Bayesian Analysis of the Mixture of Frechet Distribution under Different Loss Functions

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Abstract
This paper has to do with 3-component mixture of the Frechet distributions when the shape parameter is known under Bayesian viewpoint. The type-I right censored sampling scheme is considered due to its extensive use in reliability theory and survival analysis. Taking different non-informative and informative priors, Bayes estimates of the parameter of the mixture model along with their posterior risks are derived under squared error loss function, precautionary loss function and DeGroot loss function. In case, no or little prior information is available, elicitation of hyper parameters is given. In order to study numerically, the execution of the Bayes estimators under different loss functions, their statistical properties have been simulated for different sample sizes and test termination times. A real life data example is also given to illustrate the study.

Keywords: Bayes Estimators, Censoring, Informative prior, Loss Functions, Posterior Risks.

1. Introduction
Frechet distribution was introduced by a French mathematician named Maurice Frechet (1878, 1973) who had determined before one possible limit distribution for the largest order statistic in 1927. The Frechet distribution has been manifested to be helpful for modeling and analysis of several extreme events ranging from accelerated life testing to earthquakes, floods, rain fall, sea currents and wind speed.

Applications of the Frechet distribution in many fields given in Harlow (2002) showed that it is an important distribution for modeling the statistical behavior of materials properties for a variety of engineering implementation. In hydrology, the Frechet distribution is applied to extreme events such as annually maximum one day rainfalls and river discharges. Nadarajah and Kotz (2008) described the sociological models based on Frechet random variables. Zaharim et al. (2009) applied Frechet distribution for analyzing the wind speed data. Chatterjee and Chatterjee (2012) used Frechet distribution to measure ultrasonic pulse. Abbas et al. (2012) described the comparison methods for Frechet distribution with known shape.
Anwar et al. (2014) used Frechet distribution to study analysis of accelerated life testing by using Geometric process. Abbas et al. (2015) discussed the analysis of Frechet distribution using the reference priors. The Bayesian and Maximum likelihood estimators are compared via simulation study.

Several types of data are encountered in everyday life, regarding simple data, grouped data, truncated data, censored data and progressively censored data. Censoring is an inevitable part of the lifetime data. A valuable account of censoring is given in Gijbels (2010) and Kalbfleish and Prentice (2011). There are different sorts of censoring schemes, including right, left and interval censoring, single or multiple censoring and type-I and type-II censoring. Kundu and Howlader (2010) discussed the Bayesian inference and prediction of the IW distribution for type-II censored data. Shi and Yan (2010) derived the empirical Bayes estimates of the two parameter exponential distribution under type-I censoring. Saleem et al. (2010) discussed Bayesian analysis on the power function mixture distribution using type-I censored data. Ali (2015) described the 2-component mixture of the inverse Rayleigh distributions under Bayesian framework. Aslam et al. (2015) presented 3-component mixture of Rayleigh distributions, properties and estimation under the Bayesian framework.

Inspired by above mentioned applications of mixture models, we intend to study Bayesian analysis of a 3-component mixture of the Frechet distributions with unknown mixing proportions. The parameters of component distributions are assumed to be unknown. Four different priors and three different loss functions are used for the Bayesian analysis. Moreover, we consider an ordinary type-I right censored sampling schemes.

The structure of this article is as follows. The Frechet mixture model along with its likelihood function is formulated in section 2. The expressions for posterior distributions using the non-informative and informative priors are derived in section 3. In section 4, the Bayes estimators and posterior risks using the uniform, the Jeffreys', the exponential and the inverse levy priors under squared error loss function (SELF), precautionary loss function (PLF) and DeGroot loss function (DLF) are presented. The elicitation of hyperparameters is given in section 5. In section 6, the limiting expressions of the Bayes estimators and their posterior risks are derived. The simulation study and the real data applications are presented in section 7 and 8, respectively. This article concludes with a brief discussion in section 9.

2. 3-Component mixture of the Frechet distributions

The probability density function (p.d.f) and the cumulative distribution function (c.d.f) of the Frechet distribution for a random variable $X$ are given by:

$$f(x; \alpha, \beta) = \frac{\alpha}{\beta} \left( \frac{\beta}{x} \right)^{\alpha+1} \exp \left[ - \left( \frac{\beta}{x} \right)^{\alpha} \right], x > 0$$

(1)

Where the parameter $\alpha > 0$ determines the shape of the distribution and $\beta > 0$ is the scale parameter.

$$F(x) = \exp \left[ - \left( \frac{\beta}{x} \right)^{\alpha} \right], x > 0$$

(2)
When the shape parameter $\alpha = 1$, then the above p.d.f and c.d.f will become as:

$$f_m(x; \beta_m) = \left(\frac{\beta_m}{x^2}\right) \exp \left[-\left(\frac{\beta_m}{x}\right)\right]; x \geq 0, \beta_m > 0, m = 1, 2, 3$$  \hfill (3)

$$F_m(x) = \exp \left[-\left(\frac{\beta_m}{x}\right)\right]$$ \hfill (4)

A finite 3-component mixture model with the unknown mixing proportions $p_1$ and $p_2$ is:

$$f(x) = p_1 f_1(x) + p_2 f_2(x) + (1 - p_1 - p_2) f_3(x), p_1, p_2 \geq 0, p_1 + p_2 \leq 1$$ \hfill (5)

$$f(x, \beta_1, \beta_2, \beta_3, p_1, p_2) = p_1 \left(\frac{\beta_1}{x^2}\right) \exp \left[-\left(\frac{\beta_1}{x}\right)\right] + p_2 \left(\frac{\beta_2}{x^2}\right) \exp \left[-\left(\frac{\beta_2}{x}\right)\right]$$
$$+ (1 - p_1 - p_2) \left(\frac{\beta_3}{x^2}\right) \exp \left[-\left(\frac{\beta_3}{x}\right)\right]; p_1, p_2 \geq 0, p_1 + p_2 \leq 1$$ \hfill (6)

While the c.d.f of 3-component mixture model is:

$$F(x) = p_1 F_1(x) + p_2 F_2(x) + (1 - p_1 - p_2) F_3(x)$$ \hfill (7)

$$F(x) = p_1 \exp \left[-\left(\frac{\beta_1}{x}\right)\right] + p_2 \exp \left[-\left(\frac{\beta_2}{x}\right)\right] + (1 - p_1 - p_2) \exp \left[-\left(\frac{\beta_3}{x}\right)\right]$$ \hfill (8)

2.1. The Likelihood Function. Suppose $n$ units from the 3-component mixture of Frechet distributions are used in a life testing experiment with fixed test termination time $t$. Let $r$ units out of $n$ units failed until fixed test termination time $t$ and the remaining $(n-r)$ units are still working. According to Mendenhall and Hader (1958), there are many practical situations in which the failed objects can be pointed out easily as subset of subpopulation-I, subpopulation-II or subpopulation-III. Out of $r$ units, suppose $r_1$, $r_2$ and $r_3$ units belong to subpopulation-I, subpopulation-II or subpopulation-III respectively and such that $r = r_1 + r_2 + r_3$. Now we define $x_{lk}, 0 < x_{lk} < t$ be the failure time of $k$th unit belonging to the $l$th subpopulation, where $l = 1, 2, 3$ and $k = 1, 2, ..., r_l$. For a 3-component mixture model, the likelihood function can be written as:

$$L(f|x) = \prod_{l=1}^{3} \prod_{k=1}^{r_l} p_1 f_1(x_{lk}) + p_2 f_2(x_{lk}) + \left(1 - p_1 - p_2\right) f_3(x_{lk}) \bigg| F(t)$$ \hfill (9)

After simplification, the likelihood function of 3-component mixture of Frechet distributions is given:

$$L(\phi|x) \propto \beta_1^{r_1} \beta_2^{r_2} \beta_3^{r_3} \sum_{i=0}^{n-r} \sum_{j=0}^{i} \sum_{l=0}^{j} (-1)^i \binom{n-r}{i} \binom{i}{j} \binom{j}{l}$$
$$\times \exp \left\{-\beta_1 \left(\sum_{k=1}^{r_1} x_{1k}^{-1} + \frac{j - l}{t}\right)\right\} \exp \left\{-\beta_2 \left(\sum_{k=1}^{r_2} x_{2k}^{-1} + \frac{j - l}{t}\right)\right\}$$
$$\times \exp \left\{-\beta_3 \left(\sum_{k=1}^{r_3} x_{3k}^{-1} + \frac{j - l}{t}\right)\right\} p_1^{-i-j+r_1} p_2^{-l+r_2} (1 - p_1 - p_2)^{l+r_3}$$ \hfill (10)

Where $X = (x_1, x_{12}, ..., x_{1r_1}, x_{21}, x_{22}, ..., x_{2r_2}, x_{31}, x_{32}, \ldots, x_{3r_3}, \hat{e}, x_{3r_3})$ are the observed failure times for the uncensored observations and $f = (b_1, b_2, b_3, p_1, p_2)$.
3. The posterior distribution using the non-informative and the informative priors

In this section, posterior distributions of parameters given data, say x, are derived using the non-informative (Uniform and Jeffreys') and the informative (Exponential and Inverse Levy) priors.

3.1. The posterior distribution using the Uniform Prior (UP). When elicitation of hyper parameters is difficult or little prior information is given, then usually the non-informative prior is assumed to be the UP. Ups over the intervals $(0, \infty)$ and $(0, 1)$ are taken for the parameters $(\beta_1, \beta_2, \beta_3)$ of Frechet distribution and for the mixing proportions $(p_1, p_2)$ respectively. With these settings, joint prior distribution of parameters $(\beta_1, \beta_2, \beta_3, p_1, p_2)$, is given by:

$$\pi(\phi) \propto 1; \beta_1, \beta_2, \beta_3 > 0, p_1, p_2 \geq 0, p_1 + p_2 \leq 1$$

The joint posterior distribution of parameters $\beta_1, \beta_2, \beta_3, p_1$ and $p_2$ given data $x$ assuming the UP is:

$$g_1(\phi | x) = \frac{L(\phi | x)\pi_1(\phi)}{\int_\phi L(\phi | x)\pi_1(\phi) d\phi}$$

$$g_1(\phi | x) = C_1^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^{r} \sum_{l=0}^{i} (-1)^i\binom{n-r}{i} \binom{i}{j} \binom{j}{l} \beta_1^{A_{i1}-1} \beta_2^{A_{i2}-1} \beta_3^{A_{i3}-1} \exp(-\beta_1 M_{i1}) \exp(-\beta_2 M_{i2}) \exp(-\beta_3 M_{i3}) (1 - p_1 - p_2)^{C_{i1}-1}$$

where

$$A_{i1} = n + i, A_{i2} = r + i, A_{i3} = r + i, M_{i1} = \frac{1}{a} x_{1k}^{-1} + \frac{i - j}{t}, M_{i2} = \frac{1}{a} x_{2k}^{-1} + \frac{j - l}{t},$$

$$M_{i3} = \frac{1}{a} x_{3k}^{-1} + \frac{l}{t}, A_{01} = r + i + n + 1, B_{01} = j - l + r + 1, C_{01} = n + i + r + 1, C_1 = \frac{n-r}{a} \binom{n-r}{i} \binom{i}{j} \binom{j}{l} \beta_1^{A_{i1}-1} \beta_2^{A_{i2}-1} \beta_3^{A_{i3}-1} \exp(-\beta_1 M_{i1}) \exp(-\beta_2 M_{i2}) \exp(-\beta_3 M_{i3}) (1 - p_1 - p_2)^{C_{i1}-1}$$

