High-order exceptional point in a cavity magnonics system

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We propose to realize the pseudo-Hermiticity in a cavity magnonics system consisting of the Kittel modes in two small yttrium-iron-garnet spheres coupled to a microwave cavity mode. The effective gain of the cavity can be achieved using the coherent perfect absorption of the two input fields fed into the cavity. With certain constraints of the parameters, the Hamiltonian of the system has the pseudo-Hermiticity and its eigenvalues can be either all real or one real and other two constituting a complex-conjugate pair. By varying the coupling strengths between the two Kittel modes and the cavity mode, we find the existence of the three-order exceptional point in the parameter space, in addition to the usual second-order exceptional point existing in the system with parity-time symmetry. Also, we show that these exceptional points can be demonstrated by measuring the output spectrum of the cavity.

I. INTRODUCTION

By harnessing the advantages of different components, the hybrid quantum systems have potential applications in quantum information. Among various hybrid systems, the cavity magnonics system has received increasing interest in recent years. This hybrid system consists of magnons in a small yttrium iron garnet (YIG) sample coupled to microwave photons in a cavity. Originating from the high spin density and the strong spin-spin exchange interactions, the Kittel mode in the YIG sample can possess both a long coherence time and a low damping rate, making the cavity magnonics system easy to reach the strong-coupling regime and even possible to reach the ultrastrong-coupling regime. Moreover, owing to the merits of high tunability and good coherence, the cavity magnonics system has become a promising platform to implement various novel phenomena, such as the magnon gradient memory, bistability of cavity-magnon polaritons, cavity spintronics, and cooperative polariton dynamics. In addition, it was experimentally shown that the magnons in the small YIG sample can couple to the optical photons, phonons, and superconducting qubit. This makes it promising to produce the magnon-photon-phonon entanglement in cavity magnonics.

As stated in the textbook of quantum mechanics, the Hamiltonian of a closed quantum system must be Hermitian to have a real energy spectrum. However, any realistic quantum systems are actually open systems. Under certain conditions, they may be effectively modeled by the non-Hermitian Hamiltonians. In Refs. 34–36, Mostafazadeh proposed the pseudo-Hermiticity for the non-Hermitian Hamiltonian of the system: If a Hamiltonian $H$ with a discrete spectrum satisfies $H^\dagger = UHU^{-1}$, where $\dagger$ denotes the Hermitian adjoint and $U$ is a linear Hermitian operator, the Hamiltonian $H$ is pseudo-Hermitian and its eigenvalues are either real or complex conjugate pairs. The pseudo-Hermiticity is an interesting topic in non-Hermitian physics, which can give rise to rich exotic phenomena in different subjects of physics (e.g., quantum chaos and quantum phase transitions, Dirac particles in gravitational fields, Maxwell’s equations, anisotropic $XY$ model, dynamical invariants, and so on).

Obviously, the Hermiticity of the Hamiltonian, $H^\dagger = H$, is a special case of the pseudo-Hermiticity, with $U$ being a unit operator. Also, the $\mathcal{PT}$-symmetric Hamiltonian is another subset of the pseudo-Hermitian Hamiltonian, where the Hamiltonian $H$ satisfies $[H, \mathcal{PT}] = 0$, with $\mathcal{P}$ and $\mathcal{T}$ being the parity and time operators, respectively. Hereafter, the pseudo-Hermiticity mentioned below excludes both the Hermiticity and the $\mathcal{PT}$ symmetry. The exceptional point related to the quantum phase transition of the spontaneous $\mathcal{PT}$-symmetry breaking, which is also called the second-order exceptional point (EP$_2$), has been studied in various non-Hermitian systems, including the optomechanical systems, coupled waveguides, coupled optical microresonators, cavity magnonics systems, and superconducting circuit-QED systems. Besides EP$_2$, high-order exceptional points may occur in non-Hermitian systems. Specifically, an $n$th-order exceptional point (EP$_n$) corresponds to the coalescence of $n$ eigenvalues in a non-Hermitian linear system. Higher-order exceptional points are more complicated but can exhibit richer physical phenomena. For instance, a higher-order exceptional point has much richer topological characteristics in coupled acoustic resonators and can further enhance the sensitivity of the sensors in photonic molecules. To the best of our knowledge, there is no study on both the pseudo-Hermiticity without the $\mathcal{PT}$ symmetry and the related higher-order exceptional point in a cavity magnonics system.

