Scar States in Deconfined Lattice Gauge Theories

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The weak ergodicity breaking induced by quantum many-body scars (QMBS) represents an intriguing concept that has received great attention in recent years due to its relation to unusual non-equilibrium behaviour. Here we reveal that this phenomenon can occur in a previously unexplored regime of a lattice gauge theory, where QMBS emerge due to the presence of an extensive number of local constraints. In particular, by analyzing the gauged Kitaev model, we provide an example where QMBS appear in a regime where charges are deconfined. By means of both numerical and analytical approaches, we find a variety of scarred states far away from the regime where the model is integrable. The presence of these states is revealed both by tracing them directly from the analytically reachable limit, as well as by quantum quenches showing persistent oscillations for specific initial states.

Introduction.— Thermalization properties of isolated quantum systems are currently under intensive investigations in different areas of modern quantum physics [1–4]. In this context, a huge attention has been recently devoted towards the study of a large variety of Hamiltonians where the ergodicity is weakly broken [5–24]. In particular, in such models quenches from carefully designed initial states reveal persistent many-body revivals that apparently contradict ergodicity [5, 25]. The reason for such an absence of thermalization turns out to be the presence of specific eigenstates called quantum many-body scars (QMBS) characterized, in particular, by a sub-volume entanglement law [6–14, 21, 24, 26]. It is worth noting that similar regular states have been previously identified [27] for chaotic quantum billiards in relation to semiclassical periodic orbit quantization [28, 29]. Moreover the concept of scarred symmetry, closely related to QMBS, has been introduced in studies of hydrogen atom in a strong magnetic field [30].

With the advent of a new generation of cold-atom quantum simulators [31–35], the weak ergodicity breaking manifested by QMBS has been experimentally detected in constrained spin [5] and bosonic [36] models. Crucially, both realizations can be viewed as lattice gauge theories (LGTs) where the energy constraints are induced by the Gauss law fixing the relation between gauge and charge variables. Motivated by the recent progress achieved in the last years implementing LGTs in quantum simulators [37–41], an impressive theoretical effort has been devoted towards a better understanding of simple gauge-invariant theories [17, 18, 42–53]. In this direction, the connection between Gauss law and QMBS has recently gained attention both in U(1) [16, 54] and Z2 [11] LGTs. Here indeed, the weak ergodicity breaking associated to the slow oscillatory dynamics can be interpreted as a string inversion phenomenon. Crucially, it has to be underlined that all the LGTs, where QMBS have been identified, are characterized by a charge confinement. This regime implies that only particle-antiparticle bound states exist and therefore charges can be observed in composite structures only. These effective pairs, together with an emergent new symmetry, generate the slow down of quantum dynamics and have been shown to be intricately connected to the presence of QMBS [55]. It thus appears natural to wonder whether confinement is a prerequisite to observe QMBS in LGTs. In this letter, we tackle this question by investigating the recently introduced gauged 1D Kitaev model [16, 56, 57], whose ground state displays a confined phase as well as a regime where charges are not bounded in pairs, thus describing a deconfined phase. We study its low entanglement states and show that QMBS are present in the ergodic deconfined phase and are absent in the ergodic confined regime of the model. Importantly, we are able to continuously track QMBS down from the analytic prediction valid in the quasi-integrable regime, and therefore providing their partial classification.

Model and observables.— The Hamiltonian of the p-wave superconducting Kitaev chain minimally coupled
to a $Z_2$ gauge field introduced in [16, 56, 57] reads
\[
H = -t \sum_j (c_j^\dagger - c_j) \sigma_j^{\tau_{j+1/2}} (c_{j+1}^\dagger + c_{j+1}) - \mu \sum_j (c_j^\dagger c_j - \frac{1}{2}) - h \sum_j \sigma_j^{\tau_{j+1/2}}. \quad (1)
\]

Here, $c_j^\dagger$ ($c_j$) denotes the fermionic creation (annihilation) operator and $t$ describes the tunneling and pairing production/annihilation processes mediated by the $Z_2$ gauge field $\sigma_j^{\tau_{j+1/2}}$ defined on the links between nearest neighbors sites. Fluctuations in the gauge field are induced by the electric field $\sigma_j^{\tau_{j+1/2}}$ of strength $h$ and the number of fermions is fixed by the chemical potential $\mu$.

