Lie-algebraic approach to the theory of polynomial solutions.

III. Differential equations in two real variables and general outlook

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ABSTRACT

Classification theorems for linear differential equations in two real variables, possessing eigenfunctions in the form of the polynomials (the generalized Bochner problem) are given. The main result is based on the consideration of the eigenvalue problem for a polynomial elements of the universal enveloping algebras of the algebras $sl_3(\mathbb{R})$, $sl_2(\mathbb{R}) \oplus sl_2(\mathbb{R})$ and $gl_2(\mathbb{R}) \bowtie \mathbb{R}^{r+1}$, $r > 0$ taken in the "projectivized" representations (in differential operators of the first order in two real variables) possessing an invariant subspace. General insight to the problem of a description of linear differential operators possessing an invariant sub-space with a basis in polynomials is presented. Connection to the recently-discovered quasi-exactly-solvable problems is discussed.

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Take the eigenvalue problem
\[ T \varphi(x, y) = \epsilon \varphi(x, y) \] (0)
where \( T \) is a linear differential operator of two real variables \( x, y \) and \( \epsilon \) is the spectral parameter.

**Definition.** Let us give the name of the *generalized Bochner problem* to the problem of classification of the differential equations (0) possessing several eigenfunctions in the form of polynomials.

In the paper [1] a general method has been formulated for generating eigenvalue problems for linear differential operators, linear matrix differential operators and linear finite-difference operators in one and several variables possessing polynomial solutions. The method was based on considering the eigenvalue problem for the representation of a polynomial element of the universal enveloping algebra of the Lie algebra in a finite-dimensional, ‘projectivized’ representation of this Lie algebra [1].

In two previous papers [2, 3] it has been proven that in this approach the consideration of algebras \( sl_2(\mathbb{R}) \), \( sl_2(\mathbb{R})_q \), \( osp(2, 2) \) in projectivized representations (in differential operators of the first order) provides in general both necessary and sufficient conditions for the existance of polynomial eigenfunctions for ordinary linear differential operators, finite-difference operators in one variable [2] and differential operators in two variables (one real and one Grassmann) (or, equivalently, 2 x 2 matrix differential operators) [3], respectively. Particularly, it manifested the classification theorems, which imply the solution of the original Bochner problem (1929) posed for ordinary differential equations. In the present paper similar classification theorems will be given for finite-order linear differential operators in two real variables in connection to the algebras \( sl_3(\mathbb{R}) \), \( sl_2(\mathbb{R}) \oplus sl_2(\mathbb{R}) \) and \( gl_2(\mathbb{R}) \bowtie \mathbb{R}^m \). Also an outlook on the problem for a general linear differential operators will be given.

For future considerations, we define the space of all polynomials in \( x, y \) of finite degree as
\[ P_{N,M} = \langle 1, x^1, y^1, x^2, xy, y^2, \ldots, x^N y^M \rangle \] (1)
where \( N, M \) are non-negative integers, \( x, y \in \mathbb{R} \) are real variables. It is convenient to illustrate the space of polynomials of finite degree through the

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Newton diagrams. In order to do this, let us consider two-dimensional plane and each point with the integer coordinates \((k, p)\) put in correspondence with the monomial \(x^k y^p\).

1 Polynomials in two real variables

1.1 Polynomials of the first type

Now let us describe the projectivized representation of the algebra \(sl_3(\mathbb{R})\) in the differential operators of the first order acting on functions of two real variables. It is easy to show that the generators have the form

\[
J_3^1 = y^2 \partial_y + x y \partial_x - n y , \quad J_2^1 = x^2 \partial_x + x y \partial_y - n x ,
\]

\[
J_3^2 = -y \partial_x , \quad J_1^2 = -\partial_x , \quad J_3^3 = -\partial_y , \quad J_2^3 = -x \partial_y ,
\]

\[
J_d = y \partial_y + 2x \partial_x - n , \quad \tilde{J}_d = 2y \partial_y + x \partial_x - n ,
\]

where \(x, y\) are the real variables and \(n\) is a real number. If \(n\) is a non-negative integer, the representation becomes finite-dimensional of the dimension \((1 + n)(1 + n/2)\). The invariant sub-space has a polynomial basis and is presented as a space of all polynomials of the following type

\[
P_n^{(I)} = \langle 1; x, y; x^2, xy, y^2; \ldots; x^n, x^{n-1}y, \ldots, xy^{n-1}, y^n \rangle
\]

or, graphically, (3) is given by the Newton diagram of the figure 1.

![Fig. 1. Graphical representation (Newton diagram) of the space \(P_n^{(I)}\) (see (3)).](image-url)
Definition. Let us name a linear differential operator of the \( k \)-th order a **quasi-exactly-solvable of the first type** \( T^{(I)}_k(x, y) \), if it preserves the space \( P^{(I)}_n \). Correspondingly, the operator \( E^{(I)}_k(x, y) \in T^{(I)}_k(x, y) \), which preserves the infinite flag \( P^{(I)}_0 \subset P^{(I)}_1 \subset P^{(I)}_2 \subset \ldots \subset P^{(I)}_n \subset \ldots \) of spaces of all polynomials of the type (3), is named an **exactly-solvable of the first type**.

**LEMMA 1.1** Take the space \( P^{(I)}_n \).

(i) Suppose \( n > (k - 1) \). Any quasi-exactly-solvable operator of the first type of the \( k \)-th order \( T^{(I)}_k(x, y) \), can be represented by a \( k \)-th degree polynomial of the generators (2). If \( n \leq (k - 1) \), the part of the quasi-exactly-solvable operator \( T^{(I)}_k(x, y) \) of the first type containing derivatives in \( x, y \) up to the order \( n \) can be represented by a \( n \)-th degree polynomial in the generators (2).

(ii) Inversely, any polynomial in (2) is a quasi-exactly solvable operator of the first type.

(iii) Among quasi-exactly-solvable operators of the first type there exist exactly-solvable operators of the first type \( E^{(I)}_k(x, y) \subset T^{(I)}_k(x, y) \).

**Comment 1.** If we define the universal enveloping algebra \( U_g \) of a Lie algebra \( g \) as the algebra of all polynomials in generators, then the meaning of the Lemma is the following: \( T^{(I)}_k(x, y) \) at \( k < n + 1 \) is simply an element of the universal enveloping algebra \( U_{sl_3(R)} \) of the algebra \( sl_3(R) \) in representation (2). If \( k \geq n + 1 \), then \( T_k(x, y) \) is represented as a polynomial of the \( n \)-th degree in (2) plus \( B \frac{\partial^{n+1}}{\partial x^{m+1} \partial y^m} \), where \( m = 0, 1, \ldots, (n + 1) \) and \( B \) is any linear differential operator of the order not higher than \( (k - n - 1) \).

