Exact vortex solutions of the complex sine-Gordon theory on the plane

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Abstract

We construct explicit multivortex solutions for the first and second complex sine-Gordon equations. The constructed solutions are expressible in terms of the modified Bessel and rational functions, respectively. The vorticity-raising and lowering Bäcklund transformations are interpreted as the Schlesinger transformations of the fifth Painlevé equation.

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Motivation. Recently there has been an upsurge of interest in the complex sine-Gordon equation. Originally derived in the reduction of the $O(4)$ nonlinear $\sigma$-model [1] and a theory of dual strings interacting through a scalar field [2], this equation reappeared in a number of field-theoretic [3] and fluid dynamical [4] contexts. The equation was shown to be completely integrable [1,5,6], and the multisoliton solutions were constructed in a variety of forms, both over vanishing [7,8] and nonvanishing backgrounds [9,10]. The study of its quantized version started in [7,11,12] and received a new impetus recently [13] when it was realized that the complex sine-Gordon theory may be reformulated in terms of the gauged Wess-Zumino-Witten action and interpreted as an integrably deformed $SU(2)/U(1)$-coset model [14].

The complex sine-Gordon theory can be conveniently defined by its action functional,

$$E_{SG-1} = \int \left[ \left| \nabla \psi \right|^2 + (1 - \left| \psi \right|^2)^2 \right] \frac{d^2x}{1 - \left| \psi \right|^2}. \quad (1)$$

The subscript 1 serves to distinguish this model from another integrable complexification of the sine-Gordon theory, the so-called complex sine-Gordon-2:

$$E_{SG-2} = \int \left[ \left| \nabla \psi \right|^2 + \frac{1}{2}(1 - \left| \psi \right|^2)^2 \right] d^2x. \quad (2)$$

The latter system was derived in ref. [15] as the bosonic limit of a generalized supersymmetric sine-Gordon equation and, independently, in ref. [16]. Quantum mechanically, the above two complex sine-Gordon models were shown to be the only $O(2)$-symmetric theories whose $S$-matrix is factorizable at the tree level [12].

In all previous analyses the complex sine-Gordon equations were considered in the $(1+1)$-dimensional Minkowski space-time. In the present Letter we study these two models in the 2-dimensional Euclidean space. One reason for this is that they define integrable perturbations of Euclidean conformal field theories; more precisely, eqs.(1)-(2) arise as reductions of the $SU(2)_N$ gauged Wess-Zumino-Witten model perturbed by a multiplet of primary fields (by $\Phi^{(1)}$ and $\Phi^{(2)}$, respectively) [14][17]. They are closely related to important two-dimensional lattice systems, viz. $Z_N$ parafermion models perturbed by the first and second thermal operators, respectively [18].
Another motivation for studying solutions of the Euclidean complex sine-Gordon equations comes from a remarkable similarity between eqs. (1)-(2) and several phenomenological lagrangians of condensed matter physics, in particular the Ginsburg-Landau expansion of the free energy in the theory of phase transitions,

\[ E_{GL} = \int \left[ |\nabla \psi|^2 + \frac{1}{2} (1 - |\psi|^2)^2 \right] d^2 x, \quad (3) \]

and the energy of the Heisenberg ferromagnet with easy-plane anisotropy [20]:

\[ E_{FM} = \int \left[ (\nabla \alpha)^2 + \sin^2 \alpha (\nabla \beta)^2 + \cos^2 \alpha \right] d^2 x. \quad (4) \]

(Hence we will be using the words “action” and “energy” interchangeably in what follows.)

To see that (1)-(2) are relatives of (4), one writes \( \psi = \sin \alpha e^{i\beta} \) and \( \psi = \sqrt{2} \sin(\alpha/2) e^{i\beta} \), transforming eqs. (1) and (2) into

\[ E_{SG-1} = \int \left[ (\nabla \alpha)^2 + \tan^2 \alpha (\nabla \beta)^2 + \cos^2 \alpha \right] d^2 x \quad (5) \]

and

\[ E_{SG-2} = \frac{1}{2} \int \left[ (\nabla \alpha)^2 + 4 \tan^2 \frac{\alpha}{2} (\nabla \beta)^2 + \cos^2 \alpha \right] d^2 x, \quad (6) \]

respectively.

