Energy Contents of Gravitational Waves in Teleparallel Gravity

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Abstract

The conserved quantities, that are, gravitational energy-momentum and its relevant quantities are investigated for cylindrical and spherical gravitational waves in the framework of teleparallel equivalent of General Relativity using the Hamiltonian approach. For both cylindrical and spherical gravitational waves, we obtain definite energy and constant momentum. The constant momentum shows consistency with the results available in General Relativity and teleparallel gravity. The angular momentum for cylindrical and spherical gravitational waves also turn out to be constant. Further, we evaluate their gravitational energy-momentum fluxes and gravitational pressure.

Keywords: Teleparallel Gravity, Gravitational Waves, Energy-Momentum.

1 Introduction

Gravitational waves are extremely elusive because gravity is a very weak force. These waves affect the matter, from which they pass through, in a quite negligible manner which makes them difficult to detect. Gravitational waves, by definition, have zero energy-momentum tensor and hence their existence was questioned. However, the theory of General Relativity (GR)
predicts the existence of gravitational waves, rising from the relation between space and time. Indeed this problem arises because energy is not well-defined in GR (as the strong equivalence principle refutes the energy localization of the gravitational field). According to Synge [1], energy of the gravitational field should be localizable independent of any observer. Bondi [2] argued that non-localizable form of energy is inadmissible in GR. This disputable point is the origin of a long-lasting discussion on the energy and momentum transported by a gravitational wave.

Scheidegger [3] raised question about the well-defined existence of gravitational radiations. Ehlers and Kundt [4] resolved this problem for gravitational waves by analyzing a sphere of test particles in the path of plane-fronted gravitational waves. They showed that these particles acquired a constant momentum from the waves. Weber and Wheeler [5] gave the similar discussion for cylindrical gravitational waves. Qadir and Sharif [6] explored an operational approach, embodying the same principle, to show that gravitational waves impart momentum. One of us [7] found energy-momentum using various prescriptions that provide acceptable results for different cosmological models and gravitational waves.

It was suggested [8, 9] that the energy-momentum problem might provide more better results in the framework of teleparallel equivalent of General Relativity (TEGR). Møller [10] was the first who observed that the tetrad description of the gravitational field could lead to a better expression for the gravitational energy-momentum than does GR. Sharif and Nazir [11] investigated energy of cylindrical gravitational waves in GR and teleparallel gravity and found inconsistency results for the two theories. Andrade et al. [12] considered the localization of energy in Lagrangian framework of TEGR. Maluf et al. [13] derived an expression for the gravitational energy, momentum and angular momentum using the Hamiltonian formulation of TEGR [14]. The same authors [15] evaluated the energy flux of gravitational waves in the framework of TEGR. Maluf and Ulhoa [16] showed that gravitational energy-momentum of plane-fronted gravitational waves is non-positive.

In this paper, we use the Hamiltonian approach in TEGR to evaluate energy and its contents for cylindrical and spherical gravitational waves. In the next section, some basic concepts of TEGR and energy-momentum expressions are given. Section 3 is devoted for the evaluation of energy and its related quantities for cylindrical gravitational waves. In section 4, we calculate these quantities for spherical gravitational waves. Summary and discussion is presented in the last section.
### 2 Hamiltonian Approach: Energy-Momentum in Teleparallel Theory

The Riemannian metric in terms of a non-trivial tetrad \( e^a_{\mu} \) is written as

\[
g_{\mu\nu} = \eta_{ab} e^a_{\mu} e^b_{\nu}, \tag{1}\]

where the tetrad field and its inverse satisfy the relation

\[
e^a_{\mu} e_b^\mu = \delta^a_b, \quad e^a_{\mu} e^\mu_a = \delta^\nu_\mu. \tag{2}\]

We denote the spacetime indices by Greek alphabets \( (\mu, \nu, \rho, ...) \) and tangent space indices by Latin alphabets \( (a, b, c, ...) \) and these run from 0 to 3. Time and space indices are denoted according to \( \mu = 0, i, a = (0), (i) \). The torsion tensor is defined as

\[
T^a_{\mu\nu} = \partial_\mu e^a_{\nu} - \partial_\nu e^a_{\mu}, \tag{3}\]

which is related to the Weitzenböck connection \[17\]

\[
\Gamma^\lambda_{\mu
u} = e^a_\lambda \partial_\nu e^a_{\mu}. \tag{4}\]

