FREDHOLMNESS AND COMPACTNESS OF TRUNCATED TOEPLITZ AND HANKEL OPERATORS

R. V. BESSONOV

Abstract. We prove the spectral mapping theorem \( \sigma_e(A_\varphi) = \varphi(\sigma_e(A_z)) \) for the Fredholm spectrum of a truncated Toeplitz operator \( A_\varphi \) with symbol \( \varphi \) in the Sarason algebra \( C + H^\infty \) acting on a coinvariant subspace \( K_\theta \) of the Hardy space \( H^2 \). Our second result says that a truncated Hankel operator on the subspace \( K_\theta \) generated by a one-component inner function \( \theta \) is compact if and only if it has a continuous symbol. We also suppose a description of truncated Toeplitz and Hankel operators in Schatten classes \( S^p \).

1. Introduction

Truncated Toeplitz and Hankel operators are compressions of the standard Toeplitz and Hankel operators on the Hardy space \( H^2 \) to its coinvariant subspaces \( K_\theta \). More precisely, consider the shift operator \( S : f \mapsto zf \) on the Hardy space \( H^2 \) in the open unit disk \( \mathbb{D} \) of the complex plane \( \mathbb{C} \). By A. Beurling theorem, invariant subspaces of \( S \) have the form \( \theta H^2 \), where \( \theta \) is an inner function in \( \mathbb{D} \). Accordingly, the subspaces \( K_\theta = H^2 \ominus \theta H^2 \) are invariant under the backward shift operator \( S^* : f \mapsto f(0) - \overline{f} \) on \( H^2 \). Fix an inner function \( \theta \) and consider \( H^2 \) and \( K_\theta \) as subspaces of the space \( L^2 = L^2(\mathbb{T}) \) on the unit circle \( \mathbb{T} \). Let \( P_\theta \) and \( \overline{zK_\theta} = \{ f \in L^2 : \overline{zf} \in K_\theta \} \), correspondingly. Take \( \varphi \in L^2 \) and define the operators

\[
A_\varphi : f \mapsto P_\theta(\varphi f), \quad \Gamma_\varphi : f \mapsto P_\theta(\varphi f),
\]

on the dense subset \( K_\theta \cap L^\infty \) of the space \( K_\theta \). The operator \( A_\varphi : K_\theta \to K_\theta \) is called the truncated Toeplitz operator; \( \Gamma_\varphi : K_\theta \to \overline{zK_\theta} \) is the truncated Hankel operator with symbol \( \varphi \). For recent results on truncated Toeplitz operators see survey [8]. The present paper is closely related to [1], [2], [4], [21], [23]. Let us now discuss in details two results formulated in the abstract.

A spectral mapping theorem. The restricted shift operator \( S_\theta = A_z \) on \( K_\theta \) is the simplest non-trivial case of Sz.-Nagy–Foias model [25] for Hilbert space contractions. Textbook [15] by N. K. Nikolski is devoted to the study of different properties of this operator. Defining \( \varphi(S_\theta) = A_\varphi \) for bounded analytic functions \( \varphi \in H^\infty \), we obtain the \( H^\infty \)-functional calculus for the operator \( S_\theta \). Let \( \sigma(\theta) \) denote the spectrum of the inner function \( \theta \), that is, the set of all points \( \xi \) in the closed unit

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disk such that \( \lim \inf_{|z|<1, z \to \zeta} |\theta(z)| = 0 \). Then (see Section III.3 in [15]) for every 
function \( \varphi \in H^\infty \) we have
\[
\sigma(A_\varphi) = \varphi(\sigma(\theta)),
\]
where \( \sigma(A_\varphi) \) denotes the spectrum of the operator \( A_\varphi \). Formula (1) can be regarded
as a spectral mapping theorem for the \( H^\infty \)-functional calculus of the operator \( S_\theta \).
Indeed, for every \( \varphi \in H^\infty \) we have \( \sigma(\varphi(S_\theta)) = \varphi(\sigma(\theta)) = \varphi(\sigma(S_\theta)) \).

Proof of formula (1) relies on the celebrated Corona theorem for the algebra \( H^\infty \)
of bounded analytic functions in the unit disk \( \mathbb{D} \). On the other hand, by much
more elementary technique one can show that for every continuous function \( \varphi \) on
the unit circle \( \mathbb{T} \) we have
\[
\sigma_c(A_\varphi) = \varphi(\sigma(\theta) \cap \mathbb{T}),
\]
where \( \sigma_c(A_\varphi) \) is the Fredholm (or essential) spectrum of \( A_\varphi \), see Section V.4 in [15]
or [9]. Denote by \( C \) the set of all continuous functions on \( \mathbb{T} \). D. Sarason [22] proved
that the set
\[
C + H^\infty = \{ \varphi_1 + \varphi_2, \ \varphi_1 \in C, \ \varphi_2 \in H^\infty \}
\]
is the closed subalgebra of \( L^\infty \).
We compute the essential spectrum of truncated Toeplitz operators with symbols in \( C + H^\infty \).

**Theorem 1.** Let \( \theta \) be an inner function, and let \( A_\varphi \) be the truncated Toeplitz
operator on \( K_\theta \) with symbol \( \varphi \in C + H^\infty \). Then \( \sigma_c(A_\varphi) = \varphi(\sigma(\theta) \cap \mathbb{T}) \).

The main step in the proof of Theorem 1 is an application of the corona theorem
for the Sarason algebra \( C + H^\infty \) obtained in 2007 by R. Mortini and B. Wick [14].

Let \( B(K_\theta)/S^\infty(K_\theta) \) be the Calkin algebra of all bounded operators on \( K_\theta \)
modulo compact operators. We will denote its elements by \([T]\). Note that for \( T \in B(K_\theta) \)
we have \( \sigma_c(T) = \sigma([T]) \) by the definition of the essential spectrum. It not difficult
to check that the mapping \( \varphi \mapsto [A_\varphi] \) is the contractive homomorphism from \( C + H^\infty \)
to \( B(K_\theta)/S^\infty(K_\theta) \). Hence if we define the \( C + H^\infty \)-functional calculus of \([S_\theta]\) by
\( \varphi([S_\theta]) = [A_\varphi] \) for \( \varphi \in C + H^\infty \), then Theorem 1 becomes the spectral mapping
theorem for this calculus.

**Compact truncated Hankel operators.** The classical result by Hartman [10]
says that a Hankel operator \( H_\varphi : H^2 \to zH^2 \) is compact if and only if it has a
continuous symbol. Its modern proof (see, e.g., Section 1.5 in [17]) is based on the
Nehari theorem \( \|H_\varphi\| = \text{dist}_{L^\infty}(\varphi, H^\infty) \) and the fact that for every function \( \varphi \in C \)
we have
\[
\text{dist}_{L^\infty}(\varphi, H^\infty) = \text{dist}_{L^\infty}(\varphi, H^\infty \cap C).
\]
Formula (3) follows easily from properties of the Poisson kernel. Another proof
is due to D. Sarason. It uses a corollary of brothers Riesz theorem on analytic
measures, namely, the duality relation \( (C/A)^* = L^\infty/H^\infty \), where \( A = H^\infty \cap C \) is
the disk algebra. Both proofs can be found in Section VII of [13]. As we will see,
the duality approach works for the following analogue of (3): for every \( \varphi \in C \) we have
\[
\text{dist}_{L^\infty}(\varphi, F_\theta) = \text{dist}_{L^\infty}(\varphi, F_\theta \cap C),
\]
where \( F_\theta \) denotes the closure of the set \( \overline{\theta zH^\infty + H^\infty} = \{ \overline{\theta f_1 + f_2}, \ f_1, f_2 \in H^\infty \} \) in
\( w^* \)-topology of the space \( L^\infty \). It worth be mentioned that the elementary argument
based on a convolution with the Poisson kernel gives (4) only in the case where \( \theta \)
is a finite Blaschke product, see Section 4.
An inner function $\theta$ is said to be one-component if the set $\{z \in \mathbb{D} : |\theta(z)| < \varepsilon\}$ is connected for a positive number $\varepsilon < 1$. It follows from the results of [1] that truncated Hankel operators acting on the space $K_\theta$ generated by a one-component inner function $\theta$ satisfy the two-sided estimate $\|\Gamma_\varphi\| \approx \text{dist}_{L^\infty}(\varphi, F^2)$ of Nehari type. Using this estimate and formula (4) for the inner function $\theta^2$, we obtain the following theorem.