The Ginsburg-Landau free energy (3) is minimized by the Gross-Pitaevski vortices originally discovered in the context of superfluidity [19]. These are topological solitons of the form \( \psi(x, y) = \Phi_n(r)e^{in\theta} \), where \( \Phi_n \to 1 \) as \( r \to \infty \). Although these important solutions were obtained numerically and in various asymptotic regimes, no analytic expressions for the Gross-Pitaevski vortices are available. Similarly, eq. (4) is minimized by magnetic vortices [20], and again, these are available only numerically. The aim of this note is to demonstrate that the Euclidean complex sine-Gordon equations also exhibit topological soliton solutions. Unlike the Gross-Pitaevski vortices and unlike their magnetic counterparts, the vortices of eqs. (1) and (2) can be found exactly, and in a closed analytic form. Consequently, the significance of the complex sine-Gordon equations on the plane stems from the fact that they
provide a laboratory for studying analytic properties of vortices and their phenomenology in a wide class of condensed matter models.

We construct these solutions in two different ways: (i) by means of an auto-Bäcklund transformation resulting from the spinor representation of the complex sine-Gordon theory, and (ii) via the Schlesinger transformation of the fifth Painlevé equation,

\[ W_{rr} + \frac{1}{r} W_r - \left( \frac{1}{W - 1} + \frac{1}{2W} \right) W_r^2 = \frac{(W - 1)^2}{r^2} \left( \alpha W + \frac{\beta}{W} \right) + \frac{\gamma}{r} W + \delta W \frac{W + 1}{W - 1}, \]  

(7)

which arises in a self-similar reduction of eqs.(1) and (2).

**Vortices via Bäcklund transformation.** The complex sine-Gordon-1 equation,

\[ \nabla^2 \psi + \frac{(\nabla \psi)^2 \psi}{1 - |\psi|^2} + \psi (1 - |\psi|^2) = 0, \]  

(8)

admits an equivalent representation in terms of the Euclidean spinor field, \( \Psi = (u,v)^T \): [10]

\[ i \bar{\partial} u + v - |u|^2 v = 0, \]  

(9a)

\[ i \partial v + u - |v|^2 u = 0. \]  

(9b)

[Here \( \partial = \partial/\partial z, \bar{\partial} = \partial/\partial \bar{z} \) and \( z = (x + iy)/2 \).] This is nothing but the Euclidean version of the massive Thirring model; the corresponding action functional has the form

\[
E_{Th} = \int \left[ i \Psi^\dagger \gamma_i \partial_i \Psi + \Psi \partial^\dagger \Psi - \frac{1}{4} (\Psi^\dagger \gamma_i \Psi)^2 - 1 + c.c. \right] d^2 x \\
= \int \left( i \bar{\partial} v + i \partial u + |u|^2 + |v|^2 - 1 + c.c. \right) d^2 x.
\]  

(10)

Since as one can easily check both \( u \) and \( v \) satisfy eq.(8), the Thirring model (10) can be regarded as a Bäcklund transformation between two different solutions of eq.(8). Here we confine ourselves to multivortex solutions of the form \( \psi = \Phi_n(r)e^{in\theta} \), where \( (r, \theta) \) are polar coordinates on the plane and \( \Phi_n(r) \) is a real function satisfying

\[
\frac{d^2 \Phi_n}{dr^2} + \frac{1}{r} \frac{d \Phi_n}{dr} + \frac{\Phi_n}{1 - \Phi_n^2} \left[ \left( \frac{d \Phi_n}{dr} \right)^2 - \frac{n^2}{r^2} \right] + \Phi_n \left( 1 - \Phi_n^2 \right) = 0.
\]  

(11)

