Isoperimetric inequalities for the first Aharonov-Bohm eigenvalue of the Neumann and Steklov problems

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Abstract. We discuss isoperimetric inequalities for the magnetic Laplacian on bounded domains of \( \mathbb{R}^2 \) endowed with an Aharonov-Bohm potential. When the flux of the potential around the pole is not an integer, the lowest eigenvalue for the Neumann and the Steklov problems is positive. We generalize the classical inequalities of Szegö-Weinberger, Brock and Weistock to the lowest eigenvalue of this particular magnetic operator, the model domain being a disk with the pole at its center. We consider more generally domains in the plane endowed with a rotationally invariant metric, which include the spherical and the hyperbolic case.

Keywords: Magnetic Laplacian, Aharonov-Bohm magnetic field, ground state, Neumann problem, Steklov problem, isoperimetric inequality

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1 Introduction

The question of the \textit{isoperimetric inequalities} for the eigenvalues of the Laplacian (in particular for the first nonzero eigenvalue) is a long standing problem. Let us give a short and partial summary in the case of bounded domains of the Euclidean space which will be the main topic of the present paper. It began with the celebrated Faber-Krahn inequality [13]: for Dirichlet boundary conditions, among the open bounded domains of given volume, the \textit{first} eigenvalue is minimized by the ball. For spaces of constant curvature the result can be found in [6]. For Neumann boundary conditions, among the bounded open domains of given volume with Lipschitz boundary, the \textit{second} eigenvalue

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(i.e., the first nonzero) is maximized by the ball. This is the Szegö-Weinberger inequality \[15, 16\]. This inequality has been extended to bounded domains in spaces of constant curvature by Ashbaugh and Benguria \[2\] (see also \[3\]). For Robin boundary conditions with positive parameter, the ball also realizes the minimum \[7\]. For other operators, similar results exist. For the Steklov problem, the second eigenvalue (the first nonzero) is maximized by the ball among the bounded open domains of given volume with Lipschitz boundary. This is the inequality of Brock \[4\]. However, if we consider the domains of \(\mathbb{R}^2\) with boundary of given length, the second eigenvalue is maximized by the disk only among the simply connected domains. This is the inequality of Weinstock \[17\]. There exist annuli with larger second eigenvalue. Again, we refer to \[3, 6\] for more discussion and generalizations. Note that, even if we will not go in this direction, the maximization or minimization of higher eigenvalues is intensively studied, see for example \[7\] for the second nonzero eigenvalue of the Neumann problem and \[12\] for the third eigenvalue of the Robin problem.

In this paper we will be concerned by magnetic operators on domains of \(\mathbb{R}^2, S^2\) and of the hyperbolic space \(H^2\). Most of the time, the first eigenvalue of this kind of problem is strictly positive and its study difficult. For example, for the magnetic Laplacian with constant non zero magnetic field and Dirichlet boundary condition in \(\mathbb{R}^2\), it is known that the first eigenvalue is minimized by the disk among open domain of given area: this was shown in \[9\] and the proof is quite involved. To our knowledge, a similar result is not known in \(S^2\) or \(H^2\). However, for the magnetic Laplacian with constant magnetic field and magnetic Neumann boundary condition, it is no longer true that the disk maximizes the first eigenvalue: even for simply connected domains, the question is open, see \[10, Question 1, Remark 2.4 and Proposition 3.3\]. More information can be found also in \[11, §4 and §5\].

In this paper, we will first consider the first eigenvalue of the magnetic Laplacian on bounded domains of \(\mathbb{R}^2\) with Neumann boundary conditions and with an Aharonov-Bohm type magnetic field (that is curvature 0 and one singularity). The fact that the curvature is 0 makes the problem more tractable. We show that, among all domains of given area, the disk with the singularity of the magnetic field at the center is the unique maximizer of the first eigenvalue, which is positive provided that the flux is not an integer. We obtain also similar results for domain of \(S^2\) or \(H^2\).

We will also introduce a Steklov magnetic operator with an Aharonov-Bohm type magnetic field for domains in \(\mathbb{R}^2\) and show two isoperimetric inequalities for the first eigenvalue, which, again, is positive if the flux is not an integer. These correspond to the inequalities of Weinstock and of Brock \[4, 17\].

We remark that when the flux of the Aharonov-Bohm field is integer, the spectrum of the magnetic Laplacian reduces to that of the standard Laplacian. Therefore the first Neumann eigenvalue is zero. On the other hand, when the flux is not an integer, the first
eigenvalue is strictly positive, therefore the problem of its maximization among domains of fixed measure is well-posed. Note that when $x_0$ belongs to the unbounded connected component of $\Omega^c$, then the flux is zero.

We finally remark that the Faber-Krahn inequality for the magnetic Dirichlet problem with Aharonov-Bohm potential is trivial: the first eigenvalue is minimized by that of the usual Laplacian on the disk, among all bounded domains of given area.

2 Notation and statement of results

Let $\Omega$ be a smooth bounded domain of $\mathbb{R}^2$ with a distinguished point $x_0 = (a, b)$ and consider the one-form

$$A_0 = -\frac{x_2}{(x_1 - a)^2 + (x_2 - b)^2}dx_1 + \frac{x_1}{(x_1 - a)^2 + (x_2 - b)^2}dx_2. \quad (1)$$

The one-form $A_\nu = \nu A_0$ will be called Aharonov-Bohm potential with pole $x_0$ and flux $\nu$. Note that $A$ is smooth, closed, co-closed (hence harmonic) on $\mathbb{R}^2 \setminus \{x_0\}$, and is singular at $x_0$; it gives rise to a zero magnetic field ($B = dA_\nu = 0$). Let $\Delta_{A_\nu}$ be the magnetic Laplacian with potential $A_\nu$: it is the operator

$$\Delta_{A_\nu} u = \Delta u + |A_\nu|^2 u + 2i \langle \nabla u, A_\nu \rangle$$

acting on complex valued functions $u$ (the sign convention is that $\Delta u = -\sum_j \partial^2 x_j x_j u$). Of course, we can always assume that $x_0$ is the origin.

We will also consider the case when the ambient space is a two-dimensional manifold of revolution $(M, g)$ with pole $x_0$ and polar coordinates $(r, t)$, where $r$ is the distance to $x_0$. In this case, we consider the form $A_0 = dt$, which is harmonic (closed and co-closed), and with flux 1 around $x_0$.

In this paper, we consider the eigenvalue problem for $\Delta_{A_\nu}$ with magnetic Neumann conditions:

$$\begin{cases} 
\Delta_{A_\nu} u = \lambda u, & \text{in } \Omega, \\
\langle \nabla u - iuA_\nu, N \rangle = 0, & \text{on } \partial \Omega,
\end{cases} \quad (2)$$

and also the magnetic Steklov eigenvalue problem:

$$\begin{cases} 
\Delta_{A_\nu} u = 0, & \text{in } \Omega, \\
\langle \nabla u - iuA_\nu, N \rangle = \sigma u, & \text{on } \partial \Omega.
\end{cases} \quad (3)$$

Here $N$ is the outer unit normal to $\partial \Omega$. With abuse of notation we still denote by $A_\nu$ the potential dual to the 1-form $A_\nu$. We also denote by $\nabla^{A_\nu} u$ the vector field

$$\nabla^{A_\nu} u = \nabla u - iuA_\nu$$

3
which is called *magnetic gradient*. Therefore, the magnetic Neumann condition reads $\langle \nabla^{A_\nu} u, N \rangle = 0$, while the magnetic Steklov condition is $\langle \nabla^{A_\nu} u, N \rangle = \sigma u$.

We will prove in Appendix A that each of these two problems admits an infinite discrete sequence of eigenvalues of finite multiplicity. We will denote by $\lambda_1(\Omega, A_\nu)$ the first eigenvalue of Problem (2) and by $\sigma_1(\Omega, A_\nu)$ the first eigenvalue of Problem (3). These two eigenvalues are non-negative for all $\nu \in \mathbb{R}$ and are strictly positive if and only if $\nu \notin \mathbb{Z}$ (in particular, when $x_0 \in \Omega^c$ and the flux of $A_0$ is 0 in $\Omega$, we set $\nu = 0$); in particular, when $\nu \in \mathbb{Z}$, the two spectra of problems (2) and (3) reduce to the corresponding spectra of the Laplacian $\Delta$ (i.e., when $A_\nu = 0$, see Appendix A).

In the sequel, we will often suppose that $\nu \notin \mathbb{Z}$, so $\lambda_1(\Omega, A_\nu)$ and $\sigma_1(\Omega, A_\nu)$ are both positive.

The first result is an analogue of the well-known Szegö-Weinberger upper bound for the first eigenvalue of the Neumann problem [15, 16].

Through all the paper, by $|\Omega|$ we denote the Lebesgue measure of a smooth bounded domain $\Omega$, and by $|\partial \Omega|$ the length of its boundary.

**Theorem 1.** Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^2$, or $\mathbb{H}^2$, and let $A_\nu$ be the Aharonov-Bohm potential with pole at $x_0$ and flux $\nu$. Let $B(x_0, R)$ be the disk with center $x_0$ and radius $R$ such that $|B(x_0, R)| = |\Omega|$. Then

$$\lambda_1(\Omega, A_\nu) \leq \lambda_1(B(x_0, R), A_\nu);$$

(4)

if $\nu \notin \mathbb{Z}$, equality holds if and only if $\Omega = B(x_0, R)$.

This theorem will be a consequence of a more general result about an isoperimetric inequality for Schrödinger operators on revolution manifolds with pole $x_0$ and radial potential $V$ that we will present in Section 3. In fact, Theorem 1 holds also in this setting, under suitable hypothesis on the function describing the density of the Riemannian metric in standard polar coordinates.

The case of the sphere $S^2$ is more involved. We are able to show a similar result to Theorem 1 only if the domain is contained in a hemisphere centered at the pole $x_0$.

**Theorem 2.** Let $\Omega$ be a smooth domain contained in a hemisphere centered at $x_0$, and let $A_\nu$ be the Aharonov-Bohm potential with pole at $x_0$ and flux $\nu$. Let $B(x_0, R)$ be the disk in $S^2$ with center $x_0$ and radius $R$ such that $|B(x_0, R)| = |\Omega|$. Then

$$\lambda_1(\Omega, A_\nu) \leq \lambda_1(B(x_0, R), A_\nu);$$

(5)

if $\nu \notin \mathbb{Z}$, equality holds if and only if $\Omega = B(x_0, R)$.

Note that the analogous result for the second eigenvalue of the Neumann Laplacian is proved in [2]. However, for simply connected domains we can do better.
Theorem 3. Let $\Omega$ be a smooth simply connected domain in $\mathbb{S}^2$ with $|\Omega| \leq 2\pi$ and $-x_0 \notin \Omega$, and let $A_\nu$ be the Aharonov-Bohm potential with pole at $x_0$ and flux $\nu$. Let $B(x_0, R)$ be the disk in $\mathbb{S}^2$ with center $x_0$ and radius $R$ such that $|B(x_0, R)| = |\Omega|$. Then

$$\lambda_1(\Omega, A_\nu) \leq \lambda_1(B(x_0, R), A_\nu);$$

(6)

if $\nu \notin \mathbb{Z}$, equality holds if and only if $\Omega = B(x_0, R)$.

The second result is the analogous of Brock’s inequality [4] for the first Steklov eigenvalue on planar domains:

Theorem 4. Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^2$ and let $A_\nu$ be the Aharonov-Bohm potential with pole at $x_0$ and flux $\nu$. Let $B(x_0, R)$ be the disk with center $x_0$ and radius $R$ such that $|B(x_0, R)| = |\Omega|$. Then

$$\sigma_1(\Omega, A_\nu) \leq \sigma_1(B(x_0, R), A_\nu) = \frac{\sqrt{\pi}}{\sqrt{|\Omega|}} \inf_{k \in \mathbb{Z}} |\nu - k|.$$

(7)

If $\nu \notin \mathbb{Z}$, equality holds if and only if $\Omega = B(x_0, R)$.

It is natural to ask what happens for the second eigenvalue of (2) and (3), at least on planar domains. One immediately observes that Theorems 1 and 4 no longer hold, in the sense that the ball punctured at the origin is not a maximiser. In fact, $\lambda_2(B(x_0, R), A_\nu) = \left(\frac{z'_1 - \inf_{k \in \mathbb{Z}} |\nu - k|}{R^2}\right)^2 < \left(\frac{z'_{1,1}}{R^2}\right)^2$ when $\nu \notin \mathbb{Z}$. Here $z'_{\mu,1}$ denotes the first positive zero of the derivative of the Bessel function $J_\mu$ (see Appendix B.3). We recall that $\frac{(z'_{1,1})^2}{R^2}$ is exactly the second Neumann eigenvalue of the Laplacian on a ball of radius $R$. Analogously, we have $\sigma_2(B(x_0, R), A_\nu) = \frac{1}{R} - \inf_{k \in \mathbb{Z}} |\nu - k| < \frac{1}{R}$ (see Appendix B.4), and $\frac{1}{R}$ is the second Steklov eigenvalue of the Laplacian on $B(x_0, R)$. Therefore we are left with the following

Open problem 1. Among all smooth bounded domains of fixed volume in $\mathbb{R}^2$, the second Neumann (Steklov) Aharonov-Bohm eigenvalue is maximized by the second Neumann (Steklov) eigenvalue of the Laplacian on the ball.

