ASYMPTOTICS OF KARHUNEN-LOÈVE EIGENVALUES FOR SUB-FRACTIONAL BROWNIAN MOTION AND ITS APPLICATION

CHUN-HAO CAI, JUN-QI HU, AND YING-LI WANG

Abstract. In the present paper, the Karhunen-Loève eigenvalues for a sub-fractional Brownian motion are considered. Rigorous large \( n \) asymptotics for those eigenvalues are shown, based on functional analysis method. By virtue of these asymptotics, along with some standard large deviations results, asymptotical estimates for the small \( L^2 \)-ball probabilities for a sub-fractional Brownian motion are derived. By the way, asymptotic analysis on the Karhunen-Loève eigenvalues for the corresponding “derivative” process is also established.

1. Introduction

The eigenproblem for a centred stochastic process \( X = (X(t))_{t \in [0,1]} \) over a probability space \((\Omega, \mathcal{F}, P)\) with covariance function \( K(s, t) = \mathbb{E}[X(s)X(t)] \) consists of finding all pairs \((\lambda, \varphi)\) satisfying the equation

\[
K \varphi = \lambda \varphi
\]

in \( L^2([0,1]) \), where the corresponding linear operator is defined by

\[
(K \varphi)(t) \triangleq \int_0^1 K(s, t) \varphi(s) \, ds, \quad \forall t \in [0,1].
\]

If \( K(s, t) \) is square integrable, then \( K : L^2([0,1]) \to L^2([0,1]) \) is self-adjoint, positive and converge to zero after arranged in decreasing order. The corresponding normalised eigenfunctions \( \{\varphi_n\} \) form a complete orthonormal basis in \( L^2([0,1]) \).

In addition, if \( (X(t))_{t \in [0,1]} \) is a square integrable process with zero mean and continuous covariance, there exists a Karhunen-Loève expansion(cf. [1]). More precisely, it admits a representation over \([0,1]\) as a uniformly \( L^2(\Omega) \)-convergent series:

\[
X(t) = \sum_{n=0}^{\infty} \sqrt{\lambda_n} \xi_n \varphi_n(t),
\]

where \( \{\xi_n\} \) are orthonormal(i.e. \( \mathbb{E}[\xi_j \xi_k] = \delta_{jk} \) random variables in \( L^2(\Omega) \) with zero mean. Since the Karhunen-Loève expansion is an influential tool in analysing the properties of stochastic processes, \( \{\lambda_n\} \) are also called Karhunen-Loève eigenvalues for \((X(t))_{t \in [0,1]}\).

There are many applications relevant to the eigenproblems for stochastic processes: asymptotics of the small ball probabilities(cf. [9]), sampling from heavy tailed distributions(cf. [11]) and so on.

On most occasions, such kind of eigenvalues and eigenfunctions are notoriously hard to find explicitly. One exception is for the standard Brownian motion \( B = \ldots \)

2020 Mathematics Subject Classification. 60G15, 60G22, 47B40.

Key words and phrases. sub-fractional Brownian motion; Karhunen-Loève Eigenvalues; small \( L^2 \)-ball estimates.
(B_t)_{t \in [0,1]}$, where
\[ \lambda_n = \frac{1}{(n + 1/2)^2\pi^2} \tag{1.4} \]
and
\[ \varphi_n(t) = \sqrt{2} \sin((n + 1/2)\pi t) \tag{1.5} \]
for \( n = 0, 1, 2, \ldots \). This problem can be easily solved by reducing (1.1) to a simple boundary value problem for an ordinary differential equation(cf. [1]).

A widely used extension of Brownian motion is fractional Brownian motion \( B^H = (B^H_t)_{t \in [0,1]} \). Its covariance function is
\[ K^H(s,t) = \frac{1}{2} (s^{2H} + t^{2H} - |s-t|^{2H}), \tag{1.6} \]
where \( H \in (0,1) \) is called its Hurst exponent. The case \( H = \frac{1}{2} \) corresponds to Brownian motion. There are some important properties of fractional Brownian motion. For examples, it has self-similarity and stationary increments(cf. [2]). The eigenproblem of fractional Brownian motion has been discussed in several papers(cf. [5, 6]).

The author in [5] used functional analysis method, and obtained the asymptotics of the eigenvalues for fractional Brownian motion. The following is just a rephrasing of one of his results:

**Theorem 1.1** (J. C. Bronski, 2003). For the fractional Brownian motion with Hurst exponent \( H \in (0,1) \), its Karhunen-Loève eigenvalues satisfies the large \( n \) asymptotics
\[ \lambda_n = \frac{\sin(\pi H)\Gamma(2H + 1)}{(n\pi)^{2H+1} + o\left(n^{\frac{(2H+2)(4H+2)}{4H+5} + \delta}\right)} \tag{1.7} \]
for every \( \delta > 0 \), where \( \Gamma \) denotes the usual Euler gamma function.

The authors in [6] converted the eigenproblem for fractional Brownian motion into an integro-algebraic systems by using Laplace transform, and solved it by taking the inversion of the Laplace transform. And some other processes derived by Brownian motion like Brownian bridge(cf. [7]), the Ornstein-Uhlenbeck process (cf. [8]), etc., can be solved in a similar way. Compared to [5], the profile of eigenpair analysed with the method in [6] is more complete and accurate.

Similar to the fractional Brownian motion, the sub-fractional Brownian motion also presents the properties of self-similarity and long-range dependence (when the Hurst exponent \( H > \frac{1}{2} \)). Different from the fractional Brownian motion, there is an additional term \(|t + s|^{2H}\) in its covariance and the increment is not stationary as a result. From this point of view, it was expected that the idea in [6] could also work for sub-fractional Brownian motion. However, it seems that it DOES NOT work for sub-fractional Brownian motion because of loss of some translation structure.

Consequently, the main results in this paper are based on the idea in [5]. It should be pointed out that there are some flaws in [5]. To some extent, the results(see Remark 4.1 in Section 4.) in this paper are supplements and corrections of the ones in [5].

This paper is organised as follows. In the next section, the asymptotics of eigenvalues for sub-fractional Brownian motion and its derivative process are going to be stated. As an application of those results, the small ball estimates for sub-fractional Brownian motion will be presented in Section 3., but its proof will be omitted since it’s just a duplication of the one in [5]. Section 4 will be concluded with the details of the proofs of the main results.
2. The main results

Sub-fractional Brownian motion (sfBm) $B^H_{sub} = (B^H_{sub}(t))_{t \in [0,1]}$ is a centred long-range dependence Gaussian process. Like fractional Brownian motion, its covariance function is

$$K^H_{sub}(s,t) = s^{2H} + t^{2H} - \frac{1}{2}[(s + t)^{2H} + |s - t|^{2H}]$$  \hspace{1cm} (2.1)

with an exponent $H \in (0,1)$. The case $H = \frac{1}{2}$ also corresponds to Brownian motion.

To some extent, sfBm is intermediate between Brownian motion and fractional Brownian motion (cf. [4]). This is reflected in the nonstationarity and correlation of the increments and the covariance of the non-overlapping intervals. The increments on non-overlapping intervals are more weakly correlated than fractional Brownian motion and the covariance decays polynomially at a higher rate.