In this work, we investigate the high-order exceptional point in a cavity magnonics system by designing an effective pseudo-Hermitian Hamiltonian without the $\mathcal{PT}$ symmetry. In our proposal, the hybrid system is composed of two small YIG spheres placed in a microwave cavity, where the Kittel mode in each YIG sphere is strongly coupled to the cavity mode. In order to realize the pseudo-Hermiticity of the Hamiltonian, a gain of the cavity is needed, which can be effectively achieved using the coherent perfect absorption (CPA) of the two input fields fed into the cavity via two ports. In addition to the
usual EP, we find the three-order exceptional point (EP3) in the parameter space. Moreover, we show that the EP3 can be observed via measuring the total output spectrum of the cavity, where the CPA frequencies are found to be coincident with the real energy spectrum of the hybrid system.

Our work brings the study of cavity magnonics systems to the interesting pseudo-Hermitian physics. In previous works, 44–63 exceptional points were realized in either the $PT$-symmetric system or the non-Hermitian system without the pseudo-Hermiticity. Our work provides an initial study to the high-order exceptional point in a cavity magnonics system owning the pseudo-Hermitian Hamiltonian without the $PT$ symmetry. In contrast to Ref. 48, we design a more sophisticated system and show that the CPA can also occur in the absence of the $PT$ symmetry. Also, our proposed hybrid system may be harnessed to explore exotic phenomena of the high-order exceptional point (e.g., the topological properties and the perturbation amplification) in the future.

II. THE MODEL

The proposed cavity magnonics system consists of two YIG spheres (YIG 1 and YIG 2) and a three-dimensional (3D) microwave cavity, as schematically shown in Fig. 1, where the considered magnon mode (i.e., the Kittel mode) in each YIG sphere couples to the same cavity mode via the collective magnetic-dipole interaction. When each Kittel mode is in the low-lying excitations and only one cavity mode is considered, the total Hamiltonian of this hybrid system can be written as \[^3\text{-}^7\]

$$H = \omega_0 a_i^\dagger a_i + \omega_1 b_i^\dagger b_i + \omega_2 b_i^\dagger b_i + g_1 (a_i^\dagger b_i + b_i^\dagger a_i) + g_2 (a_i^\dagger b_2 + b_2^\dagger a_i),$$

where $a$ and $a_i^\dagger$ ($b_j$ and $b_j^\dagger$, $j = 1, 2$) are the annihilation and creation operators of the cavity mode (the Kittel mode in the $j$th YIG sphere), $\omega_0$ and $\omega_j$ are the corresponding frequencies of these modes, and $g_j$ is the coupling strength between the cavity photons and the magnons in the $j$th YIG sphere. When two input fields $a_i^{(\text{in})}$ and $a_2^{(\text{in})}$ with the same frequency are fed into the microwave cavity via ports 1 and 2, the dynamics of the hybrid system is governed by the following quantum Langevin equations: \[^64\]

$$\dot{a} = -i(\omega_0 - i(\kappa_1 + \kappa_2 + \kappa_{\text{int}}))a - ig_1 b_1 - ig_2 b_2 + \sqrt{2\kappa_1 a_1^{(\text{in})}} + \sqrt{2\kappa_2 a_2^{(\text{in})}},$$

$$\dot{b}_j = -i(\omega_j - i\gamma_j)b_j - ig_j a,$$

where $\kappa_{\text{int}}$ is the intrinsic decay rate of the cavity mode and $\kappa_i$ is the decay rate of the cavity mode due to the $i$th port ($i = 1, 2$). Then, the total decay rate of the cavity mode is $\kappa_1 + \kappa_2 + \kappa_{\text{int}}$. The Kittel mode in the $j$th YIG sphere has a damping rate $\gamma_j$ and no input field is applied to the Kittel mode. According to the input-output theory, \[^64\] we can connect the intra-cavity field $a$ with the input field $a_i^{(\text{in})}$ and output field $a_i^{(\text{out})}$ via

$$a_i^{(\text{in})} + a_i^{(\text{out})} = \sqrt{2\kappa_i} a_i.$$

at each port $i$.

A. Effective Hamiltonian

With appropriate parameters, the CPA may occur in the hybrid system (see Sec. IIB), with no output fields going out from ports 1 and 2, i.e., $a_i^{(\text{out})} = 0$. In this case, Eq. (3) becomes

$$a_i^{(\text{in})} = \sqrt{2\kappa_i} a_i.$$ (4)

Substituting Eq. (4) into Eq. (2), we obtain

$$\dot{a} = -i(\omega_0 - i(\kappa_1 + \kappa_2 + \kappa_{\text{int}}))a - ig_1 b_1 - ig_2 b_2,$$

$$\dot{b}_j = -i(\omega_j - i\gamma_j)b_j - ig_j a.$$ (5)

