ON A FAMILY OF SELF-AFFINE IFS WHOSE ATTRACTORS HAVE A NON-FRACTAL TOP
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Abstract. Let \( 0 < \lambda < \mu < 1 \) and \( \lambda + \mu > 1 \). In this note we prove that for the vast majority of such parameters the top of the attractor \( A_{\lambda,\mu} \) of the IFS \( \{(\lambda x, \mu y), (\mu x + 1 - \mu, \lambda y + 1 - \lambda)\} \) is the graph of a continuous, strictly increasing function. Despite this, for most parameters, \( A_{\lambda,\mu} \) has a box dimension strictly greater than 1, showing that the upper boundary is not representative of the complexity of the fractal. Finally, we prove that if \( \lambda \mu \geq 2^{-1/6} \), then \( A_{\lambda,\mu} \) has a non-empty interior.

1. Introduction

Self-affine iterated function systems (IFS) are well studied. When such an IFS is given by a single matrix, e. g., \( \{Mx, Mx + u\} \), it appears that all of its boundary is fractal, though there are no rigorous results in this direction, to our best knowledge. The purpose of this note is to present a family of two-dimensional IFS for which their attractors have a different kind of boundary for the top and the bottom. In particular, their tops are not fractal.

Assume \( 0 < \lambda < \mu < 1 \) and \( \lambda + \mu > 1 \). Put
\[
T_0(x, y) = (\lambda x, \mu y), \quad T_1(x, y) = (\mu x + 1 - \mu, \lambda y + 1 - \lambda).
\]
Let \( A_{\lambda,\mu} \) denote the attractor for the IFS \( \{T_0, T_1\} \). Notice that \( A_{\lambda,\mu} \subset [0,1] \times [0,1] \) – see Figure 1. Based upon visual inspection of such sets, one would expect that \( A_{\lambda,\mu} \) would have dimension strictly greater than 1. Despite this, it also surprisingly appears that the top of this IFS is one-dimensional. This is in stark contrast with the family of IFS \( \{(\lambda x, \mu y), (\lambda x + 1 - \lambda, \mu y + 1 - \mu)\} \) studied in detail in [3].

Put
\[
\partial_{\text{top}}(A_{\lambda,\mu}) = \{(x, y) \in A_{\lambda,\mu} : \forall (x, y') \in A_{\lambda,\mu} \text{ we have } y' \leq y\}.
\]

We will define a closed subset \( G \subset \{\lambda, \mu : \lambda + \mu > 1, 0 < \lambda < \mu < 1\} \) in Section 3 for which \( \partial_{\text{top}}(A_{\lambda,\mu}) \) is strictly increasing and continuous.

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This set $G$ has the property that it is at least 98.3% of the parameter space $\{(\lambda, \mu) : \lambda + \mu > 1, 0 < \lambda < \mu < 1\}$.

We have three main results. The first is

**Theorem 1.1.** For all $(\lambda, \mu) \in G$ we have the set $\partial_{\text{top}}(A_{\lambda,\mu})$ is the graph of a continuous, strictly increasing function.

It appears computationally that we can construct a $G$ arbitrarily close to the full parameter space. From this we make the

**Conjecture 1.2.** For all $0 < \lambda < \mu < 1$ with $\lambda + \mu > 1$ the set $\partial_{\text{top}}(A_{\lambda,\mu})$ is the graph of a continuous, strictly increasing function.

We have

**Theorem 1.3.** There exists $(\lambda, \mu)$ with $0 < \lambda < \mu < 1, \lambda + \mu > 1$ such that the set $A_{\lambda,\mu}$ has dimension strictly greater than 1.

In fact Theorem 1.3 is stronger than this. We give a range of parameters, making up 91.8% of the parameter space $\{(\lambda, \mu) : 0 < \lambda < \mu < 1, \lambda + \mu > 1\}$ for which $A_{\lambda,\mu}$ has dimension strictly greater than 1. In fact the range of parameters that satisfy both Theorem 1.1 and 1.3 makes up 91.3% of the parameter space. Unfortunately the technique used in Theorem 1.3 probably cannot be extended arbitrarily close to 100%, as we will discuss later. We observe that if $\lambda \mu < 1/2$ then we necessarily have $\dim(A_{\lambda,\mu}) < 2$ and hence all points are boundary points. This reinforces the observation that the upper boundary of $A_{\lambda,\mu}$ is not representative of the boundary of $A_{\lambda,\mu}$.

