A NOTE ON THE BOUNDEDNESS OF DOOB MAXIMAL OPERATORS
ON A FILTERED MEASURE SPACE

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ABSTRACT. Let $M$ be the Doob maximal operator on a filtered measure space and let $\nu$ be an $A_p$ weight with $1 < p < +\infty$. We try proving that

$$\|Mf\|_{L^p(\nu)} \leq p'\nu\|f\|_{L^p(\nu)},$$

where $1/p + 1/p' = 1$. Although we do not find an approach which gives the constant $p'$, we obtain that

$$\|Mf\|_{L^p(\nu)} \leq p\frac{1}{p-1}p'\nu\|f\|_{L^p(\nu)},$$

with $\lim_{p \to +\infty} p\frac{1}{p-1} = 1$.

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1. Introduction

Let $M$ be the Doob maximal operator on a filtered measure space. For $1 < p < +\infty$, it is well known (see e.g. [10]) that

\[ \|Mf\|_{L^p} \leq p'|f|_{L^p}, \]

where $1/p + 1/p' = 1$ and $p'$ is the best constant. Let $v$ be an $A_p$ weight with $1 < p < +\infty$. Tanaka and Terasawa [14] proved that

\[ \|Mf\|_{L^p(v)} \leq C[v]_{A_p}^{1/p - 1} \|f\|_{L^p(v)}, \]

where $C$ is independent of $v$.

For a Euclidean space with a dyadic filtration, the dyadic maximal operator is the above Doob maximal operator. For the dyadic maximal operator, the constant $1/(p-1)$ is the optimal power on $[v]_{\Lambda_p}$ (see e.g. [11] or [9]). It follows that the constant $1/(p-1)$ is also the optimal power on $[v]_{\Lambda_p}$ for the Doob maximal operator $M$.

In this note, we estimate the constant $C$ in (1.2). Substituting $v = 1$ into (1.2), we get (1.1). Thus, we conjecture that the constant $C$ equals $p'$ in (1.2). But we do not find an approach which gives the constant $C = p'$. Our results are as follows.

**Theorem 1.3.** Let $v$ be a weight and $1 < p < \infty$. We have the inequality

\[ \|Mf\|_{L^p(v)} \leq C[f]_{L^p(v)}, \]

if and only if $v \in A_p$. Moreover, if we denote the smallest constant in (1.4) by $\|M\|$, we have

\[ [v]_{A_p} \leq \|M\|^p \]

and

\[ \|M\| \leq p^{1/p - 1} p'[v]_{A_p}^{1/p - 1}. \]

**Remark 1.7.** The content of Theorem 1.3 is (1.6). In order to prove (1.6), we use different approaches as follows:

1. Motivated by the proof of [9, Theorem B], we get $C = p^{1/p - 1} p'$.
2. Using the construction of principal sets [14] and the conditional sparsity [3], we have $C = \alpha^2 \eta^{(p'-1)p'}$, where $\alpha$, $\eta$ are the constants in the construction of principal sets (Appendix A).
3. Long [10, Theorem 6.6.3] qualitatively evaluated $\|M\|$. Modifying Long’s proof, we have $C = p^{1/p - 1} p'$ which is the same as (1).

Approaches (1) and (3) both use the boundedness of Doob maximal operator twice and give the same estimation $C = p^{1/p - 1} p'$. Approach (2) depends on the conditional sparsity.
and the boundedness of Doob maximal operator. Letting $\sigma = v^{p-1}$ and $f = h\sigma$, we can rewrite (1.4) as

$$\|M(h\sigma)\|_{L^p(v)} \leq C\|h\sigma\|_{L^p(\sigma)}.$$  

Cao and Xue [1] (see also the references therein) used the atomic decomposition to study weighted theory on the Euclidean space, but we do not know whether it is possible on the filtered measure space.

This paper is organized as follows. Sect. 2 consists of the preliminaries for this paper. In Sect. 3 we give the proof of Theorem 1.3, and in Sect. 4 we compare $p^{n-1}$ with $\alpha^2\eta(p'-1)$. In order to keep track the constants in our paper, we modify the construction of principal sets in Appendix A.

2. Preliminaries

The filtered measure space was discussed in [6, 14], which is abstract and contains several kinds of spaces. For example, a doubling metric space with systems of dyadic cubes was introduced in Hytönen and Kairema [4]. In order to develop discrete martingale theory, a probability space endowed with a family of $\sigma$-algebra was considered in Long [10]. In addition, a Euclidean space with several adjacent systems of dyadic cubes was mentioned in Hytönen [7]. Because the filtered measure space is abstract, it is possible to study these spaces together([5, 12, 13]). As is well known, Lacey, Petermichl and Reguera [8] studied the shift operators, which is related to the martingale theory on a filtered measure space. When Hytönen [7] solved the conjecture of $A_2$, those operators are very useful.

2.1. Filtered Measure Space. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $\mathcal{F}^0 = \bigcup\{E : E \in \mathcal{F}, \mu(E) < +\infty\}$. As for $\sigma$-finite, we mean that $\Omega$ is a union of $(E_i)_{i \in \mathbb{Z}} \subset \mathcal{F}^0$. We only consider $\sigma$-finite measure space $(\Omega, \mathcal{F}, \mu)$ in this paper. Let $\mathcal{B}$ be a sub-family of $\mathcal{F}^0$ and let $f : \Omega \rightarrow \mathbb{R}$ be measurable on $(\Omega, \mathcal{F}, \mu)$. If for all $B \in \mathcal{B}$, we have $\int_B |f|d\mu < +\infty$, then we say that $f$ is $\mathcal{B}$-integrable. The family of the above functions is denote by $L^1(\mathcal{B}, \mu)$.