3.2. The posterior distribution using the Jeffreys' prior (JP).

According to Jeffreys (1946, 1998) and Berger (1985), the JP is defined as

$$\rho(B_m) \propto \sqrt{I(B_m)}, m = 1, 2, 3$$

where $I(B_m) = E_{B_m} \frac{\partial^2 f(B_m | \theta)}{\partial \theta^2}$ is the Fisher's information matrix. The prior distributions of the mixing proportions $p_1$ and $p_2$ are again taken to be the uniform over the interval $(0, 1)$. Under the assumption of independence of all parameters, the joint prior distribution of $(\beta_1, \beta_2, \beta_3, p_1, p_2)$ is:

$$\pi_2(\phi) \propto \frac{1}{\beta_1 \beta_2 \beta_3}, \beta_1, \beta_2, \beta_3 \geq 0, p_1, p_2 \geq 0, p_1 + p_2 \leq 1$$
The joint posterior distribution of parameters $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$, $p_1$ and $p_2$ given data $\mathbf{x}$ assuming the JP is:

$$g_2(\phi \mid \mathbf{x}) = \frac{L(\phi \mid \mathbf{x})\pi_2(\phi)}{\int_0^{\infty} L(\phi \mid \mathbf{x})\pi_2(\phi) \, d\phi}$$

$$g_2(\phi | \mathbf{x}) = C_2^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^{i} \sum_{l=0}^{j} (-1)^i \binom{n-r}{i} \binom{i}{j} \binom{j}{l} \beta_1^{A_{12}-1} \beta_2^{A_{22}-1} \beta_3^{A_{32}-1} \exp(-\beta_1 M_{12}) \exp(-\beta_2 M_{22}) \times \exp(-\beta_3 M_{32}) p_1^{A_{01}-1} p_2^{A_{02}-1} (1 - p_1 - p_2)^{C_{03}-1}$$

(12)

where

$$A_{12} = \eta, A_{22} = r_2, A_{32} = r_3, M_{12} = \frac{\eta}{\lambda} \sum_{k=1}^{i} x_k^{-1} + \frac{i - j}{t} M_{22} = \frac{\eta}{\lambda} \sum_{k=2}^{j} x_k^{-1} + \frac{j - l}{t}$$

$$M_{32} = \frac{\eta}{\lambda} \sum_{k=3}^{r} x_k^{-1} + \frac{l}{t} A_{02} = i - j + \eta, B_{02} = j - l + r_2, C_{02} = l + r_3 + \eta$$

$$C_2 = \frac{\eta}{\lambda} \sum_{i=0}^{r} \sum_{j=0}^{i} \sum_{l=0}^{j} (-1)^i \binom{i}{j} \binom{j}{l} \beta_1^{A_{12}} \beta_2^{A_{22}} \beta_3^{A_{32}} \frac{\pi_2(\phi)}{M_{12}^{A_{12}} M_{22}^{A_{22}} M_{32}^{A_{32}}}$$

3.3. The posterior distribution using the Exponential prior (EP). As an informative prior, we take the exponential prior for the component parameters $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$ and Bivariate Beta prior for proportion parameters $p_1$, $p_2$. Symbolically, it can be written as: $\mathbf{b}_1 \sim \text{Exponential}(k_1)$, $\mathbf{b}_2 \sim \text{Exponential}(k_2)$, $\mathbf{b}_3 \sim \text{Exponential}(k_3)$ and $p_1$, $p_2 \sim \text{Bivariate Beta}(a, b, c)$. Again assuming independence of all parameters, the joint prior distribution of $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, p_1, p_2)$ is given by:

$$\pi_3(\phi) \propto \exp(-\beta_1 k_1) \exp(-\beta_2 k_2) \exp(-\beta_3 k_3) p_1^{a-1} p_2^{b-1} (1 - p_1 - p_2)^{c-1}$$

The joint posterior distribution of parameters $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$, $p_1$ and $p_2$ given data $\mathbf{x}$ is:

$$g_3(\phi \mid \mathbf{x}) = \frac{L(\phi \mid \mathbf{x} \pi_3(\phi)}{\int_0^{\infty} L(\phi \mid \mathbf{x})\pi_3(\phi) \, d\phi}$$

$$g_3(\phi | \mathbf{x}) = C_3^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^{i} \sum_{l=0}^{j} (-1)^i \binom{n-r}{i} \binom{i}{j} \binom{j}{l} \beta_1^{A_{13}-1} \beta_2^{A_{23}-1} \beta_3^{A_{33}-1} \exp(-\beta_1 M_{13}) \exp(-\beta_2 M_{23}) \times \exp(-\beta_3 M_{33}) p_1^{A_{01}-1} p_2^{A_{02}-1} (1 - p_1 - p_2)^{C_{03}-1}$$

(13)

where

$$A_{13} = \eta + l, A_{23} = r_2 + l, A_{33} = r_3 + l, M_{13} = \frac{\eta}{\lambda} \sum_{k=1}^{i} x_k^{-1} + \frac{i - j}{t} + k_1 M_{23} = \frac{\eta}{\lambda} \sum_{k=2}^{j} x_k^{-1} + \frac{j - l}{t} + k_2$$

$$M_{33} = \frac{\eta}{\lambda} \sum_{k=3}^{r} x_k^{-1} + \frac{l}{t} A_{03} = i - j + \eta, B_{03} = j - l + r_2 + b, C_{03} = l + r_3 + c$$

$$C_3 = \frac{\eta}{\lambda} \sum_{i=0}^{r} \sum_{j=0}^{i} \sum_{l=0}^{j} (-1)^i \binom{i}{j} \binom{j}{l} \beta_1^{A_{13}} \beta_2^{A_{23}} \beta_3^{A_{33}} \frac{\pi_3(\phi)}{M_{13}^{A_{13}} M_{23}^{A_{23}} M_{33}^{A_{33}}}$$
3.4. The posterior distribution using the Inverse Levy prior (ILP). As an informative prior, we take the Inverse Levy prior for the component parameters $\hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3$ and Bivariate Beta prior for proportion parameters $p_1, p_2$. Symbolically, it can be written as: $\tilde{\beta}_1 \sim \text{Inverse Levy}(a_1), \tilde{\beta}_2 \sim \text{Inverse Levy}(a_2), \tilde{\beta}_3 \sim \text{Inverse Levy}(a_3)$ and $p_1, p_2 \sim \text{Bivariate Beta}(a, b, c)$. Again assuming independence of all parameters, the joint prior distribution of $(\tilde{\beta}_1, \tilde{\beta}_2, \tilde{\beta}_3, p_1, p_2)$ is given by:

$$
\pi_4(\phi) \propto \beta_1^{-\frac{a_1}{2}} \exp\left(-\frac{a_1 \beta_1}{2}\right) \beta_2^{-\frac{a_2}{2}} \exp\left(-\frac{a_2 \beta_2}{2}\right) \beta_3^{-\frac{a_3}{2}} \exp\left(-\frac{a_3 \beta_3}{2}\right) p_1^{a_1-1} p_2^{a_2-1} (1 - p_1 - p_2)^{a_3-1}
$$

The joint posterior distribution of parameters $\tilde{\beta}_1, \tilde{\beta}_2, \tilde{\beta}_3, p_1$ and $p_2$ given data $x$ is:

$$
g_4(\phi | x) = \frac{L(\phi | x) \pi_4(\phi)}{\int_0^1 L(\phi | x) \pi_4(\phi) d\phi}
$$

$$
g_4(\phi | x) = C_4^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^{l} \sum_{k=0}^{M_1} \binom{n-r}{i} \binom{l}{j} \binom{M_1}{k} \times \beta_1^{A_1-1} \beta_2^{A_2-1} \beta_3^{A_3-1} \exp(-\beta_1 M_1) \exp(-\beta_2 M_2) \exp(-\beta_3 M_3) p_1^{A_1-1} p_2^{A_2-1} (1 - p_1 - p_2)^{A_3-1}
$$

where

$$
A_1 = n + \frac{1}{2}, A_2 = r_2 + \frac{1}{2}, A_3 = r_3 + 1, M_1 = \frac{\eta}{a_1} \sum_{k=1}^{a_1} x_k^{a_3} + \frac{r_1}{t} + \frac{a_1}{2}, M_2 = \frac{\eta}{a_2} \sum_{k=1}^{a_2} x_k^{a_3} + \frac{r_2}{t} + \frac{a_2}{2}, M_3 = \frac{\eta}{a_3} \sum_{k=1}^{a_3} x_k^{a_3} + \frac{r_3}{t} + \frac{a_3}{2},
$$

$$
C_4 = \prod_{i=0}^{n-r} \prod_{j=0}^{l} \prod_{k=0}^{M_1} \prod_{l=0}^{M_2} \prod_{i=0}^{M_3} \prod_{M_1}^{M_2} \prod_{M_2}^{M_3} \frac{G(A_1) G(A_2) G(A_3)}{M_1^{A_1} M_2^{A_2} M_3^{A_3}}
$$

4. Bayes estimators and posterior risks using the UP, the JP, the Exponential and Inverse Levy prior under SELF, PLF and DLF

If $\hat{\beta}$ is a Bayes estimator, then $r(\hat{\beta})$ is called posterior risk and is defined as:

$$
r(\hat{\beta}) = E_{\beta \mid x} [r(\hat{\beta})].
$$

Our purpose, in this study, is to look for efficient Bayes estimates of the different parameters. For this purpose, three different loss functions, namely SELF, PLF and DLF used to obtain Bayes estimators and their posterior risks. The SELF, defined as $L(b, d) = (b - d)^2$, was introduced by Legendre to develop the least squares theory. Norstrom (1996) discussed an asymmetric PLF and also introduced a special case of general class of PLFs, which is defined as $L(b, d) = \frac{(b - d)^2}{d}$. While the DLF is presented by DeGroot (2005) and is defined as $L(b, d) = \frac{b - d}{c}$. For a given prior, the Bayes estimator and posterior risk under SELF are calculated as: $\hat{\beta} = E_{\beta \mid x}(b)$ and $r(\hat{\beta}) = E_{\beta \mid x}(b^2) - E_{\beta \mid x}(b)^2$ respectively. Similarly, the Bayes estimators and posterior risks with PLF and DLF are given by: $\hat{\beta} = E_{\beta \mid x}(b^2)$ and $r(\hat{\beta}) = E_{\beta \mid x}(b^2) - 2E_{\beta \mid x}(b)$. Also, $\hat{\beta} = E_{\beta \mid x}(b^2)$ and $r(\hat{\beta}) = 1 - \frac{E_{\beta \mid x}(b)^2}{E_{\beta \mid x}(b^2)}$, respectively.
4.1. The Bayes estimators and posterior risks using the UP, the JP and IP under SELF. The Bayes estimators and posterior risks using the UP, the JP and IP for parameters $\beta_1, \beta_2, \beta_3, p_1$ and $p_2$ under SELF are obtained with their respective marginal posterior distributions are given below:

$$
\hat{\beta}_{1e} = C_v^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^{l} \sum_{l=0}^{(n-r)} (-1)^i \binom{n-r}{i} \binom{j}{l} \frac{\Gamma(A_{1e} + 1) \Gamma(A_{2e})}{M_{1e}^{A_{1e}+1} M_{2e}^{A_{2e}+1}} \\
\times \frac{\Gamma(A_{3e})}{M_{3e}^{A_{3e}}} B(A_{0e}, C_{0e}) B(B_{0e}, A_{0e} + C_{0e})
$$

$$
\hat{\beta}_{2e} = C_v^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^{l} \sum_{l=0}^{(n-r)} (-1)^i \binom{n-r}{i} \binom{j}{l} \frac{\Gamma(A_{1e}) \Gamma(A_{2e} + 1)}{M_{1e}^{A_{1e}} M_{2e}^{A_{2e}+1}} \\
\times \frac{\Gamma(A_{3e})}{M_{3e}^{A_{3e}}} B(A_{0e}, C_{0e}) B(B_{0e}, A_{0e} + C_{0e})
$$

$$
\hat{\beta}_{3e} = C_v^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^{l} \sum_{l=0}^{(n-r)} (-1)^i \binom{n-r}{i} \binom{j}{l} \frac{\Gamma(A_{1e}) \Gamma(A_{2e}) \Gamma(A_{3e})}{M_{1e}^{A_{1e}} M_{2e}^{A_{2e}} M_{3e}^{A_{3e}}} \\
\times \frac{\Gamma(A_{3e} + 1)}{M_{3e}^{A_{3e}+1}} B(A_{0e}, C_{0e}) B(B_{0e}, A_{0e} + C_{0e})
$$

$$
\hat{p}_{1e} = C_v^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^{l} \sum_{l=0}^{(n-r)} (-1)^i \binom{n-r}{i} \binom{j}{l} \frac{\Gamma(A_{1e}) \Gamma(A_{2e}) \Gamma(A_{3e})}{M_{1e}^{A_{1e}} M_{2e}^{A_{2e}} M_{3e}^{A_{3e}}} \\
\times B(B_{0e}, C_{0e}) B(A_{0e} + 1, B_{0e} + C_{0e})
$$

