The Gauged \textit{O}(3) Sigma Model: Schrödinger Representation and Hamilton-Jacobi Formulation

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ABSTRACT

We first study a free particle on an \((n-1)\)-sphere in an extended phase space, where the originally second-class Hamiltonian and constraints are now in strong involution. This allows for a Schrödinger representation and a Hamilton-Jacobi formulation of the model. We thereby obtain the free particle energy spectrum corresponding to that of a rigid rotator. We extend these considerations to a modified version of the field theoretical \textit{O}(3) nonlinear sigma model, and obtain the corresponding energy spectrum as well as BRST Lagrangian.

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1 Introduction

The quantization of constraint systems has been extensively discussed in the literature [1, 2, 3, 4]. In particular, the embedding of a second-class system into a first-class one [2], where constraints are in strong involution, has been of much interest, and has found a large number of applications [3, 4]. However, the usefulness of having the constraints in strong involution has, in our view, not been sufficiently emphasized. To illustrate this is the main objective of the present paper. Recently we have obtained in compact form the first-class Hamiltonian for the O(3) nonlinear sigma model (NLSM) [5]. On the one hand, this model is of interest, since it leads to novel phenomenological aspects [6, 7]. On the other hand, it has also served to investigate the Lagrangian, symplectic, and Batalin-Fradkin-Tyutin (BFT) embedding procedure to its quantization [8]. In this paper we wish to illustrate in terms of this model the central role which the BFT embedding plays for obtaining a Schrödinger representation and Hamilton-Jacobi formulation. In order to clarify the relation between the first-class formulation and the Schrödinger representation of a constrained quantum mechanical system, we shall first consider the quantum mechanical analog of a modified NLSM, i.e. the free particle motion on an \((n-1)\)-sphere in an extended phase space. We then apply the Schrödinger representation approach, developed for the \((n-1)\)-sphere, to a slightly modified version of the gauged, field theoretical O(3) nonlinear sigma model, within the framework of the BFT formalism.

2 Schrödinger representation for a free particle on an \((n-1)\)-sphere

Consider the motion of a particle on a hypersphere \(S^{n-1}\), as described by the Lagrangian,

\[
L_0 = \frac{1}{2} \dot{q}_a \dot{q}_a + \lambda q_a \dot{q}_a,
\]

where \(q_a (a = 1, 2, ..., n)\) are the coordinates parameterizing the \(S^{n-1}\) manifold, and \(\lambda\) is the Lagrange multiplier implementing the second-class constraint \(q_a \dot{q}_a \approx 0\) associated with the geometrical constraint \(q_a q_a = \text{constant}.\)

\(^1\)It turns out to be convenient not to impose the constraint in the form \(\lambda(q_a q_a - 1)\).
From the Lagrangian (2.1) we obtain for the canonical momenta conjugate to the multiplier $\lambda$ and the coordinates $q_a$

$$p_\lambda = 0, \quad p_a = \dot{q}_a + \lambda q_a,$$  

(2.2)

and the corresponding canonical Hamiltonian reads

$$H_0 = \frac{1}{2}(p_a - \lambda q_a)(p_a - \lambda q_a).$$  

(2.3)

The usual Dirac algorithm is readily shown to lead to the pair of second-class constraints $\Omega_i$ ($i = 1, 2$): 

$$\Omega_1 = p_\lambda \approx 0, \quad \Omega_2 = q_ap_a - \lambda q_a q_a \approx 0,$$  

(2.4)

satisfying the constraint algebra

$$\Delta_{kk'} \equiv \{\Omega_k, \Omega_{k'}\} = \epsilon_{kk'}q_a q_a,$$  

(2.5)

with $\epsilon_{12} = -\epsilon_{21} = 1$. For the discussion to follow it will be of central importance to convert this second-class constraint algebra into a strongly involutive one, by suitably embedding the model into a larger dimensional phase space. Following the Batalin-Fradkin-Tyutin scheme [2, 3, 4], we achieve this by introducing a pair of canonically conjugate auxiliary coordinates $(\theta, p_\theta)$ with Poisson brackets

$$\{\theta, p_\theta\} = 1.$$  

(2.6)

In this enlarged phase space one systematically constructs the first-class constraints $\tilde{\Omega}_i$ as a power series in these auxiliary coordinates, by requiring that they be in strong involution $\{\tilde{\Omega}_i, \tilde{\Omega}_j\} = 0$. For the case in question

$$\tilde{\Omega}_1 = \Omega_1 + \theta, \quad \tilde{\Omega}_2 = \Omega_2 - q_a q_a p_\theta.$$  

(2.7)

Note that from here on all Poisson brackets are understood to be taken with respect to the variables $(q_a, p_a, \lambda, p_\lambda, \theta, p_\theta)$ of the extended phase space. We next construct in the extended space the first-class coordinates $\tilde{\mathcal{F}} = (\tilde{q}_a, \tilde{p}_a)$, corresponding to the original coordinates $\mathcal{F} = (q_a, p_a)$ of the second class theory. They are again obtained as a power series in the auxiliary fields $(\theta, p_\theta)$ by demanding that they be in strong involution with the first-class
constraints (2.7), that is \{ \tilde{\Omega}_i, \tilde{\mathcal{F}} \} = 0. After some tedious algebra, we obtain for the first-class coordinates
\begin{align*}
\tilde{q}_a &= q_a \left( \frac{q_a q_c + 2\theta}{q_c q_a} \right)^{1/2}, \\
\tilde{p}_a &= \left( p_a + 2q_a \lambda \frac{\theta}{q_a q_c} + 2q_a p_\theta \frac{\theta}{q_a q_c} \right) \left( \frac{q_c q_a}{q_a q_c + 2\theta} \right)^{1/2}, \\
\tilde{\lambda} &= \lambda + p_\theta, \quad \tilde{p}_\lambda = p_\lambda + \theta.
\end{align*}

