A PROOF OF THE Riemann Hypothesis USING THE REMAINDER TERM OF THE DIRICHLET ETA FUNCTION.

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Abstract. The Dirichlet eta function can be divided into \( n \)-th partial sum \( \eta_n(s) \) and remainder term \( R_n(s) \). We focus on the remainder term which can be approximated by the expression for \( n \). And then, to increase reliability, we make sure that the error between remainder term and its approximation is reduced as \( n \) goes to infinity. According to the Riemann zeta functional equation, if \( \eta(\sigma + it) = 0 \) then \( \eta(1 - \sigma - it) = 0 \). In this case, \( n \)-th partial sum also can be approximated by expression for \( n \). Based on this approximation, we prove the Riemann hypothesis.

1. Introduction

The Riemann hypothesis conjectured by Bernhard Riemann in 1859 states that the real part of every nontrivial zeros of the Riemann zeta function is \( \frac{1}{2} \). The Riemann zeta function is the function of the complex variable \( s \), which converges for any complex number having \( \Re(s) > 1 \) [1].

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}
\]

The Riemann hypothesis discusses zeros outside the region of convergence of this series, so it must be analytically continued to all complex \( s \) [2]. This statement of the problem can be simplified by introducing the Dirichlet eta function, also known as the alternating zeta function. The Dirichlet eta function is defined as [1]

\[
\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = \left(1 - \frac{1}{2^{s-1}}\right) \zeta(s)
\]

Since \( \eta(s) \) converges for all \( s \in \mathbb{C} \) with \( \Re(s) > 0 \), one need not consider analytic continuation (see p. 55-56 of [4]). The Dirichlet eta function extends the Riemann zeta function from \( \Re(s) > 1 \) to the larger domain \( \Re(s) > 0 \), excluding the zeros \( s = 1 + \frac{2\pi}{\ln 2} i (n \in \mathbb{Z}) \). The Riemann hypothesis is equivalent to the statement that all the zeros of the Dirichlet eta function falling in the critical strip \( 0 < \Re(s) < 1 \) lie on the critical line \( \Re(s) = \frac{1}{2} \) (see p. 49 of [4]).

In the strip \( 0 < \Re(s) < 1 \) the Riemann zeta function satisfies the functional equation [2, 3] related to values at the points \( s \) and \( 1 - s \).

\[
\zeta(s) = 2^s\pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1 - s) \zeta(1 - s)
\]

where \( \Gamma(s) \) is the gamma function. The functional equation shows that the Riemann zeta function have the infinitely zeros, called the trivial zeros, at the negative even
integers. But the functional equation do not tell us about the zeros of the Riemann zeta function in the strip $0 < \Re(s) < 1$. Actually there are zeros in the strip and they are called nontrivial zeros. Calculation of some number of these nontrivial zeros show that they are lying exactly on the line $\Re(s) = \frac{1}{2}$ \[5\].

2. The remainder term of the Dirichlet eta function

Let $s = \sigma + it$, where $0 < \sigma < 1$ and $\sigma, t \in \mathbb{R}$. The Dirichlet eta function can be written as

\[
\eta(s) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^s} + \sum_{k=n+1}^{\infty} \frac{(-1)^{k-1}}{k^s} = \eta_n(s) + R_n(s)
\]

where $\eta_n(s)$ is the $n$-th partial sum and $R_n(s)$ is the sum of remainder term. $\eta_n(s)$ and $R_n(s)$ converge to $\eta(s)$ and zero respectively, as $n \to \infty$.

\[
\lim_{n \to \infty} \eta_n(s) = \eta(s), \quad \lim_{n \to \infty} R_n(s) = 0
\]

The expand form of the remainder terms are represented as follows.

\[
-R_{n-1}(s) = (-1)^n \left\{ \frac{1}{n^s} - \frac{1}{(n+1)^s} + \frac{1}{(n+2)^s} - \cdots \right\}
\]

\[
R_n(s) = (-1)^n \left\{ \frac{1}{(n+1)^s} - \frac{1}{(n+2)^s} + \frac{1}{(n+3)^s} - \cdots \right\}
\]

\[
-R_{n+1}(s) = (-1)^n \left\{ \frac{1}{(n+2)^s} - \frac{1}{(n+3)^s} + \frac{1}{(n+4)^s} - \cdots \right\}
\]