$$
\hat{p}_{2e} = C_v^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^{l} \sum_{l=0}^{(n-r)} (-1)^i \binom{n-r}{i} \binom{j}{l} \frac{\Gamma(A_{1e}) \Gamma(A_{2e}) \Gamma(A_{3e})}{M_{1e}^{A_{1e}} M_{2e}^{A_{2e}} M_{3e}^{A_{3e}}} \\
\times B(A_{0e}, C_{0e}) B(B_{0e} + 1, A_{0e} + C_{0e})
$$

$$
\rho(\hat{\beta}_{1e}) = C_v^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^{l} \sum_{l=0}^{(n-r)} (-1)^i \binom{n-r}{i} \binom{j}{l} \frac{\Gamma(A_{1e} + 2) \Gamma(A_{2e})}{M_{1e}^{A_{1e}+2} M_{2e}^{A_{2e}}} \\
\times \frac{\Gamma(A_{3e})}{M_{3e}^{A_{3e}}} B(A_{0e}, C_{0e}) B(B_{0e}, A_{0e} + C_{0e}) - (\hat{\beta}_{1e})^2
$$

$$
\rho(\hat{\beta}_{2e}) = C_v^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^{l} \sum_{l=0}^{(n-r)} (-1)^i \binom{n-r}{i} \binom{j}{l} \frac{\Gamma(A_{1e}) \Gamma(A_{2e} + 2)}{M_{1e}^{A_{1e}} M_{2e}^{A_{2e}+2}} \\
\times \frac{\Gamma(A_{3e})}{M_{3e}^{A_{3e}}} B(A_{0e}, C_{0e}) B(B_{0e}, A_{0e} + C_{0e}) - (\hat{\beta}_{2e})^2
$$

$$
\rho(\hat{\beta}_{3e}) = C_v^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^{l} \sum_{l=0}^{(n-r)} (-1)^i \binom{n-r}{i} \binom{j}{l} \frac{\Gamma(A_{1e}) \Gamma(A_{2e})}{M_{1e}^{A_{1e}} M_{2e}^{A_{2e}}} \\
\times \frac{\Gamma(A_{3e} + 2)}{M_{3e}^{A_{3e}+2}} B(A_{0e}, C_{0e}) B(B_{0e}, A_{0e} + C_{0e}) - (\hat{\beta}_{3e})^2
$$

$$
\rho(\hat{p}_{1e}) = C_v^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^{l} \sum_{l=0}^{(n-r)} (-1)^i \binom{n-r}{i} \binom{j}{l} \frac{\Gamma(A_{1e}) \Gamma(A_{2e})}{M_{1e}^{A_{1e}} M_{2e}^{A_{2e}}} \\
\times \frac{\Gamma(A_{3e})}{M_{3e}^{A_{3e}}} B(B_{0e}, C_{0e}) B(A_{0e} + 2, B_{0e} + C_{0e}) - (\hat{p}_{1e})^2
$$
where \( v=1 \) for the UP, \( v=2 \) for the JP, \( v=3 \) for the EP and \( v=4 \) for the ILP.

4.2. The Bayes estimators and posterior risks using the UP, the JP and IP under PLF. Norstrom (1996) discussed an asymmetric PLF and also introduced a special case of general class of PLFs, which is defined as

\[
\ell(b,d) = \frac{(b-d)^2}{d}.
\]

The Bayes estimator and posterior risk are:

\[
\hat{\beta}_1 = \left\{ C_v^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^{i} \sum_{l=0}^{j} (-1)^i \binom{n-r}{i} \binom{i}{j} \binom{j}{l} \frac{\Gamma(A_{1v}) \Gamma(A_{2v})}{M_{1v}^{A_{1v}} M_{2v}^{A_{2v}}} \right. \\
\left. \times \frac{\Gamma(A_{3v})}{M_{3v}^{A_{3v}}} B(A_{0v}, C_{0v}) B(B_{0v} + 2, A_{0v} + C_{0v}) \right\}^{1/2}
\]

\[
\hat{\beta}_2 = \left\{ C_v^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^{i} \sum_{l=0}^{j} (-1)^i \binom{n-r}{i} \binom{i}{j} \binom{j}{l} \frac{\Gamma(A_{1v}) \Gamma(A_{2v} + 2)}{M_{1v}^{A_{1v}} M_{2v}^{A_{2v} + 2}} \right. \\
\left. \times \frac{\Gamma(A_{3v})}{M_{3v}^{A_{3v}}} B(A_{0v}, C_{0v}) B(B_{0v} + 2, A_{0v} + C_{0v}) \right\}^{1/2}
\]

\[
\hat{\beta}_3 = \left\{ C_v^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^{i} \sum_{l=0}^{j} (-1)^i \binom{n-r}{i} \binom{i}{j} \binom{j}{l} \frac{\Gamma(A_{1v}) \Gamma(A_{2v} + 2)}{M_{1v}^{A_{1v}} M_{2v}^{A_{2v} + 2}} \right. \\
\left. \times \frac{\Gamma(A_{3v})}{M_{3v}^{A_{3v}}} B(A_{0v}, C_{0v}) B(B_{0v} + 2, A_{0v} + C_{0v}) \right\}^{1/2}
\]

\[
\hat{p}_{1v} = \left\{ C_v^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^{i} \sum_{l=0}^{j} (-1)^i \binom{n-r}{i} \binom{i}{j} \binom{j}{l} \frac{\Gamma(A_{1v}) \Gamma(A_{2v})}{M_{1v}^{A_{1v}} M_{2v}^{A_{2v}}} \right. \\
\left. \times \frac{\Gamma(A_{3v})}{M_{3v}^{A_{3v}}} B(B_{0v} + 2, A_{0v} + C_{0v}) \right\}^{1/2}
\]

\[
\hat{p}_{2v} = \left\{ C_v^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^{i} \sum_{l=0}^{j} (-1)^i \binom{n-r}{i} \binom{i}{j} \binom{j}{l} \frac{\Gamma(A_{1v}) \Gamma(A_{2v})}{M_{1v}^{A_{1v}} M_{2v}^{A_{2v}}} \right. \\
\left. \times \frac{\Gamma(A_{3v})}{M_{3v}^{A_{3v}}} B(A_{0v}, C_{0v}) B(B_{0v} + 2, A_{0v} + C_{0v}) \right\}^{1/2}
\]
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\[
\rho \left( \beta_1 \right) = 2 \left\{ C_v^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^{r} \sum_{l=0}^{j} (-1)^i \binom{n-r}{i} \binom{r}{j} \frac{\Gamma(A_{1v} + 2)}{M_{1v}^{A_{1v}+2}} \right. \\
\left. - 2 \frac{\Gamma(A_{2v}) \Gamma(A_{3v})}{M_{2v}^{A_{2v}} M_{3v}^{A_{3v}}} B(A_{0v}, C_{0v}) B(B_{0v}, A_{0v} + C_{0v}) \right\}^{\frac{1}{2}} \\
\rho \left( \beta_2 \right) = 2 \left\{ C_v^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^{r} \sum_{l=0}^{j} (-1)^i \binom{n-r}{i} \binom{r}{j} \frac{\Gamma(A_{1v} + 1)}{M_{1v}^{A_{1v}+1}} \right. \\
\left. - 2 \frac{\Gamma(A_{2v}) \Gamma(A_{3v})}{M_{2v}^{A_{2v}} M_{3v}^{A_{3v}}} B(A_{0v}, C_{0v}) B(B_{0v}, A_{0v} + C_{0v}) \right\}^{\frac{1}{2}} \\
\rho \left( \beta_3 \right) = 2 \left\{ C_v^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^{r} \sum_{l=0}^{j} (-1)^i \binom{n-r}{i} \binom{r}{j} \frac{\Gamma(A_{1v})}{M_{1v}^{A_{1v}}} \right. \\
\left. - 2 \frac{\Gamma(A_{2v}) \Gamma(A_{3v} + 2)}{M_{2v}^{A_{2v}+2} M_{3v}^{A_{3v}+2}} B(A_{0v}, C_{0v}) B(B_{0v}, A_{0v} + C_{0v}) \right\}^{\frac{1}{2}} \\
\rho \left( \beta_1 \right) = 2 \left\{ C_v^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^{r} \sum_{l=0}^{j} (-1)^i \binom{n-r}{i} \binom{r}{j} \frac{\Gamma(A_{1v})}{M_{1v}^{A_{1v}}} \right. \\
\left. - 2 \frac{\Gamma(A_{2v}) \Gamma(A_{3v} + 1)}{M_{2v}^{A_{2v}+1} M_{3v}^{A_{3v}+1}} B(A_{0v}, C_{0v}) B(B_{0v}, A_{0v} + C_{0v}) \right\}^{\frac{1}{2}} \\
\rho \left( \beta_1 \right) = 2 \left\{ C_v^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^{r} \sum_{l=0}^{j} (-1)^i \binom{n-r}{i} \binom{r}{j} \frac{\Gamma(A_{1v})}{M_{1v}^{A_{1v}}} \right. \\
\left. - 2 \frac{\Gamma(A_{2v}) \Gamma(A_{3v} + 2)}{M_{2v}^{A_{2v}+2} M_{3v}^{A_{3v}+2}} B(B_{0v}, C_{0v}) B(A_{0v} + 2, B_{0v} + C_{0v}) \right\}^{\frac{1}{2}} \\
\rho \left( \beta_1 \right) = 2 \left\{ C_v^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^{r} \sum_{l=0}^{j} (-1)^i \binom{n-r}{i} \binom{r}{j} \frac{\Gamma(A_{1v})}{M_{1v}^{A_{1v}}} \right. \\
\left. - 2 \frac{\Gamma(A_{2v}) \Gamma(A_{3v})}{M_{2v}^{A_{2v}} M_{3v}^{A_{3v}}} B(B_{0v}, C_{0v}) B(A_{0v} + 1, B_{0v} + C_{0v}) \right\}^{\frac{1}{2}} 
\]
4.3. The Bayes estimators and posterior risks using the UP, the JP and IP under DLF. The Bayes estimators and posterior risks using the UP, the JP and the IP for parameters $\theta_1, \theta_2, \theta_3, \alpha$ and $\beta$ under DLF are:

$$\hat{\beta}_1 = \frac{C_v^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^{i} \sum_{l=0}^{j} (-1)^i \binom{n-r}{i} \binom{i}{j} \binom{j}{l} \frac{\Gamma(A_{1v})}{M_{1v}^{A_{1v}}} + 2}{\frac{\Gamma(A_{2v})}{M_{2v}^{A_{2v}}} \frac{\Gamma(A_{3v})}{M_{3v}^{A_{3v}}} B(A_{0v}, C_{0v}) B(B_{0v}, A_{0v} + C_{0v})}$$

$$\hat{\beta}_2 = \frac{C_v^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^{i} \sum_{l=0}^{j} (-1)^i \binom{n-r}{i} \binom{i}{j} \binom{j}{l} \frac{\Gamma(A_{1v})}{M_{1v}^{A_{1v}}} + 1}{\frac{\Gamma(A_{2v} + 2)}{M_{2v}^{A_{2v}+2}} \frac{\Gamma(A_{3v})}{M_{3v}^{A_{3v}}} B(A_{0v}, C_{0v}) B(B_{0v}, A_{0v} + C_{0v})}$$
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\[
\hat{\beta}_{3v} = \frac{\left\{ \sum_{i=0}^{n-r} \sum_{j=0}^{i} \sum_{l=0}^{j} (-1)^i \binom{n-r}{i} \binom{i}{j} \frac{\Gamma(A_{1v})}{M_{1v}^{A_{1v}}} \right\}}{\frac{\Gamma(A_{2v})}{M_{2v}^{A_{2v}}} \frac{\Gamma(A_{3v} + 2)}{M_{3v}^{A_{3v} + 2}} B(A_{0v}, C_{0v}) B(B_{0v}, A_{0v} + C_{0v})}
\]

\[
\hat{p}_{1v} = \frac{\left\{ \sum_{i=0}^{n-r} \sum_{j=0}^{i} \sum_{l=0}^{j} (-1)^i \binom{n-r}{i} \binom{i}{j} \frac{\Gamma(A_{1v})}{M_{1v}^{A_{1v}}} \right\}}{\frac{\Gamma(A_{2v})}{M_{2v}^{A_{2v}}} \frac{\Gamma(A_{3v} + 1)}{M_{3v}^{A_{3v} + 1}} B(A_{0v}, C_{0v}) B(A_{0v} + 2, B_{0v} + C_{0v})}
\]

