Quantum Stress Tensor Fluctuations and their Physical Effects

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Abstract

We summarize several aspects of recent work on quantum stress tensor fluctuations and their role in driving fluctuations of the gravitational field. The role of correlations and anticorrelations is emphasized. We begin with a review of the properties of the stress tensor correlation function. We next consider some illuminating examples of non-gravitational effects of stress tensors fluctuations, specifically fluctuations of the Casimir force and radiation pressure fluctuations. We next discuss passive fluctuations of spacetime geometry and some of their operational signatures. These include luminosity fluctuations, line broadening, and angular blurring of a source viewed through a fluctuating gravitational field. Finally, we discuss the possible role of quantum stress tensor fluctuations in the early universe, especially in inflation. The fluctuations of the expansion of a congruence of comoving geodesics grows during the inflationary era, due to non-cancellation of anticorrelations that would have occurred in flat spacetime. This results in subsequent non-Gaussian density perturbations and allows one to infer an upper bound on the duration of inflation. This bound is consistent with adequate inflation to solve the horizon and flatness problems.
I. INTRODUCTION

As is well-known, the classical stress tensor, $T_{\mu\nu}$, is both the source of the gravitational field in general relativity theory, and the quantity which describes stresses on material objects. In quantum field theory, the stress tensor becomes an operator whose expectation value is formally infinite, and needs to be renormalized. In Minkowski spacetime, this is usually accomplished by simply subtracting the vacuum expectation value, and replacing the stress tensor operator by its normal-ordered version:

$$\mathcal{T}_{\mu\nu} := T_{\mu\nu} - \langle T_{\mu\nu} \rangle_0,$$

where $\langle \rangle_0$ denotes an expectation value in the Minkowski vacuum state. This amounts to defining the zero of energy density to be at the vacuum level. This allows for states with local negative energy densities, although the total energy must be non-negative, and the regions of negative energy density are severely constrained by quantum inequalities [1]. In curved spacetime, the renormalization of $\langle T_{\mu\nu} \rangle$ is more complicated and involves renormalization of the cosmological constant, Newton’s constant, and the coefficients of counterterms quadratic in the curvature.

Our concern will not be with issues of renormalization of $\langle T_{\mu\nu} \rangle$ or with quantum violation of classical energy conditions, but rather with fluctuations of the stress tensor operator about its mean value. That there must be such fluctuations in all realizable quantum states follows from the fact that these states are never eigenstates of the stress tensor operator.

The gravitational effects of stress tensor fluctuations have been discussed by several authors in recent years [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. In this paper, we will discuss a selection of topics relating to the basic character of stress tensor fluctuations, their role in creating fluctuating forces on material bodies, and especially their role in gravitational physics.

II. THE STRESS TENSOR CORRELATION FUNCTION

The basic object which we will need to study in order to understand stress tensor fluctuations, will be the correlation function,

$$C_{\mu\nu\alpha\beta}(x, x') = \langle T_{\mu\nu}(x) T_{\alpha\beta}(x') \rangle - \langle T_{\mu\nu}(x) \rangle \langle T_{\alpha\beta}(x') \rangle.$$

This object is independent of the choice of renormalization of $\langle T_{\mu\nu} \rangle$, as $C_{\mu\nu\alpha\beta}(x, x')$ is unchanged if we shift the stress tensor by a c-number. It is, however, singular in the coincidence limit $x' \to x$, even if $\langle T_{\mu\nu} \rangle$ is finite. It is often useful to decompose the correlation function into three parts with differing singularities. Here we consider only the case of Minkowski spacetime, but an analogous decomposition may be defined in curved spacetimes.

The stress tensor for a free quantum field is a sum of terms, each quadratic in field operators or derivatives of field operators. Consider the stress tensor for a bosonic field, which is a sum of terms of the form $T(x) = \phi_1(x)\phi_2(x)$: Now consider products of operators of the form of $T$. It may be shown using Wick’s theorem that

$$T(x) T(x') = S_0 + S_1 + S_2,$$

where

$$S_0 = \langle \phi_1(x)\phi_1(x') \rangle_0 \langle \phi_2(x)\phi_2(x') \rangle_0 + \langle \phi_1(x)\phi_2(x') \rangle_0 \langle \phi_2(x)\phi_1(x') \rangle_0.$$
\[ S_1 =: \phi_1(x)\phi_1(x') : \langle \phi_2(x)\phi_2(x') \rangle_0 + : \phi_1(x)\phi_2(x') : \langle \phi_2(x)\phi_1(x') \rangle_0 + : \phi_2(x)\phi_1(x') : \langle \phi_2(x)\phi_1(x') \rangle_0, \tag{5} \]

and
\[ S_2 =: \phi_1(x)\phi_2(x)\phi_1(x')\phi_2(x') : . \tag{6} \]

Thus the operator product \( T(x)T(x') \) consists of a purely vacuum part \( S_0 \), a fully normal-ordered part \( S_2 \), and a part \( S_1 \) which is a cross term between the vacuum and normal-ordered parts.

