Semi-Invariants for Gentle String Algebras

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ABSTRACT OF DISSERTATION

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Abstract

This thesis is devoted to the study of the geometry of representation spaces of string algebras. For each irreducible component $C$ of a representation space of a gentle string algebra, we give an algorithm to determine the ring of semi-invariant functions on $C$. We show that these rings are semigroup rings (even coordinate rings of toric varieties) whose generators and relations can be described as walks on a particular graph. In addition, we determine the canonical decompositions of the modules in $C$. This decomposition allows a general discussion of the generating semi-invariants via Schofield’s construction. This decomposition can be used to describe certain important GIT quotients for particular choices of $C$. 
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For Jen
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5.2 GIT Quotients
Since their inception in the early nineteen-seventies, quivers and their representations have been the topic of great scrutiny. While quivers were initially developed to study problems arising in linear algebra, it quickly became apparent that they could be used to study more general problems in the study of modules over finite-dimensional algebras [20]. In particular, every finite-dimensional associative algebra over an algebraically closed field is Morita equivalent (i.e., has an isomorphic module category) to the path algebra of a quiver modulo relations. Furthermore, modules over path algebras can be viewed as representations of the quiver.

Drozd [17] showed that the module categories of finite-dimensional algebras can fall into one of two levels of complexity called tame and wild. Algebras of tame type have module categories which one could hope to describe; that is, module categories whose indecomposable objects could be classified, whereas for algebras of wild type, there is little hope of such description. (Algebras with only finitely many non-isomorphic indecomposable modules are tame in this dichotomy, although we sometimes refer to them as finite type.)

It was Gabriel [20] who proved that (connected) quivers whose path algebras have only
finitely many indecomposable modules are those whose underlying graphs are simply-laced Dynkin diagrams. One year later, Donovan-Freislich and Nazarova ([15], [31]) independently showed that the quivers of tame type are those whose underlying graphs are extended Dynkin diagrams (otherwise known as Euclidean). In all other cases, the path algebra is wild.

Gentle string algebras are a generally well-behaved class of algebras, which are special cases of (special) biserial algebras. They are of tame representation type ([9], [40]), but exhibit non-polynomial growth, meaning that the number of one-parameter families of indecomposables increases exponentially with the dimension (in contrast to the situation of quivers with no relations). The indecomposable modules have been classified ([5], [40]), and the combinatorial nature of their description makes these algebras an obvious set of algebras on which to test or disprove conjectures concerning tameness (see [16], for example). The Auslander-Reiten theory for these algebras is also well-known and very combinatorial ([5]).

More recently, string algebras have appeared in connection with cluster algebras arising from surfaces ([2], [18], [27], [30]) and in the description of quiver Grassmanians ([6]).

Instead of considering questions pertaining to the module category, it is interesting to ask about the module varieties (or representation spaces) for these algebras. These are affine varieties admitting an action of a product of general linear groups. If \((Q, I)\) is a bound quiver, and \(\beta\) is a dimension vector, we denote by \(\text{Rep}_{Q,I}(\beta)\) the variety of representations of \(Q\) of dimension \(\beta\), which is not necessarily irreducible (unless \(I = 0\)). Among many natural questions arising in this context a few are the following:

a. What can be said about singularities of these varieties? Alternatively, what types of singularities appear in orbit closures? ([1], [4])

b. Can an appropriate geometric quotient be constructed for the group action? If so, what does its structure imply about representation type? ([7])
c. What do the rings of invariant regular functions under the aforementioned product of general linear group (or some subgroup) imply about representation type? ([38], [37])

This type of analysis was carried out by Weyman and Skowroński for representation spaces for quivers without relations ([38]) and for canonical algebras ([37]). The work led them to conjecture that in general, an algebra is tame if and only if all rings of semi-invariants are complete intersections (as is true in the cases they considered). In 2010, Kraśkiewicz and Weyman found particular gentle string algebras for which this was not the case [26].

In this thesis, we work exclusively over an algebraically closed field \( k \) of characteristic 0. We will explicitly determine the irreducible components of representation space \( \text{Rep}_{Q,I}(\beta) \) when \((Q,I)\) is a gentle string algebra, and give a combinatorial procedure to construct generators and relations of the rings of semi-invariant functions. It will turn out that components will be parameterized by maps \( r : Q_1 \to \mathbb{N} \) satisfying certain properties, which will be called rank maps. The first main theorem concerns the degrees of the generators and relations on the rings of semi-invariants of \( \text{Rep}_{Q,I}(\beta, r) \), relative to the grading induced by the embedding \( \text{Rep}_{Q,I}(\beta, r) \hookrightarrow \text{Rep}_{Q}(\beta) \).

**Theorem 1**

*The ring of semi-invariant functions on \( \text{Rep}_{Q,I}(\beta, r) \) is a semigroup ring with generators in degree at most

\[
\sum_{a \in Q_1} 2 \binom{r(a) + 1}{2}
\]

and relations in degree at most

\[
\sum_{a \in Q_1} 8 \binom{r(a) + 1}{2}.
\]
This theorem is shown by defining a more general class of so-called matching semigroups. The determination of degree bounds for these matching semigroups consists of constructing a graph in such a way that certain walks on the graph correspond to elements in the semigroup, and from which relations are clear (see section 3.4). The degree bounds for these matching semigroups are sharp, however they may not give rise to sharp bounds on these rings of semi-invariants.

A powerful tool in studying the geometry of these representation spaces is the notion of the canonical decomposition of a dimension vector $\beta$ with respect to some irreducible component $\text{Rep}_{Q,I}(\beta, r)$ of $\text{Rep}_{Q,I}(\beta)$ (see section 4). Inherent in the canonical decomposition is a dense subset whose elements are referred to as generic representations (in contrast to the notion of generic due to Ringel, for example in [32]).

In chapter 4, we construct a family of modules $V_{Q,I}(\beta, r)$ for each component of a representation space $\text{Rep}_{Q,I}(\beta, r) \subset \text{Rep}_{Q,I}(\beta)$. The work of Crawley-Boevey and Schröer [10] gives homological criteria to determine when a module is generic based on data about its direct summands. We use these criteria to show the following theorem.

**Theorem 2**

*The union of the orbits of $V_{Q,I}(\beta, r)$ is the set of generic representations in $\text{Rep}_{Q,I}(\beta, r)$. In particular, it is dense in its irreducible component.*

In proving theorem 1, the combinatorics of Schur modules are used heavily. While this does provide a concrete description of the ring of semi-invariants, there is another construction of semi-invariants for representations of quivers—the so-called Schofield semi-invariants—that were extended to representations of bound quivers by Derksen and Weyman in [13]. In chapter 5, we will focus on dimension vectors $\beta$ and rank maps $r$ for which $V_{Q,I}(\beta, r)$ consists of representations whose indecomposable direct summands are bands,
and determine the GIT quotients for fixed weights. It is conjectured that in tame type, these should always be products of projective spaces. We show that this holds for for components whose generic representation is an indecomposable band.

**Theorem 3**

*Suppose that the generic representation in $\text{Rep}_{Q,I}(\beta, r)$ is an indecomposable band module. Let $\chi$ be the weight $\langle \langle \beta, - \rangle \rangle$. Then the GIT-quotient of the set of $\chi$-semi-stable points of $\text{Rep}_{Q,I}(\beta, r)$ (with respect to the group $\text{PGL}_Q(\beta)$) is isomorphic to $\mathbb{P}^1$.***
Chapter 2

Preliminaries

2.1 Quivers and Representations

In this section, we will review basic notions of quivers, bound quivers, and representations. The primary reference is the text “Elements of the Representation Theory of Associative Algebras” ([3]).

A quiver $Q = (Q_0, Q_1)$ is a directed graph with vertices $Q_0$ and arrows $Q_1$. We denote by $ta$ (resp. $ha$) the tail (resp. head) of an arrow $a \in Q_1$. For each vertex $x \in Q_0$, we also introduce the paths $e_x$ of length zero concentrated at $x$. The path algebra $kQ$ of $Q$ is the vector space with basis consisting of paths in $Q$ and multiplication given by concatenation of paths. Namely

$$p \cdot q := \begin{cases} pq & \text{if } h(q) = t(p) \\ 0 & \text{otherwise.} \end{cases}$$

(In the above, the head and tail of a path are defined in the obvious way). Notice that $kQ$ is an associative algebra, which is finite dimensional if and only if $Q$ is finite and without oriented cycles. Finally, it is a graded algebra with grading given by length of paths.
An admissible relation in $Q$ is a $k$-linear combination of paths of length at least two, 
\[ \rho = \sum_{i=1}^{n} \lambda_i w_i, \] where $\lambda_i \in k$, and $w_1, w_2, \ldots, w_n$ have common tail and head, and a zero relation is a relation with $n = \lambda_1 = 1$. A quiver $Q$ together with a family $\{\rho_i\}_{i \in J}$ of admissible relations is called a bound quiver or quiver with relations, and the algebra $kQ/\langle \rho_i \rangle_{i \in J}$ is called a bound quiver algebra.

2.1.1 Representations

A (finite dimensional) representation of a quiver $Q$ is an assignment of a (finite dimensional) $k$-vector space $V_x$ to each vertex $x \in Q_0$, and a linear map $V_a : V_{ta} \to V_{ha}$ to each arrow. All representations will be assumed finite-dimensional. The category of representations of $Q$, denoted $\text{Rep}_Q$, is the category whose objects are representations of $Q$ with morphisms defined as follows: if $V, W$ are representations of $Q$, a morphism $\varphi : V \to W$ consists of a $Q_0$-tuple of linear maps $\varphi_x : V_x \to W_x$ such that for every $a \in Q_1$, the following diagram commutes:

\[
\begin{array}{ccc}
V_{ta} & \xrightarrow{V_a} & V_{ha} \\
\downarrow{\varphi_{ta}} & & \downarrow{\varphi_{ha}} \\
W_{ta} & \xrightarrow{W_a} & W_{ha}
\end{array}
\]

If $p = a_m \ldots a_1$ is a path in $Q$, and $V$ is a representation of $Q$, then $V_p := V_{a_m} \cdots V_{a_1}$ is defined to the composition of the maps along the path. If $\rho = \sum_{i=1}^{n} \lambda_i w_i \in kQ$ is an admissible relation, then by definition $V_\rho = \sum_{i=1}^{n} \lambda_i V_{w_i}$. Notice that since $\rho$ is admissible, $V_\rho$ is a sum of linear maps on a common domain and codomain.

If $\{\rho_i\}_{i \in J}$ is a set of admissible relations, let $I = \langle \rho_i \rangle_{i \in J}$ be the associated ideal of relations. Then the category of representations of the bound quiver $(Q, I)$, denoted $\text{Rep}_{Q,I}$, is the subcategory of $\text{Rep}_Q$ whose objects are representations $V$ of $Q$ such that $V(\rho_i) = 0$ for all $i \in J$. The dimension vector of a representation $V$ is the vector $\dim V \in \mathbb{Z}^{Q_0}$. 

with \((\dim V)_x = \dim_k(V_x)\).

The category of representations of a bound quiver is equivalent to the category of modules over the bound quiver algebra. Given this equivalence, we will speak interchangeably about representations of an algebra and representations of its (bound) quiver.

Being equivalent to a category of modules, the categories of representations of quivers (or bound quivers) inherit a direct sum operation, and are Krull-Schmidt categories. A fundamental problem in the representation theory of algebras is to classify all indecomposable objects (with respect to this operation). To that end, there are two main classes of algebras defined below.

**Definition 2.1.1.** A (bound) quiver algebra \(A \cong kQ/I\) is called **tame** if for each dimension vector \(d \in \mathbb{N}^{Q_0}\), there are finitely many one-parameter families of \(A\)-\(k[t]\)-bimodules \(M_1, \ldots, M_h\) where the \(M_i\) are finitely generated free (right) \(k[t]\)-modules, such that all but finitely many \(d\)-dimensional indecomposable \(A\)-modules \(M\) are of the form \(M \cong M_i \otimes_{k[t]} k[t]/(t - \lambda)\) for some \(i\) and \(\lambda \in k\). In particular, if there are finitely many indecomposable modules, then \(A\) is tame. The algebra \(A\) is called **wild** if the category of finitely generated \(A\)-modules contains as a subcategory the category of finitely generated \(k\langle x, y \rangle\)-modules.

Drozd showed in [17] that every algebra is of exactly one of the two above types.

The Euler form associated to the algebra \(kQ/I\) plays a crucial role both in the representation theory of \(A\) and the study of semi-invariants. In general, for a finite-dimensional algebra \(A\) (over an algebraically closed field) with finite global dimension, we can identify the Grothendieck group \(K_0(A)\) with \(\mathbb{Z}^{Q_0}\). This identification takes the class of a module \([V]\) to its dimension vector (and is then extended linearly). Now \(K_0(A)\) can be endowed
with a bilinear form defined on $\mathbb{N}^n \times \mathbb{N}^n$ by the following:

$$\langle \langle \dim V, \dim V' \rangle \rangle = \sum_{i \geq 0} (-1)^i \dim \text{Ext}_A^i(V, V')$$

and then extended by linearity to $K_0(A) \times K_0(A)$. Denote the matrix of this bilinear form by $E_A$. We also write $q_A$ to denote the associated quadratic form $q_A(\beta) = \langle \langle \beta, \beta \rangle \rangle$.

### 2.2 Module Varieties

We now introduce some geometric objects which will be the primary objects of study. Let $Q$ be a quiver, and let $\beta \in \mathbb{N}^{Q_0}$ be a dimension vector. Define by $\text{Rep}_Q(\beta)$ the set

$$\text{Rep}_Q(\beta) := \{ V \in \text{Rep}_Q \mid \dim V = \beta \}.$$  

Fixing a basis at each vertex, this set can be identified with

$$\prod_{a \in Q_1} \text{Hom}_k(k^{\beta_{ta}}, k^{\beta_{ha}}) = \prod_{a \in Q_1} \text{Mat}_{\beta_{ha} \times \beta_{ta}}(k)$$

where $\text{Mat}_{k,l}(k)$ is the set of $k \times l$ matrices with entries in $k$. This latter identification makes it clear that $\text{Rep}_Q(\beta)$ is affine space of dimension $\sum_{a \in Q_1} \beta_{ta} \beta_{ha}$. This variety will be called the **variety of representations of $Q$ of dimension vector $\beta$**. If $(Q, I)$ is a bound quiver, then define by $\text{Rep}_{Q,I}(\beta) \subset \text{Rep}_Q(\beta)$ the subvariety consisting of representations of $(Q, I)$ of dimension vector $\beta$, i.e.,

$$\text{Rep}_{Q,I}(\beta) = \{ M \in \prod_{a \in Q_1} \text{Mat}_{\beta_{ha} \times \beta_{ta}}(k) \mid M_\rho = 0 \forall \rho \in I \}.$$  

Notice that the entries of the matrix $M_\rho$ are polynomials in the entries of the matrices $M_a$, and recall that $I$ is a finitely generated ideal. Therefore, $\text{Rep}_{Q,I}(\beta)$ is an algebraic set in
$\text{Rep}_Q(\beta)$, although may fail to be irreducible.

If $\beta$ is a dimension vector for some quiver $Q$, denote by $\text{GL}_Q(\beta)$ the product $\prod_{x \in Q_0} \text{GL}(\beta_x)$, and $\text{SL}_Q(\beta) = \prod_{x \in Q_0} \text{SL}(\beta_x)$. Both algebraic groups act by simultaneous change of basis on $\text{Rep}_Q(\beta)$ (resp. $\text{Rep}_{Q,I}(\beta)$) as follows:

$$(g_x)_{x \in Q_0}, (M_a)_{a \in Q_1} = (g_h a M_a g_h^{-1})_{a \in Q_1}.$$ 

In fact, $\text{GL}_Q(\beta)$ preserves irreducible components, so the coordinate ring of each irreducible component is a $\text{GL}_Q(\beta)$-module.

### 2.3 Semi-Invariants

Let $G$ be a linear algebraic group, and $X$ be a rational $G$-variety. Denote by $k[X]$ the algebra of regular functions on $X$. Let $X(G)$ be the group of (multiplicative) characters of $G$.

**Definition 2.3.1.** For $\chi \in X(G)$, the space of semi-invariants of weight $\chi$ is the vector space

$$\text{SI}(G, X)_\chi = \{ f \in k[X] \mid g. f = \chi(g) \cdot f \text{ for all } g \in G \}.$$ 

The algebra of semi-invariants is defined to be $	ext{SI}(G, X) := \bigoplus_{\chi \in X(G)} \text{SI}(G, V)_\chi$.

We will work specifically with the group $G = \text{GL}_Q(\beta)$, so let us recall pertinent material from the representation theory of $\text{GL}(V)$ when $V$ is a finite-dimensional $k$ vector space of dimension $n$. Denote by $\mathbb{Z}_+^n$ the set of **dominant integral weights**, that is, non-increasing integer sequences of length $n$. It is well-known ([19], [41]) that the irreducible rational representations of $\text{GL}(V)$ are parameterized by $\mathbb{Z}_+^n$. For $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}_+^n$, denote by $S_{\lambda} V$ the corresponding representation, called a **Schur module**. In particular, $S_{(1,1,\ldots,1,0,\ldots,0)} V = \bigwedge^l V$ the $l$-th exterior power (where $l$ is the number of ones) and
\[ S_{(l,0,...,0)} V = S_l V \] the \( l \)-th symmetric power of \( V \). Therefore, as a \( \text{GL}(V) \) module we can decompose \( k[X] \):

\[ k[X] = \bigoplus_{\lambda \in \mathbb{Z}_l^n} m(\lambda, X) S_{\lambda} V. \]

Notice that \( X(\text{GL}(V)) = \{ \det^a \mid a \in \mathbb{Z} \} \) consists of integer powers of the determinant function, which can be be identified with \( \mathbb{Z} \) itself. Denote by \( \chi_a \) the corresponding character and \( \lambda_a = (a, a, \ldots, a) \). Then \( \text{SI}(\text{GL}(V), X)_{\chi_a} = m(\lambda_a, X) S_{\lambda_a} V \). In particular

\[ \text{SI}(\text{GL}(V), X) \cong k[X]^{\text{SL}(V)}, \]

where the latter is the ring of regular functions invariant under the action of the special linear subgroup \( \text{SL}(V) \).

Notice that each irreducible component \( Z \) of \( \text{Rep}_{Q,I}(\beta) \) is a rational \( \text{GL}_Q(\beta) \)-variety, and as such it is a rational \( \text{GL}(\beta_x) \) variety for each \( x \in Q_0 \). Therefore, the coordinate ring \( k[Z] \) is a rational \( \text{GL}_Q(\beta) \)-module. Denoting by \( \Lambda = \{ \Lambda \in \prod_{x \in Q_0} \mathbb{Z}_{\beta_x} \} \) the set of dominant integral weights for \( \text{GL}_Q(\beta) \), we have

\[ k[Z] = \bigoplus_{\Lambda \in \Lambda} m(\Lambda, Z) \bigotimes_{x \in Q_0} S_{\lambda(x)} V_x \]

where \( V_x = k^{\beta_x} \). Furthermore, \( X(\text{GL}_Q(\beta)) \) can be identified with \( \mathbb{Z}^{Q_0} \) as products of integer powers of the determinant functions at each vertex. Denote by \( \chi_\underline{a} \) the character corresponding to the vector \( \underline{a} \in \mathbb{Z}^{Q_0} \). Then \( \text{SI}(\text{GL}_Q(\beta), Z)_{\chi_\underline{a}} = m(\lambda_\underline{a}, X) \bigotimes_{x \in Q_0} S_{\lambda_\underline{a}_x} V_x \) and, in particular,

\[ \text{SI}(\text{GL}_Q(\beta), Z) \cong k[Z]^{\text{SL}_Q(\beta)}. \]

In the subsequent chapters, we will always work over \( \text{GL}_Q(\beta) \). For notational simplicity, we will denote the ring of semi-invariants of the irreducible component \( Z \) of \( \text{Rep}_{Q,I}(\beta) \) by \( \text{SI}_{Q,I}(\beta, Z) \).
2.3.1 Schofield Semi-invariants

The difficulty in applying the above techniques is that there may not exist an explicit description of the coordinate ring of a $G$ variety by Schur modules. An alternate point of view was developed by Schofield [33] and Derksen-Weyman [13].

Suppose that $V$ is a $kQ/I$-module of projective dimension one with projective resolution $V \xleftarrow{\delta_0} P(0) \xleftarrow{\delta_0} P(1)$. For any $kQ/I$-module $W$, we can consider the map $d^V_W : \text{Hom}_{kQ/I}(P(0), W) \to \text{Hom}_{kQ/I}(P(1), W)$ obtained by applying the functor $\text{Hom}_{kQ/I}(-, W)$ to this presentation. Notice that the kernel of $d^V_W$ is precisely $\text{Hom}_{kQ/I}(V, W)$, and the cokernel is $\text{Ext}^1_{kQ/I}(V, W)$. Thus, if $\langle \langle \dim V, \dim W \rangle \rangle = 0$, then after choosing bases, $d^V_W$ is a square matrix. Furthermore, if either $\text{Hom}_{kQ/I}(V, W) = 0$ or $\text{Ext}^1_{kQ/I}(V, W) = 0$, then $d^V_W$ is an isomorphism, so has non-zero determinant.

We can consider $c^V : \text{Rep}_{Q,I}(\beta) \to k$ to be the map with $c^V(W) = \det d^V_W$. Schofield constructed these functions in [33] and showed that they are semi-invariants. Furthermore, we have the following proposition due to Derksen and Weyman.

**Proposition 2.3.2.** [13] Suppose $Z \subset \text{Rep}_{Q,I}(\beta)$, and $\langle \langle \dim V, \beta \rangle \rangle = 0$. The function $c^V$ is a semi-invariant of weight $\langle \langle \dim V, - \rangle \rangle$ and if $Z$ is a faithful component (that is $\text{ann}_{kQ}(Z) = I$), then $\text{SI}_{Q,I}(\beta, Z)_{\langle \langle \alpha, - \rangle \rangle}$ is spanned by the set of all $c^V$ such that

a. $V$ is of projective dimension 1;

b. $\dim V = \alpha$.

2.3.2 GIT Quotients

The study of moduli spaces for finite dimensional algebras was initiated by King [24], although the general definition of GIT quotients goes back to Mumford [29]. The central problem is to determine a variety whose points parametrize orbits of a group acting on
another variety. In general, such a space is elusive, so one settles for parameterizing only certain orbits. For this section’s notation, we refer to the quiver notes by Derksen-Weyman [14]. Let \((Q, I)\) be a bound quiver, \(\beta\) a dimension vector, and \(Z \subset \text{Rep}_{Q,I}(\beta)\) an irreducible component of the representation space. Suppose that \(\chi\) is a character for \(\text{GL}_{Q}(\beta)\) (which can be considered as an element in \(\mathbb{Z}^{Q_0}\)). Let \(\text{GL}_{Q}(\beta)\chi\) be the kernel of the map \(\chi\) when viewed as a character.

If \(\chi\) is not divisible by the characteristic of \(k\), then

\[
R_{\chi} := k[Z]^{\text{GL}_{Q}(\beta)\chi} = \bigoplus_{n \geq 0} \text{SI}_{Q,I}(\beta, Z)_{n, \chi}.
\]

Notice that \(R_{\chi} \subset k[Z]\), so there is a surjective morphism \(\Psi : Z \to \text{Spec}(R_{\chi})\). Let \(Z_{\chi}^{ss}\) be the set of representations \(V \in Z\) such that there is a function \(f \in R_{\chi}\) with \(f(V) \neq 0\). These are called the \(\chi\)-semi-stable points of \(Z\). In this case, \(\Psi^{-1}(0)\) is the complement of \(Z_{\chi}^{ss}\) in \(Z\). If \(Y := \text{Proj}(R)\), then we have the following commuting diagram:

\[
\begin{array}{ccc}
Z_{\chi}^{ss} & \to & X \setminus \{0\} \\
& \searrow & \downarrow \\
& & Y
\end{array}
\]

The map \(\pi : Z_{\chi}^{ss} \to Y\) is a geometric quotient called the \textbf{GIT-quotient} of \(Z_{\chi}^{ss}\) by \(\text{PGL}(\beta)\) the product of projective general linear groups.

### 2.4 String and band modules

A particularly important class of modules in \(\text{Rep}_{Q,I}\) will be the string and band modules, which we introduce following the conventions of Crawley-Boevey [8]. If \(T\) is an arbitrary quiver, a \textit{bound quiver morphism} \(F = (F_0, F_1) : T \to (Q, I)\) consists of a set maps \(F_0 : T_0 \to Q_0, F_1 : T_1 \to Q_1\) such that
• for each \( a \in Q_1 \), \( h(F_1(a)) = F_0(h(a)) \) and \( t(F_1(a)) = F_0(t(a)) \);

• if \( h(a) = h(b) \) or \( t(a) = t(b) \) then \( F_1(a) \neq F_1(b) \);

• for all paths \( p \) on \( T \), \( F(p) \) is a path in \((Q, I)\) (that is it does not pass through any relations).

A bound quiver morphism induces a functor \( F_* : \text{Rep}_T \to \text{Rep}_{Q, I} \) so that \((F_* V)_x = \bigoplus_{y \in F^{-1}(x)} V_y\) if \( x \in Q_0 \) or \( x \in Q_1 \).

Let \( T \) be a tree, that is a finite quiver whose underlying graph is acyclic, and let \( 1_T \) be the representation of \( T \) consisting of a one-dimensional vector space at each vertex and with every acting by the identity transformation. The representations \( F_*(1_T) \) are called tree modules, and are indecomposable if \( T \) is connected as a result of Gabriel [22]. If \( T \) is a chain (every vertex has valence at most 2), then \( F_*(1_T) \) is called a string module.

Let \( B \) be any orientation of \( \tilde{A}_n \) for any \( n \). Label the vertices \( 1, \ldots, n, n+1 \), and arrows \( a_1, \ldots, a_{n+1} \) such that \( t(a_1) = 1 \) and for each \( i = 1, \ldots, n+1 \), the arrows \( a_i, a_{i+1} \) share a vertex. For \( m \in \mathbb{N} \), and any vector space automorphism \( \varphi : k^m \to k^m \), let \( \varphi_B \) be the representation of \( B \) with \((\varphi_B)_x = k^m\), and

\[
(\varphi_B)_{a_i} = \begin{cases} 
\text{Id}_m & i \neq n \\
\varphi & i = n.
\end{cases}
\]

The representations \( F_*(\varphi_B) \) are called band modules. Let \( c = c_n c_{n-1} \ldots c_1 \) be the sequence of elements in the letters \( Q_1 \cup Q_1^{-1} \) with

\[
c_i = \begin{cases} 
F(a_i) & \text{if } t(a_i) = i \\
F(a_i)^{-1} & \text{otherwise}.
\end{cases}
\]
In this way, the word $c$ can be viewed as a cyclic path on $Q$. If $c$ cannot be written as the power of any smaller word (i.e., $c \neq (c')(c') \ldots (c')$ for any cyclic word $c'$), and $(k^m, \varphi)$ is an indecomposable $k[t, t^{-1}]$-module, then $F_*(\varphi_B)$ is indecomposable.

### 2.5 Gentle String Algebras

In this section, we collect some results concerning string algebras. Most of the details can be found in [5]. In addition, Schröer has two articles ([34], [35]) containing nice introductions.

**Definition 2.5.1.** A bound quiver algebra $kQ/I$ is called a **string algebra** if $I$ is generated by paths and the following conditions hold:

a. For each $x \in Q_0$, the number of arrows $a \in Q_1$ with $ha = x$ (resp. $ta = x$) is at most 2.

b. Given an arrow $\beta \in Q_1$, there is at most one arrow $\gamma \in Q_1$ with $t\gamma = h\beta$ (resp. at most one arrow $\alpha \in Q_1$ with $t\alpha = h\beta$) such that $\gamma\beta \notin I$ (resp. $\beta\alpha \notin I$).

c. For each $\beta \in Q_1$, there is a bound $n(\beta)$ such that every path $p$ of length greater than $n(\beta)$ whose first or final arrow is $\beta$ is in $I$.

A string algebra is called **gentle** if it satisfies the additional conditions:

d. Given an arrow $\beta \in Q_1$, there is at most one arrow $\gamma \in Q_1$ with $t\gamma = h\beta$ (resp. at most one arrow $\alpha \in Q_1$ with $t\beta = h\alpha$) such that $\gamma\beta \in I$ (resp. $\beta\alpha \in I$).

e. $I$ is generated by paths of length two.

**Proposition 2.5.2 ([5],[36]).** Every indecomposable representation of a string algebra $kQ/I$ is either a string module or a band module.
2.6 Varieties of Complexes

The varieties of complexes were defined and studied by DeConcini-Strickland in [11] and later extended to cyclic complexes by Mehta-Trivedi [28]. These are varieties whose points correspond to complexes of vector spaces of some fixed dimension. Here we recall the related material concerning these varieties including their irreducible components, and a description of their coordinate rings.

