HEISENBERG MODEL IN PSEUDO–EUCLIDEAN SPACES

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Abstract
We construct analogues of the classical Heisenberg spin chain model (or the discrete Neumann system) on pseudo–spheres and light–like cones in the pseudo–Euclidean spaces and show their complete Hamiltonian integrability. Further, we prove that the Heisenberg model on a light–like cone leads to a new example of integrable discrete contact system.

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1 Introduction
A pseudo–Euclidean space $E^{k,l}$ of signature $(k,l)$, $k, l \in \mathbb{N}$, $k + l = n$, is the space $\mathbb{R}^n$ endowed with the scalar product

$$\langle x, y \rangle = \sum_{i=1}^{k} x_i y_i - \sum_{i=k+1}^{n} x_i y_i \quad (x, y \in \mathbb{R}^n).$$

A vector $x \in E^{k,l}$ is called space–like, time–like, light–like, if $\langle x, x \rangle$ is positive, negative, or zero, respectively. Denote by $(\cdot, \cdot)$ the Euclidean inner product in $\mathbb{R}^n$ and let

$$E = \text{diag}(\tau_1, \ldots, \tau_n) = \text{diag}(1, \ldots, 1, -1, \ldots, -1),$$

where $k$ diagonal elements are equal to 1 and $l$ to $-1$. Then $\langle x, y \rangle = (Ex, y)$, for all $x, y \in \mathbb{R}^n$.

We consider a discrete system on a pseudo–sphere and on the light-like cone given by

$$S_{c}^{n-1} = \{ \langle q, q \rangle = (q, Eq) = c \},$$

c = \pm 1 and, respectively, $c = 0$. The system is defined by the action functional

$$S[q] = \sum_{k} L(q_k, q_{k+1}),$$

with the discrete Lagrangian

$$L(q_k, q_{k+1}) = \langle q_k, Jq_{k+1} \rangle = (q_k, EJq_{k+1}),$$
where $q = (q_k), k \in \mathbb{Z}$ is a sequence of points on $S^{n-1}_c$ and $J = \text{diag}(J_1, \ldots, J_n), J_i \neq 0, i = 1, \ldots, n$. The equations of the stationary configuration have the form

$$\frac{\partial L(q_k, q_{k+1})}{\partial q_k} + \frac{\partial L(q_{k-1}, q_k)}{\partial q_k} = \lambda_k E q_k, \quad k \in \mathbb{Z},$$

(1.1)

that is,

$$q_{k+1} + q_{k-1} = \lambda_k J^{-1} q_k, \quad k \in \mathbb{Z},$$

(1.2)

where the multipliers are determined by the constraints

$$c = \langle q_{k+1}, q_{k+1} \rangle = \langle q_{k-1}, q_{k-1} \rangle - 2\lambda_k \langle J^{-1} q_k, q_{k-1} \rangle + \lambda_k^2 \langle J^{-2} q_k, q_k \rangle.$$  

(1.3)

Thus either $\lambda_k = 0$ or, in the case $\langle J^{-2} q_k, q_k \rangle \neq 0$, we have another solution

$$\lambda_k = 2 \langle J^{-1} q_k, q_{k-1} \rangle / \langle J^{-2} q_k, q_k \rangle.$$  

(1.4)

We consider the dynamics given by the second expression, which is defined outside the singular set $\langle J^{-2} q_k, q_k \rangle = 0$.

For $n = 3$ and the Euclidean case, the functional defines the energy of a classical spin chain in the Heisenberg model \cite{11, 12}. Also, in the Euclidean case for arbitrary $n$ the equations represent a discretisation of the classical Neumann system \cite{9, 10}. So we refer to (1.2), (1.4) as a Heisenberg model, or a discrete Neumann system in a pseudo–Euclidean space $E^{k,l}$.

We present a matrix Lax representation of the mapping, show that it is symplectic, and prove its complete integrability. Recently, a related elliptical billiard problem in pseudo–Euclidean spaces was studied in \cite{6, 7, 2}. The light–like billiard flow provides a natural example of a discrete contact completely integrable system \cite{1, 7, 3}. We shall prove that the Heisenberg model on a light–like cone leads to an integrable contact system as well.