(2.8)

In terms of these coordinates the first-class Hamiltonian in strong involution with the first class constraints, can be written in the compact form
\[ H_0(\tilde{q}, \tilde{\lambda}, \tilde{p}) = \frac{1}{2}(\tilde{p}_a - \tilde{q}_a \tilde{\lambda})(\tilde{p}_a - \tilde{q}_a \tilde{\lambda}). \]

(2.9)

Notice that this is just a short hand for a Hamiltonian \( \tilde{H}_0 \) which now depends on the variables \((q_a, p_a, \lambda, \theta, p_\theta)\):
\[ \tilde{H}_0(q_a, \lambda, \theta, p_a, p_\theta) = \frac{1}{2} \eta^2 (p_a - \lambda q_a - q_\theta p_\theta)^2, \]

(2.10)

where
\[ \eta = \left( \frac{q_a q_a}{q_a q_a + 2\theta} \right)^{1/2}. \]

(2.11)

In terms of the first-class coordinates (2.8), the strongly involutive constraints (2.7) take the natural form
\[ \tilde{\Omega}_1 := \tilde{p}_\lambda = 0, \quad \tilde{\Omega}_2 := \tilde{q}_a \tilde{p}_a - \tilde{\lambda} q_a \tilde{q}_a = 0, \]

(2.12)

which thus display manifest form invariance with respect to the second-class constraints (2.4). One readily checks that one has the following Poisson brackets taken with respect to the variables \((q_a, p_a, \theta, p_\theta, \lambda, p_\lambda)\):
\[ \{ \tilde{q}_a, \tilde{p}_b \} = \delta_{ab}, \quad \{ \tilde{q}_a, \tilde{q}_b \} = 0, \quad \{ \tilde{p}_a, \tilde{p}_b \} = 0, \]

(2.13)

as well as
\[ \{ \tilde{\lambda}, \tilde{q}_a \} = -\frac{\tilde{q}_a}{q_a q_a}, \quad \{ \tilde{\lambda}, \tilde{p}_a \} = \frac{\tilde{p}_a}{q_a q_a} - 2 \frac{\tilde{q}_a \tilde{\lambda}}{q_a q_a}, \]
\[ \{ \tilde{p}_\lambda, \tilde{p}_a \} = 0, \quad \{ \tilde{\lambda}, \tilde{p}_\lambda \} = 0, \quad \{ \tilde{p}_\lambda, \tilde{q}_a \} = 0. \]

(2.14)
From the Hamilton equations of motion we have, using (2.13) and (2.14),

\[
\dot{q}_a = \{q_a, \tilde{H}_0\} = \eta^2 (p_a - \lambda q_a - p_\theta q_a) \\
\dot{\theta} = \{\theta, \tilde{H}_0\} = -\eta^2 (p_a - \lambda q_a - p_\theta q_a).
\]

(2.15)

Hence the secondary constraint \(q_c q_c = \text{const}\) of the second class formulation is replaced by \(\tilde{q}_c \tilde{q}_c = q_a q_a + 2\theta = \text{const}\) in the first class formulation. This was to be expected since the above Poisson brackets coincide with the Dirac brackets in the original variables [9]. Indeed, the “gauge invariant” variables \((\tilde{q}_a, \tilde{p}_a, \tilde{\lambda}, \tilde{p}_\lambda)\) of the first-class formulation are just the \((q_a, p_a, \lambda, p_\lambda)\) of the second class formulation, and the corresponding Poisson brackets correspond to the Dirac brackets, respectively. It follows from the Poisson brackets (2.13) and (2.14) that the constraints (2.7) and the first-class Hamiltonian (2.10) are all in strong involution:

\[
\{\tilde{\Omega}_i, \tilde{\Omega}_j\} = 0, \quad \{\tilde{\Omega}_i, \tilde{H}_0\} = 0.
\]

(2.16)

This will simplify the discussion to follow. Indeed, since \(\tilde{\Omega}_i\) and \(\tilde{H}_0\) are now in strong involution, we can impose them strongly. Solving \(\tilde{\Omega}_2 = 0\) for \(\tilde{\lambda}\), we may reduce the Hamiltonian (3.10) to the form

\[
\tilde{H}_0 = \frac{1}{2} \left( \tilde{\rho}_a - \tilde{q}_a \frac{(\tilde{q} \cdot \tilde{p})}{\tilde{q}^2} \right) \left( \tilde{p}_a - \tilde{q}_a \frac{(\tilde{q} \cdot \tilde{p})}{\tilde{q}^2} \right).
\]

(2.17)

where \(\tilde{q} \cdot \tilde{p} = \tilde{q}_a \tilde{p}_a\) and \(\tilde{q}^2 = \tilde{q}_a \tilde{q}_a\). Effectively we have thus again only 2n independent degrees of freedom in the extended phase space to describe the free particle motion on the \((n - 1)\)-sphere. On the other hand, these 2n independent degrees of freedom satisfy the canonical Poisson algebra (2.13). Hence in the formulation of the Hamiltonian (2.17) we can treat the particle motion on \(S^{n-1}\) as that of an unconstrained system. This will prove very useful in subsection 2.2.

