FINITE QUANTUM PHYSICS AND NONCOMMUTATIVE GEOMETRY

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Abstract

Conventional discrete approximations of a manifold do not preserve its nontrivial topological features. In this article we describe an approximation scheme due to Sorkin which reproduces physically important aspects of manifold topology with striking fidelity. The approximating topological spaces in this scheme are partially ordered sets (posets). Now, in ordinary quantum physics on a manifold $M$, continuous probability densities generate the commutative C*-algebra $C(M)$ of continuous functions on $M$. It has a fundamental physical significance, containing the information to reconstruct the topology of $M$, and serving to specify the domains of observables like the Hamiltonian. For a poset, the role of this algebra is assumed by a noncommutative C*-algebra $\mathcal{A}$. As noncommutative geometries are based on noncommutative C*-algebras, we therefore have a remarkable connection between finite approximations to quantum physics and noncommutative geometries. Various methods for doing quantum physics using $\mathcal{A}$ are explored. Particular attention is paid to developing numerically viable approximation schemes which at the same time preserve important topological features of continuum physics.

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1 Introduction

Experience teaches us that realistic physical theories are complicated and require approximations for extraction of their predictions. A powerful approximation method, particularly effective for numerical work, is the discretisation of continuum physics where manifolds are replaced by a lattice of points. It has acquired a central role in the study of fundamental physical theories such as QCD [1] or Einstein gravity [2].

A notable limitation of such discretisations is their poor ability to preserve the topological properties of continuum theories. Thus in these approximations a manifold is typically substituted by a set of points with discrete topology. The latter is entirely incapable of describing any significant topological attribute of the continuum, this being equally the case for both local and global properties. There is for example no nontrivial concept of winding number when manifolds are modelled by discrete points and hence also no way to associate solitons with winding numbers in these approximations.

Some time ago, Sorkin [3] studied a very interesting method for finite approximations of manifolds by certain point sets in detail. [See also ref. [4].] These sets are partially ordered sets (posets) and have the ability to reproduce important topological features of the continuum with remarkable fidelity. Subsequent research [5] developed these methods of Sorkin and others and made them usable for approximate computations in quantum physics. They could thus become viable alternatives to computational schemes like those in lattice QCD [1].

In this article, after a review of the poset approximation scheme in Section 2, we explore its properties in a novel direction. In quantum physics on a manifold $M$, a fundamental role is played by the (C*--) algebra $\mathcal{C}(M)$ of continuous functions on $M$. Indeed, it is possible to recover $M$, its topology and even its $C^\infty$-structure when this algebra and a distinguished subalgebra are given [4, 6, 7, 8, 9, 10]. For this reason, it is also possible to rewrite quantum theories on $M$ substituting this algebra for $M$, the tools for doing such calculations efficiently also being readily available [8, 9, 10]. All this material on $\mathcal{C}(M)$ is reviewed in Section 3 with particular attention to its physical meaning.

The special role of $\mathcal{C}(M)$ for manifolds suggests that it is of basic interest to know the algebra $\mathcal{A}$ replacing $\mathcal{C}(M)$ when $M$ is approximated by a poset. As we shall see in Section 4, $\mathcal{A}$ is an infinite-dimensional noncommutative C*-algebra, the poset and its topology being recoverable from the knowledge of $\mathcal{A}$. Noncommutative geometries are built using noncommutative C*-algebras [8, 9, 10]. In this way we discover the striking result that topologically meaningful finite approximations to manifolds lead to quantum physics based on noncommutative geometries.

It bears emphasis that this conclusion emerges in a natural manner while approximating conventional quantum theory. Therefore the interest in noncommutative geometry for a physicist need not depend on unusual space-time topologies like the one used by Connes and Lott [10] in building the standard model. Furthermore these quantum models on posets are of independent interest and not just as approximations to continuum theories,
as they provide us with a whole class of examples with novel geometries.

In Sections 4 and 5, we also discuss many aspects of quantum physics based on $\mathcal{A}$, drawing on known mathematical methods of the noncommutative geometer and the $C^*$-algebraist.

The $C^*$-algebras for our posets are as a rule inductive limits \[6\] of finite dimensional algebras, being examples of “approximately finite dimensional” algebras \[11, 12\]. Therefore we can approximate $\mathcal{A}$ by finite dimensional algebras and in particular by a commutative finite dimensional algebra $\mathcal{C}(\mathcal{A})$. Their elements can be regarded as continuous “functions” (or rather, as sections of a certain bundle) on the poset. They too encode the topology of the latter. The algebra $\mathcal{C}(\mathcal{A})$ is also strikingly simple, so that it is relatively easy to build a quantum theory using $\mathcal{C}(\mathcal{A})$. In Section 6, we describe these approximations and argue also that the approximation by $\mathcal{C}(\mathcal{A})$ can be obtained from a gauge principle.

Section 7 deals with a concrete example having nontrivial topological features, namely the poset approximation to a circle. We establish that global topological effects can be captured by poset approximations and algebras $\mathcal{C}(\mathcal{A})$ by showing that the “$\theta$-angle” for a particle on a circle can also be treated using $\mathcal{C}(\mathcal{A})$.

Section 8 concerns the sense in which the algebras $\mathcal{A}$ and $\mathcal{C}(\mathcal{A})$ are continuous “functions” on their posets. While this discussion is conceptually important, it was not undertaken earlier to prevent interruption of the main flow of ideas. The article concludes with Section 9.

This article is an expanded version of the material covered by the last lecture of A.P. Balachandran at the XV Autumn School on “Particle Physics in the Nineties” [Lisbon, 11-16 October 1993]. The material covered by the remaining lectures are available elsewhere. [Chapters 8 and 20 of \[13\], and also \[14\].] It is also an expanded version of the talk at the International Colloquium on Modern Quantum Field Theory II [Tata Institute of Fundamental Research, Bombay, 5-11 January, 1994].

2 The Finite Approximation

Let $M$ be a continuous topological space like for example the sphere $S^N$ or the Euclidean space $R^N$. Experiments are never so accurate that they can detect events associated with points of $M$, rather they only detect events as occurring in certain sets $O_\lambda$. It is therefore natural to identify any two points $x, y$ of $M$ if every set $O_\lambda$ containing either point contains the other too. Let us assume that the sets $O_\lambda$ cover $M$,

\[ M = \bigcup_\lambda O_\lambda, \]

and write $x \sim y$ if $x$ and $y$ are not separated or distinguished by $O_\lambda$ in the sense above:

\[ x \sim y \text{ means } x \in O_\lambda \Leftrightarrow y \in O_\lambda \text{ for every } O_\lambda. \]
Then $\sim$ is an equivalence relation, and it is reasonable to replace $M$ by $M/\sim\equiv P(M)$ to reflect the coarseness of observations. It is this space, obtained by identifying equivalent points [and with the quotient topology explained later], that will be our approximation for $M$.

We assume that the number of sets $O_\lambda$ is finite when $M$ is compact so that $P(M)$ is an approximation to $M$ by a finite set in this case. When $M$ is not compact, we assume instead that each point has a neighbourhood intersected by only finitely many $O_\lambda$ so that $P(M)$ is a “finitary” approximation to $M$. We also assume that each $O_\lambda$ is open and that

$$U = \{O_\lambda\}$$

is a topology for $M$ [15]. This implies that $O_\lambda \cup O_\mu$ and $O_\lambda \cap O_\mu \in U$ if $O_{\lambda,\mu} \in U$. Now experiments can isolate events in $O_\lambda \cup O_\mu$ and $O_\lambda \cap O_\mu$ if they can do so in $O_\lambda$ and $O_\mu$ separately, the former by detecting an event in either $O_\lambda$ or $O_\mu$, and the latter by detecting it in both $O_\lambda$ and $O_\mu$. The hypothesis that $U$ is a topology is thus conceptually consistent.

These assumptions allow us to isolate events in certain sets of the form $O_\lambda \setminus [O_\lambda \cap O_\mu]$ which may not be open. This means that there are in general points in $P(M)$ coming from sets which are not open in $M$.

In the notation we employ, if $P(M)$ has $N$ points, we sometimes denote it by $P_N(M)$.

Let us illustrate these considerations for a cover of $M = S^1$ by four open sets as in Fig. 1(a). In that figure, $O_{1,3} \subset O_2 \cap O_4$. Figure 1(b) shows the corresponding discrete space $P_4(S^1)$, the points $x_i$ being images of sets in $S^1$. The map $S^1 \to P_4(S^1)$ is given by

$$O_1 \to x_1, \quad O_2 \setminus [O_2 \cap O_4] \to x_2,$$

$$O_3 \to x_3, \quad O_4 \setminus [O_2 \cap O_4] \to x_4.$$  

(2.4)

Now $P(M)$ inherits the quotient topology from $M$ [13]. It is defined as follows. Let $\Phi$ be the map from $M$ to $P(M)$ obtained by identifying equivalent points. An example of $\Phi$ is given by (2.4). In the quotient topology, a set in $P(M)$ is declared to be open if its inverse image for $\Phi$ is open in $M$. It is the finest topology compatible with the continuity of $\Phi$. We adopt it hereafter as the topology for $P(M)$.

This topology for $P_4(S^1)$ can be read off from Fig.1, the open sets being

$$\{x_1\}, \quad \{x_3\}, \quad \{x_1, x_2, x_3\}, \quad \{x_1, x_4, x_3\},$$

and their unions and intersections (an arbitrary number of the latter being allowed as $P_4(S^1)$ is finite).

A partial order $\preceq$ [4, 16, 17] can be introduced in $P(M)$ by declaring that $x \preceq y$ if every open set containing $y$ contains also $x$. It then becomes a partially ordered set or a poset. For $P_4(S^1)$, this order reads

$$x_1 \preceq x_2, \quad x_1 \preceq x_4; \quad x_3 \preceq x_2, \quad x_3 \preceq x_4,$$

(2.6)
where we have omitted writing the relations \( x_j \preceq x_j \).

Later, we will write \( x \prec y \) to indicate that \( x \preceq y \) and \( x \neq y \).

In a Hausdorff space \([15]\), there are open sets \( O_x \) and \( O_y \) containing any two distinct points \( x \) and \( y \) such that \( O_x \cap O_y = \emptyset \). A finite Hausdorff space necessarily has the discrete topology where each point is an open set. So \( P(M) \) is not Hausdorff. But it is what is called \( T_0 \) \([15]\), where for any two distinct points, there is an open set containing at least one of these points and not the other. For \( x_1 \) and \( x_2 \) of \( P_4(S^1) \), the open set \( \{x_1\} \) contains \( x_1 \) and not \( x_2 \), but there is no open set containing \( x_2 \) and not \( x_1 \).

Any poset can be represented by a Hasse diagram constructed by arranging its points at different levels and connecting them using the following rules: 1) If \( x \prec y \), then \( y \) is higher than \( x \). 2) If \( x \prec y \) and there is no \( z \) such that \( x \prec z \prec y \), then \( x \) and \( y \) are connected by a line called a link.

In case 2), \( y \) is said to cover \( x \).

The Hasse diagram for \( P_4(S^1) \) is shown in Fig. 2.

The smallest open set \( O_x \) containing \( x \) consists of all \( y \) preceding \( x \) \((y \preceq x)\) so that the closure of the singleton set \( \{y\} \) contains \( x \). In the Hasse diagram, it consists of \( x \) and all points we encounter as we travel along links from \( x \) to the bottom. In Fig. 2, this rule gives \( \{x_1, x_2, x_3\} \) as the smallest open set containing \( x_2 \), just as in \((2.5)\).

As another example, consider the Hasse diagram of Fig. 3 for a two-sphere poset \( P_6(S^2) \) derived in \([3]\). Its open sets are generated by

\[
\{x_1\}, \quad \{x_3\}, \quad \{x_1, x_2, x_3\}, \quad \{x_1, x_4, x_3\},
\]

\[
\{x_1, x_2, x_5, x_4, x_3\}, \quad \{x_1, x_2, x_6, x_4, x_3\},
\]

(2.7)

by taking unions and intersections.

As one more example, Fig. 4 shows a cover of \( S^1 \) by \( 2N \) open sets \( O_j \) and the Hasse diagram of its poset \( P_{2N}(S^1) \).

Next we define the notion of rank.

