BOUNDEDNESS AND STABILIZATION IN A TWO-SPECIES CHEMOTAXIS SYSTEM WITH SIGNAL ABSORPTION

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ABSTRACT. This paper is concerned with the Neumann initial-boundary value problem for the two-species chemotaxis system with consumption of chemoattractant

\[
\begin{aligned}
&u_t = \Delta u - \chi_1 \nabla \cdot (u \nabla w), \quad x \in \Omega, \ t > 0, \\
v_t = \Delta v - \chi_2 \nabla \cdot (v \nabla w), \quad x \in \Omega, \ t > 0, \\
w_t = \Delta w - (\alpha u + \beta v)w, \quad x \in \Omega, \ t > 0
\end{aligned}
\]

in a smooth bounded domain \( \Omega \subset \mathbb{R}^n \ (n \geq 2) \), where the parameters \( \chi_1, \chi_2, \alpha \) and \( \beta \) are positive. It is proved that if

\[
\max\{\chi_1, \chi_2\} \|w(x, 0)\|_{L^\infty(\Omega)} < \sqrt{\frac{2}{n \pi}}
\]

the problem possesses a unique global classical solution that is uniformly bounded. Moreover, we prove that

\[
u(x, t) \to \frac{1}{|\Omega|} \int_\Omega u(x, 0), \quad v(x, t) \to \frac{1}{|\Omega|} \int_\Omega v(x, 0) \quad \text{and} \quad w(x, t) \to 0 \quad \text{as} \ t \to \infty
\]

uniformly with respect \( x \in \Omega \).

1. INTRODUCTION

In this paper, we deal with the Neumann initial-boundary value problem for the chemotaxis system

\[
\begin{aligned}
&u_t = \Delta u - \chi_1 \nabla \cdot (u \nabla w), \quad x \in \Omega, \ t > 0, \\
v_t = \Delta v - \chi_2 \nabla \cdot (v \nabla w), \quad x \in \Omega, \ t > 0, \\
w_t = \Delta w - (\alpha u + \beta v)w, \quad x \in \Omega, \ t > 0, \\
\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, \quad x \in \partial \Omega, \ t > 0, \\
u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad w(x, 0) = w_0(x), \quad x \in \Omega
\end{aligned}
\]

in the bounded domain \( \Omega \subset \mathbb{R}^n \ (n \geq 2) \) with smooth boundary, where \( \chi_1, \chi_2, \alpha \) and \( \beta \) are positive constants, the unknown functions \( u = u(x, t) \) and \( v = v(x, t) \) stand for population densities and \( w = w(x, t) \) denotes the concentration of chemoattractant. The symbol \( \frac{\partial}{\partial \nu} \) represents differentiation with respect to the outward normal \( \nu \) on \( \partial \Omega \). The initial data \( u_0, v_0 \) and \( w_0 \) are given positive functions satisfying

\[
u_0 \in C^0(\Omega), \quad v_0 \in C^0(\Omega) \quad \text{and} \quad w_0 \in W^{1,q}(\Omega)
\]

with some \( q > n \).

The model (1.1) is used in mathematical biology to describe the movement of two populations in respond to the concentration gradient of one common chemical signal. It is a generalization of
the famous Keller-Segel system [8]

\[
\begin{cases}
  u_t = \Delta u - \chi \nabla \cdot (u \nabla w), \\
  w_t = \Delta w - uw,
\end{cases}
\]  

(1.3)

which has been studied from a mathematical viewpoint in the last years. It is proved that if \( n \leq 2 \) or

\[
\chi \|w(x,0)\|_{L^\infty(\Omega)} < \frac{1}{6(n+1)}
\]

in higher dimensions \( n \geq 3 \), the problem (1.3) possesses a unique global classical solution which is bounded and satisfies

\[
u(x,t) \to \frac{1}{|\Omega|} \int_\Omega u_0 \quad \text{and} \quad w(x,t) \to 0 \quad \text{as} \ t \to \infty\)

(1.4)

uniformly with respect to \( x \in \Omega \) [14,15,26]. Moreover, the problem admits at least one global weak solution which is eventually smooth and enjoys the convergence properties (1.4) in bounded convex domain \( \Omega \subset \mathbb{R}^3 \) [15]. Recently, some blow-up properties of (1.1), including blow-up criteria for the local classical solution, lower global blow-up estimate on \( \|u\|_{L^\infty(\Omega)} \) and local non-degeneracy property for the blow-up points, have been obtained in [6]. Furthermore, global solutions and the stabilization for the corresponding variants of (1.3), such as chemotaxis-consumption systems with tensor-valued sensitivities [9,11,22,25] and singular sensitivities [24], system (1.3) with logistic source [1,10] or coupled chemotaxis-fluid system [12,20,21,23] have also been investigated. For more results on the model variations of (1.3), we refer to the recent survey [2] and the references therein.