\[
\hat{p}_{2v} = \frac{\left\{ \sum_{i=0}^{n-r} \sum_{j=0}^{i} \sum_{l=0}^{j} (-1)^i \binom{n-r}{i} \binom{i}{j} \frac{\Gamma(A_{1v})}{M_{1v}^{A_{1v}}} \right\}}{\frac{\Gamma(A_{2v})}{M_{2v}^{A_{2v}}} \frac{\Gamma(A_{3v})}{M_{3v}^{A_{3v}}} B(A_{0v}, C_{0v}) B(B_{0v} + 1, A_{0v} + C_{0v})}
\]
\[
\rho(\hat{\beta}_1) = 1 - \frac{C^{-1}_v \sum_{i=0}^{n-r} \sum_{j=0}^i \sum_{l=0}^j (-1)^i \binom{n-r}{i} \binom{i}{j} \binom{j}{l} \Gamma(A_{1v} + 1) M_{1v}^{A_{1v} + 1}}{B(A_{0e}, C_{0e}) B(B_{0e}, A_{0e} + C_{0e})^2}
\]
\[
\rho(\hat{\beta}_2) = 1 - \frac{C^{-1}_v \sum_{i=0}^{n-r} \sum_{j=0}^i \sum_{l=0}^j (-1)^i \binom{n-r}{i} \binom{i}{j} \binom{j}{l} \Gamma(A_{1v} + 2) M_{1v}^{A_{1v} + 2}}{B(A_{0e}, C_{0e}) B(B_{0e}, A_{0e} + C_{0e})^2}
\]
\[
\rho(\hat{\beta}_3) = 1 - \frac{C^{-1}_v \sum_{i=0}^{n-r} \sum_{j=0}^i \sum_{l=0}^j (-1)^i \binom{n-r}{i} \binom{i}{j} \binom{j}{l} \Gamma(A_{1v} + 1) M_{1v}^{A_{1v} + 1}}{B(A_{0e}, C_{0e}) B(B_{0e}, A_{0e} + C_{0e})^2}
\]
\[
\rho(\hat{\beta}_4) = 1 - \frac{C^{-1}_v \sum_{i=0}^{n-r} \sum_{j=0}^i \sum_{l=0}^j (-1)^i \binom{n-r}{i} \binom{i}{j} \binom{j}{l} \Gamma(A_{1v} + 1) M_{1v}^{A_{1v} + 1}}{B(A_{0e}, C_{0e}) B(B_{0e}, A_{0e} + C_{0e})^2}
\]
Elicitation of Hyper-parameters

Elicitation is the main task for subjective Bayesian. The complete procedure for quantifying the prior information in the form of prior distribution is precisely known as the elicitation. Aslam (2003) presented different methods of elicitation based on prior predictive distribution for the elicitation of the hyper-parameters. In this study, we use the method of elicitation using prior predictive distribution based on the predictive probabilities. In this approach, confidence levels of the prior predictive are gained for the particular intervals of the random variables. The set of hyper parameters, for which the difference between the elicited probabilities and the expert predictive probabilities is minimum, is considered.

5.1. Elicitation of hyper-parameters using the Exponential Prior. For eliciting the hyper-parameters, prior predictive distribution (PPD) is used. The PPD for a random variable $X$ is:

$$p(x) = \int \phi \ p(x|\phi) \pi_3 (\phi) \, d\phi$$

$$p(x) = \frac{1}{x^2 (a + b + c)} \left[ \frac{a k_1}{(k_1 + x - 1)^2} + \frac{b k_2}{(k_2 + x - 1)^2} + \frac{c k_3}{(k_3 + x - 1)^2} \right]$$

We choose the prior predictive probabilities, satisfying the laws of probability, to elicit the hyper-parameters of the prior density. Using the prior predictive distribution, we consider the six intervals (0,1), (1,2), (2,3), (3,4), (4,5) and (5,6) with probabilities 0.73, 0.11, 0.05, 0.02, 0.02, and 0.01 respectively, given an expert opinion. The following nine equations are derived from the above information:

$$\frac{1}{x^2 (a + b + c)} \int_0^1 \left[ \frac{a k_1}{(k_1 + x - 1)^2} + \frac{b k_2}{(k_2 + x - 1)^2} + \frac{c k_3}{(k_3 + x - 1)^2} \right] \, dx = 0.73$$

$$\frac{1}{x^2 (a + b + c)} \int_1^2 \left[ \frac{a k_1}{(k_1 + x - 1)^2} + \frac{b k_2}{(k_2 + x - 1)^2} + \frac{c k_3}{(k_3 + x - 1)^2} \right] \, dx = 0.11$$

$$\frac{1}{x^2 (a + b + c)} \int_2^3 \left[ \frac{a k_1}{(k_1 + x - 1)^2} + \frac{b k_2}{(k_2 + x - 1)^2} + \frac{c k_3}{(k_3 + x - 1)^2} \right] \, dx = 0.05$$

$$\frac{1}{x^2 (a + b + c)} \int_3^4 \left[ \frac{a k_1}{(k_1 + x - 1)^2} + \frac{b k_2}{(k_2 + x - 1)^2} + \frac{c k_3}{(k_3 + x - 1)^2} \right] \, dx = 0.02$$
For eliciting the hyper parameters \( k_1, k_2, k_3, a, b \) and \( c \) and the equations are simultaneously solved through the computer program developed in SAS package using the \( \text{PROC SYSLIN} \) command, the values of the hyper parameters are found to be 2.0003, 3.0030, 4.0016, 2.0103, 1.7607 and 1.50 respectively.

5.2. Elicitation of hyper parameters using the Inverse Levy Prior. The PPD using Inverse Levy prior for a random variable \( X \) is given by:

\[
p(x) = \int \frac{1}{2\sqrt{2}(a + b + c)} x^2 \left[ \frac{a\sqrt{a_1}}{(a_1 + x^{-1})^{3/2}} + \frac{b\sqrt{a_2}}{(a_2 + x^{-1})^{3/2}} + \frac{c\sqrt{a_3}}{(a_3 + x^{-1})^{3/2}} \right] d\phi
\]

Using same canon defined as above for the exponential prior, the values of the hyperparameters \( a_1, a_2, a_3, a, b \) and \( c \) are 1.9520, 2.5321, 3.7735, 0.2763, 0.1167 and 1.0.

6. Limiting Expressions. Letting \( t \to \infty \), all the observations that are assimilated in our analysis are uncensored and therefore \( r \) tends to \( n \), \( r_1 \) tends to the unknown \( n_1 \), \( r_2 \) tends to the unknown \( n_2 \) and \( r_3 \) tends to the unknown \( n_3 \). As a result, the amount of information carried in the sample expands, which results in the depletion of the variances of the estimates. The limiting (complete sample) expressions for Bayes estimators and posterior risks using the UP, the JP, the EP and the ILP under SELF, PLF and DLF are given in the Tables 1-6.

Table 1: Limiting Expressions for the Bayes Estimators as \( t \to \infty \) assuming the UP, the JP and the IP under SELF

| Parameters | UP | JP | Exponential prior | Inverse Levy Prior |
|------------|----|----|-------------------|-------------------|
| \( \hat{\beta}_1 \) | \( \frac{n_1 + 1}{\sum_{k=1}^{n_1} x_{1k}} \) | \( \frac{n_1}{\sum_{k=1}^{n_1} x_{1k}} \) | \( \frac{n_1 + 1}{\sum_{k=1}^{n_1} x_{1k} + k_1} \) | \( \frac{n_1 + 1}{\sum_{k=1}^{n_1} x_{1k} + \frac{k_1}{2}} \) |
| \( \hat{\beta}_2 \) | \( \frac{n_2 + 1}{\sum_{k=1}^{n_2} x_{2k}} \) | \( \frac{n_2}{\sum_{k=1}^{n_2} x_{2k}} \) | \( \frac{n_2 + 1}{\sum_{k=1}^{n_2} x_{2k} + k_2} \) | \( \frac{n_2 + 1}{\sum_{k=1}^{n_2} x_{2k} + \frac{k_2}{2}} \) |
| \( \hat{\beta}_3 \) | \( \frac{n_3 + 1}{\sum_{k=1}^{n_3} x_{3k}} \) | \( \frac{n_3}{\sum_{k=1}^{n_3} x_{3k}} \) | \( \frac{n_3 + 1}{\sum_{k=1}^{n_3} x_{3k} + k_3} \) | \( \frac{n_3 + 1}{\sum_{k=1}^{n_3} x_{3k} + \frac{k_3}{2}} \) |
| \( p_1 \) | \( \frac{n_1 + 1}{n + 3} \) | \( \frac{n_1}{n + 3} \) | \( \frac{n_1 + a}{n + a + b + c} \) | \( \frac{n_1 + a}{n + a + b + c} \) |
| \( p_2 \) | \( \frac{n_2 + 1}{n + 3} \) | \( \frac{n_2}{n + 3} \) | \( \frac{n_2 + b}{n + a + b + c} \) | \( \frac{n_2 + b}{n + a + b + c} \) |
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Table 2: Limiting Expressions for the Bayes Estimators as $t \to \infty$ using the UP and the JP under PLF

| Parameters | UP | JP |
|------------|----|----|
| $b_1$      | $\frac{(n_1+1)(n_2+2)}{\sum_{k=1}^{n} x_{1k}^{1}}$ | $\frac{(n_1+1)(n_2+2)}{\sum_{k=1}^{n} x_{2k}^{1}}$ |
| $b_2$      | $\frac{[(n_2+1)(n_2+2)]^{1/2}}{\sum_{k=1}^{n} x_{2k}^{1}}$ | $\frac{[(n_2+1)(n_2+2)]^{1/2}}{\sum_{k=1}^{n} x_{2k}^{1}}$ |
| $b_3$      | $\frac{[(n_3+1)(n_3+2)]^{1/2}}{\sum_{k=1}^{n} x_{3k}^{1}}$ | $\frac{[(n_3+1)(n_3+2)]^{1/2}}{\sum_{k=1}^{n} x_{3k}^{1}}$ |
| $p_1$      | $\frac{(n_1+1)(n_2+2)}{(n+3)(n+4)}^{1/2}$ | $\frac{(n_1+1)(n_2+2)}{(n+3)(n+4)}^{1/2}$ |
| $p_2$      | $\frac{(n_2+1)(n_2+2)}{(n+3)(n+4)}^{1/2}$ | $\frac{(n_2+1)(n_2+2)}{(n+3)(n+4)}^{1/2}$ |

Table 3: Limiting Expressions for the Bayes Estimators as $t \to \infty$ using the EP and the ILP under PLF

| Parameters | Exponential Prior | Inverse Levy Prior |
|------------|-------------------|-------------------|
| $b_1$      | $\frac{(n_1+n+1)}{\sum_{k=1}^{n} x_{1k}^{1}+k_1}$ | $\frac{(n_1+n+1)}{\sum_{k=1}^{n} x_{1k}^{1}+\frac{k_1}{2}}$ |
| $b_2$      | $\frac{[(n_2+n+1)(n_2+n+2)]^{1/2}}{\sum_{k=1}^{n} x_{2k}^{1}+k_2}$ | $\frac{[(n_2+n+1)(n_2+n+2)]^{1/2}}{\sum_{k=1}^{n} x_{2k}^{1}+\frac{k_2}{2}}$ |
| $b_3$      | $\frac{[(n_3+n+1)(n_3+n+2)]^{1/2}}{\sum_{k=1}^{n} x_{3k}^{1}+k_3}$ | $\frac{[(n_3+n+1)(n_3+n+2)]^{1/2}}{\sum_{k=1}^{n} x_{3k}^{1}+\frac{k_3}{2}}$ |
| $p_1$      | $\frac{(n_1+a)(n_2+a+1)}{(n+a+b+c)(n+a+b+c+1)}^{1/2}$ | $\frac{(n_1+a)(n_2+a+1)}{(n+a+b+c)(n+a+b+c+1)}^{1/2}$ |
| $p_2$      | $\frac{(n_2+b)(n_2+b+1)}{(n+a+b+c)(n+a+b+c+1)}^{1/2}$ | $\frac{(n_2+b)(n_2+b+1)}{(n+a+b+c)(n+a+b+c+1)}^{1/2}$ |