The Langevin equations in Eq. (5) can be expressed in a matrix form as

$$\dot{\mathbf{V}} = -iH_{\text{eff}} \mathbf{V},$$ (6)

where $\mathbf{V} = (a, b_1, b_2)^T$ represents a column vector and $H_{\text{eff}}$ is the effective non-Hermitian Hamiltonian of the hybrid system,

$$H_{\text{eff}} = \begin{pmatrix}
\omega_0 + ig_2 & g_1 & g_2 \\
g_1 & \omega_1 - ig_1 & 0 \\
g_2 & 0 & \omega_2 - i\gamma_2
\end{pmatrix}.$$ (7)
where $\kappa_s \equiv \kappa_1 + \kappa_2 - \kappa_{\text{int}} > 0$ represents an effective gain of the cavity mode owing to the CPA. \cite{48,63}

In the special case without YIG 2, the effective Hamiltonian $H_{\text{eff}}$ in Eq. (7) is reduced to a $2 \times 2$ matrix,

$$\tilde{H}_{\text{eff}} = \begin{pmatrix} \omega_c + i\kappa_s & g_1 \\ g_1 & \omega_1 - i\gamma_1 \end{pmatrix}. \tag{8}$$

When the system parameters satisfy $\omega_c = \omega_1$ and $\kappa_s = \gamma_1$, the binary system can possess a $PT$-symmetry, \cite{43} and the observation of its EP was reported in Ref. 48. In Eq. (7), when having both $g_1 \neq 0$ and $g_2 \neq 0$ to possess the $PT$-symmetry, the ternary system should satisfy \cite{42} $\kappa_s = 0$ and $\gamma_1 = -\gamma_2$. This is not achievable in the usual case when the Kittel modes are lossy ($\gamma_1 > 0$ and $\gamma_2 > 0$). However, as shown in Sec. III and Sec. IV, the ternary system without the $PT$-symmetry can also have the real energy spectrum and exhibit both EPs and EPs in the parameter space under the condition of pseudo-Hermiticity.

### B. CPA conditions

Using Fourier transformations $a(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a(\omega) e^{-i\omega t} d\omega$ and $b_j(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} b_j(\omega) e^{-i\omega t} d\omega$, we can convert the Langevin equations in Eq. (2) to

$$\begin{align*}
- \frac{i}{\hbar} [a(\omega_c - \omega) - i(\kappa_1 + \kappa_2 + \kappa_{\text{int}})] a - ig_1 b_1 - ig_2 b_2 \\
+ \sqrt{2\kappa_1} a^{(i\omega)} + \sqrt{2\kappa_2} b_2^{(i\omega)} = 0, \\
- i[(\omega_j - \omega) - \gamma_j] b_1 - ig_1 a = 0.
\end{align*} \tag{9}$$

From Eq. (9), the intra-cavity field is obtained as

$$a = \frac{\sqrt{2\kappa_1} a_1^{(i\omega)} + \sqrt{2\kappa_2} a_2^{(i\omega)}}{(\kappa_1 + \kappa_2 + \kappa_{\text{int}}) + i(\omega_c - \omega) + \sum(\omega)}, \tag{10}$$

where

$$\sum(\omega) = \sum_{j=1,2} g_j^2 \gamma_j + i(\omega_j - \omega). \tag{11}$$

is the self-energy due to the two Kittel modes.

Using Eq. (10) and Eq. (3), we can also obtain the output fields $a_1^{(\text{out})}$ and $a_2^{(\text{out})}$ at ports 1 and 2,

$$a_1^{(\text{out})} = \frac{2\kappa_1 a_1^{(i\omega)} + 2\sqrt{\kappa_1 \kappa_2} a_2^{(i\omega)}}{(\kappa_1 + \kappa_2 + \kappa_{\text{int}}) + i(\omega_c - \omega) + \sum(\omega)} - a_1^{(i\omega)}, \tag{12}$$

$$a_2^{(\text{out})} = \frac{2\sqrt{\kappa_1 \kappa_2} a_1^{(i\omega)} + 2\kappa_2 a_2^{(i\omega)}}{(\kappa_1 + \kappa_2 + \kappa_{\text{int}}) + i(\omega_c - \omega) + \sum(\omega)} - a_2^{(i\omega)}.$$  \tag{12}

When the CPA occurs, the two input fields are fully fed into the cavity, so $a_1^{(\text{out})} = a_2^{(\text{out})} = 0$ in Eq. (12). Solving Eq. (12) with $a_1^{(\text{out})} = 0$, we obtain three constraints:

The first constraint on the two input fields $a_1^{(i\omega)}$ and $a_2^{(i\omega)}$ is

$$a_2^{(i\omega)} = \sqrt{\kappa_1/\kappa_2} a_1^{(i\omega)}, \tag{13}$$

while the second and third constraints on the parameters of the system and the frequency of the input fields are

$$\kappa_s = \sum_{j=1,2} \frac{g_j^2}{(\omega_j - \omega_{\text{CPA}})^2 + \gamma_j^2}, \tag{14}$$

$$\omega_c - \omega_{\text{CPA}} = \sum_{j=1,2} \frac{g_j^2}{(\omega_j - \omega_{\text{CPA}})^2 + \gamma_j^2} (\omega_j - \omega_{\text{CPA}}),$$

where $\omega_{\text{CPA}}$ denotes the frequency of the two input fields in the case of the CPA. The constraint in Eq. (13) means that the two input fields should have the same phase and a specific magnitude ratio $\sqrt{\kappa_1/\kappa_1}$, which can be readily satisfied via a variable phase shifter and a variable attenuator in the experiment. \cite{48}

### III. PSEUDO-THEMERITIANT HAMILTONIAN

Below we derive the parameter conditions of the pseudo-Hermiticity for the effective Hamiltonian $H_{\text{eff}}$ in Eq. (7). For this considered Hamiltonian, there are three eigenvalues. Following Ref. 34, $H_{\text{eff}}$ becomes a pseudo-Hermitian only if its eigenvalues satisfy one of the following conditions: (i) all three eigenvalues are real, or (ii) one of the three eigenvalues is real and other two are a complex-conjugate pair. Solving $\det(H_{\text{eff}} - \Omega I) = 0$, i.e.,

$$\begin{vmatrix}
(\omega_c + i\kappa_s) - \Omega - g_1 \\
\gamma_1 & -\Omega - g_2 \\
\end{vmatrix} = 0, \tag{15}
$$

where $I$ is an identity operator, we can obtain the three eigenvalues. According to the energy-spectrum property of the pseudo-Hermitian Hamiltonian, \cite{34} both Eq. (15) and its complex-conjugate expression $\det(H_{\text{eff}}^* - \Omega I) = 0$, i.e.,

$$\begin{vmatrix}
(\omega_c - i\kappa_s) - \Omega - g_1 \\
\gamma_1 & -\Omega - g_2 \\
\end{vmatrix} = 0, \tag{16}
$$

should yield the same solutions.

By expanding the determinants in Eqs. (15) and (16) and comparing their corresponding coefficients, we find that the system parameters satisfy the following constraints:

$$\kappa_s = \gamma_1 - \gamma_2 = 0,$$

$$\Delta_1 \gamma_1 + \Delta_2 \gamma_2 = 0,$$

$$\Delta_1 \Delta_2 - \gamma_1 \gamma_2 \kappa_s + g_1^2 \gamma_2 + g_2^2 \gamma_1 = 0,$$

and the characteristic polynomial in Eq. (15) is reduced to

$$(\Omega - \omega_c)^3 + c_2 (\Omega - \omega_c)^2 + c_1 (\Omega - \omega_c) + c_0 = 0. \tag{18}$$

Here $\Delta_{1(2)} = \omega_{1(2)} - \omega_c$ is the frequency detuning between the Kittel mode 1 (2) and the cavity mode, and the coefficients $c_0$, $c_1$, and $c_2$ are given by

$$c_0 = g_1^2 \Delta_2 + g_2^2 \Delta_1 - \kappa_s (\gamma_1 \Delta_2 + \gamma_2 \Delta_1),$$

$$c_1 = \kappa_s^2 + \Delta_1 \Delta_2 - \gamma_1 \gamma_2 - g_1^2 - g_2^2,$$

$$c_2 = -\Delta_1 - \Delta_2.$$
Obviously, this is achievable in our considered system.

\[ \gamma_1 = \eta \gamma_2, \quad g_2 = k \gamma_2, \]  

where we have assumed that \( \gamma_2 \leq \gamma_1 \), i.e., \( \eta \geq 1 \). Using Eq. (20), the pseudo-Hermitian conditions in Eq. (17) become

\[ \kappa = (1 + \eta) \gamma_2, \]
\[ \Delta_2 = -\eta \Delta_1, \]
\[ \Delta_1^2 = \frac{(1 + \eta)k^2}{1 + \eta^2} \gamma_2^2 - \gamma_2^2, \]  

and the coefficients of the characteristic polynomial in Eq. (19) are

\[ c_0 = (k^2 - \eta) \gamma_2^2 \Delta_1 + (\eta^2 - 1)(1 + \eta) \gamma_2^2 \Delta_1, \]
\[ c_1 = (1 + \eta)^2 \gamma_2^2 - \eta (\Delta_1^2 + \gamma_2^2) - (1 + k^2) \gamma_2^2, \]
\[ c_2 = (\eta - 1) \Delta_1. \]  

From the last equation in Eq. (21), it follows that the coupling strength \( g_1 \) should be in an appropriate regime to ensure \( \Delta_1^2 \geq 0 \). Setting \( \Delta_1 = 0 \), the allowed minimal value \( g_{\text{min}} \) of the coupling strength \( g_1 \) is given by

\[ g_{\text{min}} = \left[ \frac{(1 + \eta)\eta}{1 + \eta k^2} \right]^{1/2} \gamma_2. \]  

Obviously, this is achievable in our considered system.

