Abstract

We study and generalize in various ways the model of rational expectation (RE) bubbles introduced by Blanchard and Watson in the economic literature. Bubbles are argued to be the equivalent of Goldstone modes of the fundamental rational pricing equation, associated with the symmetry-breaking introduced by non-vanishing dividends. Generalizing bubbles in terms of multiplicative stochastic maps, we summarize the result of Lux and Sornette that the no-arbitrage condition imposes that the tail of the return distribution is hyperbolic with an exponent $\mu < 1$. We then outline the main results of Malevergne and Sornette, who extend the RE bubble model to arbitrary dimensions $d$: a number $d$ of market time series are made linearly interdependent via $d \times d$ stochastic coupling coefficients. We derive the no-arbitrage condition in this context and, with the renewal theory for products of random matrices applied to stochastic recurrence equations, we extend the theorem of Lux and Sornette to demonstrate that the tails of the unconditional distributions associated with such $d$-dimensional bubble processes follow power laws, with the same asymptotic tail exponent $\mu < 1$ for all assets. The distribution of price differences and of returns is dominated by the same power-law over an extended range of large returns. Although power-law tails are a pervasive feature of empirical data, the numerical value $\mu < 1$ is in disagreement with the usual empirical estimates $\mu \approx 3$. We then discuss two extensions (the crash hazard rate model and the non-stationary growth rate model) of the RE bubble model that provide two ways of reconciliation with the stylized facts of financial data.
1 The model of rational bubbles

Blanchard [5] and Blanchard and Watson [6] originally introduced the model of rational expectations (RE) bubbles to account for the possibility, often discussed in the empirical literature and by practitioners, that observed prices may deviate significantly and over extended time intervals from fundamental prices. While allowing for deviations from fundamental prices, rational bubbles keep a fundamental anchor point of economic modelling, namely that bubbles must obey the condition of rational expectations. In contrast, recent works stress that investors are not fully rational, or have at most bound rationality, and that behavioral and psychological mechanisms, such as herding, may be important in the shaping of market prices [41,33,34]. However, for fluid assets, dynamic investment strategies rarely perform over simple buy-and-hold strategies [30], in other words, the market is not far from being efficient and little arbitrage opportunities exist as a result of the constant search for gains by sophisticated investors. Here, we shall work within the conditions of rational expectations and of no-arbitrage condition, taken as useful approximations. Indeed, the rationality of both expectations and behavior often does not imply that the price of an asset be equal to its fundamental value. In other words, there can be rational deviations of the price from this value, called rational bubbles. A rational bubble can arise when the actual market price depends positively on its own expected rate of change, as sometimes occurs in asset markets, which is the mechanism underlying the models of [5] and [6].

In order to avoid the unrealistic picture of ever-increasing deviations from fundamental values, Blanchard [6] proposed a model with periodically collapsing bubbles in which the bubble component of the price follows an exponential explosive path (the price being multiplied by \( a_t = \bar{a} > 1 \)) with probability \( \pi \) and collapses to zero (the price being multiplied by \( a_t = 0 \)) with probability \( 1 - \pi \). It is clear that, in this model, a bubble has an exponential distribution of lifetimes with a finite average lifetime \( \frac{\pi}{1 - \pi} \). Bubbles are thus transient phenomena. The condition of rational expectations imposes that \( \bar{a} = 1/\delta \), where \( \delta \) is the discount factor. In order to allow for the start of new bubbles after the collapse, a stochastic zero mean normally distributed component \( b_t \) is added to the systematic part of \( X_t \). This leads to the following dynamical equation

\[
X_{t+1} = a_t X_t + b_t,
\]

where, as we said, \( a_t = \bar{a} \) with probability \( \pi \) and \( a_t = 0 \) with probability \( 1 - \pi \). Both variables \( a_t \) and \( b_t \) do not depend on the process \( X_t \). There is a huge literature on theoretical refinements of this model and on the empirical detectability of RE bubbles in financial data (see [11] and [1], for surveys of this literature). Model (1) has also been explored in a large variety of contexts, for instance in ARCH processes in econometry [14], 1D random-field Ising models [13] using Mellin transforms, and more recently using extremal properties of the \( G \)-harmonic func-
tions on non-compact groups [26] and the Wiener-Hopf technique [35]. See also [37] for a short review of other domains of applications including population dynamics with external sources, epidemics, immigration and investment portfolios, the internet, directed polymers in random media.

Large $|X_k|$ are generated by intermittent amplifications resulting from the multiplication by several successive values of $|a|$ larger than one. We now offer a simple “mean-field” type argument that clarifies the origin of the power law fat tail. Let us call $p_>$ the probability that the absolute value of the multiplicative factor $a$ is found larger than 1. The probability to observe $n$ successive multiplicative factors $|a|$ larger than 1 is thus $p_>^n$. Let us call $|a_>|$ the average of $|a|$ conditioned on being larger than 1: $|a_>|$ is thus the typical absolute value of the amplification factor. When $n$ successive multiplicative factors occur with absolute values larger than 1, they typically lead to an amplification of the amplitude of $X$ by $|a_>|^n$. Using the fact that the additive term $b_k$ ensures that the amplitude of $X_k$ remains of the order of the standard deviation or of other measures of typical scales $\sigma_b$ of the distribution $P_b(b)$ when the multiplicative factors $|a|$ are less than 1, this shows that a value of $X_k$ of the order of $|X| \approx \sigma_b |a_>|^n$ occurs with probability

$$p_>^n = \exp (n \ln p_>) \approx \exp \left( \ln p_> \frac{\ln |X|}{\ln |a_>|} \right) = \frac{1}{(|X|/\sigma_b)\mu} \quad (2)$$

with $\mu = \ln p_> / \ln |a_>|$, which can be rewritten as $p_>|a_>|^\mu = 1$. Note the similarity between this last “mean-field” equation and the exact solution (8) given below. The power law distribution is thus the result of an exponentially small probability of creating an exponentially large value [39]. Expression (2) does not provide a precise determination of the exponent $\mu$, only an approximate one since we have used a kind of mean-field argument in the definition of $|a_>|$.