Eqs.(9) with \( u \) and \( v \) of the form \( u = -i \Phi_{n-1} e^{i(n-1)\theta} \) and \( v = \Phi_n e^{in\theta} \) furnish an equivalent representation for eq.(11):
− \frac{d\Phi_{n-1}}{dr} + \frac{n-1}{r} \Phi_{n-1} = (1 - \Phi_{n-1}^2)\Phi_n, \quad (12a)

\frac{d\Phi_n}{dr} + \frac{n}{r} \Phi_n = (1 - \Phi_n^2)\Phi_{n-1}, \quad (12b)

where \Phi_n and \Phi_{n-1} satisfy eq.(11) with n and n' = n − 1, respectively. When n = 1, eq.(12a) is solved by \Phi_0 = 1 and eq.(12b) becomes a Riccati equation:

\Phi'_1 + r^{-1}\Phi_1 = 1 - \Phi_1^2. \quad (13)

This equation can be linearized by writing \Phi_1 = S'/S, where S(r) satisfies the modified Bessel’s equation of zero order: S'' + S'/r − S = 0. Selecting S = I_0(r) gives the explicit form of the n = 1 vortex solution of the complex sine-Gordon theory:

\Phi_1 = \frac{I_1(r)}{I_0(r)}. \quad (14)

Here I_0(r) and I_1 = I_0'(r) are the modified Bessel functions of zero and first order, respectively. The vortex is plotted in Fig.1.

With the solution \Phi_1 at hand, eqs.(12) yield a recursion relation allowing us to construct solutions with vorticity n > 1 in a purely algebraic way:

\Phi_{n+1} = \frac{-1}{1 - \Phi_n^2} \left[ \frac{d\Phi_n}{dr} - \frac{n}{r} \Phi_n \right] = \Phi_{n-1} - \frac{2}{1 - \Phi_n^2} \frac{d\Phi_n}{dr}, \quad n \geq 1. \quad (15)

In particular, the first two higher-order vortices (shown in Fig.1) are given by

\Phi_2 = -\frac{I_0 I_2 - I_1^2}{I_0^2 - I_1^2},

\Phi_3 = \frac{(I_3 - I_1)(I_0^2 - I_1^2) + I_1(I_0 - I_2)^2}{(I_0 - I_2)(I_0 I_2 - 2I_1^2 + I_0^2)},

where we have eliminated derivatives by means of the well known relation between the modified Bessel functions of different order: I_{n+1} + I_{n-1} = 2I_n'. The asymptotic behaviour of the vortex with vorticity n is readily found from eq.(12b):

\Phi_n \sim \frac{1}{2^n n!} r^n - \frac{1}{2^{n+2}(n+1)!} r^{n+2} + O(r^{n+4}) \quad \text{as} \quad r \to 0, \quad (16)

\Phi_n \sim 1 - \frac{n}{2r} - \frac{n^2}{8r^2} + O(r^{-3}) \quad \text{as} \quad r \to \infty. \quad (17)
One consequence of eq. (17) is that the energy of the vortices diverges [cf. eq. (19) below], similarly to the energy of the Gross-Pitaevski and easy-plane ferromagnetic vortices [19,20]. (Physically, this fact simply indicates that there is a cut-off radius in the system, for example the radius of the cylindrical superfluid container, or the distance between two adjacent vortex lines.)

**Bogomol’nyi bound.** An important question is whether the vortex renders the action a minimum. Let \( n = 1 \) and rewrite eq. (1) as

\[
E_{SG-1} = \int \left| \partial \psi + |\psi|^2 - 1 \right| \frac{d^2x}{1 - |\psi|^2} + \int \nabla \cdot A \, d^2x, \tag{18}
\]

where \( A \) is a real vector field with components

\[
A_i = \ln(1 - |\psi|^2) \varepsilon_{ij} \partial_j \text{Arg} \psi + 2 \psi_i; \quad i = 1, 2,
\]

