Particular boundary condition ensures that a fermion in $d = 1 + 5$, compactified on a finite disk, manifests in $d = 1 + 3$ as massless spinor with a charge $1/2$, mass protected and chirally coupled to the gauge field.

N.S. Mankoč Borštnik

Department of Physics, FMF, University of Ljubljana, Jadranska 19, Ljubljana, 1000

H. B. Nielsen

Department of Physics, Niels Bohr Institute, Blegdamsvej 17, Copenhagen, DK-2100

Abstract

The genuine Kaluza-Klein-like theories—with no fields in addition to gravity—have difficulties with the existence of massless spinors after the compactification of some space dimensions [1]. We proposed in ref. [2] a boundary condition for spinors in $(1 + 5)$ compactified on a flat disk that ensures masslessness of spinors (with all positive half integer charges) in $d = (1 + 3)$ as well as their chiral coupling to the corresponding background gauge gravitational field. In this paper we study the same toy model, proposing a boundary condition allowing a massless spinor of one handedness and only one charge $(1/2)$ and infinitely many massive spinors of the same charge, allowing disc to be curved. We define the operator of momentum to be Hermitean on the vector space of spinor states—the solutions on a disc with the boundary.
Keywords: Unifying theories, Kaluza-Klein-like theories, mass protection mechanism, generalized Hermitean operators for momentum, higher dimensional spaces

I. INTRODUCTION

The major problem of the compactification procedure in all Kaluza-Klein-like theories with only gravity and no additional gauge fields is how to ensure that massless spinors be mass protected after the compactification. Namely, even if we start with only one Weyl spinor in some even dimensional space of \( d = 2 \) modulo 4 dimensions (i.e. in \( d = 2(2n + 1) \), \( n = 0, 1, 2, \cdots \)) so that there appear no Majorana mass if no conserved charges exist and families are allowed, as we have proven in ref. [3], and accordingly with the mass protection from the very beginning, a compactification of \( m \) dimensions gives rise to a spinor of one handedness in \( d \) with both handedness in \( (d - m) \) and is accordingly not mass protected any longer.

And besides, since the spin (or the total conserved angular momentum) in the compactified part of space will in space of \( (d - m) \) dimensions manifest accordingly a charge of both signs while in the second quantization procedure antiparticles of opposite charges appear anyhow, doubling the number of massless spinors when coming from \( d(= 2(2n + 1)) \)-dimensional space down to \( d = 4 \) (after the second quantized procedure) is not in agreement with what we observe. Therefore there must be some requirements, some boundary conditions [2], which ensure in a compactification procedure that only spinors of one handedness and one charge survive if Kaluza-Klein-like theories have some meaning, what due to the beautifulness of the idea of the gravity as the only gauge field we would hope for.

One of us [4, 5, 6, 7, 8, 9] has for long tried to unify the spin and all the charges to only the spin, so that spinors would in \( d \geq 4 \) carry nothing but (two kinds of [13]) a spin and interact accordingly with only the gauge fields of the corresponding generators of the infinitesimal transformations (of translations and two kinds of the Lorentz transformations in the space of spinors), that is with vielbeins \( f^\alpha_a \) [14] and (two kinds of) spin connections \( (\omega_{aba}, \omega_{\bar{a}ba}) \), which are the gauge fields of \( S^{ab} = \frac{i}{4}(\gamma^a\gamma^b - \gamma^b\gamma^a) \) and \( \tilde{\omega}_{aba} \), which are the gauge fields of \( \tilde{S}^{ab} = \frac{i}{4}(\tilde{\gamma}^a\tilde{\gamma}^b - \tilde{\gamma}^b\tilde{\gamma}^a) \), with \( \{\gamma^a, \nabla^b\}_+ = 0 \).

In this paper we take (as we did in ref. [2]) for the covariant momentum of a spinor

\[
p_{0a} = f^\alpha_a p_{0\alpha}, \quad p_{0\alpha} \psi = p_{\alpha} - \frac{1}{2} S^{cd} \omega_{cda}.
\]
The corresponding Lagrange density $\mathcal{L}$ for a Weyl spinor has the form

$$
\mathcal{L} = E\frac{1}{2}[(\psi^+ \gamma^0 \gamma^a p_{0a} \psi) + (\psi^+ \gamma^0 \gamma^a p_{0a} \psi)^+] 
$$

and leads to

$$
\mathcal{L} = \psi^+ \gamma^0 \gamma^a \{E(p_a - \frac{1}{2} S^{cd} \omega_{cd}) + \frac{1}{2}(p_{a}, E^a)\} \psi, 
$$

(2)

with $E = \text{det}(e^a_{\alpha})$. If we have no gravity in $d = (1 + 3)$, and for $\omega_{abc} = 0$, while $f^a_s = \delta^a_s f(\rho)$, the equations of motion follow

$$
\{E \gamma^0 \gamma^m p_m + E f \gamma^0 \gamma^s(p_s + \frac{1}{2}E f \{p_s, E f\})\} \psi = 0.
$$

(3)