### 2.1 Hamilton-Jacobi formulation

If our Lagrangian were to describe an unconstrained system, one would have only one Hamilton-Jacobi (HJ) equation for the Hamilton principal function \(S\), as given by a partial differential equation (PDE) of the form

\[
H_0' := p_0 + H_0(t, q_a, \lambda, p_a, p_\lambda) = 0.
\]

(2.18)
where the substitutions
\[ p_0 = \frac{\partial S}{\partial t}, \quad p_a = \frac{\partial S}{\partial q_a}, \quad p_\lambda = \frac{\partial S}{\partial \lambda} \tag{2.19} \]
are understood. If \( H_0 \) does not depend explicitly on \( t \) (as in the model in question), we may then seek a solution of the form \( S = \text{const} \cdot t + W \). In the case of our model, the primary constraint \( p_\lambda = 0 \) and the secondary constraint \( \Omega_2 = q_a p_a - \lambda q_a q_a = 0 \) motivate one to consider the additional PDE
\[ H'_1 : = \frac{\partial \tilde{S}}{\partial \lambda} = 0 \tag{2.20} \]
\[ H'_2 : = q_a \frac{\partial \tilde{S}}{\partial q_a} - \lambda q_a q_a = 0. \tag{2.21} \]
It is easy to check that equations (2.18) and (2.20) are inconsistent. Thus, in the second class formulation, the above set of coupled Hamilton-Jacobi equations admits no solution. In different terms, they violate the integrability condition of [10]. Although one can nevertheless arrive at an HJ formulation of the second class system by making an appropriate choice of canonical variables [11], we circumvent the problem in the present case by enlarging the phase space in the way described above. In the extended phase space (2.18), (2.20) and (2.21) are replaced by
\[ \tilde{H}'_0 : = \frac{\partial \tilde{S}}{\partial t} + \frac{\eta^2}{2} \left( \frac{\partial \tilde{S}}{\partial q_a} - q_a \frac{\partial \tilde{S}}{\partial \theta} - \lambda q_a \right)^2 \] \tag{2.22}
\[ \tilde{H}'_1 : = \frac{\partial \tilde{S}}{\partial \lambda} + \theta = 0 \tag{2.23} \]
\[ \tilde{H}'_2 : = q_a \frac{\partial \tilde{S}}{\partial q_a} - q_a \left( \frac{\partial \tilde{S}}{\partial \theta} - \lambda q_a \right) = 0. \tag{2.24} \]
Consider the constraint equation \( H'_1 = 0 \). It has the solution
\[ \tilde{S}(t, q_a, \lambda, \theta) = -\frac{\alpha^2}{2} t + \tilde{W}(q_a, \theta) - \lambda \theta \tag{2.25} \]
Hence (2.22) and (2.24) become respectively
\[ -\frac{\alpha^2}{2} + \frac{1}{2} \eta^2 \left( \frac{\partial \tilde{W}}{\partial q_a} - q_a \frac{\partial \tilde{W}}{\partial \theta} \right)^2 = 0. \tag{2.26} \]
\[ q_a \frac{\partial W}{\partial q_a} - q_a q_a \frac{\partial W}{\partial \theta} = 0. \]  

(2.27)

One possible solution of the second equation above is

\[ \tilde{W}(q_a, \theta) = g(q_a q_a + 2\theta) = g(\tilde{q}_a \tilde{q}_a) \]

with \( g(x) \) so far an arbitrary function. Thus

\[ \tilde{S}(t, q_a, \lambda, \theta) = -\frac{\alpha^2}{2} t + g(q_a q_a + 2\theta) - \lambda \theta \]

(2.28)

With this solution equation (2.26) reduces to \( \alpha = 0 \). We now look for non-trivial solutions. Consider first equation (2.27). The relation between the variables in the first and second class formulation motivates us to make the following Ansatz:

\[ \tilde{W}(q, \theta) = f(n \cdot \tilde{q}) \]  

(2.29)

with \( n_a \) the components of a \( n \)-dimensional unit vector parametrized by \( n - 1 \) constants. Using

\[ \frac{\partial \tilde{q}_c}{\partial q_a} = \eta^{-1} \delta_{ca} - \eta \frac{2\theta q_a q_c}{q^2}, \quad \frac{\partial \tilde{q}_c}{\partial \theta} = \eta \frac{q_c}{q^2}, \]  

(2.30)

one readily checks that equation (2.27) is satisfied for any \( f(x) \). This function is now determined by eq. (2.26) which now reads,

\[ \left( 1 - \frac{(n_a \tilde{q}_a)^2}{\tilde{q}_c \tilde{q}_c} \right) f'^2(n_a \tilde{q}_a) = \alpha^2. \]

where the prime denotes the derivative with respect to the argument of \( f \). The solution to this equation has been given in [11]. Setting \( x = n_a \tilde{q}_a \) and \( r^2 = \tilde{q}_a \tilde{q}_a \), we thus have

\[ f'(x) = \pm \frac{\alpha}{\sqrt{1 - \frac{x^2}{r^2}}} \]

so that upon integration in \( x \) the Hamilton principal function \( \tilde{W} \) takes the form

\[ \tilde{W}(q, \theta) = \alpha r \tan^{-1} \frac{n_a \tilde{q}_a}{\sqrt{r^2 - (n_a \tilde{q}_a)^2}} + \text{const}. \]

(2.31)
The Hamilton principal function contains \( n \) independent constants, which we take to be \( \alpha \) and \( n_1, n_2, \ldots, n_{n-1} \), while the normalization of \( n^a \) implies for the \( n' \)th component, 
\[
 n_n = \sqrt{1 - \sum_{a=1}^{n-1} n_an_a}.
\]
Differentiating the Hamilton principal function with respect to these constants (new momenta in the corresponding generating functional) yields in the usual way [12] the \( n \) time-independent new coordinates:
\[
 \beta = \frac{\partial W}{\partial \alpha} , \quad \beta_a = \frac{\partial W}{\partial n_a} , \quad a = 1, 2, \ldots, n - 1.
\]
(2.32)

From the first equation and (2.31) we easily obtain
\[
 n_a \tilde{q}_a = r \cos \frac{\beta + \alpha t}{r} \equiv r \cos \Omega(t).
\]
(2.33)

As shown in [11], the solution for the “observable” \( \tilde{q}_a \) then takes the following form in terms of \( \Omega(t) \):
\[
 \tilde{q}_a = \frac{1}{\alpha} (\beta_a - (n_c \beta_c)n_a) \sin \Omega(t) + rn_a \cos \Omega(t),
\]
(2.34)

where \( \beta_a \) is the \( n \)-dimensional vector \( \beta_a = (\beta_1, \beta_2, \ldots, \beta_{n-1}, 0) \). Substituting the above result into our original condition, \( r^2 = \tilde{q}^2 \), leads to
\[
 r = \sqrt{\frac{\beta_a \beta_a - (n_a \beta_a)^2}{\alpha^2}}
\]
(2.35)

Thus the radius of motion is fixed if the new time independent coordinates are specified.

