Relativistic Ohm and Fourier laws for binary mixtures of electrons with protons and photons

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Abstract. Binary mixtures of electrons with protons and of electrons with photons subjected to external electromagnetic fields are analyzed by using the Anderson and Witting model equation. The relativistic laws of Ohm and Fourier are determined as well as general expressions for the electrical and thermal conductivities for relativistic ionized gas mixtures. Explicit expressions for the transport coefficients are given for the particular cases: a non-relativistic mixture of protons and non-degenerate electrons; an ultra-relativistic mixture of photons and non-degenerate electrons; a non-relativistic mixture of protons and completely degenerate electrons; an ultra-relativistic mixture of photons and completely degenerate electrons and a mixture of non-relativistic protons and ultra-relativistic completely degenerate electrons.

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INTRODUCTION

The analysis of non-relativistic and relativistic ionized gases by using the Boltzmann equation is a very difficult subject, since it refers to a system of coupled nonlinear integro-differential equations for the distribution functions. Simpler model equations for the collision term have been proposed in the literature in order to overcome the difficulties of the Boltzmann integro-differential equation. The model equations simplify the structure of the collision term but maintain its basic properties. For the non-relativistic Boltzmann equation the most widely known model is the BGK model which was formulated independently by Bhatnagar, Gross and Krook [1] and Welander [2]. The first extension of the non-relativistic BGK model to the relativistic case was proposed by Marle [3]. Although the non-relativistic limiting case of the Marle’s model recovers the non-relativistic BGK model, in the case of particles with zero rest mass the relaxation time of the distribution function tends to infinity. This shortcoming was found by Anderson and Witting [4] who proposed a new model equation.

In this work we follow [5] and analyze binary mixtures of electrons and protons and of electrons and photons subjected to external electromagnetic fields within the framework of Anderson and Witting model equation. These two systems are important in astrophysics since they could describe magnetic white dwarfs or cosmological fluids in the plasma period and in the radiation dominated period. By using the Chapman-Enskog methodology we determine Ohm and Fourier laws in the presence of electromagnetic fields and general expressions for the electrical and thermal conductivities for relativistic non-degenerated and degenerate binary mixtures of electrons with protons and electrons with photons. Furthermore, explicit expressions for these coefficients are given for
the particular mixtures: (a) a non-relativistic mixture of protons and non-degenerate electrons; (b) an ultra-relativistic mixture of photons and non-degenerate electrons; (c) a non-relativistic mixture of protons and completely degenerate electrons; (d) an ultra-relativistic mixture of photons and completely degenerate electrons and (e) a mixture of non-relativistic protons and ultra-relativistic completely degenerate electrons.

RELATIVISTIC UEHLING-UHLENBECK EQUATION

Let us first consider a single relativistic quantum ideal gas in a Minkowski space characterized by metric tensor $\eta^{\alpha\beta}$ with signature $\text{diag}(1,-1,-1,-1)$. In the phase space spanned by the space-time coordinates $(x^\alpha) = (ct, x^\beta)$ and momentum four-vector $(p^\alpha) = (p^0, p^\beta)$ the state of the relativistic quantum gas is characterized by the one-particle distribution function

$$f(x^\alpha, p^\alpha) = f(x, p, t),$$

since the length of the momentum four-vector is given by $mc \sqrt{p^2 + m^2c^2}$. The number of particles at time $t$ in the volume element $d^3x$ about $x$ and with momenta in the range $d^3p$ about $p$ is given by $f(x, p, t) d^3x d^3p$.

The space-time evolution of the one-particle distribution function $f(x, p, t) = f$ in the phase space is given by the Boltzmann equation (see e.g. [6, 7])

$$p^\alpha \frac{\partial f}{\partial x^\alpha} + m \frac{\partial f K^\alpha}{\partial p^\alpha} = Q,$$  \hspace{1cm} (1)

where $m$ denotes the rest mass of the particle and $K^\alpha$ is the Minkowski force which acts on the particles of the gas. Furthermore, $Q$ is a term which takes into account the collisions of the particles. For a relativistic gas which obeys the classical statistical mechanics it is given by

$$Q = \int (f'_s f' - f_s f) F \sigma d\Omega \frac{d^3p_s}{p_s^0}. \hspace{1cm} (2)$$

In the above equation we have introduced the abbreviations $f'_s \equiv f(x, p'_s, t)$, $f' \equiv f(x, p', t)$, $f_s \equiv f(x, p_s, t)$, $f \equiv f(x, p, t)$, where $p$ and $p'_s$ denote the momenta of two particles before a binary collision and $p'$ and $p'_s$ are the corresponding momenta after collision. The pre and post collisional momentum four-vectors are connected by the energy-momentum conservation law $p^\alpha + p'^\alpha = p'^\alpha + p'^{\alpha}$. Furthermore, $F = \sqrt{(p_s^0 p^0)}^2 - m^2c^4$ is the invariant flux, which in the non-relativistic limiting case is proportional to the modulus of the relative velocity. The differential cross-section and the element of solid angle that characterize the binary collision are denoted by $\sigma$ and $d\Omega$, respectively.

The collision term $Q$ for a gas whose particles obey quantum statistics may be motivated as follows. First we note that the volume element in the phase space $d^3x d^3p$ is a scalar invariant, but when quantum effects are taken into account in a semi-classical description, we divide the volume element by $h^3$, where $h = 6.626 \times 10^{-34}$ J s is the Planck constant. Hence we write $d^3x d^3p / h^3$, which is also a scalar invariant. The term $d^3x d^3p / h^3$ may be interpreted as the number of available states in the volume element $d^3x d^3p$. For particles with spin $s$ there are more states, corresponding to the values that the spin component on a given axis can take and we have to introduce the degeneracy
factor $g_s$. Hence the number of available states is given by

$$g_s \frac{d^3x d^3p}{h^3} \quad \text{where} \quad g_s = \begin{cases} 2s + 1 & \text{for} \quad m \neq 0; \\ 2s & \text{for} \quad m = 0. \end{cases} \quad (3)$$

In quantum mechanics a system of identical particles may be described by two kinds of particles: bosons and fermions. Bosons have integral spin, obey the Bose-Einstein statistics and include mesons (pion, kaon), photons, gluons and nuclei of even mass number like helium-4. Fermions have half-integral spin, obey the Fermi-Dirac statistics and include leptons (electron, muon, tau), baryons (neutron, proton) and nuclei of odd mass number like helium-3. The main difference between bosons and fermions in quantum statistical mechanics refers to the occupation number of a state. Any number of boson particles may occupy the same state, while fermion particles obey the Pauli exclusion principle and at most one particle may occupy each state.