Let $\mathcal{B} \subset \mathcal{F}$ be a sub-$\sigma$-algebra and let $f \in L^1_{\mathcal{B}^0}(\mathcal{F}, \mu)$. Because of $\sigma$-finiteness of $(\Omega, \mathcal{B}, \mu)$ and Radon-Nikodym’s theorem, there is a unique function denoted by $\mathbb{E}(f|B) \in L^1_{\mathcal{B}^0}(\mathcal{B}, \mu)$ or $\mathbb{E}_B(f) \in L^1_{\mathcal{B}^0}(\mathcal{B}, \mu)$ such that

$$\int_B f d\mu = \int_B \mathbb{E}_B(f) d\mu, \quad \forall B \in \mathcal{B}^0.$$  

Letting $(\Omega, \mathcal{F}, \mu)$ with a family $(\mathcal{F}_i)_{i \in \mathbb{Z}}$ of sub-$\sigma$-algebras satisfying that $(\mathcal{F}_i)_{i \in \mathbb{Z}}$ is increasing, we say that $\mathcal{F}$ has a filtration $(\mathcal{F}_i)_{i \in \mathbb{Z}}$. Then, a quadruplet $(\Omega, \mathcal{F}, \mu; (\mathcal{F}_i)_{i \in \mathbb{Z}})$ is said to be a filtered measure space. It is clear that $L^1_{\mathcal{F}_i}(\mathcal{F}, \mu) \supset L^1_{\mathcal{F}_j}(\mathcal{F}, \mu)$ with $i < j$. 

Let \( L := \bigcap_{i \in \mathbb{Z}} L^1_{\mathcal{F}_i}(\mathcal{F}, \mu) \) and \( f \in L \), then \((\mathbb{E}_i(f))_{i \in \mathbb{Z}}\) is a martingale, where \( \mathbb{E}_i(f) \) means \( \mathbb{E}(f|\mathcal{F}_i) \). The reason is that \( \mathbb{E}_i(f) = \mathbb{E}_i(\mathbb{E}_{i+1}(f)) \), \( i \in \mathbb{Z} \).

2.2. Stopping Times. Let \((\Omega, \mathcal{F}, \mu; (\mathcal{F}_i)_{i \in \mathbb{Z}})\) be a \( \sigma \)-finite filtered measure space and let \( \tau : \Omega \to \{-\infty\} \cup \mathbb{Z} \cup \{+\infty\} \). If for any \( i \in \mathbb{Z} \), we have \( \{\tau = i\} \in \mathcal{F}_i \), then \( \tau \) is said to be a stopping time. We denote the family of all stopping times by \( \mathcal{T} \). For \( i \in \mathbb{Z} \), we denote \( T_i := \{\tau \in \mathcal{T} : \tau \geq i\} \).

2.3. Operators and Weights. Let \( f \in L \). The Doob maximal operator is defined by \( Mf = \sup_{i \in \mathbb{Z}} |\mathbb{E}_i(f)| \).

For \( i \in \mathbb{Z} \), we define the tailed Doob maximal operator by \( ^\star M_i f = \sup_{j \geq i} |\mathbb{E}_j(f)| \).

For \( \omega \in L \) with \( \omega \geq 0 \), we say that \( \omega \) is a weight. The set of all weights is denoted by \( L^+ \).

Let \( B \in \mathcal{F} \), \( \omega \in L^+ \). Then \( \int_{\Omega} \chi_B d\mu \) and \( \int_{\Omega} \chi_B \omega d\mu \) are denoted by \( |B| \) and \( |B|_\omega \), respectively. Now we give the definition of \( A_p \) weights.

**Definition 2.1.** Let \( 1 < p < \infty \) and let \( \omega \) be a weight. We say that the weight \( \omega \) is an \( A_p \) weight, if there exists a positive constant \( C \) such that
\[
(2.2) \quad \sup_{i \in \mathbb{Z}} \mathbb{E}_i(\omega) \mathbb{E}_i(\omega^{1-p'})^{\frac{1}{p'}} \leq C,
\]
where \( \frac{1}{p} + \frac{1}{p'} = 1 \). We denote the smallest constant \( C \) in \((2.2)\) by \([\omega]_{A_p}\).

3. Approaches of Theorem 1.3

**Proof.** We prove that \((1.4)\) implies \((1.5)\). For \( i \in \mathbb{Z} \) and \( B \in \mathcal{F}_i^0 \), we let \( f = \chi_B \). Then
\[
\mathbb{E}_i(v^{rac{1}{p}-1}) \chi_B \leq M(f\sigma) \chi_B,
\]
where \( \sigma = v^{rac{1}{p}-1} \). It follows from \((1.4)\) that
\[
\left( \int_B \mathbb{E}_i(v^{rac{1}{p}-1})^p v d\mu \right)^{\frac{1}{p}} \leq \|M\| \left( \int_{\Omega} v^{rac{1}{p}-1} \chi_B d\mu \right)^{\frac{1}{p}}.
\]

Thus
\[
\mathbb{E}_i(v^{rac{1}{p}-1})^p \mathbb{E}_i(v) \leq \|M\|^p \mathbb{E}_i(v^{rac{1}{p}-1}),
\]
which shows that
\[
[v]_{A_p} \leq \|M\|^p.
\]