**Definition 2.6.1.** For \( n \in \mathbb{N} \) and \( \beta = (\beta_1, \beta_2, \ldots, \beta_{n+1}) \in \mathbb{N}^{n+1} \), the variety of complexes of length \( n \) and dimension \( \beta \) is the set

\[
\text{Com}^n(\beta) := \{(M_i \in \text{Mat}_{\beta_i \times \beta_{i+1}}(k))_{i=1,\ldots,n} \mid M_{i+1} \cdot M_i = 0 \text{ for } i = 1, \ldots, n-1\}.
\]

Denote by \( A_{n+1}^{eq} \) the equioriented quiver of type \( A_{n+1} \) with vertices \( 1, 2, \ldots, n+1 \) and arrows \( a_i : i \to i+1 \). Let \( I = \langle a_{i+1} a_i \mid i = 1, \ldots, n-1 \rangle \), then \( \text{Com}^n(\beta) = \text{Rep}_{A_{n+1}^{eq},I}(\beta) \).

In this way, the varieties of complexes can be viewed as representation spaces for the most basic gentle string algebras. Their irreducible components can be nicely parameterized by sequences of maximal ranks in the following way. A function \( r : \{a_1, \ldots, a_n\} \to \mathbb{N} \) is called a **rank map** for \( \beta \) if \( r(a_i) + r(a_{i+1}) \leq \beta_{i+1} \) for \( i = 0, \ldots, n \) (here we define \( r(a_0) = r(a_{n+1}) = 0 \)). The set of all rank maps for \( \beta \) is a (finite) poset with \( r \leq r' \) if and only if \( r(a_i) \leq r'(a_i) \) for \( i = 1, \ldots, n \). Given a fixed rank map \( r \) for \( \beta \), we define the following subsets of \( \text{Com}^n(\beta) \):

\[
\text{Com}^n_0(\beta, r) = \{M_i \in \text{Com}^n(\beta) \mid \text{rank}_k M_i = r(a_i)\}
\]

\[
\text{Com}^n(\beta, r) = \{M_i \in \text{Com}^n(\beta) \mid \text{rank}_k M_i \leq r(a_i)\}
\]

**Proposition 2.6.2 ([10]).** Let \( r \) be a maximal rank map for \( \beta \). Then \( \text{Com}^n(\beta, r) \) is an
irreducible component of Com\(_n(\beta)\). Furthermore,

\[
\text{Com}_n(\beta) = \bigcup_{r \text{ maximal for } \beta} \text{Com}_n(\beta, r).
\]

### 2.6.1 The Coordinate Ring \(k[\text{Com}_n(\beta, r)]\)

In the same article, DeConcini-Strickland give a basis for \(k[\text{Com}_n(\beta, r)]\) in terms of multitableau. Their work prescribes a filtration on the coordinate ring whose associated graded is given by Schur modules. The early portion of this section is a recollection of Young diagrams. In the last part of this section, we describe a filtration on \(k[\text{Com}(\beta, r)]\) and its associated graded ring. For the remainder of this section, we fix \(n\), a dimension vector \(\beta\), and a maximal rank sequence \(r\) for \(\beta\).

A **Young diagram** \(\lambda\) is a sequence of non-increasing positive integers \(\lambda_1 \geq \ldots \geq \lambda_m\), \(m\) is called the number of parts of \(\lambda\). We will draw Young diagrams as a table of rows of left-justified boxes such that the \(i\)-th row has \(\lambda_i\) boxes. For a Young diagram \(\lambda\), we denote by \(\lambda'\) the transpose diagram, where \(\lambda'_i = \{i \mid \lambda_j \geq i\}\). If \(p\) be a positive integer with \(p \geq m\). Denote by \(\lfloor p - \lambda \rfloor\) the diagram with \(p\) parts and \(\lfloor p - \lambda \rfloor_j = \lambda_1 - \lambda_{p-j+1}\) (in this expression, if \(\lambda_{p-j+1}\) is not defined, then it is considered to be 0). We will call a filling standard if it is row increasing and column strictly-increasing. To a filling \(t\) of \(\lambda\), we associated a sequence of sets \(I(t) = (I(t)_1, \ldots, I(t)_{\lambda_1})\) where \(I_l = \{t_{1,l}, t_{2,l}, \ldots, t_{\lambda'_l,l}\}\).

**Definition 2.6.3.** Let \(V\) be a vector space, and \(\lambda\) a Young diagram with at most \(\text{dim } V\) parts. We will denote by \(\bigwedge^\lambda V\) the product of exterior powers of \(V\) prescribed by the columns of \(\lambda\). Namely

\[
\bigwedge^\lambda V = \bigwedge^{\lambda'_1} V \otimes \ldots \otimes \bigwedge^{\lambda'_{\lambda_1}} V.
\]

For a set \(I = \{i_1, \ldots, i_k\}\), let \(e_I = e_{i_1} \wedge \ldots \wedge e_{i_k}\). If \(t\) is a column-increasing filling of \(\lambda\) with integers from the set \(\{1, \ldots, \text{dim } V\}\), then we have the associated sequence of sets
$(I(t)_1, \ldots, I(t)_{\lambda_1})$ and the associated basis element in $\bigwedge^\lambda V$ is $e_{I(t)_1} \otimes \ldots \otimes e_{I(t)_{\lambda_1}}$.

Let $V$ be a $k$ vector space and $\lambda$ a Young diagram with at most $\dim V$ parts. Define the map

$$\text{op}_\lambda : \bigwedge^{[\dim V - \lambda]} V \to \bigwedge^\lambda V$$

as follows: if $t$ is a column-increasing filling of $[\dim V - \lambda]$, and $I(t) = (I(t)_1, \ldots, I(t)_{\lambda_1})$ is the associated sequence of sets, then take $t'$ the filling of $\lambda$ with associated sequence of sets $I(t') = (I(t')_1, \ldots, I(t')_{\lambda_1})$ such that $I(t')_j = \{1, \ldots, \dim V\} \setminus I(t)_{\lambda_1 - j + 1}$. Then

$$\text{op}_\lambda(t) := \left( \prod_{j=1}^{\lambda_1} \text{sgn}(I(t')_j, I(t)_{\lambda_1 - j + 1}) \right) t'.$$

Here, $\text{sgn}(I, J)$ is the sign of the permutation $(I, J)$ with both $I, J$ written in increasing order.

Suppose now that $V_i, V_{i+1}$ are $k$-vector spaces, and $\lambda$ is a Young diagram with at most $\min(\dim V_i, \dim V_{i+1})$ parts. Define a map $\delta^{(i)}_{\lambda} : \bigwedge^\lambda V_i \otimes \bigwedge^\lambda V_{i+1} \to k[\text{Com}(\beta, r)]$ as follows: suppose that $t_i$ is a filling of $\lambda$ from the integers $\{1, \ldots, \dim V_i\}$ with associated sequence of sets $I(t_i)$, and $t_{i+1}$ is a filling of $\lambda$ from the integers $\{1, \ldots, \dim V_{i+1}\}$ with associated sequence of sets $I(t_{i+1})$. Then

$$\delta_{\lambda} : t_i \otimes t_{i+1} \mapsto \prod_{j=1}^{\lambda_1} \Delta_{I(t_i)_j, I(t_{i+1})_j}^{(i)}.$$

(Recall that $\Delta_{I, J}^{(i)}$ is the minor of the matrix $A_i$ with columns given by $I$ and rows given by $J$.)

If $\lambda = (\lambda(1), \ldots, \lambda(n))$ is a sequence of Young diagrams such that $\lambda(i)$ has at most
min(dim \(V_i\), dim \(V_{i+1}\)) parts, then take

\[
\delta_{\lambda} : \bigotimes_{i=1}^{n} \left( \bigwedge \lambda(i) V_i \otimes \bigwedge [\beta_{i+1} - \lambda(i)] V_{i+1} \right) \to k[\text{Com}(\beta, r)]
\]

To be the composition of the map

\[
\bigotimes_{i=1}^{n} (\text{id} \otimes \text{op}_{\lambda(i)}) : \bigotimes_{i=1}^{n} \left( \bigwedge \lambda(i) V_i \otimes \bigwedge [\beta_{i+1} - \lambda(i)] V_{i+1} \right) \to \bigotimes_{i=1}^{n} \left( \bigwedge \lambda(i) V_i \otimes \bigwedge \lambda(i) V_{i+1} \right)
\]

and the map

\[
\bigotimes_{i=1}^{n} \delta^{(i)}_{\lambda} : \bigotimes_{i=1}^{n} \left( \bigwedge \lambda(i) V_i \otimes \bigwedge \lambda(i) V_{i+1} \right) \to k[\text{Com}(\beta, r)].
\]

**Remark 2.6.1.** If \(\lambda(i)\) has more than \(r(i)\) parts for some \(i\), then image \(\delta_{\lambda} = 0\) on \(\text{Com}(\beta, r)\) since one factor is the an \(r(i) + l \times r(i) + l\) minor of \(A_i\), and rank \(A_i \leq r(i)\) by definition of \(\text{Com}(\beta, r)\).

**Definition 2.6.4.** Let \(\Lambda_n(\beta, r)\) be the set of sequence of partitions \((\lambda(1), \ldots, \lambda(n))\) such that \([\beta_{i+1} - \lambda(i)]_{\lambda(i)} \leq \lambda(i+1)'\). I.e., the first column of \([\beta_{i+1} - \lambda(i)]\) is shorter than the last column of \(\lambda(i+1)\). If \(\lambda \in \Lambda_n(\beta, r)\), denote by \([\lambda(i+1) : \lambda(i)]\) the Young diagram with

\[
[\lambda(i+1) : \lambda(i)]_j = [\beta_{i+1} - \lambda(i)]_j + \lambda(i+1)_j.
\]

Diagrammatically, this is simply juxtaposing the diagrams \(\lambda(i)\) and \([\beta_{i+1} - \lambda(i)]\), which is still a Young diagram by definition of \(\Lambda_n(\beta, r)\). We will also write \(\lambda(1) = [\lambda(1) : \lambda(0)]\) and \([\beta_{n+1} - \lambda(n)] = [\lambda(n+1) : \lambda(n)]\) for the degenerate cases.

A filling of the diagrams \([\lambda(1) : \lambda(0)], [\lambda(2) : \lambda(1)], \ldots, [\lambda(n) : \lambda(n-1)], [\lambda(n+1) : \lambda(n)]\) is the same as a filling of all diagrams \(\lambda(i)\) and \([\beta_{i+1} - \lambda(i)]\) for \(i = 1, \ldots, n\) and is called a multitableau.
Definition 2.6.5. For two partitions $\lambda, \mu$, we define $\lambda \preceq \mu$ if $(\lambda'_1, \ldots, \lambda'_n) \succeq (\mu'_1, \ldots, \mu'_n)$. Extend this to a partial order on $\Lambda_n(\beta, r)$ with $\lambda \preceq \mu$ if
\[
(\lambda(1) : \lambda(0)], [\lambda(2) : \lambda(1)], \ldots, [\lambda(n+1) : \lambda(n)], [\lambda(n+1) : \lambda(n)]) \succeq ([\mu(1) : \mu(0)], [\mu(2) : \mu(1)], \ldots, [\mu(n+1) : \mu(n)], [\mu(n+1) : \mu(n)])
\]
in the lexicographical order.

Definition 2.6.6. Suppose that $\lambda$ and $\mu$ are partitions with $r_1, r_2$ parts, respectively, and $V$ is a vector space of dimension $n$. Then we denote by $S_{(\lambda, -\mu)} V$ the Schur module $S_{(\lambda_1, \ldots, \lambda_r, 0, \ldots, 0, -\mu_{r_2}, -\mu_{r_2-1}, \ldots, -\mu_1)} V$, where we include $n - (r_1 + r_2)$ zeros in the indexing vector. Furthermore, we will write $-\mu$ for the vector $(-\mu_{r_2}, \ldots, -\mu_2, -\mu_1)$.

Proposition 2.6.7 ([11]). Denote by $F_\lambda = \sum_{\mu \in \Lambda_n(\beta, r), \mu \preceq \lambda} \text{image} \delta_\mu$, and $F_{<\lambda} = \sum_{\mu \in \Lambda_n(\beta, r), \mu < \lambda} \text{image} \delta_\mu$. Then $F_\lambda / F_{<\lambda}$ has a basis given by standard fillings of the diagrams $[\lambda(i+1) : \lambda(i)]$ for $i = 0, \ldots, n$. A collection of fillings of this sequence of diagrams is called a multitableau of shape $\lambda$. Furthermore,
\[
F_\lambda / F_{<\lambda} \cong \bigotimes_{i=1}^n S_{(\lambda(i), -\lambda(i-1))} V_i.
\]

The above proposition is proven by showing that if $t_\lambda$ is a multitableau of shape $\lambda$, then $\delta_\lambda(t_\lambda)$ can be written, modulo terms in $F_{<\lambda}$, as a linear combination of standard multitableau of shape $\lambda$.

The content of a multitableau $t$ of shape $\lambda$ is the sequence of vectors $(\kappa^1, \ldots, \kappa^{n+1})$ where
\[
\kappa^i_j = \#\{ \text{boxes in } [\lambda(i) : \lambda(i-1)] \text{ that are filled with the integer } j \}\]
Corollary 2.6.8 ([11]). Suppose that $t$ is a non-standard multitableau of shape $\lambda$. Then

$$\delta_\lambda(t) = s(t) + y(t)$$

where $s(t)$ is a linear combination of standard multitableaux of the same content as $t$, and $y(t) \in \mathcal{F}_{\prec \lambda}$.

Proposition 2.6.9 ([11]). $k[\Com(\beta, r)] = \bigcup_{\lambda \in \Lambda_n(\beta, r)} \mathcal{F}_\lambda$.

So every function in $k[\Com(\beta, r)]$ is in a $\mathcal{F}_\lambda$. Next we show that the $\mathcal{F}_\lambda$ form a filtration. For two elements $\lambda, \mu \in \Lambda_n(\beta, r)$, let $\lambda + \mu$ be the sequence of diagrams with

$$(\lambda + \mu)(i)_j = \lambda(i)_j + \mu(i)_j. \quad (2.6.1)$$

Proposition 2.6.10. Suppose that $t_\lambda$ and $t_\mu$ are multitableaux of shapes $\lambda$ and $\mu$, respectively. Then

$$\delta_\lambda(t_\lambda) \cdot \delta_\mu(t_\mu) \in \mathcal{F}_{\lambda+\mu}.$$ 

Proof. It suffices to show this when $\mu$ consists of a single column, i.e.,

$$\mu(i) = (1, 1, \ldots, 1, 0, 0, \ldots)$$

for some $i$ and $\mu(j) = 0$ otherwise. Thus, $\delta_\lambda(t_\mu) = \Delta^{(i)}_{I,J}$ for some sets $I \subset \{1, \ldots, \beta_i\}$, $J \subset \{1, \ldots, \beta_{i+1}\}$. Therefore, $\delta_\lambda(t_\lambda) \cdot \delta_\mu(t_\mu) = \delta_\lambda(t_\lambda) \cdot \Delta^{(i)}_{I,J}$. Now notice that $\lambda + \mu$ is the sequence diagrams which is the same as $\lambda$ except for $(\lambda + \mu)(i)$ which has an extra column of height $j$. Take multitableau of shape $(\lambda + \mu)$ so that all entries not corresponding to the extra column are the same as in the filling $t_\lambda$, and all entries in the columns corresponding to the extra column are taken from $t_\mu$. Denoting by $t_{\lambda+\mu}$ this filling, we have that $\delta_{\lambda+\mu}(t_{\lambda+\mu}) = \delta_\lambda(t_\lambda) \cdot \delta_\mu(t_\mu)$. In short, each column of the sequence $\mu$ can be absorbed into $\lambda$ until the
result is the sequence $\lambda + \mu$. □

**Corollary 2.6.11 ([11]).** The set $\{F_\lambda\}_{\lambda \in \Lambda}$ is a filtration of $k[\text{Com}(\beta, r)]$ and the associated graded algebra is

$$\text{gr}_\Lambda (k[\text{Com}(\beta, r)]) = \bigoplus_{\lambda \in \Lambda} \bigotimes_{i=1}^n S_{(\lambda(i), -\lambda(i-1))} V_i.$$  

(For the definition of the Schur modules $S_\lambda V$, we refer to [41] Chapter 2.1.)
Chapter 3

Explicit Description of the Rings of Semi-invariants

In the following chapter, we give an explicit description of the rings of semi-invariants for components of representation spaces of gentle string algebras. This begins by recognizing that these representation spaces are products of varieties of complexes as discussed in section 2.6. This allows us to give a filtration on the coordinate rings of these components whose associated graded algebra is a direct sum of Schur modules. By analyzing this decomposition, we can in fact exhibit a grading on the rings of semi-invariants. These rings are then shown to be semigroup rings of a particular sort, which we can investigate to determine degree bounds on generators and relations.

3.1 Colorings of quivers

As stated in the introduction, the determination of irreducible components for representation spaces can be difficult. However, for a large class of zero-relation algebras, the problem can be solved using varieties of complexes. This is achieved by coloring the arrows of the
quiver in such a way that the composition of any two same-colored arrows in $Q$ is in the ideal $I$.

**Definition 3.1.1.** Let $S$ be some finite set, whose elements we call colors. A **coloring** of a quiver $Q$ is a set map $c : Q_1 \rightarrow S$ satisfying the condition that $c^{-1}(s)$ is a direct path for each $s \in S$. The ideal $I_c \subset kQ$ associated to a coloring is defined to be the ideal $I_c = \langle ba \mid a, b \in Q_1, h(a) = t(b), c(a) = c(b) \rangle$. If $Q$ is a quiver with coloring $c$, we will write $\text{Rep}_{Q,c}$ in lieu of $\text{Rep}_{Q,I_c}$.

**Remark 3.1.1.** Notice that the varieties of complexes can be realized as $A_{n+1}^{eq}/I_c$ where $c$ is the coloring $c : \{a_1, \ldots, a_n\} \rightarrow \{1\}$ on $A_{n+1}^{eq}$.

For each color $s \in S$, let $Q(s) = (Q(s)_0, Q(s)_1)$ be the subquiver of $Q$ consisting of the arrows of color $s$ and vertices to which they are incident. Suppose that $c^{-1}(s)$ has $n(s)$ arrows. Notice, then, that $Q(s) = A_{n(s)+1}^{eq}$, and let $I(s)$ be the ideal generated by all length-two paths in $Q(s)$. If $\beta$ is a dimension vector for $Q$, let $\beta|_s$ be the restriction of $\beta$ to $Q(s)$.

**Proposition 3.1.2.** Let $c$ be a coloring of $Q$, and $\beta$ be a dimension vector. There is an isomorphism of affine varieties

$$\prod_{s \in S} \text{Com}_{n(s)+1}(\beta|_s) \cong \text{Rep}_{Q,c}(\beta)$$

**Proof.** This isomorphism is best viewed as a composition. First, it is clear by definition that

$$\prod_{s \in S} \text{Com}_{n(s)+1}(\beta|_s) \cong \prod_{s \in S} \text{Rep}_{Q(s),I(s)}(\beta|_s)$$

by remark 3.1.1. Now define the map $\prod_{s \in S} \text{Rep}_{Q(s),I(s)}(\beta|_s) \rightarrow \text{Rep}_{Q,c}(\beta)$ in the following way. Since every arrow $a$ is in a unique $Q(s)_1$ (namely $s = c(a)$), an element of $\prod_{s \in S} \text{Rep}_{Q(s),I(s)}(\beta|_s)$ consists of a linear map $V_a$ for each $a \in Q_1$ such that if $c(a) = c(b) = s$
and $h(a) = t(b)$, then $V_b \cdot V_a = 0$. This is precisely the data of a representation of $(Q, I_c)$. □

This isomorphism allows a description of the irreducible components by generalizations of rank maps. Namely, with the notation as above, a rank map $r : Q_1 \to \mathbb{N}$ is a function such that for each $s \in S$, $r|_s : Q(s)_1 \to \mathbb{N}$ is a rank map for the $\beta|_s$ (on the variety $\text{Com}_{n(s)+1}(\beta|_s)$). Let $\leq$ be the partial ordering on rank maps with $r \leq r'$ if $r|_s \leq r'|_s$ for each $s \in S$.

For a given rank map $r$, we have the following sets in $\text{Rep}_{Q,c}(\beta)$:

\[
\text{Rep}_{Q,c}(\beta, r) = \{ V \in \text{Rep}_{Q,c}(\beta) \mid \text{rank}_k V_a \leq r(a) \text{ for each } a \in Q_1 \}
\]

\[
\text{Rep}^o_{Q,c}(\beta, r) = \{ V \in \text{Rep}_{Q,c}(\beta) \mid \text{rank}_k V_a = r(a) \text{ for each } a \in Q_1 \}
\]

We finish this section by pointing out two important facts about the representations spaces of quivers with colored relations. The first important fact gives a parameterization of irreducible components, and the second is a note about the geometry of these irreducible components. Both facts rely on the fact that these spaces are viewed as products of varieties of complexes.

**Proposition 3.1.3.** The irreducible components of the variety $\text{Rep}_{Q,c}(\beta)$ are the sets $\text{Rep}_{Q,c}(\beta, r)$ where $r$ is maximal under the aforementioned partial ordering. Furthermore, $\text{Rep}^o_{Q,c}(\beta, r)$ is an open (therefore dense) subset of $\text{Rep}_{Q,c}(\beta, r)$.

**Proposition 3.1.4.** For each dimension vector $\beta$ and each maximal rank sequence $r$, the variety $\text{Rep}_{Q,c}(\beta, r)$ is normal and Cohen-Macaulay with rational singularities.

### 3.1.1 Coordinate rings for algebras admitting a coloring

In section we exploit the filtration on the coordinate ring of the varieties of complexes (section 2.6.1) to construct a filtration on the coordinate ring of $\text{Rep}_{Q,c}(\beta, r)$. Denote by
\( \mathcal{X} \subset Q_0 \times S \) the set of pairs \((x, s)\) consisting of a vertex and a color incident to that vertex. For each such pair, we denote by \(i(x, s)\) (resp. \(o(x, s)\)) the arrow of color \(s\) whose head (resp. tail) is \(x\). If no such arrow exists, write \(\emptyset\). A vertex will be called lonely if there is exactly one element \((x, s) \in \mathcal{X}\), and coupled if there is more than one.

**Definition 3.1.5.** Let \(\Lambda(Q, c, \beta, r)\) be the set of functions \(\lambda : Q_1 \to \mathcal{P}\) (where \(\mathcal{P}\) is the set of Young diagrams or partitions) such that \(\lambda(a)\) has at most \(r(a)\) non-zero parts. We write \(\Lambda\) when the parameters are understood. If \(p(s) = p_{m_s}^s \ldots p_1^s\) is the full path of color \(s\), then write \(\lambda^s\) for the sequence of diagrams \(\lambda^s = (\lambda(p_1^s), \ldots, \lambda(p_{m_s}^s))\).

We can now extend the map \(\hat{\delta}\) from section 2.6.1. Suppose \(\lambda \in \Lambda(Q, c, \beta, r)\). Define by

\[
\hat{\delta}_\lambda : \bigotimes_{s \in S} \left( \bigotimes_{i=1}^{m_s} \lambda(p_i^s) \bigotimes [\beta_{\lambda(p_i^s)} - \lambda(p_i)] \right) \to k[\text{Rep}_{Q,c}(\beta, r)]
\]

which is simply the product of the maps \(\hat{\delta}_{\lambda^s}\). It will be convenient to denote the domain of this map by \(\bigwedge^\lambda V\), and let \(V_x = k^{eta_x}\). Extending the partial ordering from 2.6.5, for \(\lambda, \mu \in \Lambda(Q, c, \beta, r)\), \(\lambda \leq \mu\) if and only if \(\lambda^s \leq \mu^s\) for each \(s \in S\). Finally, denote by

\[
\mathcal{F}_\lambda = \sum_{\mu \preceq \lambda} \text{im}(\hat{\delta}_\mu) \quad \text{and} \quad \mathcal{F}_{<\lambda} = \sum_{\mu < \lambda} \text{im}(\hat{\delta}_\mu)
\]

**Proposition 3.1.6.** Let \(r\) be a maximal rank map for \(\beta\). Then \(\{\mathcal{F}_\lambda \mid \lambda \in \Lambda(Q, c, \beta, r)\}\) is a filtration of \(\text{Rep}_{Q,c}(\beta, r)\) relative to the partial order just described. Furthermore,

\[
\mathcal{F}_\lambda / \sum_{\mu < \lambda} \mathcal{F}_\mu \cong \bigotimes_{(x, s) \in \mathcal{X}} S_{\lambda(x, s)} V_x
\]

where \(\lambda(x, s) = (\lambda(o(x, s)), -\lambda(i(x, s)))\).
Corollary 3.1.7. For \(Q, c, \beta, r\) as above,

\[
gr_{\Lambda, \leq}(k[\text{Rep}_{Q,c}(\beta, r)]) \cong \bigoplus_{\lambda \in \Lambda} \bigotimes_{(x,s) \in X} S_{\lambda(x,s)} V_x
\]  

As a result of the above remarks, we can give a basis for \(k[\text{Rep}_{Q,c}(\beta, r)]\) via standard multitableaux by generalizing the procedure described by DeConcini and Strickland in the case of the varieties of complexes. The following definitions will introduce the notation necessary for this generalization.

Definition 3.1.8. Let \(\lambda \in \Lambda(Q, c, \beta, r)\). For each element \((x, s) \in X\), denote by \([\lambda_{x,s}]\) the partition with \([\lambda_{x,s}]_j = \lambda(o(x, s))_j + (\beta_x - \lambda(i(x, s)))_j\). This can be viewed as adjoining the partitions \((\beta_x - \lambda(i(x, s)))\) and \(\lambda(o(x, s))\) left-to-right.

This is the natural generalization of the notation \([\lambda(i) : \lambda(i - 1)]\) in section 2.6.1, so we expect to build a basis from fillings of these diagrams.

Definition 3.1.9. Let \(\lambda \in \Lambda(Q, c, \beta, r)\). A multitableau of shape \(\lambda\) is a column-strictly-increasing filling of each of the diagrams \([\lambda_{x,s}]\) for \((x, s) \in X\). A multitableau is called standard if each filling of each diagram is a standard filling. The content \(\kappa\) of a filling of \(\lambda\) is the collection of vectors \(\kappa_{x,s} \in \mathbb{N}^{\beta_x}\) with \((\kappa_{x,s})_j = \#\{\text{occurrences of } j \text{ in the filling of } [\lambda_{x,s}]\}\).

Using the same conventions as in section 2.6.1, we can see that \(\bigwedge^\lambda V\) has basis given by multitableaux of shape \(\lambda\). In the subsequent section, we will determine explicit elements of \(\bigwedge^\lambda V\) whose image under \(\hat{\delta}_\lambda\) is a semi-invariant function.

3.1.2 Application to gentle string algebras

Of particular interest herein are the colorings that give rise to string algebras. It turns out that the triangular string algebras that admit colorings are precisely the gentle string alge-
bras. Furthermore, all triangular gentle string algebras admit a coloring (see the following proposition).

**Proposition 3.1.10.** Suppose that \( kQ/I \) is a triangular gentle string algebra. Then there is a coloring \( c \) of \( Q \) such that \( I_c = I \).

**Proof.** Let \( S \) be a set of arrows \( a \in Q_1 \) with the property that there is no \( b \in Q_1 \) with \( h(b) = t(a) \) and \( ab \in I \) (since \( Q \) has no cycles, so every arrow incident to a sink is in \( S \)). To avoid confusion, let us denote by \( s_a \in S \) the element corresponding to such an \( a \in Q_1 \). For each element \( s_a \in S \), let \( p(a) = p_{l(a)}(a) \ldots p_1(a) \) be the longest path with \( p_1(a) = a \) and \( p_{i+1}(a)p_i(a) \in I \). Notice first that the length is bounded since \( Q \) is acyclic. Additionally, this path is unique and well-defined since for each arrow \( p_i(a) \) there is at most one arrow \( p_{i+1}(a) \) such that \( p_{i+1}(a)p_i(a) \in I \). Take \( c : Q_1 \to S \) to be the map with \( c(p_i(a)) = s_a \) for each \( i = 1, \ldots, l(a) \). By definition of the gentle string algebras, for each \( b \) there is at most one arrow \( a \) with \( ha = tb \) and \( ba \notin I \). Therefore, since \( I \) is generated by paths of length 2, so \( I_c = I \). \( \square \)

Now in a gentle string algebra, by definition, there are at most two colors incident to each color. Therefore, we can interpret the corollary 3.1.7 in the following way.