As to the problem (1.1) with Lotka-Volterra competitive kinetics

\[
\begin{cases}
  u_t = \Delta u - \chi_1 \nabla \cdot (u \nabla w) + \mu_1 u(1 - u - a_1 v), \\
  v_t = \Delta v - \chi_2 \nabla \cdot (v \nabla w) + \mu_2 v(1 - a_2 u - v), \\
  w_t = \Delta w - (\alpha u + \beta v) w, \\
  \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, \\
  u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad w(x,0) = w_0(x), \quad x \in \Omega,
\end{cases}
\]

it is shown that solutions exist globally and remain bounded if either \( n = 2 \) [5] or

\[
\chi_i \|w_0\|_{L^\infty(\Omega)} < \frac{\pi}{\sqrt{n+1}}, \quad i = 1,2
\]

in the case \( n \geq 3 \) [16], and if \( a_1, a_2 \in (0,1) \) these solutions satisfy

\[
(u(\cdot,t), v(\cdot,t), w(\cdot,t)) \to \left( \frac{1-a_1}{1-a_1a_2}, \frac{1-a_2}{1-a_1a_2}, 0 \right) \quad \text{in} \ L^\infty(\Omega)
\]

and if \( a_1 \geq 1 > a_2 \),

\[
(u(\cdot,t), v(\cdot,t), w(\cdot,t)) \to (0,1,0) \quad \text{in} \ L^\infty(\Omega)
\]

as \( t \to \infty \) [5,16]. On the other hand, the global existence and stabilization of (weak) solutions to the two-species chemotaxis-fluid system with Lotka-Volterra competitive kinetics have also been established, see e.g. [3–5,7]. However, to the best of our knowledge, the problem (1.1) seems not be investigated yet in the literature.

In the present paper, we prove global existence, boundedness and stabilization of classical solutions to the problem (1.1). Our main results are stated as follows.
Theorem 1.1. Let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be bounded domain with smooth boundary and let the parameters $\chi_1, \chi_2, \alpha, \beta > 0$. If
$$\chi_i \|u_0\|_{L^\infty(\Omega)} < \sqrt{\frac{2}{n}}, \quad i = 1, 2,$$
then for any initial data $u_0$, $v_0$ and $w_0$ satisfying (1.2), the problem (1.1) possesses a unique global classical solution which is bounded in $\Omega \times (0, \infty)$.

The second result concerns stabilization of the solution provided by Theorem 1.1.

Theorem 1.2. Under the assumptions of Theorem 1.1, the global classical solution of (1.1) satisfies
$$\|u(\cdot, t) - \tilde{u}_0\|_{L^\infty(\Omega)} \to 0,$$
$$\|v(\cdot, t) - \tilde{v}_0\|_{L^\infty(\Omega)} \to 0,$$
$$\|w(\cdot, t)\|_{L^\infty(\Omega)} \to 0$$
as $t \to \infty$, where $\tilde{u}_0 := \frac{1}{|\Omega|} \int_{\Omega} u_0$ and $\tilde{v}_0 := \frac{1}{|\Omega|} \int_{\Omega} v_0$.

The rest of the paper is organized as follows. In Section 2, we list some preliminaries. Section 3 is devoted to the proof of Theorem 1.1. Finally, we prove Theorem 1.2 in Section 4.

2. Preliminaries

As a preparation to the proof, we first state one result concerning local existence of classical solution to the problem (1.1).

Lemma 2.1. Let $u_0$, $v_0$ and $w_0$ satisfy (1.2), and let $\chi_1, \chi_2, \alpha, \beta > 0$. Then there exist $T_{\text{max}} \leq \infty$ and a uniquely determined triple $(u, v, w)$ of nonnegative functions
$$u \in C^0(\bar{\Omega} \times [0, T_{\text{max}})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\text{max}})),$$
$$v \in C^0(\bar{\Omega} \times [0, T_{\text{max}})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\text{max}})),$$
$$w \in C^0(\bar{\Omega} \times [0, T_{\text{max}})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\text{max}})) \cap L^\infty(\Omega, T_{\text{max}}); W^{1,q}(\Omega)),$$
which solves (1.1) in the classical sense. If $T_{\text{max}} < \infty$, then
$$\|u(t)\|_{L^\infty(\Omega)} + \|v(t)\|_{L^\infty(\Omega)} + \|w(t)\|_{W^{1,q}(\Omega)} \to \infty \quad \text{as} \quad t \nearrow T_{\text{max}}.$$
Furthermore, the solution $(u, v, w)$ satisfies
$$\|u(t)\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)} \quad \text{for all} \quad t \in (0, T_{\text{max}}) \quad (2.1)$$
and
$$\|v(t)\|_{L^1(\Omega)} = \|v_0\|_{L^1(\Omega)} \quad \text{for all} \quad t \in (0, T_{\text{max}}) \quad (2.2)$$
as well as
$$0 \leq w \leq \|w_0\|_{L^\infty(\Omega)} \quad \text{in} \quad \bar{\Omega} \times [0, T_{\text{max}}). \quad (2.3)$$

Proof. The local existence and regularity of the solution is based on standard contraction mapping arguments and parabolic regularity theory, which can be found in [19, Lemma 1.1](see also [14, Lemma 2.1]). Integrating the first and the second equations in (1.1), we immediately obtain
$$\frac{d}{dt} \int_{\Omega} u(x, t) dx = 0 \quad \text{and} \quad \frac{d}{dt} \int_{\Omega} v(x, t) dx = 0$$
which yield (2.1) and (2.2). The statement (2.3) follows from an application of the maximum principle.