### IV. EP3 IN THE CA VITY MAGNONICS SYSTEM

In this section, we study the EP3 in both symmetric and asymmetric cases by solving the characteristic polynomial in Eq. (18) under the pseudo-Hermitian conditions of the system’s parameters and demonstrate that this EP3 can be observable via measuring the total output spectrum of the cavity. Assuming that the pseudo-Hermitian system has an EP3 at \( \Omega \equiv \Omega_{\text{EP3}} \) and the corresponding critical parameters are denoted as \( g_1 \equiv g_{\text{EP3}} \) and \( \Delta_1 \equiv \Delta_{\text{EP3}} \), we can rewrite the secular equation in Eq. (18) as

\[ (\Omega - \Omega_{\text{EP3}})^3 = 0 \]  

at the EP3. Comparing the coefficients of Eq. (18) and Eq. (24), we can link the coalescence eigenvalue \( \Omega = \Omega_{\text{EP3}} \) to the parameters of the system,

\[ -3(\Omega_{\text{EP3}} - \omega_c) = (\eta - 1) \Delta_{\text{EP3}}, \]
\[ 3(\Omega_{\text{EP3}} - \omega_c)^2 = (1 + \eta) \gamma_2^2 - \eta (\Delta_{\text{EP3}}^2 + \gamma_2^2) - (1 + k^2) \gamma_2^2, \]
\[ -\Omega_{\text{EP3}} - \omega_c)^3 = (k^2 - \eta) \gamma_2^2 \Delta_{\text{EP3}} + (\eta^2 - 1)(1 + \eta) \gamma_2^2 \Delta_{\text{EP3}}. \]  

The first equation in Eq. (25) gives the corresponding eigenvalue at the EP3,

\[ \Omega_{\text{EP3}} = \omega_c + \frac{1}{3} (1 - \eta) \Delta_{\text{EP3}}. \]  

### A. The symmetric case of \( \gamma_1 = \gamma_2 \)

When the two Kittel modes have identical damping rates \( \gamma_1 = \gamma_2 \) (i.e., \( \eta = 1 \)), the coalescence eigenvalue in Eq. (26) becomes \( \Omega_{\text{EP3}} = \omega_c \) and the last two equations in Eq. (25) can be simplified to

\[ \Delta_{\text{EP3}}^2 + (1 + k^2) \gamma_2^2 \Delta_{\text{EP3}} - 3 \gamma_2^2 = 0, \]  

\[ (k^2 - 1) \gamma_2^2 \Delta_{\text{EP3}} = 0. \]  

Solving Eq. (27) under the pseudo-Hermitian conditions in Eq. (21) and ignoring the trivial solution, we can analytically express the critical parameters as

\[ g_{\text{EP3}} = \frac{2}{\sqrt{3}} \gamma_2, \quad \Delta_{\text{EP3}} = \frac{1}{\sqrt{3}} \gamma_2, \]  

and the obtained ratio \( k \) in Eq. (20) is \( k = 1 \).

In such a case with \( \gamma_1 = \gamma_2 \) and \( g_1 = g_2 \) (i.e., \( \eta = k = 1 \)), the secular equation in Eq. (18) can be rewritten as

\[ [(\Omega - \omega_c)^3 - (3g_1^2 - 4\gamma_2^2)][\Omega - \omega_c] = 0. \]  

The corresponding three eigenvalues of the effective pseudo-Hermitian Hamiltonian \( H_{\text{eff}} \) are

\[ \Omega_0 = \omega_c, \quad \Omega_+ = \omega_c \pm \sqrt{3g_1^2 - 4\gamma_2^2}, \]  

in the region \( g_1 \geq g_{\text{min}} \). Now, \( g_{\text{min}} \) in Eq. (23) becomes \( g_{\text{min}} = \gamma_2 \), which is smaller than \( g_{\text{EP3}} = \frac{2}{\sqrt{3}} \gamma_2 \). Clearly, the eigenvalue \( \Omega_0 \) is real for any allowed values of \( g_1 \) (i.e., \( g_1 \geq g_{\text{min}} \)), while the two eigenvalues \( \Omega_+ \) are real for \( 3g_1^2 - 4\gamma_2^2 > 0 \) (i.e., \( g_1 > g_{\text{EP3}} \)) and complex for \( 3g_1^2 - 4\gamma_2^2 < 0 \) (i.e., \( g_{\text{min}} \leq g_1 \leq g_{\text{EP3}} \)). When \( 3g_1^2 - 4\gamma_2^2 = 0 \) (i.e., \( g_1 = g_{\text{EP3}} \)), the three eigenvalues \( \Omega_+ \) and \( \Omega_0 = \Omega_{\text{EP3}} = \omega_c \). Because \( g_{\text{min}} < g_{\text{EP3}} \), the EP3 is experimentally observable in this symmetric case.

