Numerical Simulation of Natural Convection between two elliptical cylinders: Influence of Rayleigh number and Prandtl number

A. Bouras\textsuperscript{a}, M. Djezzar\textsuperscript{b} and C. Ghernoug\textsuperscript{b}

\textsuperscript{a} Université de M’sila Algérie, karimbouras2006@yahoo.fr
\textsuperscript{b} Laboratoire Physique Energétique Faculté des Sciences Exactes, Université I Constantine Algérie

Abstract

This paper reports the numerical study of the phenomenon of the natural convection in a space annulus, situated between two horizontal confocal elliptic cylinders. This space is filled by a Newtonian fluid. The Prandtl number and the Rayleigh number vary. By using the Boussinesq approximation and the vorticity-stream function formulation, the flow is modeled by the differential equations with the derivative partial: continuity and momentum equations are expressed in a frame of reference known as "elliptical", to facilitate the writing of the boundary conditions and to transform the curvilinear field into a rectangular one. The two cylindrical walls of the enclosure are kept isotherms, $T_1$ for the internal wall and $T_2$ for the external wall, with $T_1>T_2$. A computer code was developed, the latter uses finite volumes, for the discretization of the equations and in order to show its reliability, the resulting results from the latter are compared with other similar results existing in the literature and the effect of the values of Rayleigh and Prandtl numbers on the results obtained that it is qualitatively or quantitatively, is examined.

Keywords : natural convection; elliptic cylinders; Rayleigh number; isotherms; Prandtl number

1. Introduction:

In the past two decades, buoyancy-driven flows in enclosed spaces have been studied extensively. Many authors have investigated the particular problem of symmetric free convection between concentric, horizontal circular cylinders and few authors have investigated the same problem between confocal horizontal elliptic cylinders. The problem has been approached analytically for small and large Rayleigh numbers by employing various methods to solve the coupled elliptic Navier-Stokes and energy equations. In order to solve the problem of naturally convective flow in a two-dimensional annulus, a numerical study of natural convection in horizontal elliptic cylinder was studied by Bello-Ochende [1] and Siegel [2] also analyzed the effect of buoyancy on heat transfer in a rotating tube.
Mack and Bishop [3] made a study in an annular space ranging between two horizontal concentric cylinders. They employed a power series truncated at the third power of the Rayleigh number to represent the stream function and temperature variables. Kuehn et al. [4] compiled a comprehensive review of the available experimental results for natural convection heat transfer between horizontal concentric and eccentric cylinders and proposed correlating equations using a conduction boundary-layer model. El-Sherbiny [5] investigated numerically the natural convection in air between two infinite horizontal concentric cylinders at different constant temperatures. The study covered a wide range of the Rayleigh number, Ra from $10^2$ to $10^6$, and the Radius Ratio, (RR) was changed between 1.25 and 10. Djezzar and Al [6], [7] simulated the case of natural convection in a space annulus between two elliptic confocal cylinders; they could detect multi-cellular flows, for certain geometries when the number of Grashof increases. Mikhail and al. [8] studied the effects of four types of influential factors such as the Rayleigh number Ra = $10^4$, $5.10^4$, $10^5$, the Prandtl number Pr = 0.7, 7.0, the thermal conductivity ratio and the inclination angle. Chatterjee and Biswas [9] examined the effect of planar confinement on heat transfer over the range $1 \leq \text{Re} \leq 30$, for a fixed value of $L/d = 5$ and over a wide range of Prandtl numbers up to Pr = 1000. M. A. Teamah [10] studied the effect of the heater length, Rayleigh number, Prandtl number and buoyancy ratio on both average Nusselt and Sherwood number in a Square Cavity. Schreiber and Singh [11] studied the cases in horizontal confocal elliptical cylinders oriented at an arbitrary angle with respect to the gravity vector in the same coordinate system. Elshamy et al [12] studied numerically the case in the horizontal confocal elliptical annulus and developed some practical correlations for the average Nusselt number. Chmaissem et al [13] simulated the case of natural convection in an annular space: having a horizontal axis bounded by circular and elliptical isothermal cylinders. Cheng and Chao [14] employed the body-fitted coordinate transformation method to generate a non-staggered curvilinear coordinate system and performed numerical study for some horizontal eccentric elliptical annuli.

