GENERALIZED BSDEs WITH RANDOM TIME HORIZON IN A PROGRESSIVELY ENLARGED FILTRATION

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Abstract

We study generalized backward stochastic differential equations (BSDEs) up to a random time horizon \( \vartheta \), which is not a stopping time, under minimal assumptions regarding the properties of \( \vartheta \). In contrast to existing works in this area, we do not impose specific assumptions on the random time \( \vartheta \) and we study the existence of solutions to BSDEs and reflected BSDEs with a random time horizon through the method of reduction. In addition, we also examine BSDEs and reflected BSDEs with a \( \lambda \)\-l\( \lambda \)\-lag driver where the driver is allowed to have a finite number of common jumps with the martingale part.

**Keywords**: BSDE, reflected BSDE, nonlinear evaluation, random time, enlargement of filtration

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1 Introduction

Our work is motivated by the arbitrage-free pricing of European and American style contracts with the counterparty credit risk and, more generally, problems of mitigation of financial and insurance risks triggered by an extraneous event. We focus on a study of BSDEs and reflected BSDEs up to a finite random time horizon \( \vartheta \), while imposing minimal assumptions on a random time \( \vartheta \), which is not an \( \mathcal{F} \)-stopping time with respect to a reference filtration \( \mathcal{F} \). In contrast to existing papers (see, e.g., Ankirchner et al. [1], Kharroubi and Lim [29], Crépey and Song [9], Dumitrescu et al. [11], and Grigorova et al. [21]), we do not make any of simplifying assumptions frequently encountered in papers on the theory of progressive enlargement of filtration, such as: the immersion hypothesis, Jacod's equivalence hypothesis, the postulate \((C)\) of continuity of all \( \mathcal{F} \)-martingales, or the postulate \((A)\) of avoidance of all \( \mathcal{F} \)-stopping times by \( \vartheta \). The only assumption we make is that the Azéma supermartingale of \( \vartheta \) with respect to \( \mathcal{F} \) (see Definition 2.2) is a strictly positive process, although we also show that this assumption can be relaxed so our results apply to a larger class of random times, (see the class \( \mathcal{K} \) in Section 5).

Stimulated by the recent paper by Choulli et al. [7] on the martingale representation theorem in the progressive enlargement of a given filtration \( \mathcal{F} \) with observations of occurrence of a random time \( \vartheta \), which is henceforth denoted as \( \mathcal{G} \), we study \( \mathcal{G} \)-adapted BSDEs and \( \mathcal{G} \)-adapted reflected BSDEs (RBSDEs) with an \( \mathcal{F}_\vartheta \)-measurable terminal value at a random time horizon \( \vartheta \). The crucial difference between the \( \mathcal{G} \) BSDEs and \( \mathcal{G} \) RBSDEs introduced in Definitions 3.1 and 10.1, respectively, and various classes of BSDEs previously studied in the existing literature is that the driver is assumed to be a \( \mathcal{G} \)-measurable process and the integrand against the compensated jump martingale \( N^{\omega,\mathcal{G}} \) given by equation (2.2) is assumed to be \( \mathcal{F} \)-optional, rather than \( \mathcal{G} \)-predictable or, equivalently, \( \mathcal{F} \)-predictable. Our main purpose in this work is to apply the method of reduction in the study of existence of a solution to the \( \mathcal{G} \) BSDE and \( \mathcal{G} \) RBSDE given by equations (3.2) and (10.1), respectively. It should be acknowledged that the idea of reduction of a BSDE in a given filtration to a more tractable BSDE in a shrunken filtration has already been explored in papers by Kharroubi and Lim [29] and Crépey and Song [9, 10] but the authors of these papers worked under various simplifying assumptions about the setup under study and did not consider the reflected case.

The first main contribution of this work is that we demonstrate that the idea of reduction can also be applied to the \( \mathcal{G} \) RBSDE (10.1). To be more precise, we show that the \( \mathcal{G} \) RBSDE with \( \mathcal{F}_\vartheta \)-measurable terminal value can be solved up to a random horizon \( \vartheta \) by first solving the corresponding reduced \( \mathcal{F} \) RBSDE and then constructing a solution to the original \( \mathcal{G} \) RBSDE by combining a solution to the \( \mathcal{F} \) RBSDE with an appropriate adjustment to the terminal value at time \( \vartheta \). In particular, we analyze in detail the required adjustment at \( \vartheta \) in the case when the driver of the \( \mathcal{G} \) RBSDE is a discontinuous \( \mathcal{F}_\vartheta \)-predictable process. Furthermore, since the simplifying conditions \((C)\) and \((A)\) of continuity of all \( \mathcal{F} \)-martingales and the avoidance of all \( \mathcal{F} \)-stopping times by \( \vartheta \) are not imposed, we allow the reference filtration \( \mathcal{F} \) to support discontinuous martingales and, in addition, we also cover the situation where a random time \( \vartheta \) has an overlap with \( \mathcal{F} \)-stopping times. As a consequence, in the càdlàg case, the reduced \( \mathcal{F} \) BSDE obtained in our setup is such that the driver and the martingale part may share common jumps, which means that the reduced \( \mathcal{F} \) BSDE has a fairly general form that was not well studied in the existing literature.

Hence, as a second contribution of this work, we show how to construct a solution to the reduced \( \mathcal{F} \) BSDE for which the driver and the martingales appearing in a BSDE may have a finite number of common jumps. Our approach relies on a detailed analysis of the appropriate intermediate BSDE with a \( \mathcal{F}_\vartheta \)-adapted driver. To illustrate our method more explicitly, we also show that a solution to an intermediate BSDE with a \( \mathcal{F}_\vartheta \)-predictable driver can be obtained by adapting the existing results from Essaky et al. [14] and Ren and El Otmani [36] who studied a particular class of BSDEs with a continuous driver.

We stress that the method presented here is not to solve directly \( \mathcal{G} \) BSDEs (or \( \mathcal{G} \) RBSDEs) through fixed point arguments by making appropriate technical assumptions on the solution space, the generator and the driver. Instead, our aim is to show that one can reduce the \( \mathcal{G} \) BSDE to a more...
manageable $\mathcal{F}$ BSDE, which can be solved by making use of the solution to an intermediate BSDEs with lággläd driver. Then we show that the solution of the intermediate BSDE with lággläd driver can be obtained by a careful analysis of jumps, in a similar spirit to that of Essaky et al. [14], Klimsiak et al. [28] and Confortola et al. [8], and making use of existing results on BSDEs and RBSDEs with continuous drivers (such as those in [12, 14, 16, 17, 33, 36]) to deal with solutions on stochastic intervals between jumps.

The structure of the paper is as follows. In Section 2, we first introduce the notation and recall some auxiliary results from the theory of enlargement of filtration (see, e.g., Aksamit and Jeanblanc [4]). Next, in view of recent works on RBSDEs with irregular barriers (see, e.g., Grigorova et al. [22, 23] and Klimsiak et al. [28]) and to demonstrate the generality of our methodology, we introduce in Definition 3.1 the notion of the lággläd $G$ BSDE (see also Definition 10.1 for the lággläd $G$ RBSDE).

In Section 4, we show how the reward process can be reduced and then discuss, in Sections 5 and 6, some possible extensions of the setup given by Assumption 3.1.

Sections 7–9 are devoted to the issues of reduction of the $G$ BSDE (3.2) and construction of its solution. We first demonstrate in Proposition 7.1 that the $G$ BSDE can be effectively reduced to a corresponding equation in the filtration $\mathcal{F}$. Subsequently, we then show in Proposition 11.1 that a solution to the $G$ BSDE (3.2) can be constructed from the corresponding reduced $\mathcal{F}$ BSDE by first solving the coupled equations (8.6)–(8.7). To examine the existence of a solution to the reduced $\mathcal{F}$ BSDE, we first show that the $\mathcal{F}$ BSDE (8.8)–(8.9) can be transformed to the $\mathcal{F}$ BSDE (8.10)–(8.11), which is more tractable and whose solution can be used to address the issue of well-posedness of the coupled equations (8.8)–(8.9). Concrete situations where the coupled equations (8.8)–(8.9) possess a unique solution are studied in Section 9 in the special case of a Brownian filtration (see Proposition 9.1).

In Sections 10–12, we are concerned with analogous issues for $G$ RBSDEs and we first show that the method of reduction can be used to reduce the $G$ RBSDE to the $\mathcal{F}$ RBSDE. As shown in Proposition 10.1, the main new feature in the reflected case is that the $G$-predictable reflection can be uniquely reduced to the $\mathcal{F}$-predictable reflection, which is required to meet the appropriately modified Skorokhod conditions. We then show in Proposition 11.1 that, in principle, a solution to the $G$ RBSDE can be constructed from a solution to the reduced $\mathcal{F}$ RBSDE. The existence of a solution to the $G$ RBSDE in the Brownian case is studied in Section 12 where we give in Proposition 12.1 sufficient conditions for the existence of a solution to the $\mathcal{F}$ RBSDE in the case of the Brownian filtration $\mathcal{F}$.

In Sections 13–14, we deviate from the previous sections and, for given a filtration $\mathcal{F}$, we focus on the BSDE (13.1) and the RBSDE (14.1) with the feature that the driver is lággläd and may shares jumps with the driving martingale. As mentioned before, even when the driver is càdlàg, there appears to be a gap in the existing literature on BSDEs when the driver shares common jumps with the driving martingale and thus we develop a jump-adapted method to solve BSDEs of this general form. Our method hinges on two steps. We first show, through a careful analysis of right-hand jumps, that the problem of solving the BSDE (13.1) on the whole interval $[0, \tau]$ can be addressed by solving a recursive system of càdlàg BSDEs (13.2) and then stitching together the solutions to that system. In the second step, we show in Proposition 13.1 that a solution to a càdlàg BSDE can be obtained from a solution of an intermediate lággläd BSDE (13.6), which in turn can be solved by first solving a recursive system of càdlàg BSDEs (13.8) with a continuous driver, which are given on intervals defined by the right-hand jumps of lággläd BSDE and, once again, appropriately aggregating these solutions. We argue that a reduction to the case of a continuous driver is important since it allow us to use existing results on well-posedness of BSDEs with a continuous driver. Concrete instances of our approach in a Brownian-Poisson filtration are presented in Examples 13.1 and 13.2.

In the final section, we show that an analogous method can be applied to study the existence of a solution to the RBSDE (14.1) with a lággläd driver using results for RBSDEs with a continuous driver. The main difference here is that we need to take care of the adjustment to the reflection process at the right-hand jumps and provide a rigorous check that the appropriate Skorokhod conditions are satisfied. The main result, Proposition 14.1, is complemented by an explicit illustration in a Poisson filtration given in Example 14.1.
2 Setup and notation

Regarding the background knowledge, for the general theory of stochastic processes, we refer to He et al. [24] and the reader interested in stochastic calculus for optional semimartingales is referred to Gal’čuk [19]. For more details on the theory of random times and enlargement of filtration with applications to problems arising in financial mathematics (such as credit risk modeling or insider trading), the interested reader may consult the monograph by Aksamit and Jeanblanc [4] and the recent paper by Jeanblanc and Li [26].

We start by introducing the notation and recalling some fundamental concepts associated with modeling of a random time and the associated notion of the progressive enlargement of a reference filtration.

We assume that a strictly positive and finite random time \( \vartheta \), which is defined on a probability space \((\Omega, \mathcal{G}, P)\), as well as some reference filtration \( \mathcal{F} \) are given. Then the enlarged filtration \( \mathcal{G} \) is defined as the progressive enlargement of \( \mathcal{F} \) by observations of \( \vartheta \) (see, e.g., [4]) and thus a random time \( \vartheta \), which is not necessarily an \( \mathcal{F} \)-stopping time and belongs to the set of all finite \( \mathcal{G} \)-stopping times, denoted as \( \overline{\vartheta} \). We emphasize that the filtrations \( \mathcal{F} \) and \( \mathcal{G} \) are henceforth supposed to satisfy the usual conditions of \( \mathcal{F} \)-completeness and right-continuity.

We will use the following notation for classes of processes adapted to the filtration \( \mathcal{F} \):

- \( \mathcal{O}(\mathcal{F}) \), \( \mathcal{P}(\mathcal{F}) \), \( \overline{\mathcal{P}}(\mathcal{F}) \) and \( \mathcal{R}(\mathcal{F}) \) are the classes of all real-valued, \( \mathcal{F} \)-optional, \( \mathcal{F} \)-predictable, \( \mathcal{F} \)-strongly predictable and \( \mathcal{F} \)-progressively measurable processes, respectively;
- \( \mathcal{O}_{\mathcal{d}}(\mathcal{F}) \), \( \mathcal{P}_{\mathcal{d}}(\mathcal{F}) \), \( \overline{\mathcal{P}}_{\mathcal{d}}(\mathcal{F}) \) and \( \mathcal{R}_{\mathcal{d}}(\mathcal{F}) \) are the classes of all \( \mathbb{R}^d \)-valued, \( \mathcal{F} \)-optional, \( \mathcal{F} \)-predictable, \( \mathcal{F} \)-strongly predictable and \( \mathcal{F} \)-progressively measurable processes, respectively;
- \( \mathcal{M}(\mathcal{F}) \) (respectively, \( \mathcal{M}_{\text{loc}}(\mathcal{F}) \)) is the class of all \( \mathcal{F} \)-martingales (respectively, \( \mathcal{F} \)-local martingales);
- \( \mathcal{M}^o(\mathcal{F}) \) (respectively, \( \mathcal{M}^o_{\text{loc}}(\mathcal{F}) \)) is the class of all \( \mathcal{F} \)-martingales (respectively, \( \mathcal{F} \)-local martingales), which are stopped at \( \vartheta \).

A stochastic process \( X \) with sample paths possessing right-hand limits is said to be \( \mathcal{F} \)-strongly predictable if it is \( \mathcal{F} \)-predictable and the process \( X_+ \) is \( \mathcal{F} \)-optional (Definition 1.1 in [19]). An analogous notation is used for various classes of \( \mathcal{G} \)-adapted processes. For instance, \( \mathcal{P}(\mathcal{G}) \) denotes the class of all \( \mathcal{G} \)-predictable processes, \( \mathcal{M}_{\text{loc}}^o(\mathcal{G}) \) is the class of all \( \mathcal{G} \)-local martingales, which are stopped at the random time \( \vartheta \), etc.

In order to simplify the notation, we denote by \( X \circ Y \) the usual Itô stochastic integral of \( X \) with respect to a \( (\mathcal{c}\mathcal{d}\mathcal{a}\mathcal{g}) \) semimartingale \( Y \), that is, \( (X \circ Y)_t := \int_0^t X_s dY_s \), while we also write \( (X \circ Y)_t := \int_{[0,t]} X_s dY_s \) so that the process \( X \circ Y \) is left-continuous as the integration is done over the interval \([0, t]\). Due to the potential presence of a jump of \( Y \) at time zero, we have that \( (X \circ Y)_t = (X \circ Y)_{t-} + X_0 \Delta Y_0 \) where, by the usual convention, \( Y_{t-} = 0 \) so that \( \Delta Y_0 = Y_0 \).

Let us recall from Gal’čuk [19] the notation pertaining to a pathwise decomposition of a \( \mathcal{c}\mathcal{d}\mathcal{a}\mathcal{g} \) process. If \( C \) is an \( \mathcal{F} \)-adapted, \( \mathcal{c}\mathcal{d}\mathcal{a}\mathcal{g} \) process, then we write \( C = C^c + C^d + C^g \) where the process \( C^c \) is continuous, the \( \mathcal{c}\mathcal{d}\mathcal{a}\mathcal{g} \) process \( C^d \) equals \( C^d := \sum_{0 \leq s \leq t} (C_s - C_{s-}) \) and the \( \mathcal{c}\mathcal{d}\mathcal{a}\mathcal{g} \) process \( C^g \) is given by \( C^g := \sum_{0 \leq s < t} (C_s - C_{s-}) \). This also means that \( C = C^r + C^g \) where the \( \mathcal{c}\mathcal{d}\mathcal{a}\mathcal{g} \) process \( C^r \) satisfies \( C^r = C - C^g = C^c + C^d \). Notice that if \( C \) is a \( \mathcal{c}\mathcal{d}\mathcal{a}\mathcal{g} \) process, then manifestly \( C^d = 0 \) and thus \( C^r = C^c \) is a continuous process. Similarly, if \( C \) is a \( \mathcal{c}\mathcal{d}\mathcal{a}\mathcal{g} \) process, then \( C^g = 0 \) and thus \( C = C^r \).

For a fixed random time \( \vartheta \), we define the indicator process \( A \in \mathcal{O}(\mathcal{G}) \) by \( A := 1_{[\vartheta, \infty]} \) so that \( A_t = 1_{\{ t \leq \vartheta \}} \) for all \( t \in \mathbb{R}_+ \) and we denote by \( A^p \) (respectively, \( A^o \)) the dual \( \mathcal{F} \)-predictable projection (respectively, the dual \( \mathcal{F} \)-optional projection) of \( A \). The BMO \( \mathcal{F} \)-martingales \( m \) and \( n \) associated with the processes \( A^o \) and \( A^p \), respectively, are given by the following definition.

**Definition 2.1.** Let \( m_t := \mathbb{E}(A^o_{\infty} \mid \mathcal{F}_t) \) so that \( m_{\infty} = A^o_{\infty} \) and let \( n_t := \mathbb{E}(A^p_{\infty} \mid \mathcal{F}_t) \) so that \( n_{\infty} = A^p_{\infty} \).
Following Azéma [6], we introduce the \( \mathbb{F} \)-supermartingales \( G \) and \( \tilde{G} \) associated with \( \vartheta \).

**Definition 2.2.** The càdlàg process \( G \in \mathcal{O}(\mathbb{F}) \), which is given by the equality \( G_t := \mathbb{P}(\vartheta > t \mid \mathcal{F}_t) \), is called the Azéma supermartingale of \( \vartheta \) with respect to \( \mathbb{F} \). The \( \mathcal{O} \)-supermartingale \( \tilde{G} \in \mathcal{O}(\mathbb{F}) \), which is given by the equality \( \tilde{G}_t := \mathbb{P}(\vartheta \geq t \mid \mathcal{F}_t) \), is called the Azéma optional supermartingale of \( \vartheta \) with respect to \( \mathbb{F} \).

For the reader’s convenience, we recall some important properties of Azéma supermartingales \( G \) and \( \tilde{G} \) (see, e.g., [4]).

**Lemma 2.1.** (i) We have that \( G = n - A^p = m - A^o \) and \( \tilde{G} = m - A^o \) and thus

\[
G_t = \mathbb{E}(A^o_{\infty} - A^p_t \mid \mathcal{F}_t) = \mathbb{E}(A^o_{\infty} - A^o_t \mid \mathcal{F}_t), \quad \tilde{G}_t = \mathbb{E}(\tilde{A}^o_{\infty} - \tilde{A}^o_t \mid \mathcal{F}_t).
\]

(ii) The processes \( G \) and \( \tilde{G} \) satisfy \( \tilde{G}_- = G_- \) and \( \tilde{G}_+ = G_+ = G \).

(iii) The inequality \( G \geq \tilde{G} \) holds and the equalities \( \tilde{G} - G = ^o(1_{[\vartheta]} - \Delta A^o) = \Delta A^o \) and \( \tilde{G} - G_- = \Delta m \) are valid.

Observe that the equality \( G = n - A^p \) gives the Doob-Meyer decomposition in the filtration \( \mathbb{F} \) of the bounded \( \mathbb{F} \)-supermartingale \( G \). From the classical theory of enlargement of filtration, it is well known that the \( \mathbb{G} \)-martingale \( N^{p,G} \) from the Doob-Meyer decomposition in the filtration \( \mathbb{G} \) of the bounded \( \mathbb{G} \)-submartingale \( A \) can be represented as follows

\[
N^{p,G} := A - 1_{[0,\vartheta]}G_{\vartheta}^{-1} \cdot A^p.
\]  

Furthermore, it was shown in Choulli et al. [7] (see Theorem 2.3 therein) and Jeanblanc and Li [26] that the following process \( N^{o,G} \) is a \( \mathbb{G} \)-martingale with the integrable variation

\[
N^{o,G} := A - 1_{[0,\vartheta]}G_{\vartheta}^{-1} \cdot A^o = A - 1_{[0,\vartheta]} \Gamma
\]

where the \( \mathbb{F} \) hazard process of \( \vartheta \) is defined by \( \Gamma := \tilde{G}_{\vartheta}^{-1} \cdot A^o \) (see Jeanblanc and Li [26]). The processes \( N^{p,G} \) and \( N^{o,G} \) are both known to belong to the class \( \mathcal{M}(\mathbb{G}) \) but their properties are quite different; in particular, \( N^{p,G} \) is not necessarily a pure default martingale (see Definition 2.2 in Choulli et al. [7]) whereas \( N^{o,G} \) has that property.

We will also make use of the following general result, which is due to Aksamit et al. [2].

**Proposition 2.1.** Let the \( \mathbb{F} \)-stopping time \( \tilde{\eta} \) be defined by the following equality \( \tilde{\eta} := \inf\{t \in \mathbb{R} \mid \tilde{G}_t > G_t = 0\} \). If \( M \) is an \( \mathbb{F} \)-local martingale, then

\[
M^\vartheta - 1_{[0,\vartheta]}G_{\vartheta}^{-1} \cdot [M,m] + 1_{[0,\vartheta]} \cdot (\Delta M)_{\tilde{\eta}} 1_{[\tilde{\eta},\infty]} \]

is a \( \mathbb{G} \)-local martingale stopped at \( \vartheta \).