Although the technique does not appear to extend to all parameters $(\lambda, \mu)$, we still believe

**Conjecture 1.4.** For all $0 < \lambda < \mu < 1$ with $\lambda + \mu > 1$ the set $A_{\lambda,\mu}$ has dimension strictly greater than 1.

Lastly, using a technique from [4] we have

**Theorem 1.5.** For all $\lambda \mu \geq 2^{-1/6}$ we have $A_{\lambda,\mu}$ has non-empty interior.

In Section 2 we give a prove of Theorem 1.1. We also introduce a subset $B_{\lambda,\mu} \subset A_{\lambda,\mu}$ upon which the definition of $G$ is based. A computational investigation of $G$ is given in Section 3. Sections 4 and 5 prove Theorems 1.3 and 1.5 respectively.

# 2. Proof of Theorem 1.1

We will prove this result in two steps. The first is to show that $A_{\lambda,\mu}$ contains a strictly increasing continuous function going from $(0, 0)$ to $(1, 1)$ with some additional properties. This will be the set $B_{\lambda,\mu}$ and is described in Lemma 2.1.

After this we will introduce a map $\mathcal{R}$ which has $\partial_{\text{top}}(A_{\lambda,\mu})$ as an attractor, and further whose iterates on $B_{\lambda,\mu}$ are continuous increasing
functions with the same additional properties as $B_{\lambda, \mu}$. This is done in Lemma 2.3.

This second step requires an additional property on $B_{\lambda, \mu}$ which conjecturally is true for all $0 < \lambda < \mu < 1$, $\lambda + \mu > 1$, and computationally is true for at least 98.3% of such pairs $(\lambda, \mu)$. The set where this additional property is true is called $G$. See Definition 2.2 for a precise definition.

Put

$$S_0(x, y) = \begin{cases} T_0(x, y), & \text{if } \lambda x + \mu y \leq 1 \\ (0, 0), & \text{otherwise.} \end{cases}$$

$$S_1(x, y) = \begin{cases} T_1(x, y), & \text{if } \mu x + \lambda y \geq \lambda + \mu - 1 \\ (1, 1), & \text{otherwise.} \end{cases}$$

The attractor of $\{S_0, S_1\}$ is not unique. For example, the pair $\{(0, 0), (1, 1)\}$ is fixed under this map. It is clear that if we have two different attractors of $\{S_0, S_1\}$, then their union is also an attractor. Further, all attractors are contained in $[0, 1] \times [0, 1]$. As such there is a maximal attractor, which we define as $B_{\lambda, \mu}$. Clearly, $B_{\lambda, \mu} \subset A_{\lambda, \mu}$.

**Lemma 2.1.** The attractor $B_{\lambda, \mu}$ is the graph of a continuous function, i.e., for any $x \in [0, 1]$ there exists a unique $y \in [0, 1]$ such that $(x, y) \in B_{\lambda, \mu}$. This function is strictly increasing.

**Proof.** Let $X = [0, 1] \times [0, 1]$. Notice that $S_0(X) \cap S_1(X)$ is a segment on $x + y = 1$. Put

$$Y_n = \bigcup_{i_1, \ldots, i_n \in \{0, 1\}^n} S_{i_1} \ldots S_{i_n}(X).$$

Then $Y_n$ is a union of $2^n$ polygons such that their interiors are disjoint – see Figure 2.1. It is worth noting that these polygons may be the
points (0, 0) or (1, 1). One can show by induction that each non-trivial polygon is either a pentagon or a hexagon. These polygons are ordered: for any two of them, one’s upper right corner is higher than the other’s (see Figure 2.1). Also, $S_{i_1} \ldots S_{i_n}(X)$ is higher than $S_{j_1} \ldots S_{j_n}(X)$ iff $i_1 \ldots i_n \succeq j_1 \ldots j_n$.

Finally, any intersection of $Y_n$ with any horizontal or vertical line is an interval; this intersection involves only a bounded number of polygons. This follows from the fact that when we go from $Y_n$ to $Y_{n+1}$, we cut out a certain proportion of each polygon both horizontally and vertically—so we will have that any polygon will be strictly higher and to the right or strictly lower and to the left from any other polygon except a number of them which depends on $\lambda$ and $\mu$ only.

It is easy to see that $Y_n \to B_{\lambda,\mu}$ in the Hausdorff metric. □

A key property of $B_{\lambda,\mu}$ that we will exploit is that $T_0(1, 1)$ and $T_1(0, 0)$ are below $B_{\lambda,\mu}$. Unfortunately, although this appears to be computationally true for all $0 < \lambda < \mu < 1$ with $\lambda + \mu > 1$, a general proof is not known.

