On the virial coefficients of nonabelian anyons

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Abstract

We study a system of nonabelian anyons in the lowest Landau level of a strong magnetic field. Using diagrammatic techniques, we prove that the virial coefficients do not depend on the statistics parameter. This is true for all representations of all nonabelian groups for the statistics of the particles and relies solely on the fact that the effective statistical interaction is a traceless operator.
In two spatial dimensions the group relevant to the quantum statistics of particles is the braid group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, rather than the permutation group. As a result, the possibility for non-standard statistics exists. A well-studied case is (abelian) anyons, transforming in a unitary abelian representation of the braid group. Anyons in the lowest Landau level, in particular, are relevant to the quantum Hall effect and constitute realizations of ideal exclusion statistics.

A natural generalization is nonabelian anyon statistics, based on nonabelian representations of the braid group. These would be the anyonic analogs of parastatistics. Just as abelian anyons can be thought of as ordinary (bosonic or fermionic) particles interacting through a non-dynamical abelian gauge field, nonabelian anyons can be thought of as particles carrying internal degrees of freedom in some irreducible representation $R$ of a nonabelian group $SU(n)$ and interacting through an appropriate non-dynamical nonabelian gauge field. What fixes the statistics, then, is the group $SU(n)$, the representation $R$ and the coupling strength $g$ of the internal degrees of freedom to the gauge field.

A field-theoretic approach to achieving such statistics, in analogy to the abelian case, is to couple the particles to a nonabelian gauge field with a Chern-Simons action. Such a particle-field model was proposed by Verlinde. $g$, then, is essentially the inverse of the coefficient of the Chern-Simons term and, as such, inherits the quantization condition

$$g = \frac{2}{n}, \quad n \text{ integer} \quad (1)$$

This condition does not seem to be crucial for the purely first-quantized approach and, at any rate, will not play any role in this paper.

It is of interest to derive the thermodynamics and statistical mechanical properties of nonabelian anyons in order to probe the possibility of new physics deriving from the nonabelian nature of the system. In a recent paper, Isakov, Lozano and Ouvry examined these questions for the simplest case of $SU(2)$ anyons in the fundamental (spin-half) representation. They found that the virial coefficients up to the fifth one do not depend on the statistics parameter $g$. They conjectured that this holds for all the coefficients and posed the question of a possible underlying symmetry that explains this vanishing dependence.

The purpose of this note is to give a complete proof of the independence of all virial coefficient of this model on the statistics parameter $g$, valid for any group and any representation. It is based on a diagrammatic expression of the cluster coefficients which is useful in deriving them in a simple way and reveals their scaling properties with the volume. It will be apparent that the only feature of the statistical interaction which is relevant for this result is that it is a traceless operator in the space of internal degrees of freedom of the particles.

We repeat here the main results for the system that will be used in this paper, as presented in [14]. The model consists of $N$ non-interacting spinless particles on the plane with internal degrees of freedom transforming in some finite-dimensional representation $R$ of $SU(n)$.
unitary irreducible representation $R$ of $SU(n)$. (We shall refer to these degrees of freedom as flavor.) In the gauge where the Hamiltonian of the particles is free, the nontrivial statistics manifests in the wavefunction of the system, which is not single valued. Under an exchange of particles following a path belonging to an element of the braid group the wavefunction transforms in some nonabelian representation of the braid group parametrized by the irrep $R$ of $SU(n)$ and a statistics parameter $g$. In principle an abelian part can also be included, parametrized by a second coupling constant $\alpha$, endowing the particles with (abelian) anyonic statistics. The contributions of the abelian and nonabelian part to the statistical mechanism decouple, however, as will be apparent in the sequel, so we are not going to be concerned with the abelian part. The particles are taken as bosons as far as their abelian statistics is concerned.