In the next section, we recall how bubbles appear as possible solutions of the fundamental pricing equation and play the role of Goldstone modes of a price-symmetry broken by the dividend flow. We then describe the Kesten generalization of rational bubbles in terms of random multiplicative maps and present the fundamental result [28] that the no-arbitrage condition leads to the constraint that the exponent of the power law tail is less than 1. We then present an extension to arbitrary multidimensional random multiplicative maps: a number $d$ of market time series are made linearly interdependent via $d \times d$ stochastic coupling coefficients. We show that the no-arbitrage condition imposes that the non-diagonal impacts of any asset $i$ on any other asset $j \neq i$ has to vanish on average, i.e., must exhibit random alternative regimes of reinforcement and contrarian feedbacks. In contrast, the diagonal terms must be positive and equal on average to the inverse of the discount factor. Applying the results of renewal theory for products of random matrices to stochastic recurrence equations (SRE), we extend the theorem of [28] and demonstrate that the tails of the unconditional distributions associated with such $d$-dimensional bub-
ble processes follow power laws (i.e., exhibit hyperbolic decline), with the same asymptotic tail exponent $\mu < 1$ for all assets. The distribution of price differences and of returns is dominated by the same power-law over an extended range of large returns. In order to unlock the paradox, we briefly discuss the crash hazard rate model [21,23] and the non-stationary growth model [40]. We conclude by proposing a link with the theory of speculative pricing through a spontaneous symmetry-breaking [38].

We should stress that, due to the no-arbitrage condition that forms the backbone of our theoretical approach, correlations of returns are vanishing. In addition, the multiplicative stochastic structure of the models ensures the phenomenon of volatility clustering. These two stylized facts, taken for granted in our present approach, will not be discussed further.

2 Rational bubbles of an isolated asset [28]

2.1 Rational expectation bubble model and Goldstone modes

We first briefly recall that pricing of an asset under rational expectations theory is based on the two following hypothesis : the rationality of the agents and the “no-free lunch” condition.

Under the rational expectation condition, the best estimation of the price $p_{t+1}$ of an asset at time $t+1$ viewed from time $t$ is given by the expectation of $p_{t+1}$ conditionned upon the knowledge of the filtration $\{F_t\}$ (i.e. sum of all available information accumulated) up to time $t$ : $E[|F_t]$.

The “no-free lunch” condition imposes that the expected returns of every assets are all equal under a given probability measure $Q$ equivalent to the historical probability measure $P$. In particular, the expected return of each asset is equal to the return $r$ of the risk-free asset (which is assumed to exist), and thus the probability measure $Q$ is named the risk neutral probability measure.

Putting together these two conditions, we are led to the following valuation formula for the price $p_t$ :

$$p_t = \delta \cdot E_Q[p_{t+1}|F_t] + d_t \quad \forall \{p_t\}_{t \geq 0},$$  \hspace{1cm} (3)

where $d_t$ is an exogeneous “dividend”, and $\delta = (1 + r)^{-1}$ is the discount factor. The first term in the r.h.s. quantifies the usual fact that something tomorrow is less valuable than today by a factor called the discount factor. Intuitively, the second term, the dividend, is added to express the fact that the expected price tomorrow
has to be decreased by the dividend since the value before giving the dividend incorporates it in the pricing.

The “forward” solution of (3) is well-known to be the fundamental price

\[ p_t^f = \sum_{i=0}^{+\infty} \delta^i \cdot E_Q[d_{t+i} | \mathcal{F}_t]. \]  

(4)

It is straightforward to check by replacement that the sum of the forward solution (4) and of an arbitrary component \( X_t \)

\[ p_t = p_t^f + X_t , \]  

(5)

where \( X_t \) has to obey the single condition of being an arbitrary martingale:

\[ X_t = \delta \cdot E_Q[X_{t+1} | \mathcal{F}_t] \]  

(6)

is also the a solution of (3). In fact, it can be shown [18] that (5) is the general solution of (3).

Here, it is important to note that, in the framework of the Blanchard and Watson model, the speculative bubbles appear as a natural consequence of the valuation formula (3), i.e., of the no free-lunch condition and the rationality of the agents. Thus, the concept of bubbles is not an addition to the theory, as sometimes believed, but is entirely embedded in it.

Notice also that the component \( X_t \) in (5) plays a role analogous to the Goldstone mode in nuclear, particle and condensed-matter physics [9,10]. Goldstone modes are the zero-wavenumber zero-energy modal fluctuations that attempt to restore a broken symmetry. For instance, consider a Bloch wall between two semi-infinite magnetic domains of opposite spin directions selected by opposite magnetic field at boundaries far away. At non-zero temperature, capillary waves are excited by thermal fluctuations. The limit of very long-wavelength capillary modes correspond to Goldstone modes that tend to restore the translational symmetry broken by the presence of the Bloch wall [42].

In the present context, as shown in [38], the

\[ p \rightarrow -p \quad \text{parity symmetry} \]  

(7)

is broken by the “external” field embodied in the dividend flow \( d_t \). Indeed, as can be seen from (3) and its forward solution (4), the fundamental price is identically zero in absence of dividends. Ref. [38] has stressed the fact that it makes perfect sense to think of negative prices. For instance, we are ready to pay a (positive) price
for a commodity that we need or like. However, we will not pay a positive price to get something we dislike or which disturb us, such as garbage, waste, broken and useless car, chemical and industrial hazards, etc. Consider a chunk of waste. We will be ready to buy it for a negative price, in order words, we are ready to take the unwanted commodity if it comes with cash. Positive dividends imply positive prices, negative dividends lead to negative prices. Negative dividends correspond to the premium to pay to keep an asset for instance. From an economic viewpoint, what makes a share of a company desirable is its earnings, that provide dividends, and its potential appreciation that give rise to capital gains. As a consequence, in absence of dividends and of speculation, the price of share must be nil and the symmetry (7) holds. The earnings leading to dividends $d$ thus act as a symmetry-breaking “field”, since a positive $d$ makes the share desirable and thus develop a positive price.

