need to approximate DSS data in $L^2_{\text{loc}}$ by divergence free DSS vector fields in $L^3_w$ and also characterize discrete self-similarity on $\mathbb{R}^3 \times (0, \infty)$ in terms of a neighborhood of the origin.

4.1 Approximation of DSS data in $L^2_{\text{loc}}$

Lemma 4.1. Let $f \in L^2_{\text{loc}}(\mathbb{R}^3; \mathbb{R}^3)$ be a given divergence free $\lambda$-DSS vector field for some $\lambda > 0$. There exists a sequence of divergence free $\lambda$-DSS vector fields $\phi^{(k)}$ so that $\phi^{(k)} \in L^3_w(\mathbb{R}^3)$ and $\|\phi^{(k)} - f\|_{L^2(B_1)} \to 0$ as $k \to \infty$ ($B_1$ is the ball of radius 1 centered at the origin).

The main difficulty in proving this lemma is that each $f^{(k)}$ must be divergence free. We thus need to use the Bogovskii map which we presently recall, see [2].

Lemma 4.2. Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$, $2 \leq n < \infty$. There is a linear map $\Psi$ that maps a scalar $f \in L^q(\Omega)$ with $\int_{\Omega} f = 0$, $1 < q < \infty$, to a vector field $v = \Psi f \in W^{1,q}_0(\Omega; \mathbb{R}^n)$ and

$$\text{div} \, v = f, \quad \|v\|_{W^{1,q}_0(\Omega)} \leq c(\Omega, q) \|f\|_{L^q(\Omega)}.$$

The map $\Psi$ is independent of $q$ for $f \in C^\infty_c(\Omega)$. 

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Proof of Lemma 4.1. Let $Z_0(x)$ be $C^\infty(\mathbb{R}^3)$ satisfy

$$Z_0(x) = \begin{cases} 
1 & |x| > 1 \\
\text{radial, increasing} & \lambda^{-1} \leq |x| \leq 1 . \\
0 & |x| < \lambda^{-1}.
\end{cases}$$

Note that $\nabla \cdot (Z_0 f) = f \cdot \nabla Z_0$ - i.e. $Z_0 f$ is not divergence free. We can correct this using Lemma 4.2 with $q = 2$ for the scalar $-f \cdot \nabla Z_0$ noting that $f$ is locally square integrable and

$$\int -f \cdot \nabla Z_0 \, dx = 0,$$

because $f$ is divergence free. Denote by $\Phi_0$ the image of $-f \cdot \nabla Z_0$ under a Bogovskii mapping with domain $\{ x : \lambda^{-1} \leq |x| \leq 1 \}$. Then, $\Phi_0 \in W^{1,2}_0(B_1 \setminus B_{\lambda^{-1}})$ and

$$\nabla \cdot (Z_0 f + \Phi_0) = 0.$$

Let $Z_i(x) = Z_0(x/\lambda^i)$ and $\Phi_i(x) = \lambda^{-i} \Phi_0(\lambda^{-i} x)$ for all $i \in \mathbb{Z}$. It follows that

$$\nabla \cdot (Z_i f + \Phi_i) = 0,$$

for all $i \in \mathbb{Z}$. Note that $\text{supp}(Z_i - Z_{i+2}) = \{ x : \lambda^{i-1} \leq |x| \leq \lambda^{i+2} \}$. Let

$$f_i = \frac{1}{2} (Z_i - Z_{i+2}) f + \frac{1}{2} (\Phi_i - \Phi_{i+2}).$$

Then each $f_i$ is divergence free and supported on $B_{\lambda^{i+2}} \setminus B_{\lambda^{i-1}}$. Furthermore,

$$f = \sum_{i \in \mathbb{Z}} f_i,$$

where convergence is understood in the point-wise sense for all $x \neq 0$. To confirm this note that, if $x$ satisfies $\lambda^i \leq |x| < \lambda^{i+1}$ then $x \in \text{supp}(Z_j - Z_{j+2})$ if and only if $j \in \{ i - 1, i, i + 1 \}$. It follows that $\sum_{j \in \mathbb{Z}} (Z_j - Z_{j+2}) f = 2$. On the other hand, $\text{supp} \Phi_j = \{ x : \lambda^{j-1} \leq |x| \leq \lambda^j \}$ and, therefore, $\sum_{j \in \mathbb{Z}} (\Phi_j - \Phi_{j+2}) = \Phi_{i+1}(x) - \Phi_{i+1}(x) = 0$. It follows that $f = \sum_{i \in \mathbb{Z}} f_i$.