In order to prove \((1.6)\), we provide the three approaches which we mentioned in Remark 1.7.
Approach (1). It is clear that
\[
\mathbb{E}_n(f) = \left( \frac{1}{\mathbb{E}_n(v)} \left( \frac{1}{\mathbb{E}_n(f)} \right)^{p-1} \right)^{1/p} = \left( \frac{1}{\mathbb{E}_n(v)} \right)^{1/p} \left( \frac{1}{\mathbb{E}_n(f)} \right)^{1/(p-1)} \leq [v]_{\Lambda_p}^{1/p} M^v(f) \left( v^{-1} M^p(f) \right)^{1/(p-1)}.
\]
Then we have
\[
M(f) \leq [v]_{\Lambda_p}^{1/p} M^v(f) \left( v^{-1} M^p(f) \right)^{1/(p-1)}.
\]
Using the boundedness of Doob maximal operators \( M^v \) and \( M^\sigma \), we obtain
\[
\|M(f)\|_{L^p(v)} \leq [v]_{\Lambda_p}^{1/p} \|M^v(f) \left( v^{-1} M^\sigma(f) \right)^{1/(p-1)}\|_{L^p(v)} \leq p^{1/p}[v]_{\Lambda_p}^{1/p} \|M^\sigma(f)\|_{L^p(\sigma)} \leq p^{1/p} [v]_{\Lambda_p}^{1/p} \|f\|_{L^p(v)}.
\]

Approach (2). For \( i \in \mathbb{Z}, k \in \mathbb{Z} \) and \( \Omega_0 \in \mathcal{F}_1^0 \), we denote
\[
P_0 = \{ \alpha^{k-1} < \mathbb{E}(f^\sigma|\mathcal{F}_i) \leq \alpha^k \} \cap \Omega_0.
\]
We claim that
\[
(3.9) \quad \left( \int_{P_0} M_i(f^\sigma)^p v d\mu \right)^{1/p} \leq \alpha^2 \eta^{(p'-1)/p'} [v]_{\Lambda_p}^{1/p} \left( \int_{P_0} f^p v d\mu \right)^{1/p},
\]
where \( \alpha, \eta \) are the constants in the construction of principal sets (Appendix A). To see this, denote \( h = f^\sigma \chi_{P_0} \). For the above \( i \), \( P_0 \) and \( h \), we construct principal sets. Then, Lemma A.3 shows that
\[
(3.10) \quad \int_{P_0} M_i(f^\sigma)^p v d\mu \leq \alpha^{2p} \sum_{P \in P} \alpha^{p(K_2(P)-1)}\int_{E(P)} f^p v d\mu.
\]
To estimate \( |E(P)|_v \). For the sake of simplicity, we denote \( E_{\mathcal{F},\mathcal{K}_1(P)}(\cdot) \) by \( E_P(\cdot) \) without confusion. We now estimate \( |E(P)|_v \) as follows:
\[ |E(P)_v| \leq |P|_v = \int_P \mathbb{E}_P(v) \, d\mu \]

\[ = \int_P \mathbb{E}_P(v)^{p'} \mathbb{E}_P(v)^{1-p'} \mathbb{E}_P(\sigma)^p \mathbb{E}_P(\sigma)^{-p} \, d\mu \]

\[ = \int_P \mathbb{E}_P(v)^{p'} \mathbb{E}_P(\sigma)^p \mathbb{E}_P(v)^{1-p'} \mathbb{E}_P(\sigma)^{-p} \, d\mu. \]

In the view of the definition of \( A_p \) and the construction of \( \mathcal{P} \), we have

\[ |E(P)_v| \leq [v]_{A_p}^{p'} \int_P \mathbb{E}_P(v)^{1-p'} \mathbb{E}_P(\sigma)^{-p} \, d\mu \]

\[ \leq \eta^{p(p'-1)} [v]_{A_p}^{p'} \int_P \mathbb{E}_P(v)^{1-p'} \mathbb{E}_P(\sigma)^{-p} \mathbb{E}_P(\chi_{E(P)})^{p(p'-1)} \, d\mu \]

\[ = \eta^{p(p'-1)} [v]_{A_p}^{p'} \int_P \mathbb{E}_P(v)^{1-p'} \mathbb{E}_P(\sigma)^{-p} \mathbb{E}_P(\chi_{E(P)})^{p(p'-1)} \, d\mu. \]

Noting that the conditional expectation satisfies Hölder’s inequality, we have

\[ |E(P)_v| \leq \eta^{p(p'-1)} [v]_{A_p}^{p'} \int_P \mathbb{E}_P(v)^{1-p'} \mathbb{E}_P(\sigma)^{-p} \]

\[ \times \mathbb{E}_P(v \chi_{E(P)})^{p(p'-1)} \mathbb{E}_P(\sigma \chi_{E(P)}) \, d\mu \]

\[ \leq \eta^{p(p'-1)} [v]_{A_p}^{p'} \int_P \mathbb{E}_P(\sigma)^{-p} \mathbb{E}_P(\sigma \chi_{E(P)}) \, d\mu. \]