The Lagrangian density in TTEGR is given by \[14\]

\[
L = -\kappa e \left( \frac{1}{4} T^{abc} T_{abc} + \frac{1}{2} T^{abc} T_{bac} - T^a T_a \right) - L_M \equiv -\kappa e \Sigma^{abc} T_{abc} - L_M; \tag{5}\]

where \( \kappa = 1/16\pi \) and \( e = \text{det}(e^a_{\mu}) \). The tensor \( \Sigma^{abc} \) is defined as

\[
\Sigma^{abc} = \frac{1}{4} \left( T^{abc} + T^{bac} - T^{cab} \right) + \frac{1}{2} \left( \eta^{ac} T^b - \eta^{ab} T^c \right) \tag{6}\]

which satisfies the antisymmetric property, i.e., \( \Sigma^{abc} = -\Sigma^{acb} \). \( L_M \) denotes the Lagrangian density for the matter fields, \( T_a = T^b_{ba} \) and \( T_{abc} = e^b_{\mu} e^c_{\nu} T_{a\mu\nu} \). The variation of the Lagrangian \( L \) with respect to \( e^a_{\mu} \) yields the field equations

\[
e_{a\lambda} e_{b\mu} \partial_\nu (e \Sigma^{b\lambda\nu}) - e (\Sigma^{b\nu} a T_{b\nu\mu} - \frac{1}{4} e_{a\mu} T_{bcd} \Sigma^{bcd}) = \frac{1}{4\kappa} e T_{a\mu}, \tag{7}\]

where

\[
\frac{\delta L_M}{\delta e_{a\mu}} = e T_{a\mu}. \]

The total Hamiltonian density is \[18\]

\[
H(e_{ai}, \Pi_{ai}) = e_{a0} C^a + \alpha_{ik} \Gamma^{ik} + \beta_k \Gamma^k + \partial_k (e_{a0} \Pi^{a0}), \tag{8}\]
where $C^a$, $\Gamma^{ik}$ and $\Gamma^k$ are primary constraints, $\alpha_{ik}$ and $\beta_k$ are the Lagrangian multipliers defined as $\alpha_{ik} = \frac{1}{2}(T_{i0k} + T_{k0i})$ and $\beta_k = T_{00k}$. $C^a$ is given by a total divergence in the form $C^a = -\partial_i \Pi^{ai} + H^a$, where

$$\Pi^{ai} = -4\kappa e \Sigma^{ai}$$

is the momentum canonically conjugated to $e_{ai}$. The term $-\partial_i \Pi^{ai}$ is identified as the energy-momentum density \[13\]. The total energy-momentum is defined as

$$P^a = -\int_V d^3x \partial_i \Pi^{ai},$$

where $V$ is an arbitrary space volume.

The constraint

$$\Gamma^{ik} = -\Gamma^{ki} = 2\Pi^{[ik]} - 2\kappa e[-g^{im} g^{kj} T^0_{mj} + (g^{im} g^{0k} - g^{km} g^{0i}) T^j_{mj}] = 0$$

gives

$$2\Pi^{[ik]} = 2\kappa e[-g^{im} g^{kj} T^0_{mj} + (g^{im} g^{0k} - g^{km} g^{0i}) T^j_{mj}]$$

which is referred to as the angular momentum density. Consequently, the angular momentum is defined as

$$M^{ik} = 2\int_V d^3x\Pi^{[ik]} = 2\kappa \int_V d^3x e[-g^{im} g^{kj} T^0_{mj} + (g^{im} g^{0k} - g^{km} g^{0i}) T^j_{mj}].$$