**Theorem 2.** Let $\theta$ be a one-component inner function. Then a truncated Hankel operator $\Gamma : K_\theta \to K_\theta$ is compact if and only if $\Gamma = \Gamma_\varphi$ for some $\varphi \in C$. Moreover, one can choose $\varphi \in C$ so that $\|\Gamma\| \leq \|\varphi\|_{L^\infty} \leq c_0\|\Gamma\|$, where the constant $c_0$ depends only on $\theta$.

In Section 4 we combine Theorem 2 with some other characterizations of compact truncated Hankel operators. We also present a conjecture on truncated Hankel operators in Schatten classes $S^p(K_\theta)$. We expect that their description is possible in terms of a mean oscillation with respect to the Clark measure of the inner function $\theta^2$.

## 2. Proof of Theorem 1

Let us recall some definitions. A bounded operator $T$ on the Hilbert space $H$ is called Fredholm if its range $\text{Ran} \, T$ is the closed subspace of $H$, $\dim \ker T < \infty$, and $\dim \ker T^* < \infty$. A basic Fredholm theory says that $T$ is Fredholm if and only if there are bounded operators $L$, $R$ and compact operators $K_L$, $K_R$ on $H$ such that $LT = I + K_L$, $TR = I + K_R$, where $I$ denotes the identical operator on $H$. The essential (Fredholm) spectrum of $T$, $\sigma_e(T)$, is the set of all $\lambda \in \mathbb{C}$ such that the operator $T - \lambda I$ is not Fredholm. Define the continuous spectrum $\sigma_c(T)$ of $T$ as the set of all $\lambda \in \mathbb{C}$ such that there exists a non-compact sequence $\{x_n\} \subset H$ for which $\lim_n \|(T - \lambda I)x_n\| = 0$. It is clear that $\sigma_c(T) \subset \sigma_e(T)$. Moreover, one can check that $\sigma_e(T) = \sigma_e(T) \cup \sigma_e(T^*)$.

We will prove that $\sigma_e(A_\varphi) \subset \varphi(\sigma(\theta) \cap \mathbb{T}) \subset \sigma_c(A_\varphi)$ for every $\varphi \in C + H^\infty$. Since functions $\varphi \in \mathbb{C} + H^\infty$ are not defined everywhere on $\mathbb{T}$, we need a definition of the image $\varphi(\sigma(\theta) \cap \mathbb{T})$. Let $m$ be the normalized Lebesgue measure on the unit circle $\mathbb{T}$. Put

$$\varphi(\sigma(\theta) \cap \mathbb{T}) = \{\zeta \in \mathbb{C} : \liminf_{z \to \zeta} \{\frac{1}{|z|^2} - |z|^2\} = 0\},$$

where $\hat{\varphi}$ denotes the Poisson transform of $\varphi$,

$$\hat{\varphi}(z) = \int_\mathbb{T} \varphi(\xi) \frac{1 - |z|^2}{1 - \xi \bar{z}} \, dm(\xi), \quad z \in \mathbb{D}.$$

For a function $\varphi \in C$ thus defined set $\varphi(\sigma(\theta) \cap \mathbb{T})$ coincides with the usual image of $\sigma(\theta) \cap \mathbb{T}$ by $\varphi$.

Proof of Theorem 1 is based on the following result [14].

**Theorem.** (R.Mortini, B. Wick) Let $f_1, \ldots, f_n$ be a family of functions in $C + H^\infty$ such that for some $\varepsilon > 0$ and $r < 1$ we have

$$|\hat{f}_1(z)| + \ldots + |\hat{f}_n(z)| > \varepsilon, \quad r \leq |z| < 1.$$

Then there exist functions $g_1, \ldots, g_n$ in $C + H^\infty$ such that $f_1 g_1 + \ldots + f_n g_n = 1$ almost everywhere on the unit circle $\mathbb{T}$. 
The corresponding statement in [14] (Theorem 1.1) is given in slightly different terms. However, that statement can be easily reduced to the above formulation by noting that \( f \in \mathcal{C} + H^\infty \) is invertible if and only if there exists \( r < 1 \) such that \( |\hat{f}(z)| > \varepsilon \) for some \( \varepsilon > 0 \) and all \( z \in \mathbb{D} \) with \( r \leq |z| < 1 \). Alternatively, one can check that the proof in [14] works for our reformulation without any essential changes.

**Lemma 2.1.** Let \( \theta \) be an inner function and let \( \varphi \in \mathcal{C} + H^\infty \). Then for the truncated Toeplitz operator \( A_\varphi : K_\theta \to K_\theta \) we have \( \varphi(\sigma(\theta) \cap \mathbb{T}) \subset \sigma_c(A_\varphi) \).

**Proof.** The proof of lemma is a modification of arguments in Section III.3 of [15]. Let \( \varphi \in \mathcal{C} + H^\infty \) and let \( \zeta \in \varphi(\sigma(\theta) \cap \mathbb{T}) \). Then there exists a sequence \( \{\lambda_n\} \subset \mathbb{D} \) such that

\[
\lim_{n \to \infty} (|\varphi(\lambda_n) - \zeta| + |\theta(\lambda_n)|) = 0.
\]

We can assume that \( \lambda_n \) converge to a point \( \lambda_\infty \in \mathbb{T} \) and that \( |\theta(\lambda_n)| < \frac{1}{4} \) for all \( n \).

For \( \lambda \in \mathbb{D} \) denote by \( \tilde{k}_\lambda \) the function \( \tilde{k}_\lambda = \frac{\theta - \theta(\lambda)}{z - \lambda} \). It is easy to check that \( \tilde{k}_\lambda \in K_\theta \) and \( \|\tilde{k}_\lambda\|^2 = \frac{1 - |\theta(\lambda)|^2}{(1 - |\lambda|)^2} \). We claim that

\[
\lim_{n \to \infty} \frac{\|A_\varphi - \zeta I\tilde{k}_{\lambda_n}\|}{\|\tilde{k}_{\lambda_n}\|} = 0.
\]

Consider functions \( \varphi_2 \in \mathcal{C} \), \( \varphi_2 \in H^\infty \) such that \( \varphi = \varphi_1 + \varphi_2 \). Put \( \zeta_1 = \varphi_1(\lambda_\infty) \) and \( \zeta_2 = \zeta - \zeta_1 \). We have

\[
\frac{\|(A_{\varphi_1} - \zeta_1 I)\tilde{k}_{\lambda_n}\|^2}{\|\tilde{k}_{\lambda_n}\|^2} \leq \frac{\|(\varphi_1 - \zeta_1)\tilde{k}_{\lambda_n}\|^2}{\|\tilde{k}_{\lambda_n}\|^2} \leq 8 \int_{\mathbb{T}} |\varphi_1(z) - \zeta_1|^2 \frac{1 - |\lambda_n|^2}{|z - \lambda_n|^2} \, dm(z).
\]

Since the function \( |\varphi_1 - \zeta_1|^2 \) is continuous on \( \mathbb{T} \) and vanishes at \( \lambda_\infty = \lim \lambda_n \), the last integral tends to zero as \( n \to \infty \). Next, for every \( \lambda_n \) the function \( \frac{\varphi_2 - \varphi_2(\lambda_n)}{z - \lambda_n} \) belongs to \( H^2 \), hence

\[
A_{\varphi_2}\tilde{k}_{\lambda_n} = P_0 \left( \frac{\varphi_2 - \varphi_2(\lambda_n)}{z - \lambda_n} \right) = -\theta(\lambda_n)P_0 \left( \frac{\varphi_2 - \varphi_2(\lambda_n)}{z - \lambda_n} \right) + \varphi_2(\lambda_n) \frac{\theta - \theta(\lambda_n)}{z - \lambda_n}.
\]

Using the estimate \( \|\varphi_2 - \varphi_2(\lambda_n)\| \leq \frac{4\|\varphi_2\|_{\infty}^2}{1 - |\lambda_n|^2} \), we obtain

\[
\frac{\|(A_{\varphi_2} - \zeta_2 I)\tilde{k}_{\lambda_n}\|^2}{\|\tilde{k}_{\lambda_n}\|^2} \leq 8|\theta(\lambda_n)|^2\|\varphi_2\|_{L^\infty}^2 + |\varphi_2(\lambda_n) - \zeta_2|^2.
\]

Both summands in the right hand side tend to zero as \( n \to \infty \). Combining estimates (6) and (7), we see that (5) holds. Since \( \lim |\theta(\lambda_n)| = 0 \), the sequence \( \tilde{k}_{\lambda_n}/\|\tilde{k}_{\lambda_n}\| \) is non-compact (use the fact that the sequence \( \sqrt{1 - |\lambda_n|^2}/(1 - |\lambda_n|^2) \) is non-compact in \( H^2 \)). It follows that \( \varphi(\sigma(\theta) \cap \mathbb{T}) \subset \sigma_c(A_\varphi) \).

