Stochastic (Approximate) Proximal Point Methods: Convergence, Optimality, and Adaptivity

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Abstract

We develop model-based methods for solving stochastic convex optimization problems, introducing the approximate-proximal point, or AProx, family, which includes stochastic subgradient, proximal point, and bundle methods. When the modeling approaches we propose are appropriately accurate, the methods enjoy stronger convergence and robustness guarantees than classical approaches, even though the model-based methods typically add little to no computational overhead over stochastic subgradient methods. For example, we show that improved models converge with probability 1 and enjoy optimal asymptotic normality results under weak assumptions; these methods are also adaptive to a natural class of what we term easy optimization problems, achieving linear convergence under appropriate strong growth conditions on the objective. Our substantial experimental investigation shows the advantages of more accurate modeling over standard subgradient methods across many smooth and non-smooth optimization problems.

1 Introduction

In this paper, we develop and analyze a family of model-based methods, moving beyond naive stochastic gradient methods, for solving the stochastic convex optimization problem

\[ \minimize_{x} \; F(x) = \mathbb{E}_{P}[f(x; S)] = \int_{S} f(x; s) dP(s) \]

subject to \( x \in \mathcal{X} \).

In problem (1), the set \( S \) is a sample space, and for each \( s \in S \), the function \( f(\cdot; s) : \mathbb{R}^{n} \to \mathbb{R} \) is a closed convex function, and \( \mathcal{X} \subset \mathbb{R}^{n} \) is closed convex.

Stochastic minimization problems, in which an optimizer has access to samples \( S_i \) drawn independently and identically distributed from \( P \) and uses these samples to minimize \( F \), have applications in numerous fields, including machine learning, statistical estimation, and simulation-based optimization [64, 22, 54]. The current accepted methodology for such problems is the stochastic (sub)gradient method [65, 64, 42, 8, 53], which Robbins and Monro [48] originally developed for smooth stochastic approximation problems, which iterates as follows: beginning at an initial point \( x_1 \), iteratively draw \( S_k \) \( \text{\iid} \) \( P \) and update

\[ x_{k+1} := x_{k} - \alpha_k g_k \quad \text{for some} \; g_k \in \partial f(x_k; S_k). \]  

The stochastic gradient method enjoys guarantees on convergence [65, 42] and widespread empirical success in large-scale convex and non-convex stochastic optimization [64, 8, 53, 15, 33]. In spite of this success, there are notable difficulties with the stochastic subgradient method (2): it is sensitive to stepsize selection; it can diverge on objectives, such as \( F(x) = x^4 \), that do not obey its convergence criteria; and it is rarely adaptive to nuanced aspects of

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problem difficulty. Engineers thus waste time and computation dealing with these issues and finding appropriate stepsizes, which cascades into additional practical challenges.

An alternative to treating the stochastic gradient method (2) (SGM) as a noisy approximation to gradient descent is to view it as minimizing a sequence of random models of the functions $F$ and $f$, and we leverage this view here. In this context, SGM makes a linear approximation to the instantaneous function $f$ around the point $x_k$, setting

$$f_{x_k}(x; S_k) := f(x_k; S_k) + \langle g_k, x - x_k \rangle$$

and choosing $x_{k+1}$ to minimize the regularized model $f_{x_k}(x; S_k) + \frac{1}{2\alpha_k} \|x - x_k\|^2_2$. More sophisticated models are plausible—most familiar is the stochastic proximal point method [50, 29, 6, 27, 7], which makes no approximation, using $f_{x_k}(x; s) = f(x; s)$ and iterating

$$x_{k+1} = \arg\min_{x \in X} \left\{ f(x; S_k) + \frac{1}{2\alpha_k} \|x - x_k\|^2_2 \right\}.$$  \hfill (3)

This modeling perspective is important in non-stochastic optimization, where (for example) Newton, Gauss-Newton, bundle, and trust-region methods [e.g. 24, 9, 43, 44] explicitly build sequences of easier-to-minimize models while minimizing the global function $F$. A substantial body of work investigates this modeling perspective in non-convex optimization [10, 17], and recent work by Davis and Drusvyatskiy [13] demonstrates convergence results for appropriate models in weakly convex stochastic optimization, motivating our approach.

We show how to extend this modeling perspective to stochastic convex optimization problems, leveraging it to build a new family of algorithms for solving problem (1), which, in homage to the stochastic proximal point iteration (3), we call the aProx (approximate proximal point) algorithms, with substantially better theoretical guarantees and empirical performance than naive stochastic subgradient methods. The aProx algorithms iterate as follows: for $k = 1, 2, \ldots$, we draw a random $S_k \overset{iid}{\sim} P$, then update the iterate $x_k$ by minimizing a regularized approximation to $f(\cdot; S_k)$, setting

$$x_{k+1} := \arg\min_{x \in X} \left\{ f_{x_k}(x; S_k) + \frac{1}{2\alpha_k} \|x - x_k\|^2_2 \right\}.$$  \hfill (4)

The function $f_x(\cdot; s)$ is a model of $f(\cdot; s)$ at the point $x$, meaning that $f_x$ satisfies the following three conditions on its structure and local approximation properties for $f$:

(C.i) The function $y \mapsto f_x(y; s)$ is convex and subdifferentiable on its domain.

(C.ii) The model $f_x$ satisfies the equality $f_x(x; s) = f(x; s)$ and

$$f_{x}(y; s) \leq f(y; s) \quad \text{for all } y.$$

(C.iii) At $y = x$, we have the containment

$$\partial_y f_x(y; s) |_{y = x} \subset \partial_x f(x; s).$$

Davis and Drusvyatskiy [13] and Duchi and Ruan [19] consider similar modeling conditions, and they inspire our treatment here. We give examples of such models in Section 2, with graphical illustration in Figure 1.

The aProx methodology (4) is flexible in that it allows many possible modeling choices. As we shall see, though stochastic gradient (2) and proximal point (3) methods are both special
cases, they possess quite different behavior. Thus, it is interesting to provide conditions on the accuracy of the models $f_x$, in addition to (C.i)–(C.iii), that imply stronger convergence guarantees than those available for stochastic gradient and other simple methods. To describe our contributions at a high level, we list two assumptions that we frequently make.

**Assumption A1.** The set $\mathcal{X}^* := \arg\min_{x \in \mathcal{X}} \{F(x)\}$ is non-empty, and there exists $\sigma^2 < \infty$ such that for each $x^* \in \mathcal{X}^*$ and selection $f'(x^*; s) \in \partial f(x^*; s)$, we have $\mathbb{E}[\|f'(x^*; S)\|^2] \leq \sigma^2$.

**Assumption A2.** There exists an increasing function $G_{\text{big}} : \mathbb{R}^+ \rightarrow [0, \infty)$ such that for all $x \in \mathcal{X}$ and each selection $f'(x; s) \in \partial f(x; s)$, $\mathbb{E}[\|f'(x; S)\|^2] \leq G_{\text{big}}(\text{dist}(x, \mathcal{X}^*))$.

Assumption A2 makes no restrictions on the growth of the function $G_{\text{big}}$, so the second moment of the subgradient $f'(x; S)$ may grow arbitrarily. This contrasts with typical assumptions for stochastic subgradient methods [e.g. 65, 42] which assume uniform boundedness or second-moment conditions on subgradients. Within this context, we take three thrusts.

1. First, in Section 3, we develop conditions for the stability of iterates (4) under Assumption A1, meaning that they remain in an explicit bounded neighborhood of the optimal solution set $\mathcal{X}^*$ of problem (1). We leverage this stability to prove strong convergence guarantees for the APROX iteration (4) even for functions with substantial variation in their gradient estimates (e.g. the gradient may grow super-exponentially in $\|x\|$) to which results on stochastic gradient methods do not apply. As a consequence of these results, we extend Polyak and Juditsky’s analysis of averaged stochastic gradient methods to all APROX models (4)—showing asymptotic normality with optimal covariance under weaker conditions than those necessary for classical situations—so long as the iterates are bounded, highlighting the importance of this stability.

2. In our second thrust, in Section 4, we study the performance of APROX methods (4) for what we term easy problems. In these problems there exists a shared minimizer $x^*$ common to all the sampled functions. This assumption is strong, yet many problems are indeed easy: in statistical learning, work of Belkin et al. [3, 2] shows that functions that perfectly interpolate the observed data (suffering zero loss on the observations) can still achieve optimal statistical convergence guarantees; Kaczmarz algorithms solve over-parameterized linear systems with consistent solutions [55, 41, 40]; the problem of finding a point in the intersection of convex sets assumes there indeed exists a point in each of them [34, 1]. By incorporating a simple lower bound condition—basically, that if $f$ is non-negative, any model $f_x$ should also be non-negative—in addition to conditions (C.i)–(C.iii), we show how APROX can adapt to these easy problems and achieve (near) linear convergence guarantees, even in stochastic settings, using methods with no additional computational complexity beyond stochastic gradient methods.

3. Our final set of results, in Section 5, provide non-asymptotic convergence guarantees. In particular, we show that any APROX method (4) recovers standard convergence guarantees of stochastic gradient and proximal point methods, and we show how stochastic proximal point methods enjoy fast convergence under (restricted) strong convexity with only weak moment conditions on $\partial f(x; s)$, further emphasizing the advantages of accurate modeling.

In addition to our theoretical results, we perform substantial simulations. Our experiments consider a wide range of smooth, non-smooth, and super-polynomially-growing convex problems: regression with squared and absolute losses, logistic and poisson regression, and
projection problems onto intersections of halfspaces (relating these to classification problems).
The common refrain in each of these is that even slightly improved APROX models are much more robust to stepsize choice than stochastic gradient methods, and more careful modeling allows fast convergence in a much broader range of problems, including those with moderately poor conditioning where stochastic gradient methods fail.

1.1 Related work

We situate our paper in relation to classical and modern work on stochastic optimization problems. Stochastic gradient methods are classical, beginning with the development by Robbins and Monro [48] in the 1950s [46, 47, 65, 42, 64, 30]. A number of authors recognize the challenges associated with stepsize selection and instability of stochastic gradient methods: in the case of smooth strongly convex minimization, Nemirovski et al. [42] show how a slightly mis-specified stepsize can cause arbitrarily slow convergence guarantees. More recent work, for example that of Mouldines and Bach [39], shows that—even when assumptions sufficient for their convergence hold—stochastic gradient methods can exhibit transient divergent behavior, where \( \mathbb{E}[\|x_k - x^*\|_2^2] \) grows exponentially with an initial stepsize choice \( \alpha_1 \).

In effort to alleviate some of these issues, there is substantial recent work on more careful approaches to stochastic optimization problems. Of most relevance to our work are stochastic proximal point methods (3), which use the true function \( f_y(s) = f(y; s) \) in the iteration (4). Bertsekas [6] analyzes stochastic proximal point algorithms in an incremental framework (that is, when \( S = \{1, \ldots, m\} \) is a finite set), showing convergence results similar to subgradient methods, while Kulis and Bartlett [29] and Karampatziakis and Langford [27] give theoretical and empirical results in online convex optimization settings, demonstrating regret bounds similar to classical results [65]. Toulis and collaborators [57, 58] study stochastic proximal point algorithms and convergence guarantees for their final iterates—a different approach than we take. Their results, however, assume that the functions under consideration are both globally Lipschitz and globally strongly convex, which is a contradiction, severely limiting the applicability of their results.\(^2\) Patrascu and Necoara [45] also give analyses for stochastic proximal point algorithms, providing non-asymptotic convergence results under the assumption that either each function \( f(\cdot; s) \) is Lipschitz or strongly convex [45, Assumptions 1 & 9]; these assumptions fail for a number of interesting problems (including linear regression with \( f(x; (a,b)) = \frac{1}{2}((a,x) - b)^2 \)), though Patrascu and Necoara’s results also apply to sets \( \mathcal{X} \) that are intersections of simpler sets \( \mathcal{X} = \mathcal{X}_1 \cap \cdots \cap \mathcal{X}_m \).

Ryu and Boyd [51] also investigate the stochastic proximal point method, making arguments on its stability stronger than classical results for stochastic gradient methods. Under Assumption A1, Ryu and Boyd show that the stochastic proximal point method guarantees \( \mathbb{E}[\|x_{k+1} - x^*\|_2^2] \leq \mathbb{E}[\|x_1 - x^*\|_2^2] + \sigma \sum_{i=1}^k \alpha_i \), so that the iterates do not diverge exponentially (as they can for stochastic gradient methods [39]). Yet this result is not enough to provide strong stability, boundedness, or any convergence guarantees; nor is it enough to explain the better empirical stability of stochastic proximal point methods.

Notation For a convex function \( f \), \( \partial f(x) \) denotes its subgradient set at the point \( x \), and \( f'(x) \in \partial f(x) \) denotes an arbitrary element of the subdifferential. Throughout, we let \( x^* \) denote a minimizer of problem (1) and \( \mathcal{X}^* = \text{argmin}_{x \in \mathcal{X}} F(x) \) its optimal set. We let

\(^2\)Their analysis uses both assumptions explicitly and frequently; if one weakens the assumptions and applies their arguments, their convergence guarantees exhibit the same potential for exponential divergence that stochastic gradient methods exhibit.
Figure 1. (a) Models of the function $f(x) = \log(1 + e^{-x})$: a linear model (5) built around the point $x_0$ and truncated model (7) built around the point $x_1$. (b) The multi-line (or bundle) model, maximum of linear functions, as in the iteration (8). The lighter lines represent individual linear approximations, the darker line their maximum.

$\mathcal{F}_k := \sigma(S_1, \ldots, S_k)$ denote the $\sigma$-field generated by the first $k$ random variables $S_i$. Under iteration (4), $x_k \in \mathcal{F}_{k-1}$ for all $k$.

2 Methods

We begin our contributions by introducing different natural models for stochastic convex optimization problems, as well as a few conditions in addition to (C.i)–(C.iii) that we can use to demonstrate new aspects of convergence for the aProx family. While the stochastic proximal point method (3) satisfies all the conditions in the paper, in some situations it may be expensive or challenging to implement exactly. With that in mind, we provide a catalogue of a few models to serve as a reference for the remainder of the paper.

**Stochastic subgradient methods:** The starting point for any model-based methods are the simple first-order models. As we discuss in the introduction, the stochastic subgradient method uses the model

$$f_x(y; s) := f(x; s) + \langle f'(x; s), y - x \rangle,$$

where $f'(x; s) \in \partial f(x; s)$ is an arbitrary element of the subdifferential. The model (5) satisfies conditions (C.i)–(C.iii) by convexity.

**Proximal point methods:** The stochastic proximal point method uses the “model”

$$f_x(y; s) := f(y; s),$$

that is, the true function. The model (6) satisfies all the conditions we provide.

**Truncated models:** The first condition beyond (C.i)–(C.iii) builds out of the simple observation that, if one is minimizing a nonnegative function (for example, in most machine learning
and statistical applications with a loss function), then a priori a model of the function that takes negative values cannot be accurate. If \( f(x; s) \geq 0 \) for all \( x \), a better approximation to \( f \) than the linear model (5) is to take \( f'(x; s) \in \partial f(x; s) \) and define

\[
f_x(y; s) := [f(x; s) + \langle f'(x; s), y - x \rangle]_+,
\]

where \( f'(x; s) \in \partial f(x; s) \). More generally, we may consider models that provide a lower guarantee:

(C.iv) For all \( s \in \mathcal{S} \), the models \( f_x(\cdot; s) \) satisfy

\[
f_x(y; s) \geq \inf_{z \in \mathcal{X}} f(z; s).
\]

Thus, if we are given an oracle that, for each fixed \( s \in \mathcal{S} \), can compute the minimal value \( \inf_{z \in \mathcal{X}} f(z; s) \), we may consider the truncated models

\[
f_x(y; s) := \max \left\{ f(x; s) + \langle f'(x; s), y - x \rangle, \inf_{z \in \mathcal{X}} f(z; s) \right\}.
\]

See Figure 1(a) for an illustration of this model.

Many statistical, machine learning, and signal-processing examples support this model, because for any individual sample \( s \) we have \( \inf_{z \in \mathcal{X}} f(z; s) = 0 \). For example, in linear regression, \( s = (a, b) \in \mathbb{R}^n \times \mathbb{R} \), and \( \inf_{z} (a, z - b)^2 = 0 \) when \( a \neq 0 \). In logistic regression [22], we have \( s = (a, b) \in \mathbb{R}^n \times \{-1, 1\} \), and \( f(x; (a, b)) = \log(1 + \exp(-b(a, x))) \) satisfies \( \inf_{z} f(z; (a, b)) = 0 \) unless \( a = 0 \). Support vector machines use \( f(x; (a, b)) = [1 - b(a, x)]_+ \), which again has infimal value 0. Even in more complex scenarios, these infimal values may be easy to compute; see, for example, our discussion of poisson regression in Section 3.3, Example 3.