The same decomposition holds for the correlation function, which can be written as a sum of normal-ordered, cross and vacuum terms:
\[ C_{\mu\nu\alpha\beta}(x, x') = C_{\mu\nu\alpha\beta}^{\text{NO}}(x, x') + C_{\mu\nu\alpha\beta}^{\text{cross}}(x, x') + C_{\mu\nu\alpha\beta}^{\text{vac}}(x, x'). \tag{7} \]

Here \( C_{\mu\nu\alpha\beta}^{\text{NO}}(x, x') \) is state-dependent and finite as \( x' \to x \), the cross term is also state-dependent and singular,
\[ C_{\mu\nu\alpha\beta}^{\text{cross}}(x, x') \sim \frac{1}{(x - x')^4}, \tag{8} \]
and the vacuum term is state-independent and singular
\[ C_{\mu\nu\alpha\beta}^{\text{vac}}(x, x') \sim \frac{1}{(x - x')^8}. \tag{9} \]

The singularities in the correlation function should not be a cause for concern, as physical observables are integrals of the correlation function, and can be defined by an integration by parts procedure. In this sense, \( C_{\mu\nu\alpha\beta}^{\text{NO}}(x, x') \) is well-defined as a distribution. To illustrate the basic idea, consider the integral
\[ \int_a^b \frac{f(x)}{(x - c)^n} dx, \tag{10} \]
where \( a < c < b \). We may use the identity
\[ \frac{1}{(x - c)^n} = (-1)^{n-1}(n - 1)! \frac{d^n}{dx^n} \ln(x - c), \tag{11} \]

\[ \int_a^b \frac{f(x)}{(x - c)^n} dx = -(n - 1)! \int_a^b f^{(n)}(x) \ln(x - c) dx + \text{surface terms}. \tag{12} \]

The last integral contains only an integrable singularity, and the surface terms are evaluated away from the singularity at \( x = c \). Thus if the function \( f \) and its first \( n \) derivatives are finite, then the integral is well defined. An alternative approach is to use dimensional regularization, in which case the integrals of \( C_{\mu\nu\alpha\beta}(x, x') \) are finite in the limit of four spacetime dimensions [13].
III. FLUCTUATIONS OF FORCES ON MATERIAL BODIES

Before turning to the main topic of this paper, it is informative to give a brief summary of a closely related subject. Just as the classical stress tensor may be used, for example, to compute electromagnetic forces on dielectric bodies, the quantum stress tensor describes the quantum fluctuations in these forces. An example is the Casimir force, the mean value of which is given by a suitably defined expectation value of $\langle T_{\mu\nu} \rangle$. This force is expected to undergo fluctuations around this mean value [14, 15, 16, 17]. However these fluctuations are too small to be readily observable. For example, in Ref. [17], the fluctuations of the Casimir-Polder force on a polarizable atom near a reflecting plate was calculated. It was found that the atom undergoes Brownian motion in the sense that its mean squared velocity shifts due to the presence of the plate. The transverse component shifts by

$$
\langle \Delta v_x^2 \rangle = \frac{47}{768} \frac{\hbar^2 \alpha^2}{m^2 z^8}
$$

and the longitudinal component by

$$
\langle \Delta v_z^2 \rangle = -\frac{3787}{3840} \frac{\hbar^2 \alpha^2}{m^2 z^8}.
$$

Here $z$ is the distance to the plate, $m$ is the mass of the atom, and $\alpha$ is its static polarizability in Lorentz-Heaviside units. The negative sign in the longitudinal component seems to imply a reduction in the velocity dispersion of the wavepacket of a quantum particle localized near the plate. Equations (13) and (14) represent a sum of a fully normal ordered (with respect to the Minkowski vacuum) contribution, and a cross term. However, in both cases, the dominant contribution is that of the cross term. The effective temperature associated with transverse component is

$$
T_{\text{eff}} \approx 10^{-1} K \left( \frac{m_H}{m} \right) \left( \frac{10^{-8} \text{cm}}{z} \right)^8 \left( \frac{\alpha}{\alpha_H} \right)^2,
$$

where $m_H$ and $\alpha_H$ are the mass and static polarizability of atomic hydrogen, respectively. Although the effect is small, it might be observable if sufficiently small values of $z$ could be attained.