In this paper, the eigenproblems for the following two operators

$$(K^H_{sub}\varphi)(t) = \int_0^1 (s^{2H} + t^{2H} - \frac{1}{2}[(s + t)^{2H} + |s - t|^{2H}])\varphi(s) \, ds, \hspace{1cm} (2.2)$$

$$(\tilde{K}^H_{sub}\varphi)(t) = \int_0^1 H(2H - 1)(|s - t|^{2H-2} - (s + t)^{2H-2})\varphi(s) \, ds \hspace{1cm} (2.3)$$

are studied. The operator in (2.2) for $H \in (0,1)$ is related to sfBm itself, and the one in (2.3) for $H \in (\frac{1}{2},1)$ corresponds to the formal derivative of the sfBm. In fact, the operator $\tilde{K}^H_{sub}$ determines the correlation structure of Wiener integrals of square integrable deterministic functions through the formula

$$E[\int_0^1 \int_0^1 f \, dB^H_{sub}\int_0^1 g \, dB^H_{sub}] = \int_0^1 f(t)(\tilde{K}^H_{sub}g)(t) \, dt. \hspace{1cm} (2.4)$$

To the best of the authors’ knowledge, those eigenproblems haven’t been rigorously considered before. Borrowed the idea from [5], rough asymptotics of eigenvalues of sub-fractional Brownian motion are derived:

**Theorem 2.1.** The Karhunen-Loève eigenvalues of sub-fractional Brownian motion with exponent $H \in (0,1)$ satisfies

- Case $0 < H < -\frac{1+\sqrt{7}}{8}$:

  $$\lambda_n = \frac{\gamma H}{n^{2H+1}} + o(n^{-\frac{(2H+2)(4H+3)}{4H+5} + \delta}) \hspace{1cm} (2.5)$$

  for every $\delta > 0$ and $n \gg 1$;

- Case $\frac{1+\sqrt{7}}{8} \leq H < 1$:

  $$\lambda_n = \frac{\gamma H}{n^{2H+1}} + O(n^{-3}) \hspace{1cm} (2.6)$$

  for every $n \gg 1$, where $\gamma_H = \frac{2\sin(\pi H)\Gamma(2H+1)}{\pi(4H+3)}$.

Specifically, given an orthonormal basis in $L^2([0,1])$, the operator $K^H_{sub}$ in (2.2) over $L^2([0,1])$ is of a representation as a linear operator over $\ell^2$, which is essentially an infinite-dimensional matrix. Asymptotic analysis on matrix elements are performed, based on some technical lemmas, some of which (see Lemma 4.3) are improvements of the ones in [5]. Afterwards, Theorem 2.1 is obtained in terms of the theory of compact operators. However, asymptotics in Theorem 2.1 are rough by simple observation or through numerical simulation, although the details of the simulation are not provided here.

Next conclusion is about the eigenvalues of the derivative process of sub-fractional Brownian. Unlike [6], the case $H \in (0,\frac{1}{2})$ is skipped.
Theorem 2.2. The Karhunen-Loève eigenvalues of the derivative process of sub-fractional Brownian with $H \in (\frac{1}{2}, 1)$ satisfies
\[
\lambda_n = \frac{\kappa_H}{n^{2H-1}} + o(n^{-\frac{2H(4H-1)}{2H+1} + \delta})
\]  
for every $\delta > 0$ and $n \gg 1$, where $\kappa_H = \frac{2\sin(\pi H)\Gamma(2H+1)}{\pi^{2H}}$.

3. An application: Small $L^2$-ball estimate

Small ball estimate is an interesting topic in probability theory, and also has important applications in statistical mechanic models. It yields estimates of the probability that some stochastic process $X = (X(t))_{t \in [0,1]}$ will lie inside a ball of radius $\varepsilon$ in a certain given norm $|| \cdot ||$. As for the $L^2([0,1])$-norm, if $X$ is a centred Gaussian process with continuous covariance, there holds
\[
||X||_{L^2}^2 = \int_0^1 X(t)^2 \, dt = \sum_{n=0}^{\infty} \lambda_n \xi_n^2,
\]  
where $\{\xi_n\}$ are i.i.d. $N(0, 1)$ random variables. Just as pointed out in [5], a crucial quantity to derive small ball estimate for fractional Brownian motion is the determinant
\[
D_H(\lambda) = \prod_{n=0}^{\infty} (1 + 2\lambda \lambda_n)
\]  
which is a variant of Fredholm determinant of $K_H$.

Now, the small $L^2([0,1])$-ball estimate of sfBm is going to be carried out. Let $\subD_{sub}^H(\lambda)$ be the corresponding determinant with respect to sfBm. Note that the dominant terms of the eigenvalues of sfBm(see Theorem 2.1) are just double of the ones of fractional Brownian motion(see Theorem 1.1) when $n \gg 1$. Through a slight modification of the proof of Corollary 1 in Appendix C of [5], the logarithmic asymptotics of $\subD_{sub}^H(\lambda)$ read

Lemma 3.1. There holds

- Case $H \in \left(0, \frac{1+\sqrt{77}}{8}\right)$:
  \[
  \log(\subD_{sub}^H(\lambda)) = \frac{(4\sin(\pi H)\Gamma(2H+1))\pi^{2H+1}}{\sin(\frac{\pi H}{2H+1})} \lambda^{\frac{1}{2H} + \delta} + o(\lambda^{\frac{2H-1}{2H+1} + \delta})
  \]  
  for every $\delta > 0$ and $\lambda \gg 1$;
- Case $H \in \left[\frac{1+\sqrt{77}}{8}, 1\right)$:
  \[
  \log(\subD_{sub}^H(\lambda)) = \frac{(4\sin(\pi H)\Gamma(2H+1))\pi^{2H+1}}{\sin(\frac{\pi H}{2H+1})} \lambda^{\frac{1}{2H} + \delta} + O(\lambda^{\frac{2H-1}{2H+1}})
  \]  
  for every $\lambda \gg 1$.

Thereafter, the small $L^2([0,1])$-ball estimate of sfBm can be directly established by using standard large deviations calculation and de Bruijn’s exponential Tauberian theorem(cf. [3]), which is exactly the same procedure as the one in [5].

Theorem 3.2. For $0 < \varepsilon \ll 1$, the small ball probability $P(||B_{sub}^H||_{L^2}^2 \leq \varepsilon)$ of a sub-fractional Brownian motion satisfies

- Case $H \in (0, \frac{1+\sqrt{77}}{8})$:
  \[
  \log(P(||B_{sub}^H||_{L^2}^2 \leq \varepsilon)) = -H \left(\frac{2\sin(\pi H)\Gamma(2H+1)}{(2H+1)\sin(\frac{\pi}{2H+1})^{2H+1}}\right) \varepsilon^{\frac{1}{2H}} + o(\varepsilon^{\frac{2H+1}{2H+1} + \delta})
  \]  
  for every $\delta > 0$;
that, if $\mathcal{A}$ will be finished in 5 steps. In the first 4 steps, \( \ell \) compact (Hilbert-Schmidt etc.) in matrix \((\operatorname{operator norm})\), the linear operator \( \mathcal{A} \) in the sequel. Now, the eigenfunctions for Brownian motion could be rewritten as

\[
\{ \phi_n \} \quad \text{(1.5)}
\]

Proof of Theorem 4.1. Throughout this section, the eigenfunctions \( \{ \phi_n \} \) in (1.5) are chosen as an orthonormal basis in \( L^2([0,1]) \). Therefore, any bounded linear operator \( K \) over \( L^2([0,1]) \) is one-to-one corresponding to the operator \( \mathcal{A} \) over \( \ell^2 \) with the same operator norm. The linear operator \( \mathcal{A} \) over \( \ell^2 \) is essentially an infinite-dimensional matrix \((A_{m,n})\), whose element is given by

\[
A_{m,n} = \int_0^1 \int_0^1 K(x,y)\phi_m(x)\phi_n(y) \, dx \, dy. \tag{4.1}
\]

Actually, such kind of mapping \( K \) to \( \mathcal{A} \) is a topologically isomorphism. It implies that, if \( K \) is compact (Hilbert-Schmidt etc.) in \( L^2([0,1]) \), then \( \mathcal{A} = (A_{m,n}) \) is also compact (Hilbert-Schmidt etc.) in \( \ell^2 \); Vice versa.

