OPTIMAL ORDER EXECUTION UNDER PRICE IMPACT:
A HYBRID MODEL

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ABSTRACT. In this paper we explore optimal liquidation in a market populated by a number of heterogeneous market makers that have limited inventory-carrying and risk-bearing capacity. We derive a reduced form model for the dynamic of their aggregated inventory considering a proper scaling limit. The resulting price impact profile is shown to depend on the characteristics and relative importance of their inventories. The model is flexible enough to reproduce the empirically documented power law behavior of the price impact function. For any choice of the market makers characteristics, optimal execution within this modeling approach can be recast as a linear-quadratic stochastic control problem in which the value function and the associated optimal trading rate can be obtained semi-explicitly subject to solving a differential matrix Riccati equation. Numerical simulations are conducted to illustrate the performance of the resulting optimal liquidation strategy in relation to standard benchmarks. Remarkably, they show that the increase in performance is determined by a substantial reduction of higher order moment risk.

Keywords: Optimal execution, Price impact, Inventory cost, Order fill uncertainty, Stochastic optimal control.

JEL Classification: C61, G10, G11.

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Price formation in real markets is the outcome from a complex interaction between traders with heterogenous trading technologies and specialized intermediaries that promote exchange by providing liquidity at a cost. In the traditional no arbitrage pricing framework, prices are completely inelastic to demand shocks so that microstructure of price formation is irrelevant. This modeling approach has a number of important drawbacks. First, it is unable to discriminate among order execution algorithms. Second, as an implication, any information on the relationship between transacted quantities and price variations is completely missing.

A key role in shaping the (possibly optimal) mechanics of order execution is played by the so called price impact function. It describes how the execution of an order impacts the transacted prices during its course of execution. There’s a vast literature analyzing its empirical shape, see for instance Bacry et al. (2015) and the references therein, proving that it increases as a power law during execution and, when the execution is complete, it reverts back to certain level, usually differing from the price at the inception.

This empirical fact is usually not encoded within the conventional modeling approaches that are in use to assess the best execution strategies. For example, in the Almgren-Chriss model the impact profile is a time-weighted-average-price (TWAP) execution which is inconsistent with empirical observations in that the profile is linear and it shows no price reversion after completion. Vice-versa, traditional models known under the name of transient impact models, produce an impact curve that misses to reproduce the permanent impact effect.

Empirical analysis has also highlighted that market makers post two-way prices to buy or sell inventories to maximize their expected utility of wealth. Liquidity suppliers and market markers profit from providing immediacy to less patient investors, but have limited inventory-carrying and risk-bearing capacity. Their main risk exposure consists of imbalances in proceeds and inventories accumulated through transactions that might eventually be realized in presence of adverse price movements.

This implies that it is possible to put the price gap between mid-prices and transacted prices in direct connection with the inventory costs of a spectrum of market makers. This price gap arguably is determined by following major components: temporary, permanent and transitory price impact, and processing costs. In this paper, we jointly model temporary, permanent and transitory price impact cost components. Processing costs are neglected since it mostly consists of fee or rebate and has almost no price impact.

We consider a trader whose goal is to design an order execution schedule to optimally liquidate a position under the price impact model. We assume that the trader considers as performance criterion the risk-adjusted profit-and-loss (P&L) plus a \(^1\) risk component that is given by the quadratic variation of the P&L process during execution. Finally, the risk-adjusted P&L is further penalized by a term incurred from a final block trade at the acquisition/liquidation horizon, should there be a block trade required to fully close the position.

More recently, Graewe and Horst (2017) analyze optimal liquidation in markets where both instantaneous and transitory price impact when only absolutely continuous trading strategies are admissible while Neuman and Voß (2022) analyze the liquidation problem in the presence of linear temporary and transient price impact and price predictability.

The novel aspect of this paper relies on the fact that we explicitly take into account the transitory impact on prices of a multiplicity of market makers’ inventories. In particular, we apply a proper scaling procedure to the optimal inventory dynamics found by Guéant, Lehalle, and Fernandez-Tapia (2013) and derive a reduced form, in the so-called hydrodynamic limit, for the market maker’s

\(^1\)Negation of the transaction P&L is equivalent to what is usually referred to as the implementation shortfall in the trading and order execution industry/community, see Section 2.4.
optimal inventory dynamics. The reduced form model reproduces the large scale behavior of the original model and consists of a simple mean reverting process that is therefore characterized uniquely by three parameters, mean reversion, long term rate and volatility. In practice, the transitory component of the price process is further split into the sum of contributions from netted inventories of individual market makers.

Then we adopt this reduced form description and compute the shape of the price impact function in terms of the aggregated inventory of a crowd of market makers with heterogeneous degree of mean reversion under the assumption that there’s a term directly related to the global volume imbalance that affects directly the price dynamics.

The resulting price impact profile is shaped by the distribution of market maker’s mean reversions. The resulting model has sufficient flexibility to reproduce the empirical shape of the observed price-impact functions while preserving analytical tractability of the resulting optimal liquidation problem.\(^2\)

For any market configuration, the computation of the optimal execution policy is reduced to a (non-standard) linear-quadratic problem and thus is analytically solvable.

In particular, following a strategy similar to the one put forward in Bouchaud et al. (2003) and considering a model with a proper distribution of market makers, the proposed modelling approach can accomodate the power-law shape of the impact function that has been empirically detected and generates a closed form solution for the corresponding trader’s optimal liquidation policy.

In the numerical section of the paper we run a comparative analysis among liquidation policies. We show that the optimal strategy improves the performance over more traditional ones since it can take properly into account also higher order moment risk. Simulations show that a large reduction in costs arising from variance, skewness and higher order moment risk components can be achieved accepting only a slight increase in the mean expected cost. The most efficient tradeoff is captured by the model thanks to the presence of two key modeling features. First, the investor’s position dynamics includes a small dynamic uncertainty component. Second, the performance indicator includes quadratic variation component as risk. Then the numerical simulation illustrates how the resulting optimal liquidation strategy takes properly into account the uncertain nature of the trading outcome and optimizes trader’s action accordingly. The rest of the paper is organized as follows. We frame the model underlying the optimal execution problem under price impact in Section 2. Section 3 recasts the optimal order execution problem as an LQ stochastic control problem and summarizes its solution and the associated optimal trading strategies. Technical proofs and verification theorems are collected in Section 3.3. Numerical illustrations and discussion of the results are reported in Section 4. Section 5 concludes the paper.

2. Model setup

Throughout the paper, \((\Omega, \mathcal{F}, \mathbb{P})\) denotes a complete probability space equipped with a filtration describing the information structure \(\mathcal{F} := \{\mathcal{F}_t\}_{t \in [0,T]}\) - where \(t\) is the time variable and \(T > 0\) the fixed finite liquidation horizon, \(\{B_S(t), B_Q(t), B_X(t)\}_{t \in [0,T]}\) is a three-dimensional uncorrelated Brownian motion defined on \((\Omega, \mathcal{F}, \mathbb{P})\), and \(\mathbb{F}\) is the filtration generated by the trajectories of the above Brownian motion, completed with all \(\mathbb{P}\)-null measure sets of \(\mathcal{F}\).

In a continuous time setting, we outline the model and describe the problem faced by an investor when liquidating a given amount of shares of certain stock within the time interval \([0, T]\), in a market where liquidity provision is operated by market makers facing inventory risk, i.e., the risk related to the signed quantity of shares they hold.

\(^2\)Whether this representation result is a mathematical artifact or empirically relevant is a question that can be answered only on a purely empirical basis and is left for future research.
2.1. Reduced form inventory dynamics. In this subsection we briefly introduce the market maker’s optimal management inventory problem as derived in Guéant, Lehalle, and Fernandez-Tapia (2013). Then we introduce the reduced form of market makers’ inventory dynamics that will be considered in this paper.

A market maker provides liquidity by posting two-way prices at which she is willing to buy or sell, i.e., the bid and ask prices respectively. A market maker faces the risk stemming from the uncertainty in the value of their holdings in the asset, this is the so called inventory risk. It was first examined theoretically in Garman (1976), Stoll (1978), Amihud and Mendelson (1980), Ho and Stoll (1981), and more recently in Avellaneda and Stoikov (2008) as well as Guéant, Lehalle, and Fernandez-Tapia (2013). We briefly introduce the framework set up by Avellaneda and Stoikov (2008) and in Guéant, Lehalle, and Fernandez-Tapia (2013), and state the elegant result obtained in Guéant, Lehalle, and Fernandez-Tapia (2013, p. 487).

Denote by $S^b$ and $S^a$ the bid and ask prices, respectively, and by $\delta^b$ and $\delta^a$ the difference between the quotes and the reference price $S$, i.e., $\delta^b := S - S^b$ and $\delta^a := S^a - S$. The market maker’s inventory problem is to continuously quote the two prices $S^a$ and $S^b$, or equivalently to determine the spreads $\delta^a$ and $\delta^b$ respectively, in order to maximize their expected utility of wealth which consists of proceeds and inventories accumulated through transactions with other investors in the market within either a finite or infinite horizon. In the works of Avellaneda and Stoikov (2008) and Guéant, Lehalle, and Fernandez-Tapia (2013), the reference price $S$ is assumed following an arithmetic Brownian motion, driftless or constant drift. The arrival of market buy and market sell orders are assumed independent Poisson processes. A market maker’s objective is to maximize their expected utility of terminal wealth consisting of proceeds and inventory throughout a market making horizon. By further assuming exponential utility $U$ given by

$$U(w) = \frac{1}{\nu} \left( 1 - e^{-\nu w} \right)$$

and exponential arrival rates $\theta_a$ and $\theta_b$ for market buy and sell orders as functions of $\delta^a$ and $\delta^b$, respectively, given by

$$\theta_a(\delta^a) = A e^{-\kappa \delta^a}, \quad \theta_b(\delta^b) = A e^{-\kappa \delta^b},$$

for some constants $A$ and $\kappa$, Guéant, Lehalle, and Fernandez-Tapia (2013) obtained the following elegant approximate optimal spreads $\delta^b$ and $\delta^a$

$$\delta^b \approx \frac{1}{\nu} \ln \left( 1 + \frac{\nu}{\kappa} \right) + \left( q + \frac{1}{2} - \frac{\mu}{\nu \sigma^2} \right) \sqrt{\frac{\sigma^2 \nu}{2 \kappa A} \left( 1 + \frac{\nu}{\kappa} \right)^{1+\frac{\nu}{\kappa}}},$$

$$\delta^a \approx \frac{1}{\nu} \ln \left( 1 + \frac{\nu}{\kappa} \right) + \left( -q + \frac{1}{2} + \frac{\mu}{\nu \sigma^2} \right) \sqrt{\frac{\sigma^2 \nu}{2 \kappa A} \left( 1 + \frac{\nu}{\kappa} \right)^{1+\frac{\nu}{\kappa}}},$$

where $\mu$ and $\sigma$ denote respectively the drift and volatility of $S$, $q$ the market maker’s current inventory.