The following lemma is a generalization of [25, Lemma 3.2], which asserts that a bound for $\|u\|_{L^p(\Omega)}$ and $\|v\|_{L^p(\Omega)}$ with $p > \frac{n}{2}$ for all $t \in (0, T_{\text{max}})$ can guarantee the global existence and boundedness of classical solutions to (1.1).
Lemma 2.2. Suppose that the initial data \( u_0, v_0 \) and \( w_0 \) satisfy (1.2) and the parameters \( \chi_1, \chi_2, \alpha, \beta > 0 \). Let \( p \geq 1 \). If the first and second components of solution satisfy
\[
\sup_{t \in (0, T_{\max})} \left( \|u(t)\|_{L^p(\Omega)} + \|v(t)\|_{L^p(\Omega)} \right) < \infty,
\]
for some \( p > \frac{n}{2} \), then the solution of (1.1) is global in time. Moreover, the solution fulfills
\[
\sup_{t \in (0, \infty)} \left( \|u(t)\|_{L^\infty(\Omega)} + \|v(t)\|_{L^\infty(\Omega)} + \|w(t)\|_{W^{1,q}(\Omega)} \right) < \infty.
\]

Proof. Since our assumption \( q > n \) and for each fixed \( p > \frac{n}{2} \) there holds
\[
\frac{np}{(n-p)_+} = \begin{cases}
\infty, & \text{if } p \geq n, \\
\frac{np}{n-p} > n, & \text{if } \frac{n}{2} < p < n,
\end{cases}
\]
it is possible to find \( 1 < p_0 < q \) fulfilling
\[
n < p_0 < \frac{np}{(n-p)_+},
\]
which enables us to choose \( k > 1 \) such that
\[
n < kp_0 < \frac{np}{(n-p)_+} \quad \text{and} \quad kp_0 \leq q.
\]
We shall argue by contradiction. Assume that \( T_{\max} < \infty \). Applying the variation-of-constants formula
\[
w(\cdot, t) = e^{t\Delta} w_0 - \int_0^t e^{(t-s)\Delta} (\alpha u(\cdot, s) + \beta v(\cdot, s)) w(\cdot, s) ds,
\]
we get
\[
\|
abla w(\cdot, t)\|_{L^{p_0}(\Omega)} \leq \left\| \nabla e^{t\Delta} w_0 \right\|_{L^{p_0}(\Omega)} + \int_0^t \left\| \nabla e^{(t-s)\Delta} (\alpha u(\cdot, s) + \beta v(\cdot, s)) w(\cdot, s) \right\|_{L^{p_0}(\Omega)} ds
\]
for all \( t \in (0, T_{\max}) \). By standard smoothing estimates for the Neumann heat semigroup [18, Lemma 1.3], (2.3) and \( kp_0 \leq q \) due to (2.6), we obtain positive constants \( C_1 \) and \( C_2 \) such that
\[
\|
abla w(\cdot, t)\|_{L^{p_0}(\Omega)} \leq C_1 \|w_0\|_{W^{1,q}(\Omega)} + C_2 \int_0^t \left( 1 + (t-s)^{-\frac{1}{2}} \left[ \frac{1}{2} \frac{4}{p} - \frac{1}{kp_0} \right] \right) e^{-\frac{\lambda_1}{2}(t-s)}
\times \left\| (\alpha u(\cdot, s) + \beta v(\cdot, s)) w(\cdot, s) \right\|_{L^p(\Omega)} ds
\leq C_1 \|w_0\|_{W^{1,q}(\Omega)} + C_2 \|w_0\|_{L^\infty(\Omega)} \int_0^t \left( 1 + (t-s)^{-\frac{1}{2}} \left[ \frac{1}{2} \frac{4}{p} - \frac{1}{kp_0} \right] \right) e^{-\frac{\lambda_1}{2}(t-s)}
\times \left( \|u(\cdot, s)\|_{L^p(\Omega)} + \|v(\cdot, s)\|_{L^p(\Omega)} \right) ds
\]
for all \( t \in (0, T_{\max}) \). Here and below, \( \lambda_1 > 0 \) denotes the first nonzero eigenvalue of \(-\Delta\) in \( \Omega \) under homogeneous Neumann boundary conditions. Since \( \frac{1}{2} + \frac{1}{2} \left[ \frac{4}{p} - \frac{1}{kp_0} \right] < 1 \) by the right-hand side of the first inequality in (2.6) and (2.4), we can take constants \( C_3 > 0 \) and \( C_4 > 0 \) fulfilling
\[
\|
abla w(\cdot, t)\|_{L^{p_0}(\Omega)} \leq C_1 \|w_0\|_{W^{1,q}(\Omega)} + C_3 \left( \sup_{t \in (0, T_{\max})} \left( \|u(t)\|_{L^p(\Omega)} + \|v(t)\|_{L^p(\Omega)} \right) \right)
\leq C_4 \quad \text{for all } t \in (0, T_{\max}).
\]