As required in LGTs, the model (1) is invariant under local gauge transformation generated by the Gauss operator $G_j = \sigma_{j-1/2} (-1)_{\mu} \sigma_{j+1/2}$ where $[H, G_j] = 0$ and $[G_i, G_j] = 0$. The physical states are thus those satisfying the Gauss law, $G_j |\psi\rangle = \pm |\psi\rangle$, for all $j$ sites [58].

For periodic boundary conditions the Hamiltonian (1) can be written in terms of gauge invariant spin-1/2 operators
\[
X_{i+\frac{1}{2}} = \sigma_{i+\frac{1}{2}}^x, \quad Y_{i+\frac{1}{2}} = (c_i^\dagger - c_i) \sigma_{i+\frac{1}{2}}^y (c_{i+1}^\dagger + c_{i+1}), \quad Z_{i+\frac{1}{2}} = (c_i^\dagger - c_i) \sigma_{i+\frac{1}{2}}^z (c_{i+1}^\dagger + c_{i+1}). \quad (2, 3, 4)
\]

Upon this transformation, the gauged Kitaev model corresponds to the quantum Ising model with both transverse and longitudinal fields
\[
H = \sum_{i=1}^L \frac{\mu}{2} Z_i Z_{i+1} - h X_i - h Z_i. \quad (5)
\]

Notice that the model is now defined on a dual lattice, where the index $i$ corresponds to the links of the original model (1). For $\mu > 0$, the phase diagram of the model (5), as shown in Fig. 1, is characterized by the presence of antiferromagnetic (AFM) order for $0 \leq t/2, h \lesssim \mu$ and a paramagnetic (PM) phase for other choices of finite $t$ and $h$ [59]. The AFM phase turns out to be of great interest since it supports domain wall excitations associated to the antiferromagnetic order caused by the spontaneous breaking of the $Z_2$ Ising and translational symmetries. In 1D these last two features imply the domain wall deconfinement. Therefore, in the gauged Kitaev Hamiltonian (1), such an AFM order corresponds to charge deconfinement (CD), where fermions are free to expand without any string tension. On the other hand, the PM regime corresponds to a phase characterized by charge confinement (CC), where fermions appear only as bound pairs.

In the limit $t \ll \mu$ it is possible to perform a Schrieffer-Wolff transformation in agreement with [56], for a higher order expansion see [60], with
\[
S = \frac{-it}{2\mu} \sum_j \left\{ \frac{1 + Z_{j-1}}{2} Y_j \left( \frac{1 + Z_{j+1}}{2} \right) - \frac{1 - Z_{j-1}}{2} Y_j \left( \frac{1 - Z_{j+1}}{2} \right) \right\}, \quad (6)
\]

to obtain an effective Hamiltonian $H_{eff} = e^S H e^{-S} = H + [S, H] + O(t^2)$. Notice that $S$ is chosen in such a way that the new terms commute with the unperturbed part of the Hamiltonian $\sum_{i=1}^L \frac{\mu}{2} Z_i Z_{i+1}$ to the leading order in the expansion, thus preserving its block-diagonal structure while also including all virtual processes within each sector. The derived effective Hamiltonian,
\[
H_{eff} = \sum_{i=1}^L \frac{\mu}{2} Z_i Z_{i+1} - h Z_i - \frac{t}{2} (X_i - Z_{i-1} X_i Z_{i+1}) \quad (7)
\]
turns out to be the well known model discussed in detail in [13], where two towers of QMBS have been identified. The latter are of the form
\[
|S^k_n\rangle = \frac{1}{n! \sqrt{N(L, n)}} (\Omega^k)^n |\Omega^1\rangle, \quad (8)
\]
with $k = 1, 2$, $|\Omega^1\rangle = |0\cdots 0\rangle$, $|\Omega^2\rangle = |1\cdots 1\rangle$, and $(\Omega^k)^i = \sum_{\alpha=\pm1} (\alpha)^{i} P_{-\alpha}^k P_{\alpha}^k$, with $\alpha_{1/2} = -j, +$, and projection operators $P_{\alpha}^k = (1 - (-1)^{k} \sigma_j^z)^{2}/2$. More precisely, such QBMS describe $n$-magnon and $n$-antimagnon excitations for $k = 1$ and $k = 2$, respectively [13].

As already pointed out, the Schrieffer-Wolff transformation described above links the models (5) and (1) with the effective spin model (7) only for $t \ll \mu$. In this limit, the states given by Eq. (8) become true QMBS of Hamiltonian (7), as it was studied in Ref. [13]. It appears natural to wonder whether, following the states (8) in the parameter space, it is possible to find scarred states also in the gauged Kitaev chain beyond the $t \ll \mu$ limit.

