DOUBLE CANONICAL BASES

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Abstract. We introduce a new class of bases for quantized universal enveloping algebras $U_q(g)$ and other doubles attached to semisimple and Kac-Moody Lie algebras. These bases contain dual canonical bases of upper and lower halves of $U_q(g)$ and are invariant under many symmetries including all Lusztig’s symmetries if $g$ is semisimple. It also turns out that a part of a double canonical basis of $U_q(g)$ spans its center.

Contents

1. Introduction and main results 2
2. Equivariant Lusztig’s Lemma and bases of Heisenberg and Drinfeld doubles 8
  2.1. An equivariant Lusztig’s Lemma 8
  2.2. Double bases of Heisenberg and Drinfeld doubles 9
3. Dual canonical bases and proofs of Theorems 1.3, 1.5, 1.10 and 1.19 13
  3.1. Bicharacters, pairings, lattices and inner products 13
  3.2. Dual canonical bases 14
  3.3. Proofs of Theorems 1.3, 1.5 and 1.10 16
  3.4. Colored Heisenberg and quantum Weyl algebras and their bases 17
  3.5. Invariant quasi-derivations 18
4. Examples of double canonical bases 21
  4.1. Double canonical basis of $U_q(sl_2)$ 22
  4.2. Action on a double basis for $sl_2$ 25
  4.3. Some elements in double canonical bases in ranks 2 and 3 27
  4.4. Examples of central elements 30
5. Bar-equivariant braid group actions 31
  5.1. Invariant braid group action on Drinfeld double 31
  5.2. Elements $T_w$, quantum Schubert cells and their bases 33
  5.3. Proof of Theorem 3.11 34
  5.4. Braid group action on elements of $B_{n,r}$ 35
  5.5. Braid group action for $U_q(sl_2)$ 37
  5.6. Wild elements of a double canonical basis 38
Appendix A. Drinfeld and Heisenberg doubles 38
  A.1. Nichols algebras 38
  A.2. Bar and star involutions 39
  A.3. Pairing and quasi-derivations 40
  A.4. Double smash products 42
  A.5. Bialgebra pairings and doubles of bialgebras 43
  A.6. Bosonisation of Nichols algebras 45
  A.7. Drinfeld double 47
  A.8. Diagonal braidings 51

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1. Introduction and main results

The goal of this paper is to construct a canonical basis $B_\mathfrak{g}$ of a quantized enveloping algebra $U_q(\mathfrak{g})$ where $\mathfrak{g}$ is a semisimple or a Kac-Moody Lie algebra. For instance, if $\mathfrak{g} = sl_2$, then $B_\mathfrak{g}$ is given by

$$B_{sl_2} = \{ q^{n(m_- - m_+)} K^n C^{(m_0)} F^m - E^m | n \in \mathbb{Z}, m_0, m_\pm \in \mathbb{Z}_{\geq 0}, \min(m_-, m_+) = 0 \},$$

(1.1)

where we used a slightly non-standard presentation of $U_q(sl_2)$ (obtained from the more familiar one by rescaling generators $E \mapsto (q^{-1} - q)E$, $F \mapsto (q - q^{-1})F$)

$$U_q(sl_2) := \langle E, F, K^\pm \rangle : KEK^{-1} = q^2 E, KFK^{-1} = q^2 F, EF - FE = (q^{-1} - q)(K - K^{-1}).$$

Here the $C^{(m)}$ are central elements of $U_q(sl_2)$ defined by $C^{(0)} = 1, C = C^{(1)} = EF - q^{-1}K - qK^{-1} = FE - qK - q^{-1}K^{-1}$ and $C \cdot C^{(m)} = C^{(m+1)} + C^{(m-1)}$ for $m \geq 1$.

We call $B_{sl_2}$ double canonical because of the following remarkable properties (we will explain later, in §4.1, the reason why we must use Chebyshev polynomials $C^{(m)}$ instead of $C^m$).

1. Each element of $B_{sl_2}$ is homogeneous and is fixed by the bar-involution $u \mapsto \overline{u}$, which is the $Q$-anti-automorphism of $U_q(sl_2)$ given by $\overline{q} = q^{-1}$, $\overline{E} = E$, $\overline{F} = F$, $\overline{K} = K$.

2. $B_{sl_2}$ is invariant, as a set, under the $Q(q)$-linear anti-automorphisms $u \mapsto u^*$ and $u \mapsto u^t$ given respectively by $E^* = E$, $F^* = F$, $K^* = K^{-1}$ and $E^t = F$, $F^t = E$, $K^t = K$; and under the rescaled Lusztig’s symmetry $T$ given by $T(E) = qFK^{-1}$, $T(F) = q^{-1}KE$, $T(K) = K^{-1}$.

3. Each monomial in $E, F, K^\pm$ is in the $\mathbb{Z}_{\geq 0}[q, q^{-1}]$-span of $B_{sl_2}$.

4. $B_{sl_2}$ is compatible with the filtered mock Peter-Weyl components $J_s = \sum_{r=0}^s (ad U_q(sl_2))(K^r)$ (see e.g. [9]), where $ad$ denotes an adjoint action of the Hopf algebra $U_q(sl_2)$ on itself.

Remark 1.1. It should be noted that this basis is rather different from Lusztig’s canonical basis since the latter is in the modified quantized enveloping algebra $U_q^{\bullet}(sl_2)$, as defined in [14, §23.1.1] and we are not aware of any relationship between these bases. It would also be interesting to compare our bases with the ones announced by Fan Qin in [16, 17]. Finally, it should be noted that John Foster constructed in [8] a basis of $U_q(sl_2)$ which differs from (1.1) in that Chebyshev polynomials $C^{(m)}$ are replaced by $C^m$.

We establish properties of $B_{sl_2}$ in §§4.1.1.2 and 5.5.

To construct $B_\mathfrak{g}$ for any symmetrizable Kac-Moody Lie algebra $\mathfrak{g}$ we need some notation. Fix a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ and let $\mathfrak{g}_+ := \mathfrak{g} \oplus \mathfrak{h}$, which we view as the Drinfeld double of the Borel subalgebra $\mathfrak{b}_+ = \mathfrak{n}_+ \oplus \mathfrak{h}$. Let $U_q(\mathfrak{g})$ be the quantized enveloping algebra of $\mathfrak{g}$ of adjoint type over $k = \mathbb{Q}(q^2)$. Thus, $U_q(\mathfrak{g}_+)$ is the $k$-algebra generated by the $E_i, F_i, K_{\pm i}, i \in I$ subject to the relations: $K = (K_{+i}, K_{-i}) = i \in I$ is commutative and

$$[E_i, F_j] = \delta_{ij}(q_i^{-1} - q_j)(K_{+i} - K_{-i}), \quad K_{\pm i}E_j = q_i^{\pm a_{ij}} E_j K_{\pm i}, \quad K_{\pm i}F_j = q_i^{-\pm a_{ij}} F_j K_{\pm i},$$

(1.2)

$$\sum_{r,s \geq 0, r+s = 1-a_{ij}} (-1)^r F_i^{(r)} E_j F_i^{(s)} = \sum_{r,s \geq 0, r+s = 1-a_{ij}} (-1)^r E_j^{(r)} F_i^{(s)} F_j = 0$$

(1.3)

for all $i, j \in I$, where $A = (a_{ij})_{i,j \in I}$ is the Cartan matrix of $\mathfrak{g}$, the $d_i$ are positive integers such that $DA = (d_i a_{ij})_{i,j \in I}$ is symmetric, $q_i = q^{d_i}$, $X_i^{(k)} := \prod_{s=1}^k \langle s \rangle q_i^{-1} X_i^k$ and $\langle s \rangle_v = v^s - v^{-s}$. 

A.9. Drinfeld double in the diagonal case

References
Denote by $U^\pm$ to construct the double canonical basis $B_q^\pm$ of $U_q(g)$ (see Theorem 1.3 below). Furthermore, using a natural embedding of k-vector spaces $\iota_+: \mathcal{H}^+_q(g) \hookrightarrow U_q(g)$, which splits the canonical projection $\pi_+: U_q(g) \twoheadrightarrow \mathcal{H}^+_q(g)$ and the Lusztig’s lemma variant again, we build the **double canonical basis** $B_q^\pm$ of $U_q(g)$ out of $\iota_+(B_q^\pm)$. Finally, the desired basis $B_q^\pm$ is just the image of $B_q^\pm$ under the canonical projection $U_q(g) \twoheadrightarrow U_q(g) = U_q(g)/\langle K_{+i}K_{-i} - 1, i \in I \rangle$.

More precisely, by a slight abuse of notation we denote by $E_i, F_i, K_{+i} (\text{respectively} K_{-i})$ the images of $E_i, F_i, K_i (\text{respectively} K_i)$ under the canonical projection $\pi_+: U_q(g) \twoheadrightarrow \mathcal{H}^+_q(g)$ (respectively under $\pi_-: U_q(g) \twoheadrightarrow \mathcal{H}^-_q(g)$). It is obvious (and well-known) that, applying $\pi_\pm$ to the triangular decomposition $U_q(g) = K_- \otimes K_+ \otimes U_q^- \otimes U_q^+$, where $U_q^- = \langle F_i: i \in I \rangle$, $U_q^+ = \langle E_i: i \in I \rangle$, $K_\pm = \langle K_{\pm i}: i \in I \rangle$, one obtains a triangular decomposition

$$\mathcal{H}_q^\pm(g) = K_\pm \otimes U_q^- \otimes U_q^+.$$

Let $\hat{\Gamma} = \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\leq 0}$ and let $\{\alpha_{-i}, \alpha_{+i}\}_{i \in I}$ be the standard basis of $\hat{\Gamma}$. Then it is easy to see that $\hat{U}_q(g)$ and $\mathcal{H}^+_q(g)$ are graded by $\hat{\Gamma}$ via $\deg_\hat{\Gamma} \left( E_i = \alpha_{+i}, \deg_\hat{\Gamma} F_i = \alpha_{-i} \right.$ and $\deg_\hat{\Gamma} K_{\pm i} = \alpha_{+i} + \alpha_{-i}$. Denote by $K_+$ (respectively, $K_-$) the submonoid of $K_\pm$ generated by the $K_{+i} (\text{respectively} K_{-i})$, $i \in I$ and let $K = K_+ \times K_-$. Denote by $B_{n_\pm}$ the dual canonical basis of $U_q(g)$ (see [14] Chapter 14 and Section 3 for the details) i.e. the upper global crystal basis of $\mathcal{H}_q^\pm(g)$ (resp. $\mathcal{H}_q^\mp(g)$). By definition, each element of $B_{n_\pm}$ is homogeneous and is fixed under the involutive $\mathbb{Q}$-linear anti-automorphism $\tau$ of $U_q(g)$ determined by $q^{r^**} = q^{-\frac{r^**}{2}}$, $E_i = E_i, F_i = F_i, K_{\pm i} = K_{\pm i}$. For instance, if $g = sl_2$ then $B_{n_+} = \{ E^r : r \in \mathbb{Z}_{\geq 0} \}$ and $B_{n_-} = \{ F^r : r \in \mathbb{Z}_{\geq 0} \}$.

We have an action of the algebra $K$ on $U_q(g)$ defined by

$$K_{\pm i} \triangleright x := q_i^{\frac{1}{2}} \alpha_i^\vee(\deg x) K_{\pm i} x,$$

where $\alpha_i^\vee \in \text{Hom}_\mathbb{Z}(\hat{\Gamma}, \mathbb{Z})$ is defined by $\alpha_i^\vee(\alpha_{\pm j}) = \pm a_{ij}$ and $x \in U_q(g)$ is homogeneous. This action is more suitable for our purposes than the left multiplication due to the following easy property

$$\overline{K \triangleright x} = \overline{K} \circ \overline{x}, \quad K \in K, \; x \in U_q(g).$$

Note that this action, as well as the involution $\overline{\cdot}$, factors through to a $K_\pm$-action and an anti-involution $\overline{\cdot}$ on $\mathcal{H}_q^\pm(g)$ via the canonical projection $\pi^\pm: U_q(g) \twoheadrightarrow \mathcal{H}_q^\pm(g)$ and (1.5) holds.

We will show (Propositions 2.7 and 3.13) that for each pair $(b_{-}, b_{+}) \in B_{n_-} \times B_{n_+}$ there exists a unique monic $d_{b_{-}, b_{+}} \in \mathbb{Z}[q, q^{-1}]$ of minimal degree such that in $U_q(g)$ one has

$$d_{b_{-}, b_{+}}(b_+b_- - b_-b_+) \in \sum_{K \in K \setminus \{1\}, b_{\pm} \in B_{n_{\pm}}} \mathbb{Z}[q, q^{-1}] d_{b_{\pm}, b_{\mp}} K \circ (b'_+ b'_-) .$$

It turns out all $d_{b_{-}, b_{+}}$ are, up to a power of $q$, products of cyclotomic polynomials in $q$ (Proposition 3.9) and that for $g$ semisimple $d_{b_{-}, b_{+}} = 1$ for all $b_{\pm} \in B_{n_{\pm}}$ (Theorem 3.11). Some examples are shown in [14, 3].
Main Theorem 1.3. For any \((b_-, b_+) \in B_{n_-} \times B_{n_+}\) there is a unique element \(b_- \circ b_+ \in H_q^+(g)\) fixed by \(\tau\) and satisfying
\[
b_- \circ b_+ - d_{b_-, b_+} b_- b_+ = \sum q\mathbb{Z}[q] d_{b'_-, b'_+} K_+ \circ (b'_- b'_+)
\]
where the sum is over \(K_+ \in K_+ \setminus \{1\}\), \(b'_ \in B_{n_+}\) such that \(\deg b'_- b'_+ + \deg K_+ = \deg b_- b_+\).

We prove this theorem in Section 3 using a variant of Lusztig’s Lemma (Proposition 2.3) which we refer to as the equivariant Lusztig’s Lemma.

Corollary 1.4. The set \(B^+_q \coloneqq \{ K \circ (b_- \circ b_+) : (b_-, b_+) \in B_{n_-} \times B_{n_+}, K_+ \in K_+ \}\) is a \(\tau\)-invariant \(\mathbb{Q}(q^2)\)-linear basis of \(H_q^+(g)\).

We call \(B^+_q\) the double canonical basis of \(H_q^+(g)\) (the double canonical basis \(B^-_q\) of \(H_q^-(g)\) is defined verbatim, with \(q\) replaced by \(q^{-1}\)).

Furthermore, we have a natural, albeit not \(\tau\)-equivariant, inclusion \(\iota_+ : H_q^+(g) = K_+ \otimes U_q^- \otimes U_q^+ \hookrightarrow K_- \otimes (K_+ \otimes U_q^- \otimes U_q^+) = U_q(\tilde{g})\).

Main Theorem 1.5. For any \((b_-, b_+) \in B_{n_-} \times B_{n_+}\) there is a unique element \(b_- \circ b_+ \in U_q(\tilde{g})\) fixed by \(\tau\) and satisfying
\[
b_- \circ b_+ - \iota_+ (b_- \circ b_+) = \sum q^{-1}\mathbb{Z}[q] K \circ \iota_+ (b'_- b'_+)
\]
where the sum is taken over \(K \in K \setminus K_+\) and \(b'_ \in B_{n_+}\) such that \(\deg b'_- b'_+ + \deg K = \deg b_- b_+\).

We prove this Theorem in Section 2 using the equivariant Lusztig’s Lemma (Proposition 2.3).

Corollary 1.6. The set \(B_{\tilde{g}} \coloneqq \{ K \circ (b_- \circ b_+) : (b_-, b_+) \in B_{n_-} \times B_{n_+}, K \in K \}\) is a \(\mathbb{Q}(q^2)\)-basis of \(U_q(\tilde{g})\).

We call \(B_{\tilde{g}}\) the double canonical basis of \(U_q(\tilde{g})\).

Remark 1.7. Note that \(B_{\tilde{g}}\) contains both bases \(B_{n_\pm}\) as subsets and therefore has a “dual flavor”.

Let \(U_q(\tilde{g}, J)\) (respectively, \(U_q(J, \tilde{g})\), \(J \subset I\) be the subalgebra of \(U_q(\tilde{g})\) generated by the \(KU_q^+\) and \(F_j\), \(j \in J\) (respectively, \(KU_q^-\) and \(E_j\), \(j \in J\)) and let \(U_q(J_-, \tilde{g}, J_+) = U_q(\tilde{g}, J_+) \cap U_q(J_-, \tilde{g})\), \(J_\pm \subset I\). The following is immediate.

Theorem 1.8. For any \(J_\pm \subset I\), \(B_{\tilde{g}} \cap U_q(J_-, \tilde{g}, J_+)\) is a basis of \(U_q(J_-, \tilde{g}, J_+)\).

Remark 1.9. Analogously to the classical \((q = 1)\) case (cf. e.g. [10]), it is natural to call \(U_q(J_-, \tilde{g}, J_+)\) quantum bi-parabolic (or seaweed) algebras.

As one should expect from a canonical basis, \(B_{\tilde{g}}\) is preserved, as a set, by various symmetries of \(U_q(\tilde{g})\). First, let \(x \mapsto x^t\) and \(x \mapsto x^*\) be the \(\mathbb{Q}(q^2)\)-linear anti-automorphism of \(U_q(\tilde{g})\) defined by \(E_i^t = F_i\), \(F_i^t = E_i\), \((K_{\pm i})^t = K_{\mp i}\) and \(E_i^* = E_i\), \(F_i^* = F_i\), \((K_{\pm i})^* = K_{\mp i}\).

Then \(B_{\tilde{g}}^t = B_{\tilde{g}}\) while \(*\) preserves both \(B_{n_\pm}\) as sets.

Theorem 1.10. \(B_{\tilde{g}}^t = B_{\tilde{g}}\). More precisely, for all \(b_\pm \in B_{n_\pm}\), \(K \in K\) be have \((K \circ (b_- \circ b_+))^t = K \circ (b_\pm^t \circ (b_-)^t)\).

We prove this Theorem in Section 2.

Conjecture 1.11. \(B_{\tilde{g}}^* = B_{\tilde{g}}\). More precisely, for all \(b_\pm \in B_{n_\pm}\), \(K \in K\) be have \((K \circ (b_- \circ b_+))^* = K^* \circ (b_-)^* \circ (b_+)^*\).

Remark 1.12. It is easy to see that this conjecture implies that \(B_{\tilde{g}}\) can also be obtained by replacing \(H_q^+(g)\) with \(H_q^-(g)\) and interchanging \(q\) and \(q^{-1}\) in Theorems 1.3 and 1.5.
It turns out that $B_q$ and $\hat{B}_q$ are preserved by appropriately modified Lusztig’s symmetries. First of all, set $\hat{U}_q(\hat{g}) = U_q(\hat{g})[k^{-1}]$. Clearly, $\cdot$, $\hat{\cdot}$, and $\ast$ extend naturally to that algebra.

**Theorem 1.13.** (a) For each $i \in I$ there exists a unique automorphism $T_i$ of $\hat{U}_q(\hat{g})$ which satisfies $T_i(K_{\pm j}) = K_{\pm j}K_{\pm i}^{-a_{ij}}$ and

$$T_i(E_j) = \begin{cases} q_i^{-1}K_i^{-1}F_i, & i = j \\ \sum_{r+s=-a_{ij}} (-1)^r q_i^{s+1/2a_{ij}} E_i^{(r)} E_j^{(s)}, & i \neq j \end{cases}$$

$$T_i(F_j) = \begin{cases} q_i^{-1}K_i^{-1}E_i, & i = j \\ \sum_{r+s=-a_{ij}} (-1)^r q_i^{s+1/2a_{ij}} F_i^{(r)} F_j^{(s)}, & i \neq j \end{cases}$$

(b) For all $x \in \hat{U}_q(\hat{g})$, $T_i(x) = T_i(x \cdot 1) = T_i(x \hat{\cdot} 1)$ and $T_i(x) = T_i(x)^\ast = T_i(x)^\dagger = T_i^{-1}(x^\dagger)$.

(c) The $T_i$, $i \in I$ satisfy the braid relations on $\hat{U}_q(\hat{g})$, that is, they define a representation of the Artin braid group $Br_{\hat{g}}$ of $\hat{g}$ on $\hat{U}_q(\hat{g})$.

We prove this Theorem in Section 5.

**Remark 1.14.** Since for each $i \in I$, $T_i$ preserves the ideal $\mathfrak{J} = (K_{\pm j}K_{j \cdot 1} - 1 : j \in I)$, $T_i$ factors through to an automorphism of $U_q(\hat{g}) = U_q(\hat{g}) / \mathfrak{J}$ which, for $x \in U_q(\hat{g})$ homogeneous, equals $q_i^{1/2 \alpha_i(\deg x)} T_{n_{i-1}}^m(x)$ where $T_{n_{i-1}}^m$ is one of Lusztig’s symmetries defined in [14, §37.1] (see Lemma 5.2).

Clearly, $\cdot$ extends to the group generated by $K$ acting on $\hat{U}_q(\hat{g})$. Then the set $\hat{B}_q := K^{-1} \cdot B_q$ is a $\hat{\cdot}$-invariant basis of $\hat{U}_q(\hat{g}) = U_q(\hat{g})[k^{-1}]$.

**Conjecture 1.15.** Let $\hat{g}$ be semisimple. Then for all $i \in I$, $T_i(\hat{B}_q) = \hat{B}_q$. In other words, $Br_{\hat{g}}$ acts on $\hat{B}_q$ by permutations.

We prove supporting evidence for this conjecture in Section 5. In view of Remark 1.14, the conjecture implies that $T_i(B_q) = B_q$.

If $\hat{g}$ is infinite dimensional, this does not hold for all elements of $\hat{B}_q$ (see Example 5.6). To amend this conjecture we introduce the following notion. We say that $b \in B_q$ is tame if $T_i(b) \in \hat{B}_q$ for all $i \in I$. We prove (Theorem 5.12) that all elements of $B_{n, \pm}$ are tame.

**Conjecture 1.16.** If $b \in \hat{B}_q$ is tame then $T(b) \in \hat{B}_q$ for all $T \in Br_{\hat{g}}$.

We provide supporting evidence for this conjecture in Section 5. We show some of it below for which more notation is necessary. Let $W$ be the Weyl group of $\hat{g}$. Following [14, §39.4.4], for each $w \in W$ define $T_w \in Br_{\hat{g}}$ recursively as $T_{w_1} = T_i$ and $T_{w_2} = T_{w_1} T_{w_2}$ for any non-trivial reduced factorization $w = w'w''$, $w', w'' \in W$ (see [5.14] for the details). Define the quantum Schubert cell $U_q^+(w)$ and $U_q^-(w)$, $w \in W$ by $U_q^+(w) := T_{w'(KU_q^-)} \cap U_q^+$ and $U_q^-(w) := U_q^- \cap T_{w^{-1}}(KU_q^-)$. Clearly, these are subalgebras of $U_q^\pm$. For $g$ semisimple we prove (Proposition 5.3) that $U_q^+(w)$ coincides with the subspace $U_q^+(w,1)$ of $U_q^+$ defined by Lusztig ([14, §40.2]), and we expect this to hold for all $\hat{g}$ (Conjecture 5.3). Let $B_{n, \pm}(w) = B_{n, \pm} \cap U_q^+(w)$ (since, conjecturally, $U_q^+(w,1) = U_q^+(w)$, by [12, Theorem 4.22] $B_{n, \pm}(w)$ is a basis of $U_q^+(w)$). The following refines Conjecture 1.15.

**Conjecture 1.17.** $T_{w^{-1}}(B_{n, \pm}(ww')) \subset K^{-1} \cdot B_{n, \pm}(w) \cdot B_{n, \pm}(w')$ for all $w, w' \in W$ such that the factorization $ww'$ is reduced.

**Remark 1.18.** Note also that this conjecture implies that $K^{-1} \cdot B_{n, \pm}(w) \cdot B_{n, \pm}(w')$ is a basis in the double Schubert cell $KU_q^-(w)U_q^+(w') = T_{w^{-1}}(U_q^+(ww'))$. 
Another application of our construction is a double canonical basis in each quantum Weyl algebra $A_q^\epsilon(g)$. Given a function $\epsilon : I \to \{+,-\}$, $\epsilon(i) = \epsilon_i$, let $A_q^\epsilon(g)$ be a $k$-algebra generated by the $x_i$, $y_i \in I$ subject to the following relations

$$x_i y_i - y_i x_i = \epsilon_i (q_i^{-1} - q_i), \quad x_i y_j = q_i^{\epsilon_i \delta_{ij}} a_{ij} y_j x_i,$$

$$\sum_{r+s = 1 - a_{ij}} (-1)^r q_i^{r \epsilon_j \delta_{ij} - s a_{ij}} x_i^{-r} y_i x_i^{-1} = 0 = \sum_{r+s = 1 - a_{ij}} (-1)^r q_i^{r \epsilon_j \delta_{ij} - s a_{ij}} y_i^{-s} y_j y_i^{-1}, \quad i \neq j. \quad (1.6)$$

We will show (see Proposition 3.17) that each $A_q^\epsilon(g)$ is naturally a subalgebra of a Heisenberg algebra $\mathcal{H}_q^\epsilon(g)$ which “interpolates” between $\mathcal{H}_q^+(g)$ and $\mathcal{H}_q^-(g)$ (see 3.4 for the details) and obtain the following result.

**Theorem 1.19.** Each quantum Weyl algebra $A_q^\epsilon(g)$ has a double canonical basis $B_{n_+} \circ \epsilon B_{n_-}$.

We prove this Theorem in 3.4.

**Remark 1.20.** In fact, the $A_q^\epsilon(g)$ are closely related to braided Weyl algebras (see e.g. [11]). Note that algebras $A_q^\epsilon(g)$ and $A_q^{\epsilon'}(g)$ are not (anti)isomorphic if $\epsilon \neq -\epsilon'$. Thus, the resulting bases $B_{n_+} \circ \epsilon B_{n_+}$ and $B_{n_-} \circ \epsilon B_{n_-}$ are rather different. To the best of our knowledge, these bases admit an alternative description similar to that in Theorem 1.3 only when $\epsilon$ is a constant function, i.e. $\epsilon_i = +$ (respectively, $\epsilon_i = -$) for all $i \in I$.

Next we discuss the properties of the decomposition of elements of the natural basis of $U_q(\hat{g})$ with respect to $B_{\hat{g}}$. Define $C_{b_-, b_+, K}^{b_-, b_+, K} \in k$ for all $b_\pm, b'_\pm \in B_{n_\pm}$ and $K \in K$ by

$$d_{b_-, b_+} b_- b_+ = \sum_{b_-, b_+, K} C_{b_-, b_+, K}^{b_-, b_+, K} K \diamond b'_- \cdot b'_+.$$

Then Main Theorem 1.5 immediately implies that $C_{b'_-, b_+}^{b_-, b_+, K} \in \mathbb{Z}[q, q^{-1}]$. These Laurent polynomials play the role similar to that of Kazhdan-Lusztig polynomials due to the following conjectural result.

**Conjecture 1.21.** If $g$ is semisimple then $C_{b'_-, b_+}^{b_-, b_+, K} \in \mathbb{Z}_{\geq 0}[q, q^{-1}]$ for all $b_\pm, b'_\pm \in B_{n_\pm}, K \in K$.

We provide some examples in Section 4.

**Remark 1.22.** It is well-known (cf. [13]) that if the Cartan matrix of $g$ is symmetric then the structure constants of $B_{n_\pm}$ belong to $\mathbb{Z}_{\geq 0}[q^{1/2}, q^{-1/2}]$. However, we expect that Conjecture 1.21 holds even for those $g$ (with non-symmetric Cartan matrix) for which such positivity fails.

Next we discuss the relation between the adjoint action of $U_q(\hat{g})$ on itself and the double canonical basis. We expect that the basis $\hat{B}_{\hat{g}}$ is perfect in the sense of the following extension of Definition 5.30 from [3].

**Definition 1.23.** Let $\mathcal{Y}$ be a $k$-vector space with linear endomorphisms $e_i$, $i \in I$ and functions $\epsilon_i : \mathcal{Y} \setminus \{0\} \to \mathbb{Z}$ such that $\epsilon_i(e_i(v)) = \epsilon_i(v) - 1$ for all $v \notin \ker e_i$. We say that a basis $B$ of $\mathcal{Y}$ is perfect if for all $i \in I$ and $b \in B$ either $e_i(b) = 0$ or there exists a unique $b' \in B$ with $\epsilon_i(b') = \epsilon_i(b) - 1$ such that

$$e_i(b) \in k^x b' + \sum_{b'' \in B : \epsilon_i(b''') < \epsilon_i(b')} k b''.$$

It follows from [3] that for any $\lambda \in \mathbb{Z}^I$ there is an action of $U_q(g)$ on $\hat{U}_q(\hat{g})$ (obtained by twisting the usual adjoint action of $\hat{U}_q(\hat{g})$) and determined by, for all $i \in I$, $K_i(x) := q_i^{\lambda_i} x F_i(x) := q_i^{\lambda_i} (\lambda_i, \alpha_i^\vee(x)) F_i x - q_i^{\lambda_i} (\lambda_i, \alpha_i^\vee(x)) x F_i, \quad E_i(x) := K_i^{-1} \circ (q_i^{\lambda_i} E_i x - q_i^{\lambda_i} x E_i), \quad (1.7)$
where for $x \in \bar{U}_q(\mathfrak{g})$ we abbreviate $\alpha^\vee(x) := \alpha'(\deg^\vee_x)$ This action clearly preserves the subalgebra $U_q(\mathfrak{g})[K_{+1}] \subset \bar{U}_q(\mathfrak{g})$ and its ideal generated by the $K_{i-1}, i \in I$, hence descends to $\mathcal{H}_q^{\pm}(\mathfrak{g})[K_{+1}]$.

**Conjecture 1.24.** In the notation of Theorem 1.7.9 the bases $K_{+}^1 \circ B_{n+} \circ B_{n-}$ are perfect with respect to the action of $U_q(\mathfrak{g})$ on $\mathcal{H}_q^{\pm}(\mathfrak{g})[K_{+1}]$.

We verify this conjecture for $\mathfrak{g} = \mathfrak{sl}_2$ in 1.2. Since $K_{+1}^1 \circ B_{n-} \circ B_{n+}$ lifts to the basis $\hat{B}_\mathfrak{g}$, we propose the following extension of Conjecture 1.24.

**Conjecture 1.25.** Under the action of $\hat{U}_q(\mathfrak{g})$ on itself the basis $\hat{B}_\mathfrak{g}$ is perfect.

We prove this conjecture for $\mathfrak{g} = \mathfrak{sl}_2$ in 1.2.

We conclude with the discussion of the compatibility of $\hat{B}_\mathfrak{g}$ with various centralizers. Given $\mu \in \mathbb{Z}^I$, define $\mu$-centralizers $Z^\mu_\pm$ of $U_q^\pm$ in $\hat{U}_q(\mathfrak{g})$ by

$$Z^\mu_+ = \{u \in \hat{U}_q(\mathfrak{g}) : E_i u = q^{\mu_i} u E_i, i \in I\}, \quad Z^\mu_- = \{u \in \hat{U}_q(\mathfrak{g}) : F_i u = q^{-\mu_i} u F_i, i \in I\}$$

(alternatively, one can consider centralizers inside a suitable extension of $U_q(\mathfrak{g})$ by appropriate radicals of $K_{+1}$ in the spirit of [13] §3.1). It is well-known that $Z^\mu_+ \subset K$ unless $\mathfrak{g}$ is finite-dimensional and $\mu = \lambda + w_0 \lambda$ where we identify $\mathbb{Z}^I$ with the weight lattice $P$ of $\mathfrak{g}$ and $\lambda \in P^+$ is a dominant weight. Here $w_0$ is the longest element of the Weyl group of $\mathfrak{g}$. Since the adjoint action is essentially given by $2\lambda$-commutators, the following is a corollary of Conjecture 1.25.

**Corollary 1.26.** (of Conjecture 1.25). Let $\mathfrak{g}$ be finite dimensional. Then for any $\lambda \in P^+$, $Z^\pm_{\lambda + w_0 \lambda} \cap \hat{B}_\mathfrak{g}$ is a basis of $Z^\pm_{\lambda + w_0 \lambda}$.

It is well-known that the space $Z^+_{\lambda + w_0 \lambda} \cap U_q^+$ is one-dimensional. Remarkably, it contains a (necessarily unique) element $v_\lambda$ of $B_{n+}$. Using $v_\lambda$, we construct in 1.4 several elements of $Z^+_{\lambda + w_0 \lambda} \cap \hat{B}_\mathfrak{g}$ and in particular, elements of $\lambda$-center $C_\lambda := Z^+_{\lambda + w_0 \lambda} \cap Z^-_{\lambda + w_0 \lambda}$. It is well-known (see e.g. [9]) that $C_\lambda$ is one-dimensional and contains a unique element $C_\lambda$ satisfying $C_\lambda - v_\lambda^\vee v_\lambda \in \hat{U}_q(\mathfrak{g})(K \setminus \{1\})$.

The importance of elements $C_\lambda$, $\lambda \in P^+$ is exhibited by the following result (which is apparently well-known, as was confirmed to us by A. Joseph in a private communication).

**Proposition 1.27.** ([9]). Let $\mathfrak{g}$ be semisimple.

(a) The subring $C^0_\mathfrak{g}$ of $U_q(\mathfrak{g})$ generated by the $C_\lambda$, $\lambda \in P^+$ is the polynomial ring in the $C_{\omega_i}$ where $\{\omega_i\}_{i \in I}$ is the basis of fundamental weights in $P^+$.

(b) The assignments $|V_\lambda| \mapsto C_\lambda$, $\lambda \in P^+$ define an isomorphism of the Grothendieck ring $K_0(\mathfrak{g})$ of the category of finite dimensional $\mathfrak{g}$-modules onto $C^0_\mathfrak{g}$.

Corollary 1.26 implies that $C_\lambda \in \hat{B}_\mathfrak{g}$ or more precisely that $C_\lambda = v_\lambda^\vee \bullet v_\lambda \in B_{\mathfrak{g}}$. In particular, this yields most of the basis 1.11 for $\mathfrak{g} = \mathfrak{sl}_2$. Thus, the canonical basis of the Grothendieck ring of the category of finite dimensional $\mathfrak{g}$-modules identifies with a subset of the double canonical basis $B_{\mathfrak{g}}$ and so $B_{\mathfrak{sl}_2}$ contains (the canonical basis of) all Schur polynomials $s_\lambda$.

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2. Equivariant Lusztig’s Lemma and bases of Heisenberg and Drinfeld doubles

2.1. An equivariant Lusztig’s Lemma. Let $\Gamma$ be an abelian monoid and let $R$ be a unital $\Gamma$-graded ring $R = \bigoplus_{x \in \Gamma} R_x$ where $R_0$ is central in $R$. Suppose that $\tilde{\cdot}$ is an involution of abelian groups on $E^n$ canonical projection.

Lemma 2.2. The following Lemma is obvious.

An equivariant Lusztig’s Lemma.

2.1. Suppose now that $E$ is also free as an $R_0$-module. Since $\hat{E}$ and $E$ are free as $R_0$-modules and the canonical projection $\pi : \hat{E} \rightarrow E$ is a morphism of $\Gamma$-graded $R_0$-modules, it admits a homogeneous splitting $\iota : E \rightarrow \hat{E}$.

Define a relation $\prec$ on $\Gamma$ by $\alpha \prec \beta$ if there exists $\gamma \in \Gamma \setminus \{0\}$ such that $\alpha + \gamma = \beta$. Assume that there exists a function $\ell : \Gamma \rightarrow \mathbb{Z}_{\geq 0}$ such that for all $\gamma \in \Gamma$, $\gamma_0 \prec \gamma_1 \prec \cdots \prec \gamma_1 \prec \gamma$ implies that $s \leq \ell(\gamma)$. For example, this assumption holds for every monoid $\Gamma$ which admits a character $\chi : \Gamma \rightarrow \mathbb{Z}_{\geq 0}$ with $\chi(\gamma) > 0$ if $\gamma \neq 0$, which is the case for $\Gamma = \mathbb{Z}_I$ where $I$ is finite. We will call such a monoid $\Gamma$ bounded. If $\Gamma$ is bounded then, in particular, $\preceq$ is a partial order and $0$ is the unique minimal element of $\Gamma$.

Lemma 2.1. Let $\Gamma$ be a bounded monoid. Then $\iota(E)$ generates $\hat{E}$ as an $R$-module.

Proof. We have $\hat{E} = \iota(E) \oplus R_+ \hat{E}$ as $\Gamma$-graded $R_0$-modules, hence $\hat{E}_\gamma = \iota(E_\gamma) \oplus (R_+ \hat{E})_\gamma$ for all $\gamma \in \Gamma$. We prove by induction on $(\Gamma, \prec)$ that $\hat{E}_\gamma \subset R\iota(E)$. Since $0 \in \Gamma$ is minimal, $\hat{E}_0 \cap R_+ \hat{E} = 0$ hence $\hat{E}_0 \subset \iota(E)$ and the induction begins. For the inductive step, let $\gamma \in \Gamma \setminus \{0\}$ and assume that $\bigoplus_{\alpha < \gamma} \hat{E}_\alpha \subset R\iota(E)$. Then $\hat{E}_\gamma \cap R_+ \hat{E} \subset \bigoplus_{\alpha + \beta = \gamma : \alpha \in \Gamma \setminus \{0\}} R_\alpha \hat{E}_\beta \subset R\iota(E)$ by the induction hypothesis. Thus, $\hat{E}_\gamma = \iota(E_\gamma) \oplus (R_+ \hat{E})_\gamma \subset R\iota(E)$. $
$
From now on we will assume that $\Gamma$ is bounded.

Let $E$ be a homogeneous basis of $E$ satisfying $\bar{e} = e$ for all $e \in E$. Clearly

$$\overline{\iota(e) - \iota(e)} \in R_+ \hat{E}. $$

The following Lemma is obvious.