Because \( E(P) \) is a subset of \( P \) and \( \alpha^{p(p'-1)} \chi_P \leq \mathbb{E}_P(h) \chi_P \), we obtain that

\[ \int_{E(P)} \alpha^{p(K_2(P))^{-1}} v \, d\mu \leq \eta^{p(p'-1)} [v]_{A_p}^{p'} \int_P \mathbb{E}_P(\sigma)^{p(p'-1)} \mathbb{E}_P(\chi_{E(P)} \sigma) \, d\mu \]

\[ = \eta^{p(p'-1)} [v]_{A_p}^{p'} \int_P \mathbb{E}_P(\sigma)^{p(p'-1)} \mathbb{E}_P(\chi_{E(P)} \sigma) \, d\mu, \]

where we have used \( \mathbb{E}_P(f \sigma) = \mathbb{E}_P^\sigma(f) \mathbb{E}_P(\sigma) \). Then

\[ \int_{E(P)} \alpha^{p(K_2(P))^{-1}} v \, d\mu \leq \eta^{p(p'-1)} [v]_{A_p}^{p'} \int_P \mathbb{E}_P^\sigma(f)^{p} \mathbb{E}_P(\chi_{E(P)} \sigma) \, d\mu \]

\[ = \eta^{p(p'-1)} [v]_{A_p}^{p'} \int_P \mathbb{E}_P^\sigma(f)^{p} \mathbb{E}_P(\chi_{E(P)} \sigma) \, d\mu \]

\[ \leq \eta^{p(p'-1)} [v]_{A_p}^{p'} \int_P \mathbb{E}_P^\sigma(f \chi_{P_0})^p \chi_{E(P)} \sigma \, d\mu \]

\[ = \eta^{p(p'-1)} [v]_{A_p}^{p'} \int_{E(P)} \mathbb{E}_P^\sigma(f \chi_{P_0})^p \sigma \, d\mu. \]
It follows from (3.10) and the boundedness of Doob maximal operator $M^\sigma$ that
\[
\int_{P_0} \ast M_i(f\sigma)^{p'}vd\mu \leq a^{2p} \eta^{p[p'-1]}[v]^{p'}_{\Lambda_p} \sum_{P \in \mathcal{P}} \int_{E(P)} M^\sigma(f|x_{P_0})^p \sigma d\mu
\]
\[
\leq a^{2p} \eta^{p[p'-1]}[v]^{p'}_{\Lambda_p} \sum_{P \in \mathcal{P}} \int_{E(P)} M^\sigma(f|x_{P_0})^p \sigma d\mu
\]
\[
\leq a^{2p} \eta^{p[p'-1]}(p')^p[v]^{p'}_{\Lambda_p} \int_{P_0} f^p \sigma d\mu,
\]
which implies (3.9). Furthermore,
\[
\int_{\Omega} \ast M_i(f\sigma)^{p'}vd\mu = \sum_{k \in \mathbb{Z}} \int_{\{a^{k-1} < E(f|\mathcal{F}_i) \leq a^k\} \cap \Omega_0} \ast M_i(f\sigma)^{p'}vd\mu
\]
\[
\leq a^{2p} \eta^{p[p'-1]}(p')^p[v]^{p'}_{\Lambda_p} \sum_{k \in \mathbb{Z}} \int_{\{a^{k-1} < E(f|\mathcal{F}_i) \leq a^k\} \cap \Omega_0} f^p \sigma d\mu
\]
\[
\leq a^{2p} \eta^{p[p'-1]}(p')^p[v]^{p'}_{\Lambda_p} \int_{\Omega} f^p \sigma d\mu.
\]
Noting that $(\Omega, \mathcal{F}, \mu)$ is a $\sigma$-finite measure space, we obtain that
\[
\left( \int_{\Omega} \ast M_i(f\sigma)^{p'}vd\mu \right)^{\frac{1}{p'}} \leq a^{p} \eta^{p[p'-1]}(p')^p[v]^{p'}_{\Lambda_p} \left( \int_{\Omega} f^p \sigma d\mu \right)^{\frac{1}{p'}}.
\]
Because $\ast M_i(\cdot) \uparrow M_i(\cdot)$ as $i \downarrow -\infty$, then
\[
\left( \int_{\Omega} M(f\sigma)^{p'}vd\mu \right)^{\frac{1}{p'}} \leq a^{p} \eta^{p[p'-1]}(p')^p[v]^{p'}_{\Lambda_p} \left( \int_{\Omega} f^p \sigma d\mu \right)^{\frac{1}{p'}}.
\]

**Approach (3).** For $f \in L^p(vd\mu)$, $b > 1$ and $k \in \mathbb{Z}$, we define stopping times
\[
\tau_k = \inf\{n : |f_n| > b^k\}.
\]
Then we denote
\[
A_{k,j} := \{\tau_k < \infty\} \cap \{b^j < \mathbb{E}(\sigma|\mathcal{F}_{\tau_k}) \leq b^{j+1}\}
\]
and
\[
B_{k,j} := \{\tau_k < \infty, \tau_{k+1} = \infty\} \cap \{b^j < \mathbb{E}(\sigma|\mathcal{F}_{\tau_k}) \leq b^{j+1}\}, j \in \mathbb{Z}.
\]
It follows that $A_{k,j} \subset \mathcal{F}_{\tau_k}$, $B_{k,j} \subset A_{k,j}$. It is clear that $\{B_{k,j}\}_{k,j}$ is a family of disjoint sets and
\[
\{b^k \leq Mf \leq b^{k+1}\} = \{\tau_k < \infty, \tau_{k+1} = \infty\} = \bigcup_{j \in \mathbb{Z}} B_{k,j}, k \in \mathbb{Z}.
\]