Next by the variation-of-constants formula
\[
u(\cdot, t) = e^{t\Delta} u_0 - \chi_1 \int_0^t \nabla e^{(t-s)\Delta} u(\cdot, s) \nabla w(\cdot, s) ds,
\]
we have
\[ \|u(\cdot, t)\|_{L^\infty(\Omega)} \leq \|e^{t\Delta}u_0\|_{L^\infty(\Omega)} + \chi_1 \int_0^t \|\nabla e^{(t-s)\Delta} u(\cdot, s)\nabla w(\cdot, s)\|_{L^\infty(\Omega)} ds \]
for all \( t \in (0, T_{\text{max}}) \). In view of the maximum principle and smoothing estimates for the Neumann heat semigroup [18, Lemma 1.3], we obtain \( C_5 > 0 \) satisfying
\[ \|u(\cdot, t)\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\Omega)} + C_5 \int_0^t \left(1 + (t-s)^{-\frac{1}{2} - \frac{s}{4p\rho_0}}\right) e^{-\lambda_1(t-s)} \|u(\cdot, s)\nabla w(\cdot, s)\|_{L^{p\rho_0}(\Omega)} ds \quad (2.8) \]
for all \( t \in (0, T_{\text{max}}) \). Here by the Hölder inequality, interpolation inequality, (2.1) and (2.7), we can find find \( C_6 > 0 \) such that
\[ \|u(\cdot, s)\nabla w(\cdot, s)\|_{L^{p\rho_0}(\Omega)} \leq \|u(\cdot, s)\|_{L^{k'\rho_0}(\Omega)} \|\nabla w(\cdot, s)\|_{L^{k\rho_0}(\Omega)} \leq \|u(\cdot, s)\|^{1-r}_{L^r(\Omega)} \|\nabla w(\cdot, s)\|^{1-\sigma}_{L^{\sigma}(\Omega)} \]
for all \( s \in (0, T_{\text{max}}) \), where \( k' \) is the dual exponent of \( k \) and \( r = 1 - \frac{1}{k'\rho_0} \in (0, 1) \). Inserting this into (2.8), it follows that
\[ \sup_{t \in (0, T)} \|u(\cdot, t)\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\Omega)} + C_7 \sup_{t \in (0, T)} \|u(\cdot, t)\|^{r\rho_0}_{L^\infty(\Omega)} \quad \text{for all } T \in (0, T_{\text{max}}) \]
with
\[ C_7 = C_5 C_6 \int_0^\infty \left(1 + \sigma^{-\frac{1}{2} - \frac{s}{4p\rho_0}}\right) e^{-\lambda_1 \sigma} d\sigma \]
is finite thanks to the left-hand side of (2.5). Therefore, there exists a constant \( C_8 > 0 \) such that
\[ \|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C_8 \quad \text{for all } t \in (0, T_{\text{max}}). \]
Arguing similarly as above, we see that
\[ \|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C_9 \quad \text{for all } t \in (0, T_{\text{max}}). \]
with some \( C_9 > 0 \). This is a contradiction to Lemma 2.1. Hence we complete the proof. \( \Box \)

For the proof of the main result we also need the following technical lemma which provides the desired weight function.

**Lemma 2.3.** Let \( \varepsilon \in (0, 1) \) and \( p > 1 \). Define the function
\[ \varphi(s) := e^{z(s)}, \quad 0 \leq s \leq M, \]
where
\[ z(s) := \frac{b}{2c}s + \frac{\sqrt{4ac - b^2}}{2c} \int_0^s \tan \left( \frac{\sqrt{4ac - b^2}}{2d} \tau + \arctan \frac{b}{\sqrt{4ac - b^2}} \right) d\tau \]
with \( a = (p - 1)^2 \), \( b = -4(p - 1)\varepsilon \), \( c = \frac{4}{p}(1 + (p - 1)\varepsilon) \) and \( d = \frac{2}{p}(p - 1)(1 - \varepsilon) \). If there holds
\[ M < \frac{2}{\sqrt{p}} \sqrt{\frac{1 - \varepsilon}{1 + p\varepsilon}} \left( \frac{\pi}{2} + \arctan \sqrt{\frac{p}{1 + (p - 1)\varepsilon - p\varepsilon^2}} \right), \quad (2.9) \]
then the function \( \varphi(s) \) is well defined and satisfies the following conditions
\[ \varphi'(s) \geq 0, \quad (2.10) \]
\[ 1 \leq \varphi(s) \leq \varphi(M), \quad (2.11) \]
\[ \frac{1}{p} \varphi''(s) - \varphi'(s) \geq 0 \quad (2.12) \]
and
\[ |(p - 1)\varphi(s) - 2\varphi'(s)| - 2\sqrt{(p - 1)(1 - \varepsilon)}\varphi(s) \left( \frac{1}{p}\varphi''(s) - \varphi'(s) \right) = 0 \quad (2.13) \]
for all \(0 \leq s \leq M\).