In analogy with the abelian case, we can perform a singular nonabelian gauge transformation that makes the wavefunction single-valued and bosonic, at the expense of introducing a gauge field coupling the particles. We also introduce an external strong constant magnetic field $B = 2\omega_c/e$, as well as an external rotationally invariant harmonic oscillator potential of frequency $\omega$ (which serves as a ‘box’ to bound the particles). Upon extracting from the wavefunction an analytic nonabelian factor that accounts for its short-distance analytic and braiding behavior and a gaussian factor, we are left with an effective Hamiltonian reading

$$H = \sum_i \left( -2\overline{\partial}_i \partial_i - (\omega_t - \omega_c) z_i \overline{\partial}_i - (\omega_t + \omega_c) z_i \overline{\partial}_i + \omega_t \right) - 2g \sum_{i < j} T^A_i T^A_j \left( \frac{\overline{\partial}_i - \overline{\partial}_j}{z_i - z_j} - \frac{\omega_t - \omega_c}{2} \right)$$

(2)

In the above $z = x + iy$ is a complex coordinate on the plane, $\partial \equiv \partial/\partial z$ is the corresponding derivative, and $\omega_t^2 = \omega_c^2 + \omega^2$. $T^A_i$ are generators of the group $SU(n)$ in the $R$-representation, each acting in the flavor space of particle $i$; so $T^A_i$ are $d_R \times d_R$ dimensional matrices and $A = 1, \ldots, n^2 - 1$. Summation over repeated indices is always implied.

All homogeneous analytic wavefunctions are eigenstates of the above Hamiltonian. When $B > 0$ the analytic wavefunctions become degenerate in the pure magnetic field limit $\omega \to 0$ and constitute the lowest Landau level (LLL) for the system. For large $B$ all higher levels will acquire a large gap and decouple. Good analytic behavior of the wavefunction near coincidence points in that case requires $g > 0$. Conversely, for $B < 0$ we can extract an anti-analytic nonabelian factor from the wavefunction and arrive at an analogous expression for $H$. In that case, it is the anti-analytic wavefunctions that constitute the LLL and we must have $g < 0$. From now on we will consider the case $B, g > 0$, the opposite one being similar. We shall also assume that $g$ is not too big, so that no new states descend to the LLL from the excited spectrum.

The end result is that on states in the LLL the Hamiltonian assumes the form

$$H = H_o + S$$

(3)
where $H_o$ is the Hamiltonian of a non-interacting bosonic system and $S$ is the statistics part coupling the internal degrees of freedom of the particles:

$$H_o = N\omega_t + (\omega_t - \omega_c) \sum_i z_i \partial_i, \quad S = g(\omega_t - \omega_c) \sum_{i<j} T_i^A T_j^A$$

(4)

The spectrum of the above Hamiltonian can be easily obtained. $H_o$ essentially counts the degree of homogeneity in $z_i$ of the analytic wavefunction, which can then be chosen a homogeneous polynomial in $z_i$. $S$ can be expressed as

$$S = g(\omega_t - \omega_c) \left[ \left( \sum_i T_i^A \right)^2 - \sum_i \left( T_i^A \right)^2 \right]$$

(5)

Under the total flavor group with generators $T^A = \sum_i T_i^A$, states transform in the tensor product of $N$ $R$-irreps $R \times \cdots \times R$, which can be decomposed into irreducible components. On states transforming under an irreducible representation $R_t$ of the total flavor, $S$ becomes

$$S = g(\omega_t - \omega_c) \left[ C_2(R_t) - NC_2(R) \right]$$

(6)

where $C_2(R)$, $C_2(R_t)$ are the quadratic Casimir of $R$ and $R_t$. The total wavefunction must carry the $R_t$ representation of total flavor and be symmetric under total particle exchange (coordinate and internal degrees of freedom). This calls for some group theory for constructing the states [14]. From this spectrum the partition function, cluster and virial coefficients can be calculated.

This approach was followed in [14] (for $R$ the spin-half of $SU(2)$) and the first few virial coefficients in the thermodynamic limit were thus calculated. We shall take here an alternative route, based on a diagrammatic expansion. The facts central to the derivation are:

1. The cluster and virial coefficients at the thermodynamic limit can be calculated by taking the strength of the external potential to zero (corresponding to taking the volume $V$ of the ‘box’ to infinity). The correct scaling limit for the $k$-th cluster coefficient is [15] [16]

$$\frac{1}{k\beta(\omega_t - \omega_c)} \rightarrow V \frac{\omega_c}{\pi}$$

(7)

2. The statistical interaction $S$ is of order $1/V$.

3. The statistical interaction $S$ is a sum of two-body terms, each of which is traceless with respect to the internal space of each particle.