**Relatively accurate models:** Now we consider models with an additional condition on accuracy, which may allow easier updates than the exact model (3). As we see in the sequel (Theorem 1), this condition is sufficient for strong stability and convergence guarantees for any APROX method using the model-based updates (4). We require a bit more notation. Let \( f_{x_0}(\cdot; s) \) be a model centered at \( x_0 \) satisfying Conditions (C.i)–(C.iii). For \( \alpha > 0 \) define

\[
x_\alpha := \arg\min_{x \in \mathcal{X}} \left\{ f_{x_0}(x; s) + \frac{1}{2\alpha} \|x - x_0\|^2 \right\},
\]

which is the result of a single update (leaving dependence on \( s \) implicit). Then we consider

(C.v) For some \( \epsilon > 0 \), there exists a function \( C : \mathcal{S} \rightarrow \mathbb{R}_+ \) with \( \mathbb{E}[C(S)] < \infty \) such that for all \( x_0 \in \mathcal{X} \), the updated point \( x_\alpha \) and model \( f_{x_\alpha}(\cdot; s) \) satisfy

\[
f(x_\alpha; s) \leq f_{x_0}(x_\alpha; s) + \frac{1-\epsilon}{2\alpha} \|x_\alpha - x_0\|^2 + C(s)\alpha.
\]

Condition (C.v) complements the lower bound condition (C.ii). We term Condition (C.v) a “relative” accuracy because the necessary approximation scales with \( \alpha \) so that higher accuracy is necessary as \( \alpha \downarrow 0 \), though \( \|x_\alpha - x_0\| = O(\alpha) \) by standard results (and our own to come) [17].

While it is not clear a priori how to guarantee that a particular model satisfies Condition (C.v), one approach is to use bundle-type methods [24, 56], identical to Kelley’s cutting plane method [28], which we term the “multi-line” model. In this situation, we begin from the
linear model \( f_\alpha^0(y; s) = f(x; s) + \langle f'(x; s), y - x \rangle \), iteratively construct the lower piecewise-linear models

\[
x_\alpha^k := \arg\min_{y \in \mathcal{X}} \left\{ f_\alpha^{k-1}(y) + \frac{1}{2\alpha} \| y - x \|_2^2 \right\}
\]

for \( k \geq 1 \). Let \( f_\alpha^k(y) := \max \left\{ f_\alpha^{k-1}(y; s), f(x_\alpha^k; s) + \langle f'(x_\alpha^k; s), y - x_\alpha^k \rangle \right\} \).

Whenever the iterate \( x_\alpha^k \) satisfies Condition (C.v), we may terminate the iteration, as \( f_\alpha^{k-1}(y) \) satisfies Conditions (C.i)–(C.iii) by construction. While we do not address this in this paper, it is possible to bound the number of iterations to achieve a solution satisfying Condition (C.v), as each step solves a strongly convex optimization problem (cf. [56]).

3 Stability and its consequences

The first of our main thrusts, upon which we focus in this section, is the stability and boundedness of the iterates for stochastic proximal point methods and their relatives in the APROX family of methods. These stability guarantees are in strong contrast to standard stochastic subgradient methods, which may diverge for problems on which APROX methods converge.

We begin with our definition of stability. In this definition, we let \( \mathcal{A} \) denote the set of positive stepsize sequences \( \{\alpha_k\} \) with \( \sum_k \alpha_k^2 < \infty \). We call a pair \( (\mathcal{F}, \mathcal{P}) \) a collection of problems if \( \mathcal{P} \) is a collection of probability measures on a sample space \( \mathcal{S} \), and \( \mathcal{F} \) is a collection of functions \( f : \mathcal{X} \times \mathcal{S} \to \mathbb{R} \), where \( f(\cdot; s) \) is convex. We make the following definition.

**Definition 3.1.** An algorithm generating iterates \( x_k \) according to the model-based update (4) is stable for the collection of problems \( (\mathcal{F}, \mathcal{P}) \) if for all \( f \in \mathcal{F} \) and \( \mathcal{P} \in \mathcal{P} \) defining \( F(x) = \mathbb{E}_P(f(x; S)) \) and \( \mathcal{X}^* = \arg\min_{x \in \mathcal{X}} F(x) \), we have both

\[
\sup_{\alpha \in \mathcal{A}} \sup_{k \in \mathbb{N}} \mathbb{E}\left[ \text{dist}(x_k, \mathcal{X}^*)^2 \right] < \infty \tag{9a}
\]

and, for all stepsize sequences \( \{\alpha_k\} \in \mathcal{A} \),

\[
\sup_k \text{dist}(x_k, \mathcal{X}^*) < \infty \quad \text{with probability 1}. \tag{9b}
\]

Standard stochastic gradient methods fail for many situations in which Assumptions A1 and A2 hold; even in easier situations in which the objective is smooth and in which there is no noise, (sub)gradient methods can be unstable. Let us demonstrate this via a few counterexamples; in the sequel, we show how appropriately accurate models alleviate the issues in the examples. (Moulines and Bach [39] provide additional examples, but we present a few here to highlight the challenges and benefits of the APROX family.)

**Example 1** (Instability for quadratics): Let \( F(x) = \frac{1}{2} x^2 \). Then the gradient method iterates

\[
x_{k+1} = (1 - \alpha_k)x_k.
\]

Let us assume that \( \alpha_k = \alpha_0 k^{-\beta} \), and let \( \alpha_0 \geq 3K^\beta \) for some \( K \in \mathbb{N} \). Then assuming \( x_1 \neq 0 \), for all \( k \leq K \) we have \( |x_{k+1}| = |1 - \alpha_k||x_k| \geq 2|x_k| \), so that \( |x_{k+1}| \geq 2^k \) for \( k \leq K \). Of course, classical convergence guarantees for the (stochastic) gradient method guarantee that \( x_k \) will converge eventually, but even for smooth quadratics, gradient descent fails the stability condition (9a).

The boundedness condition (9b) is often easier to satisfy. Indeed, classical results coupled with the Robbins-Siegmund supermartingale convergence theorem [49] guarantee that the
stochastic subgradient method satisfies \(\sup_k \text{dist}(x_k, \mathcal{X}^*) < \infty\) whenever

\[
\mathbb{E}\left[\left\|f'(x; S)\right\|^2\right] \leq C_0 + C_1 \text{dist}(x, \mathcal{X}^*)^2 \quad \text{for all } x \in \mathcal{X},
\]

which typical smoothness or boundedness conditions imply (see, for example, the papers and books [49, 47, 4]). Stochastic (approximate) proximal point approaches allow us to move beyond these quadratic growth assumptions; in contrast, for objectives for which the gradients grow more than quadratically, even condition (9b) typically fails for gradient methods.

**Example 2** (Divergence for non-quadratics): Let \(F(x) = \frac{1}{4}x^4\), and consider any sequence of stepsizes \(\alpha_k > 0\) satisfying \(\alpha_{k+1} \geq \frac{1}{4}\alpha_k\) for all \(k\), and let \(x_{k+1} = x_k - \alpha_k f'(x_k)\) be generated by the gradient method. Then whenever the initial iterate \(x_1\) satisfies \(|x_1| \geq \sqrt{3/\alpha_1}\), we have \(|x_k| \geq 2|x_{k-1}|\) for all \(k\), so \(|x_k| \geq 2^k|x_1|\) for all \(k \in \mathbb{N}\). \(\diamondsuit\)

When the objective grows faster than polynomially, even worse behavior is possible; for example, for the objective \(F(x) = (e^x + e^{-x})\), for any polynomially decreasing stepsize sequence, if \(x_1\) is large enough we have the super-exponential divergence \(|x_k| \geq 2^k|x_1|\). While stylized, these examples highlight the difficulty of naive application of gradient methods.

### 3.1 Stability of (approximate) proximal methods

The starting point of almost all of what follows are sufficient conditions on the models we use to guarantee stability as in Definition 3.1. For this first result, in addition to the three conditions (C.i)–(C.iii), we assume the models are accurate at their updated points, that is, Condition (C.v). The full model (stochastic proximal point method) satisfies these conditions, but as we briefly mention in Section 2, so too do bundle methods (the multi-line model (8)) as well as other approximations. In the theorem, recall the \(\sigma\)-field \(\mathcal{F}_k := \sigma(S_1, \ldots, S_k)\).

**Theorem 1.** Let Assumption A1 hold and \(x_k\) be generated by the iteration (4) with any model satisfying Conditions (C.i)–(C.iii) and (C.v). Then for all \(x^* \in \mathcal{X}^*\),

\[
\mathbb{E}\left[\left\|x_{k+1} - x^*\right\|^2 \mid \mathcal{F}_{k-1}\right] \leq \left\|x_k - x^*\right\|^2 + \alpha_k^2 \left(\frac{\sigma^2}{\epsilon} + \mathbb{E}[C(S)]\right).
\]

Before providing the proof of the theorem, which we do in Section 3.1.1, we present a few of its consequences for stability. By taking \(x^*\) to be the projection of \(x_k\) onto \(\mathcal{X}^*\), Theorem 1 implies that

\[
\mathbb{E}[^\text{dist}(x_{k+1}, \mathcal{X}^*)^2 \mid \mathcal{F}_{k-1}] \leq \mathbb{E}[\left\|x_{k+1} - x^*\right\|^2 \mid \mathcal{F}_{k-1}] \leq \text{dist}(x_k, \mathcal{X}^*)^2 + \alpha_k^2 \left(\frac{\sigma^2}{\epsilon} + \mathbb{E}[C(S)]\right).
\]

A few somewhat more consequential corollaries, at least from the perspective of our stability Definition 3.1, follow. We first have

**Corollary 3.1.** Let the conditions of Theorem 1 hold. For each \(k \in \mathbb{N}\),

\[
\mathbb{E}\left[\text{dist}(x_{k+1}, \mathcal{X}^*)^2\right] \leq \mathbb{E}\left[\text{dist}(x_1, \mathcal{X}^*)^2\right] + \left(\frac{\sigma^2}{\epsilon} + \mathbb{E}[C(S)]\right) \sum_{i=1}^k \alpha_i^2.
\]

A second corollary establishes that, indeed, the iterates of appropriately accurate APROX methods are stable. To state this result, we require the classical Robbins-Siegmund almost supermartingale convergence lemma.
Lemma 3.1 ([49]). Let \( A_k, B_k, C_k, D_k \geq 0 \) be non-negative random variables adapted to the filtration \( \mathcal{F}_k \) and satisfying \( \mathbb{E}[A_{k+1} \mid \mathcal{F}_k] \leq (1 + B_k)A_k + C_k - D_k \). Then on the event \( \{ \sum_k B_k < \infty, \sum_k C_k < \infty \} \), there is a random \( A_\infty < \infty \) such that \( A_k \to A_\infty \) and \( \sum_k D_k < \infty \).

By applying Theorem 1 and with \( A_k = \text{dist}(x_{k+1}, \mathcal{X}^*)^2 \), \( B_k = 0, D_k = 0 \), and \( C_k = \alpha_{k+1}^2(\sigma^2/\epsilon + \mathbb{E}[C(S)]) \) in Lemma 3.1, we have

**Corollary 3.2.** Let the conditions of Theorem 1 hold and assume \( \sum_k \alpha_k^2 < \infty \). Then

\[
\sup_{k \in \mathbb{N}} \text{dist}(x_k, \mathcal{X}^*) < \infty
\]

and \( \text{dist}(x_k, \mathcal{X}^*) \) converges to some finite value with probability 1.

Combining Corollaries 3.1 and 3.2, we see that the stochastic proximal point method and its APROX variants—as long as they satisfy the accuracy condition (C.v)—are stable according to Definition 3.1. This is in contrast to stochastic gradient methods and their relatives, which, as we discuss following Definition 3.1, can be unstable even for relatively simple problems.

### 3.1.1 Proof of Theorem 1

We now return to the promised proof of Theorem 1. In giving the proof, we present lemmas on the progress of individual iterates for subsequent use. These results are typical of Lyapunov-type arguments for convergence of stochastic gradient methods [65, 42].

**Lemma 3.2.** Let \( g \) be convex and subdifferentiable on a closed convex set \( \mathcal{X} \) and let \( \beta > 0 \). Then for all \( x_0, x_1, y \in \mathcal{X} \), and \( g'(y) \in \partial g(y) \),

\[
g(y) - g(x_1) \leq \langle g'(y), y - x_0 \rangle + \frac{1}{2\beta} \|x_1 - x_0\|^2 + \frac{\beta}{2} \|g'(y)\|^2
\]

**Proof.** By the first-order conditions for convexity, we have

\[
g(y) - g(x_1) \leq \langle g'(y), y - x_1 \rangle = \langle g'(y), y - x_0 \rangle + \langle g'(y), x_0 - x_1 \rangle
\]

\[
\leq \langle g'(y), y - x_0 \rangle + \frac{1}{2\beta} \|x_1 - x_0\|^2 + \frac{\beta}{2} \|g'(y)\|^2,
\]

where the second line uses the Fenchel-Young inequality. \( \square \)

We also have the following lemma, which gives a one-step progress guarantee for any algorithm using models satisfying Conditions (C.i)–(C.iii).

**Lemma 3.3.** Let Condition (C.i) hold. In each step of the method (4), for any \( x \in \mathcal{X} \),

\[
\frac{1}{2} \|x_{k+1} - x\|^2 \leq \frac{1}{2} \|x_k - x\|^2 - \alpha_k [f_{x_k}(x_{k+1}; S_k) - f_{x_k}(x; S_k)] - \frac{1}{2} \|x_k - x_{k+1}\|^2.
\]

**Proof.** By the first-order conditions for convex optimization, for some \( g_k \in \partial f_{x_k}(x_{k+1}; S_k) \) we have that \( \langle \alpha_k g_k + (x_{k+1} - x_k), y - x_{k+1} \rangle \geq 0 \) for all \( y \in \mathcal{X} \). Setting \( y = x \), we obtain

\[
\alpha_k \langle g_k, x_{k+1} - x \rangle \leq \langle x_{k+1} - x_k, x - x_{k+1} \rangle = \frac{1}{2} \left( \|x_k - x\|^2 - \|x_{k+1} - x\|^2 - \|x_{k+1} - x_k\|^2 \right).
\]

As \( f_{x_k}(x; S_k) \geq f_{x_k}(x_{k+1}; S_k) + \langle g_k, x - x_{k+1} \rangle \) by Condition (C.i), this gives the result. \( \square \)
With Lemmas 3.2 and 3.3 in place, we can prove the theorem. Let \( x^* \in \mathcal{X} \) be an otherwise arbitrary optimal point. Applying Lemma 3.3 with \( x = x^* \), we have
\[
\frac{1}{2} \|x_{k+1} - x^*\|^2 \leq \frac{1}{2} \|x_k - x^*\|^2 - \alpha_k \{f_k(x_{k+1}; S_k) - f_k(x^*; S_k)\} - \frac{1}{2} \|x_k - x_{k+1}\|^2 \\
\leq \frac{1}{2} \|x_k - x^*\|^2 - \alpha_k \{f(x_{k+1}; S_k) - f(x^*; S_k)\} - \frac{\epsilon}{2} \|x_k - x_{k+1}\|^2 + C(S_k)\alpha_k^2 \\
\leq \frac{1}{2} \|x_k - x^*\|^2 - \alpha_k \{f(x_{k+1}; S_k) - f(x^*; S_k)\} - \frac{\epsilon}{2} \|x_k - x_{k+1}\|^2 + C(S_k)\alpha_k^2,
\]
where inequality (i) is a consequence of the accurate model condition (C.v) and (ii) because \( f_k(x^*; s) \leq f(x^*; s) \) by the lower model condition (C.ii). Now, we apply Lemma 3.2 with \( x_1 = x_{k+1}, \ x_0 = x_k, \ y = x^*, \) and \( \beta = \frac{\alpha_k}{\epsilon} \) to find
\[
\frac{1}{2} \|x_{k+1} - x^*\|^2 \leq \frac{1}{2} \|x_k - x^*\|^2 + \alpha_k \|f'(x^*; S_k)\| x^* - x_k) + \frac{\alpha_k^2}{2\epsilon} \|f'(x^*; S_k)\|^2 + C(S_k)\alpha_k^2
\]
for all \( f'(x^*; S_k) \in \partial f(x^*; S_k) \).

For some \( F'(x^*) \in \partial F(x^*) \), we have \( \langle F'(x^*), y - x^* \rangle \geq 0 \) for all \( y \in \mathcal{X} \). As our choice of \( f'(x^*; s) \in \partial f(x^*; s) \) above was arbitrary, we may take \( f'(x^*; S_k) \) so that \( E[f'(x^*; S_k)] = F'(x^*) \) for any desired \( F'(x^*) \in \partial F(x^*) \) (cf. [5]). Thus, taking expectations with respect to \( F_{k-1} \),
\[
\frac{1}{2} E[\|x_{k+1} - x^*\|^2 | F_{k-1}] \leq \frac{1}{2} \|x_k - x^*\|^2 + \frac{\alpha_k^2}{2\epsilon} E\left[\|f'(x^*; S_k)\|^2\right] + E[C(S)]\alpha_k^2 + \alpha_k \langle F'(x^*), x^* - x_k \rangle.
\]
As \( \langle F'(x^*), x^* - x_k \rangle \leq 0 \), we obtain the theorem.

