Lattice Isometries and K3 Surface Automorphisms: Salem Numbers of Degree 20

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Abstract

This article extends Bayer-Fluckiger’s theorem on characteristic polynomials of isometries on an even unimodular lattice to the case where the isometries have determinant −1. As an application, we show that the logarithm of every Salem number of degree 20 is realized as the topological entropy of an automorphism of a nonprojective K3 surface.

1 Introduction

Let non-negative integers $r$ and $s$ be given. Which polynomial $F(X) \in \mathbb{Z}[X]$ can occur as the characteristic polynomial of an isometry on an even unimodular lattice of signature $(r, s)$? It is known that if $(r, s)$ is the signature of an even unimodular lattice then $r \equiv s \mod 8$. In the following we assume this congruence. For a monic polynomial $F$ with $F(0) \neq 0$, we define $F^*(X) := F(0)^{-1}X^{\deg F}F(X^{-1})$ and say that $F$ is $*$-symmetric if $F = F^*$. In this case, the constant term $F(0)$ is 1 or −1, so we say that $F$ is $+1$-symmetric or $-1$-symmetric according to the value $F(0)$ (see §2.1 for a slightly more formal definition). Gross and McMullen [GM02] raised the above question and gave the following necessary conditions essentially: If $F(X) \in \mathbb{Z}[X]$ is the characteristic polynomial of an isometry on an even unimodular lattice of signature $(r, s)$ then $F$ is a $*$-symmetric polynomial of even degree;

$$r, s \geq m(F)$$

and if $F(1)F(-1) \neq 0$ then $r \equiv s \equiv m(F) \mod 2$, (Sign)

where $m(F)$ is the number of roots $\lambda$ of $F$ with $|\lambda| > 1$ counted with multiplicity; and

$$|F(1)|, |F(-1)| \text{ and } (-1)^{\deg F}/2 F(1)F(-1) \text{ are all squares.}$$

(Square)

For an irreducible $*$-symmetric polynomial $F$ of even degree, they speculated that these conditions are sufficient and showed that if the assumption (Square) is replaced by the assumption $|F(1)| = |F(-1)| = 1$, then these are sufficient. Afterwards, Bayer-Fluckiger and Taelman [BT20] showed that the speculation is correct using a local-global theory.

Bayer-Fluckiger [Ba20, Ba21, Ba22] proceeded to the case where polynomials are reducible and $+1$-symmetric. In this case, the above conditions are not sufficient as pointed out in [GM02]. She showed that the condition (Square) is necessary and sufficient for the existence of an even unimodular $\mathbb{Z}_p$-lattice having a semisimple isometry with characteristic polynomial $F$ for each prime $p$. Moreover, she gave a sufficient and necessary condition for the local-global principle to hold.

Let $F \in \mathbb{R}[X]$ be a $*$-symmetric polynomial with the condition (Square). If $t$ is an isometry with characteristic polynomial $F$ on an $\mathbb{R}$-inner product space $V$ of signature $(r, s)$, then $V$ decomposes as $V = \bigoplus_f V(f; t)$, where $V(f; t) := \{v \in V \mid f(t)^Nv = 0 \text{ for some } N \geq 0\}$ and $f$ ranges over the irreducible factors of $F$ in $\mathbb{R}[X]$. The index $\text{idx}_t$ of $t$ is a map from the set of irreducible $*$-symmetric factors of $F$ to $\mathbb{Z}$ defined by $\text{idx}_t(f) = rf - sf$ where $(rf, sf)$ is the signature of $V(f; t)$. The set of maps expressed as $\text{idx}_t$ for some $t$ as above is denoted by $\text{Idx}_{r,s}(F)$, and a map in $\text{Idx}_{r,s}(F)$ are called an index map.

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(see §2.4 for the precise definition). We refer to an isometry with characteristic polynomial $F$ and index $i \in \text{Idx}_{r,s}(F)$ as an $(F,i)$-isometry for short.

Now, let $F \in \mathbb{Z}[X]$ be a $+1$-symmetric polynomial of even degree with the conditions $\text{(Sign)}$ and $\text{(Square)}$, and let $i \in \text{Idx}_{r,s}(F)$ be an index map. Bayer-Fluckiger introduced a group $\Omega$ and a homomorphism $\text{ob} : \Omega \to \mathbb{Z}/2\mathbb{Z}$, which are determined by $F$ and $i$, and showed that there exists an even unimodular lattice of signature $(r,s)$ having a semisimple $(F,i)$-isometry if and only if the obstruction map $\text{ob}$ is the zero map.

This article extends her work to the case where $F$ is $*$-symmetric, which covers the $-1$-symmetric case. We will redefine the group $\Omega$ and the map $\text{ob}$ to work in the general case and show the following theorem. We will call the map $\text{ob}$ the obstruction map.

**Theorem 1.1.** Let $F \in \mathbb{Z}[X]$ be a $*$-symmetric polynomial of even degree with the conditions $\text{(Sign)}$ and $\text{(Square)}$, and let $i \in \text{Idx}_{r,s}(F)$ be an index map. Then there exists an even unimodular lattice of signature $(r,s)$ having a semisimple $(F,i)$-isometry if and only if the obstruction map $\text{ob} : \Omega \to \mathbb{Z}/2\mathbb{Z}$ vanishes.

As an application, we show:

**Theorem 1.2.** Let $F \in \mathbb{Z}[X]$ be a $*$-symmetric polynomial of even degree $2n$ with the condition $\text{(Square)}$. If neither the multiplicity of $X - 1$ nor that of $X + 1$ in $F$ is 1, then there exists a semisimple isometry on an even unimodular lattice of signature $(n,n)$ with characteristic polynomial $F$.

The question mentioned at the beginning is related to the study of automorphisms on K3 surfaces. For any K3 surface $X$, that is, a simply connected compact complex surface with a nowhere vanishing holomorphic 2-form, the middle cohomology group $H^2(X,\mathbb{Z})$ with the intersection form is an even unimodular lattice of signature $(3,19)$. Such a lattice is called a K3 lattice, which is uniquely determined up to isomorphism. Let $\Lambda$ be a K3 lattice. We can define an additional structure on $\Lambda$ which is called a K3 structure. An isometry $t$ on $\Lambda$ preserving a given K3 structure ‘lifts’ to an automorphism on a K3 surface. In other words, there exists an automorphism $\varphi$ on a K3 surface $X$ such that the induced homomorphism $\varphi^* : H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{Z})$ can be identified with the isometry $t$. This is a consequence of the Torelli theorem and surjectivity of the period mapping.

There are many studies of dynamics on K3 surfaces using the lifting, see [Mc02, Og10, Mc11, Mc16, Ba21, Ba22] for instance. This article focuses on the topological entropy. It is known that the topological entropy of a K3 surface automorphism $\varphi$ coincides with the logarithm of the spectral radius $\lambda(\varphi^*)$ of $\varphi^* : H^2(X,\mathbb{C}) \to H^2(X,\mathbb{C})$, and that $\lambda(\varphi^*)$ is a Salem number (see §2.1 unless $\lambda(\varphi^*) = 1$ (or the entropy equals 0)). Let us say that a Salem number $\lambda$ is projectively (resp. nonprojectively) realizable if there exists an automorphism on a projective (resp. nonprojective) K3 surface with entropy $\log \lambda$. We remark that the degree of such a Salem number is an even integer between 2 and 20 (resp. 4 and 22).

McMullen proved that the Lehner number $\approx 1.7628$, which is the smallest known Salem number, is nonprojectively realizable in [Mc11], and projectively realizable in [Mc16]. Moreover, Bayer-Fluckiger and Taelman showed in [Ba21, Ba22] that a Salem number of degree 22 is nonprojectively realizable if and only if its minimal polynomial satisfies the condition $\text{(Square)}$, and Bayer-Fluckiger proved in [Ba22] that all Salem numbers of degree 4, 6, 8, 12, 14 or 16 are nonprojectively realizable, using her theorems on characteristic polynomials of isometries on an even unimodular lattice. Along this line, we show:

**Theorem 1.3.** All Salem numbers of degree 20 are nonprojectively realizable.

To prove this theorem, we will use Theorem 1.1 to show that the polynomial of the form $(X - 1)(X + 1)S(X)$ for each Salem polynomial $S(X)$ of degree 20 is realizable as the characteristic polynomial of an isometry on a K3 lattice. See Theorem 0.3 for a more general consequence of Theorem 1.1. We remark that the polynomial $(X - 1)(X + 1)S(X)$ is $-1$-symmetric, so Theorem 1.3 is a benefit of extending Bayer-Fluckiger’s theory. The cases of degree 10 and degree 18 are still open, but there are some criteria for realizability and it is known that the smallest Salem number of degree 18 is not nonprojectively realizable, see [Ba22]. Note that the lattice theoretic approach brings more subtle problems in the projective case, see [Mc16]. We refer to [Ba20] for other results on entropy spectra.

The organization of this article is as follows. We review fundamental facts on inner product spaces in §2, and on lattice theory in §3. These sections contain no new results. In §4 we reproduce local theory on even unimodular lattices and characteristic polynomials of isometries, which is given by Bayer-Fluckiger
in [Ba21]. Moreover, we extend her theory to isometries of determinant $-1$ on an even unimodular $\mathbb{Z}_2$-lattice. For this, Theorem 2.10 is crucial. In §3 we give a necessary and sufficient condition for a pair $(F,\iota)$ of a polynomial $F$ and an index map $\iota$ to be realized on an even unimodular lattice. In §4 we reformulate the obstruction group and map, and establish Theorem 1.3, extending her theory. Theorem 1.3 is also proved in this section. In the last section 5 we deal with entropy problems for K3 surface automorphisms, and prove Theorem 1.3.

2 Inner products and isometries

Let $K$ be a field of characteristic $\neq 2$. An inner product space $(V, b)$ over $K$ is a pair of a finite dimensional $K$-vector space $V$ and an inner product $b : V \times V \to K$, that is, a nondegenerate symmetric bilinear form. We may write either of $V$ or $b$ for $(V, b)$. Two inner product spaces $(V, b)$ and $(V', b')$ are isomorphic if there exists a linear isomorphism $\varphi : V \to V'$ satisfying $b(x, y) = b'(\varphi(x), \varphi(y))$ for all $x, y \in V$. The group of isometries on $V$ is denoted by $O(V)$.

2.1 Symmetric polynomials

For a polynomial $F(X) \in K[X]$, define $F^\vee(X) := X^\deg F F(X^{-1}) \in K[X]$. A polynomial $F$ is $\epsilon$-symmetric if $F(X) = \epsilon F^\vee(X)$ where $\epsilon = \pm 1$. Such a polynomial occurs naturally as the characteristic polynomial of an isometry (see [2.2]). For a monic polynomial $F(X) \in K[X]$ with $F(0) \neq 0$, define

$$F^*(X) := F(0)^{-1} X^\deg F F(X^{-1}) = F(0)^{-1} F^\vee(X),$$

and we say that $F$ is $*$-symmetric if $F = F^*$. A monic polynomial $F(X) \in K[X]$ with $F(0) \neq 0$ is $*$-symmetric if and only if $F$ is $+1$-symmetric or $-1$-symmetric. Following [Ba15] we say that a $*$-symmetric polynomial $F$ is of

- **type 0** if $F$ is a product of powers of $(X - 1)$ and of $(X + 1)$;
- **type 1** if $F$ is a product of powers of $+1$-symmetric irreducible monic polynomials of even degrees;
- **type 2** if $F$ is a product of polynomials of the form $GG^*$, where $G$ is monic, irreducible and $G^* \neq G$.

Every $*$-symmetric polynomial $F$ admits a unique factorization $F = F_0 F_1 F_2$, where $F_i$ is of type $i$ for $i = 0, 1, 2$ (see [Ba15] Proposition 1.3). This is true even if char $K = 2$. We refer to the factor $F_i$ as the **type $i$ component** of $F$. For a $*$-symmetric polynomial $F$, we have for example:

- If $F$ has no type 0 component, that is, $F(1)F(-1) \neq 0$, then $F$ is of even degree and $+1$-symmetric.
- $F$ is $-1$-symmetric if and only if $F$ has the factor $X - 1$ with odd multiplicity.

We remark that if $K = \mathbb{Q}$ and $F \in \mathbb{Z}[X]$ then the factorization $F = F_0 F_1 F_2$ in $\mathbb{Q}[X]$ is the same as that in $\mathbb{Z}[X]$, thanks to Gauss’s lemma.

Typical examples of $*$-symmetric polynomials which appear in this article are cyclotomic polynomials and Salem polynomials. A cyclotomic polynomial is the minimal polynomial of a root of unity, and a Salem polynomial is the minimal polynomial of a **Salem number**, that is, a real algebraic unit $\lambda > 1$ whose conjugates other than $\lambda^{\pm 1}$ lie on the unit circle in $\mathbb{C}$. We allow Salem numbers of degree 2 following [Mo02], see [Sa83] p. 26 for Salem’s definition.

2.2 Inner products

Let $(V, b)$ be an inner product space over $K$. If $e_1, \ldots, e_d$ is a basis of $V$, the matrix $(b(e_i, e_j))_{ij}$ is called the **Gram matrix** of $e_1, \ldots, e_d$. A Gram matrix of $V$ is the Gram matrix of some basis. The **determinant** of $(V, b)$ is the element of $K^\times/K^\times 2$ represented by $\det G$, where $G$ is any Gram matrix of $V$. Here $K^\times$ is the group of invertible elements and $K^\times 2 := \{a^2 \mid a \in K^\times\}$. The determinant of $(V, b)$ is denoted by $\det b$, and we define the **discriminant** $\text{disc} b$ of $(V, b)$ to be $(-1)^d(d-1)/2 \det b \in K^\times/K^\times 2$, where $d$ is the dimension of $V$.

It is well known that every inner product space has an orthogonal basis, that is, a basis of which the Gram matrix is diagonal. Based on this fact, the Hasse-Witt invariant is defined as follows. First of all, we
denote by Br(K) the Brauer group of K, and by (a, b) ∈ Br(K) the Brauer class of the quaternion algebra defined by a, b ∈ K× (see e.g. [Se85] Chapter 8 for Brauer groups). Let e1, . . . , ed be an orthogonal basis of (V, b) with the Gram matrix diag(a1, . . . , ad). Then we define
\[ \epsilon(b) := \sum_{i<j} (a_i, a_j) \in \text{Br}(K), \]
where we regard Br(K) as an additive group. The element \( \epsilon(b) \) does not depend on the choice of the orthogonal basis. We call \( \epsilon(b) \) the Hasse-Witt invariant of \( b \). The Hasse-Witt invariant of any 1-dimensional inner product space is defined to be 0.

If (V, b) and (V′, b′) are two inner product spaces, we have \( \epsilon(b \oplus b′) = \epsilon(b) + \epsilon(b′) + (\det b, \det b′) \). This formula implies the following lemma.

**Lemma 2.1.** Let \( V_1, \ldots, V_k \) be K-vector spaces, and let \( b_j \) and \( b'_j \) be inner products on \( V_j \) for \( j = 1, \ldots, k \). If \( \det b_j = \det b'_j \) for all \( j \), then we have
\[ \epsilon \left( \bigoplus_{j=1}^k b_j \right) - \sum_{j=1}^k \epsilon(b_j) = \epsilon \left( \bigoplus_{j=1}^k b'_j \right) - \sum_{j=1}^k \epsilon(b'_j). \]

If \( K \) is a local field, Hasse-Witt invariants will often be considered to take values in \{0, 1\} = \mathbb{Z}/2\mathbb{Z} because the image of \( \epsilon \) is a subgroup of order 2 in Br(K). Suppose that \( K \) is the field of rational numbers \( \mathbb{Q} \), and let \( v \) be a place. In this case Hasse-Witt invariants over the \( v \)-adic field \( \mathbb{Q}_v \) or over \( \mathbb{Q}_\infty = \mathbb{R} \) will be denoted by \( \epsilon_v : \{ \text{inner products over } \mathbb{Q}_v \} \to \mathbb{Z}/2\mathbb{Z} \). Here \( \infty \) denotes the infinite place of \( \mathbb{Q} \). Furthermore, for an inner product \( b \) over \( \mathbb{Q} \), we will write \( \epsilon_v(b) \) for \( \epsilon_v(b \otimes \mathbb{Q}_v) \).

### 2.3 Isometries

Let \( (V, b) \) be an inner product space and \( t \in O(V) \) an isometry with characteristic polynomial \( F \). As mentioned earlier, the polynomial \( F \) is *-symmetric (see [GM02] Proposition A.1) and its proof). Let us define
\[ V(f; t) := \{ v \in V \mid f(t)^N v = 0 \text{ for some } N \geq 0 \} \]
for \( f \in K[X] \). Then we have an orthogonal direct sum decomposition
\[ V = \bigoplus_{f \in I_0 \cup I_1} V(f; t) \oplus \bigoplus_{(f, f^*) \in I_2} [V(f; t) \oplus V(f^*; t)], \]
where \( I_i \) denotes the set of irreducible factors of the type \( i \) component of \( F \) for \( i = 0, 1, 2 \). Furthermore, for each \( f \in I_2 \), the component \( V(f; t) \oplus V(f^*; t) \) is split, i.e., \( V(f; t) \oplus V(f^*; t) \) has a Gram matrix of the form
\[ \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \]
where \( I \) denotes the identity matrix, see [Me09] §3.

There is a relation between the determinant \( \det b \) and the polynomial \( F \).

**Lemma 2.2.** If \( F(1)F(-1) \neq 0 \) then \( \det b = F(1)F(-1) \) in \( K^\times /K^\times 2 \). In particular \( \det b|_{V(f; t)\oplus V(f^*; t)} = (-1)^{\deg f} \) in \( K^\times /K^\times 2 \) for \( f \in I_2 \).

**Proof.** See [Ba15] Corollary 5.2.]}

### 2.4 Algebras with involution and hermitian forms

In this article a \( K \)-algebra means an associative unital \( K \)-algebra. We always assume that \( K \)-algebras are finite dimensional over \( K \). Moreover, \( K \)-algebras will often be equipped with an involution. If \( A = (A, \sigma) \) is a \( K \)-algebra with an involution \( \sigma : A \to A \), define the fixed subalgebra \( A^\sigma \) to be \( \{ x \in A \mid \sigma(x) = x \} \).

Two \( K \)-algebras \( A = (A, \sigma) \) and \( A' = (A', \sigma') \), where \( \sigma \) and \( \sigma' \) are involutions, are isomorphic if there exists an isomorphism \( \varphi : A \to A' \) as \( K \)-algebra satisfying \( \varphi \circ \sigma = \sigma' \circ \varphi \).
Let \( A = (A, \sigma) \) be a \( K \)-algebra with involution and \( M \) an \( A \)-module. A map \( h : M \times M \to A \) is called a hermitian form on \( M \) if \( h \) is \( A \)-linear in the first variable and \( h(x, y) = \sigma h(y, x) \) for all \( x, y \in M \). If \( M \) is free over \( A \), a Gram matrix and the determinant of \( h \) are defined in the same way as for an inner product. The determinants of hermitian forms take values in

\[
\text{Tw}(A, \sigma) := (A^\sigma)^\times / N_{A/A^\sigma}(A^\times),
\]
which will be called the twisting group of \((A, \sigma)\), where \( N_{A/A^\sigma} : A^\times \to (A^\sigma)^\times \) is the norm map.

Let \( f \in K[X] \) be a \(*\)-symmetric polynomial of even degree and suppose that \( f \) is irreducible or of the form \( gg^* \) for some irreducible monic polynomial \( g \) satisfying \( g \neq g^* \). The \( K \)-algebra \( E := K[X]/(f) \) has an involution \( \sigma \) defined by \( \alpha \mapsto \alpha^{-1} \), where \( \alpha = X + (f) \in E \) is the image of \( X \). Let \( M \) be a direct product of finite copies of \( E \). We can regard \( M \) as an \( E \)-module and also as a \( K \)-vector space. Hermitian forms on \( M \) over \( E \) and inner products on \( M \) over \( K \) which make \( \alpha : M \to M \) (multiplication by \( \alpha \)) an isometry are related as follows.

**Lemma 2.3.** Let \( f \) and \( M \) be as above, and assume that \( f \) is separable. If \( b \) is an inner product on the \( K \)-vector space \( M \) such that \( \alpha \) becomes an isometry, then there exists one and only one hermitian form \( h \) on \( M \) over \( E \) such that \( b = \text{Tr}_{E/K} \circ h \). Here \( \text{Tr}_{E/K} \) is the trace map.