In particular, if \( \tilde{G} \) is a strictly positive process then for any \( \mathbb{F} \)-local martingale \( M \), we set

\[
\tilde{M} := M - G^{-1} \cdot [M,m],
\]

so that the process \( \tilde{M}^\vartheta \) is a \( \mathbb{G} \)-local martingale stopped at \( \vartheta \). If, in addition, all \( \mathbb{F} \)-martingales are continuous (that is, if the so-called Assumption (C) holds, for instance, when \( \mathbb{F} \) is a Brownian filtration), then \( \mathcal{O}(\mathbb{F}) = \mathcal{P}(\mathbb{F}) \) and thus the equalities \( G = G_- \) and \( A^o = A^p \) are valid so that also \( N^{o,G} = N^{p,G} \). Then equality (2.4) becomes

\[
\tilde{M} = M - G^{-1} \cdot (M,m).
\]
3 BSDEs with a random time horizon

Our study of BSDEs with a random time horizon is conducted within the following setup.

**Assumption 3.1.** We assume that we are given the following objects:
(i) a probability space \((\Omega, \mathcal{G}, \mathbb{P})\) endowed with a filtration \(\mathbb{F}\);
(ii) a random time \(\vartheta\) such that the Azéma supermartingale \(G\) (hence also \(G_-\) and the Azéma optional supermartingale \(G\)) is a strictly positive process;
(iii) the class of all finite \(\mathcal{G}\)-stopping times \(\hat{T}\) where \(\mathcal{G}\) denotes the progressive enlargement of \(\mathbb{F}\) with a random time \(\vartheta\);
(iv) the bounded processes \(X, R \in \mathcal{O}(\mathbb{F})\), which are used to define the bounded reward process \(\hat{X} \in \mathcal{O}(\mathcal{G})\) through the following expression
\[
\hat{X} := X1_{[0, \vartheta]} + R\vartheta1_{[\vartheta, \infty]};
\]
(v) a real-valued \(\mathcal{G}\)-martingale \(N^{o,G}\) associated with \(\vartheta\) and given by (2.2);
(vi) an \(\mathbb{R}\)-valued, \(\mathcal{F}\)-local martingale \(M\), which is assumed to have the predictable representation property (PRP) for the filtration \(\mathbb{F}\);
(vii) an \(\mathbb{R}^k\)-valued, \(\mathcal{G}\)-adapted process \(\hat{D} = (\hat{D}^1, \hat{D}^2, \ldots, \hat{D}^k)\) where \(\hat{D}^i\) is a linear combination of a ládlág \(\mathcal{G}\)-strongly predictable process of finite variation and a ládlág \(\mathbb{F}\)-optional process of finite variation; and
(viii) a mapping \(\widehat{F} = (\widehat{F}^r, \widehat{F}^g)\) where mappings \(\widehat{F}^r: \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}^k\) and \(\widehat{F}^g: \Omega \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}^k\) are such that, for any fixed \((y, z, u) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}\), the process \((\widehat{F}^r(y, z, u))_{t \geq 0}\) belongs to \(\mathcal{P}_k(\mathcal{G})\) and the process \((\widehat{F}^g_t(y, z, u))_{t \geq 0}\) belongs to \(\mathcal{O}_k(\mathcal{G})\).

We are in a position to introduce a particular class of BSDEs with a random time horizon. For the sake of brevity, they will be called \(\mathcal{G}\) BSDEs, as opposed to the associated \(\mathbb{F}\) BSDEs, which are introduced in Section 7.

**Definition 3.1.** For a fixed \(\bar{\tau} \in \hat{T}\), we say that a triplet \((\bar{Y}, \bar{Z}, \bar{U})\) is a solution on \([0, \bar{\tau} \wedge \vartheta]\) to the \(\mathcal{G}\) BSDE
\[
\hat{Y}_t = \hat{X}_{\bar{\tau} \wedge \vartheta} - \int_{[t, \bar{\tau} \wedge \vartheta]} \hat{F}^r_s(\hat{Y}_s, \hat{Z}_s, \hat{U}_s) d\hat{D}_s - \int_{[t, \bar{\tau} \wedge \vartheta]} \hat{F}^g_s(\hat{Y}_s, \hat{Z}_s, \hat{U}_s) d\hat{D}^g_s - \int_{[t, \bar{\tau} \wedge \vartheta]} \hat{Z}_s d\hat{M}^\vartheta_s - \int_{[t, \bar{\tau} \wedge \vartheta]} \hat{U}_s dN^{o,G}_s
\]
if \(\bar{Y} \in \mathcal{O}(\mathcal{G})\) is a làdlág process, the processes \(\bar{Z} \in \mathcal{P}_d(\mathcal{G})\) and \(\bar{U} \in \mathcal{O}(\mathbb{F})\) are such that the stochastic integrals in the right-hand side of (3.2) are well defined and equality (3.2) is satisfied on the stochastic interval \([0, \bar{\tau} \wedge \vartheta]\).

The process \(\hat{D}\) from Assumption 3.1 (vii) and the mapping \(\widehat{F} = (\widehat{F}^r, \widehat{F}^g)\) from Assumption 3.1 (viii) are henceforth called the *driver* and the *generator* of the \(\mathcal{G}\) BSDE, respectively. The processes \(N^{o,G}\) and \(\hat{M}^\vartheta\), given by equations (2.2) and (2.4), respectively, are orthogonal \(\mathcal{G}\)-local martingales stopped at \(\vartheta\) and they are referred to as *driving martingales*. For explicit integrability conditions, which ensure that the stochastic integral \(\bar{U} \cdot N^{o,G}\) is a \(\mathcal{G}\)-local martingale, see Theorem 2.13 in Choulli et al. [7]. To the best of our knowledge, the issue of well-posedness of the \(\mathcal{G}\) BSDE (3.2) is not addressed in the existing comprehensive literature on BSDEs and thus our aim is to contribute to the theory of BSDEs by filling that gap.

In the next definition, we implicitly make the natural postulate of well-posedness of the \(\mathcal{G}\) BSDE (3.2) in a suitable space of stochastic processes, which can be left unspecified at this stage.

**Definition 3.2.** The nonlinear evaluation \(\hat{E}\) is the collection of mappings \(\hat{E} = \{\hat{E}_{\hat{\sigma}, \hat{\tau}} | \hat{\sigma}, \hat{\tau} \in \hat{T}; \hat{\sigma} \leq \hat{\tau}\}\) where for every \(\hat{\sigma}, \hat{\tau} \in \hat{T}\) such that \(\hat{\sigma} \leq \hat{\tau}\) we have \(\hat{E}_{\hat{\sigma}, \hat{\tau}}(\hat{X}_{\hat{\tau}}) := \hat{Y}_{\hat{\tau} \wedge \vartheta}\) where the triplet \((\hat{Y}, \hat{Z}, \hat{U})\) is a unique solution to the \(\mathcal{G}\) BSDE (3.2) on the interval \([0, \hat{\tau} \wedge \vartheta]\).
4 Reduction of the reward process

Let $\mathcal{T}$ denote the class of all finite $\mathbb{F}$-stopping times and, for any fixed $\tau \in \mathcal{T}$, let the stopped filtration $\mathbb{F}^\tau$ be given by $\mathbb{F}^\tau := (\mathcal{F}^\tau_t)_{t \geq 0}$. We will now examine the structure of the reward process $\hat{X}$ specified by (3.1). We claim that, for any $\bar{\tau} \in \mathcal{T}$, there exists $\tau \in \mathcal{T}$ such that $\hat{X}_{\bar{\tau}} = \hat{X}_{\tau \wedge \rho} = \hat{X}_{\tau}$. First, it is clear from (3.1) that the process $\mathcal{X}$ is stopped at $\rho$ so that $\mathcal{X} = \mathcal{X}_\rho$, which immediately implies that $\hat{X}_{\bar{\tau}} = \hat{X}_{\tau \wedge \rho}$. Hence, by using also the well known property that for any $\bar{\tau} \in \mathcal{T}$ there exists $\tau \in \mathcal{T}$ such that $\tau \land \rho = \bar{\tau} \land \rho$, we obtain the following equalities

$$\hat{X}_{\bar{\tau}} = \mathcal{X}_{\tau \wedge \rho} = X_{\tau \wedge \rho} \cdot 1_{\{\tau \wedge \rho < \rho\}} + R_{\rho} \cdot 1_{\{\tau \wedge \rho \geq \rho\}} = \mathcal{X}_{\tau \wedge \rho} = \hat{X}_{\tau} \land \rho$$

so that $\hat{X}_{\bar{\tau}} = \hat{X}_{\tau \wedge \rho} = \hat{X}_{\tau}$ for some stopping time $\tau \in \mathcal{T}$, as was required to show.

**Lemma 4.1.** For any $\tau \in \mathcal{T}$, there exists $X(\tau) \in \mathcal{O}(\mathbb{F}^\tau)$ such that the equality $\hat{X}_{\tau} = X(\tau)$ holds.

**Proof.** It suffices to observe that

$$\hat{X}_{\tau} = R_{\rho} \cdot 1_{[\rho, \infty)}(\tau) + X_{\tau} \cdot 1_{[0, \rho]}(\tau) = R_{\rho} \cdot 1_{[0, \tau]}(\rho) + X_{\tau} \cdot 1_{[\tau, \infty]}(\rho) = X_{\rho}(\tau)$$

(4.1)

where, for any fixed $\tau$, the $\mathbb{F}$-adapted process $X(\tau)$ is given by

$$X(\tau) := R_{\rho} \cdot 1_{[0, \rho]} + X_{\tau} \cdot 1_{[\tau, \infty]} = R^\tau + (X_{\tau} - R_{\tau}) \cdot 1_{[\tau, \infty]}.$$  

(4.2)

Since the processes $X$ and $R$ are assumed to be $\mathbb{F}$-optional, by Lemma 3.53 in He et al. [24], the process $X(\tau)$ is $\mathbb{F}^\tau$-optional, although it is not a càdlàg process, in general. □

Since $\hat{X}_{\tau}$ is $\mathcal{G}_{\tau}$-measurable and $X_{\rho}(\tau)$ is $\mathcal{G}_{\rho}$-measurable, by part (3) of Theorem 3.4 in He et al. [24], the random variable $\hat{X}_{\tau} = X_{\rho}(\tau)$ is $\mathcal{G}_{\tau \land \rho}$-measurable or, more precisely, it is $\mathcal{F}^\rho_{\tau \land \rho}$-measurable and $\mathcal{F}^\rho_{\rho} \subset \mathcal{G}_{\tau \land \rho}$. In view of equalities (4.1), we will freely interchange $\hat{X}_{\bar{\tau}}, \hat{X}_{\tau}$ and $X_{\rho}(\tau)$.

**Proposition 4.1.** Given two $\mathcal{G}$-stopping times $\hat{\sigma}, \hat{\tau}$ such that $\hat{\sigma} \leq \hat{\tau}$, there exists $\sigma \leq \tau$ where $\sigma, \tau \in \mathcal{T}$ are such that $\sigma \land \rho = \hat{\sigma} \land \rho$, $\tau \land \rho = \hat{\tau} \land \rho$ and $\mathcal{E}_{\hat{\sigma}, \hat{\tau}}(\hat{X}_{\hat{\tau} \land \rho}) = \mathcal{E}_{\sigma, \tau}(X_{\rho}(\tau))$.

**Proof.** First, we observe that there exists $\sigma$ such that $\hat{\sigma} \land \rho = \sigma \land \rho$. Furthermore, if the inequality $\sigma \leq \tau$ fails to hold, then we can take $\sigma' = \sigma \cup \tau$ and observe that $\sigma' \land \rho = \sigma \land \tau \land \rho = \sigma \land \hat{\tau} \land \rho = \hat{\sigma} \land \rho$. Using the fact that there exists $\sigma \leq \tau$ such that $\sigma \land \rho = \hat{\sigma} \land \rho$ and $\tau \land \rho = \hat{\tau} \land \rho$, we obtain the following equalities

$$\mathcal{E}_{\hat{\sigma}, \hat{\tau}}(\hat{X}_{\hat{\tau} \land \rho}) = X_{\rho}(\tau) - \int_{[\sigma \land \rho, \tau \land \rho]} \hat{F}_{\tau}^y(\hat{Y}_s, \hat{Z}_s, \hat{U}_s) d\hat{\mathcal{G}}_s - \int_{[\sigma \land \rho, \tau \land \rho]} \hat{F}_{\tau}^g(\hat{Y}_s, \hat{Z}_s, \hat{U}_s) d\hat{\mathcal{G}}_s^g - \int_{[\sigma \land \rho, \tau \land \rho]} \hat{Z}_s d\hat{\mathcal{M}}_s^g - \int_{[\sigma \land \rho, \tau \land \rho]} \hat{U}_s d\mathcal{N}_s^{G, \rho}$$

(4.3)

and thus we conclude that $\mathcal{E}_{\hat{\sigma}, \hat{\tau}}(\hat{X}_{\hat{\tau} \land \rho}) = \mathcal{E}_{\sigma, \tau}(X_{\rho}(\tau)).$ □

5 The case of a general Azéma supermartingale

Let us make some comments on the possibility of relaxing Assumption 3.1(ii), that is, allowing the Azéma supermartingale $G$ of $\rho$ to hit zero. As an example, let us first consider a random time of the form $\rho = \rho' \wedge T_1$ where the Azéma supermartingale of $\rho'$ is a strictly positive process and $T_1$ is an $\mathbb{F}$-stopping time. Then the Azéma supermartingale of $\rho$ jumps to zero at $T_1$ and it is not hard to check that all results can be extended to that case by replacing the terminal time $\tau$ by $\tau' = \tau \wedge T_1$.
and \( \vartheta \) by \( \vartheta' \). Since \( G \) is a nonnegative supermartingale, we have that (see, e.g., Theorem 2.62 in He et al. [24])
\[
\eta := \inf\{t \in \mathbb{R}_+ | G_t = 0\} = \inf\{t \in \mathbb{R}_+ | G_{t^-} = 0\} = \lim_{n \to \infty} \eta_n
\]
where \( \eta_n := \inf\{t \in \mathbb{R}_+ | G_t \leq 1/n\} \). It is known that \( G = 0 \) on \( [\eta, \infty] \) and, from Lemma 2.14 of [4], we have that \([0, \vartheta \] \subset \{G > 0\} \) and \([0, \vartheta[ \subset \{G > 0\} \) so that \( \vartheta \leq \eta \).

In order to weaken Assumption 3.1(ii), we introduce the class of random times \( \vartheta \) such that the Azéma supermartingale \( G' \) of \( \vartheta' := \vartheta_{\{\vartheta < \eta\}} \) is strictly positive on the interval \([0, \eta]\). Specifically, we set (recall that \( \tilde{G}_- = G_- \))
\[
K := \{\vartheta | \tilde{G}_\vartheta > 0\}
\]
and we will argue that Assumption 3.1(ii) can be replaced by the postulate that \( \vartheta \) belongs to \( K \). It is clear that the class \( K \) is nonempty and contains not only all random times with a strictly positive Azéma supermartingale, but also their minimum with any \( \mathbb{F} \)-stopping time. Observe that if a random time \( \vartheta \) belongs to \( K \) then clearly
\[
\tilde{\eta} := \inf\{t \in \mathbb{R}_+ | \tilde{G}_{t^-} > \tilde{G}_t = 0\} = \eta_{\{G_{t^-} > \tilde{G}_{t^-} = 0\}} = \infty
\]
and thus the term \( \mathbb{I}_{[0, \eta]} \cdot (\Delta M_\eta \mathbb{I}_{[\tilde{G}, \infty]} \mathbb{P}) \) in the G-semimartingale decomposition (2.3) of an arbitrary \( \mathbb{F} \)-local martingale \( M \) is in fact null.

**Lemma 5.1.** Let \( \vartheta \) be a random time with the Azéma optional supermartingale \( \tilde{G} \) and let \( \vartheta' := \vartheta_{\{\vartheta < \eta\}} \). Then \( \vartheta = \vartheta' \land \eta \) and the Azéma optional supermartingale \( G' \) of \( \vartheta' \) is given by \( G' = \tilde{G}^\eta + \mathbb{I}_{[\eta, \infty]} \cdot H \) where \( H = \sigma(\mathbb{I}_{\{\vartheta > \eta\}}) \). Furthermore, we have that \( (A')^\eta = (A')^\eta \) where \( A' := \mathbb{I}_{[\vartheta, \infty]} \).

**Proof.** We first observe that \( \vartheta' := \vartheta_{\{\vartheta < \eta\}} = \vartheta \mathbb{I}_{\{\vartheta < \eta\}} + \infty \mathbb{I}_{\{\vartheta = \eta\}} \) where in the second equality we have used the inequality \( \vartheta \leq \eta \). Using also the equalities \( \{\vartheta < \eta\} = \{\vartheta' < \eta\} \) and \( \{\vartheta = \eta\} = \{\vartheta' = \eta\} \), we obtain
\[
\vartheta = \vartheta \mathbb{I}_{\{\vartheta \leq \eta\}} + \vartheta \mathbb{I}_{\{\vartheta < \eta\}} = \vartheta' \mathbb{I}_{\{\vartheta < \eta\}} + \eta \mathbb{I}_{\{\vartheta = \eta\}} = \vartheta' \mathbb{I}_{\{\vartheta < \eta\}} + \eta \mathbb{I}_{\{\vartheta \geq \eta\}} = \vartheta' \land \eta.
\]
Next, to compute the Azéma optional supermartingale of \( \vartheta' \), we observe that for all \( t > 0 \),
\[
\mathbb{P}(\vartheta' < t | \mathcal{F}_t) = \frac{\mathbb{P}(\vartheta < t, \vartheta < \eta | \mathcal{F}_t)}{\mathbb{P}(\vartheta < t | \mathcal{F}_t)} = \mathbb{P}(\vartheta < t, \vartheta = \eta | \mathcal{F}_t) - \mathbb{P}(\vartheta < t, \vartheta = \eta | \mathcal{F}_t)
\]
\[
= 1 - \tilde{G}_t \mathbb{I}_{\{t \leq \eta\}} - \mathbb{P}(\vartheta = \eta | \mathcal{F}_t) \mathbb{I}_{\{t < \eta\}}
\]
and hence \( \tilde{G}' = \tilde{G}^\eta + \mathbb{I}_{[\eta, \infty]} \cdot H \) where \( H = \sigma(\mathbb{I}_{\{\vartheta = \eta\}}) \). The last assertion follows from the uniqueness of the Doob-Meyer-Mertens decomposition of \( G' \). It is worth noting that \( (G')^\eta = (\tilde{G})^\eta \) and \( (G')^\eta = G \mathbb{I}_{[0, \eta]} + \tilde{G}_\eta \mathbb{I}_{[\eta, \infty]} \).

It is important to observe that if \( \vartheta \in K \) then the supermartingale \( G' \) and the optional supermartingale \( G'' \) are strictly positive on \([0, \tau']\) where \( \tau' := \vartheta \land \eta \). Furthermore, the equalities \( \tilde{X}_\tau = \tilde{X}_\tau \land \vartheta = \tilde{X}_\tau \land \vartheta \land \eta = X_{\vartheta'}(\tau') \) hold. We conclude that, on the one hand, all our arguments used to address the case of a strictly positive Azéma supermartingale are still valid when the pair \( (\vartheta, \tau) \) is replaced by \( (\vartheta', \tau') \). For instance, the BSDE in Proposition 4.1 would not change since \( \vartheta \leq \eta \), whereas in Section 7 the terminal date can be changed to \( \tau' = \tau \land \eta \). On the other hand, however, if a random time \( \vartheta \) is not in \( K \) then technical issues involving either an explosion of integrals or ill-defined terminal condition at time \( \eta \) may arise.

6 Extended terminal condition

Let us make some comments on a possible extension of the terminal condition in the \( G \) BSDE. Since the processes \( X \) and \( R \) are assumed to be \( \mathbb{F} \)-optional, Lemma 4.1 implies that \( X(\tau) \) is an \( \mathbb{F}' \)-optional
process and \( X_\vartheta(\tau) \) is \( \mathcal{F}_\vartheta^\tau \)-measurable. This implies that in our formulation of the \( \mathbb{G} \) BSDE (3.2) and the \( \mathbb{G} \) RBSDE (10.1) the terminal condition is measurable with respect to \( \mathcal{F}_\vartheta^\tau \subset \mathcal{G}_{\tau \wedge \vartheta} \). It is also worth noting that

\[
\sigma(V_{\vartheta}^\tau \mid V \in \mathcal{O}(\mathbb{F})) = \mathcal{F}_{\tau \wedge \vartheta} \subset \mathcal{F}_\vartheta^\tau := \sigma(V_{\vartheta} \mid V \in \mathcal{O}(\mathbb{F}^\tau)).
\]

Since we do not consider all \( \mathcal{G}_{\tau \wedge \vartheta} \)-measurable terminal conditions, the multiplicity in the martingale representation property established in Theorem 2.22 of Choulli et al. [7] can be taken to be equal to two, which in fact gives a partial motivation for Definition 3.1 of a solution to the \( \mathbb{G} \) BSDE.