\( \square \)

**Lemma 2.2.** Let \( \theta \) be an inner function. The mapping \( \varphi \mapsto [A_\varphi] \) is the contractive homomorphism from \( \mathcal{C} + H^\infty \) to \( B(K_\theta)/S^\infty(K_\theta) \).

**Proof.** Clearly, the mapping \( \varphi \mapsto [A_\varphi] \) is linear and \( \|[A_\varphi]\| \leq \|A_\varphi\| \leq \|\varphi\|_{L^\infty} \).

We have \( A_{\varphi_1}A_{\varphi_2} = A_{\varphi_1\varphi_2} \) for all \( \varphi_1, \varphi_2 \in H^\infty \), see Section III.2 in [15]. Also, it was observed in [9] that if \( \varphi_1 \in \mathcal{C} \) and \( \varphi_2 \in L^\infty \), then \( A_{\varphi_1}A_{\varphi_2} = A_{\varphi_1\varphi_2} + K_1 \) and \( A_{\varphi_2}A_{\varphi_1} = A_{\varphi_1\varphi_2} + K_2 \), where \( K_1, K_2 \) are compact operators on \( K_\theta \). This yields the statement of the lemma.

\( \square \)
Lemma 2.3. Let \( \theta \) be an inner function and let \( \varphi \in \mathbb{C} + H^\infty \). Then for the truncated Toeplitz operator \( A_\varphi : K_\theta \to K_\theta \) we have \( \sigma(e(A_\varphi)) \subset \varphi(\sigma(\theta) \cap \mathbb{T}) \).

Proof. Take a point \( \zeta \in \mathbb{C} \setminus \varphi(\sigma(\theta) \cap \mathbb{T}) \). Let us show that the operator \( A_\varphi - \zeta I \) on \( K_\theta \) is Fredholm. We have

\[
|\hat{\varphi}(z) - \zeta| + |\theta(z)| > \varepsilon, \quad z \in U_\delta \cap \mathbb{D},
\]

for some neighbourhood \( U_\delta \) of the closed set \( \sigma(\theta) \cap \mathbb{T} \). From the definition of \( \sigma(\theta) \) we see that estimate (8) holds for all \( z \) with \( |z| \geq r \), where \( r < 1 \) is a positive number depending on \( \theta \) and \( \varepsilon \). Thus, we can apply the corona theorem for the algebra \( \mathbb{C} + H^\infty \) and find functions \( g_1, g_2 \in \mathbb{C} + H^\infty \) such that

\[
(\varphi - \zeta)g_1 + \theta g_2 = 1
\]

almost everywhere on the unit circle \( \mathbb{T} \). Note that \( A_\theta \) is the zero operator on \( K_\theta \).

Using equation (9) and Lemma 2.3 we see that

\[
A_{g_1}(A_\varphi - \zeta I) = I + K_L, \quad (A_\varphi - \zeta I)A_{g_1} = I + K_R,
\]

for some compact operators \( K_L, K_R \) on \( K_\theta \). It follows that \( \zeta \in \mathbb{C} \setminus \sigma_e(A_\varphi) \), as required. \( \square \)

Proof of Theorem 1. By Lemma 2.1 and Lemma 2.3 we have the following chain of inclusions: \( \sigma_e(A_\varphi) \subset \varphi(\sigma(\theta) \cap \mathbb{T}) \subset \sigma_e(A_\varphi). \) Since \( \sigma_e(T) \subset \sigma_e(T) \) for every bounded operator \( T \), we have \( \sigma_e(A_\varphi) = \varphi(\sigma(\theta) \cap \mathbb{T}) \). In particular, for \( \varphi = z \) we get \( \sigma_e(A_z) = \sigma(\theta) \cap \mathbb{T} \) and hence \( \sigma_e(A_z) = \varphi(\sigma_e(A_z)) \). The theorem is proved. \( \square \)

Next proposition describes the kernel of the homomorphism \( \varphi \mapsto A_\varphi \) from \( \mathbb{C} + H^\infty \) to \( B(K_\theta)/S^\infty(K_\theta) \). It is an extension of the Sarason theorem on compact truncated Toeplitz operators with symbols in \( H^\infty \), see Section VIII.3 in [15].

Proposition 2.1. Let \( \theta \) be an inner function and let \( \varphi \in \mathbb{C} + H^\infty \). Then the truncated Toeplitz operator \( A_\varphi : K_\theta \to K_\theta \) is compact if and only if \( \varphi \in \theta \mathbb{C} + \theta H^\infty \).

Proof. Let \( P_- \) denote the orthogonal projector on \( L^2 \) to the subspace \( \mathbb{C} H^2 \). Consider the operator \( T : g \mapsto \theta P_-(\theta g) - P_-g \) on \( L^2 \). It is easy to see that \( Tg = 0 \) for \( g \in \mathbb{C} H^2 \oplus \theta H^2 \) and \( Tg = g \) for \( g \in K_\theta \). Hence, we have \( T = P_\theta \). Define the Hankel operator \( H_\psi : H^2 \to \mathbb{C} H^2 \) with symbol \( \psi \in L^\infty \) by \( H_\psi : f \mapsto P_-(\psi f) \). By Hartman’s theorem, \( H_\psi \) is compact if and only if \( \psi \in \mathbb{C} + H^\infty \), see Section 1.5 in [17]. For every function \( f \in K_\theta \) we have

\[
A_\varphi f = P_\theta(\varphi f) = \theta P_-(\theta \varphi f) - P_- (\varphi f) = \theta H_{\theta \varphi} f - H_{\varphi} f.
\]

Assume that \( \varphi \in \mathbb{C} + H^\infty \) is such that \( A_\varphi \in S^\infty(K_\theta) \). Consider a weakly convergent sequence \( g_n \in H^2 \) with zero limit. Put \( f_n = P_\theta g_n \). From formula (10) we see that

\[
\lim ||H_{\theta \varphi} f_n|| = 0
\]

because the operators \( A_\varphi, H_{\theta \varphi} \) are compact. Consider the sequence \( \theta h_n = g_n - f_n \) of functions in \( \theta H^2 \). Note that \( h_n \in H^2 \) converge weakly to zero. Since \( \varphi \in \mathbb{C} + H^\infty \), we have \( \lim ||H_{\theta \varphi} \theta h_n|| = \lim ||H_{\theta \varphi} h_n|| = 0 \). It follows that \( \lim ||H_{\theta \varphi} g_n|| = 0 \) for every weakly convergent to zero sequence \( \{g_n\} \subset H^2 \). Hence the operator \( H_{\theta \varphi} : H^2 \to \mathbb{C} H^2 \) is compact and therefore \( \varphi \in \theta \mathbb{C} + \theta H^\infty \).