3.2 Convergence of aProx methods

The key consequence of Theorem 1 is that the iterates \( x_k \) remain bounded with probability 1. This boundedness by itself does not guarantee convergence; however, in this section, we show that as a consequence of this boundedness, any algorithm satisfying Conditions (C.i)–(C.iii) is convergent, even under very weak assumptions. Assumptions A1 and A2 are insufficient to guarantee convergence of subgradient methods, which—as our examples show—may diverge without uniform boundedness conditions on the subgradients. As the next proposition shows, however, any stable enough member of the aProx family of methods—meaning that Eq. (9b) holds in Definition 3.1—guarantees convergence with probability 1. Throughout this section and Section 3.3, we without comment make the assumption that the stepsizes \( \alpha_k \) satisfy the standard summability conditions
\[
\alpha_k > 0 \text{ for all } k, \quad \sum_{k=1}^{\infty} \alpha_k = \infty, \quad \text{and} \quad \sum_{k=1}^{\infty} \alpha_k^2 < \infty.
\]

Proposition 1. Let Assumptions A1 and A2 hold. Let the iterates \( x_k \) be generated by any method satisfying Conditions (C.i)–(C.iii), and additionally assume that the iterates are bounded: with probability 1, \( \sup_k \|x_k\| < \infty \). Then \( \sum_k \alpha_k (F(x_k) - F(x^*)) < \infty \). If in addition
\[
\Gamma(\epsilon) := \inf_{x \in \mathcal{X}} \{F(x) - F(x^*) \mid \text{dist}(x, \mathcal{X}^*) \geq \epsilon \} > 0
\]
for all \( \epsilon > 0 \), then \( \text{dist}(x_k, \mathcal{X}^*) \xrightarrow{a.s.} 0 \).
Deferring the proof of the proposition temporarily, we provide some discussion. A simple sufficient condition for $\Gamma(\epsilon) > 0$ to hold for $\epsilon > 0$ is that $\mathcal{X}^*$ is compact. Convex functions are Lipschitz on compacta [24], so $F$ achieves its minimal value on the compact annulus $A_\epsilon^* := \{x \mid 2\epsilon \geq \text{dist}(x, \mathcal{X}^*) \geq \epsilon\}$ at some point $x_k \in A_\epsilon^*$, and $\inf_x \{F(x) \mid \text{dist}(x, \mathcal{X}^*) \geq \epsilon\} > F^*$.

Proposition 1 also implies an asymptotic convergence rate on (weighted averages of) the iterates $x_k$. Indeed, let $\{\gamma_k\}_{k=1}^{\infty} \subset \mathbb{R}_+$ be a non-decreasing sequence with $\gamma_k \geq 0$. Then define the weighted averages $\bar{x}_k = \sum_{i=1}^{k} \gamma_i \alpha_i x_i / (\sum_{i=1}^{k} \gamma_i \alpha_i)$. We have

**Corollary 3.3.** Let the conditions of Proposition 1 hold. Then with probability 1,

$$\limsup_{k \to \infty} \frac{1}{\gamma_k} \left( \sum_{i=1}^{k} \gamma_i \alpha_i \right) [F(\bar{x}_k) - F^*] = 0.$$

**Proof** We have $(\sum_{i=1}^{k} \gamma_i \alpha_i) (F(\bar{x}_k) - F^*) \leq (\sum_{i=1}^{k} \gamma_i \alpha_i) (F(x_i) - F^*)$ by Jensen’s inequality. By Kronecker’s lemma, as $\sum_i \alpha_i (F(x_i) - F^*) < \infty$ and $\{\gamma_k\}$ is non-decreasing, we have $\gamma_k^{-1} \sum_{i=1}^{k} \gamma_i \alpha_i (F(x_i) - F^*) \xrightarrow{a.s.} 0$.

For example, taking $\gamma_k = \alpha_k^{-1}$, we obtain that the average $\bar{x}_k = \frac{1}{k} \sum_{i=1}^{k} x_i$ satisfies

$$k \alpha_k (F(\bar{x}_k) - F^*) \xrightarrow{a.s.} 0.$$

To prove Proposition 1, we present a lemma giving a one-step progress guarantee for any method satisfying Conditions (C.i)–(C.iii).

**Lemma 3.4.** Let Conditions (C.i)–(C.iii) hold and let $x_k$ be generated by the updates (4). Then for any $x \in \mathcal{X}$,

$$\frac{1}{2} \|x_{k+1} - x\|_2^2 \leq \frac{1}{2} \|x_k - x\|_2^2 - \alpha_k [f(x_k; S_k) - f(x; S_k)] + \frac{\alpha_k^2}{2} \|f'(x_k; S_k)\|_2^2.$$

**Proof** Using Lemma 3.3, it suffices to show that for any $\alpha > 0$ and $x_0, x_1, x \in \mathcal{X}$

$$-\alpha [f_{x_0}(x_1; s) - f_{x_0}(x; s)] - \frac{1}{2} \|x_1 - x_0\|_2^2 \leq -\alpha [f(x_0; s) - f(x; s)] + \frac{\alpha^2}{2} \|f'(x_0; s)\|_2^2.$$

To see this, note that

$$-f_{x_0}(x_1; s) + f_{x_0}(x; s) = -[f_{x_0}(x_0; s) - f_{x_0}(x; s)] + f_{x_0}(x_0; s) - f_{x_0}(x_1; s)$$

$$(C.\text{iii}) \leq -[f_{x_0}(x_0; s) - f_{x_0}(x; s)] + \langle f'(x_0; s), x_0 - x_1 \rangle$$

$$(C.\text{ii}) \leq -[f(x_0; s) - f(x; s)] + \langle f'(x_0; s), x_0 - x_1 \rangle.$$

Then we use that for any vector $v$, $\alpha \langle v, \Delta \rangle - \frac{1}{2} \|\Delta\|_2^2 \leq \frac{\alpha^2}{2} \|v\|_2^2$, which gives the result. □

**Proof of Proposition 1** By Assumption A2, for all $x$ such that $\|x - x^*\| \leq r$, $\mathbb{E} [\|f'(x; S)\|_2^2] \leq G_{\text{big}}(r)$. Take $x^*$ as the projection of $x_k$ onto $\mathcal{X}^*$. Then Lemma 3.4 implies that

$$\mathbb{E} [\|x_{k+1} - x^*\|_2^2 \mid F_{k-1}] \leq \text{dist}(x_k, \mathcal{X}^*)^2 - 2\alpha_k (F(x_k) - F^*) + \alpha_k^2 G_{\text{big}}(\|x_k - x^*\|).$$

On the event that $\sup_k \text{dist}(x_k, \mathcal{X}^*) < \infty$, we have $\sum_k \alpha_k^2 G_{\text{big}}(\text{dist}(x_k, \mathcal{X}^*)) < \infty$, and so the Robbins-Siegmund Lemma 3.1 implies that $\text{dist}(x_k, \mathcal{X}^*) \xrightarrow{a.s.} D_\infty$ for some finite random variable $D_\infty$ and $\sum_k \alpha_k (F(x_k) - F(x^*)) < \infty$. If $\Gamma(\epsilon) > 0$, then a simple argument by contradiction shows that $D_\infty = 0$ with probability 1, as $\sum_k \alpha_k = \infty$. □
3.3 Asymptotic normality

Without additional conditions, it is challenging to provide more precise convergence guarantees than those of Corollary 3.3, which (at best) provides an asymptotic rate scaling as $1/(\alpha_k k)$ for $\alpha_k \propto k^{-\beta}$. With this in mind, we now turn to a result on the asymptotic normality of averaged iterates, extending earlier results of Polyak and Juditsky [47]. In most analyses of stochastic convex optimization problems in which one might expect some type of asymptotic normality, typical assumptions are that the random functions $f(\cdot; s)$ have globally Lipschitz gradients ([e.g. [47, Sec. 5], [21, 31, 51]], which is reasonable when $\mathcal{X}$ is compact, but frequently fails over non-compact sets $\mathcal{X}$. In contrast, we consider

**Assumption A3.** The functions $F$ and $f$ satisfy the following.

(i) The function $F$ is $C^2$ in a neighborhood of $x^* = \arg\min_{x \in \mathcal{X}} F(x)$, and $\nabla^2 F(x^*) > 0$.

(ii) There exists $\epsilon > 0$ such that $f(\cdot; s)$ is $L(s)$-smooth on the set $\mathcal{X}_s^* := \{x | \|x - x^*\| \leq \epsilon\}$, meaning that $x \mapsto \nabla f(x; s)$ is $L(s)$ Lipschitz on $\mathcal{X}_s^*$, and $E[|S|^2] = L^2 < \infty$.

Assumption A3 says that in a neighborhood of $x^*$, the random functions $f$ have smooth gradients with probability 1. We will apply Assumption A3 in conjunction with Assumption A2, which enforces a type of local Lipschitz continuity of $f$ and $F$. The typical Lipschitz conditions on $\nabla f$ in the literature imply Assumption A2: if we make the assumption that $x \mapsto \nabla f(x; s)$ is $L_r(s)$ Lipschitz on $\mathcal{X}_s^* = \{x \in \mathcal{X} | \|x - x^*\| \leq r\}$, where the smoothness constant $L_r$ is square integrable for any finite $r$, Assumption A2 holds as long as $E[\|\nabla f(x^*; S)\|^2] < \infty$, as

$$\|\nabla f(x; s)\| \leq \|\nabla f(x^*; s)\| + \|\nabla f(x^*; s) - \nabla f(x; s)\| \leq \|\nabla f(x^*; s)\| + L_r(s) \|x - x^*\|,$$

and we have the moment bound $G_{\text{big}}(r) \leq 2E[\|\nabla f(x^*; S)\|^2] + 2E[L_r(S)^2]r^2$ in Assumption A2.

To further motivate Assumption A3, we provide a brief example.

**Example 3 (Poisson regression):** In problems with count data $b_1, b_2, \ldots, b_m \in \mathbb{N}$, we may wish to predict counts based on a covariate vector $a_i \in \mathbb{R}^n$. In this case, a standard model is poisson regression model, a generalized linear model [37, 22]. In this case, we model a count $b \in \mathbb{N}$ conditional on a vector $a \in \mathbb{R}^n$ as coming from a poisson distribution with parameter $e^{a_i x}$, that is, $p(b \mid a, x) = e^{-\lambda} \lambda^b / b!$ for $\lambda = e^{a_i x}$.

Then the negative log likelihood is $f(x; (a, b)) = -\log(p(b \mid a, x) = \log(b!) + \exp(\langle a, x \rangle) - b(a, x)$. In this case, it is easy to compute the truncated model (7), as $f$ satisfies

$$\inf_z f(x; (a, b)) = \log(b!) + \inf_t \{e^t - bt\} = \log(b!) + b - b \log b.$$

In this case, we have $f'(x; (a, b)) = ae^{a_i x} - ba$, and using $|e^t - e^s| \leq e^{\max\{s, t\} |s - t|}$ shows that $f$ satisfies Assumption A3 as soon as we have the covariance condition $\text{Cov}(a) > 0$ and $E[e^{|a||a|/2}] < \infty$ for $r < \infty$. Such moment conditions are not sufficient for SGM to converge. \(\diamond\)

We then have the following theorem, whose proof is long and somewhat involved, so we present it in Appendix A.

**Theorem 2.** Let Assumptions A1–A3 hold. Let the iterates $x_k$ be generated by any method satisfying Conditions (C.i)–(C.iii) with stepsizes $\alpha_k = \alpha_0 k^{-\beta}$ for some $\beta \in (1/2, 1)$ and $\alpha_0 > 0$. Assume additionally that the iterates are bounded: with probability 1, $\sup_k \|x_k\| < \infty$. Then

$$\frac{1}{\sqrt{k}} \sum_{i=1}^{k} (x_i - x^*) \overset{d}{\to} \mathcal{N}(0, \nabla^2 f(x^*)^{-1} \text{Cov}(\nabla f(x^*; S))\nabla^2 f(x^*)^{-1}).$$
To prove Theorem 2, we use two main insights. The first is that, if the iterates remain bounded, then Proposition 1 guarantees convergence. The second is a gradient approximation result that shows that even if the models in the iterations (4) are non-smooth, they locally behave as first-order Taylor approximations to the functions $f$, and thus eventually the iterates approximate the stochastic gradient method on quadratics. From this, we can apply the techniques of Polyak and Juditsky [47] to guarantee asymptotic normality.

The asymptotic convergence guarantee in Theorem 2 is unimprovable. It achieves the local asymptotic minimax bound for stochastic optimization [18] (the analogue of the standard Fisher information in classical statistical problems [32, 60, Sec. 8.7]). In contrast to stochastic gradient schemes, however, Theorem 2 requires essentially only that the iterates remain bounded. In the end, then, the important consequence of the aProx family is that, with appropriately accurate models, we can guarantee stability (Definition 3.1). By leveraging these stability guarantees, we can then show that model-based iteration schemes (4) are (asymptotically) optimal.

4 Fast convergence for easy problems

The stability and asymptotic results in Section 3 provide some evidence for the benefits of using better models within stochastic optimization problems: if the models are accurate enough that the iterates remain bounded, then we obtain a number of asymptotic optimality results even when standard stochastic gradient methods diverge. In this section, we study a different collection of problems, which we term easy optimization problems. More precisely, we say that a stochastic minimization problem is easy if there are shared global minimizers:

**Definition 4.1.** Let $F(x) := \mathbb{E}_P[f(x; S)]$. Then $F$ is an easy optimization problem if for $\mathcal{X}^* := \arg\min_{x \in \mathcal{X}} F(x)$, for each $x^* \in \mathcal{X}^*$ and $P$-almost all $s \in S$ we have

$$\inf_{x \in \mathcal{X}} f(x; s) = f(x^*; s).$$

Definition 4.1 places strong restrictions on the class of functions we consider, but a number of examples satisfy its conditions, and we detail them more carefully in Section 4.3. Other researchers have considered similar conditions to Definition 4.1; for example, Schmidt and Le Roux [52] study stochastic optimization problems of the form $F(x) = \frac{1}{m} \sum_{i=1}^m f_i(x)$ where $\nabla f_i(x^*) = 0$ for all $i$. We briefly enumerate a few examples to motivate what follows, returning in Section 4.3 to flesh them out fully.

**Example 4** (Overdetermined linear systems): In an overdetermined linear system, we have a matrix $A \in \mathbb{R}^{m \times n}$, with $m \geq n$, and we wish to solve $Ax = b$, where we assume the system of equalities is feasible, and $A$ is rank $n$. Letting $a_i \in \mathbb{R}^n$ denote the rows of $A$, both objectives $F(x) = \frac{1}{2m} \|Ax - b\|_2^2$ and $F(x) = \frac{1}{m} \|Ax - b\|_1$ satisfy Definition 4.1, where we take samples $s = (a_i, b_i) \in \mathbb{R}^n \times \mathbb{R}$, and it is clear that the unique solution $x^*$ to $Ax = b$ satisfies $f(x^*; (a_i, b_i)) = 0$ for all $i$. See Section 4.3.1. ◊

**Example 5** (Finding a point in the intersection of convex sets): Let $C_1, C_2, \ldots, C_m$ be closed convex sets with non-empty intersection $\mathcal{X}^* := \cap_{i=1}^m C_i$. Then the objective $F(x) = \frac{1}{m} \sum_{i=1}^m \text{dist}(x, C_i)$ is convex, and treating the sample space $\mathcal{S} = \{1, \ldots, m\}$, we have $F(x) = 0$ and $f(x; i) := \text{dist}(x, C_i) = 0$ for all $i$ if and only if $x \in \mathcal{X}^*$. See Section 4.3.2. ◊

**Example 6** (Data interpolation): A more involved example arises out of recent results in statistical machine learning. In this area, substantial recent success in deep learning [33, 63]
arises out of models that fit a training sample of data perfectly. In settings more amenable to analysis, Belkin et al. [2, 3] study statistical algorithms that minimize convex losses and perfectly interpolate the data, that is, given a sample \{S_1, \ldots, S_m\} drawn i.i.d. from an underlying population, they find points \(x^*\) satisfying \(f(x^*; S_i) = \inf_x f(x; S_i)\) for each \(i = 1, \ldots, m\). See Section 4.3.3. \diamond 

Given Examples 4, 5, and 6, it is of interest to investigate algorithmic aspects of the aProx family of algorithms when we have problems satisfying Definition 4.1. In this case, we show that any model satisfying the local approximation conditions (C.i)–(C.iii) and the additional lower bound condition (C.iv) possesses strong adaptivity and convergence properties. This is in contrast to subgradient methods, which (given a very precise stepsize choice) can exhibit fast convergence, but are typically non-adaptive.

To highlight the types of models we consider, without loss of generality, we may assume that \(\inf_{x \in X} f(x; s) = 0\), as given an oracle that provides the value \(\inf f(\cdot; s)\) we can replace \(f\) with \(f(\cdot; s) - \inf f(\cdot; s)\). The results in this section thus apply to the lower-truncated model (7),

\[ f_s(y; s) := [f(x; s) + \langle f'(x; s), y - x \rangle]_+ \].

The updates for this model are easy to compute when \(X = \mathbb{R}^n\); indeed, in this case, the guarded model (7) yields the update

\[ x_{k+1} = x_k - \min \left\{ \alpha_k, \frac{f(x_k; S_k)}{\|f'(x_k; S_k)\|_2^2} \right\} f'(x_k; S_k). \]

This update is reminiscent of the classical Polyak subgradient method [46], which chooses “optimal” stepsizes in the subgradient method when the value \(F(x^*)\) is known.

In the remainder of this section, we analyze the performance of the aPROX family of models on easy problems, showing that in a number of settings, these methods even enjoy linear convergence. The starting point of each of our results is the following lemma, whose proof we defer to Section B.1, that shows that if a problem has shared minimizers as in Definition 4.1, iterates of any method satisfying the lower bound condition (C.iv) are guaranteed to make progress toward the optimal set.