For the sake of simplicity, denote

\[
m^* = (m + \frac{1}{2})\pi, \quad n^* = (n + \frac{1}{2})\pi, \quad m,n = 0,1,2,\cdots \tag{4.2}
\]

in the sequel. Now, the eigenfunctions for Brownian motion could be rewritten as

\[
\phi_n(t) = \sqrt{2}\sin (n^* t), \quad n = 0,1,2,\cdots \tag{4.3}
\]

It’s ready to prove the main results in this paper.

4.1. Proof of Theorem 2.1. Here, the eigenproblem is \( K^H_{\text{sub}}\phi = \lambda\phi \). The proof will be finished in 5 steps. In the first 4 steps, \( H \in (\frac{1}{2},1) \) is imposed temporarily, but this condition will be dropped off in Remark 4.4.

Step 1. Obviously, for the operator \( K^H_{\text{sub}} \) in \( L^2([0,1]) \), there exists a linear operator \( A^H_{\text{sub}} = (A^H_{\text{sub}})_{m,n} \) in \( \ell^2 \) with the same operator norm, whose element is

\[
(A^H_{\text{sub}})_{m,n} = \int_0^1 \int_0^1 2 \left[ x^{2H} + y^{2H} - \frac{1}{2} (x+y)^{2H} + |x-y|^{2H} \right] \sin(n^* x) \sin(m^* y) \, dx \, dy. \tag{4.4}
\]

It’s easy to see that for every \( m,n = 0,1,2,\cdots \), there holds

\[
(A^H_{\text{sub}})_{m,n} = \frac{2H(2H-1)}{n^* m^*} \int_0^1 \int_0^1 (|x-y|^{2H-2}-(x+y)^{2H-2}) \cos(n^* x) \cos(m^* y) \, dx \, dy. \tag{4.5}
\]

by utilising the integration by parts since \( H > \frac{1}{2} \). Splitting the right hand side of (4.5) into two integrals, the linear operator \( A^H_{\text{sub}} \) has a decomposition \( A^H_{\text{sub}} = 2A - A^{(1)} \), where their corresponding elements share the same relations, i.e., \((A^H_{\text{sub}})_{m,n} = 2A_{m,n} - A_{m,n}^{(1)} \) for every \( m,n = 0,1,2,\cdots \).

Step 2. It’s worthy mentioning that the linear operator \( \mathcal{A} = (A_{m,n}) \) in \( \ell^2 \) with its elements of the forms

\[
A_{m,n} = \frac{H(2H-1)}{n^* m^*} \int_0^1 \int_0^1 |x-y|^{2H-2} \cos(n^* x) \cos(m^* y) \, dx \, dy \tag{4.6}
\]

has been discussed in [5]. There exists a decomposition \( A = D + O \), where the linear operator \( D \) and \( O \) are corresponding to infinite-dimensional matrices whose
elements are respectively as a leading order diagonal piece and a higher order off-diagonal piece of \((A_{m,n})\). Accurately speaking, according to the proof of Theorem 1 in Appendix A in [5], there hold

\[ D_{n,m} = \left( \frac{\sin(\pi H)\Gamma(2H+1)}{n^{2H+1}} + O\left(\frac{1}{n^{2(H+1)}}\right) \right) \delta_{n,m}, \quad (4.7) \]

\[ O_{n,m} = \frac{\cos(\pi H)\Gamma(2H+1)}{n^m(m^{n+(1)n^m+m})} \left( \frac{1}{n^{2H-1}} + \frac{1}{m^{2H-1}} \right) + O\left(\frac{1}{n^{2m^{2H}}}\right), \quad (4.8) \]

for \(m,n \gg 1\), where \(O_{n,n} = 0\) for every \(n = 1, 2, \ldots\).

**Remark 4.1.** By applying Lemma 4.3 below, it is accidently found that the remainder order of \(D_{n,n}\) in [5] (or see (4.7) above) is not correct, while (4.31) is the right one in stead.

In order to obtain the exact rate of convergence of \(O\), (4.8) could be rewritten as

\[ O_{n,m} = \begin{cases} \frac{\cos(\pi H)\Gamma(2H+1)}{n^m(m^{n+(1)n^m+m})} \left( \frac{1}{n^{2H-1}} + \frac{1}{m^{2H-1}} \right) + O\left(\frac{1}{n^{2m^{2H}}}\right) & m + n \text{ even} \\ \frac{\cos(\pi H)\Gamma(2H+1)}{n^m(m^{n+(1)n^m+m})} \left( \frac{1}{n^{2H-1}} + \frac{1}{m^{2H-1}} \right) + O\left(\frac{1}{n^{2m^{2H}}}\right) & m + n \text{ odd} \end{cases} \]

(4.9)

It’s sufficient to discuss the case of \(m > n \gg 1\) because of the symmetry with respect to the subscripts \(m\) and \(n\) in (4.6). Whenever \(m + n\) is even or not, it is clear that

\[ \frac{1}{n^m(m^{n+(1)n^m+m})} \left( \frac{1}{n^{2H-1}} \right) = \pm \frac{1}{m^{2n^{2H}}} \pm \frac{1 \pm \left(\frac{n}{m}\right)^{2H-1}}{m^{2n^{2H}}} \]

which leads to

\[ O_{n,m} \equiv \frac{1}{m^{2n^{2H}}}, \quad m > n \gg 1 \]

(4.11)

by noticing the boundedness of \(f(t) = \frac{1 \pm t^{2H-1}}{1 \pm t^2}\) in \(t \in (0, 1)\).

**Step 3.** It’s time to deal with the linear operator \(A^{(1)} = (A^{(1)}_{n,m})\), whose element is

\[ A^{(1)}_{n,m} = \frac{2H(2H-1)}{n^m(m^{n+(1)n^m+m})} \int_0^1 \int_0^1 (x+y)^{2H-2} \cos(n^x \cos(m^y) \cos(m^y) dx dy, \quad (4.12) \]

Simply decompose \(A^{(1)}\) into \(A^{(1)} = D^{(1)} + O^{(1)}\) as done in Step 2, where \(D^{(1)} = A^{(1)} - D^{(1)}\).

