(Almost) Contact (3) (Metric) Structure(s) and Transverse $SU(3)$ Structures Associated with $\mathcal{M}$-Theory Dual of Thermal QCD at Intermediate Coupling

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Abstract

Relevant to top-down $\mathcal{M}$-theory dual of large-$N$ thermal QCD-like theories at intermediate coupling, induced by the $G_2$ structure supported by the closed $M_7$ - a warped product of the $\mathcal{M}$-theory circle and a non-Kähler six-fold which is the warped product of the thermal circle with a non-Einsteinian deformation of $T^{1,1}$ - we obtain explicitly (Almost) Contact (3) (Metric) Structure(s) and a three-tuple of $SU(3)$ structures along the “transverse” six-fold induced by the AC(3)MS on $M_7$. (Assuming the arguments to also hold for non-closed manifolds, similar structures can be obtained on the non-closed $\tilde{M}_7$ - a warped product of the $\mathcal{M}$-theory circle and a non-Kähler cone-fold.) The aforementioned ACS do(es) not yield a Contact Structure. We separately obtain an explicit Contact Structure on the seven-fold $M_7$, and then obtain an explicit transverse $SU(3)$ structure from the same. These results are expected to be very useful in the context of classification of non-supersymmetric geometries relevant to top-down holographic thermal QCD at intermediate coupling.

One of us (AM) dedicates this paper to the memory of Professor Asoke Nath Mitra - an unparalleled academic and sentient being

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1 Introduction

$G_2$-structure manifolds are ubiquitous to fluxed $\mathcal{M}$-theory compactifications [1]. Manifolds with Almost Contact Structures (which induce a complex structure on the “transverse geometry”) could be thought of as odd-dimensional analogs of even-dimensional manifolds equipped with almost complex structures. The obstruction of the existence of nowhere vanishing vector fields on closed manifolds essential to the existence of ACS, is non-vanishing Euler characteristic. As the Euler characteristic of odd-dimensional manifolds is vanishing, therefore odd-dimensional manifolds may support ACS. Seven-dimensional manifolds equipped with $G_2$-structure are known to provide an Almost Contact 3 Structure (AC(3)S). The existence of nowhere-vanishing vector fields implies that the structure group of the tangent bundle is reducible which has an implication on the amount of SUSY preserved by the vacuum. Also, a seven-fold supporting a $G_2$ structure can be shown to admit an Almost Contact Metric Structure (ACMS) and this implies that the structure group reduces to $SU(3)$. In fact, there exist at least three non-zero vector fields on a manifold with $G_2$ structure which provides the manifold with an AC3MS.

The use of $G$-structure torsion classes is a very useful tool for classifying, specially non-Kähler geometries. A complete classification of the $SU(n)$ structures relevant to non-supersymmetric string vacua, does not exist [2].

In the context of $SU(3)$-structure manifolds, non-supersymmetric type II vacua on $SU(3)$-structure manifolds were studied in [3] and classified using calibrations in [4], and similar solutions in heterotic string theory were obtained in [5] - see [6] for $G_2$ structures relevant to non-supersymmetric vacua in heterotic($\mathcal{M}$-) SUGRA.

A classification of $SU(3)/G_2/Spin(7)/Spin(4)$ structures relevant to non-supersymmetric (UV-complete) string theoretic dual of large-$N$ thermal QCD-like theories, and its $\mathcal{M}$-theory uplift, has been missing in the literature. This is what we initiated in [7] wherein we worked out the $SU(3)$-structure torsion classes of the relevant non-Kähler conifold $M_6 = \text{relevant to the type IIa SYZ mirror,}$ the $G_2$-structure torsion classes of the relevant seven-fold $M_7 = S^1 \times_w M_6$, i.e. a warped product of the $\mathcal{M}$-theory circle and $M_6$, as well as the $SU(4)$-structure and $Spin(7)$-structure torsion classes of the eight-fold $M_8$ - a warped product of the thermal $S^1$ and $M_7$ - relevant to the $\mathcal{M}$-Theory uplift. Table 1 summarizes the $G$-structure torsion classes’ results.

We continue in this work wherein we obtain explicitly (Almost) Contact (Metric)(3)Structures and associated transverse $SU(3)$ structures corresponding to seven/six-folds relevant to the $\mathcal{M}$-
theory inclusive of $O(R^4)$ terms, dual of large-$N$ thermal QCD at intermediate coupling as obtained in [8], [7].

**Main result of the paper:** In this paper we prove the proposition (partly numerically):

**Proposition 1:** The seven-fold $M_7$, with coframes $e^{a=1,...,7}$, which is the warped product of the $\mathcal{M}$-theory circle and a non-Kähler six-fold that is the warped product of the thermal circle and a non-Einsteinian generalization of $T^{1,1}$ which figures in the $\mathcal{M}$-theory uplift of the type IIB dual of large-$N$ thermal QCD (the latter as constructed in [9], with its type IIA SYZ mirror and leading order (in a derivative expansion) $\mathcal{M}$-theory uplift constructed in [8] and $O(R^4)$ corrections worked out in [7]), near the Ouyang embedding of the flavor $D7$-branes (A4) in the parent type IIB dual of large-$N$ thermal QCD in the limit of very small Ouyang embedding parameter, and $N = 100, M$(number of fractional $D3$-branes in [9])$= N_f$(number of flavor $D7$-branes in [9])$= 3, g_s = 0.1$:

1. supports a $G_2$ structure with the non-trivial torsion classes $\tau$ given by $\tau = \tau_1 \oplus \tau_2 \oplus \tau_3$;

2. supports an Almost Contact (3) Metric Structure $(\sigma^1, \sigma^2, \sigma^3) = (e^1, e^7, -e^2)$ with associated $(R^1, R^2, R_1 \times \Phi R^2)$ which does not correspond to Contact (3) Structure;

3. supports a Contact (3) Structure: $(\sigma^1, \sigma^2, \sigma^3) = (\alpha_1 e^1 + \alpha_3 e^3 + \alpha_7 e^7, \beta_1 e^1 + \beta_4 e^4 + \beta_7 e^7, \sigma^3)$ with $\sigma^3 : \sigma^3(R^1 \times \Phi R^2) = 1$

4. inherits a transverse $SU(3)$ structure $(\Omega^{(a)}_+, \Omega^{(a)}_-)$ wherein $\Omega^{(a)}_+ = \Phi - \sigma^{(a)} \wedge \omega^{(a)}_\Phi, \Omega^{(a)}_- = \sigma^{(a)} \wedge \left(\frac{1}{2} \omega^{(a)}_\Phi \wedge \omega^{(a)}_\Phi - s^\tau \Phi\right), \alpha = 1, 2, 3$ from the AC(3)S constructed in 2.

| S. No. | Manifold | G-Structure | Non-Trivial Torsion Classes |
|--------|----------|-------------|----------------------------|
| 1.     | $M_6 =$ non-Kähler conifold | $SU(3)$ | $T^{HIA}_{SU(3)} = W_1 \oplus W_2 \oplus W_3 \oplus W_4 \oplus W_5 : W_4 \sim W_5$ |
| 2.     | $M_7 = S^1 \times_w M_6$ | $G_2$ | $T^{M}_{G_2} = W_1 \oplus W_7 \oplus W_{14} \oplus W_{27}$ |
| 3.     | $M_8 = S^1 \times_w M_7$ | $SU(4)$ | $T^{M}_{SU(4)} = W_2 \oplus W_3 \oplus W_5$ |
| 4.     | $M_8$ | $Spin(7)$ | $T^{M}_{Spin(7)} = W_1 \oplus W_2$ |

Table 1: IR G-Structure Classification of Six-/Seven-/Eight-Folds in the type IIA/\mathcal{M}-Theory Duals of Thermal QCD-Like Theories (at High Temperatures) and for $r \in IR$.
5. inherits a transverse $SU(3)$ structure from the $C(3)S$ constructed in 3, \( \left( \Omega_+^{(\alpha)}, \Omega_-^{(\alpha)} \right) \), where 
\[ \Omega_+^{(\alpha)} = \Phi - \alpha^{(\alpha)} \wedge \omega^{(\alpha)} \] 
and, e.g., for $\alpha = 1$, a three-parameter \( \left( \Lambda_{145}^{(1)}, \Lambda_{156}^{(1)}, \Lambda_{456}^{(1)} \right) \) family of:

\[ \Omega_- = \Lambda_{AMC} e^{ABC} = \Lambda_{1b0c0} e^{1b0c0} + \Lambda_{3b0c0} e^{3b0c0} + \Lambda_{7b0c0} e^{7b0c0} + \Lambda_{a0b0c0} e^{a0b0c0}, \]

where $a_0, b_0, c_0 = 2, 4, 5, 6$ and $\Lambda_{456}$ is the only linearly independent non-vanishing $\Lambda_{a0b0c0}$.

The remainder of the paper which is based on the proof of the aforementioned Proposition 1, is organised as follows. Section 2, via two subsections, summarizes in 2.1 the $M$-theory dual of large-$N$ thermal QCD and the “MQGP limit”, and (Almost) Contact (3)(Metric) Structure(s) in 2.2. In Lemma 1 in 3, we provide an explicit $G_2$ structure on $M_7 = S_1^4 \times_w M_6, M_6 = S_{\text{thermal}}^4 \times_w M_5$ where $w$ implies a warped product and $M_5$ is a non-Einsteinian generalization of $T^{1,1}$, and evaluate the four $G_2$-structure torsion classes of $M_7$. In 4, explicit Almost Contact 3-Structures are constructed in Lemma 2 in 4.1. In Lemma 3 in 4.2, it is shown that the Almost Contact Structure on $M_7$ constructed in 4.1 is in fact an Almost Contact Metric Structure. In Lemma 4 in 5, we provide an explicit Contact (3) Structure on $M_7$. In 6.1.1, we provide an explicit transverse $SU(3)$-structure arising from the $G_2$ structure constructed in Lemma 5 in 3. In Lemma 6 in 6.1.2, we provide an explicit transverse $SU(3)$-structure arising from the Contact Structure constructed in Lemma 4. There are four appendices. Appendix A quotes from [7] the co-frames for the non-Einsteinian generalization of $T^{1,1}$. Appendix B lists out the independent components of $\Omega^1_a, \Omega^7_a, \Omega^a_{bc}$ that figure in $de^1 = \Omega^1_a e^a, de^7 = \Omega^7_a e^a, de^a = \Omega^a_{bc} e^b \wedge e^c, a = 2, ... , 6$. In appendix C, expressions for $\Phi, *\Phi, *d\Phi, *d * \Phi$ relevant to 3, are listed out. Appendix D has details of the computation of the transverse $SU(3)$-structure valued $\Omega_-$.

2 String/$M$-Theory Setup and (Almost) Contact (3) Metric Structures [(A)C(3)MS] Basics

2.1 $M$-Theory Uplift of Large-$N$ Thermal QCD and the MQGP Limit

Holographic dual of thermal QCD-like theories at finite coupling was successfully constructed in [8, 10]. Since one requires finite coupling on gauge theory side therefore one is required to study the strong coupling limit of string theory i.e. $M$-theory to be consistent with gauge - gravity duality. Authors did the same in the ‘MQGP limit’ defined as [8, 10]:

\[ g_s \lesssim 1; N_f, M \equiv O(1), N \gg 1, \frac{g_s M^2}{N} \ll 1. \] (1)

$M$-theory uplift of type IIB string dual [9] was obtained by first constructing type IIA Strominger-Yau-Zaslow mirror symmetry of type IIB string theory implemented via a triple T duality along
a delocalized special Lagrangian (sLag) $T^3$ — which could be identified with the $T^2$-invariant sLag of \[11\] with a large base $B(r, \theta_1, \theta_2)$ \[10, 12\], and then uplifted to $\mathcal{M}$-theory using Witten's prescription. Type IIB string dual involves $N$ color $D3$-branes ($\mathbb{R}^{1,3}(t, x^{1,2,3}) \times \{r = 0\}$), $M$ $D5(\mathbb{R}^{1,3}(t, x^{1,2,3}) \times \{r = 0\}) \times S^2(\theta_1, \phi_1) \times \text{NP}_{S^2(\theta_1, \phi_2)}$-branes and $D7$ ($\mathbb{R}^{1,3}(t, x^{1,2,3}) \times \{r = 0\} \times S^2(\theta_1, \phi_1) \times \text{SP}_{S^2(\theta_2, \phi_2)}$)-branes and $N_f$ flavor $D7$ ($\mathbb{R}^{1,3}(t, x^{1,2,3}) \times \mathbb{R}_+ (r \in [\mu_{\text{Ouyang}}^2, \epsilon, r_{\text{UV}}]) \times S^3(\theta_1, \phi_1, \psi) \times \text{NP}_{S^2(\theta_2, \phi_2)}$) and $D7$ ($\mathbb{R}^{1,3}(t, x^{1,2,3}) \times \mathbb{R}_+ (r \in [\mathcal{R}_{D5}/D5 - \epsilon, r_{\text{UV}}]) \times S^3(\theta_1, \phi_1, \psi) \times \text{SP}_{S^2(\theta_2, \phi_2)}$)-branes \footnote{1}. SYZ mirror symmetry is triple T-duality along three isometry directions ($\phi_1, \phi_2, \psi$). By performing first T-duality along $\psi$ direction, one obtains $N$ $D4$ branes which are wrapping $\psi$ direction and $M$ $D4$-branes straddling a pair of orthogonal NS5-branes. Further, from T-dualities along $\phi_1$ and $\phi_2$, one obtains a pair of Taub-Nut spaces and $N$ $D6$ branes. Effect of triple T-dualities on the flavor $D7$ branes is that $D7$ branes are replaced by $D6$-branes. The $\mathcal{M}$-theory mirror of the type IIA mirror yields KK monopoles (variants of Taub-NUT spaces).

Therefore, we can see that there are no branes in $\mathcal{M}$-theory uplift and we have $\mathcal{M}$-theory on a $G_2$-structure manifold with fluxes.

The $\mathcal{N} = 1, D = 11$ supergravity action inclusive of $\mathcal{O}(t^6_p)$ terms, is hence given by:

$$S_{D=11} = \frac{1}{2 \kappa_{11}^2} \left[ \int_{M_{11}} \sqrt{G}R + \int_{\partial M_{11}} \sqrt{h} K - \frac{1}{2} \int_{M_{11}} \sqrt{G} G_4^2 - \frac{1}{6} \int_{M_{11}} C_3 \wedge G_4 \wedge G_4 \right] + \frac{(4 \pi \kappa_{11}^2)^{\frac{3}{2}}}{(2 \pi)^{4} 3^2 2^{13}} \left( \int_{\mathcal{M}} d^{11}x \sqrt{G} M \left( J_0 - \frac{1}{2} E_8 \right) + 3^2 2^{13} \int C_3 \wedge X_8 + \int t_8 G^2 R^3 + \cdot \right) - S^{\text{ct}}; \quad (2)$$

where:

$$J_0 = 3 \cdot 2^8 (R^{H MNK} R_{PMNQ} R^R_{RKP} + \frac{1}{2} R^{H MNK} R_{PMNQ} R^R_{RKP})$$

$$E_8 = \frac{1}{3!} \epsilon^{ABCM_1N_1 \ldots M_4N_4} \epsilon_{ABCM_1N_1 \ldots M_4N_4} R^{M_1N_1^r} R_{M_1N_1} \ldots R_{M_4N_4}^r$$

$$t_8 g^2 G^2 R^3 = t_8 M_1 \ldots M_4 \delta^{i_1 \ldots i_N} G_{M_1}^{i_1 \ldots i_N} G_{M_2}^{N_2} R_{M_5M_6}^{N_3} R_{M_5M_6}^{N_4} R_{M_7M_8}^{N_5}$$

$$\kappa_{11}^2 = \frac{(2 \pi)^4 t_8^0}{2}; \quad (3)$$

$\kappa_{11}^2$ being related to the eleven-dimensional Newtonian coupling constant, $G = dC$ with $C$ being the $\mathcal{M}$-theory three-form potential with the four-form $G$ being the associated four-form field strength and $S^{\text{ct}}$ being the counter-terms needed for renormalization at $\mathcal{O}(R^4)$.

Seeking a completion of the 1-loop $\mathcal{O}(R^4)$ in the presence of NS-NS $B$ in type IIA compatible with T duality, and thus defining the torsionful spin connection, $\Omega = \Omega \pm \frac{1}{2} \mathcal{H}$, $\mathcal{H}^{ab} = \mathcal{H}_{\mu}^{ab} dx^\mu$, 

1 $S^2(\theta_2, \phi_2)$ is blown up $S^2$ in the conifold geometry, NP and SP stand for north pole and south pole of the resolving $S^2$, and $a$ is the resolution parameter.
and $\overline{X}_8 \equiv \frac{X_8(R(\Omega_+)) + X_8(R(\Omega_-))}{2}$, where $R(\Omega_+) = R(\Omega) + \frac{1}{2} d\mathcal{H} + \frac{1}{2} \mathcal{H} \wedge \mathcal{H}$, the ten dimensional $\overline{X}_8$ shifts by an exact form $[13]^2$:

$$\overline{X}_8 = \frac{1}{192(2\pi)^4} \left[ \left( \text{tr} R^4 - \frac{1}{4} (\text{tr} R^2)^2 \right) + \frac{d}{2} \left( \frac{1}{16} \text{tr} (2\mathcal{H}^3 (\mathcal{H} R + R \mathcal{H}) + \mathcal{H} R^2 \mathcal{H}) - \frac{1}{8} (\text{tr} R^2 \mathcal{H} \mathcal{H} + 2 \text{tr} \mathcal{H} R \text{tr} \mathcal{H}) \right) + \frac{1}{32} \text{tr} \mathcal{H} \mathcal{H}^8 \right] \right].$$

(4)

Defining the $O(1, 10)$-valued one-form $G^{abc} \equiv 4G_{\mu\nu\lambda\rho} dx^\mu e^\alpha e^\beta e^\gamma$, the $\mathcal{M}$-theory uplift of the first two lines of (4) of type IIA, yields $[13]$:

$$B_2 \wedge \overline{X}_8 \rightarrow \frac{1}{192(2\pi)^4} \left[ C \wedge \left( \text{tr} R^4 - \frac{1}{4} (\text{tr} R^2)^2 \right) + G \wedge \left( \frac{1}{4} \left( R^{ab} G_{cd} \nabla G_{de} + 2 R^{ab} G^{bce} R^{cd} G_{de} + R^{ab} G^{bce} \nabla G_{de} \right) \right) - \frac{1}{24} \left( \text{tr} R^2 \wedge G^{abc} \nabla G_{bc} + 6 R^{ab} G^{bce} R^{cd} G_{de} + \ldots \right) \right].$$

(5)

In this paper as in $[7]$, we restrict ourselves only to the first line in (5). Given that the same was shown to vanish $[8]$, perhaps to be T-duality invariant, the sum of the terms in the second and third lines of (5) too may yield zero. We have not proven the same.

The $D = 11$ action in (2) is holographically renormalizable by construction of appropriate counter terms $S^{\text{ct}}$. It can be shown $[14]$ that the bulk on-shell $D = 11$ supergravity action inclusive of $O(R^4)$-corrections is given by:

$$S_{D=11}^{\text{on-shell}} = -\frac{1}{2} \left[ -2 S_{\text{EH}}^{(0)} + 2 S_{\text{GHY}}^{(0)} + \beta \left( \frac{20}{11} S_{\text{EH}}^{(0)} - 2 \int_{M_1} \sqrt{-g} R^{(0)} + 2 S_{\text{GHY}}^{(0)} - \frac{2}{11} \int_{M_1} \sqrt{-g} g^{MN} \delta J_0 \frac{\delta g_{MN}}{\delta g_{(0)}} \right) \right].$$

(6)

The UV divergences of the various terms in (6) are of the following types:

$$\int_{M_1} \sqrt{-g} R \bigg|_{\text{UV-divergent}}, \quad \int_{\partial M_1} \sqrt{-h K} \bigg|_{\text{UV-divergent}} \sim r_{\text{UV}}^4 \log r_{\text{UV}},$$

$$\int_{M_1} \sqrt{-g} g^{MN} \delta J_0 \frac{\delta g_{MN}}{\delta g_{(0)}} \bigg|_{\text{UV-divergent}} \sim \frac{r_{\text{UV}}^4}{\log r_{\text{UV}}}. \quad (7)$$

$^2$When wedged with $C$ it will be understood that the metric, curvature, etc. are eleven-dimensional and when wedged with $B$, ten dimensional.
It can be shown [14] that an appropriate linear combination of the boundary terms: $\int_{\partial M_1} \sqrt{-h} K \bigg|_{r=r_{UV}}$ and $\int_{\partial M_1} \sqrt{-h} h^{\mu \nu} \frac{\partial J_0}{\partial h^{\mu \nu}} \bigg|_{r=r_{UV}}$ serves as the appropriate counter terms to cancel the UV divergences as given in (7).

The EOMS are:

$$R_{MN} - \frac{1}{2} g_{MN} R = -\frac{1}{12} \left( G_{MPQR} G^{PQR}_N - \frac{g_{MN}}{8} G_{PQRS} G^{PQRS} \right)$$

$$= -\beta \left[ \frac{g_{MN}}{2} \left( J_0 - \frac{1}{2} E_8 \right) + \frac{\delta}{\delta g^{MN}} \left( J_0 - \frac{1}{2} E_8 \right) \right],$$

$$d * G = \frac{1}{2} G \wedge G + 3^2 2^{13} (2\pi)^4 \beta X_8,$$

(8)

where [15]:

$$\beta \equiv \frac{(2\pi)^4 (\kappa_{11}^2) \Phi}{(2\pi)^4 3^2 2^{12}} \sim l_p^6,$$

(9)

$R_{MNPQ}$, $R_{MN}$, $R$ in (2)/(8) being respectively the elven-dimensional Riemann curvature tensor, Ricci tensor and the Ricci scalar.

Now, it was shown in [7] that if one makes an ansatz:

$$g_{MN} = g_{MN}^{(0)} + \beta g_{MN}^{(1)},$$

$$C_{MNP} = C_{MNP}^{(0)} + \beta C_{MNP}^{(1)},$$

(10)

one can self-consistently, set $C_{MNP}^{(1)} = 0$. Further, as proved in [7] (as Lemma 1), in the neighborhood of the Ouyang embedding of flavor $D7$-branes (A4) (that figure in the type IIB string dual of thermal QCD-like theories at high temperatures [9]) effected by working in the neighborhood of small $\theta_{1,2}$ (assuming a vanishingly small Ouyang embedding parameter), in the MQGP limit (1), $\lim_{N \to \infty} \frac{E_8}{\theta_0} = 0, \lim_{N \to \infty} \frac{t_8 G^2 R^3}{E_8} = 0$. Therefore, $E_8$ and $t_8^2 G^2 R^3$-contributions (were) are disregarded (in [7]).

2.2 (A)C(3)MS

Let $Y$ be an odd dimensional ((non-)closed) Riemannian manifold with metric $g$, then if it admits the existence of an endomorphism $J$ of the tangent bundle $TY$, a unit vector field $R$ (with respect to the metric $g$) and a one form $\sigma$ which satisfy

$$J^2 = -1 + R \otimes \sigma, \quad \sigma(R) = 1$$
$Y$ is said to admit an almost contact structure $(J, R, \sigma)$ (ACS) and $\sigma$ is called the contact potential [16]. The structure group of the tangent space reduces to $U(n) \times 1$ where $2n + 1$ is the dimension of $Y$.

The ACS is said to be a contact structure if
\[ \sigma \wedge d\sigma \neq 0 \quad \forall \text{points } \in Y. \]

A Riemannian manifold $Y$ with an ACS $(J, R, \sigma)$ has an almost contact metric structure $(J, R, \sigma, g)$ (ACMS) if,
\[ g(Ju, Jv) = g(u, v) - \sigma(u)\sigma(v), \quad \forall u, v \in \Gamma(TY). \]

The fundamental two-form $\omega$ of the almost contact manifold is then defined as,
\[ \omega(u, v) = g(Ju, v), \quad \forall u, v \in \Gamma(TY) \]
and satisfies
\[ \sigma \wedge \omega \neq 0. \]

The ACMS determines a foliation $\mathcal{F}_R$ of $Y$ by one-dimensional integral curves of $R$ w.r.t. $G_2$ metric: $ds^2 = \sigma^2 + ds^2_\perp$, where $ds^2_\perp$ is the metric on the transverse geometry of $\mathcal{F}_R$ induced by the ACMS on $Y$.

An almost contact 3-structure (AC3S) on a manifold $Y$ is defined by three distinct ACS $(J^\alpha, R^\alpha, \sigma^\alpha), \alpha = 1, 2, 3$ on $Y$ which satisfy [17],
\[ J^\gamma = J^\alpha J^\beta - R^\alpha \otimes \sigma^\beta = J^\beta J^\alpha - R^\beta \otimes \sigma^\alpha \]
\[ R^\alpha = J^\alpha (R^\beta) = -J^\beta (R^\alpha) \]
\[ \sigma^\gamma = \sigma^\alpha \circ J^\beta = -\sigma^\beta \circ J^\alpha \]
\[ \sigma^\alpha (R^\beta) = \sigma^\beta (R^\alpha) = 0 \]
where $\{\alpha, \beta, \gamma\}$ are cyclic permutation of $\{1, 2, 3\}$. An AC3S consisting of three contact structures satisfying (11), defines a 3-Sasakian geometry [18].

