Improved Alternating Direction Implicit Method

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Abstract. The alternating direction implicit method (ADI) is a common classical numerical method that was first introduced to solve the heat equation in two or more spatial dimensions and can also be used to solve parabolic and elliptic partial differential equations as well. In this paper, we introduce an improvement to the alternating direction implicit (ADI) method to get an equivalent scheme to Crank-Nicolson differences scheme in two dimensions with the main feature of ADI method. The new scheme can be solved by similar ADI algorithm with some modifications. A numerical example was provided to support the theoretical results in the research.

1. Introduction:
The alternating direction implicit (ADI) method was first proposed in the first place for partial differential parabolic equations in two spatial dimensions by D. Peaceman and H. Rachford in 1955 [1], they produce the ADI method to solve multidimensional petroleum simulators reservoir, which is between the multi-scale many types of systems which that require implicit discretization. For solving the problem of any useful size, memory-efficient, fast converging methods are needed to solve the large linear equations that arise at each time step [2]. Although computers at that time were of limited capacity, they were able to use this method to solve the problem of heat diffusion in two spatial dimensions. Later ADI method developed to solve other problems and became a significant approach in numerical methods to solve different type of partial differential equations in two or more dimensions [3, 4, 5].

Consider the two-dimensional heat equation:

\[
\frac{\partial u(x,y,t)}{\partial t} = \sigma \left( \frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2} \right)
\]

\[
u(x,y,0) = \varphi(x,y) \quad (x,y) \in \Omega \cup \partial \Omega
\]

\[
u(x,y,t) = \psi(x,y) \quad (x,y) \in \partial \Omega
\]

Where \((x,y) \in \Omega, \Omega = \{(x,y) | 0 < x < 1, 0 < y < 1\}, \sigma \) is a positive constant. We will consider the rectangle domain \(0 < x < 1, 0 < y < 1\) with Dirichlet boundary conditions, so that \(u(x,y,t)\) is given at all rectangular boundary points, for all \(t > 0\) and an initial condition \(u(x,y,0)\) is given. The region is covered with a uniform rectangular grid of points, with a spacing \(h = \Delta x\) in the \(x\)-axis and \(k = \Delta y\) in the \(y\)-axis, where \(h = \frac{1}{N_x}, k = \frac{1}{N_y}, \Delta t = \tau, N_x, N_y\) are positive integer numbers, Which
denote the approximated solution is then the finite difference $u_{lm}^n = u(lh, mk, n\tau) = u(x, y, t)$, $l = 0, 1, 2, ..., N_x$, $m = 0, 1, 2, ..., N_y$, for simplicity suppose ($N_x = N_y = N$).

The explicit finite difference schemes are used for solving such problems but these schemes are conditionally stable so the time the step must take a small value to achieve the stability conditions, while the implicit finite difference schemes are unconditionally stable but these schemes lead to a linear large system of equations must be solved, solving $(N - 1)^2$ linear equations.

The mentioned method to solve the heat conduction equation is the Crank-Nicolson method which it like implicit method need the same number of equations in implicit method to solve at every time step. But with the ADI method we need to solve $(N - 1)$ systems of linear equations and every system consist of $(N - 1)$ of linear equations. After fifty years of their pioneering work on alternating direction implicit methods, D. Peaceman and H. Rachford attended a conference organized to honor them and celebrate a legacy that continues to grow [2].

2. Theory and Calculations:

To explain the advantages of the ADI method for the parabolic equation we will consider the explicit scheme, the implicit and the Crank Nicolson finite difference Scheme for equation (1) with their basic properties.

2.1. Explicit Finite Difference Scheme:

The simplest difference analog to equation (1) is the explicit finite difference scheme which can be found by replacing the time derivative $\frac{\partial u}{\partial t}$ use the difference forward approximation at the point $(x, y, t)$ and the space derivatives $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$ with the central difference approximation at the same grid point, then the explicit finite difference scheme has the following difference equation

$$u_{lm}^{n+1} - u_{lm}^n = \sigma \left( \frac{1}{h^2} \delta_x^2 u_{lm}^n + \frac{1}{k^2} \delta_y^2 u_{lm}^n \right)$$

