The nature and boundary of the floating phase in a dissipative Josephson junction array

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(Dated: October 30, 2018)

We study the nature of correlations within, and the transition into, the floating phase of dissipative Josephson junction arrays. Order parameter correlations in this phase are long-ranged in time, but only short-ranged in space. A perturbative RG analysis shows that, in arbitrary spatial dimension, the transition is controlled by a continuous locus of critical fixed points determined entirely by the local topology of the lattice. This may be the most natural example of a line of critical points existing in arbitrary dimensions.

PACS numbers: 74.81.Fa, 71.10.Hf, 74.78.Na

I. INTRODUCTION

Systems where phases and phase transitions specific to a particular dimension are hidden in a higher dimensional manifold have recently come under renewed focus. Of special importance is the case where the hidden phases and transitions are governed by spatially local, or zero-dimensional, modes. A higher dimensional manifold with such local systems embedded in it may retain some of the properties that are effectively zero-dimensional.

It has been shown recently that this is indeed the case in a dissipative Josephson junction array in any dimension. For a system of Josephson-coupled superconducting grains in a metallic substrate, the dissipation is modelled by shunting resistors across the Josephson junctions. The phase diagram of this model in the dissipation (α)-nearest-neighbor Josephson coupling (V) plane shows a “floating” phase at small α and V in which V is “irrelevant” in the renormalization group sense, so that the phases of the superconducting order parameters on different grains are quantum disordered and the system is metallic. For larger α or V, V grows to strong coupling and the system orders into a superconductor. This phase diagram is shown in Fig. 1.

In addition, it has been shown recently that all longer-range Josephson couplings, that may couple the spatially separated junctions in the array, are irrelevant in the metallic phase as long as V is irrelevant. However, they all become relevant simultaneously with V at the same value of dissipation, αc, so that there are no further transitions within the ordered phase; the ordered phase is a global superconductor. It has also been indicated that in the disordered metallic phase, the superconducting grains have long range in time, but only short range in space, correlations at zero temperature. The metallic phase has been called ‘floating’, the long-time-correlated zero-dimensional systems on the grains ‘float’ over each other.

In the present paper we build on our earlier work by explicitly calculating the important correlation functions in the floating phase. By carefully including all relevant and dangerously irrelevant variables, we show that the long-time correlation functions of the order parameters on individual grains are indeed finite, whereas the spatially long-range correlation functions are exactly zero at zero temperature. At nonzero temperatures, the spatial correlation functions are exponentially decaying with a finite correlation length. Thus, each grain constitutes a dynamic local system on its own, and they are only short-range-correlated in space. It is then clear that the dissipation-driven quantum phase transition (QPT) between the metallic floating phase and the global superconductor is governed by the same (0 + 1)-dimensional physics as in the ‘quantum to classical’ phase tran-
tion studied in the context of a single quantum particle in a double well or cosine-periodic potential. Above \( \alpha = \alpha_c \), the quantum fluctuations of the individual Josephson phases are effectively quenched by dissipation. Hence, in the absence of any quantum fluctuations, any infinitesimal \( V \) orders the extended system into a global superconductor.

We further study this phase transition by computing the relevant renormalization group (RG) equations up to third order in \( V \). While linear order recursion relations for \( V \) and \( \alpha \) only show the existence of the critical \( \alpha_c \) to separate the metal from the superconductor, the higher order corrections reveal a continuous locus of critical points in the \((V - \alpha)\) plane. This locus (see Fig. 1) separates the metal (\( V \) irrelevant) from the superconductor (\( V \) relevant), determining a phase boundary on the \((V - \alpha)\) plane that approaches the \( V \)-axis for the spatial dimension \( D > 1 \). The bending of the phase boundary toward the \( V \)-axis is also what is physically expected and experimentally observed. The functional form of the locus is determined entirely by the local topology of the lattice and does not depend on the global dimension. If the number of sides of the minimum closed loop on the lattice is three, for example in the 2D triangular or 3D face-centered-cubic lattice, the locus is a straight line making an angle with the \( \alpha = 0 \) line close to \( V = 0 \). For the number of sides in the minimum closed loop greater than three, for example in 2D square or 3D simple cubic lattice, the locus is a parabola close to \( V = 0 \). Since the forms of these loci do not depend on the global dimension in any way, they exist in any \( D \); this is a demonstration of a locus of critical points in any dimension; there is no upper-critical-dimension for this problem. These results point to the intriguing possibility that the global transition in an extended system is ultimately driven by local physics at the level of a single junction. For \( D = 1 \), the line of critical points remains vertical, parallel to the \( \alpha = 0 \) line. This is analogous to the corresponding result in the zero-dimensional problem. Apart from intrinsic theoretical interest, these results also help unravel the nature of the floating phase and experimental observations thereof. Local quantum critical points and their implications have also been widely discussed in the context of the cuprate high-temperature superconductors and heavy-fermion materials.

In section II we will give the preliminaries by introducing the model and sketching the first order recursion relation for \( V \). This will enable us to quickly identify the metallic floating phase as the one with \( \alpha < \alpha_c \), and the superconductor with \( \alpha > \alpha_c \). In section III we will discuss the correlation functions in the floating phase. Some of the details are relegated to appendix A. In section IV we will compute the corrections to the linear order recursion relation for \( V \) and derive the forms of the loci of critical points for the triangular and square lattices in \( D = 2 \). Together with appendix B, which addresses the contribution to the flows from third order cumulant expansion of the partition function, this completes the analysis of the perturbative renormalization group up to third order in \( V \) for the problem. The results are summarized in Fig. 1 which also determines the phase boundary. In section V we will discuss possible experimental consequences of our results. Finally, the paper is summarized and concluded in section VI.

II. THE MODEL

We start with the following action describing a Josephson junction array coupled to dissipation in any dimension,

\[
S/\hbar = \int_0^\beta \left[ \frac{C}{2} \sum_i \left( \frac{\partial \theta_i}{\partial \tau} \right)^2 + V \sum_{\langle i,j \rangle} [1 - \cos \Delta \theta_{ij}(\tau)] \right] d\tau + \frac{\alpha}{4\pi} \sum_{\langle i,j \rangle} \sum_n |\omega_n| |\Delta \hat{\theta}_{ij}(\omega_n)|^2 \quad (1)
\]

Here the sum \( \langle i,j \rangle \) is over nearest-neighbor pairs, and we will refer to \( \Delta \theta_{ij} = (\theta_i - \theta_j) \) as the phase difference on the bond \( \langle i,j \rangle \). \( \Delta \hat{\theta}_{ij}(\omega_n) \) is a Fourier component of \( \Delta \theta_{ij}(\tau) \), and \( \omega_n = \pm 2\pi n/\beta \), with \( n \) an integer and \( \beta \) the inverse-temperature. \( C \), the capacitance of the superconducting grains, is assumed to be small, and so \( E_0 \sim 1/C \) constitutes the largest energy scale in the problem. The dimensionless variable \( \alpha = R_Q/R \), where \( R \) is the shunt resistance and \( R_Q = h/4e^2 \) is the quantum of resistance, couples the Josephson-phases \( \Delta \theta_{ij} \) to a local dissipative heat bath, with an Ohmic dissipation.