and \( \psi = \psi_1 + i\psi_2 \). Assume our fields are such that \( |\psi|^2 < 1 \); then the first term in (18) attains its minimum at solutions to the “Bogomol’nyi equation” \( \partial \psi = 1 - |\psi|^2 \). This is exactly our eq. (9b) with \( v = \psi \) and \( u = -i \); its vortex solution is given by eq. (14). The second integral in (18) represents the divergent part of the action; it can be written as a flux through a circle of the radius \( R \to \infty \). Perturbing the vortex \( \psi = \Phi_1(r)e^{i\theta} \) by a function \( \delta \psi \) decaying faster than \( 1/r \) at infinity will not affect this part; the flux is uniquely determined by the vortex asymptotes:

\[
\oint_{C_R} A \cdot n \, dl = 2\pi(2R - \ln R - 1) + O(R^{-1}). \tag{19}
\]

Consequently, the \( n = 1 \) vortex saturates the minimum of the action in the class of functions with \( |\psi|^2 < 1 \).

The importance of the last inequality should be specially emphasized. Without the condition \( |\psi|^2 < 1 \) being imposed, one could construct a perturbation \( \tilde{\psi}(x, y) \) of the vortex satisfying \( |\tilde{\psi}| = 1, \nabla \tilde{\psi} = 0 \) on some closed curve on the \((x, y)\)-plane which does not enclose the origin. Taking then \( |\tilde{\psi}| \gg 1 \) in the interior of this contour, the action (1) could be made arbitrarily negative.
It is interesting to note that the first-order equations (9) with generic \( u \) and \( v \) can also be interpreted as the Bogomol’nyi limit for some more general system with twice as many degrees of freedom. The corresponding action functional is

\[
E[u, v] = \int \left( \frac{\|\nabla u\|^2}{1 - |u|^2} + 1 - |u|^2 \right) d^2 x + \int \left( \frac{\|\nabla v\|^2}{1 - |v|^2} + 1 - |v|^2 \right) d^2 x + E_{Th},
\]

(20)

where \( E_{Th} \) is the Thirring action (10). Clearly, any solution to eq.(9) is automatically a solution to the second-order system (20). The action (20) can be written as

\[
E[u, v] = \int \frac{|i \partial u + v(1 - |u|^2)|^2}{1 - |u|^2} d^2 x + \int \frac{|i \partial v + u(1 - |v|^2)|^2}{1 - |v|^2} d^2 x + \int \nabla \cdot A d^2 x, \tag{21}
\]

where \( A_i = \ln(1 - |v|^2) \epsilon_{ij} \partial_j \text{Arg} v - \ln(1 - |u|^2) \epsilon_{ij} \partial_j \text{Arg} u \). Assuming, again, that \(|u|^2, |v|^2 < 1\), the lower bound of the action (20-21) is saturated by solutions to eqs.(9).

Some properties of the complex sine-Gordon vortices receive a natural interpretation when the equation is reformulated as a \( \sigma \)-model on a two-dimensional surface \( \Sigma \) embedded in a three-dimensional space \((n_1, n_2, n_3)\). The metric on \( \Sigma \) is \( ds^2 = d\alpha^2 + \tan^2 \alpha d\beta^2 \) [see eq.(5)]. In order for \( \Sigma \) to be smooth, the space \((n_1, n_2, n_3)\) has to be pseudoeuclidean and the surface noncompact; in fact it looks like an asymptotically conical infinite bowl:

\[
n_1 + i n_2 = \tan \alpha e^{i \beta},
\]

\[
n_3 = \frac{1}{q} - \tanh^{-1} q,
\]

\[
q = \frac{\cos \alpha}{(1 + \cos^2 \alpha)^{1/2}}.
\]

Here \( 0 \leq \alpha < \pi/2, \ 0 \leq \beta < 2\pi \). In terms of \( n_i \), the lagrangian (8) reads

\[
E_{SG-1} = \int \left[ \|\nabla n_1\|^2 + (\nabla n_2)^2 - (\nabla n_3)^2 + (1 + n_1^2 + n_2^2)^{-1} \right] d^2 x.
\]

As \( r \to \infty \), all three components of the vortex field, \( n_1, n_2 \) and \( n_3 \), tend to infinity. Consequently, the vortices map a noncompactified \((x, y)\)-plane onto a noncompact surface — this accounts for their infinite energy. We also acknowledge the role of the condition \(|\psi|^2 < 1\), which characterizes solutions admitting the \( \sigma \)-model interpretation.

