3-MANIFOLDS
WITH POSITIVE FLAT CONFORMAL STRUCTURE

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Abstract. In this paper, we consider a closed 3-manifold $M$ with flat conformal structure $C$. We will prove that if the Yamabe constant of $(M, C)$ is positive, then $(M, C)$ is Kleinian.

1. Introduction and Main Theorem

In 1988, Schoen and Yau [19] gave a final resolution for the Yamabe Problem (cf. [3, 15, 18]). In [19, Proposition 3.3], they also proved that any closed $n$-manifold with flat conformal structure of positive Yamabe constant is Kleinian, provided that $n \geq 4$. Moreover, under the assumption that an extended Positive Mass Theorem holds (but a proof has not yet appeared), they showed that the above assertion still holds even when $n = 3$ (see [19, Proposition 4.4] and the paragraph just before it). On the other hand, there are enormous examples of closed 3-manifolds with flat conformal structures which are not Kleinian (see [8, Remark 7.4]).

The purpose of this brief note is to prove the above assertion for the remaining case $n = 3$.

Theorem 1.1. Let $M$ be a closed 3-manifold with flat conformal structure $C$. If its Yamabe constant is positive, then $(M, C)$ is Kleinian.

This assertion can be obtained by an argument in the proof of [1, the second assertion of Theorem 1.4], which is a combination of a result [19, Proposition 4.2], a positive mass theorem [1, the first assertion of Theorem 1.4] (different from the one Schoen and Yau mentioned in [19]) and a classification of 3-manifolds with positive scalar curvature [7, 10, 11]. Here, we will explicitly give a proof of it (see also Remark 2.2 below).

The remaining sections are organized as follows. Section 2 contains some necessary definitions and preliminary geometric results. Section 3 is devoted to the proof of Theorem 1.1.

2. Preliminaries

Let $M$ be a closed 3-manifold, that is, a compact 3-manifold without boundary. To simplify the presentation and the argument, we always assume that $\dim M = 3$.

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throughout this paper. For each conformal class $C$ on $M$, the Yamabe constant $Y(M,C)$ of $(M,C)$ is defined by

$$Y(M,C) := \inf_{g \in C} E(g), \quad E(g) := \frac{\int_M R_g d\mu_g}{\text{Vol}_g(M)^{1/3}},$$

where $R_g$, $\mu_g$ and $\text{Vol}_g(M)$ denote respectively the scalar curvature, the volume element of $g$ and the volume of $(M,g)$. It is a finite-valued conformal invariant of $C$. The Yamabe constant $Y(M,C)$ is positive if and only if there exists a positive scalar curvature metric $g \in C$ (cf. [3]). A remarkable theorem [22, 20, 2, 17, 19] of Yamabe, Trudinger, Aubin and Schoen asserts that each conformal class $C$ contains a minimizer $\tilde{g}$ of $E|_C$, called a Yamabe metric (or a solution of the Yamabe Problem), which is of constant scalar curvature

$$R_{\tilde{g}} = Y(M,C) \cdot \text{Vol}_{\tilde{g}}(M)^{-2/3}.$$ 

Let $M_\infty$ be an infinite covering of $M$. We shall say that the fundamental group $\pi_1(M)$ of $M$ has a descending chain of finite index subgroups tending to $\pi_1(M_\infty)$ if it satisfies the following: There exists a family of subgroups $\{\Gamma_i\}_{i \geq 1}$ of $\pi_1(M)$ such that

(i) each $\Gamma_i$ is of finite index in $\pi_1(M)$ with $\Gamma_i \supseteq \pi_1(M_\infty)$,

(ii) $\pi_1(M) = \Gamma_1 \supseteq \Gamma_2 \supseteq \cdots \supseteq \Gamma_i \supseteq \Gamma_{i+1} \supseteq \cdots$,

(iii) $\bigcap_{i=1}^\infty \Gamma_i = \pi_1(M_\infty)$.