**Proof.** First of all we remark that for any \(\varepsilon \in (0, 1)\)
\[ 4ac - b^2 = \frac{16(p - 1)^2}{p} (1 + (p - 1)\varepsilon - p\varepsilon^2) > 0. \]
Owing to (2.9), for \(0 \leq s \leq M\) we get
\[ \sqrt{\frac{4ac - b^2}{2d}} s + \arctan \frac{b}{\sqrt{4ac - b^2}} \geq -\arctan \sqrt{\frac{p}{1 + (p - 1)\varepsilon - p\varepsilon^2}} > -\frac{\pi}{2} \]
and
\[ \frac{\sqrt{4ac - b^2}}{2d} s + \arctan \frac{b}{\sqrt{4ac - b^2}} \leq \frac{\sqrt{p}}{2} \sqrt{\frac{1 + p\varepsilon}{1 - \varepsilon}} M - \arctan \sqrt{\frac{p}{1 + (p - 1)\varepsilon - p\varepsilon^2}} \]
\[ < \frac{\sqrt{p}}{2} \sqrt{\frac{1 + p\varepsilon}{1 - \varepsilon}} \left( \frac{2}{\sqrt{p}} \sqrt{\frac{1 - \varepsilon}{1 + p\varepsilon}} \left( \frac{\pi}{2} + \arctan \sqrt{\frac{p}{1 + (p - 1)\varepsilon - p\varepsilon^2}} \right) \right) \]
\[ - \arctan \sqrt{\frac{p}{1 + (p - 1)\varepsilon - p\varepsilon^2}} \]
\[ = \frac{\pi}{2}. \]

Therefore, the function \(\varphi(s)\) is well defined for \(0 \leq s \leq M\). A direct computation reveals that
\[ \varphi'(s) = \varphi(s)z'(s) \]
\[ = \varphi(s) \left( -\frac{b}{2c} + \frac{\sqrt{4ac - b^2}}{2c} \left( \tan \left( \frac{\sqrt{4ac - b^2}}{2d} s + \arctan \frac{b}{\sqrt{4ac - b^2}} \right) \right) \right) \]
\[ \geq \varphi(s)z'(0) \]
\[ = 0 \quad \text{for all} \ 0 \leq s \leq M. \]

Then the relation (2.11) is obvious according to (2.10). Since \(\varphi''(s) = \varphi(s) \left( z''(s) + (z'(s))^2 \right)\) and
\[ z''(s) = \frac{4ac - b^2}{4cd} \left( 1 + \tan^2 \left( \frac{\sqrt{4ac - b^2}}{2d} s + \arctan \frac{b}{\sqrt{4ac - b^2}} \right) \right) \]
\[ = \frac{4ac - b^2}{4cd} \left( 1 + \frac{4c^2}{4ac - b^2} \left( z'(s) + \frac{b}{2c} \right)^2 \right) \]
\[ = \frac{1}{d} \left( a + bz'(s) + c(z'(s))^2 \right), \]
we have
\[
\frac{1}{p} \varphi''(s) - \varphi'(s) = \left( \frac{1}{p} \left( (z''(s) + (z'(s))^2) - z'(s) \right) \right) \varphi(s)
\]
\[
= \frac{1}{p} \left( \left( \frac{c}{d} + 1 \right) (z'(s))^2 + \left( \frac{b}{d} - p \right) z'(s) + \frac{a}{d} \right) \varphi(s)
\]
\[
= \frac{1}{4(p-1)(1-\varepsilon)} \varphi(s)( (p-1) - 2z'(s))^2
\]
which proves (2.12) and (2.13). 

\[\square\]

3. Global existence. Proof of Theorem 1.1

With the preliminaries at hand, we are now prepared to prove global existence and boundedness in (1.1). In the following lemma we establish \(L^p\)-bounds for \(u\). The method of the proof is a modification of an idea in [17] (see also, e.g. [1, 14, 25]).