Now let \( \varphi = \theta \varphi_1 + \theta \varphi_2 \), where \( \varphi_1 \in \mathbb{C} \) and \( \varphi_2 \in H^\infty \). Then we have \( A_\varphi = A_{\theta \varphi_1} \). Moreover, from Lemma 2.2 we see that \( A_{\theta \varphi_1} = A_\theta A_{\varphi_1} + K \) for a compact operator \( K \) on \( K_\theta \). Since \( A_\theta = 0 \), the operator \( A_{\theta \varphi_1} \) is compact. \( \square \)
Remark. Proposition [2.1] can be reformulated in the following way: the truncated Hankel operator $\Gamma_\theta : K_\theta \to zK_\theta$ with symbol $\varphi \in \overline{\theta(C + H^\infty)}$ is compact if and only if $\varphi \in \mathcal{C} + H^\infty$ (see Lemma [3.3]). In general, the assumption $\varphi \in \overline{\theta(C + H^\infty)}$ can not be omitted: one can construct an inner function $\theta$ and a rank-one truncated Hankel operator on $K_\theta$ which has no bounded symbols [2]. In Theorem [2] we prove that if $\theta$ is a one-component inner function, then every compact truncated Hankel operator on $K_\theta$ has a continuous symbol.

3. Proof of Theorem [2]

The proof of Theorem [2] splits into series of lemmas. The main analytic ingredient is due to T. Wolff [26].

Theorem (T. Wolff). Denote $QC = \overline{(\mathcal{C} + H^\infty)} \cap (\mathcal{C} + H^\infty)$ and $QA = QC \cap H^\infty$. For every $f \in L^\infty$ there exists an outer function $g \in QA$ such that $gf \in QC$.

Invariant subspaces of the backward shift operator $S^*$ on the Hardy space $H^1$ have the form $K_\theta^1 = H^1 \cap z\overline{\theta H^1}$, where $\theta$ is an inner function. It follows that from the definition that a function $f \in L^1$ belongs to $K_\theta^1$ if $\int_T fzh \, dm = 0$ and $\int_T f\overline{\theta h} \, dm = 0$ for all $h \in H^\infty$, where $m$ denotes the normalized Lebesgue measure on the unit circle $T$. Recall that $F_\theta$ is the closure of the set $\overline{\theta H^\infty} + H^\infty = \{ \theta f_1 + f_2, f_1, f_2 \in H^\infty \}$ in the $w^*$-topology of the space $L^\infty$. The following lemma is close to Remark 4.1 in [24].

Lemma 3.1. Let $\theta$ be an inner function. Then $(\mathcal{C} / (F_\theta \cap \mathcal{C}))^* = K_\theta^1 \cap zH^1$. In particular, $K_\theta^1 \cap zH^1$ is closed in the $w^*$-topology of the space $zH^1$.

Proof. Let $\Phi$ be a continuous linear functional on $\mathcal{C} / (F_\theta \cap \mathcal{C})$. By the Riesz-Markov theorem, there exists a complex measure $\mu$ on the unit circle $T$ such that

$$\Phi(\varphi) = \int_T \varphi \, d\mu, \quad \varphi \in \mathcal{C},$$

and $\Phi(\varphi) = 0$ for all $\varphi \in F_\theta \cap \mathcal{C}$. This measure $\mu$ can be chosen so that its total variation equals $\|\Phi\|$. For all integers $k \geq 0$ we have $\Phi(z^k) = 0$. Hence $\mu = h \, dm$ for a function $h \in zH^1$ by the brothers Riesz theorem. Using Wolff’s theorem, find an outer function $g$ such that $\theta g \in QC$. Note that $\theta g \in H^\infty$ and therefore $\theta g \in QA$. For every analytic polynomial $p$ we have $\theta gp \in QA$ because QA is an algebra. Since $QA \subset \overline{\mathcal{C} + H^\infty}$, one can find functions $\varphi \in \mathcal{C}$, $\psi \in H^\infty$ such that $\overline{\varphi + \psi} = \theta gp$. By the construction, we have $\varphi \in F_\theta \cap \mathcal{C}$. Hence,

$$\int_T \overline{\theta gp} \, h \, dm = \int_T (\varphi + \psi) \, h \, dm = \int_T \varphi \, h \, dm = \Phi(\varphi) = 0.$$

Since $g$ is outer, functions of the form $gp$, where $p$ is an analytic polynomial, form the dense subset of $H^\infty$ in the $w^*$-topology of $L^\infty$. We see that

$$\int_T \mathcal{T} \cdot \overline{\theta h} \, dm = 0$$

for all $f \in H^\infty$. It follows that $\overline{\theta h} \in zH^1$ and hence $h$ lies in $zH^1 \cap z\overline{\theta H^1} = K_\theta^1 \cap zH^1$. By our choice of the measure $\mu = h \, dm$, we have $\|h\|_{L^1} = \|\Phi\|$. Thus, we have the isometric inclusion $(\mathcal{C} / F_\theta)^* \subset K_\theta^1 \cap zH^1$. To obtain the inverse inclusion, observe that for $h \in K_\theta^1 \cap zH^1$ the linear functional $\Phi_h : \varphi \mapsto \int_T \varphi \, h \, dm$ is continuous on $\mathcal{C}$ and vanishes on $F_\theta \cap \mathcal{C}$. The latter also shows that $K_\theta^1 \cap zH^1$ is the annihilator...
Lemma 3.2. Let \( \theta \) be an inner function. Then for every \( \varphi \in \mathcal{C} \) we have
\[
\text{dist}_{L^\infty}(\varphi, \mathcal{F}_\theta) = \text{dist}_{L^\infty}(\varphi, \mathcal{F}_\theta \cap \mathcal{C}).
\]

Proof. For every inner function \( \theta \) we have \((K^1_\theta \cap zH^1)^* = L^\infty/\mathcal{F}_\theta \) by a standard duality argument. Indeed, this follows from the simple fact that \( \mathcal{F}_\theta \) is the annihilator of \( K^1_\theta \cap zH^1 \) in \( L^\infty \). Hence \((\mathcal{C}/(\mathcal{F}_\theta \cap \mathcal{C}))^{**} = L^\infty/\mathcal{F}_\theta \) by Lemma 3.1. Consider the canonical embedding
\[
\mathcal{C}/(\mathcal{F}_\theta \cap \mathcal{C}) \hookrightarrow L^\infty/\mathcal{F}_\theta
\]
which sends the factor-class \( \varphi + \mathcal{F}_\theta \cap \mathcal{C} \) from \( \mathcal{C}/(\mathcal{F}_\theta \cap \mathcal{C}) \) to the factor-class \( \varphi + \mathcal{F}_\theta \) in \( L^\infty/\mathcal{F}_\theta \). By a general theory of Banach spaces, this embedding is isometric, see Section II.3.19 in [7]. Now the conclusion is evident. \( \square \)

To use some results of [1], we need to relate truncated Hankel and Toeplitz operators. This is the subject of the following lemma.

Lemma 3.3. Let \( \theta \) be an inner function and let \( \varphi \in L^2 \). Consider the truncated Hankel operator \( \Gamma_\varphi : K_\theta \to zK_\theta \) and the truncated Toeplitz operator \( A_{\theta \varphi} : K_\theta \to K_\theta \). For all \( f \in K_\theta \cap L^\infty \) we have \( \Gamma_\varphi f = \theta A_{\theta \varphi} f \).

Proof. As we have seen in the proof of Proposition 2.1 \( P_\theta g = \theta P_-(\overline{\theta g}) - P_-g \) for all \( g \in L^2 \). Analogously, one can check that \( P_\theta g = P_-g - \overline{\theta P_- (\theta g)} \). Using this two formulas, we obtain
\[
\overline{\theta A_{\theta \varphi}} f = \overline{\theta (P_- (\overline{\theta \varphi f}) - P_- (\theta \varphi f))} = P_- (\varphi f) - \overline{\theta P_- (\theta \varphi f)} = \Gamma_\varphi f
\]
for all \( f \in K_\theta \cap L^\infty \), as required. \( \square \)

Lemma 3.4. Let \( \theta \) be an inner function. Then \( \mathcal{F}_\theta = (\overline{\theta H^2 + H^2}) \cap L^\infty \). Consequently, for \( \varphi \in L^\infty \) the truncated Hankel operator \( \Gamma_\varphi : K_\theta \to zK_\theta \) is the zero operator if and only if \( \varphi \) belongs to \( \mathcal{F}_\theta = (\overline{\theta H^2 + H^2}) \cap L^\infty \).

