Shear-free perfect fluids with linear equation of state

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Abstract
We prove that shear-free perfect fluid solutions of Einstein’s field equations must be either expansion-free or rotation-free (as conjectured by Ellis and Treciokas) for all linear equations of state \( p = w \rho \) except for \( w \in \{-\frac{1}{5}, -\frac{1}{6} - \frac{1}{11}, -\frac{1}{21}, \frac{1}{15}, \frac{1}{4}\} \).

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1. Introduction

Perfect fluids are the most commonly used matter sources for the Einstein field equations. The hyperbolicity of the coupled Einstein–Euler system is by now a classical matter [7]. However, the explicit construction or the characterization of solutions under further ‘symmetry’ assumptions may raise challenging questions. Two simplifying, but fairly general and observationally defensible assumptions will be of interest here: the fluid is barotropic (i.e. its pressure \( p \) depends uniquely on the energy density \( \rho \)) and shear-free (i.e. fluid’s velocity is a transversally conformal vector). The former is a thermodynamic condition which implies constant entropy, the latter assures the isotropy of local motion, and both are common features of standard spacetimes, such as the FRW and Gödel models.

In 1950, Gödel announced the first result about the structure of solutions in this context [13]: a shear-free pressure-less fluid (in particular a congruence of timelike geodesics) in a spatially homogeneous spacetime of type IX cannot both expand (i.e. have non-zero divergence velocity) and rotate (i.e. have non-zero vorticity 2-form, which measure the failure of being hypersurface orthogonal). Gödel’s result was later proved [16] to remain true without the homogeneity assumption and it can be seen as a timelike analogue of the Goldberg–Sachs theorem [19]. After proving that the same conclusion holds for radiation fluids, Ellis and Treciokas [38] formulated the shear-free fluid conjecture as follows:
If the velocity vector field of a self-gravitating barotropic perfect fluid \( p + \rho \neq 0 \) and \( p = p(\rho) \) is shear-free, then either the expansion or the rotation of the fluid vanishes.

If true, this would contrast Newtonian cosmology where expanding and rotating (shear-free) solutions are known and can avoid singularity formation [26].

Many partial results have been obtained by specializing (i) the curvature (divergence-free electric or magnetic part of Weyl curvature [5, 40], Petrov types N [3] and III [4]), (ii) kinematical quantities (parallel vorticity and acceleration [41], functionally dependent expansion and energy density [22] or expansion and rotation scalar [36], fluid flow parallel with a conformal vector field [9]), or (iii) the equation of state \( p = w\rho \), with \( w = 0, \pm \frac{1}{3}, \frac{1}{9} \) [16, 21, 35, 38, 39]). Moreover the conjecture holds true when the equations are linearized about a FRW background [27]. Further aspects supporting the conjecture can be found in [11, 17].

Independently, several results of similar flavour have been discovered in Riemannian geometry. They are all related to the notion of \((r-)\) harmonic morphism (see [1] for a general account).

Let us recall that the relativistic barotropic perfect fluids (whether self-gravitating or not) can benefit from a variational treatment if we define the action functional in terms of a submersion from the (four-dimensional) spacetime having its fibres tangent to the fluid velocity, \( U \). This effective field theory approach, initiated in [6, 12, 37], has been mathematically settled in [8] and renewed in recent physics literature, e.g. [14, 15]. In particular, it turns out [34] that a shear-free barotropic perfect fluid is determined by a (locally defined) horizontally conformal submersion which is a critical point of the action functional. In particular, for linearly related pressure and energy density this leads to an \( r\)-harmonic morphism [23] with one-dimensional fibres tangent to \( U \).

In Riemannian signature and dimension at least four, precisely in the one-dimensional fibres case, \( r\)-harmonic morphisms on Einstein manifolds and \((r \geq 2)\)-harmonic morphisms on constant curvature spaces have been classified [2, 24, 29, 30]: they are all either of Killing type (with the unit vertical vector \( U \) divergence/expansion-free), of warped product type (with \( U \) self-parallel and non-rotating, since orthogonal to a foliation by hypersurfaces), or (only on Einstein 4-manifolds) of a third type which implies Ricci flatness. A similar result holds also on conformally-flat manifolds [31]. When the codomain is of constant curvature, the three types provide us with constructions of Einstein metrics in dimension four: the Killing type leads to the Gibbons–Hawking ansatz, the warped product type to the warped product construction, and the third type essentially to Eguchi–Hanson metric.

Adopting the standpoint of \( r\)-harmonic morphisms, in this paper we prove the shear-free fluid conjecture for all linear equations of state \( p = w\rho \) except for six values of \( w \) which require a case by case analysis. Together with the four values already settled, they form the full set of exceptional values of the present proof.

2. Preliminaries

In this section we present the terminology related to perfect fluids and introduce some basic results needed latter on. We begin with a summary of notations.

2.1. Notations and conventions

Throughout the paper \((M, g)\) denotes a connected time-oriented Lorentz four-dimensional manifold with metric signature \((- , + , + , + )\), usually referred as \textit{spacetime}. 
We denote by \( \mathcal{L} \) the Lie derivative, by \( \nabla \) the Levi-Civita connection of \((M, g)\), and we use the following sign conventions for the curvature tensor field \( R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z \), and \( \Delta f = \text{div} (\text{grad} f) \) for the Laplacian on functions. The covariant Hessian of a function is \( \text{Hess}_f(X, Y) = X (Y (f)) - (\nabla_X Y)(f) \) is the Hessian of \( f \). The Ricci and scalar curvature of \((M, g)\) are denoted by \( \text{Ric} \) and \( \text{Scal} \), respectively.

Given \( X \in T TM, X^\perp \) is the 1-form defined by \( X^\perp (Y) = g(X, Y) \) for all \( Y \in T TM \); conversely given the 1-form \( \alpha, \alpha^\perp \) is the vector defined by \( \alpha(Y) = g(\alpha, Y) \) for all \( Y \in T TM \).

Given a 2-form \( \Omega \) on \((M, g)\), the interior product with a vector \( X \) is \( t_X \Omega = \Omega(X, \cdot) \) and its co-differential is defined in analogy with Riemannian case by \( \delta \Omega(X) = -\sum_{i=1}^4 \epsilon_i (\nabla_{e_i} \Omega)(e_i, X) \), where \( \{e_1, \ldots, e_4\} \) is an orthonormal frame with \( g(e_i, e_i) = \epsilon_i = \pm 1 \).

The metric \( g \) on \( M \) induces a (pointwise) metric on the bundle of \( p \)-covariant tensors (or \( p \)-forms) on \( M \):
\[
(\mathcal{A}, \mathcal{B}) = \frac{1}{2} \sum_{i_1, \ldots, i_p = 1}^4 \epsilon_{i_1} \ldots \epsilon_{i_p} \mathcal{A}(e_{i_1}, \ldots, e_{i_p}) \mathcal{B}(e_{i_1}, \ldots, e_{i_p}),
\]

that provides us with the norm \( |\mathcal{A}| \) of such object. The notation \( \varphi^* \mathcal{A} \) refers to the usual pullback of \( \mathcal{A} \) by the mapping \( \varphi \).

### 2.2. Perfect fluids coupled with gravity

Let \((M, g)\) be a four-dimensional spacetime.

**Definition 1.** ([25, 28]) A triple \((U, p, \rho)\) is called (relativistic) perfect fluid on \((M, g)\) if:

(i) \( U \) is a timelike future-pointing unit vector field on \( M \), called the flow vector field (or normalized 4-velocity),

(ii) \( \rho, p: M \to \mathbb{R} \) are real functions, called mass (energy) density and pressure, respectively,

(iii) the stress–energy tensor of the fluid is conserved:
\[
\text{div} (p g + (\rho + p) U^\alpha \otimes U^\alpha) = 0,
\]
where \( U^\alpha(X) = g(U, X) \), for all \( X \) tangent to \( M \).

If at (iii) we impose the Einstein field equations to be satisfied
\[
\text{Ric} - \frac{1}{2} \text{Scal} \cdot g = p g + (\rho + p) U^\alpha \otimes U^\alpha,
\]
then \((U, p, \rho)\) is called perfect fluid coupled to gravity (or self-gravitating).

Given a unit timelike vector \( U \) on \( M \), we denote by \( \mathcal{V} \) the (vertical) foliation spanned by \( U \) and by \( \mathcal{H} \) its (horizontal) orthogonal complement \( \mathcal{V}^\perp \) in the tangent bundle of \( M \). The orthogonal projector on \( \mathcal{H} \) (or \( \mathcal{V} \)) of a vector field \( X \) will be denoted by \( X^\mathcal{H} \) (or \( X^\mathcal{V} \)). Then any vector \( X \) has the splitting: \( X = X^\mathcal{H} - U^\alpha \otimes U^\alpha \).

Condition (iii) of definition 1 splits the fluid equations into:
\[
\begin{align*}
(\rho + p) \nabla_{U} U + \text{grad}^g p &= 0 \quad \text{(Euler equations),} \\
(\rho + p) \text{div} U + U(\rho) &= 0 \quad \text{(energy conservation).}
\end{align*}
\]

The coupling with gravity (1) imposes the following block-diagonal form of the Ricci tensor:
\[
\begin{align*}
\text{Ric}(U, U) &= \frac{1}{2} (\rho + 3p), \quad \text{Ric}(X, U) = 0, \quad \forall X \perp U, \\
\text{Ric}(X, Y) &= \frac{1}{2} (\rho - p) g(X, Y), \quad \forall X, Y \perp U.
\end{align*}
\]

Notice that \( \text{Scal} = \rho - 3p \).

Let \( \nabla U \) be the tensor defined as \( \langle \nabla U \rangle(X, Y) = g(\nabla_X U, Y) \), for all \( X, Y \).
Proposition 1. The covariant derivative of a timelike unit vector field $U$ decomposes as follows

$$\nabla U + U^\rho \otimes (\nabla U)_\rho = \omega + \sigma + \frac{1}{2} \text{div} U \, g^{\lambda \mu},$$

where $g^{XY}(X,Y) = g(X^\lambda, Y^\lambda)$ is the horizontal metric and

$$\omega(X,Y) = \frac{1}{2} d\rho(X^\lambda, Y^\lambda),$$

$$\sigma(X,Y) = \frac{1}{2} \left( \mathcal{L}_U g - \frac{3}{2} \text{div} U \, g \right) (X^\lambda, Y^\lambda).$$

Proof. Notice that Koszul formula can be rewritten as follows:

$$g(\nabla X U, Y) = \frac{1}{2} d\rho(X,Y) + \frac{1}{2} (\mathcal{L}_U g)(X,Y), \quad \forall X, Y.$$

Since $g(\nabla X U, Y) = g(\nabla X \subset U, Y^\lambda) - U^\rho(X) g(\nabla U, Y)$, applying Koszul formula on the first right-hand term yields the result. □

Definition 2 (Kinematical quantities). For a relativistic fluid $(U, \rho, \gamma)$, $\nabla U$ is the acceleration (vector), $\text{div} U$ is the expansion rate (scalar), $\omega$ is the rotation rate or vorticity (2-form) and $\sigma$ is the shear rate (symmetric and traceless 2-covariant tensor).

Remark 1. In cosmology [18], the shear measures the rate of change of relative distance of neighbouring galaxies (so the shear-free condition expresses the isotropy of the relative recession motion, allowing the red shift and the cosmic microwave background radiation to be anisotropic), expansion represents the rate of volume expansion of a sphere of galaxies ($\frac{1}{2} \text{div} U$ is the Hubble parameter), and the vorticity is the rate of change of relative position, so measuring the rotation of the ‘nearby’ matter about an observer moving with the matter, relative to an inertial frame.

In kinetic theory shear-free assumption characterizes [38] the velocity of collision-dominated gases with isotropic distribution function, under the Einstein–Boltzmann equations.