Step 3.1. To calculate the elements of \(A^{(1)} = (A^{(1)}_{n,m})\), firstly divide the square \([0,1] \times [0,1]\) into two sub-domains \(I_1, I_2\) (see Figure 1), where \(I_1\) represents the triangle enclosed by the lines \(x = 0, y = 0\) and \(x + y = 1\); \(I_2\) the triangle enclosed by \(x = 1, y = 1\) and \(x + y = 1\). It leads to

\[ \int_0^1 \int_0^1 (x+y)^{2H-2} \cos(n^x \cos(m^y) \cos(m^y) dx dy, \quad (4.13) \]

Through the change of variables

\[ \begin{cases} u = x + y \\ v = x - y, \end{cases} \]

(4.14)

\[ \tag{4.14} \]

The notation \(f \asymp g\) means \(f\) and \(g\) are the same order of magnitude.
it maps $I_1$ and $I_2$ to $J_1$ and $J_2$ respectively. By changing of variables in double integration, it implies that
\[
2 \int_{0}^{1} \int_{0}^{1} (x + y)^{2H-2} \cos(n^* x) \cos(m^* y) \, dx \, dy = \left( \int_{0}^{1} d \int_{-u}^{u} + \int_{1}^{2} d \int_{u-2}^{-u+2} \right) u^{2H-2} \cos(n^* \left( \frac{u + v}{2} \right)) \cos(m^* \left( \frac{u - v}{2} \right)) \, dv.
\]
\[
(4.15)
\]
It’s convenient to denote two integral terms on the right hand side of (4.15) by $Q_{1n,m}$ and $Q_{2n,m}$. Combined with the formulae for trigonometric functions, it implies
\[
\begin{align*}
Q_{1n,m} &= \frac{1}{2} \int_{0}^{1} u^{2H-2} \, du \int_{-u}^{u} \left( \cos\left( \frac{m^* + n^* u}{2} \right) - \cos\left( \frac{m^* - n^* u}{2} \right) \right) \, dv, \\
Q_{2n,m} &= \frac{1}{2} \int_{1}^{2} u^{2H-2} \, du \int_{u-2}^{-u+2} \left( \cos\left( \frac{m^* + n^* u}{2} \right) - \cos\left( \frac{m^* - n^* u}{2} \right) \right) \, dv.
\end{align*}
\]
\[
(4.16)
\]
\[
(4.17)
\]
Substituting the above identities into $A^{(1)}_{n,m}$, it can be deduced that
\[
A^{(1)}_{n,m} = \frac{H(2H - 1)}{n^* m^*} (Q_{1n,m} + Q_{2n,m}).
\]
\[
(4.18)
\]
Step 3.2. Since the singularity among the integrands in $A^{(1)}_{m,n}$ only occurs at $(0, 0)$, it seems that the contribution of $Q_{1n,m}$ should be much greater than the one of $Q_{2n,m}$. To see it, the following integral identities will be needed.

**Lemma 4.2.** Let $a$ be a real number, there hold
\[
\begin{align*}
\int_{1}^{2} u^a \cos(\omega u) \, du &= \frac{2^a \sin(2\omega) - \sin \omega}{\omega} + O\left( \frac{1}{\omega^2} \right), \\
\int_{1}^{2} u^a \sin(\omega u) \, du &= -\frac{2^a \cos(2\omega) - \cos \omega}{\omega} + O\left( \frac{1}{\omega^2} \right)
\end{align*}
\]
for $\omega \gg 1$.

**Proof.** It’s necessary to prove the first identity since the second could be proved in a similar way. Noticing that
\[
\frac{1}{\omega} \int_{1}^{2} d(u^a \sin(\omega u)) = \frac{2^a \sin(2\omega) - \sin \omega}{\omega}
\]
\[
(4.21)
\]
it implies that
\[
\int_{1}^{2} u^a \cos(\omega u) \, du = \frac{2^a \sin(2\omega) - \sin \omega}{\omega} - \frac{a}{\omega} \int_{1}^{2} u^{a-1} \sin(\omega u) \, du.
\]
\[
(4.22)
\]
Combining with the second mean value theorem for Riemann integrals, the desired result is obtained.

To calculate $Q_{m,n}^2$, the order of $\int_1^2 u^{2H-2} \sin(m^*u) \, du$ needs to be estimated. By setting $a = 2H - 2$ and $\omega = m^*$, Lemma 4.2 gives

$$\int_1^2 u^{2H-2} \cos(m^*u) \, du = \frac{(-1)^{m+1}}{m^*} + O\left(\frac{1}{m^2}\right),$$

$$\int_1^2 u^{2H-2} \sin(m^*u) \, du = \frac{2^{2H-2}}{m^*} + O\left(\frac{1}{m^2}\right)$$

for $m \gg 1$. Base on same idea, the order of remainder term can be improved. For example, it’s true that

$$\int_1^2 u^{2H-1} \cos(m^*u) \, du = -\frac{2H - 1}{m^*} \int_1^2 u^{2H-2} \sin(m^*u) \, du - \frac{(-1)^m}{m^*}$$

$$= \frac{(-1)^{m+1}}{m^*} + O\left(\frac{1}{m^2}\right)$$

for $m \gg 1$. Moreover, there holds

$$\frac{1}{m^* - n^*} \int_1^{m^* - n^*} u^{2H-2} (\sin(m^*u) - \sin(n^*u)) \, du = \frac{1}{m^* - n^*} \left(\frac{2^{2H-2}}{m^*} - \frac{2^{2H-2}}{n^*}\right) + O\left(\frac{1}{mn}\right)$$

for $m > n \gg 1$.

**Lemma 4.3.** If $a \in (0, 1)$, there hold

$$\int_0^1 x^{a-1} \cos(\omega x) \, dx = \frac{\Gamma(a) \cos(\frac{\pi a}{2})}{\omega^a} + \frac{\sin \omega}{\omega} + O\left(\frac{1}{\omega^2}\right),$$

$$\int_0^1 x^{a-1} \sin(\omega x) \, dx = \frac{\Gamma(a) \sin(\frac{\pi a}{2})}{\omega^a} - \frac{\cos \omega}{\omega} + O\left(\frac{1}{\omega^2}\right)$$

for $\omega \gg 1$.

The proof of Lemma 4.3 is postponed in the next subsection. By setting $a = 2H - 1$ and $\omega = m^*$, the two identities in Lemma 4.3 are turned into

$$\int_0^1 x^{2H-2} \cos(m^*x) \, dx = \frac{\Gamma(2H - 1) \sin(\pi H)}{m^{2H-1}} + \frac{(-1)^m}{m^*} + O\left(\frac{1}{m^2}\right),$$

$$\int_0^1 x^{2H-2} \sin(m^*x) \, dx = -\frac{\Gamma(2H - 1) \cos(\pi H)}{m^{2H-1}} + O\left(\frac{1}{m^*}\right)$$

for $m \gg 1$.

On the one hand, it was mentioned in Remark 4.1 that the asymptotics in (4.7) are not correct. As a matter of fact, using Lemma 4.3, correct ones could be deduced. That is, the diagonal part of the matrix corresponding to fractional Brownian motion can be revised as

$$D_{n,n} = \frac{\sin(\pi H) \Gamma(2H + 1)}{n^{2H+1}} + \frac{(-1)^n}{n^{3}} + O\left(\frac{1}{n^4}\right)$$

for $n \gg 1$.

On the other hand, by using the integration by parts (see The proof of Lemma 4.3) and the second mean value theorem for Riemann integrals, it’s valid that

$$\int_0^1 u^{2H-1} \cos(m^*u) \, du = \frac{(-1)^m}{m^*} + \frac{\Gamma(2H) \cos(\pi H)}{m^{2H}} + O\left(\frac{1}{m^3}\right),$$

$$\int_0^1 u^{2H-1} \sin(m^*u) \, du = \frac{(-1)^m}{m^*} + \frac{\Gamma(2H) \sin(\pi H)}{m^{2H}} + O\left(\frac{1}{m^3}\right).$$
for \( m \gg 1 \). Furthermore, there holds
\[
\frac{1}{m^* - n^*} \int_0^1 x^{2H-2} \sin(m^* x) \, dx = -\frac{\Gamma(2H - 1) \cos(\pi H)}{m^* - n^*} \left( \frac{1}{m^{*2H-1}} - \frac{1}{n^{*2H-1}} \right) + O\left( \frac{1}{mn} \right)
\]
for \( m > n \gg 1 \).