**Corollary 3.1.11.** Let \( kQ/I_c \) be a gentle string algebra, \( \beta \) a dimension vector, and \( r \) a maximal rank map. If \( x \) is a coupled vertex, let \( (x,s_1(x)) \) and \( (x,s_2(x)) \) be the two elements in \( \mathcal{X} \) with first coordinate \( x \). Then

\[
\text{gr}_{\Lambda(Q,c,\beta,r)}(k[\text{Rep}_{Q,c}(\beta, r)]) \cong \bigoplus_{\lambda \in \Lambda(Q,c,\beta,r)} S_{\lambda(x,s)}V_x \otimes S_{\lambda(x,s_1(x))}V_x \otimes S_{\lambda(x,s_2(x))}V_x
\]

(3.1.3)

In particular, at each vertex is the tensor product of at most two Schur modules. This restriction allows for a combinatorial description of the rings of semi-invariants.
3.2 Semi-Invariant Functions in $k[\text{Rep}_{Q,c}(\beta, r)]$

Fix a gentle string algebra $kQ/I_c$, a dimension vector $\beta$, and a maximal rank sequence $r$ for $\beta$. We denote by $M_\lambda$ the term $\mathcal{F}_\lambda/\mathcal{F}_{<\lambda}$ for $\lambda \in \Lambda$. With this notation, we may write $\text{gr}(k[\text{Rep}_{Q,c}(\beta, r)]) \cong \bigoplus_{\lambda \in \Lambda} M_\lambda$. We are interested in the ring $\text{SI}_{Q,c}(\beta, r) := k[\text{Rep}_{Q,c}(\beta, r)]^{\text{SL}_Q(\beta)}$. In the forthcoming, we will show that $\text{SI}_{Q,c}(\beta, r)$ is isomorphic to a semigroup ring. We do so by defining a basis $\{m_\lambda\}_{\lambda \in \Lambda' \subset \Lambda}$ for $\text{SI}_{Q,c}(\beta, r)$ and then exhibiting the multiplication on said basis.

**Definition 3.2.1.** Let $\Lambda_{SI}(Q, c, \beta, r)$ be the set of elements $\lambda$ in $\Lambda(Q, c, \beta, r)$ such that $M_\lambda$ contains a semi-invariant for $\text{GL}_Q(\beta)$. As usual, we write $\Lambda_{SI}$ if the parameters are understood.

**Proposition 3.2.2.** Let $\lambda \in \Lambda$. Then $\lambda \in \Lambda_{SI}$ if and only if there is a vector $\sigma(\lambda) \in \mathbb{Z}_{Q_0}$ such that for each $x \in Q_0$, we have

$$\lambda(x, s_1)_i + \lambda(x, s_2)_{\beta_x+1-i} = \sigma(\lambda)_x \quad i = 1, \ldots, \beta_x, \tag{3.2.1}$$

(here if $x$ is a lonely vertex, then the second summand is suppressed, i.e., $\lambda(x, s)_i = \sigma(\lambda)_x$ for $i = 1, \ldots, \beta_x$). Furthermore, if $\lambda \in \Lambda_{SI}$, then the space of semi-invariants in $M_\lambda$ is one-dimensional.

**Proof.** The decomposition of the tensor product of two Schur modules is given by the Littlewood-Richardson rule (see [41] proposition 2.3.1). Applying this to equation 3.1.3, we see that there is an $\text{SL}_Q(\beta)$-invariant (meaning that the Schur module appearing as a factor at $x$ is a height-$\beta_x$ rectangle for each $x$) if and only if the system of equations in the proposition hold. \qed

**Corollary 3.2.3.** $\Lambda_{SI}$ is a semigroup under the $+$ operation as defined on partitions in equation 2.6.1.
Proof. Indeed, if $\sigma(\lambda)$ and $\sigma(\mu)$ are the vectors in $\mathbb{Z}^{Q_0}$ satisfying proposition 3.2.2 for the sequences $\lambda, \mu \in \Lambda$, then $\sigma(\lambda) + \sigma(\mu)$ is the vector satisfying the proposition for the sequence $\lambda + \mu$. \hfill $\Box$

Remark 3.2.1. Recall the definition of $[\lambda_{x,s}]$ in 3.1.8. We will collect some useful points:

a. this notation allows us to rewrite the domain of the map 3.1.1 in the form

$$\bigotimes_{(x,s) \in \mathcal{X}} [\lambda_{x,s}] \bigwedge V_x$$

b. Using this notation, we can restate proposition 3.2.2, namely that $\lambda \in \Lambda_{SI}$ if and only if there is a vector $\sigma(\lambda) \in \mathbb{Z}^{Q_0}$ such that

i. For each lonely element $(x,s) \in \mathcal{X}$ (i.e., with no other color passing through $x$),

$$[\lambda_{x,s}]'_i = \beta_x, \ i = 1, \ldots, \sigma(\lambda)_x$$

ii. For each coupled pair $(x,s_1), (x,s_2) \in \mathcal{X}$, $[\lambda_{x,s_1}]'_i + [\lambda_{x,s_2}]'_i = \beta_x$ for $i = 1, \ldots, \sigma(\lambda)_x$.

This restatement will be useful for defining a map whose image consists of semi-invariants.

Definition 3.2.4. For $\lambda \in \Lambda_{SI}$, define the following maps:

i. If $(x,s) \in \mathcal{X}$ is a lonely pair, then let

$$\Delta^\lambda_{x} : \bigwedge V_x \to \bigwedge V_x$$

be the identity map for $i = 1, \ldots, \sigma(\lambda)_x$ (since, by the above remark, $\beta_x = [\lambda_{x,s}]'_i$);

ii. If there is a coupled pair $(x,s_1), (x,s_2) \in \mathcal{X}$, then take

$$\Delta^\lambda_{x} : \bigwedge V_x \to \bigwedge V_x \otimes \bigwedge V_x$$
to be the diagonalization map (since, by the above remark, the sum of the two powers
is precisely $\beta_x$).

We collect these maps into the map $\Delta^\lambda$ in the following way:

$$\Delta^\lambda := \bigotimes_{x \in Q_0} \bigotimes_{i=1}^{\sigma(\lambda)_x} \Delta^\lambda_{x,i} : \bigotimes_{x \in Q_0} \left( \bigwedge^{\beta_x} V_x \right)^{\sigma(\lambda)_x} \rightarrow \bigotimes_{(x,s) \in \mathcal{X}} \left[\lambda_{x,s}\right] \bigwedge V_x.$$  (3.2.2)

Notice that $\Delta^\lambda$ is a $\text{GL}(\beta)$-equivariant map, since both identity and diagonalization
are such. Fixing a basis for each space $V_x$, and let $e$ be the corresponding basis element of
$\bigotimes_{x \in Q_0} \left( \bigwedge^{\beta_x} V_x \right)^{\sigma(\lambda)_x}$ (note that this space is one-dimensional).

**Definition 3.2.5.** Denote by

$$m_\lambda = \hat{\delta}_\lambda \Delta^\lambda(e).$$

This is unique up to scalar multiple.

**Proposition 3.2.6.** For $\lambda \in \Lambda_{SI}$, the function $m_\lambda$ is a semi-invariant of weight $\sigma(\lambda)$.
Furthermore, $\overline{m}_\lambda \neq 0 \in \mathcal{F}_\lambda / \mathcal{F}_{\prec \lambda}$.

The first statement is evident since both $\hat{\delta}_\lambda$ and $\Delta^\lambda$ are $\text{GL}(\beta)$-equivariant homomorphisms, and the weight is clear from the action on the domain of the map. We delay
the proof of the second statement for a brief description of the straightening relations in
$k[\text{Rep}_{Q,c}(\beta, r)]$ relative to fillings of Young diagrams, since the description of $m_\lambda$ is not
given in terms of *standard* multitableaux. We will come back to this proof when we can
show that there is a *standard* multitableau of shape $\lambda$ whose coefficient is non-zero in $m_\lambda$.
The following is simply a generalization of the material in section 2.6.1. We record these
statements as corollaries to DeConcini and Strickland.

**Corollary 3.2.7 ([11]).** If $t_\lambda$ is a filling of $\lambda$, then

$$\hat{\delta}_\lambda(t_\lambda) = s(t_\lambda) + y(t_\lambda)$$
where \( y(t_\lambda) \in F_{<\lambda} \) and \( s(t_\lambda) \) is a linear combination of standard fillings of the same content as \( t_\lambda \).

**Corollary 3.2.8** ([11]). If \( t_\lambda \) and \( t_\mu \) are fillings of shape \( \lambda, \mu \), then

\[
\hat{\delta}_\lambda(t_\lambda) \cdot \hat{\delta}_\mu(t_\mu) \in F_{\lambda+\mu}.
\]

**proof of proposition 3.2.6.** It remains to be shown that \( \overline{m}_\lambda \neq 0 \) in \( F_\lambda / F_{<\lambda} \). For a filling \( t_\lambda \) of \( \lambda \), let \( I(t_\lambda)_{x,s,i} \) be the set of entries in the \( i \)-th column of \( [\lambda_{x,s}] \). Notice that \( \Delta^\lambda(\xi) \) is the sum of all fillings \( t_\lambda \) of \( \lambda \) satisfying the property that \( I(t_\lambda)_{x,s,i} \cup I(t_\lambda)_{x,s',\sigma(\lambda)_x-i+1} = \{1, \ldots, \beta_x\} \), call this property \((\ast)\). Pick one distinguished element from each coupled pair \((x, s), (x, s') \in \mathfrak{X}\). Consider the filling \( t_\lambda^0 \) of \( \lambda \) with \( I(t_\lambda^0)_{x,s,i} = \{1, 2, \ldots, [\lambda_{x,s}]_i^r\} \) whenever \((x, s)\) is the distinguished element in the coupled pair and \( I(t_\lambda^0)_{x,s',i} = \{\beta_x, \beta_x - 1, \ldots, [\lambda_{x,s'}]_{i+1}^r\} \). This filling satisfies the property \((\ast)\) above so it appears with non-zero coefficient (namely 1) in \( m_\lambda \). Notice that this filling is standard. We will show that the content of this filling is unique among fillings appearing with non-zero coefficient in \( \Delta^\lambda(\xi) \), so after straightening the other fillings, this distinguished filling cannot be canceled. Indeed, the content of this filling is \( \kappa(t_\lambda^0)_{x,s} = ([\lambda_{x,s}]_{1}, [\lambda_{x,s}]_{2}, \ldots) \) if \((x, s)\) is the distinguished pair, and \( (\kappa(t_\lambda^0)_{x,s})_{\beta_x-j+1} = [\lambda_{x,s}]_j \) otherwise. This content uniquely determines the filling \( t_\lambda^0 \), so indeed \( \hat{\delta}_\lambda(t_\lambda^0) \) appears with non-zero coefficient in \( \overline{m}_\lambda \).

**Theorem 3.2.9.** The ring of semi-invariants \( SL(Q,c,\beta,r) \) is isomorphic to the semigroup ring \( k[\Lambda_{SI}(Q,c,\beta,r)] \).

**Proof.** We have already shown that there is a (vector space) homomorphism

\[
m : k[\Lambda_{SI}(Q,c,\beta,r)] \rightarrow SL(Q,c,\beta,r)
\]

where \( m(\lambda) = m_\lambda \).
Claim 1: $m$ is injective.

Suppose that $y = m(\sum_{\lambda \in T} a_{\lambda} \lambda) = \sum_{\lambda \in T} a_{\lambda} m_{\lambda} = 0 \in \text{SI}_{Q,c}(\beta, r)$, where $T$ is a finite subset of $\Lambda_{SI}$. Let $\max(T)$ be the set of maximal elements in $T$ under the partial order $\preceq$ defined on $\Lambda$. Then $y \in \sum_{\lambda \in \max(T)} F_{\lambda}$ Now for each $\mu \in \max(T)$ there is a surjection

$$\varphi_{\mu} : \sum_{\lambda \in \max(T)} F_{\lambda} \rightarrow M_{\mu}$$

given by the quotient of this space by the subspace $F_{\prec \mu} + \sum_{\lambda \in \max(T) \setminus \mu} F_{\lambda}$. Given that $y$ is a semi-invariant, its image under this map is $a_{\mu} m_{\mu}$, since the space of semi-invariants in $M_{\lambda}$ is one dimensional. By assumption, this is 0, and since $m_{\mu} \neq 0$, we must have that $a_{\mu} = 0$ for all $\mu \in \max(T)$, contradicting the choice of $\max(T)$.

Claim 2: The map $m$ is surjective.

This fact exploits the same methods as the previous claim: we show that the maximal $\lambda$ appearing in a semi-invariant must be elements of $\Lambda_{SI}$, and subtract the corresponding semi-invariant $m_{\lambda}$ and are left with a semi-invariant function with smaller terms. Suppose that $y \in \text{SI}_{Q,c}(\beta, r)$, and write $y = \sum_{\lambda \in T} a_{\lambda} x_{\lambda}$ where $T \subset \Lambda$ is a finite subset (recall that $k[\text{Rep}_{Q,c}(\beta, r)]$ has a basis given by standard fillings of all $\lambda \in \Lambda$, and take $x_{\lambda}$ to be the summands corresponding to $\lambda$). Let $\max(T)$ again be the maximal elements in $T$ under the partial order $\preceq$. Notice that the collection of empty partitions is indeed an element of $\Lambda_{SI}$, so we will proceed by induction on $\text{height}(T)$ defined to be the length of the longest chain joining both the empty partition and an element of $\max(T)$. For $\text{height}(T) = 0$, $m$ is a constant, which is the image of the same constant under the map $m$. For $\mu \in \max(T)$, notice that $\varphi_{\mu}(y) = a_{\mu} m_{\mu}$ must be a semi-invariant in $\text{gr}_{\lambda, \preceq}(\text{Rep}_{Q,c}(\beta, r))$, so $\mu \in \Lambda_{SI}$. Therefore, for each $\mu \in \max(T)$, $a_{\mu} m_{\mu} = b_{\mu} m_{\mu}$. In particular, $a_{\mu} x_{\mu} - b_{\mu} m_{\mu} \in F_{\prec \mu}$. Now let

$$y_1 = y - \sum_{\mu \in \max(T)} b_{\mu} m_{\mu}.$$
By the above remarks, then, \( y_1 = \sum_{\lambda \in T_1} a'_\lambda x_\lambda \) where \( T_1 = \{ \lambda \prec \max(T) \} \). As the difference of semi-invariants, \( y_1 \) is itself a semi-invariant, and \( \text{height}(T_1) < \text{height}(T) \). By induction, then, \( y_1 = \sum_{\lambda \in \Lambda_{SI}} b_\lambda m_\lambda \), so

\[
y = \left( \sum_{\mu \in \max(T)} b_\mu m_\mu \right) + \left( \sum_{\lambda \in T_1} b_\lambda m_\lambda \right).
\]

**Claim 3:** \( m \) is a semigroup homomorphism.

This is proven directly. It has already been shown that \( m_\lambda \cdot m_\mu \in F_{\lambda + \mu} \). Now \( \Delta^\lambda(\epsilon) \) is a linear combination of all multitableau of shape \( \lambda \) such that \( I(t_\lambda)_{x,s,i} \cup I(t_\lambda)_{x,s',\sigma(\lambda)_{x-i+1}} = \{1, \ldots, \beta_x\} \). The coefficient of each multitableau is the sign of the permutation taking the sequence \( (I(t_\lambda)_{x,s,i}, I(t_\lambda)_{x,s',\sigma(\lambda)_{x-i+1}}) \) into increasing order. Now consider \( \Delta^{\lambda+\mu}(\epsilon) \). We will simply show a bijection between pairs \( t_\lambda, t_\mu \), summands in \( \Delta^\lambda(\epsilon) \) and \( \Delta^\mu(\epsilon) \), respectively, and summands in \( \Delta^{\lambda+\mu}(\epsilon) \), and show that the signs agree. To this end, consider \( [\lambda + \mu]_{x,s} \). Recall that this is the shape given by adjoining \( (\beta_x - (\lambda + \mu)(i(x,s))) \) and \( (\lambda + \mu)(o(x,s)) \). Notice that by definition of \( (\lambda + \mu)(o(x,s)) \), we can choose indices \( 1 \leq i_1 < i_2 < \ldots < i_{\lambda(o(x,s))_1} \leq (\lambda + \mu)(o(x,s))_1 \) such that

\[
((\lambda + \mu)(o(x,s))'_{i_1}, (\lambda + \mu)(o(x,s))'_{i_2}, \ldots, (\lambda + \mu)(o(x,s))'_{i_{\lambda(o(x,s))_1}}) = (\lambda(o(x,s))_{1}, \ldots, \lambda(o(x,s))_{\lambda(o(x,s))_1}).
\]

This is easiest to see in a picture:

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+-----------------------
|                       |
|                       |
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In fact, the entire shape \( [\lambda + \mu]_{x,s} \) can be partitioned into columns in such a way that
the gray columns constitute \([\lambda_{x,s}]\) and those in white constitute \([\mu_{x,s}]\). Now for each distinguished pair \((x, s) \in X\), choose such a partition of the columns, and partition the columns of the other shapes \([\lambda_{x,s'}]\) accordingly, namely if the column \(i\) of \([(\lambda + \mu)_{x,s}]\) is colored gray, then the \(\sigma(\lambda + \mu) - i + 1\) column of \([(\lambda + \mu)_{x,s'}]\) is colored gray as well. Fixing this partition of the columns, we have that a multitableau of shape \((\lambda + \mu)\) gives rise uniquely to a multitableau of shape \(\lambda\) (given by gray columns), and a multitableau of shape \(\mu\), and every pair of multitableau of shapes \(\lambda\) and \(\mu\) determine a filling of \((\lambda + \mu)\) by the same partitioning of the columns. So indeed \(\Delta^{\lambda+\mu}(e)\) consists of a linear combinations of all products of pairs of multitableau of shapes \(\lambda\) and \(\mu\). Furthermore, since the sign is calculated by taking the product of the signs given by reordering columns, it is evident that the sign of the product agrees with the sign in \(\Delta^{\lambda+\mu}(e)\). \(\square\)

### 3.3 Combinatorics: The Semigroup \(\Lambda_{SI}(Q, c, \beta, r)\)

In this section, we determine the structure of the semigroup \(\Lambda_{SI}\). As we have shown above,

\[
\text{SI}_{Q,c}(\beta, r) \cong k[\Lambda_{SI}(Q, c, \beta, r)].
\]

We will exhibit a grading on \(k[\Lambda_{SI}]\), and show that \(k[\Lambda_{SI}]\) is a polynomial ring over a sub-semigroup ring which we denote by \(k[U]\). For this section, we fix a quiver \(Q\), a coloring \(c\), a dimension vector \(\beta\), and a maximal rank sequence \(r\). For ease of presentation we will write \(\Lambda = \Lambda(Q, c, \beta, r)\) and \(\Lambda_{SI}\) similarly.

**Definition 3.3.1.** Let \(\{\alpha_i^x \mid x \in Q_0 \ i = 1, \ldots, \beta_x - 1\}\) be the simple roots for the group \(\text{SL}_Q(\beta)\). I.e., for \(\lambda \in \Lambda\), \(\alpha_i^x(\lambda(x, s)) := \lambda(x, s)_i - \lambda(x, s)_{i+1}\)

**Proposition 3.3.2.** The element \(\lambda \in \Lambda_{SI}\) if and only if both of the following hold:
• For every coupled vertex $x$ with $(x, s_1), (x, s_2) \in \mathcal{X}$, and every $i = 1, \ldots, \beta - 1$,

\[
\alpha_i^x(\lambda(x, s_1)) = \alpha_{\beta-1-i}^x(\lambda(x, s_2));
\]

• For every lonely vertex $x$, say $(x, s) \in \mathcal{X}$,

\[
\alpha_i^x(\lambda(x, s)) = 0.
\]

Proof. Indeed, the equality in the proposition holds if and only if

\[
\lambda(x, s_1(x))_i - \lambda(x, s_1(x))_{i+1} = \lambda(x, s_2(x))_{\beta-1-i} - \lambda(x, s_2(x))_{\beta-1-i+1}
\]
\[
\Leftrightarrow \lambda(x, s_1(x))_i + \lambda(x, s_2(x))_{\beta-1-i} = \lambda(x, s_1(x))_{i+1} + \lambda(x, s_2(x))_{\beta-1-i+1}
\]
\[
\Leftrightarrow \lambda(x, s_1(x))_i + \lambda(x, s_2(x))_{\beta-1-i} = \lambda(x, s_1(x))_j + \lambda(x, s_2(x))_{\beta-1-j} := \sigma_x
\]

This is precisely the set of conditions given by proposition 3.2.2.

To organize the equations that arise from proposition 3.3.2, we will set up some notation and define a graph whose vertices are simple roots, with multiplicity.

**Definition 3.3.3.**

a. Denote by $\Sigma = \Sigma(Q, c, \beta)$ the set of labeled simple roots $\{\alpha_i^{(x, s)} \mid (x, s) \in \mathcal{X}, \ i = 1, \ldots, \beta - 1\}$ (namely the simple roots from above but with multiplicity for the colors included).

b. For each $\lambda \in \Lambda$, define the function $f_\lambda : \Sigma \to \mathbb{N}$ by

\[
f_\lambda(\alpha_i^{(x, s)}) := \alpha_i^{(x, s)}(\lambda(x, s)).
\]

c. Define the partition equivalence graph, written $\text{PEG}(Q, c, \beta, r)$ to be the graph with
vertices given by the set $\Sigma$ and the following edges:

i. for each coupled vertex $x \in Q_0$, with associated pair $(x, s_1), (x, s_2) \in \mathcal{X}$ say, and each $i = 1, \ldots, \beta_x - 1$, define an edge $\alpha_i^{(x, s_1)} \xrightarrow{} \alpha_{\beta_x - i}^{(x, s_2)}$.

ii. for each arrow $a : x \rightarrow y$, and each $i = 1, \ldots, r(a) - 1$, define an edge $\alpha_i^{(x, s)} \xrightarrow{} \alpha_{\beta_y - i}^{(y, s)}$.

In words, edges of the first type connect labeled simple roots arising from the same $\text{SL}(\beta_x)$, i.e., from the same vertex, and edges of the second type connect simple roots along colors. For this reason we may call edges of the second type colored edges.

**Proposition 3.3.4.** Let $\lambda \in \Lambda$. Then $\lambda \in \Lambda_{SI}$ if and only if $f_\lambda(\alpha) = f_\lambda(\alpha')$ whenever $\alpha$ and $\alpha'$ are in the same connected component of the PEG and $f_\lambda(\alpha) = 0$ if $\alpha$ corresponds to a root at a lonely vertex.

**Proof.** Let $a \in Q_1$ be an arrow of color $s$ with $ta = x$ and $ha = y$. Then $\lambda \in \Lambda$ implies that $f_\lambda \left( \alpha_i^{(x, s)} \right) = f_\lambda \left( \alpha_{\beta_y - i}^{(y, s)} \right)$, i.e., $f_\lambda(\alpha) = f_\lambda(\alpha')$ whenever $\alpha, \alpha'$ are connected by a colored edge. This is so because if $\lambda \in \Lambda$, then

$$f_\lambda \left( \alpha_i^{(x, s)} \right) = \lambda(x, s)_i - \lambda(x, s)_{i+1}$$

$$= \lambda(a)_i - \lambda(a)_{i+1}$$

$$= (-\lambda(a)_{i+1}) - (-\lambda(a)_i)$$

$$= \lambda(y, s)_{\beta_y - i} - \lambda(y, s)_{\beta_y - i+1}$$

$$= f_\lambda \left( \alpha_{\beta_y - i}^{(y, s)} \right).$$

But proposition 3.3.2 shows that $M_\lambda$ contains a semi-invariant if and only if $f_\lambda(\alpha) = f_\lambda(\alpha')$ whenever $\alpha, \alpha'$ are linked by an edge of type (i). Therefore, $\lambda \in \Lambda_{SI}$ if and only if equality holds for all roots in the same connected component. \qed
Proposition 3.3.5. Let $K_1, \ldots, K_l$ be the list of connected components in $\text{PEG}(Q, c, \beta, r)$, and let $\{\alpha(i)\}_{i=1, \ldots, l}$ be some set of elements in $\Sigma$ such that the vertex corresponding to $\alpha(i)$ is in the component $K_i$ for each $i$. For any vector $g = (g_1, \ldots, g_l) \in \mathbb{N}^l$, let $V_g$ be the vector space with basis $\{m_{\lambda} \mid \lambda \in \Lambda_{SI}, f_{\lambda} \alpha(i) = g_i\}$. Then

$$k[\Lambda_{SI}] = \bigoplus_{g \in \mathbb{N}^l} V_g$$

is a graded direct sum decomposition of the semigroup ring $k[\Lambda_{SI}]$. In other words, $k[\Lambda_{SI}]$ has a multigrading by the connected components of $\text{PEG}(Q, c, \beta, r)$.

This follows immediately from the description of the semigroup structure of $\Lambda_{SI}$ above and proposition 3.3.4.

Definition 3.3.6. Let $E = E_{Q,c}(\beta, r)$ be the set of elements in $\Sigma$ whose corresponding vertices are endpoints for the PEG associated to $(Q, c, \beta, r)$. For an element $e \in E$ which is contained in the string, write $\Theta(e)$ for the distinct second endpoint contained in this string (we do not consider an isolated vertex to be a string). Clearly $\Theta : E \to E$ is an involution.

In fact, we can explicitly describe $E$.

Proposition 3.3.7. Each endpoint of the PEG is of one of the following two mutually exclusive forms:

I. if $x$ is coupled and $(x, s) \in \mathfrak{X}$, then $\alpha^{(x, s)}_i$ is an endpoint for $r(o(x, s)) \leq i \leq \beta_x - r(i(x, s));$

II. if $x$ is lonely and $(x, s) \in \mathfrak{X}$, then $\alpha^{(x, s)}_i$ is an endpoint for $1 \leq i \leq \beta_x$.

Proof. This is a consequence of the definition 3.3.3. We will call the edges that connect roots on the same vertex of the quiver non-colored, and those that connect roots on different
vertices of the quiver colored. If \( x \) is lonely then there can only possibly be colored edges containing any of the elements \( \alpha_i^{(x,s)} \), and by definition, each vertex can be contained in at most one such. If, however, \( x \) is coupled and \((x,s) \in \mathfrak{X}\), then each vertex \( \alpha_i^{(x,s)} \) is incident to precisely one non-colored edge. Those with \( i < r(o(x,s)) \) or \( i > \beta_x - r(i(x,s)) \) are also incident to a colored edge by definition. For \( r(o(x,s)) \leq i \leq \beta_x - r(i(x,s)) \), there are no colored edges incident to \( \alpha_i^{(x,s)} \). \( \square \)

We will use the endpoints of the strings to find a system of equations so that each positive integer-valued solution of the system will correspond to an element \( \lambda \in \Lambda_{SI} \).

**Remark 3.3.1.** Below lists the endpoints in \( \{\alpha_i^{(x,s)}\}_{i=1}^{\beta_x} \) and calculates the values of \( f_\lambda \) on such endpoints. In order to write the system of equations mentioned above in a compact form, we also label these possibilities:

a. If \( r(o(x,s)) + r(i(x,s)) = \beta_x \) for some \((x,s) \in \mathfrak{X}\), then \( \alpha_{r(o(x,s))}^{(x,s)} \) is the unique endpoint in this set. For this endpoint, we have

\[
f_\lambda \left( \alpha_{r(o(x,s))}^{(x,s)} \right) = \lambda(o(x,s))_{r(o(x,s))} + \lambda(i(x,s))_{r(i(x,s))}.
\]

We will denote this endpoint by \((o(x,s),i(x,s))\).

b. If \( r(o(x,s)) + r(i(x,s)) < \beta_x \) for some \((x,s) \in \mathfrak{X}\), then \( \alpha_{r(o(x,s))}^{(x,s)} \) is an endpoint, and

\[
f_\lambda \left( \alpha_{r(o(x,s))}^{(x,s)} \right) = \lambda(o(x,s))_{r(o(x,s))}.
\]

We will denote this endpoint by the arrow \( o(x,s) \).

c. If \( r(o(x,s)) + r(i(x,s)) < \beta_x \) for some \((x,s) \in \mathfrak{X}\), then \( \alpha_{\beta_x - r(i(x,s))}^{(x,s)} \) is an endpoint, and

\[
f_\lambda \left( \alpha_{\beta_x - r(i(x,s))}^{(x,s)} \right) = \lambda(i(x,s))_{r(i(x,s))}.
\]
Such an endpoint will be denoted by the arrow $i(x,s)$.

d. Finally, if $r(o(x,s)) < i < \beta_x - r(i(x,s))$, or $(x,s)$ has no mirror and $i \neq r(o(x,s))$, $i \neq \beta_x - r(i(x,s))$, then

$$f_\lambda(\alpha_i^{(x,s)}) = 0.$$ 

Such endpoints will be denoted by the symbol $0_i^{(x,s)}$.

Thus, an endpoint can be of type Ia, Ib, Ic, Id, or type IIa, IIb, IIc, IId.

