Sustaining Moore’s Law Through Inexactness
A Mathematical Foundation

John Augustine, 1* Krishna Palem, 2 Parishkrati 1

1Department of Computer Science and Engineering,
Indian Institute of Technology Madras, Chennai, India.
2Department of Computer Science, Rice University, Houston, USA.

*Corresponding author; E-mail: augustine@iitm.ac.in

Abstract

Inexact computing aims to compute good solutions that require considerably less resource – typically energy – compared to computing exact solutions. While inexactness is motivated by concerns derived from technology scaling and Moore’s law, there is no formal or foundational framework for reasoning about this novel approach to designing algorithms. In this work, we present a fundamental relationship between the quality of computing the value of a boolean function and the energy needed to compute it in a mathematically rigorous and general setting. On this basis, one can study the tradeoff between the quality of the solution to a problem and the amount of energy that is consumed. We accomplish this by introducing a computational model to classify problems based on notions of symmetry inspired by physics. We show that some problems are symmetric in that every input bit is, in a sense, equally important, while other problems display a great deal of asymmetry in the importance of input bits. We believe that our model is novel and provides a foundation for inexact Computing. Building on this, we show that asymmetric problems allow us to invest resources favoring the important bits – a feature that can be leveraged to design efficient inexact algorithms. On the negative side and in contrast, we can prove that the best inexact algorithms for symmetric problems are no better than simply reducing the resource investment uniformly across all bits. Akin to classical theories concerned with space and time complexity, we believe the ability to classify problems as shown in our paper will serve as a basis for formally reasoning about the effectiveness of inexactness in the context of a range of computational problems with energy being the primary resource.

Many believe that the exponential scaling afforded by Moore’s law [Moore 1965] is reaching its limits as transistors approach nanometer scales. Many of these limitations are based on physics based limits ranging over thermodynamics and electromagnetic noise [Kish 2002] and optics [Ito and Okazaki 2000]. Given that information technology is the prime beneficiary of Moore’s law, computers, memories, and related chip technologies are likely to be affected the most. Given the tremendous value of sustaining Moore’s law through information technology in a broad sense, much effort has gone into sustaining Moore’s law, notably through innovations in material science and electrical engineering. Given the focus on information technology, a central tenet of these innovations has been to preserve the behavior of CMOS transistors and computing systems built from them.

While many of these innovations revolve around non-traditional materials such as graphene [Novoselov et al. 2004, Chodos 2004] supplementing or even replacing CMOS [Anthony 2014], exciting developments based on alternate and potentially radical models of computing have also emerged. Notable examples
include DNA [Adleman, 1994, Boneh et al., 1996], and quantum computing frameworks [Benioff, 1980, Feynman, 1982, Deutsch, 1985]. However, these exciting approaches and alternate models face a common and potentially steep hurdle to becoming deployable technologies leading to the preeminence of CMOS as the material of choice. This brings the importance of Moore’s law back to the fore and consequently, in the foreseeable future, the centrality of CMOS to growth in information technologies remains.

The central theme of this paper is to develop a coherent theoretical foundation with the goal of reconciling these competing concerns. On the one hand, continuing with CMOS centric systems is widely believed to result in hardware that is likely to be erroneous or function incorrectly in part. On the other hand, dating back to the days of Alan Turing [1936] and explicitly tackled by von Neumann [1956], a computer — the ubiquitous information technology vehicle — has an unstated expectation that it has to function correctly. This expectation of computers always functioning correctly as an essential feature is at the very heart of our alarm about the doomsday scenario associated with the end of Moore’s law. For if one can use computers with faulty components as they are with concomitant but acceptable errors in the computation, we could continue to use CMOS transistors albeit functioning in a potentially unreliable regime.