Assume $f_0^{(k)}$ is a sequence of divergence free vector fields in $C_0^\infty(B_{\lambda^2} \setminus B_{\lambda^{-1}})$ so that $f_0^{(k)} \to f_0$ in $L^2(B_{\lambda^2} \setminus B_{\lambda^{-1}})$. Let $\phi_i^{(k)} = \lambda^{-i} f_0^{(k)}(\lambda^{-i} x)$. Then the vector field

$$\phi^{(k)} = \sum_{i \in \mathbb{Z}} \phi_i^{(k)},$$

is a divergence free, $\lambda$-DSS vector field, and satisfies

$$|\phi^{(k)}(x)| \leq c_k |x|^{-1},$$

(where the proportionality constants $c_k$ are not uniformly bounded with respect to $k$). Hence, $\phi^{(k)} \in L^3_w$. We finish by arguing that $\phi^{(k)} \to f$ in $L^2(B_1)$. We know that
\[ \int_{B_{\frac{1}{2}} \setminus B_{\lambda^{-1}}} (\phi_0^{(k)} - f)^2 \, dx \to 0 \text{ as } k \to \infty. \]

Using the definition of \( \phi^{(k)} \) and the fact that \( f \) is discretely self-similar we have, letting \( A_i = B_{\lambda^i} \setminus B_{\lambda^{i-1}} \), that

\[ \int_{B_1} (\phi^{(k)} - f)^2 \, dx = \sum_{i \leq 0} \int_{A_i} (\phi^{(k)} - f)^2 \, dx = \sum_{i \leq 0} \lambda^i \int_{A_0} (\phi^{(k)} - f)^2 \, dx = \frac{\lambda}{\lambda - 1} \int_{A_0} (\phi^{(k)} - f)^2 \, dx. \]

In \( A_0 \), we have \( \phi^{(k)} - f = \sum_{i=-2}^0 (\phi_i^{(k)} - f_i) \). Thus

\[ \|\phi^{(k)} - f\|_{L^2(A_0)} \leq \sum_{i=-2}^0 \|\phi_i^{(k)} - f_i\|_{L^2(A_0)} \]

\[ = \sum_{k=0}^2 \lambda^{-k/2} \|\phi_0^{(k)} - f_0\|_{L^2(A_k)} \leq 3\|\phi_0^{(k)} - f_0\|_{L^2(B_{\frac{1}{2}} \setminus B_{\lambda^{-1}})}, \]

which completes the proof. \( \square \)

### 4.2 DSS solutions in a neighborhood of the origin

In the introduction we saw that any time-periodic solution \( u \) to (1.10) corresponds to a DSS solution \( v \) after the change of variables (1.9). Distributionally, \( u \) is a time-periodic solution to (1.10) if and only if

\[ \int_{s'}^{s' + T} ((u, \partial_s f) - (\nabla u, \nabla f) + (\frac{1}{2} u + \frac{1}{2} y \cdot \nabla u - u \cdot \nabla u, f)) \, ds = 0, \]  

(4.1)

holds for all \( s' \in \mathbb{R} \) and \( f \in \mathcal{D}_T \) where \( \mathcal{D}_T \) denotes the collection of all smooth divergence free vector fields in \( \mathbb{R}^3 \times \mathbb{R} \) which are time periodic with period \( T \) and whose supports are compact in space. In [3], this definition was used with \( s' = 0 \) since the goal was to extend a solution on \([0, T]\) to \( \mathbb{R} \) using periodicity. The same modification can be made here based on the observations that if \( u \) satisfies (4.1) then \( u \) can be extended to a time-periodic solution on \( \mathbb{R} \) and if \( u \) is a time-periodic solution on \( \mathbb{R} \) then \( u \) satisfies (4.1).

Since there is a one to one correspondence between time-periodic solutions to (1.10) and DSS solutions, an equivalent characterization of DSS solutions is obtained by reformulating (4.1) in the physical variables. For \( f \in \mathcal{D}_T \) let \( \zeta_f(x, t) = t^{-1} f(y, s) \). Note \( \zeta_f(x, t) = \lambda^2 \zeta_f(\lambda x, \lambda^2 t) \). Then, \( v \) is \( \lambda \)-DSS if and only if

\[ \int_{t}^{\lambda^2 t} ((v, \partial_t \zeta_f) - (\nabla v, \nabla \zeta_f) - (v \cdot \nabla v, \zeta_f)) \, d\tau = 0, \]  