(2)

where $\delta_x^2 u_{lm}^n = u_{l+1,m}^n - 2u_{lm}^n + u_{l-1,m}^n$, $\delta_y^2 u_{lm}^n = u_{l,m+1}^n - 2u_{lm}^n + u_{l,m-1}^n$ and can be rewritten as:

$$u_{lm}^{n+1} = (1 + r_x \delta_x^2 + r_y \delta_y^2)u_{lm}^n$$

(3)

where $r_x = \frac{\sigma \tau}{h^2}$ and $r_y = \frac{\sigma \tau}{k^2}$. This is the explicit scheme which can be solved explicitly for $u_{lm}^{n+1}$, it is stable with conditions and the stability condition is

$$r_x + r_y \leq \frac{1}{2}$$

(4)

For the case $h = k$ the condition of stability becomes

$$r_x = \frac{\sigma \tau}{h^2} \leq \frac{1}{4}$$

(5)

That it is as restrictive twice as the one dimensional case [6, 7].

2.2 Implicit Finite Difference Scheme:

The implicit finite difference scheme can be obtained by replacing the derivative time $\frac{\partial u}{\partial t}$ using forward difference approximation at the point $(x, y, t)$ and the space derivatives $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$ with the central difference approximation at the grid point $(x, y, t_{n+1})$, then the implicit finite difference scheme has the following difference scheme

$$u_{lm}^n = u_{lm}^{n+1} - r_x \delta_x^2 u_{lm}^{n+1} - r_y \delta_y^2 u_{lm}^{n+1}$$

(6)

or
The implicit scheme is unconditionally stable [6, 7], but leads to large number of linear equations which are more difficult to solve than the explicit scheme.

2.3 Crank Nicolson Finite Difference Scheme:

It is another implicit difference scheme and can be found by replacing the time derivative $\frac{\partial u}{\partial t}$ using forward difference approximation at the point $(x_t, y_m, t_n)$ and the space derivatives $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$ with the central difference approximation at the two grid points $(x_t, y_m, t_n)$ and $(x_t, y_m, t_{n+1})$ and take the average then the Crank Nicolson difference scheme given by:

$$u_{t,m}^{n+1} - u_{t,m}^{n} = \frac{1}{2}[(r_x \delta_x^2 + r_y \delta_y^2)u_{t,m}^{n} + (r_x \delta_x^2 + r_y \delta_y^2)u_{t,m}^{n+1}]$$

which can be rewritten as

$$\left[1 - \frac{1}{2}(r_x \delta_x^2 + r_y \delta_y^2)\right]u_{t,m}^{n+1} = \left[1 + \frac{1}{2}(r_x \delta_x^2 + r_y \delta_y^2)\right]u_{t,m}^{n}$$

3. ADI method:

The ADI method is a finite difference method for two dimensional (or more) heat flow and diffusion problems. The main idea of the ADI method is to divide the scheme from $t$ to $t + \Delta t$ into two steps, in the first half step, from $t$ to $t + \frac{\Delta t}{2}$, treating one of the spatial derivatives implicitly (say $\frac{\partial^2 u}{\partial x^2}$) and treating the other derivative (say $u_{y,m}$) explicitly, this lead to the difference equation:

$$u_{t,m}^{n+\frac{1}{2}} - u_{t,m}^{n} = \frac{r_x}{2} \delta_x^2 u_{t,m}^{n+\frac{1}{2}} + \frac{r_y}{2} \delta_y^2 u_{t,m}^{n}$$

The matrix of the unknowns $u_{t,m}^{n+\frac{1}{2}}$ will appearing in (10) as a block tridiagonal linear algebraic system of equations and that can solved by the algorithm of tridiagonal linear system. For the second step reverse the treating of the spatial derivatives, i.e. from $t + \frac{\Delta t}{2}$ to $t + 1$, treating $\frac{\partial^2 u}{\partial x^2}$ explicitly and treating $\frac{\partial^2 u}{\partial y^2}$ implicitly and this lead to the second difference equation:

$$u_{t,m}^{n+1} - u_{t,m}^{n+\frac{1}{2}} = \frac{r_x}{2} \delta_x^2 u_{t,m}^{n+1} + \frac{r_y}{2} \delta_y^2 u_{t,m}^{n+\frac{1}{2}}$$

The unknowns $u_{t,m}^{n+1}$ in (11) will appearing like equation (10) as a block tridiagonal linear system of algebraic equations and can be solved by the same algorithm.

The two equations (10) and (11) consist the ADI scheme. The ADI Scheme is unconditional stability with simplicity in calculation. Nowadays there are many versions of the method, with applications to elliptic and hyperbolic problems as well as to systems of parabolic equations.

