CLOSED SUBGROUPS OF FREE PROFINITE MONOIDS ARE PROJECTIVE PROFINITE GROUPS

JOHN RHODES AND BENJAMIN STEINBERG

Abstract. We prove that the class of closed subgroups of free profinite monoids is precisely the class of projective profinite groups. In particular, the profinite groups associated to minimal symbolic dynamical systems by Almeida are projective. Our result answers a question raised by Lubotzky during the lecture of Almeida at the Fields Workshop on Profinite Groups and Applications, Carleton University, August 2005. We also prove that any finite subsemigroup of a free profinite monoid consists of idempotents.

1. Introduction

It has long been an open question whether closed subgroups of free profinite monoids must be projective profinite groups. At one time it was even hoped that maximal subgroups would be free profinite groups. Almeida recently associated to each minimal symbolic dynamical system a maximal subgroup of a free profinite monoid, which is a conjugacy invariant of the system [2]. Almeida gave sufficient conditions for the associated subgroup to be a free profinite group and, in particular, showed this to be the case for the group associated to a Sturmian system [2]. He also provided an example of a dynamical system leading to a maximal subgroup that is not free profinite [2]. Almeida lectured about this work at the Fields Workshop on Profinite Groups and Applications, Carleton University, August 2005. Lubotzky asked at the end of the talk whether all closed subgroups of a free profinite monoid are projective. In this paper, we provide a positive answer to Lubotzky’s question by showing that closed subgroups of free profinite monoids are indeed projective. In fact, a more general result holds.

Almeida and Weil [4, 5] defined a pseudovariety of groups $H$ to be arborescent if whenever $1 \to A \to G \to H \to 1$ is a short exact sequence of groups with $A, H \in H$ and $A$ abelian, one has $G \in H$. The origin of the terminology is that if $H$ is non-trivial, then being arborescent is equivalent to having the Cayley graph of each free pro-$H$ group being a pro-$H$ tree [4, 5].

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If $H$ is a pseudovariety of groups, then $\Pi$ denotes the pseudovariety of monoids whose subgroups belong to $H$. This note proves that if $H$ is an arborescent pseudovariety of groups, then any closed subgroup of a free pro-$\Pi$ monoid is a projective profinite group and that any finite subsemigroup of a free pro-$\Pi$ monoid is an idempotent semigroup. In particular, by considering the pseudovariety of all finite groups and the trivial pseudovariety, this result applies to free profinite monoids and free profinite aperiodic monoids.

A semigroup is termed aperiodic if all its subgroups are trivial.

Before stating precise results, let us first recall some definitions. A profinite group $G$ is called projective if whenever one has a diagram of groups (called an embedding problem)

$$
\begin{array}{c}
G \\
\downarrow \varphi \\
1 \longrightarrow K \longrightarrow A \longrightarrow B \longrightarrow 1
\end{array}
$$

with $A$ finite and $\varphi$ a continuous epimorphism, there is a continuous lift $\tilde{\varphi} : G \to A$, called a weak solution to the embedding problem [19], such that

$$
\begin{array}{c}
G \\
\downarrow \tilde{\varphi} \\
1 \longrightarrow K \longrightarrow A \longrightarrow B \longrightarrow 1
\end{array}
$$

commutes. If $H$ is a pseudovariety of groups, then a pro-$H$ group is called $H$-projective if each embedding problem (1.1) with $A \in H$ has a weak solution (1.2); see [19]. It turns out that for arborescent pseudovarieties $H$, being projective and $H$-projective coincide (see Corollary 8).

If $V$ and $W$ are pseudovarieties of monoids, then $V \ast W$ denotes the pseudovariety of monoids generated by semidirect products of the form $V \rtimes W$ with $V \in V$ and $W \in W$. If $V$ is a pseudovariety of semigroups and $W$ is a pseudovariety of monoids, then their Mal’cev product $V \odot W$ is the pseudovariety generated by monoids $M$ admitting a homomorphism $\varphi : M \to N$ with $N \in W$ such that $e\varphi^{-1} \in V$ for each idempotent $e \in N$. A finite semigroup $S$ is called completely regular if each element $s \in S$ satisfies $s^m = s$ for some $m > 0$. A finite semigroup is called a band if each element is idempotent. A band is precisely the same thing as an aperiodic completely regular semigroup. Let $\text{Ab}$ denote the pseudovariety of finite abelian groups and $A$ denote the pseudovariety of finite aperiodic monoids. Our main results are then the following two theorems.

