Study on nonlinear parametric vibration of footbridge considering time-lag effect

De-yuan Deng¹, Guo-rui Chen², Zhou Chen³*, Xiao-qing Zhang³, Jian-xin Lu¹

¹China Construction Steel Structure Corp. Ltd, Guangzhou, China
²Zhe Jiang Jiaogong Highway Maintenance Co., LTD, Hangzhou, China
³Foshan University, School of Transportation and Civil Engineering & Architecture, Foshan, China

*Corresponding author e-mail: gscz19861985@126.com

Abstract: Under the action of pedestrian dynamic load, a pedestrian bridge with large span and light flexibility will show nonlinear characteristics of the structure, and there is a time lag phenomenon in the feedback of the pedestrian's force on the pedestrian bridge. Therefore, based on the derivation of the nonlinear dynamic model of the parametric vibration of pedestrian bridge, the time lag effect is introduced. Taking the Millennium Bridge of London as an example, the influence of time-lag on parametric vibration of pedestrian bridge is analyzed. By using Galerkin method and multi-scale perturbation method, theoretical and numerical analysis of large vibration induced by parametric vibration and forced vibration of pedestrian bridge is carried out. The amplitude frequency curve of nonlinear parametric vibration with time-lag effect is obtained, and it is concluded that the time-lag of nonlinear parametric vibration in Millennium Bridge has no effect on the amplitude of the structure, but only affects the peak arrival time and the time of stable amplitude. And the conclusion is proved by the numerical analysis.

1 Introduction

In order to achieve a beautiful shape and to beautify the landscape, more and more large-span soft structures are used in the design and construction of the footbridge. For example, the main span of Millennium Bridge in London is 144 meters, and the main span of the S-type pedestrian bridge of Mianyang bridge in Sichuan reaches 200 meters, so some footbridges will show the nonlinear characteristics of the structure under the dynamic loading of pedestrians. At present, the vibration of the footbridge is mainly concentrated on linear vibration, and there are few studies on nonlinear vibration. In the process of the large amplitude vibration of footbridge, the geometric nonlinearity often occurs, and the amplitude of the vibration cannot be obtained in the linear vibration model. Therefore, it is necessary to consider the geometric nonlinearity of the pedestrian bridge and study it. Nakamura considered the situation that pedestrians stop or slow down in the case of large swings. Based on the pedestrian-bridge interaction, the nonlinear variation of modal resonance excitation force is proposed [1]. Sun Li-min and Yuan Xu-bin considered the self-excitation of the synchronous crowd and the forced vibration of the unsynchronized crowd. Assuming that the synchronization probability and the amplitude are nonlinear, a nonlinear model of human-induced vibration was established [2]. Blekherman considered Millennium Bridge London as a double pendulum and studied its nonlinear internal resonance [3]. Liu Long considered the Millennium Bridge as a bifilar pendulum model, and
proposes that the vertical force can also cause lateral dynamic buckling [4].

In the process of pedestrian walking, the force of the pedestrian to bridge cannot get the reaction on the footbridge immediately, there is a time-lag, which is the phenomenon of time-lag. There are three main reasons for the time-lag. One is the influence of the pedestrian’s own factors, that is, the reaction time of people. The second is the contact between the pedestrian and the bridge, the pace of pedestrian adjustment and the vibration of the bridge cannot be reached at the same time. The third is the parameters of the footbridge itself, such as the type, span, mass, stiffness of the footbridge, the characteristics and thickness of the bridge deck material, etc. The results show that the thinner and harder the contact material of pedestrian and bridge, the smaller the time lag [7,8]. At present, the research on time-lag mainly focuses on the use of time-lag for vibration control, and the time-lag phenomenon helps to reduce the vibration of the structure and change the stability and bifurcation behavior of structural linear and nonlinear vibration [9-11]. However, there are few considerations about the phenomenon of time-lag in human-induced bridge vibration. In the present reported literature, only Liu Long and Zhen bin analysis the time-lag of forced vibration to help reduce the vibration of the footbridge by using the Nakamura model [7,8]. Therefore, it is necessary to study the vibration of human-induced bridges when considering the phenomenon of time lag and analyze the influence of time lag effect on the vibration of human bridges. Based on the measured data, the relationship between the dynamic load coefficient and the velocity is fitted in this paper. A nonlinear lateral parameter vibration model based on the relationship between the force and the velocity is proposed and established. The effect of time-lag on the vibration of the bridge is carried out in the case of Millennium Bridge in London.

