On the Hall algebra of semigroup representations over $\mathbb{F}_1$

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Abstract Let $A$ be a finitely generated semigroup with 0. An $A$-module over $\mathbb{F}_1$ (also called an $A$-set), is a pointed set $(M, \ast)$ together with an action of $A$. We define and study the Hall algebra $H_A$ of the category $C_A$ of finite $A$-modules. $H_A$ is shown to be the universal enveloping algebra of a Lie algebra $n_A$, called the Hall Lie algebra of $C_A$. In the case of $(t)$—the free monoid on one generator $(t)$, the Hall algebra (or more precisely the Hall algebra of the subcategory of nilpotent $(t)$-modules) is isomorphic to Kreimer’s Hopf algebra of rooted forests. This perspective allows us to define two new commutative operations on rooted forests. We also consider the examples when $A$ is a quotient of $(t)$ by a congruence, and the monoid $G \cup \{0\}$ for a finite group $G$.

1 Introduction

The aim of this paper is to define and study the Hall algebra of the category of set-theoretic representations of a semigroup. Classically, Hall algebras have been studied in the context abelian categories linear over finite fields $\mathbb{F}_q$. Given such a category $\mathcal{A}$, finitary in the sense that $\text{Hom}(M, N)$ and $\text{Ext}^1(M, N)$ are finite-dimensional $\forall M, N \in \mathcal{A}$ (and therefore finite sets), we may construct from $\mathcal{A}$ an associative algebra $\mathbb{H}_A$ defined over the integers $1$, called the Ringel-Hall algebra of $\mathcal{A}$. As a $\mathbb{Z}$-module, $\mathbb{H}_A$ is freely generated by the isomorphism classes of objects in $\mathcal{A}$, and its structure constants are expressed in terms of the number of extensions between objects. Explicitly, if $M$ and $N$ denote two isomorphism classes, their product in $\mathbb{H}_A$ is given by

$$M \ast N = \sum_{R \in \text{Iso}(\mathcal{A})} p^R_{M, N} R \tag{1}$$

1 It is common to include a twist which makes this algebra over $\mathbb{Q}(\nu)$, where $\nu^2 = q$. 

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where \( \text{Iso}(A) \) denotes the set of isomorphism classes in \( A \), and
\[
\mathbb{P}_{M,N}^R = \# \{ [L \subset R, L \simeq N, R/L \simeq M] \}
\]  
(2)

Denoting by \( \text{Aut}(M) \) the automorphism group of \( M \in \text{Iso}(A) \), it is easy to see that
\[
\mathbb{P}_{M,N}^R \mid \text{Aut}(M) \mid \text{Aut}(N)
\]
counts the number of short exact sequences of the form
\[
0 \to N \to R \to M \to 0,
\]  
(3)

showing that \( \mathbb{H}_A \) encodes the structure of extensions in \( A \). Under additional assumptions on \( A \), \( \mathbb{H}_A \) can be given the structure of a Hopf algebra (see [19]).

A closer examination of formula (1) and the description of \( \mathbb{P}_{M,N}^R \) in terms of short exact sequences (9) reveals that it makes sense in situations where \( A \) is not abelian, or even additive. It suffices that \( A \) be an exact category in the sense of Quillen, satisfying certain finiteness conditions (see [11]), or possibly a non-additive analogue thereof (see [8] for a very general framework). One such example is the category of set-theoretic representations of a semigroup (for other examples of Hall algebras in a non-additive context see [14,22,23,25]). Given a finitely generated semigroup \( A \) possessing and absorbing element 0, we define a (finite) \( A \)-module to be a finite pointed set \((M, \ast)\) equipped with an action of \( A \). Maps between \( A \)-modules are defined to be maps of pointed sets compatible with the action of \( A \), and we denote the resulting category by \( C_A \). \( C_A \) possesses many of the good properties of an abelian category, such as the existence of zero object, small limits and co-limits, and in particular kernels and co-kernels. We may therefore talk about short exact sequences in \( C_A \).

At the same time, \( C_A \) differs crucially from an abelian category in that it is not additive, and morphisms \( f : M \to N \) are not necessarily normal, meaning that the natural map \( \text{Im}(f) \to \text{coim}(f) \) is not in general an isomorphism. This means that the correspondence between the definition of \( \mathbb{P}_{M,N}^R \) in (2) and the count of short exact sequences breaks down. In fact, given \( M, N \) in \( C_A \), there will in general be infinitely many distinct short exact sequences of the form (9). We may however pass to the subcategory \( C_A^N \) of \( C_A \) consisting of the same objects, but with only normal morphisms. The requirement that \( f \) be normal is easily seen to be equivalent to the property that the fiber \( f^{-1}(n) \) over an element \( n \in N \) contain at most one element with the exception of \( f^{-1}(\ast) \). If \( L \subset R \) is a sub-module, then all morphisms in the short exact sequence
\[
0 \to L \to R \to R/L \to 0
\]
are normal, and conversely, any normal short exact sequence corresponds to the data in (2) (up to automorphisms of the kernel and co-kernel). Thus, in \( C_A^N \), the correspondence between the two descriptions of \( \mathbb{P}_{M,N}^R \) which holds in the abelian setting is restored. Furthermore, in \( C_A^N \) there are only finitely many extensions between any two objects, and so we may define \( \mathbb{H}_{C_A^N} \) as in the abelian case. \( \mathbb{H}_{C_A^N} \) may be further equipped with a co-commutative co-multiplication and antipode (after tensoring with \( \mathbb{Q} \)), which gives it the structure of a graded connected co-commutative Hopf algebra. The Milnor-Moore theorem shows that \( \mathbb{H}_{C_A^N} \) is isomorphic to the enveloping algebra \( U(n_{C_A^N}) \) of the Lie algebra \( n_{C_A^N} \) of its primitive elements, which correspond to indecomposable \( A \)-modules. To summarize:

**Theorem 1** There is an isomorphism of Hopf algebras \( \mathbb{H}_{C_A^N} \simeq U(n_{C_A^N}) \), where \( \mathbb{H}_{C_A^N} \) is the Hall algebra of the category \( C_A^N \) of finite \( A \)-modules with normal morphisms, and \( n_{C_A^N} \) is the Lie sub-algebra spanned by indecomposable \( A \)-modules.
This construction may be re-cast in the yoga of the “field with one element”, denoted $\mathbb{F}_1$ (for more on $\mathbb{F}_1$, see [2, 3, 5–7, 10, 15, 16, 21, 26]). The basic principle in working with $\mathbb{F}_1$ is that one loses any additive structure and only multiplication remains. Semigroups are therefore analogues of (possibly non-unital) algebras over $\mathbb{F}_1$, monoids analogues of unital algebras, and pointed sets analogues of vector spaces. The study of semigroup actions on pointed sets is therefore the $\mathbb{F}_1$-analogue of the representation theory of algebras over a field. The above result shows that the category of representations of “an algebra over $\mathbb{F}_1$” leads, via the Hall algebra construction, to a Hopf algebra in a way analogous to the situation over $\mathbb{F}_q$.