**Proof.** The proof is based on the proof of irreducibility (3) and then on the application of Burnside theorem. 

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3I appreciate to J.Frohlich and M.Shubin for suggestion this way of the proof, which is easy and natural unlike the technical proof originally given at [2]. Similarly, one can prove the analogous lemmas for the case of linear differential operators in one variable, (see Lemma 1 in [2]), finite-difference operators in one variable (Lemma 4 in [2]) and differential operators in one real and one Grassmann variable (Lemma 1 in [3]).
Let us introduce the grading of the generators (2) in the following way. The generators are characterized by the two-dimensional grading vectors

\[ \text{deg}(J_1^3) = (+1, 0), \quad \text{deg}(J_2^1) = (0, +1), \]
\[ \text{deg}(J_2^3) = (-1, +1), \quad \text{deg}(J_3^2) = (+1, -1), \]
\[ \text{deg}(J_4) = (0, 0), \quad \text{deg}(\tilde{J}_d) = (0, 0), \]
\[ \text{deg}(J_3^1) = (0, -1), \quad \text{deg}(J_1^2) = (-1, 0). \]  

(4)

Apparently, the grading vector of a monomial in the generators (2) can be defined by the grading vectors of the generators by the rule

\[ \text{deg}T \equiv \text{deg}\left[(J_1^3)^{n_{13}}(J_2^1)^{n_{12}}(J_3^2)^{n_{23}}(J_d)^{n_d}(\tilde{J}_d)^{n_{31}}(J_1^2)^{n_{32}}(J_2^1)^{n_{31}}(J_1^2)^{n_{32}}\right] = \]
\[ (n_{13} + n_{32} - n_{23} - n_{31}, n_{12} + n_{23} - n_{32} - n_{21}) \equiv (\text{deg}_x T, \text{deg}_y T) \]  

(5)

Here the \(n\)'s can be arbitrary non-negative integers.

**Definition.** Let us name the grading of a monomial \(T\) in generators (2) the number

\[ \text{deg}(T) = \text{deg}_x T + \text{deg}_y T. \]

We will say, that a monomial \(T\) possesses positive grading, if this number is positive. If this number is zero, then a monomial has zero grading. The notion of grading allows one to classify the operators \(T_k(x, y)\) in a Lie-algebraic sense.

**Lemma 1.2** A quasi-exactly-solvable operator \(T_k^{(I)} \subset U_{sl(3)}(R)\) either has no terms of positive grading, iff it is an exactly-solvable operator.

It is worth noting that among exactly-solvable operators there exists a certain important class of degenerate operators, which preserve an infinite flag of spaces of all homogeneous polynomials

\[ \hat{P}_n^{(I)} = \langle x^n, x^{n-1}y, \ldots, xy^{n-1}, y^n \rangle \]  

(6)

(represented by a horizontal line in Fig.1).

**Lemma 1.3** Linear differential operator \(T_k(x, y)\) preserves the infinite flag \(\hat{P}_0^{(I)} \subset \hat{P}_1^{(I)} \subset \hat{P}_2^{(I)} \subset \ldots \subset \hat{P}_n^{(I)} \subset \ldots\) of spaces of all polynomials of the type (6), iff it is an exactly-solvable operator having terms of zero grading.
only. Any operators of such a type can be represented as a polynomial in the generators $J_3^2, J_2^3, J_d, \tilde{J}_d$ (see (2)), which form the algebra $so_3 \oplus R$. If such an operator contains only terms with zero grading vectors, this operator preserves any space of polynomials.

**THEOREM 1.1** Let $n$ be non-negative integer. In general, the eigenvalue problem for a linear symmetric differential operator in two real variables $T_k(x,y)$:

$$T_k(x,y)\varphi(x,y) = \varepsilon \varphi(x,y)$$ (7)

has $(n + 1)(n/2 + 1)$ eigenfunctions in the form of a polynomial in variables $x, y$ belonging to the space (3), iff $T_k(x,y)$ is a symmetric quasi-exactly-solvable operator of the first type. The problem (7) has an infinite sequence of eigenfunctions in the form of polynomials of the form (3), iff the operator is a symmetric exactly-solvable operator.

This theorem gives a general classification of differential equations

$$\sum_{m=0}^{k} \sum_{i=0}^{m} a_{i,m-i}(x,y) \frac{\partial^m \varphi(x,y)}{\partial x^i \partial y^{m-i}} = \varepsilon \varphi(x,y)$$ (8)

having at least one eigenfunction in the form of polynomial in $x, y$ of the type (3). In general, the coefficient functions $a_{i,m-i}(x,y)$ have quite cumbersome functional structure and we do not display them here (below we will give their explicit form for $T_2(x,y)$). They are polynomials in $x, y$ of the order $(k + m)$ and always contain a general inhomogeneous polynomial of the order $m$ as a part. The explicit expressions for those polynomials are obtained by substituting (2) into a general, the $k$-th order polynomial element of the universal enveloping algebra $U_{sl_3(R)}$ of the algebra $sl_3(R)$. Thus, the coefficients in the polynomials $a_{i,m-i}(x,y)$ can be expressed through the coefficients of the $k$-th order polynomial element of the universal enveloping algebra $U_{sl_3(R)}$.

The number of free parameters of the polynomial solutions is defined by the number of parameters characterizing a general, $k$-th order polynomial element of the universal enveloping algebra $U_{sl_3(R)}$. In counting free parameters a certain ordering of generators should be fixed to avoid double counting due to commutation relations. Some relations between generators should be taken into account, specifically for the given representation (2), like

$$J_2^1 J_d - 2J_2^1 \tilde{J}_d - 3J_3^1 J_1^3 = n J_2^1$$
between quadratic expressions in generators (and the ideals generated by them). For the case of exactly-solvable problems, the coefficient functions 
\( a_{i,m-i}(x, y) \) take the form
\[
 a_{i,m-i}(x, y) = \sum_{p,q=0}^{p+q \leq m} a_{i,m-i,p,q} x^p y^q
\] (10)
with arbitrary coefficients.

Now let us proceed to the case of the second-order differential equations. The second-order polynomial in the generators (2) can be represented as such
\[
 T_2 = c_{\alpha\beta,\gamma\delta} J^\alpha_\gamma J^\beta_\delta + c_{\alpha\beta} J^\alpha_\beta + c, \quad (11)
\]
where we imply summation over all repeating indices; $\alpha, \beta, \gamma, \delta$ correspond to the indices of operators in (2) and for the Cartan generators we suppose both indices simulate $d$ or $\tilde{d}$, all $c$’s are set to be real numbers. Taking (9) into account, it is easy to show, that $T_2$ is characterized by 36 free parameters. Substituting (2) into (11), we obtain the explicit form of the second-order-quasi-exactly-solvable operator

$$ T_2^{(I)}(x, y) = $$

$$ [x^2 P_{2,2}^{xx}(x, y) + x P_{2,1}^{xx}(x, y) + \tilde{P}_{2,0}^{xx}(x, y)] \frac{\partial^2}{\partial x^2} + $$

$$ [xy P_{2,2}^{xy}(x, y) + P_{3,1}^{xy}(x, y) + \tilde{P}_{2,0}^{xy}(x, y)] \frac{\partial^2}{\partial x \partial y} + $$

$$ [y^2 P_{2,2}^{yy}(x, y) + y P_{2,1}^{yy}(x, y) + \tilde{P}_{2,0}^{yy}(x, y)] \frac{\partial^2}{\partial y^2} + $$

$$ [x P_{2,2}^{x}(x, y) + P_{2,1}^{x}(x, y) + \tilde{P}_{1,0}^{x}(x, y)] \frac{\partial}{\partial x} + $$