(We use $s, t$, or $\sigma, \tau$, to denote the two compactified dimensions ($x^5$ and $x^6$ for the flat and the Einstein index, respectively), $m, n$, to denote the flat experimentally observed $1 + 3 = d - m$ dimensions, and $a, b$, and $\alpha, \beta$, to denote all the flat and Einstein indices, respectively.) The authors of this work found a way out of the “Witten’s no go theorem” for a toy model of $M^{(1+3)} \times$ a flat finite disk in $(1 + 5)$-dimensional space [2] by postulating a particular boundary condition, which allows a spinor to carry only one handedness after the compactification—but many charges (all positive integers). Massless spinors then chirally couple to the corresponding background gauge gravitational field, which solves equations of motion for a free field, linear in the Riemann curvature, while the current through the boundary for the massless and all the massive solutions is equal to zero. In ref. [2] the boundary condition was written in a covariant way as

$$
\hat{\mathcal{R}} \psi \big|_{\text{wall}} = 0, \quad \hat{\mathcal{R}} = \frac{1}{2} (1 - i n^{(\rho)}_a n^{(\phi)} \gamma^a \gamma^b), \quad \hat{\mathcal{R}}^2 = \hat{\mathcal{R}}
$$

(4)

with $n^{(\rho)} = (0, 0, 0, 0, \cos \phi, \sin \phi)$, $n^{(\phi)} = (0, 0, 0, 0, -\sin \phi, \cos \phi)$, which are the two unit vectors perpendicular and tangential to the boundary of the disk at $\rho_0$, respectively. The operator $\hat{\mathcal{R}}$ is a projector. It can for the above choice of the two vectors $n^{(\rho)}$ and $n^{(\phi)}$ be written as

$$
\hat{\mathcal{R}} = [-] = \frac{1}{2} (1 - i \gamma^5 \gamma^6).
$$

(5)

In Appendix A properties of the Clifford algebra objects $^{ab}_{\pm}$, $^{ab}_{\mp}$ are presented. The boundary condition requires that only massless states (fulfilling equations of motion (Eq.(3))) of one (let us say right) handedness with respect to the compactified disk are allowed. Accordingly massless states of only one handedness are allowed in $d = (1 + 3)$ for each charge. Since the realistic theory, leading to massless states of only one charge is desirable, we search for
boundary conditions, which would lead to one massless state of one handedness and only one charge.

In this paper:
i. We formulate the boundary condition allowing one massless state of one handedness and only one charge and infinitely many massive states with the same charge so that to each mass only one state corresponds.

ii. We define the operator for momentum $p^a$ so that it becomes Hermitean on the vector space of states fulfilling the boundary conditions and we comment on the orthogonality relations of these states.

iii. We study the properties of states on a curved disc with the boundary.

II. BOUNDARY CONDITIONS

The boundary condition, presented in ref. [3], allows massless spinors of only one handedness but of all positive half integer charges. To resemble with our toy model a ”realistic theory” at $d=(1+3)$ the massless spinor must manifest in $d=(1+3)$ only one charge. The boundary condition on a disc which does lead to massless spinors of only one handedness and only one charge and which leads to infinite many massive spinors of the same charge is as follows

$$\hat{R}' \psi|_{\text{wall}} = 0,$$

$$\hat{R}' = \frac{1}{2} \left[ [-56] + e^{i\theta} n^{(\phi)} a n^{(\phi)} b \gamma^b n^{(\phi)} c n^{(\phi)} d p_d + \text{h.c.} \right] (6)$$

with $[-]= (1 - i n^{(\phi)} a n^{(\phi)} b \gamma^a \gamma^b)$, while $\theta$ is a parameter. If taking $n^{(\phi)} a = (0, 0, 0, 0, \cos \phi, \sin \phi)$ and $n^{(\phi)} a = (0, 0, 0, 0, -\sin \phi, \cos \phi)$, $\hat{R}'$ in Eq.(6) simplifies to

$$\hat{R}' = [-56] + \cos \theta \left[ +56 \right] \left( -\frac{i}{\rho} \frac{\partial}{\partial \rho} \frac{\partial}{\partial \phi} \right).$$

The two conditions, the one of Eq.(7) and that of Eq.(5), coincide for $\theta = (2k+1) \pi/2$, where $k$ is an integer. However, while $\hat{R}$ is a projector, $\hat{R}'$ is not. Both, $\hat{R}$ and $\hat{R}'$, are Hermitean. Accordingly $\hat{O} = I - 2\hat{R}$, with $\hat{O}^2 = I$ and $\hat{O}^\dagger = \hat{O}$, is unitary, while $\hat{O}' = I - 2\hat{R}'$ is not unitary. We shall see that when requiring that solutions of equations of motion obey the condition $\hat{R}'^k \psi|_{\rho=\rho_0} = 0$, for $k = 1$, the same states are projected out for any $k$. In this sense also $\hat{O}'$ manifests unitarity on the states of solutions (obeying boundary conditions on the boundary, where both operators only apply).
III. EQUATIONS OF MOTION AND SOLUTIONS

We study two cases:
i. We assume that the two dimensional space of coordinates \( x^5 \) and \( x^6 \) is a Euclidean plane \( M^{(2)} \) (with no gravity) so that \( f^\sigma_\sigma = \delta^\sigma_\sigma, \omega_{56} = 0 \), with the rotational symmetry around the origin,

ii. We assume that space of \( x^5 \) and \( x^6 \) is curved with the zweibein \( f^\sigma_\sigma = \delta^\sigma_\sigma f(\rho) \), with \( \rho \), defined by \( x^5 = \rho \cos \phi, \ x^6 = \rho \sin \phi \), so that the rotational symmetry around the axis perpendicular to the plane of \( x^5 \) and \( x^6 \) is preserved.