In this work we propose a simulation which uses the finite volumes method described by Patankar [15], with the elliptic coordinates quoted by Moon [16] and the vorticity-stream function formulation illustrated by Nogotov [17] to solve the equations governing the phenomenon under study.

| Nomenclature                        | Definition                                                                 |
|-------------------------------------|---------------------------------------------------------------------------|
| $C_p$                               | Specific heat at constant pressure (j.kg$^{-1}$.K$^{-1}$)                  |
| $e_1, e_2$                          | Eccentricities of ellipses                                                |
| $Nu, Nua$                           | Local and average Nusselt number                                          |
| $g$                                 | Gravitational acceleration (m.s$^{-2}$)                                   |
| $Ra$                                | Rayleigh number                                                          |
| $H$                                 | Dimensional metric coefficient (m)                                        |
| $Pr$                                | Number of Prandtl defined by $S$                                          |
| $S_b$                               | Source Term                                                              |
| $T_1$                               | Hot’s wall Temperature (K)                                                |
| $T_2$                               | Cold’s wall Temperature (K)                                               |
| $\Delta T$                          | Temperature difference ($T_1$-$T_2$) (K)                                 |
| $V_\eta, V_\theta$                  | Components speed according to $\eta$ and $\theta$ directions (m.s$^{-1}$) |

| Greek Symbols                        | Definition                                                                 |
|--------------------------------------|---------------------------------------------------------------------------|
| $\alpha$                             | Inclination angle (°)                                                     |
| $\beta$                              | Thermal expansion coefficient (K$^{-1}$)                                 |
| $\lambda$                            | Thermal conductivity of the fluid (W.m$^{-1}$)                            |
| $\nu$                                | Kinematic viscosity (m$^2$.s$^{-1}$)                                      |
| $\rho$                               | Density of the fluid (kg.m$^{-3}$)                                       |
| $x, y$                               | Cartesian coordinates (m)                                                |
| $\eta, \theta, Z$                    | Elliptic coordinates                                                     |
| $\Psi$                               | Stream function (m$^2$.s$^{-1}$)                                         |
| $\omega$                             | Vorticity (s$^{-1}$)                                                     |
2. Theoretical analysis:

In fact let us consider a space annular, filled of a Newtonian fluid, located between two confocal elliptic cylinders of horizontal axes. The figure 1 represents a cross-section of the system.

![Fig. 1: A cross-section of the system](image)

Both elliptic internal and external walls are isothermal, kept at temperatures $T_1$ and $T_2$ respectively. With $T_1 > T_2$ the physical properties of the fluid are constant, apart from the density $\rho$ whose variations are at the origin of the natural convection. Viscous dissipation is neglected, just as the radiation (emissive properties of the two walls being neglected). We admit that the problem is bidimensionnal, permanent and laminar. It occurs in this space a natural convection that we propose to study numerically. The natural convection equations within the framework of the Boussinesq approximation are written:

$$\text{div} \vec{V} = 0 \quad (1)$$

$$\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \text{grad} ) \vec{V} = \frac{\rho}{\rho_0} g + \frac{\nabla \rho}{\rho_0} \quad (2)$$

$$\frac{\partial T}{\partial t} + (\vec{V} \cdot \text{grad} ) T = \frac{\lambda}{\rho_c} \nabla^2 T \quad (3)$$

It is convenient to define a reference frame such as the limits of the system result in constant values of the coordinates. The passage of the Cartesian coordinates $(x, y)$ to the elliptic coordinates $(\eta, \theta)$ is obtained by the following relations:

$$x = a.ch(\eta)\cos(\theta)$$

$$y = a.sh(\eta)\sin(\theta) \quad (4)$$

After of course, the introduction of the stream function, in order to check the equation of continuity identically. And by posing the following adimensional quantities:

$$D_h = a \text{ (arbitrarily selected focal distance)}$$

$$H = \frac{h}{D_h}, \quad V_\eta = \frac{V_\eta}{D_h}, \quad V_\theta = \frac{V_\theta}{D_h}, \quad \omega^+ = \omega \frac{D_h^2}{\nu}, \quad \psi^+ = \frac{\psi}{\nu} \quad \text{and} \quad T^+ = \frac{T - T_2}{T_1 - T_2}$$

The equations (1), (2) and (3) become:

$$V_\eta \frac{\partial (H V^+)}{\partial \eta} + V_\theta \frac{\partial (H V^+)}{\partial \theta} = 0 \quad (5)$$
\[ \frac{V_0^+}{H} \frac{\partial \psi^+}{\partial \eta} + \frac{V_0^+}{H} \frac{\partial \psi^+}{\partial \theta} = \frac{Ra \cdot Pr}{H} \left[ F(\eta, \theta) \cos(\alpha) - G(\eta, \theta) \sin(\alpha) \right] \frac{\partial T^+}{\partial \eta} - \left[ F(\eta, \theta) \sin(\alpha) + G(\eta, \theta) \cos(\alpha) \right] \frac{\partial T^+}{\partial \theta} \right] \]  

\[ + \frac{1}{H^2} \left( \frac{\partial^2 \omega^+}{\partial \eta^2} + \frac{\partial^2 \omega^+}{\partial \theta^2} \right) \]

\[ H V_\eta^+ \frac{\partial T^+}{\partial \eta} + H V_\theta^+ \frac{\partial T^+}{\partial \theta} = \frac{1}{Pr} \left( \frac{\partial^2 T^+}{\partial \eta^2} + \frac{\partial^2 T^+}{\partial \theta^2} \right) \]

\[ \omega^+ = -\frac{1}{H^2} \left( \frac{\partial^2 \psi^+}{\partial \eta^2} + \frac{\partial^2 \psi^+}{\partial \theta^2} \right) \]