2 Nonlinear parametric vibration model of Millennium Bridge

For a beam structure subjected to lateral and longitudinal pedestrian loads, as shown in Fig. 1, the micro segment at the distance of x at the end of the beam is assumed to be perpendicular to the axis of the beam before and after the deformation of the cross section. This paper considers the influence of pedestrians on the bridge on the self-weight of the bridge, the influence of the distributed population on the damping of the bridge, and uses the elastic mechanics to analyze the force of the micro-segment in the figure, as shown in Fig. 2, the equilibrium equations for the longitudinal and lateral motions of the centroid of the microsegment according to the displacement method are as follows:

\[
\left( \rho_A + m_{px} \right) \frac{\partial^2 u}{\partial t^2} ds + (\mu_1 + \rho_c \gamma) \frac{\partial u}{\partial t} ds = \frac{\partial}{\partial s} \left( N \cos \alpha + Q \sin \alpha \right) ds + f_h ds \tag{1}
\]

\[
\left( \rho_A + m_{px} \right) \frac{\partial^2 w}{\partial t^2} ds + (\mu_2 + \rho_c \gamma) \frac{\partial w}{\partial t} ds = \frac{\partial}{\partial s} \left( N \sin \alpha - Q \cos \alpha \right) ds + f_l ds \tag{2}
\]

Where \( u(x, t) \) is the longitudinal displacement, \( w(x, t) \) is the lateral displacement, \( N(x, t) \) is the axial force of the tangent direction of the neutral layer after the beam deformation, \( Q(x, t) \) is the shear force, \( \rho_k \) is the mass of the main beam unit length of the footbridge, \( m_{px} \) is the quality of pedestrians per meter on the bridge, \( \mu_1, \mu_2 \) are the vertical and horizontal damping coefficient, \( \rho_c \) is the density of the crowd walking on the bridge, \( \gamma \) is the influence coefficient of the crowd damping, \( \alpha \) is the corner of the cross section. \( f_h, f_l \) are the pedestrian forces of a vertical and horizontal direction unit length. The expressions are as follows:
\[ f_l(t) = \lambda a_{11} m_p g \cos(\omega_p t) \]  \hspace{1cm} (3)\\
\[ f_h(t) = \lambda a_{h1} m_p g \cos(\omega_p t) + \lambda a_{h2} m_p g \cos(\omega_p t) = \lambda \sqrt{a_{h1}^2 + a_{h2}^2} m_p g \sin(\omega_p t - \arctan \frac{a_{h1}}{a_{h2}}) \]  \hspace{1cm} (4)\\

Where \( a_{11} \) is the first-order lateral dynamic load coefficient; it is the pedestrian dynamic load coefficient related to the vibration of the footbridge, which is related to pedestrians and bridges, and different pedestrians and different bridges have different values. \( a_{h1} \) is the first-order longitudinal dynamic load coefficient; \( a_{h2} \) is the second-order longitudinal dynamic load coefficient; \( \omega_p \) is the longitudinal and lateral walking frequency of pedestrians. \( \lambda \) is pedestrian synchronous adjustment ratio, \( \lambda \) ignoring the longitudinal dynamic load factor related to pedestrian bridge vibration.