**Theorem 1.** Let $V$ be a pseudovariety of monoids with the property that $(V \cap \text{Ab}) \ast V = V$. Then every closed subgroup of a free pro-$V$ monoid is a projective profinite group, and in particular torsion-free.

**Theorem 2.** Let $V$ be a pseudovariety of monoids such that $A \odot V = V$. Then every finite subsemigroup of a free pro-$V$ monoid is completely regular.
Since pseudovarieties of the form $\mathbf{H}$ with $\mathbf{H}$ arborescent evidently satisfy the hypotheses of Theorems 1 and 2, we deduce our main result:

**Corollary 3.** If $\mathbf{H}$ is an arborescent pseudovariety of groups, then every closed subgroup of a free pro-$\mathbf{H}$ monoid is a projective profinite group and each finite subsemigroup of a free pro-$\mathbf{H}$ monoid is a band.

It is well known that a profinite group is projective if and only if it is a closed subgroup of a free profinite group [19]. As the natural continuous projection from the free profinite monoid onto the free profinite group splits [3, 17], it in fact follows that projective profinite groups are precisely the closed subgroups of free profinite monoids.

The paper is organized as follows. The first section gives a proof that closed subgroups of projective groups are projective, highlighting the main idea underlying Theorem 1. The next two sections of the paper prove Theorems 1 and 2. The final section gives an application to projective semigroups.

## 2. Projective profinite groups

The usual proof (c.f. [19]) that a closed subgroup of a projective profinite group is projective goes roughly as follows. First one shows that a projective profinite group embeds in a free profinite group. Then one shows that projectivity of closed subgroups of free profinite groups reduces to the case of clopen subgroups. Clopen subgroups are then shown to be free using Hall’s theorem [11] and the Nielsen-Schreier theorem. We discovered an elementary proof of this result using the monomial map [12, 24] (sometimes called the Krasner-Kaloujnine embedding [15]). Luis Ribes pointed out to us that this same proof scheme was used in [8]. Nonetheless, we provide the proof as it relies on two lemmas that we also use in the monoidal context. The first lemma contains the key idea for both the group and monoid cases.

**Lemma 4.** Let $G$ be a profinite group and suppose that one has an embedding problem (1.1) which has no weak solution. Suppose that $\tilde{B}$ is a finite group with $B \leq \tilde{B}$. Then one can find a diagram of finite groups

\[
\begin{array}{cccc}
1 & \longrightarrow & \tilde{K} & \longrightarrow & \tilde{A} & \overset{\tilde{\alpha}}{\longrightarrow} & \tilde{B} & \longrightarrow & 1 \\
& & & & & \downarrow{\varphi} & & & \\
& & & & & 1
\end{array}
\]

such that $\ker \tilde{\alpha}$ is a subgroup of $K^n$ for some $n$ and there is no continuous homomorphism $\tilde{\varphi} : G \to \tilde{A}$ lifting $\varphi$ (i.e. so that $\tilde{\varphi}\tilde{\alpha} = \varphi$).

**Proof.** Let $H$ be the quotient of $\tilde{B}$ by the kernel of the action on the set $\tilde{B}/B$ of right cosets of $B$. Then there is a well-known embedding

\[
\tilde{B} \hookrightarrow B \wr (\tilde{B}/B, H) = B^{\tilde{B}/B} \rtimes H,
\]

*We are grateful to Luis Ribes for directing our attention to [8].
see [9, 12, 15, 18, 24]. The construction is as follows. Choose coset representatives $\overline{Bg}$ for each right coset of $B$ so that $\overline{B} = 1$. Let us write $[g]$ for the image in $H$ of $g \in \overline{B}$. Then the embedding takes $g \in \overline{B}$ to $(f_g, [g])$ where $(Bb)f_g = Bbg(Bbg)^{-1}$. In particular, if $g \in B$, then $Bf_g = g$. Thus the map $(f_g, [g]) \mapsto Bf_g$ is an isomorphism $\tau$ from the copy of $B$ in $\overline{B}$ to $B$.

