Proper 3-orientations of bipartite planar graphs with minimum degree at least 3

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Abstract

In this short note, we show that every bipartite planar graph with minimum degree at least 3 has proper orientation number at most 3.

Keywords. proper orientation, planar graph, bipartite graph

1 Introduction

For basic terminology in graph theory undefined in this paper, refer to [4]. Let $G$ be a simple graph. An orientation $\sigma$ of $G$ is a digraph obtained from $G$ by replacing each edge by exactly one of the two possible arcs with the same end-vertices. Let $E_\sigma(G)$ be the resulting arc set. For $v \in V(G)$, the indegree of $v$ in $\sigma$, denoted by $d^-_\sigma(v)$, is the number of arcs in $E_\sigma(G)$ incoming to $v$. We denote by $\Delta^-(\sigma)$ the maximum indegree of $\sigma$. An orientation $\sigma$ of $G$ is proper if $d^-_\sigma(u) \neq d^-_\sigma(v)$ for every $uv \in E(G)$. The proper orientation number of $G$, denoted by $\chi^+(G)$, is the minimum integer $k$ such that $G$ admits a proper $k$-orientation. Proper orientation number is defined by Ahadi and Dehghan [1], and see the related research [2, 3, 6]. Knox et al. [6] showed the following.

**Theorem 1** ([6], Theorem 3). Let $G$ be a 3-connected bipartite planar graph. Then $\chi^+(G) \leq 5$.

The main theorems of this paper are as follows. The maximum average degree $\text{Mad}(G)$ of a graph $G$ is defined as

$$\text{Mad}(G) = \max \left\{ \frac{2|E(H)|}{|V(H)|} : H \text{ is a subgraph of } G \right\}.$$

**Theorem 2.** Let $k$ be a positive integer and $G = G[X, Y]$ be a bipartite graph with $\text{Mad}(G) \leq 2k$. If $\deg(x) \geq k + 1$ for every $x \in X$, then $\chi^+(G) \leq k + 1$.

Theorem 2 partially answers Problem 5 in [2], which asks whether $\chi^+(G)$ can be bounded by a function of $\text{Mad}(G)$. Using Theorem 2 we can state the following. The minimum degree of $G$ is denoted by $\delta(G)$.

**Theorem 3.** Let $G$ be a bipartite planar graph with $\delta(G) \geq 3$. Then $\chi^+(G) \leq 3$.

Both of the bounds of $\delta(G)$ and $\chi^+(G)$ in Theorem 3 are tight; see Theorems 5 and 6, respectively. The following corollary immediately follows from Theorem 3 which is the improvement of Theorem 1.

**Corollary 4.** Let $G$ be a 3-connected bipartite planar graph. Then $\chi^+(G) \leq 3$.

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Furthermore, for quadrangulations, which are bipartite planar graphs whose each face is bounded by a 4-cycle, the orientation number is completely determined as follows.

**Theorem 5.** Let $G$ be a quadrangulation with $\delta(G) \geq 3$. Then $\chi^-(G) = 3$ unless $G$ is isomorphic to the 3-cube $Q_3$. If $G$ is isomorphic to $Q_3$, then $\chi^-(G) = 2$.

Finally, we show the tightness of the minimum degree condition in Theorem 3. Note that, when $\delta(G) = 1$, Araujo et al. [2, Corollary 2] showed that there exists a tree $T$ with $\chi^-(T) = 4$.

**Theorem 6.** There exist bipartite planar graphs $G$ with $\delta(G) = 2$ and $\chi^-(G) = 4$.

## 2 Proofs of main theorems

The proof of Theorem 2 is very simple.

**Proof of Theorem 2.** Let $G = G'[X, Y]$ be a bipartite graph with $\text{Mad}(G) \leq 2k$ satisfying $\deg(x) \geq k+1$ for every $x \in X$. By Hakimi’s result [5], every graph $G$ with $\text{Mad}(G) \leq 2k$ has a $k$-orientation $\sigma$. Let $S_\sigma = \{(u, v) \in E(G) \mid u \in X\}$. For every $x \in X$, since $\deg(x) \geq k+1$, we can switch the orientation of some edges in $S_\sigma$ so that the indegree of $x$ is exactly $k+1$. In the resulting orientation $\sigma'$ of $G$, observe that $d^-_{\sigma'}(y) \leq k$ for every $y \in Y$. Thus, $\sigma'$ is a proper $(k+1)$-orientation of $G$.

To prove Theorem 3 we use the following well-known lemma.

**Lemma 7.** Let $G$ be a bipartite planar graph with $n \geq 4$ vertices. Then $|E(G)| \leq 2n - 4$ with equality holding if and only if $G$ is a quadrangulation.

**Proof of Theorem 3.** Let $G$ be a bipartite planar graph with $\delta(G) \geq 3$. By Lemma 7, $G$ has maximum average degree less than 4. Thus, $\chi^-(G) \leq 3$ by Theorem 2.