$$ [y P_{2,2}^{y}(x, y) + P_{2,1}^{y}(x, y) + \tilde{P}_{1,0}^{y}(x, y)] \frac{\partial}{\partial y} + $$

$$ [P_{2,2}^{0}(x, y) + P_{1,1}^{0}(x, y) + \tilde{P}_{0,0}^{0}(x, y)] $$ (12)

where $P_{k,m}^{c}(x, y)$ and $\tilde{P}_{k,m}^{c}(x, y)$ are homogeneous and inhomogeneous polynomials of the order $k$, respectively, the index $m$ numerates them, the superscript $c$ characterizes the order of derivative, which this coefficient function corresponds to. For the case of the second-order-exactly-solvable operator $E_2^{(I)}(x, y)$, the structure of coefficient function is similar to (12), except for
the fact that all tildeless polynomials disappear. If we denote the number of free parameter of the operator $T$ with the symbol $\text{par}(T)$, then it is easy to show that

$$
\text{par}(T_2^{(I)}(x, y)) = 36 .
$$

while for the case of an exactly-solvable operator (an infinite sequence of polynomial eigenfunctions in (8))

$$
\text{par}(E_2^{(I)}(x, y)) = 25 .
$$

An important particular case is when the quasi-exactly-solvable operator $T_2^{(I)}(x, y)$ possesses two invariant sub-spaces of the type (3). This situation is described by the following lemma:

**LEMMA 1.4** Suppose in (11) there are no terms of grading 2:

$$
c_{12,12} = c_{13,13} = c_{12,13} = 0
$$

and if there exists some coefficients $c$’s and a non-negative integer $N$ such that the conditions

$$
c_{12} = (n - N - m)c_{12,d} + (n - 2N + m)c_{12,d} + (N - m)c_{12,32} ,
$$

$$
c_{13} = (n - N - m)c_{13,d} + (n - 2N + m)c_{13,d} + mc_{13,23} ,
$$

are fulfilled at all $m = 0, 1, 2, \ldots, N$, then the operator $T_2^{(I)}(x, y)$ preserves both $\mathcal{P}_n$ and $\mathcal{P}_N$, and $\text{par}(T_2^{(I)}(x, y)) = 31$.

Now let us proceed to the important item: under what conditions on the coefficients in (11) the second-order-quasi-exactly-solvable operators can be

\footnote{Recall, that for the case of the second-order-quasi-exactly-solvable differential operator in one real variable, the number of free parameters was equal to 9 (see [3]), for the case of one real and one Grassmann variables this number was 25 (see [3]).}

\footnote{Recall, that for the case of the second-order-exactly-solvable differential operator in one real variable, the number of free parameters was equal to 6 (see [3]), for the case of one real and one Grassmann variables this number is 17 (see [3]).}
reduced to a form of the Schroedinger operator after some gauge transformation
\[ f(x, y)e^{t(x,y)}T_2(x, y)e^{-t(x,y)} = -\Delta_g + V(x, y) \]  \( \text{(15)} \)
where \( f, t, V \) are some functions in \( \mathbb{R}^2 \), and \( \Delta_g \) is the Laplace-Beltrami operator with some metric tensor \( g_{\mu\nu}; \mu, \nu = 1, 2 \). This is a difficult problem for which there is yet no complete solution. In Refs. [4, 6] a few multi-parametric examples were constructed. However, there is a quite general situation, for which a rather wide class of the solutions can be obtained.

The algebra \( sl_3(\mathbb{R}) \) in realization (2) contains the algebra \( so_3(\mathbb{R}) \) as a sub-algebra
\[
J^1 = (1 + y^2)\partial_y + xy\partial_x - ny, \quad J^2 = (1 + x^2)\partial_x + xy\partial_y - nx, \\
J^3 = x\partial_y - y\partial_x
\] \( \text{(16)} \)
If the parameter \( n \) is a non-negative integer (and coincides with that in (2)), the same finite-dimensional invariant sub-space \( P_n^1 \) (see (3) and Fig.1) as in the original \( sl_3(\mathbb{R}) \) occurs. For this case a corresponding finite-dimensional representation (3) is reducible and unitary. It has been proven [4, 7], that any symmetric bilinear combination of generators (16), \( T_2 = c_{\alpha\beta}J^\alpha J^\beta, \quad c_{\alpha\beta} = c_{\beta\alpha} \), can be reduced to a form of the Laplace-Beltrami operator plus a scalar function. In general, the metric tensor \( g_{\mu\nu} \) is not degenerate \( ^7 \). Generically, the potential \( V(x, y) \) is given by a rational function and has no dependence on the spin \( n \) (see (16)). However, if the matrix \( c_{\alpha\beta} \) has one vanishing eigenvalue, a certain mysterious relation appears [4, 8]
\[
V(x, y, \{c\}) = \frac{3}{16}R(x, y, \{c\}), \quad \text{(17)}
\]
where \( R(x, y, \{c\}) \) is the scalar curvature calculated through the metric tensor attached in the Laplace-Beltrami operator. This can imply that the corresponding Schroedinger operator has a purely geometrical nature! The real meaning of this fact is still not understood.

So, we arrive to the two-dimensional exactly-solvable Schroedinger equations. As a consequence of the quadratic Casimir operator for \( so_3(\mathbb{R}) \) in

\[ ^6 \text{Our further consideration will be restricted the case } f = 1 \text{ only} \]

\[ ^7 \text{Degeneracy of } g_{\mu\nu} \text{ occurs, for example, if } c_{\alpha\beta} \text{ has two vanishing eigenvalues} \]
the form (16) being non-trivial and commuting with $T_2$, the functional space of $T_2$ is subdivided into the finite-dimensional blocks corresponding to the irreducible reps of $so_3(\mathbb{R})$.

### 1.2 Polynomials of the second type

In Ref.[2] we studied the quasi-exactly-solvable operators in one real variable. It turned out that the solution to this problem was found using a connection with the projectivized representation of the algebra $sl_2(\mathbb{R})$. As a natural step in developing the original idea, let us consider the projectivized representation of the direct sum of two algebras $sl_2(\mathbb{R})$.

The algebra $sl_2(\mathbb{R}) \oplus sl_2(\mathbb{R})$ taken in projectivized representation acts on functions of two real variables. The generators have the form (see e.g.[4])

\[
J^+_x = x^2 \partial_x - nx, \quad J^+_y = y^2 \partial_y - my \\
J^0_x = x \partial_x - \frac{n}{2}, \quad J^0_y = y \partial_y - \frac{m}{2} \\
J^-_x = \partial_x, \quad J^-_y = \partial_y
\]

where $x, y$ are the real variables and $n, m$ are non-negative integers. There exists the finite-dimensional representation of dimension $(n+1)(m+1)$. Evidently, the invariant sub-space has a polynomial basis and is presented as a space of all polynomials with the Newton diagram shown in Fig.2. We denote this space as $\mathcal{P}^{(II)}_{n,m}$.