$Y$ admitting an AC3S must have dimensionality $4n + 3, n \in \mathbb{Z}^+$, and the structure group reduces to $Sp(n) \times 1_3$. An almost contact metric 3-structure (AC3MS) on a Riemannian manifold $Y$ with metric $g$ is an AC3S satisfying,
\[ g(J^\alpha u, J^\alpha v) = g(u, v) - \sigma^\alpha(u)\sigma^\alpha(u), \quad \forall u, v \in \Gamma(TY) \]
for $\alpha \in 1, 2, 3$. With $J^\alpha(u) = R^\alpha \times_\Phi u$, one can see that with $R^\alpha R^\beta = \delta^{\alpha\beta}, R^1, R^2, (R^1 \times R^2)$ provide an AC3S.
3 $G_2$-Structure of

$M_7 = S^1_M \times_w \left(S^1_{\text{thermal}} \times_w M_5\right) \bigg|_{\text{Ouyang-embedding}}$ (parent type IIB)$\cap|\mu_{\text{Ouyang}}| \ll 1$

Given that the adjoint of $SO(7)$ decomposes under $G_2$ as $21 \rightarrow 7 \oplus 14$ where 14 is the adjoint representation of $G_2$, one obtains four $G_2$-structure torsion classes:

$$T \in \Lambda^1 \otimes g_2^\perp = W_1 \oplus W_7 \oplus W_{14} \oplus W_{27} = \tau_0 \oplus \tau_1 \oplus \tau_2 \oplus \tau_3,$$

(11)

$g_2^\perp$ being the orthogonal complement of $g_2$, the subscript $a$ in $W_a$ denoting the dimensionality of the torsion class $W_a$, and $p$ in $\tau_p$ denoting the rank of the associated differential form. We now proceed to determine the $\tau_p$’s of the seven-fold $M_7 = S^1_M \times_w (S^1_{\text{thermal}} \times_w M_5)$, $M_5$ being a non-Einsteinian generalization of $T^{1,1}$ and close to the Ouyang embedding (A4) of the flavor D7-branes in the parent type IIB dual in the limit of very small Ouyang embedding parameter limit ($|\mu_{\text{Ouyang}}| \ll 1$).

Lemma 1: Near the Ouyang embedding locus of the flavor D7-branes (A4) assuming the modulus of the Ouyang embedding parameter $|\mu_{\text{Ouyang}}|$ to be very small and the $\psi = 2n\pi, n = 0, 1, 2$-coordinate patches, in the MQGP limit (2.1), $\tau (M_7) = \tau_1 \oplus \tau_2 \oplus \tau_3$.

Proof: Setting,

$$e^1 = \sqrt{G_{xx}^M} dx^0,$$

(12)

and using the following definitions from [7],

$$d\theta_{i=1/2} = \sum_{a=2}^{6} \Theta_{ia} e^a,$$

$$dx = \sum_{a=2}^{6} X_a e^a,$$

$$dy = \sum_{a=2}^{6} \gamma_a e^a,$$

$$dz = \sum_{a=2}^{6} Z_a e^a,$$

(13)

one can show that:

$$de^a = \Omega_{abc} e^b \wedge e^c,$$

(14)
where \( a, b, c = 2, \ldots, 6 \) and,
\[
\Omega^\alpha_\omega(r = \text{constant} \in \text{IR}) = \partial_{[\theta_2 e^a_{\theta_1}]} \Theta_2 \Theta_1 c + \partial_{[x e^a_{\theta_1}]} Y_b \Theta_1 c + \partial_{[y e^a_{\theta_1}]} Z_b \Theta_1 c + \partial_{[x e^a_{\theta_1}]} X_b \Theta_2 c
\]
\[+ \partial_{[y e^a_{\theta_2}]} Y_b \Theta_2 c + \partial_{[x e^a_{\theta_2}]} Z_b \Theta_2 c + \partial_{[y e^a_{\theta_2}]} X_b \Theta_2 c + \partial_{[x e^a_{\theta_2}]} Z_b Y_c + \partial_{[x e^a_{\theta_2}]} Z_b Y_c. \quad (15)\]

- Similarly,
\[
de^{4}(r = \text{constant} \in \text{IR}) = \Omega^\alpha_\alpha e^a_1, \quad (16)\]
where,
\[
\Omega^\alpha_\alpha(r = \text{constant} \in \text{IR}) = \frac{1}{2 G^M_{x^{10} x^1}} \left( \partial_{[\theta_2 G^M_{x^{10} x^1} \Theta_1 a} + \partial_{[x G^M_{x^{10} x^1} \Theta_2 a} + \partial_{[y G^M_{x^{10} x^1} X_a}
\]
\[+ \partial_{[y G^M_{x^{10} x^1} Y_a} + \partial_{[x G^M_{x^{10} x^1} Z_a} \right). \quad (17)\]

- Also, defining
\[
e^{7} = \sqrt{G^M_{x^{10} x^10} dx^{10}}, \quad (18)\]
\[
de^{7}(r = \text{constant} \in \text{IR}) = \Omega^\gamma_\gamma e^a_7, \quad (19)\]
where,
\[
\Omega^\gamma_\gamma(r = \text{constant} \in \text{IR}) = \frac{1}{2 G^M_{x^{10} x^10}} \left( \partial_{[\theta_2 G^M_{x^{10} x^10} \Theta_1 a} + \partial_{[x G^M_{x^{10} x^10} \Theta_2 a} + \partial_{[y G^M_{x^{10} x^10} X_a}
\]
\[+ \partial_{[y G^M_{x^{10} x^10} Y_a} + \partial_{[x G^M_{x^{10} x^10} Z_a} \right). \quad (20)\]

The results of appendices B and C yield:
\[
\Phi \wedge \ast\gamma d\Phi = e^{127456} (2 \Omega^7_{2} - 2 \Omega^2_{23} - 2 \Omega^5_{23}) + e^{127356} (-2 \Omega^7_{2} + \Omega^3_{23} - \Omega^5_{23}) + e^{127346} (2 \Omega^7_{2} + \Omega^2_{23} + 2 \Omega^5_{23})
\]
\[+ e^{127345} (-2 \Omega^2_{23} - \Omega^5_{23}) + e^{234567} (\Omega^2_{23} + \Omega^3_{24} + \Omega^5_{23}) + e^{123456} (-\Omega^2_{23} - \Omega^3_{23} + \Omega^5_{23}) + e^{134567} \Omega^5_{23}
\]
\[\approx e^{234567} \Omega^3_{24} - e^{123456} \Omega^3_{23}, \quad (21)\]

implying,
\[
\tau_1 = \frac{1}{12} \ast\gamma (\Phi \wedge \ast\gamma d\Phi) \sim e^{4} \Omega^3_{24} + e^{7} \Omega^3_{23}. \quad (22)\]

Also \(^3\),
\[
\tau_0 = d\Phi \wedge \ast\gamma \Phi = 0. \quad (23)\]

To obtain \(\tau_2\), one notes that:
\[
\tau_1 \wedge d\Phi \sim (e^{27} + e^{35} - e^{46}) \Omega^3_{24} - J \Omega^3_{23}, \quad (24)\]
where \(J = e^{12} + e^{34} + e^{56}\).

Similarly, to obtain \(\tau_3\), one notes that:
\[
\tau_1 \wedge \ast\gamma \Phi \sim (e^{2} \wedge J + e^{457} + e^{367}) \Omega^3_{24} - (e^{246} - e^{235} - e^{456}) \Omega^3_{23}. \quad (25)\]

\(^3\)Thanks to S. Sarkar for pointing out an error in a previous version of this result.
4 (A)C(3)(M)S

In this section, in 4.1, we provide explicit construction of Almost Contact (3) Structures at fixed IR-valued $r$ supported on $M_7$ (which is a warped product of the $\mathcal{M}$-theory circle and $M_6$ where $M_6$ is a warped product of the thermal circle and a non-Einsteinian generalization of $T^{1,1}$). As a small aside, we also provide a similar construction of AC(3)S for $\tilde{M}_7$ which is a warped product of the $\mathcal{M}$-theory circle and a non-Kähler conifold $\tilde{M}_6$. In 4.2, we show that the ACS of 4.1 is in fact an Almost Contact Metric Structure (ACMS).

4.1 AC(3)S arising from $\tau_{i=0,2,3}(M_7)$

From (12), (18) and (A1), we arrive at the following lemma:

Lemma 2: Near the Ouyang embedding of the flavor $D7$-branes in the parent type IIB dual of $[9]$ near the $\psi = 2n\pi, n = 0, 1, 2$-coordinate patches in the MQGP limit, one obtains AC(3)MS $|_{\text{MQGP-Ouyang}}\cap \psi = 2n\pi - \text{coordinate patch}$:

$$R^1 = \sqrt{G_{\mathcal{M}rr}^0} \partial_r,$$
$$R^2 = \sqrt{G_{\mathcal{M}x^{10}x^{10}}} \partial_{x^{10}},$$
$$R^3 = \Phi^\mu \sqrt{G_{\mathcal{M}(00)G_{\mathcal{M}}x^{10}x^{10}}} \partial_\mu = -G^{\mu\nu}_{\mathcal{M}} \sqrt{G_{\mathcal{M}(00)G_{\mathcal{M}}x^{10}x^{10}}} (e^{217})_{\nu x^0 x^{10}} \partial_\mu = -G^{\mu\nu}_{\mathcal{M}} e^2 \partial_\mu. \quad (26)$$

Further, $\sigma^\alpha = G^{\mathcal{M}M}_{mn} R^m_\alpha dx^m$ implying

$$\sigma^1, \sigma^2, \sigma^3 = (e^1, e^7, -e^2); \quad (27)$$

the fundamental two-forms $\omega^\alpha = d\sigma^\alpha = i_{R^\alpha} \Phi$, which using (14), (16) and (19), can be evaluated. We demonstrate the existence of a C(3)S in 5.

One can similarly see that near the Ouyang embedding of the flavor $D7$-branes in the parent type IIB dual of [9] near the $\psi = 2n\pi, n = 0, 1, 2$-coordinate patches, the non-compact $\tilde{M}_7$ - a warped product of the $\mathcal{M}$-theory circle and a non-Kähler conifold $\tilde{M}_6$ - supports an AC(3)S:

$$R^1 = \sqrt{G_{\mathcal{M}rr}^0} \partial_r,$$
$$R^2 = \sqrt{G_{\mathcal{M}x^{10}x^{10}}} \partial_{x^{10}},$$
$$R^3 = \Phi^\mu \sqrt{G_{\mathcal{M}rr}^0G_{\mathcal{M}}x^{10}x^{10}} (e^{217})_{\nu x^0 x^{10}} \partial_\mu = -G^{\mu\nu}_{\mathcal{M}} e^2 \partial_\mu. \quad (28)$$

Further, $\sigma^\alpha = G^{\mathcal{M}M}_{mn} R^m_\alpha dx^m$; fundamental two-forms $\omega^\alpha = d\sigma^\alpha = i_{R^\alpha} \Phi$:

$$\sigma^1, \sigma^2, \sigma^3 = (e^1, e^7, -e^2). \quad (29)$$
4.2 AC(3)MS on $M_7$

In this section we prove the lemma:

**Lemma 3:** The AC(3)S of 4.1 in fact corresponds to AC(3)MS.

**Proof:** We will explicitly show the existence of ACMS for $(R^1, \sigma^1)$ in (27).

An we know, ACS is ACMS provided $g(Ju, Jv) = g(u, v) - \sigma(u)\sigma(v)$. We now show that the same is satisfied by $(R^1, \sigma^1)$. In components, the aforementioned requirement is written as:

$$g_{mn}J^m_{n_1}J^n_{n_2} = g_{mn} - \sigma_{mn}$$

$$or \quad g_{mn}\Phi_{m_1}^n\Phi_{n_1}^m R^1_{(a)} R^1_{(b)} = g_{mn} - \sigma_{mn} \sigma_{n_1}.$$  \hspace{1cm} (30)

- $m_1 = n_1 = x^0$: The RHS of (30) in this case is null. The LHS of (30) will be proportional to $g_{mn}\Phi_{x^0}^m\Phi_{x^0}^n = 0$ and therefore (30) checks out.

- $m_1 = n_1 = x^{10}$: The RHS of (30) in this case is $g_{x^{10}x^{10}}$. The LHS of (30) is:

$$g_{mn}\Phi_{x^{10}x^{10}}\Phi_{x^{10}x^{10}} (R^{x^0})^2 = g_{mn} (e^{217})_{x^{10}x^{10}} (e^{217})_{x^{10}x^{10}} (R^{x^0})^2 = g_{mn} e^2_\sigma e^2_n g_{x^{10}x^{10}} = g_{x^{10}x^{10}}.$$  \hspace{1cm} (31)

Therefore, (30) checks out.

- $m_1, n_1 \neq x^0, x^{10}$ The LHS of (30) yields $g_{m_1 n_1}$. The LHS of (30) obtains:

$$g_{mn}\Phi_{m_1}^n\Phi_{n_1}^m (R^{x^0})^2 = \left(e^{27} + e^{35} - e^{46}\right)_{mn} \left(e^{27} + e^{35} - e^{46}\right)_{n_1} (e^1_{x^0})^2 (R^{x^0})^2.$$  \hspace{1cm} (32)

Now (32) will involve:

(a) $(e^{27})_{m_1 n_1} (e^{27})_{n_1} = e^2_m e^2_n E^{2n}e^7_{n_1} - e^2_n e^7_{m_1} E^{2n}e^7_{n_1} + e^7_{m_1} E^{n}e^2_{n_1} = 0 - 0 + e^2_{m_1} e^2_{n_1};$

(b) $(e^{35})_{m_1 n_1} (e^{35})_{n_1} = e^3_n E^{3n}e^5_{m_1} e^5_{n_1} + e^5_n E^{5n}e^5_{m_1} e^5_{n_1} - e^3_n E^{5n}e^5_{m_1} e^5_{n_1} = e^5_{m_1} e^5_{n_1} + e^3_n e^5_{m_1} - 0;$

(c) $(e^{46})_{m_1 n_1} (e^{46})_{n_1} = e^4_n E^{4n}e^6_{m_1} e^6_{n_1} + e^6_n E^{6n}e^6_{m_1} e^6_{n_1} - e^4_n E^{6n}e^6_{m_1} e^6_{n_1} = e^6_{m_1} e^6_{n_1} + e^4_n e^6_{m_1} - 0.$  \hspace{1cm} (33)

One hence obtains:

$$LHS = \left(\sum_{a=2}^6 c_{m_1} c_{n_1} (e^1_{x^0})^2 (R^{x^0})^2 = g_{m_1 n_1},$$  \hspace{1cm} (34)

and therefore, (30) checks out.
\[ m_1 = x^0, n_1 = x^{10} \] The LHS of (30) then becomes proportional to \( \Phi_{m_1 x^0} \Phi_{n_1 x^{10}} = 0 \) and the RHS of (30) is the difference of \( g_{x^0 x^{10}} = 0 \) and \( \sigma_{x^0}^{(1)} \sigma_{x^{10}}^{(1)} = 0 \). Therefore, (30) checks out. Similarly, for \( m_1 = x^{10}, n_1 = x^0 \).

One can similarly argue that the LHS and RHS of (30) vanish identically for \( m_1/n_1 = x^0, n_1/m_1 \neq x^{0,10} \).

\[ m_1 = x^{10}, n_1 \neq x^{0,10} \] The LHS of (30) becomes

\[ g^{nn} \Phi_{m_1 x^{10}} \Phi_{n_1 x^0} \left( R^{a_0} \right)^2 = g^{nn} (e^{27})_{n_1 x^0} (e^{37} + e^{35} - e^{46})_{n_1} \] (35)

Now,

\[ a. \quad g^{nn} (e^{27})_{n_1 x^{10}} (e^{27})_{n_1} = g^{nn} (e^{37} e_n e_{n_1} e_1 e_{10} - e^{27} e_n e_{n_1} e_1 x^{10} - e^{7} e_n e_{n_1} e_1 x^{10} + e^{27} e_n e_{n_1} e_1 x^{10} = 0) \]

\[ b. \quad g^{n_1 n_1 x^{10}} e^{35} = g^{n_1 n_1 x^{10}} e^{35} = 0 \] (as \( g^{m n} e_m e_n = \delta^{a b} \)).

Also, as \( g_{x^0 x^{10}} = \sigma_{x^0}^{(1)} = 0 \), (30) checks out.

One can similarly argue that the ACS corresponding to \( \sigma_{2,3} \) of (27), correspond to ACMSs.

5 \textbf{C(3)S on } M_7

As \( \omega \wedge \omega^3 = 0 \) in 4.1 in the \( \psi = 2n\pi, r \) -constant-coordinate patches, the same does not provide a contact structure. In this section, we provide an explicit construction of a contact structure in the \( \psi = 2n\pi \)-coordinate patch.

5.1 \textbf{Construction of Contact (3) Structure}

We now prove the lemma:

\textit{Lemma 4:} Near the \( \psi = 2n\pi \)-coordinate patch, \( (\sigma^1, \sigma^2, \sigma^3) = (\alpha_1 e^1 + \alpha_3 e^3 + \alpha_7 e^7, \beta_1 e^1 + \beta_4 e^4 + \beta_7 e^7, \sigma^3) \),
\( R^2 \) = 1 with
\[
R^2_1 = \frac{\alpha_3 e^3_{\theta_1}}{G^{M}_{\theta_0 \theta_1}};
R^2_2 = \frac{\alpha_3 (e^3_{\theta_2} G^{M}_{x \theta_1} - e^3_{\theta_1} G^{M}_{x \theta_2})}{G^{M}_{y \theta_2} G^{M}_{y \theta_1}};
R^2_3 = \frac{\alpha_3 (e^3_{y} G^{M}_{x \theta_1} - e^3_{\theta_1} G^{M}_{x y})}{G^{M}_{y \theta_2} G^{M}_{y \theta_1}};
\]
\[
R^{\theta_1}_{10} = \frac{\alpha_3 (-e^3_{\theta_0} G^{M}_{x \theta_1} G^{M}_{y \theta_2} - e^3_{\theta_1} G^{M}_{x \theta_1} G^{M}_{y \theta_2} - e^3_{\theta_2} G^{M}_{x \theta_1} G^{M}_{y \theta_2} + e^3_{y} G^{M}_{x \theta_1} G^{M}_{x \theta_2} + e^3_{y} G^{M}_{x \theta_1} G^{M}_{y \theta_2})}{G^{M}_{x \theta_1} G^{M}_{x \theta_2} G^{M}_{y \theta_1} G^{M}_{y \theta_2}};
R^{\theta_2}_{10} = \frac{\alpha_3 (e^3_{\theta_1} G^{M}_{x \theta_1} G^{M}_{y \theta_2} + e^3_{y} G^{M}_{x \theta_1} G^{M}_{y \theta_2} - e^3_{y} G^{M}_{x \theta_1} G^{M}_{x \theta_2})}{G^{M}_{x \theta_1} G^{M}_{x \theta_2} G^{M}_{y \theta_1} G^{M}_{y \theta_2}},
\]
\( \alpha_1 \sim \alpha_3 \sim \alpha_7; \beta_1 \sim \beta_4 \sim \beta_7 \) and \( R_2 = R_1(e^3 \rightarrow e^4, \alpha_3 \rightarrow \beta_4) \), provides a Contact 3-Structure.

**Proof:**

- To find a contact structure, one makes the ansatz:
  \[
  \sigma^1 = \alpha_1 e^1 + \alpha_3 e^3 + \alpha_7 e^7,
  \]
  which, using (13)-(20), implies:
  \[
  \omega^1 = d\sigma^1 = \alpha_1 \Omega_{a1} e^{a1} + \alpha_3 \Omega_{bc} e^{bc} + \alpha_7 \Omega_{a7} e^{a7}, \quad a, b, c = 2, \ldots, 6;
  \]
  \[
  \sigma^1 \wedge (\omega^1)^3 \sim \alpha_1 \alpha_3^2 \alpha_7 \Omega_{23}^{12} \Omega_{61}^{0} e^{1234567} \neq 0.
  \]
  \( \sigma^1(\omega^1)^3 \) implies (in the \( \psi = 2n\pi \)-coordinate patch):
  \[
  \sigma^1_r = G^{M}_{x \theta_0 \theta_1} R^{\theta_0}_{10} = \alpha_1 \sqrt{G^{M}_{x \theta_0 \theta_1}} \quad \text{implying} \quad R^{\theta_0}_{10} = \frac{\alpha_1}{\sqrt{G^{M}_{x \theta_0 \theta_1}}};
  \]
  \[
  \sigma^1_{x^{10}} = G^{M}_{x^{10} \theta_0} R^{\theta_0}_{10} = \alpha_7 \sqrt{G^{M}_{x^{10} \theta_0}} \quad \text{implying} \quad R^{\theta_0}_{10} = \frac{\alpha_7}{\sqrt{G^{M}_{x^{10} \theta_0}}};
  \]
  \[
  \sigma^1_{\theta_1} = G^{M}_{\theta_1 \theta_1} R^{\theta_1}_{10} = \alpha_3 e^3, \quad \ldots \equiv x, y, z (G^{M}_{\theta_1 \theta_1}(\psi = 2n\pi) = 0);
  \]
  \[
  \sigma^1_{\ldots \beta} = G^{M}_{\ldots \beta} R^{\beta}_{10} = \alpha_3 e^3, \quad \ldots \equiv x, y, z; \beta = x, y, z, \theta_1, \theta_2.
  \]
  Also,
  \[
  G_{mn} R^{m}_{10} R^{n}_{10} = 1, \quad m, n = r, x^{10}, \theta_{1,2}, x, y, z.
  \]
  The last equation in (40) and (41) are solved to yield (37).
  \[
  \sigma^1(R_1) = 1 \text{ implies:}
  \]
  \[
  \alpha_1^2 + \alpha_3 R_1 e^3 + \alpha_7^2 = 1.
  \]
where, \( r = \langle r \rangle \in \text{IR} \) either as valued in [19] or \( R.e^2(\langle r \rangle) = 0 \) arising from \( i_r \Phi = \omega^\alpha, \alpha = 1, 2, 3 - 54, 52 \) in particular.

Similarly, making an ansatz:

\[
\sigma^2 = \beta_1 e^1 + \beta_4 e^4 + \beta_7 e^7, \tag{43}
\]

implying

\[
\omega^2 = d\sigma^2 = \beta_1 \Omega^a e^{a1} + \beta_4 \Omega^4 e^{bc} + \beta_7 \Omega^a e^{a7};
\]

\[
\sigma^2 \land (\omega^2)^3 \sim \beta_1 \beta_4 \beta_7 \Omega^4 \Omega^2 \Omega^{56} \Omega^{61} e^{1234567}. \tag{44}
\]

Now, (43) implies:

\[
\sigma^2_{x^0} = G^M_{x^0,0} R^x_2 = \beta_1 \sqrt{G^M_{x^0,0}}, \text{ implying } R^x_2 = \frac{\alpha_1}{\sqrt{G^M_{x^0,0}}};
\]

\[
\sigma^2_{x^{10}} = G^M_{x^{10},x^{10}} R^x_2 = \beta_7 \sqrt{G^M_{x^{10},x^{10}}}, \text{ implying } R^x_2 = \frac{\beta_7}{\sqrt{G^M_{x^{10},x^{10}}}};
\]

\[
\sigma^2_{\theta_i} = G^M_{\theta_i, \theta_i} R^\theta_2 = \beta_4 e^4, \ldots \equiv x, y, z (G^M_{\theta_i, \theta_i} (\psi = 2\pi) = 0);
\]

\[
\sigma^2_{\varphi} = G^M_{\varphi, \varphi} R^\varphi_2 = \beta_7 e^7, \ldots \equiv x, y, z; \beta = x, y, z, \theta_1, \theta_2. \tag{45}
\]

Also,

\[
G^M_{mn} R^m_2 R^n_2 = 1, \ m, n = x^0, x^{10}, \theta_1, 2, x, y, z;
\]

\[
\sigma^1(R_2) = \sigma^2(R_1) = 0: G^M_{mn} R^m_1 R^n_2 = 0, \tag{46}
\]

along with

\[
R_2 = R_1 (e^3 \rightarrow e^4, \ \alpha_3 \rightarrow \beta_4), \tag{47}
\]

implies: \( \beta^2_1 + \beta_4 R_2 e^4 + \beta^2_7 = 1 \). The second constraint in (46), implies:

\[
\alpha_1 \beta_1 + \alpha_7 \beta_7 + \Sigma_{\sigma^1(R_2)} (N, M, N_f; r_h) = 0, \tag{48}
\]

where,

\[
\Sigma_{\sigma^1(R_2)} (N, M, N_f; r_h) \equiv G^M_{\sigma_1} (R^\theta_2 R^\theta_1 + R^\theta_1 R^\theta_2) + G^M_{\sigma_2} (R^\varphi_2 R^\varphi_1 + R^\varphi_1 R^\varphi_2)
+ G^M_{\sigma_3} (R^\varphi_2 R^\varphi_1 + R^\varphi_1 R^\varphi_2) + G^M_{\sigma_4} (R^\varphi_2 R^\varphi_1 + R^\varphi_1 R^\varphi_2)
+ G^M_{\sigma_5} (R^\varphi_2 R^\varphi_1 + R^\varphi_1 R^\varphi_2) + G^M_{\sigma_6} (R^\varphi_2 R^\varphi_1 + R^\varphi_1 R^\varphi_2).
\tag{49}
\]
One can show,

\[ \Sigma_{\sigma^1(R_2)}(N, M, N_f; r_h) \sim -\beta_2^2 \left( \kappa_{g_1}^{4; 1; \beta} \right)^2 \kappa_{g_1}^{4; 1; \beta} \frac{r^4}{(r^2 - 3d^2)^2} \frac{(1 + \alpha_3^3) N^{9/5}}{\alpha_{g_1}^4} \]

\[ + \mathcal{O} \left( \frac{N^{13/10}}{(A(r))^{\frac{3}{4}} (|\log r|^3 \log r_h)} \right). \tag{50} \]

In the IR and in the MQGP limit, \(|\log r|, |\log r_h| \sim N^{1/3}\) as used earlier, and hence,

\[ \Sigma_{\sigma^1(R_2)}(N, M, N_f; r_h) \sim (1 + \alpha_3^3) N^{9/5}, \]

or globally, \(\Sigma_{\sigma^1(R_2)}(N, M, N_f; r_h) \sim \frac{1}{N^{11/5}}\) and is thus large-\(N\) suppressed. Thus, (48) can be approximated by \(\alpha_1 \beta_1 + \alpha_7 \beta_7 = 0\). One expects that the \(\sigma^3\) dual to \(R^3 = R^1 \times \Phi R^2\) also provides a Contact Structure.