(4.2)

for all \( t > 0 \) and \( f \in \mathcal{D}_T \), since (4.1) is just (4.2) in similarity variables. Note that \( (v, \zeta_f)|_{\tau = \lambda t} = (v, \zeta_f)|_{\tau = t} \). It follows that, if \( v \) is a solution to (1.1) that satisfies (4.2) for \( t = 1 \), then \( v|_{\tau \in [1, \lambda^2]} \) can be extended to a \( \lambda \)-DSS solution for all positive times.
Fix $k \in \mathbb{Z}$ and let $Q_k = B_{\lambda^k}(0) \times (0, \lambda^{2k})$. Our goal is to give a third characterization of discrete self-similarity on $Q_k$. Let $f \in \mathcal{D}_T$ be given and $\zeta_f$ be as above. Let $R$ be large enough so that, for all $t \in [1, \lambda^2]$, the support of $\zeta_f(t)$ is a subset of $B_R(0)$ and choose $m = m(f) \in \mathbb{Z}$ so that $R/\lambda^m < \lambda^k$ and $\lambda^{2-2m} < \lambda^{2k}$. It follows that

$$B_{R/\lambda^m}(0) \times [\lambda^{-2m}, \lambda^{2-2m}] \subset Q_k.$$ 

Extend $\zeta_f$ to all $t > 0$ using the following scaling: For $(x, t) \in \mathbb{R}^3 \times (0, \infty)$, let

$$\zeta_f(x, t) = \lambda^{2i} \zeta_f(\lambda^ix, \lambda^{2i}t),$$

where $i$ is chosen so that $\lambda^{2i}t \in [1, \lambda^2]$. Since $\zeta_f|_{\mathbb{R}^3 \times [1, \lambda^2]}$ is compactly supported in space, its spatial support shrinks as $t \to 0^+$. In particular, for $t \in [\lambda^{-2m}, \lambda^{2-2m}]$, supp $\zeta_f \subset Q_k$. For $m \in \mathbb{Z}$, let

$$\mathcal{D}_{Q_k}^m = \{ \phi \in C^\infty(\mathbb{R}^3 \times (0, \infty)) : \text{supp } \phi|_{\mathbb{R}^3 \times [\lambda^{-2m}, \lambda^{2-2m}]} \subset Q_k \}
\quad \text{and } \forall (x, t) \in \mathbb{R}^3 \times (0, \infty), \exists f \in \mathcal{D}_T \text{ such that } \phi(x, t) = \zeta_f(x, t).} \tag{4.3}$$

It is easy to see that, $\cup_{m \in \mathbb{Z}} \mathcal{D}_{Q_k}^m = \mathcal{D}_T$.

Re-scaling (4.2) gives

$$\int_{\lambda^{-2m}}^{\lambda^{2-2m}} \left((v, \partial_t \zeta_f) - (\nabla v, \nabla \zeta_f) - (v \cdot \nabla v, \zeta_f)\right) dt' = 0, \tag{4.4}$$

where $t' = t/\lambda^{2m}$ and the innerproducts are taken with respect to the re-scaled spatial variable $x' = x/\lambda^m$. In particular, the integral is computed over a subset of $Q_k$ and is identical to the same integral with $\zeta_f$ replaced by $\phi$ for some $\phi \in \mathcal{D}_{Q_k}^m$. Thus, if $v$ is a solution to (1.1), and $\phi \in \mathcal{D}_{Q_k}^m$ for some $m \in \mathbb{Z}$, then (4.2) is satisfied if and only if (4.4) is satisfied for the $f \in \mathcal{D}_T$ for which $\zeta_f = \phi$. This leads to the following extendability property: If $v$ is a solution to (1.1) on $Q_k$ and satisfies (4.4) for every $m \in \mathbb{Z}$ and $\phi \in \mathcal{D}_{Q_k}^m$, then $v$ can be extended to a discretely self-similar solution on $\mathbb{R}^3 \times (0, \infty)$; in other words, if a solution is DSS in a neighborhood of the origin, then it can be extended to a DSS solution on $\mathbb{R}^3 \times (0, \infty)$.