From the following slightly reformulated Yano’s identity [43]

$$\text{Ric}(X,Y) = \text{div}(\nabla X Y) - X(\text{div} Y) - \frac{1}{2} (\mathcal{L}_X g, \mathcal{L}_Y g) + \frac{1}{2} (dX^\rho, dY^\rho),$$

for all vectors $X, Y$ tangent to a (semi-)Riemannian manifold, we obtain

Proposition 2 (Raychaudhuri equation [32]). Let $U$ be a unit timelike vector on $(M,g)$. Then the following identity holds:

$$U(\text{div} U) = -\frac{1}{2} (\text{div} U)^2 + 2 (|\omega|^2 - |\sigma|^2) + \text{div}(\nabla U) - \text{Ric}(U, U).$$

In this context, the shear-free fluid conjecture reads: if equation (1) is satisfied by a perfect fluid with $p = p(\rho)$, $p + \rho \neq 0$ and $\sigma = 0$, then $(\text{div} U) \omega = 0$. We shall treat the special case described below.

2.3. Shear-free linear-barotropic fluids

On a spacetime $(M,g)$ ($\dim M = 4$), consider a shear-free perfect fluid with linear equation of state $p = w \rho$ ($w \in \mathbb{R}$), which is the $\gamma$-law used in cosmology. Around a point where $\rho$ does not vanish (assume $\rho > 0$), we can write it in the form $(U, p = \frac{w+1}{w-1} \lambda^r, \rho = \lambda^r)$, where $\lambda$ is some positive function (locally) on $M$ ($\lambda^{-1}$ is the length scale) and $r = 3(w + 1)$.

The fluid equations (2) become, respectively

$$-\nabla_U U + (3 - r) \text{grad} \lambda (\ln \lambda) = 0,$$

(8)
\[
\text{div} U + 3U (\ln \lambda) = 0.
\] (9)

In this case, the conditions (3) for the fluid to be coupled with gravity turn out to be

\[
\text{Ric}(U, U) = \frac{r - 2}{2} \lambda', \quad \text{Ric}(X, U) = 0, \quad \forall X \perp U,
\]

\[
\text{Ric}(X, Y) = \frac{6 - r}{6} g(X, Y), \quad \forall X, Y \perp U
\] (10)

and therefore \( \text{Scal} = (4 - r) \lambda' \).

A crucial notion in the sequel are basic tensors.

**Definition 3.** (1) A section \( \varsigma \) of \( \mathcal{H} \) is called basic if \( (L_W \varsigma)^H = 0 \) for all \( W \in \Gamma(\mathcal{V}) \).

In particular, a function \( f \) on \( M \) is basic if \( \vartheta (f) = 0 \), and a horizontal vector field \( X \) on \( M \) is basic if \( [W, X]^H = 0 \), for all \( W \in \Gamma(\mathcal{V}) \). See the appendix section for further properties of basic quantities.

Let us define the fundamental vector field of \( (U, p = \frac{r - 3}{2} \lambda', \rho = \lambda') \) by

\[
V = \lambda^{3-r} U.
\] (11)

Analogously to [29] (see also [1, p 341]), we have

**Proposition 3.** The Euler equation (8) is equivalent to

\[
[V, X] = 0,
\] (12)

for all basic vector fields \( X \) on \( M \).

**Proof.** Let \( X \) be a basic vector field. Then \([V, X]\) is vertical. Since

\[
g([V, X], U) = \lambda^{3-r} g((3 - r) \text{grad} \ln \lambda - \nabla_U X),
\]

we conclude that \([V, X] = 0\) if and only if Euler equation is satisfied. \(\square\)

Let \( \vartheta \) be the 1-form dual to \( V \), defined by \( \vartheta (X) = -\lambda^{2(r-3)} g(X, V) \) for all \( X \), and let \( \Omega = d \vartheta \).

**Lemma 1.** Let \( (U, p = \frac{r - 3}{2} \lambda', \rho = \lambda') \) be a shear-free perfect fluid. Then

(i) \( \Omega(X, Y) = \lambda^{3-r} g([X, Y], V) \) for all \( X, Y \in \Gamma(\mathcal{H}) \);

(ii) \( \Omega = -2\lambda^{3-r} \omega \) (\( \Omega \) is proportional to the rotation rate);

(iii) \( \vartheta \Omega = 0 \), for all \( W \in \Gamma(\mathcal{V}) \) (\( \Omega \) is a horizontal 2-form)

(iv) \( \Omega = 0 \) if and only if \( \mathcal{H} \) is integrable;

(v) \( \mathcal{L}_W \Omega = 0 \) or, equivalently, \( W(\Omega(X, Y)) = 0 \), for all \( X, Y \) basic vectors and for all \( W \in \Gamma(\mathcal{V}) \) (\( \Omega \) is a basic 2-form);

(vi) for any basic vector field \( X \), the following function is basic:

\[
\lambda^{-2} [\delta \Omega(X) + (r - 4) \Omega(X, \text{grad} \ln \lambda)].
\]

1 In other words, for the 4-velocity of a perfect fluid, being hypersurface orthogonal is equivalent with being vorticity-free (i.e. having zero rotation).
Proof. (i) and (ii) are immediate using only the definitions. For (iii) we can check that \( \Omega(V, X) = 0 \) for a basic vector \( X \) by employing equation (12). (iv) is the Frobenius’ theorem upon applying (iii). (v) is the consequence of Cartan’s formula \( \mathcal{L}_\psi \Omega = t_\psi d\Omega + d(t_\psi \Omega) \).

To prove the last assertion (vi), let \( \{X, Y, Z\} \) be orthogonal basic vectors from a preferred frame (see remark below). By direct computation we obtain

\[
\delta \Omega(X) = \lambda^2 \left[ (4 - r)Y (\ln \lambda) \Omega(X, Y) + Y (\Omega(X, Y)) + \lambda^2 g([Z, Y], Z) \Omega(X, Y) \right],
\]

\[
\delta \Omega(Y) = \lambda^2 \left[ (4 - r)X (\ln \lambda) \Omega(Y, X) - X (\Omega(X, Y)) - \lambda^2 g([Z, X], Z) \Omega(X, Y) \right],
\]

\[
\delta \Omega(Z) = -\lambda^2 g([X, Y], Z) \Omega(X, Y).
\]

Since \( [X, Y]^t_i \), \([X, Z]^t_i \) and \([Y, Z]^t_i \) are basic vectors, the functions \( \lambda^2 g([X, Y], Z) \), \( \lambda^2 g([Z, X], Z) \) and \( \lambda^2 g([Z, Y], Z) \) are basic. Using also the fact that \( \Omega \) is basic cf (v), we obtain the conclusion for \( X, Y \) and \( Z \), and therefore for any basic vector. \( \square \)

Remark 2 (Preferred orthonormal frames, cf [1, 29]). Let us choose a local orthogonal frame \( \{X, Y, Z\} \) of basic horizontal vector fields. The shear-free hypothesis allows us to suppose that their lengths satisfy \( |X| = |Y| = |Z| = 1/\lambda \). Moreover, since \( \Omega \) is basic, we may suppose \( Z \) chosen such that \( t_2 \Omega = 0 \), that is collinear with vorticity vector \( (*\gamma \omega)^3 \).

Alternatively \( \{X, Y, Z\} \) can be seen as a local orthonormal frame of eigenvectors for the basic field \( \Omega^2 \) of self-adjoint positive semi-definite endomorphisms of \( (\mathcal{H}, \tilde{\Omega} = \lambda^2 g^i_t) \) defined by \( \tilde{\Omega}(\Omega^3(A), B) = \langle t_3 \Omega, t_3 \Omega \rangle \) for horizontal fields \( A \) and \( B \).

In particular, the contractions \( t_3 \Omega \) and \( t_4 \Omega \) are both basic and orthogonal, i.e. \( \langle t_\chi \Omega, t_\gamma \Omega \rangle = 0 \). Since \( d\Omega(X, Y, Z) = 0 \), we also have

\[
(\mathcal{L}_{\phi^3})\Omega(X, Y) = 0.
\]

Moreover, \( |X|^2 = \lambda^2 \Omega(X, X)^2 \) and \( |Y|^2 = |Y|^2 = \lambda^2 \Omega(X, Y)^2 \).

We call \( \{U, \lambda X, \lambda Y, \lambda Z\} \) a preferred orthonormal frame. This is related to a Fermi propagated triad choice, as it was done in [39].

Remark 3 (Affine-barotropic fluids). Consider the slightly more general case of self-gravitating perfect fluids with an affine equation of state \( p = w\rho + c, \ w \in \mathbb{R} \setminus \{-1\}, \ c \in \mathbb{R} \) (that is \( dp/d\rho = w \)). Defining \( \Lambda = -\frac{w}{w + 1} \), we see that \( \langle U, \tilde{\gamma} = \rho = \Lambda \rangle \) is a linear-barotropic fluid \( \langle \tilde{\rho} = \rho \rangle \) satisfying Einstein equation with cosmological constant:

\[
\text{Ric} - \frac{1}{2} \text{Scal} \cdot g = \tilde{\rho} g + (\tilde{\rho} + \tilde{\rho}) U^r \otimes U^r - \Lambda g.
\]

By choosing \( \lambda \) and \( r \) as before, in such a way as to respect the conditions above, namely as being positive and as \( w > -1 \) we can equivalently work with \( \langle U, \tilde{\rho} = \langle \frac{w}{w + 1} \lambda \rangle \rangle \) in the previous set-up except for the coupling with gravity condition (10) that should be replaced with

\[
\text{Ric}(U, U) = \frac{r - 2}{2} \lambda^r - \Lambda, \quad \text{Ric}(X, U) = 0, \quad \forall X \perp U,
\]

\[
\text{Ric}(X, Y) = \left( \frac{6 - r}{6} \lambda^r + \Lambda \right) g(X, Y), \quad \forall X, Y \perp U.
\]

2.4. Effective field theory and \( r \)-harmonic morphisms

The main idea of the effective field theory approach is to associate to the 4-velocity \( U \) of the fluid, a (locally defined) submersion \( \varphi : (M, g) \to (N, h) \) from the four-dimensional spacetime into a Riemannian 3-manifold such that \( U \in \ker d\varphi \). Then the action functional for gravitation with barotropic perfect fluid sources is of the following form

\[
\mathcal{E}(g, \varphi) = \int_M \{\text{Scal} + F(|\wedge^3 d\varphi|^2)\} \text{vol}_g.
\]
For shear-free vectors, the associated submersion is \textit{horizontally conformal} (i.e. \( \varphi^* h = \lambda^2 g \)) with \( \lambda \) called \textit{dilation}; see [1, proposition 2.5.11] for further details. In particular, a shear-free linear barotropic fluid will correspond to a horizontally conformal critical point of the previous action for \( F(x) = x^\rho \) that turns out [34] to be an \( r \)-harmonic morphism.

Recall [23] that an \( r \)-\textit{harmonic morphism} \( \varphi \) is characterized as a horizontally conformal critical point of the action functional \( \int_M|d\varphi|^2 \text{vol}_g \).

According to [2, 24], fluid’s equations (8) and (9) are the necessary and sufficient conditions for \( U \) to be tangent to the fibres of a (locally defined) \( r \)-harmonic morphism \( \varphi \), with dilation \( \lambda \). The corresponding Euler–Lagrange equations are given precisely by (8), while (9) is an identity fulfilled by any horizontally conformal submersion (see [34]).

By definition, an \( r \)-harmonic morphism with one-dimensional fibers is of \textit{Killing} type if the gradient of the dilation is horizontal (i.e. \( U(\lambda) = 0 \)), and of \textit{warped product} type if it is horizontally homothetic (i.e. the gradient of the dilation is vertical) with totally geodesic fibres and integrable horizontal distribution \( \mathcal{H} \). Notice that, in the first case the fundamental vector field (11) is Killing, and in the second case, that \( \varphi \) is \( r \)-harmonic for all values of \( r \) (in particular, it is harmonic).

3. Consequences of the Einstein equations with shear-free fluid sources

The very existence of a shear-free perfect fluid constrains the spacetime geometry, in particular the Ricci curvature (proposition 4) that, in addition, must have block-diagonal form (3) by the Einstein field equations. For a shear-free fluid with linear equation of state \( p = \frac{\rho}{\rho + 3} \), with \( \rho = \rho' \) and \( r \neq 2 \), we see these curvature restrictions as prescribed conditions on the second order derivatives of \( \ln \lambda \). This will provide compatibility conditions at the level of third order derivatives, in the form of polynomial relations in first derivatives only.