The study of $\langle t \rangle$-modules may be seen as linear algebra over $\mathbb{F}_1$, since given a field $k$, the monoid algebra over $k$ is the polynomial ring $k[t]$. To a $\langle t \rangle$-module $M$ we may attach a directed graph $\Gamma$ whose vertices correspond to (non-zero) elements of $M$, and whose directed edges are $\{m \to t \cdot m\}$. We give a description of the possible graphs, and identify the nilpotent $\langle t \rangle$-modules with rooted forests. The latter form a full subcategory $\mathcal{C}_{\langle t \rangle,nil}$ of $\mathcal{C}_{\langle t \rangle}$, and we point out that the Hall algebra $\mathbb{H}_{\mathcal{C}_{\langle t \rangle,nil}}$ is isomorphic to the enveloping algebra of Kreimer’s Lie algebra of rooted trees ([4, 13]), which encodes the combinatorial structure of perturbative renormalization in quantum field theory. The monoid $\langle t \rangle$ is the path monoid of the Jordan quiver, and so this observation may also be interpreted in the context of quiver representations over $\mathbb{F}_1$. It is worth remarking that over a finite field $\mathbb{F}_q$, the Hall algebra of nilpotent representations of the Jordan quiver is isomorphic to the Hopf algebra of symmetric functions $\Lambda$ (see [19]). $\mathbb{H}_{\mathcal{C}_{\langle t \rangle,nil}}$ is therefore an $\mathbb{F}_1$ analogue of $\Lambda$.

This paper is organized as follows. In Sect. 2 we recall basic facts about semigroups, monoids, and their (set-theoretic) representations. We will quite often represent semigroup actions by graphs, and so start by fixing notation. Given a (possibly directed) graph $\Gamma$, we denote by $V(\Gamma)$ its vertex set, and by $E(\Gamma)$ its edge set. For a directed edge $e \in E(\Gamma)$, let $s(e)$ and $t(e)$ denote its initial and terminal vertices, respectively.

**Definition 1** By a semigroup we will always mean a multiplicatively written semigroup $A$ with absorbing element $0$, satisfying

$$0 \cdot a = a \cdot 0 = 0 \quad \forall a \in A.$$
A monoid is semigroup $A$ together with identity element $1$, satisfying
\[ 1 \cdot a = a \cdot 1 = a \quad \forall a \in A. \]

A morphism $f : A \to B$ of semigroups is a multiplicative map preserving 0. If $A$ and $B$ are monoids, we require the map to preserve 1 as well.

We denote the category of semigroups by $\mathcal{M}_0$, and view it as the $\mathbb{F}_1$-analogue of the category of associative (but not necessarily unital) algebras. Given a general semigroup $B$ not necessarily possessing 1, we may adjoin to it an identity to obtain a monoid $A = B \cup \{1\}$ in the above sense (by the same procedure, we may adjoin to a general semigroup not possessing an absorbing element $a_0$). We say that $A$ is finitely generated if there exists a finite collection $\{a_1, \ldots, a_n\} \in A$ such that every element of $A$ can be written as word in $a_1, \ldots, a_n$.

Examples:

1. The free semigroup $\langle x_1, \ldots, x_n \rangle$ on $n$ generators, which consists of 0 and all words in the letters $x_1, \ldots, x_n$ under concatenation.
2. The monoid $\mathbb{F}_1 = \{0, 1\}$, sometimes called the field with one element.
3. The free commutative monoid on one generator $\langle t \rangle = \{0, 1, t, t^2, t^3, \ldots\}$.
4. Any group $G$ is automatically a semigroup. We obtain a monoid $G = G \cup \{0\}$ by adjoining 0.
5. For a ring $R$, we obtain its multiplicative monoid $R^\times$ by forgetting the additive structure.

Given a ring $R$, there exists a base-change functor
\[ \otimes_{\mathbb{F}_1} R : \mathcal{M}_0 \to \mathcal{R} - \text{alg} \]
(4)
to the category of $R$-algebras, defined by setting
\[ A \otimes_{\mathbb{F}_1} R := R[A] \]
where $R[A]$ is the monoid algebra:
\[ R[A] := \left\{ \sum r_ia_i \mid a_i \in A, r_i \in R \right\} / \langle 0 \rangle \]
with multiplication induced from the monoid multiplication.

Definition 2 A congruence on a semigroup $A$ is an equivalence relation $\sim$ on $A$ such that if $x, y, u, v \in A$ and $x \sim y, u \sim v$, then $xu \sim yv$. We denote by $\bar{x}$ the image of $x \in A$ in $A/\sim$. $A/\sim$ inherits a semigroup structure with $\bar{x} \cdot \bar{y} := \bar{xy}$.

Example 1 Consider the free monoid $\langle t \rangle$. For any $x \in \langle t \rangle$, $n \in \mathbb{N}$, the equivalence relation generated by $t^{k+n} \sim t^k x, \ k \geq 0$ is a congruence on $\langle t \rangle$. It is easy to see (see Sect. 4.2) that every congruence on $\langle t \rangle$ is of this form.

2.1 A-modules

Definition 3 Let $A$ be a semigroup. An $A$-module is a pointed set $(M, *)$ equipped with an action of $A$. More explicitly, an $A$-module structure on $(M, *)$ is given by a map
\[ A \times M \to M \]
\[ (a, m) \to a \cdot m \]
satisfying
\[ (a \cdot b) \cdot m = a \cdot (b \cdot m), \quad 1 \cdot m = m, \quad 0 \cdot m = *, \quad \forall a, b, \in A, \ m \in M \]
A morphism of $A$-modules is given by a pointed map $f : M \to N$ compatible with the action of $A$, i.e. $f(a \cdot m) = a \cdot f(m)$. The $A$-module $M$ is said to be finite if $M$ is a finite set, in which case we define its dimension to be $\dim(M) = |M| - 1$ (we do not count the basepoint, since it is the analogue of 0). We say that $N \subset M$ is an $A$-submodule if it is a (necessarily pointed) subset of $M$ preserved by the action of $A$. $A$ always possesses the trivial module $\{\ast\}$, which will be referred to as the zero module.