With:

\[ h = a \sqrt{\frac{sh^2(\eta) + \sin^2(\theta)}{sh^2(\eta) + \sin^2(\theta)}} \]

\[ F(\eta, \theta) = \frac{sh(\eta) \cos(\theta)}{\sqrt{sh^2(\eta) + \sin^2(\theta)}}, \quad G(\eta, \theta) = \frac{ch(\eta) \sin(\theta)}{\sqrt{sh^2(\eta) + \sin^2(\theta)}} \]

The boundary conditions are the following ones:

- Hot inner wall (\( \eta = \eta_i = \text{cst} \)):
  \[ V_0^+ = V_0^- = \frac{\partial \psi^+}{\partial \eta} = \frac{\partial \psi^+}{\partial \theta} = 0 \quad \text{and} \quad T_1^+ = 1 \]

- Cold outer wall (\( \eta = \eta_e = \text{cst} \)):
  \[ V_0^+ = V_0^- = \frac{\partial \psi^+}{\partial \eta} = \frac{\partial \psi^+}{\partial \theta} = 0 \quad \text{and} \quad T_2^+ = 0 \]

3. Numerical Method

To solve equations (6) and (7) with the associated boundary conditions, we consider a numerical solution by the finite volumes method, exposed by Patankar [15]. The power law scheme was used for the discretization. For equation (8), we consider a numerical solution by the centered differences method.

The figure (2) shows physical and computational domain.

\[ \frac{N_T}{H} \left. \frac{\partial T^+}{\partial \eta} \right|_{\eta = \text{cst}} \quad \text{Nu} = \frac{1}{\theta_{\text{in}} - \theta_1} \int_{\theta_1}^\theta \text{Nu} \, d\theta \]

Once the temperature distribution is available, the local Nusselt number and average in the physical domain is defined as:

\[ \frac{\text{Nu}}{H} \left| \frac{\partial T^+}{\partial \eta} \right|_{\eta = \text{cst}} \quad \text{Nu} = \frac{1}{\theta_{\text{in}} - \theta_1} \int_{\theta_1}^\theta \text{Nu} \, d\theta \]
4. Results and discussions:

For the validation of our mathematical model, we compared our numerical results obtained for the Nusselt number at the internal and external cylinders walls (\(\text{Nu}_1, \text{Nu}_2\)) in the case of natural convection heat transfer between concentric cylinders with those of Elshamy et al [12] who studied numerically natural heat transfer for air bounded by two confocal horizontal elliptic cylinders. The present work and the work of Elshamy et al. [12] are compared in Table 1. For two cases we can notice that these values are in a good agreement.

Table 1: Comparison of average Nusselt number of Ref. [12] with our results:

| \(e_1\) | \(e_2\) | \(\alpha\) | \(R_a\) | \(\text{Nu (intern wall)}\) | \(\text{Nu (intern wall)}\) | \(\text{Nu (extern wall)}\) | \(\text{Nu (extern wall)}\) |
|--------|--------|-----------|--------|----------------|----------------|----------------|----------------|
| 0.86   | 0.4    | 90°       | 10³    | 3.72           | 3.68           | 1.37           | 1.35           |
| 0.86   | 0.4    | 90°       | 4.10⁴  | 4.86           | 5.34           | 1.80           | 1.93           |

4.1. Effect of Rayleigh Number

The figures (3-5) represent the isotherms and streamlines for different values of Rayleigh number \(Ra\) when \(\alpha=0°\). We note that these isotherms and these streamlines are symmetrical about the median fictitious vertical plane.

The isotherms of figure (3) corresponding to \(Ra = 50\) are parallel and concentric closed curves which coincide well with the wall profiles, in this case the temperature distribution is simply decreasing from the hot wall to the cold wall. The streamlines of the same figure show that the flow is organized in two main cells that rotate very slowly in opposite directions. The values of the Stream function are very low. In this case heat transfer takes mainly by conduction at the heated wall. Therefore when the Rayleigh number is weak, as being lower or equal to 50, the heat transfer is essentially conductive. For \(Ra=10^3\), (fig. 4) the isothermal lines change in a symmetrical way compared to the vertical axis, and the values of the stream function mentioned on the same figure increase also, which translates a transformation of the conductive transfer to the convective transfer.