Finally, we prove the analogue of Weinstock’s inequality [17]:

Theorem 5. Let $\Omega$ be a smooth bounded and simply connected domain in $\mathbb{R}^2$ and let $A_\nu$ be the Aharonov-Bohm potential with pole at $x_0$ and flux $\nu$. Let $B(x_0, R)$ be the disk with center $x_0$ and radius $R$ such that $|\partial B(x_0, R)| = |\partial \Omega|$. Then

$$\sigma_1(\Omega, A_\nu) \leq \sigma_1(B(x_0, R), A_\nu) = \frac{2\pi}{|\partial \Omega|} \inf_{k \in \mathbb{Z}} |\nu - k|.$$

(8)

If $\nu \notin \mathbb{Z}$, equality holds if and only if $\Omega = B(x_0, R)$.
Note that the upper bounds of Theorems 4 and 5 correctly reduce to zero whenever the flux is an integer.

We stated Theorem 5 for planar domains, however it extends to any Riemannian surface with boundary.

Preliminary calculations show that inequality (8) holds for all circular annuli in $\mathbb{R}^2$, but it fails in the case of long cylinders. In fact, when $\Omega = \mathbb{S}^1 \times (-L, L)$, the first Steklov eigenvalue is given by $\inf_{k \in \mathbb{Z}} |\nu - k| \tanh (\inf_{k \in \mathbb{Z}} |\nu - k| L)$, hence (8) does not hold for $L > L_0$, with $L_0$ sufficiently large. We are left with the following

**Open problem 2.** Does inequality (8) hold for all doubly connected domains of the plane?

The present paper is organized as follows. In Section 3 we prove an isoperimetric inequality for the first (positive) Neumann eigenvalue of the Schrödinger operator $\Delta + V$ on domains in manifolds of revolution, under suitable hypothesis on the potential $V$ and on the density of the Riemannian metric (Theorem 6). In Section 4 Theorem 6 is applied to the magnetic Neumann spectrum. In particular, in Subsection 4.1 Szegö-Weinberger inequality is proved for planar domains (Theorem 9). In Subsection 4.2 it is proved for manifolds of revolution (Theorem 10). In Subsection 4.3 it is proved for domains in the hyperbolic plane (Theorem 11). In Subsection 4.4 it is proved for spherical domains contained in a hemisphere centered at the pole (Theorem 13). In Section 5 we prove the isoperimetric inequality for spherical simply connected domains with area less than $2\pi$.

In Section 6 we prove Brock’s inequality for planar domains (Theorem 16), while in Section 7 we prove Weinstock’s inequality for planar domains (Theorem 17). In Appendix A we provide the basic spectral theory for problems (2) and (3). Appendix B contains a more explicit description of the eigenvalues and the eigenfunctions of the magnetic Neumann and Steklov problems on disks in manifolds of revolution (see Appendices B.1 and B.2). These facts, which have an interest on their own, are crucial for the proofs of the main Theorems. In Appendices B.3 and B.4 we describe the eigenfunctions and eigenvalues on disks in $\mathbb{R}^2$. Finally, in Appendix C we prove the conformal invariance of the Aharonov-Bohm energy which is crucial in the proof of Weinstock’s inequality.

### 3 Isoperimetric inequality for Schrödinger operators

In this section, $\Omega$ will be a bounded smooth domain in a $n$-dimensional manifold of revolution $(M, g)$ with pole $x_0$. With $D$ we denote the diameter of $M$ (which can be infinite) and with $D_\Omega$ we denote the diameter of $\Omega$.

We recall that a smooth $n$-dimensional Riemannian manifold $(M, g)$ with a distinguished point $x_0$ is called a *revolution manifold with pole $x_0$* if $M \setminus \{x_0\}$ is isometric to $(0, D] \times \mathbb{S}^{n-1}$.
whose metric is, in normal coordinates based at the pole, \( g = dr^2 + \Theta(r)^2 g_{S^{n-1}} \), for \( r \in (0, D) \). Here \( \Theta(0) = \Theta''(0) = 0, \Theta'(0) = 1 \), and \( g_{S^{n-1}} \) is the standard metric on the \( n-1 \)-dimensional sphere. The density of the Riemannian metric on \( M \) in normal coordinates is given by \( \sqrt{\det g} = \Theta^{-1}(r) = \theta(r) \).

It is known that, for space forms of constant curvature \( K = 0, -1, 1 \) we have:

\[
\theta(r) = \begin{cases} 
  r^{n-1} & \text{if } K = 0 \\
  \sinh^{n-1}(r) & \text{if } K = -1 \\
  \sin^{n-1}(r) & \text{if } K = 1
\end{cases}
\]

In general, we have \( \theta > 0 \) on \((0, D)\). We refer to Appendix \[B\] for more information on manifolds of revolution.

We discuss here an isoperimetric inequality for the first eigenvalue of the Schrödinger operator:

\[
\begin{cases} 
  \Delta u + Vu = \lambda u, & \text{in } \Omega \\
  \langle \nabla u, N \rangle = 0, & \text{on } \partial \Omega.
\end{cases}
\] (9)

Note that the results of this section can be applied to manifolds of revolution of any dimension \( n \geq 2 \).

**Assumptions on the potential \( V \).**

1. The potential \( V \) is smooth on \( M \setminus \{x_0\} \), non-negative and radial with respect to \( x_0 \), that is, \( V = V(r) \);

2. \( V \) is non-increasing on \((0, D_\Omega)\): \( V'(r) \leq 0 \) on \((0, D_\Omega)\);

3. \( \theta'V' + 2V^2\theta \leq 0 \) on \((0, R)\), where \( R > 0 \) is such that \( |B(x_0, R)| = |\Omega| \);

4. there exists a first eigenfunction \( u \) of (9) on \( B(x_0, R) \) which is non-negative, radial and non-decreasing in the radial direction: \( u' \geq 0 \).

We consider the following number:

\[
\lambda_1(\Omega, V) = \inf_{\theta \neq u \in H^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 + Vu^2}{\int_\Omega u^2},
\] (10)

where \( H^1(\Omega) = \{ u \in H^1(\Omega) : V^{1/2}u \in L^2(\Omega) \} \), and \( H^1(\Omega) \) is the standard Sobolev space of square integrable functions with square integrable weak first derivatives. Since \( V \) is non-negative, the infimum in (10) exists and is non-negative. We are ready to state the main result of this section.
Theorem 6. Let $\Omega$ be a smooth bounded domain in a manifold of revolution $M$ with pole at $x_0$. Let $B = B(x_0, R)$ be the ball centered at $x_0$ with the same volume of $\Omega$. Let Assumptions 1-4 hold. Then

$$\lambda_1(\Omega, V) \leq \lambda_1(B(x_0, R), V).$$

Equality holds if and only if $\Omega = B(x_0, R)$.

If the bottom of the spectrum of (9) is discrete, the number $\lambda_1(\Omega, V)$ is the first eigenvalue. This is the case of regular potentials (e.g., $V \in L^{n/2}$ for $n \geq 3$ or $V \in L^{1+\delta}$, $\delta > 0$ for $n = 2$), but also of singular potentials of the form $\frac{V^2}{r^2}$ (inverse-square potentials). In both these cases, the whole spectrum is purely discrete and made of non-negative eigenvalues of finite multiplicity diverging to $+\infty$.

Let now $B(x_0, R)$ be the ball of radius $R$ centered at the pole $x_0$ and assume that there exists a first eigenfunction of (9) on $B(x_0, R)$ which is non-negative (and therefore radial, as $V$ is radial) and non-decreasing with respect to $r$. Let us denote this function by $u = u(r)$. It satisfies

$$\begin{cases}
u'' + \frac{\theta'}{\theta}u' + (\lambda - V)u = 0, & \text{in } (0, R) \\
u'(R) = 0,
\end{cases} \tag{11}$$

where $\lambda = \lambda_1(B(x_0, R), V)$ is the first eigenvalue.

In order to prove Theorem 6 we need the following lemma:

Lemma 7. Let $u = u(r)$ be a solution of (11) such that $u \geq 0$ and $u' \geq 0$ on $(0, R)$. Let

$$F(r) = u'(r)^2 + Vu(r)^2.$$

If $V' \leq 0$ on $(0, R)$ and $\theta'V' + 2V^2\theta \leq 0$ on $(0, R)$, then one has:

$$F'(r) \leq 0$$

on $(0, R)$.

Proof. One has:

$$F' = 2u'u'' + V'u^2 + 2Vu'u' = 2\nu'\left(-\frac{\theta'}{\theta}u' - (\lambda - V)u\right) + V'u^2 + 2Vu'$$

$$= -2\frac{\theta'}{\theta}u^2 - 2\lambda uu' + V'u^2 + 4Vu' \leq -2\frac{\theta'}{\theta}u^2 + V'u^2 + 4Vu'$$
because \( u \geq 0 \) and \( u' \geq 0 \). Now:
\[
V'u^2 + 4Vu'u' = V'(u + 2\frac{V}{V'}u')^2 - 4\frac{V^2}{V'}u'^2,
\]
and we have:
\[
F' \leq -2\left(\frac{\theta'}{\theta} + 2\frac{V^2}{V'}\right)u'^2 + V'(u + 2\frac{V}{V'}u')^2.
\]
As \( V' \leq 0 \) we conclude:
\[
F' \leq -2\left(\frac{\theta'}{\theta} + 2\frac{V^2}{V'}\right)u'^2.
\]
If \( \theta'V' + 2V^2\theta \leq 0 \) then, dividing by \( \theta V' \) (which is non-positive) we indeed have
\[
\frac{\theta'}{\theta} + 2\frac{V^2}{V'} \geq 0
\]
which guarantees that \( F' \leq 0 \).

\(\square\)

**Proof of Theorem 6.** Define the radial function \( f : M \to \mathbb{R} \) as follows:
\[
f(r) = \begin{cases} u(r) & \text{for } r \leq R \\ u(R) & \text{for } r \geq R \end{cases}
\]
We note that, by construction, \( f|_\Omega \in H^1_V(\Omega) \), therefore it is possible to use as test function in (10).

We start by observing that, by assumption \( |\Omega \cap B^c| = |\Omega^c \cap B| \), so that, since \( u \) is increasing, we have \( u(r) \leq u(R) \) and
\[
\int_\Omega f^2 \geq \int_B u^2
\]
In fact:
\[
\int_\Omega f^2 = \int_{\Omega \cap B} f^2 + \int_{\Omega \cap B^c} f^2 = \int_{\Omega \cap B} u^2 + u(R)^2 |\Omega \cap B^c| = \int_{\Omega \cap B} u^2 + u(R)^2 |\Omega^c \cap B| \geq \int_{\Omega \cap B} u^2 + \int_{\Omega \cap B} u^2 = \int_{\Omega} u^2 \quad (12)
\]
We have to control the energy. Since \( F(r) = u^2(r) + V(r)u^2(r) \) is decreasing, \( f \) is constant, equal to \( u(R) \) on \( \Omega \cap B^c \), \( V' \leq 0 \) on \((0,D_\Omega)\), and \( u(R) \leq u(r) \) on \( B^c \):
\[
\int_{\Omega \cap B^c} |\nabla f|^2 + Vf^2 = u(R)^2 \int_{\Omega \cap B^c} V \leq u(R)^2 V(R) |\Omega \cap B^c| = u(R)^2 V(R) |\Omega^c \cap B| \leq F(R) |\Omega^c \cap B| \leq \int_{\Omega \cap B} F, \quad (13)
\]
where we have used the monotonicity of $V$ in the first inequality and the monotonicity of $F$ in the last inequality. Therefore, from (12), (13) and from the fact that $F = |\nabla f|^2 + V f^2 = |\nabla u|^2 + Vu^2$ on $B$, we deduce:

$$
\lambda_1(\Omega, \Delta + V) \int_B u^2 \leq \lambda_1(\Omega, \Delta + V) \int_\Omega f^2 \leq \int_\Omega |\nabla f|^2 + V f^2
$$

$$
= \int_{\Omega \cap B} |\nabla f|^2 + V f^2 + \int_{\Omega \cap B^c} |\nabla f|^2 + V f^2 \leq \int_{\Omega \cap B} F + \int_{\Omega \cap B^c} F = \int_B F = \int_B |\nabla u|^2 + Vu^2 = \lambda_1(B, \Delta + V) \int_B u^2
$$

and the assertion follows. \hfill \Box

Remark 8. Note that Assumption 3 may look quite involved. However, when we will apply Theorem 6 to the particular case of the Aharonov-Bohm operator we will choose $V = \nu_2 \theta_2$ and condition 3 will take a simpler and more natural form (see Theorem 10).