**Proof.** This statement for an irreducible \( f \) is in [Mi69, Lemma 1.1] and the same proof is valid for \( f = gg^* \). Note that separability of \( f \) guarantees that the trace map is non-zero. \( \square \)

### 2.5 Inner Product Spaces over \( \mathbb{R} \)

In this subsection we introduce an invariant of an isometry on a real inner product space called the index (cf. [Ba21, Section 6]). Let \((V, b)\) be an inner product space over \( \mathbb{R} \) of signature \((r, s)\). The *index* of \((V, b)\) is \( r - s \) and we denote this by \( \text{idx } V \) or \( \text{idx } b \). Notice that \( r = (\dim V + \text{idx } V)/2 \) and \( s = (\dim V - \text{idx } V)/2 \). Let \( t \in O(V) \) be an isometry with characteristic polynomial \( F \in \mathbb{R}[X] \). The symbol \( I_t(\mathbb{R}) \) denotes the set of irreducible factors (in \( \mathbb{R}[X] \)) of the type \( i \) component of \( F \) for \( i = 0, 1, 2 \). Note that any \( f \in I_t(\mathbb{R}) \) is expressed as \( f(X) = X^2 - (\delta + \delta^{-1})X + 1 \) for some \( \delta \in \mathbb{T} \setminus \{ \pm 1 \} \), where \( \mathbb{T} := \{ \delta \in \mathbb{C} \mid |\delta| = 1 \} \). The *index of \( t \)*, denoted by \( \text{idx}_t(f) = \text{idx } V(f; t) \). The reason for not including \( I_2(\mathbb{R}) \) in the domain of definition is in (ii) of the following proposition.

**Proposition 2.4.** Let \((r_f, s_f)\) be the signature of \( V(f; t) \) for \( f \in I_1(\mathbb{R}) \), and that of \( V(f; t) \oplus V(f^*; t) \) for \( f \in I_2(\mathbb{R}) \). Then we have

(i) \( r_f \equiv s_f \equiv 0 \mod 2 \) for each \( f \in I_1(\mathbb{R}) \); and

(ii) \( r_f = s_f \) (i.e. \( \text{idx } V(f; t) \oplus V(f^*; t) = 0 \)) for each \( f \in I_2(\mathbb{R}) \).

In particular, we have

\[
r, s \geq m(F) \quad \text{and} \quad F(1)F(-1) \neq 0 \quad \text{then} \quad r \equiv s \equiv m(F) \mod 2 \tag{\text{Sign}}
\]

where \( m(F) \) is the number of roots \( \lambda \) of \( F \) with \( |\lambda| > 1 \) counted with multiplicity.

**Proof.** (i) is a consequence of Lemma 2.3; see the proof of [Ba15, Proposition 8.1 (a)]. (ii) follows from the fact that \( V(f; t) \oplus V(f^*; t) \) is split for \( f \in I_2(\mathbb{R}) \). \( \square \)

Conversely, let \( F \in \mathbb{R}[X] \) be a \(*\)-symmetric polynomial of degree \( d \), and \( r, s \) non-negative integers with \( r + s = d \), and assume that the condition \( \text{SIGN} \) holds. We denote by \( n_+, n_- \) and \( n_f \) the multiplicities of \( X - 1, X + 1 \) and \( f \in I_1(\mathbb{R}) \) in \( F \) respectively. Let us define \( \text{Id}_{r,s}(F) \) to be the set of maps \( \iota : I_0(\mathbb{R}) \cup I_1(\mathbb{R}) \to \mathbb{Z} \) such that

\[
\iota(X + 1) \equiv n_+ \mod 2 \quad \text{and} \quad -n_- \leq \iota(X + 1) \leq n_+ ; \tag{1}
\]

\[
\iota(f) \text{ is even, } -2n_f \leq \iota(f) \leq 2n_f, \quad \text{and} \quad \eqref{2}
\]

\[
\sum_{f \in I_0(\mathbb{R}) \cup I_1(\mathbb{R})} \iota(f) = r - s. \tag{3}
\]
Proposition 2.4 implies that \( \text{id}_t \) belongs to \( \text{Id}_r,F \) for any isometry \( t \) with characteristic polynomial \( F \) on a \( d \)-dimensional inner product space over \( \mathbb{R} \) of signature \((r,s)\). We call a map in \( \text{Id}_r,F \) an index map.

**Proposition 2.5.** Let \( F \) and \( r,s \) be as above with the condition \( \text{sign}(F) \). For any index map \( t \in \text{Id}_r,F \), there exists an inner product space having a semisimple isometry with characteristic polynomial \( F \) and index \( t \).

**Proof.** See the proof of [Ba15, Proposition 8.1 (b)] and [Ba21, Proposition 7.1]. \( \square \)

### 3 Unimodular lattices and equivariant Witt groups

In this section, we review some terms and known results of lattice theory.

#### 3.1 Lattices

Let \( R \) be a Dedekind domain and \( K \) its field of fractions. A **lattice** over \( R \) (or \( R \)-lattice) is a pair \((\Lambda, b)\) of a finitely generated free \( R \)-module \( \Lambda \) and an inner product \( b : \Lambda \times \Lambda \rightarrow K \). Let \( \Lambda = (\Lambda, b) \) be a lattice over \( R \). The lattice \( \Lambda' := \{ y \in \Lambda \otimes_R K \mid b(x,y) \in R \text{ for all } x \in \Lambda \} \) is called the dual lattice of \( \Lambda \). The lattice \((\Lambda, b)\) is said to be \( R \)-**valued** if \( b \) takes values in \( R \). It is obvious that \( \Lambda \) is \( R \)-valued if and only if \( \Lambda \subset \Lambda' \). We say that \( \Lambda \) is **unimodular** if \( \Lambda' = \Lambda \), and even if \( b(x,x) \in 2R \) for all \( x \in \Lambda \) and odd otherwise. Note that if \( 2 \) is a unit of \( R \) then any lattice is even.

Let \( \mathbb{Z}_2 \) denote the ring of 2-adic integers. We will use the following facts, see e.g. [OM73, §106 A].

**Proposition 3.1.** Any even unimodular \( \mathbb{Z}_2 \)-lattice is of even rank \( 2n \) and isomorphic to \( U^{\oplus n} \) or \( U^{\oplus n-1} \oplus V \), where \( U \) and \( V \) are \( \mathbb{Z}_2 \)-lattices of rank 2 which have Gram matrices

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
2 & 1 \\
1 & 2 \\
\end{pmatrix}
\]

respectively. In particular, its discriminant equals 1 or \(-3 \) in \( \mathbb{Q}_2^x/\mathbb{Q}_2^{x^2} \).

**Proposition 3.2.** Let \( r \) and \( s \) be non-negative integers. There exists an even unimodular \( \mathbb{Z} \)-lattice of signature \((r,s)\) if and only if \( r \equiv s \mod 8 \).

#### 3.2 Equivariant Witt groups

In this subsection we review equivariant Witt groups, see [BT20] for more details. Let \( K \) be a field and \( G \) a group. The group ring \( K[G] \) has a \( K \)-linear involution \( \sigma \) induced by \( g \mapsto g^{-1} \) for all \( g \in G \). If \( G \) is commutative then \( \sigma \) is a \( K \)-algebra involution. We will mostly consider the case where \( G \) is the infinite cyclic group. A \( K[G] \)-**bilinear form** or just \( K[G] \)-**form** is a pair \((V, b)\) of a \( K[G] \)-module \( V \) which is finite dimensional over \( K \) and an inner product \( b \) on \( V \) over \( K \) satisfying

\[
b(ax, y) = b(x, \sigma(a)y) \quad \text{for all } x, y \in V \text{ and } a \in K[G].
\]

If \((V, b)\) is a \( K[G] \)-form then each \( g \in G \) acts on \( V \) as an isometry. Hence, a \( K[G] \)-form \((V, b)\) can be regarded as a triple \((V, b, \rho)\) consisting of a \( K \)-vector space \( V \), an inner product \( b \) on \( V \), and an ‘orthogonal’ representation \( \rho : G \rightarrow \text{O}(V) \) of \( G \). In this case, the \( K[G] \)-form \((V, b, \rho)\) is also denoted by \((V, b, \rho)\).

The **equivariant Witt group** \( W_G(K) \) is defined as an analog of the usual Witt group \( W(K) \) of \( K \) (see [BT20, Definition 3.3]). The Witt class represented by a \( K[G] \)-form \((V, b, \rho)\) is denoted by \([V, b, \rho]\) (or \([V, b, \rho] \)). For an irreducible representation \( \rho : G \rightarrow \text{GL}(V) \) on a finite dimensional vector space \( V \), we define \( W_G(K; \rho) \) as the subgroup of \( W_G(K) \) generated by classes which can be written as \([V, b, \rho] \) for some inner product \( b \) on \( V \). Then we have a decomposition

\[
W_G(K) = \bigoplus_{\rho} W_G(K; \rho)
\]

where \( \rho \) ranges over the isomorphism classes of finite dimensional irreducible representations of \( G \).
The theory of (equivariant) Witt groups can be used to discuss the existence of a unimodular lattice in an inner product space over a discrete valuation field. Assume that $K$ is a discrete valuation field with valuation $v$, valuation ring $\mathcal{O}$, and residue class field $k$. A $K[G]$-form $V$ is bounded if $V$ contains a $G$-stable lattice, that is, a lattice $\Lambda$ over $\mathcal{O}$ in $V$ such that $K\Lambda = V$ and $g\Lambda = \Lambda$ for all $g \in G$. We denote by $W^b_G(K)$ the subgroup of $W_G(K)$ generated by Witt classes of bounded $K[G]$-forms.

Let $(V, b)$ be a bounded $K[G]$-form. Then there exists an almost unimodular lattice $\Lambda$ in $V$, that is, a lattice in $V$ satisfying $\pi\Lambda^\vee \subset \Lambda \subset \Lambda^\vee$, where $\pi$ is a uniformizer of $K$. For an almost unimodular lattice $\Lambda$ we can define a $k[G]$-form $(\Lambda^\vee/\Lambda, b)$ by

$$b: \Lambda^\vee/\Lambda \times \Lambda^\vee/\Lambda \to \pi^{-1}\mathcal{O}/\mathcal{O} \to \mathcal{O}/\pi\mathcal{O} = k.$$ 

Let $\partial: W^b_G(K) \to W_G(k)$ denote the map sending the class of a bounded $K[G]$-form $(V, b)$ to the class of the $k[G]$-form $(\Lambda^\vee/\Lambda, b)$ for some almost unimodular lattice $\Lambda$ in $V$. We remark that $\dim \partial [V, b] \equiv v(\det b) \bmod 2$. See [BT20, Theorem B] for the following theorem.

**Theorem 3.3.** The map $\partial: W^b_G(K) \to W_G(k)$ is a well-defined homomorphism. A bounded $K[G]$-form $(V, b)$ contains a $G$-stable almost unimodular lattice if and only if $\partial [V, b] = 0$.

Here we prepare some notations related to the infinite cyclic group $\Gamma$ and $K[\Gamma]$-forms, and will use them throughout this article.

**Notation 3.4.** The symbol $\Gamma$ denotes the infinite cyclic group. Let $\gamma$ be a fixed generator of $\Gamma$, and let $\rho$ be the orthogonal representation on an inner product space $(V, b)$ defined by $\gamma \mapsto t \in \mathcal{O}(V)$ for a given $t$. Then we write $(V, b, t)$ for $(V, b, \rho)$. Furthermore, if the representation $\rho$ is a direct sum of copies of an irreducible representation $\chi$, we write $W_{T}(K; t)$ for $W_{T}(K; \chi)$.

For any field $K$, the subgroups $W_{T}(K; 1)$ and $W_{T}(K; -1)$ of $W_{T}(K)$ are isomorphic to the usual Witt group $W(K)$. We recall the structure of Witt groups of finite fields. See e.g. [Ser5, §2.3] for the following proposition.

**Proposition 3.5.** Let $k$ be a finite field.

(i) If $\char k = 2$ then sending $\omega \in W(k)$ to $\dim \omega \bmod 2 \in \mathbb{Z}/2\mathbb{Z}$ gives an isomorphism $W(k) \cong \mathbb{Z}/2\mathbb{Z}$.

(ii) If $\char k \neq 2$ then

$$W(k) \cong \begin{cases} 
\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & \text{if } \char k \equiv 1 \bmod 4 \\
\mathbb{Z}/4\mathbb{Z} & \text{if } \char k \equiv 3 \bmod 4,
\end{cases}$$

and $\omega \in W(k)$ is the trivial class if and only if $\dim \omega \equiv 0 \bmod 2$ and $\disc \omega = 1 \bmod k^2/k^2$.

4 Local theory

Let $K$ be a non-archimedean local field of characteristic 0. We denote by $v_K, \mathcal{O}_K, \mathfrak{m}_K$ the normalized valuation, valuation ring, and maximal ideal respectively, and fix a uniformizer $\pi_K$. Similar notation will be used for an extension of $K$. The residue class field of $K$ is denoted by $k$.

We can describe a necessary and sufficient condition for a unimodular $\mathcal{O}_K$-lattice having a semisimple isometry with characteristic polynomial $F$ to exist, in terms of the valuations of $F(1)$ and $F(-1)$. We adopt the convention that $v_K(0) \equiv 0 \bmod 2$.

**Theorem 4.1.** [Ba21, Theorem 4.1]. Let $F(X) \in \mathcal{O}_K[X]$ be a $*$-symmetric polynomial of even degree. The following are equivalent:

(i) There exists a unimodular $\mathcal{O}_K$-lattice having a semisimple isometry with characteristic polynomial $F$.

(ii) $v_K(F(1)) \equiv v_K(F(-1)) \equiv 0 \bmod 2$ if $\char k \neq 2$ and $v_K(F(1)F(-1)) \equiv 0 \bmod 2$ if $\char k = 2$.

Moreover, if $K$ is the 2-adic field $\mathbb{Q}_2$ we have:
\textbf{Theorem 4.2.} Let $F(X) \in \mathbb{Z}_2[X]$ be a $*$-symmetric polynomial of even degree $2n$. There exists an even unimodular $\mathbb{Z}_2$-lattice of discriminant 1 having a semisimple isometry with characteristic polynomial $F$ if and only if $F$ satisfies the following conditions:

(a) $v_2(F(1)) \equiv v_2(F(-1)) \equiv 0 \mod 2$; and

(b) If $F(1)F(-1) \neq 0$ then $(-1)^n F(1)F(-1) = 1 \in \mathbb{Q}_2^* / \mathbb{Q}_2^2$.

This is an extension of Theorems 5.1 and 5.2 of [16a21] to the case where $F$ is $*$-symmetric, which covers the $-1$-symmetric case. The proof of this theorem will be given in the last part of this section. Theorems 4.1 and 4.2 yield an important corollary (cf. [16a21, Proposition 8.1]).

\textbf{Corollary 4.3.} Let $F \in \mathbb{Z}[X]$ be a $*$-symmetric polynomial of even degree $2n$. There exists an even unimodular $\mathbb{Z}_p$-lattice having a semisimple isometry with characteristic polynomial $F$ for each prime $p$ if and only if $F$ satisfies the following condition:

$$|F(1)|, |F(-1)| \quad \text{and} \quad (-1)^n F(1)F(-1)$$

are all squares. \hspace{20mm} (Square)

\textit{Proof.} Assume that there exists an even unimodular $\mathbb{Z}_p$-lattice $\Lambda_p$ having a semisimple isometry with characteristic polynomial $F$ for each prime $p$. Then $|F(1)|$ and $|F(-1)|$ are squares because $v_p(F(\pm 1)) \equiv 0 \mod 2$ for any prime $p$ by Theorems 4.1 and 4.2, where $v_p$ is the $p$-adic valuation. If $F(1)F(-1) = 0$ then $F$ satisfies (Square). Let $F(1)F(-1) \neq 0$. Then $|(-1)^n F(1)F(-1)| = |F(1)| |F(-1)|$ is a square in $\mathbb{Q}^*$ and thus $(-1)^n F(1)F(-1) = 1$ or $-1$ in $\mathbb{Q}_2^* / \mathbb{Q}_2^2$. On the other hand, we have $(-1)^n F(1)F(-1) = \text{disc} \Lambda_2 \in \{1, -3\}$ by Lemma 4.3 and Proposition 5.1 and hence $(-1)^n F(1)F(-1) = 1$ in $\mathbb{Q}_2^* / \mathbb{Q}_2^2$. These mean that $(-1)^n F(1)F(-1) = 1$ in $\mathbb{Q}^* / \mathbb{Q}^2$. The converse is straightforward from Theorems 4.1 and 4.2. \hfill $\square$

In §4.2, for a given $*$-symmetric polynomial $F \in \mathcal{O}_K[X]$, we define a $K$-vector space $M$ having a semisimple linear map $\alpha : M \to M$ with characteristic polynomial $F$. Then we consider when $M$ admits an inner product $b$ such that $\alpha$ becomes an isometry on $(M, b)$ and $\partial[M, b, \alpha]$ vanishes in $W_T(k)$ (recall Theorem 3.3). For this purpose, in §4.1 we make some preparations in a slightly more general situation.

\textbf{4.1 The map $\partial_{M, \alpha} : \text{Tw}(E, \sigma) \to W_T(k)$}

Let $E$ be a $K$-algebra with a nontrivial involution $\sigma$, and assume that the fixed algebra $E^\sigma$ is a field and one of the following holds:

(sp) $E \cong E^\sigma \times E^\sigma$ where the involution on the right-hand side is the transposition of the first and second components.

(ur) $E$ is an unramified extension field of $E^\sigma$.

(rm) $E$ is a ramified extension field of $E^\sigma$.

One can see that $\text{Tw}(E, \sigma)$ is a trivial group if $E$ is of type (sp), and that $\text{Tw}(E, \sigma) \cong \mathbb{Z} / 2\mathbb{Z}$ if $E$ is of type (ur) or (rm).

Let $M$ be a finitely generated free $E$-module. A hermitian form on $M$ is uniquely determined by its dimension and determinant. More precisely, the following holds.

\textbf{Proposition 4.4.} Sending a hermitian form $h$ on $M$ to its determinant $\det h$ gives rise to a one-to-one correspondence between the isomorphism classes of hermitian forms on $M$ and the elements of the twisting group $\text{Tw}(E, \sigma)$.

\textit{Proof.} Surjectivity is obvious and see [16a21] Example 10.1.6 (iii) for injectivity. \hfill $\square$

If $b : M \times M \to K$ is an inner product which can be written as $b = \text{Tr}_{E/K} \circ h$ for some hermitian form $h$ over $E$, Proposition 4.1 implies that the isomorphism class of $b$ is determined by $\det h$.

\textbf{Notation 4.5.} Let $\lambda \in \text{Tw}(E, \sigma)$. The symbol $b[\lambda]$ denotes an inner product of the form $\text{Tr}_{E/K} \circ h$ on $M$, where $h$ is a hermitian form with $\det h = \lambda$. The isomorphism class of $b[\lambda]$ is uniquely determined by $\lambda$. 

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We now fix $\alpha \in \mathcal{O}_K^*$ satisfying $\alpha \sigma(\alpha) = 1$. Then $\alpha : M \to M$ is an isometry on $(M, b[\lambda])$ for any $\lambda \in \text{Tw}(E, \sigma)$, and the triple $(M, b[\lambda], \alpha)$ becomes a $K[\Gamma]$-form. Hence, we obtain the map

$$\partial_{M, \alpha} : \text{Tw}(E, \sigma) \to W_T(k), \lambda \mapsto \partial[M, b[\lambda], \alpha]$$

where $\partial$ is the map mentioned in Theorem 3.3. The purpose of this subsection is to describe the image of this map.

We begin with the case where $M$ is of rank one, that is, $M = E$. In this case, for each $\lambda \in (E^\sigma)^\times$, the inner product $b_\lambda : E \times E \to K$ defined by

$$b_\lambda(x, y) = \text{Tr}_{E/K}(\lambda x \sigma(y)) \quad \text{for } x, y \in E$$

is a typical example of $b[\lambda]$. Notice that if $E$ is of type (sp) then $[E, b_\lambda, \alpha] = 0$ in $W_T(K)$ for any $\lambda \in \text{Tw}(E, \sigma)$. In particular, the map $\partial_{E, \alpha}$ is zero. Let $E$ be of type (ur) or (rm). This means that $E$ is a field. Let $l$ denote the residue class field of $E$. The field $\overline{l}$ is also the residue class field of the maximal unramified extension field of $E/K$, which is denoted by $L$. The involution $\sigma_1$ on $l$ (which may be trivial). For any $x \in \mathcal{O}_E$, the image of $x$ under $\mathcal{O}_E \to l$ is denoted by $\overline{x}$. Let $\delta = v_E(\mathcal{O}_E/K)$ be the valuation of the different ideal $\mathcal{D}_E/K$ of $E/K$.