There is a natural epimorphism $\psi: A \rtimes B/B \rightarrow B$ induced by $\alpha$, namely $(h, [g]) \psi = (h\alpha, [g])$. Also $\ker \psi \cong K^{B/B}$ since it consists of all pairs $(h, [1])$ with $(Bg)h \in \ker \alpha = K$, all $Bg \in \overline{B}/B$. So let $\overline{A} = \overline{B}\psi^{-1}$ and $\overline{\alpha} = \psi|_{\overline{A}}$. Then $\ker \overline{\alpha}$ is isomorphic to a subgroup of $K^{B/B}$. Let $A' = B\overline{\alpha}^{-1}$. We claim that the map $\rho: A' \rightarrow A$ given by $(f, [g]) \mapsto Bf$ is a homomorphism such that the diagram

\[
A' \xrightarrow{\overline{\alpha}} B \subset \overline{B} \rtimes B/H \xrightarrow{\tau} A \xrightarrow{\rho} B
\]  

(2.2)

commutes. Let us assume this for the moment and complete the proof. If $\overline{\varphi}$ lifts $\varphi$ in (2.1), then $\overline{\varphi}\rho$ solves the original embedding problem (1.1). Therefore, $\varphi$ has no lift by the hypothesis.

Clearly (2.2) commutes by definition of $\rho$ and $\tau$, so we just need to check that $\rho$ is a homomorphism. Indeed, if $(f, [g])$ and $(f', [g'])$ are in $A'$ (and so we may take $g, g' \in B$), then $(f, [g])(f', [g']) = (f[g]f', [gg'])$. As $g \in B$,

\[
((f, [g])(f', [g'])))\rho = B(f[g]f') = Bf \cdot Bg f' = Bf \cdot Bf' = (f, [g])\rho(f', [g'])\rho.
\]

This completes the proof. 

Our next lemma is a reduction on the types of embedding problems one must solve to establish projectivity.

**Lemma 5.** Let $G = \lim_{\leftarrow i \in I} G_i$ with the $G_i$ finite continuous quotients of $G$ and let $\pi_i: G \rightarrow G_i$ be the projection. Consider an embedding problem as per (1.1). Then there is an index $i$ and an epimorphism $\psi: G_i \rightarrow B$ such that the diagram

\[
A \times_{\psi} G_i \xrightarrow{\alpha'} G_i \xrightarrow{\varphi} G
\]

\[
A \xrightarrow{\psi} A \xrightarrow{\alpha} B \xrightarrow{\varphi} G
\]

(2.3)

commutes, where $\psi^*: A \times_{\psi} G_i \rightarrow A$ is the pullback of $\psi$ along $\alpha$. Moreover, $\ker \alpha' \cong \ker \alpha$.

In particular, $G$ is a projective profinite group if and only if all embedding problems (1.1) with $B = G_i$ and $\varphi = \pi_i$, some $i \in I$, have a weak solution.
Proof. Since $B$ is finite, $\varphi$ factors through one of the continuous projections $\pi_i : G \to G_i$ via an epimorphism $\psi : G_i \to B$. Consider the pullback $A \times \psi G_i = \{(a, g) \in A \times G_i \mid a\alpha = g\psi\}$ of $\psi$ along $\alpha$. Then the projections $\alpha'$ to $G_i$ and $\psi^*$ to $A$ are epimorphisms as $\alpha$ and $\psi$ are epimorphisms. Also, ker $\alpha' = \{(a, 1) \mid a\alpha = 1\psi = 1\} \cong \ker \alpha$. This proves the existence of (2.3).

The final statement follows as a weak solution to the embedding problem for $\alpha'$ in (2.3) leads to a weak solution to the embedding problem (1.1). □

**Corollary 6.** Closed subgroups of projective profinite groups are projective. In particular, projective profinite groups are torsion-free.