**Lemma 4.1.** Let \(F\) be an easy stochastic minimization problem (Definition 4.1). Let \(x_k\) be generated by the updates (4) using a model satisfying Conditions (C.i)–(C.iv). Then for any \(x^* \in X^*\),

\[ \frac{1}{2} \|x_{k+1} - x^*\|^2 \leq \frac{1}{2} \|x_k - x^*\|^2 - \frac{1}{2} [f(x_k; S_k) - f(x^*; S_k)] \min \left\{ \alpha_k, \frac{f(x_k; S_k) - f(x^*; S_k)}{\|f'(x_k; S_k)\|_2^2} \right\}. \]

In the next two sections, we use this lemma to derive conditions on the growth of \(F\) that we can leverage for fast convergence of aPROX models. We return to our examples in Section 4.3, demonstrating that common problems satisfy the assumptions in Sections 4.1 and 4.2.

### 4.1 Sharp growth with shared minimizers

For our first set of problems, we consider objectives that exhibit sharp growth away from the optimal set \(X^*\); classical and recent optimization literature highlights the importance of such conditions for the convergence of deterministic optimization methods [11, 17]. (We provide explicit examples in Section 4.3.) As we shall see, these conditions are sufficient to guarantee
linear convergence of APROX models in stochastic settings. We begin with the following assumption, which we use for its ease of applicability following the structure of the progress guarantee in Lemma 4.1.

**Assumption A4 (Expected sharp growth).** There exist constants \( \lambda_0, \lambda_1 > 0 \) such that for all \( \alpha \in \mathbb{R}_+ \) and \( x \in \mathcal{X} \) and \( x^* \in \mathcal{X}^* \),

\[
\mathbb{E} \left[ \min \left\{ \alpha [f(x; S) - f(x^*; S)], \frac{(f(x; S) - f(x^*; S))^2}{\|f'(x; S)\|^2_2} \right\} \right] \geq \text{dist}(x, \mathcal{X}^*) \min \{\lambda_0 \alpha, \lambda_1 \text{dist}(x, \mathcal{X}^*)\}.
\]

While Assumption A4 is somewhat complex, a simple sufficient condition for it to hold is that there exist constants \( \lambda > 0 \) and \( p > 0 \) such that

\[
\mathbb{P}(f(x; S) - f(x^*; S) \geq \lambda \text{dist}(x, \mathcal{X}^*)) \geq p
\]

and \( \mathbb{E}[\|f'(x; S)\|^2_2] \leq M^2 \) for \( x \in \mathcal{X} \), which is reasonably easy to check (for example, using the Paley-Zygmund inequality and Mendelson’s small-ball conditions [38]).

Under these conditions, we have the following fast convergence guarantee for the APROX family of methods; we provide the proof in Section B.2.

**Proposition 2.** Let Assumption A4 hold and \( x_k \) be generated by the stochastic iteration (4) using any model satisfying Conditions (C.i)–(C.iv), where the stepsizes \( \alpha_k \) satisfy \( \alpha_k = \alpha_0 k^{-\beta} \) for some \( \beta \in (-\infty, 1) \). Define \( K_0 := \left\lfloor (\lambda_0 \alpha_0 / (\lambda_1 \text{dist}(x_1, \mathcal{X}^*)))^{1/\beta} \right\rfloor \). Then

\[
\mathbb{E}[\text{dist}(x_{k+1}, \mathcal{X}^*)^2] \leq \begin{cases} 
\exp \left(-\lambda_1 \min\{k, K_0\} - \frac{\lambda_0}{\text{dist}(x_1, \mathcal{X}^*)} \sum_{i=K_0+1}^k \alpha_i \right) \text{dist}(x_1, \mathcal{X}^*)^2 & \text{if } \beta \geq 0 \\
\exp \left(-\lambda_1 [k - K_0]_+ - \frac{\lambda_0}{\text{dist}(x_1, \mathcal{X}^*)} \sum_{i=1}^{k \wedge K_0} \alpha_i \right) \text{dist}(x_1, \mathcal{X}^*)^2 & \text{if } \beta < 0
\end{cases}
\]

and with probability 1, we have the linear convergence

\[
\limsup_{k \to \infty} \frac{\text{dist}(x_k, \mathcal{X}^*)^2}{(1 - \lambda)^k} < \infty.
\]

Without careful stepsize choices, even non-stochastic subgradient methods do not achieve such convergence guarantees. Indeed, consider the simple objective \( F(x) = |x| \), which certainly satisfies the sharp growth conditions, and apply the subgradient method \( x_{k+1} = x_k - \alpha_k \text{sign}(x_k) \) (where we treat \( \text{sign}(0) = +1 \)). Then the convergence can be no faster than \( O(\alpha_k) \); this is the typical jamming behavior of subgradient methods. In contrast, Proposition 2 shows that by leveraging the knowledge that \( \inf_x F(x) = 0 \), we achieve linear convergence.

### 4.2 Quadratic growth with shared minimizers

As an alternative to sharp growth conditions, we also consider optimization problems that exhibit quadratic growth—strong convexity—away from their optima. As is the case for the sharpness conditions in Section 4.1, strong convexity conditions play an important role in the analysis and implementation of methods for convex optimization [24, 46, 43, 17] as well as stochastic optimization problems [23, 20]. It is thus of interest to develop an understanding of the behavior of the APROX family of methods under strong convexity conditions, so that we make the following assumption (which is slightly weaker than strong convexity).
**Assumption A5** (Quadratic growth with shared minimizers). There exist constants $\lambda_0, \lambda_1 > 0$ such that for all $x \in X$ and $\alpha > 0$,

$$
\mathbb{E} \left[ (f(x; S) - f(x^*; S)) \min \left\{ \alpha, \frac{f(x; S) - f(x^*; S)}{\|f'(x; S)\|_2^2} \right\} \right] \geq \min \{ \lambda_0 \alpha, \lambda_1 \} \text{dist}(x, X^*)^2.
$$

To give some intuition for the assumption, we note that it holds if $f$ is differentiable, there exist constants $0 < c, C < \infty$ such that with probability $p > 0$ we simultaneously have

$$
\|\nabla f(x; S)\|^2_2 \leq C \|\nabla F(x)\|^2_2 \quad \text{and} \quad f(x; S) - f(x^*; S) \geq c(F(x) - F(x^*)),
$$

and the two conditions that $x \mapsto \nabla F(x)$ is $L$-Lipschitz and $F(x) - F(x^*) \geq c_0 \text{dist}(x, X^*)^2$. In this case the Lipschitz condition implies $F(x^*) \leq F(y) \leq F(x) + \langle \nabla F(x), y - x \rangle + \frac{L}{2} \|y - x\|^2_2$, and setting $y = x - \frac{1}{L} \nabla F(x)$ gives $F(x) - F(x^*) \geq \frac{1}{2L} \|\nabla F(x)\|_2^2$ or $\|\nabla F(x)\|_2 \geq \frac{1}{f(x) - f(x^*)}$. Thus, under these conditions,

$$
\mathbb{E} \left[ (f(x; S) - f(x^*; S)) \min \left\{ \alpha, \frac{f(x; S) - f(x^*; S)}{\|f'(x; S)\|_2^2} \right\} \right] \\
\geq p(F(x) - F(x^*)) \min \left\{ \alpha, \frac{F(x) - F(x^*)}{C\|\nabla F(x)\|_2^2} \right\} \geq p c_0 \text{dist}(x, X^*)^2 \min \{ \alpha, (2CL)^{-1} \}.
$$

Under the quadratic growth Assumption A5, whenever the problem is easy (Definition 4.1) Lemma 4.1 implies the following proposition, which gives nearly linear convergence of the aProx family whenever the lower bound condition (C.iv) holds.

**Proposition 3.** Let Assumption A5 hold and $x_k$ be generated by the stochastic iteration (4) by any model satisfying Conditions (C.i)–(C.iv), where the stepsizes $\alpha_k$ satisfy $\alpha_k = \alpha_0 k^{-\beta}$ for some $\beta \in (-\infty, \infty)$. Define $K_0 = \lceil (\lambda_0 \alpha_0 / \lambda_1)^{1/\beta} \rceil$. If $\beta \geq 0$, then

$$
\mathbb{E}[\text{dist}(x_{k+1}, X^*)^2] \leq \exp \left( -\lambda_1 \min\{k, K_0\} - \lambda_0 \sum_{i=K_0+1}^{k} \alpha_i \right) \text{dist}(x_1, X^*)^2,
$$

while if $\beta < 0$, then

$$
\mathbb{E}[\text{dist}(x_{k+1}, X^*)^2] \leq \exp \left( -\lambda_1 \max\{k - K_0, 0\} - \lambda_0 \sum_{i=1}^{K_0} \alpha_i \right) \text{dist}(x_1, X^*)^2.
$$

Under similar strong convexity assumptions, Schmidt and Le Roux [52] and Ma et al. [36] show that stochastic gradient methods can achieve linear or near-linear convergence for easy convex optimization problems. As is typical in the analysis of stochastic gradient methods, however, this requires precise stepsize choices that reflect typically unknown constants, such as global Lipschitz conditions and the strong convexity parameter $\lambda_0$. In contrast, the aProx family of methods is adaptive to the easiness of the problem—achieving optimal asymptotic behavior (as in Section 3) while providing strong finite-sample guarantees and nearly linear convergence (Proposition 3) when problems satisfy strong growth conditions.

### 4.3 Examples of easy problems with strong growth

We now return to the three examples 4–6 fitting our framework of easy (Definition 4.1) problems with shared minimizers, exhibiting quantitative growth conditions for each.
4.3.1  Overdetermined linear systems and Kaczmarz algorithms

Kaczmarz-like algorithms [55, 34, 41, 40] for overdetermined linear systems, as in Example 4, are effective, solving feasible systems $Ax = b$ (where $A \in \mathbb{R}^{m \times n}, m \geq n$) using careful stochastic gradient steps on the objective $\|Ax - b\|_2^2$ to achieve fast convergence. We consider instead the mean absolute error $F(x) := \frac{1}{m} \|Ax - b\|_1$. As we show presently, the objective $F(x) = \frac{1}{m} \sum_{i=1}^m |\langle a_i, x \rangle - b_i|$ typically satisfies the sharp growth condition A4, so that the APROX method (4) using the truncated model (7) achieves linear convergence. The convergence rates we describe are looser than those that more sophisticated analyses of Kaczmarz algorithms yield [55, 41, 40], but in this case, we have the additional benefit that the APROX methods are adaptive to the problem at hand and still convergent outside of linear systems.

Let us provide conditions sufficient to demonstrate Assumption A4. Let the vectors $a_i$ be drawn independently from a distribution with $\|a_i\|_2 \leq M$, and assume for small $\varepsilon > 0$ there exists $\ell > 0$ such that $\mathbb{P}(|\langle a_i, v \rangle| \geq \varepsilon \|v\|_2) \geq \ell$ for $v \in \mathbb{R}^n$. Letting $b_i = \langle a_i, x^* \rangle$ and $f_i(x) = |\langle a_i, x \rangle - b_i|$, we have the following lemma; see Appendix B.4 for its proof.

**Lemma 4.2.** Let the preceding conditions on the vectors $a_i$ hold. There exists a numerical constant $C < \infty$ such that for $c > 0$ and $t \geq 0$, if we define

$$\lambda_0 := c \left( p_c - C \sqrt{\frac{n + t}{m}} \right) \quad \text{and} \quad \lambda_1 := \frac{c^2}{M^2} \left( p_c - C \sqrt{\frac{n + t}{m}} \right),$$

then with probability at least $1 - e^{-t}$ over the randomness in the $a_i$, simultaneously for all $x$

$$\frac{1}{m} \sum_{i=1}^m \min \left\{ \alpha [f_i(x) - f_i(x^*)], \frac{(f_i(x) - f_i(x^*))^2}{\|f_i(x)\|_2^2} \right\} \geq \|x - x^*\|_2 \min \{\lambda_0 \alpha, \lambda_1 \|x - x^*\|_2\}.$$

That is, with high probability over the choice of $A \in \mathbb{R}^{m \times n}$, Assumption A4 holds with parameters $\lambda_0$ and $\lambda_1$. As a more concrete example, suppose the vectors $a_i$ are uniform on $\sqrt{n} \cdot S^{n-1}$, the sphere of radius $\sqrt{n}$. Then $\mathbb{P}(|\langle a_i, v \rangle| \geq \frac{1}{2} \|v\|_2) \geq \frac{1}{2}$, so that $\lambda_1 \gtrsim 1/n$ with high probability, and Proposition 2 implies that $\limsup_k \|x_k - x^*\|^2/(1 - C/n)^k < \infty$. Roughly, then, $O(1) \cdot n \log \frac{1}{\varepsilon}$ iterations of the APROX update (4) are sufficient to achieve $\varepsilon$-accuracy in the solution of $Ax = b$, each of which requires time $O(n)$, yielding a total operation count of $O(1) \cdot n^2 \log \frac{1}{\varepsilon}$. This is a less precise version of the bound Strohmer and Vershynin [55, Sec. 2.1] attain for Kaczmarz methods on well-conditioned problems; we achieve it without specialization beyond truncated APROX models.

4.3.2  Finding a point in the intersection of convex sets

We return now to Example 5, building connections with randomized projection algorithms [1, 34, 35]. As in the example, let $C_1, C_2, \ldots, C_m$ be closed convex sets, where $\mathcal{X}^* := \bigcap_{i=1}^m C_i$ is non-empty. The conditioning of the problem of finding a point in this intersection is related to the relative ratio of $\text{dist}(x, X)$ to the maximum distance to any of the sets $C_i$ (cf. [34, 35]). As a simple special case, if the sets $C_i$ are all halfspaces of the form $C_i = \{x \mid \langle a_i, x \rangle \leq b_i\}$, then the Hoffman error bound [25] implies that $\|Ax - b\|_{\infty} \geq c \text{dist}(x, \mathcal{X}^*)$ for a constant $c > 0$; this result also holds in infinite dimensions under a constraint qualification [26]. Abstracting away the particular form of the sets $C_i$, consider the objective $F(x) = \frac{1}{m} \sum_{i=1}^m \text{dist}(x, C_i)$, and assume an analog of the Hoffman bound, that is, for that constant $\lambda > 0$ we have

$$\text{dist}(x, \mathcal{X}^*) \geq \max_i \text{dist}(x, C_i) \geq \lambda \text{dist}(x, \mathcal{X}^*),$$  \hspace{1cm} (10)
where the first inequality always holds, and the constant $\lambda > 0$ is analogous to a condition number [35, 34]. For example, if $C_i = \{x \mid (a_i, x) \leq b_i\}$ are halfspaces with $\|a_i\| = 1$, and we only have two halfspaces, then $\lambda = \sqrt{(1 + \langle a_1, a_2 \rangle)/2}$ satisfies inequality (10).

To understand the growth properties of $F(x) = \frac{1}{m} \sum_{i=1}^{m} \text{dist}(x, C_i)$, recall [24, Ex. VI.3.3] that $\nabla \text{dist}(x, C_i) = (x - \pi_{C_i}(x))/\|\pi_{C_i}(x) - x\|_2$ for $x \notin C_i$, where $\pi_{C_i}(x)$ denotes projection onto $C_i$, and $\partial \text{dist}(x, C_i)$ consists of those $v$ in the normal cone to $C_i$ at $x$ with $\|v\|_2 \leq 1$ if $x \in C_i$, so that the subgradients of the component functions $\text{dist}(x, C_i)$ have norm bounded by 1. Then letting $I$ be uniform in $\{1, \ldots, m\}$, we obtain for any $\alpha > 0$ that

$$
\mathbb{E}[\text{dist}(x, C_I) \min\{\alpha, \text{dist}(x, C_I)\}] = \frac{1}{m} \sum_{i=1}^{m} \text{dist}(x, C_i) \min\{\alpha, \text{dist}(x, C_i)\}
$$

$$
\geq \frac{1}{m} \max_{i} \text{dist}(x, C_i) \min\{\alpha, \max_{i} \text{dist}(x, C_i)\}
$$

$$
\geq \text{dist}(x, X^*) \min\left\{\frac{\alpha \lambda}{m}, \frac{\lambda}{m} \text{dist}(x, X^*)\right\}.
$$

In particular, Assumption A4 holds with constant $\lambda_1 = \lambda/m$, and thus Proposition 2 implies the convergence $\limsup_k \text{dist}(x_k, X^*)^2/(1 - \lambda/m)^k < \infty$. Roughly $\frac{m}{\lambda} \log \frac{1}{\epsilon}$ steps are sufficient to find a point $x_k$ that is within distance $\epsilon$ of the set $X^*$ using the APROX family.