Over the past decade, this unorthodox approach to using a computer and related hardware such as memory built out of faulty components, and used in this potentially faulty mode, referred to as inexact computing, has emerged as a viable alternative to coping with the Moore’s law cliff. Palem and Lingamneni [2013] and Palem [2014] (and references therein) provide a reasonable overview of inexact computing practice. At its core, the counterintuitive thesis behind inexactness is to note that, perhaps surprisingly, working with faulty components can in fact result in computing systems that are thermodynamically more efficient [Palem, 2003a,b, Korkmaz et al., 2006]. This approach simultaneously appeals to another hurdle facing the sustenance of Moore’s law. Quite often referred to as the energy-wall or power-wall, energy dissipation has reached such prohibitive levels that being able to cope with it is the predominant concern in building computer systems today. For example, to quote from an article from the New York Times [Markoff, 2015] about the potential afforded through inexact computing: “If such a computer were built in today’s technologies, a so-called exascale computer would consume electricity equivalent to 200,000 homes and might cost $20 million or more to operate.”

While individual technological artifacts demonstrating the viability of inexact computing might be many, a coherent understanding of how to design algorithms — essential to using inexact computing in large scale — and understand the inherent limits to the power of this idea are not there. Such characterizations are typically the purview of theoretical computer science, where questions of designing efficient algorithms, and inherent limits to being able to design efficiently are studied. While algorithm design is concerned with finding efficient ways of solving problems, inherent limits allow us to understand what is not possible under any circumstance within the context of a mathematically well-defined model. Understanding what is inherent to computing in abstract terms has been a significant part of these enquiries, and has evolved into the field referred to as computational complexity [Arora and Barak, 2009, Moore and Mertens, 2011]. In this paper, we present a computational complexity theoretic foundation to characterizing inexactness, and to the best of our knowledge for the first time.

As in classical complexity theory, the atomic object at the heart of our foundation is a bit of information. However, inexactness allows something entirely novel: each bit is characterized by two attributes or dimensions, a cost and a quality. Historically, physical energy was the cost and the probability of correctness was the quality [Palem, 2003b, 2005]. As shown in Figure 1 (originally reported in [Korkmaz et al., 2006]), under this interpretation, a cost versus quality relationship was measured in the context of physically constructed CMOS gates. More recently, Frustaci et al. [2015] have presented a voltage-scaled SRAM along with a characterization of the energy/error tradeoff, where, unsurprisingly, we see that the bitcell error rate
(BER) drops exponentially as $V_{dd}$ increases.

![Figure 1: The quality-cost relationship referred to as the energy-probability relationship (e-p relationship) of a CMOS switch from [Akgul et al., 2006, Korkmaz et al., 2006], where the energy spent $e$ increases exponentially with increasing probability of correctness $p$. Specifically, from [Korkmaz et al., 2006], the probability the operation is correct $p = 1 - \frac{1}{2} \text{erfc} \left( \frac{V_{dd}}{\sqrt{2} \sigma} \right)$, where $V_{dd}$ is the supply voltage, $\sigma$ is the standard deviation of additive gaussian noise, and the complementary error function $\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-u^2} du$.}

From a historical perspective, our work here builds naturally on the entire theme of computing or reasoning in the presence of uncertainty [Feige et al., 1994, Kenyon and King, 1994], which is concerned about computing reliably in the presence of erroneous or uncertain information. In this context, the bits in question have one of our two dimensions, quality alone. As a bit is read, depending on the (unspecified) circumstance, it can be erroneous with a certain probability. However, early work in inexactness (see Figure 1) showed that error or quality can in fact be a parameter that can be related through the underlying physics (thermodynamics) to a cost, namely energy. Thus, our work can be viewed as an extension of classical theoretical foundations to reasoning about uncertainty, to one where we can trade uncertainty with cost: less uncertain being typically much more expensive! In this sense, the degree of uncertainty of cost is an attribute that we can trade-off based on what we wish to pay, as opposed to an externally imposed quantity that we are forced to live with — the theme of prior work in this domain.