Proof. Suppose that $H$ is a closed subgroup of a projective profinite group $G$. Write $G = \lim_{\leftarrow i \in I} G_i$ with the $G_i$ finite quotients and let $\pi_i : G \to G_i$ be the projection. As $H$ is closed, $H = \lim_{\leftarrow i \in I} H_i$ where $H_i = H\pi_i$. Suppose that $H$ is not projective. Then by Lemma 5 there is an embedding problem

$$
\begin{array}{ccc}
1 & \longrightarrow & K \\
\downarrow \pi_i & & \downarrow \alpha \\
H_i & \longrightarrow & A \\
\end{array}
$$

having no weak solution. Taking $B = H_i$ and $\tilde{B} = G_i$ in Lemma 4 we obtain a diagram

$$
\begin{array}{ccc}
1 & \longrightarrow & \tilde{K} \\
\downarrow \pi_i & & \downarrow \tilde{\alpha} \\
H & \longrightarrow & \tilde{A} \\
\end{array}
$$

so that $\pi_i$ cannot be lifted to $\tilde{A}$. It follows that the embedding problem

$$
\begin{array}{ccc}
1 & \longrightarrow & \tilde{K} \\
\downarrow \pi_i & & \downarrow \tilde{\alpha} \\
G & \longrightarrow & \tilde{A} \\
\end{array}
$$

has no weak solution, contradicting that $G$ is projective.

Since any finite subgroup of a projective profinite group is closed, it follows that such a subgroup must be projective. Since a cyclic group of prime order $p$ is not projective (as the canonical projection $\mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p^2\mathbb{Z}$ does not split) we conclude that projective profinite groups are torsion-free. □

Now we wish to show that if $H$ is an arborescent pseudovariety of groups, then the $H$-projective groups are projective and that one only needs to consider very special embedding problems. According to [19], a pseudovariety of groups $H$ is called saturated if whenever $G$ is a finite group such that its Frattini quotient $G/\Phi(G)$ belongs to $H$, then $G \in H$. It is shown in [14, Satz III.3.8] (see also [19, Example 7.6.5]) that if $p$ is a prime dividing the order of the Frattini subgroup $\Phi(G)$, then $p$ divides the order of the Frattini quotient.
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$G/\Phi(G)$. Since $\Phi(G)$ is nilpotent it follows that if $H$ is arborescent, then $H$ is saturated.

The importance of saturated pseudovarieties is that groups from these pseudovarieties lift. The following is [19, Lemma 7.6.6].

**Lemma 7.** Let $H$ be a saturated pseudovariety of groups and let $\varphi : G \to H$ be an epimorphism of finite groups with $H \in H$. Then there exists $M \leq G$ such that $M \in H$ and $M \varphi = H$.

The next corollary follows from well-known results, which can be found in [19], that will be used for our monoidal results.

**Corollary 8.** Let $H$ be an arborescent pseudovariety of groups and let $G$ be a pro-$H$ group. Then the following are equivalent:

1. $G$ is projective;
2. $G$ is $H$-projective;
3. All embedding problems (1.1) for $G$ with $K, B \in H$ and $K$ an elementary abelian $p$-group have a weak solution.

**Proof.** The implications (1) implies (2) implies (3) are trivial (the last uses that $H$ is arborescent and so $A \in H$). For (3) implies (1), we use that in order to show that $G$ is projective, it suffices by [19, Theorem 7.5.1] and [19, Proposition 7.5.4] to establish the existence a weak solution for all embedding problems (1.1) for $G$ with $A$ finite and $K$ an elementary abelian $p$-group. But given such an embedding problem, we have $B = G \varphi \in H$ since $G$ is pro-$H$. Lemma 7 allows us to replace $A$ by a subgroup $M$ belonging to $H$. Then $M \cap K$ is an elementary abelian $p$-group in $H$, as required. \[\square\]

3. PROOF OF THEOREM [1]

We proceed to prove Theorem [1]. If $X$ is a topological space, then we denote by $\hat{F}_V(X)$ the free pro-$V$ monoid on $X$ [1, 5, 18]. Let $G$ be a closed subgroup of $\hat{F}_V(X)$. Denote by $H$ the pseudovariety of all groups belonging to $V$. Then $H$ is arborescent and $G$ is pro-$H$. Suppose that $G$ is not projective. Then by Corollary 8 there is an embedding problem (1.1), with $K, B \in H$ and $K$ an elementary abelian $p$-group, having no weak solution. Since $G$ is closed, $G$ is the projective limit of its images under the inverse system of continuous finite quotients of $\hat{F}_V(X)$. Lemma 5 then implies that we may assume without loss of generality that $\varphi : G \to B$ is the restriction of a continuous surjective homomorphism $\hat{F}_V(X) \to M$ with $M \in V$.