**Reductions to the Painlevé-V.** The transformation

\[
\Phi_n = \frac{1 + W}{1 - W},
\]
reduces eq. (11) to the fifth Painlevé equation (7) with coefficients

$$\alpha = n^2/8, \quad \beta = -n^2/8, \quad \gamma = 0, \quad \delta = -2.$$  \hspace{1cm} (22)

For $\gamma = c(1 - a - b)$, where $a^2 = 2\alpha$, $b^2 = -2\beta$, and $c^2 = -2\delta$, eq. (7) admits a reduction \[21\] to a Riccati equation

$$W_r = r^{-1}(W - 1)(aW + b) + cW.$$  \hspace{1cm} (23)

The above relation between the coefficients is in place for $n = 1$; in terms of the vortex modulus $\Phi_1$, eq. (23) turns out to be nothing but our eq. (13). Next, the Schlesinger transformations of the Painlevé-V to itself \[22, 23\] have the form

$$W_r = r^{-1}(W - 1)(aW + b) + cW\frac{1 + \hat{W}}{1 - \hat{W}},$$  \hspace{1cm} (24a)

$$-\hat{W}_r = r^{-1}(\hat{W} - 1)(\hat{a}\hat{W} + \hat{b}) + c\hat{W}\frac{1 + W}{1 - W}.$$  \hspace{1cm} (24b)

Here $W$ and $\hat{W}$ satisfy eq. (7) with the coefficients $(\alpha, \beta, \gamma, \delta)$ and $(\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \delta)$, respectively, where $\hat{a}^2 = 2\hat{\alpha}$, $\hat{b}^2 = -2\hat{\beta}$, $\hat{\gamma} = c(b - a)$, and $2\hat{\alpha} = a + b - 1 - \gamma/c$, $2\hat{b} = a + b - 1 + \gamma/c$. With $\alpha$, $\beta$ and $\gamma$ as in eq. (22), eqs. (24) amount to the vorticity-raising transformations (12).

We conclude the discussion of the complex sine-Gordon-1 equation by mentioning that it would be natural to expect its vortex solutions (confined to a finite region on the plane) to arise as degenerate cases of its $N$-soliton solutions \[10\] (which have the form of $N$ intersecting infinite folds). This kind of correspondence between two-dimensionally localized “lumps” and one-dimensional multisolitons exists, for example, in the Kadomtsev-Petviashvili equation \[24\]. Surprisingly, the only two-dimensionally localized bounded solution resulting from the “degeneration” of the generic two-soliton solution of eq. (8) is discontinuous at the origin: $\psi = (X^2 - \sinh^2 Y)(X^2 + \sinh^2 Y)^{-1}$. Here $X + iY = e^{i\alpha}(x + iy)$, and $\alpha$ is an arbitrary constant angle.

**Vortices of the complex sine-Gordon-2.** The complex sine-Gordon-2 results from the variation of eq. (2):

$$\nabla^2 \psi + \frac{(\nabla \psi)^2 \overline{\psi}}{2 - |\psi|^2} + \frac{1}{2} \psi(1 - |\psi|^2)(2 - |\psi|^2) = 0.$$  \hspace{1cm} (25)
The multivortex Ansatz \( \psi = \Phi_n(r) e^{in\theta} = Q_n^{1/2}(r) e^{in\theta} \) takes it to

\[
\frac{d^2 Q_n}{dr^2} + \frac{1}{r} \frac{d Q_n}{dr} + \frac{1 - Q_n}{Q_n(Q_n - 2)} \left( \frac{d Q_n}{dr} \right)^2 + Q_n(1 - Q_n)(2 - Q_n) + \frac{(a_n^2 - b_n^2)Q_n}{r^2(2 - Q_n)} + \frac{4a_n^2(1 - Q_n)}{r^2Q_n(2 - Q_n)} = 0,
\]