**Definition 2.2.** Define

$$G := \{(\lambda, \mu) : T_0(1, 1) \text{ and } T_1(0, 0) \text{ are below } B_{\lambda,\mu}\}$$

In Section 3 we discuss how one can find regions in $G$, and provide a link to data demonstrating that $G$ is at least 98.3% of the $0 < \lambda < \mu < 1$ with $\lambda + \mu > 1$.

We now introduce one last function, going from the set of non-empty compact sets to non-empty compact sets by

$$\mathcal{R}(A) = \partial_{\text{top}}(T_0(A) \cup T_1(A)).$$

We observe that $\partial_{\text{top}}(A_{\lambda,\mu})$ is fixed by this map. It is not true in general if $A$ is a continuous function that $\mathcal{R}(A)$ will also be a continuous function.
Lemma 2.3. Assume $(\lambda, \mu) \in G$. Define $R_n = R^{[n]}(B_{\lambda,\mu})$. We have

1. $R_n$ is a continuous increasing function
2. $R_{n-1} \leq R_n$ in the sense that for $(x, y) \in R_{n-1}$ there exists a $y' \geq y$ such that $(x, y') \in R_n$.
3. $R_n \leq \partial_{\text{top}}(A_{\lambda,\mu})$.
4. $R_n \to \partial_{\text{top}}(A_{\lambda,\mu})$ as $n \to \infty$.
5. $\partial_{\text{top}}(A_{\lambda,\mu})$ has no jump discontinuities and is strictly increasing.

Proof. To see (2) and (3), observe that $R_n \subset \bigcup_{a \in \{0,1\}^n} T_a(B_{\lambda,\mu}) \subset A_{\lambda,\mu}$ and $\partial_{\text{top}}(R_{n-1}) = R_{n-1}$.

We prove (1) by induction. We observe that $R_0 = B_{\lambda,\mu}$ is a continuous increasing curve with the property that $T_0(1,1)$ and $T_1(0,0)$ are below the curve $R_0$. We see that $R_n \subset T_0(R_{n-1}) \cup T_1(R_{n-1})$. Hence $T_0(R_{n-1})$ is a continuous increasing curve from $(0,0)$ to $T_0(1,1) = (\lambda,\mu)$. Further, as $T_0(1,1)$ is below $R_{n-1}$ which in turn is below $R_n$ we have that $T_0(1,1)$ is below $T_1(R_{n-1})$. This implies that the curve $R_n$ is continuous and increasing at $x = \lambda$, as $T_1(R_{n-1})$ is continuous and increasing at $x = \lambda$.

As similar observation can be made for $T_1(0,0)$. Hence $R_n$ is increasing and continuous.

We have that (1) follows from the observation that $R_n = \partial_{\text{top}} \left( \bigcup_{a \in \{0,1\}^n} T_a(B_{\lambda,\mu}) \right)$ and $\lim_{n \to \infty} \left( \bigcup_{a \in \{0,1\}^n} T_a(B_{\lambda,\mu}) \right) = A_{\lambda,\mu}$ in the Hausdorff topology.

Lastly, to see (5), let $M$ be the supremum of the jump discontinuities of $\partial_{\text{top}}(A_{\lambda,\mu})$. We note that $\lambda M$ is the supremum of the jump discontinuities of $R(\partial_{\text{top}}(A_{\lambda,\mu})) = \partial_{\text{top}}(A_{\lambda,\mu})$. Hence $M = 0$ and $\partial_{\text{top}}(A_{\lambda,\mu})$ has no jump discontinuities. As $\partial_{\text{top}}(A_{\lambda,\mu})$ is symmetric about $\lambda + \mu = 1$ we see that it is strictly increasing.

Remark 2.4. It is tempting to believe that $R(B_{\lambda,\mu}) = B_{\lambda,\mu}$. This is unfortunately not always the case. In Figure 2.2 we show the image of $T_0(B_{0.4,0.9})$ and $T_1(B_{0.4,0.9})$, magnified near the region of intersection.

Theorem 1.1 is proved.

3. Computational results on $G$

We will first prove a special case, and then discuss how this can be extended.