Proof. We first check that \( \mathcal{F} = (\overline{\theta H^2 + H^2}) \cap L^\infty \) is closed in the \( w^* \)-topology of the space \( L^\infty \). Assume that \( \varphi_n \in \mathcal{F} \) are such that \( \lim_n \int \varphi_n f \, dm = \int \varphi f \, dm \) for every \( f \in L^1 \). Then \( \varphi \in L^\infty \) and for every function \( f \in K_\theta \cap zH^2 \) we have \( \int \varphi f \, dm = 0 \) because \( K_\theta \subset L^1 \) and \( \int \varphi_n f \, dm = 0 \) for all \( n \). It follows that \( \varphi \) belongs to the orthogonal complement of \( K_\theta \cap zH^2 \) in the space \( L^2 \), that is, \( \varphi \in \mathcal{F} \). The same argument shows that \( f \in L^1 \) is such that \( \int f \varphi \, dm = 0 \) for all \( \varphi \in \mathcal{F}_\theta \) if and only if \( \int f \varphi \, dm = 0 \) for all \( \varphi \in \mathcal{F} \). Since \( \mathcal{F} \) and \( \mathcal{F}_\theta \) are \( w^* \)-closed, we have \( \mathcal{F} = \mathcal{F}_\theta \). The second part of the statement is a direct consequence of Theorem 3.1 in [23] and Lemma 3.3. \( \square \)

Lemma 3.5. Let \( \theta \) be an inner function. Truncated Hankel operators with continuous symbols are compact and norm-dense in the set of all compact truncated Hankel operators on \( K_\theta \).

Proof. By Lemma 3.3 it is sufficient to show that truncated Toeplitz operators with symbols in \( \theta \mathcal{C} \) are compact and norm-dense in the set of all compact truncated Toeplitz operators on \( K_\theta \). The first claim (compactness) follows from Proposition 2.1. For \( \lambda \in \mathbb{D} \) denote \( k_\lambda = \frac{1 - \overline{\theta \lambda}}{1 - \lambda z} \) and \( \hat{k}_\lambda = \frac{\theta - \theta \lambda}{z - \lambda} \). Then \( k_\lambda, \hat{k}_\lambda \in K_\theta \).
and the rank-one operator \( T_\lambda : h \mapsto (h, k_\lambda) \tilde{k}_\lambda \) is the truncated Toeplitz operator with symbol \( \varphi_\lambda = \frac{\varphi}{\tilde{k}} \), see Section 5 in [12]. Moreover, by Corollary 5.1 in [11] the linear span of the set \( \{ T_\lambda, \lambda \in \mathbb{D} \} \) is norm-dense in the set of all compact truncated Toeplitz operators on \( K_\theta \). Since \( \varphi_\lambda \in \mathcal{C} \) for each \( \lambda \in \mathbb{D} \), the lemma is proved. \( \square \)

Next lemma is a consequence of Corollary 2.5 in [11].

**Lemma 3.6.** Let \( \theta \) be a one-component inner function. Then every bounded truncated Hankel operator \( \Gamma \) on \( K_\theta \) has a bounded symbol \( \varphi \in L^\infty \). Moreover, there exists a constant \( c_\theta \) depending only on \( \theta \) such that \( \| \Gamma \| \leq \text{dist}_{L^\infty}(\varphi, \mathcal{F}_{\theta^2}) \leq c_\theta \| \Gamma \| \) for every symbol \( \varphi \in L^\infty \) of \( \Gamma \).

**Proof.** Let \( \mathcal{H}_\theta \) be the set of all bounded truncated Hankel operators on \( K_\theta \). It follows from Theorem 4.2 in [12] and Lemma 3.3 that \( \mathcal{H}_\theta \) is closed in the weak operator topology. In particular, \( \mathcal{H}_\theta \) is the Banach space with respect to the operator norm. By Corollary 2.5 of [1] and Lemma 3.3 every bounded truncated Hankel operator \( \Gamma \) on \( K_\theta \) has a bounded symbol \( \varphi_\Gamma \). The set of all bounded symbols of \( \Gamma \) equals \( \varphi_\Gamma + \mathcal{F}_{\theta^2} \) by Lemma 3.3. Hence the linear mapping \( \Gamma \mapsto \| \varphi_\Gamma \| \) from \( \mathcal{H}_\theta \) to \( L^\infty/\mathcal{F}_{\theta^2} \) is correctly defined and bounded by the closed graph theorem. It follows that there exists a constant \( c_\theta \) depending only on \( \theta \) such that \( \text{dist}_{L^\infty}(\varphi, \mathcal{F}_{\theta^2}) \leq c_\theta \| \Gamma \| \) for every bounded symbol \( \varphi \) of \( \Gamma \). On the other hand, we have \( \| \Gamma \| \leq \text{dist}_{L^\infty}(\varphi, \mathcal{F}_{\theta^2}) \) because \( \| \Gamma_\varphi \psi \| \leq \| \varphi + \psi \|_{L^\infty} \) and \( \psi_\Gamma = 0 \) for every \( \psi \in \mathcal{F}_{\theta^2} \). \( \square \)

**Proof of Theorem 2.** Let \( \theta \) be a one-component inner function and let \( \Gamma \) be a compact truncated Hankel operator from \( K_\theta \) to \( K_r \). Fix a number \( \varepsilon > 0 \) and put \( c_\varepsilon = (1 + \varepsilon) \). By Lemma 3.5 one can find compact truncated Hankel operators \( \Gamma_{\phi_k} \) with symbols \( \phi_k \in \mathcal{C} \) such that

\[
\Gamma = \sum_k \Gamma_{\phi_k}, \quad \sum \| \Gamma_{\phi_k} \| \leq c_\varepsilon \| \Gamma \|.
\]

It follows from Lemma 3.2 for the inner function \( \theta^2 \) that there exist functions \( \tilde{\phi}_k \in \mathcal{C} \) such that \( \phi_k - \tilde{\phi}_k \in \mathcal{F}_{\theta^2} \) and \( \| \tilde{\phi}_k \|_{L^\infty} \leq c_\varepsilon \text{dist}_{L^\infty}(\phi_k, \mathcal{F}_{\theta^2}) \). By Lemma 3.6 we have

\[
\sum_k \| \tilde{\phi}_k \|_{L^\infty} \leq c_\varepsilon \sum_k \text{dist}_{L^\infty}(\phi_k, \mathcal{F}_{\theta^2}) \leq c_\varepsilon c_\theta \sum_k \| \phi_k \| \leq c_\varepsilon^2 c_\theta \| \Gamma \|.
\]

Consider the function \( \tilde{\varphi} = \sum_k \tilde{\phi}_k \). By the construction, \( \tilde{\varphi} \) is continuous on \( \mathbb{T} \) and we have \( \Gamma = \sum \Gamma_{\phi_k} = \sum \Gamma_{\tilde{\phi}_k} = \Gamma_{\tilde{\varphi}} \), that is, \( \tilde{\varphi} \) is the symbol of the operator \( \Gamma \). Clearly, we have \( \| \Gamma \| = \| \tilde{\varphi} \|_{L^\infty} \). On the other hand, \( \| \tilde{\varphi} \|_{L^\infty} \leq c^2_\varepsilon c_\theta \| \Gamma \| \) by estimate (11).