In order to have a clear basis for our discussion on computing, let us define a computational problem as evaluating a Boolean function $f : \{0,1\}^n \rightarrow \mathbb{Z}$. Although quite elementary, such Boolean functions possess the ability to succinctly encode input/output structure of problems in computing without compromising any of the mathematical rigor needed for careful analysis [O’Donnell, 2014]. Evaluating a Boolean function $f$ can be represented as a truth table $T_f$ with $2^n$ rows corresponding to each possible $n$-bit vector, and $n + 1$ columns; we will use $c_f(i, j)$ (or just $c(i, j)$ when clear from context) to denote the $j$th element in the $i$th row of $T_f$. The first $n$ columns correspond to the $n$ input bit positions and the $(n+1)$th column represents the output. To facilitate measuring the quality of the solution we produce, we view our output as a number in $\mathbb{Z}$. Since we are interested in inexact computing, let us suppose that an algorithm outputs $f'(I)$ for an input string $I \in \{0,1\}^n$, which could be different from the correct output $f(I)$. Then, the absolute difference $|f(I) - f'(I)|$ captures the magnitude of error. Although algorithms we consider in this paper are deterministic, it must be noted that the magnitude of error will be a random variable because input bits
can be read incorrectly with some probability.

Let us consider some elementary examples of Boolean functions. The OR problem for instance takes \( n \) bits as input and outputs a 1 except when all input bits are zeros. The Unary Evaluation problem (or UE in short) outputs the number of 1’s in the input, while Binary Evaluation problem (or BE in short) outputs \( \sum_{i=0}^{n-1} 2^i b_i \), where \((b_{n-1}, b_{n-2}, \ldots, b_0)\) is the input bit string. These problems can be succinctly captured in truth table representation as shown in Table 1.

| \( b_2 \) | \( b_1 \) | \( b_0 \) | OR | UE | BE |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 1 | 1 | 1 |
| 0 | 1 | 0 | 1 | 1 | 2 |
| 0 | 1 | 1 | 1 | 2 | 3 |
| 1 | 0 | 0 | 1 | 1 | 4 |
| 1 | 0 | 1 | 1 | 2 | 5 |
| 1 | 1 | 0 | 1 | 2 | 6 |
| 1 | 1 | 1 | 1 | 3 | 7 |

Table 1: Truth table indicating the input and output for OR, UE, and BE.

Let us now impose the two dimensions of cost and quality upon computational problems. Let us suppose that we invest a cost (or energy) \( e_i \) on each bit \( b_i \). Under finite values of \( e_i \), there will be a loss in quality of the bits. Therefore, our view of the input bits will be restricted to an approximate bit vector \((b'_0, b'_1, \ldots, b'_n)\). We then (inspired by [Akgul et al., 2006]) model the quality in probabilistic terms as:

\[
\Pr[b_i \neq b'_i] = 2^{-e_i}.
\]

Thus, under a finite energy budget, the outcome will be an approximation with the quality of the approximation increasing as the energy budget is increased.

Let us now consider a thought experiment that brings out two different modes in which algorithms can operate via a novel and elegant use of permutation groups. Our algorithm must assign each input bit with an energy value such that the total energy is within some finite budget. Suppose an adversary has the power to permute the energy values (but not the associated bits) by choosing a permutation \( \pi \) uniformly at random from some permutation group \( G \) (with the two extremes, the identity permutation \( I_n \) and the symmetric group \( S_n \), being of most interest). This will imply that the energy values associated with each bit could change. The algorithm is aware of \( G \), but not the exact permutation \( \pi \) chosen by the adversary. The question that comes to mind now is: how much can the adversary affect the outcome? Clearly, a well-designed optimal algorithm will try to compensate for this adversarial intrusion in some way, but can it succeed in mitigating the effects of this adversarial permutation?

Let us consider the OR problem. If any one of the input bits is a 1, the correct output is a 1, and in this sense, every bit has equal impact on the output. Therefore, intuitively, an optimal algorithm under a finite energy budget must allocate equal energy to every bit. (We will formally prove this shortly.) When the energy values across the input bits are the same, the adversary is rendered toothless, and cannot impact the quality of the outcome in anyway. Therefore, we say that the OR problem is symmetric under \( G \). In fact, it is straightforward to see that OR is symmetric even under the symmetric group \( S_n \) that captures all
$n!$ permutations. With a little thought, one can also surmise that the Unary Evaluation problem is also symmetric under $S_n$.