In general, the multiplicity in Theorem 2.22 of Choulli et al. [7] is equal to three and thus it would be possible to consider, when a (bounded) terminal condition \( \zeta \) is \( \mathcal{G}_{\tau \wedge \vartheta} \)-measurable, a more general \( \mathbb{G} \) BSDE driven by \( \mathcal{M}^\vartheta, N^0, \mathcal{G} \) and a pure jump martingale yielding an additional ‘correction term’ at the terminal time \( \tau \wedge \vartheta \). To be more specific, in view of Proposition 4.1, one could study the extended \( \mathbb{G} \) BSDE of the form

\[
\hat{Y}_t = \zeta - \int_{[t, \tau \wedge \vartheta]} \hat{F}^\vartheta_s(\hat{Y}_s, \hat{Z}_s, \hat{U}_s, \hat{J}_s(\tau)) \, d\hat{D}_s^\vartheta - \int_{[t, \tau \wedge \vartheta]} \hat{F}^\vartheta_s(\hat{Y}_s, \hat{Z}_s, \hat{U}_s, \hat{J}_s(\tau)) \, d\hat{D}_s^\vartheta + \int_{[t, \tau \wedge \vartheta]} \hat{Z}_s \, d\hat{M}^\vartheta_s - \int_{[t, \tau \wedge \vartheta]} \hat{U}_s \, dN^0_s, \mathcal{G}
\]

where \( \hat{Z} \) is \( \mathbb{F} \)-predictable, \( \hat{U} \) is \( \mathbb{F} \)-optional, \( \hat{J}(\tau) \) is \( \mathbb{F}^\tau \)-progressively measurable and \( \mathbb{E}(\hat{J}_\vartheta(\tau) \mid \mathcal{F}_\vartheta) = 0 \).

It is not difficult to check that the last condition implies that \( \hat{J}(\tau) \cdot \mathbb{A} \) is a \( \mathbb{G} \)-local martingale, provided that a suitable integrability condition is satisfied by \( \hat{J}(\tau) \).

A detailed study of the \( \mathbb{G} \) BSDE given by equation (6.1) is beyond the scope of this work since its practical applications are unclear. Let us only point out that if the generator \( \hat{F} \) in (6.1) does not depend on \( \hat{J} \), then one can formally reduce (6.1) to (3.2) and show that a solution to (6.1) can be obtained from a solution to (3.2). To this end, we will need the following auxiliary result.

**Lemma 6.1.** Assume that \( \zeta \) is bounded and \( \mathcal{G}_{\tau \wedge \vartheta} \)-measurable. Then

(i) there exists a process \( X'(\tau) \in \mathcal{P}_r(\mathbb{F}^\tau) \) such that \( \zeta = X'_\vartheta(\tau) \),

(ii) there exists a process \( X(\tau) \in \mathcal{O}(\mathbb{F}^\tau) \) such that \( X_\vartheta(\tau) = \mathbb{E}(X'_\vartheta(\tau) \mid \mathcal{F}_\vartheta) \).

**Proof.** To show the first assertion, we note that since \( \zeta \in \mathcal{G}_{\tau \wedge \vartheta} \)-measurable, there exists a process \( \hat{H} \in \mathcal{O}(\mathbb{G}) \) such that \( \zeta = \hat{H}_{\tau \wedge \vartheta} \) and thus also, by Proposition 2.11 in Aksamit and Jeanblanc [4], a process \( H \in \mathcal{O}(\mathbb{F}) \) such that \( \hat{H} \mathbb{1}_{[0, \vartheta]} = H \mathbb{1}_{[0, \vartheta]} \).

Furthermore, since \( \mathcal{G}_{\tau \wedge \vartheta} \subset \mathcal{G}_\vartheta = \mathcal{F}_\vartheta^\tau \), there exists a process \( H' \in \mathcal{P}_r(\mathbb{F}) \) such that \( \zeta = \hat{H}_{\tau \wedge \vartheta} = H'_\vartheta \) (see Lemma B.1 in Aksamit et al. [3], which is obtained by modifying Proposition 5.3 (b) in Jeulin [27]). We thus have the equalities

\[
H'_\vartheta \mathbb{1}_{(\tau < \vartheta)} = \hat{H} \mathbb{1}_{(\tau < \vartheta)} = H \mathbb{1}_{(\tau < \vartheta)}
\]

and we can define the \( \mathbb{F}^\tau \)-progressively measurable process

\[
X'(\tau) := H'_\vartheta \mathbb{1}_{[0, \tau]} + H \mathbb{1}_{[\tau, +\infty[}
\]

which satisfies \( X'_\vartheta(\tau) = \zeta \). For the second assertion, we note that since \( \mathbb{1}_{[0, \tau]} \) and \( H \mathbb{1}_{[\tau, +\infty[} \) belong to \( \mathcal{O}(\mathbb{F}) \), we have

\[
\mathbb{E}(X'_\vartheta(\tau) \mid \mathcal{F}_\vartheta) = \mathbb{E}(H'_\vartheta \mid \mathcal{F}_\vartheta) \mathbb{1}_{[0, \tau]}(\vartheta) + H \mathbb{1}_{[\tau, +\infty[}(\vartheta).
\]

By Proposition 2.21 in Choulli et al. [7] there exists an \( \mathbb{F} \)-optional process \( X \) such that \( X_\vartheta = \mathbb{E}(H'_\vartheta \mid \mathcal{F}_\vartheta) \). It now suffices to set \( X(\tau) := X \mathbb{1}_{[0, \tau]} + H \mathbb{1}_{[\tau, +\infty[} \) and observe that the process \( X(\tau) \) belongs to \( \mathcal{O}(\mathbb{F}^\tau) \).

By applying Lemma 6.1 to an integrable, \( \mathcal{G}_{\tau \wedge \vartheta} \)-measurable random variable \( \zeta \), we can rewrite \( \zeta = X'_\vartheta(\tau) = X_\vartheta(\tau) - X_\vartheta(\tau) + X_\vartheta(\tau) \). Therefore, in view of (6.2), we have \( \hat{J}(\tau) := X'(\tau) - X(\tau) = (H' - X) \mathbb{1}_{[0, \tau]} \), which shows that \( \hat{J}(\tau) = \hat{J}(\tau) \mathbb{1}_{[0, \tau]} \).
Then the BSDE (6.1) becomes

\[
\hat{Y}_t = X_\vartheta(\tau) - \int_{[t,\tau]} \hat{F}_s(\hat{Y}_s, \hat{Z}_s, \hat{U}_s) \, d\hat{D}_s - \int_{[t,\tau]} \hat{G}_s(\hat{Y}_s, \hat{Z}_s, \hat{U}_s) \, d\hat{D}_s^g - \int_{[t,\tau]} \hat{Z}_s \, d\hat{M}_s^g - \int_{[t,\tau]} \hat{U}_s \, dN_s^{\vartheta,G}
\]

where we have used the equalities

\[
\int_{[t,\tau]} \hat{J}_s(\tau) \, dA_s = \hat{J}_\vartheta(\tau) 1_{\{t < \vartheta \leq \tau\}} = \hat{J}_\vartheta(\tau).
\]

We conclude that if \((\hat{Y}, \hat{Z}, \hat{U})\) is a solution to the \(G\) BSDE (3.2), but with \(\tau\) replaced by \(\tau\), then \((\hat{Y}, \hat{Z}, \hat{U}, \hat{J}(\tau))\) solves the BSDE (6.1) with a \(G_{\tau,\vartheta}\)-measurable terminal condition. Similar arguments can be applied to the case of the RBSDE (10.1). However, in the case of generators depending on \(\hat{J}(\tau)\), the proper form of the adjustment to the terminal condition would be more complicated and its computation would involve the generators \(\hat{F}\) and \(\hat{F}^g\).

7 Reduction of a solution to the \(G\) BSDE

Our goal is to show that the BSDE (3.2) has a solution, which can be obtained in two steps. In the reduction step, the filtration is shrunk from \(G\) to \(F\) and the BSDE (3.2) is analyzed through an associated reduced BSDE in the filtration \(F\). In the construction step, we show that a solution to the reduced BSDE can be lifted from \(F\) to \(G\) in order to obtain a solution to the BSDE (3.2). Notice that in Sections 7 and 8 the random times \(\vartheta\) and \(\tau\) are fixed throughout.

We first establish some preliminary lemmas related to the concept of shrinkage of filtration. In the main result of this section, Proposition 7.1, we give an explicit representation for the \(F\) BSDE associated with the \(G\) BSDE (3.2). We work here under Assumption 7.1, which will be relaxed in Section 8 where an explicit construction of a solution to the \(G\) BSDE is proposed and analyzed.

**Assumption 7.1.** A solution \((\hat{Y}, \hat{Z}, \hat{U})\) to the BSDE (3.2) exists on the stochastic interval \([0, \hat{\tau} \land \vartheta]\) or, equivalently, on the interval \([0, \tau \land \vartheta]\) where \(\tau \in \mathcal{T}\) is such that \(\tau \land \vartheta = \hat{\tau} \land \vartheta\).

Our present goal is to analyze the consequences of Assumption 7.1. We start by recalling that there exist a unique \(F\)-optional process \(Y\) and a unique \(F\)-predictable process \(Z\) such that the equalities \(\hat{Y} 1_{[0,\vartheta]} = Y 1_{[0,\vartheta]}\) and \(\hat{Z} 1_{[0,\vartheta]} = Z 1_{[0,\vartheta]}\) are valid. Moreover, \(Y_\tau = X_\tau\) and the process \(Y\) and \(Z\) are given by

\[
Y = \circ(1_{[0,\vartheta]}^\tau) G^{-1}, \quad Z = \circ(1_{[0,\vartheta]}^\tau) \hat{Z} G^{-1}.
\]

Similarly, in view of Assumption 3.1(vii) and Lemma 7.2 below, there exists an right continuous \(F\)-adapted process \(D^r\) and a left-continuous \(F\)-adapted process \(D^g\) such that \(\hat{D}^r 1_{[0,\vartheta]} = D^r 1_{[0,\vartheta]}\) and \(\hat{D}^g 1_{[0,\vartheta]} = D^g 1_{[0,\vartheta]}\). Finally, it is clear that \(\hat{X} 1_{[0,\vartheta]} = X 1_{[0,\vartheta]}\). We shall refer to \(\tau\), \(Y\), \(Z\), \(D^r\), \(D^g\) and \(X\) as the \(F\)-reduction of \(\hat{\tau}\), \(\hat{Y}\), \(\hat{Z}\), \(\hat{D^r}\), \(\hat{D^g}\) and \(\hat{X}\).

In the following, we slightly abuse the notation and we again denote by \(Y\) and \(Z\) the stopped processes \(Y := Y_\tau\) and \(Z := Z_\tau\). Recall that the component \(\hat{U}\) in a solution to (3.2) is assumed to be an \(F\)-optional process and thus \(\hat{U}\) is equal to its \(F\)-reduction \(U\) so that, trivially, \(\hat{U} = U\) and, once again, we will write \(U := U_\tau\).

To show more explicitly how the process \(Y\) is computed, we observe that

\[
\hat{E}_{\hat{\tau}}(\hat{X}) 1_{[0,\tau]} 1_{[0,\vartheta]} = \hat{E}_{\hat{\tau}}(\hat{X}) 1_{[0,\tau]} 1_{[0,\vartheta]} = \hat{E}_{\tau}(X_\theta(\tau)) 1_{[0,\tau]} 1_{[0,\vartheta]} = Y 1_{[0,\tau]} 1_{[0,\vartheta]}.
\]
Therefore, by applying the \( F \)-optional projection operator, we obtain
\[
o(\tilde{E}, \tau) (\tilde{X}_{\tau}) \mathbb{I}_{[0, \tau]} \mathbb{I}_{[0, \sigma]} = \no(\tilde{E}, \tau) (X_{\sigma}) \mathbb{I}_{[0, \tau]} \mathbb{I}_{[0, \sigma]} = Y G \mathbb{I}_{[0, \tau]}.
\]

A general representation of \( Y \) can then be obtained on \([0, \tau]\) by noticing that for any \( F \)-stopping time \( \sigma \)
\[
Y_{\sigma} G_{\sigma} \mathbb{I}_{(\sigma \leq \tau)} = \mathbb{E}(\tilde{E}_{\sigma} (X_{\sigma}) \mathbb{I}_{(\sigma \leq \tau)} \mid \mathcal{F}_{\sigma}) \mathbb{I}_{(\sigma \leq \tau)}.
\]

Our next goal is to provide a more explicit computation of the right-hand side in the above equality (see Lemma 7.4).

**Remark 7.1.** Suppose that Assumption 3.1(ii) is relaxed and we postulate instead that \( \vartheta \in \mathcal{K} \) where the class \( \mathcal{K} \) is defined by (5.1). Then the modified terminal condition would be \( \tilde{X}_{\tau \land \vartheta \land \eta} = \tilde{X}_{\tau \land \vartheta \land \eta} \) and the reduced terminal condition would become \( X_{\tau \land \eta} G_{\tau \land \eta} \). Finally, the terminal condition for \( Y \) would be \( X_{\tau \land \eta} \) instead of \( X_{\tau} \). Hence, it would be enough to replace \( \tau \) with \( \tau' = \tau \land \eta \) and study the \( F \)-BSDE on the interval \([0, \tau']\), rather than \([0, \tau]\).

The following result can be deduced from Proposition 2.11 in Aksamit and Jeanson [4].

**Lemma 7.1.** For every \((y, z, u) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}\) there exists an \( \mathbb{R}^k \)-valued, \( F \)-predictable process \( F_{\tau}^r(y, z, u) \) such that \( \tilde{F}^r_{\tau} (y, z, u) \mathbb{I}_{(\vartheta \geq t)} = F_{\tau}^r (y, z, u) \mathbb{I}_{(\vartheta \geq t)} \) for every \( \tau \geq 0 \). For every \((y, z, u) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}\) there exists an \( \mathbb{R}^k \)-valued, \( F \)-optional process \( F^g(y, z, u) \) such that \( \tilde{F}^g_{\tau} (y, z, u) \mathbb{I}_{(\vartheta > t)} = F_{\tau}^g (y, z, u) \mathbb{I}_{(\vartheta > t)} \) for every \( \tau \geq 0 \).

To reduce the driver \( \hat{D} \) and later the reflection in the \( G \) RBSDE, we prove the following result. Notice that a similar result was established in Jeanblanc et al. [25] in the case where the partition of the space \( \Omega \times [0, \infty) \) was independent of time.

**Lemma 7.2.** Let \( \hat{D} = \hat{D}^r + \hat{D}^g \) be an \( G \)-adapted, càdlàg, increasing process. Then there exists an \( F \)-optional, càdlàg, increasing process \( D^r \) and an \( F \)-predictable, càdlàg, increasing process \( D^g \) such that \( D^r = \hat{D}^r \) on \([0, \vartheta]\) and \( D^g = \hat{D}^g \) on \([0, \vartheta]\). If \( \hat{D} \) is a \( G \)-strongly predictable increasing process, then \( D^r \) can be chosen such that it is an \( F \)-predictable, càdlàg, increasing process and \( D^r = \hat{D}^r \) on \([0, \vartheta]\).

**Proof.** Since \( \hat{D}^r \) belongs to the class \( \mathcal{O}(G) \), there exists an \( F \)-optional process \( D^r \) such that \( \hat{D}^r \mathbb{I}_{[0, \vartheta]} = D^r \mathbb{I}_{[0, \vartheta]} \) (see the first equality in (7.1)). Since the optional projection of a càdlàg processes is again a càdlàg process, the process \( D^r \) is càdlàg on the set \( \{ G > 0 \} = \Omega \times [0, \infty) \) where the last equality is clear since we have assumed that \( G \) is strictly positive.

To show that the process \( D^r \) is increasing, we observe that, for every \( s \leq t \),
\[
D^r_{t} \mathbb{I}_{(\vartheta > t)} = \hat{D}^r_{t} \mathbb{I}_{(\vartheta > t)} \geq \hat{D}^r_{s} \mathbb{I}_{(\vartheta > t)} = \hat{D}^r_{s} \mathbb{I}_{(\vartheta > t)} \mathbb{I}_{(\vartheta > t)} = D^r_{s} \mathbb{I}_{(\vartheta > t)}.
\]

Then, by taking the \( \mathcal{F}_t \) conditional expectation of both sides, we deduce that the process \( D^r \) is increasing on the set \( \{ G > 0 \} = \Omega \times [0, \infty) \).

Furthermore, since the process \( \hat{D}^g \) is càdlàg and thus belongs to the class \( \mathcal{P}(G) \), there exists an \( F \)-predictable process \( D^g \) such that \( \hat{D}^g \mathbb{I}_{[0, \vartheta]} = D^g \mathbb{I}_{[0, \vartheta]} \) (see the second equality in (7.1)). The rest of the proof is similar to the case of \( D^r \) except that we now use the properties of the \( F \)-predictable projection, rather than the \( F \)-optional projection.

Finally, in the case where \( \hat{D} \) is \( F \)-strongly predictable, from the decomposition \( \hat{D} = \hat{D}^r + \hat{D}^g \) and the fact that \( \hat{D} \) and \( \hat{D}^g \) belong to \( \mathcal{P}(G) \), we deduce that \( \hat{D} \) belongs to \( \mathcal{P}(G) \). Thus there exists an \( F \)-predictable process \( D^r \) such that \( \hat{D}^r \mathbb{I}_{[0, \vartheta]} = D^r \mathbb{I}_{[0, \vartheta]} \). This implies that \( \hat{D}^r \mathbb{I}_{[0, \vartheta]} = D^r \mathbb{I}_{[0, \vartheta]} \) and, by taking the \( F \)-optional projection, we deduce from similar arguments as before, that on the set \( \{ G > 0 \} \) the process \( D^r \) is increasing and càdlàg. \( \square \)
Remark 7.2. Clearly if the process \( \hat{D} \) is \( \mathbb{F} \)-adapted then the equality \( \hat{D}^r = D^r \) holds everywhere and not only on \([0, \vartheta] \). We remark here that an \( \mathbb{F} \)-adapted driver \( \hat{D} \) can have certain practical interpretations. For example, one can take \( \hat{D} \) to be the hazard process, that is \( \hat{D} = \Gamma = \hat{G}^{-1} \cdot A^o \), and this can be interpreted as a way to introduce ambiguity in the recovery and the default intensity (see, for instance, Fadina and Schmidt [15]).

The next result is an immediate consequence of Lemma 7.2 and equations (3.1) and (3.2). To alleviate the notation, we will frequently write \( \hat{F}^s_r(\cdot) = \hat{F}^r_s(\cdot, \hat{Z}, \hat{U}), \hat{F}^g_s(\cdot) = \hat{F}^g_s(\cdot, \hat{Z}, \hat{U}), F^r_s(\cdot) = F^r_s(\cdot, Z_s, U_s) \) and \( F^g_s(\cdot) = F^g_s(\cdot, Z_s, U_s) \).

**Lemma 7.3.** The following equalities are satisfied, for every \( t \in \mathbb{R}_+ \) on the event \( \{t \leq \tau \} \cap \{t < \vartheta \} \),

\[
\mathbb{E}\left(1_{\{\vartheta > t\}} \int_{[t, \tau]} \hat{F}^r_s \, d\hat{D}^r_s \mid \mathcal{F}_t\right) = \mathbb{E}\left(\int_{[t, \tau]} F^r_s(Y_s) \, dD^r_s \mid \mathcal{F}_t\right)
\]

\[
+ \mathbb{E}\left(\int_{[t, \tau]} (F^r_s(R_s) - F^r_s(Y_s)) \Delta D^r_s \, dA^o_s \mid \mathcal{F}_t\right)
\]

and, on the event \( \{t < \tau \} \cap \{t < \vartheta \} \),

\[
\mathbb{E}\left(1_{\{\vartheta > t\}} \int_{[t, \tau]} 1_{\{\vartheta > s\}} \hat{F}^g_s \, d\hat{D}^g_s \mid \mathcal{F}_t\right) = \mathbb{E}\left(\int_{[t, \tau]} G_s F^g_s(Y_s) \, dD^g_s \mid \mathcal{F}_t\right).
\]

**Proof.** Using Lemma 7.2, the equalities \( \hat{Y}_o 1_{\{\vartheta \leq t\}} = R_o 1_{\{\vartheta \leq t\}} \) and \( \mathbb{P}(\vartheta = s \mid \mathcal{F}_s) = \hat{G}_s - G_s = \Delta A^o_s \) and noticing that \( \{\vartheta \geq s\} \cap \{\vartheta > t\} \) for \( s > t \), we obtain

\[
\mathbb{E}\left(1_{\{\vartheta > t\}} \int_{[t, \tau]} 1_{\{\vartheta \geq s\}} \hat{F}^r_s(\hat{Y}_s) \, d\hat{D}^r_s \mid \mathcal{F}_t\right)
\]

\[
= \mathbb{E}\left(\int_{[t, \tau]} 1_{\{\vartheta > s\}} F^r_s(Y_s) \, dD^r_s \mid \mathcal{F}_t\right) + \mathbb{E}\left(\int_{[t, \tau]} 1_{\{\vartheta = s\}} F^r_s(R_s) \, dD^r_s \mid \mathcal{F}_t\right)
\]

\[
= \mathbb{E}\left(\int_{[t, \tau]} G_s F^r_s(Y_s) \, dD^r_s \mid \mathcal{F}_t\right) + \mathbb{E}\left(\int_{[t, \tau]} F^r_s(R_s) \Delta A^o_s \, dD^r_s \mid \mathcal{F}_t\right)
\]

\[
= \mathbb{E}\left(\int_{[t, \tau]} \hat{G}_s F^r_s(Y_s) \, dD^r_s \mid \mathcal{F}_t\right) + \mathbb{E}\left(\int_{[t, \tau]} (F^r_s(Y_s) - F^r_s(R_s)) \Delta D^r_s \, dA^o_s \mid \mathcal{F}_t\right).
\]

Similarly, again from Lemma 7.2, we have

\[
\mathbb{E}\left(1_{\{\vartheta > t\}} \int_{[t, \tau]} 1_{\{\vartheta > s\}} \hat{F}^g_s(\hat{Y}_s) \, d\hat{D}^g_s \mid \mathcal{F}_t\right) = \mathbb{E}\left(\int_{[t, \tau]} G_s F^g_s(Y_s) \, dD^g_s \mid \mathcal{F}_t\right),
\]

which gives the required result. □

To simplify further computations we define, for every \( (y, z, u) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \) and \( t \geq 0 \),

\[
\hat{F}^r_t(y, z, u) := F^r_t(y, z, u) + (F^r_t(R_t, z, u) - F^r_t(y, z, u)) G^{-1}_t \Delta A^o_t.
\]

By combining Lemmas 4.1 and 7.3 with equality (3.2), we obtain the following result.

