Comments on two papers of Clément and Gal’tsov

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We comment on physical inconsistences of the Clément-Gal’tsov approach to Smarr’s mass formula in the presence of magnetic charge. We also point out that the results of Clément and Gal’tsov involving the NUT parameter are essentially based on the known study (dating back to 2006) of the Demiański-Newman solutions which was not cited by them.

In the paper [1], Clément and Gal’tsov considered the mass and angular momentum distributions in the dyonic Kerr-Newman (KN) black-hole spacetime [2, 3] to get the results different from those earlier obtained for this spacetime in [4]. The preprint [4] was later published under a slightly different title [5] better reflecting the topic of the special issue of Classical and Quantum Gravity on black holes and electromagnetic fields, and the paper [1] was not mentioned there because the physical inconsistences in the formulas (4.8) and (4.14) of [1] were so glaring, that we hoped Clément and Gal’tsov would be able to detect these themselves. However, it appears that the aforementioned authors were pretty sure about the correctness of their results because in the recent paper [6] they have extended their approach further to the solutions with the NUT parameter [7], hinting in passing that the title change of the preprint [4] might have had something to do with the critical tone of their previous work [1]. Therefore, we now feel ourselves obliged to respond the Clément and Gal’tsov’s critique, and in what follows we will comment on the physical inconsistences of the papers [1, 6]; moreover, we will also point out a research article whose results have been appreciably used (but not cited) in the paper [6].

We start by noting that in [4] it was shown how the magnetic charge can be elegantly introduced into the well-known Smarr mass formula [8] for black holes, and the extended formula was applied to several dyonic black-hole systems. The paper [1] of Clément and Gal’tsov addresses a technical issue of the evaluation of mass by hole systems. The paper [1] was not mentioned there because the physical inconsistences in the formulas (4.8) and (4.14) of [1] were so glaring, that we hoped Clément and Gal’tsov would be able to detect these themselves. However, it appears that the aforementioned authors were pretty sure about the correctness of their results because in the recent paper [6] they have extended their approach further to the solutions with the NUT parameter [7], hinting in passing that the title change of the preprint [4] might have had something to do with the critical tone of their previous work [1]. Therefore, we now feel ourselves obliged to respond the Clément and Gal’tsov’s critique, and in what follows we will comment on the physical inconsistences of the papers [1, 6]; moreover, we will also point out a research article whose results have been appreciably used (but not cited) in the paper [6].

A simple inspection of formulas (1) in the subextreme case ($M^2 > a^2 + Q^2 + P^2$), however, reveals that the model proposed and advocated by Clément and Gal’tsov as alternative to the usual interpretation of $M$ (the mass fully confined inside the central body) has several frankly unphysical features. First, the semi-infinite strings introduced in [1] have different masses $M_{S±}$, which apparently contradicts the equatorial symmetry of the dyonic KN solution (see [11] for the definition of the equatorially symmetric electrovac spacetimes) requiring $M_{S+} = M_{S−}$. Moreover, it is easy to see that for small values of the magnetic charge $P$ the masses $M_{S+}$ and $M_{S−}$ of the two strings can even take opposite signs,

$$\sigma = \sqrt{M^2 - a^2 - Q^2 - P^2},$$

which satisfy the relation $M_H + M_{S+} + M_{S−} = M$, where $M$ stands for the total mass (the remaining parameters $a$, $Q$ and $P$ are, respectively, the ratio of the total angular momentum $J$ and total mass $M$, the electric charge and the magnetic charge).

![FIG. 1: Distribution of mass and angular momentum along the symmetry axis in the Clément-Gal’tsov model of the dyonic KN solution. The three different parts of the symmetry axis are: $z > \sigma$ (part $S+$, the upper part of the axis), $-\sigma < z < \sigma$ (part $H$, the horizon) and $z < -\sigma$ (part $S−$, the lower part of the axis).](image-url)
which introduces undesirable negative masses into a well-behaved solution. Mention also that the parameter \( a \) in the Clément-Galtsov treatment does not represent the total angular momentum per unit mass calculated on the horizon because the parts \( S_{\pm} \) of the symmetry axis have zero angular momenta and nonzero masses, thus contradicting Carter’s interpretation [3] of the dyonic KN solution.