A second example of force fluctuation is the quantum fluctuation of radiation pressure. This is expected to be a significant source of noise in future generations of laser interferometer detectors of gravity waves. For the case of light in a coherent state, this effect was first analyzed by Caves [18, 19], using an approach based upon fluctuation in photon numbers. It was studied by the present authors [20] using the quantum stress tensor, where it was shown that the radiation pressure fluctuations arise entirely from the cross term in the correlation function. This follows from the fact that for a single mode coherent state $|z\rangle$,

$$
\langle z | : T_{\mu\nu} T_{\rho\sigma} : | z \rangle = \langle z | : T_{\mu\nu} : | z \rangle \langle z | : T_{\rho\sigma} : | z \rangle,
$$

and hence the fully normal ordered term vanishes, and the vacuum term is independent of the state of the radiation field. If a free mirror of mass $m$ is subjected to a laser beam moving in the $x$-direction for a time $\tau$, then the variance in the mirror’s velocity is

$$
\langle \Delta v^2 \rangle = \frac{1}{m^2} \int_0^\tau dt \int_0^\tau dt' \int_A da \int_A da' \langle T_{xx}(x)T_{xx}(x') \rangle_{\text{cross}}.
$$
FIG. 1: A Michelson interferometer with several bounces in each arm. The different illuminated spots on each mirror have correlated radiation pressure fluctuation.

Here $\int_A da$ denotes an integral over the area of the mirror. If the laser beam has linear polarization in the $y$-direction, then it may be shown that

$$\langle T_{xx}(x)T_{xx}(x') \rangle_{\text{cross}} = \langle :B_z(x)B_z(x'):\rangle \langle B_z(x)B_z(x')\rangle_0,$$

where $\langle :B_z(x)B_z(x'):\rangle$ is a finite state-dependent factor, and $\langle B_z(x)B_z(x')\rangle_0$ is the vacuum magnetic field two-point function in the presence of the mirror. Although the latter function is singular as $x' \to x$, the integral in Eq. (17) is finite, and may be evaluated to find

$$\langle \Delta v^2 \rangle = 4 \frac{A\omega \rho}{m^2 \tau},$$

where $A$ is the mirror’s area, $\omega$ is the angular frequency of the laser beam, and $\rho$ is the energy density in the beam. This agrees with Caves’ [18, 19] result using photon number fluctuations.

One remarkable aspect of the radiation pressure fluctuations is that the fluctuations on different bounces in an interferometer are correlated. A Michelson interferometer is illustrated in Fig. 1. The laser beam is split and subsequently bounces $b$ times in each arm, illuminating $b$ different spots on each mirror in the process. If the pressure fluctuations at each spot were uncorrelated, the variance in a mirror’s velocity would be proportional to $b^2$, whereas in fact it is proportional to $b$. In a photon number approach these correlations come from the fact that an fluctuation in photon number in a wavepacket is preserved as the packet bounces in the interferometer. In the stress tensor approach, the correlations of different spots in one arm are encoded in the vacuum two-point function, $\langle B_z(x)B_z(x')\rangle_0$. In the presence of mirrors, the usual lightcone singularity follows the path of the beam in the interferometer.
IV. ANTI-CORRELATIONS AND THE PROBABILITY DISTRIBUTION

In the previous section, we saw an example in which stress tensor fluctuations exhibit strong correlations. There are also examples of anticorrelations which we now discuss. First consider the case of electric field fluctuations in the Minkowski vacuum. A charged particle coupled to the fluctuating electric field should undergo Brownian motion. However, if successive field fluctuations were uncorrelated, then the particle’s velocity should undergo a random walk, and in the absence of damping we would have \( \langle v^2 \rangle \propto t \). However, this would clearly violate energy conservation, as there is no source of energy when the radiation field is in its vacuum state. The resolution of this apparent paradox is that successive velocity fluctuations must be anticorrelated. The particle can acquire an energy \( E \) from a field fluctuation, but will lose it again on a timescale no longer than \( \bar{\hbar}/E \). This effect can be analyzed clearly in the case where one considers the effects of a reflecting plate, so that one is calculating the change in \( \langle v^2 \rangle \) due to the plate. This was done in Ref. [21]. Consider a particle of mass \( m \) and charge \( e \) which is at rest at \( t = 0 \). At time \( t \), the variance of the \( i \)-component of the velocity is

\[
\langle \Delta v_i^2 \rangle = \frac{e^2}{m^2} \int_0^t \int_0^t \langle E_i(x, t_1) E_i(x, t_2) \rangle_R dt_1 dt_2 ,
\]

where \( \langle E_i(x, t_1) E_i(x, t_2) \rangle_R \) is the shift in the electric field correlation function due to the plate. One finds that all components of \( \langle \Delta v_i^2 \rangle \) either vanish or approach a nonzero constant in the limit that \( t \to \infty \). The result that \( \langle \Delta v_i^2 \rangle \) does not grow in time comes from the fact that

\[
\int_0^\infty \langle E_i(x, t_1) E_i(x, t_2) \rangle_R dt_1 = 0 ,
\]

which is the mathematical expression of the anticorrelations. Similar anticorrelations prevent the growth of the atom’s velocity dispersion in Eqs. (13) and (14).