**Step 3.3.** Calculate \( Q_{n,m}^1 \) and \( Q_{n,m}^2 \) in the case of \( m > n \gg 1 \). Firstly, Using the fundamental theorem for Riemann integrals in (4.16), it implies that
\[
Q_{n,m}^1 = \frac{1}{m^* + n^*} \int_0^1 u^{2H-2} (\sin(m^* u) + \sin(n^* u)) \, du
\]
which gives
\[
Q_{n,m}^1 = -\frac{\Gamma(2H - 1) \cos(\pi H)}{m^* + n^*} \left( \frac{1}{m^{*2H-1}} + \frac{1}{n^{*2H-1}} \right) + O\left( \frac{1}{m^*} \right) + O\left( \frac{1}{n^*} \right)
\]
\[
\]
in terms of Lemma 4.3 (see (4.30) and (4.33)). Observing that for \( m > n \gg 1 \),
\[
\frac{1}{m^* + n^*} (O\left( \frac{1}{m^*} \right) + O\left( \frac{1}{n^*} \right)) = O\left( \frac{1}{mn} \right)
\]
it means that
\[
Q_{n,m}^1 = -\frac{\Gamma(2H - 1) \cos(\pi H)}{m^* + n^*} \left( \frac{1}{m^{*2H-1}} + \frac{1}{n^{*2H-1}} \right) - \frac{\Gamma(2H - 1) \cos(\pi H)}{m^* - n^*} \left( \frac{1}{m^{*2H-1}} - \frac{1}{n^{*2H-1}} \right) + O\left( \frac{1}{mn} \right)
\]
i.e. the order of \( Q_{n,m}^1 \) is the same as \( m^{-1} n^{-(2H-1)} \).

Next goal is to calculate \( Q_{n,m}^2 \). It’s clear that
\[
Q_{n,m}^2 = \frac{(-1)^{m+n+1}}{m^* + n^*} \int_0^1 u^{2H-2} (\sin(m^* u) + \sin(n^* u)) \, du
\]
which leads to
\[
Q_{n,m}^2 = \frac{(-1)^{m+n+1}}{m^* + n^*} \left( \frac{2^{2H-2}}{m^*} + \frac{2^{2H-2}}{n^*} \right) + O\left( \frac{1}{mn} \right)
\]
in terms of Lemma 4.2 (see (4.24) and (4.26)). After all, it gives that
\[
Q_{n,m}^2 = O\left( \frac{1}{mn} \right)
\]
which verifies that the contribution of \( Q_{n,m}^2 \) is smaller than the one of \( Q_{n,m}^1 \).

Since
\[
\frac{O_{n,m}^{(1)}}{O_{n,m}^{(1)}} = A_{n,m}^{(1)} = H(2H - 1) \frac{Q_{n,m}^1 + Q_{n,m}^2}{n^* m^*}
\]
for \( m \neq n \), \( O_{n,m}^{(1)} \) is the same order as \( m^{-2} n^{-2H} \) for \( m > n \gg 1 \).

**Step 3.4.** Calculate \( Q_{n,m}^1 \) and \( Q_{n,m}^2 \) in the case of \( m = n \gg 1 \). At first, (4.16) could be transformed into
\[
Q_{m,m}^1 = \int_0^1 u^{2H-1} \cos(m^* u) \, du + \frac{1}{m^*} \int_0^1 u^{2H-2} \sin(m^* u) \, du
\]
which gives
\[
Q_{m,m}^1 = \frac{(-1)^m}{m^*} + \frac{\cos(\pi H)}{m^{2H}}(\Gamma(2H) - \Gamma(2H - 1)) + O\left(\frac{1}{m^3}\right) \quad (4.43)
\]
by virtue of (4.32) and (4.30). Secondly, it’s valid that
\[
Q_{m,m}^2 = -\int_1^2 u^{2H-1} \cos(m^*u) \, du + 2\int_1^2 u^{2H-2} \cos(m^*u) \, du + \frac{1}{m^*} \int_1^2 u^{2H-2} \sin(m^*u) \, du
\]
which implies that
\[
Q_{m,m}^2 = \frac{(-1)^{m+1}}{m^*} + O\left(\frac{1}{m^2}\right) \quad (4.45)
\]
by using (4.23), (4.24) and (4.25).

\[
D_{m,m}^{(1)} = A_{m,m}^{(1)} = \frac{H(2H - 1)}{m^{2H+2}} (Q_{m,m}^1 + Q_{m,m}^2)
\]
(4.46)

it implies that
\[
D_{m,m}^{(1)} = \frac{(H - 1) \cos(\pi H) \Gamma(2H + 1)}{m^{2H+2}} + O\left(\frac{1}{m}\right)
\]
(4.47)
i.e. $D_{m,m}^{(1)}$ is the same order as $m^{-2H-2}$ for $m \gg 1$.

**Step 4.** Summarise all asymptotic information for $A_{sub}$. Noting that $A_{sub} = 2A - A^{(1)}$ and $A^{(1)} = D^{(1)} + O^{(1)}$, $A_{sub}$ has also a decomposition $A_{sub} = D_{sub} + O_{sub}$ just like the linear operator $A$ in Step 2, if $D_{sub} = 2D - D^{(1)}$ and $O_{sub} = 2O - O^{(1)}$ are set.

The orders of the elements of $A_{sub}$ are as follows: As for the diagonal piece, combined with (4.47), it gives
\[
(D_{sub})_{m,m} = \frac{2\sin(\pi H) \Gamma(2H + 1)}{m^{2H+1}} + \frac{(-1)^m}{m^{3}} + \frac{(H - 1) \cos(\pi H) \Gamma(2H + 1)}{m^{2H+2}} + O\left(\frac{1}{m}\right)
\]
(4.48)
for $m \gg 1$; As for the off-diagonal piece, noticing (4.11) and (4.41), it implies
\[
(O_{sub})_{n,m} \asymp \frac{1}{m^{2m^2H}}
\]
(4.49)
for $m > n \gg 1$.

**Remark 4.4.** During processing the proof of Theorem 2.1, $H \in (\frac{1}{2}, 1)$ is imposed. In fact, $A_{n,m}$ (see (4.4)) is holomorphic with respect to the variable $H$ in $(0, 1)$, so are $D_{n,m}$ and $O_{n,m}$. Moreover, the first three terms on the right hand side of (4.48) are holomorphic in $H \in (0, 1)$, so is the remaining term in (4.48). In terms of the principle of analytic continuation, (4.48) is still valid for $H \in (0, 1)$. Same argument works for off-diagonal piece in the case of $H \in (0, 1)$.

**Step 5.** It’s clear that $D_{sub}$ is self-adjoint, positive and compact in $\ell^2$. For any fixed $\beta \in (0, 1)$, $D_{sub}^{\beta}$ is well-defined by spectral decomposition theorem. Hence, $O_{sub}$ can be turned into
\[
O_{sub} = D_{sub}^{\beta} \tilde{O}_{sub} D_{sub}^{\beta} \quad (4.50)
\]
where
\[
\tilde{O}_{sub} = D_{sub}^{-\beta} O_{sub} D_{sub}^{-\beta} \quad (4.51)
\]
The order of the elements of $D_{sub}^{\beta}$ is $m^{\beta(2H+1)}$ for $m \gg 1$, so the order of the ones of $\tilde{O}_{sub}$ is $n^{(2H+1)\beta - 2m^{2H+1} \beta - 2H}$ for $m > n \gg 1$. If $\beta \in (0, \frac{1}{2})$, the elements of $\tilde{O}_{sub}$ are square summable. Therefore, $\tilde{O}_{sub}$ is a Hilbert-Schmidt operator.
thus compact). The eigenvalues of \( \hat{O}_{\text{sub}} \) are square summable and thus (arranged in order of decreasing magnitude) satisfy
\[
|\lambda_n(\hat{O}_{\text{sub}})| \lesssim n^{-\frac{1}{2}}.
\] (4.52)

Finally, recall the following two results (cf. [10]).