To establish the existence of a dissipation-tuned quantum phase transition (QPT), we employ a frequency-shell RG which is perturbative in \( V \). Let’s divide the field \( \theta_i(\tau) \) into slow and fast components \( \theta_{is}(\tau) \) and \( \theta_{if}(\tau) \), such that \( \theta_{is}(\tau) = \frac{1}{\sqrt{\beta}} \sum_{|\omega_n| \leq \omega_c/\beta} \hat{\theta}_i(\omega_n)e^{i\omega_n \tau} \), and \( \theta_{if}(\tau) \) is given by a similar expression where the sum is between \( \omega_c/\beta < |\omega_n| \leq \omega_c \). Here, \( \omega_c \sim E_0 \) is the high-frequency cut-off of the frequency integrals, and \( b > 1 \) is a frequency rescaling factor.

We can now write the partition function \( Z \) as a functional integral,

\[
Z = N \prod_i \int D\theta_i(\tau) e^{-S/\hbar} = N' \prod_i \int D\theta_{is}(\tau)e^{-S_0/h + \ln \langle e^{-S'/\hbar} \rangle_{\alpha}} \quad (2)
\]

Here \( N \) and \( N' \) are normalization constants, \( S_0^f \) is the slow-frequency component of the quadratic part of the action containing the first and the third terms in Eq. 1, \( S' \) contains the second term, and \( \langle ... \rangle_{\alpha} \) denotes average with respect to the fast component of the quadratic part. After computing the averages, we rescale \( \tau \), \( \tau' = \tau/b \), to restore the original frequency cut-off, and then redefine the coupling-constants to complete the renormalization. The dissipation term is dimensionless, and so it
is held fixed in RG. The first term in Eq. (1) then has
\(\tau\)-dimension \(-1\). Since it is not renormalized by any of
the other terms, the coupling-constant \(C\) renormalizes to
zero by power counting. As we will show in the next sec-
tion, we can put this term equal to zero for the pur-
poses of the RG, since inclusion of it does not change any of
the results. For computations of some correlation functions
in the floating phase, however, \(C\) is dangerously irrele-
vant; some of these functions will depend singularly on
\(C\).

The average in Eq. (2) can be easily performed in the
leading order in \(V\),

\[
\langle \exp [V \sum_{(i,j)} \int_0^\beta d\tau \cos \Delta \theta_{ij}(\tau)] \rangle_{0f} = \exp \frac{1}{2} \sum_{i,j} \int_0^\beta d\tau \sum_{\epsilon = \pm 1} e^{i \epsilon \Delta \theta_{ij}(\tau)} \langle \exp [\Delta \theta_{ij}(\tau)] \rangle_{0f} = \exp \frac{1}{2} \int_0^\beta d\tau \cos \Delta \theta_{ij}(\tau) e^{-\frac{1}{2} \langle (\Delta \theta_{ij}(\tau))^2 \rangle_{0f}},
\]

where,

\[
\langle (\Delta \theta_{ij}(\tau))^2 \rangle_{0f} = \frac{1}{2} \frac{1}{z_0} \sum_{\omega_n / b < |\omega_n| \leq \omega_c} \frac{1}{|\omega_n|} = 2 \ln b \quad z_0 = z/2,
\]

Here \(z_0 = z/2\), where \(z\) is the coordination number of
the lattice. In deriving Eq. (4) we have taken \(C\) to be zero,
which, as mentioned before, is allowed for the present
purpose. After rescaling the time, \(\tau' = \tau / b\), and taking
\(b = e^t\), where \(t > 0\) is infinitesimal, we end up with a
linear order RG flow equation for \(V\),

\[
\frac{dV}{d\beta} = (1 - \frac{1}{z_0}) V
\]

It is clear from this equation that \(\alpha = \alpha_c = 1/z_0\) is
a critical fixed point for the flow of \(V\); for \(\alpha < \alpha_c\), \(V\)
scales to zero, while for \(\alpha > \alpha_c\), \(V\) grows. In the phase
where \(V\) scales to zero, the junction-phases are quantum-
disordered and the system is metallic by construction due
to the existence of the slutt resistors. When \(V\) flows to
higher values, the system is phase-ordered and supercon-
cducting.

Now we show that in the metallic phase, all longer-
ranged Josephson couplings beyond the nearest-neighbor
coupling \(V\), which can couple spatially separated \(\Delta \theta_{ij}\)’s
on the array if included in the starting action, are irrele-
vant. These couplings are described by the general term,

\[
S_f / \hbar = - \sum_{si} \sum_j \int_0^\beta d\tau J([s_i]) \cos [\sum_j s_j \theta(\vec{r}_i + \vec{d}_j, \tau)]
\]

Here, \(\theta(\vec{r}_i, \tau)\)’s are the order parameter phases in the
superconducting grains, the \(\vec{d}_i\)’s are arbitrary vectors sepa-
rating the lattice points on the lattice and \(s_i\) is an integer-valued function of the layer number \(i\) satisfying \(\sum_i s_i = 0\). The last constraint ensures the absence of an
external field, which implies “rotation invariance” under
adding the same constant to all of the \(\theta\)’s. Note that
the special case \(s_0 = +1, \vec{d}_0 = \vec{0}, s_1 = -1, \vec{d}_1 = \vec{a}_\gamma,\)
where \(\vec{a}_\gamma\) is a nearest neighbor vector, is just the nearest neighbor
Josephson coupling \(V\) in Eq. (1).

Calculations analogous to that leading to Eq. (5) give,
to first order in \(J\),

\[
\frac{dJ([s_i])}{d\beta} = (1 - \frac{1}{\alpha}) J([s_i]),
\]

with

\[
\Gamma([s_i]) \equiv \sum_{i,j} s_i s_j U(\vec{\delta}_i - \vec{\delta}_j).
\]

The “potential” \(U(\vec{r}) \equiv \frac{1}{\pi} \sum_i e^{i \vec{q} \cdot \vec{r}} / f_{\vec{q}},\) with

\[
J_{\vec{q}} \equiv \sum_i (1 - e^{i \vec{q} \cdot \vec{a}_\gamma}),\]

where the sum over \(\gamma\) is over all nearest neighbors.
It is straightforward to show that \(U(\vec{r})\) is the
“lattice Coulomb potential” of a unit negative charge
at the origin, with the zero of the potential set at \(\vec{r} = \vec{0}\).
That is, \(U(\vec{r})\) satisfies the “lattice Poisson equation”:

\[
\sum_i U(\vec{r} - \vec{a}_\gamma) - z U(\vec{r}) = -\delta_{\vec{r}, \vec{0}}.\]

For a symmetrical (e.g., square, hexagonal, cubic) lattice,
the left-hand side is just the “lattice Laplacian,” appro-
aching \(\nabla^2 U \times O(a^2)\) where \(a \equiv |\vec{a}_\gamma|\).