**Lemma 4.6.** Let $\lambda \in (E^\sigma)^\times$.

(i) If $v_E(\lambda) + \delta$ is even, set $n = -(v_E(\lambda) + \delta)/2$ and $\Lambda = \mathfrak{m}_E^n$. Then $\Lambda$ is an $\alpha$-stable unimodular lattice in $(E, b_\lambda)$.

(ii) If $v_E(\lambda) + \delta$ is odd, set $n = -(v_E(\lambda) + \delta - 1)/2$ and $\Lambda = \mathfrak{m}_E^n$. Then we have $\Lambda^\sigma = \mathfrak{m}_E^{n-1}$, and in particular, $\Lambda$ is an $\alpha$-stable almost unimodular lattice in $(E, b_\lambda)$. Moreover, there is an isomorphism $(\Lambda^\sigma/\Lambda, b_{\lambda}) \cong (l, b_u, \alpha)$ as $k[\Gamma]$-forms, where $u \in \mathcal{O}_L$ is the $\sigma$-invariant unit defined by $u := u_\lambda := \text{Tr}_{E/L}(\lambda x_k \pi_K^{-1} \sigma(\pi_K^{-1}))$ and $b_u$ is the inner product defined by

$$b_u(\overline{x}, \overline{y}) = \text{Tr}_{l/K}(\overline{ux_\lambda}(\overline{y})) \quad \text{for } x, y \in \mathcal{O}_L. \quad (4)$$

**Proof.** See Corollary 6.2 and Proposition 6.3 of [31T29]. \qed

**Lemma 4.7.** If $\alpha$ is neither 1 nor $-1$, then $\sigma_1$ is nontrivial and $l = k(\alpha)$.

**Proof.** We have

$$\overline{\alpha} \neq 1, -1 \iff \overline{\alpha}^2 - 1 \neq 0 \iff \overline{\alpha} - \overline{\alpha}^{-1} \neq 0 \iff \overline{\alpha} \neq \sigma_1(\alpha).$$

Let $\alpha \neq 1, -1$. Then $\sigma_1$ is nontrivial, which implies that $l^{\sigma_1}$ is a proper subfield of $l$ and $[l : l^{\sigma_1}] = 2$. If $[l : k(\alpha)] \geq 2$ then we would get

$$l^{\sigma_1} : k = [l : k]/2 \geq [l : k]/[l : k(\alpha)] = [k(\alpha) : k].$$

This means that $k(\alpha) \subset l^{\sigma_1}$ but this inclusion contradicts $\alpha \neq \sigma_1(\alpha)$. Hence we get $[l : k(\alpha)] = 1$ and $l = k(\alpha)$. \qed

In the case (ur) we have:

**Proposition 4.8.** Assume that $E/E^\sigma$ is unramified.

(i) If $\alpha \neq \pm 1$ then $\text{im } \partial_{E, \alpha} = \{0, [k(\alpha), b_1, \alpha] \} \subset W_T(k)$ and $[k(\alpha), b_1, \alpha]$ has order 2. Here $b_1$ is the inner product defined by equation 4 for $\overline{u} = 1$.

(ii) If $\alpha = \pm 1$ then $\text{im } \partial_{E, \alpha} = \{\omega \in W_T(k; \pm 1) \mid \text{dim } \omega \equiv 0 \text{ mod } 2\}$.

In particular $\text{dim } \partial_{E, \alpha}(\lambda) \equiv 0 \text{ mod } 2$ for any $\lambda \in \text{Tw}(E, \sigma)$.

**Proof.** Let $\lambda \in (E^\sigma)^\times$. Note that the class of $\lambda$ in $\text{Tw}(E, \sigma)$ is uniquely determined by the parity of $v_E(\lambda) = v_E^\sigma(\lambda)$ since $E/E^\sigma$ is unramified. If $v_E(\lambda) + \delta$ is even then $\partial_{E, \alpha}(\lambda) = 0$ by Lemma 3.7 (i). Assume that $v_E(\lambda) + \delta$ is odd. Then $\partial_{E, \alpha}(\lambda) = [l, b_\lambda, \alpha]$ by Lemma 4.6 (ii), where $u = u_\lambda$.

If $\alpha \neq \pm 1$, the involution $\sigma_1$ is nontrivial by Lemma 3.7. This implies that all hermitian forms on $l$ (over the $k$-algebra $l$) are isomorphic to one another (see [31C83 Example 10.1.6 (i)]), and their induced
inner products over $k$ are isomorphic to $b_1$. Thus, we have $2\partial_{E,\alpha}(\lambda) = 0$. Furthermore, the $k[\Gamma]$-module $l$ is irreducible because $l = k(\bar{\alpha})$ by Lemma 4.7. Hence $\partial_{E,\alpha}(\lambda) = [l, b_u, \bar{\alpha}] = [k(\bar{\alpha}), b_1, \bar{\alpha}]$ is not 0 and has order 2.

If $\bar{\alpha} = \pm 1$ then
\[
\text{im} \partial_{E,\alpha} \subset \{ \omega \in W_T(k; \pm 1) \mid \dim \omega \equiv 0 \mod 2 \}
\] (5)

because $[l : k] = [l' : k] = 2[l' : k] \in 2\mathbb{Z}$, where $l'$ is the residue class field of $E'$. If char $k = 2$ then the right-hand side of (5) is $\{0\}$ and the statement is clear. Let char $k \neq 2$. We have to prove that $\partial_{E,\alpha}(\lambda) = [l, b_u, \bar{\alpha}] \in W_T(k; \pm 1)$ is nontrivial, and it is sufficient to show that $\text{disc } b_u \neq 1$ in $k^\times/k^{\times 2}$. We have
\[
\text{disc } b_u = N_{l/k}(\bar{u}) \text{disc } b_1 = N_{l'/k}(\bar{u})^2 \text{disc } b_1 = \text{disc } b_1
\]
in $k^\times/k^{\times 2}$. One can verify that $(\text{disc } b_1)^{#k^\times/2} \neq 1$ to show that $\text{disc } b_1$ is not a square.

In the case (rm) we have:

Proposition 4.9. Assume that $E/E'$ is ramified. Then $\bar{\alpha} = 1$ or $-1$ and in particular $W_T(k; \bar{\alpha}) \cong W(k)$.

If char $k \neq 2$, then $v_E(\lambda) + \delta$ is odd for any $\lambda \in (E')^\times$, and we have
\[
\text{im} \partial_{E,\alpha} = \{ \omega \in W_T(k; \bar{\alpha}) \mid \dim \omega \equiv [l : k] \mod 2 \}.
\]
If char $k = 2$ then for any $\lambda \in (E')^\times$ we have
\[
\partial_{E,\alpha}(\lambda) = \begin{cases} 0 & \text{if } [l : k] \delta \text{ is even} \\ (1) & \text{if } [l : k] \delta \text{ is odd.} \end{cases}
\]

Here (1) is a 1-dimensional inner product over $k$ whose Gram matrix is the $1 \times 1$ matrix (1).

Proof. See Lemma 6.5 and Propositions 6.6 and 6.7 of [BT20].

We have now described the image of the map $\partial_{M,\alpha}$ in the case where $M$ is of rank one. In the general case, we obtain the following theorem.

Theorem 4.10. Let $M$ be a free $E$-module of rank $m$.

(i) If $E$ is of type (sp) then $\partial_{M,\alpha}$ is the zero map.

(ii) If $E$ is of type (ur) then
\[
\text{im} \partial_{M,\alpha} = \begin{cases} \{0, [k(\bar{\alpha}), b_1, \bar{\alpha}]\} & \text{if } \alpha \neq \pm 1 \\ \{ \omega \in W_T(k; \pm 1) \mid \dim \omega \equiv 0 \mod 2 \} & \text{if } \alpha = \pm 1. \end{cases}
\]

(iii) If $E$ is of type (rm) then $\bar{\alpha} = 1$ or $-1$, and moreover,

- if char $k \neq 2$ then $\text{im} \partial_{M,\alpha} = \{ \omega \in W_T(k; \bar{\alpha}) \mid \dim \omega \equiv m[l : k] \mod 2 \}$;
- if char $k = 2$ then for any $\lambda \in \text{Tw}(E, \sigma)$ we have
\[
\partial_{M,\alpha}(\lambda) = \begin{cases} 0 & \text{if } m[l : k] \delta \text{ is even} \\ (1) & \text{if } m[l : k] \delta \text{ is odd.} \end{cases}
\]

Proof. (i) is clear. (ii) and (iii) follow from Propositions 6.8 and 6.9 respectively. □
4.2 Even unimodular lattice with isometry

Let $F \in \mathcal{O}_K[X]$ be a $*$-symmetric polynomial of even degree $2n$. The polynomial $F$ can be expressed as $F(X) = (X - 1)^{n_+}(X + 1)^{n_-}$, where $n_+, n_- \in \mathbb{Z}_{\geq 0}$ and $f(X) \in \mathcal{O}_K[X]$ with $f(1)f(-1) \neq 0$. Furthermore, the polynomial $f$ decomposes in $\mathcal{O}_K[X]$ as

$$f(X) = \prod_{w \in W_{sp}} (f_w(X)f_w^*(X))^{n_w} \times \prod_{w \in W'} f_w(X)^{n_w},$$

where $W_{sp}$ and $W'$ are index sets, and $n_w$ is the multiplicity of $f_w$ in $f$. Moreover, each factor $f_w$ is irreducible and indexed as follows: if $w \in W_{sp}$ then $f_w \neq f_w^*$; if $w \in W'$ then $f_w = f_w^*$.

Now we define an algebra $M$. First, set $M^\pm := (K[X]/(X \mp 1))^{n_{\pm}}$. Next, for each $w \in W := W_{sp} \cup W'$, set

$$E_w := \begin{cases} K[X]/(f_w) & \text{if } f_w = f_w^* \\ K[X]/(f_w f_w^*) & \text{if } f_w \neq f_w^* \end{cases}$$

and $M_w := (E_w)^{n_w}$. Finally, define

$$M := M^+ \times M^- \times \prod_{w \in W} M_w.$$

We denote by $\alpha$ the image of $X$ in $M$, and by $\alpha_w$ the image of $X$ in $M_w$. It is obvious that the linear map $\alpha : M \to M$ is a semisimple automorphism with characteristic polynomial $F$. In addition, the $K$-algebra $M$ has an involution $\sigma$ defined by $\alpha \mapsto \alpha^{-1}$. It is clear that

$$\sigma M^\pm = M^\pm, \quad \sigma E_w = E_w \text{ for } w \in W,$$

and $\sigma$ acts on $M^\pm$ as the identity and on $E_w$ nontrivially. The restriction of $\sigma$ to a $\sigma$-invariant subalgebra will also be denoted by $\sigma$.

We remark that there is a decomposition $W = W_{sp} \sqcup W_{ur} \sqcup W_{rm}$ (or $W' = W_{ur} \sqcup W_{rm}$), where

$$W_{sp} := \{w \in W \mid f_w \neq f_w^* \text{ (and } E_w \text{ is of type (sp))} \},$$

$$W_{ur} := \{w \in W \mid f_w = f_w^* \text{ and } E_w \text{ is of type (ur)} \},$$

$$W_{rm} := \{w \in W \mid f_w = f_w^* \text{ and } E_w \text{ is of type (rm)} \}.$$

Furthermore, by Theorem 4.10 (iii), if $\text{char } k \neq 2$ then $W_{rm}$ decomposes as $W_{rm} = W_+ \sqcup W_-$, where $W_{\pm} := \{w \in W_{rm} \mid \alpha_w = \pm 1\}$. In this subsection we write $v = v_K$ simply.

**Lemma 4.11.** Let $w \in W$. For any $\lambda_w \in \text{Tw}(E_w, \sigma)$ we have:

(i) If $w \in W_{sp}$ then $\dim \partial_{M_w, \alpha_w}(\lambda_w) \equiv v(f_w(1)f_w^*(1)) \equiv v(f_w(-1)f_w^*(-1)) \equiv 0 \bmod 2$.

(ii) If $w \in W_{ur}$ then $\dim \partial_{M_w, \alpha_w}(\lambda_w) \equiv v(f_w(1)) \equiv v(f_w(-1)) \equiv 0 \bmod 2$.

(iii) Suppose that $w \in W_{rm}$.

- If $\text{char } k \neq 2$ and $\alpha_w = \pm 1$ then $\dim \partial_{M_w, \alpha_w}(\lambda_w) \equiv n_+ v(f_w(1)) \bmod 2$.

- If $\text{char } k = 2$ then $\dim \partial_{M_w, \alpha_w}(\lambda_w) \equiv n_- v(f_w(1)) + n_+ v(f_w(-1)) \bmod 2$.

In particular, if $\text{char } k \neq 2$ then

$$\sum_{w \in W_{\pm}} \dim \partial_{M_w, \alpha_w}(\lambda_w) \equiv v(f(\pm 1)) \bmod 2$$

for any $(\lambda_w)_{w \in W_{rm}}$.

**Proof.** (i) is clear. If $w \in W_{ur}$ then $\dim \partial_{M_w, \alpha_w}(\lambda_w) \equiv 0 \bmod 2$ by Theorem 4.11 and $v(f_w(\pm 1)) = [l_w : k] v_E(1 \pm \alpha_w) \equiv 0 \bmod 2$, where $l_w$ is the residue class field of $E_w$. Suppose that $w \in W_{rm}$. If $\text{char } k \neq 2$ and $\alpha_w = \pm 1$ then $\dim \partial_{M_w, \alpha_w}(\lambda_w) \equiv n_+ [l_w : k] \bmod 2$ by Theorem 4.10. Because $[l_w : k] \equiv v(f_w(\pm 1)) \bmod 2$ by [BT20] Lemma 6.8, we obtain the desired congruence. Similarly, Theorem 4.10 and [BT20] Lemma 6.8 show the desired congruence in the case $\text{char } k = 2$. □
The following proposition is a reformulation of a part of [Ba21 Proposition 4.3].

**Proposition 4.12.** Let \( k \neq 2 \). Assume that \( v(F(1)) \equiv v(F(-1)) \equiv 0 \mod 2 \). Then there exists an inner product \( b \) on \( M \) such that \((M,b)\) contains an \( \alpha \)-stable unimodular \( \mathcal{O}_K \)-lattice. Furthermore,

- if \( F(\pm 1) = 0 \) then such an inner product can be chosen to satisfy \( \det M^\pm = u_\pm f(\pm 1) \) for any given \( u_\pm \in \mathcal{O}_K^* \); and
- if \( W_{tr} = \emptyset \) then such an inner product can be chosen for \((M_w,b_w)\) to contain an \( \alpha \)-stable unimodular \( \mathcal{O}_K \)-lattice for each \( w \in W \). Here \( b_w \) is the restriction of \( b \) to \( M_w \).

**Proof.** If \( F(\pm 1) = 0 \) we choose an inner product \( b^\pm \) on \( M^\pm \) whose Gram matrix is \( \text{diag}(u_\pm f(\pm 1),1,\ldots,1) \). For each \( w \in W_{sp} \cup W_{ur} \), choose \( \lambda_w \in Tw(E_w,\sigma) \) satisfying \( \partial_{M_w,\alpha_w}(\lambda_w) = 0 \). This is possible by Theorem 4.10. For any \((\lambda_w)_{w \in W^\pm}\) we have by Lemma 4.11

\[
\dim \partial \left( \bigoplus_{w \in W^\pm} (M_w, b[\lambda_w], \alpha_w) \right) \equiv v(f(\pm 1)) \equiv v(\dim \partial[M^\pm, b^\pm, \pm 1]) \mod 2.
\]

Then we can choose \((\lambda_w)_{w \in W^\pm}\) satisfying \( \partial[\bigoplus_{w \in W^\pm} (M_w, b[\lambda_w], \alpha_w)] = -\partial[M^\pm, b^\pm, \pm 1] \) by Theorem 4.10. The inner product \( b := b^+ \oplus b^- \oplus \bigoplus_{w \in W} b[\lambda_w] \) on \( M \) satisfies \( \partial[M,b,\alpha] = 0 \). This implies that \((M,b)\) contains an \( \alpha \)-stable unimodular \( \mathcal{O}_K \)-lattice by Theorem 5.3. The latter part of this proposition is obvious by the above construction. \( \square \)

We consider an analog of Proposition 4.12 when \( K = \mathbb{Q}_2 \). Our purpose is to show Theorem 4.16 which is a generalization of [Ba21 Theorem 5.1] to the case where \( F \) is \( s \)-symmetric. We denote by \( v(sn(t)) \) the spinor norm of an isometry \( t \), see [OM73 §55] or [Za92] for definition.

**Lemma 4.13.** Let \((\Lambda,b)\) be an even unimodular \( \mathbb{Z}_2 \)-lattice. For any isometry \( t : \Lambda \rightarrow \Lambda \), we have \( v(sn(t)) \equiv 0 \mod 2 \) if \( \det t = 1 \), and \( v(sn(t)) \equiv 1 \mod 2 \) if \( \det t = -1 \).

**Proof.** Let \( t \) be an isometry on \( \Lambda \). If \( \det t = 1 \) then \( v(sn(t)) \equiv 0 \mod 2 \) by [BT20 Proposition 8.6]. Let \( \det t = -1 \). We can choose \( r \in \Lambda \) satisfying \( v(b(r,r)) = 1 \) by Proposition 5.4. Let \( s_r \) denote the reflection defined by \( r \). Then \( \det(s_r \circ t) = 1 \) and hence \( v(sn(s_r \circ t)) \equiv 0 \mod 2 \). Because \( v(sn(s_r)) \equiv 1 \) we get \( v(sn(t)) \equiv 1 \mod 2 \). \( \square \)

**Proposition 4.14.** Let \((V,b)\) be an inner product space over \( \mathbb{Q}_2 \), and \( t \) an isometry on \( V \). There exists a \( t \)-stable even unimodular lattice of discriminant \( 1 \) in \( V \) if and only if the following conditions hold:

(i) \( V \) contains a \( t \)-stable unimodular lattice.

(ii) \( V \) contains an even unimodular lattice of discriminant \( 1 \).

(iii) \( v(sn(t)) \equiv 0 \mod 2 \) if \( \det t = 1 \) and \( v(sn(t)) \equiv 1 \mod 2 \) if \( \det t = -1 \).

**Proof.** The basic idea of the proof is similar to that of [BT20 Theorem 8.1]. If there exists a \( t \)-stable even unimodular lattice of discriminant \( 1 \) in \( V \), then the conditions (i) and (ii) are obvious, and (iii) holds by Lemma 4.13.

Let the three conditions hold. There exists a \( t \)-stable unimodular lattice \( \Lambda_0 \) in \( V \) by (i). If \( \Lambda_0 \) is even then we are done. Assume that \( \Lambda_0 \) is odd. We can take an even unimodular lattice \( \Lambda_1 \) of discriminant \( 1 \) in \( V \) by (ii). Proposition 5.1 implies that there exist vectors \( 2e, f \in \Lambda_1 \) satisfying

\[
b(2e,f) = b(f,f) = 0 \quad \text{and} \quad b(2e,f) = 1.
\]

Set \( H := (\mathbb{Z}_2(2e) + \mathbb{Z}_2 f)^\perp \subset \Lambda_1 \) and \( \Lambda'_0 := H + \mathbb{Z}_2(e+f) + \mathbb{Z}_2(e-f) \) in \( V \). Because \( \Lambda'_0 \) is an odd unimodular lattice in \( V \), we may assume that \( \Lambda'_0 = \Lambda_0 \) by [OM73 Theorem 93:29]. We now define lattices \( \Lambda_2 \) and \( \Lambda \) in \( V \) by \( \Lambda_2 := H + \mathbb{Z}_2 e + \mathbb{Z}_2(2f) \) and by \( \Lambda := H + \mathbb{Z}_2(2e) + \mathbb{Z}_2(2f) \) respectively. Then
\(\Lambda_2\) is an even unimodular lattice different from \(\Lambda_1\), and \(\Lambda\) is contained in \(\Lambda_0, \Lambda_1\) and \(\Lambda_2\). We remark that \(\Lambda\) can be written as \(\Lambda = \{x \in \Lambda_0 \mid b(x, x) \in 2\mathbb{Z}_2\}\), so \(t\) preserves \(\Lambda\). There is no unimodular lattice containing \(\Lambda\) other than \(\Lambda_0, \Lambda_1\) and \(\Lambda_2\) because there is a natural one-to-one correspondence between the \(\mathbb{Z}_2\)-valued lattices containing \(\Lambda\) and the isotropic subgroups of \(\Lambda' / \Lambda \cong \mathbb{Z}_2 \times \mathbb{Z}_2\). Hence, we have \(t\Lambda_1 = \Lambda_1\) or \(t\Lambda_1 = \Lambda_2\). We need to show that \(t\Lambda_1 = \Lambda_1\). Suppose to the contrary that \(t\Lambda_1 = \Lambda_2\). Let \(s : V \to V\) denote the reflection defined by \(e - f\). Because \(s\) maps \(\Lambda_2\) to \(\Lambda_1\), the composition \(s \circ t\) preserves \(\Lambda_1\). On the other hand, by the equation \(\text{sn}(t) = b(e - f, e - f) = -1\) and the assumption (iii), we would have

\[
\text{v(sn}(s \circ t)\text{)} = \text{v(sn}(t)\text{)} = \begin{cases} 0 & \text{if } \det t = 1 \\ 1 & \text{if } \det t = -1 \end{cases} = \begin{cases} 0 & \text{if } \det(s \circ t) = -1 \\ 1 & \text{if } \det(s \circ t) = 1 \end{cases}
\]

mod 2. However, this contradicts Lemma 4.13. Therefore, we have \(t\Lambda_1 = \Lambda_1\).