**Lemma 7.4.** The process \((Y, Z, U)\) satisfies on \([0, \tau] \)

\[
Y_t = G_t^{-1} \mathbb{E}\left(X_r G_t + \int_{[t, \tau]} R_s \, dA^o_s - \int_{[t, \tau]} \hat{G}_s \hat{F}^r_s \, dD^r_s - \int_{[t, \tau]} G_s F^g_s \, dD^g_s \mid \mathcal{F}_t\right)
\]

where we denote \( \hat{F}^r_s = \hat{F}^r_s(Y_s, Z_s, U_s) \) and \( F^g_s = F^g_s(Y_s, Z_s, U_s) \).

**Proof.** For any fixed \( \mathbb{F} \)-stopping time \( \tau \), we denote by \( K(\tau) \) the \( \mathbb{F} \)-martingale given by

\[
K_t(\tau) := \mathbb{E}\left(X_r G_t + (R \cdot A^o)_t - (\hat{G} \hat{F}^r \cdot D^r)_t - (G F^g \cdot D^g)_t \mid \mathcal{F}_t\right).
\]

Then the \( \mathbb{F} \)-optional process \( Y \) has the following representation on \([0, \tau] \)

\[
Y_t = G_t^{-1} \left( K_t(\tau) - (R \cdot A^o)_{t \wedge} + (\hat{G} \hat{F}^r \cdot D^r)_{t \wedge} + (G F^g \cdot D^g)_{t \wedge} \right)
\]

with \( Y_\tau = X_\tau \) and thus the asserted equality holds. □
For brevity, we set $C := \hat{G}F^r \cdot D^r + GF^g \cdot D^g_+$ and we note that equality (7.5) can be rewritten as follows
\[ Y = G^{-1}(K(\tau) - R \cdot A^o + C). \] (7.6)
In addition, we define $\tilde{m} := m - \hat{G}^{-1} \cdot [m, m]$ and
\[ \tilde{K}(\tau) := K(\tau) - \hat{G}^{-1} \cdot [K(\tau), m]. \]
To express the dynamics of the process $Y$ in terms of $\tilde{m}$ and $\tilde{K}(\tau)$, we will use the following immediate consequence of Lemma 15.3 from the appendix.

**Lemma 7.5.** If $C = C^r + C^g$ is a càdlàg process of finite variation and the process $Y$ is given by $Y = G^{-1}(K - R \cdot A^o + C)$ for some $\mathbb{F}$-martingale $K$, then
\[ Y = Y_0 + \hat{G}^{-1} \cdot C^r + G^{-1} \cdot C^g - (R - Y) \cdot \Gamma - Y_\tau G_\tau^{-1} \cdot \tilde{m} + G_\tau^{-1} \cdot \tilde{K}(\tau). \]
where $\tilde{K} := K - \hat{G}^{-1} \cdot [K, m]$.

By applying Lemma 7.5 to equality (7.6) and using Lemma 7.2, we obtain the following corollary.

**Corollary 7.1.** The process $D = D^r + D^g$ is a càdlàg process of finite variation and
\[ Y = Y_0 + \hat{F}^r \cdot D^r + F^g \cdot D^g - (R - Y) \cdot \Gamma - Y_\tau G_\tau^{-1} \cdot \tilde{m} + G_\tau^{-1} \cdot \tilde{K}(\tau). \]
Assumption 3.1(vi) yields the existence $\mathbb{F}$-predictable processes $\psi^{Y,Z}$ and $\nu$ such that
\[ K(\tau) = \psi^{Y,Z} \cdot M, \quad m = \nu \cdot M. \] (7.7)

The next proposition is an immediate consequence of Corollary 7.1 combined with (7.7). As before, we write $F^r_s(\cdot) := F^r(\cdot, Z_s, U_s)$ and $F^g_s(\cdot) := F^g(\cdot, Z_s, U_s)$ and we give an explicit representation for the $\mathbb{F}$ BSDE associated with the $\mathbb{G}$ BSDE (3.2).

It is worth noting that Proposition 7.1 extends several results from the existing literature where the method of reduction was studied in a particular framework and under additional assumptions, such as the immersion hypothesis or the simplifying conditions (A) or (C).

**Proposition 7.1.** If the triplet $(\hat{Y}, \hat{Z}, \hat{U})$ is a solution to the $\mathbb{G}$ BSDE (3.2), then the triplet $(Y, Z, U)$ where $U = \hat{U}$ satisfies on $[0, \tau]$
\[
Y_t = X_t - \int_{[t, \tau]} F^r_s(Y_s) \, dD^r_s - \int_{[t, \tau]} F^g_s(Y_s) \, dD^g_s + \int_{[t, \tau]} z_s \, d\hat{M}_s \\
+ \int_{[t, \tau]} [R_s - Y_s - (F^r_s(R_s) - F^r_s(Y_s)) \Delta D^r_s] \, d\Gamma_s
\]
where the process $z$ is given by $z_t := G_t^{-1}(\psi^{Y,Z}_t - Y_{t-} \nu_t)$.

In the above representation of the $\mathbb{F}$ BSDE associated with the $\mathbb{G}$ BSDE, we can clearly identify the reduced generators $F^r$ and $F^g$ and the form of the adjustment, which is integrated with respect to the hazard process $\Gamma = \hat{G}^{-1} \cdot A^o$ of a random time $\vartheta$.

# 8 Construction of a solution to the $\mathbb{G}$ BSDE

In this section, we no longer postulate that a solution to the BSDE (3.2) exists, which means that Assumption 7.1 is relaxed. Our goal is to show that a solution to the $\mathbb{F}$ BSDE (8.6) can be expanded to obtain a solution to the BSDE (3.2) if equation (8.7) has an $\mathbb{F}$-optional solution $U$. We stress that equations (8.6) and (8.7) are coupled, in the sense that they need to be solved jointly in order
to construct a solution to the BSDE (3.2). Obviously, the issues of existence and uniqueness of a solution \((Y, Z, U)\) to equations (8.6) and (8.7) need to be studied under additional assumptions on the generator and all other inputs to the BSDE (3.2). In the next result, we denote by \(\hat{U}\) an arbitrary prescribed \(\mathbb{F}\)-optional process and we do not use Lemma 7.3.

**Lemma 8.1.** For a given process \(\hat{U} \in \mathcal{O}(\mathbb{F})\), let \((Y, Z)\) be an \(\mathbb{R} \times \mathbb{R}^d\)-valued, \(\mathbb{F}\)-adapted solution to the BSDE, on \([0, \tau]\),

\[
Y_t = X_\tau - \int_{[t, \tau]} F^r_s(Y_s, Z_s, \hat{U}_s) \, dD^r_s - \int_{[t, \tau]} F^g_s(Y_s, Z_s, \hat{U}_s) \, dD^g_s + \int_{[t, \tau]} Z_s \, d\hat{M}_s
\]

and let the \(\mathbb{G}\)-adapted process \(\hat{Y}\) be given by

\[
\hat{Y}_t = Y_0 + \mathbbm{1}_{[0, \tau]} \hat{Y}^r + \mathbbm{1}_{[0, \tau]} Y^g + (R_\sigma - Y_{\sigma-}) \mathbbm{1}_{[\sigma, \infty]} \mathbbm{1}_{\{\sigma \geq \sigma\}}
\]

Then \((\hat{Y}, \hat{Z}) := (\hat{Y}, Z^\theta)\) is a \(\mathbb{G}\)-adapted solution to the BSDE, on \([0, \tau \wedge \theta]\),

\[
\hat{Y}_t = X_{\tau \wedge \theta} - \int_{[t, \tau \wedge \theta]} \hat{F}^r_s(\hat{Y}_s, \hat{Z}_s, \hat{U}_s) \, dD^r_s
\]

\[
- \int_{[t, \tau \wedge \theta]} \hat{F}^g_s(\hat{Y}_s, \hat{Z}_s, \hat{U}_s) \, dD^g_s + \int_{[t, \tau \wedge \theta]} \hat{Z}_s \, d\hat{M}_s
\]

\[
- \int_{[t, \tau \wedge \theta]} [R_s - Y_s - (F^r_s(R_s, \hat{Z}_s, \hat{U}_s) - F^r_s(Y_s, \hat{Z}_s, \hat{U}_s)) \Delta D^r_s] \, dN^G_s
\]

**Proof.** Let us write \(C^r := \hat{F}^r \cdot D^r\) and \(C^g := F^g \cdot D^g\). From (8.1) and (8.2), we can deduce that the following equalities hold

\[
\hat{Y}_{\tau \wedge \theta} = X_{\tau \wedge \theta} \mathbbm{1}_{\{\tau < \theta\}} + R_\theta \mathbbm{1}_{\{\tau \geq \theta\}}
\]

and

\[
\hat{Y}_t = Y_0 + Z_{\tau \wedge \theta} \mathbbm{1}_{[0, \tau \wedge \theta]} \hat{M} + (R - Y) \mathbbm{1}_{[0, \tau \wedge \theta]} \Gamma + \mathbbm{1}_{[0, \tau \wedge \theta]} \mathbbm{1}_{[0, \tau \wedge \theta]} C^r + \mathbbm{1}_{[0, \tau \wedge \theta]} \mathbbm{1}_{[0, \tau \wedge \theta]} C^g + (R - Y_{\tau -}) \mathbbm{1}_{[0, \tau \wedge \theta]} \cdot A
\]

Using again (8.1), we obtain

\[
(Y - Y_{\tau -}) \mathbbm{1}_{[0, \tau \wedge \theta]} \cdot A = Z_{\tau \wedge \theta} \mathbbm{1}_{[0, \tau \wedge \theta]} \Gamma + \mathbbm{1}_{[0, \tau \wedge \theta]} \mathbbm{1}_{[0, \tau \wedge \theta]} C^r
\]

Thus, by replacing \((R - Y_{\tau -})\) by \((R - Y)\) in the last term of (8.4) and using the equality \(N^G = A - \mathbbm{1}_{[0, \tau \wedge \theta]} \cdot \Gamma\) (see (2.2)), we see that \(\hat{Y}\) is equal to

\[
Y_0 + Z_{\tau \wedge \theta} \mathbbm{1}_{[0, \tau \wedge \theta]} \hat{M} + (R - y) \mathbbm{1}_{[0, \tau \wedge \theta]} \cdot N^G + \mathbbm{1}_{[0, \tau \wedge \theta]} \mathbbm{1}_{[0, \tau \wedge \theta]} \cdot C^r + \mathbbm{1}_{[0, \tau \wedge \theta]} \mathbbm{1}_{[0, \tau \wedge \theta]} \cdot C^g
\]

To establish (8.3), it now remains to show that

\[
\mathbbm{1}_{[0, \tau \wedge \theta]} \cdot C^r = \hat{F}^r(\hat{Y}) \mathbbm{1}_{[0, \tau \wedge \theta]} \cdot D^r - (F^r(R) - F^r(Y)) \Delta D^r \mathbbm{1}_{[0, \tau \wedge \theta]} \cdot N^G
\]

where the variables \(Z\) and \(\hat{U}\) are suppressed.

To this end, by using the fact that the equalities \(F^r = \hat{F}^r\) and \(Y = \hat{Y}\) hold on \([0, \theta]\) and \(\hat{Y}_{[\theta]} = R \mathbbm{1}_{[\theta]}\), we deduce from (7.3) that

\[
I_1 := \hat{F}^r(Y, \hat{U}) \mathbbm{1}_{[0, \tau]} \mathbbm{1}_{[0, \theta]} \cdot D^r = \hat{F}^r(Y) \mathbbm{1}_{[0, \tau \wedge \theta]} \cdot D^r - F^r(R) \mathbbm{1}_{[0, \tau]} \mathbbm{1}_{[0, \theta]} \cdot D^r
\]

\[
+ \Delta \Gamma (F^r(R) - F^r(Y)) \mathbbm{1}_{[0, \tau]} \mathbbm{1}_{[0, \theta]} \cdot D^r
\]
and
\[ I_2 := \hat{F}^r(Y)1_{[0,\vartheta]} \hat{I}_1 \hat{D}^r = \left[ F^r(Y) + \Delta \Gamma (F^r(R) - F^r(Y)) \right] 1_{[0,\tau]} \hat{I}_1 \hat{D}^r. \]

Consequently, since \( N^{\alpha_G} = A - 1_{[0,\vartheta]} \hat{\Gamma} \) is a process of finite variation stopped at \( \vartheta \) and \( 1_{[\vartheta]} = \Delta A \), we conclude that
\[ 1_{[0,\tau \wedge \vartheta]} \hat{C}^r = I_1 + I_2 \]
\[ = \hat{F}^r(\hat{Y})1_{[0,\tau \wedge \vartheta]} \hat{D}^r - 1_{[0,\tau]} (F^r(R) - F^r(Y)) (\Delta A - \Delta \hat{\Gamma} 1_{[0,\vartheta]}) \hat{D}^r \]
\[ = \hat{F}^r(\hat{Y})1_{[0,\tau \wedge \vartheta]} \hat{D}^r - 1_{[0,\tau]} (F^r(R) - F^r(Y)) \Delta \hat{D}^r \hat{F}^r \hat{I}_{[0,\tau \wedge \vartheta]} N^{\alpha_G}, \]

which shows that equality (8.5) is valid. \( \square \)

**Remark 8.1.** In the case where \( R \) belongs to the class \( \mathcal{P}(\mathbb{F}) \), one can modify the above proof by noticing that \( Q := A^p - A^p \) is a finite variation \( \mathbb{F} \)-martingale
\[ R\hat{G}^{-1} \cdot (A^p - A^p) = -R(G^{-1} \cdot Q + \hat{G}^{-1} \cdot Q - G^{-1} \cdot Q) = -R\hat{G}^{-1} \cdot \hat{Q}, \]

which, when stopped at \( \vartheta \), is a \( \mathbb{G} \)-martingale. Then, in view of the predictable representation property of \( M \), this term will contribute to the \( \mathbb{G} \)-martingale term \( \hat{M} \) in (8.1).

The following proposition is a consequence of Lemma 8.1. It shows that a solution to the \( \mathbb{G} \) BSDE can be constructed by first solving the coupled equations (8.6)–(8.7). Recall that we denote \( F^r_s(\cdot) := F^r_s(\cdot, Z_s, U_s), F^g_s(\cdot) := F^g_s(\cdot, Z_s, U_s), \hat{F}^r_s(\cdot) := \hat{F}^r_s(\cdot, \hat{Z}_s, \hat{U}_s) \) and \( \hat{F}^g_s(\cdot) := \hat{F}^g_s(\cdot, \hat{Z}_s, \hat{U}_s) \).

**Proposition 8.1.** Assume that \((Y, Z, U)\) is a solution to the BSDE on \([0,\tau] \)
\[ Y_t = X_t - \int_{[t,\tau]} F^r_s(Y_s) dD^r_s - \int_{[t,\tau]} F^g_s(Y_s) dD^g_s + \int_{[t,\tau]} Z_s d\hat{M}_s \]
\[ + \int_{[t,\tau]} \left[ R_s - Y_s - (F^r_s(R_s) - F^r_s(Y_s)) \Delta \hat{D}^r_s \right] d\Gamma_s \]  
where the \( \mathbb{F} \)-optional process \( U \) satisfies the following equality, for all \( t \in \mathbb{R}_+ \),
\[ \int_{[0,\vartheta]} U_s dN^{\alpha_G}_s = \int_{[0,\vartheta]} \left[ R_s - Y_s - (F^r_s(R_s) - F^r_s(Y_s)) \Delta \hat{D}^r_s \right] dN^{\alpha_G}_s. \]  
(8.7)

Then the triplet \((\hat{Y}, \hat{Z}, \hat{U}) := (\hat{Y}, Z^g, U)\) where the process \( \hat{Y} \) is given by
\[ \hat{Y} := Y_0 + 1_{[0,\vartheta]} \hat{Y}^r + 1_{[0,\vartheta]} \hat{Y}^g + (R_0 - Y_0 -) 1_{[\vartheta,\infty]} 1_{[t \wedge \vartheta]} \]
is a solution to the BSDE (3.2) on \([0, \vartheta \wedge \tau]\), that is,
\[ \hat{Y}_t = \hat{X}_{t \wedge \vartheta} - \int_{[t, \tau \wedge \vartheta]} \hat{P}_s^g(\hat{Y}_s) d\hat{D}^g_s - \int_{[t, \tau \wedge \vartheta]} \hat{F}^g_s(\hat{Y}_s) d\hat{D}^g_s + \int_{[t, \tau \wedge \vartheta]} \hat{Z}_s d\hat{M}_s - \int_{[t, \tau \wedge \vartheta]} \hat{U}_s dN^{\alpha_G}_s. \]

In the next step, we will examine the existence of a solution to the coupled equations (8.6)–(8.7). Specifically, we seek a triplet \((\hat{Y}, Z, U)\) of processes that satisfy, for all \( t \in [0, \tau] \),
\[ Y_t = X_t - \int_{[t,\tau]} F^r_s(Y_s) dD^r_s - \int_{[t,\tau]} F^g_s(Y_s) dD^g_s + \int_{[t,\tau]} Z_s dM_s \]
\[ + \int_{[t,\tau]} Z_s G^{-1} \nu_s d[M, M]_s + \int_{[t,\tau]} \left[ R_s - Y_s - (F^r_s(R_s) - F^r_s(Y_s)) \Delta \hat{D}^r_s \right] d\Gamma_s \]  
(8.8)

and, for all \( t \in \mathbb{R}_+ \) (notice that (8.9) is manifestly stronger than (8.7))
\[ U_t = R_t - Y_t - (F^r_t(R_t) - F^r_t(Y_t)) \Delta \hat{D}^r_t. \]  
(8.9)
To examine the existence of a solution to the coupled equations (8.8)–(8.9), we introduce the transformed equations (8.10)–(8.11). Our goal is to remove the quadratic variation term $G^{-1} \cdot [m, M] = G^{-1} \nu \cdot [M, M]$ from (8.8) and place $\nu$ inside the generators $F^r$ and $F^g$, which are assumed to be bounded. In that way, we avoid the need to check the appropriate growth conditions when applying the existing results on well-posedness of BSDEs.

We define the linear transformation $\tilde{Y} := GY$, $\tilde{Z} := GZ + G^{-1} \tilde{Y} \nu$ and $\tilde{U} := U$. Hence we obtain the transformed generators

$$\tilde{F}_s^r(y, z, u) := \bar{G}_s F_s^r(G_s^{-1}y, G_s^{-1}(z - G_s^{-1}y\nu_s), u)$$

and

$$\tilde{F}_s^g(y, z, u) := \bar{G}_s F_s^g(G_s^{-1}y, G_s^{-1}(z - G_s^{-1}y\nu_s), u)$$

and we denote $\tilde{F}_s^r(\cdot) := \tilde{F}_s^r(\cdot, \tilde{Z}_s, \tilde{U}_s)$ and $\tilde{F}_s^g(\cdot) := \tilde{F}_s^g(\cdot, \tilde{Z}_s, \tilde{U}_s)$. Observe that if a solution $(Y, Z, U) \in \mathcal{O}(\mathbb{F}) \times \mathcal{P}_d(\mathbb{F}) \times \mathcal{O}(\mathbb{F})$ to (8.8)–(8.9) exists, then $(\tilde{Y}, \tilde{Z}, \tilde{U}) \in \mathcal{O}(\mathbb{F}) \times \mathcal{P}_d(\mathbb{F}) \times \mathcal{O}(\mathbb{F})$ satisfies the following coupled equations, for all $t \in [0, \tau],$

$$\tilde{Y}_t = G_\tau X_\tau - \int_{[t, \tau]} \tilde{F}_s^r(\tilde{Y}_s) \, dD_s^r - \int_{[t, \tau]} \tilde{F}_s^g(\tilde{Y}_s) \, dD_s^g + \int_{[t, \tau]} \tilde{Z}_s \, dM_s + \int_{[t, \tau]} [\bar{G}_s R_s - \Delta \tilde{F}_s^r \Delta D_s^r] \, d\Gamma_s$$

(8.10)

and, for all $t \in \mathbb{R}_+$

$$\tilde{U}_t = R_t - \tilde{Y}_t G_t^{-1} - \bar{G}_t^{-1} \Delta \tilde{F}_t^r \Delta D_t^r$$

(8.11)

where we denote $\Delta \tilde{F}_s^r := \bar{F}_s^r(G_s R_s) - \bar{F}_s^r(\tilde{Y}_s)$. In the reverse, a solution $(Y, Z, U)$ to the coupled equations (8.8)–(8.9) can be obtained from a solution $(\tilde{Y}, \tilde{Z}, \tilde{U})$ to the coupled equations (8.10)–(8.11) by setting $Y := G^{-1} \tilde{Y}$, $Z := G^{-1}(\tilde{Z} - G^{-1} \tilde{Y} \nu)$ and $U := \tilde{U}$.