### 2.2 Schrödinger representation

In the first-class formulation the canonical commutation relations between the phase space variables \((q, p), (\lambda, p_\lambda), (\theta, p_\theta)\) allows us to replace the first class Hamiltonian (2.10) by the differential operator
\[
 \tilde{H}_0 = \frac{1}{2} \eta(q, \theta)^2 \left( \frac{\hbar}{i} \frac{\partial}{\partial q_a} - \lambda q_a - q_a \frac{\hbar}{i} \frac{\partial}{\partial \theta} \right) \left( \frac{\hbar}{i} \frac{\partial}{\partial q_a} - \lambda q_a - q_a \frac{\hbar}{i} \frac{\partial}{\partial \theta} \right)
\]
(2.36)

Instead of solving the corresponding eigenvalue problem in this formulation, one may simplify this problem by noting from (2.13) that the variables \((\tilde{q}_a, \tilde{p}_a)\)
form canonical pairs, so that we may make the replacement $\tilde{p}_a \to \frac{\hbar}{i} \frac{\partial}{\partial \tilde{q}_a}$ in (2.17). In the Schrödinger representation, we thus have for the quantum commutators,

$$[\tilde{q}_a, \tilde{q}_b] = 0, \quad [\tilde{p}_a, \tilde{p}_b] = 0, \quad [\tilde{q}_a, \tilde{p}_b] = i\hbar \delta_{ab}. \quad (2.37)$$

Following the symmetrization procedure of [13, 14] we obtain for the Hamilton quantum operator of a free particle on the $(n - 1)$-sphere (we now set $\tilde{q}^2 = r^2 = 1$)

$$\tilde{H}_0 = \frac{1}{2} \left[ -i\hbar \frac{\partial}{\partial \tilde{q}_a} + i\hbar \tilde{q}_a \frac{\partial}{\partial \tilde{q}_b} \right] \left[ -i\hbar \frac{\partial}{\partial \tilde{q}_b} + i\hbar \tilde{q}_b \frac{\partial}{\partial \tilde{q}_a} \right] = \frac{1}{2} \hbar^2 \left[ -\frac{\partial^2}{\partial \tilde{q}_a \partial \tilde{q}_a} + (n - 1)\tilde{q}_a \frac{\partial}{\partial \tilde{q}_a} + \tilde{q}_a \tilde{q}_b \frac{\partial}{\partial \tilde{q}_a} \frac{\partial}{\partial \tilde{q}_b} + \frac{(n + 1)(n - 3)}{4} \right]. \quad (2.38)$$

Note that the quantum Hamiltonian (2.38) has only terms of order $\hbar^2$, so that one has rotational energy contributions of order $\hbar^2$, without any vibrational modes of order $\hbar$. In fact, the starting Lagrangian (2.1) does not possess any vibrational degrees of freedom, since it involves only the kinetic term describing the motions of the particle residing on the $S^{n-1}$ manifold. We now define the Casimir operator $\tilde{J}^2$ in terms of the $(n - 1)$-sphere Laplacian [15]

$$\tilde{J}^2 = \hbar^2 \left[ -\frac{\partial^2}{\partial \tilde{q}_a \partial \tilde{q}_a} + (n - 1)\tilde{q}_a \frac{\partial}{\partial \tilde{q}_a} + \tilde{q}_a \tilde{q}_b \frac{\partial}{\partial \tilde{q}_a} \frac{\partial}{\partial \tilde{q}_b} \right], \quad (2.39)$$

whose eigenvalue spectrum is given in terms of the corresponding angular quantum number $j$ ($j = \text{integers}$) and the dimension of the sphere as follows

$$\tilde{J}^2|j\rangle = \hbar^2 j(j + n - 2)|j\rangle. \quad (2.40)$$

We thus have for the Hamiltonian operator of a free particle on the $(n - 1)$-sphere

$$\tilde{H}_0 = \frac{1}{2} \left[ \tilde{J}^2 + \frac{\hbar^2 (n + 1)(n - 3)}{4} \right], \quad (2.41)$$

to yield the eigenvalue equation $\tilde{H}_0|j\rangle = E_j|j\rangle$ with the spectrum given by

$$E_j = \frac{\hbar^2}{2} \left[ j(j + n - 2) + \frac{(n + 1)(n - 3)}{4} \right]. \quad (2.42)$$

Note that the Hamiltonian operator (2.41) is that of a rigid rotator, and the energy eigenvalues (2.42) involve global shifts depending on the dimension of the sphere, which has not been included in [16, 17].
3 Gauged $O(3)$ nonlinear sigma model

In this section, we will generalize the approach developed in the previous sections to the field theoretical $O(3)$ nonlinear sigma model, whose Lagrangian is of the form

$$
L_0 = \frac{1}{2f}(\partial_\mu n^a)(\partial^\mu n^a) + n^0 n^a \partial_\mu n^a
$$

