Dimensionally Reduced Yang–Mills Theories in Noncommutative Geometry

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We study a class of noncommutative geometries that give rise to dimensionally reduced Yang–Mills theories. The emerging geometries describe sets of copies of an even dimensional manifold. Similarities to the D-branes in string theory are discussed.
1 Introduction

Connes’ theory of noncommutative geometry (NCG) has enabled physicists to study configuration spaces with a geometry decisively more general than that of the usually considered differentiable manifolds [1]. In these constructions geometry gives naturally rise to a characteristic field theory defined on the considered noncommutative space. Not all field theories, however, can yet be obtained this way. Those that can, include the classical field theory formulations of the General Relativity, the Standard Model, grand unified theories and some space–time supersymmetric theories in four dimensions [2, 3, 4]. These studies have lead to the understanding of the geometric origin of the Higgs’ scalar field in the Standard Model.

All of these NCG’s are modifications of the idea of expanding a manifold into a collection of its copies (called p-branes in the following) in a nontrivial way. In this context, the NCG origin of a field theory strongly constrains the form of the scalar potential [5]. In this article we shall consider a particular subclass of these NCG models, namely those that give rise to a dimensionally reduced Yang–Mills theory. The idea of dimensional reduction may seem foreign to NCG at the first sight but a closer look at it leads to a natural generalization of an embedding of geometric objects into a larger, partly compactified space. The compactified directions turn out to appear as relations between the embedded objects – no reference to an actual embedding space is necessary.

This work was originally motivated by the apparent similarity of the above mentioned NCG’s and the D(irichlet) p-branes that appear in string theory. These are in $D$ dimensions embedded $p + 1$-dimensional surfaces on which the boundaries of the open string world sheets are constrained [6]. The vanishing of the 1-loop beta-functions of the open strings is equivalent to the fact that the classical equations of motion of the D-branes (coming from a Born–Infeld action) are satisfied. The emerging low energy effective field theory is to the 2nd order in the field strength a dimensionally reduced Yang–Mills theory. The similarity of the topologies of the two systems leads one to ask whether there indeed is a NCG capable of producing the expected low energy field theory and the geometric relations between the D-branes.

In this work we shall study NCG’s that give rise to these field theories, but describe a geometry – in the sense of measuring distances – that does not coincide with the D-brane
results. These problems were already pointed out in Ref. [7]. In what follows we study this particular class of NCG’s on its own right, and point out many of the similarities to and differences from the D-branes. An other appearance of NCG in string theory was considered in Ref. [8], where the M-theory membranes were studied from the NCG point of view. After the completion of the present work we received Ref. [9] where similar issues were considered.

2 Connes’ NCG

Let us start by considering a collection of copies of a Riemannian manifold \((M, g)\) in NCG [1, 10]. In Riemannian geometry one would embed the copies of \(M\) in a bigger space in order to address questions concerning their mutual relations. In NCG this is not necessary, but the formalism is particularly well suited for studying geometry in spaces that are of the form \(M \times \mathbb{Z}_n\), \(\mathbb{Z}_n\) being a discrete set of points. The basic data needed for this are a K-cycle \((\mathcal{H}, \mathcal{D}, \Gamma)\) on a \(*\)-algebra \(\mathcal{A}\). In the present case \(\mathcal{A}\) can be taken to be the algebra of \(M_n(\mathbb{C})\)-valued (smooth) functions on \(M\), \(\mathcal{H}\) the Hilbert space of square integrable spinors on \(M\) tensored with the \(n\)-dimensional representation space of \(M_n(\mathbb{C})\), \(\Gamma\) the Clifford grading and

\[ \mathcal{D} = \mathcal{D} \otimes 1 + \Gamma \otimes N, \]  

where \(N \in M_n(\mathbb{C})\) is an anti-Hermitian matrix (with a vanishing diagonal) of dimension mass.

2.1 The metric

The measurement of distances will turn out to be the distinguishing factor between our NCG models and the D-brane models, so we shall need to consider it at some length.