**Level dynamics.**— In order to investigate this point, we first determine in which regime of parameters the model (5) can be considered as ergodic, and thus where the regular states may be called QMBS. Ergodicity may be revealed by the adjacent mean gap-ratio $r$ [61] between subsequent level spacings $\Delta_i$,
\[
r_i = \frac{\min(\Delta_i, \Delta_{i+1})}{\max(\Delta_i, \Delta_{i+1})}, \quad (9)
\]
where $r \simeq 0.531$ corresponds to the fully ergodic regime (as described by the Gaussian Orthogonal Ensemble (GOE) [62]) and $r \simeq 0.386$ indicates the near-integrable regime [61, 63]. As shown in Fig. 1, our model shows strong indications of near integrable behavior for $h \ll \mu$ and for a wide range of values of $h$. Outside this region, the model is expected to be non-integrable, and
FIG. 1. (a) Contour plot of the mean gap ratio $r$ for $L = 16$. The black bold line here represents the phase transition between the confined (CC) and deconfined (DC) phases [57]. The dashed lines correspond to paths taken in the parameter space and the red crosses indicate points where nonergodic properties of the tracked states are lost. (b) The average of the two lowest entanglement entropies of states in the middle of the spectrum. (c) Panel (i) shows the gap ratio $r$ as a function of $t$. Panels (ii), (iii) and (iv) shows the entanglement entropy ($S$) for different tracked QMBS. The values of $t$ in (i),(ii), (iii) and (iv) refer to path I. (d) Time dependence of the fidelity relative to an initial state $|\psi(0)\rangle$ and evolved with (5) at $h = 0.5$ and different values of $t$ for $L = 16$ ($\mu = 1$).

thus QMBS can appear. The next step is then to identify the $|S_n^k\rangle$ states for the Hamiltonian (5) with parameters $t, h \ll \mu$, e.g. for $t_0 = h_0 = 0.001$ (we set $\mu = 1$ in the following), with the aim of tracking such initial states $|S_n^k(t_0, h_0)\rangle$ following their possible deformations induced by making small changes of the parameters, $t = t_0 + \delta t$ and $h = h_0 + \delta h$.

Every time we update the parameters, we diagonalize the Hamiltonian in the symmetry sector with momentum $p = 0$ and parity +1 (we consider $\mu$ even only) and find the new candidate $|E_n(t, h)\rangle$ by maximizing the overlap $O = \langle |E_n(t, h)\rangle |S_n^k(t_0, h_0)\rangle \rangle^2$. For an isolated level the new state is accepted, $|S_n^k\rangle$ is updated, and we repeat the procedure. Special care is taken to diabatically cross narrow avoided crossings [64].

Figure 1 shows two illustrative paths tracking the QMBS states identified as $|S_n^k(t, h)\rangle$. The path indicated as I remains in the deconfined phase, while path II crosses the phase boundary into the confined phase region. The paths are followed until the scarred character is lost, and the end points are marked as crosses in the $(t, h)$ plane of Fig. 1(a).

The loss of the scarred nature is also evident from Fig. 1(b), which shows the average of the two lowest entanglement entropies in the energy region of interest. Clearly, in the region above the dashed line, the lowest entanglement entropy is more than half of its maximum possible value, ruling out the existence of QMBS there.