To build a contrast, let us consider the Binary Evaluation problem. When the adversary is restricted to $I_n$, an optimal algorithm will assign more energy to bit $b_{n-1}$ as its “impact” on the output can be as high as $2^{n-1}$. However, when the adversary is empowered with $S_n$, the algorithm cannot favor bit $b_{n-1}$ over $b_0$. Therefore, the adversary can significantly affect the quality of the output. Thus, Binary Evaluation is said to be an asymmetric problem.

There is a curious perhaps even striking connection between the formulation above and the role that symmetries and asymmetries played in physics at the turn of the last century. Perhaps the earliest work that historians point to as a basis of this connection is the work of Pierre Curie more than a century ago widely recognized as Curie’s principle [Chalmers, 1970]. Informally speaking, he ties the appearance of phenomena to when a system is “transformed.” We can interpret this to mean that for a change of some type to occur in the system, we need an absence of symmetries. Physicists tend to think of outcomes or effects in terms of phenomena and so another way of interpreting Curie’s powerful concept is to note that for a phenomenon to occur or exist, there must be inherent asymmetries in the system. In our own case, the example of Boolean evaluation mentioned above has a curious analogical connection since the quality of the output is the basis for observing change and, as noted above, asymmetries are an essential part of being able to observe change—since symmetric case of the OR function for example will not exhibit any change. Thus, in our case also, the existence of asymmetries is inherently necessary to observe changes in the “quality” of what is being computed. We note in passing that Curie’s work is the first that we are aware of which formally, in a mathematical sense, captures symmetries and asymmetries using a group theoretic formulation. This approach is also reflected in our own work where we use permutation groups as a basis for characterizing symmetries and asymmetries, as outlined above through the three examples. We remark in passing that several conditions must hold for Curie’s principle to be applicable, which physicists have documented extensively, and our analogical remark is a substantial simplification of the concept.

Following Curie, several historical figures in physics pursued the use of group theoretic symmetries and asymmetries as formal tools in physics with perhaps another analogical connection to our symmetric case—the OR problem described above. Best characterized in the work of Emmy Noether [1918] and again based on a group theoretic foundation—Lie groups to be specific—paraphrased, Noether’s theorem states that a physical system that embodies symmetries will result in (physical) quantities to be preserved or conserved under transformations. Thus, changes in systems which embody such symmetries will not yield observable changes in physical quantities of interest such as momentum and energy. In our own symmetric case, for example in the OR problem, we can interpret the quality of the output to be preserved under the (permutation) transformation of the energy vector stated above. Consequently, we can conclude that for symmetric functions in our sense, the quality is conserved under such transformations. We wish to add that our own framework in this paper was inspired by the style of thinking central to symmetries and symmetry breaking in physics as exemplified by the two cases discussed above, but that the connection is analogical—we do not wish to imply any novel insights in the physics domain based on our work presented in the sequel.

While we illustrated symmetry and asymmetry at the two extremes (OR and UE at one end and BE at the other), we can clearly envision a host of intermediate problems for which the symmetry is broken at various levels. We capture the level to which the symmetry can be broken by a parameter called the measure of broken symmetry or MoBS, which we formally define shortly. We show that the MoBS for symmetric problems like OR and UE is 1, but exponential in $n$ for asymmetric problems like BE. (See Table 2 for a complete listing of results.)
Let us suppose that \( A \) is aware of the permutation group \( P \). We emphasize that this is not the case. We will go on to show that the best algorithm in the blindfolded setting elegantly corresponds to the traditional computational model in which equal energy is invested in each bit.