We are now ready to state our first result:

**Proposition 2.1.** Consider the quoting rules given in (2). Then scaling the parameter $A$ by $A \frac{h^2}{\nu^2}$ in the arrival rates, as well as scaling down the Poisson processes $N^b$ and $N^a$ to $hN^b$ and $hN^a$ respectively, then in the limit as $h \to 0$, the evolution of market maker’s inventory converges to an Ornstein-Uhlenbeck process:

$$\begin{cases}
    dQ(t) = \theta(\bar{q}^0 - Q_i(t))dt + \sigma_Q dB(t), & t \in [0,T] \\
    Q(0) = 0
\end{cases}$$

with mean reversion $\theta = 2c_1c_2\kappa$, long term mean $\bar{q}^0 = \frac{\mu}{\nu \sigma^2}$, and volatility $\sigma_Q = 2\sqrt{c_1}$. 

Proof. See Appendix A. □

This rescaling limit, the so-called hydrodynamic limiting procedure, considers the optimal inventory dynamics in the limit $h \to 0$ assuming that in this limit the arrival rate diverges but the marginal impact of each transaction becomes negligible.

The resulting reduced form dynamics is the one relevant for the trader that is missioned to liquidate a large position. The scaling procedure removes any reference to the micro-structure mechanics of the order execution. For example, this is the situation faced by a trader that delegates the optimization of the order selection to a high-frequency specialized intermediary or to an algorithmic procedure.

The second important assumption we introduce is that the market is populated by heterogeneous market makers, each one characterized by the same reduced form dynamics but different characteristics. The dynamics of the $i$–th market maker’s inventory $Q_i$ will follow an Ornstein-Uhlenbeck process:

$$
\begin{align*}
\{dQ_i(t) &= \theta_i(\tilde{q}_i(t) - Q_i(t))dt + \sigma_{Q_i}dB(t), \quad t \in [0,T] \\
Q_i(0) &= 0 \quad \text{for } i = 1, 2, \ldots, n,
\end{align*}
$$

where

- The initial inventory $Q_i(0), i = 1, \ldots, n,$ is assumed 0 for simplicity;
- All the $Q_i$’s are driven by the same Brownian motion $B_Q$.
- The capacity of the $i$th market maker is proxied by $\tilde{q}_i(t) = \tilde{q}_i^1v(t) + \tilde{q}_i^0$, assuming $\tilde{q}_i^1$ and $\tilde{q}_i^0$ constants.

While the reduced form derived in the previous Proposition corresponds to $\tilde{q}_i^1 = 0$, in the model formulation we consider also the possibility of a feedback, i.e. that the trading rate of market maker $i$ is selected based on the agent’s trading rate $v: \Omega \times [0,T] \to \mathbb{R}$ which will be regarded as the main control variable in the formulation of the agent’s liquidation problem.

The linear coefficient $\tilde{q}_i^1$ reflects the $i$th market maker’s direct reaction to the investor’s liquidation rate, whereas the constant term $\tilde{q}_i^0$ can be interpreted as an upfront capacity set up by the $i$th market maker as a passive attempt to maintain their inventory close to a fixed capacity level during execution. Upon completion of order execution, they all revert their inventory back to zero.

The relative importance of each market makes in the market is set by a weight $\nu_i > 0, i = 1, \ldots, n$, such that $\sum_{i=1}^{n} \nu_i = 1$. Then, we can define a netted aggregate market volume of orders:

$$
Q^M(\cdot) := \sum_{i=1}^{n} \nu_i Q_i(\cdot).
$$

Every transaction (buy or sell) is assumed traded with any of the market makers. Then $Q^M$ will have a dynamics given by the following equation

$$
\begin{align*}
\{dQ^M(t) &= \sum_{i=1}^{n} \nu_i \theta_i(\tilde{q}_i(t) - Q_i(t))dt + \sigma_Q^M dB(t), \quad t \in [0,T] \\
Q^M(0) &= 0,
\end{align*}
$$

with $\sigma_Q^M := \sum_{i=1}^{n} \nu_i \sigma_{Q_i}$.

2.2. Investor’s trading strategy. As in Cheng, Di Giacinto, and Wang (2017, 2019), the evolution of the investor’s position $X$ is assumed to satisfy the following stochastic differential equation

$$
\begin{align*}
\{dX(t) &= -v(t)dt + m dB_X(t), \quad t \in [0,T] \\
X(0) &= x_0 > 0,
\end{align*}
$$

(4)
where \( m \geq 0 \) measures the magnitude of the uncertainty, while the quantity \( x_0 \) represents the initial position to be liquidated by \( T \). The non-controlled diffusive component in the dynamics of the investor’s position \( X \) in (4), takes into account that in real situations the agent’s wealth process \( X \) includes a small uncertainty component. The recent paper Carmona and Leal (2022), see also Carmona and Webster (2019), produces compelling econometric evidence about the existence of a non-zero quadratic variation component in the time series of various institutional investors’ positions of Toronto Stock-Exchange that are publicly available. From a modeling point of view such term is well characterized: it represents the uncertainty that affects also quantities, i.e. the inventory of the trader, beyond prices. Note that the trader can only control the rate of trading, i.e., the drift of the process. Hence introducing this term, the optimal liquidation strategy will also take into account the risk arising from the quantity uncertainty. Traditional model dynamics are recovered in the limit of \( m \to 0 \).

While it would be analytically feasible to consider a non-zero level of correlation between price and quantities, we follow Carmona and Leal (2022) (see Remark 1 of that paper) and solve the model for the zero correlation case our modeling approach does not explicitly model the price formation mechanics. In fact, as discussed in Carmona and Webster (2019) a limit order book execution would imply a positive correlation while a market order would imply a negative correlation. As previously stated, we assume a reduced form expression of inventory dynamics that abstracts from the effective market microstructure of the order execution.

2.3. Traded price dynamics. The dynamic evolution of the fair price \( S \) of the stock is assumed to be

\[
\begin{align*}
\begin{cases}
   dS(t) &= \gamma dX(t) - \phi dQ^M(t) + \mu dt + \sigma_S dB_S(t), \quad t \in [0,T], \\
   S(0) &= s_0 > 0,
\end{cases}
\end{align*}
\]

namely, in integral form,

\[
S(t) = s_0 + \mu t + \sigma_S B_S(t) + \gamma (X(t) - x_0) - \phi Q^M(t), \quad t \in [0,T].
\]

(5)

In other words, the fair price \( S \) is driven by an Arithmetic Brownian motion with drift equal to \( \mu \geq 0 \) and volatility \( \sigma_S > 0 \), along with an inventory cost equal to \( \phi \geq 0 \) and a linear permanent impact with parameter \( \gamma \geq 0 \). More specifically, the permanent impact is modeled taking into account the continuous version of Almgren and Chriss (2000) model, whereas the inventory cost is framed following Guéant, Lehalle, and Fernandez-Tapia (2013, Subsection 5.2, p. 487): the term \( \phi Q^M(t) \) quantifies the price pressure determined by the total inventories carried by the market makers at time \( t \).

Taking into account the dynamics of the position \( X \) and the inventory \( Q^M \) as given by (3) and (4), respectively, the dynamics of the fair value price \( S \) can be written as

\[
\begin{align*}
\begin{cases}
   dS(t) &= \left[ \mu - \phi \sum_{i=1}^n \nu_i \theta_i(\bar{q}_i(t) - Q_i(t)) - \gamma v(t) \right] dt - \phi \sigma_Q^M dB_Q(t) + \gamma m dX(t) + \sigma_S dB_S(t), \\
   S(0) &= s_0 > 0,
\end{cases}
\end{align*}
\]

\( t \in [0,T], \)

or in integral form as

\[
S(t) = s_0 + \mu t - \int_0^t \left[ \gamma v(u) + \phi \sum_{i=1}^n \nu_i \theta_i(\bar{q}_i(t) - Q_i(u)) \right] du - \phi \sigma_Q^M B_Q(t) + \gamma m B_X(t) + \sigma_S B_S(t).
\]
Following Almgren and Chriss (2000), the transacted price $\tilde{S}$ consists of the fair price and a slippage referred to as temporary impact

$$\tilde{S}(t) = S(t) - \eta v(t), \quad t \in [0, T],$$

that is, the transacted price reflects a temporary impact given by a linear function of the current trading rate of market order $v$ with size $\eta > 0$. Note that $\eta v(t)$ is regarded as the informational cost, at time $t$ such as in the Kyle (1985) model. From the above relations, we can derive the following transacted price expression

$$\tilde{S}(t) = s_0 + \mu t + \sigma_S B_S(t) - \eta v(t) - \frac{\gamma}{2} \int_0^t v(u) du + \gamma m B_X(t) - \phi \int_0^t \sum_{i=1}^n \nu_i \theta_i (\bar{q}_i - \bar{Q}_i(u)) du - \phi \sigma^M Q(t), \quad t \in [0, T]$$

The transient contribution determined as a function of the aggregate volume as determined by the population of market makers changes the shape of the price-impact function. In the case of a single market maker, it is easy to verify that the resulting price-impact curve has an exponential shape. The composition of multiple modulated exponential terms offers a natural approximation tool that may be used to reproduce a more realistic, power law shape of the price impact function. This 'heterogenous agents' approach that produces a power law response is well-know by now. It has been discussed in the context of volatility models by Andersen and Bollerslev (1997) and in the context of a propagator model in Bouchaud et al. (2003). See also Ortu et al. (2020) and the references inside for a review of a similar decomposition which has been introduced and is used in applied econometric analysis of discrete time series.

2.4. **Profit and loss.** Following Almgren and Chriss (2000, Section 2.4, p. 10), the profit and loss (P&L) $\Pi^0(t)$ of a trading strategy earned over the time interval $[0, t], t \leq T$, is defined as

$$\Pi^0(t) := X(t) (S(t) - S(0)) + \int_0^t (S(u) - \tilde{S}(u)) dX(u). \quad (7)$$

The P&L defined above can be decomposed in a self-financing strategy contribution and a slippage component. They generalize the usual self-financing relationships of frictionless markets to make it compatible with markets with frictions, including the presence of the uncertainty in the agent inventory as defined in our model. More details about the decomposition of the P&L formula can be found in Cheng, Di Giacinto, and Wang (2019, Appendix A.1 pp. 1673–1674).