In order to incorporate the statistics of bosons and fermions into the collision term, we begin to analyze fermions and note that due the Pauli exclusion principle, the phase space is completely occupied if the number of the particles in $d^3x d^3p$ is equal to the number of available states $f d^3x d^3p = g_s d^3x d^3p / h^3$, so that $f = g_s / h^3$. Hence, $(1 - fh^3 / g_s)$ gives the number of vacant states in the phase space. If the number of particles that enter the volume element $d^3x d^3p$ in phase space, as a consequence of a binary collision, is proportional to $f'f_*$ this quantity must be multiplied by the number of vacant states which is proportional to $(1 - fh^3 / g_s)(1 - f_*h^3 / g_s)$. Hence the following substitution in the collision term of the Boltzmann equation must be consider:

$$f'f_* \mapsto f'f_* (1 - \frac{fh^3}{g_s}) (1 - \frac{f_*h^3}{g_s}). \quad (4)$$

On the basis of the same reasoning we have to substitute

$$ff_* \mapsto ff_* (1 - \frac{f'h^3}{g_s}) (1 - \frac{f'*h^3}{g_s}), \quad (5)$$

for the particles that leave the volume element $d^3x d^3p$ in phase space.

To include the apparent attraction between the boson particles – due to the statistics of indistinguishable particles with no restrictions on the occupation of a state – the factor $(1 - fh^3 / g_s)$ must be replaced by $(1 + fh^3 / g_s)$. Hence we can write from the Boltzmann equation (1) and from the above conclusions the relativistic Uehling-Uhlenbeck equation (for the non-relativistic Uehling-Uhlenbeck equation see [8])

$$p^\alpha \frac{\partial f}{\partial x^\alpha} + m \frac{\partial f K^\alpha}{\partial p^\alpha} = \int \left[ f'f_*(1 + \varepsilon \frac{f'h^3}{g_s}) (1 + \varepsilon \frac{f_*h^3}{g_s}) 
- f_*f (1 + \varepsilon \frac{f'h^3}{g_s}) (1 + \varepsilon \frac{f_*h^3}{g_s}) \right] F \sigma d\Omega \frac{d^3p_*}{p_{\geq 0}}. \quad (6)$$
where \( \varepsilon \) is defined through

\[
\varepsilon = \begin{cases} 
+1 & \text{for Bose-Einstein statistics;} \\
-1 & \text{for Fermi-Dirac statistics;} \\
0 & \text{for Maxwell-Boltzmann statistics.}
\end{cases}
\] (7)

At equilibrium the number of particles that enter and leave the volume element in the phase space must be equal to each other, so that the quantity within the brackets in (6) must vanish. Equivalently, \( \ln \left( \frac{f^{(0)}}{1 + \varepsilon f^{(0)} h^3 / g_s} \right) \) must be a summational invariant – i.e., a function that obeys the relationship \( \psi + \psi' = \psi' + \psi \) – where \( f^{(0)} \) denotes the equilibrium distribution function. For summational invariants there exists the following theorem (see e.g. [7]): A continuous and differentiable function of class \( C^2 \) \( \psi(p^\alpha) \) is a summational invariant if and only if it is given by \( \psi(p^\alpha) = A + B^\alpha p^\alpha \), where \( A \) is an arbitrary scalar and \( B^\alpha \) an arbitrary four-vector that do not depend on \( p^\alpha \). Hence we have

\[
\ln \left( \frac{f^{(0)}}{1 + \varepsilon f^{(0)} h^3 / g_s} \right) = -(A + B^\alpha p^\alpha), \quad \text{or} \quad f^{(0)} = \frac{g_s / h^3}{e^{-\left( A + B^\alpha p^\alpha / g_s \right)} \pm 1},
\] (8)

where \( a = -A - \ln(g_s / h^3) \).

For the determination of \( a \) and \( B^\alpha \) we refer to [7]. Here we give only the results that \( a = \mu/kT \) and \( B^\alpha = U^\alpha/kT \), where \( \mu \) is the chemical potential, \( T \) the temperature, \( k \) the Boltzmann constant, and \( U^\alpha \) the four-velocity (with \( U^\alpha U_\alpha = c^2 \)). Hence, the equilibrium distribution function reads

\[
f^{(0)} = \frac{g_s / h^3 e^{\frac{\mu}{kT} - \frac{U^\alpha p^\alpha}{g_s}}}{1 + \varepsilon f^{(0)} h^3 / g_s}, \quad f^{(0)} = \frac{g_s / h^3}{e^{-\left( A + B^\alpha p^\alpha / g_s \right)} \pm 1},
\] (9)

when \( \varepsilon = 0 \) and \( \varepsilon = \mp 1 \), respectively. The relativistic Maxwell-Boltzmann distribution function \( (9)_1 \) was obtained by Jüttner [9] in 1911 and the relativistic Fermi-Dirac (+) and Bose-Einstein (−) distribution function \( (9)_1 \) was deduced by him [10] in 1928.

The extension of the Uehling-Uhlenbeck equation to a mixture of \( r \) constituents is straightforward. We introduce an one-particle distribution function for each constituent of the mixture \( f_a \equiv f(x, p_a, t) \) \((a = 1, \ldots, r)\) which must satisfy the equation

\[
p_a^\alpha \frac{\partial f_a}{\partial x^\alpha} + \frac{q_a}{c} F^{\alpha\beta} p_{ab} \frac{\partial f_a}{\partial p_a^\beta} = \sum_{b=1}^r \int \left[ f_b f_a^0 \left( 1 + \varepsilon_a f_a h^3 / g_s \right) - f_b h^3 / g_s \right] F_{ba} \sigma_{ab} d\Omega_{ba} \frac{d^3 p_b}{p_{b0}}.
\] (10)

Above it was supposed that the external force that acts on the particles of electric charge \( q_a \) is of electromagnetic nature. In this case the Minkowski force reads

\[
K_a^\alpha = \frac{q_a}{c} F^{\alpha\beta} \frac{p_a^\beta}{m_a},
\] (11)
where $F^{\alpha\beta}$ is the electromagnetic field tensor.

Now we introduce the moments of the distribution function, which are the partial particle four-flow $N^\alpha_a$ and the partial energy-momentum tensor $T^\alpha\beta_a$. They are defined through:

$$
N^\alpha_a = c \int p^\alpha_a f_a \frac{d^3p_a}{p_a0}, \quad T^\alpha\beta_a = c \int p^\alpha_{a\beta} f_a \frac{d^3p_a}{p_a0}.
$$

The corresponding quantities for the mixture read

$$
N^\alpha = \sum_{a=1}^r N^\alpha_a, \quad T^\alpha\beta = \sum_{a=1}^r T^\alpha\beta_a.
$$

In the analysis of ionized gases it is also important to introduce the electric charge four-vector $J^\alpha$, which is defined in terms of the partial particle four-flows $N^\alpha_a$ and of the partial electric charges $q_a$ as

$$
J^\alpha = \sum_{a=1}^r q_a N^\alpha_a.
$$

The balance equations for the particle four-flow and of the energy-momentum tensor of the mixture are obtained by multiplying (10) by $c$ and $cp^\alpha_a$, respectively, and by summing the resulting equations, yielding

$$
\partial_\alpha N^\alpha = 0, \quad \partial_\beta T^{\alpha\beta} = \frac{1}{c} F^{\alpha\beta} \sum_{a=1}^r q_a N_{a\beta} = \frac{1}{c} F^{\alpha\beta} J_\beta.
$$

Equation (15) is the conservation law for the particle four-flow of the mixture. Equation (15) when compared with the balance equation for the energy-momentum tensor of the electromagnetic field $T_{em}^{\alpha\beta}$ has an opposite sign on its right-hand side. However, if we denote the energy-momentum tensor of (15) by an index pt – that refers to the particles – we get the conservation law (see Landau and Lifshitz [11]):

$$
\partial_\alpha (T^{\alpha\beta}_{pt} + T_{em}^{\alpha\beta}) = 0,
$$

which means that the sum of the energy-momentum tensors of the particles and of the electromagnetic field satisfies a conservation equation.