In this section we collect the relevant equations and their compatibility conditions that are useful in the proof of the conjecture. Analogously to [29] (see also [1, p 343]) we have

**Proposition 4** (Ricci curvature restrictions). \textit{Let} \((M, g)\) be a four-dimensional spacetime and \( \varphi : (M, g) \rightarrow (N, h) \) an \( r \)-harmonic morphism of dilation \( \lambda \), into a three-dimensional Riemannian manifold. If the fibres are tangent to the timelike unit vector field \( U \), then the following identities hold for all horizontal vectors \( X \) and \( Y \):

\[
\text{Ric}(U, U) = (3 - r) \Delta \ln \lambda + (6 - r) U(U(\ln \lambda)) + 3(r - 4) U(\ln \lambda)^2 + \frac{\lambda^{2(3-r)}}{2} |\Omega|^2, \quad (13)
\]
\[
\text{Ric}(X, U) = 2X(U(\ln \lambda)) - \frac{\lambda^{3-r}}{2} [\delta \Omega(X) + 2(3 - r) \Omega(X, \text{grad} \ln \lambda)], \quad (14)
\]
\[
\text{Ric}(X, Y) = \varphi^* \text{Ric}^N(X, Y) + (r - 2) \text{Hess}_{\text{inh}}(X, Y)
\]
\[-[(r - 3)^2 + 1]X(\ln \lambda) Y(\ln \lambda) + \frac{\lambda^{2(3-r)}}{2} |\Omega|^2, \quad (15)
\]

\(+\Delta \ln \lambda + (r - 2) [\text{grad} \ln \lambda]^2 g(X,Y).\]

\textbf{Proof.} The first identity is simply another way of writing Raychaudhuri equation (7) in the shear-free case, by using equations (8) and (9).

The second identity can be derived directly from Yano’s formula (6), taking into account the decomposition (4). This formula is known in physics literature as shear-divergence identity or \((0, \alpha)\)—equation and it is one of the standard constraint equations, cf e.g. [33].

The third identity is obtained by taking the trace of the following formula ([20, theorem 2.2.3]), true for any horizontally conformal map \( \varphi : (M, g) \rightarrow (N, h) \) and for all horizontal vectors \( X, Y, Z, T \) (see also [1, p 320])
\( \langle R(X, Y)Z, T \rangle = \lambda^{-2} \langle R^{\lambda}(d\varphi(X), d\varphi(Y))d\varphi(Z), d\varphi(T) \rangle \)

\(- \langle X(\ln \lambda)Y - Y(\ln \lambda)X, T(\ln \lambda)Z - Z(\ln \lambda)T \rangle + \langle Y, Z \rangle H_{\text{in}, \lambda}(X, T) \)

\(- \langle X, Z \rangle H_{\text{in}, \lambda}(Y, T) + \langle X, T \rangle H_{\text{in}, \lambda}(Y, Z) - \langle Y, T \rangle H_{\text{in}, \lambda}(X, Z) \)

\(+ \frac{1}{4} \{ \langle I(X, Z), I(Y, T) \rangle - \langle I(Y, Z), I(X, T) \rangle + 2 \langle I(X, Y), I(Z, T) \rangle \}

\(+ \langle \langle Y, Z \rangle \langle X, T \rangle - \langle X, Z \rangle \rangle \rangle [\text{grad} \ln \lambda] \rangle \)

where \( I(X, Y) = [X, Y]^\lambda \), combined with the propagation of shear:

\( R(U, X, Y, Z) + \frac{1}{2} (\mathcal{L}_{\nu}U)g(X, Y) - \frac{1}{4} \left[ U(\text{div}U) + \frac{1}{4} (\text{div}U)^2 \right] g(X, Y) \)

\(+ \left\langle \nu X, \nu Y \right\rangle + g(\nabla_{\nu}U, X)g(\nabla_{\nu}U, Y) = 0, \quad \forall X, Y \in \mathcal{H}. \)

\( \Box \)

Using the gravity coupling condition (10) and Section 2.4, we obtain

**Corollary 1.** Let \( (U, p = \frac{\lambda}{\lambda'}, \rho = \lambda') \) be a shear-free perfect fluid coupled with gravity on \( (M, g) \). Then the identities (13), (14) and (15) hold with the left-hand side replaced by \( \frac{\lambda}{\lambda'} \lambda' \), \( 0 \) and \( \frac{\lambda}{\lambda'} \lambda' g(X,Y) \), respectively, where \( N \) is endowed with a metric \( h \) such that the projection along \( U, \Phi : (M, g) \rightarrow (N, h) \), satisfies \( \Phi^* h = \lambda^2 g^N \).

Taking the (horizontal) trace of (15) and combining with (13) gives:

**Corollary 2 (Trace constraints).** Let \( (U, p = \frac{\lambda}{\lambda'}, \rho = \lambda') \) be a shear-free perfect fluid coupled with gravity on \( (M, g) \). Then the following identities must hold

\[
\Delta \ln \lambda = \frac{r - 2}{2} U(\ln \lambda)^2 - \frac{r - 4}{3} \lambda' + \frac{3r - 14}{24} \lambda^6 - 2|\Omega|^2 + \frac{(2r - 5)(r - 6)}{6} |\text{grad}^N \ln \lambda|^2 + \frac{r - 6}{12} \lambda^2 \text{Scal}^N. \tag{16}
\]

\[
U(U(\ln \lambda)) = \frac{r - 5}{2} U(\ln \lambda)^2 + \frac{2r - 5}{6} \lambda' - \frac{3r - 5}{24} \lambda^6 - 2|\Omega|^2 + \frac{(2r - 5)(r - 3)}{6} |\text{grad}^N \ln \lambda|^2 - \frac{r - 3}{12} \lambda^2 \text{Scal}^N. \tag{17}
\]

### 3.1. Necessary conditions in a preferred frame

Let \( \{X, Y, Z, U\} \) be a preferred orthonormal frame. In the following we will obtain some necessary conditions for the existence of a shear-free fluid \( (U, p = \frac{\lambda}{\lambda'}, \rho = \lambda') \) coupled with gravity in terms of these basic vector fields \( X, Y \) and \( Z \).

Condition (14) is equivalent to

\[
X(V(\ln \lambda)) = \frac{\lambda^{2(4 - r)}}{4} [\beta(X) + (10 - 3r)Y(\ln \lambda)\Omega(X,Y)] + (3 - r)X(\ln \lambda)V(\ln \lambda),
\]

\[
Y(V(\ln \lambda)) = \frac{\lambda^{2(4 - r)}}{4} [\beta(Y) - (10 - 3r)X(\ln \lambda)\Omega(X,Y)] + (3 - r)Y(\ln \lambda)V(\ln \lambda),
\]

\[
Z(V(\ln \lambda)) = \frac{\lambda^{2(4 - r)}}{4} \beta(Z) + (3 - r)Z(\ln \lambda)V(\ln \lambda), \tag{18}
\]

where \( \beta(T) := \lambda^{-2} [\delta \Omega(T) + (r - 4)\Omega(T, \text{grad} \ln \lambda)] \) is basic whenever \( T \) is a basic vector, according to lemma 1.
Assume $r \neq 2$ and let $A(r) = \frac{(r-3)^2+1}{r-2}$ and $B(r) = 1 + \frac{(2r-5)(r-6)}{6(r-2)}$. Condition (15) combined with the trace constraints gives:

\[ 
\text{Hess}_{\ln}(X, X) = (A(r) - B(r))X(\ln \lambda)^2 - B(r)[Y(\ln \lambda)^2 + Z(\ln \lambda)^2] + \frac{1}{2} \lambda^{-2}U(\ln \lambda)^2 
\]

\[ + \frac{1}{6} \lambda^{r-2} - \frac{3r-2}{24(r-2)} \lambda^{8-2r} \Omega(X, Y)^2 \]

\[ - \frac{1}{r-2} \left( \psi^* \text{Ric}^N(X, X) + \frac{r-6}{12} \text{Scal}^N \right), \]

(19)

the analogous equation for $\text{Hess}_{\ln}(Y, Y)$ and

\[ 
\text{Hess}_{\ln}(Z, Z) = (A(r) - B(r))Z(\ln \lambda)^2 - B(r)[Y(\ln \lambda)^2 + Z(\ln \lambda)^2] + \frac{1}{2} \lambda^{-2}U(\ln \lambda)^2 
\]

\[ + \frac{1}{6} \lambda^{r-2} - \frac{3r-14}{24(r-2)} \lambda^{8-2r} \Omega(X, Y)^2 \]

\[ - \frac{1}{r-2} \left( \psi^* \text{Ric}^N(Z, Z) + \frac{r-6}{12} \text{Scal}^N \right). \]

(20)

Condition (15) on pairs of orthogonal vectors gives:

\[ \text{Hess}_{\ln}(X, Y) - A(r)X(\ln \lambda)Y(\ln \lambda) + \frac{1}{r-2} \psi^* \text{Ric}^N(X, Y) = 0 \]

(21)

and similar equations for $(X, Z)$ and $(Y, Z)$.

The previous equations prescribe second order derivatives of $\ln \lambda$ in terms of its first derivatives. Differentiating them along $V$ (i.e. 'propagating') and using commutation (12) will provide us with compatibility conditions purely in terms of first derivatives.

By taking the derivative along $V$ of the equation (cf (15))

\[ 
\text{Hess}_{\ln}(X, X) - A(r)X(\ln \lambda)^2 + \frac{1}{r-2} \psi^* \text{Ric}^N(X, X) 
\]

\[ = \text{Hess}_{\ln}(Y, Y) - A(r)Y(\ln \lambda)^2 + \frac{1}{r-2} \psi^* \text{Ric}^N(Y, Y), \]

(22)

inserting $X(V(\ln \lambda))$, $Y(V(\ln \lambda))$ from (18) and $X(Y(\ln \lambda))$ from (21), and simplifying the result using again (22), we obtain:

\[ (3 - r)V(\ln \lambda)((5 - r - A(r))(X(\ln \lambda)^2 - Y(\ln \lambda)^2) \]

\[ + \frac{1}{r-2} [\psi^* \text{Ric}^N(Y, Y) - \psi^* \text{Ric}^N(X, X)] \]

\[ \lambda^{8-2r} \]

\[ = - \frac{4}{2(10 - 3r)(13 - 3r - A(r))\Omega(X, Y)X(\ln \lambda)Y(\ln \lambda)} \]

\[ + [(15 - 3r - 2A(r))\beta(X) + (10 - 3r)Y(\Omega(X, Y)]X(\ln \lambda) \]

\[ + [- (15 - 3r - 2A(r))\beta(Y) + (10 - 3r)X(\Omega(X, Y))]Y(\ln \lambda) \]

\[ + (10 - 3r)\lambda^2 g(\nabla_X Y + \nabla_Y X, Z)\Omega(X, Y)Z(\ln \lambda) \]

\[ + X(\beta(X)) - Y(\beta(Y)) - \frac{2(10 - 3r)}{r - 2} \Omega(X, Y)\psi^* \text{Ric}^N(X, Y) \]

\[ + \lambda^2 g([X, Y], Y)\beta(X) + \lambda^2 g([X, Y], X)\beta(Y) \]

\[ + \lambda^2 g([X, Z], X) \beta(X)] \beta(Z). \]

(23)

Analogously, by propagating (i.e. taking the $V$-derivative of) the following equation (cf (15))

\[ 
\text{Hess}_{\ln}(X, X) - A(r)X(\ln \lambda)^2 + \frac{1}{r-2} \psi^* \text{Ric}^N(X, X) + \frac{1}{2(r-2)} \lambda^{8-2r} \Omega(X, Y)^2 
\]

\[ = \text{Hess}_{\ln}(Z, Z) - A(r)Z(\ln \lambda)^2 + \frac{1}{r-2} \psi^* \text{Ric}^N(Z, Z), \]

(24)
we obtain
\[
(3-r)V(\ln \lambda)((5-r-A(r))[X(\ln \lambda)^2-Z(\ln \lambda)^2] \\
+ \frac{1}{r-2}[\psi^*Ric^N(Z, Z) - \psi^*Ric^N(X, X)] + \\
\frac{3r^2 - 20r + 40 \lambda^{5-2r}}{4(r-2)}V(\ln \lambda)\Omega(X, Y)^2
\]
\[
= -\frac{\lambda^{5-2r}}{4}[(10-3r)(13-3r-A(r))\Omega(X, Y)X(\ln \lambda)Y(\ln \lambda) \\
+ [(15-3r-2A(r))\beta(X) + (10-3r)\lambda^2 g([Z, Y], Z)\Omega(X, Y)]X(\ln \lambda) \\
+ (10-3r)[\Omega(X, Y)] - \lambda^2 g([Z, X], Z)\Omega(X, Y)]Y(\ln \lambda) \\
+ [-15-3r-2A(r))\beta(Z) + (10-3r)\lambda^2 g(\nabla_Y Z, \Omega(X, Y))]Z(\ln \lambda) \\
+ X(\beta(X)) - Z(\beta(Z)) - \frac{10-3r}{r-2}\Omega(X, Y)\psi^*Ric^N(X, Y) \\
- \lambda^2 g([Z, X], Z)\beta(X) + \lambda^2 (g([X, Y], X) - g([Z, Y], Z))\beta(Y) \\
+ \lambda^2 g([X, Z], X)\beta(Z)].
\] (25)

Notice that, inside the brackets of both (23) and (25), the polynomial expressions in the derivatives of \(\ln \lambda\) have basic coefficients.