Taking into account (3), (4) and (6), the P&L over the time interval $[0, t]$ can be rewritten as

$$\Pi^0(t) = X(t) S(t) - x_0 s_0 - \int_0^t S(u) dX(u) + \int_0^t \eta v(u) dX(u)$$

Furthermore, for any given trading strategy $v(\cdot)$, the P&L can be calculated explicitly as

$$\Pi^0(t) = \frac{\gamma}{2} (X^2(t) - x_0^2) + \frac{\gamma}{2} m^2 t - \phi X(t) Q^M(t) + \int_0^t (-\eta v^2(u) - \phi Q^M(u)v(u) + \mu X(u)) du + m \int_0^t (\eta v(u) + \phi Q^M(u)) dB_X(u) + \sigma_S \int_0^t X(u) dB_S(u),$$

or, equivalently, by applying the integration by parts formula
\[ \Pi^0(t) = \gamma m^2 t + \int_0^t \left\{ -\eta v^2(u) - \gamma v(u) X(u) + \left[ \mu - \phi \sum_{i=1}^n \theta_i (\bar{q}_i - Q_i(u)) \right] X(u) \right\} du \]

\[ - \int_0^t \phi \sigma_Q^2 X(u) dB_Q(u) + m \int_0^T (\eta v(u) + \gamma X(u)) dB_X(u) + \sigma_S \int_0^T X(u) dB_B(u). \tag{8} \]

2.4.1. Penalty of final block trade. Following Cheng, Di Giacinto, and Wang (2017, 2019), we take into consideration as a penalty for a final block trade of size \( x \) at time \( t \),

\[ f(x) := \beta x^2, \quad \beta > 0. \tag{9} \]

Thus, the P&L posterior to the final block trade is given by

\[ \Pi^0(T) - \beta X^2(T), \]

should there be \( X(T) \) shares of the stock remaining to be executed at the horizon \( T \).

2.4.2. Quadratic variation. We also choose to penalize the expected P&L by its quadratic variation as in Forsyth et al. (2012). From (8) the instantaneous quadratic variation \( QV[\Pi^0(t)] \) of P&L \( \Pi^0(t) \) at time \( t \in [0, T] \), is given by

\[ dQV[\Pi^0(t)] = \phi^2(\sigma_Q^2)^2 X^2(t) dt + m^2 (\eta v(t) + \gamma X(t))^2 dt + \sigma_S^2 X^2(t) dt \]

\[ = \left[ m^2 \eta^2 v^2(t) + 2m^2 \eta \gamma X(t) v(t) + (\phi^2(\sigma_Q^2)^2 + m^2 \gamma^2 + \sigma_S^2) X^2(t) \right] dt. \tag{10} \]

Therefore, the P&L posterior to the final block trade and penalized by its quadratic variation for some risk aversion parameter \( \lambda \geq 0 \) reads

\[ \Pi^0(T) - \beta X^2(T) - \lambda QV[\Pi^0(T)] \]

\[ = -\beta X^2(T) + \frac{\gamma}{2} \left( X^2(T) - x_0^2 \right) + \frac{\gamma}{2} m^2 T - \phi X(T) Q^M(T) \]

\[ + \int_0^T \left\{ -\eta v^2(u) - \phi Q^M(u) v(u) + \mu X(u) \right\} du \]

\[ + m \int_0^T (\eta v(u) + \phi Q^M(u)) dB_X(u) + \sigma_S \int_0^T X(u) dB_B(u) \]

\[ - \lambda \int_0^T \left[ m^2 \eta^2 v^2(t) + 2m^2 \eta \gamma X(t) v(t) + (\phi^2(\sigma_Q^2)^2 + m^2 \gamma^2 + \sigma_S^2) X^2(t) \right] dt \]

\[ = \gamma m^2 T - \frac{\gamma}{2} x_0^2 + \left( \frac{\gamma}{2} - \beta \right) X^2(T) - \phi X(T) Q^M(T) \]

\[ + \int_0^T \left\{ -\eta v^2 - \phi Q^M v_t - \bar{\xi} X_t v_t - \psi X_t^2 + \mu X_t \right\} dt \]

\[ + m \int_0^T (\eta v(u) + \phi Q^M(u)) dB_X(u) + \sigma_S \int_0^T X(u) dB_B(u) \]

where

\[ \bar{\eta} := \eta(1 + \lambda m^2 \eta), \tag{12a} \]

\[ \bar{\xi} := 2\gamma \lambda m^2 \eta, \tag{12b} \]

\[ \psi := \lambda \left( \phi^2(\sigma_Q^2)^2 + m^2 \gamma^2 + \sigma_S^2 \right). \tag{12c} \]
3. The optimal control problem

The order execution problem can be naturally recast as a linear quadratic (LQ) stochastic control problem. To this end, denote by $A = \text{diag}(-\theta_1, \cdots, -\theta_n, 0)$ the $(n+1) \times (n+1)$ diagonal matrix whose diagonal entries starting in the upper left corner being $-\theta_1, \ldots, -\theta_n, 0$, and

$X := \begin{bmatrix} Q_1 \\ \vdots \\ Q_n \\ X \end{bmatrix}$, $a := \begin{bmatrix} \theta_1 q_1^1 \\ \vdots \\ \theta_n q_n^1 \\ -1 \end{bmatrix}$, $b := \begin{bmatrix} \theta_1 q_1^0 \\ \vdots \\ \theta_n q_n^0 \\ 0 \end{bmatrix}$, $\Sigma := \begin{bmatrix} \sigma q_1 & 0 \\ \sigma q_2 & 0 \\ \vdots & \vdots \\ \sigma q_n & 0 \\ 0 & m \end{bmatrix}$, $B := \begin{bmatrix} BQ \\ BX \end{bmatrix}$, $x := \begin{bmatrix} 0 \\ 0 \\ \vdots \\ x \end{bmatrix}$.

The state equation is governed by the following non-homogenous linear stochastic differential equation

$$
\begin{align*}
dX(t) &= (AX(t) + av(u) + b) du + \Sigma dB(u), \quad u \in [t, T], \\
X(t) &= x, \quad x \in \mathbb{R}^{n+1}.
\end{align*}
$$

(13)

The set of admissible controls $V_{ad}$ is defined by

$$V_{ad}(t, x) := \{v : [t, T] \times \Omega \to \mathbb{R} | v \in \mathcal{H}_F^2(t, T; \mathbb{R})\},$$

where $\mathcal{H}_F^2(t, T; \mathbb{R})$ denotes the set of $F$-progressively measurable $\mathbb{R}$-valued processes $\{H(u)\}_{u \in [t, T]}$ such that $\mathbb{E}\left[ \int_t^T |H(u)|^2 du \right] < +\infty$. Observe that, for any $v(\cdot) \in V_{ad}(t, x)$, the above state equation admits a unique (strong) solution $X(\cdot) := X(\cdot; t, x, v(\cdot))$.

3.1. The objective functional. Given the initial data $(t, x) \in [0, T] \times \mathbb{R}^{n+1}$, the investor’s objective functional $J$ reads as

$$J(t, x; v(\cdot)) := \mathbb{E}\left[ \Pi'(T) - \beta X^2(T) - \lambda QV \left[ \Pi'(T) \right] \right]$$

where

$$\Pi'(T) := \Pi^0(T) - \Pi^0(t),$$

namely, $\Pi'(T)$ denotes the P&L earned over the time interval $[t, T]$. Therefore, taking into account (10), (11), (12), and temporarily ignoring the constant term $\gamma m^2$, the objective functional $J$ can be written as

$$J(t, x; v(\cdot)) := \mathbb{E}\left[ X'(T)G X(T) + \int_t^T \left( 2v(u)k' X(u) - \bar{\eta}v^2(u) - \psi X^2(u) + \mu X(u) \right) du \right],$$

where

$$G := \begin{bmatrix} 0 & \cdots & 0 & -\frac{\phi \nu_1}{2} \\ 0 & \cdots & 0 & -\frac{\phi \nu_2}{2} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & -\frac{\phi \nu_n}{2} \end{bmatrix}, \quad k := \begin{bmatrix} -\phi \nu_1 \\ \vdots \\ -\phi \nu_n \\ -\xi \end{bmatrix}.$$ 

Our goal is thus to find the solution to the following linear-quadratic (LQ) stochastic control problem

**Problem 3.1.** For any initial data $(t, x) \in [0, T] \times \mathbb{R}^{n+1}$, maximize $J(t, x; v(\cdot))$ subject to the state equation (13) over $v(\cdot) \in V_{ad}(t, x)$. 
The optimal solution to the control problem, i.e., the value function \( W \), is defined as
\[
\begin{align*}
W(t, x) &:= \sup_{v(\cdot) \in \mathcal{V}_w(t, x)} J(t, x; v(\cdot)), \quad \forall (t, x) \in [0, T] \times \mathbb{R}^{n+1} \\
W(T, x) &:= x' G x, \quad \forall x \in \mathbb{R}^{n+1},
\end{align*}
\] (14)

3.2. The HJB equation. By standard arguments in stochastic control theory, the value function \( W \) satisfies a Hamilton-Jacobi-Bellman (HJB) equation. In our framework, the HJB equation in \([0, T] \times \mathbb{R}^{n+1}\) with terminal boundary condition is given by
\[
\begin{align*}
\frac{\partial}{\partial t} w(t, x) + \mathcal{H} (x, Dw(t, x), D^2 w(t, x)) &= 0, \quad (t, x) \in [0, T] \times \mathbb{R}^{n+1},
\end{align*}
\] (15)
\[
\begin{align*}
w(T, x) &= x' G x, \quad \forall x \in \mathbb{R}^{n+1},
\end{align*}
\] (15)
where \( Dw \) and \( D^2 w \) denote respectively the gradient and the Hessian matrix of \( w \) with respect to state variable \( x \), and \( \mathcal{H} \) is the Hamiltonian given by
\[
\mathcal{H} (x, Dw(t, x), D^2 w(t, x)) := \sup_{\varphi \in \mathcal{V}^*} \mathcal{H}_{cv} (x, Dw(t, x), D^2 w(t, x); \varphi), \quad (t, x) \in [0, T] \times \mathbb{R}^{n+1},
\] (16)
with \( \mathcal{H}_{cv} \) representing the Hamiltonian current-value defined as
\[
\mathcal{H}_{cv} (x, p, P, v) := 2vk'x - \tilde{\eta}v^2 - \psi x^2 + \mu x + p' (Ax + av + b) + \frac{1}{2} \text{tr} [\Sigma \Sigma' P],
\] (17a)
for \((x, p, P) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times \mathcal{S}_n^+ \), \( v \in \mathbb{R} \), where \( \mathcal{S}_n^+ \) is the set of \((n + 1) \times (n + 1)\) symmetric matrices.

Given \((x, p, P) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times \mathcal{S}_n^+ \), the function \( v \rightarrow \mathcal{H}_{cv} (x, p, P ; v) \) has a unique maximum point over \( \mathbb{R} \) given by
\[
v^* (x, p) = \frac{2k'x + a'p}{2\tilde{\eta}}.
\] (17b)

The HJB equation associated to the stochastic control problem (3.1) reduces to
\[
\begin{align*}
\frac{\partial}{\partial t} w(t, x) + \frac{1}{4\tilde{\eta}} \left( 2k'x + a'Dw(t, x) \right)^2 &- \psi x^2 + \mu x + (Ax + b)' Dw(t, x) + \frac{1}{2} \text{tr} [\Sigma \Sigma' D^2 w(t, x)] = 0, \quad (t, x) \in [0, T] \times \mathbb{R}^{n+1},
\end{align*}
\] (17a)
with terminal condition
\[
w(T, x) = x' G x, \quad \forall x \in \mathbb{R}^{n+1}.
\] (17b)

The value function \( W \) defined in (14) may be characterized as the unique solution to (17).