Let $e \in B$ be the identity and let $R = \{ m \in M \mid mM = eM \}$ denote the $R$-class [10] of $B$ in $M$. We consider the Schützenberger representation of $M$ on $R$. Namely, $M$ acts on the right of $R$ by partial transformations via

$$r \cdot m = \begin{cases} rm & \text{if } rm \in R \\ \text{undefined} & \text{if } rm \notin R \end{cases}$$  \hspace{1cm} (3.1)
Let $N$ be the quotient of $M$ by the kernel of this action. Then $B$ maps injectively into $N$, since it clearly acts faithfully on itself and hence on $R$. Therefore, by replacing $M$ with $N$, we may assume without loss of generality that $M$ acts faithfully on the right of $R$.

Let $H$ be the maximal subgroup of $M$ containing $B$. So $H$ is the group of units of the monoid $eMe$. By Lemma [4] with $\tilde{B} = H$, we can find a diagram

$$
\begin{array}{c}
1 \\
\rightarrow \\
\longrightarrow \\
\longrightarrow
\end{array}
\quad
\begin{array}{cc}
\tilde{K} \\
\rightarrow \\
\tilde{A} \\
\rightarrow \\
\longrightarrow
\end{array}
\quad
\begin{array}{c}
\alpha \\
\varphi \\
\longrightarrow \\
\longrightarrow
\end{array}
\quad
\begin{array}{c}
H \\
\rightarrow
\end{array}
\quad
\begin{array}{c}
1
\end{array}
$$

(3.2)

such that $\varphi$ has no lift and $\tilde{K}$ is an elementary abelian $p$-group (being a subgroup of a direct power of $K$).

In order to reuse the key idea from the proof of Lemma [4] we need an embedding of $M$ into a wreath product of $H$ with some monoid. The embedding we use is due to Schützenberger [20, 21]. In [20], Schützenberger proved that $H$ acts freely on the left of $R$ by automorphisms of the right action (3.1) of $M$ on $R$; see [7, Theorem 2.22] and [18]. Hence $M$ acts on the right of the orbit set $Q = H \setminus R$ by partial transformations.$^1$ Set $Q^* = Q \cup \{\star\}$ where $\star \notin Q$. One can define an action of $M$ on the right of $Q^*$ by total functions by defining, for $r \in R$ and $m \in M$,

$$
Hr \cdot m = \begin{cases} 
Hrm & \text{if } rm \in R \\
\star & \text{if } rm \notin R
\end{cases}
$$

and by defining $\star \cdot m = \star$. Let $N$ be the quotient of $M$ by the kernel of this action. Let $H^0 = H \cup \{\star\}$ (viewed, say, as a submonoid of the group ring of $H$). Then there is an embedding $M \hookrightarrow H^0 \wr (Q^*, N) = (H^0)^Q \times N$, where $n \in N$ acts on $f : Q^* \to H^0$ by $x^nf = xnf$ (see [16, Proposition 8.2.17]).

Let us briefly describe the embedding. We use $[m]$ for the image of $m \in M$ in $N$. Choose a representative $\overline{Hr}$ for each orbit of $H$ on $R$ so that $\overline{H} = e$. Then $m \mapsto (f_m, [m])$ where $f_m = 0$ and, for $r \in R$, $Hrf_m = 0$ if $Hr \cdot m = \star$ and otherwise is the unique element of $H$ such that $\overline{Hr} \cdot m = (Hr)f_m \overline{Hr} \cdot m$.

The existence and uniqueness of this element comes from the fact that $H$ acts freely on the left of $R$ by automorphisms of the right action (3.1). Notice that if $h \in H$, then $h \mapsto (f_h, [h])$ where $Hf_h = h$ since $\overline{H} \cdot h = h = h\overline{H} = h\overline{H}$. That is, the map $\tau$ sending $(f_h, [h])$ to $Hf_h$ is an isomorphism from the copy of $H$ in $M \leq H^0 \wr (Q^*, N)$ to $H$.

