A second order expansion of the separatrix map for trigonometric perturbations of a priori unstable systems

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Abstract

In this paper we study a so-called separatrix map introduced by Zaslavskii-Filonenko [ZF68] and studied by Treschev and Piftankin [Tre98, Tre02, Pif06, PT07]. We derive a second order expansion of this map for trigonometric perturbations. As an application, combining with results of [CK15], in [KZZ15] we describe a class of nearly integrable deterministic systems with stochastic diffusive behavior. More exactly, we show that distributions given by deterministic evolution of certain random initial conditions weakly converge to a diffusion process on the line.

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1 Introduction

The main goal of this paper is to derive a second order expansion of a so-called separatrix map for a class of nearly integrable systems. In nearly integrable Hamiltonian systems with one and a half degree of freedom this map was introduced by Zaslavskii and Filonenko in [ZF68]. Shilnikov [Sil65], using a very similar geometric idea, studied it in a neighborhood of homoclinic orbits without restriction to Hamiltonian structure or closeness to integrability. Treschev and Piftankin estimated error terms in the traditional version of the separatrix map and studied the multidimensional situation [Tre98, Tre02, Pif06].

It is becoming clear that the separatrix map is a powerful tool to analyze the dynamics in a neighborhood of homoclinic orbits to a normally hyperbolic invariant manifold and instabilities of Hamiltonian systems (see the survey [PT07] and the papers by Treschev [Tre04, Tre12]). For this reason in [KZZ15], using results of this paper, we perform an indepth analysis of the phenomenon of global instabilities in nearly integrable Hamiltonian systems. Usually this phenomenon, discovered by Arnold [Arn64], is called Arnold diffusion.

The purpose of this paper is to have detailed studies of the multidimensional separatrix map, proposed in [Tre02], and to compute more precise formulas for the reminder terms.

The main motivation for this work is to study certain stochastic diffusive behavior for nearly integrable deterministic systems. Such stochastic behavior was
conjectured by Chirikov [Chi79] in the 1970s. Combining results in [CK15] with the result in this paper in [KZZ15] we present a class of nearly integrable deterministic Hamiltonian systems, usually called the generalized Arnold example, such that deterministic evolution of the associated Hamiltonian flow of a certain random initial condition weakly converges to a diffusion process on the line. The set of initial conditions “producing” diffusive behavior is a fractal set in the phase space with a certain Bernoulli measure supported it. Examples of nearly integrable systems with stochastic diffusive behavior was constructed by Marco and Sauzin (see [MS04 Sau06]).

We emphasize that the previously known results on Arnold diffusion for the generalized Arnold example or, more generally, for or an apriori chaotic systems, or an apriori unstable systems, or an apriori stable systems (see [Ber08, BKZ11, BT91, BK93, CY04, CY09, CZ13, Che13, DdlLS00, DdlLS06, DH09, DdlLS13, FGKP11, GT08, GdlL06, GKL14, Kal03, KMV04, KL08a, KL08b, KS12, KZ12, KLS14, MS02, Mat91a, Mat91b, Mat93, Mat96, Mat03, Mat08, Moe96, Pif06, Tre04, Tre12]) establish existence of “special diffusing” orbits. Here we study deterministic evolution of a continuum of orbits and if orbits are randomly chosen with respect to a certain measure distributions converge to a diffusion process on the line.

We start with the set up. Consider a Hamiltonian system

\[ H_\varepsilon(I, \varphi, p, q, t) = H_0(I, p, q) + \varepsilon H_1(I, \varphi, p, q, t), \]

where \( I \in \mathbb{R}^n \) are actions, \( \varphi \in \mathbb{T}^n \) are angles and \( (p, q) \) belong to an open domain \( D \subset \mathbb{R}^2 \). Even if not written explicitly, the Hamiltonian \( H_1 \) may depend on the parameter \( \varepsilon \). Fix a bounded open set \( D \subset \mathbb{R}^n \).

We assume that for every fixed \( I \in D \), the Hamiltonian \( H_0(I, p, q) \) has a saddle at \( (p, q) = (0, 0) \) with two separatrix loops (see Fig. [1]). In [Tre02] it is assumed the following hypothesis.

\[ H_1 \]

The function \( H \) is \( C^5 \)-smooth in all arguments while \( H_0 \) is real-analytic in \( p, q \), and \( C^5 \)-smooth in \( I \).

We consider the alternative assumption.

\[ H_1' \]

The function \( H_0 \) is \( C^r \) for \( r \geq 50 \) and \( H \) is \( C^s \)-smooth in all arguments for \( s \geq 6 \) and \( r \geq 8s + 2 \).

That is, we admit lower regularity on \( H_0 \). On \( H_1 \) we assume one degree more of regularity. It is needed to have the better estimates of the separatrix map. For a first order analysis of the separatrix map, it would suffice \( s \geq 5 \) and \( r \geq 42 \), \( r \geq 8s + 2 \).
For all points $I^0 \in \mathcal{D}$, the function $H(I^0, p, q)$ has a non-degenerate saddle point at $(p, q) = (p_0, q_0)$ smoothly depending on $I$. For all $I^0 \in \mathcal{D}$, $(p_0, q_0)$ belongs to a connected component of the set $\{(p, q) : H_0(I^0, p, q) = H_0(I^0, p_0, q_0)\}$. Moreover, $(p_0, q_0)$ is the unique critical point of $H_0(I^0, p, q)$ in this component.

**Remark 1.1.** Using Prop. 1, [Tre02], if one assumes that the saddle is at a certain point $(p, q) = (p_0, q_0)$ which depends smoothly on $I$, then, one can perform a symplectic change of coordinates so that the critical point is at $(p, q) = (0, 0)$ for all $I \in \mathcal{D}$. After such a coordinate change $\mathcal{C}^r$ in $H_1$ is replaced by $\mathcal{C}^{r-2}$.

We denote the loops of the “eight” given by Hypothesis $\textbf{H2}$ by $\gamma^\pm(I^0)$. These loops have the natural orientation generated by the flow of the system (see Fig. 1). We can define an orientation in $\mathcal{D}$ by the coordinate system $(p,q)$.

For all $I^0 \in \mathcal{D}$, the natural orientation of $\gamma^\pm(I^0)$ coincides with the orientation of the domain, i.e. the motion of the separatrices is counterclockwise.

In [Tre02], Treschev defines the separatrix map for Hamiltonians satisfying this hypotheses and obtains a formula for this map with certain remainder terms. The goal of this paper is to refine Treschev formulas in several aspects.

Note that the formulas for the separatrix map present certain differences in what are called a non-resonant and a resonant regime (see below). Treschev provides global formulas for the separatrix map. Here we separate the two regimes and give more precise formulas. The refinements we do are the following.
• For the non-resonant regime we compute the separatrix map up to 2nd order in $\varepsilon$.

• For the resonant regime we give formulas in slow-fast variables and we show that the system possesses an almost first integral.

We obtain formulas for general Hamiltonians, but also pay attention to the particular case of the generalized Arnold example. Namely, consider models $H_0 + \varepsilon H_1$ of the form

$$
H_0(I, p, q) = \frac{I^2}{2} + \frac{p^2}{2} + (\cos q - 1),
$$

$$
H_1(I, \varphi, p, q, t) = (\cos q - 1)P(e^{i\varphi}, e^{it}),
$$

where $P(e^{i\varphi}, e^{it})$ is a real valued trigonometric polynomial. The unperturbed $H_0$ defines the product of a rotor and a pendulum. Our method also applies to perturbations of the form

$$
H_1(I, \varphi, p, q, t) := P(I, p, q, \exp(i\varphi), \exp(it)),
$$

where $P$ is a real trigonometric polynomial of order $N$ in $\varphi$ and $t$, i.e. it has the form:

$$
\sum_{|k_i| \leq N, i=1,2} h_{k_1, k_2}(p, q, I) \exp ik_1\varphi + k_2t,
$$

for some $N \in \mathbb{Z}_+$. For the classical Arnold example $P = \cos \varphi + \cos t$. The separatrix map for this particular case is computed in Section A.

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2 The separatrix map: Treschev’s results

We devote this section to define the separatrix map and state the results obtained in [Tre02].

We want to define a separatrix map for points whose $(p, q)$-components are “near” the unperturbed separatrices (see the shaded region on Fig. 2 below).
For the unperturbed system there is a normally hyperbolic invariant cylinder \( \Lambda = \{ p = q = 0 \} \) and for each \( I^0 \in \mathcal{D} \) we have invariant tori
\[
\Lambda(I^0) := \{ I = I^0, p = q = 0 \} = \Lambda \cap \{ I = I^0 \}.
\]
These tori are (partially) hyperbolic, i.e. there are expanding and contracting directions dominating the other directions. There exist two asymptotic manifolds,
\[
\hat{\Gamma}^\pm(I^0) \subset \{ (I^0, \varphi, p, q, t) : \varphi, t \in \mathbb{T}, H_0(I^0, p, q) = 0 \}
\]
\[
\hat{\Gamma}^\pm(I^0) = \{ I^0 \} \times \mathbb{T} \times \gamma^\pm \times \mathbb{T},
\]
where \( \gamma^\pm \) are the two separatrices in the \((p, q)\) plane. The manifolds \( \hat{\Gamma}^\pm(I^0) \) consist of unperturbed solutions that approach \( \Lambda(I^0) \).

Assume that \( \mathcal{D} \) is an open connected domain with compact closure \( \overline{\mathcal{D}} \). In what follows, we consider the dynamics of the non-perturbed system in a neighbourhood of the unstable and stable manifolds of the cylinder \( \Lambda \):
\[
\hat{\Gamma} = \bigcup_{I \in \overline{\mathcal{D}}} \left( \hat{\Gamma}^+(I) \cap \hat{\Gamma}^-(I) \right).
\]
This neighbourhood contains the most interesting part of the perturbed dynamics. It is convenient to pass to the time one map \( T_\varepsilon \): for any point \((I, \varphi, p, q)\)
\[
T_\varepsilon : (I, \varphi, p, q) \rightarrow (I(1), \varphi(1), p(1), q(1)),
\]
where \((I(t), \varphi(t), p(t), q(t))\) is the solution of the Hamiltonian system \( H_\varepsilon \) with initial conditions \((I(0), \varphi(0), p(0), q(0)) = (I, \varphi, p, q)\). The map \( T_0 \) has 1-dimensional hyperbolic tori \( L(I) = \pi(\Lambda(I)) \), where the map \( \pi : (I, \varphi, p, q, t) \rightarrow (I, \varphi, p, q) \) is the natural projection. Let \( \Sigma^\pm(I) = \pi(\hat{\Gamma}^\pm(I)) \) be the invariant manifolds (see Fig. 2). We define the separatrix map \( S\mathcal{M}_\varepsilon \) corresponding to \( T_\varepsilon \) in a neighbourhood of the set
\[
\Sigma = \bigcup_{I \in \overline{\mathcal{D}}} \left( \Sigma^+(I) \cap \Sigma^-(I) \right) = \pi(\hat{\Gamma}).
\]

Let \( U \) be a small neighbourhood of the set \( \bigcup_{I \in \overline{\mathcal{D}}} L(I) \) and let \( U \) be a neighbourhood of \( \Sigma \) (see Figure 2 where \( \Sigma \) is pictured in the case \( n = 0 \)). If \( U \) is sufficiently small, then \( U \setminus \overline{U} \) consists of two connected components \( U^+ \) and \( U^- \), and, thus, \( \Sigma^\pm \subset U^+ \cup U^- \).

Consider a point \( z \in U^+ \cup U^- \). Let \( k_1 = k_1(z) \) be the minimal positive integer such that \( T_\varepsilon^{k_1}(z) \notin U^+ \cup U^- \) and let \( k_2 = k_2(z) \) be the minimal positive integer such that \( k_2 > k_1 \) and \( T_\varepsilon^{k_2}(z) \in U^+ \cup U^- \). The trajectory \( T_\varepsilon^{k}(z) \) leaves \( U^+ \cup U^- \) at \( k = k_1 \), and it returns to \( U^+ \cup U^- \) at \( k = k_2 \). A point \( z \) is said to be good if \( k_2 < +\infty \) and \( T_\varepsilon^{k_2-1}(z) \in \mathcal{U} \). Setting
\[
\mathcal{U}_\varepsilon = \{ z \in U^+ \cup U^- : z \text{ is good} \}
\]
we obtain maps
\[ SM_\varepsilon(\cdot, k_2(\cdot) + k) : U_\varepsilon \to U^+ \cup U^- , \]
\[ SM_\varepsilon(z, k_2(z) + k) = T_\varepsilon^{k_2(z)+k}(z). \]
Here \( k \in \{0, 1, \ldots \} \) is a parameter, and it is assumed that
\[ T_\varepsilon^{k_2(z)+1}(z), \ldots, T_\varepsilon^{k_2(z)+k}(z) \in U^+ \cup U^- . \]
In [PT07] it is set \( t_+ = k_2(z)+k \). Since neither \( U^+ \) nor \( U^- \) serves as a fundamental domain, it is leaves a considerable freedom for \( k \). We would like to avoid this freedom. By analogy with the pendulum case (see Fig. 3) we let \( \Lambda^+ \) and \( \Lambda^- \) be hyperplanes in \( \mathbb{A} \times \mathbb{A} \times \mathbb{T} \ni (I, \varphi, p, q, t) \) whose projection onto the \( (p, q) \)-component are curves going from one of the boundaries \( |H_0| < c \) to another and transversal to the upper and lower separatrix loops. We denote by \( \Delta^\pm \) the subdomain of \( U_\varepsilon \) between the curves \( \Lambda^\pm \) and \( T_\varepsilon(\Lambda^\pm) \) (see Fig. 3). We choose such \( k \) above that \( T_\varepsilon^{k_2(z)+k}(z) \in \Delta^+ \cup \Delta^- \).

To provide formulas for the separatrix map we need to set up some notation. We define
\[ E(I) = H_0(I, 0, 0), \quad \nu(y) = \partial_I E(I) : D \to \mathbb{R}^n. \]
The Hamiltonian \( H_1(I, \varphi, 0, 0, t) \) has the Fourier expansion
\[ H_1(I, \varphi, 0, 0, t) = \sum_{k \in \mathbb{Z}^{n+1}} H_1^k(I) e^{2\pi ik \cdot (\varphi, t)}. \]
Figure 3: The fundamental domain

Let $\psi : \mathbb{R} \rightarrow [0, 1]$ be a $C^\infty$ function such that $\psi(r) = 0$ for any $|r| \geq 1$ and $\psi(r) = 1$ for any $|r| \leq 1/2$. We define

$$
\mathcal{H}_1(I, \phi, x) = \sum_{k \in \mathbb{Z}^n+1} \psi(k \cdot (I, 1)) H^k_1(I) e^{2\pi ik \cdot (\phi, x)}
$$

(3)

for some constant $\beta > 0$.

To obtain quantitative estimates, we follow [Tre02] and use skew norms, defined as follows. Let $K \subset \mathbb{R}^m$ be a compact set and $j \in \mathbb{N}$. Then, for functions $f \in C^j(\overline{D} \times K)$ we define

$$
\|f(I, z)\|^{(b)}_j = \max_{0 \leq l' + l'' \leq j} b^{l'} \left| \partial^{l' + l''} f \right|,
$$

where $l'' = l''_1 + \cdots + l''_m$. It is assumed that $f$ can take values in $\mathbb{R}^s$, where $s$ is an arbitrary positive integer. The norms $\| \cdot \|^{(b)}_j$ are anisotropic, and the variables $r$ play a special role in these norms because the additional factor $b$ corresponds to the derivatives with respect to $r$. Obviously, $\| \cdot \|^{(1)}_r$ is the usual $C^r$-norm. This norm is similar to the skew-symmetric norm introduced in [KZ12, section 7.2].

For a function $f \in C^r(\overline{D} \times K)$ and $g \in C^0(\overline{D} \times K)$ we say that

$$
f = O^{(b)}_0(g) \text{ if } \|f\|^{(b)}_r \leq C |g|^k,
$$

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where \( C \) does not depend on \( b \). For brevity we write
\[
\| \cdot \|_r = \| \cdot \|_{r,b}^{(s)}, \quad \mathcal{O}^{(b)} = \mathcal{O}^{(b)}_{1}, \quad \mathcal{O}^{k} = \mathcal{O}^{(b)}_{k}.
\]

First, we state the result obtained in \([\text{Tre02}]\). He sets \( b = \varepsilon^{1/4} \).