A point \( x \) of a poset \( P \) can be assigned a rank \( r(x) \) as follows. A point of \( P \) is regarded as of rank 0 if it converges to no point, or is a highest point. Let \( P_1 \) be the poset got from \( P \) by removing all rank zero points and their links. The highest points of \( P_1 \) are assigned rank 1. We continue in this way to rank all points.

The rank of a poset is just the maximum rank occurring among the points. [This definition is commonly used only for “rankable” posets, a concept we will not need and will not define in this article]

We conclude this section by illustrating one of the remarkable properties of a poset approximation, namely its ability to accurately reproduce the fundamental group of the manifold. This we do by indicating that the fundamental group \([13]\) of \( P_4(S^1) \) is \( \mathbb{Z} \) \([3]\). This group is obtained from continuous maps of \( S^1 \) to \( P_4(S^1) \), or equivalently, from such
maps of \([0,1]\) to \(P_4(S^1)\) with the same value at 0 and 1. Figure 5 shows maps like this. The maps shown are continuous, the inverse images of open sets being open \([13]\). The map in Fig. 5(a) can be deformed to the constant map and has zero winding number. The image in Fig. 5(b) “winds once around” \(P_4(S^1)\). The map in this figure has winding number one and is not homotopic to the map in Fig. 5(a). It leads in a conventional way to the generator of \(\mathbb{Z}\). Of course \(\mathbb{Z}\) is also \(\pi_1(S^1)\).

3 Topology from Quantum Physics

In conventional quantum physics, the configuration space is generally a manifold when the number of degrees of freedom is finite. If \(M\) is this manifold and \(\mathcal{H}\) the Hilbert space of wave functions, then \(\mathcal{H}\) consists of all square integrable functions on \(M\) for a suitable integration measure. A wave function \(\psi\) is only required to be square integrable. There is no need for \(\psi\) or the probability density \(\psi^*\psi\) to be a continuous function on \(M\). Indeed there are plenty of noncontinuous \(\psi\) and \(\psi^*\psi\). Wave functions of course are not directly observable, but probability densities are, and the existence of noncontinuous probability densities have potentially disturbing implications. If all states of the system are equally available to preparation, which is the case if all self-adjoint operators are equally observable, then clearly we cannot infer the topology of \(M\) by measurements of probability densities.

It may also be recalled in this connection that any two infinite-dimensional (separable) Hilbert spaces \(\mathcal{H}_1\) and \(\mathcal{H}_2\) are unitarily related. \([\text{Choose an orthonormal basis } \{h_n^{(i)}\}, (n = 0,1,2,\ldots) \text{ for } \mathcal{H}_i (i = 1,2). \text{ Then a unitary map } U : \mathcal{H}_1 \rightarrow \mathcal{H}_2 \text{ from } \mathcal{H}_1 \text{ to } \mathcal{H}_2 \text{ is defined by } Uh_n^{(1)} = h_n^{(2)}.]\) They can therefore be identified or thought of as the same. Hence the Hilbert space of states in itself contains no information whatsoever about the configuration space.

It seems however that not all self-adjoint operators have equal status in quantum theory. Instead, there seems to exist a certain class of privileged observables \(\mathcal{PO}\) which carry information on the topology of \(M\) and also have a special role in quantum physics. This set \(\mathcal{PO}\) contains operators like the Hamiltonian and angular momentum, and particularly also the set of continuous functions \(C(M)\) on \(M\), vanishing at infinity if \(M\) is noncompact.

In what way is the information on the topology of \(M\) encoded in \(\mathcal{PO}\)? To understand this, recall that an unbounded operator such as a typical Hamiltonian \(H\) cannot be applied on all vectors in \(\mathcal{H}\). Instead, it can be applied only on vectors in its domain \(D(H)\), the latter being dense in \(\mathcal{H}\) \([19]\). In ordinary quantum mechanics, \(D(H)\) typically consists of twice-differentiable functions on \(M\) with suitable fall-off properties at \(\infty\) in case \(M\) is noncompact. In any event, what is important to note is that if \(\psi, \chi \in D(H)\) in elementary quantum theory, then \(\psi^*\chi \in C(M)\). A similar property holds for the domain \(D\) of any unbounded operator in \(\mathcal{PO}\): If \(\psi, \chi \in D\), then \(\psi^*\chi \in C(M)\). It is thus in the nature of
these domains that we must seek the topology of $M$. [1]

We have yet to remark on the special physical status of $PO$ in quantum theory. Let $E$ be the intersection of the domains of all operators in $PO$. Then it seems that the basic physical properties of the system, and even the nature of $M$, are all inferred from observations of the privileged observables on states associated with $E$. [2]

This discussion shows that for a quantum theorist, it is quite important to understand clearly how $M$ and its topology can be reconstructed from the algebra $C(M)$. Such a reconstruction theorem already exists in the mathematical literature. It is due to Gel’fand and Naimark [6], and is a basic result in the theory of $C^*$-algebras. Its existence is reassuring and indicates that we are on the right track in imagining that it is $PO$ which contains information on $M$ and its topology.

The following however must be noted: The above theorem uses specific steps to reconstruct $M$ and its topology. It is not so clear that we actually achieve this reconstruction from physical experience using the same steps.

We next explain the Gel’fand-Naimark results briefly.

A $C^*$-algebra $A$, commutative or otherwise, is an algebra with a norm $\| \cdot \|$ and an antilinear involution $^*$ such that $\| a \| = \| a^* \|$, $\| a^*a \| = \| a^* \| \| a \|$ and $(ab)^* = b^*a^*$ for $a, b \in A$. $A$ is also assumed to be complete in the given norm. Examples of $C^*$-algebras are: 1) The algebra of $n \times n$ matrices $T$ with $T^*$ the hermitian conjugate of $T$, and $\| T \|^2 = $ the largest eigenvalue of $T^*T$; 2) The algebra $C(M)$ of continuous functions on $M$, with $^*$ denoting complex conjugation and the norm given by the supremum norm, $\| f \| = \sup_{x \in M} |f(x)|$.

In discussing the reconstruction theorem, it is not useful to imagine that $C(M)$ is given as a set of actual functions on $M$, for that presupposes that $M$ is already known. Rather, it is better to suppose that we are given a commutative $C^*$-algebra $C$. We must then reconstruct a topological space $M$ from $C$ such that $C(M) = C$. We would also like $M$ to be unique up to homeomorphisms.

So we assume that we are given such a $C$. Let $M$ denote the space of [equivalence classes of] irreducible representations (IRR’s), also called the structure space, of $C$. [The trivial IRR given by $C \to \{0\}$ is not included in $M$.] The $C^*$-algebra $C$ being commutative, every IRR is one-dimensional. Hence if $x \in M$ and $f \in C$, the image $x(f)$ of $f$ in the IRR defined by $x$ is a complex number. Writing $x(f)$ as $f(x)$, we can therefore regard $f$ as a complex-valued function on $M$ with the value $f(x)$ at $x \in M$. We thus get the interpretation of $C$ as $C$-valued functions on $M$.

We next topologise $M$ by declaring that the set of zeros of each $f \in C$ is a closed

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*Our point of view about the manner in which topology is inferred from quantum physics was developed in collaboration with G. Marmo and A. Simoni.

†Note in this connection that any observable of $PO$ restricted to $E$ must be essentially self-adjoint [19]. This is because if significant observations are all confined to states given by $E$, they must be sufficiently numerous to determine the operators of $PO$ uniquely.
set. [This is natural to do since the set of zeros of a continuous function is closed.] The topology of \( M \) is generated by these closed sets. It is also called the hull kernel or Jacobson topology.

Gel’fand and Naimark also show that \( \mathcal{C} = \mathcal{C}(M) \) and that the requirement \( \mathcal{C} = \mathcal{C}(N) \) uniquely fixes the manifold \( N \) up to homeomorphisms. In this way, we recover the topological space \( M \), uniquely up to homeomorphisms, from the algebra \( \mathcal{C} \).

We next briefly indicate how we can do quantum theory starting from \( \mathcal{C}(M) = \mathcal{C} \).

Elements of \( \mathcal{C} \) are observables, they are not quite wave functions. The set of all wave functions form a Hilbert space \( \mathcal{H} \). Our first step in constructing \( \mathcal{H} \), essential for quantum physics, is the construction of the space \( \mathcal{E} \) which will serve as the common domain of all the privileged observables.

The simplest choice for \( \mathcal{E} \) is \( \mathcal{C} \) itself. With this choice, \( \mathcal{C} \) acts on \( \mathcal{E} \), as \( \mathcal{C} \) acts on itself by multiplication. The presence of this action is important as the privileged observables must act on \( \mathcal{E} \). Further, for \( \psi, \chi \in \mathcal{E} \), \( \psi^* \chi \in \mathcal{C} \) exactly as we want.

Now Gel’fand and Naimark have established that it is possible to integrate over the structure space \( M \) of \( \mathcal{C} \). A scalar product \( (\cdot, \cdot) \) for elements of \( \mathcal{E} \) can therefore be defined by choosing an integration measure \( d\mu \) on \( M \) and setting

\[
(\psi, \chi) = \int_M d\mu(x)(\psi^* \chi)(x).
\]

The completion of the space \( \mathcal{E} \) using this scalar product gives the Hilbert space \( \mathcal{H} \).

The final set-up for quantum theory here is conventional. What is novel is the shift in emphasis to the algebra \( \mathcal{C} \). It is from this algebra that we now regard the configuration \( M \) and its topology as having been constructed.

There is of course no reason why \( \mathcal{E} \) should always be \( \mathcal{C} \). Instead it can consist of sections of a vector bundle over \( M \) with a \( \mathcal{C} \)-valued positive definite sesquilinear form \( <\cdot, \cdot> \). \( <\cdot, \cdot> \) is positive definite if \( <\alpha, \alpha> \) is a nonnegative function for any \( \alpha \in \mathcal{E} \) which identically vanishes iff \( \alpha = 0 \).] The scalar product is then written as

\[
(\psi, \chi) = \int_M d\mu(x) <\psi, \chi>(x).
\]

The completion of \( \mathcal{E} \) using this scalar product as before gives \( \mathcal{H} \).

There is an algebraic construction of nontrivial \( \mathcal{E} \) from \( \mathcal{C} \) which is meaningful even for noncommutative \( \mathcal{C} \). We will explain this construction in Section.

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\( ^3 \) We remark that more refined attributes of \( M \) such as a \( C^\infty \)-structure can also be recovered using only algebras if more data are given. For the \( C^\infty \)-structure, for example, we must also specify an appropriate subalgebra \( C^\infty(M) \) of \( \mathcal{C}(M) \). The \( C^\infty \)-structure on \( M \) is then the unique \( C^\infty \)-structure for which the elements of \( C^\infty(M) \) are all the \( C^\infty \)-functions.

\( ^4 \) Differentiability requirements will in general further restrict \( \mathcal{E} \). We will as a rule ignore such details in this article.
4 Finite Approximations and Noncommutative Geometry

4.1 The Algebra $\mathcal{A}$ for a Poset

In the preceding section, it has been argued that the algebra $\mathcal{C}(M)$ plays a basic role in quantum theory on a manifold $M$. It is hence of interest to enquire about the algebra $\mathcal{A}$ replacing $\mathcal{C}(M)$ when $M$ is approximated by a poset $P$.

Let us first recall a few definitions and results from operator theory \cite{19} before outlining an answer to this question. An operator in a Hilbert space is said to be of finite rank if the orthogonal complement of its null space is finite dimensional. It is thus essentially like a finite dimensional matrix as regards its properties even if the Hilbert space is infinite dimensional. An operator $k$ in a Hilbert space is said to be compact if it can be approximated arbitrarily closely in norm by finite rank operators. Let $\lambda_1, \lambda_2, \ldots$ be the eigenvalues of $k^*k$ for such a $k$, with $\lambda_{i+1} \leq \lambda_i$ and an eigenvalue of multiplicity $n$ occurring $n$ times in this sequence. [Here and in what follows, $*$ denotes the adjoint for an operator.] Then $\lambda_n \to 0$ as $n \to \infty$. It follows that the operator $\mathbb{1}$ in an infinite dimensional Hilbert space is not compact.

The set $\mathcal{K}$ of all compact operators $k$ in a Hilbert space is a $C^*$-algebra. It is a two-sided ideal in the $C^*$-algebra $\mathcal{B}$ of all bounded operators \cite{6,20}. The $*$-operation here for both $\mathcal{K}$ and $\mathcal{B}$ is hermitian conjugation.