Without considering the influence of the rotation of the beam, the shear force \( Q(x, t) \) and the bending moment \( M(x, t) \) meet the following relations:

\[ Q = \frac{\partial M}{\partial x} = \cos \alpha \frac{\partial M}{\partial x} \]  \hspace{1cm} (5)\\

Substituting equation (5) into (1) and (2):

\[ (\rho A + m_p) \frac{\partial^2 u}{\partial t^2} + (\mu_1 + \rho_c \gamma) \frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left( N \cos \alpha + \frac{\partial M}{\partial x} \cos \alpha \sin \alpha \right) \cos \alpha + f_h(t) \]  \hspace{1cm} (6)\\
\[ (\rho A + m_p) \frac{\partial^2 w}{\partial t^2} + (\mu_2 + \rho_c \gamma) \frac{\partial w}{\partial t} = \frac{\partial}{\partial x} \left( N \sin \alpha - \frac{\partial M}{\partial x} \cos^2 \alpha \right) \cos \alpha + f_l(t) \]  \hspace{1cm} (7)\\

According to the deformation of the beam, keeping the second-order truncation can be obtained:

\[ \sin \theta = \frac{\partial w}{\partial x}, \quad \cos \theta \approx 1 \]  \hspace{1cm} (8)\\

According to the assumption of equation (8), the axial and bending moments are obtained by dividing the cross-sectional area of the beam:

\[ N(x, t) = \iint \sigma(x, z, t) dA = E \iint \left( \frac{\partial u(x, t)}{\partial x} + \frac{1}{2} \frac{\partial w(x, t)}{\partial x} \right)^2 \cos \theta (x, t) dA = EA \left[ \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right] \]  \hspace{1cm} (9)\\
\[ M(x, t) = \iint \sigma(x, z, t) z dA = E \iint z \left( \frac{\partial u(x, t)}{\partial x} + \frac{1}{2} \left( \frac{\partial w(x, t)}{\partial x} \right)^2 \right) \cos \theta (x, t) dA = EI \frac{\partial \theta}{\partial x} \]  \hspace{1cm} (10)\\

From \( \tan \theta = \frac{\partial w}{\partial x} \) it may be deduced that: \( \frac{\partial \theta}{\partial x} = \cos^2 \theta \frac{\partial^2 w}{\partial x^2} \), so:

\[ M(x, t) = EI \frac{\partial^2 w}{\partial x^2} \cos^2 \theta \]  \hspace{1cm} (11)\\

Here take the second-order truncation, so:

\[ \cos \theta \approx 1 - \left( \frac{\partial w}{\partial x} \right)^2 \]  \hspace{1cm} (12)\\

Substituting equations (11) and (12) into equations (6) and (7) yields a dynamic equation containing only longitudinal and lateral displacements as unknowns:

\[ (\rho A + m_{px}) \frac{\partial^2 u}{\partial t^2} + (\mu_1 + \rho_c \gamma) \frac{\partial u}{\partial t} = EA \frac{\partial^2 u}{\partial x^2} + EA \frac{\partial w}{\partial x} \frac{\partial^2 w}{\partial x^2} + EI \frac{\partial^4 w}{\partial x^4} \] \hspace{1cm} (13)\\
\[ + EI \frac{\partial^4 w}{\partial x^4} - 7EI \frac{\partial^3 w}{\partial x^3} \frac{\partial^2 w}{\partial x^2} \frac{\partial w}{\partial x} - 4EI \frac{\partial^2 w}{\partial x^2} \frac{\partial w}{\partial x} + f_h(t) \]

\[ (\rho A + m_{px}) \frac{\partial^2 w}{\partial t^2} + (\mu_2 + \rho_c \gamma) \frac{\partial w}{\partial t} + EI \frac{\partial^4 w}{\partial x^4} = EA \frac{\partial^2 u}{\partial x^2} + EA \frac{\partial u}{\partial x} \frac{\partial^2 w}{\partial x^2} + 3 \frac{3}{2} EA \left( \frac{\partial w}{\partial x} \right)^2 \frac{\partial^2 w}{\partial x^2} \] \hspace{1cm} (14)\\