**Theorem 2.1** ([\text{Tre02}]). Let conditions \([\text{H1-H3}]\) hold. Then, there exist \( C \) smooth functions
\[
\lambda, \kappa^\pm, \mu^\pm : \mathcal{D} \to \mathbb{R}, \quad \Theta^\pm : \mathcal{D} \times \mathbb{T}^{n+1} \to \mathbb{R},
\]
and canonical coordinates \((\eta, \xi, h, \tau)\) such that the following conditions hold.

- \( \omega = d\eta \wedge d\xi + dh \wedge d\tau \).
- \( \eta = I + \mathcal{O}(\varepsilon^{1/4})(\varepsilon^{3/4}, H_{0} - E(I)), \xi + \nu(\eta) = \varphi + f, h = H_{0} + \mathcal{O}(\varepsilon^{1/4})(\varepsilon^{3/4}, H_{0} - E(I)), \)
  where \( f \) denotes a function depending only on \((I, p, q, \varepsilon)\) satisfying
  \[
f(I, 0, 0, 0) = 0^{+}.
\]
- Define
  \[
  w_0^\sigma = h^* - E(\eta^*) - \varepsilon H_{1}(\eta^*, \xi + \nu(\eta^*)\tau + \mu^\sigma(\eta^*)).
  \]
  For any \( \sigma = \{+, -\} \) and \((\eta^*, \xi, h^*, \tau)\) such that
  \[
  |\tau| < c^{-1}, \quad c < |w_0^\sigma| < \varepsilon^{7/8},
  \]
  the separatrix map \((\eta^*, \xi^*, h^*, \tau^*) = S\mathcal{M}(\eta, \xi, h, \tau)\) is defined implicitly as follows
  \[
  \eta^* = \eta - \varepsilon \partial_\xi \Theta(\eta^*, \xi, \tau) - \frac{\partial_\xi w_0^\sigma}{\lambda} \log \left| \frac{\kappa^\sigma w_0^\sigma}{\lambda} \right| + \mathcal{O}_2
  \]
  \[
  \xi^* = \xi + \mu^\sigma + \varepsilon \partial_\eta \Theta(\eta^*, \xi, \tau) + \frac{\partial_\eta w_0^\sigma}{\lambda} \log \left| \frac{\kappa^\sigma w_0^\sigma}{\lambda} \right| + \mathcal{O}_1
  \]
  \[
  h^* = h - \varepsilon \partial_h \Theta(\eta^*, \xi, \tau) - \frac{\partial_h w_0^\sigma}{\lambda} \log \left| \frac{\kappa^\sigma w_0^\sigma}{\lambda} \right| + \mathcal{O}_2
  \]
  \[
  \tau^* = \tau + \bar{\tau} + \frac{\partial_h w_0^\sigma}{\lambda} \log \left| \frac{\kappa^\sigma w_0^\sigma}{\lambda} \right| + \mathcal{O}_1
  \]

- \( \sigma^* = \sigma \text{ sgn} w_0^\sigma \),

\( \lambda, \kappa^\pm \) and \( \mu^\pm \) are functions of \( \eta^*, \bar{\tau} \) is an integer such that
\[
|\tau + \bar{\tau} + \frac{\partial_h w_0^\sigma}{\lambda} \log \left| \frac{\kappa^\sigma w_0^\sigma}{\lambda} \right| | < c^{-1}
\]
and \( \mathcal{O}_1 = \mathcal{O}(\varepsilon^{1/4})(\varepsilon^{7/8}) \log^2 \varepsilon, \mathcal{O}_2 = \mathcal{O}(\varepsilon^{1/4})(\varepsilon^{5/4}) \log^2 \varepsilon. \)

\(^{1}\)One can show that \( f = \mathcal{O}(\varepsilon^{1/4})(w_0^\sigma + \varepsilon) \).
Remark. Recall that the Hamiltonian $H_\varepsilon$ has a normally hyperbolic invariant cylinder $\Lambda_\varepsilon$ close to $\Lambda_0 = \{p = q = 0\}$. Dynamics of $H_\varepsilon$ restricted to $\Lambda_\varepsilon$ is often called inner dynamics. Dynamics of orbits belonging to the intersection of $W^s(\Lambda_\varepsilon)$ and $W^u(\Lambda_\varepsilon)$ is called outer dynamics and thoroughly studied in [DdlLS06, DdlLS08].

It is helpful to have Figure 3 in mind.

An heuristic explanation of the separatrix map is the following: every orbit starting in the fundamental region $\Delta$ and sufficiently close to the stable manifold $W^s(\Lambda_\varepsilon)$ has three regimes:

1. approaching the cylinder $\Lambda_\varepsilon$
2. evolution near $\Lambda_\varepsilon$
3. departing away from $\Lambda_\varepsilon$.

In each of these there regimes we have:

1. Straightening invariant manifolds trivialized the first regime.
2. The Hamiltonian $w$ describes the evolution in the second regime with an error.
3. The splitting potential describes the third regime with an error.

The regime 2) is reasonable to call the inner dynamics as orbits near the cylinder $\Lambda_\varepsilon$ can be shadowed by orbits inside of the cylinder.

The regime 1)+3) is reasonable to call the outer dynamics. The outer dynamics to the leading order is well described by the splitting potential.

Look now at the formula (7). The separatrix map has contribution from the $w_0$, which is the inner dynamics, and from $\Theta$, which is the splitting potential.

2.0.1 Formulas for functions for $\lambda, \kappa^\pm, \mu^\pm$

The functions $\lambda > 0$, $\kappa^\pm > 0$ and $\mu^\pm \in \mathbb{R}$ are defined by the unperturbed Hamiltonian $H_0$ as follows. Hypothesis H2 implies that both eigenvalues of the matrix

$$\Lambda(I) = \begin{pmatrix} -\partial_{pq}H_0(I,0,0) & -\partial_{qq}H_0(I,0,0) \\ \partial_{pp}H_0(I,0,0) & \partial_{pq}H_0(I,0,0) \end{pmatrix}$$

are real and the trace of this matrix is equal to 0 for all $I$. We denote by $\lambda(I)$ the positive eigenvalue of this matrix.

We denote by $\gamma^\pm(I,\cdot) : \mathbb{R} \rightarrow \mathbb{R}^2$ the time parameterization of the upper and lower separatrices of $H_0$ in the level of energy $H_0(I,p,q) = H_0(I,0,0)$. We denote by $a^\pm(I)$ the left eigenvectors of the matrix $\Lambda(I)$, that is, $a_+\Lambda = \lambda a_+$ and
\( a_- \Lambda = -\lambda a_-, \) such that the \( 2 \times 2 \) matrix with \( a_\pm \) as columns has determinant equal to one.

Then, we define

\[
\mu^\pm(I) = \int_{-\infty}^{+\infty} (-\nu(I) + \partial_t H_0(I, \gamma^\pm(I, t))) \, dt
\]

(10)

\[
\kappa^\pm(I) = \lim_{t \to +\infty} \left[ \langle a_+(I), \gamma^\pm(I, -t) \rangle \langle a_-(I), \gamma^\pm(I, t) \rangle e^{2\lambda(I)t} \right].
\]

(11)

To define the functions \( \Theta^\sigma \), we have to introduce some notation. We define

\[
\partial = \nu(I) \partial_\varphi + \partial_\tau.
\]

(12)

Fix \( \beta > 0 \). The inverse operator \( \partial^{-1} \) is defined on the “resonant” space

\[
\text{Res} = \{ f : \mathcal{D} \times \mathbb{T}^{n+1} : f^{k,k_0} = 0 \text{ if } |\langle k, \nu(I) \rangle + k_0| \leq \beta/2 \},
\]

where \( f^{k,k_0} \) are the Fourier coefficients of \( f \). Then, for \( f \in \text{Res} \), we have that \( \partial^{-1} f \) is well defined and satisfies

\[
\partial^{-1} f = O^*(\beta^{-1} f).
\]

As we have already mentioned, Treschev in [Tre02] chooses \( \beta = \varepsilon^{1/4}\).

We define

\[
\vartheta^\sigma(I, \varphi, \varpi) = -\partial^{-1} \left[ H_1(I, \varphi, 0, 0, \varpi) - \overline{H}_1(I, \varphi, \varpi) \right].
\]

(13)

Any solution of the unperturbed system lying on \( \hat{\Gamma}^\pm(I) \) has the form

\[
(I, \varphi, p, q)(t) = \Gamma^\sigma(I, \varphi, \varpi + t)
\]

with

\[
\Gamma^\sigma(I, \varphi, \varpi) = (I, \varphi + \nu(I)\varpi + \chi^\pm(I, \varpi, \varpi))
\]

where \( \chi^\pm(I, \varpi) \) are solutions of

\[
\dot{\chi}^\pm(I, t) = -\nu(I) + \partial_t H_0(I, \gamma^\pm(I, t)), \quad \lim_{t \to -\infty} \chi^\pm(I, t) = 0.
\]

By the definition of \( \mu^\sigma \) in [10],

\[
\mu^\pm(I) = \lim_{t \to +\infty} \chi^\pm(I, t).
\]

We define

\[
H^\sigma_\pm(I, \varphi, \varpi, t) = H_1(\Gamma^\sigma(I, \varphi, t - \varpi)) - H_1(I, \varphi + \nu t + \chi^\sigma(I, \pm\infty), 0, 0, t - \varpi).
\]
Note that $H^\sigma_{\pm}(I, \varphi, \tau, t)$ tend to zero exponentially as $t \to \pm \infty$.

Then,

$$\Theta^\sigma(I, \varphi, \tau) = \vartheta^\sigma(I, \varphi, -\tau) - \vartheta^\sigma(I, \varphi + \mu^\sigma, -\tau) - \int_{-\infty}^{0} H^\sigma_{-}(I, \varphi, \tau, t) \, dt - \int_{0}^{\infty} H^\sigma_{+}(I, \varphi, \tau, t) \, dt. \tag{14}$$

In [Tre02] these functions are called splitting potentials. The integral term is the classical Melnikov potential (also often called the Poincaré function).

As we have explained the purpose of this paper is to refine the formulas given by Theorem 2.1.

3 The finite harmonics setting

To get more precise formulas for the separatrix map in the finite harmonics setting, we consider different regions where it is defined. These regions overlap each other.

First fix some notation. Take a function $f : T^n \times \mathbb{R}^n \times \mathbb{R}^2 \times T \to \mathbb{R}$ with Fourier series

$$f = \sum_{k \in \mathbb{Z}^{n+1}} f^k(I, p, q) e^{2\pi i k \cdot (\varphi, t)}.$$

Define $N$ as

$$N(f) = \{ k \in \mathbb{Z}^{n+1} : f^k \neq 0 \}$$

and

$$N^{(2)}(f) = \{ k \in \mathbb{Z}^{n+1} : k = k_1 + k_2, k_1, k_2 \in N(f) \}.$$

Consider the non-resonant region (NRR), which stays away from the resonances created by the harmonics in $N^{(2)}(H_1)$.

Define

$$NRR_\beta = \{ I : \forall k \in N^{(2)}(H_1), |k \cdot (\nu(I), 1)| \geq \beta \} \tag{15}$$

for a fixed parameter $\beta$. The complement of the non-resonant zone is build up by the different resonant zones associated to the harmonics in $N^{(2)}(H_1)$. Fix $k \in N^{(2)}(H_1)$, then we define the resonant zone

$$Res^k_\beta = \{ I : |k \cdot (\nu(I), 1)| \leq \beta \}. \tag{16}$$

The parameter $\beta$ in both regions will be chosen differently, so that the different zones overlap.

We abuse notation and we redefine the norms in (4) as

$$\| \cdot \|_r^* = \| \cdot \|^{(\beta)}_r, \quad \mathcal{O}^{(b)} = \mathcal{O}^{(b)}_1, \quad C^*_k = C^{(\beta)}_k.$$

for fixed $\beta > 0$. Now we can give formulas for the separatrix map in both regions.
3.1 The separatrix map in the non-resonant regime

The main result of this section is Theorem 3.1 which gives refined formulas for the separatrix map in the non-resonant zone (see [15]). To state it we need to define an auxiliary function \( w \). This \( w \) is a slight modification of the function \( w_0 \) given in [5].

Consider a function \( g(\eta, r) \). It is obtained in Section 4.1 by applying Moser’s normal form to \( H_0 \). This \( g \) satisfies \( g(\eta, r) = \lambda(\eta) r + \mathcal{O}(r^2) \), where \( \lambda \) is the positive eigenvalue of the matrix [9]. Therefore, \( g \) is invertible with respect to the second variable for small \( r \). Somewhat abusing notation, call \( g^{-1} \) the inverse of \( g \) with respect to the second variable. Then, we define the function \( w \) by

\[
w(\eta, h) = g^{-1}_{\epsilon}(\eta, h - E(\eta)).
\]

Theorem 3.1. Fix \( \beta > 0 \) and \( 1 \geq a > 0 \). Let conditions [H1’,H2,H3] hold for some \( r \geq 50, s \geq 6, r \geq 8s + 2 \). Then for \( \epsilon \) sufficiently small there exist \( c > 0 \) independent of \( \epsilon \) and a \( C^{s-4} \) canonical coordinates \( (\eta, \xi, h, \tau) \) such that in the non-resonant zone \( NRR_\beta \) the following conditions hold:

- the canonical form \( \omega = dh \wedge d\xi + dh \wedge d\sigma \);
- \( \eta = I + \mathcal{O}_1(\epsilon) + \mathcal{O}_2^*(H_0 - E(I)), \xi + \nu(\eta) \tau = \varphi + f, h = H_0 + \mathcal{O}_1^*(\epsilon) + \mathcal{O}_2^*(H_0 - E(I)), \) where \( f \) denotes a function depending only on \( (I, p, q, \epsilon) \) and such that \( f = \mathcal{O}(w_0^2 + \epsilon), f(I, 0, 0, 0) = 0 \).
- In these coordinates \( \mathcal{S}M_c \) has the following form. For any \( \sigma \in \{-, +\} \) and \( (\eta^*, h^*) \) such that

\[
c^{-1} \epsilon^{1+a} < |w(\eta^*, h^*)| < c\epsilon, \quad |\sigma| < c^{-1}, \quad c < |w(\eta^*, h^*)| e^{\lambda(\eta^*)} < c^{-1},
\]

the separatrix map \( (\eta^*, \xi^*, h^*, \tau^*) = \mathcal{S}M_c(\eta, \xi, h, \tau) \) is defined implicitly as follows

\[
\eta^* = \eta - \epsilon M_1^\sigma, \eta + \epsilon^2 M_2^\sigma, \eta + \mathcal{O}_3^*(\epsilon) |\log \epsilon| \xi^* = \xi + \partial_1 \Phi^\sigma(\eta, w(\eta^*, h^*)) + \partial_2 w(\eta^*, h^*) |\log |w(\eta^*, h^*)|| + \partial_2 \Phi^\sigma(\eta^*, w(\eta^*, h^*)) + \mathcal{O}_1^*(\epsilon) |\log \epsilon| h^* = h - \epsilon M_1^\tau, \tau + \epsilon^2 M_2^\tau, \tau + \mathcal{O}_3^*(\epsilon) \tau^* = \tau + \tilde{l} + \partial_2 h w(\eta^*, h^*) |\log |w(\eta^*, h^*)|| + \partial_2 \Phi^\sigma(\eta^*, w(\eta^*, h^*)) + \mathcal{O}_1^*(\epsilon) |\log \epsilon|,
\]

where \( w \) is the function defined in [17], \( M_1^\sigma \) and \( \Phi^\pm \) are some \( C^{s-4} \) functions and \( \tilde{l} \) is an integer satisfying [18]. The functions \( M_1^\sigma \) are evaluated at \( (\eta^*, \xi, h^*, \tau) \).

\(^2\)The subindex is to emphasize that the inverse is performed with respect to the variable \( r \).
This theorem is proven in Section 7.

**Remark 3.2.** The change of coordinates in the above Theorem is ε-close (in the $C^2$-norm) to the system of coordinates obtained in Theorem 2.1.

The function $M^*_i$ and $\Phi^\pm$ are defined in Lemmas 6.1 and 7.2 respectively.

