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Computational Lower Bounds for Colourful Simplicial Depth

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COMPUTATIONAL LOWER BOUNDS FOR COLOURFUL SIMPLICIAL DEPTH

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ABSTRACT. The colourful simplicial depth problem in dimension $d$ is to find a configuration of $(d + 1)$ sets of $(d + 1)$ points such that the origin is contained in the convex hull of each set (colour) but contained in a minimal number of colourful simplices generated by taking one point from each set. A construction attaining $d^2 + 1$ simplices is known, and is conjectured to be minimal. This has been confirmed up to $d = 3$, however the best known lower bound for $d \geq 4$ is $\lceil \frac{(d+1)^2}{2} \rceil$. A promising method to improve this lower bound is to look at combinatorial octahedral systems generated by such configurations. The difficulty to employing this approach is handling the many symmetric configurations that arise. We propose a table of invariants which exclude many of partial configurations, and use this to improve the lower bound in dimension 4.

1. Introduction

A colourful configuration is the union of $(d + 1)$ sets, or colours, $S_0, S_1, \ldots, S_d$ of $(d + 1)$ points in $\mathbb{R}^d$. We are interested in the colourful simplices formed by taking the convex hull of a set containing one point of each colour. The colourful simplicial depth problem is to find a colourful configuration, with each $S_i$ containing the origin $0$ in the interior of its convex hull, minimizing the number of colourful simplices containing $0$. We denote this minimum by $\mu(d)$. Computing $\mu(d)$ can be viewed as refining Bárány’s Colourful Carathéodory Theorem [Bár82] whose original version gives $\mu(d) \geq 1$, and $\mu(d) \geq d + 1$ with a useful modification. The question of computing $\mu(d)$ was studied in [DHST06], which showed $\mu(2) = 5$, that $2d \leq \mu(d) \leq d^2 + 1$ for $d \geq 3$ and that $\mu(d)$ is even when $d$ is odd. The lower bound has since been improved by [BM07], who verified the conjecture for $d = 3$, [ST08] and [DSX11] which includes the current strongest bound of $\mu(d) \geq \lceil \frac{(d+1)^2}{2} \rceil$ for $d \geq 4$.

One motivation for colourful simplicial depth is to establish bounds on ordinary simplicial depth. A point $p \in \mathbb{R}^d$ has simplicial depth $k$ relative to a set $S$ if it is contained in $k$ closed simplices generated by $(d + 1)$ sets of $S$. This was introduced by Liu [Liu90] as a statistical measure of how representative $p$ is of $S$. See [Gro10, MW12, Kar12, KMS12] for recent progress on this problem. We remark also that the colourful simplicial depth of a point is the number of solutions to a colourful linear program in the sense of [BO97] and [DHST08].

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Our strategy, following [CDSX11], is to show that a particular hypergraph whose edges correspond to the colourful simplices containing 0 in a configuration cannot exist.

1.1. Preliminaries. A colourful configuration is a collection of \((d+1)\) sets \(S_0, S_1, \ldots, S_d\) of \((d+1)\) points in \(\mathbb{R}^d\). Without loss of generality we assume that the points in \(S \cup \{0\}\) are in general position. Recall that \(\mu(d)\) represents the minimal number of colourful simplices generated by one point from each \(S_i\) and containing 0 in any \(d\)-dimensional colourful configuration with \(0 \in \cap_{i=1}^{d+1} \text{conv}(S_i)\).

We take the simplices to be closed and remark that the minimum should be attained.

Bárány’s Colourful Carathéodory Theorem [Bár82] gives that some colourful simplex containing 0 exists in a colourful configuration with \(0 \in \cap_{i=1}^{d+1} \text{conv}(S_i)\), and in fact that that every point in \(S\) is part of some such colourful simplex containing 0. This guarantees that \(\mu(d) \geq d + 1\). A colourful configuration of [DHST06] in \(\mathbb{R}^d\) has only \(d^2 + 1\) colourful simplices containing 0, thus \(\mu(d) \leq d^2 + 1\). It is known that \(\mu(1) = 2\) (trivial), \(\mu(2) = 5\) [DHST06], \(\mu(3) = 10\) [BM07] and that \(\mu(d) \geq \lceil (d+1)^2 \rceil\) for \(d \geq 4\) [DSX11].