The theorem is proved. \( \square \)

**Example.** Let \( \theta \) be a finite Blaschke product. Then one can obtain Lemma 3.2 (and hence Theorem 2) using an elementary convolution argument. Indeed, take a function \( \varphi \in \mathcal{C} \) and consider \( f \in \mathcal{F}_{\theta} \) such that

\[
\| \varphi - f \|_{L^\infty} = \text{dist}_{L^\infty}(\varphi, \mathcal{F}_{\theta}).
\]

Since the subspace \( \mathcal{F}_{\theta} \) has finite codimension in \( L^\infty \), we have \( f = f_1 + \theta f_2 \) for some functions \( f_1, f_2 \in H^\infty \). For \( g \in L^\infty \) and \( 0 < r < 1 \) denote by \( g_r \) the continuous function \( z \mapsto \tilde{g}(rz) \) in the closed unit disc. Note that \( \| g_r \| \leq \| g \| \) because \( g_r \) is the convolution of \( g \) and the Poisson kernel \( \frac{1-r^2}{|1-r|^2} \) which is of unit norm in \( L^1(\mathbb{T}) \). Hence,

\[
\| \varphi_r - f_r \|_{L^\infty} \leq \| \varphi - f \|_{L^\infty}.
\]
We have \( \lim_{r \to 1} \| \varphi - \varphi_r \|_{L^\infty} = 0 \) and \( \lim_{r \to 1} \| \theta - \theta_r \|_{L^\infty} = 0 \) because the functions \( \varphi, \theta \) are continuous on the unit circle \( T \). Put \( \tilde{f}_r = f_{1-r} + \overline{\theta f_r} \). Then we have

\[
\text{dist}_{L^\infty}(\varphi, \mathcal{F}_\theta \cap C) \leq \lim_{r \to 1} \| \varphi - \tilde{f}_r \|_{L^\infty} = \lim_{r \to 1} \| \varphi_r - \tilde{f}_r \|_{L^\infty} \leq \| \varphi - \tilde{f} \|_{L^\infty}.
\]

This shows that \( \text{dist}_{L^\infty}(\varphi, \mathcal{F}_\theta \cap C) \leq \text{dist}_{L^\infty}(\varphi, \mathcal{F}_\theta) \). The opposite inequality is obvious. \( \square \)

**Remark.** The main problem with the above proof in the general case is that the equality \( \lim_{r \to 1} \| \theta - \theta_r \|_{L^\infty} = 0 \) is false if \( \theta \) is not a finite Blaschke product.

4. **Truncated Toeplitz and Hankel Operators in Schatten Classes**

Let \( H_1, H_2 \) be separable Hilbert spaces, and let \( 0 < p < \infty \). Recall that a compact operator \( T : H_1 \to H_2 \) belongs to the Schatten class \( S^p = S^p(H_1, H_2) \) if the quantity

\[
\| T \|_{S^p} = \left( \sum s_k(T)^p \right)^{1/p}
\]

is finite, where \( s_k(T) \) are the singular values of \( T \) (that is, \( s_k(T) \) are the eigenvalues of the compact selfadjoint operator \( |T| \)).

In Section 4.1 we introduce some singular integral operators closely related to truncated Hankel operators. In Section 4.2 we collect several characterizations of compact truncated Hankel operators, extending Theorem 2. Next in Section 4.3 we present a conjecture related to truncated Toeplitz and Hankel operators in Schatten classes \( S^p \), \( 0 < p < \infty \).

4.1. **A class of singular integral operators.** Fix a complex number \( \alpha \) such that \( |\alpha| = 1 \). Let \( \sigma_\alpha \) denote the corresponding Clark measure of \( \theta \), that is, the positive measure on the unit circle \( T \) such that

\[
\text{Re} \left( \frac{\alpha + \theta(z)}{\alpha - \theta(z)} \right) = \int_T \frac{1 - |z|^2}{|1 - \xi z|^2} \, d\sigma_\alpha(\xi), \quad z \in \mathbb{D}.
\]

Formula (12) allows us to identify functions in \( K_\theta \) and their traces in \( L^2(\sigma_\alpha) \).

Consider the unitary operator \( H_\alpha = V_\alpha V_\alpha^{-1} \) acting from \( L^2(\sigma_\alpha) \) to \( L^2(\sigma_{-\alpha}) \). From formula (13) we see that for every \( f \in L^2(\sigma_\alpha) \) and \( g \in L^2(\sigma_{-\alpha}) \) with separated supports we have

\[
\langle H_\alpha f, g \rangle_{L^2(\sigma_{-\alpha})} = 2 \int_T \left( \int_T \frac{f(\xi)}{1 - \xi \zeta} \, d\sigma_\alpha(\xi) \right) \frac{\overline{g(\zeta)}}{\zeta} \, d\sigma_{-\alpha}(\zeta),
\]

where \( \lim_{r \to 1} \| \varphi - \varphi_r \|_{L^\infty} = 0 \) and \( \lim_{r \to 1} \| \theta - \theta_r \|_{L^\infty} = 0 \) because the functions \( \varphi, \theta \) are continuous on the unit circle \( T \). Put \( \tilde{f}_r = f_{1-r} + \overline{\theta f_r} \). Then we have

\[
\text{dist}_{L^\infty}(\varphi, \mathcal{F}_\theta \cap C) \leq \lim_{r \to 1} \| \varphi - \tilde{f}_r \|_{L^\infty} = \lim_{r \to 1} \| \varphi_r - \tilde{f}_r \|_{L^\infty} \leq \| \varphi - \tilde{f} \|_{L^\infty}.
\]

This shows that \( \text{dist}_{L^\infty}(\varphi, \mathcal{F}_\theta \cap C) \leq \text{dist}_{L^\infty}(\varphi, \mathcal{F}_\theta) \). The opposite inequality is obvious. \( \square \)

**Remark.** The main problem with the above proof in the general case is that the equality \( \lim_{r \to 1} \| \theta - \theta_r \|_{L^\infty} = 0 \) is false if \( \theta \) is not a finite Blaschke product.
Thus, $H_\alpha$ is the unitary Hilbert transform from $L^2(\sigma_\alpha)$ to $L^2(\sigma_{-\alpha})$; in the discrete setting it was characterized by Yu. Belov, T. Mengestie, and K. Seip in [3].

We will need the following technical lemma.

**Lemma 4.1.** Let $\Omega : L^2(\sigma_\alpha) \times L^2(\sigma_{-\alpha}) \to \mathbb{C}$ be a sesquilinear form defined on pairs of Lipschitz functions $f \in L^2(\sigma_\alpha)$, $g \in L^2(\sigma_{-\alpha})$ with dist(supp $f$, supp $g$) > 0. Assume that $|\Omega(f,g)| \leq c \|f\|_{L^2(\sigma_\alpha)} \|g\|_{L^2(\sigma_{-\alpha})}$ for all such $f$, $g$ and some $c \geq 0$. Then there exists the unique linear bounded operator $T : L^2(\sigma_\alpha) \to L^2(\sigma_{-\alpha})$ such that $\Omega(f,g) = (Tf,g)_{L^2(\sigma_{-\alpha})}$ for all pairs $f, g$ from the domain of definition of $\Omega$.

**Proof.** Since the measures $\sigma_\alpha$, $\sigma_{-\alpha}$ are mutually singular, for every pair of functions $f \in L^2(\sigma_\alpha)$, $g \in L^2(\sigma_{-\alpha})$ one can find Lipschitz functions $f_n \in L^2(\sigma_\alpha)$, $g_n \in L^2(\sigma_{-\alpha})$ with separated supports and such that $\lim \|f - f_n\|_{L^2(\sigma_\alpha)} = 0$, $\lim \|g - g_n\|_{L^2(\sigma_{-\alpha})} = 0$. This observation shows that the domain of definition $\text{dom} \, \Omega$ of $\Omega$ is dense in $L^2(\sigma_\alpha) \times L^2(\sigma_{-\alpha})$. Also it is easy to check that $\Omega$ is continuous on $\text{dom} \, \Omega$. Now we can extend $\Omega$ to the whole space $L^2(\sigma_\alpha) \times L^2(\sigma_{-\alpha})$ by continuity and find a linear bounded operator $T : L^2(\sigma_\alpha) \to L^2(\sigma_{-\alpha})$ such that $\Omega(f,g) = (Tf,g)_{L^2(\sigma_{-\alpha})}$.

Lemma 4.1 shows that the operator $H_\alpha$ is completely determined by its sesquilinear form (14). The same is true for the singular operators $C_\varphi$ that will be defined below.