Note that the sets of finite rank, compact and bounded operators are all the same in a finite dimensional Hilbert space. All operators in fact belong to any of these sets in finite dimensions.

The construction of $\mathcal{A}$ rests on the following result from the representation theory of $\mathcal{K}$. The representation of $\mathcal{K}$ by itself is irreducible \cite{6}. It is the only IRR of $\mathcal{K}$ up to equivalence. [The trivial representation where the whole algebra is represented by zero is ignored here and hereafter.]

The simplest nontrivial poset is $P_2 = \{p, q\}$ with $q \prec p$. It is shown in Fig.6. It is the poset for the interval $[r, s]$ ($r < s$) where the latter is covered by the open sets $[r, s]$ and $[r, s]$. The map from subsets of $[r, s]$ to the points of $P_2$ is

$$\{s\} \to p, \quad [r, s] \to q. \quad (4.1)$$

The algebra $\mathcal{A}$ for $P_2$ is

$$\mathcal{A} = \mathbb{C} \mathbb{1} + \mathcal{K}, \quad (4.2)$$

the Hilbert space on which the operators of $\mathcal{A}$ act being infinite dimensional.

We can see this result from the fact that $\mathcal{A}$ has only two IRR’s and they are given by

$$p : \lambda \mathbb{1} + k \to \lambda := (\lambda \mathbb{1} + k)(p),$$
\[ q : \lambda \mathbb{I} + k \rightarrow \lambda \mathbb{I} + k := (\lambda \mathbb{I} + k)(q). \] 

(4.3)

This remark about IRR’s becomes plausible if it is remembered that \( K \) has only one IRR.

Now we can use the hull kernel topology for the set \( \{p, q\} \). For this purpose, consider the “function” \( k \). It vanishes at \( p \) and not at \( q \), so \( p \) is closed. Its complement \( q \) is hence open. So of course is the whole space \( \{p, q\} \). The topology of \( \{p, q\} \) is thus given by Fig. 6(a) and is that of the \( P_2 \) poset just as we want.

We remark here that an equivalent topology can be defined for finite structure spaces as follows: Let \( I_x \) be the kernel for the IRR \( x \). It is the (two-sided) ideal mapped to 0 by the IRR \( x \). We set \( x \prec y \) if \( I_x \subset I_y \) thereby converting the space of IRR’s into a poset. The hull kernel topology is the topology of this poset.

In our case, \( I_p = K, I_q = \{0\} \subset I_p \) and hence \( q \prec p \). This gives Fig. 6(a).

We next consider the \( \lor \) poset. It can be obtained from the following open cover of the interval \([0, 1]\):

\[ [0, 1] = \bigcup_{\lambda} O_{\lambda} : \]

\[ O_1 = [0, 2/3], \quad O_2 = [1/3, 1], \quad O_3 = [1/3, 2/3]. \]  

(4.4)

The map from subsets of \([0, 1]\) to the points of the \( \lor \) poset in Fig. 7(a) is given by

\[ [0, 1/3] \rightarrow \alpha, \quad [1/3, 2/3] \rightarrow \gamma, \quad [2/3, 1] \rightarrow \beta. \]

(4.5)

Let us next find the algebra \( \mathcal{A} \) for the \( \lor \) poset of Fig. 7(a). The \( \lor \) has two arms 1 and 2. The first step in the construction is to attach an infinite-dimensional Hilbert space \( \mathcal{H}_i \) to each arm \( i \) as in Figs. 8(a) and 8(b). To a \( \lor \) with arms \( i, i+1 \), attach the algebra \( \mathcal{A}_i \) with elements \( \lambda_i \mathcal{P}_i + \lambda_{i+1} \mathcal{P}_{i+1} + k_{i,i+1} \). Here \( \lambda_i, \lambda_{i+1} \) are any two complex numbers, \( \mathcal{P}_i, \mathcal{P}_{i+1} \) are orthogonal projectors on \( \mathcal{H}_i \) and \( \mathcal{H}_{i+1} \) in the Hilbert space \( \mathcal{H}_i \oplus \mathcal{H}_{i+1} \) and \( k_{i,i+1} \) is any compact operator in \( \mathcal{H}_i \oplus \mathcal{H}_{i+1} \). This is as before. But now, for glueing the various \( \lor \)’s together, we also impose the condition that \( \lambda_j = \lambda_k \) if lines \( j \) and \( k \) meet at top. The algebra \( \mathcal{A} \) is then the direct sum of \( \mathcal{A}_i \)’s with this condition:

\[ \mathcal{A} = \bigoplus \mathcal{A}_i \text{ with } \lambda_j = \lambda_k \text{ if lines } j, k \text{ meet at top.} \]

(4.7)
Figures 8(a) and (b) also show the values of an element \( a = \oplus \lambda_i P_i + \lambda_{i+1} P_{i+1} + k_{i,i+1} \) at the different points of two typical rank one posets.

There is a systematic construction of \( \mathcal{A} \) for any poset (that is, any “finite \( T_0 \) topological space”) which generalizes the preceding constructions for two-level posets. It is explained in Fell and Doran [6] and will not be described here. Actually the poset does not uniquely fix its algebra as there are in general many \( \mathbb{C}^* \)-algebras with the same poset as structure space [21]. [However a Hausdorff structure space such as a manifold does indeed do so.] The Fell-Doran choice seems to be the simplest. We will call it \( \mathcal{A} \) and adopt it in this paper.

4.2 How to do Quantum Theory using \( \mathcal{A} \)

The noncommutative algebra \( \mathcal{A} \) is an algebra of observables. It replaces the algebra \( \mathcal{C}(M) \) when \( M \) is approximated by a poset. We must now find the space \( \mathcal{E} \) on which \( \mathcal{A} \) acts, convert \( \mathcal{E} \) into a pre-Hilbert space and therefrom get the Hilbert space \( \mathcal{H} \) by completion.

Incidentally, as \( \mathcal{A} \) is supposed to generalize \( \mathcal{C}(M) \), it is also pertinent to enquire about the sense in which the elements of \( \mathcal{A} \) are continuous functions on the poset. We postpone the clarification of this point to Section 8 in order not to interrupt the present discussion.

Now as \( \mathcal{A} \) is noncommutative, it turns out to be important to specify if \( \mathcal{A} \) acts on \( \mathcal{E} \) from the right or the left. We will take the action of \( \mathcal{A} \) on \( \mathcal{E} \) to be from the right, thereby making \( \mathcal{E} \) a right \( \mathcal{A} \)-module.

The simplest model for \( \mathcal{E} \) is obtained from \( \mathcal{A} \) itself. As for the scalar product, note that \( (\xi^* \eta)(x) \) is an operator in a Hilbert space \( \mathcal{H}_x \) if \( \xi, \eta \in \mathcal{A} \) and \( x \in \text{poset} \). We can hence find a scalar product \( (\cdot, \cdot) \) by first taking its operator trace \( Tr \) on \( \mathcal{H}_x \) and then summing it over \( \mathcal{H}_x \) with suitable weights \( \rho_x \):

\[
(\xi, \eta) = \sum_x \rho_x Tr(\xi^* \eta)(x), \quad \rho_x \geq 0.
\]  

(4.8)

As remarked in Section 3, there is no need for \( \mathcal{E} \) to be \( \mathcal{A} \). It can be any space with the following properties:

(1) It is a right \( \mathcal{A} \)-module. So, if \( \xi \in \mathcal{E} \) and \( a \in \mathcal{A} \), then \( \xi a \in \mathcal{E} \).

(2) There is a positive definite “sesquilinear” form \( < \cdot, \cdot > \) on \( \mathcal{E} \) with values in \( \mathcal{A} \). That is, if \( \xi, \eta \in \mathcal{E} \), and \( a \in \mathcal{A} \), then

a) \[
< \xi, \eta > \in \mathcal{A} , \quad < \xi, \eta >^* = < \eta, \xi > , \quad < \xi, \xi > \geq 0 \quad \text{and} \quad < \xi, \xi > = 0 \Leftrightarrow \xi = 0.
\]  

(4.9)
Here “$<\xi,\xi> \geq 0$” means that it can be written as $a^*a$ for some $a \in \mathcal{A}$.

b)

$$<\xi, \eta a > = <\xi, \eta > a, \quad <\xi a, \eta > = a^* <\xi, \eta > .$$

(4.10)

The scalar product is then given by

$$(\xi, \eta) = \sum_x \rho_x Tr <\xi, \eta > (x) .$$

(4.11)

As $\xi^* \eta (x), <\xi, \eta > (x), <\xi, \eta a > (x)$ or $<a \xi, \eta > (x)$ may not be of trace class [20], there are questions of convergence associated with (4.8) and (4.11). We presume that these traces must be judiciously regularized and modified (using for example the Dixmier trace [8, 9, 10]) or suitable conditions put on $\mathcal{E}$ or both. (See also Section 5.2.) But we will not address such questions in detail in this article.

When $\mathcal{A}$ is commutative and has structure space $\mathcal{M}$, then an $\mathcal{E}$ with the properties described consists of sections of hermitian vector bundles over $\mathcal{M}$. Thus, the above definition of $\mathcal{E}$ achieves a generalization of the familiar notion of sections of hermitian vector bundles to noncommutative geometry.

In the literature [8, 9, 10], a method is available for the algebraic construction of $\mathcal{E}$. It works both when $\mathcal{A}$ is commutative and noncommutative. In the former case, Serre and Swan [8, 9, 10] also prove that this construction gives (essentially) all $\mathcal{E}$ of physical interest, namely all $\mathcal{E}$ consisting of sections of vector bundles. It is as follows. Consider $\mathcal{A} \otimes \mathbb{C}^N \equiv \mathcal{A}^N$ for some integer $N$. This space consists of $N$-dimensional vectors with coefficients in $\mathcal{A}$ (that is, with elements of $\mathcal{A}$ as entries). We can act on it from the left with $N \times N$ matrices with coefficients in $\mathcal{A}$. Let $e = [e^i_j]$ be such a matrix which is idempotent, $e^2 = e$, and self-adjoint, $<e \xi, \eta > = <\xi, e^* \eta >$. Then, $e \mathcal{A}^N$ is an $\mathcal{E}$, and according to the Serre-Swan theorem, every $\mathcal{E}$ [in the sense above] is given by this expression for some $N$ and some $e$ for commutative $\mathcal{A}$. An $\mathcal{E}$ of the form $e \mathcal{A}^N$ is called a “projective module of finite type” or a “finite projective module”.

Note that such $\mathcal{E}$ are right $\mathcal{A}$-modules. For, if $\xi \in e \mathcal{A}^N$, it can be written as a vector $(\xi^1, \xi^2, \ldots, \xi^N)$ with $\xi^i \in \mathcal{A}$ and $e^i_j \xi^j = \xi^i$. The action of $a \in \mathcal{A}$ on $\mathcal{E}$ is

$$\xi \rightarrow \xi a = (\xi^1 a, \xi^2 a, \ldots, \xi^N a) .$$

(4.12)

With this formula for $\mathcal{E}$, it is readily seen that there are many choices for $<\cdot, \cdot >$. Thus let $g = [g_{ij}], g_{ij} \in \mathcal{A}$, be an $N \times N$ matrix with the following properties: a) $g^i_j = g_{ji}$; b) $\xi^* g_{ij} \xi^j \geq 0$ and $\xi^* g_{ij} \xi^j = 0 \Leftrightarrow \xi = 0$. Then, if $\eta = (\eta^1, \eta^2, \ldots, \eta^N)$ is another vector in $\mathcal{E}$, we can set

$$<\xi, \eta > = \xi^* g_{ij} \eta^j .$$

(4.13)

In connection with (4.13), note that the algebras $\mathcal{A}$ we consider here generally have unity. In those cases, the choice $g_{ij} \in \mathbb{C}$ is a special case of the condition $g_{ij} \in \mathcal{A}$. But if $\mathcal{A}$ has no unity, we should also allow the choice $g_{ij} \in \mathbb{C}$.
The minimum we need for quantum theory is a Laplacian $\Delta$ and a potential function $W$, as a Hamiltonian can be constructed from these ingredients. We now outline how to write $\Delta$ and $W$.

Let us first look at $\Delta$, and assume in the first instance that $E = A$.