(26)

where \( a_n = 0 \) and \( b_n = -2n \). Next, the substitution

\[
Q_n = 2(1 - W)^{-1}
\]

transforms eq. (26) to the Painlevé equation (7) with coefficients

\[
\alpha = 0, \quad \beta = -2n^2, \quad \gamma = 0, \quad \delta = 2.
\]

This time, in order to construct the multivortex solutions we apply the Schlesinger transformation (24) twice. This leads to a recurrent relation

\[
Q^{(k-1)} = (2 - Q^{(k)}) \left\{ 1 - \frac{2(a_k + b_k - 1)Q^{(k)}}{r Q_r^{(k)} - 2a_k + (a_k + b_k)Q^{(k)}} \right\},
\]

(27)

where \( Q^{(k)} \) and \( Q^{(k-1)} \) satisfy eq. (26) with the parameters \( (a_k, b_k) \) and \( (a_{k-1}, b_{k-1}) \), respectively. Here \( a_{k-1} = a_k - 1 \) and \( b_{k-1} = b_k - 1 \). Starting with a trivial solution \( Q^{(0)} = 1 \) arising for \( a_0 = -b_0 = n \), and using eq. (27) \( n \) times, we end up with a solution \( Q_n = Q^{(-n)} \) which satisfies eq. (26) with \( a_n = 0 \) and \( b_n = -2n \) and the boundary condition \( Q_n \to 1 \) as \( r \to \infty \).

These solutions are given by rational functions; in particular, the first three multivortices (see Fig. 1) read

\[
Q_1 = \frac{r^2}{r^2 + 4};
\]

\[
Q_2 = \frac{r^4(r^2 + 24)^2}{r^8 + 64r^6 + 1152r^4 + 9216r^2 + 36864};
\]

\[
Q_3 = \frac{r^6(r^6 + 144r^4 + 5760r^2 + 92160)^2 D_3^{-1}}{4};
\]

\[
D_3 = r^{18} + 324r^{16} + 41472r^{14} + 2820096r^{12} + 114130944r^{10} + 2919628800r^8 + 50960793600r^6 + 611529523200r^4 + 4892236185600r^2 + 19568944742400.
\]

The energy of the complex sine-Gordon-2 vortices is logarithmically divergent.
Concluding remarks. The Ginsburg-Landau expansion (3) is regarded as a central postulate in the phenomenological theory of phase transitions; however, for some systems eqs.(1)-(2) may happen to provide a more adequate description. In fact, the difference is not as big as one might think. Assuming, for instance, $|\psi|^2 \leq 1$, eq.(2) can be rewritten as

$$E_{SG-2} \approx \int \left[ |\nabla \psi|^2 + \frac{1}{2} (1 - |\psi|^2)^2 + \frac{|\nabla \psi|^2 |\psi|^2}{2} + ... \right] d^2 x;$$

(28)

this is different from (3) only in the third term which is small both when $\psi \sim 0$ and when $|\psi| \sim 1, \nabla \psi \sim 0$. More importantly, the complex sine-Gordon models provide a unique opportunity for studying a number of analytic properties which are common to a wide class of vortex-bearing systems. These include the correct Ansatz for two spatially separated vortices, the vortex-phonon scattering matrix and so on; our present construction of coaxial multivortices is hopefully but a first step in this direction. Finally, one may see the complex sine-Gordon vortices as a starting point in the perturbative construction of the corresponding solutions of the Ginsburg-Landau and ferromagnet models.

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Figure Caption

Fig.1  The vortex solutions with $n = 1, 2$ and $3$. 
Solid lines: complex sine−Gordon−1
Dotted lines: complex sine−Gordon−2