If the Euclidean plane is curved on \( S^2 \) with the radius \( \rho_0 \) and the rotational symmetry around the axis perpendicular to the plane of \( x^5 \) and \( x^6 \), then \( f(\rho) = 1 + (\frac{\rho}{\rho_0})^2 \). From \( ds^2 = e^{\sigma_\tau}dx^\sigma dx^\tau = f^{-2}(d\rho^2 + \rho^2 d\phi^2) \) we find \( E = f^{-2}(\rho) \).

Wave functions describing spinors in \((1 + 5)\)-dimensional space demonstrating \( M^{(1+3)} \times \) a disk symmetry are required to obey Eq.(3).

The most general solution for a free particle in \( d = (1 + 5) \) should be written as a superposition of all four \((2^6/2-1)\) states of a single Weyl representation. The reader can see in Appendix [A] technical details about how to write a Weyl representation in terms of the Clifford algebra objects after making a choice of the Cartan subalgebra, for which we take: \( S^{03} \), \( S^{12} \), \( S^{56} \). In our technique [10] one spinor representation—the four states which are the eigenstates of the chosen Cartan subalgebra—are the following four products of projections \( a_b \) and nilpotents \( k \):

\[
\begin{align*}
\varphi_1^1 &= 56^{03}1^2 (+) (+i)(+) \psi_0, \\
\varphi_2^1 &= 56^{03}1^2 (+) [-i][-] \psi_0, \\
\varphi_1^2 &= 56^{03}1^2 [-][-i](+) \psi_0, \\
\varphi_2^2 &= 56^{03}1^2 [-][-i][-] \psi_0,
\end{align*}
\]

where \( \psi_0 \) is a vacuum state. If we write the operators of handedness in \( d = (1 + 5) \) as \( \Gamma^{(1+5)} = \gamma_0 \gamma_1 \gamma_2 \gamma_3 \gamma_5 \gamma_6 \) \((= 2^3 i S^{03} S^{12} S^{56})\), in \( d = (1+3) \) as \( \Gamma^{(1+3)} = -i \gamma_0 \gamma_1 \gamma_2 \gamma_3 \) \((= 2^2 i S^{03} S^{12})\) and in the two dimensional space as \( \Gamma^{(2)} = i \gamma_5 \gamma_6 \) \((= 2 S^{56})\), we find that all four states are left handed with respect to \( \Gamma^{(1+5)} \), with the eigenvalue \(-1\), the first two states are right handed and the second two states are left handed with respect to \( \Gamma^{(2)} \), with the eigenvalues \(1\) and \(-1\), respectively, while the first two are left handed and the second two right handed with
respect to $\Gamma^{(1+3)}$ with the eigenvalues $-1$ and $1$, respectively. Taking into account Eq.\((8)\) we may write the most general wave function $\psi^{(6)}$ obeying Eq.\((3)\) in $d = (1 + 5)$ as

$$\psi^{(6)} = A (\psi^{(4)}_{(-)} + B [-] \psi^{(4)}_{(-)},)$$

where $A$ and $B$ depend on $x^5$ and $x^6$, while $\psi^{(4)}_{(+)}$ and $\psi^{(4)}_{(-)}$ determine the spin and the coordinate dependent parts of the wave function $\psi^{(6)}$ in $d = (1 + 3)$

$$\psi^{(4)}_{(+)} = \alpha_+ (\frac{6}{3})^{12} + \beta_+ [\frac{6}{3}]^{12},$$

$$\psi^{(4)}_{(-)} = \alpha_- [-]^{12} + \beta_- [-]^{12}.$$  

Using $\psi^{(6)}$ in Eq.\((3)\) we recognize the following expressions as the mass terms:

$$\frac{\alpha_+}{\alpha_-} (p^0 - p^3) - \frac{\beta_+}{\alpha_-} (p^4 + p^3) = m, \frac{\alpha_+}{\beta_-} (p^0 - p^3) + \frac{\beta_+}{\alpha_-} (p^4 - p^3) = m,$$

$$\frac{\beta_+}{\alpha_-} (p^0 - p^3) + \frac{\alpha_+}{\beta_-} (p^4 - p^3) = m.$$  

(One can notice that for massless solutions ($m = 0$) the $\psi^{(4)}_{(+)}$ and $\psi^{(4)}_{(-)}$ decouple.) We end up with the equations of motion for $A$ and $B$ as follow

$$-2 i f (\frac{\partial}{\partial z} + \frac{\partial \ln \sqrt{E f}}{\partial z}) B + m A = 0,$$

$$-2 i f (\frac{\partial}{\partial z} + \frac{\partial \ln \sqrt{E f}}{\partial z}) A + m B = 0,$$

where $z := x^5 + i x^6 = \rho e^{i \phi}$, $\bar{z} := x^5 - i x^6 = \rho e^{-i \phi}$ and $\frac{\partial}{\partial z} = \frac{1}{2} (\frac{\partial}{\partial \rho} + \frac{\partial}{\partial \psi}) = e^{-i \phi} (\frac{\partial}{\partial \rho} - \frac{i}{\rho} \frac{\partial}{\partial \phi})$, $\frac{\partial}{\partial \bar{z}} = \frac{1}{2} (\frac{\partial}{\partial \rho} - \frac{i}{\rho} \frac{\partial}{\partial \phi}) = e^{i \phi} (\frac{\partial}{\partial \rho} + \frac{i}{\rho} \frac{\partial}{\partial \phi}).$  

We can rewrite Eq.\((11)\) in a more compact form as follows

$$-2 i f (\frac{\partial}{\partial z} (B \sqrt{E f}) + m (A \sqrt{E f}) = 0,$$

$$-2 i (\frac{\partial}{\partial \bar{z}} (A \sqrt{E f}) + m (B \sqrt{E f}) = 0.$$  

Having the rotational symmetry around the axis perpendicular on the fifth and the sixth dimension we require that $\psi^{(6)}$ is the eigenfunction of the total angular momentum operator $M^{56}$

$$M^{56} \psi^{(6)} = (n + \frac{1}{2}) \psi^{(6)}, \quad M^{56} = x^5 p^6 - x^6 p^5 + S^{56}.$$  

Then $A = A_\phi (\rho) e^{i n \phi}$ and $B = B_\phi (\rho) e^{i (n+1) \phi}$.