### 4.3.3 Interpolation problems

Finally, we consider Example 6, which involves the solution of statistical machine learning problems in which one interpolates the data. Recent work by Belkin and collaborators [2, 3, 36] suggests that in a number of statistical machine learning problems, it is possible to achieve zero error on a training sample while still achieving optimal convergence rates for the population objective. To give the simplest possible example, we study underparameterized least-squares. Our data comes in $m$ pairs $(a_i, b_i) \in \mathbb{R}^n \times \mathbb{R}$, and $f_i(x) = \frac{1}{2}(\langle a_i, x \rangle - b_i)^2$, where $n > m$. Then Belkin et al. [2, 3, 36] (see also [63, Sec. 5]) argue that the minimum norm interpolant

$$
x^* := \arg\min_{x \in \mathbb{R}^n} \{\|x\|_2 \mid \langle a_i, x \rangle = b_i \text{ for } i = 1, \ldots, m\}
$$

possesses certain statistical optimality properties while exhibiting strong empirical prediction performance. An equivalent formulation is to take any loss $\ell : \mathbb{R} \to \mathbb{R}$ with $\ell(0) = 0$ and $\ell(t) > 0$ for $t \neq 0$ and write

$$
x^* = \arg\min_{x \in \mathbb{R}^n} \left\{\sum_{i=1}^{m} \ell(\langle a_i, x \rangle - b_i) \mid x \in \text{span}\{a_1, \ldots, a_m\}\right\}.
$$

In our case, the truncated models (7) guarantee that the iterates of the APROX family lie in the span of the vectors $a_i$. With our choice $f_i$ above, if we let $M := \max_i \|a_i\|_2$ then

$$
\frac{1}{m} \sum_{i=1}^{m} f_i(x) \min\left\{\alpha, \frac{f_i(x)}{\|\nabla f_i(x)\|_2}\right\} \geq \frac{1}{2m} \|A(x - x^*)\|_2^2 \min\left\{\alpha, \frac{1}{M^2}\right\}.
$$

Let $USV^T = A$ be the singular value decomposition of $A$, so

$$
\|A(x - x^*)\|_2^2 = \sum_{i=1}^{m} \sigma_i(A)^2 \langle v_i, x - x^* \rangle^2 \geq \sigma_m(A)^2 \|VV^T(x - x^*)\|_2^2.
$$
As \( x^* \in \text{span}\{a_i\} \), whenever \( x \in \text{span}\{a_i\} \), we have \( \|VV^T(x-x^*)\|_2 = \|x-x^*\|_2 \), and Assumption A5 holds for such \( x \): if \( I \) is uniform on \( \{1, \ldots, m\} \) then

\[
\mathbb{E} \left[ (f_I(x) - f_I(x^*)) \min \left\{ \alpha, \frac{f_I(x) - f_I(x^*)}{\|\nabla f_I(x)\|_2^2} \right\} \right] \geq \frac{\sigma_m(A)^2}{m} \min \{\alpha, M^{-2}\} \|x-x^*\|_2^2.
\]

Random matrices \( A \in \mathbb{R}^{m \times n} \) with independent rows typically satisfy an inequality of the form

\[
\sigma_m(A) \gtrsim \mathbb{E}[\|a\|_2^2]^{1/2}(1 - \sqrt{m/n}) \quad \text{with high probability (cf. [62])}.
\]

Assuming that \( M^2 \lesssim \mathbb{E}[\|a\|_2^2] \), we see that in this case, Assumption A5 then holds with constants \( \lambda_0 \gtrsim \frac{1}{m} \mathbb{E}[\|a\|_2^2] \) and \( \lambda_1 \gtrsim \frac{1}{m} \).

Summarizing, under typical scenarios, Proposition 3 guarantees the following

**Corollary 4.1.** Consider the underdetermined least squares problem above, where the matrix \( A \) has rows with constant norm \( \|a_i\|_2 = \sqrt{n} \) and \( \sigma_m(A) \gtrsim \sqrt{m} \), and let \( x_k \) be generated by the iteration (4) with the truncated model (7) and stepsizes \( \alpha_k = \alpha_0 k^{-\beta} \) for some \( \beta > 0 \). Then there exists a constant \( C \) depending on \( n, m, \alpha_0, \) and \( \beta \) such that for all \( k \in \mathbb{N} \)

\[
\mathbb{E}[\|x_k - x^*\|_2^2] \leq C \max \left\{ \exp \left( -\frac{k}{m} \right), \exp \left( -\frac{\alpha_0}{1-\beta} k^{1-\beta} \right) \right\} \|x_1 - x^*\|_2^2.
\]

Corollary 4.1 shows that the iterates exhibit nearly linear convergence for the interpolation problem we consider. With a more careful stepsize choice, gradient methods achieve linear convergence [52, 36, Section 4]. However, in our scenario, we achieve these convergence rates without careful stepsizes choices, while the method still enjoys convergence in other problems; the recommended choices of stepsize in other work [36, 52] cause stochastic gradient methods to be non-convergent except on the narrower set of “easy” problems they consider.

## 5 Non asymptotic convergence results

For our final set of theoretical results, we provide two propositions on the non-asymptotic convergence of APROX methods. We begin with a treatment of minimization of Lipschitz functions, showing that all APROX models are sufficient for the typical convergence guarantees; after this, we show that for any functions exhibiting a type of strong convexity, the stochastic proximal point method achieves strong non-asymptotic guarantees.

### 5.1 Convergence on Lipschitz functions

In contrast to our results and assumptions in Section 3, in most treatments of stochastic convex optimization, one makes some type of Lipschitzian assumptions on the random functions \( f \) (e.g. [65, 42, 6]). In this case, as we show presently, any method using the model based iteration (4) and satisfying Conditions (C.i)–(C.iii) achieves the familiar convergence rates of the stochastic subgradient method.

**Assumption A6.** There exists \( M^2 < \infty \) such that for each \( x \in \mathcal{X} \), the functions \( f \) satisfy \( \mathbb{E}[\|f'(x; S)\|_2^2] \leq M^2 \).

Under this Lipschitzian assumption, we have the following extension of the well-known results on convergence of the stochastic gradient method [65, 42] (also generalizing Bertsekas [6]).
Proposition 4. Let Assumptions A1 and A6 hold, and let the iterates $x_k$ be generated by algorithm (4) by any model satisfying Conditions (C.i)–(C.iii). Define $\bar{x}_k = (\sum_{i=1}^k \alpha_i)^{-1} \sum_{i=1}^k \alpha_i x_i$. Then

$$\mathbb{E}[F(\bar{x}_k)] - F(x^*) \leq \frac{1}{2} \frac{\|x_0 - x^*\|^2_2}{\sum_{i=1}^k \alpha_i} + \frac{M^2}{2} \sum_{i=1}^k \alpha_i.$$ 

If $\mathcal{X}$ is compact with $R := \sup_{x \in \mathcal{X}} \|x - x^*\|_2$, then the average $\bar{x}_k := \frac{1}{k} \sum_{i=1}^k x_i$ satisfies

$$\mathbb{E}[F(\bar{x}_k)] - F(x^*) \leq \frac{R^2}{2k\alpha_k} + \frac{M^2}{2k} \sum_{i=1}^k \alpha_i.$$ 

The proof is a more or less standard application of Lemma 3.4, so we defer it to Appendix C.1.

5.2 Strong convexity

We present one final theoretical result, showing how the stochastic proximal point method (the full model (6)) achieves reasonable non-asymptotic convergence guarantees even when the functions $f$ may be non-Lipschitz and non-smooth, as long as they obey a (restricted) strong convexity condition. The convergence guarantees we present here are impossible with standard stochastic gradient methods, which can diverge under the assumptions we consider.

Assumption A7 (Restricted strong convexity). The functions $f(\cdot; s)$ are strongly convex with respect to the matrix $\Sigma(s) \succeq 0$, that is,

$$f(y; s) \geq f(x; s) + \langle f'(x; s), y - x \rangle + \frac{1}{2} (x - y)^T \Sigma(s)(x - y)$$

for all $f'(x; s) \in \partial f(x; s)$. The matrix $\Sigma$ satisfies $\mathbb{E}[\Sigma(S)] \succeq \lambda_{\min} I_{n \times n}$, where $\lambda_{\min} > 0$.

We now present an analysis of the stochastic proximal point method under Assumption A7. We begin with a technical lemma, which provides a guarantee on the one-step progress of the method.

Lemma 5.1. Let Assumption A7 hold and the iterates $x_k$ be generated by the stochastic proximal point method (3). Define

$$\Sigma_k := \mathbb{E} \left[ \frac{1}{1 + 2\alpha_k \lambda_{\max}(\Sigma(S))} \Sigma(S) \right].$$

Then

$$\frac{1}{2} \mathbb{E} \left[ \|x_{k+1} - x^*\|^2_2 \mid \mathcal{F}_{k-1} \right] \leq \frac{1}{2} (x_k - x^*)^T (I - \alpha_k \Sigma_k) (x_k - x^*) + \alpha_k^2 \mathbb{E} \left[ \|f'(x^*; S)\|^2_2 \right].$$

Lemma 5.1 is somewhat technical, and we present its proof in Appendix C.2.

Lemma 5.1 guarantees that the proximal-point method makes progress whenever the restricted strong convexity conditions hold, irrespective of smoothness of the objective functions. First, we have $0 < \Sigma_0 \preceq \Sigma_k$ for all $k \in \mathbb{N}$, and defining $\lambda_k := \lambda_{\min}(\Sigma_k) > 0$, we have $\lambda_k \uparrow \lambda_{\infty} := \lambda_{\min}(\mathbb{E}[\Sigma(S)])$ when $\alpha_k \downarrow 0$. Under the conditions of Lemma 5.1 and Assumption A1, we thus have

$$\mathbb{E} \left[ \|x_{k+1} - x^*\|^2_2 \mid \mathcal{F}_{k-1} \right] \leq (1 - \alpha_k \lambda_k) \|x_k - x^*\|^2_2 + \alpha_k^2 \sigma^2 \leq (1 - \alpha_k \lambda_0) \|x_k - x^*\|^2_2 + \alpha_k^2 \sigma^2.$$
Applying this inequality recursively gives

$$\mathbb{E}[\|x_{k+1} - x^*\|_2^2] \leq \prod_{i=1}^{k} (1 - \alpha_i \lambda_i) \left\| x_1 - x^* \right\|_2^2 + \sum_{i=1}^{k} \alpha_i^2 \prod_{j=i+1}^{k} (1 - \alpha_j \lambda_j) \sigma^2,$$

where we note that $\alpha_j \lambda_j < 1$ for all $j$. Then a standard inductive argument \cite{47} (see also the technical Lemma A.7 in Sec. A.2) implies the next proposition.

**Proposition 5.** Let Assumptions A1 and A7 hold, and let $x_k$ be generated by the stochastic proximal point method (3) with stepsizes $\alpha_k = \alpha_0 k^{-\beta}$ for some $\beta \in (0, 1)$. Then for a numerical constant $C < \infty$,

$$\mathbb{E}[\|x_{k+1} - x^*\|_2^2] \leq \exp \left( -\lambda_0 \sum_{i=1}^{k} \alpha_i \right) \left\| x_1 - x^* \right\|_2^2 + C \cdot \frac{\sigma^2}{\lambda_0} \alpha_k \cdot \log k.$$

An asymptotic argument \cite{47} gives convergence $\mathbb{E}[\|x_k - x^*\|_2^2] \lesssim \frac{\sigma^2}{\lambda_0} \alpha_k$ for large $k$. For stochastic proximal point methods, if the objective is strongly convex, choosing $\alpha_k = C/k$ for a large constant $C$ yields asymptotic convergence bounds on $\|x_k - x^*\|_2^2$ of the form $\frac{\sigma^2}{\lambda_0} \frac{1}{k}$. In comparison to convergence results available for stochastic gradient methods \cite{46, 42, 39}, Proposition 5 holds whenever the subgradient $f'(x^*; S)$ has finite second moment, and we require this only at the point $x^*$; moreover, it holds no matter the stepsize sequence.

6 Experiments

The final component of this paper is an empirical evaluation of the aProx methods. Our goal in the experiments is to evaluate the relative merits of different approximate models in the iteration (4), and accordingly, we consider the four approximations below.

(i) Stochastic gradient method (SGM): uses the linear model (5).

(ii) Proximal: uses the full model (6).

(iii) Truncated: Uses the lower truncated model (7).

(iv) Multi-line: Uses the multi-line (bundle) model (8) with two lines, that is, with $i = 1$.

We present several experiments, each comparing different aspects of the aProx models. Throughout, we consider step size sequences of the form $\alpha_k = \alpha_0 k^{-\beta}$, where $\beta \in (1/2, 1)$. We wish to evaluate the robustness and stability of each of the models (5)–(8) for different problem scenarios, investigating both well- and poorly-conditioned problems, as well as problems satisfying the “easy” conditions of Definition 4.1. Before we begin the experimental results proper, let us provide some guidance toward expected results. Roughly, for standard smooth statistical problems with globally Lipschitz gradients (such as linear regression problems with noise), we expect the methods to have fairly similar performance—stochastic gradient, proximal point, truncated, and multi-line models are all asymptotically normal with optimal covariance. However, as problems become either (i) easier—closer to satisfying Definition 4.1—or (ii) more poorly conditioned or harder because of large or unbounded Lipschitz constants, we expect stochastic gradient methods to (i) converge more slowly or (ii) be substantially more sensitive to stepsize choices.
For each of our experiments, we conduct a careful comparison of the methods. Within each experiment, we run each model-based iteration (4) for \( K \) total iterations across multiple initial stepsizes \( \alpha_0 \). For a fixed accuracy \( \epsilon > 0 \), we record the number of steps \( k \) required to achieve \( F(x_k) - F(x^*) \leq \epsilon \), reporting these times (where we terminate each run at iteration \( K \)). We perform \( T \) experiments for each initial stepsize choice, reporting the median of the time-to-\( \epsilon \)-accuracy, as well as displaying 90% confidence intervals.

### 6.1 Linear Regression

![Figure 2](image_url)

**Figure 2.** The number of iterations to achieve \( \epsilon \)-accuracy versus initial stepsize \( \alpha_0 \) for linear regression with \( m = 1000, n = 40 \), and condition number \( \kappa(A) = 1 \). (a) The noiseless setting with \( \sigma = 0 \). (b) Noisy setting with \( \sigma = \frac{1}{2} \).

![Figure 3](image_url)

**Figure 3.** The number of iterations to achieve \( \epsilon \)-accuracy versus initial stepsize \( \alpha_0 \) for linear regression with \( m = 1000, n = 40 \), and condition number \( \kappa(A) = 15 \). (a) The noiseless setting with \( \sigma = 0 \). (b) Noisy setting with \( \sigma = \frac{1}{2} \).
In our linear regression experiments, we let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $F(x) = \frac{1}{2m} \|Ax - b\|_2^2$, where in each individual experiment we generate $x^* \sim \mathcal{N}(0, I_n) \in \mathbb{R}^n$ and set $b = Ax^* + \sigma v$ for $v \sim \mathcal{N}(0, I_m)$. We choose $\sigma$ differently depending on the experiment, setting $\sigma = 0$ in noiseless experiments and $\sigma = \frac{1}{2}$ otherwise. We generate $A$ as $A = QD$, where $Q \in \mathbb{R}^{m \times n}$ has uniformly random orthogonal columns, and $D = \text{diag}(1, 1 + (\kappa - 1)/(n - 1), \ldots, \kappa)$ is a diagonal matrix with linearly spaced entries between 1 and a desired condition number $\kappa \geq 1$.

In this case, we have $f(x; (a_i, b_i)) = \frac{1}{2}(\langle a_i, x \rangle - b_i)^2$.

In Figure 2, we plot the results of our experiments on well-conditioned problems, which use matrices $A$ with condition number $\kappa(A) = 1$, while Figure 3 shows identical results except that we use condition number $\kappa(A) = 15$. Plot (a) of each figure demonstrates the results for the noiseless setting with $\sigma = 0$. In Fig. 2(a), we see the expected result that the stochastic gradient method has good performance for a precise range of stepsizes in $[10^{-1}, 1]$, while the better approximations of the proximal point (6) and the truncated (7) models yield better convergence over a range of stepsizes with six orders of magnitude. The multi-line model (8) shows somewhat more robustness than SGM, but for large stepsizes also exhibits some oscillation. In the noisy cases, plots (b) in each figure, the results are similar, except that the full proximal model is somewhat less robust to stepsize choice; roughly, in the stochastic proximal point (SPPM) case, the model trusts the instantaneous function too much, and overfits to the noise at each iteration.

Figure 3 tells a similar story to Figure 2, except that the stochastic gradient method is essentially not convergent: in no experiment did it ever achieve accuracy below $\epsilon = 0.05$ in the noisy or noiseless settings. This problem is not in any real sense challenging: a condition number of $\kappa(A) = 15$ is not particularly poorly conditioned [59], yet stochastic gradient methods exhibit very poor behavior. These plots suggest that the reliance on stochastic gradient methods in much of the statistical and machine learning literature may be misplaced.

### 6.2 Absolute Loss Regression

Figure 4. The number of iterations to achieve $\epsilon$-accuracy as a function of the initial step size $\alpha_0$ for absolute loss regression with $m = 1000$, $n = 40$, and condition number $\kappa(A) = 1$. (a) Noiseless setting with $\sigma = 0$. (b) Noisy setting with $\sigma = \frac{1}{2}$.