Below we check the CPA conditions in Eq. (14). For \( \eta = k = 1 \), the CPA conditions are reduced to

\[ [(\omega_{\text{CPA}} - \omega_c)^3 - (3g_1^2 - 4\gamma_2^2)][(\omega_{\text{CPA}} - \omega_c)^2 - 3g_1^2] = 0, \]
\[ [(\omega_{\text{CPA}} - \omega_c)^2 - 3g_1^2][((\omega_{\text{CPA}} - \omega_c)^2 - g_1^2)] = 0, \]  

under the pseudo-Hermitian conditions in Eq. (21). Solving the above equations, we obtain the three CPA frequencies

\[ \omega_{\text{CPA}}^{(0)} = \omega_c, \quad g_1 \geq g_{\text{min}}; \]
\[ \omega_{\text{CPA}}^{(1)} = \omega_c \pm \sqrt{3g_1^2 - 4\gamma_2^2}, \quad g_1 \geq g_{\text{EP3}}. \]  

Comparing Eq. (32) with Eq. (30), we find that the CPA frequencies are coincident with the eigenvalues of the hybrid system when the eigenvalues \( \Omega_+ \) and \( \Omega_0 \) are real. However, for the complex eigenvalues, the CPA goes to disappear.
FIG. 2. The ratio $k = g_2/g_1$ of the coupling strengths $g_2$ and $g_1$ versus the ratio $\eta = \gamma_1/\gamma_2$ of the Kittel-mode damping rates $\gamma_1$ and $\gamma_2$.

**B. The asymmetric case of $\gamma_1 \neq \gamma_2$**

In the experiment, it is difficult to have two Kittel modes with the same damping rates, because the Kittel-mode damping rate is not tunable. Thus, it is useful to investigate the EP\(_3\) in the asymmetric case of $\gamma_1 \neq \gamma_2$ (i.e., $\eta \neq 1$). With the pseudo-Hermitian conditions in Eq. (21) and the conditions of the EP\(_3\) in Eq. (25), we find that the parameter $k$ satisfies the following expression:

$$
\left[ 1 + \frac{1 + \eta k^2}{(1 + \eta)^2} \right] \left[ 1 + \frac{1 + \eta k^2}{(1 + \eta)^2} - \frac{27(k^2 - \eta)}{(\eta - 1)^3} \right] = \left[ 1 + \frac{27(1 + \eta)^2}{(\eta - 1)^2} \right] \left[ 1 + \frac{1 + \eta k^2}{(1 + \eta)^2} - \frac{27(k^2 - \eta)}{(\eta - 1)^3} \right],
$$

(33)

where $\eta \neq 1$, and the critical parameters are

$$
g_{\text{EP}3} = \frac{1 + \eta k^2}{(1 + \eta)^2} + \frac{3(1 + k^2)}{1 + \eta + \eta^2} \Omega/2, $$

$$
\Delta_{\text{EP}3} = \left[ 1 + \frac{1 + \eta k^2}{(1 + \eta)^2} g_{\text{EP}3} - \frac{7}{2} \right]^{1/2}.
$$

(34)

With the expressions of $g_{\text{min}}$ and $g_{\text{EP}3}$ in Eqs. (23) and (34), respectively, we obtain the relation

$$
g_{\text{EP}3} = g_{\text{min}} \sqrt{1 + 27\eta^2(2 + \eta)^2(1 + 2\eta)^{-2}},
$$

(35)

where we eliminate the parameter $k$ via Eq. (33). Obviously, $g_{\text{EP}3} > g_{\text{min}}$, so the EP\(_3\) is achievable in the experiment.

Using Eq. (33), we plot in Fig. 2 the ratio $k = g_2/g_1$ of the coupling strengths versus the ratio $\eta = \gamma_1/\gamma_2$ of the damping rates. It can be seen that $k$ decreases from 1 to 0.3 as $\eta$ varies from 1 to 3, which means that the coupling strengths should satisfy the relation $g_1 > g_2$ in the case of $\gamma_1 > \gamma_2$ (because $g_1 = \eta_2 g_2$ when $g_1 = g_2/k$) to observe EP\(_3\) in our proposed system. Different from the symmetric case with $\gamma_1 = \gamma_2$ and $g_1 = g_2$ (i.e., $\eta = k = 1$), it is difficult to analytically solve the secular equation in Eq. (18) as well as the CPA conditions in Eq. (14) for $\gamma_1 \neq \gamma_2$ and $g_1 \neq g_2$, but we can numerically solve them.