Following from
\[
\mathbb{E}(f|\mathcal{F}_{\tau_k}) = \mathbb{E}^\sigma(f\sigma^{-1}|\mathcal{F}_{\tau_k}) \mathbb{E}(\sigma|\mathcal{F}_{\tau_k}),
\]
we have
\[ b^{kp} \leq \essinf_{A_{k,j}} \mathbb{E}(f|\mathcal{F}_{\tau_k})^p \]
\[ \leq \essinf_{A_{k,j}} \mathbb{E}(f\sigma^{-1}|\mathcal{F}_{\tau_k})^p \esssup_{A_{k,j}} \mathbb{E}(\sigma|\mathcal{F}_{\tau_k})^p \]
\[ \leq b^p \essinf_{A_{k,j}} \mathbb{E}(f\sigma^{-1}|\mathcal{F}_{\tau_k})^p |B_{k,j}|^{-1} \int_{B_{k,j}} \mathbb{E}(\sigma|\mathcal{F}_{\tau_k})^p v d\mu. \]

Applying the \( A_p \) condition
\[ 1 \leq \mathbb{E}(v|\mathcal{F}_{\tau}) \mathbb{E}(\sigma|\mathcal{F}_{\tau})^{-p} \leq [v]_{A_p}, \quad \forall \tau, \]
we have
\[ \mathbb{E}(\sigma|\mathcal{F}_{\tau_k})^p \leq [v]_{A_p}^{-p} \mathbb{E}(v|\mathcal{F}_{\tau_k})^{-p'} = [v]_{A_p}^{-p} \mathbb{E}(v^{-1}|\mathcal{F}_{\tau_k})^{p'}. \]

It follows that
\[ \int_{\Omega} (Mf)^p v d\mu = \sum_{k \in \mathbb{Z}} \int_{\{b^k < Mf \leq b^{k+1}\}} (Mf)^p v d\mu \]
\[ \leq b^p \sum_{k \in \mathbb{Z}} \int_{\{b^k < Mf \leq b^{k+1}\}} b^{kp} v d\mu \]
\[ = b^p \sum_{k \in \mathbb{Z}, j \in \mathbb{Z}} \int_{B_{k,j}} b^{kp} v d\mu \]
\[ \leq b^{2p} [v]_{A_p}^{-p} \sum_{k \in \mathbb{Z}, j \in \mathbb{Z}} \essinf_{A_{k,j}} \mathbb{E}(f\sigma^{-1}|\mathcal{F}_{\tau_k})^p \int_{B_{k,j}} \mathbb{E}(v^{-1}|\mathcal{F}_{\tau_k})^{p'} v d\mu. \]

Letting \( X := \mathbb{Z}^2 \) and
\[ \vartheta(k, j) := \int_{B_{k,j}} \mathbb{E}(v^{-1}|\mathcal{F}_{\tau_k})^{p'} v d\mu, \]
we have that \( \vartheta \) is a measure on \( X \). For \( f \in L^p(v d\mu) \) and \( \lambda > 0 \), we denote
\[ T_f(k, j) := \essinf_{A_{k,j}} \mathbb{E}(f\sigma^{-1}|\mathcal{F}_{\tau_k})^p, \]
\[ \mathbb{E}_\lambda := \{(k, j) : \essinf_{A_{k,j}} \mathbb{E}(f\sigma^{-1}|\mathcal{F}_{\tau_k})^p > \lambda\}, \]
\[ G_\lambda := \bigcup_{(k, j) \in \mathbb{E}_\lambda} A_{k,j}. \]
It follows that
\[
\|\{Tf > \lambda\}\|_\theta = \sum_{(k,j) \in \mathcal{E}} \int_{B_{k,j}} \mathbb{E}^v(\nu^{-1}|\mathcal{F}_{tk})^p' \nu d\mu \\
\leq \sum_{(k,j) \in \mathcal{E}} \int_{B_{k,j}} \mathbb{E}^v(\nu^{-1}\chi_{G_\lambda}|\mathcal{F}_{tk})^p' \nu d\mu \\
\leq \int_{G_\lambda} \left( M^v(\nu^{-1}\chi_{G_\lambda}) \right)^{p'} \nu d\mu.
\]

For \( \tau = \inf \{ n : \mathbb{E}^\sigma(f_{\sigma^{-1}}|\mathcal{F}_n)^p > \lambda \} \), we obtain \( G_\lambda \subseteq \{ M^\sigma(f_{\sigma^{-1}})^p > \lambda \} = \{ \tau < \infty \} \).

In view of the boundedness of Doob maximal operator \( M^v \), we get that
\[
\|\{Tf > \lambda\}\|_\theta \leq \int_{G_\lambda} \left( M^v(\nu^{-1}\chi_{G_\lambda}) \right)^{p'} \nu d\mu \\
\leq \int_{\{\tau < \infty\}} \left( M^v(\nu^{-1}\chi_{\{\tau < \infty\}}) \right)^{p'} \nu d\mu \\
\leq p^p' \|\{\tau < \infty\}\|_\sigma \\
= p^p' \|\{M^\sigma(f_{\sigma^{-1}})^p > \lambda\}\|_\sigma.
\]