Lemma 2.2. Let $\mathcal{R} \subset R$, $1 \in \mathcal{R}$. The following are equivalent:

(i) $\{rt(e) : (r, e) \in \mathcal{R} \times \mathcal{E}\}$ is an $R_0$-basis of $\hat{E}$

(ii) As an $R_0$-module, $R = \text{Ann}_R \iota(E) \oplus \bigoplus_{r \in \mathcal{R}} R_0r$.

Given a homogeneous element $x$ of $R$, $\hat{E}$ or $E$ we denote its degree by $|x|$.

Proposition 2.3. Suppose that $R_0 = \mathbb{Z}[\nu, \nu^{-1}]$ and that $\bar{\cdot} : R_0 \rightarrow R_0$ is the unique ring automorphism satisfying $\bar{\nu} = \nu^{-1}$. Fix an $R_0$-module splitting $\iota : E \rightarrow \hat{E}$ of the canonical projection $\tilde{E} \rightarrow \hat{E} \cong E$. Suppose that there exists a subset $\mathcal{R} \subset R$ of homogeneous elements containing $1$ such that

(i) As an $R_0$-module, $R = \text{Ann}_R \iota(E) \oplus \bigoplus_{r \in \mathcal{R}} R_0r$;

(ii) For all $r \in \mathcal{R}$, $e \in \mathcal{E}$

$$\overline{\iota(e) - \iota(e)} \in \sum_{(r', e') \in \mathcal{R} \times \mathcal{E}} R_0r' \iota(e')$$
Then for each \((r, e) \in \mathcal{R} \times \mathcal{E}\) there exists a unique \(C_{r,e} \in \hat{\mathcal{E}}\) such that \(C_{r,e} = C_{r,e}\)
\[
C_{r,e} - r\iota(e) \in \sum_{(r',e') \in \mathcal{R} \times \mathcal{E}: |r'| + |e'| = |r| + |e|, |e'| < |e|} \nu \mathbb{Z}[\nu] r\iota(e').
\]

In particular, the set \(\mathcal{B}_{\mathcal{R}, \mathcal{E}} := \{C_{r,e} : (r, e) \in \mathcal{R} \times \mathcal{E}\}\) is an \(R_0\)-basis of \(\hat{\mathcal{E}}\).

**Proof.** Define a relation \(<\) on \(\mathcal{R} \times \mathcal{E}\) by \((r', e') < (r, e)\) if \(|e'| < |e|\) and \(|r'| + |e'| = |r| + |e|\). It is easy to see that \((r', e') < (r, e)\) implies that \(0 < |r'|\) (otherwise \(|e'| = |r| + |e|\) hence \(|e| \leq |e'| < |e|\)). Then \(<\) is a partial order and all assumptions of \([5, \text{Theorem 1.1}]\) for \(L := \mathcal{R} \times \mathcal{E}\), \(\mathcal{A} = \hat{\mathcal{E}}\) and \(E_{(r, e)} = r\iota(e)\), \((r, e) \in \mathcal{L}\) are satisfied. Thus, the assertion follows from the aforementioned result. \(\square\)

We conclude this section with a discussion of some symmetries of the \(\mathcal{B}_{\mathcal{R}, \mathcal{E}}\) constructed in Proposition 2.3. Consider the data \((R, \mathcal{R}, \hat{\mathcal{E}}, \mathcal{E}, \iota)\) satisfying the assumptions of Proposition 2.3.

**Definition 2.4.** We say that a homogeneous \(\iota\)-equivariant \(R_0\)-module automorphism \(\psi\) of \(\hat{\mathcal{E}}\) is triangular if there exists a permutation \(\phi\) of \(\mathcal{R}\) with \(\phi(1) = 1\) and a permutation \(\overline{\psi}\) of \(\mathcal{E}\) such that
\[
\psi(r\iota(e)) - \phi(r)\iota(\overline{\psi}(e)) = \sum_{(r',e') \in \mathcal{R} \times \mathcal{E}: |r'| + |e'| = |r| + |e|, |e'| < |e|} \nu \mathbb{Z}[\nu] r\iota(e').
\]

Using the same argument as in the proof of Proposition 2.3, we conclude that in all non-zero terms in the right-hand side we have \(0 < |r'|\).

**Lemma 2.5.** Suppose that \(\psi : \hat{\mathcal{E}} \rightarrow \hat{\mathcal{E}}\) is triangular. Then
\[
\psi(C_{r,e}) = C_{\phi(r), \overline{\psi}(e)}, \quad r \in \mathcal{R}, e \in \mathcal{E}.
\]

**Proof.** Since \(\psi\) commutes with \(\iota\), \(\psi(C_{r,e}) = \psi(C_{r,e})\). Applying \(\psi\) to (2.2) we obtain
\[
\psi(C_{r,e}) - \psi(r\iota(e)) = \sum_{(r',e') \in \mathcal{R} \times \mathcal{E}: |r'| + |e'| = |r| + |e|, |e'| < |e|} \nu \mathbb{Z}[\nu] \psi(r\iota(e')).
\]

Applying (2.3) to the left and the right hand side we conclude that
\[
\psi(C_{r,e}) - \phi(r)\iota(\overline{\psi}(e)) = \sum_{(r',e') \in \mathcal{R} \times \mathcal{E}: |r'| + |e'| = |r| + |e|, |e'| < |e|} \nu \mathbb{Z}[\nu] \psi(r\iota(e')).
\]

Proposition 2.3 then implies that \(\psi(C_{r,e}) = C_{\phi(r), \overline{\psi}(e)}\). \(\square\)

2.2. Double bases of Heisenberg and Drinfeld doubles. In this section we will use the notation and the setup of §8.1.3-A.9

Let \(\Gamma\) be a bounded abelian monoid as defined in 2.1. Let \(\mathbb{k} = \mathbb{Q}(\nu)\), \(R_0 = \mathbb{Z}[\nu, \nu^{-1}]\). Let \(H = \mathbb{k}[\hat{\Gamma}]\) be the monoidal algebra of \(\hat{\Gamma} = \Gamma \oplus \Gamma\) with a basis \(\{K_{\alpha_-\alpha_+} : \alpha_\pm \in \Gamma\}\) and let \(R = \bigoplus_{\alpha_\pm \in \Gamma} R_0 K_{\alpha_-\alpha_+}\).

Let \(V^\pm = \bigoplus V_{\alpha}^\pm\) be \(\Gamma\)-graded vector spaces. We regard \(V^+\) (respectively, \(V^-\)) as a right (respectively, left) Yetter-Drinfeld module over the localization \(\hat{H}\) of \(H\) with respect to the multiplicative set of all monomials in the \(K_{\alpha_-\alpha_+}\), \(\alpha_\pm \in \Gamma\) (see §8.3-A.9 for the details). Let \((\cdot, \cdot) : V^- \otimes V^+ \rightarrow \mathbb{k}\) be a pairing such that \((V^-_{\alpha}, V^+_\beta) = 0, \alpha \neq \beta\) and \((\cdot, \cdot)_{|_{V_{\alpha}^- \otimes V_{\alpha}^+}}\) is non-degenerate. Set \(\Gamma_0 = \{\alpha \in \Gamma : V_{\alpha}^\pm \neq 0\}\) and assume that \(\Gamma\) is generated by \(\Gamma_0\). Let \(\chi : \Gamma \times \Gamma \rightarrow R_0^\times = \pm \nu^Z\) be a symmetric bicharacter.

Given \(t_+, t_- \in \mathbb{k}\), let \(\mathcal{U}_{\chi, t_+, t_-}(V^-, V^+)\) be the algebra \(\mathcal{U}_{\chi}(V^-, V^+)\) defined in §8.9 with \((\cdot, \cdot)_\pm = t_\pm (\cdot, \cdot)\). We have in \(\mathcal{U}_{\chi, t_+, t_-}(V^-, V^+)\)
\[
K_{\alpha_-\alpha_+} v^+ = \frac{\chi(\alpha_+, \deg v^+)}{\chi(\alpha_-, \deg v^+)} \nu^+ K_{\alpha_-\alpha_+}, \quad K_{\alpha_-\alpha_+} v^- = \frac{\chi(\alpha_-, \deg v^-)}{\chi(\alpha_+, \deg v^-)} \nu^- K_{\alpha_-\alpha_+},
\]
and
\[ [v^+ , v^-] = t_- K_{\deg v^-, 0} (v^-, v^+) - t_+ K_{0, \deg v^+} (v^-, v^+), \] (2.5)
for all \( v^\pm \in \mathcal{B}(V^\pm) \) homogeneous and \( \alpha_\pm \in \Gamma \). We regard \( \mathcal{U}_{\chi,t_- ,t_+}(V^- , V^+) \) as graded by \( \hat{\Gamma} \) with deg\( \hat{\Gamma} v^+ = (0, \deg v^+), \) deg\( \hat{\Gamma} v^- = (\deg v^-, 0) \) and deg\( \hat{\Gamma} K_{\alpha_-, \alpha_+} = (\alpha_- + \alpha_+, \alpha_- + \alpha_+) \), where \( v^\pm \in V^\pm \) are homogeneous and \( \alpha_\pm \in \Gamma \).

Denote
\[ \mathcal{H}^0_{\chi}(V^-, V^+) := \mathcal{U}_{\chi,0,0}(V^-, V^+), \]
\[ \mathcal{H}^{-}_{\chi}(V^-, V^+) = \mathcal{U}_{\chi,1,0}(V^-, V^+), \quad \mathcal{H}^{-}_{\chi}(V^-, V^+) := \mathcal{U}_{\chi,0,1}(V^-, V^+), \]
\[ \mathcal{U}_{\chi}(V^-, V^+) = \mathcal{U}_{\chi,1,1}(V^-, V^+). \]
Thus, all these algebras have the same underlying vector space, namely \( \mathcal{B}(V^-) \otimes H \otimes \mathcal{B}(V^+) \) and differ only in the cross relations between \( \mathcal{B}(V^-) \) and \( \mathcal{B}(V^+) \).

Let \( \bar{\circledast} : k \rightarrow k \) be the unique field involution defined by \( \bar{\nu} = \nu^{-1} \). Fix its extension to \( V^\pm \) preserving the grading and assume that \( \langle v^-, v^+ \rangle = \langle v^-, v^+ \rangle \), \( v^\pm \in V^\pm \). Assume also that \( \chi \) satisfies \( \chi(\alpha, \alpha^\prime) = \chi(\alpha, \alpha^\prime)^{-1} \) for all \( \alpha, \alpha^\prime \in \Gamma \). Then all algebras described above admit an antilinear \( \bar{\cdot} \)-anti-involution extending \( \bar{\circledast} : V^\pm \rightarrow V^\pm \) and satisfying \( \bar{K}_{\alpha, \alpha^\prime} = K_{\alpha, \alpha^\prime} \), \( \alpha, \alpha^\prime \in \Gamma \).

Assume that \( \chi(\alpha, \alpha^\prime) \in \nu^{2\mathbb{Z}} \) for all \( \alpha, \alpha^\prime \in \Gamma \) and let \( \chi^{1/2} : \Gamma \times \Gamma \rightarrow \pm \nu^\mathbb{Z} \) be a bicharacter satisfying \( \chi^{1/2}(\alpha, \alpha^\prime)^2 = \chi(\alpha, \alpha^\prime), \alpha, \alpha^\prime \in \Gamma \). Extend \( \chi^{1/2} \) to a bicharacter of \( \hat{\Gamma} \) via
\[ \chi^{1/2}((\alpha_-, \alpha_+), (\beta_-, \beta_+)) = \frac{\chi^{1/2}(\alpha_+, \beta_+)}{\chi^{1/2}(\alpha_-, \beta_-)} \]
where \( t : \hat{\Gamma} \rightarrow \hat{\Gamma} \) is defined by \( (\alpha_-, \alpha_+) t = (\alpha_+, \alpha_-) \), \( \alpha_\pm \in \Gamma \). The following Lemma is obvious.

**Lemma 2.6.** For all \( t_\pm \in k \), (2.6) defines a structure of a left \( H \)-module on \( \mathcal{U}_{\chi,t_- ,t_+}(V^-, V^+) \) satisfying
\[ K_{\alpha_-, \alpha_+} \circ x = (\chi^{1/2}((\alpha_-, \alpha_+), (\deg^\hat{\Gamma} x)))^{-1} K_{\alpha_-, \alpha_+} x \]
\[ = (\chi^{1/2}((\alpha_-, \alpha_+), (\deg^\hat{\Gamma} x)))^{-1} x K_{\alpha_-, \alpha_+}, \] (2.6)
where \( t : \hat{\Gamma} \rightarrow \hat{\Gamma} \) is defined by \( (\alpha_-, \alpha_+) t = (\alpha_+, \alpha_-) \), \( \alpha_\pm \in \Gamma \). The following Lemma is obvious.

It should be noted, however, that \( \mathcal{U}_{\chi,t_- ,t_+}(V^-, V^+) \) is not an \( H \)-module algebra with respect to the \( \circ \) action.

We will now use Proposition 2.8 to construct a basis in \( \mathcal{H}^\chi(V^-, V^+) \) starting from a natural basis in \( \mathcal{H}^0_{\chi}(V^-, V^+) \) and then use the resulting basis to obtain a basis in \( \mathcal{U}_{\chi}(V^-, V^+) \). First we need to construct a suitable “initial basis”.

**Proposition 2.7.** Let \( B_+ \) (respectively, \( B_- \)) be a \( \Gamma \)-homogeneous basis of \( \mathcal{B}(V^+) \) (respectively, of \( \mathcal{B}(V^-) \)). Then
\( a \) There exists a unique \( d : B_- \times B_+ \rightarrow \mathbb{Z}[\nu + \nu^{-1}], \) \( (b_-, b_+) \mapsto d_{b_-, b_+} \) such that for all \( b_\pm \in B_\pm \)
we have in \( \mathcal{U}_{\chi,t_-, t_+}(V^-, V^+) \)
\[ d_{b_-, b_+}(b_+ b_- - b_- b_+) \in \sum_{(\alpha_-, \alpha_+) \in \Gamma \setminus \{(0,0)\}, b_\pm \in B_\pm} R_0 d_{b_-, b_+} K_{\alpha_-, \alpha_+} \circ (b'_- b'_+), \]
and the degree of \( d_{b_-, b_+} \) in \( \nu + \nu^{-1} \) is minimal and the highest coefficient is positive and minimal.
(b) If \( \Delta(B_\pm) \in R_0 B_\pm \otimes B_\pm \) and \( (B_-, B_+), (B_-, \mathcal{S}^{-1}(B_+)) \subset R_0 \) then \( d_{b_-, b_+} = 1 \) for all \( (b_-, b_+) \in B_- \times B_+ \).
Proof. We may assume, without loss of generality, that the element of $B_\pm$ of degree 0 is 1. We have

$$\left(\Delta \otimes 1\right)\Delta (b_\pm) = \sum_{b'_\pm, b''_\pm, b'''_\pm \in B_\pm} C_{b'_\pm, b''_\pm, b'''_\pm}^{b'_\pm, b''_\pm, b'''_\pm} b'_\pm \otimes b''_\pm \otimes b'''_\pm, \quad C_{b'_\pm, b''_\pm, b'''_\pm}^{b'_\pm, b''_\pm, b'''_\pm} \in k, \quad C_{b'_\pm, b''_\pm, b'''_\pm}^{1, b_\pm} = 1,$$

and $C_{b'_\pm, b''_\pm, b'''_\pm}^{b'_\pm, b''_\pm, b'''_\pm} = 0$ unless $\deg b'_\pm + \deg b''_\pm + \deg b'''_\pm = \deg b_\pm - \deg b'_\pm$. Given a homogeneous element $u_\pm \in B(V^\pm)$, denote its $Z_{\geq 0}$-degree by $|u_\pm|$. Then (A.41) implies that

$$b_+ \cdot b_- = \sum_{b'_\pm, b''_\pm, b'''_\pm \in B_\pm} \left(\chi(b''_-, b'_+) \chi(b''_+, b'''_+) \chi(b''_-, b'_+)\right)^{-1} \times$$

$$C_{b'_+, b''_+, b'''_+}^{b'_+, b''_+, b'''_+} C_{b'_-, b''_-, b'''_-}^{b'_-, b''_-, b'''_-} |b'_+| |b''_+| |b'''_+| \langle b'_+, \xi(b'''_+)^{-1} \rangle \langle b''_+, b'''_+ \rangle K_{\deg b'''_+, b''_+} b''_+ K_0 \deg b'''_+ (2.7)$$

where

$$F_{b'_-, b'_+, b''_+, b'''_+, \alpha_- + \alpha_+}^{b''_-, b'''_+} = \sum_{b'_\pm, b''_\pm, b'''_\pm \in B_\pm : \deg b''_\pm = \alpha_\pm} \left(\chi(b''_+, b'''_+) \chi(b''_-, b'''_-) \chi(b''_-, b'_+)\right)^{-1} \times$$

$$C_{b'_+, b''_+, b'''_+}^{b'_+, b''_+, b'''_+} C_{b'_-, b''_-, b'''_-}^{b'_-, b''_-, b'''_-} |b'_+| |b''_+| |b'''_+| \langle b'_+, \xi(b'''_+)^{-1} \rangle \langle b''_+, b'''_+ \rangle.$$

Since $\Gamma$ and hence $\widehat{\Gamma}$ is bounded, we can now construct $d$ inductively. For $\deg b_- = \deg b_+ = 0$ we set $d_{b_-, b_+} = 1$. We need the following

Lemma 2.8. For any finite subset $F \subset \mathbb{Q}(\nu)$ there exists a unique $d(F) \in \mathbb{Z}[\nu + \nu^{-1}]$ such that $d(F)F \subset \mathbb{Z}[\nu, \nu^{-1}]$, the degree of $d(F)$ in $\nu + \nu^{-1}$ is minimal and the highest coefficient of $d(F)$ is positive and minimal. Moreover, if all poles of elements of $F$ are roots of unity then $d(F) = c(\nu + \nu^{-1} - 2)^{m_1} (\nu + \nu^{-1} + 2)^{m_2} \prod_{k \geq 3} (\nu - e^{2\pi i/k})^{m_k}$ with $c, m_k \in \mathbb{Z}_{\geq 0}, c \neq 0$, where $\Phi_k$ is the $k$th cyclotomic polynomial and $\varphi(k)$ is the Euler function.

Proof. Let $F = \{f_1, \ldots, f_r\}$, $f_i = a_i/b_i$ where $a_i, b_i \in \mathbb{Z}[\nu]$ and are coprime. Then there exists a unique $f \in \mathbb{Z}[\nu]$ of minimal degree such that $f f_i \in \mathbb{Z}[\nu]$ for all $1 \leq i \leq r$, namely, $f$ is the least common factor of the $b_i$. Write $f = c \prod_{j=1}^{r} p_j^{m_j}$, where $c \in \mathbb{Z}$, each $p_j \in \mathbb{Z}[\nu]$ is irreducible and $m_j \in \mathbb{Z}_{>0}$. We may assume without generality that $c$ as well as the highest coefficient of each of the $p_j$ is positive. Given an irreducible $p \in \mathbb{Z}[\nu]$ of positive degree, define

$$\overline{p} = \begin{cases} q^{-\deg p} p, & \overline{p} = \nu^{-\deg p} p \text{ and } \deg p \text{ is even} \\ p, & \text{otherwise.} \end{cases}$$

Then $\overline{p} \in \mathbb{Z}[\nu + \nu^{-1}]$ and is irreducible in that ring. It follows that $d(F) := c \prod_{j=1}^{r} \overline{p}_j^{m_j}$ has the desired properties. This proves the first assertion.

If the only zeroes of all the $b_i, 1 \leq i \leq r$ are roots of unity then the only non-constant irreducible factors of $f$ are cyclotomic polynomials. Clearly, $\Phi_1 = \nu + \nu^{-1} - 2$ and $\Phi_2 = \nu + \nu^{-1} + 2$. Since $\varphi(k)$ is even for all $k \geq 3$, it follows that $\overline{\Phi_k} = \nu^{-1/2} e^{2\pi i/k}\Phi_k$.

Denote $F_{b_-, b_+} := \{d_{b'_-, b'_+}^{b''_-, b'''_+} : b'_+ \in B_\pm, (\alpha_- + \alpha_+) \in \widehat{\Gamma} \setminus \{(0,0)\}\}$. Then $F_{b_-, b_+}$ is finite and we set $d_{b_-, b_+} = d(F_{b_-, b_+})$. Then by the above computation

$$d_{b_-, b_+}(b_+ b_- - b_- b_+) \in \sum_{(\alpha_- + \alpha_+) \in \Gamma \setminus \{(0,0)\}, b'_+ \in B_\pm} R_0 d_{b'_-, b'_+} K_{\alpha_- + \alpha_+} b'_- b'_+.$$

It remains to observe that $R_0 K_{\alpha_- + \alpha_+} b'_- b'_+ = R_0 K_{\alpha_- + \alpha_+} \circ b'_- b'_+$. The uniqueness of $d$ is obvious.
Part (b) is immediate from (2.8).

**Theorem 2.9.** Suppose that \( B_\pm \) are \( \Gamma \)-homogeneous bases of \( B(V^\pm) \) and \( \overline{b}_\pm = b_\pm \) for all \( b_\pm \in B_\pm \). Then for each \( (b_-, b_+) \in B_- \times B_+ \) there exist

(a) a unique element \( b_- \circ b_+ \in \mathcal{H}_\chi^+(V^-, V^+) \) such that \( \overline{b_- \circ b_+} = b_- \circ b_+ \) and

\[
\begin{align*}
b_- \circ b_+ - d_{b_-, b_+} b_- b_+ &\in \sum_{\alpha \in \Gamma \setminus \{0\}, b'_- \in B_\pm : \deg b'_- + \alpha = \deg b_\pm} \nu Z[\nu] d_{b'_-, b'_+} K_{0, \alpha} \circ b'_- b'_+;
\end{align*}
\]

The elements \( \{K_{\alpha_-, \alpha_+} \circ (b_- b_+) : \alpha_\pm \in \Gamma, b_\pm \in B_\pm\} \) form a \( \Gamma \)-invariant basis of \( \mathcal{H}_\chi^+(V^-, V^+) \).

(b) a unique element \( b_- \bullet b_+ \in \mathcal{U}_\chi(V^-, V^+) \) such that \( \overline{b_- \bullet b_+} = b_- \bullet b_+ \) and

\[
\begin{align*}
b_- \bullet b_+ - b_- \circ b_+ &\in \sum_{\alpha_-, \alpha_+ \in \Gamma, b'_- \in B_\pm : \deg b'_- + \alpha_- = \deg b_\pm, \alpha_+ \neq 0} \nu Z[\nu]^{-1} |K_{\alpha_-, \alpha_+} \circ b'_- b'_+|.
\end{align*}
\]

The elements \( \{K_{\alpha_-, \alpha_+} \circ (b_- \bullet b_+) : \alpha_\pm \in \Gamma, b_\pm \in B_\pm\} \) form a \( \Gamma \)-invariant basis of \( \mathcal{U}_\chi(V^-, V^+) \).

**Proof.** To prove (a), we apply Proposition 2.3 with the following data. Let \( \hat{E} \) be the free \( R_0 \)-module generated by \( \{d_{b_-, b_+} K_{\alpha_-, \alpha_+} \circ (b_- b_+) : b_\pm \in B_\pm, \alpha_\pm \in \Gamma\} \), which is clearly a \( \hat{\Gamma} \)-graded \( R_0 \)-module via the \( \circ \) action. Then \( E \) identifies with the \( R_0 \)-submodule of the algebra \( \mathcal{H}_\chi^0(V^-, V^+) \) generated by \( E := \{d_{b_-, b_+} b_- b_+ : b_\pm \in B_\pm\} \). Let \( \iota \) be the identity map of vector spaces \( \mathcal{H}_\chi^0(V^-, V^+) \to \mathcal{H}_\chi^+(V^-, V^+) \). Let \( R = \{K_{\alpha_-, \alpha_+} : \alpha_\pm \in \Gamma\} \). Then in \( \mathcal{H}_\chi^+(V^-, V^+) \) we have by Proposition 2.7

\[
\begin{align*}
d_{b_-, b_+} K_{\alpha_-, \alpha_+} \circ (b_- b_+)
\end{align*}
\]

Thus, all assumptions of Proposition 2.3 are satisfied and hence for each \( (\alpha_-, \alpha_+) \in \Gamma \times \Gamma, (b_-, b_+) \in B_- \times B_+ \) there exists a unique element \( C_{\alpha_-, \alpha_+, b_- b_+} \in \mathcal{H}_\chi^+(V^-, V^+) \) such that \( C_{\alpha_-, \alpha_+, b_- b_+} = C_{\alpha_-, \alpha_+, b_- b_+} \) and

\[
\begin{align*}
C_{\alpha_-, \alpha_+, b_- b_+} - d_{b_-, b_+} K_{\alpha_-, \alpha_+} \circ b_- b_+ &\in \sum_{\alpha' \neq \alpha_+, \alpha'_+ \in \Gamma, b'_- \in B_\pm : \deg b'_- + \alpha'_+ = \deg b_\pm} \nu Z[\nu] d_{b'_-, b'_+} K_{\alpha'_-, \alpha'_+} \circ b'_- b'_+.
\end{align*}
\]

Set \( b_- \circ b_+ = C_{0,0,b_- b_+} \). Then \( K_{\alpha_-, \alpha_+} \circ b_- \circ b_+ \) has the same properties as \( C_{\alpha_-, \alpha_+, b_- b_+} \) hence they coincide. This completes the proof of part (a).

To prove part (b), we again employ Proposition 2.3. Let \( \hat{E} \) be the free \( R_0 \)-submodule of \( \mathcal{U}_\chi(V^-, V^+) \) generated by \( \{K_{\alpha_-, \alpha_+} \circ b_- \circ b_+ : \alpha_\pm \in \Gamma, b_\pm \in B_\pm\} \), which is clearly a \( \hat{\Gamma} \)-graded \( R_0 \)-module. Then \( E \) identifies with the free \( R_0 \)-submodule of \( \mathcal{H}_\chi^+(V^-, V^+) \) generated by \( \{K_{\alpha,0} \circ b_- \circ b_+ : \alpha \in \Gamma, b_\pm \in B_\pm\} \). Let \( R = \{K_{\alpha,0} : \alpha \in \Gamma\} \). By part (a) we have

\[
\begin{align*}
d_{b_-, b_+} b_- b_+ - b_- \circ b_+ &\in \sum_{\alpha \in \Gamma \setminus \{0\}} \nu Z[\nu] K_{0, \alpha} \circ b_- \circ b_+.
\end{align*}
\]
Together with Proposition 2.7 this implies that in $U_\chi(V^-, V^+)$
\[
\bar{d}_{b_-,b_+} b_- b_+ \in d_{b_-,b_+} b_- b_+ + \sum_{\nu' \in B_+(\alpha_-,\alpha_+) \in \Gamma \setminus \{(0,0)\}} R_0 d_{\nu', b_+} K_{\alpha_-,\alpha_+} \circ b_- b_+ \\
= b_- \circ b_+ + \sum_{\nu' \in B_+(\alpha_-,\alpha_+) \in \Gamma \setminus \{(0,0)\}} R_0 K_{\alpha_-,\alpha_+} \circ b_- \circ b_+,
\]
which together with (a) yields
\[
\bar{b_-} \circ b_+ \in d_{b_-,b_+} b_- b_+ + \sum_{\nu' \in B_+(\alpha_-,\alpha_+) \in \Gamma \setminus \{(0,0)\}} R_0 d_{\nu', b_+} K_{0,\alpha} \circ b_- b_+ \\
= b_- \circ b_+ + \sum_{\nu' \in B_+(\alpha_-,\alpha_+) \in \Gamma \setminus \{(0,0)\}} R_0 K_{\alpha_-,0} \circ (K_{0,\alpha_+} \circ b_- \circ b_+).
\]
Note that in the last sum only terms with $\alpha_- \neq 0$ may occur with non-zero coefficients, since $b_- \circ b_+$ is $\sim$-invariant in $H^+_\chi(V^-, V^+)$. Thus, all assumptions of Proposition 2.3 are satisfied, and, using it with $\nu$ replaced by $\nu^{-1}$ we obtain the desired basis. The rest of the argument is essentially the same as in part (m) and is omitted.

Remark 2.10. In view of Remark 1.12 it would be interesting to compare our elements $b_- \bullet b_+$ with those obtained by interchanging $\nu$ and $\nu^{-1}$ in and/or $H^+_{\chi}(V^-, V^+)$ with $H^-_{\chi}(V^-, V^+)$ in Theorem 2.9.

Choose bases $B_+^0 = \{E_i\}_{i \in I}$ of $V^+$ and $B_0 = \{F_i\}_{i \in I}$ of $V^-$ such that $\deg E_i = \deg F_i$, $i \in I$; thus, $\Gamma_0 = \{\deg E_i\}_{i \in I}$. Assume that $\bar{E}_i = E_i$ and $\bar{F}_i = F_i$. Let $t$ be the unique anti-involution $\xi$, as defined in Lemma 1.34, such that $E_i^t = F_i$, $F_i^t = E_i$.

Proposition 2.11. Let $B_\pm$ be a $\Gamma$-homogeneous basis of $B(V^+)$ consisting of $\sim$-invariant and containing $B_+^0$ and let $B_- = B_\pm t$. Then for all $b_{\pm} \in B_\pm$, $\alpha_{\pm} \in \Gamma$ we have
\[
(K_{\alpha_-,\alpha_+} \circ b_- \circ b_+)^t = K_{\alpha_-,\alpha_+} \circ b_+^t \circ b_-^t, \quad (K_{\alpha_-,\alpha_+} \circ b_- \bullet b_+)^t = K_{\alpha_-,\alpha_+} \circ b_+^t \bullet b_-^t.
\]
Proof. Since $\deg b_\pm^t = \deg b_\pm$, we have in $H^+_{\chi}(V^-, V^+)$
\[
(K_{\alpha_-,\alpha_+} \circ b_- \circ b_+)^t = (\bar{\chi}(\alpha_-,\alpha_+), (\deg b_-, \deg b_+))^{-1} b_\pm^t b_-^t K_{\alpha_-,\alpha_+} = K_{\alpha_-,\alpha_+} \circ b_\pm^t b_-^t.
\]
Thus, the anti-automorphism $^t$ of $H^+_{\chi}(V^-, V^+)$ is triangular in the sense of Definition 2.4, hence $(K_{\alpha_-,\alpha_+} \circ b_- \circ b_+)^t = K_{\alpha_-,\alpha_+} \circ b_\pm^t \circ b_-^t$. This implies that the anti-automorphism $^t$ of $U_\chi(V^-, V^+)$ is also triangular in the sense of Definition 2.4 with $\nu$ replaced by $\nu^{-1}$, and the second assertion follows.

3. Dual canonical bases and proofs of Theorems 1.3, 1.5, 1.10 and 1.19

3.1. Bicharacters, pairings, lattices and inner products. Let $k = \mathbb{Q}(q^{\frac{1}{2}})$ and let $R_0 = Z[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$. Let $g$ be a symmetrizable Kac-Moody Lie algebra and let $A = (a_{ij})_{i,j \in I}$ be its Cartan matrix. Fix positive integers $d_i$, $i \in I$ such that $d_ia_{ij} = a_jd_j$, $i, j \in I$. Let $\Gamma = \bigoplus_{i \in I} Z_{\geq 0} \alpha_i$ be a free abelian monoid, $\hat{\Gamma} = \Gamma \oplus \Gamma$ and set $\alpha_{-i} = (\alpha_i, 0), \alpha_{+i} = (0, \alpha_i) \in \Gamma$. We will sometimes denote the first (respectively, second) copy of $\Gamma$ in $\hat{\Gamma}$ by $\hat{\Gamma}_-$ (respectively, $\hat{\Gamma}_+$). Let $K$ be the monoidal
algebra of $\tilde{\Gamma}$ with the basis $\{K_{\alpha_+,\alpha_-} : \alpha_\pm \in \Gamma\}$ and denote $K_{\pm i} := K_{\alpha_\pm}$. The monoid $\Gamma$ and hence $\tilde{\Gamma}$ clearly afford a sign character (cf. A.8).

Define a symmetric bicharacter $\cdot : \tilde{\Gamma} \times \tilde{\Gamma} \to \mathbb{Z}$ by $\alpha_i \cdot \alpha_j = d_i a_{ij}$ and set $\chi(\alpha, \alpha') = q^{\alpha \cdot \alpha'}$, $\alpha, \alpha' \in \Gamma$. It is easy to see that $\alpha \cdot \alpha \in 2\mathbb{Z}$ for all $\alpha \in \Gamma$. Furthermore, let $\eta : \tilde{\Gamma} \to \mathbb{Z}_{\geq 0}$ be the character defined by $\eta(\alpha_i) = d_i$, $i \in I$. We extend $\cdot$ to a bicharacter of $\tilde{\Gamma}$ via $\alpha_i \cdot \alpha_j = -d_i a_{ij}$, $i, j \in I$ and $\eta$ to a character of $\tilde{\Gamma}$ via $\eta(\alpha_{\pm i}) = \eta(\alpha_i)$, $i \in I$. Define $\gamma : \tilde{\Gamma} \to \mathbb{Z}$ by $\gamma(\alpha) = \frac{1}{2} \alpha \cdot \alpha - \eta(\alpha)$, $\alpha \in \Gamma$. Then
\[
\gamma(\alpha_i) = 0, \quad \gamma(\alpha + \alpha') = \gamma(\alpha) + \gamma(\alpha'), \quad i \in I, \alpha, \alpha' \in \Gamma.
\]
This implies that $\gamma : \Gamma \to \mathbb{Z}$ defined by $\gamma(\alpha) = q^{\gamma(\alpha)}$, $\alpha \in \Gamma$ is the function discussed in A.8.

Let $V^+ = \bigoplus_{i \in I} kE_i$, $V^- = \bigoplus_{i \in I} kF_i$. We regard $V^\pm$ as $\Gamma$-graded with deg $E_i = \deg F_i = \alpha_i$. It is well-known (cf. [14, Chapter 1] and §A.3) that $U_\pm^q$ is the Nichols algebra $B(V^\pm, \Psi^\pm)$ where the braiding $\Psi^\pm$ is defined via the bicharacter $\chi$ as in A.8.

Define a pairing $\langle \cdot, \cdot \rangle : V^- \otimes V^+ \to k$ by $\langle F_i, E_j \rangle = \delta_{ij} (q_i - q_i^{-1})$. Then $\langle \cdot, \cdot \rangle$ extends to a pairing of braided Hopf algebras $U_\pm^q \otimes U_\pm^q \to k$ (see §A.3). The algebra $U_{\chi, t_- t_+}(V^-, V^+)$, $t_\pm \in k$, is then $\tilde{\Gamma}$-graded as in §2.2.

**Proposition 3.1.** The algebra $U_q(\tilde{\mathfrak{g}})$ is isomorphic to $U_{\chi}(V^-, V^+) = U_{\chi, 1.1}(V^-, V^+)$ while $\mathcal{H}_\pm^q(\mathfrak{g})$ identify with the subalgebra of $\mathcal{H}_\chi^q(V^-, V^+)$ generated by the $K_{\pm i}$ (respectively, $K_{-i}$), $E_i$ and $F_i$, $i \in I$, in the notation of §2.2.

**Proof.** After [14] Proposition 1.4.3, §1.3 hold in $B(V^\pm)$, while (A.36) yield (1.2). Thus, $U_{\chi}(V^-, V^+)$ is a $\tilde{\Gamma}$-graded quotient of $U_q(\tilde{\mathfrak{g}})$, and it remains to observe that their homogeneous subspaces have the same dimensions. The assertion about $\mathcal{H}_\pm^q(\mathfrak{g})$ is proved similarly. \hfill $\square$

Define $\tilde{\cdot} : V^+ \to V^\pm$ as the unique antilinear map satisfying $E_i = E_i$, $F_i = F_i$, $i \in I$. Then $\langle v^-, \tilde{v}^+ \rangle = -\langle v^-, v^+ \rangle$, hence by Lemma [A.34] $U_q(\tilde{\mathfrak{g}})$ admits an antilinear antinvolution $\tilde{\cdot}$ preserving the generators, an antinvolution $\cdot$ preserving the $E_i$ and the $F_i$, $i \in I$ and satisfying $K_{\pm i} \cdot = K_{\mp i}$, $i \in I$, and an antinvolution $t$ which restricts to antiisomorphisms $U_\pm^q \to U_\mp^q$ such that $E_i^t = F_i$, $F_i^t = E_i$, and preserves the $K_{\pm i}$, $i \in I$. In particular, $\cdot$ is an antinvolution which restricts to isomorphisms $U_\pm^q \to U_\mp^q$.

Let $\mathbb{Z}U^+$ (respectively, $\mathbb{Z}U^-$) be the $\mathbb{Z}[q, q^{-1}]$-subalgebra of $U_\pm^q$ (respectively, $U^-_\pm$) generated by the $E_i^{(n)}$ (respectively, $F_i^{(n)}$), $i \in I$, $n \in \mathbb{Z}_{\geq 0}$; thus, $\mathbb{Z}U^\pm$ is the preimage under $\psi_\pm$ of the subalgebra of $U_\pm^q$ generated by the usual divided powers ([14] §1.4.7)). Define

\[
U_+^\mathbb{Z} = \{x \in U_+^q : \langle x, U^- \rangle \subset \mathbb{Z}[q, q^{-1}]\}, \quad U_-^\mathbb{Z} = \{x \in U_-^q : \langle x, U^+ \rangle \subset \mathbb{Z}[q, q^{-1}]\}
\]

**Proposition 3.2.** $U_\pm^\mathbb{Z}$ is a $\mathbb{Z}[q, q^{-1}]$-subalgebra of $U_\pm^q$ satisfying $\Delta(U_\pm^\mathbb{Z}) \subset U_\pm^\mathbb{Z} \otimes \mathbb{Z}[q, q^{-1}] U_\pm^\mathbb{Z}$.

**Proof.** We prove the statements for $U_+^\mathbb{Z}$ only, the argument for $U_-^\mathbb{Z}$ being similar. Let $R = \mathbb{Z}[q, q^{-1}]$.