Consider our example $\lambda = 0.4$ and $\mu = 0.9$ from before. Consider an infinite word $a = (a_i)_{i=1}^\infty \in \{0,1\}^\mathbb{N}$. We define $pt_a$ as the limit $\lim_{n \to \infty} T_{a_1} \circ T_{a_2} \circ \cdots \circ T_{a_n}$. We note that the limit it independent of the point upon which we act.

Let $(x_1, y_1) = pt_{(0)^\infty}$ and $(x_2, y_2) = pt_{(1)^\infty}$.

We make two claims.

1. $(x_1, y_1) \in B_{0.4,0.9}$. 


Figure 2.2. The set $T_0(B_{0.4,0.9}) \cup T_1(B_{0.4,0.9})$, magnified in the neighbourhood of the intersection area.

(2) $x_1 < x_2$ and $y_1 > y_2$.

These two claims are sufficient to prove $(0.4, 0.9) \in G$. To see this, we note that $B_{0.4,0.9}$ is a continuous increasing function bounding $\partial_{top}(A_{0.4,0.9})$ from below.

To see the first claim, we notice that

$$x_1 = \frac{\lambda(\mu - 1)}{\lambda\mu - 1} \quad \text{and} \quad y_1 = \frac{\mu(\lambda - 1)}{\lambda\mu - 1}$$

We see that

$$x_1 + y_1 = \frac{2\lambda\mu - \lambda - \mu}{\lambda\mu - 1} = 0.90625 < 1.$$  

We further see that

$$T_1(x_1, y_1) = \left( \frac{\mu - 1}{\lambda\mu - 1}, \frac{\lambda - 1}{\lambda\mu - 1} \right)$$

We further have that

$$\frac{\mu - 1}{\lambda\mu - 1} + \frac{\lambda - 1}{\lambda\mu - 1} = \frac{\lambda + \mu - 2}{\lambda\mu - 1} = 1.09375 > 1$$

We easily see that $(x_1, y_1) \in Y_0$, and by induction we have that $(x_1, y_1) \in Y_n$ for all $n$. This proves that $(x_1, y_1) \in B_{0.4,0.9} = \cap Y_n$.

The second claim follows as

$$x_2 = 1 - \mu = 0.1 > x_1 = 0.0625$$

and

$$y_2 = 1 - \lambda = 0.4 < x_1 = 0.84375.$$  

We notice that the inequalities needed to ensure this result are true for more than this specific value of $\lambda$ and $\mu$. In particular, so long as $pt_{(01)\infty}$ is below the line $x + y = 1$, and $T_1(pt_{(01)\infty})$ is above the line $x + y = 1$ we have that $pt_{(01)\infty}$ is in $B_{\lambda,\mu}$. Similar, the necessary inequality between $pt_{(01)\infty}$ and $pt_{1(0)\infty}$ can be easily checked for ranges of $\lambda$ and $\mu$. For example, we can easily show a more general result that for all $(\lambda, \mu) \in [3/8, 7/16] \times [7/8, 15/16]$ that $pt_{(01)\infty}$ is on $B_{\lambda,\mu}$.
and that the necessarily in equality holds for $pt_{(01)}\infty$ and $pt_{(10)}\infty$. That is, $[3/8, 7/16] \times [7/8, 15/16] \subset G$.

We computationally search for regions $R$ and eventually periodic $a \in \{0, 1\}$ such that

1. $pt_a \in B$ for $(\lambda, \mu) \in R$
2. $pt_a$ satisfies the desired inequality with one of $pt_{0(1)}\infty$ or $pt_{1(0)}\infty$.

This data is collected on [2].

A graph of the proven regions is given in Figure 3.1. Each rectangle indicates a different region with a (potentially) different eventually periodic word $a$. Some of these regions are very small, with a width of $1/2^{12}$.

**4. Proof of Theorem 1.3**

Consider again our example with $\lambda = 0.4$ and $\mu = 0.9$. Let $(x_0, y_0) = \left(\frac{\lambda(\mu-1)}{\lambda\mu-1}, \frac{\mu(\lambda-1)}{\lambda\mu-1}\right)$, the solution to $T_0T_1(x_0, y_0) = (x_0, y_0)$. Let $X = [x_0, 1] \times [y_0, 1]$. Consider the sub-IFS generated by $\{T_0T_1, T_1\}$.

It is easy to see that $T_0T_1(X) \subset X$, $T_1(X) \subset X$ and $T_0T_1(X) \cap T_1(X) = \emptyset$. Hence this sub-IFS satisfies the rectangular open set condition. See Figure 4.1.