Let us now consider how algorithm \( A \) can compute \( f \). We interpret the behavior of \( A \) as an attempt to decipher either the input row \( i \) that was fed in as the input or another row that produces the same output. Let us suppose that \( A \) concludes that the input row is \( A(i) \). We restrict \( A \) to behave deterministically even though the input row that it actually sees is random due to the application of \( E^{A}_\sigma \). Therefore, we can view \( c(A(i), n) \) as a random variable over the set of all legal output values, i.e., all values in the \( n \)th column of \( T_f \), but the randomness is over the probability space induced by \( E^{A}_\sigma \) and \( P^A \). Thus, the worst case probability that \( A \) is incorrect is \( \max \Pr[c(A(i), n) \neq c(i, n)] \) and the quality of the algorithm \( Q(A) \) (which in this case can be interpreted as the expected number of correct executions in the worst case taken over all input rows before we see an incorrect execution) is simply

\[
\min_{i} \left(1/\Pr[c(A(i), n) \neq c(i, n)]\right).
\]

In the clairvoyant setting, each bit \( j \) is read incorrectly with probability \( 2^{-e_j} \) and therefore, a clairvoyant algorithm can assign energy in proportion to the correctness required for bit \( j \). Let us now consider the consequence of blindfolding an algorithm.

**Claim 1.1 (Blindfolding Claim).** When we blindfold an algorithm, the probability that any bit is read incorrectly is at least 2\(^{-E/n}\), where \( E = \sum_j e_j \).

### Table 2: The MoBS of problems showing dramatic difference under asymmetric situations.

| Problem Name                  | MoBS |
|-------------------------------|------|
| OR Problem                    | 1    |
| Unary Evaluation              | 1    |
| Binary Evaluation             | \(2^{\Omega(n)}\) |
| Comparison two \( k \)-bit numbers | \(2^{\Omega(k)}\) |
| Sorting \( k \)-bit numbers   | \(2^{\Omega(k)}\) |

### 1 Computational Model and a Related Property

Going beyond traditional notions of algorithms designed for exact inputs, in our model of computation, we empower our algorithms with the ability (within well-defined bounds) to statistically alter noise characteristics in the input data and adapt its computational steps accordingly. We allow an algorithm \( A \) to work within an energy budget \( E \). Given \( E \), the algorithm must specify an energy vector \( E^A = (e_0, e_1, \ldots, e_{n-1}) \) (with \( \sum_j e_j \leq E \)) that will be used for reading the \( n \) input bits. Moreover, the algorithm is also aware of the permutation group \( P^A \) that will be employed for permuting the energy vector. Let \( \sigma \) be a permutation drawn uniformly at random from \( P \). Let us denote the energy vector \( E^A \) permuted by \( \sigma \) as \( E^A_\sigma = (e_{\sigma(0)}, e_{\sigma(1)}, \ldots, e_{\sigma(n-1)}) \). Algorithm \( A \) does not have direct access to the input row \( i \) in the truth table, but rather receives the input row after the energy vector \( E^A_\sigma \) has been applied to \( i \). More precisely, each cell \( c(i, j) \) is read correctly with probability \( 1 - 2^{-e_{\sigma(j)}} \) and incorrectly with probability \( 2^{-e_{\sigma(j)}} \). Note that while \( A \) is aware of the permutation group \( P^A \), the exact permutation \( \sigma \) is not revealed to \( A \). When \( P^A \) contains only the identity permutation, we say that \( A \) is clairvoyant because the adversary is incapable of hiding or altering the association between energy values and bit positions. Otherwise, we say that \( A \) is blindfolded by \( P^A \). When \( P^A = S_n \), the symmetric group defined on all \( n! \) permutations, we simply say that \( A \) is blindfolded. One may astutely wonder if the use of the permutation group is somehow restricting the adversary. We emphasize that this is not the case. We will go on to show that the best algorithm in the blindfolded setting elegantly corresponds to the traditional computational model in which equal energy is invested in each bit.