**Proof of Theorem 6.** Let $G[X, Y]$ be a quadrangulation with $\delta(G) \geq 3$, where $|X| \geq |Y|$. Note that $n = |X| + |Y| \geq 8$ and $|Y| \geq 3$. By Theorem 3, $\chi^-(G) \leq 3$. We show that $G$ does not admit a proper 2-orientation or is isomorphic to $Q_3$. Since $|E(G)| = 2n - 4$ by Lemma 7, if there exists a proper 2-orientation $\sigma$ of $G$, then the number of vertices of indegree 2 in $\sigma$ is at least $n - 4$. Since two vertices of indegree 2 in $\sigma$ cannot be adjacent in $G$, $|Y| \leq 4$. If $|Y| = 3$, then three vertices $x_1, x_2, x_3$ in $X$ must be adjacent to all vertices in $Y$, and hence there exists a complete bipartite graph $K_{3,3}$, contrary to the planarity of $G$. If $|Y| = 4$, let $Y = \{y_1, y_2, y_3, y_4\}$.

Case 1. Suppose that there exist two vertices $x_1, x_2 \in X$ such that each of them is adjacent to the same three vertices in $Y$, say $y_1, y_2, y_3$ (see Figure 1). We may assume that $y_4$ is in the face $f = x_1y_1x_2y_2$ by symmetry. Then the other vertices in $X$ must be in $f$ to be adjacent to at least 3 vertices in $Y$ by the planarity of $K_{2,3}$. So the degree of $y_3$ is exactly 2, which contradicts to $\delta(G) \geq 3$.

![Figure 1: The complete bipartite graph $K_{2,3}$.](image)
Case 2. Suppose that there exists no such pair of vertices in $X$. In this case one can see that $|X| = 4$ and that $G$ is isomorphic to $Q_3$. It is easy to show that $\chi^-(Q_3) = 2$ (see also [2, Proposition 2]).

**Proof of Theorem** Consider the bipartite planar graph (quadrangulation) $G$ defined as follows (see Figure 2). Let $V(G) = \{a_{ijk}\} \cup \{b_{ij}, c_{ij}\} \cup \{d_{ik}\} \cup \{p_i, q_i\} \cup \{s, t\}$, where $1 \leq i, j \leq 4$, $1 \leq k \leq 7$, and

$$E(G) = \{a_{ijk}b_{ij}, a_{ijk}c_{ij} | 1 \leq i, j \leq 4, 1 \leq k \leq 7\} \cup \{b_{ij}p_i, b_{ij}q_i, c_{ij}p_i, c_{ij}q_i | 1 \leq i, j \leq 4\} \cup \{d_{ik}p_i, d_{ik}q_i | 1 \leq i \leq 4, 1 \leq k \leq 7\} \cup \{p_is, p_it, q_is, q_it | 1 \leq i \leq 4\}.$$

Suppose that $\chi^-(G) \leq 3$ and let $\sigma$ be a proper 3-orientation of $G$. Thoughout the proof, an indegree means one in $\sigma$.

Step 1. Consider the graph induced by $\{a_{ijk}\} \cup \{b_{ij}, c_{ij}\}$ for fixed $i, j$. Since $d^+_{\sigma}(b_{ij}) \leq 3$ and $d^-_{\sigma}(c_{ij}) \leq 3$, there exists a vertex of indegree 2 in $\{a_{ijk}\}$. So $b_{ij}$ and $c_{ij}$ cannot be indegree 2 for all $1 \leq i, j \leq 4$.

Step 2. Consider the graph induced by $\{a_{ijk}\} \cup \{b_{ij}, c_{ij}\} \cup \{d_{ik}\} \cup \{p_i, q_i\}$ for a fixed $i$. If all vertices in $\{b_{ij}, c_{ij}\}$ have indegree at most 1, then at least one of the indegrees of $p_i$ and $q_i$ must be at least 4, a contradiction. By this fact and Step 1, there exists a vertex of indegree 3 in $\{b_{ij}, c_{ij}\}$. So $p_i$ and $q_i$ cannot be indegree 3 for all $1 \leq i \leq 4$. Moreover, since there exists a vertex of indegree 2 in $\{d_{ik}\}$ by the same argument in Step 1, $p_i$ and $q_i$ cannot be indegree 2 for all $1 \leq i \leq 4$.

Step 3. By Steps 1 and 2, all of $p_1, \ldots, p_4, q_1, \ldots, q_4$ have indegree at most 1. This leads to the fact that at least one of the indegrees of $s$ and $t$ must be at least 4, a contradiction.
Thus, $\chi(G) \geq 4$. It is easy to show that $\chi(G) \leq 4$. (An infinite family of such graphs are easy to construct; add some vertices $a_{118}, \ldots, a_{11k}$ of degree 2 to $G$ for example.)

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