The equation (13) and (14) are the longitudinal and transverse coupling dynamic equation after the geometric nonlinear effect of the footbridge. Because the longitudinal stiffness of the beam is far greater than the lateral stiffness, the longitudinal and lateral coupling vibration of the beam is not considered, that is, the effect of the lateral motion of the beam on the longitudinal motion is ignored, and the equation becomes a linear vibration equation:

\[ (\rho A + m_{px}) \frac{\partial^2 u}{\partial t^2} + (\mu_1 + \rho_c \gamma) \frac{\partial u}{\partial t} - EA \frac{\partial^2 u}{\partial x^2} = f_h \]  \hspace{1cm} (15)\\

Excluding the longitudinal inertia of the beam, then the axial force is:\( N(x, t) = EA \frac{\partial u}{\partial x} = -f_h l \).
therefore
\[
\frac{\partial \gamma}{\partial x} = EA \frac{\partial^2 u}{\partial x^2} = 0 \tag{16}
\]

Therefore, considering only its lateral vibration, the equation is as follows:
\[
(\rho A + m_p) \frac{\partial^2 w}{\partial t^2} + (\mu_2 + \rho_c \gamma) \frac{\partial w}{\partial t} + EI \frac{\partial^4 w}{\partial x^4} = EA \frac{\partial^2 u}{\partial x^2} + EA \frac{\partial u}{\partial x} \frac{\partial^2 w}{\partial x^2} + \frac{3}{2} EA \left( \frac{\partial w}{\partial x} \right)^2 \frac{\partial^2 w}{\partial x^2} + 6EI \frac{\partial^3 w}{\partial x^3} \frac{\partial w}{\partial x} + 2EI \left( \frac{\partial^2 w}{\partial x^2} \right)^3 + f(t) \tag{17}
\]

Equation (17) is a nonlinear parameter vibration equation considering the influence of pedestrians on the quality and damping of footbridges. The nonlinear factors mainly come from two aspects. First, the modal quality is composed of the mass of the footbridge and the quality of the pedestrians on the bridge. Second, the geometric nonlinearity of the pedestrian bridge structure.

3 Lateral nonlinear vibration analysis of mid-span bridge in London Millennium Bridge

3.1 Lateral nonlinear vibration analysis of mid-span bridge in London Millennium Bridge

For the reason that the large lateral vibration of Millennium Bridge in London is explained mainly from the angle of parameter vibration. It is forced vibration if the second-order factor is considered, and the second-order frequency is not in the range of lateral swinging frequency when pedestrians walk normally, so only the first order mode is considered. Let the first-order mode function be \( \phi(x) = \sin \frac{\pi x}{L} \), and the solution of the equation can be written as:
\[
w(x,t) = w_1(t) \sin \frac{\pi x}{L} \tag{18}
\]

Where \( w_1(t) \) is the generalized coordinate of the first-order mode. Substituting equation (18) into equation (17) and discretizing it by Galerkin method, the vibration differential equation of the single-mode of the Millennium Bridge considering the influence of the crowd on the bridge quality is obtained, so that it satisfies the following formula:
\[
\ddot{w}_1(x,t) + \dot{\gamma}_1 \dot{w}_1(x,t) + \omega_1^2 w_1(x,t) - \beta_1 x_1(x,t) \cos(\omega_p t) + \beta_2 x_1^3(x,t) - F_0 \cos(\omega_p t) = 0 \tag{19}
\]

Where \( \dot{\gamma}_1 = 2\zeta_0 \omega_1, \omega_1 = \frac{EI\pi^4}{16 \rho (\rho A + m_p)}, \beta_1 = \frac{\pi^2 \lambda_1 m_p g}{81 \rho^2 (\rho A + m_p)}, \beta_2 = \frac{10EI\pi^6 + 3\pi^4 E A}{81 \rho^2 (\rho A + m_p)}, F_0 = \frac{4\lambda_1 m_p g}{\pi (\rho A + m_p)}. \)