**Note** This structure is called an A-act in [12] and an A-set in [5].

We denote by $\mathcal{C}_A$ the category of finite $A$-modules. It is the $F_1$ analogue of the category of a finite-dimensional representations of an algebra. In particular, a $F_1$-module is simply a pointed set, and will be referred to as a vector space over $F_1$.

Given a morphism $f : M \to N$ in $\mathcal{C}_A$, we define the image of $f$ to be

$$\text{Im}(f) := \{n \in N | \exists m \in M, f(m) = n\}.$$ 

For $M \in \mathcal{C}_A$ and an $A$-submodule $N \subset M$, the quotient of $M$ by $N$, denoted $M/N$ is the $A$-module

$$M/N := M \setminus N \cup \{\ast\},$$

i.e. the pointed set obtained by identifying all elements of $N$ with the base-point, equipped with the induced $A$-action.

We recall some properties of $\mathcal{C}_A$, following [5,12], where we refer the reader for details:

1. The trivial $A$-module $0 = \{\ast\}$ is an initial, terminal, and hence zero object of $\mathcal{C}_A$.
2. Every morphism $f : M \to N$ in $\mathcal{C}_A$ has a kernel $\ker(f) := f^{-1}(\ast)$.
3. Every morphism $f : M \to N$ in $\mathcal{C}_A$ has a cokernel $\text{coker}(f) := M/\text{Im}(f)$.
4. The categorical co-product of a finite collection $\{M_i\}, i \in I$ in $\mathcal{C}_A$ exists, and is given by the wedge product

$$\bigvee_{i \in I} M_i = \bigsqcup_{i \in I} M_i/\sim$$

where $\sim$ is the equivalence relation identifying the basepoints. We will denote it by

$$\bigoplus_{i \in I} M_i$$

5. The categorical product of a finite collection $\{M_i\}, i \in I$ in $\mathcal{C}_A$ exists, and is given by the Cartesian product $\prod M_i$, equipped with the diagonal $A$-action. It is clearly associative, it is however not compatible with the co-product in the sense that $M \times (N \oplus L) \not\cong M \times N \oplus M \times L$.

6. The category $\mathcal{C}_A$ possesses a reduced version $M \wedge N$ of the Cartesian product $M \times N$, called the smash product. $M \wedge N := M \times N / M \vee N$, where $M$ and $N$ are identified with the $A$-submodules $\{(m, \ast)\}$ and $\{(*, n)\}$ of $M \times N$, respectively. The smash product inherits the associativity from the Cartesian product, and is compatible with the co-product—i.e. $M \wedge (N \oplus L) \simeq M \wedge N \oplus M \wedge L$.

7. $\mathcal{C}_A$ possesses finite limits and co-limits.

8. If $A$ is commutative, $\mathcal{C}_A$ acquires a monoidal structure called the tensor product, denoted $M \otimes_A N$, and defined by

$$M \otimes_A N := M \times N / \sim_\otimes$$

where $\sim_\otimes$ is the equivalence relation generated by $(a \cdot m, n) \sim_\otimes (m, a \cdot n)$ for all $a \in A, m \in M, n \in N$. Note that $(\ast, n) = (0 \cdot \ast, n) \sim_\otimes (\ast, 0 \cdot n) = (\ast, \ast)$, and likewise.
(m, *) ∼ N (*, *). This allows us to rewrite the tensor product as $M \otimes_A N = M \wedge N / \sim_\otimes$, where $\sim_\otimes$ denotes the equivalence relation induced on $M \wedge N$ by $\sim_\otimes$. We have

$$M \otimes_A N \cong N \otimes_A M,$$

$$(M \otimes_A N) \otimes_A L \cong M \otimes_A (N \otimes_A L),$$

$$M \otimes_A (L \oplus N) \cong (M \otimes_A L) \oplus (M \otimes_A N).$$

(9) Given $M$ in $\mathcal{C}_A$ and $N \subset M$, there is an inclusion-preserving correspondence between flags $N \subset L \subset M$ in $\mathcal{C}_A$ and $A$-submodules of $M/N$ given by sending $L$ to $L/N$. The inverse correspondence is given by sending $K \subset M/N$ to $\pi^{-1}(K)$, where $\pi : M \to M/N$ is the canonical projection. This correspondence has the property that if $N \subset L \subset L' \subset M$, then $(L'/N)/(L/N) \cong L'/L$.

These properties suggest that $\mathcal{C}_A$ has many of the properties of an abelian category, without being additive. It is an example of a quasi-exact and belian category in the sense of Deitmar and a proto-exact category in the sense of Dyckerhof–Kapranov. Let $\text{Iso}(\mathcal{C}_A)$ denote the set of isomorphism classes in $\mathcal{C}_A$, and by $\overline{M}$ the isomorphism class of $M \in \mathcal{C}_A$.

**Definition 4** (1) We say that $M \in \mathcal{C}_A$ is indecomposable if it cannot be written as $M = N \oplus L$ for non-zero $N, L \in \mathcal{C}_A$.

(2) We say $M \in \mathcal{C}_A$ is irreducible or simple if it contains no proper sub-modules (i.e those different from * and $M$).

It is clear that every irreducible module is indecomposable. We have the following analogue of the Krull-Schmidt theorem:

**Proposition 1** Every $M \in \mathcal{C}_A$ can be uniquely decomposed (up to reordering) as a direct sum of indecomposable $A$-modules.

**Proof** Let $\Omega_M$ be the directed colored graph with vertex set $V(\Omega_M) := M \setminus \ast$, and edge set $E(\Omega_M) := \{m \to a \cdot m | a \in A, a \cdot m \neq \ast\}$, where the edge $m \to a \cdot m$ is colored by $a$. It is clear that $\Omega_{M \oplus N} = \Omega_M \bigsqcup \Omega_N$ (5)

Let $\Omega_M = \Gamma_1 \cup \Gamma_2 \ldots \cup \Gamma_k$ be the decomposition of $\Omega_M$ into connected components, and $M_{\Gamma_i}$ the subset of $M$ corresponding to $\Gamma_i$, together with the basepoint *. Then $M = \bigoplus M_{\Gamma_i}$, and it is clear from 5 that each $M_{\Gamma_i}$ is indecomposable. The uniqueness of the decomposition is immediate. □

**Remark 1** Suppose $M = \bigoplus_{i=1}^k M_i$ is the decomposition of an $A$-module into indecomposables, and $N \subset M$ a submodule. It then immediately follows that $N = \bigoplus (N \cap M_i)$.