Let us now consider how algorithm \( A \) can compute \( f \). We interpret the behavior of \( A \) as an attempt to decipher either the input row \( i \) that was fed in as the input or another row that produces the same output. Let us suppose that \( A \) concludes that the input row is \( A(i) \). We restrict \( A \) to behave deterministically even though the input row that it actually sees is random due to the application of \( E^A_\sigma \). Therefore, we can view \( c(A(i), n) \) as a random variable over the set of all legal output values, i.e., all values in the \( n \)th column of \( T_f \), but the randomness is over the probability space induced by \( E^A_\sigma \) and \( P^A \). Thus, the worst case probability that \( A \) is incorrect is \( \max_i \Pr[c(A(i), n) \neq c(i, n)] \) and the quality of the algorithm \( Q(A) \) (which in this case can be interpreted as the expected number of correct executions in the worst case taken over all input rows before we see an incorrect execution) is simply

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\min_i \left(1/\Pr[c(A(i), n) \neq c(i, n)]\right).
\]

In the clairvoyant setting, each bit \( j \) is read incorrectly with probability \( 2^{-e_j} \) and therefore, a clairvoyant algorithm can assign energy in proportion to the correctness required for bit \( j \). Let us now consider the consequence of blindfolding an algorithm.

**Claim 1.1 (Blindfolding Claim).** When we blindfold an algorithm, the probability that any bit is read incorrectly is at least 2\(^{-E/n}\), where \( E = \sum_j e_j \).
Proof. As a result of the blindfolding, each of the $n$ entries in the energy vector is applied with equal probability and therefore, $\Pr[b_j \neq b_j'] = \sum_j (1/n)2^{-e_{\sigma(j)}}$, where $\sigma$ is a random permutation.

Since the arithmetic mean is at least the geometric mean, $\sum_j (1/n)2^{-e_{\sigma(j)}} \geq \left(\Pi_{j=0}^{n-1}2^{-e_{\sigma(j)}}\right)^{1/n} = 2^{-E/n}$ as claimed.

The implication of this claim is that the probability of error in each bit cannot be improved by employing an energy vector with non-uniform energy entries. This means that the best algorithm under the blindfolded setting corresponds to the current trend in computing whereby equal effort or energy is expended in reading each bit, but the clairvoyant setting opens up the possibility for variations. To capture a sense of the price we pay when employing uniform energy vectors, we define the measure of broken symmetry (MoBS) for computing $f$ as

$$\text{MoBS}(f) = \max_E \max_i \frac{\Pr[c(\text{BF}_E(i), n) \neq c(i, n)]}{\Pr[c(\text{CV}_E(i), n) \neq c(i, n)]},$$

where $\text{CV}_E$ and $\text{BF}_E$ are the optimal clairvoyant and optimal blindfolded algorithms, resp., while both are restricted to an energy budget of $E$. When $\text{MoBS}(f) = 1$ for the problem of computing some function $f$, then, we can infer that computing $f$ is identical under both the clairvoyant and the blindfolded setting. This reveals an inherent symmetry in the problem.

2 Symmetric Problems

To see how the notion of MoBS helps us understand the applicability of non-uniform energy distributions, let us consider two fundamental canonical problems. Consider first the Unary Evaluation problem (in short UE problem) that requires us to count the number $k$ of bits set to 1. Consider the optimal clairvoyant algorithm $\text{CV}$ and let us say that it employs energy vector $E_{\text{CV}} = (e_0, e_1, \ldots, e_{n-1})$. Let $(b_0, b_1, \ldots, b_{n-1})$ be the input vector and $(b'_0, b'_1, \ldots, b'_{n-1})$ be the vector read by the algorithm $\text{CV}$ under the energy vector $E_{\text{CV}}$.