Where \( \omega_1 \) is the frequency of the structure considering the impact of mass quality, which is related to the number of people on the deck. In this paper, according to the data measured by Dallard, the linear relationship between the lateral force of the pedestrian on the bridge and the lateral vibration velocity of the structure where the pedestrian is located is made. Therefore, this paper proposes a geometrically nonlinear parametric vibration analysis, in which the force acting on the pedestrian bridge is linearly related to the lateral vibration velocity of the pedestrian bridge. At the same time, since the lateral motion of the pedestrian is harmonic motion, therefore:
\[
f_l(t) = \lambda \left[ \alpha_{l1} + \alpha_l \frac{\partial w(x,t)}{\partial t} \right] m_p g \cos(\omega_p t) \tag{20}
\]

Where \( \alpha_{l1} \) is the fixed platform dynamic coefficient, \( \alpha_l \) is a dynamic load factor related to the lateral velocity of the footbridge. Different bridges have different values, and different \( \alpha_l \) will cause different amplitudes of the footbridge.

According to the experimental data in [6], this paper fits the linear relationship between the dynamic load coefficient and the velocity. According to the result of curve fitting: \( \alpha_{l1} = 0.04; \alpha_l = 0.7. \)
3.2 Approximate solution of parametric resonance

The multiscale method is used to solve the equation (19). Since the pedestrian excitation frequency is not near \( \omega_1 \), small parameters are introduced for the damping term, the parameter excitation term and the nonlinear term, and the following scale transformation is performed:

\[
0 \leq \varepsilon \ll 1, \quad \hat{\zeta}_1 = \varepsilon \zeta_1, \quad \hat{\beta}_1 = \varepsilon \beta_1, \quad \hat{\beta}_2 = \varepsilon \beta_2, \quad \mu = 0
\]

The simultaneous equation (20), then the equation (19) can be rewritten as:

\[
\ddot{w}_1(t) + \varepsilon \zeta_1 \dot{w}_1(t) - \varepsilon \zeta_2 \cos(\omega_p t) \dot{w}_1(t) + \omega_1^2 w_1(t) - \varepsilon \beta_1 w_1(t) \cos(\omega_p t) + \varepsilon \beta_2 w_1^3(t) - \varepsilon F_0 \cos(\omega_p t) = 0 \tag{21}
\]

Where \( \hat{\zeta}_2 = \varepsilon \zeta_2, \quad \hat{\zeta}_1 = \frac{\lambda \varepsilon \mu m_0 \rho}{\rho A^3 m_p} \). The third item of the above equation is the parameter vibration of the crowd’s force acting on the footbridge due to the vibration speed of the footbridge. The fifth item is the parameter vibration caused by the longitudinal load of the pedestrian, and the seventh item is the forced vibration of the pedestrian walking.

4 Nonlinear parameter vibration analysis considering time-lag effect

4.1 Nonlinear parametric vibration theory considering time-lag effect

Based on the nonlinear dynamic model of the parametric vibration of the footbridge derived from the above, the time lag effect is introduced. Considering that the action of pedestrian forces on the bridge has a lag process, the influence of the lag on the parametric vibration of the pedestrian bridge is analyzed. The time lag that the pedestrian's effect on the footbridge is reflected on the bridge is \( \tau \). Ignoring the influence of the longitudinal direction, equation (22) is the nonlinear vibration equation of the lateral parameters of the beam bridge considering the time-lag phenomenon:

\[
\ddot{w}_1(t) + \varepsilon \zeta_1 \dot{w}_1(t) - \varepsilon \zeta_2 \cos(\omega_p(t - \tau)) \dot{w}_1(t) + \omega_1^2 w_1(t) + \varepsilon \beta_2 w_1^3(t) - \varepsilon F_0 \cos(\omega_p(t - \tau)) = 0 \tag{22}
\]