Let $\tilde{\alpha}^0 : \tilde{A}^0 \to H^0$ be the induced surjective homomorphism. The reader easily verifies that each local monoid of the derived category$^4$ of $D_{\tilde{\alpha}^0}$ is $\tilde{K} = \ker \tilde{\alpha}$ except the local monoid at $0$, which is trivial. Hence $D_{\tilde{\alpha}^0}$ divides an elementary abelian $p$-group by the locality of group pseudovarieties [23].

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$^1$The set $Q$ is in bijection with the set of $\mathcal{L}$-classes of the $f$-class of $H$ [7, 10, 16, 18].

$^2$The reader just interested in the proof of Corollary may ignore arguments involving derived categories.
Next, we consider the natural surjective homomorphism
\[ \psi : \tilde{A}^0 \to (Q^*, N) \to H^0 \to (Q^*, N) \]
induced by \( \tilde{\alpha}^0 \) — so \((f, n) \mapsto (f\tilde{\alpha}^0, n)\). Set \( M' = M\psi^{-1} \) and consider the surjective homomorphism \( \lambda = \psi|_M : M' \to M \). Since \( D_{\tilde{\alpha}^0} \) divides an elementary abelian \( p \)-group, [22, Theorem 7.1] implies that the derived category of \( \psi \), and hence of its restriction \( \lambda \) (using [22, Proposition 5.12]), divides an elementary abelian \( p \)-group. As \( V \) contains a cyclic group of order \( p \) and \( M \in V \), we have \( M' \in (V \cap \text{Ab}) * V = V \) by the Derived Category Theorem [23].

Let \( A' = H\lambda^{-1} \) be the preimage of \( H \) in \( M' \). We claim that the map \( \rho : A' \to A \) given by \((f, [h]) \mapsto H f \) is a homomorphism such that the diagram

\[
\begin{array}{ccc}
A' & \xrightarrow{\lambda} & H \\
\rho \downarrow & & \downarrow \tau \\
\tilde{A} & \xrightarrow{\tilde{\alpha}} & H
\end{array}
\] (3.3)

commutes. Let us assume this for the moment and complete the proof. Since \( F_V(X) \) is free, the natural projection \( F_V(X) \to M \) (whose restriction to \( G \) is \( \varphi \)) can be lifted to a continuous homomorphism \( \eta : F_V(X) \to M' \). Then \( G\eta \leq A' \) and \( \eta \rho \) lifts \( \varphi \), contradicting the non-existence of a lift in (3.2).

Since (3.3) commutes by definition of \( \rho \) and \( \tau \), we just need to check that \( \rho \) is a homomorphism. Well, if \((f, [h]) \) and \((f', [h']) \) are in \( A' \) (and so we may take \( h, h' \in H \)), then \((f, [h])(f', [h']) = (f[h]f', [hh']) \). Since \( h \in H \), we have \( ((f, [h])(f', [h']))\rho = H(f[h]f') = Hf \cdot Hh f' = Hf \cdot Hf' = (f, [h])\rho(f', [h'])\rho \). This completes the proof of Theorem [2].

4. Proof of Theorem [2]

This proof uses the Henckell-Schützenberger expansion (see [6,13] for details). If \( X \) is a set, then \( X^* \) denotes the free monoid on \( X \). If \( M \) is an \( X \)-generated monoid, then one can define a congruence \( \equiv \) on \( X^* \) by \( w_1 \equiv w_2 \) if and only if, for each factorization \( w_1 = u_1 v_1 \), there is a factorization \( w_2 = u_2 v_2 \) with \([u_1]_M = [u_2]_M \), \([v_1]_M = [v_2]_M \), and conversely. The quotient \( \overline{M} = M/\equiv \) is finite if \( M \) is finite and there is a natural surjective homomorphism \( \eta : \overline{M} \to M \) of \( X \)-generated monoids. Moreover, \( \eta \) is an aperiodic morphism, meaning that \( e\eta^{-1} \in A \) for each idempotent \( e \in M \) (see [6,13]). In particular, if \( A \boxtimes V = V \), then \( M \in V \) implies \( \overline{M} \in V \). The monoid \( \overline{M} \) is called the Henckell-Schützenberger expansion of \( M \).