### 4.3 Construction of DSS solutions

**Proof of Theorem 1.2.** Fix $\lambda > 1$ and assume $v_0 \in L^2_{\text{loc}}$ is a divergence free $\lambda$-DSS vector field. Let $\{v_0^{(k)}\}$ be the sequence of vector fields $\{\phi^{(k)}\}$ from Lemma 4.1 applied to $v_0$. Then, the values $\|v_0^{(k)}\|_{L^2(B_1)}$ are uniformly bounded and $\|v_0^{(k)} - v_0\|_{L^2(B_1)} \to 0$ as $k \to \infty$. Since $v_0^{(k)} \in L^3_w$ and is $\lambda$-DSS, by [3] there exists a $\lambda$-DSS local Leray solution $v_k$ to (1.1) and an associated pressure $\pi_k$ having initial data $v_0^{(k)}$ for every $k \in \mathbb{N}$. By Proposition 3.1, $v_k$ are uniformly bounded in $L^\infty(0, T; L^2(B_1)) \cap L^2(0, T; H^1(B_1))$ (hence also in $L^{10/3}(0, T; L^{10/3}(B_1))$) for some $T$ which depends only on $\lambda$ and $\|v_0^{(k)}\|_{L^2(B_1)}$. As usual (cf. [3, 14, 18]), there exists a distribution $v$ and a subsequence of $\{v_k\}$ (still indexed by $k$ for simplicity) so that $v_k$ converges to $v$ in the weak star topology on $L^\infty(0, T; L^2(B_1))$, in the weak topology on $L^2(0, T; H^1(B_1))$, and in...
$L^2(0,T; L^2(B_1))$. Since they are uniformly bounded in $L^{10/3}(0,T; L^{10/3}(B_1))$, they also converge in $L^q(0,T; L^q(B_1))$ for any $q < 10/3$. By the pressure estimate (3.2) in Proposition 3.1, $\pi_k$ are uniformly bounded in $L^{3/2}(0,T; L^{3/2}(B_1))$ by $C(\lambda, ||v_0||_{L^2(B_1)})$ and, therefore, we may extract a subsequence which converges weakly to a distribution $\pi \in L^{3/2}(0,T; L^{3/2}(B_1))$.

Fix $\kappa \in \mathbb{Z}$ so that $\lambda^\kappa < 1$ and $\lambda^{2\kappa} < T$. Then, $Q_\kappa = B_{\lambda^\kappa} \times (0, \lambda^{2\kappa}) \subset B_1 \times (0,T)$. Therefore $v_k$ satisfies (1.1) on $Q_\kappa$ and satisfies (4.4) for every $m \in \mathbb{Z}$ and $\phi \in \mathcal{D}'_{Q_\kappa}$. Thus, $v$ can be extended to a DSS solution on $\mathbb{R}^3 \times (0, \infty)$ (which we still denote by $v$).

For compact subsets $K$ of $B_1$, we automatically have $\lim_{t \to 0^+} ||v - v_0||_{L^2(K)} = 0$. For a general compact subset $K$ of $\mathbb{R}^3$, we have $K' = \lambda^m K \subset B_1$ for some $m \in \mathbb{Z}$, and

$$\int_K |v(x,t) - v_0(x)|^2 \, dx = \lambda^{-m} \int_{K'} |v(x', \lambda^{2m} t) - v_0(x')|^2 \, dx'.$$

It follows that $\lim_{t \to 0^+} ||v(t) - v_0||_{L^2(K)} = 0$ for every compact set $K \subset \mathbb{R}^3$. A similar re-scaling argument also implies that $v \in L^{\infty}(0,T'; L^2(K)) \cap L^2(0,T'; H^1(K))$ and $\pi \in L^{3/2}(0,T'; L^{3/2}(K))$ for any $T' > 0$ and compact subset $K$ of $\mathbb{R}^3$.

To confirm that $v$ satisfies the local energy inequality, first note that each $v_k$ satisfies the local energy inequality

$$\int |v_k|^2 \phi \, dx + 2 \int \int |\nabla v_k|^2 \phi \, dx \, dt$$

$$\leq \int |v_0^{(k)}|^2 \phi \, dx + \int \int |v_k|^2 (\partial_t \phi + \Delta \phi) \, dx \, dt + \int \int (|v_k|^2 + 2\pi_k) (v_k \cdot \nabla \phi) \, dx \, dt,$$

for all non-negative $\phi \in C_c^\infty(\mathbb{R}^3 \times \mathbb{R}_+^3)$. Furthermore, the right hand sides of the energy inequality for $v^{(k)}$ converge to the right hand side of the energy inequality for $v$ as $k \to \infty$ while the left hand-sides are lower semi-continuous (cf. [6, (A.51)]). The local energy inequality for $v$ plainly follows.

Finally, note that $\pi_k$ satisfies the formula (3.3). Applying Lemma 2.1 to the above sequence and limit implies that $\pi$ satisfies the desired pressure formula in $L^{3/2}(0,T; L^{3/2}(B_1))$. Re-scaling establishes the formula in $L^{3/2}_{loc}(\mathbb{R}^3 \times (0, \infty))$. \hfill \Box

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