We can write using the field equations as

$$\frac{d}{dt}[\int_V d^3x \partial_i \Pi^{ai}] = -\Phi^a_g - \Phi^a_m,$$

where

$$\Phi^a_g = \int_S dS_j \phi^{aj}, \quad \Phi^a_m = \int_S dS_j (ee^a \mu T^{j\mu})$$

are the $a$ components of the gravitational and matter energy-momentum flux. The quantity

$$\phi^{aj} = \kappa e e^{aj} (4\Sigma^{bcj} T_{bc\mu} - \delta^j_{\mu} \Sigma^{bc\nu} T_{bcd})$$

represent the $a$ component of the gravitational energy-momentum flux density in $j$ direction. In terms of the gravitational energy-momentum, Eq.\[13\] takes the form

$$\frac{dP^a}{dt} = -\Phi^a_g - \Phi^a_m.$$
For the vacuum spacetime, the above equation reduces to

$$\frac{dP^a}{dt} = -\Phi^a_g = - \int_S dS_j \phi^{aj}. \quad (17)$$

If we take $a = (i) = (1), (2), (3)$, then

$$\frac{dP^{(i)}}{dt} = \int_S dS_j (-\phi^{(ij)}). \quad (18)$$

The left hand side of the above equation has the character of force while the density $(-\phi^{(ij)})$ is considered as a force per unit area, or pressure density. Thus Eq. (18) has the nature of the gravitational pressure.

3 Cylindrical Gravitational Waves

The line element of cylindrical gravitational waves given by Einstein and Rosen is [5]

$$ds^2 = -e^{2(\gamma - \psi)} dt^2 + e^{2(\gamma - \psi)} d\rho^2 + \rho^2 e^{-2\psi} d\phi^2 + e^{2\psi} dz^2, \quad (19)$$

where the arbitrary functions, $\gamma = \gamma(\rho, t)$, $\psi = \psi(\rho, t)$, satisfy the vacuum field equations

$$\psi'' + \frac{1}{\rho} \psi' - \ddot{\psi} = 0, \quad \gamma' = \rho(\psi'^2 + \ddot{\psi}^2), \quad \dot{\gamma} = 2\rho \dot{\psi} \dot{\psi}', \quad (20)$$

dot and prime represent differentiation with respect to $t$ and $\rho$ respectively. The tetrad field, satisfying Eqs. (11) and (2) is

$$e^a_{\mu}(t, \rho, \phi) = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & A \cos \phi & -\rho C \sin \phi & 0 \\ 0 & A \sin \phi & \rho C \cos \phi & 0 \\ 0 & 0 & 0 & B \end{pmatrix}. \quad (21)$$

Here $A = e^{(\gamma - \psi)}$, $B = e^\psi$, $C = e^{-\psi}$ and its determinant is

$$e = \text{det}(e^a_{\mu}) = \rho A^2.$$
The non-zero components of the torsion tensor are

\[ T_{(0)01} = A', \quad T_{(1)01} = \dot{A} \cos \phi, \quad T_{(1)02} = -\rho \dot{C} \sin \phi, \]
\[ T_{(1)12} = (A - C - \rho C') \cos \phi, \quad T_{(2)01} = \dot{A} \sin \phi, \quad T_{(2)02} = \rho \dot{C} \cos \phi, \]
\[ T_{(2)12} = -(A - C - \rho C') \cos \phi, \quad T_{(3)03} = \dot{B}, \quad T_{(3)13} = B' \quad (22) \]

which give rise to

\[ T_{001} = AA', \quad T_{101} = A\dot{A}, \quad T_{202} = \rho^2 C\dot{C}, \]
\[ T_{212} = \rho C(C - A + \rho C'), \quad T_{303} = B\dot{B}, \quad T_{313} = BB'. \quad (23) \]