Lemma 2.1. The remainder term of $\eta(s)$ satisfy the following limit as $n \to \infty$.

\[
\lim_{n \to \infty} \frac{-R_{n-1}(s)}{R_n(s)} = \lim_{n \to \infty} \frac{-R_{n+1}(s)}{R_n(s)} = 1
\]

Proof. Consider the recurrence relation,

\[
R_n(s) - R_{n-1}(s) = \sum_{k=n+1}^{\infty} \frac{(-1)^{k-1}}{k^s} - \sum_{k=n}^{\infty} \frac{(-1)^{k-1}}{k^s} = \frac{(-1)^n}{n^s}
\]

\[
R_n(s) - R_{n+1}(s) = \sum_{k=n+1}^{\infty} \frac{(-1)^{k-1}}{k^s} - \sum_{k=n+2}^{\infty} \frac{(-1)^{k-1}}{k^s} = \frac{(-1)^n}{(n+1)^s}
\]

Thus, we obtain the following relation.

\[
\frac{R_n(s) - R_{n-1}(s)}{R_n(s) - R_{n+1}(s)} = \frac{(n+1)^s}{n^s}
\]

Taking the limit as $n \to \infty$, we have

\[
\lim_{n \to \infty} \frac{R_n(s) - R_{n-1}(s)}{R_n(s) - R_{n+1}(s)} = \lim_{n \to \infty} \frac{1 - \frac{1}{n+1}}{1 - \frac{R_{n+1}(s)}{R_n(s)}} = \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^s = 1
\]

Thus, we have

\[
\lim_{n \to \infty} \frac{-R_{n-1}(s)}{R_n(s)} = \lim_{n \to \infty} \frac{-R_{n+1}(s)}{R_n(s)} = 1
\]
Lemma 2.2. For sufficiently large $n$, the remainder term of $\eta(s)$ can be approximated as

$$R_n(s) = \sum_{k=n+1}^{\infty} \frac{(-1)^{k-1}}{k^s} \approx \frac{(-1)^n}{2(n + 0.5)^s}$$

Proof. Separate the $R_n(s)$ and $R_{n+1}(s)$ into real and imaginary parts and change the index of summation $k$ so that it would start from 1. Then we have,

$$R_n(s) = \sum_{k=1}^{\infty} \frac{\cos(t \ln(n + k)) - i \sin(t \ln(n + k))}{(-1)^{n+k-1}(n + k)^{\sigma}}$$

$$-R_{n+1}(s) = \sum_{k=1}^{\infty} \frac{\cos(t \ln(n + k + 1)) - i \sin(t \ln(n + k + 1))}{(-1)^{n+k}(n + k + 1)^{\sigma}}$$

For every $\epsilon > 0$ there are natural numbers $N_1$ and $N_2$ such that $n > N_1$ implies $|t \ln(n + k + 1) - t \ln(n + k)| < \epsilon$ for all $t \in \mathbb{R}$, and $n > N_2$ implies $|(n + k + 1)^{-\sigma} - (n + k)^{-\sigma}| < \epsilon$. Let $N = \max\{N_1, N_2\}$. By the choice of $N$, $n > N$ implies $|\Re[R_n(s) - \{-R_{n+1}(s)\}]| < \epsilon$ and $|\Im[R_n(s) - \{-R_{n+1}(s)\}]| < \epsilon$. Thus, it follows that $R_n(s) \approx -R_{n+1}(s)$ for sufficiently large $n$.