Observe that BSDEs (8.8) and (8.10) have the lagläd driver $D = (D^r, D^g)$, which may share common jumps with the martingale $M$. To the best of our knowledge, there is a gap in the literature on BSDEs of this form and thus we develop in Section 13 a jump-adapted methodology to solve such BSDEs under specific assumptions.

**Remark 8.2.** If a random time $\vartheta$ is an $F$-pseudo-stopping time (see Nikeghbali and Yor [31] and Aksamit and Li [5]), then $m = 1$ and thus $\nu = 0$ in (7.7) so that we can deal directly with (8.8). Also, under Assumption (C), it is possible to eliminate the term $G^{-1} \cdot [m, M]$ using the Girsanov transform in $\mathbb{G}$ if $\vartheta$ is an invariance time, as introduced and studied in Crépey and Song [10]. We point out that in [10], the authors have worked under a predictable setup, rather than an optional one, as we do in the present work. Furthermore, the $\mathbb{G}$ BSDE in [9] was stopped strictly before $\vartheta$ (specifically, at $\vartheta - $), rather than at $\vartheta$. Therefore, the Girsanov theorem was in fact applied in [9] to eliminate the term $G^{-1} \cdot [m, M]$, which is known to coincide with $G^{-1} \cdot [m, M]$ under Assumption (C).

In principle, one could attempt to mimic the approach developed in [10] by studying a new class of random times for which the drift term $G^{-1} \cdot [m, M]$ can be removed or, in other words, to introduce a family of random times for which there exists an equivalent probability measure under which they are $F$-pseudo-stopping times. However, this issue is beyond the scope of the current work and here we assume that either the generator is bounded or $\vartheta$ is an $F$-pseudo-stopping time.

### 9 Solution to the $\mathbb{G}$ BSDE in the Brownian case

To illustrate our approach from Sections 7 and 8, we first use results from Essaky et al. [14] to show that in the case of the Brownian filtration, if $F^g = 0$ and $U$ does not appear in the right-hand side of (8.9) or (8.11), then a unique solution $(\tilde{Y}, \tilde{Z}, \tilde{U})$ to (8.10)–(8.11) exists and thus a unique solution to $(Y, Z, U)$ to (8.8)–(8.9) exists as well. Specifically, we now take $M = W$ to be a $d$-dimensional
Wiener process in its natural filtration $\mathcal{F}$ and we consider below the coupled equations (8.8)–(8.9) of $F^g = 0$. Our goal is to demonstrate that, under some natural assumptions, these equations have a unique solution $(Y, Z, U) \in \mathcal{O}(\mathcal{F}) \times \mathcal{P}_d(\mathcal{F}) \times \mathcal{O}(\mathcal{F})$.

To simplify the notation, we write $D$ and $F$ instead of $(D^r, 0)$ and $(F^r, 0)$, respectively, and we consider the situation where $D = (D^1, D^2) = (\langle W \rangle, \Gamma)$ and $F = (F^1, F^2) = (F^1(y, z, u), F^2(y))$ for $(y, z, u) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$ and $s \in [t, \tau]$. Then the BSDE (8.10) becomes

$$
\begin{align*}
\bar{Y}_t &= G_x X_t + \int_{[t, \tau]} \bar{F}^1_s(\bar{Y}_s) d(W)_s - \int_{[t, \tau]} \bar{F}^2_s(\bar{Y}_s) d\Gamma_s - \int_{[t, \tau]} Z_s dW_s \\
&\quad + \int_{[t, \tau]} [\bar{G}_s R_s - (F^2_s(G_s R_s) - F^2_s(\bar{Y}_s)) \Delta \Gamma_s] d\Gamma_s
\end{align*}
$$

(9.1)

where $\bar{F}^1_s(\bar{Y}_s) := \bar{F}^1_s(\bar{Y}_s, \bar{Z}_s, \bar{U}_s)$ and equation (8.11) has an explicit solution given by

$$
\bar{U} = (R - \bar{Y} G^{-1} - (\bar{F}^2(\bar{G} R) - \bar{F}^2(\bar{Y}))\bar{G}^{-1} \Delta \Gamma).
$$

(9.2)

We point out that, in the above, our choice of $D^2 = \Gamma$ for simplicity of presentation.

**Proposition 9.1.** Assume that:

(i) for every $t \in \mathbb{R}_+$, the map $F^1_t : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ is bounded and Lipschitz continuous;

(ii) for every $t \in \mathbb{R}_+$, the map $F^2_t : \mathbb{R} \to \mathbb{R}$ is bounded, Lipschitz continuous and decreasing;

(iii) the dual $\mathcal{F}$-optional projection $\Lambda^c$ has a finite number of discontinuities.

Then the BSDE (9.1) has a solution $(\bar{Y}, \bar{Z})$ and a solution $(Y, Z)$ to (8.8) is obtained by setting $Y := G^{-1} \bar{Y}$ and $Z := G^{-1}(\bar{Z} - G^{-1} \bar{Y})$.

**Proof.** To establish the existence of a solution $(\bar{Y}, \bar{Z})$ to (9.1), we will apply Theorem 2.1 in Essaky et al. [14] to the data $(P, R, \theta, F)$. We observe that the BSDE (9.1) is a special case of equation (2.1) in [14], which has the following form

$$
\begin{align*}
\bar{Y}_t &= G_x X_t + \int_{[t, \tau]} f(s, \bar{Y}_s, \bar{Z}_s, \bar{U}_s) ds + \int_{[t, \tau]} g(s, \bar{Y}_s) d\bar{A}_s \\
&\quad + \sum_{t < \tau} h(s, \bar{Y}_s, \bar{Z}_s, \bar{U}_s) - \int_{[t, \tau]} \bar{Z}_s dW_s.
\end{align*}
$$

(9.3)

Indeed, (9.1) can be recovered from (9.3) if we set $\bar{A} := \Gamma^c$ where $\Gamma^c$ is the continuous part of $\Gamma$ and define the mappings $f, g$ and $h$ as follows

$$
\begin{align*}
f(s, \bar{Y}_s, \bar{Z}_s, \bar{U}_s) &:= -\bar{F}^1_s(\bar{Y}_s, \bar{Z}_s, \bar{U}_s), \\
g(s, \bar{Y}_s) &:= \bar{G}_s R_s - \bar{F}^2_s(\bar{Y}_s), \\
h(s, \bar{Y}_s, \bar{Z}_s) &:= (\bar{G}_s R_s - \bar{F}^2_s(\bar{Y}_s)) \Delta \Gamma_s - (F^2_s(G_s R_s) - F^2_s(\bar{Y}_s)) (\Delta \Gamma_s)^2
\end{align*}
$$

where $\bar{U}$ is given by (9.2) and the mapping $h$ is in fact independent of the variable $\bar{Y}_s$.

To apply Theorem 2.1 from Essaky et al. [14], it suffices to check that Assumptions (A.1)–(A.4) on page 2151 of [14] are satisfied. To check Assumption (A.1), we note that the mapping $F^1$ is assumed to be continuous and, since it is also bounded, there exists a constant $C > 0$ such that $|F^1| \leq C < C(1 + |z|)$, which shows that the linear growth condition is satisfied. Next, from the fact that $|\Delta \Gamma| \leq 1$, we deduce that $|g|$ is bounded and thus Assumption (A.2) holds as well. Finally, $h$ is bounded and continuous and, since the process $\Gamma$ is assumed to have a finite number of jumps, it is enough to check condition (c) in Assumption (A.3).

To this purpose, we observe that $0 \leq \Delta \Gamma \leq 1$ and thus the mapping

$$
\begin{align*}
y \mapsto y + h(s, y) &= y - \Delta \Gamma_s \bar{F}^2_s(y)(1 - \Delta \Gamma_s) + \Delta \Gamma_s (\bar{G}_s R_s - \Delta \Gamma_s \bar{F}^2_s(G_s R_s))
\end{align*}
$$

is nondecreasing and continuous. Finally, the Mokobodski condition postulated in (A.4) is trivially satisfied as we deal here with the BSDE with no reflecting boundaries. Thus, by applying Theorem 2.1 in [14] with $T$ replaced by $\tau \in T$, we obtain the existence of a maximal solution $(\bar{Y}, \bar{Z})$ to (9.1).
Example 9.1. Let us consider a special case where \( \vartheta \) is an \( \mathbb{F} \)-pseudo-stopping time (see Nikeghbali and Yor [31]) and thus the process \( \nu \) in (7.7) vanishes. We can suppose that the mapping \( z \mapsto F^1_t(y, z, u) \) has a linear (quadratic) growth and thus, since in the case of a Brownian filtration the equality \( G = G_- \) holds, we obtain
\[
|F^1_t(y, z)| = G_s^{-1} F^1_s(G_s^{-1} y, G_s^{-1}(z - G_s^{-1} y_v), u) \leq G_{s-}(1 + G_{s-}^{-1}|z|) \leq (1 + |z|),
\]
which shows that the boundedness of \( F^1 \) postulated in Proposition 9.1 can be relaxed.

10 Reduction of a solution to the \( \mathbb{G} \) RBSDE

Our goal in this section is to study the properties of solutions to the \( \mathbb{G} \) RBSDE with a random time horizon \( \vartheta \).

Definition 10.1. A quadruplet \((\hat{Y}, \hat{Z}, \hat{U}, \hat{L})\) is a solution on the interval \([0, \bar{\tau} \wedge \vartheta]\) to the \( \mathbb{G} \) RBSDE
\[
\hat{Y}_t = \hat{X}_{\bar{\tau} \wedge \vartheta} - \int_{[t, \bar{\tau} \wedge \vartheta]} \hat{F}_s^r(\hat{Y}_s, \hat{Z}_s, \hat{U}_s) d\hat{D}^r_s - \int_{[t, \bar{\tau} \wedge \vartheta]} \hat{F}_s^g(\hat{Y}_s, \hat{Z}_s, \hat{U}_s) d\hat{D}^g_s - \int_{[t, \bar{\tau} \wedge \vartheta]} \hat{Z}_s \alpha dM_s - \int_{[t, \bar{\tau} \wedge \vartheta]} \hat{U}_s dN^{\vartheta, \mathbb{G}}_s - (\hat{L}_{\bar{\tau} \wedge \vartheta} - \hat{L}_t)
\]
(10.1)
if \( \hat{Y} \in \mathcal{O}(\mathbb{G}) \) is a \( \mathcal{G} \)-adapted process such that \( \hat{Y} \geq \hat{X} \), the processes \( \hat{Z} \in \mathcal{P}_d(\mathbb{G}) \) and \( \hat{U} \in \mathcal{O}(\mathbb{F}) \) are such that the stochastic integrals in the right-hand side of (3.2) are well defined, \( \hat{L} = \hat{L}^\vartheta \) is a \( \mathcal{G} \)-adapted, increasing, and \( \mathbb{G} \)-strongly predictable process with \( \hat{L}_0 = 0 \) and has the decomposition \( \hat{L} = \hat{L}^r + \hat{L}^\vartheta \) where the processes \( \hat{L}^r \) and \( \hat{L}^\vartheta \) obey the Skorokhod conditions
\[
(1_{\{\hat{Y} = \hat{X} \}} \cdot \hat{L}^\vartheta)_{\bar{\tau}} = (1_{\{\hat{Y} = \hat{X} \}} \cdot \hat{L}^r)_{\bar{\tau}} = 0
\]
and equality (10.1) is assumed to hold on \([0, \bar{\tau} \wedge \vartheta]\).

As in Section 7, in order to show that (10.1) has a solution, we will first reduce a solution to the \( \mathbb{G} \) RBSDE (10.1) to a solution of an associated RBSDE in filtration \( \mathbb{F} \). Subsequently, we show how a solution to the reduced \( \mathbb{F} \) RBSDE, which can be shown to exist under suitable assumptions, can be employed to construct a solution to the \( \mathbb{G} \) RBSDE (10.1). Again, we first work under the following temporary postulate, which will be relaxed in Section 11.

Assumption 10.1. A solution \((\hat{Y}, \hat{Z}, \hat{U}, \hat{L})\) to the \( \mathbb{G} \) RBSDE (10.1) exists.

Following the approach developed for the non-reflected case, we decompose \( \hat{Y} \) into the pre-default and post-default components
\[
\hat{Y}_t 1_{\{\vartheta > t\}} + \hat{Y}_t 1_{\{\vartheta \leq t\}} = \hat{Y}_t 1_{\{\vartheta > t\}} + \hat{X}_{\vartheta \wedge \tau} 1_{\{\vartheta \leq t\}} = \hat{Y}_t 1_{\{\vartheta > t\}} + \hat{X}_t 1_{\{\vartheta \leq t\}} = G_t^{-1} \mathbb{E}(\hat{Y}_t 1_{\{\vartheta > t\}} | \mathcal{F}_t) 1_{\{\vartheta > t\}} + R_\vartheta 1_{\{\vartheta \leq t\}}.
\]
To compute the component \( G_t^{-1} \mathbb{E}(\hat{Y}_t 1_{\{\vartheta > t\}} | \mathcal{F}_t) \) we proceed similarly to the non-reflected case. The new feature here is the use of Lemma 7.2 in order to obtain a reduction of the \( \mathbb{G} \)-strongly predictable, increasing process \( \hat{L} \). Computations in this section are similar to those in Section 7, except for the presence of the reflection process and thus in the following we will focus on new elements. To proceed, similarly to (7.4), we set
\[
K_t(\tau) := \mathbb{E}(X_{\tau} G_\tau + (\hat{F}^r \cdot D)_\tau + (\hat{F}^g \cdot D^g_\tau)_\tau | \mathcal{F}_t) + \mathbb{E}((R + L) \cdot A^o)_\tau + L_\tau G_\tau | \mathcal{F}_t
\]
(10.2)
where \( \hat{F}^r \) is given by (7.3). From Assumption 3.1, we deduce the existence of \( \mathbb{F} \)-predictable processes \( \psi^{Y,Z,L} \) and \( \nu \) such that \( K(\tau) = \psi^{Y,Z,L} \cdot M \) and \( m = \nu \cdot M \).
Proposition 10.1. The process $Y = o(\tilde{Y} \mathbb{1}_{[0,\theta]} )G^{-1}$ satisfies the $\mathbb{F}$ RBSDE, on $[0, \tau]$, 

$$
Y_t = X_t - \int_{[t, \tau]} F^\tau_s(Y_s) \, dD^\tau_s - \int_{[t, \tau]} F^\theta_s(Y_s) \, dD^\theta_s + \int_{[t, \tau]} z_s \, d\tilde{M}_s \\
+ \int_{[t, \tau]} [R_s - Y_s - (F^\tau_s(\tau_s) - F^\tau_s(Y_s))\Delta D^\tau_s] \, d\tilde{G}_s - (L- - L_t)
$$

where $Y \geq X, U = \tilde{U}$, the process $Z$ is the $\mathbb{F}$-reduction of the process $\tilde{Z}$ given by (7.1), $z_t := G^{-1}_t(\psi_t^{Y,Z} - Y_{t-\theta_t})$ is an $\mathbb{F}$-predictable process and $L$ is an $\mathbb{F}$-strongly predictable, increasing process such that $L = \tilde{L}$ on $[0, \theta]$ and the Skorokhod conditions are satisfied, that is, $(\mathbb{1}_{\{Y \neq X\}} \cdot L^\tau)_\tau = (\mathbb{1}_{\{Y \neq X\}} \cdot L^\theta)_\tau = 0$.

Proof. In view of Lemmas 4.1, 7.2 and 7.3 the generators $\hat{F}^\tau, \hat{F}^\theta$ and the increasing process $\hat{L}$ can be reduced to the filtration $\mathbb{F}$ to obtain, on the event $\{\tau \geq t\}$,

$$
E(\tilde{Y}_t \mathbb{1}_{\{\theta \geq t\}} | F_t) = E(P_t \mathbb{1}_{\{\theta \geq t\}} + R_\theta \mathbb{1}_{\{t < \theta \leq \tau\}} + (\bar{G}\hat{F}^\tau \cdot D^\tau)_{\tau} - (\bar{G}\hat{F}^\tau \cdot D^\tau)_t \\
+ (Gf^\theta \cdot D^\theta)_r - (Gf^\theta \cdot D^\theta)_t + (L_\theta - L_t)\mathbb{1}_{\{\theta \geq t\}} + (L_\theta - L_t)\mathbb{1}_{\{t < \theta \leq \tau\}} | F_t)
$$

Next, an application of the laccording product rule to $LG$ and the equalities $\bar{G} = G_- + \Delta m$ and $L = L_- + \Delta L^\tau$ yield

$$
LG = L_- \cdot m - L_- \cdot A^\circ + G_- \cdot L^\tau + G \cdot L^\theta + \Delta G \cdot \Delta L^\tau \\
= L_- \cdot m - L_- \cdot A^\circ + \bar{G} \cdot L^\tau + G \cdot L^\theta.
$$

By combining these computations, we conclude that

$$
YG = K(\tau) - \bar{G}\hat{F}^\tau \cdot D^\tau - Gf^\theta \cdot D^\theta - R_- \cdot A^\circ + \bar{G} \cdot L^\tau + G \cdot L^\theta
$$

where $Y := G^{-1} o(\tilde{Y} \mathbb{1}_{[0, \theta]} )$ and $K(\tau)$ is given by (10.2).

The backward dynamics of $Y$ can now be computed from Corollary 7.1

$$
Y_t = X_t - \int_{[t, \tau]} \tilde{F}_s(Y_s) \, dD_s^\tau - \int_{[t, \tau]} F^\theta_s(Y_s) \, dD_s^\theta + \int_{[t, \tau]} z_s \, dM_s \\
+ \int_{[t, \tau]} [R_s - Y_s] \, d\tilde{G}_s + \int_{[t, \tau]} [G^{-1}_s - z_s \, d[M, n]_s - (L_\tau - L_t)]
$$

and thus, after rearranging and using (7.3), we obtain the asserted BSDE.

It remains to check that the appropriate Skorokhod conditions are met by the process $L$. Recall that the Skorokhod conditions satisfied by $\tilde{L}^\tau$ and $\tilde{L}^\theta$ are

$$
(\mathbb{1}_{\{\tilde{Y} \neq X\}} \cdot \tilde{L}^\tau)_{\tau \wedge \theta} = (\mathbb{1}_{\{\tilde{Y} \neq X\}} \cdot \tilde{L}^\theta)_{\tau \wedge \theta} = 0.
$$

By integrating the first equality in (10.3) with respect to $G^{-1}$, we obtain

$$
(G^{-1} \mathbb{1}_{[0, \theta]} \mathbb{1}_{\{\tilde{Y} \neq X\}} \cdot \tilde{L}^\tau)_\tau = 0.
$$

The equality $\tilde{Y} \mathbb{1}_{[0, \theta]} = Y \mathbb{1}_{[0, \theta]}$ implies that $Y_- \mathbb{1}_{[0, \theta]} = Y_- \mathbb{1}_{[0, \theta]}$ and $(\tilde{L}^\tau)^\theta = (L^\tau)^\theta$. Consequently, since $X_- = X_- \mathbb{1}_{[0, \theta]} + R_\theta \mathbb{1}_{[0, +\infty)}$, we get

$$
(G^{-1} \mathbb{1}_{[0, \theta]} \mathbb{1}_{\{\tilde{Y} \neq X\}} \cdot \tilde{L}^\tau)_\tau = (G^{-1} \mathbb{1}_{[0, \theta]} \mathbb{1}_{\{Y \neq X\}} \cdot L^\tau)_\tau = 0.
$$
Then, by taking the expectation and using the property of the dual \(\mathbb{F}\)-predictable projection, we obtain
\[
\mathbb{E}((G^{-1}\mathbb{1}_{[0,\vartheta]}\mathbb{1}_{\{Y_{\neq X}\}} \cdot L^r)_{\tau}) = \mathbb{E}((\mathbb{1}_{\{Y_{\neq X}\}} \cdot L^r)_{\tau}),
\]
which implies that \(L^r\) obeys the first Skorokhod condition, that is, \((\mathbb{1}_{\{Y_{\neq X}\}} \cdot L^r)_{\tau} = 0\). Similarly, to check the second Skorokhod condition, we integrate the second equality in (10.3) with respect to \(G^{-1}\) and use the equality \(\hat{L}^q_{\tau} \mathbb{1}_{[0,\vartheta]} = L^q_{\tau} \mathbb{1}_{[0,\vartheta]}\) to obtain
\[
(G^{-1}\mathbb{1}_{[0,\vartheta]}\mathbb{1}_{\{\hat{Y}_{\neq X}\}} \cdot \hat{L}^q_{\tau})_{\tau} = (G^{-1}\mathbb{1}_{[0,\vartheta]}\mathbb{1}_{\{Y_{\neq X}\}} \cdot L^q_{\tau})_{\tau}.
\]
By taking the expectation and using the property of the dual \(\mathbb{F}\)-optional projection, we obtain the equality \(\mathbb{E}((\mathbb{1}_{\{Y_{\neq X}\}} \cdot L^q_{\tau})_{\tau} = 0\), which in turn implies that \((\mathbb{1}_{\{Y_{\neq X}\}} \cdot L^q_{\tau})_{\tau} = 0\). \(\square\)

11 Construction of a solution to the \(\mathbb{G}\) RBSDE

In this section, we relax the postulate that a solution \((\hat{Y}, \hat{Z}, \hat{U}, \hat{L})\) exists. In the next result, we again denote by \(\hat{U}\) an arbitrary prescribed \(\mathbb{F}\)-adapted process and we do not use Lemma 7.3.

