Vector and Axial Nucleon Form Factors:

A Duality Constrained Parameterization

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Abstract. We present new parameterizations of vector and axial nucleon form factors. We maintain an excellent description of the form factors at low momentum transfers, where the spatial structure of the nucleon is important, and use the Nachtmann scaling variable ξ to relate elastic and inelastic form factors and impose quark-hadron duality constraints at high momentum transfers where the quark structure dominates. We use the new vector form factors to re-extract updated values of the axial form factor from neutrino experiments on deuterium. We obtain an updated world average value from τ, d and pion electroproduction experiments of $M_A = 1.014 \pm 0.014$ GeV/c². Our parameterizations are useful in modeling neutrino interactions at low energies (e.g. for neutrino oscillations experiments). The predictions for high momentum transfers can be tested in the next generation electron and neutrino scattering experiments.

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1 Introduction

The nucleon vector and axial elastic form factors have been measured for more than 50 years in $e^-N$ and $\nu N$ scattering. At low $Q^2$, a reasonable description of the proton and neutron elastic form factors is given by the dipole approximation. The dipole approximation is a lowest-order attempt to incorporate the non-zero size of the proton into the form factors. The approximation assumes that the proton has a simple exponential spatial charge distribution, $\rho(r) = \rho_0 e^{-r/r_0}$, where $r_0$ is the scale of the proton radius. Since the form factors are related in the non-relativistic limit to the Fourier transform of the charge and magnetic moment distribution, the above $\rho(r)$ yields the dipole form defined by: $G^{V,A}_D(Q^2) = C^{V,A} / \left(1 + Q^2/M_0^2\right)^2$. Here $C^{V,A} = (1, g_A)$, $g_A = -1.267$, $M_V = 0.71$ (GeV/c)², and $M_A$ is the axial mass.

Since $M_A$ is not equal to $M_V$, the distribution of electric and axial charge are different. However, the magnetic moment distributions were assumed to have the same spatial dependence as the charge distribution (i.e., form factor scaling). Recent measurements from Jefferson Lab show that the ratio of $\frac{\mu_p}{2\sqrt{2}}$ falls at high $Q^2$ challenging the validity of form factor scaling [1] and resulting in new updated parameterizations of the form factors [2]. In this paper we present parameterizations that simultaneously satisfy constraints at low $Q^2$ where the spatial structure of the nucleon is important, and at high $Q^2$ where the quark structure is important. A violation of form-factor scaling is expected from quark-hadron duality. We use our new vector form factors to re-extract updated values of the axial form factor from a re-analysis of previous neutrino scattering data on deuterium and present a new parameterization for the axial form factor within the framework of quark-hadron duality.

2 New Parametrization

The new parameterizations presented in this paper are referred to as the duality based “BBBA07” parameterization. Our updated parameterizations feature the following: (1) Improved functional form that adds an additional $Q^2$ dependence using the Nachtmann scaling variable ξ to relate elastic and inelastic form factors. For elastic scattering ($x = 1$) $\xi_{p,n,N} = \sqrt{\frac{1+1/\xi_{p,n,N}}{1+1/\xi_{p,n,N}}}$, while satisfying quark-hadron duality constraints at high-$Q^2$. (2) Yield the same values as Arrington and Sick [4] for $Q^2 < 0.64$(GeV/c)², while satisfying quark-hadron duality constraints at high-$Q^2$.

For vector form factors our fit functions are $A_N(\xi)$ (i.e. $A_{Ep}(\xi^p)$, $A_{Mp}(\xi^p)$, $A_{En}(\xi^n)$, $A_{Mn}(\xi^n)$) multiplying an updated Kelly[3] type parameterization of one of the
Fig. 1. Ratios of $G_E$ (a), $G_M/\mu_p$ (b), $G_{En}$ (c) and $G_{Mn}/\mu_n$ (d) to $G_D$. The short-dashed line in each plot is the old Kelly parameterizations (old Galster for $G_{En}$). The solid line is our new BBA07.25 parameterization for $\frac{Q^2}{\mu} = 0.0$, and the long-dashed line is BBA07.43 for $\frac{Q^2}{\mu} = 0.2$. The values of $\xi$ and the corresponding values of $Q^2$ are shown on the bottom and top axis.