An element $a \in A$ defines the operator $\oplus_x a(x)$ on the Hilbert space $\mathcal{H} = \oplus_x \mathcal{H}_x$, the map $a \to \oplus_x a(x)$ giving a faithful representation of $A$. So let us identify $a$ with $\oplus_x a(x)$ and $A$ with this representation of $A$ for the present.

In noncommutative geometry [8, 9, 10], $\Delta$ is constructed from an operator $D$ with specific properties on $\mathcal{H}$. The operator $D$ must be self-adjoint and the commutator $[D, a]$ must be bounded for all $a \in A$:

$$D^* = D, \quad [D, a] \in \mathcal{B} \quad \text{for all} \quad a \in A. \quad (4.14)$$

Given $D$, we construct the ‘exterior derivative’ of any $a \in A$ by setting

$$da = [D, a] := [D, \oplus_x a_x]. \quad (4.15)$$

Note that $da$ need not be in $A$, but it is in $\mathcal{B}$.

Next we introduce a scalar product on $\mathcal{B}$ by setting

$$(\alpha, \beta) = \text{Tr}[\alpha^* \beta], \quad \text{for all} \quad \alpha, \beta \in \mathcal{B}, \quad (4.16)$$

the trace being in $\mathcal{H}$. [Restricted to $A$, it becomes (4.11) with $\rho_x = 1$. This choice of $\rho_x$ is made for simplicity and can readily dispensed with. See also the comment after (4.11)]

Let $p$ be the orthogonal projection operator on $A$ for this scalar product:

$$p^2 = p^* = p, \quad pa = a \quad \text{if} \quad a \in A, \quad po = 0 \quad \text{if} \quad (a, \alpha) = 0 \quad \text{for all} \quad a \in A. \quad (4.17)$$

The Laplacian $\Delta$ on $A$ is defined using $p$ as follows.

We introduce the adjoint $\delta$ of $d$ by writing

$$(da', da) = (a', \delta da), \quad a', a \in A. \quad (4.18)$$

At this point we may be tempted to call $\Delta a = -\delta da$, following the definition of the Laplacian on functions in manifold theory. But that would not be quite correct since (4.18) fixes only the component of $-\delta da$ in $A$. For instance, (4.18) suggests the formula $[D, [D, a]]$ for $\delta da$, but the former may not be in $A$. But $p\delta da$ is in $A$ and is uniquely fixed to be $p[D, [D, a]]$. Thus we define $\Delta$ on $A$ by

$$\Delta a = -p[D, [D, a]] . \quad (4.19)$$
As for $W$, it is essentially any element of $\mathcal{A}$. (There may be restrictions on $W$ from positivity requirements on the Hamiltonian.) It acts on a wave function $a$ according to $a \rightarrow aW$, where $(aW)(x) = a(x)W(x)$.

A possible Hamiltonian $H$ now is $-\lambda \Delta + W$, $\lambda > 0$, while a Schrödinger equation is

$$i\frac{\partial a}{\partial t} = -\lambda \Delta a + aW.$$  \hspace{1cm} (4.20)

When $\mathcal{E}$ is a nontrivial projective module of finite type over $\mathcal{A}$, it is necessary to introduce a connection and “lift” $d$ from $\mathcal{A}$ to an operator $\nabla$ on $\mathcal{E}$. Let us assume that $\mathcal{E}$ is obtained from the construction described before (4.12). In that case the definition of $\nabla$ proceeds as follows.

Because of our assumption, an element $\xi \in \mathcal{E}$ is given by $\xi = (\xi^1, \xi^2, \cdots, \xi^N)$ where $\xi^i \in \mathcal{A}$ and $e^i_j \xi^j = \xi^i$. Thus $\mathcal{E}$ is a subspace of $\mathcal{A} \otimes \mathbb{C}^N := \mathcal{A}^N$:

$$\mathcal{E} \subseteq \mathcal{A}^N,$$

$$\mathcal{A}^N = \{(a^1, \cdots, a^N) : a^i \in \mathcal{A}\}. \hspace{1cm} (4.21)$$

Here we regard $a^i$ as operators on $\mathcal{H}$. Now $\mathcal{A}^N$ is a subspace of $\mathcal{B} \otimes \mathbb{C}^N := \mathcal{B}^N$ where $\mathcal{B}$ consists of bounded operators on $\mathcal{H}$. Thus

$$\mathcal{E} \subseteq \mathcal{A}^N \subseteq \mathcal{B}^N,$$

$$\mathcal{B}^N = \{ (\alpha^1, \cdots, \alpha^N) : \alpha^i = \text{bounded operator on } \mathcal{H} \}. \hspace{1cm} (4.22)$$

Let us extend the scalar product $(\cdot, \cdot)$ on $\mathcal{E}$ [given by (4.13) and (4.11)] to $\mathcal{B}^N$ by setting

$$<\alpha, \beta> = \alpha^i g_{ij} \beta^j,$$

$$(\alpha, \beta) = Tr <\alpha, \beta> \quad \text{for } \alpha, \beta \in \mathcal{B}^N. \hspace{1cm} (4.23)$$

Next, having fixed $d$ on $\mathcal{A}$ by a choice of $D$ as in (4.14), we define $d$ on $\mathcal{E}$ by

$$d\xi = (d\xi^1, d\xi^2, \cdots, d\xi^N). \hspace{1cm} (4.24)$$

Note that $d\xi$ may not be in $\mathcal{E}$, but it is in $\mathcal{B}^N$:

$$d\xi \in \mathcal{B}^N. \hspace{1cm} (4.25)$$

A possible $\nabla$ for this $d$ is

$$\nabla\xi = ed\xi + \rho \xi, \hspace{1cm} (4.26)$$

where

a) $e$ is the matrix introduced earlier,
b) \( \rho \) is an \( N \times N \) matrix with coefficients in \( B \):

\[
\rho = [\rho^i_j] , \quad \rho^i_j \in B ,
\]

(4.27)

c) \[
\rho = e^\rho e ,
\]

(4.28)

and

d) \( \rho \) is anti-hermitian:

\[
< \rho \alpha, \beta > + < \alpha, \rho \beta > = 0 .
\]

(4.29)

Note that if \( \hat{\rho} \) fulfills all conditions but c), then \( \rho = e^\hat{\rho} e \) fulfills c) as well.

Having chosen a \( \nabla \), we try defining \( \nabla^* \nabla \) using

\[
(\nabla \xi, \nabla \eta) = (\xi, \nabla^* \nabla \eta) , \quad \xi, \eta \in E
\]

(4.30)

where \( (\cdot, \cdot) \) is defined by (4.23). But as in the case of \( A \), this does not fully determine \( \nabla^* \nabla \eta \). What is fully determined is \( q \nabla^* \nabla \eta \) where \( q \) is the orthogonal projector on \( E \) for the scalar product \( (\cdot, \cdot) \). We thus define \( \Delta \) on \( E \) by

\[
\Delta \eta = -q \nabla^* \nabla \eta , \quad \eta \in E .
\]

(4.31)

Of course, \( \Delta \eta \in E \).

A potential \( W \) is an element of \( A \). It acts on \( E \) according to the rule (1) following (4.8).

A Hamiltonian as before has the form \( -\lambda \Delta + W \), \( \lambda > 0 \). It gives the Schrödinger equation

\[
\frac{i}{\hbar} \frac{\partial \xi}{\partial t} = -\lambda \Delta \xi + \xi W , \quad \xi \in E .
\]

(4.32)

We will not try to find explicit examples for \( \Delta \) here. That task will be taken up for a simple problem in Section 7.

We will conclude this section by pointing out an interesting property of states for posets. It does not seem possible to localize a state at the rank zero points [unless they happen to be isolated points, both open and closed]. We can see this for example from Fig. 7(b) which shows that if a probability density vanishes at \( \gamma \), then \( \lambda_i \) (and \( k_{12} \)) are zero and therefore they vanish also at \( \alpha \) and \( \beta \). It seems possible to show in a similar way that localization in an arbitrary poset is possible only at open sets. We will briefly interpret this result in the context of simplicial decompositions in Section 6.2.
5 Structural Results on $\mathcal{A}$ Suggest Simple Models for Quantum States

5.1 Simple Models for $\mathcal{E}$

As mentioned earlier, the $C^*$-algebras for our posets are generally of a type called approximately finite dimensional (AF) \[1\], \[2\]. [See also Section 5.1.] They have nice structural properties. In particular, they have very simple presentations in terms of their maximal commutative subalgebras, a fact which can be exploited to develop relatively transparent models for $\mathcal{E}$.

Let us start with some definitions \[1\], \[2\]. The commutant $A'$ of a subalgebra $A$ of $\mathcal{A}$ consists of all elements of $\mathcal{A}$ commuting with all elements of $A$:

$$A' = \{ x \in \mathcal{A} : xy = yx , \ \forall \ y \in A \} . \quad (5.1)$$

A maximal commutative subalgebra $C$ of $\mathcal{A}$ is a commutative $C^*$-subalgebra of $\mathcal{A}$ with $C' = C$.

The $C^*$-algebras $\mathcal{A}$ we shall now consider have a unity $1$. We therefore have the concepts of the inverse and unitary elements for $\mathcal{A}$: by definition the inverse $a^{-1}$ of $a$ fulfills $a^{-1}a = aa^{-1} = 1$, while $u^*u = uu^* = 1$ if $u$ is a unitary element of $\mathcal{A}$.

Let $C$ be a maximal commutative subalgebra of $\mathcal{A}$ and let $U$ be the normalizer of $C$ among the unitary elements of $\mathcal{A}$:

$$U = \{ u \in \mathcal{A} \mid u^*u = 1 ; \ u^*cu \in C \ \text{if} \ c \in C \} . \quad (5.2)$$

Then the algebra generated by $C$ and $U$ is $\mathcal{A} \ [1\ [2\]$, the algebra $\mathcal{A}$ for a poset being AF. If $M_1, M_2, \ldots$, are subsets of the $C^*$-algebra $\mathcal{A}$, and we denote by $< M_1, M_2, \ldots, >$ or by $< \bigcup_n M_n >$ the smallest $C^*$-subalgebra of $\mathcal{A}$ containing $\bigcup_n M_n$, then the above result can be written as

$$\mathcal{A} = < C, \ U > . \quad (5.3)$$

A property of $U$ significant for us is that if $u \in U$, then $u^* \in U$, so that $U$ is a unitary group.

Let $\hat{\mathcal{C}}$ be the space of IRR’s or the structure space of $\mathcal{C}$. It is a countable set for the AF algebra $\mathcal{A}$.

Let $\ell^2(\hat{\mathcal{C}})$ be the Hilbert space of square summable functions on $\hat{\mathcal{C}}$:

$$(g, h) = \sum_x g(x)^*h(x) < \infty , \ \forall g, h \in \ell^2(\hat{\mathcal{C}}) . \quad (5.4)$$

We will now see that $\mathcal{A}$ can be realized as operators on $\ell^2(\hat{\mathcal{C}})$. 

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For this purpose, note first that each \( x \in \hat{C} \) defines an ideal \( I_x \) of \( C \):

\[
I_x = \{ f \in C \mid f(x) = 0 \} .
\]  

(5.5)

[Here we regard elements of \( C \) as functions on \( \hat{C} \). Also, all ideals are two-sided.] Such ideals are called primitive ideals. They have the following properties for the abelian algebra \( C \):
a) Every ideal is contained in a primitive ideal, and a primitive ideal is maximal, that is it is contained in no other ideal; b) A primitive ideal \( I \) uniquely fixes a point \( x \) of \( \hat{C} \) by the requirement \( I_x = I \). Thus \( \hat{C} \) can be identified with the space \( \text{Prim}(\hat{C}) \) of primitive ideals.

Now if \( I_x \in \text{Prim}(\hat{C}) \) and \( c \in C \), then \( cu^*I_xu = u^*[ucu^*]I_xu = u^*I_xu \) since \( ucu^* \in I_x \). Similarly \( u^*I_xuc = u^*I_xu \). Hence \( u^*I_xu \) is an ideal. That being so, there is a primitive ideal \( I_y \) containing \( u^*I_xu \), \( u^*I_xu \subseteq I_y \). Hence \( I_x \subseteq uI_yu^* \). Since \( uI_yu^* \) is an ideal too, we conclude that \( I_x = uI_yu^* \) or \( u^*I_xu = I_y \). Calling

\[
y := u^*x = u^{-1}x ,
\]  

(5.6)

we thus get an action of \( U \) on \( \hat{C} \).