For our second set of experiments, we consider regression with an absolute loss, using
$F(x) = \frac{1}{m} \|Ax - b\|_1$ and $f(x; (a, b)) = |\langle a, x \rangle - b|$, which is non-smooth but Lipschitz. We generate $A$ and $b$ identically to the linear regression experiments in Sec. 6.1. Based on our results in Section 4.3.1, we expect that in the noiseless case, the proximal-point (6), truncated (7), and multi-line (8) methods should have very fast convergence, which is indeed what we see in Figure 4(a). (For this objective, the multi-line model functions identically to the stochastic proximal point method.) When the problems are easy, even when they are non-smooth, a more careful (even simple truncated) model yields substantial improvements over naive linear models (5). In Figure 4(b), we display results with noise with standard deviation $\sigma = \frac{1}{2}$; in this case, the better approximations to $f$ achieve convergence for a slightly larger range of stepsizes—statistically significant as their confidence regions exhibit—but the difference is less substantial. Experiments with condition numbers $\kappa(A) = 15$ yielded results completely parallel to those in Figure 3.

6.3 Logistic Regression

We now turn to classification experiments, beginning with a logistic regression experiment. In logistic regression, widely used for fitting models for binary classification in statistics and machine learning [22], we have data pairs $(a_i, b_i) \in \mathbb{R}^n \times \{\pm 1\}$, and we wish to minimize

$$F(x) := \frac{1}{m} \sum_{i=1}^{m} f(x; (a_i, b_i)) \quad \text{where} \quad f(x; (a, b)) = \log \left( 1 + e^{-b \langle a, x \rangle} \right).$$

We generate the data as follows: we sample $a_i \overset{iid}{\sim} N(0, I_n)$ and $u^* \sim N(0, I_n)$. We then define the labels $b_i = \text{sign}(\langle a_i, u^* \rangle)$; in the noisy setting we flip each label’s sign independently with probability $p$.

We present the results of this experiment in Figures 5 and 6, including plots for both the noiseless (perfectly separated) and noisy cases (plots (a) and (b) in each figure, respectively), where Fig. 6 displays results when the condition number of the data matrix $A$ is $\kappa(A) = 15$. These plots demonstrate similar results to those in the preceding sections, though there are a few differences. First, in the noiseless setting, there is no optimizer $x^*$, as the optimal value is
Figure 6. The number of iterations to achieve $\epsilon$-accuracy as a function of the initial step size $\alpha_0$ for logistic regression with parameters $n = 40$, $m = 1000$, and condition number $\kappa(A) = 15$. (a) Noiseless experiment. (b) Labels flipped with probability $p = .01$.

$\lim_{t \to \infty} F(tu^*) = 0$, yet still we see the benefits of the more accurate models in Figures 5(a) and 6(a). Moreover, the truncated and proximal models exhibit a wider range of convergent stepsizes than the simple stochastic gradient method does even in the noisy case.

6.4 Multi Class Hinge Loss

Figure 7. The number of iterations to achieve $\epsilon$-accuracy as a function of the initial step size $\alpha_0$ for multi class hinge loss with parameters $n = 15$, $m = 2000$, $K = 10$ and (a) label randomization probability $p = 0$ and (b) label randomization probability $p = .01$.

In our second classification experiment, we focus on a somewhat more complex multi-class setting, using the multi-class hinge loss [12]. In this setting, we receive $m$ vectors $a_i \in \mathbb{R}^n$ and a correct label $\ell_i \in [K]$ for each $i$, where $K$ is the number of classes. We wish to find a
classifier, represented as a collection of $K$ vectors $X = [x_1 \cdots x_K] \in \mathbb{R}^{n \times K}$ that minimizes

$$F(X) = \frac{1}{m} \sum_{i=1}^{m} \max_{j \neq \ell_i} \left[ 1 + \langle a_i, x_j - x_{\ell_i} \rangle \right]_+.$$ 

In the case that the data is separable with a positive margin, meaning that there exists $X^*$ such that $\langle a_i, x_{\ell_i} \rangle \geq 1 + \langle a_i, x_j \rangle$ for all $j \neq \ell_i$, this problem is equivalent to that of finding a point in the intersection of halfspaces (recall Section 4.3.2). Accordingly, we expect to see fast convergence for the truncated (7) and stochastic proximal-point (6) models for all large enough stepsizes in these experiments.

To generate the data, we draw vectors $a_i \overset{iid}{\sim} \mathcal{N}(0, I_n)$, then generate an “optimal” classifier $U^* \in \mathbb{R}^{n \times K}$ with i.i.d. $\mathcal{N}(0, 1)$ entries. In the non-noisy setting, we set $\ell_i = \text{argmax}_j \langle u^*_j, a_i \rangle$, while in the noisy setting, for each $i \in \{1, \ldots, m\}$ we resample a value $\ell_i$ uniformly at random with probability $p$. We present the results of this experiment in Figure 7. The experiment reinforces the conclusions of the previous experiments: better models (4) in the aPROX family are significantly more robust to the step size values and achieve generally faster convergence than more naive subgradient methods.

### 6.5 Poisson Regression

![Figure 8.](image)

The number of iterations to achieve $\epsilon$-accuracy as a function of the initial step size $\alpha_0$ for Poisson regression with parameters $n = 40$, and $m = 1000$.

For our final experiment, we consider a Poisson regression problem, as we discuss in Example 3, for which classical results on stochastic approximation do not apply. In this case, we model counts $b_i \in \mathbb{N}$ as coming from a distribution $p(b \mid a, x) = \exp(-e^{\langle a, x \rangle}) \exp(b \langle a, x \rangle) / b!$, giving loss $f(x; (a, b)) = \exp(\langle a, x \rangle) - b(a, x)$. We generate the data by first drawing $u \sim \sqrt{n} \cdot \text{Uni}(S^{n-1})$, then drawing $a_i \overset{iid}{\sim} \mathcal{N}(0, (1/n)I_n)$ and $b_i \sim \text{Poisson}(e^{\langle a_i, u \rangle})$. In this case, $\nabla f(x; (a, b)) = a e^{\langle a, x \rangle} - ab$, which scales exponentially in $x$, thus breaking the typical assumptions for convergence of stochastic gradient methods. In this experiment, computing the proximal update is somewhat non-trivial, as it involves minimizing a quadratic plus exponen-
tial term. Yet, as Example 3 shows, it is quite simple to implement the truncated model (7) by computing $\inf_{x} f(x; (a, b))$.

We present the results of this experiment in Figure 8. It is surprising to us that the stochastic gradient method converges at all on this problem, but with low noise scenarios and small enough stepsizes, it seems that SGM does not leave a region around zero and so is convergent. The perhaps more satisfying result is that the truncated model (7), in spite of its substantially easier calculation, enjoys convergence nearly as robust as that of the stochastic proximal point method.

A Proof of Theorem 2

The proof of the theorem proceeds in a series of lemmas, each of which requires some work. Roughly, the outline is as follows: first, we develop a recursion for the iterates $x_{k+1}$, which parallels a noisy gradient recursion, except that the errors implicitly depend on the next iterate. This allows a decomposition (see Eq. (13)) of $\frac{1}{k} \sum_{i=1}^{k} (x_i - x^*)$ into a leading term, which is obviously asymptotically normal, and several error terms. Two of these errors are standard stochastic approximation errors (similar, for example, to those in Polyak and Juditsky [47]), though we require care in showing they are negligible (see Lemmas A.2, A.1, and A.3). The last error term involves subgradients of the models $f_{x_k}(:, S_k)$ at the point $x_{k+1}$, causing an implicit and potentially non-smooth dependence in the errors. To address this, we provide a gradient comparison result (Lemma A.4), based on Davis et al. [14], which shows that even if the method generating the iterates $x_k$ uses a non-smooth approximation to $f(:, S_k)$, near $x_k$ the subgradients of the model approximate $\nabla f$. This allows us to adapt the results of Polyak and Juditsky [47] on asymptotic optimality of stochastic gradient methods, because the boundedness assumption allows us to eliminate error terms that would otherwise dominate a stochastic gradient method.

Let $\Delta_k = x_k - x^*$, and for $H = \nabla^2 F(x^*)$, define the remainder term $R(x) = \nabla F(x) - H(x - x^*)$, so that $\|R(x)\| = O(\|x - x^*\|^2)$ as $x \to x^*$. We perform an expansion to rewrite the implicit iteration $x_{k+1} = x_k - \alpha_k f'_x(x_k; S_k)$. Defining the localized (sub)gradient errors

$$\zeta_k = \zeta(x_k, S_k) := (\nabla f'(x_k; S_k) - \nabla f(x^*; S_k)) - (F'(x_k) - \nabla F(x^*)),$$

we obtain

$$x_{k+1} = x_k - \alpha_k f'_x(x_k+1; S_k)$$

$$= x_k - \alpha_k \left[ H(x_k - x^*) + R(x_k) + f'(x_k; S_k) - F'(x_k) + f'_x(x_k+1; S_k) - f'(x_k; S_k) \right]$$

$$= x_k - \alpha_k H(x_k - x^*) - \alpha_k \nabla f(x^*; S_k) - \alpha_k [R(x_k) + \zeta_k + \epsilon_k].$$

Subtracting $x^*$ to use $\Delta_k = x_k - x^*$, the implicit iteration (12) becomes

$$\Delta_{k+1} = (I - \alpha_k H)\Delta_k - \alpha_k \nabla f(x^*; S_k) - \alpha_k [R(x_k) + \zeta_k + \epsilon_k].$$

Defining the matrices

$$B^k_i := \alpha_i \sum_{j=i}^{k} \prod_{l=i+1}^{j} (I - \alpha_l H) \quad \text{and} \quad A^k_i := B^k_i - H^{-1},$$

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Polyak and Juditsky [47, Lemma 2] show that

$$\sqrt{k}\Delta_k = \frac{1}{\sqrt{k}} \sum_{i=1}^{k} H^{-1} \nabla f(x^*; S_k)$$

(13)

and additionally \(\sup_{i,k} \|B_k\| < \infty\) and \(\lim_{k} \frac{1}{k} \sum_{i=1}^{k} \|A_i\| = 0\). Evidently, equality (13) implies the theorem as soon as we show each of the terms except \(k^{-1/2} H^{-1} \sum_{i=1}^{k} \nabla f(x^*; S_k)\) are \(o_P(1)\). We thus bound each of the terms in (13) in turn.

**Lemma A.1.** Under the conditions of Theorem 2, \(\|\Delta_k\| \overset{a.s.}{\to} 0\), \(\sum_{k=1}^{\infty} \alpha_k (F(x_k) - F(x^*)) < \infty\), and \(\sum_{k=1}^{\infty} \alpha_k \|\Delta_k\|^2 < \infty\).

**Lemma A.2.** Under the conditions of Theorem 2, \(\frac{1}{\sqrt{k}} \sum_{i=1}^{k} \|R(x_i)\| \overset{a.s.}{\to} 0\).

**Lemma A.3.** Under the conditions of Theorem 2, \(\frac{1}{\sqrt{k}} \sum_{i=1}^{k} \zeta_i \overset{a.s.}{\to} 0\) and \(\frac{1}{\sqrt{k}} \sum_{i=1}^{k} A_i^k \zeta_i \overset{a.s.}{\to} 0\).

See Sections A.1, A.2, and A.3 for proofs of each of these lemmas, respectively.

Lastly, we control the implicit and modeling errors \(\varepsilon_k = f'_x(x_{k+1}; S_k) - \nabla f(x_k; S_k)\). For this, we begin with a gradient comparison lemma, which is a corollary of Davis et al. [14, Thm. 6.1]. See Section A.4 for a proof of the result.

**Lemma A.4.** Let \(f\) and \(h\) be convex and subdifferentiable on \(\mathbb{R}^n\) and \(\varepsilon > 0\), \(r < \infty\). Assume that on the set \(\{x \mid \|x - x^*\| \leq \varepsilon\}\), the function \(f\) has L-Lipschitz gradient. Assume additionally that \(f \geq h\) and at the point \(x_0\), \(h'(x_0) \in \partial h(x_0)\). Then for all \(x\) and \(h'(x) \in \partial h(x)\), if \(\|x - x^*\| \leq \varepsilon/4\) and \(\|x_0 - x^*\| \leq \varepsilon/4\),

$$\|\nabla f(x) - h'(x)\| \leq 2L \|x - x_0\|.$$

Key to the application of Lemma A.4 is that individual iterates move very little.

**Lemma A.5.** Let Conditions (C.i) and (C.iii) hold. Then \(\|x_k - x_{k+1}\| \leq \alpha_k \|f'(x_k; S_k)\|\) for some \(f'(x_k; S_k) \in \partial f(x_k; S_k)\).

**Proof** Let \(g_{k+1} \in \partial f(x_{k+1}; S_k)\) satisfy \(\langle \alpha_k g_{k+1} + (x_{k+1} - x_k), y - x_{k+1} \rangle \geq 0\) for all \(y \in X\). Then, using \(\langle g_k - g_{k+1}, x_k - x_{k+1} \rangle \geq 0\) for any \(g_k \in \partial f(x_k; S_k) \subset \partial f(x_k; S_k)\) and setting \(y = x_k\), we have \(\alpha_k \langle g_k, x_k - x_{k+1} \rangle \geq \alpha_k \langle g_{k+1}, x_k - x_{k+1} \rangle \geq \|x_k - x_{k+1}\|^2\). Cauchy-Schwarz gives the result. \(\square\)

With Lemmas A.4 and A.5 in place, we can control the final error terms in the expansion (13). We provide the proof of the next lemma in Section A.5.

**Lemma A.6.** Under the conditions of Theorem 2, \(\frac{1}{\sqrt{k}} \sum_{i=1}^{k} \|\varepsilon_i\| \overset{a.s.}{\to} 0\).

This completes the proof of Theorem 2.

### A.1 Proof of Lemma A.1

The first two results are consequences of Proposition 1. By the local strong convexity of \(F\) (Assumption A3(i)), for all \(r \in \mathbb{R}_+\) there exists \(\lambda_r > 0\) such that that \(\|x - x^*\| \leq r\) implies \(F(x) - F(x^*) \geq \frac{\lambda_r}{2} \|x - x^*\|^2\). Thus for some (random) \(r < \infty\), we have \(\sum_k \alpha_k (F(x_k) - F(x^*)) \geq \frac{\lambda_r}{2} \sum_k \alpha_k \|\Delta_k\|^2\), which gives the last result of the lemma.
A.2 Proof of Lemma A.2

By Assumption A.3, $R(x) = \nabla F(x) - H(x - x^*)$ satisfies $R(x) = O(\|x - x^*\|^2)$ as $x \to x^*$, and thus for each $r \in \mathbb{R}_+$, there exists some $C_r < \infty$ such that $\|x - x^*\| \leq r$ implies $\|R(x)\| \leq C_r \|x - x^*\|^2$. Now, for $r \in \mathbb{R}_+$ define the stopping time

$$\tau_r := \inf\{k \in \mathbb{N} \mid \|\Delta_k\| \geq r\},$$

so that $\{\tau_r \geq k\} \in \mathcal{F}_k$, as $x_{k+1} \in \mathcal{F}_k$. Then using Lemma 3.4 exactly as in the proof of Proposition 1, we have

$$\mathbb{E}[\|\Delta_{k+1}\|^2 \{\tau_r > k + 1\} \mid \mathcal{F}_{k-1}] \leq \mathbb{E}[\|\Delta_{k+1}\|^2 \{\tau_r > k\} \mid \mathcal{F}_{k-1}]$$

$$\leq 1 \{\tau_r > k\} \left(\|\Delta_k\|^2 - 2\alpha_k (F(x_k) - F^*) + \alpha_k^2 G_{\text{big}}(r)\right).$$

Again using the local strong convexity of $F$ (Assumption A3), there exists $\lambda_r > 0$ such that $F(x_k) - F^* \geq (\lambda_r/2) \|x_k - x^*\|^2$ on the event $\{\tau_r > k\}$, so we obtain

$$\mathbb{E}[\|\Delta_{k+1}\|^2 \{\tau_r > k + 1\} \mid \mathcal{F}_{k-1}] \leq 1 \{\tau_r > k\} \left(1 - \alpha_k \lambda_r\right) \|\Delta_k\|^2 + \frac{\alpha_k^2}{2} G_{\text{big}}(r).$$

A technical lemma, whose proof we defer to section A.6, helps to control this term.

**Lemma A.7.** Let $\alpha_k = \alpha_0 k^{-\beta}$ for some $\beta \in (0,1)$ and $\alpha_0 > 0$, and let $\lambda > 0$ and $\rho > \frac{1}{2}$. Define $p_k := \sum_{i=1}^{k} \alpha_0^i \prod_{j=i+1}^{k} [1 - \lambda \alpha_j]$. Then $\limsup_k p_k / (\alpha_0^{\rho-1} \log k) < \infty$. If additionally $\lambda \alpha_1 \leq 1$, then there exists a numerical constant $C < \infty$ such that

$$p_k \leq C \alpha_0^{\rho} \left(\frac{1}{k} + \frac{1}{k^{\rho \beta}}\right) + C \frac{\log k}{\lambda^{\rho} \alpha_k^{-1}}.$$

A recursive application of inequality (15) and (see also [47, Pg. 851]) yields the bound

$$\mathbb{E}[\|\Delta_k\|^2 \{\tau_r > k\}] \leq o(\alpha_k) + C \sum_{i=1}^{k} \alpha_1^i \prod_{j=i+1}^{k} [1 - \alpha_j]$$

for constants $0 < c, C < \infty$, which may depend on $r$, and therefore Lemma A.7 implies

$$\mathbb{E}[\|\Delta_k\|^2 \{\tau_r > k\}] \leq C_r \alpha_k \cdot \log k$$

for a finite $C_r < \infty$. Once we note that $\|R(x)\| \leq C_r \|x - x^*\|^2$, the remainder of the argument is completely identical to that of Polyak and Juditsky [47, Pg. 851], which demonstrates that

$$\sum_k \|\Delta_k\|^2 / \sqrt{k} < \infty$$

with probability 1 once there exists a (random) $r < \infty$ such that $\tau_r = \infty$, then applies the Kronecker lemma.