Let $\text{Rep}(A) := \mathbb{Z}[\overline{M}]/I$, $\overline{M} \in \text{Iso}(\mathcal{C}_A)$, where $I$ is the sub-group generated by differences $\overline{M} \oplus \overline{N} - \overline{M} - \overline{N}$. The fact that the symmetric monoidal operations $\wedge, \otimes$ (when defined) are compatible with $\oplus$ shows that they descend to $\text{Rep}(A)$. More precisely:

**Definition 5** Let $A$ be a semigroup.

(1) Let $\text{Rep}^\wedge(A)$ denote the commutative ring obtained from $\text{Rep}(A)$ using the product induced by the smash product on $\mathcal{C}_A$. 

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(2) If $A$ is commutative, let $\text{Rep}^\otimes(A)$ denote the commutative ring obtained from $\text{Rep}(A)$ using the product induced by the tensor product on $\mathcal{C}_A$.

Given a ring $R$, we obtain a base-change functor (which by abuse of of notation, we will denote by the same symbol as 4)

$$\otimes_{F_1} R : \mathcal{C}_A \to R[A] - \text{mod}$$

(6)

to the category of $R[A]$-modules defined by setting

$$M \otimes_{F_1} R := \bigoplus_{m \in M, m \neq *} R \cdot m$$

i.e. the free $R$-module on the non-zero elements of $M$, with the $R[A]$-action induced from the $A$-action on $M$.

2.2 Exact sequences

A diagram $M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$ in $\mathcal{C}_A$ is said to be exact at $M_2$ if $\text{Ker}(g) = \text{Im}(f)$. A sequence of the form

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is a short exact sequence if it is exact at $M_1$, $M_2$ and $M_3$.

One key respect in which $\mathcal{C}_A$ differs from an abelian category is the fact that given a morphism $f : M \to N$, the induced morphism $M/\text{Ker}(f) \to \text{Im}(f)$ need not be an isomorphism. This defect also manifests itself in the fact that the base change functor $\otimes_{F_1} R : \mathcal{C}_A \to R[A] - \text{mod}$ fails to be exact. $\mathcal{C}_A$ does however contain a (non-full) subcategory which is well-behaved in this sense, and which we proceed to describe.

**Definition 6** A morphism $f : M \to N$ is normal if every fibre of $f$ contains at most one element, except for the fibre $f^{-1}(\ast)$ of the basepoint $\ast \in N$.

It is straightforward to verify that this condition is equivalent to the requirement that the canonical morphism $M/\text{Ker}(f) \to \text{Im}(f)$ be an isomorphism, and that the composition of normal morphisms is normal.

**Definition 7** Let $\mathcal{C}_N^A$ denote the subcategory of $\mathcal{C}_A$ with the same objects as $\mathcal{C}_A$, but whose morphisms are restricted to the normal morphisms of $\mathcal{C}_A$. A short exact sequence in $\mathcal{C}_N^A$ is called admissible.

**Remark 2** In contrast to $\mathcal{C}_A$, $\mathcal{C}_N^A$ is typically neither (small) complete nor co-complete. However, $\otimes_{F_1} R$ is exact on $\mathcal{C}_N^A$ for any ring $R$. Note that $\text{Iso}(\mathcal{C}_A) = \text{Iso}(\mathcal{C}_N^A)$, since all isomorphisms are normal.

**Lemma 1** Let $A$ be a semigroup and $\mathcal{C}_N^A$ as above.

(1) For $M, N \in \mathcal{C}_N^A$, $|\text{Hom}_{\mathcal{C}_N^A}(M, N)| < \infty$

(2) Suppose $A$ is finitely generated. For $M, N \in \mathcal{C}_N^A$, there are finitely many admissible short exact sequences

$$0 \to M \xrightarrow{f} L \xrightarrow{g} N \to 0$$

(7)

up to isomorphism.
**Proof** (1) This is obvious, since $M, N$ are finite sets.

(2) Let $n \in \mathbb{N}$. We begin by showing that up to isomorphism, there are finitely many $K \in \mathcal{C}^N_A$ such that $\text{dim}(K) = n$. The action of $A$ on $K$ can be specified by giving the action of each of the generators, which corresponds to an element of $\text{Hom}_{pSet}(K, K)$. The latter is a finite set, and so the claim follows. Since $\text{dim}(L) = \text{dim}(M) + \text{dim}(N)$, it follows that $L$ can belong to only finitely many isomorphism classes in $\mathcal{C}^N_A$. The result now follows from (1).

$\square$

**Remark 3** The requirement that $A$ be finitely generated in part (2) of Lemma 1 is necessary. It is easy to see that for a free semigroup on countably many generators the structure constants are not well-defined.

### 2.2.1 The Grothendieck group

We may use the category $\mathcal{C}^N_A$ to attach to $A$ an invariant $K_0(A)$—the Grothendieck group of $\mathcal{C}^N_A$. Let

$$K_0(A) := \mathbb{Z}[\overline{M}] / J,$$

where $J$ is the subgroup generated by $\overline{L} - \overline{M} - \overline{N}$ for all admissible short exact sequences (7).

### 3 The Hall algebra of $\mathcal{C}^N_A$

Let $A$ be a finitely generated semigroup. In this section, we define the Hall algebra of the category $\mathcal{C}^N_A$. In order to off-load notation, we will denote it by $\mathbb{H}_A$ rather than the more cumbersome $\mathbb{H}_{\mathcal{C}^N_A}$ used in the introduction. For more on Hall algebras see [17–19].

As a vector space:

$$\mathbb{H}_A := \{ f : \text{Iso}(\mathcal{C}_A^N) \to \mathbb{Q} | \# \text{supp}(f) < \infty \}.$$ 

We equip $\mathbb{H}_A$ with the convolution product

$$f \star g(\overline{M}) = \sum_{N \subset M} f(\overline{M}/N)g(\overline{N}),$$

where the sum is over all $A$ sub-modules $N$ of $M$ (in what follows, it is conceptually helpful to fix a representative of each isomorphism class). Note that Lemma 1 and the finiteness of the support of $f, g$ ensures that the sum in (3) is finite, and that $f \star g$ is again finitely supported.