Consider the recurrence relation (see (9) and (7))

$$R_n(s) + \{-R_{n-1}(s)\} = \frac{(-1)^n}{n^s}$$

$$R_n(s) + \{-R_{n+1}(s)\} = \frac{(-1)^n}{(n + 1)^s}$$

For all $\epsilon > 0$, there exist $\delta > 0$ such that for all $n > N$ that satisfy $|R_n(s) - \{-R_{n-1}(s)\}| < \delta$ and $|R_n(s) - \{-R_{n+1}(s)\}| < \delta$, it follows that

$$\left| R_n(s) - \frac{(-1)^n}{2n^s} \right| < \epsilon \quad \text{and} \quad \left| R_n(s) - \frac{(-1)^n}{2(n + 1)^s} \right| < \epsilon$$

In this paper, we select the value of 0.5 between 0 and 1 in order to reduce the approximation error.

$$R_n(s) = \sum_{k=n+1}^{\infty} \frac{(-1)^{k-1}}{k^s} \approx \frac{(-1)^n}{2(n + 0.5)^s}$$

Now, in order to confirm the relationship between $R_n(s)$ and $\frac{(-1)^n}{2(n + 0.5)^s}$, consider the relative error.

Lemma 2.3. The relative error $\epsilon$ between the remainder term of $\eta(s)$ and its approximation $\frac{(-1)^n}{2(n + 0.5)^s}$ converge to zero as $n \to \infty$.

$$\epsilon = \lim_{n \to \infty} \left| \frac{R_n(s) - \frac{(-1)^n}{2(n + 0.5)^s}}{R_n(s)} \right| = 0$$
Proof. Consider the recurrence relation, (see (7))
\[ R_n(s) - R_{n+1}(s) = \frac{(-1)^n}{(n+1)^s} \]
Dividing both sides by \( R_n(s) \) and taking the limit as \( n \to \infty \), then we get the following limit.
\[ \lim_{n \to \infty} \left( 1 - \frac{R_{n+1}(s)}{R_n(s)} \right) = \lim_{n \to \infty} \frac{(-1)^n}{(n+1)^s} \frac{1}{R_n(s)} \]
By the result of the Lemma 2.1, we have
\[ (10) \lim_{n \to \infty} \frac{(-1)^n}{(n+1)^s} \frac{1}{R_n(s)} = 2 \]
Let \( F_n(s) = \frac{(-1)^n}{(n+1)^s} R_n(s) \), then \( R_n(s) = \frac{(-1)^n}{(n+1)^s} F_n(s) \) and \( \lim_{n \to \infty} F_n(s) = 2 \). Thus,
\[ \epsilon = \lim_{n \to \infty} \left| \frac{R_n(s) - \frac{(-1)^n}{(n+0.5)^s} F_n(s)}{R_n(s)} \right| = \lim_{n \to \infty} \left| 1 - \frac{\frac{(-1)^n}{(n+0.5)^s}}{(n+1)^s} \right| = 0 \]
\[ \square \]
For example, in order to check the Lemma 2.2, we perform a numerical calculation. Let \( s = 0.1234 + 56.789i \) (random value) and \( T_n(s) = \frac{(-1)^n}{(n+0.5)^s} R_n(s) \). Then \( R_n(s) \) and \( T_n(s) \) for four values (\( n = 10^8, n = 10^{10}, n = 10^{12}, n = 10^{14} \)) are given below. The significant figure of a number may be underlined.

\[ R_{10^8}(s) = -0.0514080530118374690874425376 \cdots 
-0.0030012424674281955071656693 \cdots i \]
\[ T_{10^8}(s) = -0.051408053011835941392302721 \cdots 
-0.0030012424674281160677214641 \cdots i \]
\[ R_{10^{10}}(s) = +0.0220754313015916605572779244 \cdots 
-0.0190708103417423704219739001 \cdots i \]
\[ T_{10^{10}}(s) = +0.0220754313015916604699783035 \cdots 
-0.0190708103417423703431444260 \cdots i \]
\[ R_{10^{12}}(s) = -0.0014437322549038780686126642 \cdots 
+0.0164629022496889818808209350 \cdots i \]
\[ T_{10^{12}}(s) = -0.0014437322549038780686122279 \cdots 
+0.0164629022496889818808142859 \cdots i \]
\[ R_{10^{14}}(s) = -0.0059111117596716499309061036 \cdots 
-0.0072599141694530105681646539 \cdots i \]
\[ T_{10^{14}}(s) = -0.0059111117596716499309061036 \cdots 
-0.0072599141694530105681646536 \cdots i \]
Define the relative errors for the real and imaginary parts of the complex error function in forms

\[
\epsilon_r = \left| \frac{\Re[R_n(s)] - \Re[T_n(s)]}{\Re[R_n(s)]} \right| \\
\epsilon_i = \left| \frac{\Im[R_n(s)] - \Im[T_n(s)]}{\Im[R_n(s)]} \right|
\]

Then the relative errors for the above eight values are given in table 1.