Now, we make a formula for the valuation of the spinor norm of an isometry.

**Lemma 4.15.** Let \(t\) be an isometry on an inner product space \((V, b)\) over \(\mathbb{Q}_2\) of even dimension, and let \(F(X) = (X - 1)^{n_+}(X + 1)^{n_-}f(X)\) be its characteristic polynomial where \(n_+, n_- \in \mathbb{Z}_{\geq 0}\) and \(f(1)f(-1) \neq 0\). Let \(V(-1; t)\) denote the eigenspace of \(t\) corresponding to \(-1\). Then we have

\[
\text{v(sn}(t)\text{)} \equiv \text{v(det } b\mid_{V(-1; t)}\text{) + v(f(-1))) mod 2. (7)}
\]

**Proof.** The Zassenhaus formula \(Z_{262}\) implies that

\[
\text{sn}(t) = \text{det } b\mid_{V(-1; t)} \cdot \text{det } \left( \frac{1 - t}{2} \right)_{V(-1; t)}^{1}\text{ in } \mathbb{Q}_2^x/\mathbb{Q}_2^x,
\]

and we have

\[
\text{det } \left( \frac{1 - t}{2} \right)_{V(-1; t)}^{1} = (-2)^{\dim(V) - n_-} \text{det } (-1 - t|_{V(-1; t))}) = (-2)^{n_- + n_+} f(-1) = f(1)
\]

in \(\mathbb{Q}_2^x/\mathbb{Q}_2^x\) since \(n_+ + n_-\) is even. Therefore, we get equation (7). □

**Theorem 4.16.** Let \(K = \mathbb{Q}_2\), and let \(F \in \mathbb{Z}_2[X]\) be a \(2\)-symmetric polynomial of even degree \(2n\). Assume that

(a) \(v(F(1)) \equiv v(F(-1)) \equiv 0 \text{ mod 2}\); and

(b) if \(F(1)F(-1) \neq 0\) then \((-1)^n F(1)F(-1) = 1\) in \(\mathbb{Q}_2^x/\mathbb{Q}_2^x\).

Then there exists an inner product \(b\) on \(M\) such that \((M, b)\) contains an \(n\)-stable even unimodular \(\mathbb{Z}_2\)-lattice. Furthermore, if \(F(1) = F(-1) = 0\) then such an inner product can be chosen to satisfy

\[
\text{det } M^n = \begin{cases} u_\pm f(\pm 1) & \text{if } n_+ \text{ is even} \\ 2u_\pm f(\pm 1) & \text{if } n_+ \text{ is odd} \end{cases}
\]

for any given \(u_+, u_- \in \mathbb{O}_K^x\) such that \(u_+ u_- = (-1)^n\).

**Proof.** We take an inner product \(b\) on \(M\) as follows. First, set

\[
D_\pm = \begin{cases} u_\pm f(\pm 1) & \text{if } n_+ \text{ is even} \\ 2u_\pm f(\pm 1) & \text{if } n_+ \text{ is odd} \end{cases}
\]

and take inner products \(b^+\) on \(M^+\) and \(b^-\) on \(M^-\) satisfying

\[
\text{det } b^+ = \begin{cases} (-1)^n f(1)f(-1) & \text{if } F(1) = 0 \text{ and } F(-1) \neq 0 \\ D_+ & \text{if } F(1) = F(-1) = 0, \end{cases}
\]

\[
\text{det } b^- = \begin{cases} (-1)^n f(1)f(-1) & \text{if } F(1) \neq 0 \text{ and } F(-1) = 0 \\ D_- & \text{if } F(1) = F(-1) = 0. \end{cases}
\]

(8)
Next, for each \( w \in \mathcal{W}_{\text{ap}} \cup \mathcal{W}_{\text{ur}} \) we fix \( \lambda_w \in \text{Tw}(E_w, \sigma) \) satisfying \( \partial_{M_w, \alpha_w}(\lambda_w) = 0 \). This is possible by Theorem 4.10. Furthermore, we choose \( \lambda_w \in \text{Tw}(E_w, \sigma) \) arbitrarily for each \( w \in \mathcal{W}_{\text{rm}} \) and define an inner product on \( M \) by \( b := b^+ \oplus b^- \oplus \bigoplus_{w \in \mathcal{W}} b[\lambda_w] \). Notice that \( \alpha \) is an isometry with respect to \( b \).

Claim 1: For any \( b \) chosen as above, the inner product space \((M, b)\) with the isometry \( \alpha \) satisfies the conditions (i) and (iii) in Proposition 4.13.

For (i), it is enough to show that \( \partial[M, b, \alpha] = 0 \) by Theorem 4.3. We have

\[
\dim \partial[M^+ \oplus M^-, b^+ \oplus b^-, \alpha] = v(\det(b^+ \oplus b^-)) \equiv v((-1)^n f(1)(f(-1)) \equiv v(f(1)f(-1)) \pmod 2.
\]

On the other hand, by Lemmas 4.11 and 2.2 we have \( \partial \) determined by (7), (8), and the assumption (a). If \( b \) contains a unimodular lattice, and the unimodular lattice is even (see the proof of [BT20, Proposition 9.1])

(ii) contains a unimodular lattice, and the unimodular lattice is even (see the proof of [BT20, Proposition 9.1]).

In general, there is a unique \( \eta_w \in \{0, 1\} \) for each \( w \in \mathcal{W}_{\text{rm}} \) so that any inner product space over \( \mathbb{Q}_2 \) of dimension 2m, discriminant 1, and Hasse-Witt invariant \( \eta_w \) contains an even unimodular \( \mathbb{Z}_2 \)-lattice. This is a consequence of Proposition 3.1. Assume that \( \mathcal{W}_{\text{rm}} \neq \emptyset \), and let \( u_0 \in \mathcal{W}_{\text{rm}} \). Choose \( \lambda_{u_0} \in \text{Tw}(E_{u_0}, \sigma) \) which is different from \( \lambda_{u_0} \), and define \( \tilde{\lambda} := b^+ \oplus b^- \oplus b[\lambda_{u_0}] \oplus \bigoplus_{w \neq u_0} b[\lambda_w] \). Because \( \epsilon_2(b) \neq \epsilon_2(\tilde{b}) \), we have \( \epsilon_2(\tilde{b}) = \eta_{u_0} \) or \( \epsilon_2(\tilde{b}) = \eta_{u_0} \). This means that the condition (ii) holds for \((M, b)\) or \((M, \tilde{b})\).

Let \( \mathcal{W}_{\text{rm}} = \emptyset \). Since \( \partial \left( \bigoplus_{w \in \mathcal{W}} M_w, \bigoplus_{w \in \mathcal{W}} [\lambda_w], \alpha \right) = 0 \), the space \( \left( \bigoplus_{w \in \mathcal{W}} M_w, \bigoplus_{w \in \mathcal{W}} [\lambda_w] \right) \) contains a unimodular lattice, and the unimodular lattice is even (see the proof of [HT20, Proposition 9.1]). Therefore it is sufficient to show that \((M^+ \oplus M^-, b^+ \oplus b^-)\) contains an even unimodular lattice.

Case I. \( n_+ > 2 \) or \( n_- > 2 \). Suppose that \( n_+ > 2 \). Then there exists an inner product \( b^+ \) on \( M^+ \) with \( \det b^+ = \det b^+ \) and \( \epsilon_2(b^+) \neq \epsilon_2(b^-) \). Because \( \epsilon_2(b^+ \oplus b^-) = \eta_{n_++n_-} \) or \( \epsilon_2(b^+ \oplus b^-) = \eta_{n_++n_-} \), either \((M^+ \oplus M^-, b^+ \oplus b^-) \) or \((M^+ \oplus M^-, b^+ \oplus b^-) \) contains an even unimodular lattice. Similarly, if \( n_- > 2 \) then \((M^+ \oplus M^-, b^+ \oplus b^-) \) contains an even unimodular lattice for a suitable \( b^+ \).

Case II. \( (n_+, n_-) = (2, 2) \). If \( D_+ \neq -1 \) or \( D_- \neq -1 \) in \( \mathbb{Q}_2^2/\mathbb{Q}_2^2 \) then we can choose \( b_+ \) or \( b_- \) to get \( \epsilon_2(b_+ \oplus b_-) = \eta_{n} \) as in Case I. If \( D_+ = D_- = -1 \) in \( \mathbb{Q}_2^2/\mathbb{Q}_2^2 \) then \( b^+ \) and \( b^- \) are isomorphic to the hyperbolic plane and contain even unimodular lattices respectively.
Case III. \((n_+, n_-) = (2, 0)\) or \((0, 2)\). A similar proof of Case II works, and \(b^+\) or \(b^-\) contains an even unimodular lattice if we choose \(b^+\) or \(b^-\) suitably.

Case IV. \((n_+, n_-) = (1, 1)\). Since \(b\) has discriminant 1 and \(\bigoplus_{w \in \mathcal{W}} M_w, \bigoplus_{w \in \mathcal{W}} b|\mathcal{W}\) contains an even unimodular lattice, we have \(\text{disc}(b^+ \oplus b^-) = \text{disc}(\bigoplus_{w \in \mathcal{W}} b|\mathcal{W}) = 1\) or \(-3\). If \(\text{disc}(b^+ \oplus b^-) = 1\) then \(b^+ \oplus b^-\) is isomorphic to the hyperbolic plane and contains an even unimodular lattice. Let \(\text{disc}(b^+ \oplus b^-) = -3\). Lemma 4.11 implies that \(v(f(1)) \equiv 0 \mod 2\) since \(\mathcal{W}_m = \emptyset\). The Hasse-Witt invariant of \(b^+ \oplus b^-\) can be calculated as

\[
\epsilon_2(b^+ \oplus b^-) = (D_+, D_-) = (D_+, -D_+) = (2u_+(f(1)), -3) = 1,
\]

and this means that \(b^+ \oplus b^-\) is isomorphic to the lattice \(V\) in Proposition 3.1. Thus, the space \((M^+ \oplus M^-, b^+ \oplus b^-)\) contains an even unimodular lattice.

In any case, the space \((M^+ \oplus M^-, b^+ \oplus b^-)\) contains an even unimodular lattice if we choose \(b^+\) and \(b^-\) suitably. This completes the proof of Claim 2.

Claims 1 and 2 mean that \((M, b)\) with the isometry \(\alpha\) satisfies the conditions (i)–(iii) in Proposition 4.14 for a suitable inner product \(b\), which implies that \((M, b)\) contains an \(\alpha\)-stable even unimodular lattice. The latter part of this theorem is obvious by the above construction. \(\square\)

Finally, we prove Theorem 4.2 as promised.

Proof of Theorem 4.2 Let \((\Lambda, b)\) be an even unimodular lattice of rank \(2n\) and discriminant 1 having a semisimple isometry \(t\) with characteristic polynomial \(Ft\). If \(\text{det} t = -1\) i.e., \(F\) is \(-1\)-symmetric, we have \(F(1) = F(-1) = 0\). Thus, the conditions \((a)\) and \((b)\) are clear. Let \(\text{det} t = 1\). We can write \(F\) as \(F(X) = (X - 1)^{n+}(X + 1)^{n-} f(X)\) using even integers \(n_+, n_-\) and \(f \in \mathbb{Z}_2[X]\) with \(f(1)f(-1) \neq 0\). Equation (9) and Proposition 4.14 imply that

\[
v(\text{det} b|_{V(-1,t)}) + v(f(-1)) \equiv 0 \mod 2.
\]

On the other hand, we have

\[
v(\text{det} b|_{V(1,t)}) + v(f(1)) + v(f(-1)) \equiv v(\text{det} b) \equiv 0 \mod 2
\]

by Lemma 2.2 and hence

\[
v(\text{det} b|_{V(1,t)}) + v(f(1)) \equiv 0 \mod 2.
\]

If \(F(-1) \neq 0\) then \(v(F(-1)) \equiv v(f(-1)) \equiv 0 \mod 2\) by equation (9). Similarly, if \(F(1) \neq 0\) then \(v(F(-1)) \equiv v(f(-1)) \equiv 0 \mod 2\) by equation (10). Therefore, we get the condition \((a)\). Let \(F(1)F(-1) \neq 0\). Then we have

\[
(1)^n F(1) F(-1) = (1)^n \text{det} b = \text{disc} b = 1 \quad \text{in} \quad \mathbb{Q}_2^\times / \mathbb{Q}_2^\times 2
\]

by Lemma 2.2. This is the condition \((b)\). The converse follows from Theorem 4.10. \(\square\)

5 Local-global principle

Let \(F \in \mathbb{Z}[X]\) be a \(*\)-symmetric polynomial of even degree \(2n\), and let \(r, s \in \mathbb{Z}_{\geq 0}\) be non-negative integers such that \(r + s = 2n\) and \(r \equiv s \mod 8\). Assume that the condition \((\Sigma)\) holds, and let \(\iota \in \text{Id}_{\mathbb{Q}_2, \iota}(F)\) be an index map. We refer to an isometry with characteristic polynomial \(F\) and index \(\iota\) as an \((F, \iota)\)-isometry for short. In this section, we establish a necessary and sufficient condition for an even unimodular lattice of signature \((r, s)\) having a semisimple \((F, \iota)\)-isometry to exist (Theorem 5.2).

We start by constructing a vector space having a semisimple automorphism with characteristic polynomial \(F\), as in 4.2. Let \(F_i\) be the type \(i\) component of \(F\) for \(i = 0, 1, 2\): \(F = F_0 F_1 F_2\). The product \(F_1 F_2\) is sometimes abbreviated to \(F_{12}\). The symbol \(I_i\) denotes the set of irreducible factors of \(F_i\). Let \(n_+\) and \(n_-\) be the multiplicities of \((X - 1)\) and \((X + 1)\) in \(F_0\) respectively, and \(n_f\) be the multiplicity of \(f \in I_1 \cup I_2\) in \(F_{12}\). Set

\[
M^\pm := [\mathbb{Q}[X]/(X \mp 1)]^{n_\pm}, \quad M^0 := M^+ \times M^-,
\]

\[
e^f := \begin{cases} 
\mathbb{Q}[X]/(f) & \text{for } f \in I_1 \\
\mathbb{Q}[X]/(ff^*) & \text{for } f \in I_2,
\end{cases} \quad M^f := (e^f)^{n_f} \text{ for } f \in I_1 \cup I_2,
\]

\[
M^1 := \prod_{f \in I_1} M^f, \quad M^2 := \prod_{(f, f^*) \in I_2} M^f, \quad M := M^0 \times M^1 \times M^2.
\]
Let \( \alpha \) denote the image of \( X \) in \( M \), and \( \sigma \) the involution defined by \( \alpha \mapsto \alpha^{-1} \). If we regard \( M \) as a \( \mathbb{Q} \)-vector space, the \( \mathbb{Q} \)-linear map \( \alpha : M \to M \) is a semisimple automorphism with characteristic polynomial \( F \). We will consider when \( M \) admits an inner product such that \( \alpha \) becomes an isometry preserving an even unimodular lattice.

**Notation 5.1.** We use the following notations.

- The set of places of \( \mathbb{Q} \) is denoted by \( \mathcal{V} \).
- The set of places of \( E^f,\sigma := (E^f)^\sigma \) above \( v \in \mathcal{V} \) is denoted by \( \mathcal{W}(f;v) \) for \( f \in I_1 \cup I_2 \).
- Let \( K \) be an algebraic number field, and \( v \) its place. For a \( K \)-algebra \( A \), we write \( A_v \) for \( A \otimes K K_v \), where \( K_v \) is the completion of \( K \) with respect to \( v \). For a \( K \)-algebra \( A^\bullet \) with a superscript, such as \( M^\pm, M^f, \) or \( E^f, \) we abbreviate \( (A^\bullet)_v \) to \( A_v^\bullet \).

Let \( I \) denote the set of irreducible \( \ast \)-symmetric factors of \( F \), that is, \( I = I_0 \cup I_1 \). If \( v \in \mathcal{V} \) is a place and \( b_v \) is an inner product on \( M_v \), then the symbol \( b_v^f \) denotes the restriction of \( b_v \) to \( M_f = M^f \otimes \mathbb{Q}_v \) for each \( f \in I \). Here, we understand that \( M^X_{\pm 1} = M^\pm \). We will show the following theorem.

**Theorem 5.2.** Let \( F \in \mathbb{Z}[X], r, s \in \mathbb{Z}_{\geq 0} \) and \( \iota \in \text{Id}_{r,s}(F) \) be as stated at the beginning of this section. The following are equivalent:

1. There exists an even unimodular lattice of signature \( (r,s) \) having a semisimple \((F,\iota)\)-isometry.
2. There exists a family \( \{b_v\}_{v \in \mathcal{V}} \) of inner products on \( M_v \) such that each \( b_v \) has the properties

\[
\alpha : M_v \to M_v \text{ is an isometry with respect to } b_v ;
\]

if \( v \neq \infty \) then there exists an \( \alpha \)-stable even unimodular \( \mathbb{Z}_v \)-lattice in \( (M_v, b_v) \), and

if \( v = \infty \) then the isometry \( \alpha \) on \( (M_\infty, b_\infty) \) has index \( \iota \);

\[
\det b_v^{X^{\pm 1}} = \begin{cases} 
(-1)^{(n_+ - \iota(X+1))/2} \cdot |F_{12}(\pm 1)| & \text{if } n_+ \text{ is even} \\
(-1)^{(n_+ - \iota(X+1))/2} \cdot 2 |F_{12}(\pm 1)| & \text{if } n_+ \text{ is odd} 
\end{cases} \quad \text{in } \mathbb{Q}_v^X / \mathbb{Q}_v^{X^2},
\]

and that almost all \( \epsilon_v(b_v^f) \) equal 0 (i.e. \( \# \{ (v,f) \in \mathcal{V} \times I \mid \epsilon_v(b_v^f) = 1 \} < \infty \)) and

\[
\sum_{v \in \mathcal{V}} \epsilon_v(b_v^f) = 0 \quad \text{for all } f \in I.
\]

To prove this theorem, we need to consider localizations of each \( M^f \). Let \( v \in \mathcal{V} \). We have

\[
M_v^\pm = [\mathbb{Q}[X]/(X \mp 1)]^{n_\pm} \otimes \mathbb{Q}_v = [\mathbb{Q}_v[X]/(X \mp 1)]^{n_\pm},
\]

and

\[
M_f^\pm = M^f \otimes_{E^f,\sigma} E_w^{f,\sigma} = [E^f \otimes_{E^f,\sigma} E_w^{f,\sigma}]^{n_f} = [E_w^{f,\sigma}]^{n_f} \quad (11)
\]

for \( f \in I_1 \cup I_2 \) and \( w \in \mathcal{W}(f;v) \). Note that there exists the canonical isomorphism between the completion \( (E^f,\sigma)_w \) and the fixed subalgebra \( (E^f)^\sigma_w \) of \( E^f = E^f \otimes_{E^f,\sigma} (E^f,\sigma)_w \). They are identified and denoted by \( E_w^{f,\sigma} \).