- Requiring \(i_{R} \Phi = \omega^a\), we work out constraints on \(\alpha_{1,3,7}\) that figure in (38) corresponding to \(R^3\) of (40) and (37). Similar constraints can be worked out for \(R^2\) of (47) and \(R^3 = R^1 \times \Phi R^2\). In the following “\(\sim\)” implies equality up to numerical multiplicative constants of \(\mathcal{O}(1)\).

As,

\[ i_{R^1} \Phi = \alpha_7 J + \alpha_1 e^{27} - R e^2 e^{17} + R e^3 e^{47} - R e^4 e^{57} + R e^5 e^{67} - R e^6 e^{57} + \alpha_1 e^{35} - R e^3 e^{15} + R e^5 e^{13} - \alpha_1 e^{46} + R e^4 e^{16} - R e^6 e^{14} - R e^2 e^{36} + R e^3 e^{26} - R e^6 e^{23} - R e^2 e^{45} + R e^4 e^{25} - R e^5 e^{24}. \tag{51} \]

The following (wherein \(m = 2, \ldots, 6\)) follow from (51):

\[ R^m_i e_m = 0. \tag{52} \]

Now, to estimate the \(r\) around which \(R_1 e^2 \sim 0\), we will expand all \(r\)-dependent terms except \(\log r\) in \(R_1 e^2\) about \(r = r_h\) to yield:

\[ R^m_i e_m \sim 10^{-4} \alpha_3 \left( \lambda_3^2 \log N^2 + \lambda_5 \log N (31.32 \log r - 0.25 \log N) + 68.73 \log r \right) \]

\[ \log r^2 r_h^2 (N_f (\log N - 3 \log r))^{2/3} \]

\[ + 10^{-6} \times \mathcal{O}(r - r_h), \tag{53} \]

\(\lambda_5\) being the parameter parametrizing the one-parameter family of eigenvectors relevant to the diagonalization of the \(M_5(\theta_{1,2}, x, y, z)\)-metric in [7]. Assuming \(\lambda_5 \gg 1\) and in the large-\(N\) limit, one sees from (53) vanishes in the IR-valued \(r\) in the neighborhood of:

\[ \log \langle r \rangle = -\mathcal{O}(10^{-2}) \lambda_5 \log N \]

or

\[ \langle r \rangle \sim N^{-\mathcal{O}(10^{-2}) \lambda_5} R_{D5/D5} \equiv r_h + \delta. \tag{54} \]
\[ \alpha_7 = -\alpha_1 \Omega_2^1 = \alpha_3 \Omega_{34}^3 = \alpha_3 \Omega_{56}^3. \]  

(55)

Now, \( \alpha_7 = -\alpha_1 \Omega_2^1 \) in (55) implies:

\[ \alpha_7 = \alpha_1 \frac{N^{7/20} \alpha_{\theta_2}}{(\log \langle r \rangle)^2 \alpha_{\theta_1}^2}; \]  

(56)

for \( N = 10^2 \) and \( \langle r \rangle \) as of (54) and \( \lambda_5 \) of (61), (56) implies \( \alpha_7 \sim \mathcal{O}(1) \alpha_1 \).

\( \alpha_7 = \alpha_3 \Omega_{34}^3 \) in (56) implies:

\[ \alpha_7 = 0.1 g_s^{3/4} (\log N)^3 M^3 N_f^2 (\log \langle r \rangle)^3 (-0.7 + \langle r \rangle \log \langle r \rangle) (0.004 + \langle r \rangle \log \langle r \rangle) (\log N + 263 \langle r \rangle \log \langle r \rangle) \]  

\[ \frac{N^{3/20} \alpha_{\theta_2}^2}{\alpha_{\theta_1}^2}, \]  

(57)

which at (54) and (61) for \( N = 10^2 \) implies \( \langle r \rangle \log \langle r \rangle \sim -0.1 \), and thus for \( M = N_f = 3, g_s = 0.1 \):

\[ \alpha_7 \sim \alpha_3. \]  

(58)

Similarly, \( \Omega_{56}^3(g_s = 0.1, M = N_f = 3, N = 100, \langle r \rangle : (54) \cap (61)) = \mathcal{O}(1) \), implying again \( \alpha_7 \sim \mathcal{O}(1) \alpha_3 \).

\[ \alpha_7 \Omega_2^5 = R_1^m e_1^1 x_0^1 = \alpha_1. \]  

(59)

From (59) and (55), one sees that consistency requires \( \Omega_2^5 \sim \Omega_2^4 \sim \mathcal{O}(1) \), i.e.,

\[ N^{7/20} \alpha_{\theta_2} \sim (\log \langle r \rangle)^2, \]  

(60)

and near (54),

\[ \lambda_5 \approx 10^2 N^{7/40} \sqrt{\frac{\alpha_{\theta_2}}{\alpha_1}}. \]  

(61)

\[ \alpha_{11} \Omega_3^1 = -R_1^m e_m^3. \]  

(62)

Using (60)-(61), one obtains:

\[ \mathcal{O}(10) \alpha_1 = \lambda_5 \frac{\alpha_3}{(\log \langle r \rangle)^{2/3}} = \mathcal{O}(10^2) \frac{\alpha_3 (\log \langle r \rangle)^{1/3}}{\log N}, \]  

(63)

implying:

\[ \alpha_1 \sim \alpha_3 \frac{(\log \langle r \rangle)^{1/3}}{\log N}. \]  

(64)
\[ \alpha_1 \Omega_6^1 = R_1^m e_4^1, \]  
(65)

which implies:
\[
\frac{\alpha_1}{\alpha_2} N^{1/20} \log N \sim \frac{\alpha_3}{\alpha_2} \frac{N^{3/5} \delta}{(r)^2 \log N (\log (r))^{11/3}}. 
\]  
(66)

Using (64), (66) implies:
\[
N \frac{N^{7/40}}{\log N} \frac{\sqrt{\frac{N^3}{\alpha_2}}}{\alpha_2} (\log N)^3 \sim \mathcal{O}(1). 
\]  
(67)

For \( N \sim 10^2 \), one notes (67) is satisfied by \( \sqrt{\frac{N^3}{\alpha_2}} = \mathcal{O}(1) \).

\[ \alpha_1 \Omega_4^1 = R_1^m e_6^1, \]  
(68)

which implies:
\[
\alpha_1 \frac{(\log N)^2}{N^{1/4}} \sim \alpha_3 \frac{\log N}{(\log (r))^{8/3}}. 
\]  
(69)

Using (64), (69) implies:
\[
N^{1/4} \frac{\mathcal{O}(10)}{(\log N)^3} \sim N^{11/40} \log N \frac{\alpha_3^3}{\alpha_2^{3/2}} 
\]  
(70)

must be \( \mathcal{O}(1) \), which is indeed the case for \( N = 10^2 \).

\[ \alpha_1 \Omega_3^1 = -R_1^m e_5^1, \]  
(71)

which implies
\[
\alpha_1 \times \mathcal{O}(1) = \mathcal{O}(10^3) \frac{\alpha_3}{(\log r)^{11/3}}. 
\]  
(72)

Using (64), (73) implies:
\[
\alpha_1 \times \mathcal{O}(1) = \mathcal{O}(10^3) \alpha_1 \frac{\log N}{(\log (r)^4} \sim \mathcal{O}(10^3) \frac{\alpha_1}{N^{7/10} (\log N)^3}, 
\]  
(73)

which should be \( \mathcal{O}(1) \) - the same checks out for \( N = 10^2 \).

\[ \alpha_7 \Omega_4^7 = R_1^m e_3^1, \]  
(74)

One can show (74) is equivalent to (62).
\[ \alpha_7 \Omega_7^2 = -R_1^m \epsilon_m, \]  

which implies:

\[ \alpha_7 \times \mathcal{O}(10^2) = \frac{\alpha_3}{\log \langle r \rangle} \frac{\alpha_{\theta_1}^{5/3}}{\alpha_{\theta_2}^{5/6}} N^{17/40} \sqrt{r_{\theta_2} \theta_1^{\frac{1}{3}}} (\log N)^{11/3} \]  

(76)

For \( N = 10^2 \), (76) obtains:

\[ \alpha_7 \sim \alpha_3. \]  

(77)

\[ \alpha_7 \Omega_5^7 = -R_1^m \epsilon_m, \]  

which implies:

\[ \alpha_7 \sim \alpha_3 \sim \alpha_3 \frac{\log N}{(\log \langle r \rangle)^{11/3}} = \mathcal{O}(10) \alpha_3 \frac{\log N \alpha_{\theta_1}^{8/3}}{N^{77/120} \alpha_{\theta_2}^{11/6}} N=10^2 \mathcal{O}(1), \]  

(79)

for \( \alpha_3 = \mathcal{O}(1) \), and hence checks out.

\[ \alpha_7 \Omega_6^7 = R_1^m \epsilon_m, \]  

(80)

which using (77) obtains:

\[ \frac{\alpha_7}{\mathcal{O}(10)} \sim \alpha_3 = \frac{\alpha_3 \mathcal{O}(10^2)}{(\log \langle r \rangle)^{11/3}} \sim \frac{10^2}{N^{77/120} \alpha_{\theta_2}^{11/6}} \sim \alpha_3. \]  

(81)

\[ \alpha_3 \Omega_{25}^3 = -R_1^m \epsilon_m, \]  

(82)

which yields:

\[ \alpha_3 \frac{g_{s}^{7/4} N^{1/10} \log N (\log \langle r \rangle)^4 \langle r \rangle}{\alpha_{\theta_2}^{2}} \frac{N^{7/20} \alpha_{\theta_2}^{2}}{\alpha_{\theta_1}^{2} (\log \langle r \rangle)^2} \sim g_{s}^{7/4} N^{4/5} \sqrt{r_{\theta_2} \theta_1^{\frac{1}{3}}} \log N \frac{1}{\alpha_{\theta_1}^2} \left( \frac{\sqrt{\alpha_{\theta_2}}}{\alpha_{\theta_1}} \right)^4 \sim \alpha_3. \]  

(83)

So, one chooses \( \alpha_{\theta_1} : \frac{\sqrt{\alpha_{\theta_2}}}{\alpha_{\theta_1}} = \mathcal{O}(1) \cap (83) \), which one can effect for \( N = 10^2 \).

\[ \alpha_3 \Omega_{26}^3 = -R_1^m \epsilon_m, \]  

(84)
which obtains:
\[
\alpha_3 N^{9/20} M N_f \log(r) \frac{\delta}{\theta_h} \sim \alpha_3 M N_f \left( \frac{N^{7/20} \alpha \theta_2}{\alpha \theta_1 (\log(r))^2} \right) \frac{\alpha \theta_1}{\alpha \theta_2} (\log(r))^3 \frac{\delta}{\theta_h}
\]
\[
\sim M N_f N^{21/40 - 7/40 \sqrt{\log N}} \alpha_3 \times \alpha_3
\]
\[
\sim \frac{\lambda_5 \alpha_3}{(\log(r))^{2/3} (3 N_f)^{2/3}} \sim 10^2 \left( \frac{N^{7/20} \alpha \theta_2}{\alpha \theta_1 (\log(r))^2} \right) \frac{1}{(3 N_f)^{2/3}} \frac{\alpha \theta_1}{\alpha \theta_2} \frac{\log(r)}{\log N}
\]
\[
\sim \frac{10^2}{(3 N_f)^{2/3} \log N} N^{7/120} \alpha_3 \sim \alpha_3,
\]
which checks out for \(N = 10^2\).

\[
\alpha_3 \Omega_{24}^3 = -R_1^m e_m^5,
\]
which obtains:
\[
\alpha_3 \frac{M N_f}{N^{1/4}} (\log N)^2 \log(r) \sim \frac{M N_f}{N^{3/40}} (\log N)^2 \sqrt{\frac{\alpha \theta_2}{\alpha \theta_1}}
\]
\[
\sim 10^3 \frac{\alpha_3}{(\log(r))^{11/6}} \frac{\alpha \theta_2}{\alpha \theta_1}^{11/6},
\]
which checks out for \(M = N_f = 3, N = 100\).

\[
\alpha_3 \Omega_{23}^3 = -R_1^m e_m^6,
\]
that obtains:
\[
-\alpha_3 \frac{N^{9/20} (\log N)^2 (\log(r))^2 \langle r \rangle (0.9 - \langle r \rangle \log(r))}{\alpha \theta_1 \alpha \theta_2} \sim -\alpha_3 \frac{N^{9/20} (\log N)^2 (\log(r))^2 \langle r \rangle (0.9 - \langle r \rangle \log(r))}{\alpha \theta_1 \alpha \theta_2}
\]
\[
\sim -\alpha_3 \frac{N^{7/20} \alpha \theta_2}{(\log(r))^2 \alpha \theta_1 \alpha \theta_2} \sim -\alpha_3 \frac{1}{N^{1/10}} \frac{N^{7/15}}{\alpha \theta_2} \frac{1}{\alpha \theta_1}
\]
\[
\sim -10 \alpha_3 \log N \frac{\alpha \theta_1}{\alpha \theta_2} \sim -10 \alpha_3 \log N \frac{\alpha \theta_2}{\alpha \theta_1} \frac{\alpha \theta_1}{\alpha \theta_2},
\]
which checks out for \(N = 100\).

\[
\alpha_3 \Omega_{36}^3 = R_1^m e_m^2,
\]
Now,
\[
\Omega_{36}^3 \sim -\alpha_3 \frac{N^{9/20} (\log N)^2 (\log(r))^2 \langle r \rangle (0.1 + \langle r \rangle \log(r))}{\alpha \theta_1 \alpha \theta_2}.
\]
Now, $\langle r \rangle \log \langle r \rangle$ for $N = 10^2$ is close to -0.1 and hence $\Omega^3_{36} \sim 0$ just like $R^m_1 e^2_m$.

\[ \alpha_3 \Omega^3_{45} = -R^m_1 e^2_m. \]  \hspace{1cm} (92)

Now, \[ \Omega^3_{45} \sim 10^{-4} \frac{M^3 N^3 (\log N)^2 (\log \langle r \rangle)^3}{N^{3/20} \alpha^2_{\theta_2}}, \]  \hspace{1cm} (93)

which for $N = 10^2$ and $\langle r \rangle$ given by (54), is much less than unity; this mirrors the vanishing of $r^m_1 e^2_m$.

The gist of this subsection hence is:

\[ \alpha_1 \sim \alpha_3 \sim \alpha_7. \]  \hspace{1cm} (94)

6 \hspace{0.5cm} SU(3) Structure on a Manifold with $G_2$ Structure $\Leftrightarrow$ Reduction of $G_2$ Structure to SU(3) Structure

A seven-fold $M_7$ admitting a $G_2$ structure $\Phi$ admits an almost contact metric structure (ACMS) and thereby reduces the structure group to SU(3) [20]. The reason stems from the fact that any such manifold admits a nowhere vanishing vector field $R$ which can be normalized with respect to the metric $g_\Phi$. (In the context of supersymmetric theories) A nowhere-vanishing vector field $R$ on $(Y, \Phi)$ and nowhere-vanishing spinor $\eta$ (implicit in the choice of $G_2$ structure) induce a second spinor $R\eta$ which together can be used to construct an SU(3) structure [21].

Alternatively, the reduction of the structure group from $G_2$ to SU(3) can be effected by construction of the latter from $\Phi_{G_2}$ and ACS:

**Proposition** [2]: The ACMS induces a reduction of the $G_2$ structure to an SU(3) structure $(\omega_\Phi, \Omega)$ on the transverse geometry of the foliation with $\omega_\Phi$ being the fundamental 2-form on $Y$ ($\omega_\Phi = i_R \Phi$) and $\Omega$ the transverse $(3,0)$-form w.r.t. $J(J(u) = R \times_\Phi u, \forall u \in \Gamma(TY))$. $(\omega_\Phi, \Omega)$ are determined by the ACS decomposition of the $G_2$ structure $\Phi$ on $Y$:

\[ \Phi = \sigma \wedge \omega_\Phi + \Omega^+, \]
\[ \psi = *_7 \Phi = -\sigma \wedge \Omega^- + \frac{1}{2} \omega_\Phi \wedge \omega_\Phi. \]  \hspace{1cm} (95)
6.1 $M_7|_{\text{Ouyang-embedding}[\text{parent type IIB}]\cap|\rho_{\text{Ouyang}}|\ll 1}$

In this subsection, via two lemmas, we will construct an explicit transverse $SU(3)$ structure in 6.1.1 from the ACMS of 4.2, and then from a CMS of 5.1 in 6.1.2.

6.1.1 Transverse $SU(3)$ Structure from ACMS

**Lemma 5:** $M_7$, near the Ouyang embedding in the limit of very small limit of the Ouyang embedding parameter and near the $\psi = 2n\pi$, $n = 0, 1, 2$-coordinate patch, inherits a transverse $SU(3)$ structure $(\Omega^{(a)}, \Omega^{(a)})$ wherein $\Omega^{(a)}_+ = \Phi - \sigma^{(a)} \wedge \omega^{(a)}_\Phi$, $\Omega^{(a)}_- = \sigma^{(a)} \wedge (\frac{1}{2} \omega^{(a)}_\Phi \wedge \omega^{(a)}_\Phi - *7\Phi)$, $a = 1, 2, 3$ from the AC(3)S constructed above.

**Proof:** Using (27), (14), (16) and (19), one hence obtains,

\[
\begin{align*}
\Omega^{(1)}_+ &= \Phi - e^1 \wedge \omega^{(1)}_\Phi, \\
\Omega^{(1)}_- &= e^1 \wedge (\frac{1}{2} \omega^{(1)}_\Phi \wedge \omega^{(1)}_\Phi - *7\Phi); \\
\Omega^{(2)}_+ &= \Phi - e^7 \wedge \omega^{(2)}_\Phi, \\
\Omega^{(2)}_- &= e^7 \wedge (\frac{1}{2} \omega^{(2)}_\Phi \wedge \omega^{(2)}_\Phi - *7\Phi); \\
\Omega^{(3)}_+ &= \Phi + e^2 \wedge \omega^{(3)}_\Phi, \\
\Omega^{(3)}_- &= -e^2 \wedge (\frac{1}{2} \omega^{(3)}_\Phi \wedge \omega^{(3)}_\Phi - *7\Phi). \tag{96}
\end{align*}
\]

6.1.2 Transverse $SU(3)$ Structure from CMS

**Lemma 6:** $M_7$, near the Ouyang embedding in the limit of very small limit of the Ouyang embedding parameter and near the $\psi = 2n\pi$, $n = 0, 1, 2$-coordinate patch, inherits a transverse $SU(3)$ structure from the C(3)S constructed in 3, $(\Omega^{(a)}_+, \Omega^{(a)}_-)$, where $\Omega^{(a)}_+ = \Phi - \sigma^{(a)} \wedge \omega^{(a)}_\Phi$ and, e.g., for $\alpha = 1$, a three-parameter $(\Lambda_{145}^{(1)}, \Lambda_{156}^{(1)}, \Lambda_{456})$ family of:

\[\Omega_+ = \Lambda_{AMC} e^{ABC} = \Lambda_{1b_0c_0}^{(1)} e^{1b_0c_0} + \Lambda_{2b_0c_0}^{(1)} e^{2b_0c_0} + \Lambda_{3b_0c_0}^{(1)} e^{3b_0c_0} + \Lambda_{7b_0c_0}^{(1)} e^{7b_0c_0} + \Lambda_{a_0b_0c_0} e^{a_0b_0c_0},\]

where $a_0, b_0, c_0 = 1, 2, 3, 4, 5, 6$ and $\Lambda_{456}$ is the only linearly independent non-vanishing $\Lambda_{a_0b_0c_0}$.

**Proof:** Making the following ansatz for $\Omega_+$:

\[
\begin{align*}
\Omega_- &= \Lambda_{AMC} e^{ABC} = \Lambda_{1b_0c_0}^{(1)} e^{1b_0c_0} + \Lambda_{13b_0c_0}^{(2)} e^{13b_0c_0} + \Lambda_{17c_0}^{(3)} e^{17c_0} \\
&+ \Lambda_{137}^{(4)} e^{137} + \Lambda_{38c_0}^{(1)} e^{38c_0} + \Lambda_{37c_0}^{(3)} e^{37c_0} + \Lambda_{78c_0}^{(1)} e^{78c_0} + \Lambda_{4b_0c_0} e^{a_0b_0c_0}, \tag{97}
\end{align*}
\]
where $a_0, b_0, c_0 = 2, 4, 5, 6$.