The quantity \(\Gamma\) in (7) and (8) is then just
equal to the potential energy of a neutral (since \(\sum_i s_i = 0\))
plasma of quantized (since the \(s_i\)’s are integers) charges \(s_i\)
on the lattice. The most relevant \(J([s_i])\) is clearly the one
that corresponds, in this Coulomb analogy, to the lowest
interaction energy. Note that strictly speaking \(\Gamma\)
corresponds to twice this energy, because the sum in \(\Gamma\)
double counts. Apart from the trivial configuration in
which all the \(s_i = 0\), the lowest energy configuration is
clearly the one in which there are two equal and op-
posite unit magnitude charges on nearest neighbor sites:
i.e., \(s_0 = +1, \vec{d}_0 = \vec{0}, s_1 = -1, \vec{d}_1 = \vec{a}_\gamma\). As we discussed
earlier, this corresponds to the nearest-neighbor Joseph-
son coupling in equation (1). Thus, it is established that
that coupling is, indeed, the most relevant, as we asserted
earlier. Furthermore, using simple symmetry arguments,
we can show that for a symmetric lattice (e.g., square,
hexagonal, cubic), where all nearest-neighbor sites are
equivalent, \(U(\vec{a}_\gamma) = -\frac{1}{2}\) which recovers the recursion
relation (6) for \(V\). As a result, all other couplings are
irrelevant for \(\alpha \leq \alpha_c = 1/z_0\); hence, they affect neither
the floating phase nor the transition between it and the
\((D + 1)\)-dimensionally coupled phase.

Thus, the metallic phase is in a sense spatially decou-
pled. This decoupling of space, but finite correlation in
time, can be more clearly seen by the correlation func-
tions in the floating phase discussed in the next section.
III. CORRELATION FUNCTIONS IN THE FLOATING PHASE

In this section, we first show that the variable $C$ is dangerously irrelevant in $D \leq 2$ for some correlation functions in the floating phase, but not for the ones required for the RG. Including all relevant and dangerously irrelevant variables, we will then compute various long-time correlation functions (Eqs. 13, 14, 15, 16) and show that they are finite. Finally, the spatially long-range correlations (Eqs. 19, 21) will be shown to be exponentially decaying with finite correlation lengths at a non-zero temperature and exactly zero in the limit of zero temperature. Thus, the intrinsically local and dynamic character of the floating phase will be established, indicating that the dissipation-driven QPT between this phase and the global superconductor is governed by the same $(0 + 1)$-dimensional physics as in the corresponding zero-dimensional transition.\cite{3,10,11,12}

From the gaussian part of the action in Eq. 11 we find that

$$\langle |\theta(k\omega_n)|^2 \rangle = \frac{1}{C\omega_n^2 + (\alpha/2\pi)f(k)|\omega_n|}. \quad (9)$$

Here $f(k) = \sum_{\delta} (e^{ik.\delta} - 1)^2$, where $\delta$'s constitute the set of nearest neighbor vectors. Since $f(k) \sim z_0k^2$ for small $k$, it is easy to see that in dimension $D \leq 2$, wavenumber integration of the right-hand-side (RHS) of Eq. 9 diverges in the infrared for $C \rightarrow 0$. This divergence forces us to include a non-zero $C$ for the correct evaluation of some of the correlation functions in the floating phase, even though $C$ is an irrelevant variable in the RG sense. The recursion relations, however, remain unaffected as we show below.

The only correlation functions we need for the purposes of the RG involve the variables defined on a bond, $\Delta\theta_{ij}(\tau)$, the simplest of which is given in Eq. 9 where we have taken $C$ to be zero. In calculations higher order in $V$, we will need higher order correlators $\sim \langle \Delta\theta_{ij}(\tau)\Delta\theta_{kl}(\tau) \rangle_{ij}$, still among the bond-variables. For such functions, it is easy to see that including a non-zero $C$ on the RHS of Eq. 9 only amounts to introducing a high-frequency cut-off $E_0 \sim 1/C$ in the frequency integrals. In the case of $\langle \Delta\theta_{ij}(\tau)^2 \rangle$, summing over $j$ over the nearest-neighbor vectors assuming symmetry among the nearest neighbors, dividing by $z_0$, and using Eq. 9 we get in $D = 1$,

$$\langle \Delta\theta_{ij}(\tau)^2 \rangle = \frac{1}{z_0} \int_0^\Lambda \frac{dk}{2\pi} \sum_{\frac{1}{2} \leq \omega_n} f(k) \langle |\theta(k\omega_n)|^2 \rangle$$

$$= \frac{1}{z_0} \int_0^\Lambda \frac{dk}{2\pi} \sum_{\frac{1}{2} \leq \omega_n} C\omega_n^2 + (\alpha/2\pi)f(k)|\omega_n|'$$

where $\Lambda$ is a momentum cut-off. It is easy to see that since $f(k)$ appears both in the numerator and the denominator, the infrared divergence in the wavenumber integration is eliminated. Since in the limit of small $k$ the integral is zero, $f(k)$ is order one in the integral, hence $C\omega_n^2$ can be neglected with respect to $(\alpha/2\pi)f(k)|\omega_n|$ for small $\omega_n$. An explicit evaluation of the integral shows the same logarithmically divergent behavior as in Eq. 9 \(\langle \Delta\theta_{ij}(\tau)^2 \rangle = \frac{1}{z_0} \ln(\beta E_0)\), where $E_0 = K/C$. Hence, for the purposes of the RG, taking $C = 0$ in the expression for $\langle |\theta(k,\omega_n)|^2 \rangle$ and taking a compensatory high-frequency cut-off $\sim 1/C$ are indeed justified.

From $\langle \Delta\theta_{ij}(\tau)^2 \rangle$, we can calculate

$$\langle \exp(iq\Delta\theta_{ij}(\tau)) \rangle = \frac{1}{(\beta E_0)^{q^2/2\pi^2}}. \quad (10)$$

Here, $q^2 = \min_{n \in Z} (q - n)^2$, and $q$ is a real number. Note that for $q = 1$, the simplest such function that one can construct, the RHS of Eq. 10 does not vanish at zero temperature since the corresponding $q'$ vanishes. This is analogous to a single paramagnetic spin developing a non-zero expectation value in the presence of an applied external field. In the present case, the Josephson coupling $V$ acts as the field inducing a non-zero expectation value of $e^{i\Delta\theta_{ij}}$, even in the complete absence of all other couplings. This expectation value does not constitute an order parameter since the broken symmetry is not spontaneous, but merely induced by the applied field. It forces us, however, to introduce the parameter $q$ to get correlation functions which do decay algebraically as $T \rightarrow 0$.

The result for $q' \rightarrow q$ is a well-known result in surface roughening,\cite{13,14} which we shall sketch here for the sake of completeness. The result, $\langle \Delta\theta_{ij}(\tau)^2 \rangle = 2\ln(\beta E_0)/z_0\alpha$, correctly gives the low-temperature fluctuations of $\Delta\theta_{ij}$ throughout the floating phase, up to an unimportant additive constant. Since this is valid irrespective of the actual bare value of $V$, we can apply it even in the limit of large $V$. But since in this limit $\Delta\theta_{ij}$ is quantized in integral multiples of $2\pi$, $\Delta\theta_{ij} = 2\pi n$ with $n$ an integer, we can write for the function in Eq. 10.

$$\langle \exp[iq\Delta\theta_{ij}(\tau)] \rangle = \sum_{n=-\infty}^{\infty} P(n) \exp(2\pi iqn)$$

$$= \sum_{n=-\infty}^{\infty} \left[ \frac{\pi z_0\alpha}{\ln(\beta E_0)} \right]^{1/2} \exp \left[ 2\pi iqn - \frac{n^2\pi^2 z_0\alpha}{\ln(\beta E_0)} \right]. \quad (11)$$