**Lemma 11.1.** Let a process \(\hat{U} \in \mathcal{O}(\mathbb{F})\) be given and let \((Y, Z, L)\) be an \(\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}\)-valued, \(\mathbb{F}\)-adapted solution to the \(\mathbb{F}\) RBSDE, on \([0, \tau]\),
\[
Y_t = X_t - \int_{[t,\tau]} F_s^r(Y_s, Z_s, \hat{U}_s) dD_s^r - \int_{[t,\tau]} G^q_s(Y_s, Z_s, \hat{U}_s) dD_s^q + \int_{[t,\tau]} \left[ R_s - Y_s - (F_s^r(Y_s, Z_s, \hat{U}_s) - F_s^r(Y_s, \hat{Z}_s, \hat{U}_s)) \Delta D_s^r \right] d\Gamma_s
\]
where \(Y \geq X\) and \(L\) is an \(\mathbb{F}\)-strongly predictable, increasing process such that the Skorokhod conditions \((\mathbb{1}_{\{Y_{\neq X}\}} \cdot L^r)_{\tau} = (\mathbb{1}_{\{Y_{\neq X}\}} \cdot L^q_{\tau})_{\tau} = 0\) hold and \(L_0 = 0\). Then the triplet \((\hat{Y}, \hat{Z}, \hat{L}) := (\hat{Y}, Z^0, L^0)\) where \(\hat{Y}\) is given by
\[
\hat{Y} := Y_0 + \mathbb{1}_{[0,\vartheta]} \cdot Y^r + \mathbb{1}_{[0,\vartheta]} \cdot Y^q + (R_\vartheta - Y_\vartheta) \mathbb{1}_{[\vartheta,\infty]} \mathbb{1}_{(\tau \geq \vartheta)}
\]
is a solution to the \(\mathbb{G}\) RBSDE, on \([0, \tau \land \vartheta]\),
\[
\hat{Y}_t = X_{t \land \vartheta} - \int_{[t,\tau \land \vartheta]} \hat{F}_s^r(\hat{Y}_s, \hat{Z}_s, \hat{U}_s) dD_s^r - \int_{[t,\tau \land \vartheta]} \hat{G}_s^q(\hat{Y}_s, \hat{Z}_s, \hat{U}_s) dD_s^q + \int_{[t,\tau \land \vartheta]} \left[ R_s - Y_s - (F_s^r(Y_s, \hat{Z}_s, \hat{U}_s) - F_s^r(Y_s, \hat{Z}_s, \hat{U}_s)) \Delta D_s^r \right] d\Gamma_s
\]
where \(\hat{Y} \geq X\) and \(\hat{L} = L^0\) is a \(\mathbb{G}\)-strongly predictable, increasing process such that the Skorokhod conditions \((\mathbb{1}_{\{\hat{Y}_{\neq \hat{X}}\}} \cdot \hat{L})_{\tau \land \vartheta} = (\mathbb{1}_{\{\hat{Y}_{\neq \hat{X}}\}} \cdot \hat{L}^q_{\tau \land \vartheta}) = 0\) are valid and \(\hat{L}_0 = 0\).

**Proof.** It suffices to set \(C^r := \hat{G}^r \cdot D^r + L^r\) and \(C^q := F^q \cdot D^q + L^q\) in the proof of Lemma 8.1. The required Skorokhod conditions are also met since
\[
(\mathbb{1}_{[0,\vartheta]} \mathbb{1}_{\{\hat{Y}_{\neq X}\}} \cdot L^r)_{\tau} = (\mathbb{1}_{[0,\vartheta]} \mathbb{1}_{\{Y_{\neq X}\}} \cdot L^r)_{\tau} = 0
\]
and
\[
(\mathbb{1}_{[0,\vartheta]} \mathbb{1}_{\{\hat{Y}_{\neq X}\}} \cdot L^q_{\tau}) = (\mathbb{1}_{[0,\vartheta]} \mathbb{1}_{\{Y_{\neq X}\}} \cdot L^q_{\tau})_{\tau} = 0
\]
where we have used the following equalities: \(\hat{Y} \mathbb{1}_{[0,\vartheta]} = Y \mathbb{1}_{[0,\vartheta]}, \hat{Y} \mathbb{1}_{[0,\vartheta]} = Y \mathbb{1}_{[0,\vartheta]}\) and \(X\). \(\square\)
Proposition 11.1. Let \((Y, Z, U, L)\) be a solution to the \(\mathbb{F}\) RBSDE, on \([0, \tau]\),
\[
Y_t = X_\tau - \int_{[t, \tau]} F^r_s(Y_s) \, dD^r_s - \int_{[t, \tau]} F^g_s(Y_s) \, dD^g_s + \int_{[t, \tau]} Z_s \, d\tilde{M}_s
\]
(11.1)
where \(Y \geq X\) is an \(\mathbb{F}\)-strongly predictable, increasing process with \(L_0 = 0\) and such that the Skorokhod conditions \((1(Y_{-} \neq X_{-}) \cdot L^r)_{\tau} = (1(Y \neq X) \ast L^g_\tau)_{\tau} = 0\) are obeyed, and the \(\mathbb{F}\)-optional process \(U\) satisfies, for all \(t \in \mathbb{R}_+\),
\[
\int_{[0, t]} U_s \, dN^{o, G}_s = \int_{[0, t]} \left[R_s - Y_s - (F^r_s(R_s) - F^r_s(Y_s)) \Delta D^r_s\right] dN^{o, G}_s
\]
(11.2)
where \(\tilde{Y} \geq X\) and \(\tilde{L} = L^g\) is a \(\mathbb{G}\)-strongly predictable, increasing process such that the Skorokhod conditions \((1(Y_{-} \neq \tilde{X}_{-}) \cdot \tilde{L}^r)_{\tau} = (1(Y \neq \tilde{X}) \ast \tilde{L}^g_{\tau})_{\tau} = 0\) hold and \(L_0 = 0\).

Proof. The assertion of the proposition follows from Lemma 11.1 and similar arguments as used in Section 8 (see, in particular, the proof of Lemma 8.1).

We now focus on the existence of a solution to the coupled equations (11.1)–(11.2) from Proposition 11.1. As in Section 8, we define the linear transformation
\[
\tilde{Y} := G Y, \quad \tilde{Z} := G Z + G^{-1} \tilde{Y} \nu, \quad \tilde{U} := U, \quad \tilde{L}^r := G \ast L^r, \quad \tilde{L}^g := G \ast L^g
\]
and the transformed generators \(\tilde{F}^r\) and \(\tilde{F}^g\). It is easy to check that if a solution \((Y, Z, U, L) \in O(\mathcal{F}) \times \mathcal{P}d(\mathcal{F}) \times O(\mathcal{F}) \times \mathcal{P}(\mathcal{F})\) to the coupled equations (11.1)–(11.2) exists, then \((\tilde{Y}, \tilde{Z}, \tilde{U}, \tilde{L}) \in O(\mathcal{F}) \times \mathcal{P}d(\mathcal{F}) \times O(\mathcal{F}) \times \mathcal{P}(\mathcal{F})\) satisfies the following coupled equations (11.4)–(11.5)
\[
\tilde{Y}_t = G_{\tau} X_\tau - \int_{[t, \tau]} \tilde{F}^r_s(\tilde{Y}_s, \tilde{Z}_s, \tilde{U}_s) \, dD^r_s - \int_{[t, \tau]} \tilde{F}^g_s(\tilde{Y}_s, \tilde{Z}_s, \tilde{U}_s) \, dD^g_s + \int_{[t, \tau]} \tilde{Z}_s \, d\tilde{M}_s
\]
(11.4)
and
\[
\int_{[0, t]} \tilde{U}_s \, dN^{o, G}_s = \int_{[0, t]} \left[R_s - \tilde{Y}_s G^{-1} - \tilde{G}^{-1}_{s} \Delta F^r_s \Delta D^r_s\right] dN^{o, G}_s
\]
(11.5)
where
\[
\Delta F^r_s := \tilde{F}^r_s(G_{s} R_s, \tilde{Z}_s, \tilde{U}_s) - \tilde{F}^r_s(\tilde{Y}_s, \tilde{Z}_s, \tilde{U}_s)
\]
and \(\tilde{L} = \tilde{L}^r + \tilde{L}^g\) satisfies
\[
(1(Y_{-} \neq G_{-} X_{-}) \ast \tilde{L}^r)_{\tau} = (1(Y \neq G_{-} X) \ast \tilde{L}^g_{\tau})_{\tau} = 0.
\]
In the reverse, a solution \((Y, Z, U, L)\) to equations (11.1)–(11.2) can be obtained from a solution \((\tilde{Y}, \tilde{Z}, \tilde{U})\) to equations (11.4)–(11.5) by setting \(Y := G^{-1} \tilde{Y}, Z := G^{-1}(\tilde{Z} - G^{-1} \tilde{Y} \nu), U := \tilde{U}, L^r := \tilde{G}^{-1} \ast \tilde{L}^r\) and \(L^g := G^{-1} \ast \tilde{L}^g\).
12 Solution to the $\mathbb{G}$ RBSDE in the Brownian case

We proceed here similarly to Section 9. Let $M = W$ be a $d$-dimensional Wiener process and $\mathbb{F}$ be its natural filtration. We consider the coupled equations (11.1)–(11.2) with $F^g = 0$ and a càdlàg lower barrier $X$. The goal is again to demonstrate that, in some specific settings, the coupled equations (11.4)–(11.5) possess a unique solution $(Y, Z, U, L)$.

As before, we write $D$ and $F$ instead of $D^r$ and $F^r$, respectively, and we consider the case where $D = (D^1, D^2) = ((W), \Gamma)$ and, for all $(y, z, u) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$,

$$F = (F^1, F^2) = (F^1(y, z, u), F^2(y)).$$

Then the BSDE (11.1) is reduced to

$$
\begin{align*}
\bar{Y}_t &= G_t X_t - \int_{[t, \tau]} \bar{F}^1_s \, d\langle W \rangle_s - \int_{[t, \tau]} \bar{F}^2_s(Y_s) \, d\Gamma_s - \int_{[t, \tau]} \bar{Z}_s \, dW_s \\
&\quad + \int_{[t, \tau]} \left[ \bar{G}_s R_s - (F^2_s(G_s R_s) - F^2_s(Y_s)) \Delta \Gamma_s \right] \, d\Gamma_s - (L_\tau - L_t)
\end{align*}
$$

(12.1)

where $\bar{F}^1_s := \bar{F}^1_s(\bar{Y}_s, \bar{Z}_s, \bar{U}_s)$ and equation (11.2) has a solution $\bar{U}$ given by equality (9.2).

The following result gives sufficient conditions for existence of a solution to $\mathbb{F}$ RBSDEs (11.1) and (12.1) in the case of the Brownian filtration $\mathbb{F}$.

**Proposition 12.1.** Assume that:
(i) for every $t \geq 0$, the map $\bar{F}^1_t : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ is bounded and Lipschitz continuous;
(ii) for every $t \geq 0$, the map $\bar{F}^2_t : \mathbb{R} \to \mathbb{R}$ is bounded, Lipschitz continuous and decreasing;
(iii) the dual $\mathbb{F}$-optional projection $A^\circ$ has a finite number of discontinuities;
(iv) the process $X$ is càdlàg.

Then the RBSDE (12.1) has a solution $(\bar{Y}, \bar{Z}, \bar{L})$ and a solution $(Y, Z, L)$ to (11.1) can be obtained by setting $Y := G^{-1} \bar{Y}$, $Z := G^{-1}(\bar{Z} - G^{-1} \bar{Y} \nu)$ and $L := G^{-1} \cdot \bar{L}$.

**Proof.** Similarly to the proof of Proposition 9.1, in order to obtain a solution $(\bar{Y}, \bar{Z}, \bar{L})$ to the BSDE (9.1), we apply Theorem 2.1 in [14] to the data $(X, R, \partial, F)$. We note that (9.1) is a special case of equation (2.1) in [14] of the form

$$
\begin{align*}
\bar{Y}_t &= G_t X_t + \int_{[t, \tau]} f(s, \bar{Y}_s, \bar{Z}_s) \, ds + \int_{[t, \tau]} g(s, \bar{Y}_s) \, dA_s - \int_{[t, \tau]} \bar{Z}_s \, dW_s \\
&\quad + \sum_{t < s \leq \tau} h(s, \bar{Y}_{s-}, \bar{Z}_s) - (\bar{L}_\tau - \bar{L}_t)
\end{align*}
$$

(12.1)

where (12.1) can be recovered if we set $\bar{A} := \Gamma^c$ (the continuous part $\Gamma$) and

$$
\begin{align*}
 f(s, \bar{Y}_s, \bar{Z}_s) := -\bar{F}^1_s(\bar{Y}_s, \bar{Z}_s, \bar{U}_s), \quad g(s, \bar{Y}_s) := \bar{G}_s R_s - \bar{F}^2_s(Y_s),
\end{align*}
$$

and

$$
\begin{align*}
 h(s, \bar{Y}_{s-}, \bar{Z}_s) := (\bar{G}_s R_s - \bar{F}^2_s(Y_s)) \Delta \Gamma_s - (\bar{F}^2_s(G_s R_s) - \bar{F}^2_s(Y_s)) (\Delta \Gamma_s)^2
\end{align*}
$$

where $\bar{U}$ is given by (9.2). Notice that once again $h$ does not depend on $\bar{Y}_{s-}$. As was explained in the proof of Proposition 9.1, the assumptions in Theorem 2.1 of [14] are satisfied and thus a solution $(\bar{Y}, \bar{Z}, \bar{L})$ exists. Finally, we observe that the $\mathbb{F}$-predictable, increasing process $L := G^{-1} \cdot \bar{L}$ clearly obeys the Skorokhod conditions since

$$
(\mathbb{1}_{\{Y_\tau \neq X_\tau\}} \cdot L)_\tau = (\mathbb{1}_{\{\bar{Y}_\tau \neq G^{-1}(\bar{Y}_\tau)\}} G^{-1} \cdot \bar{L})_\tau = 0
$$

and thus the proof is completed. \qed
BSDEs with a làglàg driver and common jumps

In the last part, we shall work in a general setting and study solutions to BSDEs where the driver is làglàg and may share common jumps with the driving martingale. To the best of our knowledge, such results are not yet available in the literature. More specifically, given a filtration \( \mathbb{F} \), we propose a method of solving the \( \mathbb{F} \) BSDE

\[
v_t = \xi_t - \int_{[t,\tau]} f^r_s(v_s, z_s) \, dD^r_s - \int_{[t,\tau]} f^g_s \, dD^g_s - \int_{[t,\tau]} z_s \, dM_s
\]  

where \( f^r_s := f^r_s(v_{s-}, v_s, z_s) \) and \( f^g_s := f^g_s(v_s, v_{s+}) \). In particular, we observe that in the case where either \( F^g \) in (8.8) does not depend on \( U \) and \( Z \) or \( U \) in (8.9) can be solved and does not depend on \( Z \) (see, for example, Section 9), then the BSDEs (8.8) and (8.10) can be obtained as a special case of the above BSDE (13.1).

We present below a jump-adapted method of transforming the làglàg BSDE given by (13.1) to a system of more tractable càdlàg BSDEs, which in turn can be further converted into a system of càdlàg BSDEs with a continuous driver. In some special cases, a solution to the latter BSDE can be obtained using results from the existing literature.

**Step 1. From a làglàg to càdlàg driver.** For simplicity, in the following we denote \( D := D^r + D^g \) so that \( D \) is a làglàg process of finite variation. We suppose that the times of right-hand jumps of \( D \) (that is, the moments when \( \Delta^+ D > 0 \)) are given by the family \( (T_i)_{i=1,2,\ldots} \) of \( \mathbb{F} \)-stopping times and we denote \( S_0 := 0, S_i := T_i \wedge \tau \) and \( S_{p+1} := \tau \). We note that at each \( S_i \) we have that \( v_s - v = \int_{T_i} f^g_s(v_s, v_{s+}) \, \Delta D^g_s \), which shows that if the value of \( v_s \) is already known, then the value of \( v \) can be obtained as a solution to that equation. This leads to the observation that a solution \((v, z)\) to (13.1) can be created by first solving iteratively, for every \( i = 0, 1, \ldots, p \), the following càdlàg BSDE on each stochastic interval \([S_i, S_{i+1}]\)

\[
v^i_t = \xi^i_t - \int_{[t,S_{i+1}]} f^r_s(v^i_s, z^i_s) \, dD^r_s - \int_{[t,S_{i+1}]} z^i_s \, dM_s
\]

where an \( \mathcal{F}_{S_{i+1}} \)-measurable random variable \( \xi^i_t \) is given by the system of equations, for every \( i = 0, 1, \ldots, p-1 \),

\[v^{i+1}_{S_{i+1}} - \xi^i_t = f^g_{S_{i+1}}(\xi^i_t, v^{i+1}_{S_{i+1}}) \, \Delta D_{S_{i+1}}^g \]

with \( \xi^p = \xi_\tau \). Then a solution \((v, z)\) to the làglàg BSDE (13.1) is constructed by setting

\[
v := \sum_{i=0}^{p} v^i \mathbb{1}_{[S_i, S_{i+1}]} \quad z := \sum_{i=0}^{p} z^i \mathbb{1}_{[S_i, S_{i+1}]}.
\]

**Step 2. From a càdlàg to continuous driver.** In view of (13.2), in the following we focus on showing that the càdlàg BSDE can be solved, under certain assumptions about the driver and filtration. We now consider the situation where the filtration \( \mathbb{F} \) can support discontinuous martingales (e.g., the Brownian-Poisson filtration) and the driver \( D \) is possibly discontinuous. More specifically, we study the càdlàg BSDE of the form

\[
y_t = \xi_t - \int_{[t,\tau]} f^r_s(y_{s-}, y_s, z_s) \, dD^r_s - \int_{[t,\tau]} z_s \, dM_s
\]

where a solution \((y, z) \in \mathcal{C}(\mathbb{F}) \times \mathcal{P}_d(\mathbb{F})\) is such that \( y \) is a càdlàg process.

**Remark 13.1.** Our interest in the BSDE (13.3) is motivated by the need to understand the well-posedness of the pre-default BSDE, which is obtained in a nonlinear reduced-form model without postulating that either condition (C) or (A) holds. The discontinuity in \( D^r \) stems from the discontinuity of the hazard process \( \Gamma \) and, in some financial applications, the introduction of the nonlinearity can be interpreted as a way to introduce ambiguity in the recovery and the default intensity (see, for instance, Fadina and Schmidt [15]).
In the following, we suppose that $\xi_\tau$ is bounded and $\mathcal{F}_\tau$-measurable and we consider a more general BSDE

$$y_t = \xi_\tau - \int_{[t, \tau]} f^r_s(y_{s-}, y_s, z_s) dD^c_s - \sum_{t < s \leq \tau} h(s, y_{s-}, y_s) - \int_{[t, \tau]} z_s dM_s$$

(13.4)

where $f^r_s := f^r_s(y_{s-}, y_s, z_s)$ and $D^c$ is the continuous part of the process $D^r$.

To recover the BSDE (13.3) from (13.4), it suffices to set $h(s, y_{s-}, y_s) := f^r_s(y_{s-}, y_s) \Delta D^r_s$. Consequently, we henceforth suppose that $h = 0$ outside the graph of a finite set of $\mathbb{F}$-predictable stopping times $(T_i)_{i=1,2,...,p}$ and we denote $S_0 := 0$, $S_i := T_i \wedge \tau$ and $S_{p+1} := \tau$.

**Remark 13.2.** Note that a sufficient assumption for the jumps of $D^r$ to be $\mathbb{F}$-predictable stopping times is to postulate that $D^r$ is an $\mathbb{F}$-predictable, increasing process. Furthermore, observe that the condition that $(T_i)_{i=1,2,...,p}$ are $\mathbb{F}$-predictable stopping times can be relaxed if the mapping $h$ does not depend on $y_-$.