Assume that $Y(M,C) > 0$. Take a positive scalar curvature metric $g \in C$ and any point $p \in M$. Then, there exists the normalized Green’s function $G_p$ for $L_g$ with a pole at $p$, that is,

$$L_g G_p = c_0 \cdot \delta_p \quad \text{on} \quad M \quad \text{and} \quad \lim_{q \to p} (q,p) G_p(q) = 1.$$ 

Here, $L_g := -8\Delta_g + R_g$, $c_0 > 0$ and $\delta_p$ stand respectively for the conformal Laplacian, a specific universal positive constant and the Dirac $\delta$-function at $p$. Assume also that the covering $P_\infty : M_\infty \to M$ is normal. Let $g_\infty$ denote the lift of $g$ to $M_\infty$, and $p_\infty$ a point in $M_\infty$ with $P_\infty(p_\infty) = p$. Then, there exists uniquely also a normalized minimal positive Green’s function $G_\infty$ on $M_\infty$ for $L_{g_\infty} := -8\Delta_{g_\infty} + R_{g_\infty}$ with pole at $p_\infty$ (cf. [19]), which satisfies the following:

$$(P_\infty)^* G_p = \sum_{\gamma \in \mathcal{G}} G_\infty \circ \gamma \quad \text{on} \quad M_\infty.$$ 

Here, $\mathcal{G}$ stands for the group of deck transformations for the normal covering $M_\infty \to M$. Set

$$g_{\infty,AF} := G_4^* \cdot g_\infty \quad \text{on} \quad M_\infty^* := M_\infty - \{p_\infty\}.$$ 

Then, $g_{\infty,AF}$ defines a scalar-flat, asymptotically flat metric on $M_\infty^*$ (cf. [15]). Note that this asymptotically flat 3-manifold $(M_\infty^*, g_{\infty,AF})$ has infinitely many singularities created by the ends of $M_\infty^*$. However, the mass $m_{ADM}(g_{\infty,AF})$ of $(M_\infty^*, g_{\infty,AF})$ can be defined in the usual way (cf. [11]). Note also that the positive mass theorem for asymptotically flat 3-manifolds with singularities does not always hold (see [11] Remark 1.5-(2) for instance).

Once this is understood, the following positive mass theorem holds as a special case of [11] the first assertion of Theorem 1.4]:
Proposition 2.1. Let \((M, C)\) be a closed 3-manifold with \(Y(M, C) > 0\). Let \((M_\infty, g_\infty)\) be a normal infinite Riemannian covering of \((M, g)\) such that \(\pi_1(M)\) has a descending chain of finite index subgroups tending to \(\pi_1(M_\infty)\), where \(g \in C\) is a positive scalar curvature metric and \(g_\infty\) is its lift to \(M_\infty\). For any point \(p_\infty \in M_\infty\), let \(G_\infty\) denote the normalized minimal positive Green’s function on \(M_\infty\) with pole at \(p_\infty\). Then, the asymptotically flat 3-manifold \((M_\infty, g_\infty, \sigma, AF)\) has nonnegative mass

\[ m_{\text{ADM}}(g_\infty, \sigma, AF) \geq 0. \]

Remark 2.2. Assume that \(M = \#\ell(S^1 \times S^2)\) for \(\ell \geq 2\) and \(M_\infty\) is its universal covering. Notice that \(M_\infty\) is spin. For each small \(\sigma > 0\), consider the complete metric \(g_{\sigma, AF} := (G_\infty + \sigma)^4 \cdot g_\infty\) with \(R_{g_{\sigma, AF}} \geq 0\) on \(M_\infty^+\) (cf. [19] Proposition 4.4]). Then, only one end of \((M_\infty^+, g_{\sigma, AF})\) is asymptotically flat and the other infinitely many ends are merely complete. Gilles Carron and the referee kindly pointed out that Witten’s approach [21] (cf. [16]) to the Positive Mass Theorem is still valid for the family \(\{(M_\infty^+, g_{\sigma, AF})\}_{0 < \sigma < 1}\). It implies that a more general positive mass theorem than Proposition 2.1 is a folk theorem for experts in this field, and Theorem [14] is too. But Proposition 2.1 itself is a complete form, and hence, by using it, we will give here an explicit and self-contained proof of Theorem [14].