Here $P(n)$ is the gaussian distribution function for $n$ with $\langle n \rangle = \frac{1}{\sigma_n} \langle \Delta\theta_{ij}(\tau) \rangle = 0$, the standard deviation $\sigma_n = \sqrt{\frac{1}{2\pi} \langle \Delta\theta_{ij}(\tau)^2 \rangle}$, and the sum over $n$, instead of an integral over a continuous variable, takes care of the quantization condition on $\Delta\theta_{ij}$. Doing this sum using the Poisson summation formula gives

$$\langle \exp[iq\Delta\theta_{ij}(\tau)] \rangle = \sum_{s=-\infty}^{\infty} (\beta E_0)^{(q-s)^2/(z_0\alpha)}. \quad (12)$$

Clearly, the dominant term in this sum as $T \rightarrow 0$ is the one with the smallest value of $(q-s)^2$. Keeping just this
term yields Eq. 10. This shows that this correlation function indeed goes to zero as the temperature goes to zero. The unequal-time correlation function of the bond-variable is also algebraic,

$$\langle \exp[iq(\Delta \theta_{ij}(\tau) - \Delta \theta_{ij}(0))] \rangle = \frac{1}{(2\pi\epsilon)^{2q^2/\alpha z_0}}.$$  \hfill (13)

The correlation function of the bond variables algebraically decaying in time, Eq. 13, is valid in all dimensions. The same is not true, however, with the correlation functions of the site variables, $\theta_i(\tau)$, where $i$ is a grain index, since now for $D \leq 2$ the wavenumber integration of $\langle \theta(\mathbf{k}, \omega) \rangle^2$ is divergent as $C \to 0$. For $D > 2$, the effect coming from $C$ is subleading and the long-time correlations remain algebraically decaying. In $D = 3$, for the spatially local but unequal in time correlator we get, ignoring unimportant constant factors,

$$\langle \exp[iq(\theta_i(\tau) - \theta_i(0))] \rangle \sim \frac{1}{(2\pi\epsilon)^{2q^2/\alpha z_0}},$$  \hfill (14)

where $\Lambda$ is the momentum cut-off. In $D = 2$, the same correlation function is given by,

$$\langle \exp[iq(\theta_i(\tau) - \theta_i(0))] \rangle \sim \exp \left[ -\frac{q^2}{4\pi\alpha z_0} (\ln \frac{\tau\alpha z_0}{2\pi C})^2 \right],$$  \hfill (15)

while in $D = 1$ it is,

$$\langle \exp[iq(\theta_i(\tau) - \theta_i(0))] \rangle \sim \exp \left[ -\sqrt{(2)q^2} \left( \frac{\tau}{\pi\alpha z_0 C} \right)^{1/2} \right].$$  \hfill (16)

Even though the long-time behavior is algebraic only for the bond-variables, Eq. 13 and for the site variables only for $D = 3$ and above, Eq. 14 temporally far-separated sites are still correlated for $D \leq 2$ according to Eqs. 15, 16. However, long-range spatial correlation functions of the grains, $\langle \exp[iq(\theta_i(\tau) - \theta_{i+r}(\tau))] \rangle$, are identically zero at zero temperature in any dimension. Spatially far-separated grains are completely uncorrelated, each one of them constitutes a dynamic local system on its own. In this sense, the temporal and spatial behaviors of the system are completely decoupled at $T = 0$. It is also a manifestation of the intrinsic local nature of the problem. At nonzero $T$, the spatial correlation functions are exponentially decaying in all dimensions; the grains are only short-range-correlated.

As we show in appendix A, the correlation function of the phases of the order parameters on the grains follow the simple scaling law,

$$\langle \theta_i(\tau)\theta_{i+r}(\tau) \rangle = \frac{2\pi r^{2-D}}{\alpha z_0} g(r/\xi),$$  \hfill (17)

where, $\xi$ is the correlation length at nonzero $T$, $\xi = (\alpha z_0/2\pi CT)^{1/2}$. As we will see below, even though $\xi$ diverges as $T \to 0$, there is very little correlation left between the order parameters themselves at low temperatures, and eventually the order parameter correlation function vanishes at $T = 0$. In $D = 3$, we find,

$$g(r/\xi) = \frac{1}{2\pi^2} \frac{\xi}{r} \exp(-r/\xi), \quad r >> \xi$$

$$= \frac{1}{2\pi^2} \ln(\xi/r), \quad r << \xi$$  \hfill (18)

This gives for the correlation function of the order parameters,

$$\langle \exp[iq(\theta_i(\tau) - \theta_{i+r}(\tau))] \rangle_c = T \frac{2\pi^2}{\pi\alpha z_0} \frac{q^2}{\pi\alpha z_0 r} e^{-\frac{q^2}{\pi\alpha z_0 r}}, \quad r >> \xi$$

$$= T \frac{2\pi^2}{\pi\alpha z_0} \left[ \frac{\xi}{r} \frac{q^2}{\pi\alpha z_0 r} - 1 \right], \quad r << \xi$$  \hfill (19)

where the subscript $c$ denotes the connected piece of the correlation function. It is clear that as $T \to 0$, these functions vanish for large $r$. Hence, there is no correlation among the spatially far separated grains at zero temperature. In $D = 2$, we get for the scaling function $g(r/\xi)$,

$$g(r/\xi) = \frac{1}{\sqrt{(2\pi^3)}} \frac{(\xi)}{r^2} \exp(-r/\xi), \quad r >> \xi$$

$$= \frac{1}{2\pi^2} [\ln(2\xi/r)]^2, \quad r << \xi$$  \hfill (20)

The order parameter correlation function becomes,

$$\langle \exp[iq(\theta_i(\tau) - \theta_{i+r}(\tau))] \rangle_c = \exp \left[ -\frac{q^2 (\ln \frac{\alpha z_0}{2\pi C})^2}{4\pi\alpha z_0} \right] \left( \frac{2}{\pi} \right)^{1/2} \frac{q^2}{\alpha z_0} \frac{\xi}{r} \exp(-r/\xi), \quad r >> \xi$$

$$= \exp \left[ -\frac{q^2 (\ln \frac{\alpha z_0}{2\pi C})^2}{4\pi\alpha z_0} \right] \left( \frac{\exp([q^2 \ln(r/2\xi)]^2)}{\pi\alpha z_0} - 1 \right), \quad r << \xi$$  \hfill (21)

It is straightforward to check that this function is also zero at $T = 0$. Finally, in $D = 1$, $g(r/\xi)$ is given by,

$$g(r/\xi) = \frac{1}{\pi} \frac{\xi}{r} \exp(-r/\xi), \quad r >> \xi$$

$$= -\frac{1}{\pi} \ln(\xi/r) + \frac{1}{\pi} \xi \exp(-r/\xi), \quad r << \xi$$  \hfill (22)
The corresponding order parameter correlation function is,

\[
\langle \exp[iq(\theta_i(\tau) - \theta_{i+r}(\tau))] \rangle_c = \exp\left[ -\frac{q^2 \sqrt{2(2\beta)}}{\sqrt{\pi \alpha z_0 C}} \right] \left( \frac{2q^2}{\alpha z_0} \right)^{\xi^2} \exp(-r/\xi), \quad r >> \xi
\]

\[
= (r/\xi)^{(2q^2 r)/(\alpha z_0)} - \exp\left[ -\frac{q^2 \sqrt{2(2\beta)}}{\sqrt{\pi \alpha z_0 C}} \right], \quad r << \xi
\]

(23)

This function also approaches zero as \( T \to 0 \). Thus, we have established that the spatially far-separated grains are uncorrelated at \( T = 0 \), and correlated only with exponentially decaying correlations at a non-zero temperature.