**Definition 3.3.8.** For any $\lambda \in \Lambda$, define $u_\lambda : Q_1 \to \mathbb{N}$ to be the function $u_\lambda(a) = \lambda(a)_{r(a)}$. For any function $u : Q_1 \to \mathbb{N}$, let $\varphi_u : E \to \mathbb{N}$ be the function defined as follows:

$$\varphi_u(e) = \begin{cases} 
  u(i(x,s)) + u(o(x,s)) & \text{if } e \text{ is of type (Ia) and labeled } (o(x,s), i(x,s)) \\
  u(o(x,s)) & \text{if } e \text{ is of type (Ib) and labeled } o(x,s) \\
  u(i(x,s)) & \text{if } e \text{ is of type (Ic) and labeled } i(x,s) \\
  0 & \text{if } e \text{ is of type (Id) or (II).}
\end{cases}$$

We call $\varphi_u$ the **companion function** to $u$.

We will denote by $U = U(Q, c, \beta, r)$ the set of functions $u : Q_1 \to \mathbb{N}$ such that $\varphi_u(e) = \varphi_u(\Theta(e))$ for all $e \in E$. Notice that $U$ is a semigroup with respect to the usual addition of functions.

**Proposition 3.3.9.** If $\lambda \in \Lambda_{SI}$ then $u_\lambda \in U(Q, C, \beta, r)$.

**Proof.** This is clear from proposition 3.3.4, together with the fact that if $\lambda \in \Lambda_{SI}$, and $x$ is a lonely vertex, $(x,s) \in \mathcal{X}$, then $f_\lambda(\alpha_i^{(x,s)}) = 0$ for $i = 1, \ldots, \beta_x$. □

Notice that from $u_\lambda$ one can calculate the values of $f_\lambda(\alpha)$ whenever $\alpha \in \Sigma$ is in a string of the PEG.
**Definition 3.3.10.** Denote by \( Y = Y(Q, c, \beta, r) \) the set of maps \( y : \{\text{bands in } \Sigma\} \to \mathbb{N} \).

For any \( u \in U \) and \( y \in Y \), take \( \lambda_{u,y} : Q_1 \to \mathcal{P} \) to be the map defined by the following conditions:

\[
\lambda_{u,y}(a)_{r(a)} = u(a)
\]

\[
\alpha(\lambda_{u,y}) = \begin{cases} 
\varphi_u(e) & \text{if } e \text{ is an endpoint of the string containing } \alpha \\
y(b) & \text{if } \alpha \text{ is contained in the band } b.
\end{cases}
\]

**Remark 3.3.2.** Let us summarize the results above:

i. The set \( U \) is a semigroup with respect to the usual addition of functions,

ii. \( \lambda_{u,y}(a) \) has at most \( r(a) \) non-zero parts, so \( \lambda_{u,y} \in \Lambda \),

iii. \( \text{image}((u, y) \mapsto \lambda_{u,y}) \subseteq \Lambda_{SI} \).

**Proposition 3.3.11.** The map \( (u, y) \mapsto \lambda_{u,y} \) is a semigroup isomorphism between \( U \times Y \) and \( \Lambda_{SI} \).

**Proof.** We construct an inverse explicitly. For any \( \lambda \in \Lambda_{SI} \), define \((u_\lambda, y_\lambda)\) as follows:

\[
u_\lambda(a) := \lambda(a)_{r(a)}
\]

\[
y_\lambda(b) := f_\lambda(\alpha) \text{ for any } \alpha \text{ in the band } b.
\]

It is routine that \( u_{\lambda(u,y)} = u \) and \( y_{\lambda(u,y)} = y \), so this is indeed a bijection, and it is clear that the composition operation in \( U \times Y \) is preserved under this map. \( \square \)

**Corollary 3.3.12.** We have the following ring isomorphism

\[
SI_{Q,c}(\beta, r) \cong k[U(Q, c, \beta, r)][y_b]_{B \in \{\text{bands in } \Sigma\}},
\]

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that is, \( S\Omega_{\mathcal{C}}(\beta, r) \) is a polynomial ring over the semigroup ring \( k[U] \).

**Proposition 3.3.13.** The semigroup \( U(Q, C, \beta, r) \) is a sub-semigroup of \( \mathbb{N}^{Q_1} \), satisfying the following:

a. \( U(Q, C, \beta, r) = \{(u_a)_{a \in Q_1} \in \mathbb{N}^{Q_1} | \varphi_u(e) = \varphi_u(\Theta(e)) \text{ for } e \in E\} \),

b. \( \varphi_u(e) = \sum_{a \in Q_1} c_a^e u_a \) with \( c_a^e \in \{0, 1\} \) for each endpoint \( e \in E \),

c. \( u_a \) appears with nonzero coefficient in at most two functions \( \varphi_u \). I.e., for each \( a \in Q_1 \), there are at most two endpoints \( e_1, e_2 \in E \) with \( c_a^{e_1} = c_a^{e_2} = 1 \).

**Proof.** (a) is simply the definition of \( U(Q, C, \beta, r) \), reformulated as a sub-semigroup of \( \mathbb{N}^{Q_1} \), while (b) is the definition of the control equations. Recall that \( \varphi_u(e) = u(a) + u(b) \), \( u(a) \) or 0 for any endpoint \( e \), and since the quiver is acyclic, \( a \neq b \), so the coefficient on any summand is at most 1. To show (c), we recall that \( r \) is a maximal rank sequence for \( \beta \). This implies that if \( e_1 \) is an endpoint of type (Ia) labeled \( (a, b) \) (in which case \( \varphi_u(e_1) = u_b + u_a \)), then the only other type of endpoint labeled with an \( a \) is either another of type (Ia) labeled \( (c, a) \), or one of type (Ic) labeled \( a \). (Similarly the only other type of endpoint labeled with a \( b \) is either another of type (Ia) labeled \( (b, c) \), or of type (Ib) labeled \( b \)).

### 3.4 Matching Semigroups

Fix \( kQ/I \) a gentle string algebra \( \beta \) a dimension vector vector and rank sequence, together with its PEG\( (Q, c, \beta, r), \Sigma \). We will define a general class of sub-semigroups of \( \mathbb{N}^l \), of which all \( U(Q, c, \beta, r) \) are members. We will then describe a general procedure for determining generators and relations for these semigroup rings by means of a graph, and show that the generators of these semigroups occur in multidegree at most 2. First, however, we exhibit some structure enjoyed by \( k[U] \).
Theorem 3.4.1. The semigroup ring $k[U]$ is the coordinate ring of an affine toric variety.

Proof. Let $k[X_a]_{a \in Q_1}$ be the polynomial ring on the arrows of $Q_1$, and let $\mathfrak{e}$ be the set of strings in $\Sigma$. Suppose that the PEG has the following endpoints: $\{e_1^{(s)}, e_2^{(s)}\}_{s \in \mathfrak{e}}$. Then we define the action of $(k^*)^\mathfrak{e}$ on $k[X_a]_{a \in Q_1}$ as follows: suppose that $(t_s)_{s \in \mathfrak{e}} \in (k^*)^\mathfrak{e}$, then

$$(t_s)_a X_a^{u(a)} := t_s^{\varphi_u(e_1^{(s)}) - \varphi_u(e_2^{(s)})} \prod_{a \in Q_1} X_a^{u(a)}.$$ 

A polynomial $p \in k[X_a]_{a \in Q_1}$ is invariant with respect to this action if and only if its monomial terms are, so it suffices to assume $p$ is a monomial. Suppose that a monomial $\prod_{a \in Q_1} X_a^{u(a)}$ is invariant with respect to each $t_s$. Then for each endpoint pair $\{e_1^{(s)}, e_2^{(s)}\}$, we have

$$t_s \cdot \prod_{a \in Q_1} X_a^{u(a)} = t_s^{\varphi_u(e_1^{(s)}) - \varphi_u(e_2^{(s)})} \prod_{a \in Q_1} X_a^{u(a)} = \prod_{a \in Q_1} X_a^{u(a)},$$

so $\varphi_u(e_1^{(s)}) = \varphi_u(e_2^{(s)})$ for $s \in \mathfrak{e}$. Therefore, such a monomial is invariant with respect to the action if and only if $u \in U$. Then clearly $k[U] = k[X_a]^{(k^*)^\mathfrak{e}}$ is the invariant ring with respect to this torus action. 

Definition 3.4.2. Let $\{f_i : \mathbb{N}^l \to \mathbb{N}\}_{i=1,...,2m}$ be a collection of $\mathbb{N}$-linear functions

$$f_i(x_1, \ldots, x_l) = \sum_{j=1}^l c_i^j x_j$$

satisfying the following properties:

a. $c_i^j \in \{0, 1\}$ for all $i = 1, \ldots, 2m$, $j = 1, \ldots, l$;

b. $c_i^j \neq c_{i+m}^j$ for $i = 1, \ldots, m$, $j = 1, \ldots, l$ (i.e., the equations $f_i(x_1, \ldots, x_l) = f_{i+m}(x_1, \ldots, x_l)$ are reduced);

c. for $j = 1, \ldots, l$, $\# \{i \mid c_i^j \neq 0, i = 1, \ldots, 2m\} \leq 2$ (i.e., each variable $x_j$ appears with non-zero coefficient in at most two functions $f_i$).
The semigroup

\[ U(\{f_i\}_{i=1,\ldots,2m}) := \{u = (u_1, \ldots, u_l) \in \mathbb{N}^l \mid f_i(u) = f_{m+i}(u), \ i = 1, \ldots, m\} \]

is called a matching semigroup if the functions \( f_i \) satisfy the conditions (a)-(c).

The following is the main theorem of this section.

**Theorem 3.4.3.** Suppose that \( U = U(\underline{f}) \subset \mathbb{N}^l \) is a matching semigroup with \( \underline{f} = \{f_i\}_{i=1,\ldots,2m} \). Then \( U \) is generated by vectors \( \underline{u} = (u_1, \ldots, u_l) \) with the property that \( f_i(u) \leq 2 \) for \( i = 1, \ldots, 2m \). In particular, \( u_i \leq 2 \).

In order to prove this theorem, we construct a graph \( G(\underline{f}) \) and interpret certain walks on this graph as elements in \( U \).

**Definition 3.4.4.** Let \( G(\underline{f}) \) be the multigraph with two types of edges, solid and dotted, on the vertices \( \{1, \ldots, 2m\} \), with a solid edge

\[ i \rightarrow k \text{ whenever } c_i^j = c_k^j = 1, \ i \neq k, \]

a solid loop

\[ i \rightarrow i \text{ whenever } i \text{ is the unique integer for which } c_i^j = 1, \]

and dotted edges \( i \rightarrow m + i \) for \( i = 1, \ldots, m \). We define a function \( L : \text{Edges}(G(\underline{f})) \rightarrow \{1, x_1, \ldots, x_l\} \) with

\[ L(E) = \begin{cases} 
1 & \text{if } E \text{ is a dotted edge} \\
x_j & \text{if } E \text{ is the edge containing } i, k \text{ arising from the condition } c_i^j = c_k^j = 1.
\end{cases} \]
In depicting this graph, we will indicate the labeling as a decoration on the appropriate edge. Heuristically, each vertex $i$ stands for a function $f_i$. A vertex $i$ is contained in a solid edge labeled $x_j$ if $x_j$ appears with non-zero coefficient in $f_i$, and the vertices corresponding to functions on either side of a defining equation of $U(f)$ are joined by a dotted edge. The name matching semigroup arises from the fact that the dotted edges form a perfect matching for the graph $G(f)$. Moreover, while each vertex is contained in exactly one dotted edge, it can be contained in several solid edges: as many as non-zero coefficients in the linear function to which it corresponds.

A walk on $G(f)$ is a sequence of vertices and edges $w = v_nE_nv_{n-1}E_{n-1}\ldots E_1v_0$ such that $V(E_i) = \{v_i, v_{i-1}\}$ (i.e., the vertices of $E_i$ are precisely the two surrounding it in the sequence). To each such walk, associate an integer vector $u(w) \in \mathbb{N}^l$ with

$$u(w)_j = \#\{k \mid \text{the edge } E_k \text{ is labeled } x_j\}.$$ 

A walk will be called alternating if $E_k, E_{k-1}$ are of different edge types for $k \in [n]$. Such a walk will be called a string if both $E_1$ and $E_n$ are loops, and a band if $v_0 = v_n, E_0, E_n$ are different edge types, and none of the $E_i$ are loops. Henceforth, we will refer to “alternating” strings and bands simply as strings and bands.

**Lemma 3.4.1.** Suppose that $w$ is a string or band. Then $u(w) \in U$.

**Proof.** Without loss of generality, assume $i \leq m$. Notice that if $w$ is a string, then

$$f_i(u(w)) = \#\{j \in \{1, \ldots, n-1\} \mid v_j \text{ is the vertex } i\},$$

while if $w$ is a band, then

$$f_i(u(w)) = \#\{j \in \{1, \ldots, n\} \mid v_j \text{ is the vertex } i\}.$$
But \(w\) is alternating, so every occurrence of the vertex \(i\) is either immediately preceded or succeeded by an occurrence of the vertex \(i + m\), so
\[
f_i(u(w)) = \# \{ j \mid v_j \text{ is the vertex } i + m \} = f_{i+m}(u(w))
\]
as claimed.

\begin{lemma}
\textbf{Lemma 3.4.2.} \(G(f)\) contains no alternating two-cycles.
\end{lemma}

\begin{proof}
If the edge labeled \(x_1\) contains two vertices \(i, i + m\) which are both contained in a single dotted edge, then
\[
f_i(z) = x_1 + \sum c^i_j x_j = x_1 + \sum c^i_{i+m} x_j = f_{i+m}(z),
\]
contradicting definition 3.4.2 (b).
\end{proof}

\begin{lemma}
\textbf{Lemma 3.4.3.} A matching semigroup \(U = U(f)\) is generated by the set
\[
\{ u(w) \mid w \text{ is either a string or a band on } G(f) \}.
\]
\end{lemma}

\begin{proof}
Let \(\leq\) be the coordinate-wise partial order on \(U\). We will show that for each \(0 \neq u \in U\), there is a non-trivial alternating walk \(w\), which is either a string or a band, and an element \(u' \in U\) such that
\begin{enumerate}
  \item \(u' \leq u\),
  \item \(u = u(w) + u'\).
\end{enumerate}

\textbf{Case 1:} Suppose that \(u_{j_1} \neq 0\) for some \(j_1\) for which \(x_{j_1}\) is a loop. We inductively construct a sequence of alternating walks
\[
t_k = v_{2k} E_{2k} v_{2k-1} \ldots v_1 E_1 v_0
\]
with \(L(E_1) = x_{j_1}\) satisfying the following:
\begin{enumerate}
  \item \(0 < u(t_k) < u(t_{k+1}) < u\)
  \item \(f_{v_{2k-1}}(u - u(t_k)) + 1 = f_{v_{2k}}(u - u(t_k))\)
\end{enumerate}
(3) \( f_i(u - u(t_k)) = f_{i+m}(u - u(t_k)) \) whenever \( \{i, i + m\} \neq \{v_{2k}, v_{2k-1}\} \).

Let \( E_1 \) be the edge with \( L(E_1) = x_{j_1}, v_0 = v_1 \) the unique vertex contained in this loop, \( E_2 \) the dotted edge containing \( v_1 \), and \( v_2 \) the unique second vertex contained in \( E_2 \).

**Claim 1:** \( t_1 \) satisfies (1)-(3).

**Proof.** \( u(t_1)_{j_1} = 1 \), so immediately \( u(t_1) > 0 \). Furthermore, \( u(t_1)_{j'} = 0 \) for \( j' \neq j_1 \), and since \( c_{v_1}^{j_1} = 1, c_{v_2}^{j_1} = 0, f_{v_2}(u(t_1)) = 0 \). On the other hand, \( f_{v_2}(u) = f_{v_1}(u) > 0 \) by assumption, so \( u(t_1) < u \), and (1) is proven.

As for (2) and (3), \( f_{v_1}(u - u(t_1)) = f_{v_1}(u) - 1 = f_{v_2}(u) - 1 = f_{v_2}(u - u(t_1)) - 1 \) since \( u \in U \). Furthermore, if \( \{i, i + m\} \neq \{v_2, v_1\} \), then \( c_i^j = c_{i+m}^j = 0 \) since \( f_{v_1} \) is the unique function in which \( x_j \) appears with non-zero coefficient (as \( E_1 \) is a loop). Therefore, \( f_i(u - u(t_1)) = f_i(u) = f_{i+m}(u) = f_{i+m}(u - u(t_1)) \), proving (3).

**Claim 2:** If \( t_k = v_{2k}E_{2k}v_{2k-1} \ldots v_1E_1v_0 \) satisfies (1)-(3), and there is no \( E_s \) for \( s = 2, \ldots, 2k \) with \( E_s \) a loop, then there are two possibilities:

a. There is a loop \( E_{2k+1} \) containing the vertex \( v_{2k} \) such that the walk \( w := v_{2k}E_{2k+1}t_k \) is an alternating string and \( u(w) \leq u \);

b. There is a solid edge \( E_{2k+1} \) which is not a loop such that \( t_{k+1} = v_{2k+2}E_{2k+2}v_{2k+1}E_{2k+1}t_k \) is an alternating walk satisfying (1)-(3).

Before proving this dichotomy, we note that this proves the following: if \( u \in U \) such that \( u_j \neq 0 \) with \( x_j \) a loop, then there is a an alternating string such that \( u - u(w) \in U \). Indeed, \( u(t_k) < u(t_{k+1}) < u \) by (1), so there must be a \( t_k \) such that \( u(t_k) < u \) and for which there is a loop \( E_{2k+1} \) such that \( w \) as defined in (a) is an alternating string and \( u(w) \leq u \).

**Proof.** Suppose that \( t_k = v_{2k}E_{2k}v_{2k-1} \ldots v_1E_1v_0 \) contains no loops other than \( E_1 \), satisfies
(1)-(3), and does not satisfy (a). By property (2),

\[ f_{v_{2k-1}}(u - u(t_k)) + 1 = f_{v_{2k}}(u - u(t_k)) = \sum_{j: c_{v_{2k}}^j \neq 0} u_j - u(t_k)_j. \]

Since \( f_{v_{2k-1}}(u - u(t_k)) \geq 0 \), there must be a \( j_k \) such that \( u_{j_k} > u(t_k)_{j_k} \) and \( c_{v_{2k}}^j = 1 \). In terms of the graph, then, there is a solid edge \( E_{2k+1} \) (which is not a loop since \( t_k \) does not satisfy (a)) with \( L(E_{2k+1}) = x_{j_k} \) containing the vertex \( v_{2k} \). Let \( v_{2k+1} \) be the distinct second vertex contained in \( E_{2k+1} \), \( E_{2k+2} \) the unique dotted edge containing \( v_{2k+1} \), and \( v_{2k+2} \) the distinct second vertex contained in \( E_{2k+2} \). Let \( t_{k+1} = v_{2k+2} E_{2k+2} v_{2k+1} E_{2k+1} t_k \). We claim that \( t_{k+1} \) satisfies (1)-(3).

Notice that \( u(t_k)_{j_k+1} = u(t_{k+1}) \) (as \( t_{k+1} \) has an additional occurrence of the edge labeled \( x_{j_k} \), and \( u(t_k)_{j'} = u(t_{k+1})_{j'} \) for \( j' \neq j_k \). Therefore, \( 0 < u(t_k) < u(t_{k+1}) \) and \( u(t_{k+1})_{j'} \leq u_{j'} \) for \( j' \neq j_k \). Furthermore, \( u_{j_k} > u(t_k)_{j_k} \) from above, so \( u_{j_k} \geq u(t_{k+1})_{j_k} + 1 = u(t_{k+1})_{j_k} \), so \( u(t_{k+1}) \leq u \). We will show in the course of proving (2) that \( u(t_{k+1}) \notin U \), implying we cannot have equality, so \( u(t_{k+1}) < u \) as claimed.

Recall that since \( v_{2k+1} \) contains the edge labeled \( x_{j_k} \), \( c_{v_{2k+1}}^j = 1 \). By lemma 3.4.2, then, \( c_{v_{2k+2}}^{j_k} = 0 \). Furthermore, \( f_{v_{2k+1}}(u - u(t_k)) = f_{v_{2k+2}}(u - u(t_k)) \) since \( t_k \) satisfies condition (3). Therefore

\[
\begin{align*}
    f_{v_{2k+1}}(u - u(t_{k+1})) &= f_{v_{2k+1}}(u - u(t_k)) - 1 \\
    &= f_{v_{2k+2}}(u - u(t_k)) - 1 \\
    &= f_{v_{2k+2}}(u - u(t_{k+1})) - 1
\end{align*}
\]

proving (2).
Finally, if \( \{i, i + m\} = \{v_{2k-1}, v_{2k}\} \), then

\[
f_{v_{2k-1}}(u - u(t_{k+1})) = f_{v_{2k-1}}(u - u(t_k)) = f_{v_{2k}}(u - u(t_k)) - 1 = f_{v_{2k}}(u - u(t_{k+1})),
\]

while if \( \{i, i + m\} \not\subset \{v_{2k-1}, v_{2k}, v_{2k+1}, v_{2k+2}\} \), then

\[
f_i(u - u(t_{k+1})) = f_i(u - u(t_k)) = f_{i+m}(u - u(t_k)) = f_{i+m}(u - u(t_{k+1})),
\]

proving (3).

Case 2: Now suppose for all \( j \) such that \( x_j \) is a loop, we have that \( u_j = 0 \). Take \( j_1 \) with \( u_{j_1} \neq 0 \) (possible since \( u \neq 0 \)). Let \( v_0, v_1 \) be the vertices (taken in some order) contained in the edge labeled \( x_j \), \( E_1 \) this edge, \( E_2 \) the dotted edge containing \( v_1 \) and \( v_2 \) the other end of this edge. Call this walk \( t_1 \). Notice that \( v_2 \neq v_0 \) by lemma 3.4.2. We can again recursively define alternating walks \( t_k \) starting with \( t_1 \) satisfying the following: if \( v_0 \neq v_{2k} \), then

1. \( 0 < u(t_k) < u(t_{k+1}) \leq u \)
2. \( f_{v_{2k-1}}(u - u(t_k)) + 1 = f_{v_{2k}}(u - u(t_k)) \)
3. \( f_i(u - u(t_k)) = f_{i+m}(u - u(t_k)) \) whenever \( \{i, i + m\} \not\subset \{v_{2k}, v_{2k-1}, v_0\} \).
4. \( t_k \) can be extended to an alternating walk \( t_{k+1} \) which is either an alternating band with \( u(t_{k+1}) \leq u \) or \( t_k \) satisfies (1)-(3).

Thus, completely analogously to Case 1, there must be a \( t_k \) that is a band. As the proof is nearly verbatim of the proof of Case 1, we omit it. Therefore, \( u = \sum u(w_i) \) for \( w_i \) some strings or bands.

Notice that it is possible that \( f_i = 0 \) for some index \( i \leq m \) (say), while \( f_{i+m} = \sum c_{i+m}^j x_j \) with some \( c_{i+m}^j \neq 0 \) for some \( j \). It may not be clear why if \( w \) is alternating string or band, then \( u(w)_j = 0 \), which would be required if \( u(w) \in U \). However, if \( f_i = 0 \), then there are
no solid edges containing the vertex $i$. Any alternating path passing through the solid edge labeled $x_j$ would then pass through the dotted edge between $i + m$ and $i$. Since the walk couldn’t finish at that vertex, it would immediately pass back through the dotted edge, contradicting the alternating property of the walk.

**Definition 3.4.5.** A string or band $w$ is called irreducible if there does not exist a pair of non-trivial strings or bands $w'$, $w''$ satisfying $u(w) = u(w') + u(w'')$.

Clearly $U$ is generated by \{ $u(w) \mid w$ is an irreducible alternating string or band \}.

**Lemma 3.4.4.** If $w$ is an irreducible string or band, then $f_i(u(w)) \leq 2$ for $i = 1, \ldots, 2m$.

**Proof.** Suppose that $w = v_n E_n \ldots E_1 v_0$ is an irreducible string or band, and $f_i(u(w)) \geq 3$ for some $i = 1, \ldots, m$ (in particular, $f_{i+m}(u(w)) \geq 3$). This implies that the vertex $i$ appears in the set $\{ v_1, \ldots, v_{n-1} \}$ at least thrice. Let $E$ be the dotted edge containing the vertices $i$ and $i + m$. Recall that in an alternating path, each occurrence of the vertex $i$ is immediately succeeded or immediately preceded by an occurrence of $i + m$. Let $1 \leq k_1 < k_2 < k_3 \leq n - 1$ be the first three integers such that $v_{k_j} = i$, and $1 \leq l_1 < l_2 < l_3 \leq n - 1$ the first three such that $v_{l_j} = i + m$. Suppose without loss of generality that $k_1 < l_1$. We claim that if $k_2 < l_2$ or $l_3 < k_3$, then $w$ is not irreducible. In this case, $k_2 < l_2$ implies that $w$ contains a sub-band, namely

$$w = \ldots v_{l_2} E (v_{k_2} E_{k_2} \ldots v_{l_1} E v_{k_1}) \ldots$$

In a diagram (although the graph is undirected, the sequence of edges and vertices of the
walk will be indicated with arrows):

(Here the thinner arc connecting the two bottom vertices represents an alternating walk that starts and ends with dotted edges.) This contradicts the assumption of irreducibility, so $k_2 > l_2$, and the same contradiction implies that $k_3 < l_2$, so we have that $k_1 < l_1 < l_2 < k_2 < k_3 < l_3$. But now we have that

$$w = \ldots v_{l_3} (E v_{k_3} E_{k_3} \ldots E_{k_2+1} v_{k_2} E_{l_2} E_{l_2} \ldots E_{l_1+1} v_{l_1}) E v_{k_1} \ldots$$

which contains the parenthesized band. In diagram form:

again contradicting irreducibility of $w$. \qed
proof of theorem 3.4.3

$U\left(f\right)$ is generated by the $u(w)$ for $w$ irreducible strings and bands, and for such walks, $f_i(u(w)) \leq 2$ for $i = 1, \ldots, 2m$ by lemma 3.4.4. This concludes the proof.

□

The presentation of $U\left(f\right)$ using walks on a graph allows us to determine the relations in the ring $k[U\left(f\right)]$ as well. Let $W\left(f\right)$ be the free semigroup generated by the irreducible paths $w_i$ on $G\left(f\right)$, and extend the function $u$ to $W\left(f\right)$ linearly. Let $\sim_W$ be the kernel equivalence of this map, i.e., $A \sim_W B$ if and only if $u(A) = u(B)$. The relation $\sim_W$ is a semigroup congruence, so $W\left(f\right)/\sim_W$ is a semigroup isomorphic to $U\left(f\right)$, and $k[U\left(f\right)]$ is isomorphic to $k[W\left(f\right)]/I_W$ where $I_W$ is generated by all elements $t_w - t_{w'}$ for $w \sim_W w'$.

Remark 3.4.5. Notice that since $\sim_W$ is a semigroup congruence, one has cancellation. That is $a + b \sim_W a + c$ if and only if $b \sim_W c$. This can be recognized immediately from the definition of $\sim_W$.

Definition 3.4.6.

![Diagram](image)

(a) X-Configuration about $E$  
(b) H-Configuration about $E, E'$

Figure 3.1: Relations in Graphical Form

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• A walk $P$ is called a *partial string* if its first edge is a loop and its last edge is solid;

• Suppose that $P_1, Q_1$ are partial strings as in the configuration of figure 3.1a. We will often abbreviate by $Q_1P_1$ the alternating string obtained by joining $Q_1$ and $P_1$ by the edge $E$.

• Suppose that $P_1, X_1$ are alternating walks as in figure 3.1b. Then we write $X_1P_1$ for the alternating band obtained by joining $P_1$ and $X_1$ along the edges $E$ and $E'$.

• Let $\sim_X$ be the minimal semigroup equivalence containing the relations:
  
  i. $Q_1P_1 + Q_2P_2 \sim Q_2P_1 + Q_1P_2$ for every collection $P_1, P_2, Q_1, Q_2$ of partial strings in an $X$-configuration (figure 3.1a) on $G(f)$;
  
  ii. $X_1P_1 + X_2P_2 \sim X_1X_2 + P_1P_2$ for every collection of alternating walks $X_1, X_2, P_1, P_2$, none containing loops, in an $H$-configuration (figure 3.1b) on $G(f)$.

**Remark 3.4.6.** Notice that for a given pair $P, Q$ of partial strings as in figure 3.1a (or a pair of alternating walks $X_1, P_1$ as in figure 3.1b), $QP$ (resp. $X_1P_1$) may not be irreducible even while $Q, P$ (resp. $X_1, P_1$) contain no sub-bands.

**Proposition 3.4.7.** The equivalence relations $\sim_W$ and $\sim_X$ coincide.

**Proof.** Notice that if two elements are equivalent under $\sim_X$, then they are equivalent under $\sim_W$, as can be seen on the relations that generate the semigroup.