Table 4: Limiting Expressions for the Bayes Estimators as $t \to \infty$ using the UP and the JP, EP and ILP under DLF

| Parameters | UP | JP | Exponential Prior | Inverse Levy Prior |
|------------|----|----|-------------------|-------------------|
| $b_1$      | $\frac{n_1+2}{\sum_{k=1}^{n} x_{1k}^{1}}$ | $\frac{n_1+2}{\sum_{k=1}^{n} x_{1k}^{1}}$ | $\frac{n_1+2}{\sum_{k=1}^{n} x_{1k}^{1}}$ | $\frac{n_1+2}{\sum_{k=1}^{n} x_{1k}^{1}+\frac{a_1}{2}}$ |
| $b_2$      | $\frac{n_2+2}{\sum_{k=1}^{n} x_{2k}^{1}}$ | $\frac{n_2+2}{\sum_{k=1}^{n} x_{2k}^{1}}$ | $\frac{n_2+2}{\sum_{k=1}^{n} x_{2k}^{1}}$ | $\frac{n_2+2}{\sum_{k=1}^{n} x_{2k}^{1}+\frac{a_2}{2}}$ |
| $b_3$      | $\frac{n_3+2}{\sum_{k=1}^{n} x_{3k}^{1}}$ | $\frac{n_3+2}{\sum_{k=1}^{n} x_{3k}^{1}}$ | $\frac{n_3+2}{\sum_{k=1}^{n} x_{3k}^{1}}$ | $\frac{n_3+2}{\sum_{k=1}^{n} x_{3k}^{1}+\frac{a_3}{2}}$ |
| $p_1$      | $\frac{n_1+2}{n+4}$ | $\frac{n_1+2}{n+4}$ | $\frac{n_1+2}{n+4}$ | $\frac{n_1+2}{n+4}$ |
| $p_2$      | $\frac{n_2+2}{n+4}$ | $\frac{n_2+2}{n+4}$ | $\frac{n_2+2}{n+4}$ | $\frac{n_2+2}{n+4}$ |
### Table 5: Limiting Expressions for the Posterior risks as \( t \rightarrow \infty \) using the UP and the JP, EP and ILP under SELF

| Parameters | UP | JP | Exponential Prior | Inverse Levy Prior |
|------------|----|----|-------------------|-------------------|
| \( \beta_1 \) | \( \frac{n_1+1}{(\sum_{k=1}^{n_1} x_{1k})} \) | \( \frac{n_1}{(\sum_{k=1}^{n_1} x_{1k})} \) | \( \frac{n_1+1}{(\sum_{k=1}^{n_1} x_{1k} + k_1)^2} \) | \( \frac{n_1+1}{(\sum_{k=1}^{n_1} x_{1k} + a_1)^2} \) |
| \( \beta_2 \) | \( \frac{n_2}{(\sum_{k=1}^{n_2} x_{2k})} \) | \( \frac{n_2}{(\sum_{k=1}^{n_2} x_{2k})} \) | \( \frac{n_2+1}{(\sum_{k=1}^{n_2} x_{2k} + k_2)^2} \) | \( \frac{n_2+1}{(\sum_{k=1}^{n_2} x_{2k} + a_2)^2} \) |
| \( \beta_3 \) | \( \frac{n_3}{(\sum_{k=1}^{n_3} x_{3k})} \) | \( \frac{n_3}{(\sum_{k=1}^{n_3} x_{3k})} \) | \( \frac{n_3+1}{(\sum_{k=1}^{n_3} x_{3k} + k_3)^2} \) | \( \frac{n_3+1}{(\sum_{k=1}^{n_3} x_{3k} + a_3)^2} \) |
| \( p_1 \) | \( \frac{(n_1+1)(n_2+n_3+2)}{(n+3)^2(n+4)} \) | \( \frac{(n_1+1)(n_2+n_3+2)}{(n+3)^2(n+4)} \) | \( \frac{(n_1+1)(n_2+n_3+b+c)}{(n+a+b+c)^2(n+a+b+c+1)} \) | \( \frac{(n_1+1)(n_2+n_3+b+c)}{(n+a+b+c)^2(n+a+b+c+1)} \) |
| \( p_2 \) | \( \frac{(n_2+1)(n_2+n_3+2)}{(n+3)^2(n+4)} \) | \( \frac{(n_2+1)(n_2+n_3+2)}{(n+3)^2(n+4)} \) | \( \frac{(n_2+1)(n_2+n_3+b+c)}{(n+a+b+c)^2(n+a+b+c+1)} \) | \( \frac{(n_2+1)(n_2+n_3+b+c)}{(n+a+b+c)^2(n+a+b+c+1)} \) |

### Table 6: Limiting Expressions for the Posterior risks as \( t \rightarrow \infty \) using the UP and the JP under PLF

| Parameters | UP | JP |
|------------|----|----|
| \( \beta_1 \) | \( \frac{2(n_1+1)}{(\sum_{k=1}^{n_1} x_{1k})} \left\{ \frac{(n_1+2)\frac{1}{2}}{(n_1+1)\frac{1}{2}} - 1 \right\} \) | \( \frac{2(n_1+1)}{(\sum_{k=1}^{n_1} x_{1k})} \left\{ \frac{(n_1+1)\frac{1}{2}}{(n_1+1)\frac{1}{2}} - 1 \right\} \) |
| \( \beta_2 \) | \( \frac{2(n_2+1)}{(\sum_{k=1}^{n_2} x_{2k})} \left\{ \frac{(n_2+2)\frac{1}{2}}{(n_2+1)\frac{1}{2}} - 1 \right\} \) | \( \frac{2(n_2+1)}{(\sum_{k=1}^{n_2} x_{2k})} \left\{ \frac{(n_2+1)\frac{1}{2}}{(n_2+1)\frac{1}{2}} - 1 \right\} \) |
| \( \beta_3 \) | \( \frac{2(n_3+1)}{(\sum_{k=1}^{n_3} x_{3k})} \left\{ \frac{(n_3+2)\frac{1}{2}}{(n_3+1)\frac{1}{2}} - 1 \right\} \) | \( \frac{2(n_3+1)}{(\sum_{k=1}^{n_3} x_{3k})} \left\{ \frac{(n_3+1)\frac{1}{2}}{(n_3+1)\frac{1}{2}} - 1 \right\} \) |
| \( p_1 \) | \( \frac{2(n_1+a)}{(n+a+b+c)} \left\{ \frac{(n_1+a+1)\frac{1}{2}}{(n_1+a)\frac{1}{2}} - 1 \right\} \) | \( \frac{2(n_1+a)}{(n+a+b+c)} \left\{ \frac{(n_1+a)\frac{1}{2}}{(n_1+a)\frac{1}{2}} - 1 \right\} \) |
| \( p_2 \) | \( \frac{2(n_2+b)}{(n+a+b+c)} \left\{ \frac{(n_2+b+1)\frac{1}{2}}{(n_2+b)\frac{1}{2}} - 1 \right\} \) | \( \frac{2(n_2+b)}{(n+a+b+c)} \left\{ \frac{(n_2+b)\frac{1}{2}}{(n_2+b)\frac{1}{2}} - 1 \right\} \) |

### Table 7: Limiting Expressions for the Posterior risks as \( t \rightarrow \infty \) using the EP and ILP under PLF

| Parameters | Exponential Prior | Inverse Levy Prior |
|------------|-------------------|-------------------|
| \( \beta_1 \) | \( \frac{2(n_1+1)}{(\sum_{k=1}^{n_1} x_{1k} + k_1)} \left\{ \frac{(n_1+2)\frac{1}{2}}{(n_1+1)\frac{1}{2}} - 1 \right\} \) | \( \frac{2(n_1+1)}{(\sum_{k=1}^{n_1} x_{1k} + a_1)} \left\{ \frac{(n_1+1)\frac{1}{2}}{(n_1+1)\frac{1}{2}} - 1 \right\} \) |
| \( \beta_2 \) | \( \frac{2(n_2+1)}{(\sum_{k=1}^{n_2} x_{2k} + k_2)} \left\{ \frac{(n_2+2)\frac{1}{2}}{(n_2+1)\frac{1}{2}} - 1 \right\} \) | \( \frac{2(n_2+1)}{(\sum_{k=1}^{n_2} x_{2k} + a_2)} \left\{ \frac{(n_2+1)\frac{1}{2}}{(n_2+1)\frac{1}{2}} - 1 \right\} \) |
| \( \beta_3 \) | \( \frac{2(n_3+1)}{(\sum_{k=1}^{n_3} x_{3k} + k_3)} \left\{ \frac{(n_3+2)\frac{1}{2}}{(n_3+1)\frac{1}{2}} - 1 \right\} \) | \( \frac{2(n_3+1)}{(\sum_{k=1}^{n_3} x_{3k} + a_3)} \left\{ \frac{(n_3+1)\frac{1}{2}}{(n_3+1)\frac{1}{2}} - 1 \right\} \) |
| \( p_1 \) | \( \frac{2(n_1+a)}{(n+a+b+c)} \left\{ \frac{(n_1+a+1)\frac{1}{2}}{(n_1+a)\frac{1}{2}} - 1 \right\} \) | \( \frac{2(n_1+a)}{(n+a+b+c)} \left\{ \frac{(n_1+a)\frac{1}{2}}{(n_1+a)\frac{1}{2}} - 1 \right\} \) |
| \( p_2 \) | \( \frac{2(n_2+b)}{(n+a+b+c)} \left\{ \frac{(n_2+b+1)\frac{1}{2}}{(n_2+b)\frac{1}{2}} - 1 \right\} \) | \( \frac{2(n_2+b)}{(n+a+b+c)} \left\{ \frac{(n_2+b)\frac{1}{2}}{(n_2+b)\frac{1}{2}} - 1 \right\} \) |
7. Simulation Study

A comprehensive simulation study was conducted in order to explore the performance of the Bayes estimators, impact of sample size and censoring rate to be appropriate for the model. Samples of sizes n=25, 40, 55 are generated from a 3-component mixture of the Frechet distributions with various set of the parametric values \( \beta_1, \beta_2, \beta_3, p_1 \) and \( p_2 \) fixed as \(( \beta_1, \beta_2, \beta_3, p_1, p_2 ) = (0.50, 1.0, 1.50, 0.30, 0.50), (1.50, 1.0, 0.50, 0.50, 0.30) \). For fixed sample size, test termination time and set of parameters, the simulation is repeated 1000 times and the results are then averaged. Sample of sizes \( p_1n, p_2n \) and \( (1-p_1-p_2)n \) are chosen randomly from first component density \( f_1(x; \theta_1) \), second component density \( f_2(x; \theta_2) \) and third component density \( f_3(x; \theta_3) \), respectively.

Table 8: Limiting Expressions for the Bayes Posterior risks as \( t \to \infty \) using the UP, the JP, the EP and the ILP under DLF

| Parameters | UP | JP | EP | ILP |
|------------|----|----|----|----|
| \( \beta_1 \) | \( \frac{n_1+1}{(n_1+1)(n_1+2)} \) | \( \frac{n_2}{n_1(n_1+1)} \) | \( \frac{n_1+1}{(n_1+1)(n_1+2)} \) | \( \frac{n_1+0.5}{(n_1+0.5)(n_1+1)} \) |
| \( \beta_2 \) | \( \frac{n_2+1}{(n_2+1)(n_2+2)} \) | \( \frac{n_2}{n_3(n_2+1)} \) | \( \frac{n_2+1}{(n_2+1)(n_2+2)} \) | \( \frac{n_2+0.5}{(n_2+0.5)(n_2+1)} \) |
| \( \beta_3 \) | \( \frac{n_3+1}{(n_3+1)(n_3+2)} \) | \( \frac{n_3}{n_3(n_3+1)} \) | \( \frac{n_3+1}{(n_3+1)(n_3+2)} \) | \( \frac{n_3+0.5}{(n_3+0.5)(n_3+1)} \) |
| \( p_1 \) | \( 1 - \frac{(n_1+1)(n_1+4)}{(n_1+2)(n_1+3)} \) | \( 1 - \frac{(n_1+1)(n_1+4)}{(n_1+2)(n_1+3)} \) | \( 1 - \frac{(n_1+a)(n+a+b+c+1)}{(n_1+a+1)(n+a+b+c)} \) | \( 1 - \frac{(n_1+a)(n+a+b+c+1)}{(n_1+a+1)(n+a+b+c)} \) |
| \( p_2 \) | \( 1 - \frac{(n_2+1)(n_2+4)}{(n_2+2)(n_2+3)} \) | \( 1 - \frac{(n_2+1)(n_2+4)}{(n_2+2)(n_2+3)} \) | \( 1 - \frac{(n_2+b)(n+a+b+c+1)}{(n_2+b+1)(n+a+b+c)} \) | \( 1 - \frac{(n_2+b)(n+a+b+c+1)}{(n_2+b+1)(n+a+b+c)} \) |

8. A Real Life Data Application

Crowder et al. (1994) reported the data on fiber failure strength. The breaking strength of fiber section of lengths 5, 12, 30 and 45mm. To elucidate the proposed methodology, we take the data on 3-component, namely 5, 12 and 30mm, respectively. The values are right
censored at 4.0 i.e. \( t=4.0 \). The sample statistics required to evaluate the proposed estimates are as follows:

\[
\begin{align*}
\sum_{k=1}^{r_1} x_{1k}^{-1} &= 5.3125, & \sum_{k=1}^{r_2} x_{2k}^{-1} &= 8.9005, \\
\sum_{k=1}^{r_3} x_{3k}^{-1} &= 13.5522.
\end{align*}
\]

Bayes estimates and Bayes Posterior risks using the UP, the JP, the EP and the ILP under SELF, PLF and DLF are in Table 17 given in appendix.