| $G_{Ep}^{Kelly}$ | $a_1$ | $b_1$ | $b_2$ | $b_3$ | $\chi^2$/ndf |
|------------------|-------|-------|-------|-------|--------------|
| $G_{Ep}^{Kelly\text{-}upd}$ | -0.24 | 10.98 | 12.82 | 21.97 | 0.78         |
| $G_{Mp}^{Kelly\text{-}upd}$ | 0.1717 | 11.26 | 19.32 | 8.33 | 1.03         |

Table 1. Parameters for $G_{Ep}^{Kelly}$ and $G_{Mp}^{Kelly\text{-}upd}$. Our parameterization employs the as-published Kelly parameterization to $G_{Ep}^{Kelly}$ and an updated set of parameters for $G_{Mp}^{Kelly\text{-}upd}(Q^2)$ that includes the recent BLAST results.

Proton form factors. The Kelly parameterization is:

$$G_{Ep}^{Kelly}(Q^2) = \frac{\sum_{k=0}^{m} a_k r_p^k}{1 + \sum_{k=1}^{m+2} b_k r_p^k},$$

where $a_0 = 1$ and $m = 1$.

In our analysis, we use all the datasets used by Kelly, updated to include the recent BLAST results, to fit $G_{Ep}, G_{En}, G_{Mp}/\mu_p, \text{and } G_{Mn}/\mu_n (\mu_p = 2.7928, \mu_n = -1.930)$. Our parameterization employs the published Kelly functional form to $G_{Ep}^{Kelly}$, and an updated set of parameters for $G_{Mp}^{Kelly\text{-}upd}(Q^2)$. The parameters used for $G_{Ep}^{Kelly}$ and $G_{Mp}^{Kelly\text{-}upd}$ are listed in Table 1 and $A_N(\xi)$ is given by

$$A_N(\xi) = \sum_{j=1}^{n} P_j(\xi)$$

$$P_j(\xi) = p_j \prod_{k=1,k\neq j}^{n} \frac{\xi - \xi_k}{\xi_j - \xi_k}.$$  

Each $P_j$ is a LaGrange polynomial in $\xi$. The $\xi_j$ are equidistant “nodes” on an interval [0, 1], and $p_j$ are the fit parameters that have an additional property $A_N(\xi_j) = p_j$. The functional form $A_N(\xi)$ (for $G_{Ep}, G_{Mp}, G_{En}, \text{and } G_{Mn}$) is used with seven $p_j$ parameters at $\xi_0 = 0, 1/6, 1/3, 1/2, 2/3, 5/6, \text{and } 1.0$. In the fitting procedure described below, the parameters of $A_N(\xi)$ are constrained to give the same vector form factors as the recent low $Q^2$ fit of Arrington and Sick for $Q^2 < 0.64(\text{GeV/c})^2$ (as that analysis includes coulombs corrections which modify $G_{Ep}$, and two photon exchange corrections which modify $G_{Mp}$ and $G_{Mn}$). Since the published form factor data do not have these corrections, this constraint is implemented by including additional “fake” data points for $Q^2 < 0.64(\text{GeV/c})^2$. 

\[ G_{Mp}^2 / \mu_p = A_{Mp}(\xi^p) \times G_D^U(Q^2) \]

\[ G_{Ep}^2 / \mu_p = A_{Ep-dipole}(\xi^p) \times G_D^U(Q^2) \]

\[ G_{En}^2 / \mu_p = A_{En-dipole}(\xi^p) \times G_D^U(Q^2) \]

where we use our updated parameters in the Kelly parametrization and ensure that \( dG_{En} / dQ^2 \) at for \( Q^2 = 0 \) is in agreement with measurements. For convenience, we also provide fits for the form factors \( E_{En} \) and \( A_{En} \) that give very close to the same values, but use the dipole form instead:

\[ G_E(Q^2) = A_{Ep-dipole}(\xi^p) \times G_D^U(Q^2) \]

\[ G_M(Q^2) / \mu_p = A_{Mp-dipole}(\xi^p) \times G_D^U(Q^2) \]

The values \( A(\xi) = p_1 \) at \( \xi = 0 \) (\( Q^2 = 0 \)) for \( G_{Mm}, G_{En}, G_{Mn} \) are set to 1.0. The value \( A(\xi) = p_7 \) at \( \xi = 1 \) \((Q^2 \to \infty)\) for \( G_{Mm} \) and \( G_{Ep} \) is set to 1.0.