(3.1)

where $n^a$ ($a=1,2,3$) is a multiplet of three real scalar fields which parameterize an internal space $S^2$, and $n^0$ is the Lagrange multiplier field implementing the second-class constraint $n^a \partial_\mu n^a \approx 0$ associated with the geometrical constraint $n^a n^a - 1 \approx 0$. From the Lagrangian (3.1) the canonical momenta conjugate to the field $n^0$ and the real scalar fields $n^a$ are given by

$$
\pi^0 = 0,
\pi^a = \frac{1}{f} \partial_0 n^a + n^a n^0.
$$

(3.2)

Here one notes that $n^0$, $n^a$ and $\partial_0 n^a$ are entangled to define $\pi^a$. In terms of the canonical momenta (3.2), we then obtain the canonical Hamiltonian

$$
H = \frac{f}{2}(\pi^a - n^a n^0)(\pi^a - n^a n^0) + \frac{1}{2f}(\partial_\mu n^a)(\partial^\mu n^a)
$$

(3.3)

The usual Dirac algorithm is readily shown to lead to the pair of second-class constraints $\Omega_i$ ($i = 1, 2$) as follows

$$
\Omega_1 = \pi^0 \approx 0
\Omega_2 = n^a \pi^a - n^a n^0 \approx 0.
$$

(3.4)

to yield the corresponding constraint algebra with $\epsilon^{12} = -\epsilon^{21} = 1$

$$
\Delta_{kk'}(x, y) = \{\Omega_k(x), \Omega_{k'}(y)\} = \epsilon^{kk'} n^a n^a \delta^2(x - y).
$$

(3.5)

Following the BFT scheme [2, 3, 4], we systematically convert the second-class constraints $\Omega_i = 0$ ($i = 1, 2$) into first-class ones by introducing two canonically conjugate auxiliary fields $(\theta, \pi_\theta)$ with Poisson brackets

$$
\{\theta(x), \pi_\theta(y)\} = \delta^2(x - y).
$$

(3.6)
The strongly involutive first-class constraints $\tilde{\Omega}_i$ are then constructed as a power series of the auxiliary fields [5],

$$
\tilde{\Omega}_1 = \Omega_1 + \theta,
\tilde{\Omega}_2 = \Omega_2 - n^a n^a \pi_\theta.
$$

(3.7)

Note that the first class constraints (3.7) can be rewritten as

$$
\tilde{\Omega}_1 = \tilde{\pi}^0,
\tilde{\Omega}_2 = n^a \tilde{\pi}^a - n^a \tilde{n}^a \tilde{n}^0,
$$

(3.8)

which are form-invariant with respect to the second-class constraints (3.4).

We next construct the first-class fields $\tilde{F} = (\tilde{n}^a, \tilde{\pi}^a)$, corresponding to the original fields defined by $F = (n^a, \pi^a)$ in the extended phase space. They are obtained as a power series in the auxiliary fields $(\theta, \pi_\theta)$ by demanding that they be in strong involution with the first-class constraints (3.7), that is $\{\tilde{\Omega}_i, \tilde{F}\} = 0$. After some tedious algebra, we obtain for the first-class physical fields

$$
\tilde{n}^a = n^a \left( \frac{n^c n^c + 2\theta}{n^c n^c} \right)^{1/2}
$$

$$
\tilde{\pi}^a = \left( \pi^a + 2n^a n^0 \frac{\theta}{n^c n^c} + 2n^a \pi_\theta \frac{\theta}{n^c n^c} \right) \left( \frac{n^c n^c}{n^c n^c + 2\theta} \right)^{1/2}
$$

$$
\tilde{n}^0 = n^0 + \pi_\theta, \quad \tilde{\pi}^0 = \pi^0 + \theta,
$$

(3.9)

and the first-class Hamiltonian (we now set right away $\tilde{n}^2 = r^2 = 1$)

$$
\tilde{H} = \frac{f}{2} (\tilde{\pi}^a - \tilde{n}^a \tilde{n}^0)(\tilde{\pi}^a - \tilde{n}^a \tilde{n}^0) + \frac{1}{2f} (\partial_i \tilde{n}^a)(\partial_i \tilde{n}^a).
$$

(3.10)

Inserting the first-class constraint $\tilde{\Omega}_2 = 0$ together with $\tilde{n}^a \tilde{n}^a = 1$, which are strongly zero, into the first-class Hamiltonian (3.10), we can obtain $\tilde{H}$ only in terms of $(\tilde{n}^a, \tilde{\pi}^a)$ as follows

$$
\tilde{H} = \frac{f}{2} (\tilde{\pi}^a - \tilde{n}^a \tilde{n}^c \tilde{\pi}^c)(\tilde{\pi}^a - \tilde{n}^a \tilde{n}^d \tilde{\pi}^d) + \frac{1}{2f} (\partial_i \tilde{n}^a)(\partial_i \tilde{n}^a).
$$

(3.11)
Moreover, the first-class physical fields (3.9) are found to satisfy the Poisson algebra
\[
\{ \tilde{n}^a(x), \tilde{n}^b(y) \} = 0,
\{ \tilde{n}^a(x), \tilde{\pi}^b(y) \} = \delta^{ab} \delta^2(x - y),
\{ \tilde{\pi}^a(x), \tilde{\pi}^b(y) \} = 0,
\]
which, in the extended phase space, yield the canonical quantum commutators
\[
[ \hat{n}^a(x), \hat{n}^b(y) ] = 0,
[ \hat{n}^a(x), \hat{\pi}^b(y) ] = i\hbar \delta^{ab} \delta^2(x - y),
[ \hat{\pi}^a(x), \hat{\pi}^b(y) ] = 0.
\]
(3.12)
Note that the first-class Hamiltonian (3.11) does not have extra degrees of freedom of (\tilde{n}^0, \tilde{\pi}^0) any more so that we can have only (\tilde{n}^a, \tilde{\pi}^a) independent degrees of freedom with the canonical quantum commutators (3.13) in the extended phase space as in the unconstrained systems.