Let \( S \) be a finite subsemigroup of \( F_V(X) \). By finiteness, one can find a continuous surjective homomorphism \( \varphi : \overline{F_V(X)} \to M \) with \( M \in V \) such

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1If one is just interested in Corollary [3] then it is easy to verify directly that each subgroup of \( M' \) is an extension of an elementary abelian \( p \)-group by a subgroup of \( M \).
that $\varphi|_{S}$ is injective. If $Y = X\varphi$ (which is a finite set), then $\varphi$ must factor through $\widehat{F}_{V}(Y)$ and so we deduce that $S$ is a finite subsemigroup of $\widehat{F}_{V}(Y)$. So without loss of generality, we may assume that $X$ is finite.

Suppose that $S$ is not completely regular. We shall arrive at a contradiction. As $S$ is not completely regular, there is an element $\sigma \in S$, which is not a group element, such that $\sigma^{2}$ is a group element. Then $\sigma^{2} = \sigma^{2+k}$ for some $k > 0$. Without loss of generality we may assume that $k \geq 2$. Now $\tilde{M} \in V$ by our hypothesis and so $\varphi$ must factor continuously through $\eta: \tilde{M} \to M$.

Since $X^{*}$ is dense in $\widehat{F}_{V}(X)$, we can find a word $w \in X^{*}$ such that $[w]_{\tilde{M}} = [\sigma]_{\tilde{M}}$. Then we have $[w^{2}]_{\tilde{M}} = [\sigma^{2}]_{\tilde{M}} = [\sigma^{k}]_{\tilde{M}} = [w^{k}]_{\tilde{M}}$. Hence there is a factorization $w^{k} = w^{k_{1}}x_{1}y_{1}w^{k_{2}}$ with $x, y \in X^{*}$ such that $w = xy$, $k_{1}, k_{2} \geq 0$, $k_{1} + k_{2} + 1 = k$ and $[w^{k_{1}}x]_{M} = [w]_{M}$ and $[yw^{k_{2}}]_{M} = [w]_{M}$. Therefore, we have the equations:

$$\begin{align*}
\left[\sigma^{k_{1}}\right]_{M}[x]_{M} &= [\sigma]_{M} \\
[y]_{M} \left[\sigma^{k_{2}}\right]_{M} &= [\sigma]_{M}.
\end{align*}$$

(4.1)

As $k_{1} + k_{2} + 1 = k \geq 4$, it cannot be the case that both $k_{1}, k_{2} \leq 1$. We conclude from (4.1) that $[\sigma]_{M} \not\rightarrow [\sigma^{2}]_{M}$ and so $[\sigma]_{M}$ is a group element $[7, 10, 16]$. Since $\varphi|_{S}$ is injective, $\sigma$ itself is a group element, a contradiction. This completes the proof of Theorem 2.

5. An application to projective semigroups

A finite semigroup $S$ is said to be projective [18] if any onto homomorphism $\varphi: T \to S$, with $T$ finite, splits (i.e. there exists $\psi: S \to T$ with $\psi\varphi = 1_{S}$). We write $\widehat{X}^{+}$ for the free profinite semigroup generated by $X$.

**Lemma 9.** Let $S$ be an $X$-generated finite projective semigroup. Then the canonical continuous projection $\varphi: \widehat{X}^{+} \to S$ splits and so $S$ embeds in $\widehat{X}^{+}$.

**Proof.** Write $\widehat{X}^{+} = \lim_{\leftarrow} S_{i}$ where the $S_{i}$ are $X$-generated finite semigroups; we may assume that each $S_{i}$ maps onto $S$ via the projection $\varphi_{i}: S_{i} \to S$. It’s easy to see that the isomorphism $\text{Hom}(S, \widehat{X}^{+}) \cong \lim_{\rightarrow} \text{Hom}(S, S_{i})$ sends the splittings of $\varphi$ to the inverse limit of the splittings of the $\varphi_{i}$. As the inverse limit of finite non-empty sets is non-empty, this completes the proof.

Since $\widehat{X}^{+}$ embeds in the free profinite monoid on $X$, we obtain as a consequence of Lemma 9 and Corollary 3 the following theorem.

**Theorem 10.** Every finite projective semigroup is a band.

There are many examples of non-trivial projective semigroups, such as chains of idempotents or left/right zero semigroups [18].

**References**

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