Consider the measure $\nu_{a^2} = (\alpha_\sigma + \sigma_{-\alpha})/2$. Observe that $\nu_{a^2}$ is the Clark measure of the inner function $\theta^2$:

$\text{Re} \left( \frac{\alpha^2 + \theta^2(z)}{\alpha^2 - \theta^2(z)} \right) = \int_T \frac{1 - |z|^2}{1 - \xi \overline{z}} \, d\nu_{a^2}(|\xi|), \quad z \in \mathbb{D}.$

Take a function $\varphi \in L^2(\nu_{a^2})$. Let $M_\varphi$ be the densely defined operator of multiplication by $\varphi$ on $L^2(\nu_{a^2})$. Consider the commutator $C_\varphi = H_\alpha M_\varphi - M_\varphi H_\alpha$ as an operator acting from $L^2(\sigma_\alpha)$ to $L^2(\sigma_{-\alpha})$. More precisely, for every pair of functions with separated supports, $f \in L^\infty(\sigma_\alpha)$, $g \in L^\infty(\sigma_{-\alpha})$, define

$$(C_\varphi f, g)_{L^2(\sigma_{-\alpha})} = \int_T \left( \int_T \frac{\varphi(\xi) - \varphi(z)}{1 - \overline{\xi} z} \, d\alpha(\xi) \right) \overline{g(\xi)} \, d\alpha_{-\alpha}(\xi).$$

Finally, introduce the unitary operator $\tilde{V}_{-\alpha} : \overline{\mathbb{K}_{\theta}} \to L^2(\sigma_{-\alpha})$ which takes a function $z f \in \overline{\mathbb{K}_{\theta}}$ into $z \cdot \overline{V}_{-\alpha} f \in L^2(\sigma_{-\alpha})$. Next lemma shows that the singular operators $C_\varphi$ are unitarily equivalent to the truncated Hankel operators on $\mathbb{K}_{\theta}$.

**Lemma 4.2.** Let $\theta$ be an inner function and let $\varphi \in K_{\theta^2}$. Consider the operators $\Gamma_\varphi : K_{\theta} \to \overline{\mathbb{K}_{\theta}}$ and $C_\varphi : L^2(\sigma_\alpha) \to L^2(\sigma_{-\alpha})$. We have $\Gamma_\varphi = \tilde{V}_{-\alpha}^{-1} C_\varphi V_{\alpha}$. In other words, for all $f, g$ in $K_{\theta}$ such that $f = f_1$ in $L^2(\sigma_\alpha)$ and $g = g_1$ in $L^2(\sigma_{-\alpha})$ for some Lipschitz functions $f_1, g_1$ on $T$ with separated supports we have

$$(\Gamma_\varphi f, \overline{g})_{L^2} = (C_\varphi f, \overline{g})_{L^2(\sigma_{-\alpha})}. \quad (15)$$

In particular, the operators $\Gamma_\varphi, C_\varphi$ are bounded (compact, of Schatten class $S^p$) or not simultaneously and their norms coincide.

**Proof.** At first, let us check that the sesquilinear forms in (15) are correctly defined. For this we need to show that $f \in K_{\theta} \cap L^\infty$ and $\varphi \in L^2(\nu_{a^2})$. The inclusion $f \in K_{\theta} \cap L^\infty$ follows from (13) and the assumption that $f = f_1$ in $L^2(\sigma_\alpha)$.
for a Lipschitz function $f_1$ on $T$. We have $\varphi \in L^2(\nu_{\alpha_2})$ because $\nu_{\alpha_2}$ is the Clark measure for the inner function $\theta^2$ and $\varphi \in K_\theta$. Next, since both $\varphi$ and $zfg$ belong to $K_\theta$, we have

$$\langle \Gamma_\varphi f, \overline{zg} \rangle_{L^2} = \langle \varphi f, \overline{zg} \rangle_{L^2} = \langle zfg, \overline{\varphi} \rangle_{L^2} = \langle zfg, \overline{\varphi} \rangle_{L^2(\nu_{\alpha_2})}.$$

Using (13) and the fact that $\theta(z) = \alpha$ for $\sigma_\alpha$-almost all $z \in T$ in the sense of angular boundary values, we obtain

$$\langle \xi f g, \overline{\varphi} \rangle_{L^2(\sigma_\alpha)} = 2 \int_T \xi f_1(\xi) \varphi(\xi) \int_T \frac{g_1(\zeta)}{1 - \zeta \xi} d\sigma_{-\alpha}(\zeta) d\sigma_\alpha(\xi)$$

$$= 2 \int \int \frac{\varphi(\xi)}{\xi - \xi} f_1(\xi) g_1(\zeta) d\sigma_{-\alpha}(\zeta) d\sigma_\alpha(\xi).$$

Note that all integrals in the above formula converge absolutely because the supports of $f_1$ and $g_1$ are separated. Analogously, we have

$$\langle \xi f g, \overline{\varphi} \rangle_{L^2(\sigma_{-\alpha})} = 2 \int \int \frac{\varphi(\xi)}{\zeta - \xi} f_1(\xi) g_1(\zeta) d\sigma_{-\alpha}(\xi) d\sigma_{-\alpha}(\zeta).$$

Summing up this two formulas, we get

$$\langle zfg, \overline{\varphi} \rangle_{L^2(\nu_{\alpha_2})} = \int \int \frac{\varphi(\xi)}{\xi - \xi} f_1(\xi) g_1(\zeta) d\sigma_{-\alpha}(\zeta) d\sigma_{-\alpha}(\xi) = \langle C_{\varphi} f, \overline{zg} \rangle_{L^2(\nu_{\alpha_2})},$$

and formula (13) follows. The second part of the statement is a consequence of Lemma 4.1.

\hspace{1cm} \square

4.2. Compact truncated Hankel operators. Consider a truncated Hankel operator $\Gamma_\varphi : K_\theta \to \overline{zK_\theta}$ with symbol $\varphi \in L^2$. Denote by $\varphi_s$ the orthogonal projection of $\varphi$ to the subspace $\overline{K_\theta z \cap zH^2}$ of $L^2$. It is easy to check that $\Gamma_\varphi = \Gamma_{\varphi_s}$. The function $\varphi_s$ is called the standard symbol of $\Gamma_\varphi$. Standard symbols of truncated Hankel operators play the same role as the anti-analytic symbols of usual Hankel operators on $H^2$. In particular, it is possible to describe compact truncated Hankel operators in terms of the mean oscillation properties of its standard symbols.

Let $\nu$ be a measure on the unit circle $T$. For a function $\varphi \in L^1(\nu)$ and $\varepsilon > 0$ denote

$$M_\varepsilon(\varphi) = \sup \left\{ \frac{1}{\nu(\Delta)} \int_\Delta |\varphi - \langle \varphi \rangle_{\Delta, \nu}| d\nu : \Delta \text{ is an arc of } T \text{ with } 0 < \nu(\Delta) \leq \varepsilon \right\},$$

where $\langle \varphi \rangle_{\Delta, \nu} = \frac{1}{\nu(\Delta)} \int_\Delta \varphi d\nu$. Define the space $\text{VMO}(\nu)$ of functions of vanishing mean oscillation with respect to $\nu$ by $\text{VMO}(\nu) = \{ \varphi \in L^1(\nu) : \lim_{\varepsilon \to 0} M_\varepsilon(\varphi) = 0 \}$.

We are in position to state the description of compact truncated Hankel and Toeplitz operators.

**Proposition 4.1.** Let $\theta$ be a one-component inner function and let $\Gamma_\varphi : K_\theta \to \overline{zK_\theta}$ be a truncated Hankel operator with the standard symbol $\varphi \in \overline{K_\theta z \cap zH^2}$. The following assertions are equivalent:

1. $\Gamma_\varphi : K_\theta \to \overline{zK_\theta}$ is compact;
2. $A_{\varphi_s} : K_\theta \to K_\theta$ is compact;
3. $C_{\varphi} : L^2(\sigma_\alpha) \to L^2(\sigma_{-\alpha})$ is compact;
4. $\lim_n \| \Gamma_\varphi - \Gamma_{\varphi_n} \| = 0$ for some operators $\Gamma_{\varphi_n} : K_\theta \to \overline{zK_\theta}$ of finite rank;
5. $\varphi \in C + \theta^2H^2 + H^2$;
6. $\varphi \in \text{VMO}(\nu_{\alpha_2})$. 