In the case \(n = 1\) we are considering a single copy of \(M\). In this case the geodesic distance between two points \(p, q \in M\) can be calculated from

\[ d(p, q) = \sup_{a \in \mathcal{A}} \left\{ |a(p) - a(q)|; \quad ||[\mathcal{D}, a]|| \leq 1 \right\}, \]  

where the norm \(||\ ||\) is the operator norm in \(\text{End } \mathcal{H}\). This metric is the same as the one found by looking for shortest paths using the metric tensor \(g\) [1]. The term \(a(p) = \chi(a)\)
should be seen as a character of the algebra $\mathcal{A}$, i.e. a homomorphism from $\mathcal{A}$ to complex numbers. In the case $n > 1$ we consequently can speak about points of a noncommuting manifold with a function algebra $\mathcal{A}$ if we know what the characters are. If $\mathcal{A}$ only contains diagonal matrices, the algebra is still Abelian, and we can construct characters by simply taking a particular diagonal entry at a particular point on the manifold

$$\chi(a) = a(p)_{ii}, \quad a \in \mathcal{A}, \ p \in M, \ i = 1, \ldots, n. \quad (3)$$

The distance between two points $p$ and $q$ on two copies labeled by $i$ and $j$ is now given with the help of the characters

$$\chi(a) = \text{tr}(H^i a)(p) \quad \text{and} \quad \psi(a) = \text{tr}(H^j a)(q), \quad (4)$$

where $(H^i)_{nm} = \delta_i^n \delta_{nm}$, as

$$d(\chi, \psi) = \sup_{a \in \mathcal{A}} \left\{ |\chi(a) - \psi(a)|; \ |[D, a]| \leq 1 \right\}. \quad (5)$$

The matrices $H^i$ are Cartan elements of the Lie algebra $\mathfrak{u}(n)$ in the fundamental representation. The distances between the p-branes according to Eq. (5) seem to be associated to root vectors of $\mathfrak{su}(n)$. The $\mathfrak{u}(n)$ Cartan element proportional to unity $1_n$ is then naturally associated to the movement of the center of mass described by $\mathfrak{u}(1) \subset \mathfrak{u}(n)$. In the case of Lie-algebras $g = \mathfrak{sp}(2r)$ and $\mathfrak{so}(2r)$ the diagonal elements are of the form $\text{diag}(A, -A^T)$. Each p-brane appears thus twice: one might say that there is a mirror in the noncommutative space. We now have characters

$$\chi(a) = \text{tr}(P_+ H^i a)(p), \quad (6)$$

where $P_+ = \text{diag}(1_r, 0)$ and $P_- = \text{diag}(0, 1_r)$ are projections to different sides of the mirror, and $H^i \in \mathfrak{h}$ is a Cartan element of the Lie-algebra $g$. In this way one may try to extend $g$ to the exceptional Lie algebras and to the Kač–Moody algebras. The latter possibility might enable one to include even the winding modes of string theory into the NCG approach.

In the direct non-Abelian generalization the functions $\chi$ introduced above fail to be homomorphisms. Let us, however, postulate that those functions $\chi : \mathcal{A} \rightarrow \mathbb{C}$ that reduce to characters in the diagonal subalgebra $\mathcal{A}_0 \subset \mathcal{A}$ are to be associated with distances in
the genuinely non-Abelian NCG. The distances then depend on the choice of the Cartan subalgebra, and thus explicitly break global gauge invariance. This is actually natural, since the global gauge invariance here is the counterpart of the Lorentz invariance of the compactified space, also broken by the introduced p-branes.

2.2 Differential geometry in NCG

In order to address questions that concern geometry, we shall need the generalized differential forms of NCG: The differential algebra of forms over a noncommutative space is given in terms of the tensor algebra of $\mathcal{A}$. A p-form $\alpha = a^0 da^1 \ldots da^p \in \Omega^p \mathcal{A}$ is represented as an operator on $\mathcal{H}$ by

$$\pi(\alpha) = a^0[D, a^1] \ldots [D, a^p].$$

(7)