There are several possible explanations for the origin of the physically unrealistic formulas (1). At the first try, the appearance of the additional term in the mass integral (3.11) of [1] leading to the above (1) could be attributed to the clearly erroneous equations (3.2) of [1] defining the magnetic scalar potential \( u \) (\( A'_\varphi \) in the notation of [3]). At the same time, even if the calculations of Clément and Galtsov are somehow correct, the presence of the term involving the product \( A_\varphi u \), \( A_\varphi \) being the magnetic component of the electromagnetic 4-potential, must not really produce any effect on the usual physical interpretation of the dyonic KN solution because there are arguments in favor of vanishing of such a term. Indeed, taking into account that the potential \( A_\varphi \) of a magnetic dipole vanishes on the \( S_{\pm} \) parts of the symmetry axis, one naturally comes to the idea that in the case of a magnetic monopole the respective \( A_\varphi \) can also be made equal to zero on \( S_{\pm} \) if one treats the Dirac string as a “gauge artifact” [2], which allows for choosing an appropriate value of the integration constant \( b_0 \) in the expression of \( A_\varphi \) on each part of the symmetry axis. Then the potential \( A_\varphi \) of the dyonic KN solution, namely,

\[
A_\varphi = b_0 - Py - a(1 - y^2)A_t, 
\tag{2}
\]

where \( A_t \) is the electric potential and \( y \) the ellipsoidal coordinate, will take zero value on \( S_+ \) \((y = +1)\) after choosing \( b_0 = P \), while on the lower part of the symmetry axis \( S_- \) \((y = -1)\) the potential \( A_\varphi \) vanishes at \( b_0 = -P \). Consequently, in this case both \( M_{S_+} \) and \( M_{S_-} \) also become zeros, which is consistent with the regularity of the metric on \( S_{\pm} \). Obviously, this approach is equivalent to calculating \( M_{S_+} \) (and \( M_H \) too) by means of the usual Tomimatsu’s mass integral.

Furthermore, it is worth noting (setting aside the quantum aspects of magnetic monopoles) that classically the electric and magnetic charges are expected to exhibit similar properties [13], in particular with respect to the singularity structure of their physical fields. In general relativity, within the framework of Ernst’s formalism of complex potentials [14], this similarity manifests itself through the invariance of the Ernst equations under the duality rotation of the electromagnetic potential \( \Phi \rightarrow e^{i\alpha}\Phi \), \( \alpha = \text{const} \), so that the same metrics can describe geometries induced either by an electric or magnetic charge, or by both. A good evidence of similarity among the two charges is provided by the dyonic Reissner-Nordström solution [2], for which the energy density of the electromagnetic field \(-T^t_t\) can be shown to have the form

\[
\frac{Q^2 + P^2}{8\pi r^4},
\tag{3}
\]

\( r \) being the radial coordinate, and one can see that the magnetic charge contributes into the electromagnetic energy on an equal footing with the electric charge. Moreover, the electric and magnetic charges of the dyonic KN solution are both located inside the horizon, so we see no plausible physical reasons to consider that they must affect differently the distribution of mass in the solution.

It should be also pointed out that, while constructing exact solutions, a proper choice of the integration constants is of paramount importance for the correct physical interpretation of the solutions. In the stationary vacuum case, at least two metric functions are defined up to additive constants, the choice of which is determined by the boundary conditions, and it is precisely for the physical reasons, say, the Kerr metric [15] has only two arbitrary real parameters instead of four. In the case of stationary electrovac spacetimes, an additional integration constant may arise in the expression of the electromagnetic potential, and it is clear that its choice must be congruent with the geometrical and physical properties of the metric. Apparently, in the paper [1] Clément and Galtsov were unable to resolve a rather nontrivial and subtle problem of the parameter choice in the potential \( A_\varphi \) in the presence of spurious singularities, and they elaborated and presented an absolutely weird interpretation of the dyonic KN black-hole solution, which can hardly be justified even by the yet hypothetical status of magnetic charges.