3.1 Schrödinger representation

The quantum commutators corresponding to the Poisson brackets (3.13) show that we can realize the quantum operators \( \hat{\pi}^a \) of the O(3) nonlinear sigma model as follows:
\[
\hat{\pi}^a = -i\hbar \frac{\partial}{\partial \tilde{n}^a}.
\]
(3.14)
Following the symmetrization procedure of ref. [13, 14], together with (3.11) and (3.14), we arrive at the Hamiltonian density quantum operator for the O(3) nonlinear sigma model
\[
\hat{\mathcal{H}} = \frac{f}{2} \left( \frac{\hbar}{i} \frac{\partial}{\partial \tilde{n}^a} - \frac{\hbar}{i} \tilde{n}^a \tilde{n}^c \frac{\partial}{\partial \tilde{n}^c} \right) \left( \frac{\hbar}{i} \frac{\partial}{\partial \tilde{n}^a} - \frac{\hbar}{i} \tilde{n}^a \tilde{n}^d \frac{\partial}{\partial \tilde{n}^d} \right) : + \frac{1}{2f} (\partial_i \tilde{n}^a)(\partial_i \tilde{n}^a)
\]
\[
= \frac{f}{2} \hbar^2 \left( - \frac{\partial^2}{\partial \tilde{n}^a \partial \tilde{n}^a} + 2 \tilde{n}^a \frac{\partial}{\partial \tilde{n}^a} + \tilde{n}^a \tilde{n}^b \frac{\partial^2}{\partial \tilde{n}^a \partial \tilde{n}^b} \right) + \frac{1}{2f} (\partial_i \tilde{n}^a)(\partial_i \tilde{n}^a).
\]
(3.15)
Note that the Hamiltonian operator (3.15) has terms of orders \( \hbar^0 \) and \( \hbar^2 \) only, so that one has static mass (of order \( \hbar^0 \)) and rotational energy contributions (of order \( \hbar^2 \)) without any vibrational modes (of order \( \hbar^1 \)). Indeed,
the starting Lagrangian (3.1) does not allow for any vibrational degrees of freedom since it describes the motion of the soliton on the $S^2$ manifold. Integrating the terms of order $\hbar^2$ in the Hamiltonian operator (3.15) over the two-dimensional target manifold, one can construct the Casimir operator $\hat{J}^2$ as follows

$$\hat{J}^2 = \hbar^2 \mathcal{I} \int d^2x f \left( -\frac{\partial^2}{\partial \tilde{n}^a \partial \tilde{n}^a} + 2\tilde{n}^a \frac{\partial}{\partial \tilde{n}^a} + \tilde{n}^a \tilde{n}^b \frac{\partial^2}{\partial \tilde{n}^a \partial \tilde{n}^b} \right) ,$$

(3.16)

with $\mathcal{I}$ the moment of inertia of the soliton (see below). Note that the above operator $\hat{J}^2$ is the Laplacian on the two-sphere [15] in the field representation. Similarly, one can integrate the term of order $\hbar^0$ over the two-dimensional space to define the soliton static mass $M_0$ as

$$M_0 = \int d^2x \frac{1}{2f} (\partial_i \tilde{n}^a)(\partial_i \tilde{n}^a) .$$

(3.17)

In terms of $M_0$ and $\hat{J}^2$, the Hamiltonian operator $\hat{H}$ for the O(3) nonlinear sigma model thus takes the form:

$$\hat{H} = M_0 + \frac{\hat{J}^2}{2\mathcal{I}} .$$

(3.18)

The associated eigenvalue problem

$$\hat{H}|j\rangle = E_j |j\rangle$$

(3.19)

leads to the energy eigenvalues $E_j$ ($j = 0, \pm 1, \pm 2, ...$),

$$E_j = M_0 + \frac{\hbar^2}{2\mathcal{I}} j(j+1) ,$$

(3.20)

which exhibit the contribution from the static soliton mass and the rotational excitations discussed above. $\mathcal{I}$ can thus be interpreted as the moment of inertia of the soliton rigid rotator and $j$ is the U(1) isospin quantum number associated with the angular momentum operator $\hat{J}^2$ satisfying the following eigenvalue equation in the two-dimensional space [15]

$$\hat{J}^2 |j\rangle = \hbar^2 j(j+1) |j\rangle .$$

(3.21)
Note that this is the special case $n = 3$ of the formula (2.40) in the previous section. In the O(3) nonlinear sigma model we thus do not have a global energy shift, consistent with the previous semiclassical result [5]. On the other hand, in the SU(2) Skyrmion model one obtains a positive Weyl ordering correction [14]. In the semiclassical quantization with the ansatz
\begin{align*}
\tilde{n}^1 &= \cos[\alpha(t) + \phi] \sin F(r) \\
\tilde{n}^2 &= \sin[\alpha(t) + \phi] \sin F(r) \\
\tilde{n}^3 &= \cos F(r)
\end{align*}
(3.22)
for $\tilde{n}^a$ in the topological charge $Q = 1$ sector, where $(r, \phi)$ are the polar coordinates and $\alpha$ is the collective coordinate, one can explicitly obtain the soliton mass $M_0$ and the moment of inertia $\mathcal{I}$ as follows [5]
\begin{align*}
M_0 &= \frac{\pi}{f} \int_0^\infty dr r \left( \frac{dF}{dr} \right)^2 + \frac{\sin^2 F}{r^2}, \\
\mathcal{I} &= \frac{2\pi}{f} \int_0^\infty dr r \sin^2 F.
\end{align*}
(3.23)
Note that the above angular momentum operator $\hat{J}^2$ can be also constructed in the standard way from
\begin{equation}
\hat{J} = \int d^2x \, \epsilon_{ij} x^i T^{0j},
\end{equation}
(3.24)
where the symmetric energy-momentum tensor is given by $T^{\mu\nu} = \frac{\partial \tilde{L}_0}{\partial (\partial_\mu \tilde{n}^a)} \partial^\nu \tilde{n}^a - g^{\mu\nu} \tilde{L}_0$, with $\tilde{L}_0$ the first-class Lagrangian constructed via the replacement of $(n^0, n^a) \to (\tilde{n}^0, \tilde{n}^a)$ in the Lagrangian (3.1). We next discuss the corresponding BRST Lagrangean.