**Lemma 5.3.** Let \( p \) be a prime. We have

\[
M_p^f = \prod_{f \in I_1} \prod_{w \in \mathcal{W}(f;p)} M_f^w, \quad M_p^\pm = \prod_{(f^\pm) \in I_2} \prod_{w \in \mathcal{W}(f;p)} M_w^f
\]

and each \( M_f^w \) is the direct product of \( n_f \) copies of \( E_w^f \). Furthermore, each \( E_w^f \) satisfies one of \((sp), (ur)\) and \((rm)\) in \( \mathbf{4} \).
Proof. By equation (11) the algebra $M_f$ is the direct product of $n_f$ copies of $E_w^f$, and

$$M_p^f = \left( \prod_{f \in I_1} M_f \right) \otimes \mathbb{Q}_p = \prod_{f \in I_1} \left[ E_w^f \right]^{n_f} \otimes \mathbb{Q}_p$$

$$= \prod_{f \in I_1} \left( \prod_{w \in W(f;p)} E_w^f \right)^{n_f} = \prod_{f \in I_1} \prod_{w \in W(f;p)} M_w^f.$$ A similar calculation shows that $M_p^2 = \prod_{\{f, f'\} \subset I_2} \prod_{w \in W(f;p)} M_w^f$. The latter part of the argument is straightforward. □

This lemma shows that $M_p^1$ and $M_p^2$ decompose into factors $M_w^f$, which can be seen as $E_w^f$-modules discussed in (1). We will use the notation $W_{sp}(f;v), W_{w}(f;v), W_{rm}(f;v), W_{\pm}(f;v) \subset W(f;v)$ as in (1). Note that $W(f;v) = W_{sp}(f;v)$ for any $f \in I_2$.

5.1 Necessity

We prove the necessity (i) ⇒ (ii) of Theorem 5.2 Let $(\Lambda, b)$ be an even unimodular lattice of signature $(r, s)$ having a semisimple $(F, t)$-isometry $t$. We identify $\Lambda \otimes \mathbb{Q}$ with the algebra $M$ regarding $t$ as $\alpha$. Then, a family $\{b_v\}_{v \in V}$ of inner products on $M_v$ can be defined naturally by $b_v := b \otimes \mathbb{Q}_v$ for each $v \in V$. It is clear that each $b_v$ has the properties (P1) and (P2).

We then verify the property (P3). Let $b$ denote the restriction of $b$ to $M_s\pm$. Notice that the inner product $b_s \otimes \mathbb{Q}_v$ on $M_{\pm}$ is the same as $b_v^{\pm\pm} = b_v |_{M_s\pm \times M_s\pm}$ for any $v \in V$. We write $b_v^{\pm\pm}$ for this inner product.

Lemma 5.4. We have $\det b_{\pm} = D_{\pm}$ in $\mathbb{Q}^x/\mathbb{R}^{x^2}$, where

$$D_{\pm} := \begin{cases} (-1)^{(n_{\pm} - (n_{\mp} + 1))/2} |F_{12}(\pm 1)| & \text{if } n_{\pm} \text{ is even} \\ (-1)^{(n_{\pm} - (n_{\mp} + 1))/2} \cdot 2 |F_{12}(\pm 1)| & \text{if } n_{\pm} \text{ is odd}. \end{cases}$$

(12)

Proof. Since we have $\det b_{\pm} = (-1)^{(n_{\pm} - (n_{\mp} + 1))/2} \in \mathbb{R}^+ / \mathbb{R}^{x^2}$, it is sufficient to show that $v_p(\det b_{\pm}) \equiv v_p(F_{12}(\pm 1)) \mod 2$ for each odd prime $p$ and

$$v_2(\det b_{\pm}) \equiv \begin{cases} v_2(F_{12}(\pm 1)) & \text{if } n_{\pm} \text{ is even} \\ 1 + v_2(F_{12}(\pm 1)) & \text{if } n_{\pm} \text{ is odd} \end{cases} \mod 2.$$ Note that $v_p(\det b_{\pm}) \equiv \dim \partial[M_{\pm}^f, b_{\pm}, t] \mod 2$. Let $p$ be an odd prime. Since $\partial[M, b, t]$ is the trivial class in $W_T(\mathbb{F}_p)$, so is its image under the projection $W_T(\mathbb{F}_p) \rightarrow W_T(\mathbb{F}_p, \pm 1)$. Thus

$$v_p(\det b_{\pm}) \equiv \dim \partial[M_{\pm}^f, b_{\pm}, t] \equiv \dim \partial \left[ \bigoplus_{f \in I_1} \bigoplus_{w \in W_{\pm}(f;p)} (M_w^f b_{\pm} |_{M_s\pm \times M_s\pm}) \right] \equiv v_p(F_{12}(\pm 1)) \mod 2$$

by Lemma 4.11. Let $p = 2$. It follows from (7) that

$$v_2(\det b^-) \equiv v_2(su(l)) + v_2(F_{12}(-1)) \equiv \begin{cases} v_2(F_{12}(-1)) & \text{if } n_{\pm} \text{ is even} \\ 1 + v_2(F_{12}(-1)) & \text{if } n_{\pm} \text{ is odd} \end{cases} \mod 2.$$ (13)

mod 2. If $b^1$ and $b^2$ denote the restrictions of $b$ to $M^1$ and $M^2$ respectively, then $\det(b^+ \oplus b^-) = \det(b) \det(b^1 \oplus b^2) = \det(b^1 \oplus b^2)$ in $\mathbb{Q}^x / \mathbb{R}^{x^2}$. This implies that

$$v_2(\det b^+) + v_2(\det b^-) \equiv v_2(F_{12}(1)) + v_2(F_{12}(-1)) \mod 2$$

by Lemma 2.2. Combining this and (13) yields

$$v_2(\det b^+) \equiv \begin{cases} v_2(F_{12}(1)) & \text{if } n_{\pm} \text{ is even} \\ 1 + v_2(F_{12}(1)) & \text{if } n_{\pm} \text{ is odd} \end{cases} \mod 2.$$ Thus the proof is complete. □
This lemma shows that $\det b^\pm_v = D_\pm$ in $\mathbb{Q}_L^\times / \mathbb{Q}_E^\times$ for each $v \in \mathcal{V}$, which is nothing but [123]. Let $b^f$ denote the restriction of $b$ to $M_f$ for each $f \in I$. Then $b^f_v = b^f_v \otimes \mathbb{Q}_v$, which implies that almost all $\epsilon_v(b^f_v)$ equal 0 and $\sum_{v \in \mathcal{V}} \epsilon_v(b^f_v) = 0$ for each $f \in I$. Therefore $\{b_v\}_v$ is the required family, and the proof of (i) $\Rightarrow$ (ii) is complete.

5.2 Twisting groups as Brauer groups

In order to prove the implication (ii) $\Rightarrow$ (i) of Theorem 5.2, we relate twisting groups to Brauer groups. In this subsection, we work with a somewhat more general setting. Let $K$ be a field of characteristic $\neq 2$, and $E$ an extension field of $K$ with a nontrivial involution $\sigma$. We denote by $(\lambda, \sigma) \in \text{Br}(E^\sigma)$ the Brauer class of the cyclic algebra defined from $\lambda \in (E^\sigma)^\times$ and the cyclic extension $E/E^\sigma$ (see e.g. [Sc85, Chapter 8, §12]). The map $(E^\sigma)^\times \to \text{Br}(E^\sigma)$ defined by $\lambda \mapsto (\lambda, \sigma)$ induces the following exact sequence:

$$1 \to \text{Tw}(E, \sigma) \xrightarrow{\iota_{\sigma}} \text{Br}(E^\sigma) \xrightarrow{\text{Res}_{E/E^\sigma}} \text{Br}(E)$$

where $\text{Res}_{E/E^\sigma}$ is the restriction map.

If $K$ is a global field, we denote by $\mathcal{W}$ the set of all places of $E^\sigma$. For each $w \in \mathcal{W}$, the inclusion $E^\times \to E^\sigma_w^\times$ induces a map $\text{Tw}(E, \sigma) \to \text{Tw}(E_w, \sigma)$. Let $w \in \mathcal{W}$. If $E_w = E \otimes_{E^\sigma} E_w^\sigma$ is a field, then $\text{Tw}(E_w, \sigma)$ is of order 2, and in particular, there exists a unique isomorphism $\theta_w : \text{Tw}(E_w, \sigma) \to \mathbb{Z}/2\mathbb{Z}$. If $E_w \cong E_w^\sigma \times E_w^\sigma$ then $\text{Tw}(E_w, \sigma)$ is a trivial group. In this case $\theta_w : \text{Tw}(E_w, \sigma) \to \mathbb{Z}/2\mathbb{Z}$ denotes the trivial map.

Proposition 5.5. Assume that $K$ is a global field. The sequence

$$1 \to \text{Tw}(E, \sigma) \xrightarrow{\iota_{\sigma}} \text{Br}(E^\sigma) \xrightarrow{\text{Res}_{E/E^\sigma}} \text{Br}(E)$$

is exact.

Proof. This follows from the exact sequence

$$0 \to \text{Br}(L) \xrightarrow{\bigoplus_{w \text{place}} \text{Br}(L_w)} \bigoplus_{w \text{place}} \text{Br}(L_w) \xrightarrow{\sum_{w \text{place}}} \mathbb{Q}/\mathbb{Z} \to 0$$

for $L = E$ and $E^\sigma$, see [BT20, Theorem 5.7].

We also need the following formula for Hasse-Witt invariants.

Lemma 5.6. Assume that $K$ is a local field, and let $M$ be a finitely generated free $E$-module. Then

$$\epsilon((M, b[\lambda]) = \epsilon((M, b[1])) + \text{Cor}_{E^\sigma/K}(\lambda, \sigma),$$

where $\text{Cor}_{E^\sigma/K} : \text{Br}(E^\sigma) \to \text{Br}(K)$ is the corestriction map.

Proof. This is a special case of [HCM03, Theorem 4.3] if $M$ is of rank 1. The result in the general case follows from that in the rank one case.

5.3 Sufficiency

To obtain Theorem 5.2, we prove the sufficiency (ii) $\Rightarrow$ (i).

Proof of Theorem 5.2. The implication (i) $\Rightarrow$ (ii) is proved in [5.1]. We show (ii) $\Rightarrow$ (i). Let $\{b_v\}_{v \in \mathcal{V}}$ be the family mentioned in (ii). We write $b^\pm_v = b^\pm_v \mathbb{X}_{v}$ for short. Since $\det b^\pm_v = D_\pm$ for each $v \in \mathcal{V}$ and $\sum_{v \in \mathcal{V}} \epsilon_v(b^\pm_v) = 0$, there exists an inner product $B^\pm$ on $M^\pm$ such that $B^\pm_v \cong b^\pm_v$ for each $v \in \mathcal{V}$ by [Sc85, Chapter IV, Proposition 7]. Let $f \in I_1$ and $v \in \mathcal{V}$. For each $w \in \mathcal{W}(f; v)$ the restriction of $b_v$ to $M^\pm_w$ is denoted by $b^f_v$. Because the automorphism $\alpha : M^\pm_w \to M^\pm_w$ is an isometry with respect to $b^f_v$, we can write $b^f_v = b[\lambda^f_v]$ for some $\lambda^f_v \in \text{Tw}(E^\sigma_w, \sigma)$ for each $w \in \mathcal{W}(f; v)$. By Lemmas 5.6 and 2.4 we have

$$\sum_{w \in \mathcal{W}(f; v)} \text{Cor}_{E^\sigma_w/Q_v}(\lambda^f_w, \sigma) = \sum_{w \in \mathcal{W}(f; v)} (\epsilon_v(b[\lambda^f_v]) + \epsilon_v(b[1])) = \epsilon_v(b^f_v) + \epsilon_v(b[1]).$$
Summing over \( v \in V \) yields \( \sum_{v \in V} \sum_{w \in \mathcal{W}(f,v)} \text{Cor}_{E_w^{\sigma}/Q_w}(\lambda'_w, \sigma) = 0 \). By combining this with the commutative diagram
\[
\begin{array}{ccc}
\text{Tw}(E^f_w, \sigma) \xrightarrow{\theta_w} \text{Br}(E^f_w, \sigma) & \xrightarrow{\text{Cor}_{E_w^{\sigma}/Q_w}} & \text{Br}(Q_w) \\
\mathbb{Z}/2\mathbb{Z} \times \frac{1}{2} & \xrightarrow{\text{inv}} & \mathbb{Q}/\mathbb{Z} \xrightarrow{\text{id}} \mathbb{Q}/\mathbb{Z}
\end{array}
\]
we get \( \sum_{v \in V} \sum_{w \in \mathcal{W}(f,v)} \theta_w(\lambda'_w) = 0 \). Thus, by Proposition \( \text{(5.5)} \) there exists \( \lambda' \in \text{Tw}(E^f, \sigma) \) such that its image under \( \text{Tw}(E^f, \sigma) \to \text{Tw}(E^f_w, \sigma) \) equals \( \lambda'_w \) for any place \( w \) of \( E^f \). We take a hermitian form \( h^f : M^f \times M^f \to \mathbb{E}^f \) satisfying \( \det h^f = \lambda^f \) and \( \text{id}x h^f = \iota(f)/2 \). This is possible by \( \text{(Sc5) Theorem 10.6.9]} \).

Now we define an inner product \( B \) on \( M \) to be \( B^+ \oplus B^- \oplus B^1 \oplus B^2 \), where \( B^1 = \bigoplus_{f \in F} \text{Tr}_{E^f/Q} \circ h^f \) and \( B^2(x,y) = \text{Tr}_{\mathbb{Q}/\mathbb{Z}}(x\sigma(y)) \). Since \( B_v \cong b^v \) for each \( v \in V \), the property \( \text{(2)} \) implies that there exists an \( \alpha \)-stable even unimodular \( \mathbb{Z}_p \)-lattice \( \Lambda_p \) in \( (M_p, B_p) \) for each prime \( p \). We may assume that \( \Lambda_p \) coincides with the image of the direct sum of the ring of integers in \( M \) under \( M \to M_p \) for almost all \( p \).
Then \( \Lambda := \{ x \in M \mid \text{the image of } x \text{ under } M \to M_p \text{ belongs to } \Lambda_p \text{ for each prime } p \} \) is an \( \alpha \)-stable even unimodular \( \mathbb{Z} \)-lattice in \( (M, B) \). Since \( \alpha \) is a semisimple \( (F, \iota) \)-isometry, the proof is complete.

\section{Local-global obstruction}

Let \( F \in \mathbb{Z}[X], r, s \in \mathbb{Z}_{>0} \) and \( \iota \in \text{Id}_{r,s}(F) \) be the same as in the previous section, and let \( B \) denote the set of families \( \{b_v\}_{v \in V} \) of inner products on \( M_v \) such that each \( b_v \) has the properties \( \text{(1)} \) and \( \text{(2)} \) and that \#\{\( v, f \) \in \text{V} \times I \mid \epsilon_v(b^f_v) = 1 \} \) is finite.

\begin{proposition}
If \( F \) satisfies the condition \text{(Square)} then \( B \) is not empty.
\end{proposition}

\begin{proof}
We can take an inner product \( b_\infty \) on \( M_\infty \) satisfying \( \text{(1)} \sim \text{(2)} \) by Proposition \( \text{(2.5)} \). Let \( p \) be a prime. By the assumption \( \text{(Square)} \), we have \( v_p(F(1)) \equiv v_p(F(-1)) \equiv 0 \mod 2 \) and \( (-1)^{s}F(1)F(-1) = 1 \mod 2 \). Hence there exists an inner product \( b_p \) on \( M_p \) satisfying \( \text{(1)} \sim \text{(2)} \) by Proposition \( \text{(1.12)} \) or Theorem \( \text{(1.10)} \). Moreover, if \( p \neq 2 \) and \( p \) is unramified in \( E^f/Q \), then we may assume that \( (M^f_p, b^f_p) \) contains an \( \alpha \)-stable (even) unimodular lattice for each \( f \in I_1 \cup I_2 \). In this case we have \( \epsilon_v(b^f_v) = 0 \) for all \( f \in I_1 \cup I_2 \) because any unimodular \( \mathbb{Z}_p \)-lattice \( (p \neq 2) \) has the trivial Hasse-Witt invariant. Furthermore, because the image of \( \partial[M^\pm_p, b^\pm_p, \pm 1] = \partial[M_p, b_p, \alpha] \) under the projection \( W^f_1(\mathbb{E}^f) \to W^f_1(\mathbb{E}^f_p; \pm 1) \) equals \( 0 \), the space \( (M^\pm_p, b^\pm_p) \) contains an unimodular \( \mathbb{Z}_p \)-lattice and we have \( \epsilon_p(b^f_p) = 0 \) again.

Let \( \{b_v\}_v \) be a family chosen as above. Almost all primes satisfy \( \text{(2)} \) and thus \#\{\( v, f \) \in \text{V} \times I \mid \epsilon_v(b^f_v) = 1 \} < \infty \).

In the following we assume that \( F \) satisfies the condition \text{(Square)}.

\subsection{Obstruction group and obstruction map}

Set \( C(I) := \{ \gamma : I \to \mathbb{Z}/2\mathbb{Z} \} = \mathbb{Z}/2\mathbb{Z} \oplus I \) where \( I = I_0 \cup I_1 \), and define a map \( \eta : B \to C(I) \) by \( \{b_v\}_v \mapsto (f \mapsto \sum_{v \in V} \epsilon_v(b^f_v)) \).

Theorem \( \text{(5.2)} \) means that there exists an even unimodular lattice of signature \( (r, s) \) having a semisimple \( (F, \iota) \)-isometry if and only if there exists a family \( \beta = \{b_v\}_v \) such that \( \eta(\beta) = 0 \). We define a group and map, which will be called the \textit{obstruction group} and \textit{obstruction map}, to describe when such a family exists. We remark that \( \gamma \in C(I) \) is the zero map if and only if \( \gamma \cdot c = 0 \) for all \( c \in C(I) \), where \( \gamma \cdot c := \sum_{f \in F} \gamma(f)c(f) \).

First, for two distinct elements \( f, g \in I \) we define a set \( \Pi_{f,g} \) of prime numbers as follows:
• For \( f, g \in I_1 \), a prime number \( p \) belongs to \( \Pi_{f,g} \) if and only if
  - there exist irreducible \( * \)-symmetric factors \( \phi \) and \( \psi \) of \( f \) and \( g \) in \( \mathbb{Z}_p[X] \) respectively which are divisible in \( \mathbb{F}_p[X] \) by a common irreducible, \( * \)-symmetric polynomial.

• For \( f \in I_1 \), a prime number \( p \) belongs to \( \Pi_{f,X+1} = \Pi_{X+1,f} \) if and only if
  - \( n_\pm \geq 3 \), or \( n_\pm = 2 \) and \( D_\pm \neq -1 \) in \( \mathbb{Q}_p^* / \mathbb{Q}_p^2 \) where \( D_\pm \) is defined in \([12] \); and
  - there exists an irreducible \( * \)-symmetric factor \( \phi \) of \( f \) in \( \mathbb{Z}_p[X] \) which is divisible by \( X \neq 1 \) in \( \mathbb{F}_p[X] \).

In particular, if \( n_\pm = 1 \) then \( \Pi_{f,X+1} = \Pi_{X+1,f} = \emptyset \) for all \( f \in I_1 \). If \( n_\pm = 2 \) then \( \Pi_{f,X+1} \) depends on \( \iota \) because so does \( D_\pm \).

Then, we define an equivalence relation \( \sim \) on \( C(I) \) as the one generated by the following relation \( R \):

\[
R(\gamma, \gamma') \iff \gamma' = \gamma + 1_{(f,g)} \text{ for some } f, g \in I \text{ such that } \Pi_{f,g} \text{ is not empty.}
\]

Here \( 1_H \) is the characteristic function of \( H \subset I_1 \).

**Theorem 6.2.** The image \( \im \eta \subset C(I) \) of \( \eta \) coincides with an equivalence class with respect to \( \sim \).

This theorem will be proved in the next subsection. Set \( C_\sim(I) := \{ c \in C(I) \mid c(f) = c(g) \text{ if } \Pi_{f,g} \neq \emptyset \} \). Notice that if \( \gamma \sim \gamma' \) then \( \gamma \cdot c = \gamma' \cdot c \) for any \( c \in C_\sim(I) \). Hence, the map

\[
C_\sim(I) \to \mathbb{Z}/2\mathbb{Z}, \quad c \mapsto \eta(\beta) \cdot c
\]

is defined independently of the choice of \( \beta \in \mathcal{B} \) by Theorem 6.2. The following proposition shows that this map factors through the obstruction group \( \Omega := C_\sim(I)/\{ \text{constant maps} \} \). The induced homomorphism \( \Omega \to \mathbb{Z}/2\mathbb{Z} \) is called the obstruction map and denoted by \( \ob \).