**Lemma 4.5** (Porter and Stirling). If \( T, K \) are compact and \( K \) is self-adjoint then the eigenvalues of \( T^*KT \) satisfy
\[
|\lambda_n(T^*KT)| \leq \min_{j \in \{1, \ldots, n\}} |\lambda_j(K)| \lambda_{n-j+1}(T^*T).
\] (4.53)

**Lemma 4.6** (Porter and Stirling). If \( K_1, K_2 \) are compact and self-adjoint then we have
\[
\lambda_n(K_1 + K_2) \leq \min_{j \in \{1, \ldots, n\}} [\lambda_{n-j}(K_1) + |\lambda_j(K_2)|].
\] (4.54)

Given any \( \delta \in (0, \frac{1}{2}) \), by setting \( \beta = \frac{1}{2} - \delta \), it’s true that
\[
|\lambda_n(O_{\text{sub}})| = |\lambda_n(D_{\text{sub}}^\beta \hat{O}_{\text{sub}} D_{\text{sub}}^\beta)| \leq |\lambda_{n-j}(\hat{O}_{\text{sub}})| |\lambda_j(D_{\text{sub}}^\beta)| \lesssim n^{-\frac{1}{2}} n^{-2\delta(2H+1)} = n^{-2H - \frac{2}{3} + (4H+2)\delta}
\] in terms of Lemma 4.5. Since \( \delta \in (0, \frac{1}{2}) \) is arbitrarily chosen, the above inequality can be rewritten as
\[
|\lambda_n(O_{\text{sub}})| \lesssim n^{-2H - \frac{2}{3} + \delta}.
\] (4.55)

Now, Lemma 4.6 yields
\[
\lambda_n(A_{\text{sub}}) \leq |\lambda_{n-n^\alpha}(D_{\text{sub}})| + |\lambda_n(O_{\text{sub}})|
\]
\[
\quad \leq \frac{\gamma_H}{n^{2H+1}} (1 + \frac{n^{\alpha}}{n-n^{\alpha}})^{2H+1} + O((n-n^{\alpha})^{-3} + O(n^{-\alpha(2H+\frac{3}{2}-\delta)})
\]
\[
\quad = \frac{\gamma_H}{n^{2H+1}} (1 + (2H+1) \frac{n^{\alpha}}{n-n^{\alpha}} + O(\frac{n^{2\alpha}}{(n-n^{\alpha})^2}))
\] (4.56)
\[
\quad + O(n^{-\alpha(2H+\frac{3}{2}-\delta)}) + O(n^{-\alpha(2H+\frac{3}{2}-\delta)})
\]
\[
\quad = \frac{\gamma_H}{n^{2H+1}} + O(n^{-2H-2+\alpha}) + O(n^{-3}) + O(n^{-\alpha(2H+\frac{3}{2}-\delta)})
\]
by setting \( K_1 = D_{\text{sub}}, K_2 = O_{\text{sub}} \) and \( j = n^{\alpha} \). Letting \( 2H + 2 - \alpha = \alpha(2H + \frac{3}{2}) \) (i.e. \( \alpha = \frac{2H+\frac{5}{2}}{2} \)) and making use of the arbitrariness of \( \delta \in (0, \frac{1}{2}) \), it implies that
\[
\lambda_n(A_{\text{sub}}) \leq \frac{\gamma_H}{n^{2H+1}} + o\left(n^{-\frac{(2H+2)(4H+3)}{4H+5} \delta}\right) + O(n^{-3}).
\] (4.57)

There are two cases:

1. If \( \frac{(2H+2)(4H+3)}{4H+5} < 3 \) (i.e. \( 0 < H < \frac{1+\sqrt{73}}{8} \)), there holds
\[
\lambda_n(A_{\text{sub}}) \leq \frac{\gamma_H}{n^{2H+1}} + o\left(n^{-\frac{(2H+2)(4H+3)}{4H+5} \delta}\right);
\] (4.58)

2. If \( \frac{(2H+2)(4H+3)}{4H+5} \geq 3 \) (i.e. \( \frac{1+\sqrt{73}}{8} \leq H < 1 \)), there holds
\[
\lambda_n(A_{\text{sub}}) \leq \frac{\gamma_H}{n^{2H+1}} + O(n^{-3}).
\] (4.59)

Repeating the above argument with \( K_1 = A_{\text{sub}}, K_2 = -O_{\text{sub}} \) gives

1. If \( 0 < H < \frac{1+\sqrt{73}}{8} \), there holds
\[
\lambda_n(A_{\text{sub}}) \geq \frac{\gamma_H}{n^{2H+1}} + o\left(n^{-\frac{(2H+2)(4H+3)}{4H+5} \delta}\right);
\] (4.60)
4.2. **Proof of Lemma 4.3.** It’s sufficient to prove the first identity. The proof of the second identity is similar to the first one. First, by changing of variable in integration, it can be deduced that
\[
\int_0^1 x^{a-1} \cos(\omega x) \, dx = \frac{1}{\omega^a} \left( \int_0^{+\infty} t^{a-1} \cos t \, dt - \int_{-\infty}^0 t^{a-1} \cos t \, dt \right). \tag{4.62}
\]
On the one hand, by using contour integration, it’s easy to verify
\[
\int_0^{+\infty} t^{a-1} \cos t \, dt = \Gamma(a) \cos \left( \frac{\pi}{2} a \right). \tag{4.63}
\]
On the other hand, by using the integration by parts and the second mean value theorem for Riemann integrals, it’s valid that
\[
\int_{-\infty}^0 t^{a-1} \cos t \, dt = -\omega^{a-1} \sin \omega - (a-1) \int_{-\infty}^0 t^{a-2} \sin t \, dt = -\omega^{a-1} \sin \omega + O(\omega^{a-2}). \tag{4.64}
\]
The proof is completed. \(\square\)

4.3. **Proof of Theorem 2.2.** Following the lines in the proof of Theorem 2.1, it’s easy to justify Theorem 2.2. Here the sketch of its proof will be given and the different part from the steps in the proof of Theorem 2.1 will be emphasised. Step 3.2. in the proof of Theorem 2.2 is skipped since technical lemmas have been exhibited there.

Formally speaking, the covariance function \(\tilde{K}_{\text{sub}}^H\) is the “mixed partial derivative” of \(K_{\text{sub}}^H\). From the point of view of the general white noise theory(cf. [2]), the sfBm is the integral process of the one related to \(\tilde{K}_{\text{sub}}^H\) in rigorous sense, since
\[
\int_0^t \int_0^s \tilde{K}_{\text{sub}}^H(x, y) \, dx \, dy = K_{\text{sub}}^H(s, t) \tag{4.65}
\]
if \(H > \frac{1}{2}\). Hence, it’s reasonable to study the eigenproblem \(\tilde{K}_{\text{sub}}^H \varphi = \lambda \varphi\).