Next note that \( C \) in general has unitary elements and hence \( U \cap C \neq \emptyset \). Now \( U \cap C \) is a normal subgroup of \( U \). We can in fact write \( U \) as the semidirect product \( \times \) of the group \( U \cap C \) with a group \( U \) isomorphic to \( U/[U \cap C] \):

\[
U = [U \cap C] \times U .
\]  

(5.7)

Hence, by (5.3),

\[
\mathcal{A} = < C , U > .
\]  

(5.8)

This result is of great interest for us.

The group \( U \) can be explicitly constructed in cases of our interest. We will do so below for the two-point and \( \vee \) posets. The general result for any rank one poset follows easily therefrom.

It is a striking theorem of [12] that \( \mathcal{A} \) can be realized as operators on \( \ell^2(\hat{C}) \) using the formul\ae

\[
(h \cdot f)(x) = h(x)f(x) ,
\]

\[
(h \cdot u)(x) = h(u^*x) , \quad \forall f \in C , u \in U , h \in \ell^2(\hat{C}) .
\]  

(5.9)

We have shown the action as multiplication on the right in order to be consistent with the convention in Section 4.2. Also the dot has been introduced in writing this action for a reason which will immediately become apparent.

This realization of \( \mathcal{A} \) can give us simple models for \( \mathcal{E} \). To see this, first note that we had previously used \( \mathcal{A} \) or \( e\mathcal{A}^e \) as models for \( \mathcal{E} \). But as elements of \( C \) are functions on \( \hat{C} \) just like \( h \), we now discover that they are also \( \mathcal{A} \)-modules in view of (5.9), the relation between the dot product of (5.9) and the algebra product (devoid of the dot) being

\[
c \cdot f = cf ,
\]

\[
c \cdot u = ucu^{-1} .
\]  

(5.10)
The verification of (5.10) is easy.

Thus $\mathcal{C}$ itself can serve as a simple model for $\mathcal{E}$.

We may be able to go further along this line since certain finite projective modules over $\mathcal{C}$ may also serve as $\mathcal{E}$. Recall for this purpose that such a module is $EC^N$ where $E$ is an $N \times N$ matrix with coefficients in $\mathcal{C}$, which is idempotent and self-adjoint [$E^2 = E$, $E^* = E$, where $E_j^i = E_i^j$]. [Cf. Section 4.2] A vector in this module is $\xi = (\xi_1, \xi_2, \cdots, \xi_N)$ with $\xi^i = E_j^i a^j$, $a^j \in \mathcal{C}$. Now consider the action $\xi \rightarrow \xi \cdot u$ where $(\xi \cdot u)^i = u(E_j^i a^j)u^{-1}$.

The vector $\xi \cdot u$ remains in $EC^N$ if

$$uE_j^i u^{-1} = E_j^i,$$

that is, $uEu^{-1} = E$. (5.11)

Since $\mathcal{C}$ anyway acts on $EC^N$, we get an action of $\mathcal{A}$ on $EC^N$ when (5.11) is fulfilled. Thus $EC^N$ is a model for $\mathcal{E}$ when $E$ satisfies (5.11).

The scalar product for $\ell^2(\hat{\mathcal{C}})$ written above may not be the most appropriate one and may require modifications or regularization as we shall see in Section 5.2. We only mention that the problem with (5.4) arises because elements of $\mathcal{E}$ must belong to $\ell^2(\hat{\mathcal{C}})$, a restriction which may be too strong to give an interesting $\mathcal{E}$ from $\mathcal{C}$ or an interesting finite projective module thereon.

### 5.2 The Two-Point Poset

We will illustrate the implementation of these ideas for the two-point, the $\lor$ and finally for any rank one poset. That should be enough to see how to use them for a general poset.

We will treat the two-point poset first. Its algebra is (4.2). In its self-representation $q$, it acts on a Hilbert space $\mathcal{H}(= \mathcal{H}_q)$. Choose an orthonormal basis $h_n$ ($n = 1, 2, \cdots$) for $\mathcal{H}$ and let $\mathcal{P}_n$ be the orthogonal projection operator on $Ch_n$. The maximal commutative subalgebra is then

$$\mathcal{C} = \langle 1, \bigcup_n \mathcal{P}_n \rangle.$$ (5.12)

The structure space of $\mathcal{C}$ is

$$\hat{\mathcal{C}} = \{1, 2, \cdots; \infty\},$$ (5.13)

where

a) $n$ : $1 \rightarrow 1 := 1(n)$,

$$\mathcal{P}_m \rightarrow \delta_{mn} := \mathcal{P}_m(n);$$ (5.14)

b) $\infty$ : $1 \rightarrow 1 := 1(\infty)$,

$$\mathcal{P}_m \rightarrow 0 := \mathcal{P}_m(\infty).$$ (5.15)
The topology of $\hat{\mathcal{C}}$ is the one given by the one-point compactification of $\{1, 2, \cdots\}$ by adding $\infty$. A basis of open sets for this topology is

$$\{n\} \ ; \ n = 1, 2, \cdots$$

$$\mathcal{O}_k = \{m \mid m \geq k\} \bigcup \{\infty\} \ .$$

(5.16)

A particular consequence of this topology is that the sequence $1, 2, \cdots, \infty$, converges to $\infty$. This topology is identical to the hull kernel topology \[6\]. Thus for instance, the zeros of $P_n$ and $1 - \sum_{i=1}^{k-1} P_i$ are $\{1, 2, \cdots, \hat{n}, n + 1, \cdots, \infty\}$ and $\{1, 2, \cdots, k - 1\}$, respectively, where the hatted entry is to be omitted. These being closed in the hull kernel topology, their complements, which are the same as (5.16), are open as assumed above.

The group $U$ is generated by transpositions $u(i, j), i \neq j$ of $h_i$ and $h_j$:

$$u(i, j) h_i = h_j \ , \ u(i, j) h_j = h_i \ , \ u(i, j) h_k = h_k \ \text{if} \ k \neq i, j .$$

(5.17)

Since the ideals of $n$ and $\infty$ are

$$I_n = \{P_1, P_2, \cdots, P_n, P_{n+1}, \cdots\} \ ,$$

$$I_\infty = \{P_1, P_2, \cdots\} \ ,$$

(5.18)

we find,

$$u(i, j)^* I_i u(i, j) = I_j \ , \ u(i, j)^* I_j u(i, j) = I_i \ ,$$

$$u(i, j)^* I_k u(i, j) = I_k \ \text{if} \ k \neq i, j \ ,$$

$$u(i, j)^* I_\infty u(i, j) = I_\infty \ ,$$

(5.19)

and

$$u(i, j) i = j \ , \ u(i, j) j = i \ ,$$

$$u(i, j) k = k \ \text{if} \ k \neq i, j \ ,$$

$$u(i, j) \infty = \infty .$$

(5.20)

It is worth noting that the representation (5.9) of $\mathcal{A}$ splits into a direct sum of the IRR's $p, q$ for the two-point poset. The proof is as follows: $\infty$ being a fixed point for $U$, the functions supported at $\infty$ give an $\mathcal{A}$-invariant one-dimensional subspace. It carries the IRR $p$ by (4.3) and (5.15). And since the orbit of $n$ under $U$ is $\{1, 2, \cdots\}$, the functions vanishing at $\infty$ give another invariant subspace. It carries the IRR of $p$ of $\mathcal{A}$ corresponds to the point $s$ which restricted to $\mathcal{C}$ remains IRR. We can think of $1, 2, \cdots, \infty$ as a subdivision of $[r, s]$ into points. Then $P_n$ can be regarded as the restriction to $\hat{\mathcal{C}}$ of a smooth function.
on \([r, s]\) with the value 1 in a small neighbourhood of \(n\) and the value zero at all \(m \neq n\) and \(\infty\). In contrast, \(1\) is the function with value 1 on the whole interval. Hence it has value 1 at all \(n\) and \(\infty\) as in (5.14-5.15). This interpretation is illustrated in Fig 9.

As mentioned previously, there is a certain difficulty in using the scalar product (5.4) for quantum physics. For the two-point poset, it reads

\[
(g, h) = \sum_n g(n)^* h(n) + g(\infty)^* h(\infty),
\]

where \(\infty\) is the limiting point of \(\{1, 2, \cdots\}\). Hence, if \(h\) is a continuous function, and \(h(\infty) \neq 0\), then \(\lim_{n \to \infty} h(n) = h(\infty) \neq 0\), and \((h, h) = \infty\). It other words, continuous functions in \(\ell^2(\hat{\mathcal{C}})\) must vanish at \(\infty\). This is in particular true for probability densities found from \(\mathcal{E}\). It is as though \(\infty\) has been deleted from the configuration space in so far as continuous wave functions are concerned.

There are two possible ways out of this difficulty. a) We can try regularization and modification of (5.4) using some such tool as the Dixmier trace [8, 9, 10]; b) we can try changing the scalar product for example to \((\cdot, \cdot)_\epsilon', \epsilon > 0\), where

\[
(g, h)_{\epsilon}' = \sum_n \frac{1}{n^{1+\epsilon}} g(n)^* h(n) + g(\infty)^* h(\infty),
\]

the choice of \(\epsilon\) being at our disposal.

There are minor changes in the choice of \(u(i, j)\) if this scalar product is adopted.

5.3 The \(\lor\) Poset and General Rank One Posets

In the case of the \(\lor\) poset, there are Hilbert spaces \(\mathcal{H}_1\) and \(\mathcal{H}_2\) for each arm, \(\mathcal{A}\) being the algebra (4.6) acting on \(\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2\). After choosing orthonormal basis \(h_n^{(i)}\), \(i = 1, 2\), \(n = 1, 2, \cdots\), where the superscript \(i\) indicates that the basis element corresponds to \(\mathcal{H}_i\), and orthogonal projectors \(\mathcal{P}_n^{(i)}\) on \(\mathcal{C}h_n^{(i)}\), the algebra \(\mathcal{C}\) can be written as

\[
\mathcal{C} = \langle \bigcup_n \mathcal{P}_n^{(1)}; \mathcal{P}_2; \bigcup_n \mathcal{P}_n^{(2)} > .
\]

Here \(\mathcal{P}_i\) are projection operators on \(\mathcal{H}_i\).

The group \(U\) as before is generated by transpositions of basis elements.

The space \(\hat{\mathcal{C}}\) consists of two sequences \(n^{(1)}\), \(n^{(2)}\) \((n = 1, 2, \cdots)\) and two points \(\infty^{(1)}\), \(\infty^{(2)}\), with \(n^{(i)}\) converging to \(\infty^{(i)}\):

\[
\hat{\mathcal{C}} = \{ n^{(i)}; \infty^{(i)} ; i = 1, 2 ; n = 1, 2, \cdots \} .
\]

Their meaning is explained by

\[
\mathcal{P}_i (n^{(j)}) = \delta_{ij} , \\
\mathcal{P}_m (n^{(j)}) = \delta_{ij} \delta_{mn} ,
\]

(5.25)
\[
\mathcal{P}_i(\infty^{(j)}) = \delta_{ij}, \\
\mathcal{P}^{(i)}_m(\infty^{(j)}) = 0.
\] (5.26)

The visual representation of \(\hat{\mathcal{C}}\) is presented in Fig. 10(a).

The remaining discussion of Section 5.2 is readily carried out for the \(\forall\) poset as also for a general rank 1 poset. So we content ourself by showing the structure of \(\hat{\mathcal{C}}\) for a \(\forall\) and a circle poset in Figs. 10(b),(c).

6 Finite Dimensional and Commutative Approximations

6.1 Why Approximate \(\mathcal{A}\)

There are several good reasons to try to simplify the preceding approach based on the algebra \(\mathcal{A}\). The leading reason is a practical one: \(\mathcal{A}\) is infinite dimensional, and it will greatly ease the pain of numerical work if it can be approximated by finite dimensional algebras. A second reason is that our experience with \(\mathcal{A}\) is limited and there are important technical and physical issues to be resolved before it can be effectively used as a computational tool. The divergence of the scalar product (5.21) for constant nonzero functions is an example of the sort of technical issues encountered for any poset while a question, both conceptual and physical, relates to the meaning of the group \(U\). For if for example the points \(n\) of Section 5.2 have the interpretation we proposed, then the significance of \(U\) gets unclear. This is because it permutes these points and alters their order in the interval \([r, s]\) and can not therefore be interpreted in terms of homeomorphisms of \([r, s]\). This question too appears for any poset.

