Stability enhancement of high Prandtl number chaotic convection in an anisotropic porous layer with feedback control

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Abstract. The chaotic dynamical behaviour of thermal convection in an anisotropic porous layer subject to gravity, heated from below and cooled from above, is studied based on theory of dynamical system in the presence of feedback control. The extended Darcy model, which includes the time derivative has been employed in the momentum equation to derive a low dimensional Lorenz-like equation by using Galerkin-truncated approximation. The classical fourth-order Runge-Kutta method is used to obtain the numerical solution in order to exemplify the dynamics of the nonlinear autonomous system. The results show that stability enhancement of chaotic convection is feasible via feedback control.

1. Introduction
Chaotic convection of fluid in a saturated porous medium has been studied intensively for its significance in industrial applications. Most of the systems in engineering industry exhibit some kind of chaotic behaviour such as stabilization in semiconductor lasers, turbulent flows of fluids in chemical reactors and mechanical vibrations on aircraft structure. The effect of anisotropy in permeability and thermal conductivity on convective flow in a rectangular porous layer is studied and resulted in Nusselt number decreases continuously as permeability increases [1]. Next, the natural convection with opposing horizontal heat and solute gradients in an anisotropic porous cavity has been explored using the Darcy model [2]. The linear and nonlinear thermal instability in an anisotropic saturated porous layer with internal heat source is explored and found that Rayleigh number increases as the anisotropic parameters increase thus stabilize the system [3]. Moreover, it was reported that the mechanical anisotropy parameter actually advanced the convection which contradict to the thermal anisotropy parameter as it delayed the convection [4]. Later, the proportional relationships between scaled Rayleigh number and scaled mechanical and thermal anisotropic parameters have been presented [5].

Several researchers have realized that chaos can literally be advantageous in many situations and as it inevitably presents, it can frequently be managed to acquire desired results. The first model-free chaos control, known as OGY (Ott-Grebogi-Yorke) method, forces some trajectory to reach a particular unstable periodic orbit via the linearization of a Poincaré map [6]. Subsequently, a time-delayed feedback control has been introduced and it could be applied in order to stabilize the chaotic system [7],[8]. Also, it was found that the Lorenz chaos could be
stabilized through the use of a controller with input saturation, even when the system external distraction exists [9]. Later, a robust controller has been proposed to stabilize dynamical system of unstable periodic orbits and chaos control can be performed even in conditions where high uncertainties are implicated [10]. Besides, it was discovered that amount of controller gain is proportional to scaled Rayleigh number after applying the feedback control strategy [11]–[13]. Recently, the effects of feedback control on chaotic and periodic dynamics for small and moderate Prandtl number has been studied in a porous layer subject to vibration [14].

In the present study, the work of [5] on the chaotic nature of anisotropic parameters is developed further to include consideration of the influence of feedback control. Truncated Galerkin approximation was applied to the governing equations for thermal convection in an anisotropic porous cavity subject to gravity and heated from below for high Prandtl number in order to arrive at a low dimensional model.

2. Mathematical structure of the problem

We consider an anisotropic square cavity in horizontal porous-saturated layer of depth $H_*$ and width $L_*$ with stress free boundaries, which is heated from below and cooled from above as presented in Fig. 1. A Cartesian co-ordinate system is used such that the vertical axis $z$ is collinear with gravity, i.e. $e_g = e_z$. The actuators are placed at the bottom heated surface and the sensors placed at the top plate of the fluid-saturated porous layer are used to detect the departure of the surface temperature from its conductive state as inspired from the proportional feedback control [15]. The determination of a control, $\hat{q}(t)$, can be performed using the proportional-integral-differential (PID) controller [16].

$$\hat{q}(t) = r + K[e(t)], \quad \epsilon(t) = \epsilon(t) = \hat{m}(t) - m(t),$$

where $r$ is the calibration of the control, $e(t)$ an error or deviation from the state measurement, $\epsilon(t)$ from some desired or reference value, $m(t)$, $K = C_p + C_D/ + C_I \int_0^t$ with $C_p$ as the proportional gain, $C_D$ differential gain, and $C_I$ integral gain. Based on (1), for one sensor plane and proportional feedback control, the actuator rectifies the heated surface temperature using a proportional relationship between the upper ($z = z_u$) and lower ($z = z_l$) thermal boundaries for the perturbation field, thus:

$$\theta(x, z_l, t) - \theta(x, z_u) = 0.5 + K[\theta(x, z_u, t) - \theta(x, z_u)],$$

or, equivalently,

$$\theta'(x, z_l, t) = 0.5 + K\theta'(x, z_u, t),$$

where $\theta'$ denotes the deviation of the fluid’s temperature from its conductive value and $K$ is the scalar gain controller.