**Lemma 3.1.** Let \( p > 1 \) and \( \varepsilon \in (0,1) \). Assume that the initial functions \( u_0, v_0 \) and \( w_0 \) fulfill (1.2). If the assumption (2.9) holds with \( M = \max \{ \chi_1, \chi_2 \} \|w_0\|_{L^\infty(\Omega)} \), then there exists \( C(p, \varepsilon) > 0 \) such that for each \( \varepsilon \in (0,1) \) the first and second components of the solution for the system (1.1) satisfy
\[
\|u(\cdot, t)\|_{L^p(\Omega)} \leq C(p, \varepsilon) \quad \text{for all} \quad t \in (0, T_{\text{max}})
\]
and
\[
\|v(\cdot, t)\|_{L^p(\Omega)} \leq C(p, \varepsilon) \quad \text{for all} \quad t \in (0, T_{\text{max}}).
\]

**Proof.** Let \( \varphi(s) \) be the function defined in Lemma 2.3 and set \( w_1 := \chi_1 w \). Using the first equation and the third equation in (1.1), we integrate by parts to obtain
\[
\frac{1}{p} \frac{d}{dt} \int_\Omega u^p \varphi(w_1) = \int_\Omega u^{p-1} \varphi(w_1) u_t + \frac{1}{p} \int_\Omega u^p \varphi'(w_1)(w_1)_t
\]
\[
= \int_\Omega u^{p-1} \varphi(w_1)(\Delta u - \nabla \cdot (u \nabla w_1)) + \frac{1}{p} \int_\Omega u^p \varphi'(w_1)(\Delta w_1 - (\alpha u + \beta v)w_1)
\]
\[
= -(p-1) \int_\Omega u^{p-2} \varphi(w_1)|\nabla u|^2 - \int_\Omega u^{p-1} \varphi'(w_1) \nabla u \cdot \nabla w_1
\]
\[
+ (p-1) \int_\Omega u^{p-1} \varphi(w_1) \nabla u \cdot \nabla w_1 + \int_\Omega u^p \varphi'(w_1)|\nabla w_1|^2
\]
\[
- \int_\Omega u^{p-1} \varphi'(w_1) \nabla u \cdot \nabla w_1 - \frac{1}{p} \int_\Omega u^p \varphi''(w_1)|\nabla w_1|^2
\]
\[
- \frac{1}{p} \int_\Omega u^p \varphi'(w_1)(\alpha u + \beta v)w_1
\]
for all \( t \in (0, T_{\text{max}}) \). Fix \( \varepsilon \in (0,1) \). Due to (2.10), it follows that
\[
\frac{1}{p} \frac{d}{dt} \int_\Omega u^p \varphi(w_1) + (p-1)\varepsilon \int_\Omega u^{p-2} \varphi(w_1)|\nabla u|^2
\]
\[
\leq -(p-1)(1-\varepsilon) \int_\Omega u^{p-2} \varphi(w_1)|\nabla u|^2 + \int_\Omega |(p-1)\varphi(w_1) - 2\varphi'(w_1)|u^{p-1}|\nabla u||\nabla w_1|
\]
\[
- \int_\Omega \left( \frac{1}{p} \varphi''(w_1) - \varphi'(w_1) \right) u^p |\nabla w_1|^2 \quad \text{for all} \quad t \in (0, T_{\text{max}, \varepsilon}).
\]
Applying (2.1), (2.11), the Gagliardo-Nirenberg inequality and Young’s inequality, we can find $C_1 > 0$ and $C_2 > 0$ such that
\[
\int_\Omega u^p \varphi(w_1) \leq \varphi \left( c_1 \|w_0\|_{L^\infty(\Omega)} \right) \|u^\frac{p}{2}\|_{L^2(\Omega)}^2
\leq C_1 \left( \|\nabla u^\frac{p}{2}\|_{L^2(\Omega)}^{2a} \|u^\frac{p}{2}\|_{L^2(\Omega)}^{2(1-a)} + \|u^\frac{p}{2}\|_{L^2(\Omega)}^2 \right)
\leq \frac{4(p-1)}{p^2} \varepsilon \|\nabla u^\frac{p}{2}\|_{L^2(\Omega)}^2 + C_2
\leq (p-1)\varepsilon \int_\Omega u^{p-2} \varphi(w_1)|\nabla u|^2 + C_2
\]
with
\[
a = \frac{\frac{p}{2} - \frac{1}{n}}{\frac{p}{2} + \frac{1}{n}} \in (0, 1).
\]
This combined with (3.3) implies
\[
\frac{1}{p} \frac{d}{dt} \int_\Omega u^p \varphi(w_1) + \int_\Omega u^p \varphi(w_1)
\leq -(p-1)(1-\varepsilon) \int_\Omega u^{p-2} \varphi(w_1)|\nabla u|^2 + \int_\Omega |(p-1)\varphi(w_1) - 2\varphi'(w_1)|u^{p-1}|\nabla u||\nabla w_1|
\leq \int_\Omega \left( \frac{1}{p} \varphi''(w_1) - \varphi'(w_1) \right) u^p |\nabla w_1|^2 + C_2 \quad \text{for all } t \in (0, T_{\text{max}}).
\]
By (2.12), we can rewrite it as
\[
\frac{1}{p} \frac{d}{dt} \int_\Omega u^p \varphi(w_1) + \int_\Omega u^p \varphi(w_1)
\leq -(p-1)(1-\varepsilon) \int_\Omega u^{p-2} \varphi(w_1)|\nabla u|^2 - \left( \frac{1}{p} \varphi''(w_1) - \varphi'(w_1) \right) u^\frac{p}{2} |\nabla w_1|^2
\quad \Phi(w_1) u^{p-1} |\nabla u||\nabla w_1| + C_2
\leq C_2 \quad \text{for all } t \in (0, T_{\text{max}}),
\]
where
\[
\Phi(w_1) := |(p-1)\varphi(w_1) - 2\varphi'(w_1)| - 2 \sqrt{(p-1)(1-\varepsilon)\varphi(w_1) \left( \frac{1}{p} \varphi''(w_1) - \varphi'(w_1) \right) u^\frac{p}{2} |\nabla w_1|^2}
= 0
\]
for all $0 \leq w_1 \leq c_1 \|w_0\|_{L^\infty(\Omega)}$ due to (2.13). Therefore, in view of a standard ODE argument, we prove (3.1). Using the second equation and the third equation in (1.1), proceeding similarly as for (3.1), we get (3.2) and hence completes the proof. 