Now we exploit the commutation of the covariant second order horizontal derivatives. First notice that, since \([X, Z]\) is horizontal, (14) gives us \([X, Z](U(\ln \lambda)) = \frac{\partial}{\partial \lambda}(\beta([X, Z]) + (10-3r)\Omega([X, Z], X)X(\ln \lambda) + \Omega([X, Z], Y)Y(\ln \lambda)).\] By means of this relation and of equation (18), we can replace the mixed (horizontal, vertical) second order derivatives of \(\ln \lambda\) in each term of the identity

\[X(\Omega(\ln \lambda)) - Z(\Omega(\ln \lambda)) - [X, Z]\Omega(\ln \lambda) = 0,\]

then derive the first two terms, and substitute \(Z(Y(\ln \lambda))\) by means of (21), to obtain after simplifications (using remark 2)

\[
(3-r)(10-3r)\Omega(X, Y)Y(\ln \lambda)Z(\ln \lambda) \\
+ [(5-r)\beta(Z) - (10-3r)\Omega(\nabla_Y Z, X)]X(\ln \lambda) \\
- (10-3r)\Omega(X, [Z, Y])Y(\ln \lambda) \\
- [-5-r(\beta(X) - (10-3r)\lambda^2 g([Z, Y], Z)\Omega(X, Y)]Z(\ln \lambda) \\
+ d\beta(X, Z) + \frac{10-3r}{r-2}\Omega(X, Y)\psi^*Ric^N(Y, Z) = 0,
\] (26)

whose left-hand term is a polynomial expression in \(X(\ln \lambda), Y(\ln \lambda)\) and \(Z(\ln \lambda)\) with basic coefficients, denoted for simplicity as follows:

\[bY(\ln \lambda)Z(\ln \lambda) + b_1X(\ln \lambda) + b_2Y(\ln \lambda) + b_3Z(\ln \lambda) + b_0.\]

Similarly, from \(Y(Z(V(\ln \lambda))) - Z(Y(V(\ln \lambda))) = [Y, Z](V(\ln \lambda))\) (or simply by permuting \(X\) and \(Y\) in the above relation) we obtain

\[
-\frac{2\Omega(X, Y)X(\ln \lambda)Z(\ln \lambda) + }{r-2} \\
+ (10-3r)\Omega([Z, X], Y)X(\ln \lambda) \\
+ [(5-r)\beta(Z) - (10-3r)\Omega(\nabla_Y Z, Y)]Y(\ln \lambda) \\
- [-5-r(\beta(Y) + (10-3r)\lambda^2 g([Z, X], Z)\Omega(X, Y)]Y(\ln \lambda) \\
+ d\beta(Y, Z) - \frac{10-3r}{r-2}\Omega(X, Y)\psi^*Ric^N(X, Z) = 0,
\] (27)
whose left-hand term has basic coefficients too and is written down as:

\[-bX(\ln \lambda)Z(\ln \lambda) + c_1X(\ln \lambda) + c_2Y(\ln \lambda) + c_3Z(\ln \lambda) + c_0.\]

Finally, replace \(Y(V(\ln \lambda)), X(V(\ln \lambda)), \) and \([X, Y]^{\alpha\beta}(V(\ln \lambda))\) by means of (18) and (14), in the identity

\[X(Y(V(\ln \lambda))) - Y(X(V(\ln \lambda))) - [X, Y]V(\ln \lambda) = 0,\]

then eliminate the resulted second order derivatives of \(\ln \lambda\) by means of (17) and (19), to obtain

\[\Omega(X, Y)C(r)[X(\ln \lambda)^2 + Y(\ln \lambda)^2] + D(r)Z(\ln \lambda)^2 + d_1X(\ln \lambda) + d_2Y(\ln \lambda) + d_3Z(\ln \lambda) + L(\lambda^{-2}, \lambda^{-2r}) + 5(4 - r)\Omega(X, Y)\lambda^{-2\gamma}U(\ln \lambda)^2 = 0,\]

where \(C(r) = \frac{2(r - 1)(r - 1)(r - 3)}{(r - 2)(r - 1)}, D(r) = \frac{2(r - 4)(r - 5)(3r - 10)}{(r - 2)(r - 1)},\)

\[L(\lambda^{-2}, \lambda^{-2r}) = \frac{(r - 4)(r - 5)(3r - 10)}{(r - 2)^2} \frac{\gamma^2}{6(r - 2)},\]

\[d_1 = (5 - r)\beta(Y) - (10 - 3r)\lambda^2U(\ln \lambda)\Omega(X, Y),\]

\[d_2 = -(5 - r)\beta(X) - (10 - 3r)\lambda^2U(\ln \lambda)\Omega(X, Y)\] (notice that \(d_i\)’s are basic functions).

4. Strategy of the proof

Given a perfect fluid \((U, P = \frac{r - 3}{3}\rho, \rho = \lambda)\) (or equivalently a local \(r\)-harmonic morphism) on a four-dimensional spacetime \((M, g)\) satisfying (10), we aim to prove that at a point where \(\lambda \neq 0\) and for \(r \neq 0\) we have either \(U(\ln \lambda) = 0\) (no expansion) or \(\Omega = 0\) (no rotation). Let \(X, Y\) and \(Z\) be basic vectors defining a preferred orthonormal frame at such point (we shall retain this convention throughout the rest of paper). The starting point is the observation\(^2\) that if the conjecture is true, then \(X(\ln \lambda), Y(\ln \lambda)\) and \(Z(\ln \lambda)\) have to be basic (possibly all equal to zero). Moreover, a converse result holds (proposition 5) providing us with an equivalent form of the conjecture which turns out to be more tractable. Indeed, equations (26) and (27) allow us to express \(X(\ln \lambda)\) and \(Y(\ln \lambda)\) as rational functions (with basic coefficients) of \(Z(\ln \lambda)\), provided that \(D = bX^2Z(\ln \lambda)^2 - b(c_1 - b_2)Z(\ln \lambda) + b_1c_2 - b_2c_1\) is not zero, and then to produce a polynomial equation with basic coefficients in one variable, \(Z(\ln \lambda)\), which will be constrained to be basic (together with \(X(\ln \lambda)\) and \(Y(\ln \lambda)\)). So the proof will split up into two major cases (\(D \neq 0\) and \(D = 0\)) to be treated with independent methods. Nevertheless they have in common the following basic tools that will help us to conclude in each case.

**Lemma 2.** If a function \(f\) on \(M\) satisfies \(a_nf^n + \cdots + a_1f + a_0 = 0\), where \(n \in \mathbb{N}\) and \(a_i\)’s are all basic functions, then either \(f\) is basic or \(a_i = 0\) for all \(i = 0, 1, \ldots, n\).

**Proof.** Iterate the derivative along \(V\) of the polynomial equation. At each step we have either \(V(f) = 0\) or an equation of smaller degree is satisfied. If \(f\) is not basic, i.e. \(V(f) \neq 0\), then after \(n\) derivations we obtain \(a_0 = 0\). \(\square\)

**Lemma 3.** Let \(\lambda\) be a positive function on \(M\) satisfying \(\sum_{i=1}^n \eta_i\lambda^p_i = 0\), where \(p_i \in \mathbb{R}\) and \(\eta_i\)’s are basic functions. Fix a ‘reference coefficient’ \(\eta_0 \neq 0\). If \(p_j \neq p_0\) for all \(j \neq i_0\), then \(\lambda\) is basic.

\(^2\) This requires a (straightforward but not so short) proof that we omit here.
Proof. Suppose that \( \lambda \) is not basic, so that \( V(\ln \lambda) \neq 0 \). Divide the equation \( \sum \eta_i \lambda^{\eta_i} = 0 \) by \( \lambda^{\eta_i} \), where \( \eta_i \) is any exponent not equal to \( p_i \), and take the \( V \)-derivative of the resulting equation. Then divide the new equation by \( \lambda^{\eta_i} V(\ln \lambda) \), where \( \eta_i \) is any exponent not equal to \( p_i \) or \( \eta_j \), and take the \( V \)-derivative again. Iterate this procedure until the \( V \)-derivative has been taken a total of \( (n-1) \) times and all powers \( \lambda^{\eta_i} \) of \( \lambda \) except \( \lambda^{\eta_0} \) have been divided out. The resulting equation is \( \eta_0 \prod_{j \neq i} (p_{i_0} - p_j) = 0 \) which contradicts the stipulation that the exponents \( p_j \) \( (j \neq i) \) are different from \( p_{i_0} \).

Let \( R_0 = \{2, 3, 4, \frac{10}{3}\} \). Recall that the conjecture is true for \( r \in R_0 \).

Proposition 5. Let \((U, \rho = \frac{e^{-2}}{\rho} \lambda^r, r = \lambda^r)\) be a shear-free perfect fluid coupled with gravity on \((M, g)\) and let \( \{X, Y, Z\} \) be basic vectors from a preferred orthonormal frame at a point of \( M \).

(i) If \( Z(\ln \lambda) \) is basic and \( Z(\ln \lambda) \neq 0 \), then either the expansion or the rotation of the fluid vanishes.

(ii) If \( X(\ln \lambda) \) and \( Y(\ln \lambda) \) are basic and \( X(\ln \lambda)^2 + Y(\ln \lambda)^2 \neq 0 \), then either the expansion or the rotation of the fluid vanishes.

Proof. Since \( r \in R_0 \) are already settled, we may assume that \( r \notin R_0 \).

(i) By hypothesis \( Z(\ln \lambda) \) is basic, so \( Z(V(\ln \lambda)) = 0 \) due to (12). Equation (18) becomes:

\[
0 = \frac{\lambda^{2(4-r)}}{4} \beta(Z) + (3-r)Z(\ln \lambda)V(\ln \lambda).
\]

Since \( Z(\ln \lambda) \neq 0 \) (by hypothesis), we have

\[
V(\ln \lambda) = -\frac{\lambda^{2(4-r)}}{4} \beta(Z) (3-r)Z(\ln \lambda).
\]

Differentiating this equation along the vector \( X \), implies:

\[
X(V(\ln \lambda)) = \frac{\lambda^{2(4-r)}}{4} ((8 - 2r)X(\ln \lambda) + X(f))
\]

where the function \( f = -\frac{\beta(Z)}{(3-r)Z(\ln \lambda)} \) is basic. Inserting \( X(V(\ln \lambda)) \) from (18) in the above equality gives us an equation with basic coefficients in \( X(\ln \lambda) \) and \( Y(\ln \lambda) \):

\[
(5 - r)fX(\ln \lambda) - (10 - 3r)\Omega(X, Y)Y(\ln \lambda) = \beta(X) - X(f).
\]

(31)

Analogously we obtain

\[
(10 - 3r)\Omega(X, Y)X(\ln \lambda) + (5 - r)fY(\ln \lambda) = \beta(Y) - Y(f).
\]

(32)

Since \( \beta(Z) = -\lambda^2\Omega(X, Y)g([X, Y], Z) \) the discriminant of the linear system formed by (31) and (32) is

\[
\Delta = \Omega(X, Y)^2 \left( (10 - 3r)^2 + (5 - r)^2 \frac{\lambda^4g([X, Y], Z)^2}{(3-r)^2Z(\ln \lambda)^2} \right).
\]

Let us suppose that \( \Omega(X, Y) \neq 0 \). Then \( \Delta \neq 0 \) (since \( r \neq \frac{10}{3} \)) and \( X(\ln \lambda) \) and \( Y(\ln \lambda) \) are basic functions as solutions of a linear system with basic coefficients.