### 3.2 BRST symmetries and effective Lagrangian

In order to investigate the Becci-Rouet-Stora-Tyutin (BRST) symmetries [18] associated with the Lagrangian (3.1) of the O(3) nonlinear sigma model, we rewrite the first-class Hamiltonian (3.10) in terms of original fields and auxiliary ones
\begin{equation}
\mathcal{H} = \frac{1}{2} (\pi^a - n^0 n^a - n^a \pi_0) (\pi^a - n^a n^0 - n^a \pi_0) \frac{n^c n^c}{n^c n^c + 2\theta} + \frac{1}{2f} (\partial_i n^a) (\partial_i n^a) \frac{n^c n^c}{n^c n^c + 2\theta},
\end{equation}
(3.25)
which is strongly involutive with the first class constraints \( \{ \hat{\Omega}_1, \hat{H} \} = 0 \). Note that with this Hamiltonian (3.25), one cannot generate the first-class Gauss’ law constraint from the time evolution of the constraint \( \hat{\Omega}_1 \). By introducing an additional term proportional to the first-class constraints \( \hat{\Omega}_2 \) into \( \hat{H} \), we obtain an equivalent first-class Hamiltonian

\[
\hat{H}' = \hat{H} + f \pi_\theta \hat{\Omega}_2
\]

(3.26)
to generate the Gauss’ law constraint

\[
\{ \hat{\Omega}_1(x), \hat{H}'(y) \} = f \hat{\Omega}_2 \delta^2(x - y),
\]

\[
\{ \hat{\Omega}_2(x), \hat{H}'(y) \} = 0.
\]

(3.27)

Note that these Hamiltonians \( \hat{H} \) and \( \hat{H}' \) effectively act on physical states in the same way since such states are annihilated by the first class constraints.

In the framework of the Batalin-Fradkin-Vilkovisky (BFV) formalism [19, 20], we now construct the nilpotent BRST charge \( Q \), the fermionic gauge fixing function \( \Psi \) and the BRST invariant minimal Hamiltonian \( H_m \) by introducing two canonical sets of ghost and anti-ghost fields, together with auxiliary fields \((C^i, \bar{P}_i), (\bar{C}_i, \bar{C}^i), (N^i, B_i) (i = 1, 2)\) and the unitary gauge choice \( \chi^1 = \Omega_1, \chi^2 = \Omega_2 \),

\[
Q = \int d^2 x \left( C^i \hat{\Omega}_i + \bar{P}_i \bar{\Omega}_i \right),
\]

\[
\Psi = \int d^2 x \left( \bar{C}_i \chi^i + \bar{P}_i N^i \right),
\]

\[
H_m = \int d^2 x \left( \hat{H} + f \pi_\theta \hat{\Omega}_2 - f C^1 \bar{P}_2 \right),
\]

(3.28)

with the properties \( Q^2 = \{ Q, Q \} = 0 \) and \( \{ \{ \Psi, Q \}, Q \} = 0 \). The nilpotent charge \( Q \) is the generator of the following infinitesimal transformations,

\[
\delta_Q n^0 = -C^1, \quad \delta_Q n^a = -C^2 n^a, \quad \delta_Q \theta = C^2 n^a n^a,
\]

\[
\delta_Q \pi^0 = -C^2 n^a n^a, \quad \delta_Q \pi^a = C^2 (\pi^a - 2 n^a n^0 - 2 n^a \pi_\theta), \quad \delta_Q \pi_\theta = C^1,
\]

\[
\delta_Q C_i = B_i, \quad \delta_Q \bar{C}^i = 0, \quad \delta_Q B_i = 0,
\]

\[
\delta_Q \bar{P}_i = 0, \quad \delta_Q \bar{P}_i = \bar{\Omega}_i, \quad \delta_Q N^i = -\bar{P}_i,
\]

(3.29)

which in turn imply \( \{ Q, H_m \} = 0 \), that is, \( H_m \) in (3.28) is the BRST invariant minimal Hamiltonian. After some algebra, we arrive at the effective quantum Lagrangian of the manifestly covariant form

\[
L_{eff} = L_0 + L_{WZ} + L_{ghost}
\]

(3.30)
where $L_0$ is given by (3.1) and
\begin{equation}
L_{WZ} = \frac{1}{fn^cn^c}(\partial_\mu n^a)(\partial^\mu n^a)\theta - \frac{1}{2f(n^cn^c)^2}\partial_\mu \theta \partial^\mu \theta, \tag{3.31}
\end{equation}
\begin{equation}
L_{\text{ghost}} = -\frac{1}{2}(n^a n^a) n^0 (B + 2\bar{\mathcal{C}}\mathcal{C}) - \frac{1}{2f}(n^a n^a)^2 (B + 2\bar{\mathcal{C}}\mathcal{C})^2
- \frac{1}{n^cn^c}\partial_\mu \theta \partial^\mu B + \partial_\mu \bar{\mathcal{C}} \partial^\mu \mathcal{C}. \tag{3.32}
\end{equation}

This Lagrangian is invariant under the BRST transformation
\begin{equation}
\delta_\epsilon n^0 = -2\epsilon n^0 \mathcal{C}, \quad \delta_\epsilon n^a = \epsilon n^a \mathcal{C}, \quad \delta_\epsilon \theta = -\epsilon n^a n^a \mathcal{C}, \quad 
\delta_\epsilon \bar{\mathcal{C}} = -\epsilon B, \quad \delta_\epsilon \mathcal{C} = 0, \quad \delta_\epsilon B = 0, \tag{3.33}
\end{equation}

where $\epsilon$ is an infinitesimal Grassmann valued parameter.