**Step 1.** The matrix element related to the linear operator \(\tilde{K}_{\text{sub}}^H\) becomes
\[
(\tilde{A}_{\text{sub}})_{n,m} = 2H(2H-1) \int_0^1 \int_0^1 (|x-y|^{2H-2}-(x+y)^{2H-2}) \sin(n^*x) \sin(m^*y) \, dx \, dy. \tag{4.66}
\]
By splitting the right hand side of the above identity into two integrals, \((\tilde{A}_{\text{sub}})_{n,m}\) has a decomposition \((\tilde{A}_{\text{sub}})_{n,m} = 2\tilde{A}_{n,m} - \tilde{A}_{n,m}^{(1)}\), where \(\tilde{A}_{n,m}\) corresponds to the part of fractional Brownian noise(cf. [5, 6]).

To calculate \(\tilde{A}_{n,m}\), through imitating the method in [5], \([0, 1] \times [0, 1]\) can be represented by a parallelogram \(V_1\) minus two triangles \(V_2\) and \(V_3\)(see Figure 2), where \(V_1\) is enclosed by the lines \(y = 0, y = 1, y - x = 1\) and \(y - x = -1\); \(V_2\) enclosed by \(x = 0, y = 0\) and \(y - x = 1\); \(V_3\) enclosed by \(x = 1, y = 1\) and \(y - x = -1\). By denoting
\[
\tilde{R}_{n,m}^{(1)} = \int_{V_1} (|x-y|^{2H-2} \sin(n^*x) \sin(m^*y) \, dx \, dy \tag{4.67}
\]
\[
\tilde{R}_{n,m}^{(2)} = \int_{V_2 \cup V_3} (|x-y|^{2H-2} \sin(n^*x) \sin(m^*y) \, dx \, dy, \tag{4.68}
\]
(2) If \(\frac{1+\sqrt{8}}{8} \leq H < 1\), there holds
\[
\lambda_n(\tilde{A}_{\text{sub}}) = \frac{\gamma_H}{n^{2H+1}} + O(n^{-3}). \tag{4.61}
\]

The proof is completed. \(\square\)
By changing the variables $u = y - x, \quad v = y$ in double integral, it can be deduced that

$$\tilde{R}_{n,m}^4 = \int_0^1 \sin(m^*v) \sin(n^*v) \, dv \int_{-1}^1 |u|^{2H-2} \cos(n*u) \, du. \quad (4.70)$$

To calculate $\tilde{R}_{n,m}^{2,3}$, first of all, perform the mapping $V_2$ to $V_3$ by changing of variables $x' = 1 - x, \quad y' = 1 - y$. Then the integral over this region is greatly simplified under the change of variables $u = x + y, \quad v = x - y$, which gives

$$\tilde{R}_{n,m}^{2,3} = \left(\frac{-1}{2}\right)^{m+n+1} \int_0^1 u^{2H-2} \int_{-u}^u \cos(n^*v - \frac{u}{2}) + (-1)^{m+n}m^* \frac{v+u}{2} \right) \, dv. \quad (4.71)$$

To calculate $\tilde{A}_{n,m}^{(1)}$, two sub-domains $I_1, I_2$ are chosen just as done in Step 3.1 of Section 4.1. Designating

$$\tilde{Q}_{n,m}^1 = \int_{I_1} (x+y)^{2H-2} \sin(n^*x) \sin(m^*y) \, dx \, dy \quad (4.72)$$

it gives

$$\tilde{A}_{n,m}^{(1)} = 2H(2H-1)(\tilde{Q}_{n,m}^1 + \tilde{Q}_{n,m}^2). \quad (4.73)$$

**Step 2.** Now, it’s time to extract diagonal and off-diagonal information from $(\tilde{A}_{sub})_{n,m}$. By setting $(\tilde{D}_{sub})_{n,m} = (\tilde{A}_{sub})_{n,m} \delta_{n,m}$, and $(\tilde{O}_{sub})_{n,m} = (\tilde{A}_{sub})_{n,m} - (\tilde{D}_{sub})_{n,m}$, a decomposition $(\tilde{A}_{sub})_{n,m} = (\tilde{D}_{sub})_{n,m} + (\tilde{O}_{sub})_{n,m}$ is obtained. Moreover, there hold

$$(\tilde{D}_{sub})_{n,m} = 2H(2H-1)(\tilde{R}_{n,m}^1 - \tilde{R}_{n,m}^{2,3} - \tilde{Q}_{n,m}^1 - \tilde{Q}_{n,m}^2), \quad (4.74)$$

$$(\tilde{O}_{sub})_{n,m} = 2H(2H-1)(\tilde{R}_{n,m}^1 - \tilde{R}_{n,m}^{2,3} - \tilde{Q}_{n,m}^1 - \tilde{Q}_{n,m}^2), \quad n \neq m. \quad (4.75)$$

The details for handling $\tilde{A}_{n,m}$ part will be emphasised, but the ones for $\tilde{A}_{n,m}^{(1)}$ will be omitted except for the conclusions.

**Step 2.1.** Calculate $\tilde{A}_{n,m}$ and $\tilde{A}_{n,m}^{(1)}$ in the case of $m > n \gg 1$. Simple calculations show that

$$\tilde{R}_{n,m}^1 = 0. \quad (4.76)$$

$$\tilde{R}_{n,m}^{2,3} = \left(\frac{-1}{m^* + (-1)^{m+n}n^*}\right) \int_0^1 u^{2H-2} \sin(m^*u) \sin(n^*u) \, du. \quad (4.77)$$
4.26

Summarise the asymptotic information for the matrix elements of $\tilde{A}_{\text{sub}}$. The asymptotics for diagonal piece of $(\tilde{A}_{\text{sub}})_{n,m}$ is

$$(\tilde{D}_{\text{sub}})_{n,m} = \frac{2\Gamma(2H + 1)\sin(\pi H)}{n^{2H - 1}} \cdot \frac{(4H + 1)\Gamma(2H + 1)\cos(\pi H)}{2m^{2H}} + O\left(\frac{1}{m^2}\right)$$

if $m \gg 1$, and the ones for off-diagonal piece is

$$(\tilde{O}_{\text{sub}})_{n,m} \approx \frac{1}{mn^{2H - 1}}$$

Using (4.24), (4.26), (4.30) and (4.33), it implies that

$$\tilde{R}_{n,m}^1 - \tilde{R}_{n,m}^{2,3} - \tilde{Q}_{n,m}^1 - \tilde{Q}_{n,m}^2 = 3(-1)^{m+n+1}\Gamma(2H - 1)\cos(\pi H) \left(\frac{1}{m^{2H - 1}} + (-1)^{m+n+1}\frac{1}{m^{2H - 1}}\right)$$

$$+ (-1)^{m+n+1}\Gamma(2H - 1)\cos(\pi H) \left(\frac{1}{m^{2H - 1}} + (-1)^{m+n+1}\frac{1}{m^{2H - 1}}\right) + O\left(\frac{1}{mn}\right).$$

Using the same techniques as Step 3.3. in Lemma 4.3, it’s obvious that $\tilde{R}_{n,m}^1 - \tilde{R}_{n,m}^{2,3} - \tilde{Q}_{n,m}^1 - \tilde{Q}_{n,m}^2 \approx m^{-1}n^{2H - 1}$.