The functions $\Phi^\sigma$ are the generalizations of the functions $\mu^\sigma$ and $\kappa^\sigma$. Indeed, they satisfy

$$\partial_\eta \Phi^\sigma (\eta, r) = \mu^\sigma (\eta) + O_2^\sigma (r) \quad \text{and} \quad e^{\partial_r \Phi^\sigma (\eta, r)} = \kappa^\sigma (\eta) + O_2^\sigma (r).$$

Moreover, the functions $M^\sigma_{i,j}$ satisfy

$$M^\sigma_{1,1} = \partial_\xi \Theta^\sigma + O_2^\sigma (w), \quad M^\sigma_{1,2} = \partial_\tau \Theta^\sigma + O_2^\sigma (w),$$

where $\Theta^\sigma$ is the (Melnikov) split potential given in (14) (see Theorem 2.1).

**Remark 3.3.** We compare this Theorem with Theorem 2.1 and the Remark afterward. Notice that the inner dynamics in non-resonant zones can be made integrable and essentially decoupled from the outer dynamics. This is reflected in the fact that the $\eta$ and the $h$-components have no contribution from $w$ as the inner dynamics is integrable.

The splitting potential contributions and the integrable contributions to rotation of the angular components are essentially the same.

Consider the following useful example:

$$H_\varepsilon (I, \varphi, p, q, t) = \frac{I^2}{2} + \frac{p^2}{2} + (\cos q - 1) + \varepsilon H_1 (I, \varphi, t).$$

Notice that the perturbation does not affect the pendulum. As the result the perturbed system is the product of the pendulum and a perturbed rotor. Notice that $\overline{H}_1 (I, \varphi, t) = H_1 (I, \varphi, t)$ and it non-vanishing. Moreover, the split potential $\Theta^\sigma$ vanishes. Since $w^\sigma_0$, defined in (5), measures the duration time $\tilde{t}$ of the separatrix map $S\mathcal{M}_\varepsilon$ (see (6)), it has to be independent of angles of the rotor and time. This implies that in (5) after substitution of $I, p, q, \varphi$ there is a cancellation of $\varepsilon H_1$ with $\varepsilon$-terms coming from $(I - \eta)$ and $(H_0 - h)$. This cancellation is not so easy to see.

### 3.2 The separatrix map in the resonant zones

To give refined formulas in the resonant zones (see (16)) we restrict to Hamiltonian systems of two and a half degrees of freedom. Namely, $I$ and $\varphi$ are one dimensional.
We consider the resonance \((\nu(I), 1) \cdot (k_0, k_1) = 0\), \((k_0, k_1) \in N^{(2)}(H_1) \subset \mathbb{Z}^2\) and we assume that it is located at \(I = 0\). Let \(A\) be the variable conjugate to time. Perform a change to slow-fast variables. Namely, consider the following

\[
(J, \theta, D, t) = \left( \frac{I}{k_0}, k_0 \varphi + k_1 t, A - \frac{k_1}{k_0} I, t \right)
\]

which is symplectic. We obtain the following Hamiltonian:

\[
\tilde{H}(J, \theta, p, q, t) = \tilde{H}_1 \left( k_0 I, k_0^{-1}(\theta - k_1 t), p, q, t \right).
\]

We consider a slight modification of the functions \(w_0^\sigma\) in (5),

\[
w_0^\sigma = h^* - E(\eta^*) - \varepsilon H_1(\eta^*, \xi + \nu(\eta^*)(\tau + \bar{t}) + \mu^*(\eta^*)).
\]

**Theorem 3.4.** Fix \(\beta > 0\) and \(1 \geq a > 0\). Let conditions [H1’,H2,H3] hold for some \(r \geq 50\), \(s \geq 6\), \(r \geq 8s + 2\). Then for \(\varepsilon\) sufficiently small there exist \(c > 0\) independent of \(\varepsilon\) and a \(C^{s-4}\) canonical coordinates \((\eta, \xi, h, \tau)\) such that in the resonant zone \(\text{Res}^\beta_0\) the following conditions hold:

- the canonical form \(\omega = dh \wedge d\xi + dh \wedge d\tau\);

- \(\eta = I + \mathcal{O}_1(\varepsilon, H_0 - E(I)), \xi + \nu(\eta) = \varphi + f, h = H_0 + \mathcal{O}_1(\varepsilon, H_0 - E(I)),\)

  where \(f\) denotes a function depending only on \((I, p, q, \varepsilon)\) and such that \(f(I, 0, 0, 0) = 0\) and \(f = \mathcal{O}(w_0^\sigma + \varepsilon)\).

- In these coordinates \(SM\) has the following form. For any \(\sigma \in \{-, +\}\) and \((\eta^*, h^*)\) such that

  \[
c^{-1} \varepsilon^{1+a} < |w_0(\eta^*, h^*)| < c \varepsilon, \quad |\tau| < c^{-1}, \quad c < |w_0(\eta^*, h^*)| e^{\lambda(\eta^*)\bar{t}} < c^{-1},
\]

the separatrix map \((\eta^*, \xi^*, h^*, \tau^*) = SM(\eta, \xi, h, \tau)\) is defined implicitly as follows

\[
\eta^* = \eta - \frac{\partial w_0^\sigma}{\lambda} \log |\kappa^\sigma w_0^\sigma| + B^{\eta, \sigma}(\eta, \xi, \tau, \bar{t}) + \mathcal{O}^*(\varepsilon)
\]

\[
\xi^* = \xi + \mu^* + \frac{\partial \nu w_0^\sigma}{\lambda} \log |\kappa^\sigma w_0^\sigma| + B^{\xi, \sigma}(\eta, \xi, \tau, \bar{t}) + \mathcal{O}^*(\varepsilon)
\]

\[
h^* = h - \frac{\partial w_0^\sigma}{\lambda} \log |\kappa^\sigma w_0^\sigma| + B^{\eta, \sigma}(\eta, \xi, \tau, \bar{t}) + \mathcal{O}^*(\varepsilon)
\]

\[
\tau^* = \tau + \bar{t} + \frac{\partial h w_0^\sigma}{\lambda} \log |\kappa^\sigma w_0^\sigma| + B^{\eta, \sigma}(\eta, \xi, \tau, \bar{t}) + \mathcal{O}^*(\varepsilon)
\]

\[
\sigma^* = \sigma \ \text{sgn} w_0^\sigma,
\]

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where $w_0^\sigma$ is defined in (19) and the functions $B^\sigma$ are defined in Lemma 8.1 and satisfy
\[ B^{\eta,\sigma}, B^{\xi,\sigma} = O^*(\varepsilon \log \varepsilon), \quad B^{\tau,\sigma} = O^*(\varepsilon \log^2 \varepsilon). \]

Moreover, the evolution of the variable $D$, conjugate to the time $t$, satisfies,
\[ D^* = D + B^D,\sigma(\eta, \xi, \tau, t) + O^*(\varepsilon^{5/3}), \]
for certain function $B^D,\sigma$ satisfying $B^D,\sigma = O^*(\varepsilon \log \varepsilon)$.

This theorem is proven in Section 8.

**Remark 3.5.** We compare this Theorem with Theorem 2.1 and the Remark afterward. Notice that both the inner and the outer dynamics are nontrivial and $\varepsilon$-coupled due to resonant terms. Since transition time of the separatrix map is $O(\log \varepsilon)$ this gives a term of order $O(\varepsilon \log \varepsilon)$ in the action components, which dominates contribution of the Melnikov function.

Note that in the aforementioned $\varepsilon$-coupling between the inner and the outer dynamics vanishes in the generalized Arnold example and the Melnikov function gives dominant contribution (see Theorem 8.3).

### 3.3 Main steps of the proof and structure of the paper

In this section we sketch the derivation of the separatrix map done by Treschev [Tre02]. At each step we refer to the section where it is done in this paper. Recall that the separatrix map $\mathcal{S}\mathcal{M}_\varepsilon$ is a return map to a certain fundamental region $U^+ \cup U^-$ (see Fig. 2). Its computation consists of six steps:

- Moser’s normal form for the unperturbed pendulum near the separatrices (see the blue and the yellow part on Figure 4, left and Section 4).
- Moser’s normal form for the perturbed pendulum in the colored part of Figure 4, left (see Section 4).
- The transition map from one yellow region to another one in the variables given by the Moser’s normal form of the unperturbed system (see Section 5).
- The regions $U^+$ (resp. $U^-$) on Figure 4, left are overlapping regions in the original coordinates (see Figure 4, right).
- Compute the gluing maps (Section 6).
- Computation of the composition of the transition map in Moser’s normal form variables with the gluing map (Section 7).
4 Normal forms

4.1 Moser normal form close to the torus

We start, as in [Tre02], by performing the classical Moser normal form [Mos56] to the unperturbed Hamiltonian \( H_0 \). To obtain a finitely smooth version of this result we apply a result [BdlLW96].

**Lemma 4.1.** Let \( H_0 \) is \( C^r \) satisfying \( H1' \) and \( H2 \). For \( I \in \mathcal{D} \) and \( (p,q) \) close to \( (0,0) \) there exists a system of coordinates \((I, s, x, y)\) such that

\begin{itemize}
  \item the change \( \mathcal{F}_0 : (I, s, x, y) \rightarrow (I, \varphi, p, q) \) \( C^\ell \) smooth and symplectic with \( \ell \geq (r - 4)/5 \).
  \item in the variables \( H_0 \circ \mathcal{F}_0 = \mathcal{H}_0(I, xy) \) is \( C^{\ell+1} \) smooth. We write it as
    \[ \mathcal{H}_0(I, xy) = E(I) + g(I, xy) \]
    where \( g(I, xy) = \lambda(I)xy + O_2(xy) \).
\end{itemize}

**Proof.** In Lemma 1, [Tre02] the assumption is that \( H_0 \) is analytic in \( p, q \). To relax this assumption we use Theorem 1.1 [BdlLW96]. Recall that by Remark 1.1 we can assume that the saddle is at \( (p, q) = (0, 0) \) for all \( I \) at the expense of losing two derivatives.

Let \( f \) be a \( C^{r-2} \) diffeomorphism with the fixed point at the origin \( f(0) = 0 \). Let \( N \) be a symplectic polynomial map such that \( N(0) = 0 \) and the \( k \)-jet of \( f \) and \( N \) coincide at 0, i.e. \( D^j f(0) = D^j N(0), \ j = 0, \ldots, k \). Let \( 1 \leq \ell < kA - B \) and \( r - 2 > 2k + 4 \) for some integer \( \ell \). Then \( N \) and \( f \) are \( C^\ell \) conjugate, i.e. there is a \( C^\ell \) diffeomorphism \( h \) such that \( h^{-1} \circ f \circ h = N \) near the origin, \( h(0) = 0 \).
In our case for each fixed $I$ we have a 2-dimensional symplectic map $f$ with a saddle fixed point. By the remark after Theorem 1.1 of [BdlLW96], $A = 1/4$, $B = 1 - 2A = 1/2$. Thus, $1 \leq \ell < \frac{k-2}{4} < \frac{r-10}{8}$.

We consider the expansion in $\varepsilon$ of the perturbation of the Hamiltonian (1), namely,

$$H_1 = H_{11} + \varepsilon H_{12} + \mathcal{O}(\varepsilon^2)$$

and define

$$\mathcal{H}_\varepsilon = \mathcal{H}_0 + \varepsilon \mathcal{H}_1 + \varepsilon^2 \mathcal{H}_2 + \mathcal{O}(\varepsilon^3) \quad (20)$$

with

$$\mathcal{H}_1(I, s, x, y, t) = H_{11} \circ \mathcal{F}_0(I, s, x, y, t), \quad \mathcal{H}_2(I, s, x, y, t) = H_{12} \circ \mathcal{F}_0(I, s, x, y, t).$$

Since we assume that $r/8 - 5/4 \geq s$ we have that $\mathcal{H}_0$ is $C^{s+1}$, $\mathcal{H}_1$ is $C^s$ and $\mathcal{H}_2$ is $C^{s-1}$. Moreover $\mathcal{O}(\varepsilon^3) = \mathcal{O}_{C^{s-2}}(\varepsilon^3)$.

4.2 Normal forms near the separatrices

We need to extend Moser normal form to the region where $|xy|$ is small (see the shaded/colored region Fig. 4 on the right/left respectively). This normal form applies to both the non-resonant and resonant regimes. Since we want to make a second order analysis of the separatrix map we need a more precise normal form than in [Tre02]. In this normal form there are two sources of error.

- *Expansion in the small parameter $\varepsilon$:* we need to do two steps of normal form instead of one to reduce the size of the remainders.

- *Powers of $xy$.* Treschev only performs normal form to remove the terms in the perturbation which are independent of the product $xy$. He takes

$$\varepsilon^{5/4} |\log \varepsilon| \lesssim |xy| \lesssim \varepsilon^{7/8}.$$  

We want to remove terms up to the first order in $xy$. Assume that

$$\varepsilon^{1+a} \lesssim |xy| \lesssim \varepsilon \quad \text{with } 1 \geq a \geq 0. \quad (21)$$

We perform the change of coordinates by the Lie Method. First, we proceed formally and then we compute the estimates. We consider the expansion of the Hamiltonian $\mathcal{H}$ given in (20) and of a Hamiltonian of the form $\varepsilon W = \varepsilon W_0 + \varepsilon^2 W_1$. 


Call $\Phi$ the time-one map associated to the flow of $\varepsilon W$. This change is symplectic. Moreover,

$$H_\varepsilon \circ \Phi = H_\varepsilon + \varepsilon \{H, W\} + \varepsilon^2 \{\{H, W\}, W\} + O(\varepsilon^3)$$

$$= H_0 + \varepsilon (H_1 + \{H_0, W_0\}) + \varepsilon^2 (\{H_0, W_1\} + \{H_1, W_0\} + \{\{H_0, W_0\}, W_0\} + H_2) + O(\varepsilon^3).$$

Now we look for suitable $W_0$ and $W_1$.

First, compute $W_0$ and then $W_1$. To compute $W_0$, split the Hamiltonian $H_1$, defined in (20), in the following way

$$H_1(I, s, x, y, t) = \overline{H}_1(I, s, t) + H^{(1)}(I, s, t) + H^{(2)}(I, s, y, t) + H^{(3)}(I, s, x, t)$$

$$+ xy \left( \overline{H}_2(I, s, t) + H^{(4)}(I, s, t) + H^{(5)}(I, s, y, t) + H^{(6)}(I, s, y, t) \right)$$

$$+ O_2(xy)$$

where

$$\overline{H}_2(I, s, t) = \sum_{k \in \mathbb{Z}^{n+1}} \psi \left( k \cdot (\nu(I), 1) \beta \right) \partial_{xy} H^k_1(I, 0, 0) e^{2\pi ik \cdot (s, t)},$$

(22)

$\psi$ is the bump function introduced before (3) and

$$H^{(1)}_1(I, s, t) = H_1(I, s, 0, 0, t) - \overline{H}_1(I, s, t)$$

$$H^{(2)}_1(I, s, t) = H_1(I, s, x, 0, t) - H_1(I, s, 0, 0, t)$$

$$H^{(3)}_1(I, s, t) = H_1(I, s, 0, y, t) - H_1(I, s, 0, 0, t)$$

$$H^{(4)}_1(I, s, t) = \partial_{xy} H_1(I, s, 0, 0, t) - \overline{H}_2(I, s, t)$$

$$H^{(5)}_1(I, s, t) = \partial_{xy} H_1(I, s, x, 0, t) - \partial_{xy} H_1(I, s, 0, 0, t)$$

$$H^{(6)}_1(I, s, t) = \partial_{xy} H_1(I, s, 0, y, t) - \partial_{xy} H_1(I, s, 0, 0, t)$$

where $\overline{H}_1$ is defined in (3). The functions $\overline{H}_1$, $H^{(j)}_1$, $j = 1, 2, 3$ are $C^s$ whereas the functions $H^{(j)}_1$, $j = 4, 5, 6$ are $C^{s-2}$.

The next lemma contains the first step of the normal form. In the next two lemmas we denote by $\{\cdot, \cdot\}_{(x,y)}$, the Poisson bracket with respect to the conjugate variables ($x, y$).

**Lemma 4.2.** There exists a $C^{s-2}$ smooth solution $W_0(I, s, x, y, t)$ given as the sum

$$\sum_{j=1}^{6} W^{(j)}_0(I, s, x, y, t)$$

with $W^{(j)}_0(I, s, 0, t) = 0$ for $j = 2, \ldots, 6$ of the equation

$$(\nu(I) + \partial_I g(I, xy)) \partial_s W_0 + \partial_s W_0 + \{g(I, xy), W_0\}_{(x,y)} + \sum_{j=1}^{3} H^{(j)}_1 + xy \sum_{j=4}^{6} H^{(j)}_1 = 0.$$
The functions $W_0$ satisfies
\[ W_0 = \mathcal{O}^*(\beta^{-1}), \]
where $\mathcal{O}^*$ is defined in \([12]\).