A colourful configuration defines a \((d+1)\)-uniform hypergraph on \(S = \cup_{i=1}^{d+1} S_i\) by taking edges corresponding to the vertices of 0 containing colourful simplices. We will call a hypergraph that arises from a colourful configuration with \(0 \in \cap_{i=1}^{d+1} \text{conv}(S_i)\) a configuration hypergraph. The Colourful Carathéodory Theorem gives that any configuration hypergraph must satisfy:

**Property 1.1.** Every vertex of a configuration hypergraph belongs to at least one of its edges.

Fix a colour \(i\). We call a set \(t\) of \(d\) points that contains exactly one point from each \(S_j\) other than \(S_i\) an \(\hat{i}\)-transversal. That is to say, an \(\hat{i}\)-transversal \(t\) has \(t \cap S_i = \emptyset\) and \(|t \cap S_j| = 1\) for \(i \neq j\). We call any pair of disjoint \(\hat{i}\)-transversals an \(\hat{i}\)-octahedron; these may or may not generate a cross-polytope \((d\)-dimensional octahedron\) in the geometric sense that their convex hull is a cross-polytope with same coloured points never adjacent in the skeleton of the polytope.

A key property of colourful configurations is that for a fixed \(\hat{i}\)-octahedron \(\Omega\), the parity of the number of colourful simplices containing 0 formed using points from \(\Omega\) a point of colour \(i\) does not depend on which point of colour \(i\) is chosen. This is a topological fact that corresponds to the fact that 0 is either “inside” or “outside” the octahedron, see for instance the Octahedron Lemma of [BM07] for a proof. Figure 1 illustrates this in a two dimensional case where 0 is at the centre of a circle that contains points of the three colours.

We carry the definitions of \(\hat{i}\)-transversals and \(\hat{i}\)-octahedra over to the hypergraph setting. Then any hypergraph arising from a colourful configuration must satisfy:

**Property 1.2.** For any octahedron \(\Omega\) of a hypergraph arising from a colourful configuration, the parity of the set of edges using points from \(\Omega\) and a fixed point \(s_i\) for the \(i\)th coordinate is the same for all choices of \(s_i\).

Consider a hypergraph whose vertices are \(S = \cup_{i=0}^{d} S_i\) and whose edges have exactly one element from each set. If the hypergraph satisfies Property 1.2 we call it an octahedral system, if it additionally satisfies Property 1.1 we call it a octahedral system without isolated
vertex. A colourful configuration with $0 \in \cap_{i=1}^{d+1} \text{conv}(S_i)$ and $k$ colourful simplices containing $0$ has a configuration hypergraph that is an octahedral system without isolated vertex with $k$ edges. Then any lower bound for the number of edges in an octahedral system without isolated vertex with $(d+1)$ colours is also a lower bound $\nu(d)$ for $\mu(d)$. It is an interesting question whether there are any octahedral systems without isolated vertex not arising from any colourful configurations, and if not whether $\nu(d) < \mu(d)$ for some $d$. This purely combinatorial approach was originally suggested by Bárány.

2. A table of invariants

Octahedral systems have the advantage of being combinatorial and finite. In principle for any particular $d$ and $k$ we can check if there exists an octahedral system without isolated vertex on $S = \cup_{i=0}^{d} S_i$ with up to $k$ edges by generating all the (finitely many) hypergraphs with up to $k$ edges, each containing one element from each $S_i$ and then testing if they satisfy Properties 1.1 and 1.2. The difficulty, of course, lies in the sheer number of such hypergraphs, and in verifying Property 1.2 efficiently.