Our main result on the boundedness of the solution in this paper is an immediate consequence of Lemma 2.2 and Lemma 3.1.

**Proof of Theorem 1.1.** In view of Lemma 2.2 and Lemma 3.1, it is suffices to verify (2.9) with $M = M_i := c_i \|w_0\|_{L^\infty(\Omega)}$, some $p_i > \frac{n}{2}$ and fixed $\varepsilon_i \in (0, 1)$ for $i = 1, 2$. Since by our assumption $M_i < \sqrt{\frac{\pi^2}{n}}$, we can choose
\[
\varepsilon_i := \frac{\pi^2 - \frac{n}{2} M_i^2}{2 \left( \pi^2 + \left( \frac{n}{2} M_i^2 \right) \right)} \in (0, 1)
\]
such that
\[ M_i = \frac{2}{\sqrt{p_i}} \sqrt{\frac{1 - \varepsilon_i}{1 + \varepsilon_i} \frac{\pi}{2}} < \frac{2}{\sqrt{\frac{1 - \varepsilon_i}{1 + \varepsilon_i} \frac{\pi}{2}}} \]
for \( i = 1, 2 \). So that there exists \( p_i > \frac{n}{2} \) such that
\[ M_i = \frac{2}{\sqrt{p_i}} \sqrt{\frac{1 - \varepsilon_i}{1 + \varepsilon_i} \frac{\pi}{2}} < \frac{2}{\sqrt{\frac{1 - \varepsilon_i}{1 + \varepsilon_i} \frac{\pi}{2}}} \]
for \( i = 1, 2 \). This completes the proof. \( \square \)

4. Stabilization. Proof of Theorem 1.2

In this section we consider the asymptotic behavior of the global bounded solutions to (1.1).

**Lemma 4.1.** Let the assumptions in Theorem 1.1 hold. Then there exists a constant \( C > 0 \) such that
\[ \int_0^\infty \int_\Omega |\nabla u|^2 \leq C \quad \text{and} \quad \int_0^\infty \int_\Omega |\nabla v|^2 \leq C. \]  \hspace{1cm} (4.1)

**Proof.** Multiplying the third equation in (1.1) by \( w \) and integrating by parts, we obtain
\[ \frac{1}{2} \frac{d}{dt} \int_\Omega w^2 + \int_\Omega |\nabla w|^2 = -\int_\Omega (\alpha u + \beta v) w \quad \text{for all} \ t > 0. \]
Since \( \alpha, \beta, u, v \) and \( w \) are all nonnegative, a time integration over \((0, T)\) yields
\[ \int_0^T \int_\Omega |\nabla w|^2 \leq \frac{1}{2} \int_\Omega w_0^2 \quad \text{for all} \ T > 0. \]  \hspace{1cm} (4.2)
We respectively test the first and second equations in (1.1) by \( u \) and \( v \) to obtain
\[ \frac{1}{2} \frac{d}{dt} \int_\Omega u^2 + \int_\Omega |\nabla u|^2 = \chi_1 \int_\Omega u \nabla u \cdot \nabla w \leq \frac{1}{2} \int_\Omega |\nabla u|^2 + \frac{1}{2} \chi_1^2 \|u\|^2_{L^\infty(\Omega \times (0, \infty))} \int_\Omega |\nabla w|^2 \]
and
\[ \frac{1}{2} \frac{d}{dt} \int_\Omega v^2 + \int_\Omega |\nabla v|^2 = \chi_2 \int_\Omega v \nabla v \cdot \nabla w \leq \frac{1}{2} \int_\Omega |\nabla v|^2 + \frac{1}{2} \chi_2^2 \|v\|^2_{L^\infty(\Omega \times (0, \infty))} \int_\Omega |\nabla w|^2 \]
for all \( t > 0 \), where we have used Young’s inequality. Integrating over \((0, T)\) and applying (4.2) readily imply (4.3). \( \square \)

Based on the parabolic regularity argument [11, Lemma 4.3], we have the following Hölder estimates of \( u \) and \( v \).