4 Application to the Aharonov-Bohm spectrum

4.1 Aharonov-Bohm spectrum on domains of $\mathbb{R}^2$

Let $A_\nu = \nu dt$ be the harmonic potential, expressed in polar coordinates $(r, t)$ around a fixed point $x_0 \in \mathbb{R}^2$, having flux $\nu$ and pole $x_0$. Namely, $A_\nu = \nu A_0$ where $A_0$ is defined in (1). We will prove in Appendix B.3 that if we assume $\nu \in (0, \frac{1}{2}]$ (which we can always do by gauge invariance, see Appendix A.4), the first eigenvalue $\lambda_1$ on the disk $B(x_0, R) \subseteq \mathbb{R}^2$ is positive and there exists a first eigenfunction which is real and radial, say $u = u(r)$. In fact, up to a multiplicative constant:

$$
u u(r) = J_\nu(c r)
$$

where $J_\nu$ denotes the Bessel function of the first kind and order $\nu$,

$$
c = \frac{z_{\nu, 1}'}{R},
$$

and $z_{\nu, 1}'$ is the first zero of $J_\nu'$. Note that, since $\nu \notin \mathbb{Z}$, $J_\nu(0) = 0$. The first eigenvalue of $\Delta_{A_\nu}$ in $B(x_0, R)$ is $\lambda_1(B(x_0, R), A) = c^2$. As $cr \leq z_{\nu, 1}'$ on $(x_0, R)$ we see that $u' \geq 0$ on $(x_0, R)$.

As $u$ is real and radial, $\langle \nabla u, A_\nu \rangle = 0$; as $|A_\nu|^2 = \frac{\nu^2}{r^2}$ we conclude that:

$$
\Delta_{A_\nu} u = \Delta u + Vu,
$$

and the assertion follows. \hfill \Box
where
\[ V = \frac{\nu^2}{\theta^2}. \]

Note that we are in the case of a Schrödinger operator with potential \( V \), and moreover \( \lambda_1(B(x_0, R), A_\nu) = \lambda_1(B(x_0, R), V) \) with \( V = \frac{\nu^2}{\theta^2} \). As \( \theta(r) = r \), we see that the condition
\[ \theta' + 2V^2 \theta \leq 0 \]
amounts to the condition \( \nu \leq 1 \), which holds true. Since also \( V' \leq 0 \), from Theorem 6 we deduce:

**Theorem 9.** Let \( \Omega \) be a smooth bounded domain in \( \mathbb{R}^2 \) and let \( B(x_0, R) \) be the disk of the same measure of \( \Omega \) centered at \( x_0 \). Then
\[ \lambda_1(\Omega, A_\nu) \leq \lambda_1(B(x_0, R), A_\nu); \]
if \( \nu \notin \mathbb{Z} \), equality holds if and only if \( \Omega = B(x_0, R) \).

### 4.2 Aharonov-Bohm spectrum on domains of manifolds of revolution

We take a 2-dimensional manifold of revolution \( M^2 \) with pole \( x_0 \), and \( \theta \) the density of the Riemannian metric in polar coordinates \((r, t)\) around the pole.

On a smooth bounded domain \( \Omega \) of \( M^2 \) we have that the spectrum of \( \Delta_{A_\nu} \) with Neumann condition, namely problem (2), is made of an increasing sequence of non-negative eigenvalues of finite multiplicity diverging to \( +\infty \) (see Appendix A).

As in the previous subsection, for all radial functions \( u = u(r) \) we have
\[ |A_\nu|^2 = \frac{\nu^2}{\theta^2}, \quad \langle \nabla u, A_\nu \rangle = 0, \]
therefore \( \Delta_{A_\nu} \) applied to a real, radial function \( u = u(r) \), writes:
\[ \Delta_{A_\nu} u = \Delta u + Vu \]
where \( V = \frac{\nu^2}{\theta^2} \).

In Appendix B.1 we will prove that for a disk centered at the pole \( B(x_0, R) \) the spectrum can be written in terms of that of the union a countable family of Sturm-Liouville problems indexed by an integer \( k \) (see Lemma 25). In particular, the first eigenvalue, denoted by \( \lambda_1(\Omega, A_\nu) \), is non-negative, and is positive if and only if \( \nu \notin \mathbb{Z} \).

As explained in Appendix B.1 thanks to gauge invariance we can take \( \nu \in (0, \frac{1}{2}] \) and in that case the first eigenfunction is real and radial. We denote it by \( u = u(r) \). Moreover,
we prove in Appendix B.1 that $u > 0$ and $u' > 0$ for all $R \in (0, \bar{R})$, where $\bar{R}$ is the first zero of $\theta'$ (see Theorem 29).

We can apply again Theorem 6, taking $V = \frac{\nu^2}{\theta^2}$. The conditions

$$V' \leq 0, \quad \theta'V' + 2V^2\theta \leq 0$$

reduce to the conditions

$$\theta' \geq 0, \quad \theta'^2 \geq \nu^2.$$ 

We therefore have the following:

**Theorem 10.** Let $\Omega$ be a smooth bounded domain with diameter $D_\Omega$ in a revolution manifold $M^2$ with pole $x_0$, and with density $\theta$, and let $B(x_0, R)$ be the disk centered in $x_0$ with the same volume as $\Omega$. Assume that  

i) $\theta' \geq 0$ on $(0, D_\Omega)$,  

ii) $\theta'^2 \geq \nu^2$ on $(0, R)$.  

Then  

$$\lambda_1(\Omega, A_\nu) \leq \lambda_1(B(x_0, R), A_\nu).$$ 

If $\nu \notin \mathbb{Z}$, equality holds if and only if $\Omega = B(x_0, R)$.

### 4.3 Hyperbolic plane

Let $\mathbb{H}^2$ be the standard hyperbolic plane of constant curvature $-1$. Let us fix a pole $x_0 \in \mathbb{H}^2$, and let $\theta$ denote the density of the Riemannian metric around $x_0$. We have

$$\theta(r) = \sinh(r),$$

where $r$ is the distance to $x_0$. Since $\theta'(r) > 0$ for all $r > 0$ and $\theta'(r)^2 = \cosh(r)^2 > 1 > \nu^2$ for all $r > 0$, Theorem 11 holds for any $R$. That is, the Szegö-Weinberger inequality for $\Delta_{A_\nu}$ holds for all domains of the hyperbolic plane (actually, for arbitrary constant curvature $-K$):

**Theorem 11.** Let $\Omega$ be a smooth bounded domain in $\mathbb{H}^2$, and let $x_0 \in \mathbb{H}^2$ be a fixed point. Let $B(x_0, R)$ be the disk centered in $x_0$ with the same volume as $\Omega$. Then

$$\lambda_1(\Omega, A_\nu) \leq \lambda_1(B(x_0, R), A_\nu).$$

If $\nu \notin \mathbb{Z}$, equality holds if and only if $\Omega = B(x_0, R)$.  

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Note that from Theorem 10 the result for planar domains follows immediately. In fact, when \( M^2 = \mathbb{R}^2 \), then \( \theta(r) = r \) and the hypothesis of Theorem 10 are satisfied by any \( R > 0 \).

**Corollary 12.** Let \( B(p, R) \) be a disk in \( \mathbb{R}^2 \) or \( \mathbb{H}^2 \), punctured at \( x_0 \in B(p, R) \). Then
\[
\lambda_1(B(p, R), A_\nu) \leq \lambda_1(B(x_0, R), A_\nu),
\]
that is, among all disks of the same measure, the first eigenvalue is maximized by the disk punctured at its center.

### 4.4 Sphere

For the standard sphere \( S^2 \) of curvature 1 we have
\[
\theta(r) = \sin(r),
\]
hence Theorem 10 can be applied to domains \( \Omega \) such that \( |\Omega| \leq |B(x_0, \pi/3)| \) and contained in a hemisphere centered at \( x_0 \).

The condition \( |\Omega| \leq |B(x_0, \pi/3)| \) ensures \( \theta'^2 \geq \nu^2 \) on \( B(x_0, R) \), and therefore \( F' \leq 0 \). Recall that \( F(r) = u'(r)^2 + V(r)u^2(r) \), with \( u = u(r) \) a real and radial eigenfunction associated to \( \lambda_1(B(x_0, R), A_\nu) \). In particular \( u \) solves (11) with \( \lambda = \lambda_1(B(x_0, R), A_\nu) \), and can be chosen positive and increasing on \((0, R)\) (see Theorem 29). Here \( B(x_0, R) \) is the ball of the same measure of \( \Omega \).

The condition that \( \Omega \) is contained in a hemisphere centered at \( x_0 \) ensures that \( \theta' \geq 0 \) on \( \Omega \cup B(x_0, R) \). This implies that the first eigenfunction on \( B(x_0, R) \) is real, radial, and non-decreasing (see Theorem 29). These two conditions are exactly those appearing in the proofs of Theorem 6 and Lemma 7.

However, in the case \( V = \frac{\nu^2}{r^2} \), we prove in Theorem 30 that the function \( F \) is decreasing for all \( R \in (0, \bar{R}) \) under suitable assumptions, even if \( \theta' < \nu \) for some \( R \in (0, \bar{R}) \). Recall that \( \bar{R} \) is the first zero of \( \theta' \). In particular, the assumptions of Theorem 30 for \( S^2 \) reduce to

i) \( \theta' > 0 \)

ii) \( \nu(\nu + 1)\theta^2 - \nu^2 + \nu^2(\theta')^2 + \nu\theta\theta'' \geq 0 \),

which are clearly satisfied as long as \( R \leq \frac{\pi}{2} \). In view of this, we only need to assume that \( \Omega \) is contained in a hemisphere centered at the pole \( x_0 \). Then we have

**Theorem 13.** Let \( \Omega \) be a smooth bounded domain contained in a hemisphere centered at \( x_0 \). Let \( B(x_0, R) \) be the disk centered in \( x_0 \) with the same volume as \( \Omega \). Then
\[
\lambda_1(\Omega, A_\nu) \leq \lambda_1(B(x_0, R), A_\nu).
\]
If $\nu \notin \mathbb{Z}$, equality holds if and only if $\Omega = B(x_0, R)$.

Note that this result coincides with that of Ashbaugh-Benguria [2]. In fact they prove that among all spherical domains contained in a hemisphere of fixed measure, the disk maximizes the first non-zero Neumann eigenvalue.

**Corollary 14.** Let $B(p, R) \subset S^2$ be a spherical disk punctured at $x_0 \in B(p, R)$. If $R \leq \frac{\pi}{4}$, then

$$\lambda_1(B(p, R), A_\nu) \leq \lambda_1(B(x_0, R), A_\nu),$$

that is, among all disks of the same measure and radius smaller than $\frac{\pi}{4}$, the first eigenvalue is maximized by the disk punctured at its center.

**5 Szegö’s isoperimetric inequality on spheres**

Theorem 13 is valid for all bounded domains which are contained in a hemisphere centered at the pole $x_0$ of $A_\nu$. We have used the Weinberger’s argument for its proof. If we take the Szegö’s point of view, we are able to extend the result to the class of simply connected domains on the sphere with area less than $2\pi$. Namely, we have:

**Theorem 15.** Let $\Omega$ be a bounded and simply connected domain in $S^2$ with $|\Omega| \leq 2\pi$ and $-x_0 \notin \overline{\Omega}$, and let $B(x_0, R)$ be the disk in $S^2$ centered at $x_0$ with $|B(x_0, R)| = |\Omega|$. Then

$$\lambda_1(\Omega, A_\nu) \leq \lambda_1(B(x_0, R), A_\nu).$$

If $\nu \notin \mathbb{Z}$, equality holds if and only if $\Omega = B(x_0, R)$.

**Proof.** Through the stereographic projection $f$ we identify a point $re^{it} \in S^2$ ($r$ is the distance to $x_0$, and $t$ is the angular coordinate) with $z = f(re^{it}) = \tan(r/2)e^{it} \in \mathbb{C}$. Then, for any function $v$ defined on $\Omega$ we have

$$\int_{\Omega} v = \int_{f(\Omega)} (v \circ f^{-1}) \frac{4}{(1 + |z|^2)^2} dz.$$

Moreover,

$$|\Omega| = \int_{f(\Omega)} \frac{4}{(1 + |z|^2)^2} dz, \quad |\partial \Omega| = \int_{f(\partial \Omega)} \frac{2}{1 + |z|^2} dz.$$

Let $u$ be an eigenfunction associated to $\lambda_1(B(x_0, R), A_\nu)$. As proved in Theorem 29, we know that $u = u(r)$ is real, radial, and can be chosen such that $u, u' > 0$ on $(0, R)$. In fact, from the our assumptions we have $R \leq \frac{\pi}{2}$; thus Theorem 29 applies. Then, $u \circ f^{-1}$ is positive and increasing as well.
Let now \( g : f(B(x_0, R)) \to f(\Omega) \) be a conformal map with \( f(0) = 0 \). To simplify our notation, we will set \( \tilde{\Omega} = f(\Omega) \) and \( \tilde{B} = f(B(x_0, R)) \). Note that \( \tilde{B} \) is a ball centered at 0 of radius \( T := \tan(R/2) \).