We further observe that the projection of this sub-IFS onto the $x$-axis is the interval $[x_0, 1]$, hence dimension 1. Lastly, we see that the
contractions are both of the form $(x, y) \to (ax + b, cx + d)$ where $a > c$. Hence by Feng and Wang [1] we can compute the dimension for this sub-IFS.

In this case $s \approx 1.244273660$ which satisfies

$$(\lambda \mu)^s + \mu^{s-1} \lambda = 1.$$ 

As the dimension of the full IFS is strictly less than 2, we see that it has no interior. Hence $\dim(K) = \dim(\partial(K)) \geq 1.244273660 > 1$.

More generally, let $w_1, w_2, \ldots, w_n \in \{0, 1\}^*$ such that $|w_i|_1 \geq |w_i|_0$ for $i = 1, 2, \ldots, n$, with at least one of the inequalities being strict. Let $(x_i, y_i)$ be the fixed point of $T_{w_i}$ for $i = 1, \ldots, n$. Define $x_{\text{min}} = \min(x_1, x_2, \ldots, x_n)$, and similarly $x_{\text{max}}, y_{\text{min}}$ and $y_{\text{max}}$. Define $X = [x_{\text{min}}, x_{\text{max}}] \times [y_{\text{min}}, y_{\text{max}}]$. We see by construction that $T_{w_i}(X) \subset X$. If we have that

- $T_{w_i}(X) \cap T_{w_j}(X) = \emptyset$ for $i \neq j$,
- The projection of the attractor of $\{T_{w_1}, T_{w_2}, \ldots, T_{w_n}\}$ onto the first coordinate is $[x_{\text{min}}, x_{\text{max}}]$,
- The projections onto the first coordinate have non-trivial overlap for some $T_{w_i}$ and $T_{w_j}$, $i \neq j$.

then the same argument will hold. That is, by the rectangular open set condition $\{T_{w_1}, T_{w_2}, \ldots, T_{w_n}\}$ has dimension greater than 1. To see this we have from Feng and Wang the dimension satisfies

$$\sum b_i^{s-1} a_i = 1.$$
The left hand side is a decreasing function with respect to $s$ and evaluates to a value strictly greater than 1 as $s = 1$, hence $s > 1$.

We use this argument with the sets Extending this argument to

$$\{T_0 T_1^m, T_1^n\} \quad \text{and} \quad \{T_0 T_1^m, T_1 T_0, T_1^2 T_0, \ldots, T_1^n T_0\}.$$ 

This covers greater than 91.8% of the parameter space. See Figure 4.2.

One problem with this technique, is it doesn’t seem to cover all cases. In particular, for $\lambda = 0.45, \mu = 0.6$ we cannot find a combination of $m$ and $n$ such that $\{T_0 T_1^m, T_1^n\}$ or $\{T_0 T_1^m, T_1 T_0, T_1^2 T_0, \ldots, T_1^n T_0\}$ has the desired properties. We have also searched more generally for this particular case. Letting $L$ being the set of all words $w$ of length up to 20 where $|w|_1 \geq |w|_0$, we have searched through all subsets of $L$ for possible proof using this technique and found none. Computationally, the dimension of $A_{0.45,0.6}$ appears to be 1.08.

Visually there seems to be a natural limit to these techniques, and a visible gap between $\lambda + \mu = 1$ and the cases that can be proved.

5. Proof of Theorem 1.5

We will prove a more general result.

**Theorem 5.1.** Let a two-dimensional IFS $\Phi$ be $\{T_0 x = M_0 x, T_1 x = M_1 x + u\}$, where $M = M_0 M_1 = M_1 M_0$. If $M$ is not scalar and
$|\det M| \geq 1/\sqrt{2}$, then the attractor of $\Phi$ has a non-empty interior. If $M$ is scalar, the same result holds if $|\det M^2 M_1| \geq 1/\sqrt{2}$.

Proof. If $M$ is not scalar, then we consider the sub-IFS $\{T_0T_1, T_1T_0\}$. Both maps are given by the same matrix $M$, whence the first claim follows from the main result of our previous work [4].

If $M$ is scalar, then we consider $\{T_0T_1T_0, T_1^2T_1\}$, also with the same matrix and apply the same result. □

Return to Theorem 1.5. Both matrices here are diagonal so they commute. Their product is scalar, so we apply the second case of the previous theorem. We thus get the condition $(\lambda\mu)^3 \geq 1/\sqrt{2}$.

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