The following result is immediate from [14] Lemma 1.4.1

**Lemma 3.3.** $\mathbb{Z}U^\pm$ is an $R$-subalgebra of $U^\pm$ satisfying $\Delta(\mathbb{Z}U^\pm) \subset \mathbb{Z}U^\pm \otimes_R \mathbb{Z}U^\pm$.

Since $U_\pm^\mathbb{Z}$ is a direct sum of free $R$-modules of finite length, $U_\pm^\mathbb{Z}$ is canonically isomorphic to the graded $\text{Hom}_R(\mathbb{Z}U^-, R)$, which immediately implies the proposition. \hfill $\square$

**3.2. Dual canonical bases.** Let $\psi : U_q(\tilde{\mathfrak{g}}) \to U_q(\tilde{\mathfrak{g}})$ be the homomorphism defined by $E_i \mapsto (q_i - q_i^{-1})^{-1} E_i$, $F_i \mapsto (q_i - q_i^{-1})^{-1} F_i$, $K_{\pm i} \mapsto K_{\pm i}$. Denote by $\psi_\pm$ its restrictions to $U_\pm^q$. Clearly, the images of generators of $U_q(\tilde{\mathfrak{g}})$ under $\psi$ satisfy the relations of the "standard" presentation of $U_q(\tilde{\mathfrak{g}})$; for example

\[
[\psi(E_i), \psi(F_j)] = \delta_{ij} \frac{K_{+i} - K_{-i}}{q_i - q_i^{-1}}.
\]}
Let $B^\text{can}$ be the preimage under $\psi^-$ of Lusztig’s canonical basis of $U_q^-$ ([14] Chapter 14]). By [14] Theorem 14.4.3], $B^\text{can}$ is a $\mathbb{Z}[q,q^{-1}]$-basis of $U_q^-$. If $g = \mathfrak{sl}_2$, $B^\text{can} = \{F^{(r)} : r \in \mathbb{Z}_{\geq 0}\}$.

Let $\langle , \rangle : U_q^- \otimes U_q^- \rightarrow \mathbb{k}$ be the pairing defined in [A.9] with $\xi$ being the anti-involution $t$ descibed above. Since $\langle , \rangle$ is non-degenerate and restricts to non-degenerate bilinear forms on finite dimensional graded components of $U_q^-$, for each $b \in B^\text{can}$ there exists a unique $\delta_b \in U_q^-$ such that $\langle \delta_b, b' \rangle = \delta_{bb'}$ for all $b' \in B^\text{can}$.

**Definition 3.4.** The dual canonical basis $B_{\mathfrak{n}^+}$ of $U_q^+$ is the set $\{\delta_b : b \in B^\text{can}\}$. The dual canonical basis $B_{\mathfrak{n}^+}$ of $U_q^+$ is defined as $B_{\mathfrak{n}^+} = B_{\mathfrak{n}^+}^\dagger$.

This definition is justified by the following Lemma.

**Lemma 3.5.** For all $b_\pm \in B_{\mathfrak{n}_\pm}$, $\delta_b = b_\pm$ and $b_\pm^* \in B_{\mathfrak{n}_\pm}$.

**Proof.** Note that for all $b \in B^\text{can}$, $\overline{\psi(b)} = \psi(b)$, hence $\delta_b = \text{sgn}(b)b$. Moreover, $b^* \in B^\text{can}$ by [14] §14.4]. It remains to apply (A.43). \hfill \square

**Proposition 3.6.** The set $\{q^{-\frac{1}{2}}\gamma^{(\deg b_\pm)}b_\pm : b_\pm \in B_{\mathfrak{n}_\pm}\}$ is a $\mathbb{Z}[q,q^{-1}]$-basis of $U^+_\mathfrak{z}$.

**Proof.** It suffices to prove that $\{q^{-\frac{1}{2}}\gamma^{(\deg b)}\delta_b^*: b \in B^\text{can}\}$ generates $U^+_\mathfrak{z}$ as a $\mathbb{Z}[q,q^{-1}]$-module. Let $b, b' \in B^\text{can}$. Then $\quad q^{-\frac{1}{2}}\gamma^{(\deg b)}\langle b', \delta_b^* \rangle = \langle \delta_b, b' \rangle = \delta_{bb'}$.

Therefore, $q^{-\frac{1}{2}}\gamma^{(\deg b)}\delta_b^* \in U^+_\mathfrak{z}$. Let $x \in U^+_\mathfrak{z}$ and write $x = \sum_{b \in B^\text{can}} q^{-\frac{1}{2}}\gamma^{(\deg b)}c_b \delta_b^* \gamma^{(\deg t)}c_b = 0$. Then for all $b \in B^\text{can}$, $\quad \langle b, x \rangle = \sum_{b' \in B^\text{can}} q^{-\frac{1}{2}}\gamma^{(\deg b')}c_b \gamma^{(\deg t)}c_b = c_b$.

Thus, $c_b \in \mathbb{Z}[q,q^{-1}]$ for all $b \in B^\text{can}$. \hfill \square

Then Propositions (3.2) (3.4) and (3.1) imply the following.

**Corollary 3.7.** The structure constants $\hat{C}^{b_\pm b'_\pm}_{b_\pm' b'_\pm}, C^{b_\pm b'_\pm}_{b_\pm' b'_\pm}, c_\pm, b_\pm, b'_\pm, b''_\pm \in B_{\mathfrak{n}_\pm}$, defined by

$$ b_\pm b'_\pm = q^{\frac{1}{2}}\gamma^{(\deg b_\pm)}\delta_b^* \sum b''_\pm \in B_{\mathfrak{n}_\pm} \hat{C}^{b''_\pm}_{b_\pm b'_\pm} b''_\pm, \quad \Delta(b_\pm) = \sum b'_\pm b''_\pm \in B_{\mathfrak{n}_\pm} q^{\frac{1}{2}}\gamma^{(\deg b_\pm)}\gamma^{(\deg b''_\pm)} C^{b''_\pm}_{b_\pm b'_\pm} b''_\pm \otimes b''_\pm $$

belong to $\mathbb{Z}[q,q^{\pm 1}]$.

It follows immediately from the above Corollary that for any $b_\pm \in B_{\mathfrak{n}_\pm}$

$$ \Delta(b_\pm) = \sum_{b'_\pm b''_\pm b''_\pm \in B_{\mathfrak{n}_\pm}} q^{\frac{1}{2}}\gamma^{(\deg b'_\pm)}\gamma^{(\deg b''_\pm)} C^{b''_\pm}_{b_\pm b'_\pm} b''_\pm \otimes b''_\pm $$

where $\hat{C}^{b''_\pm}_{b_\pm b'_\pm} = \sum b_\pm \in B_{\mathfrak{n}_\pm} \hat{C}^{b'_\pm b''_\pm}_{b_\pm b'_\pm} C^{b''_\pm}_{b'_\pm b''_\pm} \in \mathbb{Z}[q,q^{\pm 1}]$.

**Remark 3.8.** It is easy to check that for any $b, b', b'' \in B^\text{can}$ we have

$$ bb' = \sum_{b'' \in B^\text{can}} C^{b''}_{b b''} b''_\pm \otimes b''_\pm $$

and

$$ \Delta(b''_\pm) = \sum_{b, b'' \in B^\text{can}} C^{b''}_{\delta b b''} \gamma^{(\deg t)} b \otimes b''_\pm. $$

After [14] §14.4.14, these structure constants are Laurent polynomials in $q$.

**Proposition 3.9.** $\langle B_{\mathfrak{n}_-}, B_{\mathfrak{n}_+} \rangle \subset \langle U^-_\mathfrak{z}, U^+_\mathfrak{z} \rangle \subset \mathbb{Z}[q,q^{-1}, \Phi_k^{-1} : k > 0]$, where $\Phi_k \in \mathbb{Z}[q]$ is the $k$th cyclotomic polynomial.
Proof. Indeed, it is immediate from properties of $\langle \cdot, \cdot \rangle$ that $\langle F_{j_1}^{(b_1)} \cdots F_{j_s}^{(b_s)} E_{i_1}^{(a_1)} \cdots E_{i_r}^{(a_r)} \rangle \in \mathbb{Z}[q, q^{-1}]$ for any $(i_1, \ldots, i_r), (j_1, \ldots, j_s) \in \mathbb{I}^*$ and for any $a = (a_1, \ldots, a_r) \in \mathbb{Z}_{\geq 0}^r, b = (b_1, \ldots, b_s) \in \mathbb{Z}_{\geq 0}^s$. Therefore,

$$\langle F_{j_1}^{(b_1)} \cdots F_{j_s}^{(b_s)} E_{i_1}^{(a_1)} \cdots E_{i_r}^{(a_r)} \rangle \in R' := \mathbb{Z}[q, q^{-1}, \Phi_k^{-1}, k > 0].$$

This implies that $\langle z U^-, z U^+ \rangle \subset R'$. We need the following, apparently well-known result.

**Lemma 3.10.** Let $\alpha \in \Gamma$. Let $B_{\alpha}^-$ be any basis of $z U_-^\alpha = \{ u \in z U^- : \text{deg } u = \alpha \}$ and set $G_\alpha = \langle (b', b''') \rangle_{b', b'' \in B_{\alpha}}$ be the corresponding Gram matrix. Then det $G_\alpha = q^n \prod_k \Phi_k(q)^{a_k}$ where $a_k \in \mathbb{Z}$ and $n \in \mathbb{Z}$.

Proof. It well-known ([14]) that the specialization of the form $\langle \cdot, \cdot \rangle$ for any $q = \zeta$, where $\zeta \in \mathbb{C}^*$ is not a root of unity, is well defined and non-degenerate. Thus, det $G_\alpha$ is a rational function of $q$ whose zeroes and poles are roots of unity and zero. This implies det $G_\alpha = cq^n \prod_k \Phi_k(q)^{a_k}$ where $c \in \mathbb{Q}$ and $a_k \in \mathbb{Z}$. It remains to prove that $c = 1$. To prove this claim, note that by [14, Theorem 14.2.3] and properties of $\langle \cdot, \cdot \rangle$, for any $b \in B_{\text{can}}$, there exists $\tilde{b} \in q^2 b$ such that for all $b', b'' \in B_{\text{can}}$, $\langle b', b'' \rangle = \delta_{b, b'} + q^{-1} \mathbb{Z}[q^{-1}]$. This in turn implies that for $B_{\alpha}^- = \{ b : b \in B_{\text{can}}, \text{deg } b = \alpha \}$, det $G_\alpha \in \mathbb{Z}^1 + q^{-1} \mathbb{Z}[q^{-1}]$. Since det $G_\alpha$ is, up to a power of $q^2$, independent of the choice of basis $B_{\alpha}^-$, it follows that $c = 1$. □

Now, let $B_{+, \alpha}$ be any basis of $(U^+_\alpha)_\alpha = \{ u \in U_+ \} : \text{deg } u = \alpha \}$ and let $B_{-}^\alpha$ be the dual basis of $B_{+, \alpha}$ with respect to $\langle \cdot, \cdot \rangle$. Then the Gram matrix $G_\alpha' = \langle (b'_+, b'_+) \rangle_{b'_+, b'_+ \in B_{+, \alpha}}$ satisfies $G_\alpha' = G^{-1}_\alpha$ over $\mathbb{Q}(q)$. As $\langle z U^-, z U^+ \rangle \subset R'$, all entries of $G_\alpha$ are in $R'$, while (det $G_\alpha$)$^{-1} \in R'$ by Lemma 3.10. Therefore, all entries of $G_\alpha'$ are in $R'$.

To prove the second inclusion note that by Proposition 3.6 we have for all $b_\pm \in B_{\alpha}\pm$

$$\langle q^{-\frac{1}{2} \gamma(\text{deg } b_-)} b_- q^{-\frac{1}{2} \gamma(\text{deg } b_+)} b_+ \rangle = q^{-\frac{1}{2} \gamma(\text{deg } b_+) + \frac{1}{2} \gamma(\text{deg } b_-)} \langle b_-, b_+ \rangle \in R'$$

since $\langle b_-, b_+ \rangle \neq 0$ implies that $\text{deg } b_+ = \text{deg } b_-$. □

For $g$ semisimple we can strengthen Proposition 3.9 as follows

**Theorem 3.11.** If $g$ is semisimple then $\langle B_{n-}, B_{n+} \rangle \subset \langle U^-_Z, U^+_Z \rangle = \mathbb{Z}[q, q^{-1}]$.

We prove this Theorem in Section 5. We expect that the converse is also true: if $\langle U^-_Z, U^+_Z \rangle \subset \mathbb{Z}[q, q^{-1}]$ then $g$ is semisimple (see Lemma 4.14 and Example 4.16).

**Remark 3.12.** We can conjecture that $\langle U^-_Z, U^+_Z (w) \rangle \subset \mathbb{Z}[q, q^{-1}]$ where $w \in W$ and $U^+_Z (w)$ is the corresponding Schubert cell.

### 3.3. Proofs of Theorems 1.3, 1.5 and 1.10

First, we need a stronger version of Proposition 2.7.

**Proposition 3.13.** Let $d : B_{n-} \times B_{n+} \to \mathbb{Z}[\nu + \nu^{-1}], \nu = q^{\frac{\Gamma}{2}}, \text{ be defined as in Proposition 2.7.}$ Then for all $b_\pm \in B_{n-}, d_{b_-, b_+} = \prod_{k \geq 3}(q^{\frac{1}{2} \gamma(k)} \Phi_k(q))^{a_k}, a_k \in \mathbb{Z}_{\geq 0}, \text{ and, in particular, is monic. Moreover, in } U_q(\hat{g})$ we have

$$d_{b_-, b_+}(b_+ b_- - b_- b_+) \in \sum_{(\alpha_-, \alpha_+) \in \mathbb{N} \times \nu} \mathbb{Z}[q, q^{-1}] d_{b'_-, b'_+} K_{\alpha_-, \alpha_+} \circ b'_- b'_+.$$

Proof. By (2.14)

$$b_+ b_- = \sum_{b'_\pm \in B_{\pm}, \alpha_\pm \in \Gamma : \text{deg } b'_\pm + \alpha_+ + \alpha_- = \text{deg } b_\pm} F_{b'_-, b'_+}^{b_-, b_+} K_{\alpha_-, \alpha_+} b'_- b'_+ (3.3)$$
where by (2.8), (3.2), Lemma A.28 and (A.35)

\[ F_{b_{-}, b_{+}}^{b^{\prime}_{-}, b^{\prime}_{+}, \alpha, \pm} = \sum_{b^{\prime}_{-}, b^{\prime}_{+} \in B_{\pm} \atop \deg b^{\prime}_{\pm} = \alpha_{\pm}} \frac{\chi_{\frac{1}{2}}(b^{\prime}_{-}, b^{\prime}_{+}) \chi_{\frac{1}{2}}(b^{\prime}_{-}, b^{\prime}_{+}) \chi_{\frac{1}{2}}(b^{\prime}_{-}, b^{\prime}_{+}) \chi_{\frac{1}{2}}(b^{\prime}_{-}, b^{\prime}_{+}) \chi_{\frac{1}{2}}(b^{\prime}_{-}, b^{\prime}_{+})}{\chi_{\frac{1}{2}}(b^{\prime}_{-}, b^{\prime}_{+}) \chi_{\frac{1}{2}}(b^{\prime}_{-}, b^{\prime}_{+}) \chi_{\frac{1}{2}}(b^{\prime}_{-}, b^{\prime}_{+}) \chi_{\frac{1}{2}}(b^{\prime}_{-}, b^{\prime}_{+})} \tilde{C}_{b_{-}, b_{+}}^{b^{\prime}_{-}, b^{\prime}_{+}, \alpha, \pm} (b^{\prime}_{-}, b^{\prime}_{+}, b^{\prime}_{-}, b^{\prime}_{+}) \]

\[ = \text{sgn}(\alpha_{+}) q^{-\frac{1}{2} \alpha_{+} \deg b^{\prime}_{+}} \tilde{C}_{b_{-}, b_{+}}^{b^{\prime}_{-}, b^{\prime}_{+}, \alpha, \pm} (b^{\prime}_{-}, b^{\prime}_{+}, b^{\prime}_{-}, b^{\prime}_{+}) \]

Since \( \tilde{C}_{b_{-}, b_{+}}^{b^{\prime}_{-}, b^{\prime}_{+}, \alpha, \pm} \in \mathbb{Z}[q, q^{-1}] \), by Proposition 3.9 we have \( \tilde{F}_{b_{-}, b_{+}}^{b^{\prime}_{-}, b^{\prime}_{+}, \alpha, \pm} \in \mathbb{Z}[q, q^{-1}, \Phi^{-1}_{k} : k > 2] \). Thus, by Lemma 2.8 we can choose \( \mathfrak{d}_{b_{-}, b_{+}} \) to satisfy the first assertion. Since by (3.3)

\[ b_{+} b_{-} = \sum_{b^{\prime}_{-}, b^{\prime}_{+} \in B_{\pm}, \alpha_{\pm} \in \Gamma : \deg b_{\pm} + \alpha_{\pm} = \deg b_{\pm}} \tilde{F}_{b_{-}, b_{+}}^{b^{\prime}_{-}, b^{\prime}_{+}, \alpha, \pm} K_{\alpha, \pm} \circ b^{\prime}_{-} b^{\prime}_{+}, \]

the second assertion is now immediate. \( \square \)

**Proofs of Theorems 1.3, 1.5** We apply Theorem 2.9 with the data from (3.1)

- \( \kappa = \mathbb{Q}(\nu) \), \( R_{0} = \mathbb{Z}[\nu, \nu^{-1}] \), \( \nu = q^{\frac{1}{2}} \)
- \( \Gamma = \tilde{\Gamma} = \bigoplus_{i \in I} \mathbb{Z}[\nu, \nu^{-1}]_{0} \)
- \( K_{0, \alpha} = K_{\alpha, \alpha} = K_{\alpha, 0} = K_{0, \alpha} \), \( i \in I \)
- \( V^{+} = \bigoplus_{i \in I} kE_{i}, V^{-} = \bigoplus_{i \in I} kF_{i} \)
- \( \tilde{\cdot} \) is determined by \( \tilde{E}_{i} = E_{i}, \tilde{F}_{i} = F_{i}, i \in I \)
- \( \chi_{(\alpha, \alpha)} = q^{\alpha_{i}}, \langle E_{i}, F_{i} \rangle = \delta_{ij}(q_{i} - q_{j}^{-1}) \), \( i, j \in I \)

Then \( U_{q}^{+}(\tilde{\mathfrak{g}}) = U_{q}^{+}(V^{-}, V^{+}) \) while \( H_{q}^{+}(\tilde{\mathfrak{g}}) \) identifies with the subalgebra of \( H_{\chi}^{+}(V^{-}, V^{+}) \) generated by the \( K_{\alpha, \alpha} \), \( E_{i}, F_{i}, i \in I \). Applying Theorem 2.9(13), we obtain elements \( b_{\pm} \circ b_{\pm} \in H_{q}^{+}(\tilde{\mathfrak{g}}) \) which proves Theorem 1.3. Theorem 1.5 then follows from Theorem 2.9(13). It remains to prove that all coefficients in the decompositions of invariant bases with respect to the initial ones in Theorems 1.3 and 1.5 are polynomials in \( q \) or \( q^{-1} \) and not just in \( q^{\pm \frac{1}{2}} \). But this is immediate from Proposition 3.13. \( \square \)

**Proof of Theorem 1.10** This is immediate from Proposition 2.11 since the anti-involution \( ^{t} \) of \( U_{q}^{+}(\tilde{\mathfrak{g}}) \) satisfies \( B_{n_{\pm}^{t}} = B_{n^{\pm}} \). \( \square \)

### 3.4 Colored Heisenberg and quantum Weyl algebras and their bases.

Let \( \tilde{H}_{q}^{\pm}(\tilde{\mathfrak{g}}) \) be the \( \kappa \)-algebra generated by \( U_{q}^{+} \) and \( L_{i}^{-1} \), \( i \in I \) where

\[ L_{i}E_{i} = q^{-\frac{1}{2} e_{ij}(\alpha_{i})} E_{i} L_{i}, \quad L_{i}F_{i} = q^{-\frac{1}{2} e_{ij}(\alpha_{i})} F_{i} L_{i} \]

and

\[ [E_{i}, F_{j}] = \delta_{ij} e_{i} L_{i}^{2}(q_{i}^{-1} - q_{i}). \]

Note that \( \tilde{H}_{q}^{\pm}(\tilde{\mathfrak{g}}) \) admits a \( \kappa \)-anti-linear anti-involution \( \tilde{\cdot} \) extending \( \tilde{\cdot} : U_{q}^{\pm} \to U_{q}^{\pm} \) and satisfying \( L_{i}^{\pm} = L_{i}^{-1} \) and an anti-involution \( ^{t} \) extending the anti-isomorphisms \( U_{q}^{\pm} \to U_{q}^{\mp} \) discussed above and preserving the \( L_{i}^{\pm} \), \( i \in I \). The following is obvious.

**Lemma 3.14.** (a) The assignments \( E_{i} \mapsto E_{i}, F_{i} \mapsto F_{i}, K_{\pm, i} \mapsto L_{i}^{2}, K_{\mp, i} \mapsto 0 \), \( i \in I \) define a homomorphism of algebras \( \psi : U_{q}^{+}(\tilde{\mathfrak{g}}) \to \tilde{H}_{q}^{\pm}(\tilde{\mathfrak{g}}) \).
(b) \( \hat{\mathcal{H}}^\varepsilon_q(g) \) is generated by \( \text{Im} \psi^\varepsilon \) and \( L_i^{-1} \), \( i \in I \) and has the triangular decomposition \( \hat{\mathcal{H}}^\varepsilon_q(g) = U_q^- \otimes \mathcal{L} \otimes U_q^+ \), where \( \mathcal{L} \) is the subalgebra generated by \( L_i^{\pm 1} \), \( i \in I \).

(c) \( \psi^\varepsilon \) commutes with \( \bar{\cdot} \) and \( \cdot \).

(d) The set \( B_{n-} \cdot \bullet \cdot B_{n+} := \psi^\varepsilon(B_{n-} \bullet B_{n+}) \) is linearly independent and \( L \cdot B_{n-} \cdot \bullet \cdot B_{n+} \), where \( L \) is the multiplicative subgroup of \( \mathcal{L} \) generated by the \( L_i^{\pm 1} \), \( i \in I \), is a basis of \( \hat{\mathcal{H}}^\varepsilon_q(g) \).

Note that \( \hat{\mathcal{H}}^\varepsilon_q(g) \) is graded by the group \( Q := \mathbb{Z}^I \) with \( \text{deg}_Q E_i = \text{deg}_Q F_i = \text{deg}_Q L_i = \alpha_i = -\text{deg}_Q L_i^{-1} \), where \( \{\alpha_i\}_{i \in I} \) is the standard basis of \( \mathbb{Z}^I \). Let \( \hat{\mathcal{H}}^\varepsilon_q(g)_{0} \) be the subalgebra of elements of degree 0.

**Lemma 3.15.** There exists a unique projection \( \tau : \hat{\mathcal{H}}^\varepsilon_q(g) \to \hat{\mathcal{H}}^\varepsilon_{q}(g)_{0} \) commuting with \( \bar{\cdot} \) such that \( \tau(x) \in q^{\frac{1}{2}a_i} \prod_{i \in I} L_i^{-n_i} x \) for \( x \) homogeneous with \( \text{deg}_Q x = \sum_{i \in I} n_i \alpha_i \).

Let \( A^\varepsilon_q(g) \) be the \( k \)-algebra with presentation (1.3). The following Lemma is easily checked.

**Lemma 3.16.** The algebra \( A^\varepsilon_q(g) \) admits an anti-linear anti-involution \( \bar{\cdot} \) defined on generators by \( \bar{x}_i = x_i, \bar{y}_i = y_i, \) and an anti-involution \( \cdot \) defined by \( x^t = y_i, y^t = x_i \).

**Proposition 3.17.** The assignments \( x_i \mapsto q^{\frac{1}{2}a_i} L_i^{r-1} E_i, y_i \mapsto q^{\frac{1}{2}t} F_i L_i^{-1} \) define an isomorphism of algebras \( j_\varepsilon : A^\varepsilon_q(g) \to \hat{\mathcal{H}}^\varepsilon_q(g)_{0} \) which commutes with \( \bar{\cdot} \) and \( \cdot \). Moreover, \( A^\varepsilon_q(g) \) has a triangular decomposition \( A^\varepsilon_q(g) = U^e_{q,+} \otimes U^e_{q,-} \) where \( U^e_{q,+} \) (respectively, \( U^e_{q,-} \)) is the subalgebra of \( A^\varepsilon_q(g) \) generated by the \( x_i \) (respectively, \( y_i \)), \( i \in I \).

**Proof.** Let \( X_i = L_i^{-1} E_i, Y_i = F_i L_i^{-1} \). Then in \( \hat{\mathcal{H}}^\varepsilon_q(g) \) we have

\[
0 = \sum_{r+s=1-a_i} (-1)^r E_i^{(r)} E_j X_j^{(s)} = \sum_{r+s=1-a_i} (-1)^r (L_i X_i)^{(s)} L_j X_j (L_i X_i)^{(r)}
\]

\[
= \sum_{r+s=1-a_i} (-1)^r q_i^{-r \epsilon_i(j) + \epsilon_j} L_i^{(r)} X_i^{(s)} L_j X_j L_i^{(r)}
\]

\[
= L_i^{1-a_i} L_j \sum_{r+s=1-a_i} (-1)^r q_i^{-r \epsilon_i(j) + \epsilon_j} L_i^{(r)} X_i^{(s)} X_j X_j^{(r)}
\]

\[
= q_i^{-\epsilon_i(1-a_i)^2} L_i^{1-a_i} L_j \sum_{r+s=1-a_i} (-1)^r q_i^{-r (r \epsilon_i + \epsilon_j) a_i} X_i^{(s)} X_j X_j^{(r)}
\]

This implies that

\[
\sum_{r+s=1-a_i} (-1)^r q_i^{r \epsilon_i a_i \delta_i, -1} X_i^{(s)} X_j X_j^{(r)} = 0.
\]

Thus, the \( X_i \) satisfy the defining identity of \( A^\varepsilon_q(g) \). Since \( Y_i = X_i^t \), the identity for the \( Y_i \) is now immediate. The remaining identities are trivial. Thus, \( j_\varepsilon \) is a well-defined homomorphism of algebras \( A^\varepsilon_q(g) \to \hat{\mathcal{H}}^\varepsilon_q(g) \) and its image clearly lies in \( \hat{\mathcal{H}}^\varepsilon_q(g)_{0} \). Since the defining relations of \( U^e_{q,+} \) are the only relations in the subalgebra of \( \hat{\mathcal{H}}^\varepsilon_q(g)_{0} \) generated by the \( \{X_i\}_{i \in I} \), it follows that the restrictions of \( j_\varepsilon \) to \( U^e_{q,+} \) are injective. Since the corresponding subalgebras quasi-commute, the assertion follows. \( \square \)

Now we have all necessary ingredients to prove Theorem 1.19.

**Proof of Theorem 1.19.** It follows from Lemma 3.15 and Proposition 3.17 that \( \tau(B_{n-} \cdot \bullet \cdot B_{n+}) \) is a basis of \( \hat{\mathcal{H}}^\varepsilon_q(g)_{0} \). Then \( B_{n-} \circ \bullet B_{n+} := j_\varepsilon^{-1} \tau(B_{n-} \cdot \bullet \cdot B_{n+}) \) is the desired basis of \( A^\varepsilon_q(g) \). \( \square \)
3.5. Invariant quasi-derivations. Following Lemma [A.32] and also [14] Proposition 3.1.6, define \( k \)-linear endomorphisms \( \partial_i, \partial_i^{op}, i \in I \) of \( U^+_q \) by
\[
[F_i, x^+] = (q_i - q_i^{-1})(K_{i+} \circ \partial_i(x^+) - K_{-i} \circ \partial_i^{op}(x^+)), \quad x^+ \in U^+_q.
\]
(3.4)

Then
\[
[x^-, E_i] = (q_i - q_i^{-1})(K_{i+} \circ \partial_i(x^{-t}) - K_{-i} \circ \partial_i^{op}(x^{-t})), \quad x^- \in U^-_q.
\]

The operators \( \partial_i, \partial_i^{op} \) are locally nilpotent since \( \deg \partial_i(x) = \deg x - \alpha_i \), hence we can define a function \( \ell_i : U^+_q \to \mathbb{Z}_{\geq 0} \) by
\[
\ell_i(x^+) = \max\{k \in \mathbb{Z}_{\geq 0} : \partial_i^k(x^+) \neq 0\}, \quad x^+ \in U^+_q.
\]

Lemma 3.18. For all \( x^+, y^+ \in U^+_q \), \( i \in I \) we have
(a) \( \partial_i(x^+) = \partial_i^{op}(x^+) \), \( \partial_i^{op}(x^+) = \partial_i^{op}(x^+) \)
(b) \( \partial_i(x^+) = (\partial_i^{op}(x^+))^* \)
(c) \( \partial_i, \partial_i^{op} \) are quasi-derivations, namely for \( x^+, y^+ \in U^+_q \) homogeneous we have
\[
\partial_i(x^+ y^+) = q_i^{\frac{1}{2}}\alpha_i^{(\deg y^+)} \partial_i(x^+) y^+ + q_i^{\frac{1}{2}}\alpha_i^{(\deg x^+)} x^+ \partial_i(y^+),
\]
\[
\partial_i^{op}(x^+ y^+) = q_i^{\frac{1}{2}}\alpha_i^{(\deg y^+)} \partial_i^{op}(x^+) y^+ + q_i^{\frac{1}{2}}\alpha_i^{(\deg x^+)} x^+ \partial_i^{op}(y^+).
\]
(3.5)

(d) \( \partial_i \partial_i^{op} = \partial_i^{op} \partial_i \) for all \( i, j \in I \)
(e) \( \partial F_i(x^+) = (q_i - q_i^{-1})q_i^{\frac{1}{2}}\alpha_i^{(\deg x^+ - \alpha_i)} \partial_i^{op}(x^+) \), \( \partial F_i^{op}(x^+) = (q_i - q_i^{-1})q_i^{\frac{1}{2}}\alpha_i^{(\deg x^+ - \alpha_i)} \partial_i(x^+) \).

Proof. Parts (a)–(b) and (3.5) are immediate consequences of the definition, while (c) follows from (3.5) by induction on \( n \). Parts (d) and (e) are easy to deduce from (c) and [A.32] by induction on \( \deg x^+ \).

We will frequently use the notation \( X_i^{(n)} := (q_i - q_i^{-1})^n X_i^{(a)} \) for every symbol \( X_i, i \in I \) such that \( X_i^{(n)} \) makes sense.

Corollary 3.19. If \( x^+, y^+ \in U^+_q \) are homogeneous then for all \( n \geq 0 \)
\[
\partial_i^{(a)}(x^+ y^+) = \sum_{a+b=n} q_i^{\frac{1}{2}}\alpha_i^{(a \deg y^+ - b \deg x^+)} \partial_i^{(a)}(x^+) \cdot \partial_i^{(b)}(y^+)
\]
(3.6)

In particular,
\[
\partial_i^{(top)}(x^+ y^+) = q_i^{\frac{1}{2}}\alpha_i^{(\ell_i(x^+) \deg y^+ - \ell_i(y^+) \deg x^+)} \partial_i^{(top)}(x^+) \partial_i^{(top)}(y^+),
\]
where \( \partial_i^{(top)}(x^+) = \partial_i^{(\ell_i(x^+))}(x^+) \).

Define \( \partial_i^{-}, \partial_i^{-op} : U^-_q \to U^-_q \) by \( \partial_i^{-}(x) = \partial_i(x^t) \) and \( \partial_i^{-op}(x) = \partial_i^{op}(x^t) \), \( x \in U^-_q \). Then \( \ell_i : U^-_q \to \mathbb{Z}_{\geq 0} \) and \( (\partial_i^{-})^{(top)} \) are defined accordingly. We will sometimes use the notation \( \partial_i^{+}, \partial_i^{+op} \) for \( \partial_i, \partial_i^{op} \).

Lemma 3.20. For all \( x, y \in U^-_q \) and \( k \in \mathbb{Z}_{\geq 0} \)
\[
\langle (\partial_i^{-})^{(k)}(x), y \rangle = \langle x, F_i^{(k)} y \rangle, \quad \langle (\partial_i^{-op})^{(k)}(x), y \rangle = \langle x, y F_i^{(k)} \rangle.
\]
(3.7)
Proof. It is sufficient to show that $(q_i - q_i^{-1})\langle \partial_i (x^t)^r, y \rangle = \langle x, F_i y \rangle$. Then an obvious induction yields $(q_i - q_i^{-1})^n \langle (\partial_i^a)^n (x^t), y \rangle = \langle x, F_i^n y \rangle$ and the assertion follows. We have

\[
(q_i - q_i^{-1})\langle \partial_i (x^t)^r, y \rangle = (q_i - q_i^{-1}) q^{-\frac{1}{2} \gamma(\deg y)} \langle y, \partial_i (x^t)^r \rangle = (q_i - q_i^{-1}) q^{-\frac{1}{2} \gamma(\deg y)} \langle y, \partial_i^a (x^t)^r \rangle
\]

\[
= q^{-\frac{1}{2} \gamma(\deg y)} \langle y, \partial_i (x^t)^r \rangle - q^{-\frac{1}{2} \gamma(\deg y)} \langle y, \partial_i (x^t)^r \rangle (F_i, x^t) = \langle x, F_i y \rangle.
\]

The second identity follows from the first since $\langle x^*, y \rangle = \langle x, y^* \rangle$. \qed

**Example 3.21.** Recall ([14], 14.5.3)] that $F_i^{(n)} \in B^{\text{can}}$ for all $i \in I, n \in \mathbb{Z}_{\geq 0}$. Clearly, $\langle x, F_i^{(n)} \rangle = 0$ unless $x \in \mathbb{k} F_i^n$. Since $\partial_i (F_i^{(n)}) = (n)q_i F_i^{n-1}$, it follows from Lemma 3.20 that $\langle F_i^n, F_i^{(n)} \rangle = 1$, hence $F_i^n = \delta_{i,n} \in B_{n^\pm}$. We will need some properties of $B_{n^\pm}$ with respect to $\partial_i^\pm$ which we gather in the following proposition

**Proposition 3.22.** Let $b^\pm \in B_{n^\pm}$. Then

(a) For all $r \in \mathbb{Z}_{\geq 0}$,

\[
\langle \partial_i^-(r) (b_-), \partial_i^-(r)(b_-) \rangle = \sum_{b'_- \in B_{n^\pm}} \check{C}_{b'_-}^{F_i, b_-} b'_-, \quad \langle \partial_i^-(r)^{\text{op}} (b_-) \rangle = \sum_{b'_- \in B_{n^\pm}} \check{C}_{b'_-}^{F_i, b_-} b'_-, \quad \langle \partial_i^+(r) (b_+), \partial_i^+(r)(b_+) \rangle = \sum_{b'_+ \in B_{n^\pm}} \check{C}_{b'_+}^{F_i, b_+} b'_+ \quad \langle \partial_i^+(r)^{\text{op}} (b_+) \rangle = \sum_{b'_+ \in B_{n^\pm}} \check{C}_{b'_-}^{F_i, b_+} b'_+,
\]

where $\check{C}_{b'_-}^{F_i, b_-}$, $\check{C}_{b'_-}^{F_i, b_-}$ are defined in Corollary 3.3. Thus, in particular, $(\partial_i^-(r) (B_{n^\pm}) \subset \mathbb{Z}[q, q^{-1}] B_{n^\pm}$.

(b) $(\partial_i^-(\text{top}) (b_-), (\partial_i^\pm \text{op})(\text{top})(b_\pm) \in B_{n^\pm}$. Moreover, for each $b^\pm \in B_{n^\pm} \cap \ker \partial_i^\pm$ and for each $n \in \mathbb{Z}_{\geq 0}$ there exists a unique $b^\pm \in B_{n^\pm}$ such that $\partial_i^\pm (b^\pm) = b^\pm$ and $\ell_i(b^\pm) = n$.

**Proof.** To prove (a), note that by Lemma 3.20, Remark 3.3 and Example 3.21 we have for any $b^\pm \in B_{n^\pm}$

\[
\langle \partial_i^-(r) (b_-), \partial_i^-(r)(b_-) \rangle = \sum_{b'_- \in B_{n^\pm}} \langle (\partial_i^-(r) (b_-), b'_- \rangle \delta_{b'_-} = \sum_{b'_- \in B_{n^\pm}} \langle \delta_{b'_-}, F_i^r (b_-) \rangle \delta_{b'_-}
\]

\[
= \sum_{b'_- \in B_{n^\pm}} \check{C}_{b'_-}^{F_i, b_-} \langle \delta_{b'_-}, b'_- \rangle \delta_{b'_-} = \sum_{b'_- \in B_{n^\pm}} \check{C}_{b'_-}^{F_i, b_-} \delta_{b'_-} \delta_{b'_-}.
\]

The remaining identities are proved similarly.