The value \( A(\xi) = p_9 \) at \( \xi = 1 \) for \( G_{Mn} \) and \( G_{En} \) are fixed by constraints from quark-hadron duality. Quark-hadron duality implies that the ratio of neutron to proton magnetic form factors should be the same as the ratio of the corresponding inelastic structure functions \( F_{2p} \) in the \( \xi = 1 \) limit. (Here \( F_2 = \sum \xi G_i q_i(\xi) \))

\[ \frac{G_{Mn}^2}{G_{Mm}^2} = \frac{F_{2n}}{F_{2p}} = 1 + \frac{4 a_2}{4 + 4 a_1} = \left( \frac{\mu_n^2}{\mu_p^2} \right) A_{Mn}^2(\xi = 1) \]

We ran fits with two different values of \( a_1 \) at the \( \xi = 1 \) limit: \( a_1 = 0 \) and 0.2 (corresponding to \( F_{2n} / F_{2p} = 0.25 \) and 0.4286). The fit utilizing \( a_1 = 0 \) is \( A_{Mn}^{20} \), and the fit utilizing \( a_1 = 0.2 \) is \( A_{Mn}^{43} \). The final parameters for both cases of \( a_1 \) are given in Table 2 (or download computer code[21]). The difference between these two sets is indicative of the theoretical error of our parametrization. Our parameterization are within the error band of recent theoretical fits based in dispersion relations[9]. Since our fits are constrained to give the same vector form factors as the recent low \( Q^2 \) fit of Arrington and Sick[4] for \( Q^2 < 0.64(\text{GeV}/c)^2 \), they are in agreement with the experimental measurements of the proton and neutron rms radii. (Note that as discussed in reference[10], the nucleon rms radius should be determined from fitting a polynomial of second order to the low \( Q^2 \) form factors. The commonly used polynomial of first order yields radius values which are too small).

The value \( A(\xi) = p_3 \) at \( \xi = 1 \) for \( G_{En} \) is set by another duality-motivated constraint. \( R \) is defined as the ratio of deep-inelastic longitudinal and transverse structure functions. For inelastic scattering, as \( Q^2 \to \infty \), \( R_n = R_p \). If we assume quark-hadron duality, the same should be true for the elastic form factors at \( \xi = 1 \) \((Q^2 \to \infty)\) limit:

\[ R_n \left( x = 1; Q^2 \right) = \frac{4 M_n^2}{Q^2} \frac{G_{En}^2}{G_{Mn}^2} \]

Table 2. Fit parameters for \( A_N(\xi) \), the LaGrange portion of the new parameterization. Note \( A_{Mn}^{20}, A_{En}^{20}, A_{En}^{43} \) are constrained to have \( A_N = 0 \) at \( \xi = 1 \), and \( A_{Mn}^{20}, A_{En}^{43} \), are constrained to have \( A_N = 0.2 \).

| \( \xi, Q^2 \) | \( p_1 \) | \( p_2 \) | \( p_3 \) | \( p_4 \) | \( p_5 \) | \( p_6 \) | \( p_7 \) |
|----------------|-------|-------|-------|-------|-------|-------|-------|
| \( A_{Ep} \)   | 1.0000 | 0.9927 | 0.9898 | 0.9975 | 0.9812 | 0.9340 | 1.0000 |
| \( A_{Mp} \)   | 1.0000 | 0.9927 | 0.9898 | 0.9975 | 0.9812 | 0.9340 | 1.0000 |
| \( A_{Ep-dipole} \) | 1.0000 | 0.9927 | 0.9898 | 0.9975 | 0.9812 | 0.9340 | 1.0000 |
| \( A_{Mp-dipole} \) | 1.0000 | 0.9927 | 0.9898 | 0.9975 | 0.9812 | 0.9340 | 1.0000 |
| \( A_{Mn}^{20} \) | 1.0000 | 0.9927 | 0.9898 | 0.9975 | 0.9812 | 0.9340 | 1.0000 |
| \( A_{Mn}^{43} \) | 1.0000 | 0.9927 | 0.9898 | 0.9975 | 0.9812 | 0.9340 | 1.0000 |
| \( A_{En}^{20} \) | 1.0000 | 0.9927 | 0.9898 | 0.9975 | 0.9812 | 0.9340 | 1.0000 |
| \( A_{En}^{43} \) | 1.0000 | 0.9927 | 0.9898 | 0.9975 | 0.9812 | 0.9340 | 1.0000 |
| \( A_{FA} \)   | 1.0000 | 0.9927 | 0.9898 | 0.9975 | 0.9812 | 0.9340 | 1.0000 |