IV. RG AT SECOND ORDER IN \( V \)

To establish the existence of a locus of critical points in the \( \alpha - V \) plane, we need to compute the higher-order corrections, if any, to the first order recursion relations for \( V \), Eq. (3). Performing the cumulant expansion of Eq. (2) in second order in \( V \), we get

\[
\exp \left[ \frac{V^2}{2} \left( \int_0^\beta d\tau \int_0^\beta d\tau' \sum_{\langle i, j \rangle} \cos \Delta \theta_{ij}(\tau) \cos \Delta \theta_{ij}(\tau') \right) \right] = \exp\left[ e^{-\frac{2im\alpha}{\beta}} \left( \frac{V^2}{4} \int_0^\beta d\tau \int_0^\beta d\tau' \sum_{\langle i, j \rangle \langle k, l \rangle} \left[ \cos(\Delta \theta_{ij}(\tau)) \cos(\Delta \theta_{kl}(\tau')) - \cos(\Delta \theta_{ij}(\tau)) \cos(\Delta \theta_{kl}(\tau')) \right] \right) \right]
\]

(24)

To avoid the generation of spurious long-ranged behavior in the correlation function of the fast modes, \( \langle \Delta \theta_{ij}(\tau) \Delta \theta_{kl}(\tau') \rangle_{0f} \), we need to adopt a smooth cut-off prescription in our frequency-shell RG. For the unequal-time correlation function on the same bond, this can be written as

\[
G_{ij,ij}^f ((\tau - \tau'), b) \equiv \langle \Delta \theta_{ij}^f(\tau) \Delta \theta_{ij}^f(\tau') \rangle_{0f} = \frac{1}{z_0 \beta} \sum_{|\omega_n| < \omega_c} \frac{\exp(i\omega_n(\tau - \tau'))}{\alpha/2\pi|\omega_n|} f(\omega_n/(\omega_c^2 b))
\]

(25)

where \( f(x) \) is a smoothing function with the properties \( f(x) \to 0 \) for \( x << 1 \), and \( f(x) \sim 1 \) for \( x >> 1 \). The choice of \( f(x) \) is somewhat arbitrary as long as it entails the properties, \( G_{ij,ij}^f ((\tau - \tau'), b) \to 0 \) for \( |\tau - \tau'| \to \infty \), and \( G_{ij,ij}^f ((\tau - \tau'), b) \to 0 \) for \( b \to 1 \), to the correlation function of the fast modes. A specific choice is given in Ref. 12, where \( G \) goes to zero exponentially with \( |\tau - \tau'| \) for large \( |\tau - \tau'| \), and \( G_{ij,ij}^f(0, b = \frac{2\ln b}{\xi_0}) \). This does not modify any of our previous calculations and we will assume that \( G_{ij,kl}^f(\tau = \tau', b) \) is short-ranged in \( |\tau - \tau'| \). The \( \tau \) and \( \tau' \) in the integrals of Eq. (24) are now constrained to be close to each other and we can use gradient-expansion in \( (\tau - \tau') \) to simplify the terms within parenthesis. It is clear that the second term does not renormalize \( V \). The first term, however, upon gradient expansion generates \( \cos(\Delta \theta_{ij}(\tau) + \Delta \theta_{kl}(\tau)) \sum_{\langle i, j \rangle} \cos(\Delta \theta_{ij}(\tau)) \) summed over all bonds \( \langle i, j \rangle \) and \( \langle k, l \rangle \).

It is clear that there is no renormalization of \( V \) in second order in one dimension. In the sum over \( \langle i, j \rangle \) and \( \langle k, l \rangle \), if \( \langle i, j \rangle = \langle k, l \rangle \), one generates a term \( \sim \exp(2\Delta \theta_{ij}(\tau)) \) which is irrelevant close to \( \alpha_c \). The second term also produces an irrelevant term, \( \sim \left. \frac{\partial \Delta \theta_{ij}(\tau)}{\partial \tau} \right|_{\tau = \tau'} \). For \( \langle i, j \rangle \neq \langle k, l \rangle \), too, it is easy to see that there is no renormalization of \( V \), since \( \cos(\Delta \theta_{ij} + \Delta \theta_{kl}) \) does not produce \( \cos(\Delta \theta_{mn}) \). In one dimension there is no closed loop on the lattice to do this. Contrast this with \( D = 2 \) triangular lattice, where because of the existence of closed loops in the lattice cos(\( \Delta \theta_{mn} \)) is produced from cos(\( \Delta \theta_{ij} + \Delta \theta_{kl} \)) when the bonds \( \langle i, j \rangle, \langle k, l \rangle, \langle m, n \rangle \) form the three sides of a minimum triangle. Similar effects exist for any lattices in \( D > 1 \). Coming back to \( D = 1 \), we have checked explicitly, as shown in the appendix, that there is no renormalization of \( V \) also at the third order. Thus we speculate, and confirm up to third order expansion in \( V \), that in \( D = 1 \) there is no correction at all at any higher order to the linear order recursion relation Eq. (4). This was conjectured in the context of the corresponding 0-dimensional problem in Ref. 12, here we have extended it to one dimension. Notice that in Eq. (4) because of the existence of the \( C \)-term, the one dimensional problem does not map onto the zero dimensional problem in any obvious way. We will see below that for \( D > 1 \) there are corrections to Eq. (6) at higher orders.

In 2D triangular array, and in fact in arbitrary dimensions where the minimum closed loop is a triangle, there is a contribution at order \( V^2 \) to the recursion relation for \( V \). This comes from the first term in (24), where, in the sum over \( \langle k, l \rangle \), when \( \langle k, l \rangle \) is the bond adjacent to \( \langle i, j \rangle \) in a minimum triangle, the sum over \( \langle i, j \rangle \) and \( \langle k, l \rangle \) produces a single sum over all bonds in the lattice. This is illustrated by Fig.(1a). If \( \langle i, j \rangle = < 1, 2, > \), and
Using recursion relations for in turn generate higher order corrections to the recursion function values of these correlation functions are somewhat arbitrary since they depend on the choice of the smoothing function \( f(x) \). Assuming they are finite only close to the linear order in \( \tau \), where \( \tau > 3 \), the order \( \tau \) integral in Eq. (26) is given by \( \tau_0 \). Using \( G_{12,12}(0) = \frac{2 \ln b}{3 \tilde{\alpha} \tilde{\alpha}} \), and rescaling \( \tau \), we find the recursion relation for \( V \) at second order in \( \tilde{\alpha} \)

\[
\frac{dV}{dl} = V(1 - \frac{1}{2 \tilde{\alpha} \tilde{\alpha}}) + C_1 V^2,
\]