It is noted that the results gained from real data are compatible with simulation results. The results declare that the execution of the informative prior is better than the non-informative priors. It is also examined that execution of DLF preferred for estimating the component parameters, while SELF better for estimating the proportion parameters.

9. Final Remarks

In this study, the Bayesian estimation of 3-component mixture of the Frechet distributions has been considered assuming the case when the shape parameter is known based on type-I censored data. The purpose of this paper is to find out the appropriate combinations of prior distributions and loss functions to estimate the parameters of the 3-component mixture of the Frechet distributions. We conducted all-encompassing simulation study to find out the relative performance of the Bayes estimators when the shape parameter is assumed to be known. From simulated results, we observed that an increase in the sample size and test termination time provides better Bayes estimators. Furthermore, as sample size increases (decreases) the posterior risks of Bayes estimators decreases (increases) for a fixed test termination time. Also, the DLF is perceived as an appropriate choice for estimating component parameters and SELF is expedient for estimating the proportion parameters. Finally, we deduce that the EP is apt prior in order to estimate the component parameters. When SELF is used, the EP is an appropriate prior for proportion parameters. The similar pattern is examined for the JP when non-informative priors are contemplated.

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## Appendix

### Table 9: Bayes estimates (BEs) and posterior risks (PRs) of 3-component mixture of Frechet distributions using the UP under SELF, PLF and DLF with $\beta_1 = 0.50, \beta_2 = 1.00, \beta_3 = 1.50, p_1 = 0.30, p_2 = 0.50$ and \( t = 15, 20 \)

| t   | n   | Loss Functions | UP  |
|-----|-----|----------------|-----|
|     |     |                | \( \hat{E}_1 \) | \( \hat{E}_2 \) | \( \hat{E}_3 \) | \( \hat{E}_4 \) | \( \hat{E}_5 \) |
| 25  | 25  | SELF           | BE  | 0.64377 | 1.15686 | 2.27870 | 0.32202 | 0.46107 |
|     |     | PLF            | BE  | 0.68530 | 1.22045 | 2.36739 | 0.33451 | 0.47026 |
|     |     | DLF            | BE  | 0.71542 | 1.26668 | 2.74792 | 0.34659 | 0.48023 |
| 15  | 40  | SELF           | BE  | 0.59278 | 1.07921 | 1.89146 | 0.30288 | 0.48644 |
|     |     | PLF            | BE  | 0.61128 | 1.14019 | 2.03863 | 0.31102 | 0.49268 |
|     |     | DLF            | BE  | 0.63787 | 1.14462 | 2.17595 | 0.31951 | 0.49756 |
| 55  | 55  | SELF           | BE  | 0.56527 | 1.06674 | 1.82731 | 0.29273 | 0.49864 |
|     |     | PLF            | BE  | 0.58922 | 1.09999 | 1.87374 | 0.29925 | 0.50284 |
|     |     | DLF            | BE  | 0.59925 | 1.09831 | 2.0063  | 0.30568 | 0.50654 |
| 20  | 40  | SELF           | BE  | 0.63481 | 1.16243 | 2.29449 | 0.32138 | 0.46218 |
|     |     | PLF            | BE  | 0.61961 | 1.11891 | 2.01203 | 0.31088 | 0.49390 |
|     |     | DLF            | BE  | 0.71154 | 1.25977 | 2.65438 | 0.34577 | 0.48139 |
| 55  | 55  | SELF           | BE  | 0.57336 | 1.06672 | 1.79499 | 0.29350 | 0.49884 |
|     |     | PLF            | BE  | 0.58012 | 1.08996 | 1.83687 | 0.29923 | 0.50346 |
|     |     | DLF            | BE  | 0.64474 | 1.15587 | 2.1441  | 0.31853 | 0.49929 |

### Loss Functions

- **UP**
  - \( \beta_1 \)
  - \( \beta_2 \)
  - \( \beta_3 \)
  - \( p_1 \)
  - \( p_2 \)
  - \( t \)

### Table Details

- **BE**: Bayes estimates
- **PR**: Posterior risks
- **SELF**: Self-normalized loss function
- **PLF**: Partial loss function
- **DLF**: Dual loss function
Table 10: Bayes estimates (BEs) and posterior risks (PRs) of 3-component mixture of Frechet distributions using the UP under SELF, PLF and DLF with $\widehat{t}$, $\widehat{u}$, $\widehat{v}$, $\widehat{\beta}$ and $\widehat{\gamma}$.

| t   | n  | Loss Functions | UP  |
|-----|----|----------------|-----|
|     |    |                | $\widehat{F}$ | $\widehat{E}$ | $\widehat{F}$ | $\widehat{E}$ | $\widehat{F}$ | $\widehat{E}$ |
| 25  | 20 | SELF           | BE  | 1.74859 | 1.29527 | 0.75497 | 0.46006 | 0.32346 |
|     |    |                | PR  | 0.270075 | 0.225689 | 0.130652 | 0.009132 | 0.008040 |
| 15  | 40 | PLF            | BE  | 1.83555 | 1.35025 | 0.82037 | 0.47208 | 0.33400 |
|     |    |                | PR  | 0.138592 | 0.14421 | 0.12496 | 0.01961 | 0.024479 |
| 55  | 20 | DLF            | BE  | 1.90234 | 1.44896 | 0.91036 | 0.48072 | 0.34776 |
|     |    |                | PR  | 0.074547 | 0.104087 | 0.146394 | 0.041797 | 0.072502 |
| 20  | 40 | SELF           | BE  | 1.65716 | 1.19814 | 0.65269 | 0.48610 | 0.30248 |
|     |    |                | PR  | 0.143936 | 0.129094 | 0.056772 | 0.006090 | 0.005145 |
| 55  | 20 | PLF            | BE  | 1.70319 | 1.22146 | 0.67259 | 0.49308 | 0.31114 |
|     |    |                | PR  | 0.081165 | 0.092916 | 0.070894 | 0.012431 | 0.016763 |
| 20  | 40 | DLF            | BE  | 1.72312 | 1.30093 | 0.73093 | 0.49883 | 0.32006 |
|     |    |                | PR  | 0.047247 | 0.074745 | 0.102658 | 0.025266 | 0.053523 |
| 55  | 20 | SELF           | BE  | 1.60162 | 1.14253 | 0.59521 | 0.49681 | 0.29308 |
|     |    |                | PR  | 0.089013 | 0.086113 | 0.032680 | 0.003948 | 0.003298 |
| 20  | 40 | PLF            | BE  | 1.64833 | 1.17848 | 0.63863 | 0.50219 | 0.30031 |
|     |    |                | PR  | 0.057169 | 0.069427 | 0.051312 | 0.009029 | 0.012717 |
| 55  | 20 | DLF            | BE  | 1.65163 | 1.21835 | 0.65325 | 0.50761 | 0.30658 |
|     |    |                | PR  | 0.034042 | 0.058143 | 0.079678 | 0.017355 | 0.041676 |
| 20  | 40 | SELF           | BE  | 1.74164 | 1.26138 | 0.75824 | 0.46195 | 0.32211 |
|     |    |                | PR  | 0.263732 | 0.2034 | 0.132482 | 0.008973 | 0.007877 |
| 55  | 20 | PLF            | BE  | 1.82546 | 1.34552 | 0.82642 | 0.47233 | 0.33397 |
|     |    |                | PR  | 0.136515 | 0.141848 | 0.12488 | 0.019230 | 0.024031 |
| 20  | 40 | DLF            | BE  | 1.86253 | 1.45122 | 0.85807 | 0.48077 | 0.34732 |
|     |    |                | PR  | 0.073767 | 0.102882 | 0.145164 | 0.040847 | 0.071070 |
| 55  | 20 | SELF           | BE  | 1.66985 | 1.18325 | 0.65066 | 0.48769 | 0.30177 |
|     |    |                | PR  | 0.143726 | 0.123538 | 0.056842 | 0.005962 | 0.005027 |
| 20  | 40 | PLF            | BE  | 1.68955 | 1.23244 | 0.69396 | 0.49364 | 0.31069 |
|     |    |                | PR  | 0.079878 | 0.092856 | 0.072807 | 0.012222 | 0.016509 |
| 55  | 20 | DLF            | BE  | 1.72569 | 1.26653 | 0.71998 | 0.49894 | 0.31941 |
|     |    |                | PR  | 0.046851 | 0.073979 | 0.101862 | 0.024806 | 0.052370 |
| 20  | 40 | SELF           | BE  | 1.60023 | 1.14286 | 0.60400 | 0.49943 | 0.29279 |
|     |    |                | PR  | 0.094254 | 0.085622 | 0.034621 | 0.004502 | 0.003697 |
| 55  | 20 | PLF            | BE  | 1.62303 | 1.1672 | 0.61780 | 0.50322 | 0.29982 |
|     |    |                | PR  | 0.056158 | 0.067970 | 0.049419 | 0.008894 | 0.012476 |
| 20  | 40 | DLF            | BE  | 1.65811 | 1.20219 | 0.65917 | 0.50655 | 0.30703 |
|     |    |                | PR  | 0.030623 | 0.057547 | 0.003026 | 0.002020 | 0.041193 |
Table 11: Bayes estimates (BEs) and posterior risks (PRs) of 3-component mixture of Frechet distributions using the JP under SELF, PLF and DLF with $\beta_1 = 0.50$, $\beta_2 = 1.00$, $\beta_3 = 1.50$, $p_1 = 0.30$, $p_2 = 0.50$ and $t = 15, 20$.