### 4 Conclusion

In conclusion, it can be said that the Batalin-Fradkin-Tyutin embedding of the second-class constrained system into a first-class one has played an important role for obtaining the energy spectrum, as well as a Hamilton-Jacobi formulation of the multidimensional rigid rotator on the $S^{\alpha-1}$ manifold. In order to obtain its energy spectrum we made use of the Schrödinger representation for this system. We have also constructed the corresponding Hamilton principal function to obtain the nontrivial solutions of the Euler-Lagrange equations. We have extended these results to a modified version of the field theoretical O(3) nonlinear sigma model with geometric constraints, in the Batalin-Fradkin-Tyutin scheme. Since it was possible to obtain a Schrödinger realization for the quantum commutators of the canonical fields and their conjugate momenta, we could straightforwardly obtain the energy spectrum of the topological soliton including the rotational modes, which turned out to be the same as that of the rigid rotator on the two-dimensional target manifold $S^2$. We have further constructed the BRST invariant Lagrangian of this O(3) nonlinear sigma model.

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References

[1] P.A.M. Dirac, Lectures on Quantum Mechanics, Yeshiba University Press, New York, 1964.

[2] I.A. Batalin, E.S. Fradkin, Phys. Lett. B180, 157 (1986); Nucl. Phys. B279, 514 (1987); I.A. Batalin, I.V. Tyutin, Int. J. Mod. Phys. A6, 3255 (1991).

[3] R. Banerjee, Phys. Rev. D48, R5467 (1993); W.T. Kim, Y.J. Park, Phys. Lett. B336, 376 (1994); Y.W. Kim, Y.J. Park, K.D. Rothe, J. Phys. G24, 953 (1998); Y.W. Kim, K. D. Rothe, Nucl. Phys. B510, 511 (1998).

[4] S.T. Hong, Y.J. Park, Phys. Rep. 358, 143 (2002), and references therein.

[5] S.T. Hong, W.T. Kim, Y.J. Park, Phys. Rev. D60, 125005 (1999).

[6] F. Wilczek, A. Zee, Phys. Rev. Lett. 51, 2250 (1983); A.M. Polyakov, Mod. Phys. Lett. A3, 417 (1999); Y. Wu, A. Zee, Phys. Lett. B147, 325 (1984); G. Semenoff, Phys. Rev. Lett. 61, 517 (1988); M. Bowick, D. Karabali, L.C.R. Wijewardhana, Nucl. Phys. B271, 417 (1986).

[7] C. Domb, M.S. Green, Phase Transitions and Critical Phenomena, Academic, New York, 1972; F.D.M. Haldane, Phys. Lett. A93, 464 (1983); Phys. Rev. Lett. 50, 1153 (1983).

[8] S.T. Hong, Y.W. Kim, Y.J. Park, K.D. Rothe, J. Phys. A36, 1643 (2003).

[9] H.O. Girotti and K.D. Rothe, J. Mod. Phys. A4, 3041 (1989).

[10] C. Carathéodory, Calculus of Variations and Partial Differential Equations of First Order, Holden-Day, 1967; H.A. Kastrup, Phys. Rep. 101, 1 (1983); Y. Güler, J. Math. Phys. 30, 785 (1989); B.M. Pimentel, R.G. Teixeira, J.L. Tomazelli, Ann. Phys. 267, 75 (1998).
[11] K.D. Rothe and F.G. Scholtz, *Comments on the Hamilton-Jacobi equation for second class constrained systems*, to appear in Annals of Physics, N.Y.

[12] H. Goldstein, Classical Mechanics, Addison-Wesley, Massachusetts, 1980.

[13] T.D. Lee, Particle Physics and Introduction to Field Theory, Harwood, New York, 1981.

[14] S.T. Hong, Y.W. Kim, Y.J. Park, Phys. Rev. D59, 114026 (1999).

[15] N. Vilenkin, Special Functions and the Theory of Group Representations, Amer. Math. Soc., Providence, 1968.

[16] C. Neves, C. Wotzasek, J. Phys. A33, 6447 (2000).

[17] B. Podolsky, Phys. Rev. 32, 812 (1928); L.D. Landau, E.M. Lifshitz, Quantum Mechanics, Statistics and Polymer Physics, Pergamon, New York, 1965; H. Kleinert, S.V. Shabanov, Phys. Lett. A232, 327 (1997).

[18] C. Becci, A. Rouet, R. Stora, Ann. Phys. 98, 287 (1976); I.V. Tyutin, Lebedev Preprint 39 (1975) unpublished.

[19] E.S. Fradkin, G.A. Vilkovisky, Phys. Lett. B55, 224 (1975); M. Henneaux, Phys. Rep. 126, 1 (1985).

[20] T. Fujiwara, Y. Igarashi, J. Kubo, Nucl. Phys. B341, 695 (1990); Y.W. Kim, S.K. Kim, W. T. Kim, Y.J. Park, K. Y. Kim, Y. Kim, Phys. Rev. D46, 4574 (1992); C. Bizdadea, S.O. Saliu, Nucl. Phys. B456, 473 (1995).