### 3.1 Energy, Momentum and Angular Momentum

The components of energy-momentum density, \(-\partial_i \Pi^{ai}\), for the cylindrical gravitational waves are found by using Eqs. (6) and (9)

\[ -\partial_i \Pi^{(0)i} = -\partial_1[-2\kappa e^{(\psi - \gamma)}(e^{\gamma} - 1)] = 2\kappa \partial_1[e^{(\psi - \gamma)}(e^{\gamma} - 1)], \]
\[ -\partial_2 \Pi^{(1)i} = -\partial_2(2\kappa \dot{e}^{\psi} \sin \phi) = -2\kappa \partial_2(\dot{e}^{\psi} \sin \phi), \]
\[ -\partial_3 \Pi^{(2)i} = -\partial_3(-2\kappa \dot{e}^{\psi} \cos \phi) = 2\kappa \partial_3(\dot{e}^{\psi} \cos \phi), \]
\[ -\partial_3 \Pi^{(3)i} = -\partial_3[-2\kappa \rho e^{-\psi}(\dot{\gamma} - 2\dot{\psi})] = 2\kappa \partial_3[\rho e^{-\psi}(\dot{\gamma} - 2\dot{\psi})]. \quad (24) \]

Using these values in Eq. (10) and integration over a cylindrical region of an arbitrary length \(L\) and radius \(\rho\) gives energy and momentum

\[ P^{(0)} = \frac{1}{4}Le^{(\psi - \gamma)}(e^{\gamma} - 1), \quad P^{(i)} = 0. \quad (25) \]

The angular momentum for cylindrical gravitational waves becomes constant.

### 3.2 Energy-Momentum Flux

The components of gravitational energy flux density for cylindrical gravitational waves, obtained by using Eq. (15), are

\[ \phi^{(0)1} = -2\kappa \dot{\psi} e^{(\psi - \gamma)}(2\psi' \rho + e^{\gamma} - 1), \quad \phi^{(0)2} = 0 = \phi^{(0)3}. \quad (26) \]

Consequently, the gravitational energy flux becomes

\[ \Phi_g^{(0)} = \frac{1}{4}Le^{(\psi - \gamma)}\{\dot{\psi}(e^{\gamma} - 1) + \dot{\gamma}\} + \text{constant}. \quad (27) \]
The components of gravitational momentum flux

\[
\Phi_g^{(1)} = -2\kappa L \sin \phi \int e^\psi (\dot{\psi}^2 - \psi'^2)(1 - \rho \psi')d\rho + \text{constant},
\]

\[
\Phi_g^{(2)} = 2\kappa L \cos \phi \int e^\psi (\dot{\psi}^2 - \psi'^2)(1 - \rho \psi')d\rho + \text{constant},
\]

\[
\Phi_g^{(2)} = \frac{1}{4} \int e^{-\psi} \{\rho^2 \psi'(\dot{\psi}^2 - \psi'^2) + 2\rho \psi'^2 + \psi'(e^\gamma - 1)\}d\rho
- \frac{1}{4}e^{(\gamma - \psi)} + \text{constant} \quad (28)
\]

are obtained by making use of the components of gravitational momentum flux densities

\[
\phi^{(1)} = 2\kappa \gamma' e^\psi \cos \phi, \quad \phi^{(2)} = -2\kappa e^\psi \sin \phi(\dot{\psi}^2 - \psi'^2)(1 - \rho \psi'), \quad \phi^{(3)} = 0, \quad (29)
\]

\[
\phi^{(2)} = 2\kappa \gamma' e^\psi \sin \phi, \quad \phi^{(2)} = 2\kappa e^\psi \cos \phi(\dot{\psi}^2 - \psi'^2)(1 - \rho \psi'), \quad \phi^{(3)} = 0, \quad (30)
\]

\[
\phi^{(3)} = 0 = \phi^{(3)}, \quad \phi^{(3)} = 2\kappa e^{-\psi} \{\rho^2 \psi'(\dot{\psi}^2 - \psi'^2) + 2\rho \psi'^2 + \psi'(e^{\gamma} - 1) - \gamma' e^\gamma\} \quad (31)
\]

in Eq.(15) for \(a = (1), (2)\) and \((3)\) respectively. Here the energy-momentum flux represents the transfer of energy-momentum in cylindrical gravitational waves. Notice that Maluf et al. \[15\] also found exactly the same energy but slightly different energy flux.