where \( C_1 \) is a positive constant \( C_1 = \tau_0/4 \tilde{\alpha} \tilde{\alpha} \). In 2D square lattice, and also in a lattice in any dimension where the minimum closed loop has a number of sides greater than three, the first non-zero contribution above the linear order in \( V \) occurs at order \( V^3 \). Interestingly, this can already be seen at the level of second order cumulant expansion of the interactions. This is illustrated by Fig. (1b). Note that in the square lattice, the first term in Eq. (24), upon the usual gradient expansion, generates a next-nearest-neighbor interaction \( J \). In a consistent RG-treatment, \( J \) (and in fact all such longer-ranged Josephson interactions) should be included in the starting action. As has been shown before, they are all irrelevant in the floating phase as long as \( V \) is irrelevant. However, as soon as \( V \) becomes relevant at \( \alpha_c \), their recursion relations pick up source terms of order \( V^2 \). They in turn generate higher order corrections to the recursion relations for \( V \). It is clear that the lowest order correction comes from the next-nearest-neighbor interaction \( J \). Including this term in the starting action in Eq. (11), and with the help of Fig. (1b) and Eq. (4), we find for the recursion relation for \( J \),

\[
\frac{dJ}{dl} = (1 - \frac{1}{2 \tilde{\alpha} \tilde{\alpha}})J + (\frac{0.19 \tau_0}{\tilde{\alpha} \tilde{\alpha}}) V^2,
\]

where \( \tau_0 \) is defined via \( \int df d(\tau - \tau') [\exp(-G_{12,23}(\tau - \tau')) - 1] = \tau_0 [\exp(-G_{12,23}(0)) - 1] \). By an exactly analogous calculation, we get for the recursion relation for \( V \),

\[
\frac{dV}{dl} = (1 - \frac{1}{2 \tilde{\alpha} \tilde{\alpha}})V + (\frac{0.63 \tau_0''}{\tilde{\alpha} \tilde{\alpha}}) V J,
\]

where \( \tau_0'' \) is defined by a third integral, \( \int df d(\tau - \tau') [\exp(-G_{13,34}(\tau - \tau')) - 1] = \tau_0'' [\exp(-G_{13,34}(0)) - 1] \).

Solving for \( \frac{dV}{dl} = 0 \) at \( \alpha = \alpha_c \), we find \( J \sim V^2 \), and putting it back in Eq. (26) we get,

\[
\frac{dV}{dl} = (1 - \frac{1}{2 \tilde{\alpha} \tilde{\alpha}})V + C_2 V^3,
\]

where \( C_2 \) is a positive constant given in terms of the time cut-offs, \( C_2 = \frac{0.46 \tau_0''}{\tilde{\alpha} \tilde{\alpha}} \).

There is a possibility of an order \( V^3 \) term appearing in the recursion relation for \( V \) at third order expansion of the interactions for the square lattice. This contribution may come from a term of the form

\[
\frac{1}{24} \int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \int_0^\beta d\tau_3 \sum_{(i,j,k,l)} \sum_{(m,n)} \cos[\Delta \theta_{ij}^s(\tau_1) + \Delta \theta_{kl}^s(\tau_2) + G_{ijkl,mn}(\tau_2 - \tau_3) + G_{mn,ij}^l(\tau_3 - \tau_1) - 1],
\]

which appears in the third order cumulant expansion of Eq. (24). We have checked explicitly, as shown in the appendix [B] that this does not happen. In fact, the contribution to Eq. (30) from third order cumulant expansion is precisely zero. Thus there is no cancellation of the order \( V^3 \) term we have found above. Note that for any lattice, where the minimum closed loop has number of sides greater than three, the order \( V^3 \) term appears via the generation of the next-nearest-neighbor interaction in the RG. Thus 2D honeycomb lattice, for example, will also have a recursion relation for \( V \) similar to Eq. (30).

Combined with the flow equation for \( \alpha, \alpha \frac{d\alpha}{dt} = 0 \), which is true in our perturbative treatment close to \( \alpha_c \) for all lattice types, we have sketched in Fig. (1) the perturbative RG flows for 1D, 2D triangular lattice and 2D square lattice.
V. EXPERIMENTAL CONSEQUENCES

Certain aspects of our theory can be seen in a recent experiment (see also references therein), although some caution must be exercised. In this paper, a phase transition is shown to exist at \( \alpha_c = 1/z_0 \), for small values of \( V/E_0 \), where \( z_0 = 2 \) for the square lattice fabricated in the experiment. This is consistent with the predictions first given in Ref. 6 and later in Ref. 13 from a single junction approach. Moreover, the experimental phase diagram, especially the absence of a dissipation-driven transition above a certain value of \( V/E_0 \), bears a remarkable resemblance with what can be trivially inferred from our Fig. 1. There are two issues that deserve further attention: (1) the quantum disordered state of the resistively shunted Josephson junction array is metallic, not an insulator, as inferred in this experiment. This is simply because the metallic shunts will short the current when the superconducting phase difference between the junctions become incoherent, unless disorder effects in two dimensions turn this metallic behavior to an insulating behavior. However, the experimental system is claimed to be free of disorder. This metallic behavior was not tracked down in this experiment, but only the insulating behavior of the unshunted array; further experiments can clearly be revealing; (2) the correct model of the capacitances must also include longer ranged contributions, as they are obviously present in the experiment.

Given the novelty of the floating phase, further experiments will be a worthwhile effort. For values of \( R > R_Q = \frac{1}{\alpha_c} \), we expect the system to be in the floating phase, where the grains are only short-range-correlated in space. We expect the current-voltage characteristic in this case to be a non-universal power law controlled by dissipation \( \alpha \), similar to the results obtained previously \(^{14}\) in the context of isolated junctions. For \( R \) smaller than the quantum of resistance, the system will be in the superconducting state, and the temperature-dependent non-universal power law will be similar to the classical XY model due to vortex unbinding in the presence of an imposed current. \(^{14}\) Recall that \( V \), along with all other longer range Josephson couplings,\(^{14}\) flow to larger values in this regime. An experimentally testable prediction for a non-universal exponent \( \nu \) controlling the critical scaling of a characteristic temperature \( T_{ch}, T_{ch} \propto (\delta V)^{\nu} \) with \( \delta V \equiv |V-V_c| \) and \( V_c \), the critical value of \( V \) on the \( (V-\alpha) \) plane, was made in our earlier work \(^{14}\). If verified, this will be a clear cut signature of the phase transition studied here. By varying the strengths of the Josephson coupling and the shunt resistors, the critical locus and hence the phase boundary on the \( (V-\alpha) \) plane can also be directly examined.

The ideas presented here assume significance also in a broader context. In a bid to understand some puzzling experimental observations in the context of heavy fermion materials \(^{15}\) and the cuprate superconductors \(^{14}\), QPT’s are sought where extended systems may go through global phase transitions which are ultimately driven by inherent local ones. The transition we have studied in the dissipative Josephson junction array provides just such an example, albeit only in spirit, since we have not treated the fermions here. It will be interesting to extend the ideas and results found here to fermionic problems to make contact with a wide range of systems of experimental interest.