The converse is proven by induction. Suppose that $A \sim_W B$ for some $A, B \in W(f)$. We will show that $A \sim_X B$. Notice that the function $u : W(f) \to U(f)$ induces a partial order on $W(f)$ via $A' \preceq A$ if and only if $u(A') \leq u(A)$. Notice that for any $A$, the set \{0 \preceq A' \preceq A\} is finite, so we can induct on $u(A)$.

For $u(A) = 0$, the proposition is clear: $u(A) = 0$ implies $u(B) = 0$, so $A = B = 0$, which are trivially equivalent under $\sim_X$. Now suppose that the implication holds for all
\(A' \prec A\). We can assume, without loss of generality, that \(a_0 \neq 0\) while \(b_0 = 0\), since otherwise cancellation would allow us to express the equivalence under \(\sim_w\) for \(A' \prec A\), which, by induction, would imply equivalence under \(\sim_x\). We state the following lemma and delay the proof in order to show that the proposition follows from it.

**Lemma 3.4.7.** With all of the above assumptions, \(B \sim_x w_0 + B'\) for some \(B' \in W(f)\).

Assuming that the claim holds, then by the first paragraph of the proof, \(B \sim_w w_0 + B'\). By transitivity, then \(A = w_0 + A' \sim_w w_0 + B'\). But \(\sim_w\) is a semigroup congruence, so the aforementioned equivalence holds if and only if \(A' \sim_w B'\). By inductive hypothesis, then, \(A' \sim_x B'\). Therefore, \(A = w_0 + A' \sim_x w_0 + B' \sim_x B\) as desired.

**Proof of Lemma 3.4.7.** For two strings \(w, w'\), choose a longest partial string common to both \(w, w'\), and denote it by \((w||w')\). (This may not be unique, but we simply choose one such for each pair of strings.) Let \(l(w||w')\) be the length of this partial string (notice that \(l(w||w')\) is odd since the first and last edges are solid and the walk is alternating).

**Case 1:** Suppose that \(w_0\) is a string. Let \(j\) be an index such that \(u(w_0)_j > 0\) and \(x_j\) is a loop. Since \(u(A)_j > 0\) and \(A \sim_w B\), we must have that \(u(B)_j > 0\), so there exists a string \(w_i\) such that \(u(w_i)_j > 0\), \(b_i \neq 0\), and such that \(l(w_0||w_i)\) is maximal. We show the following: if \(w_i \neq w_0\), then \(B \sim_x \Phi(B)\) in such a way that there is a walk \(w_{i_2}\) appearing with non-zero coefficient in \(\Phi(B)\) such that \(l(w_0||w_{i_2}) > l(w_0||w_i)\). Since the length of \(w_0\) if finite, there must be an \(N > 0\) such that \(w_0\) appears with non-zero coefficient in \(\Phi^N(B)\). Since equivalence under \(\sim_x\) implies equivalence under \(\sim_w\), then, we have that \(A \sim_w \Phi^N(B)\), so \(A \sim_w w_0 + B'\) for some \(B'\), as desired.

Let \(v\) be the last vertex in \((w_0||w_i)\), \(E\) the dotted edge containing said vertex, \(v'\) the other vertex contained in \(E\), and \(Q\) the partial string such that \(Q(w_0||w_i) = w_i\). This is demonstrated in the diagram below, where the walk \(w_0\) is depicted in black, and \(w_i\) is in
Now $x_{j_1}$ appears in $w_0$, so $u(B)_{j_1} = u(A)_{j_1} > 0$, implying that there is a walk $w_{i_1}$ with non-zero coefficient appearing in $B$ with $u(w_{i_1}) \neq 0$. There are three subcases:

(A) $w_{j_1}$ is the (unique) walk appearing in $B$ with this property, then $x_{j_1}$ is an edge in $Q$;

(B) $w_{i_1}$ is not $w_{j_1}$, and is an alternating string;

(C) $w_{i_1}$ is an alternating band.

Subcase A: This case impossible, for suppose that $w_{j_1}$ indeed contains $x_{j_1}$. Said edge cannot be the first solid edge in $Q$, or else $x_{j_1}E(w_0||w_{i_1})$ would be a partial string common to both $w_0$ and $w_{i_1}$ with length one greater than $(w_0||w_{i_1})$, contradicting the definition. Otherwise, $w_{i_1}$ takes one of the following two forms:

$$w_{i_1} = \ldots E x_{j_1} \ldots E(w_0||w_{i_1})$$
$$w_{i_1} = \ldots x_{j_1} ECE(w_0||w_{i_1}),$$

where $C$ is an alternating walk starting with the vertex $v'$ and ending with $v$. In the former case, the walk $w_{i_1}$ could be written in the form $\ldots E \ldots x_{j_1} E(w_0||w_{i_1})$. But $x_{j_1} E(w_0||w_{i_1})$ has greater length than $(w_0||w_{i_1})$. Contradiction. Finally, in the latter case, $w_{i_1}$ is not an irreducible walk since $EC$ is a band, so $w_{i_1} = \ldots x_{j_1} E(w_0||w_{i_1}) + EC$, and the first
summand is an alternating string with \( l(w_0||\ldots x_{j_1} E(w_0||w_{i_1})) > l(w_0||w_{i_1}) \), contradicting the choice of \( w_{i_1} \).

**Subcase B:** Now we have \( w_{i_1} \) an alternating string containing the edge \( x_{j_1} \). Let \( Q' \) be the partial string in \( w_{i_1} \) containing \( x_{j_1} \) and not \( E \), and \( P' \) the partial string such that \( Q'P' = w_{i_1} \). This is depicted in the diagram below:

I.e., \( Q'P' + Q(w_0||w_{i_1}) \) appears in \( B \). Notice that this is an \( X \)-configuration about \( E \), so \( Q'P' + Q(w_0||w_{i_1}) \sim_X Q'(w_0||w_{i_1}) + QP' \). Take \( \Phi(B) = B - (Q'P' + Q(w_0||w_{i_1}))) + (Q'(w_0||w_{i_1}) + QP') \). Then \( \Phi(B) \sim_X B \) and \( \Phi(B) \) contains a summand, namely \( Q'(w_0||w_{i_1}) \), with \( l(w_0||(Q'(w_0||w_{i_1}))) > l(w_0||w_{i_1}) \) as claimed.
Subcase C: Finally, if \( w_{l_1} = PEx_{j_1} \) is a band, then we are in the following situation:

In this case, we can define \( w_{i_2} = QEPx_{j_1}E(w_0||w_{i_1}) \) (caution: this walk is not irreducible). Then \( l(w_0||w_{i_2}) > l(w_0||w_{i_1}) \), as desired.

Case 2: Now suppose that \( w_0 \) is an alternating band. Notice that we can assume (by symmetry) that there are no strings appearing as summands in \( B \). Again, for some band \( w \) we will denote by \( (w_0||w) \) any of the longest alternating paths contained in both \( w_0 \) and \( w \). Let \( y_1 \) be some solid edge contained in \( w_0 \). Since \( u(w_0)_{y_1} \neq 0 \), there must be a band \( w_{i_1} \) appearing in \( B \) passing through this edge. This is depicted below, again the black edges
form the band $w_0$ and the gray edges are from $w_{i_1}$.

Fix an orientation on $w_0$, and suppose that $y_2$ is the first edge in $w_0$ (in the chosen orientation) which is not contained in $w_{i_1}$ as in the diagram. But $u(B)_{y_2} \neq 0$, so there must be a band $w_{i_1}$ containing this edge. By the same reasoning as the proof of case A for strings, if this band were $w_{i_1}$ (i.e., if $w_{i_1}$ contained $y_2$), then $w_{i_1}$ could be rewritten so as to contain a longer common subpath with $w_0$. Therefore, this path is distinct from $w_{i_1}$. There are two cases:

Subcase A: $w_{i_1}$ contains all other edges in $w_0$ as in the diagram including that labeled $y_2$: 
then \(w_{i_1}\) and \(w_{i_2}\) are in an \(H\)-configuration.

\[P \sim X w_{i_1} + w_{i_2}\]

since \(w_{i_1} = EX_1E'P_1\), and \(w_{i_2} = EX_2E'P_2\). Therefore

\[w_{i_1} + w_{i_2} = EX_1E'P_1 + EX_2E'P_2\]
\[= EX_1E'P_2 + EX_2E'P_1\]
\[= w_0 + EX_2E'P_1.\]

As such, \(B \sim_X w_0 + B'\) with \(u(B') = u(B) - u(w_0) < u(B)\).

Subcase B: \(w_{i_1}\) does not contain all other edges in \(w_0\):

Let \(X\) be the subpath common to both \(w_{i_1}\) and \(w_{i_2}\) as above, \(P_1\) and \(P_2\) the paths such that \(w_{i_1} = P_1X\) and \(w_{i_2} = P_2X\), respectively. Then \(w_{i_1} + w_{i_2} = XP_1XP_2\) is an alternating
band (although clearly not irreducible). Furthermore, \( l(w_0|XP_1XP_2) > l(w_0|w_i) \). Since the length of \( w_0 \) is finite, iteration of this will introduce an \( H \)-configuration as in case A within \( l(w_0) \) steps.

\[
\square
\]

\[
\square
\]

### 3.5 Degree Bounds

It is a simple consequence of section 2.6.1 that for \( \lambda \in \Lambda_{SI}(Q,c,\beta,r) \), the function \( m_\lambda \) is of degree

\[
\sum_{a \in Q_{1i}} |\lambda(a)|
\]

under the usual grading on the polynomial ring. We will use this and the map \((u,y) \mapsto \lambda_{u,y}\) to give degree bounds on the generators and relations for \( SI_{Q,c}(\beta,r) \). Recall that there is a second grading on \( SI_{Q,c}(\beta,r) \), as in proposition 3.3.5, given by the connected components of the partition equivalence graph. The first corollary relates to this grading.

**Corollary 3.5.1.** The generators for \( SI_{Q,c}(\beta,r) \) occur in multi-degrees bounded by \( \varphi_{\lambda,e}(e) \leq 2 \) and \( y_\lambda(e) \leq 1 \).

As for degree bounds in the polynomial ring, we have the following result:

**Corollary 3.5.2.** The generators for \( SI_{Q,c}(\beta,r) \) occur in total degrees bounded by

\[
2 \sum_{a \in Q_0} \left( \frac{r(a) + 1}{2} \right).
\]

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Proof. Since $\lambda(a)_r(a) \leq 2$, and $\lambda(a)_{i+1} \leq \lambda(a)_i \leq \lambda(a)_{i+1} + 2$, we have

$$\deg(m_\lambda) = \sum_{a \in Q_1} |\lambda(a)| \leq \sum_{a \in Q_1} \sum_{i=1}^{r(a)} 2i$$

$$= 2 \sum_{a \in Q_1} \binom{r(a) + 1}{2}.$$

\[\square\]

**Corollary 3.5.3.** The relations for $SI_{Q,c}(\beta, r)$ occur in total degrees bounded by

$$8 \sum_{a \in Q_1} \binom{r(a) + 1}{2}.$$

Proof. We may assume that in an $X$-relation, none of the arms contains a subband, so by theorem 3.4.3, we have that for each arm $u(a) \leq 2$ and $\varphi_u(e) \leq 2$ for any $e$. Therefore, on $P_1P_2 \cdot Q_1Q_2$, the bounds become $u(a) \leq 8$ and $\varphi_u(e) \leq 8$. The bound is derived similarly to the previous corollary. The same technique works for $H$-relations as well, so the bound is as desired. \[\square\]
Chapter 4

Generic Modules

We call the decomposition \( \beta = \beta(1) + \ldots + \beta(s) \) the canonical or generic decomposition of \( \beta \) (with respect to a fixed irreducible component \( Z \) of \( \text{Rep}_{Q,I}(\beta) \)) if the generic representation in \( Z \) can be written as a direct sum \( V(1) \oplus \ldots \oplus V(s) \) of indecomposables such that \( V(i) \) has dimension \( \beta(i) \). Kac ([23] 2.24) points out that such a decomposition always exists, although in many the explicit description is unknown. In this chapter, we describe the generic modules in representation spaces for gentle string algebras.

4.1 The Up-and-Down Graph

In this section, we construct a graph for each irreducible component of \( \text{Rep}_{Q,c}(\beta) \) when \((Q, c)\) is a gentle string algebra. In section 4.2 we will construct a module from each such graph.

Denote by \( \mathcal{X} \subset Q_0 \times S \) the set of pairs \((x, s)\) such that there is an arrow \( a \) of color \( s \) incident to the vertex \( x \). We define a sign function, which will dictate how the graph is constructed.

**Definition 4.1.1.** A sign function on \((Q, c)\) is a map \( \epsilon : \mathcal{X} \to \{ \pm 1 \} \) such that if \((x, s_1), (x, s_2)\)
are distinct elements in \( X \), then \( \epsilon(x, s_1) = -\epsilon(x, s_2) \).

The following lemma is not used in the remainder of the article, but is recorded here for completeness.

**Lemma 4.1.1.** If there are no isolated vertices in \( Q \), then there are \( 2^{|Q_0|} \) sign functions on \((Q, c)\).

**Proof.** Let \( \mathcal{E} \) be the set of all sign functions on \((Q, c)\). We will define a bijection between this space and \( \{\pm 1\}^{|Q_0|} \). Namely, for each \( x \in Q_0 \), select a color \( s_x \in C \) such that \((x, s_x) \in X\). If \( \epsilon \) is a sign function, denote by \( \epsilon \in \{\pm 1\}^{|Q_0|} \) the vector with \( \epsilon_x = \epsilon(x, s_x) \). For \( \epsilon \in \{\pm 1\}^{|Q_0|} \), let \( \epsilon : X \to \{\pm 1\} \) be the extension of the map \( \epsilon \) by

\[
\epsilon(x, s) = \begin{cases} 
\epsilon(x, s_x) & \text{if } s = s_x \\
-\epsilon(x, s_x) & \text{otherwise}
\end{cases}
\]

These maps are mutual inverses, so indeed \(|\mathcal{E}| = |\{\pm 1\}^{|Q_0|}| = 2^{|Q_0|}\). \(\square\)

**Definition 4.1.2.** Fix a quiver \( Q \) with coloring \( c \), a dimension vector \( \beta \), and a maximal rank map \( r \). For any sign function \( \epsilon \) on \((Q, c)\), denote by \( \Gamma_{Q,c}(\beta, r, \epsilon) \) the graph with vertices \( \{v^x_i \mid x \in Q_0 \land i = 1, \ldots, \beta_x\} \) and edges as follows (see figure 4.1 for a visual depiction):

- For each arrow \( a \in Q_1 \) and each \( i = 1, \ldots, r(a) \)
  - a. \( v^{ta}_i \longrightarrow v^{ha}_i \) if \( \epsilon(ta, c(a)) = 1, \epsilon(ha, c(a)) = -1 \),
  - b. \( v^{ta}_i \longrightarrow v^{ha}_{\beta_h a - i + 1} \) if \( \epsilon(ta, c(a)) = 1, \epsilon(ha, c(a)) = 1 \),
  - c. \( v^{ta}_{\beta_h a - i + 1} \longrightarrow v^{ha}_i \) if \( \epsilon(ta, c(a)) = 1, \epsilon(ha, c(a)) = -1 \),
  - d. \( v^{ta}_{\beta_h a - i + 1} \longrightarrow v^{ha}_{\beta_h a - i + 1} \) if \( \epsilon(ta, c(a)) = -1, \epsilon(ha, c(a)) = 1 \).

We will call the graph \( \Gamma_{Q,c}(\beta, r, \epsilon) \) an **up-and-down graph**.
Such a graph comes equipped with a map $w : \text{Edges}(\Gamma_{Q,c}(\beta, r, \epsilon)) \to Q_1$ where $w(e) = a$ if $e$ is an edge arising from the arrow $a$. The vertices $v^x_i$ will be referred to as the vertices concentrated at level $x$. Figure 4.1 depicts the various edge configurations in $\Gamma_{Q,c}(\beta, r, \epsilon)$ for different choices of $\epsilon$ at the tail and head of an arrow.

**Proposition 4.1.3.** Let $\Gamma_{Q,c}(\beta, r, \epsilon)$ be an up-and-down graph. Then

a. If a vertex is contained in two edges $e, e'$, then $c(w(e)) \neq c(w(e'))$;

b. Each vertex in $\Gamma_{Q,c}(\beta, r, \epsilon)$ is contained in at most two edges (therefore $\Gamma$ consists of string and band components).

c. A connected component of an up-and-down graph is again an up-and-down graph.
Proof. For part (a), suppose that \( v_i^x \) is a vertex in \( \Gamma \) incident to two edges \( e \) and \( e' \) where \( c(w(e)) = c(w(e')) = s \). It is clear from the definition of the edges that \( w(e) \neq w(e') \). Since there is at most one outgoing and at most one incoming arrow of color \( s \) relative to \( x \), it can be assumed that \( w(e) = a_1 \) and \( w(e') = a_2 \) where \( h(a_1) = t(a_2) = x \) and \( c(a_i) = s \).

Suppose that \( \epsilon(x, s) = 1 \) (the other case is identical). Then by definition 4.1.2, \( i \leq r(a_2) \), and \( i \geq \beta_x - r(a_1) + 1 \). But \( r \) is a rank map, so \( \beta_x \geq r(a_1) + r(a_2) \). Therefore, \( i \geq r(a_2) + 1 \) and \( i \leq r(a_2) \), a contradiction. For part (b), if a vertex \( v_i^x \) in \( \Gamma \) is contained in three edges, then by part (a) the arrows corresponding to the edges are of three different colors, and all incident to \( x \), which is false by assumption that \( kQ/I_c \) is a gentle string algebra. Finally, suppose that \( \gamma \) is a connected component of \( \Gamma_{Q,c}(\beta, r, \epsilon) \). Let us suppose that \( \gamma \) has \( \beta_x' \) vertices at level \( x \) for each \( x \in Q_0 \), and has \( r'(a) \) edges labeled \( a \) for each \( a \in Q_1 \).

Then \( \gamma = \Gamma_{Q,c}(\beta', r', \epsilon) \) (this is not simply isomorphism of graphs, but one that preserves the labeling of edges and levels of vertices). Let us label the vertices in \( \Gamma_{Q,c}(\beta', r', \epsilon) \) by \( \{ w_i^x \mid x \in Q_0, i = 1, \ldots, \beta_x' \} \). Let \( f : \Gamma_{Q,c}(\beta', r', \epsilon) \rightarrow \Gamma_{Q,c}(\beta, r, \epsilon) \) be the homomorphism of graphs defined as follows: \( f : w_i^x \mapsto v_{\gamma_i(x)}^x \) where \( \gamma_i(x) \) is the \( i \)-th vertex in \( \gamma \) at level \( x \). It is clear that the image of this map is precisely the graph \( \gamma \), and that \( f \) gives a bijection between \( \Gamma_{Q,c}(\beta', r', \epsilon) \) and \( \gamma \).

Remark 4.1.2. It is worth noting that distinct sign functions give rise to a different numbering on the vertices of the graph \( \Gamma \), but do not change the graph structure. In fact, if \( \epsilon \) and \( \epsilon' \) differ in only one vertex, \( x \) (say), the graphs \( \Gamma_{Q,c}(\beta, r, \epsilon) \) and \( \Gamma_{Q,c}(\beta, r, \epsilon') \) differ only by applying the permutation \( i \mapsto \beta_x - i + 1 \) to the vertices \( \{ v_i^x \mid i = 1, \ldots, \beta_x \} \). We will soon see that the families of modules arising from different choices of \( \epsilon \) coincide.

Here we collect some technical definitions and notations to be used concerning these graphs. We will extend the terminology of Butler and Ringel ([5]) slightly. Let \( \Gamma_{Q,c}(\beta, r, \epsilon) \) be an up and down graph. A vertex \( v_j^x \) is said to be \textbf{above} (resp. \textbf{below}) \( v_j^x \) if \( j > j' \) (resp. \( j < j' \)). We will depict the graphs of \( \Gamma_{Q,c}(\beta, r, \epsilon) \) in such a way that above and
A vertex \( v_j \) in \( \Gamma_{Q,c}(\beta, r, \epsilon) \) will be referred to as a \textbf{source} (resp. \textbf{target}) if \( t(w(e)) = x \) (resp. \( h(w(e)) = x \)) for every edge \( e \) containing it. A \textbf{2-source} (resp. \textbf{2-target}) is a source (resp. target) incident to exactly two edges. We will denote the sets of such vertices by \( S(\Gamma), T(\Gamma), S^2(\Gamma), \) and \( T^2(\Gamma), \) respectively.

To a path \( p = v_{i_0}^{x_0} e_n \ldots v_{i_1}^{x_1} e_1 v_{i_0}^{x_0} \) on \( \Gamma_{Q,c}(\beta, r, \epsilon) \), we will associate a sequence \( A(p) \) of elements in the set alphabet \( Q_1 \cup Q_1^{-1} \) (that is the formal alphabet with characters consisting of the arrows and their inverses), with

\[
A(p)_i = \begin{cases} 
  w(e_i) & \text{if } t(w(e_i)) = x_{i-1} \\
  w(e_i)^{-1} & \text{if } t(w(e_i)) = x_i
\end{cases}
\]

Such a path \( p \) will be called \textbf{direct} (resp. \textbf{inverse}) if \( A(p) \) is a sequence of elements in \( Q_1 \) (resp. \( Q_1^{-1} \)).

Finally, a path \( p \) will be called \textbf{left positive} (resp. \textbf{left negative} if \( A(p)_n \in Q_1 \) and \( \epsilon(x_n, c(e_n)) = 1 \) (resp. \(-1\)). Analogously the path is called \textbf{right positive} (resp. \textbf{right negative}) if \( A(p)_1 \in Q_1 \) and \( \epsilon(x_0, c(e_0)) = 1 \) (resp. \(-1\)).

\textbf{Example 4.1.3.} Consider the quiver below with coloring indicated by type of arrow:

```
\begin{figure}
\centering
\begin{tikzpicture}
  \node (1) at (0,0) {1};
  \node (2) at (2,1) {2};
  \node (3) at (4,0) {3};
  \node (4) at (0,-2) {4};
  \node (5) at (2,-2) {5};
  \node (6) at (4,-2) {6};

  \draw[->, thick, blue] (1) -- node[above] {$r_1$} (2);
  \draw[->, thick, red] (2) -- node[below] {$r_2$} (3);
  \draw[->, thick, green] (1) -- node[left] {$g_1$} (4);
  \draw[->, thick, red] (3) -- node[right] {$g_2$} (5);
  \draw[->, thick, green] (4) -- node[below] {$b_1$} (5);
  \draw[->, thick, blue] (4) -- node[below] {$p_1$} (6);
  \draw[->, thick, red] (5) -- node[above] {$p_2$} (6);
\end{tikzpicture}
\end{figure}
```

Let us say that the color of the arrow \( a_i \) is \( a \) in the above picture. Let \( \beta, r \) be the pair
depicted in the following:

and $\epsilon^{-1}(1) = \{(1, g), (2, p), (3, g), (4, b), (5, b), (6, p)\}$ (so $\epsilon^{-1}(-1)$ is the complement in $X$).

Then $\Gamma_{Q,c}(\beta, r, \epsilon)$ takes the following form:

i. The vertices $v_1^{(1)}, v_2^{(1)}, v_3^{(1)}, v_4^{(1)}, v_4^{(4)}$ are sources, and $v_3^{(2)}, v_2^{(5)}, v_1^{(3)}, v_1^{(6)}, v_2^{(6)}$ are targets.

ii. The path $v_2^{(6)}e_2v_1^{(2)}e_1v_3^{(1)}$ with $w(e_1) = r_1$ and $w(e_2) = p_2$ is a direct path that is left positive (since $\epsilon(6, p_2) = 1$), and right negative; while $v_1^{(4)}e_2v_3^{(2)}e_1v_1^{(1)}$ with $w(e_1) = p_1$, $w(e_2) = r_1$ is not a direct path.

4.1.1 Some Combinatorics for Up-and-Down Graphs

The proof of the main theorem requires an explicit description of the projective resolution of the modules arising from up-and-down graphs. In this section, we collect some technical lemmas concerning the structure of the graphs $\Gamma_{Q,c}(\beta, r, \epsilon)$ to be used in describing the
projective resolution.

Lemma 4.1.4. Let $v_j^x$ be a vertex in $\Gamma_{Q,c}(\beta, r, \epsilon)$, and suppose that

$$p = v_j^x e_{i_1} v_{i_1}^{x_{i_1}} \ldots v_{i_k}^{x_{i_k}} e_{i_k}^x v_i^y$$

is a left direct path ending in $v_j^x$.

A. If $p$ is left negative direct, and $v_j^x$ is above $v_j^x$, then there is a left negative direct path

$$p' = v_j^x e_{i_1} v_{i_1}^{x_{i_1}} \ldots v_{i_k}^{x_{i_k}} e_{i_k}^x v_i^y$$

with $A(p') = A(p)$. Furthermore,

A1. $v_i^y$ is above $v_i^y$ if and only if $\epsilon(y, c(w(e'_1))) = 1$;

A2. $v_i^y$ is below $v_i^y$ if and only if $\epsilon(y, c(w(e'_1))) = -1$.

B. If $p$ is left positive direct, and $v_j^x$ is below $v_j^x$, then there is a left positive direct path

$$p' = p' = v_j^x e_{i_1} v_{i_1}^{x_{i_1}} \ldots v_{i_k}^{x_{i_k}} e_{i_k}^x v_i^y$$

with $A(p') = A(p)$. Furthermore,

B1. $v_i^y$ is below $v_i^y$ if and only if $\epsilon(y, c(w(e'_1))) = -1$;

B2. $v_i^y$ is above $v_i^y$ if and only if $\epsilon(y, c(w(e'_1))) = 1$.

Proof. We will prove this lemma by induction on the length of $p$. Suppose that $p = v_j^x e_{i_1} v_i^y$ with $A(p) = a$. If $p$ is left negative direct, then $\epsilon(x, c(a)) = -1$. By definition of the graph $\Gamma$, then, $j \leq r(a)$. But $v_j^x$ is above $v_j^x$ if and only if $j' < j$. By definition 4.1.2 (a), (c), there is an edge $e'_1$ terminating at $v_j^x$ labeled $a$, so $p' = v_j^x e_{i_1}^x v_i^y$. If $\epsilon(y, c(a)) = 1$, $i = j$ and $i' = j'$, so indeed $i' < i$, implying that $v_i^y$ is above $v_i^y$. On the other hand, if $\epsilon(x, c(a)) = -1$, 75
then \( i = \beta x - j + 1 \) and \( i' = \beta x - j' + 1 \), so \( i' > i \), and \( v^y_i \) is below \( v^y_{i'} \). The other direction is also clear for [A1] and [A2].

Now suppose that \( p \) is left positive direct of length one, i.e., \( \epsilon(x, c(a)) = 1 \). Write \( p = v^x_j e_1 v^y_i \). By definition of \( \Gamma \), then, \( j \geq \beta x - r(a) + 1 \). Suppose that \( j = \beta x - \hat{j} + 1 \), and \( j' = \beta x - \hat{j'} + 1 \). Since \( v^x_j \) is below \( v^x_{j'} \), we have that \( j' = \beta x - \hat{j'} + 1 > \beta x - \hat{j} + 1 = j \), so \( \hat{j'} < \hat{j} \). Indeed, \( \hat{j'} > \beta x - r(a) + 1 \), so by definition 4.1.2 (b) or (d), there is an edge \( e'_1 \) labeled \( a \) terminating at \( v^x_{j'} \). \( \epsilon(y, c(a)) = -1 \) if and only if \( i = \beta y - \hat{j} + 1 \) and \( i' = \beta y - \hat{j'} + 1 \), i.e., \( i' > i \), so \( v^y_{i'} \) is above \( v^y_i \). By the first step, there is a path \( v^x_j e_1 v^y_{i-1} \) with \( w(e_1) = w(e'_1) \).

Case 1: \( \epsilon(y_{i-1}, c(w(e'_1))) = 1 \) if and only if \( v^y_{i-1} \) is above \( v^y_{j-1} \). But by proposition 4.1.3, \( \epsilon(y_{i-1}, c(w(e_1))) = -\epsilon(y_{i-2}, c(w(e_{i-1}))) \), so

\[
\tilde{p} = v^y_{j-1} e_{i-1} \ldots e_1 v^y_i
\]

is left negative direct. So by the inductive hypothesis, since \( \tilde{p} \) is of length \( l - 1 \), we have a path

\[
\tilde{p}' = v^y_{j'-1} e'_{i-1} \ldots e'_1 v^y_{i'}
\]

with \( A(\tilde{p}) = A(\tilde{p}') \). Taking \( p' = e'_1 \tilde{p}' \), we have a left negative direct path \( p' \) terminating in \( v^x_{j'} \). Again, by the inductive step, \( v^y_i \) is above (resp. below) \( v^y_{i'} \) if and only if \( \epsilon(y, c(w(e_1))) = 1 \) (resp. \(-1\)).