A conformal 3-manifold \((M, C)\) is said to be locally conformally flat if, for any point \(p \in M\), there exists a metric \(\mathfrak{G} \in C\) such that \(\mathfrak{G}\) is flat on some neighborhood of \(p\). A conformal class \(C\) on \(M\) is called a flat conformal structure if \((M, C)\) is locally conformally flat. In [14], Kuiper proved that, for a simply connected locally conformally flat 3-manifold \((X, C')\), there is a conformal immersion into \((S^3, C_0)\) called a developing map, which is unique up to composition with a Möbius transformation of \((S^3, C_0)\). Therefore, the universal covering of a locally conformally flat manifold \((M, C)\) admits a developing map. Here, \((S^3, C_0)\) denotes the 3-sphere \(S^3\) with the conformal class \(C_0 := [g_0]\) of the standard metric \(g_0\) of constant curvature one. \((M, C)\) is called Kleinian if \((M, C)\) is conformal to \(\Omega/\Gamma\) for some open set \(\Omega\) of \(S^3\) and some discrete subgroup \(\Gamma\) of the conformal transformation group \(\text{Conf}(S^3, C_0)\), which leaves \(\Omega\) invariant and acts freely and properly discontinuously on \(\Omega\). Note that, if the developing map of the universal covering of a locally conformally flat manifold \((M, C)\) is injective, then \((M, C)\) is Kleinian.

With this understanding, the following criterion also holds as a special case of [19] Proposition 4.2:

Proposition 2.3. Let \((M, C)\) be a closed 3-manifold with \(Y(M, C) > 0\), and \((\widetilde{M}, \widetilde{g})\) be the universal Riemannian covering of \((M, g)\), where \(g \in C\) is a positive scalar curvature metric. For any point \(\tilde{p} \in \widetilde{M}\), let \(G\) denote the normalized minimal positive Green’s function on \(\widetilde{M}\) for \(L_{\widetilde{g}}\) with pole at \(\tilde{p}\), and \((\widetilde{M} - \{\tilde{p}\}, \widetilde{g}_{AF} = G^4 \cdot \widetilde{g})\) the asymptotically flat 3-manifold as above. If the mass \(m_{\text{ADM}}(\widetilde{g}_{AF})\) is nonnegative, then the developing map of \((\widetilde{M}, \widetilde{g})\) is injective. In particular, \((M, C)\) is Kleinian.

Remark 2.4. We remark that the mass \(m_{\text{ADM}}(\widetilde{g}_{AF})\) is equal to the ADM energy \(E\) of \((\widetilde{M} - \{\tilde{p}\}, \widetilde{g}_{AF})\) appearing in [19] page 64 up to a positive constant.

3. Proof of main theorem

Proof of Theorem 1.1 Consider the universal covering \(\widetilde{M}\) of \(M\) and denote the lift of the flat conformal structure \(C\) by \(\widetilde{C}\). If \(|\pi_1(M)| < \infty\), then \((\widetilde{M}, \widetilde{C})\) is conformal
to $(S^3, C_0)$ by Kuiper’s Theorem \[14\]. Hence, $(M, C)$ is Kleinian. From now on, we assume that $|\pi_1(M)| = \infty$, that is, the degree of the covering map $P: \tilde{M} \to M$ is infinite.

Take a unit-volume Yamabe metric $g \in C$, and consider its lift $\tilde{g} \in \tilde{C}$ to $\tilde{M}$. Note that $R_{\tilde{g}} = R_{\tilde{g}} = Y(M, C) > 0$. Take any base points $p \in M, \tilde{p} \in \tilde{M}$ satisfying $P(\tilde{p}) = p$, and fix them. Then, let $\tilde{G}$ denote the normalized minimal positive Green function on $M$ for $L_{\tilde{g}}$ with pole at $\tilde{p}$, and the mass $m_{\text{ADM}}(\tilde{g}_{\text{AF}})$ of the asymptotically flat 3-manifold $(\tilde{M} - \{\tilde{p}\}, \tilde{g}_{\text{AF}} := G^4 \cdot \tilde{g})$.