**Lemma 2** The convolution product $\star$ is associative.

**Proof** Suppose $f, g, h \in \mathbb{H}_A$. Then

$$(f \star (g \star h))(\overline{M}) = \sum_{N \subset M} f(\overline{M}/N)(g \star h)(\overline{N})$$

$$= \sum_{N \subset M} f(\overline{M}/N) \left( \sum_{L \subset N} g(\overline{N}/L)h(\overline{L}) \right)$$

$$= \sum_{L \subset N \subset M} f(\overline{M}/N)g(\overline{N}/L)h(\overline{L})$$

$\square$
whereas
\[
((f \star g) \star h)(M) = \sum_{L \subseteq M} (f \star g)(M/L)h(L)
\]
\[
= \sum_{L \subseteq M} \left( \sum_{K \subseteq M/L} f((M/L)/K)g(K) \right) h(L)
\]
\[
= \sum_{L \subseteq N \subseteq M} f(M/N)g(N/L)h(L),
\]
where in the last step we have used the fact (see property 9 of Sect. 2.1) that there is an inclusion-preserving bijection between sub-modules \(K \subseteq M/L\) and sub-modules \(N \subseteq M\) containing \(L\), under which \(N/L \cong K\). This bijection is compatible with taking quotients in the sense that \((M/L)/K \cong M/N\).

\[\square\]

\(\mathbb{H}_A\) is spanned by \(\delta\)-functions \(\delta_M \in \mathbb{H}_A\) supported on individual isomorphism classes, and so it is useful to make explicit the multiplication of two such elements. We have
\[
\delta_M \star \delta_N = \sum_{R \in \text{Iso}(C^A_N)} \mathbb{P}^R_{M,N} \delta_R
\]
where
\[
\mathbb{P}^R_{M,N} := \#\{|L \subseteq R, L \cong N, R/L \cong M|\}
\]
As explained in the introduction,
\[
\mathbb{P}^R_{M,N} |\text{Aut}(M)| |\text{Aut}(N)|
\]
counts the isomorphism classes of admissible short exact sequences of the form
\[
0 \rightarrow N \rightarrow R \rightarrow M \rightarrow 0,
\]
where \(\text{Aut}(M)\) is the automorphism group of \(M\).

We may also equip \(\mathbb{H}_A\) with a co-multiplication
\[
\Delta : \mathbb{H}_A \rightarrow \mathbb{H}_A \otimes \mathbb{H}_A
\]
given by \(\Delta(f)(M, N) := f(M \oplus N)\). The co-multiplication \(\Delta\) is clearly co-commutative.

**Lemma 3** The following holds in \(\mathbb{H}_A\):

1. \(\Delta\) is co-associative: \((\Delta \otimes \text{Id}) \circ \Delta = (\text{Id} \otimes \Delta) \circ \Delta\)
2. \(\Delta\) is compatible with \(\star\): \(\Delta(f \star g) = \Delta(f) \star \Delta(g)\).

**Proof** (1) We have
\[
((\Delta \otimes \text{Id}) \circ \Delta(f))(M, N, \bar{J}) = \Delta(f)(M \oplus N, \bar{J})
\]
\[
= f(M \oplus N \oplus \bar{J})
\]
\[
= \Delta(f)(M, N \oplus \bar{J})
\]
\[
= ((\text{Id} \otimes \Delta) \circ \Delta(f))(M, N, \bar{J}).
\]
By linearity, it suffices to establish this property for \( f = \delta_M, g = \delta_N \). We have
\[
\Delta(\delta_M \star \delta_N)(A, B) = (\delta_M \star \delta_N)(A \oplus B) = \# \{ X \subset A \oplus B \mid X \simeq N, (A \oplus B)/X \simeq M \}.
\]
while
\[
\Delta(\delta_M) \star \Delta(\delta_N)(A, B) = \# \{ A_1 \subset A, B_1 \subset B \mid A_1 \oplus B_1 \simeq N, (A/A_1 \oplus B/B_1) \simeq M \}.
\]
A bijection between the two sets on the right is given by
\[
X \rightarrow (A_1 = X \cap A, B_1 = X \cap B)
\]
and
\[
(A_1, B_1) \rightarrow A_1 \oplus B_1.
\]
Note that this makes crucial use of the property described in the remark after Proposition 1.

\[ \square \]

\( \mathbb{H}_A \) carries a natural grading by \( \mathbb{Z}_{\geq 0} \) corresponding to the dimension of \( M \in \mathcal{C}_A^N \). With this grading, \( \mathbb{H}_A \) becomes a graded, connected, co-commutative bialgebra, and thus automatically a Hopf algebra. By the Milnor-Moore Theorem, \( \mathbb{H}_A \) is isomorphic to \( U(\mathfrak{n}_A) \)—the universal enveloping algebra of \( \mathfrak{n}_A \), where the latter is the Lie algebra of its primitive elements. and the definition of the co-multiplication implies that \( \mathfrak{n}_A \) is spanned by \( \delta_M \) for isomorphism classes \( M \) of indecomposable \( A \)-modules, with bracket
\[
[\delta_M, \delta_N] = \delta_M \star \delta_N - \delta_N \star \delta_M.
\]
We have thus proved:

**Theorem 1** Let \( A \) be a finitely generated semigroup. There is an isomorphism of Hopf algebras \( \mathbb{H}_A \simeq U(\mathfrak{n}_A) \), where \( \mathbb{H}_A \) is the Hall algebra of the category \( \mathcal{C}_A^N \) of finite \( A \)-modules with normal morphisms, and \( \mathfrak{n}_A \) is the Lie sub-algebra spanned by \( \delta_M \) for \( M \) indecomposable.

### 4 Examples

#### 4.1 The monoid \( \langle t \rangle \) and linear algebra over \( \mathbb{F}_1 \)

Let \( \langle t \rangle \) denote the free monoid on one generator, i.e. \( \langle t \rangle := \{0, 1, t, t^2, t^3 \ldots \} \). Given a \( \langle t \rangle \)-module \( M \), the action of the generator \( t \) yields a map of pointed sets \( (\mathbb{F}_1, \text{vector spaces}) \)
\[
t : M \rightarrow M,
\]
and conversely giving such a map equips \( M \) with a \( \langle t \rangle \)-module structure. For a field \( k \), linear algebra over \( k \) is the study of modules over the polynomial ring \( k[t] \), which is the base-change \( \langle t \rangle \otimes_{\mathbb{F}_1} k \). Thus we may view the study of \( \langle t \rangle \)-modules as linear algebra over \( \mathbb{F}_1 \).