Step 2.2. Calculate $\tilde{A}_{n,m}$ and $\tilde{A}_{\text{sub}}^{(1)}$ in the case of $m = n \gg 1$. It’s easy to check that

$$\tilde{R}_{n,m}^2 = \int_0^1 \sin(m^* v) \sin(m^* v) \dd v \int_{-1}^1 |u|^{2H - 2} \cos(m^* u) \dd u$$

$$= 2\left[\frac{\Gamma(2H - 1)\sin(\pi H)}{m^{2H - 1}} + (-1)^m + O\left(\frac{1}{m^2}\right)\right]$$

$$\tilde{R}_{n,m}^{2,3} = -\frac{1}{m^*} \int_0^1 u^{2H - 2} \sin(m^* u) \dd u$$

$$= \frac{\Gamma(2H - 1)\cos(\pi H)}{m^{2H}} + O\left(\frac{1}{m^3}\right).$$

Following the similar procedures as Step 3. in Section 4.1, there hold

$$\tilde{Q}_{n,m}^1 = \frac{1}{2} \int_0^1 u^{2H - 1} \cos(m^* u) \dd u - \frac{1}{2m^*} \int_0^1 u^{2H - 2} \sin(m^* u) \dd u.$$

$$\tilde{Q}_{n,m}^2 = \frac{1}{2m^*} \int_1^2 u^{2H - 2} \sin(m^* u) \dd u + \frac{1}{2} \int_1^2 u^{2H - 1} \cos(m^* u) \dd u - \int_1^2 u^{2H - 2} \cos(m^* u) \dd u.$$

Using (4.23), (4.24), (4.25) and (4.32), it leads to

$$\tilde{R}_{n,n}^1 - \tilde{R}_{n,n}^{2,3} - \tilde{Q}_{n,n}^1 - \tilde{Q}_{n,n}^2 = \frac{2\Gamma(2H - 1)\sin(\pi H)}{n^{2H - 1}} \cdot \frac{(2\Gamma(2H + 1)\cos(\pi H)}{2n^{2H}} + O\left(\frac{1}{n^2}\right).$$

Step 2.3. Summarise the asymptotic information for the matrix elements of $\tilde{A}_{\text{sub}}$. The asymptotics for diagonal piece of $(\tilde{A}_{\text{sub}})_{n,m}$ is

$$(\tilde{D}_{\text{sub}})_{n,m} = \frac{2\Gamma(2H + 1)\sin(\pi H)}{n^{2H - 1}} \cdot \frac{(4H + 1)\Gamma(2H + 1)\cos(\pi H)}{2n^{2H}} + O\left(\frac{1}{mn^2}\right)$$
if $m > n \gg 1$.

Step 3. Noticing that $\hat{D}_{\text{sub}}$ is self-adjoint and positive, given any $\beta \in (0, 1)$, $\hat{O}_{\text{sub}}$ can also be written as $\hat{O}_{\text{sub}} = \hat{D}_{\text{sub}}^{-\beta} \hat{O}_{\text{der}} \hat{D}_{\text{sub}}^{-\beta}$ with $\hat{O}_{\text{der}} = \hat{D}_{\text{sub}}^{-\beta} \hat{O}_{\text{sub}} \hat{D}_{\text{sub}}^{-\beta}$. Since the order of the elements of $\hat{D}_{\text{sub}}^{-\beta}$ is $m^{\beta(2H-1)}$ when $m \gg 1$, the order of the ones of $\hat{O}_{\text{der}}$ is $m^{(2H-1)\beta - 1} m^{(2H-1)\beta - 2H + 1}$ when $m > n \gg 1$. If $\beta \in (0, \frac{1}{2})$, it’s easy to verify whether the elements of $\hat{O}_{\text{der}}$ are square summable. In fact,

$$\sum_{m>n} (\hat{O}_{\text{der}})^2_{m,n} \leq \sum_{m>n} m^{2(2H-1)\beta - 2} n^{2(2H-1)\beta - 4H + 2}$$

$$= \sum_{n} n^{2(2H-1)\beta - 4H + 2} \sum_{m=n+1}^{\infty} m^{2(2H-1)\beta - 2} \lesssim \sum_{n} n^{4(2H-1)\beta - 4H+1}.$$  

The square summability of $\hat{O}_{\text{der}}$ is verified since $4(2H-1)\beta - 4H + 1 \in (-4H+1, -1)$ when $\beta \in (0, \frac{1}{2})$. Therefore, $\hat{O}_{\text{der}}$ is a Hilbert-Schmidt operator (and thus compact). Using Lemma 4.5, it is immediately obtained that

$$|\lambda_n(\hat{O}_{\text{sub}})| \lesssim n^{-2H + \frac{1}{2} + \delta}. \quad (4.87)$$

Setting $K_1 = \hat{D}_{\text{sub}}, K_2 = \hat{O}_{\text{sub}}$ and $j = n^\alpha$ in Lemma 4.6, it can be deduced that

$$\lambda_n(\hat{\Delta}_{\text{sub}}) \leq \frac{K_{H}}{n^{2H-1}} + O(n^{-2H+\alpha}) + O(n^{-\alpha(2H-\frac{1}{2}-\delta)}). \quad (4.88)$$

Choosing $2H - \alpha = \alpha(2H - \frac{1}{2})$ (i.e. $\alpha = \frac{2H}{2H + \frac{1}{2}}$), it implies that

$$\lambda_n(\hat{\Delta}_{\text{sub}}) \leq \frac{K_{H}}{n^{2H-1}} + o(n^{-2H(\frac{1}{2H} + 1)+\delta}). \quad (4.89)$$

Repeating the argument with $K_1 = \hat{\Delta}_{\text{sub}}, K_2 = -\hat{O}_{\text{sub}}$ gives

$$\lambda_n(\hat{\Delta}_{\text{sub}}) \geq \frac{K_{H}}{n^{2H-1}} + o(n^{-2H(\frac{1}{2H} + 1)+\delta}). \quad (4.90)$$

The proof is completed.  

5. ACKNOWLEDGEMENT

Ying-Li WANG, one of the authors, would like to thank Professor Ping HE and Assistant Professor Qing-Hua WANG for their patient discussion.

REFERENCES

[1] Ash, R. B., Gardner, M. F.: Topics in Stochastic Processes. New York: Academic Press, (1975).
[2] Biagini, F., Hu, Y., Oksendal, B., Zhang, T.: Stochastic Calculus for Fractional Brownian Motion and Applications, London: Springer, (2008).
[3] Bingham, N. H., Goldie, C. M., Teugels, J. L.: Regular Variation. Cambridge University Press, (1989).
[4] Bojdecki, T., Gorostiza, L. G., Talarczyk, A.: Sub-fractional Brownian motion and its relation to occupation times, Statistics & Probability Letters 69, (2004), 405-419.
[5] Bronski, J. C.: Asymptotics of Karhunen-Loeve eigenvalues and tight constants for probability distributions of passive scalar transport, Communications in Mathematical Physics 238, (2003), 563-582.

[6] Chigansky, P., Kleptsyna, M.: Exact asymptotics in eigenproblems with fractional covariance operators, Stochastic Processes and their Applications 128, (2018), 2007-2059.

[7] Chigansky, P., Kleptsyna, M., Marushkevych, D.: On the eigenproblem for Gaussian bridges, Bernoulli 26(3), (2020), 1706-1726.

[8] Chigansky, P., Kleptsyna, M.: Exact spectral asymptotics of fractional processes, (2018), arXiv preprint 1802.09045v2.

[9] Li, W. V., Shao, Q. M.: Stochastic processes: theory and methods, Handbook of Statist., Vol.19, Elsevier, New York (2001), 533-597.

[10] Porter, D., Stirling, D. S. G.: Integral Equations. Cambridge: Cambridge University Press, (1990).

[11] Veillette, M. S., Taqqu, M. S.: Properties and numerical evaluation of the rosenblatt distribution, Bernoulli 19(3), (2013), 982-1005.