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Case 2: $\epsilon(y_{l-1}, c(w(e_l))) = -1$ if and only if $v_{j_{l-1}}^{y_{l-1}}$ is below $v_{j_{l-1}}^{y_{l-1}}$. By proposition 4.1.3, 

$\epsilon(y_{l-1}, c(w(e_l))) = -\epsilon(y_{l-2}, c(w(e_{l-1})))$, so

$$\tilde{p} = v_{j_{l-1}}^{y_{l-1}}e_{l-1}\ldots eiev_{i}^{y}$$

is left positive direct. By the inductive hypothesis, there is a path

$$\tilde{p}' = v_{j_{l-1}}^{y_{l-1}}e_{l-1}'\ldots e_{1}'v_{i}'^{y}$$

with $A(\tilde{p}) = A(\tilde{p}')$. Taking $p' = v_{j_{l-1}}^{y_{l-1}}e_{l-1}'\tilde{p}'$, we have a left negative direct path $p'$ terminating in $v_{j_{l-1}}^{y_{l-1}}$. By the hypothesis, $v_{j_{l-1}}^{y_{l-1}}$ is above (resp. below) $v_{i}^{y}$ if and only if $\epsilon(y, c(w(e_{1}))) = 1$ (resp. $-1$).

The same argument hold if $p$ is left positive direct, interchanging the terms ‘above’ and ‘below’.

\[\square\]

Here we collect some properties that determine what types of extremal vertices occur in which levels.

**Lemma 4.1.5.** Let $\Gamma_{Q,c}(\beta, r, \epsilon)$ be an up-and-down graph, and let $a_{1}, a_{2}, b_{1}, b_{2} \in Q_{1}$ be colored arrows as indicated in the figure:

\[\begin{array}{c}
    a_{1} \quad a_{2} \\
    v_{b_{1}} \quad v_{b_{2}} \\
    y
\end{array}\]

i. If $v_{y}^{j_{l}}$ is a 2-source (resp. 2-target), then $r(a_{1}) + r(b_{1}) > \beta_{y}$ (resp. $r(a_{2}) + r(b_{2}) > \beta_{y}$);

ii. Let $m_{1} = \max\{r(a_{1}), r(b_{2})\}$ and $m_{2} = \max\{r(b_{1}), r(a_{2})\}$. Then if $v_{j_{l}}^{y}$ is an isolated vertex, $m_{1} + m_{2} < \beta_{y}$ (in particular, there are neither 2-sources not 2-targets vertices at level $y$);
iii. If \( v^y_i \) is a 1-target contained in an edge labeled by \( a_1 \) (resp. \( b_2 \)), then \( r(b_1) + r(b_2) < \beta_x \) (resp. \( r(a_1) + r(a_2) < \beta_x \)).

Proof. We prove only (iii), since the others are similar. Suppose that \( v^y_i \) is a 1-target contained in an edge labeled \( a_1 \). If \( r(b_1) + r(b_2) = \beta_y \), then each vertex at level \( y \) would be contained in an edge (either labeled \( b_1 \) or \( b_2 \)), including \( v^y_i \). But this contradicts the assumption.

Lemma 4.1.6. Suppose that there is a sequence of arrows \( a_1, a_2, a_3 \in Q_1 \) with \( c(a_1) = c(a_2) = c(a_3) \), \( h(a_1) = t(a_2) = x_1 \), and \( h(a_2) = t(a_3) = x_2 \). If \( r \) is a maximal rank map, then we have the following:

i. if \( r(a_1) + r(a_2) < \beta_{x_1} \) then \( r(a_2) + r(a_3) = \beta_{x_2} \)

ii. if \( r(a_2) + r(a_3) < \beta_{x_2} \) then \( r(a_1) + r(a_2) = \beta_{x_1} \).

Proof. Suppose that both \( r(a_1) + r(a_2) < \beta_{x_1} \) and \( r(a_2) + r(a_3) < \beta_{x_2} \). Define by \( r^+ \) the rank map with \( r^+(a_2) = r(a_2) + 1 \) and \( r^+(b) = r(b) \) otherwise. \( r^+ \) is a rank map and \( r^+ > r \), contradicting the assumption of maximality of \( r \). \( \square \)

4.2 Up-and-Down Modules

We will now define a module (or family of modules) \( V_{Q,c}(\beta, r) \) based on two additional parameters, later proving that the isomorphism class of this module (or family of modules) is independent of these parameters. Fix \( Q, c, \beta, r, \epsilon \) as described above. Recall that proposition 4.1.3 guarantees \( \Gamma_{Q,c}(\beta, r, \epsilon) \) is comprised of strings and bands. Let \( B(\Gamma) \) be the set of bands and fix a function \( \Theta : B(\Gamma) \to \text{Vert}(\Gamma_{Q,c}(\beta, r, \epsilon)) \) with \( \Theta(b) \) a target contained in the band \( b \).

Definition 4.2.1. For \( \mu \in (k^*)^{B(\Gamma)} \), denote by \( V_\mu := V_{Q,c}(\beta, r, \Theta)_\mu \) the representation of \( Q \) given by the following data. The space \( (V_\mu)_x \) is a \( \beta_x \)-dimensional \( k \) vector space together
with a fixed basis \( \{ e_j^+ \}_{j=1, \ldots, \beta} \). The linear map \((V_\mu)_a : (V_\mu)_{ta} \to (V_\mu)_{ha}\) is defined as follows: if \( v_j^{ta} \) and \( v_k^{ha} \) are joined by an edge \( e \) labeled \( a \), then

\[
(V_\mu)_a : e_j^{ta} \mapsto \begin{cases} 
\mu_b e_k^{ha} & \text{if there is a band } b \text{ with } \Theta(b) = v_k^{ha} \text{ and } \epsilon(ha, c(a)) = 1 \\
 e_k^{ha} & \text{otherwise.}
\end{cases}
\]

If there is no such edge, then \((V_\mu)_a : e_j^{ta} \mapsto 0\). If there are no bands in \( \Gamma_{Q,c}(\beta, r, \epsilon) \), denote by \( V_{Q,c}(\beta, r, \epsilon, \Theta) \) the subset of \( \text{Rep}_Q(\beta) \) containing this module. If there are bands, then denote by \( V_{Q,c}(\beta, r, \epsilon, \Theta) \) the set of all modules \( V_{Q,c}(\beta, r, \epsilon, \Theta)_\mu \) for \( \mu \in (k^*)^{B(\Gamma)} \).

**Example 4.2.1.** Continuing with example 4.1.3, let \( b \) be the unique band in \( \Gamma_{Q,c}(\beta, r, \epsilon) \), and take \( \Theta(b) = v_1^{(6)} \). For \( \mu \in k^* \), the module \( V_{Q,c}(\beta, r, \epsilon, \Theta)_\mu \) is given by the following:

![Diagram](image)

**Proposition 4.2.2.** Every representation in the set \( V_{Q,c}(\beta, r, \epsilon, \Theta) \) is a representation of the gentle string algebra \((Q, c)\).

**Proof.** If \( a_1, a_2 \in Q_1 \) are arrows with \( ha_1 = ta_2 \) and \( c(a_1) = c(a_2) \), then by proposition 4.1.3 there is no path \( v_1 \xrightarrow{e_1} v_2 \xrightarrow{e_2} v_3 \) in \( \Gamma \) with \( w(e_1) = a_1 \) and \( w(e_2) = a_2 \). Therefore,
\(a_2a_1(e_i^2) = 0\) for all \(x \in Q_0, i = 1, \ldots, \beta_x\). Since \(I_c\) is generated by precisely these relations, each module in \(V_{Q,c}(\beta, r, \epsilon, \Theta)\) is indeed a \(kQ/I_c\) module. \qedhere

The definition appears highly dependent on both \(\epsilon\) and the choice of distinguished vertices \(\Theta\). In the following proposition, we show that the family does not depend on \(\Theta\).

**Proposition 4.2.3.** The family \(V_{Q,c}(\beta, r, \epsilon, \Theta)\) does not depend on the choice of vertices \(\Theta\).

**Proof.** This proof is a simple consequence of [5] (cf. Theorem page 161). Indeed, suppose that \(b\) is a band component of some \(V_{Q,c}(\beta, r, \epsilon, \Theta)\). Denote by \(V_{b,\mu_b}\) the submodule corresponding to this band, and \(\omega_b\) a cyclic word in \(Q_1 \cup Q_1^{-1}\) which yields this band. Recall that Butler and Ringel produce, for each such cyclic word, a functor \(F_{\omega_b}\) from the category of pairs \((V, \varphi)\) with \(V\) a \(k\)-vector space and \(\varphi : V \to V\) and automorphism, to the category \(\text{Rep}_{Q,c}\). The indecomposable module \(V_{b,\mu_b}\) is isomorphic to the image under this functor of the pair \((k, \mu_b)\) where \(\mu_b : x \mapsto \mu_b \cdot x\). Butler and Ringel show that the family \(V_{b,\mu_b}\) for \(\mu_b \in k^*\) is independent of cyclic permutation of the word \(\omega_b\). (They show the image of the functor \(F_{\omega_b}\) itself is independent cyclic permutations of \(\omega_b\).) Therefore, any choice of vertices \(\Theta\) yields the same family \(V_{Q,c}(\beta, r, \epsilon)\). \qedhere

Henceforth, we drop the argument \(\Theta\), and when necessary we make a particular choice of said function.

### 4.2.1 Main Theorem and Consequences

The following statement allows us to show that the representations \(V_{Q,c}(\beta, r)\) are generic. The remainder of the article will be primarily concerned with proving this theorem.

**Theorem 4.2.4.** Let \(B(\Gamma)\) be the set of bands for the graph \(\Gamma_{Q,c}(\beta, r, \epsilon)\).
a. Suppose that $\mu, \mu' \in (k^*)^{B(\Gamma)}$ with $\mu_b \neq \mu'_b$ for all $b, b' \in B(\Gamma)$. Then

$$\dim \text{Ext}^1_{kQ/I_c}(V_{Q,c}(\beta, r, \epsilon)_\mu, V_{Q,c}(\beta, r, \epsilon)_{\mu'}) = 0.$$ 

b. Suppose that $\Gamma_{Q,c}(\beta, r, \epsilon)$ consists of a single band component. Then

$$\dim \text{Ext}^1_{kQ/I_c}(V_{Q,c}(\beta, r, \epsilon)_\mu, V_{Q,c}(\beta, r, \epsilon)_{\mu}) = 1.$$ 

**Corollary 4.2.5.** If $B(\Gamma) = \emptyset$, i.e., $\Gamma_{Q,c}(\beta, r, \epsilon)$ consists only of strings, then the unique element $V \in V_{Q,c}(\beta, r)$ has a Zariski open orbit in $\text{Rep}_{Q,c}(\beta, r)$.

**Proof.** If there are indeed no band components in $\Gamma_{Q,c}(\beta, r, \epsilon)$, then by theorem 4.2.4 part (a), we have $\text{Ext}^1_{kQ/I_c}(V_{Q,c}(\beta, r, \epsilon), V_{Q,c}(\beta, r, \epsilon)) = 0$. The corollary then follows by ([21] Corollary 1.2, [39]). 

If there are band components, then the analogous corollary is more subtle, although the result is essentially the same. Namely that the union of the orbits of all elements in $V_{Q,c}(\beta, r)$ is dense in its irreducible component. The proof relies on some auxiliary results due to Crawley Boevey-Schröer [10], and so we exhibit those first. Let $kQ/I$ be an arbitrary quiver with relations. Suppose that $C_i \subset \text{Rep}_{(Q,I)}(\beta(i))$ are $\text{GL}_Q(\beta(i))$-stable subsets for some collection of dimension vectors $\beta(i), i = 1, \ldots, t$, and denote by $\beta = \sum_i \beta(i)$ the sum of the dimension vectors. Define by $C_1 \oplus \ldots \oplus C_t$ the $\text{GL}_Q(\beta)$-stable subset of $\text{Rep}_{kQ/I}(\beta)$ given by the set of all $\text{GL}_Q(\beta)$ orbits of direct sums $M_1 \oplus \ldots \oplus M_t$ with $M_i \in C_i$.

**Theorem 4.2.6** (Theorem 1.2 in [10]). For an algebra $kQ/I$, $C_i \subset \text{Rep}_{(Q,I)}(\beta(i))$ irreducible components and $t$ defined as above, the set $C_1 \oplus \ldots \oplus C_t$ is an irreducible component of $\text{Rep}_{(Q,I)}(\beta)$ if and only if

$$\text{ext}^1_{kQ/I}(C_i, C_j) = 0$$

for all $i \neq j.$
Corollary 4.2.7. In general, \( \bigcup_{\mu \in (k^*)^{B(\Gamma)}} O_{\nu} \) is dense in \( \text{Rep}_{Q,c}(\beta, r) \).

Proof. Enumerate the connected components of \( \Gamma_{Q,c}(\beta, r, \epsilon), c_1, \ldots, c_t \) with \( c_i \) a band for \( i = 1, \ldots, l \) and a string for \( i = l + 1, \ldots, t \). Let \( \beta|_i, r|_i \) be the restrictions of \( \beta \) and \( r \), respectively, to the \( i \)-th connected component. (By proposition 4.1.3, each connected component is itself an up-and-down graph, so is associated with a dimension vector and maximal rank map.) Let \( C_i = \text{Rep}_{Q,c}(\beta|_i, r|_i) \), which is an irreducible component by 3.1.3. Notice that \( V_{Q,c}(\beta|_i, r|_i) \) is an irreducible component if \( c_i \) is a string and \( V_{Q,c}(\beta|_i, r|_i) \mu_i \) if \( c_i \) is a band. Thus, the \( C_i \) are irreducible and, assuming theorem 4.2.4 is true, \( \text{ext}^1_{kQ/I_c}(C_i, C_j) = 0 \), so \( \text{Rep}_{Q,c}(\beta, r) = C_1 \oplus \ldots \oplus C_t \).

Thus, all that remains to be shown is that if \( c_i \) is a band, then the union of the orbits of all elements in \( V_{Q,c}(\beta|_i, r|_i) \) contains an open set. Indeed, if this is the case, then denoting by \( S_i \) the set \( \text{GL}(\beta|_i) \cdot V_{Q,c}(\beta|_i, r|_i) \) we have

\[
C_1 \oplus \ldots \oplus C_t = S_1 \oplus \ldots \oplus S_t.
\]

Suppose that \( \beta \) is a dimension vector and \( r \) is a maximal rank map such that \( \Gamma_{Q,c}(\beta, r, \epsilon) \) is a single band. Let \( V_{\mu} = V_{Q,c}(\beta, r)_{\mu} \), and denote by \( O_{V_{\mu}} \) the \( \text{GL}(\beta) \)-orbit of \( V_{\mu} \). From Kraft (2.7 [25]), there is an embedding

\[
T_{V_{\mu}}(\text{Rep}_{Q,c}(\beta, r))/T_{V_{\mu}}(O_{V_{\mu}}) \hookrightarrow \text{Ext}^1(V_{\mu}, V_{\mu})
\]

where \( T_M(X) \) denotes the tangent space in \( X \) at \( M \). By theorem 4.2.4, then

\[
\dim T_{V_{\mu}}(\text{Rep}_{Q,c}(\beta, r)) - \dim T_{V_{\mu}}(O_{V_{\mu}}) \leq 1.
\]

Claim. \( V_{\mu} \) is a non-singular point in \( \text{Rep}_{Q,c}(\beta, r) \) (and in \( O_{V_{\mu}} \)).
Proof. Consider the construction of \( V_{Q,c}(\beta, r) \) as a specific choice of embedding a product of varieties of complexes into \( \text{Rep}_{Q,c}(\beta, r) \), i.e.,

\[
\prod_{s \in S} \text{Com}(\beta, r, s) \cong \text{Rep}_{Q,c}(\beta, r)
\]

For each \( x \in Q_0 \), let \( \sigma(x) \) be the matrix of the map (in the distinguished basis of \( V_{Q,c}(\beta, r) \)) corresponding to the permutation \((1, \beta x)(2, \beta x - 1)\ldots\). If \( V \in \prod_{s \in S} \text{Com}(\beta, r, s) \), then define by \( \varphi(V) \) the element of \( \text{Rep}_{Q,c}(\beta, r) \) with

\[
\varphi(V)_a = \begin{cases} 
V_a & \text{if } \epsilon(ta, c(a)) = 1 = -\epsilon(ha, c(a)) \\
\sigma(ha)V_a & \text{if } \epsilon(ta, c(a)) = 1 = \epsilon(ha, c(a)) \\
V_a\sigma(ta) & \text{if } \epsilon(ta, c(a)) = -1 = \epsilon(ha, c(a)) \\
\sigma(ha)V_a\sigma(ta) & \text{if } \epsilon(ta, c(a)) = -1 = -\epsilon(ha, c(a))
\end{cases}
\]

The map \( \varphi \) is an isomorphism, since \( \sigma(x) \in \text{GL}(\beta x) \) for each \( x \). Furthermore, \( \text{rank}_k V_a = \text{rank}_k \varphi(V)_a \). Therefore, since \( \text{rank}_k(V_{Q,c}(\beta, r)_a) = r(a) \), there is a \( V \in \prod_{s \in S} \text{Com}(\beta, r, s) \) with \( \varphi(V) = V_{Q,c}(\beta, r) \), and \( \text{rank}_k V_a = r(a) \). \( \text{Com}(\beta, r, s) \) has a dense open orbit, given by the those complexes \( W \) such that \( \text{rank}_k W_a = r(a) \). Thus, \( \prod_{s \in S} \text{Com}(\beta, r, s) \) is smooth at \( V \), and so \( \text{Rep}_{Q,c}(\beta, r) \) is smooth at \( V_{Q,c}(\beta, r) \).

Hence, we have the following:

\[
\dim(\text{Rep}_{Q,c}(\beta, r)) - \dim(\mathcal{O}_V) = \dim T_{V_{\mu}}(\text{Rep}_{Q,c}(\beta, r)) - \dim T_{V_{\mu}}(\mathcal{O}_V) \leq 1.
\]

If the difference is 0, then \( \overline{\mathcal{O}_V} \) is a closed set of the same dimension as \( \dim(\text{Rep}_{Q,c}(\beta, r)) \), so these are equal. On the other hand, if the difference is 1, then \( X := \bigcup_{\mu \in k^*} \overline{\mathcal{O}_V} \) is a closed set. For \( t \in k^* \), \( V_{\mu+t} \not\cong V_{\mu} \) and \( V_{\mu+t} \in X \). Therefore, \( \dim T_{V_{\mu}} X \geq \dim T_{V_{\mu}} \mathcal{O}_V + 1 \), and

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therefore \( \dim T_{V,X} = \dim(\text{Rep}_{Q,c}(\beta, r)) \). Since \( X \) is closed, \( X = \text{Rep}_{Q,c}(\beta, r) \).

In order to prove theorem 4.2.4, we will explicitly describe the projective resolution of \( V_{Q,c}(\beta, r) \) for any \( \beta, r \), and then apply the appropriate Hom-functor to the resolution.

### 4.2.2 Projective resolutions of \( V_{Q,c}(\beta, r) \) and the \( \text{EXT} \)-graph

The summands in the projective resolutions of \( V_{Q,c}(\beta, r) \) depend on a number of characteristics of the graph \( \Gamma_{Q,c}(\beta, r, \epsilon) \). We collect the pertinent characteristics in the following list.

**Definition 4.2.8.** Let \( \Gamma = \Gamma_{Q,c}(\beta, r, \epsilon) \) be a fixed up-and-down graph.

a. Denote by \( \text{ISO}(\Gamma) \) the set of isolated vertices in \( \Gamma \);

b. Denote by \( S^1(\Gamma) \) (resp. \( T^1(\Gamma) \)) the set of sources (resp. targets) of degree one in \( \Gamma \). These will be referred to as 1-sources (resp. 1-targets).

c. For a vertex \( v_j^x \in T(\Gamma) \), we denote by \( l^+(v_j^x) \) (resp. \( l^-(v_j^x) \)) the longest left positive (resp. left negative) direct path in \( \Gamma \) terminating in \( v_j^x \) (if such a path exists). Similarly, for a vertex \( v_j^x \in S(\Gamma) \), denote by \( r^+(v_j^x) \) (resp. \( r^-(v_j^x) \)) the longest right positive (resp. right negative) direct path initiating in \( v_j^x \).

d. For a vertex \( v_j^x \in T(\Gamma) \), we denote by \( l^+(v_j^x) \), (resp. \( l^-(v_j^x) \)) the source at the other end of \( l^+(v_j^x) \) (resp. \( l^-(v_j^x) \)). Similarly, for a vertex \( v_j^x \in S(\Gamma) \), we denote by \( r^+(v_j^x) \) (resp. \( r^-(v_j^x) \)) the target at the other end of \( r^+(v_j^x) \) (resp. \( r^-(v_j^x) \)).

e. If \( v_j^x \in S^1 \cup T^1 \), let \( p_j^x \) be the direct path of maximal length containing \( v_j^x \);

f. If \( v_j^x \in S^1 \cup T^1 \), denote by \([v_j^x]_1 \in Q_1 \) be the arrow with the property that \( t([v_j^x]_1) = x \) and \( c([v_j^x]_1) = c(w(e)) \) where \( e \) is the edge in \( p_j^x \) containing \( v_j^x \).
g. Furthermore, recursively define the arrows $[v_j^x]_l$ with $t([v_j^x]_l) = h([v_j^x]_{l-1})$, and $c([v_j^x]_l) = c([v_j^x]_1)$.

h. Suppose $v_j^x \in \text{ISO}$. Denote by $[v_j^x]_1^+$ (resp. $[v_j^x]_1^-$) the arrow (if such exists) with $t([v_j^x]_1^+) = x$ and $e(x, c([v_j^x]_1^)) = \delta$. Again, recursively define $[v_j^x]_l^\delta$ with $t([v_j^x]_l^\delta) = h([v_j^x]_{l-1}^\delta)$, and $c([v_j^x]_l^\delta) = c([v_j^x]_1^\delta)$.

i. In case $[v_j^x]_l$ or $[v_j^x]_l^\pm$ fails to exist, write $h([v_j^x]_l) := \emptyset$ (or $h([v_j^x]_l^\pm) := \emptyset$), and let $P_0$ be the zero object. (This is nothing more than notation to write the projective resolution of up-and-down modules in a more compact form.)

**Example 4.2.2.** Referring again to example 4.1.3, we have the following aspects:

i. $\text{ISO}(\Gamma) = \emptyset$;

ii. $v_3^{(1)}$ is a 1-source, and $p_3^{(1)}$ is the path $v_2^{(6)} e_2 v_1^{(2)} e_1 v_3^{(1)}$ where $w(e_1) = r_1$ and $w(e_2) = p_2$;

iii. $r^+(v_1^{(1)}) = v_1^{(6)}$, and $r^-(v_1^{(1)}) = v_3^{(2)}$.

iv. Since $\epsilon((6), b_2) = -1$, we have $lp^-(v_1^{(6)}) = b_2 g_1$. Similarly, $rp^-(v_1^{(1)}) = b_2 g_1$.

To illustrate the situation (e)-(h), consider the dimension vector and rank sequence below:

The associated up-and-down graph is given by

\[
v_1^{(1)} \xrightarrow{r_1} v_1^{(2)} \quad v_2^{(2)}
\]
In this case, $v_1^{(1)} \in S^1$, and $[v_1^{(1)}]_1 = r_2$ since the longest path containing $v_1^{(1)}$ is $v_1^{(2)}_1 r_1 v_1^{(1)}$, $c(r_1) = c(r_2)$, and $t(r_2) = (2)$. The vertex $v_2^{(2)}$ is isolated, and in this case, $[v_2^{(2)}]_1^+ = p_2$ and $[v_2^{(2)}]_1^- = r_2$.

We are now prepared to exhibit the projective resolution in the general case. Notice that the simple factor modules of $V_{Q,c}(\beta, r, \epsilon, \Theta)_\mu$ are $S_x$ for $v_j^x \in S(\Gamma)$.

**Proposition 4.2.9.** The following is a projective resolution of $V_{Q,c}(\beta, r, \epsilon, \Theta)_\mu$ is:

$\cdots \rightarrow P(V_\mu)_2 \xrightarrow{\delta (V_\mu)_1} P(V_\mu)_1 \xrightarrow{\delta (V_\mu)_0} P(V_\mu)_0 \rightarrow V_\mu \rightarrow 0$

where

$P(V_\mu)_0 = \bigoplus_{v_j^x \in S(\Gamma)} P_x$

$P(V_\mu)_1 = \bigoplus_{v_i^y \in T^2} P_y \bigoplus \bigoplus_{v_j^x \in S^1 \cup T^1} P_{h([v_j^x]_1^+)} \bigoplus \bigoplus_{v_j^x \in \text{ISO}} P_{h([v_j^x]_1^+)} \bigoplus P_{h([v_j^x]_1^-)}$

$P(V_\mu)_l = \bigoplus_{v_j^x \in T^1 \cup S^1} P_{h([v_j^x]_l^+)} \bigoplus \bigoplus_{v_j^x \in \text{ISO}} P_{h([v_j^x]_l^+)} \bigoplus P_{h([v_j^x]_l^-)}$

and where the differential is given by the following maps (we write $P_{x,j}$ for the projective $P_x$ arising from $v_j^x$):

1. If $v_i^y \in T^2$, $v_i^{y+} = l^+(v_i^y)$ and $v_i^{y-} = l^-(v_i^y)$, then the map $\delta (V_\mu)_0$ restricts to

$$
\begin{bmatrix}
    l p^+(v_i^y) \\
    -\mu_0 l p^-(v_i^y)
\end{bmatrix} \rightarrow P_{y^+,i^+} \oplus P_{y^-,i^-}
$$

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if \( v_i^y = \Theta(b) \) for some band \( b \), and

\[
P_{y,i} \begin{bmatrix} lp^+(v_i^y) \\ -lp^-(v_i^y) \end{bmatrix} P_{y,i}^+ \oplus P_{y,-i}^-
\]

otherwise.

ii. If \( v_i^y \in T^1 \), \( p_i^y \) is the longest direct path terminating at \( v_i^y \), and \( v_j^x \) is the source at the other end of \( p_i^y \), then the restriction of \( \delta(V_\mu)_0 \) to \( P_{h([v_i^y]_1)} \) is given by

\[
P_{h([v_i^y]_1)} \begin{bmatrix} [v_i^y]_1 A(p_i^y) \end{bmatrix} P_{x,j}.
\]

iii. If \( v_i^y \in ISO \), then restriction of \( \delta(V_\mu)_0 \) to \( P_{h([v_i^y]_1^+)} \oplus P_{h([v_i^y]_1^-)} \) is given by

\[
P_{h([v_i^y]_1^+)} \oplus P_{h([v_i^y]_1^-)} \begin{bmatrix} [v_i^y]_1^+ \\ [v_i^y]_1^- \end{bmatrix} P_{y,i}.
\]

iv. If \( v_i^y \in T^1 \cup S^1 \), then the restriction of \( \delta(V_\mu)_l \) to \( P_{h([v_i^y]_l+1)} \) is

\[
P_{h([v_i^y]_l+1)} \begin{bmatrix} [v_i^y]_l \end{bmatrix} P_{h([v_i^y]_l)}.
\]

v. If \( v_i^y \in ISO \), then \( \delta(V_\mu)_l \) restricted to \( P_{h([v_i^y]_l^+,l+1)} \) is

\[
P_{h([v_i^y]_l^+,l+1)} \begin{bmatrix} [v_i^y]_l^+ \end{bmatrix} P_{h([v_i^y]_l^+,l+1)}.
\]

We now apply the functor Hom\((- , V_\nu) \) to the complex \( P(V_\mu)_\bullet \). Recall that we have a
fixed basis for the spaces \( (V_\nu)_x \) for each \( x \in Q_0 \), namely \( \{ e_{x}^{1}, \ldots, e_{x}^{\beta} \} \), relative to which the arrows act by the description given by the graph \( \Gamma_{Q,c}(\beta, r, \epsilon) \). So we take \( \{ v_{x}^{x} \otimes e_{j}^{h([v_{x}^{x}]_l)} \} \) the basis for \( \text{Hom}(P_{h([v_{x}^{x}]_l)}, V_\nu) \) for \( v_{x}^{x} \in S^1 \cup T^1 \), and \( \{ v_{x}^{x} \otimes e_{j}^{h([v_{x}^{x}]_l)} \} \) for \( v_{x}^{x} \in \text{ISO} \) and \( t = +, - \), relative to the aforementioned bases.

We will construct a graph \( \text{EXT} \) whose vertices correspond to a fixed basis for \( \text{Hom}(P(V_\mu)_x, V_\nu) \) as described above. We will partition the vertices into subsets \( \text{EXT}(i) \) for \( i = 0, 1, \ldots \) called levels. From this graph the homology of the complex can be easily read.