![Newton Diagram](image_url)

**Fig. 2.** Graphical representation (Newton diagram) of the space $\mathcal{P}^{(II)}_{n,m}$.
**Definition.** Let us name a linear differential operator of the $k, p$-th order, containing derivatives in $x$ and $y$ up to $k$-th and $p$-th orders, respectively, a quasi-exactly-solvable of the second type, $T_{k,p}^{(II)}(x, y)$, if it preserves the space $P_{n,m}^{(II)}$. Correspondingly, the operator $E_{k,p}^{(II)}(x, y) \in T_{k,p}^{(II)}(x, y)$, which preserves either the infinite flag $P_{0,m}^{(II)} \subset P_{1,m}^{(II)} \subset P_{2,m}^{(II)} \subset \ldots \subset P_{n,m}^{(II)} \subset \ldots$, or the infinite flag $P_{n,0}^{(II)} \subset P_{n,1}^{(II)} \subset P_{n,2}^{(II)} \subset \ldots \subset P_{n,m}^{(II)} \subset \ldots$ of spaces of all polynomials, is named an exactly-solvable of the type $2_x$ or $2_y$, respectively.

**Lemma 2.1** Take the space $P_{n,m}^{(II)}$.

(i) Suppose $n > (k - 1)$ and $m > (p - 1)$. Any quasi-exactly-solvable operator of the second type $(k, p)$-th order $T_{k,p}^{(II)}(x, y)$, can be represented by a $(k, p)$-th degree polynomial of the generators (18). If $n \leq (k - 1)$ and/or $m \leq (p - 1)$, the part of the quasi-exactly-solvable operator $T_{k,p}^{(II)}(x, y)$ of the second type containing derivatives in $x, y$ up to the order $n, m$, respectively, can be represented by a $(n, m)$-th degree polynomial in the generators (18).

(ii) Inversely, any polynomial in (18) is a quasi-exactly solvable operator of the second type.

(iii) Among quasi-exactly-solvable operators of the second type there exist exactly-solvable operators of the second type $E_{k,p}^{(II)}(x, y) \subset T_{k,p}^{(II)}(x, y)$.

The proof is analogous to the proof of Lemma 1.1 and is based on the irreducibility of $P_{n,m}^{(II)}$ and Burnside theorem.

Similarly, as has been done for the algebra $sl_3(\mathbb{R})$, one can introduce the notion of grading:

\[
\deg(J_x^+)= (+1, 0) , \quad \deg(J_y^+) = (0, +1) ,
\]
\[
\deg(J_x^0) = (0, 0) , \quad \deg(J_y^0) = (0, 0) ,
\]
\[
\deg(J_x^-) = (-1, 0) , \quad \deg(J_y^-) = (0, -1) .
\]

(19)

So a leading derivative has a form $\frac{\partial^{(k+p)}}{\partial x^k \partial y^p}$. Also we will use a notation through this section $T_N(x, y)$ implying that in general all derivatives of the order $N$ are presented.
Apparently, the grading vector of a monomial in the generators (18) can be
defined by the grading vectors (19) of the generators by the rule
\[
\tilde{\deg} T \equiv \deg[(J^+_x)^{n_{x+}}(J^0_x)^{n_{x0}}(J^-_x)^{n_{x-}}(J^+_y)^{n_{y+}}(J^0_y)^{n_{y0}}(J^-_y)^{n_{y-}}] = \\
(n_{x+} - n_{x-}, n_{y+} - n_{y-}) \equiv (\deg_x(T), \deg_y(T)) 
\]
Here the \(n\)'s can be arbitrary non-negative integers.

**Definition.** Let us name the \(x\)-grading, \(y\)-grading and grading of a
monomial \(T\) in generators (18) the numbers \(\deg_x(T), \deg_y(T)\) and \(\deg(T) = \deg_x(T) + \deg_y(T)\), respectively. We say that a monomial \(T\) possesses positive
\(x\)-grading \((y\)-grading, grading\), if the number \(\deg_x(T), \deg_y(T), \deg(T)\) is positive. If \(\deg_x(T)(\deg_y(T), \deg(T)) = 0\), then a monomial has zero
\(x\)-grading \((y\)-grading, grading\).

The notion of grading allows one to classify the operators \(T^{(II)}_{k,p}(x, y)\) in a
Lie-algebraic sense.

**Lemma 2.2** The quasi-exactly-solvable operator \(T^{(II)}_{k,p}(x, y)\) of the second
type preserves the infinite flag \(\mathcal{P}^{(II)}_{0,0} \subset \mathcal{P}^{(II)}_{1,1} \subset \mathcal{P}^{(II)}_{2,2} \subset \ldots \subset \mathcal{P}^{(II)}_{n,m} \subset \ldots \) of spaces of all the polynomials, iff it is an exactly-solvable operator of the
type \(2_x\) having no terms of positive \(x\)-grading, \(\deg_x > 0\).

The quasi-exactly-solvable operator \(T^{(II)}_{k,p}(x, y)\) preserves the infinite flag \(\mathcal{P}^{(I)}_{n,0} \subset \mathcal{P}^{(I)}_{n,1} \subset \mathcal{P}^{(I)}_{n,2} \subset \ldots \subset \mathcal{P}^{(I)}_{n,m} \subset \ldots \) of all spaces of the polynomials, iff
it is an exactly-solvable operator of the type \(2_y\) having no terms of positive \(y\)-grading, \(\deg_y > 0\).

If a quasi-exactly-solvable operator of the second type contains no terms
of positive grading, this operator preserves the infinite flag \(\mathcal{P}^{(I)}_0 \subset \mathcal{P}^{(I)}_1 \subset \mathcal{P}^{(I)}_2 \subset \ldots \subset \mathcal{P}^{(I)}_n \subset \ldots \) of spaces of all the polynomials of the type (3) and is
attached to the exactly-solvable operator of the first type.

**Theorem 2.1** Let \(n,m\) be non-negative integers. In general, the
eigenvalue problem (7) for a linear symmetric differential operator in two
real variables \(T_{k,p}(x, y)\) has \((n + 1)(m + 1)\) eigenfunctions in the form of a
polynomial in variables \(x, y\) belonging to the space \(\mathcal{P}^{(I)}_{n,m}\), iff \(T_{k,p}(x, y)\) is a
quasi-exactly-solvable symmetric operator of the second type. The problem
(7) has an infinite sequence of eigenfunctions in the form of polynomials
belonging the space $\mathcal{P}_{n,m}^{(II)}$ at fixed $m$ ($n$), iff the operator is an exactly-solvable symmetric operator of the type $2_x (2_y)$.