**Proof.** We take $W_0 = \sum_{j=1}^{6} W_0^{(j)}$ and solve the equations
\[
(\nu(I) + \partial_I g(I, xy)) \partial_s W_j^{(j)} + \partial_t W_j^{(j)} + \{g(I, xy), W_0^{(j)}\}_{(x,y)} + \mathcal{H}_1^{(j)} = 0, \quad j = 1, 2, 3
\]
\[
(\nu(I) + \partial_I g(I, xy)) \partial_s W_j^{(j)} + \partial_t W_j^{(j)} + \{g(I, xy), W_0^{(j)}\}_{(x,y)} = \mathcal{H}_1^{(j)} = 0, \quad j = 4, 5, 6.
\]
Each equation is solved as follows. For the first and fourth ones, we just have
\[
(\nu(I) + \partial_I g(I, xy)) \partial_s W_1^{(1)} + \partial_t W_1^{(1)} + \mathcal{H}_1^{(1)} = 0
\]
\[
(\nu(I) + \partial_I g(I, xy)) \partial_s W_4^{(4)} + \partial_t W_4^{(4)} + \mathcal{H}_1^{(4)} = 0.
\]
Thus, we invert the operator
\[ \tilde{\partial} := (\nu(I) + \partial_I g(I, xy)) \partial_s + \partial_t \]
by using the Fourier expansion and inverting for each Fourier coefficient. Note that this operator and the operator $\partial$ in \([12]\) satisfy $\tilde{\partial} - \partial = \mathcal{O}(xy)$. Moreover, recall that $\mathcal{H}_0$ is $C^{s+1}$ and therefore so is $g$. Then, $W_0^{1,4}$ are $C^{s-2}$ and satisfy
\[ W_0^1 = \partial^{-1} \mathcal{H}_1^{(1)} = \mathcal{O}^*(\beta^{-1}) \quad \text{and} \quad W_0^4 = \mathcal{O}^*(xy\beta^{-1}). \]

For the others, we use the characteristics method to obtain
\[ W_0^2 = -\int_{-\infty}^{0} \mathcal{H}_1^{(2)}(I, s + (\nu(I) + \partial_I g(I, xy))t', ye^{\partial_2 g(I, xy)t'}, t + t') dt' \]
\[ W_0^3 = -\int_{0}^{+\infty} \mathcal{H}_1^{(3)}(y, s + (\nu(I) + \partial_I g(I, xy))t', xe^{-\partial_2 g(I, xy)t'}, t + t') dt' \]
\[ W_0^5 = -xy \int_{-\infty}^{0} \mathcal{H}_1^{(5)}(I, s + (\nu(I) + \partial_I g(I, xy))t', ye^{\partial_2 g(I, xy)t'}, t + t') dt' \]
\[ W_0^6 = xy \int_{0}^{+\infty} \mathcal{H}_1^{(6)}(I, s + (\nu(I) + \partial_I g(I, xy))t', xe^{-\partial_2 g(I, xy)t'}, t + t') dt'. \]
Thus, they are all $C^{s-2}$.

Now we solve the second order equation. Define
\[ \tilde{\mathcal{H}}_2 = \{\mathcal{H}_1, W_0\} + \{\{\mathcal{H}_0, W_0\}, W_0\} + \mathcal{H}_2, \]
where $\mathcal{H}_2$ is Hamiltonian defined in (20). Using the equation for $W_0$, given in Lemma 4.2, one has that $\tilde{\mathcal{H}}_2$ is $C^{s-3}$.

Let $\tilde{\mathcal{H}}_2(k,0,0)$, $k \in \mathbb{Z}^{n+1}$ denote the Fourier coefficients of $\mathcal{H}_2$ in $s$ and $t$. We split $\tilde{\mathcal{H}}_2$ in several terms, as done for $\mathcal{H}_1$, in the following way

$$\tilde{\mathcal{H}}_2(I,s,x,y,t) = \tilde{\mathcal{H}}_3(I,s,t) + \tilde{\mathcal{H}}^{(1)}_2(I,s,t) + \tilde{\mathcal{H}}^{(2)}_2(I,s,y,t) + \tilde{\mathcal{H}}^{(3)}_2(I,s,x,t) + \mathcal{O}^*(xy)$$

with

$$\tilde{\mathcal{H}}_3(I,s,t) = \sum_{k \in \mathbb{Z}^{n+1}} \psi \left( \frac{k \cdot (\nu(I),1)}{\beta} \right) \tilde{\mathcal{H}}^k_2(I,0,0) e^{2\pi i k \cdot (s,t)}$$

and

$$\begin{align*}
\tilde{\mathcal{H}}^{(1)}_2(I,s,t) &= \tilde{\mathcal{H}}_2(I,s,0,0,t) - \tilde{\mathcal{H}}_3(I,s,t) \\
\tilde{\mathcal{H}}^{(2)}_2(I,s,t) &= \tilde{\mathcal{H}}_2(I,s,x,0,t) - \tilde{\mathcal{H}}_3(I,s,0,0,t) \\
\tilde{\mathcal{H}}^{(3)}_2(I,s,t) &= \tilde{\mathcal{H}}_2(I,s,0,y,t) - \tilde{\mathcal{H}}_3(I,s,0,0,t).
\end{align*}$$

All these terms are $C^{s-3}$.

**Lemma 4.3.** There exists a $C^{s-3}$ smooth solution $W_1(I,s,x,y,t)$ given as the sum $\sum_{j=1}^3 W^{(j)}_1(I,s,x,y,t)$ with $W^{(j)}_1(I,s,0,t) = 0$ for $j = 2, 3$ of the equation

$$(\nu(I) + \partial_I g(I,xy)) \partial_s W_1 + \partial_t W_1 + \{g(I,xy), W_1\}_{(x,y)} + \sum_{j=1}^3 \mathcal{H}^{(j)}_2 = 0.$$ 

Moreover, $W_1 = \mathcal{O}^*(\beta^{-3})$, where $\mathcal{O}^*$ is defined in (4).

**Proof.** As in the proof of Lemma 4.2, we take $W_1 = \sum_{j=1}^3 W^{(j)}_1$ and solve the equations

$$(\nu(I) + \partial_I g(I,xy)) \partial_s W^{(j)}_1 + \partial_t W^{(j)}_1 + \{g(I,xy), W^{(j)}_1\}_{(x,y)} + \mathcal{H}^{(j)}_2 = 0, \quad j = 1, 2, 3.$$ 

Each equation is solved in the same way as the first order in Lemma 4.2. One can see that $\tilde{\mathcal{H}}_2$ satisfies $\tilde{\mathcal{H}}_2 = \mathcal{O}^*(\beta^{-2})$. This implies that $W_1 = \mathcal{O}^*(\beta^{-3})$ (one can have more precise bounds for each $W^{(j)}_1$).

Denote by

$$\Phi : (\tilde{I}, \tilde{s}, \tilde{x}, \tilde{y}, \tilde{t}) \longrightarrow (I, s, x, y, t)$$

the time-one map associated to the flow of the Hamiltonian $\varepsilon W = \varepsilon W_0 + \varepsilon^2 W_1$. This change is $C^{s-4}$ and symplectic. In the next two lemmas we analyze the change of coordinates and the transformed Hamiltonian.
Lemma 4.4. The change $\Phi$ is $C^{s-4}$ and satisfies the equations

$$I = \tilde{I} + \varepsilon M_1^I + \varepsilon^2 M_2^I + O^*(\varepsilon^3 \beta^{-5})$$
$$= \tilde{I} + \varepsilon \partial_z W_0 + \varepsilon^2 (\partial_z W_1 + \{\partial_z W_0, W_0\}) + O^*(\varepsilon^3 \beta^{-5})$$
$$s = \tilde{s} + \varepsilon M_1^s - \varepsilon^2 M_2^s + O^*(\varepsilon^3 \beta^{-6})$$
$$= \tilde{s} - \varepsilon \partial_t W_0 - \varepsilon^2 (\partial_t W_1 + \{\partial_t W_0, W_0\}) + O^*(\varepsilon^3 \beta^{-6})$$
$$x = \tilde{x} + \varepsilon M_1^x + \varepsilon^2 M_2^x + O^*(\varepsilon^3 \beta^{-5})$$
$$= \tilde{x} + \varepsilon \partial_y W_0 + \varepsilon^2 (\partial_y W_1 + \{\partial_y W_0, W_0\}) + O^*(\varepsilon^3 \beta^{-5})$$
$$y = \tilde{y} + \varepsilon M_1^y + \varepsilon^2 M_2^y + O^*(\varepsilon^3 \beta^{-5})$$
$$= \tilde{y} - \varepsilon \partial_x W_0 - \varepsilon^2 (\partial_x W_1 + \{\partial_x W_0, W_0\}) - O^*(\varepsilon^3 \beta^{-5})$$

Moreover,

$$M_1^z = O^*(\beta^{-1}), \quad z = I, x, y \quad \text{and} \quad M_1^s = O^*(\beta^{-2})$$

and

$$M_2^z = O^*(\beta^{-3}), \quad z = I, x, y \quad \text{and} \quad M_2^s = O^*(\beta^{-4}).$$

We also have

$$xy = \tilde{x}\tilde{y} + \varepsilon M_1^r + \varepsilon^2 M_2^r + O^*(\varepsilon^3 \beta^{-5}),$$

with

$$M_1^r = O^*(\beta^{-1}), \quad \text{and} \quad M_2^r = O^*(\beta^{-3}).$$

The inverse change is of the same form, that is

$$\tilde{I} = I - \varepsilon M_1^I + \varepsilon^2 \tilde{M}_1^I + O^*(\varepsilon^3 \beta^{-5})$$
$$\tilde{s} = s - \varepsilon M_1^s - \varepsilon^2 \tilde{M}_1^s + O^*(\varepsilon^3 \beta^{-6})$$
$$\tilde{x} = x - \varepsilon M_1^x + \varepsilon^2 \tilde{M}_1^x + O^*(\varepsilon^3 \beta^{-5})$$
$$\tilde{y} = y - \varepsilon M_1^y + \varepsilon^2 \tilde{M}_1^y + O^*(\varepsilon^3 \beta^{-5})$$

and

$$\tilde{x}\tilde{y} = xy - \varepsilon M_1^r + \varepsilon^2 \tilde{M}_1^r + O^*(\beta^{-5} \varepsilon^3).$$

The terms $\tilde{M}_2^z$ satisfy the same estimates as $M_2^z$.

Proof. It is enough to recall that

$$\tilde{I} = I + \varepsilon \{I, W\} + \varepsilon^2 \{\{I, W\}, W\} + \ldots.$$ 

To compute the remainder, one has to estimate

$$\{\{z, W_0\}, W_0\} + \{\{z, W_0\}, W_1\} + \{\{z, W_1\}, W_0\}, \quad z = I, s, x, y.$$ 

Using the estimates for $W_0$ and $W_1$, one obtains the bounds for the remainder. \hfill \square
Now we can apply this symplectic change of coordinates to the Hamiltonian $H_\epsilon$ given by Lemma 4.1.

**Lemma 4.5.** The Hamiltonian $H_\epsilon \circ \Phi$ is $C^{s-4}$ and is of the following form

$$H_\epsilon \circ \Phi(\hat{I}, \hat{s}, \hat{x}, \hat{y}, \hat{t}) = E(\hat{I}) + g(\hat{I}, \hat{x}\hat{y}) + \epsilon \tilde{H}_1(\hat{I}, \hat{s}, \hat{t}) + \epsilon \hat{x}\hat{y} \tilde{H}_2(I, s, t) + \epsilon^2 \hat{s} \tilde{H}_3(\hat{I}, \hat{s}, \hat{t}) + \mathcal{O}^*(\epsilon^3 \beta^{-4} + \epsilon^2 \beta^{-2} \hat{x}\hat{y} + \epsilon (\hat{x}\hat{y})^2).$$

5 Transition near the singularity dynamics in the normal form

We compute the equation associated to the Hamiltonian given in Lemma 4.5. We drop the hats to simplify notation. Following [Tre02], consider a region for the initial conditions of the form

$$U_* = \{(I, s, x, y) : c_* < |x| < c_*^{-1}, c_0^{-1}(\epsilon \beta^{-1} + |xy|)^2 \log^2(xy) \leq |xy| \leq \kappa_* \}$$

(25)

and a final time $\bar{t}$ with

$$c_* \leq |y_*| e^{\partial_1 g(I^*, \rho) \bar{t}} \leq c_*^{-1}. \quad \text{(26)}$$

Take, as in [Tre02], $c^*$ and $c_0$ are independent of $\epsilon$ and $\kappa_* \sim \epsilon$.

The Hamiltonian obtained in Lemma 4.5 has different formulas in non-resonant and resonant zones. We first analyze in the non-resonant zone and later in the resonant one, which are defined in (15) and (16) respectively.

5.1 The non-resonant regime

Recall that we study trigonometric perturbations. In the non-resonant zone (15), analyzing (3), (22), and (24) we have that $\tilde{H}_j = 0$ for $j = 1, 2, 3$. Therefore, we have the Hamiltonian

$$\mathcal{H}_* \circ \Phi(\hat{I}, \hat{s}, \hat{x}, \hat{y}, \hat{t}) = E(\hat{I}) + g(\hat{I}, \hat{x}\hat{y}) + \mathcal{O}^*(\epsilon^3 \beta^{-4} + \epsilon^2 \beta^{-2} \hat{x}\hat{y} + \epsilon (\hat{x}\hat{y})^2),$$

(26)

which is $C^{s-4}$ and is integrable up to order 3.

**Lemma 5.1.** Suppose that for some $(I_*, s_*, x_*, y_*) \in U_*$ (see (25)) and $\bar{t} \in \mathbb{R}$,

$$c_* \leq |y_*| e^{\partial g(I^*, \rho) \bar{t}} \leq c_*^{-1}.$$ 

where $\rho = |x_*y_*|.$
Then,

\[
\begin{align*}
    s(t) &= s^* + (\nu(I^*) + \partial_t g(I^*, \rho)) t + \mathcal{O}^* \left( \varepsilon^3 \beta^{-5} + \varepsilon^2 \beta^{-2} \rho + \varepsilon \rho^2 \right) \log^2 \rho \\
    I(t) &= I^* + \mathcal{O}^* \left( \varepsilon^3 \beta^{-4} + \varepsilon^2 \beta^{-2} \rho + \varepsilon \rho^2 \right) |\log \rho| \\
    x(t) &= x_* e^{-\partial_t g(I^*, \rho) t} + \mathcal{O}^* \left( \varepsilon^3 \beta^{-4} + \varepsilon^2 \beta^{-2} \rho + \varepsilon \rho^2 \right) \left( 1 + \mathcal{O}^* \left( \varepsilon^3 \beta^{-4} + \varepsilon^2 \beta^{-2} \rho + \varepsilon \rho^2 \right) \right) \\
    y(t) &= y_* e^{\partial_t g(I^*, \rho) t} \left( 1 + \mathcal{O}^* \left( \varepsilon^3 \beta^{-4} + \varepsilon^2 \beta^{-2} \rho + \varepsilon \rho^2 \right) \right) \\
    x(t)y(t) &= x_* y_* + \mathcal{O}^* \left( \varepsilon^3 \beta^{-4} + \varepsilon^2 \beta^{-2} \rho + \varepsilon \rho^2 \right) \left| \log \rho \right|.
\end{align*}
\]

Proof. The proof of this lemma is a direct consequence of the particular form of the equations associated to Hamiltonian \([26]\). Indeed, one can easily see that

\[
\frac{d}{dt}(xy) = \mathcal{O}^* \left( \varepsilon^3 \beta^{-4} + \varepsilon^2 \beta^{-2} xy + \varepsilon (xy)^2 \right).
\]

Therefore, one can easily see that

\[
|x(t)y(t) - x^* y^*| = \mathcal{O}^* \left( \varepsilon^3 \beta^{-4} + \varepsilon^2 \beta^{-2} \rho + \varepsilon \rho^2 \right) |\log \rho|.
\]

Taking this into account, we have

\[
\dot{I} = \mathcal{O}^* \left( \varepsilon^3 \beta^{-4} + \varepsilon^2 \beta^{-2} \rho + \varepsilon \rho^2 \right)
\]

which leads to the formula for \(I(t)\). Using that \(I\) is almost constant, one can easily deduce the formulas for the other variables. \(\square\)

5.2 The resonant regime

To analyze the resonance recall that we focus on the case of two and a half degrees of freedom. Namely, \(s \in \mathbb{T}\) and \(I \in \mathbb{R}\). Perform a change to slow-fast variables.