**Proposition 6.3.** We have \( \eta(\beta) \cdot 1_I = 0 \) for any \( \beta \in \mathcal{B} \).

**Proof.** Let \( B^\pm \) be an inner product on \( M^\pm \) whose Gram matrix is diag(\( D^\pm, 1, \cdots, 1 \)), and \( B^f \) an inner product on \( M^f \) defined by

\[
B^f(x,y) = \Tr_{M^f/\mathbb{Q}}(x\sigma(y)) \quad \text{for } x, y \in M^f,
\]

for each \( f \in I_1 \cup I_2 \). Then, set \( B := B^+ \oplus B^- \oplus \bigoplus_{f \in I_1} B^f \oplus \bigoplus_{f \in I_2} B^f \). Let \( \beta = \{ b_v \}_{v \in \mathcal{B}} \). By Lemma 2.1, we have \( \epsilon_v(b_v) - \sum_{f \in I} \epsilon_v(b^f_v) = \epsilon_v(B) - \sum_{f \in I} \epsilon_v(B^f) \). Summing over \( v \in V \) yields

\[
\sum_{v \in V} \epsilon_v(b_v) - \sum_{v \in V} \sum_{f \in I} \epsilon_v(b^f_v) = \sum_{v \in V} \epsilon_v(B) - \sum_{v \in V} \sum_{f \in I} \epsilon_v(B^f) = 0
\]

since \( B \) and \( B^f \) are global objects. Furthermore, we have \( \sum_{v \in V} \epsilon_v(b_v) = 0 \) because \( b_v \cong \Lambda_{r,s} \otimes \mathbb{Q}_v \) for all \( v \in V \), where \( \Lambda_{r,s} \) is an even unimodular \( \mathbb{Z} \)-lattice of signature \( (r, s) \), see [BT20] §10. Therefore we get \( \eta(\beta) \cdot 1_I = \sum_{v \in V} \sum_{f \in I} \epsilon_v(b^f_v) = 0 \).

If there exists \( \beta \in \mathcal{B} \) with \( \eta(\beta) = 0 \) then the obstruction map \( \ob : \Omega \to \mathbb{Z}/2\mathbb{Z} \) is zero. To prove Theorem 6.1 we also need the converse. We start by describing the equivalence class in \( C(I) \) containing \( 0 \). Let \( G(F) \) denote the graph such that the vertices are all elements in \( I \), and that two distinct vertices \( f, g \in I \) are joined if and only if \( \Pi_{f,g} \neq \emptyset \). Then \( C_\sim(I) \) is the set of maps in \( C(I) \) which are constant on each connected component of \( G(F) \). We remark that the graph \( G(F) \) may depend on \( \iota \).

**Proposition 6.4.** The subset \( C_\sim(I)^\perp := \{ \gamma \in C(I) \mid \gamma \cdot c = 0 \text{ for any } c \in C_\sim(I) \} \) is the equivalence class containing \( 0 \) in \( C(I) \).

**Proof.** Let \( \mathcal{E} \) denote the equivalence class containing \( 0 \). We start by showing the inclusion \( \mathcal{E} \subset C_\sim(I)^\perp \). Let \( f, g \in I \) be distinct elements with \( \Pi_{f,g} \neq \emptyset \). For any \( c \in C_\sim(I) \) we have

\[
1_{(f,g)} \cdot c = c(f) + c(g) = 2c(f) = 0,
\]

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which implies that \(1_{\{f,g\}} \in C_\infty(I)^{1/2}\). Since any element of \(E\) can be expressed as a sum of maps \(1_{\{f,g\}}\) with \(\Pi_{f,g} \neq 0\), we have \(E \subset C_\infty(I)^{1/2}\).

Let us prove the reverse inclusion \(C_\infty(I)^{1/2} \subset E\). We claim that for any nonzero \(\gamma \in C_\infty(I)^{1/2}\) there exists \(\gamma' \in C_\infty(I)^{1/2}\) such that \(\gamma' \sim \gamma\) and \(#\text{Supp}(\gamma') < #\text{Supp}(\gamma)\), where \(\text{Supp}(\gamma) := \{f \in I \mid \gamma(f) \neq 0\}\). To prove this, take an element \(f \in \text{Supp}(\gamma)\) and consider the connected component \(H \subset I\) in \(G(F)\) containing \(f\). Since \(1_H \in C_\infty(I)\) we have \(\sum_{h \in H} \gamma(h) = \gamma \cdot 1_H = 0\). This implies that \(H\) has an element \(g\) such that \(g \neq f\) and \(\gamma(g) \neq 0\). Then there exists a path from \(f\) to \(g\) in \(G(F)\) via \(h_1, h_2, \ldots, h_k \in H\), where \(\Pi_{f,h_i}, \Pi_{h_i,h_{i+1}}\) (\(i = 1, \ldots, k - 1\)), and \(\Pi_{h_k,g}\) are not empty. This means that

\[
\gamma' := \gamma + 1_{\{f,h_1\}} + 1_{\{h_1,h_2\}} + \cdots + 1_{\{h_{k-1},h_k\}} + 1_{\{h_k,g\}} \in C_\infty(I)^{1/2}
\]

is equivalent to \(\gamma\). Since \(\text{Supp}(\gamma') = \text{Supp}(\gamma) \setminus \{f, g\}\), we get \(#\text{Supp}(\gamma') = #\text{Supp}(\gamma) - 2 < #\text{Supp}(\gamma)\). This completes the proof of the claim.

Let \(\gamma \in C_\infty(I)^{1/2}\). Then we can see that \(\gamma \sim 0\) by repeatedly applying the claim proved above. Therefore we obtain \(C_\infty(I)^{1/2} \subset E\).

This proposition shows that the obstruction map vanishes only if the image \(\text{im} \eta \subset C(I)\) coincides with the equivalence class containing \(0\), or equivalently, there exists a family \(\beta \in \mathcal{B}\) with \(\eta(\beta) = 0\).

**Proof of Theorem 6.4** To summarize the discussion so far, we obtain Theorem 6.4.

---

### 6.2 Image of the map \(\eta : \mathcal{B} \to C(I)\)

The purpose of this subsection is to prove Theorem 6.2. We keep the notation of the previous subsection. For \(f \in I_1\), a prime \(p\) and \(w \in \mathcal{W}(f; p) \setminus \mathcal{W}_{sp}(f; p)\), the symbol \(f_w\) will denote the irreducible factor of \(f\) in \(\mathbb{Z}_p[X]\) corresponding to the place \(w\). Recall the classification theorem of inner products over the \(p\)-adic field \(\mathbb{Q}_p\). Two inner products over \(\mathbb{Q}_p\) are isomorphic if and only if they have the same dimension, same determinant, and same Hasse-Witt invariant, see [20, Chapter IV, Theorem 7]. We begin with the following lemma. For a prime \(p\) and an inner product \(b_p^\pm\) on \(M_p^\pm\), we write \(\partial[b_p^\pm] = \partial[M_p^\pm, b_p^\pm, \pm 1]\) for short.

**Lemma 6.5.** Let \(p\) be a prime.

(i) Let \(p \neq 2\), and let \(b_p^\pm, \hat{b}_p^\pm\) be inner products on \(M_p^\pm\) having the same determinant. Assume that \(n_p > 1\). Then \(\partial[b_p^\pm] = \partial[\hat{b}_p^\pm]\) if and only if \(\epsilon_p(b_p^\pm) = \epsilon_p(\hat{b}_p^\pm)\).

(ii) Let \(f \in I_1, w \in \mathcal{W}(f; p)\), and \(L_w, \hat{L}_w \in \text{Tw}(E_w^f, \sigma)\). If \(w \notin \mathcal{W}_{rm}(f; 2)\) then the following are equivalent:

(a) \(\lambda_w = \hat{\lambda}_w\).

(b) \(\epsilon_p(b[\lambda_w]) = \epsilon_p(b[\hat{\lambda}_w])\).

(c) \(\partial[M_w^f, b[\lambda_w], \partial_w^f] = \partial[M_w^f, b[\hat{\lambda}_w], \partial_w^f]\).

Here \(\partial_w^f\) denotes the image of \(X\) in \(M_w^f\) or \(E_w^f\).

**Proof.** (i). This follows by directly computing the value of the map \(\partial : W_I(\mathbb{Q}_p, \pm 1) \to W_I(\mathbb{F}_p, \pm 1)\) for a representative of each isomorphism class for the inner products on \(M_p^\pm\).

(ii). The implication (a) \(\Rightarrow\) (b) is obvious. If (b) holds, then \(b[\hat{\lambda}_w]\) and \(b[\lambda_w]\) are isomorphic because they have the same dimension, same determinant, and same Hasse-Witt invariant. In particular, the condition (c) holds. If \(w \notin \mathcal{W}_{rm}(f; 2)\), Theorem 6.10 implies that \(\partial[M_w^f, \partial_w^f]\) is injective. Therefore, we have (c) \(\Rightarrow\) (a).

**Proposition 6.6.** For any \(\beta \in \mathcal{B}\) and \(f, g \in I\) with \(\Pi_{f,g} \neq 0\), there exists \(\hat{\beta} \in \mathcal{B}\) such that \(\eta(\beta) + 1_{\{f,g\}} = \eta(\hat{\beta})\). In particular, the image \(\text{im} \eta \subset C(I)\) contains an equivalence class.

**Proof.** Let \(\beta = \{b_v\}_v \in \mathcal{B}\). Take \(f, g \in I\) with \(\Pi_{f,g} \neq 0\), and let \(p \in \Pi_{f,g}\). Assume that \(f, g \in I_1\). By the definition of \(\Pi_{f,g}\), there exist places \(v_0 \in \mathcal{W}(f; p) \cap \mathcal{W}_{sp}(f; p)\) and \(u_0 \in \mathcal{W}(g; p) \cap \mathcal{W}_{sp}(g; p)\) such that \(f_{u_0} \mod p\) and \(g_{u_0} \mod p\) have a common irreducible, \(\ast\)-symmetric factor \(h \in \mathbb{F}_p[X]\). Note that \(h\) is the
minimal polynomial of $\tilde{a}_{w_0}^0$ and $\tilde{a}_{u_0}^0$, which implies that there exists an irreducible representation $\chi$ of $\Gamma$ over $\mathbb{F}_p$ such that $\partial[M_{w_0}^f, b_{w_0}^f, \alpha_{w_0}^f]$ and $\partial[M_{u_0}^g, b_{u_0}^g, \alpha_{u_0}^g]$ are in $W_1(\overline{\mathbb{F}_p}; \chi)$.

According to the decompositions $M_{w_0}^f = \bigoplus_{u \in W(f,p)} M_{w_0}^f$ and $M_{u_0}^g = \bigoplus_{w \in W(g,p)} M_{u_0}^g$, we can write

\[ b_p^f = \bigoplus_{w \in W(f,p)} b_{w_0}^f, \quad b_p^g = b_0^\lambda \] for some $\lambda_{w_0} \in \text{Tw}(E_{w_0}^f, \sigma)$,

\[ b_p^g = \bigoplus_{w \in W(g,p)} b_{u_0}^g, \quad b_p^g = b_0^\lambda \] for some $\lambda_{u_0} \in \text{Tw}(E_{u_0}^g, \sigma)$

by Lemma 2.3. Let $\tilde{\lambda}_{w_0}^f \in \text{Tw}(E_{w_0}^f, \sigma)$ and $\tilde{\lambda}_{u_0}^g \in \text{Tw}(E_{u_0}^g, \sigma)$ be elements satisfying $\tilde{\lambda}_{w_0}^f \neq \lambda_{w_0}$ and $\tilde{\lambda}_{u_0}^g \neq \lambda_{u_0}$, and set

\[
\begin{align*}
\tilde{b}_p^f &:= b[\tilde{\lambda}_{w_0}^f] \oplus \bigoplus_{w \neq w_0} b_{w_0}^f, \\
\tilde{b}_p^g &:= b[\tilde{\lambda}_{u_0}^g] \oplus \bigoplus_{w \neq u_0} b_{u_0}^g, \\
\tilde{\beta} &:= \{ \tilde{b}_v \} \quad \text{where } \tilde{b}_v = b_v \text{ for } v \neq p.
\end{align*}
\]

Here $b_p^f$ and $b_p^g$ are the restrictions of $b_p$ to $M_{w_0}^f$ and $M_{u_0}^g$ respectively.

Claim 1: We have $\tilde{\beta} \in B$. Since $\tilde{b}_v = b_v$ for any $v \neq p$, each $\tilde{b}_v$ ($v \neq p$) has the properties (P1)–(P3), and $\# \{(v, f) \in V \times I \mid \varepsilon_v(b_v^f) = 1\}$ is finite. Furthermore, (P1) and (P3) are obvious for $b_p$. Therefore, it suffices to show that $\tilde{b}_p$ has the property (P2). By using Theorem 4.10 and Lemma 6.5 some calculations show that

\[
\partial[M_{w_0}^f, b[\tilde{\lambda}_{w_0}^f], \alpha_{w_0}^f] + \partial[M_{u_0}^g, b[\tilde{\lambda}_{u_0}^g], \alpha_{u_0}^g] - \partial[M_{w_0}^f, b[\lambda_{w_0}], \alpha_{w_0}^f] - \partial[M_{u_0}^g, b[\lambda_{u_0}], \alpha_{u_0}^g] = 0.
\]

Then

\[
\partial[M_p, \tilde{b}_p, \alpha] = \partial[M_p, b_p, \alpha] - \partial[M_{w_0}^f, b[\tilde{\lambda}_{w_0}^f], \alpha_{w_0}^f] - \partial[M_{u_0}^g, b[\tilde{\lambda}_{u_0}^g], \alpha_{u_0}^g] \\
+ \partial[M_{w_0}^f, b[\lambda_{w_0}], \alpha_{w_0}^f] + \partial[M_{u_0}^g, b[\lambda_{u_0}], \alpha_{u_0}^g] = 0,
\]

which implies that $(M_p, \tilde{b}_p)$ contains an $\alpha$-stable unimodular lattice by Theorem 3.3. If $p$ is odd then the lattice is even and we are done. Let $p = 2$. We use Proposition 4.11. The condition (i) of Proposition 4.11 has already been proved. The condition (ii) follows from the formula (7). For the condition (ii), it is sufficient to show that $\tilde{b}_2 \cong b_2$. We have

\[
\varepsilon_2(\tilde{b}_2) - \varepsilon_2(b_2) = \varepsilon_2(b[\tilde{\lambda}_{w_0}^f]) + \varepsilon_2(b[\tilde{\lambda}_{u_0}^g]) - \varepsilon_2(b[\lambda_{w_0}^f]) = 0
\]

by Lemmas 2.3 and 6.3 (even if $w_0 \in W_{x}^f(1; 2)$ or $u_0 \in W_{x}^g(2; 2)$). Hence $\tilde{b}_2$ and $b_2$ have the same dimension, same determinant, and same Hasse-Witt invariant, which means that $\tilde{b}_2 \cong b_2$ as required.

Therefore $(M_2, 2)$ contains an $\alpha$-stable even unimodular lattice, that is, $b_2$ has the property (P2). This completes the proof of Claim 1.

Claim 2: We have $\eta(\beta) + 1_{(f, g)} = \eta(\tilde{\beta})$. It is obvious that $\eta(\beta)(k) = \eta(\tilde{\beta})(k)$ for $k \neq f, g$, and we have

\[
\eta(\beta)(f) - \eta(\beta)(g) = \varepsilon_p(\tilde{b}_p^f) - \varepsilon_p(\tilde{b}_p^g) = \varepsilon_p(b_{\tilde{\lambda}_{w_0}^f}) - \varepsilon_p(b_{\lambda_{w_0}^f}) = 1.
\]

The same calculation yields $\eta(\tilde{\beta})(g) - \eta(\tilde{\beta})(g) = 1$, and thus we arrive at the claim.

The proof for the case $f, g \in I_1$ has been completed now. Let $f(X) = X \neq 1$. In this case, define an inner product $b_p^f = b_p^k$ to satisfy $\det b_p^k = D_\pm$ and $\varepsilon_p(b_p^k) \neq \varepsilon_p(b_p^k)$, and set

\[
\tilde{b}_p := \tilde{b}_p^k \oplus b_p^k \oplus \bigoplus_{k \in I \setminus (f, g)} b_p^k \oplus b_p^k \quad \text{and} \quad \tilde{\beta} := \{ \tilde{b}_v \} \quad \text{where } \tilde{b}_v = b_v \text{ for } v \neq p.
\]

Here $\tilde{b}_p^k$ is defined as in (7) if $g \in I_1$ and as above if $g \in I_0$. Then the above two claims hold similarly and the proof is complete.
Proposition 6.7. The image \( \eta \subset C(1) \) is contained in an equivalence class. In other words \( \eta(\beta) \sim \eta(\tilde{\beta}) \) for any \( \beta, \tilde{\beta} \in \mathcal{B} \).

Proof. Let \( \beta = \{b_v\}_v, \tilde{\beta} = \{\tilde{b}_v\}_v \in \mathcal{B} \), and set \( V(\beta, \tilde{\beta}) := \{v \in V \mid \eta_v(\beta) \neq \eta_v(\tilde{\beta})\} \), where \( \eta_v(\beta) = (f \mapsto \varepsilon_v(b'_v)) \in C(1) \). Note that \( \eta(\beta)(f) = \sum_{v \in V} \eta_v(\beta)(f) \) for any \( \beta \in \mathcal{B} \) and \( f \in \mathcal{I} \). Since almost all \( \varepsilon_v(b'_v) \) and \( \varepsilon_v(\tilde{b}'_v) \) are trivial and \( b_\infty \cong b_\infty \), the set \( V(\beta, \tilde{\beta}) \) is a finite set of primes. If \( \# V(\beta, \tilde{\beta}) = 0 \), then \( \eta(\beta) = \eta(\tilde{\beta}) \) and there is nothing to prove. By induction on \( \# V(\beta, \tilde{\beta}) \), it is sufficient to show that there exists \( \tilde{\beta} \in \mathcal{B} \) such that \( \eta(\tilde{\beta}) \sim \eta(\beta) \) and \( V(\beta, \tilde{\beta}) \subset V(\beta, \tilde{\beta}) \).

Claim 1: There exists \( \tilde{\beta} = \{\tilde{b}_v\}_v \in \mathcal{B} \) such that \( \eta(\tilde{\beta}) \sim \eta(\beta) \) and \( \tilde{\beta} \subset V(\beta, \tilde{\beta}) \). Let \( p \in V(\beta, \tilde{\beta}) \).

Then \( \tilde{\beta} \) belongs to \( \mathcal{B} \) as in Claim 1 of the proof of Proposition 6.6. Furthermore

\[
\eta(\tilde{\beta})(X - 1) - \eta(\beta)(X - 1) = \varepsilon_p(\tilde{b}'_p) - \varepsilon_p(b'_p) = 1, \\
\eta(\tilde{\beta})(f) - \eta(\beta)(f) = \varepsilon_p(\tilde{b}'_p) - \varepsilon_p(b'_p) = 1
\]

by Lemma 6.5 and \( \eta(\tilde{\beta})(k) = \eta(\beta)(k) \) for all \( k \in \mathcal{I} \setminus \{X - 1, f\} \). These equations mean that \( \eta(\tilde{\beta}) = \eta(\beta) + 1_{X - 1, f} \). Since \( \Pi_{X - 1, f} \) contains the prime \( p \) and \( p \) is non-empty, we get \( \eta(\tilde{\beta}) \sim \eta(\beta) \). It is clear that \( V(\beta, \tilde{\beta}) \subset V(\beta, \tilde{\beta}) \) and \( \partial[b'_p] = \partial[\tilde{b}'_p] \) by the definition of \( \tilde{\beta} \). If \( \partial[b'_p] \neq \partial[\tilde{b}'_p] \) then we repeat a similar procedure to obtain \( \partial[b'_p] = \partial[\tilde{b}'_p] \). The proof of Claim 1 is complete.

We assume that \( \partial[b'_p] = \partial[\tilde{b}'_p] \) and \( \partial[b'_p] = \partial[\tilde{b}'_p] \) without loss of generality by Claim 1. Set \( D_p(\beta, \tilde{\beta}) := \bigcup_{f \in \mathcal{I}_1} \{w \in V(f; p) \mid \partial[w'_w] \neq \partial[\tilde{w}'_w]\} \).