We can reduce the search space by exploiting the many combinatorial symmetries in such hypergraphs and considering only configurations that satisfy certain normalizations. However, this alone is not sufficient to improve the known lower bounds even for $d = 4$. We thus turn our attention to how to use Property 1.2 effectively. Since a given configuration has very many octahedra, in fact $(d+1)^{d(d+1)/2}$, we expect that most unstructured hypergraphs fail Property 1.2 for many octahedra.

There are so many octahedra that it would be difficult to verify Property 1.2 explicitly for non-trivial octahedral systems. However, we can often quickly detect violations of Property 1.2 since hypergraphs which are not octahedral systems may fail Property 1.2 for most octahedra. Even more, we will show that certain partial systems cannot satisfy Property 1.2 even with the addition of a number of edges.

2.1. The large table. We begin by fixing an arbitrary colour as colour 0 and an arbitrary 0-transversal. We can label the points in each set from 0 to $d$ and without loss of generality take the transversal to contain the 0 point of each set. For convenience we write edges as a string of $(d+1)$ numbers and transversals as string of $d$ numbers with * corresponding to the omitted colour. Thus the 0-transversal considered is *00…0.

There are $d^d$ possible octahedra formed by choosing a second 0-octahedron disjoint from *00…0. For a hypergraph of colourful edges on $S$, we generate a $d^d \times (d+1)$ table
whose rows are indexed by the octahedra containing \(*00\ldots0\) and whose columns are indexed by the points of colour 0. The entries of the table are the parity of the number of edges using points from the octahedron corresponding to the row and the point of colour 0 corresponding to the column. We call this the large table. We observe that Property 1.2 implies that if the hypergraph is an octahedral system, the rows of the large table must be constant – either all zeros or all ones.

For any hypergraph if we build the large table, by taking the view that the entries in the 0 column are correct we can produce a score for the hypergraph by simply counting the entries in the remaining columns that do not agree with the 0 column entry. So any octahedral system will have a score of 0, while the hypergraph that consists only of the edge 000\ldots0 has the maximum possible score of \(d^{d+1}\). Let \(z(e)\) be the number of zeros in hypergraph edge \(e\). Then we have:

**Lemma 2.1.** Adding or deleting edge \(e\) to an octahedral system changes its score by at most \(d^{\frac{z(e)}{d-1}}\).

**Proof.** Consider first the case where the zero element of \(e\) is \(j \neq 0\). Then the only entries of the large table that change are those in column \(j\). Further, an entry will change if and only if the remaining elements of \(e\) lie in the octahedron corresponding to the row. Then the affected rows are those that whose non-zero transversal agrees with the non-zero elements of \(e\). For zero elements of \(e\) any choice of transversal element is allowed. This gives the requisite number of affected entries.

In the case where the 0th element of \(e\) is 0, elements of column 0 are changed, thus changing the correctness of the remaining \(d\) entries in the affected rows, which themselves do not change. The number of rows affected again depends on the number of zero elements in the remainder of \(e\). This is one less than the total number of zero elements in \(e\), again giving the requisite number of affected entries. \(\square\)

Lemma 2.1 allows us to make the following useful observation. Call an edge \(e\) of a hypergraph isolated if there is no other edge that differs from \(e\) only in a single coordinate. Then:

**Lemma 2.2.** An octahedral system with \(d^2\) or fewer edges must not contain any isolated edges.

**Proof.** Without loss of generality we take the isolated edge to be 000\ldots0 and consider the large Table formed. The hypergraph consisting of the unique edge 000\ldots0 has score \(d^{d+1}\). Now keep track of the score as additional edges are added one at a time. Since 000\ldots0 is isolated, all subsequent edges have at most \(d - 1\) zeros, and thus reduce the score by at most \(d^{d-1}\). So at least \(\frac{d^{d+1}}{d^{d-1}} = d^2\) additional edges are required to reduce the score to zero. The claim follows. \(\square\)