**Table 1. Relative Errors for \( \epsilon_r \) and \( \epsilon_i \)**

| \( n \)  | \( \epsilon_r \)     | \( \epsilon_i \)     |
|---------|-----------------|-----------------|
| \( 10^8 \) | \( 4.0362 \times 10^{-14} \) | \( 2.5151 \times 10^{-14} \) |
| \( 10^{10} \) | \( 3.9546 \times 10^{-18} \) | \( 4.1335 \times 10^{-18} \) |
| \( 10^{12} \) | \( 3.0220 \times 10^{-22} \) | \( 4.0388 \times 10^{-22} \) |
| \( 10^{14} \) | \( 3.3835 \times 10^{-26} \) | \( 4.1323 \times 10^{-26} \) |

In the table 1, \( \epsilon_r \) and \( \epsilon_i \) are reduced as \( n \) goes to infinity.

Lemma 2.2 and 2.3 show that \( R_n(s) \) can be approximated by \( \frac{(-1)^n}{2(n+0.5)^s} \). So, \( R_n(s) \) can be written as

\[
R_n(s) = \frac{(-1)^n}{2(n+0.5)^s} + \epsilon_n(s)
\]

where \( \epsilon_n(s) \) is error term and \( \epsilon_n(s) \) converges to zero, as \( n \to \infty \).

\[
\lim_{n \to \infty} \epsilon_n(s) = 0
\]

Lemma 2.3 can be written by \( \epsilon_n(s) \) as follow.

\[
\epsilon = \lim_{n \to \infty} \left| \frac{R_n(s) - \frac{(-1)^n}{2(n+0.5)^s}}{R_n(s)} \right| = \lim_{n \to \infty} \left| \frac{\epsilon_n(s)}{R_n(s)} \right| = 0
\]

In addition, dividing both sides of (11) by \( \frac{(-1)^n}{2(n+0.5)^s} \) and taking the limit as \( n \to \infty \), then we get the following limit.

\[
\lim_{n \to \infty} R_n(s) \frac{2(n+0.5)^s}{(-1)^n} = 1 + \lim_{n \to \infty} \epsilon_n(s) \frac{2(n+0.5)^s}{(-1)^n}
\]

By using the (10), we have

\[
\lim_{n \to \infty} \epsilon_n(s)(n+0.5)^s = 0
\]

**Lemma 2.4.** Let \( s = \sigma + it \) where \( \sigma \) is constant on \( 0 < \sigma < 1 \) and \( t \in \mathbb{R} \), then the Dirichlet eta function is converges uniformly.
Proof.

\[ |\eta_n(s) - \eta(s)| = |R_n(s)| = \left| \frac{(-1)^n}{2(n + 0.5)^s} \right| \leq \frac{1}{n^s} \]

Since \( \frac{1}{n^s} \to 0 \) as \( n \to \infty \), given any \( \epsilon > 0 \) there exist \( N \in \mathbb{N} \), depending only on \( \epsilon \) and \( n \), such that

\[ 0 \leq \frac{1}{n^s} < \epsilon \quad \text{for all } n > N \]

It follows that

\[ \left| \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^s} - \eta(s) \right| < \epsilon \quad \text{for all } t \in \mathbb{R} \text{ and all } n > N \]

which proves that the Dirichlet eta function converges uniformly on constant \( \sigma \). \( \square \)

Lemma 2.5. Let \( s = \sigma + it \) denote the nonzeros of \( \eta(s) \) where \( \sigma \) is constant on \( 0 < \sigma < 1 \) and \( t \in \mathbb{R} \), then \( \frac{n(1-s)}{\eta(s)} \) is converges uniformly.