A.3 Proof of Lemma A.3

We begin with the first statement. By Lemma A.1, we have $\sum_k \alpha_k \|\Delta_k\|^2 < \infty$. Now we apply Dembo [16, Exercise 5.3.35], which states that if $M_k$ is an $\mathcal{F}_k$-adapted martingale and $b_k \uparrow \infty$ are non-random, if $\sum_k b_k^2 \mathbb{E}[\|M_k - M_{k-1}\|^2 \mid \mathcal{F}_{k-1}] < \infty$ then $b_k^{-1} M_k \overset{a.s.}{\rightarrow} 0$. Recall the definition (11) of $\zeta_k = \zeta(x_k, S_k)$ for $\zeta(x, s) = (f'(x; s) - \nabla f(x^*; s)) - (F'(x) - \nabla F(x^*))$, where $f'(x; s) \in \partial f(x; s)$ and $F'(x) \in \partial F(x)$ (which are both singleton sets near $x^*$ by Assumption A3). Then if $\|x - x^*\| \leq \epsilon$, Assumption A3(ii) guarantees that $\|\zeta(x, s)\| \leq (L(s) + \mathbb{E}[L(S)]) \|x - x^*\|$, while if $\|x - x^*\| \leq r$ we have $\|\zeta(x, s)\| \leq \|f'(x; s)\| + \|\nabla f(x^*; s)\| + G_{\text{big}}(r)$ by Assumption A2. Combining these inequalities, we have that whenever $\|x - x^*\| \leq r$, $\|\zeta(x, s)\|^2 \leq C_r \|x - x^*\|^2$
for a constant $C_r < \infty$. Recall the stopping times (14), $\tau_r = \inf\{k \mid \|\Delta_k\| \geq r\}$, and define the truncated variables

$$\zeta_i^{(r)} := \zeta_i 1\{\tau_r > i\} \in \mathcal{F}_i. \quad (17)$$

With probability 1, there is some finite $r$ such that $\zeta_i^{(r)} = \zeta_i$ for all $i$ by assumption in Theorem 2. Moreover, $E[\zeta_i^{(r)} \mid \mathcal{F}_{i-1}] = E[\zeta_i(x, S_i) \mid \mathcal{F}_{i-1}] 1\{\tau_r > i\} = 0$, and

$$\sum_i \frac{1}{i} E[\|\zeta_i^{(r)}\|^2 \mid \mathcal{F}_{i-1}] \leq C_r \sum_i \frac{1}{i} \|\Delta_i\|^2 \leq C_r \sum_i \alpha_i \|\Delta_i\|^2 < \infty$$

with probability 1 (by Lemma A.1). This gives $k^{-1/2} \sum_{i=1}^k \zeta_i \overset{a.s.}{\to} 0$ as desired, and also shows that $k^{-1/2} \sum_{i=1}^k H^{-1}\zeta_i \overset{a.s.}{\to} 0$.

We now turn to the second claim of the lemma. We first argue that for all $r < \infty$, $k^{-1/2} \sum_{i=1}^k A_i^{k\zeta_i^{(r)}} \overset{P}{\to} 0$, and then use continuity to argue that the sequence converges almost surely, which implies the result as $k^{-1/2} \sum_{i=1}^k A_i^{k\zeta_i^{(r)}} = k^{-1/2} \sum_{i=1}^k A_i^k \zeta_i$ with probability 1 for some $r \in \mathbb{N}, r < \infty$. We begin with the convergence in probability. First, recall inequality (16) in the proof of Lemma A.2, which states that $E[\|\Delta_k\|^2 1\{\tau_r > k\}] \leq C_r \alpha_k \log k$. Then for $r < \infty$, we have that

$$E[\|\zeta_i^{(r)}\|^2] = E \left[ E[\|\zeta_i(x, S_i)\|^2 \mid \mathcal{F}_{i-1}] 1\{\tau_r > i\} \right] \leq E[C_r \|x_i - x^*\|^2 1\{\tau_r > i\}] \leq C_r \alpha_i \log i$$

for a constant $C_r < \infty$ whose value may change from line to line. Thus

$$E \left[ \left\| \frac{1}{\sqrt{k}} \sum_{i=1}^k A_i^{k\zeta_i^{(r)}} \right\|^2 \right] = \frac{1}{k} \sum_{i=1}^k \|A_i^k\|^2 E[\|\zeta_i^{(r)}\|^2] \leq \frac{C_r}{k} \sum_{i=1}^k \|A_i^k\|,$$

where we have used that $\zeta_i^{(r)}$ still form a martingale difference sequence. This converges to zero [47, Lemma 2].

Now, we come to the second claim on the almost sure convergence. Recall the definition $B_i^k = \alpha_i \sum_{j=i}^k \prod_{l=i+1}^j (I - \alpha_l H)$, so that $A_i^k = B_i^k - H^{-1}$. For $r < \infty$, define the sequence

$$Z_{k,r} := \sum_{i=1}^k B_i^k \zeta_i^{(r)}.$$

The lemma will be proved if we can show that for any finite $r$, we have $k^{-1/2} Z_{k,r} \overset{a.s.}{\to} 0$. Note that $B_i^{k+1} - B_i^k = \alpha_i \prod_{j=i+1}^{k+1} (I - \alpha_j H)$, and so if we define

$$W_i^k = \prod_{j=i}^k (I - \alpha_j H), \quad V_{k,r} := \sum_{i=1}^k \alpha_i W_i^{k+1} \zeta_i^{(r)},$$

then $V_{k,r} \in \mathcal{F}_k$ and

$$Z_{k,r} = Z_{k-1,r} + V_{k-1,r} + \alpha_k \zeta_k^{(r)} = \sum_{i=1}^{k-1} V_{i,r} + \sum_{i=1}^k \alpha_i \zeta_i^{(r)} \quad (18).$$

The second sum $\sum_{i=1}^k \alpha_i \zeta_i^{(r)}$ is a square-integrable martingale with summable squared increments, so it converges with probability 1 [16, Ex. 5.3.35], and $1/\sqrt{k} \sum_{i=1}^k \alpha_i \zeta_i^{(r)} \overset{a.s.}{\to} 0$. It thus
suffices to show that $\frac{1}{\sqrt{k}} \sum_{i=1}^{k} V_{i,r}$ converges almost surely, which we do by showing that it is a Cauchy sequence. Now, we note that

$$\mathbb{E}[\|V_{k,r}\|^2] = \sum_{i=1}^{k} \alpha_i^2 \mathbb{E}[W_{i+1}^k \| \cdot \|_r^2] \leq C_r \sum_{i=1}^{k} \alpha_i^2 \exp \left( -c \sum_{j=i+1}^{k+1} \alpha_j \right) \alpha_i \log i,$$

where we have used that $\mathbb{E}[\|s_i\|^2] \lesssim \alpha_i \log i$ as above. As $\alpha_i^3 \log i \ll \alpha_i^{3-\epsilon}$ for all $\epsilon > 0$, Lemma A.7 implies that for all $\epsilon > 0$,

$$\mathbb{E}[\|V_{k,r}\|^2] \leq \frac{C_r}{k^{2\beta-\epsilon}}. \quad (19)$$

We use inequality (19) to demonstrate that the sequence $T_{k,r} := \frac{1}{\sqrt{k}} \sum_{i=1}^{k} V_{i,r}$ is Cauchy. Indeed, we have for all $\epsilon > 0$ that

$$\mathbb{E}[\|T_{k,r} - T_{k+1,r}\|] \leq \left| \frac{1}{\sqrt{k+1}} - \frac{1}{\sqrt{k}} \right| \sum_{i=1}^{k} \mathbb{E}[\|V_{i,r}\|] + \frac{1}{\sqrt{k+1}} \mathbb{E}[\|V_{k+1,r}\|]$$

$$\leq \frac{C}{k^{3/2}} \sum_{i=1}^{k} \frac{1}{i^{\beta-\epsilon}} + \frac{1}{\sqrt{k}} \frac{1}{k^{\beta-\epsilon}} + \frac{1}{\sqrt{k}} \leq Ck^{-\beta-1/2+\epsilon}.$$

As $\beta \in (\frac{1}{2}, 1)$, we have $\sum_k \mathbb{E}[\|T_{k,r} - T_{k+1,r}\|] \ll \sum_k k^{-\beta-1/2+\epsilon} < \infty$ for sufficiently small $\epsilon > 0$. Thus for all $r < \infty$, the sequence $T_{k,r}$ is Cauchy with probability 1, so that $k^{-1/2} \sum_{i=1}^{k} V_{k,r}$ converges. As we know that $k^{-1/2} \sum_{i=1}^{k} V_{k,r} \overset{P}{\rightarrow} 0$ by the previous arguments, the limit point must be zero.

### A.4 Proof of Lemma A.4

The result is a consequence of Davis et al. [14]. A simplified variant of their theorem follows:

**Lemma A.8** (Theorem 6.1 [14]). Let $f$ and $h$ be convex and assume there exists a function $u$ such that $0 \leq f(y) - h(y) \leq u(y)$ for all $y$. Then for any $x$ and $\gamma > 0$, there exists $\hat{x}$ such that for all $h'(x) \in \partial h(x)$, there is $f'(\hat{x}) \in \partial f(\hat{x})$ such that and

$$\|x - \hat{x}\| \leq 2\gamma \quad \text{and} \quad \|h'(x) - f'(\hat{x})\| \leq \frac{u(x)}{\gamma}.$$

In our context, we note that by assumption,

$$f(y) \leq f(x_0) + \langle \nabla f(x_0), y - x_0 \rangle + \frac{L}{2} \|x_0 - y\|^2$$

if $\|x_0 - x^*\| \leq \epsilon$ and $\|y - x^*\| \leq \epsilon$, while otherwise certainly $f(y) < \infty$. Noting that our function $h$ satisfies $h(y) \geq h(x_0) + \langle h'(x_0), y - x_0 \rangle = f(x_0) + \langle f'(x_0), y - x_0 \rangle$, we may take the upper bound function

$$u(y) = \begin{cases} \frac{L}{2} \|y - x_0\|^2 & \text{if } \|y - x^*\| \leq \epsilon, \|x_0 - x^*\| \leq \epsilon \\ +\infty & \text{otherwise,} \end{cases}$$

so that $0 \leq f(y) - h(y) \leq u(y)$ for all $y$. 

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When \( \|x - x^*\| \leq \epsilon/4 \) and \( \|x_0 - x^*\| \leq \epsilon/4 \). In this case, we choose \( \gamma = \frac{1}{2} \|x - x_0\| \leq \epsilon/4 \), so that \( \|x - \hat{x}\| \leq 2\gamma \leq \epsilon/2 \) implies that \( \|\hat{x} - x^*\| \leq 3\epsilon/4 < \epsilon \). Thus, we have

\[
\|h'(x) - \nabla f(x)\| \leq \|h'(x) - \nabla f(\hat{x})\| + \|\nabla f(\hat{x}) - \nabla f(x)\| \\
\leq \frac{L}{2\gamma} \|x - x_0\|^2 + L\|x - \hat{x}\| \leq \frac{L}{2\gamma} \|x - x_0\|^2 + 2L\gamma
\]

by Lemma A.8 and our definition of \( u \). The choice \( \gamma = \frac{1}{2} \|x - x_0\| \) gives the lemma.

### A.5 Proof of Lemma A.6

In the implicit iteration (12), we know by Lemma A.4 that for \( x_k = f'_{x_k}(x_{k+1}; S_k) - \nabla f(x_k; S_k) \), if both \( x_k \) and \( x_{k+1} \) belong to \( \{x : \|x - x^*\| \leq \epsilon/4\} \), then

\[
\|\varepsilon_k\| \leq 2L(S_k) \|x_k - x_{k+1}\| \overset{(*)}{\leq} 2\alpha_k L(S_k) \|\nabla f(x_k; S_k)\| \\
\leq \alpha_k L(S_k)^2 + \alpha_k \|\nabla f(x_k; S_k)\|^2,
\]

where inequality \( (*) \) follows by the single step guarantee in Lemma A.5.

We show that each of the two terms in inequality (20) has appropriately small sum. We have \( \mathbb{E}[L^2(S)] < \infty \) and \( \mathbb{E}\sum_k \alpha_k k^{-1/2} L^2(S_k) \|S_k\| < \infty \), so that the Kronecker lemma implies \( k^{-1/2} \sum_{i=1}^k \alpha_i \|L^2(S_i)\|^{a,\epsilon} \to 0 \). For the second term, let \( \epsilon > 0 \) be such that \( \sum_k \alpha_k k^{-1/2+\epsilon} < \infty \), which must exist as \( \alpha_k = \alpha_0 k^{-\beta} \) for \( \beta \in (1/2, 1) \). Define

\[
Z_k := \frac{1}{k^{1/2-\epsilon}} \sum_{i=1}^k \alpha_i \|f'(x_i; S_i)\|^2,
\]

which is adapted to \( \mathcal{F}_k \). Then

\[
\mathbb{E}[Z_{k+1} | \mathcal{F}_k] \leq \frac{k^{1/2-\epsilon}}{(k+1)^{1/2-\epsilon}} Z_k + (k + 1)^{-1/2+\epsilon} \alpha_{k+1} \|f'(x_{k+1}; S_{k+1})\|^2 | \mathcal{F}_k \\
\leq Z_k + (k + 1)^{-1/2+\epsilon} \alpha_{k+1} \|f'(x_{k+1}; S_{k+1})\|^2 \overset{a,\epsilon}{\rightarrow} 0,
\]

the last inequality following by Assumption A2. On the event that \( \sup_k \|\Delta_k\| < \infty \), we have \( \sum_k k^{-1/2+\epsilon} \alpha_k G_{\text{big}}(\|\Delta_k\|) < \infty \), so that the Robbins-Siegmund Lemma 3.1 applies, and thus \( Z_k \overset{a,\epsilon}{\rightarrow} Z_\infty \) for some finite random variable \( Z_\infty \). Thus \( k^{-\epsilon} Z_k = \frac{1}{\sqrt{k}} \sum_{i=1}^k \alpha_i \|f'(x_i; S_i)\|^{2 \overset{a,\epsilon}{\rightarrow} 0} \).

The definition (4) of the iteration for \( x_{k+1} \) implies that \( f'_{x_k}(x_{k+1}; S_k) = \alpha_k \|x_k - x_{k+1}\| \leq \|\nabla f(x_k; S_k)\| \). Thus, we obtain

\[
\frac{1}{\sqrt{k}} \sum_{i=1}^k \|\varepsilon_i\| \leq \frac{2}{\sqrt{k}} \sum_{i=1}^k 1 \left\{ \|x_i - x^*\| \geq \epsilon/4 \text{ or } \|x_{i+1} - x^*\| \geq \epsilon/4 \right\} \|f'(x_i; S_i)\| \\
+ \frac{1}{\sqrt{k}} \sum_{i=1}^k (\alpha_i L(S_i)^2 + \alpha_i \|f'(x_i; S_i)\|^2),
\]

where the second term is a consequence of inequality (20). Because \( x_k \overset{a,\epsilon}{\rightarrow} x^* \) (Lemma A.1), both of these terms converge to zero with probability 1.
A.6 Proof of Lemma A.7

Because \( \alpha_k \) is decreasing in \( k \), there exists some \( K_0 \) such that \( k > K_0 \) implies \( \lambda \alpha_k \leq 1 \), so that defining \( C_0 = \prod_{i=1}^{K_0} (|1 - \lambda \alpha_i| \vee 1) \exp(\lambda \sum_{i=1}^{K_0} \alpha_i) \), we have

\[
\prod_{j=i+1}^{k} |1 - \lambda \alpha_j| \leq C_0 \exp \left( -\lambda \sum_{j=i+1}^{k} \alpha_j \right). \tag{21}
\]

Using that \( \alpha_j = \alpha_0 j^{-\beta} \), we have \( \sum_{j=i+1}^{k} \alpha_j \geq \alpha_0 \frac{c}{1-\beta} (k^{1-\beta} - (i+1)^{1-\beta}) \) for a numerical constant \( c > 0 \). Consequently, by our definition of \( p_k \) we have

\[
p_k \leq C_0 \sum_{i=1}^{k_0} \alpha_i^\rho \exp \left( -c \frac{\lambda \alpha_0}{1-\beta} (k^{1-\beta} - (i+1)^{1-\beta}) \right).
\]

Now, for \( B > 0 \) and \( k_0 = B k^\beta \log k \), we have for a numerical constant \( c \) that \( k^{1-\beta} - (i+1)^{1-\beta} \geq c B \log k \) for all \( i \leq k_0 \). Taking \( B \geq c (1-\beta)/(\lambda \alpha_0) \), where \( c \) is a numerical constant, we obtain

\[
\lambda \sum_{j=i+1}^{k} \alpha_j \geq c \frac{\lambda \alpha_0}{1-\beta} (k^{1-\beta} - (i+1)^{1-\beta}) \geq \log k
\]

for all \( i \leq k_0 \). Returning to the definition of \( p_k \), we thus obtain

\[
p_k \leq C_0 \sum_{i=1}^{k_0} \alpha_i^\rho \exp(- \log k) + C_0 \sum_{i=k_0+1}^{k} \alpha_i^\rho \leq c \frac{C_0 \alpha_0^\rho}{k} \int_1^k t^{-\rho \beta} dt + \frac{c C_0}{\alpha_0 \lambda} \alpha_k^\rho \cdot k^\beta \log k
\]

\[
= c \left[ \frac{C_0 \alpha_0^\rho}{k} \frac{1}{1-\rho \beta} \left( k^{1-\rho \beta} - 1 \right) + \frac{C_0 \log k}{\lambda} \alpha_k^{\rho-1} \right].
\]

Finally, if \( \lambda \alpha_1 \leq 1 \), then in inequality (21) we may take \( C_0 = 1 \), giving the second result of the lemma.