It is now clear that the addition of the bubble $X_t$, which can be anything but for the martingale condition (6), is playing the role of the Goldstone modes restoring the broken symmetry: the bubble price can wander up or down and, in the limit where it becomes very large in absolute value, dominate over the fundamental price, restoring the independence with respect to dividend. Moreover, as in condensed-matter physics where the Goldstone mode appears spontaneously since it has no energy cost, the rational bubble itself can appear spontaneously with no dividend.

The “bubble” Goldstone mode turns out to be intimately related to the “money” Goldstone mode introduced by Bak et al. [3]. Ref. [3] introduces a dynamical many-body theory of money, in which the value of money in equilibrium is not fixed by the equations, and thus obeys a continuous symmetry. The dynamics breaks this continuous symmetry by fixating the value of money at a level which depends on initial conditions. The fluctuations around the equilibrium, for instance in the presence of noise, are governed by the Goldstone modes associated with the broken symmetry. In apparent contrast, a bubble represents the arbitrary deviation from fundamental valuation. Introducing money, a given valuation or price is equivalent to a certain amount of money. A growing bubble thus corresponds to the same asset becoming equivalent to more and more cash. Equivalently, from the point of view of the asset, this can be seen as cash devaluation, i.e., inflation. The “bubble” Goldstone mode and the “money” Goldstone mode are thus two facets of the same fundamental phenomenon: they both are left unconstrained by the valuation equations.

2.2 The no-arbitrage condition and fat tails

Following [28], we study the implications of the RE bubble models for the unconditional distribution of prices, price changes and returns resulting from a general discrete-time formulation extending (1) by allowing the multiplicative factor
\( a_t \) to take arbitrary values and be i.i.d. random variables drawn from some non-degenerate probability density function (pdf) \( P_a(a) \). The model can also be generalized by considering non-normal realizations of \( b_t \) with distribution \( P_b(b) \) with \( E_P[b_t] = 0 \), where \( E_P[\cdot] \) is the unconditionnal expectation with respect to the probability measure \( P \).

Provided \( E_P[\ln a] < 0 \) (stationarity condition) and if there is a number \( \mu \) such that \( 0 < E_P[|b|^\mu] < +\infty \), such that

\[
E_P[|a|^\mu] = 1
\]  
(8)

and such that \( E_P[|a|^\mu \ln |a|] < +\infty \), then the tail of the distribution of \( X \) is asymptotically (for large \( X \)'s) a power law [24,16]

\[
P_X(X) \, dX \approx \frac{C}{|X|^{1+\mu}} \, dX ,
\]  
(9)

with an exponent \( \mu \) given by the real positive solution of (8).

Rational expectations require in addition that the bubble component of asset prices obeys the “no free-lunch” condition

\[
\delta \cdot E_Q[X_{t+1}|F_t] = X_t
\]  
(10)

where \( \delta < 1 \) is the discount factor and the expectation is taken conditional on the knowledge of the filtration (information) until time \( t \). Condition (10) with (1) imposes first

\[
E_Q[a] = 1/\delta > 1 ,
\]  
(11)

and then

\[
E_P[a] > 1 ,
\]  
(12)

on the distribution of the multiplicative factors \( a_t \).

Consider the function

\[
M(\mu) = E_P[a^\mu] .
\]  
(13)

It has the following properties

(1) \( M(0) = 1 \) by definition,
(2) \( M'(0) = E_p[\ln a] < 0 \) from the stationarity condition,

(3) \( M''(\mu) = E_p[(\ln a)^2|a|^{\mu}] > 0 \), by the positivity of the square,

(4) \( M(1) = 1/\delta > 1 \) by the no-arbitrage result (12).

\( M(\mu) \) is thus convex and is shown in figure 1. This demonstrate that \( \mu < 1 \) automatically (see [28] for a detailed demonstration). It is easy to show [28] that the distribution of price differences has the same power law tail with the exponent \( \mu < 1 \) and the distribution of returns is dominated by the same power-law over an extended range of large returns [28], as shown in figure 2. Although power-law tails are a pervasive feature of empirical data, these characterizations are in strong disagreement with the usual empirical estimates which find \( \mu \approx 3 \) [43, 27, 32, 19, 17]. Lux and Sornette [28] concluded that exogenous rational bubbles are thus hardly reconciliable with some of the stylized facts of financial data at a very elementary level.

3 Generalization of rational bubbles to arbitrary dimensions [29]

3.1 Generalization to several coupled assets

In reality, there is no such thing as an isolated asset. Stock markets exhibit a variety of inter-dependences, based in part on the mutual influences between the USA, European and Japanese markets. In addition, individual stocks may be sensitive to the behavior of the specific industry as a whole to which they belong and to a few other indicators, such as the main indices, interest rates and so on. Mantegna et al. [31, 7] have indeed shown the existence of a hierarchical organization of stock interdependences. Furthermore, bubbles often appear to be not isolated features of a set of markets. For instance, ref. [15] tested whether a bubble simultaneously existed across the nations, such as Germany, Poland, and Hungary, that experienced hyperinflation in the early 1920s. Coordinated bubbles can sometimes be detected. One of the most prominent example is found in the market appreciations observed in many of the world markets prior to the world market crash in Oct. 1987 [4]. Similar intermittent coordination of bubbles have been detected among the significant bubbles followed by large crashes or severe corrections in Latin-American and Asian stock markets [22]. It is therefore desirable to generalize the one-dimensional RE bubble model (1) to the multi-dimensional case. One could also hope a priori that this generalization would modify the result \( \mu < 1 \) obtained in the one-dimensional case and allow for a better adequation with empirical results. Indeed, 1d-systems are well-known to exhibit anomalous properties often not shared by higher dimensional systems. Here however, it turns out that the same result \( \mu < 1 \) holds, as we shall see.