For comparing ADI with the Crank Nicolson scheme consider equation (10), that rewritten as:

$$\left(1 - \frac{r_x}{2} \delta_x^2\right)u_{t,m}^{n+\frac{1}{2}} = \left(1 + \frac{r_y}{2} \delta_y^2\right)u_{t,m}^{n}$$

or

$$u_{t,m}^{n+\frac{1}{2}} = \left(1 - \frac{r_x}{2} \delta_x^2\right)^{-1} \left(1 + \frac{r_y}{2} \delta_y^2\right)u_{t,m}^{n}$$

similarly, equation (11) can be rewritten as:

$$u_{t,m}^{n+1} = \left(1 + \frac{r_x}{2} \delta_x^2\right)u_{t,m}^{n+\frac{1}{2}} + \frac{r_y}{2} \delta_y^2 u_{t,m}^{n+\frac{1}{2}}$$
from equation (13) and equation (14) we can get
\begin{equation}
(1 + \frac{r_x}{2} \delta_x^2) \left(1 - \frac{r_x}{2} \delta_x^2 \right)^{-1} \left(1 + \frac{r_y}{2} \delta_y^2 \right) u_{t,m}^n + \frac{r_x}{2} \delta_x^2 u_{t,m}^{n+1}
\end{equation}

By simplifying the equation we get
\begin{equation}
(1 - \frac{r_y}{2} \delta_y^2) u_{t,m}^{n+1} = \left(1 + \frac{r_x}{2} \delta_x^2 \right) \left(1 - \frac{r_x}{2} \delta_x^2 \right)^{-1} \left(1 + \frac{r_y}{2} \delta_y^2 \right) u_{t,m}^n
\end{equation}
or
\begin{equation}
(1 - \frac{r_x}{2} \delta_x^2) \left(1 - \frac{r_y}{2} \delta_y^2 \right) u_{t,m}^{n+1} = \left(1 + \frac{r_x}{2} \delta_x^2 \right) \left(1 + \frac{r_y}{2} \delta_y^2 \right) u_{t,m}^n
\end{equation}
with more simplifying we get
\begin{equation}
\left[1 - \frac{1}{2} (r_x \delta_x^2 + r_y \delta_y^2) + \frac{1}{4} r_x r_y \delta_x^2 \delta_y^2\right] u_{t,m}^{n+1} = \left[1 + \frac{1}{2} (r_x \delta_x^2 + r_y \delta_y^2) + \frac{1}{4} r_x r_y \delta_x^2 \delta_y^2\right] u_{t,m}^n
\end{equation}
Now by comparing equation (18) with the Crank Nicolson scheme, equation (9) we can notice that
ADI consider
\begin{equation}
\left(\frac{1}{4} r_x r_y \delta_x^2 \delta_y^2\right) u_{t,m}^{n+1} = \left(\frac{1}{4} r_x r_y \delta_x^2 \delta_y^2\right) u_{t,m}^n
\end{equation}
in two time steps, \(n\) and \(n+1\).