**LANDAU-LIFSHITZ DECOMPOSITION**

The decomposition of the partial particle four-flow and of the partial energy-momentum tensor proceeds by introducing the four-velocity $U^\alpha$ and the projector $\Delta^{\alpha\beta}$ defined by

$$
\Delta^{\alpha\beta} = \eta^{\alpha\beta} - \frac{1}{c^2} U^\alpha U^\beta, \quad \text{such that} \quad \Delta^{\alpha\beta} U_\beta = 0.
$$
In the Landau-Lifshitz description \[12\] the partial particle four-flow and the partial energy-momentum tensor may be decomposed according to

\[
N^\alpha_a = n_a U^\alpha + J^\alpha_a - \frac{n_a q^\alpha}{nh},
\]

\[
T_{\alpha \beta}^a = p^a_{(\alpha \beta)} - (p_a + \varpi_a) \Delta^\alpha \beta + \frac{1}{c^2} U^\alpha \left( q^\beta_a + h_a J^\beta_a - \frac{n_a h_a}{nh} q^\beta_a \right)
\]

\[+ \frac{1}{c^2} U^\beta \left( q^\alpha_a + h_a J^\alpha_a - \frac{n_a h_a}{nh} q^\alpha_a \right) + e_a n_a \frac{1}{c^2} U^\alpha U^\beta.\]

(19)

Above we have introduced the following quantities for the constituent \(a\) in the mixture: particle number density \(n_a\), diffusion flux \(J^\alpha_a\), pressure deviator \(p^a_{(\alpha \beta)}\), pressure \(p_a\), non-equilibrium pressure \(\varpi_a\), heat flux \(q^\alpha_a\), energy per particle \(e_a\) and enthalpy per particle \(h_a = e_a + p_a/n_a\). The corresponding quantities for the mixture are given by the sums

\[
n = \sum_{a=1}^{r} n_a, \quad p^a_{(\alpha \beta)} = \sum_{a=1}^{r} p^a_{(\alpha \beta)}, \quad p = \sum_{a=1}^{r} p_a, \quad \varpi = \sum_{a=1}^{r} \varpi_a,
\]

\[
e n = \sum_{a=1}^{r} n_a e_a, \quad q^\alpha = \sum_{a=1}^{r} (q^\alpha_a + h_a J^\alpha_a), \quad nh = \sum_{a=1}^{r} n_a h_a.
\]

(20)

The sum of (18) and (19) over all constituents of the mixture lead to the following decompositions of the particle four-flow and energy-momentum tensor of the mixture

\[
N^\alpha = n U^\alpha - \frac{q^\alpha}{h}, \quad T_{\alpha \beta} = p^a_{(\alpha \beta)} - (p + \varpi) \Delta^\alpha \beta + \frac{e n}{c^2} U^\alpha U^\beta,
\]

(22)

thanks to the constraint that there exist only \((r - 1)\) partial diffusion fluxes that are linearly independent for a mixture of \(r\) constituents, namely,

\[
\sum_{a=1}^{r} J^\alpha_a = 0.
\]

(23)

We may also define the electric current four-vector \(I^\alpha\) in terms of the partial diffusion fluxes \(J^\alpha_a\) and of the partial electric charges \(q_a\) as

\[
I^\alpha = \sum_{a=1}^{r} q_a J^\alpha_a.
\]

(24)

We refer to the works of de Groot and Suttorp \[13\] and of van Erkelens and van Leeuwen \[14\] and decompose the electromagnetic field tensor \(F^{\alpha \beta}\) into one part which is parallel to the four-velocity \(U^\alpha\) and another which is perpendicular to it, i.e.

\[
F^{\alpha \beta} = \frac{1}{c^2} \left( F^{\alpha \gamma} U_\gamma U^\beta - F^\beta \gamma U_\gamma U^\alpha \right) + \Delta^\alpha \gamma F^{\gamma \delta} \Delta^\beta \delta.
\]

(25)

Furthermore, by introducing the tensors \(E^\alpha\) and \(B^{\alpha \beta}\) defined by

\[
E^\alpha = \frac{1}{c} F^{\alpha \beta} U_\beta, \quad B^{\alpha \beta} = -\Delta^\alpha \gamma F^{\gamma \delta} \Delta^\beta \delta,
\]

(26)
we may write the electromagnetic field tensor as
\[ F_{\alpha\beta} = \frac{1}{c} \left( E^\alpha U^\beta - E^\beta U^\alpha \right) - B_{\alpha\beta}. \]  
(27)

If we consider a local Lorentz rest frame where \((U^\alpha) = (c,0)\), equations (26) imply that
\[ (E^\alpha) = (0,E), \quad B^{0\alpha} = B^{\alpha 0} = 0, \quad B^{ij} = -c e^{ijk} B^k, \]  
(28)
and we can identify \(E^\alpha\) with the electric field \(E\) and \(B_{\alpha\beta}\) with the magnetic flux induction \(B\).

Due to the fact that \(F_{\alpha\beta}\) is an antisymmetric tensor \(F_{\alpha\beta} U^\alpha U^\beta = 0\), it follows from (26) and (27) the relationships
\[ E_\alpha U^\alpha = 0, \quad B_{\alpha\beta} U^\beta = 0, \quad \text{and} \quad B^{\alpha\beta} = -B^{\beta\alpha}. \]  
(29)

**CHAPMAN-ENSKOG METHOD**

Since we are interested to derive the laws of Fourier and Ohm for a binary mixture of electrons and protons and of electrons and photons, we have to made some simplifications of our model, which are enumerated below:

1. the electric current four-vector (24) for a binary mixture of electrons \((a = e)\) and protons \((a = p)\) may be written as
\[ I^\alpha = -2e J^\alpha_e, \]  
(30)
since the relationship between the diffusion fluxes reads \(J^\alpha_p = -J^\alpha_e\) and the electric charges are given by \(q_e = -e, q_p = e\), with \(e\) denoting the elementary charge. Furthermore, we shall analyze the so-called Lorentzian plasma [15] where the collisions between the electrons may be neglected in comparison with the collisions between the electrons and protons. A Lorentzian plasma must fulfill the condition that the mass of one constituent is much larger than the mass of the other constituent. Here we have that \(m_p/m_e \approx 1836\), where \(m_e\) and \(m_p\) denote the electron and proton masses, respectively. Moreover, we shall assume a locally neutral system where \(q_en_e + q_pn_p = 0\), which implies that \(n_e = n_p\);