**Definition 3.2** (Classical solution). A function \( w \) is called a classical solution of (17) if
(i) \( w \in C^{1,2}([0, T] \times \mathbb{R}^{n+1}; \mathbb{R}) \)
(ii) \( w \) satisfies pointwise in classical sense (17a) (the derivatives with respect to time variable at \( t = 0 \) and \( t = T \) have to be intended respectively as right and left derivative),
(iii) \( w \) satisfies the boundary condition (17b). \( \square \)

**Lemma 3.3** (Solution to HJB equation). Let \( \beta > \frac{\gamma}{2} \). Define
\[
w(t, x) = x' R(t) x + r'(t) x + \varphi(t), \quad \forall (t, x) \in [0, T] \times \mathbb{R}^{n+1},
\] (18)
where \( R \in C([t,T]; S^{n+1}) \), \( r \in C([t,T]; \mathbb{R}^{n+1}) \), and \( \varphi \in C([t,T]; \mathbb{R}) \) are the unique solution to the following terminal value problem (TVP), for some \( t \in [0,T] \),

\[
\begin{aligned}
\dot{R} - \psi e_{n+1} e'_{n+1} + \frac{1}{\bar{\eta}} \left\{ Raa' R + (\bar{\eta} A + ak')' R + R(\bar{\eta} A + ak') + kk' \right\} &= 0 \quad (19a) \\
\dot{r} + A'r + 2Rb + \mu e + \frac{1}{\bar{\eta}} (Raa' + ka') r &= 0 \quad (19b) \\
\dot{\varphi} + \text{tr}(\Sigma \Sigma'R) + b'r + \frac{1}{4\bar{\eta}} r'aa'r &= 0 \quad (19c)
\end{aligned}
\]

with terminal condition

\[
R(T) = G, \quad r(T) = 0, \quad \varphi(T) = 0, \quad (20)
\]

where for notational simplicity we suppressed the dependence on \( t \) for the functions \( R, r, \) and \( \varphi \) unless necessary, and \( e_{n+1} \) denotes the unit vector \( e_{n+1} = [0 \cdots 0 1] \in \mathbb{R}^{n+1} \). Then the quadratic function \( w \) in the state variable \( x \) is a classical solution to the HJB equation (17).

**Proof.** Substituting (18) into (15) with \( p = Dw(t,x) = 2Rx + r \), \( P = D^2w(t,x) = 2R \) and completing the square for \( v \), the Hamiltonian current value \( H_{cv} \) reads as

\[
H_{cv}(x, Dw(t,x), D^2w(t,x); v) = -\bar{\eta} \left[ v - \frac{1}{2\bar{\eta}} (2k + 2Ra)' x - \frac{1}{2\bar{\eta}} a' r \right]^2 + \frac{1}{4\eta} \{ (2k + 2Ra)' x + a' r \}^2 \quad (21)
\]

Since \( \bar{\eta} > 0 \), from (21) we clearly see that the function \( v \to H_{cv}(x, Dw, D^2w; v) \) has a unique maximum point \( v^* \) over \( \mathbb{R} \) given by

\[
v^* = \frac{2(k + Ra)' x + a' r}{2\bar{\eta}}, \quad (t,x) \in [0,T] \times \mathbb{R}^{n+1},
\]

and the HJB equation reads as

\[
x'Rx + x'r + \dot{\varphi} = \frac{1}{4\bar{\eta}} \{ (2k + 2Ra)' x + a' r \}^2 - \psi x^2 + \mu x + (2Rx + r)'(Ax + b) + \text{tr}[\Sigma \Sigma'R]
\]

with terminal condition

\[
w(T, x) = x'Gx, \quad \forall x \in \mathbb{R}^{n+1}.
\]

Finally, by comparing the coefficients, we obtain the Riccati equation (19a), the linear term (19b), and the HJB equation as (19c), respectively, with

\[
R(T) = G, \quad r(T) = 0, \quad \varphi(T) = 0.
\]

\[\Box\]

We remark that the system of ODEs (19a) satisfied by \( R \) is a matrix Riccati differential equation whereas the ODEs for \( r \) and \( \varphi \) are linear. Thus, the existence and uniqueness for \( r \) and \( \varphi \) are straightforward so long as we have those for \( R \). The flow of Riccati equation (19a) can be linearized by doubling the dimension of the problem. This is due to the fact that a Riccati ODE solution belongs to a quotient manifold (see Grasselli and Tebaldi (2008) for further details). The explicit linearization procedure and the closed form solution to (19a) follows from Da Fonseca, Grasselli, and Tebaldi (2008). It is interesting to notice however that the explicit computation is made non-trivial by the fact the computation of matrix exponentials is complicated by the presence of degenerate matrices. Computation of \( R \) is linearized thanks to the following lemma.
Lemma 3.4. (Solution to the matrix Riccati equation)
The solution to the matrix Riccati equation (19a) for $R$ solves the linear system
\[ RN(t) = M(t), \quad t \in [0, T] \]
where the $(n+1) \times (n+1)$ matrices $M$ and $N$ satisfy the following system of linear ODEs
\[ \frac{d}{dt} \begin{bmatrix} M(t) \\ N(t) \end{bmatrix} = \begin{bmatrix} -\left( A + \frac{1}{\eta} ak' \right)' & \psi ee' - \frac{1}{\eta} kk' \\ \frac{1}{\eta} aa' & A + \frac{1}{\eta} ak' \end{bmatrix} \begin{bmatrix} M(t) \\ N(t) \end{bmatrix}, \quad t \in [0, T] \tag{22} \]
with terminal conditions $M(T) = G$ and $N(T) = I$. Moreover, the solution to the linear system can be written as
\[ \begin{bmatrix} M(t) \\ N(t) \end{bmatrix} = e^{-(T-t)\Psi} \begin{bmatrix} G \\ I \end{bmatrix}, \]
where
\[ \Psi := \begin{bmatrix} -\left( A + \frac{1}{\eta} ak' \right)' & \psi ee' - \frac{1}{\eta} kk' \\ \frac{1}{\eta} aa' & A + \frac{1}{\eta} ak' \end{bmatrix}. \]
and $I$ is an $(n+1) \times (n+1)$ identity matrix.

Proof. By straightforward substitution one can show that if $M$ and $N$ satisfy the system of ODEs (22), then a solution to the linear system $RN(t) = M(t)$ solves (19a). \qed

We present closed form expressions for optimal trading rates $v^*$ in this case, in the next Corollary 3.7 and Corollary 3.9.

3.3. The verification theorem and the optimal feedback policy. The aim of this section is to prove a verification theorem stating that the function $w$ defined in (18) is actually the value function and giving an optimal feedback strategy for the stochastic LQ problem 3.1.

Let $(t, x) \in [0, T] \times \mathbb{R}^{n+1}$, take into account (16) and Lemma 3.3. The feedback map $G$ reads as
\[ (t, x) \mapsto G(t, x) := \frac{2k'x + a'Dw(t, x)}{2\eta}. \]
By virtue of Lemma 3.3 we have
\[ Dw(t, x) = 2R(t)x + r(t), \quad \forall (t, x) \in [0, T] \times \mathbb{R}^{n+1}, \]
the above feedback map becomes
\[ G(t, x) = \frac{2(k + R(t)a')x + a'r(t)}{2\eta}, \quad (t, x) \in [0, T] \times \mathbb{R}^{n+1}. \tag{23} \]

The corresponding closed loop equation is
\[
\begin{cases} 
    dX(u) = \left( A + \frac{1}{\eta} (ak' + aR(u)) \right) X(u) + \frac{1}{2\eta} aa'r(u) + b \, du + \Sigma dB(u), & u \in [t, T], \\
    X(t) = x, & x \in \mathbb{R}^{n+1}.
\end{cases}
\tag{24}
\]

Lemma 3.5 (Solution to closed loop equation). For every $(t, x) \in [0, T] \times \mathbb{R}^{n+1}$ there exists a unique $\mathcal{F}^t$-progressively measurable process $X_G(\cdot; t, x) \in L^2(\Omega \times [t, T]; \mathbb{R}^{n+1})$ solution to (24).
Proof. The proof of the existence and uniqueness of $X_G(\cdot; t, x)$ is due to the Lipschitz continuity of the map $G$ and it is a rather standard result (see, e.g., Karatzas and Shreve (1991, Chapter 5, Theorem 2.5, p. 287, and Theorem 2.9, p. 289)).

By applying standard arguments we obtain the following result.

**Theorem 3.6** (Verification theorem and optimal feedback). Let $\beta > \frac{\gamma}{2}$. Then the function $w$ given in (18) is the value function $W$ defined in (14), namely,

$$W(t, x) = w(t, x), \quad \forall (t, x) \in [0, T] \times \mathbb{R}^{n+1}.$$ 

Furthermore, $v^*(\cdot) \in V_{ad}(t, x)$ is optimal for the initial point $(t, x) \in [0, T] \times \mathbb{R}^{n+1}$ if and only if

$$v^* (u) = G(u, X_G(u; t, x)), \quad u \in [t, T],$$

where $G$ is given by (23) and $X_G(\cdot; t, x)$ is the unique solution to the closed loop equation (24), i.e.,

$$v^* (u) = \frac{2(k + R(u) a' X_G(u; t, x) + a' r(u))}{2\eta}, \quad u \in [t, T].$$

In particular, the above feedback strategy $v^*$ is the unique optimal strategy.

**Proof.** By Lemma 3.3, the function $w$ defined in (18) is a classical solution to the HJB equation (17). Let $(t, x) \in [0, T] \times \mathbb{R}^{n+1}$, $v(\cdot) \in V_{ad}(t, x)$, and apply the Dynkin formula to the corresponding state trajectory $X(\cdot) := X(\cdot; t, x, v(\cdot))$ with the function $w$. We obtain

$$w(T, X(T)) - w(t, x) = \mathbb{E} \left[ \int_t^T \left\{ \frac{\partial}{\partial t} w(u, X(u)) + (AX(u) + av(u) + b'Dw(u, X(u)) + \frac{1}{2} tr [\Sigma \Sigma' D^2 w(u, X(u))] du \right\} \right],$$

i.e.,

$$w(t, x) = \mathbb{E} \left[ X(T)'GX(T) - \int_t^T \left\{ \frac{\partial}{\partial t} w(u, X(u)) + (AX(u) + av(u) + b'Dw(u, X(u)) + \frac{1}{2} tr [\Sigma \Sigma' D^2 w(u, X(u))] du \right\} \right].$$

Recalling that $w$ solves the original HJB equation (17), we may write the fundamental identity

$$w(t, x) = J(t, x; v(\cdot)) + \mathbb{E} \left[ \int_t^T (H(X(u), Dw(u, X(u)), D^2 w(u, X(u))) - H_{cv}(X(u), Dw(u, X(u)), D^2 w(u, X(u)); v(u))) du \right],$$

obtaining

$$w(t, x) \geq J(t, x; v(\cdot)).$$

As the above inequality holds for every $v(\cdot) \in V_{ad}(t, x)$ and $H(\cdot) \geq H_{cv}(\cdot)$ for every $v \in \mathbb{R}$, thus $w \geq W$.