Analogous effects can be found in quantum stress tensor fluctuations in the Minkowski vacuum state. Suppose that the energy density along the worldline of an inertial observer is sampled using a sampling function \( g(t, t_0) \), which is centered about \( t = t_0 \) and has a characteristic width \( a \). Defined the sampled energy density by

\[
S(t_0) = \int_{-\infty}^\infty dt g(t, t_0) : T_{tt}(t) :
\]

Now perform two successive measurements, the first with the sampling function \( g(t, 0) \), and the second with the function \( g(t, t_0) \), and then define the quantity

\[
\langle S(t_0)S(0) \rangle = \int_{-\infty}^\infty dt \int_{-\infty}^\infty dt' g(t, t_0) g(t', 0) C(t, t') ,
\]

where \( C(t, t') = C_{ttt}(t, t') \) is the energy density correlation function evaluated at coincident points in space, but distinct times. The function \( \langle S(t_0)S(0) \rangle \) describes the correlation between the two successive measurements, the first of the energy density in an interval around \( t = 0 \), and the second in an interval around \( t = t_0 \). The explicit evaluation of the integrals in Eq. (23) requires an integration by parts and is performed in Ref. [22] for several sampling functions. The results reveal that \( \langle S(t_0)S(0) \rangle > 0 \) for \( t_0 \ll a \), but then becomes negative as \( t_0 \) increases. It may undergo several damped oscillations, but approaches zero for \( t_0 \gg a \). (See, for example Fig. 5 in Ref. [22].) The physical interpretation is this behavior is that
nearly overlapping measurements are positively correlated, but measurements taken somewhat further apart in time become anticorrelated. There is an equal amount of correlation and anticorrelation in the sense that
\[ \int_0^\infty dt_0 \langle S(t_0)S(0) \rangle = 0. \] (24)

These results show that quantum fluctuations in the Minkowski vacuum state exhibit strict anticorrelations as well as correlations. We will see later that in curved spacetime it is possible to upset the anticorrelations.

An issue related to the anticorrelations is that of the probability distribution for stress tensor fluctuations. Let \( \bar{T} \) be a renormalized stress tensor component averaged over some spacetime region. One would like to construct a probability function \( P(\bar{T}) \) which gives the probability of finding a given value of \( \bar{T} \) in a measurement. This is an unsolved problem which is currently being investigated. A preliminary account of this work was summarized in Ref. [23]. The function \( P(\bar{T}) \) is asymmetric, as typically odd moments such as \( \langle \bar{T}^3 \rangle \) are nonzero. This means that the distribution cannot be Gaussian. Furthermore, it is expected that \( P(\bar{T}) = 0 \) for \( \bar{T} < T_{QI} \), where \( T_{QI} \) is a negative lower bound determined by quantum inequalities. However, there is no upper bound. Because \( \langle \bar{T} \rangle = 0 \), the mean of the distribution is at zero, but the area to the left of \( \bar{T} = 0 \) exceeds that to the right. This means that if one makes a measurement of the average energy density in a given region of spacetime, the result will be negative more frequently than positive, but when positive the result typically has a greater magnitude.

V. PASSIVE QUANTUM GRAVITY

In general, the spacetime geometry should be subjected both to fluctuations in the intrinsic degrees of freedom of gravity, the active fluctuations, as well as the effects of quantum stress tensor fluctuations, the passive fluctuations. In this paper, we are concerned only with the latter. Even without consideration of the active fluctuations, we can have a restricted theory with only the passive gravitational field fluctuations. Here we will outline such a theory on a flat background, following Ref. [24]. We consider only the contribution of the vacuum term in the stress tensor correlation function. For the electromagnetic field, the correlation function may be written as
\[
C_{(V)}^{\mu\nu\sigma\lambda}(x, x') = 4 (\partial_\mu \partial_\nu D)(\partial_\sigma \partial_\lambda D) + 2 g_{\mu\nu} (\partial_\sigma \partial_\lambda D)(\partial_\alpha \partial^\alpha D) + 2 g_{\sigma\lambda} (\partial_\mu \partial_\alpha D)(\partial_\nu \partial^\alpha D) - 2 g_{\mu\sigma} (\partial_\nu \partial_\lambda D)(\partial_\alpha \partial^\alpha D) - 2 g_{\lambda\nu} (\partial_\mu \partial_\sigma D)(\partial_\alpha \partial^\alpha D) - 2 g_{\nu\lambda} (\partial_\mu \partial_\sigma D)(\partial_\sigma \partial^\alpha D) + (g_{\nu\sigma} g_{\mu\lambda} + g_{\nu\lambda} g_{\mu\sigma} - g_{\mu\nu} g_{\sigma\lambda})(\partial_\rho \partial_\alpha D)(\partial^\rho \partial^\alpha D),
\]
where
\[ D = D(x - x') = \frac{1}{4\pi^2(x - x')^2} \] (26)
is the Hadamard (symmetric two-point) function for the massless scalar field. A similar result for the case of the scalar field has been given by Martin and Verdaguer [25]. Let \( h_{\mu\nu} \)
be a metric perturbation due to the stress tensor $T_{\mu\nu}$. In a gauge in which $\partial_\nu h^{\mu\nu} = h^\mu_\mu = 0$, the linearized Einstein equation becomes