B Proofs of fast convergence on easy problems

B.1 Proof of Lemma 4.1

We assume without loss of generality that \( f(x^*; s) = 0 \) for all \( x^* \in X^* \), as we may replace \( f \) with \( f - \inf f \). By Lemma 3.3, the update (4) satisfies

\[
\frac{1}{2} \| x_{k+1} - x^* \|_2^2 \leq \frac{1}{2} \| x_k - x^* \|_2^2 + \alpha_k [f_{x_k}(x^*; S_k) - f_{x_k}(x_{k+1}; S_k)] - \frac{1}{2} \| x_{k+1} - x_k \|_2^2.
\]

For shorthand, let \( g_k = f'(x_k; S_k) \) and \( f_k = f(x_k; S_k) \). Now, note that \( f_{x_k}(x^*; S_k) \leq f(x^*; S_k) = 0 \), and by Condition (C.iv) we have \( f_{x_k}(x_{k+1}; S_k) \geq [f_k + \langle g_k, x_{k+1} - x_k \rangle]_+ \). Thus we have

\[
\frac{1}{2} \| x_{k+1} - x^* \|_2^2 \\
\leq \frac{1}{2} \| x_k - x^* \|_2^2 + \alpha_k [f(x^*; S_k) - [f_k + \langle g_k, x_{k+1} - x_k \rangle]_+] - \frac{1}{2} \| x_{k+1} - x_k \|_2^2. \tag{22}
\]
If we let \( \tilde{x}_{k+1} \) denote the unconstrained minimizer

\[
\tilde{x}_{k+1} = \arg\min_x \left\{ f_k + \langle g_k, x - x_k \rangle + \frac{1}{2\alpha_k} \| x - x_k \|^2 \right\} = x_k - \lambda_k g_k \quad \text{for} \quad \lambda_k := \min \left\{ \alpha_k, \frac{f_k}{\| g_k \|^2} \right\},
\]

then because \( x_{k+1} \in \mathcal{X} \) we have

\[
-\alpha_k f_{x_k}(x_{k+1}; S_k) - \frac{1}{2} \| x_{k+1} - x_k \|^2 \leq -\alpha_k f_{x_k}(\tilde{x}_{k+1}; S_k) - \frac{1}{2} \| \tilde{x}_{k+1} - x_k \|^2.
\]

By inspection, the guarded stepsize \( \lambda_k \) guarantees that \([f_k + \langle g_k, \tilde{x}_{k+1} - x_k \rangle]_+ = f_k - \lambda_k \| g_k \|^2\), and thus inequality (22) (setting \( f(x^*; S_k) = 0 \)) yields

\[
\frac{1}{2} \| x_{k+1} - x^* \|^2 \leq \frac{1}{2} \| x_k - x^* \|^2 - \alpha_k f_{x_k}(\tilde{x}_{k+1}; S_k) - \frac{1}{2} \| \tilde{x}_{k+1} - x_k \|^2
\]

\[
= \frac{1}{2} \| x_k - x^* \|^2 - \lambda_k f_k + \frac{\lambda_k^2}{2} \| g_k \|^2.
\]

We have two possible cases: whether \( f_k / \| g_k \|^2 \leq \alpha_k \). In the case that \( f_k / \| g_k \|^2 \leq \alpha_k \), we have \( \lambda_k = \frac{f_k}{\| g_k \|^2} \) and so \(-\lambda_k f_k + \lambda_k^2 \| g_k \|^2 / 2 = -\frac{f_k^2}{2} \| g_k \|^2 \). In the alternative case that \( f_k / \| g_k \|^2 > \alpha_k \), we have \( \lambda_k = \alpha_k \) and \( \alpha_k \| g_k \|^2 / 2 \leq \alpha_k f_k / 2 \). Combining these cases gives

\[
\frac{1}{2} \| x_{k+1} - x^* \|^2 \leq \frac{1}{2} \| x_k - x^* \|^2 - \frac{1}{2} \min \left\{ \alpha_k f_k, \frac{f_k^2}{\| g_k \|^2} \right\},
\]

which is the desired result.

### B.2 Proof of Proposition 2

We adopt a bit of shorthand notation. Let \( D_k = \text{dist}(x_k, \mathcal{X}^*) \), so \( D_k \in \mathcal{F}_{k-1} \). Then Lemma 4.1 implies that under Assumption A4, we have

\[
\mathbb{E}[D^2_{k+1} \mid \mathcal{F}_{k-1}] \leq D^2_k - \min \left\{ \lambda_0 \alpha_k D_k, \lambda_1 D^2_k \right\} = \max \left\{ (1 - \lambda_1), (1 - \lambda_0 \alpha_k / D_k) \right\} D^2_k
\]

\[
\leq \max \left\{ (1 - \lambda_1), (1 - \lambda_0 \alpha_k / D_1) \right\} D^2_k, \quad (23)
\]

where we have used that \( D_1 \geq D_k \) for all \( k \geq 1 \) by Lemma 4.1. Inequality (23) immediately implies that if \( \beta \geq 0 \), then

\[
K_0 = \sup \{ k \in \mathbb{N} \mid \lambda_0 \alpha_k > \lambda_1 D_1 \} = \left\lfloor \frac{\lambda_0 \alpha_0}{\lambda_1 D_1} \right\rfloor^{1/\beta}
\]

is the index for which \( k \geq K_0 \) implies \( \lambda_0 \alpha_k / D_1 \leq \lambda_1 \). This gives the first result of the proposition, \( \mathbb{E}[D^2_{k+1}] \leq \exp(-\lambda_1 \min\{k, K_0\} - \frac{\lambda_0}{D_1} \sum_{i=K_0+1}^k \alpha_i)D^2_1 \) if \( \beta \geq 0 \). For \( \beta < 0 \), the same choice of \( K_0 \) gives \( \mathbb{E}[D^2_{k+1}] \leq \exp(-\lambda_1 [k - K_0]_+ - \frac{\lambda_0}{D_1} \sum_{i=1}^{k \wedge K_0} \alpha_i)D^2_1 \).

For the second result, we prove only the case that \( \beta > 0 \), as the other case is similar. Note that \( \sum_{i=1}^k \alpha_i \geq k^{1-\beta} \), so for any \( \epsilon > 0 \) we have for constants \( 0 < c, C < \infty \) depending on \( \alpha_0 \), \( \beta \), \( \lambda_0 \) and \( \lambda_1 \) that

\[
\sum_{k=1}^{\infty} \mathbb{P}(D_k > \epsilon \alpha_k) \leq \frac{1}{\epsilon} \sum_{k=1}^{\infty} \exp \left( C - ck^{1-\beta} + \log \frac{1}{\alpha_k} \right) < \infty.
\]
The Borel Cantelli lemma implies that $D_k/\alpha_k \to 0$. The first inequality (23) implies that if $V_k = D^2_{k+1}/(1 - \lambda_1)^{k+1}$, then

$$
\mathbb{E}[V_k \mid \mathcal{F}_{k-1}] \leq \frac{D^2_k}{(1 - \lambda)^k} \max \left\{ 1, \frac{1 - \alpha_k/D_k}{1 - \lambda_1} \right\} \leq \left( 1 + \left[ \frac{1 - \alpha_k/D_k}{1 - \lambda_1} \right]_+ \right) V_{k-1}.
$$

As $[(1 - \alpha_k/D_k)/(1 - \lambda_1)]_+ = 0$ eventually with probability 1 and is $\mathcal{F}_{k-1}$-measurable, the Robbins-Siegmund Lemma 3.1 implies that $V_k \to V_\infty$ for some $V_\infty \in \mathbb{R}_+$.

**B.3 Proof of Proposition 3**

Under Assumption A5, the distance recursion for $D_k := \text{dist}(x_k, \mathcal{X}^*)$ in Lemma 4.1 then becomes $D_k \leq D_{k-1}$ and

$$
\mathbb{E}[D_{k+1}^2 \mid \mathcal{F}_{k-1}] \leq \max \{1 - \lambda_0 \alpha_k, 1 - \lambda_1\} D_k^2.
$$

The claim follows by algebraic manipulations.

**B.4 Proof of Lemma 4.2**

Let $\Delta = x - x^*$ for shorthand, noting that $f_i(x^*) = 0$, and note that for any $c > 0$ we have

$$
\frac{1}{m} \sum_{i=1}^{m} \min \left\{ \alpha f_i(x), \frac{f_i(x)^2}{\|f_i(x)\|^2_2} \right\} \geq \frac{1}{m} \sum_{i=1}^{m} \min \left\{ \alpha \|a_i\|, \frac{|\langle a_i, \Delta \rangle|^2}{\|a_i\|^2} \right\}
$$

$$
\geq \frac{1}{m} \sum_{i=1}^{m} c \|\Delta\|_2 1 \{ |\langle a_i, \Delta \rangle| \geq c \|\Delta\|_2 \} \min \left\{ \alpha, \frac{c \|\Delta\|_2}{M^2} \right\},
$$

where the final inequality uses that $\|a_i\|_2 \leq M$ by assumption. Now, the $a_i \in \mathbb{R}^n$ are independent from some distribution on $\mathbb{R}^n$ satisfying $\mathbb{P}(|\langle a_i, \Delta \rangle| \geq c \|\Delta\|_2) \geq p_c > 0$ by assumption. Using that the VC-dimension of linear functions $v \mapsto \langle a_i, v \rangle$ is $O(n)$ a standard VC argument [61, Ch. 2.6] implies that for a numerical constant $C < \infty$, for any $t \geq 0$ with probability at least $1 - e^{-t}$ over the random draw of the $a_i$, we have

$$
\frac{1}{m} \sum_{i=1}^{m} 1 \{ |\langle a_i, \Delta \rangle| \geq c \|\Delta\|_2 \} \geq p_c - C \sqrt{\frac{n + t}{m}}
$$

simultaneously for all $\Delta \in \mathbb{R}^n$. This implies the lemma.

**C Proofs of results from Section 5**

**C.1 Proof of Proposition 4**

The proposition follows nearly directly from Lemma 3.4. Indeed, applying that lemma, for any $x^* \in \mathcal{X}^*$ we have

$$
\frac{1}{2} \mathbb{E}[\|x_{k+1} - x^*\|^2_2 \mid \mathcal{F}_{k-1}] \leq \frac{1}{2} \|x_k - x^*\|^2_2 - \alpha_k [F(x_k) - F(x^*)] + \frac{\alpha_k^2}{2} M^2,
$$

where we have used Assumption A6 and that $x_{k+1} \in \mathcal{F}_{k-1}$. Rearranging, we have

$$
\alpha_k \mathbb{E}[F(x_k) - F(x^*)] \leq \frac{1}{2} \mathbb{E} \left[ \|x_k - x^*\|^2_2 - \|x_{k+1} - x^*\|^2_2 \right] + \frac{\alpha_k^2}{2} M^2.
$$
Summing and telescoping yields \( \sum_{i=1}^{k} \alpha_i \mathbb{E}[ F(x_i) - F(x^*) ] \leq \frac{1}{2} \| x_1 - x^* \|_2^2 + \frac{1}{2} \sum_{i=1}^{k} \alpha_i^2 M^2 \), and dividing by \( \sum_{i=1}^{k} \alpha_i \) and using convexity gives the first result of the proposition.

For the second result, we rearrange the first display above to see that

\[
\mathbb{E}[ F(x_k) - F(x^*) ] \leq \frac{1}{2 \alpha_k} \mathbb{E}[ \| x_k - x^* \|_2^2 - \| x_{k+1} - x^* \|_2^2 ] + \frac{\alpha_k}{2} M^2.
\]

Summing this quantity, we obtain

\[
\sum_{i=1}^{k} \mathbb{E}[ F(x_i) - F(x^*) ] \leq \sum_{i=2}^{k} \left( \frac{1}{2 \alpha_i} - \frac{1}{2 \alpha_{i-1}} \right) \mathbb{E}[ \| x_i - x^* \|_2^2 ] + \frac{1}{2 \alpha_1} \| x_1 - x^* \|_2^2 + \frac{M^2}{2} \sum_{i=1}^{k} \alpha_i.
\]

Noting that \( \mathbb{E}[ \| x_i - x^* \|_2^2 ] \leq R^2 \) by assumption, dividing by \( k \) and applying Jensen’s inequality to \( F(\bar{x}_k) \leq \frac{1}{k} \sum_{i=1}^{k} F(x_i) \) gives the result.

### C.2 Proof of Lemma 5.1

Let \( \Sigma_k = \Sigma(S_k) \), so that for all \( g_k \in \partial f(x_{k+1}; S_k) \) and \( y \in \mathcal{X} \) we have

\[
f(y; S_k) \geq f(x_{k+1}; S_k) + \langle g_k, y - x_{k+1} \rangle + \frac{1}{2} (y - x_{k+1})^T \Sigma_k (y - x_{k+1}).
\]

Using this inequality in place of the last step of the proof of Lemma 3.3 yields

\[
\frac{1}{2} \| x_{k+1} - x^* \|_2^2 + \frac{\alpha_k}{2} (x_{k+1} - x^*)^T \Sigma_k (x_{k+1} - x^*) \leq \frac{1}{2} \| x_k - x^* \|_2^2 - \alpha_k [ f(x_{k+1}; S_k) - f(x^*; S_k) ] - \frac{1}{2} \| x_k - x_{k+1} \|_2^2.
\]

Using that

\[
\begin{align*}
\frac{1}{2} (x_{k+1} - x^*)^T \Sigma_k (x_{k+1} - x^*) = & \frac{1}{2} (x_k - x^*)^T \Sigma_k (x_k - x^*) + \frac{1}{2} (x_{k+1} - x_k)^T \Sigma_k (x_{k+1} - x_k) + (x_{k+1} - x_k)^T \Sigma_k (x_k - x^*) \\
\geq & \frac{1 - \eta}{2} (x_k - x^*)^T \Sigma_k (x_k - x^*) + \frac{\eta - 1}{2 \eta} (x_{k+1} - x_k)^T \Sigma_k (x_{k+1} - x_k)
\end{align*}
\]

for all \( \eta > 0 \), inequality (24) implies

\[
\frac{1}{2} \| x_{k+1} - x^* \|_2^2 \leq \frac{1}{2} (x_k - x^*)^T (I - \alpha_k (1 - \eta) \Sigma_k) (x_k - x^*) - \alpha_k [ f(x_{k+1}; S_k) - f(x^*; S_k) ] - \frac{\alpha_k (\eta - 1)}{\eta} (x_k - x_{k+1})^T (I + \frac{\alpha_k (\eta - 1)}{\eta} \Sigma_k)(x_k - x_{k+1})
\]

for all \( \eta > 0 \).

We wish to guarantee that the last term in the above expansion is negative enough; the choice \( \eta_k = \frac{2 \alpha_k \lambda_{\text{max}}(\Sigma_k)}{1 + 2 \alpha_k \lambda_{\text{max}}(\Sigma_k)} \in (0, 1) \) is sufficient to guarantee \( I + \frac{\alpha_k (\eta_k - 1)}{\eta_k} \Sigma_k \succeq \frac{1}{2} I \). Substituting this choice of \( \eta_k \) above, we obtain that

\[
\frac{1}{2} \| x_{k+1} - x^* \|_2^2 \leq \frac{1}{2} (x_k - x^*)^T (I - \alpha_k (1 - \eta_k) \Sigma_k) (x_k - x^*) - \alpha_k [ f(x_{k+1}; S_k) - f(x^*; S_k) ] - \frac{1}{4} \| x_k - x_{k+1} \|_2^2.
\]
Applying Lemma 3.2 with the choices $y = x^*$, $x_1 = x_{k+1}$, $x_0 = x_k$, $\beta = 2\alpha_k$, and $g = f(\cdot; S_k)$ implies
\[
\frac{1}{2} \|x_{k+1} - x^*\|_2^2 \\
\leq (x_k - x^*)^T (I - \alpha_k(1 - \eta_k)\Sigma_k) (x_k - x^*) + 2\alpha_k \langle f'(x^*; S_k), x^* - x_k \rangle + \alpha_k^2 \|f'(x^*; S_k)\|_2^2.
\]
Taking expectations and using the reasoning in the proof of Theorem 1 that $\mathbb{E}[\langle f'(x^*; S), x^* - x \rangle] \leq 0$ for all $x$ gives the desired result.

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