Proof.

\[
\left| \frac{\eta_n(1-s)}{\eta_n(s)} - \frac{\eta(1-s)}{\eta(s)} \right| \leq \left| \frac{\eta_n(1-s)}{\eta_n(s)} - \frac{\eta(1-s)}{\eta_n(s)} \right| + \left| \frac{\eta(1-s)}{\eta_n(s)} - \frac{\eta(1-s)}{\eta(s)} \right|
\]

\[ = \left| \frac{\eta_n(1-s) - \eta(1-s)}{\eta_n(s)} \right| + \left| \eta(1-s) \right| \left| \frac{1}{\eta_n(s)} - \frac{1}{\eta(s)} \right| \]

Since \( \eta(s) \) converges uniformly, for every \( \epsilon > 0 \) we can choose \( N_1, N_2 \in \mathbb{N} \) such that \( n \in N_1 \) implies

\[ \left| \frac{\eta_n(1-s) - \eta(1-s)}{\eta_n(s)} \right| < \frac{\epsilon}{2} \]

for all \( \Im(s) \in \mathbb{R}, \eta(s) \neq 0 \) and \( n \in N_2 \) implies

\[ \left| \eta(1-s) \right| \left| \frac{1}{\eta_n(s)} - \frac{1}{\eta(s)} \right| < \frac{\epsilon}{2} \]

Let \( N=\max\{N_1, N_2\} \). By the choice of \( N \), \( n \geq N \) implies

\[ \left| \frac{\eta_n(1-s) - \eta(1-s)}{\eta_n(s)} \right| + \left| \eta(1-s) \right| \left| \frac{1}{\eta_n(s)} - \frac{1}{\eta(s)} \right| < \epsilon \]

So,

\[ \left| \frac{\eta_n(1-s)}{\eta_n(s)} - \frac{\eta(1-s)}{\eta(s)} \right| < \epsilon \]

for all \( n > N \). \( \square \)

Lemma 2.6. If a sequence of continuous function \( \eta_n(s) : A \to \mathbb{C} \) converges uniformly on \( A \subset \mathbb{C} \), then \( \eta(s) \) is continuous on \( A \).

Proof. Suppose that \( c = \sigma + iu \in A \) denote the nonzeros of \( \eta(s) \) where \( u \in \mathbb{R} \) and \( \epsilon > 0 \). For every \( n \in \mathbb{N} \)

\[
\left| \frac{\eta(1-s)}{\eta(s)} - \frac{\eta(1-c)}{\eta(c)} \right| < \left| \frac{\eta(1-s)}{\eta(s)} - \frac{\eta_n(1-s)}{\eta_n(s)} \right| + \left| \frac{\eta_n(1-s)}{\eta_n(s)} - \frac{\eta_n(1-c)}{\eta_n(c)} \right| + \left| \frac{\eta_n(1-c)}{\eta_n(c)} - \frac{\eta(1-c)}{\eta(c)} \right| + \left| \frac{\eta(1-c)}{\eta(c)} - \frac{\eta(1-s)}{\eta(s)} \right|
\]
By the uniform convergence of \( \eta(s) \), we can choose \( n \in \mathbb{N} \) such that
\[
\left| \frac{\eta(1-s)}{\eta(s)} - \frac{\eta_n(1-s)}{\eta_n(s)} \right| < \frac{\epsilon}{3} \quad \text{for all } t \in \mathbb{R}, \text{ if } n > N
\]
and for such an \( n \) it follows that
\[
\left| \frac{\eta(1-s)}{\eta(s)} - \frac{\eta(1-c)}{\eta(c)} \right| < \left| \frac{\eta_n(1-s)}{\eta_n(s)} - \frac{\eta_n(1-c)}{\eta_n(c)} \right| + \frac{2\epsilon}{3}
\]
Since, \( \eta_n(s) \) if continuous on \( A \), there exist \( \delta > 0 \) such that
\[
\left| \frac{\eta_n(1-s)}{\eta_n(s)} - \frac{\eta_n(1-c)}{\eta_n(c)} \right| < \frac{\epsilon}{3} \quad \text{if } |s-c| < \delta \text{ and } s \in A
\]
This prove that \( \eta(s) \) is continuous. \( \square \)

This result can be interpreted as justifying an “exchange in the order of limits”

(14) \[
\lim_{n \to \infty} \lim_{t \to u} \frac{\eta_n(1-\sigma - it)}{\eta_n(\sigma + it)} = \lim_{t \to u} \lim_{n \to \infty} \frac{\eta_n(1-\sigma - it)}{\eta_n(\sigma + it)}
\]

3. A PROOF OF THE RIEMANN HYPOTHESIS

In 1914 Godfrey Harold Hardy proved that \( \zeta(\frac{1}{2}+it) \) has infinitely many nontrivial zeros [7].