FIG. 3. The eigenvalues of the effective Hamiltonian $H_{\text{eff}}$ in Eq. (7) versus the coupling strength $g_1$ between the cavity mode and the Kittel mode in Fig. 1. Note that there is no pseudo-Hermiticity for the system with $g_1 < g_{\text{min}}$ (green regions). In each figure, the dashed red and dotted blue lines denote the eigenvalues $\Omega_{\pm}$ and the solid black line denotes the eigenvalue $\Omega_{0}$. (a) and (c) The real and imaginary parts of the eigenvalues $\Omega_{\pm}$, and $\Omega_{0}$ versus $g_1$, in the symmetric case of $\eta = 1$, and $\omega_{\text{cav}} = 2.25$ MHz and $\gamma_1/\gamma_2 = 1.5$ MHz. (b) and (d) The real and imaginary parts of the eigenvalues $\Omega_{\pm}$, and $\Omega_{0}$ versus $g_1$, in the asymmetric case of $\eta = 2$ and $k = 0.494$, where $\omega_{\text{cav}} = 3$ MHz and $\gamma_1/\gamma_2 = 3$ MHz. Other parameters are chosen to be $\gamma_2/\gamma_1 = \kappa_{\text{cav}}/2\pi = 1.5$ MHz.

In Fig. 3, we plot the energy spectra of the effective Hamiltonian $H_{\text{eff}}$ in Eq. (7) versus the coupling strength $g_1$ in the symmetric and asymmetric cases of $\gamma_1 = \gamma_2$ and $\gamma_1 = 2\gamma_2$ (i.e., $\eta = 1$ and $\eta = 2$), respectively. Note that no eigenvalue exists when $g_1 < g_{\text{min}}$ (see the green regions), because there is no pseudo-Hermiticity for the system. Figures 3(a) and 3(c) show the real and imaginary parts of the eigenvalues $\Omega_{\pm}$ and $\Omega_{0}$ given in Eq. (30) versus $g_1$ for $\eta = 1$, with the critical coupling strength $g_{\text{EP}3}/2\pi = 1.732$ MHz. The eigenvalues have different characteristics in the two regions: $g_{\text{min}} \leq g_1 < g_{\text{EP}3}$ and $g_1 > g_{\text{EP}3}$. When $g_{\text{min}} \leq g_1 < g_{\text{EP}3}$, the eigenvalues $\Omega_{\pm}$ are a complex-conjugate pair (see the dashed red and dotted blue lines) and $\Omega_{0}$ is real (see the solid black lines). It is clear that the three eigenvalues $\Omega_{\pm}$ and $\Omega_{0}$ coalesce to $\Omega_{\text{EP}3} = \omega_{\text{cav}}$ at $g_1 = g_{\text{EP}3}$ (i.e., the EP\(_3\)). For $g_1 > g_{\text{EP}3}$, all three eigenvalues $\Omega_{\pm}$ and $\Omega_{0}$ are real.

In the asymmetric case of $\gamma_1 \neq \gamma_2$ (where we choose $\eta = 2$), the corresponding real and imaginary parts of the eigenvalues $\Omega_{\pm}$ and $\Omega_{0}$ versus the coupling strength $g_1$ are shown in Figs. 3(b) and 3(d), respectively. We note that there are two critical coupling strengths $g_{\text{EP}2}/2\pi = 3.394$ MHz and $g_{\text{EP}2}/2\pi = 3.600$ MHz. The eigenvalue $\Omega_{0}$ is real for any allowed values of the coupling strength $g_1 \geq g_{\text{min}}$ (see the solid black lines) and $\Omega_{\pm}$ are complex (real) for $g_{\text{min}} \leq g_1 < g_{\text{EP}3}$ and $g_{\text{EP}3} < g_1 < g_{\text{EP}2}$ ($g_1 = g_{\text{EP}3}$ and $g_1 \geq g_{\text{EP}2}$) (see the dashed red and dotted blue lines). In this case, in addition
to the EP3 at \( g_1 = g_{EP3} \), where the three eigenvalues \( \Omega_3 \) and \( \Omega_2 \) are coalescent, there is the EP2 at \( g_1 = g_{EP2} \), where the two eigenvalues \( \Omega_2 \) are coalescent. This is different from the symmetric case.