2. the electric current four-vector (24) for a binary mixture of electrons \((a = e)\) and photons \((a = \gamma)\), reduces to
\[ I^\alpha = -e J^\alpha_e, \]  
(31)
due to the fact that the electric charge of the photons is zero \((q_\gamma = 0)\). Furthermore, the collisions between electrons can also be neglected in comparison to the collisions between electrons and photons, which is the Compton scattering;

3. the partial heat fluxes of the protons and of the photons are negligible in comparison with the partial heat flux of the electrons so that we can write from (21)\(_2\) that the heat flux of the mixture reduces to
\[ q^\alpha = q^\alpha_e + (h_e - h_b)J^\alpha_e, \quad \text{with} \quad b = p, \gamma. \]  
(32)
For simplicity we shall adopt the Anderson and Witting model equation [4] for the electrons instead of using the relativistic Uehling-Uhlenbeck equation (10). Hence, by taking into account the above considerations we write the space-time evolution of the distribution function for the electrons as

$$p_e^\alpha \frac{\partial f_e}{\partial x^\alpha} = \frac{e}{c} F^{\alpha\beta} p_e^\beta \frac{\partial f_e}{\partial p_e^\alpha} = -\frac{U^\alpha}{c^2 \tau_{eb}} (f_e - f_e^{(0)}),$$

(33)

where $\tau_{eb}$ with $b = p$ or $b = \gamma$ is the mean free time between collisions of electrons-protons or electrons-photons, respectively. In the above equation $f_e^{(0)}$ is the equilibrium distribution function of the electrons which reads

$$f_e^{(0)} = \frac{2}{\hbar^3} \exp \left( -\frac{\mu_e}{kT} + \frac{U^\alpha p_e^\alpha}{kT} \right) + 1,$$

(34)

by considering that the electrons obey the Fermi-Dirac statistics. Above, $T$ denotes the temperature of the mixture, $\mu_e$ the chemical potential of the electrons and the factor 2 refers to the degeneracy factor of the electrons.

Once we know the equilibrium distribution function of the electrons we may calculate the values of the fields at equilibrium: particle number density $n_e$, energy density $n_e e_e$ and pressure $p_e$ defined by

$$n_e = \frac{1}{c^2} U^\alpha N_e^\alpha = \frac{1}{c^2} U^\alpha c \int p_e^\alpha f_e^{(0)} \frac{d^3 p_e}{p_e},$$

(35)

$$n_e e_e = \frac{1}{c^2} U^\alpha U^\beta T_e^{\alpha\beta} = \frac{1}{c^2} U^\alpha U^\beta c \int p_e^\alpha p_e^\beta f_e^{(0)} \frac{d^3 p_e}{p_e},$$

(36)

$$p_e = -\frac{1}{3} \Delta^\alpha\beta T_e^{\alpha\beta} = -\frac{1}{3} \Delta^\alpha\beta c \int p_e^\alpha p_e^\beta f_e^{(0)} \frac{d^3 p_e}{p_e}. $$

(37)

The calculation proceeds as follows: we consider a local Lorentz rest system where $U^\alpha = (c, 0)$ so that the particle number density of the electrons (35) reduces to

$$n_e = \int \frac{2}{\hbar^3} \exp \left( -\frac{\mu_e}{kT} + \frac{c p_e^0}{kT} \right) + 1 |p_e|^2 \sin \psi d\chi d\psi d|p_e|,$$

(38)

where we have introduced the spherical coordinates $0 \leq \psi \leq \pi$, $0 \leq \chi \leq 2\pi$ and $0 \leq |p_e| < \infty$. Now we change the integration variable by introducing a new variable $\vartheta$ defined through

$$|p_e| = m_e c \sinh \vartheta, \quad \text{so that} \quad \frac{c p_e^0}{kT} = \zeta_e \cosh \vartheta,$$

(39)

where $\zeta_e = m_e c^2 / kT$ is the ratio between the electron rest energy $m_e c^2$ and the thermal energy of the gas $kT$. When $\zeta_e \gg 1$ the electron behaves as a non-relativistic gas, while when $\zeta_e \ll 1$ it behaves as an ultra-relativistic gas. The change of variables and the integration of (38) in the angles $\chi$ and $\psi$ leads to

$$n_e = 8\pi \left( \frac{m_e c}{\hbar} \right)^3 \int_0^\infty \frac{\sinh^2 \vartheta \cosh \vartheta d\vartheta}{\exp(-\mu_e^* + \zeta_e \cosh \vartheta) + 1} = \frac{8\pi}{h^3} (m_e c)^3 J_{21}(\zeta_e, \mu_e^*).$$

(40)
In the above equation we have introduced the electron chemical potential \( \mu_e^* = \mu_e/kT \) in units of \( kT \) and the integral \( J_{nm}(\zeta_e, \mu_e^*) \) defined by

\[
J_{nm}(\zeta_e, \mu_e^*) = \int_0^\infty \frac{\sinh^m \vartheta \cosh^n \vartheta d \vartheta}{\exp(-\mu_e^* + \zeta_e \cosh \vartheta) + 1}.
\] (41)

Following the same methodology we get that

\[
n_e e = \frac{8\pi}{\hbar^2} m_e^4 c^5 J_{22}(\zeta_e, \mu_e^*), \quad p_e = \frac{8\pi}{\hbar^2} m_e^4 c^5 J_{40}(\zeta_e, \mu_e^*).
\] (42)

Now we shall determine from (33) the non-equilibrium distribution function for the electrons by adopting the Chapman-Enskog methodology. For that purpose we search for a solution of the form

\[
f_e = f_e^{(0)} + \phi_e,
\] (43)

where the deviation from the equilibrium distribution function is considered to be a small quantity, i.e., \(|\phi_e| \ll 1\). If we insert (43) into the Boltzmann equation (33) we get

\[
p_e^\alpha \frac{\partial f_e^{(0)}}{\partial x^\alpha} - \frac{e}{c} p_e^\alpha p_e^\beta \frac{\partial f_e^{(0)}}{\partial p_e^\alpha} - \frac{e}{c} p_e^\alpha p_e^\beta \frac{kT}{c^2} \frac{\partial \phi_e}{\partial p_e^\alpha} = - \frac{U^\alpha}{c^2} \frac{p_e^\alpha}{\delta p_e^\alpha} e, \quad \phi_e,
\] (44)

where we have not taken into account the term \( \partial \phi_e / \partial x^\alpha \), since it is not our aim in deriving constitutive equations which are functions of second-order derivatives (Burnett equations). The above equation can be written as