The solution using results of Appendix D, along with addition of a regulator $\epsilon_{1745}$ (as) in $\epsilon_{1745} + \alpha_3 \Lambda_{145}^{(1)} = \frac{2a_0}{\alpha_1} \Lambda_{745}^{(1)}$ that arises from (D9)(3) and (D11)(4), is given by:

\[
\Lambda_{125}^{(1)} = -\frac{(0.02\alpha_1 \epsilon_{1745} R e^{-\eta_{1745}} \Lambda_{724}^{(1)}}{R e^6(-R e^4 + R e^5)},
\]

\[
\Lambda_{126}^{(1)} = -(\Lambda_{125}^{(1)} R e^5)/R e^6,
\]

\[
\Lambda_{146}^{(1)} = -(R e^5(\Lambda_{125}^{(1)} R e^2 + \Lambda_{156}^{(1)} R e^6))/(R e^4 R e^6),
\]

\[
\Lambda_{324}^{(1)} = 0,
\]

\[
\Lambda_{325}^{(1)} = (0.5\Lambda_{125}^{(1)})/\alpha_1,
\]

\[
\Lambda_{326}^{(1)} = -(0.5\Lambda_{125}^{(1)} R e^5)/(\alpha_1 R e^6),
\]

\[
\Lambda_{345}^{(1)} = (0.5\Lambda_{145}^{(1)})/\alpha_1,
\]

\[
\Lambda_{346}^{(1)} = -(0.5\Lambda_{145}^{(1)} R e^5)/(\alpha_1 R e^6),
\]

\[
\Lambda_{356}^{(1)} = (0.5(\Lambda_{125}^{(1)} R e^2 + \Lambda_{145}^{(1)} R e^4))/(\alpha_1 R e^6),
\]

\[
\Lambda_{724}^{(1)} = \epsilon_{1724} : |\epsilon_{1724}| \ll 1,
\]

\[
\Lambda_{725}^{(1)} = (-\Lambda_{724}^{(1)} R e^4 + 2\alpha_7 \Lambda_{325}^{(1)} R e^6)/(R e^5(0.6\Lambda_{325}^{(1)} R e^5)/R e^5,
\]

\[
\Lambda_{726}^{(1)} = (\alpha_7 \Lambda_{126}^{(1)})/\alpha_1,
\]

\[
\Lambda_{745}^{(1)} = (\alpha_7(2\epsilon_{1745} + \Lambda_{145}^{(1)}))/\alpha_1,
\]

\[
\Lambda_{746}^{(1)} = \frac{1}{\Lambda_{125}^{(1)} \Lambda_{724}^{(1)}} \left((2\alpha_7 \Lambda_{126}^{(1)} \Lambda_{325}^{(1)} \Lambda_{724}^{(1)} + 2\alpha_7 \Lambda_{126}^{(1)} \Lambda_{345}^{(1)} \Lambda_{724}^{(1)} - 2\alpha_7 \Lambda_{126}^{(1)} \Lambda_{325}^{(1)} \Lambda_{725}^{(1)} + \Lambda_{126}^{(1)} \Lambda_{725}^{(1)} 2 + 2\epsilon_{1745} \Lambda_{724}^{(1)} \Lambda_{726}^{(1)}\right) = -(\Lambda_{745}^{(1)} R e^5)/R e^6),
\]

\[
\Lambda_{756}^{(1)} = \frac{1}{\Lambda_{125}^{(1)} \Lambda_{724}^{(1)} 2} \left(-4\alpha_7^2 \Lambda_{126}^{(1)} \Lambda_{325}^{(1)} \Lambda_{345}^{(1)} \Lambda_{724}^{(1)} + 2\alpha_7 \Lambda_{126}^{(1)} \Lambda_{325}^{(1)} \Lambda_{724}^{(1)} \Lambda_{725}^{(1)} + 2\alpha_7 \Lambda_{126}^{(1)} \Lambda_{345}^{(1)} \Lambda_{724}^{(1)} \Lambda_{725}^{(1)} - 2\alpha_7 \Lambda_{126}^{(1)} \Lambda_{325}^{(1)} \Lambda_{724}^{(1)} \Lambda_{725}^{(1)} 2 + \Lambda_{126}^{(1)} \Lambda_{725}^{(1)} 3 - 4\epsilon_{1745} \Lambda_{325}^{(1)} \Lambda_{724}^{(1)} \Lambda_{726}^{(1)} + 2\epsilon_{1745} \Lambda_{724}^{(1)} \Lambda_{725}^{(1)} \Lambda_{726}^{(1)}\right) = (\Lambda_{745}^{(1)} R e^4)/R e^6,
\]

\[
\Lambda_{324}^{(3)} = (-\alpha_1 \Lambda_{124}^{(1)} - \alpha_7 \Lambda_{724}^{(1)})/R e^3,
\]

\[
\Lambda_{326}^{(1)} = -(0.5\Lambda_{125}^{(1)} R e^5)/(\alpha_1 R e^6),
\]
\[
\Lambda_{325}^{(3)} = \frac{1}{(R.e^3(R.e^4 - R.e^5))} \left( (\alpha_7 \Lambda_{724}^{(1)} - 0.02 \alpha_7^2 \epsilon_{1745} \kappa_{\log, \text{in}} \Lambda_{724}^{(1)} (R.e^4)^2 \right.
\]
\[
+ \alpha_7(-2 \alpha_7 \Lambda_{325}^{(1)} - 1.1 \Lambda_{724}^{(1)} R.e^4 R.e^5 + R.e^5 (-0.02 \alpha_7^2 \epsilon_{1745} \kappa_{\log, \text{in}} \Lambda_{724}^{(1)} R.e^2 + 2 \alpha_7^2 \Lambda_{325}^{(1)} R.e^5) \right),
\]
\[
\Lambda_{326}^{(3)} = \frac{(\alpha_7^2 \Lambda_{724}^{(1)} R.e^5 - \alpha_7^2 \Lambda_{126}^{(1)} R.e^6)}{\alpha_1 R.e^3 R.e^6},
\]
\[
\Lambda_{345}^{(3)} = \frac{\alpha_7^2(-2 \epsilon_{1745} - \Lambda_{145}^{(1)} - \alpha_7^2 \Lambda_{145}^{(1)} - \alpha_1 \Lambda_{456} R.e^6}{\alpha_1 R.e^3} \sim \frac{\alpha_7^2(-\Lambda_{145}^{(1)} - \alpha_7^2 \Lambda_{145}^{(1)} - \alpha_1 \Lambda_{456} R.e^6)}{\alpha_1 R.e^3},
\]
\[
\Lambda_{346}^{(3)} = \frac{1}{\Lambda_{125}^{(1)} \Lambda_{724}^{(1)} R.e^3 R.e^4 R.e^6} \left( -\alpha_1 \Lambda_{125}^{(1)} ^2 \Lambda_{724}^{(1)} R.e^2 R.e^5 + ((\alpha_7(-1 \Lambda_{125}^{(1)} \Lambda_{725}^{(1)} 2
\]
\[
+ \alpha_7 \Lambda_{126}^{(1)} (-2 \Lambda_{325}^{(1)} \Lambda_{724}^{(1)} - 2 \Lambda_{345}^{(1)} \Lambda_{724}^{(1)} + 2 \Lambda_{325}^{(1)} \Lambda_{725}^{(1)}) - 2 \epsilon_{1745} \Lambda_{724}^{(1)} \Lambda_{726}^{(1)})
\]
\[
+ \Lambda_{125}^{(1)} \Lambda_{724}^{(1)} \Lambda_{456}^{(1)} R.e^5 R.e^4 + 1 \alpha_1 \Lambda_{125}^{(1)} \Lambda_{156}^{(1)} \Lambda_{724}^{(1)} R.e^5 R.e^6) \right) \sim \frac{\Lambda_{456}^{(1)} R.e^5}{R.e^3},
\]
\[
\Lambda_{356}^{(3)} = \frac{1}{\Lambda_{125}^{(1)} \Lambda_{456}^{(1)} R.e^3 R.e^4 R.e^6} \left( 4 \alpha_7^2 \Lambda_{126}^{(1)} \Lambda_{325}^{(1)} \Lambda_{345}^{(1)} \Lambda_{724}^{(1)} - \alpha_1 \Lambda_{125}^{(1)} \Lambda_{156}^{(1)} \Lambda_{724}^{(1)} 2
\]
\[
+ \alpha_7^2(\Lambda_{126}^{(1)} \Lambda_{725}^{(1)} (-2 \Lambda_{325}^{(1)} \Lambda_{126}^{(1)} + 2 \Lambda_{345}^{(1)} \Lambda_{724}^{(1)} + 2 \Lambda_{325}^{(1)} \Lambda_{725}^{(1)})
\]
\[
+ 4 \epsilon_{1745} \Lambda_{325}^{(1)} \Lambda_{724}^{(1)} \Lambda_{726}^{(1)} + \alpha_7(-\Lambda_{126}^{(1)} \Lambda_{125}^{(1)} 3 - 2 \epsilon_{1745} \Lambda_{724}^{(1)} \Lambda_{725}^{(1)} \Lambda_{726}^{(1)})
\]
\[
- \Lambda_{125}^{(1)} \Lambda_{724}^{(1)} \Lambda_{456}^{(1)} R.e^4) \right) \sim \frac{-\alpha_1 \Lambda_{156}^{(1)} + \Lambda_{456}^{(1)} R.e^4 + \frac{(0.02 \alpha_7^2 \epsilon_{1745} \kappa_{\log, \text{in}} \Lambda_{456}^{(1)} (R.e^4)^2 + R.e^2 R.e^5))}{(\alpha_1(-R.e^4 + R.e^5)(R.e^6))}}{R.e^3}.
\]

(98)

We hence see that (setting \( \epsilon_{1724}, \epsilon_{1745} \) to zero), one obtains a three-parameter \((\Lambda_{145}^{(1)}, \Lambda_{156}^{(1)}, \Lambda_{456}^{(1)})\) family of solutions.

As regards \( \Lambda_{756}^{(1)} \), on one hand:

\[
\Lambda_{756}^{(1)} \sim \frac{(2 \epsilon_{1745} \kappa_{\log, \text{in}} \Lambda_{724}^{(1)} ((R.e^4)^2 + R.e^2 R.e^5)^2)}{R.e^5(-R.e^4 + R.e^5)^2 R.e^6);}
\]

(99)
using:

\[ R.e^2 \sim 0, \]
\[ R.e^3 \sim - \frac{\mathcal{O}(1)\alpha_3\lambda_5}{(\log(r)/N_f)^{2/3}} \log(r) \sim -N^{7/40}\alpha_0, \lambda_5 \sim 10^2N^{7/40} \log N \]
\[ R.e^4 \sim - \frac{\alpha_3\delta N^{3/5}}{\mathcal{O}(1)g_s^{7/2} \log N (\log(r))^{3/2} M^2 N_f^{8/3} \langle r \rangle^2} \log(r) \sim -N^{7/40} \alpha_0, \lambda_5 \sim 10^2N^{7/40} \frac{\alpha_0}{\log N} \]
\[ R.e^5 \sim - \frac{\alpha_3 N^{7/40}}{\mathcal{O}(1)} \frac{\alpha g}{\log N} \sim -7/24 \]
\[ R.e^6 \sim - \frac{\alpha_3 \log N N^{7/40} \frac{\alpha g}{\log N}}{\mathcal{O}(1)} \left( \alpha g N_f^{2/5} \right)^{5/3} \sim - R.e^5 \]

and given that \( N^{11/30} < 10.7 \) for \( N = 100 \), one obtains:

\[ \Lambda^{(1)}_{756} \sim \frac{\epsilon_{1745}^{2}e_{\log r,IR}^{4}N}{g_{s}^{14}}. \]  

One the other hand:

\[ \Lambda^{(1)}_{756} = \frac{\Lambda^{(1)}_{745}R.e^4}{R.e^6} \sim \frac{N^{1/4}}{g_s^{7/2}} \Lambda^{(1)}_{745}. \]  

Assuming,

\[ \epsilon_{1745}^{2}e_{\log r,IR}^{4} \sim \mathcal{O}(1), \]
\[ \Lambda^{(1)}_{745} \sim \frac{N^{3/4}}{g_s^{21/2}}, \]  

one may obtain consistency between (\( 102 \)) and \( \Lambda^{(1)}_{745} \sim \frac{\alpha e}{\alpha_1} \).

Substituting (\( 98 \)) into the remaining two equations:

\[ \alpha_7 \Lambda^{(1)}_{325} - \alpha_3 \Lambda^{(1)}_{725} - \frac{10^5 e^{-107/20 \alpha_0}}{\alpha_9^4}, \]  

and

\[ \alpha_3 \Lambda^{(1)}_{124} - \alpha_1 \Lambda^{(1)}_{324} - \mathcal{O}(10^3) \alpha_0^2 = 0, \]  

substituting \( \Lambda^{(1)}_{324} = 0 \) from (\( 98 \)). For \( N = 10^2 \) and \( \alpha_0 \sim 8 \) for an appropriate value of \( \Lambda^{(1)}_{124} \) obtained by solving (\( 105 \)) for \( \alpha_0 \), (\( 104 \)) is approximately numerically satisfied.
7 Summary

Continuing with the program of classification of non-supersymmetric geometries relevant to M-theory dual of large large-N thermal QCD at intermediate coupling initiated in [7], we obtained the $G_2$-structure torsion classes of the relevant seven-fold $M_7$ which is the M-theory circle times the warped product $S^1_{\text{thermal}} \times_w M_5$, $M_5$ being non-Einsteinian deformation of $T^{1,1}$:

$$\tau \in \tau_1 \oplus \tau_2 \oplus \tau_3.$$  \hspace{1cm} (106)

We then explicitly showed that the aforementioned $M_7$ supports an Almost Contact 3 Structure. Subsequently, we verified that this AC(3)S is in fact an Almost Contact 3 Metric Structure (AC3MS). It turned out that the abovementioned Almost Contact Structure is in fact not a Contact Structure. We then showed that it is possible to explicitly construct a Contact Structure on the seven-fold $M_7$.

Using a proposition of [2], we not only obtained a transverse $SU(3)$ structure by a reduction of the $G_2$ structure obtained earlier induced by aforementioned Almost Contact Metric Structure, but also a three-parameter family of transverse $SU(3)$ structures by a reduction of the $G_2$ structure induced by the Contact Structure obtained earlier.

Combining the Proposition of [7] and this paper, we hence arrive at the following (combined) proposition:

**Proposition 2:**

- The non-Kähler warped six-fold $M_6$, obtained as a cone over a compact five-fold $M_5$ (a non-Einsteinian generalization of $T^{1,1}$), that appears in the type IIA background corresponding to the M-theory uplift of thermal QCD-like theories at high temperatures, in the neighborhood of the Ouyang embedding (with vanishingly small embedding parameter) ([22]) of the parent type IIB flavor D7-branes (that figure in the type IIB string dual of thermal QCD involving $N$ D3-branes, $M$ fractional D3-branes and $N_f$ flavor D7-branes “wrapping” $\mathbb{R}_{>0} \times S^3$) effected by working in the neighborhood of small $\theta_{1,2}$ ($\theta_{1,2} \in [0, \pi]$), in the "MQGP limit" inclusive of the $O(l_p^6)$, ($l_p$ being Planck length) corrections,

- is a non-complex manifold (though the deviation from $W^{SU(3)}_{1,2} = 0$ being $N$-suppressed)

- proved in [7],

- $W^{SU(3)}_4 \sim W^{SU(3)}_5$ (upon comparison with [23], interpreted as “almost” supersymmetric [in the large-$N$ limit]) - proved in [7].

- The $G_2$-structure torsion classes of the seven-fold $M_7$ (a cone over the M-theory-$S^1$-fibration over $M_5$: $p_1^2(M_{11}) = p_2(M_{11}) = 0$, are: $\kappa_{G_2}^{0} = W_{14}^{G_2} \oplus W_{27}^{G_2}$ - proved in [7].

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• $M_8$, a warped product of the thermal $S^3$ and $M_7$: $W_{M_8}^{SU(4)/Spin(7)} = W_2^{SU(4)} \oplus W_3^{SU(4)} \oplus W_5^{SU(4)} / W_1^{Spin(7)} \oplus W_2^{Spin(7)}$ - proved in [7].

• (Proved in this work - Proposition 1) $M_7|_{r=\text{const} \in IR}$

  - has the following non-trivial $G_2$-structure torsion classes: $\kappa_0^{G_2} = W_7^{G_2} \oplus W_{14}^{G_2} \oplus W_{27}^{G_2}$
  - supports an Almost Contact (3) Metric Structure $(\sigma^1, \sigma^2, \sigma^3) = (e^1, e^7, -e^2)$ [and associated $(R^1, R^2, R^1 \times \Phi R^2)$], but the same does not correspond to Contact (3) Structure;
  - supports a Contact (3) Structure: $(\sigma^1, \sigma^2, \sigma^3) = (\alpha_1 e^1 + \alpha_3 e^5 + \alpha_7 e^7, \beta_1 e^1 + \beta_4 e^4 + \beta_7 e^7, \sigma^3)$ with $\sigma^3 : \sigma^3(R^1 \times \Phi R^2) = 1$
  - inherits a transverse SU(3) structure $\left(\Omega_+^{(\alpha)}, \Omega_-^{(\alpha)}\right)$ wherein $\Omega_+^{(\alpha)} = \Phi - \sigma^{(\alpha)} \wedge \omega^{(\alpha)}_\Phi$, $\Omega_-^{(\alpha)} = \sigma^{(\alpha)} \wedge \left(\frac{1}{2} \omega^{(\alpha)}_\Phi \wedge \omega^{(\alpha)}_\Phi - * \Phi\right)$, $\alpha = 1, 2, 3$ from the abovementioned AC(3)S.
  - inherits a transverse SU(3) structure from the aforementioned C(3)S, $\left(\Omega_+^{(\alpha)}, \Omega_-^{(\alpha)}\right)$, where $\Omega_+^{(\alpha)} = \Phi - \sigma^{(\alpha)} \wedge \omega^{(\alpha)}_\Phi$ and, e.g., for $\alpha = 1$, a three-parameter $(\Lambda_{145}^{(1)}, \Lambda_{156}^{(1)}, \Lambda_{456}^{(1)})$ family of:

$$\Omega_- = \Lambda_{AMC}e^{ABC} = \Lambda_{1b_0c_0}^{(1)} e_{b_0c_0} + \Lambda_{3b_0c_0}^{(1)} e_{3b_0c_0} + \Lambda_{7b_0c_0}^{(1)} e_{7b_0c_0} + \Lambda_{a_0b_0c_0}a_{a_0b_0c_0},$$

where $a_0, b_0, c_0 = 2, 4, 5, 6$ and $\Lambda_{456}$ is the only linearly independent non-vanishing $\Lambda_{a_0b_0c_0}$.

The results are summarized in the following table:
| S. No. | Structure Type | Description | Reference in the paper wherefrom Structure obtained |
|-------|----------------|-------------|--------------------------------------------------|
| 1.    | $G_2$-structure torsion classes $\tau$ | $\tau \in \tau_1 \oplus \tau_2 \oplus \tau_3$ | Section 3 |
| 2.    | AC(3)S | $(\sigma^1, \sigma^2, \sigma^3) = (e^1, e^7, -e^2)$ | (26)-(27) in 4.1 |
| 3.    | AC(3)MS | $g_{mn}J^m_{\gamma_1}J^n_{\gamma_2} = g_{m_1n_1} - \sigma_{m_1}\sigma_{n_1}$ | 4.2 |
| 4.    | C(3)S | $\sigma^1 = \alpha_1 e^1 + \alpha_2 e^3 + \alpha_\tau e^7$  
$\sigma^2 = \beta_1 e^1 + \beta_4 e^4 + \beta_\tau e^7$  
$\sigma^3 : \sigma^3(R^1 \times R^2) = 1$ | 5 |
| 5.    | Transverse $SU(3)$  
3-Structure from AC(3)S | | (95) (96) in 6.1.1 |
| 6.    | Transverse $SU(3)$ Structure from Contact Structure | $(\Omega_+|_{(97)}, \Omega_-)$ | (95), (98) in 6.1.2, App. D |

Table 2: Summary of Results for $M_7$ : a warped product of the $M$-theory circle and the warped product $S^1_{\text{thermal}} \times_w M_5$, $M_5$ being a non-Einsteinian deformation of $T^{1,1} - G_2$ Structure torsion classes, (A)C(3)(M)S and Transverse $SU(3)$ Structures for $N \sim 10^2, M = N_f = 3, g_s \sim 0.1$

**Acknowledgements**

AM is partly supported by a Core Research Grant number SER-1829-PHY from the Science and Engineering Research Board, Govt. of India. GY is supported by a Senior Research Fellowship.
(SRF) from the Council of Scientific and Industrial Research, Govt. of India. Some of the results were presented by one of us (AM) in a seminar at UC Santa Barbara this fall. We thank S. Sarkar for verifying some results of 3 and 4.1 as part of his Master’s project.

A Co-frames near $\psi = 2n\pi, n = 0, 1, 2$-patches

Quoting from [7], the coframes diagonalizing the five-fold which locally is $\mathbb{R}_{>0} \times M_{5}, M_{5}$ being a non-Einsteinian deformation of $T^{1,1}$, near $\psi = 2n\pi, n = 0, 1, 2$-coordinate patches, relevant to this paper, are given by:

$$e^1 = \sqrt{g_{rr}} dr,$$

$$e^2 = \sqrt{\kappa_{1,1;0}} \csc^2 (\theta_2) + \kappa_{3,0;0} \frac{r^6 \sin^6 \left(\frac{\theta_2}{2}\right) \sin \theta_2}{g_{rr} \log N^3 M^{1/3} N^{3/3} \log \left(\frac{r}{N} \right)} + \kappa_{3,1;0} \frac{\sqrt{r \left(C_{\phi} C_{\mu}ight)}}{\sin \frac{\theta_2}{2}}$$

$$\times \left[ \begin{array}{c}
\frac{d\theta_1}{\kappa_{1,1;0}} g_{7/4 M \sin \theta_2 \csc \theta_2} \left(2 \theta_2 \log (r) \left(3a^2 \log (r) + 0.08r \right) \right) + \kappa_{3,2;0} \frac{\sqrt{r \left(C_{\phi} C_{\mu}ight)}}{\sin \frac{\theta_2}{2}}
\end{array} \right]$$

$$+ d\beta_2 \left( \kappa_{2,1;0} g_{7/4} M \sin^2 \left(\theta_2\right) \csc \theta_2 \left(2 \theta_2 \log (r) \left(3a^2 \log (r) + 0.08r \right) \right) + \kappa_{3,2;0} \frac{\sqrt{r \left(C_{\phi} C_{\mu}\right)}}{\sin \frac{\theta_2}{2}} \right)$$

$$+ dx \left( \kappa_{2,1;0} g_{7/2 M^2 \sin^2 \left(\theta_2\right) \csc \theta_2 \left(2 \theta_2 \log (r) \left(3a^2 \log (r) + 0.08r \right) \right) - \kappa_{3,1;0} \frac{\sqrt{r \left(C_{\phi} C_{\mu}\right)}}{\sin \frac{\theta_2}{2}} \right)$$

$$+ dz \left( \kappa_{2,1;0} \sin \left(\theta_2\right) - \kappa_{3,1;0} \frac{\sqrt{r \left(C_{\phi} C_{\mu}\right)}}{\sin \frac{\theta_2}{2}} \right) \right]$$

$$e^3 = \sqrt{\kappa_{1,1;0}} \csc^2 (\theta_2) + \kappa_{3,0;0} \frac{r^6 \sin^6 \left(\frac{\theta_2}{2}\right) \sin \theta_2}{g_{rr} \log N^3 M^{1/3} N^{3/3} \log \left(\frac{r}{N} \right)} - \kappa_{3,1;0} \frac{\sqrt{r \left(C_{\phi} C_{\mu}\right)}}{\sin \left(\frac{\theta_2}{2}\right) \csc \theta_2}$$

$$\times \left[ \begin{array}{c}
\frac{d\theta_1}{\kappa_{1,1;0}} 4g_{7/4 M \sin^2 \left(\theta_2\right) \csc \theta_2 \left(2 \theta_2 \log (r) \left(0.00065 a^2 \log (r) + 0.000014 r \right) \right) + \kappa_{3,1;0} \frac{\sqrt{r \left(C_{\phi} C_{\mu}\right)}}{\sin \left(\frac{\theta_2}{2}\right) \csc \theta_2}
\end{array} \right]$$

$$+ d\beta_2 \left( \kappa_{2,1;0} g_{7/4 M \sin^2 \left(\theta_2\right) \csc \theta_2 \left(2 \theta_2 \log (r) \left(0.00065 a^2 \log (r) + 0.000014 r \right) \right) + \kappa_{3,1;0} \frac{\sqrt{r \left(C_{\phi} C_{\mu}\right)}}{\sin \left(\frac{\theta_2}{2}\right) \csc \theta_2} \right)$$

$$+ dx \left( 1 - \kappa_{3,1;0} \sin \left(\frac{\theta_2}{2}\right) \csc \theta_2 - \kappa_{3,1;0} \frac{\sqrt{r \left(C_{\phi} C_{\mu}\right)}}{\sin \left(\frac{\theta_2}{2}\right) \csc \theta_2} \right)$$

$$+ dz \left( \kappa_{2,1;0} \sin \theta_2 - \kappa_{3,1;0} \frac{\sqrt{r \left(C_{\phi} C_{\mu}\right)}}{\sin \left(\frac{\theta_2}{2}\right) \csc \theta_2} \right) \right]$$

(A1)
\[
e^4 = \sqrt{\kappa_{1,0}^{4} \csc^2 \left( \theta_2 \right) - \kappa_{2,0}^{4} \frac{r^6 \sin^6 \left( \theta_2 \right) \sin^6 \left( \theta_1 \right)}{g_{13}^6 \log N M N_j^3 N_j^3 \log^3 \left( r \right)} + \kappa_{1,0}^{4} \frac{\sqrt{N} C_{0}^{(13)} C_{q}^{(1)}}{\sin^2 \left( \theta_2 \right)} \left( \frac{g_{s}^{7/4} \log N M N_j^3 \left( 0.005a^2 - 0.012a^2 r^2 + 0.004r^4 \right) \log(r)}{g_{13}^{15/4} \log N M M_j^2 N_j^2 \sqrt{\sin \left( \theta_1 \right)} \log^2 \left( r \right)} \right) + d\theta_1 \sqrt{N} \left( \kappa_{0,1}^{4} \frac{\sqrt{N} C_{0}^{(13)} C_{q}^{(1)}}{g_{s}^{15/4} \log N M N_j^3 N_j^3 \log^3 \left( r \right)} - \kappa_{2,0}^{4} \frac{\sqrt{N} C_{0}^{(13)} C_{q}^{(1)}}{g_{13}^{7/4} \log N M N_j^3 N_j^3 \log^3 \left( r \right)} \right) \right) 
\]

\[
e^5 = \sqrt{\kappa_{1,0}^{5} \csc^2 \left( \theta_2 \right) + \kappa_{2,0}^{5} \frac{r^6 \sin^6 \left( \theta_2 \right) \sin^6 \left( \theta_1 \right)}{g_{13}^6 \log N M N_j^3 N_j^3 \log^3 \left( r \right)} + \kappa_{1,0}^{5} \frac{\sqrt{N} C_{0}^{(13)} C_{q}^{(1)}}{\sin^2 \left( \theta_2 \right)} \left( \frac{g_{s}^{7/4} \log N M N_j^3 \left( 0.000092 - 0.00004a^2 \right) \log \left( \theta_1 \right) \csc \left( \theta_2 \right) \log^2 \left( r \right)}{g_{13}^{7/4} \log N M N_j^3 N_j^3 \log^3 \left( r \right)} \right) \right) 
\]

\[
e_\beta^{(3)} \equiv \begin{cases} 
\kappa_{1,0}^{5} \csc^2 \left( \theta_2 \right) + \kappa_{2,0}^{5} \frac{r^6 \sin^6 \left( \theta_2 \right) \sin^6 \left( \theta_1 \right)}{g_{13}^6 \log N M N_j^3 N_j^3 \log^3 \left( r \right)} + \kappa_{1,0}^{5} \frac{\sqrt{N} C_{0}^{(13)} C_{q}^{(1)}}{\sin^2 \left( \theta_2 \right)} \left( \frac{g_{s}^{7/4} \log N M N_j^3 \left( 0.000092 - 0.00004a^2 \right) \log \left( \theta_1 \right) \csc \left( \theta_2 \right) \log^2 \left( r \right)}{g_{13}^{7/4} \log N M N_j^3 N_j^3 \log^3 \left( r \right)} \right) 
\end{cases} 
\]

\[
e_\beta^{(4)} \equiv \begin{cases} 
\kappa_{1,0}^{5} \csc^2 \left( \theta_2 \right) + \kappa_{2,0}^{5} \frac{r^6 \sin^6 \left( \theta_2 \right) \sin^6 \left( \theta_1 \right)}{g_{13}^6 \log N M N_j^3 N_j^3 \log^3 \left( r \right)} + \kappa_{1,0}^{5} \frac{\sqrt{N} C_{0}^{(13)} C_{q}^{(1)}}{\sin^2 \left( \theta_2 \right)} \left( \frac{g_{s}^{7/4} \log N M N_j^3 \left( 0.000092 - 0.00004a^2 \right) \log \left( \theta_1 \right) \csc \left( \theta_2 \right) \log^2 \left( r \right)}{g_{13}^{7/4} \log N M N_j^3 N_j^3 \log^3 \left( r \right)} \right) 
\end{cases} 
\]

\[
e_\beta^{(5)} \equiv \begin{cases} 
\kappa_{1,0}^{5} \csc^2 \left( \theta_2 \right) + \kappa_{2,0}^{5} \frac{r^6 \sin^6 \left( \theta_2 \right) \sin^6 \left( \theta_1 \right)}{g_{13}^6 \log N M N_j^3 N_j^3 \log^3 \left( r \right)} + \kappa_{1,0}^{5} \frac{\sqrt{N} C_{0}^{(13)} C_{q}^{(1)}}{\sin^2 \left( \theta_2 \right)} \left( \frac{g_{s}^{7/4} \log N M N_j^3 \left( 0.000092 - 0.00004a^2 \right) \log \left( \theta_1 \right) \csc \left( \theta_2 \right) \log^2 \left( r \right)}{g_{13}^{7/4} \log N M N_j^3 N_j^3 \log^3 \left( r \right)} \right) 
\end{cases} 
\]