We claim that $E_{\text{CV}}$ is in fact the uniform vector in which each $e_j = E/n$, where $E = \sum_j e_j$ is the total energy budget. For the sake of contradiction, let us assume an arbitrary $E_{\text{CV}}$. Consider bit $j$. Since $\Pr[b_j \neq b'_j] = 2^{-e_j}$, we can extend this to stating that $\Pr[b_j = 1] = b'_j(1 - 2^{-e_j}) + (1 - b'_j)2^{-e_j}$. Define Poisson trial variable $X_j$ that takes the value 1 with probability $b'_j(1 - 2^{-e_j}) + (1 - b'_j)2^{-e_j}$ and 0 otherwise. Since each bit $j$ is read independently of other bits, the algorithm can be viewed as optimizing how well $X = \sum_j X_j$ estimates $\sum_j b_j$. Since $X$ is the sum of $n$ Poisson trials, we can get the optimal estimation by minimizing the variance of $X$. Notice that $\text{Var}[X_j] = (1 - 2^{-e_j})2^{-e_j}$. Notice that for any two positions $j$ and $j'$, their combined variance $\text{Var}[X_j] + \text{Var}[X'_{j'}] = (1 - 2^{-e_j})2^{-e_j} + (1 - 2^{-e'_{j'}})2^{-e'_{j'}}$ is minimized when $e_j = e'_{j'}$ when $e_j + e'_{j'}$ is fixed. This can be easily extended to computing $\text{Var}[X]$ where we can show that the variance is minimized when $e_j = E/n$ for every $j$. We can therefore claim that $\text{CV}$ uses a uniform energy vector. Thus, $\text{MoBS}(\text{Unary Evaluation}) = 1$ because the same optimal algorithm can be employed in the blindfolded setting.

More generally, $\text{MoBS}(f) = 1$ for all symmetric Boolean functions $f$ where the outcome $f(I)$ equals $f(\sigma(I))$, where $\sigma$ is any permutation of the input bits $I$. [O’Donnell 2014]. The OR problem discussed earlier is a canonical example. But the set of symmetric problems is larger than the set of symmetric Boolean functions. Consider for example, the Tribes problem where (in a simplified sense), the $n$ input bits are partitioned into two tribes: the first $n/2$ bits and the second $n/2$ bits. The output of the Tribes function is a one iff at least one of the tribes consists of all 1 bits. This function is not a symmetric Boolean function.
as is witnessed by $\text{Tribes}(0011)=1$ while $\text{Tribes}(0101)=0$. However, the problem of evaluating the Tribes function is a symmetric problem.

Figure 2: A Venn diagram showing the various classes of computational problems that evaluate Boolean functions.

3 Asymmetric Problems

Thankfully, most real-world problems have quite the opposite structure wherein there is significant benefit to employing non-uniform energy vectors. To illustrate this, consider Binary Evaluation (or BE for short) where we have to compute $\sum_{j=0}^{n-1} 2^j b_j$, which arguably is the most fundamental problem. Clearly, the $(n-1)$th bit has impact $2^j$, which is significantly more important than the 0th bit and therefore displays a marked difference from the Unary Evaluation. One can intuitively see that non-uniform energy distribution ought to provide significant improvement.

In order to formalize this intuition about Binary Evaluation, consider an energy budget of $E = n(n+1)/2$. The blindfolded algorithm will use $e = (n+1)/2$ units of energy for each bit. The behavior of the blindfolded algorithm can be viewed as essentially evaluating $\sum_j w_j (b'_j (1 - 2^{-e}) + (1 - b'_j) 2^{-e})$. Without loss in generality, if we assume that bit $b'_{n-1} = 0$, then, the expected evaluated quantity — taking just the most significant bit into account and also noting that $w_{n-1} = 2^{n-1} = 2^{n-1} - 2^{-(n+1)/2} = 2^{(n-3)/2}$. However, in the clairvoyant setting, we can set $e_j = j + 1$, which respects the energy budget constraint. The expected error is $\sum_j w_j 2^{-j-1} = n/2$. Under characteristic quality function defined as the reciprocal of the expected error, therefore, $\text{MoBS(BE)} = \Omega(2^{n/2}/n)$. (This analysis is based on the unpublished work by Chakrapani and Palem[2011].)