To prove (b), note that since $B_{n^\pm} = B_{n^\pm}^\prime$ and $B_{n^\pm}^\prime = B_{n^\pm}$, it suffices to prove that $(\partial_i^-(\text{top}) (b_-) \in B_{n^\pm}$. Following [14], §14.3, denote $B_{n^\pm}^\text{can} = B_{n^\pm} \cap F_i^\text{top}$ and $B_{n^\pm}^\text{can} = B_{n^\pm}^\text{can} \setminus B_{n^\pm}^\text{can}$. It follows from [14], §14.3] that for all $i \in I$, $B_{n^\pm}^\text{can} = \bigcup_{r \geq 0} B_{i,n}^r$. Let $b^\pm \in B_{n^\pm}^\text{can}$ and let $n = \ell_i(b^\pm)$, $u = (\partial_i^-(\text{top}) (b^\pm) = (\partial_i^-(r) (b^\pm)$. Then $u \in \ker \partial_i^-$ which, by Lemma 3.20 is orthogonal to $B_{i,n}^\text{can}, s > 0$. Thus, we can write

\[
\sum_{b'_- \in B_{n^\pm}^\text{can} \setminus b'_-} (\delta_{b'_-}, b'_- \rangle \delta_{b'_-} = \sum_{b'_- \in B_{n^\pm}^\text{can} \setminus b'_-} (\delta_{b'_-}, F_i^r (b_-) \rangle \delta_{b'_-}.
\]

By [14], Theorem 14.3.2, for each $b^\prime \in B_{i,n}^\text{can} \setminus b'_- \in \sum_{r > n} \mathbb{Z}[q, q^{-1}] B_{i,n}^\text{can}$. Using Lemma 3.20 we conclude that for any $b'' \in B_{i,n}^\text{can} \setminus b'_- \in \sum_{r > n} \mathbb{Z}[q, q^{-1}] B_{i,n}^\text{can}$. We have

\[
\langle \partial_i^-(r) (b_-), \partial_i^-(r)(b_-) \rangle = \sum_{b'_- \in B_{n^\pm}^\text{can} \setminus b'_-} (\delta_{b'_-}, b'_- \rangle \delta_{b'_-} = \sum_{b'_- \in B_{n^\pm}^\text{can} \setminus b'_-} (\delta_{b'_-}, F_i^r (b_-) \rangle \delta_{b'_-}.
\]
\( \langle \delta_b, b' \rangle \in \langle \delta_b, F_i^{(r)} U_q^{-} \rangle = \langle (\partial_i^{(r)}) (\delta_b), U_q^{-} \rangle = 0. \) Thus,
\[
u = \sum_{b' \in B_{i,0}^{\text{can}}} \langle \delta_b, \pi_{i,n}(b') \rangle \delta_{b'}.
\]
Note that, since \( \nu \neq 0 \), we cannot have \( \langle \delta_b, \pi_{i,n}(b') \rangle = 0 \) for all \( b' \in B_{i,0}^{\text{can}} \). Since \( \langle \delta_b, b' \rangle = \delta_{b,b'} \), we conclude that there exists a unique \( b' \in B_{i,0}^{\text{can}} \) such that \( \pi_{i,n}(b') = b \) and then \( \nu = (\partial_i^{(r)}) (\text{top}) (\delta_b) = \delta_{b'} \). Since \( \pi_{i,n} : B_{i,0}^{\text{can}} \rightarrow B_{i,n}^{\text{can}} \) is a bijection by [14 Theorem 14.3.2], the assertion follows.

Let \( b_+ \in B_{n+} \) and let \( r \leq \ell_i(b_+) \). By the above Proposition, there exists a unique \( b'_+ \) such that \( \ell_i(b'_+) = r \) and \( \partial_i^{(r)}(b'_+) = \partial_i^{(r)}(b_+) \). This implies that for each \( b_+ \in B_{n+} \) and each \( 0 \leq r \leq \ell_i(b_+) \) there exists a unique element of \( B_{n+} \), denoted \( \partial_i^{(r)}(b_+) \) such that \( \ell_i(\partial_i^{(r)}(b_+)) = \ell_i(b_+) \) and \( r \) and
\[
\partial_i^{(r)}(b_+) = \left( \frac{\ell_i(b_+)}{r} \right)_q \partial_i^{(r)}(b_+) \in \sum_{b'_+ \in B_{n+} : \ell_i(b'_+) < \ell_i(b_+) - r} \mathbb{Z}[q, q^{-1}] b'_+.
\]
(3.8)
The correspondence \( b_+ \mapsto \partial_i^{(r)}(b_+) \) is a bijection. In particular, \( \partial_i^{(r)}(b_+) = \partial_i^{(r)}(b_+) \).

**Example 3.23.** Let \( x^+ \in U_q^+ \). Using induction on \( r \) we obtain from (3.4)
\[
x^+ F_i^r = \sum_{r',r'' \geq 0, r' + r'' \leq r} (-1)^{r'} q_i^{-(r'')} \binom{r'}{2} (r', r'') (r + r') q_i! \binom{r}{r'} \times
\]
\[
K_{r'}^r K_{r''}^{r'} \circ (F_i^{r-r'} b_+^r \partial_i^{(r)} \partial_i^{(r')}(x^+)).
\]
In particular, if \( b_+ \in B_{n+} \cap \ker \partial_i^{(r)} \) then we have
\[
F_i^r b_+ = F_i^r b_+ + \sum_{r > 0} (-1)^r q_i^{-(r')} \binom{r'}{r} q_i! \binom{r}{r'} \sum_{b'_+ \in B_{n+} \cap \ker \partial_i^{(r)} \ell_i(b'_+) \leq \ell_i(b_+) - r} C_{b_+}^{b'_+} K_{r'}^{r'} \circ (F_i^{r-r'} b_+^r).
\]
This implies that \( F_i^r \bullet b_+ = F_i^r \circ b_+ \) is the unique \( \sim \)-invariant element of \( U_q(\mathfrak{g}) \) of the form
\[
F_i^r b_+ + \sum_{r' = 1} \sum_{b'_+ \in B_{n+} \cap \ker \partial_i^{(r')} \ell_i(b'_+) \leq \ell_i(b_+) - r'} C_{b_+}^{b'_+} K_{r'}^{r'} \circ (F_i^{r-r'} b_+^r), \quad C_{b_+}^{b'_+} \in q \mathbb{Z}[q].
\]
(3.10)
Similarly, if \( b_+ \in \ker \partial_i \) then \( F_i^r \circ b_+ = F_i^r b_+ \) and \( F_i^r \bullet b_+ \) is the unique \( \sim \)-invariant element of \( U_q(\mathfrak{g}) \) of the form
\[
F_i^r b_+ + \sum_{r' = 1} \sum_{b'_+ \in B_{n+} \cap \ker \partial_i \ell_i(b'_+) \leq \ell_i(b_+) - r'} \tilde{C}_{b_+}^{b'_+} K_{r'}^{r'} \circ (F_i^{r-r'} b_+^r), \quad \tilde{C}_{b_+}^{b'_+} \in q^{-1} \mathbb{Z}[q^{-1}].
\]
(3.11)
The coefficients \( \tilde{C}_{b_+}^{b'_+} \) can be expressed inductively, but in general it is not possible to write an explicit formula for them.

4. **Examples of Double Canonical Bases**

We fix some notation which will be used repeatedly throughout the rest of the paper. Let
\[
[a]_{\nu} = \frac{\nu^a - 1}{\nu - 1}, \quad [a]_{\nu}! = \prod_{j=1}^{a} [j]_{\nu}, \quad \left[ \begin{array}{c} a \\ n \end{array} \right]_{\nu} = \frac{[a]_{\nu} [a-1]_{\nu} \cdots [a-n+1]_{\nu}}{[n]_{\nu}!}
\]
(4.1)
\[(a)\nu = \frac{\nu^a - \nu^{-a}}{\nu - \nu^{-1}}, \quad (a)\nu! = \prod_{j=1}^{a} (j)\nu, \quad \left(\begin{array}{c} a \\ n \end{array}\right)_\nu = \frac{(a)\nu(a-1)\nu \cdots (a-n+1)\nu}{(n)\nu!}\]  

(4.2)

and

\[\langle a\rangle_\nu = \nu^a - \nu^{-a}, \quad \langle a\rangle_\nu! = \prod_{j=1}^{a} \langle j\rangle_\nu.\]  

(4.3)

In particular, \(X_i^{(n)} = X_i^n/(q)_n!\). We always use the convention that \[\begin{array}{c} a \\ n \end{array}\nu = 0 = \left(\begin{array}{c} a \\ n \end{array}\right)_\nu\] if \(n < 0\) or \(n > a \geq 0\). If \(a, n\) are non-negative integers, then all expressions in (4.1) lie in \(1 + \nu Z_{\geq 0}[\nu]\) while all expressions in (4.2) are in \(Z_{>0}[\nu + \nu^{-1}]\). Clearly,

\[\langle a\rangle_\nu \geq \nu^{a-1}(a)_\nu = \nu^{a-1}(\nu - \nu^{-1})^{-1}(a)_\nu,\]

hence

\[\langle a\rangle_\nu! \geq \nu^{a}(\frac{a}{n})_\nu = \nu^{a-n}(\frac{a}{n})_\nu \cdot \frac{(a)\nu \cdots (a-n+1)\nu}{(n)\nu!}\]  

(4.4)

thus, there is no need to introduce “angular” \(\nu\)-binomial coefficients. Finally,

\[\left[\begin{array}{c} a \\ n \end{array}\nu \right] = \nu^{n(n-a)} \left[\begin{array}{c} a \\ n \end{array}\right]_\nu = \nu^{2n(n-a)} \left[\begin{array}{c} a \\ n \end{array}\nu \right].\]  

4.1. **Double canonical basis of \(U_q(\mathfrak{sl}_2)\).** In this section we explicitly compute the double canonical basis in \(\mathcal{H}_q^+(\mathfrak{g})\) and \(U_q(\mathfrak{g})\) for \(\mathfrak{g} = \mathfrak{sl}_2\).

**Lemma 4.1.** In \(\mathcal{H}_q^+(\mathfrak{sl}_2)\) we have

\[K_+^a \circ F^b \circ E^c = \sum_{j \geq 0} (-1)^j q^j(b-c+1) \left[\begin{array}{c} \min(b, c) \\ j \end{array}\right] q^2 K_+^{a+j} \circ F^{b-j} E^{c-j}, \quad a, b, c \in Z_{\geq 0}.\]

**Proof.** Let

\[b_{m,n} = \sum_{j \geq 0} (-1)^j q^j(m-n+1) \left[\begin{array}{c} \min(m, n) \\ j \end{array}\right] q^2 K_+^j \circ F^{m-j} E^{n-j}.\]

By definition, \(b_{m,n} - F^m E^n \in \sum_{j > 0} q \mathbb{Z}[q] K_+^j \circ F^{m-j} E^{n-j}\), hence by Theorem 1.3 it is sufficient to prove that \(b_{m,n} = \overline{b_{m,n}}\). We claim that

\[C_+^a = b_{a,a} = \sum_{j \geq 0} (-q)^j \left[\begin{array}{c} a \\ j \end{array}\right] q^2 K_+^j \circ F^{a-j} E^{a-j}.\]  

(4.5)

where \(C_+ = b_{1,1} = FE - qK_+\). Observe that \(C_+\) is central in \(\mathcal{H}_q^+(\mathfrak{g})\), since \(C_+ F = F(FE + (q^{-1} - q)K_+) - qK_+ F = FC_+\) and similarly \([E, C_+] = 0\). Then

\[b_{a,a} C_+ = \sum_{j \geq 0} (-1)^j q^j \left[\begin{array}{c} a \\ j \end{array}\right] q^2 K_+^j F^{a-j} E^{a-j} (FE - qK_+)\]

\[= \sum_{j \geq 0} (-q)^j \left[\begin{array}{c} a \\ j \end{array}\right] q^2 K_+^j (F^{a+1-j} E^{a+1-j} - q^{2(a-j)+1} K_+ F^{a-j} E^{a-j})\]

\[= \sum_{j \geq 0} (-q)^j \left[\begin{array}{c} a \\ j \end{array}\right] + q^{2(a-j)+1} \left[\begin{array}{c} a \\ j-1 \end{array}\right] q^2 \right] K_+^j F^{a+1-j} E^{a+1-j} = b_{a+1,a+1} + 1.\]

It is easy to see now that \(b_{m,n} = F^{\max(m-n,0)} C_+^{\min(m,n)} F^{\max(n-m,0)},\) \(m, n \in Z_{\geq 0}\). Since \(C_+ = FE - q^{-1} K_+ = FE + (q^{-1} - q)K_+ - q^{-1} K_+ = C_+\) and \(C_+\) is central, it follows that all the \(b_{m,n}\) \(m, n \in Z_{\geq 0}\) are \(\sim\)-invariant. \(\square\)
Thus, the double canonical basis of $\mathcal{H}_q^+(\mathfrak{sl}_2)$ is

$$\mathbf{B}^+_{\mathfrak{sl}_2} = \{ K^a_+ \circ F^m_+ C^m_+ E^m_+ : a, m, m_0 \in \mathbb{Z}, \min(m_+, m_-) = 0 \}.$$ 

Let $C^{(0)} = 1$, $C^{(1)} = C = C_+ - q^{-1}K_- = FE - qK_+ - q^{-1}K_-$ and define inductively

$$C^{(m+1)} = CC^{(m)} - K_+ K_- C^{(m-1)}, \quad m \geq 1.$$ 

Note that $C$ is central in $U_q(\mathfrak{g})$ and $\sim$-invariant, hence $\overline{C^{(m)}} = C^{(m)}$. It follows directly by induction on $m$ that

**Lemma 4.2.** For all $m, k \geq 0$

$$F^k C^{(m)} = \sum_{a,b \geq 0} (-1)^{a+b} q^{(k+1)(a-b)} \begin{bmatrix} m-a & b \ a \\ b & m-b \end{bmatrix}_{q^{-2}} K^a_+ K^b_+ \circ F^{m+k-a-b} E^{m-a-b}.$$ 

$$C^{(m)} E^k = \sum_{a,b \geq 0} (-1)^{a+b} q^{(k+1)(a-b)} \begin{bmatrix} m-a & b \ a & m-b \end{bmatrix}_{q^{-2}} K^a_+ K^b_+ \circ F^{m-a-b} E^{m+k-a-b} \quad (4.6)$$

**Proposition 4.3.** For all $m \geq 0$,

$$C^{(m)} = \sum_{0 \leq i, j, i+j \leq m} (-1)^i q^{-j-i^2} \begin{bmatrix} m-i & j \ j & i \end{bmatrix}_{q^{-2}} K^i_+ K^j_+ E^{m-i-j} \circ E^{m-i-j}, \quad m \geq 0. \quad (4.7)$$

In particular, $C^{(m)} = F^m \cdot E^m \in \mathbf{B}^-_{\mathfrak{sl}_2}$.

**Proof.** Let $\iota : \mathcal{H}_q^+(\mathfrak{g}) \to U_q(\mathfrak{g})$ be the natural inclusion of vector spaces. One can show by induction on $k$ that in $U_q(\mathfrak{g})$

$$\iota(C^k) C = \iota(C^{k+1}) - q^{-2k-1} K_- \iota(C^k) + (1 - q^{-2k}) K_+ K_- \iota(C^{k-1}). \quad (4.8)$$

Denote by $X_m$ the right hand side of (4.7). It follows from (4.8) that

$$X_m C = K_+ K_- X_{m-1}$$

$$= \sum_{i,j \geq 0} (-1)^i q^{-j-i^2} \begin{bmatrix} m-i & j \ j & i \end{bmatrix}_{q^{-2}} (K^i_+ K^j_+ \iota(C^{m+1-i-j})$$

$$- q^{-2(m-i-j)-1} K^i_+ K^{i+1}_+ \iota(C^{m-i-j}) + (1 - q^{-2(m-i-j)}) K^{i+1}_+ K^{j+1}_- \iota(C^{m-i-j}))$$

$$+ \sum_{i,j \geq 0} (-1)^{i+1} q^{-j-i^2} \begin{bmatrix} m-1-i & j \ j & i \end{bmatrix}_{q^{-2}} K^{i+1}_+ K^{j+1}_- \iota(C^{m+1-i-j})$$

$$= \sum_{i,j \geq 0} (-1)^{i} q^{-j-i^2} \begin{bmatrix} m-i & j \ j & i \end{bmatrix}_{q^{-2}} (K^i_+ K^j_+ \iota(C^{m-i-j}) - q^{-2(m-i-j)-1} K^i_+ K^{j+1}_- \iota(C^{m-i-j})$$

$$+ q^{-2(j-i)} \begin{bmatrix} j & i-1 \ i-1 & j \end{bmatrix}_{q^{-2}} K^i_+ K^j_+ \iota(C^{m+1-i-j})$$

$$= \sum_{i,j \geq 0} (-1)^{i} q^{-j-i^2} \begin{bmatrix} m+1-i & j \ j & i \end{bmatrix}_{q^{-2}} K^i_+ K^j_+ \iota(C^{m+1-i-j}) = X_{m+1}$$

$$= X_m + \sum_{i,j \geq 0} (-1)^{i+1} q^{-j-i^2} \begin{bmatrix} m-i & j \ j & i \end{bmatrix}_{q^{-2}} K^{i+1}_+ K^{j+1}_- \iota(C^{m-i-j})$$
For all \( m \geq 0 \),

\[
\begin{bmatrix} j - 1 \\ i \end{bmatrix} q^{-2(i-j)} + q^{-2(j-i)} \begin{bmatrix} j - 1 \\ i - 1 \end{bmatrix} q^{-2} = \begin{bmatrix} j \\ i \end{bmatrix} q^{-2},
\]

\[
q^{-2(m-i-j+1)} \begin{bmatrix} m - i \\ j - 1 \end{bmatrix} q^{-2} + \begin{bmatrix} m - i \\ j \end{bmatrix} q^{-2} = \begin{bmatrix} m + 1 - i \\ j \end{bmatrix} q^{-2}.
\]

Thus, we conclude that \( X_m \) satisfies the same relation as \( C^{(m)} \). Since \( X_0 = 1 \) and \( X_1 = C \), we conclude that \( X_m = C^{(m)} \) for all \( m \geq 0 \). The second assertion is now immediate by Theorem 1.5, since \( C^{(m)} = C^{(m)} \) and by (4.7),

\[
C^{(m)} = F^m \circ E^m \in \bigoplus_{j \geq 0} \sum_{0 \leq i \leq \min(j,m-j)} q^{-1} \mathbb{Z}[q^{-1}] K^i_+ K^i_- F^{m-i-j} \circ E^{m-i-j}.
\]

\[\square\]

**Corollary 4.4.** For all \( m_-, m_+ \geq 0 \),

\[
F^{m_-} \cdot E^{m_+} = \sum_{0 \leq i \leq j} (-1)^{j-i} q^{-j-i^2-(j-i)(m_-+m_-)} [m-i] \begin{bmatrix} j \\ i \end{bmatrix} q^{-2} K^i_+ K^i_- \circ F^{m_- - i - j} \circ E^{m_+ - i - j},
\]

where \( m = \min(m_+, m_-) \).

Combining Lemma 4.2 and Proposition 4.3 and using (4.4) we obtain the following identity.

**Proposition 4.5.** For all \( m, a, b \geq 0 \) with \( a + b \leq m \) we have in \( \mathbb{Z}[\nu] \)

\[
\sum_{r=0}^{\min(a,b)} (-1)^r \nu^{(r)} \frac{[m-r]_\nu!}{[a-r]_\nu! [b-r]_\nu! [r]_\nu!} = \nu^{ab} [m-a-b]_\nu \begin{bmatrix} m-a \\ a \end{bmatrix}_\nu \begin{bmatrix} m-b \\ b \end{bmatrix}_\nu
\]

Our preceding computations, together with Theorem 1.5, immediately yield the following

**Proposition 4.6.** For all \( m_\pm \in \mathbb{Z}_{\geq 0} \),

\[
F^{m_-} \cdot E^{m_+} = \sum_{0 \leq a+b \leq m} (-1)^{a+b} q^{[m-m_+ - m_- + 1)(a-b)]} \begin{bmatrix} m-a \\ b \end{bmatrix} q^{-2} \begin{bmatrix} m-b \\ a \end{bmatrix} q^2 K^a_+ K^b_- \circ F^{m-a-b} \circ E^{m+a-b}
\]

where \( m = \min(m_+, m_-) \). Thus, the double canonical basis in \( U_q(\mathfrak{sl}_2) \) is given by \( \{ K^a_+ K^b_- \circ F^{m} \circ C^{(m_0)} \circ E^{m_+} : a_\pm, m_\pm, m_0 \in \mathbb{Z}_{\geq 0}, \min(m_+, m_-) = 0 \} \).

An easy induction shows that

\[
C^{(a)} C^{(b)} = \sum_{j=0}^{\min(a,b)} (K^- K^+) C^{(a+b-2j)}
\]

(4.10)

**Lemma 4.7.** For all \( n \geq 0 \) we have

\[
F^n E^n = \sum_{r=0}^{n} \sum_{j=0}^{r} \binom{n}{r-j} \binom{r}{j} C^{(n-r)}
\]

\[
E^n F^n = \sum_{r=0}^{n} \sum_{j=0}^{r} \binom{n}{r-j} \binom{r}{j} C^{(n-r)}
\]

where \( c_{0,0}^{(n)} = 1, c_{r,j}^{(n)} = c_{r,-j}^{(n)} \in \mathbb{Z}_{\geq 0}[q, q^{-1}] \). In particular, Conjecture 1.27 holds for \( g = \mathfrak{sl}_2 \).
Lemma 4.9. That for any two elements $b, b'$ of $\mathcal{B}_{\mathfrak{sl}_2}$, $bb'$ decomposes as a linear combination of elements of the same basis with coefficients being Laurent polynomials in $q$ with positive coefficients. However, this fact is special for $\mathfrak{sl}_2$ and is unlikely to hold in greater generality.

4.2. Action on a double basis for $\mathfrak{sl}_2$. We now consider the action (1.7) on the double canonical basis of $U_q(\mathfrak{sl}_2)$. We denote the corresponding operators by $E_\lambda, F_\lambda$.

Lemma 4.9. Let $\lambda \in \mathbb{Z}$. Then for all $m_+ > m_-$

$$F_\lambda(F^{m_-} \bullet E^{m_+}) = (\lambda + 2m_+ - 2m_-)qK_+ F^{m_-} \bullet E^{m_+} + (\lambda + m_+ - m_-)qK_- F^{m_-} \bullet E^{m_+} - (\lambda + m_+ - m_-)qF^{m_-} \bullet E^{m_+} + \frac{1}{2}\lambda qK_- F^{-m_-} \bullet E^{-m_-}$$

where we use the convention that $F^r \bullet E^s = 0$ if $r < 0$ or $s < 0$, while for $m_+ \leq m_-$

$$F_\lambda(F^{m_-} \bullet E^{m_+}) = (\lambda + m_+ - m_-)q^{m_-} F^{-m_-} \bullet E^{m_+}$$

Furthermore, for all $m_+ \geq m_-$

$$E_\lambda(F^{m_-} \bullet E^{m_+}) = (\lambda)qK_+ F^{-m_-} \bullet E^{-m_-}$$
while for all \( m_+ < m_- \)
\[
E_\lambda(F^{m_-} \cdot E^{m_+}) = \langle \frac{1}{2} \lambda + m_- - m_+ \rangle_q F^{m_- - 1} \cdot E^{m_+} \\
+ \langle \frac{1}{2} \lambda \rangle_q K_- \cdot F^{m_- - 2} \cdot E^{m_- - 1} + \langle \frac{1}{2} \lambda \rangle_q K_+^{-1} \cdot F^{m_-} \cdot E^{m_+ + 1} \\
+ \langle \frac{1}{2} \lambda + m_- - m_+ \rangle_q K_+^{-1} K_- \cdot F^{m_- - 1} \cdot E^{m_+}
\]
(4.14)

Proof. It is an easy consequence of \((4.6)\) that
\[
F^k = CE^{k-1} + q^k K_+ \cdot E^{k-1} + q^{-k} K_- \cdot E^{k-1},
\]
\[
E^k F = CE^{k-1} + q^{-k} K_+ \cdot E^{k-1} + q^k K_- \cdot E^{k-1}, \quad k \geq 1
\]
Suppose first that \( m_+ > m_- \). Then \( F^{m_-} \cdot E^{m_+} = C^{m_-} E^{m_+ - m_-} \) and
\[
F_\lambda(C^{m_-} E^{m_+ - m_-}) = \langle \frac{1}{2} \lambda + m_- - m_+ \rangle C^{m_-} E^{m_+ - m_-} = q^{-\frac{1}{2} \lambda - (m_+ - m_-)} C^{m_-} E^{m_+ - m_-} F
\]
\[
= \langle \frac{1}{2} \lambda + m_- - m_+ \rangle q C^{m_-} E^{m_+ - m_- - 1} + \langle \frac{1}{2} \lambda + 2(m_- - m_+) \rangle q K_+ \cdot C^{m_-} E^{m_+ - m_- - 1}
\]
\[
+ \langle \frac{1}{2} \lambda \rangle_q K_- \cdot C^{m_-} E^{m_+ - m_- - 1}
\]
Then it remains to use \( C^{m_-} = C^{m_- + 1} + K_+ K_- C^{m_- - 1} \). If \( m_+ \leq m_- \) then \( F^{m_-} \cdot E^{m_+} = F^{m_- - m_+} C^{m_+} \) and
\[
E_\lambda(F^{m_-} \cdot E^{m_+}) = \langle q^{-\frac{1}{2} (m_- - m_+)} \rangle q^{-\frac{1}{2} + m_- - m_+} C^{m_-} F^{m_- - m_+} \cdot E^{m_+} = \langle \frac{1}{2} \lambda + m_- - m_+ \rangle q F^{m_- - m_+} \cdot E^{m_+}.
\]
The identities involving \( E_\lambda \) are proved similarly.

Corollary 4.10. If \( k_+ = \frac{1}{2} + \max(0, m_- - m_+) \) is a non-negative integer then \( k_+ = \max\{k \geq 0 : E^{k}_\lambda(F^{m_-} \cdot E^{m_+}) \neq 0\} \). Similarly, if \( k_- = \frac{1}{2} + m_- - m_+ + \max(0, m_+ - m_-) \) is a non-negative integer then \( k_- = \max\{k \geq 0 : F^{k}_\lambda(F^{m_-} \cdot E^{m_+}) \neq 0\} \).

Proof. We prove only the first statement, the proof of the second one being similar. If \( m_- \leq m_+ \) then by an obvious induction we obtain
\[
E_\lambda^s(F^{m_-} \cdot E^{m_+}) = \langle \frac{1}{2} \lambda \rangle q \cdots \langle \frac{1}{2} \lambda - s + 1 \rangle_q K_+^{s} \cdot F^{m_-} \cdot E^{m_+},
\]
which is zero if and only if \( \lambda \in 2\mathbb{Z}_{\geq 0} \) and \( s \geq \frac{1}{2} + 1 \). If \( m_- > m_+ \) then each term in the right hand side of \((4.14)\) is of the form \( K \cdot F^a \cdot E^b \) with \( a - b = m_- - m_+ - 1 \) and the term with the largest coefficient is \( F^{m_- - 1} \cdot E^{m_+} \). Thus,
\[
E^{m_- - m_+}_\lambda(F^{m_-} \cdot E^{m_+}) = \langle \frac{1}{2} + m_- - m_+ \rangle q \cdots \langle \frac{1}{2} + 1 \rangle_q F^{m_-} \cdot E^{m_+} + \cdots
\]
where the remaining terms are of the form \( K \cdot F^a \cdot E^b \) with the coefficients being of the form \( \prod_{j=0}^{s}(\frac{1}{2} + k - j)_q \) with \( k < m_- - m_+ \). It follows that \( E^{k}_\lambda(F^{m_-} \cdot E^{m_+}) = 0 \) only if \( \frac{1}{2} \lambda + m_- - m_+ \in \mathbb{Z}_{\geq 0} \) and \( s > \frac{1}{2} \lambda + m_- - m_+ \).

Define
\[
\varepsilon^\lambda(F^{m_-} \cdot E^{m_+}) = \frac{1}{2} + \max(0, m_- - m_+)
\]
Then we obtain the following

Corollary 4.11. For all \( \lambda \in \mathbb{Z}, \ m_1, m_2 \in \mathbb{Z}_{\geq 0} \)
\[
E_\lambda(F^{m_-} \cdot E^{m_+}) = \langle \varepsilon^\lambda(F^{m_-} \cdot E^{m_+}) \rangle_q b + \sum_{b' : \varepsilon^\lambda(b') < \varepsilon^\lambda(F^{m_-} \cdot E^{m_+})} c_{b'} \cdot b'
\]
where
\[
b = \begin{cases} \langle F^{m_- - 1} \cdot E^{m_+} \rangle, & m_- > m_+ \\
K_+^{-1} \cdot F^{m_-} \cdot E^{m_+ + 1}, & m_- \leq m_+ 
\end{cases}
\]
4.3. Some elements in double canonical bases in ranks 2 and 3. We will need explicit formulae for some elements of dual canonical bases for computational purposes. We already listed the most obvious ones in Example 3.21.

**Example 4.12.** It is easy to see, extending [14] §14.5.4, that the elements $F_i^{(r)} F_j^{(s)} F_i^{(n-s)}$, $0 \leq s \leq n \leq 2$ are in $\mathbf{B}^{\text{can}}$ and form a basis of the homogeneous component of $U_q^{-}$ of degree $na_{-j} + \alpha_{-j}$. Let $F_i^{(r)} F_j^{(s)} = \delta F_i^{(r)} F_j^{(s)}$, $r, s, 0 \leq r + s - a_{ij}$ and let $E_i^{(r)} j^{(s)} = (F_i^{(r)} j^{(s)})^d$. We summarize their properties in the following Lemma, which is proved by direct computations based on Lemma 3.20.

**Lemma 4.13.** (a) For all $k, l \geq 0$, $k + l < -a_{ij}$ we have

$$F_i F_{k+j} = q^{l+\frac{1}{2}a_{ij}} F_{k+j} + q^{-k-\frac{1}{2}a_{ij}} F_{k+j+l+1}, \quad F_{k+j} F_i = q^{-l-\frac{1}{2}a_{ij}} F_{k+j+l} + q^{k+\frac{1}{2}a_{ij}} F_{k+j+l+1}$$

(b) For all $r, s \geq 0$, $r + s \leq -a_{ij}$ we have

$$\Delta_i F_i^{(r)} j^{(s)} = \sum_{r', r'' = r} q^{r'(r'' + s + \frac{1}{2}a_{ij})} F_{q^{r'} r''} F_i^{(r')} \otimes F_i^{(r'' s')} + \sum_{s_s' + s_s'' = s} q^{s''(s' + r + \frac{1}{2}a_{ij})} (s'' q^{s'} q^{r'}) F_i^{(s' s'')} \otimes F_i^{(s'' s')}.
$$

(c) For all $s, r, s', r' \geq 0$, $s + r = s' + r' \leq -a_{ij}$, we have

$$\langle F_i^{(r)} j^{(s)} F_i^{(r') j^{(s')}} F_i^{(r'' s')} E_i^{(s' s'')} F_i^{(r' s')} \rangle = (-1)^{s + s'} q^{s''(r' + s') - (a_{ij} + r' - 1)} p_s' r, r'(q) \frac{1}{1 + \sum_{l=0}^{2s'} q_s' r_q^{s' + 2a_{ij} - 2l}} (s' q^{r'}) q_s' r_q^{s' + 2a_{ij} - 2l} \in \mathbb{Z}_{\geq 0}[q, q^{-1}]$$

where

$$p_s' r, r'(q) = \prod_{l=0}^{\min(s', r)} q^{-s'(r' + s' + 2a_{ij} - 2)} (s' q^{r'}) q_s' r_q^{s' + 2a_{ij} - 2l} \in \mathbb{Z}_{\geq 0}[q, q^{-1}]$$

The following Lemma provides a partial converse to Theorem 3.11.

**Lemma 4.14.** Suppose that $\langle b_-, b_+ \rangle \in \mathbb{Z}[q, q^{-1}]$ for all $b_+ \in \mathbf{B}_n$. Then for every $i \neq j$, $a_{ij} a_{ji} < 4$.

**Proof.** We may assume without generality that $a_{ij}, a_{ji} \neq 0$ and $|a_{ij}| \geq |a_{ji}|$ hence $d_i \leq d_j$. Then by Lemma 4.13, $\langle F_i, E_j \rangle = (q^i - q^{-i})(q^j - q^{-j})/(q^{a_{ij}} - q^{-a_{ij}})$ which can only be in $\mathbb{Z}[q, q^{-1}]$ if $d_j |a_{ij}| = d_i |a_{ij}| \leq d_i + d_j$. Therefore, $|a_{ij}| \leq 1 + d_i / d_j < 2$, hence $a_{ji} = -1$ and $d_j = -d_i a_{ij}$. Suppose that $|a_{ij}| \geq 4$. Applying Lemma 4.13 again we obtain

$$\langle F_i, E_j \rangle = \frac{(q^i - q^{-1})(q^j - q^{-1})(q^{a_{ij}} - q^{-a_{ij}}) (q^{a_{ij}+1} - q^{-a_{ij}-1})}{(q^{a_{ij}} - q^{-a_{ij}}) (q^{a_{ij}+1} - q^{-a_{ij}-1})} = \frac{(q^i - q^{-1})(q^j - q^{-1})}{q^{a_{ij}+1} - q^{-a_{ij}-1}}.$$

This cannot be a Laurent polynomial if $|a_{ij}| > 4$ by the degree considerations, while for $a_{ij} = -4$ we have $\langle F_i, E_j \rangle (q^4 - 1)/(q^4 + 1) \not\in \mathbb{Z}[q, q^{-1}]$. Thus, $|a_{ij}| \leq 3$.

From now on, given $f = \sum_j a_j \nu^j \in \mathbb{Z}[\nu, \nu^{-1}]$, let $[f]_+ = \sum_{j>0} a_j \nu^j$ and $[f]_- = \sum_{j<0} a_j \nu^j$. We will now consider some examples in rank 2.