(A2)
\[ e^{\theta} = \sqrt{\kappa_{1,0;0}^6 \csc^2(\theta_2) + \kappa_{2,0;0}^6 r^6 \sin^{a_1}(\theta_2) \sin^{a_2}(\theta_1)} - \kappa_{1,0;1}^6 \sqrt{\frac{C_{1/2}}{C_{z \bar{z}}}} \csc \theta_2 \]

\[
\times \left[ d\theta_1 \left( \kappa_{\theta_1;1,0;0}^6 \frac{g_s \beta^{7/2} \log MNJ \left( 3.49a - 11.6a^2 \right) \sin (\theta_1) \csc (\theta_2) \log (r)}{\sqrt{N_r^2}} + \kappa_{\theta_1;1,0;0}^6 \frac{\beta^{7/4} C_{1/2} \csc \left( \frac{7/4}{M N J} \left( 0.064a^2 - 5.8a^2 \right) \csc (\theta_2) \log (r) \right)}{\sqrt{N_r^2}} \sin (\theta_1) \right) \right. \\
+ d\theta_2 \left( \kappa_{\theta_2;2,1;0}^6 g_s \beta^{7/4} MNJ \sin^2(\theta_1) \csc^2(\theta_2) \log (r) (0.5a^2 \log (r) + 0.014a^2) + \kappa_{\theta_2;2,1;0}^6 \frac{\beta^{7/4} C_{1/2} \csc \left( \frac{7/4}{MNJ} \sqrt{\sin (\theta_1)} \csc^2(\theta_2) \log (r) (2.9a^2 \log (r) + 0.1r) \right)}{\sqrt{N_r^2}} \right) \\
+ d\theta \left( \kappa_{\theta;1;1;0}^6 \sin^2(\theta_1) \csc(\theta_2) \log (r) (15.625a^4 \log (r) + 0.875a^2r \log (r) + 0.01225r^2) \right. \\
\left. \sqrt{N_r^2}/N_r \right) \\
\left. + d\psi \left( 1 - \kappa_{\theta;1;0;0}^6 \frac{g_s \beta^{7/2} M^2 N_r^2 r^2 \sin^4(\theta_1) \csc^4(\theta_2) \log^2(r)}{\sqrt{N_r^2}} \right) \right]

+ d\left( \kappa_{\theta_1;1;0;0}^6 \sin^2(\theta_1) \csc(\theta_2) \log (r) (15.625a^4 \log (r) + 0.875a^2r \log (r) + 0.01225r^2) \right. \\
\left. \sqrt{N_r^2}/N_r \right) \\
\left. + d\left( \kappa_{\theta_2;2,1;0}^6 \sin^2(\theta_1) \csc(\theta_2) \log (r) (15.625a^4 \log (r) + 0.875a^2r \log (r) + 0.01225r^2) \right) \right].
\]  

(A3)

In (A1) - (A3), \( \kappa_{\theta;1,2}^6 \ll 1 \). Except for \( e^4 \), however, all the rest have an IR-enhancement factor involving some power of \( \log r \) appearing in the contributions picked up from the \( O(R^4) \) terms. Further, these contributions also receive near-Ouyang-embedding enhancements around small \( \theta_{1,2} \) - which also provide the most dominant contributions to all the terms of the action. Also, \( \kappa_{\theta;1;0}^6 \gg 1 \) but are accompanied by IR-suppression factors involving exponents of \( r \) along with near-Ouyang-embedding enhancements around small \( \theta_{1,2} \).

Now, (A1) - (A3) can be inverted - in this paper, for simplicity, one restricts to the Ouyang embedding:

\[ (r^6 + 9a^2r^4)^{1/2} e^{\frac{\beta}{2}(\psi - \psi_1 - \psi_2)} \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} = \mu, \]  

(A4)

\( \mu \) being the Ouyang embedding parameter assuming \( |\mu| \ll r^{3/2} \), effected, e.g., by working near the \( \theta_1 = \frac{\alpha_1}{N}, \theta_2 = \frac{\alpha_2}{N}, \psi = 2n\pi, n = 0, 1, 2, \) $r$ in the IR estimated as $r (\in \text{IR}) \sim N^{-\frac{1}{2}}$ with $f_r \sim 1$ [24], [14]. One can then show, that by working in the neighborhood of \( (\theta_{10} = \frac{\alpha_{10}}{N^2}, \theta_{20} = \frac{\alpha_{20}}{N^2}, \psi = 2n\pi, n = 0, 1, 2, \) the $\beta$-dependent terms are given by: \( O(10^{-14}) \sqrt{\beta} + O(10^{-13}) \sqrt{\beta} + O(10^{-10}) \beta^{3/2} + O(1) \beta \). Therefore, e.g., by choosing $\beta \sim O(10^{-19})$, one sees that the most dominant terms are the $\beta^{3/2}$ and the $\beta$ terms which are both of the same order. This is hence the reason why we write explicitly down corrections in the co-frames up to $O(\beta^{3/2})$ and this hence captures the essence of the exact calculation up to $O(\beta)$. 

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In this appendix, we list out the expressions for \( \Omega^1_a, \Omega^7_a, \Omega^a_{bc} \) inclusive of \( \mathcal{O}(\beta) \)-corrections, relevant to 3 and 5.1.

**\( \Omega^1_a \) Components**

\[
\begin{align*}
\Omega^1_a & = \frac{880 N^{7/20} \alpha_2 (2\alpha^2 + r^2)}{M \alpha_2^3 N_f g_h^{7/4} \log(r) (2 \hat{A})} - \beta^{1/4} \left( \frac{\hat{\omega}_{12} C_\mu \sqrt{C_{11}^{(1)} / \alpha_2}}{\sqrt{\alpha_2^{(1)} \sqrt{\alpha_2} \log(r) (2 \hat{A})}} \right) \\
\Omega^3_a & = \frac{56.7618 N^{7/20} \alpha_2}{M \alpha_2^3 N_f g_h^{7/4} \log(r) (2 \hat{A})} - \beta^{1/4} \left( \frac{\hat{\omega}_{12} C_\mu \sqrt{C_{11}^{(1)} / \alpha_2}}{\sqrt{\alpha_2^{(1)} \sqrt{\alpha_2} \log(r) (2 \hat{A})}} \right) \\
\Omega^4_a & = \frac{15.6835 M (r^2 - 2.85714a^2) N_f g_h^{7/4} \log(N) \log(r)}{\log(r) (2 \hat{A})} - \beta^{1/4} \left( \frac{\hat{\omega}_{12} C_\mu \sqrt{C_{11}^{(1)} / \alpha_2}}{\sqrt{\alpha_2^{(1)} \sqrt{\alpha_2} \log(r) (2 \hat{A})}} \right) \\
\Omega^6_a & = \frac{-88.0597 N^{7/20} \alpha_2}{M \alpha_2^3 N_f g_h^{7/4} \log(r) (2 \hat{A})} + \beta^{1/4} \left( \frac{\hat{\omega}_{12} C_\mu \sqrt{C_{11}^{(1)} / \alpha_2}}{\sqrt{\alpha_2^{(1)} \sqrt{\alpha_2} \log(r) (2 \hat{A})}} \right) \\
\Omega^8_a & = 1.29 M \sqrt{\alpha^2 - 0.5r^2} N_f g_h^{7/4} \log(N) \log(r)
\end{align*}
\]

\[
\begin{align*}
+ \beta^{1/4} \left( \frac{\hat{\omega}_{12} C_\mu \sqrt{C_{11}^{(1)} / \alpha_2}}{\sqrt{\alpha_2^{(1)} \sqrt{\alpha_2} \log(r) (2 \hat{A})}} \right)
\end{align*}
\]

where \( \hat{A} \equiv \log \left( \frac{N}{\sqrt{9 \alpha^2 r^2 + 4 \alpha^2}} \right) \).

**\( \Omega^7_a \) Components**

\[
\begin{align*}
\Omega^7_a & = \frac{-160 N^{7/20} \alpha_2 (2\alpha^2 + r^2)}{M \alpha_2^3 N_f g_h^{7/4} \log(r) (2 \hat{A})} - \beta^{1/4} \hat{\Omega}_{72} \left( \frac{C_\mu \sqrt{C_{11}^{(1)} / \alpha_2}}{\sqrt{\alpha_2^{(1)} \sqrt{\alpha_2} \log(r) (2 \hat{A})}} \right) \\
\hat{\Omega}_{72} & = \left( r N_f g_h \left( a^2 \log(r) \left( \hat{A} \right) - 0.01r \log(9a^2 r^4 + r^6) + 0.03 \log(N) - 0.07 \log(\alpha_2 \alpha_2) \right) \right)
\end{align*}
\]

\[
\begin{align*}
+ \beta^{1/4} \hat{\Omega}_{73} \left( \frac{C_\mu \sqrt{C_{11}^{(1)} / \alpha_2}}{\sqrt{\alpha_2^{(1)} \sqrt{\alpha_2} \log(N) (2 \hat{A})}} \right) \\
\hat{\Omega}_{73} & = \left( r N_f g_h \left( a^2 \log(r) \left( \hat{A} \right) - 0.01r \log(9a^2 r^4 + r^6) + 0.03 \log(N) - 0.07 \log(\alpha_2 \alpha_2) \right) \right)
\end{align*}
\]

\[
\begin{align*}
\times \left( \frac{N_f g_h (2.5a^2 + 0.3r^2) \log(\alpha_2 \alpha_2) + (-2a^2 - 0.1r^2) \log(N) + (0.6a^2 + 0.1r^2) \log(9a^2 r^4 + r^6) - 0.4r^2) - 1.7r^2}{N_f g_h (2.5a^2 + 0.3r^2) \log(\alpha_2 \alpha_2)} \right)
\end{align*}
\]

\[
\begin{align*}
\hat{\Omega}_{74} & = \left( 5870.7 N^{7/20} \alpha_2 \right) \left( \frac{C_\mu \sqrt{C_{11}^{(1)} / \alpha_2}}{\sqrt{\alpha_2^{(1)} \sqrt{\alpha_2} \log(N) (2 \hat{A})}} \right) \\
- \beta^{1/4} \hat{\Omega}_{74} \left( \frac{C_\mu \sqrt{C_{11}^{(1)} / \alpha_2}}{\sqrt{\alpha_2^{(1)} \sqrt{\alpha_2} \log(N) (2 \hat{A})}} \right) \\
\hat{\Omega}_{74} & = \left( 5870.7 N^{7/20} \alpha_2 \right) \left( \frac{C_\mu \sqrt{C_{11}^{(1)} / \alpha_2}}{\sqrt{\alpha_2^{(1)} \sqrt{\alpha_2} \log(N) (2 \hat{A})}} \right)
\end{align*}
\]

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\[ \Omega_7^2 = \frac{176.121 N_{\Omega}^{7/40} \alpha_{\Omega}}{M_r \alpha_{\Omega}^2 \sqrt{g_5 \log(r)}} \left( \frac{\alpha_{\Omega}}{\alpha_{b_2}} \right)^{1/2} \left[ \log(N) (33.3a^2 \log(r) + r) + (-16.7a^3 \log(r) - 0.3r) \log (9a^2r^4 + r^6) - 66.7a^2 \log(r) \log (\alpha_{\Omega} \alpha_{b_2}) \right] \\
- 2.3r \log(\alpha_{\Omega} \alpha_{b_2}) - \beta^{1/4} \omega_75 \left[ \frac{10C_3 \sqrt{C_{zz}^{1/4} N_{\Omega}^{17/20} \sqrt{g_5}}} {\alpha_{b_2}^2 \sqrt{g_5 \log(N)}} \left( \frac{\alpha_{\Omega}}{\alpha_{b_2}} \right) \right] \times \frac{N_{\Omega} g_{x} (-19a^2 + 2.5r^2) \log (\alpha_{\Omega} \alpha_{b_2}) + (9a^2 + 11.1r^2) \log(N) + (-4.9a^2 - 0.6r^2) \log (9a^2r^4 + r^6) + 28a^2 + 3.1r^2) + 120a^2 + 14r^2} {N_{\Omega} g_{x} (-20a^2 + 2r^2) \log (\alpha_{\Omega} \alpha_{b_2}) + (10a^2 + r^2) \log(N) + (-5a^2 - 0.5r^2) \log (9a^2r^4 + r^6) + 3.1r^2 + 14.1r^2} \\
\Omega_5^2 = \frac{12.9M \sqrt{N_{\Omega} g_{x}^{7/4} \log(N)}} {r^2 \alpha_{\Omega}^2} \left( \frac{\alpha_{\Omega}}{\alpha_{b_2}} \right)^{1/2} \left[ -0.2a^2 \log (\alpha_{\Omega}) - 0.2a^2 \log (\alpha_{b_2}) + \log(N) (a^2 \log(r) + 0.1a^2 - 0.05r^2) - 2a^2r \log(r) \log (\alpha_{\Omega}) \right] \\
+ \left( -0.5a^2 \log(r) - 0.05a^2 + 0.02r^2 \right) \log (9a^2r^4 + r^6) - 2a^2r \log(r) \log (\alpha_{b_2}) + 0.1r^2 \log (\alpha_{\Omega}) + 0.1r^2 \log (\alpha_{b_2}) \right] \\
- \beta^{1/4} \omega_76 \left[ \frac{10C_3 \sqrt{C_{zz}^{1/4} N_{\Omega}^{17/20} \sqrt{g_5}}} {\alpha_{b_2}^2 \sqrt{g_5 \log(N)}} \left( \frac{\alpha_{\Omega}}{\alpha_{b_2}} \right) \right] \left[ 6.6a^2 \log (\alpha_{\Omega}) + 6.6a^2 \log (\alpha_{b_2}) + (-3.3a^2 - 0.4r^2) \log(N) \\
+ (1.6a^2 + 0.2r^2) \log (9a^2r^4 + r^6) + 0.7r^2 \log (\alpha_{\Omega}) + 0.7r^2 \log (\alpha_{b_2}) \right] \frac{1}{D} \\
\] 

where \( \tilde{D} \equiv (a^2 + 0.1r^2) \tilde{A} \).

\( \Omega_{bc}^2 \) Components (i) \( \Omega_{bc}^2 \):

\[ \Omega_{23}^2 = \frac{12.8N^{3/4} \alpha_{\Omega} (5.8a^2 + r^2)} {M_r \alpha_{\Omega}^2 \sqrt{g_5 \log(r) \sqrt{g_5}}} + \beta^{1/4} \omega_{23} \left[ \frac{0.5C_3 \sqrt{C_{zz}^{1/4} N_{\Omega}^{17/20} \sqrt{g_5}}} {r^2 \alpha_{\Omega}^2 \alpha_{b_2}^2 \sqrt{g_5}} \left( \frac{\alpha_{\Omega}}{\alpha_{b_2}} \right) \right] \times \left( -5.7 \times 10^3 a^2 r \log(r) + 0.3a^2 - 3.1 \times 10^2 r^2 \right) \]

\[ \Omega_{24}^2 = \frac{2.6N^{3/20} \alpha_{\Omega} \log(N) \log(r)} {M_r \alpha_{\Omega}^2 \sqrt{g_5 \log(r) \sqrt{g_5}}} \left( -13.6a^4 + 4.54545a^2r^2 + (1363.6a^4 - 227.3a^2r^3) \log(r) + r^4 \right) \]

\[ + \beta^{1/4} \omega_{24} \left[ - \frac{1.5C_3 \sqrt{C_{zz}^{1/4} N_{\Omega}^{19/20} \sqrt{g_5}}} {r^4 \alpha_{\Omega}^2 \alpha_{b_2}^2 \sqrt{g_5}} \left( \frac{\alpha_{\Omega}}{\alpha_{b_2}} \right) \right] \left( -8.3a^4 - 2.2a^2r^2 + r^4 \right) \]

\[ \Omega_{25}^2 = \frac{9.9N^{3/4} \alpha_{\Omega} (9.7a^2 + r^2)} {M_r \alpha_{\Omega}^2 \sqrt{g_5 \log(r) \sqrt{g_5}}} + \beta^{1/4} \omega_{25} \left[ \frac{4.7 \times 10^2 C_3 \sqrt{C_{zz}^{1/4} N_{\Omega}^{17/4} \sqrt{g_5}}} {r^4 \alpha_{\Omega}^2 \alpha_{b_2}^2 \sqrt{g_5 \log(N)}} \left( \frac{\alpha_{\Omega}}{\alpha_{b_2}} \right) \right] \times \log(N) \left( 0.05a^4 + 1.3a^2 r^2 + (12.09a^2 r^3 - 1.2a^4 r) \log(r) + r^4 + 0.007r^4 \right) \]

\[ \Omega_{26}^2 = \frac{4.1N^{3/20} \alpha_{\Omega} (2a^2 + r^2)} {M_r \alpha_{\Omega}^2 \sqrt{g_5 \log(r) \sqrt{g_5}}} + \beta^{1/4} \omega_{26} \left[ \frac{0.5C_3 \sqrt{C_{zz}^{1/4} N_{\Omega}^{17/20} \sqrt{g_5}}} {r^4 \alpha_{\Omega}^2 \alpha_{b_2}^2 \sqrt{g_5}} \left( \frac{\alpha_{\Omega}}{\alpha_{b_2}} \right) \right] \left( -17a^4 + 1.5 \times 10^2 a^2 r^3 \log(r) - 4.5a^2 r^2 + 76r^4 \right) \]

\[ \Omega_{27}^2 = \frac{M N^{3/20} \alpha_{\Omega} \log(N) \log(r)} {\alpha_{\Omega}^2 \sqrt{g_5 \log(N)}} + \beta^{1/4} \omega_{27} \left[ \frac{0.5C_3 \sqrt{C_{zz}^{1/4} N_{\Omega}^{19/20} \sqrt{g_5}}} {r^4 \alpha_{\Omega}^2 \alpha_{b_2}^2 \sqrt{g_5}} \left( \frac{\alpha_{\Omega}}{\alpha_{b_2}} \right) \right] \left( 20a^2 r^2 + (590a^4 r - 140a^2 r^3) \log(r) - 5r^4 \right) \]

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\[ \sum_{25}^{3} = \frac{16.7N^{3/4}a_{03}}{M^{\alpha_{03}}_{\frac{3}{4}}N_{\frac{1}{2}}g_{\frac{3}{4}}} \log(r) + \beta^{1/4} \omega_{25}^{3} \left[ \frac{450C_{q} \sqrt{C_{zz}(1)} N^{5/4} \left( 0.03 a^{2} r + (1.1 a^{4} + 37.5 a^{2} r^{2}) \log(r) + r^{3} \right)}{r^{3} \alpha_{03}^{1/2} \alpha_{02}^{4} \sqrt{g_{s}}} \right] \]

\[ \sum_{36}^{3} = \frac{-6.9N^{3/4}a_{02}}{M^{\alpha_{03}}_{\frac{3}{4}}N_{\frac{1}{2}}g_{\frac{3}{4}}} \log(r) + \beta^{1/4} \omega_{36}^{3} \left[ \frac{C_{q} \sqrt{C_{zz}(1)} N^{5/4} \left( -0.1 a^{2} r^{2} + (22 a^{2} r^{3} - 4.1 a^{4} r) \log(r) + r^{4} \right)}{r^{4} \alpha_{03}^{1/2} \alpha_{02}^{4} \sqrt{g_{s}}} \right] \]

\[ \sum_{35}^{3} = \frac{1.6M N^{3/20}N_{\frac{1}{2}}g_{\frac{3}{4}} \log^{2}(N) \log(r) + \beta^{1/4} \omega_{35}^{3} \left[ \frac{C_{q} \sqrt{C_{zz}(1)} N^{19/20}}{r^{4} \alpha_{03}^{1/2} \alpha_{02}^{4} \sqrt{g_{s}}} \log(N) \right]}{r^{3} \alpha_{03}^{1/2} \alpha_{02}^{4} \sqrt{g_{s}}} \times \left( 0.04 a^{2} r^{2} \log(N) + \log(r) \left( a^{4} r \log(N) + a^{2} r^{3}(-0.2 \log(N) - 0.7) \right) + r^{4}(-0.008 \log(N) - 0.013) \right) \]

\[ \sum_{36}^{3} = \frac{-1.4M N^{3/20} \left( r^{2} - 3 a^{2} \right) N_{\frac{1}{2}}g_{\frac{3}{4}} \log^{2}(N) \log(r) - \beta^{1/4} \omega_{36}^{3} \left[ \frac{C_{q} \sqrt{C_{zz}(1)} N^{19/20} \left( 20.5 a^{2} r \log(r) + r \right)}{r^{4} \alpha_{03}^{1/2} \alpha_{02}^{4} \sqrt{g_{s}}} \log(N) \right]}{r^{3} \alpha_{03}^{1/2} \alpha_{02}^{4} \sqrt{g_{s}}} \]

\[ \sum_{35}^{3} = \frac{10.6N^{3/4}a_{02}}{M^{\alpha_{03}}_{\frac{3}{4}}N_{\frac{1}{2}}g_{\frac{3}{4}}} \log(r) + \beta^{1/4} \omega_{35}^{3} \left[ \frac{0.5C_{q} \sqrt{C_{zz}(1)} N^{5/4} \left( 22 a^{2} r + (-3.2 \times 10^{2} a^{2} r^{2}) \log(r) - 1.5 \times 10^{2} r^{3} \right)}{r^{3} \alpha_{03}^{1/2} \alpha_{02}^{4} \sqrt{g_{s}}} \right] \]

(B3)

In (B3), \[ \left| \tilde{\omega}_{25}^{3} \right| \ll 1. \]