We now give the rules for the path-integral representation of the system. (For a more detailed discussion see [17].) The $N$-body partition function $Z_N$ can be expressed as a many-body path integral in periodic Euclidean time $\beta$. For short, we shall call such path-integral configurations diagrams. Since the particles are identical, the configuration at time $\tau = \beta$ can be any permutation of the one at $\tau = 0$. This means that the paths of particles can braid and interchange as they go round the
periodic time direction. Such periodically connected paths of \( p \) particles constitute one ‘thread’ wrapping \( p \) times around the time circle. Appropriate symmetry factors must be included in each diagram to avoid overcounting of degrees of freedom.

Further, since the particles have color degrees of freedom, each path is also labeled by a color index \( a = 1, \ldots, d_R \). Summation over all possible values of such indices is assumed.

The interaction \( S \) can be taken into account perturbatively. It is two-body and instantaneous, so each insertion corresponds to coupling two distinct particle paths at a given time. Since it acts on the flavor space of the two particles, it will change the flavor index on the two paths before and after the interaction, say from \( a \) to \( b \) on one and from \( c \) to \( d \) on the other. The strength of this interaction is given by the matrix element

\[
S_{ab;cd} = g(\omega_t - \omega_c)(T^A)_{ab}(T^A)_{cd}
\]

The symmetry factors of diagrams with such insertions are modified, since the paths connected by \( S \) are obviously singled out. A typical configuration for the path integral in the case of five particles and two insertions of \( S \) is depicted in fig. 1.a, where the constraints of periodicity for the paths and their flavors have been taken into account.

![Fig. 1.a: A typical diagram with five particles (solid lines) and two interactions (dashed lines). It consists of two topologically connected parts dynamically interconnected.](image)

For our purposes only the topology and connectivity of these diagrams will be important, so we will depict them in the simplified fashion of fig. 1b.
Fig. 1.b: The same diagram as above in a simplified, topologically equivalent depiction.

The grand partition function for the system $\mathcal{Z}$ is given by the sum of the $N$-body partition functions for all $N$ weighted by fugacity factors $z^N = e^{\mu \beta N}$ with $\mu$ the chemical potential:

$$\mathcal{Z} = \sum_{N=0}^{\infty} Z_N z^N$$

(9)

As such, it is the sum of all many-body diagrams. The grand potential $\Omega$ is the logarithm of $\mathcal{Z}$ and, by the standard argument, it will be given by the sum of all connected diagrams. Two parts of a diagram are disconnected if the particle paths of each diagram do not mix with the other and if there are no interactions $S$ coupling the two diagrams. The coefficients $b_k$ of the expansion of $\Omega$ in powers of $z$ are the cluster coefficients:

$$\Omega = \sum_{k=1}^{\infty} b_k z^k$$

(10)

Therefore, the $k$-th cluster coefficient $b_k$ is simply the sum of all connected $k$-particle diagrams.

We come, now, to the question of determining $b_k$ in the thermodynamic limit. We must isolate, in the class of $k$-body connected diagrams, the leading contribution in $V$ (or, equivalently, in $(\omega_l - \omega_c)^{-1}$), which, for a proper extensive behavior, must be of order $V$. To achieve this, note that each topologically connected part of a diagram, consisting of a single thread looping $p$ times, in the absence of interaction insertions is of order $V$. Indeed, this is simply the $p$-th cluster coefficient of noninteracting bosons coming in $d_R$ flavors, which is properly of order $V$. (Alternatively, if the infrared regulator were a flat ‘box’ rather than an oscillator potential, the factor $V$ would come from the translation invariance of the connected diagram within the box.)

Thus, if a diagram contains $q$ topologically connected components, it will be, a priori, of order $V^q$. For it to be connected, there must be enough insertions of $S$ to connect the $q$ components to each other. We must have a minimum number of $q - 1$
insertions in order to fully connect the components in a tree-like topology (fig. 2). Since each insertion $S$ contributes a factor $1/V$, such minimally connected diagrams are of order $V$. Any further insertion of $S$ will give a sub-leading in $V$ diagram. In fact, by simple topological counting arguments, we see that the number of loops in non-minimally connected diagrams counts the sub-leading powers of $1/V$.

**Fig. 2:** A typical diagram of leading order $V$, consisting of five components minimally connected by four interactions in a tree topology. Each of its three endpoints carries a unique flavor index.