First, assume that $a_{ji} = -1$ (in particular, this includes all subdiagrams of rank 2 for $g$ semisimple and all affine cases except those of rank 2). Then $d_j = d_i, a_{ij} = -d_i$ and by Lemma 4.13,

$$[E_i^{(s)} F_j^{(s)}] = [E_j^{(s)} F_j^{(s)}] = \frac{1}{(d-1)q_{i} \cdots (d-s+1)q_{i}} (K_{i+1}^{s} K_{(s-1)} - K_{s}^{s} K_{(s-1)})$$

hence $d F_i^{(s)} E_j^{(s)} = d F_j^{(s)} E_j^{(s)}$ and $F_i^{(s)} E_j^{(s)} - d F_i^{(s)} E_j^{(s)} E_j^{(s)} F_i^{(s)} = F_j^{(s)} E_j^{(s)} - d F_j^{(s)} E_j^{(s)} F_j^{(s)} E_j^{(s)}$, while

$$F_{ij} \bullet E_{ij} = F_{ij} E_{ij} - q_i K_{i+1} K_{i+1} - q_i ^{-1} K_{i+1} K_{-i},$$

$$F_{ij} \bullet E_{ij} = F_{ij} E_{ij} - q_i K_{i+1} K_{i+1} - q_i ^{-1} K_{i+1} K_{-i}.$$
and for \( d > 2 \)
\[
F_{i\bar{j}} \cdot E_{i\bar{j}} = \begin{cases} 
(d - 1)q_i F_{i\bar{j}}E_{i\bar{j}} - q_i^2 K_{i+j}^2 - q_i^{-2} K_{-i-j}^2, & d \text{ even} \\
\frac{1}{2}(d - 1)q_i^2 F_{i\bar{j}}E_{i\bar{j}} - q_i K_{i+j}^2 - q_i^{-1} K_{-i-j}^2, & d \text{ odd}
\end{cases}
\]
while for \( d > 3 \)
\[
F_{i\bar{j}} \cdot E_{i\bar{j}} = \begin{cases} 
\frac{(d - 1)}{2}q_i F_{i\bar{j}}E_{i\bar{j}} - q_i^3 K_{i+j}^3 - q_i^{-3} K_{-i-j}^3, & d = 0 \pmod{3} \\
\frac{(d - 1)}{2}q_i^2 F_{i\bar{j}}E_{i\bar{j}} - q_i^2 K_{i+j}^3 - q_i^{-2} K_{-i-j}^3, & d = 1 \pmod{3} \\
\frac{(d - 2)}{2}q_i^3 F_{i\bar{j}}E_{i\bar{j}} - q_i K_{i+j}^3 - q_i^{-1} K_{-i-j}^3, & d = 2 \pmod{3} \\
\frac{(d - 2)}{2}q_i^3 F_{i\bar{j}}E_{i\bar{j}} - q_i^3 K_{i+j}^3 - q_i^{-3} K_{-i-j}^3, & d = 5 \pmod{3}
\end{cases}
\]
Note that if \( d \leq 2 \) then \( F_{ij} \in B_{n+} \); for \( d = 2 \) we also have \( F_{ij} \in B_{n-} \) for all \( k \in \mathbb{Z}_{\geq 0} \). Then we can use (1.9) to compute \( F_{ij}^{m+} \cdot E_{ij}^{mn} \) (respectively, \( F_{ij}^{m-} \cdot E_{ij}^{mn} \)) for all \( m_\pm \in \mathbb{Z}_{\geq 0} \). Similarly, we obtain
\[
F_{ij} \circ E_{ji} = F_{ij}E_{ji} - q_i^2 K_{i+j} F_i E_i + (q_i^{d+1} - [q_i^{d-1}]_+) K_{i+j} K_{i+j},
\]
\[
F_{ij} \bullet E_{ji} = F_{ij} \circ E_{ji} - q_i^{-1} K_{i-j} F_j \circ E_j + q_i^{-1-d} K_{i-j} K_{i-j},
\]
\[
F_{ji} \circ E_{ij} = F_{ji} E_{ij} - q_i K_{i+j} F_j E_j + q_i^{d+1} K_{i+j} K_{i+j},
\]
\[
F_{ji} \bullet E_{ij} = F_{ji} \circ E_{ij} - q_i^{-d} K_{i-j} F_i E_i + ([q_i^{d-1}]_+ - [q_i^{d-1}]_- K_{i-j} K_{i-j} K_{i-j}.
\]
If \( d_i = d_j, a_{ij} = a_{ji} = -a \) we obtain
\[
[E_{i\bar{j}}^s, F_{i\bar{j}}^s] = [E_{i\bar{j}}^s, F_{i\bar{j}}^s] = -\frac{(1)_{q_i} \langle s \rangle_{q_i}!}{\langle a_{ij} \rangle_{q_i} \cdots \langle a_{s+1} \rangle_{q_i}} (K_{i+j}^s K_{i+j} - K_{i-j}^s K_{i-j}),
\]
and so for all \( 1 \leq s \leq a \)
\[
F_{i\bar{j}} \cdot E_{i\bar{j}} - \left(\begin{array}{c} a \\ s \end{array}\right)_{q_i} F_{i\bar{j}}^s E_{i\bar{j}} = -q_i K_{i+j}^s K_{j-i} - q_i^{-1} K_{-i-j}^s K_{i-j} = F_{ji}^s \bullet E_{ji}^s - \left(\begin{array}{c} a \\ s \end{array}\right)_{q_i} F_{ji}^s E_{ji}^s.
\]
We also have
\[
F_{ij} \circ E_{ji} = (a_{ij})_q F_{ij} F_j - q_i^2 K_{ij} F_i E_i + (q_i^{a-1} - [q_i^{a-1}]_+) K_{i+j} K_{i+j},
\]
\[
F_{ij} \bullet E_{ji} = F_{ij} \circ E_{ji} - q_i^{-a} K_{i-j} F_j \circ E_j - [q_i^{1-a}]_-(K_{i-j} K_{i+j} K_{i-j}).
\]
Furthermore, for \( a > 1 \)
\[
F_{ij}^a \circ E_{ji}^a = \left(\begin{array}{c} a \\ q_i \end{array}\right) F_{ij}^a E_{ji}^a - (q_i^a + [q_i^{a-2}]_+) (a_{ij})_q K_{i+j} F_j E_j + (q_i^a (1 + [q_i^{2a-4}]_+)) K_{i+j} F_i E_i
\]
\[
- (q_i^3 + [q_i^3 - q_i^2]_{q_i^{2a-4}}) K_{i+j}^2 F_i E_i
\]
\[
F_{ij}^a \bullet E_{ji}^a = F_{ij}^a \circ E_{ji}^a - q_i^{-1-2(a)} [q_i^{2a-2}]_+ K_{i+j} F_j \circ E_j
\]
\[
+ q_i^2 K_{i-j} K_{i+j} F_j \circ E_j + (q_i^{2a} + q_i^{-2(a-1)} - q_i^{-2(a-1)}_+) K_{i-j} K_{-i-j} F_j \circ E_j
\]
\[
+ (q_i^{2a+1} + q_i^{2a+3-2\delta_{a,2}} - q_i^{-3} - [q_i^{1-2(a)}]_- K_{i-j} K_{i-j} F_j \circ E_j
\]
\[
+ q_i^{-1}(1 - \delta_{a,2}) K_{i-j} K_{i+j}^2 + q_i^{2a+3} - q_i^{2a+1} - [q_i^{1-2(a)}]_- K_{i-j} K_{i-j} F_j \circ E_j
\]
\[
F_{ij}^a \circ E_{ji}^a = \left(\begin{array}{c} a \\ q_i \end{array}\right) F_{ij}^a E_{ji}^a - q_i^{1+2(a)} [q_i^{2a}]_+ K_{i+j} F_j E_j^2 + (q_i^a + q_i^{2(a-1)} - [q_i^{2(a)}]_+) K_{i+j} K_{i+j} F_i E_i
\]
\[
+ (q_i^{2a-3} - q_i^{2a-1} - [q_i^{2a}]_+) K_{i+j}^2 F_i E_i
\]
\[
F_{ij}^a \bullet E_{ji}^a = F_{ij}^a \circ E_{ji}^a - (q_i^a + [q_i^{a+2}]_-) K_{i-j} F_j \circ E_j + q_i^{-2(1 + [q_i^{2a-4}]_-) K_{i-j}^2 F_j \circ E_j
\]
$+ [q_i^{-2+2(a)}]_2 K_{-i} K_{i+j} F_i \circ E_i - [q_i^{-1(a)}]_2 K_{-i} K_{i+j}$
$+ (-q_i^{-3}[q_i^{-2a+4}]_2 + q_i^{-2}([q_i^{-2a+5}]_2 - q_i^{-1})) K^2_{-i} K_{j} + (q_i^{-1}[q_i^{-2a+4}]_2 + q_i^{-1(a)}) K_{-j} K^2_{i+j}$

$F_{iji} \circ E_{ij2} = \left(\frac{a}{2}\right)_{q_i} F_{iji} E_{ij2} - q_i^{-1} K_{i} K_{j} F_i E_i + (q_i^{a} - [q_i^{a-2}]_2) K^2_{i+j} K_{j}$

$F_{iji} \bullet E_{ij2} = F_{iji} \circ E_{ij2} - q_i^{-1(a)} K_{-i} F_{ji} \circ E_{ji} - [q_i^{-2(a)}]_2 K_{-i} K_{i+j} + (q_i^{-a} - [q_i^{-2(a)}_2]_2 K^2_{i-j}$

$F_{iji} \circ E_{ij3} = (a)_{q_i} F_{iji} E_{ij3} - q_i^{-2} K_{-i} K_{i+j} F_i E_i - q_i^{-1} (1 + q_i^{2a}) K^2_{i+j} K_{j}$

$F_{iji} \bullet E_{ij3} = F_{iji} \circ E_{ij3} - q_i^{-2(a-1)} K_{-i} F_{ji} \circ E_{ji} - [q_i^{-3(a)}_2]_2 K_{-i} K_{i+j} + (q_i^{-a} - [q_i^{-3(a)}_2]_2 K^3_{i-j}$

Example 4.15. Let $d_1 = d_2 = 1$ and $a_{ij} = -2$. After [14] §14.5.5], the elements of degree $2(\alpha_{-i} + \alpha_{-j})$ in $B^{an}$ are

$F_i^{(2)} F_j^{(2)} F_i^{(1)} F_j^{(1)} F_i^{(1)} F_j^{(1)} F_i^{(1)} F_j^{(2)} - F_i^{(2)} F_j^{(2)}$

as well as three more elements obtained from these by applying the automorphism which interchanges $F_i$ and $F_j$. The corresponding elements of $B_{an}$ are, respectively,

$F_{ij2} = (q^2(2)_{q} F_i F_j F_i F_j + (2 q - (2)_{q}) F_i F_j F_i F_j + (q - q^{-1})(F_i F_j F_i F_j + F_j F_i F_j F_i))$
$+ ((2)_{q} + 2 q^{-3}) F_i F_j F_i F_j - q^{-2} (2)_{q} F_i^{2} F_j^{2} F_i^{2} F_j^{2} / ((q - q^{-1})(q^2 - q^{-2})(q^4 - q^{-4}))$

$F_{ij3} = (F_i^{2} F_j^{2} + F_j^{2} F_i^{2} + F_j^{2} F_i^{2} F_j^{2} + (3)_{q} (F_i^{3} F_j^{3} F_j F_j - F_j^{3} F_j F_j - F_j F_j F_j F_j + F_j F_j F_j F_j)) / ((q - q^{-1})(q^2 - q^{-2}))$

$F_{ijj} = (q^{-2} (2)_{q} F_i^{2} F_j^{2} F_i^{2} F_j^{2} + (q - q^{-1})(F_i F_j F_i F_j + F_j F_i F_j F_i)) + q^4 (2 q^{-3} + (2)_{q}) F_i F_j F_j F_j - q^{-4} (2 q^{3} + (2)_{q}) F_j F_j F_j F_j / ((q - q^{-1})(q^2 - q^{-2})(q^4 - q^{-4}))$

Set $E_a = F_{a^{*}}$. Since $d_{F_{ji}, E_{ji}} = (2)_{q}$ by the previous example we have

$F_{ij2} \circ E_{ij2} = (2)_{q} (4)_{q} F_{ij2} E_{ij2} + (q - q^{-1}) (2)_{q} K_{i} K_{j} F_{ij} E_{ji} - 2 q^{-2} K^2_{i+j} K_{j}$

$F_{ij2} \bullet E_{ij2} = F_{ij2} \circ E_{ij2} + (q^{-1} - q^{-3}) K_{-i} K_{j} F_{ij} \circ E_{ji} - 2 q^{-2} K^2_{i-j} - q^{-2} K_{-i} K_{j} K_{i+j} K_{j}$

Similarly,

$F_{ij3} \circ E_{ij3} = (2)_{q} F_{ij3} E_{ij3} + (q - q^{-3}) K_{i} K_{j} F_{ij} E_{ji} + (q^2 - q^{-3}) (2)_{q} K_{i} K_{j} F_{ij} E_{ji}$
$+ (q^2 - q^{-3}) K_{i} K_{j} F_{ij} E_{ji} + (q^2 - q^{-3}) K_{i} K_{j} F_{ij} E_{ji}$

$F_{ij} \circ E_{ij3} = (2)_{q} (4)_{q} F_{ij} E_{ij3} + (q - q^{-3}) (2)_{q} K_{i} K_{j} F_{ij} E_{ji} - 2 q^{-2} K^2_{i+j} K_{j}$

$F_{ij} \bullet E_{ij3} = F_{ij} \circ E_{ij3} + (q^{-1} - q^{-3}) K_{-i} K_{j} F_{ij} \circ E_{ji} - 2 q^{-2} K^2_{i-j} - q^{-2} K_{-i} K_{j} K_{i+j} K_{j}$

$F_{ijj} = (2)_{q} (4)_{q} F_{ijj} E_{ijj} + (q - q^{-3}) (2)_{q} K_{i} K_{j} F_{ij} E_{ji} - 2 q^{-2} K^2_{i+j} K_{j}$

$F_{ijj} \bullet E_{ijj} = F_{ijj} \circ E_{ijj} + (q^{-1} - q^{-3}) K_{-i} K_{j} F_{ij} \circ E_{ji} - 2 q^{-2} K^2_{i-j} - q^{-2} K_{-i} K_{j} K_{i+j} K_{j}$. 


Example 4.16. Let $\mathfrak{g} = \widehat{sl}_3$, that is, $I = \{1, 2, 3\}$ and $a_{ij} = a_{ji} = -1$ for all $i \neq j$. For $\{i, j, k\} = \{1, 2, 3\}$ let

$$F_{ijk} = ((q - q^{-1})(q^3 - q^{-3}))^{-1} \left( q^\frac{2}{3} ((2)_q F_k F_j F_i - F_j F_k F_i - F_k F_i F_j) + q^{-\frac{2}{3}} ((2)_q F_i F_j F_k - F_i F_k F_j - F_j F_i F_k) \right).$$

Then $F_{ijk} = \delta_{F_k^{(1)} F_j^{(1)} F_i^{(1)}}$. We have

$$F_{ijk} \bullet E_{ijk} = (3)_q F_{ijk} E_{ijk} - q^2 K_{i+j} K_{i+k} - q^{-2} K_{-i} K_{-j} K_{-k}$$

$$F_{ijk} \bullet E_{ikj} = (3)_q F_{ijk} E_{ikj} - q K_{i+j} K_{j+k} - q^{-1} K_{-i} K_{-j} K_{-k}$$

$$F_{ijk} \bullet E_{ikj} = (3)_q F_{ijk} E_{ikj} - q K_{i+j} K_{j+k} - q^{-1} K_{-i} K_{-j} K_{-k}$$

$$F_{ijk} \circ E_{jki} = (3)_q F_{ijk} E_{jki} - q^3 K_{i+j} K_{j+k} F_i E_i + (q^4 - q^2) K_{i+j} K_{j+k} K_{i+k} K_{i+j} K_{j+k}$$

$$F_{ijk} \bullet E_{kji} = F_{ijk} \circ E_{kji} - q^{-3} K_{i} F_{j} K_{j} K_{j} K_{j} K_{j} + (q - q^{-2}) K_{i} K_{j} K_{j} K_{j} K_{j} K_{j}$$

$$F_{ijk} \bullet E_{kji} = F_{ijk} \circ E_{kji} - q^{-3} K_{i} F_{j} K_{j} K_{j} K_{j} + (q - q^{-2}) K_{i} K_{j} K_{j} K_{j} K_{j} K_{j}$$

$$F_{ijk} \circ E_{kij} = F_{ijk} \circ E_{kij} - q^{-3} K_{i} F_{j} K_{j} K_{j} K_{j} K_{j} + (q - q^{-2}) K_{i} K_{j} K_{j} K_{j} K_{j} K_{j}$$

These examples show that we can have $d_{b_-, b_+} \neq 1$ even if all subdiagrams of the Dynkin diagram of rank 2 are of finite type.

4.4. Examples of central elements. We already saw that for $\mathfrak{g} = sl_2$ most elements of the double canonical basis of $U_q(\mathfrak{g})$ are obtained from its part which is contained in the center. Let $\mathfrak{g} = sl_3$. Then $B_{n+}$ consists of elements

$$b_+(a) := q^{\frac{1}{3}(a_1 - a_2)(a_1 a_2 - a_2)} E_1^{a_1} E_2^{a_2} F_1^{a_2} E_2^{a_1}, \quad a = (a_1, a_2, a_{12}) \in \mathbb{Z}_{\geq 0}^3, \min(a_1, a_2) = 0.$$

Set $b_-(a) = b_+(a)^{st}$. Then the elements $C_i := F_{ij} \bullet E_{ji}, \{i, j\} = \{1, 2\}$ described in [4,3] are quasi-central, namely

$$E_i C_i = q C_i E_i, \quad E_i C_i = q^{-1} C_i E_i, \quad F_i C_i = q^{-1} C_i F_i, \quad F_j C_i = q C_i F_j.$$

We have

$$b_-(0, 0, 1, 1) \circ b_+(0, 0, 1, 1) = b_-(0, 0, 1, 1) b_+(0, 0, 1, 1)$$

$$- q K_{1+} b_-(0, 1, 1, 0) b_+(0, 1, 0, 1) - q K_{2+} b_-(1, 0, 0, 1) b_+(1, 0, 1, 0)$$

$$+ (q^2 + q^4) K_{1+} K_{2+} F_{12} E_{21} + F_{21} E_{12} + q^3 K_{1+} K_{2+} (F_{12} E_{12} + F_{21} E_{21})$$

$$- q^3 K_{1+} K_{2+}^2 F_1 E_1 - q^3 K_{1+} K_{2+}^2 F_2 E_2 + q^4 K_{1+} K_{2+}^2.$$

$$b_-(0, 0, 1, 1) \bullet b_+(0, 0, 1, 1) = b_-(0, 0, 1, 1) \circ b_+(0, 0, 1, 1)$$

$$- q^{-1} K_{-2} b_-(0, 1, 0, 1) \circ b_-(0, 1, 1, 0) - q^{-1} K_{-2} b_-(1, 0, 0, 1) \circ b_+(1, 0, 0, 1)$$

$$+ (q^{-2} + q^{-4}) K_{-2} F_{12} E_{21} + F_{21} E_{12} + q^{-3} K_{-2} K_{-2} (F_{12} E_{12} + F_{21} E_{21})$$

$$- q^{-3} K_{-2} K_{-2} F_1 E_1 - q^{-3} K_{-2} K_{-2} F_2 E_2 + q^{-4} K_{-2} K_{-2}^2.$$

This element is central in $U_q(\mathfrak{g})$ and satisfies $b_-(0, 0, 1, 1) \bullet b_+(0, 0, 1, 1) = C_1 C_2 - K_{-2} K_{-2} K_{-2}$. 

ARKADY BERENSTEIN AND JACOB GREENSTEIN
If \( g = \mathfrak{sp}_4 \), we obtain two central elements \( C_i, \ i = 1, 2 \) as \( C_1 = F_{121} \cdot E_{121} \), where

\[
F_{121} \circ E_{121} = F_{121}E_{121} - qK_1K_2F_{12}E_{21} + q^3K_1K_2^2F_{12}E_1 - q^4K_1^2K_2^2
\]

\[
F_{121} \cdot E_{121} = F_{121} \circ E_{121} - q^{-1}K_2 F_{12} \circ E_{12} - q^{-3}K_1K_2 F_{12} \circ E_1 + q^{-4}K_1^2K_2^2
\]

and \( C_2 = F_{2112} \cdot E_{2112} \), where \( E_{2112} = E_2E_{12} - q^2E_{12}^2, F_{2112} = E_{2112}^t \) and

\[
F_{2112} \circ E_{2112} = F_{2112}E_{2112} - q^2K_2F_{2112}E_{12} + (q^5 + q^3)K_1K_2F_{2112}E_{12}
\]

\[
- q^4K_1^2K_2F_{2112}E_2 + q^6K_2^2
\]

\[
F_{2112} \cdot E_{2112} = F_{2112} \circ E_{2112} - q^{-2}K_2F_{2112} \circ E_{2112} + (q^{-5} + q^{-3})K_1K_2F_{2112} \circ E_{211}
\]

\[
- q^{-4}K_1^2K_2F_{2112} \circ E_2 + q^{-6}(K_1K_2)^2 + q^{-4}K_1K_1K_2K_2^2.
\]

Finally, let \( g = \mathfrak{sl}_4 \) and for any \( \{i, j, k\} = \{1, 2, 3\}, \{j, k\} \neq \{1, 3\} \) set

\[
E_{ijk} = (q - q^{-1})(q^{1/2}E_{ijk}E_i - q^{-1/2}E_{ijk}E_j), \quad F_{ijk} = E_{ijk}^t,
\]

where \( E_{132} = E_{312} \). Let \( E_{213} = E_{132}^t, F_{213} = E_{132} \). Then we have

\[
F_{ijk} \circ E_{kji} = F_{ijk}E_{kji} - qK_{i+k}F_{ijk}E_{ji} + q^2K_{i+k}K_{i+k}F_{ijk}E_i - q^3K_{i+k}K_{i+k}E_k
\]

\[
F_{ijk} \cdot E_{kji} = F_{ijk} \circ E_{kji} - q^{-1}K_{-i}F_{kji} \circ E_{kji} + q^{-2}K_{-i}K_{-i}F_{kji} \circ E_{kji} - q^{-3}K_{-i}K_{-i}K_{-i}K_{-i}
\]

Let \( C_{ijk} = F_{ijk} \cdot E_{ijk} \). Then \( E_iC_{ijk}E_i, E_kC_{ijk}E_k = q^{-1}C_{ijk}E_k \) and \( [E_j, C_{ijk}] = 0 \). Applying \( t \) we conclude that \( C_{ijk} \) is quasi-central.

5. Bar-equivariant braid group actions

5.1. Invariant braid group action on Drinfeld double. Denote by \( U'_q(\bar{g}) \) the quotient of \( k[z_i^{\pm 1} : i \in I] \otimes_k U_q(\bar{g}) \) by the ideal generated by \( z_i^2 \otimes 1 - 1 \otimes K_iK_{-i} \). It is easy to see that \( \bar{\cdot} \) extends to an \( \mathbb{Q} \)-linear anti-involution of \( U'_q(\bar{g}) \) by \( \bar{z}_i = z_i \). Then it is immediate that the set

\[
B'_q = \{(\prod_{i \in I} z_i^{a_i})b : b \in B_q, a_i \in \mathbb{Z}\}
\]

is a \( \bar{\cdot} \)-invariant basis of \( U'_q(\bar{g}) \). In the sequel we use the presentation of \( U_q(\bar{g}) \) obtained from \( (1.2) \) and \( (1.3) \) by replacing \( K_{i+1} \) with \( K_i^{\pm 1} \). The following Lemma is immediate.

**Lemma 5.1.** (a) The assignments \( E_i \mapsto E_i, F_i \mapsto F_i, K_{\pm i} \mapsto K_i^{\pm 1}, z_i \mapsto 1 \) extends to a surjective homomorphism of algebras \( \phi : U'_q(\bar{g}) \rightarrow U_q(\bar{g}) \).

(b) The assignments \( E_i \mapsto E_i z_i^{-1}, F_i \mapsto F_i, K_{\pm i} \mapsto K_{\pm i} z_i^{-1} \) extends to an injective homomorphism of algebras \( \iota : U_q(\bar{g}) \rightarrow U'_q(\bar{g}) \) which splits \( \phi \).

Clearly, there exists a unique anti-involution \( \bar{\cdot} \) on \( U'_q(\bar{g}) \) which commutes with \( \iota \) and \( \phi \). It is also easy to see that there exists a unique basis \( B_q \) of \( U_q(\bar{g}) \) such that \( \iota(B_q) = B'_q \cap \iota(U_q(\bar{g})) \). Clearly \( B_q = \phi(B'_q) \) and each element of \( B_q \) is fixed by \( \bar{\cdot} \). From now on we refer to \( B_q \) as the **double canonical basis** of \( U_q(\bar{g}) \).

Given \( \alpha_\pm \in \Gamma \), set \( A_\pm^{1/2} K_{\alpha_-, \alpha_+}^t (x) = \chi^{1/2}((\alpha_-, \alpha_+), \deg x) x \) for \( x \in U_q(\bar{g}) \) homogeneous. Let \( Q \) be the free abelian group generated by the \( \alpha_i, i \in I \) and let \( \hat{Q} = Q \oplus \hat{Q} \) and \( \hat{\Gamma} \) is a submonoid of \( Q \) (respectively, \( \hat{Q} \)). Extend \( \alpha_i^\gamma \in \text{Hom}_Z(\Gamma, \mathbb{Z}) \) to elements of \( \text{Hom}_Z(Q, Z) \) in a natural way. The Weyl group \( W \) of \( g \) acts on \( Q \) and hence on \( \hat{Q} \) via \( s_i(\alpha) = \alpha - \alpha_i^\gamma(\alpha) \alpha_i, i \in I \).

**Lemma 5.2.** In the presentation \( (1.2) \) of \( U'_q(\bar{g}) \), we have \( T_i(z_j) = z_j z_i^{a_{ij}}, i, j \in I \)

\[
T_i(K_{\pm j}) = \begin{cases} K_{\pm j} z_j^{-2}, & i = j \\ K_{\pm j} K_{\pm i}^{-a_{ij}}, & i \neq j \end{cases}
\]
and

\[ T_i(E_j) = \begin{cases} 
K_{-i}\zeta_i^{-2} \circ F_i, & i = j \\
\sum_{r+s = -a_{ij}} (-1)^r q_i^{s+\frac{1}{2}a_{ij}} F_i^{(r)} E_j E_i^{(s)}, & i \neq j 
\end{cases} \]

\[ T_i(F_j) = \begin{cases} 
K_{+i}\zeta_i^{-2} \circ E_i, & i = j \\
\sum_{r+s = -a_{ij}} (-1)^r q_i^{s+\frac{1}{2}a_{ij}} F_i^{(r)} F_j E_i^{(s)}, & i \neq j 
\end{cases} \]

Moreover, the \( T_i \) satisfy the braid relations, commute with \( \tau \) and satisfy \( T_i* = \tau T_i^{-1} \), \( T_i \circ \tau = \tau \circ T_i^{-1} \).

Proof. Recall that our presentation of \( U_q(\mathfrak{g}) \) is obtained from the standard one by rescaling \( E_i \mapsto (q_i^{-1} - q)^{-1} E_i, \ F_i \mapsto (q_i - q_i^{-1})^{-1} F_i \) for all \( i \in I \). In this presentation the symmetries \( T''_{i,1}, T''_{i,-1} \) of \( U_q(\mathfrak{g}) \) defined in [14 §37.1.3] are given by \( T''_{i,1}(K_j) = K_j K_{-a_{ij}} = T''_{i,-1}(K_j) \),

\[ T''_{i,-1}(E_j) = \begin{cases} 
F_i K_i^{-1}, & i = j \\
\sum_{r+s = -a_{ij}} (-1)^r q_i^{s} E_i^{(r)} E_j E_i^{(s)}, & i \neq j 
\end{cases} \]

\[ T''_{i,-1}(F_j) = \begin{cases} 
K_i E_i, & i = j \\
\sum_{r+s = -a_{ij}} (-q_i)^{t} F_i^{(r)} F_j F_i^{(s)}, & i \neq j 
\end{cases} \]

and

\[ T'_{i,1}(E_j) = \begin{cases} 
K_i F_i, & i = j \\
\sum_{r+s = -a_{ij}} (-1)^s q_i^{s} E_i^{(r)} E_j E_i^{(s)}, & i \neq j 
\end{cases} \]

\[ T'_{i,1}(F_j) = \begin{cases} 
E_i K_i^{-1}, & i = j \\
\sum_{r+s = -a_{ij}} (-q_i)^{-t} F_i^{(s)} F_j F_i^{(r)}, & i \neq j 
\end{cases} \]

By [14 Proposition 37.1.2] \( T'_{i,1}, T''_{i,-1} \) are automorphisms of \( U_q(\mathfrak{g}) \) while by [14 Theorem 39.4.3] they satisfy the braid relations of the braid group of \( \mathfrak{g} \). Also, \( T''_{i,-1} = (T'_{i,1})^{-1} \). It is easy to see that \( T''_{i,-1}(E_j), T''_{i,-1}(E_j) \) and \( T'_{i,1}(E_j), T'_{i,1}(F_j), i \neq j \), are given on \( U_q(\mathfrak{g}) \) by the same formula as on \( U_q(\mathfrak{g}) \). Furthermore we have

\[ z_i T''_{i,-1}(E_i) = T''_{i,-1}(E_i z_i^{-1}) = F_i K_{-i} z_i^{-1} \]

\[ z_i T'_{i,1}(E_i) = T'_{i,1}(E_i z_i^{-1}) = K_{+i} F_i z_i^{-1} \]

whence \( T''_{i,-1}(E_i) = F_i K_{-i} z_i^{-2} \) and \( T'_{i,1}(E_i) = K_{+i} z_i^{-2} F_i \). Similarly, \( T''_{i,-1}(F_i) = K_{+i} z_i^{-1} E_i z_i^{-1} = K_{+i} z_i^{-2} E_i \) and \( T'_{i,1} = F_i K_{-i} z_i^{-2} \). Finally,

\[ z_i T''_{i,-1}(K_{\pm i}) = T''_{i,-1}(K_{\pm i} z_i^{-1}) = K_{\mp i} z_i^{-1} = z_i T'_{i,1}(K_{\mp i}) \]

whence \( T''_{i,-1}(K_{\pm i}) = K_{\mp i} z_i^{-2} = T'_{i,1}(K_{\mp i}) \), while \( z_j^{-1} z_i a_{ij} T''_{i,-1}(K_{\pm j}) = K_{\pm j} z_j^{-1} (K_{\pm i} z_i^{-1})^{-a_{ij}} \).

Define \( T_i(x) = T''_{i,-1}(\text{Ad} \hat{z}^2_{i} K_{i}(x)) = \text{Ad} \hat{z}^2_{i} K_{i}(T''_{i,-1}(x)), x \in U_q(\mathfrak{g}) \). Then we have \( T_i^{-1}(x) = T'_{i,1}(\text{Ad} \hat{z}^2_{i} K_{i}(x)) \). It is easy to see that \( T_i \) is given on generators by the formulae from Lemma [5.2]

For example, \( T_i(E_j) = q_i K_{+i} z_i^{-2} F_i = K_{+i} z_i^{-2} \circ F_i \). Thus, in particular, \( T_i \) is an automorphism of \( U_q(\mathfrak{g}) \). Clearly, \( \overline{T_i(E_j)} = T_i(E_j) \), while for \( j \neq i \)

\[ \overline{T_i(E_j)} = \sum_{r+s = -a_{ij}} (-1)^{s} q_i^{-s+\frac{1}{2}a_{ij}} E_i^{(s)} E_j E_i^{(r)} \sum_{r+s = -a_{ij}} (-1)^{r} q_i^{s+\frac{1}{2}a_{ij}} E_i^{(r)} E_j E_i^{(s)} = T_i(E_j) \]
where we used that $E^{(k)} = (-1)^k E^{(k)}$. The remaining identities are checked similarly. The identities involving $*$ and $\ell$ can be checked using the explicit formulae for $T_i^{-1} = T_i^{\ell} \circ \text{Ad} K_i$.

It remains to prove that the $T_i$ satisfy the braid relations. For, let $w$ be an element of the Weyl group of $\mathfrak{g}$ and let $w = s_{i_1} \cdots s_{i_r}$ be its reduced decomposition. It is sufficient to prove that $T_{i_1} \circ \cdots \circ T_{i_r}$ depends only on $w$ and not on the reduced decomposition. This holds for Lusztig symmetries $T_{i_1}^{T_r} T_{i_r}^{-1}$ by [14, §39.4.4], whence for each $w \in W$ one has a well-defined automorphism $T''_{w,-1}$ of $U_q(\mathfrak{g})$ satisfying $T''_{w,-1} = T_{i_1,-1}^{T_r,-1} T_{i_r,-1}$. We have

$$T_{i_1} \circ \cdots \circ T_{i_r}(x) = \text{Ad} \frac{1}{2} K_{i_1}^{s_{i_1} \cdots s_{i_r} (\alpha_{i_r})_0} \circ \cdots \circ \text{Ad} \frac{1}{2} K_{i_r}^{s_{i_1} \cdots s_{i_r} (\alpha_{i_r})_0} T_{i_r,-1} = \text{Ad} \frac{1}{2} K_{s_{i_1} \cdots s_{i_r} (\alpha_{i_r})_0} T_{w,-1}.$$  

It is well-known that $\sum_{r=1}^{\ell} s_{i_1} \cdots s_{i_r} (\alpha_{i_r}) = \sum_{\beta \in R_{\pm} \cap w(-R_{\pm})} \beta$, where $R_{\pm} \subset Q$ denotes the set of positive roots of $\mathfrak{g}$, depends only on $w$ and not on its reduced decomposition. Therefore, the right hand side depends only on $w$. \hfill $\square$

**Proof of Theorem 1.13**: Note that $\hat{U}_q(\mathfrak{g})$ embeds into $U'_q(\mathfrak{g})$ via $E_i \mapsto E_i$, $F_i \mapsto F_i$, $K_{\pm i} \mapsto K_{\pm i}$, $K_{\pm 1} \mapsto K_{\pm 1} z_i^{-2}$ for all $i \in I$. All assertions of Theorem 1.13 are then immediate consequences of Lemma 5.2. \hfill $\square$

In particular, for each $w \in W$, we have a unique automorphism $T_w$ of $U_q(\mathfrak{g})$ such that $T_{s_i} = T_i$ and $T_w T_w' = T_{w,w'}$ for any reduced decomposition $w = w''w'$, $w', w'' \in W$. It follows from Lemma 5.2 that for all $x \in U_q(\mathfrak{g})$

$$T_w(x) = T_w(x^*), \quad T_w(x^*) = (T_{w_1}^{-1}(x))^*, \quad T_w(x^t) = (T_{w_1}^{-1}(x))^t. \quad (5.1)$$

Furthermore, we have for $x \in U_q(\mathfrak{g})$ homogeneous

$$T_w(x) = \chi^{1/2}((\langle w \rangle, 0, w \text{deg}_{\hat{\Gamma}} x) T_{w_1}^{-1}(x) = \chi^{1/2}((0, \langle w^{-1} \rangle, \text{deg}_{\hat{\Gamma}} x) T_{w_1}^{-1}(x), \quad (5.2)$$

where $\langle w \rangle = \sum_{\beta \in R_{\pm} \cap w(-R_{\pm})} \beta$ and the action of $W$ on $\Gamma$ is extended to $\hat{\Gamma}$ diagonally.

5.2. Elements $T_w$, quantum Schubert cells and their bases. Let $\mathfrak{g}$ be any Kac-Moody Lie algebra. Given $w \in W$ define

$$U^+_q(w) = T_w(KU^-_q) \cap U^+_q.$$  

Let $i = (i_1, \ldots, i_m)$, $m = \ell(w)$, be such that $w = s_{i_1} \cdots s_{i_m}$ is a reduced decomposition. Then for $a = (a_1, \ldots, a_m) \in \mathbb{Z}_{\geq 0}^m$ define

$$E_i^a := E_{i_1}^{a_1} T_{s_{i_1}} (E_{i_2}^{a_2}) \cdots T_{s_{i_1} \cdots s_{i_m}} (E_{i_m}^{a_m}). \quad (5.3)$$

It follows from [14] and (5.1) that for all $w \in W$, $i \in I$ such that $\ell(ws_i) = \ell(w) + 1$, we have

$$T_w(E_i), \quad T_{w_1}^{-1}(E_i) \in U^+_q, \quad T_{w_1}^{-1}(E_i) \in U^-_q.$$  

Thus, the $E_i^a \in U^+_q$. It follows from [14] Proposition 40.2.1] that the $E_i^a$ are linearly independent. Let $U^+_q(w, 1)$ be the $\mathbb{k}$-subspace of $U^+_q$ spanned by the $E_i^a$, $a \in \mathbb{Z}_{\geq 0}^m$.

**Proposition 5.3**: If $\mathfrak{g}$ is semisimple then $U^+_q(w) = U^+_q(w, 1)$.

**Proof**: We need the following

**Lemma 5.4**: For any Kac-Moody Lie algebra $\mathfrak{g}$, $U^+_q(w, 1) \subset U^+_q(w)$.

**Proof**: Since $U^+_q(w, 1)$ is contained in the subalgebra of $U^+_q$ generated by the $T_{w_1}(E_{i_r})$, $1 \leq r \leq m$, where $w = s_{i_1} \cdots s_{i_{r-1}}$, it suffices to prove that $T_{w_1}(T_{w_1}(E_{i_r})) \in KU^-_q$, $1 \leq r \leq m$. Indeed, write $w = u_r s_{i_r} v_r$ where $v_r = s_{i_{r+1}} \cdots s_m$. Since $\ell(w) = \ell(u_r) + \ell(v_r) + 1$, we have by (5.1)

$$T_{w_1}(T_{w_1}(E_{i_r})) = T_{v_r}^{-1}(T_{v_r}(E_{i_r})) = T_{v_r}^{-1}(K_{v_r}^{-1} F_{v_r}) \in KU^-_q. \quad \square$$
To prove the inclusion $U^+_q(w) \subset U^+_q(w, 1)$ for $g$ semisimple, let $w_0$ be the longest element in $W$ and set $w' = w^{-1}w_0$. Since $\ell(w) + \ell(w') = \ell(w_0)$, we can choose a reduced word $i$ for $w_0$ which is the concatenation of reduced words $i$ and $i'$ for $w$ and $w'$ respectively. Then by [16, Corollary 40.2.2], monomials $E^a_{1i}; a \in \mathbb{Z}_0 \in \mathbb{Z}_0^T$ form a basis of $U^+_q$. Observe that $E^a_{1i} = E^a_{1i}T_w(E^a_{1i'})$ is in $U^+_q(w)T_w(E^a_{1i'})$. Let $u \in U^+_q(w)$. Then we can write $u = \sum a' \in \mathbb{Z}_0^T c_a' T_w(E^a_{1i'})$, where $c_a' \in U^+_q(w)$. Then

$$T^{-1}(u) = \sum a' \in \mathbb{Z}_0^T c_a' T^{-1}(E^a_{1i'}).$$

By definition of $U^+_q(w)$, $T^{-1}(c_a') \in KU^{-}$. Note that the triangular decomposition $U_q(\mathfrak{g}) \cong K \otimes U^- \otimes U^+_q$ implies that the $E^a_{1i}$ are linearly independent over $KU^{-}$. Therefore, $T^{-1}(u) \in KU^{-}$ if and only if $c_a' = 0$ unless $a' = 0$. $\square$

Let $U^+_q(w)' = T(w)(U^+_q) \cap U^+_q$. The proof of Proposition 5.3 suggests the following.

**Conjecture 5.5.** For any $g$ we have a unique (tensor) factorization $U^+_q = U^+_q(w) \cdot U^+_q(w)'$. In particular, $U^+_q(w, 1) = U^+_q(w)$.

5.3. **Proof of Theorem 3.11.** We will often need the following identity, which is an immediate consequence of [16, Lemma 1.4.4]

$$\langle F^1_i, F^1_j \rangle = q_i^{Z}(r) q_j!, \quad r \in \mathbb{Z}_{\geq 0}, i \in I. \quad (5.5)$$

Let $w \in W$ and let $w = s_{i_1} \cdots s_{i_m}$ be its reduced decomposition. Denote $i = (i_1, \ldots, i_m) \in I^m$ and set $w_r = s_{i_1} \cdots s_{i_r} \leq 0 \leq r \leq m$. Given $a \in \mathbb{Z}_{\geq 0}^I$, let $\mu_i(a) := q^{-\frac{1}{2}} \sum a_{w_r}^{-1})^{-1} \alpha_{w_r}$ (cf. (5.2)). Let $U^+_w(w, 1) = U^+_w(w, 1) \cap U^+_Z$. We need the following Lemma.

**Lemma 5.6.** The elements $\{\mu_i(a)F^1_i : a \in \mathbb{Z}_{\geq 0}^T\}$ (respectively, $\{\mu_i(a)F^1_i : a \in \mathbb{Z}_{\geq 0}^T\}$) form a $\mathbb{Z}[q, q^{-1}]$-linearly independent over $\mathbb{Z}_{\geq 0}$ for $U^+_Z(w, 1)$ (respectively, $U^+_Z(w, 1)$). Moreover,

$$\langle \mu_i(a)F^1_i, \mu_i(a')F^1_i \rangle \in \mathbb{Z}[q, q^{-1}]$$

and equals zero unless $a = a'$.

**Proof.** Set

$$\tilde{E}^a_{1i} = E^{(a)}_{1i}T_{w_{1i-1}}(\tilde{E}^{a_2}_{1i}) \cdots T_{w_{m-1i-1}}(\tilde{E}^{a_m}_{1i}).$$

Then by (5.2), $\tilde{E}^a_{1i} = \mu_i(a)^{-1}(\prod_{r=1}^m(a_{w_r}))^{-1}E^a_{1i}$. We also set $\tilde{E}^a_{1i} = \tilde{F}^a_{1i}$. It follows from [16, Proposition 41.1.4] that the monomials $\{\tilde{F}^a_{1i} : a \in \mathbb{Z}_{\geq 0}^T\}$ (respectively, $\tilde{E}^a_{1i}$) form a $\mathbb{Z}[q, q^{-1}]$-basis of $\mathbb{Z}U^+_Z(w, 1)$ (respectively, $\mathbb{Z}U^-_Z(w, 1)$), where $\mathbb{Z}U^+_Z(w, 1) = \mathbb{Z}U^+_Z \cap \mathbb{Z}U^-_Z(w, 1)$. Moreover, it follows from [16, Proposition 38.2.3] and (5.5) that

$$\langle \tilde{F}^a_{1i}, \tilde{F}^a_{1i} \rangle = \delta_{a, a'} \sum_{r=1}^m a_r \eta^{n(w_r-1)}(\alpha_{w_r}) \prod_{r=1}^m(q_{\alpha_{w_r}})^{N} = \delta_{a, a'} \sum_{r=1}^m a_r \eta^{n(w_r-1)}(\alpha_{w_r}) \prod_{r=1}^m(q_{\alpha_{w_r}})^{N}.$$
Proposition 3.22(b) that for any element $b$ such that $\partial^x$, where for any $h$ hence by Proposition 3.22(a) and Corollary 3.7 Braid group action on elements of $B$.