In the case of several assets, rational pricing theory again dictates that the fun-
damental price of each individual asset is given by a formulalike (3), where the specific dividend flow of each asset is used, with the same discount factor. The corresponding forward solution (4) is again valid for each asset. The general solution for each asset is (5) with a bubble component $X_t$ different from an asset to the next. The different bubble components can be coupled, as we shall see, but they must each obey the martingale condition (6), component by component. This imposes specific conditions on the coupling terms, as we shall see.

Following this reasoning, we can therefore propose the simplest generalization of a bubble into a “two-dimensional” bubble for two assets $X$ and $Y$ with bubble prices respectively equal to $X_t$ and $Y_t$ at time $t$. We express the generalization of the Blanchard-Watson model as follows:

$$X_{t+1} = a_t X_t + b_t Y_t + \eta_t$$  \hspace{1cm} (14)

$$Y_{t+1} = c_t X_t + d_t Y_t + \epsilon_t$$  \hspace{1cm} (15)

where $a_t$, $b_t$, $c_t$ and $d_t$ are drawn from some multivariate probability density function. The two additive noises $\eta_t$ and $\epsilon_t$ are also drawn from some distribution function with zero mean. The diagonal case $b_t = c_t = 0$ for all $t$ recovers the previous one-dimensional case with two uncoupled bubbles, provided $\eta_t$ and $\epsilon_t$ are independent.

Rational expectations require that $X_t$ and $Y_t$ obey both the “no-free lunch” condition (10), i.e., $\delta \cdot E_Q[X_{t+1}|F_t] = X_t$ and $\delta \cdot E_Q[Y_{t+1}|F_t] = Y_t$. With (14,15), this gives

$$\left(E_Q[a_t] - \delta^{-1}\right) X_t + E_Q[b_t] Y_t = 0,$$  \hspace{1cm} (16)

$$E_Q[c_t] X_t + \left(E_Q[d_t] - \delta^{-1}\right) Y_t = 0,$$  \hspace{1cm} (17)

where we have used that $\eta_t$ and $\epsilon_t$ are centered. The two equations (16,17) must be true for all times, i.e. for all values of $X_t$ and $Y_t$ visited by the dynamics. This imposes $E_Q[b_t] = E_Q[c_t] = 0$ and $E_Q[a_t] = E_Q[d_t] = \delta^{-1}$. We are going to retrieve this result more formally in the general case.

3.2 General formulation

A generalization to arbitrary dimensions leads to the following stochastic random equation (SRE)

$$X_t = A_t X_{t-1} + B_t$$  \hspace{1cm} (18)
where \((X_t, B_t)\) are \(d\)-dimensional vectors. Each component of \(X_t\) can be thought of as the price of an asset above its fundamental price. The matrices \((A_t)\) are identically independent distributed \(d \times d\)-dimensional stochastic matrices. We assume that \(B_t\) are identically independent distributed random vectors and that \((X_t)\) is a causal stationary solution of (18). Generalizations introducing additional arbitrary linear terms at larger time lags such as \(X_{t-2}, \ldots\) can be treated with slight modifications of our approach and yield the same conclusions. We shall thus confine our demonstration on the SRE of order 1, keeping in mind that our results apply analogously to arbitrary orders of regressions.

In the following, we denote by \(|\cdot|\) the Euclidean norm and by \(||\cdot||\) the corresponding norm for any \(d \times d\)-matrix \(A\)

\[
||A|| = \sup_{|x|=1} |Ax|.
\]  

(19)

Technical details are given in [29].

3.3 The no-free lunch condition

The valuation formula (3) and the martingale condition (6) given for a single asset easily extends to a basket of assets. It is natural to assume that, for a given period \(t\), the discount rate \(r_t(i)\), associated with asset \(i\), are all the same. In frictionless markets, a deviation for this hypothesis would lead to arbitrage opportunities. Furthermore, since the sequence of matrices \(\{A_t\}\) is assumed to be i.i.d. and therefore stationary, this implies that \(\delta_t\) or \(r_t\) must be constant and equal respectively to \(\delta\) and \(r\).

Under those conditions, we have the following proposition:

**Proposition 1**

*The stochastic process*

\[
X_t = A_t X_{t-1} + B_t
\]  

(20)

*satisfies the no-arbitrage condition if and only if*

\[
E_Q[A] = \frac{1}{\delta} I_d.
\]  

(21)

The proof is given in [29] in which this condition (21) is also shown to hold true under the historical probability measure \(\mathbb{P}\).
The condition (21) imposes some stringent constraints on admissible matrices $A_t$. Indeed, while $A_t$ are not diagonal in general, their average must be diagonal. This implies that the off-diagonal terms of the matrices $A_t$ must take negative values, sufficiently often so that their averages vanish. The off-diagonal coefficients quantify the influence of other bubbles on a given one. The condition (21) thus means that the average effect of other bubbles on any given one must vanish. It is straightforward to check that, in this linear framework, this implies an absence of correlation (but not an absence of dependence) between the different bubble components $E[X^{(k)}X^{(\ell)}] = 0$ for any $k \neq \ell$.

In contrast, the diagonal elements of $A_t$ must be positive in majority in order for $E_{\mathbb{P}}[A_{ii}] = \delta^{(i)-1}$, for all $i$’s, to hold true. In fact, on economic grounds, we can exclude the cases where the diagonal elements take negative values. Indeed, a negative value of $A_{ii}$ at a given time $t$ would imply that $X_t^{(i)}$ abruptly change sign between $t - 1$ and $t$, what does not seem to be a reasonable financial process.

3.4 Renewal theory for products of random matrices

In the following, we will consider that the random $d \times d$ matrices $A_t$ are invertible matrices with real entries. We use the theorem 2.7 of Davis et al. [12], which synthetized Kesten’s theorems 3 and 4 in [24], to the case of real valued matrices. The proof of this theorem is given in [25]. We stress that the conditions listed below do not require the matrices $(A_n)$ to be non-negative. Actually, we have seen that, in order for the rational expectation condition not to lead to trivial results, the off-diagonal coefficients of $(A_n)$ have to be negative with sufficiently large probability such that their means vanish.