Later on in this section, we will propose an interpretation of \(U\). But first, we will describe a sequence of finite dimensional approximations to \(\mathcal{A}\). The existence of these finite dimensional approximations is exactly what characterizes \(\mathcal{A}\) as an approximately finite dimensional, or AF, algebra. The leading nontrivial approximation here is commutative while the succeeding ones are not. The commutative approximation \(\mathcal{C}(\mathcal{A})\) has a suggestive physical interpretation for rank one posets. Further these approximations correctly capture the topology of the poset. Being also free of divergences because of their finite dimensionality, they can thus provide us with excellent models to initiate practical calculations, and to gain experience and insight into noncommutative geometry in the quantum domain.
6.2 The Approximations

The algebra \( A \) for the two point poset of Fig. 6 (a) is \( \mathbb{C}I + \mathcal{K} \). Consider the following sequence of \( C^* \)-algebras of increasing dimension, the \( * \)-operation being hermitian conjugation:

\[
A_1 = \mathbb{C}I_{1 \times 1}, \\
A_2 = \mathbb{C}I_{2 \times 2} + M(1, \mathbb{C}), \\
A_3 = \mathbb{C}I_{3 \times 3} + M(2, \mathbb{C}), \\
\ldots \\
A_n = \mathbb{C}I_{n \times n} + M(n - 1, \mathbb{C}), \\
\ldots
\]

(6.1)

Here \( I_{n \times n} \) is the \( n \times n \) unit matrix while \( M(n, \mathbb{C}) \) is isomorphic to the \( C^* \)-algebra of \( n \times n \) complex matrices. A typical element of \( A_{n+1} \) is

\[
a_{n+1} = \begin{bmatrix} m_{n \times n} & 0 \\ 0 & \lambda \end{bmatrix},
\]

(6.2)

where \( m_{n \times n} \) is an \( n \times n \) complex matrix and \( \lambda \) is a complex number. Note that the subalgebra \( M(n, \mathbb{C}) \) consists of matrices of the form (6.2) with the last row and column zero.

The algebra \( A_n \) is seen to approach \( A \) as \( n \) becomes bigger and bigger. We can make this intuitive observation more precise as thus: There is an inclusion

\[
F_{n+1,n} : A_n \rightarrow A_{n+1}
\]

(6.3)

given by

\[
a_n = \begin{bmatrix} m_{n-1 \times n-1} & 0 \\ 0 & \lambda \end{bmatrix} \rightarrow \begin{bmatrix} m_{n-1 \times n-1} & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}.
\]

(6.4)

It is a \( * \)-homomorphism [6] since

\[
F_{n+1,n}(a_{n-1}^*) = [F_{n+1,n}(a_{n-1})]^*.
\]

(6.5)

Thus the sequence

\[
A_1 \rightarrow A_2 \rightarrow \ldots
\]

(6.6)

gives a directed system of \( C^* \)-algebras. Its inductive limit is \( A \) as is readily proved using the definitions in [6].

We must now associate appropriate representations to \( A_n \) which will be good approximations to the two-point poset.

The algebra \( A_1 \) is trivial. Let us ignore it. All the remaining algebras \( A_n \) have the following two representations:
a) The one-dimensional representation $p_n$ with

$$p_n : a_n \to \lambda .$$

(6.7)

b) The defining representation $q_n$ with

$$q_n : a_n \to a_n .$$

(6.8)

It is clear that these representations approach the representations $p$ and $q$ of $A$ as $n \to \infty$.

The ideal $I_{p_n}$ for $p_n$ is

$$M_{n-1} = \left\{ \begin{bmatrix} m_{n-1 \times n-1} & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

(6.9)

while the ideal for $q_n$ is

$$I_{q_n} = \{0\} .$$

(6.10)

Hence $I_{q_n} \subset I_{p_n}$ and the hull kernel topology [3] gives the two-point poset $\{p_n, q_n\}$ with $q_n \prec p_n$. This is shown in Fig. 11 (a). It is exactly the same as the poset in Fig. 6 (a).

Thus the preceding two representations of $A_n$ form a topological space identical to the poset of $A$.

All this suggests that it is good to approximate $A$ by $A_n$, and regard its representations $p_n$ and $q_n$ as constituting the configuration space.

In our previous discussions, either involving the algebra $C$ or the algebra $A$, we considered only their IRR’s. But the representation $q_n$ of $A_n$ is not IRR. It has the invariant subspace

$$C \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} .$$

(6.11)

In this respect we differ from the previous sections in our treatment of $A_n$.

Now consider $A_2$. It is a commutative algebra with elements

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \equiv (\lambda_1, \lambda_2) , \lambda_i \in C .$$

(6.12)

In this way, we can achieve a commutative simplification of $A$.

The cover $[r, s] \cup [r, s]$ of $[r, s]$ giving the two-point poset is not a good one in a technical sense, one of its open sets $([r, s])$ being entirely contained in the other $([r, s])$. There is no good interpretation of this algebra for this reason. We therefore now turn to the $\vee$ poset obtainable from a good cover of the interval as in Section 4.1.
For the \( \forall \) poset, the Hilbert space defining \( \mathcal{A} \) is realized as the direct sum of two Hilbert spaces. Accordingly, we construct the approximation \( \mathcal{A}_{n+1} \) of \( \mathcal{A} \) by considering
\[ D^{n+1} = C^{n+1} \oplus C^n . \] (6.13)

A vector in this space is the direct sum
\[ x \oplus y , \quad x = (x^1, x^2, \ldots, x^{n+1}), \quad y = (y^1, y^2, \ldots, y^{n+1}) . \] (6.14)

We take the scalar product for \( D^{n+1} \) to be
\[ (x' \oplus y', x \oplus y) = \sum_i [x'(i)^\ast x(i) + y'(i)^\ast y(i)] . \] (6.15)

Note also that \( D^{n+1} \) has the subspace
\[ E^{n+1} = \{ x \oplus y : x^{n+1} = y^{n+1} = 0 \} \] (6.16)
which is isomorphic to \( C^n \oplus C^n \).

Now let \( P_1 \) be the orthogonal projection on the first \( C^{n+1} \) of \( D^{n+1} \) and \( P_2 \) on the second. Let also \( L[2n, C] = \{ L_{2n} \} \) be the set of all linear operators on \( D^{n+1} \) mapping \( E^{n+1} \) to \( E^{n+1} \) and its orthogonal complement to \( \{0\} \). Then
\[ \mathcal{A}_{n+1} = \{ \lambda_1 P_1 + \lambda_2 P_2 + L_{2n} : \lambda_i \in C, L_{2n} \in L[2n, C] \} = C P_1 + C P_2 + L[2n, C] . \] (6.17)

The term \( L[2n, C] \) is absent here if \( n = 0 \).

As before, there is a \( * \)-homomorphism
\[ F_{n+1,n} : \mathcal{A}_n \to \mathcal{A}_{n+1} . \] (6.18)

Here the image of \( L[2(n-1), C] \subset \mathcal{A}_n \) is being realized as a subalgebra of \( L(2n, C) \) in a natural way while the images of \( P_i \) in \( \mathcal{A}_n \) are the corresponding \( P_i \) in \( \mathcal{A}_{n+1} \). We thus have a directed system of \( C^* \)-algebras. Its inductive limit is \( \mathcal{A} \) showing that \( \mathcal{A}_n \) approximates \( \mathcal{A} \).

The algebras \( \mathcal{A}_n \) have the following three representations:

\begin{align*}
a) \quad & \alpha_n : \quad a_n \to \lambda_1 , \\
b) \quad & \beta_n : \quad a_n \to \lambda_2 , \\
c) \quad & \gamma_n : \quad a_n \to a_n . \end{align*} (6.19)

Here
\[ a_n = \lambda_1 P_1 + \lambda_2 P_2 + L_{2(n-1)} \text{ for } n \geq 2 , \]
\[ a_1 = \lambda_1 P_1 + \lambda_2 P_2 . \] (6.20)
Note that $\alpha_n$ and $\beta_n$ are abelian IRR’s while $\gamma_n$ is not IRR just like $q_n$.

Now the kernels of these representations are

\[
\begin{align*}
I_{\alpha_n} &= C\mathcal{P}_2 + L[2(n-1), C], \\
I_{\beta_n} &= C\mathcal{P}_1 + L[2(n-1), C], \\
I_{\gamma_n} &= \{0\}.
\end{align*}
\]  

(6.21)

Since

\[
I_{\gamma_n} \subset I_{\alpha_n} \text{ and } I_{\gamma_n} \subset I_{\beta_n},
\]

we set

\[
\gamma_n \prec \alpha_n, \; \gamma_n \prec \beta_n.
\]  

(6.22)

The poset that results is shown in Fig. 12. It is again the $\lor$ poset, suggesting that $A_n$ and its representations $\alpha_n, \beta_n, \gamma_n$ are good approximations for our purposes.

Now the $C^*$-algebra

\[
A_1 = \{a_1 = \lambda_1\mathcal{P}_1 + \lambda_2\mathcal{P}_2 ; \; \lambda_i \in C\}
\]

\[
= C\mathcal{P}_1 + C\mathcal{P}_2.
\]  

(6.24)

is commutative and its representations $\alpha_1, \beta_1, \gamma_1$ also capture the poset topology correctly. Is it possible to interpret $\lambda_i$?

For this purpose, let us remember that the points of a manifold $M$ are closed, and so are the top or rank zero points of the poset. The latter somehow approximate the former. Since the values of $a_1$ at the rank zero points are $\lambda_1$ and $\lambda_2$, we can regard $\lambda_i$ as the values of a continuous function on $M$ when restricted to this discrete set. The role of the bottom poset point and the value of $a_1$ there is to somehow glue the top points together and generate a nontrivial approximation to the topology of $M$.

We can explain this interpretation further using simplicial decomposition. Thus the interval $[0, 1]$ has a simplicial decomposition with $[0]$ and $[1]$ as zero-simplices and $[0, 1]$ as the one-simplex. Assuming that experimenters can not resolve two points if every simplex containing one contains also the other, they will regard $[0, 1]$ to consist of the three points $\alpha_1 = [0], \beta_1 = [1]$ and $\gamma_1 = ]0, 1[$. There is also a natural map from $[0, 1]$ to these points as in Section 2. Introducing the quotient topology on these points following that section, we get back the $\lor$ poset. In this approach then, $\lambda_1$ and $\lambda_2$ are the values of a continuous function at the two extreme points of $[0, 1]$ whereas the association of $\lambda_1\mathcal{P}_1 + \lambda_2\mathcal{P}_2$ with the open interval is necessary to cement the extreme points together in a topologically correct manner.

[We remark here that the simplicial decomposition of any manifold yields a poset in the manner just indicated. Recall also the remark at the end of Section 14.2 that a probability density can not be localized at rank zero points. This result appears eminently

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reasonable in the context of a simplicial decomposition where rank zero points are points of the manifold. Reasoning like this also suggests that localization must in general be possible only at the subsets of the poset representing the open sets of \( M \). That seems in fact to be the case. For, as remarked earlier, localization seems possible only at the open sets of this poset and the latter correspond to open sets of \( M \).

We have mentioned in Section 4.1 that the construction of \( A \) for any poset is known \([6]\). It is possible to obtain its approximations \( A_n \) including its commutative simplification \( C(A) \) therefrom. It is also possible to find the representations we must use in conjunction with \( A_n \) and \( C(A) \).

We emphasize that the algebra with the minimum number of degrees of freedom correctly reproducing the poset and its topology seems to be \( C(A) \).

### 6.3 Abelianization from Gauge Invariance

We have already pointed out that the meaning of the algebra \( U \subset A \) is not very clear, even though it is essential to reproduce the poset as the structure space of \( A \).

But if its role is just that and nothing more, is it possible to reduce \( A \) utilizing \( U \) or a suitable subgroup of \( U \) in some way and get the algebra \( C(A) \)? The answer seems to be yes in all interesting cases. We will now show this result and argue also that this subgroup can be interpreted as a gauge group.

Let us start with the two-point poset. It is an “uninteresting” example for us where our method will not work, but it is a convenient example to illustrate the ideas.

The condition we impose to reduce \( A \) here is that the observables must commute with \( U \). The commutant \( U' \) of \( U \) in \( A \) is just \( C\mathbb{I} \). The algebra \( A \) thus gets reduced to a commutative algebra, although it is not the algebra we want.