Therefore
\[
\int_\Omega (Mf)^p \nu d\mu \leq b^{2p[v]_{\Lambda_p}^{p'}} \int_\chi Tf d\theta = b^{2p[v]_{\Lambda_p}^{p'}} \int_0^\infty \|\{Tf > \lambda\}\|_\theta d\lambda \\
\leq b^{2p} p^p' [v]_{\Lambda_p}^{p'} \int_0^\infty \|\{M^\sigma(f_{\sigma^{-1}})^p > \lambda\}\|_\sigma d\lambda \\
= b^{2p} p^p' [v]_{\Lambda_p}^{p'} \int_\Omega M^\sigma(f_{\sigma^{-1}})^p \sigma d\mu.
\]

Using the boundedness of Doob maximal operator \( M^\sigma \), we conclude that
(3.30) \[
\int_\Omega (Mf)^p \nu d\mu \leq b^{2p} p^p' [v]_{\Lambda_p}^{p'} \int_\Omega |f|^p \nu d\mu.
\]

Taking limit as \( b \to 1^+ \) in (3.30), we have
\[
\|Mf\|_{L^p(\nu)} \leq p^p' [v]_{\Lambda_p}^{p'} \|f\|_{L^p(\nu)}.
\]

4. Comparison of \( p^{\frac{1}{p'-1}} \) and \( a^2 \eta^{(p'-1)} \)

We compare \( p^{\frac{1}{p'-1}} \) with \( a^2 \eta^{(p'-1)} \) in this section, where \( a > 1 \) and \( \eta = \frac{a}{a-1} \) are the constants in the construction of principal sets (Appendix A). We split our comparison into two theorems, Theorem 4.1 and Theorem 4.2.
Theorem 4.1. For $1 < p < +\infty$, let $\varphi(a) = a^2 \eta[p^{(p-1)}]$. Then we have
\[
\min_{a>1} \varphi(a) = \varphi\left(\frac{2p-1}{2p-2}\right).
\]

Proof. We deal with $\ln \varphi(a)$. Then
\[
\ln \varphi(a) = 2 \ln a + \frac{1}{p-1} \ln \frac{a}{a-1}.
\]
It is easy to check $\lim_{a \to 1^+} \ln \varphi(a) = \lim_{a \to +\infty} \ln \varphi(a) = +\infty$. We have
\[
\left( \ln \varphi(a) \right)' = \frac{2}{a} + \frac{1}{a(p-1)} - \frac{1}{(a-1)(p-1)}.
\]
It is clear that the unique $a_0 = \frac{2p-1}{2p-2}$ solves equation $\left( \ln \varphi(a) \right)' = 0$ and $a_0 = \frac{2p-1}{2p-2} > 1$. Thus
\[
\min_{a>1} \varphi(a) = \varphi\left(\frac{2p-1}{2p-2}\right) = \left(\frac{2p-1}{2p-2}\right)^2 \left(2p-1\right)^\frac{1}{p-1}.
\]

It follows from Theorem 4.1 that the minimum of $\varphi(a)$ is a function of $p$. Then we denote the minimum $\left(\frac{2p-1}{2p-2}\right)^2 \left(2p-1\right)^\frac{1}{p-1}$ and the constant $\frac{2p-1}{2p-2}$ by $\phi(p)$ and $\psi(p)$, respectively. Because of $\frac{2p-1}{2p-2} > 1$ and $2p-1 > p$, we have $\phi(p) \geq \psi(p)$. Now we study limits of $\phi(p)$ and $\psi(p)$ in the following Theorem 4.2.

Theorem 4.2. Let $\phi$ and $\psi$ as above. Then
\[
\lim_{p \to 1^+} \phi(p) = +\infty, \quad \lim_{p \to 1^+} \psi(p) = e
\]
and
\[
\lim_{p \to 1^+} \phi(p) = \lim_{p \to +\infty} \psi(p) = 1.
\]
Moreover
\[
\lim_{p \to +\infty} \frac{\ln \phi(p)}{\ln \psi(p)} = 1.
\]

Proof. Because
\[
\lim_{p \to 1^+} \ln \phi(p) = \lim_{p \to 1^+} 2 \ln \left(\frac{2p-1}{2p-2}\right) + \lim_{p \to 1^+} \frac{1}{p-1} \ln(2p-1) = +\infty,
\]
and
\[
\lim_{p \to +\infty} \ln \phi(p) = \lim_{p \to +\infty} 2 \ln \left(\frac{2p-1}{2p-2}\right) + \lim_{p \to +\infty} \frac{1}{p-1} \ln(2p-1) = 0,
\]
we have $\lim_{p \to 1^+} \phi(p) = +\infty$ and $\lim_{p \to +\infty} \phi(p) = 1$, respectively.

Similarly, we get $\lim_{p \to 1^+} \psi(p) = e$ and $\lim_{p \to +\infty} \psi(p) = 1$. 

