INVARANTS IN C ∪ P-GEOMETRY

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Abstract. This note describes the construction of c ∪ p -invariant differential operators on statistical manifolds, i.e. of operators canonically associated to a geometry which synthetizes the properties of conformal and projective geometries.

1. Introduction

Let (S, g, t) be a statistical manifold, i.e. a Riemannian manifold (S, g) endowed with a fully symmetric 3 covariant tensor field t, the skewness tensor. This tensor has been introduced to formalize the notion of statistical curvature through the introduction of a family {α∇}α∈R of affine connections defined as follows [1][2]

\[ \tilde{\nabla}_x Y = \nabla_x Y - \frac{\alpha}{2} (t \cdot g^{-1})(X, Y), \]  

(1.1)

for any couple X, Y of vector fields over S, \( \tilde{\nabla} \) denoting the canonical Levi-Civita connection and the dot (.) an obvious contraction. In fact \( \tilde{\nabla} \) is the unique torsion-free connection satisfying

\[ \tilde{\nabla} g = \alpha t, \]  

(1.2)

and \( (\tilde{\nabla}, g) \) form a Codazzi pair [3], i.e. Codazzi equations are satisfied

\[ (\tilde{\nabla}_x g)(Y, Z) = (\tilde{\nabla}_y g)(X, Z). \]

In a previous paper [4], we have dealt with the so-called c ∪ p-geometry, where the c ∪ p symbol refers to a conformal implementation of the projective geometry in which the properties of both conformal and projective geometries are preserved, as it can be seen from the following

Definition. Let \( \alpha \) be a real number. Statistical manifolds \( (S, g, t) \) and \( (\tilde{S}, \tilde{g}, \tilde{t}) \) are \( \alpha \)-c ∪ p-related if, for some positive smooth function \( \eta \) over \( S \),

\[ \tilde{g} = \eta g, \quad \tilde{t} = \eta (t + \text{sym}(g \otimes \psi)) \]

where \( \psi \) is an exact 1-form satisfying \( d(\text{Log } \eta) = -\alpha \psi \).

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Proposition. For any real $\alpha$, the families of $\alpha$-connections of two $\alpha$-$c \cup p$-related statistical manifolds are projectively related, i.e. the corresponding 1-form connections satisfy the following relation
\[
\varphi^\alpha - \varphi^\alpha = -\alpha (\theta \otimes \psi + (\psi \cdot \theta) \text{Id}),
\]
and the curvature 2-forms are such that
\[
\tilde{\Phi}^\alpha - \Phi^\alpha = \alpha \theta \wedge \tilde{\nabla} \psi + \alpha^2 (\theta \otimes \psi) \wedge (\psi \cdot \theta).
\]

The invariants of the $c \cup p$-structure themself are described in [4], here we want to study the closely related question of invariance of auxiliary objects associated to the $c \cup p$-structure.

2. $c \cup p$-invariant second order differential operators

As the conformal one’s, the $c \cup p$-geometry depends on the (conformal) $\eta$ factor only, so we take up the step traced by the conformal experience by saying that a differential operator $D$, depending on a choice of the couple $(g,t)$, is $c \cup p$-invariant of type $(r;s)$ if the operator $\tilde{D}$, corresponding to the rescaled metric $\tilde{g}$ and to the modified tensor $\tilde{t}$, is such that $\tilde{D}f = \eta^s Df$, where $f$ is a density of weight $r$, section of a suitable line bundle, and $f = \eta^rf$, $r, s \in \mathbb{R}$.

Let us consider the $\alpha$-Hessian $\tilde{\nabla} \otimes d$ acting on functions on $S$, we look for the existence of a modification of it, form invariant for the $c \cup p$-geometry. Knowing that conformal invariant operators may be expressed in terms of the Levi-Civita connection and the Ricci curvature of a metric in the conformal class [5], we study the operator $\tilde{\nabla} \otimes d + k \tilde{Ric}^\alpha$, $k \in \mathbb{R}$, where $\tilde{Ric}^\alpha$ denotes the Ricci tensor of the $\alpha$-connection.

Starting from Rel(1.4) it can be shown that the Ricci tensors of two $\alpha$-$c \cup p$-related connections are such that
\[
\tilde{\text{Ric}}^\alpha = \text{Ric}^\alpha + (n - 1)\alpha \{ \tilde{\nabla} \psi + \alpha (\psi \otimes \psi) \},
\]
where $n = \text{dim} S$.