This leads to a Hamiltonian which is almost a first integral, namely, its dependent terms are small.

Fix \((k_0, k_1) \in \mathbb{Z}^2\). Assume that the resonance \((\nu(I), 1) \cdot (k_0, k_1) = 0\), \((k_0, k_1) \in \mathcal{N}^{(2)}(H_1) \subset \mathbb{Z}^2\) is located at \(I = 0\). Call \(A\) the variable conjugate to time. Then, the change

\[
(J, \theta, D, t) = \left( \frac{I}{k_0}, k_0 s + k_1 t, A - \frac{k_1}{k_0} I, t \right)
\]

is symplectic. Substitute and obtain the following Hamiltonian. We drop the hats to simplify notations.

\[
\tilde{\mathcal{H}}(J, \theta, x, y, t) = \tilde{E}(J) + \tilde{g}(J, xy) + \varepsilon \tilde{H}_1(J, \theta, xy) + \mathcal{O}^* \left( \varepsilon^3 \beta^{-4} + \varepsilon^2 \beta^{-2} xy + \varepsilon (xy)^2 \right).
\]

where

\[
\tilde{E}(J) = E(k_0 J) + k_1 J,
\]

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where the functions \( \tilde{E}(0) = 0 \), \( \tilde{g}(J, xy) = g(k_0 J, xy) \)

and

\[
\tilde{H}_1(J, \theta, xy) = \sum_{j=-N}^N \left( \tilde{H}_1^{(j k_0, j k_1)}(k_0 J) + x y \tilde{H}_2^{(j k_0, j k_1)}(k_0 J) + \varepsilon \tilde{H}_3^{(j k_0, j k_1)}(k_0 J) \right) e^{2 \pi i j \theta}.
\]

We use this system of coordinates to analyze the flow in the resonant zones. Recall that by construction \( \tilde{v}(0) = 0 \).

**Lemma 5.2.** Suppose that for some \( (J_*, \theta_*, x_*, y_*) \in \mathcal{U}_* \) and \( \tilde{I} \in \mathbb{R} \),

\[
c_* \leq |y_*| e^{\varepsilon g(J_* \rho \tilde{I})} \leq c_*^{-1}.
\]

where \( \rho = |x_* y_*| \).

Then,

\[
\begin{align*}
\theta(\tilde{I}) &= \theta^* + (\nu(I^*) + \partial_1 g(I^*, \rho))^\tilde{I} + \varepsilon F_1(I^*, \theta^*, \tilde{I}) + \varepsilon^2 F_2(I^*, \theta^*, \tilde{I}) \\
&\quad + \varepsilon^3 \mathcal{O}^* (\beta^{-1} + |\log \rho|)^{\frac{1}{2}} \log^2 \rho + \mathcal{O}_3^* (\varepsilon \beta^{-1} + \rho) \log^2 \rho \\
J(\tilde{I}) &= J^* + \varepsilon G_1(J^*, \theta^*, \tilde{I}) + \varepsilon^2 G_2(J^*, \theta^*, t) \\
&\quad + \varepsilon^3 \log \rho \mathcal{O}^* (\beta^{-1} + |\log \rho|)^4 + (1 + \mathcal{O}_3^* (\varepsilon \beta^{-1} + \rho)) \log^2 \rho \\
x(\tilde{I}) &= x_* e^{-(\partial_2 g(I^*, \rho))^\tilde{I} + \varepsilon \Phi(J^*, \theta^*, \tilde{I})} + \mathcal{O}_3^* (\varepsilon \beta^{-1}, \rho) \log^2 \rho \\
y(\tilde{I}) &= y_* e^{\varepsilon \partial_2 g(I^*, \rho)^\tilde{I} + \varepsilon \Phi(J^*, \theta^*, \tilde{I}) (1 + \mathcal{O}_3^* (\varepsilon \beta^{-1}, \rho) \log^2 \rho)
\end{align*}
\]

where the functions \( F_i \) and \( G_i \) are defined below through integrals of the perturbed Hamiltonian and

\[
\Phi(J^*, \theta^*, \tilde{I}) = \int_0^\tilde{I} \left( \tilde{H}_2(J^*, \theta^* + \nu(I^*) t + \partial_2 g(J^*, \rho) t + \partial_{12} g(J^*, \rho) G_1(J^*, \theta^*, t) \right) dt
\]

Let \( D \) be the variable conjugate to the time \( t \). Then we have

\[
D(\tilde{I}) = D^* + \mathcal{O}^* (\varepsilon^3 \beta^{-4} + \varepsilon^2 \beta^{-2} \rho + \varepsilon \rho^2) |\log \rho|.
\]

**Proof.** We have the equations

\[
\begin{align*}
\dot{\theta} &= \nu(J) + \partial_1 g(J, xy) + \varepsilon \partial_1 \tilde{H}_1(J, \theta, t) + \varepsilon xy \partial_1 \tilde{H}_2(J, \theta, t) + \varepsilon^2 \partial_1 \tilde{H}_3(J, \theta, t) \\
&\quad + \mathcal{O}^* (\varepsilon^3 \beta^{-4} + \varepsilon^2 \beta^{-2} xy + \varepsilon (xy)^2) \\
\dot{J} &= -\varepsilon \partial_\theta \tilde{H}_1(J, \theta, t) - \varepsilon xy \partial_\theta \tilde{H}_2(J, \theta, t) - \varepsilon^2 \partial_{\theta} \tilde{H}_3(J, \theta, t) \\
&\quad + \mathcal{O}^* (\varepsilon^3 \beta^{-4} + \varepsilon^2 \beta^{-2} xy + \varepsilon (xy)^2) \\
\dot{y} &= \partial_1 g(J, xy) y + \varepsilon \partial_1 \tilde{H}_2(J, \theta, t) + \mathcal{O}_2^* (\varepsilon \beta^{-1}, xy) \\
\dot{x} &= -\partial_1 g(J, xy) x + \varepsilon \partial_1 \tilde{H}_2(J, \theta, t) + \mathcal{O}_2^* (\varepsilon \beta^{-1}, xy).
\end{align*}
\]
We also have
\[
\frac{d}{dt}(xy) = O^* \left( \varepsilon^3 \beta^{-4} + \varepsilon^2 \beta^{-2} xy + \varepsilon (xy)^2 \right).
\]

Now we compute estimates for this flow. Call \((I^*, s^*, q^*, p^*)\) to the initial point and recall \(\rho = x^* y^*\). Then,
\[
|xy - x^* y^*| = O^* \left( \varepsilon^3 \beta^{-4} + \varepsilon^2 \beta^{-2} \rho + \varepsilon \rho^2 \right) |\log \rho|.
\]

This implies that the first orders of \(\dot{s}\) and \(\dot{y}\) are independent of \(x\) and \(y\), since only depend on \(\rho\). Indeed, we have
\[
\dot{\theta} = \nu(J) + \partial_J g(J, \theta) + \varepsilon \partial_J H_1(J, \theta, t) + \varepsilon \rho \partial_J H_2(J, \theta, t)
\]
\[
+ \varepsilon^2 \partial_J H_3(J, \theta, t) + O^* \left( \varepsilon^3 \beta^{-4} + \varepsilon^2 \beta^{-2} \rho + \varepsilon \rho^2 \right) |\log \rho|
\]
\[
\dot{J} = - \varepsilon \partial_\rho H_1(J, \theta, t) - \varepsilon \rho \partial_\rho H_2(J, \theta, t)
\]
\[
- \varepsilon^2 \partial_\rho H_3(J, \theta, t) + O^* \left( \varepsilon^3 \beta^{-4} + \varepsilon^2 \beta^{-2} \rho + \varepsilon \rho^2 \right).
\]

These equations can be solved perturbatively in powers of \(\varepsilon\). We look for solutions of the following form and we take \(t^* = 0\)
\[
\theta = \theta^* + \nu(J^*) t + \partial_J g(J^*, \rho) t + \varepsilon F_1(J^*, \theta^*, t) + \varepsilon^2 F_2(J^*, \theta^*, t) + \text{h.o.t}
\]
\[
J = J^* + \varepsilon G_1(J^*, \theta^*, t) + \varepsilon^2 G_2(J^*, \theta^*, t) + \text{h.o.t}
\]

Plugging these expressions into the equation, we obtain the following equations at each order.
\[
\dot{F}_1 = (\nu'(J^*) + \partial_J^2 g(J^*, \rho) \nu(J^*)) G_1
\]
\[
+ \left( \partial_J H_1 + \rho \partial_J H_2 \right) (J^*, \theta^* + \nu(J^*) t + \partial_J g(J^*, \rho) t, t)
\]
\[
\dot{G}_1 = - \left( \partial_\rho H_1 + \rho \partial_\rho H_2 \right) (J^*, \theta^* + \nu(J^*) t + \partial_J g(J^*, \rho) t, t)
\]

We can first solve the second equation, taking
\[
G_1(J^*, \theta^*, \bar{t}) = - \int_0^\bar{t} \left( \partial_\rho H_1 + \rho \partial_\rho H_2 \right) (J^*, \theta^* + \nu(J^*) t + \partial_J g(J^*, \rho) t, t) dt.
\]

and plugging this into the first equation, we obtain \(F_1[3]\)
\[
F_1(J^*, \theta^*, \bar{t}) = - (\nu'(J^*) + \partial_J^2 g(J^*, \rho)) \int_0^\bar{t} G_1(J^*, \theta^* + \nu(J^*) t + \partial_J g(J^*, \rho) t, t) dt
\]
\[
+ \int_0^\bar{t} \left( \partial_J H_1 + \rho \partial_J H_2 \right) (J^*, \theta^* + \nu(J^*) t + \partial_J g(J^*, \rho) t, t) dt.
\]

\(^3\)The terms \(F_1\) does not appear in [Tre02]
Proceeding analogously, one can compute $F_2$ and $G_2$ using that they are solution of
\[
\dot{F}_2 = (\nu'(J^*) + \partial_J^2 G(\rho, J^*))G_2 \\
+ (\nu''(J^*) + \partial_J^3 G(\rho, J^*)) \left( G_1(J^*, \theta^*, t) \right)^2 \\
+ \partial_J^2 \left( \underbar{H}_1 + \rho \underbar{H}_2 \right) (J^*, \theta^*) + \nu(J^*) t + \partial_J g(J^*, \rho) t, t) G_1(J^*, \theta^*, t) \\
+ \partial_J \theta \left( \underbar{H}_1 + \rho \underbar{H}_2 \right) (J^*, \theta^*) + \nu(J^*) t + \partial_J g(J^*, \rho) t, t) F_1(J^*, \theta^*, t) \\
+ \partial_J \underbar{H}_3 (J^*, \theta^*) + \nu(J^*) t + \partial_J g(J^*, \rho) t, t) \\
\dot{G}_2 = - \partial_J \theta \left( \underbar{H}_1 + \rho \underbar{H}_2 \right) (J^*, \theta^*) + \nu(J^*) t + \partial_J g(J^*, \rho) t, t) G_1(J^*, \theta^*, t) \\
- \partial_J \theta \left( \underbar{H}_1 + \rho \underbar{H}_2 \right) (J^*, \theta^*) + \nu(J^*) t + \partial_J g(J^*, \rho) t, t) F_1(J^*, \theta^*, t) \\
- \partial_J \underbar{H}_3 (J^*, \theta^*) + \nu(J^*) t + \partial_J g(J^*, \rho) t, t)
\]

From its definition we can deduce that the functions $F_1$ and $G_1$ satisfy

\[
F_1 = \mathcal{O}^* \left( \log^2 \rho + \beta^{-1} \log \rho \right), \quad G_1 = \mathcal{O}^* \left( \log \rho \right).
\]

Note that bigger term in $F_1$ depends on the choice of $\beta$.

For the second order
\[
F_2 = \mathcal{O}^* \left( \beta^{-2} \log^2 \rho + \beta^{-1} \log^3 \rho + \log^4 \rho \right), \quad G_2 = \mathcal{O}^* \left( \beta^{-1} \log^2 \rho + \log^3 \rho \right).
\]

Analyzing the remainders, one can compute the size of the higher order terms of the evolution of $J$ and $s$ given in Lemma 3.2.

Now we analyze the flow for the $x$ and $y$ variables. Recall that the conditions on the initial $y_*$ and on the time $\tau$ imply

\[
\frac{\tau^2}{\rho} \leq e^{\partial_\rho g(J^*, \rho) \tau} \leq \frac{1}{\rho c_s^2}. \tag{28}
\]

Using the almost conservation of $xy$ we have
\[
\dot{y} = \left( \partial_r g(\rho, J) + \varepsilon \underbar{H}_2(J, \theta, t) + \Psi_1(t) \right) y + \Theta_1(t) \\
\dot{x} = - \left( \partial_r g(\rho, J) + \varepsilon \underbar{H}_2(J, \theta, t) + \Psi_2(t) \right) x + \Theta_2(t).
\]

where $\Psi_i = \mathcal{O}^*_2(\beta^{-1} \varepsilon, \rho)$ and $\Theta_i = \mathcal{O}^*_3(\beta^{-1} \varepsilon, \rho) | \log \rho |$.

Thus,

\[
\begin{align*}
x(t) &= x_0 e^{-\int_0^t \left( \partial_r g(\rho, J(t)) + \varepsilon \underbar{H}_2(J(t), \theta(t), t) + \Psi_1(t) \right) dt} \\
+ \int_0^t e^{-\int_0^\sigma \left( \partial_r g(\rho, J(t)) + \varepsilon \underbar{H}_2(J(t), \theta(t), t) + \Psi_1(t) \right) dt} \Theta_1(\sigma) d\sigma.
\end{align*}
\]
Using \(28\),
\[
x(t) = x_* e^{-\int_0^t (\partial_r g(\rho,J(t)) + \epsilon \Phi(J^*,\theta^*,t)) \, dt} + O_3^{*}(\beta^{-1} \varepsilon, \rho) \log^2 \rho.
\]

Now we expand \(J(t)\) and \(\theta(t)\) with the formulas already obtained. Then,
\[
x(t) = x_* e^{-\int_0^t (\partial_r g(\rho,J^*) + \epsilon \Phi(J^*,\theta^*,t)) \, dt} + O_3^{*}(\beta^{-1} \varepsilon, \rho) \log^2 \rho
\]
with
\[
\Phi(J^*,\theta^*,t) = \int_0^t (H_2(J^*,\theta^*) + \nu(J^*) t + \partial_J g(\rho,J^*) t, t) + \partial_r g(\rho,J^*) G_1(J^*,\theta^*,t)) \, dt.
\]
Using this estimate and \(27\), we can deduce the following formulas for \(y(t)\),
\[
y(t) = y_* e^{\partial_r g(\rho,J^*) t + \epsilon \Phi(J^*,\theta^*,t)} \left(1 + O_3^{*}(\beta^{-1} \varepsilon, \rho) \log^2 \rho\right).
\]

6 The gluing maps

We compute the gluing maps. First in Section 6.1, we consider the unperturbed Hamiltonian \(H_0\) given in Lemma 4.1. Then, we consider the full Hamiltonian in the non-resonant setting (Section 6.2) and resonance setting (Section 6.3).

6.1 Unperturbed gluing maps

The unperturbed gluing maps are computed in Section 5 of [Tre02]. We include here the computation to make the paper self contained.

We consider the gluing maps
\[
S^+: \{|y| \text{ is small, } x > 0\} \rightarrow \{y > 0, |x| \text{ is small}\}
\]
\[
S^-: \{|y| \text{ is small, } x > 0\} \rightarrow \{y < 0, |x| \text{ is small}\}.
\]

The gluing maps of the unperturbed system must satisfy the following properties

(i) are symplectic

(ii) preserve \(I\) and \(xy\).
Lemma 6.1 \((\text{Tre02})\). Properties (i), (ii) and \((29)\) imply that the gluing maps

\[
(I^\pm, s^\pm, x^\pm, y^\pm) = S^\pm(I, s, x, y)
\]

are of the form

\[
\begin{align*}
I^\pm &= I \\
s^\pm &= s + \partial_I \Phi^\pm(I, xy) \\
y^\pm &= x^{-1} e^{-\partial_r \Phi^\pm(I, xy)} \\
x^\pm &= x^2 y e^{\partial_r \Phi^\pm(I, xy)}
\end{align*}
\]

for some functions \(\Phi^\pm\).