We set
\[
\hat{u} := u \circ f^{-1} \circ g^{-1} \circ f
\]
Then \( \hat{u} \) is a function defined in \( \Omega \), the conformal transplantation of \( u \) through the map \( f^{-1} \circ g^{-1} \circ f \). The conformal invariance of the Aharonov-Bohm energy, proved in Appendix C tells that
\[
\int_{\Omega} |\nabla^{A_{\nu}} \hat{u}|^2 = \int_{B(x_0, R)} |\nabla^{A_{\nu}} u|^2,
\]
then \( \hat{u} \in H^1_{A_{\nu}}(\Omega) \) and it is a suitable test function for the min-max principle (28) for \( \lambda_1(\Omega, A_{\nu}) \). In particular
\[
\lambda_1(\Omega, A_{\nu}) \leq \frac{\int_{B(x_0, R)} |\nabla^{A_{\nu}} u|^2}{\int_{\Omega} \hat{u}^2}.
\]
If
\[
\int_{\Omega} \hat{u}^2 \geq \int_{B(x_0, R)} u^2
\]
then we conclude from (14)
\[
\lambda_1(\Omega, A_{\nu}) \leq \frac{\int_{B(x_0, R)} |\nabla^{A_{\nu}} u|^2}{\int_{B(x_0, r)} u^2} = \lambda_1(B(x_0, R), A_{\nu}).
\]
which is what we want. Note that (15) is equivalent to
\[
\int_{\tilde{B}} U |g'|^2(\sigma \circ g) \geq \int_{\tilde{B}} U \sigma
\]
where
\[
U = (u \circ f^{-1})^2
\]
and \( \sigma(z) = \frac{4}{(1+|z|^2)^2} \). We note that \( U' > 0 \) on \((0, \pi) = (0, \tan(R/2))\). Let us set \( B_r = B(0, r) \) the disk in \( \mathbb{C} \) centered at 0 of radius \( r \). In particular \( \tilde{B} = B_T = B_{\tan(R/2)} \). We define
\[
a(r) := \int_{B_r} |g'|^2(\sigma \circ g), \quad v(r) = \int_{B_r} \sigma = 2\pi \int_0^r \frac{4r}{(1+r^2)^2} dr = \frac{4\pi r^2}{1+r^2}.
\]
Then, since \( |\Omega| = |B(x_0, R)| \), we have \( a(T) = v(T) \).
We write a differential inequality for $a(r)$:

$$a'(r) = \int_{\partial B_r} |g'|^2 (\sigma \circ g) \geq \left( \frac{\int_{\partial B_r} |g'| (\sigma \circ g)^{1/2}}{2\pi r} \right)^2$$  \hspace{1cm} (17)$$

If $\Omega_r = (g \circ \sigma)^{-1}(B_r) \in S^2$, then

$$|\partial \Omega_r| = \int_{\partial B_r} |g'| (\sigma \circ g)^{1/2}, \quad |\Omega_r| = \int_{B_r} |g'|^2 (\sigma \circ g) = a(r)$$

The isoperimetric inequality $|\partial \Omega_r|^2 \geq |\Omega_r| (4\pi - |\Omega_r|)$ holds for spherical domains, hence from (17)

$$a'(r) \geq \frac{a(r)(4\pi - a(r))}{2\pi r}$$  \hspace{1cm} (18)$$

Then, the function

$$r \mapsto \frac{a(r)}{r^2(4\pi - a(r))}$$

is not decreasing as long as $a(r) \leq 4\pi$, just take the derivative and use (18). We see now that

$$\frac{v(r)}{r^2(4\pi - v(r))} = 1$$

for all $r$, and in particular, since $v(T) = a(T)$,

$$\frac{a(r)}{r^2(4\pi - a(r))} \leq \frac{a(T)}{T^2(4\pi - a(T))} = 1$$

and then

$$a(r) \leq \frac{4\pi r^2}{1 + r^2} = \int_{B_r} \sigma = v(r).$$

This gives the desired result. In fact, since $U$ is radial with $U' > 0$,

$$\int_{\tilde{B}} U |g'|^2 (\sigma \circ g) = \int_0^T U(r) a'(r) dr$$

$$= U(T)a(T) - \int_0^T U'(r) a(r) = U(T)v(T) - \int_0^T U'(r) a(r)$$

$$\geq U(T)v(T) - \int_0^T U'(r)v(r) = \int_{\tilde{B}} U\sigma. \hspace{1cm} (19)$$

This proves (16) and then the isoperimetric inequality for $\lambda_1(\Omega, A_\nu)$. Finally, if equality holds in the isoperimetric inequality for $\lambda_1(\Omega, A_\nu)$, then equality holds in the isoperimetric
inequality $|\partial \Omega_r|^2 \geq |\Omega_r|(4\pi - |\Omega_r|)$ used in (17) for all $r$, hence all $\Omega_r$ are spherical disks, and $\Omega = \Omega_T$ as well.

6 Brock’s theorem

We now focus on the magnetic Steklov problem on a bounded smooth domain $\Omega \subset \mathbb{R}^2$, namely problem (3). The min-max principle for the first eigenvalue reads

$$
\sigma_1(\Omega, A_\nu) = \inf_{0 \neq u \in H^1_{A_\nu}(\Omega, \mathbb{C})} \frac{\int_{\Omega} |\nabla^{A_\nu} u|^2}{\int_{\partial \Omega} |u|^2},
$$

(20)

where $H^1_{A_\nu}(\Omega, \mathbb{C})$ denotes the standard magnetic Sobolev space (see Appendix A for the precise definition).

We have Brock’s Theorem for $\sigma_1(\Omega, A_\nu)$:

**Theorem 16.** Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^2$, $x_0 \in \mathbb{R}^2$ a fixed pole, and let $B(x_0, r)$ be the disk with the same measure of $\Omega$. Let $\nu \in (0, \frac{1}{2}]$. Then:

$$
\sigma_1(\Omega, A_\nu) \leq \sigma_1(B(x_0, R), A_\nu) = \frac{\sqrt{\pi} \nu}{|\Omega|^{\frac{1}{2}}}.
$$

Equality holds if and only if $\Omega = B(x_0, r)$.

**Proof.** From (20) we have:

$$
\sigma_1(\Omega, A_\nu) \leq \frac{\int_{\Omega} |\nabla^{A_\nu} u|^2}{\int_{\partial \Omega} |u|^2}
$$

for all $u \in H^1_{A_\nu}(\Omega, \mathbb{C})$. Then we choose $u = r^\nu$ which is the first eigenfunction for any disk centered at $x_0$ (see Appendix B.4),

$$
|\nabla^{A_\nu} u|^2 = 2\nu^2 r^{2\nu - 2}.
$$

In particular, since $\nu \in (0, \frac{1}{2}]$,

$$
\int_{\Omega} |\nabla^{A_\nu} u|^2 = 2\nu^2 \int_{\Omega} r^{2\nu - 2} \leq 2\nu^2 \int_{B(0, R)} r^{2\nu - 2} = 2\pi \nu R^{2\nu}.
$$

In fact,

$$
\int_{\Omega \cap B(0, R)^c} r^{2\nu - 2} \leq R^{2\nu - 2} |\Omega \cap B(0, R)^c| = R^{2\nu - 2} |\Omega^c \cap B(0, R)| \leq \int_{\Omega^c \cap B(0, R)} r^{2\nu - 2}.
$$
Here $R = \frac{|\Omega|^{1/2}}{\pi^{1/2}}$ because $B(x_0, R)$ has the same volume of $\Omega$.

We recall a well-known fact: for all $p \geq 0$,

$$\int_{\partial \Omega} r^p \geq 2\pi^{1/2} |\Omega|^{p+1/2}.$$  

When $p = 0$ this is just the classical isoperimetric inequality. For $p > 0$ this inequality says that the infimum of $\int_{\partial \Omega} r^p$ among all domains with fixed measure is attained by the ball centered at $x_0$, which is the unique minimizer. This result is proved in [1]. Using $u$ as test function for $\sigma_1(\Omega, A_\nu)$ and the isoperimetric inequality above with $p = 2\nu$ we obtain

$$\sigma_1(\Omega, A_\nu) \leq \frac{2\nu \pi R^{2\nu}}{2\pi^{1/2} |\Omega|^{2\nu+1/2}},$$

that is

$$|\Omega|^{\frac{1}{2}} \sigma_1(\Omega, A_\nu) \leq \pi^{\frac{1}{2}} \nu.$$  

(22)

By gauge invariance (see Appendix A.4), if $\nu \notin (0, \frac{1}{2}]$, we can replace $\nu$ in (21) and (22) by $\inf_{k \in \mathbb{Z}} |\nu - k|$.

7  Weinstock’s isoperimetric inequality

**Theorem 17.** Let $\Omega$ be bounded simply connected domain in $\mathbb{R}^2$, $x_0 \in \mathbb{R}^2$ be a fixed pole, and let $B(x_0, r)$ the disk with the same perimeter of $\Omega$. Let $\nu \in (0, \frac{1}{2}]$. Then:

$$\sigma_1(\Omega, A_\nu) \leq \sigma_1(B(x_0, R), A_\nu) = \frac{2\pi}{|\partial \Omega|} \nu.$$  

Equality holds if and only if $\Omega = B(x_0, R)$.

**Proof.** Assume for simplicity that $x_0 = 0$. Take the unique conformal map $\Phi : \Omega \to B$, where $B$ is the unit disk centered at the origin, and with $\Phi(0) = 0$, and fix the eigenfunction $u = r^\nu$ of the unit disk, associated to $\sigma_1(B, A_\nu) = \nu$. We refer to Appendix B.4 for more details. We take as test-function

$$\hat{u} = u \circ \Phi.$$  

Then:

$$\sigma_1(\Omega, A) \int_{\partial \Omega} |\hat{u}|^2 = \sigma_1(\Omega, \hat{A}) \int_{\partial \Omega} |\hat{u}|^2 \leq \int_{\Omega} |d^A \hat{u}|^2 = \int_D |d^A u|^2 = \sigma_1(D, A) \int_{\partial D} |u|^2 = 2\pi \nu.$$
where, on the first line, we used gauge invariance (Lemma 37) and in the third we used the conformal invariance of the magnetic energy (Lemma 36). Here $\hat{A} = \Phi^* A$. On the other hand, $\hat{u} = 1$ on $\partial \Omega$ so that

$$\int_{\partial \Omega} |\hat{u}|^2 = |\partial \Omega|$$

The conclusion is

$$\sigma_1(\Omega, A_\nu) \leq \frac{2\pi \nu}{|\partial \Omega|}$$

as asserted.

\[\square\]

A Setting

In this section we prove that problems (2) and (3) admit purely discrete spectrum made of non-negative eigenvalues of finite multiplicity diverging to $+\infty$. Through all this section $\Omega$ will denote a bounded domain in $\mathbb{R}^2$. We will also prove the gauge invariance property of the two problems.

A.1 Functional setting

Let $H^1(\Omega, \mathbb{C})$ be the standard Sobolev space of complex-valued functions. As for problem (2), we shall assume that $\Omega$ is such that the embedding $H^1(\Omega, \mathbb{C}) \subset L^2(\Omega, \mathbb{C})$ is compact. As for problem (3) we shall assume that the trace operator $\gamma_0 : H^1(\Omega, \mathbb{C}) \to L^2(\partial \Omega, \mathbb{C})$ is compact. In both cases the compactness is guaranteed if $\Omega$ is smooth (Lipschitz is enough).

Let $x_0 \in \mathbb{R}^2$ and let $C^\infty_{x_0}(\Omega, \mathbb{C})$ be the space of smooth functions on $\Omega$ vanishing in a neighborhood of $x_0$. We introduce the magnetic Sobolev space $H^1_{A}(\Omega, \mathbb{C})$ defined as the closure of \{ $u \in C^\infty_{x_0}(\Omega, \mathbb{C}) : \nabla^A u, u \in L^2(\Omega, \mathbb{C})$ \} with respect to the norm

$$\|u\|^2_A := \int_{\Omega} |\nabla^A u|^2 + |u|^2, \quad \forall u \in C^\infty_{x_0}(\Omega, \mathbb{C}) : \nabla^A u, u \in L^2(\Omega, \mathbb{C}).$$

To simplify the notation, through all this section we shall denote by $A$ the Aharonov-Bohm potential $\nu A_0$ with pole at $x_0$. We will make use of equivalent norms, suitable to our problems. One fundamental equivalent norm is given in the following proposition.