Let us similarly analyze Comparison, the problem of comparing two $k = n/2$ bit numbers $x$ and $y$ to evaluate which one is larger. In the inexact computing perspective, it is more important to distinguish the two numbers when they are far apart, i.e., when $|x - y|$ is large, than when they are very
close to each other. With that perspective in mind, we define the quality function to be $1/(|x - y| \cdot \Pr[x \text{ and } y \text{ are incorrectly compared}])$. Let us consider an energy budget of $k(k + 1)/2$. Under the blindfolded setting, each bit-wise comparison will get $(k + 1)/2$ units of energy, but in the quality optimal setting, we can assign $j + 1$ units of energy for comparing $j$th bit positions, $0 \leq j \leq k - 1$. Extending the argument from BE, we will get $\text{MoBS}(\text{Comparison}) = 2\Omega(n)$.

Let us now consider the Sorting problem that takes $L$ numbers $(x_1, x_2, \ldots, x_L)$, each $k$ bits long, and reorders them in non-decreasing order. This will make the total number of input bits $n = kL$. Consider a pair of numbers $x_{\ell_1}$ and $x_{\ell_2}$, $\ell_1 \neq \ell_2$. As in the case of Comparison, we can tolerate the two numbers being wrongly ordered in the output sequence when $|x_{\ell_1} - x_{\ell_2}|$ is small, but not when the difference is large. Thus, as a natural extension to the quality function defined for Comparison, we consider the following as the quality function for Sorting wherein the probability of error in the ordering of a pair of numbers is weighted by their magnitude difference:

$$Q_{\text{Sorting}} = \sum_{\ell_1 < \ell_2} \frac{1}{(|x_{\ell_1} - x_{\ell_2}| \cdot \Pr[x_{\ell_1} \text{ and } x_{\ell_2} \text{ are wrongly ordered}] \cdot 2^{-j}}. \quad (1)$$

Consider the input in which $L/2$ of the numbers are of the binary form $(100 \cdots 0)$ and the remaining are of the form $(000 \cdots 0)$. There are at least $L^2/4$ pairs that we call expensive pairs for which the absolute magnitude difference is $2^{k-1}$. Let us now consider the expensive pairs under the blindfolded and the quality optimal scenarios. Extending the ideas from comparison, their probabilities of wrongly comparing an expensive pair under the blindfolded setting is at least $2 - (k+1)/2$. In the quality optimal setting, however, the probability will reduce to $2^{-k}$. Substituting these probability values into Equation 1 and subsequently into the formula for MoBS, we get:

$$\text{MoBS}(\text{Sorting}) \geq \sum_{\ell_1 \text{ and } \ell_2 \text{ are expensive}} \frac{|x_{\ell_1} - x_{\ell_2}| \cdot 2^{-(k+1)/2}}{\sum_{\ell_1 \text{ and } \ell_2 \text{ are expensive}} |x_{\ell_1} - x_{\ell_2}| \cdot 2^{-k}} = 2\Omega(k). \quad (2)$$

**Remarks**

Our work is built on the complexity theoretic philosophy of understanding the inherent resource needs of problems under various computational models, and classifying them based on those insights. From a utilitarian perspective, we now know that symmetric problems will not lead to significant gains from the principles of inexact computing. Thankfully, most real world problems display quite a bit of asymmetry where, we believe, opportunities abound, thus opening the door for future work in optimization techniques a practitioner may employ. Moreover, hardware that is capable of trading error for energy is not just a theoretical possibility, but a reality with several groups spanning industry and academia actively pursuing their fabrication. In fact, our model with its emphasis on memory errors is in fact a reflection of current pursuits in memory technology [Frustaci et al., 2015].

Finally, for clarity, we have focussed on problems where the quality of bits are probabilistically independent of each other, but this is somewhat simplistic as evidenced for example by the effect of carry propagating through adders [Chakrapani, 2008, Parishkrati, 2016]. Arbitrary Boolean functions are likely to be riddled with much more complex dependencies and we believe that there is significant scope for future work.
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