A NEW CHARACTERIZATION OF COMMUTATIVE ARTINIAN RINGS

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Abstract. Let $R$ be a commutative Noetherian ring. It is shown that $R$ is Artinian if and only if every $R$-module is good, if and only if every $R$-module is representable. As a result, it follows that every nonzero submodule of any representable $R$-module is representable if and only if $R$ is Artinian. This provides an answer to a question which is investigated in [1].

1. Introduction

All rings considered in this paper are assumed to be commutative with identity. There are several characterizations of Artinian rings. In particular, it is known that a Noetherian ring $R$ is Artinian if and only if every prime ideal of $R$ is maximal. In this article, we present a new characterization of Artinian rings according to the notions of primary decomposition and (its dual) secondary representation. To do so, we need to introduce a generalization of the notions Hopficity and co-Hopficity.

In [2] V. A. Hiremath introduced the concept of Hopficity for $R$-modules. The dual notion is defined by K. Varadarajan [6]. An $R$-module $M$ is said to be Hopfian (resp. co-Hopfian) if any surjective (resp. injective) $R$-homomorphism is automatically an isomorphism. We refer the reader to [6] for reviewing the most important properties of Hopfian and co-Hopfian $R$-modules. We extend these definitions as follows: An $R$-module $M$ is said to be semi Hopfian (resp. semi co-Hopfian) if for any $x \in R$, the endomorphism of $M$ induced by multiplication by $x$ is an isomorphism, provided it is surjective (resp. injective). Clearly any Hopfian (resp. co-Hopfian) $R$-module is semi-Hopfian (resp. semi co-Hopfian). Also, it is obvious that $R$, as an $R$-module, is Hopfian (resp. co-Hopfian) if and only if it is semi Hopfian (resp. semi co-Hopfian).
As the main result of this note, we establish the following characterization of Artinian rings.

**Theorem 1.1.** Let \( R \) be a commutative Noetherian ring. Then the following are equivalent:

i) \( R \) is Artinian.

ii) Every nonzero \( R \)-module is good.

iii) Every \( R \)-module is semi Hopfian.

iv) Every nonzero \( R \)-module is representable.

iv') Every nonzero Noetherian \( R \)-module is representable.

v) Every \( R \)-module is semi co-Hopfian.

v') Every Noetherian \( R \)-module is semi co-Hopfian.

v") Every Noetherian \( R \)-module is co-Hopfian.

vi) Every nonzero \( R \)-module is Laskerian.

In [1] the following question was investigated: When are submodules of representable \( R \)-modules representable? In that paper [1, Theorem 2.3], it is shown that this is the case, when \( R \) is Von Neumann regular. For a Noetherian ring \( R \), we prove that every nonzero submodules of any representable \( R \)-modules is representable if and only if \( R \) is Artinian (see 2.4).

## 2. The proof of the main theorem

Recall that a nonzero \( R \)-module \( M \) is called *good*, if its zero submodule possesses a primary decomposition. A nonzero \( R \)-module \( S \) is said to be *secondary*, if for any \( x \in R \), the map induced by multiplication by \( x \) is either surjective or nilpotent. We say the \( R \)-module \( M \) is *representable*, if there are secondary submodules \( S_1, S_2, \ldots, S_k \) of \( M \) such that \( M = S_1 + S_2 + \cdots + S_k \). The two notions primary decomposition and secondary representation are dual concepts. We refer the reader to [3, Appendix to §6], for more details about secondary representation. Also, recall that an \( R \)-module \( M \) is said to be *Laskerian*, if any submodule of \( M \) is an intersection of a finite number of primary submodules.

**Lemma 2.1.** i) Every finitely generated \( R \)-module is Hopfian.

ii) Every Artinian \( R \)-module is co-Hopfian.

iii) Every good \( R \)-module is semi Hopfian.

iv) Every representable \( R \)-module is semi co-Hopfian.
Proof. i) See [7, Proposition 1.2].

ii) is well known and can be checked easily.

iii) Let \( x \in R \) be an \( M \)-coregular element of \( R \). Let \( 0 = \bigcap_{i=1}^n Q_i \) be a primary decomposition of the zero submodule of \( M \). Fix \( 1 \leq i \leq n \). Since \( Q_i \) is a proper submodule of \( M \) and \( \frac{M}{Q_i} \xrightarrow{x} \frac{M}{Q_i} \) is either injective or nilpotent, it follows that \( x \) is \( \frac{M}{Q_i} \)-regular. Now, if \( xm = 0 \) for some element \( m \) in \( M \), then for each \( i \), it follows that \( xm \in Q_i \) and so \( m \in Q_i \). Hence \( m = 0 \), and so \( x \) in \( M \)-regular as required.

iv) is similar to (iii). □

Example 2.2. Let \( N \) be a nonzero co-Hopfian \( R \)-module. Set \( M = \bigoplus_{i \in \mathbb{N}} N \). Then \( M \) is semi co-Hopfian, but it is not co-Hopfian. To this end define the \( R \)-homomorphism \( \psi : M \to M \) by \( \psi(m_1, m_2, \ldots) = (0, m_1, m_2, \ldots) \) for all \( (m_1, m_2, \ldots) \in M \). Then \( \psi \) is injective, while it is not surjective.