For all $\lambda \in \mathcal{L}$.

Proof of Theorem 3.11. Suppose that $g$ is semisimple and that $w = w_0$ is the longest element in $W$. Then $U^+_{\lambda}(w_0) = U^+_{\lambda}$ and by Lemma 5.6, $U_{\lambda}$ admits a pair of bases $B_\pm$ such that $\langle B_-, B_+ \rangle \subseteq \mathbb{Z}[q, q^{-1}]$. Thus, $\langle U_{\lambda}^-, U_{\lambda}^+ \rangle = \mathbb{Z}[q, q^{-1}]$. The same argument as in the proof of Proposition 3.9 shows that $\langle B_{n-}, B_{n+} \rangle \subseteq \mathbb{Z}[q, q^{-1}]$.

5.4. Braid group action on elements of $B_{n+}$. Retain the notation of 3.5. It follows from Proposition 3.22(b) that for any element $b_+ \in B_{n+}$, and $r \in \mathbb{Z}_{\geq 0}$ there exists a unique $b'_+ \in B_{n+}$ such that $\tilde{\partial}_i^{(\text{top})}(b_+) = \tilde{\partial}_i^{(\text{top})}(b'_+)$ and $\ell_i(b'_+) = \ell_i(b_+) + r$. We denote this element by $\tilde{\partial}_i^{-r}(b_+)$. Observe that

$$\tilde{\partial}_i^{-r}(b_+) = \tilde{\partial}_i^{-r-\ell_i(b_+)} \partial_i^{(\text{top})}(b_+).$$

(5.6)

Proposition 5.7. For all $b_+ \in B_{n+} \cap \ker \partial_i$, $r \in \mathbb{Z}_{\geq 0}$, $i \in I$ we have

$$E_i^r \circ b_+ - \tilde{\partial}_i^{-r}(b_+) \in \sum_{b'_+ \in B_{n+} : \ell_i(b'_+) < r} q\mathbb{Z}[q]b'_+,$$

(5.7)

where for any $x \in U^+_q$ and $r \in \mathbb{Z}_{\geq 0}$ homogeneous we denote

$$E_i^r \circ x := q_i^{-\frac{1}{2}r\alpha_i^\vee(x)} E_i^r x,$$

$$\alpha_i^\vee(x) := \alpha_i^\vee(\deg x).$$

Proof. First, note that for $b_+ \in \ker \partial_i$, $\ell_i(E_i^r \circ b_+) = r = \ell_i(\tilde{\partial}_i^{-r}(b_+))$ and by Corollary 3.19

$$\partial_i^{(r)}(E_i^r \circ b_+ - \tilde{\partial}_i^{-r}(b_+)) = \partial_i^{(\text{top})}(b_+) - \partial_i^{(\text{top})}(b_+) = 0,$$

hence by Proposition 3.22(a) and Corollary 3.7

$$E_i^r \circ b_+ - \tilde{\partial}_i^{-r}(b_+) \in \sum_{b'_+ \in B_{n+} : \ell_i(b'_+) < r} \mathbb{Z}[q, q^{-1}]b'_+.$$

(5.8)

Given $\lambda = (\lambda_i)_{i \in I} \in \mathcal{L}$ and $i \in I$, define $k$-linear operators on $U^+_q$ by

$$F_{i, \lambda}(x) = \frac{q_i^{-\lambda_i + \frac{1}{2} \alpha_{i}^\vee(x)} E_i x - q_i^{-\frac{1}{2} \alpha_{i}^\vee(x) + \lambda_i} x E_i}{q_i - q_i^{-1}}, \quad K_{i, \lambda}(x) = q_i^{\lambda_i - \alpha_{i}^\vee(x)} x.$$

The following result is well-known (see e.g. [2, Section 3]).

Lemma 5.8. For any $\lambda \in \mathcal{L}$, the assignments $E_i \mapsto \partial_i$, $F_i \mapsto F_{i, \lambda}$, $K_i \mapsto K_{i, \lambda}$ define a structure of a $U_q(\mathfrak{g})$-module on $U^+_q$. Moreover, the submodule $\mathcal{V}_\lambda$ of $U^+_q$ generated by $1$ is simple and if $\lambda \in \mathcal{L}_{\geq 0}$ then $\mathcal{V}_\lambda = \{ x \in U^+_q : \ell_i(x^+) \leq \lambda_i \}$ and is integrable.

Remark 5.9. Here we use the “standard” generators of $U_q(\mathfrak{g})$.

We need the following technical fact which is easy to check by induction.

Lemma 5.10. For all $\lambda \in \mathcal{L}$, $r \in \mathbb{Z}_{\geq 0}$ and $x \in U^+_q$ homogeneous

$$q_i^{\frac{1}{2} \alpha_{i}^\vee(x) - 1/2} F_{i, \lambda}(x) = (1 - q_i^2)^{-r} q_i^{-\frac{1}{2} \alpha_{i}^\vee(x)} \sum_{k=0}^{r} (-1)^k q_i^{2k(2\lambda_i - \alpha_{i}^\vee(x) - 2r + 2) + k(k-1)} \frac{r!}{k!} q_i^k x E_i^k.$$

This immediately implies that

$$E_i^r \circ x = q_i^{\frac{1}{2} \alpha_{i}^\vee(x) - 1/2} (1 - q_i^2)^{-r} F_{i, \lambda}(x)$$

$$+ q_i^{-\frac{1}{2} \alpha_{i}^\vee(x)} \sum_{k=1}^{r} (-1)^{k+1} q_i^{k(2\lambda_i - \alpha_{i}^\vee(x) + k - 2r + 1)} \frac{r!}{k!} q_i^k x E_i^k.$$  

(5.9)
Let \( b_+ \in \mathbf{B}_{n+} \cap \ker \partial_i \). It follows by an obvious induction from [13 Proposition 5.3.1] that

\[
q_i^{r \varphi_i(b_+)-\binom{r+1}{2}}(1-q_i^2)^r F_i^{\ell_i(b_+)}(b_+) = \sum_{t=0}^{r-1} \left( 1-q_i^{2(\varphi_i(b_+)-t)}) \tilde{\partial}_i^{-r}(b_+) + \sum_{b'_+ \in \mathbf{B}_{n+} : \ell_i(b'_+) < r} qQ[q]b'_+ ,
\]

where \( \varphi_i(b_+) = \lambda_i - \alpha_i^\vee(b_+) \). Combining this identity with (5.9) we obtain

\[
E_i^r \circ b_+ = \sum_{k=1}^r (-1)^{k+1} q_i^{k(2\lambda_i - \alpha_i^\vee(b_+)+k-2r+1)} E_i^{-k} b_+ E_i^k + \sum_{b'_+ \in \mathbf{B}_{n+}} qQ[q]b'_+.
\]

By Corollary 3.7 we have for all \( 1 \leq k \leq r \)

\[
q_i^{-\frac{1}{2} \lambda_i^\vee(b_+)+k} \sum_{b'_+ \in \mathbf{B}_{n+}} q_{b'_+} C_{b'_+,r,k}^r E_i^{-k} b_+ E_i^k = q_i^{2\lambda_i} \sum_{b'_+ \in \mathbf{B}_{n+}} qZ_{q}[\mathbf{B}_{n+}] \text{ and } C_{b'_+,r,k}^r \in \mathbb{Z}[q,q^{-1}].
\]

Since only finitely many terms in this sum are non-zero, there exists \( \lambda_i \in \mathbb{Z}_{\geq 0}, \lambda_i \geq \alpha_i^\vee(b_+) + r \) such that \( q_i^{2\lambda_i} C_{b'_+,r,k}^r \in q\mathbb{Z}[q] \) for all \( b'_+ \in \mathbf{B}_{n+}, 1 \leq k \leq r \). Therefore, it follows from (5.10) that

\[
E_i^{r} \circ b_+ - \tilde{\partial}_i^{-r}(b_+) \in \sum_{b'_+ \in \mathbf{B}_{n+}} q\mathbb{Z}[q]b'_+.
\]

It remains to apply (5.8).

\[ \square \]

**Corollary 5.11.** For any \( b_+ \in \mathbf{B}_{n+} \) we have

\[
\begin{align*}
b_+ - E_i^{\ell_i(b_+)} \circ \partial_i^{(top)}(b_+) & \in \sum_{b'_+ \in \mathbf{B}_{n+} \cap \ker \partial_i, 0 \leq r < \ell_i(b_+)} q\mathbb{Z}[q]E_i^r \circ b'_+. \\
\end{align*}
\]

**Proof.** It follows from the Theorem that the elements \( \{ E_i^r \circ b_+ : b_+ \in \mathbf{B}_{n+} \cap \ker \partial_i, r \geq 0 \} \) form a \( \mathbb{Z}[q] \)-basis of the lattice \( \mathbb{Z}[q] \mathbf{B}_{n+} \) and the transfer matrix is unitriangular with off-diagonal elements in \( q\mathbb{Z}[q] \). Then the inverse matrix has the same property.

\[ \square \]

We can now prove the following

**Theorem 5.12.** Let \( b_+ \in \mathbf{B}_{n+} \). Then \( T_i(b_+) = K_i^{-\ell_i(b_+)} \circ F_i^{\ell_i(b_+)} \circ T_i((\partial_i^{(top)}(b_+)) \). In particular, all elements of \( \mathbf{B}_{n+} \) are tame.

**Proof.** We need the following crucial corollary of [15 Theorem 1.2].

**Proposition 5.13.** \( T_i \) induces a bijection \( \mathbf{B}_{n+} \cap \ker \partial_i \to \mathbf{B}_{n+} \cap \ker \partial_i^{op} \).

**Proof.** It follows from [13 Lemma 38.1.3 and Proposition 38.1.6] that \( T_i^{\nu} \) induces an isomorphism of algebras \( \ker \partial_i = \ker \partial_{F_i}^{op} \to \ker \partial_{F_i}^{op} = \ker \partial_{F_i} \). Moreover [15 Theorem 1.2] implies that if \( b \in \mathbf{B}_{can} \cap \ker \partial_i^{op} \) then \( T_i^{\nu}(b^{rt}) \in (\mathbf{B}_{can})^{rt} \cap \ker \partial_i^{op} \). Now, let \( b_+ = \delta_b^{rt} \in \mathbf{B}_{n+} \cap \ker \partial_i \) and \( b' \in \mathbf{B}_{can} \cap \ker \partial_{E_i}^{op} \). Then it follows from (5.2) and [14 Proposition 38.2.1] that \( \delta_{b,b'} = \langle \delta_b, b' \rangle = q^{\nu}(q_i^r T_i(\delta^{rt}_b)^{rt}, T_i^{\nu}(b^{rt})^{rt}) = q^{\nu}(q_i^r T_i(\delta^{rt}_b)^{rt}, b'^{rt}) \), where \( b'^{rt} \in \mathbf{B}_{can} \cap \ker \partial_{E_i} \) and \( \nu \in \mathbb{Z} \) depends only on the degree of \( b \). This implies that \( T_i(\delta^{rt}_b) = q^{-\frac{1}{2} \nu} \delta^{rt}_b \). But since \( T_i \) commutes with \( ^\vee \), it follows that \( \nu = 0 \).

\[ \square \]
We have, for any $r > 0$, $b'_+ \in B_{n_+} \cap \ker \partial_i$

$$T_i(E_i^r \circ b'_+) = q_i^{-\frac{1}{2}} r\alpha_i^\vee (\deg b'_+) \cdot T_i(E_i^r T_i(b'_+)) = q_i^{-\frac{1}{2}} r\alpha_i^\vee (\deg b_+) \cdot (K_{i+}^{-r} \circ T_i(b'_+))$$

$$= q_i^{-\frac{1}{2}} r\alpha_i^\vee (\deg b_+) - \frac{1}{2} r\alpha_i^\vee (s(\deg b_+)) \cdot K_{i+}^{-r} \circ (F_i^r T_i(b_+)) = K_{i+}^{-r} \circ (F_i^r T_i(b_+)).$$

Then applying $T_i$ to (5.11) yields

$$T_i(b_+) = K_{i+}^{-\ell_i(b_+)} \circ F_i^{\ell_i(b_+)}(T_i(\partial_i^{(\text{top})(b_+)})) + \sum_{0 \leq r < \ell_i(b_+), b'_+ \in \ker \partial_i \cap B_{n_+}} D_{b'_+.i}^{b_+} K_{i+}^{-r} \circ (F_i^{r T_i}(b'_+))$$

$$= K_{i+}^{-\ell_i(b_+)} \circ (F_i^{\ell_i(b_+)}(T_i(\partial_i^{(\text{top})(b_+)}))) + \sum_{0 < r < \ell_i(b_+), b'_+ \in \ker \partial_i \cap B_{n_+}} D_{b'_+.i}^{b_+} K_{i+}^{-r} \circ (F_i^{\ell_i(b_+)-r T_i}(b'_+)).$$

Since $\partial_i$ commutes with the $T_i$, this element is $\partial_i$-invariant. Since all $D_{b'_+.i}^{b_+} \in qZ[q]$ and $T_i(b'_+) \in B_{n_+} \cap \ker \partial_i$, $T_i(b'_+) = K_{i+}^{-\ell_i(b_+)} \circ F_i^{\ell_i(b_+)} \circ T_i(\partial_i^{(\text{top})(b_+)}))$ by Theorem 1.3. But since for $b_+ \in \ker \partial_i$, $T_i(x) \in U_q(\mathfrak{g})$ and in $H_q^+(\mathfrak{g})$ coincides that $F_i^{\ell_i(b_+)} \circ T_i(\partial_i^{(\text{top})(b_+)})) = F_i^{\ell_i(b_+)} \circ T_i(\partial_i^{(\text{top})(b_+)}))$ by Theorem 1.5.

**Example 5.14.** We now use the above Theorem to compute $F_i^r \bullet b_+$, $r \geq 0$, $b_+ \in B_{n_+} \cap \ker \partial_i$ for $\mathfrak{g} = sl_3$. Retain the notation of [4.7]. Then

$$B_{n_+} \cap \ker \partial_i = \{b_+(0, a_2, a_{12}, 0) : a_2, a_{12} \in \mathbb{Z}_{\geq 0}\}.$$ Since $T_i(E_2) = E_{21}$, $T_i^{-1}(E_{12}) = E_2$ we have $T_i^{-1}(b_+(0, a_2, a_{12}, 0)) = b_+(0, a_{12}, 0, a_2)$. Then $F_i^r \bullet b_+(0, a_2, a_{12}, 0) = K_{i+}^{-1} \circ T_i(\partial_i^{-r}(b_+(0, a_2, a_{12}, 0, a_2))).$ Since

$$\ell_i(0, a_2, a_{12}, 0, a_2) = a_{12}, \quad \ell_i(0, a_2, a_{12}, 0, a_2) = a_2 + a_{12},$$

we conclude that

$$\partial_i^{-r}(b_+(0, a_{12}, 0, a_2)) = \begin{cases} b_+(0, a_{12} - r, r, a_2), & 0 \leq r \leq a_{12} \\ b_+(r - a_{12}, 0, a_2), & r > a_{12} \end{cases}.$$ Since

$$b_+(a_2', a_{12}', a_{12}, a_{21}) = \sum_{t=0}^{a_{21}} (-1)^t q^t(a_2' + a_{21} + 1) \cdot \left[ \begin{array}{c} a_{12} \\ t \end{array} \right] q^2 E_1^{a_2' + a_{21} - t} \circ b_+(0, a_2' + a_{21} - t, 0, a_{12} + t),$$

we obtain

$$F_i^r \bullet b_+ (0, a_2, a_{12}, 0) = \sum_{t=0}^{\min(r, a_{12})} (-1)^t q^t(a_{12} - r + 1) \cdot \left[ \begin{array}{c} \min(r, a_{12}) \\ t \end{array} \right] K_{i+}^t \circ F_i^{r-t} b_+(0, a_2 + t, a_{12} - t, 0).$$

Then it is easy to see that $T_2(F_i^r \bullet b_+(0, a_2, a_{12}, 0)) = K_{i+}^{-a_2} \circ b_-(0, a_2, 0, r) \bullet E_1^{a_{12}} = (K_{i+}^{-a_2} \circ F_i^{a_{12}} \bullet b_+(0, a_2, r, 0))^t.$

5.5. Braid group action for \( U_q(\mathfrak{sl}_2) \). Retain the notation from §4.1

**Lemma 5.15.** We have \( T(C^{(r)}) = (K_+ K_-)^{-r} C^{(r)} \), \( r \geq 0 \).

**Proof.** Note that \( T(K_+ K_-) = (K_+ K_-)^{-1} \), \( T(C) = (K_- K_+)^{-1} C \). Then by induction hypothesis we have

\[
T(C^{(r+1)}) = T(C) T(C^{(r)}) - (K_+ K_-)^{-1} T(C^{(r-1)}) = (K_- K_+)^{-r-1} (C^{(r)} - K_- K_+ C^{(r-1)}) = (K_- K_+)^{-r-1} C^{(r+1)}.
\]

Since \( T(E^m) = K_+^{-m} \circ F^m \) and \( T(F^n) = K_-^{-n} \circ E^n \) we obtain

\[
T(K_+^{-a} K_-^a \circ F^m - \bullet E^{m+n}) = K_+^{-a} - m K_+^{-a} \circ F^n C^{(m)} = K_+^{-a} - m K_+^{-a} \circ F^n \circ m \bullet E^m.
\]

and similarly \( T(K_+^{-a} K_-^a \circ F^m + \bullet E^{m+n}) = K_+^{-a} - m K_+^{-a} \circ F^n \circ m \bullet E^m \). Thus,

\[
T(K_+^{-a} K_-^a \circ F^m - \bullet E^{m+n}) = K_+^{-a} - m K_+^{-a} \circ F^n \circ m \bullet E^m + a_\pm \in \mathbb{Z}, m_\pm \in \mathbb{Z}_0.
\]

Thus, if we parametrize an element of our basis by \( (a_+, a_-, m_+, m_-) \) with \( a_\pm \in \mathbb{Z}, m_\pm \in \mathbb{Z}_0 \), we obtain an action of \( \mathbb{Z} \) on \( \mathbb{Z}^2 \times \mathbb{Z}_0^2 \) given by \( (a_+, a_-, m_+, m_-) \mapsto (-a_+ + m_+, -a_--m_-, m_+, m_-) \).

**5.6. Wild elements of a double canonical basis.** Assume that \( a_{ij} = a_{ji} = -a \), \( d_i = d_j = 1 \), \( a \geq 2 \) and consider elements \( F_{ij} \bullet E_{ji} \) computed in §4.1 Then for \( a = 2 \) we have

\[
T_i(F_{ij} \bullet E_{ij}) = K_+^{-1} F_{ij} \bullet E_{ij} + K_-^{-1} F_{ji} \bullet E_{ji},
\]

while for \( a = 3 
T_i(F_{ij} \bullet E_{ij}) = (3q) K_+^{-1} F_{ij} \bullet E_{ij} + (2q) K_-^{-1} F_{ij} \bullet E_{ij} + (2q) K_+ K_+ K_j F_i \bullet E_i + K_+ K_j K_i F_j \bullet E_{ij} + K_-^{-1} K_3^{-1} K_3 K_j.
\]

**Appendix A. Drinfeld and Heisenberg doubles**

**A.1. Nichols algebras.** Let \( k \) be a field, let \( V \) be a \( k \)-vector space and let \( \Psi = \Psi_V : V \otimes V \to V \otimes V \) be a braiding, that is, \( \Psi \) is invertible and \( \Psi_{1,2} \Psi_{2,3} \Psi_{1,2} = \Psi_{2,3} \Psi_{1,2} \Psi_{2,3} \) as endomorphisms of \( V \otimes^3 \), where

\[
\Psi_{i,i+1} id_V \otimes (\Psi \otimes id_V)^{(n-i-1)} \in \operatorname{End}(V \otimes^n), \quad 1 \leq i < n.
\]
Define \( [n]\Psi, [n]\Psi! \in \operatorname{End}(V \otimes^n), n \in \mathbb{Z}_0 \), by

\[
[n]\Psi = \Psi_{i,i+1} \Psi_{n-1,n} + \Psi_{n-1,n} \Psi_{n-2,n-1} + \cdots + \Psi_{1,1},
\]

\[
[n]\Psi! = ((1/\Psi \otimes \Psi^{(n-1)}) \circ (2/\Psi \otimes \Psi^{(n-2)}) \circ \cdots \circ [n]\Psi \Psi!.
\]

In particular, \( [0]\Psi! = 1 \) and \( [1]\Psi! = id_V \). Furthermore, given \( \sigma \) in the symmetric group \( S_n \), let \( \Psi_\sigma = \Psi_{i_1,i_1+1} \cdots \Psi_{i_r,i_r+1} \) where \( \sigma = (i_1,i_1+1) \cdots (i_r,i_r+1) \) is a reduced expression. A standard argument shows that \( \Psi_\sigma \) depends only on \( \sigma \) and not on the reduced expression. In particular, if \( \ell(\sigma) + \ell(\tau) = \ell(\sigma\tau) \), where \( \ell(\sigma) \) denotes the length of any reduced expression of \( \sigma \) as a product of elementary transpositions, then \( \Psi_\sigma \Psi_\tau = \Psi_{\sigma\tau} \). Then

\[
[n]\Psi! = \sum_{\sigma \in S_n} \Psi_\sigma.
\]

Let \( \sigma_0 : (1, \ldots, n) \mapsto (n, \ldots, 1) \) be the longest element of \( S_n \). Since \( \ell(\sigma) + \ell(\sigma_0^{-1}) = \ell(\sigma) + \ell(\sigma_0) \), it follows that \( \Psi_\sigma \Psi_{\sigma_0} = \Psi_{\sigma_0} \Psi_\sigma = \Psi_{\sigma_0^{-1}} \Psi_\sigma \). This implies that

\[
[n]\Psi! = \Psi_{\sigma_0} [n]\Psi! = [n]\Psi! \Psi_{\sigma_0}.
\]

Also, by §5 Proposition 5.5 or §7 Proposition 4.17, \( \Psi_{\sigma_0} \Psi_\tau = \Psi_{\sigma_0 \tau \sigma_0} \Psi_{\sigma_0} \) for all \( \tau \in S_n \), hence

\[
[n]\Psi! \Psi_{\sigma_0} = \Psi_{\sigma_0} [n]\Psi!.
\]
Let \( r, s > 0 \). Then the element \( \Psi_\sigma \in \text{End}(V^{\otimes (r+s)}) \) where \( \sigma : (1, \ldots, r+s) \mapsto (s+1, \ldots, r+s, 1, \ldots, s) \) defines a braiding \( \Psi_{V^{\otimes r},V^{\otimes s}} \).

The tensor algebra \( T(V) = \bigoplus_{n \geq 0} V^{\otimes n} \) of \( V \), where \( V^{\otimes 0} = \mathbb{k} \), is the free associative algebra generated by \( V \). The braiding \( \Psi \) extends to a braiding \( \Psi_{T(V)} : T(V) \otimes T(V) \to T(V) \otimes T(V) \) via \( \Psi_{T(V)}|_{V^{\otimes n} \otimes V^{\otimes m}} = \Psi_{V^{\otimes n},V^{\otimes m}} \). Then \( T(V) \otimes T(V) \) can be endowed with a braided algebra structure via \( m_{T(V) \otimes T(V)} := (m_{T(V)} \otimes m_{T(V)}) \circ (\text{id}_{T(V)} \otimes \Psi_{T(V)} \otimes \text{id}_{T(V)}) \) where \( m_{T(V)} : T(V) \otimes T(V) \to T(V) \) is the multiplication map. Furthermore, \( T(V) \) becomes a braided bialgebra with the coproduct defined by \( \Delta(v) = v \otimes 1 + 1 \otimes v \), \( v \in V \), and the counit defined by \( \varepsilon(1) = 1 \), \( \varepsilon(v) = 0 \), \( v \in V \).

The Woronowicz symmetrizer \( \text{Wor}(\Psi) : T(V) \to T(V) \) is the linear map defined by
\[
\text{Wor}(\Psi)|_{V^{\otimes n}} = [n]_\Psi!.
\]

It turns out (cf. [311]) that \( \ker \text{Wor}(\Psi) \) is a bi-ideal of \( T(V) \). Note that (A.1) implies that \( \ker \text{Wor}(\Psi) = \ker \text{Wor}(\Psi^{-1}) \).

**Definition A.1.** The quotient \( T(V)/\ker \text{Wor}(\Psi) \) is called the Nichols-Woronowicz algebra \( \mathcal{B}(V, \Psi) \) of \( (V, \Psi) \).

The algebra \( \mathcal{B}(V, \Psi) \) is thus a braided bialgebra, where the braiding \( \Psi_{\mathcal{B}(V, \Psi)} \) on \( \mathcal{B}(V, \Psi) \otimes \mathcal{B}(V, \Psi) \) is induced by \( \Psi_{T(V)} \). By construction, \( \mathcal{B}(V, \Psi) \) is \( \mathbb{Z}_{\geq 0} \)-graded, \( \mathcal{B}^r(V, \Psi) \) being the canonical image of \( V^{\otimes r} \). Since \( V \cap \ker \text{Wor}(\Psi) = 0 \), \( V \) identifies with its canonical image in \( \mathcal{B}(V, \Psi) \) and can be shown to coincide with the space of primitive elements in \( \mathcal{B}(V, \Psi) \).

The braided antipode \( S_\Psi \) on \( T(V) \) is defined by \( S_\Psi|_{V^{\otimes n}} = (-1)^n \Psi_\sigma \) where \( \sigma : (1, \ldots, n) \mapsto (n, \ldots, 1) \) is the longest permutation in \( S_n \). It satisfies the usual properties, namely
\[
m \circ (S_\Psi \otimes 1) \circ \Delta = \varepsilon, \quad \Delta \circ S_\Psi = (S_\Psi \otimes S_\Psi) \circ \Psi_{T(V)} \circ \Delta, \quad S_\Psi \circ m = m \circ \Psi_{T(V)} \circ (S_\Psi \otimes S_\Psi)
\]
where \( m = m_{T(V)} \) (see for example [311] \S 9.4.6). By (A.1), \( S_\Psi \) preserves \( \ker \text{Wor}(\Psi) \) hence factors through to a map \( \overline{S_\Psi} : \mathcal{B}(V, \Psi) \to \mathcal{B}(V, \Psi) \) satisfying (A.3).

### A.2. Bar and star involutions.
Let \( \bar{\cdot} : \mathbb{k} \to \mathbb{k} \) be a field involution and fix an additive involutive map \( \bar{\cdot} : V \to V \) satisfying \( \bar{\bar{v}} = \bar{v} \) \( v \in V \), \( x \in \mathbb{k} \) (we will call such a map anti-linear). There is a unique anti-linear algebra homomorphism \( \bar{\cdot} : T(V) \to T(V) \) whose restriction to \( \mathbb{k} \) and \( V \) coincides with the corresponding \( \bar{\cdot} \). We say that \( \Psi \) is unitary if \( \bar{\bar{\sigma}} = \Psi^{-1} \). If \( \Psi \) is unitary then, by (A.1), \( \ker \text{Wor}(\Psi) = \ker \text{Wor}(\Psi) \), hence \( \bar{\cdot} \) factors through to an anti-linear algebra involution of \( \mathcal{B}(V, \Psi) \).

**Proposition A.2.** \( \Psi_{T(V)}(\bar{\cdot} \otimes \bar{\cdot}) \circ \Delta = \bar{\Delta} = \Delta \circ \bar{\cdot} \). Moreover, the same identity holds for \( \mathcal{B}(V, \Psi) \).

**Proof.** Let \( u \in V^{\otimes n}, n \geq 0 \). We prove that \( \Psi_{T(V)}(\bar{u}_{(1)} \otimes \bar{u}_{(2)}) = \Delta(\bar{u}) \) by induction on \( n \). The identity is clear for \( u \in V \). Furthermore, take \( u \in V^{\otimes r}, v \in V^{\otimes s} \). Then
\[
\Delta(uv) = \Delta(\bar{u}) = \bar{\Delta}(\bar{v}) = \Psi_{T(V)}(\bar{\bar{u}}_{(1)} \otimes \bar{u}_{(2)}) \Psi_{T(V)}(\bar{\bar{v}}_{(1)} \otimes \bar{v}_{(2)})
\]
\[
= (m_{T(V)} \otimes m_{T(V)}) \circ (1 \otimes \Psi_{T(V)} \otimes 1)(\Psi_{T(V)} \otimes \Psi_{T(V)})(\bar{u}_{(1)} \otimes \bar{u}_{(2)} \otimes \bar{v}_{(1)} \otimes \bar{v}_{(2)})
\]
On the other hand,
\[
\Psi_{T(V)}(\bar{\cdot} \otimes \bar{\cdot})\Delta(uv) = \Psi_{T(V)}(\bar{\bar{\sigma}} \otimes \bar{\bar{\sigma}})(m_{T(V)} \otimes m_{T(V)})(1 \otimes \Psi_{T(V)} \otimes 1)(\bar{u}_{(1)} \otimes \bar{u}_{(2)} \otimes \bar{v}_{(1)} \otimes \bar{v}_{(2)})
\]
\[
= \Psi_{T(V)}(m_{T(V)} \otimes m_{T(V)})(1 \otimes \Psi_{T(V)}^{-1} \otimes 1)(\bar{\bar{u}}_{(1)} \otimes \bar{\bar{u}}_{(2)} \otimes \bar{\bar{v}}_{(1)} \otimes \bar{\bar{v}}_{(2)}).
\]
So, the first assertion follows from the commutativity of the diagram
\[
\begin{array}{ccc}
U_1 \otimes U_2 \otimes U_3 \otimes U_4 & \xrightarrow{\text{id}_{U_1} \otimes \Psi_{U_2,U_3} \otimes \text{id}_{U_4}} & U_1 \otimes U_3 \otimes U_2 \otimes U_4 \\
\Psi_{U_1,U_2,U_3} & & \\
U_3 \otimes U_4 \otimes U_1 \otimes U_2 & \xleftarrow{\text{id}_{U_3} \otimes \Psi_{U_1,U_4} \otimes \text{id}_{U_2}} & U_3 \otimes U_1 \otimes U_4 \otimes U_2
\end{array}
\]
where \( U_i = V^\otimes r_i, r_1 + r_2 = r, r_3 + r_4 = s \). The second assertion is immediate. \( \square \)

Let \( \tau_n \in \operatorname{End}(V^\otimes n) \) be the map satisfying \( v_1 \otimes \cdots \otimes v_n \mapsto v_n \otimes \cdots \otimes v_1, v_i \in V \). We say that \( \Psi \) is self-transposed if \( \Psi = \tau_2 \Psi \tau_2 \). Define \( ^* \in \operatorname{End}(T(V)) \) by \( ^*|_{V^\otimes n} = \tau_n \). Then \( ^* \) is the unique anti-automorphism of \( T(V) \) whose restriction to \( k \) and \( V \) is the identity. Since for a self-transposed \( \Psi \) we have \( \tau_n \Psi \tau_n = \Psi_{n-i,n-i+1}, 1 \leq i \leq n - 1 \), it follows that
\[
\tau_n \Psi \sigma \tau_n = \Psi_{\sigma \sigma_0 \sigma}, \quad \sigma \in S_n. \tag{A.4}
\]
This implies that \([n]_\Psi \circ \tau_n = \tau_n \circ [n]_\Psi \) hence \( ^* \) preserves \( \ker \operatorname{Wor}(\Psi) \) and so factors through to an anti-automorphism of \( B(V, \Psi) \).

**Lemma A.3.** Suppose that \( \Psi \) is self-transposed. Then \( \Delta^* = \Delta^* \circ \Delta^{op} \) on \( T(V) \), where \( \Delta^{op}(u) = \hat{u}_{(2)} \otimes \hat{u}_{(1)} \) in Sweedler’s notation, \( u \in T(V) \). Moreover, the same identity holds on \( B(V, \Psi) \).

**Proof.** The assertion clearly holds for \( v \in V \). Let \( u \in V^\otimes r, v \in V^\otimes s \). By the induction hypothesis,
\[
\Delta((uv)^*) = \Delta(v^*) \Delta(u^*) = (\hat{u}_{(2)} \otimes \hat{u}_{(1)}) (\hat{u}_{(2)} \otimes \hat{u}_{(1)}) = (m_{T(V)} \otimes m_{T(V)}) (\hat{u}_{(2)} \otimes \hat{u}_{(1)}) (\hat{u}_{(2)} \otimes \hat{u}_{(1)}) = \Delta^* \Delta^{op}(uv). \tag{A.4}
\]
By \[(A.4)\] we have \( \Psi_{T(V)} \circ ^* = \Delta^{op} \circ \Delta^{T(V)} \), hence
\[
\Delta((uv)^*) = \Delta(m_{T(V)} \otimes m_{T(V)}) (\hat{u}_{(2)} \otimes \hat{u}_{(1)}) (\hat{u}_{(2)} \otimes \hat{u}_{(1)}) = \Delta^* \Delta^{op}(uv). \tag{A.5}
\]

**A.3. Pairing and quasi-derivations.** Let \( V^* \) be another \( k \)-vector space with a braiding \( \Psi^* : V^* \otimes V^* \to V^* \otimes V^* \). Suppose that there exists a pairing \( \langle \cdot, \cdot \rangle : V^* \otimes V \to k \) and let \( \langle \cdot, \cdot \rangle' \) be the natural pairing \( T(V^*) \otimes T(V) \to k \) defined by
\[
\langle f \otimes \cdots \otimes f_r, v_1 \otimes \cdots \otimes v_r \rangle' = \prod_{k=1}^{r} \langle f_k, v_k \rangle, \quad f_k \in V^*, v_k \in V, 1 \leq k \leq r,
\]
while \( \langle (V^\otimes r) \otimes (V^\otimes s) \rangle = 0 \) if \( r \neq s \). If \( \Psi^* \) is the adjoint of \( \Psi \) with respect to \( \langle \cdot, \cdot \rangle |_{V^* \otimes 2 \otimes V^* \otimes 2} \) define \( \langle \cdot, \cdot \rangle : T(V^*) \otimes T(V) \to k \) by
\[
\langle f, u \rangle = \langle f, \operatorname{Wor}(\Psi)(u) \rangle' = \langle \operatorname{Wor}(\Psi^*)(f), u \rangle', \quad f \in T(V^*), \quad u \in T(V).
\]

The following Lemma is standard.

**Lemma A.4.** Suppose that \( \Psi^* \) is the adjoint of \( \Psi \). Then
(a) for all \( f, f' \in T(V^*), v, v' \in T(V) \) we have
\[
\langle ff', v \rangle = \langle f, \hat{u}_{(2)} \rangle \langle f', \hat{u}_{(2)} \rangle, \quad \langle f, vv' \rangle = \langle f, \hat{u}_{(1)} \rangle \langle v, \hat{u}_{(2)} \rangle \langle v', \hat{u}_{(2)} \rangle,
\]
where \( \Delta(v) = \hat{u}_{(1)} \otimes \hat{u}_{(2)} \) and \( \Delta(f) = \hat{f}_{(1)} \otimes \hat{f}_{(2)} \) in Sweedler’s notation.
(b) Let \( \langle \cdot, \cdot \rangle : V^* \otimes V \to k \) be non-degenerate. Then \( \ker \operatorname{Wor}(\Psi) = \{ v \in T(V) : \langle T(V^*), v \rangle = 0 \} \), and \( \ker \operatorname{Wor}(\Psi^*) = \{ f \in T(V^*) : \langle f, T(V) \rangle = 0 \} \). In particular, \( \langle \cdot, \cdot \rangle \) induces a non-degenerate pairing \( \langle \cdot, \cdot \rangle : B(V^*, \Psi^*) \otimes B(V, \Psi) \to k \) satisfying (W).

**Remark A.5.** The same construction works if \( \Psi^* \) is the adjoint of \( \Psi^{-1} \).
Lemma A.6. $\langle f, S_\Psi(u) \rangle = \langle S_\Psi^\ast(f), u \rangle$ for all $f \in T(V^\ast), \ u \in T(V)$ (respectively, $f \in B(V^\ast, \Psi^\ast), \ u \in B(V, \Psi)$).

Proof. We may assume, without loss of generality, that $f \in V^\ast \otimes \mathbb{n}$, $u \in V^\otimes \mathbb{n}$. Let $\sigma_0$ be the longest permutation in $\mathbb{S}_n$.

Then

$$\langle f, S_\Psi(u) \rangle = (-1)^n \langle f, [n]_\Psi \Psi_{\sigma_0}(u) \rangle' = (-1)^n \langle f, \Psi_{\sigma_0}[n]_\Psi \Psi_{\sigma_0}(u) \rangle' = (-1)^n \langle \Psi_{\sigma_0}^\ast(f), u \rangle = \langle S_\Psi^\ast(f), u \rangle,$$

where we used (A.2). The assertion for Nichols algebras is now immediate.

\hspace{1cm}$\square$

Proposition A.7. (a) Suppose that $\Psi$ is unitary and $\langle \cdot, \cdot \rangle$ satisfies $\langle f, v \rangle = -\langle f, v \rangle$, $f \in V^\ast, \ v \in V$. Then for all $f \in T(V^\ast), \ v \in T(V)$ we have $\langle f, v \rangle = \langle f, S_\Psi^{-1}(v) \rangle$ and $S_\Psi^{-1}(v) = S_\Psi(v)$.

(b) Suppose that $\Psi$ is self-transposed. Then $\langle f, v \rangle = \langle f, \Psi v \rangle$ for all $f \in T(V^\ast), \ v \in T(V)$.