Spinors which manifest masslessness in $d = (1 + 3)$ must obey the equations \((11)\) and \((12)\) for $m = 0$. One easily sees that for $m = 0$ and any $f$, which determines the curvature in the fifth and the sixth dimension ($f^{(5)} = \delta^{(5)} \frac{\partial}{\partial \rho}$, while $\omega_{abs} = 0$), we find the solution of Eq.\((12)\)
with the total angular momentum in the fifth and the sixth dimension equal to \( n + 1/2 \) as follows

\[
\psi_0^{(6) n+1/2} = \alpha_n \, z^n \, \sqrt{f} \, 56 \, \psi_0^{(4)(+)} + \beta_n \, z^n \, \sqrt{f} \, [-] \, \psi_0^{(4)(-)}.
\] (14)

There is a solution for any positive integer \( n \) and there is obviously no mass protection, since the solution for any chosen \( n \) is the superposition of the left and the right handed components in \( d = (1 + 3) \). Taking into account the boundary condition of Eq.(4) one sees that \( \beta_n \) must be zero for all \( n \), accordingly \( \psi_0^{(6) n+1/2} = \alpha_n \, z^n \, \sqrt{f} \, 56 \, \psi_0^{(4)(+)} \) for any \( f(\rho) \), which means that only solutions of one handedness in \( d = (1 + 3) \) are allowed, assuring the mass protection mechanism in \( d = (1 + 3) \). However, all the positive integers \( n \) are allowed (we require \( n \geq 0 \) to ensure the integrability of solutions at the origin).

Taking into account the boundary condition of Eq.(7) we see that \( \beta_n \) must still be zero for any \( n \), while now the condition \( \hat{\mathcal{R}}' \psi_0^{(6) n+1/2}|_{\rho=\rho_0} = 0 \) leads to the condition

\[
n(n + \rho \frac{\partial \ln \sqrt{f}}{\partial \rho})|_{\rho=\rho_0} = 0.
\] (15)

In the case that a disc is flat with the boundary at \( \rho = \rho_0 \) we get that only \( n = 0 \) fulfills the boundary condition of Eqs.(7,15). If a disk is curved on a sphere with radius \( \rho_0 \) and we put a boundary at \( \rho = \rho_0 \), then \( f = (1 + (\frac{\rho}{\rho_0})^2) \) and the boundary condition requires \( n(n + \frac{1}{2}) = 0 \). Again the only solution is \( n = 0 \), since \( n \) is an integer. More general \( f \) would lead to rational or irrational numbers (and so would a boundary at some other \( \rho_1 \neq \rho_0 \)), so that we can conclude that \( n = 0 \) is the only solution even if the disk is not flat.

Therefore for \( m = 0 \) we get as the only solution for any curvature \( f \) \((f^s_5 = f \delta^s_5, \omega_{56s} = 0, s = 5, 6; \sigma = 5, 6,)\)

\[
\psi_0^{(6) 1/2} = a_0 \, 56 \, \psi_0^{(4)(+)}
\] (16)

and accordingly this massless state is mass protected and manifests spin 1/2 (as we shall see) as the only charge in \( d = (1 + 3) \) (Eq.(13)).

In the massive case \((m \neq 0)\) we get for \( f = 1 \) the following solutions of Eq.(12)

\[
\psi_m^{(6) n+1/2} = \mathcal{N}_n \, e^{im\phi} \left\{ J_n \, 56 \, \psi_0^{(4)(+)} - ie^{i\phi} \, J_{n+1} \, [+] \, \psi_0^{(4)(-)} \right\};
\] (17)

where \( J_n \) are the Bessel’s functions of the first order, which depend on \( \rho \), while \( \mathcal{N}_n \) determines the normalization \[3\].
If we require that the boundary condition of Eq. (5) should be fulfilled, then \( J_{n+1}|_{\rho=\rho_0} = 0 \) and the zeros of \( J_{n+1} \) determine for each \( n \) and each zero of \( J_n \) a (different) mass \( m \).