However, to avoid the case where $\pi(\alpha) = 0$ but $\pi(d\alpha) \neq 0$ one should properly consider the equivalence classes

$$\pi_D(\alpha) \in \Omega^p_D \mathcal{A} = \pi[\Omega^p \mathcal{A}/(\ker \pi + d \ker \pi)].$$

(8)

The concept of a fiber bundle has its generalization in NCG, as well. Here we only need the trivial bundle over $\mathcal{A}$ that has a connection $d + \rho$, where $\rho \in \Omega^1 \mathcal{A}$, and a curvature $\vartheta = d\rho + \rho^2 \in \Omega^2 \mathcal{A}$. We choose $\rho^* = -\rho$ so that $\vartheta^* = \vartheta$. Such a connection 1-form can be given as a (formal) sum

$$\rho = \sum a^n db^n,$$

(9)

where the sequence $(a^n, b^n)$ is chosen to satisfy $\sum a^n b^n = 1$.

We can define, under suitable conditions, an inner product $\langle | \rangle$ in $\Omega^* \mathcal{A}$ by setting

$$\langle \alpha | \beta \rangle = \text{Tr}^+(\pi(\alpha^* \beta)|D|^{-p}),$$

(10)

where the trace is as in [10] and $p = \text{dim} M$. This can be used to project $\Omega^* \mathcal{A} \rightarrow \Omega^*_D \mathcal{A}$ by choosing a representative of $\alpha$ in $\Omega^*_D \mathcal{A}$ such that it minimizes the norm $\langle \alpha | \alpha \rangle$.

The NCG-version of a Yang–Mills action is

$$S_{YM}(\rho) = \langle \vartheta(\rho) | \vartheta(\rho) \rangle_D = \text{Tr}^+(\pi_D(\vartheta(\rho))^2|D|^{-p}).$$

(11)

Since $\pi$ is assumed a faithful representation of $\mathcal{A}$ on $\mathcal{H}$ we write $\pi(a) \equiv a$, $a \in \mathcal{A}$.
This can be evaluated further in the cases that we shall consider: The result is
\[
S_{YM}(\rho) = C_p \int_M \text{tr}(\pi_D(\vartheta(\rho))^2),
\]

where \(C_p\) is a dimension dependent constant and \(\text{tr}\) is taken over the finite matrix indices.

3 The construction

Our aim is to construct a NCG that produces a Yang–Mills–Higgs theory with a scalar potential and a fermionic sector that match those of a dimensionally reduced Yang–Mills theory. It turns out to be necessary to consider mass matrices that are a tensor product of a Clifford algebra and an arbitrary matrix algebra. This also automatically leads to the right fermion sector. Our construction is essentially a generalization of \cite{4} where four-dimensional supersymmetric field theories where studied in NCG.

Let \(M_p\) be a \(p+1\) dimensional compact spin-manifold with a Euclidean metric \(g_{\mu\nu}\) and \(p\) odd. The Hermitian generators of its Clifford algebra are \(\gamma^\mu\), and the Hermitian chirality operator \(\gamma_{p+2}\) satisfies \(\gamma_{p+2}^2 = 1\). The spin-connection \(D_\mu\) operates on the square integrable spinors in \(L^2(S(M_p))\). Let further \(\Sigma^a, a = 1, \ldots, \tilde{p}\) be \(s \times s\)-matrix-valued functions that satisfy
\[
\{\Sigma^a, \Sigma^b\} = 2g^{ab},
\]

where \(g^{ab}\) are scalar functions on \(M_p\). The matrix \((g^{ab}) > 0\) has an inverse \((g_{ab})\). These two algebrae can be naturally combined into a Clifford algebra with generators
\[
\Gamma^\mu = \gamma^\mu \otimes 1, \quad \mu = 0, \ldots, p
\]
\[
\Gamma^{a+p} = \gamma_{p+2} \otimes \Sigma^a, \quad a = 1, \ldots, \tilde{p}.
\]

Also this Clifford algebra has a Hermitian chirality operator for even \(\tilde{p}\)
\[
\Gamma_{D+1} = \gamma_{p+2} \otimes \Sigma_{\tilde{p}+1}, \quad D = \tilde{p} + p + 1.
\]