So far we concluded that $b_k$ will be given by the sum of all minimally connected $k$-particle diagrams with any number of components $q$ ($1 \leq q \leq k$). Now comes the final observation: each tree must have two or more ‘endpoints,’ that is, components where only one insertion of $S$ connects. The entire thread of this diagram must clearly carry the same flavor index $a$; thus, the corresponding matrix element for the insertion $S$ connecting to this diagram is $S_{aa;bc}$, with $b, c$ the flavor indices connecting at the other end of the insertion. Upon summing over $a$ we have

$$S_{aa;bc} = g(\omega_t - \omega_c)(T^A)_{aa}(T^A)_{bc} = g(\omega_t - \omega_c)(T^A)_{bc} \operatorname{tr}T^A = 0$$  

Therefore, all such diagrams vanish. The only surviving diagram is the one with a single topologically connected component and no $S$ insertions, which reproduces the cluster coefficient of free bosons with $d_R$ flavors. Since virial coefficients are uniquely expressed in terms of cluster coefficients, we have proved that the virial coefficients of the system are independent of their nonabelian statistics, for any $R$ of any $SU(n)$.

The above reasoning can also be used to show that the contribution of the abelian part is the same as in the absence of the nonabelian part. An abelian part can be included by appending a $U(1)$ generator $T^0 = Q$ to the $T^A$, proportional to the unit matrix. The trace in the above insertion, then, would give

$$\sum_a S_{aa;bc} = g(\omega_t - \omega_c)(T^0)_{aa}(T^0)_{bc} = gd_R Q^2 (\omega_t - \omega_c) \delta_{bc}$$  

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This is a contribution proportional to an anyonic parameter $\alpha = gQ^2d_R$. The effect of the insertion on the flavor indices of the remaining diagrams is effectively suppressed (since, due to $\delta_{bc}$, $b = c$). Repeating the argument with all endpoint graphs, we eventually reduce the whole graph to a set of components with non-interacting flavor indices and the standard abelian statistics interaction between the components. The cluster coefficients are simply $d_R$ times the single-flavor anyonic coefficients.

We remark here that the above techniques can be used to easily obtain the sub-leading in $1/V$ contributions to the cluster and virial coefficients. To each component of the diagram at least two $S$ insertions are attached (else the diagram vanishes by the previous argument). Summing over the flavor indices on a component where $m$ insertions attach gives a term proportional to

$$g^m(\omega_t - \omega_c)^m D_{R}^{A_1 \cdots A_m}$$

(13)

where the $m$-index symbol $D_R$ is

$$D_{R}^{A_1 \cdots A_m} = \text{tr}(T^{A_1} \cdots T^{A_m})$$

(14)

and the total diagram involves multiplying the $D$-symbols of each component and contracting the group indices $A_i$ according to the connectivity of the components through $S$-insertions. A diagram with $q$ components and $q + s - 1$ insertions will be of subleading order $1/V^s$ and of order $g^{q+s-1}$ in the statistics parameter. Since $q$ can range from 1 to $k$ for a $k$-particle diagram we conclude that the $1/V^s$ correction to the cluster coefficient $b_k$ will be a polynomial in $g$ with powers ranging from $g^s$ to $g^{k+s-1}$.

The task of calculating the above corrections simplifies further in the special case that $R$ is the fundamental of $SU(n)$. In that case, a well-known completeness relation simplifies $S$:

$$S_{ab,cd} = g(\omega_t - \omega_c) \sum_A (T^A)_{ab}(T^A)_{cd} = g(\omega_t - \omega_c) \frac{1}{2}(\delta_{ad}\delta_{cb} - \frac{1}{n}\delta_{ab}\delta_{cd})$$

(15)

So $S$ is the sum of a part that simply interchanges the flavor indices of the strands that it couples plus a part proportional to the identity operator. The evaluation of diagrams in this case becomes a simple matter with no group theory required.

Having described the above, we should still point out that the $1/V$ corrections obtained this way are specific to the ‘harmonic box’ regularization of the system. They are, thus, likely not universal and of little interest.

Concluding, the results of this paper are somewhat disappointing, since they indicate that no new physics are expected at the thermodynamic limit from any nonabelian statistics of particles at the LLL. The result seems completely generic since it relies on little else than the very nonabelian nature of the statistics, that is, the vanishing of the trace of its generators. Still it is expected that nonabelian statistics will influence the properties of systems not in the LLL. A calculation of
the properties of such systems along the lines presented here may be an interesting endeavor.

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