Proposition 18. If $\nu \notin \mathbb{Z}$, the norm $\|u\|_A$ is equivalent to the following norm:

$$\|u\|^2_A := \int_{\Omega} |\nabla^A u|^2 + |u|^2 + \frac{|u|^2}{|x - x_0|^2}.$$

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If \( \Omega \) does not contain the pole \( x_0 \), then the equivalence is immediate to check. In any case, the proof of Proposition 18 is standard, and follows from the well-known Hardy-type inequality proved in [14]:

**Lemma 19.** For any \( R > 0 \) and any \( u \in H^1(B(x_0, R), \mathbb{C}) \) we have

\[
\int_{B(x_0, R)} |\nabla^A u|^2 \, dx \geq C^2 \int_{B(x_0, R)} \frac{|u|^2}{|x - x_0|^2},
\]

where \( C = \inf \{ |\nu - k| : k \in \mathbb{Z} \} \). In particular, if \( \nu \in (0, \frac{1}{2}] \), then \( C = \nu \).

**Proof.** Inequality (23) is proved in [14]. We recall briefly here the proof. We use polar coordinates \( (r, \theta) \) in \( \mathbb{R}^2 \) centered at \( x_0 \), and express \( \int_{B(x_0, R)} |\nabla^A u|^2 \, dx \) in this new coordinate system. We have

\[
\int_{B(x_0, R)} |\nabla^A u|^2 \, dx = \int_0^R \int_0^{2\pi} \left( |\partial_r u|^2 + \frac{1}{r^2} |\partial_\theta u - i \nu u|^2 \right) r \, d\theta \, dr.
\]

We focus on \( \int_0^{2\pi} |\partial_\theta u - i \nu u|^2 \, d\theta \). We estimate

\[
\inf_{0 \neq u \in H^1(S^1, \mathbb{C})} \frac{\int_0^{2\pi} |\partial_\theta u - i \nu u|^2 \, d\theta}{\int_0^{2\pi} r^2 |u|^2 \, d\theta}.
\]

This infimum corresponds to the first eigenvalue, which we denote by \( \mu_1 \), of

\[
-\partial_{\theta^2} u + 2i\nu \partial_\theta u + \nu^2 u = \mu u
\]

on \( S^1 \). A set of \( L^2(S^1, \mathbb{C}) \)-normalized eigenfunctions is given by \( (2\pi)^{-1/2} e^{ik \theta} \), \( k \in \mathbb{Z} \), with corresponding eigenvalue \( \mu = (k - \nu)^2 \). If \( \nu \notin \mathbb{Z} \), then \( \mu_1 = \inf \{ (k - \nu)^2 : k \in \mathbb{Z} \} \).

This concludes the proof.

We recall the following well-known result

**Lemma 20.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) with Lipschitz boundary. Then the norm \( \|u\|^2_{H^1(\Omega, \mathbb{C})} \) is equivalent to the following norm:

\[
\|u\|^2_\Omega := \int_{\Omega} |\nabla u|^2 + \int_{\partial \Omega} |u|^2,
\]

for all \( u \in H^1(\Omega, \mathbb{C}) \). With abuse of notation, we write \( \int_{\partial \Omega} |u|^2 \) in place of \( \int_{\partial \Omega} |\gamma_0(u)|^2 \).

A consequence of Lemmas 18 and 20 is the following
Lemma 21. For \( \nu \not\in \mathbb{Z} \), the norm \( \|u\|_A \) is equivalent to the following norms:

\[
\|u\|^2_{A,\theta} := \int_\Omega |\nabla A u|^2 + \int_{\partial \Omega} |u|^2,
\]

and

\[
\|u\|^2_{A',\theta} := \int_\Omega |\nabla u|^2 + \int_{\partial \Omega} \frac{|u|^2}{|x-x_0|^2} + \int_{\partial \Omega} |u|^2.
\]

Remark 22. Lemma 18 essentially says that, if \( \nu \not\in \mathbb{Z} \)

\[
H^1_A(\Omega, \mathbb{C}) = \left\{ u \in H^1(\Omega, \mathbb{C}) : \frac{u}{|x-x_0|} \in L^2(\Omega, \mathbb{C}) \right\}.
\]

In particular, \( H^1_A(\Omega, \mathbb{C}) \subset H^1(\Omega, \mathbb{C}) \).

### A.2 The Neumann eigenvalue problem

Problem (2) is understood in the weak sense, namely, find a function \( u \in H^1_A(\Omega, \mathbb{C}) \) and number \( \lambda \in \mathbb{C} \) such that

\[
\int_\Omega (\nabla A u, \nabla A \phi) = \lambda \int_\Omega u \bar{\phi}, \quad \forall \phi \in H^1_A(\Omega, \mathbb{C}).
\]

We rewrite it as

\[
\int_\Omega (\nabla A u, \nabla A \phi) + \int_\Omega u \bar{\phi} = (\lambda + 1) \int_\Omega u \bar{\phi}, \quad \forall \phi \in H^1_A(\Omega, \mathbb{C}).
\]

Since the quadratic form on the left-hand side of (27) is bounded and coercive on \( H^1_A(\Omega, \mathbb{C}) \) and the embedding \( H^1_A(\Omega, \mathbb{C}) \subset L^2(\Omega, \mathbb{C}) \) is compact, problem (27) is recast to an eigenvalue problem for a compact self-adjoint operator on the Hilbert space \( H^1_A(\Omega, \mathbb{C}) \). Standard Spectral Theory implies that problem (27) admits a sequence of eigenvalues

\[-\infty < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \nearrow +\infty.\]

Moreover, there exists a Hilbert basis of \( H^1_A(\Omega, \mathbb{C}) \) of eigenfunctions of (27). The eigenfunctions can be chosen to form an orthonormal basis of \( L^2(\Omega, \mathbb{C}) \) as well.

We finally recall the variational characterization of the eigenvalues:

\[
\lambda_k = \min_{U \subset H^1_A(\Omega, \mathbb{C})} \max_{\dim U = k} \frac{\int_\Omega |\nabla A u|^2}{\int_\Omega |u|^2}.
\]

In particular, \( \lambda_1 \geq 0 \). If \( \nu \not\in \mathbb{Z} \) and \( x_0 \in \Omega \), we deduce from Lemma 19 that \( \lambda_1 > 0 \). On the other hand, if \( \nu \in \mathbb{Z} \), then \( e^{i\nu \theta} \in H^1_A(\Omega, \mathbb{C}) \) is an eigenfunction corresponding to the
eigenvalue $\lambda = 0$. We deduce by gauge invariance (see Appendix A.4) that in this case the eigenvalues of (2) coincide with those of the Neumann Laplacian (i.e., $\nu = 0$). If $x_0$ belongs to the unbounded component of $\Omega^c$, then the flux of $A_\nu$ is zero, hence $A_\nu = \nabla \phi$ with $\Delta \phi = 0$ on $\Omega$, and the eigenvalues of (2) coincide with those of the Neumann Laplacian. If $x_0$ belongs to some bounded component of $\Omega^c$, the flux is $\nu$, and if it is not an integer, inequality (23) with $C > 0$ holds with $B(x_0, R)$ replaced by an annular region $A \subset \Omega$ such that $x_0$ belongs to the bounded component of $A^c$ (one immediately realizes that (23) holds if we replace $B(x_0, R)$ by $B(x_0, R) \setminus B(x_0, r)$ for any $0 < r < R$). Then $\lambda_1 > 0$.

A.3 The Steklov eigenvalue problem

Problem (3) is understood in the weak sense, namely, find a function $u \in H^1_A(\Omega, \mathbb{C})$ and number $\sigma \in \mathbb{C}$ such that
\[
\int_{\Omega} \langle \nabla^A u, \nabla^A \phi \rangle = \sigma \int_{\partial \Omega} u \bar{\phi}, \quad \forall \phi \in H^1_A(\Omega, \mathbb{C}).
\]

We rewrite it as
\[
\int_{\Omega} \langle \nabla^A u, \nabla^A \phi \rangle + \int_{\partial \Omega} u \bar{\phi} = (\sigma + 1) \int_{\partial \Omega} u \bar{\phi}, \quad \forall \phi \in H^1_A(\Omega, \mathbb{C}). \tag{29}
\]

For the study of problem (29) we consider the space $H^1_A(\Omega, \mathbb{C})$ endowed with the equivalent norm $\| u \|_{A,\partial}$.

Since the quadratic form on the left-hand side of (29) is bounded and coercive on $H^1_A(\Omega, \mathbb{C})$ (with the equivalent norm $\| u \|_{A,\partial}$, see Lemma 21) and the trace operator $\gamma_0 : H^1_A(\Omega, \mathbb{C}) \to L^2(\partial \Omega, \mathbb{C})$ is compact, problem (29) is recast to an eigenvalue problem for a compact self-adjoint operator on the Hilbert space $H^1_A(\Omega, \mathbb{C})$. Standard Spectral Theory implies that problem (29) admits a sequence of eigenvalues
\[-\infty < \sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_k \leq \cdots \nearrow +\infty.
\]

Moreover, there exists a Hilbert basis of $H^1_{A,h}(\Omega, \mathbb{C})$ of eigenfunctions of (29), where
\[
H^1_{A,h}(\Omega, \mathbb{C}) = \left\{ u \in H^1_A(\Omega, \mathbb{C}) : \int_{\Omega} \langle \nabla^A u, \nabla^A \phi \rangle = 0, \quad \forall \phi \in H^1_A(\Omega, \mathbb{C}) : \gamma_0(u) = 0 \right\}.
\]

The eigenfunctions can be chosen to form a orthonormal basis of $L^2(\partial \Omega, \mathbb{C})$ as well.

We finally recall the variational characterization of the eigenvalues:
\[
\sigma_k = \min_{U \subset H^1_A(\Omega, \mathbb{C})} \max_{0 \neq u \in U} \frac{\int_{\Omega} |\nabla^A u|^2}{\int_{\partial \Omega} |u|^2}. \tag{30}
\]
In particular, \(\sigma_1 \geq 0\). As in the Neumann case, when \(x_0 \in \Omega\), \(\sigma_1 > 0\) if and only if \(\nu \notin \mathbb{Z}\).

If \(\nu \in \mathbb{Z}\) or \(x_0\) belongs to the unbounded connected component of \(\Omega^c\), the eigenvalues coincide with those of the usual Steklov problem (i.e., \(\nu = 0\)). If \(\nu \notin \mathbb{Z}\) and \(x_0\) belongs to some bounded connected component of \(\Omega^c\), then \(\sigma_1 > 0\).

We finally remark that the discussion contained in this section applies with no essential modifications to the case of bounded domains in two-dimensional Riemannian manifolds \(M^2\), and also to the case in which multiple singularities (of Aharonov-Bohm type) occur, i.e., the domain is punctured in many points.

### A.4 Gauge invariance

It is well-known that, if \(A_\nu\) and \(A_\nu'\) have fluxes \(\nu, \nu'\) which differ by an integer, i.e., \(\nu - \nu' \in \mathbb{Z}\), then \(\Delta_{A_\nu}\) and \(\Delta_{A_\nu'}\) are unitarily equivalent. In fact, one can define a multivalued function

\[
\psi(x) = \int_{y_0}^x (A_\nu - A_\nu')
\]

where \(y_0\) is any reference point (but not the pole). By the given condition, \(\psi\) is multivalued but \(e^{i\psi}\) is well-defined. Gauge invariance identity is:

\[
\Delta_{A_\nu} e^{-i\psi} = \Delta_{A_\nu} - \nabla \psi e^{i\psi} = e^{-i\psi} \Delta_{A_\nu}.
\]

One also observes that, if \(\nabla^{A_\nu} u = \nabla u - iu A_\nu\),

then one has the identity:

\[
\nabla^{A_\nu - \nabla \psi} (e^{-i\psi} u) = e^{-i\psi} \nabla^{A_\nu} u.
\]

From this one gets easily that, if \(u\) is an eigenfunction of the magnetic Neumann problem (2) associated to \(\lambda\), then \(e^{-i\psi} u\) is an eigenfunction of the corresponding problem with potential \(A_\nu - \nabla \psi\), associated to the same eigenvalue. The same assertion holds for the magnetic Steklov problem (3).