Given \( M \in \mathcal{C}_A^N \), we may attach to it an oriented graph \( \Gamma_M \), with \( V(\Gamma_M) := M \setminus \{ * \} \) (i.e. the non-zero elements of \( M \)), and the oriented edges
\[
E(\Gamma_M) := \{ m \rightarrow t \cdot m \mid m \in M, m \neq * \}.
\]
Every vertex in \( \Gamma_M \) therefore has at most one out-going edge. We have \( \Gamma_{M \oplus N} = \Gamma_M \bigsqcup \Gamma_N \), and it follows from this that the connected components of \( \Gamma_M \) correspond to its indecomposable factors. We proceed to characterize the possible graphs \( \Gamma_M \). Suppose that \( M \) is indecomposable. There are two possibilities for the action of \( t \) on \( M \):

\[ \square \]
(1) \( \exists n \in \mathbb{N} \) such that \( t^n \cdot m = \ast \forall m \in M \)—in this case we say that \( M \) and \( t \) are nilpotent

(2) \( \exists m \in M \) such that \( t^n \cdot m \neq \ast \forall n \in \mathbb{N} \). Since \( \{t^k \cdot m\}_{k \in \mathbb{N}} \subseteq M \) is finite, this implies that there are \( n_1, n_2 \in \mathbb{N} \) such that \( t^{n_1} \cdot m = t^{n_2} \cdot m \).

Thus, in the first case, \( \Gamma_M \) is a tree, and in the second, it contains an oriented cycle.

**Example 2** Examples of \( \Gamma_M \):

If \( \Gamma \) is a (not necessarily oriented) graph, we may describe a path \( \sigma \) in \( \Gamma \) by the ordered tuple \([v_1, v_2, \ldots, v_k]\) of consecutive vertices \( v_i \in \Gamma \) encountered along the way (i.e. in this notation, \( v_1 \) is the starting vertex and \( v_k \) the final one). The opposite path \([v_k, \ldots, v_1]\) will be denoted \( \overline{\sigma} \). A cycle \([v_1, v_2, \ldots, v_k, v_1]\) \( k \geq 3 \) will be called minimal if \( v_i \neq v_j \) for \( 1 \leq i \neq j \leq k \). Given a vertex \( v \in \Gamma_M \) with \( a \) incoming and \( b \) outgoing edges, we will call the ordered pair \((a, b)\) its type. Since each vertex has at most one outgoing edge, the possible types are \((a, 0)\) and \((a, 1), a \in \mathbb{Z}_{\geq 0}\). Note that a vertex of type \((a, 0)\) corresponds to an element \( m \in M \) such that \( t \cdot m = \ast \). We will denote by \( \gamma_M \) the un-oriented graph underlying \( \Gamma_M \), and by \( h_1(\gamma_M) = h_1(\Gamma_M) \) their first betti number. A path in \( \gamma_M \) is said to be correctly oriented if it arises from an oriented path in \( \Gamma_M \).

**Lemma 4** If \( \Gamma_M \) is connected (hence \( M \) indecomposable), it contains at most one vertex of type \((a, 0)\).

**Proof** Suppose that \( u, w \) are two vertices of type \((a, 0)\), and \( \sigma = [u, v_1, \ldots, v_k, w] \) is a path in \( \gamma_M \) from \( u \) to \( w \), which we may assume to contain no cycles (i.e. each vertex occurs once). Note that neither \( \sigma \) nor \( \overline{\sigma} \) arise from oriented paths in \( \Gamma_M \), since their initial edges are traversed in a direction opposite to their orientation in \( \Gamma_M \). It follows that at least one of \( v_1, \ldots, v_k \) must have at least two out-going edges in \( \Gamma_M \), yielding a contradiction.

**Lemma 5** If \( \sigma = [v_1, v_2, \ldots, v_k, v_1] \) is a minimal cycle in \( \gamma_M \), then either \( \sigma \) or \( \overline{\sigma} \) is correctly oriented.

**Proof** Suppose not. Then one of the vertices \( v_i \) must have at least two incoming or out-going edges. It is easy to see a vertex with at least two incoming edges forces the existence of another one with at least two out-going edges, yielding a contradiction.

**Lemma 6** If \( h_1(\Gamma_M) > 0 \), then \( \Gamma_M \) contains exactly one oriented cycle (i.e. \( h_1(\Gamma_M) = 1 \)).

**Proof** Suppose \( h_1(\Gamma_M) > 1 \). Then by the previous lemma we may find two distinct oriented cycles \( \sigma_1, \sigma_2 \). These cannot share a vertex, since an oriented cycle is determined by any of its vertices. We may thus find pair of vertices \( u \in \sigma_1, w \in \sigma_2 \), and a path \( \tau = [u, v_1, \ldots, v_k, w] \) in \( \gamma_M \) connecting \( u \) to \( w \). Moreover, \( \tau \) may be chosen free of cycles, and such that \( v_1, \ldots, v_k \) are disjoint from \( \sigma_1, \sigma_2 \). It is clear that \( \tau \) is not correctly oriented since \([u, v_1]\) is not but \([v_k, w]\) is. This forces the existence of a vertex in \( \tau \) with two out-going edges, yielding a contradiction.
Suppose now that $h_1(\Gamma) = 1$, and let $\sigma$ be the unique oriented cycle. Denote by $\Gamma_M/\sigma$ (resp. $\gamma_M/\sigma$) the directed (resp. un-directed) graph obtained by collapsing $\sigma$ to a point, and by $r$ the vertex of $\Gamma_M/\sigma$ and $\gamma_M/\sigma$ corresponding to the collapsed $\sigma$. It follows from the previous lemma that $h_1(\Gamma_M/\sigma) = 0$, so that $\Gamma_M/\sigma$ is a tree. This shows that in $\gamma_M$, there is a unique shortest path $\tau_{v\sigma} = [v, v_1, \ldots, v_k, w]$ from any vertex $v \in \Gamma_M\setminus\sigma$ to the cycle $\sigma$, having the property that $w \in \sigma$, but $v_1, \ldots, v_k \in \Gamma_M\setminus\sigma$. I claim that $\tau_{v\sigma}$ is correctly oriented. This follows since $[v_k, w]$ is correctly oriented, and so if $\tau_{v\sigma}$ is not, one of $v_1, \ldots, v_k$ would have to have at least two outgoing edges. Thus, $\Gamma_M/\sigma$ is canonically a rooted tree, with root $r$, and this shows that in $\Gamma_M$, the there is a unique shortest path $\tau_{v\sigma}$ from any vertex $v \in \Gamma_M\setminus\sigma$ to the cycle $\sigma$, having the property that $w \in \sigma$, but $v_1, \ldots, v_k \in \Gamma_M\setminus\sigma$. We proceed to describe the $\langle t \rangle$-submodules of a $\langle t \rangle$ module $M$ in terms of $\Gamma_M$. By Remark 1, it suffices to describe the sub-modules of $M$ when $M$ is indecomposable, hence $\Gamma_M$ connected. Such an $N$ then corresponds to an oriented sub-graph $\Gamma_N \subset \Gamma_M$ with the property that any oriented path in $\Gamma_M$ starting at a vertex of $\Gamma_N$ stays in $\Gamma_N$—this clearly being equivalent to the condition that $N$ is $\langle t \rangle$-invariant. We call such a $\Gamma_N$ invariant.