The extended-Darcy equations that comprise the time derivative term is engaged for the momentum equation. The governing equations are given by

$$\nabla \cdot \vec{q} = 0,$$

$$\frac{1}{\varepsilon} \frac{\partial \vec{q}}{\partial t} = -\frac{1}{\rho_0} \nabla p + \frac{\rho}{\rho_0} \vec{g} - \frac{\nu}{P_p} \vec{q},$$

$$\frac{\partial T}{\partial t} + (\vec{q} \cdot \nabla) T = \nabla \cdot (\kappa \nabla \cdot T)$$

By applying the Boussinesq approximation to the equation of state, we obtain

$$\rho = \rho_0 [1 - \alpha_T (T - T_0)]$$
where \( \vec{q} \) is the velocity of fluid in porous medium, \( \varepsilon \) the porosity, \( P \) the permeability of the porous domain, \( \kappa \) the thermal diffusivity, \( T \) the temperature, \( \vec{g} \) the gravitational acceleration, \( \nu \) the kinematic viscosity. The height of the layer \( H_* \) was used for scaling the variables \( x, y, z \) and \( H^2/\kappa \) for scaling the time \( t_* \). Accordingly, \( x = x_*/H_* \), \( y = y_*/H_* \), \( z = z_*/H_* \) and \( t = t_*/\kappa/\frac{H^2}{\kappa} \). The time derivative term was included in Darcy’s equation (5).

The governing equations can be presented in terms of a stream function defined by \( u = \frac{\partial \psi}{\partial z} \) and \( w = -\frac{\partial \psi}{\partial x} \), as for convective rolls having axes parallel to the shorter dimension (i.e. \( y \)) when \( v = 0 \). Applying the curl (\( \nabla \times \)) operator on Eq. (5) yields the following PDEs:

\[
\begin{align*}
\left[ \frac{1}{V_\alpha} \frac{\partial}{\partial t} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right) + \left( \frac{\partial^2}{\partial x^2} + \frac{1}{\xi} \frac{\partial^2}{\partial z^2} \right) \right] \psi + RaT \frac{\partial T}{\partial x} &= 0, \\
\frac{\partial}{\partial t} - \eta \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial z^2} T &= -\frac{\partial \psi}{\partial x} + \frac{\partial (\psi, T)}{\partial (x, z)},
\end{align*}
\]

(8)

(9)

The set of partial differential Eqs. (8) and (9) form a nonlinear coupled system with the boundary conditions: \( T(1) = -0.5, T(0) = 0.5 + KT(1) \) and \( \partial T/\partial x = 0 \) at \( x = 0 \) and \( x = Ar \) will accept a basic motionless conduction solution and has been solved by using Galerkin method.

3. Method of solution

In order to obtain the solution of the nonlinear coupled system of partial differential Eqs. (8) and (9), we represent the stream function and temperature in the form

\[
\begin{align*}
\psi &= A_{11} \sin \left( \frac{\pi x}{Ar} \right) \sin(\pi z), \\
T &= \frac{1}{2} \left[ 1 - K - (2 - K)z \right] + B_{11} \cos \left( \frac{\pi x}{Ar} \right) \sin(\pi z) + B_{02} \sin(2\pi z).
\end{align*}
\]

(10)

(11)

Substituting Eqs. (10) and (11) into Eqs. (8) and (9), multiplying the equations by the orthogonal eigenfunctions corresponding to Eqs. (10) and (11) and integrating them yields a set of ODEs for the time evolution of the amplitudes:

\[
\begin{align*}
\frac{\partial A_{11}}{\partial \tau} &= -\frac{V_\alpha \gamma}{\pi^2} \left[ A_{11} + Ra \frac{\pi \theta}{\theta^3} B_{11} \right], \\
\frac{\partial B_{11}}{\partial \tau} &= -B_{11} - \frac{(2 - K)}{2\pi \theta} A_{11} - \frac{1}{\theta} A_{11} B_{02}, \\
\frac{\partial B_{02}}{\partial \tau} &= \frac{1}{2\theta} A_{11} B_{11} - 4\gamma B_{02},
\end{align*}
\]

(12)

(13)

(14)

Rescaling the amplitudes with respect to their convective fixed points provided the following set of scaled equations which are equivalent to Eqs. (12)–(14):

\[
\begin{align*}
\dot{X} &= \alpha \zeta (Y - X), \\
\dot{Y} &= \frac{1}{\zeta} \left[ \frac{1}{2} (2 - K) RX - \zeta XY - \left( \frac{1}{2} (2 - K) R - \zeta X \right) XZ \right], \\
\dot{Z} &= 4\gamma (XY - Z),
\end{align*}
\]

(15)

(16)