Now, consider $X_G(\cdot) := X(\cdot; t, x, v^*(\cdot))$ and apply the fundamental identity to $X_G(\cdot)$ with function $w$. Taking into account Lemma 3.3 and (23) we see that the feedback map maximizes at any time $t \in [0, T]$ the Hamiltonian current value. Thus, in this case we have $w(t, x) = J(t, x; v^*(\cdot))$, which shows that

$$w(t, x) = W(t, x) = J(t, x; v^*(\cdot)).$$
By the uniqueness of the solution to the closed loop equation (24) stated in Lemma 3.5 and the Lipschitz continuity of $G$, the feedback strategy $v^*(\cdot)$ is admissible, that is, $v^*(\cdot) \in V_{ad}(t, x)$. Furthermore, an optimal strategy must satisfy (25) since $W = w$ and (27) holds. Finally, the uniqueness of the optimal strategy is a consequence of the characterization (25) and uniqueness of solution to the closed loop equation (24).

3.4. Optimal trading strategy and closed form solutions to the matrix Riccati equation. Let $X_G(\cdot; t, x) := [Q_G(\cdot), X_G(\cdot)]'$ the solution to the closed loop equation (24). The following two corollaries hold.

**Corollary 3.7** ($\bar{q}_1 = 0$ and $\lambda > 0$). Let $\bar{q}_1 = 0$. In this case, $a = -e_{n+1}$. Assume $(\frac{2+\xi}{2} - \beta)^2 > \psi\bar{\eta}$ and $\beta - \frac{2+\xi}{2} > 0$. Define the constant $\bar{\alpha}$ by

$$\bar{\alpha} := \sqrt[\psi\bar{\eta}]\sinh^{-1} \left\{ \frac{\sqrt{\psi\bar{\eta}}}{\sqrt{(\frac{2+\xi}{2} - \beta)^2 - \psi\bar{\eta}}} \right\}. \quad (28)$$

Thus, we have

$$\sinh\left(\sqrt[\psi\bar{\eta}]{\bar{\alpha}}\right) = \frac{\sqrt{\psi\bar{\eta}}}{\sqrt{(\frac{2+\xi}{2} - \beta)^2 - \psi\bar{\eta}}} \quad \text{and} \quad \cosh\left(\sqrt[\psi\bar{\eta}]{\bar{\alpha}}\right) = \frac{\beta - \frac{\gamma + \xi}{2}}{\sqrt{(\frac{2+\xi}{2} - \beta)^2 - \psi\bar{\eta}}}. \quad (29)$$

Let $\zeta := \sqrt[\psi\bar{\eta}]{\bar{\alpha}}$. The optimal trading rate $v^*$ in this case is given by the feedback form

$$v^*(u) = \zeta \coth(\zeta \{T - u + \bar{\alpha}\}) X_G(u) - \frac{\phi}{2\eta} \nu' Q_G(u) + \frac{\phi}{2\eta} \nu' \{\Lambda^u(T) Q_G(u) + \Lambda^u_0(T) \bar{q}_0\}$$

$$+ \frac{\mu}{2\sqrt{\psi\bar{\eta}}} \left\{ \frac{\cosh(\zeta\bar{\alpha})}{\sinh(\zeta(T-u+\bar{\alpha}))} - \coth(\zeta(T-u+\bar{\alpha})) \right\},$$

where $\Lambda^u(T)$ and $\Lambda^u_0(T)$ are time dependent $n \times n$ matrices defined by

$$\Lambda^u(T) := \frac{\sinh(\zeta\bar{\alpha})}{\sinh(\zeta(T-u+\bar{\alpha}))} e^{(u-T)\Theta} \left[ I - \left( I - \frac{1}{\zeta^2} \Theta^2 \right)^{-1} \right] \left( \Theta \coth(\zeta\bar{\alpha}) \right)$$

$$+ \left( I - \frac{1}{\zeta^2} \Theta^2 \right)^{-1} \left( - \frac{1}{\zeta} \left( I - \frac{1}{\zeta^2} \Theta^2 \right)^{-1} \Theta \coth(\zeta(T-u+\bar{\alpha})) \right) \quad (29)$$

and

$$\Lambda^u_0(T) := \frac{\sinh(\zeta\bar{\alpha})}{\sinh(\zeta(T-u+\bar{\alpha}))} \left[ I - e^{(u-T)\Theta} \right] \left[ I - \left( I - \frac{1}{\zeta^2} \Theta^2 \right)^{-1} \right] \left( \Theta \coth(\zeta\bar{\alpha}) \right)$$

$$+ \frac{1}{\zeta} \left( I - \frac{1}{\zeta^2} \Theta^2 \right)^{-1} \Theta \left[ \frac{\cosh(\zeta\bar{\alpha})}{\sinh(\zeta(T-u+\bar{\alpha}))} + \coth(\zeta(T-u+\bar{\alpha})) \right]$$

$$- \frac{1}{\zeta^2} \left( I - \frac{1}{\zeta^2} \Theta^2 \right)^{-1} \Theta^2 \left[ 1 - \frac{\sinh(\zeta\bar{\alpha})}{\sinh(\zeta(T-u+\bar{\alpha}))} \right] \quad (30)$$
Proof. The proof is lengthy and tedious. We postpone it to Section B in Appendix. □

Remark 3.8. In the above corollary we assumed the conditions \((\frac{2 + \xi}{2} - \beta)^2 > \psi \eta\) and \(\beta - \frac{2 + \xi}{2} > 0\) that are sufficient for our discussion but not necessary. All the other cases can be also discussed with minor modifications.

Note that, when \(\phi = 0\), i.e., price impact not taking into account the inventory cost component, the optimal trading rate \(v^*\) given above recovers the optimal trading rate obtained in Cheng, Di Giacinto, and Wang (2017, p. 62, eq. (4.7)) which reads apparently, in the current notations,

\[
v^*(u) = \zeta \coth(\zeta \{T - u + \tilde{\alpha}\}) X_G(u) + \frac{\mu}{2\sqrt{\psi \eta}} \left\{ \frac{\cosh(\zeta \tilde{\alpha})}{\sinh(\zeta (T - u + \tilde{\alpha}))} - \coth(\zeta (T - u + \tilde{\alpha})) \right\}.
\]

Optimal trading strategy for a risk neutral trader, i.e., \(\lambda = 0\), when \(\tilde{q}_1 = 0\) can be further reduced and the final expression becomes much neater. We summarize the result in the following corollary whose proof will be omitted since it can be obtained by repeating the procedure as in Corollary 3.7 with \(\lambda = 0\).

Corollary 3.9 (\(\tilde{q}_1 = 0\) and \(\lambda = 0\)). Let \(\lambda = 0\) and \(\tilde{q}_1 = 0\). In this case, we have \(\tilde{\eta} = \eta, \psi = \tilde{\xi} = 0, \) and \(a = -e_{n+1}\). The optimal trading rate is given in closed form by

\[
v^*(u) = \frac{\phi}{2\eta} (\nu' \tilde{q}_0 - \nu' Q_G(u)) + \frac{\phi}{2\eta} \frac{\nu'}{T - u + \alpha} \left[ \alpha e^{-(T-u)\Theta} + (e^{-(T-u)\Theta} - I) \Theta^{-1} \right] (Q_G(u) - \tilde{q}_0)
\]

\[+ \frac{X_G(u)}{T - u + \alpha} - \frac{\mu}{4\eta} \left( T - u + \alpha - \frac{\alpha^2}{T - u + \alpha} \right).\]

Note that the optimal trading rate \(v^*\) depends on the remaining shares to be liquidated \(x\) and the discrepancy between the current inventory \(q\) and its long term mean \(\tilde{q}_0\). Thus, the trader is suggested to take into account the traded volume while liquidating his position optimally. We remark that apparently when \(\phi = 0\), i.e., price impact disregarding the inventory cost component, then the optimal trading rate \(v^*\) reduces to

\[
v^*(u) = \frac{X_G(u)}{T - u + \alpha} - \frac{\mu}{4\eta} \left( T - u + \alpha - \frac{\alpha^2}{T - u + \alpha} \right)
\]

which recovers the optimal trading rate in Cheng, Di Giacinto, and Wang (2017, p. 58, eq. (3.3)) .

4. Numerical examples

In order to illustrate the performance of the optimal liquidation strategy obtained in Theorem 3.6 and gain some economic insight, in this section we run a number of numerical tests to assess the marginal improvement of the current approach with respect to known, benchmark execution strategies.

The investor’s target is assumed to be the liquidation within one day of the amount of \(x_0 = 200,000\) shares of a certain stock. Parameters for price impact are selected so as to be in line with those in Almgren and Chriss (2000) and Cheng, Di Giacinto, and Wang (2017, 2019): \(\gamma\) and \(\eta\) are set as \(\gamma = 2.5 \times 10^{-7}\) and \(\eta = 2.5 \times 10^{-6}\). We set the order-fill-uncertainty parameters as \(m = x_0 \times 10\% = 20,000\). This is equivalent to say that the execution risk may generate on average a deviation from the submission path of 10\%. The initial stock price level, irrelevant for the implementations of the strategies under consideration, is assumed \(S_0 = 50\), price volatility is set to \(\sigma_S = 0.5\) and for simplicity expected annual return for the stock is set to zero.
The spectrum of market makers consists of 10 market makers indexed by the mean-reverting
rates $\theta_i = i$ for $i = 1, 2, \ldots, 10$. The weight $\nu_i$ is chosen as a discretized gamma distribution
with degrees of freedom 3. Specifically,
$$
\nu_i = \frac{\Gamma(i, 3)}{\sum_{n=1}^{10} \Gamma(n, 3)},
$$
where $\Gamma(\cdot, 3)$ denotes the probability density function for gamma distribution with degrees of free-
dom 3. The long term means of the Ornstein-Uhlenbeck processes are assumed
$$
\bar{q}_i(t) = \frac{v(t)}{100 \theta_i} + \frac{1}{10 \theta_i}
$$
for $i \in \{1, 2, \ldots, 10\}$. The rationale for this choice is that the higher the market maker’s mean-
reverting rate, the faster the market maker is able to quickly reduce its inventories and the closer
the long term mean of the inventory to zero. In fact, the market makers’ objective is to carry no
position overnight in average and are committed to maintain their long term expected inventory
as small as possible. Recalling that the moment generating function of a gamma distribution is a
power function, it is easy to verify that this choice implies a power law decay of the price impact
function. The parameters are summarized in Table 1.