By using (30), the trace constraint (17) becomes

\[
\lambda^{8-2r} \left( \frac{3r - 5}{24} \Omega(X, Y)^2 - \frac{2r - 5}{6} \Omega(X, Y)^2 \right) + \frac{2r - 5}{6} \lambda^{r-2}
\]

\[
= \frac{6}{(2r - 5)(r - 3)} \left(X(\ln \lambda)^2 + Y(\ln \lambda)^2 + Z(\ln \lambda)^2\right) - \frac{r - 3}{12} \text{Scal} = 0,
\]

that is a linear equation in \( \lambda^{8-2r} \) and \( \lambda^{r-2} \) with basic coefficients. Since the first two coefficients cannot cancel simultaneously, by applying lemma 3 we conclude that \( U(\ln \lambda) = 0 \).
(ii) Supposing $X(\ln \lambda) \neq 0$, from (18) we have
\[
V(\ln \lambda) = \frac{\lambda^{2(4-r)}}{4} \left( (10 - 3r)Y(\ln \lambda)\Omega(X, Y) + \frac{1}{3} - \frac{5}{3} \lambda^{r-2} \right).
\]

The function $f = -\frac{\beta(X)(10 - 3r)Y(\ln \lambda)\Omega(X, Y)}{(3-r)X(\ln \lambda)}$ is basic according to our hypothesis. Suppose $f \neq 0$ (otherwise $V(\ln \lambda) = 0$ and the proof ends). Analogously to the previous case (i), by differentiating (33) along the vector $Z$ and inserting $Z(V(\ln \lambda))$ from (18) we obtain
\[
(5 - r)fZ(\ln \lambda) = \beta(Z) - Z(f).
\]

If $r \neq 5$, then $Z(\ln \lambda)$ is basic. Inserting $U(\ln \lambda) = \frac{\lambda^r}{r}f$ in the trace constraint (17), we obtain again an equation of the form $\lambda^r \cdot \text{basic} + \frac{1}{2}\lambda^{r-2} \cdot \text{basic} = 0$; then, like in the case (i), we obtain $U(\ln \lambda) = 0$.

If $r = 5$, let us suppose that $Z(\ln \lambda)$ is not basic and show that this leads to a contradiction. From (21) we deduce that $g(\nabla_X Y, Z) = 0$, which inserted in (25) together with (33) gives us: $-fZ(\ln \lambda)^2 + \frac{3}{4} f\Omega(X, Y)^2 + \beta(Z)Z(\ln \lambda) + \text{basic} = 0$. Derive along $V$ this equation (and reinsert it into the result) to obtain $f^2Z(\ln \lambda)^2 - \beta(Z)fZ(\ln \lambda) + \text{basic} = 0$. According to lemma 2 this implies $f = 0$, contradiction.

\begin{remark}
The above result is a stronger version of the fact that, around a regular point, if $\text{grad}^Y \lambda^{r-2}$ is basic, then an $r$-harmonic morphism of dilation $\lambda$ can be rendered $(2-)$harmonic by a conformal change of codomain metric (so the problem reduces to the settled case $r = 2$).

$\text{grad}^Y \lambda^{r-2}$ basic means that $X(\ln \lambda)$, $Y(\ln \lambda)$ and $Z(\ln \lambda)$ are all basic.
\end{remark}

5. The case $D \neq 0$

Recall that $D = b^2 Z(\ln \lambda)^2 - b(c_1 - b_2)Z(\ln \lambda) + (b_1 c_2 - b_2 c_1)$. In this case the key observation is that $X(\ln \lambda), Y(\ln \lambda)$ are rational functions with basic coefficients of $Z(\ln \lambda) \neq 0$.

This allows, upon propagation of (26) and (27), to obtain a polynomial equation in $Z(\ln \lambda)$ that leads us via lemma 2 to the conclusion that $Z(\ln \lambda)$ is basic and the conjecture is true according to proposition 5. We mention that the proof makes use of the previously known fact [41] that the conjecture holds in the special case of aligned vorticity and acceleration, i.e. $X(\ln \lambda) = Y(\ln \lambda) = 0$ (see [33] for a covariant proof).

Assume $\Omega \neq 0$, $D \neq 0$ and $r \notin \mathbb{R}_0$. From (26) and (27) we obtain
\[
X(\ln \lambda) = \frac{1}{D}(bc_1 Z(\ln \lambda)^2 + (b_2 c_3 - b_3 c_2 + bc_0)Z(\ln \lambda) + b_2 c_0 - b_0 c_3),
\]
\[
Y(\ln \lambda) = \frac{1}{D}(-bb_0 Z(\ln \lambda)^2 + (b_0 c_1 - b_1 c_3 - bb_0)Z(\ln \lambda) + b_0 c_1 - b_1 c_0).
\]

Notice that we can also assume $Z(\ln \lambda) \neq 0$; otherwise from (34) we deduce that $X(\ln \lambda)$, $Y(\ln \lambda)$ are both basic and either proposition 5 applies (if one of them is not zero) or $\text{grad} \ln \lambda \in \mathcal{V}$ (if they are both zero) and therefore $U$ is irrational (if $\lambda \neq \text{constant}$), contradicting the assumption $\Omega \neq 0$, or $U(\ln \lambda) = 0$ (if $\lambda = \text{constant}$).

By differentiating equation (26) along $V$, inserting second order derivatives from (18) and simplifying the result by means of (26) and (27), we obtain
\[
(3 - r) V(\ln \lambda)\{-b_1 X(\ln \lambda) - b_2 Y(\ln \lambda) - b_3 Z(\ln \lambda) - 2b_0\}
\]
\[
= \frac{\lambda^{8-2r}}{4}((10 - 3r)\Omega(X, Y)(c_1 + b_2)X(\ln \lambda)
+ [(10 - 3r)\Omega(X, Y)(c_2 - b_1) - b^2(\lambda^r)]Y(\ln \lambda)
+ [(10 - 3r)\Omega(X, Y)c_3 - b^2(\lambda^r)]Z(\ln \lambda)
+ (10 - 3r)\Omega(X, Y)c_0 - b_3 \beta(X) - b_2 \beta(Y) - b_3 \beta(Z)),
\]
while differentiating (27) along $V$ gives us

$$(3 - r)V(\ln \lambda) \{-c_1 X(\ln \lambda) - c_2 Y(\ln \lambda) - c_3 Z(\ln \lambda) - 2c_0\}
= \frac{\lambda^{8-2r}}{4} \left[\{(10 - 3\beta)\Omega(X, Y)(c_2 - b_1) + b\beta(Z)\}X(\ln \lambda)
- (10 - 3\beta)\Omega(X, Y)(c_1 + b_2)Y(\ln \lambda)
- (10 - 3\beta)\Omega(X, Y)b_3 - b\beta(X)Z(\ln \lambda)
- (10 - 3\beta)\Omega(X, Y)b_0 - c_1\beta(X) - c_2\beta(Y) - c_3\beta(Z)\}. \quad (36)$$

Eliminating $V(\ln \lambda)$ and $\lambda^{8-2r}$ from equations (35) and (36) and inserting $X(\ln \lambda)$ and $Y(\ln \lambda)$ from (34) gives us a sixth degree polynomial equation with basic coefficients in $Z(\ln \lambda): \mathcal{P}(Z(\ln \lambda)) = 0$ (see appendix for the explicit form). According to lemma 2, either $Z(\ln \lambda)$ is basic or the coefficients of $\mathcal{P}$ are all vanishing. In the former case proposition 5 applies and the conjecture is true, while in the latter we shall obtain a contradiction (except for three values of $r$).

Let us suppose that $Z(\ln \lambda)$ is not basic and that the coefficients of $\mathcal{P}$ are all zero, in particular the leading one:

$$b_3^2 + c_2^2 - \frac{r - \frac{4}{2}}{r - 2} (b_3\beta(X) + c_3\beta(Y)) = 0. \quad (37)$$

Lemma 4. If $D \neq 0$, $\Omega \neq 0$ and $Z(\ln \lambda)$ is not basic, then $b_2 + c_1 = 0$ and $b_1 - c_2 = 0$.

Proof. Substitute $X(\ln \lambda)$, $Y(\ln \lambda)$ from (34) and $X(Z(\ln \lambda))$, $Y(Z(\ln \lambda))$ from (21), in equation (22) to obtain a seventh degree polynomial equation with basic coefficients in $Z(\ln \lambda)$ whose leading coefficient $b^6(b_2 + c_1)$. Since $Z(\ln \lambda)$ is not basic, $b_2 + c_1$ must vanish according to lemma 2.

Analogously, substituting $X(\ln \lambda)$, $Y(\ln \lambda)$ from (34) in equation (21), then summing with the corresponding relation with $X$ and $Y$ permuted, we obtain again a seventh degree polynomial equation in $Z(\ln \lambda)$ with basic coefficients, the leading one being $b^6(b_1 - c_2)$. ∎

This lemma allows us to eliminate $b_1$ and $b_2$ in the following computations. According to (34) and (35), $V(\ln \lambda) = \frac{\lambda^{8-2r}}{2}Q$, where $Q = (b^2\tilde{c}_3 Z(\ln \lambda)^3 + \cdots)/((r - 3)b^2b_3Z(\ln \lambda)^3 + \cdots)$ is a rational function in $Z(\ln \lambda)$ with basic coefficients. Plugging this into (25) together with $X(\ln \lambda)$ and $Y(\ln \lambda)$ from (34) gives us:

$$\frac{\lambda^{8-2r}}{4} = \frac{b^6u(r)\tilde{c}_3 Z(\ln \lambda)^9 + \cdots}{b^6v(r)\Omega(X, Y)^2\tilde{c}_3 Z(\ln \lambda)^7 + \cdots} \quad \text{(lower degree terms)},$$

where $\tilde{c}_3 := c_3 - \frac{r - 4}{r - 2} \beta(Y)$, $u(r) = \frac{(r - 3)(r - 4)(2r - 5)}{2(r - 2)}$ and $v(r) = \frac{3r^2 - 20r + 40}{2(r - 2)}$.

Suppose $r \neq 5/2$ to have $u(r) \neq 0$ (notice that $v(r) \neq 0$). Replacing $V(\ln \lambda)$ and $\lambda^{8-2r}$ with their (rational) expression in terms of $Z(\ln \lambda)$ in the identity $V(\frac{\lambda^{8-2r}}{2}) = (8 - 2r)\frac{\lambda^{8-2r}}{2}V(\ln \lambda)$ and using (18) gives a polynomial equation with basic coefficients in $Z(\ln \lambda)$ whose leading term vanishes if and only if $b_3 = 0$ or $\tilde{c}_3 = 0$. A similar argument, starting from (36), imposes $c_3 = 0$ or $b_3 = 0$, where $\tilde{c}_3 := c_3 - \frac{r - 4}{r - 2} \beta(X)$.

Due to (37) we have to consider only the following two sub-cases (a detailed account is provided in the auxiliary files, see appendix).
5.1. Sub-case $b_3 = c_3 = 0$

Returning to the polynomial equation $\mathcal{P} = 0$ mentioned above, we obtain $b_0 \beta(X) + c_0 \beta(Y) = 0$ from the fifth degree coefficient term. Inserting it in the 4th and third degree coefficient terms we obtain respectively

$$2(b_0^2 + c_0^2)(10 - 3r)\Omega(X, Y) + 3c_2(b_0 \beta(Y) - c_0 \beta(X)) = 0,$$

$$c_1[7(b_0^2 + c_0^2)(10 - 3r)\Omega(X, Y) + 10c_2(b_0 \beta(Y) - c_0 \beta(X))] = 0. \quad (38)$$

If $c_1 \neq 0$, the system (38) implies $b_0 = c_0 = 0$; therefore $X(\ln \lambda) = Y(\ln \lambda) = 0$, that is we are in the known case of aligned vorticity and acceleration where the conjecture is true [41], so $V(\ln \lambda) = 0$ that contradicts our assumption that $Z(\ln \lambda)$ is not basic.