### 3.3 Gravitational Pressure

In order to calculate gravitational pressure for cylindrical gravitational waves, we use Eq.(18) and confine the considerations to a surface along radial direction, i.e.,

\[
\frac{dP^{(i)}}{dt} = \int_S dS_1 (-\phi^{(i)}) \quad (32)
\]

Using the components of gravitational momentum flux density \(\phi^{(i)}\)

\[
\phi^{(1)} = 2\kappa \cos \phi(\gamma' e^\psi), \quad \phi^{(2)} = 2\kappa \sin \phi(\gamma' e^\psi), \quad \phi^{(3)} = 0, \quad (33)
\]

in Eq.(32) and taking the unit vector \(\hat{r} = (\cos \phi, \sin \phi, 0)\), it follows that

\[
\frac{dP}{dt} = -2\kappa(\gamma' e^\psi) \int_S d\phi d\gamma \hat{r}. \quad (34)
\]
Conversion of surface element $d\phi dz$ into spherical polar coordinates, we have

$$\frac{dP}{dt} = -2\kappa(\gamma'e^\psi) \int_S \rho \sin \theta d\theta d\phi \hat{r}. \quad (35)$$

Integration over a small solid angle $d\Omega = \sin \theta d\theta d\phi$ of constant radius $\rho$ gives

$$\frac{dP}{dt} = -2\kappa(\rho \gamma'e^\psi) \Delta \Omega \hat{r}. \quad (36)$$

Replacing $dt \rightarrow d(ct)$, $\kappa = \frac{1}{16\pi} \rightarrow \frac{c^4}{16\pi G}$ in the above equation, we obtain

$$\frac{dP}{dt} = -(\gamma'e^\psi) \frac{c^4}{8\pi G\rho} (\rho^2 \Delta \Omega) \hat{r}. \quad (37)$$

The quantity $-(\gamma'e^\psi)c^4/8\pi G\rho$ on the right hand side of this equation gives the gravitational pressure exerted on the area element ($\rho^2 \Delta \Omega$). This equation can also be written as

$$\frac{d}{dt} \left( \frac{P}{M} \right) = -(\gamma'e^\psi) \frac{c^4}{8\pi GM} \rho \Delta \Omega \hat{r}. \quad (38)$$

The left hand side corresponds to acceleration which can be taken as the gravitational acceleration field acting on the solid angle $\Delta \Omega$ at a radial distance $\rho$.

## 4 Spherical Gravitational Waves

The line element describing the gravitational waves with spherical wavefronts is [19]

$$ds^2 = e^{-M}(-dt^2 + d\rho^2) + e^{-U}(e^{-V}d\phi^2 + e^Vdz^2), \quad (39)$$

where $M$, $U$ and $V$ are arbitrary functions depending on $t$ and $\rho$. The vacuum field equations imply that $e^{-U}$ and $V$ satisfy the wave equation

$$(e^{-U})_{tt} - (e^{-U})_{\rho\rho} = 0, \quad V_{tt} - U_tV_t - V_{\rho\rho} + U_\rho V_\rho = 0. \quad (40)$$

Equations for $M$ are

$$U_{tt} - U_{\rho\rho} = \frac{1}{2}(U_t^2 + U_\rho^2 + V_t^2 + V_\rho^2) - U_t M_t - U_\rho M_\rho = 0, \quad (41)$$

$$2U_{t\rho} = U_t U_\rho - U_t M_\rho - U_\rho M_t + V_t V_\rho. \quad (42)$$
The tetrad field corresponding to the metric (39) is

\[ e^a_{\mu}(t, \rho, \phi) = \begin{pmatrix} e^{-\frac{M}{2}} & 0 & 0 & 0 \\ 0 & e^{-\frac{M}{2}} \cos \phi & e^{-\frac{(U+V)}{2}} \sin \phi & 0 \\ 0 & e^{-\frac{M}{2}} \sin \phi & e^{-\frac{(U+V)}{2}} \cos \phi & 0 \\ 0 & 0 & 0 & e^{\frac{(U+V)}{2}} \end{pmatrix} \]  