**Lemma 4.2.** Under the assumptions of Theorem 1.1, there exists \( \theta > 0 \) and \( C > 0 \) such that
\[ \|u\|_{C^{\frac{\theta}{2}}(\Omega \times [t, t+1])} \leq C \quad \text{and} \quad \|v\|_{C^{\frac{\theta}{2}}(\Omega \times [t, t+1])} \leq C \]  \hspace{1cm} (4.3)
for all \( t \geq 1 \).
Proof. We first note that
\[(\nabla u - \chi_1 u \nabla w) \cdot \nabla u = |\nabla u|^2 - \chi_1 u \nabla u \cdot \nabla w \geq \frac{1}{2} |\nabla u|^2 - \frac{1}{2} \chi_1^2 \|u\|_{L^\infty(\Omega \times (0, \infty))}^2 |\nabla w|^2\]
and
\[|\nabla u - \chi_1 u \nabla w| \leq |\nabla u| + \chi_1 \|u\|_{L^\infty(\Omega \times (0, \infty))} |\nabla w|\].
Since \[|\nabla w|^2 \in L^\infty((0, \infty); L^2)\] with \(q > n\), an application of standard parabolic regularity estimates \[13, \text{Theorem 1.3}\] yields \(\theta_1 > 0\) and \(C_1 > 0\) such that
\[\|u\|_{C^{\theta_1, \frac{\theta_1}{2}}(\tilde{\Omega} \times [t, t+1])} \leq C_1 \text{ for all } t \geq 1.\]
By a similar reasoning, we have \(\theta_2 > 0\) and \(C_2 > 0\) fulfilling
\[\|v\|_{C^{\theta_2, \frac{\theta_2}{2}}(\tilde{\Omega} \times [t, t+1])} \leq C_2 \text{ for all } t \geq 1.\]
This completes the proof. □

In the proof of the stabilization of the solution, we shall need the following common statement which is proved in \[5, \text{Lemma 4.6}\].

**Lemma 4.3.** Let \(n \in C^0(\bar{\Omega} \times [0, \infty))\) satisfy that there exist \(C > 0\) and \(\theta \in (0, 1)\) such that
\[\|n\|_{C^{\theta, \frac{\theta}{2}}(\bar{\Omega} \times [t, t+1])} \leq C \text{ for all } t \geq 1.\]
If
\[\int_0^\infty \int_{\Omega} (n(x, t) - N)^2 < \infty\]
with some constant \(N > 0\), then we have
\[n(\cdot, t) \to N \text{ in } C^0(\bar{\Omega}) \text{ as } t \to \infty.\]

With the above lemmas at hand, we are now in the position to prove our main result on stabilization of solutions.

**Proof of Theorem 1.2.** We apply the Poincaré inequality
\[\left\|\varphi - \frac{1}{|\Omega|} \int_{\Omega} \varphi\right\|_{L^2(\Omega)}^2 \leq C(n, \Omega) \|\nabla \varphi\|_{L^2(\Omega)}^2, \quad \varphi \in W^{1,2}(\Omega)\]
and (4.3) to get \(C_1 > 0\) such that
\[\int_0^\infty \int_{\Omega} |u(x, t) - \bar{u}_0|^2 \leq C(n, \Omega) \int_0^\infty \int_{\Omega} |\nabla u|^2 \leq C_1\]
and
\[\int_0^\infty \int_{\Omega} |v(x, t) - \bar{v}_0|^2 \leq C(n, \Omega) \int_0^\infty \int_{\Omega} |\nabla v|^2 \leq C_1.\]
By means of Lemma 4.2 and Lemma 4.3 we thereby obtain that
\[u(\cdot, t) \to \bar{u}_0 \quad \text{and} \quad v(\cdot, t) \to \bar{v}_0 \quad \text{in } C^0(\bar{\Omega})\]
as \(t \to \infty\). Next we prove the decay property of \(w\). Aided by (4.4), we have \(T > 0\) such that
\[u(\cdot, t) \geq \frac{\bar{u}_0}{2} \quad \text{and} \quad v(\cdot, t) \geq \frac{\bar{v}_0}{2}\]
for all $t \geq T$. Then from the third equation in (1.1), we get

$$w_t \leq \Delta w - \frac{1}{2}(\alpha \bar{u}_0 + \beta \bar{v}_0)w$$

for all $t \geq T$,

which yields

$$w(\cdot, t) \leq \|w(\cdot, T)\|_{L^\infty(\Omega)} e^{-\frac{1}{2}(\alpha \bar{u}_0 + \beta \bar{v}_0)(t-T)}$$

for all $t \geq T$,

and thereby completes the proof.

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