In \((\text{Tre02})\) it is also shown that \(\partial_I \Phi^\pm(I, r) = \mu^\pm(I) + \mathcal{O}(r)\) and \(e^{\partial_r \Phi^\pm(I, r)} = \kappa^\pm(I) + \mathcal{O}(r)\). Notice that the maps \(S^\pm\) are \(C^s\).

Moreover, since \(\rho = xy\), \(x^\pm = \mathcal{O}(\rho)\) is always small and will contribute to second orders only. One can consider also the inverse change for \((x, y)\). Since \(xy = x^\pm y^\pm\), it is given by

\[
\begin{align*}
y &= y^\pm \rho e^{-\partial_r \Phi^\pm(I, \rho)} \\
x &= (y^\pm)^{-1} \rho e^{\partial_r \Phi^\pm(I, \rho)}
\end{align*}
\]

So, \(y\) also is small since \(y = \mathcal{O}(\rho)\).

6.2 Gluing maps in non-resonant zones

Now express these gluing maps in the normal form coordinates \((\hat{T}, \hat{s}, \hat{x}, \hat{y})\). We use the formulas for the normal form coordinates obtained in Section 4.2.

The gluing maps have the following form. Each term can be expressed in terms of the Hamiltonian \(W\) associated to the normal form variables from Section 4.2.

\[
\begin{align*}
\hat{T}^\pm &= \hat{T} + \varepsilon(M_1^1 \circ S^\pm - M_1^1) + \varepsilon^2(M_2^1 \circ S^\pm - \hat{M}_2^1) + \mathcal{O}^*(\varepsilon^{-5}) \\
\hat{s}^\pm &= \hat{s} + \partial_I \Phi^\pm(I + \varepsilon(M_1^1 \circ S^\pm - M_1^1), \hat{x} \hat{y} + \varepsilon(M_1^1 \circ S^\pm - M_1^1)) \\
&+ \varepsilon(M_1^1 \circ S^\pm - M_1^1) + \mathcal{O}^*(\varepsilon^{-4}) \\
\hat{y}^\pm &= \hat{x}^{-1} e^{-\partial_r \Phi^\pm(I, \hat{x} \hat{y})} + \mathcal{O}^*(\varepsilon^{-1}) \\
\hat{x}^\pm &= e^{\partial_r \Phi^\pm(I, \hat{x} \hat{y})} (\hat{x} \hat{y} + \varepsilon(M_1^1 \circ S^\pm - M_1^1)) + \mathcal{O}^*(\rho^2 + \rho \varepsilon^{-1} + \varepsilon^2 \beta^{-3})
\end{align*}
\]

Moreover,

\[
\begin{align*}
\hat{y}^\pm \hat{x}^\pm &= \hat{y} \hat{x} + \varepsilon(M_1^1 \circ S^\pm - M_1^1) + \varepsilon^2(M_2^1 \circ S^\pm - \hat{M}_2^1) + \mathcal{O}^*(\varepsilon^{-5}).
\end{align*}
\]
6.3 Gluing maps in resonances

We consider the slow-fast variables \((\theta, I, p, q)\) and we extend the phase space with the conjugate variables \((J, t)\), where \(J\) is the variable introduced in Lemma 5.2. Moser normal form does not alter the variables \((J, t)\) (see Lemma 4.1). Then, in the slow-fast variables, we have the gluing map

\[
(J^\pm, \theta^\pm, y^\pm, x^\pm, D^\pm, t^\pm) = S^\pm(J, \theta, x, y, D, t)
\]

with

\[
J^\pm = J
\]

\[
\theta^\pm = \theta + \partial_J \tilde{\Phi}^\pm(J, xy)
\]

\[
y^\pm = x^{-1} e^{-\partial_r \tilde{\Phi}^\pm(J, xy)}
\]

\[
x^\pm = x^2 y e^{\partial_r \tilde{\Phi}^\pm(J, xy)}
\]

\[
D^\pm = D
\]

\[
t^\pm = t
\]

where

\[
\tilde{\Phi}^\pm(I, r) = \Phi^\pm(k_0 I, r).
\]

From now on, we drop the tilde in \(\Phi^\pm\) to simplify the notation. We also abuse notation and we consider the normal form change of coordinates given by the generating function \(W\) expressed in slow-fast variables (recall that all these changes of coordinates are symplectic).

Now we express the gluing map in the normal form (slow-fast) coordinates. As before, for the first four coordinates we have

\[
\hat{t}^\pm = \hat{t} + \varepsilon (M_1^I \circ S^\pm - M_1^I) + \varepsilon^2 (M_2^I \circ S^\pm - \widetilde{M}_2^I) + O^\ast(\varepsilon^3 \beta^{-5})
\]

\[
\hat{\theta}^\pm = \hat{\theta} + \partial_J \tilde{\Phi}^\pm(J + \varepsilon (M_1^I \circ S^\pm - M_1^I), \hat{x} \hat{y} + \varepsilon (M_1^I \circ S^\pm - M_1^I))
\]

\[
+ \varepsilon (M_1^I \circ S^\pm - M_1^I) + O^\ast(\varepsilon^2 \beta^{-4})
\]

\[
\hat{y}^\pm = \hat{x}^{-1} e^{-\partial_r \tilde{\Phi}^\pm(J, \hat{x} \hat{y})} + O^\ast(\varepsilon \beta^{-1})
\]

\[
\hat{x}^\pm = e^{\partial_x \tilde{\Phi}^\pm(J, \hat{x} \hat{y})} (\hat{x} \hat{y} + \varepsilon (M_1^I \circ S^\pm - M_1^I)) + O^\ast(\rho^2 + \rho \varepsilon \beta^{-1} + \varepsilon^2 \beta^{-3})
\]

and,

\[
\hat{y}^\pm \hat{x}^\pm = \hat{y} \hat{x} + \varepsilon (M_1^I \circ S^\pm - M_1^I) + \varepsilon^2 (M_2^I \circ S^\pm - \widetilde{M}_2^I) + O^\ast_3(\varepsilon \beta^{-5}).
\]

Now we compute the two last components. Since \(W\) does not depend on \(J\), we have that \(\hat{t}^\pm = \hat{t}\). For the \(D\) component, note that proceeding as in Lemma 4.4 we have

\[
\hat{D} = D + \varepsilon \partial_t W_0 + \varepsilon^2 (\partial_t W_1 + \{\partial_t W_0, W_0\}) + \ldots
\]
Note that in this formula the Poisson bracket can either denote the Poisson bracket with respect to the six variables or with respect to the four first variables, since $w_0$ is independent of $D$. Then, we have

$$\hat{D}^\pm = \hat{D} + \varepsilon (M_1^D \circ S - M_1^D) + \mathcal{O}(\varepsilon^2 \beta^{-3}),$$

where $M_1^D = \partial t w_0$, which by Lemma 4.2 satisfies

$$M_1^D = \mathcal{O}^\ast (\beta^{-1}). \quad (30)$$

7 The separatrix map in the non-resonant regime

We use the results in the previous Sections 5.1 and 6.2 to look for formulas of the separatrix map in the non-resonant regime (15). First we compose the flow in normal form coordinates and the gluing map. We obtain the separatrix map in the $(I, s, y, x)$ coordinates. Later we look for a good system of coordinates which will transform $(y, x)$ to a certain symplectic flow-box coordinates around the former separatrix.

Consider $g^T_\varepsilon$ analyzed in Lemma 5.1 and the gluing map analyzed in Section 6.2. We have the following

**Lemma 7.1.** The composition of the two maps $F = g^T_\varepsilon \circ G^\sigma_\varepsilon$ is given by

$$F(I, s, y, x) = (F_I(I, s, y, x), F_s(I, s, y, x), F_y(I, s, y, x), F_x(I, s, y, x))$$

with

$$F_I = I + \varepsilon P_1^I + \varepsilon^2 P_2^I + \mathcal{O}^\ast (\varepsilon^3 \beta^{-4} + \varepsilon^2 \beta^{-2} \rho + \varepsilon \rho^2) |\log \rho|$$

$$F_s = s + \partial I \Phi^\pm (I, xy) + (\nu(I) + \partial I g(I, xy)) \overline{t} + \varepsilon P_1^s + \mathcal{O}^\ast (\varepsilon^2 \beta^{-4}) |\log \rho|$$

$$F_y = x^{-1} \exp(-\partial r g(I + \varepsilon P_1^y, xy + \varepsilon P_1^r) \overline{t} - \partial r \Phi^\pm (I, xy)) \times$$

$$(1 + \mathcal{O}_2^\ast (\varepsilon, \rho)|\log \rho|) + \mathcal{O}_2^\ast (\varepsilon, \rho)$$

$$F_x = \exp(\partial r g(I + \varepsilon P_1^y, xy + \varepsilon P_1^r) \overline{t} + \partial r \Phi^\pm (I, xy)) \times$$

$x(xy + \varepsilon P_1^r) (1 + \mathcal{O}_2 (\varepsilon, \rho)|\log \rho|)$,

where

$$P_1^I = M_1^I \circ S^\pm - M_1^I$$

$$P_2^I = M_2^I \circ S^\pm - \tilde{M}_2^I$$

and

$$P_1^s = M_1^s \circ S^\pm - M_1^s + \partial I^2 \Phi^\pm (I, xy) (M_1^I \circ S^\pm - M_1^I) +$$

$$\partial r^2 \Phi^\pm (I, xy) (M_1^r \circ S^\pm - M_1^r) + \overline{t} \partial r (\nu(I) + \partial r g(I, xy)) (M_1^r \circ S^\pm - M_1^r) +$$

$$\partial r^2 g(I, xy) (M_1^r \circ S^\pm - M_1^r)$
Moreover
\[ \mathcal{F}_x \mathcal{F}_y = xy + \varepsilon \mathcal{P}_1^r + \varepsilon^2 \mathcal{P}_2^r + \mathcal{O}_3^r (\varepsilon \beta^{-1}, \rho) \]
where
\[ \mathcal{P}_1^r = M_1^r \circ S - M_1^r \]
\[ \mathcal{P}_2^r = M_2^r \circ S - \tilde{M}_2^r \]

Then,
\[ \mathcal{P}_1^1, \mathcal{P}_1^r = \mathcal{O}^*(\beta^{-1}), \quad \mathcal{P}_2^1, \mathcal{P}_2^r = \mathcal{O}^*(\beta^{-3}), \quad \text{and} \quad \mathcal{P}_1^* = \mathcal{O}^*(1) |\log \rho| \]

The proof of this lemma is a direct consequence of the results in Lemma 5.1 and Section 6.2.

Now we look for a coordinate change near the former separatrix such that formulas for the separatrix map are as simple as possible. In comparison with [Tref02], we want to point out two main differences. First, since we are away from resonances, we have \( \mathbf{H}_j = 0 \), \( j = 1, 2, 3 \). This simplifies the formulas. On the other hand, we want to have a more precise dependence on \( xy \) since we are doing a higher order analysis. This second fact implies that we have to slightly modify the change of coordinates.

We look for a symplectic change
\[
\begin{pmatrix}
\xi \\
\eta \\
\tau \\
h
\end{pmatrix} = \Upsilon^{-1} \begin{pmatrix}
s \\
I \\
y \\
x
\end{pmatrix}
\]

The function
\[ I \mapsto g(I, r) \]
is invertible with respect to \( r \) in a neighborhood of \( r = 0 \) (recall that \( \partial_r g(I, 0) = \lambda(I) > 0 \)). Recall that we have denoted by \( g_r^{-1}(I, r) \) the inverse function with respect to the second coordinate \( r \). We consider the following generating function.
\[ S(\eta, s, h, q) = \eta s + g_r^{-1}(\eta, h - E(\eta)) \log |y|. \tag{31} \]

This generating function induces the following change of coordinates
\[
\begin{pmatrix}
\xi \\
\eta \\
\tau \\
h
\end{pmatrix} = \begin{pmatrix}
s - \partial_r g_r^{-1}(I, g(I, xy))(\nu(I) + \partial_I g(I, xy)) \log |y| \\
I \log |y|/\partial_r g(I, xy) \\
\partial_r g(I, xy) \\
E(I) + g(I, xy)
\end{pmatrix}
\]
Lemma 7.2. The separatrix map, has the following form

\[
\begin{pmatrix}
  s \\
  I \\
y \\
x
\end{pmatrix}
= \Upsilon
\begin{pmatrix}
  \xi \\
  \eta \\
  \tau \\
h
\end{pmatrix}
= \left( \begin{array}{c}
  \xi + [\nu(\eta) + \partial_\eta g^{-1}(\eta, g_\tau^{-1}(h - E(\eta)))] \tau \\
  \eta \\
  \sigma \exp\left( \frac{\tau}{\partial_\tau g^{-1}(\eta, h - E(\eta))} \right) \\
  g_\tau^{-1}(\eta, h - E(\eta)) \sigma \exp\left( -\frac{\tau}{\partial_\tau g^{-1}(\eta, h - E(\eta))} \right)
\end{array} \right)
\]  

(32)

Note that the \( y \) component does not get modified by this change of coordinates and that this change of coordinates does not depend on \( \varepsilon \).

We express the separatrix map in these coordinates. We use the change \( \Upsilon \) to write down the formulas. From now on we omit the dependence on \( \beta \). Recall that \( \beta > 0 \) is a fixed parameter independent of \( \varepsilon \). Due to the previous analysis, smoothness of the separatrix map obeys our estimate in Theorem 3.1.

\[
\eta^* = \eta + \varepsilon \mathcal{P}_1 \circ \Upsilon + \varepsilon^2 \mathcal{P}_2 \circ \Upsilon + \mathcal{O}_3(\varepsilon, \rho) |\log \rho|
\]

\[
\xi^* = \xi + \partial_\tau \Phi^*(\eta, g_\tau^{-1}(\eta, h - E(\eta))) - \frac{\nu(\eta) + \partial_\tau g(\eta, g_\tau^{-1}(\eta, h - E(\eta)))}{\partial_\tau g(\eta, g_\tau^{-1}(\eta, h - E(\eta)))} \log R
\]

\[
+ \mathcal{O}_1(\varepsilon + \rho)(|\log \varepsilon| + |\log \rho|)
\]

\[
h^* = h + \varepsilon \left( [\nu(\eta) + \partial_\tau g(\eta, g_\tau^{-1}(\eta, h - E(\eta)))] \mathcal{P}_1 \circ \Upsilon
\]

\[
+ \partial_\tau g(\eta, g_\tau^{-1}(\eta, h - E(\eta))) \mathcal{P}_1 \circ \Upsilon \right) + \varepsilon^2 \mathcal{P}_2 + \mathcal{O}_3(\varepsilon, \rho)
\]

\[
\tau^* = \tau + 1 \frac{\log R + \mathcal{O}_1(\varepsilon + \rho)(|\log \varepsilon| + |\log \rho|)}{\partial_\tau g(\eta, g_\tau^{-1}(\eta, h - E(\eta)))}
\]

where

\[
R = e^{\partial_\tau \Phi^*(\eta, g_\tau^{-1}(\eta, h - E(\eta)))} \left( g_\tau^{-1}(\eta^*, h^* - E(\eta^*)) + \mathcal{O}^*_2 \right)
\]

(33)

and

\[
\mathcal{P}_2^h = (\nu(\eta) + \partial_\tau g(\eta, g_\tau^{-1}(\eta, h - E(\eta)))) \mathcal{P}_2 \circ \Upsilon
\]

\[
+ \partial_\tau g(\eta, g_\tau^{-1}(\eta, h - E(\eta))) \mathcal{P}_2 \circ \Upsilon
\]

\[
+ \frac{1}{2} \left( \partial_\tau^2 g(\eta, g_\tau^{-1}(\eta, h - E(\eta))) \left( \mathcal{P}_1 \circ \Upsilon \right)^2
\]

\[
+ \frac{1}{2} \partial_\tau^2 g(\eta, g_\tau^{-1}(\eta, h - E(\eta))) \left( \mathcal{P}_1 \circ \Upsilon \right)^2.
\]

Proof. We start with the \( \eta \) component. We have that

\[
\eta^* = y^* = y + \varepsilon \mathcal{P}_1 + \varepsilon^2 \mathcal{P}_2 + \mathcal{O}_3(\varepsilon, \rho) |\log \rho|
\]

Then, it is enough to apply the change \( \Upsilon \) defined in (32) to obtain the formula for \( \eta^* \).