**Definition 8** An admissible cut on an oriented graph $\Gamma$ is collection $\Phi \subset E(\Gamma)$ of edges of $\Gamma$ such that at most one member of $\Phi$ is encountered at most once along any oriented path in $\Gamma$. An admissible cut is called simple if $\Phi$ consists of a single edge.

**Example 3** The following are examples of admissible cuts on the graphs from Example 2. The cut edges are indicated by dashed lines.

![Diagram](image)

**Remark 4** It follows immediately that an admissible cut may not include any edges that lie along an oriented cycle.

**Lemma 7** If $M$ is indecomposable, then sub-modules $N \subset M$ correspond to admissible cuts on $\Gamma_M$.

**Proof** Suppose $N \subset M$, and let $\Phi_N \subset E(\Gamma_M)$ be the collection of edges joining a vertex of $\Gamma_M\setminus\Gamma_N$ to one in $\Gamma_N$. I claim that $\Phi_N$ is an admissible cut. If not, then there exists an oriented path $\sigma$ in $M$ and two distinct edges $e_1, e_2 \in \Phi_N$ that lie along it, where $e_1$ occurs before $e_2$. Let $\sigma' \subset \sigma$ be the sub-path of $\sigma$ starting at $t(e_1)$ and ending at $s(e_2)$. The existence of $\sigma'$ contradicts the fact that $\Gamma_N$ is invariant.
Conversely, suppose that $\Phi \subset E(\Gamma_M)$ is an admissible cut, and $\Gamma'_M$, the oriented graph obtained from $\Gamma_M$ by removing the edges in $\Phi$. Let $\Gamma_N$ be the connected component of $\Gamma'_M$ containing the root (if $\Gamma_M$ is a tree), or a point on the cycle (if $\Gamma_M$ contains an oriented cycle). Then it is clear that $\Gamma_N$ is an invariant sub-graph.

**Remark 5** It follows from Remark 4 and Proposition 2 that $\Gamma'_M$ consists of two connected components exactly when the admissible cut $\Phi$ is simple. In this case, denote by $Rt_\Phi(M)$ the component of $\Gamma'_M$ containing either the root or cycle, and by $Lf_\Phi(M)$ the remaining component.

### 4.1.1 $\mathbb{H}_{\langle t \rangle}$ and Kreimer’s Hopf algebra of rooted trees

We may now give a simple description of the Hall algebra $\mathbb{H}_{\langle t \rangle}$ in terms of the combinatorics of the graphs $\Gamma_M$. Recall that $\mathbb{H}_{\langle t \rangle} \simeq \bigcup(n_{\langle t \rangle})$, where $n_{\langle t \rangle}$ is the Lie algebra spanned by $\delta_M$ for indecomposable $\langle t \rangle$-modules $M$. Combining equation 8 and Remark 5 we obtain:

$$\delta_M \ast \delta_N = \delta_{M \oplus N} + \sum_{R \in \text{IndMod}} n(R, M, N)\delta_R$$

(10)

where $n(R, M, N)$ denotes the number of simple cuts $\Phi$ on $\Gamma_R$ such that $Rt_\Phi(R) \simeq \Gamma_N$ and $Lf_\Phi \simeq \Gamma_M$, and IndMod denotes the set of isomorphism classes of indecomposable modules. The Lie bracket in $n_{\langle t \rangle}$ is therefore given by

$$[\delta_M, \delta_N] = \sum_{R \in \text{IndMod}} (n(R, M, N) - n(R, N, M))\delta_R$$

(11)

The nilpotent $\langle t \rangle$-modules form a full subcategory of $C^N_{\langle t \rangle}$, closed under extensions, which we denote by $C^N_{\langle t \rangle, \text{nil}}$. We may therefore consider the Hall algebra $\mathbb{H}_{\langle t \rangle, \text{nil}}$ of $C^N_{\langle t \rangle, \text{nil}}$. If $M$ is nilpotent and indecomposable, $\Gamma_M$ is a rooted tree, and so the bracket (11) equips the vector space spanned by these with a Lie algebra structure. This Lie algebra first appeared in the work of Dirk Kreimer (see [4, 13]) on the combinatorial structure of perturbative renormalization in quantum field theory.

**Remark 6** It is a classical result (see [19]) that when $A$ is the category of finite-dimensional nilpotent $\mathbb{F}_q[t]$-modules, $\mathbb{H}_A$ is isomorphic to the Hopf algebra of symmetric functions, which we denote by $\Lambda$. As an algebra,

$$\Lambda \simeq \mathbb{Q}[e_i], \ i \in \mathbb{N},$$

where the collection of $e_i$ may be taken to be a variety of symmetric polynomials (for instance, elementary symmetric functions, power sums etc.). From this perspective, rooted forests can be viewed as the $\mathbb{F}_1$-analogue of symmetric functions. It is known furthermore [1, 9] that $\mathbb{H}_K^*$ is a free non-commutative algebra.

### 4.1.2 $\text{Rep}^\Lambda(\langle t \rangle)$, $\text{Rep}^\otimes(\langle t \rangle)$ and combinatorial operations on forests

Given $M, N$ in $C^N_{\langle t \rangle}$, it is natural to ask how the graphs $\Gamma_{M \wedge N}, \Gamma_{M \oplus N}$ are related to $\Gamma_M$ and $\Gamma_N$, i.e. for an explicit description of the ring structure in $\text{Rep}^\Lambda(\langle t \rangle), \text{Rep}^\otimes(\langle t \rangle)$. In particular, since the subcategory $C^N_{\langle t \rangle, \text{nil}}$ is closed under all three operations, we obtain two commutative combinatorial operations on rooted forests. As the examples below suggest, these are non-trivial:
4.2 The monoids $\langle t \rangle / \sim$

Recall from Example 1 that for any $x \in \langle t \rangle$, $n \in \mathbb{N}$, the equivalence relation generated by $t^{k+n} \sim t^k x$, $k \geq 0$ is a congruence. To see that these are all possible congruences on $\langle t \rangle$, observe that $A = \langle t \rangle / \sim$ is naturally a $\langle t \rangle$-module, generated by $1$. $\langle t \rangle / \sim$ therefore has at most one vertex of type $(0, 1)$. It follows from the classification of possible $\Gamma$’s above that $\Gamma / \sim$ is either a ladder tree (if $x = 0$), or a cycle with a single (possibly empty) ladder tree attached (if $x \neq 0$).