If $c_1 = 0$, the coefficient of the second degree term gives us

$$c_2^2 \left[ \frac{15r - 32}{5(r - 2)}(b_0^2 + c_0^2)(10 - 3r)\Omega(X, Y) + 4c_2(b_0 \beta(Y) - c_0 \beta(X)) \right] = 0,$$

where we have used $\beta(Z) = -\frac{2}{2r - 1}c_2$, consequence of the fact established in lemma 4. So either $c_2 = 0$ (and, from (38) we obtain again $b_0 = c_0 = 0$, so $X(\ln \lambda) = Y(\ln \lambda) = 0$) or $c_2 \neq 0$ and the system formed by the above equation and the first equation (38) implies again $b_0 = c_0 = 0$ (so $X(\ln \lambda) = Y(\ln \lambda) = 0$) except for $r = \frac{16}{5}$.

5.2. Sub-case $\tilde{b}_3 = \tilde{c}_3 = 0$

In this case the system formed by the cancellation of coefficients of $\mathcal{P}$ can be solved in $c_1$ and $c_2$ and, after a case by case analysis, we obtain (except for $r = \frac{12}{5}$)

$$X(\ln \lambda) = \frac{\beta(Y)}{(10 - 3r)\Omega(X, Y)}, \quad Y(\ln \lambda) = -\frac{\beta(X)}{(10 - 3r)\Omega(X, Y)}.$$

Plugging this into (18) we see that either $X(\ln \lambda)$, $Y(\ln \lambda)$ are both zero (the case of aligned vorticity and acceleration) or $V(\ln \lambda) = 0$, which contradicts the assumption that $Z(\ln \lambda)$ is not basic.

In conclusion, if $D \neq 0$, the conjecture is true for $r \notin \left\{ \frac{12}{5}, \frac{5}{2}, \frac{16}{5} \right\}$.

6. The case $D = 0$

In this case we deal essentially with the situation when the vorticity is orthogonal to the acceleration, $Z(\ln \lambda) = 0$. Since the compatibility conditions (26) and (27) become trivial, the remaining one, (28), comes into play. By successive propagation of this condition we can eliminate all terms in $X(\ln \lambda)$ and $Y(\ln \lambda)$ and obtain a quadratic equation in $V(V(\ln \lambda))$ and $V(\ln \lambda)^2$ with coefficients involving only $\lambda$ and basic quantities. This quadratic equation conserves its form upon iterated $V$-derivations, so the terms in $V(\ln \lambda)$ can be eliminated and finally it results an equation for which lemma 3 applies to conclude that the fluid must be expansion-free (once the rotation is non-zero).

Assume that $D = 0$ and $r \notin \mathcal{R}_0$. According to lemma 2, either $b = 0$ (that is $\Omega = 0$ and the proof ends) or $Z(\ln \lambda)$ is basic. If $Z(\ln \lambda)$ is basic and $Z(\ln \lambda) \neq 0$, proposition 5 applies and we obtain the conclusion. The remaining case is $Z(\ln \lambda) = 0$.

For the rest of this section, let us suppose moreover that $Z(\ln \lambda) = 0$ and $\Omega(X, Y) \neq 0$.

From (18) we see that $\beta(Z) = 0$ that is $g([X, Y], Z) = 0$ (recall that $\beta(Z) = -\lambda^2 g([X, Y], Z) \Omega(X, Y)$). Equation (26) reduces to

$$\lambda^2 g(\nabla_X Y, Z)X(\ln \lambda) + \lambda^2 g([Z, Y], Y)Y(\ln \lambda) = \frac{b_0}{(10 - 3r)\Omega(X, Y)}, \quad (39)$$
while equation (27) reduces to
\[
\lambda^2 g([Z, X], X)X(\ln \lambda) + \lambda^2 g(\nabla_X Y, Z)Y(\ln \lambda) = -\frac{c_0}{(10 - 3r)\Omega(X, Y)}, \tag{40}
\]

Notice that the system formed by (39) and (40) is dependent (\(D = 0\)).

Therefore two situations have to be considered: in (39) at least one of the (basic) coefficients of \(X(\ln \lambda)\) and \(Y(\ln \lambda)\) is not vanishing, or both are zero.

### 6.1. Sub-case \(g(\nabla_X Y, Z)^2 + g([Z, Y], Y)^2 \neq 0\)

Suppose \(g(\nabla_X Y, Z) \neq 0\) (the other choice is similar). According to (39), \(X(\ln \lambda)\) is a linear function of \(Y(\ln \lambda)\) with basic coefficients: \(X(\ln \lambda) = aY(\ln \lambda) + b\). Taking the \(V\)-derivative of this relation and employing (18) we obtain:
\[
V(\ln \lambda) = \frac{\lambda^{8-2r}}{4(r-3)b}[\beta(aX - Y) + (10 - 3r)\Omega(X, Y)[(a^2 + 1)Y(\ln \lambda) + ab]],
\]
where we supposed \(b \neq 0\) (otherwise the conclusion is immediate). Plugging this into equation (23) we obtain a third degree polynomial in \(Y(\ln \lambda)\) with basic coefficients. According to lemma 2, either \(Y(\ln \lambda)\) is basic or the coefficients are all vanishing. If \(Y(\ln \lambda)\) is basic then \(X(\ln \lambda)\) is basic and we can apply proposition 5 to conclude. If this is not the case, then the leading coefficient \((a^2 - 1)(10 - 3r)(3 - r)(5 - r - A(r))\Omega(X, Y)\) must vanish, so \(r = 5/2\) (a case not covered by our proof) or \(a = \pm 1\). But in the last case, the coefficient second degree term is \(8(r-4)/(10 - 3r)(3 - r)b\Omega(X, Y)\) and it cannot be zero, contradiction.

### 6.2. Sub-case \(g(\nabla_X Y, Z) = 0\) and \(g([Z, Y], Y) = 0\)

Inserting this in (40) we obtain that either \(X(\ln \lambda)\) is basic, or \(g([Z, X], X) = 0\).

Firstly, let us suppose that \(g([Z, X], X) \neq 0\) (so \(X(\ln \lambda)\) is basic). From (18) we have:
\[
\lambda^{2(4-r)}/4[\beta(X) + (10 - 3r)Y(\ln \lambda)\Omega(X, Y)] + (3 - r)X(\ln \lambda)Y(\ln \lambda) = 0. \tag{41}
\]

If \(X(\ln \lambda) = 0\), then \(Y(\ln \lambda) = -\beta(X)(10 - 3r - 4A(\ln \lambda))\Omega(X, Y)\) is basic, so either proposition 5 applies (if \(Y(\ln \lambda) \neq 0\)), or grad \(\ln \lambda \in V\) (if \(Y(\ln \lambda) = 0\)). In the latter case \(U\) is hypersurface orthogonal (so irrotational, a contradiction) if \(\lambda \neq \text{constant}\), or \(U(\ln \lambda) = 0\) if \(\lambda\) is constant.

If \(X(\ln \lambda) \neq 0\), then eliminating \(V(\ln \lambda)\) and \(\lambda^{2(4-r)}/4\) from (23) and (41) we obtain a third degree polynomial equation in \(Y(\ln \lambda)\) with basic coefficients whose leading coefficient is \((10 - 3r)(3 - r)(5 - r - A(r))\Omega(X, Y)\). Again lemma 2 imposes that either \(Y(\ln \lambda)\) is basic (so proposition 5 applies) or the leading coefficient is zero, i.e. \(r = 5/2\) (excepted case).

Secondly, let us suppose that \(g([Z, X], X) = 0\). This remaining sub-case can be resumed by the following assumptions:

- \(Z(\ln \lambda) = 0\);
- \(g(\nabla_X Y, Z) = g(\nabla_Y X, Z) = 0\);
- \(g([Z, X], X) = g([Z, Y], Y) = 0 \Leftrightarrow Z(\Omega(X, Y)) = 0, \text{div}Z = 0\).

In this last sub-case \(Z\) itself is shear-free and divergence-free\(^3\). Moreover its three-dimensional orthogonal distribution is integrable (so \(Z\) is aligned with a gradient vector) and minimal. It also follows from the above conditions that \(\text{Hess}_{\ln \lambda}(Z, X) = \text{Hess}_{\ln \lambda}(Z, Y) = 0\), \(\varphi^*\text{Ric}^N(Z, X) = \varphi^*\text{Ric}^N(Z, Y) = 0\) and \(d\beta(Z, X) = d\beta(Z, Y) = 0\).

\(^3\) We are close to the situation in which there exists a Killing vector parallel to the vorticity, a notorious case for difficulty despite the drastic simplifications it brings about, see [11].
Since, in this case, $\text{Hess}_{\Omega_1}(Z, Z) = -X(\ln \lambda)^2 - Y(\ln \lambda)^2 + \delta_1 X(\ln \lambda) + \delta_2 Y(\ln \lambda) + \lambda^{-2} U(\ln \lambda)^2$, equation (20) becomes:

$$(1 - B(r))X(\ln \lambda)^2 + Y(\ln \lambda)^2 - \delta_1 X(\ln \lambda) - \delta_2 Y(\ln \lambda)$$

$$- \frac{1}{2} \lambda^{-2} U(\ln \lambda)^2 + \frac{1}{6} \lambda^{-2} - \frac{3r - 14}{24(r - 2)} \lambda^{8 - 2r} \Omega(X, Y)^2$$

$$- \frac{1}{r - 2} \left( \varphi^* \text{Ric}^N(Z, Z) + \frac{r - 6}{12} \text{Scal}^N \right) = 0,$$

where we introduced the notations $\delta_1 = \lambda^2 g([Z, X], Z)$ and $\delta_2 = \lambda^2 g([Z, Y], Z)$.

Recall that $\beta(X) = Y(\Omega(X, Y)) + \delta_2 \Omega(X, Y)$ and $\beta(Y) = -X(\Omega(X, Y)) - \delta_1 \Omega(X, Y)$ (see the proof of lemma 1). Divide (28) by $\Omega(X, Y)$, multiply (42) with $(10 - 3r)$ and take their sum to obtain:

$$(C(r) + (10 - 3r)(1 - B(r))) \left[ X(\ln \lambda)^2 + Y(\ln \lambda)^2 \right]$$

$$+ \frac{15 - 4r}{\Omega(X, Y)} [\beta(Y)X(\ln \lambda) - \beta(X)Y(\ln \lambda)]$$

$$+ \frac{30 - 7r}{2} \lambda^{-2} U(\ln \lambda)^2 + \frac{11r - 30}{6} \lambda^{-2} - \frac{60 + 44r - 21r^2}{24(r - 2)} \lambda^{8 - 2r} \Omega(X, Y)^2$$

$$+ \frac{36 + 12r - 7r^2}{12(r - 2)} \text{Scal}^N - \frac{2(10 - 3r)}{r - 2} \varphi^* \text{Ric}^N(Z, Z) + \frac{d\beta^N(X, Y)}{\Omega(X, Y)} = 0. \quad (43)$$

A detailed account of the computations in the rest of this section is provided in the auxiliary files; see also appendix.

From (18) we deduce the following propagation equations:

$$V X(\ln \lambda)^2 + Y(\ln \lambda)^2 = \frac{\lambda^{8 - 2r}}{4} [\beta(X)X(\ln \lambda) + \beta(Y)Y(\ln \lambda)]$$

$$+ (3 - r)V X(\ln \lambda) \left( X(\ln \lambda)^2 + Y(\ln \lambda)^2 \right),$$

$$V (\beta(Y)X(\ln \lambda) - \beta(X)Y(\ln \lambda)) = (10 - 3r) \Omega(X, Y) \frac{\lambda^{8 - 2r}}{4} [\beta(X)X(\ln \lambda) + \beta(Y)Y(\ln \lambda)]$$

$$+ (3 - r)V X(\ln \lambda) [\beta(Y)X(\ln \lambda) - \beta(X)Y(\ln \lambda)],$$

$$V (\beta(X)X(\ln \lambda) + \beta(Y)Y(\ln \lambda))$$

$$= \frac{\lambda^{8 - 2r}}{4} [\beta(X)^2 + \beta(Y)^2 - (10 - 3r) \Omega(X, Y)[\beta(Y)X(\ln \lambda) - \beta(X)Y(\ln \lambda)]]$$

$$+ (3 - r)V X(\ln \lambda) [\beta(X)X(\ln \lambda) + \beta(Y)Y(\ln \lambda)].$$

The ‘cyclic’ property of the $V$-derivatives of the linear term $\beta(Y)X(\ln \lambda) - \beta(X)Y(\ln \lambda)$ allows us to eliminate it by taking the derivative of equation (43) twice along $V$.