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For the $h$ component we have,

$$h^* = E(I^*) + g(I^*, x^*y^*)$$

$$= E(I) + g(I, xy)$$

$$+ (\nu(I) + \partial_I g(I, xy))(\varepsilon P_I^l + \varepsilon^2 P_I^2) + \partial_g(I, xy)(\varepsilon P_I^l + \varepsilon^2 P_I^2)$$

$$+ \frac{\varepsilon^2}{2} (\partial_I \nu(I) + \partial_I^2 g(I, xy))(P_I^l)^2 + \frac{\varepsilon^2}{2} \partial_I^2 g(I, xy)(P_I^r)^2 + O_3(\varepsilon, \rho) |\log \rho|$$

Apply the change $\Upsilon$, one obtains the formula for $h^*$.

To compute the $\tau$ component we use the following identity,

$$g_{r}^{-1}(\eta^*, h^* - E(\eta^*)) = g_{r}^{-1}(\eta, h - E(\eta)) + \varepsilon P_{r}^l + O_2^r.$$ 

We also have

$$\tau^* = \frac{\log |y^*|}{\partial_r g(I^*, x^*y^*)}$$

$$= \log |y| + \partial_r g(I + \varepsilon P_I^l, xy + \varepsilon P_I^r) + \log \left( e^{\partial_r \Phi_{\pm}(I, xy)}(xy + \varepsilon P_I^r) \right) + O_2 |\log \rho|.$$

Then, we obtain

$$\tau^* = \tau + \bar{t} + \frac{1}{\partial_r g(\eta, g_{r}^{-1}(\eta, h - E(\eta)))} \log R + O_1^r(\varepsilon + \rho)(| \log \varepsilon | + | \log \rho |),$$

where $R$ is the function introduced in [33].

Proceeding analogously, one can compute the $\xi$ component,

$$\xi^* = \xi + \partial_\Phi(\eta, g_{r}^{-1}(\eta, h - E(\eta))) - \frac{\nu(\eta) + \partial_I g(\eta, g_{r}^{-1}(\eta, h - E(\eta)))}{\partial_r g(\eta, g_{r}^{-1}(\eta, h - E(\eta)))} \log R$$

$$+ O_1(\varepsilon + \rho)(| \log \varepsilon | + | \log \rho |).$$

Note that $\partial_I g(\eta, g_{r}^{-1}(\eta, h - E(\eta)))$ satisfies

$$\partial_I g(\eta, g_{r}^{-1}(\eta, h - E(\eta))) = O_1^r(\varepsilon, \rho).$$

Therefore,

$$\xi^* = \xi + \partial_\Phi(\eta, g_{r}^{-1}(\eta, h - E(\eta))) - \frac{\nu(\eta)}{\partial_r g(\eta, g_{r}^{-1}(\eta, h - E(\eta)))} \log R$$

$$+ O_1(\varepsilon + \rho)(| \log \varepsilon | + | \log \rho |).$$

This completes the derivation of the separatrix map in the non-resonant case.
Theorem 3.1 is a direct consequence of this lemma. Note that the first orders in the action components in Theorem 3.1 satisfy
\[ M_{\sigma,1}^1 = \mathcal{P}_1^I \circ \Upsilon \]
\[ M_{\sigma,1}^2 = \left[ \nu(\eta) + \partial_I g(\eta, g_r^{-1}(\eta, h - E(\eta))) \right] \mathcal{P}_1^I \circ \Upsilon + \partial_r g(\eta, 0) \mathcal{P}_1^r \circ \Upsilon. \]

We finish this section by identifying the order \( \varepsilon \) of the separatrix map in terms of the Melnikov potential. This allows to compare our results with the results in [Tre02].

**Lemma 7.3.** The functions \( M_{\sigma,1}^1 \) and \( M_{\sigma,1}^2 \) satisfy
\[ M_{\sigma,1}^1 = \partial_\eta \Theta + \mathcal{O}(h - E(\eta)), \quad M_{\sigma,1}^2 = \partial_\nu \Theta + \mathcal{O}(h - E(\eta)) \]
where \( \Theta \) is the Melnikov potential defined in (14).

**Proof.** In Lemma 5.1 we have seen that
\[ M_{\sigma,1}^1 = \mathcal{P}_1^I \circ \Upsilon = \mathcal{P}_1^I \circ \mathcal{S} \circ \Upsilon - \mathcal{P}_1^I \circ \Upsilon. \]

We start by analyzing the function \( M_1^I = \partial_I W_0 \) given in Lemma 4.4.

By the definition of the function \( W_0 \) in 4.2, we have that \( W_0 = W_{00} + \mathcal{O}(xy) \) where \( W_{00} \) is defined as follows. Recall that \( \partial_I g(I, r) = \mathcal{O}(r) \) and \( \partial_r g(I, r) = \lambda(I) + \mathcal{O}(r) \) with \( \lambda(I) > 0 \). The function \( W_{00} \) can be split as
\[ W_{00} = W_{00}^1 + W_{00}^2 + W_{00}^3, \]
where
\[ W_{00}^1 = \partial_0^{-1} \mathcal{H}_1^{(1)} \]
\[ W_{00}^2 = - \int_{-\infty}^{0} \mathcal{H}_1^{(2)}(I, s + \nu(I)t', ye^\lambda(I)t', t + t') dt' \]
\[ W_{00}^3 = - \int_{0}^{+\infty} \mathcal{H}_1^{(3)}(I, s + \nu(I)t', xe^{-\lambda(I)t'}, t + t') dt' \]
where
\[ \partial^{-1}(f) = \partial^{-1} \left( \sum_{(k,k_0) \in \mathbb{Z}^{n+1}} f_{k,k_0}(I)e^{2\pi i (ks + k_0 t)} \right) = \]

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\[ \sum_{(k,k_0) \in \mathbb{Z}^{n+1}} \frac{f^{k,k_0}(I)}{ik\nu(I) + k_0} e^{2\pi i (k s + k_0 t)}, \]

which is well defined in the non-resonant zone.

Therefore, \( M'_1 = \partial_s W^1_{00} + \partial_s W^2_{00} + \partial_s W^3_{00} + \mathcal{O}(x y) \). Now, in Section 6.1 we have seen that \( y^\tau, x = \mathcal{O}(x y) \). Thus,

\[ M'_1 \circ \mathcal{S} - M'_1 = \partial_s W^1_{00} \circ \mathcal{S} - \partial_s W^1_{00} - \partial_s W^2_{00} + \partial_s W^3_{00} \circ \mathcal{S} + \mathcal{O}(x y). \]

Moreover, as shown in [Tre02] and recalled in Section 6.1, the function \( \Phi^\pm \) satisfies \( \partial_I \Phi^\pm (I, r) = \mu^\pm (I) + \mathcal{O}(r) \) and \( e^{\partial_I \Phi^\pm (I, r)} = \kappa(I) + \mathcal{O}(r) \). Then, we obtain

\[ M'_1 \circ \mathcal{S} - M'_1 = \partial_s W^1_{00}(I, s + \mu^\sigma (I), t) - \partial_s W^1_{00}(I, s, t) + \partial_s W^2_{00}(I, s, y, t) + \mathcal{O}(x y) \]

\[ = \partial_s W^1_{00}(I, s + \mu^\sigma (I), t) - \partial_s W^1_{00}(I, s, t) + \partial_s W^3_{00}(I, s, t) + \partial_s W^2_{00}(I, s, y, t) + \mathcal{O}(x y). \]

Now it only remains to apply the change of coordinates \( \Upsilon \). The change \( \Upsilon \) satisfies

\[ \Upsilon(\xi, \eta, h, \tau) = (\xi + \nu(\eta)\tau, \eta, \sigma e^{\lambda(\eta)\tau}, 0) + \mathcal{O}(h - E(\eta)). \]

We apply this change of coordinates to each term. We obtain first

\[ \partial_s W^1_{00}(I, s + \mu^\sigma (I), 0) - \partial_s W^1_{00}(I, s, 0)]_{(I,s,0)=\Upsilon(\eta,\xi,\tau)} = \]

\[ \vartheta(\eta, \xi + \nu(\eta)\tau, 0) - \vartheta(\eta, \xi + \nu(\eta)\tau + \mu^\sigma(\eta), 0), \]

where \( \vartheta \) is the function introduced in [13].

For the two other terms, it is enough to use to see

\[ \partial_s W^2_{00}(I, s, y, 0)]_{(I,s,0)=\Upsilon(\eta,\xi,\tau)} = \]

\[ \partial_s W^2_{00}(\eta, \xi + \nu(\eta)\tau + \mu^\sigma(\eta), 0) \]

\[ \partial_s W^3_{00}(I, s, (\kappa(I)\gamma)^{-1}, 0)]_{(I,s,0)=\Upsilon(\eta,\xi,\tau)} = \]

\[ \partial_s W^3_{00}(\eta, \xi + \nu(\eta)\tau + \mu^\sigma(\eta), \kappa(\eta)^{-1} \sigma e^{-\lambda(\eta)\tau}, 0). \]

**Lemma 7.4.** The following identity is satisfied,

\[ \vartheta(\eta, \xi + \nu(\eta)\tau, 0) - \vartheta(\eta, \xi + \nu(\eta)\tau + \mu^\sigma(\eta)\tau, 0) \]

\[ -\partial_s W^2_{00}(\eta, \xi + \nu(\eta)\tau, \sigma e^{\lambda(\eta)\tau}, 0) \]

\[ + \partial_s W^3_{00}(\eta, \xi + \nu(\eta)\tau + \mu^\sigma(\eta), \kappa(\eta)^{-1} \sigma e^{-\lambda(\eta)\tau}, 0) = \]

\[ \vartheta(\eta, \xi, -\tau) - \vartheta(\eta, \xi + \mu^\sigma(\eta), \tau) \]

\[ + \partial_s W^2_{00}(\eta, \xi, \sigma, -\tau) + \partial_s W^3_{00}(\eta, \xi + \mu^\sigma(\eta), \kappa(\eta)^{-1} \sigma, -\tau). \]
Proof. This lemma follows from the definition (13). Indeed, one can expand both sides into Fourier series and match them.

From this lemma, one can easily deduce the statement of Lemma 7.3.

8 The separatrix map in the neighborhood of resonances

In the resonant regime we only compute the system up to first order. Thus, we follow closely [Tre02]. The main difference is that our resonant region is much larger than in [Tre02]. Indeed, our $\beta$ is fixed, whereas in [Tre02] he considers $\beta = \varepsilon^{1/4}$. Here we also compute the evolution of the variable $D$, defined in (18), by the separatrix map.

First, we compose the flow in the normal form coordinates and the gluing map. We obtain the separatrix map in the $(J, \theta, x, y, t)$ coordinates. We also pay attention to $D$, the variable conjugate to the time $t$.

We denote the composition of the two maps $F = g_t \circ G_\varepsilon$ by

$$F(J, \theta, x, y, t) = (F_J(J, \theta, x, y), F_\theta(J, \theta, x, y), F_x(J, \theta, x, y), F_y(J, \theta, x, y)).$$

The map $F$ is independent of $t$ because we choose the initial time as $t = 0$.

Denote by $F_D$ the image of $D$ under the composition of the gluing map and the flow.

For the $J$ and $D$ components one can easily see that

$$F_J = J + \varepsilon P^J_1 + \varepsilon^2 \mathcal{O}^*(\beta \log^2 \rho, \log^3 \rho)$$

$$F_D = D + \varepsilon P^D_1 + \varepsilon^2 \mathcal{O}^*(1)$$

where

$$P^J_1 = G_1 + M^J_1 \circ S - M^J_1$$

$$P^D_1 = M^D_1 \circ S - M^D_1,$$

where the function $G_1$ has been introduced in Section 6.3 and the functions $M^z$ have been used to define the gluing maps in Section 6.3.

From Lemma 4.4, Lemma 5.2 and (30), we can deduce that

$$P^J_1 = \mathcal{O}^*(\log \rho), \quad P^D_1 = \mathcal{O}^*(1).$$

For the $\theta$ variable we have

$$F_\theta = \theta + \partial J \Phi(J, xy) + (\nu(J) + \partial J g(J, xy)) \bar{t}$$

$$+ \varepsilon P^\theta_1 + \varepsilon^2 \mathcal{O}^*(\log^4 \rho + \beta^{-1} \log^3 \rho + \log^2 \rho)$$
where
\[ P_1^\theta = F_1 + M_1^\circ S - M_1^\circ + \partial_\theta^2 \Phi^\pm(J, xy)(M_1^\circ S - M_1^\circ) + \tilde{t} (\partial \nu(J) + \partial_\theta^2 g(J, xy)) (M_1^\circ S - M_1^\circ) + \tilde{t} \partial_\theta^2 g(J, xy)(M_1^\circ S - M_1^\circ) \]

Using Lemmas 4.4 and 5.2 one can check that \( P_1^\theta = O^\ast(\log^2 \rho + \beta^{-1} \log \rho) \).

For the \( x \) and \( y \) components we have,
\[ F_x = \exp(-(\partial g(J, \varepsilon P_1, xy + \varepsilon P_1)\bar{t} + \Phi(J, \theta, \bar{t}) + \partial_\theta \Phi^\pm(J, xy)) \times (1 + O^\ast_2(\varepsilon, \rho)(\log \rho)) + O^\ast_3(\varepsilon, \rho) \]
\[ F_y = \exp((\partial g(J, \varepsilon P_1, xy + \varepsilon P_1)\bar{t} + \Phi(I, \theta, \bar{t}) + \partial_\theta \Phi^\pm(J, xy)) \times (y(xy + \varepsilon P_1) + E(J, \theta, \bar{t}) + \sigma x e^{\lambda(y)\tau} + \varepsilon \tau \bar{H}_1(J, \theta) \log |y| + O^\ast(\varepsilon, xy) \times (\lambda(J)xy + \varepsilon \bar{H}_1(J, \theta) + \varepsilon \lambda(J)(\nu(J), \partial_\theta \Phi^\pm(J, xy)) \log |y| + O^\ast_2(\varepsilon, xy) \times \left( \begin{array}{c} \xi \\ \eta \\ \tau \\ h \end{array} \right) \]

Moreover,
\[ \mathcal{S}(J, \xi, x, \tau) = J\xi + E(J)\tau + \sigma x e^{\lambda(y)\tau} + \varepsilon \tau \bar{H}_1(J, \xi + \nu(y)\tau). \] (34)

Recall that after switching to slow-fast variables, \( \bar{H}_1 \) does not depend on \( t \). This implies that this change of variables is still symplectic if we extend it by the identity to the \( (D, t) \) variables.

We have then the following changes
\[
\left( \begin{array}{c} \xi \\ \eta \\ \tau \\ h \end{array} \right) = \left( \begin{array}{c} \theta - \frac{\nu(J)}{\lambda(J)} \log |y| + \varepsilon \frac{1}{\lambda(J)} \bar{H}_1(J, \theta) \log |y| + O^\ast(\varepsilon, xy) \\ J + \varepsilon \frac{\lambda(J)}{\lambda(J)} \partial_\theta \bar{H}_1(J, \theta) \log |y| + O^\ast_2(\varepsilon, xy) \\ E(J) + \lambda(J)xy + \varepsilon \bar{H}_1(J, \theta) + \varepsilon \lambda(J)(\nu(J), \partial_\theta \Phi^\pm(J, xy)) \log |y| + O^\ast_2(\varepsilon, xy) \end{array} \right)
\]
and

\[
\begin{pmatrix}
\theta \\
J \\
x \\
y
\end{pmatrix} = \Upsilon \begin{pmatrix}
\xi \\
\eta \\
\tau \\
h
\end{pmatrix} = \left( \begin{array}{c}
\xi + \nu(\eta)\tau + \varepsilon\tau \tilde{H}_1(\eta, \xi + \nu(\eta)\tau) + \varepsilon\mathcal{O}(\varepsilon, h - E) \\
\eta + \varepsilon\tau \partial_\xi \tilde{H}_1(\eta, \xi + \nu(\eta)\tau) + \varepsilon\mathcal{O}(\varepsilon \beta^{-1}, h - E) \\
\frac{\sigma}{\lambda(\eta)} e^{-\lambda(\eta)\tau} (h - E(\eta) - \varepsilon\tilde{H}_1(\eta, \xi + \nu(\eta)\tau)) + \varepsilon\mathcal{O}(\varepsilon \beta^{-1}, h - E) \\
\sigma e^{\lambda(\eta)\tau} + \mathcal{O}(\varepsilon, h - E)
\end{array} \right)
\]

We call \( \Upsilon_0 \) the first order of this change of coordinates, defined by

\[
\begin{pmatrix}
\theta \\
J \\
x \\
y
\end{pmatrix} = \Upsilon_0 \begin{pmatrix}
\xi \\
\eta \\
\tau \\
h
\end{pmatrix} = \left( \begin{array}{c}
\xi + \nu(\eta)\tau \\
\eta \\
\tau \\
h
\end{array} \right) + \varepsilon \mathcal{O}(\varepsilon, h - E) \]

Theorem 3.4 can be rephrased as the following lemma.