If we require that the boundary condition of Eq. (7) is fulfilled, then only \( n = 0 \) is the solution. In this case we namely have \( J_{n+1}|_{\rho=\rho_0} = 0 \) and \( n \left( \frac{1}{\rho} \frac{\partial J_n}{\partial \rho} \right)|_{\rho=\rho_0} = 0 \). It turns out that \( n = 0 \) is the only possibility, since \( J_{n+1}|_{\rho=\rho_0} = 0 = \frac{\partial J_n}{\partial \rho}|_{\rho=\rho_0} \) is true only for \( n = 0 \). This relation is fulfilled for infinitely many masses \( m_i = \alpha_1/\rho_0, i = 1, \ldots \), where index \( i \) determines the successive number of a zero of \( J_1 \) at \( \rho = \rho_0 \). There are accordingly infinitely many massive solutions, which obey the equations of motion (Eq. (12)) for \( f = 1 \) and the boundary condition of Eq. (7), all having eigenvalue of \( M^{56} \) equal to \( 1/2 \)

\[
\psi_{m_i}^{(6)1/2} = N_i \left( J_0(\alpha_1\rho/\rho_0) \right)^{56} \psi_{(+)m_i}^{(4)} - iJ_1(\alpha_1\rho/\rho_0) \left[ - \right] \psi_{[-]m_i}^{(4)}. \tag{18}
\]

For \( f = (1 + (\rho/\rho_0)^2) \) the solutions of Eq. (12) obeying the boundary condition of Eq. (7) can not be found among the known functions, but we still know that they have the eigenvalue of \( M^{56} \) equal to \( 1/2 \) and we also guess that they behave pretty like the two Bessel’s functions in Eq. (18).

**IV. CURRENT THROUGH THE WALL**

The current perpendicular to the wall can be written as

\[
n^{(\rho)\gamma}\gamma_s = \psi^\dagger \gamma^0 \gamma^s \psi = \psi^\dagger \hat{j}_\perp \psi, \quad \hat{j}_\perp = -\gamma^0 \left\{ e^{-i\phi} \left( + \right) + e^{i\phi} \left( - \right) \right\}. \tag{19}
\]

For physically acceptable cases when spinors are localized inside the disk the current through the wall must be equal to zero

\[
\{ \psi^\dagger \hat{j}_\perp \psi \}|_{\rho=\rho_0} = 0. \tag{20}
\]

One easily checks that—since the current operator \( \hat{j}_\perp \) changes the handedness of a state—in the massless case (since massless states of only one handedness exist (Eq. (16))) and all the massive cases (the product of \( A \) and \( B \) appears in the current and \( B \) is zero, in particular in Eq. (18) \( J_1|_{\rho=\rho_0} = 0 \) the current through the wall is for both types of the boundary conditions equal to zero.
V. HERMITICITY OF THE OPERATORS AND ORTHOGONALITY OF SOLUTIONS

The operators \( s \) (and consequently also \((\gamma^s p_s)^2\)) are not Hermitean on the space of solutions which have nonzero values on the boundary \( p = p_0 \), since then \( \int d^2x p_s(\psi_i^\dagger \psi_j) \neq 0 \).

We define therefore a new operator \( \hat{p}_s \). We take care of a flat disc with the boundary.

Statement: The operators \( \hat{p}_s \),

\[
\hat{p}_s = i\left\{ \frac{\partial}{\partial x^s} - \frac{1}{2} \frac{x^s}{\rho} \delta(\rho - \rho_0)\right\} [+] ,
\]

are Hermitean on the vector space of solutions presented in Eqs.(16,18).

Proof: Since the expectation value of \( \hat{p}_s \) is zero between the massless states (Eq.(16)) and so is also between the massless and all the massive states (Eq.(18)), we check the Hermiticity of the operator \((\gamma^s \hat{p}_s)^2 = \hat{p}_s p_s + \frac{1}{2} \left\{ \left( \frac{\partial}{\partial \rho} + \frac{1}{\rho} \right) \delta(\rho - \rho_0) + \delta(\rho - \rho_0) \left( \frac{\partial^2}{\partial \rho^2} - i \frac{\partial}{\partial \rho} \right) \right\} \), where \( p_s p_s = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho^2} \left( \int d^2x \psi_i^\dagger (\gamma^s p_s)^2 \psi_j = \int d^2x \psi_i^\dagger \psi_j m^2 \delta_{ij} \right) \). We ought to check only the \( x^5 \) and \( x^6 \) part, with the corresponding spin components included. We obtain for the expectation values of \((\gamma^s \hat{p}_s)^2 \) between the massless and a massive state the values:

\[
\int d^2x Tr_{56}(J_{6i}^{56})^{\dagger}[(\gamma^s \hat{p}_s)^2 \hat{p}_s] = \pi \frac{\alpha_i}{\rho_0}(\rho J_{1i})|_{\rho = \rho_0} = \int d^2x Tr_{56}[(\gamma^s \hat{p}_s)^2 J_{6i}]^{\dagger} = \pi \left( -\frac{\alpha_i}{\rho_0} \rho J_{1i} \right)|_{\rho = \rho_0} = 0 .
\]

due to the properties of the Bessel functions \( J_{1i} = -\frac{\alpha_i}{\alpha_1} \frac{\partial J_{0i}}{\partial \rho} \) with \( J_{1i}(\alpha_1) = 0 \) and the property of the delta function \( \int_0^{\rho_0} g(\rho) \frac{\partial \rho}{\partial \rho} = -\frac{\partial g(\rho)}{\partial \rho} |_{\rho = \rho_0} \), where \( g(\rho) \) is any smooth function of \( \rho \). Accordingly the massless state is orthogonal to all the massive states. Taking into account that for the Bessel functions \( \frac{\partial J_{0i}}{\partial \rho} = \frac{\alpha_i}{\rho_0} \alpha_1 \frac{\partial J_{0i}}{\partial \rho} \) one finds that for \( i \neq k \) it follows

\[
\int d^2x Tr_{56}(J_{6i}^{56} - iJ_{1i} | [-] e^{i\phi}|[(\gamma^s \hat{p}_s)^2 (J_{6k}^{56} - iJ_{1k}^{56} | [-] e^{i\phi})] = \]

\[
2\pi \int_0^{\rho_0} \rho d\rho \left[ -(m_k)^2 (J_{6i} J_{1i} J_{1k} J_{6k}) + \frac{1}{2} (-\rho J_{1i} J_{6k} | \frac{\alpha_{1k}}{\rho_0}) |_{\rho = \rho_0} \right] = \]

\[
2\pi (\rho J_{6i} J_{1k} + \rho J_{1i} J_{6k}) |_{\rho = \rho_0} = 0 ,
\]

since \( J_{1k}(\alpha_{1k}) = 0 \). We checked accordingly the Hermiticity of the operator \( \rho (\gamma^s \hat{p}_s)^2 \) on the vector space of the massive states and correspondingly also the orthogonality of these states.