**Theorem 1**

Let $(A_n)$ be an i.i.d. sequence of matrices in $GL_d(\mathbb{R})$ satisfying the following set of conditions that we state in a heuristic manner (see [29] for technical details). Provided that the following conditions hold,

$H1$ : stationarity condition,
$H2$ : ergodicity,
$H3$ : intermittent amplification of the random matrices,
$H4$ : the fattailness of the distribution is not controlled by that of the additive part $B_t$, then,

- there exists a unique solution $\mu \in (0, \mu_0]$ to the equation

$$\lim_{n \to \infty} \frac{1}{n} \ln E_{\mathbb{P}}[||A_1 \cdots A_n||^{\mu}] = 0,$$

(22)

- If $(X_n)$ is the stationary solution to the stochastic recurrence equation in (18)
then $X$ is regularly varying with index $\mu$. In other words, the tail of the marginal distribution of each of the components of the vector $X$ is asymptotically a power law with exponent $\mu$.

The equation (22) determining the tail exponent $\mu$ reduces to (8) in the one-dimensional case, which is simple to handle. In the multi-dimensional case, the novel feature is the non-diagonal nature of the multiplication of matrices which does not allow in general for an explicit equation similar to (8).

It is important to stress that the tails of the distribution of returns for all the components of the bubble decrease with the same tail index $\mu$. This model thus provides a natural setting for rationalizing the well-documented empirical observation that the exponent $\mu$ is found to be approximately the same for most assets. The constraint on its value discussed in the next paragraph does not diminishes the value of this remark, as explained in section 5.

3.5 Constraint on the tail index

The first conclusion of the theorem above shows that the tail index $\mu$ of the process $(X_t)$ is driven by the behavior of the matrices $(A_t)$. We will then state a proposition in which we give a majoration of the tail index.

**Proposition 2**

A necessary condition to have $\mu > 1$ is that the spectral radius (largest eigenvalue) of $E_P[A]$ be smaller than 1:

$$
\mu > 1 \implies \rho(E_P[A]) < 1.
$$

(23)

The proof is given in [29]. This proposition, put together with Proposition 1 above, allows us to derive our main result. We have seen in section 3.3 from Proposition 1 that, as a result of the no-arbitrage condition, the spectral radius of the matrix $E_P[A]$ is greater than 1. As a consequence, by application of the converse of Proposition 2, we find that the tail index $\mu$ of the distribution of $(X)$ is smaller than 1. This result does not rely on the diagonal property of the matrices $E_P[A_t]$ but only on the value of the spectral radius imposed by the no-arbitrage condition.

This result generalizes to arbitrary $d$-dimensional processes the result of [28]. As a consequence, $d$-dimensional rational expectation bubbles linking several assets suffer from the same discrepancy compared to empirical data as the one-dimensional bubbles. It would therefore appear that exogenous rational bubbles are hardly reconcilable with some of the most fundamental stylized facts of financial data at a very elementary level.
At this stage, we have to ask the question: what is wrong with the model of rational bubbles? Two alternative answers are explored below: either we believe in the standard valuation formula and we are led to extend the restricted framework described by the Blanchard and Watson’s model; or we believe in the existence of the bubbles within their framework and we have to generalize the valuation formula. In the next two sections, we will discuss these two points of view.

4 The crash hazard rate model \[21,23\]

In the stylised framework of a purely speculative asset that pays no dividends - i.e with zero fundamental price- and in which we ignore information asymmetry and the market-clearing condition, the price of the asset equals the price of the bubble and the valuation formula (3) leads to the familiar martingale hypothesis for the bubble price:

$$\text{for all } t' > t \quad \delta_{t \to t'} E_Q[X(t')|\mathcal{F}_t] = X(t).$$

This equation is nothing but a generalisation of equation (6) to a continuous time formulation, in which $\delta_{t \to t'}$ denotes the discount factor from time $t$ to time $t'$.

We consider a general bubble dynamics given by

$$dX = m(t) X(t) \, dt - \kappa X(t) \, dj ,$$

where $m(t)$ can be any nonlinear causal function of $X$ itself. We add a jump process $j$ to capture the possibility that the bubble exhibits a crash. $j$ is thus zero before the crash and one afterwards. The random nature of the crash occurrence is modeled by the cumulative distribution function $Q(t)$ of the time of the crash, the probability density function $q(t) = dQ/dt$ and the hazard rate $h(t) = q(t)/[1 - Q(t)]$. The hazard rate is the probability per unit of time that the crash will happen in the next instant provided it has not happened yet, i.e:

$$E_Q[dj|\mathcal{F}_t] = h(t)dt .$$

Expression (25) assumes that, during a crash, the bubble drops by a fixed percentage $\kappa \in (0, 1)$, say between 20 and 30% of the bubble price.

Using $E_Q[X(t + dt)|\mathcal{F}_t] = (1 + rdt)X(t)$, where $r$ is the riskless discount rate, taking the expectation of (25) conditioned on the filtration up to time $t$ and using equation (26), we get

$$E_Q[dX|\mathcal{F}_t] = m(t)X(t)dt - \kappa X(t)h(t)dt = rX(t)dt ,$$

13
which yields:

\[ m(t) - r = \kappa h(t). \] (28)

If the crash hazard rate \( h(t) \) increases, the return \( m(t) - r \) above the riskless interest rate increases to compensate the traders for the increasing risk. Reciprocally, if the dynamics of the bubble shoots up, the rational expectation condition imposes an increasing crash risk in order to ensure the absence of arbitrage opportunities: the risk-adjusted return remains constant equal to the risk-free rate. The corresponding equation for the bubble price, conditioned on the crash not to have occurred, is:

\[
\log \left[ \frac{X(t)}{X(t_0)} \right] = rt + \kappa \int_{t_0}^{t} h(t')dt'
\] before the crash. (29)

The integral \( \int_{t_0}^{t} h(t')dt' \) is the cumulative probability of a crash until time \( t \). This gives the logarithm of the bubble price as the relevant observable. It has successfully been applied to the 1929 and 1987 Wall Street crashes up to about 7.5 years prior to the crash [36,23].