| t | n | Loss Functions | JP |
|---|---|----------------|----|
|   |   |                | $\hat{E}_1$ | $\hat{E}_2$ | $\hat{E}_3$ | $\hat{E}_4$ | $\hat{E}_5$ |
|   |   | SELF           | BE 0.58091 | 1.07457 | 1.85935 | 0.32217 | 0.46147 |
|   |   | PR 0.051092    | 0.109223 | 0.951593 | 0.007807 | 0.009032 |
| 25 | 25 | PLF            | BE 0.60021 | 1.13312 | 2.08241 | 0.33307 | 0.47157 |
|   |   | PR 0.070939    | 0.092292 | 0.381584 | 0.023884 | 0.019457 |
| 15 | 40 | DLF            | BE 0.65665 | 1.18461 | 2.34335 | 0.34626 | 0.48170 |
|   |   | PR 0.114117    | 0.079845 | 0.176529 | 0.070452 | 0.041015 |
| 55 | 55 | SELF           | BE 0.55630 | 1.05499 | 1.67388 | 0.30230 | 0.48745 |
|   |   | PR 0.029581    | 0.060950 | 0.43985 | 0.004982 | 0.006022 |
|   |   | PLF            | BE 0.56735 | 1.077 | 1.83786 | 0.31062 | 0.49246 |
|   |   | PR 0.045768    | 0.053800 | 0.221624 | 0.016264 | 0.012335 |
|   |   | DLF            | BE 0.59180 | 1.11057 | 1.95818 | 0.319287 | 0.49886 |
|   |   | PR 0.079044    | 0.049363 | 0.117900 | 0.051770 | 0.024977 |
| 25 | 25 | SELF           | BE 0.53448 | 1.02133 | 1.66279 | 0.29362 | 0.49907 |
|   |   | PR 0.019683    | 0.039942 | 0.298843 | 0.003652 | 0.004498 |
|   |   | PLF            | BE 0.55075 | 1.04966 | 1.74973 | 0.29920 | 0.50327 |
|   |   | PR 0.033769    | 0.037669 | 0.158084 | 0.012287 | 0.008977 |
|   |   | DLF            | BE 0.57305 | 1.07198 | 1.8412 | 0.30605 | 0.50608 |
| 40 | 55 | SELF           | BE 0.57603 | 1.07749 | 1.87336 | 0.32097 | 0.46305 |
|   |   | PR 0.049727    | 0.109074 | 0.943486 | 0.007725 | 0.008929 |
|   |   | PLF            | BE 0.57162 | 1.08094 | 1.80660 | 0.31099 | 0.49397 |
|   |   | PR 0.045693    | 0.053343 | 0.217134 | 0.016094 | 0.012096 |
|   |   | DLF            | BE 0.63484 | 1.16903 | 2.27552 | 0.34558 | 0.48200 |
|   |   | PR 0.113312    | 0.079047 | 0.173688 | 0.069736 | 0.040230 |
| 20 | 25 | SELF           | BE 0.54322 | 1.0624 | 1.73359 | 0.30293 | 0.48638 |
|   |   | PR 0.027587    | 0.061416 | 0.464869 | 0.004935 | 0.005931 |
|   |   | PLF            | BE 0.55228 | 1.05684 | 1.72704 | 0.29962 | 0.50377 |
|   |   | PR 0.033652    | 0.037662 | 0.154077 | 0.012207 | 0.008851 |
|   |   | DLF            | BE 0.58417 | 1.10718 | 1.91391 | 0.31895 | 0.49892 |
|   |   | PR 0.078584    | 0.048943 | 0.116287 | 0.051297 | 0.024487 |
| 55 | 40 | SELF           | BE 0.54156 | 1.02793 | 1.64473 | 0.29325 | 0.49877 |
|   |   | PR 0.020176    | 0.039843 | 0.283740 | 0.003626 | 0.004297 |
|   |   | PLF            | BE 0.53905 | 1.03189 | 1.67272 | 0.30641 | 0.49593 |
|   |   | PR 0.025535    | 0.029513 | 0.119855 | 0.009916 | 0.006883 |
|   |   | DLF            | BE 0.56019 | 1.07439 | 1.82138 | 0.30513 | 0.50741 |
|   |   | PR 0.060170    | 0.035413 | 0.086891 | 0.040516 | 0.017614 |
Table 12: Bayes estimates (BEs) and posterior risks (PRs) of 3-component mixture of Frechet distributions using the JP under SELF, PLF and DLF with $\beta_1 = 1.50, \beta_2 = 1.00, \beta_3 = 0.50, p_1 = 0.50, p_2 = 0.30$ and $t = 15, 20, 55$.

| Loss Functions | t  | n  | JP | \( \hat{\beta}_1 \) | \( \hat{\beta}_2 \) | \( \hat{\beta}_3 \) | \( \hat{p}_1 \) | \( \hat{p}_2 \) |
|----------------|----|----|----|-----------------|-----------------|-----------------|----------------|----------------|
| SELF           | 25 | 40 | BE  | 1.62451         | 1.15324         | 0.61133         | 0.46153        | 0.32283        |
|                |    |    | PR  | 0.252026        | 0.201082        | 0.108391        | 0.009131       | 0.008023       |
| PLF            | 15 | 40 | BE  | 1.70086         | 1.21865         | 0.69166         | 0.47090        | 0.33565        |
|                |    |    | PR  | 0.139487        | 0.145222        | 0.124433        | 0.019688       | 0.024467       |
| DLF            | 55 | 40 | BE  | 1.77249         | 1.30880         | 0.75414         | 0.48225        | 0.34643        |
|                |    |    | PR  | 0.080069        | 0.116407        | 0.171114        | 0.041300       | 0.072567       |
| SELF           | 25 | 40 | BE  | 1.58907         | 1.10680         | 0.58480         | 0.48657        | 0.30299        |
|                |    |    | PR  | 0.139066        | 0.118858        | 0.051264        | 0.006084       | 0.005144       |
| PLF            | 15 | 40 | BE  | 1.61493         | 1.14689         | 0.61894         | 0.49342        | 0.31113        |
|                |    |    | PR  | 0.080890        | 0.094580        | 0.073166        | 0.009072       | 0.016798       |
| DLF            | 55 | 40 | BE  | 1.58564         | 1.14535         | 0.60851         | 0.50466        | 0.30500        |
|                |    |    | PR  | 0.036722        | 0.056727        | 0.085868        | 0.018976       | 0.042790       |
| SELF           | 25 | 40 | BE  | 1.62234         | 1.15064         | 0.60662         | 0.46212        | 0.32254        |
|                |    |    | PR  | 0.249683        | 0.199318        | 0.098794        | 0.008987       | 0.007898       |
| PLF            | 15 | 40 | BE  | 1.71172         | 1.20341         | 0.68101         | 0.47208        | 0.33446        |
|                |    |    | PR  | 0.138772        | 0.142427        | 0.121547        | 0.019336       | 0.024137       |
| DLF            | 55 | 40 | BE  | 1.78693         | 1.33034         | 0.76499         | 0.48152        | 0.34689        |
|                |    |    | PR  | 0.079564        | 0.114999        | 0.170062        | 0.040795       | 0.071248       |
| SELF           | 25 | 40 | BE  | 1.58633         | 1.08764         | 0.57886         | 0.48664        | 0.30311        |
|                |    |    | PR  | 0.136908        | 0.112561        | 0.049910        | 0.005969       | 0.005047       |
| PLF            | 15 | 40 | BE  | 1.62130         | 1.14751         | 0.60402         | 0.49260        | 0.31195        |
|                |    |    | PR  | 0.080508        | 0.093075        | 0.070809        | 0.012214       | 0.016453       |
| DLF            | 55 | 40 | BE  | 1.65336         | 1.19010         | 0.64195         | 0.49868        | 0.32036        |
|                |    |    | PR  | 0.049082        | 0.079401        | 0.113552        | 0.024710       | 0.052125       |
| SELF           | 25 | 40 | BE  | 1.57172         | 1.05855         | 0.55101         | 0.49892        | 0.29354        |
|                |    |    | PR  | 0.093999        | 0.077687        | 0.031318        | 0.004417       | 0.003700       |
| PLF            | 15 | 40 | BE  | 1.58530         | 1.10819         | 0.57266         | 0.50527        | 0.29999        |
|                |    |    | PR  | 0.065463        | 0.060803        | 0.039898        | 0.004309       | 0.009323       |
| DLF            | 55 | 40 | BE  | 1.61441         | 1.12986         | 0.60430         | 0.50667        | 0.30620        |
|                |    |    | PR  | 0.037014        | 0.061531        | 0.085999        | 0.019852       | 0.038715       |
Table 13: Bayes estimates (BEs) and posterior risks (PRs) of 3-component mixture of Frechet distributions using the EP under SELF, PLF and DLF with $b_1 = 0.50$, $b_2 = 1.0$, $b_3 = 1.50$, $k_1 = 2.0003$, $k_2 = 3.0030$, $k_3 = 4.0016$, $a = 2.0103$, $b = 1.7607$, $c = 1.50$, $p_1 = 0.30$, $p_2 = 0.50$, $t = 15, 20$

| t | n | Loss Functions | EP |
|---|---|---|---|
| **SELF** | BE | $\hat{b}_1$ | 0.55551 |
| **PR** | $\hat{b}_2$ | 0.92274 |
| **EP** | $\hat{b}_3$ | 0.82568 |
| | $\hat{p}_1$ | 0.33441 |
| | $\hat{p}_2$ | 0.45521 |
| 25 | PLF | BE | 0.038928 |
| | $\hat{b}_1$ | 0.071285 |
| | $\hat{b}_2$ | 0.126697 |
| | $\hat{b}_3$ | 0.007427 |
| | $\hat{p}_1$ | 0.008348 |
| **PR** | 0.038928 |
| 40 | DLF | BE | 0.60325 |
| | $\hat{b}_1$ | 0.96364 |
| | $\hat{b}_2$ | 0.88972 |
| | $\hat{b}_3$ | 0.34532 |
| | $\hat{p}_1$ | 0.46587 |
| **PR** | 0.063094 |
| 55 | SELF | BE | 0.101987 |
| | $\hat{b}_1$ | 0.073424 |
| | $\hat{b}_2$ | 0.152642 |
| | $\hat{b}_3$ | 0.0062567 |
| | $\hat{p}_1$ | 0.038909 |
| **PR** | 0.073225 |
| 20 | PLF | BE | 0.53918 |
| | $\hat{b}_1$ | 0.94815 |
| | $\hat{b}_2$ | 0.97882 |
| | $\hat{b}_3$ | 0.31189 |
| | $\hat{p}_1$ | 0.48209 |
| **PR** | 0.025079 |
| 40 | DLF | BE | 0.56027 |
| | $\hat{b}_1$ | 0.97349 |
| | $\hat{b}_2$ | 1.03919 |
| | $\hat{b}_3$ | 0.31946 |
| | $\hat{p}_1$ | 0.48807 |
| **PR** | 0.041728 |
| 55 | SELF | BE | 0.53334 |
| | $\hat{b}_1$ | 0.96646 |
| | $\hat{b}_2$ | 1.07127 |
| | $\hat{b}_3$ | 0.29926 |
| | $\hat{p}_1$ | 0.49509 |
| **PR** | 0.018262 |
| 20 | PLF | BE | 0.58569 |
| | $\hat{b}_1$ | 0.99463 |
| | $\hat{b}_2$ | 1.10532 |
| | $\hat{b}_3$ | 0.32711 |
| | $\hat{p}_1$ | 0.49421 |
| **PR** | 0.073225 |
| 40 | DLF | BE | 0.57275 |
| | $\hat{b}_1$ | 0.98655 |
| | $\hat{b}_2$ | 0.98501 |
| | $\hat{b}_3$ | 0.35626 |
| | $\hat{p}_1$ | 0.47369 |
| **PR** | 0.073225 |
Table 14: Bayes estimates (BEs) and posterior risks (PRs) of 3-component mixture of Frechet distributions using the EP under SELF, PLF and DLF with $b_1 = 1.50$, $b_2 = 1.0$, $b_3 = 0.50$, $k_1 = 2.0003$, $k_2 = 3.0030$, $k_3 = 4.0016$, $a = 2.0103$, $b = 1.7607$, $c = 1.50$, $p_1 = 0.50$, $p_2 = 0.30$, $t = 15, 20$.