Claim 2: There exists \( \tilde{\beta} \in \mathcal{B} \) such that \( \eta(\tilde{\beta}) \sim \eta(\beta), V(\beta, \tilde{\beta}) \subset V(\beta, \tilde{\beta}) \) and \( D_p(\beta, \tilde{\beta}) = \emptyset \). Use induction on \( \# D_p(\beta, \tilde{\beta}) \), the case \( \# D_p(\beta, \tilde{\beta}) = 0 \) being obvious. Let \( \# D_p(\beta, \tilde{\beta}) > 0 \), and choose \( f \in \mathcal{I}_1 \) and \( w_0 \in V(f; p) \) satisfying \( \partial[w'_w] \neq \partial[\tilde{w}'_w] \). There is an irreducible representation \( \chi \) of \( \mathcal{G} \) such that \( \partial[w'_w], \partial[\tilde{w}'_w] \subset W(\mathbb{F}_p; \chi) \). Since the images of \( \partial[M_p, b_p, \alpha] \) and \( \partial[M_p, \tilde{b}_p, \alpha] \) under the projection \( W(\mathbb{F}_p) \to W(\mathbb{F}_p; \chi) \) are the trivial class, we can take \( g \in \mathcal{I}_1 \) and \( u_0 \in V(g; p) \) satisfying \( \partial[u'_u] \neq \partial[\tilde{u}'_u] \) in \( W(\mathbb{F}_p; \chi) \). Set

\[
\tilde{b}'_w := b'_w + \bigoplus_{w \neq u} b'_w, \\
\tilde{b}'_w := \tilde{w}'_w + \bigoplus_{w \neq u} b'_w, \\
b'_p := b'_p + \bigoplus_{w \neq u} b'_w \oplus b'_w, \\
b'_p := \tilde{b}'_w + \bigoplus_{w \neq u} b'_w \oplus b'_w, \\
\tilde{\beta} := \{\tilde{b}_v\}_v, \quad \text{where } \tilde{b}_v = b_v \text{ for } v \neq p.
\]

then we have \( \tilde{\beta} \in \mathcal{B}, \eta(\tilde{\beta}) = \eta(\beta) + 1_{f,g} \sim \eta(\beta), \) and \( V(\beta, \tilde{\beta}) \subset V(\beta, \tilde{\beta}) \) as in Claim 1. Because \( \# D_p(\beta, \tilde{\beta}) = \# D_p(\beta, \tilde{\beta}) - 2 \), we arrive at the claim by induction.

Now we take \( \tilde{\beta} \in \mathcal{B} \) mentioned in Claim 2. If \( p \neq 2 \) then the equation \( \eta_\beta(\tilde{\beta}) = \eta_p(\tilde{\beta}) \) follows from \( \partial[b'_p] = \partial[\tilde{b}'_p] \), \( \partial[b'_p] = \partial[\tilde{b}'_p] \), \( D_p(\beta, \tilde{\beta}) = \emptyset \), and Lemma 6.5. Therefore we get \( \eta(\tilde{\beta}) \sim \eta(\beta) \) and \( V(\beta, \tilde{\beta}) \subset V(\beta, \tilde{\beta}) \).
Let $p = 2$. If $\eta_2(\tilde{\beta}) \neq \eta_2(\hat{\beta})$ then $V(\tilde{\beta}, \hat{\beta}) \subset V(\beta, \hat{\beta}) \setminus \{2\} \subset V(\beta, \hat{\beta})$ and we are done. If $\eta_2(\tilde{\beta}) \neq \eta_2(\hat{\beta})$ then assume that $\beta = \tilde{\beta}$ without loss of generality. Then there exists $f \in I$ such that $\eta_2(\beta)(f) \neq \eta_2(\hat{\beta})(f)$. Suppose that $f \in I_1$. Then we have

$$1 = \epsilon_2(\hat{b}_2) - \epsilon_2(b_2) = \sum_{w \in \mathcal{W}(f; 2)} \epsilon_2(\hat{b}_{w, 2}) - \sum_{w \in \mathcal{W}(f; 2)} \epsilon_2(b_{w, 2}),$$

and this implies that there exists $w_0 \in \mathcal{W}(f; 2)$ satisfying $\epsilon_2(\hat{b}_{w_0}) \neq \epsilon_2(b_{w_0})$. Moreover, the place $w_0$ must be in $\mathcal{W}_{rm}(f; 2)$ by Lemma 6.5. In particular $\partial[b_{w_0}], \partial[\hat{b}_{w_0}] \in \mathcal{W}_{r, s}(F; 2; 1)$. On the other hand, the equation

$$0 = \epsilon_2(\hat{b}_2) - \epsilon_2(b_2) = \sum_{k \in \mathcal{I}} \epsilon_2(\hat{b}_k) - \sum_{k \in \mathcal{I}} \epsilon_2(b_k)$$

implies that there exists $g \in I$ such that $\epsilon_2(\hat{b}_2) \neq \epsilon_2(b_2)$. Suppose that $g \in I_1$, then there exists $w_0 \in \mathcal{W}_{rm}(g; 2)$ satisfying $\epsilon_2(\hat{b}_{w_0}) \neq \epsilon_2(b_{w_0})$ as above. Once again, we define $\hat{\beta}$ as in (13). In the case $f \in I_0$ or $g \in I_0$, we also define $\beta$ as in (13). Then $\beta \in \mathcal{B}$ and $\eta(\beta) \sim \eta(\beta)$ as before. Moreover, because

$$\{k \in I \mid \eta_2(\beta)(k) \neq \eta_2(\hat{\beta})(k)\} = \{k \in I \mid \eta_2(\beta)(k) \neq \eta_2(\hat{\beta})(k)\} \setminus \{f, g\},$$

we can assume that $\eta_2(\beta) = \eta_2(\hat{\beta})$ by induction. Thus, we obtain $V(\tilde{\beta}, \hat{\beta}) \subset V(\beta, \hat{\beta}) \setminus \{2\} \subset V(\beta, \hat{\beta})$, which completes the proof.

Proof of Theorem 6.2: Propositions 6.6 and 6.7 mean that the image of $\eta$ coincides with an equivalence class. This is Theorem 6.2. □

6.3 Applications

As an application of Theorem 1.3, we begin with the following theorem, which plays a crucial role in proving Theorem 1.3.

**Theorem 6.8.** Let $r$ and $s$ be non-negative integers with $r \equiv s \mod 8$, and $S \in \mathbb{Z}[X]$ an irreducible *-symmetric polynomial of degree $r + s - 2$. Assume that $F(X) := (X - 1)(X + 1)S(X)$ satisfies the condition $[\text{Sign}]$. Then there exists an even unimodular lattice of signature $(r, s)$ having a semisimple $(F, \iota)$-isometry for any $\iota \in \text{Id}_{r, s}(F)$.

**Proof.** Fix $\iota \in \text{Id}_{r, s}(F)$ arbitrarily, and let $\beta = \{b_v\}_v \in \mathcal{B}$. Note that we have $I = \{X - 1, X + 1, S\}$. For each $v \in \mathcal{V}$, we have $e_v(b_v) = 0$ since $b_v$ is 1-dimensional. This implies that $\eta(\beta)(X - 1) = \eta(\beta)(X + 1) = 0$, and furthermore $\eta(\beta)(S) = \eta(\beta) - 1 = 0$ by Proposition 6.3. Therefore $\eta(\beta) = 0$, and the obstruction map vanishes. This means that there exists an even unimodular lattice of signature $(r, s)$ having a semisimple $(F, \iota)$-isometry by Theorem 1.1. □

The rest of this section is devoted to the proof of Theorem 1.2.

**Lemma 6.9.** Let $f \in \mathbb{Z}[X]$ be a +1-symmetric polynomial with $f(1)f(-1) \neq 0$, and $p$ a prime. If $f$ has no irreducible *-symmetric factor in $\mathbb{Z}_p[X]$ which is divisible by $X \mp 1$ in $\mathbb{F}_p[X]$, then $v_p(f(\pm 1)) \equiv 0 \mod 2$, and moreover, if $p = 2$ then $(1)^{(\deg f)/2}f(1)f(-1) = 1$ or $-3$ in $\mathbb{Q}_2^\times/\mathbb{Q}_2^\times 2$.

**Proof.** We have a decomposition $f(X) = g(X)h(X)$ in $\mathbb{Z}_p[X]$ such that $(g \mod p) = (X \mp 1)^{\deg g}$ in $\mathbb{F}_p[X]$ and $h(\pm 1) \not\equiv 0 \mod p$. The assumption of the lemma means that $g$ is of type 2 (in $\mathbb{Z}_p[X]$), so $g$ is expressed as $g = k^* k$ for some $k \in \mathbb{Z}_p[X]$. In this case we have

$$v_p(g(\pm 1)) = v_p(k(\pm 1)k^*(\pm 1)) = v_p(k(\pm 1)k(0)^{-1}(\pm 1)^{\deg k}k(1)) \equiv 0 \mod 2$$

and thus

$$v_p(f(\pm 1)) = v_p(g(\pm 1)) \equiv 0 \mod 2.$$

Moreover, if $p = 2$ then

$$(1)^{(\deg g)/2}g(1)g(-1) = (-1)^{(\deg g)/2}k(1)k^*(1)k(-1)k^*(-1) = 1 \text{ in } \mathbb{Q}_2^\times/\mathbb{Q}_2^\times 2.$$
Let $\phi \in \mathbb{Z}_2[X]$ be the trace polynomial of $h$, that is, the monic polynomial defined by the equation $h(X) = X^{(\deg h)/2} \phi(X + X^{-1})$. Then we have

$$h(1) - (-1)^{(\deg h)/2}h(-1) = \phi(2) - \phi(-2) \equiv 0 \mod 4$$

and thus $(-1)^{(\deg h)/2}h(1)h(-1) \equiv h(1)^2 \equiv 1 \mod 4$. This means that $(-1)^{(\deg h)/2}h(1)h(-1) = 1$ or $-3$ in $\mathbb{Q}_2^* / \mathbb{Q}_2^*$. Since $g$ is of type 2, we obtain

$$(-1)^{(\deg f)/2}f(1)f(-1) = (-1)^{(\deg g)/2}g(1)g(-1) \cdot (-1)^{(\deg h)/2}h(1)h(-1) = 1 \text{ or } -3$$
in $\mathbb{Q}_2^* / \mathbb{Q}_2^*$. \qed

Let $F \in \mathbb{Z}[X]$ be a *-symmetric polynomial of even degree $2n$ with the condition \textbf{Square}, and assume that $n_+ \neq 1$ and $n_- \neq 1$, where $n_{\pm}$ denotes the multiplicity of $X \mp 1$ in $F$. In addition, let $r,s \in \mathbb{Z}_{\geq 0}$, $\ell \in \text{Id}_{x_{\pm}}(F)$ and $D_{\pm}$ be as in \textbf{Square}. For a subset $H = \{f_1, \ldots, f_{\ell}\} \subset I$, we may identify $H$ with the polynomial $\prod_{j=1}^{\ell} f_j^{n_j}(X)$, where $n_j$ is the multiplicity of $f_j$ in $F$. In particular, we write $H(\pm 1) = \prod_{j=1}^{\ell} f_j^{n_j}(\pm 1)$ and $\deg H = \sum_{j=1}^{\ell} n_j \deg f_j$. Recall that we define the graph $G(F)$ in \textbf{Square}.

**Proposition 6.10.** Let $F$ and $\ell$ be as above. Each connected component of $G(F)$ has even degree and satisfies the condition \textbf{Square}.

**Proof.** We begin with the case $F(1)F(-1) \neq 0$. Let $H$ be a connected component of $G(F)$. The degree of $H$ is even since $H$ has no type 0 component. First, suppose that $|H(\pm 1)|$ are not a square. Then there would exist a prime $p$ such that $v_p(H(\pm 1)) \equiv 1 \mod 2$. Furthermore, there would exist $g \in I \setminus H$ satisfying $v_p(g(\pm 1)) \equiv 1 \mod 2$ because $v_p(F(\pm 1))$ is even. Therefore, Lemma \textbf{Square} shows that $H$ and $g$ have irreducible *-symmetric factors in $\mathbb{Z}[X]$ which are divisible by $X \mp 1$ in $\mathbb{Z}[X]$. This means that there exists $f \in H$ such that $p \not\in f \setminus g$. Hence the factor $g$ is connected to $H$ in $G(F)$, that is, $g \in H$. This is a contradiction, so $|H(\pm 1)|$ and $|H(-1)|$ are squares.

Next, suppose that $|H(\pm 1)|$ and $|H(-1)|$ are squares, we would have $(-1)^{(\deg H)/2}H(1)H(-1) = 1 \mod 2$. On the other hand, there exists $g \in I \setminus H$ such that $(-1)^{(\deg H)/2}g(1)g(-1) \not\in \{1, -3\}$ in $\mathbb{Q}_2^* / \mathbb{Q}_2^*$ because $(-1)^n F(F(-1))$ is a square (in Q, and thus in Q2). These mean that $H$ and $g$ have an irreducible *-symmetric factors in $\mathbb{Z}[X]$ which are divisible by $X - 1$ in $\mathbb{F}_2[X]$. By Lemma \textbf{Square} we have $|H(\pm 1)|$ and $|H(-1)|$ are squares. Thus $g$ is connected to $H$ in $G(F)$. This is a contradiction, so $(-1)^{(\deg H)/2}H(1)H(-1)$ is a square. We have now proved the case $F(1)F(-1) \neq 0$.

Let us proceed to the case $F(1)F(-1) = 0$. We denote by $G(F)'$ the graph obtained by removing $X - 1$ and $X + 1$ from $G(F)$. Let $H$ be a connected component of $G(F)'$, and $H$ the connected component of $G(F)$ containing $K$. From the assumption that $n_+ \neq 1$ and $n_- \neq 1$, it follows that deg $H$ is even. If $K$ satisfies \textbf{Square} then so does $H$, and we are done. Let $|K(\pm 1)|$ be not a square. Then $v_p(K(\pm 1)) \equiv 1 \mod 2$ for some prime $p$. Notice that there is no connected component $K'$ of $G(F)'$ satisfying $v_p(K'(\pm 1)) \equiv 1 \mod 2$, since otherwise $K'$ would be connected to $K$. If $n_{\pm} \geq 3$ then we see easily that $X \mp 1$ is connected to $K$ in $G(F)$ by Lemma \textbf{Square}. This means that $X \mp 1 \not\in H$, and hence $H(\pm 1) = 0$ is a square. If $n_{\pm} = 2$ then

$$v_p(D_{\pm}) = v_p(F_{12}(\pm 1)) \equiv v_p(K(\pm 1)) \equiv 1 \mod 2,$$

and in particular, $D_{\pm} \not\equiv 1$ in $\mathbb{Q}_2^* / \mathbb{Q}_2^*$. Therefore $X \mp 1$ is connected to $K$ in $G(F)$, and $H(\pm 1) = 0$ is a square. To summarize, if $|K(1)|$ or $|K(-1)|$ is not a square then $H$ satisfies the condition \textbf{Square}.

The remaining case is where $|K(1)|$ and $|K(-1)|$ are squares but $(-1)^{(\deg K)/2}K(1)K(-1) = -1 \mod 2$. If $n_+ \geq 3$ or $n_- \geq 3$ then Lemma \textbf{Square} implies that $2 \in \Pi_{X - 1,f}$ or $2 \in \Pi_{X + 1,f}$ for some $f \in K$, and thus $X - 1 \not\in H$ or $X + 1 \not\in H$. Hence $H$ satisfies the condition \textbf{Square}, and we are done. Let $(n_+, n_-) = (2, 2), (2, 0)$ or $(0, 2)$, and let $K'$ be the complement of $K$ in $G(F)'$. Lemma \textbf{Square} implies that $(-1)^{(\deg K')/2}K'(1)K'^{-1}(-1) = 1$ or $-3$ in $\mathbb{Q}_2^* / \mathbb{Q}_2^*$, since otherwise $K'$ would be connected to $K$. Hence

$$(-1)^{(\deg F_{12})/2}F_{12}(1)F_{12}(-1) = (-1)^{(\deg K)/2}K(1)K(-1) \cdot (-1)^{(\deg K')/2}K'^{-1}(1)K'^{-1}(-1) = -1 \text{ or } 3$$
in $\mathbb{Q}_2^X/\mathbb{Q}_2^Z$, which implies that
\[
D_+D_- = (-1)^{(n_+ - 1)(X-1)/2}(-1)^{(n_+ - 1)(X+1)/2} |F_{12}(1)F_{12}(-1)|
\]
\[
= (-1)^{1}F_{12}(1)F_{12}(-1)
\]
\[
= (-1)^{1}F_{12}(1)F_{12}(-1)
\]
\[
= (-1)^{(n_+ - n_-)/2}(-1)^{(\deg F_{12})/2}F_{12}(1)F_{12}(-1)
\]
\[
= \begin{cases} 
1 	ext{ or } -3 & \text{if } (n_+, n_-) = (2, 0) \text{ or } (0, 2) \\
-1 	ext{ or } 3 & \text{if } (n_+, n_-) = (2, 2) 
\end{cases}
\]
in $\mathbb{Q}_2^X/\mathbb{Q}_2^Z$. In particular, in $\mathbb{Q}_2^X/\mathbb{Q}_2^Z$, we have $D_+ \neq -1$ if $(n_+, n_-) = (2, 0)$ and $D_- \neq -1$ if $(n_+, n_-) = (0, 2)$, and $D_+ \neq -1$ or $D_+ \neq -1$ if $(n_+, n_-) = (2, 2)$. These imply that $X - 1 \in H$ or $X + 1 \in H$, and therefore, the component $H$ satisfies the condition \text{[Square]}.

For a polynomial $f \in \mathbb{Z}[X]$ with $f(1)f(-1) \neq 0$, let $e(f) \in \{1, -1\}$ denote the sign of $(-1)^{(\deg f)/2} f(1)f(-1)$. We remark that if $f$ satisfies \text{[Square]} then $e(f) = 1$.

\textbf{Lemma 6.11.} Let $F$ be as above. Then $F$ satisfies the condition \text{[Sign]} for $(r, s) = (n, n)$. Moreover, a map $i_0$ from $I_0(\mathbb{R})$ to $Z$ satisfying the condition \text{[1]} in \text{[2]} and the equation $i_0(X - 1) + i_0(X + 1) = e(F_{12}) - 1$ can be prolonged to an index map in $\text{Idx}_{n,n}(F)$. Here, if $X \pm 1 \notin I_0(\mathbb{R})$ then we understand that $i_0(X \mp 1) = 0$.

\textbf{Proof.} The inequality $n \geq m(F)$ is obvious. Let $\phi$ be the trace polynomial of $F_{12}$. We remark that there is a 1:2 correspondence between the roots of $\phi$ which are in the interval $(-2, 2)$ and those of $F_{12}$ which lie on the unit circle. Because the sign of $\phi(2)\phi(-2) = e(F_{12})$, the number of roots of $F_{12}$ which lie on the unit circle is $1 - e(F_{12}) \mod 4$. Hence, if $F(1)F(-1) \neq 0$ (i.e. $F = F_{12}$) then
\[
m(F) \equiv n - (1 - e(F_{12})) \equiv n \mod 2.
\]
This means that $F$ satisfies \text{[Sign]} for $(n, n)$.

Moreover, the fact that the number of roots of $F_{12}$ which lie on the unit circle is $1 - e(F_{12}) \mod 4$ implies that we can define a map $i_1 : I_1(\mathbb{R}) \to Z$ satisfying the condition \text{[2]} in \text{[3]} and the equation $i_0(X - 1) + i_0(X + 1) = e(F_{12}) - 1$ can be prolonged to an index map in $\text{Idx}_{n,n}(F)$ because this map satisfies equation \text{[3]} for $(r, s) = (n, n)$.

Note that the graph $G(F)$ is determined by the polynomial $F$ and values $i(X - 1)$ and $i(X + 1)$ where $i$ is an index map, even if $n_+ = 2$ or $n_- = 2$. We now prove Theorem 1.2.

\textbf{Proof of Theorem 1.2.} It is sufficient to show that we can define an index map in $\text{Idx}_{n,n}(F)$ such that the following holds for each connected component $H$ of $G(F)$:
\[
H \text{ is realized as the characteristic polynomial of a semisimple isometry}
\]
of an even unimodular lattice of signature $(d, d)$, where $d := (\deg H)/2$.

We begin with the case where $n_+ \neq 2$ and $n_- \neq 2$. In this case, the graph $G(F)$ depends on $F$ only, so does the set of connected components of $G(F)$. Let $H$ be a connected component of $G(F)$, and put $d = (\deg H)/2$. Proposition 6.10 implies that $H$ satisfies the condition \text{[Square]}, and Lemma 6.11 implies that $H$ satisfies the condition \text{[Sign]} for $(d, d)$. The obstruction group of $H$ is trivial for any index map since $H$ is connected, and then we have \text{[1]} by Theorem 1.1. Therefore, if we choose $i_H \in \text{Idx}_{d,d}(H)$ for each connected component $H$, then the sum $\oplus_H i_H \in \text{Idx}_{n,n}(F)$ is the required index map.