2.2. **The small table.** For some cases that we are interested in it is possible to compute the score for the large table in reasonable time, but for the purpose of quickly showing that a given hypergraph is not an octahedral system (and may be far from one), it is effective to initially focus on a subset of the rows. One such small subset are the \(d\) rows generated
by transversals \( *11\ldots1, *22\ldots2, \ldots, *dd\ldots d \). We call the restriction of the large table to these \( d \) rows the small table. Note that the initial numberings are arbitrary, and the composition of the small table depends on this numbering, which we may fix later as part of our search algorithm.

The advantage of focusing on this \( d \times (d + 1) \) table is that the entries are relatively independent. Only edges of the form \( x00\ldots0 \) can change more than one entry of this table. After accounting for such edges, each entry can be only be affected by the \( 2^d - 1 \) edges that are on the relevant octahedron with the given final coordinate.

3. Enumeration Strategy and Computation

In this section we describe how the small and large tables, combined with symmetry breaking strategies, can be used to search efficiently for small octahedral systems. This strategy was implemented [Xie12] in Python version 2.6 on an AMD Opteron Processor 8356 core (2.3G Hz) and is able to prove that \( \mu(4) \geq 14 \) in about 30 days of CPU time. This improves by 1 the bound of [DSX11].

Our strategy is enumerative: we try to build an octahedral system without isolated vertex by adding one edge at a time. At the start we reduce our choices by using symmetry. We then seek to add edges that are required by the small table. Subsequently we seek to add edges that are required by Lemma 2.2, and only as a last resort do we consider arbitrary edges. As we branch we attempt to quickly identify partial configurations that cannot extend to a sufficiently small octahedral system.

3.1. Initial assumptions. We begin by fixing an initial colour 0 and transversal \( t_0 = *00\ldots0; \) the tables we consider are with respect to \( t_0 \). We then use the results of [DSX11], to break the problem into several cases based on \( l \), the number of edges containing \( t_0 \), \( b \) the number of the small table octahedra that have odd parity, and \( j \) the minimum number of transversals covering any point of colour 0. Recall that the small table octahedra are those formed by \( t_0 \) with each of \( t_i = *ii\ldots i \) for \( i = 1, 2, \ldots, d \).

It is clear that for any octahedral system without isolated vertex and with \( d^2 \) or fewer edges we must have \( 1 \leq l, b, j \leq d \) and that the number of edges is at least \( j(d + 1) \). Further, [DSX11] shows that we must have \( j + b \geq d + 1 \), and that the number of edges must be at least \( (b + l)(d + 1) - 2bl \), as well as at least \( dl + 1 \) assuming that the colour 0 is chosen to minimize \( l \) and that \( l \geq \frac{d+2}{2} \). This last fact allows us to assume that \( l \leq d - 1 \) if we have an octahedral system without isolated system with less than \( d^2 + 1 \) edges.

3.2. Details of enumeration. In the following we describe an enumeration that excludes \( \mu(4) = 13 \). This improves the bound of [DSX11] by 1. To rule out possible octahedral systems without isolated vertex of size 13 (or 14), it is sufficient to consider cases where \( j = 1 \) or \( j = 2 \), which in turn means \( b = 3 \) or \( b = 4 \). In the case \( b = 3 \), we have at least \( 15 - l \) simplices, so \( l = 2 \) or \( l = 3 \), and in the case \( b = 4 \), we have \( 20 - 3l \) so \( l = 3 \); we would need to consider additional cases for \( l \) and \( b \) to rule out 14. In summary, we need to rule out systems where the triple \((l, b, j)\) is one of \((3, 4, 2), (3, 4, 1), (3, 3, 2), (2, 3, 2)\).