$$\partial_\alpha \partial^\alpha h_{\mu\nu} = -16\pi T_{\mu\nu}$$

(27)
in units in which $G = 1$, where $G$ is Newton’s constant. One may solve this equation as an integral involving the retarded Green’s function, and use the result to construct the metric correlation function $\langle h^{\mu\nu}(x)h^{\rho\sigma}(x') \rangle$. Remarkably, this correlation function can be expressed as a local function,

$$\langle h^{\mu\nu}(x)h^{\rho\sigma}(x') \rangle = -\frac{1}{60\pi^2} \left[ 4 \partial_\mu \partial_\nu \partial_\rho \partial_\sigma S + 2 (g^{\mu\rho} \partial_\sigma \partial_\lambda + g^{\mu\lambda} \partial_\rho \partial_\sigma) \partial_\alpha \partial^\alpha S - 3 (g^{\mu\sigma} \partial_\nu \partial_\lambda + g^{\mu\lambda} \partial_\nu \partial_\sigma + g^{\nu\sigma} \partial_\mu \partial_\lambda + g^{\nu\lambda} \partial_\mu \partial_\sigma) \partial_\alpha \partial^\alpha S + 3 (g^{\mu\sigma} g^{\nu\lambda} + g^{\nu\sigma} g^{\mu\lambda}) (\partial_\alpha \partial^\alpha) S - 2 g^{\mu\nu} g^{\rho\sigma} (\partial_\alpha \partial^\alpha)^2 S \right],$$

(28)

where

$$S = \ln^2[\mu^2(x - x')^2],$$

(29)
and $\mu$ is an arbitrary constant. The significance of Eq. (28) is that expresses the metric correlation function as a sum of total derivatives. This allows us to readily evaluate observable quantities associated with metric fluctuations using integration by parts, as these quantities are expressible as integrals involving the correlation function.

VI. SOME PHYSICAL EFFECTS OF PASSIVE GEOMETRY FLUCTUATIONS

In this section, we will summarize three physical phenomena which in principle could be caused by quantum geometry fluctuations near a flat background.

A. Luminosity Fluctuations

The image of a distant source viewed through a fluctuating medium will undergo variations in apparent luminosity. This effect is the cause of the familiar “twinkling” of stars viewed thought the earth’s atmosphere. In principle, the same effect will arise due to geometry fluctuations. A quantitative discussion of this effect was given in Ref. [26]. The basic idea is that luminosity fluctuations are related to the expansion parameter, $\theta$, of a bundle of geodesics. If $k^\mu$ is the tangent vector to the geodesics, $\theta$ is defined by $\theta = k^\mu_\mu$, or equivalently as the logarithmic derivative of the cross sectional area of the bundle,

$$\theta = \frac{d\log(A/A_0)}{d\lambda},$$

(30)

where $\lambda$ is the affine parameter for the geodesics. The expansion satisfies the Raychaudhuri equation,

$$\frac{d\theta}{d\lambda} = -R_{\mu\nu}k^\mu k^\nu - a \theta^2 - \sigma_{\mu\nu} \sigma^{\mu\nu} + \omega_{\mu\nu} \omega^{\mu\nu}. $$

(31)

Here $\sigma^{\mu\nu}$ is the shear and $\omega^{\mu\nu}$ is the vorticity of the congruence. The constant $a = 1/2$ for null geodesics, and $a = 1/3$ for timelike geodesics. The use of the Raychaudhuri equation
as a Langevin equation for fluctuating spacetimes was proposed by Moffat [27]. We are interested in situations where shear, vorticity and $\theta^2$ may be neglected, in which case

$$\frac{d\theta}{d\lambda} = -R_{\mu\nu}k^\mu k^\nu,$$  \hfill (32)

where $R_{\mu\nu}$ is determined by the fluctuating stress tensor by

$$R_{\mu\nu} = 8\pi \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^\rho_\rho\right).$$  \hfill (33)

For photons on a nearly flat background, we may take the affine parameter to be the coordinate time in our frame. The luminosity fluctuations may be shown to be an integral of the expansion correlation function

$$\left\langle \left(\frac{\Delta L}{L}\right)^2 \right\rangle = \int_0^s \int_0^s dt' dt'' \left[ \left\langle \theta(t') \theta(t'') \right\rangle - \left\langle \theta(t') \right\rangle \left\langle \theta(t'') \right\rangle \right],$$  \hfill (34)

where $s$ is the flight distance. However, in many cases [26] this integral is approximately

$$\left\langle \left(\frac{\Delta L}{L}\right)^2 \right\rangle \propto s^2 \left\langle (\Delta \theta)^2 \right\rangle.$$  \hfill (35)

The variance of $\theta$ may be computed as an integral of the Ricci tensor correlation function

$$\langle \theta^2 \rangle - \langle \theta \rangle^2 = \langle (\Delta \theta)^2 \rangle = \int_0^{\lambda_0} d\lambda \int_0^{\lambda_0} d\lambda' K_{\mu\nu\alpha\beta}(\lambda, \lambda') k^\mu(\lambda) k^\nu(\lambda) k^\alpha(\lambda') k^\beta(\lambda'),$$  \hfill (36)

where $K_{\mu\nu\alpha\beta}$ is the Ricci tensor correlation function, which is algebraically related to the stress tensor correlation function, Eq. (2).