Proof of theorem 1.1. (i) \( \Rightarrow \) (ii) Let \( M \) be an \( R \)-module. Since \( R \) is Artinian, it is representable as an \( R \)-module. Hence \( M \simeq \text{Hom}_R(R, M) \) is good, by [4, Theorem 2.8]. The implications (ii) \( \Rightarrow \) (iii), (iv) \( \Rightarrow \) (v) and (iv)' \( \Rightarrow \) (v)' follow, by 2.1.

Now we prove (iii) \( \Rightarrow \) (v). Let \( M \) be an \( R \)-module and \( D(\cdot) = \text{Hom}_R(\cdot, E) \), where \( E \) is an injective cogenerator of \( R \). Let \( x \in R \) be such that the map \( M \xrightarrow{x} M \) is injective. Then the map \( D(M) \xrightarrow{x} D(M) \) is surjective and it is also injective, because \( D(M) \) is semi Hopfian. But this implies that \( x \) is \( M \)-coregular, as the functor \( D(\cdot) \) is faithfully exact. Hence \( M \) is semi co-Hopfian.

(v)' \( \Rightarrow \) (i) Suppose the contrary and assume that \( p \subset m \) is a strict containment of prime ideals of \( R \). Let \( x \in m \setminus p \). Then \( x \) is \( R/p \)-regular, but it is not \( R/p \)-coregular. We achieved at a contradiction. Therefore every prime ideal of \( R \) is maximal and so \( R \) is Artinian.

Next, we prove (i) \( \Rightarrow \) (iv). Since \( R \) is Artinian, \( \text{Max}(R) \) is finite. Let \( \text{Max}(R) = \{m_1, \ldots, m_k\} \). There are \( m_i \)-primary ideals \( a_i \) of \( R \) such that \( R \simeq \prod_{i=1}^k R/a_i \). Let \( F = \bigoplus_{j \in J} R \) be an arbitrary free \( R \)-module. Set \( S_i = \bigoplus_{j \in J} R/a_i \) for \( i = 1, 2, \ldots, k \). Then

\[
F \simeq \bigoplus_{j \in J} (\prod_{i=1}^k R/a_i) \simeq \prod_{i=1}^k S_i = \bigoplus_{i=1}^k S_i.
\]

It follows that for each \( i = 1, 2, \ldots, k \), the \( R \)-module \( S_i \) is \( m_i \)-secondary and hence \( F \) is representable. But any \( R \)-module is homeomorphic image of some free \( R \)-module.
and so the conclusion follows. Note that one can check easily that any nonzero quotient of a representable $R$-module is also representable.

It follows from [8, Theorem] that the statements (i) and (v”) are equivalent. Let $N$ be a proper submodule of an $R$-module $M$. Then $N$ possesses a primary decomposition if and only if the $R$-module $M/N$ is good. Thus (ii) and (vi) are equivalent. Now, because the implications (iv) $\Rightarrow$ (iv') and (v) $\Rightarrow$ (v') are clearly hold, the proof is complete. □

**Corollary 2.3.** Let $M$ be an $R$-module such that the ring $R/\text{Ann}_R M$ is Artinian. Then $M$ is both good and representable.

**Proof.** Set $S = R/\text{Ann}_R M$. Then $M$ possesses the structure of an $S$-module in a natural way. A subset $N$ of $M$ is an $R$-submodule of $M$ if and only if it is an $S$-submodule of $M$. Thus it is straightforward to see that $M$ is good (resp. representable) as an $R$-module if and only if it is good (resp. representable) as an $S$-module. Now the conclusion follows by 1.1. □

**Proposition 2.4.** Let $R$ be a Noetherian ring. The following statements are equivalent:

i) Every nonzero submodule of any representable $R$-module is representable.

ii) $R$ is Artinian.

**Proof.** (ii) $\Rightarrow$ (i) is clear by 1.1.

(i) $\Rightarrow$ (ii) By [5], any nonzero injective module over a commutative Noetherian ring is representable. Since any $R$-module can be embedded in an injective $R$-module, it follows that all nonzero $R$-modules are representable. Therefore by the implication (iv) $\Rightarrow$ (i) of 1.1, it follows that $R$ is Artinian. □

A commutative ring $R$ is said to be Von Neumann regular, if for each element $a \in R$, there exists $b \in R$ such that $a = a^2 b$. In [1, Theorem 2.3], it is shown that over a commutative Von Neumann regular ring $R$ every nonzero submodule of a representable $R$-module is representable. Since commutative Artinian rings are Noetherian, we can deduce the following result, by 2.4.

**Corollary 2.5.** Let $R$ be a commutative Von Neumann regular ring. Then $R$ is Noetherian if and only if it is Artinian.
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