We proceed to the case $n_+ = 2$ or $n_- = 2$. We deal with the case $n_+ = 2$ only because the case $n_- = 2$ is similar. For $i_0 : I_0(\mathbb{R}) \rightarrow Z$ mentioned in Lemma 6.11, let $G(F;i_0)$ denote the graph determined from $(F, i)$ for an extension $i \in \text{Idx}_{n,n}(F)$ of $i_0$, which is independent of the choice of $i$. If we can define $i_0 : I_0(\mathbb{R}) \rightarrow Z$ mentioned in Lemma 6.11 so that \text{[1]} holds for the connected components containing $X - 1$ and $X + 1$ of $G(F;i_0)$ respectively, then, as above, \text{[2]} holds for the others and we obtain the required index map. Therefore, it suffices to define such a map $I_0(\mathbb{R}) \rightarrow Z$. We remark that the connected component containing $X - 1$ and that containing $X - 1$ may coincide.
First, let \(n_+ = 2\) and \(n_- = 2\). If \(|F_{12}(1)| = 1\) in \(\mathbb{Q}_2^\times /\mathbb{Q}_2^\times 2\), we set \(t_0(X - 1) = -2\) and \(t_0(X + 1) = e(F_{12}) + 1\). Then \(D_+ = 1 \neq 1\) in \(\mathbb{Q}_2^\times /\mathbb{Q}_2^\times 2\), and in particular \(2 \in \Pi_{-1,1,1}\). Thus \(X - 1\) and \(X + 1\) are in a common connected component \(H\) of \(G(F; t_0)\). Because any connected component \(H^0\) other than \(H\) satisfies (Square) by Proposition 6.10 and in particular \(e(H^0) = 1\), we have \(e(H^0) = e(F_{12})\) where \(H^0 = H \setminus \{X - 1, X + 1\}\). Therefore \(t_0\) can be prolonged to a map in \(\text{Idx}_{d,d}(H)\) by Lemma 6.11 where \(d = (\deg H)/2\). Then (3) holds for \(H\) since the obstruction group of \(H\) is trivial. If \(|F_{12}(1)| \neq 1\) in \(\mathbb{Q}_2^\times /\mathbb{Q}_2^\times 2\), we set \(t_0(X - 1) = 0\) and \(t_0(X + 1) = e(F_{12}) - 1\). Then, as above, the set \(\Pi_{-1,1,1}\) has the prime 2, and we obtain (3) for the connected component containing \(X - 1\) and \(X + 1\). Thus we are done.

Next, let \(n_+ = 2\) and \(n_- = 0\). In this case we define \(t_0(X - 1) = e(F_{12}) - 1\). If \(H\) is the connected component containing \(X - 1\), then we get \(e(H \setminus \{X - 1\}) = e(F_{12})\) as above. Moreover \(t_0\) can be prolonged to a map in \(\text{Idx}_{d,d}(H)\), and the condition (3) holds for \(H\) as above. Therefore we are done.

Finally, let \(n_+ = 2\) and \(n_- = 2\). If \(|F_{12}(-1)|\) is neither 1 nor \(-1\) in \(\mathbb{Q}_2^\times /\mathbb{Q}_2^\times 2\), we set

\[
t_0(X - 1) = -2 \quad \text{and} \quad t_0(X + 1) = e(F_{12}) + 1 \quad \text{if} \quad |F_{12}(1)| = 1 \quad \text{in} \quad \mathbb{Q}_2^\times /\mathbb{Q}_2^\times 2,
\]

\[
t_0(X - 1) = 0 \quad \text{and} \quad t_0(X + 1) = e(F_{12}) - 1 \quad \text{if} \quad |F_{12}(1)| \neq 1 \quad \text{in} \quad \mathbb{Q}_2^\times /\mathbb{Q}_2^\times 2.
\]

Then we can get \(D_+ \neq -1\) and \(D_- \neq -1\) in \(\mathbb{Q}_2^\times /\mathbb{Q}_2^\times 2\) and \(2 \in \Pi_{-1,1,1}\). This implies as above that (3) holds for the connected component containing \(X - 1\) and \(X + 1\). We can also finish the proof similarly in the case where \(|F_{12}(1)| \neq 1\) in \(\mathbb{Q}_2^\times /\mathbb{Q}_2^\times 2\).

Suppose that \(|F_{12}(1)| = 1\) and \(|F_{12}(-1)| = 1\) in \(\mathbb{Q}_2^\times /\mathbb{Q}_2^\times 2\). We only deal with this case, because one can finish the proof similarly in the remaining cases, i.e., the cases \((|F_{12}(1)|, |F_{12}(-1)|) = (1, -1), (-1, 1)\), or \((-1, -1)\) in \(\mathbb{Q}_2^\times /\mathbb{Q}_2^\times 2\). If \(|F_{12}(1)| = 1\), we set \(t_0(X - 1) = 2\) and \(t_0(X + 1) = -2\). Then, as above, we have \(2 \in \Pi_{-1,1,1}\) and (3) holds for the connected component containing \(X - 1\) and \(X + 1\). Let \(e(F_{12}) = -1\), and let \(G(F)'\) denote the graph obtained by removing \(X - 1\) and \(X + 1\) from \(G(F)\). We remark that \(G(F)\) may depend on the choice of \(t_0\) but \(G(F)\) is independent of it. As discussed in the proof of Proposition 6.10 if \(K\) is a connected component of \(G(F)\) such that \(|K(\pm 1)|\) is not a square then \(X \mp 1\) is connected to \(K\) in \(G(F)\) independently of the choice of \(t_0\). Let \(K_+\) denote the union of connected components \(K \subset I_1\) of \(G(F)\) such that \(|K(\pm 1)|\) is not a square. If \(K_+ \cap K_- \neq \emptyset\) then there exists a connected component \(K\) which contains an element of \(K_+ \cap K_-\). In this case, \(X - 1\) and \(X + 1\) are connected via \(K\) and contained in a common connected component \(H\) of \(G(F)\) (independently of the choice of \(t_0\)). Hence we obtain (3) for \(H\) as required. Assume that \(K_+ \cap K_- = \emptyset\), and set \(K_0 := I_1 \setminus (K_+ \cup K_-)\). There are the following four cases:

(a) \(e(K_+) = 1, e(K_0) = 1\) and \(e(K_-) = -1\).

(b) \(e(K_+) = 1, e(K_0) = -1\) and \(e(K_-) = 1\).

(c) \(e(K_+) = -1, e(K_0) = 1\) and \(e(K_-) = 1\).

(d) \(e(K_+) = -1, e(K_0) = -1\) and \(e(K_-) = -1\).

Notice that if \(e(K_0) = -1\), that is, in the cases (b) and (d), then it follows from Lemma 6.9 that each connected component \(K\) of \(K_0\) has an irreducible \(s\)-symmetric factor in \(\mathbb{Z}_2[X]\) which is divisible by \(X - 1\) in \(\mathbb{F}_2[X]\). Moreover, if we set \(t_0(X + 1) = -2\) then \(X + 1\) and \(K\) are connected in \(G(F; t_0)\) since \(D_- = 1 \neq -1\) in \(\mathbb{Q}_2^\times /\mathbb{Q}_2^\times 2\).

In the case (a), (b) or (c), we set

\[
t_0(X - 1) = \begin{cases} 0 & \text{in the case (a) or (b)} \\ -2 & \text{in the case (c)} \end{cases} \quad \text{and} \quad t_0(X + 1) = \begin{cases} -2 & \text{in the case (a) or (b)} \\ 0 & \text{in the case (c)} \end{cases}.
\]

If \(H_+\) and \(H_-\) denote the connected components of \(G(F; t_0)\) containing \(X - 1\) and \(X + 1\) respectively (\(H_+\) and \(H_-\) may coincide), then Lemma 6.11 shows that \(t_0\) can be prolonged to index maps in \(\text{Idx}_{d,d}(H_+)\) and in \(\text{Idx}_{d,d}(H_-)\) respectively, where \(d_\pm = \deg H_\pm\). Therefore, we can obtain (3) for \(H_+\) and \(H_-\), and we are done.

Let us proceed to the case (d), and let \(\phi\) be the trace polynomial of \(K_+\). We remark that \(\phi(2)\) and \(\phi(-2)\) have different signs since \(e(K_+) = -1\), and that

\[
|\phi(2)| = |K_+(1)| = |F_{12}(1)| = 1 \quad \text{in} \quad \mathbb{Q}_2^\times /\mathbb{Q}_2^\times 2 \quad \text{and}
\]

\[
|\phi(-2)| = |K_+(1)| = 1 \quad \text{is a square in} \quad \mathbb{Q}_2^\times .
\]
These imply that
\[ (-1)^{(\deg K_+)}/2 K_+(1) K_+(-1) = \phi(2) \phi(-2) = -|\phi(2)\phi(-2)| = -1 \quad \text{in} \quad \mathbb{Q}^+_2 / \mathbb{Q}^*_2. \]
Then, Lemma 6.9 shows that there are connected components \( K \) of \( K_+ \) and \( K' \) of \( K_0 \) such that \( K \) and \( K' \) are connected in \( G(F)' \). Now, we define \( i_0 : I_0(\mathbb{R}) \to \mathbb{Z} \) by \( i_0(X - 1) = 0 \) and \( i_0(X + 1) = -2 \). Then \( X - 1, K, K' \) and \( X + 1 \) are connected and contained in a common connected component of \( G(F; i_0) \), and we obtain \( \square \) for this component. This completes the proof.

The above proof is not valid if \( n_+ = 1 \) or \( n_- = 1 \), but Theorem 1.2 seems to hold even if \( n_+ = 1 \) or \( n_- = 1 \).

### 7 Automorphisms of K3 surfaces

A **K3 surface** is a simply connected compact complex surface with a nowhere vanishing holomorphic 2-form. We prove Theorem 1.3 in this section.

#### 7.1 Lifting

Let \( X \) be a K3 surface with a nowhere vanishing holomorphic 2-form \( \omega_X \). The middle cohomology group \( H^2(X, \mathbb{Z}) \) has the intersection form \( \langle \cdot, \cdot \rangle \), which makes \( H^2(X, \mathbb{Z}) \) an even unimodular lattice of signature \((3,19)\). Such a lattice is called a **K3 lattice**, which is uniquely determined up to isomorphism. The form \( \langle \cdot, \cdot \rangle \) is extended on \( H^2(X, \mathbb{C}) \) as a hermitian form. The **Hodge structure** is the direct sum decomposition
\[
H^2(X, \mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)
\]
where \( H^{2,0}(X) = \mathbb{C}\omega_X \), \( H^{0,2}(X) = \mathbb{C}\overline{\omega} \), and \( H^{1,1}(X) = \left( H^{2,0}(X) \oplus H^{0,2}(X) \right)' \). Here \( \overline{\omega} \) is the complex conjugate. The real part \( H^{1,1}_R(X) = \{ x \in H^{1,1}(X) \mid \overline{x} = x \} \) of \( H^{1,1}(X) \) is of signature \((1,19)\), and thus \( C_X := \{ x \in H^{1,1}_R(X) \mid \langle x, x \rangle > 0 \} \) has exactly two connected components. The one containing a Kähler class is called the **positive cone** and denoted by \( C^+_X \). The intersection \( P_X := H^2(X, \mathbb{Z}) \cap H^{1,1}(X) \) is called the **Picard lattice** of \( X \). Set \( \Delta_X := \{ x \in P_X \mid \langle x, x \rangle = -2 \} \) and \( \Delta^+_X := \{ x \in \Delta_X \mid x \text{ is effective} \} \). Then the Kähler cone \( K_X \) is written as
\[
K_X = \{ x \in C^+_X \mid \langle x, r \rangle > 0 \quad \text{for all} \quad r \in \Delta^+_X \},
\]
see [Hm16] Chapter 8, Theorem 5.2. Notice that the Kähler cone \( K_X \) is a Weyl chamber with respect to the root system \( \Delta_X \).

Conversely, we can define a Hodge structure and a Kähler cone on a K3 lattice formally. Let \( (\Lambda, b) \) be a K3 lattice, and set \( \Lambda_{\mathbb{C}} = \Lambda \otimes \mathbb{C} \). A vector \( \omega \in \Lambda_{\mathbb{C}} \) such that the signature of \( \mathbb{C}\omega \oplus \mathbb{C}\overline{\omega} \subset \Lambda_{\mathbb{C}} \) is \((2,0)\) gives the decomposition
\[
\Lambda_{\mathbb{C}} = H^{2,0} \oplus H^{1,1} \oplus H^{0,2} \quad \text{where} \quad H^{2,0} = \mathbb{C}\omega, \quad H^{1,1} = v, \quad H^{0,2} = \mathbb{C}\overline{\omega}. \tag{17}
\]
Such a decomposition or such a vector \( \omega \) is called a **Hodge structure** on \( \Lambda \). Fix a Hodge structure \( \omega \in \Lambda_{\mathbb{C}} \) and one connected component \( C^+ \) of \( C := \{ x \in H^{1,1}_R \mid b(x, x) > 0 \} \) in the real part \( H^{1,1}_R \) of \( H^{1,1} \). We define the **Picard lattice** \( P \) of \( \Lambda \) to be \( \Lambda \cap H^{1,1} \). Then the set \( \Delta := \{ x \in P \mid b(x, x) = -2 \} \) is a root system. A Kähler cone of \( \Lambda \) is a Weyl chamber in \( C^+ \). If \( K \) is a Kähler cone of \( \Lambda \) then the pair \( (\omega, K) \) is called a **K3 structure** of \( \Lambda \).

The following theorem is a well-known consequence of the Torelli theorem and surjectivity of period mapping. See [Mc11] §6 for instance.

**Theorem 7.1.** Let \( \Lambda \) be a K3 lattice equipped with a K3 structure \((\omega, K)\), and \( t \) an isometry on \( \Lambda \) preserving the K3 structure. Then there exists a K3 surface \( X \), an automorphism \( \varphi \) on \( X \), and a lattice isometry \( \tau : H^2(X, \mathbb{Z}) \to \Lambda \) such that \( \tau(\omega_X) \in \mathbb{C}\omega, \quad \tau(K_X) = K \), and the diagram
\[
\begin{array}{ccc}
H^2(X, \mathbb{Z}) & \xrightarrow{\varphi^*} & H^2(X, \mathbb{Z}) \\
\downarrow \tau & & \downarrow \tau \\
\Lambda & \xrightarrow{t} & \Lambda
\end{array}
\]
commutes.
7.2 Entropy

Let \( X \) be a K3 surface, and \( \varphi \) an automorphism on \( X \). It is known that the topological entropy of \( \varphi \) is given by \( \log \lambda(\varphi^*) \), where \( \lambda(\varphi^*) \) is the spectral radius of \( \varphi^* : H^2(X) \to H^2(X) \), that is, the maximum absolute value of the eigenvalues of \( \varphi^* \). On the other hand, the characteristic polynomial of \( \varphi^* \) is either

- a product of cyclotomic polynomials, or
- a product of one Salem polynomial and a product of cyclotomic polynomials.

Hence the entropy of \( \varphi \) is 0 or the logarithm of a Salem number, see [Mc02, Section 3].

A polynomial \( F(X) \) of degree 22 is called a complemented Salem polynomial if \( F(X) \) can be expressed as \( F(X) = S(X)C(X) \), where \( S(X) \) is a Salem polynomial and \( C(X) \) is a product of cyclotomic polynomials. In this case, \( S \) is called the Salem factor of \( F \). The facts mentioned above imply that if a K3 surface automorphism \( \varphi \) has positive entropy, then the characteristic polynomial of \( \varphi^* \) is a complemented Salem polynomial.

Definition 7.2. A Salem number \( \lambda \) is projectively (resp. nonprojectively) realizable if there exists a projective (resp. nonprojective) K3 surface and an automorphism on it of topological entropy log \( \lambda \).

Notation 7.3. Let \( F \) be a complemented Salem polynomial with Salem factor \( S \). For each root \( \delta \) of \( S \) with \( |\delta| = 1 \), the symbol \( \iota_\delta \) denotes the index map in \( \text{Id}_X \) defined by

\[
\iota_\delta(f) = \begin{cases} 
2 & \text{if } f(X) = X - (\delta + \delta^{-1})X + 1 \\
-2n_f & \text{if } f(X) = X - (\zeta + \zeta^{-1})X + 1 \text{ for some } \zeta \in T \setminus \{\pm 1, \delta \pm 1\} \\
-n_f & \text{if } f(X) = X \mp 1,
\end{cases}
\]

where \( n_f \) is the multiplicity of \( f \) in \( F \) and \( T = \{\delta \in \mathbb{C} | |\delta| = 1\} \).

The nonprojective case is tractable thanks to the following proposition.

Proposition 7.4. Let \( \lambda \) be a Salem number with \( 4 \leq \deg \lambda \leq 22 \), and \( S \) its minimal polynomial. The following are equivalent:

(i) \( \lambda \) is nonprojectively realizable.

(ii) There exists a conjugate \( \delta \) of \( \lambda \) with \( |\delta| = 1 \) and a complemented Salem polynomial \( F \) with Salem factor \( S \) such that a K3 lattice has a semisimple \( (F, \iota_\delta) \)-isometry.

Proof. (i) \( \Rightarrow \) (ii). Let \( \varphi \) be an automorphism on a nonprojective K3 surface \( X \) with entropy log \( \lambda \). Then \( \varphi^* : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z}) \) is a semisimple isometry on a K3 lattice with characteristic polynomial \( F \), where \( F \) is a complemented Salem polynomial with Salem factor \( S \). Let \( \delta \in T \) be the root of \( F \) such that \( \varphi^* \omega_X = \delta \omega_X \). Set \( \Lambda(S; \varphi^*) := \{ x \in H^2(X, \mathbb{Z}) | S(\varphi^*)x = 0 \} \). If \( \delta \) were a root of unity, then \( \Lambda(S; \varphi^*) \) would be orthogonal to \( H^{2,0}(X) \oplus H^{0,2}(X) \), and

\[
\Lambda(S; \varphi^*) \subset H^{1,1}(X) \cap H^2(X, \mathbb{Z}) = P_X
\]

would hold. In particular, the Picard lattice \( P_X \) contains an element whose self intersection number is positive. This contradicts the nonprojectivity of \( X \). Therefore \( \delta \in T \) must be a root of \( S \), and this means that \( \text{Id}_X \varphi^* = \iota_\delta \).

(ii) \( \Rightarrow \) (i). Let \( \delta \) be a conjugate of \( \lambda \) with \( |\delta| = 1 \), and let \( F(X) = S(X)C(X) \) be a complemented Salem polynomial with Salem factor \( S \). Suppose that a K3 lattice \( \Lambda \) has a semisimple \( (F, \iota_\delta) \)-isometry \( t \). Then, an eigenvector \( \omega \in \Lambda_\omega \) of \( t \) corresponding to \( \delta \) defines the Hodge structure \( \mathbb{H} \) preserved by \( t \), since \( \iota_\delta(X^2 - (\delta + \delta^{-1})X + 1) = 2 \). In this case, the Picard lattice is written as \( P = \{ x \in \Lambda | C(t)x = 0 \} \) by [LT22, Theorem 7.4], and in particular, \( P \) is negative definite. Let us fix a Kähler cone \( K \). In general, the isometry \( t \) maps \( K \) to another chamber. However, the Weyl group, that is, the subgroup of \( O(P) \) generated by all root reflections, has a unique element \( w \) such that \( w(t(K)) = K \), because the Weyl group acts on the set of Weyl chambers simply transitively. Here \( w \) is extended to an isometry on \( \Lambda \) by letting \( w \) act on \( P^+ \) as the identity. Then, the composition \( w \circ t \) preserves the K3 structure \( (\omega, K) \) and has spectral radius \( \lambda \). Theorem 7.4 means that there exists a K3 surface \( X \) and an automorphism \( \varphi \) on \( X \) with entropy log \( \lambda \). Furthermore, the K3 surface \( X \) is nonprojective because its Picard lattice is negative definite. \( \square \)
We finally prove Theorem 1.3.

Proof of Theorem 1.3. Let $\lambda$ be a Salem number of degree 20, and $S$ its minimal polynomial. Fix a conjugate $\delta$ of $\lambda$ with $|\delta| = 1$. Then, a K3 lattice has a semisimple $((X - 1)(X + 1)S(X), \iota, \delta)$-isometry by Theorem 6.8. This implies that $\lambda$ is nonprojectively realizable by Proposition 7.4. \hfill \Box

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