The luminosity fluctuations caused by the stress tensor fluctuations in a thermal bath of scalar particles is [26]

$$\left(\frac{\Delta L}{L}\right)_{rms} = 0.02 \left(\frac{s}{10^{26} \text{cm}}\right)^{\frac{3}{2}} \left(\frac{T}{10^6 \text{K}}\right)^{\frac{7}{2}} = 10^{-3} \left(\frac{s}{10^{6} \text{km}}\right)^{\frac{3}{2}} \left(\frac{T}{1 \text{GeV}}\right)^{\frac{7}{2}}.$$  \hfill (37)

This result holds when the wavelength of the radiation from the source is short compared to the typical wavelength of the particles in the thermal bath. It shows that although the effects of spacetime geometry fluctuations are small, there are conceivable circumstances in which they are nonzero.

**B. Line Broadening and Angular Blurring**

In addition to luminosity fluctuations, there are other possible effects of a fluctuating gravitational field upon light propagation. These include the broadening of spectral lines and the angular blurring of images. These two effect may both be expressed in terms of integrals of the Riemann tensor correlation function [28]. This geometric construction relies upon the fact that the change in a vector, when parallel transported around a closed path in curved spacetime, is the integral of the Riemann tensor over the enclosed surface.
Consider two observers who are initially at rest with respect to one another and have 4-velocity $t^\mu$. Suppose that two successive photons are sent from one observer to the other. If there is classical gravitational field present, then there will be a fractional frequency shift of $\xi = \Delta \omega / \omega$. If the gravitational field fluctuates, then this fractional shift has a variance of

$$\delta \xi^2 = \langle (\Delta \xi)^2 \rangle - \langle \Delta \xi \rangle^2 = \int da \int da' C_{\alpha\beta\mu\nu \gamma \delta \rho \sigma}(x, x') t^\alpha k^\beta t^\mu k^\nu t^\gamma k^\rho t^\sigma,$$

where $k^\mu$ is the photon wavevector and $C_{\alpha\beta\mu\nu \gamma \delta \rho \sigma}(x, x')$ is the Riemann tensor correlation function:

$$C_{\alpha\beta\mu\nu \gamma \delta \rho \sigma}(x, x') = \langle R_{\alpha\beta\mu\nu}(x) R_{\gamma \delta \rho \sigma}(x') \rangle - \langle R_{\alpha\beta\mu\nu}(x) \rangle \langle R_{\gamma \delta \rho \sigma}(x') \rangle.$$

The fluctuation in the angle of the source in a direction specified by a unit spacelike vector $s^\mu$ is given by a similar expression,

$$\delta \Theta^2 = \langle (\Delta \Theta)^2 \rangle - \langle \Delta \Theta \rangle^2 = \int da \int da' C_{\alpha\beta\mu\nu \gamma \delta \rho \sigma}(x, x') s^\alpha k^\beta t^\mu k^\nu s^\gamma k^\rho t^\sigma.$$

There is a crucial difference between luminosity fluctuations on the one hand and both line broadening and angular blurring on the other. The expansion fluctuations, and hence the luminosity fluctuations, are determined by the Ricci tensor correlation function, and hence arise in leading order only for passive geometry fluctuations. The other two effects, being determined by the Riemann tensor fluctuations, can occur for active fluctuations as well. However, in many cases of passive fluctuations, all three effects are of the same order. Thus for a thermal bath, Eq. (37) also gives an estimate of the magnitude of the line broadening and angular blurring effects.

C. Black Hole Fluctuations

Fluctuations in both the Hawking radiation and in black hole horizons have been discussed by several authors. In particular, fluctuations in the outgoing flux at infinity were calculated in Ref. [29]. This fluctuation, in dimensionless terms, is of order unity. This can be understood by recalling that a radiating black hole emits on average one particle in a time of order $M$. Thus the variance in this number is also of order unity. The calculation of the mass fluctuations of evaporating black holes requires consideration of the backreaction, as recently noted by Hu and Roura [30]. This leads to a larger result than the estimate given in Ref. [29] because a positive fluctuation in the outgoing radiation causes the black hole’s mass to decrease and hence for it to radiate more rapidly.

Fluctuation of the horizon are more difficult to calculate, and are a topic of ongoing research [30, 31, 32].