VI. SUMMARY AND CONCLUSION

In this paper, we have clarified in detail the nature of the correlation functions within, and the transition into, the floating phase of a dissipative Josephson junction array. We have shown that the long-time correlations in the floating phase are finite locally for each superconducting grain, but the spatially long-range correlations are identically zero at \( T = 0 \). At any nonzero temperature, the spatial correlation functions are exponentially decaying with a finite correlation length, indicating locally dynamic grains which are only short-range correlated in space. These correlations help identify the dissipation-driven metal to superconductor transition in the extended system of the array with the ‘quantum to classical’ dynamic transition existing at the level of a single junction \(^{12}\) the local transition drives the global one.

Further, we have shown that the transition in the extended system is controlled by a locus of critical points in the \( \alpha - V \) plane with continuously varying critical exponents. To derive our results, we computed the perturbative recursion relations for the nearest neighbor coupling \( V \) up to third order in \( V \). Since \( V \) is the assumed small parameter, the calculation is valid only close to the critical value of \( \alpha = \alpha_c \) needed for ordering with infinitesimal Josephson coupling. We have found no correction to the linear-order recursion relation, Eq. 5 for \( D = 1 \). In this case, then, the fixed line close to \( \alpha_c \) is parallel to the \( V \)-axis. This is a result similar to that obtained for the corresponding zero-dimensional problem treated in Ref. \(^{12}\). Here we have extended their result to \( D = 1 \).

For any \( D > 1 \), however, there are higher order corrections to Eq. \(^{5}\). Coupled with the recursion relation for \( \alpha, \frac{d\alpha}{d\alpha} = 0 \), the locus of critical points bends toward the \( V \)-axis determining a phase boundary. This locus is a straight line in any \( D > 1 \) as long as the number of sides in a minimum closed loop is three, it’s a parabola if that number is greater than three. The functional form of the locus, which determines the phase boundary between the metal and the superconductor, is entirely determined by the local topology of the lattice, and not on the global dimension. Consequently, the locus of critical points exists in any dimension; there is no upper-critical-dimension for the problem. All of these results are consistent with the global transition being driven by the inherent local one.

The correlations in the floating phase and the phase boundary studied here can be experimentally probed by measuring the current-voltage characteristics with the resistance of the shunt resistors continuously tuned. In a
broader context, taken together with our earlier results that all longer-range Josephson interactions induced by the lattice among a set of periodically spaced local systems, which each goes through individual local transitions with dissipation as the tuning parameter, are irrelevant in the RG sense, our results here assume significance in the context of the search for local quantum criticality in extended systems.

VII. ACKNOWLEDGEMENTS

This work was supported by the NSF under grants: DMR-0411931, DMR-0132555 and DMR-0132726. We thank S. Kivelson, B. Spivak, and T. Kirkpatrick for discussions, and the Aspen Center for Physics for their hospitality while a portion of this work was being done. S. T. also thanks the Department of Physics and the Institute of Theoretical Science at the University of Oregon for their hospitality when a part of this work was being completed.

APPENDIX A: SPATIAL CORRELATION FUNCTIONS IN THE FLOATING PHASE

To compute the spatial correlation functions of the phase variables at a temperature \( \frac{1}{\beta} \), we need to evaluate the following integral in frequency and wavenumber space:

\[
\langle \theta_i(\tau)\theta_i(\tau) \rangle = \sum_{\frac{\pi}{\xi} \leq |\omega_n|} \frac{1}{|\omega_n|} \int_0^L \frac{dk}{2\pi} \frac{\exp(ikr)}{2\pi^d \alpha' f(k) + C|\omega_n|}.
\]

where \( \alpha' = \alpha/2\pi \) for notational convenience. Noticing that only \( k \sim 1/r \) modes will contribute significantly to the integral, for large values of \( r \), we expand the function \( f(k) \sim z_0 k^2 \) in the denominator. The \( k \)-space integral is then simply the Fourier-transform of the Yukawa potential at wavenumber space.

1. Correlation function in \( D=3 \)

In \( D = 3 \), doing the \( k \)-space integral, which simply gives the three dimensional Yukawa potential, and after a change of variable, we get

\[
\langle \theta_i(\tau)\theta_i(\tau) \rangle = \frac{1}{2\pi^2 \alpha' z_0} \frac{1}{r} \int_0^\infty \frac{du}{u \exp(-u)},
\]

where \( \xi^{-1} \equiv \xi(T)^{-1} = \sqrt{\frac{CT}{\alpha' z_0}} \). The \( u \)-integral produces the exponential-integral function and the exact correlation function is given by,

\[
\langle \theta_i(\tau)\theta_i(\tau) \rangle = -\frac{1}{2\pi^2 \alpha' z_0} \frac{1}{r} Ei(-r/\xi).\]

Taking the asymptotic values of \( Ei(x) \) for large and small \( |x| \), we end up with Eq. 15. With the help of the result \( \langle \theta_i(\tau) \rangle = \frac{\lambda}{2}\pi \alpha' z_0 \ln(\beta) \), we get Eq. 16.

2. Correlation function in \( D=2 \)

In \( D = 2 \), the angular integration in \( k \)-space produces \( J_0(kr) \), where \( J_0(z) \) is the Bessel function of the first kind of order zero. The remaining part of the \( k \)-space integral produces the function \( 2\pi K_0((\alpha' z_0)^2 r) \), where \( K_0(z) \) is simply related to the Hankel function of the first kind \( H_0^1(z) \). After a change of variable, we end up with the integral,

\[
\langle \theta_i(\tau)\theta_i(\tau) \rangle = \frac{1}{\pi^2 \alpha' z_0} \int_0^\infty \frac{du}{u} K_0(u). \tag{A4}
\]

For \( r >> \xi \), we use the asymptotic expansion of the function \( K_0(u) \) for large \( u \), and after doing the \( u \)-integral we get the result

\[
\langle \theta_i(\tau)\theta_i(\tau) \rangle_{r>>\xi} = \frac{1}{\pi^2 \alpha' z_0} \left( \frac{\xi}{2r} \right)^{3/4} \exp(-r/2\xi) \times W_{-3/4,-1/4}(r/\xi), \tag{A5}
\]

where \( W_{\lambda,\mu}(z) \) is the Whittaker function. Asymptotically expanding the Whittaker function for \( r >> \xi \), we end up with the first line of Eq. 20. In the limit \( r << \xi \), we notice that the integral in Eq. 20 is dominated by small \( u \). Expanding \( K_0(u) \) for small \( u \), and doing the \( u \)-integral we get the remaining part of Eq. 20. With the additional result in \( D = 2 \), \( \langle \theta_i(\tau) \rangle^2 = \frac{1}{8\pi^2 \alpha' z_0} (\ln \frac{\beta \alpha' z_0}{C})^2 \), we finally get Eq. 21.

3. Correlation function in \( D=1 \)

In \( D = 1 \), after doing the \( k \)-space integral and a change of variable, we get,

\[
\langle \theta_i(\tau)\theta_i(\tau) \rangle = \frac{r}{\pi \alpha' z_0} \int_{r/\xi}^\infty \frac{du}{u^2} \exp(-u) = \frac{r}{\pi \alpha' z_0} [Ei(-r/\xi) + \frac{\xi}{r} e^{-\xi}], \tag{A6}
\]

Asymptotically expanding the exponential-integral function for large \( r/\xi \), the first term in the expansion cancels the second term within the square bracket of Eq. A6.