| t  | n  | Loss Functions | EP       |
|----|----|----------------|----------|
|    | 25 | SELF           | BE 1.36721 0.85140 0.46772 0.46301 0.32171 |
|    |    | PR 0.157862 0.089797 0.041536 0.008446 0.007398 |
|    | PLF| BE 1.4189 0.92149 0.50909 0.47024 0.33399 |
|    |    | PR 0.107054 0.098496 0.077229 0.018129 0.022563 |
|    | DLF| BE 1.47634 0.97137 0.56335 0.48053 0.34477 |
|    |    | PR 0.073811 0.10423 0.146148 0.038094 0.066913 |
|    | 15 | SELF           | BE 1.40294 0.93721 0.48385 0.48530 0.30402 |
|    |    | PR 0.101245 0.075063 0.029030 0.005763 0.004883 |
|    | PLF| BE 1.43686 0.95059 0.52373 0.49043 0.31218 |
|    |    | PR 0.068436 0.072228 0.054999 0.011804 0.015799 |
|    | DLF| BE 1.49100 0.99448 0.54173 0.49686 0.32017 |
|    |    | PR 0.047121 0.074741 0.102588 0.023844 0.050124 |
|    | 55 | SELF           | BE 1.43967 0.92606 0.50277 0.49547 0.29381 |
|    |    | PR 0.076162 0.055728 0.023097 0.004015 0.003477 |
|    | PLF| BE 1.47659 0.96460 0.50801 0.50167 0.30105 |
|    |    | PR 0.051274 0.056849 0.041014 0.010081 0.013542 |
|    | DLF| BE 1.49007 1.00929 0.54053 0.49697 0.30087 |
|    |    | PR 0.034447 0.058223 0.079346 0.019807 0.036419 |
|    | 25 | SELF           | BE 1.3559 0.88419 0.47244 0.46168 0.32347 |
|    |    | PR 0.154326 0.095135 0.042023 0.008297 0.007299 |
|    | PLF| BE 1.42931 0.91622 0.51409 0.47185 0.33300 |
|    |    | PR 0.106651 0.097322 0.077748 0.017816 0.022290 |
|    | DLF| BE 1.48645 0.97129 0.54757 0.47943 0.34577 |
|    |    | PR 0.073451 0.102901 0.145475 0.037700 0.065653 |
|    | 20 | SELF           | BE 1.4347 0.91166 0.48972 0.48624 0.30334 |
|    |    | PR 0.104301 0.070267 0.029489 0.005681 0.004792 |
|    | PLF| BE 1.45956 0.94762 0.51860 0.49190 0.31141 |
|    |    | PR 0.068724 0.071368 0.054277 0.011574 0.015574 |
|    | DLF| BE 1.49852 0.99278 0.53416 0.49705 0.31990 |
|    |    | PR 0.046634 0.073803 0.101863 0.023495 0.049426 |
|    | 55 | SELF           | BE 1.43816 0.93488 0.50274 0.49735 0.29321 |
|    |    | PR 0.075478 0.056123 0.023135 0.003680 0.003524 |
|    | PLF| BE 1.45763 0.95834 0.51495 0.50311 0.30110 |
|    |    | PR 0.047920 0.055543 0.041162 0.002523 0.000927 |
|    | DLF| BE 1.48625 0.99030 0.53579 0.50456 0.30614 |
|    |    | PR 0.034065 0.057383 0.078288 0.012819 0.021566 |
Table 15: Bayes estimates (BEs) and posterior risks (PRs) of 3-component mixture of Frechet distributions using the ILP under SELF, PLF and DLF with \( b_1 = 0.50, b_2 = 1.0, b_3 = 1.50, a_1 = 1.9520, a_2 = 2.5321, a_3 = 3.7735, a = 0.2763, b_0 = 0.1167, c = 1.0, p_1 = 0.30, p_2 = 0.50, t = 15, 20.\)

| t | n | Loss Functions | ILP |
|---|---|----------------|-----|
|   |   |                | \( \hat{b}_1 \) | \( \hat{b}_2 \) | \( \hat{b}_3 \) | \( \hat{p}_1 \) | \( \hat{p}_2 \) |
| 25 | 40 | SELF           | BE 0.56523 | 1.01139 | 1.12826 | 0.31465 | 0.45912 |
|    |    | PR 0.044554    | 0.090911 | 0.267451 | 0.008230 | 0.009621 |
|    |    | PLF            | BE 0.59757 | 1.04628 | 1.23254 | 0.32907 | 0.46956 |
|    |    | PR 0.065972    | 0.081507 | 0.211980 | 0.025600 | 0.020770 |
|    |    | DLF            | BE 0.63355 | 1.0946 | 1.33595 | 0.34240 | 0.47914 |
|    |    | PR 0.107596    | 0.076576 | 0.164423 | 0.076680 | 0.044163 |
| 40 | 55 | SELF           | BE 0.53336 | 1.00966 | 1.23515 | 0.29845 | 0.48646 |
|    |    | PR 0.018927    | 0.38060 | 0.169564 | 0.003838 | 0.004994 |
|    |    | PLF            | BE 0.57394 | 1.02998 | 1.31238 | 0.30605 | 0.49318 |
|    |    | PR 0.044519    | 0.049916 | 0.152275 | 0.017071 | 0.012805 |
|    |    | DLF            | BE 0.59161 | 1.06348 | 1.40867 | 0.31462 | 0.49928 |
|    |    | PR 0.076175    | 0.047963 | 0.112568 | 0.055395 | 0.026073 |
| 55 | 25 | SELF           | BE 0.57062 | 1.00137 | 1.12997 | 0.31516 | 0.45869 |
|    |    | PR 0.044086    | 0.087920 | 0.263408 | 0.008132 | 0.009446 |
|    |    | PLF            | BE 0.59099 | 1.03025 | 1.30701 | 0.29622 | 0.50312 |
|    |    | PR 0.032747    | 0.036346 | 0.121035 | 0.015682 | 0.014800 |
|    |    | DLF            | BE 0.57377 | 1.04171 | 1.42649 | 0.30118 | 0.50569 |
|    |    | PR 0.058687    | 0.034896 | 0.085874 | 0.044595 | 0.019809 |
| 20 | 40 | SELF           | BE 0.54405 | 1.00478 | 1.24465 | 0.29727 | 0.48533 |
|    |    | PR 0.026580    | 0.053027 | 0.20577 | 0.005089 | 0.006168 |
|    |    | PLF            | BE 0.56536 | 1.02789 | 1.33523 | 0.30662 | 0.49226 |
|    |    | PR 0.043443    | 0.049528 | 0.152538 | 0.016898 | 0.012658 |
|    |    | DLF            | BE 0.58566 | 1.05776 | 1.41234 | 0.31451 | 0.49793 |
|    |    | PR 0.075514    | 0.047664 | 0.110136 | 0.054559 | 0.025622 |
| 55 | 20 | SELF           | BE 0.53292 | 1.0021 | 1.31814 | 0.28968 | 0.49703 |
|    |    | PR 0.018825    | 0.037525 | 0.168365 | 0.003640 | 0.004276 |
|    |    | PLF            | BE 0.54706 | 1.02466 | 1.37435 | 0.29611 | 0.50310 |
|    |    | PR 0.032300    | 0.035835 | 0.118065 | 0.012781 | 0.008715 |
|    |    | DLF            | BE 0.56482 | 1.03511 | 1.43552 | 0.29579 | 0.48353 |
|    |    | PR 0.058186    | 0.034708 | 0.084337 | 0.040373 | 0.016322 |
Table 16: Bayes estimates (BEs) and posterior risks (PRs) of 3-component mixture of Frechet distributions using the ILP under SELF, PLF and DLF with \( b_1 = 1.50, b_2 = 1.0, b_3 = 0.50, a_1 = 1.9520, a_2 = 2.5321, a_3 = 3.7735, a = 0.2763, b = 0.1167, c = 1.0, p_1 = 0.50, p_2 = 0.30, t = 15, 20.\)

| t  | n  | Loss Functions | BE  | PR  | ILP  |
|----|----|----------------|-----|-----|------|
|    |    | SELF           | \( \hat{\beta}_1 \) | \( \hat{\beta}_2 \) | \( \hat{\beta}_3 \) | \( \hat{\beta}_1 \) | \( \hat{\beta}_2 \) |
| 25 | 25 | 1.50074        | 0.20167 | 0.121795 | 0.077116 |
|    |    | 1.0113         | 0.139007 | 0.121661 | 0.109754 |
|    |    | 0.52618        | 0.059989 | 0.097373 | 0.158355 |
|    |    | 0.46474        | 0.009709 | 0.020884 | 0.043828 |
|    |    | 0.30720        | 0.008305 | 0.026615 | 0.081399 |
| 40 | 40 | 1.50733        | 0.121982 | 0.074183 | 0.048309 |
|    |    | 1.04203        | 0.084245 | 0.077631 | 0.107985 |
|    |    | 0.56073        | 0.037530 | 0.026403 | 0.026486 |
|    |    | 0.49484        | 0.006323 | 0.009709 | 0.006501 |
|    |    | 0.30991        | 0.005249 | 0.026670 | 0.065501 |
| 55 | 55 | 1.50871        | 0.048111 | 0.035235 | 0.035235 |
|    |    | 1.09042        | 0.059862 | 0.082028 | 0.059862 |
|    |    | 0.59615        | 0.037530 | 0.020493 | 0.020493 |
|    |    | 0.50119        | 0.006323 | 0.047445 | 0.026670 |
| 20 | 20 | 1.54108        | 0.190802 | 0.051611 | 0.035235 |
|    |    | 1.05272        | 0.062028 | 0.077631 | 0.107985 |
|    |    | 0.55675        | 0.037530 | 0.026645 | 0.026486 |
|    |    | 0.50402        | 0.006323 | 0.023686 | 0.006501 |
|    |    | 0.29269        | 0.005249 | 0.026670 | 0.065501 |
| 40 | 40 | 1.55419        | 0.119934 | 0.051611 | 0.035235 |
|    |    | 1.06093        | 0.118052 | 0.076170 | 0.059862 |
|    |    | 0.56901        | 0.048111 | 0.047445 | 0.020493 |
|    |    | 0.50656        | 0.037530 | 0.047445 | 0.026670 |
|    |    | 0.29809        | 0.006323 | 0.023686 | 0.006501 |
| 55 | 55 | 1.41089        | 0.121835 | 0.051611 | 0.035235 |
|    |    | 1.05272        | 0.118052 | 0.076170 | 0.059862 |
|    |    | 0.55675        | 0.048111 | 0.047445 | 0.020493 |
|    |    | 0.50402        | 0.037530 | 0.023686 | 0.006501 |
|    |    | 0.29269        | 0.026670 | 0.026670 | 0.065501 |

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Table 17: Bayes estimates (BEs) and posterior risks (PRs) of 3-component mixture of Frechet distributions using the UP, the JP, the EP and the ILP under SELF, PLF and DLF with Crowder (1994) mixture data.

| Prior | Loss Functions | $\hat{\beta}_1$ | $\hat{\beta}_2$ | $\hat{\beta}_3$ | $\hat{\beta}_4$ | $\hat{\beta}_5$ |
|-------|----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| **UP** | SELF           | 4.40085         | 3.52976         | 2.89294         | 0.25687         | 0.33460         |
|        | PR             | 0.81291         | 0.398116        | 0.21769         | 0.002437        | 0.002781        |
|        | PLF            | 4.49226         | 3.58571         | 2.93033         | 0.26157         | 0.33873         |
|        | PR             | 0.182818        | 0.111902        | 0.074766        | 0.009401        | 0.008261        |
|        | DLF            | 4.58557         | 3.64254         | 2.96819         | 0.26636         | 0.34292         |
|        | PR             | 0.040282        | 0.030964        | 0.025352        | 0.035618        | 0.024239        |
| **JP** | SELF           | 4.22572         | 3.42494         | 2.82415         | 0.25659         | 0.33463         |
|        | PR             | 0.784679        | 0.387813        | 0.21316         | 0.002438        | 0.002786        |
|        | PLF            | 4.31757         | 3.48109         | 2.86164         | 0.26130         | 0.33876         |
|        | PR             | 0.183695        | 0.112312        | 0.074980        | 0.009416        | 0.008273        |
|        | DLF            | 4.41141         | 3.53817         | 2.89963         | 0.26609         | 0.34295         |
|        | PR             | 0.042093        | 0.032003        | 0.026030        | 0.035712        | 0.024273        |
| **EP** | SELF           | 3.29247         | 2.72831         | 2.28724         | 0.26750         | 0.33895         |
|        | PR             | 0.137801        | 0.086027        | 0.058819        | 0.007456        | 0.006175        |
|        | PLF            | 3.36137         | 2.72831         | 2.28724         | 0.26750         | 0.33895         |
|        | PR             | 0.137801        | 0.086027        | 0.058819        | 0.007456        | 0.006175        |
|        | DLF            | 3.43171         | 2.77201         | 2.31703         | 0.27128         | 0.34207         |
|        | PR             | 0.040575        | 0.031283        | 0.025551        | 0.027679        | 0.018135        |
| **ILP** | SELF          | 3.68196         | 3.06762         | 2.5327          | 0.25202         | 0.33045         |
|        | PR             | 0.589974        | 0.308466        | 0.16967         | 0.002605        | 0.003082        |
|        | PLF            | 3.76123         | 3.11749         | 2.56598         | 0.25714         | 0.33508         |
|        | PR             | 0.158527        | 0.099745        | 0.066555        | 0.010233        | 0.009262        |
|        | DLF            | 3.8422          | 3.16817         | 2.5997          | 0.26236         | 0.33978         |
|        | PR             | 0.041704        | 0.031739        | 0.025769        | 0.039401        | 0.027451        |