VII. STRESS TENsOR FLUCTUATIONS IN INFLATION

In this section, we will summarize the results of Ref. [33], where the effect of electromagnetic stress tensor fluctuations in inflation were studied. Take the spacetime to be a spatially flat Robertson-Walker universe

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2).$$
In a general spacetime, the conservation law for a perfect fluid can be written as\[34, 35\]

\[
\dot{\rho} + (\rho + p) \theta = 0,
\]

where \(\theta\) is the expansion of the congruence of worldlines for the observers who measure the energy density \(\rho\) and pressure \(p\). In the case of the unperturbed Robertson-Walker universe,

\[
\theta = \theta_0 = 3 \frac{\dot{a}}{a},
\]

leading to the usual form of the conservation law. Our interest is in fluctuations of \(\theta\), which will in turn lead to fluctuations in \(\rho\).

Consider a conformally invariant field, such as the electromagnetic field, and assume that the shear and vorticity terms in the Raychaudhuri equation can be neglected, but retain the \(\theta^2\) term. It may be shown that the expansion correlation function can be expressed as an integral of the stress tensor correlation function as

\[
\langle \theta(t_1) \theta(t_2) \rangle - \langle \theta(t_1) \rangle \langle \theta(t_2) \rangle = (8\pi)^2 a^{-2}(t_1) a^{-2}(t_2) \int_{t_0}^{t_1} dt \int_{t_0}^{t_2} dt' a^2(t') C_{\mu \nu \rho}.
\]

The energy density correlation function can be obtained by a conformal transformation from flat spacetime:

\[
C_{\mu \nu \rho}(x, x') = a^{-4}(t) a^{-4}(t') E,
\]

where \(E\) is the flat space vacuum energy density correlation function. It is convenient to convert from comoving time \(t\) to conformal time \(\eta\), where \(dt = a d\eta\). The \(\theta\)-correlation function now becomes

\[
\langle \theta(\eta_1) \theta(\eta_2) \rangle - \langle \theta(\eta_1) \rangle \langle \theta(\eta_2) \rangle = \frac{(8\pi)^2}{a^2(\eta_1) a^2(\eta)} \int_{\eta_1}^{\eta} d\eta \int_{\eta_1}^{\eta_2} d\eta' \frac{d\eta'}{a(\eta')} E(\Delta \eta, r),
\]

where \(\Delta \eta = \eta - \eta'\) and \(r = |x - x'|\) is the coordinate space separation of the pair of points at which \(\theta\) is measured. Here we assume that the \(\theta\)-fluctuations vanish at \(\eta = \eta_0\). For the electromagnetic field

\[
E_{em} = \frac{(r^2 + 3 \Delta \eta^2)^2}{4\pi^4(r^2 - \Delta \eta^2)^6}.
\]

Consider inflation followed by a radiation-dominated universe. We take the scale factor to be

\[
a(\eta) = \frac{1}{1 - H \eta}, \quad \eta_0 \leq \eta \leq 0,
\]

and

\[
a(\eta) = 1 + H \eta, \quad \eta \geq 0.
\]

Thus reheating occurs on the \(\eta = 0\) surface. If we assume that the perfect fluid has the equation of state \(p = w \rho\), then the conservation law yields the density fluctuations in the post-inflationary era:

\[
\left\langle \left( \frac{\delta \rho}{\rho} \right)^2 \right\rangle = (8\pi)^2 (1 + w)^2 \int_0^{\eta_0} \frac{d\eta_1}{a(\eta_1)} \int_0^{\eta_0} \frac{d\eta_2}{a(\eta_2)} \left[ \langle \theta(\eta_1) \theta(\eta_2) \rangle - \langle \theta(\eta_1) \rangle \langle \theta(\eta_2) \rangle \right].
\]
Here the integrations begin at $\eta = 0$ because the classical fluid is assumed to be created then. The power spectrum of density perturbations, $P_k$, is defined by

$$\left\langle \left( \frac{\delta \rho}{\rho} \right)^2 \right\rangle = \int d^3 k e^{i \mathbf{k} \cdot \Delta \mathbf{x}} P_k(\eta_s).$$  \hspace{1cm} (51)

Remarkably, one finds that the effect of the quantum stress tensor fluctuations upon the density perturbations grows as the duration of inflation increases. For $H|\eta_0| \gg 1$, one finds that the variance of the expansion at the end of inflation is given by

$$\left\langle (\Delta \theta)^2 \right\rangle = \left\langle \theta(0) \theta(0) \right\rangle - \left\langle \theta(0) \right\rangle \left\langle \theta(0) \right\rangle \approx \frac{8H^2|\eta_0|^2}{5\pi^2r^6}. \hspace{1cm} (52)$$

This result reveals that the expansion fluctuations grow with the duration of inflation. In the flat space limit, $H \rightarrow 0$, there would be no growth. We can understand the lack of growth in flat spacetime as being due to anticorrelations. The growth in deSitter spacetime means that the cancellations that would have occurred in flat spacetime have been upset. This is implemented by the factors of $1/a$ which appear in the integrand of Eq. (46), and reflect the time dependent geometry of an expanding universe.