\[
\frac{-2}{\hbar^2} \exp\left(-\frac{\mu_e}{kT} + \frac{U^\alpha p_e^\alpha}{kT^2}\right) \left[ \frac{1}{c^2} (p_e^\alpha U^\alpha) \right] \left[ D \left( \frac{\mu_e}{kT} \right) + \frac{p_e^\beta U^\beta}{kT^2} DT \right]
\]

\[
+ \frac{p_e^\beta U^\beta}{kT^2} p_e^\alpha \left[ \nabla^\alpha T - \frac{T}{\c^2} DU^\alpha \right] - \frac{e}{c} kT \frac{p_e^\alpha}{\delta p_e^\alpha} E^\alpha - \frac{kT}{c} \nabla^\alpha \left( \frac{\mu_e}{kT} \right) \]

\[- \frac{p_e^\alpha p_e^\beta}{kT} \nabla^\beta U^\alpha \right\} = \frac{U^\gamma p_e^\gamma}{c^2 \tau_{eb}} \left[ 1 + \frac{m_e c}{U^\delta p_e^\delta} \frac{\omega_e \tau_{eb}}{B} \right] B^\alpha \beta p_e^\beta \frac{\partial}{\partial p_e^\alpha} \phi_e, \quad \phi_e,
\] (45)

where we have not considered the term \( E^\alpha \partial \phi_e / \partial p_e^\alpha \), since it refers also to a second-order term. Furthermore, we have introduced in the above equation the electron cyclotron frequency \( \omega_e = eB/m_e \) – where \( B \) is the modulus of the magnetic flux induction – and the differential operators \( D \equiv U^\alpha \partial / \partial x^\alpha \) and \( \nabla^\alpha \equiv \Delta^\alpha / \partial \beta \). In this work we are interested in the derivation of the laws of Fourier and Ohm, so that we can restrict ourselves to the thermodynamic forces that are four-vectors, namely

\[
\nabla^\alpha T \equiv \left[ \nabla^\alpha T - \frac{T}{c^2} DU^\alpha \right] \quad \text{and} \quad \mathcal{E}^\alpha \equiv \left[ E^\alpha - \frac{kT}{c} \nabla^\alpha \left( \frac{\mu_e}{kT} \right) \right], \quad (46)
\]

the first being a combination of a temperature gradient and an acceleration, while the second refers to a combination of an external electric field and a gradient of the
chemical potential of the electrons. Hence, we obtain from (45) that the deviation from
the distribution function may be written as
\[ \phi_e = A^\alpha \left\{ \frac{p^\beta U^\beta_e}{kT^2} \nabla^\alpha T - \frac{e}{kT} \mathcal{E}^\alpha \right\}. \]  

(47)

Up to terms in \((\omega_e \tau_e / B)^2\) the four-vector \(A^\alpha\) is given by
\[ A^\alpha = \frac{-2}{h^3} \exp \left( -\frac{\mu_e}{kT} + \frac{U^\alpha_e p_e}{kT} \right) \frac{c^2 \tau_e}{U^\gamma p_e^\gamma} \left[ \eta^\alpha^\beta - \frac{m_e c}{U^\delta p^\delta_e} \left( \frac{\omega_e \tau_e}{B} \right) B^\alpha^\beta \right. 
\]
\[ + \left. \left( \frac{m_e}{U^\delta p^\delta_e} \right)^2 \left( \frac{\omega_e \tau_e}{B} \right)^2 B^\alpha^\gamma B^\gamma^\beta \right] p_e^\beta. \]

(48)

Equation (47) together with (48) represent the deviation of the distribution function of the electrons as a function of thermodynamic forces that are four-vectors. We shall use the distribution function (43) in the following section in order to determine the laws of Ohm and Fourier.

**OHM AND FOURIER LAWS**

The determination of the diffusion flux \(J_e^\alpha\) and the heat flux \(q_e^\alpha\) of the electrons proceeds by noting that (12), (18) and (19) lead to
\[ \frac{\hbar}{\hbar} J_e^\alpha - \frac{n_e}{nh} q_e^\alpha = \Delta_e^\alpha N_e^\beta = \Delta_e^\alpha \int c p_e^\beta f_e \frac{d^3 p_e}{p_e^0}, \]

(49)

\[ \frac{\hbar}{\hbar} J_e^\alpha + \frac{n_e}{nh} q_e^\alpha = \Delta_e^\alpha U_\gamma T_e^\gamma = \Delta_e^\alpha \int c p_e^\beta p_e^\gamma f_e \frac{d^3 p_e}{p_e^0}. \]

(50)

The insertion of the distribution function of the electrons (43) together with (47) and (48) into (49) and (50) and integration of the resulting equations, implies a system of equations for \(J_e^\alpha\) and \(q_e^\alpha\) which is used to determine the heat flux of the mixture (32) and the electric current four-vector (30) or (31). From this system of equations it follows Fourier and Ohm laws
\[ q^\alpha = \Lambda^\alpha^\beta \nabla^\beta T + \Omega^\alpha^\beta \mathcal{E}_\beta, \quad I^\alpha = \sigma^\alpha^\beta \mathcal{E}_\beta + \Omega^\alpha^\beta \nabla^\beta T, \]

(51)

respectively. Above \(\Lambda^\alpha^\beta\) is a tensor associated with the thermal conductivity, \(\sigma^\alpha^\beta\) is the electrical conductivity tensor, while the tensors \(\Omega^\alpha^\beta\) and \(\Omega^\alpha^\beta\) are related with cross effects. We may represent the general expressions for the above mentioned tensors as
\[ \{\Lambda, \Omega, \Omega, \sigma\} = \{a_1, b_1, c_1, d_1\} \eta^\alpha^\beta + \{a_2, b_2, c_2, d_2\} B^\alpha^\beta \]
\[ + \{a_3, b_3, c_3, d_3\} B^\alpha^\gamma B^\gamma^\beta, \]

(52)
where the scalar coefficients $a_1$ through $d_3$ are given below:

1. Coefficients associated with $\Lambda^{\alpha \beta}$

$$
a_1 = \frac{8\pi m e c^2 \tau_{eb} h}{3h^3 h_b kT^2} \left( J_{4_1} - \frac{h_b}{m_e c^2} J_{4_0} \right),
$$

(53)

$$
a_2 = \frac{8\pi m e c^2 \tau_{eb} h}{3h^3 h_b kT^2} \left( J_{4_0} - \frac{h_b}{m_e c^2} J_{4_1} \right) \left( \frac{\omega_e \tau_{eb}}{cB} \right),
$$

(54)

$$
a_3 = \frac{8\pi m e c^2 \tau_{eb} h}{3h^3 h_b kT^2} \left( J_{4_1} - \frac{h_b}{m_e c^2} J_{4_2} \right) \left( \frac{\omega_e \tau_{eb}}{cB} \right)^2.
$$

(55)