Fig. 2. The constraint used in fitting \( G_{En} \) stipulates that \( G_{En}^2 / G_{Mm}^2 = G_{Ep}^2 / G_{Mm}^2 \) at high \( \xi \). The solid line is \( G_{Ep} / G_{Mm} \), and the short-dashed line is \( G_{En} / G_{Mm} \).
Table 3. $M_A$ (GeV/$c^2$) values published by $\nu_p$-deuteron experiments [13] and updated corrections $\Delta M_A$ when re-extracted with updated $BBBA2007_{25}$ form factors, and $g_A=1.267$. Also shown is updated $M_A$ from $\pi_p$Hydrogen $\rightarrow \mu^- n$ [14].

| Experiment | $v_{\nu}d\rightarrow\mu^-p$ $p_d$ | QE events | $Q^2$ range (GeV/$c^2$) | $E_\nu$ GeV | $g_A$, $M_A^2$ used | $M_A$ (published) | $\Delta M_A$ FF,RC | $M_A^{\text{updated}}$ (GeV/$c^2$) |
|------------|----------------------------------|------------|--------------------------|-------------|---------------------|------------------|----------------|------------------|
| Mann75     | 166                              | .05 - .16  | 0.7                      | Bartl, $G_{en}=0$ | 1.23, 84$^2$       | .95 $\pm$ .12    | -.026, .002    | 0.972            |
| Barish77   | 500                              | .05 - .16  | 0.7                      | Olsen, $G_{en}=0$ | 1.23, 84$^2$       | .95 $\pm$ .09    | -.030, .002    | 0.972            |
| Miller82,77,73 | 1737                          | .05 - .25  | 0.7                      | Olsen, $G_{en}=0$ | 1.23, 84$^2$       | 1.00 $\pm$ .05    | -.036, .002    | 0.972            |
| Baker81     | 1138                             | .06 - .30  | 1.6                      | Olsen, $G_{en}=0$ | 1.23, 84$^2$       | 1.07 $\pm$ .06    | -.028, .002    | 0.972            |
| Kitagaki80  | 2654                             | .10 - .30  | 1.6                      | Olsen, $G_{en}=0$ | 1.23, 84$^2$       | 1.05 $\pm$ .05    | -.051, .001    | 0.9723           |
| Ollsn,G     | 362                              | 1.1 - .20  | 20                      | Olsen, $G_{en}=0$ | 1.23, 84$^2$       | 1.05 $\pm$ .10    | -.036, .002    | 0.9723           |
| Allasia90   | 552                              | -1.3-75    | 20                      | dipole, $G_{en}=0$ | 1.256, 84$^2$       | 1.080 $\pm$ .08   | -.080, .002    | 0.9723           |

$A_{25,43}^{\text{p}}(\xi=1) = \frac{b}{a} \times \left(\frac{1 + \frac{d}{2a}}{4 + \frac{d}{2a}}\right)^{1/2}$

where $b/a = 1.7/3.3$. As there are two parameter sets $A_{25,43}^{\text{p}}(\xi)$, we have produced two parameter sets $A_{25,43}^{\text{p}}(\xi)$ as shown in Table 2.

The new form factors $G_{Ep}$, $G_{Mp}/\mu_p$, $G_{Mn}/\mu_n$, and $G_{En}$ are plotted in Figure 1 as ratios to the dipole form $G_D$. As seen in Table 2, $A_N(\xi)$ is not needed for $G_{Mp}$ as it is very close to 1.0. For $G_{Ep}$, it yields a correction of 1% at low $Q^2$ (because it is required to agree with the fits of Arrington and Sick [11]), which includes two photon exchange and Coulomb corrections. For $G_{En}$ and $G_{Mn}$ it is used to impose quark-hadron duality asymptotic constraints. Figure 2 shows plots of the data and fits to $G_{En}$ and $G_{Mp}$, respectively.

![Graph showing data and fits](image)

3 Re-extraction of Axial Form Factor

Using our updated $BBBA2007_{25}$ form factors and an updated value $g_A=1.267$, we perform a complete reanalysis of published quasielastic [13] (QE) data on deuteron ($v_{\nu}p \rightarrow \mu^- p$) using the procedure described in detail in ref. [11]. We extract new values of $M_A$ with updated form factors (FF) and also include radiative corrections [5] (RC). Although of lower statistical significance, for completeness we also include all available antineutrino data on hydrogen targets [13].