Table 1: Selected parameters in the numerical simulation.

| Volatilities | Price impact | Block trade penalty | Inventory | Risk aversion |
|--------------|--------------|---------------------|-----------|---------------|
| $\sigma_S = 0.5$ | $\gamma = 2.5 \times 10^{-7}$ | $\beta = 100 \eta$ | $\phi = 100 \eta$ | $\lambda = 0$ |
| $\sigma_M = 0.1$ | $\eta = 2.5 \times 10^{-6}$ | | | $\lambda = 0.001$ |
| $m = 20,000$ | | | | |

Simulations for evaluating the objective functional (11) were conducted by applying the following
strategy: first we focus on a comparison restricted to the performance of different strategies in a
risk-neutral setup where $\lambda = 0$ and then we move on to the case where the agent is risk averse
and the risk aversion coefficient $\lambda$ is set to 0.001. When $\lambda = 0$ we compare the optimal strategy
in Theorem 3.6, with two strategies: the one denoted TWAP obtained by setting $v(t) = \frac{X(t)}{T}$, and
a second one denoted by adapted TWAP (for risk neutral trader) which is the optimal one when
$\phi = 0$ and $\lambda = 0$ (the case corresponding to the analysis of Cheng, Di Giacinto, and Wang (2017))
$$
\nu(t) = \frac{X(t)}{T - t + \alpha},
$$
where $\alpha = \frac{2\eta}{\eta + \gamma}$ and $X(t)$ denotes the remaining shares to be liquidated at time $t$.

Then we move to the more relevant case where the risk aversion is set to a finite level $\lambda = 0.001$
and we consider, in addition to the previous benchmarks also the Almgren-Chriss one.

For a risk neutral trader, i.e., $\lambda = 0$, Figures 1 through 3 exhibit respectively the expected
remaining positions $X(\cdot)$ during liquidation, the histograms of objective functionals, as well as
the histograms of the terminal shares $X(\cdot)$ prior to a final block trade. Solid lines indicate kernel
density estimate for the histograms. The adapted TWAP in this setting reads
$$
v(t) = \frac{X(t)}{1 - t - 0.01005},
$$
In this case, we observe that, since the expected optimal trading trajectory is pretty close to
TWAP strategy, the histograms from applying adapted TWAP and the optimal strategies are
Figure 1. Expected trading trajectory during the course of execution for risk neutral trader. Optimal in red and TWAP in blue.

Figure 2. Histogram with kernel density estimate of objective functional for risk neutral trader. Optimal in green, TWAP in blue, and adapted TWAP in red.

almost identical. Note that Figure 2 shows that the histogram associated with TWAP generates a distribution of performances that is severely left skewed, showing that the major loss w.r.t. the optimal strategy is driven by higher moment risk. Likewise, in Figure 3 we report the histograms of terminal position $X(T)$ prior to final block trade. Those for adapted TWAP and the optimal strategy are more concentrated than TWAP, again indicating that adapted strategies substantially reduce dispersion. Note also that the optimal strategy achieves a lower mean size of the final block liquidation size with respect to the adapted TWAP, thus proving that the optimal strategy produces a systematic reduction of the average final block trade, that is determined by properly taking into account the transient price impact determined by the market maker inventories’ management.
Figure 3. Histogram with kernel density estimate for terminal position $X_T$ to be liquidated by a block trade for risk neutral trader. Optimal in green, TWAP in blue, and adapted TWAP in red.

Then we set $\lambda = 0.001$ and consider a risk averse trader. Figures 4 through 6 exhibit respectively the expected remaining positions $X(\cdot)$ during liquidation, the histograms of objective functionals, as well as the histograms of the terminal shares $X(\cdot)$ prior to a final block trade. Solid lines indicate kernel density estimates for the histograms. For reader’s convenience, we recall that the Almgren-Chriss strategy (consisting of the number of units to be sold) is given by

$$X(t) = x_0 \frac{\sinh(\kappa(T-t))}{\sinh(\kappa T)}.$$  

Figure 4. Expected trading trajectory during the course of execution for risk averse trader. Optimal in red, TWAP in blue, and Almgren-Chriss in orange.
where } \kappa := \sqrt{\frac{\lambda \sigma^2}{\eta}}. \text{ In our context, } T = 1 \text{ and } \kappa = 10. \text{ The expected optimal trading strategies in Figures 4 show that the optimal one is substantially deviating from TWAP type strategies and is closer to the Almgren and Chriss one. The histogram of the distribution of the achieved objective functional levels for different strategies in Figure 5 shows the dramatic decline of TWAP-like strategies’ performance and the high level of left skewness and higher moment risk generated also by the Almgren-Chriss one. Finally, looking at Figure 6 it is also evident that, despite the marginal change in the strategy, the histograms of terminal position } X(T) \text{ prior to final block trade}
for adapted TWAP and the optimal strategies are more concentrated than TWAP and the Almgren-Chriss strategies, indicating that adapted strategies are able to hedge optimally risk arising from uncertainty of order fills.

5. Conclusions

We introduced in this article a price impact model that takes into account a contribution representing the price pressure driven by market makers’ inventories’ risk. The resulting model is expected to be flexible enough to capture some well known stylized features of the empirically documented price response behavior during order execution and simple enough to be analytically tractable. The numerical illustration shows that the resulting optimal liquidation policy provides substantial performance improvements in relation to higher moment risk that emerges by analyzing the performance statistics.

Clearly, an effective characterization of the optimal policy based on public information about the market structure is still out of reach, in fact under a normal condition market maker inventories cannot be observed. However it is worth observing that this one interesting direction of development of the current framework will consist in integrating this optimization approach within a partial information scheme. Then, in light of the linear-quadratic nature of the optimal solution, it is easy to envisage the possibility to extend the model including also a procedure that filters inventories’ size from publicly available information to achieve the goal of reproducing a realistic price impact function as an outcome of optimal behavior of all the market participants.

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References

Almgren, Robert and Neil Chriss (2000). “Optimal execution of portfolio transactions”. Journal of Risk 3(2), pp. 5–39.
Amihud, Yakov and Haim Mendelson (1980). “Dealership market: Market-making with inventory”. Journal of Financial Economics 8(1), pp. 31–53.
Andersen, Torben G. and Tim Bollerslev (1997). “Heterogeneous Information Arrivals and Return Volatility Dynamics: Uncovering the Long-Run in High Frequency Returns”. The Journal of Finance 52(3), pp. 975–1005.
Avellaneda, Marco and Sasha Stoikov (2008). “High-frequency trading in a limit order book”. Quantitative Finance 8(3), pp. 217–224.
Bacry, Emmanuel, Adrian Iuga, Matthieu Lasnier, and Charles-Albert Lehalle (2015). “Market impacts and the life cycle of investors orders”. Market Microstructure and Liquidity 1(02), p. 1550009.
REFERENCES

Bouchaud, Jean-Philippe, Yuval Gefen, Marc Potters, and Matthieu Wyart (2003). “Fluctuations and response in financial markets: the subtle nature of random price changes”. *Quantitative finance* 4(2), pp. 176–190.

Carmona, René and Laura Leal (2022). “Optimal Execution with Quadratic Variation Inventories”. *Quantitative Finance*. Accepted.

Carmona, René and Kevin Webster (2019). “The self-financing equation in limit order book markets”. *Finance and Stochastics* 23(3), pp. 729–759.

Cheng, Xue, Marina Di Giacinto, and Tai-Ho Wang (2017). “Optimal execution with uncertain order fills in Ahmeng-Chriss framework”. *Quantitative Finance* 17(1), pp. 55–69.

Cheng, Xue, Marina Di Giacinto, and Tai-Ho Wang (2019). “Optimal execution with dynamic risk adjustment”. *Journal of the Operational Research Society* 70(10), pp. 1662–1677.

Da Fonseca, Josué, Martino Grasselli, and Claudio Tebaldi (2008). “A multifactor volatility Heston model”. *Quantitative Finance* 8(6), pp. 591–604.

Forsyth, Peter A., J. Shannon Kennedy, Shu Tong Tse, and Heath Windcliff (2012). “Optimal trade execution: A mean quadratic variation approach”. *Journal of Economic Dynamics and Control* 36(12), pp. 1971–1991.

Garman, Mark B. (1976). “Market microstructure”. *Journal of Financial Economics* 3(3), pp. 257–275.

Graewe, Paulwin and Ulrich Horst (2017). “Optimal trade execution with instantaneous price impact and stochastic resilience”. *SIAM Journal on Control and Optimization* 55(6), pp. 3707–3725.

Grasselli, Martino and Claudio Tebaldi (2008). “Solvable Affine Term Structure Models”. *Mathematical Finance* 18(1), pp. 135–153.

Gueant, Olivier, Charles-Albert Lehalle, and Joaquin Fernandez-Tapia (2013). “Dealing with the inventory risk: a solution to the market making problem”. *Mathematics and Financial Economics* 7(4), pp. 477–507.

Ho, Thomas and Hans R. Stoll (1981). “Optimal dealer pricing under transactions and return uncertainty”. *Journal of Financial Economics* 9(1), pp. 47–73.

Karatzas, Ioannis and Steven E. Shreve (1991). *Brownian Motion and Stochastic Calculus*. Second Edition. New York: Springer-Verlag.

Kyle, Albert S. (1985). “Continuous Auctions and Insider Trading”. *Econometrica* 53(6), pp. 1315–1335.

Neuman, Eyal and Moritz Voß (2022). “Optimal signal-adaptive trading with temporary and transient price impact”. *SIAM Journal on Financial Mathematics*. To appear.

Ortu, Fulvio, Federico Severino, Andrea Tamoni, and Claudio Tebaldi (2020). “A persistence-based Wold-type decomposition for stationary time series”. *Quantitative Economics* 11(1), pp. 203–230.

Stoll, Hans R. (1978). “The supply of dealer services in securities markets”. *Journal of Finance* 33(4), pp. 1133–1151.

APPENDIX A. CONVERGENCE TO ORNSTEIN-UHLENBECK PROCESS

We show in this appendix that the dynamics of market maker’s inventory when following the approximately optimal quoting rule given in eq.s (2) converges to an Ornstein-Uhlenbeck process in the limit as $h$ approaches zero.

Recall that $S^b$ and $S^a$ denote the bid and ask prices, respectively, and $\delta^b$ and $\delta^a$ the difference between the quotes and the reference price $S$, i.e., $\delta^b := S - S^b$ and $\delta^a := S^a - S$. The reference price $S$ is assumed following an arithmetic Brownian motion with drift $\mu \geq 0$ and volatility $\sigma > 0$. Let $N_t^b$ and $N_t^a$ be the Poisson processes that represent respectively the cumulative market sell
and market buy orders up to time $t$. Thus, market maker’s inventory $q_t$ at time $t$ is given by $q_t = N_t^b - N_t^a$. The arrival rates $\theta^a$ and $\theta^b$ for market orders are given in (1).