Finally, we obtain
\[
\lim_{p \to +\infty} \frac{\ln \phi(p)}{\ln \psi(p)} = \lim_{p \to +\infty} \frac{2 \ln\left(\frac{2p-1}{2p-2}\right) + \frac{1}{p-1} \ln(2p-1)}{\ln p}
\]
\[
= \lim_{p \to +\infty} \frac{2(p-1) \ln\left(\frac{2p-1}{2p-2}\right) + \ln(2p-1)}{\ln p}
\]
\[
= \lim_{p \to +\infty} \frac{2(p-1) \ln\left(\frac{2p-1}{2p-2}\right)}{\ln p} + \lim_{p \to +\infty} \frac{\ln(2p-1)}{\ln p}
\]
\[
= 0 + 1 = 1.
\]
\[\square\]

**Remark 4.7.** We give further properties of \(\phi(p)\) and \(\psi(p)\).

1. We claim that the function \(\phi(p)\) is decreasing on \((1, +\infty)\). Writing \(\phi_1(p) = \left(\frac{2p-1}{2p-2}\right)^2\) and \(\phi_2(p) = (2p-1)^{\frac{1}{p-1}}\), we will show that \(\phi_1(p)\) and \(\phi_2(p)\) are both decreasing on \((1, +\infty)\). Combining this with \(0 < \phi_1(p)\) and \(0 < \phi_2(p)\), we obtain that \(\phi(p)\) is decreasing on \((1, +\infty)\). We now check that \(\phi_1(p)\) and \(\phi_2(p)\) are both decreasing.

For \(\phi_1(p)\) with \(p \in (1, +\infty)\), it is clear that
\[
\phi_1(p) = \left(\frac{2p-1}{2p-2}\right)^2 = (1 + \frac{1}{2p-2})^2.
\]
Thus \(\phi_1(p)\) is decreasing on \((1, +\infty)\).

For \(\phi_2(p)\) with \(p \in (1, +\infty)\), to consider
\[
\ln \phi_2(p) = \frac{1}{p-1} \ln(2p-1).
\]
It is clear that
\[
\left(\ln \phi_2(p)\right)' = \frac{1}{(p-1)^2} \left(\frac{2}{2p-1}(p-1) - \ln(2p-1)\right)
\]
\[
= \frac{1}{(p-1)^2} \left(\frac{2(p-1)}{2p-1} - \ln(2p-1)\right).
\]
Using the Mean Value Theorem, we have
\[
\ln(2p-1) = \ln(2p-1) - \ln 1 = \frac{1}{\xi}(2p-1 - 1) = \frac{1}{\xi}(2p-1),
\]
where \(\xi \in (1, 2p-1)\). It follows that
\[
\ln(2p-1) > \frac{2(p-1)}{2p-1},
\]
which implies \(\left(\ln \phi_2(p)\right)' < 0\). Thus \(\phi_2(p)\) is decreasing on \((1, +\infty)\).
2. We claim that the function $\psi(p)$ is decreasing on $(1, +\infty)$. It suffices to show that $\psi'(p) < 0$. We have

$$\psi'(p) = \frac{\psi(p)}{(p-1)^2} \left( 1 - \frac{1}{p} + \ln \frac{1}{p} \right).$$

It is clear that $\psi'(p) < 0$ if and only if $1 - \frac{1}{p} + \ln \frac{1}{p} < 0$. Let $s(t) = 1 - t + \ln t$ with $t \in (0, 1]$. Because of $s'(t) = \frac{1}{t} - 1 > 0$ on $(0, 1)$, the function $s(t)$ is strictly increasing on $(0, 1]$. It follows from $s(1) = 0$ that $s(t) < 0$ on $(0, 1)$. That is $1 - \frac{1}{p} + \ln \frac{1}{p} < 0$ with $p > 1$. Thus $\psi(p)$ is decreasing on $(1, +\infty)$.

At the end of Section 4, we check our work with graphing device in Figure 1.

![Figure 1. Computer confirmation of $\phi(p)$ and $\psi(p)$](image)

**Appendix A. Construction of Principal Sets**

The construction of principal sets first appeared in Tanaka and Terasawa [14], and Chen, Zhu, Zuo and Jiao [2, 3] found the conditional sparsity of the construction, which is new and useful. We will use the construction of principal sets. Because we keep track the constants of the conditional sparsity, we will give the modifications in the construction of principal sets in this Appendix.

For $i \in \mathbb{Z}$, $h \in L^+$, $a > 1$ and $k \in \mathbb{Z}$, stopping times are defined by

$$\tau := \inf \{ j \geq i : \mathbb{E}(h|\mathcal{F}_j) > a^{k+1} \}.$$ 

Let

$$P_0 := \{ a^{k-1} < \mathbb{E}(h|\mathcal{F}_i) \leq a^k \} \cap \Omega_0,$$
where $\Omega_0 \in \mathcal{F}_i^0$, then $P_0 \in \mathcal{F}_i^0$. We denote $\mathcal{K}_1(P_0) := i$ and $\mathcal{K}_2(P_0) := k$. Then we define $\mathcal{P}_1 := \{P_0\}$, which is the first generation $\mathcal{P}_1$. Now we show how to define the second one. Let

$$\tau_{P_0} := \tau \chi_{P_0} + \infty \chi_{P_0^c},$$

where $P_0^c = \Omega \setminus P_0$. Let $P$ be a subset of $P_0$ with $\mu(P) > 0$. If there is $i < j$ and $k + 1 < j$ such that

$$P = \{a^{i-1} < \mathbb{E}(h|\mathcal{F}_j) \leq a^j\} \cap \{\tau_{P_0} = j\} \cap P_0$$

we say that $P$ is a principal set of $P_0$. We denote $K_1(P) := j$ and $K_2(P) := l$. Letting $\mathcal{P}(P_0)$ be the family of the above principal sets of $P_0$, we say that $\mathcal{P}_2 := \mathcal{P}(P_0)$ is the second generation.