We now focus our attention specifically on path I. Here, we find that QMBS exist in the whole region described by I, where we start our tracking from a nearly integrable region (as indicated in Fig. 1) and follow the $t = 0.2h$ line up to $(t, h) = (0.1, 0.5)$. Up to these values, the gap ratio $r$ reaches almost the GOE level (blue color in Fig. 1). This path is followed upwards up to the point $(t, h) = (0.3, 0.5)$ where, therefore, QMBS states still exist. The observables helping us to characterize the system along this path are depicted in Fig. 1(c). Panel (i) shows the mean gap ratio $r$ along the path, calculated for the states in the middle of the spectrum. The starting point lies in the almost fully non-ergodic regime. However, the gap ratio $r$ steadily increases along the line $t = 0.2h$, reaching the GOE value. The mean gap ratio drops slightly while crossing the transition line to the confined phase. While the $r$ value is unique in the middle of the spectrum, each QMBS has to be tracked individually. This procedure can be analyzed by calculating the half-chain entanglement entropy $S = -Tr[\rho_{L/2} \ln \rho_{L/2}]$ [64] for the tracked states. Note that this quantity behaves similarly for the three cases corresponding to dif-
different eigenstates shown in panels (ii) to (iv) of Fig. 1(c). One starts with a low entanglement entropy which remains quite low throughout $t = 0.1$ and persists further into the GOE-like spectrum, thereby confirming the tracked states to be true QMBS. The apparent sub-volume entropy of the states tracked starts to increase around $t \approx 0.3$, marking the loss of the scarred nature from this point onwards. As discussed, the presence of QMBS can be further revealed by the persistent time oscillations of an out-of-equilibrium configuration. In order to unveil this aspect, we prepare an initial state given by the equal superposition of two states of the form (8) for small $t, h$ values, $|\psi(0)\rangle = \frac{1}{\sqrt{2}} (|S_0^1\rangle + |S_2^1\rangle)$. After this initialization, we let this state evolve with Hamiltonian (5) and we calculate the fidelity $F(\tau) = |\langle \psi(0)|\psi(\tau)\rangle|^2$, where $\tau$ is the time (in units of inverse $\mu$). At $t = 0.25$, we observe persistent oscillations of $F(\tau)$ as expected for QMBS states and this behavior persists until the end of path I. Beyond this limit ($t > 0.3$), low entanglement states disappear and $F(\tau)$ shows irregular oscillations around the mean value of about $1/L^2$, as expected for thermal states.

Figure 2 visualizes the presence of QMBS by showing the value of $S$ for all the eigenstates at $t = 0.2$ and $h = 0.5$, where the system is in the weak ergodic regime. The entropy still reveals a finger-like structure indicating the existence of a hidden, unidentified by us, symmetries (which is beyond the scope of the present model). States enclosed in circles are those tracked up from the near integrable limit. The members of the antimagron-like family $S_n^6$, denoted by green circles, have very low entanglement entropies as compared to other states of similar energy. They are thus truly QMBS. The inset reveals a sub-volume scaling of the entanglement entropy of $S_2^2$ state. The magnon excitations, on the other hand, dissolve among other states (in the high density of states region).

So far we have solely concentrated on path I, the properties of the followed states are quite similar for path II – c.f. Fig. 3. One can find QMBS like characteristics along this path as well, however, the $r$ value remains near Poissonian value indicating non-ergodic dynamics. Once one enters the region with GOE-like statistics (beyond the end of path II), the entanglement entropy of the followed states rapidly increases reaching almost the GOE limit.

In summary, motivated by recent predictions of finding QMBS states in a confined regime [55, 65] of LGTs, we investigated the thermalization properties of the gauged Kitaev chain. We observe that deep in the confined phase even states with the smallest entanglement entropy have relatively large, volume-law values indicating lack of QMBS. On the other hand, for relatively large values of the electric field, some states reveal low entanglement entropy. We can identify these states by following them from the very small $t, h$ values for which analytic predictions are available [13]. Such states result to be true QMBS and are found uniquely in the deconfined phase, which turns to be a weakly ergodic regime characterized by the GOE-like mean gap ratio. The presence of QMBS has been further verified by studying their time dynamics. By building an initial state given by the superposition of two QMBS, the fidelity of the state show pro-
nounced oscillations with no sign of thermalization. The adiabatic following of the states breaks down at the end points of our chosen paths due to energy level mixing, where interestingly QMBS also disappear. In conclusion, our results unambiguously reveal that QMBS occur even in deconfined regimes of LGTs, thus paving the way toward a deeper understanding of the connection existing between lack of thermalization and local symmetries.

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FIG. 4. The energy levels along the path-I. The regularly spaced levels split following the line $t = 0.2h$ up to $t = 0.1$. From there $h$ remains constant, $h = 0.5$ while $t$ increases up to $t \approx 0.3$ which is the point where all the high density states have high entanglement entropy ($S$).

SUPPLEMENTARY MATERIAL FOR SCAR STATES IN DECONFINED $Z_2$ LATTICE GAUGE THEORIES

Entanglement Entropy

We use the half-chain entanglement entropy, $S$, as a useful measure of eigenstates properties. The half-chain entanglement entropy is defined as the von Neumann entropy of the reduced density matrix $\rho_{L/2}$

$$S = -Tr[\rho_{L/2}\ln\rho_{L/2}], \quad (S.1)$$

where $\rho_{L/2} = Tr_{1,..,L/2}|\psi\rangle\langle\psi|$ is obtained by tracing out half of the system for eigenstate $|\psi\rangle$. The typical state in the ergodic regime obeys RMT prediction given by $S_{RMT} = (L/2)\ln(2) + (1/2 + \ln(1/2))/2 - 1/2$ [66–68].