Proof. Assertions (1), (2), (3) and (4) are equivalent for all inner functions \( \theta \).
Indeed, the equivalence (1) \( \iff \) (2) and (1) \( \iff \) (3) was proved in Lemma 3.3 and Lemma 4.2 correspondingly. Evidently, (4) implies (1). The proof of Lemma 3.5 shows that (1) implies (4). By Theorem 3.1 in [23] and Lemma 3.3, we have \( \Gamma_\psi = 0 \) for \( \psi \in L^2 \) if and only if \( \psi \in \overline{\mathcal{H}^2} + \mathcal{H}^2 \). From Lemma 3.5, we see that (5) yields (1). Now assume that \( \theta \) is a one-component inner function. Then (1) and (5) are equivalent by Theorem 2. Equivalence (1) \( \iff \) (6) was proved in Proposition 4.1 in [4]. \( \square \)

Remark. In paper [21] R. Rochberg studied some discrete singular integral operators in connection with Toeplitz and Hankel operators on the Paley-Wiener space. That operators \( \tilde{C}_\varphi : L^2(\sigma) \to L^2(\sigma) \) are defined by the formula
\[
\tilde{C}_\varphi : f \mapsto \int_{\mathbb{R}\setminus\{\xi\}} \frac{\varphi(x) - \varphi(y)}{x-y} f(\xi) d\sigma(\xi),
\]
where \( \sigma \) is the counting measure on the set of integers \( \mathbb{Z} \) (note that \( \sigma \) is the Clark measure \( \sigma_1 \) of the inner function \( e^{2\pi i \xi} \) in the upper half-plane of the complex plane). It follows from the results by V. V. Kapustin that for a general inner function \( \theta \) with the discrete Clark measure \( \sigma = \sigma_\alpha \), every bounded operator \( \tilde{C}_\varphi \) on \( L^2(\sigma) \) is unitarily equivalent to the difference of a truncated Toeplitz operator \( A_\psi \) on \( K_\theta \) and a certain wave operator related to \( A_\psi \). For more details, see Section 3 in [12] and Section 2 in [11].

4.3. Truncated Hankel operators in Schatten classes \( S^p \). Let \( \nu \) be a finite measure on the unit circle \( \mathbb{T} \) and let \( f \in L^1(\nu) \). Take a positive integer \( r \). For every arc \( \Delta \) of \( \mathbb{T} \) define the mean oscillation of \( f \) of order \( r \) by
\[
\text{osc}(f, \nu, \Delta, r) = \frac{1}{\nu(\Delta)} \int_{\mathbb{T}} |f - f_{\Delta,r}| d\nu(\xi),
\]
where \( f_{\Delta,r} \) is a polynomial of degree at most \( r \) such that
\[
\int_{\mathbb{T}} f_{\Delta,r} e^{ik\xi} d\nu(\xi) = 0, \quad k = 0, 1, \ldots, r.
\]
If \( \nu(\Delta) = 0 \), we put \( \text{osc}(f, \nu, \Delta, r) = 0 \). For \( p \in (0, \infty) \) let \( r_p \) be the integer part of the number \( 1/p \). Denote by \( J \) the family of all dyadic subarcs of the unit circle \( \mathbb{T} \). It is known that the classical Besov space \( B_p = B_{p,1/p}^0(\mathbb{T}) \) can be defined in terms of the mean oscillation as the set of all functions \( f \in L^1 \) such that
\[
\|f\|_{B_p} = \left( \sum_{\Delta \in J} \text{osc}(f, m, \Delta, r_p)^p \right)^{1/p} < \infty,
\]
where \( m \) is the Lebesgue measure on \( \mathbb{T} \). See Theorem 1 in [6] or Lemma 9.9 in [20] for the equivalence of the above definition of \( B_p \) and the classical one. By Peller’s theorem, the Hankel operator \( H_\varphi : \mathcal{H}^2 \to \overline{\mathcal{H}^2} \) is in the class \( S^p \), \( 0 < p < \infty \), if and only if its anti-analytic symbol \( \bar{P}_{-\varphi} \) belongs to \( B_p \), see Chapter 6 in [17].

Now let \( \nu \) be a discrete measure with isolated atoms on the unit circle \( \mathbb{T} \), and let \( \text{supp}_\nu \) be the set of all points \( \xi \in \text{supp} \nu \) such that \( \nu(\{\xi\}) = 0 \). Assume that \( \nu(\text{supp}_\nu) \) \( \nu \) \( \nu \) \( \nu \) \( \nu \) \( \nu \) \( \nu \) \( 0 \). For a subset \( E \) of \( \mathbb{T} \) we will denote by \( \text{int} E \) the interior of \( E \). Let \( J_\nu \) be the family of all dyadic subarcs of \( \mathbb{T} \setminus \text{supp}_\nu \). More precisely, a closed arc
$\Delta \subset T$ is in $J_\nu$ if the interior of $\Delta$ is contained in a connected component $I$ of the open set $T \setminus \supp \nu$ and there exists an integer $k \geq 0$ such that

$$I = \operatorname{int} \bigcup_{i=1}^{2^k} \Delta_i$$

for some rotated copies $\Delta_i$ of $\Delta$ with disjoint interiors; $\Delta_1 = \Delta$. Define the Besov space $B_p(\nu)$ as the set of all functions $f \in L^1(\nu)$ such that

$$\|f\|_{B_p(\nu)} = \left( \sum_{\Delta \in J_\nu} \operatorname{osc}(f, \nu, \Delta, r_p) \right)^{1/p} < \infty.$$ 

Our conjecture on truncated Toeplitz and Hankel operators in Schatten classes $S^p$ reads as follows.

**Conjecture.** Let $\theta$ be a one-component inner function and let $\Gamma_\varphi : K_\theta \to \bar{z}K_\theta$ be a truncated Hankel operator with the standard symbol $\varphi \in K_{\theta^2} \cap zH^2$. The following assertions are equivalent for each $p \in (0, \infty)$:

1. $\Gamma_\varphi : K_\theta \to \bar{z}K_\theta$ is in $S^p$;
2. $A_{\theta, \varphi} : K_\theta \to K_\theta$ is in $S^p$;
3. $C_{\varphi} : L^2(\sigma_\alpha) \to L^2(\sigma_{-\alpha})$ is in $S^p$;
4. $\Gamma_\varphi = \sum_n a_n \Gamma_{\varphi_n}$ for some rank-one operators $\Gamma_{\varphi_n} : K_\theta \to \bar{z}K_\theta$ of unit norm and a sequence $\{a_n\} \subset \mathbb{C}$ such that $\sum_n |a_n|^p < \infty$;
5. $\varphi \in B_p + \theta^2H^2 + H^2$;
6. $\varphi \in B_p(\nu_{\alpha^2})$.

As before, assertions (1), (2), and (3) are equivalent for all inner functions $\theta$. It is also easy to see that each of (4) and (5) implies (1). Consider the case where $\theta$ is the inner function of the form $\theta : z \mapsto e^{i \pi z}$ in the upper half-plane of the complex plane. In this situation (1) is equivalent to

$$\varphi \in B_p[0, 2\pi],$$

where $B_p[0, 2\pi]$ is a Besov class associated with the interval $[0, 2\pi]$. This result is due to R. Rochberg for $1 \leq p < \infty$ (Theorems 5.1, 5.2 in [21]) and to V. Peller [16] for $0 < p < 1$. Also, R. Rochberg [21] proved that for the operators $\tilde{C}_{\varphi}$ in (15) the assertion

$$\tilde{C}_{\varphi} : L^2(\sigma) \to L^2(\sigma) \text{ is in } S^p \ (1 < p < \infty);$$

is equivalent to assertion (6) with $\nu_{\alpha^2} = \nu_{1} = \sigma$. His proof does not involve Hankel operators.

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St. Petersburg State University (7-9, Universitetskaya nab., 199034, St. Petersburg, Russia), St. Petersburg Department of Steklov Mathematical Institute of Russian Academy of Science (27, Fontanka, 191023, St. Petersburg, Russia), and School of Mathematical Sciences, Tel Aviv University (69978, Tel Aviv, Israel)

E-mail address: bessonov@pdmi.ras.ru