In the subsequent paper [6] Clément and Galtsov extended their specific ideas about the magnetic charge to a NUT generalization of the dyonic KN solution. The nonzero NUT parameter endows the metric with a pair of semi-infinite singularities located on the symmetry axis which, in contradistinction from the fictitious singularities of the potential \( A_\varphi \), describing the magnetic monopole, do affect the mass distribution in the solution. Here, it would be worthwhile noting that the first study of the physical properties of the NUT singularity was undertaken by Bonnor [16] with the aid of an approximation method, and he interpreted it as a massless source of finite angular momentum. This actually erroneous interpretation was rectified only 36 years later in the paper [17], where it was rigorously proven that the NUT singularity is massive and carries infinite angular momentum; besides, there exists a unique choice of the integration constant at which the total angular momentum of the NUT solution takes finite (zero) value, and it corresponds to the case of two counter-rotating semi-infinite singularities attached to the nonrotating central body. Later, the properties of the NUT singularities in the more general metrics were also analyzed [18], and in this respect we would like to mention that the so-called Kerr-NUT and dyonic Kerr-Newman-NUT solutions were both obtained for the first time by Demiański.
and Newman [19], who also gave the name ‘Kerr-NUT’
to their vacuum spacetime.

Since the original form of the Demiański-Newman
(DN) 5-parameter electrovac solution is not quite suitable
for applications, in the papers [18] another representa-
tion of the DN metric was worked out within the frame-
work of the extended N-soliton electrovac spacetime [21].
In [18] the choice of the integration constant at which
the two DN solutions have finite angular momentum was
established, and the distributions of mass and angular
momentum along the symmetry axis were studied sepa-
rately in the vacuum and electrovac cases. Surprisingly,
in the recent paper [18], Clément and Gal’tsov have pre-
ated a fairly similar analysis of the mass and angular
momentum distributions in the Kerr-NUT (vacuum DN)
and dyonic KN-NUT (electrovac DN) solutions, and for
their purpose they made use of the representations ob-
tained in the paper [18] for the DN spacetimes. How-
ever, in their paper they do not give any reference on
the work [18], neither they cite the original paper [19] of
Demiański and Newman where the solutions were first
constructed. With regard to the electrovac DN solution
we would only like to point out that the basic formulas
(3.52)-(3.55) of [6] are precisely formulas (3), (10) of [18]
in which Clément and Gal’tsov performed the following
formal redefinitions of the parameters:

\[ a \rightarrow -a, \quad \nu \rightarrow n, \quad q \rightarrow -q, \quad b \rightarrow p, \]
\[ C_1 \rightarrow 0, \quad C_2 \rightarrow 0. \]  

(4)

Apparently, the results of Clément and Gal’tsov involving
the magnetic charge and NUT parameter are plagued
with the same problems as already discussed earlier in the
case of the dyonic KN solution, so no further comments
on that are really needed.

As far as the Kerr-NUT solution is concerned, the pa-
per [18] deserves special remarks to be made. First, it is
very clear that the form (3.40) of the Kerr-NUT met-
ric given in [6] is identical with the form (10) of [18]; of
course, (3.40) was not obtained from (2.9)-(2.11) of [6] by
two successive coordinate transformations, as affirmed by
Clément and Gal’tsov, but rather by just setting to zero
the charge parameters \( q \) and \( p \) in the electrovac DN
(dyonic KN-NUT) solution, like this was also done in [18].
Moreover, the fact that the coordinates \( x \) and \( y \) are erro-
neously called in [6] the prolate spheroidal coordinates is
a clear indication that Clément and Gal’tsov do not re-
ally understand the notion of the generalized spheroidal
coordinates \( x \) and \( y \) introduced for the DN solutions in
[20] to cover both the real and pure imaginary sectors
of the quantity \( \sigma = \sqrt{m^2 + n^2 - a^2 - q^2 - p^2} \), where all
five parameters can take arbitrary real values. In the
usual prolate spheroidal coordinates, \( \sigma \) appears as an ar-
bbitrary real parameter.

Second, all the formulas given in subsection 3.3 of [6]
for the distributions of mass and angular momentum in
the Kerr-NUT spacetime had already been obtained in
section 3 of [18] even for arbitrary values of the integra-
tion constant \( C \) (in [6]) entering the expression of the
metric function \( \omega \). An important physical message of the
paper [18] was that the NUT parameter always intro-
duces negative mass via the semi-infinite singularities.

Third, Clément and Gal’tsov consider in [6] what they
call “a symmetric Misner string configuration”, cor-
responding to \( s = 0 \) by analogy with the pure NUT case
[17]. However, unlike in the latter case, in the generic
Kerr-NUT solution there is no any “symmetric” configu-
ration of two semi-infinite singularities at any \( s \) because
the Kerr rotational parameter \( a \) introduces the asym-
metry into the combined Kerr-NUT metric (due to the
counter-rotation of strings). Therefore, in principle the
choice \( s = 0 \) leading to the finite angular momentum in
the Kerr-NUT solution needs a rigorous justification,
which was actually done in [18], and the unequal masses
of the two semi-infinite singularities in this case clearly
show that the singularity structure defined by \( s = 0 \) is
asymmetric indeed. Moreover, as was shown in [18], the
aggregate mass of the two singularities in the \( s = 0 \) con-
figuration is a negative quantity, which invalidates, in
our opinion, the importance of the Kerr-NUT solution
for thermodynamics.