**Remark 13.3.** Suppose that $p = 1$ and denote $S = S_1$. Let us assume that a solution to (13.4) on $[S, \tau]$ has already been found and our goal is to construct its extension to the interval $[0, \tau]$. We observe that if $(y, z)$ is a solution to (13.4), then

$$y_t = y_0 + \int_{[0,t]} f^r_s(y_{s-}, y_s, z_s) dD^c_s + \sum_{0 < s \leq t} h(s, y_{s-}, y_s) + \int_{[0,t]} z_s dM_s$$

and hence the jump of the càdlàg process $y$ at time $S$ satisfies

$$\Delta y_S := y_S - y_{S-} = h(S, y_{S-}, y_S) + z_S \Delta M_S.$$  

(13.5)

By taking the conditional expectation of both sides of (13.5) with respect to $\mathcal{F}_{S-}$, we obtain the following equation

$$y_{S-} = \mathbb{E}(y_S - h(S, y_{S-}, y_S) \mid \mathcal{F}_{S-}),$$

which, at least in principle, can be solved for $y_{S-}$ under suitable additional assumptions. Subsequently, one could compute $z_S$ from equality (13.5). However, if one decides to proceed in that way, then to solve the BSDE (13.4) on $[0, S]$ one would need to solve (13.4) on $[0, S]$ and thus to study the BSDE driven by the martingale $M$ stopped at $S_-$. Since this would be quite cumbersome, we propose in Proposition 13.1 an alternative method where this difficulty is circumvented.

To show the existence of a solution to the BSDE (13.4), we introduce an auxiliary làglàd BSDE

$$v_t = \xi_\tau - h(\tau, v_{\tau-}, \xi_\tau) - \int_{[t, \tau]} f^l_s(v_{s-}, v_s, z_s) dD^c_s - \int_{[t, \tau]} z_s dM_s$$

(13.6)

where a solution $(v, z) \in \mathcal{O}(\mathbb{F}) \times \mathcal{P}_d(\mathbb{F})$ is such that $v$ is a làglàd process.

**Proposition 13.1.** Let $v$ be a làglàd process such that $(v, z)$ is a solution to the BSDE (13.6) on $[0, \tau]$. Then $(y, z)$ where $y := v_+ \mathbb{1}_{[0, \tau]} + h(\tau, v_{\tau-}, \xi_\tau) \mathbb{1}_{[\tau, \tau]}$ is a solution to the càdlàg BSDE (13.4) on $[0, \tau]$.

**Proof.** Suppose that $(v, z)$ is a solution to (13.6). It is clear from (13.6) that the left-hand and right-hand jumps of $v$ are given by $\Delta v = z \Delta M$ and $\Delta^+ v = h(\cdot, v_-, v_+)$, respectively. By the optional sampling theorem, we have that $\mathbb{E}(v_S \mid \mathcal{F}_{S-}) = v_{S-}$ for any $\mathbb{F}$-predictable stopping time $S$. Therefore, if the random variable $v_{S+}$ is known, then the $\mathcal{F}_S$-measurable random variable $v_S$ is a solution to the equation

$$\Delta^+ v_S := v_{S+} - v_S = h(S, \mathbb{E}(v_S \mid \mathcal{F}_{S-}), v_{S+}).$$

(13.7)
If we set \( y := v_+ \) on \([0, \tau]\), then \( y_- = v_- \) and thus

\[
\Delta y_\tau = y_\tau - y_\tau = y_{s_+} - y_{s_-} = v_{s_+} - v_{s_-} = \Delta^+ v_S + \Delta v_S
\]

which coincides with (13.5). In the next step, we take inspiration from the proof of Theorem 3.1 in Essaky et al. [14] and rewrite (13.4) into

\[
v_t = \xi_\tau - h(\tau, v_\tau, \xi_\tau)\Delta^*_\tau - \int_{[\tau, \tau]} f^*_s(v_{s_-}, v_s, z_s) dD^*_s - \int_{[\tau, \tau]} z_s dM_s - \sum_{1 \leq i < \tau} \Delta^+ v_i.
\]

(13.6) Furthermore, since \( \Delta y_i \) for every \( i \),

\[
\text{Note that if the recovery process } R = F\text{-predictable (so that one can use } A^p \text{ instead of } A^\circ \text{ and } D^f \text{ is chosen to be have } F\text{-predictable jumps (for instance, if } D^f = (\langle M \rangle, \mathcal{G}^{-1} \cdot A^p)), \text{ then the transformed BSDE (8.10) has the form (13.4) and } h \text{ vanishes outside the graph of a family of } F\text{-predictable stopping times. In that case, assuming that } \xi^t \text{ can be solved in (13.9), we would be able}
\]

\[
\text{to consider jumps of a size } h \text{ depending on } v_-.
\]

**Remark 13.4.** Note that if the recovery process \( R \) is \( F\)-predictable (so that one can use \( A^p \) instead of \( A^\circ \)) and \( D^f \) is chosen to be have \( F\)-predictable jumps (for instance, if \( D^f = (\langle M \rangle, \mathcal{G}^{-1} \cdot A^p) \)), then the transformed BSDE (8.10) has the form (13.4) and \( h \) vanishes outside the graph of a family of \( F\)-predictable stopping times. In that case, assuming that \( \xi^t \) can be solved in (13.9), we would be able to consider jumps of a size \( h \) depending on \( v_- \).

**Example 13.1.** Let us show that if appropriate conditions are imposed on the inputs \( (f^*, D^*, M) \), then a unique solution \((v^i, z^i)\) to (13.8) can be obtained on each interval \([S_i, S_{i+1}]\) for \( i = 0, 1, \ldots, p\) and hence a solution \((y, z)\) to (13.4) can be constructed as well. In the following, we assume that the process \( \langle M \rangle \) is continuous, the function \( h \) does not depend on \( v_- \) and

\[
f^*(v^t, v, z) \cdot D^* = f(v^t, v, z) \cdot \langle M \rangle + g(v) \cdot B
\]

where \( B \) is an \( F\)-adapted, bounded, continuous, increasing process and \( f \) and \( g \) are some real-valued mappings satisfying appropriate measurability conditions. We note that, as \( h \) does not depend on \( v_- \), the assumption that the jump times of \( D^* \) (and hence also \((S_i)_{i=1,\ldots,p}\)) are \( F\)-predictable stopping times can be relaxed. Furthermore, the right-hand jumps of the process \( v \) are given by \( \Delta^+_t v_t = h(t, v_t) \).

We thus need to analyze the following càdlàg BSDE with a continuous driver, on each stochastic interval \([S_i, S_{i+1}]\) for every \( i = 0, 1, \ldots, p\),

\[
dv^i_t = -f_t(v^i_{t-}, v^i_t, z^i_t) \, d\langle M \rangle_t - g(t, v^i_t) \, dB_t - z^i_t \, dM_t,
\]

\[
v^i_{S_{i+1}} = v^i_{S_{i+1}} - h(S_{i+1}, v^i_{S_{i+1}}),
\]

with the terminal condition \( v^i_{S_{p+1}} = \xi_\tau \).
Observe that in the case of a Brownian-Poisson filtration $\mathbb{F}$, the existence and uniqueness of a family of solutions $(v^i, z^i)$ can be deduced from Theorem 53.1 in Pardoux [33] under the postulate that $f, g$ and $h$ are bounded and Lipschitz continuous functions, the process $B$ is bounded, and $M = (W, \bar{N})$ where $W$ is a Brownian motion and $\bar{N}$ is an independent compensated Poisson process.

**Example 13.2.** Let the filtration $\mathbb{F}$ be the Brownian-Poisson filtration. We consider below an example given in Gapeev et al. [20] of a supermartingale $J$ valued in $(0, 1]$ which is the solution to the SDE

$$dJ_t = -\lambda J_t dt + \frac{b}{\sigma} J_t (1 - J_t) dW_t, \quad J_0 = 1.$$  

The process $J$ takes a multiplicative form $J_t = Q_t e^{-\lambda t}$ where $Q$ satisfy

$$Q_t = 1 + \int_0^t \frac{b}{\sigma} (1 - J_u) Q_u \, dW_u.$$  

For a fixed $p \in (0, 1)$, we consider the supermartingales

$$\tilde{G}_t = J_t \mathbb{I}_{\{t \leq T_1\}} + p J_t \mathbb{I}_{\{T_1 < t\}} = J_t - (1 - p) J_t \mathbb{I}_{\{T_1 < t\}},$$

$$G_t = J_t - (1 - p) J_t \mathbb{I}_{\{T_1 \leq t\}},$$

and we observe that, by an application of the Itô formula, we have

$$\tilde{G}_t = 1 + \int_0^t \frac{b}{\sigma} G_u (1 - J_u) \, dW_u - \int_0^t \lambda G_u \, du - (1 - p) J_t \mathbb{I}_{\{T_1 < t\}}.$$  

We know from Jeanblanc and Li [26] that it is possible to construct a random time $\tau$ such that the Azéma optional supermartingale and the Azéma supermartingale associated with $\tau$ are given by $\tilde{G}$ and $G$, respectively. In the present example, the equality $\tilde{G} = G$ holds and the martingale $m$, the dual $\mathbb{F}$-optional projection $A^\circ$ and the hazard process $\Gamma = (\tilde{G}^{-1} \cdot A^\circ)$ associated with $\tau$ are given by the following expressions

$$m_t = 1 + \int_0^t \frac{b}{\sigma} G_u (1 - J_u) \, dW_u,$$

$$A^\circ_t = \int_0^t \lambda G_u \, du + (1 - p) J_t \mathbb{I}_{\{T_1 \leq t\}},$$

$$\Gamma_t = \lambda t + (1 - p) \mathbb{I}_{\{T_1 \leq t\}},$$

so that $\Gamma_t^\circ = \lambda t$ and $\Gamma_t = (1 - p) \mathbb{I}_{\{T_1 \leq t\}}$.

The stopping time $T_1$ can be viewed as a shock to the underlying financial asset and $\tau$ is the timing of a default event. The parameter $p \in (0, 1)$ can be regarded as the conditional probability that the default event occurs at $T_1$ given that the default event has not occurred before $T_1$. In the following, we denote the compensated Poisson process by $\bar{N}$ and for the ease of presentation we set $D^r = \Gamma / (1 - p)$ and $D^d = \Gamma^d / (1 - p) = \mathbb{I}_{[T_1, \infty[}$. Furthermore, we suppose the generators $F^r$ and $F^d$ does not depend on $Y_\cdot$ and the coupled equation (8.6)-(8.7) reduces to a single BSDE given by

$$Y_t = X_T - \int_{[t, T]} \frac{F^r_s(Y_s)}{1 - p} \, d\Gamma_s - \int_{[t, T]} \frac{F^d_s(Y_s)}{1 - p} \, d\Gamma^d_s + \int_{[t, T]} \frac{b}{\sigma} (1 - J_s) \bar{N}_s \, ds - \int_{[t, T]} Z^i_s \, dW_s$$

$$- \int_{[t, T]} Z^2_s \, d\bar{N}_s + \int_{[t, T]} \left[ R_s - Y_s - (F^r_s(R_s) - F^r_s(Y_s)) \mathbb{I}_{[T_1, 1]}(s) \right] \, d\Gamma_s.$$  

On the set $\{T_1 \leq T\}$, we observe that the driver of the above BSDE has only one jump at time $T_1$ and thus on the stochastic interval $[T_1, T]$ we need only to find the solution $(y, u)$ where $u = (u^1, u^2)$ to the BSDE,

$$y_t = X_T - \int_t^T \left[ \frac{\lambda F^r_s(y_s)}{1 - p} - \frac{b(1 - J_s) u^1_s}{\sigma} - \frac{\lambda (R_s - y_s)}{1 - p} \right] ds$$

$$+ \int_t^T u^1_s \, dW_s + \int_t^T u^2_s \, d\bar{N}_s. \quad (13.10)$$
At the jump time $T_1$, the right jump of $Y$ is given by $\Delta^+ Y_{T_1} = F^\sigma_{T_1} (Y_{T_1})$ and the quantity $Y_{T_1}$ is obtained by solving the equation $y_{T_1} - F^\sigma_{T_1} (Y_{T_1}) = Y_{T_1}$.

Given that $Y_{T_1}$ can be obtained, we see that one is required to solve the càdlàg BSDE, on the stochastic interval $[0, T_1]$, 

$$Y_t = Y_{T_1} - \int_{t \in [t, T_1]} \frac{F^\sigma_s(Y_s)}{1 - p} \, d\Gamma_s + \int_{t \in [t, T_1]} \frac{b(1 - G_s) Z^1_s \, dW_s - \int_{t \in [t, T_1]} Z^2_s \, d\tilde{N}_s}{\sigma}$$

$$+ \int_{t \in [t, T_1]} [R_s - Y_s - (F^r_s(R_s) - F^r_s(Y_s)) \mathbb{1}_{[T_1]}(s)] \, d\Gamma_s,$$

which is a GBSDE where the martingale term and the driver can share a common jump at $T_1$. Again, we observe that the driver jumps only at $T_1$ with the jump size given by

$$h(T_1, Y_{T_1}) := F^\sigma_{T_1} (Y_{T_1}) - [R_{T_1} - Y_{T_1} - (F^r_{T_1} (R_{T_1}) - F^r_{T_1} (Y_{T_1})) (1 - p)].$$

Therefore, the adjusted terminal condition $v_{T_1}$ at $T_1$ equals

$$v_{T_1} := Y_{T_1} - h(T_1, Y_{T_1}) = R_{T_1} - F^r_{T_1} (R_{T_1}) - p [R_{T_1} - Y_{T_1} - (F^r_{T_1} (R_{T_1}) - F^r_{T_1} (Y_{T_1}))].$$

and we see that we need to solve the following BSDE with a continuous driver, on the stochastic interval $[0, T_1]$,

$$v_t = v_{T_1} - \int_{t \in [t, T_1]} \left[ \frac{\lambda F^r_s(v_s)}{1 - p} + \frac{b(1 - J_s) z^1_s}{\sigma} + \frac{\lambda (R_s - v_s)}{1 - p} \right] \, ds$$

$$- \int_{t \in [t, T_1]} z^1_s \, dW_s - \int_{t \in [t, T_1]} z^2_s \, d\tilde{N}_s. \quad (13.11)$$

To this end, let $Y_{T_1}$ be a solution to the equation $y_{T_1} - F^\sigma_{T_1} (Y_{T_1}) = Y_{T_1}$. Then a solution $(Y, Z)$ where $Z = (Z^1, Z^2)$ on the whole interval $[0, T]$ can be obtained by setting

$$Y := v^1_{[0, T_1]} + h(T_1, Y_{T_1}) \mathbb{1}_{[T_1]} + y \mathbb{1}_{T_1, T},$$

$$Z' := z^1_{[0, T_1]} + u \mathbb{1}_{T_1, T}.$$

Let us now consider the set $\{T_1 > T\}$. Since there are no jumps before $T$, it suffices to find $(v, z)$ in $(13.11)$ on the whole interval $[0, T]$ with the terminal condition $v_T = X_T$. 

To better visualise the jump-adapted method outlined above, we give a graphical illustration

[Graph showing the solution for $(v, z)$ and $(y, u)$ with the terminal condition $v_T = X_T$.]

Here we point out that since $(1 - J)$ is bounded by one, when considering $(13.10)$ and $(13.11)$, we do not need to study the transformed BSDE given in $(8.10)$-$(8.11)$. This is because, given appropriate assumptions on $F^r$, the linear growth conditions in $z$ can be easily verified here.
14 RBSDEs with a lâgâd driver and common jumps

Following the structure of Section 13, given a filtration $\mathbb{F}$, we focus on $\mathbb{F}$ RBSDEs of the form

$$v_t = \xi_t - \int_{[t, \tau]} f_s^l(v_{s-}, v_s, z_s) \, dD_s^l - \int_{[t, \tau]} f_s^g(v_s, v_{s+}) \, dD_s^g$$

$$- \int_{[t, \tau]} z_s \, dM_s + l_t^l - l_t^g - l_t^g$$  \hspace{1cm} (14.1)

where $l^l$ and $l^g$ satisfy $(\mathbb{1}_{\{v_{t-}\neq -\xi_t\} \cdot l^l})_\tau = (\mathbb{1}_{\{v_{t-}\neq -\xi_t\} \ast l^g})_\tau = 0$. We observe that in the case where $F^g$ in (11.1) does not depend on $U$ and $Z$ or that $U$ in (11.2) can be solved and does not depend on $Z$ (for an example, see Section 12), then both BSDE (11.1) and (11.4) can be obtained as a special case of the above BSDE (14.1). Similar to the non-reflected case, we present below a jump-adapted method to reduce the lâgâd RBSDE (14.1) to a system of càdlâg RBSDEs, which can be further reduced to a system of càdlâg RBSDEs with continuous drivers.

Step 1. From a lâgâd to càdlâg driver. By examining the right-hand jumps of $v$, that is, $\Delta^+ v$, and the Skorokhod condition satisfied by $l^g$, we observe that $\Delta^+ v$ and $\Delta l^g$ must satisfy the conditions

$$v_+ - v = f^g(v_+, v) \Delta D_+^g + \Delta l_+^g, \quad (v - \xi)\Delta l_+^g = 0,$$

which in turn implies that

$$\Delta l_+^g = (\xi - (v_+ - f^g(v, v_+) \Delta D_+^g))^+, \quad v = \xi \lor (v_+ - f^g(v, v_+) \Delta D_+^g).$$

We thus see that, at the jump times of $l^g$ and $D_+^g$, the quantity $v$ can be obtained by solving the second equation (of course, assuming that a solution exists) and $\Delta l_+^g$ can be obtained by substitution. In particular, if $f^g$ does not depend on $v$, then it is clear that we have

$$\Delta l_+^g = (\xi - (v_+ - f^g(v_+, v_+) \Delta D_+^g))^+, \quad v = \xi \lor (v_+ - f^g(v_+, v_+) \Delta D_+^g).$$

These arguments lead to the observation that a solution $(v, z, l)$ to (14.1) can be obtained by solving iteratively, for every $i = 0, 1, \ldots, p$, the following càdlâg RBSDE on $[S_i, S_{i+1}]

$$v_i^l = \xi^l - \int_{[t, S_{i+1}]} h_s^l(v_{s-}, v_s, z_s) \, dD_s^l - \int_{[t, S_{i+1}]} z_s^l \, dM_s + l_{i+1}^l - l_i^l$$  \hspace{1cm} (14.2)

where the càdlâg increasing process $l^i$ obeys the Skorokhod condition $(\mathbb{1}_{\{\xi = -\xi_t\} \ast l^i})_\tau = 0$ and $(\xi^i, \Delta l^g_{S_{i+1}})$ are $\mathcal{F}_{S_{i+1}}$-measurable random variables such that $\Delta l^g_{S_{i+1}} = 0$, $\xi^p = \xi_\tau$ and for $i = 0, 1, \ldots, p - 1,

$$\Delta l^g_{S_{i+1}} = \left(\xi^+_{S_{i+1}} - (v_{S_{i+1}}^+ - h_{S_{i+1}}^g(\xi^i, v_{S_{i+1}}^i) \Delta D^g_{S_{i+1}})\right)^+, \quad \xi^i = \xi_{S_{i+1}} \lor (v_{S_{i+1}}^i - h_{S_{i+1}}^g(\xi^i, v_{S_{i+1}}^i) \Delta D^g_{S_{i+1}}).$$

Then a global solution $(v, z, l)$ where $l = l^l + l^g$ is obtained by setting

$$v = v_0 + \sum_{i=0}^p v_i^l \mathbb{1}_{[S_i, S_{i+1}]}, \quad z = z_0 + \sum_{i=0}^p z_i^l \mathbb{1}_{[S_i, S_{i+1}]},$$

$$l^l = \sum_{i=0}^p (l_{S_i}^l - l_{i+1}^l) \mathbb{1}_{[S_i, S_{i+1}]}, \quad l^g = \sum_{i=1}^p \Delta l^g_{S_{i+1}} \mathbb{1}_{[S_i, S_{i+1}]};$$

where $l_0^l = 0$, $v_0 = v_0^0$ and $z_0 = z_0^0$.

Step 2. From a càdlâg to continuous driver. In view of the càdlâg RBSDE (14.2), we study the RBSDE of the form

$$y_t = \xi_t - \int_{[t, \tau]} f_s^r(y_{s-}, y_s, z_s) \, dD_s^r - \int_{[t, \tau]} z_s \, dM_s + l_t - l_t^l$$  \hspace{1cm} (14.3)
where a solution \( (y, z, l) \in \mathcal{O}(\mathbb{F}) \times \mathcal{P}_d(\mathbb{F}) \times \mathcal{P}(\mathbb{F}) \) is such that \( y \) is a càdlàg process and \( l \) is a càdlàg, increasing process such that \( (1_{\{y \neq \xi_\cdot\}} \cdot l)_\tau = 0 \) and \( l_0 = 0 \). In the following, we consider a more general RBSDE of the form
\[
y_t = \xi_t - \int_{[t, \tau]} f^r_s(y_s, y_s, z_s) dD^c_s - \sum_{t \leq s < \tau} h(s, y_s, y_s) - \int_{[t, \tau]} z_s dM_s + l_\tau - l_t \tag{14.4}
\]
where a solution \( (y, z, l) \in \mathcal{O}(\mathbb{F}) \times \mathcal{P}_d(\mathbb{F}) \times \mathcal{P}(\mathbb{F}) \) is such that \( y \) is a càdlàg and \( l \) is a càdlàg increasing process such that \( (1_{\{y \neq \xi_\cdot\}} \cdot l')_\tau = 0 \) and \( l_0 = 0 \).