The next example is a “good” one, it is the example of the \( \lor \) poset. The group \( U \) here has two commuting subgroups \( U^{(1)} \) and \( U^{(2)} \). \( U^{(i)} \) is generated by the transpositions \( u(k,l; i) \) which permute only the basis elements \( h_i^{(k)} \) and \( h_i^{(l)} \):

\[
\begin{align*}
  u(k,l; i)h_i^{(k)} &= h_i^{(l)}, \quad u(k,l; i)h_i^{(l)} = h_i^{(k)}; \\
  u(k,l; i)h_m^{(j)} &= h_m^{(j)} \quad \text{if} \quad m \notin \{k,l\}. (6.25)
\end{align*}
\]

These are thus operators acting along each arm of \( \lor \), but do not act across the arms of \( \lor \). The full group \( U \) is generated by \( U^{(1)} \) and \( U^{(2)} \) and the elements transposing \( h_i^{(1)} \) and \( h_i^{(2)} \).

Let us now require that the observables commute with \( U^{(1)} \) and \( U^{(2)} \). They are given by the commutant of \( \langle U^{(1)}, U^{(2)} \rangle \), the latter being

\[
\langle U^{(1)}, U^{(2)} \rangle' = CP_1 + CP_2 \quad (6.26)
\]
in the notation of (4.6). This algebra being isomorphic to (6.24), we get the result we want.

We can also find the correct representations to use in conjunction with (6.26). They are isomorphic to the IRR’s of $\mathcal{A}$ when restricted to $\langle U^{(i)}, U^{(2)} \rangle'$. This is an obvious result.

The procedure for finding the algebra $\mathcal{C}(\mathcal{A})$ and its representations of interest for a general poset now follows. Associated with each arm $i$ of a poset, there is a subgroup $U^{(i)}$ of $U$. It permutes the projections, or equivalently the IRR’s of $\mathcal{A}$ when restricted to $\langle U^{(i)}, U^{(2)} \rangle'$. This is an obvious result.

The algebra $\mathcal{C}(\mathcal{A})$ is then the commutant of $\langle \bigcup_i U^{(i)} \rangle'$:

$$\mathcal{C}(\mathcal{A}) = \langle \bigcup_i U^{(i)} \rangle' .$$

(6.27)

The representations of $\mathcal{C}(\mathcal{A})$ of interest are isomorphic to the restrictions of IRR’s of $\mathcal{A}$ to $\mathcal{C}(\mathcal{A})$.

In gauge theories, observables are required to commute with gauge transformations. In an analogous manner, we here require the observables to commute with the transformations generated by $U^{(i)}$. The group generated by $U^{(i)}$ thus plays the role of the gauge group in the approach outlined here.

7 A Particle on a Circle

A circle $S^1 = \{ e^{i\phi} \}$ is an infinitely connected space. It has the fundamental group $\mathbb{Z}$. Its universal covering space $\mathbb{R}^1$ is the real line $\mathbb{R}^1 = \{ x : -\infty < x < \infty \}$. The fundamental group $\mathbb{Z}$ acts on $\mathbb{R}^1$ according to

$$x \rightarrow x + N , \ N \in \mathbb{Z} .$$

(7.1)

The quotient of $\mathbb{R}^1$ by this action is $S^1$, the projection map $\mathbb{R}^1 \rightarrow S^1$ being

$$\pi : \mathbb{R}^1 \rightarrow S^1 : x \rightarrow e^{ix} .$$

(7.2)

Now the domain of a typical Hamiltonian for a particle on $S^1$ need not consist of smooth functions on $S^1$. Rather it can be obtained from functions $\psi_\theta$ on $\mathbb{R}^1$ transforming by an IRR

$$\rho_\theta : N \rightarrow e^{iN\theta}$$

(7.3)

of $\mathbb{Z}$ according to

$$\psi_\theta(x + N) = e^{iN\theta} \psi_\theta(x) .$$

(7.4)
The domain \( D_\theta(H) \) for a typical Hamiltonian \( H \) then consists of these \( \psi_\theta \) restricted to a fundamental domain \( 0 \leq x \leq 1 \) for the action of \( \mathbb{Z} \) and subjected to a differentiability requirement:

\[
D_\theta(H) = \{ \psi_\theta : \psi_\theta(1) = e^{i\theta} \psi_\theta(0) ; \quad \frac{d\psi_\theta(1)}{dx} = e^{i\theta} \frac{d\psi_\theta(0)}{dx} \} .
\] (7.5)

[In addition, of course, if \( dx \) is the measure on \( S^1 \) for defining the scalar product of wave functions, then \( H\psi_\theta \) must be square integrable for this measure.] We obtain a distinct quantization, called the \( \theta \)-quantization, for each choice of \( e^{i\theta} \).

As we have shown earlier [5], there are similar quantization possibilities for a circle poset as well. The fundamental group of a circle poset is \( \mathbb{Z} \). Its universal covering space is the poset of Fig. 13. Its quotient, for example by the action

\[
N : \quad x_j \to x_{j+3N} ,
\]

\[
x_j = a_j \text{ or } b_j \text{ of Fig. 13} , \quad N \in \mathbb{Z}
\] (7.6)
gives the circle poset of Fig. 8(b).

In our previous article [5], we argued that the poset analogue of \( \theta \)-quantization can be obtained from complex functions \( f \) on the poset of Fig. 13 transforming by an IRR of \( \mathbb{Z} \):

\[
f(x_{j+3}) = e^{i\theta} f(x_j) .
\] (7.7)

While answers such as the spectrum of a typical Hamiltonian came out correctly, this approach was nevertheless affected by a serious defect: continuous complex functions on a connected poset are constants, so that our wave functions can not be regarded as continuous.

This defect can now be repaired by using the algebra \( \mathcal{C}(A) \) for a circle poset and the corresponding algebra \( \overline{\mathcal{C}}(A) \) for Fig. 13. [See also next section.] If the circle poset is taken to be the one in Fig. 8(b), then

\[
\mathcal{C}(A) = \{ c = (\lambda_1, \lambda_2) \oplus (\lambda_2, \lambda_3) \oplus (\lambda_3, \lambda_1) : \lambda_i \in \mathbb{C} \} ,
\] (7.8)

and

\[
\overline{\mathcal{C}}(A) = \{ \overline{c} = \oplus(\mu_j, \mu_{j+1}) : \mu_k \in \mathbb{C} \} .
\] (7.9)

Here the values of \( c \) at the points of Fig. 8(b) are obtained by setting \( k_{ij} = 0 \) and replacing \( \lambda_j P_j + \lambda_k P_k \) by \( (\lambda_j, \lambda_k) \) in that Figure. As for the values of \( \overline{c} \) at the points of Fig. 13, they are given by

\[
\overline{c}(a_j) = \mu_j ; \quad \overline{c}(b_j) = (\mu_j, \mu_{j+1}) .
\] (7.10)

The analogue of (7.7) is now the condition

\[
\mu_{j+3} = e^{i\theta} \mu_j
\] (7.11)
while wave functions $\chi_\theta$ are obtained by restricting their domains to $\{a_j, b_j : 1 \leq j \leq 3\}$. The result is the analogue of the “finite projective module” $\mathcal{E}$ we encountered earlier. Let us call it here as $\mathcal{E}_\theta$. It is

$$\mathcal{E}_\theta = \{ \chi_\theta, \chi'_\theta, \ldots : \chi_\theta = (\mu_1, \mu_2) \oplus (\mu_2, \mu_3) \oplus (\mu_3, e^{i\theta} \mu_1), \chi'_\theta = (\mu'_1, \mu'_2) \oplus (\mu'_2, \mu'_3) \oplus (\mu'_3, e^{i\theta} \mu'_1), \ldots : \mu_i, \mu'_i \in \mathbb{C} \}.$$  \hfill (7.12)

Here $\mathcal{E}_\theta$ is a $\mathcal{C}(\mathcal{A})$-module, with the action of $c$ on $\chi_\theta$ given by

$$\chi_\theta c = (\mu_1 \lambda_1, \mu_2 \lambda_2) \oplus (\mu_2 \lambda_2, \mu_3 \lambda_3) \oplus (\mu_3 \lambda_3, e^{i\theta} \mu_1 \lambda_1).$$ \hfill (7.13)

It has a sesquilinear form $\langle \cdot, \cdot \rangle$ valued in $\mathcal{C}(\mathcal{A})$:

$$\langle \chi'_\theta, \chi_\theta \rangle := (\mu'_1 \mu_1, \mu'_2 \mu_2) \oplus (\mu'_2 \mu_2, \mu'_3 \mu_3) \oplus (\mu'_3 \mu_3, \mu'_1 \mu_1).$$ \hfill (7.14)

The associated scalar product is

$$\langle \chi'_\theta, \chi_\theta \rangle = \sum_{j=1}^{3} \{ \langle \chi'_\theta, \chi_\theta \rangle(a_j) + \langle \chi'_\theta, \chi_\theta \rangle(b_j) \}.$$ \hfill (7.15)

We have yet to consider the Laplacian $\Delta$. We first define it on $\mathcal{C}(\mathcal{A})$ by setting

$$\Delta \tilde{c} = \frac{1}{\epsilon^2} \left\{ \bigoplus (\mu_{j-1} + \mu_{j+1} - 2 \mu_j ; \mu_j + \mu_{j+2} - 2 \mu_{j+1}) \right\},$$ \hfill (7.16)

with $\epsilon$ a positive constant. We then restrict $j$ to $1 \leq j \leq 3$ and impose the boundary condition (7.11). This gives us $\Delta$ on $\mathcal{E}_\theta$. The result is very similar to the one we previously had. Notice especially that the two entries in $\Delta \tilde{c}$ can be interpreted in terms of the action of the standard discretized version of the Laplacian acting on $\mu$’s.

The solutions of the eigenvalue problem

$$\Delta \chi_\theta = \lambda \chi_\theta$$ \hfill (7.17)

are

$$\lambda = \lambda_k = \frac{2}{\epsilon^2} (\cos k - 1), \chi^{(k)}_\theta(a_j) = A^{(k)} e^{ikj} + B^{(k)} e^{-ikj}, A^{(k)}, B^{(k)} \in \mathbb{C}$$ \hfill (7.18)

where $k = m \frac{2\pi}{3} + \frac{\theta}{3}, m = 1, 2, 3$. [The expression for $\chi^{(k)}_\theta(b_j)$ follows from $\chi^{(k)}_\theta(a_j)$.] These are exactly our answers in ref. [3] but for one significant difference. In ref. [3], the operator $\Delta$ did not mix the values of the wave function at points of rank zero and rank one, resulting in a double degeneracy of eigenvalues. That unphysical degeneracy has now been removed because of a better treatment of continuity properties. The latter prevents
us from giving independent values to continuous probability densities [cf. Section 8] at these two kinds of points.

It is possible to obtain (7.16) using the formula (4.19). But we will not describe that approach here, as the preceding expression for \( \Delta \) can be easily guessed at.

We have now illustrated several crucial ideas of Section 4.2 using \( E_\theta \) and \( C(A) \).

8  Posets as Configuration Spaces and their Continuous “Functions”

In our treatment of quantum physics, there is the underlying notion that algebras like \( A \) and \( C(A) \) correspond to the algebra \( C(M) \) of continuous functions on a manifold \( M \). In the case of \( C(M) \), there is a topology on \( M \) and on the target space \( C \), and the continuity of elements of \( C(M) \) is inferred therefrom. In contrast, for posets, we have not defined the topology of the target space until now. We have not therefore yet established the continuity of \( A, A_n \) and \( C(A) \) in a sense similar to the continuity of \( C(M) \). Our remarks in Section 7 on the lack of continuity of wave functions in our previous work, and its presence in the current approach, serve to highlight the significance of this notion. We now turn to the task of defining the target space topology and establishing this notion.

Let us consider the algebra \( A \) to be specific, the treatment of the other algebras being similar. If \( a \in A \) and \( x \) is a point of its poset \( P \), then \( a(x) \) is a bounded operator on a Hilbert space \( H_x \). Let \( A_x \) denote the set of all these operators \( a(x) \) as \( a \) runs over \( A \). Then \( a \) has values in the space \( \bigcup_x A_x \). It is a section of the bundle

\[
E = \bigcup_x A_x \quad (8.1)
\]

over the poset. It is this \( E \) that we must topologize, and then verify that \( A \) consists of continuous maps of \( P \) to \( E \).