The NCG’s we are interested in are given by the K-cycle \((\mathcal{H}, \mathcal{D}, \Gamma)\) on the *-algebra \(\mathcal{A}\)
\[
\mathcal{A} = C^\infty(M_p, C) \otimes U(g)
\]
\[ \mathcal{H} = L^2(S(M_p)) \otimes C^s \otimes L \otimes C^k \]  
\[ \mathcal{D} = \Gamma^\mu D_\mu \otimes 1_L \otimes 1_k + \Gamma^a \otimes S_a \otimes K \]  
\[ \Gamma = \Gamma_{D+1} \otimes 1_L \otimes 1_k. \]

Here \( U \) denotes the universal enveloping algebra of a Lie algebra, \( S_a \in \mathcal{V} \), \( \mathcal{V} \subset \mathfrak{g} \) is a (finite) subset containing anti-Hermitian elements of a Lie-algebra \( \mathfrak{g} \) of a Lie-group \( G \) and the Hermitian matrix \( K \) mixes the \( k \) fermion flavours included in the Hilbert space.

Notice that \( U(u(n)) = M_n(C) \) in the fundamental representation. The operator \( \mathcal{D} \) is anti-Hermitian on \( \mathcal{H} \), and it anticommutes with \( \Gamma \). The algebra \( \mathcal{A} \) acts in \( \mathcal{H} \) through the (irreducible) representation \( R \) in the \( n \)-dimensional space \( L \) as

\[ \pi : \mathcal{A} \rightarrow \text{End}(\mathcal{H}); \ a \mapsto 1_{S(M_p)} \otimes 1_s \otimes R(a) \otimes 1_k. \]

In the following all tensor products, unit matrices and explicit summations as in Eq. (9) will be omitted.

### 3.1 The NCG dimensional reduction

The geometric content of the theory can be elucidated by considering differentials

\[ \pi(da) = \phi a + \Gamma^a \left( (S_a, \alpha) a^\alpha E^\alpha - S_a^\alpha (a, \alpha) E^\alpha + S_a^\alpha a^\beta \varepsilon_{\alpha, \beta} E^{\alpha + \beta} + S_a^\alpha a^{-\alpha} H^\alpha \right) \]

in the Cartan–Weyl basis. Summation is assumed over repeated indices and \((,\) is the inner product in the root space. Consider in particular the case

\[ S_a = S_a^\alpha (E^\alpha - E^{-\alpha}), \]

where the roots belong to the lattice \( \Phi \) of \( \mathfrak{su}(n) \). Let \( a \in \mathcal{A}_0 \) be diagonal and denote \( \alpha = e^i - e^j \). A differential of \( \pi(a) \) becomes

\[ \pi(da) = \Gamma^\mu \partial_\mu a^i H^i + \Gamma^a S_a^\alpha (a^j - a^i) (E^\alpha + E^{-\alpha}). \]

One can view \( i, j \) as labels of two lattice points a distance \( \epsilon_a \) apart from each other where

\[ S_a^\alpha = \frac{\epsilon_a}{\epsilon_a^2} \delta_a^\alpha. \]
Taking the limit $\epsilon \to 0$ with $a^i - a^j = \mathcal{O}(\epsilon)$ one reduces the differential to the standard form
\[ \bar{a}(x + h) - \bar{a}(x) \equiv \text{tr}_{\text{Cliff}}(\hbar \pi(da)) = Da \cdot h + \mathcal{O}(h)^2 \] 
(26)
where $h = (h_\mu, \epsilon_a)$ and the arrows refer to a basis of the vector space $\mathbf{g}$. Moving from the diagonal element $a^i$ to the element $a^j$ has thus an interpretation as motion in a larger space to a direction corresponding to the root $e^i - e^j$. Depending on the choice of the matrices $S_a$ there is an index $a = 1, \ldots, \tilde{p}$ that corresponds to this direction. The only novelty here is that the vector space $\mathbf{g}$’s basis elements do not commute. The theory is thus indeed $D$ dimensional: the additional $\tilde{p}$ directions appear in a complicated way in the mutual relations of the p-branes.