The power spectrum of density perturbations also grows as $|\eta_0|^2$,

$$P_k(\eta_s) \approx \frac{32|\eta_0|^2k^3}{15\pi} \ln^2[a(\eta_s)] (1 + w)^2 \ell_p^4,$$  \hspace{1cm} (53)

where $\ell_p$ is the Planck length. If we consider the effects of the modes within a finite bandwidth, $\Delta k \approx k$, then we find an estimate of the associated density perturbations

$$\left( \frac{\delta \rho}{\rho} \right)_{rms} \approx \frac{\ell_p^2|\eta_0| k^3 \ln[a(\eta_s)]}{2^{12}}.$$  \hspace{1cm} (54)

Here $1/a(\eta_s)$ is the redshift factor between reheating and the last scattering surface, so that

$$a(\eta_s) \approx \frac{E_R}{1\text{eV}},$$  \hspace{1cm} (55)

where $E_R$ is the reheating energy scale. Note that $k = a_{now} k_p$. Let $k_p = 2\pi/\lambda$ correspond to the typical intergalactic separation today, $\lambda \approx 2\text{Mpc}$, or $k_p \approx 10^{-24}\text{cm}^{-1}$. On this scale, we must have

$$\left( \frac{\delta \rho}{\rho} \right)_{rms} < 10^{-4},$$  \hspace{1cm} (56)

which leads to an upper bound on the duration of inflation of

$$H |\eta_0| < 10^{79} \left( \frac{10^{12}\text{GeV}}{E_R} \right).$$  \hspace{1cm} (57)

This bound is considerably greater than the amount of inflation needed to solve the horizon and flatness problems, $H |\eta_0| > 10^{23}$. The density perturbations described by Eq. (54) can be understood as arising from a kinematic effect. The $\theta$ fluctuations which accumulate during inflation cause differential expansion rates at the reheating surface, $\eta = 0$, which are

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transmitted through this surface to the radiation dominated universe. The density variations then arise from differential redshifting.

It is possible to obtain a stronger bound by examining the dynamics of the inflaton field, \( \varphi \). The expansion \( \theta \) appears in the inflaton equation of motion, so that \( \theta \) fluctuations create \( \varphi \) fluctuations. These are in addition to the intrinsic quantum fluctuations of \( \varphi \), which are usually considered. In Ref. \[33\], it was shown that the expansion induced density perturbations are

\[
\left( \frac{\delta \rho}{\rho} \right)_{\text{rms}} \approx \sqrt{k^3 P_k} \approx 10^{-2} \frac{f_p^2 H|\eta_0|}{t_R} k.
\]

(58)

With the same choice of scale as above, we find

\[
H |\eta_0| < 10^{45} \left( \frac{10^{12}\text{GeV}}{E_R} \right)^3.
\]

(59)

This is considerably more restrictive than Eq. (57), but is still compatible with adequate inflation to solve the horizon problem.

VIII. SUMMARY

We have seen that quantum stress tensor fluctuations can have several physical effects. Because the stress tensor describes the forces which the electromagnetic field exerts on a material body, stress tensor fluctuations result in effects such as Casimir force fluctuations and radiation pressure fluctuations. These systems can be useful analog models for gravitational field fluctuations and are potentially amenable to study in the laboratory.

A crucial feature of stress tensor fluctuations is the presence of subtle correlations and anticorrelations. In flat spacetime, the anticorrelations often lead to the exact cancellation of the effects of the fluctuations, as in the case illustrated in Eq. (24). However, in curved spacetime these cancellations can be altered, as in the case of deSitter spacetime where the variance of the expansion parameter \( \theta \) grows in time.

Quantum stress tensor fluctuations lead to passive fluctuations of the gravitational field, which are in addition to the active fluctuations coming from the quantization of the gravitational field itself. These fluctuations around a flat background are described by a metric correlation function, which for the electromagnetic field in the vacuum state is given by Eq. (28). In more general quantum states for the matter field, one can have various physical phenomena produce by gravitational field fluctuations, These include effects on the propagation of signals through the fluctuating background, such as luminosity fluctuations, line broadening, and angular blurring of images. All of these effects can be described in terms of integrals of the Riemann tensor correlation function, and for passive spacetime fluctuations, in terms of the stress tensor correlation function. Although the stress tensor correlation function is formally singular in the limit of coincident points, it is a well defined distribution in the sense that its integrals can be evaluated using integration by parts.

Passive geometry fluctuations can play a role in both black hole evaporation and in the early universe. We have seen how the effects of the fluctuations of the electromagnetic field can cause the expansion of the comoving geodesics in an inflationary universe to undergo growing fluctuations. These expansion fluctuations in turn lead to density perturbations in the post-inflationary universe that are potentially observable. These perturbations have a
non-scale invariant and non-Gaussian nature, and must hence constitute at most a small fraction of the primordial density fluctuations. This allows one to place constraints on the duration of inflationary expansion.

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