Now there are already topologies \( T(P) \) and \( T(A) \) for \( P \) and \( A \). The latter is given by the norm \( || \cdot || \), the \( \epsilon \)-neighborhood of \( a \) being all \( b \) such that \( ||b - a|| < \epsilon \). These topologies being given, there is not much freedom in the choice of the topology for \( E \). There is in fact a canonical way to find a topology \( T(E) \) for \( E \) from those of \( P \) and \( A \). Let us now describe it.

The Cartesian product \( P \times A \) has the topology \( T(P) \times T(A) \) where a basis of open sets is given by Cartesian products of open sets in \( P \) and \( A \). There is also a map

\[
\phi : P \times A \rightarrow E \quad (8.2)
\]

defined by

\[
\phi : (x, a) \rightarrow (x, a(x)) \quad (8.3)
\]
It is called the evaluation map. The topology $\mathcal{T}(E)$ is the finest topology compatible with the continuity of $\phi$. A set $U$ in $E$ is open in this topology if its inverse image $\phi^{-1}(U)$ is open in $P \times A$.

Note that the way we define the topology for $E$ here and the way we defined it for finite spaces approximating manifolds in Section 2 are similar. $\mathcal{T}(E)$ is in fact the quotient topology \[15\] for the map $\phi$.

We can now show that any element $a$ in $A$ defines a continuous map from $P$ to $E$ in this topology. For this purpose consider the subset $S = P \times \{a\}$ of $P \times A$. It inherits the induced topology \[15\] from $P \times A$, its open sets being the Cartesian product of open sets of $P$ with $\{a\}$. The restriction $\phi|_S$ of $\phi$ to $S$ is a continuous map from $S$ to $E$. Now consider the map $a : P \to E$ given by $x \to (x, a(x))$. If $a^{-1}(U)$ is not open for an open set $U$ in $E$, then $\phi|_S^{-1}(U) = a^{-1}(U) \times \{a\}$ is also not open. That being a contradiction, the continuity of $a$ is established.

We will now verify this result explicitly for the $\lor$ poset and the algebra $\mathcal{C}(A)$, the verification in any other case of interest being similar.

According to (6.24) and Fig. 12(b), a typical element $c_1$ of $\mathcal{C}(A)$ [or $A_1$] for the $\lor$ poset is $\lambda_1 p_1 + \lambda_2 p_2$. Its $\epsilon$-neighborhood is defined as stated previously.

Consider the point $(\alpha_1, \lambda_1)$ in $E$. Its inverse image is

$$\{\alpha_1\} \times \{\lambda_1 p_1 + \mu_2 p_2 : \mu_2 \in C\} \cdot (8.4)$$

Any open set containing (8.5) contains the open sets

$$\{\alpha_1, \gamma_1\} \times \{\lambda_1' p_1 + \mu_2' p_2 : ||(\lambda_1' - \lambda_1) p_1 + (\mu_2' - \mu_2) p_2|| = \sup(\{|\lambda_1' - \lambda_1|, |\mu_2' - \mu_2|\}) < \epsilon\} \cdot (8.6)$$

for all $\mu_2 \in C$ and some $\epsilon > 0$. Hence it contains the open set

$$U^{(1)}_\epsilon = \{\alpha_1, \gamma_1\} \times \{\lambda_1' p_1 + \mu_2 p_2 : |\lambda_1' - \lambda_1| < \epsilon, \mu_2 \in C\} \cdot (8.7)$$

for some $\epsilon > 0$.

Now consider

$$\phi(U^{(1)}_\epsilon) \subset E \cdot (8.8)$$

Its inverse image is easily seen to be $U^{(1)}_\epsilon$. As the latter is open, we conclude that

a) $\phi(U^{(1)}_\epsilon)$ is open in $E$;

b) Any open set containing $(\alpha_1, \lambda_1)$ contains $\phi(U^{(1)}_\epsilon)$ for some $\epsilon > 0$.

But $\phi(U^{(1)}_\epsilon)$ also contains $(\gamma_1, \lambda_1 p_1 + \mu_2 p_2)$ for every $\mu_2$. Hence by definition the closure $\{(\gamma_1, \lambda_1 p_1 + \mu_2 p_2)\}$ of the set $\{(\gamma_1, \lambda_1 p_1 + \mu_2 p_2)\}$ contains $(\alpha_1, \lambda_1)$. In other
words, \((\alpha_1, \lambda_1)\) is a limit point of \(\{(\gamma_1, \lambda_1 \mathcal{P}_1 + \mu_2 \mathcal{P}_2)\}\). Let us indicate this fact using an arrow (to suggest convergence) as follows:

\[
(\gamma_1, \lambda_1 \mathcal{P}_1 + \mu_2 \mathcal{P}_2) \rightarrow (\alpha_1, \lambda_1), \quad \forall \mu_2 \in C. \tag{8.9}
\]

We find similarly that

c) Any open set containing \((\beta_1, \lambda_2)\) also contains the open set \(\phi(U_\epsilon^{(2)})\) for some \(\epsilon > 0\), where

\[
U_\epsilon^{(2)} = \{\beta_1, \gamma_1\} \times \{\mu_1 \mathcal{P}_1 + \lambda_2 \mathcal{P}_2 : |\lambda_2' - \lambda_2| < \epsilon, \mu_1 \in C\}; \tag{8.10}
\]

d) \((\beta_1, \lambda_2) \in \{(\gamma_1, \mu_1 \mathcal{P}_1 + \lambda_2 \mathcal{P}_2)\}\) or

\[
(\gamma_1, \mu_1 \mathcal{P}_1 + \lambda_2 \mathcal{P}_2) \rightarrow (\beta_1, \lambda_2), \quad \forall \mu_1 \in C. \tag{8.11}
\]

Furthermore as one can readily see, if \((\gamma_1, \Lambda) \rightarrow (\alpha_1, \lambda_1)\) and \((\gamma_1, \Lambda) \rightarrow (\beta_1, \lambda_2)\), then \(\Lambda = \lambda_1 \mathcal{P}_1 + \lambda_2 \mathcal{P}_2\).

The open sets analogous to \(\phi(U_\epsilon^{(i)})\) containing \((\gamma_1, \lambda_1 \mathcal{P}_1 + \lambda_2 \mathcal{P}_2)\) look different because \(\gamma_1\) is open in \(P\). They are

\[
\phi(W_\epsilon) \tag{8.12}
\]

where

\[
W_\epsilon = \{\gamma_1\} \times \{\mu_1 \mathcal{P}_1 + \lambda_2 \mathcal{P}_2 : \sup(|\mu_1 - \lambda_1|, |\mu_2 - \lambda_2|) < \epsilon\}. \tag{8.13}
\]

Any open set containing \((\gamma_1, \lambda_1 \mathcal{P}_1 + \lambda_2 \mathcal{P}_2)\) also contains \(\phi(W_\epsilon)\) for some \(\epsilon > 0\).

We next show the following two results:

a) Every element \(c_1 = \lambda_1 \mathcal{P}_1 + \lambda_2 \mathcal{P}_2\) of \(\mathcal{C}(\mathcal{A})\) defines a continuous map.

b) If a map \(P \rightarrow E\) is continuous, it is necessarily a member of \(\mathcal{C}(\mathcal{A})\).

As for (a), \(c_1\) is continuous by definition if the inverse image \(c_1^{-1}(X)\) of any open set \(X \subseteq E\) is open in \(P\). Also by definition, \(c_1^{-1}(X)\) is the set in \(P\) with an image under \(c_1\) contained in \(X\): \(c_1[c_1^{-1}[X]] \subseteq X\). Now \(c_1^{-1}[\phi(U_\epsilon^{(1)})] = \{\alpha_1, \gamma_1\}, c_1^{-1}[\phi(U_\epsilon^{(2)})] = \{\beta_1, \gamma_1\}\) and \(c_1^{-1}[\phi(W_\epsilon)] = \{\gamma_1\}\) are all open, proving that \(c_1\) is continuous.

As for (b), let \(\sigma : P \rightarrow E\) be a continuous map with \(\sigma(\alpha_1) = \lambda_1, \sigma(\beta_1) = \lambda_2\). As \(\sigma\) is continuous and \(\gamma_1 \rightarrow \alpha_1\), we have \(\sigma(\gamma_1) \rightarrow \lambda_1\). Similarly \(\sigma(\gamma_1) \rightarrow \lambda_2\). Hence by a remark above, \(\sigma(\gamma_1) = \lambda_1 \mathcal{P}_1 + \lambda_2 \mathcal{P}_2\) and \(\sigma = \lambda_1 \mathcal{P}_1 + \lambda_2 \mathcal{P}_2\).

It is worth noting in conclusion that not all maps \(\Sigma : P \rightarrow E\) are continuous. Thus suppose that \(\Sigma(\alpha_1) = \lambda_1, \Sigma(\beta_1) = \lambda_2\) and \(\Sigma(\gamma_1) = \mu_1 \mathcal{P}_1 + \mu_2 \mathcal{P}_2\) where \(\mu_1 \neq \lambda_1\) and/or \(\mu_2 \neq \lambda_2\). Then \(\Sigma\) is not continuous.
9 Final Remarks

In this article, we have reviewed a physically well-motivated approximation method to continuum physics based on partially ordered sets or posets. These sets have the power to reproduce important topological features of continuum physics with striking fidelity, and that too with just a few points. In addition, as discussed in the previous pages, there is also a remarkable connection of posets to noncommutative geometry. It is our impression that this connection is quite deep, and can lead to powerful and novel schemes for numerical approximations which are also topologically faithful. They seem in particular to be capable of describing solitons and the analogues of QCD $\theta$-angles. Much work of course remains to be done, but there are already persuasive indications of the fruitfulness of the ideas sketched in this article for finite quantum physics.

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Fig. 1. (a) shows an open cover for the circle $S^1$ and (b) the resultant discrete space $P_4(S^1)$. $\Phi$ is the map (2.4).

Fig. 2. The Hasse diagram for the circle poset $P_4(S^1)$. 
Fig. 3. The Hasse diagram for the two-sphere poset $P_6(S^2)$.

Fig. 4. In (a) is shown a covering of $S^1$ by open sets $O_j$ with $O_3 = O_2 \cap O_4$, $O_5 = O_4 \cap O_6$, ..., $O_1 = O_{2N} \cap O_2$. (b) is the Hasse diagram of its poset.
The figures show continuous maps of $S^1$ to $P_4(S^1)$. The dark lines correspond to closed intervals of $S^1$ and the light lines to open intervals. The map in (a) has zero winding number, and the map in (b) has winding number 1.

Fig. 5.

Fig. 6. (a) is the poset for the interval $[0, 1]$ when covered by the open sets $[0, 1]$ and $[0, 1]$. (b) shows the values of a generic element $\lambda \mathbb{1} + k$ of its algebra $\mathcal{A}$ at its two points $p$ and $q$. 
Fig. 7. (a) shows the $\vee$ and the association of an infinite dimensional Hilbert space $\mathcal{H}_i$ to each of its arms. (b) shows the values of a typical element $a = \lambda_1 P_1 + \lambda_2 P_2 + k_{12}$ of its algebra at its three points.

Fig. 8. These figures show how the Hilbert spaces $\mathcal{H}_i$ are attached to the arms of two rank one posets. They also show the values of a generic member $a$ of their algebras $\mathcal{A}$ at their points.
Fig. 9. The figure shows the division of $[0, 1]$ into an infinity of points $1, 2, \ldots$, which get increasingly dense towards $\infty$ or $p$. The point $\infty$ being a limit point of $1, 2, \ldots$, is distinguished by a star. According to the suggested interpretation, these points and $\infty$ correspond to IRR's of $C$ while $q = \{1, 2, \ldots\}$ and $p = \infty$ correspond to IRR's of $A$. 
Fig. 10. The figures show the structure space of $\hat{C}$ for three typical rank one posets.
Fig. 11. (a) is the poset for the algebra $A_n$ of (6.1) while (b) shows the values of a typical element $q_n$ of this algebra at its two points $p_n$ and $q_n$.

Fig. 12. (a) is the poset for the algebra $A_n$ defined by (6.17) while (b) shows the values of a typical element $a_n$ of this algebra at its three points $\alpha_n, \beta_n, \gamma_n$. 
Fig. 13. The figure shows the universal covering space of a circle poset.
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