(c) Suppose that the assumptions of (a) and (i) hold. Then $\langle f, v \rangle = \langle f, S_\Psi^{-1}(v) \rangle$ and $S_\Psi^{-1}(v) = S_\Psi(v)$ for all $f \in T(V^\ast), \ v \in T(V)$.

(d) Identities (a)-(c) hold in corresponding Nichols algebras.

Proof. To prove (a) we use induction on the degree in $T(V)$. The induction base is given by the assumption. Suppose that the identity is established for all $f \in V^\ast \otimes r, \ v \in V^\otimes r, \ r < n$. Note that the induction hypothesis implies

$$\langle f \otimes \tilde{g}, u \otimes \tilde{v} \rangle = \langle f \otimes \tilde{g}, (S_\Psi^{-1} \circ S_\Psi^{-1})(u \otimes v) \rangle, \ f \in V^\ast \otimes r, \ g \in V^\otimes s, \ u \in V^\otimes s, \ v \in V^\otimes s, \ 0 < r + s \leq n.$$

Furthermore, $S_\Psi^{-1} = S_{\Psi^{-1}}$. Hence for all $u \in V^\otimes \mathbb{n}, \ f \in V^\ast \otimes r, \ g \in V^\otimes s$ with $0 < r + s = n$ we have

$$\langle f \otimes \tilde{g}, \tilde{u} \rangle = \langle f \otimes \tilde{g}, \Delta(u) \rangle = \langle f \otimes \tilde{g}, \Psi T(V)(\tilde{\cdot} \otimes \tilde{\cdot})(\tilde{\cdot} \otimes \tilde{\cdot})\Delta(u) \rangle$$

$$= \langle f \otimes g, (S_\Psi^{-1} \circ S_\Psi^{-1})(\Psi T(V) \Delta(u)) \rangle = \langle f \otimes g, (S_\Psi^{-1} \circ S_\Psi^{-1})(\Psi T(V) \Delta(u)) \rangle = \langle f \otimes g, S_\Psi^{-1}(v) \rangle,$$

where we used (A.3) and Proposition (A.2). The identity $S_\Psi^{-1}(v) = S_{\Psi^{-1}}(v)$ is a direct consequence of the unitarity of $\Psi$ and the definition of $S_\Psi$.

To prove (b), we also use induction on the degree in $T(V)$. The induction base is obvious. Suppose that the identity is established for all $f \in V^\ast \otimes r, \ v \in V^\otimes r, \ r < n$. Then for all $u \in V^\otimes \mathbb{n}, \ f \in V^\ast \otimes r, \ g \in V^\otimes s$ with $0 < r + s = n$ we have

$$\langle (fg)^\ast, u \rangle = \langle g^\ast f^\ast, u \rangle = \langle g^\ast f^\ast, \Delta(u) \rangle = \langle g^\ast, \Psi T(V)(\tilde{\cdot} \otimes \tilde{\cdot})(\tilde{\cdot} \otimes \tilde{\cdot})\Delta(u) \rangle$$

$$= \langle f \otimes g, (\tilde{\Psi}^{-1} \circ \tilde{\Psi}^{-1})(\tilde{\Psi} T(V) \Delta(u)) \rangle = \langle f \otimes g, (\tilde{\Psi}^{-1} \circ \tilde{\Psi}^{-1})(\tilde{\Psi} T(V) \Delta(u)) \rangle = \langle f, u^\ast \rangle,$$

where we used Lemma A.3.

To prove (c) note that by (a) and (i) we have

$$\langle f, v \rangle = \langle f, S_\Psi^{-1}(\Psi v) \rangle.$$

Let $f \in V^\ast \otimes \mathbb{n}, \ g \in V^\otimes \mathbb{n}$. Then $S_\Psi^{-1}(\Psi v) = (-1)^n \tau_n \Psi^{-1} \tau_n(v) = (-1)^n \Psi^{-1} \tau_n(v) = S_\Psi^{-1}(v)$, where we used (A.4). Part (d) is immediate.

\hspace{1cm}$\square$

Suppose that for every $n > 0$, there exists an invertible $L_n \in \text{End}(V^\otimes \mathbb{n})$ such that $L_n = (-1)^n S_\Psi \circ \ast, \ L_n \circ \ast = \ast \circ L_n^{-1}$ and $L_n \circ \ast = \ast \circ L_n$. Let $L \in \text{End}(T(V))$ be the linear operator defined by $L |_{V^\otimes \mathbb{n}} = L_n$ and define $(\cdot, \cdot) : B(V^\ast, \Psi^\ast) \otimes B(V, \Psi) \rightarrow \mathbb{k}$ by

$$(f, v) = \langle f, L^{-1}(v) \rangle.$$

Lemma A.8. Suppose that $\Psi$ is self-transposed and unitary. Then for all $f \in B^r(V^\ast, \Psi^\ast), \ v \in B^s(V, \Psi)$ we have

$$\langle f, v \rangle = (-1)^{\delta_{r,s}}(f, v).$$
Suppose that the actions $H$ and $\Delta$ are locally nilpotent.

\begin{proof}
Let $f \in B^r(V^*, \Psi^*)$, $v \in B^r(V, \Psi)$, the case $r \neq s$ being trivial. Then
\[
\langle f, \overline{v} \rangle = \overline{\langle f, L^*_{r-1}(v) \rangle} = (-1)^r \langle f, L^*_{r-2}(L_r(v)) \rangle = (-1)^r \langle f, L^*_{r-1}(v) \rangle = (-1)^r \langle f, v \rangle.
\]
\end{proof}

Given $f \in B(V^*, \Psi^*)$, $v \in B(V, \Psi)$ define $k$-linear operators $\partial_f, \partial_v^\circ : B(V, \Psi) \to B(V, \Psi)$, $\partial_v, \partial_v^\circ : B(V^*, \Psi^*) \to B(V^*, \Psi^*)$ by
\[
\partial_v(g) = \langle g_1, v \rangle g_2, \quad \partial_v^\circ(g) = g_1 \langle g_2, v \rangle, \quad g \in B(V, \Psi)
\]
and
\[
\partial_f(u) = \langle f, u_1 \rangle u_2, \quad \partial_f^\circ(u) = u_1 \langle f, u_2 \rangle, \quad u \in B(V, \Psi).
\]
Then for all $f, g \in B(V^*, \Psi^*)$, $u, v \in B(V, \Psi)$
\[
\langle f, u \rangle = \langle \partial_v(f), v \rangle = \langle \partial_v^\circ(f), u \rangle
\]
\[
\langle f, u \rangle = \langle g, \partial_f(u) \rangle = \langle f, \partial_f^\circ(u) \rangle.
\]

The definitions immediately imply that if $f \in B(V^*, \Psi^*)$, $v \in B(V, \Psi)$ are homogeneous then $\partial_f$, $\partial_f^\circ$, $\partial_v$, $\partial_v^\circ$ are homogeneous. Moreover, if say $f \in B^r(V^*, \Psi^*)$, $v \in B^k(V, \Psi)$ then $\partial_f(v), \partial_f^\circ(v) \in \sum_{\rho = 0}^{r-k} B^r(V, \Psi)$ and $\partial_f(v), \partial_f^\circ(v) \in \sum_{\rho = 0}^{r-k} B^r(V^*, \Psi^*)$. Thus, $\partial_f, \partial_f^\circ, \partial_v, \partial_v^\circ$ are locally nilpotent.

\begin{proposition}
$D_{H, H^\cop}(C)$ is an associative algebra. Moreover, $H$ and $C$ identify with subalgebras of $D_{H, H^\cop}(C)$.
\end{proposition}

\begin{proof}
The only non-trivial identity to check is $h \cdot (cc') = (h \cdot c) \cdot c'$ for all $h \in H$, $c, c' \in C$. We have
\[
(h \cdot c) \cdot c' = (h_1 \cdot (h_2 \cdot (h_3 \cdot c')) = (h_1 \cdot (h_2 \cdot (h_3 \cdot c')) \cdot (h_3 \cdot c') = (h_1 \cdot ((h_4 \cdot (h_5 \cdot c')) = (h_2) \cdot (h_3 \cdot c')).
\]
\end{proof}

\begin{remark}
Note that if $\triangleright$ (respectively, $\triangleright$) is trivial, that is $h \triangleright c = \varepsilon_H(h)c$ (respectively, $h \triangleright c = \varepsilon_H(h)c$), then $D_{H, H^\cop}(C) = C \rtimes H$ (respectively, $C \rtimes H^\cop$).
\end{remark}

Suppose now that $C$ is a bialgebra and that $\Delta_C, \varepsilon_C$ are homomorphisms of $H^\cop \otimes H$-modules, where $H^\cop \otimes H$ acts naturally on $C \otimes C$ and $k$. Thus, $\Delta_C(h \triangleright h' \triangleright c) = (h' \triangleright c_{(1)}) \otimes (h \triangleright c_{(2)})$ and $\varepsilon_C(h \triangleright h' \triangleright c) = \varepsilon_C(c)\varepsilon_H(h)\varepsilon_H(h')$ for all $c \in C$, $h, h' \in H$.

\begin{proposition}
Suppose that the actions $\triangleright$, $\triangleright$ satisfy
\[
h_2 \cdot c_{(1)} \otimes h_1 \triangleright c_{(2)} = \varepsilon_H(h)\Delta(c), \quad c \in C, h \in H.
\]
\end{proposition}

Then $D_{H, H^\cop}(C)$ is a bialgebra with the comultiplication and the counit defined by $\Delta(c \cdot h) = \Delta_C(c) \cdot \Delta^\cop_H(h)$ and $\varepsilon(c \cdot h) = \varepsilon_C(c)\varepsilon_H(h)$, $c \in C$, $h \in H$, and $C$, $H^\cop$ identify with its subbialgebras. If both $C$ and $H$ are Hopf algebras and
\[
S_C(h \triangleright h' \triangleright c) = S_H^{-1}(h') \triangleright h \triangleright S_C(c), \quad c \in C, h, h' \in H
\]
then $D_{H, H^\cop}(C)$ is a Hopf algebra with the antipode defined by $S(c \cdot h) = S_H^{-1}(h) : S_C(c)$ and $C$, $H^\cop$ identify with its Hopf subalgebras.
Proof. We need to check that \( \Delta(h \cdot c) = \Delta_H^\op(h) \cdot \Delta_C(c) \) for all \( c \in C, h \in H \). Indeed
\[
\Delta(h \cdot c) = \Delta((h_1 \triangleright h_2 \triangleright c) \cdot h_2) = \Delta_C(h_1 \triangleright h_2 \triangleright c) \cdot (h_2 \otimes h_2)
\]
\[
= (h_1 \triangleright c_1 \otimes h_1 \triangleright c_2) \cdot (h_2 \otimes h_2) = \varepsilon(h_2)(h_3 \triangleright c_1 \otimes h_1 \triangleright c_2) \cdot (h_4 \otimes h_2)
\]
\[
= (h_1 \triangleright h_6 \triangleright c_1) \cdot h_6 \otimes (h_1 \triangleright h_3 \triangleright c_2) \cdot h_2
\]
\[
= h_2 \cdot (c_1 \otimes h_1) \cdot (c_2) = \Delta_H^\op(h) \cdot \Delta_C(c).
\]
The property of \( \varepsilon \) is obvious. For the antipode, we have
\[
S(h \cdot c) = S_H^{-1}(h_2) \cdot S_C(h_1 \triangleright h_3 \triangleright c) = S_H^{-1}(h_2) \cdot S_H^{-2}(h_3) \triangleright h_1 \triangleright S_C(c)
\]
\[
= S_H^{-1}(h_4) S_H^{-2}(h_5) \triangleright S_H^{-1}(h_2) h_1 \triangleright S_C(c) \cdot S_H^{-1}(h_3)
\]
\[
= S_C(c) \cdot S_H^{-1}(h).
\]
Denote \( H^\op \) the opposite algebra and coalgebra of \( H \). Note that we can endow \( H^\op \otimes C^\op \) with an associative algebra structure via
\[
c \cdot h = h_2 (h_1 \triangleright h_3 \triangleright c).
\]
Denote the resulting algebra \( \mathcal{D}_{H^\op, H^\op}(C^\op) \). The following proposition is immediate.

**Proposition A.12.** The map \( \tau : C \otimes H \rightarrow H \otimes C, c \otimes h \mapsto h \otimes c \) is an isomorphism of algebras \( \mathcal{D}_{H^\op, H^\op}(C^\op) \rightarrow \mathcal{D}_{H^\op, H^\op}(C^\op) \). Moreover, if \( \text{[A.7]} \) and \( \text{[A.8]} \) hold then \( \tau \) is an isomorphism of Hopf algebras \( \mathcal{D}_{H^\op, H^\op}(C^\op) \rightarrow \mathcal{D}_{H^\op, H^\op}(C^\op) \).

Let \( \overline{\cdot} \) be a field involution on \( \mathbb{k} \) and suppose that it extends to an anti-linear anti-involutions of algebras \( C \) and \( H \). Assume that \( \overline{\cdot} \) is an anti-linear involution of coalgebras for \( H \). Note that then we have \( \overline{S_H(h)} = S_H^{-1}(h), h \in H \). Extend \( \overline{\cdot} \) to an anti-linear map \( \mathcal{D}_{H^\op, H^\op}(C) \rightarrow \mathcal{D}_{H^\op, H^\op}(C) \) by
\[
\overline{c \cdot h} = \overline{h} \cdot \overline{c}.
\]

**Lemma A.13.** Suppose that
\[
\overline{h_1 \triangleright h_2 \triangleright c} = \varepsilon_H(h) \overline{c} = \overline{h_2 \triangleright h_3 \triangleright c}, \quad h \in H, c \in C.
\]
Then \( \overline{\cdot} \) is an anti-linear anti-involution of the algebra \( \mathcal{D}_{H^\op, H^\op}(C) \).

**Proof.** We have
\[
\overline{h \cdot c} = \overline{(h_1 \triangleright h_3 \triangleright c) \cdot h_2} = \overline{h_2} \cdot \overline{(h_1 \triangleright h_3 \triangleright c)} = \overline{h_2} \cdot \overline{(h_1 \triangleright h_3 \triangleright c)} \cdot \overline{h_3}
\]
\[
= \varepsilon_H(h_1) \overline{h_2} \cdot \overline{(h_1 \triangleright c)} \cdot \overline{h_3} = \varepsilon_H(h_1) \overline{\varepsilon_H(h)} = \overline{\overline{\varepsilon_H(h)}} = \overline{\overline{h}} = \overline{h}.
\]
This shows that \( \overline{\cdot} \) is a well-defined anti-linear anti-involution of \( \mathcal{D}_{H^\op, H^\op}(C) \).

**Remark A.14.** It is easy to check that \( \text{[A.9]} \) holds if
\[
\overline{h \triangleright c} = S_H^{-1}(h) \triangleright \overline{c}, \quad \overline{h \triangleright c} = S_H(h) \triangleright \overline{c}, \quad h \in H, c \in C.
\]

A.5. Bialgebra pairings and doubles of bialgebras. We will now consider a special case of the double smash product construction. Given bialgebras \( H \) and \( C, \phi \in \text{Hom}_k(C \otimes H, k) \) is said to be a bialgebra pairing if
\[
\phi(cc', h) = \phi(c, h_1) \phi(c', h_2), \quad \phi(c, hh') = \phi(c_1, h) \phi(c_2, h')
\]
\[
\phi(c, 1) = \varepsilon_C(c), \quad \phi(1, h) = \varepsilon_H(h),
\]
for all \( c, c' \in C \) and \( h, h' \in H \). If both \( C \) and \( H \) are Hopf algebras, a bialgebra pairing \( \phi \) is called a Hopf pairing if
\[
\phi(S_C(c), h) = \phi(c, S_H(h)), \quad c \in C, h \in H.
\]
Given a bialgebra pairing \( \phi : C \otimes H \to k \), define
\[
 h \triangleright c = c(1) \phi(c(2), h), \quad c \triangleleft h = c(2) \phi(c(1), h), \quad c \triangleright h = h(1) \phi(c, h(2)), \quad h \triangleleft c = h(2) \phi(c, h(1)) \tag{A.11}
\]

The following is easily checked.

**Lemma A.15.** Let \( \phi, \phi' \) be two bialgebra pairings \( C \otimes H \to k \). Then \( \triangleright, \triangleleft \) define a structure of an \( H \)- (respectively, a \( C \)-) bimodule algebra on \( C \) (respectively, on \( H \)). Moreover,
\[
 \Delta_C(h \triangleright c \triangleleft h') = (c(1) \triangleleft h') \otimes (h \triangleright c(2)). \tag{A.12}
\]

Given two bialgebra pairings \( \phi_+, \phi_- : C \otimes H \to k \) define \( \mathcal{D}_{\phi_+, \phi_-}(C, H) \) as \( \mathcal{D}_{H^\text{cop}}(C) \) where

\[
 h \triangleright c = h \triangleright_\phi c \quad \text{and} \quad h \triangleright c = c \triangleleft S_H^{-1}(h). \quad \text{Thus, in } \mathcal{D}_{\phi_+, \phi_-}(C, H) \text{ we have}
\]
\[
h \cdot c = (c(2) \cdot h(2)) \phi_-(c(3), S_H^{-1}(h(3))) \phi_+(c(1), h(1))
\]
\[
= (h(1) \triangleright_\phi c \triangleleft_\phi S_H^{-1}(h(3))) \cdot h(2) = c(2) \cdot (S_C^{-1}(c(1)) \triangleright_\phi h \triangleleft_\phi c(3)) \tag{A.13}
\]

We abbreviate \( \mathcal{D}_\phi(C, H) = \mathcal{D}_{\phi_+, \phi_-}(C, H) \)

**Proposition A.16.** Let \( H \) be a Hopf algebra, \( C \) be a bialgebra and \( \phi, \phi_\pm : C \otimes H \to k \) be bialgebra pairings.
\[
\begin{align*}
(\text{a}) & \quad \mathcal{D}_{\phi_+, \phi_-}(C, H) \text{ is an associative algebra and } C, H \text{ identify with its subalgebras.} \\
(\text{b}) & \quad \mathcal{D}_\phi(C, H) \text{ is a bialgebra and } C, H^\text{cop} \text{ identify with its sub-bialgebras. Moreover, if } C \text{ is Hopf} \\
& \quad \text{algebras and } \phi \text{ is a Hopf pairing then } \mathcal{D}_\phi(C, H) \text{ is a Hopf algebra.}
\end{align*}
\]

**Proof.** Part \( \text{(a)} \) is immediate from Proposition \( \text{A.9} \). To prove \( \text{(b)} \) note that by \( \text{(A.12)} \) we only need to check that \( \text{(A.7)} \) and \( \text{(A.8)} \) hold. Indeed
\[
h(2) \triangleright_\phi c(1) \otimes c(2) \triangleleft_\phi S_H^{-1}(h(1)) = \phi(c(2), h(2)) \phi(c(3), S_H^{-1}(h(1))) c(1) \otimes c(4)
\]
\[
= \phi(c(2), h(2) S_H^{-1}(h(1))) c(1) \otimes c(3) = \epsilon_H(h) c(1) \otimes c(2) = \epsilon_H(h) \Delta(c).
\]

Finally, to prove \( \text{(A.8)} \) note that
\[
S_C(h \triangleright c \triangleleft S_H^{-1}(h')) = \phi(c(3), h) \phi(c(1), S_H^{-1}(h')) S_C(c(2))
\]
\[
= \phi(S_C(c(3)), S_H^{-1}(h)) \phi(S_C(c(1)), S_H^{-2}(h')) S_C(c(2)) = S_H^{-2}(h') \triangleright_\phi S_C(c) \triangleleft S_H^{-1}(h). \quad \Box
\]

Note the following useful identity in \( \mathcal{D}_{\phi_+, \phi_-}(C, H) \)
\[
c \cdot h = h(2) \cdot (S_H^{-1}(h(1)) \triangleright_\phi c \triangleleft_\phi h(3)), \quad c \in C, \ h \in H. \tag{A.14}
\]

The compatibility conditions from Lemma \( \text{A.13} \) read
\[
\overline{c(1)} \phi_+(\overline{c(2)}, \overline{h(2)}) \phi_-(c(3), h(1)) = \epsilon_H(\overline{h}) \overline{c} = \overline{c(3)} \phi_-(\overline{c(2)}, S_H^{-1}(\overline{h(1)})) \phi_-(c(1), S_H^{-1}(\overline{h(2)}) \tag{A.15}
\]
and are satisfied if
\[
\phi_\pm(\overline{c}, \overline{h}) = \phi_\pm(c, S_H^{-1}(h)), \quad c \in C, \ h \in H.
\]
A.6. Bosonisation of Nichols algebras. Suppose that $V$ is a left Yetter-Drinfeld module over a Hopf algebra $H$ with the comultiplication $\Delta_H$ and the antipode $S_H$. That is, $V$ is a left $H$-module with the action denoted by $\triangleright$ and a left $H$-comodule with the co-action $\delta : V \to H \otimes V$. We use the Sweedler-type notation $\delta(v) = v(1) \otimes v(0)$. The action and co-action are compatible, that is
\[
\delta(h \triangleright v) = h(1) v(1) S_H(h(3)) \otimes h(2) \triangleright v(0), \quad h \in H, \ v \in V, \quad (A.16)
\]
where $(\Delta_H \otimes 1) \Delta_H = h(1) \otimes h(2) \otimes h(3)$.

The category $H \mathcal{YD}$ of left Yetter-Drinfeld modules over $H$ is a braided tensor category with the braiding $\Psi : V \otimes W \to W \otimes V$ being given by
\[
\Psi_{V,W}(v \otimes w) = v(1) \triangleright w \otimes v(0), \quad v \in V, \ w \in W. \quad (A.17)
\]
Note that
\[
\Psi^{-1}_{V,W}(w \otimes v) = v(0) \otimes S_H^{-1}(v(1)) \triangleright w. \quad (A.18)
\]
In particular, $T(V)$ is a braided Hopf algebra in the category $H \mathcal{YD}$. We will denote the corresponding Nichols algebra by $\mathcal{B}(V)$.

Consider now the algebra $T(V) \rtimes H = T(V) \otimes H$ with the cross-relation
\[
h \cdot u = (h(1) \triangleright u) \cdot h(2). \quad (A.19)
\]
It has a co-algebra structure defined by
\[
\Delta(v) = v \otimes 1 + \delta(v), \ \Delta(h) = \Delta_H(h), \ \epsilon(v) = 0, \ \epsilon(h) = \epsilon_H(h), \ v \in V, \ h \in H. \quad (A.20)
\]
It is easy to check, using (A.16), that this comultiplication and counit extend to homomorphisms of respective algebras.

**Lemma A.17.** Let $u \in T(V)$. Then
\[
\Delta(u) = u(1) u(2)^{-1} \otimes u(2),
\]
where $\Delta(u) = u(1) \otimes u(2)$.

**Proof.** For $v \in V$ there is nothing to prove. Suppose that the identity holds for all $u \in V^{\otimes r}, r < n$. Let $u \in V^{\otimes r}, v \in V^{\otimes s}, r, s > 0, r + s = n$. Then
\[
\Delta(uv) = \Delta(u) \Delta(v) = (u(1) \otimes 1) \Psi(u(2) \otimes u(1))(1 \otimes u(2)) = u(1)(u(2)^{-1} \triangleright u(1)) \otimes u(2),
\]
whence
\[
\Delta(uv) = u(1)(u(2)^{-1} \otimes u(2))(u(1)u(2)^{-1} \otimes u(2)) = u(1)(u(2)^{-1} \otimes u(2)) \otimes u(2) = u(1) \otimes u(2)^{-1} \otimes u(2)^0.
\]

Denote by $\mathcal{S}$ the braided antipode on $T(V)$ corresponding to the braiding $\Psi_{V,V}$. Note that $\mathcal{S}$ is a morphism in the category $H \mathcal{YD}$ hence commutes with the action and the co-action of $H$. Define $S : T(V) \rtimes H \to T(V) \rtimes H$ by
\[
S(uh) = S_H(h^{-1}) \mathcal{S}(u(0)). \quad (A.21)
\]

**Lemma A.18.** $S$ is an antipode for $T(V) \rtimes H$. Moreover, $S$ is invertible and
\[
S^{-1}(uh) = S_H^{-1}(h) \mathcal{S}^{-1}(u(0)) S_H^{-1}(u(1)), \quad u \in T(V), \ h \in H. \quad (A.22)
\]
Proof. By definition, we have $S(uh) = S(h)S(u)$, $u \in T(V)$, $h \in H$. Furthermore, using \( \text{A.16} \), we obtain

$$S(hu) = S((h(1) \triangleright u)h(2)) = S_H(h(1) \triangleright u)(-1)h(2)S((h(1) \triangleright u)(0))$$

$$= S_H(h(1)u^{-1}S_H(h(3))h(4))S((h(2) \triangleright u)(0)) = S_H(h(1)u^{-1}S(h(2) \triangleright u)(0))$$

To prove that $S$ is an anti-endomorphism of $T(V) \times H$, it remains to show that $S(uv) = S(v)S(u)$ for all $u, v \in T(V)$. Indeed,

$$S(uv) = S_H((uv)(-1)S((uv)(0))) = S_H(v(-1))S_H(u(-1))(S(u(0))(1) \triangleright S(v(0)))(S(u(0))(0))$$

$$= S_H(v(-1))S_H(u(-2))(u(1) \triangleright S(u(0)))S(v(0))$$

We have

$$m(S \otimes 1)\Delta(u) = S_H((u(1) \otimes 1)u(2)) = S_H(u(1) \otimes 1)S(u(0)) = S_H(((\Delta(u)(-1)) \varepsilon(u(0))) = \varepsilon(u).$$

On the other hand,

$$m(1 \otimes S)\Delta(u) = u(1) \otimes S(u(2)) = S_H(u(1) \otimes S(u(2)))S(u(0)) = u(1)\varepsilon_H(u(2))S(u(0)) = u(1)S(u(2)) = \varepsilon(u).$$

Define $\tilde{S} : T(V) \times H \to T(V) \times H$ by $\tilde{S}(uh) = S^{-1}_H(h)S^{-1}(u(0))S^{-1}_H(u(-1))$. Then we have

$$\tilde{S}(hu) = \tilde{S}((h(1) \triangleright u)h(2)) = S^{-1}_H(h(2))S^{-1}_H((h(1) \triangleright u)(0))S^{-1}_H((h(1) \triangleright u)(-1))$$

Now

$$\tilde{S}(uh) = S(S^{-1}_H(h)S^{-1}_H(0)S^{-1}_H(u(-1))) = u(-1)S(S(u(0))h)$$

while

$$\tilde{S}(uh) = S(S_H(u(-1))hS(u(0))) = S(S(u(0)))u^{-1}h$$

Thus, $\tilde{S}$ is the inverse of $S$. \( \square \)

Observe that $\text{ker } \text{Wor}(\Psi)$ is a bi-ideal in $T(V) \times H$. In particular, we can consider the quotient of $T(V) \times H$ by that ideal which is isomorphic to $B(V) \times H$. Clearly, Lemmata \( \text{A.17} \) and \( \text{A.18} \) hold in $B(V) \times H$.

Let $\tilde{c}$ be a field involution on $\mathbb{k}$ and fix its extension to $V$ as in \( \text{A.12} \). Suppose that $h \triangleright v = S^{-1}_H(h) \triangleright \overline{v}$ and that $(\cdot \otimes \cdot) \circ \delta \circ \tilde{c} = \delta$.

Lemma A.19. Suppose that $\Psi$ is self-transposed. Then $\Psi$ is also unitary, that is $\overline{\delta} \circ \tilde{c} = \Psi^{-1} \circ \tilde{c}$.
Proof. Since $\Psi$ is self-transposed, it follows that

$$u^{(-1)} \triangleright v \otimes u^{(0)} = v^{(0)} \otimes u^{(-1)} \triangleright u$$

(A.23)

Applying $\bar{\cdot} \otimes \bar{\cdot}$ to both sides yields

$$(\bar{\cdot} \otimes \bar{\cdot}) \circ \Psi(u \otimes v) = \bar{\Psi}(u^{(-1)} \triangleright v \otimes u^{(0)}) = \bar{\Psi}(v^{(0)} \otimes u^{(-1)} \triangleright u) = \bar{\Psi}(v^{(0)} \otimes S^1_H(v^{(-1)}) \triangleright \bar{u}) = \Psi^{-1}(\bar{v} \otimes \bar{v}),$$

where we used (A.18). □

Thus, if (A.23) holds, $B(V)$ admits the anti-linear anti-involution $\bar{\cdot}$. Then by Lemma A.21 and Remark A.10, $\bar{\cdot}$ extends uniquely to an anti-linear anti-involution on $B(V) \rtimes H$ such that $v \cdot h = \bar{h} \cdot \bar{v}$, $v \in V$, $h \in H$. Thus, we obtain the following

Lemma A.20. Suppose that $\Psi : V \otimes V \rightarrow V \otimes V$ is self-transposed, $\bar{\cdot}$ commutes with the co-action on $V$ and $\bar{\cdot}$ extends uniquely to an anti-linear anti-involution on $B(V) \rtimes H$.

A.7. Drinfeld double. Let $C$, $H$ be Hopf algebras and fix a Hopf pairing $\xi : C \otimes H \rightarrow \mathbb{k}$. Let $V$ (respectively, $V^*$) be an object in $H \mathcal{D}$ (respectively, in $H \mathcal{D}$). Then we have a right $C$-module (respectively, $H$-module) structure on $V$ (respectively, $V^*$) defined by

$$f \triangleleft h = \xi(f^{(-1)}, h) h^{(0)}, \quad v \triangleleft c = \xi(v^{(-1)}, v^{(0)}) v^{(0)}, \quad f \in V^*, \quad v \in V, \quad c \in C, \quad h \in H.$$

(A.24)

Assume that a pairing $\langle \cdot , \cdot \rangle : V^* \otimes V \rightarrow \mathbb{k}$ satisfies

$$\langle f, v \triangleright w \rangle = \langle f \triangleright v, w \rangle, \quad \langle c \triangleright f, v \rangle = \langle f, v \triangleleft c \rangle, \quad f \in V^*, \quad v \in V, \quad c \in C, \quad h \in H.$$  

(A.25)

Lemma A.21. Suppose that (A.25) holds. Then the braiding $\Psi^*$ is the adjoint of $\Psi$ with respect to $\langle \cdot , \cdot \rangle' : V^* \otimes V \rightarrow \mathbb{k}$ in the notation of (A.3).

Proof. We need to show that for all $f, g \in V^*$, $u, v \in V$

$$\langle \Psi^*(f \otimes g), u \otimes v \rangle' = \langle f \otimes g, \Psi(u \otimes v) \rangle'$$

which, by the definition of $\Psi$ and $\Psi^*$ is equivalent to

$$\langle f^{(-1)} \triangleright g, u \rangle \langle f^{(0)}, v \rangle = \langle f, u^{(-1)} \triangleright v \rangle \langle g, u^{(0)} \rangle.$$  

But, using (A.24) and (A.25) we obtain

$$\langle f^{(-1)} \triangleright g, u \rangle \langle f^{(0)}, v \rangle = \langle g, u \triangleleft f^{(-1)} \rangle \langle f^{(0)}, v \rangle = \langle g, u^{(0)} \rangle \xi(f^{(-1)}, u^{(-1)}) \langle f^{(0)}, v \rangle = \langle g, u^{(0)} \rangle \langle f, u^{(-1)} \triangleright v \rangle.$$  

Thus, we can define the pairing $\langle \cdot , \cdot \rangle : T(V^*) \otimes T(V) \rightarrow \mathbb{k}$ as in (A.3). Note that (A.25) holds for all $f \in T(V^*)$, $v \in T(V)$. Clearly, we can replace $T(V)$, $T(V^*)$ by the corresponding Nichols algebras.

It should be noted that $V$ is not an $H$-$C$ bimodule with respect to the actions $\triangleright$ and $\triangleleft$. Given $c \in C$, $h \in H$ define, for all $v \in V$, $f \in V^*$

$$v \triangleleft (c \cdot h) = S_H^{-1}(h) \triangleright (v \triangleleft c), \quad (c \cdot h) \triangleright f = c \triangleright (f \triangleleft S_H^{-1}(h)).$$

(A.26)

Lemma A.22. $V$ (respectively, $V^*$) is a right (respectively, left) Yetter-Drinfeld module over $\mathcal{D}_\xi(C,H)$, with the right coaction on $V$ defined by $\delta_H(v) = v^{(0)} \otimes v^{(-1)}$, the left coaction on $V^*$ defined by $\delta(f) = f^{(-1)} \otimes f^{(0)}$ and the left (right) action defined by (A.26).
Proof. Let \( c \in C, \ h \in H \) and \( v \in V \). By definition, we have \( v \triangleleft (c \cdot h) = (v \triangleleft c) \triangleleft h \). On the other hand,

\[
(v \triangleleft h) \triangleleft c = \xi(c, \left(S_H^{-1}(h) \triangleright v\right)^{(1)}(S_H^{-1}(h) \triangleright v)^{(0)}) = \xi(c, S_H^{-1}(h(3)))v^{(1)}h(1)S_H^{-1}(h(2)) \triangleright v^{(0)}
\]

\[
= \xi(c(1), S_H^{-1}(h(3)))\xi(c(2), v^{(1)})S_H^{-1}(h(2)) \triangleright v^{(0)}
\]

\[
= \xi(c(1), S_H^{-1}(h(3)))\xi(c(3), h(1))S_H^{-1}(h(2)) \triangleright (v \triangleleft c(2))
\]

\[
= \xi(c(1), S_H^{-1}(h(3)))\xi(c(3), h(1))S_H^{-1}(h(2)) \triangleright (v \triangleleft (c(2) \cdot h)) = v \triangleleft (h \cdot c).
\]

Thus, (A.26) defines a right \( D_C(H) \)-module structure on \( V \). It remains to verify that this action is compatible with the right co-action. Recall that \( H^{\text{cop}} \) identifies with a sub-bialgebra of \( D_C(H) \), hence we only need to check the compatibility condition for \( c \in C \). We have

\[
(v^{(0)} \triangleleft c(2)) \otimes S_C(c(1))v^{(1)}c(3) = (v^{(0)} \triangleleft c(2)) \otimes S_C(c(1))c(4)v^{(2)}\xi(c(3), S_H^{-1}(v^{(1)}))\xi(c(5), v^{(3)})
\]

\[
= v^{(0)} \otimes S_C(c(1))c(4)v^{(2)}\xi(c(3), S_H^{-1}(c(2), v^{(1)})\xi(c(5), v^{(4)})
\]

\[
= v^{(0)} \otimes S_C(c(1))c(2)v^{(1)}\xi(c(3), v^{(2)}) = v^{(0)} \otimes v^{(1)}\xi(c, v^{(2)}) = \xi(c, v^{(1)})\delta_R(v^{(0)})
\]

The assertion for \( V^* \) is proved similarly. \hfill \Box

Definition A.23. Fix pairings \( \langle \cdot, \cdot \rangle_\pm : V^* \otimes V \to k \) such that (A.25) holds for both of them. The algebra \( U_k(V^*, C, V, H) \) is generated by \( V, V^*, D_C(H) \) subject to the following relations

(i) The subalgebra generated by \( V \) (respectively, \( V^* \)) and \( D_C(H) \) is isomorphic to \( D_C(H) \rtimes B(V) \) (respectively, \( B(V^*) \rtimes D_C(H) \))

(ii) \([v, f] = f(-1)\langle f(0), v\rangle_+ - v(-1)\langle f, v(0)\rangle_- \), \( f \in V^*, \ v \in V \).

Proposition A.24. The algebra \( U_k(V^*, C, V, H) \) is isomorphic to the braided double \( B(V^*) \rtimes D_C(H) \rtimes B(V) \) in the sense of \( \text{[3]} \) and admits a triangular decomposition. In particular, if \( \langle \cdot, \cdot \rangle_\cdot \) equals zero, \( U_k(V^*, C, V, H) \) is the Heisenberg double.

Proof. Define \( \beta : V^* \otimes V \to D_C(H) \) by \( \beta = \beta_+ - \beta_- \) where \( \beta_+(f, v) = f(-1)\langle f(0), v\rangle_+ \) and \( \beta_-(f, v) = v(-1)\langle f, v(0)\rangle_- \), \( f \in V^*, \ v \in V \). Then in \( U_k(V^*, C, V, H) \) we have \([v, f] = \beta(f, v) \). Thus, \( U_k(V^*, C, V, H) \) is a braided double. By \( \text{[3] Theorem A} \), it remains to prove that

\[
x_{(1)}\beta_+(f, v \triangleleft x_{(2)}) = \beta_+(x_{(1)} \triangleright f, v)x_{(2)}, \quad x \in D_C(H), \ v \in V, \ f \in V^*.
\]

Using Lemma (A.22) we obtain

\[
\beta_+(x_{(1)} \triangleright f, v)x_{(2)} = (x_{(1)} \triangleright f)^{-1}\langle (x_{(1)} \triangleright f)^{(0)}, v\rangle_+x_{(2)} = x_{(1)}f^{-1}S(x_{(3)})x_{(4)}x_{(2)} \triangleright f^{(0)}, v\rangle_+
\]

\[
= x_{(1)}f^{-1}\langle f^{(0)}, v \triangleleft x_{(2)}\rangle_+ = x_{(1)}\beta_+(f, v \triangleright x_{(2)}),
\]

while

\[
x_{(1)}\beta_-(f, v \triangleleft x_{(2)}) = \langle f, v^{(0)} \triangleleft x_{(3)}\rangle_+x_{(1)}S(x_{(2)})v^{(1)}x_{(4)} = \langle f, v^{(0)} \triangleleft x_{(1)}\rangle_-v^{(-1)}x_{(2)}
\]

\[
= v^{(-1)}\langle x_{(1)} \triangleright f, v^{(0)}\rangle_-x_{(2)} = \beta_-(x_{(1)} \triangleright f, v)x_{(2)}. \hfill \Box
\]

We now obtain another presentation of \( U_k(V^*, C, V, H) \). Given a pairing \( \langle \cdot, \cdot \rangle : V^* \otimes V \to k \) satisfying (A.25), define \( \phi : B(V^*) \rtimes C \otimes B(V) \rtimes H \to k \) by \( \phi(fc, vh) = \langle f, v\rangle \xi(c, h) \).