The higher the probability of a crash, the faster the bubble must increase (conditional on having no crash) in order to satisfy the martingale condition. Reciprocally, the higher the bubble, the more dangerous is the probability of a looming crash. Intuitively, investors must be compensated by a higher return in order to be induced to hold an asset that might crash. This is the only effect that this model captures. Note that the bubble dynamics can be anything and the bubble can in particular be such that the distribution of returns are fat tails with an exponent \( \mu \approx 3 \) without loss of generality [2].

Ilinski [20] raised the concern that the martingale condition (24) leads to a model which “assumes a zero return as the best prediction for the market.” He continues: “No need to say that this is not what one expects from a perfect model of market bubble! Buying shares, traders expect the price to rise and it is reflected (or caused) by their prediction model. They support the bubble and the bubble support them!”.

In other words, Ilinski [20] criticises a key economic element of the model [21,23]: market rationality. This point is captured by assuming that the market level is expected to stay constant (up to the riskless discount rate) as written in equation (24). Ilinski claims that this equation (24) is wrong because the market level does not stay constant in a bubble: it rises, almost by definition.

This misunderstanding addresses a rather subtle point of the model and stems from the difference between two different types of returns:

1. The unconditional return is indeed zero as seen from (24) and reflects the fair game condition.
The conditional return, conditioned upon no crash occurring between time $t$ and time $t'$, is non-zero and is given by equation (28). If the crash hazard rate is increasing with time, the conditional return will be accelerating precisely because the crash becomes more probable and the investors need to be remunerated for their higher risk.

Thus, the expectation which remains constant in equation (24) takes into account the probability that the market *may* crash. Therefore, *conditionally* on staying in the bubble (no crash yet), the market must rationally rise to compensate buyers for having taken the risk that the market *could* have crashed.

The market price reflects the equilibrium between the greed of buyers who hope the bubble will inflate and the fear of sellers that it may crash. A bubble that goes up is just one that could have crashed but did not. The model [21,23] is well summarised by borrowing the words of another economist: “(...) the higher probability of a crash leads to an acceleration of [the market price] while the bubble lasts.” Interestingly, this citation is culled from the very same article by Blanchard [5] that Ilinski [20] cites as an alternative model more realistic than the model [21,23]. We see that this is in fact more of an endorsement than an alternative.

A simple way to incorporate a different level of risk aversion into the model [21,23] is to say that the probability of a crash in the next instant is perceived by traders as being $K$ times bigger than it objectively is. This amounts to multiplying our hazard rate $h(t)$ by $K$, and once again this makes no substantive difference as long as $K$ is bounded away from zero and infinity. Risk aversion is a central feature of economic theory, and it is generally thought to be stable within a reasonable range, associated with slow-moving secular trends such as changes in education, social structures and technology. Ilinski [20] rightfully points out that risk perceptions are constantly changing in the course of real-life bubbles, but wrongfully claims that the model [21,23] violates this intuition. In this model, risk perceptions do oscillate dramatically throughout the bubble, even though subjective aversion to risk remains stable, simply because it is the *objective degree of risk that the bubble may burst* that goes through wild swings. For these reasons, the criticisms put forth by Ilinski, far from making a dent in the economic model [21,23], serve instead to show that it is robust, flexible and intuitive.

To summarize, the crash hazard rate model is such that the price dynamics can be essentially arbitrary, and in particular such that the corresponding returns exhibit a reasonable fat tail. A jump process for crashes is added, with a crash hazard rate such that the rational expectation condition is ensured.
5 The non-stationary growth model [40]

In the previous section, we have presented a model which assumes that the fundamental valuation formula remains valid and have generalized Blanchard and Watson’s framework by reformulating the rational expectation condition with a jump crash process. We now consider the second viewpoint which consists in rejecting the validity of the valuation formula while keeping the decomposition of the price of an asset into the sum of a fundamental price and a bubble term. In this aim, we present a possible modification of the rational bubble model of Blanchard and Watson, recently proposed in [40], which involves an average exponential growth of the fundamental price at some return rate \( r_f > 0 \) larger than the discount rate.

5.1 Exponentially growing economy

Recall that (5) shows that the observable market price is the sum of the bubble component \( X_t \) and of a “fundamental” price \( p^f_t \)

\[
p_t = p^f_t + X_t .
\] (30)

Thus, waiving off the valuation formula (3), let us assume that the fundamental price \( p^f_t \) is growing exponentially as

\[
p^f_t = p_0 e^{r_f t}
\] (31)

at the rate \( r_f \) and the bubble price is following (1).

Note that this formulation is compatible with the standard valuation formula as long as \( r_f < r \), provided the the cash-flow \( d_t \) at time \( t \) also grows with the same exponential rate \( r_f \), i.e : \( d_t = d_0 e^{r_f t} \). Indeed, the standard valuation formula then applies and leads to

\[
p^f_t = \sum_{k=0}^{\infty} \frac{d_k e^{k r_f}}{e^{k r}} \simeq \frac{d_0}{r - r_f} e^{r_f t}
\] (32)

up to the first order in \( r_f - r \). The discussion of the case \( r_f > r \), for which (32) loses its meaning, is the subject of the sequel in which we follow [38].

Putting (1) and (31) together with (30), we obtain

\[
p_{t+1} = p^f_{t+1} + a_t X_t + b_t = a_t p_t + (e^{r_f} - a_t) p^f_t + b_t .
\] (33)
Replacing $p^f_t$ in (30) by $p_0e^{r_ft}$ given in (31) leads to

$$p_t = e^{r_ft} (p_0 + \hat{a}_t).$$  \hspace{1cm} (34)$$

where we have defined the “reduced” bubble price following

$$\hat{a}_{t+1} = a_te^{-r_ft} \hat{a}_t + e^{-r_ft} e^{-r_ft}b_t.$$ \hspace{1cm} (35)

Thus, if we allow the additive term $b_t$ in (1) to also grow exponentially as

$$b_t = e^{r_ft}\hat{b}_t,$$ \hspace{1cm} (36)

where $\hat{b}_t$ is a stationary stochastic white noise process, we obtain

$$\hat{a}_{t+1} = a_te^{-r_ft} \hat{a}_t + e^{-r_ft} \hat{b}_t,$$ \hspace{1cm} (37)

which is of the usual form. Intuitively, the additive term represents the background of “normal” fluctuations around the fundamental price (31). Their “normal” fluctuations have thus to grow with the same growth rate in order to remain stationary in relative value.