Solving equation (22) using the multi-scale method, the approximate solution of the equation (22) is as follows:

\[
w_{11}(t) = u_0(T_0, T_1) + \varepsilon u_1(T_0, T_1) \tag{23}
\]

\[
w_{11r}(t) = u_{0r}(T_0, T_1) + \varepsilon u_{1r}(T_0, T_1) \tag{24}
\]

Substituting the equations (23) and (24) into the equation (22), and making the coefficients of the small parameters \( \varepsilon \) on both sides of the equation equal, the following linear partial differential equations can be obtained.

\[
\varepsilon^0: \quad D_0^2 u_0 + \omega_1^2 u_0 = 0 \tag{25}
\]

\[
\varepsilon^1: \quad D_0^2 u_1 + \omega_1^2 u_1 = -2D_0 D_2 u_0 - \mu_1 D_2 u_0 + \mu_2 D_0 u_0 \cos(\omega_p(t - \tau)) - \beta_2 \mu_1^2 + F_0 \cos(\omega_p(t - \tau)) \tag{26}
\]

The solution of the above equation (25) is as follows:

\[
u_0(T_0, T_1) = A(T_1)e^{j\omega_1 T_0} + \tilde{A}(T_1)e^{-j\omega_1 T_0} \tag{27}
\]

The time-lag term is expressed as follows:

\[
u_{0r}(T_0, T_1) = A_r(T_1)e^{j\omega_1 (T_0 - \tau)} + \tilde{A}_r(T_1)e^{-j\omega_1 (T_0 - \tau)} \tag{28}
\]
Write $\mu_2 D_2 u_0 \tau \cos(\omega_p(t - \tau))$, $F_0 \cos(\omega_p(t - \tau))$ as the plural form $\frac{1}{2} \mu_2 D_2 u_0 \tau e^{j \omega_p(\tau_0 - \tau)} + cc$, $\frac{1}{2} F_0 e^{j \omega_p(\tau_0 - \tau)} + cc$, and substitute equations (27) and (28) into equation (26) to get:

$$D_2^2 u_1 + \omega_1^2 u_1 = -2D_1 A_j \omega_1 + \mu_2 A_j \omega_1 e^{j \omega_1 T_0} + \frac{\mu_2 j \omega_1}{2} \left( A_T e^{j((\omega_p + \omega_1) T_0 - \tau)} - A_T e^{j((\omega_p - \omega_1) T_0 - \tau)} \right) - \beta_2 (A^3 e^{3j \omega_1 T_0} + 3A^2 A_T e^{j \omega_1 T_0} + F_0 e^{j \omega_p (\tau_0 - \tau)} + cc) \quad (29)$$

Where $cc$ is the conjugate plural of the previous expression. From the equation (29), it is known that the system has parametric vibration at $\omega_p \approx 2 \omega_1$, and the system is forced vibration at $\omega_p \approx \omega_1$. Because the parametric resonance of 1:2 and the forced vibration of 1:1 are not possible at the same time, the following study considers the 1:2 parametric vibration when considering time-lag and feedback adjustment effects.

Introduce a new excitation frequency tuning parameter $\sigma$, and let:

$$\omega_p = 2 \omega_1 + \varepsilon \sigma \quad (30)$$

The conditions for the elimination of the secular term can be obtained by formula (29):

$$2D_1 A_j \omega_1 + \mu_2 A_j \omega_1 + \frac{\mu_2 j \omega_1}{2} A_T e^{j \omega_1 T_1} e^{-j(\omega_1 + \varepsilon \sigma) \tau} + 3 \beta_2 A^2 A_T = 0 \quad (31)$$