This theorem gives a general classification of differential equations (8), having at least one eigenfunction in the form of a polynomial in $x, y$ of the type $\mathcal{P}_{n,m}^{(II)}$. In general, the coefficient functions $a_{i,m-i}^{(m)}(x, y)$ in (8) have a quite cumbersome functional structure and we do not display them here (below we will give their explicit form for $T_2^{(II)}(x, y)$). They are polynomials in $x, y$ of the order $(k+m)$. The explicit expressions for those polynomials are obtained by substituting (18) into a general, the $k$-th order polynomial element of the universal enveloping algebra $\mathcal{U}_{sl_2(R)\oplus sl_2(R)}$ of the algebra $sl_2(R) \oplus sl_2(R)$ . Thus, the coefficients in the polynomials $a_{i,m-i}^{(m)}(x, y)$ can be expressed through the coefficients of the $k$-th order polynomial element of the universal enveloping algebra $\mathcal{U}_{sl_2(R)\oplus sl_2(R)}$. The number of free parameters of the polynomial solutions is defined by the number of parameters characterizing a general, $k$-th order, polynomial element of the universal enveloping algebra $\mathcal{U}_{sl_2(R)\oplus sl_2(R)}$. In counting free parameters a certain ordering of generators should be fixed to avoid double counting due to commutation relations. Also some relations between generators should be taken into account, specifically for the given representation (18), like $[\begin{array}{lllll} 13 \\ 9 \end{array}]$

\begin{align*}
J_x^+ J_x^- - J_x^0 J_x^0 &= \left( \frac{n}{2} + 1 \right) J_x^0 = -\frac{n}{2} (\frac{n}{2} - 1) \\
J_y^+ J_y^- - J_y^0 J_y^0 &= \left( \frac{m}{2} + 1 \right) J_y^0 = -\frac{m}{2} (\frac{m}{2} - 1)
\end{align*}

(21)

between quadratic expressions in generators (and the ideals generated by them) $[\begin{array}{lllll} 14 \end{array}]$

Now let us proceed to the case of the second-order differential equations. The second-order polynomial in the generators (18) can be represented as such

\begin{align*}
T_2 &= c_{x\alpha}^{x\beta} J_x^\alpha J_x^\beta + c_{y\alpha}^{x\beta} J_y^\alpha J_x^\beta + c_{y\alpha}^{x\beta} J_x^\alpha J_y^\beta + c_{x\alpha}^{y\beta} J_x^\alpha + c_{y\beta}^{y\alpha} J_y^\alpha + c
\end{align*}

where we imply summation over all repeating indices and $\alpha, \beta = \pm, 0$; and all $c$'s are set to be real numbers. Taking (21) in account, it is easy to show that

\footnote{For this case they correspond to quadratic Casimir operators}
\( T_2 \) is characterized by 26 free parameters. Substituting (18) into (22), we obtain the explicit form of the second-order-quasi-exactly-solvable operator

\[
T_{2(II)}(x, y) =
\tilde{P}_{4,0}^{xx}(x) \frac{\partial^2}{\partial x^2} + [x^2 y^2 \tilde{P}_{0,2}^{xy} + x y \tilde{P}_{1,2}^{xy}(x, y) + \tilde{P}_{2,0}^{xy}(x, y)] \frac{\partial^2}{\partial x \partial y} + \tilde{P}_{4,0}^{yy}(y) \frac{\partial^2}{\partial y^2} +

[\tilde{P}_{3,1}^x(x) + y \tilde{P}_{2,0}^x(x)] \frac{\partial}{\partial x} + [\tilde{P}_{3,1}^y(y) + x \tilde{P}_{2,0}^y(y)] \frac{\partial}{\partial y} + \tilde{P}_{2,0}^0(x, y)
\]

where \( P_{k,m}^c(x, y) \) and \( \tilde{P}_{k,m}^c(x, y) \) are homogeneous and inhomogeneous polynomials of the order \( k \), respectively, the index \( m \) numerates them, and the superscript \( c \) characterizes the order of the derivative corresponding to this coefficient function. For the case of the second-order-exactly-solvable operator of \( 2_x \) type

\[
E_{2(II)}(x, y) = 
\tilde{Q}_{2,0}^{xx}(x) \frac{\partial^2}{\partial x^2} + [x \tilde{Q}_{2,1}^{xy}(y) + \tilde{Q}_{2,0}^{xy}(y)] \frac{\partial^2}{\partial x \partial y} + \tilde{Q}_{4,0}^{yy}(y) \frac{\partial^2}{\partial y^2} +

\tilde{Q}_{1,1}^x(x) + y \tilde{Q}_{1,0}^x(x)] \frac{\partial}{\partial x} + \tilde{Q}_{1,1}^y(y) + x \tilde{Q}_{1,0}^y(y)] \frac{\partial}{\partial y} + \tilde{Q}_{2,0}^0(y)
\]

For the case of the second-order-exactly-solvable operator of \( 2_y \) type the functional form is similar to (24) with the interchange \( x \leftrightarrow y \).

It is easy to show that the number of free parameters are \( 10 \)

\[
\text{par}(T_{2(II)}(x, y)) = 26 .
\]

\footnote{Recall, that for the case of the second-order-quasi-exactly-solvable differential operator in one real variable, the number of free parameters was equal to 9 (see \textsuperscript{2}), for the case of one real and one Grassmann variables this number was 25 (see \textsuperscript{3}). Also, for the case of the second-order-exactly-solvable differential operator in one real variable, the number of free parameter was equal to 6 (see \textsuperscript{2}), for the case of one real and one Grassmann variables this number is 17 (see \textsuperscript{3}) (see also p.9 for comparison to the case of the first type polynomials).}
\[ \text{par}(E_{2}^{II}(x, y)) = 20. \]

Similar to the previous cases, there is a very important particular case of quasi-exactly-solvable operators of the second order, where they possess two invariant sub-spaces.

**Lemma 2.3** Suppose in (22) there are no terms of \( x \)-grading 2:
\[ c_{xx}^{xx} = 0 \]  
and if there exists some coefficients \( c \)'s and a non-negative integer \( N \) such that the conditions
\[ c_{xx}^{xy} = 0, \]
\[ c_{xy}^{xx} = 0, \]
\[ c_{xx}^{x} = (n/2 - N)c_{xx}^{xx} + (m/2 - k)c_{xx}^{xy}, \]
are fulfilled at all \( k = 0, 1, 2, \ldots, m \), then the operator \( T_{2}^{II}(x, y) \) preserves both \( P_{n,m}^{II} \) and \( P_{N,m}^{II} \), besides that \( \text{par}(T_{2}^{II}(x, y)) = 22 \).

Generically, the question of the reduction of quasi-exactly-solvable operator of the second type \( T_{2}^{II}(x, y) \) to the form of the Schroedinger operator is still open. In the papers [4, 6] several multi-parametrical families of those Schroedinger operators were constructed. As in the case of the first type quasi-exactly-solvable operators, corresponding Schroedinger operators contain in general the non-trivial Laplace-Beltrami operator.