For \(Ra=10^4\), the isothermal lines change and end up adopting the shape of a mushroom. The distribution of the temperature is decreasing from the hot wall towards the cold wall. The direction of the deformation of the isotherms is in conformity within the meaning of rotation of the streamlines. In laminar mode the particles which take off from the hot wall on the level of the axis of symmetry, the isothermal lines “arch” and move away from the wall to this place. The convective process is the predominate one. In figure (6) it is observed that the heat transfer increased as the Rayleigh number increased, the convection is dominant; the circulating cells are very strong.

![Fig.3: Isotherms and streamlines for Ra = 50 and Pr=0.702](image-url)
Fig. 4: Isotherms and streamlines for $Ra = 10^3$ and $Pr=0.702$

Fig. 5: Isotherms and streamlines for $Ra = 10^4$ and $Pr=0.702$

Fig. 6: Isotherms and streamlines for $Ra = 2.10^4$ and $Pr=0.702$
4.2. Variation of the local Nusselt number on the inner and outer wall:

Figure 7 illustrates the variation of the local Nusselt number on the inner and outer wall and allowing us to note that the minimum Nusselt number is reached and the angular position \( \theta = 90^\circ \), which is in accordance with figure 6 which shows that the two vortices meet at this spot in the fluid away from this wall. On the outer wall, the changes are reversed.

![Fig.7: Variation of the local Nusselt number on the inner and outer wall.](image)

4.3. Effect of Prandtl Number:

4.3.1. Effect of Prandtl number on the streamlines and isothermals

Figure 8 illustrates the effect of Prandtl number \( Pr \), on the streamlines and isothermals. To highlight on the effect of \( Pr \), the Rayleigh number is kept constant \( Ra = 10^5 \), at \( Pr = 0.25 \) the flow shows weak clockwise cell with maximum strength \( \Psi_{max} = 1.33 \). As the Prandtl number increases to \( Pr = 0.7 \) the flow shows also weak clockwise cell with maximum strength \( \Psi_{max} = 1.38 \). Increasing the Prandtl number shows no significant effect on the isothermal lines for both values of \( Pr = 0.25 \) and 0.7.

![a)](image)
Figure 8 shows the effect of Prandtl number $Pr$ on the streamlines and isothermals for $Ra=10^5$. The effect of Prandtl number on the Streamline and Isothermal for $Ra=10^5$.

Figure 9 shows the effect of Prandtl number $Pr$, on the streamlines and isothermals for, the Rayleigh number $Ra = 4.10^4$, at Prandtl number $Pr= 0.25$ the flow show strong circulating cell with maximum strength $\Psi_{\text{max}} = 34.02$. As increasing the Prandtl number $Pr=0.7$ the maximum strength of the circulating cell increases $\Psi_{\text{max}} = 40.47$. The effect of Prandtl number on the Streamline and Isothermal for $Ra=4.10^4$. 

![Figure 8: a) Pr=0.25, b) Pr=0.7. The effect of Prandtl number on the Streamline and Isothermal for Ra=10^5.](image)

![Figure 9: a) Pr=0.25, b) Pr=0.7, the effect of Prandtl number on the Streamline and Isothermal for Ra=4.10^4.](image)
4.3.2. Effect of Prandtl number on the average Nusselt number

The results are sketched in figure (10): for different values of the Prandtl number, ranging between 0.3 and 0.7 at Ra = 10^4, shows the influence of the Prandtl number on the average Nusselt number. It is found that the average Nusselt number increases with increase in Prandtl number.

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Fig.10: The effect of Prandtl number on the average Nusselt number
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5. Conclusion:

The phenomenon of the natural convection in an annular space, situated between two horizontal confocal elliptic cylinders is numerically studied. The study covered a wide range of Ra from 50 to 2.10^4. We established a mathematical model representing the transfers of movement in the fluid and heat through the active walls of the enclosure. This model rests on the assumption of Boussinesq and the bidimensionnality of the flow. A computer code is developed to solve the governing equations.

It is observed that the heat transfer increased as the Rayleigh number increased. For low Rayleigh number values, the coefficient of heat transfer is dominated by the conduction. For high Rayleigh numbers, the role of the convection becomes dominating.

For low Rayleigh numbers: Ra = 10^2, the Prandtl number has no significant effect on both average Nusselt number, for the Rayleigh number >10^2, we note that the heat transfer and streamlines values increase with increasing of Prandtl number.

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