By reordering the points of colour 0, we can take the edges \( x0000 \) to be in the system for \( 0 \leq l - 1 \), and not in the system for \( l \leq x \leq 4 \). Consider the small table after including
these edges with $l = 3$, illustrated in Table 1. Now if $b = 4$, then we are requiring that the small table be comprised entirely of 1’s. So in this case the entries in the first 3 columns are correct, while the entries in the last 2 columns are incorrect.

For $(l, b, j) = (3, 4, 2)$ we proceed to enumerate configurations as follows. Since $l = 3$, we include initial edges 00000, 10000, 20000. We then add edges to correct each of the 8 entries of Table 1 which must be fixed to get the correct small table for $b = 4$. As previously remarked, adding any edge not of the form $x0000$ will change only a single entry in the small table. For instance, the entry in the first row and fourth column can be changed only by an edge of the form $3abcd$ where $a, b, c, d \in \{0, 1\}$. Given that that 30000 cannot be added to the configuration without changing $l$, there remain only 15 possible edges that change the entry, and one must be in our configuration. In fact, by reordering the colours we can take it to be one of 31000, 31100, 31110 and 31111.

We could continue to exploit symmetries in this way – for instance depending on which of the previous 4 edges is chosen, the next edge could be one of 4 to 7 edges fixing the next table entry. However, we did not do this so as to avoid extensive case analysis. Instead, we began branching on all 15 possible edges that switch a given table entry until the table is correct and the partial configuration has 11 edges.

Once we reach this stage we introduce two simple predictors that may indicate that the configuration requires several more edges. First, we look for points that are not currently included in any edge. If some colour still has $k$ uncovered points, then we require $k$ additional edges. Second, since any vertex of colour 0 must be covered by at least $j$ edges, we look to see which points of colour 0 are not contained in sufficiently many edges, and get a score $k'$ by totaling the undercounts. At the same time, we may find that all vertices of colour 0 are already covered by more than $j$ edges (especially when $j = 1$), in which case the partial configuration no longer belongs to this subcase and can be excluded. Again, we require $k'$ additional edges. If either $k$ or $k'$ is sufficiently large (in this case 3), then the current partial configuration cannot extend to an octahedral system without isolated vertex with less than 14 edges and is abandoned.

Otherwise, we examine the configuration to see if it has an isolated edge. If it contains an isolated edge $e$, the by Lemma 2.2, if the configuration is to extend to an octahedral system without isolated vertex with less than 17 edges, it must include an edge adjacent to $e$. That is, it must contain $e'$ differing from $e$ only in a single coordinate. There are only 20 such edges so we can branch on them. We then repeat the process of applying predictors and looking for an isolated edge until we either find an octahedral system without isolated vertex with less than 14 edges, or all partial configurations with fewer edges are exhausted.

|   | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 1111| 1 | 1 | 1 | 0 | 0 |
| 2222| 1 | 1 | 1 | 0 | 0 |
| 3333| 1 | 1 | 1 | 0 | 0 |
| 4444| 1 | 1 | 1 | 0 | 0 |

Table 1. The small table with $l = 3$
If we do arrive at a partial configuration with no isolated edges, then as a last resort we may have to branch on all possible edges. However, this happens infrequently enough that the enumeration ends in a reasonable time.

The remaining cases, where \((l, b, j)\) is \((3,4,1), (3,3,2)\) or \((2,3,2)\) are similar. Having exhausted all these cases, we conclude that \(\mu(4) \geq 14\).

4. Conclusions and remarks

Octahedral systems are intriguing combinatorial objects. Using the observation that configuration hypergraphs generate octahedral systems without isolated vertex, we propose a computational approach to establishing lower bounds for colourful simplicial depth. A straightforward implementation of this improves the best known lower for 4-dimensional configurations from \(\mu(4) \geq 13\) to \(\mu(4) \geq 14\).