The above analysis of linear differential operators preserving the space \( P_{n,m}^{II} \) can be naturally extended to the case of linear finite-difference operators defined through the Jackson symbol \( D \):
\[ Df(x) = \frac{f(x) - f(qx)}{(1 - q)x} + f(qx)D \]
where \( f(x) \) is a real function and \( q \) is a number, instead of continuous derivative. All above-described results hold \[\] with replacement of the algebra \( sl_{2}(\mathbb{R}) \oplus sl_{2}(\mathbb{R}) \) to the quantum algebra \( sl_{2}(\mathbb{R})_{q} \oplus sl_{2}(\mathbb{R})_{q} \) in ‘projectivized’ representation (see[1, 2]). In the limit \( q \to 1 \) all results, which can be obtained, coincide to the results of present Section.

\[11 \] with minor modifications
1.3 Polynomials of the third type

The third case, which we are going to discuss here, corresponds to the Lie algebra \( \mathfrak{gl}_2(\mathbb{R}) \ltimes \mathbb{R}^{r+1} \) (semidirect sum of \( \mathfrak{gl}_2(\mathbb{R}) \) with a \((r+1)\)-dimensional abelian ideal; the Case 24 in the classification given the paper [9]). This family of the Lie algebras, depending on an integer \( r > 0 \), can be realized in terms of the first order differential operators

\[
\begin{align*}
J^1 &= \partial_x, \\
J^2 &= x\partial_x - \frac{n}{3}, \\
J^3 &= y\partial_y - \frac{n}{3r}, \\
J^4 &= x^2\partial_x + rx y\partial_y - nx, \\
J^{5+i} &= x^i\partial_y, \quad i = 0, 1, \ldots, r,
\end{align*}
\]

where \( x, y \) are the real variables and \( n \) is a real number. If \( n \) is a non-negative integer, the representation becomes finite-dimensional. The invariant sub-space has a polynomial basis and is presented as a space of all polynomials of the form

\[
P^{(III)}_{r,n} = \sum_{i,j \geq 0} a_{ij} x^i y^j
\]

or, graphically, (28) is given by the Newton diagram of the figure 3. The general formulas for the dimension of corresponding finite-dimensional representation (28) is given by

\[
dim P^{(III)}_{r,n} = \frac{[n^2 + (r + 2)n + \alpha_{r,n}]}{2r}
\]

where for small \( r \)

\[
\begin{align*}
\alpha_{1,n} &= 2, \\
\alpha_{2,n} &= \begin{cases} 3 & \text{at odd } n \\
4 & \text{at even } n \end{cases} \\
\alpha_{3,n} &= \begin{cases} 4 & \text{at } (n+1) \text{ multiple } 3 \\
6 & \text{at other } n
\end{cases}
\end{align*}
\]

\footnote{It is worth noting that at \( r = 1 \) the algebra \( \{gl_2(\mathbb{R}) \ltimes \mathbb{R}^2 \} \subset sl_3(\mathbb{R}) \). Thus, this case is reduced to one about the first type polynomials (see Ch.1.1). Hereafter, we include \( r = 1 \) into consideration just for the purpose of completeness.}
\[ \alpha_{4,n} = \begin{cases} 5 & \text{at (n+1) multiple 4} \\ 8 & \text{at other n} \\ 9 & \text{at (n+3) multiple 4} \end{cases} \]

Actually, the numbers \( \alpha_{r,n} \) are given by all possible products \( m(2r - m) \) at \( m = 1, 2, \ldots, r \).

Fig. 3. Graphical representation (Newton diagram) of the space \( \mathcal{P}^{(III)}_{r,n} \) (see (28)).

**Definition.** Let us name a linear differential operator of the \( k \)-th order a **quasi-exactly-solvable of the \( r \)-third type**, \( T^{(r,III)}_{k}(x, y) \), if it preserves the space \( \mathcal{P}^{(III)}_{r,n} \). Correspondingly, the operator \( E^{(r,III)}_{k}(x, y) \in T^{(r,III)}_{k}(x, y) \), which preserves the infinite flag \( \mathcal{P}^{(III)}_{r,0} \subset \mathcal{P}^{(III)}_{r,1} \subset \mathcal{P}^{(III)}_{r,2} \subset \ldots \subset \mathcal{P}^{(III)}_{r,n} \subset \ldots \) of spaces of all polynomials of the type (28), is named an **exactly-solvable of the \( r \)-third type**.

**LEMMA 3.1** Take the space \( \mathcal{P}^{(III)}_{r,n} \).

(i) Suppose \( n > (k-1) \). Any quasi-exactly-solvable operator of the \( r \)-third type of the \( k \)-th order \( T^{(r,III)}_{k}(x, y) \), can be represented by a \( k \)-th degree polynomial of the generators (27). If \( n \leq (k-1) \), the part of the quasi-exactly-solvable operator \( T^{(r,III)}_{k}(x, y) \) of the \( r \)-third type containing derivatives in \( x, y \) up to the order \( n \) can be represented by a \( n \)-th degree polynomial in the generators (27).

(ii) Inversely, any polynomial in (27) is a quasi-exactly solvable operator of the \( r \)-third type.
(iii) Among quasi-exactly-solvable operators of the $r-$third type there exist exactly-solvable operators of the $r-$third type $E_k^{(r,III)}(x, y) \subset T_k^{(r,III)}(x, y)$.

The proof is analogous to the proof of the Lemma 1.1 and 2.1. It is based on irreducibility $P_{r,n}^{(III)}$ and Burnside theorem.

One can introduce the grading of the generators (27) in analogous way as has been done before for the cases of the algebra $sl_3(\mathbb{R})$ (see (4)) and $sl_2(\mathbb{R}) \oplus sl_2(\mathbb{R})$ (see (19)). All generators are characterized by the two-dimensional grading vectors

$$deg(J^1) = (-1, 0),$$

$$deg(J^2) = (0, 0), \quad deg(J^3) = (0, 0),$$

$$deg(J^4) = (1, 0),$$

$$deg(J^5) = (0, -1), \quad deg(J^6) = (1, -1), \ldots, \quad deg(J^{5+r}) = (r, -1) \quad (30)$$

Similarly as before, the grading vector of a monomial in the generators (27) can be defined through the grading vectors of the generators (27) (cf.(5),(20)).

**Definition.** Let us name the grading of a monomial $T$ in generators (27) the number

$$deg(T) = deg_x T + r deg_y T$$

(cf. the case of $sl_3(\mathbb{R})$). We will say that a monomial $T$ possesses positive grading if this number is positive. If this number is zero, then a monomial has zero grading. The notion of grading allows one to classify the operators $T_k(x, y)$ in a Lie-algebraic sense.

**Lemma 3.2** A quasi-exactly-solvable operator $T_k^{(r,III)} \subset U_{gl_2(\mathbb{R})} \otimes \mathbb{R}^{r+1}$ either has no terms of positive grading, iff it is an exactly-solvable operator.

**Theorem 3.1** Let $n$ and $(r - 1)$ be non-negative integers. In general, the eigenvalue problem (7) for a linear symmetric differential operator of the $k-$th order in two real variables $T_k(x, y)$ has a certain amount of eigenfunctions in the form of a polynomial in variables $x, y$ belonging to the space (28), iff $T_k(x, y)$ is a quasi-exactly-solvable, symmetric operator of the $r-$third type. The problem (7) has an infinite sequence of eigenfunctions in the form of polynomials belonging to (28), iff the operator is an exactly-solvable, symmetric operator of the $r-$third type.
This theorem gives a general classification of differential equations (8), having at least one eigenfunction in the form of a polynomial in $x, y$ of the type $P_{r,n}^{(III)}$. In general, the coefficient functions $a_{i,m-i}^{(m)}(x,y)$ in (8) are polynomials in $x, y$ and have a quite cumbersome functional structure and we do not display them here (below we will give their explicit form for $T_2^{(r,III)}(x,y)$). The explicit expressions for those polynomials are obtained by substituting (27) into a general, $k$-th order polynomial element of the universal enveloping algebra $U_{gl_2}(R) \prec R^{r+1}$ of the algebra $gl_2(R) \triangleright R^{r+1}$. Thus, the coefficients in the polynomials $a_{i,m-i}^{(m)}(x,y)$ can be expressed through the coefficients of the $k$-th order polynomial element of the universal enveloping algebra $U_{gl_2}(R) \prec R^{r+1}$. The number of free parameters of the polynomial solutions is defined by the number of parameters characterizing general $k$-th order polynomial element of the universal enveloping algebra. In counting free parameters a certain ordering of generators should be fixed to avoid double counting due to commutation relations. Also some relations between generators should be taken into account, specifically for a given representation (27), like