Suppose that

\[ m_{\text{ADM}}(\tilde{g}_{\text{AF}}) \geq 0. \]

Recall that we can choose the base point $\tilde{p} \in \tilde{M}$ arbitrarily. It then follows from Proposition 2.3 that the developing map of $(\tilde{M}, \tilde{C})$ is injective, and hence $(M, C)$ is Kleinian. In this case, especially $m_{\text{ADM}}(\tilde{g}_{\text{AF}}) = 0$. Therefore, it is enough to show that $m_{\text{ADM}}(g_{\text{AF}}) \geq 0$.

By combining [7, Theorem 8.1] (cf. [9]) with $Y(M, C) > 0$ (replacing $M$ by its orientable double covering if necessary), $M$ can be decomposed uniquely into prime closed 3-manifolds

\[ M = N_1 \# \cdots \# N_\ell \# \ell_2(S^1 \times S^2), \]

where $\pi_1(N_i)$ is finite for $i = 1, \ldots, \ell_1$ and $\ell_1, \ell_2$ are nonnegative integers. By applying the C-prime decomposition theorem for closed 3-manifolds with flat conformal structures [10, 11] to $(M, C)$, there exists a flat conformal structure $C_i$ on each $N_i$ $(i = 1, \ldots, \ell_1)$. Then, Kuiper’s Theorem [14] again implies that each $(N_i, C_i)$ is a nontrivial quotient of $(S^3, C_0)$. After taking an appropriate finite covering $M'$ of $M$, we have

\[ M' = \# \ell(S^1 \times S^2) \quad \text{for some } \ell \geq 1. \]

Recall that $\tilde{M}$ is the infinite universal covering of $M$. Then, there exists (uniquely) an infinite universal covering $\tilde{M} \to M'$. Moreover, since $\pi_1(M')$ is a finitely generated free group, it has a descending chain of finite index subgroups tending to $\pi_1(\tilde{M}) = \{e\}$. Let $g'$ be the lifting of $g$ to $M'$. Applying Proposition 2.1 to the normal infinite Riemannian covering $(M, \tilde{g}) \to (M', g')$, we have that

\[ m_{\text{ADM}}(g_{\text{AF}}) \geq 0. \]

This completes the proof of Theorem 1.1.

\[ \square \]

Remark 3.1. Even if we replace the positivity $Y(M, C) > 0$ in Theorem 1.1 by the nonnegativity $Y(M, C) \geq 0$, it seems that the same conclusion still holds. More precisely, we propose the following (cf. [5, 13]).

**Conjecture.** Let $M$ be a closed 3-manifold with flat conformal structure $C$. If its Yamabe constant is zero, then either (1) or (2) holds:

1. There exists a flat metric $\bar{g} \in C$.
2. There exists a smooth family $\{g_t\}_{0 \leq t \leq 1}$ of locally conformally flat metrics on $M$ such that $g_0 \in C$ and $Y(M, [g_1]) > 0$ (possibly $Y(M, [g_1]) < 0$ for some $t \in (0, 1)$).

In the case (1), the universal covering $(\tilde{M}, \tilde{C})$ of $(M, C)$ is conformal to $(S^3 - \{p_N\}, C_0)$ where $p_N := (1, 0, 0, 0) \in S^3$, and hence $(M, C)$ is Kleinian. In the case (2), Theorem 1.1 implies that $(M, [g_1])$ is Kleinian. The argument in the proof of
Theorem 1.1 also implies that there exists a torsion free subgroup $\Gamma$ of finite index in $\pi_1(M)$ such that $\Gamma$ is either a trivial group or a nontrivial finitely generated free group. Then, the virtual cohomological dimension $\text{vcd} \pi_1(M)$ of $\pi_1(M)$ is either 0 or 1 (see [6]). Therefore, $(M, [g_1])$ is a closed Kleinian 3-manifold with $\text{vcd} \pi_1(M) < 3$. The quasiconformal stability of Kleinian groups [12] Theorem 2 implies that any flat conformal structure on $M$ which is a smooth deformation of $[g_1]$ is also Kleinian; in particular $C$ is too.

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