Let us add the normalization property of the massive states

\[
\int d^2x Tr_{56}(J_{6i}^{56} - iJ_{1i}^{56} | [-] e^{i\phi}|[(\gamma^s \hat{p}_s)^2 (J_{6i}^{56} - iJ_{1i}^{56} | [-] e^{i\phi})] = \]

\[
-\pi (m_k)^2 (\rho^2 (J_{6i}^2 + J_{1i}^2)) |_{\rho = \rho_0} = \pi (\rho^2 J_{6i}^2) |_{\rho = \rho_0} ,
\]

(23)
since $m_i = \frac{\alpha_i}{\rho_0}$.

We conclude that on the space of solutions (Eqs. (16,18)) the operators $\hat{p}_s$ (Eq. (21)) are Hermitean and the solutions are orthogonal. Since we do not know the explicit expressions for solutions on the curved disc ($f \neq 1$), we do not comment orthogonality properties of these functions.

VI. PROPERTIES OF SPINORS IN $d = (1 + 3)$

To study how do spinors couple to the Kaluza-Klein gauge fields in the case of $M^{(1+5)}$, “broken” to $M^{(1+3)} \times$ a flat disk with $\rho_0$ and with the involution boundary condition, which allows only right handed spinors at $\rho_0$, we first look for (background) gauge gravitational fields, which preserve the rotational symmetry on the disk. Following ref. [2] we find for the background vielbein field

$$
e^\alpha_a = \begin{pmatrix} \delta^m_\mu & e^m_\sigma = 0 \\ e^s_\mu & e^s_\sigma \end{pmatrix}, f^\alpha_a = \begin{pmatrix} \delta^\mu_m & f^\sigma_m \\ 0 & f^\sigma_s \end{pmatrix},$$

with $f^\sigma_m = A_\mu \delta^\mu_m \varepsilon^\sigma_\tau x^\tau$ and the spin connection field

$$\omega_{st\mu} = -\varepsilon_{st} A_\mu, \quad \omega_{sm\mu} = -\frac{1}{2} F_{\mu\nu} \delta^\nu_m \varepsilon_{s\sigma} x^\sigma.$$ (25)

The $U(1)$ gauge field $A_\mu$ depends only on $x^\mu$. All the other components of the spin connection fields are zero, since for simplicity we allow no gravity in $(1 + 3)$ dimensional space.

To determine the current, coupled to the Kaluza-Klein gauge fields $A_\mu$, we analyze the spinor action

$$S = \int d^d x E \bar{\psi}^{(6)} \gamma^a P_{0a} \psi^{(6)} = \int d^d x \bar{\psi}^{(6)} \gamma^m \delta^\mu_m \delta^\nu_\mu p_{\nu} \psi^{(6)} +
\int d^d x \bar{\psi}^{(6)} \gamma^m (-) S^{em} \omega_{sm\mu} \psi^{(6)} + \int d^d x \bar{\psi}^{(6)} \gamma^s \delta^\sigma_s p_\sigma \psi^{(6)} +
\int d^d x \bar{\psi}^{(6)} \gamma^m \delta^\mu_m A_\mu (\varepsilon^\sigma_\tau x^\tau p_\sigma + S^{56}) \psi^{(6)}. $$ (26)

$\psi^{(6)}$ are solutions of the Weyl equation in $d = (1 + 5)$, $E$ is for $f^\alpha_a$ from equal to $f^{-2}$. The first term on the right hand side of Eq. (26) is the kinetic term (together with the last term defines the covariant derivative $p_{0\mu}$ in $d = (1 + 3)$). The second term on the right hand side contributes nothing when the integration over the disk is performed, since it is proportional to $x^\sigma (\omega_{sm\mu} = -\frac{1}{2} F_{\mu\nu} \delta^\nu_m \varepsilon_{s\sigma} x^\sigma).$
We end up with
\[ j^\mu = \int d^2x \overline{\psi}^{(6)}(6) \gamma^m \delta^\mu_m M^{56} \psi^{(6)} \] (27)
as the current in \(d = (1 + 3)\). The charge in \(d = (1 + 3)\) is proportional to the total angular momentum \(M^{56} = L^{56} + S^{56}\) on a disk, which for either massless or massive spinors equal to \(1/2\).

VII. CONCLUSIONS

We presented for a toy model the boundary condition which makes massless spinors which in \(M^{1+5}\) carry nothing but a spin to live in \(M^{(1+3)} \times \text{a disk with a boundary and manifest in } M^{(1+3)}\) —if massless—as a left handed spinor (with no right handed partner and accordingly mass protected), which carries only one kind of the Kaluza-Klein type of charge and chirally couples to the corresponding Kaluza-Klein gauge field (so that after the second quantization procedure a particle and an antiparticle of only that particular charge and the opposite one appear, respectively).