### 3.2 Distances

Distances calculated from Eq. (26) depend on the choice of the Cartan subalgebra. The simplest case that will turn out to be interesting from the field theory point of view as well, is $\mathcal{V} \subset \mathbf{h}$ in which case all distances become infinite.

Let us next conjugate $\mathcal{V}$ by

\[ g = \exp \left( i \frac{1}{2} \sum_j \varphi_j (E^{\alpha(j)} + E^{-\alpha(j)}) \right) \in \text{SU}(n), \] 
(27)
where the roots for simplicity satisfy the condition

\[ (\alpha^i, \alpha^j) = 2\delta^{ij}. \] 
(28)

The new Dirac operator is

\[ \mathcal{D} = \Gamma^\mu D_\mu + \Gamma^a (\text{Ad}_g S_a) K. \] 
(29)

A nontrivial lower bound for the distance between a pair of p-branes corresponding to a root $\alpha \in \Phi$ is obtained by first noticing that $\text{tr}(H^\alpha a)$ in Eq. (26) only depends on the part of $a$ proportional to $H^\alpha$. One then estimates the constraint term from above, and saturates the estimate by choosing $H^\alpha a_0 = a$ for some number $a_0 \in \mathbb{C}$. We then get

\[ d(\alpha) \geq \left( \max_j \left\{ m_j \left| (\alpha^*, \sin \varphi_j \alpha^j) \right| \right\} \right)^{-1}, \] 
(30)
up to a constant normalization, where

\[ m_j^2 = 2g^{ab}(\alpha^{(j)}, S_a)(\alpha^{(j)}, S_b) \]  

is the W-boson mass in a theory in which the adjoint representation scalars get vev’s \( \langle \Phi_a \rangle = S_a \) and \( g_{ab} \) is a metric in the space of the scalars \( \Phi^a \). This will happen in the theory at hand as well, and thus unbroken gauge symmetry \( (S_a, \alpha) = 0 \) implies infinite distance.

Notice that here (the estimate of) the metric \( d \) does not necessarily satisfy the triangle inequality. The situation with D-branes is exactly the opposite: \( d_D(\alpha) = \alpha' m_\alpha \), unbroken gauge symmetry implies zero distance and the triangle inequality is valid.

4 The field theory

We are now ready to find out to which Yang–Mills theory the present NCG model gives rise. Let us choose an anti-Hermitian 1-form \( \rho \in \Omega^1 A \) as in (9) and write

\[ \pi(\rho) = \Gamma^\mu A_\mu + \Gamma^a (A_a - S_a), \]  

where

\[ A_\mu = aD_\mu b = a\partial_\mu b \]  

\[ A_c = a[S_c, b] + S_c = aS_c b. \]

The curvature becomes

\[ \pi(\vartheta) = X + (g^{ab}A_aA_b - Z)K^2 + \frac{1}{2}(F_{\mu\nu} - T^{\kappa}_{\mu\nu}A_\kappa)\Gamma^{\mu\nu} \]

\[ + (D_\mu A_a)\Gamma^{a\kappa}K + \frac{1}{2}([A_a, A_b] - f_{ab}^c A_c - \bar{f}_{ab}^c Y_c)\Gamma^{ab} K^2, \]

where \( F_{\mu\nu} \) is the field strength of \( A_\mu, T^{\kappa}_{\mu\nu} \) are the structure constants of the algebra spun by \( \partial_\mu, f_{ab}^c \) and \( \bar{f}_{ab}^c \) are the \( g \)-structure constants and the spin-connection has become a U(\( n \))-covariant connection \( D_\mu \to D_\mu + [A_\mu, ] \). Functions \( X \) and \( Y_c \) are independent of \( A_B \), where \( B = (\mu, c) \). \( Y_c \)'s Lie-algebra index \( c \) refers to the subspace of \( g \) orthogonal to \( \text{sp } V \) under the Killing metric. The function \( Z = aCb, \) where \( C = g^{ab}S_aS_b \), is generically a free field.
If $K = 1$ the $u(1)$-part of $A_B$ couples only to fermions, as we shall see, and decouples completely if the fermions are in the adjoint representation of $g$. As was shown before, this part of the gauge group should be associated to the motion of the center of mass. We impose for simplicity the constraint $\text{tr} \ A_B = 0$. We shall also assume $T = 0$.