Next, by scaling up the parameter $A$ and abuse the use of notation) in (1) by

$$A \rightarrow \frac{A}{h^2}$$

for some fixed constant $A > 0$ and scaling down the jump size of Poisson processes by $h$, i.e.,

$$N_t^a \rightarrow hN_t^a \quad \text{and} \quad N_t^b \rightarrow hN_t^b,$$

the arrival rates becomes

$$\theta^b(\delta^b) = \frac{A}{h^2} e^{-\kappa \delta^b}, \quad \theta^a(\delta^a) = \frac{A}{h^2} e^{-\kappa \delta^a}$$

and market marker’s inventory $q_t = h(N_t^b - N_t^a)$. We have that the approximately optimal quotes in (2) transform into

$$\delta^b(q) = \frac{1}{\nu} \ln \left( 1 + \frac{\nu}{\kappa} \right) + \left( q + \frac{1}{2} - \frac{\mu}{\nu \sigma^2} \right) \sqrt{\frac{\sigma^2 \nu h^2}{2 \kappa A}} \left( 1 + \frac{\nu}{\kappa} \right)^{1+\frac{\nu}{\kappa}},$$

$$\delta^a(q) = \frac{1}{\nu} \ln \left( 1 + \frac{\nu}{\kappa} \right) - \left( q + \frac{1}{2} + \frac{\mu}{\nu \sigma^2} \right) \sqrt{\frac{\sigma^2 \nu h^2}{2 \kappa A}} \left( 1 + \frac{\nu}{\kappa} \right)^{1+\frac{\nu}{\kappa}}.$$
Appendix B. Proof of Corollary 3.7

Note that in this case the system of ODEs reduce respectively to

\[
\frac{d}{dt} \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = \begin{bmatrix} \Theta M_{11} - \frac{\xi}{2\eta} \nu M_{21} - \frac{\xi^2}{4\eta} \nu' N_{11} - \frac{\xi}{2\eta} \nu N_{21} & \Theta M_{12} - \frac{\xi}{2\eta} \nu M_{22} - \frac{\xi^2}{4\eta} \nu' N_{12} - \frac{\xi}{2\eta} \nu N_{22} \\ -\frac{\xi}{2\eta} M_{21} - \frac{\xi^2}{4\eta} \nu' N_{11} + \left(-\frac{\xi^2}{4\eta} + \psi \right) N_{21} & -\frac{\xi}{2\eta} M_{22} - \frac{\xi^2}{4\eta} \nu' N_{12} + \left(-\frac{\xi^2}{4\eta} + \psi \right) N_{22} \end{bmatrix}
\]

(31)

and

\[
\frac{d}{dt} \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} = \begin{bmatrix} -\Theta N_{11} & -\Theta N_{12} \\ \frac{1}{\eta} M_{21} + \frac{\xi}{2\eta} \nu' N_{11} + \frac{\xi}{2\eta} N_{21} & \frac{1}{\eta} M_{22} + \frac{\xi}{2\eta} \nu' N_{12} + \frac{\xi}{2\eta} N_{22} \end{bmatrix}.
\]

(32)

We have, from (32) and taking into account the terminal conditions \(N_{11}(T) = I\) and \(N_{12}(T) = 0\), that

\[
N_{11} = e^{(T-u)\Theta}, \quad N_{12} = 0.
\]

Next notice that \(M_{22}\) and \(N_{22}\) satisfy the following coupled ODEs

\[
\dot{M}_{22} = -\frac{\xi}{2\eta} M_{22} + \left(-\frac{\xi^2}{4\eta} + \psi \right) N_{22},
\]

\[
\dot{N}_{22} = \frac{1}{\eta} M_{22} + \frac{\xi}{2\eta} N_{22}.
\]

Define \(\tilde{M}_{22} = M_{22} + \frac{\xi}{2} N_{22}\). Since

\[
M_{22} = \dot{\tilde{M}}_{22} - \frac{\xi}{2} \dot{N}_{22},
\]

We have

\[
\dot{M}_{22} = -\frac{\xi}{2\eta} \tilde{M}_{22} + \psi N_{22},
\]

\[
\dot{N}_{22} = \frac{1}{\eta} \tilde{M}_{22}
\]

and

\[
\dot{\tilde{M}}_{22} = \dot{M}_{22} + \frac{\xi}{2} \dot{N}_{22}
\]

\[
= -\frac{\xi}{2\eta} \tilde{M}_{22} + \psi N_{22} + \frac{\xi}{2\eta} \tilde{M}_{22}
\]

\[
= \psi N_{22}
\]

Hence, \(\tilde{M}_{22}\) and \(N_{22}\) satisfy

\[
\dot{\tilde{M}}_{22} = \psi N_{22}
\]

\[
\dot{N}_{22} = \frac{1}{\eta} \tilde{M}_{22}.
\]
Taking into the account the terminal conditions $\tilde{M}_{22}(T) = \gamma + \xi - \beta$ and $N_{22}(T) = 1$, we obtain the solutions for $\tilde{M}_{22}$ and $N_{22}$ as

$$\tilde{M}_{22} = \left(\gamma + \xi \right) - \beta \cosh \left(\sqrt{\psi \eta} \{T - u\} \right) - \sqrt{\psi \eta} \sinh \left(\sqrt{\psi \eta} \{T - u\} \right),$$

$$N_{22} = \cosh \left(\sqrt{\psi \eta} \{T - u\} \right) - \frac{1}{\sqrt{\psi \eta}} \left(\gamma + \xi \right) \sinh \left(\sqrt{\psi \eta} \{T - u\} \right).$$

By using the notation $\tilde{\alpha}$ defined in (28), we further rewrite $\tilde{M}_{22}$ and $N_{22}$ as

$$\tilde{M}_{22} = -\sqrt{\left(\gamma + \xi \right) - \beta - \psi \eta} \cosh \left(\sqrt{\psi \eta} \{T - u + \tilde{\alpha}\} \right),$$

$$N_{22} = \frac{\sinh \left(\sqrt{\psi \eta} \{T - u + \tilde{\alpha}\} \right)}{\sinh \left(\sqrt{\psi \eta} \alpha\right)}.$$

It follows that

$$\frac{\tilde{M}_{22}}{N_{22}} = -\sqrt{\psi \eta} \coth \left(\sqrt{\psi \eta} \{T - u + \tilde{\alpha}\} \right).$$

We solve $M_{12}$ as follows. Note that, from (31) and since $N_{12} = 0$, $M_{12}$ satisfies

$$M_{12} = \Theta M_{12} - \frac{\phi}{2\eta} \nu M_{22} - \frac{\phi \xi}{4\eta} \nu N_{22} = \Theta M_{12} - \frac{\phi}{2\eta} \nu \tilde{M}_{22}$$

$$\implies \frac{d}{dt} \left\{ e^{-u\Theta} M_{12} \right\} = -\frac{\phi}{2\eta} e^{-u\Theta} \tilde{M}_{22} \nu$$

$$\implies e^{-T\Theta} M_{12}(T) - e^{-u\Theta} M_{12} = -\frac{\phi}{2\eta} \int_{u}^{T} e^{-s\Theta} \tilde{M}_{22}(s) ds \nu$$

$$\implies M_{12} = \frac{\phi}{2} e^{(u-T)\Theta} \nu + \frac{\phi}{2\eta} \int_{u}^{T} e^{(u-s)\Theta} \tilde{M}_{22}(s) ds \nu,$$

since $M_{12}(T) = -\frac{\phi}{2} \nu$. Therefore,

$$M_{12} = -\frac{\phi}{2} \left\{ e^{(u-T)\Theta} I + \frac{\zeta}{\sinh(\tilde{\alpha})} \int_{u}^{T} e^{(u-s)\Theta} \cosh \left(\zeta \{T - s + \tilde{\alpha}\} \right) ds \right\} \nu,$$

where $\zeta = \sqrt{\frac{\psi \eta}{\phi}}$ for notational simplicity. We evaluate the integral on the right hand side as follows. Since

$$\int_{u}^{T} e^{(u-s)\Theta} \cosh \left(\zeta \{T - s + \tilde{\alpha}\} \right) ds$$

$$= -\frac{1}{\zeta} \left( I - \frac{1}{\zeta^2} \Theta^2 \right)^{-1} e^{(u-T)\Theta} \sinh \left(\zeta \tilde{\alpha}\right) + \frac{1}{\zeta} \left( I - \frac{1}{\zeta^2} \Theta^2 \right)^{-1} \Theta e^{(u-T)\Theta} \cosh(\zeta \tilde{\alpha})$$

$$+ \frac{1}{\zeta} \left( I - \frac{1}{\zeta^2} \Theta^2 \right)^{-1} \sinh \left(\zeta \{T - u + \tilde{\alpha}\} \right) - \frac{1}{\zeta^2} \left( I - \frac{1}{\zeta^2} \Theta^2 \right)^{-1} \Theta \cosh(\zeta \{T - u + \tilde{\alpha}\}),$$
we obtain that

\[
M_{12} = -\frac{\phi}{2}\left\{ e^{(u-T)\Theta}I + \frac{\zeta}{\sinh(\zeta\bar{\alpha})} \int_u^T e^{(u-s)\Theta} \cosh(\zeta \{ T - s + \bar{\alpha} \}) ds \right\} \nu
\]

\[
= -\frac{\phi}{2}\left\{ e^{(u-T)\Theta}I - \left( I - \frac{1}{\zeta^2} \Theta^2 \right)^{-1} e^{(u-T)\Theta} + \frac{1}{\zeta} \left( I - \frac{1}{\zeta^2} \Theta^2 \right)^{-1} \Theta e^{(u-T)\Theta} \coth(\zeta\bar{\alpha}) \right. \\
\left. + \left( I - \frac{1}{\zeta^2} \Theta^2 \right)^{-1} \frac{\sinh \{ \zeta(T - u + \bar{\alpha}) \}}{\sinh(\zeta\bar{\alpha})} - \frac{1}{\zeta} \left( I - \frac{1}{\zeta^2} \Theta^2 \right)^{-1} \Theta \frac{\cosh(\zeta(T - u + \bar{\alpha}))}{\sinh(\zeta\bar{\alpha})} \right\} \nu
\]

and

\[
\int_u^T M_{12}(s)ds
\]

\[
= -\frac{\phi}{2}\left\{ \Theta^{-1} \left[ I - e^{(u-T)\Theta} \right] \left[ I - \left( I - \frac{1}{\zeta^2} \Theta^2 \right)^{-1} \right] + \frac{1}{\zeta} \left[ I - \frac{1}{\zeta^2} \Theta^2 \right]^{-1} \Theta \coth(\zeta\bar{\alpha}) \right. \\
\left. + \frac{1}{\zeta} \left( I - \frac{1}{\zeta^2} \Theta^2 \right)^{-1} \left\{ - \coth(\zeta\bar{\alpha}) + \frac{\cosh \{ \zeta(T - u + \bar{\alpha}) \}}{\sinh(\zeta\bar{\alpha})} \right\} \right. \\
\left. - \frac{1}{\zeta^2} \left( I - \frac{1}{\zeta^2} \Theta^2 \right)^{-1} \Theta \left[ -1 + \frac{\sinh \{ \zeta(T - u + \bar{\alpha}) \}}{\sinh(\zeta\bar{\alpha})} \right] \right\} \nu
\]