The first subleading term in the expansion produces the first line in Eq. 22. Expansion of \( Ei(r/\xi) \) for small values of the argument produces the remaining part. With the additional result that in \( D = 1 \), \( <\theta_i(\tau)\rangle^2 > = \frac{1}{\pi \sqrt{\alpha' z_0}} g^{1/2} \), we end up with Eq. 23.

Finally, by doing the change of variables \( \beta \omega = \Omega \) and \( kr = Q \) where \( r \) is the magnitude of the spatial coordinate \( r \), and taking \( 1/(\alpha' z_0) \) outside the integral in Eq. A1, we get the scaling relation Eq. 17.
APPENDIX B: RG FROM THE THIRD ORDER CUMULANT EXPANSION IN V

Here we show that the term of order $V^3$ in the recursion relation for $V$, found from the cumulant expansion of the partition function of Eq. 2 at third order, identically vanishes for any lattices in arbitrary dimensions. Using the formula for third order cumulant expansion,

$$
\langle \exp[f] \rangle = \exp(\langle f \rangle + \frac{1}{2}(\langle f^2 \rangle - \langle f \rangle^2)(1 - \langle f \rangle) + \frac{\langle f^3 \rangle}{6} - \frac{\langle f \rangle^3}{6},
$$

We get, for $f = V \int_0^\beta d\tau \sum_{\langle i,j \rangle} \cos \Delta \theta_{ij}(\tau),$

$$
\text{exp}[f] = \exp\left[V e^{-\ln b/\alpha} \int_0^\beta d\tau \sum_{\langle i,j \rangle} \cos \Delta \theta_{ij}(\tau) + \left(\frac{V^2}{4} e^{-2\ln b/\alpha} \sum_{\langle i,j \rangle} \sum_{\langle k,l \rangle} \int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \left[ \cos(\Delta \theta_{ij}(\tau_1) + \Delta \theta_{kl}(\tau_2)) \right. \right.

\times \left. (e^{-G_{ijkl}(\tau_1 - \tau_2)} - 1) + \left. \cos(\Delta \theta_{ij}(\tau_1) - \Delta \theta_{kl}(\tau_2)) \times (e^{G_{ijkl}(\tau_1 - \tau_2)} - 1) \right]\left(1 - V e^{-\ln b/\alpha} \int_0^\beta d\tau_3 \sum_{\langle m,n \rangle} \cos \Delta \theta_{mn}(\tau_3)\right) \right) + \langle f^3 \rangle_{0f} - \langle f^3 \rangle_{0f}
$$

After evaluating the correlation functions of the fast modes, $G_{ijkl}(\tau_1 \tau_2$)-s, doing gradient expansions in $(\tau_2 - \tau_1)$ and $(\tau_3 - \tau_1)$ (it can be seen by straightforwardly writing out $(\langle f^3 \rangle_{0f} - \langle f^3 \rangle_{0f})$ in the above expression that, all terms where $\tau_1, \tau_2$ and $\tau_3$ are not constrained to be close to each other cancel among themselves), and finally rescaling $\tau_1$, a general term that may contribute to renormalize $V$ at third order has the form (with $\ln b = l$),

$$
V^3 e^{-3\ln b/\alpha} \int_0^\beta d\tau_1 \sum_{\langle i,j \rangle} \sum_{\langle k,l \rangle} \sum_{\langle m,n \rangle} \cos \left(\Delta \theta_{ij}(\tau_1) + \Delta \theta_{kl}(\tau_1) + \Delta \theta_{mn}(\tau_1)\right) \times \int_{-\tau_1}^{\beta} d(\tau_2 - \tau_1)

\int_{-\tau_1}^{\beta} d(\tau_3 - \tau_1) \left(\exp(-G_{ijkl}(\tau_2 - \tau_1) - G_{klmn}(\tau_3 - \tau_2)

- G_{mnij}(\tau_3 - \tau_1)) - 1\right)
$$

Note that the $G_{ijkl}(\tau_1 \tau_2$)-s are evaluated within a thin shell $\omega c e^{-l} < |\omega| < \omega c$ around the high frequency cut-off. Using the property that they go to zero as $l \to 0$, we have to expand their exponentials appearing in the above expansion in linear order in the infinitesimal parameter $l$ to get their contributions to the recursion relation of $V$. Hence, we do this expansion in $l$ right away in all of the terms appearing in Eq. 12. We also use the trigonometric identity $\cos(A) \cos(B) = \frac{1}{2}[\cos(A + B) + \cos(A - B)]$ to simplify the second term of the argument of the exponential in Eq. 12. Then writing out $(\langle f^3 \rangle_{0f} - \langle f^3 \rangle_{0f})$ using three variables $\epsilon_1, \epsilon_2, \epsilon_3$, which take the values $\pm 1$, performing the averages with respect to the fast modes (note that no higher order correlation functions arise, only the familiar $G_{ijkl}(\tau_1 - \tau_2)$ with various combinations of the indices suffice), and using all eight permutations of the values of the $\epsilon$-variables, we get sixteen different terms at order $V^3$. Denoting $\cos \left(\Delta \theta_{ij}(\tau_1) + \Delta \theta_{kl}(\tau_2) + \Delta \theta_{mn}(\tau_3)\right)$ by $(ij + kl + mn)$ and so on, and omitting the summation and integration signs along with a multiplicative factor exp($-3\ln b/\alpha$), they are,

$$
V^3 \left(\frac{1}{8} \left(\langle i+j+kl+mn \rangle \left(\frac{2}{3} G_{ijkl}(\tau_1 - \tau_2) - \frac{1}{3} G_{ijkl}(\tau_1 + \tau_2) - \frac{1}{3} G_{klmn}(\tau_2 - \tau_3) - \frac{1}{3} G_{mnij}(\tau_3 - \tau_1)\right) + \left(\frac{2}{3} (ij + kl + mn) G_{ijkl}(\tau_1 - \tau_2) - \frac{1}{3} (kl + mn - ij) G_{klmn}(\tau_2 - \tau_3) - \frac{1}{3} (mn + ij - kl) G_{mnij}(\tau_3 - \tau_1)\right) - \left(\frac{4}{3} (ij - kl + mn) G_{ijkl}(\tau_1 - \tau_2) - \frac{1}{3} (kl + mn - ij) G_{mnij}(\tau_3 - \tau_1)\right)\right)\right).
$$

Keeping track of the signs, using the pairwise symmetries among the indices $(\langle i, j \rangle, \langle k, l \rangle, \langle m, n \rangle)$, and using the property that $G_{ijkl}(\tau_1 - \tau_2)$ is even in space and
time, it is straightforward to see that all of the terms in Eq. [34] cancel among themselves. Hence, the third order cumulant expansion of Eq. [2] does not renormalize $V$. Note that, to get this result, we did not have to assume any particular lattice-type; it’s generally true for all lattices in any dimension. Among other things, we have thus confirmed the conjecture made in Ref. [12] about vanishing of the third order term in the corresponding zero-dimensional problem. Moreover, we have established that the same is true in any higher dimension. This calculation does not imply, however, that there is no order $V^3$ correction to Eq. [5]. As we have seen in Eq. [30], the order $V^3$ term arises already at the second order cumulant level, by generating the diagonal Josephson interaction $J$ in a square lattice. In $D = 1$, however, there is no higher order correction to Eq. [5].

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