The next problem is to find a representative of $\vartheta$ in $\Omega^2_D A$. The 1-forms $\sigma \in \ker \pi$ give rise to those 2-forms $d\sigma$ that constitute the ambiguity in the choice of this representative. The ambiguity is actually just the freedom to shift the nondynamical fields at will, and choosing the representative of $\vartheta$ as suggested above amounts to eliminating the nondynamical fields $X, Y_c$ and, depending on the choice of $V$, also $Z$ by imposing their classical equations of motion.

Suppose the matrix $C$ is expressible as a linear combination

$$C = \text{tr} C \ 1_n + 2 C^a \ S_a.$$  \hspace{1cm} (36)

Then the field $Z$ is not free, and we only need to eliminate $X$ and $Y_c$. The resulting action is

$$S_{YM} = -\frac{1}{2} \int_{M_p} \left( \text{tr} F_{\mu\nu}^2 + 2 \text{tr}(D_\mu A_a)^2 - 2 \kappa \text{tr} \left( (A_a - C_a)^2 - C_a^2 - \text{tr} C \right)^2 \right. $$

$$\left. +(1 + \kappa) \text{tr} \left( ([A_a, A_b] - f_{ab}^c A_c) \mathcal{P}_{ab,de} \right)^2 \right),$$  \hspace{1cm} (37)

where $\kappa = \text{tr} \ K^4$, $K$ is normalized to $\text{tr} \ K^2 = 1$ and $\mathcal{P}$ is a projection to the subspace of $v \land w \in \text{sp} V \land \text{sp} V$ with the property $[v, w] \in \text{sp} V$. If $C$ is not of the form suggested above then the third term in Eq. (37) vanishes.

Choosing $K = 1$ and $[V, V] = 0$ the Yang–Mills action reduces to

$$S_{YM} = -\frac{1}{2} \int_{M_p} \text{tr} \left( F_{\mu\nu}^2 + 2(D_\mu A_a)^2 + [A_a, A_b]^2 \right) = -\frac{1}{2} \int_{M_p} \text{tr} F_{AB}^2. \hspace{1cm} (38)$$

This is the trivially from $D$ dimensions down to $p + 1$ dimensions reduced Yang–Mills theory.

The dynamics of the fermions $|\psi\rangle \in \mathcal{H}$ is determined by the action

$$S_F = \langle \psi | D + \pi_D(\rho) | \psi \rangle = \int_{M_p} \text{tr} \tilde{\psi} \Gamma_B D_B \psi.$$  \hspace{1cm} (39)

Under assumptions that the elements of $V$ commute and that the gauge fields be traceless this theory describes the $D$ dimensional SU($n$) Yang–Mills coupled to fermions after the
trivial dimensional reduction to $p+1$ dimensions. This is an immediate consequence of
the structure of the Dirac operator $D$.

In particular, if $D = 10$ and $\mathcal{H}$ contains Majorana–Weyl fermions or $D = 6$ and $\mathcal{H}$
contains Weyl fermions, we get in $p+1 = 4$ dimensions $N=4$ and $N=2$ supersymmetric
Yang–Mills theories, respectively [4]. For this, we let the covariant derivative $\pi_D(d + \rho)$
act on the $g$-valued fermion fields in (39) through the Lie-brackets.

5 Symmetries and classical moduli

The theory is invariant under unitary transformations $u \in U(A)$

$$ \rho \rightarrow u \rho u^* + u du^*, \quad \rho \in \Omega^1 \mathcal{A} \quad (40) $$

$$ |\psi\rangle \rightarrow R(u) |\psi\rangle, \quad |\psi\rangle \in \mathcal{H}. \quad (41) $$

In terms of its constituent fields $a^m, b^m \in \mathcal{A}$ of Eq. (33) the transformation of the connection 1-form is expressible as $(a^m, b^m) \rightarrow (ua^m, b^m u^*)$. This symmetry gives rise to the
local SU($n$) gauge symmetry.