2. Coefficients associated with $\Upsilon^{\alpha \beta}$

$$
b_1 = -\frac{8\pi m e c^2 \tau_{eb} h e}{3h^3 h_b kT} \left( J_{4_0} - \frac{h_b}{m_e c^2} J_{4_1} \right),
$$

(56)

$$
b_2 = -\frac{8\pi m e c^2 \tau_{eb} h e}{3h^3 h_b kT} \left( J_{4_1} - \frac{h_b}{m_e c^2} J_{4_2} \right) \left( \frac{\omega_e \tau_{eb}}{cB} \right),
$$

(57)

$$
b_3 = -\frac{8\pi m e c^2 \tau_{eb} h e}{3h^3 h_b kT} \left( J_{4_2} - \frac{h_b}{m_e c^2} J_{4_3} \right) \left( \frac{\omega_e \tau_{eb}}{cB} \right)^2.
$$

(58)

3. Coefficients associated with $\sigma^{\alpha \beta}$

$$
c_1 = \frac{8\pi m e c^2 \tau_{eb} n_e (Z + 1) e^2}{3h^3 n_h h_b kT} \left( J_{4_0} + \frac{n_h h_b}{n_e m_e c^2} J_{4_1} \right),
$$

(59)

$$
c_2 = \frac{8\pi m e c^2 \tau_{eb} n_e (Z + 1) e^2}{3h^3 n_h h_b kT} \left( J_{4_1} + \frac{n_h h_b}{n_e m_e c^2} J_{4_2} \right) \left( \frac{\omega_e \tau_{eb}}{cB} \right),
$$

(60)

$$
c_3 = \frac{8\pi m e c^2 \tau_{eb} n_e (Z + 1) e^2}{3h^3 n_h h_b kT} \left( J_{4_2} + \frac{n_h h_b}{n_e m_e c^2} J_{4_3} \right) \left( \frac{\omega_e \tau_{eb}}{cB} \right)^2.
$$

(61)

4. Coefficients associated with $\Omega^{\alpha \beta}$

$$
d_1 = -\frac{8\pi m e c^2 \tau_{eb} n_e (Z + 1) e}{3h^3 n_h h_b kT^2} \left( J_{4_1} + \frac{n_h h_b}{n_e m_e c^2} J_{4_0} \right),
$$

(62)

$$
d_2 = -\frac{8\pi m e c^2 \tau_{eb} n_e (Z + 1) e}{3h^3 n_h h_b kT^2} \left( J_{4_0} + \frac{n_h h_b}{n_e m_e c^2} J_{4_1} \right) \left( \frac{\omega_e \tau_{eb}}{cB} \right),
$$

(63)

$$
d_3 = -\frac{8\pi m e c^2 \tau_{eb} n_e (Z + 1) e}{3h^3 n_h h_b kT^2} \left( J_{4_1} + \frac{n_h h_b}{n_e m_e c^2} J_{4_2} \right) \left( \frac{\omega_e \tau_{eb}}{cB} \right)^2.
$$

(64)

In the above equations $J_{nm}$ represents the partial derivative of (41) with respect to the chemical potential of the electrons $\mu_e^* = \mu_e/(kT)$ in units of $kT$. Furthermore, we have introduced the abbreviation $\zeta_e = m_e c^2/(kT)$ which refers to the ratio between the rest energy of the electrons $m_e c^2$ and the thermal energy of the mixture $kT$. We note that in all above equations one has to consider $Z = 1$ for binary mixtures of electrons and protons and $Z = 0$ for binary mixtures of electrons and photons.
The thermal conductivity tensor $\lambda^{\alpha\beta}$ is obtained by eliminating $E^\alpha$ from (51), through the use of (51)$_2$ by assuming that there is no electric current. Hence, we get a relationship between $E^\alpha$ and $\nabla^\alpha T$ from (51)$_2$ which may be used to write Fourier law as

$$q^\alpha = \lambda^{\alpha\beta} \nabla_\beta T,$$

where $\lambda^{\alpha\beta} = e_1 \eta^{\alpha\beta} + e_2 B^{\alpha\beta} + e_3 B^{\alpha\gamma} B^\beta_\gamma.$ (65)

Up to terms in $[\omega_e \tau_{eb} / (cB)]^2$ the scalar coefficients $e_1$ through $e_3$ read

$$e_1 = \frac{a_1 c_1 - b_1 d_1}{c_1}, \quad e_2 = \frac{a_2 c_1^2 - b_1 (c_1 d_2 - c_2 d_1) - b_2 c_1 d_1}{c_1^2},$$

$$e_3 = \frac{a_3 c_1^3 - b_1 [d_1 (c_2^2 - c_1 c_3) - c_1 c_2 d_2] - c_2^2 (b_1 d_3 + b_3 d_1) - c_1 b_2 (c_1 d_2 - c_2 d_1)}{c_1^3}. \quad (66)$$

In order to get a better physical interpretation of the components of the tensors, it is usual in the theory of ionized gases to decompose the thermodynamic forces $\nabla^\alpha T$ and $E^\alpha$ into parts parallel, perpendicular and transverse to the magnetic flux induction. To achieve this goal, we follow van Erkelens and van Leeuwen [14] and introduce the dual $B^\alpha\beta$ of the magnetic flux induction tensor $B^{\alpha\beta}$ defined by

$$B^\alpha\beta = \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} B_{\gamma\delta}. \quad (68)$$

One may easily verify from (68) and (28) that in a local Lorentz rest frame the only non-zero components of $B^\alpha\beta$ are $B^{0i} = cB^i$ since $\tilde{B}^{ij} = 0$ and $\tilde{B}^{00} = 0.$

The desired decomposition of the thermodynamic forces into parallel $\nabla^\alpha T$, $E^\alpha$; perpendicular $\nabla^\perp T$, $E^\perp$; and transverse $\nabla^\parallel T$, $E^\parallel$ parts read

$$\mathcal{F}^\parallel = \frac{1}{(2 B^{\gamma\delta} B_{\gamma\delta})^2} B^{\alpha\beta} \tilde{B}_{\beta\gamma} \mathcal{F}^\gamma, \quad \mathcal{F}^\perp = -\frac{1}{(2 B^{\gamma\delta} B_{\gamma\delta})} B^{\alpha\beta} B_{\beta\gamma} \mathcal{F}^\gamma, \quad (69)$$

$$\mathcal{F}^\alpha = \frac{1}{(2 B^{\gamma\delta} B_{\gamma\delta})^2} B^{\alpha\beta} \mathcal{F}_\beta, \quad (70)$$

where $\mathcal{F}^\alpha$ is an abbreviation for $E^\alpha$ or $\nabla^\alpha T$. In a local Lorentz rest frame (69) and (70) reduce to

$$\mathcal{F}^\parallel = \mathcal{F}^\perp = \mathcal{F}^0 = 0, \quad \mathcal{F}_t = \frac{1}{B^2} (B \cdot \mathcal{F}) B, \quad (71)$$

$$\mathcal{F}^\perp = \frac{1}{B^2} [(B \cdot \mathcal{F}) B - (B \cdot B) \mathcal{F}], \quad \mathcal{F}_t = \frac{1}{B} (\mathcal{F} \times B), \quad (72)$$

thanks to the relationship $\sqrt{B^{\gamma\delta} B_{\gamma\delta}/2} = c \sqrt{B \cdot B} = cB.$ From the above equations it is easy to verify that $\mathcal{F}^\parallel$ is parallel to the magnetic flux induction $B$, $\mathcal{F}^\perp$ perpendicular to it while $\mathcal{F}_t$ is perpendicular to both $\mathcal{F}^\parallel$ and $\mathcal{F}^\perp$.