For people induced lateral vibration of bridges, the time-lag $\tau$ is not very large. At the very small $\varepsilon$, $A_T$ and $A_T$ can expand according to Taylor expansion as follows:

$$A_T(T_1) = A_T(\varepsilon t) = A_T(T_1 - \varepsilon t) = A_T(T_1) - \varepsilon t A_T'(T_1) + \frac{1}{2} \varepsilon^2 t^2 A_T''(T_1) \approx A_T(T_1) \quad (32)$$

The equation (31) can be written as:

$$2D_1 A_j \omega_1 + \mu_2 A_j \omega_1 + \frac{\mu_2 j \omega_1}{2} A_T(T_1) e^{j \omega_1 T_1} e^{-j(\omega_1 + \varepsilon \sigma) \tau} + 3 \beta_2 A^2 A_T = 0 \quad (34)$$

Write $A(T_1)$ into an exponential form:

$$A(T_1) = \frac{a_1(T_1)}{2} e^{j \gamma_1(T_1)} \quad (35)$$

Substituting equation (35) into (34), comparing the real and imaginary parts, and letting $\phi = \sigma T_1 - 2 \gamma_1$ to get:

$$D_1 \alpha_1 = -\frac{\xi \alpha_1}{2} - \frac{\zeta \alpha_1}{4} \cos \phi \cos((\omega_1 + \varepsilon \sigma) \tau) - \frac{\xi \alpha_1}{4} \sin \phi \sin((\omega_1 + \varepsilon \sigma) \tau) \quad (36)$$

$$D_1 \phi_1 = \sigma + \frac{\zeta \phi_1}{2} \sin \phi \cos((\omega_1 + \varepsilon \sigma) \tau) - \frac{\xi \phi_1}{2} \cos \phi \sin((\omega_1 + \varepsilon \sigma) \tau) - \frac{3 \beta_2 \alpha_1^2}{4 \omega_1} \quad (37)$$

Let $D_1 \alpha_1 = 0, D_1 \phi_1 = 0$ get the following:

$$\frac{\zeta \alpha_1}{2} = -\frac{\zeta \alpha_1}{4} \cos \phi \cos((\omega_1 + \varepsilon \sigma) \tau) - \frac{\xi \alpha_1}{4} \sin \phi \sin((\omega_1 + \varepsilon \sigma) \tau) \quad (38)$$

$$\sigma - \frac{3 \beta_2 \alpha_1^2}{4 \omega_1} = \frac{\zeta \phi_1}{2} \cos \phi \sin((\omega_1 + \varepsilon \sigma) \tau) - \frac{\xi \phi_1}{2} \sin \phi \cos((\omega_1 + \varepsilon \sigma) \tau) \quad (39)$$

Equations (38) and (39) are squared and then add together to obtain:

$$\frac{\zeta^2}{2} + \left( \sigma - \frac{3 \beta_2 \alpha_1^2}{4 \omega_1} \right)^2 = \frac{\xi^2}{4} \quad (40)$$

Equation (40) is the amplitude-frequency curve of the nonlinear parameter vibration considering the time-lag effect. Considering the value of $\beta_1$ is small, therefore, in the nonlinear parameter vibration of the Millennium Bridge, the time lag has no effect on the amplitude of the structure.

### 4.2 Numerical analysis

Taking 250 examples of Millennium Bridge as an example, the time history analysis of the parametric vibration of Millennium Bridge considering time-lag is carried out. Fig. 4 gives the time history curve of displacement response with time-lag $\tau = 0.2, \tau = 0.5, \tau = 0.8, \tau = 1.0$. The initial condition is $w(0) = 0.001, \dot{w}(0) = 0.001$. 