Since we are interested in the case \(\nu \notin \mathbb{Z}\), we can always assume, without loss of generality, that

\[
\nu \in \left(0, \frac{1}{2}\right].
\]

(31)
Aharonov-Bohm eigenvalues in two-dimensional manifolds of revolution

Let \((M^2, g)\) be a 2-dimensional compact manifold of revolution with pole \(x_0\) and diameter \(D\). We assume that the metric, in polar coordinates \((r, t)\) around the pole, is given by:

\[
g = \begin{pmatrix} 1 & 0 \\ 0 & \theta^2(r) \end{pmatrix},
\]

with \(\theta(r)\) smooth and positive. Here \(r\) is the distance to \(x_0\) (i.e., the radial coordinate). We shall always assume that

\[
\theta > 0 \text{ on } (0, D), \quad \theta'(0) = 1.
\]

We also define

\[
\bar{R} = \min\{R : \theta'(R) = 0\} \in (0, D).
\]

If such minimum does not exists, then we set \(\bar{R} = D\).

If \(f = f(r, t)\) the Laplacian writes:

\[
\Delta f = -f'' - \frac{\theta'}{\theta} f' - \frac{1}{\theta^2} \frac{\partial^2 f}{\partial t^2},
\]

where by \(f'\) we denote the derivative of \(f\) with respect to the radial variable \(r\). We consider the harmonic 1-form \(A_\nu\) which is written in polar coordinates as

\[
A = \nu dt
\]

where \(\nu\) is the flux around \(x_0\).

Lemma 23. For any \(f = f(r, t)\):

\[
\Delta_{A_\nu} f = -f'' - \frac{\theta'}{\theta} f' - \frac{1}{\theta^2} \frac{\partial^2 f}{\partial t^2} + \nu^2 \frac{\nu^2}{\theta^2} f + 2\nu \frac{\nu}{\theta^2} \frac{\partial f}{\partial t}.
\]

If \(f(r, t) = u(r)e^{ikt}\) then

\[
\frac{\partial f}{\partial t} = iku e^{ikt}, \quad \frac{\partial^2 f}{\partial t^2} = -k^2 u e^{ikt}
\]

and therefore:

\[
\Delta_{A_\nu} f = \left(-u'' - \frac{\theta'}{\theta} u' + \frac{(k - \nu)^2}{\theta^2} u\right) e^{ikt}.
\]
Proof. The magnetic Laplacian writes, for co-closed potentials \( A_\nu \):
\[
\Delta_{A_\nu} f = \Delta f + |A_\nu|^2 f + 2i\langle df, A_\nu \rangle.
\]
Normalizing the basis \( \left( \frac{\partial}{\partial r}, \frac{\partial}{\partial t} \right) \) we obtain the orthonormal basis:
\[
(e_1, e_2) = \left( \frac{\partial}{\partial r}, \frac{1}{\theta} \frac{\partial}{\partial t} \right)
\]
hence we get:
\[
|A_\nu|^2 = \frac{\nu^2}{\theta^2}
\]
and
\[
2i\langle df, A_\nu \rangle = 2i \frac{\nu}{\theta^2} \frac{\partial f}{\partial t}.
\]
The claim immediately follows. \( \square \)

**Corollary 24.** If \( f(r,t) = u(r)e^{ikt} \), then \( \Delta_{A_\nu} f = \lambda f \) reads:
\[
u'' + \frac{\theta'}{\theta} u' + \left( \lambda - \frac{(k - \nu)^2}{\theta^2} \right) u = 0.
\]
(32)

**B.1 Eigenfunctions of the Neumann problem**

We consider now problem (2) on disks \( B(x_0, R) \) of radius \( R \) centered at the pole of a manifold of revolution \( M^2 \).

Looking for eigenfunctions of the form \( f(r,t) = u(r)e^{ikt} \), \( k \in \mathbb{Z} \), we find, thanks to (32), that \( u \) is a bounded solution of the following singular Sturm-Liouville problem.
\[
\begin{cases}
u'' + \frac{\theta'}{\theta} u' + \left( \lambda - \frac{(k - \nu)^2}{\theta^2} \right) u = 0 \text{ in } (0, R), \\
u'(R) = 0.
\end{cases}
\]
(33)

If \( \nu \notin \mathbb{Z} \), we see from (33) that necessarily \( u(0) = 0 \) (and in fact, any function in \( H^1_A(\Omega, \mathbb{C}) \) vanishes at \( x_0 \)). If \( \nu \in \mathbb{Z} \), we see that any bounded solution of (33) with \( \nu = k \) satisfies \( u'(0) = 0 \), while for \( \nu \neq k \), \( u(0) = 0 \). This is exactly the case of the standard Laplacian. The condition \( u'(R) = 0 \) is the Neumann condition; in fact, \( \langle \nabla A_\nu u, N \rangle = \langle \nabla u, N \rangle = u'(R)e^{ikt} \) in the case of a disk centered at \( x_0 \).

The associated quadratic form is
\[
\int_0^R \left( u'^2 + V_k u^2 \right) \theta \, dr
\]
where \( V_k(r) = \frac{(k-\nu)^2}{\theta(r)} \). Note that, if \( \nu \in (0, \frac{1}{2}] \), then \( V_k \) is increasing in \(|k|\), hence:

\[
V_k \geq V_0,
\]

for all \( k \).

**Lemma 25.** Let \( \nu \in (0, \frac{1}{2}] \). Then:

i) For each \( k \in \mathbb{Z} \), problem (33) admits an infinite sequence of eigenvalues:

\[
0 < \lambda_{k1} < \lambda_{k2} < \cdots \leq \lambda_{kj} < \cdots \nearrow +\infty
\]

with associated bounded eigenfunctions \( u_{kj} \). All eigenvalues are simple and \( u_{kj} \) has \( j \) zeros in \((0, R)\).

ii) If \( |k| \leq |h| \) then:

\[
\lambda_{k1} \leq \lambda_{h1}.
\]

In particular, \( \min_{k,j} \{\lambda_{kj}\} = \lambda_{01} \).

**Proof.** Point i) follows from standard theory for singular Sturm-Liouville problems, since \( \theta(r) \) is smooth and positive in \((0, R]\), and \( \theta(r) \sim \theta'(0)r \) as \( r \to 0^+ \).

To prove ii) it is sufficient to consider \( k = 0 \). The claim is a consequence of the fact that \( V_h \) is increasing in \( h \). Let us take \( u = u_{h1} \), an eigenfunction associated to \( \lambda_{h1} \), as test function for \( \lambda_{01} \) in its variational characterization (which is the analogue of (10) weighted with \( \theta \)). We get

\[
\lambda_{01} \int_0^R u^2 \theta \leq \int_0^R (u'^2 + V_0 u^2) \theta \, dr \leq \int_0^R (u'^2 + V_h u^2) \theta \, dr = \lambda_{h1} \int_0^R u^2 \theta \, dr \]

\[
\square
\]

**Corollary 26.** We have

\[
\lambda_1(B(x_0, R), A_\nu) = \lambda_{01}.
\]

A first eigenfunction is given by \( u_{01}(r) \), and it is real and radial.

Note that \( \lambda_1(B(x_0, R), A_\nu) \) is not necessarily simple (this is the case of \( \nu = \frac{1}{2} \) and the unit disk, see Appendix B.3). At any rate, Corollary 26 states that we can always find a first eigenfunction which is real and radial. When \( \nu \in \mathbb{Z} \), Lemma 25 gives exactly the eigenvalues of the Neumann Laplacian, and in particular \( \lambda_{01} = 0 \).

We have the following result on completeness of the family of eigenfunctions \( \{u_{kj}(r)e^{ikt}\} \).

**Theorem 27.** Let

\[
\psi_{kj}(r,t) = u_{kj}(r)e^{ikt}
\]

for all \( k \in \mathbb{Z}, j = 1, 2, \ldots \), where \( u_{kj} \) are the eigenfunctions of (33). Then \( \{\psi_{kj}(r,t)\}_{k,j} \) is a complete system in \( L^2(B(x_0, R)) \).
Proof. If $\nu \in \mathbb{Z}$ the result is well-known: in fact the eigenfunctions $\{\psi_{kj}\}$ are the eigenfunctions of the Neumann Laplacian. Assume then $\nu \notin \mathbb{Z}$. Let us assume that the system of eigenfunctions $\{\psi_{kj}(r,t)\}_{k,j}$ is not complete. Then, there exists an eigenfunction $f$ of (2) which is orthogonal to the span of all $\psi_{kj}$, that is:
\[
\int_{B(x_0,R)} \overline{f} \psi_{hi} = 0 \tag{34}
\]
for all $h \in \mathbb{Z}, i = 1, 2, \ldots$. Expand $f(r,t)$ in a Fourier series, for every fixed $r$, and get:
\[
f(r,t) = \sum_{k \in \mathbb{Z}} a_k(r) e^{ikt}
\]
As by assumption $f \neq 0$, there exists $h \in \mathbb{Z}$ such that $a_h \neq 0$. We prove in Lemma 28 here below that, for all $k \in \mathbb{Z},$
\[
a_k(r) = c_k u_{kj}(r) \tag{35}
\]
for some $j_k = 1, 2, \ldots$ and $c_k \in \mathbb{C}, c_k \neq 0$. The conclusion follows from (35). In fact, we can write
\[
f(r,t) = \sum_{k \in \mathbb{Z}} c_k u_{kj}(r) e^{ikt} = \sum_{k \in \mathbb{Z}} c_k \psi_{kj}(r,t)
\]
and then, by the orthogonality of $\psi_{kj}$'s:
\[
\int_{B(x_0,R)} \overline{f} \psi_{hj} = c_h \neq 0 \tag{36}
\]
which contradicts (34). \quad \square

Lemma 28. Let $\{u_{kj}\}$ denote the eigenfunctions of (33) and assume that $f(r,t) = \sum_{k \in \mathbb{Z}} a_k(r) e^{ikt}$ is a nonzero eigenfunction of (2). Then, for all $k \in \mathbb{Z}$
\[
a_k(r) = c_k u_{kj}(r)
\]
for some $j_k = 1, 2, \ldots$.

Proof. We show that for each $k \in \mathbb{Z}$ the function $a_k : [0, R] \to \mathbb{R}$ is a bounded solution of
\[
\begin{cases}
a''_k + \frac{\varphi'}{\varphi} a'_k + \left(\lambda - \frac{(k-\nu)^2}{\varphi^2}\right) a_k = 0 \quad \text{in } (0, R) \\
a'_k(R) = 0.
\end{cases}
\]
In particular, since $\nu \notin \mathbb{Z}, a_k(0) = 0$. Hence, since the eigenvalues are simple, for every $k$, there exists $j = j_k$ and $c_k \in \mathbb{C}$ such that
\[
a_k(r) = c_k u_{kj}(r), \quad \lambda = \lambda_{k,jk}.
\]
If we assume \( a_k \neq 0 \), then \( c_k \neq 0 \).

First observe that \( f(x_0) = 0 \) because otherwise \( f \) is not in \( H^1_\lambda(\Omega, \mathbb{C}) \) when \( \nu \notin \mathbb{Z} \). Then:

\[
0 = f(x_0) = \sum_{k \in \mathbb{Z}} a_k(0)e^{ikt}
\]

for all \( t \). Hence, by multiplying by \( e^{-ikt} \) and integrating on \([0, 2\pi]\) we see that \( a_k(0) = 0 \).

Then \( a_k(0) = 0 \) for all \( k \). Similarly, the Neumann condition \( \frac{\partial f}{\partial r}(R, t) = 0 \) implies \( a'_k(R) = 0 \) for all \( k \).

It remains to show that \( a_k \) is an eigenfunction of the problem (33). First observe that

\[
a_k(r) = \int_0^{2\pi} f(r, t)e^{-ikt} \, dt
\]

Hence:

\[
-k^2 a_k = \int_0^{2\pi} f(r, t)\frac{\partial^2}{\partial t^2}e^{-ikt} \, dt = \int_0^{2\pi} \frac{\partial^2}{\partial t^2}f(r, t)e^{-ikt} \, dt
\]

therefore

\[
-k^2 \frac{\theta^2}{\theta^2} a_k = \int_0^{2\pi} \frac{1}{\theta^2} \frac{\partial^2}{\partial t^2}f(r, t)e^{-ikt} \, dt \tag{37}
\]

Now we know that \( \Delta_{\lambda_{\nu}} f = \lambda f \); hence

\[
\frac{1}{\theta^2} \frac{\partial^2}{\partial t^2} f = -\lambda f - f'' - \frac{\theta'}{\theta} f' + \frac{\nu^2}{\theta^2} f + 2i \frac{\nu}{\theta^2} \frac{\partial f}{\partial t}
\]

so that

\[
\int_0^{2\pi} \frac{1}{\theta^2} \frac{\partial^2}{\partial t^2} f e^{-ikt} \, dt = -\lambda \int_0^{2\pi} f e^{-ikt} \, dt - \int_0^{2\pi} f'' e^{-ikt} \, dt
\]

\[
= -\frac{\theta'}{\theta} \int_0^{2\pi} f' e^{-ikt} \, dt + \frac{\nu^2}{\theta^2} \int_0^{2\pi} f e^{-ikt} \, dt + 2i \frac{\nu}{\theta^2} \int_0^{2\pi} \frac{\partial f}{\partial t} e^{-ikt} \, dt
\]

Integrating by parts:

\[
\int_0^{2\pi} \frac{\partial f}{\partial t} e^{-ikt} \, dt = ik \int_0^{2\pi} f e^{-ikt} \, dt = ika_k.
\]

Then:

\[
\int_0^{2\pi} \frac{1}{\theta^2} \frac{\partial^2}{\partial t^2} f e^{-ikt} \, dt = -\lambda a_k - a_k'' - \frac{\theta'}{\theta} a'_k + \frac{\nu^2}{\theta^2} a_k - 2k \frac{\nu}{\theta^2}
\]
which, substituted in (37), gives:

\[ a_k'' + \frac{\theta'}{\theta} a_k' + \left( \lambda - \frac{(k - \nu)^2}{\theta^2} \right) a_k = 0 \]

which is the final assertion.