By a global gauge transformation one usually means a $u \in U(A)$ that satisfies $du = 0$.
In the present case $u$ would thus be a constant matrix that commutes with $V$. However,
let us relax this condition and consider such transformations of the algebra $\mathcal{A}$ that
become global symmetry transformations of the field theory, ie. $u \in \text{Int } G$. On the level
of the differential algebra and the choice of the K-cycle these transformations act as

$$ \omega \rightarrow \text{Ad}_u \omega, \quad \omega \in \Omega^* \mathcal{A} \quad (42) $$

$$ \mathcal{D} \rightarrow \text{Ad}_u^* \mathcal{D} = \Gamma^u \mathcal{D}_\mu + \Gamma^a \otimes \text{Ad}_u \ast S_a. \quad (43) $$

A global gauge transformation is thus essentially a change of NCG. There is consequently
a whole orbit of NCG’s that yield the same field theory. Notice, however, that since the
formula for distances is not $G$-invariant, the distances between the branes vary as we
move along the orbit of $G$ in the parameter space.

Let us consider the Yang–Mills theory of Eq. (38). The vacuum expectation values
$\langle A_a \rangle$ that minimize the potential $-\text{tr}[A_a, A_b]^2$ commute. Thus for any dimensionally
reduced Yang–Mills theory with vev’s $\langle A_a \rangle = S_a$ there is a NCG of the form suggested
above with the property $A_a = a[S_a, b] + S_a = S_a$ for the configuration $a = b = 1$. This
can be inverted by simply postulating that the coefficients $S_a$ are the vacuum expectation values of the fields $A_a$ and that the vacuum corresponds to the configuration $a = b = 1$. This is the point of view also adopted in the study of symmetry break down in Ref. [3]. The moduli space of vacua thus becomes the moduli space of NCG’s.

For $g = su(n)$, the moduli space of the considered NCG’s consists of vectors

$$ (S_a) \in \mathcal{M}_{NCG} = \text{Ad}_G(h^{n-1})/W, a = 1, \ldots, n-1, \quad (44) $$

where the group $W$ acts by permutations in the index $a$. In the fixed points of $W$ we get degeneracy in $S_a$. This means that we can choose a new basis of $\Gamma^a$’s so that some of the new coefficients $S_a$ vanish in Eq. (44). This leads to $A_a \equiv 0$ for some $a$, and to the restoration of some of the gauge symmetry.

In the Abelian limit $A \to A_0$ the gauge symmetry becomes a local $U(1)^{n-1}$ and the corresponding gauge fields $A^{\mu}$ decouple. Due to the condition $ab = 1$ we get $A_a = S_a$, if the $S_a$’s are diagonal, and the $W^{\pm\alpha}$-bosons would get a mass $m_{\alpha}$ of Eq. (31) if there were any. Even in the case that the $S_a$’s are not diagonal, the vector bosons keep out of the theory, and one sees fluctuation fields around the vev’s $\langle A_a \rangle = S_a$ only in those directions of the matrix space, where the $S_a$’s have components.

6 Conclusions

We have analysed a subclass of NCG’s that describe a collection p-branes with $p$ odd. The studied NCG’s give rise to field theories that can also be obtained by dimensional reduction. The additional dimensions enter the NCG formulation in the form of the p-branes’ mutual relations.

The metric of the noncommutative space turned out not to be uniquely determined by the commuting analogue. The latitude in its definition was related to the mutual orientations of the geometric objects. In addition, the usual matrix structure of the NCG models was obtained from an underlying group structure intimately connected with the measurement of distances. The classical moduli of the field theory could also be related to the parameters of the NCG models in a transparent manner.

The obtained field theories also describe low energy physics of D-branes. Despite the similarities of the presented models to the D-brane effective theories, there are differences
Most importantly, the distances one obtains between p-branes are here of the form $1/m_\alpha$ whereas in string theory one obtains $\alpha' m_\alpha$. Also, it is not known, how to give correct dynamics to the metric $g_{AB}$, how to incorporate winding modes of strings or how to extend the theory to higher orders in the field strength.

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