**Definition 4.2.10.** Let \( V_\mu \) be as described above. Let \( \text{EXT}(l) \) be the sets defined as follows.

\[
\text{EXT}(0) = \{ v_{y}^{x} \otimes v_{y'}^{x} \} \quad v_{y}^{x} \in S(\Gamma) \\
\text{EXT}(1) = \{ v_{y}^{x} \otimes v_{y'}^{x} \} \cup \{ v_{y}^{x} \otimes v_{y'}^{h([v_{y}^{x}]_l)} \} \quad v_{y}^{x} \in T^1 \cup S^1 \\
\text{EXT}(l) = \{ v_{y}^{x} \otimes v_{y'}^{h([v_{y}^{x}]_l)} \} \cup \{ v_{y}^{x} \otimes v_{y'}^{h([v_{y}^{x}]_l)} \} \\
\text{EXT}(l) = \{ v_{y}^{x} \otimes v_{y'}^{h([v_{y}^{x}]_l)} \} \cup \{ v_{y}^{x} \otimes v_{y'}^{h([v_{y}^{x}]_l)} \} \quad v_{y}^{x} \in \text{ISO} \\
\text{EXT}(l) = \{ v_{y}^{x} \otimes v_{y'}^{h([v_{y}^{x}]_l)} \} \cup \{ v_{y}^{x} \otimes v_{y'}^{h([v_{y}^{x}]_l)} \} \\
\text{EXT}(l) = \{ v_{y}^{x} \otimes v_{y'}^{h([v_{y}^{x}]_l)} \} \cup \{ v_{y}^{x} \otimes v_{y'}^{h([v_{y}^{x}]_l)} \} \\
\text{EXT}(l) = \{ v_{y}^{x} \otimes v_{y'}^{h([v_{y}^{x}]_l)} \} \cup \{ v_{y}^{x} \otimes v_{y'}^{h([v_{y}^{x}]_l)} \}
\]

and \( \text{EXT} \) the graph with vertices \( \bigcup_{l \geq 0} \text{EXT}(l) \) and edges given by

a. \( v_{y}^{x} \otimes v_{y'}^{x} \longrightarrow v_{y}^{y} \otimes v_{y'}^{y} \) if

\[
\text{Hom}(\delta(V_\mu)_0, V_\nu) : \quad v_{y}^{x} \otimes v_{y'}^{x} \mapsto \sum s_{j,j',x}^{i,i',y,y'} v_{y}^{x} \otimes v_{y'}^{y}
\]

with \( s_{j,j',x}^{i,i',y,y'} \neq 0 \) between levels \( \text{EXT}(0) \) and \( \text{EXT}(1) \);
4.2.3 Properties of the \(\text{EXT}\)-graph

We collect now the properties of the \(\text{EXT}\) graph that will be used to show exactness of complex \(\text{Hom}(P(V_\mu)_\bullet, V_\nu)\).

**Proposition 4.2.11.** Let \(\text{EXT}\) be the graph given above

**E1.** There is an edge

\[
\text{EXT}(0) \ni v_j^x \otimes \nu_j^x \quad \text{if} \quad v_j^x \in S^2, \quad v_i^y \in T^2, \quad v_j^x \xrightarrow{p} v_i^y \quad \text{and} \quad v_j^x \xrightarrow{p'} v_i^y
\]

in the graph \(\text{EXT}\) if \(v_j^x \in S^2, \quad v_i^y \in T^2, \quad v_j^x \xrightarrow{p} v_i^y \quad \text{and} \quad v_j^x \xrightarrow{p'} v_i^y\) are paths in \(\Gamma\) with \(A(p) = A(p')\).

**E2.** If \(v_i^y \in T^1, \quad v_j^x = l^\pm(v_i^y) \in S(\Gamma)\) and \(p = lp^\pm(v_i^y)\), then there is an edge

\[
\text{EXT}(0) \ni v_j^x \otimes \nu_j^x \quad \text{if} \quad v_j^x \xrightarrow{p} v_i^y \quad \text{is a path in} \quad \Gamma \quad \text{with} \quad A(p') = [v_i^y]_1 A(p)\].

Furthermore, there is an edge

\[
\text{EXT}(l) \ni v_i^y \otimes v_i^{h([v_i^y]_l)} \quad \text{if} \quad v_i^y \xrightarrow{e} v_i^{h([v_i^y]_{l+1})} \quad \text{in} \quad \Gamma \quad \text{with} \quad w(e) = [v_i^y]_{l+1}
\]
E3. Similarly, if \( v_i^y \in S^1 \), then there is an edge

\[
\text{EXT}(0) \ni v_i^y \otimes v_i^y \rightarrow v_i^y \otimes v_j^h([v_i^y]_{1+t}) \in \text{EXT}(1)
\]

in \( \text{EXT} \) if there is an edge \( v_i^y \rightarrow v_j^h([v_i^y]_{1+t}) \) with \( w(e) = [v_i^y]_{1+t} \). Furthermore, there is an edge

\[
\text{EXT}(l-1) \ni v_i^y \otimes v_j^h([v_i^y]_{1+t-1}) \rightarrow v_i^y \otimes v_j^h([v_i^y]_{1+t}) \in \text{EXT}(l) \quad \text{in} \ \text{EXT} \text{ if there is an edge } v_j^h([v_i^y]_{1+t-1}) \rightarrow v_j^h([v_i^y]_{1+t}) \text{ in } \Gamma \text{ with } w(e) = [v_i^y]_{1+t}.
\]

E4. Finally, if \( v_i^y \in ISO \), then there is an edge

\[
\text{EXT}(0) \ni v_i^y \otimes v_i^y \rightarrow v_i^y \otimes v_j^h([v_i^y]_{1+t}) \in \text{EXT}(1)
\]

in \( \text{EXT} \) if there is an edge \( v_i^y \rightarrow v_j^h([v_i^y]_{1+t}) \) in \( \Gamma \) with \( w(e) = [v_i^y]_{1+t} \). Furthermore, there is an edge

\[
\text{EXT}(l-1) \ni v_i^y \otimes v_i^y \rightarrow v_i^y \otimes v_j^h([v_i^y]_{1+t-1}) \in \text{EXT}(l)
\]

in \( \text{EXT} \) if there is an edge \( v_j^h([v_i^y]_{1+t-1}) \rightarrow v_j^h([v_i^y]_{1+t}) \) in \( \Gamma \) with \( w(e) = [v_i^y]_{1+t} \).

Lemma 4.2.3. There are no isolated vertices in \( \text{EXT}(1) \).

Proof. First, suppose \( v_i^y \otimes v_i^y \in \text{EXT}(1) \) (i.e., \( v_i^y \in T^2 \)). If \( i' < i \) (resp. \( i' > i \)), then by lemma 4.1.4, there is a path \( p' \) terminating at \( v_i^y \) with \( A(p') = lp^-(v_i^y) \) (resp. \( A(p') = lp^+(v_i^y) \)). Therefore, there is an edge \( v_j^y \otimes v_j^y \rightarrow v_i^y \otimes v_i^y \).

Next, suppose \( v_i^y \in T^1 \), and \([v_i^y]_1 \) exists (otherwise, no vertex \( v_i^y \otimes v_i^y \) would exist in \( \Gamma \)). Let \( p \) be the path of maximal length terminating at \( v_i^y \), and \( v_j^y \) the source at which \( p \) starts. Label the edge of \( p \) containing \( y \) by \( a_1 \), let \( b_2 := [v_j^y]_1 \), and \( b_1 \) the arrow (if it exists) with \( h(b_1) = y \) and \( c(b_1) = c(b_2) \). By lemma 4.1.5, \( r(b_1) + r(b_2) < \beta_y \). Now denote by \( b_3 \) the arrow \([v_i^y]_2 \). By lemma 4.1.6, \( r(b_2) + r(b_3) = \beta_{h_{b_2}} \), so \( v_i^y \otimes v_i^y \) is contained in an edge.
with such a label. If said label is \( b_2 \), then \( c_p^{y'} \in \text{imb}_2 A(p) \), and so \( v_i^y \otimes v_i^{h b_2} \) is contained in an edge between \( \text{EXT}(1) \) and \( \text{EXT}(0) \). Otherwise, \( b_3 c_i^{h b_2} = e_{y''} \neq 0 \). In this case, \( v_i^y \otimes v_i^{h b_2} \in \text{EXT}(1) \) and \( v_i^y \otimes v_i^{h b_2} \in \text{EXT}(2) \) are contained in an edge.

Finally, suppose that \( v_i^y \in \text{ISO} \), and let \( y' = h([v_i^y]_1^+) \) or \( h([v_i^y]_1^-) \). We will show that \( v_i^y \otimes v_i^{y'} \) is non-isolated for \( i = 1,\ldots,\beta_y' \). Note first that \( v_i^{y'} \) is non-isolated in \( \Gamma \) by lemma 4.1.6, for suppose that \( a_0 \) is the arrow (if it exists) with \( h(a_0) = y \), and \( c(a_0) = c([v_i^y]_1^+) \). By lemma 4.1.5, \( r(a_0) + r([v_i^y]_1^+) < \beta_y \), so by lemma 4.1.6, \( r([v_i^y]_1^+) + r([v_i^y]_2^+) = \beta_y' \). Therefore, there is an edge \( e \) incident to \( v_i^{y'} \) such that \( w(e) = [v_i^y]_1^+ \) or \( [v_i^y]_2^+ \). In the former case, \( v_i^y \otimes v_i^{y'} \) is contained in a common edge with a vertex in \( \text{EXT}(0) \), and in the latter case it is contained in a common edge with a vertex in \( \text{EXT}(2) \).

**Lemma 4.2.4.** All vertices in \( \text{EXT} \) are contained in at most two edges, and every vertex with label \( v_i^y \otimes v_i^{h([v_i^y])} \) for \( l \geq 1 \) is contained in at most one edge. Furthermore, the neighbor of any vertex \( v_i^y \otimes v_i^{h([v_i^y])} \) in \( \text{EXT}(l) \) is \( v_i^y \otimes v_i^{h([v_i^y]_l-1)} \) or \( v_i^y \otimes v_i^{h([v_i^y]_l+1)} \) for some \( v'' \). Therefore, the graph \( \text{EXT} \) splits into string and band components, such that the band components and strings of length greater than one occur between levels \( \text{EXT}(0) \) and \( \text{EXT}(1) \).

**Proof.** Recall from property E2 that \( v_i^y \otimes v_i^{h([v_i^y])} \) is connected by an edge to \( v_j^y \otimes v_j^{y'} \in \text{EXT}(0) \) if and only if \( v_i^y \in T^1 \), \( v_j^y \leadsto v_i^y \) is the longest left direct path in \( \Gamma \) ending at \( v_i^y \), and there is a path \( v_j^y \leadsto v_i^y \) with \( A(p') = [v_i^y]_1 A(p) \). It is clear that there is only one such vertex, if it exists. If such a path does exist, then there is no edge in \( \text{EXT} \) between \( v_i^y \otimes v_i^{h([v_i^y])} \) and \( v_i^y \otimes v_i^{h([v_i^y]_2)} \), since this would mean that \( v_i^{h([v_i^y])} \) and \( v_i^{h([v_i^y]_1)} \) are contained in an edge \( e \) in \( \Gamma \) with \( w(e) = [v_i^y]_2 \). This contradicts proposition 4.1.3, since \( v_i^{h([v_i^y])} \) would be in two edges of the same color. Otherwise, \( v_i^y \otimes v_i^{h([v_i^y])} \) is connected to the vertex \( v_i^y \otimes v_i^{h([v_i^y]_2)} \) in \( \text{EXT} \) if and only if there is an edge \( v_i^{h([v_i^y])} \leadsto v_i^{h([v_i^y]_2)} \) with \( w(e) = [v_i^y]_2 \), by property E3. By definition of the Up and Down graph, this describes a unique vertex.

As for the other vertices, the lemma is clear from property E1. \( \square \)
In terms of the complex $\text{Hom}(P(V_\mu), V_\nu)$, the above lemma says that the kernel of the map $\text{Hom}(\delta_2, V_\nu)$ is spanned by the elements $\{v_i^y \otimes v'_{i'} \mid v_i^y \in T^2, i' = 1, \ldots, \beta_y\}$ together with those of $\{v_i^y \otimes v^{h([v_i^y])}_{i'} \mid v_i^y \in T^1 \cup S^1, i' = 1, \ldots, \beta_h([v_i^y])\}$ which share no edge with vertices in $\text{EXT}(2)$.

**Lemma 4.2.5.** *No string in $\text{EXT}$ has both endpoints in $\text{EXT}(1)$.***

*Proof.* Suppose that there is a string with one endpoint $v_{j_0}^y \otimes v_{j_0}'$ in $\text{EXT}(1)$ and containing the following substring:

$$
\begin{array}{c}
\cdots \\
v_{i_2}^x \otimes v_{i_2}' \quad \quad v_{j_1}^y \otimes v_{j_1}' \\
v_{i_1}^x \otimes v_{i_1}' \quad \quad v_{j_0}^y \otimes v_{j_0}'
\end{array}
$$

with $v_{i_t}^x \otimes v_{i_t}' \in \text{EXT}(0)$ and $v_{j_s}^y \otimes v_{j_s}' \in \text{EXT}(1)$. We will show that the string does not end in the vertex $v_{j_n}^y \otimes v_{j_n}'$. Recall by definition of the graph $\text{EXT}$ that for such a string to exist, we must have paths

$$
\begin{array}{c}
\cdots \\
v_{i_2}^x \otimes v_{i_2}' \quad \quad v_{j_1}^y \otimes v_{j_1}' \\
v_{i_1}^x \otimes v_{i_1}' \quad \quad v_{j_0}^y \otimes v_{j_0}'
\end{array}
$$

with $v_{i_t}^x \otimes v_{i_t}' \in \text{EXT}(0)$ and $v_{j_s}^y \otimes v_{j_s}' \in \text{EXT}(1)$. We will show that the string does not end in the vertex $v_{j_n}^y \otimes v_{j_n}'$. Recall by definition of the graph $\text{EXT}$ that for such a string to exist, we must have paths

$$
\begin{array}{c}
\cdots \\
v_{i_2}^x \otimes v_{i_2}' \quad \quad v_{j_1}^y \otimes v_{j_1}' \\
v_{i_1}^x \otimes v_{i_1}' \quad \quad v_{j_0}^y \otimes v_{j_0}'
\end{array}
$$
in $\Gamma$. A small notational point: if $p$ is a direct path which starts (resp. ends) in the vertex $v^i_j$, with $e$ the edge of $p$ incident to said vertex, then we write $\epsilon(z, p) := \epsilon(z, c(w(e)))$.

**Case 1:** Assume that $v_{j_0}^{y_0}, v_{i_n}^{y_n} \in T^2$. Let $p_n$ be the longest left path terminating in $v_{i_n}^{y_n}$ with $p_n \neq q_n$ (this is guaranteed since $v_{i_n}^{y_n}$ is a 2-target). Similarly, let $q_0$ be the longest left path terminating in $v_{j_0}^{y_0}$ with $q_0 \neq p_0$.

**A:** If $i'_{0} < i_0$, then $\epsilon(y_0, p_0) = -1$. If not, then by lemma 4.1.4 there would be a path $q'_0$ terminating at $v_{i_0}^{y_0}$ in $\Gamma$ with $A(q'_0) = A(q_0)$. By definition of the graph $\text{EXT}$, then, there would be an other edge terminating at the vertex $v_{i_0}^{y_0} \bowtie v_{i'_0}^{y_0}$.

**A1:** If $i'_{n} > i_n$, then $\epsilon(y_n, q_n) = 1$ by lemma 4.1.4. Thus, by proposition 4.1.3, $\epsilon(y_n, p_n) = 1$. Therefore, again by lemma 4.1.4, there is a path $p'_n$ in $\Gamma$ terminating at $v_{i'_n}^{y_n}$ with $A(p'_n) = A(p_n)$, so there is another edge in $\text{EXT}$ containing the vertex $v_{i_n}^{y_n} \bowtie v_{i'_n}^{y_n}$.

**A2:** If $i'_{n} < i_n$, then $\epsilon(y_n, q_n) = 1$ by lemma 4.1.4. Thus, by proposition 4.1.3, $\epsilon(y_n, p_n) = -1$. Therefore, again by lemma 4.1.4, there is a path $p'_n$ in $\Gamma$ terminating at $v_{i'_n}^{y_n}$ with $A(p'_n) = A(p_n)$, so there is another edge in $\text{EXT}$ containing the vertex $v_{i_n}^{y_n} \bowtie v_{i'_n}^{y_n}$.

**B:** If $i'_{0} > i_0$, then $\epsilon(y_0, p_0) = 1$, by the same reasoning at Subcase A. The subcases B1 and B2 are analogous to A1 and A2.

**Case 2:** Assume that $v_{j_0}^{y_0} \in T^2$ while $v_{i_n}^{y_n} \in T^1$. We will show that Let $(i'_n)^-$ be the integer such that there is an edge $v_{(i'_n)^-}^{y_n} \xrightarrow{e} v_{i_n}^{h([v_{i_n}^{y_n}]_1)}$ in $\Gamma$ with $w(e) = [v_{i_n}^{y_n}]_1$. This is guaranteed to exist by the definition of $[v_{i_n}^{y_n}]_1$ (refer to property E2 in proposition 4.2.11).

**A:** Suppose $(i_0') < i_0$. Then $\epsilon(y_0, p_0) = -1$ by definition of $\Gamma$.

**A1:** If $(i'_n)^- < i_n$, then $\epsilon(y_n, q_n) = 1$, and since there is a path $eq_n$ in $\Gamma$ with $w(e) = [v_{i_n}^{y_n}]_1$, we must have that $\epsilon(y_n, [v_{i_n}^{y_n}]_1) = -1$. If this were the case, then
by the definition of the edges in $\Gamma$, there would be an edge $e'$ with $w(e') = [v_{i_n}^{y_n}]_1$ with one end at the vertex $v_{i_n}^{y_n}$. This contradicts the assumption that $v_{i_n}^{y_n}$ is a 1-target.

A2: Similarly, if $i_n^- > i_n$, then $\epsilon(y_n, q_n) = 1$, and since there is a path $eq_n$ in $\Gamma$ with $w(e) = [v_{i_n}^{y_n}]_1$, we have that $\epsilon(y_n, [v_{i_n}^{y_n}]_1) = -1$. If this were the case, then there would be an edge $e'$ with $w(e') = [v_{i_n}^{y_n}]_1$ with one end at the vertex $v_{i_n}^{y_n}$, contradicting the assumption of $v_{i_n}^{y_n}$ being a 1-target.

B: Suppose that $(i_0') > i_0$. Then $\epsilon(y_0, p_0) = 1$ by definition of $\Gamma$. Subcases b1 and b2 are the same as above with signs of $\epsilon$ flipped.

Case 3: Assume that $v_{i_0}^{y_0} \in T^1$ and $v_{i_n}^{y_n} \in T^2$. Let $p_n$ be the left direct path in $\Gamma$ of maximal length with endpoint $v_{i_n}^{y_n}$ and $p_n \neq q_n$ (guaranteed since the vertex is a 2-target).

As above, let $(i_0')^-$ be the integer such that there is an edge $e$ with endpoints $v_{(i_0')^-}^{y_0}$ and $v_{(i_0')^+}^{y_0}$.

A: Suppose that $(i_0')^- < i_0$. Then $\epsilon(y_0, [v_{i_0}^{y_0}]_1) = 1$, so $\epsilon(y_0, p_0) = -1$.

A1: If $i_n' < i_n$, then $\epsilon(y_n, q_n) = 1$, so $\epsilon(y_n, p_n) = -1$. But then by lemma 4.1.4, there is an edge $p_n'$ with $A(p_n) = A(p_n')$ one of whose endpoints is $v_{i_n}^{y_n}$.

A2: If $i_n' > i_n$, then $\epsilon(y_n, q_n) = 1$, $\epsilon(y_n, p_n) = 1$. By lemma 4.1.4, there is an edge $p_n'$ with $A(p_n) = A(p_n')$ one of whose endpoints is $v_{i_n}^{y_n}$.

B: If $(i_0')^- > i_0$, then the same arguments hold with the values of $\epsilon$ exchanged.

Case 4: Assume that $v_{i_0}^{y_0}, v_{i_n}^{y_n} \in T^1$.

A: Suppose $(i_0')^- < i_0$, so $\epsilon(y_0, [v_{i_0}^{y_0}]_1) = -1$ and $\epsilon(y_0, p_0) = 1$.

A1: If $(i_n')^- < i_n$, then $\epsilon(y_n, q_n) = -1$ by lemma 4.1.4. But if this were the case, then there would be an edge $e$ in $\Gamma$ with $w(e) = [v_{i_n}^{y_n}]_1$ and one of whose endpoints was $v_{i_n}^{y_n}$. This contradicts the assumption that said vertex was a 1-target.
A2: If \((i'_n) - i_n\) > \(i_n\), then \(\epsilon(y_n, q_n) = 1\) by lemma 4.1.4. If this were the case, then there would be an edge \(e\) in \(\Gamma\) with \(w(e) = [v^y_{i_n}]\) and one of whose endpoints was \(v^y_{i_n}\). This contradicts the assumption that said vertex was a 1-target.

B: If \((i'_0) - i_0\) < \(i_0\), then the same argument holds with the values of \(\epsilon\) exchanged.

\[\square\]

### 4.2.4 Homology and the EXT graph

Let us pause to interpret the above results into data concerning the maps \(\text{Hom}(\delta(V_\mu)_1, V_\nu)\) and \(\text{Hom}(\delta(V_\mu)_0, V_\nu)\). Recall that a vertex \(v_i^x \otimes v_j^y\) corresponds to the basis element \(v_i^x \otimes e_j^y\). By lemma 4.2.3, there are no isolated vertices in \(\text{EXT}(1)\), and by lemma 4.2.4, if \(\text{Hom}(\delta(V_\mu)_1, V_\nu) : v_i^x \otimes e_j^y \mapsto v'_i^x \otimes e'_j^y\), then after reordering the chosen basis, \(\text{Hom}(\delta(V_\mu)_1, V_\nu)\) takes the form

\[
\begin{bmatrix}
1 & 0 & \ldots & 0 \\
0 & * & \ldots & * \\
\vdots & \vdots & \ddots & \vdots \\
0 & * & \ldots & * \\
\end{bmatrix}
\]

In particular, \(\ker(\text{Hom}(\delta(V_\mu)_1, V_\nu))\) is precisely the span of those vertices in \(\text{EXT}(1)\) that have an edge in common with a vertex in \(\text{EXT}(0)\).

It remains to be shown that every other vertex in \(\text{EXT}(1)\) corresponds to a basis element that is in the image of \(\text{Hom}(\delta(V_\mu)_0, V_\nu)\). This will show that the image of said map equals the kernel of \(\text{Hom}(\delta(V_\mu)_1, V_\nu)\). Let us denote by \(C_1, C_2, \ldots, C_m\) the connected components of the induced subgraph on the vertices \(\text{EXT}(0) \cup \text{EXT}(1)\). Then \(\text{Hom}(\delta(V_\mu)_1, V_\nu)\) can be
written in block form:

\[
\begin{bmatrix}
\delta C_1 & 0 & \cdots & 0 \\
0 & \delta C_2 & \ddots & \\
\vdots & \ddots & \ddots & \\
0 & \cdots & 0 & \delta C_m
\end{bmatrix}
\]

Therefore, it suffices to show that each block corresponding to a connected component is surjective.

**Lemma 4.2.6.** If \( v^x_j \otimes v^y_i \in \text{EXT}(1) \) is contained in a string between levels 0 and 1, then \( v^x_j \otimes e^y_i \in \text{im}(\text{Hom}({\delta(V_\mu)}_0, V_\nu)) \).

**Proof.** Suppose that the vertex is contained in the connected component \( C_i \), and that \( C_i \) is a string. We have shown in lemma 4.2.5 that if a string is between levels 0 and 1, then either one endpoint lies in level 0 and the other in level 1, or both endpoints lie in level 0.

In the first case, \( \delta_{C_i} \) is strictly upper triangular with nonzero entries on the diagonal which must be from the set \( \{\pm 1, \pm \mu, \pm \nu\} \). Therefore, the map is invertible. In the second case, there is one more vertex in level \( \text{EXT}(0) \) than in \( \text{EXT}(1) \), and \( (\delta_{C_i})_{j,j} \neq 0 \) for each \( j \), so the given map is surjective. \( \square \)

**Lemma 4.2.7.** If \( C_i \) is a band, then \( \delta(C_i) \) is an isomorphism.

**Proof.** If a component \( C_i \) is cyclic, then it must come from the following cycles on \( \Gamma_{Q,c}(\beta, r, \epsilon) \):
In particular, by definition of $\delta(V_\mu)_1$, the matrix of $\delta(C_i)$ takes the following form:

$$
\begin{bmatrix}
1 & -\mu & 0 & 0 & \ldots & 0 \\
0 & \pm 1 & \pm 1 & 0 & \ldots & 0 \\
0 & 0 & \pm 1 & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\pm 1 & 0 & \ldots & \pm 1 \\
\end{bmatrix}
$$

where one of the diagonal entries is $\nu$, and in each row there is exactly one positive and one negative entry. Then it is an elementary exercise (expanding by the first column and calculating the determinant of upper or lower triangular matrices) to show that $\det \delta(C_i) = \pm (\mu - \nu)$. Since, by assumption, $\mu \neq \nu$, we have that $\delta(C_i)$ is nonsingular.

Now that part (a) of the theorem is proved, we move to part (b), recalled here:

**Proposition 4.2.12.** Suppose that $\Gamma_{Q,e}(\beta, r, \epsilon)$ consists of a single band component, and let $\mu \in (k^*)^{B(\Gamma)} = k^*$. Let $V_\mu = V_{Q,e}(\beta, r, \epsilon)_\mu$. Then

$$\text{Ext}^1_{kQ/I_e}(V_\mu, V_\mu) = 1.$$ 

**Proof.** The projective dimension of $V_\mu$ is one by the constructions above. Furthermore, there is exactly one band component in the graph $\mathbb{E}XT$, since there is exactly one pair of bands $b_1, b_2$ in $\Gamma$ with the $A(p_i) = A(p'_i)$ and $A(q_i) = A(q'_i)$ as in the proof of lemma 4.2.7. Therefore, the image of the restriction of the map $\text{Hom}(P(V_\mu)_\bullet, V_\mu)$ to the vectors $v_{i_k}^{x_k} \otimes e_{i_k}^{x_k}$ is in the span of the vectors $v_{i_k}^{y_k} \otimes e_{i_k}^{y_k}$. Again, as in the proof of lemma 4.2.7, the restriction
of said map to the aforementioned subspaces relative to the basis given above is

$$C = \begin{bmatrix}
-\mu \\
\pm \mu & \pm 1 \\
\pm 1 & \pm 1 \\
\pm 1 & \pm 1 & \ddots & \pm 1 & \pm 1 \\
\pm 1 & \pm 1 & \pm 1 & \pm 1 & \pm 1 & \ddots & \pm 1 & \pm 1 & \pm 1 \\
\end{bmatrix}.$$ 

Recall that in each row there is exactly one positive and one negative entry. Therefore, the sum of the last $n-1$ columns of this matrix is $[1 \pm 1 0 \ldots 0]$ where the sign of the second entry is opposite of the sign of $\pm \mu$. Therefore, the first column is in the span of the last $n-1$ columns. Column reducing gives the matrix

$$C = \begin{bmatrix}
0 \\
0 & \pm 1 \\
\pm 1 & \pm 1 \\
\pm 1 & \pm 1 & \ddots & \pm 1 & \pm 1 \\
\pm 1 & \pm 1 & \pm 1 & \pm 1 & \pm 1 & \ddots & \pm 1 & \pm 1 & \pm 1 \\
\end{bmatrix}.$$ 

The lower right $n-1 \times n-1$ minor is clearly non-zero, since it is a strictly lower triangular matrix, so this map has rank $n-1$, showing that the complex $\text{Hom}(P(V_{\mu}), V_{\mu})$ has exactly one dimensional homology at $\text{Hom}(P(V_{\mu})_1, V_{\mu})$.

**Proof of theorem 4.2.4** By lemma 4.2.6, blocks corresponding to strings on $\text{EXT}$ are surjective, and by lemma 4.2.7, blocks corresponding to bands on $\text{EXT}$ are surjective, so
the homology of the complex

$$\text{Hom}(P(V_\mu)_0, V_\nu) \rightarrow \text{Hom}(P(V_\mu)_1, V_\nu) \rightarrow \ldots$$

vanishes in the first degree.

Finally, we point out a corollary to the above proof that will be useful for describing the Schofield semi-invariants.

**Corollary 4.2.13.** Suppose that the generic module in $\text{Rep}_{Q,c}(\beta, r)$ consists of an indecomposable band module. Then $\det(\text{Hom}(\delta_0(V_\mu), V_\nu)) = \pm \mu^k \nu^l(\mu - \nu)$.