2 Description of the dynamics

Symplectic description. Let

$$P_c = S^{n-1}_c \times S^{n-1}_c \subset E^{k,l} \times E^{k,l}(q, Q): \quad \phi_1 = \langle q, q \rangle = c, \quad \phi_2 = \langle Q, Q \rangle = c.$$  

The equations (1.2), (1.4) determine the mapping

$$\Phi : P_c \rightarrow P_c$$

$(\Phi(q_{k-1}, q_k) = (q_k, q_{k+1}))$, defined outside the singular set $\langle J^{-2} Q, Q \rangle = 0$. Let

$$\alpha = \frac{\partial L(q, Q)}{\partial q} dq = E J Q dq = \sum_i \tau_i J_i Q_i dq_i.$$  

The flow of $\Phi$ preserves 2-form $\Omega = d\alpha = \sum_i \tau_i J_i dQ_i \wedge dq_i$ (see Veselov \cite{11, 12})

$$\Phi^* \Omega = \Omega.$$  

Namely, $\Phi$ preserves a form $\omega$ if the restriction of $\pi^*_1 \omega - \pi^*_2 \omega$ to the associated graph

$$\Gamma_\Phi = \{(q_1, Q_1, q_2, Q_2) \in P_c \times P_c \mid Q_1 = q_2, \quad \frac{\partial L(q_2, Q_2)}{\partial q} + \frac{\partial L(q_1, Q_1)}{\partial Q} = \lambda E Q_1\}$$  

(1.5)
equals zero, where \( \pi_1 \) and \( \pi_2 \) are projections to the first and the second factor of \( P_c \times P_c \) and \( \lambda = 2\langle J^{-1} q_1, Q_1 \rangle / \langle J^{-2} q_1, Q_1 \rangle \). At \( \Gamma_\Phi \), we have

\[
\pi_1^* \alpha - \pi_2^* \alpha = \frac{\partial L(q_1, Q_1)}{\partial q} dq_1 - \frac{\partial L(q_2, Q_2)}{\partial q} dq_2 = \frac{\partial L(q_1, Q_1)}{\partial q} dq_1 + \frac{\partial L(q_1, Q_1)}{\partial Q} dQ_1 - \lambda EQ_1 dQ_1
\]

\[
= d(\pi_1^* L) - \frac{1}{2} \lambda d(\langle Q_1, Q_1 \rangle)
\]

implying

\[
\pi_1^* \Omega - \pi_2^* \Omega = -\frac{1}{2} d\lambda \wedge d(\langle Q, Q \rangle) = 0, \quad \text{at} \quad \Gamma_\Phi.
\]

The form \( \Omega \) is symplectic on \( E^{k,l} \times E^{k,l} \). Let \( \{\cdot, \cdot\}_\Omega \) be the corresponding Poisson bracket. We have

\[
\{\phi_1, \phi_2\}_\Omega = -\frac{\partial \phi_1}{\partial Q} \cdot E^{-1} \frac{\partial \phi_2}{\partial q} + \frac{\partial \phi_1}{\partial q} \cdot E^{-1} \frac{\partial \phi_2}{\partial Q} = 4\langle q, J^{-1} Q \rangle.
\]

Therefore, the subvariety

\[
P_c^* = P_c \setminus \{\langle q, J^{-1} Q \rangle = 0\}
\]

is symplectic. Moreover, \( P_c^* \) is \( \Phi \)-invariant. Indeed, alternatively to (1.3), the Lagrange multiplier \( \lambda_k \) can be found from the expression \( \langle q_{k-1}, q_{k-1} \rangle = c \), provided \( \langle q_k, q_k \rangle = \langle q_{k+1}, q_{k+1} \rangle = c \). This leads to \( c = c - 2\langle J^{-1} q_k, q_{k+1} \rangle + \lambda_k^2 \langle J^{-2} q_k, q_k \rangle \) and the Lagrange multiplier equals

\[
\lambda_k = 2\langle J^{-1} q_{k+1}, q_k \rangle / \langle J^{-2} q_k, q_k \rangle.
\]

By combining (1.4) and (2.2) we get that

\[
K(q, Q) = \langle J^{-1} Q, q \rangle = \langle J^{-1} EQ, q \rangle
\]

is the first integral of the system and \( P_c^* \) is \( \Phi \)-invariant.