Following [3, P.804], we have

$$\mu(P_0) \leq \frac{a}{a-1} \mu(E(P_0)) =: \eta \mu(E(P_0))$$

where

$$E(P_0) := P_0 \cap \{\tau_{P_0} = \infty\} = P_0 \cap \{\tau = \infty\} = P_0 \setminus \bigcup_{P \in \mathcal{P}(P_0)} P.$$ 

Furthermore, we have $\chi_{P_0} \leq \eta \mathbb{E}(\chi_{E(P_0)}) \chi_{P_0}$, which is called the conditional sparsity of principal sets with $\eta$ (see [2, 3]).

Proceeding inductively, we obtain the next generalizations

$$\mathcal{P}_{n+1} := \bigcup_{P \in \mathcal{P}_n} \mathcal{P}(P).$$

Let

$$\mathcal{P} := \bigcup_{n=1}^{\infty} \mathcal{P}_n,$$

then the collection of principal sets $\mathcal{P}$ satisfies the following properties:

1. The sets $E(P)$ where $P \in \mathcal{P}$, are disjoint and $P_0 = \bigcup_{P \in \mathcal{P}} E(P)$;
2. $P \in \mathcal{F}_{K_1(P)}$;
3. $\chi_{P} \leq \eta \mathbb{E}(\chi_{E(P)}|\mathcal{F}_{K_1(P)}) \chi_{P}$;
4. $a^{K_2(P)-1} \mathbb{E}(h|\mathcal{F}_{K_1(P)}) \leq a^{K_2(P)}$ on $P$;
5. $\sup_{j \geq 1} \mathbb{E}_j(h \chi_{P}) \leq a^{K_2(P)+1}$ on $E(P)$;
6. $\chi_{\{K_1(P) \leq j < \tau(P)\}} \mathbb{E}_j(h) \leq a^{K_2(P)+1}$.

where $\eta = a/(a-1)$.

Now, we represent the tailed Doob maximal operator by the principal sets, which is the following lemma.
Lemma A.3. Let \( h \in L^+ \), \( a > 1 \) and \( i \in \mathbb{Z} \). For \( k \in \mathbb{Z} \) and \( \Omega_0 \in \mathcal{F}^0_i \), we let
\[
P_0 := \{ a^{k-1} < \mathbb{E}(h|\mathcal{F}_i) \leq a^k \} \cap \Omega_0.
\]
If \( \mu(P_0) > 0 \), then
\[
* \mathcal{M}_i(h) \chi_{P_0} = * \mathcal{M}_i(h \chi_{P_0}) \chi_{P_0} = \sum_{P \in P} * \mathcal{M}_i(h \chi_{P_0}) \chi_{E(P)} \leq a^2 \sum_{P \in P} a^{(K_2(P)-1)} \chi_{E(P)}.
\]

References

[1] Mingming Cao and Qingying Xue, Characterization of two-weighted inequalities for multilinear fractional maximal operator, Nonlinear Anal. 130 (2016), 214–228, DOI 10.1016/j.na.2015.10.004. MR3424618
[2] Wei Chen and Yong Jiao, Weighted estimates for the bilinear maximal operator on filtered measure spaces, J. Geom. Anal. 31 (2021), no. 5, 5309–5335. MR4244905
[3] Wei Chen, Chunxiang Zhu, Yahui Zuo, and Yong Jiao, Two-weighted estimates for positive operators and Doob maximal operators on filtered measure spaces, J. Math. Soc. Japan 72 (2020), no. 3, 795–817. MR4125846
[4] Tuomas Hytönen and Anna Kairema, Systems of dyadic cubes in a doubling metric space, Colloq. Math. 126 (2012), no. 1, 1–33. MR2901199
[5] Tuomas Hytönen and Mikko Kemppainen, On the relation of Carleson’s embedding and the maximal theorem in the context of Banach space geometry, Math. Scand. 109 (2011), no. 2, 269–284. MR2854692
[6] Tuomas Hytönen, Jan van Neerven, Mark Veraar, and Lutz Weis, Analysis in Banach spaces. Vol. I. Martingales and Littlewood-Paley theory, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 63, Springer, Cham, 2016. MR3617205
[7] Tuomas P. Hytönen, The sharp weighted bound for general Calderón-Zygmund operators, Ann. of Math. (2) 175 (2012), no. 3, 1473–1506. MR2912709
[8] Michael T. Lacey, Stefanie Petermichl, and Maria Carmen Reguera, Sharp \( A_2 \) inequality for Haar shift operators, Math. Ann. 348 (2010), no. 1, 127–141. MR2657437
[9] Andrei K. Lerner, An elementary approach to several results on the Hardy-Littlewood maximal operator, Proc. Amer. Math. Soc. 136 (2008), no. 8, 2829–2833. MR2399047
[10] Rui Lin Long, Martingale spaces and inequalities, Peking University Press, Beijing; Friedr. Vieweg & Sohn, Braunschweig, 1993. MR1224450
[11] Kabe Moen, Sharp one-weight and two-weight bounds for maximal operators, Studia Math. 194 (2009), no. 2, 163–180. MR2534183
[12] René L. Schilling, Measures, integrals and martingales, Second, Cambridge University Press, Cambridge, 2017. MR3644418
[13] Daniel W. Stroock, Probability theory, an analytic view, Cambridge University Press, Cambridge, 1993. MR1267569
[14] Hitoshi Tanaka and Yutaka Terasawa, Positive operators and maximal operators in a filtered measure space, J. Funct. Anal. 264 (2013), no. 4, 920–946. MR3004953