A substantial difference from $S_{RMT}$ for states in the middle of the spectrum can be taken as a signature of nonergodic states, although to really observe a sub-volume property of the entanglement entropy for a given state one has to study the entropy dependence on the system size (see the inset in Fig. 2 in the main text).

Breaking of Adiabaticity

The initial state $|S_n^k(t_0, h_0)\rangle$ is identified from the mapping between the Hamiltonian of quantum Ising chain in both transverse and longitudinal fields (5) and the effective Hamiltonian (7) at $t_0, h_0 = 0.001$. The Hamiltonian, $H$, (5) is then diagonalized with the small parameter change of $\delta h = 0.001$ and $\delta t = 0.0002$ and a state $|E_n(t, h)\rangle$ is identified as the state which has a maximum overlap with $|S_n^k(t_0, h_0)\rangle$ among all the eigenstates of $H(t_0 + \delta t, h_0 + \delta h)$. To adiabatically follow the suspected QMBS state $|S_n^k(t, h)\rangle$ is updated along the paths.
But this is only done so, if for the next 10 diagonalizations (for a parameter change of $\Delta h = 0.01$ and $\Delta t = 0.002$) the eigenindex of the state $|E_n(t + \Delta t, h + \Delta h)\rangle$ does not change from its value at $|E_n(t, h)\rangle$. This procedure allows us to overcome problems encountered while passing the avoided crossings (see also below). The following protocol is continued until the final value of $h$ along a given path is reached. Later we continue updating $t$ only with $\delta t = 0.001$ while keeping $h$ constant.

This procedure runs into difficulties when avoided crossings are met. In the regime corresponding to mixed regular and chaotic dynamics (as revealed statistically by the gap ratio, $r$, being between Poissonian and GOE values) the avoided crossings are isolated. Such avoided crossings may be crossed diabatically with care. Once we reach the irregular regime multiple avoided crossings typically arise and only those states can be traced that are almost decoupled in the process of changing the parameters. Such a situation is well known from chaotic, single particle, strongly scarred systems such as quantum billiards or hydrogen atom in a strong magnetic field (see e.g. [69]). While in generalized time reversal invariant systems small avoided crossings are highly probable [70, 71] overlapping avoided crossings dissolve the scar character of the state unless a periodic orbit [27] or symmetry [30] reinforces this character after the crossing. While this intuition is based on a single particle systems, we expect the similar qualitative behavior for the possible preservation of QMBS.

FIG. 5. An enlarged image of Fig. 4 – top – restricted to the parameters where the peaks in entanglement entropy of the followed state and in $r$ value for the spectrum appear - compare Fig. 1(c). Note that the range of $h$ corresponds via $t = 0.2h$ relation defining path I to $t \in [0, 0.1]$. The orange dashed lines represents the location of $t$ values for the peaks. The bottom plot shows the integrable system states energies obtained by putting $t = 0$ in the Hamiltonian, (S.2).

FIG. 6. Attempt to follow the magnon type state $S_6^1$ along path I. Once gap ratio reaches values close to GOE limit the entropy of the followed state rapidly and significantly increases indicating that the scarred character is lost.

identifies that for $t, h \ll 1$ the states are ordered by number of dimers as given by the operator $D = \sum_{i=1}^L Z_i Z_{i+1}$. Once $h > 0$ those states form a manifold split by $\sum Z_i$ operator eigenvalues. Only at the crossings of the manifolds the term $t \sum_i X_i$ mixes eigenstates of $Z_i$ leading to nontrivial avoided crossings, compare Fig. 5. It is at these avoided crossings that sharp spikes visible in the entropy $t$ scans along the path I appear. Here also the mean gap ratio shows the unusual peaks.

For larger $t \in [0.1, 0.3]$ the $r$ value is GOE-like indicating an ergodic character, in this region only the QMBS states built out of antimagnons are sustained indicating these states to be characteristically different from the other states. At $t > 0.3$ the levels become strongly mixed and the Hilbert space splitting into characteristic bands disappears. Then all the tracked states loose their QMBS characteristics.
Magnon States

The scar states built out of magnon states which were located at the middle of the spectrum (in the region of a high density of states) tracked along path I were lost on entering the ergodic regime. For the path II both the magnon and antimagnon states for high density of states were lost on entering the confined regime which is then ergodic along the path. For the path I, only antimagnon states may be followed and linked to QMBS in the ergodic regime. The example of our attempt to track a magnon state along path I is summarized in Fig. 6.