4. Improved ADI
ADI semi-implicit method is because it expresses one spatial derivative in an explicit difference scheme and the other spatial derivative in implicit difference scheme, if we have a three-dimensional problem then the ADI method will be a one-third implicit method and so on.
In this work, we introduce an improvement to the ADI method to get a finite difference scheme similar to the Crank-Nicolson scheme as follows:
\begin{equation}
\frac{u_{t,m}^{n+1} - u_{t,m}^n}{r_x \delta_x^2 u_{m+1}^{n+1} + r_y \delta_y^2 u_{t,m}^n}
\end{equation}
\begin{equation}
\frac{u_{t,m}^{n+1} - u_{t,m}^n}{r_x \delta_x^2 u_{m+1}^{n+1} + r_y \delta_y^2 u_{t,m}^n}
\end{equation}
Then by addition and divided by 2 for equations (20) and (21) we get the average:
\begin{equation}
\frac{u_{t,m}^{n+1} = \frac{1}{2} \left[ u_{t,m}^{n+1} + u_{t,m}^{n+1} \right]}
\end{equation}
The three equations (20), (21) and (22) are consist the Improved ADI scheme. The main idea in improved ADI method is of only one of the 2nd-order replaced derivatives, like ADI method, using implicit finite difference approximation in terms of unknown values of \(u\) from \((n+1)\)th level time, and the other 2nd-order derivative being replaced by an explicit finite difference approximation then solve the resulting system to get the first solution. And repeat this for the second derivative to the same time level and get the second solution, then gather the two solutions and divide them by two. With this technique, we will get a scheme similar to the Crank Nicolson scheme.
The equations (20) and (21) can be solved by tridiagonal matrix algorithm. Both \(u_{t,m}^{n+1}\) and \(u_{t,m}^{n+1}\) represent a solution of equation (1) so replaced by an \(u_{t,m}^{n+1}\), and the equations (20) and (21) can be rewritten as:
\begin{equation}
(1 - r_x \delta_x^2) u_{t,m}^{n+1} = (1 + r_y \delta_y^2) u_{t,m}^n
\end{equation}
\begin{equation}
(1 - r_y \delta_y^2) u_{t,m}^{n+1} = (1 + r_x \delta_x^2) u_{t,m}^n
\end{equation}
Add (23) with (24) to get
\[
\left[ (1 - r_x \delta_x^2) + (1 - r_y \delta_y^2) \right] u_{l,m}^{n+1} = \left[ (1 + r_x \delta_x^2) + (1 + r_y \delta_y^2) \right] u_{l,m}^n
\]
By simplifying the equation, we can get
\[
2u_{l,m}^{n+1} - r_x \delta_x^2 u_{l,m}^{n+1} - r_y \delta_y^2 u_{l,m}^{n+1} = 2u_{l,m}^n + r_x \delta_x^2 u_{l,m}^n + r_y \delta_y^2 u_{l,m}^n
\]
or
\[
2(u_{l,m}^{n+1} - u_{l,m}^n) = r_x \delta_x^2 (u_{l,m}^{n+1} + u_{l,m}^n) + r_y \delta_y^2 (u_{l,m}^{n+1} + u_{l,m}^n)
\]
\[
u_{l,m}^{n+1} - u_{l,m}^n = \frac{1}{2} r_x \delta_x^2 (u_{l,m}^{n+1} + u_{l,m}^n) + \frac{1}{2} r_y \delta_y^2 (u_{l,m}^{n+1} + u_{l,m}^n)
\]
and this led to:
\[
\left[ 1 - \frac{1}{2} (r_x \delta_x^2 + r_y \delta_y^2) \right] u_{l,m}^{n+1} = \left[ 1 + \frac{1}{2} (r_x \delta_x^2 + r_y \delta_y^2) \right] u_{l,m}^n
\]
We can notice that this is the same Crank Nicolson finite difference scheme (9). And so the improved ADI method has the same accuracy as Crank-Nicolson method.

4.1 Stability Analysis of Improved ADI method:
We will use the Von Neumann method to find the stability condition for the improved ADI finite difference scheme [8]. It is common to write
\[
u_{l,m}^n = \xi^n e^{i \beta \theta} e^{i \gamma mk}
\]
where \( \beta \) and \( \gamma \) are real spatial wave numbers and \( \xi \) is the amplification factor.

Theorem 1: The improved ADI finite difference scheme is unconditionally stable.

Proof: The finite difference scheme (20) and (21) and (22) can be rewritten as
\[
u_{l,m}^{n+1} = \frac{1}{2} \left[ u_{l,m}^{n+1} + u_{l,m}^{n+1} \right]
\]
use the expression (30) to get we obtain
\[
2\xi^n e^{i \beta \theta} e^{i \gamma m k} = 2\xi^n e^{i \beta \theta} e^{i \gamma m k} + r_x (\xi^{n+1} e^{i \beta (l-1) h} e^{i \gamma m k}) - 2\xi^{n+1} e^{i \beta \theta} e^{i \gamma m k}
\]
We consider both \( u_{l,m}^{n+1} \) and \( u_{l,m}^{n+1} \) as a solution of equation (1) in the level \( n + 1 \) so we can consider \( \xi^{n+1} \) and \( \xi^{n+1} \) as \( \xi^{n+1} \). Divided the above equation by \( \xi^n e^{i \beta \theta} e^{i \gamma m k} \) to get
\[ 2\xi = 2 + r_x (\xi e^{-ibh} - 2\xi + \xi e^{ibh}) + r_y (\xi e^{-i\beta k} - 2 + e^{i\beta k}) + r_x (\xi e^{-ibh} - 2 + e^{ibh}) + r_y (\xi e^{-i\gamma k} - 2\xi + \xi e^{i\gamma}) \]