C. The output spectrum

In this subsection, we derive the total output spectrum of the cavity for the hybrid system and show that the pseudo-Hermiticity can be observed using the output spectrum. As discussed in Sec. IIB, when the CPA occurs, the first constraint is on the two input fields \( d_1^{(in)} \) and \( d_2^{(in)} \), i.e., Eq. (13). Using this equation, the expressions of the two outgoing fields in Eq. (12) can be rewritten as

\[
\begin{align*}
    a_1^{(out)} &= S_1(\omega) a_1^{(in)}, \\
    a_2^{(out)} &= S_2(\omega) a_2^{(in)},
\end{align*}
\]

where \( S_1(\omega) \) and \( S_2(\omega) \) are the output coefficients at ports 1 and 2 for the frequency \( \omega \) of the two input fields.

\[
\begin{align*}
    S_1(\omega) &= \frac{2\kappa_1 + 2\kappa_2}{(\kappa_1 + \kappa_2 + \kappa_{int}) i(\omega_c - \omega) + \sum(\omega)} - 1, \\
    S_2(\omega) &= S_1(\omega). \tag{37}
\end{align*}
\]

Here we define a total output spectrum \( |S_{tot}(\omega)|^2 \) to characterize the input-output property of the hybrid system,

\[
|S_{tot}(\omega)|^2 = |S_1(\omega)|^2 + |S_2(\omega)|^2. \tag{38}
\]

It is easy to check that \( |S_{tot}(\omega)|^2 = 0 \) when the second and third constraints in Eq. (14) are satisfied at \( \omega = \omega_{CPA} \).

In Figs. 4(a) and 4(b), we show the total output spectrum \(|S_{tot}(\omega)|^2 \) versus the coupling strength \( g_1 \) and the frequency detuning \( \omega - \omega_c \) between the two input fields and the cavity mode, when \( \eta = 1 \) and \( \eta = 2 \), respectively. The minimum in the total output spectrum (see the blue pattern) represents the CPA, i.e., \( a_1^{(out)} = a_2^{(out)} = 0 \). As expected, the CPA frequencies are coincident with the real eigenfrequencies of the effective pseudo-Hermitian Hamiltonian \( H_{eff} \) in Eq. (7), where the real eigenvalues and the EPs are indicated by the dashed white lines and the white stars, respectively. Therefore, the energy spectra as well as the EP3 and EP2 can be demonstrated via measuring the total output spectrum of the microwave cavity.

V. DISCUSSIONS AND CONCLUSIONS

Both the 3D microwave cavity with a high \( Q \) factor (e.g., \( \kappa_{int}/2\pi \sim 1 \) MHz) and the highly-polished small YIG sphere with \( \gamma_{1,2}/2\pi \sim 1 \) MHz are experimentally available. Also, the decay rates of the cavity induced by the two ports are tunable (ranging from 0 to 8 MHz) by adjusting the intracavity pin lengths of the ports. For a saturated magnetized YIG sphere by a static magnetic field, the frequency of the Kittel mode in the YIG sphere can be further tuned in the range of tens of megahertz via the magnetic field generated by a small coil near the sphere. In Ref. 48, the YIG sphere is adhered to a thin rod placed into the cavity through a small hole of the cavity and the coupling strength between the cavity and Kittel mode can be tuned from 0 to 9 MHz by moving the rod. Moreover, a microwave signal generated by a vector network analyzer can be divided into two feeding fields needed for realizing the CPA and their magnitudes and relative phases can be adjusted using variable attenuator and phase shifter, respectively. With these achievable conditions, our proposed scheme is experimentally implementable.

In Ref. 48, the CPA was achieved for a \( \mathcal{PT} \)-symmetric system, while the CPA was investigated for an optical system without the \( \mathcal{PT} \) symmetry. In the present work, we find that the CPA is also realizable for a system with the pseudo-Hermiticity. It is known that the EPs are more complicated but has richer physics than the EPs, \( 54-62 \) which are important for the implementation of the pseudo-Hermiticity of the system. Also, as a hybrid system, the cavity magnonics system has good compatibility with phonons, \( 30 \) optical photons \( 26-29 \) and superconducting qubits \( 31,32 \). Moreover, the YIG has the intrinsic nonreciprocity. \( 27 \) These characteristics will make the cavity magnonics system useful in exploring the richer properties of the high-order exceptional...
In short, we have theoretically studied the pseudo-Hermiticity and EP$_3$ in a cavity magnonics system consisting of two small YIG spheres in a microwave cavity. Under the parameter conditions of the pseudo-Hermiticity, the effective Hamiltonian of the system has either three real eigenvalues or one real and two complex-conjugate eigenvalues. By tuning the coupling strengths between the two Kittel modes and the cavity mode, the three eigenvalues can coalesce at the EP$_3$. Also, we show that the pseudo-Hermiticity and EP$_3$ can be probed using the total output spectrum of the cavity. Our work provides an experimentally feasible scheme to realize the pseudo-Hermiticity and EP$_3$ in a hybrid quantum system.

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