In addition, replacing $p^f_t$ in (33) again by $p_0e^{r_ft}$ leads to

$$p_{t+1} = a_t p_t + e^{r_ft}[p_0(e^{r_ft} - a_t) + \hat{b}_t].$$ \hspace{1cm} (38)

The expression (38) has the same form as (1) with a different additive term $[p_0(e^{r_ft} - a_t) + \hat{b}_t]$ replacing $b_t$. The structure of this new additive term makes clear the origin of the factor $e^{r_ft}$: as we said, it reflects nothing but the average exponential growth of the underlying economy. The contributions $b_t = e^{r_ft}\hat{b}_t$ are then nothing but the fluctuations around this average growth.

5.2 The value of the tail exponent

The condition $E[\ln a] < r_f$ ensures that $E[\ln(ae^{-r_f})] < 0$ which is now the stationarity condition for the process $\hat{a}_t$ defined by (37). The conditions $0 < E[|e^{r_ft}\hat{b}_t|^\mu] < +\infty$ (which is the same condition $0 < E[|b_t|^\mu] < +\infty$ as before) and the solution of

$$E[|ae^{-r_f}|^\mu] = 1$$ \hspace{1cm} (39)

together with the constraint $E[|ae^{-r_f}|^\mu \ln |ae^{-r_f}|] < +\infty$ (which the same as $E[|a|^\mu \ln |a|] < +\infty$) leads to an asymptotic power law distribution for the reduced price variable
\( \hat{a}_t \) of the form \( P_\alpha(\hat{a}) \approx C_\alpha/|\hat{a}|^{1+\mu} \), where \( \mu \) is the real positive solution of (39). Note that the condition \( \mathbb{E}[\ln(ae^{-r_f})] < 0 \) which is \( \mathbb{E}[\ln(a)] < r_f \) now allows for positive average growth rate of the product \( a_t a_{t-1} a_{t-2} \ldots a_2 a_1 a_0 \).

Consider the illustrative case where the multiplicative factors \( a_t \) are distributed according to a log-normal distribution such that \( \mathbb{E}[\ln a] = \ln a_0 \) (where \( a_0 \) is thus the most probable value taken by \( a_t \)) and of variance \( \sigma^2 \). Then,

\[
\mathbb{E}[|ae^{-r_f}|^{\mu}] = \exp \left[-r_f \mu + \mu \ln a_0 + \mu^2 \sigma^2/2 \right]. \tag{40}
\]

Equating (40) to 1 to get \( \mu \) according to equation (39) gives

\[
\mu = 2 \frac{r_f - \ln a_0}{\sigma^2} = \frac{r_f - \ln a_0}{r - \ln a_0} = 1 + \frac{r_f - r}{r - \ln a_0}. \tag{41}
\]

We have used the notation \( 1/\delta = 1 + r \) for the discount rate \( r \) defined in terms of the discount factor \( \delta \). The second equality in (41) uses \( \mathbb{E}[a] = a_0 e^{\sigma^2/2} \).

First, we retrieve the result [28] that \( \mu < 1 \) for the initial RE model (1) for which \( r_f = 0 \) and \( \ln a_0 < 0 \). However, as soon as \( r_f > r \simeq -\ln \delta \), we get

\[
\mu > 1, \tag{42}
\]

and \( \mu \) can take arbitrary values. Technically, this results fundamentally from the structure of the process in which the additive noise grows exponentially to mimick the growth of the bubble which alleviates the bound \( \mu < 1 \). Note that \( r_f \) does not need to be large for the result (42) to hold. Take for instance an annualized discount rate \( r_y = 2\% \), an annualized return \( r_y^f = 4\% \) and \( a_0 = 1.0004 \). Expression (41) predicts \( \mu = 3 \), which is compatible with empirical data.

5.3 Price returns

The observable return is

\[
R_t = \frac{p_{t+1} - p_t}{p_t} = \frac{p_{t+1}^f - p_t^f + X_{t+1} - X_t}{p_t^f + X_t} = \chi_t \left( \frac{p_{t+1}^f - p_t^f}{p_t} + \frac{X_{t+1} - X_t}{p_t^f} \right) = \chi_t \left( r_f + \frac{\hat{a}_{t+1} - \hat{a}_t}{p_0} \right), \tag{43}
\]

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For large bubbles, we can thus distinguish two regimes: where

\[ \chi_t = \frac{p_t^f}{p_t^f + X_t} = \frac{1}{1 + (\hat{a}_t/p_0)}. \]  

(44)

In order to derive the last equality in the right-hand-side of (43), we have used the definition of the return of the fundamental price (neglecting the small second order difference between \( e^{r_f} - 1 \) and \( r_f \)). Expression (43) shows that the distribution of returns \( R_t \) of the observable prices is the same as that of the product of the random variable \( \chi_t \) by \( r_f + (\hat{a}_{t+1} - \hat{a}_t)/p_0 \). Now, the tail of the distribution of \( r_f + (\hat{a}_{t+1} - \hat{a}_t)/p_0 \) is the same as the tail of the distribution of \( \hat{a}_{t+1} - \hat{a}_t \), which is a power law with exponent \( \mu \) solution of (39), as shown rigorously in [28].