Fig. 4 Lateral displacement under different time-lag

Fig. 4 is the lateral displacement of Millennium Bridge in London at different time-lag. Comparing Figure 4, it can be seen that the response amplitude of the Millennium Bridge does not vary with the existence of time-lag effects and time-lag. However, it can be seen from the above four diagrams that the time of reaching the peak and the stable amplitude is different, and there is no obvious rule, which reflects the complexity of the time-lag effect on the peak and stable amplitude. Through numerical analysis, it can be concluded that the time-lag has no effect on the peak value and the stable amplitude, but only affects the time to reach the peak and stabilize the amplitude, which is consistent with the theoretical approximate solution. Considering the structural vibration response of time-lag and no time-lag, the number of critical numbers will be the same, which proves the correctness of theoretical analysis. This is because in the parameter vibration, the response of the structure depends on the time variable. Even if there is a time-lag effect, the influence of the time-lag effect on the amplitude can be overwhelmed with time, which is the difference between the parameter vibration and the forced vibration. This also confirms that even considering time-lag, the critical number of people obtained in the previous paragraph that does not suffer from parameter vibration is correct.

5 Conclusions

Taking Millennium Bridge in London as an example, this paper considers the effect of pedestrian mass on the frequency of pedestrian bridge. Based on the measured data, the relationship between the dynamic load coefficient and the velocity is fitted, and a nonlinear lateral parameter vibration model based on the relation between force and velocity is proposed and established. By using Galerkin method and multi-scale perturbation method, the large amplitude vibration of pedestrian bridge caused by the combined action of parametric vibration and forced vibration is analyzed theoretically and numerically. The reasons for the large vibration phenomenon of the Millennium Bridge were studied, and the following conclusions were obtained:

(1) The model can well explain that the phenomenon of large lateral vibration still occurs when the fundamental frequency of the bridge is far from the pedestrian lateral walking frequency. The theoretical calculation of critical number is close to the measured value, which shows that the
established model has certain reliability.

(2) Only considering the force lag of pedestrians on the bridge, the time lag effect has no effect on the amplitude, but has an effect on the time to reach the stable amplitude.

References

[1] Nakamura S., Kawasaki T.. Lateral vibration of footbridge by synchronized walking. Journal of Constructional Steel Research, 2006, 62(11): 1148-1160.

[2] YUAN Xu-bin. Research on pedestrian-induced vibration of footbridge. Shanghai. Tongji University, 2006.

[3] Blekherman AN. Swaying of pedestrian bridges. Journal of Bridge Engineering, 2005, 10(2): 142-50.

[4] LIU Long. Study on Crowd-beam Vertical Interaction and Mechanic of Lateral Vibration of Footbridge. Wuhan University of Technology, 2013.

[5] HU Hai-yan. Applied Nonlinear Dynamics. Beijing, Aviation Industry Press, 2000: 57-60.

[6] P. Dallard, A. J. Fitzpatrick, A. Flint, A. Low, R. M. Ridsdill Smith, M. Willford, The London Millennium Footbridge. The Structure Engineer, 2001, 79: 17-33.

[7] LIU Long, XIE Wei-ping. Influence of Time Delay on the Lateral Vibration of Footbridges Induced by Pedestrians. Journal of Civil Engineering and Management, 2013. 30(1): 6-10.

[8] Zhen Bin, Xie Wei-ping, Xu Jian, Nonlinear Analysis for the Lateral Vibration of Footbridges Induced by Pedestrians. Journal of Bridge Engineering, 2013. 18(1): 122-130.

[9] An Fang, Chen Wei-dong, Shao Min-qiang. Time-Delayed Velocity-Acceleration Feedback for Active Vibration Control of Cantilever Beam. Journal of Vibration, Measurement & Diagnosis, 2012. 32(3): 364-370.

[10] SHANG Hui-lin. Multiple Stable Motions and Their Regions of Attraction the Delay Induces in Nonlinear Dynamical Systems. Shanghai. Tongji University, 2008.

[11] Suqi Ma, Qishao Lu, Zhaosheng Feng, Double Hopf bifurcation for van der Pol-Duffing oscillator with parametric delay feedback control. J. Math. Anal. Appl., 2008. 33: 993-1007.