(ii) \[ \sum_{3}^{7} \]

\[ \sum_{23}^{3} = \frac{-0.5M N^{9/20} \left( r^{2} - 3 a^{2} \right) \left( r^{2} - 2 a^{2} \right) \left( 2 a^{2} + r^{2} \right) N_{\frac{1}{2}}g_{\frac{3}{4}} \log^{2}(N) \log(r)}{r^{6} \alpha_{03}^{1/2} \alpha_{02}^{4}} + \beta^{1/4} \omega_{23}^{3} \left[ \frac{C_{q} \sqrt{C_{zz}(1)} N^{5/4}}{r^{3} \alpha_{03}^{1/2} \alpha_{02}^{4} \sqrt{g_{s}}} \log(N) \right] \times \left( 33.3 a^{2} r \log(r) - 0.3 a^{2} + r^{2} \right) \left( r^{2} - 3.3 a^{2} \right) \log(N) - 87.7 a^{2} r \log(r) \right] \]

\[ \sum_{24}^{3} = \frac{M \left( -9.88 a^{4} + 35.2 a^{2} r^{2} + 7.24 r^{4} \right) N_{\frac{1}{2}}g_{\frac{3}{4}} \log^{2}(N) \log(r)}{\sqrt{r^{4}}} - \beta^{1/4} \omega_{24}^{3} \left[ \frac{C_{q} \sqrt{C_{zz}(1)} N^{19/20} \left( r^{2} - 3 a^{2} \right) \left( 2 a^{2} + r^{2} \right)}{r^{4} \alpha_{03}^{1/2} \alpha_{02}^{4} \sqrt{g_{s}}} \log(N) \right] \]

\[ \sum_{25}^{3} = \frac{0.3M N^{9/20} \left( r^{2} - 3 a^{2} \right) \left( r^{2} - 2 a^{2} \right) \left( 2 a^{2} + r^{2} \right) N_{\frac{1}{2}}g_{\frac{3}{4}} \log^{2}(N) \log(r)}{r^{6} \alpha_{03}^{1/2} \alpha_{02}^{4}} + \beta^{1/4} \omega_{25}^{3} \left[ \frac{C_{q} \sqrt{C_{zz}(1)} N^{5/4}}{r^{4} \alpha_{03}^{1/2} \alpha_{02}^{4} \sqrt{g_{s}}} \log(N) \right] \]

\[ \left( 100 a^{2} r \log(r) - 4 a^{2} + r^{2} \right) \left( r^{2} - 3.3 a^{2} \right) \log(N) - 87.6667 a^{2} r \log(r) \right] \]

\[ \sum_{26}^{3} = \frac{0.3M N^{9/20} \left( r^{2} - 3 a^{2} \right) \left( r^{2} - 2 a^{2} \right) \left( 2 a^{2} + r^{2} \right) N_{\frac{1}{2}}g_{\frac{3}{4}} \log^{2}(N) \log(r)}{r^{6} \alpha_{03}^{1/2} \alpha_{02}^{4}} + \beta^{1/4} \omega_{26}^{3} \left[ \frac{C_{q} \sqrt{C_{zz}(1)} N^{5/4} \left( 2 a^{2} + r^{2} \right) \left( r^{2} - 3 a^{2} \right) \log(N) - 78.9 a^{2} r \log(r) \right)}{r^{4} \alpha_{03}^{1/2} \alpha_{02}^{4} \sqrt{g_{s}}} \log(N) \right] \]

\[ \sum_{34}^{3} = \frac{-10.4M \left( 1.7 a^{2} - 0.8 r^{2} \right) \left( a^{2} - 0.3 r^{2} \right) \left( 0.2 a^{2} - 0.07 r^{2} \right) N_{\frac{1}{2}}g_{\frac{3}{4}} \log^{4}(N) \log^{3}(r)}{N^{3/20} r^{6} \alpha_{02}^{4}} \]

\[ - \beta^{1/4} \omega_{34}^{3} \left[ \frac{C_{q} \sqrt{C_{zz}(1)} N^{19/20} \left( 33.3 a^{2} r \log(r) + r^{2} \right) \left( r^{2} - 3.3 a^{2} \right) \log(N) - 87.7 a^{2} r \log(r) \right)}{r^{4} \alpha_{03}^{1/2} \alpha_{02}^{4} \sqrt{g_{s}}} \log(N) \right] \]
$$\Omega_{35} = \frac{0.5MN^{9/20} \left( r^2 - 3.3a^2 \right) \left( r^2 - 3.a^2 \right) N_f g_s^{7/4} \log^2(N) \log(r)}{r^4 a_{\beta_1}^a a_{\beta_2}^a}$$

$$- \beta^{1/4} \omega_{35} \left[ \sqrt{C_{zz}^{(1)}} C_2 N^{5/4} \left( 33.3a^2 r \log(r) + r^2 \right) \left( (r^2 - 3.3a^2) \log(N) - 87.7a^2 r \log(r) \right) \right]$$

$$\Omega_{36} = \frac{0.4MN^{9/20} \left( r^2 - 3.a^2 \right) \left( r^2 - 2.a^2 \right) N_f g_s^{7/4} \log^2(N) \log(r)}{r^4 a_{\beta_1}^a a_{\beta_2}^a}$$

$$- \beta^{1/4} \omega_{36} \left[ \sqrt{C_{zz}^{(1)}} C_2 N^{5/4} \left( a^2 - 0.3r^2 \right) \left( a^2 \log(r) + 0.03r \right) \right]$$

$$\Omega_{37} = \frac{178.2M^3 \left( r^2 - 0.3r^2 \right) \left( 0.2a^2 - 0.07r^2 \right) \left( 0.03r^2 - 0.06a^2 \right) N_f g_s^{21/4} \log^4(N) \log^2(r)}{r^4 a_{\beta_1}^a a_{\beta_2}^a \sqrt{g_s}}$$

$$\Omega_{38} = \frac{0.99M^3 \left( 3.3a^2 - r^2 \right)^2 \left( r^2 - 2.a^2 \right) N_f g_s^{21/4} \log^4(N) \log^3(r)}{N^{3/20}r^4 a_{\beta_2}^a}$$

$$+ \beta^{1/4} \omega_{38} \left[ \sqrt{C_{zz}^{(1)}} M^2 N^{13/20} \left( a^2 - 0.3r^2 \right) \left( a^2 \log(r) + 0.03r \right) \right]$$

$$\times \left( (r^2 - 3.3a^2) \log(N) - 87.7a^2 r \log(r) \right)$$

$$\Omega_{39} = \frac{0.5MN^{9/20} \left( r^2 - 3.a^2 \right) \left( r^2 - 2.a^2 \right) N_f g_s^{7/4} \log^2(N) \log(r)}{r^4 a_{\beta_1}^a a_{\beta_2}^a} + \beta^{1/4} \omega_{39} \left[ \sqrt{C_{zz}^{(1)}} N^{5/4} \left( a^2 - 0.3r^2 \right) \right]$$

In (B4), $|\omega_{3c}| \ll 1$.  

(iii) $\Omega^4_{bc}$

$$\Omega_{23} = \frac{-38.7M \left( r^2 - 2.a^2 \right) \left( 8.2a^4 - 5.a^2 r^2 + r^4 \right) N_f g_s^{7/4} \log(N) \log(r)}{\sqrt{r^2 N}} - \beta^{1/4} \omega_{23} \left[ \sqrt{C_{zz}^{(1)}} N^{37/20} \left( r^2 - 9.7a^2 \right) \right]$$

$$\Omega_{24} = \frac{-32.9a^2 M a_{\beta_1}^a \left( a^2 - 0.3r^2 \right) \left( a^2 - 0.7a^2 r^2 + 0.1r^4 \right) N_f g_s^{7/4} \log^2(N) \log^2(r)}{N^2 r^2 a_{\beta_2}^a \left( a^2 - 0.7a^2 r^2 + 0.09r^4 \right)} - \beta^{1/4} \omega_{24} \left[ \sqrt{C_{zz}^{(1)}} N^{5/4} \right]$$

$$\times \left( 50a^4 - 40a^2 r^2 + r^4 \right)$$

$$\Omega_{25} = \frac{-21.9M \left( r^2 - 2.a^2 \right) \left( 8.25a^4 - 5.a^2 r^2 + r^4 \right) N_f g_s^{7/4} \log(N) \log(r)}{\sqrt{r^2 N}} - \beta^{1/4} \omega_{25} \left[ \sqrt{C_{zz}^{(1)}} N^{37/20} \left( -7.02a^2 r^2 + r^4 \right) \right]$$

$$\Omega_{26} = \frac{-22.8M \left( r^2 - 2.a^2 \right) \left( 8.25a^4 - 5.a^2 r^2 + r^4 \right) N_f g_s^{7/4} \log(N) \log(r)}{\sqrt{r^2 N}} - \beta^{1/4} \omega_{26} \left[ \sqrt{C_{zz}^{(1)}} N^{37/20} \left( 2a^2 + r^2 \right) \right]$$

$$\boxed{36}$$
\[ \Omega_{34} = \frac{23.2M^3\alpha_2^2 r (r^2 - 2.1a^2) (r^2 - 2a^2) (8.2a^4 - 5a^2r^2 + r^4) N_f^2 g_s^{21/4} \log^3(N) \log^4(r)}{N^{17/20} \alpha_2 (11a^4 - 7.5a^2r^2 + r^4)} \]

\[ + \beta^{1/4} \tilde{\omega}_{34} \left[ \frac{C_q \sqrt{C_{41}^{(11/20)}} (1000a^4r + 100a^2r^3) \log(r) + r^4)}{r^4 \alpha_2 \sqrt{g_s}} \right] \]

\[ \Omega_{35} = -70.6M (r^2 - 2a^2) (8.2a^4 - 5a^2r^2 + r^4) N_f^2 g_s^{7/4} \log(N) \log(r) \sqrt{N} r^2 \left(11a^4 - 7.5a^2r^2 + r^4\right) \]

\[ + \beta^{1/4} \tilde{\omega}_{35} \left[ \frac{C_q \sqrt{C_{41}^{(11/20)}} (\beta^4)}{M^2r^4 \alpha_2^2 \alpha_2^3 N_f^2 g_s^{15/4} \log^2(N) \log^2(r)} \right] \]

\[ \Omega_{36} = \frac{47.3M (r^2 - 2a^2) (8.2a^4 - 5a^2r^2 + r^4) N_f^2 g_s^{7/4} \log(N) \log(r)}{\sqrt{N} r^2 \left(11a^4 - 7.5a^2r^2 + r^4\right)} \]

\[ - \beta^{1/4} \tilde{\omega}_{36} \left[ \frac{C_q \sqrt{C_{41}^{(11/20)}} (15.9a^4 \log(r) + r) (28.3a^2 \log(r) + r)}{M^2r^2 \alpha_2^2 (N_f^2 g_s^{15/4} \log(N) \log^2(r))} \right] \]

\[ \Omega_{45} = \frac{13.1M^3\alpha_2^2 (r^2 - 2a^2) (r^2 - 2a^2) (8.2a^4 - 5a^2r^2 + r^4) N_f^3 g_s^{21/4} \log^3(N) \log^4(r)}{N^{17/20} \alpha_2 (11a^4 - 7.5a^2r^2 + r^4)} \]

\[ + \beta^{1/4} \tilde{\omega}_{45} \left[ \frac{C_q \sqrt{C_{41}^{(11/20)}} (500a^4 r + 100a^2r^3) \log(r) + r^4)}{r^4 \alpha_2 \sqrt{g_s}} \right] \]

\[ \Omega_{46} = \frac{13.7M^3\alpha_2^2 (r^2 - 2.1a^2) (r^2 - 2a^2) (8.2a^4 - 5a^2r^2 + r^4) N_f^2 g_s^{21/4} \log^3(N) \log^4(r)}{N^{17/20} \alpha_2 (11a^4 - 7.5a^2r^2 + r^4)} \]

\[ - \beta^{1/4} \tilde{\omega}_{46} \left[ \frac{C_q \sqrt{C_{41}^{(11/20)}} (35.7143a^2 r \log(r) - 2.8571a^2 + r^2)}{r^2 \alpha_2 \sqrt{g_s}} \right] \]

\[ \Omega_{56} = \frac{-68.5M (r^2 - 2a^2) (8.25a^4 - 5a^2r^2 + r^4) N_f g_s^{7/4} \log(N) \log(r)}{\sqrt{N} r^2 (11a^4 - 7.5a^2r^2 + r^4)} \]

\[ + \beta^{1/4} \tilde{\omega}_{56} \left[ \frac{C_q \sqrt{C_{41}^{(11/20)}} \alpha_2^2 N_f^2 g_s^{15/4} \log^2(N) \log^2(r)}{M^2 \alpha_2^2 \alpha_2^3 N_f g_s^{15/4} \log^2(N) \log^2(r)} \right] \]

In (B5), \( |\tilde{\omega}_{\text{osc}}| \ll 1 \).

(iv) \( \tilde{\omega}_{\text{sc}} \):

\[ \tilde{\omega}_{23} = \frac{407.4N^{7/20} \alpha_2 (4.6a^2 + r^2)}{M \alpha_2^2 \alpha_2^3 N_f g_s^{4/4} \log(r)} - \beta^{1/4} \tilde{\omega}_{23} \left[ \frac{C_q \sqrt{C_{41}^{(11/20)}} \sqrt{N} a (0.01a^4 - 0.03a^2r^2 + (13.1a^2r^3 - 1.2a^4r) \log(r) + r^4)}{\sqrt{N} \alpha_2 \sqrt{g_s}} \right] \]

\[ \tilde{\omega}_{24} = \frac{M (854.5a^4 + 302.5a^2r^2 + 62.7r^4) N_f g_s^{7/4} \log^2(N) \log(r)}{\sqrt{N} r^4} \]

\[ + \beta^{1/4} \tilde{\omega}_{24} \left[ \frac{0.5C_q \sqrt{C_{41}^{(11/20)}} (a \log N - 5.03 \times 10^{-12} a^2r^2 \log(r) - 9.9 \times 10^{-14} \log(N) \log(r))}{\sqrt{g_s} \log N \alpha_2^2 \sqrt{g_s}} \right] \]

\[ \tilde{\omega}_{25} = -\frac{228.8N^{7/20} \alpha_2 (9.2a^2 + r^2)}{M \alpha_2^2 \alpha_2^3 N_f g_s^{4/4} \log(r)} + \beta^{1/4} \tilde{\omega}_{25} \left[ \frac{C_q \sqrt{C_{41}^{(11/20)}} \sqrt{N} a (0.03a^4 + 1.23a^2r^2 + (13.9a^2r^3 - 0.95a^4r) \log(r) + r^4)}{\sqrt{N} \alpha_2 \sqrt{g_s}} \right] \]

\[ \tilde{\omega}_{26} = -\frac{202.05N^{7/20} \alpha_2 (2a^2 + r^2)}{M \alpha_2^2 \alpha_2^3 N_f g_s^{4/4} \log(r)} + \beta^{1/4} \tilde{\omega}_{26} \left[ \frac{C_q \sqrt{C_{41}^{(11/20)}} \sqrt{N} a (r^4 - 0.0746269a^4)}{\sqrt{g_s} \alpha_2^2 \sqrt{g_s}} \right] \]

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\[ \Omega_{\bar{0}4} = \frac{M}{\sqrt{N}} \left( 274.7a^2 - 66.9r^2 \right) N_f g_r^{7/4} \log^2(N) \log(r) - \beta^{1/4} \omega_5^{\beta} \left[ C_\alpha \sqrt{C_{zz}^{(1)}} N^{11/20} \sqrt{\alpha_{b_1}} \left( r^2 - 1.5a^2 \right) \right] \]

\[ \Omega_{\bar{0}5} = \frac{674.08 N^{7/20} \alpha_{b_2}}{M \alpha_{b_2}^2 N_f g_r^{7/4} \log(r)} - \beta^{1/4} \omega_5^{\beta} \left[ C_\alpha \sqrt{C_{zz}^{(1)}} N^{11/20} \sqrt{\alpha_{b_1}} \left( 0.04a^2 r^2 + (1.4a^4 r + 59.8a^2 r^3) \log(r) + r^4 \right) \right] \]

\[ \Omega_{\bar{0}6} = \frac{-337.8 N^{7/20} \alpha_{b_2}}{M \alpha_{b_2}^2 N_f g_r^{7/4} \log(r)} - \beta^{1/4} \omega_5^{\beta} \left[ \frac{(a^2 - 26.7) C_\alpha \sqrt{C_{zz}^{(1)}} N^{11/20} \sqrt{\alpha_{b_1}}}{r^2 \alpha_{b_2}^2 \sqrt{g_r}} \right] \]

\[ \Omega_{\bar{0}7} = \frac{66.3 M (0.0316742a^2 + r^2) N_f g_r^{7/4} \log^2(N) \log(r)}{\sqrt{N}} - \beta^{1/4} \omega_5^{\beta} \left[ C_\alpha \sqrt{C_{zz}^{(1)}} \alpha_{b_1} \left( -3a^2 r^2 + (50.2a^2 r^3 - 100.4a^4 r) \log(r) + r^4 \right) \right] \]

\[ \Omega_{\bar{0}8} = \frac{85.2 M N_f g_r^{7/4} \log^2(N) \log(r) (302.03a^4 r^2 \log^2(r) - 2.5a^2 r^2 + (42.3a^2 r^3 - 22.7a^4 r) \log(r) + r^4)}{\sqrt{N}} - \beta^{1/4} \omega_5^{\beta} \left[ \frac{20.5a^2 r \log(r) + r^2}{r^2 \alpha_{b_2}^2 \sqrt{g_r} \log(N)} \right] \]

\[ \Omega_{\bar{0}9} = \frac{524.01 N^{7/20} \alpha_{b_2}}{M \alpha_{b_2}^2 N_f g_r^{7/4} \log(r)} - \beta^{1/4} \omega_5^{\beta} \left[ C_\alpha \sqrt{C_{zz}^{(1)}} \alpha_{b_1} \left( -0.04a^2 r + (10.1a^2 r^2 - 1.4a^4) \log(r) + r^3 \right) \right] \]

\[ \Omega_{\bar{10}} = \frac{517.5 N^{7/20} \alpha_{b_2} (4.5a^2 + r^2)}{M r^2 \alpha_{b_2}^2 N_f g_r^{7/4} \log(r)} + \beta^{1/4} \omega_5^{\beta} \left[ \frac{C_\alpha \sqrt{C_{zz}^{(1)}} \alpha_{b_1}}{r^2 \alpha_{b_2}^2 \sqrt{g_r} \log(N)} \right] \times \left( 23.9a^4 r^2 \log^2(r) + \log(N) \left( -0.008a^4 + 0.03a^2 r^2 + (a^4 r - 11.1a^2 r^3) \log(r) - 0.8r^4 \right) \right) \]

\[ \Omega_{\bar{11}} = \frac{78.1 M \left( -13.5a^4 + 4.7a^2 r^2 + r^4 \right) N_f g_r^{7/4} \log^2(N) \log(r)}{\sqrt{N}} + \beta^{1/4} \omega_5^{\beta} \left[ \frac{a^4 C_\alpha \sqrt{C_{zz}^{(1)}} \alpha_{b_1}}{r^2 \alpha_{b_2}^2 \sqrt{g_r} \log(N)} \right] \times \left( 23.9a^4 r^2 \log^2(r) + \log(N) \left( -0.04a^4 - 1.3a^2 r^2 + (a^4 r - 14.6a^2 r^3) \log(r) - 1.1r^4 \right) \right) \]

\[ \Omega_{\bar{12}} = \frac{283.8 N^{7/20} \alpha_{b_2} (9.2a^2 + r^2)}{M r^2 \alpha_{b_2}^2 N_f g_r^{7/4} \log(r)} - \beta^{1/4} \omega_5^{\beta} \left[ \frac{C_\alpha \sqrt{C_{zz}^{(1)}} \alpha_{b_1}}{r^2 \alpha_{b_2}^2 \sqrt{g_r} \log(N)} \right] \times \left( 23.9a^4 r^2 \log^2(r) + \log(N) \left( a^4 - 130.04a^2 r^3 \log(r) + 0.2a^2 r^2 - 15.2r^4 \right) \right) \]

In (B6), \[ |\omega_5^{\beta}| \ll 1. \]

(v) \[ \Omega_{\bar{0}e} \]

\[ \Omega_{\bar{23}} = \frac{517.5 N^{7/20} \alpha_{b_2} (4.5a^2 + r^2)}{M r^2 \alpha_{b_2}^2 N_f g_r^{7/4} \log(r)} + \beta^{1/4} \omega_5^{\beta} \left[ \frac{C_\alpha \sqrt{C_{zz}^{(1)}} \alpha_{b_1}}{r^2 \alpha_{b_2}^2 \sqrt{g_r} \log(N)} \right] \times \left( 23.9a^4 r^2 \log^2(r) + \log(N) \left( -0.008a^4 + 0.03a^2 r^2 + (a^4 r - 11.1a^2 r^3) \log(r) - 0.8r^4 \right) \right) \]

\[ \Omega_{\bar{24}} = \frac{78.1 M \left( -13.5a^4 + 4.7a^2 r^2 + r^4 \right) N_f g_r^{7/4} \log^2(N) \log(r)}{\sqrt{N}} + \beta^{1/4} \omega_5^{\beta} \left[ \frac{a^4 C_\alpha \sqrt{C_{zz}^{(1)}} \alpha_{b_1}}{r^2 \alpha_{b_2}^2 \sqrt{g_r} \log(N)} \right] \times \left( 23.9a^4 r^2 \log^2(r) + \log(N) \left( -0.04a^4 - 1.3a^2 r^2 + (a^4 r - 14.6a^2 r^3) \log(r) - 1.1r^4 \right) \right) \]

\[ \Omega_{\bar{25}} = \frac{283.8 N^{7/20} \alpha_{b_2} (9.2a^2 + r^2)}{M r^2 \alpha_{b_2}^2 N_f g_r^{7/4} \log(r)} - \beta^{1/4} \omega_5^{\beta} \left[ \frac{C_\alpha \sqrt{C_{zz}^{(1)}} \alpha_{b_1}}{r^2 \alpha_{b_2}^2 \sqrt{g_r} \log(N)} \right] \times \left( 23.9a^4 r^2 \log^2(r) + \log(N) \left( -0.04a^4 - 1.3a^2 r^2 + (a^4 r - 14.6a^2 r^3) \log(r) - 1.1r^4 \right) \right) \]

\[ \Omega_{\bar{26}} = \frac{283.8 N^{7/20} \alpha_{b_2} (9.2a^2 + r^2)}{M r^2 \alpha_{b_2}^2 N_f g_r^{7/4} \log(r)} - \beta^{1/4} \omega_5^{\beta} \left[ \frac{C_\alpha \sqrt{C_{zz}^{(1)}} \alpha_{b_1}}{r^2 \alpha_{b_2}^2 \sqrt{g_r} \log(N)} \right] \times \left( 23.9a^4 r^2 \log^2(r) + \log(N) \left( a^4 - 130.04a^2 r^3 \log(r) + 0.2a^2 r^2 - 15.2r^4 \right) \right) \]
\[ \Omega_6 = \frac{M (350.1a^2 - 87.8r^2) N_f g_{7/4} \log^2(N) \log(r)}{N r^2} + \beta^{1/4} \omega_{6} \left[ C \sqrt{C_{\infty}^{(1)} N^{11/20} \sqrt{\alpha_s}} \right]_N \]
\[ \times (a^4 r \log^2(r) + \log(N) (0.001a^2 r + (0.04a^4 - 0.01a^2 r^2) \log(r) - 0.0004r^3)) \]
\[ \Omega_6 = \frac{867.8N^{7/20} \alpha_s}{M \alpha_s^2 N_f g_{7/4} \log(r)} + \beta^{1/4} \omega_{6} \left[ C \sqrt{C_{\infty}^{(1)} N^{11/20} \sqrt{\alpha_s}} \right]_N \]
\[ \times (23.9a^4 r^2 \log^2(r) + \log(N) (0.03a^2 r^2 + (a^4 r + 45.2a^2 r^3) \log(r) + 0.8r^4)) \]
\[ \Omega_6 = -\frac{438.8N^{7/20} \alpha_s}{M \alpha_s^2 N_f g_{7/4} \log(r)} - \beta^{1/4} \omega_{6} \left[ C \sqrt{C_{\infty}^{(1)} N^{11/20} \sqrt{\alpha_s}} \right]_N \]
\[ \times (a^4 r \log^2(r) + \log(N) (0.001a^2 r + (0.042a^4 - 0.012a^2 r^2) \log(r) - 0.0004r^3)) \]
\[ \Omega_6 = \frac{82.8M (r^2 - 0.03a^2) N_f g_{7/4} \log^2(N) \log(r)}{N r^2} + \beta^{1/4} \omega_{6} \left[ C \sqrt{C_{\infty}^{(1)} N^{11/20} \sqrt{\alpha_s}} \right]_N \]
\[ \times (a^4 r^2 \log^2(r) + \log(N) (0.03a^2 r^2 + (a^4 r - 7.5a^2 r^3) \log(r) - 0.8r^4)) \]

(B7)

In (B7), \(|\omega_{6c}| \ll 1\).

From (B1) - (B7), one notes that in the MQGP limit,
\[ (\Omega_2)^{30} \sim (\Omega_3)^{30} \sim - (\Omega_5)^{30}, \]
\[ (\Omega_2)^{30} \sim - (\Omega_4)^{30}; \]
\[ (\Omega_3)^{30} \sim - (\Omega_4)^{30} \sim - (\Omega_5)^{30}, \]

(B9)
and,

\[
(\Omega_{23}^2)^{\phi^0} \sim - (\Omega_{25}^2)^{\phi^0} \sim - (\Omega_{26}^2)^{\phi^0} \sim (\Omega_{56}^2)^{\phi^0} \sim - (\Omega_{36}^2)^{\phi^0}, \\
(\Omega_{34}^2)^{\phi^0} \sim (\Omega_{45}^2)^{\phi^0} \sim (\Omega_{46}^2)^{\phi^0}; \\
(\Omega_{23}^3)^{\phi^0} \sim (\Omega_{24}^3)^{\phi^0} \sim (\Omega_{25}^3)^{\phi^0} \sim (\Omega_{26}^3)^{\phi^0} \sim (\Omega_{35}^3)^{\phi^0} \sim (\Omega_{36}^3)^{\phi^0} \sim (\Omega_{56}^3)^{\phi^0}, \\
(\Omega_{34}^3)^{\phi^0} \sim (\Omega_{45}^3)^{\phi^0} \sim (\Omega_{46}^3)^{\phi^0}; \\
(\Omega_{43}^4)^{\phi^0} \sim (\Omega_{45}^4)^{\phi^0} \sim (\Omega_{46}^4)^{\phi^0}; \\
(\Omega_{52}^5)^{\phi^0} \sim (\Omega_{53}^5)^{\phi^0} \sim (\Omega_{54}^5)^{\phi^0} \sim (\Omega_{56}^5)^{\phi^0}; \\
(\Omega_{24}^6)^{\phi^0} \sim (\Omega_{25}^6)^{\phi^0} \sim (\Omega_{26}^6)^{\phi^0} \sim (\Omega_{35}^6)^{\phi^0} \sim (\Omega_{36}^6)^{\phi^0} \sim (\Omega_{45}^6)^{\phi^0} \sim (\Omega_{46}^6)^{\phi^0}, \\
(\Omega_{23}^6)^{\phi^0} \sim - (\Omega_{25}^6)^{\phi^0} \sim - (\Omega_{26}^6)^{\phi^0} \sim (\Omega_{35}^6)^{\phi^0} \sim - (\Omega_{36}^6)^{\phi^0} \sim - (\Omega_{56}^6)^{\phi^0}. \tag{B10}
\]

C  $\Phi, *\Phi, d\Phi, *d\Phi, *d* \Phi$

In this appendix, we summarize the results for $\Phi, d\Phi, *7d\Phi$ and $*7d*7 \Phi$ that will be useful to obtaining the results of 3.