(43)

and its determinant is \( e = e^{(-M-U)} \). The non-vanishing components of the torsion tensor are

\[ T_{(0)01} = -\frac{M}{2} e^{-\frac{M}{2}}, \quad T_{(1)01} = -\frac{\dot{M}}{2} \cos \phi, \quad T_{(1)02} = \frac{1}{2} (\dot{U} + \dot{V}) e^{\frac{(U-V)}{2}} \sin \phi, \]

\[ T_{(2)01} = -\frac{\dot{M}}{2} e^{-\frac{M}{2}} \sin \phi, \quad T_{(2)02} = \frac{1}{2} (\dot{U} + \dot{V}) e^{\frac{(U-V)}{2}} \cos \phi, \]

\[ T_{(2)02} = \frac{1}{2} (\dot{U} + \dot{V}) e^{\frac{(U-V)}{2}} \cos \phi, \quad T_{(3)03} = \frac{1}{2} (\dot{V} - \dot{U}) e^{\frac{(U+V)}{2}}, \quad T_{(3)13} = \frac{1}{2} (V' - U') e^{\frac{(U-V)}{2}}. \]  

(44)

The tensor \( T_{\lambda\mu\nu} = e^a_\lambda T_{a\mu\nu} \) becomes

\[ T_{001} = -\frac{M}{2} e^{-\frac{M}{2}}, \quad T_{101} = -\frac{\dot{M}}{2} e^{-\frac{M}{2}}, \quad T_{202} = \frac{1}{2} (\dot{U} + \dot{V}) e^{\frac{(U-V)}{2}}, \]

\[ T_{212} = -\left\{ \frac{1}{2} (U' + V') e^{\left( -\frac{U-V}{2} \right)} + e^{-\frac{M}{2}} \right\}, \quad T_{303} = \frac{1}{2} (\dot{V} - \dot{U}) e^{\frac{(U+V)}{2}}, \quad T_{313} = \frac{1}{2} (V' - U') e^{\frac{(U-V)}{2}}. \]  

(45)

4.1 Energy, Momentum and Angular Momentum

The components of energy-momentum density for the spherical gravitational waves become

\[ -\partial_t \Pi^{(0)i} = 2\kappa \partial_t \left\{ e^{-U} \left( e^{\left( \frac{U+V}{2} \right)} + U' e^{-\frac{M}{2}} \right) \right\}, \]

\[ -\partial_i \Pi^{(1)i} = -\kappa e^{-U} \cos \phi \left\{ 2 e^{\frac{M}{2}} (U' - \dot{U} U' + \frac{\dot{M} (U+V)}{2}) + \frac{e^{\left( \frac{U-V}{2} \right)}}{2} (V - \dot{M}) \right\}, \]

\[ -\partial_i \Pi^{(2)i} = -\kappa e^{-U} \sin \phi \left\{ 2 e^{\frac{M}{2}} (U' - \dot{U} U' + \frac{\dot{M} (U+V)}{2}) + \frac{e^{\left( \frac{U-V}{2} \right)}}{2} (V - \dot{M}) \right\}, \]

\[ -\partial_t \Pi^{(3)i} = 0. \]  

(46)
Inserting these values in Eq.(10), we obtain energy and momentum as
\[ P^{(0)} = \frac{1}{4} Le^{-U} \left( e^{\frac{(U+V)}{2}} + U' e^{\frac{M}{2}} \right), \quad P^{(i)} = 0. \] (47)

Again, all the components of angular momentum turn out to be constant.

### 4.2 Energy-Momentum Flux

The components of gravitational energy flux density
\[ \phi^{(0)1} = -\kappa \left\{ e^{\frac{M}{2}} e^{-U} (V' \dot{V} - U' \dot{U} - M' \dot{M}) + e^{\frac{-(U+V)}{2}} (\dot{V} - \dot{U}) \right\}, \quad \phi^{(0)2} = 0 = \phi^{(0)3} \] (48)
give rise to gravitational energy flux
\[ \Phi^{(0)}_g = \frac{1}{8} L \left\{ e^{\frac{M}{2}} e^{-U} (V' \dot{V} - U' \dot{U} - M' \dot{M}) + e^{\frac{-(U+V)}{2}} (\dot{V} - \dot{U}) \right\} + \text{constant}. \] (49)