To recover equation (14.3) from (14.4), it suffices to set \( h(s, y_s, y_s) := f^r_s(y_s, y_s, \xi) D^c_s \). In view of this, we further suppose that \( h = 0 \) outside the graph of a finite family of \( \mathbb{F} \)-predictable stopping times \( (T_i)_{i=1, 2, \ldots, p} \) and we denote \( S_0 = 0, S_i = T_i \wedge \tau \) and \( S_{p+1} = \tau \). To examine the existence of a solution to the RBSDE (14.4), we introduce an auxiliary RBSDE
\[
v_t = \xi_t - h(\tau, v_{\tau-}, \xi_\tau) - \int_{[t, \tau]} f^r_t(v_s, v_s, z_s) dD^c_s - \sum_{t \leq s < \tau} h(s, v_s, v_{s+}) - \int_{[t, \tau]} z_s dM_s + l_\tau - l_t \tag{14.5}
\]
where a solution \( (v, z, l) \) is such that \( v \) is a làglàd, \( \mathbb{F} \)-adapted process, the process \( z \) is \( \mathbb{F} \)-predictable and the process \( l \) obeys the Skorohod condition \( (1_{\{v \neq \xi_\cdot\}} \cdot l')_\tau = 0 \) and \( l_0 = 0 \).

**Proposition 14.1.** Let \( v \) be a làglàd process such that \( (v, z, l) \) is a solution to the RBSDE (14.5) on \([0, \tau]\). Then \( (y, z, l) \) where \( y := v + 1_{[0, \tau]} + h(\tau, v_{\tau-}, \xi_\tau) 1_{[0, \tau]} \) solves the càdlàg RBSDE (14.4) on \([0, \tau]\).

**Proof.** Suppose that \( (v, z, l) \) is a solution to (14.4). It is clear from (14.4) that the left-hand and right-hand jumps of \( v \) are given by \( \Delta v = z \Delta M + \Delta l \) and \( \Delta^+ v = h(\cdot, v_{\cdot-}, v_{\cdot+}) \), respectively. Note that \( l \) must satisfy the reflection condition \( (v_{\cdot-} - \xi_{\cdot-}) \Delta l_S = 0 \) and, by the optional sampling theorem, we have that \( \mathbb{E}[v_S | \mathcal{F}_{S-}] = v_{S-} + \Delta l_S \) for any \( \mathbb{F} \)-stopping time \( S \). Then, by solving these two equations, we obtain
\[
v_{S-} = \xi_{S-} \vee \mathbb{E}[v_S | \mathcal{F}_{S-}], \quad \Delta l_S = (\xi_{S-} - \mathbb{E}[v_S | \mathcal{F}_{S-}])^+.
\]
Therefore, if the random variable \( v_{S-} \) is known, then the \( \mathcal{F}_S \)-measurable random variable \( v_S \) is a solution to the equation
\[
\Delta^+ v_S := v_{S+} - v_S = h(S, \xi_{S-} \vee \mathbb{E}[v_S | \mathcal{F}_{S-}], v_{S+}).
\]
Recall that, by assumption about \( h, \) the right-hand jump times of \( v \) are given by the family \( (T_i)_{i=1, 2, \ldots, p} \) of \( \mathbb{F} \)-predictable stopping times and we set \( S_0 = S_i = T_i \wedge \tau \) and \( S_{p+1} = \tau \). We observe that a solution \( (v, z, l) \) can be constructed by first solving by iteration, for every \( i = 0, 1, \ldots, p \), the following càdlàg RBSDE on the stochastic interval \([S_i, S_{i+1}]\)
\[
v^i_t = \xi^i_t - \int_{[t, S_{i+1}]} f^r_s(v^i_s, v^i_s, z^i_s) dD^c_s - \int_{[t, S_{i+1}]} z^i_s dM_s + l^i_{S_{i+1}} - l^i_t
\]
where, for every \( i = 0, 1, \ldots, p \), we have \( 1_{\{v \neq \xi_\cdot\}} \cdot l^i_{S_{i+1}} = 0 \) and we denote by \( \xi^i \) an \( \mathcal{F}_{S_{i+1}} \)-measurable random variable satisfying \( v^p_{S_{p+1}} = \xi^p \) and
\[
v^{i+1}_{S_{i+1}} - \xi^i = h(S_{i+1}, \xi_{S_{i+1}-} \vee \mathbb{E}[\xi^i | \mathcal{F}_{S_{i+1}-}], v^{i+1}_{S_{i+1}}).
\]
We aggregate the family of solutions \( (v^i, z^i) \) for \( i = 0, 1, \ldots, p \) by setting \( l_{-1}^0 = 0 \) and
\[
v = v^0_0 + \sum_{i=0}^p v^i 1_{[S_i, S_{i+1}]}, \quad z = z^0_0 + \sum_{i=0}^p z^i 1_{[S_i, S_{i+1}]}, \quad l = \sum_{i=0}^p (l_{S_i}^0 + l^i_{S_{i+1}}) 1_{[S_i, S_{i+1}]}.
\]
Using the Itô formula, one can check that \((v, z, l)\) satisfies the lâglâg RBSDE (14.4) and \(l\) is increasing and satisfies \((1_{\{v_\neq \xi_\cdot \}} \cdot l)_\tau = 0\). Furthermore, since \(\Delta y_S = v_{S_+} - v_{S_-}\) and the dynamics of \(y\) and \(v\) (see (14.4) and (14.5), respectively) are easily seen to coincide on \([S_i, S_{i+1}]\), we conclude that \((y, z, l) := (v_+, z, l)\) is a solution to the càdlàg RBSDE (14.4) once we made the appropriate adjustment to the last jump of size \(h\) at the terminal time \(\tau\), since \(v_\tau = y_\tau\) and \((1_{\{v_\neq \xi_\cdot \}} \cdot l)_\tau = 0\).

**Example 14.1.** Here we show that if appropriate conditions are imposed on the inputs data \((f, D, M)\), then a unique solution \((v, z, l)\) to (13.4) can be obtained on each stochastic interval \([S_i, S_{i+1}]\) for \(i = 0, 1, \ldots, p\) and thus a solution \((y, z, l)\) to (13.4) can be constructed. In the following, we assume that the process \(\langle M \rangle\) is continuous, the function \(h\) does not depend on \(v_\cdot\) and

\[
f^*(v_\cdot, v, z) \cdot D = f(v_\cdot, v, z) \cdot \langle M \rangle + g(v) \cdot C
\]

where \(C\) is an \(\mathcal{F}\)-adapted, continuous, increasing process. Furthermore, \(f\) and \(g\) are some real-valued mappings that satisfies appropriate measurability conditions.

We note that since \(h\) does not depend on \(v_\cdot\), the assumption that the jumps of \(D\) occur at \(\mathcal{F}\)-predictable stopping times can be relaxed and the right-hand jumps of \(v\) are given by \(\Delta^+ v_i = h(t, v_{i+})\). Hence one is required to solve the following càdlàg RBSDE with continuous drivers, on each stochastic interval \([S_i, S_{i+1}]\) for \(i = 0, 1, \ldots, p\),

\[
dv_i^l = -f_i(v_i^l, z_i^l) \, dt - g_i(v_i^l) \, dD_i - z_i^l \, dM_i + dl_i,
\]

\[
v_{S_{i+}+1}^s = v_{S_{i+1}}^s + h(S_{i+1}, v_{S_{i+1}}^s),
\]

where \(v_{S_{i+}+1}^s = \xi\) and \(l_i \in \mathcal{P}(\mathcal{F})\) is a càdlàg, increasing process with \(l_0^s = 0\) and such that the following equality holds

\[
1_{\{v_\neq \xi_\cdot \cap [S_i, S_{i+1}]\}} \cdot l_i = 0.
\]

In the case where \(\mathcal{F}\) is a Poisson filtration or, more generally, is generated by the Teugels martingales (see Nualart and Schoutens [32] or Schoutens and Teugels [37]), the existence and uniqueness of a solution \((v_i^l, z_i^l, l_i)\) can be obtained by an application of Theorem 5 in Ren and El Otmani [36] under the postulate that \(f, g\) and \(h\) are bounded and Lipschitz continuous functions, the process \(D\) is bounded and \(M\) is the compensated Poisson process.

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15 Appendix

We assume that the process $R$ is $F$-optional and we define the làglàg process $Q$ by

$$Q := K - R \cdot A^o + C = K - R \cdot A^o + C^r + C^g$$

where $K$ is an $F$-local martingale and $C$ is a làglàg process of finite variation. If $Y$ is a làglàg process of finite variation or, more generally, an optional semimartingale (which, by definition, is assumed to be a làglàg process), then $Y$ admits the decomposition $Y = Y^r + Y^g$ where $Y^r := \sum_{s \leq t} (Y_s - Y_s^-)$ and the càdlàg process $Y^r$ is given by $Y^r := Y - Y^g$.

**Lemma 15.1.** Assume that $G > 0$. Then the process $G^{-1}$ satisfies

$$G^{-1} = G_0 - G^{-2} \cdot \tilde{m} + G^{-1} \cdot \Gamma$$

(15.1)

where $\Gamma := \tilde{G}^{-1} \cdot A^o$ and $\tilde{m} := m - \tilde{G}^{-1} \cdot [m, m]$. Moreover, for the càdlàg process

$$Q^r := K - R \cdot A^o + C^r$$

we have that

$$[Q^r, G^{-1}] = -G^{-1} G^{-1}_- ([K, m] - [K, A^o] + [C^r, G]) - R \Delta G^{-1} \cdot A^o.$$  (15.2)
Proof. For brevity, we write \([G] := [G, G]\) and \([m] = [m, m]\). The Itô formula yields
\[
G^{-1} = G_0^{-1} - G_0^{-2} \cdot G + G^{-1}G_0^{-2} \cdot [G] = G_0^{-1} - G_0^{-2} \cdot J
\]
where \(J := G - G^{-1} \cdot [G]\). Since \(G = m - A^0\) and thus \(\Delta G = \Delta m - \Delta A^0\), we obtain
\[
[G] = [m] - [m, A^0] + [A^0, A^0] = [m] - \Delta m \cdot A^0 - (\Delta m - \Delta A^0) \cdot A^0
= [m] - \Delta m \cdot A^0 - \Delta G \cdot A^0
\]
so that
\[
J = G - G^{-1} \cdot [G] = m - A^0 - G^{-1} \cdot [m] + G^{-1} (\Delta m \cdot A^0 + \Delta G \cdot A^0).
\]
Using the equalities \(\bar{m} = m - G\) and \(\Delta m = \bar{G} - G\), we get
\[
J = \bar{m} + \bar{G}^{-1} \cdot [m] - A^0 - G^{-1} \cdot [m] - G^{-1} ((\bar{G} - G) \cdot A^0 - \Delta G \cdot A^0).
\]
Since \(\bar{G} - G = \Delta A^0\), we also have that
\[
(\bar{G}^{-1} - G^{-1}) \cdot [m] = G^{-1} \bar{G}^{-1} (G - \bar{G}) \cdot [m] = -G^{-1} \bar{G}^{-1} \Delta A^0 \cdot [m]
= -G^{-1} \bar{G}^{-1} (\Delta m)^2 \cdot A^0 = -G^{-1} \bar{G}^{-1} (\bar{G} - G)^2 \cdot A^0.
\]
Consequently,
\[
J = \bar{m} - A^0 + G^{-1} (\bar{G} - G + \Delta G - \bar{G} \cdot (\bar{G} - G)^2) \cdot A^0 = \bar{m} - G^{-1} G^2 \cdot \Gamma,
\]
which, when combined with (15.3), shows that (15.1) is valid. To establish (15.2), we first compute
\[
[Q^r, G^{-1}] = -G^{-2} \cdot [Q^r, G] + G^{-1}G^{-2} \cdot [Q^r, [G]] = -G^{-2} \cdot [Q^r, G] + G^{-1} \Delta G \cdot [Q^r, G] = -G^{-1} G^{-1} \cdot [Q^r, G].
\]
Finally, using the equalities \(\Delta G = -GG_\Delta G^{-1}\) and \(G = m - A^0\), we obtain
\[
[Q^r, G^{-1}] = -G^{-1} G^{-1} \cdot [Q^r, G] = G^{-1} G_{\Delta G^{-1}} \cdot [A^0, G] - G^{-1} G^{-1} \cdot ([K, G] + [C^r, G])
= -G^{-1} G_{\Delta G^{-1}} \cdot ([K, G] + [C^r, G]) - R \Delta G^{-1} \cdot A^0
= -G^{-1} G_{\Delta G^{-1}} \cdot ([K, m] - [K, A^0] + [C^r, G]) - R \Delta G^{-1} \cdot A^0
\]
and thus equality (15.2) is proven as well. \(\square\)

We maintain the assumption that \(G > 0\) (and thus also \(\bar{G} > 0\)) and we consider the process
\[
Y := G^{-1}Q = G^{-1}(K - R \cdot A^0 + C).
\]
Our goal is to derive the dynamics of \(Y\) in terms of \(\Gamma, \bar{m}\) and
\[
\bar{K} := K - \bar{G}^{-1} \cdot [K, m].
\]
In the proof of Lemma 15.3, we will employ the optional integration by parts formula. Recall that, by definition, any semimartingale is a càdlàg process but an optional semimartingale is not necessarily a càdlàg process although, by definition, it is a lângla process.

Let \(X = X^r + X^g\) and \(Y = Y^r + Y^g\) be lângl̓d optional semimartingales such as \(Y\) is of finite variation. Then the optional integration by parts formula reads (see Theorem 8.2 in Gal’čuk [18])
\[
XY = X_0 Y_0 + X \circ Y + Y \circ X + [X, Y]
\]
where the \textit{optional stochastic integrals} are given by

\[
X \circ Y = X_- \bullet Y^r + X \star Y^g,
\]

\[
Y \circ X = Y_- \bullet X^r + Y \star X^g,
\]

where \(X^g\) (respectively, \(Y^g\)) is the càdlàg version of the càglàd process \(X^g\) (respectively, \(Y^g\)) and the quadratic covariation \([X, Y]\) equals

\[
[X, Y]_t = \sum_{0 < s \leq t} \Delta X_s \Delta Y_s + \sum_{0 \leq s < t} \Delta^+ X_s \Delta^+ Y_s
\]

where we denote \(\Delta X_t = X_t - X_{t-}\) and \(\Delta^+ X_t = X_{t+} - X_t\).

For the reader’s convenience, we formulate a variant of the optional integration by parts formula (15.5), which holds when \(X = X^r\) is a (càdlàg) semimartingale and \(Y = Y^g\) is a càdlàg process of finite variation.

\textbf{Lemma 15.2.} Let \(X = X^r\) be a semimartingale and let \(Y = Y^g\) be a càdlàg process of finite variation. Then the process \(XY\) is làdlàg and satisfies, for every \(0 \leq s < t\),

\[
X_t Y_t = X_s Y_s + \int_{\llbracket s, t \rrbracket} Y_u dX_u + \int_{\llbracket s, t \rrbracket} X_u dY_u^g
\]  \hspace{1cm} (15.6)

where \(Y^g\) is the càdlàg version of \(Y^g\).

We will use the shorthand notation for (15.6)

\[
XY = X_0 Y_0 + Y \bullet X + X \star Y^g
\]

but all equalities in the proof of Lemma 15.3 should be understood in the sense of (15.6), meaning that all integrals with respect to a càdlàg (respectively, càglàd) process should be evaluated on the interval \([s, t]\) (respectively, on the interval \([s, t]\)) for arbitrary \(0 \leq s < t\).

\textbf{Lemma 15.3.} If the process \(Y\) is given by (15.4) where \(C\) is a làdlàg process of finite variation with the decomposition \(C = C^r + C^g\), then

\[
Y_t = Y_0 - \int_{[0,t]} (R_s - Y_s) d\Gamma_s - \int_{[0,t]} Y_s G^{-1}_s d\tilde{m}_s + \int_{[0,t]} G^{-1}_s d\tilde{K}_s + \int_{[0,t]} \tilde{G}^{-1}_s dC^r + \int_{[0,t]} G^{-1}_s dC^g
\]  \hspace{1cm} (15.7)

\textbf{Proof.} We note that \(Q\) satisfies

\[
Q = K - R \bullet A^o + C = K - R \bullet A^o + C^r + C^g = Q^r + C^g
\]

where

\[
Q^r = K - R \bullet A^o + C^r = \tilde{K} + \tilde{G}^{-1} \cdot [K, m] - R \bullet A^o + C^r.
\]

The integration by parts formulas applied to \(Y = G^{-1}Q = G^{-1}Q^r + G^{-1}C^g\) gives

\[
Y = G^{-1}Q^r + G^{-1}C^g
\]

\[
= Y_0 + Q_- \bullet G^{-1} + G^{-1} \bullet Q^r + [Q^r, G^{-1}] + G^{-1} \star C^g
\]  \hspace{1cm} (15.8)

since \(G^{-1}\) and \(Q^r\) are (càdlàg) semimartingales and thus the Itô integration by parts formula applied to the product \(G^{-1}Q^r\) yields

\[
G^{-1}Q^r = Q^r_\bullet G^{-1} + G^{-1} \bullet Q^r + [Q^r, G^{-1}]
\]
whereas the optional integration by parts formula (see Lemma 15.2) gives

\[ G^{-1}C^g = C^g \cdot G^{-1} + G^{-1} \star C^g_+ \]

From (15.2) and (15.8), we obtain

\[
Y = Y_0 - Q - (G^{-2} \cdot \tilde{m} - G^{-1} \cdot \Gamma) + \Delta A \cdot \left( K + \Delta G^{-1} \cdot [K, m] - R \cdot A^o + C^r \right) \\
- G^{-1}G^{-1} \left[ [K, m] - [K, A^o] + [C^r, G] \right] - R \Delta G^{-1} \cdot A^o + G^{-1} \star C^g_+ \\
= Y_0 - Y_- \Delta G^{-1} \cdot \tilde{m} + G^{-1} \cdot \tilde{K} + K + H
\]

where

\[
K := G^{-1} \cdot C^r + G^{-1} \star C^g_+ - G^{-1} G^{-1} \cdot [C^r, G] \\
= G^{-1} \cdot C^r + G^{-1} \star C^g_+ - G^{-1} G^{-1} \Delta G \cdot \Delta C^r = G^{-1} \cdot C^r + G^{-1} \star C^g_+
\]

and

\[
H := G^{-1} G^{-1} \Delta K \cdot A^o + G^{-1} (\tilde{G}^{-1} - G^{-1}) \cdot [K, m] \\
- R G^{-1} \cdot A^o + Y_- G^{-1} G_- G^{-1} \cdot A^o = \sum_{i=1}^4 H^i.
\]

We first recall that \( \Delta A^o = \tilde{G} - G \) and \( \Delta m = \tilde{G} - G_- \). Therefore,

\[
H^1 + H^2 = G^{-1} G^{-1} \Delta K \cdot A^o + G^{-1} (\tilde{G}^{-1} - G^{-1}) \cdot [K, m] \\
= G^{-1} G^{-1} (\Delta K \cdot A^o - \tilde{G}^{-1} \Delta A^o \cdot [K, m]) = G^{-1} G^{-1} (\Delta K \cdot A^o - \tilde{G}^{-1} \Delta K \Delta m \cdot A^o) \\
= G^{-1} G^{-1} (\Delta K - \tilde{G}^{-1} (G - G_-) \Delta K) \cdot A^o = G^{-1} G^{-1} \Delta K \cdot A^o.
\]

Next, we deduce from (15.4) that

\[
\Delta K = \Delta (YG) + R \Delta A^o - \Delta C^r = \Delta (YG) + R(\tilde{G} - G) - \Delta C^r
\]

and thus

\[
H = G^{-1} \tilde{G}^{-1} (\Delta (YG) + R(\tilde{G} - G) - \Delta C^r) \cdot A^o - R G^{-1} \cdot A^o + Y_- G^{-1} G_- G^{-1} \cdot A^o \\
= G^{-1} \tilde{G}^{-1} (YG + R(\tilde{G} - G) - \Delta C^r) \cdot A^o - R G^{-1} \cdot A^o \\
= \tilde{G}^{-1} Y \cdot A^o - G^{-1} \tilde{G}^{-1} \Delta C^r \cdot A^o - R \tilde{G}^{-1} \cdot A^o \\
= \tilde{G}^{-1} Y \cdot A^o - G^{-1} \tilde{G}^{-1} \Delta A^o \cdot C^r - R \tilde{G}^{-1} \cdot A^o \\
= (Y - R) \cdot \Gamma + (\tilde{G}^{-1} - G^{-1}) \cdot C^r.
\]

To complete the derivation of (15.7), it suffices to substitute \( K \) and \( H \) into (15.9). \( \square \)