**Lemma 8.1.** Assume that the function \( w_0 \) in (19) satisfies

\[ c^{-1}\varepsilon^{1+a} < |w_0(\eta^*, h^*)| < c\varepsilon \]

for some \( c > 0 \) and \( 1 \geq a > 0 \) independent of \( \varepsilon \). Then, there exists functions \( B^{z,\sigma} \), \( z = \eta, \xi, h, D, \tau, \sigma \), satisfying

\[ B^{z,\sigma} = \mathcal{O}(\varepsilon \log \varepsilon) \text{ for } z = \eta, h, D \text{ and } B^{z,\sigma} = \mathcal{O}(\varepsilon \log^2 \varepsilon) \text{ for } z = \xi, \tau \]

such that the separatrix map \( \mathcal{S}M_\varepsilon \) has the following form

\[
\begin{align*}
\eta^* &= \eta - \frac{\partial_\xi w_0^\sigma}{\lambda} \log \left| \frac{\kappa^\sigma w_0^\sigma}{\lambda} \right| + B^{\eta,\sigma}(\eta, \xi, \tau, \bar{t}) + \mathcal{O}(\varepsilon^{5/3}) \\
\xi^* &= \xi + \mu^* + \frac{\partial_\eta w_0^\sigma}{\lambda} \log \left| \frac{\kappa^\sigma w_0^\sigma}{\lambda} \right| + B^{\xi,\sigma}(\eta, \xi, \tau, \bar{t}) + \mathcal{O}(\varepsilon) \\
h^* &= h - \frac{\partial_\tau w_0^\sigma}{\lambda} \log \left| \frac{\kappa^\sigma w_0^\sigma}{\lambda} \right| + B^{h,\sigma}(\eta, \xi, \tau, \bar{t}) + \mathcal{O}(\varepsilon^{5/3}) \\
\tau^* &= \tau + \tilde{t} + \frac{\partial_{\bar{h}} w_0^\sigma}{\lambda} \log \left| \frac{\kappa^\sigma w_0^\sigma}{\lambda} \right| + B^{\tau,\sigma}(\eta, \xi, \tau, \bar{t}) + \mathcal{O}(\varepsilon) \\
\sigma^* &= \sigma \text{ sgn} w_0^\sigma.
\end{align*}
\]

Moreover, the evolution of the \( D \) component satisfies,

\[ D^* = D + B^{D,\sigma} + \mathcal{O}(\varepsilon). \]
Proof. One can see that
\[
\eta^* = \eta + \varepsilon \mathcal{P}_1^I \circ \Upsilon_0 - \varepsilon r \partial_\xi \mathcal{H}_1(\eta, \xi + \nu(\eta)\tau) + \varepsilon (\tau + \bar{t}) \partial_\xi \mathcal{H}_1(\eta, \xi + \nu(\eta)(\tau + \bar{t})) + \varepsilon \frac{1}{\lambda(\eta)} \partial_\xi \mathcal{H}_1(\eta, \xi + \nu(\eta)(\tau + \bar{t})) \log \left| \frac{\kappa(\eta)}{\lambda(\eta)} | w_0^\sigma | + \mathcal{O}_2^*(\varepsilon) \right| \log |\varepsilon|
\]
\[
h^* = h + \varepsilon \nu(\eta) \mathcal{P}_1^I \circ \Upsilon_0 + \varepsilon \lambda(\eta) \mathcal{P}_1^* \circ \Upsilon_0
\]
\[
+ \varepsilon \left( \mathcal{H}_1(\eta, \xi + \mu^\sigma(\eta) + \nu(\eta)(\tau + \bar{t}) - \mathcal{H}_1(\eta, \xi + \nu(\eta)\tau) \right)
\]
\[
+ \varepsilon r \nu(\eta), \mathcal{H}_1(\eta, \xi + \mu^\sigma(\eta) + \nu(\eta)(\tau + \bar{t})) - \mathcal{H}_1(\eta, \xi + \nu(\eta)\tau) \right)
\]
\[
+ \varepsilon (\nu(\eta), \mathcal{H}_1(\eta, \xi + \mu^\sigma(\eta) + \nu(\eta)(\tau + \bar{t})) \left( \bar{t} + \frac{1}{\lambda(\eta)} \log \left| \frac{\kappa(\eta)}{\lambda(\eta)} | w_0^\sigma | \right| \right)
\]
\[
+ \mathcal{O}^*(\varepsilon^{5/3})
\]

Then, it is enough to define
\[
B_{\nu, \sigma} = \varepsilon \mathcal{P}_1^I \circ \Upsilon_0 - \varepsilon r \partial_\xi \mathcal{H}_1(\eta, \xi + \nu(\eta)\tau)
\]
\[
+ \varepsilon (\tau + \bar{t}) \partial_\xi \mathcal{H}_1(\eta, \xi + \nu(\eta)(\tau + \bar{t}))
\]
\[
B_{\nu, \sigma} = \varepsilon \nu(\eta) \mathcal{P}_1^I \circ \Upsilon_0 + \varepsilon \lambda(\eta) \mathcal{P}_1^* \circ \Upsilon_0
\]
\[
+ \varepsilon (\nu(\eta), \mathcal{H}_1(\eta, \xi + \mu^\sigma(\eta) + \nu(\eta)(\tau + \bar{t}) - \mathcal{H}_1(\eta, \xi + \nu(\eta)\tau) \right)
\]
\[
+ \varepsilon (\nu(\eta), \mathcal{H}_1(\eta, \xi + \mu^\sigma(\eta) + \nu(\eta)(\tau + \bar{t})) - \mathcal{H}_1(\eta, \xi + \nu(\eta)\tau) \right)
\]
\[
+ \varepsilon (\nu(\eta), \mathcal{H}_1(\eta, \xi + \mu^\sigma(\eta) + \nu(\eta)(\tau + \bar{t})) \bar{t}
\]
\[
(37)
\]

Using the estimates for the functions \( \mathcal{P}_1^I \), one can deduce bounds for these functions.

For the \( D \) component, it is enough to recall that
\[
D^* = D + \varepsilon \mathcal{P}_1^D (I, \theta, q) + \mathcal{O}_2^*(\varepsilon)
\]
and to perform the change of variables \( \Upsilon \). Thus
\[
D^* = D + \varepsilon \mathcal{P}_1^D \circ \Upsilon_0 + \mathcal{O}_2^*(\varepsilon)
\]
and it is enough to define
\[
B_{\nu, \sigma} = \varepsilon \mathcal{P}_1^D \circ \Upsilon_0
\]
(38)

Proceeding analogously, one can compute also the formulas for the angular variables \( \xi \) and \( \tau \). \( \square \)

A The separatrix map of the generalized Arnold example

In this appendix we apply Theorems [3.3] and [3.4] to the generalized Arnold example [2]. The Arnold example presents many simplifications which imply that
many formulas turn out to be considerably simpler than in the general case. For these models all the transformations and maps are $C^\infty$.

For the Arnold example, the Hamiltonian $\tilde{H}_1$ vanishes. Nevertheless, this is not the case for $\tilde{H}_2$ and $\tilde{H}_3$, defined in (22) and (24) respectively. Thus, one has to consider the two regimes.

We first analyze the non-resonant regime as defined in (15). In this regime, $\tilde{H}_j = 0$ for $j = 1, 2, 3$. Second, we consider the resonant regime for any resonance $k \in \mathcal{N}(2)(H_1)$ as defined in (16). In these regions, $\tilde{H}_1 = 0$ thanks to the particular form of the Arnold model. Nevertheless, $\tilde{H}_2$ and $\tilde{H}_3$ do not vanish.

For the generalized Arnold example, the matrix $\Lambda$ defined in (9) is just

$$\Lambda = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

therefore its positive eigenvalue $\lambda$ is $\lambda = 1$ and therefore, it is independent of $I$.

The function $g$ defined through Moser normal form in Lemma 4.1 is independent of $I$ as well and satisfies $g(r) = r + O_2(r)$. Then, the function $w$ defined in (17) is defined by

$$w = g^{-1}\left( h - \frac{\eta^2}{2} \right).$$

Analogously, since the Moser normal form is independent of $I$, the functions $\Phi^\pm$ involved in the gluing map is also $I$ independent. Thus, the functions $\mu^\pm$ in (10).

Then, we have the following theorem.

**Theorem A.1.** Fix $\beta > 0$ and $1 \geq a > 0$. For $\varepsilon$ sufficiently small there exist $c > 0$ independent of $\varepsilon$ and a canonical system of coordinates $(\eta, \xi, h, \tau)$ such that in the non-resonant zones $\text{NRR}_\beta$ we have

$$\eta = I + O^*_1\left( \varepsilon, H_0 - \frac{f^2}{2} \right), \quad \xi + \nu(\eta)\tau = \varphi + f, \quad h = H_0 + O^*_1\left( \varepsilon, H_0 - \frac{f^2}{2} \right),$$

where $f$ denotes a function depending only on $(I, p, q, \varepsilon)$ and such that $f = O(\varepsilon)$. In these coordinates the separatrix map has the following form. For any $\sigma \in \{-, +\}$ and $(\eta^*, h^*)$ such that

$$c^{-1}\varepsilon^{1+a} < |w(\eta^*, h^*)| < c\varepsilon, \quad |\tau| < c^{-1}, \quad c < |w(\eta^*, h^*)| e^{\lambda(\eta^*)} < c^{-1},$$

the separatrix map $(\eta^*, \xi^*, h^*, \tau^*) = SM(\eta, \xi, h, \tau)$ is defined implicitly as follows

$$\eta^* = \eta - \varepsilon M^{\tau n}_1 + \varepsilon^2 M^{\tau n}_2 + O^*_3(\varepsilon \log \varepsilon),$$

$$\xi^* = \xi + \partial_\eta w(\eta^*, h^*) \left[ \log |w(\eta^*, h^*)| + (\Phi^\sigma)'(w(\eta^*, h^*)) \right] + O^*_1(\varepsilon \log \varepsilon),$$

$$h^* = h - \varepsilon M^{\tau^*}_1 + \varepsilon^2 M^{\tau^*}_2 + O^*_3(\varepsilon),$$

$$\tau^* = \tau + \ell + \partial_h w(\eta^*, h^*) \left[ \log |w(\eta^*, h^*)| + (\Phi^\sigma)'(w(\eta^*, h^*)) \right] + O^*_1(\varepsilon \log \varepsilon),$$

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where $M_1^*\text{ and }\Phi^\sigma$ are $C^\infty$ functions defined in Lemmas 6.1 and 7.2 respectively, and $I$ is an integer satisfying (8). The functions $M_1^*$ are evaluated at $(\eta^*,\xi,h^*,\tau)$ and satisfy

$$M_1^{\sigma,1} = \partial_\xi \Theta^\sigma + O_2^\sigma(w), \quad M_1^{\sigma,2} = \partial_\tau \Theta^\sigma + O_2^\sigma(w),$$

where $\Theta^\sigma$ is the Melnikov potential defined by

$$\Theta^\sigma(\eta,\xi,\tau) = \int_{-\infty}^{+\infty} H_1(\Gamma^\sigma(\eta,\xi,\tau + t), t - \tau) \, dt$$

(39)

and $\Gamma^\sigma$ are the time parameterization of the pendulum separatrices, that is

$$\Gamma^\sigma(\eta,\xi,\tau) = \left(\eta, \xi + I\tau, 4\arctan(e^{\sigma \tau}), \frac{2\sigma}{\cosh \tau}\right).$$

Proof. This theorem can be easily deduced from Theorem 3.1. One just need to recall that the Moser normal form only depends on $p$ and $q$ (that is the reason for the better estimates for the function $f$).

To deduce the formula of the Melnikov potentials $\Theta^\sigma$ defined in (14) it is enough to recall that for the generalized Arnold model (2), $\mu^\sigma = 0$ and that $H_1(I, \varphi, 0, 0, t) = 0$. $\square$

Now we analyze the resonant regions. We consider the slow fast variables $(J, \theta, D, t)$ defined in Section 3. The function $w_0$ defined in (5) just becomes

$$w_0(\eta, h) = h - \frac{I^2}{2}.$$

**Theorem A.2.** Fix $\beta > 0$, $k \in \mathcal{N}^{(2)}(H_1)$, and $1 \geq a > 0$. For $\varepsilon$ sufficiently small there exist $c > 0$ independent of $\varepsilon$ and a canonical coordinates $(\eta, \xi, h, \tau)$ such that in the resonant zone $\text{Res}_\beta$ the following conditions hold:

- the canonical form $\omega = d\eta \wedge d\xi + dh \wedge d\tau$;

- $\eta = I + O_1^*(\varepsilon, H_0 - E(I)), \xi + \nu(\eta) = \varphi + f, h = H_0 + O_1^*(\varepsilon, H_0 - E(I))$, where $f$ denotes a function depending only on $(I, p, q, \varepsilon)$ and such that $f = O(\varepsilon), f = O(w_0^\sigma + \varepsilon)$.

- In these coordinates $\mathcal{S}_M$ has the following form. For any $\sigma \in \{-, +\}$ and $(\eta^*, h^*)$ such that

$$c^{-1} \varepsilon^{a+1} < |w_0(\eta^*, h^*)| < c\varepsilon, \quad |\tau| < c^{-1}, \quad c < |w_0(\eta^*, h^*)| e^{\lambda(\sigma)^\tau} < c^{-1},$$

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the separatrix map \((\eta^*, \xi^*, h^*, \tau^*) = SM_\varepsilon(\eta, \xi, h, \tau)\) is defined implicitly as follows
\[
\eta^* = \eta + B^\eta,\sigma(\eta, \xi, \tau, t) + \mathcal{O}^*(\varepsilon^{5/3}) \\
\xi^* = \xi + \frac{\partial_\eta w^\sigma_0}{\lambda} \log \left| \frac{\kappa^\sigma w^\sigma_0}{\lambda} \right| + B^{\xi,\sigma}(\eta, \xi, \tau, t) + \mathcal{O}^*(\varepsilon) \\
h^* = h + B^{h,\sigma}(\eta, \xi, \tau, t) + \mathcal{O}^*(\varepsilon^{5/3}) \\
\tau^* = \tau + \frac{\partial_h w^\sigma_0}{\lambda} \log \left| \frac{\kappa^\sigma w^\sigma_0}{\lambda} \right| + B^{\tau,\sigma}(\eta, \xi, \tau, t) + \mathcal{O}^*(\varepsilon) \\
\sigma^* = \sigma \text{ sgn} w^\sigma_0,
\]
where the functions \(B^{z,\sigma}\) are defined in Lemma 8.1 and satisfy
\[
B^{\eta,\sigma} = \varepsilon \partial_\xi \Theta^\sigma, \quad B^{h,\sigma} = \varepsilon \partial_\tau \Theta^\sigma, \quad B^{\tau,\sigma}, B^{\xi,\sigma} = \mathcal{O}^*(\varepsilon \log \varepsilon).
\]
and \(\Theta^\sigma\) are the Melnikov potentials given in (39).

Moreover, the evolution of the variable \(D\), conjugate to the time \(t\), satisfies,
\[
D^* = D + B^D(\eta, \xi, \tau, t) + \mathcal{O}^*(\varepsilon^{5/3}).
\]
for certain function \(B^D\) satisfying \(B^D = \varepsilon \partial_t \Theta^\sigma\).

Proof. To prove this theorem is enough to use the same properties of the generalized Arnold model used in the proof of Theorem A.1 and go through Sections 5.2 and 8 to improve the estimates.

Since \(\overline{\mathbf{H}}_1 = 0\), the functions \(F_1\) and \(G_1\), defined in Section 8 also vanish. This implies better estimates for the functions \(\mathcal{P}_1^J\) and \(\mathcal{P}_1^\theta\) defined in Section 8 \(\mathcal{P}_1^J = \mathcal{O}^*(\varepsilon)\) and \(\mathcal{P}_1^\theta = \mathcal{O}^*(\varepsilon \log \varepsilon)\). Finally, since \(F_1\) and \(G_1\) vanish, one can proceed as in the non-resonant regime to see that \(B^{\eta,\sigma}\) and \(B^{h,\sigma}\) are given by the Melnikov potential. \(\square\)

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