Then, by using Rel(1.3) one gets
\[
(\tilde{\nabla}^\alpha \otimes d)(\eta^f) = \eta^r (\nabla^\alpha \otimes d) f + f (\nabla^\alpha \otimes d) \eta^r + d(\eta^r) \otimes \nabla^\alpha f + df \otimes \nabla^\alpha (\eta^r),
\]
taking into account for $d(\text{Log} \eta) = -\alpha \psi$, one establishes the following

Proposition. The modified Hessian $\tilde{\nabla} \otimes d + k \tilde{\text{Ric}}^\alpha$ is $\alpha - c \cup p$-invariant of type $(1; 1)$ for $k = \frac{1}{n-1}$.

\[
(\tilde{\nabla}^\alpha \otimes d + \frac{1}{n-1} \tilde{\text{Ric}}^\alpha) \tilde{f} = \eta (\nabla^\alpha \otimes d + \frac{1}{n-1} \text{Ric}^\alpha) f.
\]

Remark. The integrability conditions of the differential system $(\tilde{\nabla} \otimes d + k \text{Ric}^\alpha) f = 0$ are given by
\[
\text{Riem}^\alpha = k (\text{Id} \otimes \text{Ric}^\alpha - \text{Ric}^\alpha \otimes \text{Id}).
\]
Except for the flat case $Riem^\alpha = 0$, this property is verified to hold, by taking the trace, for the $c \cup p$-invariant Hessian only, and for statistical manifolds such that

$$Riem^\alpha = \frac{1}{n-1} (Id \otimes Ric^\alpha - Ric^\alpha \otimes Id).$$  \hspace{1cm} (2.4)

For instance this property is satisfied for the statistical manifold associated to the gaussian law and for the statistical manifold of multinomial distributions.

3. The $c \cup p$-Laplacian

Let us consider the trace of the Ricci modified Hessian operator

$$g^{-1}( \alpha \nabla \otimes d + k Ric^\alpha) = g^{-1}( \alpha \nabla \otimes d) + k R^\alpha$$

where $R^\alpha$ denotes the scalar $\alpha$-curvature $R^\alpha = g^{-1} (Ric^\alpha)$. This expression can be written by using the $\alpha - \text{Laplacian} = \text{div}^\alpha \text{grad}$, denoted by $\Delta$, but we have to take into account for the non-metricity of the connection; so we are led to give the

Definition. The second order differential operator given by

$$\Delta_{c \cup p} = g^{-1}( \alpha \nabla \otimes d + \frac{1}{n-1} Ric^\alpha) = \alpha \Delta + \alpha g^{-1}.t.g^{-1}.d + \frac{1}{n-1} R^\alpha,$$ \hspace{1cm} (3.1)

where each point denotes an obvious contraction due to the symmetry of implied tensor fields, is called the $c \cup p - \text{Laplacian}$.

Indeed this operator is associated to a $c \cup p$ -structure canonically as it is clear from the following

Proposition. \hspace{1cm} $\tilde{\Delta}_{c \cup p} \tilde{f} = \Delta_{c \cup p} f$, \hspace{1cm} ($\tilde{f} = \eta f$), \hspace{1cm} (3.2)

the $c \cup p - \text{Laplacian}$ is a $c \cup p - \text{invariant differential operator of type (1;0)}$.

This property stems from Rel(2.3) by noting that $\tilde{g}^{-1} = \frac{1}{\eta} g^{-1}$.

4. Quasi-$c \cup p$-invariant non-linear operators

We now wish to construct a non-linearly modified $c \cup p$- Laplacien through a self-interaction term $f^a$ with a coupling potential $\lambda$, function on $S$. We can use property (3.2) to get immediately the

Proposition. \hspace{1cm} $\tilde{\Delta}_{c \cup p} \tilde{f} + \tilde{\lambda} \tilde{f}^a = \Delta_{c \cup p} f + \lambda f^a$, \hspace{1cm} ($\tilde{f} = \eta f, \tilde{\lambda} = \lambda/\eta^a, a = \text{constante} \neq 0$). \hspace{1cm} (4.1)

It is worth noticing that there is no constraint over the non-linearity coefficient $a$ contrarily to the conformal case for which $a$ depends on the dimension of the manifold [6].
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