$$J^2 J^5 - J^1 J^6 + \frac{n}{3} J^5 = 0,$$

$$J^1 J^4 - J^2 J^2 - r J^2 J^3 - J^2 - r \left(\frac{n}{3} + 1\right) J^3 = -\frac{n}{3} \left(\frac{n}{3} + 1\right), \quad (31)$$

$$J^2 J^{6+i} + r J^3 J^{6+i} - J^4 J^{5+i} - \left(\frac{n}{3} + 1\right) J^{6+i} = 0,$$

at $i = 0, 1, 2, \ldots, (r - 1)$, \hspace{1cm} (32)

$$J^1 J^{7+i} - J^2 J^{6+i} - \left(\frac{n}{3} + 1\right) J^{6+i} = 0,$$

at $i = 0, 1, 2, \ldots, (r - 2)$, \hspace{1cm} (33)

$$J^5 J^{7+i} = \ldots = J^{5+k} J^{7+i-k},$$

at $k = 0, 1, 2, \ldots$, and $2k \leq (2 + i)$ and $i = 0, 1, 2, \ldots, (2r - 2)$ \hspace{1cm} (34)

between quadratic expressions in generators (and the ideals generated by them).
Now let us proceed to the case of the second-order differential equations.
The second-order polynomial in the generators (27) can be represented as

\[ T_2 = c_{\alpha\beta}J^\alpha J^\beta + c_{\alpha}J^\alpha + c \]  

where \( \alpha, \beta = 1, 2, \ldots, (5 + r) \) and we imply summation over all repeating indices; all \( c \)’s are set to be real numbers. Taking the relations (31)-(34) in account, one can count the number of free parameters and obtain

\[ \text{par}(T_2^{(r,III)}(x, y)) = 5(r + 4) \]  

(36)

Substituting (27) into (35), we obtain the explicit form of the second-order-quasi-exactly-solvable operator

\[ T_2^{(r,III)}(x, y) = \]

\[ \tilde{P}_{4,0}^x(x) \frac{\partial^2}{\partial x^2} + [\tilde{P}_{r+1,1}^x(x) + y\tilde{P}_{3,0}^{xy}(x)]\frac{\partial^2}{\partial x \partial y} + [\tilde{P}_{2,1}^y(x) + y\tilde{P}_{r+1,0}^{yy}(x)]\frac{\partial^2}{\partial y^2} + \\
[\tilde{P}_{3,0}^x(x)]\frac{\partial}{\partial x} + [\tilde{P}_{r+1,1}^y(x) + y\tilde{P}_{1,0}^{xy}(x)]\frac{\partial}{\partial y} + \tilde{P}_{2,0}^0(x) \]  

(37)

where \( \tilde{P}_{k,m}^c(x, y) \) are inhomogeneous polynomials of the order \( k \), the index \( m \) numerates them, and the superscript \( c \) characterizes the order of the derivative corresponding to this coefficient function. For the case of the second-order-exactly-solvable operator of \( r \)-third type

\[ E_2^{(r,III)}(x, y) = \]

\[ \tilde{P}_{2,0}^x(x) \frac{\partial^2}{\partial x^2} + [\tilde{P}_{r+1,1}^y(x) + y\tilde{P}_{1,0}^{xy}(x)]\frac{\partial^2}{\partial x \partial y} + [\tilde{P}_{2,1}^{yy}(x) + y\tilde{P}_{0,0}^{yy}(x)]\frac{\partial^2}{\partial y^2} + \\
[\tilde{P}_{1,0}^x(x)]\frac{\partial}{\partial x} + [\tilde{P}_{r+1,1}^y(x) + y\tilde{P}_{0,0}^{xy}(x)]\frac{\partial}{\partial y} + \tilde{P}_{0,0}^0(x) \]  

(38)

In this case the number of free parameters is equal to

\[ \text{par}(E_2^{(r,III)}(x, y)) = 5(r + 3) \]  

(39)

(cf.(36)).
Similar to the previous cases, there is an important particular case of $r$-third type quasi-exactly-solvable operators of the second order, where they possess two invariant sub-spaces.

**LEMMA 3.3** Suppose in (35) there are no terms of grading 2:

$$c_{44} = 0$$  \hspace{1cm} (40)

and if there exists some coefficients $c$’s and a non-negative integer $N$ such that the conditions

$$c_{4,5+r} = 0,$$

$$c_4 = \left(\frac{N}{3} - m\right)c_{24} + \left(\frac{N}{3} - k\right)c_{34},$$  \hspace{1cm} (41)

are fulfilled at all $m, k = 0, 1, 2, \ldots$ such that $m + rk = N$, then the operator $T_2^{(r,III)}(x, y)$ preserves both $P_{r,n}^{III}$ and $P_{r,N}^{III}$, and $\text{par}(T_2^{(r,III)}(x, y)) = 5r + 17$.

Generically, the question of the reduction of quasi-exactly-solvable operator of the $r$-third type $T_2^{(r,III)}(x, y)$ to the form of the Schroedinger operator is still open. Initially \[6\] several multi-parametrical families of those Schroedinger operators were constructed. As in the case quasi-exactly-solvable operators of the first and the second type, corresponding Schroedinger operators contain in general a non-trivial Laplace-Beltrami operator.

1.4 Discussion

Through all the above analysis, very crucial requirement played a role: we considered the problem in general position, assuming the spaces of all polynomials contain polynomials with all possible coefficients. If this requirement is not fulfilled, than the above connection of the finite-dimensional space of all polynomials and the finite-dimensional representation of the Lie algebra is lost and degenerate cases occur. This demands a separate investigation.

So, we described three types of finite-dimensional spaces (see Fig.1-3) with a basis in polynomials of two real variables, which can be preserved by linear differential operators. Very natural question emerges: is it possible to find linear differential operators, which preserve the space of polynomials of finite order other than shown in Fig.1,2,3?
Conjecture 1. If a linear differential operator $T_k(x, y)$ does preserve a finite-dimensional space of all polynomials other than $P_I^n, P_{n,m}^{II}, P_{r,n}^{III}$, then this operator preserves also a certain infinite flag of finite-dimensional spaces of all polynomials.

2 General consideration

The main conclusion of the considerations of the present paper and the papers [1, 2, 3] one can formulate as the following theorem:

(Main) Theorem. Take a Lie algebra $g$, realized in differential operators of the first order, which possesses a finite-dimensional irreducible representation $P$ (a finite module of smooth functions). Any linear differential operator, having the invariant subspace $P$, which coincides to the above finite module of smooth functions, can be represented by a polynomial in generators of the algebra $g$ plus an operator annihilating $P$.