The first $V$-derivative of equation (43) along $V$ (simplified by reinserting $\beta(Y)X(\ln \lambda) - \beta(X)Y(\ln \lambda)$ from (43) in the result and by using (17)) reads:

$$\frac{2(r - 3)(r - 4)(11r - 30)}{3(r - 2)} \lambda^{8 - 2r} \left[ \beta(X)X(\ln \lambda) + \beta(Y)Y(\ln \lambda) \right]$$

$$+ V X(\ln \lambda) \frac{2(r - 3)^2(r - 4)(7r - 10)}{3(r - 2)} \left[ X(\ln \lambda)^2 + Y(\ln \lambda)^2 \right]$$

$$+ \frac{2(2r - 5)}{3} \lambda^{-2} + \frac{r(3r - 14)(7r - 20)}{12(r - 2)} \lambda^{8 - 2r} \Omega(X, Y)^2$$

$$- (r - 3) \frac{8(r - 3)}{3(r - 2)} \text{Scal}^N + \frac{2(10 - 3r)}{r - 2} \varphi^* \text{Ric}^N(Z, Z) - \frac{d\beta^N(X, Y)}{\Omega(X, Y)} = 0.$$
Now, taking the derivative along $V$ of equation (44), then replacing $\beta(X)X(\ln \lambda) + \beta(Y)Y(\ln \lambda)$ from (44), $\beta(Y)X(\ln \lambda) - \beta(X)Y(\ln \lambda)$ from (43) and also $X(\ln \lambda)^2 + Y(\ln \lambda)^2$ from the trace constraint (17), gives us an equation of the following form:

$$
\alpha(r)V(V(\ln \lambda)) + \beta(r)V(V(\ln \lambda))V(\ln \lambda)^2 + \gamma(r)\beta(V(\ln \lambda))^2 + V(\ln \lambda)^2 P_1(\lambda^{8-2r}, \lambda^{r-2}) + P_3(\lambda^{8-2r}, \lambda^{r-2}) = 0,
$$

where $P_i$ are polynomial expressions with basic coefficients (see appendix for the explicit form) and we supposed $r \notin \{\frac{5}{7}, \frac{30}{17}, \frac{15}{7}\}$ to have well defined expressions in $r$.

From (17) we obtain the third order $V$-derivative of $\ln \lambda$ in the form:

$$
V(V(V(\ln \lambda))) = V(\ln \lambda)(q_1(r)V(V(\ln \lambda)) + q_2(r)\beta(V(\ln \lambda))^2 + P_0(\lambda^{8-2r}, \lambda^{r-2})),
$$

where we have used equation (44) to eliminate $\beta(X)X(\ln \lambda) + \beta(Y)Y(\ln \lambda)$ and again (17) to eliminate $X(\ln \lambda)^2 + Y(\ln \lambda)^2$ (see appendix for the explicit form). This constraint on $V(V(V(\ln \lambda)))$ implies that equation (45) conserves its general form when it is derived along $V$ (after simplification of $V(\ln \lambda)$, if not zero). More precisely, the propagation of (45), after inserting (46) and dividing by $V(\ln \lambda)$, is

$$
\alpha^{(1)}(r)V(V(\ln \lambda)) + \beta^{(1)}(r)V(V(\ln \lambda))V(\ln \lambda)^2 + \gamma^{(1)}(r)\beta(V(\ln \lambda))^2 + V(\ln \lambda)^2 P_1^{(1)}(\lambda^{8-2r}, \lambda^{r-2}) + P_3^{(1)}(\lambda^{8-2r}, \lambda^{r-2}) = 0,
$$

with

$$
\alpha^{(1)}(r) = \alpha(r)q_1(r) + 2\beta(r), \quad P_1^{(1)} = P_1 + q_2(r)P_2 - q_1(r)P_1 + \beta(r)P_0, \\
\beta^{(1)}(r) = 2\alpha(r)q_2(r) + 4\gamma(r), \quad P_2^{(1)} = P_2 + 2P_1 + 2\alpha(r)P_0, \\
\gamma^{(1)}(r) = \beta(r)q_2(r) - \gamma(r)q_1(r), \quad P_3^{(1)} = P_3 - q_1(r)P_3 + P_0P_2,
$$

where $P_i = \lambda^{6r}$.

Iterating the $V$-derivative of (45) shows that $\alpha^{(k)}, \beta^{(k)}, \gamma^{(k)}$ form geometric progressions with the same common ratio (or $\alpha^{(1)}, \beta^{(1)}, \gamma^{(1)}$ is an eigenvector of the recurrence matrix). Therefore, from the first four $V$-derivatives of (45) we obtain an overdetermined linear system in $V(\ln \lambda)^2$ and $V(V(\ln \lambda))$. Its compatibility condition (zero determinant) leads us to a fourth degree polynomial equation $P(\lambda^{8-2r}, \lambda^{r-2}) = 0$ having basic coefficients:

$$
\eta_1(r)\Omega(X, Y)^2\lambda^{8-2r+3(r-2)} + \eta_2(r)\Omega(X, Y)^{6\lambda}\lambda^{4(8-2r)} + \ldots = 0,
$$

where $\eta_1$ and $\eta_2$ are rational functions with common zeros $r = 0$, $r = \frac{10}{7}$ and $r = \frac{20}{7}$ (these are common zeros for all fourth degree coefficient terms).

Recall that $r \neq 0$ by the conjecture hypothesis and that $r \neq \frac{10}{7}$ as we have already assumed. Suppose moreover $r \neq \frac{20}{7}$. If $\eta_1(r) = 0$, we employ lemma 3 with reference coefficient $\eta_1$. Since the corresponding exponent of $\lambda$ is different from the others exponents in $P$, we conclude that $V(\ln \lambda) = 0$. If $\eta_1(r) \neq 0$ we apply lemma 3 with reference coefficient $\eta_1$ to deduce that either $V(\ln \lambda) = 0$ or $r \in \{\frac{15}{7}, \frac{20}{7}, \frac{15}{7}\}$. But in the latter case we check that lemma 3 with reference coefficient $\eta_2$ leads us to $V(\ln \lambda) = 0$.

In conclusion, if $D = 0$, the conjecture is true for $r \notin \{\frac{5}{7}, \frac{30}{17}, \frac{20}{7}, \frac{15}{7}\}$.

7. Conclusion

We have proved the following
Theorem 1. Consider a shear-free perfect fluid solution of Einstein’s field equations where the fluid pressure satisfies a barotropic equation of state of the form \( p = w \rho \) with \( w \in \mathbb{R} \setminus \left\{ -\frac{1}{5}, -\frac{1}{6}, -\frac{1}{11}, -\frac{1}{21}, \frac{1}{15}, \frac{1}{4} \right\} \).

Then the fluid is either non-rotating or non-expanding.

The conclusion of the theorem is in fact sharper. On one hand, in the expanding and non-rotating case, the acceleration vanishes (and \( \text{grad} H \ln \lambda = 0 \)) and the spacetime must be in this case a warped product of an interval with a constant curvature 3-manifold, that is a FRW model. This was proved in [10] by analysing the complete list of spacetimes that admit a shear-free non-rotating fluid, classified in terms of the Weyl curvature symmetries. Another, direct proof can be done by starting with compatibility equations as (23), (25) and the propagation of (21), and showing that \( \text{grad} H \ln \lambda = 0 \) (this requires a separate analysis for \( r \in \{ 3, 4, \frac{5}{2} \} \)).

On the other hand, in the non-expanding and rotating case, the spacetime must be stationary since the fundamental vector \( V \) is Killing.

From the perspective of associated submersions, we obtain

Corollary 3. An r-harmonic morphisms on spacetimes satisfying (10) is either of warped product type or of Killing type (with the six exceptions for \( r \)).

Concerning the exceptional values of \( w \), we notice that the two positive ones are in the physical regime defined by a speed of sound between 0 and 1, and that all excepted fluids are satisfying the strong energy condition \( \rho + 3p > 0 \). We expect that the proof can be adapted for the remaining values of \( w \) like for the already known cases \( w \in \{ 0, \pm \frac{1}{3}, \frac{1}{9} \} \) that also appeared to be exceptional in our proof.

The affine equations of state (\( dp/d\rho = \text{cst.} \)) need also a separate analysis that should be very similar in view of remark 3 (in particular it can be seen that introducing the cosmological constant does not change the statement of proposition 5). To what extent the present proof can also be adapted for nonlinear equations of state is the next step to consider.

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Appendix

A.1. Basic tensorial quantities

For the convenience of the reader we include here some useful properties of basic quantities. Let \( \varphi : M \to N \) be a submersion. Recall that the vectors tangent to \( \ker d\varphi \) are called vertical vectors, and that a vector \( X \) on \( M \) is projectable if there exists a vector field \( \overline{X} \) on \( N \) such that \( d\varphi(X_x) = \overline{X}_{\varphi(x)} \) for any \( x \in M \).

The following are well-known properties of basic vectors and functions.

Proposition 6.

(i) A linear combination of basic vector fields, with basic coefficient functions, is basic.
(ii) The horizontal part of the commutator of two basic vector fields is basic.
(iii) If \( X \) is a basic vector field and \( f \) a basic function, then \( X(f) \) is a basic function.
(iv) If $\phi : M \rightarrow N$ is a submersion (with connected fibres), then the vector field $X$ on $M$ is basic if and only if it is (horizontal and) projectable.

If $T$ is a (covariant) tensor (of degree 2) on $N$, then $\varphi^*T$ is basic, that is $W(\varphi^*T(X, Y)) = 0$ for any $W$ vertical and $X, Y$ basic vector fields on $M$. Indeed, by (iv) we have: $W(\varphi^*T(X, Y)) = W(T(X, Y) \circ \varphi) = d\varphi(W(T(X, Y))) = 0$. In particular, when $\varphi^*h = \lambda^2 \beta^3$, the function $\lambda^3 g(X, Y)$ is basic for any basic vector fields $X$ and $Y$.

Along the paper we extensively used Mathematica [42] to simplify the coefficients and to solve polynomial equations. Many of the formulae involved in the final argument in both principal cases are very long and were not included in the main text. We reproduce here some of them. Full version is available in auxiliary files.4

A.2. Details for the case $D \neq 0$

When $b_1 = c_2$ and $b_2 = -c_1$ the polynomial we employed is $\mathcal{P}(Z(\ln \lambda)) = \sum_{i=0}^{b_1} \text{Coeff}_i Z(\ln \lambda)^{i}$, where

- \text{Coeff}_0 = -b_1^2 \left[ b_1 b_2 \beta(X) + c_1 \beta(Y) \right] - (b_2^2 + c_1^2) \Omega(X, Y) (10 - 3 r),
- \text{Coeff}_1 = -b_1^2 \left[ 2 b_1^2 (b_2 b_2 \beta(X) + c_1 b_2 \beta(Y)) - 2 b_1 (2 b_2 c_1 \beta(X) + (b_2 c_2 + c_1) \beta(Y)) + 3 \left( c_1 (b_1 c_1 + c_2) - b_1 b_2 c_1 + c_2 \right) \right] \Omega(X, Y),
- \text{Coeff}_2 = -b_1^2 \left[ - (10 - 3 r) (b_2^2 + c_1^2) \right] \Omega(X, Y) + b_1^2 b_2 \beta(Z) c_2 b_1 (2 b_2 \beta(X) c_1 + 3 \beta(Y) c_2 + \beta(Z) c_3) - 2 b_1 \left( 2 b_2 \beta(Z) c_1 c_2 - 2 b_3 c_1 (3 \beta(Y) c_1 + 3 \beta(Y) c_2 + 50 b_2 \beta(Y) c_1 (10 - 3 r)) + c_1 (-6 b_1 b_2 c_1^2 + 2 b_1 b_2 c_1 c_2 - 7 b_1 b_2 c_1 \Omega(X, Y) \right) \right] \Omega(X, Y),
- \text{Coeff}_3 = b_1 \left( - c_1 c_2^2 + c_2 \beta(Z) c_2 (10 - 3 r) - 2 b_1 b_2^2 \beta(Z) c_1 c_2 b_3 (2 b_2 \beta(X) c_1 + 3 \beta(Y) c_2 + 3 b_2 c_1 \beta(Y) c_2 + 2 b_2 \beta(Z) c_1 c_2 - 7 b_2 \beta(Z) c_1 \Omega(X, Y)) \right) \Omega(X, Y).