The induced Poisson bracket on \( (P_c^*, \Omega) \) can be described by the Dirac construction (e.g., \( S \)):

\[
\{f_1, f_2\}_\Omega^D = \{f_1, f_2\}_\Omega - \frac{\{\phi_1, f_1\}_\Omega \{\phi_2, f_2\}_\Omega - \{\phi_2, f_1\}_\Omega \{\phi_1, f_2\}_\Omega}{\{\phi_1, \phi_2\}_\Omega}.
\]

**Light–like cone and contact description.** In the light–like case, multiplying (1.2) by \( q_k \), we obtain the additional invariant relation

\[
\langle J q_{k-1}, q_k \rangle + \langle J q_k, q_{k+1} \rangle = 0,
\]

which gives that \( L^2 = \langle J q, Q \rangle^2 \) is also the integral of the mapping \( \Phi \). Consider the corresponding \( (2n - 3) \)-dimensional invariant variety

\[
M_\kappa = \{\langle q, Q \rangle \in \Gamma^0 \mid L(q, Q) = \langle q, J Q \rangle = \pm \kappa \} = M_{\kappa,+} \bigcup M_{\kappa,-}, \quad \kappa \geq 0.
\]

**Theorem 2.1:** The restriction of 1-form \( \alpha \) to \( M_\kappa \) is a contact form for \( \kappa > 0 \). Moreover, \( \Phi \) is a contact transformation

\[
\Phi^* \alpha = \alpha
\]

that interchanges the components of \( M_{\kappa,+} \) and \( M_{\kappa,-} \) of \( M_\kappa \).
3 Integrability

The restriction of $\alpha$ to $M_\kappa$ is contact if $\alpha \neq 0$ and the restriction of $\Omega = d\alpha$ to the horizontal distribution

$$\mathcal{H} = \ker \alpha = \{ \xi \in T_{(q, Q)}M_\kappa | \alpha(\xi) = 0 \}$$

is nondegenerate.

Let $\tilde{X}_L$ be the Hamiltonian vector field of the discrete Lagrangian $L(q, Q)$ on $(P_0^\ast, \Omega)$. It appears that the Hamiltonian vector field $X_L = (q, -Q)$ of $L$ on $(E_{k, l}^k \times E_{k, l}^{k, l}, \Omega)$ is tangent to $P_0^\ast$, and therefore,

$$\tilde{X}_L = X_L|_{P_0^\ast} = (q, -Q).$$

From (2.6) and the definition of the Hamiltonian vector field $\tilde{X}_L$ ($i_{\tilde{X}_L} \Omega = -dL|_{P_0^\ast}$), the kernel of $\Omega$ restricted to $M_\kappa$ is proportional to $\tilde{X}_L$. Further, $\alpha(\tilde{X}_L) = (EJq, Q) = L = \pm \kappa$, $(q, Q) \in M_{\kappa, \pm}$ (2.8)

implies that $\ker d\alpha \cap \mathcal{H} = 0$ and the restriction of $d\alpha$ to $\mathcal{H}$ is nondegenerate, for $\kappa > 0$.

Next, by restricting (2.1) to $M_\kappa \times M_\kappa$, and using the fact that both, the restrictions of $dL$ and $d(\langle Q_1, Q_1 \rangle)$ to $M_\kappa$ are equal to zero, we obtain (2.7).

3 Integrability

Define

$$F = \text{diag}(1, \ldots, 1, i, \ldots, i),$$

where the first $k$ components are equal to 1, and the last $n - k$ components are equal to the imaginary unit $i$ ($F^2 = E$).

Motivated by the Moser–Veselov Lax matrix representation for the Heisenberg system in the Euclidean case [9], we obtain the following statement.

Theorem 3.1: The equations (1.2) implies the matrix equation

$$L_{k+1}(\lambda) = A_k(\lambda)L_k(\lambda)A_k^{-1}(\lambda),$$

where

$$L_k(\lambda) = J^2 + \lambda Fq_{k-1} \wedge F \cdot Jq_k - \lambda^2 \cdot Fq_{k-1} \otimes Fq_{k-1}, \quad A_k(\lambda) = J - \lambda Fq_k \otimes Fq_{k-1}.$$  

Like in [9], we have the matrix factorization $L_k = A_k^T(-\lambda)A_k(\lambda)$. Therefore, the discrete Lax representation can be seen also as interchanging of $A_k$–matrixes:

$$L_{k+1} = A_{k+1}^T(-\lambda)A_{k+1}(\lambda) = A_k(\lambda)A_k^T(-\lambda).$$

Note that in the case of a Heisenberg model on a light–like cone ($c = 0$), the $L$–matrix is linear in $\lambda$.