Inserting the components of gravitational flux densities
\[ \phi^{(1)1} = \kappa e^{\frac{M}{2}} e^{-U} \cos \phi \{ (V' - U' - M') e^{-\frac{M}{2}} e^{\frac{(U+V)}{2}} - \dot{U}^2 - U'^2 + \dot{U} \dot{M} \}, \]
\[ \phi^{(1)2} = -\frac{1}{2} \kappa e^{-\frac{(U+V)}{2}} \sin \phi \{ \dot{M} (\dot{V} - \dot{U}) + M' (U' - V') \}, \quad \phi^{(1)3} = 0, \] (50)
\[ \phi^{(2)1} = \kappa e^{\frac{M}{2}} e^{-U} \cos \phi \{ (V' - U' - M') e^{-\frac{M}{2}} e^{\frac{(U+V)}{2}} - \dot{U}^2 - U'^2 + \dot{U} \dot{M} \}, \]
\[ \phi^{(2)2} = \frac{1}{2} \kappa e^{-\frac{(U+V)}{2}} \cos \phi \{ \dot{M} (\dot{V} - \dot{U}) + M' (U' - V') \}, \quad \phi^{(2)3} = 0, \] (51)
\[ \phi^{(3)1} = 0 = \phi^{(3)2}, \]
\[ \phi^{(3)3} = \frac{1}{2} \kappa \left\{ 2 M' e^{-\frac{M}{2}} + e^{\frac{-(U+V)}{2}} \{ M' (U' + V') - \dot{M} (\dot{V} + \dot{U}) \} \right\} \] (52)
in Eq.(14), we obtain the gravitational momentum flux
\[ \Phi^{(1)}_g = -\frac{1}{2} \kappa L \sin \phi \int e^{-\frac{(U+V)}{2}} \{ \dot{M} (\dot{V} - \dot{U}) + M' (U' - V') \} d\rho + \text{constant}, \]
\[ \Phi^{(2)}_g = \frac{1}{2} \kappa L \cos \phi \int e^{-\frac{(U+V)}{2}} \{ \dot{M} (\dot{V} - \dot{U}) + M' (U' - V') \} d\rho + \text{constant}, \]
\[ \Phi^{(3)}_g = -\frac{1}{4} \kappa e^{-\frac{M}{2}} + \text{constant} + \frac{1}{16} \int e^{-\frac{(U+V)}{2}} \{ M' (U' + V') - \dot{M} (\dot{U} + \dot{V}) \} d\rho. \] (53)
4.3 Gravitational Pressure

For spherical gravitational waves, the gravitational momentum flux density components become

\[ \phi^{(1)} = \kappa e^{M} e^{-U} \cos \phi \{(V' - U' - M') e^{M/2} e^{(U+V)/2} - \dot{U}^2 - \dot{U}'^2 + \dot{U} \dot{M} \}, \]

\[ \phi^{(2)} = \kappa e^{M} e^{-U} \sin \phi \{(V' - U' - M') e^{M/2} e^{(U+V)/2} - \dot{U}^2 - \dot{U}'^2 + \dot{U} \dot{M} \}, \]

\[ \phi^{(3)} = 0. \] (54)

Substituting these values in Eq. (32) and taking the unit vector \( \hat{r} = (\cos \phi, \sin \phi, 0) \), it follows that

\[ \frac{dP}{dt} = -\kappa e^{M} e^{-U} \{(V' - U' - M') e^{M/2} e^{(U+V)/2} - \dot{U}^2 - \dot{U}'^2 + \dot{U} \dot{M} \} \int_S d\phi d\zeta \hat{r}. \] (55)

Proceeding in a similar way as for the cylindrical gravitational waves, it follows that

\[ \frac{dP}{dt} = \left[ -\frac{c^4}{16\pi \rho G} e^{M} e^{-U} \{(V' - U' - M') e^{M/2} e^{(U+V)/2} - \dot{U}^2 - \dot{U}'^2 + \dot{U} \dot{M} \} \right] \times \hat{r}(\rho^2 \Delta \Omega). \] (56)

The term in the square brackets on the right hand side of the above equation is interpreted as the gravitational pressure exerted on the area element \((\rho^2 \Delta \Omega)\). Equation (56) can be re-written as

\[ \frac{d}{dt} \left( \frac{P}{M} \right) = \frac{c^4}{16\pi GM} e^{M} e^{-U} \{(V' - U' - M') e^{M/2} e^{(U+V)/2} - \dot{U}^2 - \dot{U}'^2 + \dot{U} \dot{M} \} \rho \Delta \Omega \hat{r}. \] (57)

The left hand side of this equation can be recognized as the gravitational acceleration. We can consider it as the gravitational acceleration field that acts on the solid angle \( \Delta \Omega \).