In the following theorems we collect a few useful properties of the eigenfunction \( u = u_{01} \) associated with the first eigenvalue \( \lambda = \lambda_{01} \).

**Theorem 29.** Let \( \nu \in (0, \frac{1}{2}) \). Let \( \lambda > 0 \) denote the first eigenvalue of problem (33) with \( k = 0 \) and \( R \in (0, \bar{R}] \), and let \( u \) denote a corresponding first eigenfunction. Then the following statements hold:

i) \( u(0) = 0 \) and \( u \neq 0 \) on \((0, R)\). In particular, we can choose \( u > 0 \) on \((0, R)\).

ii) \( u' > 0 \) on \((0, R)\) and \( \lambda > \frac{\nu^2}{\bar{u}(R)^2} \).

iii) The first eigenvalue \( \lambda \) is strictly decreasing in \( R \) for \( R \in (0, \bar{R}) \). In particular \( \lambda > \bar{\lambda} \), where \( \bar{\lambda} \) is the first eigenvalue of (33) with \( R = \bar{R} \).

**Proof.** Point i) has been already discussed in Lemma 25 and follows from standard Sturm-Liouville theory and the fact that \( \theta, \theta' > 0 \) on \((0, R)\), \( \theta(r) \sim \theta'(0)r \) as \( r \to 0^+ \).

Consider now ii). Set \( N(r) := \theta(r)u'(r) \). Then \( N(0) = N(R) = 0 \) and

\[ N'(r) = \left( \frac{\nu^2}{\theta^2(r)} - \lambda \right) \theta(r)u(r). \]

Since \( N'(r) > 0 \) on \((0, \delta)\) for some \( \delta > 0 \), we deduce that the function \( N \) increases from zero and decreases to zero. It has only one maximum point in \((0, R)\). In fact, since \( \theta \) and \( u \) are strictly positive in \((0, R)\), \( N'(r) = 0 \) if and only if \( \lambda = \frac{\nu^2}{\bar{u}(r)} \). Since \( \frac{1}{\theta^2} \) is strictly decreasing in \((0, R)\), it vanishes exactly once in \((0, R)\). This implies that \( N > 0 \) on \((0, R)\) and therefore \( u' > 0 \) on \((0, R)\). Moreover \( N'(R) < 0 \).

As for iii), let us write \( \lambda = \lambda(R) \) to highlight the dependence on \( R \). Then, as \( \lambda(R) \) is simple, we derive the identity

\[ \int_0^R u^2 \theta \lambda(R) = \int_0^R \left( u'^2 + \frac{\nu^2}{\theta^2} u^2 \right) \theta \]

and obtain

\[ \lambda'(R) = \left( \frac{\nu^2}{\theta^2(R)} - \lambda(R) \right) \frac{u^2(R)\theta(R)}{\int_0^R u^2 \theta}. \]
which is strictly negative by point ii) as long as \( \theta' < 0 \), that is, \( R \in (0, \bar{R}) \). More details can be found e.g., in [8].

**Theorem 30.** Let \( \nu \in (0, \frac{1}{3}] \). Let \( \lambda > 0 \) denote the first eigenvalue of problem (33) with \( k = 0 \) and \( R \in (0, \bar{R}) \), and let \( u \) denote a corresponding eigenfunction. Let \( F(r) = u'(r)^2 + \frac{\nu^2 u(r)^2}{\theta(r)^2} \). If either

a) \( \theta'(r) \geq \nu \) on \( (0, R) \), or

b) \( \lambda \theta^2 - \nu^2 + \nu^2(\theta')^2 + \nu \theta'' \geq 0 \) on \( (0, R) \),

then \( F'(r) \leq 0 \) on \( (0, R) \). If moreover \( \theta'' \) is a solution to (33) for \( R = \bar{R} \), then \( F'(r) \leq 0 \) on \( (0, \bar{R}) \).

**Proof.** Define

\[
q(r) = \theta(r) \frac{u'(r)}{u(r)}.
\]

Then \( F = u'^2 + \frac{\nu^2 u}{\theta^2} = \frac{\nu^2}{\theta^2}(q^2 + \nu^2) \), and with standard computations we see that

\[
F' = 2qq' \frac{\nu^2}{\theta^2} + 2u'^2 \frac{\nu^2}{\theta^2}(q^2 + \nu^2)(q - \theta').
\]

Since \( u, \theta, \theta' > 0 \) on \( (0, R) \), the claim follows provided \( q' \leq 0 \) and \( q \leq \theta' \) on \( (0, R) \).

We start by proving \( q' \leq 0 \) on \( (0, R) \). We see that

\[
q' = -\theta' \lambda + \frac{\nu^2}{\theta} - \frac{q^2}{\theta}.
\]

The local behavior of \( u \) near \( r = 0^+ \) depends only on the differential equation (33), and in particular, using the Frobenius-Taylor expansion, we deduce that \( u \sim cr^\nu \) as \( r \to 0^+ \) for some \( c > 0 \). Therefore \( q(0) = \nu, q'(0) = 0 \). Clearly there exists \( a > 0 \) such that \( q \neq \nu \) on \( (0, a) \), otherwise \( q' = -\theta' \lambda \neq 0 \) on \( (0, a) \). Choose then (a possibly smaller) \( a > 0 \) such that \( q, q' \) have constant sign on \( (0, a) \) (in particular, \( q > 0 \) on \( (0, R) \)). We have \( q' < 0 \) in \( (0, a) \). Otherwise, \( q > q(0) = \nu \), but then the differential equation tells then \( q' < 0 \). Then \( q \) starts decreasing. The same argument shows that \( 0 < q < \nu \) on \( (0, R) \). If \( q' \) changes sign in \( (0, R) \), let then \( b, c, d \in (0, R) \), \( b < c < d \), be such that \( q'(b) < 0, q'(c) > 0, q'(d) < 0 \) and \( q(b) = q(c) = q(d) = \bar{q} < \nu \). Such points exist if \( q' \) changes sign since \( q(R) = 0 \). We note now that, since \( \theta' > 0 \) in \( (0, R) \), \( \theta(c) = t\theta(b) + (1-t)\theta(d) \) for some \( t \in (0, 1) \). Then, from the differential equation for \( q' \), we see that

\[
0 < q'(c) = -\lambda \theta(c) + \frac{\nu^2 - \bar{q}^2}{\theta(c)} \\
\leq t \left( -\lambda \theta(b) + \frac{\nu^2 - \bar{q}^2}{\theta(b)} \right) + (1-t) \left( -\lambda \theta(d) + \frac{\nu^2 - \bar{q}^2}{\theta(d)} \right) = tq'(b) + (1-t)q'(d) < 0,
\]
a contradiction. Then \( q' < 0 \) on \((0, R)\). Note that \( q' < 0 \) also in the case \( R = \bar{R} \).

It remains to prove that \( q \leq \theta' \). Note that, if \( \theta' \geq \nu \) on \((0, R)\), since \( q \) is decreasing and \( q(0) = \nu \), we have \( q \leq \theta' \). We have used condition \( a \). Note that this condition has already been found in Theorem 9.

Assume now \( b \) holds. Let \( R \in (0, \bar{R}) \). In order to conclude, it is sufficient to prove

\[
q \leq (\nu + \delta)\theta' \tag{38}
\]

on \((0, R)\), for some \( \delta \in (0, 1 - \nu) \). For any \( \delta > 0 \) we have \( q(0) < (\nu + \delta)\theta'(0) \) and

\[
q'(r) = G(r, q(r)),
\]

where \( G(r, q) \) is defined as

\[
G(r, q) := -\lambda \theta(r) + \frac{\nu^2}{\theta(r)} - \frac{q^2}{\theta(r)}.
\]

Note also that, since \( R \in (0, \bar{R}) \), we have from \( iii \) of Theorem 29 that \( \lambda = \bar{\lambda} + \epsilon \) for some \( \epsilon > 0 \).

Let us compute

\[
G(r, (\nu + \delta)\theta'(r)) - ((\nu + \delta)\theta'(r))' = -\lambda \theta(r) + \frac{\nu^2}{\theta(r)} - (\nu + \delta)^2 \frac{\theta'(r)^2}{\theta(r)} - (\nu + \delta)\theta''(r)
\]

\[
= -\epsilon \theta(r) - \frac{1}{\theta} \left( \bar{\lambda} \theta^2(r) - \nu^2 + \nu^2 \theta'(r)^2 + \nu \theta(r) \theta''(r) \right)
\]

\[
- \frac{\delta}{\theta} \left( \theta'(r)^2(\delta + 2\nu) + \theta''(r)\theta(r) \right)
\]

The second summand is less or equal than zero by hypothesis, therefore we can choose \( \delta \in (0, 1 - \nu) \) arbitrarily small so that

\[
G(r, (\nu + \delta)\theta'(r)) - ((\nu + \delta)\theta'(r))' < 0
\]

on \((0, R)\). From Lemma 31 here below with \( v = q \) and \( w = (\nu + \delta)\theta' \), \( a = 0 \), \( b = R \), we deduce that

\[
q < (\nu + \delta)\theta'
\]

as claimed.

In the case \( R = \bar{R} \), if \( u = \theta'\), then \( q = \nu \theta' \leq \theta' \) on \((0, \bar{R})\).

\( \square \)
Lemma 31. Suppose \( v, w \) continuous on an interval \([a, b]\) and differentiable on \((a, b]\), and let \( G : (a, b] \times \mathbb{R} \to \mathbb{R} \) continuous. Suppose that

\[
v(a) < w(a)
\]

and

\[
v'(r) - G(r, v(r)) < w'(r) - G(r, w(r))
\] (39)

on \((a, b]\). Then \( v < w \) on \([a, b]\).

**Proof.** Assume that there exists \( c \in (a, b] \) such that \( v(c) = w(c) \) and \( v - w < 0 \) on \([a, c)\). Then clearly \( v'(c) \geq w'(c) \). On the other hand, inserting \( v(c) = w(c) \) in (39), we see that \( v'(c) < w'(c) \). A contradiction. \( \square \)

### B.2 Eigenfunctions of the Steklov problem

The same results of the previous subsection hold also for the Steklov problem (3). For the reader’s convenience, we briefly resume them, even if in this note we almost only consider the Steklov problem for domains in \( \mathbb{R}^2 \).

When looking for solutions on \( B(x_0, R) \) of the form \( f(r, t) = u(r)e^{ikt} \) we get the following singular Sturm-Liouville problem, for \( k \in \mathbb{Z} \):

\[
\begin{align*}
&u'' + \frac{a'}{a} u' - \frac{(k-w)^2}{a} u = 0 \text{ in } (0, R), \\
&u'(R) = \sigma u(R).
\end{align*}
\] (40)

with the requirement that a solution is bounded near 0 (and therefore, that it is vanishes at 0 when \( \nu \notin \mathbb{Z} \)).

We have the following

**Lemma 32.** Let \( \nu \in (0, \frac{1}{2}] \). Then problem (40) admits a unique solution \( u_k(r) \) with \( u_k(0) = 0 \) for all \( k \in \mathbb{Z} \). Let us set

\[
\eta_k = \frac{u_k'(R)}{u_k(R)}.
\]

If \( |k| \leq |h| \) then:

\[
\eta_k \leq \eta_h.
\]

Then the set \( \{\sigma_k(B(x_0, R), A_{\nu})\}_{k=1}^{\infty} \) of the Steklov eigenvalues on \( B(x_0, R) \) coincides with \( \{\eta_k\}_{k \in \mathbb{Z}} \). In particular, \( \min_k \{\eta_k\} = \eta_0 \), therefore

\[
\sigma_1(B(x_0, R), A_{\nu}) = \eta_0.
\]
The eigenfunctions corresponding to an eigenvalues $\sigma = \eta_k$ of (3) are given by

$$\psi_k(r, t) = u_k(r)e^{ikt}.\]

In particular, an eigenfunction associated to $\sigma_1(B(x_0, R), A_\nu)$ is given by $\psi_0(r, t)$ which is real and radial. The restrictions of $\{\psi_k(r, t)\}_k$ to $\partial B(x_0, R)$ form a orthonormal system in $L^2(\partial B(x_0, R))$

Proof. The proof is analogous of that of Lemma 25. It is sufficient to note that a bounded solution of (40) is of the form

$$u_k(r) = e^{\int r \frac{d|k - \nu|}{|k - \nu|} ds}.\]

The last statement is straightforward. □

B.3 The magnetic Neumann spectrum of the unit disk

We consider the particular case $M = B(x_0, 1)$, where $B(x_0, 1)$ is the unit disk in $\mathbb{R}^2$ centered at $x_0$. We recall that $A = \nu dt$ and we assume $\nu \in (0, \frac{1}{2}]$. Here $(r, t)$ are the standard polar coordinates in $\mathbb{R}^2$ centered at $x_0$. In this case $\theta(r) = r$.