1.

\[
\Phi = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{246} - e^{245}, \\
d\Phi = (e^{2135} + e^{2146} + e^{3127} + e^{3146} - e^{5127} - e^{5146}) \Omega_2^1 + (e^{2734} + e^{2756} + e^{3712} + e^{3756} - e^{4712} - e^{4756} - e^{5712} - e^{5756}) \Omega_2^7 + (e^{2317} - e^{2345} - e^{2517} - e^{2536} - e^{2617} - e^{2645} + e^{3517} - e^{3617} - e^{3645} + e^{5617}) \Omega_{23}^2 + (e^{2347} + e^{2315} + e^{2547} + e^{2647} + e^{2615} + e^{3547} - e^{3526} + e^{3647} + e^{3615} + e^{5647}) \Omega_{23}^3 + e^{2415} \Omega_{24}^3 + (e^{2367} + e^{2567} + e^{3567} - e^{3524} - e^{3624} - e^{5624} + e^{2357} - e^{3214} + e^{2514} - e^{2657} + e^{2614} - e^{3657} + e^{3614} - e^{5614} - e^{5623}) \Omega_{23}^5 \approx (e^{2347} + e^{2315} + e^{2547} + e^{2647} + e^{2615} + e^{3547} - e^{3526} + e^{3647} + e^{3615} + e^{5647}) \Omega_{23}^2 + e^{2415} \Omega_{24}^2; \tag{C1}
\]
2. 

\[ *\gamma\Phi = e^{3456} + e^{1256} + e^{1234} - e^{2467} + e^{2357} + e^{1457} + e^{1367} \]

\[ d *\gamma\Phi = -e^{12457}(\Omega_1^2 + \Omega_2^7) - e^{12367}(\Omega_2^1 + \Omega_2^7 + \Omega_3^3 + \Omega_5^5) + e^{12356}(\Omega_2^1 + \Omega_2^2 + \Omega_5^5) \]

\[ -e^{13457}((\Omega_1^2 + \Omega_2^7 + \Omega_3^3 + \Omega_5^5) + e^{23457}(\Omega_2^1 + \Omega_2^2 + \Omega_3^3 + \Omega_5^5)) - e^{13567}(\Omega_2^1 + \Omega_2^3) \]

\[ +e^{23467}(\Omega_2^7 - \Omega_2^3 - \Omega_5^5) - e^{23457}(\Omega_2^7 - \Omega_2^3 - \Omega_5^5) + e^{13467}(\Omega_2^5 - \Omega_2^7) - e^{24567}(\Omega_2^2 + \Omega_3^5) + e^{14567}(\Omega_2^5 - \Omega_2^7) \]

\[ +e^{23456}(\Omega_2^3 + \Omega_5^5) - e^{12567}(\Omega_2^3 + \Omega_5^5) + e^{23567}(\Omega_2^3 + \Omega_5^5) - e^{12346}(\Omega_2^3 + \Omega_5^5) + e^{12456}(\Omega_3^4 + \Omega_5^5) \]

\[ + e^{12347} - e^{12357} - e^{23456} - e^{12356} + e^{13567} \Omega_2^5 \]

\[ \approx \left( -e^{12367} + e^{12345} - e^{13657} + e^{23456} - e^{12567} + e^{23567} - e^{12346} + e^{12456} \right) \Omega_2^5; \quad (C2) \]

3. 

\[ *\gamma d\Phi = (e^{3457} - e^{346} - e^{257} + e^{346} - e^{237}) \Omega_2^1 \]

\[ + (-e^{156} + e^{134} + e^{156} + e^{124} - e^{356} - e^{123} + e^{346} + e^{126}) \Omega_2^7 \]

\[ + (-e^{456} - e^{167} + e^{346} - e^{147} - e^{345} + e^{137} + e^{246} + e^{245} - e^{127} - e^{234}) \Omega_2^7 \]

\[ + (-e^{156} - e^{467} - e^{136} - e^{347} + e^{126} - e^{147} - e^{125} + e^{247} + e^{123}) \Omega_2^3 \]

\[ + e^{145} - 2e^{124} + e^{167} - e^{157} + e^{137} - e^{146} - e^{567} - e^{357} - e^{257} - e^{237} - e^{147} \Omega_2^3 \]

\[ \approx (e^{156} - e^{467} - e^{136} + e^{347} + e^{126} - e^{147} - e^{125} + e^{247} + e^{123}) \Omega_2^3 - e^{367}\Omega_2^3; (C3) \]

4. 

\[ *\gamma (d *\gamma\Phi) = (-e^{36} - e^{45} + e^{47} + e^{26} - e^{67} - e^{24}) \Omega_2^1 \]

\[ + (-e^{36} - e^{45} - e^{15} + e^{26} - e^{16} - e^{25}) \Omega_2^7 \]

\[ - (-e^{15} + e^{47} - e^{13} - e^{67} - e^{23} + e^{26}) \Omega_2^7 \]

\[ + (-e^{17} - e^{45} - e^{34} + e^{24} + e^{67} + e^{14} + e^{57} - e^{37}) \Omega_2^3 + e^{35}\Omega_2^3 \]

\[ + (e^{56} - e^{17} - e^{36} + e^{35} + e^{47} + e^{26} - e^{25} + e^{14} + e^{23} - e^{46} + e^{17} + e^{45} - e^{15} + e^{13} - e^{24}) \Omega_2^3 \]

\[ \approx (-e^{17} - e^{45} - e^{34} + e^{24} + e^{67} + e^{14} + e^{57} - e^{37}) \Omega_2^3 + e^{35}\Omega_2^3; \quad (C4) \]

D \quad \Omega_- : i_{R^{(1)}}\Omega_- = 0 \cap \ast\Phi = -\sigma^{(1)} \land \Omega_- + \frac{1}{2} \omega_{\Phi}^2 \]

Relevant to the explicit construction of a transverse SU(3) structure in 6 and in particular, 6.1, arising from the contact structure constructed in 5, in this appendix we discuss the details.
pertaining to solution of:

\[ i_{R(1)} \Omega_- = 0; \]
\[ *\Phi = -\sigma^{(1)} \wedge \Omega_- + \frac{1}{2} \omega_\Phi, \]  \hspace{1cm} (D1)

for \( \Omega_- \) with the understanding that \( \Omega_+ = \Phi - \sigma \wedge \omega_\Phi \) (the holomorphic three-form \( \Omega = \Omega_+ + i\Omega_- \)).

From the ansatz (97), one obtains:

\[ i_{R(1)} \Omega_- = \Lambda^{(1)}_{160c0} (R_{(1)}, e^1 e^{b0c0} - R_{(1)}, e^b e^{c01} + R_{(1)}, e^{c0} e^{b01}) + \Lambda^{(2)}_{13c0} (R_{(1)}, e^1 e^{3c0} - R_{(1)}, e^3 e^{c01} + R_{(1)}, e^{c0} e^{31}) \]
\[ + \Lambda^{(3)}_{17c0} (R_{(1)}, e^1 e^{7c0} - R_{(1)}, e^7 e^{c01} + R_{(1)}, e^{c0} e^{71}) + \Lambda^{(4)}_{137} (R_{(1)}, e^1 e^{37} - R_{(1)}, e^3 e^{71} + R_{(1)}, e^{7} e^{31}) \]
\[ + \Lambda^{(1)}_{3b0c0} (R_{(1)}, e^3 e^{b0c0} - R_{(1)}, e^{b0} e^{c01} + R_{(1)}, e^{c0} e^{b01}), \]  \hspace{1cm} (D2)

\((a_0, b_0, c_0 = 2, 4, 5, 6)\) that yields:

\[ \Lambda^{(1)}_{160c0} R_{(1)}, e^1 + \Lambda^{(3)}_{3b0c0} R_{(1)}, e^3 + \Lambda^{(1)}_{13c0} R_{(1)}, e^7 + 3 \Lambda_{a_0b_0c_0} R_{(1)}, e^{c0} = 0, \]
\[ \Lambda^{(1)}_{160c0} R_{(1)}, e^b + \Lambda^{(2)}_{13c0} R_{(1)}, e^3 + \Lambda^{(3)}_{17c0} R_{(1)}, e^7 = 0, \]
\[ \Lambda^{(2)}_{13c0} R_{(1)}, e^1 = 2 \Lambda^{(1)}_{3b0c0} R_{(1)}, e^b - \Lambda^{(3)}_{17c0} R_{(1)}, e^7 = 0, \]
\[ \Lambda^{(3)}_{17c0} R_{(1)}, e^1 + \Lambda^{(3)}_{37c0} R_{(1)}, e^3 - 2 \Lambda^{(1)}_{b0c0} R_{(1)}, e^b = 0, \]
\[ \Lambda^{(2)}_{13c0} R_{(1)}, e^c + \Lambda^{(1)}_{137} R_{(1)}, e^7 = 0, \]
\[ \Lambda^{(4)}_{137} R_{(1)}, e^3 = 0, \]
\[ \Lambda^{(3)}_{17c0} R_{(1)}, e^c = 0, \]
\[ \Lambda^{(4)}_{137} R_{(1)}, e^1 + \Lambda^{(3)}_{37c0} R_{(1)}, e^c = 0. \]  \hspace{1cm} (D3)

From \((D3)\),

\[ \Lambda^{(2)}_{13c0} = \Lambda^{(1)}_{17c0} = \Lambda^{(3)}_{37c0} = \Lambda^{(3)}_{17c0} = 0, \]
\[ \Lambda^{(4)}_{137} = 0. \]  \hspace{1cm} (D4)
Substituting

\[
\omega_0^2 = 2 \left( \alpha_1 \alpha_3 \Omega_{a} \Omega_{b} \Omega_{c} e^{a+1b+2} + \alpha_1 \alpha_7 \Omega_{a} \Omega_{b} \Omega_{c} e^{a+7b} + \alpha_3 \alpha_7 \Omega_{b} \Omega_{c} \Omega_{a} e^{b+ca+7} \right)
\]

\[+ \alpha_3^2 \Omega_{b} \Omega_{c} e^{b+ca+7} \]

\[= 2 \left( \alpha_1 \alpha_3 \Omega_{a} \Omega_{b} \Omega_{c} e^{a+1b+2} + 2 \alpha_1 \alpha_3 \Omega_{a} \Omega_{b} \Omega_{c} e^{a+3c+0} + \Omega_{b} \Omega_{c} \Omega_{a} e^{3+1b+7} \right)
\]

\[+ \Omega_{a} \Omega_{b} \Omega_{c} e^{a+13b} + 2 \alpha_3 \alpha_7 \Omega_{a} \Omega_{b} \Omega_{c} e^{3+3c+0} + \alpha_3 \alpha_7 \Omega_{a} \Omega_{b} \Omega_{c} e^{3+3c+0} \]

\[+ 2 \left( \alpha_1 \alpha_3 \Omega_{a} \Omega_{b} \Omega_{c} e^{a+1b+2} + \alpha_1 \alpha_7 \Omega_{a} \Omega_{b} \Omega_{c} e^{a+3c+0} + \alpha_3 \alpha_7 \Omega_{a} \Omega_{b} \Omega_{c} e^{3+1b+7} \right)
\]

\[+ \alpha_3^2 \Omega_{b} \Omega_{c} \Omega_{a} e^{b+ca+7} \]

\[\approx 0, \quad \text{(D5)} \]

\[\Omega_{a} \Omega_{b} \Omega_{c} e^{b+ca+7} \]

\[
\Omega_{b} \Omega_{c} \Omega_{a} e^{b+ca+7} = \left( \Omega_{24} \Omega_{56} - \Omega_{25} \Omega_{46} + \Omega_{26} \Omega_{45} \right) e^{2+456} \approx 0, \quad \text{(D6)} \]

and

\[
\sigma^1 \wedge \omega_0 = \alpha_1 \alpha_3 \left( \Omega_{a} e^{b+2} + \Omega_{a} e^{3+1} \right) + \alpha_1 \alpha_7 \left( \Omega_{a} e^{1+a+7} + \Omega_{a} e^{3+7} \right) + \alpha_3 \alpha_7 \left( \Omega_{a} e^{3+7} + \Omega_{a} e^{7+3} \right), \quad \text{(D7)}
\]
into (95) one can obtain $\Omega_+$, and the following set of conditions on $\Lambda^{(i)}_{ABC}$’s:

\begin{align*}
(i) \quad & -\alpha_1 \Lambda^{(1)}_{000} + \alpha_1 \alpha_3 \Omega^1_{\theta \Omega ^3_{[b_0]c_0]} b_0, c_0 \neq 2, 4; \\
& -\alpha_1 \Lambda^{(1)}_{024} + \alpha_1 \alpha_3 \Omega^1_{[3] \Omega ^3_{24]} = 1, \\
(ii) \quad & - \left( \alpha_1 \Lambda^{(3)}_{37} - \alpha_3 \Lambda^{(3)}_{17} + \alpha_7 \Lambda^{(3)}_{136} \right) + \frac{1}{2} \Omega^{(i)}_{\theta \Omega ^7_{[b_0]c_0]} c_0 \neq 6; \\
& - \left( \alpha_1 \Lambda^{(3)}_{376} - \alpha_3 \Lambda^{(3)}_{176} + \alpha_7 \Lambda^{(3)}_{136} \right) + \frac{1}{2} \Omega^{(i)}_{[3] \Omega ^7_{24]} = 1, \\
(iii) \quad & - \left( \alpha_3 \Lambda^{(1)}_{7b_0c_0} - \alpha_7 \Lambda^{(1)}_{3b_0c_0} \right) + \frac{1}{2} \left( 2\alpha_3 \alpha_7 \Omega^1_{\theta \Omega ^3_{[b_0]c_0]} c_0 \right) = 0, b_0, c_0 \neq 2, 5; \\
& - \left( \alpha_3 \Lambda^{(1)}_{725} - \alpha_7 \Lambda^{(1)}_{325} \right) + \frac{1}{2} \left( 2\alpha_3 \alpha_7 \Omega^1_{[3] \Omega ^3_{24]} \right) = -1, \\
(iv) \quad & -\Lambda_{ab_0c_0} + \alpha_3 \Omega^1_{\theta \Omega ^3_{[b_0]c_0]} b_0 = 0, a_0, b_0, c_0 \neq 4, 5, 6; \\
& -\Lambda_{456} + \alpha_3 \Omega^1_{[3] \Omega ^3_{24]} = 1, \\
(v) \quad & \Lambda_{ab_0c_0} + \alpha_1 \alpha_3 \Omega^1_{\theta \Omega ^3_{[b_0]c_0]} b_0 = 0, a_0, b_0, c_0 \neq 2, 5, 6; \\
& \Lambda_{256} + \alpha_1 \alpha_3 \Omega^1_{[3] \Omega ^3_{24]} = 0, \\
(vi) \quad & -\alpha_1 \Lambda^{(1)}_{7b_0c_0} + \frac{1}{2} \alpha_1 \alpha_7 \Omega^1_{[b_0] \Omega ^3_{c_0]} b_0 \right) = 0, b_0, c_0 \neq 4, 5; \\
& -\alpha_1 \Lambda^{(1)}_{745} + \frac{1}{2} \alpha_1 \alpha_7 \Omega^1_{[5] \Omega ^3_{4]} = 1, \\
(vii) \quad & \Lambda_{ab_0c_0} + \frac{1}{2} \alpha_3 \alpha_7 \Omega^3_{\theta \Omega ^3_{[b_0]c_0]} = 0, a_0, b_0, c_0 \neq 2, 4, 6; \\
& \Lambda_{246} + \frac{1}{2} \alpha_3 \alpha_7 \Omega^3_{[46] \Omega ^3_{7}} = 1. 
\end{align*}

(D8)

In (D8), assuming $\alpha_{\theta_1} \sim \mathcal{O}(1)$,
(D8) (i) implies:

1. \( \Omega_3^1 \beta^0 \Omega_2^5 \beta^0 - \Omega_6^1 \beta^0 \Omega_3^3 \beta^0 \sim \frac{10^2 N^{4/5} \log N}{\alpha_1^4} \)

\( \downarrow N = 10^2 \)

\( \frac{O(1)}{\alpha_1} \ll 1, \)

\( \alpha_3 A_{124}^{(1)} - \alpha_1 A_{324}^{(1)} \sim \frac{(\log N)^2 (-O(1)\alpha_2^2 e^{-\alpha_0 N^{7/40}} N^{5/8} g_s^{7/2} M^2 N_f^2 \log N + O(10^2)\alpha_2^2)}{\alpha_0 N^{3/40} \alpha_1^2 \alpha_2} \)

\( \downarrow N = 10^2, M = N_f = 3, g_s = 0.1 \)

\( O(10^2)\alpha_2^2, \alpha_0 = \sqrt{\alpha_2}, \)

2. \( \Omega_3^1 \beta^0 \Omega_2^5 \beta^0 - \Omega_6^1 \beta^0 \Omega_3^3 \beta^0 \sim - \frac{O(1) \delta N^{4/5} (\log N)^2}{\langle r \rangle \log \langle r \rangle} \alpha_1^4 - \frac{g_s^{7/2} (\log N)^2 \sqrt{N} M^2 N_f \langle r \rangle \log \langle r \rangle}{O(1) \alpha_0^2 \alpha_2^2} \)

\( \downarrow \delta \sim \langle r \rangle \sim N^{-N^{7/40}} \alpha_0 \log \langle r \rangle; N = 100, M = N_f = 3, g_s = 0.1, \alpha_0 = O(1) \)

\( \frac{O(10^2)}{\alpha_0} \ll 1, \alpha_1 \sim 8; \)

3. \( \Omega_3^1 \beta^0 \Omega_{15}^5 \beta^0 - \Omega_6^1 \beta^0 \Omega_{34}^3 \beta^0 \sim \frac{O(10^{-3}) g_s^{7/2} N^{1/5} (\log N)^3 M^2 N_f^2 \log \langle r \rangle}{\alpha_2^2} \)

\( \downarrow \delta \sim \langle r \rangle \sim N^{-N^{7/40}} \alpha_0 \log \langle r \rangle; N = 100, M = N_f = 3, g_s = 0.1, \alpha_0 = O(1) \)

\( - \frac{O(1)}{\alpha_1 \alpha_2^2} \ll 1; \)

4. \( \Omega_3^1 \beta^0 \Omega_{45}^5 \beta^0 - \Omega_6^1 \beta^0 \Omega_{34}^3 \beta^0 \sim - \frac{10^{-2} g_s^7 (\log N)^4 \langle \log \langle r \rangle \rangle^3}{N^{1/10} \alpha_0^2 \alpha_2^2} - \frac{O(1) g_s^{7/2} M^2 N_f^2 N_f^{1/5} (\log N)^3 \log \langle r \rangle}{\alpha_0^2 \alpha_2^2} \)

\( \downarrow \log \langle r \rangle \sim -N^{7/40} \alpha_0; N = 100, M = N_f = 3, g_s = 0.1, \alpha_0 = O(1) \)

\( \frac{O(1)}{\alpha_0^2} \ll 1; \)

5. \( \Omega_3^1 \beta^0 \Omega_{56}^5 \beta^0 - \Omega_6^1 \beta^0 \Omega_{35}^3 \beta^0 \)

\( \sim \frac{O(1) g_s^{7/2} M^2 N_f^2 \left( \frac{1}{\langle \log N \rangle} (\log N)^2 \sqrt{N} (\langle r \rangle)^2 (\log \langle r \rangle)^3 + (\log N)^3 N_f^{1/5} \log \langle r \rangle \right) \alpha_0^2 \alpha_2^2}{\alpha_0^2 \alpha_2^4} \)

\( \downarrow \langle r \rangle \sim N^{-N^{7/40}} \alpha_0 \log \langle r \rangle; N = 100, M = N_f = 3, g_s = 0.1, \alpha_0 = O(1) \)

\( \frac{O(10^2)}{\alpha_0^2} \ll 1. \)
\[ \Omega_{3}^1, \beta^0 \Omega_{24}^3, \beta^0 - \Omega_{4}^1, \beta^0 \Omega_{32}^7, \beta^0 \sim \frac{10^2 N^{4/5} \log N}{\alpha_{\theta_1}} \]
\[ \sim \frac{2g_s^{7/2} (\log N)^3 N^{1/5} N_f (\log \langle r \rangle)^3 + 1784 (\log N)^2 N^{1/10} \alpha_{\theta_2}^2}{\log \langle r \rangle \alpha_{\theta_1}^2 \alpha_{\theta_2}} \]
\[ \downarrow \langle r \rangle \sim \frac{N^{-N^{7/40}}}{\log N}, \log \langle r \rangle; \ N = 100, M = N_f = 3, g_s = 0.1, \alpha_{\theta} = \mathcal{O}(1) \]
\[ \downarrow 6868 \alpha_{\theta}^2 \]

\[ \text{(D9)} \]

- \text{(D8) (ii) implies:} 

\[ \begin{align*}
(1) \quad & \Omega_{[3]}^1, \beta^0 \Omega_{[2]}^7, \beta^0 \sim \frac{\mathcal{O}(1) N^{7/10} \alpha_{\theta}^4}{g_s^{7/2} M^2 N_f^2 (\log N - 3 \log \langle r \rangle)^2 (\log \langle r \rangle)^2} \\
& \downarrow \log \langle r \rangle - N^{7/40} \alpha_{\theta}; \ N = 100, M = N_f = 3, g_s = 0.1, \alpha_{\theta} = \mathcal{O}(1) \\
& \mathcal{O}(10^{-3}); \\
(2) \quad & \Omega_{[3]}^1, \beta^0 \Omega_{[4]}^7, \beta^0 \sim \frac{N^{1/10} \alpha_{\theta}^2 (\mathcal{O}(10)(\log N)^2(\log \langle r \rangle)^2 \alpha_{\theta_1}^2 + \frac{\mathcal{O}(10^3) N^{3/5} \alpha_{\theta_1}^2 \alpha_{\theta_2}^2}{g_s^{7/2} M^2 N_f^2})}{(\log \langle r \rangle)^4 \alpha_{\theta_1}} \\
& \downarrow \log \langle r \rangle \sim -\kappa_{\log r} N^{7/40} \alpha_{\theta}; \ N = 100, M = N_f = 3, g_s = 0.1, \alpha_{\theta} = \mathcal{O}(1) \\
& \frac{\mathcal{O}(10)}{\kappa_{\log r}} 1, \kappa_{\log r} \sim \mathcal{O}(1); \\
(3) \quad & \Omega_{[3]}^1, \beta^0 \Omega_{[5]}^7, \beta^0 \sim \frac{\alpha_{\theta_1}^2 (\mathcal{O}(10)^3) N^{7/10}}{g_s^{7/2} M^2 N_f^2 \log \langle r \rangle^4} \\
& \downarrow \log \langle r \rangle \sim -\kappa_{\log r} N^{7/40} \alpha_{\theta}; \ N = 100, M = N_f = 3, g_s = 0.1, \alpha_{\theta}, \kappa_{\log r} \sim \mathcal{O}(1) \\
& \frac{\mathcal{O}(10)}{\kappa_{\log r}} \ll 1; \\
(4) \quad & \Omega_{[3]}^1, \beta^0 \Omega_{[6]}^7, \beta^0 \sim \frac{N^{2/5} \log N}{\mathcal{O}(1)(\log \langle r \rangle)^2 \alpha_{\theta_1}^2} \xrightarrow{\mathcal{O}(10^{-2})} \frac{\mathcal{O}(10^{-2})}{\alpha_{\theta_1}^2} \ll 1, \end{align*} \]