The average of the corrected measurements of $M_A$ from Table 3 is $M_A^{\text{deuteron}} = 1.016 \pm 0.026$ GeV/$c^2$. This is in agreement with the average value of $M_A^{\text{pion}} = 1.014 \pm 0.016$ GeV/$c^2$ extracted from pion electroproduction experiments after corrections for hadronic effects [12]. The average of the $\nu_\mu$ and electroproduction values is $M_A^{\text{world-average}} = 1.014 \pm 0.004$ GeV/$c^2$.

This precise $M_A$ is smaller than the recent results (for $Q^2 > 0.25$ (GeV/$c^2$) reported by MiniBoone [18] on a carbon target ($M_A^{\text{carbon}} = 1.25 \pm 0.12$ GeV/$c^2$) and by the K2K [19] collaboration on oxygen ($M_A^{\text{oxygen}} = 1.20 \pm 0.12$ GeV/$c^2$). Both experiments use updated vector form factors. Although the collaborations attribute the larger $M_A$ to nuclear effects, there are theoretical arguments that $M_A$ in nuclear targets should be smaller [20] than (or the same [10]) in deuterium. This $M_A$ discrepancy is important for $\nu$ oscillations experiments since it affects the normalization (at high energies the QE cross section is approximately proportional to $M_A$) and non-linearity of the QE cross section, which is relevant to the extraction of $\nu$ mass difference and mixing angle.

For deep-inelastic scattering, the vector and axial parts of $F_2$ are equal. Local quark-hadron duality at large $Q^2$ implies that the axial and vector components of $F_2$ (elastic) are also equal, which yields:

$$|F_A(Q^2)_{A2=V2}|^2 = (G_E^2)^2 Q^4 + \gamma_N(G_M^2 Q^2)^2/(1 + \gamma_N),$$

where $G_E^2(Q^2) = G_{Ep}(Q^2) - G_{En}(Q^2)$, and $G_M^2(Q^2) = G_{Mp}(Q^2) - G_{Mn}(Q^2)$.

We extract values of $F_A(Q^2)$ from the differential cross sections using the procedure of ref. [11]. The overall normalization is set by the theoretical QE cross section [22]. We then do a duality based fit to $F_A(Q^2)$ (including pion electroproduction data) of the form:

$$F_A(Q^2) = A_{25}^{\text{p}}(\xi = 0) \times G_D(Q^2).$$

We impose the constraint $A_{25}^{\text{p}}(\xi = 0) = p_{12} = 1.0$. We also constrain the fit by requiring that $A_{25}^{\text{p}}(\xi = 0)$ yield $F_A(Q^2) = F_A(Q^2)_{A2=V2}$ by including additional "fake" data points for $\xi > 0.9$ ($Q^2 > 7.2$ (GeV/$c^2$)).
Figure 3(a) shows $F_A(Q^2)$ extracted from neutrino-deuterium experiments divided by $G_D^2(Q^2)$ \cite{22}. Figure 3(b) shows $F_A(Q^2)$ extracted from pion electroproduction experiments divided by $G_D^2(Q^2)$ \cite{22}. These pion electroproduction results can be directly compared to the neutrino results because they are multiplied by a factor $F_A(Q^2,M_A = 1.014 \text{GeV/c}^2)/F_A(Q^2,M_A = 1.069 \text{GeV/c}^2)$ to correct for $\Delta M_A = 0.055 \text{GeV/c}^2$ originating from hadronic effects\cite{12}. The solid line is our duality based fit. The short-dashed line is $F_A(Q^2)_{A2=1/2}$. The dashed-dot line is a constituent quark model\cite{17} prediction.

4 Conclusion

In conclusion, our new parameterizations of vector and axial nucleon form factors use quark-hadron duality constraints at high momentum transfers, and maintain a very good descriptions of the form factors at low momentum transfers. Our new parameterizations are useful in modeling $\nu$ interactions for oscillations experiments. Our predictions for $G_E(v^2)$ and $F_A(Q^2)$ at high ($Q^2$) can be tested in future $\mu - N$ and $\nu-N$ experiments. at Jefferson Laboratory and at Fermilab (MINERvA)\cite{24}.

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