We propose the boundary condition
\[ \{ \hat{\mathcal{O}} \psi = \psi \} |_{\rho = \rho_0}, \quad \hat{\mathcal{O}} = I - 2\hat{R}', \quad \hat{R}' = [-] + \cos \theta \ [+] \left( -\frac{i}{\rho} \frac{\partial}{\partial \rho} \frac{\partial}{\partial \phi} \right) \] (28)
which is in Eq.(6) presented in a Lorentz invariant way, with \(\theta\) which is an arbitrary parameter \(\neq (2k + 1)\pi/2\), and \(k\) is any integer. This boundary condition allows in the massless case only the right handed spinor to live on the disk and accordingly manifests left handedness in \(M^{(1+3)}\). The massless and the massive solutions have the eigen value of the total angular momentum in the fifth and the sixth dimension equal to \(1/2\), which then manifests as a charge in \(d = (1 + 3)\). The massless solution is mass protected. When a disk is flat, the massless solution is independent of \(x^5\) and \(x^6\), while the massive solutions are expressible in terms of the Bessel’s functions \(J_0(\alpha^i \rho/\rho_0)\) and \(J_1(\alpha^i \rho/\rho_0)\), defining masses \(m^i = \alpha^i/\rho_0\) through the requirement that the \(i - \text{th zero of } J_1\) is zero at \(\rho = \rho_0\).

We define a generalized momentum
\[ \hat{p}_s = i\left\{ \frac{\partial}{\partial x^s} - \frac{1}{2} \left( \frac{\cos \phi}{\sin \phi} \right) \delta(\rho - \rho_0) [+] \right\}, \] (29)
which is Hermitean on the vector space of states obeying equations of motion (Eq. (3)) for 
\( f_s^\sigma = \delta_s^\sigma \) and our boundary condition (Eq. (28)). Accordingly also the operator \( \gamma_s^* \hat{p}_s \gamma_t^* \hat{p}_t \) is Hermitean on the same vector space, and the states are accordingly orthogonal, with the
eigen values of this operator which demonstrate the masses of states.

The negative \(-1/2\) charge states appear only after the second quantization procedure in
agreement with what we observe.

If the disc is curved, so that \( f_s^\sigma = \delta_s^\sigma f \), with \( f = (1 + (\rho/\rho_0)^2) \) if it is on \( S^2 \) with a radius \( \rho_0 \), the solutions obeying Eqs. (3,28) have similar properties as for \( f = 1 \), but in this case we
present only the explicit expression for the massless state, while the massive ones stayed to
be determined.

VIII. ACKNOWLEDGEMENT

One of the authors (N.S.M.B.) would like to warmly thank Jože Vrabec for fruitful dis-
cussions.

APPENDIX A: SPINOR REPRESENTATION TECHNIQUE IN TERMS OF
CLIFFORD ALGEBRA OBJECTS

We define\[10\] spinor representations as superposition of products of the Clifford algebra
objects \( \gamma^a \) so that they are eigen states of the chosen Cartan sub algebra of the Lorentz
algebra \( SO(d) \), determined by the generators \( S^{ab} = i/4(\gamma^a \gamma^b - \gamma^b \gamma^a) \). By introducing the
notation
\[
\begin{align*}
(\pm i) & : = \frac{1}{2}(\gamma^a \mp \gamma^b), & [\pm i] : = \frac{1}{2}(1 \pm \gamma^a \gamma^b), & \text{for } \eta^{aa} \eta^{bb} = -1, \\
(\pm) & : = \frac{1}{2}(\gamma^a \pm i \gamma^b), & [\pm] : = \frac{1}{2}(1 \pm i \gamma^a \gamma^b), & \text{for } \eta^{aa} \eta^{bb} = 1,
\end{align*}
\]
(A1)

it can be checked that the above binomials are really “eigenvectors” of the generators \( S^{ab} \)
\[
S^{ab} (k) : = \frac{k}{2} [k]^{ab}, & S^{ab} [k] : = \frac{k}{2} [k]^{ab}.
\]

(A2)

Accordingly we have
\[
\begin{align*}
(\pm i) : = \frac{1}{2}(\gamma^0 \mp \gamma^3), & \quad [\pm i] : = \frac{1}{2}(1 \pm \gamma^0 \gamma^3), \\
03 & : = \frac{1}{2}(\gamma^0 \mp \gamma^3), & \quad 03 i : = \frac{1}{2}(1 \pm \gamma^0 \gamma^3),
\end{align*}
\]
\[ (\pm) = \frac{1}{2}(\gamma^1 \pm i\gamma^2), \quad [\pm] = \frac{1}{2}(1 \pm i\gamma^1\gamma^2), \]
\[ (\pm) = \frac{1}{2}(\gamma^5 \pm i\gamma^6), \quad [\pm] = \frac{1}{2}(1 \pm i\gamma^5\gamma^6), \]
\[ (\pm) = \frac{1}{2}(\gamma^5 \pm i\gamma^6), \quad [\pm] = \frac{1}{2}(1 \pm i\gamma^5\gamma^6), \]

with eigenvalues of \( S^{03} \) equal to \( \pm \frac{1}{2} \) for \( 03 \) and \( 03 \), and to \( \pm \frac{1}{2} \) for \( 12 \) and \( 12 \), as well as for \( \pm \) and \( [\pm] \).