Now by using the following relationship

$$(cB)^2 \eta^{\alpha\beta} = \tilde{B}^{\alpha\gamma} B^\beta_\gamma - B^{\alpha\gamma} B^\beta_\gamma, \quad (73)$$
the Fourier and Ohm laws can be rewritten in terms of \( F^\alpha \), \( F^\perp \), and \( F^t \). In fact, if we substitute (73) into the Ohm’s law (51) and Fourier’s law (65) and make use of the definitions (69) and (70), it follows that the electric current four-vector and the heat flux can be written, without the cross-effects terms, as
\[
I^\alpha = \sigma^\parallel E^\alpha + \sigma^\perp E^\perp + \sigma^t E^t, \quad q^\alpha = \lambda^\parallel \nabla^\alpha T + \lambda^\perp \nabla^\perp T + \lambda^t \nabla^t T, \tag{74}
\]
respectively. In the above equations the scalars are called the parallel, perpendicular and transverse components of the tensors, and their expressions are given by
\[
\begin{align*}
\sigma^\parallel &= c_1, & \sigma^\perp &= c_1 - c_3 (cB)^2, & \sigma^t &= c_2 (cB), \\
\lambda^\parallel &= e_1, & \lambda^\perp &= e_1 - e_3 (cB)^2, & \lambda^t &= e_2 (cB). \tag{75}
\end{align*}
\]
From the above formulas we shall obtain the parallel, perpendicular and transverse components of the electrical and thermal conductivities for binary mixtures of electrons and protons and of electrons and photons.

**ELECTRICAL AND THERMAL CONDUCTIVITIES**

**Non-degenerate electrons**

Here we shall analyze two important cases, namely: a non-relativistic mixture of protons and non-degenerate electrons and an ultra-relativistic mixture of photons and non-degenerate electrons. We note that the chemical potential of the electrons in the non-degenerate case must fulfill the condition that \( e^{-\mu^e} \gg 1 \).

1. A non-relativistic mixture of electrons and protons is identified by two conditions \( m_p/m_e \gg 1 \) and \( \zeta_e = m_e c^2/(kT) \gg 1 \). In this case the transport coefficients read
\[
\begin{align*}
\sigma^\parallel &= \frac{e^2 \tau_{ep} n_e}{m_e} \left( 1 - \frac{5}{2\zeta_e} \right), & \sigma^t &= \frac{e^2 \tau_{ep} n_e (\omega_e \tau_{ep})}{m_e} \left( 1 - \frac{5}{2\zeta_e} \right), \tag{76} \\
\sigma^\perp &= \frac{e^2 \tau_{ep} n_e}{m_e} \left[ \left( 1 - \frac{5}{2\zeta_e} \right) - (\omega_e \tau_{ep})^2 \left( 1 - \frac{15}{2\zeta_e} \right) \right], \tag{77} \\
\lambda^\parallel &= \frac{5k^2 T \tau_{ep} n_e n_p}{2m_e n_p} \left( 1 - \frac{3}{\zeta_e} \right), & \lambda^t &= \frac{5k^2 T \tau_{ep} n_e n_p (\omega_e \tau_{ep})}{2m_e n_p} \left( 1 - \frac{15}{2\zeta_e} \right), \tag{78} \\
\lambda^\perp &= \frac{5k^2 T \tau_{ep} n_e n_p}{2m_e n_p} \left[ \left( 1 - \frac{3}{\zeta_e} \right) - (\omega_e \tau_{ep})^2 \left( 1 - \frac{12}{\zeta_e} \right) \right]. \tag{79}
\end{align*}
\]

The first relativistic corrections to the transport coefficients are related to the term \( 1/\zeta_e \) and if fix our attention to the leading terms without the relativistic corrections, the electrical conductivities can be written from (76) and (77) as:
\[
\begin{align*}
\sigma^\parallel &= \frac{e^2 \tau_{ep} n_e}{m_e}, & \sigma^t &= \sigma^\parallel (\omega_e \tau_{ep}), & \sigma^\perp &\approx \frac{\sigma^\parallel}{1 + (\omega_e \tau_{ep})^2}, \tag{80}
\end{align*}
\]
since we have considered $\omega_e \tau_{ep} \ll 1$. The expressions for the electrical conductivities (80) are well-known in the theory of non-degenerate and non-relativistic ionized gases (see, for example, Cap [16]) and show their dependence on the magnetic flux induction $B$ through the electron cyclotron frequency $\omega_e$. Furthermore, the thermal conductivities (78) and (79) without the relativistic corrections become

$$
\lambda_\parallel = \frac{5k^2 T \tau_{ep} n_e}{2m_e n_p}, \quad \lambda_t = \lambda_\parallel (\omega_e \tau_{ep}), \quad \lambda_\perp \approx \frac{\lambda_\parallel}{1 + (\omega_e \tau_{ep})^2}.
$$  \hspace{1cm} (81)

Note that the expression for the parallel thermal conductivity is well-known in the theory of non-relativistic gases which follow from a BGK model equation.

2. An ultra-relativistic mixture of photons and non-degenerate electrons is characterized by the condition $\zeta_e = m_e c^2 / (kT) \ll 1$. Here the transport coefficients reduce to

$$
\sigma_\parallel = \sigma_\perp = \frac{e^2 c^2 \tau_{ep} n_e (3n_e + 4n_\gamma)}{12nkT}, \quad \sigma_t = \frac{e^2 c^2 \tau_{ep} n_e (n_e + 2n_\gamma) (\omega_e \tau_{ep}) \zeta_e}{12nkT},
$$

$$
\lambda_\parallel = \lambda_\perp = \frac{4k c^2 \tau_{ep} n_e n}{3n_e + 4n_\gamma}, \quad \lambda_t = \frac{8k c^2 \tau_{ep} n_e n^2 (\omega_e \tau_{ep}) \zeta_e}{(3n_e + 4n_\gamma)^2}.
$$

We infer from these equations that the parallel and perpendicular electrical and thermal conductivities coincide, while the transverse electrical and thermal conductivities are small quantities since they are proportional to $\zeta_e$.