$\langle t \rangle$ maps surjectively to $A$, and so any $A$-module is automatically a $\langle t \rangle$-module (which has to respect $\sim$). We can thus use the classification of graphs $\Gamma$ above to describe the Lie algebra $n_A$. For this, we need the notion of height of a rooted tree:

**Definition 9** The height of a rooted tree $T$ is the length of the longest path from leaf to root.

We distinguish two cases:

1. $x = 0$. Then $\bar{t}^n = 0$ in $A$, and so $\bar{t}$ acts nilpotently on any module. If $M$ is an indecomposable $A$-module, then $\Gamma_M$ is a rooted tree of height $\leq n - 1$ and conversely, any such rooted tree corresponds to an indecomposable $A$-module.

2. $x = t^m$, $m \geq 0$ We may assume that $m < n$ (if $m = n$ we get the identity equivalence relation, which was treated in the previous section). If $\bar{t}$ acts nilpotently on an indecomposable module $M$, then the condition $\bar{t}^n = \bar{t}^m$ implies that $\Gamma_M$ is a rooted tree of height at most $n - m - 1$. If the action of $\bar{t}$ is not nilpotent, then for any non-zero $x \in M$, $\bar{t}^m x$ must be part of a cycle of length dividing $n - m$. Thus, the possible $\Gamma_M$ are either rooted trees of height at most $n - m - 1$ or cycles of length dividing $n - m$ with rooted trees of height at most $m$ attached at the roots.

4.3 Quiver representations over $\mathbb{F}_1$

Recall that a quiver $Q$ is a directed graph (which we will assume to be finite). To $Q$ we may attach a finitely generated semigroup $A(Q)$, with generators $0, v_i, i \in V(Q)$ and $e_l, l \in E(Q)$, and relations

$$v_i v_j = \delta_{i,j} v_i$$

$$e_l v_i = 0 \quad \text{unless } i \text{ is the terminal vertex of } l, \text{ in which case } e_l v_i = e_l$$

$$v_i e_l = 0 \quad \text{unless } i \text{ is the initial vertex of } l, \text{ in which case } v_i e_l = e_l$$

$$0 \cdot x = 0 \quad \text{for any element } x$$
Informally, the non-zero elements of $A(Q)$ consist of paths in $Q$ (including the trivial paths $v_i$ corresponding to vertices of $Q$), with multiplication given by concatenation of paths when it makes sense, and 0 otherwise. Note that $A(Q)$ does not in general have a unit.

Let $Q_J$ denote the Jordan quiver, consisting of a single vertex $v$ and a single edge (loop) $e$ from $v$ to $v$. We then have $A(Q_J) \simeq \langle t \rangle$, and so $\mathbb{H}_{A(Q_J)}$ is a generalization of $\mathbb{H}_{\langle t \rangle}$ above. A detailed description of $\mathbb{H}_{A(Q_J)}$ will be given in [24].

4.4 The monoid $\overline{G}$ and the Burnside ring

Let $G$ be a finite group, and $\overline{G} = G \cup \{0\}$ the monoid obtained by adjoining to it the absorbing element 0. We proceed to describe the category $C^N_{\overline{G}}$ and its Hall algebra. Let $M$ be a $\overline{G}$-module, and $M' = M \setminus \{\ast\}$ the set obtained by removing the base-point. $M'$ carries an action of $G$, since every element of $G$ has an inverse. It follows that every $\overline{G}$-module arises from a $G$-set by adjoining a base-point. $M'$ decomposes into a disjoint union $M' = \bigsqcup_{i=1}^k M_i$ of $G$-orbits, and setting $M_i = M'_i \cup \{\ast\}$, we see that $M = \bigoplus_{i=1}^k M_i$ is the unique (up to permutation) decomposition of $M$ into irreducible hence indecomposable factors. Each orbit $M_i$ is of the form $G/H$ for a subgroup $H \subset G$, and since conjugate subgroups produce isomorphic $\overline{G}$-modules, we see that the non-trivial irreducible $\overline{G}$-modules are in bijection with conjugacy classes of subgroups of $G$. The action of $G$ on each orbit is transitive, so the notions of indecomposable and irreducible module are equivalent.

Let $\text{Conj}(G)$ denote the set of conjugacy classes of subgroups of $G$, and for $i \in \text{Conj}(G)$, denote by $H_i$ (any) subgroup belonging to the corresponding conjugacy class. Let $M_i = G/H_i \cup \{\ast\}$, viewed as a left $\overline{G}$-module.

If $N \subset M$ is a $\overline{G}$-submodule, then defining $N'$, $M'$ as above, and setting $K = M \setminus N'$, we see that $M = N \oplus K$, thus the category $C^N_{\overline{G}}$ is semi-simple in the appropriate sense. It follows that every admissible short exact sequence splits, and so if $M$ and $N$ are distinct indecomposable $\overline{G}$-modules, then in $\mathbb{H}_{\overline{G}}$,

$$\delta_M \ast \delta_N = \delta_{M \oplus N}.$$ 

It follows that $\mathbb{H}_{\overline{G}}$ is free commutative, with generators $\overline{M}_i$, $i \in \text{Conj}(G)$, and $\mathbb{H}_{\overline{G}}$ is abelian.

The ring $\text{Rep}^\wedge[\overline{G}]$ is also a free module on the $\overline{M}_i$. If $M_i \simeq G/H_i \cup \{\ast\}$ and $M_j \simeq G/H_j \cup \{\ast\}$ then $M_i \wedge M_j \simeq G/H_i \times G/H_j \cup \{\ast\}$, with $\overline{G}$ acting diagonally. It is therefore exactly the Burnside ring $\Omega(G)$ of $G$ (see [20]). $\text{Rep}^\wedge[\overline{G}]$ is not defined unless $G$ is abelian.

**Proposition 3** Let $G$ be a finite group, and $\overline{G}$ the monoid $G \cup \{0\}$.

1. The category $C^N_{\overline{G}}$ is semi-simple, in the sense that any $M \in C^N_{\overline{G}}$ can be written uniquely (up to permutation) $M \simeq \bigoplus_{j=1}^k M_{i_j}$, $i_j \in \text{Conj}(G)$.

2. $\mathbb{H}_{\overline{G}} \simeq \mathbb{Q}[\overline{M}_i]$, $i \in \text{Conj}(G)$.

3. $\text{Rep}^\wedge[\overline{G}] \simeq \Omega(G)$, where $\Omega(G)$ is the Burnside ring of $G$.

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