Thus,

\[
\frac{2}{N_{22}} \int_u^T M_{12}(s)ds
\]

\[
= -\phi \left\{ \frac{\sinh(\zeta\bar{\alpha})}{\sinh(\zeta(T - u + \bar{\alpha}))} \Theta^{-1} \left[ I - e^{(u-T)\Theta} \right] \left[ I - \left( I - \frac{1}{\zeta^2} \Theta^2 \right)^{-1} \right] \\
+ \frac{1}{\zeta} \left( I - \frac{1}{\zeta^2} \Theta^2 \right)^{-1} \Theta \coth(\zeta\bar{\alpha}) \right. \\
+ \frac{1}{\zeta} \left( I - \frac{1}{\zeta^2} \Theta^2 \right)^{-1} \left\{ - \frac{\cosh(\zeta\bar{\alpha})}{\sinh(\zeta(T - u + \bar{\alpha}))} + \coth \{ \zeta(T - u + \bar{\alpha}) \} \right\} \right. \\
\left. - \frac{1}{\zeta^2} \left( I - \frac{1}{\zeta^2} \Theta^2 \right)^{-1} \Theta \left[ 1 - \frac{\sinh(\zeta\bar{\alpha})}{\sinh(\zeta(T - u + \bar{\alpha}))} \right] \right\} \nu
\]
Hence,

\[
\frac{2}{N_{22}} \int_u^T M_{12}'(s) ds \Theta q_0 \bar{q}_0
\]

\[
= -\phi \nu' \left\{ \frac{\sinh(\zeta \bar{\alpha})}{\sinh(\zeta(T - u + \bar{\alpha}))} \left[ I - e^{(u-T)\Theta} \right] \left[ I - \left( I - \frac{1}{\zeta^2} \Theta^2 \right)^{-1} \right] + \frac{1}{\zeta} \left( I - \frac{1}{\zeta^2} \Theta^2 \right)^{-1} \Theta \coth(\zeta \bar{\alpha}) \right\}
\]

\[
+ \frac{1}{\zeta} \left( I - \frac{1}{\zeta^2} \Theta^2 \right)^{-1} \Theta \left[ -\frac{\cosh(\zeta \bar{\alpha})}{\sinh(\zeta(T - u + \bar{\alpha}))} + \coth \{ \zeta(T - u + \bar{\alpha}) \} \right]
\]

\[
- \frac{1}{\zeta^2} \left( I - \frac{1}{\zeta^2} \Theta^2 \right)^{-1} \Theta^2 \left[ 1 - \frac{\sinh(\zeta \bar{\alpha})}{\sinh(\zeta(T - u + \bar{\alpha}))} \right] \right\} \bar{q}_0.
\]

Therefore,

\[
\frac{M_{12}}{N_{22}} = -\frac{\phi}{2} \left\{ \frac{e^{(u-T)\Theta}}{N_{22}} \left[ I - \left( I - \frac{1}{\zeta^2} \Theta^2 \right)^{-1} \right] + \frac{1}{\zeta} \left( I - \frac{1}{\zeta^2} \Theta^2 \right)^{-1} \Theta \coth(\zeta \bar{\alpha}) \right\}
\]

\[
+ \left( I - \frac{1}{\zeta^2} \Theta^2 \right)^{-1} \Theta \left[ -\frac{\cosh(\zeta \bar{\alpha})}{\sinh(\zeta(T - u + \bar{\alpha}))} \right] \nu.
\]

Since

\[
N^{-1} = \left[ -\frac{1}{N_{22}} N_{11}^{-1} \frac{1}{N_{22}} \right]
\]

we have

\[
R = MN^{-1}
\]

\[
= \left[ \begin{array}{cc} M_{11} & M_{12} \\ M_{21} & M_{22} \end{array} \right] \left[ \begin{array}{c} N_{11}^{-1} \\ \frac{1}{N_{22}} \end{array} \right] = \left[ \begin{array}{cc} M_{11} N_{11}^{-1} + M_{12} N_{21} & M_{12} \\ M_{21} N_{11}^{-1} & M_{22} \end{array} \right].
\]

Thus,

\[
R_{12} = \frac{M_{12}}{N_{22}}
\]

\[
= -\frac{\phi}{2} \left\{ \frac{e^{(u-T)\Theta}}{N_{22}} \left[ I - \left( I - \frac{1}{\zeta^2} \Theta^2 \right)^{-1} \right] + \frac{1}{\zeta} \left( I - \frac{1}{\zeta^2} \Theta^2 \right)^{-1} \Theta \coth(\zeta \bar{\alpha}) \right\}
\]

\[
+ \left( I - \frac{1}{\zeta^2} \Theta^2 \right)^{-1} \Theta \left[ -\frac{\cosh(\zeta \bar{\alpha})}{\sinh(\zeta(T - u + \bar{\alpha}))} \right] \nu.
\]

The equation (19b) for \( r \) becomes

\[
\dot{r} + A' r + 2Rb + \mu e + \frac{1}{\eta} (Raa' + ka') r = 0
\]
We shall focus on the term $r_{n+1}$ only since it is the term that is relevant to the determination of the optimal trading rate

\[
\hat{r}_{n+1} + 2R_{21} \Theta \bar{q}_0 + \mu + \frac{1}{\eta} \left( R_{22} + \frac{\bar{x}}{2} \right) r_{n+1} = 0
\]

\[
\Rightarrow \hat{r}_{n+1} + 2R_{21} \Theta \bar{q}_0 + \mu + \frac{1}{\eta} M_{22} \frac{N_{22}}{N_{22}} r_{n+1} = 0
\]

\[
\Rightarrow \hat{r}_{n+1} + 2R_{21} \Theta \bar{q}_0 + \mu + \frac{N_{22}}{N_{22}} r_{n+1} = 0
\]

\[
\Rightarrow N_{22} \hat{r}_{n+1} + \dot{N}_{22} r_{n+1} = -2M_{12} \Theta \bar{q}_0 - \mu N_{22}
\]

\[
\Rightarrow -N_{22} r_{n+1} = -2 \int_{u}^{T} M_{12}(s) ds \Theta \bar{q}_0 - \mu \int_{u}^{T} N_{22}(s) ds
\]

\[
\Rightarrow r_{n+1} = \frac{2}{N_{22}} \int_{u}^{T} M_{12}(s) ds \Theta \bar{q}_0 + \frac{\mu}{N_{22}} \int_{u}^{T} N_{22}(s) ds.
\]

Note that since $N_{22} = \frac{\bar{\psi}}{2} M_{22}$, we have

\[
- \frac{1}{2\eta} M_{22} \int_{u}^{T} N_{22}(s) ds = - \frac{\mu}{2\psi\eta} \left( \frac{\bar{\psi} + \bar{\xi}}{2} - \bar{\beta} \right)
\]

\[
= - \frac{\mu}{2\psi\eta} \left[ \frac{\bar{\psi} + \bar{\xi}}{2} - \frac{\bar{\beta}}{N_{22}} + \sqrt{\psi\eta} \coth(\zeta(T - u + \bar{\alpha})) \right]
\]

\[
= - \frac{\mu}{2\sqrt{\psi\eta}} \left[ \frac{\bar{\psi} + \bar{\xi}}{2} \sinh(\zeta\bar{\alpha}) + \sinh(\zeta(T - u + \bar{\alpha})) \right]
\]

\[
= \frac{\mu}{2\sqrt{\psi\eta}} \left[ \sqrt{\psi\eta} \coth(\zeta(T - u + \bar{\alpha})) - \coth(\zeta(T - u + \bar{\alpha})) \right].
\]

Finally, the optimal trading rate $v^*$ in this case reads

\[
v^*(u) = \frac{1}{2\eta} \{2(k + R(u)a)' X_G(u) + a' r(u)\}
\]

\[
= - \frac{\phi}{2\eta} v' Q_G(u) - \frac{\bar{\psi}}{2\eta} X_G(u) - \frac{1}{\eta} (R_{12}(u) Q_G(u) + R_{22}(u) X_G(u)) - \frac{1}{2\eta} r_{n+1}
\]

\[
= - \frac{\phi}{2\eta} v' Q_G(u) - \frac{1}{\eta} M_{22} X_G(u) - \frac{1}{\eta} M_{12}^2 Q_G(u) - \frac{1}{2\eta} r_{n+1}
\]

\[
= - \frac{\phi}{2\eta} v' Q_G(u) + \sqrt{\psi\eta} \coth \left( \sqrt{\psi\eta} (T - u + \bar{\alpha}) \right) X_G(u)
\]

\[
+ \frac{\phi}{2\eta} v' \left\{ e^{u-T} \Theta \left[ I - \left( I - \frac{1}{\zeta^2} \Theta^2 \right)^{-1} + \frac{1}{\zeta} \left( I - \frac{1}{\zeta^2} \Theta^2 \right)^{-1} \Theta T \coth(\zeta\bar{\alpha}) \right] \right. \]

\[
+ \left( I - \frac{1}{\zeta^2} \Theta^2 \right)^{-1} - \frac{1}{\zeta} \left( I - \frac{1}{\zeta^2} \Theta^2 \right)^{-1} \Theta T \coth(\zeta(T - u + \bar{\alpha})) \right\} Q_G(u)
\]
\[ + \frac{\phi}{2\eta} \nu' \left\{ \frac{\sinh(\bar{\alpha})}{\sinh(\zeta(T-u+\bar{\alpha}))} \left[ I - e^{(u-T)\Theta} \right] \left[ I - \left( I - \frac{1}{\zeta^2} \Theta^2 \right)^{-1} \right] \right\} \]

\[ + \frac{1}{\zeta} \left( I - \frac{1}{\zeta^2} \Theta^2 \right)^{-1} \Theta \ coth(\zeta\bar{\alpha}) \]

\[ + \frac{1}{\zeta^2} \left( I - \frac{1}{\zeta^2} \Theta^2 \right)^{-1} \Theta^2 \left[ 1 - \frac{\sinh(\zeta\bar{\alpha})}{\sinh(\zeta(T-u+\bar{\alpha}))} \right] \bar{q}_0 \]

\[ + \frac{\mu}{2\sqrt{\psi\eta}} \left\{ \frac{\cosh(\zeta\bar{\alpha})}{\sinh(\zeta(T-u+\bar{\alpha}))} - \coth(\zeta(T-u+\bar{\alpha})) \right\} \]

\[ = \sqrt{\frac{\psi}{\eta}} \ coth \left( \sqrt{\frac{\psi}{\eta}} \{ T - u + \bar{\alpha} \} \right) X_G(u) \]

\[ + \frac{\mu}{2\sqrt{\psi\eta}} \left[ \frac{\cosh(\zeta\bar{\alpha})}{\sinh(\zeta(T-u+\bar{\alpha}))} - \coth(\zeta(T-u+\bar{\alpha})) \right] \]

\[ - \frac{\phi}{2\eta} \nu' Q_G(u) + \frac{\phi}{2\eta} [\Lambda^n(T) Q_G(u) + \Lambda^n_0(T) \bar{q}_0], \]

where $\Lambda^n(T)$ and $\Lambda^n_0(T)$ are given in (29) and (30), respectively.