We can describe the spectrum of (2) on $B(x_0, 1)$ more explicitly:

**Theorem 33.** Let $\nu \in (0, \frac{1}{2}]$. Then:

i) The spectrum of (2) on $B(x_0, 1)$ consists of the numbers:

$$\lambda_{kj}, \quad k \in \mathbb{Z}, \ j = 1, 2, ...$$

Here

$$\sqrt{\lambda_{kj}} = z'_{|k - \nu|, j}$$

where $z'_{\mu, j}$ denotes the $j$-th positive zero of $J'_\mu(\sqrt{\lambda})$ and $J_\mu(z)$ denotes the Bessel function of the first kind and order $\mu$.

ii) The eigenspace associated to $\lambda_{kj}$ is spanned by:

$$\psi_{kj}(r, t) = J_{|k - \nu|}(\sqrt{\lambda_{kj}}r)e^{ikt}$$

iii) The lowest eigenvalue is

$$\lambda_1(B(x_0, 1), A_\nu) = \lambda_{01}$$

iv) A first eigenfunction is

$$\psi_{01}(r, t) = J_\nu(\sqrt{\lambda_{01}}r),$$

it is real and radial.
v) The set of eigenfunctions \( \{ \psi_{kj} \}_{k,j} \) is complete in \( L^2(B(x_0, 1)) \).

**Proof.** We look for solution of problem (2) of the form \( f(r, t) = u(r)e^{ikt} \). Following Subsection B.1 we are recast to problem (33) for the radial part \( u \), with \( \theta(r) = r \) and \( R = 1 \). The differential equation reads

\[
  u'' + \frac{1}{r} u' + \left( \lambda - \frac{(k - \nu)^2}{r^2} \right) u = 0. \tag{41}
\]

With the substitution \( \sqrt{\lambda} r = z \), this equation can be recast to a standard Bessel equation. Therefore, a couple of linearly independent solutions of (41) is given by \( J_{|k-\nu|}(\sqrt{\lambda} r) \), \( J_{-|k-\nu|}(\sqrt{\lambda} r) \), since by hypothesis \( \nu \not\in \mathbb{Z} \). Therefore

\[
  u(r) = b_{k1} J_{|k-\nu|}(\sqrt{\lambda} r) + b_{k2} J_{-|k-\nu|}(\sqrt{\lambda} r).
\]

Note that, while \( J_{|k-\nu|} \) vanishes at the origin whenever \( \nu \not\in \mathbb{Z} \), the function \( J_{-|k-\nu|} \) is unbounded near \( r = 0 \). Any bounded solution of (41) is given by setting \( b_{k2} = 0 \). Therefore, any solution of (33) with \( \theta(r) = r \) has the form

\[
  u_k(r) = b_{k1} J_{|k-\nu|}(\sqrt{\lambda} r).
\]

We impose now the Neumann boundary condition \( u_k'(1) = 0 \). We recall that for any \( \mu > 0 \), the function \( J_{\mu} \) has an infinite set of positive zeroes, denoted by \( z_{\mu j}, j = 1, 2, ... \). Moreover, the function \( J_{\mu}' \) an infinite number of positive zeroes as well, namely \( z'_{\mu j}, j = 1, 2, ... \).

Then, points i)-v) follow from Lemma 25 and Theorem 27 with \( \sqrt{\lambda_{kj}} = z'_{|k-\nu|,j} \) and \( \psi_{kj}(r, t) = J_{|k-\nu|}(\sqrt{\lambda_{kj}} r)e^{ikt} \).

\[ \square \]

If \( \nu \in (0, \frac{1}{2}) \) then the set \( \{|k-\nu|\}_{k\in\mathbb{Z}} \) is given by \( \nu, 1-\nu, 1+\nu, 2-\nu, 2+\nu, ... \). For example, when \( \nu = \frac{1}{4} \) we have \( \frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}, ... \). If \( \nu \in (0, \frac{1}{2}) \), we immediately see that \( \lambda_1(B(x_0, 1), A_{\nu}) \) is simple, and a first eigenfunction is real and radial:

\[
  \psi_{01}(r, t) = J_\nu(\sqrt{\lambda_{01}} r).
\]

Such function is is positive and increasing on \((0, 1)\).

When \( \nu = \frac{1}{2} \), we have that the set of \( \{|k-\nu|\}_{k\in\mathbb{Z}} \) is given by

\[
  \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, ... \tag{34}
\]

and the first eigenvalue has multiplicity 2 with eigenspace spanned by the eigenfunctions with \( k = 0 \) and \( k = 1 \):

\[
  \begin{align*}
    \psi_{01}(r, t) &= J_{\frac{1}{2}}(\sqrt{\lambda_{01}} r) \\
    \psi_{11}(r, t) &= J_{\frac{1}{2}}(\sqrt{\lambda_{01}} r)e^{it}
  \end{align*}
\]
and \( \lambda_{01} = \lambda_{11} = z^\prime_{1/2,1} \) is the first zero of \( J^\prime_{1/2} \).

### B.4 The magnetic Steklov spectrum of the unit disk

In this section we investigate the spectrum of problem (3) on \( B(x_0, 1) \subset \mathbb{R}^2 \). Also in this section we assume that \( \nu \in (0, \frac{1}{2}] \).

We apply the results of Subsection B.2 with \( \theta(r) = r \) and \( R = 1 \).

**Theorem 34.** Let \( \nu \in (0, \frac{1}{2}] \). Then:

i) The spectrum of (3) on \( B(x_0, 1) \) consists of the numbers:

\[
\eta_k = |k - \nu|, \quad k \in \mathbb{Z}.
\]

ii) The eigenspace associated to \( \eta_k \) is spanned by:

\[
\psi_k(r, t) = r^{|k - \nu|} e^{ikt}
\]

iii) The lowest eigenvalue is

\[
\sigma_1(B(x_0, 1), A_\nu) = \eta_0 = \nu.
\]

iv) A first eigenfunction is

\[
\psi_1(r, t) = r^\nu,
\]

it is real and radial.

v) The restrictions of \( \{\psi_{kj}\}_{k,j} \) to \( \partial B(x_0, 1) \) is complete in \( L^2(\partial B(x_0, 1)) \).

**Proof.** It is sufficient to note that any solution to the differential equation in (40) is of the form

\[
u_k(r) = b_{k1}r^{|k - \nu|} + b_{k2}r^{-|k - \nu|},
\]

and in order to have a solution of (40) we need to impose \( b_{k2} = 0 \). The theorem now easily follows from Lemma 32.

When \( \nu \in (0, \frac{1}{2}) \), the first eigenvalue is \( \nu > 0 \). It is positive and simple and a corresponding eigenfunction is given by \( r^\nu \) and it is radial, positive and increasing on \( (0, 1) \).

We can list the Steklov eigenvalues \( \{\sigma_k(B(x_0, 1), A_\nu)\}_{k=1}^\infty \) as follows:

\[
u, 1 - \nu, 1 + \nu, 2 - \nu, 2 + \nu, ...
\]

If \( \nu = \frac{1}{2} \) all eigenvalues are double and are given by \( \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \ldots \).

In particular, \( \sigma_1(B(x_0, 1), A_\nu) = \sigma_2(B(x_0, 1), A_\nu) = \frac{1}{2} \) and two linearly independent eigenfunctions are \( u_1(x) = \sqrt{|x|}, u_2(x) = \sqrt{|x|} \left( \frac{2}{|x|} + i \frac{2}{|x|} \right) \). However, also in this case there exist a radial, real eigenfunction associated with \( \sigma_1(B(x_0, 1), A_\nu) \), positive and increasing on \( (0, 1) \).
C Conformal invariance of magnetic energy

Let \((\Omega_1, g), (\Omega_2, g)\) bet two 2-dimensional manifolds, and let
\[ \Phi : (\Omega_1, g_1) \to (\Omega_2, g_2) \]
be a conformal map between them. We can assume then that \(\Phi^* g_2 = e^{2f} g_1\), where \(f\) is the conformal factor. Let \(\omega\) be a 1-form on \(\Omega_2\). We start from:

**Lemma 35.** We have

i) In the above notation
\[ \delta_{g_1} \Phi^* \omega = e^{2f} \Phi^* (\delta_{g_2} \omega). \]
In particular, \(\omega\) is co-closed if and only if \(\Phi^* \omega\) is co-closed.

ii) For any complex 1-form \(\Omega\) on \(\Omega_2\), one has:
\[ \int_{\Omega_1} |\Phi^* \omega|^2_{g_1} d\mu_{g_1} = \int_{\Omega_2} |\omega|^2_{g_2} d\mu_{g_2}. \]

**Proof.** The first fact is standard; for the second, fix a \(g_1\)- orthonormal basis \((e_1, e_2)\) and let
\[ E_1 = d\Phi(e^{-f} e_1), \quad E_2 = d\Phi(e^{-f} e_2). \]
Using these orthonormal frames to compute the norms we end-up with the identity:
\[ |\Phi^* \omega|^2_{g_1} = e^{2f} (|\omega|^2_{g_2} \circ \Phi). \]
Integrating the identity on \(\Omega_1\) and using the change of variables formula, we obtain the assertion. \(\square\)

Now fix a potential one-form \(A\) on \(\Omega_2\). We get a potential one-form \(\hat{A} = \Phi^* A\) on \(\Omega_1\) by pull-back. Recall the magnetic gradient on \(\Omega_2\):
\[ d^A u = du - iu A \]
for a complex valued function \(u\) on \(\Omega_2\). Consider the function
\[ \hat{u} = u \circ \Phi = \Phi^* u \]
on \(\Omega_1\). Then, since \(d\Phi^* = \Phi^* d\):
\[ d^A \hat{u} = d^{\Phi^* A} \Phi^* u = d\Phi^* u - i\Phi^* u \Phi^* A = \Phi^* \left( du - iu A \right) = \Phi^* d^A u \]
Applying the \(L^2\) invariance property to \(\omega = d^A u\) we conclude with the following fact, expressing the conformal invariance of the magnetic energy.
Proposition 36. Let \( \Phi : \Omega_1 \to \Omega_2 \) be a conformal map between 2-manifolds, let \( A \) be a potential one-form on \( \Omega_2 \) and let \( \hat{A} = \Phi^* A \). For any complex function \( u \) on \( \Omega_2 \), let \( \hat{u} = u \circ \Phi \). Then we have:
\[
\int_{\Omega_1} |d^\hat{A} \hat{u}|^2 d\mu_1 = \int_{\Omega_2} |dA u|^2 d\mu_2.
\]

C.1 Aharonov-Bohm potentials

Let now \((\Omega, x_0)\) be a simply connected plane domain punctured at \(x_0\), with Aharonov-Bohm potential having pole at \(x_0\) and flux \(\nu\). We can assume without loss of generality that \(x_0\) is the origin. Then:
\[
A = \frac{\nu}{r^2} (-ydx + xdy). \tag{42}
\]
Take the unit disk \(D\) centered at the origin. By the standard theory, there is a conformal map \(\Phi : \Omega \to D\) fixing the origin. Note that the form \(\Phi^* A\) on \(\Omega\) is closed (clear) and co-closed (from previous section, because \(\Phi\) is conformal). Then, \(\Phi^* A\) is harmonic. It is also clear that \(\Phi^* A\) has flux \(\nu\) around the pole, the origin of \(\mathbb{R}^2\) (by elementary change of variable in dimension one). It follows that \(\Phi^* A\) differs from \(A\) by an exact 1-form, and gauge invariance applies. Precisely:

Lemma 37. Let \(\Omega\) be a plane domain and let \(D\) be the unit disk, both punctured at \(O \in \Omega\). Let \(\Phi : \Omega \to D\) be the unique conformal map fixing \(O\). If \(A\) is the canonical Aharonov-Bohm potential with pole \(O\) and flux \(\nu\), as in (42), then
\[
\sigma_k(\Omega, A) = \sigma_k(\Omega, \Phi^* A)
\]
for all \(k\).

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