(17)

where the dots (\( \cdot \)) denote the time derivative \( d(\cdot)/d\tau \). Eqs. (15)–(17) are equivalent to the Lorenz equations (see [17] and [18]), although with different coefficients. Setting \( K = 0 \), we recover the uncontrolled Lorenz-like system [5].
4. Results and discussion
Maple's built-in 4th-order Runge-Kutta method on double precision with stepsize 0.001 has been used for the numerical simulations of the system (15)–(17) for the time domain \(0 < t < 210\), unless otherwise specified. In all solutions, we considered the initial conditions \(X(0) = Y(0) = 0.8\) and \(Z(0) = 0.9\) and fixed the parameters \(\gamma = 0.5\), \(\alpha = 500\) \((Va = 9870)\) and taking \(R = 750, 1250\). For \(R = 750\), a low mechanical and thermal anisotropic parameters have been considered, i.e. \(\zeta = 0.8\) and \(\chi = 0.9\), correspondingly. As the Rayleigh number increased i.e. \(R = 1250\), the mechanical and thermal anisotropic parameters are also increased to \(\zeta = \chi = 1.5\).

The phase portrait diagrams depict how feedback control affect the dynamics of the thermal convection for different kinds of scaled Rayleigh number and anisotropy parameters. The sequence of occurrences leading to the stabilization of the homoclinic convective regime is presented in Figs. 2–5 in terms of projections of trajectories corresponding to \(R = 1250\). Fig. 2 shows a homoclinic chaotic regime corresponding to the uncontrolled case, \(K = 0\). It is delayed and a solitary limit cycle (period-1) occured as depicted in Fig. 3 at \(K = 0.2125\). As soon as the value of \(K\) is slightly higher, \(K = 0.2147\), we observe that a concave periodic explosion takes over as illustrated in Fig. 4. We rendered our computations up to \(K = 0.25\) and found that a steady bifurcation converging toward the fixed point occurs as shown in Fig. 5.

Periodic results are presented in Figs. 6(a)–(h) subject to time-domain \(200 < t < 400\) as the outcome of the solution obtained at different proportional controller gain values for \(R = 750\). A period-2 periodic solution appears as observed in Figs. 6(a)–(d) corresponding to \(K = 0\). Increasing the proportional controller gain to \(K = 0.3\) yields the results presented in Figs. 6(e)–(h) identifying a period-1 periodic solution. This period-halving bifurcation sequence leading to the stabilization of the periodic bifurcations as presented in Figs. 6(i)–(l) when \(K = 1.155\).
Figure 2. Phase trajectories and time domain diagram show a homoclinic bifurcation at \( K = 0 \) with parameters \( R = 1250, \alpha = 500, \gamma = 0.5, \zeta = \chi = 1.5 \), with \( 0 < t < 210 \).

Figure 3. Phase trajectories and time domain diagram show a limit cycle at \( K = 0.2125 \) with parameters \( R = 1250, \alpha = 500, \gamma = 0.5, \zeta = \chi = 1.5 \), with \( 200 < t < 400 \).
Figure 4. Phase trajectories and time domain diagram show a periodic bifurcation at $K = 0.2147$ with parameters $R = 1250, \alpha = 500, \gamma = 0.5, \zeta = \chi = 1.5$, with $0 < t < 210$.

Figure 5. Phase trajectories and time domain diagram show a steady bifurcation at $K = 0.25$ with parameters $R = 1250, \alpha = 500, \gamma = 0.5, \zeta = \chi = 1.5$, with $0 < t < 210$. 
Figure 6. Projections of phase portraits and time domain diagram for parameters $R = 750$, $\alpha = 500$, $\gamma = 0.5$, $\zeta = 0.8$, $\chi = 0.9$, with $200 < t < 400$. Phase trajectories show Figs. 6(a)–(d) a period-2 at $K = 0$; Figs. 6(e)–(h) a period-1 at $K = 0.3$; Figs. 6(i)–(l) a steady solution at $K = 1.155$. 
5. Conclusions
In this work, by using phase portrait and time domain diagrams the following findings are achieved.

(i) The proportional controller gain $K$ has tendency to inhibit the homoclinic chaos and periodic behaviour in the system.

(ii) The structural variation in transition from homoclinic chaotic convection to steady state via a solitary limit cycle is observed under the influence of various values of the controller gain $K$ when $R = 1250$, $\alpha = 500$, $\gamma = 0.5$, $\zeta = 1.5$, $\chi = 1.5$.

(iii) A sequence of period halving followed by the convergence to the final steady-state have been discovered as the proportional controller gain $K$ increased in the range $0 \leq K \leq 1.155$, at the fixed values of $R = 750$, $\alpha = 500$, and $\gamma = 0.5$ for low anisotropic parameters.

(iv) There is proportional relation between the amount of feedback control and scaled Rayleigh number.

(v) The results indicate that feedback control has stabilizing character according to the significantly slightly higher controller gain $K$ than in an uncontrolled case and will confer a tremendous advantage for controlling homoclinic chaos and periodic dynamics in many industrial applications.

6. References
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