We further find
\[ \gamma^a_{ab}(k) = \eta^{aa}[-k], \quad \gamma^b_{ab}(k) = -ik[-k], \]
\[ \gamma^a_{ab}[k] = (-k), \quad \gamma^b_{ab}[k] = -ik\eta^{aa}(-k). \]

We also find
\[ (ab)(k) = 0, \quad (ab)(-k) = \eta^{aa}[k], \quad (ab)[k] = [k], \quad (ab)[k] = 0, \]
\[ (ab)(ab)(ab)(k) = 0, \quad (ab)(k)(k)(k)(ab) = 0, \quad (ab)(-k)(-k)(ab) = 0. \]

To represent one Weyl spinor in \( d = (1 + 5) \), one must make a choice of the operators belonging to the Cartan sub algebra of 3 elements of the group \( SO(1,5) \)
\[ S^{03}, S^{12}, S^{56}. \]

Any eigenstate of the Cartan sub algebra (Eq. (A6)) must be a product of three binomials, each of which is an eigenstate of one of the three elements. A left handed spinor \( (\Gamma^{(1+5)}) = -1 \) representation with \( 2^6/2-1 \) basic states is presented in Eq. (8). For example, the state \( (+i)(+)(+\psi_0) \), where \( \psi_0 \) is a vacuum state (any, which is not annihilated by the operator in front of the state) has the eigenvalues of \( S^{03} \), \( S^{12} \) and \( S^{56} \) equal to \( \pm \frac{1}{2}, \frac{1}{2} \) and \( \frac{1}{2} \), correspondingly. All the other states of one representation of \( SO(1,5) \) follow from this one by just the application of all possible \( S'(ab) \), which do not belong to the Cartan subalgebra.

[1] E. Witten, “Search for realistic Kaluza-Klein theory”, Nucl. Phys. B 186 (1981) 412; “Fermion quantum numbers in Kaluza-Klein theories”, Princeton Technical Rep. PRINT -83-1056, October 1983.
[2] N. S. Mankoč Borštnik, H. B. Nielsen, “An example of Kaluza-Klein-like theory with boundary conditions, which lead to massless and mass protected spinors chirally coupled to gauge fields”, Phys. Lett. B 633 (2006) 771-775, hep-th/0311037, hep-th/0509101.

[3] N. S. Mankoč Borštnik, H. B. Nielsen, “Fermions with no fundamental charges call for extra dimensions”, Phys. Lett. B 644 (2007)198-202, hep-th/0608006.

[4] N. S. Mankoč Borštnik, “Spin connection as a superpartner of a vielbein”, Phys. Lett. B 292 (1992) 25.

[5] N. S. Mankoč Borštnik, “Spinor and vector representations in four dimensional Grassmann space”, J. Math. Phys. 34 (1993) 3731.

[6] N.S. Mankoč Borštnik, “Unification of spins and charges”, Int. J. Theor. Phys. 40 315 (2001).

[7] N. S. Mankoč Borštnik, “Unification of spins and charges in Grassmann space?”, Modern Phys. Lett. A 10 (1995) 587.

[8] A. Borštnik, N. S. Mankoč Borštnik, “The approach unifying spins and charges in and its predictions”, Proceedings to the Euroconference on Symmetries Beyond the Standard Model, Portorož, July 12-17, 2003, Ed. by N. Mankoč Borštnik, H. B. Nielsen, C. Froggatt, D. Lukman, DMFA Založništvo 2003, p.27-51, hep-ph/0401043, hep-ph/0401055, hep-ph/0301029.

[9] A. Borštnik Bračič, N.S. Mankoč Borštnik, “Origin of families of fermions and their mass matrices“, Phys. Rev. D 74 (2006)073013, hep-ph/0512062.

[10] N. S. Mankoč Borštnik, H. B. Nielsen, “How to generate spinor representations in any dimension in terms of projection operators”, J. of Math. Phys. 43 (2002) 5782, hep-th/0111257.

[11] N. S. Mankoč Borštnik, H. B. Nielsen, “How to generate families of spinors”, J. of Math. Phys. 44 (2003) 4817, hep-th/0303224.

[12] N. S. Mankoč Borštnik, H. B. Nielsen, “An example of Kaluza-Klein-like theories with boundary conditions, which lead massless and mass protected spinors chirally coupled to gauge fields”, Phys. Lett. B 633 (2006) 771, hep-th/0311037, hep-th/0509101.

[13] To understand the appearance of the two kinds of spin we invite the reader to look at the refs. [9, 10, 11].

[14] $f^a_\alpha$ are inverted vielbeins to $e^\alpha_a$ with the properties $e^a_\alpha f^\alpha_b = \delta^a_b$, $e^a_\alpha f^\beta_a = \delta^\beta_\alpha$. Latin indices $a, b, \ldots, m, n, \ldots, s, t, \ldots$ denote a tangent space (a flat index), while Greek indices $\alpha, \beta, \ldots, \mu, \nu, \ldots, \sigma, \tau, \ldots$ denote an Einstein index (a curved index). Letters from the beginning of both the alphabets indicate a general index $(a, b, c, \ldots$ and $\alpha, \beta, \gamma, \ldots)$, from the middle of
both the alphabets the observed dimensions 0, 1, 2, 3 (m, n, .. and μ, ν, ..), indices from the bottom of the alphabets indicate the compactified dimensions (s, t, .. and σ, τ, ..). We assume the signature $\eta^{ab} = \text{diag}\{1, -1, -1, \cdots, -1\}$. 