using the formula \((e^{i\theta} - 2 + e^{-i\theta}) = -4\sin^2 \frac{\theta}{2}\) to get

\[ 2\xi = 2 + r_x \xi \left(-4\sin^2 \frac{\beta h}{2}\right) + r_y \left(-4\sin^2 \frac{\gamma k}{2}\right) + r_x \left(-4\sin^2 \frac{\beta h}{2}\right) + r_y \xi \left(-4\sin^2 \frac{\gamma k}{2}\right) \]

rearrange the equation and divided by two to get

\[ \xi + r_x \xi \left(2\sin^2 \frac{\beta h}{2}\right) + r_y \xi \left(2\sin^2 \frac{\gamma k}{2}\right) = 1 + r_y \left(-2\sin^2 \frac{\gamma k}{2}\right) + r_x \left(-2\sin^2 \frac{\beta h}{2}\right) \]

this lead to

\[ \xi = \frac{1 - 2r_y \sin \frac{\gamma k}{2} - 2r_x \sin \frac{\beta h}{2}}{1 + 2r_x \sin \frac{\beta h}{2} + 2r_y \sin \frac{\gamma k}{2}} \quad (31) \]

For stability we require \(|\xi| \leq 1\), and from equation (31) for all values of \(r_x, r_y, \beta, \gamma\) This ratio has an absolute value less than or equal to one.

**Table 1**: Results of the example when \(\Delta x = \Delta y = \Delta t\) with \(T = 0.5\).

| \(N \times M\) | ADI Method | Improve ADI |
|----------------|------------|-------------|
|                | Average Error | Max Error | Average Error | Max Error |
| 1 10 \times 10 | 0.003791562244134 | 0.013263668239204 | 0.001359974472108 | 0.004383685897438 |
| 2 15 \times 15 | 0.001990674075311 | 0.006252252554629 | 0.000279746819906 | 0.000995690415892 |
| 3 20 \times 20 | 0.001145347137462 | 0.003448770747655 | 0.000576055621512 | 0.0039504353320 |
| 4 25 \times 25 | 0.000777710853193 | 0.00226118533172 | 0.00144191125440 | 0.000730750281131 |
| 5 30 \times 30 | 0.000541626617549 | 0.001554638768872 | 0.0008459496706 | 0.00065184988994 |
| 6 35 \times 35 | 0.000411224398929 | 0.001161040587889 | 0.00054859072335 | 0.00082448762950 |
| 7 40 \times 40 | 0.000314098601357 | 0.000875586629557 | 0.00064951419127 | 0.000604186421875 |

**Numerical Example**: For comparison between the improved ADI and ADI, we will consider the following diffusion equation in two dimensions

\[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f(x, y, t) \]

where \(f(x, y, t) = (2\pi^2 - 1)u\) on the domain \(0 < x < 1, 0 < y < 1\) with \(0 \leq t\) and the initial condition \(u(x, y, 0) = \cos(\pi x)\sin(\pi y)\) and Dirichlet boundary conditions on the rectangle in the form \(u(0, y, t) = -u(1, y, t) = e^{-\tau}\cos(\pi y), u(x, 0, t) = u(x, 1, t) = 0\). The exact solution is given by \(u(x, y, t) = e^{-\tau}\cos(\pi x)\sin(\pi y)\).

The table (1) represents the results of the example with different values of \(N \times M\) with two error measures, the average error and the maximum of errors.

**5. Results and Conclusion:**

The Improved ADI is stable with out condition and consistent with a local truncation error \(O((\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2)\) as Crank Nicolson method. Then by Lax’s Equivalence Theorem [8], the converge conditions are satisfied. The ADI method has a local truncation error \(O((\Delta t) + (\Delta x)^2 + (\Delta y)^2)\). To solve the two dimensional diffusion equation by Crank Nicolson method we need to solve a linear system of \((N - 1)^2\) equation in every time step. But The ADI techniques reduce the Number of arithmetic operation, we need to solve \((n - 1)\) linear system and every system have \((N - 1)\) linear equations at every half time step. But with improved ADI method we need to solve \((n - 1)\) linear system and every system have \((N - 1)\) linear equations two times at every time step.
The numerical examples show that the improved ADI method have a good agreement with the theoretical findings. In this paper we consider the diffusion equation in two dimensions, it can be possibly generalized and extended to elliptic and hyperbolic problems and for more than two dimensions.

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