**Theorem 3.1.** [Riemann Hypothesis] The real part of every nontrivial zeros of the Riemann zeta function is \( \frac{1}{2} \).

**Proof.** The Dirichlet eta functional equation is
\[
\eta(1-s) = \frac{(2-2^{s+1})}{(2^s-2)} \pi^{-s} \cos \left( \frac{\pi s}{2} \right) \Gamma(s) \eta(s)
\]
If \( \eta(s) \neq 0 \), then
\[
\frac{\eta(1-s)}{\eta(s)} = \frac{(2-2^{s+1})}{(2^s-2)} \pi^{-s} \cos \left( \frac{\pi s}{2} \right) \Gamma(s)
\]
The above equation has a removable discontinuity at the zeros of \( \eta(s) \).

Let \( s_0 = \sigma + it_0 \) is zero of \( \eta(s) \) and \( s = \sigma + it \) where \( \sigma \) is constant on \( \frac{1}{2} < \sigma < 1 \) and \( t \in \mathbb{R} \). For each point \( t \), we can choose the open interval \( t_0 < t < c \) where \( c \) is an arbitrary point such that \( \eta_n(\sigma + it) \) and \( \eta_n(1-\sigma - it) \) are converge uniformly. By using the Eq. (14), we have

(15) \[
\lim_{n \to \infty} \lim_{t \to t_0^+} \frac{\eta_n(1-\sigma - it)}{\eta_n(\sigma + it)} = \lim_{t \to t_0^+} \lim_{n \to \infty} \frac{\eta_n(1-\sigma - it)}{\eta_n(\sigma + it)}
\]

(i) By using the Lemma 2.2, the left-hand side of (15) is as follows.
\[
\lim_{n \to \infty} \lim_{t \to t_0^+} \frac{\eta_n(1-\sigma - it)}{\eta_n(\sigma + it)} = \lim_{n \to \infty} \frac{R_n(1-\sigma - it)}{R_n(\sigma + it)}
\]
\[
= \lim_{n \to \infty} \frac{(-1)^n}{2(n+0.5)^1-\sigma-it} \frac{(-1)^n}{2(n+0.5)^{\sigma+it}}
\]
\[
= \lim_{n \to \infty} (n+0.5)^{2\sigma-1+2it}
\]
Thus, the left-hand side of the equation diverges to infinity.

(ii) By using the Dirichlet eta functional equation, the right-hand sides of (15) is as follows.

\[
\lim_{t \to t_0^+} \lim_{n \to \infty} \frac{\eta_n(1 - \sigma - it)}{\eta_n(\sigma + it)} = \lim_{t \to t_0^+} \frac{\eta(1 - \sigma - it)}{\eta(\sigma + it)} \\
= \lim_{t \to t_0^+} \frac{(2 - 2\sigma^2 + it + 1)}{(2\sigma + it_0 - 2\sigma + it_0)} \cos \left\{ \frac{\pi(\sigma + it)}{2} \right\} \Gamma(\sigma + it) \\
= \left( \frac{2 - 2\sigma^2 + it_0 + 1}{2} \right) \frac{\pi(\sigma + it_0)}{2} \cos \left\{ \frac{\pi(\sigma + it_0)}{2} \right\} \Gamma(\sigma + it_0)
\]

Thus, the right-hand side of the equation does not diverges to infinity.

By the (i) and (ii), This is contradiction. Therefore \( \eta(s) \) does not have zeros in the strip \( \frac{1}{2} < 0 < 1 \), and \( \eta(s) \) has no zeros in the strip \( 0 < \Re(s) < \frac{1}{2} \), because all nontrivial zeros of \( \eta(s) \) were symmetric about the line \( \Re(s) = \frac{1}{2} \). In conclusion, the real part of every nontrivial zeros of the Riemann zeta function is only \( \frac{1}{2} \).

\[ \square \]

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