**Proof.** We have already shown that the restriction of the map $\text{Hom}(\delta_0(V_\mu), V_\nu)$ to the cyclic component of the $\text{EXT}$ graph is a multiple of $\mu - \nu$. Furthermore, for each of the string components, the entries on the diagonal are in the set $\{\pm 1, \pm \mu, \pm \nu\}$, and in the proof of lemma 4.2.6, we showed that these restrictions are upper-triangular. Therefore, for some powers $k, l$, the determinant is precisely $\pm \mu^k \nu^l(\mu - \nu)$.

\[\square\]

### 4.3 Higher Extension Groups

The graphical representation given above can be used to calculate higher extension groups.

For each vertex $v_j^x \in S^1 \cup T^1$, let $X_{j,x}$ be the complex

$$V_x \rightarrow^{[v_j^x]} V_{h([v_j^x]_1)} \rightarrow^{[v_j^x]_2} V_{h([v_j^x]_2)} \rightarrow^{[v_j^x]_3} \ldots$$

and if $v_j^x \in \text{ISO}$, let $X_{j,x}^+$ be the complex

$$V_x \rightarrow^{[v_j^x]^+} V_{h([v_j^x]^+_1)} \rightarrow^{[v_j^x]^+_2} \ldots,$$
and analogously for $X^{-\bar{x},\bar{j}}$. Let $h^i(X)$ be the dimension of the $i$-th homology space of the complex $X$.

**Corollary 4.3.1.** Let $\Gamma_{Q,c}(\beta, r, \epsilon)$ be an up-and-down graph for $(Q, c)$ a gentle string algebra. Then

$$\dim \operatorname{Ext}^i(V_{Q,c}(\beta, r, \epsilon)_{\mu}, V_{Q,c}(\beta, r, \epsilon)_{\nu}) = \sum_{v_j^x \in \mathbb{S}_1 \cup \mathbb{T}_1} h^i(X_{\bar{x},j}) + \sum_{v_j^x \in \text{ISO}(X_{\bar{x},j})} (h^i(X_{\bar{x},j}^+) + h^i(X_{\bar{x},j}^-)).$$

### 4.3.1 Example

We finish by exhibiting the EXT graph for example 4.1.3. Recall that we chose $\Theta(b) = v^{(6)}_1$ for the band component. By proposition 4.2.9, the projective resolution of the representation in the example is given by

$$V_{\mu} \xleftarrow{\delta_0} P_1^3 \oplus P_4 \xrightarrow{\delta_1} P_2 \oplus P_3 \oplus P_5 \oplus P_6 \xleftarrow{\delta_1} P_3$$

where

$$\delta_0 = \begin{bmatrix}
r_1 & 0 & 0 & 0 & -\mu b_2 g_1 \\
0 & 0 & -g_1 & 0 & p_2 r_1 \\
0 & 0 & 0 & g_1 & 0 \\
p_1 & -g_2 b_1 & 0 & 0 & 0 \\
0 & r_2 p_1 & b_1 & 0 & 0
\end{bmatrix}, \quad \delta_1 = \begin{bmatrix}
0 \\
g_2 \\
0 \\
0 \\
0
\end{bmatrix}.$$

The associated EXT graph is obtained by applying $\operatorname{Hom}(-, V_{\nu})$ to the resolution, so we have the complex:

$$\begin{array}{c}
(V_{\nu})_3^3 \oplus (V_{\nu})_4^2 \\
\xrightarrow{\operatorname{Hom}(\delta_0, V_{\nu})} (V_{\nu})_2 \oplus (V_{\nu})_3 \oplus (V_{\nu})_5^2 \oplus (V_{\nu})_6 \\
\xrightarrow{\operatorname{Hom}(\delta_1, V_{\nu})} (V_{\nu})_3
\end{array}$$
The \texttt{EXT} graph is depicted below, with the vertices lying in a cyclic component of the graph boxed.
Chapter 5

GIT Quotients

5.1 Dimension Combinatorics

In this section, we illustrate conditions on $\beta$ and $r$ under which the generic module of $\text{Rep}_{Q,c}(\beta, r)$ is a direct sum of band modules. Furthermore, in the case that $Q$ is acyclic, we consider some combinatorics of the Euler form $\langle\langle -, -\rangle\rangle_{kQ/I}$.

We will say that a pair $(\beta, r)$ consisting of a dimension vector and rank map is called a band pair if the generic module in $\text{Rep}_{Q,c}(\beta, r)$ is a direct sum of band modules, and called an exact pair if $r(i(x, s)) + r(o(x, s)) = \beta_x$ for every $(x, s) \in \mathcal{X}$.

**Proposition 5.1.1.** The pair $(\beta, r)$ is a band pair if and only if $(\beta, r)$ is an exact pair such that $\beta_x = 0$ for every lonely vertex $x$.

**Proof.** Suppose that $(\beta, r)$ is a band pair. First note that if $\beta_x = 0$ for a lonely vertex $x$, then there are two possibilities: if $(x, s) \in \mathcal{X}$ and one of $r(i(x, s)), r(o(x, s))$ is non-zero, then there is a vertex $v^x_i$ incident to exactly one edge (which is of color $s$), contradicting the assumption. On the other hand, if $(x, s) \in \mathcal{X}$ and both $r(i(x, s)), r(o(x, s))$ are zero. In this case, there is a vertex $v^x_i$ which is isolated, so $S_x$ is a direct summand of $V_{Q,c}(\beta, r)$. $S_x$ is not a band module, contradicting the assumption. Therefore, $\beta_x = 0$ whenever $x$
is a lonely vertex. Since \((\beta, r)\) is a band pair, each vertex \(v^x_i\) in \(\Gamma_{Q,c}(\beta, r, \epsilon)\) is incident to precisely two edges, and by proposition 4.1.3, each such vertex is contained in exactly one edge of color \(s\) if \((x, s) \in \mathcal{X}\). There are \(\beta_x\) vertices in the set \(\{v^x_i\}_{i=1,\ldots,\beta_x}\), of which \(r(i(x, s)) + r(o(x, s))\) are incident to edges of color \(s\). Therefore, \(\beta_x = r(i(x, s)) + r(o(x, s))\).

On the other hand, suppose that \(\beta_x = 0\) whenever \(x\) is a lonely vertex, and \(\beta_x = r(i(x, s)) + r(o(x, s))\) for all \(x\). This means that for each \((x, s) \in \mathcal{X}\), every vertex in the set \(\{v^x_i\}_{i=1,\ldots,\beta_x}\) is contained in exactly one edge of color \(s\). Since \(\beta_x \neq 0\), and by assumption \(x\) is not a lonely vertex, there is another element \((x, s') \in \mathcal{X}\) with \(s' \neq s\) and, by assumption, \(\beta_x = r(i(x, s')) + r(o(x, s'))\), so every vertex in the set \(\{v^x_i\}_{i=1,\ldots,\beta_x}\) is contained in exactly one edge of color \(s'\). By proposition 4.1.3, each such vertex is contained in at most two edges, and we have just shown that it is contained in at least two edges. Since every vertex is contained in exactly two edges, the graph \(\Gamma_{Q,c}(\beta, r, \epsilon)\) consists only of band components.

**Proposition 5.1.2.** Suppose that \((\beta, r)\) is a band pair such that the generic module of \(\text{Rep}_{Q,c}(\beta, r)\) is an indecomposable band. Then \((n\beta, nr)\) is a band pair and the generic module of \(\text{Rep}_{Q,c}(n\beta, nr)\) is a direct sum of \(n\) copies of \(V_{Q,c}(\beta, r)\).

**Proof.** In the course of proving theorem 4.2.4, we showed that \(\text{Ext}^1(V_{Q,c}(\beta, r)_\mu, V_{Q,c}(\beta, r)_\nu) = 0\) when \(\mu \neq \nu\). Therefore, \(\text{Rep}_{Q,c}(\beta, r) \oplus \ldots \oplus \text{Rep}_{Q,c}(\beta, r)\) is a generic component in \(\text{Rep}_{Q,c}(\beta)\) by Crawley Boevey-Schröer ([10]). But it contains an element \(V\) such that \(\text{rank}_k V_a = r(a)\) (specifically \(V = V_{Q,c}(n\beta, nr)\)), so the generic module of \(\text{Rep}_{Q,c}(n\beta, nr)\) is a direct sum of \(n\) copies of \(V_{Q,c}(\beta, r)\) as claimed.

We will now explore the Euler form on dimension vectors of generic band modules (refer to section 2.1.1 for detailed definitions). We will denote by \(E_A\) the matrix associated to this bilinear form, so that for two vertices \(x, y\), \((E_A)_{x,y} = \sum_{i \geq 0} (-1)^i \dim \text{Ext}^1_{A}(S_x, S_y)\). We will show that if \((\beta, r)\) is a band pair, then \(\langle \langle \beta, \beta \rangle \rangle = 0\). This will be exploited in section...
5.2 to determine the structure of some GIT quotients.

Suppose that $kQ/I_c$ is a gentle string algebra. Recall that if $s \in S$ is a color, then $Q_0(s)$ is defined to be the set of vertices $x \in Q_0$ such that $(x, s) \in \mathfrak{X}$. If $x, y \in Q_0(s)$, let $d_s(x, y)$ be the length of the path of color $s$ with endpoints $x, y$. Such a path clearly exists since $c^{-1}(s)$ is a direct path, and $x, y$ are vertices in this path. We can define a total order on $Q_0(s)$ by $x \leq_s y$ if $x$ appears before $y$ in the path $c^{-1}(s)$. For the remainder of this section, we fix $kQ/I_c$, a triangular gentle string algebra (i.e., such that $Q$ has no oriented cycles), and a sign function $\epsilon : \mathfrak{X} \to \{\pm 1\}$.

Proposition 5.1.3. Let $E_{kQ/I_c} = E$ be the Euler matrix for $kQ/I_c$. I.e., the matrix associated to the bilinear form $\langle \langle - , - \rangle \rangle$. Then

$$E_{x,y} = \sum_{\{s \in S \mid \{x,y\} \subset Q_0(s) \atop x \leq_s y\}} (-1)^{d_s(x,y)}$$

Proof. Notice that $S_x$ is an up-and-down module, so we have already constructed a projective resolution of $S_x$ in section 4.2. Namely, we take $[x]_0^+$ (resp. $[x]_0^-$) to be the arrow (if it exists) with $t([x]_0^+) = x$ and $\epsilon(x, c([x]_0^+)) = 1$ (resp. $t([x]_0^-) = x$ and $\epsilon(x, c([x]_0^-)) = -1$). Recursively define $[x]_{i+1}^\pm$ to be the arrow with $t([x]_{i+1}^+) = h([x]_{i-1}^+), c([x]_{i+1}^+) = c([x]_0^+)$. Taking $P(i) = P_{h([x]_i^+)} \oplus P_{h([x]_i^-)}$ (where, if either doesn’t exist, the summand is suppressed), the projective resolution of $S_x$ is

$$S_x \leftarrow P(0) \leftarrow P(1) \leftarrow \ldots \leftarrow P(i-1) \leftarrow P(i) \leftarrow \ldots$$

Applying $\text{Hom}_{kQ/I_c}(-, S_y)$ to the projective resolution above (which has finite length, since $kQ/I$ was taken to be of finite global dimension), we have the complex with 0 differential and which, in degree $i$, is a vector space of dimension equal to the number of those ver-
vertices \( h([x]_{i}^{\pm 1}) \) which are precisely \( y \). The Euler characteristic of the complex is precisely
\[
\sum_{\{s \in S : \{x, y\} \subset Q_0(s), x \leq s, y \}} (-1)^{d_s(x, y)}.
\]

Lemma 5.1.1. Suppose that \( \beta, r \) is an exact pair. Then for each \( s \in S \) and each \( y \in Q_0(s) \),
\[
\sum_{x \in Q_0(s)} (-1)^{d_s(x, y)} \beta_x = 0.
\]

Proof. Suppose that \( V \in \text{Rep}_{Q,c}(\beta, r) \) is a module in the open set (so \( \text{rank}_a V_a = r(a) \) for each \( a \in Q_1 \)). Consider the statement in terms of complexes. If \( x(s)_0, x(s)_1, \ldots, x(s)_{l(s)} \) are the vertices incident to an arrow of color \( s \) so that \( x(s)_i <_s x(s)_{i+1} \), and \( a(s)_1, \ldots, a(s)_{l(s)} \) are the arrows of color \( s \) with \( h(a(s)_i) = x(s)_i \), then the complex
\[
0 \rightarrow V_{x(s)_0} \xrightarrow{a(s)_1} V_{x(s)_1} \xrightarrow{a(s)_2} \cdots \xrightarrow{a(s)_{l(s)}} V_{x(s)_{l(s)}} \rightarrow 0
\]
is exact, so has an Euler characteristic of 0. But the Euler characteristic of the above complex is \( \sum_{i=0}^{l(s)} (-1)^{i} \beta_{x(s)_i} \). Notice that \( d_s(x(s)_i, x(s)_j) = |j - i| \), so indeed, the sum is equal (up to sign change) to \( \sum_{x \in Q_0(s)} (-1)^{d_s(x, y)} \beta_x \), so the latter expression is also zero. \( \square \)

We are now prepared to prove the main proposition.

Proposition 5.1.4. If there is a rank map \( r \) such that \( (\beta, r) \) is a band pair, then \( q(\beta) = 0 \).

Proof. Consider the symmetric form \( (\alpha, \beta) = \langle\langle \alpha, \beta \rangle\rangle + \langle\langle \beta, \alpha \rangle\rangle \), and let \( \tilde{E} = E + E^T \) be its associated matrix. For any \( s \in S \), let \( E(s) \) be the \( Q_0 \times Q_0 \) matrix with
\[
E(s)_{x,y} = \begin{cases} (-1)^{d_s(x,y)} & \text{if } \{x,y\} \subset Q_0(s), x <_s y \\ 0 & \text{otherwise} \end{cases}
\]

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Notice that $E = I + \sum_{s \in S} E(s)$, so $\tilde{E} = 2I + \sum_{s \in S} \tilde{E}(s)$, where

\[
\tilde{E}(s) := E(s) + E(s)^T = \begin{cases} 
(-1)^{d_s(x,y)} & \text{if } \{x, y\} \subset Q_0(s), \ x \neq y \\
0 & \text{otherwise}
\end{cases}
\]

By proposition 5.1.1, $(\beta, r)$ is an exact pair. From the above description, we have that

\[
(\tilde{E}(s)\beta)_y = \sum_{x \in Q_0(s) \setminus y} (-1)^{d_s(x,y)} \beta_x
\]

\[
= \left( \sum_{x \in Q_0(s)} (-1)^{d_s(x,y)} \beta_x \right) - \beta_y
\]

\[
= -\beta_y
\]

where the last equality is by lemma 5.1.1. We are now prepared to calculate $(\beta, \beta)$:

\[
(\beta, \beta) = \beta^T \tilde{E} \beta = 2 \beta^T I \beta + \sum_{s \in S} \beta^T \tilde{E}(s) \beta
\]

\[
= 2 \sum_{x \in Q_0} \beta_x^2 + \sum_{s \in S} \beta^T (\tilde{E}(s) \beta)
\]

\[
= 2 \sum_{x \in Q_0} \beta_x^2 + \sum_{s \in S} \sum_{y \in Q_0(s)} -\beta_y^2,
\]

By proposition 5.1.1, if $y$ is a lonely vertex, then $\beta_y = 0$, so if $y$ is not a lonely vertex, there are precisely two elements $s \in S$ with $y \in Q_0(s)$. Therefore,

\[
\beta^T \tilde{E} \beta = 2 \sum_{x \in Q_0} \beta_x^2 + \sum_{y \in Q_0} -2\beta_y^2
\]

\[
= 0.
\]

Since $q(\beta) = \langle \langle \beta, \beta \rangle \rangle = \frac{1}{2}(\beta, \beta)$, this concludes the proof. \qed
5.2 GIT Quotients

We now calculate the GIT-quotients for faithful band components of representation spaces. Recall that a component of $\text{Rep}_{Q,c}(\beta, r)$ is called faithful if the annihilator of its generic module is precisely the ideal $I_c$. Throughout this section, $kQ/I_c$ will denote a gentle string algebra that is triangular, that is, $Q$ has no oriented cycles. In particular, $kQ/I_c$ has finite projective dimension. We will say a pair $(\beta, r)$ is indecomposable if the generic module in $\text{Rep}_{Q,c}(\beta, r)$ is indecomposable, and faithful if the component itself is.

**Proposition 5.2.1.** Suppose that $kQ/I_c$ is a triangular gentle string algebra, and $(\beta, r)$ an indecomposable, faithful, band pair. Then $\text{SI}_{Q,c}(\beta, r, ((\beta, -)))$ is two-dimensional.

**Proof.** Let us denote by $\chi$ the weight $\langle\langle \beta, - \rangle \rangle$. Since $(\beta, r)$ is faithful, $\text{SI}_{Q,c}(\beta, r, \chi)$ is the span of the functions $c^V$ such that $\dim V = \beta$ and is of projective dimension 1 (see proposition 2.3.2). From a corollary of Derksen-Fei ([12] corollary 2.6) it follows that the set of modules of projective dimension 1 is irreducible, let us call it $Z_{p.d.1}$. We have already seen that the generic module in $\text{Rep}_{Q,c}(\beta, r)$ is of projective dimension 1, so $V_{Q,c}(\beta, r) \subset Z_{p.d.1} \subset \text{Rep}_{Q,r}(\beta, r)$. In particular $V_{Q,c}(\beta, r)$ is dense in $Z_{p.d.1}$. So we can view $c^\gamma$ as a map $Z_{p.d.1} \to \text{SI}_{Q,c}(\beta, r, \chi)$, and rephrase proposition 2.3.2 as saying that $\text{span}\{f \in \text{image } c^\gamma\} = \text{SI}_{Q,c}(\beta, r)$. Since $V_{Q,c}(\beta, r) \subset Z_{p.d.1}$ is dense, the above span is precisely equal to $\text{span}\{c^V \mid V_{Q,c}(\beta, r)\}$.

Let us write $V_\mu$ for the indecomposable element in $V_{Q,c}(\beta, r)$ corresponding to $\mu \in k^*$. Let $\nu \neq \lambda \in k^*$.

**Claim.** If $\mu \neq \nu \in k^*$, then $c^{V_\mu}$ and $c^{V_\nu}$ are independent.

Indeed, recall from corollary 4.2.13 that $c^{V_\mu}(V_x) = \mu^k x^l (\mu - x)$ for $\mu, x \in k^*$. Therefore, $c^{V_\nu}(V_\mu) = c^{V_\nu}(V_\nu) = 0$ while $c^{V_\mu}(V_\nu) \neq 0$ $c^{V_\nu}(V_\mu) \neq 0$.

**Claim.** $c^{V_\gamma} \in \text{span}\{c^{V_\mu}, c^{V_\nu}\}$ for every $\gamma \in k^*$. 

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Let
\[ \alpha_{\mu} = \left( \frac{\gamma}{\mu} \right)^k \left( \frac{\gamma - \nu}{\mu - \nu} \right) \]
\[ \alpha_{\nu} = \left( \frac{\gamma}{\nu} \right)^k \left( \frac{\gamma - \mu}{\nu - \mu} \right). \]

Then \( c^{V_\gamma} = \alpha_{\mu} c^{V_\mu} + \alpha_{\nu} c^{V_\nu} \). Indeed, we need only check this equality on \( V_z \) for all \( z \in k^* \) since each function is semi-invariant of the same weight, and the union of the orbits of these modules is dense in \( \text{Rep}_{Q,c}(\beta, r) \).

\[
\begin{align*}
\alpha_{\mu} c^{V_\mu}(V_z) + \alpha_{\nu} c^{V_\nu}(V_z) &= \left( \frac{\gamma}{\mu} \right)^k \left( \frac{\gamma - \nu}{\mu - \nu} \right) \mu^k z^l (\mu - z) + \left( \frac{\gamma}{\nu} \right)^k \left( \frac{\gamma - \mu}{\nu - \mu} \right) \nu^k z^l (\nu - z) \\
&= \frac{\gamma^k y^l}{\mu - \nu} ((\gamma - \nu)(\mu - z) - (\gamma - \mu)(\nu - z)) \\
&= \frac{\gamma^k y^l}{\mu - \nu} (\mu - \nu)(\gamma - z) \\
&= \gamma^k y^l (\gamma - z) \\
&= c^{V_\gamma}(V_z)
\end{align*}
\]

Moreover, these functions are algebraically independent, as shown in the following proposition.

**Proposition 5.2.2.** Suppose that \( f(x, y) \) is a polynomial function such that \( f(c^{V_\mu}, c^{V_\nu}) = 0 \). Then \( f(x, y) = 0 \).

**Proof.** We can also assume that \( f \) is homogeneous of minimal degree (since \( c^{V_\mu}, c^{V_\nu} \) are of the same weight, any relation must be a sum of homogeneous relations), let us call that
degree $n$. Write $f(x, y) = \sum_{i+j=n} \alpha_{i,j} x^i y^j$. Notice that

$$f(c^\nu, c^\nu)(V_\mu) = \sum_{i+j=n} \alpha_{i,j} (c^\nu(V_\mu))^i (c^\nu(V_\mu))^j$$

$$= \alpha_{0,n} (\nu^k \mu^l (\nu - \mu))^n = 0$$

So $\alpha_{0,n} = 0$, and symmetrically $\alpha_{n,0} = 0$. Therefore,

$$f(x, y) = xy \left( \sum_{i+j=n} \alpha_{i,j} x^{i-1} y^{j-1} \right).$$

Clearly $c^\nu \cdot c^\nu \neq 0$ by evaluating on any point $V_x$ with $x \neq \mu, \nu$, so $\tilde{f} = \sum_{i+j=n} \alpha_{i,j} x^{i-1} y^{j-1}$ is a relation $\tilde{f}(c^\nu, c^\nu) = 0$. This contradicts minimality of the relation $f$.

In particular, the GIT-quotient (by $\text{PGL}(\beta)$) of $\text{Rep}_{Q,c}(\beta, r)_{\chi}^{ss}$ is a projective space.

**Corollary 5.2.3.** The ring $R = \bigoplus_{n \geq 0} SI_{Q,c}(\beta, r)_{n((\beta,-))}$ is isomorphic to the polynomial ring in two variables $c^\nu, c^\nu$. Therefore, $Y = \text{Proj}(R)$ is isomorphic to $\mathbb{P}^1$. 

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Bibliography

[1] S. Abeasis, A. Del Fra, and H. Kraft. The geometry of representations of $A_m$. *Math. Ann.*, 256(3):401–418, 1981.

[2] I. Assem, T. Brüstle, G. Charbonneau-Jodoin, and P-G Plamondon. Gentle algebras arising from surface triangulations. *Algebra Number Theory*, 4(2):201–229, 2010.

[3] I. Assem, D. Simson, and A. Skowroński. *Elements of the representation theory of associative algebras. Vol. 1*, volume 65 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2006. Techniques of representation theory.

[4] G. Bobiński and G. Zwara. Normality of orbit closures for Dynkin quivers of type $A_n$. *Manuscripta Math.*, 105(1):103–109, 2001.

[5] M. C. R. Butler and C. M. Ringel. Auslander-Reiten sequences with few middle terms and applications to string algebras. *Comm. Algebra*, 15(1-2):145–179, 1987.

[6] G. Cerulli Irelli. Quiver Grassmannians associated with string modules. *J. Algebraic Combin.*, 33(2):259–276, 2011.

[7] C. Chindris. Geometric characterizations of the representation type of hereditary algebras and of canonical algebras. *Adv. Math.*, 228(3):1405–1434, 2011.

[8] W. Crawley-Boevey. Maps between representations of zero-relation algebras. *J. Algebra*, 126(2):259–263, 1989.
[9] W. Crawley-Boevey. Tameness of biserial algebras. *Arch. Math. (Basel)*, 65(5):399–407, 1995.

[10] W. Crawley-Boevey and J. Schröer. Irreducible components of varieties of modules. *J. Reine Angew. Math.*, 553:201–220, 2002.

[11] C. De Concini and E. Strickland. On the variety of complexes. *Adv. in Math.*, 41(1):57–77, 1981.

[12] H. Derksen and J. Fei. General presentations of algebras. *preprint* arxiv:math.RA/0911.4913, 2009.

[13] H. Derksen and J. Weyman. Semi-invariants for quivers with relations. *J. Algebra*, 258(1):216–227, 2002. Special issue in celebration of Claudio Procesi’s 60th birthday.

[14] H. Derksen and J. Weyman. The combinatorics of quiver representations. *preprint* arxiv:math.RT/0608288, 2006.

[15] P. Donovan and M. R. Freislich. *The representation theory of finite graphs and associated algebras*. Carleton University, Ottawa, Ont., 1973. Carleton Mathematical Lecture Notes, No. 5.

[16] P. Dowbor and A. Mróz. On a separation of orbits in the module variety for string special biserial algebras. *J. Pure Appl. Algebra*, 213(9):1804–1815, 2009.

[17] J. A. Drozd. Tame and wild matrix problems. In *Representation theory, II (Proc. Second Internat. Conf., Carleton Univ., Ottawa, Ont., 1979)*, volume 832 of *Lecture Notes in Math.*, pages 242–258. Springer, Berlin, 1980.

[18] S. Fomin, M. Shapiro, and D. Thurston. Cluster algebras and triangulated surfaces. I. Cluster complexes. *Acta Math.*, 201(1):83–146, 2008.
[19] W. Fulton. *Young tableaux*, volume 35 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1997.

[20] P. Gabriel. Unzerlegbare Darstellungen. I. *Manuscripta Math.*, 6:71–103; correction, ibid. 6 (1972), 309, 1972.

[21] P. Gabriel. Finite representation type is open. In *Proceedings of the International Conference on Representations of Algebras (Carleton Univ., Ottawa, Ont., 1974)*, Paper No. 10, Ottawa, Ont., 1974. Carleton Univ.

[22] P. Gabriel. The universal cover of a representation-finite algebra. In *Representations of algebras (Puebla, 1980)*, volume 903 of *Lecture Notes in Math.*, pages 68–105. Springer, Berlin, 1981.

[23] V. G. Kac. Infinite root systems, representations of graphs and invariant theory. *Invent. Math.*, 56(1):57–92, 1980.

[24] A. D. King. Moduli of representations of finite-dimensional algebras. *Quart. J. Math. Oxford Ser. (2)*, 45(180):515–530, 1994.

[25] H. Kraft. Geometrische methoden in der invariantentheorie. *Aspects der Mathematik*, 1984.

[26] W. Kraśkiewicz and J. Weyman. Generic decompositions and semi-invariants for string algebras. *preprint arxiv:math.RT/1103.5415*, 2011.

[27] D. Labardini-Fragoso and G. Cerulli Irelli. Quivers with potentials associated to triangulated surfaces, part iii: Tagged triangulations and cluster monomials. *preprint arxiv:math.RT/1108.1774*, 2011.

[28] V. B. Mehta and V. Trivedi. The variety of circular complexes and F-splitting. *Invent. Math.*, 137(2):449–460, 1999.
[29] D. Mumford. Picard groups of moduli problems. In *Arithmetical Algebraic Geometry (Proc. Conf. Purdue Univ., 1963)*, pages 33–81. Harper & Row, New York, 1965.

[30] G. Musiker, R. Schiffler, and L. Williams. Positivity for cluster algebras from surfaces. *Adv. Math.*, 227(6):2241–2308, 2011.

[31] L. A. Nazarova. Representations of quivers of infinite type. *Izv. Akad. Nauk SSSR Ser. Mat.*, 37:752–791, 1973.

[32] C. M. Ringel. On generic modules for string algebras. *Bol. Soc. Mat. Mexicana (3)*, 7(1):85–97, 2001.

[33] A. Schofield. Semi-invariants of quivers. *J. London Math. Soc. (2)*, 43(3):385–395, 1991.

[34] J. Schröer. On the infinite radical of a module category. *Proc. London Math. Soc. (3)*, 81(3):651–674, 2000.

[35] J. Schröer. On the Krull-Gabriel dimension of an algebra. *Math. Z.*, 233(2):287–303, 2000.

[36] A. Skowroński and J. Waschbüsch. Representation-finite biserial algebras. *J. Reine Angew. Math.*, 345:172–181, 1983.

[37] A. Skowroński and J. Weyman. Semi-invariants of canonical algebras. *Manuscripta Math.*, 100(3):391–403, 1999.

[38] A. Skowroński and J. Weyman. The algebras of semi-invariants of quivers. *Transform. Groups*, 5(4):361–402, 2000.

[39] D. Voigt. *Induzierte Darstellungen in der Theorie der endlichen, algebraischen Gruppen*. Lecture Notes in Mathematics, Vol. 592. Springer-Verlag, Berlin, 1977. Mit einer englischen Einführung.
[40] B. Wald and J. Waschbüsch. Tame biserial algebras. *J. Algebra*, 95(2):480–500, 1985.

[41] J. Weyman. *Cohomology of vector bundles and syzygies*, volume 149 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2003.
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