If $J_1^2 \neq J_2^2$, the integrals of the mapping $\Phi$ obtained from the matrix representation can be written in the form

$$f_i(q, Q) = c \cdot \tau_i q_i^2 + \sum_{j \neq i} \frac{\tau_i \tau_j (J_j Q_j q_k - q_j J_j Q_i q_k)^2}{J_j^2 - J_i^2}, \quad i = 1, \ldots, n.$$ (3.1)
Lemma 3.1: The integrals (2.3), (3.1) on $P_c$ are related by

\[ \sum f_i \equiv c^2, \quad (3.2) \]
\[ \sum \frac{1}{f_i} f_i \equiv K^2, \quad (3.3) \]

and, for $c = 0$, we have the additional relation

\[ \sum J_i^2 f_i \equiv -L^2. \quad (3.4) \]

By direct calculations, one can prove that the integrals (3.1) commute on $(P^*_c, \Omega)$:

\[ \{ f_i, f_j \}_{\Omega}^P = 0, \quad i, j = 1, \ldots, n \quad (3.5) \]

and that the only relation among them is (3.2).

Theorem 3.2: The Heisenberg model is a completely integrable discrete Hamiltonian system on $(P^*_c, \Omega)$, $c = \pm 1, 0$.

We proceed with the light–like case and consider the mapping $\Phi$ restricted to the contact manifold $(M_\kappa, \alpha)$. Recall that a vector field $Y$ is contact if it preserves the horizontal distribution $\mathcal{H}$, i.e., $L_Y \alpha = \lambda \alpha$, for some smooth function $\lambda$. The distinguish contact vector field is the Reeb vector field $Z$, uniquely defined by

\[ i_Z \alpha = 1, \quad i_Z d\alpha = 0. \]

From (2.8), the Reeb vector field reads

\[ Z|_{(q, Q)} = \pm \frac{1}{\kappa}(q, -Q), \quad (q, Q) \in M_\kappa, \pm. \]

Since the Lie derivatives of the integrals (3.1) along $X_L$, for $c = 0$, are equal to zero

\[ \mathcal{L}_{X_L} f_i = \{ f_i, L \}_{\Omega}^P = 0 \quad i = 1, \ldots, n, \]

we obtain

\[ \mathcal{L}_Z f_i = 0 \iff [Z, Y_{f_i}] = 0 \quad (3.6) \]

where $Y_{f_i}$ are the contact Hamiltonian vector fields with Hamiltonians $f_i$, $i = 1, \ldots, n$. \footnote{The mapping $\Psi : Y \mapsto f = i_Y \alpha$ establish the isomorphism between the vector spaces of contact vector fields and smooth functions on $M_\kappa$. The vector field $Y_f = \Psi^{-1}(f)$ is the contact Hamiltonian vector field with the Hamiltonian function $f$. In particular, the Hamiltonian of the Reeb vector field is $f \equiv 1$. Further, $\Psi$ is a Lie algebra isomorphism: $[f,g] = [Y_f,Y_g]$, where on the right hand side we have the usual Lie bracket of vector fields, while on the left hand side we have the Jacobi bracket on $C^\infty(M_\kappa)$ defined by $[f,g] = \mathcal{L}_Y g - g \mathcal{L}_Z f$ (e.g., see [5]).} From (3.5) and (3.6), using the theorem on isoenergetic integrability (for more details, see [4]), we get

\[ [f_i, f_j] = 0 \iff [Y_{f_i}, Y_{f_j}] = 0 \quad i, j = 1, \ldots, n. \quad (3.7) \]

Apart from (3.2), on $M_\kappa$ the integrals (3.1) have the additional relation $\sum_i J_i^2 f_i \equiv -\kappa^2$ (see (3.3)), and there are $n - 2$ independent functions among them.
Thus, the mapping $\Phi$ is a completely integrable contact transformation (see [7, 8]). The conditions (3.6), (3.7), ensure that $M_\kappa$ is almost everywhere foliated on $(n - 1)$-dimensional invariant manifolds, regular level sets of integrals $f_1, \ldots, f_n$. The associated distribution $\mathcal{F}$ is pre-Legendrian (or co-Legendrian) and it is generated by the commuting contact vector fields $Z, Y_{f_1}, \ldots, Y_{f_n}$ [7].

**Theorem 3.3:** The Heisenberg model on the light–like cone in a pseudo–Euclidean space $E^{k,l}$ is completely integrable contact system on $(M_\kappa, \alpha)$, $\kappa > 0$.

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$F$ is transversal to $\mathcal{H}$ and $\mathcal{G} = \mathcal{F} \cap \mathcal{H}$ is a maximal integrable (Legendrian) distribution of $\mathcal{H}$. 