Proof. Use of the Burnside theorem.

As a probable extension of this theorem, we assume that the following conjecture hold

Conjecture 2. If a linear differential operator $T$ acting on functions in $R^k$ does possess the only finite-dimensional invariant subspace with polynomial basis, this finite-dimensional space coincides to a certain finite-dimensional representation of some Lie algebra, realized by differential operators of the first order.

The interesting question is how to describe finite modules of smooth functions in $R^k$, which can serve as invariant sub-spaces of linear operators. Evidently, if we could found some realization of a Lie algebra $g$ in differential operators, possessing an irreducible finite module of smooth functions, one can immediately consider the direct sum of several species of $g$ acting on the functions given at the corresponding direct products of spaces, where the original algebra $g$ acts in each of them. As an example, this procedure is presented in the Section 1.2: the direct sum of two $sl_2(R)$ leads to the operators acting on the functions at $R^2$ and a rectangular $P_{n,m}^{III}$ is preserved.
Taking the direct sum of \( k \) species of \( \mathfrak{sl}_2(\mathbb{R}) \) acting at \( \mathbb{R}^k \), we arrive to \( k \)-dimensional parallelopipeds as the invariant sub-space. In this case the algebra \( \mathfrak{sl}_2(\mathbb{R}) \) plays the role of a primary algebra giving rise to a multidimensional geometrical figure (Newton diagram) as the invariant sub-space of some linear differential operators. In the space \( \mathbb{R}^1 \) there is only one such "irreducible" Newton diagram — a finite interval, in other words, a space of all polynomials in \( x \) of degree not higher than a certain integer. Thus, in \( \mathbb{R}^2 \) the rectangular Newton diagram is "reducible" stemming from the product of two intervals as one-dimensional Newton diagrams. In \( \mathbb{R}^2 \) the irreducible Newton diagrams are exhausted by different triangles, connected to the algebras \( \mathfrak{sl}_3(\mathbb{R}) \) and \( \{ \mathfrak{gl}_2(\mathbb{R}) \leq \mathbb{R}^{r+1} \} \). We could not find any other Newton diagrams in \( \mathbb{R}^2 \) (see conjecture 1).

Before to present our present knowledge about Newton diagrams for the \( \mathbb{R}^k \) case, let us recall on the regular representation (in the first order differential operators) of the Lie algebra \( \mathfrak{sl}_N(\mathbb{R}) \) given on the flag manifold, which acts on the smooth functions of \( N(N-1)/2 \) real variables \( z_{i,i+q}, i = 1, 2, \ldots, (N-1), q = 1, 2, \ldots, (N - i) \). The explicit formulas for generators are given by (see [10])

\[
D(e_i) = \frac{\partial}{\partial z_{i,i+1}} - \sum_{q=i+2}^{N} z_{i+1,q} \frac{\partial}{\partial z_{i,q}}
\]

\[
D(f_i) = \sum_{q=1}^{i} z_{q,i+1} z_{i,i+1} \frac{\partial}{\partial z_{q,i+1}} + \sum_{q=1}^{i-1} (z_{q,i+1} - z_{q,i} z_{i,i+1}) \frac{\partial}{\partial z_{q,i}}
\]

\[
\sum_{q=i+2}^{N} z_{i,q} \frac{\partial}{\partial z_{i+1,q}} - n_i z_{i,i+1}
\]

\[
D(\tilde{h}_i) = - \sum_{q=i+1}^{N} z_{q,i} \frac{\partial}{\partial z_{q,i+1}} + \sum_{q=1}^{i-1} z_{q,i} \frac{\partial}{\partial z_{q,i}} - \sum_{p=1}^{i} n_{i-p}
\]  \( (42) \)

where we use a notation \( e_i, f_i, \tilde{h}_i \) for generators of positive and negative roots, and the Cartan generators, correspondingly. If \( n_i \) are non-negative integers, the finite-dimensional irreducible representation of \( \mathfrak{sl}_N(\mathbb{R}) \) will occur in the
form of inhomogeneous polynomials in variables $z_{i,j+q}$. The highest weight vector is characterized by the integer numbers $n_i$, $i = 1, 2, \ldots, (N-1)$.

Connected to the space $\mathbb{R}^3$, at least, two irreducible Newton diagrams appear: a tetrahedron, connected to a degenerate finite-dimensional representation of $sl_4(\mathbb{R})$ (highest weight vector has two vanishing integers and one non-vanishing), and a certain geometrical figure, corresponding to the regular representation of $sl_3(\mathbb{R})$ given on the flag manifold. We do not know if those exhaust all irreducible Newton diagrams or none. For the general case, $\mathbb{R}^k$ there also exist polyhedra connected to $sl_{k+1}(\mathbb{R})$ (except one, all integers, characterizing the highest weight vector are vanishing) and geometrical figures related to some finite-dimensional regular representations of $sl_{k+1-i}(\mathbb{R})$, $i > 0$. It is easy to show since the algebra $so_{k+1}(\mathbb{R}) \subset sl_{k+1}(\mathbb{R})$ and has the polyhedron as the invariant sub-space \footnote{Corresponding finite-dimensional representation is unitary and reducible (see e.g. the discussion in the end of Section 1.1)} any quadratic polynomial in generators of the algebra $so_{k+1}(\mathbb{R})$ with real symmetric coefficients can be reduced to the form the Laplace-Beltrami operator plus a scalar function by means of a gauge transformation (see discussion in Ref. [7]).

The above procedure can be extended to the case of the Lie super-algebras and also the quantum algebras. For the former, for linear operators acting on functions of real and Grassmann variables one can appear finite-dimensional invariant subspaces others than finite-dimensional representations (see [3]). However, those subspaces are connected to the representations of polynomial elements of the universal enveloping algebra and, correspondingly, they can be constructed through finite-dimensional representations of the original super-algebra. For the latter, we could not find any irreducible (non-trivial) Newton diagrams in $\mathbb{R}^k$ for $k > 1$; they will occur, if a quantum space is considered instead of ordinary one \footnote{I am grateful to O.V.Ogievetski for discussion of this topic}. Also one can construct quasi-exactly-solvable and exactly-solvable operators considering ”mixed” algebras: a direct sum of the Lie algebras, Lie superalgebras and the quantum algebras, realized in the first order differential and/or difference operators, possessing a finite-dimensional invariant subspace.

We have described above the linear operators, which possess a finite-dimensional invariant sub-space with a polynomial basis. Certainly, by means
of diffeomorphism and gauge transformation one can obtain the operators with an invariant subspace with a non-polynomial basis (but emerging from polynomial one). The open question is: is it possible to find the operators possessing a finite-dimensional invariant sub-space with some explicit basis, which can not be reduced to a polynomial one using a diffeomorphism and gauge transformation? Up to now no examples of this type have been found (see Conjecture 2). Since the original aim of this whole investigation was the construction of the quasi-exactly-solvable Schroedinger operators \[11, 4, 7, 5, 6, 8\], a resolution of this problem plays a crucial role in leading to a complete classification of the quasi-exactly-solvable Schroedinger operators.

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