**Completely degenerate electrons**

All thermal conductivities vanish in the limit of completely degenerate electrons, since this behavior is connected with the well-known result from statistical mechanics that the heat capacity of a completely degenerate gas vanishes. For the electrical conductivities there exist three important cases to be analyzed which are: a non-relativistic mixture of protons and completely degenerate electrons; an ultra-relativistic mixture of photons and completely degenerate electrons and a mixture of non-relativistic protons and ultra-relativistic completely degenerate electrons. We proceed to analyze the electrical conductivities for these cases.

1. A non-relativistic mixture of protons and completely degenerate electrons is identified by $\zeta_e \gg 1$ and $p_F \ll m_e c$, where $p_F$ denotes the Fermi momentum of the electrons. Here we have

$$
\sigma_\parallel = \frac{8\pi e^2 \tau_{ep} p_F^3}{3m_e h^3} \left( 1 - \frac{p_F^2}{2m_e c^2} \right), \quad \sigma_t = \frac{8\pi e^2 \tau_{ep} p_F^3 (\omega_e \tau_{ep})}{3m_e h^3} \left( 1 - \frac{p_F^2}{m_e c^2} \right),
$$

$$
\sigma_\perp = \frac{8\pi e^2 \tau_{ep} p_F^3}{3m_e h^3} \left[ \left( 1 - \frac{p_F^2}{2m_e c^2} \right) - (\omega_e \tau_{ep})^2 \left( 1 - \frac{3p_F^2}{2m_e c^2} \right) \right].
$$

(84)
Let us fix our attention to the leading terms of the electrical conductivities

\[ \sigma_\parallel = \frac{8\pi e^2 \tau_e p_F^3}{3m_e h^3}, \quad \sigma_\perp = \sigma_\parallel (\omega_e \tau_e), \quad \sigma_\perp \approx \frac{\sigma_\parallel}{1 + (\omega_e \tau_e)^2}, \]

since the term \( p_F/(m_e c^2) \) is a small quantity and the condition \( \omega_e \tau_e \ll 1 \) holds. These equations show the dependence of the electrical conductivities on the magnetic flux induction \( B \) through the electron cyclotron frequency \( \omega_e \).

2. An ultra-relativistic mixture of photons and completely degenerate electrons is characterized by the conditions \( \zeta_e \ll 1 \) and \( p_F \gg m_e c \), and the electrical conductivities for this case read

\[ \sigma_\parallel = \sigma_\perp = \frac{8\pi e^2 \tau_e c^2 n_e p_F^3}{12n_k T h^3} \left( 1 + \frac{4kT n_e}{n_e c p_F} \right), \]

\[ \sigma_\parallel = \frac{8\pi e^2 \tau_e c^2 n_e \zeta_e c (\omega_e \tau_e) p_F^2}{12n_k h^3} \left( 1 + \frac{4kT n_e}{n_e c p_F} \right). \]

We infer from the above equations that the parallel and perpendicular electrical conductivities are equal to each other, while the transverse electrical conductivity is a small quantity since it is proportional to \( \zeta_e \).

3. A mixture of non-relativistic protons and ultra-relativistic completely degenerate electrons is also an important case since it could describe a white dwarf star. Here the conditions \( m_p/m_e \gg 1 \) and \( p_F \gg m_e c \) hold and the electrical conductivities become

\[ \sigma_\parallel = \sigma_\perp = \frac{8\pi e^2 \tau_e c^2 p_F^2}{3h^3} \left( 1 - \frac{m_e^2 c^2}{2p_F^2} \right), \quad \sigma_\parallel = \frac{8\pi e^2 \tau_e c^2 kT (\omega_e \tau_e) p_F}{3h^3}, \]

showing that the parallel and perpendicular conductivities coincide and that the transverse conductivity is a small quantity, since it is proportional to \( \zeta_e \ll 1 \).

**APPENDIX: INTEGRALS \( J_{nm}^* \)**

1. Non-degenerate case

In this case \( e^{-\mu_e^*} \gg 1 \) so that the integrals \( J_{nm} \) reduce to

\[ J_{nm}(\zeta_e, \mu_e^*) = \int_0^\infty e^{-\zeta_e \cosh \vartheta + \mu_e^* \sinh \vartheta \vartheta} d\vartheta, \]

and the integrals \( J_{nm}^*(\zeta_e, \mu_e^*) \equiv J_{nm}^* \) can be expressed in terms of the modified Bessel functions of the second kind \( K_n(\zeta_e) \equiv K_n \) and of their integrals \( K_i_n(\zeta_e) \equiv K_i_n \) (see Abramowitz and Stegun [17] pages 376 and 483) as follows:

\[ J_{41}^* = \frac{e^{\mu_e^*}}{2\zeta_e^2} (K_4 - K_2), \quad J_{40}^* = \frac{3e^{\mu_e^*}}{4\zeta_e} (K_3 - K_1), \quad J_{43}^* = \frac{e^{\mu_e^*}}{\zeta_e} (K_2 - K_1), \]

\[ J_{42}^* = \frac{e^{\mu_e^*}}{2\zeta_e} (K_3 + K_2 - K_1), \quad J_{44}^* = \frac{3e^{\mu_e^*}}{\zeta_e} (K_2), \]

\[ J_{4-1}^* = \frac{3e^{\mu_e^*}}{\zeta_e} (K_1 - K_3), \quad J_{4-3}^* = \frac{3e^{\mu_e^*}}{\zeta_e} (K_1 - K_4). \]
Moreover, the chemical potential of the electrons is given by

\[ \mu^e = \frac{n_e h^3}{8 \pi m_e^2 c kT K_2}. \]  

(93)

2. Completely degenerate case

The integrals \( J_{nm} \) for this case reduce to

\[ J_{nm} = \int_0^{\vartheta_F} \sinh^n \vartheta \cosh^m \vartheta d\vartheta, \quad \text{with} \quad \vartheta_F = \text{arcosh} \left( \frac{p_F}{m_e c} \right)^2, \]  

(94)

where \( p_F \) is the Fermi momentum of the electrons. The integrals \( J^\bullet_{nm} \) read

\[ J^\bullet_{41} = \frac{1}{\zeta_e} \left( \frac{p_F}{m_e c} \right)^3 \sqrt{1 + \left( \frac{p_F}{m_e c} \right)^2}, \quad J^\bullet_{4-1} = \frac{\left( \frac{p_F}{m_e c} \right)^3}{\zeta_e \sqrt{1 + \left( \frac{p_F}{m_e c} \right)^2}}, \]  

(95)

\[ J^\bullet_{40} = \frac{1}{\zeta_e} \left( \frac{p_F}{m_e c} \right)^3, \quad J^\bullet_{4-2} = \frac{\left( \frac{p_F}{m_e c} \right)^3}{\zeta_e \left[ 1 + \left( \frac{p_F}{m_e c} \right)^2 \right]^{\frac{3}{2}}}, \quad J^\bullet_{4-3} = \frac{\left( \frac{p_F}{m_e c} \right)^3}{\zeta_e \left[ 1 + \left( \frac{p_F}{m_e c} \right)^2 \right]^{\frac{3}{2}}}. \]  

(96)

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