Using \( \Lambda_{17a} = 0 \), 
\( \alpha_1 \Lambda_{376}^{(3)} + \alpha_7 \Lambda_{136}^{(3)} = -1. \) 

\[ \text{(D10)} \]
(D8) (iii) implies:

\[ \begin{array}{l}
(1) \quad 2\left(\Omega_{32}^3 \beta^0 \Omega_4^7 \beta^0 - \Omega_{34}^3 \beta^0 \Omega_2^7 \beta^0 + \Omega_{24}^3 \beta^0 \Omega_3^7 \beta^0\right) \\
\quad \sim N^{1/10} \log N \left(10^3 N^{7/10} \langle r \rangle + \alpha_2 \frac{\left(O(1)g_7^{7/2} \log(r)M^2 N^{1/10} N^2 (\log N)^2 + O(1) \log N \alpha_1^3 g_3^2\right)}{\log(r)}\right)
\end{array} \]

\[ \downarrow \log(r) \sim -N^{7/4} \alpha_\theta; \quad \alpha_\theta = O(1); \text{ retaining the most dominant term in } N \]

\[ \sim O(1) \alpha_\theta \alpha_1^4 \]

\[ \downarrow N = 100, M = N_f = 3, g_s = 0.1 \]

\[ \frac{\alpha_1^4}{\alpha_1^4 \alpha_\theta} \ll 1; \]

\[ \begin{array}{l}
(2) \quad 2\left(\Omega_{32}^3 \beta^0 \Omega_5^7 \beta^0 - \Omega_{35}^3 \beta^0 \Omega_2^7 \beta^0 + \Omega_{25}^3 \beta^0 \Omega_3^7 \beta^0\right) \\
\quad \sim O(10^2) e^{-\alpha_1 N^{7/40} N^{4/5} \log N}
\end{array} \]

\[ \downarrow \log(r) \sim -N^{7/40} \alpha_\theta; \quad N = 100, M = N_f = 3, g_s = 0.1, \alpha_\theta = O(1) \]

\[ \sim O(10^2) \]

\[ \frac{\alpha_1^4}{\alpha_1^4 \alpha_\theta} \ll 1; \]

\[ \begin{array}{l}
(3) \quad 2\left(\Omega_{32}^3 \beta^0 \Omega_6^7 \beta^0 - \Omega_{36}^3 \beta^0 \Omega_2^7 \beta^0 + \Omega_{26}^3 \beta^0 \Omega_3^7 \beta^0\right) \\
\quad \sim N^{4/5} \log N \left(O(10^2) \langle r \rangle^2\right) - \frac{O(1) g_s^7 \langle \log(r) \rangle^2 M^2 \sqrt{N N^2 \langle r \rangle (\log N)^2}}{\langle r \rangle \alpha_1^4 \alpha_2^4}
\end{array} \]

\[ \downarrow \langle r \rangle \sim \frac{N^{N^{7/40} \alpha_\theta}}{\log(N)}, \quad \log(r); \quad N = 100, M = N_f = 3, g_s = 0.1, \alpha_\theta = O(1) \]

\[ \frac{\alpha_1^4}{\alpha_1^4 \alpha_2^4} < 1; \]

\[ \begin{array}{l}
(4) \quad 2\left(\Omega_{34}^3 \beta^0 \Omega_5^7 \beta^0 - \Omega_{35}^3 \beta^0 \Omega_4^7 \beta^0 + \Omega_{35}^3 \beta^0 \Omega_3^7 \beta^0\right) \\
\quad \log(r) N^{1/5} \log N \left(-O(10^4) N^{3/5} \langle r \rangle^2 + \frac{g_7^{7/2} M^2 N^2 \left(O(10) - O(10^{-2}) \log N\right)(\log N)^2 \alpha_1^2}{\alpha_2^2}\right)
\end{array} \]

\[ \downarrow \langle r \rangle \sim \frac{N^{N^{N^{7/40} \alpha_\theta}}}{\log(N)}; \quad N = 100, M = N_f = 3, g_s = 0.1, \alpha_\theta = O(1) \]

\[ \frac{\alpha_1^4}{\alpha_1^4 \alpha_2^4} < 1; \]
\[
(5) \quad 2 \left( \Omega_{34}^3 \beta^0 \Omega_{46}^7 \beta^0 - \Omega_{36}^3 \beta^0 \Omega_{46}^7 \beta^0 + \Omega_{46}^3 \beta^0 \Omega_{36}^7 \beta^0 \right) - \frac{\mathcal{O}(10^7) N^{14/5}(r) \log N}{\alpha_{g_1}^3} - \frac{\mathcal{O}(10^{-2}) g_1^7 \log(r) M^4 N_f^2 \log N^4}{\alpha_{g_1}^3 \alpha_{g_1}} + \frac{\mathcal{O}(10) g_1^{7/2} \log(r) M^2 N_f^2 \log N^3}{\alpha_{g_1}^2 \alpha_{g_1}}.
\]

\[
\downarrow \langle r \rangle \sim N^{-N^{7/40}/\alpha_{g_1}}; \quad N = 100, M = N_f = 3, g_s = 0.1, \alpha_{g_1} = \mathcal{O}(1)
\]

\[
- \frac{10^3}{\alpha_{g_1}^3} \alpha_{g_1} \ll 1;
\]

\[
(6) \quad 2 \left( \Omega_{32}^3 \beta^0 \Omega_{6}^7 \beta^0 - \Omega_{36}^3 \beta^0 \Omega_{2}^7 \beta^0 + \Omega_{26}^3 \beta^0 \Omega_{3}^7 \beta^0 \right)
\]

\[
\sim \frac{N^{4/5} \log N \left( 353.2 \langle r \rangle^2 + 29.7 e^{-\alpha_{g_1}^3 \log N} \right)}{\alpha_{g_1}} - \frac{g_1^{7/2} \log(r) M^2 \sqrt{N N_f^2 \langle r \rangle \log N^2}}{\alpha_{g_1}^3 \alpha_{g_1}^3}
\]

\[
\downarrow \langle r \rangle \sim N^{-N^{7/40}/\alpha_{g_1}}; \quad N = 100, M = N_f = 3, g_s = 0.1, \alpha_{g_1} = \mathcal{O}(1)
\]

\[
\sim - \frac{10^3}{\alpha_{g_1}^3} \alpha_{g_1} \approx -0.2 \approx 0.
\]

\[\text{(D11)}\]

\[\bullet \quad \text{(D8) (iv) implies:}\]

\[
(1) \quad - \left( \Omega_{24}^3 \beta^0 \Omega_{35}^3 \beta^0 + \Omega_{45}^3 \beta^0 \Omega_{32}^3 \beta^0 - \Omega_{25}^3 \beta^0 \Omega_{34}^3 \beta^0 \right)
\]

\[
\sim \frac{g_1^{7/2} M^2 N_f^2 \sqrt{N^{1/5} \langle r \rangle \log N \langle r \rangle}}{\alpha_{g_1}^3 \alpha_{g_1}^3} - \frac{g_1^{7/2} M^2 N_f^2 \sqrt{N^{1/10} \langle r \rangle \log N \langle r \rangle}}{\alpha_{g_1}^3 \alpha_{g_1}^3} + \frac{\mathcal{O}(10^{-3}) \log N \log \langle r \rangle + \mathcal{O}(10^{-3}) \log \langle r \rangle}{\alpha_{g_1}^3 \alpha_{g_1}^3};
\]

\[
(2) \quad - \left( \Omega_{24}^3 \beta^0 \Omega_{35}^3 \beta^0 + \Omega_{45}^3 \beta^0 \Omega_{32}^3 \beta^0 - \Omega_{25}^3 \beta^0 \Omega_{34}^3 \beta^0 \right)
\]

\[
\sim \frac{g_1^{7/2} M^2 N_f^2 \sqrt{N^{1/5} \langle r \rangle \log N \langle r \rangle}}{\alpha_{g_1}^3 \alpha_{g_1}^3} - \frac{g_1^{7/2} M^2 N_f^2 \sqrt{N^{1/10} \langle r \rangle \log N \langle r \rangle}}{\alpha_{g_1}^3 \alpha_{g_1}^3} + \frac{\mathcal{O}(10^{-3}) \log N \log \langle r \rangle + \mathcal{O}(10^{-3}) \log \langle r \rangle}{\alpha_{g_1}^3 \alpha_{g_1}^3};
\]

\[
(3) \quad - \left( \Omega_{25}^3 \beta^0 \Omega_{36}^3 \beta^0 + \Omega_{46}^3 \beta^0 \Omega_{32}^3 \beta^0 - \Omega_{26}^3 \beta^0 \Omega_{34}^3 \beta^0 \right)
\]

\[
\sim \frac{g_1^{7/2} M^2 N_f^2 \sqrt{N^{1/10} \langle r \rangle \log N \langle r \rangle}}{\alpha_{g_1}^3 \alpha_{g_1}^3} - \frac{g_1^{7/2} M^2 N_f^2 \sqrt{N^{1/10} \langle r \rangle \log N \langle r \rangle}}{\alpha_{g_1}^3 \alpha_{g_1}^3} + \frac{\mathcal{O}(10^{-3}) \log N \log \langle r \rangle + \mathcal{O}(10^{-3}) \log \langle r \rangle}{\alpha_{g_1}^3 \alpha_{g_1}^3};
\]

\[
(4) \quad - \left( \Omega_{45}^3 \beta^0 \Omega_{36}^3 \beta^0 + \Omega_{36}^3 \beta^0 \Omega_{46}^3 \beta^0 - \Omega_{46}^3 \beta^0 \Omega_{36}^3 \beta^0 \right)
\]

\[
\sim \frac{g_1^{7/2} M^4 N_f^2 \langle r \rangle \log N^4 \log \langle r \rangle^5}{\alpha_{g_1}^2 \alpha_{g_1}^3} - \frac{g_1^{7/2} M^4 N_f^2 \langle r \rangle \log N^4 \log \langle r \rangle^5}{\alpha_{g_1}^2 \alpha_{g_1}^3} + \frac{\mathcal{O}(10^{-2}) \log N \log \langle r \rangle + \mathcal{O}(10^{-2}) \log \langle r \rangle}{\alpha_{g_1}^2 \alpha_{g_1}^3}.
\]

\[\text{(D12)}\]
Now, for $N = 10^2, M = N_f = 3, g_s = 0.1, \alpha_\theta = 3.9, \alpha_{\theta_2} = \alpha_\theta^2 \alpha_{\theta_1}^2$,

$$
\begin{align*}
(a) \quad \Lambda_{245} &= \frac{0.48}{\alpha_{\theta_1}^2} , \\
(b) \quad \Lambda_{246} &= \frac{9.06}{\alpha_\theta^2 \alpha_{\theta_1}^2} , \\
(c) \quad \Lambda_{456} &= -1 + \frac{0.26}{\alpha_\theta^2 \alpha_{\theta_1}^2} , \\
(d) \quad \Lambda_{256} &= \frac{18.15}{\alpha_\theta^6 \alpha_{\theta_1}^8} . \tag{D13}
\end{align*}
$$

- (D8) (v) implies:

$$
\begin{align*}
(1) & \Omega_4^{\beta} \cdot \Omega_4^{\beta} - \Omega_4^{\beta} \cdot \Omega_4^{\beta} + \Omega_4^{\beta} \cdot \Omega_4^{\beta} \\
& \sim \frac{g_s^{7/2} (\log N)^3 \log(r) M^2 N_f^2 / N^{1/5} (\mathcal{O}(10^{-3}) \log N - 1.1 \log(r) \log(r)) \log(r) + \mathcal{O}(10^2) (\log N)^2 N^{3/10} \alpha_\theta^2 \alpha_{\theta_1}^2)}{\log(r) \alpha_{\theta_1}^4 \alpha_\theta^2} , \\
(2) & \Omega_4^{\beta} \cdot \Omega_4^{\beta} - \Omega_4^{\beta} \cdot \Omega_4^{\beta} + \Omega_4^{\beta} \cdot \Omega_4^{\beta} \\
& \sim \frac{g_s^{7/2} (\log N)^3 M^2 N_f^2 / N^{1/5} (\mathcal{O}(10^{-2}) \log N \log(r) - \mathcal{O}(1)(r) \log(r)) - \mathcal{O}(1)(r) \log(r) / \alpha_{\theta_1}^2)}{\log(r) \alpha_{\theta_1}^4 \alpha_\theta^2} , \\
(3) & \Omega_4^{\beta} \cdot \Omega_4^{\beta} - \Omega_4^{\beta} \cdot \Omega_4^{\beta} + \Omega_4^{\beta} \cdot \Omega_4^{\beta} \\
& \sim \frac{g_s^{7/2} (\log N)^3 (\log(r)^2 M^2 N_f^2 / N^{1/5} (\mathcal{O}(10^{-5}) \log N \log(r) - \mathcal{O}(10^{-2}) \log(r) \alpha_\theta^2 \alpha_{\theta_1}^2))}{\log(r) \alpha_{\theta_1}^4 \alpha_\theta^2} , \\
(4) & \Omega_4^{\beta} \cdot \Omega_4^{\beta} - \Omega_4^{\beta} \cdot \Omega_4^{\beta} + \Omega_4^{\beta} \cdot \Omega_4^{\beta} \\
& \sim \frac{N^{1/5} (\log N)^2 / \log(r) \log(r) + \mathcal{O}(10)^{3/10} \log N \log(r) - \mathcal{O}(10^{-5}) \log N \log(r)}{\log(r) \alpha_{\theta_1}^4 \alpha_\theta^2} . \tag{D14}
\end{align*}
$$

Hence, for $N = 10^2, M = N_f = 3, g_s = 0.1, \alpha_\theta = 3.9, \alpha_{\theta_2} = \alpha_\theta^2 \alpha_{\theta_1}^2$,

$$
\begin{align*}
|\Lambda_{245}| &= \frac{1}{\mathcal{O}(1) \alpha_\theta^4 \alpha_{\theta_1}^4} \ll 1 ; \\
|\Lambda_{246}| &= \frac{\mathcal{O}(1)}{\alpha_\theta^4 \alpha_{\theta_1}^4} \ll 1 ; \\
|\Lambda_{456}| &= \frac{\mathcal{O}(10^2)}{\alpha_\theta^4 \alpha_{\theta_1}^4} \ll 1 ; \\
|\Lambda_{256}| &= \left| 1 - \frac{4350}{\alpha_\theta \alpha_{\theta_1}^4} \right| \sim 0 \text{ for appropriate } \alpha_\theta, \alpha_{\theta_1} . \tag{D15}
\end{align*}
$$

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• (D8) (vi) implies:

\[
\Omega_{[2]} \beta_{0} \Omega_{[4]} \beta_{0} \sim \frac{N^{1/10} \alpha_{0}^{2} \left( O(10) (\log N)^{2} \log(r) \alpha_{0}^{2} + \frac{O(10^{3}) \alpha_{0}^{2} \alpha_{1}^{2}}{g_{s}^{7/2} M^{2} N_{f}^{2}} \right)}{(\log(r))^{4} \alpha_{0}^{2} \alpha_{1}^{2}} \]

\[\downarrow \langle r \rangle \sim \frac{N^{-N^{7/40} \alpha_{0}}}{\log N}; \quad N = 100, M = N_{f} = 3, g_{s} = 0.1, \alpha_{0} = O(1) \]

\[\frac{O(10)}{g_{s}^{7/2} K_{\log r_{IR}}^{4}} \ll 1;\]

(2) \[\Omega_{[2]} \beta_{0} \Omega_{[5]} \beta_{0} \]

\[\downarrow \log(r) \sim -N^{7/40} \alpha_{0} K_{\log r_{IR}}; \quad N = 100, M = N_{f} = 3, g_{s} = 0.1, \alpha_{0} = O(1) \]

\[\frac{O(10^{3}) \alpha_{0}^{2}}{g_{s}^{7/2} K_{\log r_{IR}}^{4}};\]

(3) \[\Omega_{[2]} \beta_{0} \Omega_{[6]} \beta_{0} \sim \frac{N^{2/5} \log N}{O(1) (\log(r))^{2} \alpha_{0}^{2}} \]

\[\downarrow \log(r) \sim -N^{7/40} \alpha_{0} K_{\log r_{IR}}; \quad N = 100, M = N_{f} = 3, g_{s} = 0.1, \alpha_{0} = O(1) \]

\[\frac{O(10^{3}) \alpha_{0}^{2} \alpha_{1}^{2}}{g_{s}^{7/2} M^{2} N_{f}^{2}} \]

(4) \[\Omega_{[4]} \beta_{0} \Omega_{[5]} \beta_{0} \sim \frac{N^{1/10} \alpha_{0}^{2} \left( O(10) (\log N)^{2} \log(r) \alpha_{0}^{2} + \frac{O(10^{3}) \alpha_{0}^{2} \alpha_{1}^{2}}{g_{s}^{7/2} M^{2} N_{f}^{2}} \right)}{(\log(r))^{4} \alpha_{0}^{2} \alpha_{1}^{2}} \]

\[\downarrow \log(r) \sim -N^{7/40} \alpha_{0} K_{\log r_{IR}}; \quad N = 100, M = N_{f} = 3, g_{s} = 0.1, \alpha_{0} = O(1) \]

\[\frac{O(10^{5})}{K_{\log r_{IR}}^{2}} > 1;\]

(5) \[\Omega_{[4]} \beta_{0} \Omega_{[6]} \beta_{0} \sim -\frac{O(1) N^{2/5} \log N}{(\log(r))^{2} \alpha_{0}^{2}} - \frac{O(10^{-2}) g_{s}^{7/2} (\log N)^{3} M^{2} N_{f}^{2}}{N^{1/5} \alpha_{0}^{2} \alpha_{1}^{2}} \]

\[\downarrow \log(r) \sim -N^{7/40} \alpha_{0} K_{\log r_{IR}}; \quad N = 100, M = N_{f} = 3, g_{s} = 0.1, \alpha_{0} = O(1) \]

\[-\frac{O(10^{-2})}{\alpha_{0}^{2} \alpha_{1}^{2}};\]

(6) \[\Omega_{[5]} \beta_{0} \Omega_{[6]} \beta_{0} \sim -\frac{O(10^{-2}) N^{2/5} \log N}{(\log(r))^{2} \alpha_{0}^{2}} \]

\[\downarrow \log(r) \sim -N^{7/40} \alpha_{0} K_{\log r_{IR}}; \quad N = 100, M = N_{f} = 3, g_{s} = 0.1, \alpha_{0} = O(1) \]

\[-\frac{1}{O(1) \alpha_{0}^{2} \alpha_{1}^{2}} \ll 1. \quad \text{(D16)}\]

Hence,

\[\alpha_{0} \Lambda_{45}^{1, \beta_{0}} - \alpha_{1} \Lambda_{45}^{1, \beta_{0}} + \frac{10^{2}}{K_{\log r_{IR}}^{2}} = 0. \quad \text{(D17)}\]
(D8) (vii) implies:

\[
\frac{2}{N^{1/10} \log N} \left( -O(10^2) N^{7/10} \langle r \rangle + \alpha^2_{\theta_1} \left( \frac{g_{s}^{7/2} M^{2} N^{11/10} N_{f}^{2} (\log N)^{3} \log(r)}{O(10^{9}) \alpha_{\theta} \alpha_{\theta_1}} - \frac{O(10^{9}) \log N \alpha_{\theta} \alpha_{\theta_1}}{\log(r)} \right) \right)
\]

\[
\downarrow \log(r) \sim -N^{7/40} \alpha_{\theta} \alpha_{\theta_1} \alpha_{\theta_2} = \alpha_{\theta_1}^2 \alpha_{\theta_1}; \text{ Large } - N \text{ limit}
\]

\[
\frac{O(10^{-4}) N^{3/8} (\log N)^{4}}{\alpha_{\theta} \alpha_{\theta_1}^{4}} \overset{N \to 10^{2}}{\longrightarrow} \frac{1}{O(1)} \alpha_{\theta} \alpha_{\theta_1}^{4} \ll 1;
\]

\[
\frac{2}{(\log N)^{2}} \left( \frac{O(10^{-4}) N^{3/8} (\log N)^{4}}{\alpha_{\theta} \alpha_{\theta_1}^{4}} + \frac{O(1) g_{s}^{7/2} M^{2} N_{f}^{2} (\log N)^{3} \log(r)}{\alpha_{\theta} \alpha_{\theta_1}^{4}} + \frac{O(1) g_{s}^{7/2} M^{2} N_{f}^{2} (\log N)^{3} \log(r)}{\alpha_{\theta} \alpha_{\theta_1}^{4}} \right)
\]

\[
\downarrow \log(r) \sim -N^{7/40} \alpha_{\theta} \alpha_{\theta_1} \alpha_{\theta_2} \langle r \rangle; \ N = 10^{0}, M = N_{f} = 3, g_{s} = 0.1, \alpha_{\theta} = O(1)
\]

\[
\Rightarrow | \Lambda_{246} | \sim 1 - \alpha_{1} \alpha_{7} \frac{4500}{\alpha_{\theta} \alpha_{\theta_1}^{6}} \sim 0 \text{ for } \alpha_{1,7} \sim \frac{1}{O(1)} \alpha_{\theta}, \alpha_{\theta_1} \sim O(1);
\]

\[
\frac{2}{N^{1/10} \alpha_{\theta} \alpha_{\theta_1}^{6}} \ll 1;
\]

\[
\downarrow \log(r) \sim -N^{7/40} \alpha_{\theta} \alpha_{\theta_1} \alpha_{\theta_2} \langle r \rangle; \ N = 10^{0}, M = N_{f} = 3, g_{s} = 0.1, \alpha_{\theta} = O(1)
\]

\[
\frac{10^{-5} g_{s}^{7} M^{4} N_{f}^{4} (\log N)^{5} \log(r)^{3}}{N^{1/10} \alpha_{\theta} \alpha_{\theta_1}^{6}} + \frac{g_{s}^{7/2} M^{2} N^{11/5} N_{f}^{2} (\log(r)^{3} \log(r)}}{\alpha_{\theta} \alpha_{\theta_1}^{4}} \ll 1;
\]

\[
\downarrow \log(r) \sim -N^{7/40} \alpha_{\theta} \alpha_{\theta_1} \alpha_{\theta_2} \langle r \rangle; \ N = 10^{0}, M = N_{f} = 3, g_{s} = 0.1, \alpha_{\theta} \sim 3.9, \alpha_{\theta_1} \approx 8 \frac{10^{3}}{\alpha_{\theta_1}^{4}} \ll 1.
\]

(D18)
From (D12), (D14) and (D18), one notes that for $N = 100, M = N_f = 3, g_s = 0.1$,

$$|\Lambda_{245}| \sim \alpha_2 \frac{0.4}{\alpha_2 \alpha_4} \alpha_3 \alpha_7 \frac{0.4}{\alpha_7 \alpha_4} \sim \alpha_3 \frac{0.5}{\alpha_4};$$

$$|\Lambda_{246}| \sim \alpha_2 \frac{5.3}{\alpha_2 \alpha_4} \sim 1 - \alpha_3 \frac{4350}{\alpha_3 \alpha_4} \sim \alpha_3 \frac{2}{\alpha_4} \frac{2}{\alpha_4} \frac{2}{\alpha_4};$$

$$|\Lambda_{256}| \sim \alpha_2 \frac{4350}{\alpha_2 \alpha_4} \sim \alpha_3 \frac{4000}{\alpha_3 \alpha_4} \sim \alpha_3 \frac{18}{\alpha_4} \frac{18}{\alpha_4} \frac{18}{\alpha_4};$$

$$|\Lambda_{456}| \sim \alpha_2 \frac{146}{\alpha_2 \alpha_4} \sim \alpha_3 \frac{12}{\alpha_3 \alpha_4} \sim 1 - \alpha_3 \frac{0.3}{\alpha_4} \alpha_2 \frac{2}{\alpha_4} \sim 1. \quad (D19)$$

Hence, only $\Lambda_{456}$ will be taken to be non-vanishing.

(i) \quad \alpha_3 \Lambda_{124}^{(1)} - \alpha_1 \Lambda_{324}^{(1)} = 6868. \quad (D20)

(ii) \quad \frac{0.05}{\alpha_2^2} = 2; \Lambda_{27a_0}^{(3)} = 0 \text{ for } a_0 = 2, 4, 5, 6. \quad (D21)

(iii) \quad \alpha_7 \Lambda_{325}^{(1)} - \alpha_3 \Lambda_{725}^{(1)} = \frac{102681 e^{-167/20 \alpha_8}}{\alpha_4 \alpha_4} ;

\quad \alpha_1 \Lambda_{325}^{(1)} = \alpha_3 \Lambda_{125}^{(1)} \text{ (from (i))}. \quad (D22)

(vi) \quad \alpha_7 \Lambda_{145}^{(1)} - \alpha_1 \Lambda_{745}^{(1)} + \frac{108.9}{\kappa \log r R^2} = 0. \quad (D23)

$$\Lambda_{7b_02}^{(1)} R e^2 + \Lambda_{7b_04}^{(1)} R e^4 + \Lambda_{7b_05}^{(1)} R e^4 + \Lambda_{7b_06}^{(1)} R e^6 = 0. \quad (D24)$$

$$\Lambda_{13a_0}^{(2)} R e^{a_0} = 0. \quad (D25)$$

is identically satisfied if $\Lambda_{13a_0}^{(2)} = 0, a_0 = 2, 4, 5, 6$.

$$\Lambda_{\alpha \alpha_0}^{(1)}, \neq 0; \quad (D26)$$
\begin{align*}
\alpha_3 \Lambda_{125}^{(1)} &= \alpha_1 \Lambda_{325}^{(1)} \text{(from (i))}; \\
\alpha_3 \Lambda_{126}^{(1)} &= \alpha_1 \Lambda_{326}^{(1)} = \frac{\alpha_1 \alpha_3}{\alpha_7} \Lambda_{726}^{(1)} \text{(from (iii))}; \\
\alpha_3 \Lambda_{145}^{(1)} &= \alpha_1 \Lambda_{345}^{(1)} = \frac{\alpha_1 \alpha_3}{\alpha_7} \Lambda_{745}^{(1)}; \\
\alpha_3 \Lambda_{146}^{(1)} &= \alpha_1 \Lambda_{346}^{(1)} = \frac{\alpha_1 \alpha_3}{\alpha_7} \Lambda_{746}^{(1)}; \\
\alpha_3 \Lambda_{156}^{(1)} &= \alpha_1 \Lambda_{356}^{(1)} = \frac{\alpha_1 \alpha_3}{\alpha_7} \Lambda_{756}^{(1)} \quad (D27)
\end{align*}

and

\begin{equation}
\Lambda_{160a0}^{(1)} R e^{b_0} = 0. \quad (D28)
\end{equation}

Using (i):

\begin{equation}
\Lambda_{3a0b0}^{(1)} R e^{b_0} = 0. \quad (D29)
\end{equation}

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