We can ask several other questions about octahedral systems. We remark that the maximum cardinality octahedral system without isolated vertex is the set of all possible edges; if we have \((d+1)\) sets of cardinality \((d+1)\) it has size \((d+1)^{d+1}\). It is the hypergraph arising from the colourful configuration of points in \(\mathbb{R}^d\) that places the sets \(S_1, \ldots, S_{d+1}\) close to vertices \(v_1, \ldots, v_{d+1}\) respectively of a regular simplex containing \(0\).

**Question 4.1.** Can all octahedral systems without isolated vertex on \((d+1)\) sets of \((d+1)\) points arise from colourful configurations in \(\mathbb{R}^d\)?

We conclude by mentioning that many aspects of colourful simplices are just beginning to be explored. For instance, the combinatorial complexity of a system of colour simplices is analyzed in [ST12]. As far as we know the algorithmic question of computing colourful simplicial depth is untouched, even for \(d = 2\) where several interesting algorithms for computing the monochrome simplicial depth have been developed. See for instance the survey [Alo06].

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References

[Alo06] Greg Aloupis, *Geometric measures of data depth*, Data depth: robust multivariate analysis, computational geometry and applications, DIMACS Ser. Discrete Math. Theoret. Comput. Sci., vol. 72, Amer. Math. Soc., Providence, RI, 2006, pp. 147–158.

[Bár82] Imre Bárány, *A generalization of Carathéodory’s theorem*, Discrete Mathematics **40** (1982), no. 2–3, 141–152.

[BM07] Imre Bárány and Jiří Matoušek, *Quadratically many colorful simplices*, SIAM Journal on Discrete Mathematics **21** (2007), no. 1, 191–198.

[BO97] Imre Bárány and Shmuel Onn, *Colourful linear programming and its relatives*, Math. Oper. Res. **22** (1997), no. 3, 550–567.
[CDSX11] Grant Custard, Antoine Deza, Tamon Stephen, and Feng Xie, Small octahedral systems, Proceedings of the 23rd Annual Canadian Conference on Computational Geometry, Toronto, Ontario, Canada, 2011, pp. 267–272.

[DHST06] Antoine Deza, Sui Huang, Tamon Stephen, and Tamás Terlaky, Colourful simplicial depth, Discrete and Comput. Geom. 35 (2006), no. 4, 597–604.

[DHST08] ———, The colourful feasibility problem, Discrete Appl. Math. 156 (2008), no. 11, 2166–2177.

[DSX11] Antoine Deza, Tamon Stephen, and Feng Xie, More colourful simplices, Discrete Comput. Geom. 45 (2011), no. 2, 272–278.

[Gro10] Mikhail Gromov, Singularities, expanders and topology of maps. part 2: from combinatorics to topology via algebraic isoperimetry, Geom. Funct. Anal. 20 (2010), no. 2, 416–526.

[Kar12] Roman Karasev, A simpler proof of the Boros–Füredi–Bárány–Pach–Gromov theorem, Discrete Comput. Geom. 47 (2012), no. 3, 492–495.

[KMS12] Daniel Král’, Lukáš Mach, and Jean-Sébastien Sereni, A New Lower Bound Based on Gromov’s Method of Selecting Heavily Covered Points, Discrete Comput. Geom. 48 (2012), no. 2, 487–498.

[Liu90] Regina Y. Liu, On a notion of data depth based on random simplices, Ann. Statist. 18 (1990), no. 1, 405–414.

[MW12] Jiří Matoušek and Uli Wagner, On Gromov’s method of selecting heavily covered points, arXiv:1102.3515, 2012.

[ST08] Tamon Stephen and Hugh Thomas, A quadratic lower bound for colourful simplicial depth, J. Comb. Opt. 16 (2008), no. 4, 324–327.

[ST12] André Schulz and Csaba D. Tóth, The union of colorful simplices spanned by a colored point set, Computational Geometry (2012), to appear.

[Xie12] Feng Xie, Python code for octahedral system computation, available at: http://optlab.mcmaster.ca/om/csd/, 2012.

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