Lastly, the results on the physical properties of the
DN vacuum and electrovac solutions obtained in [18]
significantly improve the understanding of the NUTty
spacetimes, and we find it quite regretful and unfair that
Clément and Gal’tsov not only attempted to ascribe to
themselves the most important findings of the paper [18],
but also gave erroneous statements about some questions
well clarified nearly fifteen years ago.

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[1] G. Clément, D. Gal’tsov, On the Smarr formula for
rotating dyonic black holes, Phys. Lett. B 773 (2017) 290.
[2] E. Newman, E. Couch, K. Chinnapared, A. Exton, A.
Prakash, R. Torrence, Metric of a rotating charged mass,
J. Math. Phys. 6 (1965) 918.
[3] B. Carter, Black hole equilibrium states, in: Black Holes
(eds. C. DeWitt and B.S. DeWitt), Gordon and Breach
Science Publishers, 1972, p. 57.
[4] V.S. Manko, H. García-Compeán, Remarks on Smarr’s mass formula in the presence of both electric and magnetic charges, arXiv:1506.03870[gr-qc].
[5] V.S. Manko, H. García-Compeán, Smarr formula for black holes endowed with both electric and magnetic charges, Class. Quantum Grav. 35 (2018) 064001.
[6] G. Clément, D. Gal’tsov, On the Smarr formulas for electrovac spacetimes with line singularities, Phys. Lett. B 802 (2020) 135270.
[7] E.T. Newman, L.A. Tamburino, T. Unti, Empty-space generalization of the Schwarzschild metric, J. Math. Phys. 4 (1963) 915.
[8] L. Smarr, Mass formula for Kerr black holes, Phys. Rev. Lett. 30 (1973) 71.
[9] A. Tomimatsu, On gravitational mass and angular momentum of two black holes in equilibrium, Prog. Theor. Phys. 72 (1984) 73.
[10] F.J. Ernst, V.S. Manko, E. Ruiz, Equatorial symmetry/antisymmetry of stationary axisymmetric electrovac spacetimes, Class. Quantum Grav. 23 (2006) 4945.
[11] L.A. Pachón, J.D. Sanabria-Gómez, Note on reflection symmetry in stationary axisymmetric electrovacuum spacetimes, Class. Quantum Grav. 23 (2006) 3251.
[12] T.P. Cheng, L.F. Li, Gauge Theory of Elementary Particle Physics (Oxford: Clarendon, 1984) p. 459.
[13] J.D. Jackson, Classical Electrodynamics, 3rd Ed. (John Wiley and Sons, Inc., 1999).
[14] F.J. Ernst, New formulation of the axially symmetric gravitational field problem. II, Phys. Rev. 168 (1968) 1415.
[15] R.P. Kerr, Gravitational field of a spinning mass as an example of algebraically special metrics, Phys. Rev. Lett. 11 (1963) 237.
[16] W.B. Bonnor, A new interpretation of the NUT metric in general relativity, Proc. Camb. Philos. Soc. 66 (1969) 145.
[17] V.S. Manko, E. Ruiz, Physical interpretation of the NUT family of solutions, Class. Quantum Grav. 22 (2005) 3555.
[18] V.S. Manko, J. Martín, E. Ruiz, Singular sources in Demiański-Newman spacetimes, Class. Quantum Grav. 23 (2006) 4473.
[19] M. Demiański, E.T. Newman, A combined Kerr-NUT solution of the Einstein field equations, Bull. Acad. Polon. Sci. Math. Astron. Phys. 14 (1966) 653.
[20] J.A. Aguilar-Sánchez, A.A. García, V.S. Manko, Demianski-Newman solution revisited, Gravit. Cosmology 7 (2001) 149; arXiv:0106011[gr-qc].
[21] E. Ruiz, V.S. Manko, J. Martín, Extended N-soliton solution of the Einstein-Maxwell equations, Phys. Rev. D 51 (1995) 4192.