Lemma A.25. \( \phi \) is a Hopf pairing.
Proof. We have

\[ \phi((fc)_1, vh)\phi((fc)_2, v'h) = \langle f_1, v \rangle \langle f_2, v' \rangle \xi(f_2 c_1, h)\xi(c_2, h') \]

\[ = \langle f_1, v \rangle \langle f_2, v' \rangle \xi(f_2 h_1, h_2)\xi(c_1, h_2)\xi(c_2, h') \]

\[ = \langle f_1, v \rangle \langle f_2, h_1 \triangleright v' \rangle \xi(c, h_2 h') = \langle f, v(h_1 \triangleright v') \rangle \xi(c, h_2 h') \]

\[ = \phi(f, v(h_1 \triangleright v') h_2 h') = \phi(f, (vh) \cdot (v'h')) , \]

where we used (A.25). Similarly,

\[ \phi(f, (vh)_1)\phi(f', (vh)_2) = \langle f, v(1) \rangle \langle f', \varphi(1) \rangle \xi(c, \varphi(1)_1)\xi(c', h_2) \]

\[ = \langle f, v(1) \rangle \langle f', \varphi(2) \rangle \xi(c_1, \varphi(2)_1)\xi(c_2, h_1)\xi(c', h_2) = \langle f, \varphi(1) \rangle \langle f', \varphi(2) \rangle \xi(c_1)\xi(c_2)\xi(c', h) \]

\[ = \phi(f c_1 \triangleright f', c_2' \triangleright vh) = \phi((f c) \cdot (f' c'), vh) . \]

Clearly, \( \phi(f, 1) = \varepsilon(f)\varepsilon(c) \) while \( \phi(1, vh) = \varepsilon(v)\varepsilon_H(h) \). Finally, we have

\[ \phi(S(f c), vh) = \phi(Sc(f^{(-1)}c)S(f^{(0)}), vh) = \phi(Sc(f^{(-1)}c_2) \triangleright S(f^{(0)}), Sc(f^{(-2)}c_1), vh) \]

\[ = \langle Sc(f^{(-1)}c_2), v \rangle \xi(Sc(f^{(-2)}c_1), h) \]

\[ = \langle S(f^{(0)}), v \rangle \xi(Sc(f^{(-1)}c_2), v)\xi(Sc(f^{(-2)}c_1), h) \]

\[ = \langle f^{(0)}, S(v^{(0)}) \rangle \xi(f^{(-1)}c_1, S_H(v^{(-1)})\xi(f^{(-1)}c_1, S_H(h)) \]

\[ = \langle f^{(0)}, S(v^{(0)}) \rangle \xi(f^{(-1)}c_1, S_H(v^{(-1)})\xi(c, S_H(v^{(-2)}h)) \]

\[ = \phi(f, S_H(v^{(-1)}h_2) \triangleright S(v^{(0)}), S_H(v^{(-2)}h)) \]

\[ = \phi(f, S_H(v^{(-1)}h_2) \triangleright S(v^{(0)}), S_H(v^{(-2)}h)) \]

\[ = \phi(f, S_H(v^{(-1)}h_2) \triangleright S(v^{(0)}), S_H(v^{(-2)}h)) \]

\[ = \phi(f, S_H(v^{(-1)}h_2) \triangleright S(v^{(0)}), S_H(v^{(-2)}h)) . \]

Theorem A.26. The algebra \( U_{\xi}(V^*, C, V, H) \) is isomorphic to \( D_{\phi_+, \phi_-}(B(V^*) \times C, B(V) \times H) \) where \( \phi_+(f, vh) = \langle f, v \rangle \xi(c, h) \). In particular, for all \( v \in B(V), f \in B(V^*) \) we have in \( U_{\xi}(V^*, C, V, H) \)

\[ v \cdot f = f^{(0)}(2) \xi^{(-1)}(3) \cdot S^{(-1)}(3) \xi^{(-2)}(3) \xi^{(-2)}(3) \xi^{(-1)}(3) \xi^{(-1)}(3) - f^{(0)}(3) \]

(A.28)

Moreover, if \( \langle \cdot, \cdot \rangle_+ = \langle \cdot, \cdot \rangle_- = \langle \cdot, \cdot \rangle \) then \( U_{\xi}(V^*, C, V, H) \) is a Hopf algebra with the comultiplication defined by \( \Delta(f) = f \otimes 1 + f^{(-1)} \otimes f^{(0)}, \Delta(v) = 1 \otimes v + v^{(0)} \otimes v^{(-1)}, \Delta(c) = \Delta(c), \Delta(h) = \Delta_H(h), v \in V, f \in V^*, c \in C, h \in H \).

Proof. Let \( D = D_{\phi_+, \phi_-}(B(V^*) \times C, B(V) \times H) \). Clearly, the subalgebra of \( U := U_{\xi}(V^*, C, V, H) \) generated by \( V^* \) and \( C \) identifies with \( B(V^*) \times C \). Likewise, the subalgebra generated by \( V \) and \( H \) identifies with \( B(V) \times H \) since

\[ (h_1 \triangleright v) \cdot h_2 = h_3 \cdot (h_2 \triangleright v) = h_3 \cdot S_H(h_2)h_2 \cdot v = h \cdot v, \quad h \in H, v \in V. \]

Furthermore, in \( D \) we have, for all \( v \in V, f \in V^*, c \in C \) and \( h \in H \)

\[ v \cdot c = c^{(1)}v^{(2)}\phi_-(c_1, S_H^{-1}(v^{(3)}))\phi_+(c_3, v^{(1)}) = c^{(1)}v^{(0)}\phi_+(c_2, v^{(-1)}) \]

\[ = c^{(1)}v^{(0)}\xi(c_2, v^{(-1)}) = c^{(1)}(v \triangleleft c_2) \]

while

\[ h \cdot f = f^{(2)}h^{(2)}\phi_-(f_1, S_H^{-1}(h^{(3)}))\phi_+(f_3, h^{(1)}) = f^{(0)}h^{(1)}\phi_-(f^{(-1)}, S_H^{-1}(h_2)) = (h_2 \triangleright f) \cdot h_1 \]
and
\[ v \cdot f = f_2 v_2 \phi_- (f_0, S^{-1}(v_3)) \phi_+ (f_3, v_1) = v^{(-1)} \phi_- (f, S^{-1}(v_0)) + f \cdot v + f^{(-1)} \phi_+ (f^{(0)}, v) \]
\[ = f \cdot v + f^{(-1)} (f^{(0)}, v)_+ - v^{(-1)} (f, v^{(0)})_-. \]

Thus, all relations between generators of \( D \) hold in \( U \), hence we have a homomorphism of algebras \( D \to U \), which is clearly an isomorphism of vector spaces.

It remains to prove (A.28). Observe that Lemma A.17 implies that
\[ (\Delta \otimes 1) \Delta (v) = \Delta (\varphi_1 (u_2^{(-1)}) \otimes u_2^{(0)}) = \varphi_1 (u_2^{(0)} \otimes u_2^{(1)} \otimes u_2^{(-2)} \otimes u_2^{(3)}) = v_1 \otimes v_2 \otimes v_3 \]
and similarly for \( (\Delta \otimes 1) \Delta (f) \). Then by (A.13) and Lemma A.18 we have
\[ v \cdot f = f_2 v_2 \phi_- (f_0, S^{-1}(v_3)) \phi_+ (f_3, v_1) \]
\[ = f_2 f_3 v_2 (u_2^{(0)} \otimes u_2^{(-1)} \otimes u_2^{(1)} \otimes u_2^{(2)}) S^{-1}(u_2^{(3)}) \phi_+ (f_3, v_1) \]

The identity (A.28) can be also written in the following form
\[ v \cdot f = \begin{pmatrix} f_0^{(0)} & f_0^{(-1)} & f_0^{(3)} \end{pmatrix} \cdot \begin{pmatrix} u_2^{(0)} & u_2^{(-2)} & u_2^{(1)} \end{pmatrix} \cdot \begin{pmatrix} \Psi \end{pmatrix} \]
whence the Harish-Chandra projection for \( U \) is given by
\[ HC(f, v) = \begin{pmatrix} f_0^{(0)} & f_0^{(-1)} & f_0^{(3)} \end{pmatrix} \cdot \begin{pmatrix} u_2^{(0)} & u_2^{(-2)} & u_2^{(1)} \end{pmatrix} \cdot \begin{pmatrix} \Psi \end{pmatrix} \]

Note that if \( \langle \cdot, \cdot \rangle_- = 0 \) on \( V^* \otimes V \), we obtain
\[ v \circ f = f_1 f_2 \Psi \begin{pmatrix} u_2^{(0)} & u_2^{(-2)} & u_2^{(1)} \end{pmatrix} \begin{pmatrix} \Psi \end{pmatrix} \]
where \( \beta_+ : \mathcal{B}(V^*) \otimes \mathcal{B}(V) \to C \) is defined by
\[ \beta_+ (f, v) = f^{(-1)} (f^{(0)}, v)_+, \quad v \in B(V), f \in B(V^*). \]

We denote the corresponding braided double \( \mathcal{B}(V^*) \times C \times \mathcal{B}(V) \) by \( \mathcal{H}_+(V^*, C, V) \). Similarly, if \( \langle \cdot, \cdot \rangle_+ = 0 \) on \( V^* \otimes V \) we have
\[ v \circ f = f_1 f_2 \Psi \begin{pmatrix} u_2^{(0)} & u_2^{(-2)} & u_2^{(1)} \end{pmatrix} \begin{pmatrix} \Psi \end{pmatrix} \]

where \( \beta_- : \mathcal{B}(V^*) \otimes \mathcal{B}(V) \to H \) is defined by
\[ \beta_- (f, v) = \langle f, v \rangle_- v^{(-1)}, \quad v \in B(V), f \in B(V^*). \]

The corresponding braided double is denoted \( \mathcal{H}_-(V^*, H, V). \) Clearly, \( \mathcal{H}_+(V^*, C, V) \) naturally identify with subspaces of \( U \). Using \( \beta_\pm \) we can write
\[ HC(f, v) = \beta_+ (f_0^{(0)}, u_2^{(1)}) (S^{-1}(f_0^{(1)})) f_2 \Psi \begin{pmatrix} \Psi \end{pmatrix} \]

\[ = \beta_+ (f_0^{(0)}, u_2^{(1)}) (S^{-1}(f_0^{(1)})) f_2 \Psi \begin{pmatrix} \Psi \end{pmatrix} \]

\[ = \beta_+ (f_0^{(0)}, u_2^{(1)}) (S^{-1}(f_0^{(1)})) f_2 \Psi \begin{pmatrix} \Psi \end{pmatrix} \]

\[ = \beta_+ (f_0^{(0)}, u_2^{(1)}) (S^{-1}(f_0^{(1)})) f_2 \Psi \begin{pmatrix} \Psi \end{pmatrix} \]

\[ = \beta_+ (f_0^{(0)}, u_2^{(1)}) (S^{-1}(f_0^{(1)})) f_2 \Psi \begin{pmatrix} \Psi \end{pmatrix} \]
Consider also some special cases of \([A.28]\). If \(f \in V^*\) we have
\[
[v, f] = f((-1)(f(0), \mathfrak{u}(1)) + \mathfrak{u}(2) - \mathfrak{u}(1)(f, \mathfrak{u}(0)) - \mathfrak{u}(2)^{-1}), \quad v \in \mathcal{B}(V).
\] (A.29)

Similarly, if \(v \in V\) we have
\[
[v, f] = f((-1)(f(0), v) + f((-2)) \xi f((-1), S_H^{-1}(v(1)))(f(0), v(0)) - \\
= f((-1)(f(0), v) + (v(-1) \triangleright f(0)), v(0)) - \\
= f((-1)(f(0), v) + f((-2)) - v((-1))(f(1), v(0)) - f(2))
\] (A.30)

Let \(\tilde{\varpi}\) be a field involution of \(k\). Suppose that it extends to \(V, V^*, C\) and \(H\) and that \(\xi\) satisfies
\[
\xi_{(\varpi, h)} = \xi(c, S_H^{-1}(h))
\]
and \(c \triangleright f = S_H^{-1}(\varpi) \triangleright f, \ h \triangleright v = S_H^{-1}(h) \triangleright v\). Then \(\tilde{\varpi}\) extends to an anti-linear algebra anti-involution and coalgebra involution of \(\mathcal{D}_\xi(C, H)\). Moreover, we have
\[
\xi(f(-1), S_H^{-1}(h))f(0) = \xi(f(-1), h)f(0) = S_H(h) \triangleright f.
\]

Since \(H^{\text{cop}}\) identifies with a sub-bialgebra of \(\mathcal{D}_\xi(C, H)\), it follows that for all \(x \in \mathcal{D}_\xi(C, H)\) we have \(x \triangleright f = S^{-1}(\varpi) \triangleright f\). Assuming that \(\Psi^*\) is self-transposed, it follows from Lemma \([A.29]\) that \(\tilde{\varpi}\) extends to an anti-linear algebra anti-involution of \(B(V^*) \times \mathcal{D}_\xi(C, H)\). Similarly, \(v \triangleright \varpi = \varpi \triangleright S^{-1}(\varpi)\) for all \(x \in \mathcal{D}_\xi(C, H)\) and \(v \in V\), whence \(\tilde{\varpi}\) extends to an anti-linear algebra anti-involution of \(\mathcal{D}_\xi(C, H) \times B(V)\).

**Proposition A.27.** Suppose that \((\overline{f}, v) = -f(0), v) \in V^*, v \in V\). Then \(\tilde{\varpi}\) extends to an anti-linear algebra anti-involution of \(\mathcal{U}_\xi(V^*, C, V, H)\).

**Proof.** Define \(\tilde{\varpi}\) on \(\mathcal{U}_\xi(V^*, C, V, H)\) by \(\tilde{\varpi} : v \mapsto \varpi \cdot f, x \in \mathcal{D}_\xi(C, H), v \in \mathcal{B}(V), f \in \mathcal{B}(V^*)\). Since the restrictions of \(\tilde{\varpi}\) to \(\mathcal{D}_\xi(C, H) \times B(V)\) and \(B(V^*) \times \mathcal{D}_\xi(C, H)\) are well-defined anti-linear algebra anti-involutions, it remains to prove that \([v, f] = [f, \varpi]\) for all \(v \in V, f \in V^*\). Indeed,
\[
[f, v] = -f(-1)(f(0), v) + v(-1)(f, v(0)) - f(-1)(f(0), v) + v(-1)(f, v(0)) = [v, f].
\]

\(\square\)

### A.8. Diagonal braiding

We now consider an important special case of the constructions discussed above. Let \(\Gamma\) be an abelian monoid and fix its bicharacter \(\chi : \Gamma \times \Gamma \to k^*\). Let \(V = \bigoplus_{\alpha \in \Gamma} V_\alpha\) be a \(\Gamma\)-graded vector space over \(k\). Define a braiding \(\Psi : V \otimes V \to V \otimes V\) by \(\Psi(v \otimes v') = \chi(\alpha, \alpha') v' \otimes v\), where \(v \in V_\alpha, v' \in V_{\alpha'}\). Furthermore, let \(V^* = \bigoplus_{\alpha \in \Gamma} V^*_\alpha\) be another \(\Gamma\)-graded vector space over \(k\) and let \(\langle \cdot, \cdot \rangle : V^* \otimes V \to k\) be any pairing satisfying \(\langle V^*_\alpha, V_{\alpha'} \rangle = 0\) if \(\alpha \neq \alpha' \in \Gamma\). Then \(\langle \cdot, \cdot \rangle\) is non-degenerate provided that its restrictions to \(V^*_\alpha \otimes V_{\alpha'}\) are non-degenerate for all \(\alpha \in \Gamma\). If \(\Psi^*\) is the adjoint of \(\Psi\) with respect to the form \(\langle \cdot, \cdot \rangle\) in the notation of \([A.3]\) then it is easy to see that \(\Psi^*(f \otimes f') = \chi(\alpha', \alpha) f' \otimes f, f \in V_{\alpha'}, f' \in V_{\alpha'}\). Henceforth we will assume that \(\langle \cdot, \cdot \rangle\) is non-degenerate and denote \(\Gamma_0 = \{\alpha \in \Gamma : V_\alpha \neq 0\} = \{\alpha \in \Gamma : V^*_\alpha \neq 0\}\). We will always assume that \(\Gamma\) is generated by \(\Gamma_0\).

The algebras \(T(V), B(V, \Psi), T(V^*)\) and \(B(V^*, \Psi^*)\) are naturally \(\Gamma\)-graded. By abuse of notation we write \(\chi(x, y) = \chi(\deg x, \deg y)\) where \(x, y\) are homogeneous elements of \(T(V), B(V, \Psi)\) or \(T(V^*), B(V^*, \Psi^*)\) and \(\deg x\) denotes the degree of \(x\) with respect to \(\Gamma\). Note that if \(u \in B(V, \Psi)\) is homogeneous and \(\Delta(u) = \mathfrak{u}(1) \otimes \mathfrak{u}(2)\) in Sweedler’s notation then \(\deg u = \deg \mathfrak{u}(1) + \deg \mathfrak{u}(2)\). Furthermore, if \(u, v \in T(V)\) are homogeneous then \(\Psi(u \otimes v) = \chi(u, v) v \otimes u\) hence \(\Delta(uv) = \chi(\mathfrak{u}(2), \mathfrak{u}(1)) \mathfrak{u}(1) \otimes \mathfrak{u}(2) \mathfrak{u}(1) \otimes \mathfrak{u}(2)\).

**Lemma A.28.** For \(f \in B(V, \Psi^*), v \in B(V, \Psi)\) homogeneous, \(\langle f, v \rangle = 0\) unless \(\deg f = \deg v\).
Lemma A.29. For all \( f, g \in \mathcal{B}(V^*, \Psi^*) \) homogeneous
\[
\partial f(uv) = \chi(u, f_{(2)})^{-1} \chi(f_{(1)}, f_{(2)})^{-1} \partial_{L(1)} f(u) \partial_{L(2)} (v),
\]
\[
\partial_{op} f(uv) = \chi(f_{(1)}, v)^{-1} \chi(f_{(1)}, f_{(2)})^{-1} \partial_{op} f(u) \partial_{op} (v),
\]
\[
\partial_{u} (fg) = \chi(\Psi(2), f)^{-1} \chi(\Psi(2), \Psi(1))^{-1} \partial_{\Psi(1)} f \partial_{\Psi(2)} (g),
\]
\[
\partial_{op} (fg) = \chi(g, \Psi(2))^{-1} \chi(\Psi(2), \Psi(1))^{-1} \partial_{op} \Psi(1) f \partial_{op} \Psi(2) (g).
\] (A.31)

In particular, for all \( E_{\alpha} \in V_{\alpha}, F_{\alpha} \in V^*_{\alpha} \)
\[
\partial_{E_{\alpha}} (uv) = \partial_{F_{\alpha}} (uv) = \chi(u, f) \partial_{F_{\alpha}} (v),
\]
\[
\partial_{op} (uv) = \chi(\Psi, f) \partial_{op} (v).
\] (A.32)

Proof. We prove only the first identity; others are proved similarly.
\[
\partial f(uv) = \langle f, \Psi(1) \rangle \langle \Psi(2) \rangle = \chi(\Psi(2), \Psi(1))^{-1} \chi(\Psi(2), \Psi(1))^{-1} \partial_{\Psi(1)} f \partial_{\Psi(2)} (v),
\]
where we used that \( \chi(\Psi(1), x) \chi(\Psi(2), x) = \chi(u, x) \) for all \( x \in \mathcal{B}(V, \Psi) \) and Lemma A.28.

An obvious induction together with (A.6) then implies that
\[
\partial_{E_{\alpha}} (F_{\alpha} r) = \langle F_{\alpha}, E_{\alpha} \rangle [r]_{\alpha} F_{\alpha}^{r-1},
\]
\[
\langle F_{\alpha}, E_{\alpha} \rangle = (\langle F_{\alpha}, E_{\alpha} \rangle)^{r} [r]_{\alpha}!,
\] (A.34)

where \( [r]_{\alpha} = \sum_{j=0}^{r-1} \chi(\alpha, \alpha j) \) and \( [r]_{\alpha}! = \prod_{j=0}^{r} j \).

Clearly, \( \Psi \) is self-transposed provided that \( \chi \) is symmetric, that is \( \chi(\gamma, \gamma') = \chi(\gamma', \gamma) \) for all \( \gamma, \gamma' \in \Gamma \). In that case, if \( V_{\alpha} \) are finite dimensional for all \( \alpha \in \Gamma \), \( \mathcal{B}(V^*, \Psi^*) \) is isomorphic to \( \mathcal{B}(V, \Psi) \) as a braided bialgebra.

If \( \chi \) is symmetric, let \( v = v_1 \cdots v_r \in \mathcal{B}(V, \Psi) \) where \( v_i \in V_{\alpha_i} \) and so \( \alpha_i \in \Gamma \). The definition of the braided antipode (cf. [A.11]) then implies that
\[
\mathcal{S}(v) = \mathcal{S}_{\Psi}(v) = (-1)^{r} \left( \prod_{1 \leq i < j \leq r} \chi(\alpha_i, \alpha_j) \right) v^*.
\]

If \( \alpha_1 + \cdots + \alpha_r = \alpha'_1 + \cdots + \alpha'_s \) with \( \alpha_i, \alpha'_j \in \Gamma \), \( 1 \leq i \leq r, 1 \leq j \leq s \) implies that \( r = s \) (mod 2) and \( \prod_{1 \leq i < j \leq r} \chi(\alpha_i, \alpha_j) = \prod_{1 \leq i < j \leq s} \chi(\alpha'_i, \alpha'_j) \) (which is manifestly the case if \( \Gamma \) is freely generated by \( \Gamma_0 \)) we can define a unique character \( sgn : \Gamma \rightarrow \{ \pm 1 \} \) with \( sgn(\alpha) = -1 \), \( \alpha \in \Gamma \) and a function \( \gamma : \Gamma \rightarrow \mathbb{R}^* \) satisfying \( \gamma(\alpha) = 1 \), \( \alpha \in \Gamma_0 \cup \{ 0 \} \) and
\[
\chi(\alpha, \alpha') = \chi_{\gamma}(\alpha, \alpha') = \frac{\gamma(\alpha + \alpha')}{\gamma(\alpha) \gamma(\alpha')}, \quad \alpha, \alpha' \in \Gamma.
\]

Then for any \( v \in \mathcal{B}(V) \) homogeneous
\[
\mathcal{S}(v) = sgn(v) \gamma(v) v^*,
\] (A.35)

where we abbreviate \( sgn(v) := sgn(\deg v) \) and \( \gamma(v) := \gamma(\deg v) \). We will say that \( \Gamma \) affords a sign character if there exists a character \( sgn : \Gamma \rightarrow \{ \pm 1 \} \) satisfying \( sgn(\alpha) = -1 \), \( \alpha \in \Gamma_0 \).
Suppose that \( \gamma : V \to V \) preserves the \( \Gamma \)-grading. Then the braiding \( \Psi \) is unitary if and only if \( \chi(\alpha,\alpha') = \chi(\alpha',\alpha)^{-1} \). The following is an immediate consequence of (A.35) and Proposition A.7(c,d).

**Proposition A.30.** Suppose that \( \Gamma \) affords a sign character, \( \chi = \chi_\gamma \) with \( \gamma(\alpha) = 1 \), \( \alpha \in \Gamma_0 \cup \{0\} \) and \( \chi(\alpha,\alpha') = \chi(\alpha',\alpha)^{-1} \) for all \( \alpha, \alpha' \in \Gamma_0 \). Assume that the pairing \( \langle \cdot, \cdot \rangle : V^* \otimes V \to \mathbb{k} \) satisfies

\[
\langle F_\alpha, E_\alpha \rangle = - (F_\alpha, E_\alpha), \quad \alpha \in \Gamma_0, \quad E_\alpha \in V_\alpha, \quad F_\alpha \in V_\alpha^*.
\]

Then for all \( f \in T(V^*), u \in T(V) \) or \( f \in B(V^*, \Psi^*), u \in B(V, \Psi) \) homogeneous we have

\[
\langle f, u \rangle = \text{sgn}(u)\gamma(u)^{-1}\langle f, u \rangle.
\]

Suppose that \( \gamma(\alpha) \) is a square in \( \mathbb{k} \) for all \( \alpha \in \Gamma \) and fix \( \gamma_\frac{1}{2} : \Gamma \to \mathbb{k}^\times \). Set \( \gamma_\frac{1}{2} = \chi_\gamma \). The operator \( L_n : V \otimes \cdots \otimes V \) defined on \( u \in V \otimes \cdots \otimes V \) homogeneous by \( L_n(u) = \gamma(u)^{\frac{1}{2}}u \) clearly satisfies \( L_n(u) = (-1)^n S_\Phi(u^*), \) commutes with * and is unitary with respect to \( \gamma \). The following is straightforward corollary of Lemma A.8.

**Corollary A.31.** In the assumptions of Proposition A.30, the form \( \langle \cdot, \cdot \rangle : B(V^*, \Psi^*) \otimes B(V, \Psi) \to \mathbb{k} \) is defined for \( u \in B(V, \Psi) \) homogeneous and \( f \in B(V^*, \Psi^*) \) by \( \langle f, u \rangle = (\gamma^\frac{1}{2}(u))^{-1}\langle f, u \rangle \) and satisfies

\[
\langle f, u \rangle = \text{sgn}(u)\langle f, u \rangle
\]

and for all \( f, f' \in B(V^*, \Psi^*), u, u' \in B(V, \Psi) \) homogeneous

\[
\langle f f', u \rangle = (\gamma^\frac{1}{2}(f, f'))^{-1}(f, u_{(1)})(f', u_{(2)}), \quad \langle f, uu' \rangle = (\gamma^\frac{1}{2}(u, u'))^{-1}(f_{(1)}, u)(f_{(2)}, u').
\]

### A.9. Drinfeld double in the diagonal case

Let \( H = \mathbb{k}[\Gamma \oplus \Gamma] \cong \mathbb{k}[\Gamma] \otimes \mathbb{k}[\Gamma] \) be the monoidal bialgebra of \( \Gamma \oplus \Gamma \) with a basis \( K_{\alpha,\alpha'}, \alpha, \alpha' \in \Gamma \). Denote by \( H^+ \) (respectively, \( H^- \)) the subalgebra of \( H \) generated by the \( K_{0,\alpha} \) (respectively, \( K_{\alpha,0} \)), \( \alpha \in \Gamma \); clearly, \( H^\pm = \mathbb{k}[\Gamma] \). Let \( \widehat{H} \) (respectively, \( \widehat{H}^\pm \)) be localizations of the corresponding algebras at \( K_{\alpha,\alpha'} \) (respectively, \( K_{0,\alpha}, K_{\alpha,0} \)), \( \alpha, \alpha' \in \Gamma \). Then \( \widehat{H} \) identifies with \( \mathcal{D}_\xi(H^-, \widehat{H}^+) \) where the Hopf pairing \( \xi : \widehat{H}^- \otimes \widehat{H}^+ \to \mathbb{k} \) is defined by \( \xi_K(K_{0,0}, K_{0,\alpha'}) = \chi(\alpha', \alpha) \).

Let \( V, V^* \) be \( \Gamma \)-graded \( \mathbb{k} \)-vector spaces as in §A.8. We regard \( V \) (respectively, \( V^* \)) as left Yetter-Drinfeld \( \widehat{H}^- \) (respectively, \( \widehat{H}^+ \))-module via

\[
K_{0,\alpha} \cdot v = \chi(\alpha, \beta)v, \quad \delta(f) = K_{\alpha,0} \otimes v
\]

\[
K_{\alpha,0} \cdot f = \chi(\beta, \alpha)f, \quad \delta(f) = K_{\alpha,0} \otimes f, \quad \alpha, \beta \in \Gamma, \quad v \in V_\beta, \quad f \in V_\beta^*.
\]

Then by (A.24) we have

\[
v \cdot K_{0,0} = \chi(\alpha, \beta)v, \quad f \cdot K_{0,0} = \chi(\alpha, \beta)f,
\]

and we can regard \( V \) (respectively, \( V^* \)) as a right (respectively, left) Yetter-Drinfeld module over \( \widehat{H} \) as in (A.26), with \( v \cdot K_{\alpha,\alpha'} = \chi(\alpha', \beta)^{-1}\chi(\beta, \alpha)v \) and \( K_{\alpha,\alpha'} \cdot f = \chi(\beta, \alpha)\chi(\alpha', \beta)^{-1}f \).

Let \( \langle \cdot, \cdot \rangle_\pm \) be pairings \( V^* \otimes V \to \mathbb{k} \) satisfying the assumptions of (A.8). Clearly, (A.25) holds. Denote by \( \partial_\Gamma^+, \partial_\Gamma^{\text{top}} : B(V) \to B(V) \) and \( \partial_\Gamma^-, \partial_\Gamma^{\text{top}} : B(V^*) \to B(V^*) \), \( v \in B(V), \ f \in B(V^*) \), the linear operators corresponding to the respective pairings \( \langle \cdot, \cdot \rangle_\pm \), as defined in (A.3) Consider now the algebra \( \mathcal{U}_\chi(V^*, V) \) which is the subalgebra of \( \widehat{U}_\chi(V^*, V) = \mathcal{U}_\xi(V^*, \widehat{H}^-, V, \widehat{H}^+) \) generated by \( V^*, \ V \) and \( H^\pm \). In particular, we have the following cross-relations

\[
K_{\alpha,\alpha'} E_\beta = \chi(\alpha, \beta)^{-1}\chi(\alpha', \beta)E_\beta K_{\alpha,\alpha'}, \quad K_{\alpha,\alpha'} F_\beta = \chi(\beta, \alpha)\chi(\alpha', \beta)^{-1}F_\beta K_{\alpha,\alpha'},
\]

\[
[E_\alpha, F_\beta] = K_{\beta,0} [F_\beta, E_\alpha] - K_{0,\alpha} [F_\beta, E_\alpha]^+, \quad E_\alpha \in V_\alpha, \ F_\beta \in V_\beta^*, \ \alpha, \beta, \gamma \in \Gamma.
\]

If \( \langle \cdot, \cdot \rangle_+ = \langle \cdot, \cdot \rangle_- \) then by Theorem A.26 \( \widehat{U}_\chi(V^*, V) \) is a Hopf algebra with the comultiplication defined by

\[
\Delta(F_\alpha) = F_\alpha \otimes 1 + K_{0,\alpha} \otimes F_\alpha, \quad \Delta(E_\alpha) = 1 \otimes E_\alpha + E_\alpha \otimes K_{0,\alpha}.
\]
and the antipode
\[ S(F_\alpha) = -K_{\alpha,0}^{-1} F_\alpha, \quad S(E_\alpha) = -E_\alpha K_{0,\alpha}^{-1} \] (A.38)
for all \( \alpha \in \Gamma, \ E_\alpha \in V_\alpha \) and \( F_\alpha \in V_\alpha^* \).

**Lemma A.32.** For all \( \alpha \in \Gamma, \ E_\alpha \in V_\alpha, \ F_\alpha \in V_\alpha^* \), \( v \in B(V), \ f \in B(V^*) \) we have in \( \mathcal{U}_\chi(V^*, V) \)
\[ [v, F_\alpha] = K_{\alpha,0} \partial_{F_\alpha}^+ v - \partial_{F_\alpha}^- \partial_{E_\alpha}^+ v K_{0,\alpha}, \quad [E_\alpha, f] = \partial_{E_\alpha}^+ \partial_{E_\alpha}^- f K_{0,\alpha} - K_{0,\alpha} \partial_{E_\alpha}^- f. \] (A.39)

**Proof.** This is immediate from (A.3), (A.29), and the fact that if \( (f, E_\alpha) \neq 0 \) (respectively, \( (F_\alpha, v) \neq 0 \)) then \( \delta(f) = K_{\alpha,0} f \) (respectively, \( \delta(v) = K_{0,\alpha} v \)). \( \square \)

Given \( f \in B(V^*), \ v \in B(V) \) homogeneous, we obtain by (A.28)
\[ v \cdot f = F_{(2)} \partial_{F_{(3)}} \chi(F_{(2)}, \chi^{-1} \chi(F_{(3)}, v) \chi(F_{(3)}, \chi^{-1} (F_{(1)}, \chi^{-1} (F_{(3)}, v) \chi(F_{(1)}, 1)) + \chi(F_{(1)}, 1)) - (F_{(3)}, v)) + \]
(A.40)
Since \( (F_{(1)}, \chi^{-1} (F_{(3)}, v)) = 0 \) unless \( \deg (F_{(3)}) = \deg F_{(1)} \) this can be written in the following form
\[ v \cdot f = (\chi(F_{(2)}, F_{(1)}) \chi(F_{(2)}, F_{(3)}) \chi(F_{(3)}, F_{(1)}))^{-1} (F_{(1)}, \chi^{-1} (F_{(3)}, v) \chi(F_{(1)}, 1)) - (F_{(3)}, v) + \chi F_{(2)} \partial_{F_{(2)}} \chi(F_{(3)}, v) K_{0,\deg F_{(3)}}. \] (A.41)
where, as before, we abbreviate \( \chi(x, y) := \chi(\deg x, \deg y) \). The following Proposition generalizes (14) Proposition 3.1.7 and is an immediate consequence of (A.41) and (A.35).

**Proposition A.33.** Suppose that \( \Gamma \) affords the sign character, \( \chi = \chi_\gamma \) with \( \gamma : \Gamma \to \mathbb{k}^\times \) satisfying \( \gamma(\alpha) = 1, \ \alpha \in \Gamma_0 \cup \{0\} \). Then for all \( f \in B(V^*), \ v \in B(V) \) homogeneous we have in \( \mathcal{U}_\chi(V^*, V) \)
\[ v \cdot f = \chi_\gamma(F_{(2)}, F_{(1)}) \chi(F_{(2)}, F_{(3)}) \chi(F_{(3)}, F_{(1)}) \chi(F_{(1)}, 1))^{-1} (F_{(1)}, \chi^{-1} (F_{(3)}, v) \chi(F_{(1)}, 1)) - (F_{(3)}, v) + \chi F_{(2)} \partial_{F_{(2)}} \chi(F_{(3)}, v) K_{0,\deg F_{(3)}}. \] (A.42)
Suppose now that \( \langle \cdot, \cdot \rangle_+ = 0 \) on \( V^* \otimes V \). Then we obtain for \( v \in B(V), \ f \in B(V^*) \) homogeneous
\[ v \circ_+ f = F_{(1)} \chi \partial_{F_{(2)}} F_{(1)} - (F_{(2)}, v)) + \chi F_{(2)} \partial_{F_{(2)}} \chi(F_{(3)}, v) K_{0,\deg F_{(3)}}. \] (A.43)
We conclude this section with the following Lemma.

**Lemma A.34.** Retain the assumptions of Proposition A.33
(a) If \( \chi(\alpha, \beta) = \chi(\alpha, \beta)^{-1} \) for all \( \alpha, \beta \in \Gamma \) and \( \langle f, v \rangle = -\langle f, v \rangle \), \( f \in V^*, \ v \in V \), then \( \mathcal{U}_\chi(V^*, V) \) admits a unique anti-linear anti-involution extending \( \cdot : V \to V, \ \cdot : V^* \to V^* \) and satisfying \( K_{\alpha,\alpha'} = K_{\alpha',\alpha}, \alpha, \alpha' \in \Gamma \).
(b) Suppose that \( \langle \cdot, \cdot \rangle_+ = \langle \cdot, \cdot \rangle \). Then \( * \) extends to an anti-involution of \( \mathcal{U}_\chi(V^*, V) \) whose restrictions to \( V, V^* \) are the identity maps while \( K_{\alpha,\alpha'} = K_{\alpha',\alpha}, \alpha, \alpha' \in \Gamma \).
(c) Any pair of graded isomorphisms \( \xi : V^* \to V \) and \( \xi : V \to V^* \) satisfying \( \xi \circ \xi = \text{id}_{V^*} \) gives rise to an anti-involution \( \xi \) of \( \mathcal{U}_\chi(V^*, V) \) satisfying \( \xi(K_{\alpha,\alpha'}) = K_{\alpha',\alpha}, \alpha, \alpha' \in \Gamma \). Moreover, if the assumptions of part (a) hold and \( \xi \) commutes with \( \cdot \) then so does \( \xi \).

**Proof.** Part (a) is an immediate consequence of Proposition A.27. Part (b) follows from (A.36). To prove (c), note that \( \xi \) define isomorphisms of braided bialgebras \( \xi : B(V) \to B(V^*) \) (respectively, \( \xi : B(V^*) \to B(V) \)) such that \( \xi \circ \xi = \text{id}_{B(V^*)} \) and \( \xi \circ \xi = \text{id}_{B(V)} \). Define \( \xi : \mathcal{U}_\chi(V^*, V) \to \mathcal{U}_\chi(V^*, V) \) by \( \xi(f) = \xi(f^*), \ f \in B(V^*), \ \xi(v) = \xi(v^*), \ v \in B(V) \) and \( \xi(h) = h, h \in H \).

Given an anti-involution \( \xi \) commuting with \( \cdot \), we can define a pairing \( \langle \cdot, \cdot \rangle : B(V^*) \otimes B(V^*) \to \mathbb{k} \) by
\[ \langle f, g \rangle = \langle f, \xi(g^*) \rangle, \ f, g \in B(V^*). \]
in the above notation and that of Corollary A.31. In particular, we have for \( f \in \mathcal{B}(V^*) \) homogeneous
\[
(f, g) = \text{sgn}(f)(f, g).
\]
(A.43)

Since the braidings \( \Psi \) and \( \Psi^* \) are self-transposed in the sense of \([14, \S 1.2.3]\), \( \langle - , - \rangle \) is symmetric (note that this form is similar to the one defined in \([14, \S 1.2.3]\)).

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