5 Summary and Discussion

In this paper, we have applied the coordinate independent prescription obtained by using the Hamiltonian approach in TEGR to investigate energy-momentum distribution of gravitational waves. We have evaluated energy, momentum, angular momentum, gravitational energy-momentum flux and
gravitational pressure of cylindrical and spherical gravitational waves. For cylindrical gravitational waves, the energy expression turns out to be definite and well defined. The constant momentum corresponds to the result of GR [20] and teleparallel gravity [21]. In the case of spherical gravitational waves, we obtain well-defined energy and constant momentum which corresponds to the result of GR [7]. The angular momentum for these solutions turns out to be constant.

It is interesting to note that for cylindrical gravitational waves, we obtain

\[ P^a P^b \eta_{ab} = 0 \] if \( \gamma = 0 \) which depicts the property of a plane electromagnetic wave. If we take the background region of spherical waves \((t < \rho, \) Minkowski\) described by the solution \(U = -\ln t - \ln \rho, V = \ln t - \ln \rho\) and \(M = 0,\) Eq.(47) yields energy-momentum zero while gravitational energy-momentum flux becomes constant as expected. This is what one can expect for Minkowski spacetime. Further, we have also evaluated the gravitational pressure exerted by gravitational waves (cylindrical and spherical). This may be helpful to investigate the thermodynamics of the gravitational field.

We would like to mention here that our results show consistency with the results of different energy-momentum complexes both in GR and teleparallel gravity. Here we can express these conserved quantities such as energy, momentum and angular momentum tensor of the gravitational field covariantly. Finally, we can say that the tetrad formulism provides a more satisfactory treatment of the localization problem.

**Appendix**

The non-zero components of the tensor

\[ \Sigma^{abc} = \frac{1}{4} (T^{abc} + T^{bac} - T^{cab}) + \frac{1}{2} (\eta^{ac} T^{b} - \eta^{ab} T^{c}), \]

for

- **Cylindrical Gravitational Waves**

\[ \Sigma^{001} = \frac{1}{2\rho} e^{4(\psi - \gamma)} (e^\gamma - 1), \quad \Sigma^{202} = \frac{1}{2\rho^2} \dot{\gamma} e^{2(2\psi - \gamma)}, \]

\[ \Sigma^{212} = -\frac{1}{2\rho^2} \gamma' e^{2(2\psi - \gamma)}, \quad \Sigma^{303} = \frac{1}{2} e^{-2\gamma} (\dot{\gamma} - 2\dot{\psi}), \]

\[ \Sigma^{313} = \frac{1}{2\rho} e^{-2\gamma} \left\{ \rho(2\psi' - \gamma') + e^\gamma - 1 \right\}, \]
• Spherical Gravitational Waves

\[ \Sigma^{001} = \frac{1}{2} e^{2M} (e^{\frac{U+V}{2}} e^{-\frac{M}{2}} + U'), \quad \Sigma^{101} = -\frac{1}{2} \dot{U} e^{2M}, \]
\[ \Sigma^{202} = \frac{1}{4} e^{M} e^{(U+V)}(\dot{V} - \dot{M} - \dot{U}), \]
\[ \Sigma^{212} = \frac{1}{4} e^{M} e^{(U+V)}(U' - V' + M'), \]
\[ \Sigma^{303} = -\frac{1}{4} e^{M} e^{(U-V)}(\dot{V} + \dot{M} + \dot{U}), \]
\[ \Sigma^{313} = \frac{1}{4} e^{M} e^{(U-V)}(U' + V' + M' + 2e^{\frac{(U+V)}{2}} e^{-\frac{M}{2}}). \]

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