A.3. Details for the case $D = 0$

In equation (45),

$$
\alpha(r) = 4 \frac{(3 - r)(r - 4)(7r - 10)}{(r - 2)(2r - 5)}, \quad \beta(r) = \alpha(r) \frac{27r^2 - 257r + 510}{2(11r - 30)},
$$

$$
\gamma(r) = \alpha(r) \frac{(11 - 3 r)(3 r^2 + 23 r - 90)}{2(11r - 30)}.
$$

4 See http://umbuc.ro/prof/slobodeanu_r_a/resurse.php
and

\[ P_1(\lambda^{8-2r}, \lambda^{-2}) = k_1 \lambda^{8-2r} + k_2 \lambda^{6-r} + k_3 \lambda^{16-4r}, \]
\[ P_2(\lambda^{8-2r}, \lambda^{-r-2}) = k_1 \lambda^{8-2r} + k_2 \lambda^{6-r} + k_3 \lambda^{16-4r}, \]
\[ P_3(\lambda^{8-2r}, \lambda^{-2}) = \frac{(r - 3)(r - 4)(11r - 30)}{24(r - 2)} \Omega(X, Y)^2 \lambda^{16-4r} (m_1 \lambda^{8-2r} + m_2 \lambda^{6-r} + m_3 \lambda^{16-4r}), \]

where

\[ k_1 = \frac{(3-r)^2((2000-5940e+1760e-257r^2)+21r^4)}{3(r-2)(r-5)(11r-30)} \text{Scal}^N - \frac{(r-3)(r-5)(19r-54)}{11r-30} \Lambda_Z, \]
\[ k_2 = -\frac{21,600+39,849e-26,944r^4+8,588r^6-1312r^2+78k}{3(r-2)(r-5)(11r-30)}, \]
\[ k_3 = \frac{12,960000-49,248,000e+70,419,600r^2-53,293,200r^4+24,111,698r^6-680,6516r^8+1184,369r^{10}-117,200r^{12}+5082r^{14}}{24(r-2)(r-5)(11r-30)}, \]
\[ \ell_1 = \frac{(3-r)^2(r-3-6)}{2(r-2)} \text{Scal}^N + (3-r)\Lambda_Z, \]
\[ \ell_2 = \frac{21(120)+68r+9r^2}{4(r-2)}, \]
\[ \ell_3 = \frac{14,400-422,400e+43,969r^2-224,068,000r^4+40574r^6-8353r^8+402r^{10}}{4(r-2)(r-5)(11r-30)}, \]
\[ m_1 = -\frac{3(r-3-6)^2}{2(2r-5)(11r-30)} \text{Scal}^N + \frac{10-r}{4r-15} \Lambda_Z + \frac{\beta(x)^2+y)^2}{\Omega(X, Y)^2}, \]
\[ m_2 = -\frac{117200}{(r-2)(r-5)(11r-30)}, \]
\[ m_3 = -\frac{2}{4(r-3)(r-2)(11r-30)} \Omega(X, Y)^2, \]

where \( \Lambda_Z := 2 \frac{10-r}{r-2} \phi^R \text{Ric}^N(Z, Z) - \frac{\partial (\mathbf{F}^N(Y, Y))}{\partial \mathbf{F}^N(Y, Y)}. \)

In equation \((46)\),
\[ q_1(r) = -\frac{91r^2 - 547r + 810}{11r - 30}, \quad q_2(r) = -\frac{(3r - 11)(29r^2 - 168r + 240)}{11r - 30}, \]

and
\[ P_0(\lambda^{8-2r}, \lambda^{-2}) = \left( \frac{(3-r)^2(r-3)(10-r)(r-5)}{(11r-30)(11r-30)} \text{Scal}^N + \frac{(3-r)(r-2)(2r-5)}{2r-4(11r-30)} \Lambda_Z \right) \lambda^{8-2r} \]
\[ - \frac{(2r-5)(10-r)(17r^2 - 92r + 120)}{6r-4(11r-30)} \lambda^{6-r} \]
\[ + \frac{r(10-r)(7r-20)}{8(r-4)(11r-30)} \lambda^{16-4r} \Omega(X, Y)^2 \]

or, in short notation, \( P_0(\lambda^{8-2r}, \lambda^{-2}) = n_1 \lambda^{8-2r} + n_2 \lambda^{6-r} + n_3 \lambda^{16-4r} \).

At the level of coefficients, the recurrence relations involved in the iterated V-derivatives of equation \((47)\) read:

\[ a^{(i)} = -K(r)\alpha^{(i-1)}, \beta^{(i)} = -K(r)\beta^{(i-1)}, \gamma^{(i)} = -K(r)\gamma^{(i-1)}, j \geq 2 \]

where \( K(r) := \frac{25(r-2)(r-1)}{11r-30} \) and, for \( j \geq 1, \)

\[ k_1^{(j)} = (6 - r - q_1(r))k^{(j-1)} + q_2(r)\ell_2^{(j-1)} + \beta^{(j-1)}(r)n_2, \]
\[ k_2^{(j)} = (16 - 4r - q_1(r))k^{(j-1)} + q_2(r)\ell_2^{(j-1)} + \beta^{(j-1)}(r)m_3, \]
\[ \ell_1^{(j)} = (6 - r)\ell_2^{(j-1)} + 2k^{(j-1)} + 2\alpha^{(j-1)}(r)n_2, \]
\[ \ell_2^{(j)} = (6 - r)\ell_2^{(j-1)} + 2k^{(j-1)} + 2\alpha^{(j-1)}(r)m_3, \]

where \( k^{(0)} = k_0, \beta^{(0)} = \gamma^{(0)} = 0, \alpha^{(0)} = \alpha \) and \( \beta^{(0)} = \beta \) (the analogous recurrence relations for \( k_1^{(j)} \) and \( \ell_1^{(j)} \) have not been used).

For \( P_3^{(i)} \), the recurrence relation gets slightly more complicated since

\[ P_3^{(i)}(\lambda^{8-2r}, \lambda^{-2}) = \lambda^{16-4r} m_1^{(i)} \lambda^{8-2r} + m_2^{(i)} \lambda^{6-r} + m_3^{(i)} \lambda^{16-4r} \]
\[ + m_4^{(i)} \lambda^{2r} + m_5^{(i)} \lambda^{r-2} + m_6^{(i)} \]

where we focus on the following coefficients:

\[ m_1^{(i)}, m_2^{(i)}, m_3^{(i)}, m_4^{(i)}, m_5^{(i)}, m_6^{(i)} \]

21
Taking linear combinations of the $V$-derivatives of equation (47), we formed the overdetermined linear system in $V(\ln \lambda)^2$ and $V(V(\ln \lambda))$: $(P_s^{(j)} + K(r)P_s^{(j-1)})V(\ln \lambda)^2 + (P_s^{(j)} + K(r)P_s^{(j-1)})V(V(\ln \lambda)) + P_s^{(j)} + K(r)P_s^{(j-1)} = 0$, where $j = 2, 3, 4$. The determinant of its coefficients matrix (that must vanish) is, after dividing by $\lambda^{(8-2r)}$, a polynomial $\mathcal{P}(\lambda^{5-2r}, \lambda^{r-2})$ with basic coefficients. For instance the coefficient of $\lambda^{(8-2r)}$ is

$$\begin{vmatrix}
\ell_4^{(2)} + K(r)\ell_4^{(1)} & \ell_3^{(2)} + K(r)\ell_3^{(1)} & m_3^{(2)} + K(r)m_3^{(1)} \\
\ell_4^{(3)} + K(r)\ell_4^{(2)} & \ell_3^{(3)} + K(r)\ell_3^{(2)} & m_3^{(3)} + K(r)m_3^{(2)} \\
\ell_4^{(4)} + K(r)\ell_4^{(3)} & \ell_3^{(4)} + K(r)\ell_3^{(3)} & m_3^{(4)} + K(r)m_3^{(3)}
\end{vmatrix} \propto \Omega(X,Y)^8 \prod_{k=1}^{s} \frac{\Omega_k^{(j)}}{\Omega_{k-1}^{(j-1)}} \ln \lambda.$$
[19] Goldberg J N and Sachs R K 1962 A theorem on Petrov types Acta. Phys. Polon. 22 (Suppl) 13–23
[20] Gudmundsson S 1992 The geometry of harmonic morphisms PhD Thesis University of Leeds
[21] Lang J M 1993 Contributions to the study of general relativistic shear-free perfect fluids PhD Thesis University of Waterloo, Canada
[22] Lang J M and Collins C B 1988 Observationally homogeneous shear-free perfect fluids Gen. Rel. Grav. 20 683–710
[23] Loubeau E 2000 On p-harmonic morphisms Differ. Geom. Appl. 12 219–29
[24] Mo X 2003 The geometry of conformal foliations and p-harmonic morphisms Math. Proc. Camb. Phil. Soc. 135 321–34
[25] Misner C W, Thorne K S and Wheeler J A 1973 Gravitation (San Francisco: Freeman)
[26] Narlikar J V 1963 Newtonian universes with shear and rotation Mon. Not. R. Astron. Soc. 126 203–8
[27] Nzioki A M, Goswami R, Dunsby P K S and Ellis G F R 2011 Shear-free perturbations of Friedmann–Lemaître–Robertson–Walker universes Phys. Rev. D 84 124028
[28] O’Neill B 1983 Semi-Riemannian Geometry with Applications to Relativity (New York: Academic)
[29] Pantilie R 2002 Harmonic morphisms with one-dimensional fibres on four-dimensional Einstein manifolds Commun. Anal. Geom. 10 779–814
[30] Pantilie R and Wood J C 2002 Harmonic morphisms with one-dimensional fibres on Einstein manifolds Trans. Am. Math. Soc. 354 4229–43
[31] Pantilie R 2008 Harmonic morphisms with one-dimensional fibres on conformally-flat Riemannian manifolds Math. Proc. Camb. Phil. Soc. 145 141–51
[32] Raychaudhuri A 1955 Relativistic cosmology: I Phys. Rev. 98 1123–6
[33] Senovilla J M M, Sopuerta C F and Szekeres P 1998 Theorems on shear-free perfect fluids with their Newtonian analogues Gen. Rel. Grav. 30 389–411
[34] Slobodeanu R 2011 Perfect fluids from high power sigma models Int. J. Geom. Methods Mod. Phys. 8 1765–82
[35] Slobodeanu R 2013 Harmonic morphisms and shear-free perfect fluids coupled with gravity Ann. Mat. Pura Appl. doi:10.1007/s10231-012-0320-8
[36] Sopuerta C F 1998 Covariant study of a conjecture on shear-free barotropic perfect fluids Class. Quantum Grav. 15 1043–62
[37] Taub A 1954 General relativistic variational principle for perfect fluids Phys. Rev. 94 1468–70
[38] Treciokas R and Ellis G F R 1971 Isotropic solutions of the Einstein–Boltzmann equations Commun. Math. Phys. 23 1–22
[39] Van den Bergh N 1999 The shear-free perfect fluid conjecture Class. Quantum Grav. 16 117–29
[40] Van den Bergh N, Carminati J, Karimian H R and Huf P 2012 Shear-free perfect fluids with a solenoidal electric curvature Class. Quantum Grav. 29 105010
[41] White A J and Collins C B 1984 A class of shear-free perfect fluids in general relativity: I J. Math. Phys. 25 332–7
[42] Wolfram Research Inc. 2008 Mathematica, Version 7.0, Champaign, IL
[43] Yano K 1952 On harmonic and Killing vector fields Ann. Math. 55 38–45