It remains to show that the product of this variable \( r_f + (\hat{a}_{t+1} - \hat{a}_t)/p_0 \) by \( \chi_t \) has the same tail behavior as \( r_f + (\hat{a}_{t+1} - \hat{a}_t)/p_0 \) itself. If \( r_f + (\hat{a}_{t+1} - \hat{a}_t)/p_0 \) and \( \chi_t \) were independent, this would follow from results in [8] who demonstrates that for two independent random variables \( \phi \) and \( \chi \) with \( \text{Proba}(|\phi| > x) \approx cx^{-\kappa} \) and \( E[\chi^{\kappa+\epsilon}] < \infty \) for some \( \epsilon > 0 \), the random product \( \phi \chi \) obeys \( \text{Proba}(|\phi \chi| > x) \approx E[\chi^\kappa]x^{-\kappa} \).

However, \( r_f + (\hat{a}_{t+1} - \hat{a}_t)/p_0 \) and \( \chi_t \) are not independent as both contain a contribution from the same term \( \hat{a}_t \). However, when \( \hat{a}_t << p_0 \), \( \chi_t \) is close to 1 and the previous result should hold. The impact of \( \hat{a}_t \) in \( \chi \) becomes important when \( \hat{a}_t \) becomes comparable to \( p_0 \).

It is then convenient to rewrite (43) using (44) as

\[ R_t = \frac{r_f}{1 + (\hat{a}_t/p_0)} + \frac{\hat{a}_{t+1} - A_t}{p_0 + \hat{a}_t} = \frac{r_f}{1 + (\hat{a}_t/p_0)} + \frac{(a_t e^{-r_f} - 1)\hat{a}_t + e^{-r_f}b_t}{p_0 + \hat{a}_t}. \]  

(45)

We can thus distinguish two regimes:

- for not too large values of the reduced bubble term \( \hat{a}_t \), specifically for \( \hat{a}_t < p_0 \), the denominator \( p_0 + \hat{a}_t \) changes more slowly than the numerator of the second term, so that the distribution of returns will be dominated by the variations of this numerator \( (a_t e^{-r_f} - 1)\hat{a}_t + e^{-r_f}b_t \) and, hence, will follow approximately the same power-law as for \( \hat{a}_t \), according to the results of [8].

- For large bubbles, \( \hat{a}_t \) of the order of or greater than \( p_0 \), the situation changes, however: from (45), we see that when the reduced bubble term \( \hat{a}_t \) increases without bound, the first term \( r_f/(1 + (\hat{a}_t/p_0)) \) goes to 0 while the second term becomes asymptotically \( a_t e^{-r} - 1 \). This leads to the existence of an absolute upper bound for the absolute value of the returns.

To summarize, we expect that the distribution of returns will therefore follow a power-law with the same exponent \( \mu \) as for \( \hat{a}_t \), but with a finite cut-off (see [40] for
details). This is validated by numerical simulations shown in figure 3 taken from [40].

Thus, when the price fluctuations associated with bubbles on average grow with the mean market return $r_f$, we find that the exponent of the power law tail of the returns is no more bounded by 1 as soon as $r_f$ is larger than the discount rate $r$ and can take essentially arbitrary values. It is remarkable that this condition $r_f > r$ corresponds to the paradoxical and unsolved regime in fundamental valuation theory where the forward valuation solution (4) loses its meaning, as discussed in [38]. In analogy with the theory of bifurcations and their normal forms, ref.[38] proposed that this regime might be associated with a spontaneous symmetry breaking phase corresponding to a spontaneous valuation in absence of dividends by pure speculative imitative processes.

**Conclusion**

Despite its elegant formulation of the bubble phenomenon, the Blanchard and Watson’s model suffers from a lethal discrepancy: it does not seem to comply with the empirical data, i.e., it cannot generate power law tails whose exponent is greater than 1, in disagreement with the empirical tail index found around 3. We have summarized the demonstration that this result holds true both for a bubble defined for a single asset as well as for bubbles on any set of coupled assets, as long as the rational expectation condition holds.

In order to reconcile the theory with the empirical facts on the tails of the distributions of returns, two alternative models have been presented. The “crash hazard rate” model extends the formulation of Blanchard and Watson by replacing the linear stochastic bubble price equation by an arbitrary dynamics solely constrained by the no-arbitrage condition made to hold with the introduction of a jump process. The “growth rate model” departs more audaciously from standard economic models since it discards one of the pillars of the standard valuation theory, but putting itself firmly in the regime $r_f > r$ for which the fundamental valuation formula breaks down. In addition to allowing for correct values of the tail exponent, it provides a generalization of the fundamental valuation formula by providing an understanding of its breakdown as deeply associated with a spontaneous breaking of the price symmetry. Its implementation for multi-dimensional bubbles is straightforward and the results obtained in section 5 carry over naturally in this case. This provides an explanation for why the tail index $\mu$ seems to be the same for any group of assets as observed empirically. This work begs for the introduction of a generalized field theory which would be able to capture the spontaneous breaking of symmetry, recover the fundamental valuation formula in the normal economic case $r_f < r$ and extend it to the still unexplored regime $r_f > r$. 

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Fig. 1. Convexity of $M(\mu)$. This enforces that the exponent solution of (8) is $\mu < 1$. 
Fig. 2. Frequencies of large price changes and returns for selected bubble processes: the plots exhibit the complement of the cumulative distribution from simulations over $10^6$ time steps (triangles) and compares them with the theoretically predicted tail behavior (straight line with slope $\mu$). In all cases, the scaling behavior is found to provide a good fit over an extended range of magnitudes. However, while with price changes, the entire tail follows a Pareto law with index $\mu$, returns are characterized by deviating behavior at the highest entries. Details on the underlying bubble processes are given in [28].
Fig. 3. Double logarithmic scale representation of the complementary cumulative distribution of the “monthly” returns $R_t$ defined in (43) of the synthetic total price sum of the exponential growing fundamental price and the bubble price. The continuous (resp. dashed) line corresponds to the positive (resp. negative) returns. The distribution is well-described by an asymptotic power law with an exponent in agreement with the prediction $\mu \approx 3.3$ given by the equations (39) and (41) and shown as the straight line. The small differences between the predicted slope and the numerically generated ones are within the error bar of $\pm 0.3$ obtained from a standard maximum likelihood Hill estimation. From [40].